On the generalized Einstein–Cartan action with fermions

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Abstract
From the freedom exhibited by the generalized Einstein action proposed in Dubois-Violette and Lagraa (2010 Lett. Math. Phys. 91 83–91), we show that we can construct the standard effective Einstein–Cartan action coupled to the fermionic matter without the usual current–current interaction and therefore an effective action which does not depend either on the Immirzi parameter or on the torsion. This establishes the equivalence between the Einstein–Cartan theory and the theory of general relativity minimally coupled to the fermionic matter.

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In the last few years, the ‘tetrad-connection’ form of the Einstein gravitational action modified by the Holst term [2] has been used as a starting point of the non-perturbative quantization of gravity [3–5] (and references therein). This modified action, which depends on a new dimensionless parameter known as the Barbero–Immirzi (BI) parameter [6, 7], does not modify the classical vacuum Einstein equations. But in the presence of minimal coupling of fermionic matter a nonvanishing torsion emerges leading to equations of motion which depend on this parameter [8–10]. This nonvanishing torsion appears in the effective action under the form of a current–current interaction term with a coupling constant determined by the BI parameter. Thus, even at the classical level the effects of this parameter must be observed. This leads to bring up the question about its physical origin which is still debated in many works where different interpretations are proposed.

In vacuum, a possible interpretation of the BI parameter as an analog of the \( \theta \) parameter that describes the different sectors associated with the topological structure of large gauge transformation in the Yang–Mills theory is proposed in [11]. In [12] this analogy is investigated within the framework of the Einstein–Cartan theory nonminimally coupled to the fermionic matter where it is shown that when the equation of motion of the connection is satisfied the sum of the Holst term and the nonminimal term of the fermionic sector reduces to a term where the BI parameter becomes a coefficient of the Nieh–Yan [13] topological invariant which is a total divergence not affecting the equations of motion and therefore does not modify the
usual second-order effective action of Einstein–Cartan coupled to the fermionic matter. This topological interpretation of the BI parameter is analyzed in [14, 15] where the topological term is played by the Nieh–Yan density. It is also shown in [16] that the BI parameter is not detectable in classical theory even after nonminimal coupling of fermions. Much more, the BI parameter is promoted to be a field interacting with gravity rather than a constant in [17–20], especially in [21] where it is shown that the chiral anomaly is reabsorbed by a redefinition of the BI field. More recently a different interpretation is proposed in [22] where, from the extension of the Plebanski theory, it is shown that the BI parameter is related to the cosmological constant.

In the above-mentioned works the incorporation of the nonminimal coupling of the fermionic matter into the Einstein–Cartan theory permits us to discard the effects of the BI parameter but not the current–current interaction term in the effective action. This makes the physical predictions of the Einstein–Cartan theory different from those of the theory of general relativity which does not contain spin–spin interaction induced by a nonvanishing torsion. Let us recall that, up to now, the experimental successes of the standard model of particle physics and of the theory of general relativity seem to be in favor of the minimal coupling procedure free of the torsion of the connection.

In a previous paper [1], the Einstein–Cartan action is generalized by adding an infinity of nontrivial local actions which lead to the classical vacuum Einstein equations. In this paper we show that we can deduce, from this generalized action, the usual second-order effective action of the Einstein–Cartan theory minimally coupled to fermionic matter without a current–current interaction term nor BI parameter-dependent and therefore an Einstein–Cartan theory minimally coupled to fermionic matter equivalent to the theory of general relativity with fermions where the torsion of the connection is, is assumed to vanish.

We start with the gravity action proposed in [1] which we couple minimally to the standard real Dirac Lagrangian

\[
S(e, \omega, \Psi) = \frac{1}{2\kappa} \int_M \left( \frac{1}{2} \epsilon_{IJKL} e^I \wedge e^J \wedge \Omega^{KL} + \frac{1}{\gamma} \Theta^I \wedge \Theta_I + F(\Theta^3) \right) - \frac{i}{2} \int_M *e_I \wedge (\bar{\Psi} \gamma^I D \Psi - c.c).
\]

Here \(M\) is the four-dimensional spacetime manifold, \(I, J, \ldots \in [0, 1, 2, 3]\) are internal Lorentz indices, \(\epsilon_{IJKL}\) are the components of the totally antisymmetric Levi-Civita symbol \(\epsilon_{0123} = 1 = -\epsilon_{0123}\) and \(e^I = e^I_\mu d^\mu\) is the co-tetrad 1-form valued in the vectorial representation space endowed with the flat metric \(\eta_{IJ} = \text{diag}(-1, 1, 1, 1)\). The metric \(\eta_{IJ}\) and its inverse \(\eta^{IJ}\) are used to lower and to lift the Lorentz indices and to determine the metric \(g_{\mu\nu} = e^I_\mu e^I_\nu \eta_{IJ}\) of the tangent space of the manifold \(M\). \(\Omega^{IJ} = d\omega^{IJ} + \omega^I_K \wedge \omega^K_J\) is the curvature 2-form associated with the connection 1-form \(\omega^I_J = \omega^I_J dx^k = \omega^I_J_k \wedge e_k\) valued in the Lorentz Lie algebra and \(k = 8\pi G\) where \(G\) is Newton’s gravitational constant in the unit \(c = 1\).

In the second term of (1),

\[
\Theta^I = D e^I = d e^I + \omega^I_K \wedge e^K = \Theta^I_{KLE} e^K \wedge e^L
\]

is the torsion 2-form and \(D\) is the exterior covariant derivative. From the identity

\[
\Theta^I \wedge \Theta_I = d(e^I \wedge D e_I) + e^I \wedge D^2 e_I = d(e^I \wedge D e_I) + e^I \wedge e^K \wedge \Omega_{IK},
\]

we see that this term is equivalent, up to the surface term which is the Nieh–Yan [13] topological invariant, to the one of Holst. Hence \(\gamma\) can be identified to the BI parameter assumed to be real in the following.
In the third term of the action \((1)\), \(F\) is a \(C^\infty\) real function of the scalar \(\Theta^2\) given by
\[\Theta^I \star \Theta_J = \Theta^2 \epsilon,\]
where \(\Theta^I = \frac{1}{4} \epsilon_{ijkl} e^I \wedge e^j \wedge e^K \wedge e^K = \epsilon_{ijkl} x = \ast 1\) is the volume form and \(\epsilon = \det(\epsilon)\) is the determinant of \(e^I\).

In the tetrad formalism, the dual map \(\ast\) acts as
\[\ast(e^I \wedge \cdots \wedge e^I) = \frac{1}{(4-q)!} e^I_{l_1 \cdots l_q} l_{r_1} \cdots l_{r_q} e^I_{r_1} \wedge \cdots \wedge e^I_{r_q}\]
which by linearity determines the action \(\ast\) on any differential form.

Finally in the fourth term, \(\gamma^I\) are Dirac matrices and the exterior covariant derivative \(D\) acts on the Dirac spinors \(\Psi\) as \(D\Psi = e^I\Psi_\gamma + \omega^I_\gamma\Psi\) where \(\omega^I_\gamma = \frac{1}{2}[\gamma_I, \gamma_J]\) are the generators of the Lie algebra of the Lorentz group in the Dirac spinorial representation. ‘\(\ast\) c.c’ indicates the complex conjugate of the preceding term.

Due to the signature of the metric \(\eta_{IJ} = \frac{1}{2}(\gamma_I \gamma_J + \gamma_J \gamma_I)\), the Dirac matrices satisfy the following properties: \((\gamma_I)^2 = -I\), \((\gamma_I)^2 = I\), \(\gamma_I \gamma_J = \gamma_J \gamma_I\), \(\gamma_I \gamma_J = -\sigma_{KL}\gamma_0\) leading to \(D\Psi = d\Psi - \sigma_{IJ} \gamma^I \Psi\) where \(\Psi = \Psi^i \gamma^0\) with \(\Psi^i\) is the Hermitian conjugation of the column \(\Psi\). These Dirac matrix properties lead to the relations
\[\ast = -i \epsilon_{MKNL} \gamma^N = \gamma^N\] and \([\gamma_M, \sigma_{KL}] = \eta_{MK} \gamma^L - \eta_{ML} \gamma^K\],
where \(\gamma_5 = i \gamma_0 \gamma_1 \gamma_2 \gamma_3\) satisfying \(\gamma_5^4 = \gamma_5\) and \((\gamma_5)^2 = I\).

With respect to an arbitrary variation of the connection, the principle of least action gives the following equation:
\[
\left(\left(\frac{1}{2} \epsilon_{ijkl} + \delta_{ijkl} \gamma^I\right) \Theta^J_{NM} + \frac{1}{2} F^J_{\ast \ast \ast \ast} \Theta^J_{PQ} - \frac{k}{4} \delta_{[MN]I} \gamma^I - \frac{k}{4} \epsilon_{MN} \gamma^I \right) e^M \wedge e^M \wedge e^J = 0,
\]
where \(\delta_{ij}^I = \frac{1}{2} \left(\delta^I_{ij} - \delta^I_{ij}\right) = \delta^I_{ij}\) and \(\epsilon_{MN} = \gamma_5\gamma^I\gamma^J\) is the axial fermionic current. \(F(x) = dF(x)/dx\) denotes the derivative of \(F(x)\) where \(x = \Theta^2\).

To solve this equation, we decompose the torsion components \(\Theta^{IJK}\) into three disjoint representations of the Lorentz group as
\[
\Theta^{IJK} = \epsilon^{IJK} \Lambda^I + \sigma^I \Lambda^J + \Lambda^K,\]
where \(\epsilon^{IJK}\) is completely antisymmetric in \(I, J, K, \Lambda^I\) is the trace component and \(\Lambda^K\) has a vanishing trace and vanishing completely antisymmetric projection, i.e. \(T^{IJK} + T^{IKJ} = 0\), \(\eta_J T^{IJK} = 0\) and \(T^{IJK} + T^{IKJ} + T^{KIJ} = 0\).

From the substitution of \((4)\) into equation \((3)\), we obtain
\[
\epsilon^{RKL} \left((1 - 2F^I) \Lambda^I + \frac{2}{\gamma} \Lambda^I\right) - \left(\eta^{RKL} (2 - F^I) \Lambda^I - \frac{1}{\gamma} A^I\right) - \eta^{RL} \left(2 - F^I\right) \Lambda^I - \frac{1}{\gamma} A^K\right) + T^{IJK} \left(1 + F^I \delta^I_{KL} \delta^M_{NJ} + \frac{1}{\gamma} \epsilon_{NM} \delta^I_{[IJ]} \right) = - \frac{k}{4} \epsilon^{RKL} \Lambda^P.
\]
Contracting this equation with \(\epsilon_{RKLJ}\) and then with \(\eta_{JK}\) we get
\[
\left(1 - 2F^I\right) \Lambda^I + \frac{2}{\gamma} \Lambda^I = - \frac{k}{4} \Lambda^J\]
and
\[
(2 - F^I) \Lambda^I - \frac{1}{\gamma} A^I = 0.
\]
respectively. Inserting (6) and (7) into (5), we get

\[ T^{INM} (1 + F') \delta^R_N \delta^L_M + \frac{1}{\gamma} T^{I[JJ]} = 0. \quad (8) \]

By combining (6) with (7) we obtain the irreducible components of the torsion in terms of the axial current as

\[ \Lambda^L = -\frac{k}{8} B F', \quad A^L = \frac{k(F' - 2)}{8 B} F' \]

defined when \( B = (F' - \frac{1}{2})(F' - 2) + \frac{1}{\gamma} \neq 0 \) from which follows that the function \( F' \) is such that the image \( \text{Im}(F') \) of \( F' \) is disjoint from the roots \( r_1 \) and \( r_2 \) of the equation \( B = 0 \).

By contracting (8) with \((1 + F') \delta^R_N \delta^L_M - \frac{1}{\gamma} \epsilon_{KL} T^R_{[RT]} \) we get

\[ T^{IKL} (1 + F')^2 + \frac{1}{\gamma^2} = 0 \]

leading to

\[ T^{IKL} = 0. \quad (10) \]

Recall that the parameter \( \gamma \) and the function \( F' \) are assumed to be real in this paper. One must note, however, that the first equation of (9) violates the parity transformation. The trace component \( \Lambda^L \) of the torsion has to be a proper vector but according to this equation it turns out to be proportional to an axial spinor current. As we show below, this apparent inconsistency, caused by the Holst term which is not invariant under the parity transformation, does not affect the effective action. In fact, in its absence which corresponds to the limiting case \( \gamma \rightarrow \infty \), we deduce from (10), (7) and (6) the equations

\[ T^{IKJ} = 0, \quad \Lambda^L = 0, \quad A^L = \frac{k}{8(F' - \frac{1}{2})} F', \]

which are invariant under the parity transformation and defined when the image \( \text{Im}(F') \) of \( F' \) is disjoint of the points \(-1, 2 \) and \( \frac{1}{2} \). This condition is the same as the one of the pure gravitational sector where the Einstein–Cartan theory becomes equivalent to the theory of general relativity [1].

In order to determine the contributions coming from the different irreducible components (4) of the torsion to the action (1), we have to solve the structure equation \( D e^I = de^I + \omega^I_J e^J = \Theta^I \) by splitting the connection \( \omega^I_J \) into two parts \( \omega^I_J = \tilde{\omega}^I_J + C^I_J \), where \( \tilde{\omega}^I_J \) is the uniquely defined torsion-free \( so(3,1) \) spin connection compatible with the tetrad \( D e^I = de^I + \tilde{\omega}^I_J e^J = 0 \), and \( C^I_J = C^I_J d x^J = C^I_J e^K \) is the contortion 1-form.

To get a metric connection, \( D e^I = 0 \), the contortion must satisfy \( C^I_J + C^J_I = 0 \). It is explicitly given in terms of torsion by

\[ C^I_{JK} = (\Theta^I_J + \Theta^I_K - \Theta^I_{JK}) \]

from which we deduce

\[ D e^I = \tilde{D} e^I + C^I_J e^J = C^I_J + e^J + \Theta^I_J e^J \land e^K = \Theta^I_J e^J \land e^K = \Theta^I. \]

In terms of the irreducible decomposition (4) of the torsion, the contortion (13) reads

\[ C^I_{KL} = -e^I_{KL} \partial_p A^p + 2\delta^I_L \Lambda^K - 2\eta_{KL} \Lambda^I + 2T^I_{[KL]} \]

and

\[ \Theta^I \land \Theta_J = \Theta^2 \epsilon = (-12A_I A_I + 12A^I \Lambda_I + 2T^{IJK} T_{JK}) \epsilon, \]

where we have used the dual map (2) and \( e^I \land e^J \land e^K \land e^L = -e^{IJKL} \epsilon \).
The relation between the curvature $\Omega^{IJ} = d\omega^{IJ} + \omega^I_N \wedge \omega^N_J$ associated with the connection $\omega^{IJ}$, the contortion $C^{IJ}$ and the curvature $\tilde{\Omega}^{IJ} = d\tilde{\omega}^{IJ} + \tilde{\omega}^I_N \wedge \tilde{\omega}^N_J$ associated with the torsion-free $so(1,3)$ spin connection $\tilde{\omega}^{IJ}(e)$ is obtained by acting the square of the exterior covariant derivative $D$ on the vectorial representation $V^I$ as

$$DDV^I = \Omega^{IJ}V_J = (\tilde{\Omega}^{IJ} + \tilde{\omega}^{IJ} + \frac{1}{2} \Theta^I \wedge \Theta_J + F(\Theta^2) \epsilon).$$

(16)

In terms of the torsion-free $so(1,3)$ spin connection and of the contortion, the action (1) reads

$$S(e, \omega, \Psi) = \frac{1}{2k} \int_M \frac{1}{2} \epsilon_{IJKL} e^I \wedge e^J \wedge \tilde{\Omega}^{KL} - \frac{i}{2} \int_M \ast e_I \wedge (\overline{\Psi} y^I \tilde{D} \Psi - c.c)
+ \frac{1}{2k} \int_M \left( \frac{1}{2} \epsilon_{IJKL} e^I \wedge e^J \wedge C^N_M \wedge C^N_L + \frac{1}{y} \Theta^I \wedge \Theta_J + F(\Theta^2) \epsilon \right)
+ \frac{1}{4} \int_M e^I \wedge e^J \wedge e^K \wedge C_{IK} J_K,$$

(17)

where the contribution of the term $\tilde{D}C^{IJ}$ of (16) is ignored since it reduces to a total derivative due to the fact that $\tilde{D}e^I = 0$. In (17), the first line of the action describes the torsion-free part. The second and the third line describe the coupling of the axial fermionic current with the non-propagating torsion.

The substitution of (4) and (14) into (17) gives the explicit form of the contributions coming from the irreducible components of the torsion:

$$S(e, \omega, \Psi) = \frac{1}{2k} \int_M \frac{1}{2} \epsilon_{IJKL} e^I \wedge e^J \wedge \tilde{\Omega}^{KL} - \frac{i}{2} \int_M \ast e_I \wedge (\overline{\Psi} y^I \tilde{D} \Psi - c.c)
+ \frac{1}{2k} \int_M \left( 24 A^I A^J + \frac{1}{2} \epsilon_{IJKL} T^N_{IJ} T^N_{KL} \right) \epsilon
+ \frac{1}{2k} \int_M \left( -12 A^I A^J + 12 A^I A^J + 2 T_{IJK} T_{IJK} \right) \epsilon
+ \frac{1}{2k} \int_M \left( -24 A^I L + 6 A^I A^J - 4 T_{IJK} T_{IJK} \right) \epsilon
+ \frac{3}{4} \int_M A^I J_I \epsilon.$$

(18)

It is easy to see that an arbitrary variation of (18) with respect to $A_I$, $A^I$ and $T_{IJK}$ gives the expressions of the irreducible components of the torsion in terms of the axial current (9) and (11).

Now, we are ready to discuss the action (18). We start by verifying the usual case $F = 0$ which, from (9), gives

$$L^I = -\frac{k}{8(1 + \gamma^2)} \Lambda^I, \quad A^L = -\frac{2k}{8(1 + \gamma^2)} \Lambda^L.$$

(19)

By substituting these values of torsion components and (11) into (18), we recover the second-order tetrad action of general relativity action coupled to the fermionic matter [10]:

$$S(e, \omega, \Psi) = \frac{1}{2k} \int_M \frac{1}{2} \epsilon_{IJKL} e^I \wedge e^J \wedge \tilde{\Omega}^{KL} - \frac{i}{2} \int_M \ast e_I \wedge (\overline{\Psi} y^I \tilde{D} \Psi - c.c)
- \frac{3k}{16} \frac{\gamma^2}{(1 + \gamma^2)} \int_M J_I \epsilon,$$

which exhibits the BI parameter dependence in front of the current–current coupling.
Note that although the first equation of (19) violates the parity transformation, it leads to an invariant current–current interaction term in the effective action.

In the absence of the Holst term, we can consider any dimension $d$ of the spacetime manifold $\mathcal{M}$ where the action (1) and (18) reads respectively

\[
S(e, \omega, \Psi) = \frac{1}{2k} \int_{\mathcal{M}} (e(e_K \wedge e_L) \wedge \Omega^{KL} + F(\Theta^2)\varepsilon) - \frac{1}{2} \int_{\mathcal{M}} \ast e_J \wedge (\overline{\Psi} \gamma^J \mathcal{D} \Psi - \text{c.c})
\]

and

\[
S(e, \omega, \Psi) = \frac{1}{2k} \int_{\mathcal{M}} (e(e_K \wedge e_L) \wedge \Omega^{KL} - \frac{i}{2} \int_{\mathcal{M}} \ast e_J \wedge (\overline{\Psi} \gamma^J \mathcal{D} \Psi - \text{c.c})
\]

where $A_{IJK}$ is the completely antisymmetric component of the irreducible decomposition (4) of the torsion and $\mathcal{J}^{IJK} = \iota \overline{\Psi} [\gamma^{I}, a^{IJK}], \Psi$ is the tensorial current which is also completely antisymmetric in $I, J, K$.

An arbitrary variation of (21) with respect to $T^{IJK}, \Lambda^I$ and $A_{IJK}$ gives

\[
T^{IJK} = 0, \quad \Lambda^I = 0, \quad A_{IJK} = \frac{(-d^I_k}{8(F' - \frac{1}{2})}\mathcal{J}^{IJK}
\]

defined if the function $F$ is such that the image $\text{Im}(F')$ of $F'$ is disjoint of the points $-1, d - 2$ and $\frac{1}{2}$. By inserting these values of the torsion components into (21), we get

\[
S(e, \omega, \Psi) = \frac{1}{2k} \int_{\mathcal{M}} (e(e_K \wedge e_L) \wedge \Omega^{KL} - \frac{i}{2} \int_{\mathcal{M}} \ast e_J \wedge (\overline{\Psi} \gamma^J \mathcal{D} \Psi - \text{c.c})
\]

where we have replaced in the last term of the action (21) the tensorial current by its expression in terms of $A_{IJK}, \mathcal{J}^{IJK} = (-d^I_k}{8(F' - \frac{1}{2})}A^{IJK}$. If we try to get an action without the current–current interaction, and then an equivalence between the Einstein–Cartan theory and the theory of general relativity, we have to cancel the second line of this action which consists to solve the equation

\[
F(x) = 2x F'(x) + \frac{x^2}{2} = 0
\]

where $x = 2A^{IJK}A_{IJK}$. The smooth solution $F(x) = \frac{x}{2}$ of this equation is not acceptable because it contradicts the condition $F' - \frac{1}{2} \neq 0$ required to have a well-defined relation between the torsion and the fermionic current given in equation (22) but if we accept real nonpolynomial solutions, the general solution\(^1\) of (23) is $F(x) = \frac{x}{2} + c \sqrt{x}$ where $c$ is any real scalar function which does not depend on the torsion. This solution is defined when $x = 2A^{IJK}A_{IJK}/0$.

This case, $F' - \frac{1}{2} = \frac{c}{2\sqrt{2A^{IJK}A_{IJK}}} \neq 0$, leads to a nonsingular value of the completely antisymmetric irreducible component of the torsion

\[
\frac{(-d^I_k}{k\sqrt{2A^{NML}A_{NML}}}A^{IJK} \mathcal{J}^{IJK},
\]

\(^1\) We thank the referee for suggesting this solution.
which reduces, for \( d = 4 \), to
\[
\frac{4c}{k\sqrt{-12A K A}} A^I = \mathcal{J}^I
\]
defined if the vector \( A^I \) is time-like, i.e. \( A^I A_I \langle 0 \).

Therefore, with this nonpolynomial choice of the function \( F \), the action (20) reads
\[
S(e, \omega, \Psi) = \frac{1}{2k} \int_M \left( *(e_K \wedge e_L) \wedge \Omega^{KL} + \left( \frac{\Theta^2}{2} + c\sqrt{\Theta^2} \right) e \right) \\
- \frac{i}{2} \int_M *e_I \wedge (\bar{\Psi} y^I \bar{\mathcal{D}} \Psi - c.c)
\]
leading to the effective action
\[
S_{eff}(e, \Psi) = \frac{1}{2k} \int_M \left( *(e_K \wedge e_L) \wedge \tilde{\Omega}^{KL} \right) - \frac{i}{2} \int_M *e_I \wedge (\bar{\Psi} y^I \bar{\mathcal{D}} \Psi - c.c)
\]
which is the standard second-order tetrad form of the Einstein–Cartan theory minimally coupled to the fermionic matter without the current–current term and which does not depend either on the torsion or on the arbitrary function \( c \).

Now, if we consider the Holst term which work in four-dimensional spacetime manifold \( \mathcal{M} \) only, the relations (9) show that for the regular value \( F' = 2 \), \( B = \frac{1}{\gamma^2} \neq 0 \), we get
\[
A^I = 0, \quad T^{IJK} = 0, \quad \Lambda^I = -\frac{k\gamma}{8} \mathcal{J}^I
\]
which make the action (18) more simple. In fact, from the solution \( F(\Theta^2) = 2\Theta^2 + C' = 24\Lambda^I A_I + C' \) of the equation \( F' = 2 \), the \( F \)-term of the action (1) writes \( F(\Theta^2)e = 2\Theta^2 \wedge *\Theta_I + C' e \) where the constant is proportional to the cosmological constant. For gravity without a cosmological constant, \( C' = 0 \), the action (1) reads
\[
S(e, \omega, \Psi) = \frac{1}{2k} \int_M \left( \frac{1}{2} e_{IJKL} e^I \wedge e^J \wedge \Omega^{KL} + \frac{1}{\gamma} \Theta^I \wedge \Theta_J + 2\Theta^I \wedge *\Theta_J \right) \\
- \frac{i}{2} \int_M *e_I \wedge (\bar{\Psi} y^I \bar{\mathcal{D}} \Psi - c.c).
\]

In spite of the fact that the nonvanishing torsion is given by relation (24) which violates the parity transformation, it does not appear in the action (18). In fact by inserting \( A^I = 0, \quad T^{IJK} = 0 \) and \( F(\Theta^2) = 2\Theta^2 = 24\Lambda^I A_I \) into (25), which consist in making this insertion into (18), we see that the Holst term and the last one which describes the coupling of the torsion with the current vanish while the third and the fourth line cancel each other out to reduce to the effective action
\[
S_{eff}(e, \Psi) = \frac{1}{4k} \int_M e_{IJKL} e^I \wedge e^J \wedge \tilde{\Omega}^{KL} - \frac{i}{2} \int_M *e_I \wedge (\bar{\Psi} y^I \bar{\mathcal{D}} \Psi - c.c),
\]
which is the standard second-order tetrad form of the Einstein–Cartan theory minimally coupled to the fermionic matter without the current–current term and which does not depend either on the torsion or on the BI parameter.

Note that although the effective action (26) does not depend of the BI parameter, the Holst term is necessary to get a well-defined relation between the trace component of the torsion and the fermionic current (24). In its absence, \( \gamma \rightarrow \infty \), \( F' = 2 \) is a singular point which makes the trace component undetermined with or without fermionic matter. This undetermined trace component does not appear to have any physical significance as we see it from (18); in fact if \( \gamma \rightarrow \infty \) and if we consider the solution \( F(\Theta^2) = 2\Theta^2 \) of the equation \( F' = 2 \) the trace
component disappears but an arbitrary variation with respect to $T^{IJK}$ and $A^I$ gives $T^{IJK} = 0$ and $A^I = \frac{k}{12} J^I$ leading to a current–current interaction term $\frac{k}{12} J^I J^J$ in the effective action.

We conclude this work by emphasizing that as opposite to the usual tetrad form of the gravitational action with fermions where a torsion emerges under the form of current–current interaction which is either $\gamma$-dependent in the Holst-modified Einstein–Cartan theory or $\gamma$-independent in the presence of nonminimally coupled fermionic matter, we have shown that if, in addition to the Holst-modified Einstein–Cartan theory, we add a torsion term of the form $\frac{1}{2} \int_M \Theta^I \wedge^* \Theta_J (25)$ we obtain an effective Einstein–Cartan theory minimally coupled to the fermionic matter (26) where neither the torsion, which induces the current–current interaction, nor the BI parameter takes place. Therefore, the generalized Einstein–Cartan action (25) is equivalent to the theory of the general relativity minimally coupled to the fermionic matter. Note that this equivalence is not based on a completely vanishing torsion; its irreducible trace component does not vanish (24). This leads to a spacetime manifold with a nonvanishing torsion depending on the BI parameter which does not induce physical effects because of its absence in the effective action (26).

We have also shown that if we add to the Einstein–Cartan action a nonpolynomial term in the torsion of the form $\frac{1}{2} \int_M (\frac{\Theta^I}{2} + c\sqrt{\Theta^I}) \epsilon$ we obtain an effective action equivalent to the theory of the general relativity minimally coupled to the fermionic matter. This equivalence is not based on a completely vanishing torsion, also its irreducible completely antisymmetric component does not vanish (22). In spite of the fact that it is related to the fermionic current, it is an inert quantity which does not affect the evolution equations. Note that we can see from (22) that the presence of the fermionic matter is necessary in this case because in its absence we cannot obtain a vanishing torsion which is a condition to have an equivalence in the pure gravitational sector.

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