Calculation of the anomalous exponents in the rapid-change model of passive scalar advection to order $\varepsilon^3$.

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The field theoretic renormalization group and operator product expansion are applied to the model of a passive scalar advected by the Gaussian velocity field with zero mean and correlation function $\propto \delta(t-t’)/k^{d+\varepsilon}$. Inertial-range anomalous exponents, identified with the critical dimensions of various scalar and tensor composite operators constructed of the scalar gradients, are calculated within the $\varepsilon$ expansion to order $\varepsilon^3$ (three-loop approximation), including the exponents in anisotropic sectors. The main goal of the paper is to give the complete derivation of this third-order result, and to present and explain in detail the corresponding calculational techniques. The character and convergence properties of the $\varepsilon$ expansion are discussed; the improved “inverse” $\varepsilon$ expansion is proposed and the comparison with the existing nonperturbative results is given.

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I. INTRODUCTION

The investigation of intermittency and anomalous scaling in fully developed turbulence remains essentially an open theoretical problem. Both the natural and numerical experiments suggest that the deviation from the classical Kolmogorov theory is even more strongly pronounced for a passively advected scalar field than for the velocity field itself; see, e.g., Ref. [3] and literature cited therein. At the same time, the problem of passive advection appears easier tractable theoretically: even simplified models describing the advection by a “synthetic” velocity field with a given Gaussian statistics reproduce many of the anomalous features of genuine turbulent heat or mass transport observed in experiments. Therefore, the problem of passive scalar advection, being of practical importance in itself, may also be viewed as a starting point in studying intermittency and anomalous scaling in the turbulence as a whole. Detailed review of the recent theoretical research on the passive scalar problem and the bibliography can be found in Ref. [3].

Most progress has been achieved for Kraichnan’s rapid-change model [4]: for the first time, the anomalous exponents have been derived on the basis of a microscopic model and within controlled approximations [5–8]. In that model, the advection of a passive scalar field $\theta(x) \equiv \theta(t, \mathbf{x})$ is described by the stochastic equation

$$\nabla_t \theta = v_0 \nabla^2 \theta + f, \quad \nabla_t \equiv \partial_t + v_i \partial_i,$$  \tag{1.1}

where $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, $v_0$ is the molecular diffusivity coefficient, $\nabla^2$ is the Laplace operator, $\mathbf{v}(x) \equiv \{v_i(x)\}$ is the transverse (owing to the incompressibility) velocity field, and $f \equiv f(x)$ is an artificial Gaussian scalar noise with zero mean and correlation function

$$\langle f(x)f(x') \rangle = \delta(t-t') C(\mathbf{r}/L), \quad \mathbf{r} = \mathbf{x} - \mathbf{x}'.$$  \tag{1.2}

The parameter $L$ is an integral scale related to the noise, and $C(\mathbf{r}/L)$ is some function finite as $L \to \infty$.

In the real problem, the field $\mathbf{v}(x)$ satisfies the Navier–Stokes equation. In the rapid-change model it obeys a Gaussian distribution with zero mean and correlation function

$$\langle v_i(x)v_j(x') \rangle = D_0 \delta(t-t') (2\pi)^{-d} \int d\mathbf{k} P_{ij}(\mathbf{k}) N_k \exp[i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')], \quad N_k = \Theta(k - m) k^{-d-\varepsilon},$$  \tag{1.3}

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ is the transverse projector, $k \equiv |\mathbf{k}|$, $D_0 > 0$ is an amplitude factor, $d$ is the dimensionality of the $\mathbf{x}$ space, $\Theta(\cdots)$ is the step function and $0 < \varepsilon < 2$ is a parameter with the real (“Kolmogorov”) value $\varepsilon = 4/3$.

The infrared (IR) regularization is provided by the cut-off in the integral (1.3) from below at $k = m$, where $m \equiv 1/\ell$ is the reciprocal of another integral scale $\ell$. The anomalous exponents are independent on the precise form of the IR regularization; the sharp cut-off is the most convenient choice from the calculational viewpoints. In what follows, we shall not distinguish the two IR scales, setting $L \sim \ell$. The relations

$$D_0/v_0 = g_0 = \Lambda^\varepsilon$$  \tag{1.4}
define the coupling constant \( g_0 \) (i.e., the formal expansion parameter in the ordinary perturbation theory) and the characteristic ultraviolet (UV) momentum scale \( \Lambda \).

The issue of interest is, in particular, the behavior of the equal-time structure functions

\[
S_n(r) = \left\langle \left[ \theta(t, x) - \theta(t, x') \right]^n \right\rangle
\]

in the inertial-convective range \( \Lambda \gg 1/r \gg m \).

In the isotropic model \((1.1)\)–\((1.3)\), the odd multipoint correlation functions of the scalar field vanish, while the even equal-time functions satisfy linear partial differential equations \(\epsilon \). The solution for the pair correlation function is obtained explicitly; it shows that the structure function \(S_2 \propto r^{-n}\) is finite at \(m = 0\). The higher-order correlators are not found explicitly, but their inertial-range behavior can be extracted from the analysis of the nontrivial zero modes of the corresponding differential operators in the limits \(\epsilon \to 0\), \(1/d \to 0\), or \(\epsilon \to 2\). It was shown that the even structure functions in the inertial-convective range exhibit anomalous scaling behavior:

\[
S_n(r) \propto D_0^{-n/2} r^{n(1-\epsilon/2)} (mr)^{\Delta_n}, \quad r = |x - x'| \tag{1.6}
\]

with negative anomalous exponents \(\Delta_n\), whose first terms of the expansion in \(\epsilon\) \[5\] and \(1/d\) \[6\] have the forms

\[
\Delta_n = -n(n-2)\epsilon/2(d+2) + O(\epsilon^2) = -n(n-2)\epsilon/2 + O(1/d^2). \tag{1.7}
\]

Another quantity of interest is the local dissipation rate of scalar fluctuations, \(E(x) = \nu_0 \partial_i \theta(x) \partial_i \theta(x)\). The equal-time correlation functions of its powers in the inertial range have the forms \[1\]–\[3\]:

\[
\langle E^n(x) E^p(x') \rangle \propto (\Lambda r)^{-\Delta_{2n} - \Delta_{2p}} (mr)^{\Delta_{2n+2p}} \tag{1.8}
\]

with \(\Lambda\) from Eq. \(\epsilon\) \[5\] and \(\Delta_n\) from \[6\] \[6\]. Relations of the form \[5\]–\[6\] are characteristic of the models with multifractal behavior \[4\]–\[6\].

In Ref. \[10\] and subsequent papers \[11\]–\[15\], the field theoretic renormalization group (RG) and operator product expansion (OPE) were applied to model \(\epsilon\) \[1\]–\[3\]. In the RG approach, the anomalous scaling for the structure functions and various pair correlators is established as a consequence of the existence in the corresponding operator product expansions of “dangerous” composite operators, whose negative critical dimensions are identified with the anomalous exponents \(\Delta_n\). This allows one to construct a systematic perturbation expansion for the anomalous exponents, analogous to the well-known \(\epsilon\) expansion in the RG theory of critical behavior.

The key role in the RG and OPE approach to model \(\epsilon\) \[1\]–\[3\] is played by the critical dimensions \(\Delta_{nl}\), associated with the irreducible tensor composite operators

\[
F_{nl} = \mathcal{I} \mathcal{R} \mathcal{P} \left[ \partial_i \theta \cdots \partial_i \theta (\partial_i \theta \partial_i \theta)^p \right],
\]

where \(l\) is the number of the free vector indices and \(n = l + 2p\) is the total number of the fields \(\theta\) entering the operator; the vector indices of the symbol \(F_{nl}\) are omitted. The symbol \(\mathcal{I} \mathcal{R} \mathcal{P}\) denotes the irreducible part, obtained by subtracting the appropriate expression involving the delta symbols, such that the resulting tensor is traceless with respect to any pair of indices. In particular, \(\mathcal{I} \mathcal{R} \mathcal{P} [\partial_i \partial_i \theta^p] = \partial_i \partial_i \theta^p - \delta_{ij} (\partial_j \theta \partial_j \theta) / d\) and so on.

The dimension \(\Delta_n = \Delta_{n0}\) of the scalar operator is nothing other than the anomalous exponent in Eq. \[\epsilon\] \[6\]; see Ref. \[10\]. The dimensions with \(l \neq 0\) come into play if the forcing \(\epsilon\) \[1\] becomes anisotropic: \(\Delta_{nl}\) corresponds to the leading zero-mode contribution to the \(l\)-th term of the Legendre decomposition for the function \(S_n\); see Ref. \[3\]. They can be systematically calculated as series in \(\epsilon\):

\[
\Delta_{nl} = \sum_{k=1}^{\infty} \Delta_{nl}^{(k)} \epsilon^k \tag{1.10}
\]

with the first-order coefficient \[13\]

\[
\Delta_{nl}^{(1)} = \frac{n(n-2)}{2(d+2)} + \frac{(d+1)l(d-l-2)}{2(d-1)(d+2)}. \tag{1.11}
\]

For \(l = 0\) this gives the result of \[4\], while for \(n = 3\) and \(l = 1, 3\) the results of Refs. \[6\] are recovered. The result \[1.11\] was rederived later in Refs. \[10\]–\[17\].

The coefficients \(\Delta_{n0}^{(2)}\) and \(\Delta_{n2}^{(2)}\) were obtained in Ref. \[10\] for any \(n\) and \(d\); the result for general \(l\) is presented in \[14\]. In particular, one has
\[ \Delta_{nl}^{(2)} = n(n-2)(0.00203n - 0.02976) - l^2(0.01732n + 0.01223) \]  
(1.12a)

for \(d = 2\) and

\[ \Delta_{nl}^{(2)} = n(n-2)(0.00203n - 0.00384) - l(l+1)(0.00710n - 0.00619) \]  
(1.12b)

for \(d = 3\) (analytical results are too cumbersome and will not be given here; see Refs. [10] for \(l = 0, 2\) and [14] for general \(l\)).

The \(O(\varepsilon^3)\) contribution to \(\Delta_{nl}\) was presented in Ref. [15]:

\[ \Delta_{nl}^{(3)} = n(n-2)(0.00454n^2 + 0.06486n + 0.06505) + l^2 \left(-0.01974n^2 - 0.10423n + 0.24094 + 0.01748l^2\right) \]  
(1.13a)

for \(d = 2\) and

\[ \Delta_{nl}^{(3)} = n(n-2)(0.00140n^2 + 0.01992n + 0.03437) + l(l+1)\left[-0.00420n^2 - 0.02421n + 0.00280l(l+1) + 0.05065\right] \]  
(1.13b)

for \(d = 3\). Here, we have presented the \(O(\varepsilon^3)\) results with improved accuracy in numerical coefficients and corrected a misprint in the expression for \(d = 3\) in Ref. [13]. No analytical formula for \(\Delta_{nl}^{(3)}\) is available for general \(d\), but the numerical result of the form (1.13) can be obtained for any given \(d\). The large \(d\) limit is discussed in Sec. [VI].

Besides the calculational efficiency, an important advantage of the RG approach is its universality: it can also be applied to the case of finite correlation time or non-Gaussian advecting field; see Refs. [13]. For passively advected fluid, the statistical models of fully developed turbulence and critical dynamics might be useful not only for the rapid-change model (1.1)–(1.3) and its descendants, but also in a wider context of the statistical theory of turbulence and the bibliography can be found in Refs. [21,22].

The main goal of this paper is to give the complete and detailed derivation of the third-order result (1.13) announced in the Rapid Communication [15], and to present and explain in detail the corresponding calculational technique. It might be useful not only for the rapid-change model (1.1)–(1.3) and its descendants, but also in a wider context of the statistical models of fully developed turbulence and critical dynamics.

Another scope of the paper is to discuss the nature and convergence properties of the \(\varepsilon\) expansion. The knowledge of the three terms allows one to obtain reasonable predictions for finite values of \(\varepsilon \sim 1\) and to compare them with the existing nonperturbative results: analytical and numerical solutions of the zero-mode equations [16] and numerical simulations [24,25].

The plan of the paper is as follows. In Sec. II we recall the field theoretic formulation of the model, diagrammatic technique, renormalization and RG equations. In Sec. III, we briefly discuss the OPE, renormalization of composite operators (1.3) and their relationship to the issue of anomalous scaling. Since the “ideology” of the RG and OPE approach to the model (1.1)–(1.3) is explained in Refs. [10,13] in detail, here we confine ourselves to only the necessary information. In Sec. IV, we present the general scheme of the calculation of the critical dimensions of the operators \(F_{nl}\). In Sec. V, the calculation in the one-loop and two-loop approximations is presented in great detail. Section VI is devoted to the three-loop calculation; some results for the relevant quantities are given in the Appendix. In Sec. VII we discuss the convergence of the \(\varepsilon\) expansion, the improved inverse \(\varepsilon\) expansion and comparison with existing nonperturbative results. The main ideas of the paper are briefly reviewed in the Conclusion.

II. FIELD THEORETIC FORMULATION OF THE MODEL, DIAGRAMMATIC TECHNIQUE, RENORMALIZATION, AND RG EQUATIONS

The stochastic problem (1.1)–(1.3) is equivalent to the field theoretic model of the set of three fields \(\Phi = \{\theta, \theta', \nu\}\) with action functional

\[ \mathcal{S}(\Phi) = \theta' D_\theta \theta'/2 + \theta' \left[-\partial_t + \nu_0 \partial^2 - (\nu \partial)\right] \theta - \nu D_\nu^{-1} \nu/2 \]  
(2.1)

The first four terms in (2.1) represent the Martin–Siggia–Rose-type action for the stochastic problem (1.1), (1.2) at fixed \(\nu\), and the last term represents the Gaussian averaging over \(\nu\). Here \(D_\theta\) and \(D_\nu\) are the correlators (1.2) and (1.3), respectively, the required integrations over \(x = (t, x)\) and summations over the vector indices are understood.

The model (2.1) corresponds to a standard Feynman diagrammatic technique with the bare propagators \(\nu, \theta, \theta'\) (the line \(\theta'\theta'\) is absent). In the diagrams, these propagators are represented by the lines:
\[\langle vv \rangle_0 = \ldots \ldots, \quad \langle \theta \theta \rangle_0 = \ldots \ldots, \quad \langle \theta \theta' \rangle_0 = \ldots \ldots, \quad (2.2)\]

the slashed end of a solid line corresponds to the field \(\theta'\), the end without a slash corresponds to \(\theta\). The triple vertex \(V(\Phi) = -\theta' v_j \partial_j \theta = \partial_j \theta' \cdot v_j \theta\) (the equality holds due to the integration over \(x\) in Eq. (2.1)) is represented as

\[
V(\Phi) = -\quad j,^\prime j,^\prime \quad = \quad j,^\prime .
\]

(2.3)

The dot with the index \(j\) on the solid line denotes the differentiation \(\partial/\partial x_j\) with respect to the argument \(x\) of the end of the line attached to the vertex; the index \(j\) of the derivative is contracted with the index of the end of the line \(vv\) attached to the vertex. Owing to the transversality of the \(vv\) line, the dot can be moved onto another line, as shown in Eq. (2.3). In the momentum representation the vertex (2.3) corresponds to the factor \(-ik_j\), or, equivalently, \(+iq_j\), where \(k\) and \(q\) are the momenta flowing into the vertex via the fields \(\theta'\) and \(\theta\), respectively. The sum of the three momenta flowing into the vertex via the fields \(\theta'\) and \(\theta\) is equal to zero.

The line \(vv\) in the diagrams corresponds to correlation function (1.3), and the lines \(\theta\theta'\) and \(\theta\theta\) in model (2.1) in the \(\omega, k\) representation correspond to the bare propagators

\[
\langle \theta \theta' \rangle_0 = (-i\omega + \epsilon_k)^{-1}, \quad \langle \theta \theta \rangle_0 = C(k)(\omega^2 + \epsilon_k^2)^{-1}, \quad \epsilon_k = \nu_0 k^2,
\]

where \(C(k)\) is the Fourier transform of the function \(C\) from Eq. (1.2) and \(\omega\) and \(k\) “flow via the line from the left to the right” if the standard form of the Fourier transform with respect to the coordinate and time differences is used.

In what follows, we shall work in the \(t, k\) representation, where the propagators (2.4) have the forms

\[
\langle \theta \theta' \rangle_0 = \Theta(t - t') \exp\{-\nu_0 |t - t'| \epsilon_k\}, \quad \langle \theta \theta \rangle_0 = \{C(k)/2\epsilon_k\} \exp\{-|t - t'| \epsilon_k\};
\]

(2.5)

in \(\langle \theta \theta' \rangle_0\), \(t\) is the time argument of \(\theta\) and \(t'\) is the argument of \(\theta'\).

The model (2.1) is logarithmic (the coupling constant \(g_0\) in Eq. (1.4) is dimensionless) at \(\varepsilon = 0\), and the UV divergences have the form of poles in \(\varepsilon\) in the correlation functions of the fields \(\theta, \theta'\). Superficial UV divergences, whose removal requires counterterms, are present only in the 1-irreducible function \((\theta \theta')_1^{ir}\), and the corresponding counterterm reduces to the form \(\theta' \partial^2 \theta\); see Ref. [10]. Thus for the complete elimination of the UV divergences it is sufficient to perform the multiplicative renormalization of the parameters \(\nu_0\) and \(g_0 = D_0/\nu_0\) with the only independent renormalization constant \(Z_v\):

\[
\nu_0 = \nu Z_v, \quad g_0 = g \mu^v Z_g, \quad Z_g = Z_v^{-1} \quad (D_0 = g_0\nu_0 = g \mu^v \nu).
\]

(2.6)

Here \(\mu\) is the renormalization mass in the minimal subtraction (MS) scheme, which we always use in what follows, \(g\) and \(\nu\) are renormalized analogs of the bare parameters \(g_0\) and \(\nu_0\), and \(Z = Z(g, \varepsilon, d)\) are the renormalization constants.

In the MS scheme they have the form “1 + only poles in \(\varepsilon\.”” The last relation in (2.6) results from the absence of renormalization of the contribution with \(D_0\) in (2.1).

The renormalized action is obtained from the functional (2.1) by the substitution (2.4) and has the form

\[
S_R(\Phi) = \theta' D_0 \theta'/2 + \theta' \left[-\partial_\nu + \nu Z_v \partial^2 - (v \theta)\right] \theta - v D_0^{-1} v/2,
\]

(2.7)

where the amplitude \(D_0 = g \mu^v \nu\) from Eq. (1.3) is expressed in renormalized parameters using Eqs. (2.6).

The exact response function \(G \equiv \langle \theta \theta' \rangle\) satisfies the standard Dyson equation, which in the \(\omega, k\) representation has the form

\[
G^{-1}(\omega, p) = -i\omega + \nu_0 p^2 + \Sigma_{\theta \theta}(\omega, p),
\]

(2.8)

where the self-energy operator \(\Sigma_{\theta \theta}\) in the diagrammatic notation (2.2), (2.3) is represented as follows:

\[
\Sigma_{\theta \theta} = \quad p \quad \quad \quad \quad p \,
\]

(2.9)

The multiloop diagrams, which could be added on the right-hand side of Eq. (2.9), contain effectively closed circuits of retarded propagators \(\langle \theta \theta' \rangle_0\) and therefore vanish; it is crucial here that the correlation function \(\langle vv \rangle\) in Eq. (1.3)
is proportional to the δ function in time. Therefore, the self-energy operator is given by the one-loop approximation exactly; it is independent of ω and has the form

\[
\Sigma_{\theta'\theta}(p) = -\frac{D_0 p^2}{2(2\pi)^d} \int dk \, P_{ij}(k) N_k = -\frac{D_0 (d-1) p^2}{2d(2\pi)^d} \int dk \, N_k \equiv -\frac{D_0 (d-1) p^2}{2d} U,
\]

(2.10)

where

\[
U \equiv (2\pi)^{-d} \int dk \, N_k = C_d \int_0^\infty dk \, \kappa^{-1-\varepsilon} = C_d m^{-\varepsilon} / \varepsilon,
\]

(2.11)

with \(N_k\) from Eq. (1.3). The quantity \(S_d = 2\pi^{d/2}/\Gamma(d/2)\) is the surface area of the unit sphere in \(d\)-dimensional space. In Eq. (2.10) we have used the standard convention \(\Theta(t-t') = 1/2\) at \(t = t'\) for the step function in the \(\theta'\) line and the relation \(\langle P_{ij}(k) \rangle = \delta_{ij} (d-1)/d\) for the angular averaging of the transverse projector in \(d\) dimensions.

From the Dyson equation and Eqs. (2.10) and (2.11) it follows that the exact response function \(\langle \theta \theta' \rangle\) obtained from its bare counterpart \(\langle \theta \theta' \rangle_0\) in Eqs. (2.4), (2.5) simply by the replacement

\[
\nu_0 \to \nu_{\text{eff}} \equiv \nu_0 + C_d (d-1) D_0 \varepsilon / 2d \varepsilon.
\]

(2.12)

The renormalization constant \(Z_{\nu}\) in Eq. (2.4) can be found exactly from the requirement that the “effective diffusivity” \(\nu_{\text{eff}}\) be UV finite in renormalized theory (2.7), i.e., have no poles in \(\varepsilon\) when expressed in renormalized variables (2.6). In the MS scheme this gives

\[
Z_{\nu} = 1 - u (d-1)/2d \varepsilon, \quad u \equiv g C_d, \quad C_d = S_d / (2\pi)^d,
\]

(2.13)

where we have changed to the more convenient coupling constant \(u\). In the renormalized variables, \(\nu_{\text{eff}}\) is given by the expression

\[
\nu_{\text{eff}} = \nu \left[ 1 + \frac{u(d-1)}{2d} \cdot \left( \frac{\mu}{m} \right)^\varepsilon - 1 \right],
\]

(2.14)

which is obviously finite at \(\varepsilon \to 0\).

The basic RG equation for a multiplicatively renormalizable quantity \(F = Z_F F_R\) (correlation function, composite operator etc) has the form

\[
[D_{\text{RG}} + \gamma_F] F_R = 0, \quad D_{\text{RG}} = D_\mu + \beta \partial_\mu - \gamma_\mu D_\nu.
\]

(2.15)

Here and below, \(D_x \equiv x \partial_x\) for any variable \(x\), and the RG functions (the \(\beta\) function and the anomalous dimensions \(\gamma\)) are defined as

\[
\beta \equiv \bar{D}_\mu u, \quad \gamma_F \equiv \bar{D}_\mu \ln Z_F = \beta \partial_\mu \ln Z_F \quad \text{for any } Z_F,
\]

(2.16)

where \(\bar{D}_\mu\) denotes the operation \(\mu \partial_\mu\) at fixed bare parameters \(g_0, \nu_0\). From the definitions, the last relation in (2.4) and exact expression (2.13), for the basic RG functions one obtains:

\[
\beta(u) = u [-\varepsilon + \gamma_\nu], \quad \gamma_\nu = u (d-1)/2d.
\]

(2.17)

From Eq. (2.17) it follows that the RG equations (2.13) possess an IR stable positive fixed point:

\[
u_* = 2d\varepsilon/(d-1), \quad \beta(u_*) = 0, \quad \beta'(u_*) = \varepsilon > 0.
\]

(2.18)

This fact implies that correlation functions of model (1.1)–(1.3) in the IR region \((\Delta r \gg 1, m r \sim 1)\) exhibit scaling behavior; the corresponding critical dimensions \(\Delta[F] = \Delta_F\) can be calculated as series in \(\varepsilon\). For the basic fields and quantities, including the composite operators \(\theta^n\), the dimensions are found exactly (1):

\[
\Delta_\omega = 2 - \varepsilon, \quad \Delta_m = 1, \quad \Delta_\theta = (-1 + \varepsilon / 2), \quad \Delta[\theta^n] = n \Delta_\theta, \quad \Delta[\theta^d] = d + 1 - \varepsilon / 2
\]

(2.19)

(no corrections of order \(\varepsilon^2\) and higher). This is a consequence of the exact equality \(\gamma_\nu(u_*) = \varepsilon\), which follows from Eqs. (2.17) and (2.18). In particular, for the structure functions (1.5) relations (2.19) along with dimensionality considerations give:

\[
S_n(r) = D_0^{-n/2} r^{n(1-\varepsilon/2)} \xi_n(m r),
\]

(2.20)

with some nontrivial dependence on the IR scale \(m\) contained in the scaling functions \(\xi_n(m r)\).
III. COMPOSITE OPERATORS, OPERATOR PRODUCT EXPANSION, AND ANOMALOUS SCALING

Representations of the form \( \langle \theta_c(t, x) - \theta_c(t, x') \rangle^n \) for any scaling functions \( \xi(mr) \) describe the behavior of the correlation functions for \( Ar \gg 1 \) and any fixed value of \( mr \). The inertial range corresponds to the additional condition \( mr \ll 1 \). The form of the functions \( \xi(mr) \) at \( mr \to 0 \) is studied using the operator product expansion (OPE).

According to the OPE, the behavior of the quantities entering into the right-hand side of Eq. (1.3) for \( r = x - x' \to 0 \) and fixed \( x + x' \) is given by the infinite sum

\[
\langle \theta_c(t, x) - \theta_c(t, x') \rangle^n = \sum_F C_F(r) F \left( t, \frac{x + x'}{2} \right),
\]

where \( C_F \) are coefficients regular in \( m^2 \) and \( F \) are all possible renormalized local composite operators allowed by the symmetry (more precisely, see below).

In what follows, it is assumed that the expansion (3.1) is made in irreducible tensors (scalars, vectors and traceless tensors); the possible tensor indices of the operators \( F \) are contracted with the corresponding indices of the coefficients \( C_F \). With no loss of generality, it can also be assumed that the expansion is made in “scaling” operators, i.e., those having definite critical dimensions \( \Delta_F \).

The structure functions (1.3) are obtained by averaging Eq. (1.1) with the weight exp \( S_R \), the mean values \( \langle F \rangle \) appear on the right-hand side. Their asymptotic behavior for \( m \to 0 \) is found from the corresponding RG equations (2.15) and has the form \( \langle F \rangle \propto m^{\Delta_F} \).

From the RG representation (2.20) and the operator product expansion (1.1) we therefore find the following expression for the structure functions in the inertial range \( (Ar \gg 1, mr \ll 1) \):

\[
S_n(r) = D_0^{-n/2} r^{n(1-\varepsilon)/2} \sum_F A_F(mr) (mr)^{\Delta_F},
\]

with coefficients \( A_F \) regular in \( m^2 \).

Some general remarks are now in order.

Owing to translational invariance, the operators having the form of total derivatives give no contribution to Eq. (2.2) \( \langle \partial F(x) \rangle = \partial \langle F(x) \rangle = \partial \times \text{const} = 0. \)

In model (1.1)–(1.3), the operators with an odd number of fields \( \theta \) also have vanishing mean values; their contributions vanish along with the odd structure functions themselves (they will be “activated” in the presence of a nonzero mixed correlation function \( \langle \theta f \rangle \) or an imposed gradient of the scalar field).

If the function \( C \) in Eq. (1.2) depends only on \( r = |r| \), the model becomes \( SO(d) \) covariant and only the contributions of scalar operators enter into Eq. (3.2). Indeed, in the isotropic case the tensor indices of the mean values \( \langle F \rangle \) of the operators \( F \) in Eq. (2.1) can only be those of Kronecker delta symbols. It is impossible, however, to construct nonzero irreducible (traceless) tensor solely of the delta symbols.

In the presence of anisotropy, irreducible tensor operators acquire nonzero mean values and their contributions appear on the right-hand side of Eq. (3.2). In the simplest case of the uniaxial anisotropy, specified by an unit vector \( n \) in the correlation function (1.2), the mean value of a \( l \) th rank traceless operator is necessarily proportional to the \( l \) th rank symmetrical traceless tensor built of the vector \( n \) along with the delta symbols; its contraction with the corresponding coefficient \( A_F \) gives rise to the \( l \) th order Legendre polynomial \( P_l(z) \) with \( z = (rn)/r \). In general, the expansion in irreducible tensors in (3.1) after the averaging leads to the decomposition in the irreducible representations of \( SO(d) \), the \( l \) th sector corresponds to the contribution of the \( l \) th rank tensor composite operators.

The leading term in the \( l \) th anisotropic sector is given by the \( l \) th rank tensor operator with minimal dimension \( \Delta[F] \). The feature typical to the models describing turbulence is the existence of composite operators with negative critical dimensions; their contributions in the OPE lead to singular behavior of the scaling functions at \( mr \to 0 \), that is, to the anomalous scaling. The operators with minimal \( \Delta_F \) are those involving the maximal possible number of fields \( \theta \) and the minimal possible number of derivatives (at least for small \( \varepsilon \)). Both the problem (1.1)–(1.3) and the quantities (1.5) possess the symmetry \( \theta \to \theta + \text{const} \). It then follows that the expansion (3.1) involves only operators invariant with respect to this shift and therefore built of the \( \theta \) of the operators entering into the right-hand side of Eq. (3.1) are those which appear in the Taylor expansion, and those that admit to them in renormalization. The leading term of the Taylor expansion for \( S_n \) is the \( n \) th rank operator which can symbolically be written as \( (\partial \theta)^n \); its decomposition in irreducible tensors gives rise to operators of lower ranks. In the presence of the noise (1.2), operators of the form \( (\partial \theta)^k \) with \( k < n \) admit to them in renormalization and also appear in the OPE. Owing to the linearity of problem (1.1), operators with \( k > n \) (whose contributions would be more important) do not admit to the terms of the Taylor expansion for \( S_n \) and do not appear in the corresponding OPE.
We thus conclude that the leading terms of the inertial-range behavior are related to the critical dimensions $\Delta_{nl}$ of the infinite family of irreducible tensor composite operators $F_{nl}$ introduced in Eq. (4.1).

In general, operators (1.9) mix in renormalization. One can show that the corresponding infinite renormalization matrix

$$F_{nl} = \sum_{n'l'} Z_{nl,n'l'} F^{R}_{n'l'}$$

(3.3)

is in fact block-triangular, i.e., $Z_{nl,n'l'} = 0$ for $n' > n$, and so are the matrix of anomalous dimensions $\gamma(u) = Z^{-1} \bar{D}_\mu Z$ and the matrix of critical dimensions $\Delta = n + n\Delta_\theta + \gamma(u_*)$. It is then obvious that the dimensions $\Delta_{nl}$, given by the eigenvalues of the latter matrix, are completely determined by the finite subblocks with $n' = n$. Therefore, we can neglect all the elements of the matrix (3.3) other than $Z_{nl,nl'}$. The latter are determined by the 1-irreducible correlation functions with one operator $F_{nl}$ and $n$ fields $\theta$. The diagrams for such functions do not involve the propagator $\langle \theta \theta \rangle_0$ from Eq. (2.4) and can therefore be calculated directly in the “unforced” model without the noise (2.2), that is, without the first terms in the action functionals (2.1), (2.7). In the absence of the forcing, the model becomes $SO(d)$ covariant, the irreducible operators with different values of $l$ cannot mix in renormalization, so that the blocks $Z_{nl,nl'}$ appear diagonal.

We thus conclude that the critical dimensions $\Delta_{nl}$ are determined by the diagonal elements $Z_{nl} \equiv Z_{nl,nl}$ of the matrix (3.3):

$$\Delta_{nl} = n + n\Delta_\theta + \gamma_{nl}(u_*) = n\varepsilon/2 + \gamma_{nl}(u_*)$$

(3.4)

with $\beta$ from (2.16), $u_*$ from (2.13) and $\Delta_\theta$ from (2.13). Owing to the renormalization, the critical dimension $\Delta_{nl}$ is not equal to the simple sum of critical dimensions $\sum_{l} = (-1 + \varepsilon/2)$, $\Delta_\theta = 1$ of the fields and derivatives constituting $F_{nl}$. The elements $Z_{nl}$ can be calculated in the model without forcing, in which the operators (1.9) are renormalized multiplicatively: $F_{nl} = Z_{nl} F^{R}_{nl}$.

IV. CALCULATION OF THE CRITICAL DIMENSIONS OF OPERATORS $F_{nl}$: GENERAL SCHEME

From now on, we shall consider composite operators (1.9) in the model without the noise, that is, with $D_\theta = 0$ in the action functional (2.1). They are renormalized multiplicatively, $F_{nl} = Z_{nl} F^{R}_{nl}$, and the renormalization constants $Z_{nl} = Z_{nl}(g, \varepsilon, d)$ are determined by the requirement that the 1-irreducible correlation function

$$\langle F^{R}_{nl}(x) \theta(x_1) \ldots \theta(x_n) \rangle_{1-ir} = Z_{nl}^{-1} \langle F_{nl}(x) \theta(x_1) \ldots \theta(x_n) \rangle_{1-ir} \equiv Z_{nl}^{-1} \Gamma_{nl}(x_1, \ldots, x_n)$$

(4.1)

be UV finite in renormalized theory (2.9), i.e., have no poles in $\varepsilon$ when expressed in renormalized variables (2.6). This is equivalent to the UV finiteness of the product $Z_{nl}^{-1} \Gamma_{nl}(x; \theta)$, in which

$$\Gamma_{nl}(x; \theta) = \frac{1}{n!} \int dx_1 \ldots \int dx_n \Gamma_{nl}(x; x_1, \ldots, x_n) \theta(x_1) \ldots \theta(x_n)$$

(4.2)

is a functional of the field $\theta(x)$. In the zeroth approximation, the functional (1.2) coincides with the operator $F_{nl}(x)$, and in higher orders the kernel $\Gamma_{nl}(x; x_1, \ldots, x_n)$ is given by the zeroth order of diagrams shown in Fig. 1. The analysis of the diagrams shows that for any argument $x_\ast$, the corresponding spatial derivative is isolated as an external factor from each diagram. Using the integration by parts, these derivatives can be moved onto the corresponding fields $\theta(x_\ast)$ in Eq. (4.2), so that the quantity (1.2) can be represented as a functional of the vector field $w_i \equiv \partial_i \theta$:

$$\Gamma_{nl}(x; \theta) = \frac{1}{n!} \int dx_1 \ldots \int dx_n \tilde{\Gamma}_{nl}^{i_1 \ldots i_n}(x; x_1, \ldots, x_n) w_{i_1}(x_1) \ldots w_{i_n}(x_n).$$

(4.3)

The diagrams that determine the kernel $\tilde{\Gamma}$ in Eq. (4.3) contain only logarithmic UV divergencies. Therefore, in order to find the constant $Z_{nl}^{-1}$ it is sufficient to calculate the functional $\tilde{\Gamma}$ with some special choice of its functional argument $w_i$, namely, one can replace it by its value at the fixed point $x$, the argument of the operator $F_{nl}$ in Eqs. (1.3). Now the product $w_{i_1}(x) \ldots w_{i_n}(x)$ can be taken outside the integrals over $x_1, \ldots, x_n$ in (4.3), so that the functional $\Gamma_{nl}(x; \theta)$ turns to a local composite operator. The integration of the remaining function $\Gamma_{nl}$ over $x_1, \ldots, x_n$ gives a quantity independent of any coordinate variables, and its vector indices can only be those of Kronecker delta symbols. Their contraction with the indices of the product $w_{i_1}(x) \ldots w_{i_n}(x)$ gives rise to the original operator $F_{nl}(x)$ with
some scalar coefficient $\bar{\Gamma}$. The integration over $x_1, \ldots, x_n$ means that in the Fourier representation, the corresponding correlation function is calculated with all its momenta set equal to zero, which is always implied in what follows.

Now we turn to the derivation of practical formulas for the calculation of the constants $Z_{nl}^{-1}$ from the diagrams. For the sake of brevity, we introduce the notation:

$$Z_{nl} = Z_F, \quad F_{nl}(x) = F, \quad \Gamma_{nl}(x; \theta) \equiv \Gamma = F\bar{\Gamma}.$$  \hspace{1cm} (4.4)

Then the UV finiteness of the quantity $Z_{nl}^{-1}\Gamma_{nl}(x; \theta)$ is expressed by the relation

$$\text{Div. p. } [Z_{nl}^{-1}\Gamma] = 0,$$  \hspace{1cm} (4.5)

where “Div. p.” is the operation which selects the UV divergent part; in the MS scheme, it selects only poles in $\varepsilon$.

Classifying the contributions in model (2.7) according to the powers of the renormalized coupling constant $u$ from Eq. (2.13) gives:

$$Z_{nl}^{-1} = 1 + \sum_{k=1}^{\infty} [Z_{nl}^{-1}]_k, \quad \Gamma = \sum_{k=0}^{\infty} \Gamma_k = F + F \sum_{k=1}^{\infty} \bar{\Gamma}_k,$$  \hspace{1cm} (4.6)

where $[Z_{nl}^{-1}]_k$, $\Gamma_k$ and $\bar{\Gamma}_k$ are the contributions of order $u^k$ in the respective quantities. We substitute expressions (4.6) into Eq. (4.5) and omit the overall factor $F$; this gives:

$$\text{Div. p. } \left[Z_{nl}^{-1} + \sum_{k=1}^{\infty} \bar{\Gamma}_k\right] = 0.$$  \hspace{1cm} (4.7)

We recall that in the MS scheme one has “$Z_{nl}^{-1} = 1$ + only poles in $\varepsilon$,” so that $\text{Div. p. } \{Z_{nl}^{-1}\} = Z_{nl}^{-1} - 1$. Substituting this equality into Eq. (4.5) gives the relation

$$Z_{nl}^{-1} = 1 - \text{Div. p. } \left[Z_{nl}^{-1} \sum_{k=1}^{\infty} \bar{\Gamma}_k\right],$$  \hspace{1cm} (4.8)

which allows for the recurrent calculation of the contributions $[Z_{nl}^{-1}]_k$ in the expansion (4.6) from the quantities $\bar{\Gamma}_k$. Indeed, selecting in Eq. (4.8) terms of the same order in $u$ gives:

$$[Z_{nl}^{-1}]_1 = -\text{Div. p. } \{\bar{\Gamma}_1\},$$  \hspace{1cm} (4.9a)

$$[Z_{nl}^{-1}]_2 = -\text{Div. p. } \{\bar{\Gamma}_2 + [Z_{nl}^{-1}]_1 \bar{\Gamma}_1\},$$  \hspace{1cm} (4.9b)

$$[Z_{nl}^{-1}]_3 = -\text{Div. p. } \{\bar{\Gamma}_3 + [Z_{nl}^{-1}]_1 \bar{\Gamma}_2 + [Z_{nl}^{-1}]_2 \bar{\Gamma}_1\},$$  \hspace{1cm} (4.9c)

and so on. The relations given in Eq. (4.9) are sufficient for the three-loop calculation.

\[
\Gamma^{(1)} = \frac{1}{2}, \quad \Gamma^{(2)} = \frac{1}{2} + \frac{1}{8},
\]

\[
\text{No 21} \quad \text{No 22} \quad \text{No 31} \quad \text{No 41}
\]
\[
\Gamma^{(3)} = \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{48} + \frac{1}{2} + \frac{1}{2} + 
\]

FIG. 1. Diagrammatic representation of the function \(\Gamma\) in the three-loop approximation.

In Fig. 1, we present, along with respective symmetry coefficients, all the diagrams needed for the three-loop calculation of the function \(\Gamma\), except for those with the self-energy insertions of the form (2.9) in the \(\theta\theta'\) lines. (The symmetry coefficients which are not shown are equal to unity.) The index \(l\) in \(\Gamma^{(l)}\) denotes the number of loops (that is, independent integration momenta): \(l = 1, 2, 3\). Below each diagram we give its number, which will always be used in the following to refer to a diagram. The numbers have the forms “No XY,” where \(X\) is the number of rays in the diagram and \(Y\) is the order number (from the left to the right in Fig. 1) of the diagram in the subset with given \(X\).

The thick dots in the diagrams correspond to the vertices of the composite operator \(F\) (more precisely, see below), all the horizontal dashed lines correspond to correlation functions \(\langle vv\rangle\) from Eq. (1.3) with \(D_0 = g\nu\mu^s\), and the “rays” correspond to chains of lines \(\langle \theta\theta'\rangle_0\langle \theta\theta'\rangle_0\ldots\) in the same order on each ray: the upper end of each line corresponds to the field \(\theta\) and the lower end corresponds to \(\theta'\). For this reason, in contrast with Eq. (2.9), in Fig. 1 we do not add slashes at the ends of the lines. If desired, they can easily be restored: a slash should be added in the lower end of each line belonging to a ray. The lowest “external lines” of the diagrams in Fig. 1 correspond to the factors \(\theta\) and \(\theta'\) (and not to the propagators). We also note that the diagrams in Fig. 1 do not involve the \(\theta\theta\) lines (even in the presence of the noise).

In Fig. 1, we omitted the diagrams which are topologically possible and would be needed for the general correlation function (1.3), but in our model with the delta function in time in Eq. (1.3) they vanish due to the presence of the closed circuits of retarded propagators \(\langle \theta\theta'\rangle_0\); cf. the remark in Sec. 1 below Eq. (2.9).

The contributions of the diagrams with the self-energy insertions (2.9) will automatically be included if the propagators \(\langle \theta\theta'\rangle_0\) in Fig. 1 are taken to be exact, that is, with \(\epsilon_k = \nu_{\text{eff}} k^2\) in Eqs. (2.4), (2.5). In the renormalized variables, \(\nu_{\text{eff}}\) is given by Eq. (2.14), the zero order approximation being \(\nu_{\text{eff}} = \nu\). It is easy to see that the parameter \(\nu\) from \(\epsilon_k = \nu k^2\) enters into the final answers for the diagrams as \(\nu^{-l}\), where \(l\) is the number of loops in the diagram. It is thus sufficient to calculate the diagrams without the self-energy insertions and with \(\epsilon_k = \nu k^2\) and then introduce the additional factor\n
\[
(\nu_{\text{eff}}/\nu)^{-l} = [1 + Q]^{-l}, \quad Q = \frac{u(d-1)(\mu/m)^s - 1}{2d\varepsilon}.
\]

for any \(l\)-loop diagram; see Eq. (2.14). We stress that the replacement \(\nu \to \nu_{\text{eff}}\) is not needed in the amplitude \(D_0 = g\nu\mu^s\) of the correlation function (1.3).

Expression (2.14) corresponds to the special choice of the function \(N_k\) in Eq. (1.3). If the specific form of the IR regularization is different (for example, the function \(N_k = (k^2 + m^2)^{-d/2-\varepsilon/2}\) was used in Refs. [5,10]), the relation (2.10) for \(\Sigma_{\theta\theta}\) remains valid, but the explicit form of the integral \(U\) in Eq. (2.11) changes. Then the quantity \(Q\) in Eq. (4.10) is given by the following general relation\n
\[
Q = -\left[\Sigma_{\theta\theta} - \text{Div. p.} \Sigma_{\theta\theta}\right]/\nu p^2,
\]

which recovers the expression (4.10) for \(N_k\) from Eq. (1.3).
Introduction of the additional factors \((4.10)\) to the diagrams of \(\Gamma^{(0)}\) in Fig. 1 with \(D_0 = g
u \mu^\varepsilon\) in \((1.3)\) and \(\epsilon_k = \nu k^2\) in \((2.3)\) for the quantities \(\Gamma_n\) in Eq. \((4.9)\) gives
\[
\Gamma_1 = \Gamma^{(1)}, \quad \Gamma_2 = \Gamma^{(2)} - Q\Gamma^{(1)}, \quad \Gamma_3 = \Gamma^{(3)} - 2Q\Gamma^{(2)} + Q^2\Gamma^{(1)}. \tag{4.12}
\]
Expanding the quantity \(Q\) from Eq. \((4.11)\) in \(\varepsilon\) gives rise to contributions with the logarithms \(\ln(\mu/m)\); similar contributions also come from the factors \((\mu/m)^\varepsilon\), which naturally arise in any \(l\)-loop diagram in \(\Gamma^{(l)}\). It is well known that the renormalization constants in the MS scheme are independent of any mass parameters, like \(m\) and \(\mu\) in the case at hand. This means that, after the substitution of relations \((4.12)\) into Eqs. \((4.9)\), all contributions with \(\ln(\mu/m)\) in \(Z_F^{-1}\) will cancel each other. Such cancellation provides a good possibility to control the absence of calculational errors.

In the two-loop calculation \([10]\) we have checked the cancellation of the contributions \(\varepsilon^{-1}\ln(\mu/m)\) in \([Z_F^{-1}]_2\). There, the function \(N_L\) was taken in the form different from \((1.3)\) (see above), the \(\varepsilon\) expansion of the corresponding quantity \(Q\) in Eq. \((4.11)\) contained constant terms along with powers of the logarithms \(\ln(\mu/m)\), so that it was necessary to take into account the contribution with \(Q\) in expression \((4.12)\) for \(\Gamma_2\).

Our present choice for \(N_k\) in Eq. \((4.3)\) is much more convenient, because the corresponding integral \((2.11)\) contains only pole in \(\varepsilon\), and the \(\varepsilon\) expansion of the corresponding quantity \(Q\) in Eq. \((4.11)\) contains only powers of \(\ln(\mu/m)\) with no constant contributions. Since we know in advance that all contributions with \(\ln(\mu/m)\) in \(Z_F^{-1}\) will cancel each other, we may simply set \(\mu = m\) in the calculation of this quantity. Then all the contributions with \(\ln(\mu/m)\) vanish, in Eq. \((4.11)\) we obtain \(Q = 0\) (no corrections from the self-energy insertions are needed), in Eq. \((4.12)\) we obtain \(\Gamma_l = \Gamma^{(l)}\), and the factors \((\mu/m)^\varepsilon\) in the \(l\)-loop diagrams turn to unity. In what follows, in the calculations of the diagrams in \(\Gamma^{(l)}\) we shall retain the factors \((\mu/m)^\varepsilon\) and replace them with unity only in the last step, that is, in the calculation of \(Z_F^{-1}\).

Let us turn to the vertex of the composite operator \(F\), denoted in Fig. 1 by thick dots on the top of the diagrams. According to the general rules of the universal diagrammatic technique (see, e.g., Ref. \([26]\)), for any composite operator \(F(x)\) built of the fields \(\theta\), the vertex with \(k \geq 0\) attached lines corresponds to the vertex factor
\[
V_k(x; x_1, \ldots, x_k) \equiv \delta^k F(x)/\delta\theta(x_1) \cdots \delta\theta(x_k). \tag{4.13}
\]
The arguments \(x_1 \ldots x_k\) of the quantity \((4.13)\) are contracted with the arguments of the upper ends of the lines \(\theta\theta'\) attached to the vertex. For our operators \((1.9)\), built solely of the gradients \(w_i(x) = \partial\theta(x)/\partial x_i\) at a single spacetime point \(x\), the factors \((4.13)\) contain the product \(\partial_i \delta(x - x_1) \cdots \partial_i \delta(x - x_k)\), and the integrations over \(x_1 \ldots x_k\) are easily performed: the derivatives move onto the upper ends of the corresponding lines \(\theta\theta'\) attached to the vertex (such derivatives we denote by dots on the lines), and their arguments \(x_1 \ldots x_k\) are substituted with \(x\). After the derivatives have been moved inside the diagram, the remaining vertex factor for the operator \(F(x)\) can be understood as an usual derivative:
\[
V_{i_1 \ldots i_k}(x) = \partial^k F(x)/\partial w_{i_1}(x) \cdots \partial w_{i_k}(x). \tag{4.14}
\]
In what follows, in order to simplify the notation we shall omit the argument \(x\) and use the numerical indices \(1, \ldots, k\) instead of \(i_1 \ldots i_k\). Then the vertex factor \((1.14)\) for a \(k\)-ray diagram takes on the form
\[
V_{12\ldots} = \partial_1 \partial_2 \cdots F \quad \text{with} \quad \partial_i \equiv \partial/\partial w_i. \tag{4.15}
\]

Diagrams Nos 41, 46, 51, 61 in Fig. 1 are “factorizable” in the sense that they can be reduced to the products of blocks with a lesser number of rays. All the other diagrams in Fig. 1 will be termed “normal.”

A. Scalarization of the diagrams

The contribution of a specific diagram into the functional \(\Gamma\) in \((1.3)\) for any composite operator \(F\), built of the gradients \(w_i = \partial\theta\), is represented in the form
\[
\Gamma = V_{12\ldots} F_{12\ldots}^{ab\ldots} w_aw_b\ldots, \tag{4.16}
\]
where \(V_{12\ldots}\) is the vertex factor \((1.15)\), \(F_{12\ldots}^{ab\ldots}\) is the “internal block” of the diagram with free indices, the product \(w_aw_b\ldots\) corresponds to external lines. The numerical indices \(1, 2, \ldots\) will always be understood as \(i_1, i_2, \ldots\), their number in Eq. \((1.16)\) equals to the number of the letter indices \(a, b, \ldots\) and is determined by the number of “rays,” that is, the number of lines that attach to the vertex of the operator. These lines are given by products of the
proponents \((\theta \theta')_0\) from Eq. (2.5) with \(\epsilon_k = \nu k^2\) (see above) and are connected by the lines \(vw\) from Eq. (1.3). As an example, we present the general form of a four-ray diagram \(\Gamma\):

\[
\begin{align*}
\text{Diagram:} \quad V_{1234} I^{abcd}_{1234} w_a w_b w_c w_d \ldots
\end{align*}
\]

The thick dot on the top represents the vertex of the composite operator, it corresponds to the vertex factor \(V_{1234}\) in Eq. (4.17). The dots on the lines denote the differential operation \(\partial_i \equiv \partial/\partial x_i\); their indices are shown explicitly on the diagram. The lower external lines of the diagram correspond to factors \(\theta\); the derivatives at the external vertices \(2,3\) denoted by dots with the indices \(a, b, c, d\), act on these fields and turn them into the product \(w_a w_b w_c w_d\) with \(w_i \equiv \partial \theta/\partial x_i\). After all these differentiations have been performed, all the external momenta in the diagrams are set to zero; the IR regularization is provided by the parameter \(m\) in the function \((\nu v)\) from Eq. (1.3).

The diagrams are calculated in the time-momentum representation. The integration momenta are assigned to all internal lines; the number of independent momenta is equal to the number of \(vw\) lines. The dots with the numerical indices \(1, 2, \ldots\) on the upper lines in Eq. (4.17) correspond to the vector factors \(\pm i k_s\), coming from the vertex of the composite operator. Here \(k\) is the integration momentum flowing via the line, with the index shown near the dot \((s = 1, 2, 3, 4)\); the coefficient equals \(-i\) if the momentum “flows into the dot” and \(+i\) if the momentum “flows out.” In general, similar factors are also present inside the diagram, that is, inside the shaded block in Eq. (4.17). Collecting all such factors \(\pm i\) from the whole diagram gives certain “sign factor” \(\pm 1\), which, of course, should be taken into account in the calculations.

Since the vertex factor \(\frac{1}{4!}\) and the product \(w_a w_b \ldots\) are symmetrical with respect to any permutations of their indices, the quantity \(I^{ab\ldots}_{123}\) in Eq. (4.16) is automatically symmetrized with respect to any permutations of the letter indices \(a, b, \ldots\) and the numerical indices \(1, 2, \ldots\). In what follows, such symmetrization will always be denoted by the symbol \(\text{Sym}\).

For any fixed number of rays \(k\), the quantity \(\text{Sym} I\) is represented as a linear combination

\[
\text{Sym} I = \sum_i B_i S_i \tag{4.18}
\]

of certain basis tensor structures \(S_i = (S_i)^{ab\ldots}_{123}\) with certain numerical coefficients \(B_i\). The structures for the \(k\)-ray diagrams with \(k = 2, 3, 4\) have the forms (there are two structures for \(k = 2, 3\) and three structures for \(k = 4\))

\[
\begin{align*}
&k = 2: \quad S_1 = \text{Sym} [\delta_{1a} \delta_{2b}], \quad S_2 = \text{Sym} [\delta_{12} \delta_{ab}]; \\
&k = 3: \quad S_1 = \text{Sym} [\delta_{1a} \delta_{2b} \delta_{3c}], \quad S_2 = \text{Sym} [\delta_{12} \delta_{ab} \delta_{3c}]; \\
&k = 4: \quad S_1 = \text{Sym} [\delta_{1a} \delta_{2b} \delta_{3c} \delta_{4d}], \quad S_2 = \text{Sym} [\delta_{12} \delta_{ab} \delta_{3c} \delta_{4d}], \quad S_3 = \text{Sym} [\delta_{12} \delta_{34} \delta_{ab} \delta_{cd}]. \tag{4.19}
\end{align*}
\]

This is sufficient for the three-loop calculation, because the diagrams with \(k = 5, 6\) in Fig. 1 factorize into products of the blocks with \(k = 2, 3\).

The quantities which will be directly calculated from the diagrams are not the coefficients \(B_i\) themselves, but the following scalar quantities related to them:

\[
A_i = \text{tr} \left[ (S_i)^{ab\ldots}_{123} \text{Sym} I^{ab\ldots}_{123} \right] = \text{tr} \left[ S_i \cdot \text{Sym} I \right]. \tag{4.20}
\]

Here and below, the symbol \(\text{tr}\) denotes the contraction with respect to all repeated indices, which will not be shown explicitly.

It is therefore necessary to express the coefficients \(B_i\) in Eq. (4.18) in terms of the quantities (4.20). This is easily done: substituting Eq. (4.18) into (4.20) gives

\[
A_i = \sum_k M_{ik} B_k, \quad \text{where} \quad M_{ik} \equiv \text{tr} \left[ S_i \cdot S_k \right]. \tag{4.21}
\]

In a compact notation, \(A = MB\) (matrix \(M\) acts onto vector \(B\) and gives vector \(A\)). The symmetrical matrix \(M\) defined in Eq. (4.21) is easily calculated for any given set of structures of the form (4.19); then the corresponding inverse matrix \(M^{-1}\) is found, and the desired expressions for \(B\) in terms of \(A\) follow from the relation \(B = M^{-1} A\).
Below we give the explicit expressions for matrix elements \( M_{ik} = M_{ki} \) for the structures (4.19):

\[
\begin{align*}
  &k = 2: & M_{11} = d(d+1)/2, & M_{12} = d, & M_{22} = d^2; \\
  &k = 3: & M_{11} = d(d+1)(d+2)/6, & M_{12} = d(d+2)/3, & M_{22} = d(d+2)^2/9; \\
  &k = 4: & M_{11} = d(d+1)(d+2)(d+3)/24, & M_{12} = d(d+2)(d+3)/12, & M_{13} = d(d+2)/3, \\
  & & M_{22} = d(d+2)(d^2 + 7d + 16)/72, & M_{23} = d(d+2)^2/9, & M_{33} = d^2(d+2)^2/9. \\
\end{align*}
\]

(4.22)

It is worth noting that in the calculation of quantities like \( M_{ik} \) in Eq. (4.21) or \( A_i \) in (4.20), it is sufficient to retain the symbol \( Sym \) only in one of the cofactors, because the second is symmetrized automatically and can be replaced with one of its terms, like the expressions in square brackets in Eq. (4.19).

Inverting the matrices \( M \) in Eq. (4.22) gives the following explicit expressions of the coefficients \( B \) in terms of \( A \):

\[
\begin{align*}
  &k = 2: & B_1 &= 2\alpha[dA_1 - A_2], & B_2 &= \alpha[-2A_1 + (d+1)A_2] \quad \text{with} \quad \alpha \equiv [(d-1)d(d+2)]^{-1}, \\
  &k = 3: & B_1 &= 6\alpha[(d+2)A_1 - 3A_2], & B_2 &= 9\alpha[-2A_1 + (d+1)A_2] \quad \text{with} \quad \alpha \equiv [(d-1)d(d+2)(d+4)]^{-1}, \\
  &k = 4: & B_1 &= 24\alpha[(d+2)(d+4)A_1 - 6(d+2)A_2 + 3A_3], & B_2 &= 72\alpha[-2(d+2)A_1 + (d^2 + 3d + 6)A_2 - (d+3)A_3], \\
  & & B_3 &= 9\alpha[8A_1 - 8(d+3)A_2 + (d+3)(d+5)A_3] \quad \text{with} \quad \alpha \equiv [(d-1)d(d+1)(d+2)(d+4)(d+6)]^{-1}. \\
\end{align*}
\]

(4.23a,b,c)

It should be emphasized that the above relations (4.22) and (4.23) are independent of the specific choice of a composite operator built of the gradients \( w_i = \partial_i \theta \), which only determines the explicit form of the vertex factor (4.15).

B. Contractions of basic tensor structures

Consider the procedure of the contraction of the quantities \( I_{ab...}^{\alpha\beta...} \) in Eq. (4.16) with external factors: the vertex factor \( V_{12...} \) of the composite operator and the product \( w_a w_b \ldots \). General rules given below are valid for any local monomial \( F \) built of the gradients \( w_i = \partial_i \theta \). The operator \( F \) may have vector indices; as a rule, they will not be shown explicitly.

Substituting the decomposition (4.18) into Eq. (4.16) gives:

\[
\Gamma = \sum_i B_i V_{12...} (S_i)^{ab...}_{12...} w_a w_b \ldots = \sum_i B_i \Gamma_i, 
\]

(4.24)

where \( \Gamma_i \) are the contractions of the quantities (4.19) with the external factors:

\[
\Gamma_i = V_{12...} (S_i)^{ab...}_{12...} w_a w_b \ldots .
\]

(4.25)

Consider first the contractions

\[
(T_i)^{12...}_{12...} = (S_i)^{ab...}_{12...} w_a w_b \ldots
\]

(4.26)

of the structures \( S_i \) with the external factors \( w \). For quantities (4.19), they are easily calculated:

\[
\begin{align*}
  &k = 2: & (T_1)^{12}_{12} = w_1 w_2, & (T_2)^{12}_{12} = w^2 \delta_{12}; \\
  &k = 3: & (T_1)^{123}_{123} = w_1 w_2 w_3, & (T_2)^{123}_{123} = w^2 Sym[\delta_{12} w_3]; \\
  &k = 4: & (T_1)^{1234}_{1234} = w_1 w_2 w_3 w_4, & (T_2)^{1234}_{1234} = w^2 Sym[\delta_{12} w_3 w_4], & (T_3)^{1234}_{1234} = w^4 Sym[\delta_{12} \delta_{34}], \\
\end{align*}
\]

(4.27)

where the symbol \( Sym \) denotes the symmetrization with respect to the numerical indices.

From Eqs. (4.25) and (4.26) one obtains:

\[
\Gamma_i = V_{12...} (T_i)^{12...}_{12...},
\]

(4.28)

where the symbol \( Sym \) in the quantities (4.27) can be omitted owing to the fact that the vertices \( V_{12...} \) are symmetrical.
For a general $k$-ray diagram we need to calculate contractions of the vertex factors $V_{1...k}$ with the structures $T_{i...k}$ of the form (4.27). Their basis is provided by the set

$$w_1 \ldots w_k, \quad w^2 \delta_{12}w_3 \ldots w_k, \quad w^4 \delta_{12}\delta_{34}w_5 \ldots w_k, \quad \text{and so on.} \quad (4.29)$$

Consider first the contraction of the vertex factor (1.19) with the first monomial in Eq. (4.29), that is, the expression $L_k F$ with the operator

$$L_k \equiv w_1 \ldots w_k \partial_1 \ldots \partial_k, \quad \partial_i \equiv \partial / \partial w_i. \quad (4.30)$$

The key role in the calculation of the quantity $L_k F$ is played by the following consideration: by permutations of the factors $w$ and $\partial / \partial w_i$, the operator $L_k$ in Eq. (4.30) can be represented in the form of a polynomial in the operation $D \equiv w_i \partial / \partial w_i$. The action of the latter onto any monomial $F$ is easily found, namely, $DF = nF,$ where $n$ is the total number of the fields $w$ in the monomial $F$. By permutations of the factors $w$ and $\partial / \partial w_i$, one can easily obtain the recurrent relation $L_{k+1} = L_k(D - k)$ for the operator $L_k$, which along with the relation $L_1 = D$ gives

$$L_k = D(D - 1)(D - 2)\ldots(D - k + 1), \quad D \equiv w_i \partial / \partial w_i. \quad (4.31)$$

When this operation acts onto $F$, each symbol $D$ is replaced with the number $n$, which gives the desired coefficient:

$$L_k F = n(n - 1)(n - 2)\ldots(n - k + 1)F, \quad (4.32)$$

where $n$ is the total number of the factors $w$ in the operator $F$.

Consider now the contraction of the vertex factor (1.17) with the second structure in Eq. (4.29). The latter includes the factor $\delta_{12}$, which after the contraction with $\partial_1 \partial_2$ gives the Laplace operator $\partial^2 \equiv \partial \partial_i$ with $\partial_i \equiv \partial / \partial w_i$. This operator commutes with the other derivatives $\partial_i$ and acts directly onto $F$. Therefore the first task is the calculation of the quantity $\partial^2 F$, which is easily done for any specific monomial $F$ built of the gradients $w_i = \partial_i \theta$.

From now on, we shall confine ourselves with the family of operators $F_{nl}$ from Eq. (1.9), which in the new notation take on the form

$$F_{nl} = \mathcal{I}RP [w_1 \ldots w_l w^{2p}], \quad n = l + 2p. \quad (4.33)$$

The quantities $\partial^2 F$ for such operators are easily calculated:

$$\partial^2 F_{nl} = \lambda_{nl} F_{n-2,l}, \quad \lambda_{nl} = (n - l)(d + n + l - 2). \quad (4.34)$$

It is worth noting that in the calculation of the quantity $\partial^2 F$, there is no need to take into account contributions in which the both derivatives act onto the monomial $w_1 \ldots w_l$ in Eq. (4.33): in this case, two factors $w$ are necessarily replaced with the symbol $\delta_{is}$ (the number of free indices is preserved, and the number of factors $w$ is reduced by 2 under the action of $\partial^2$), and any contribution with $\delta_{is}$ disappears under the action of the operation $\mathcal{I}RP$ in Eq. (4.33).

It is clear that the combination of the relations (4.32) and (4.34) gives a complete solution to the problem of the calculation of the contractions of vertex factor (1.17) with basis structures of the form (4.29) for our specific operators (1.33), and that the above considerations can be directly generalized to the case of any polynomial in $w$ operator $F$. If the basis structure includes several delta symbols (e.g., two delta symbols in the third monomial in Eq. (4.29)), the operation $\partial^2$ should be applied repeatedly; one should also remember that each new Laplacian acts onto the operator which has already been modified (replacement $n \rightarrow n - 2$ in Eq. (4.34)). After all the operations $\partial^2$ have been applied, only the operation $L_k$ (4.30) with the reduced number of factors remains; it acts onto the properly modified operator $F$ according to the rule (4.32). For our operators $F_{nl}$, additional scalar factors $w^2, w^4, \ldots$ in monomials (4.29) restore the original scalar factor $w^{2p}$ in Eq. (4.33). Therefore, all the quantities $\Gamma_i$ in Eq. (1.28) are proportional to the original operator (1.3): $\Gamma_i = k_i F$, with some numerical coefficients $k_i$. They are easily found using the rules discussed above. In particular, for the 2-ray diagrams $\Gamma_1 = V_{12} w_1 w_2 = L_2 F = n(n - 1)F$, $\Gamma_2 = V_{12} w^2 \delta_{12} = w^2 \partial^2 F = \lambda_{nl} F$, and similarly for the 3-ray and 4-ray diagrams.

Substituting the relation $\Gamma_i = k_i F$ into Eq. (1.24) gives:

$$\Gamma = F \Gamma, \quad \Gamma = \sum_i k_i B_i, \quad (4.35)$$

where $k_i$ are the coefficients in front of $F$ in the contractions of the vertex factor $V_{1...k}$ with structures (4.29), numbered in the same order ($k_1$ corresponds to the structure without delta symbols, $k_2$ corresponds to the structure with one delta symbol, and so on):
The nontrivial parts from Eq. (4.16) is represented as follows:

\[ \nu' = (n - 2)(n - 3) \lambda_n, \quad \nu = (n - 4) \lambda_n, \quad k = \lambda_n \lambda_n; \tag{4.36a} \]

\[ k = 3: \quad k_1 = n(n - 1)(n - 2), \quad k_2 = (n - 2) \lambda_n; \tag{4.36b} \]

\[ k = 4: \quad k_1 = n(n - 1)(n - 2)(n - 3), \quad k_2 = (n - 2)(n - 3) \lambda_n, \quad k_3 = \lambda_{n-1} \lambda_n; \tag{4.36c} \]

\[ k = 5: \quad k_1 = n(n - 1)(n - 2)(n - 3)(n - 4), \quad k_2 = (n - 2)(n - 3)(n - 4) \lambda_n, \quad k_3 = (n - 4) \lambda_{n-2} \lambda_n; \tag{4.36d} \]

\[ k = 6: \quad k_1 = n(n - 1)(n - 2)(n - 3)(n - 4)(n - 5), \quad k_2 = (n - 2)(n - 3)(n - 4)(n - 5) \lambda_n, \]

\[ k_3 = (n - 4)(n - 5) \lambda_n; \tag{4.36e} \]

The coefficients \( k_i \) with \( k = 5 \) and \( 6 \) will be needed only for the calculation of the factorizable diagrams Nos 51, 61 with five and six rays, respectively.

The quantities \( A_i \) are calculated directly from the diagrams, then one calculates the quantities \( B_i \) using the relations (4.23), and then, finally, one calculates the desired contribution of the diagram \( \Gamma \) using the relations (4.37) and (4.36).

C. Calculation of the quantities \( A_i \)

The quantities \( I_{12...}^{ab...} \) for the diagrams \( \Gamma \) are calculated at zero external momenta; the independent integration momenta will always be denoted by \( k \) in the one-loop diagrams, \( k, q \) in all two-loop diagrams, and \( k, q, l \) in all three-loop diagrams. They are always assigned to the horizontal lines \( vv \); in the normal (not factorizable) diagrams the order is the following: \( k \) flows via the uppermost line, \( q \) flows via the next line, and \( l \) flows via the lowest line.

These rules are illustrated by Fig. 2, where the one-loop and two-loop diagrams are presented with all their momenta; the directions of the latter are shown by arrows. We denote all the propagators by solid lines, because the horizontal \( vv \) lines cannot be confused with the “rays” built of the \( \theta \theta' \) lines. The letter indices \( a, b, ... \) of the \( vv \) lines are always free (they will be contracted with the indices of the external factors \( w_a w_b, ... \)), while the numerical indices of the \( vv \) lines are always contracted with the indices of derivatives (that is, the momentum factors \( \pm i p_a \)), shown by dots on the \( \theta \theta' \) lines.

![Fig. 2. Diagrams of the function \( \Gamma \) in the one-loop and two-loop approximations in the detailed notation.](image)

We shall calculate the diagrams in the time–momentum \((t-k)\) representation. First, the elementary integrations over all times are performed. This gives certain “energy denominator” \( \phi_E \), that is, the factor \( \phi_E^{-1} \) in the integrand. The remaining integrations are those over \( d \)-dimensional momenta \((k \text{ in one-loop diagrams, } k, q \text{ in two-loop diagrams, and } k, q, l \text{ in three-loop diagrams})\). Each integration is accompanied by the factor \( D_0(2\pi)^{-d} \) with \( D_0 = g\mu^2 \), and all \( \nu 's \) will later be cancelled out with analogous factors in the energy denominators. As a result, the quantity \( I_{12...}^{ab...} \) from Eq. (4.16) is represented as follows:

\[ I_{12...}^{ab...} = \left[ D_0(2\pi)^{-d} \right] \int dk N_k \phi_{12...}^{ab...}; \tag{4.37a} \]

\[ 2 \text{ loops: } I_{12...}^{ab...} = \left[ D_0(2\pi)^{-d} \right]^2 \int dk dq N_k N_q \phi_{12...}^{ab...}; \tag{4.37b} \]

\[ 3 \text{ loops: } I_{12...}^{ab...} = \left[ D_0(2\pi)^{-d} \right]^3 \int \int \int dk dq dl N_k N_q N_l \phi_{12...}^{ab...}; \tag{4.37c} \]

with \( N_k \) from Eq. (4.3). After these scalar factors have been isolated, only the transverse projectors \( P \) remain in the \( vv \) lines. The nontrivial parts \( \phi_{12...}^{ab...} \) of the integrands in Eq. (4.37) are represented as follows:

\[ \phi_{12...}^{ab...} = C \cdot \phi_E^{-1} \cdot \phi_k \cdot I_{12...}^{ab...}; \tag{4.38} \]
where $C$ is a numerical factor, $\phi_E$ is the energy denominator, $\phi_s$ is an additional “scalar factor,” which can arise due to contractions over the internal indices, and $I^n_{12...}$ is the nontrivial “index structure” of the integrand, which remains after $\phi_s$ has been isolated.

Let us explain in more detail the calculation of the factors entering into Eq. (4.38), starting with the energy factors $\phi_E^{-1}$. Since the velocity correlation function (1.3) contains the delta function in time, the dots on the rays connected by such correlators have coincident times $t_i$. If we assign the time $t = 0$ to the uppermost vertex of a diagram (that is, the vertex of the composite operator), the independent variables will be the times $t_i < 0$ of the horizontal $vv$ lines. On any of the rays, the times $t_i$ decrease from above to below owing to the retardation property of the functions $(\theta \theta')^0$ in Eq. (2.3). For the normal (not factorizable) one and two-loop diagrams in Fig. 1, the order of the times $t_i < 0$ is determined by the form of the diagram in a unique way, that is, there is only one “temporal version.” This is also true for all the three-loop diagrams with two and three rays. However, for two of the three-loop diagrams (Nos 43, 44 in Fig. 1) there are two equivalent temporal versions (the exchange $t_2 \leftrightarrow t_3$ in diagram No 43 and $t_1 \leftrightarrow t_2$ in diagram No 44). This gives the additional factor 2 in the coefficient $C$ in Eq. (4.38) for these two diagrams.

The energy denominator $\phi_E$, obtained as the result of the integrations over all time variables, is given by the product of the factors corresponding to the “temporal cross-sections” of the diagrams between the horizontal $vv$ lines; to each cross-section corresponds the sum of “energies” $\epsilon_p = \nu p^2$ for all integrated $\theta \theta'$ lines. For example, one has $\phi_E = 2k$ for diagram No 21 in Fig. 2, $\phi_E = 2\epsilon_q \cdot 2k + q$ for diagram No 22, and $\phi_E = 2\epsilon_q [\epsilon_k + \epsilon_a + \epsilon_k + q]$ for diagram No 31.

The scalar factors $\phi_s$ in Eq. (4.38) are generated by the $vv$ lines whose vector indices are contracted with the indices of two integration momenta in the diagram. This gives rise to expressions of the form $(p P p') = p_i P_{ij} p'_j$, where $P$ is the projector that corresponds to the $vv$ line and $p, p'$ are the momenta contracted to the line (the factors $\pm i$ are not included). In what follows, we denote as $P_k$, $P_a$ and $P_l$ the transverse projectors for the momenta $k, q$ and $l$, respectively. Then the scalar factors from the diagrams in Fig. 2 take on the form $\phi_s = (q k q)$ for No 22 and $\phi_s = 1$ for Nos 21, 31, because the latter two diagrams do not contain lines $vv$ contracted to two momenta.

The numerical coefficient $C$ in Eq. (4.38) is represented as the product $C = C_1 C_2 C_3$, where $C_1$ is the symmetry coefficient of the diagram, shown in Fig. 1, $C_2 = \pm 1$ is the sign factor (see Sec. IV A) and $C_3 = 1$ or 2 is the number of the temporal versions. For the most of the diagrams one has $C_2 = +1$, but there are two diagrams with $C_2 = -1$ (Nos 44, 45). The factor $C_3$ is equal to 1 except for the diagrams Nos 43, 44 with $C_3 = 2$. In what follows, we shall always present the coefficient $C$ as the product $C = C_1 C_2 C_3$, for example, $C = 1/2 \cdot 1 \cdot 1$ for diagrams Nos 21, 22 in Fig. 2 and $C = 1 \cdot 1 \cdot 1$ for Nos 31.

Now let us turn to the quantities $I^n_{12...}$ in Eq. (4.38), which remain in the integrands after the scalar factors $\phi_s$ have been isolated. They have the same structure for all the diagrams with two and three rays and any number of loops, and two possible structures for the four-ray diagrams, namely:

\[
\begin{align*}
    k = 2: & \quad n1 = I_{12}^{ab} = x_1 x_2 P_{ab}; \\
    k = 3: & \quad n2 = I_{123}^{abc} = x_1 (x + y) y_3 a b P_{bc}; \\
    k = 4: & \quad n3 = I_{1234}^{abcd} = x_1 y_2 z_3 (x + y + z) a b c d P_{cd}; \\
    k = 4: & \quad n4 = I_{1234}^{abcd} = x_1 y_2 z_3 (x + y + z) a b c d P_{cd}.
\end{align*}
\]

(4.39)

We have introduced the numbering $n1-n4$ and in what follows, when describing the diagrams we shall only give the number of the corresponding structure, e.g., $I = n1$ with the specification of the vectors $x, y, z, \alpha, \beta$ and projectors $P, P'$. In particular, we have $I = n1$ with $x = k$, $P = P_k$ for diagram No 21 in Fig. 2, $I = n2$ with $x = k + q$, $P = P_q$ for diagram No 22, and $I = n2$ with $x = k$, $y = q$, $\alpha = P_k q$, $P = P_q$ for diagram No 31 (quantities like $\alpha = P_k q$ are understood as vectors obtained by the action of the matrix $P_k$ onto the vector $q$).

Our aim is the calculation of the quantities $A_i$ in Eq. (4.24) for each diagram in Fig. 1. They are given by the contractions of the integrals (4.37) with the structures $S_i$ from Eq. (4.19), which leads to the replacement of the factors $\phi_{12...}$ in Eq. (4.37) with the quantities $\phi_i = tr \left[ (S_i)_{12...} \cdot Sym \phi_{12...} \right]$. Substituting expression (4.38) into this construction gives $\phi_i = C \phi_E^{-1} \phi_s A_i$, where

\[
\bar{A}_i = tr \left[ S_i \cdot Sym \bar{I} \right]
\]

(4.40)

are the analogs of quantities (4.20) with the replacement $I \rightarrow \bar{I}$. As a result, from the relations (4.20), (4.37), (4.38) and (4.40) one obtains the following explicit formulas for the calculation of the coefficients $A_i$:

\[
\begin{align*}
    1 \text{ loop:} & \quad A_i = [D_0(2\pi)^{-d}] \int dk N_k C \phi_E^{-1} \phi_s \bar{A}_i, \quad (4.41a) \\
    2 \text{ loops:} & \quad A_i = [D_0(2\pi)^{-d}]^2 \int dk dq N_k N_q C \phi_E^{-1} \phi_s \bar{A}_i. \quad (4.41b)
\end{align*}
\]

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3 loops: \[ A_i = \left[ D_0(2\pi)^{-d}\right]^3 \int \int dkdq d\lambda N_k N_q N_i C \phi_E^{-1} \phi_s \bar{A}_i \] \hspace{1cm} (4.14c)

with \( \bar{A}_i \) from Eq. (4.40).

Calculation of these quantities for structures n1 and n2 from Eq. (4.38) and \( S_i \) from (4.19) gives:

\[ k = 2 \ (\bar{I} = n1) : \quad \bar{A}_1 = (xP\bar{A}), \quad \bar{A}_2 = (d-1)x^2; \quad \] \hspace{1cm} (4.42)

\[ k = 3 \ (\bar{I} = n2) : \quad 3\bar{A}_1 = (y\alpha xP\bar{A}) + (x\alpha yP\bar{A}) + ((x+y)\alpha)(xP\bar{A}), \]

\[ 9\bar{A}_2 = \left[ x^2 + 2(xy) \right] [(d-1)(y\alpha) + 2(yP\alpha)] + \left[ y^2 + 2(xy) \right] [(d-1)(x\alpha) + 2(xP\alpha)]. \] \hspace{1cm} (4.43)

Analogous expressions can also be obtained for the structures n3 and n4 in Eq. (4.39) for the four-ray diagrams; they are rather cumbersome and will be given later in Sec. VI A.

V. CALCULATION IN THE ONE-LOOP AND TWO-LOOP APPROXIMATIONS

In the one-loop approximation, we need the only diagram No 21, and in the two-loop approximation, all the four diagrams from Fig. 2 are needed. Using the general rules discussed in the previous Section, for the normal diagrams in the notation (4.38), (4.39) we obtain:

\[ \text{No 21:} \quad C = 1/2 \cdot 1 \cdot 1, \quad \phi_s = 1, \quad \phi_E = 2\epsilon_k, \]

\[ \bar{I} = n1 \quad \text{with} \quad x = k, \quad P = P_k \ [P \bar{x} = 0]; \] \hspace{1cm} (5.1)

\[ \text{No 22:} \quad C = 1/2 \cdot 1 \cdot 1, \quad \phi_s = (qP_kq), \quad \phi_E = 4\epsilon_q \epsilon_{k+q}, \]

\[ \bar{I} = n1 \quad \text{with} \quad x = k + q, \quad P = P_q; \] \hspace{1cm} (5.2)

\[ \text{No 31:} \quad C = 1 \cdot 1 \cdot 1, \quad \phi_s = 1, \quad \phi_E = 2\epsilon_q (\epsilon_k + \epsilon_q + \epsilon_{k+q}), \]

\[ \bar{I} = n2 \quad \text{with} \quad x = k, \quad y = q, \quad \alpha = P_kq, \quad P = P_q \ [(x\alpha) = 0, \quad (y\alpha) = 0]; \] \hspace{1cm} (5.3)

in the square brackets we give simple relations specific of a given set of momenta and projectors, which are useful in the calculations. We also recall that \( \epsilon_p = \nu p^2 \).

Substituting the explicit expressions (5.1), (5.3) into the general relations (4.42), (4.43) gives:

\[ \text{No 21:} \quad A_1 = 0, \quad A_2 = (d-1)k^2; \] \hspace{1cm} (5.4)

\[ \text{No 22:} \quad \bar{A}_1 = (qP_kq), \quad \bar{A}_2 = (d-1)(k + q)^2; \] \hspace{1cm} (5.5)

\[ \text{No 31:} \quad 3\bar{A}_1 = (qP_kq) (kP_eq), \quad 9\bar{A}_2 = (d-1) (qP_kq) [k^2 + 2(kq)] + 2(kP_eq) [q^2 + 2(kq)]. \] \hspace{1cm} (5.6)

A. One-loop approximation

The one-loop approximation \( \Gamma^{(1)} \) in Eq. (4.40) is determined by the only diagram No 21 in Fig. 2. Substituting the quantities known from Eqs. (4.11), (5.4) and the relation \( \epsilon_k = \nu k^2 \) into Eq. (4.41a) gives (we recall that the symmetry coefficient \( 1/2 \) is included into \( C \)):

\[ \text{No 21:} \quad A_1 = 0, \quad A_2 = [(d-1)/4]g\mu^2 U \] \hspace{1cm} (5.7)

with the integral \( U \equiv (2\pi)^{-d} \int d\lambda kN_k \) from Eq. (2.11). Substituting that expression into (5.7) gives:

\[ \text{No 21:} \quad A_1 = 0, \quad A_2 = u [(d-1)/4\epsilon] (\mu/m)^\epsilon, \quad u = gC_d \] \hspace{1cm} (5.8)

with coefficient \( C_d \) from Eq. (2.11).

From the relations (4.23a) and (5.8) we find \( B_1 = -2\alpha A_2, \quad B_2 = \alpha (d+1) A_2 \) with \( \alpha \) from (4.23a) and \( A_2 \) from (5.8); then from Eqs. (4.35) and (4.36a) we obtain the quantity \( \Gamma \) for our diagram: \( \Gamma (\text{No 21}) = \alpha A_2 [-2n(n-1)+(d+1)\lambda_n]. \)

Substituting the known expressions for \( \alpha \) and \( A_2 \) gives:

\[ \Gamma^{(1)} = \Gamma (\text{No 21}) = u (\mu/m)^\epsilon \frac{(d+1)\lambda_n - 2n(n-1)}{4\epsilon d(d+2)}. \] \hspace{1cm} (5.9)

Then the relation (4.9a) gives the final result for the \( O(u) \) contribution to the renormalization constant:

\[ Z_F^{-1} = 1 + [Z_F^{-1}]_1, \quad [Z_F^{-1}]_1 = u \frac{2n(n-1) - (d+1)\lambda_n}{4\epsilon d(d+2)}. \] \hspace{1cm} (5.10)
Consider now the two-loop diagrams Nos 22,31 in Fig. 2; the corresponding coefficients \( \tilde{A} \), are known from Eqs. \( (5.3) \) and \( (5.6) \). They include the quantities \( (kP_qk) = k^2 \sin^2 \vartheta, \ (q^2/k^2) = q^2 \sin^2 \vartheta, \ (kP_qP_qk) = -(kq)\sin^2 \vartheta, \ kq^2 = k^2q^2 \cos^2 \vartheta \), where \( \vartheta \) is the angle between the vectors \( k \) and \( q \). Substituting these expressions into Eqs. \( (5.13), (5.14) \) are expressed in terms of these integrals as follows:

\[ H_{2p} = (2\pi)^{-2d} \int dkdq N_kN_q \sin^{2p} \vartheta, \quad p = 1, 2, \]

\[ h = (2\pi)^{-2d} \int dkdq N_kN_q (kq) \sin^4 \vartheta / (k + q)^2, \]

\[ h_{2p} = (2\pi)^{-2d} \int dkdq N_kN_q (kq) \sin^{2p} \vartheta / [k^2 + q^2 + (kq)]. \]

The coefficients \( A \) in Eqs. \( (5.13), (5.14) \) are expressed in terms of these integrals as follows:

\[ A_1 = [(g\mu^2/8)(2\pi)^{-2d} \int dkdq N_kN_q k^2 \sin^4 \vartheta / (k^2 + q^2 + (kq)), \]

\[ A_2 = [(d-1)(g\mu^2/8)(2\pi)^{-2d} \int dkdq N_kN_q \sin^2 \vartheta; \]

Substituting these expressions along with the other needed information from Eqs. \( (5.2), (5.3) \) and the relation \( \epsilon_p = \nu p^2 \) into Eq. \( (4.41b) \) gives

\[ \tilde{A} = k^2 \sin^2 \vartheta, \quad \tilde{A}_2 = (d-1)(k+q)^2; \]

\[ A_1 = k^2 \sin^2 \vartheta, \quad A_2 = (d-1)(k+q)^2; \quad (5.11) \]

\[ 3A_1 = [(g\mu^2/4)(2\pi)^{-2d} \int dkdq N_kN_q k^2 \sin^4 \vartheta / (k^2 + q^2 + (kq)), \]

\[ 9A_2 = [(g\mu^2/4)(2\pi)^{-2d} \int dkdq N_kN_q \sin^2 \vartheta \{(d-1)[k^2 + 2(kq)] - 2(kq) - 4k^2 \cos^2 \vartheta\} / (k^2 + q^2 + (kq)). \quad (5.14) \]

We note that in the coefficients \( A_{1,2} \) for diagram No 31 the factor \( q^2 \) from \( \epsilon_q \) in the denominators cancels out with the analogous factor in the numerators, while in the coefficient \( A_2 \) for diagram No 22 a similar cancellation of the factor \( (k+q)^2 \) takes place.

The denominators in the integrals \( (5.13), (5.14) \) are symmetrical with respect to the permutation \( k \leftrightarrow q \). This allows one to perform the analogous symmetrization in the numerators, which is equivalent to the replacement \( k^2 \rightarrow (k^2 + q^2)/2 \) in the latter. Then the expressions simplify and reduce to linear combinations of the following standard integrals:

\[ H_{2p} = (2\pi)^{-2d} \int dkdq N_kN_q \sin^{2p} \vartheta, \quad p = 1, 2, \]

\[ h = (2\pi)^{-2d} \int dkdq N_kN_q (kq) \sin^4 \vartheta / (k + q)^2, \]

\[ h_{2p} = (2\pi)^{-2d} \int dkdq N_kN_q (kq) \sin^{2p} \vartheta / [k^2 + q^2 + (kq)], \quad p = 1, 2. \]

The two-loop quantity \( \Gamma^{(2)} \) also includes the contribution from the factorizable diagram No 41 with the symmetry coefficient \( 1/8 \); see Fig. 1. It reduces to the contraction of the vertex factor \( V_{1234} \) with the product \( T_{12}T_{34} \) of two identical one-loop blocks similar to diagram No 21. In the notation of Eqs. \( (4.24) \)–\( (4.28) \) this contribution has the form
\[ \Gamma = CV_{1244} \left[ B_1 w_1 w_2 + B_2 w^2 \delta_{12} \right] \left[ B_1 w_3 w_4 + B_2 w^2 \delta_{34} \right] \equiv F \Gamma, \]  

(5.20)

where \( B_{1,2} \) are the known coefficients for the one-loop diagram No 21 (see Sec. V A) and \( C \) is an additional symmetry coefficient; in the case at hand, \( C = 1/2 \). Let us explain its origin. By definition, the quantities \( B_{1,2} \) for diagram No 21 already include its symmetry coefficient 1/2 (in the factor \( C \) in Eq. (4.38)), which gives 1/4 for the product of two such blocks. Therefore, in order to obtain the correct symmetry coefficient 1/8 for diagram No 41, the additional coefficient \( C = 1/2 \) should be included.

Multiplying the expressions in the square brackets in Eq. (5.20) and taking into account the definition of the coefficients \( k_i \) in Eq. (4.36) gives

\[ \Gamma(\text{No 41}) = \frac{1}{2} \left[ k_1B_1^2 + 2k_2B_1B_2 + k_3B_2^2 \right] \]  

(5.21)

with coefficients \( k_{1,2,3} \) from Eq. (4.36) and known quantities \( B_{1,2} \) for the one-loop diagram No 21. In what follows, the same scheme will be used for the calculation of the contributions of the three-loop factorizable diagrams Nos 46, 51, 61; the coefficients \( k_i \) from Eqs. (4.36) and (4.36) will be involved.

We have found all terms in the two-loop expression

\[ \Gamma^{(2)} = \Gamma(\text{No 22}) + \Gamma(\text{No 31}) + \Gamma(\text{No 41}). \]  

(5.22)

Now Eq. (4.9b) is used to determine the two-loop contribution \( [Z_F^{-1}]_2 \) in the renormalization constant \( Z_F^{-1} \), with \( \Gamma^{(1)} \) from Eq. (5.3), \( \Gamma^{(2)} \) from Eq. (5.22) and \( [Z_F^{-1}]_1 \) from Eq. (5.10). It is also necessary to set \( \mu = m \) in all the diagrams of \( \Gamma^{(1,2)} \) in order to eliminate contributions with the logarithms \( \ln(\mu/m) \) (they would be cancelled with the contributions of the diagrams with self-energy insertions, which we already omitted).

All the quantities in Eq. (4.9b) are known explicitly, except for the contributions of diagrams Nos 22, 31, for which we only know the expressions of the coefficients \( A_i \), in terms of the standard integrals (5.15–5.17). In order to calculate the quantity (4.9b), we need only their divergent parts \( \sim 1/\varepsilon^2 \) and \( 1/\varepsilon \), but for the three-loop calculation we shall also need the zero-order in \( \varepsilon \) (constant) terms.

Calculation of these standard integrals is a separate task and will be discussed in the next Subsection.

C. Calculation of the two-loop integrals (5.15–5.17)

In this Subsection we denote \( n_k = k/k \) for any vector \( k \), \( d n_k \) is the area element of the unit \( d \)-dimensional sphere and \( \langle \ldots \rangle \) is the averaging over the sphere. In particular,

\[ \int dN_k \ldots = \int_{m}^{\infty} \frac{dk}{k^{1+\varepsilon}} \int dN_k \ldots = S_d \int_{m}^{\infty} \frac{dk}{k^{1+\varepsilon}} \langle \ldots \rangle \]  

(5.23)

with \( N_k \) from Eq. (1.3); \( S_d = 2\pi^{d/2}/\Gamma(d/2) \) is the surface area of the unit sphere.

For any two vectors with the angle \( \vartheta \) between them, the following formulas will be useful:

\[ \alpha_{2n} \equiv \langle \cos^{2n} \vartheta \rangle = \frac{(2n - 1)!!}{d(d + 2) \ldots (d + 2n - 2)} \]  

(5.24)

with \( n = 1, 2, \ldots \) (obviously, \( \alpha_0 \equiv (1) = 1 \)). From Eq. (5.24) one easily obtains:

\[ \langle \sin^2 \vartheta \cos^{2n} \vartheta \rangle = \frac{(d - 1)}{(d + 2n)} \alpha_{2n}, \quad \langle \sin^4 \vartheta \cos^{2n} \vartheta \rangle = \frac{(d^2 - 1)}{(d + 2n)(d + 2n + 2)} \alpha_{2n} \]  

(5.25)

with \( n = 0, 1, 2, \ldots \) and \( \alpha_{2n} \) from (5.24).

In the notation introduced above, the integrals \( H_{2p} \) in Eq. (5.17) are written in the form

\[ H_{2p} = C_d^2 \int_{m}^{\infty} \frac{dk}{k^{1+\varepsilon}} \int_{m}^{\infty} \frac{dq}{q^{1+\varepsilon}} \langle \sin^{2p} \vartheta \rangle, \quad p = 1, 2. \]  

(5.26)

Then Eq. (5.25) gives:

\[ H_2 = C_d^2 \frac{m^{-2\varepsilon}(d - 1)}{d\varepsilon^2}, \quad H_4 = C_d^2 \frac{m^{-2\varepsilon}(d^2 - 1)}{d(d + 2)\varepsilon^2}. \]  

(5.27)
Now let us turn to the integrals \(h_{2p}\) in Eq. (5.17). Expanding their integrands in \((kq)\) gives:

\[
h_{2p} = C_d^2 \sum_{l=0}^{\infty} (-1)^l \int_{m}^{\infty} \frac{dk}{k^{1+\varepsilon}} \int_{m}^{\infty} \frac{dq}{q^{1+\varepsilon}} \left( \frac{kq}{k^2 + q^2} \right)^{l+1} \langle \sin^{2p} \vartheta \cos^{l+1} \vartheta \rangle. \tag{5.28}
\]

The average \(\langle \sin^{2p} \vartheta \cos^{l+1} \vartheta \rangle\) differs from zero only for odd \(l\), and the series in Eq. (5.28) can be written as

\[
h_{2p} = -C_d^2 \sum_{n=0}^{\infty} I_n(m) \langle \sin^{2p} \vartheta \cos^{2n+2} \vartheta \rangle \tag{5.29}
\]

with the integral

\[
I_n(m) = \int_{m}^{\infty} \frac{dk}{k^{1+\varepsilon}} \int_{m}^{\infty} \frac{dq}{q^{1+\varepsilon}} \left( \frac{kq}{k^2 + q^2} \right)^{2n+2} = m^{-2\varepsilon} I_n(1). \tag{5.30}
\]

Using the identity

\[
I_n(m) = -\frac{1}{2\varepsilon} D_m I_n(m), \quad D_m \equiv m\partial/m,
\]

which follows from the last equality in Eq. (5.30), the integral \(I_n(m)\) can be represented in the form

\[
I_n(m) = m^{-2\varepsilon} \int_1^{\infty} \frac{dk}{k^{1+\varepsilon}} \left( \frac{k}{k^2 + 1} \right)^{2n+2} \tag{5.31}
\]

where the number of integrations is reduced and the pole in \(\varepsilon\) is isolated explicitly. Expanding the integrand in Eq. (5.32) in \(\varepsilon\) and neglecting the terms of order \(O(\varepsilon)\) and higher, with the desired accuracy we obtain

\[
I_n(m) \simeq m^{-2\varepsilon} \int_1^{\infty} \frac{dk k^{2n+1}}{(k^2 + 1)^{2n+2}} (\varepsilon^{-1} - \ln k) = m^{-2\varepsilon} \frac{(n!)^2}{4(2n+1)!} (\varepsilon^{-1} - B_n), \tag{5.33}
\]

where the quantities \(B_n\), needed only for the three-loop calculation, can be represented as finite sums:

\[
B_n = \ln 2 + \frac{(2n+1)!}{n!} \sum_{k=0}^{n} \frac{(-1)^k \left(1/2\right)^{n+k} - 1}{k!(n-k)!(n+k+1)^2}. \tag{5.34}
\]

For the integrals \(h_{2p}\) in Eq. (5.17) this gives:

\[
h_2 = m^{-2\varepsilon} C_d^2 (d-1) \sum_{n=0}^{\infty} \frac{n! 2^{-n-2}}{d(d+2)\ldots(d+2n+2)} (\varepsilon^{-1} + B_n); \tag{5.35a}
\]

\[
h_4 = m^{-2\varepsilon} C_d^2 (d^2-1) \sum_{n=0}^{\infty} \frac{n! 2^{-n-2}}{d(d+2)\ldots(d+2n+4)} (\varepsilon^{-1} + B_n). \tag{5.35b}
\]

Expanding the integrand in Eq. (5.16) in \(2(kq)\) and proceeding as above for \(h_{2p}\), we obtain analogous expression for the integral \(h\):

\[
h = m^{-2\varepsilon} C_d^2 (d^2-1) \sum_{n=0}^{\infty} \frac{n! 2^{-n-1}}{d(d+2)\ldots(d+2n+4)} (\varepsilon^{-1} + B_n). \tag{5.35c}
\]

The \(O(\varepsilon^{-1})\) terms in Eqs. (5.35) can be expressed in terms of the hypergeometric series (see, e.g., [27])

\[
F(a, b; c; z) \equiv 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{c(c+1)} z^2 + \ldots
\]

as follows:
\[ h_2 = \frac{m^{-2\varepsilon} C^2_d (d - 1)}{4d(d + 2)} \left( -\varepsilon^{-1} F(1, 1; d/2 + 2; 1/4) + (d + 2)C_2(d) \right), \]  
\[ h_4 = \frac{m^{-2\varepsilon} C^2_d (d^2 - 1)}{4d(d + 2)(d + 4)} \left( -\varepsilon^{-1} F(1, 1; d/2 + 3; 1/4) + (d + 4)C_4(d) \right), \]  
\[ h = \frac{m^{-2\varepsilon} C^2_d (d^2 - 1)}{2d(d + 2)(d + 4)} \left( -\varepsilon^{-1} F(1, 1; d/2 + 3; 1) + (d + 4)C(d) \right), \]  

and for the \( O(1) \) terms one has:

\[ C_2(d) = \sum_{n=0}^{\infty} \frac{n! B_{2n}}{4^n(d/2 + 1) \ldots (d/2 + 1 + n)}, \]  
\[ C_4(d) = \sum_{n=0}^{\infty} \frac{n! B_{2n}}{4^n(d/2 + 2) \ldots (d/2 + 2 + n)} = C_2(d + 2), \]  
\[ C(d) = \sum_{n=0}^{\infty} \frac{n! B_{2n}}{(d/2 + 2) \ldots (d/2 + 2 + n)}. \]  

It is worth noting that the series entering into Eqs. (5.37) and (5.38) are convergent; this fact is nontrivial for \( h \) since \( |z| = 1 \) is the convergence radius for the corresponding series.

For \( F(\ldots) \) in Eqs. (5.37) one has \( F(1, 1; d/2 + 3; 1) = (d + 4)/(d + 2) \) for any \( d \), while the expressions for \( F(\ldots) \) in Eqs. (5.37a), (5.37b) simplify only for integer \( d \); see (27). In particular, for \( d = 2 \) and \( d = 3 \) one has

\[ d = 2 : \quad F(1, 1; d/2 + 2; 1/4) = 8[-3 \ln(4/3) + 1] \approx 1.0956, \]
\[ F(1, 1; d/2 + 3; 1/4) = 6[18 \ln(4/3) - 5] \approx 1.0696, \]
\[ C_2 \approx 0.3671, \quad C_4 \approx 0.2411, \quad C \approx 0.2917; \]  
\[ d = 3 : \quad F(1, 1; d/2 + 2; 1/4) = 10(\pi \sqrt{3} - 16/3) \approx 1.0806, \]
\[ F(1, 1; d/2 + 3; 1/4) = 14(-15\pi \sqrt{3} + 82)/5 \approx 1.0613, \]
\[ C_2 \approx 0.2912, \quad C_4 \approx 0.2056, \quad C \approx 0.2412; \]

and for the other integer \( d \) analogous expressions can be obtained from the recurrent relation

\[ 3F(1, 1; d/2 + 2; 1/4) + (d + 2)F(1, 1; d/2 + 3; 1/4)/(d + 4) = 4, \]  
valid for all \( d \). In Eq. (5.39) we also included the numerical values for the coefficients \( C_{2,4} \) and \( C \) obtained from the series (5.38).

Substituting all these expressions for the integrals into Eqs. (5.18) and (5.19) gives the final answers for the coefficients \( A_i \) of the diagrams Nos 22, 31:

\[ A_1 = \frac{u^2 (\mu/m)^{2\varepsilon} (d^2 - 1)}{16d(d + 2)} \left[ \frac{1}{\varepsilon^2} + \frac{1}{(d + 2)\varepsilon} - \frac{1}{2}C(d) \right], \]
\[ A_2 = \frac{u^2 (\mu/m)^{2\varepsilon} (d - 1)^2}{16d\varepsilon^2}; \]  
\[ A_3 = \frac{u^2 (\mu/m)^{2\varepsilon} (d^2 - 1)^2}{24d(d + 2)} \left[ \frac{1}{\varepsilon^2} + \frac{F(1, 1; d/2 + 3; 1/4)}{4(d + 4)\varepsilon} - \frac{1}{8}C_4(d) \right], \]
\[ A_4 = \frac{u^2 (\mu/m)^{2\varepsilon} (d - 1)^2}{24d(d + 2)} \left[ \frac{(d - 2)(d + 3)}{3\varepsilon^2} - \frac{(d - 1)F(1, 1; d/2 + 2; 1/4)}{4\varepsilon} + \frac{(d + 1)F(1, 1; d/2 + 3; 1/4)}{3(d + 4)\varepsilon} + \frac{1}{8}(d - 1)(d + 2)C_2(d) - \frac{1}{6}(d + 1)C_4(d) \right] \]

with \( F(\ldots) \) from Eq. (5.36) and \( C, C_{2,4} \) from (5.38). From these expressions, using the standard general scheme, one obtains the contributions of the diagrams Nos 22, 31 into Eq. (5.22) and then the two-loop contribution \( [Z_F^{-1}]_2 \) into the renormalization constant \( Z_F^{-1} \), presented earlier in Ref. [10].
VI. THREE-LOOP APPROXIMATION

A. Scalarization of the three-loop diagrams

All the needed three-loop diagrams with the symmetry coefficients are given in Fig. 1. Below we shall describe them in more detail and give in a compact form the complete set of relations which allow one to express the coefficients $A_i$ in Eq. (4.20) for any given diagram in terms of the scalar integrals (4.41). We recall that all the external momenta in the diagrams are set equal to zero.

We begin with the normal (not factorizable) diagrams. For these, the integration momenta $k, q, l$ are always assigned to the horizontal lines $(vv)$ in the following order: $k$ flows via the uppermost line, $q$ flows via the middle line, and $l$ flows via the lowest line. Then in order to determine the momenta for all lines in the diagram, it is sufficient to show (by an arrow) the chosen direction of the momentum in each line. The needed derivatives $\partial$ on the lines are always implied (for the one-loop and two-loop diagrams, they were shown explicitly by dots in Fig. 2). The numerical indices $1, 2, \ldots$ of the upper dots are always chosen to increase from the left to the right (like in Eq. (4.17) and in the diagrams in Fig. 2); the positions of the letter indices $a, b, \ldots$ in the diagram are always shown explicitly. This information allows one to completely restore the configuration of the momenta and the form of the index structures (4.39) for any diagram.

Now we turn to the description of specific diagrams in this notation. For any diagram, we give the directions of the momenta, positions of the indices $a, b, \ldots$ and forms of all cofactors in Eq. (4.38) in the same form as in Eqs. (4.39). The two-ray diagram No 23:

$$C = 1/2 \cdot 1 \cdot 1, \quad \phi_s = (IP_q l)((q + l)P_k (q + l)), \quad \phi_E = 8\epsilon q_{q+1} \epsilon k_{q+1}, \quad \bar{I} = n1 \quad \text{with} \quad x = k + q + l, \quad P = P_1. \quad (6.1)$$

The three-ray normal diagrams:

No 32 : $C = 1 \cdot 1 \cdot 1, \quad \phi_s = (IP_q l), \quad \phi_E = 4\epsilon q_{q+1} (\epsilon_k + \epsilon_{q+1} + \epsilon_{k+q+1}), \quad \bar{I} = n2 \quad \text{with} \quad x = k, \quad y = q + 1, \quad \alpha = P_k (q + 1), \quad P = P_1 \quad [(x\alpha) = 0]; \quad (6.3)$

No 33 : $C = 1 \cdot 1 \cdot 1, \quad \phi_s = (IP_k (q + 1)), \quad \phi_E = 2\epsilon (\epsilon_q + \epsilon_{q+1}) (\epsilon_k + \epsilon_{k+q+1}), \quad \bar{I} = n2 \quad \text{with} \quad x = q, \quad y = k + 1, \quad \alpha = P_k q, \quad P = P_1 \quad [(x\alpha) = 0]; \quad (6.4)$

No 34 : $C = 1 \cdot 1 \cdot 1, \quad \phi_s = (aq P_k (q + 1)), \quad \phi_E = 2\epsilon (\epsilon_q + \epsilon_{q+1}) (\epsilon_k + \epsilon_{q+1} + \epsilon_{k+q+1}), \quad \bar{I} = n2 \quad \text{with} \quad x = l, \quad y = k + q, \quad \alpha = P_k q, \quad P = P_1 \quad [Px = 0]; \quad (6.5)$

No 35 : $C = 1 \cdot 1 \cdot 1, \quad \phi_s = (aq P_k l), \quad \phi_E = 2\epsilon (\epsilon_q + \epsilon_{q-1}) (\epsilon_{k+q} + \epsilon_{k+1} + \epsilon_{q-1}), \quad \bar{I} = n2 \quad \text{with} \quad x = k + l, \quad y = q - 1, \quad \alpha = P_k q, \quad P = P_1. \quad (6.6)$

The structure $\bar{I} = n2$ from Eq. (4.39) is the same for all these diagrams; the corresponding quantities $\bar{A}_i$ are determined by the general formula (4.43) in which the specific values of the vectors $x, y, z, \alpha$ and projectors $P$ for any given diagram should be substituted from Eqs. (6.3–6.6).
The four-ray normal diagrams:

![Diagrams](image)

For these, one has:

No 42:
- \( C = 1 \cdot 1 \cdot 1 \), \( \phi_s = 1 \),
- \( \phi_E = 2\alpha(\epsilon_q + \epsilon_1 + \epsilon_{q+1})(\epsilon_k + \epsilon_q + \epsilon_1 + \epsilon_{k+q+1}) \),
- \( \bar{I} = n3 \) with \( x = k, y = q, z = 1, \alpha = P_k(q+1), \beta = P_q1, P = P1 \)

\[
[(\mathbf{x}\alpha) = 0, (\mathbf{y}\beta) = 0, Pz = 0];
\]

No 43:
- \( C = 1/2 \cdot 1 \cdot 2 \), \( \phi_s = (qP_k)1 \),
- \( \phi_E = 4\alpha(\epsilon_q + \epsilon_1)(\epsilon_k + \epsilon_q + \epsilon_{k-q} + \epsilon_{k+1}) \),
- \( \bar{I} = n4 \) with \( x = q, y = 1, z = k - q, P = P1, P' = P_q \)

\[
[Py = 0, P'x = 0];
\]

No 44:
- \( C = 1 \cdot (-1) \cdot 1 \), \( \phi_s = 1 \),
- \( \phi_E = 2\alpha(\epsilon_q + \epsilon_1 + \epsilon_{q-1})(\epsilon_k + \epsilon_q + \epsilon_{k+1} + \epsilon_{k-q}) \),
- \( \bar{I} = n3 \) with \( x = k, y = 1, z = k - q, \alpha = P_kq, \beta = P_q1, P = P1 \)

\[
[(\mathbf{x}\alpha) = 0, (\mathbf{y}\beta) = 0, P(y + z) = 0];
\]

No 45:
- \( C = 1 \cdot (-1) \cdot 1 \), \( \phi_s = 1 \),
- \( \phi_E = 2\alpha(\epsilon_q + \epsilon_1 + \epsilon_{q+1})(\epsilon_k + \epsilon_q + \epsilon_{k-q} + \epsilon_{k+1}) \),
- \( \bar{I} = n3 \) with \( x = k, y = 1, z = k - q, \alpha = P_kq, \beta = P_q1, P = P1 \)

\[
[(\mathbf{x}\alpha) = 0, (\mathbf{y}\beta) = 0, P(y + z) = 0].
\]

Three diagrams have the index structure \( \bar{I} = n3 \) from Eq. (6.39) and one diagram has the structure \( \bar{I} = n4; (\mathbf{x}\alpha) = 0 \) for all the diagrams with \( \bar{I} = n3 \).

Below we give the expressions for the coefficients \( \tilde{A}_i \) for the structures \( \bar{I} = n3, n4 \), analogous to Eqs. (4.42), (4.43).

For \( \bar{I} = n3 \) from Eqs. (4.39) and (4.40) using the relation \( (\mathbf{x}\alpha) = 0 \) one obtains:

\[
12\tilde{A}_1 = (xPx)[(y\alpha)(z\beta) + (y\beta)(z\alpha)] + (yPy)(z\alpha)(x\beta) + (zPz)(x\beta)(y\alpha) +
+ 2(xPy)[(y\alpha)(z\beta) + (z\alpha)(x + y + z, \beta)] + 2(xPz)[(y\beta)(z\alpha) + (y\alpha)(x + y + z, \beta)] +
+ 2(yPx)(x\beta)(z + y, \alpha);
\]

\[
72\tilde{A}_2 = U_1 \left\{ (d - 1)(y\alpha)(z\beta) + (y\beta)(z\alpha) \right\} + 2(yP\alpha)(z\beta) + 2(zP\alpha)(y\beta) +
+ 2(yP\beta)(z\alpha) + 2(zP\beta)(y\alpha) + 2(yPz)(x\alpha) +
+ U_2 \left\{ (d - 1)(y\alpha)(x\beta) + 2(xP\alpha)(z\beta) + 2(xP\beta)(z\alpha) + 2(xPz)(x\alpha) \right\} +
+ U_3 \left\{ (d - 1)(y\alpha)(x\beta) + 2(xP\alpha)(y\beta) + 2(yP\alpha)(x\beta) + 2(yP\beta)(x\alpha) + 2(yPy)(x\alpha) \right\} +
+ 2(d - 1)\left[ (z\alpha)(z\beta)(xy) + (y\alpha)(y\beta)(zx) \right] +
+ 2\left[ (zP\alpha)(z\beta)(xy) + (yP\alpha)(y\beta)(xz) + (xP\alpha)(x\beta)(yz) + (zP\beta)(z\alpha)(xy) + (yP\beta)(y\alpha)(xz) \right] +
+ 2(\alpha\beta)\left[ (xPx)(yz) + (yPy)(xz) + (zPz)(xy) \right];
\]

\[
9\tilde{A}_3 = U_4 \left\{ (d - 1)(\alpha\beta) + 2(\alpha P\beta) \right\}.
\]

Here and below in Eq. (6.14)

\[
U_1 \equiv x^2 + 2(xy) + 2(xz), \quad U_2 \equiv y^2 + 2(xy) + 2(yz), \quad U_3 \equiv z^2 + 2(xz) + 2(yz),
\]

\[
U_4 \equiv x^2(yz) + y^2(xz) + z^2(xy) + 2(xy)(xz) + 2(xy)(yz) + 2(xz)(yz).
\]
For \( \bar{t} = n4 \) from Eq. (4.39) using the relations \( P y = 0, P' x = 0 \) in Eq. (5.3) one obtains:

\[
\begin{align*}
6 \bar{A}_1 &= (x P x)(y P' z) + 2(x P z)(y P' z) + (x P z)(y P' y), \\
36 \bar{A}_2 &= (d - 1) \left\{ U_1(y P' z) + U_2(x P z) + (x P x)(y z) + (y P' y)(y z) + (z(P + P') z)(x y) \right\} + \\
&+ 2 \left\{ U_1(y P' P z) + U_2(x P P' z) + U_3(x P P' y) + (z(P P' + P' P) z)(x y) \right\}, \\
9 \bar{A}_3 &= U_1 \left\{ (d - 1)^2 + 2 \text{tr} (P P') \right\}
\end{align*}
\]

(6.14)

with \( U_1 - U_4 \) from Eq. (6.13); \( \text{tr} (P P') \) is the trace of a matrix product.

Substituting the specific values for the vectors and projectors for any given diagram from Eqs. (6.8)–(6.11) into expressions (6.12)–(6.14) gives explicit expressions for the corresponding coefficients \( \bar{A}_i \). Then, using the additional information from (6.8)–(6.11) one obtains the desired expressions for the coefficients \( A_i \) for any diagram in the form of scalar integrals (4.41). All these simple but cumbersome technical operations are easily performed by means of a computer.

Now let us turn to the factorizable diagrams Nos 46, 51, 61. Diagram No 46 factorizes to a product of two blocks Nos 21, 22; diagram No 51 factorizes to a product of the blocks Nos 21, 31; diagram No 61 factorizes to a product of three blocks No 21. Since the quantities \( B_i \) for diagrams Nos 21, 22 already include their own symmetry coefficient 1/2, we conclude that in order to obtain the needed symmetry coefficient in Fig. 1 for diagrams Nos 46, 51 an additional symmetry coefficient (like \( C \) in Eq. (4.20)) is not needed, while for diagram No 61 it should be taken as 1/6.

Therefore, taking into account Eqs. (4.18) and (4.27), we obtain for diagrams Nos 46, 51, 61:

No 46: \( \Gamma = V_{1234} [B_1 w_1 w_2 + B_2 w^2 \delta_{12}] [B'_1 w_3 w_4 + B'_2 w^2 \delta_{34}], \) (6.15a)

No 51: \( \Gamma = V_{12345} [B_1 w_1 w_2 + B_2 w^2 \delta_{12}] [B'_1 w_3 w_4 w_5 + B'_2 w^2 \delta_{34} w_5], \) (6.15b)

No 61: \( \Gamma = [1/6] V_{123456} [B_1 w_1 w_2 + B_2 w^2 \delta_{12}] [B_1 w_3 w_4 + B_2 w^2 \delta_{34}] [B_1 w_5 w_6 + B_2 w^2 \delta_{56}], \) (6.15c)

where \( B_{1,2} \) are the known coefficients for diagram No 21, \( B'_{1,2} \) in Eq. (6.15a) are the analogous coefficients for diagram No 22, and \( B'_{1,2} \) in Eq. (6.15b) are the coefficients for diagram No 31.

Using the relations (4.38), from Eqs. (4.15) one obtains:

\[
\begin{align*}
\bar{\Gamma}(\text{No 46}) &= k_1 B_1 B'_1 + k_2 (B_1 B'_2 + B_2 B'_1) + k_3 B_2 B'_2, \\
\bar{\Gamma}(\text{No 51}) &= k_1 B_1 B'_1 + k_2 (B_1 B'_2 + B_2 B'_1) + k_3 B_2 B'_2, \\
\bar{\Gamma}(\text{No 61}) &= [1/6] [k_1 B'_1 + 3k_2 B'_2 B_2 + 3k_3 B_1 B'_2 + k_4 B'_3]
\end{align*}
\]

(6.16)

with the coefficients \( k_1 \), known from Eq. (4.38c) for diagram No 46, from Eq. (4.36a) for diagram No 51, and from Eq. (4.36c) for diagram No 61. The relations (6.16) give the desired answers for the three-loop factorizable diagrams.

**B. Calculation of the three-loop integrals and anomalous dimensions: General scheme**

Substituting the specific values for the vectors and projectors, given in (6.1), (6.3)–(6.6), (6.8)–(6.11) for all normal three-loop diagrams, into the general formulas (4.42), (4.43), (6.12)–(6.14) gives explicit expressions for the quantities \( A_i \) in Eq. (4.41) for any diagram in the form of polynomials in the scalar products (kq), (kl), (ql) and moduli \( k, q, l \) of the vectors k, q, l.

It is worth noting that for all the normal three-loop diagrams, the products \( \phi_i \bar{A}_i \) have the form of sums of monomials in \( k, q, l \) of order six. In the variables “moduli-angles,” each of these monomials contains the factor \( l^2 \), which is canceled out by the analogous factor in the energy denominator \( \epsilon_l = \nu l^2 \), present in each of these diagrams. It is also worth noting that these products involve the modulus \( k \) in the power 2 or less (otherwise the integrals over \( k \) would be divergent).

In the three-loop diagrams, we only need the poles in \( \epsilon \), that is, the contributions of order \( \epsilon^{-3}, \epsilon^{-2} \) and \( \epsilon^{-1} \). The following general scheme is used in their calculation:

1. The integrands are expanded in the set of scalar products (kq), (kl), (ql); the results are represented as multiple series in these quantities.
2. Then, the angular averaging \( \langle \ldots \rangle \) is performed with respect to the directions of the vectors k, q, l, that is, the following quantities are calculated:

\[
\langle (kq)^{n_1} (kl)^{n_2} (ql)^{n_3} \rangle
\]

(6.17)
with arbitrary integer exponents $n_i \geq 0$.

3. The next step is the integration over the moduli $k, q, l$. All the needed integrals reduce to the form:

$$
\int_{1}^{\infty} \frac{dk}{k^{1+\varepsilon}} \int_{1}^{\infty} \frac{dq}{q^{1+\varepsilon}} \int_{1}^{\infty} \frac{dl}{l^{1+\varepsilon}} \frac{k^{2n_1}q^{2n_2}l^{2n_3}}{(k^2 + q^2 + l^2)^{n_s}(q^2 + l^2)^{n_s}},
$$

(6.18)

where the integer exponents $n_i \geq 0$ satisfy the relation $n_1 + n_2 + n_3 = n_4 + n_5$, so that the integrals contain only logarithmic UV divergences for $\varepsilon \to 0$. Only the pole parts are extracted from the integrals (6.18).

4. The last step is the summation of the resulting series. They have the forms of double infinite series with the coefficients given by $n$-fold finite sums ($n \leq 5$); the number of terms in the latter increases rapidly with the order of the coefficient. This summation is the only operation which cannot be performed exactly (analytically) and is therefore the only source of errors in numerical coefficients in expressions like Eqs. (6.13).

Of course, the straightforward but cumbersome operations listed above have been performed with the aid of a computer.

The first step, the expansion in the scalar products $(kq), (kl), (ql)$, contains some conceptual subtleties, and we shall discuss it in more detail. The problem is that the plain expansion of the integrands in the powers of the scalar products in some cases leads to divergent series.

The non-polynomial dependence of the integrands on the cosines of the angles appears only from the energy factors $\phi_E^1$, whose explicit forms are given in Eqs. (6.11), (6.3), (6.4), (6.8)–(6.11). For all the diagrams, the nontrivial factors in $\phi_E$ have the forms $q^2 + l^2 + const (ql)$ and $k^2 + q^2 + l^2 + some linear combination of all scalar products, and in the expansions in powers of the scalar products, the denominators will contain powers of the quantities $(q^2 + l^2)$ and $(k^2 + q^2 + l^2)$. Such expansions converge for all cofactors in $\phi_E^1$, which do not contain the “energy” $\epsilon_{k+q+l}$ with the sum of all three momenta. The cofactors with $\epsilon_{k+q+l}$ require special consideration. They are present in diagrams Nos 23, 32, 33, 34, 42 and are proportional to the following factors:

No 23: $\phi_E^1 \propto \epsilon^{-1}_{k+q+l} \times [Q + 2S]^{-1}$

No 32: $\phi_E^1 \propto [k + \epsilon_{q+l} + \epsilon_{k+q+l}]^{-1} \times [Q + S + (ql)]^{-1}$

No 33: $\phi_E^1 \propto [q + \epsilon_{k+l} + \epsilon_{k+q+l}]^{-1} \times [Q + S + (kl)]^{-1}$

No 34: $\phi_E^1 \propto [l + \epsilon_{k+q} + \epsilon_{k+q+l}]^{-1} \times [Q + S + (kq)]^{-1}$

No 42: $\phi_E^1 \propto [k + \epsilon_{q} + \epsilon_{l} + \epsilon_{k+q+l}]^{-1} \times [Q + S]^{-1}$

(6.19)

where $Q \equiv k^2 + q^2 + l^2$ and $S \equiv (kq) + (kl) + (ql)$.

From the obvious identities $(k+q+l)^2 \geq 0, (k-q)^2 + (k-l)^2 + (q-l)^2 \geq 0, |(kq)| \leq kq, |(kl)| \leq kl$ and |$(ql)| \leq ql$ it follows that

$$
-Q/2 \leq S \leq Q, \quad |(kq)| \leq Q/2, \quad |(kl)| \leq Q/2, \quad |(ql)| \leq Q/2.
$$

(6.20)

It then follows that the factor (6.19) in diagram No 42 can be expanded in the powers of the ratio $S/Q$ with $|S/Q| \leq 1$, while for No 23 this is impossible: in the integration region, there is a subregion of the same dimension (namely, $3 \times d$) in which $2|S| > Q$. This difficulty can be circumvented by means of the shift of the point around which the expansion is made in diagram No 23:

$$
(Q + 2S)^{-1} = [3Q/2 + (4S - Q)/2]^{-1} = [2/3Q] \sum_{n=0}^{\infty} [(1 - 4S/Q)/3]^n.
$$

(6.21)

The convergence of the series in (6.21) is ensured by the inequality $|(1 - 4S/Q)/3| \leq 1$, which follows from Eq. (6.20). The equality takes place only in the $2 \times d$-dimensional subregion $k + q + l = 0$, which has zero measure in the $3 \times d$-dimensional integration region and does not spoil the convergence of the corresponding series for the integral (similar considerations ensure the convergence of the series (5.55) for the two-loop integral $h$).

For the factor (6.19) in diagram No 32 the following shift can be used:

$$
[Q + S + (ql)]^{-1} = [5Q/4 + (S + (ql) - Q/4)]^{-1}
$$

with the subsequent expansion in the powers of the ratio $|S + (ql) - Q/4|/(5Q/4)$, whose modulus is $\leq 1$ owing to inequalities (6.20). Similar expansions (with obvious modifications) can be written for diagrams Nos 33, 34.
performed in the first place, before all the other summations: only this order of summations ensures the convergence of the series.

Calculation of the angular (6.17) and momentum (6.18) integrals is a separate task; it is discussed in the next two Subsections. The results for the coefficients $A_i$ for all the normal three-loop diagrams are presented in the Appendix. The $\varepsilon^{-3}$ contributions in all diagrams have been found analytically for general space dimensionality $d$ (in order to calculate them, it is sufficient to neglect all the scalar products in the energy denominators $\phi_E$), while the $\varepsilon^{-2}$ and $\varepsilon^{-1}$ parts have been found for the physical cases $d = 2, 3$ and in the large $d$ limit; the results have been presented in Ref. [15] (of course, this calculation can be made for any fixed given value of $d$).

Now, using the standard scheme (see Sec. IV), one finds the three-loop contribution $[Z_F^{-1}]_3$ in the expansion (4.7) for the renormalization constant $Z_F^{-1}$. The anomalous dimension $\gamma_F \equiv \gamma_{n_1}$ is given by the relation $\gamma_F = -\beta_0 \ln Z_F^{-1}$; see Eq. (3.4). Substituting Eqs. (2.17) and (1.7) into the last relation expresses the anomalous dimension in the coefficients $[Z_F^{-1}]_k$. Within our accuracy one obtains

$$\gamma_F(u) = [\varepsilon - u (d - 1)/2d] \left[ [Z_F^{-1}]_1 + 2 [Z_F^{-1}]_2 - [Z_F^{-1}]_1^2 + 3 [Z_F^{-1}]_3 - 3 [Z_F^{-1}]_1 [Z_F^{-1}]_2 + [Z_F^{-1}]_1^3 \right].$$

Substituting the known expressions for $[Z_F^{-1}]_k$ into Eq. (6.22) gives the anomalous dimension $\gamma_F(u)$ to order $u^3$.

Since the quantity $\gamma_F$ is UV finite, that is, finite at $\varepsilon = 0$, the pole parts must cancel each other in Eq. (6.22). This implies some exact relations between the senior poles in the quantities $[Z_F^{-1}]_k$ with $k \geq 2$ $(\varepsilon^{-2}$ in $[Z_F^{-1}]_2$ and $\varepsilon^{-2}, \varepsilon^{-3}$ in $[Z_F^{-1}]_3$) and the $\varepsilon^{-1}$ parts of the previous orders in $u$. Such relations provide an additional possibility to control the absence of calculational errors. In fact, the knowledge of senior poles ($\varepsilon^{-k}$ with $k \geq 2$) is needed only to check this cancellation; the nonvanishing contributions in Eq. (6.24) are completely determined by the $\varepsilon^{-1}$ terms in the quantities $[Z_F^{-1}]_k$. In the MS scheme, the anomalous dimension $\gamma_F(u)$ appears independent of $\varepsilon$; this is a consequence of the relation $Z_F = 1 +$ only poles in $\varepsilon$.

Finally, the relation (6.24) with $u_a$ from Eq. (2.18) gives the $O(\varepsilon^3)$ results for the critical dimensions presented in Ref. [15] and Eqs. (1.13).

C. Angular integrations in the three-loop diagrams

The two-loop integrals (6.17) involve only two vectors $k$ and $q$ and one angle $\vartheta$ between them, so that the procedure of the angular averaging reduces there to the only standard formula (5.24). The three-loop integrals (4.41c) involve three vectors $k$, $q$ and $l$ and three angles between them, so that the calculation of the quantities (6.17) is not all that simple. Below we present the results of this calculation.

Obviously, the quantity (6.17) differs from zero only if all the three numbers $n_1, n_2, n_3$ are simultaneously even or odd. It is also clear that the quantity (6.17) is symmetrical with respect to any permutation of the exponents $n_1, n_2, n_3$ and with no loss of generality it can be assumed that $n_2$ is the minimal exponent:

$$n_2 = \min \{n_1, n_2, n_3\}$$  \hspace{1cm} (6.23)

(the notation in the formulas below is consistent with the assumption (6.23)).

The straightforward calculation (first, the averaging over the direction of one momentum, say $k$, and then the averaging over the angle between the two remaining vectors $q$ and $l$) leads to the following result for the quantity (6.17):

$$T_{n_1 n_2 n_3} = k^{n_1 + n_2} q^{n_1 + n_3} l^{n_2 + n_3} T_{n_1 n_2 n_3}$$

where the moduli of the vectors are isolated explicitly, and

$$T_{n_1 n_2 n_3} = \alpha_{n_1 + 2} \alpha_{n_2 + 3} K(n_1, n_2) \tilde{T}_{n_1 n_2 n_3}$$

with coefficients $\alpha_{2n}$ from Eq. (5.24) and

$$K(n_1, n_2) = \frac{2^{n_2} n_2! [n_1 + n_2]!/[(n_1 - n_2)/2]!}{(n_1 + 2)! [n_1)! [n_1 - n_2)/2]!} = \frac{(n_1 - n_2 + 2)(n_1 - n_2 + 4) \ldots (n_1 + n_2)}{(n_1 + 1)(n_1 + 2) \ldots (n_1 + n_2)}.$$

$$\tilde{T}_{n_1 n_2 n_3} = \sum_{k=0}^{n_2/2} Y_k, \quad Y_k = \frac{2^{-2k} n_2! [n_1 - n_2)!/2]! \alpha_{n_2 + n_3 - 2k}}{k! (n_2 - 2k)! [n_1 - n_2)/2 + k)! \alpha_{n_2 + n_3}}.$$  \hspace{1cm} (6.27)
We recall that the numbers \( n_{1,2,3} \) in these expressions have the same parity, and that \( n_2 \) is the minimal one according to Eq. (6.23). The upper limit in the sum \((6.27)\) for odd \( n_2 \) is understood as the integer part of \( n_2/2 \).

The following special values and recurrent relations for the quantity \( K \) in Eq. (6.26) are useful in the computer calculations:

\[
K(m,0) = K(m,1) = 1, \quad K(m,m) = m!/(2m-1)!, \quad K(m+1,m+1)/K(m,m) = (m+1)/(2m+1),
\]

\[
K(m+2,n)/K(m,n) = (m+1)(m+2)/(m-n+2)(m+n+1).
\]

(6.28)

For the terms \( Y_k \) in the sum \((6.27)\) one has:

\[
Y_0 = 1, \quad Y_{k+1}/Y_k = \frac{(n_2-2k)(n_2-2k-1)(d+n_2+n_3-2k-2)}{2(k+1)(n_1-n_2+2k+2)(n_2+n_3-2k-1)}.
\]

(6.29)

Using these relations, the quantities \((6.26), (6.27)\) are easily calculated by a computer.

D. Modular integrations in the three-loop diagrams

Let us turn to the calculation of the three-loop integrals (6.18) over the moduli \( k, q, l \), which arise as coefficients in the expansion of the quantities \( A_i \) in Eq. (6.41c) in the scalar products \((kq), (kl), (ql)\); see Sec. VIB. For the complete three-loop calculation, it is sufficient to know some special cases of the general integral (6.18), which we denote \( I_1-I_9 \) in what follows. Below we give the explicit answers for these integrals, with the precise specification of the indices \( n_{1-5} \), and then turn to the derivation. Only the pole parts \( \epsilon^{-3}, \epsilon^{-2} \) and \( \epsilon^{-1} \) of these integrals are given, which is sufficient for the calculation of the quantities \( A_i \) within our accuracy. We shall use the notation

\[
I\{F\} \equiv \int_1^\infty \frac{dk}{k^{1+\epsilon}} \int_1^\infty \frac{dq}{q^{1+\epsilon}} \int_1^\infty \frac{dl}{l^{1+\epsilon}} F
\]

for any function \( F = F(k, q, l) \). The integrals \( I_1-I_9 \) are as follows:

\[
I_1 \equiv I\left\{ \frac{k^{2n_1}q^{2n_2}l^{2n_3}}{(k^2+q^2+l^2)^{n_1}(q^2+l^2)^{n_3}} \right\} = \frac{(n_1-1)(n_2-1)(n_3-1)(n_4-1)}{12\epsilon (n_4-1)(n_2+n_3-1)}
\]

(6.31)

with \( n_1+n_2+n_3 = n_4+n_5, n_4 > n_1, n_{1,2,3,4} > 0 \) and \( n_5 \geq 0 \);

\[
I_2 \equiv I\left\{ \frac{q^{2n_2}l^{2n_3}}{(k^2+q^2+l^2)^{n_4}(q^2+l^2)^{n_5}} \right\} = \frac{(n_2-1)(n_3-1)}{12(n_2+n_3-1)} \left[ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left( \sum_{k=1}^{n_2+n_3-1} \frac{1}{k} - \sum_{k=1}^{n_4-1} \frac{1}{k} - \frac{1}{2} \sum_{k=1}^{n_2-1} \frac{1}{k} - \frac{1}{2} \sum_{k=1}^{n_3-1} \frac{1}{k} \right) \right]
\]

(6.32)

with \( n_2+n_3 = n_4+n_5, n_{2,3,4} > 0 \) and \( n_5 \geq 0 \) (in Eq. (6.32) and all the formulas below, any sum with the upper limit lesser than the lower one is understood as equal to zero);

\[
I_3 \equiv I\left\{ \frac{k^{2n_1}q^{2n_2}}{(k^2+q^2+l^2)^{n_4}} \right\} = \frac{(n_1-1)(n_2-1)}{12(n_4-1)} \left[ \frac{1}{\epsilon^2} - \frac{1}{2\epsilon} \left( \sum_{k=1}^{n_1-1} \frac{1}{k} + \sum_{k=1}^{n_2-1} \frac{1}{k} \right) \right]
\]

(6.33)

with \( n_1+n_2 = n_4 \) and \( n_{1,2} > 0 \);

\[
I_4 \equiv I\left\{ \frac{k^{2n_1}q^{2n_2}}{(k^2+q^2+l^2)^{n_4}(q^2+l^2)} \right\} = \frac{(n_1-1)(n_2-1)}{12\epsilon (n_4-1)} \left[ \frac{1}{\epsilon^2} - \frac{1}{2\epsilon} \left( \frac{2}{(n_2-1)} + \sum_{k=1}^{n_1-1} \frac{1}{k} + \sum_{k=1}^{n_2-2} \frac{1}{k} \right) \right]
\]

(6.34)

with \( n_1+n_2 = n_4+1, n_1 > 0 \) and \( n_2 > 1 \);

\[
I_5 \equiv I\left\{ \frac{k^2}{(k^2+q^2+l^2)^{n}} \left( \frac{q^2}{q^2+l^2} \right)^{2n} \right\} = \frac{(n-1)!!}{12(2n-1)!} \left[ \frac{2}{\epsilon^2} - \frac{1}{\epsilon} \left( 3B_{n-1} + \sum_{k=n}^{2n-1} \frac{1}{k} \right) \right]
\]

(6.35)
with \( n > 0 \) and \( B_n \) from Eq. \((6.34)\):

\[
I_6 \equiv I \left\{ \frac{k^2 q^{2(n+1)} l^{2n}}{(k^2 + q^2 + l^2)(q^2 + l^2)^{2n+1}} \right\} = \frac{1}{2} I_5;
\]

\[
I_7 \equiv I \left\{ \frac{k^2}{k^2 + q^2 + l^2} \right\} = \frac{1}{3 \varepsilon^3};
\]

\[
I_8 \equiv I \left\{ \frac{k^2 q^2}{(k^2 + q^2 + l^2)(q^2 + l^2)} \right\} = \frac{1}{6 \varepsilon^3};
\]

\[
I_9 \equiv I \left\{ \frac{q^4}{(k^2 + q^2 + l^2)(q^2 + l^2)} \right\} = \frac{1}{3 \varepsilon^3} - \frac{1}{12 \varepsilon^2} - \frac{1}{12 \varepsilon}.
\]

Now let us turn to the derivation of expressions \((6.31)–(6.39)\). The integrals \(I_{1,2}\) can be obtained from the generation function

\[
R(a, b; m) \equiv \int_0^\infty \frac{dk}{k^{1+\varepsilon}} \int_0^\infty \frac{dp}{\rho^{1+2\varepsilon}} \frac{\rho^2}{(ak^2 + b \rho^2)} = \frac{m^{-3\varepsilon}}{6b} \left[ \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \ln(b/a) + O(1) \right].
\]

We shall also need the first terms of the \(\varepsilon\) expansion of the integral

\[
J(p, q) \equiv \int_0^{\pi/2} d\varphi (\cos \varphi)^{2p-1-\varepsilon} (\sin \varphi)^{2q-1-\varepsilon} = \frac{\Gamma(p-\varepsilon/2)\Gamma(q-\varepsilon/2)}{2\Gamma(p+q-\varepsilon)} = \frac{\Gamma(p)\Gamma(q)}{2\Gamma(p+q)} \left\{ 1 + \varepsilon \left[ \psi(p+q) - \frac{1}{2} \psi(p) - \frac{1}{2} \psi(q) \right] \right\} + O(\varepsilon^2) = \frac{(p-1)!(q-1)!}{2(p+q-1)!} \left[ 1 + \varepsilon \left( \sum_{k=1}^{p+q-1} \frac{1}{k} - \frac{1}{2} \sum_{k=1}^{p-1} \frac{1}{k} - \frac{1}{2} \sum_{k=1}^{q-1} \frac{1}{k} \right) \right] + O(\varepsilon^2),
\]

where \(\Gamma(z)\) is the Euler Gamma function and \(\psi(z) = d\ln \Gamma(z)/dz\).

The calculation of the quantity \(R \equiv R(a, b; m) = m^{-3\varepsilon} R(a, b; 1)\) is similar to the derivation of Eq. \((5.33)\). The operation \(D_m \equiv m \partial / \partial m\) is applied to the double integral in Eq. \((6.40)\), which reduces it to the sum of two single integrals and explicitly isolates the pole factor \(\varepsilon^{-1}\):

\[
R = -\frac{1}{3 \varepsilon} D_m R = \frac{m^{-3\varepsilon}}{3 \varepsilon} \left[ \int_1^\infty \frac{dk}{k^{1+\varepsilon}(ak^2 + b)} + \int_1^\infty \frac{dp}{\rho^{1+2\varepsilon}} \frac{\rho^2}{(ak^2 + b \rho^2)} \right].
\]

Now one can set \(\varepsilon = 0\) in the integral over \(k\) (it remains finite at \(\varepsilon = 0\)). The pole in \(\varepsilon\) in the integral over \(\rho\) comes from large \(\rho\) and it can be isolated explicitly:

\[
\int_1^\infty \frac{dp}{\rho^{1+2\varepsilon}} \frac{\rho^2}{(a + b \rho^2)} = \frac{1}{b} \int_1^\infty \frac{dp}{\rho^{1+2\varepsilon}} - \frac{a}{b} \int_1^\infty \frac{dp}{\rho^{1+2\varepsilon}} \frac{1}{(a + b \rho^2)} = \frac{1}{2b \varepsilon} - \frac{a}{b} \int_1^\infty \frac{dp}{\rho(a + b \rho^2)} + O(\varepsilon),
\]

which immediately leads to the answer \((6.40)\).

In the integral \(I_1\) the pole contribution comes from the part of the integration region where all three momenta \(k, q, l\) simultaneously tend to infinity, and in \(I_2\) there is also a pole contribution coming from the subregion where \(q, l\) simultaneously tend to infinity at fixed finite \(k\). This means that the pole parts of the integrals \(I_{1,2}\) are not affected if one changes to the polar coordinates \(\rho, \varphi\) \((q = \rho \cos \varphi, l = \rho \sin \varphi)\) with the integration region \(0 \leq \varphi \leq \pi/2, 1 \leq \rho < \infty\), that is, the following replacement is performed:

\[
\int_1^\infty dq \int_1^\infty dl \ldots \rightarrow \int_0^{\pi/2} d\varphi \int_1^{\infty} \rho d\rho \ldots .
\]

The integration regions in these two expressions are not identical, but the values of integrals differ only by an unessential contribution, finite at \(\varepsilon \rightarrow 0\). For \(I_1\) this gives:
\[ I_1 = \int_0^{\pi/2} d\varphi (\cos \varphi)^{2n_2-1-\varepsilon} (\sin \varphi)^{2n_3-1-\varepsilon} \int_1^\infty \frac{dk}{k^{1+\varepsilon}} \int_1^\infty \frac{dp}{p^{1+2\varepsilon}} \frac{k^{2n_1} \rho^{2n_4-2n_1}}{(k^2 + \rho^2)^{n_4}}. \]

This integral can be expressed using \( J(p, q) \) from Eq. (6.41) and the derivative of the generating function \( R(a, b; m) \) from Eq. (6.42) as follows:

\[ I_1 = [(n_4 - 1)!]^{-1} J(n_2, n_3)(-\partial_a)^{n_1} (-\partial_b)^{n_4-n_1-1} R(a, b; 1)|_{a=b=1}, \quad (6.42) \]

with \( \partial_a = \partial/\partial a, \partial_b = \partial/\partial b \), which immediately leads to the answer (6.31). Note that the \( \varepsilon^{-2} \) contribution in \( R \) depends only on \( b \) (see Eq. (6.40)) and therefore vanishes in the above expression owing to the differentiation with respect to \( a \) (we recall that \( n_1 > 0 \) for \( I_1 \)). Although the integral \( I_2 \) is formally given by expression (6.42) with \( n_1 = 0 \),

\[ I_2 = [(n_4 - 1)!]^{-1} J(n_2, n_3)(-\partial_b)^{n_4-1} R(1, b; 1)|_{b=1}, \quad (6.43) \]

the \( \varepsilon^{-2} \) part of the function (6.41) survives in Eq. (6.43) owing to the absence of the derivative \( \partial_a \), so that the \( O(\varepsilon) \) contributions should be taken into account in the integral (6.43). This explains the sharp difference between the final expressions (6.31), (6.32) for the integrals \( I_1 \) and \( I_2 \).

The relations (6.33)–(6.34) follow easily from (6.31), (6.32). The result (6.33) for \( I_3 \) is obtained from Eq. (6.32), at \( n_2 = 0 \) using the symmetries of the integrand. In its turn, the result (6.34) for \( I_4 \) follows from Eqs. (6.31), (6.33) if the obvious identity \( q^{2n_2}/(q^2 + l^2) = q^{2(n_2-1)} - q^{2(n_2-1)}l^2/(q^2 + l^2) \) is substituted on its left-hand side. The result (6.35) for \( I_5 \) is obtained using the substitution \( k^2/(k^2 + q^2 + l^2) = 1 - q^2/(k^2 + q^2 + l^2) \); in the first term, the integral over \( k \) is trivial and the remaining integrations over \( l, q, l \) are performed using Eqs. (6.30), (6.33), (6.34), while the second term reduces to \( I_2 \). The integral \( I_6 \) in Eq. (6.36) reduces to \( I_5 \) after the symmetrization of the integrand with respect to \( q, l \).

The results (6.37), (6.38) follow from the obvious relation \( I\{1\} = \varepsilon^{-3} \) in Eq. (6.30) after the symmetrization of the integrands with respect to \( k, q, l \). The last relation (6.39) is obtained using the substitution \( q^2/(q^2 + l^2) = 1 - l^2/(q^2 + l^2) \); the first term coincides with \( I_7 \) and the second one is given by \( I_2 \) with \( n_2 = n_3 = n_4 = n_5 = 1 \).

### E. Anomalous exponents at large \( d \)

In this Section, we shall briefly discuss the behavior of the coefficients \( \Delta^{(k)}_{nl} \) in the expansion (1.10) for \( d \to \infty \). The model (1.1)–(1.3) has no finite upper critical dimension, above which the anomalous scaling vanishes. It disappears for infinite \( d \) [28], but reveals itself already in the \( O(1/d) \) approximation [6]. This fact confirms the relevance of the large \( d \) expansions for the issue of anomalous scaling; see also Refs. [29,30] for the discussion of the Navier–Stokes problem.

It follows from Eqs. (5.24) that for \( d \to \infty \), the angular averages \( \alpha_{2n} \equiv \langle \cos^{2n} \theta \rangle \) behave as \( \alpha_{2n} \propto d^{-n} \) (each additional factor \( \cos^2 \theta \) introduces additional smallness in 1/\( d \)). Then from Eqs. (5.24)–(5.27) for the averages (6.17) one obtains \( T_{n_1 n_2 n_3} \propto d^{-(n_1+n_2+n_3)/2} \) for \( n_{1,2,3} \) even and \( T_{n_1 n_2 n_3} \propto d^{-(n_1+n_2+n_3+1)/2} \) for \( n_{1,2,3} \) odd. This means that, in order to find the behavior of the coefficients \( \Delta_{A_1} \) for large \( d \) to any given finite order in 1/\( d \), one needs to take into account only finite number of terms in the expansion of the integrands in Eq. (4.37b) in the scalar product (\( \mathbf{kq} \)), (\( \mathbf{kq} \)), and integrands in Eq. (4.37d) in the set of scalar products (\( \mathbf{q} \)), (\( \mathbf{q} \)), (\( \mathbf{q} \)).

The \( d \) dependence of the coefficient \( \Delta_{nl}^{(1)} \) in Eq. (1.10) is known from Eq. (1.11), while the quantities \( \Delta_{nl}^{(2)}, \Delta_{nl}^{(3)} \) can be found as series in 1/\( d \) to any given order as explained above. For general \( n \) and \( l \) to order 1/\( d^2 \) we have obtained

\[ \Delta_{nl} = \varepsilon \left[ -n(n-1)(1-2d)/2d + (l/2)(1-2d+l/2)/2d \right] + 3\varepsilon^2(n-2)(n-1)/4d^2 + \varepsilon^3(n-1)(1.74988(n-2) - 0.624916l)/d^2 + O(\varepsilon^4). \]

(6.44)

Note that the \( \varepsilon^2 \) and \( \varepsilon^3 \) contributions decay for \( d \to \infty \) faster than 1/\( d \) in agreement with the \( O(1/d) \) result obtained in Ref. [6] for \( \Delta_{nl} \). Moreover, from Eq. (6.14) it follows that the leading \( O(1/d^2) \) terms in these contributions vanish for \( n = l \), so that the decay at \( d \to \infty \) becomes even faster. For \( n = l \) we have obtained

\[ \Delta_{nn} = \varepsilon n/2 + n(n-1) \left\{ \varepsilon/(d-1)(d+2) - \varepsilon^2[1 + (2n-7)/d]/d^2 - \varepsilon^2(3n-8)/2d^2 \right\} + O(\varepsilon^4) \]

(6.45)

with the accuracy of \( O(1/d^4) \).
VII. CONVERGENCE OF THE $\varepsilon$ EXPANSION, INVERSE $\varepsilon$ EXPANSION, AND COMPARISON WITH NONPERTURBATIVE RESULTS

An important issue which can be discussed on the example of the rapid-change model is that of the nature and convergence of $\varepsilon$ expansions in models of turbulence and the possibility of their extrapolation to finite values of $\varepsilon \sim 1$. The knowledge of the three terms of the $\varepsilon$ expansion in model (1.1)–(1.3) allows one to discuss its convergence properties and to obtain improved predictions for finite $\varepsilon$ in reasonable agreement with the existing nonperturbative results: analytical solution of the zero-mode equations for $n = 2$ [5], numerical solutions for $n = 3$ [6] and numerical experiments for $n = 4$ [23,24] and $n = 6$ [25].

In Figs. 3a and 3b, we show the anomalous dimension $\gamma = \gamma_{22}^\pm \equiv \gamma_{22}(u_n)$ for $d = 3$ (a) and 2 (b) in the $O(\varepsilon)$, $O(\varepsilon^3)$ and $O(\varepsilon^2)$ approximations (from above to below): the latter two obtained as simple sums of two and three terms of the $\varepsilon$ expansion, respectively. The dashed line corresponds to the exact solution by Ref. [3]; see also Ref. [31] for the special cases $d = 2$ and 3. Analogous diagrams for the cases $n = 3, l = 1$ and $n = 4, l = 0$ can be found in Ref. [12].

All these figures show that the agreement between the $\varepsilon$ expansion and nonperturbative results for small $\varepsilon$ improves when the higher-order terms are taken into account, but the deviation becomes remarkable for $\varepsilon \sim 1$ and decreasing $d$. Furthermore, the convergence of the $\varepsilon$ series appears more irregular for $d = 2$, while the forms of the nonperturbative results are not much affected by the choice of $d$.

Such behavior can be naturally understood on the example of the exact analytical result for $\gamma = \gamma_{22}^\pm$, which can be written in the form

$$
2\gamma = -(d + 2 + \varepsilon) + \sqrt{(\varepsilon + \varepsilon_+)(\varepsilon + \varepsilon_-)}, \quad \varepsilon_{\pm} = \left( d^2 + d + 2 \pm \sqrt{8d(d + 1)} \right) / (d - 1)
$$

(7.1)

with $\varepsilon_+\varepsilon_- = (d + 2)^2$. It is useful to rewrite the smaller quantity $\varepsilon_-$ in the form

$$
\varepsilon_- = (d - 1) \left\{ 1 + \left( 1 + 3d + \sqrt{8d(d + 1)} \right)^{-1} \right\}.
$$

(7.2)

From these expressions it follows that the corresponding $\varepsilon$ expansion has the finite radius of convergence $\varepsilon_-$, ranging from 0 to $\infty$ when $d$ varies from 1 to $\infty$ (in particular, $\varepsilon_- \sim 1.1$, $\varepsilon_+ \sim 14.5$ for $d = 2$ and $\varepsilon_- \sim 2.1$, $\varepsilon_+ \sim 11.9$ for $d = 3$). Hence, the naive summation of the $\varepsilon$ expansion for $\gamma$ works only in the interval $\varepsilon < \varepsilon_-$, which decreases almost linearly with $(d - 1)$. Since the singularity in Eq. (7.1) occurs for negative $\varepsilon$, it affects strongly the convergence of the $\varepsilon$ expansion but is not “visible” in the form of the exact curve for positive $\varepsilon$, in contrast with the singularities occurring at $\varepsilon = 2$ in higher-order critical dimensions [32,33,23,24].

Therefore, in order to recover the behavior of $\gamma$ from its $\varepsilon$ series for larger $\varepsilon$, it is necessary to isolate explicitly the singularity at $\varepsilon = -\varepsilon_-$ in Eq. (7.1), thus changing to a kind of improved $\varepsilon$ expansion, whose radius of convergence is determined by a more distant singularity. This can be done, for example, by introducing the new expansion parameter $x = \sqrt{\varepsilon + \varepsilon_-} - \sqrt{\varepsilon_-}$, that is, $\varepsilon = x^2 + 2x\sqrt{\varepsilon_-}$. Then Eq. (7.1) can be written in the form

$$
2\gamma = -(d + 2 + 2x\sqrt{\varepsilon_-} + x^2) + (x + \sqrt{\varepsilon_-}) \sqrt{x^2 + 2x\sqrt{\varepsilon_-} + \varepsilon_+}.
$$

(7.3)

The convergence radius of the expansion in $x$ is found from the equation $x^2 + 2x\sqrt{\varepsilon_-} + \varepsilon_+ = 0$ with the solutions $x_{\pm} = -\sqrt{\varepsilon_-} \pm i\sqrt{\varepsilon_+ - \varepsilon_-}$ and is therefore equal to $|x_{\pm}| = \sqrt{\varepsilon_+}$. For positive $x$ and $\varepsilon$, this corresponds to the convergence for $0 < x < \sqrt{\varepsilon_+}$ or, equivalently, $0 < \varepsilon < \varepsilon_+ + 2\sqrt{\varepsilon_-\varepsilon_+} = \varepsilon_+ + 2(d + 2)$.

The improvement of the convergence is illustrated by Fig. 3c, where the exact exponent $\gamma$ for $d = 2$ is shown as a function of $\varepsilon$ along with its first (x), second ($x$ and $x^2$) and third ($x$, $x^2$ and $x^3$) orders of the improved $x$ expansion, in which the variable $x$ is also expressed as a function of $\varepsilon$. One can see that the convergence of the $x$ expansion appears more regular and that its third-order approximation is hardly distinguishable from the exact result for all $0 < \varepsilon < 2$ (for $d = 3$, the agreement is even better and for this reason is not shown). It should be stressed that crucial for the improvement was not the existence of the exact solution or explicit form of the substitution $x(\varepsilon)$ but the knowledge of the character and location of the singularity which determines the convergence properties of the plain $\varepsilon$ expansion.

The difference with the models of critical phenomena, where $\varepsilon$ series are always asymptotical, can be traced back to the fact that in the rapid-change models, there is no factorial growth of the number of diagrams in higher orders of the perturbation theory. The divergence for $d \to 1$ is naturally explained by the fact that the transverse vector field does not exist in one dimension. We also recall that the RG fixed point diverges at $d = 1$; see Eq. (2.18), so that the coefficients of the $\varepsilon$ series diverge for all dimensions $\Delta n$.

It is then natural to assume that the $\varepsilon$ series for all $\Delta n$ also have finite radii of convergence with the behavior similar to that of $\varepsilon_-$ in Eq. (7.2). Therefore, in order to improve their convergence and to obtain reasonable predictions for
finite values of \(\varepsilon\), one should augment plain \(\varepsilon\) expansions by the information about the character of the singularities and their location in the complex \(\varepsilon\) plane. Such information can be extracted from the asymptotical behavior of the coefficients \(\Delta_{n}\) in Eq. (1.11) at large \(k\). To our knowledge, the large \(k\) behavior of the \(\varepsilon\) series remains an open problem for any dynamical model; the instanton analysis developed in Refs. \[22\] has mostly been concentrated on the behavior of the exponents in the limit \(n \to \infty\). One can hope that the implementation of the instanton calculus within the RG framework will give the solution of this important problem.

It turns out, however, that certain elementary considerations allow one to improve the convergence of the \(\varepsilon\) series and, at the same time, to achieve a better agreement with the nonperturbative results. Let us explain the idea on the example of the exact solution \(\gamma\). We express \(\varepsilon\) as a function of the exponent \(\gamma\) using Eq. (7.1) and expand the right-hand side of the resulting exact relation

\[
\varepsilon = \gamma (d - 1)(d + 2 + \gamma)/[2 - \gamma(d - 1)]
\]

in \(\gamma\); this gives the “inverse \(\gamma\) expansion.” It is easy to see that, for \(\varepsilon \sim 1\) and physical dimensions \(d = 2\) and \(3\), the respective values of \(\gamma\) lie within the region of convergence of the inverted series: \(\gamma < 2/(d - 1)\). The improvement of the convergence is also seen from Fig. 3d, where the exponent \(\gamma = \gamma_{d2}\) for \(d = 2\) is shown as a function of \(\varepsilon\) along with the first \((\gamma)\), second \((\gamma\) and \(\gamma^2\)) and third \((\gamma\), \(\gamma^2\) and \(\gamma^3\)) orders of the \(\gamma\) expansion, expressed in the original variable \(\varepsilon\): the approximate curves approach the exact curve from the same side and represent the exact result \(\gamma\) much better than the corresponding approximations of the plain \(\varepsilon\) expansion. The improvement is even better for \(d = 3\).

A simple explanation of the improvement follows. The convergence radius for the direct \(\varepsilon\) series is determined by the singularity in the right-hand side of Eq. (7.1), closest to the origin, that is, \(\varepsilon_c = -\varepsilon\). This square-root singularity disappears in the inverse relation (4.4), that is, the dependence of \(\varepsilon\) on \(\gamma\) in the vicinity of the corresponding point \(2\gamma_c = -d - 2 - \varepsilon_c = -d + 2 + \varepsilon\). becomes analytic. This would also happen for any singularity of the form \(\gamma - \gamma_c \propto (\varepsilon - \varepsilon_c)^{1/k}\) with any integer \(k \geq 0\).

We assume that these features are also typical to the higher-order dimensions \(\Delta_{nl}\) and construct the corresponding \(\gamma\) expansions. It is important here that \(\gamma \equiv \gamma_{nl} = O(\varepsilon)\), so that there is a one-to-one correspondence between these two expansions and three terms of the \(\gamma\) expansion can be immediately obtained from Eqs. \(\text{[1.7]}\), \(\text{[1.13]}\) and \(\text{[1.14]}\). This simple procedure leads to remarkable improvement of the convergence and, at the same time, the agreement with the numerical results, as is easily seen from Figs. 4 and 5.

In Fig. 4, we show the anomalous dimension \(\gamma \equiv \gamma_{d3}\) for \(d = 3\) (left) and \(d = 2\) (right): the \(O(\varepsilon)\) approximation, the third-order approximation of the inverse \(\gamma\) expansion, and the third-order approximation of the plain \(\varepsilon\) expansion (from above to below); the dotted lines represent the exact numerical solution by Refs. \(\text{[21]}\) \((\gamma = \lambda - 3\) in the notation of \(\text{[20]}\)).

In Figs. 5a and 5b, we show the quantities \(\Delta_n \equiv \Delta_{n0}\) in three dimensions for \(n = 6\) \((a)\) and \(n = 4\) \((b)\): the \(O(\varepsilon)\) slope, the third-order approximation of the \(\gamma\) expansion, and the third-order approximation of the plain \(\varepsilon\) expansion (from above to below); the dots connected by dashed lines represent the results of the numerical simulations by Refs. \(\text{[23]}\) \((n = 6)\) and \(\text{[24]}\) \((n = 4)\). In all these cases, the improvement in the agreement with nonperturbative results is obvious. It should be emphasized, however, that even the plain \(\varepsilon\) expansion captures some qualitative features of the dimensions \(\Delta_n\) established in the numerical simulations \(\text{[23]} \text{[24]}\); the quantity \(|\Delta_n|\) increases with \(\varepsilon\), achieves a maximum at some point inside the interval \(0 < \varepsilon < 2\) and then decreases to zero; the height of the maximum increases and its position moves to the left as \(n\) grows from 4 to 6 or \(d\) decreases from 3 to 2 \((\varepsilon_{2n} - n\varepsilon_2 = \Delta_{2n}\) in the notation of Refs. \(\text{[23]}\) \text{[24]}\)).

It is no surprise, of course, that the disagreement between the perturbative and nonperturbative results becomes rather strong for \(\varepsilon > 1\); this can be explained, for example, by the effect of the singularity at \(\varepsilon = 2\) in the exact solutions \(\text{[7]} \text{[8]} \text{[23]} \text{[24]}\) and by insufficient number of the known terms in the \(\varepsilon\) and \(\gamma\) series. For the case \(n = 4, d = 3\), the situation can be improved by an interpolation formula which takes into account the first terms of the \(\varepsilon\) expansion along with the asymptotical behavior of the dimension \(\Delta_4\) in the vicinity of the opposite edge \(\varepsilon = 2\), known from the numerical simulation \(\text{[23]} \text{[24]}\): \(\Delta_4 = -\left[0.06 (2 - \varepsilon) + 1.13 (2 - \varepsilon)^{3/2}\right]\). In particular, one can choose:

\[
\Delta_1 = -\left[c_1 \varepsilon + c_2 \varepsilon^2 + c_3 \varepsilon^3 + \ldots + c_k \varepsilon^k + \ldots\right]\left[0.06 (2 - \varepsilon) + 1.13 (2 - \varepsilon)^{3/2}\right].
\]

The first coefficients \(c_1 - c_p\) are determined by the requirement that the expansion in \(\varepsilon\) of the right-hand side of Eq. (7.3) reproduce the first \(p\) terms of the \(\varepsilon\) expansion for \(\Delta_4\), known from the RG (therefore, in practice one can only take \(p \leq 3\)). The values of these coefficients, once determined, will not change if one takes a larger value of \(p\). The remaining coefficients \(c_k\) with \(k > p\) should be chosen to reproduce the correct behavior at \(\varepsilon \to 2\). The simplest possibility is to set \(c_k = 0\) for all \(k > p + 1\); then the last remaining coefficient \(c_{p+1}\) is unambiguously determined by the relation \(2(c_1 + 4c_2 + 8c_3 + \ldots + 2^{p+1}c_{p+1}) = 1\) \((O(2 - \varepsilon)\) terms in the expression in the first square brackets produce only corrections of order \((2 - \varepsilon)^2\) to the behavior of the right-hand of Eq. (7.5) side at \(\varepsilon \to 2\) and thus
should be neglected]. This procedure gives $c_1 = 0.241$, $c_2 = 0.129$ for $p = 1$ (upper curve in Fig. 5c) and $c_1 = 0.241$, $c_2 = 0.168$, $c_3 = -0.225$, $c_4 = 0.103$ for $p = 3$ (lower curve). One can see that the inclusion of the higher orders of the $\varepsilon$ expansion improves remarkably the agreement with the “experimental” results of Refs. [23,24], shown in Fig. 5c by the dots connected by dashed lines. It is also worth noting that the value of the coefficient $c_2$ for $p = 1$ (determined by the behavior at $\varepsilon \to 2$) appears rather close to its value for $p = 3$ (determined by the $\varepsilon$ expansion), which demonstrates the robustness of the results obtained by the above procedure.

FIG. 3. Anomalous dimension $\gamma \equiv \gamma_{22}$ for $d = 3$ (a) and $d = 2$ (b,c,d). Dashed line: exact solution by Refs. [5-8]. Solid lines in Figs. 3a and 3b: first, third and second approximations of the plain $\varepsilon$ expansion (from above to below). Solid lines in Fig. 3c: first, third and second approximations of the improved $\varepsilon$ expansion (from above to below; the third approximation is practically indistinguishable from the exact solution for all $0 < \varepsilon < 2$). Solid lines in Fig. 3d: first, second and third approximations of the inverse $\gamma$ expansion (from above to below).

FIG. 4. Anomalous dimension $\gamma \equiv \gamma_{31}$ for $d = 3$ (left) and $d = 2$ (right): the $O(\varepsilon)$ approximation, the third-order approximation of the inverse $\gamma$ expansion, and the third-order approximation of the plain $\varepsilon$ expansion (from above to below). Dashed line: numerical solution by Refs. [9].

FIG. 5. Critical dimensions $\Delta_n$ for $d = 3$: $n = 6$ (a) and $n = 4$ (b,c). Dots connected by dashed lines: numerical simulations by Refs. [25] ($n = 6$) and [23,24] ($n = 4$). Solid lines in Figs. 5a,b: the $O(\varepsilon)$ slope, third-order approximation of the $\gamma$ expansion, and third-order approximation of the plain $\varepsilon$ expansion (from above to below). Solid lines in Fig. 5c: approximations for $n = 4$, obtained using the interpolation formula (7.5) with $p = 1$ (upper curve) and $p = 3$ (lower curve).
VIII. CONCLUSION

To conclude with, we have studied the inertial-range anomalous scaling of a passive scalar quantity advected by the Gaussian velocity field, white in time and self-similar in space. The corresponding stochastic problem (1.1)–(1.3) can be reformulated as a field theoretic model (2.1), which allows one to identify the anomalous exponents with the critical dimensions of certain scalar and tensor composite operators built of the scalar gradients and to calculate them within the RG and OPE approach in the form of a regular perturbation expansion, similar to the well-known $\varepsilon$ expansion in the RG theory of critical behavior.

Earlier, the anomalous exponents were presented to orders $\varepsilon^2$ [10–12,14] and $\varepsilon^3$ [15]; the main goal of the present paper has been the detailed explanation of the corresponding calculational techniques and derivation of the three-loop result, including the anisotropic sectors. Owing to the comparative universality of the RG and OPE formalism, these techniques can be applied to other models of dynamical critical phenomena and systems far from equilibrium: passive advection by the non-Gaussian velocities with finite correlation time, stochastic Navier–Stokes equation and so on; see Refs. [13,21,22].

Another scope of the paper has been the discussion of the convergence properties of the $\varepsilon$ expansion and the possibility of its extrapolation to finite values of $\varepsilon$. It was shown that the knowledge of three terms allows one to obtain reasonable predictions for finite $\varepsilon \sim 1$; even the plain $\varepsilon$ expansion captures some subtle qualitative features of the anomalous exponents established in numerical experiments.

We believe that the framework of the renormalization group and operator product expansion, the concept of dangerous composite operators and the $\varepsilon$ expansion will become the necessary elements of the appearing theory of the anomalous scaling in fully developed turbulence.

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APPENDIX A: COEFFICIENTS $A_i$ FOR THE THREE-LOOP DIAGRAMS

Below we give the pole parts of the coefficients $A_i$ from Eq. (4.20) for all normal (not factorizable) three-loop diagrams. We use the following notation:

$$A_i = u^3 (\mu/m)^3 \varepsilon \left\{ A_i^{(3)} \varepsilon^{-3} + A_i^{(2)} \varepsilon^{-2} + A_i^{(1)} \varepsilon^{-1} \right\}, \quad A_i^{(3)} = (d-1) a_i(d)/432d^2(d+2)^2.$$

Coefficients $A_i^{(1,2)}$ are given in Tables 1 and 2 for $d = 2$ and $d = 3$, respectively; coefficients $a_i(d)$ are given in Table 3 for general $d$.

| No | $A_i^{(2)} \cdot 10^3$ | $A_i^{(1)} \cdot 10^4$ | $A_i^{(2)} \cdot 10^3$ | $A_i^{(1)} \cdot 10^4$ | $A_i^{(2)} \cdot 10^3$ | $A_i^{(1)} \cdot 10^4$ | $A_i^{(2)} \cdot 10^3$ | $A_i^{(1)} \cdot 10^4$ |
|----|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| 23 | 0.23875              | -0.06103             | 0                    | -0.06103             | -0.06103             | -0.06103             | -0.06103             | -0.06103             |
| 32 | -0.56396             | 0.32532              | 0.49056              | -0.56645             | -0.56645             | -0.56645             | -0.56645             | -0.56645             |
| 33 | 1.11953              | -0.49834             | 0.93382              | -1.04911             | -1.04911             | -1.04911             | -1.04911             | -1.04911             |
| 34 | -1.05470             | -0.12495             | 0.2434               | -0.5770              | -0.5770              | -0.5770              | -0.5770              | -0.5770              |
| 35 | -0.41468             | 0.30671              | -0.63626             | 0.22330              | 0.22330              | 0.22330              | 0.22330              | 0.22330              |
| 42 | -0.00053             | -0.11026             | 0.06564              | -0.16222             | -0.16222             | -0.16222             | -0.16222             | -0.16222             |
| 43 | 0.02902              | 0.00249              | 0.03161              | -0.00715             | -0.00715             | -0.00715             | -0.00715             | -0.00715             |
| 44 | -0.03113             | -0.01477             | 0.03350              | -0.00286             | -0.00286             | -0.00286             | -0.00286             | -0.00286             |
| 45 | -0.26802             | 0.26530              | -0.02352             | 0.03554              | 0.03554              | 0.03554              | 0.03554              | 0.03554              |
TABLE II. Coefficients $A_{i}^{(1,2)}$ for $d = 3$.

| No | $A_{1}^{(2)} \cdot 10^{3}$ | $A_{1}^{(1)} \cdot 10^{3}$ | $A_{2}^{(2)} \cdot 10^{3}$ | $A_{2}^{(1)} \cdot 10^{3}$ | $A_{3}^{(2)} \cdot 10^{3}$ | $A_{3}^{(1)} \cdot 10^{3}$ |
|----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 23 | 1.03775         | -0.14683        | 0               | 0               |                 |                 |
| 32 | -0.52289        | 0.39815         | 0.21884         | -0.50345        |                 |                 |
| 33 | 1.57028         | 0.00156         | 1.56822         | -0.76955        |                 |                 |
| 34 | -1.64375        | -0.82315        | -0.43185        | -1.46450        |                 |                 |
| 35 | -0.43289        | 0.51544         | -1.07015        | 0.37916         |                 |                 |
| 42 | -0.07198        | -0.08409        | 0.02952         | -0.83453        | 0.37471         |                 |
| 43 | 0.02808         | 0.00152         | 0.03587         | -0.01186        | -0.62532        | 0.07298         |
| 44 | -0.07726        | 0.01392         | 0.00504         | 0.02341         | -0.36121        | 0.37991         |
| 45 | -0.33881        | 0.33509         | -0.10389        | 0.11756         | 0.94431         | -0.95923        |

TABLE III. Coefficients $a_i(d)$ for general $d$.

| No | $a_1(d)$ | $a_2(d)$ | $a_3(d)$ |
|----|----------|----------|----------|
| 23 | $9(d+1)(2d^2 + 2d - 5)$ | $27(d-1)^4(d+2)^4$ | — |
| 32 | $6(d+1)(d^2 + d - 3)$ | $2d^4 + 3d^3 - 3d^2 - 7d + 10$ | — |
| 33 | $-12(d+1)$ | $-4(d^2 + 2d - 2)$ | — |
| 34 | $3(d+1)(3d^2 + 3d - 8)$ | $3d^4 + 8d^3 - 13d^2 - 22d + 32$ | — |
| 35 | $-12(d+1)$ | $-4(d^2 + 2d - 2)$ | — |
| 42 | $3(d+1)(d^2 + d - 8)/4$ | $(d^4 + 6d^3 + 5d^2 - 16d + 32)/8$ | $-4(d^2 + 2d - 2)$ |
| 43 | $-6(d+1)$ | $(d^5 + d^2 - 4d + 8)/2$ | $d^4 + 4d^3 + d^2 - 6d + 8$ |
| 44 | $-6(d+1)$ | $(d^5 + d^2 - 4d + 8)/2$ | $d^4 + 4d^3 + d^2 - 6d + 8$ |
| 45 | $3(d+1)(d^2 + d - 8)/4$ | $(d^4 + 6d^3 + 5d^2 - 16d + 32)/8$ | $-4(d^2 + 2d - 2)$ |
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