Elementary Processes in Dissipative Cosmic Medium

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ABSTRACT

Quantum dynamics of a finite degrees of freedom is often much affected by interaction with the larger environment of cosmic medium. In this lecture we first review some recent developments of the theory of quantum dissipation in the linear open system. In the second part we discuss two classes of applications: decay of unstable particle in medium, and environmental effect on the parametric particle production. The first subject of particle decay may have important consequences on the scenario of baryogenesis due to a GUT heavy particle. On the other hand, the parametric resonant particle production is related to the reheating problem after inflation, and its environmental effect is related also, perhaps more importantly, to the decay of moduli fields in supergravity models.

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I Introduction

Ideal elementary process rarely occurs, presumably except in carefully prepared laboratory experiments. It might be useful to recall that even processes usually considered elementary can be regarded taking place in complex environments. This has to do with how one separates a system in question from a surrounding environment. For instance, the beta decay, the fundamental weak process of $n \rightarrow p + e + \bar{\nu}_e$, when it occurs in nuclei, is compounded by effects of nuclear strong interaction with the rest of nucleons. Presumably the most spectacular of this kind is how nucleon decay proceeds. The extremely weak process of baryon number violating process at the quark-lepton level must inevitably occur in the hadronic environment. A repeated question on the nucleon decay raised in the past is how the extremely slow process of the lifetime of order $10^{31}$ years or even larger is modified by strong interaction having the time scale of order, $10^{-23}$ seconds. Thus one must deal with the short time limit of the decay law, and one must face the fact that the exponential decay law is not exact in quantum mechanics. To my knowledge even now there is no convincing calculation of the nucleon decay rate, assuming that the law of grand unified theory is known.

In cosmology essentially every elementary process occurs in macroscopic cosmic medium. The process needs not involve a finite number of fundamental particles alone. For instance, one can consider a macroscopic quantum process such as the first order phase transition taking place in thermal environment. Here the order parameter of the symmetry, the magnitude of a Higgs field, is the system variable and it couples to essentially all light particles of the standard model that make up thermal environment. With these in mind, in cosmology there are plenty of places one has to estimate effects of medium. When one has a clear idea of how to separate the "small" system from the environment, the essential part of the problem becomes how elementary processes are modified in dissipative medium. We feel that the problem should be analyzed from the first principles of quantum mechanics, at least at the conceptual level. In condensed matter physics this problem is generically known as quantum Brownian motion or quantum dissipation.

It has become increasingly clear to us that the bulk of the past works on quantum dissipation relies on the simplified form of quantum friction, namely dissipation which is local in time. This corresponds to the exponential decay law when one examines the fate of some initial excitation. This approximation is excellent in the most
dominant phase of dissipation. But it fails both in early time and very late time behavior of the quantum system immersed in dissipative medium. Some formalism exists to deal with more general nonlocal dissipation, but they are not very useful to the problems we would like to address. Indeed, some of our new results have not been recognized in the past.

In this talk I shall first review what we have recently achieved with regard to the general framework of quantum dissipation. Then in the second part I shall discuss two applications of this formalism: particle decay process in thermal medium and parametric resonant effect in medium. These are related to some important problems in modern cosmology; baryogenesis, inflation, and the moduli problem in supergravity models.

Most results described here are already in our papers, [1], [2], [3]. But I reorganize arguments with a new perspective, and add some new material. I attempted in this written version to explain basic results in a self-contained manner.

II Basic Formalism of Quantum Dissipation

There are two standard methods for the theory of quantum dissipation. The first one is the approach based on quantum Langevin equation [4]. The other is the path integral approach initiated by Feynman and Vernon [5], [6], [7], [2]. Both methods have merits and demerits, and one should use both in clarifying specific details of relevant problems. Before going to explain some fundamentals of these approaches, let me explain how one models the dissipation for a small quantum system, being caused by interaction with a larger environment.

The basic idea taken by most approaches is that existence of a continuously many environment variables coupled to a finite number of subsystem variables is the essential part of dissipation. After all one does not know all details of the environment. Indeed, ignorance should be the privilege here. It is fortunate that dissipation, or at least something recognized as such, occurs once one makes no measurement of the environment. Thus one postulates that detailed modeling of the environment and its coupling to the small system should be unimportant to dissipative behavior of the system. After integrating out the environment variables, one should have only a few phenomenological parameters to describe dissipation,
ultimately obtainable from experiments. Despite of this phenomenological nature one should base all augments on rigorous quantum physics, and the modeling is inevitable.

One is thus led to the theory of the simplest, yst the most fundamen tal model of quantum dissipation, the linear dissipation in open systems. In this approach one models the environment by a continuously many set of harmonic oscillators of some arbitrary spectrum and couples it to the subsystem via a bilinear term. Let the subsystem variable in question be denoted by \( q \) and the environment variable by \( Q_a \). For simplicity we assume that the subsystem has one degree of freedom, but it should be evident to extend it to any finite degrees of freedom. We often use a discrete label for the environment variable, but actually we assume that there are continuously many of them. The Lagrangian of our problem consists of three parts:

\[
L = L_q[q] + L_Q[Q] + L_{\text{int}}[q,Q].
\]  

(1)

We take for the system-environment interaction the bilinear term:

\[
L_q = \frac{1}{2} \dot{q}^2 - V(q),
\]  

(2)

\[
L_Q = \frac{1}{2} \sum_a (\dot{Q}_a^2 - \omega_a^2 Q_a^2) \rightarrow \frac{1}{2} \int_{\omega_c}^{\infty} d\omega \left( \dot{Q}^2(\omega) - \omega^2 Q^2(\omega) \right),
\]  

(3)

\[
L_{\text{int}} = -q \sum_a c_a Q_a \rightarrow -q \int_{\omega_c}^{\infty} d\omega c(\omega)Q(\omega).
\]  

(4)

\( \omega_c \) taken to be positive is the smallest of the environment frequency spectrum.

As a technical aside, the renormalization of the frequency will be discussed later in appropriate places. Introduction of a counter term in relation to the renormalization as often discussed in the literature is actually the problem of how to relate the parameters of the theory to observable quantities.

The strategy of subsequent investigation is to integrate out quantum dynamics of the environment, by averaging over its initial ensemble given by some mixed state.

**IIA Operator method**

The Heisenberg equation of motion for the subsystem and the environment variable is derived from the above Lagrangian,

\[
\dot{q} + V'(q) = -\int_{\omega_c}^{\infty} d\omega c(\omega)Q(\omega, t),
\]  

(5)

\[
\dot{Q}(\omega, t) + \omega^2 Q(\omega, t) = -c(\omega)q.
\]  

(6)
The environment part is first solved as a sum of the homogeneous and the inhomogeneous solutions:

\[
Q(\omega, t) = Q_i(\omega) \cos(\omega t) + P_i(\omega) \frac{\sin(\omega t)}{\omega}
\]

\[- \frac{c(\omega)}{2\omega} \int_0^t d\tau \sin(\omega(t - \tau)) q(\tau) - \frac{c(\omega)}{2\omega} \int_t^{t_f} d\tau \sin(\omega(t - \tau)) q(\tau)
\]

\[- \frac{c(\omega)}{2\omega} \int_0^{t_f} d\tau \sin(\omega(t - \tau)) q(\tau).
\]

(7)

Here \(Q_i(\omega), P_i(\omega)\) are initial environment operators at \(t = 0\), and \(t_f(> t)\) is some arbitrary time. The system evolution is then determined by the quantum Langevin equation [8],

\[
\ddot{q} + V'(q) + 2 \int_0^t d\tau \alpha_I(t - \tau) q(\tau) = F_Q,
\]

(8)

\[
F_Q(t) = - \int_{\omega_c}^{\infty} d\omega \sqrt{r(\omega)} \left( b_i(\omega) e^{-i\omega t} + b_i^\dagger(\omega) e^{i\omega t} \right),
\]

(9)

\[
\alpha_I(\tau) = - \frac{i}{2} \int_{-\infty}^{\infty} d\omega r(\omega) e^{-i\omega \tau},
\]

(10)

\[
r(\omega) = \sum_a \frac{c^2_a}{2\omega_a} \left( \delta(\omega - \omega_a) - \delta(\omega + \omega_a) \right)
\]

\[= \int_{\omega_c}^{\infty} d\omega D(\omega) \frac{c^2(\omega)}{2\omega} - \int_{-\infty}^{-\omega_c} d\omega D(\omega) \frac{c^2(\omega)}{2\omega}.
\]

(11)

\(b_i(\omega)\) and \(b_i^\dagger(\omega)\) are the initial creation and the annihilation operators of the environment part of variables. \(D(\omega)\) is the density of states per unit frequency. The fundamental quantity \(r(\omega)\) that characterizes both the environment spectrum and its coupling to the system is called the response weight. The right hand side of eq.(8) describes the random force by the environment, and it is characterized by the following correlation function, evaluated in the initial state of environment:

\[
\langle (F_Q(\tau)F_Q(s))_{sym} \rangle_{env} = \int_{\omega_c}^{\infty} d\omega r(\omega) \cos(\omega(\tau - s)) \langle 2n_i(\omega) + 1 \rangle_{env}.
\]

(12)

Only the symmetric part of the correlation is written here. \(n_i(\omega) = b_i^\dagger(\omega)b_i(\omega)\) is the number operator of the initial environmental variable. For instance, in thermal state of temperature \(T = 1/\beta\),

\[
\langle 2n_i(\omega) + 1 \rangle_{env} = \coth(\frac{\beta\omega}{2}).
\]

(13)

To go further, we shall take dynamics of a single harmonic oscillator for the small system,

\[
L_q = \frac{1}{2} \left( \frac{dq}{dt} \right)^2 - \frac{1}{2} \omega_0^2 q^2.
\]

(14)
The analytic function that appears here is

$$\omega < \omega_c$$

along this cut. An important property of $F$ is that it can be diagonalized via real orthogonal matrix, it is more convenient to attach some physically unobservable phase factors to this transformation. The diagonal oscillator coordinates, $\tilde{Q}(\omega)$, along with inverse relations, are then given by

$$\tilde{Q}(\omega) = Q(\omega) - \sqrt{2\omega r(\omega)} F(\omega - i0^+) \left[ q - \int_{\omega_c}^\infty d\omega' \frac{\sqrt{2\omega' r(\omega')}}{\omega'^2 - \omega^2 + i0^+} Q(\omega') \right], (15)$$

$$\tilde{P}(\omega) = P(\omega) - \sqrt{2\omega r(\omega)} F(\omega - i0^+) \left[ p - \int_{\omega_c}^\infty d\omega' \frac{\sqrt{2\omega' r(\omega')}}{\omega'^2 - \omega^2 + i0^+} P(\omega') \right], (16)$$

where

$$q = - \int_{\omega_c}^\infty d\omega \sqrt{2\omega r(\omega)} F^*(\omega - i0^+) \tilde{Q}(\omega), (17)$$

$$p = - \int_{\omega_c}^\infty d\omega \sqrt{2\omega r(\omega)} F^*(\omega - i0^+) \tilde{P}(\omega), (18)$$

$$Q(\omega) = \tilde{Q}(\omega) + \sqrt{2\omega r(\omega)} \int_{\omega_c}^\infty d\omega' \frac{\sqrt{2\omega' r(\omega')}}{\omega'^2 - \omega^2 + i0^+} \tilde{Q}(\omega'), (19)$$

$$P(\omega) = \tilde{P}(\omega) + \sqrt{2\omega r(\omega)} \int_{\omega_c}^\infty d\omega' \frac{\sqrt{2\omega' r(\omega')}}{\omega'^2 - \omega^2 + i0^+} \tilde{P}(\omega'). (20)$$

The analytic function that appears here is

$$F(z) = \frac{1}{z^2 + \omega_0^2 + 2\pi G(z)}, \quad G(z) = \int_{-\infty}^\infty d\omega \frac{r(\omega)}{2\pi z - \omega}. (21)$$

This function has cuts along the real axis, $\omega > \omega_c$ and $\omega < -\omega_c$. We assume the threshold $\omega_c > 0$ such that there is a gap between the two cuts. The following discontinuity relation holds:

$$F(\omega + i0^+) - F(\omega - i0^+) = i2\pi r(\omega) F(\omega + i0^+) F(\omega - i0^+) \equiv i2\pi H(\omega), \quad (22)$$

along this cut. An important property of $F(z)$ is that in the cut $z$–plane this function is regular except on the real axis, and that it can be analytically continued into the other Riemann sheets by the discontinuity formula. In the second sheet $F(z)$ can be shown to have simple poles. The location of these poles is given by

$$z^2 - \omega_0^2 - 2\pi G(z) + 2\pi ir(z) = 0. \quad (23)$$

The analytic property thus derived is based on the presence of the gap at $-\omega_c < \omega < \omega_c$: by analytic extension through this gap the boundary values at $\omega \pm i0^+$ with
−∞ < ω < −ωc and ωc < ω < ∞ become related. Without the gap as in the case of ωc = 0, there is no unique way to analytically extend \( \overline{G}(z) \) (hence \( F(z) \)) with a given \( r(\omega) \) at \( \omega \geq 0 \). Indeed, the gapless case is difficult to deal with in complete generality. In the case of odd regular function for \( r(z) \) the analytic extension is possible. On the other hand, in cases with the spectrum weight behaving like \( r(\omega) \propto \omega^\alpha \) (with \( \alpha \) a positive non-integer) near \( \omega = 0 \), the simple analytic extension in terms of the two relevant Riemann sheets is impossible.

What happened to the subsystem spectrum? The equation that governs the real part of its energy eigenvalue squared \( \lambda \) is given by

\[
f(\lambda) = \lambda, \quad f(\lambda) \equiv \omega_0^2 + \mathcal{P} \int_{\omega_c}^{\infty} d\omega \frac{2\omega r(\omega)}{\lambda - \omega^2},
\]

with \( \mathcal{P} \) denoting the principal part of integration. The function \( f(\lambda) \) is related to the analytic function by

\[
f(\lambda) - \lambda = \Re F^{-1}(\sqrt{\lambda} + i0^+),
\]

with \( \Re \) denoting the real part. In the \( \lambda < \omega_c^2 \) region one readily derives from the monotonic behavior of \( f(\lambda) \) the condition for absence of isolated spectrum at \( 0 < \omega < \omega_c \),

\[
f(\omega_c^2^-) > \omega_c^2,
\]

with \( ^- \) indicating the limit from below. Consistency further requires \( \omega_0 > \omega_c \). When this condition is obeyed, the system frequency \( \omega_0 \) moves, with increase of the coupling \( c(\omega) \), into the second Riemann sheet of the complex \( \omega \) plane across the cut starting at \( \omega = \omega_c \). It finally settles at the zero of eq.(23). The imaginary part of this zero then gives the decay rate of any initial excitation of the subsystem, as will be made clear shortly.

When the response weight \( r(\omega) \to \) a constant as \( \omega \to \infty \), as often happens and in field theory models later discussed in particular, one has to subtract a term from integrals containing the response weight. This corresponds to renormalization of the bare frequency \( \omega_0 \) with the frequency shift,

\[
\delta \omega^2 = -2 \int_{\omega_c}^{\infty} d\omega \frac{r(\omega)}{\omega}.
\]

The renormalized frequency is given by

\[
\omega_R^2 = \omega_0^2 + \delta \omega^2 = \omega_0^2 - 2 \int_{\omega_c}^{\infty} d\omega \frac{r(\omega)}{\omega}.
\]
The eigenvalue equation written using $\omega_R^2$ is then modified to
\[ \lambda - \omega^2 R - \lambda \mathcal{P} \int_{\omega_c}^{\infty} d\omega \frac{2r(\omega)}{\omega (\lambda - \omega^2)} = 0. \] (29)

The condition of stability of the entire system requires that the smallest eigenvalue $\lambda > 0$, giving $\omega_R^2 > 0$, or equivalently
\[ \omega_0^2 > 2 \int_{\omega_c}^{\infty} d\omega \frac{r(\omega)}{\omega}, \] (30)
if the integral on the right hand side is convergent. In the discussion that follows the original subsystem frequency $\omega_0$ appears only through the renormalized $\omega_R$ or the real part of the zero (23). In this sense $\omega_0$ is not a fundamental parameter when physical consequences are discussed.

The important quantity for physical interpretation of the results that follow is the overlap probability between the true eigen operator $\tilde{Q}(\omega)$ with the original system variable $q$, which is calculable from eq.(15):
\[ |\langle q | \tilde{Q}(\omega) \rangle|^2 = 2\omega r(\omega) |F(\omega - i0^+)|^2 = 2\omega H(\omega) \]
\[ = \frac{2\omega r(\omega)}{(\omega^2 - \omega_0^2 - \Pi(\omega))^2 + (\pi r(\omega))^2}. \] (31)

We decomposed the analytic $G(\omega - i0^+)$ into the real and the imaginary parts,
\[ 2\pi G(\omega - i0^+) = \Pi(\omega) + i\pi r(\omega), \] (32)
\[ \Pi(\omega) = \mathcal{P} \int_{-\infty}^{\infty} d\omega' \frac{r(\omega')}{\omega - \omega'}, \] (33)
In the weak coupling limit of $c(\omega) \to 0$ the overlap probability (31) has the Breit-Wigner form with a complex pole at the zero of eq.(23). As will be more evident shortly, it physically means that any system excitation is damped with a decay rate,
\[ 2\gamma = \frac{\pi r(\bar{\omega})}{\bar{\omega}}, \] (34)
with $\bar{\omega}$ the real part of the pole of $F(z)$.

With complete diagonalization it should be possible to explicitly write the operator solution. The solution is
\[ q(t) = -\int_{\omega_c}^{\infty} d\omega \sqrt{2\omega r(\omega)} F^*(\omega - i0^+) \tilde{Q}(\omega, t), \]
\[ \tilde{Q}(\omega, t) = \cos(\omega t) \tilde{Q}_i(\omega) + \frac{\sin(\omega t)}{\omega} \tilde{P}_i(\omega). \] (36)
The initial values $\tilde{Q}_i, \tilde{P}_i$ are then rewritten in terms of the original variables, $q, p$ and

$$b_i(\omega) = \sqrt{\frac{\omega}{2}} Q_i(\omega) + \frac{i}{\sqrt{2\omega}} P_i(\omega), \quad b_i^\dagger.$$  \hspace{1cm} (37)

After some straightforward calculation one finds [3] that

$$q(t) = p_i g(t) + q_i \dot{g}(t) - \int_{\omega_c}^{\infty} d\omega \left( h^*(\omega, t) e^{-i\omega t} b_i(\omega) + h(\omega, t) e^{i\omega t} b_i^\dagger(\omega) \right), \hspace{1cm} (38)$$

$$p(t) = p_i \dot{g}(t) + q_i \ddot{g}(t) - \int_{\omega_c}^{\infty} d\omega \left( k^*(\omega, t) e^{-i\omega t} b_i(\omega) + k(\omega, t) e^{i\omega t} b_i^\dagger(\omega) \right), \hspace{1cm} (39)$$

where we introduced

$$g(t) = 2 \int_{\omega_c}^{\infty} d\omega H(\omega) \sin(\omega t), \hspace{1cm} (40)$$

$$h(\omega, t) = \int_0^t d\tau g(\tau) e^{-i\omega \tau}, \hspace{1cm} (41)$$

$$k(\omega, t) = \int_0^t d\omega \dot{g}(\tau) e^{-i\omega \tau} = g(t)e^{-i\omega t} + i \omega h(\omega, t). \hspace{1cm} (42)$$

These solutions, eqs. (38), (39), are fundamental to any calculation of correlators such as $\langle q(\tau)q(s) \rangle_{\text{env}}$. Moreover, it completely solves the quantum Langevin equation, (3), albeit for the limited case of system dynamics of the harmonic oscillator. The operator solution consists of two parts, one that depends on the system initial operator values, $q_i, p_i$, and the other that depends on the initial environment values. Thus the memory effect of initial system dependence is governed by the function $g(t)$.

To see more on the memory effect, let us first ignore the random force caused by environment, by taking the ensemble average over the initial environment variables that are assumed to have no expectation value, $\langle Q_i(\omega) \rangle = \langle P_i(\omega) \rangle = 0$. The expectation value of the system variable then follows [3]

$$\frac{d\langle p \rangle}{dt} = - \Omega^2(t) \langle q \rangle - C(t) \langle p \rangle, \quad \frac{d\langle q \rangle}{dt} = \langle p \rangle, \hspace{1cm} (43)$$

$$\Omega^2(t) = \frac{\ddot{g} \dot{g} - \dot{g}^2}{g \dot{g} - \ddot{g}^2}, \quad C(t) = \frac{\ddot{g} \ddot{g} - g \dddot{g}}{g \ddot{g} - \dddot{g}^2}. \hspace{1cm} (44)$$

The quantities that appear in this equation are the time dependent friction ($C(t)$) and the time dependent frequency squared ($\Omega^2(t)$), incorporating environmental effects.
Actually the ensemble average is not necessary. Using the explicit form of solution, one gets after eliminating initial \( q_i, p_i \) dependence,

\[
\frac{d^2 q}{dt^2} + \Omega^2(t) q + C(t) \frac{dq}{dt} = -\Omega^2(t) f_q - C(t) f_p - \dot{f}_p, \tag{45}
\]

\[
f_q = \int_{\omega_c}^{\infty} d\omega \sqrt{r(\omega)} \left( h^*(\omega, t) e^{-i\omega t} b_i(\omega) + (\text{h.c.}) \right), \tag{46}
\]

\[
f_p = \int_{\omega_c}^{\infty} d\omega \sqrt{r(\omega)} \left( k^*(\omega, t) e^{-i\omega t} b_i(\omega) + (\text{h.c.}) \right). \tag{47}
\]

This may be viewed as a local form of quantum Langevin equation: the operator acting on the system variable \( q \) is the local differential operator with local coefficient functions of time. Note that this form is not available in the general case of arbitrary potential. In the general case the integro-differential equation is available.

Let us note that the function \( g(t) \) has the behavior of damped oscillation,

\[
g(t) \sim \frac{1}{\bar{\omega}} \sin(\bar{\omega}t) e^{-\gamma t}, \tag{48}
\]

if one takes the Breit-Wigner form for \( H(\omega) \), or the pole approximation for \( F(z) \). When the coupling is weak, this pole approximation is excellent in many practical applications. In this case

\[
\Omega^2(t) = \bar{\omega}^2 + \gamma^2 \equiv \Omega_0^2, \quad C(t) = 2\gamma. \tag{49}
\]

Thus the effect of the environment is very simple in the pole approximation: aside from the random fluctuating environmental force, the system experiences the frequency renormalization and a constant friction:

\[
\frac{d^2 \langle q \rangle}{dt^2} + 2\gamma \frac{d\langle q \rangle}{dt} + \Omega_0^2 \langle q \rangle = 0. \tag{50}
\]

There are however physical phenomena that the pole approximation fails to describe, as will be shown later. Let us simply state here that both early time and late time behavior of the system variables are not correctly described by the pole approximation. For instance, the correct \( C(t) = O[t^3] \) as \( t \to 0^+ \) and at late times both of \( \Omega^2(t) \) and \( C(t) \) decrease with powers of time. These are not obeyed in the pole approximation. Indeed, the pole approximation violates the positivity of the density matrix at early times, as will be shown later.

The correct final behavior at \( t \to \infty \) is the power law of time, as seen in the following way. Using the discontinuity formula, one may rewrite the \( \omega \) integration
for $g(t)$, eq. (I), containing $H(\omega) = (F(\omega + i0^+) - F(\omega - i0^+))/(2\pi i)$ along the real axis into the $F(z)$ integration, the complex $z$ running both slightly above and below the cuts. The factor $\sin(\omega t)$ is replaced by $\Im e^{-i\omega t}$ in this procedure. A half of this complex contour can be deformed into the second sheet, and one thereby encounters simples poles in the second sheet. The integral for $g(t)$ may then be expressed as the sum of the pole contribution (at $z = z_0$ with $\Im z_0 < 0$ for a single pole) in the second sheet and the contribution parallel to the imaginary axis passing through $z = \omega c$, both in the first (I) and in the second (II) sheet [9]:

$$g(\tau) = \Im \left( K e^{-i\pi \omega_0 \tau} e^{\Im \omega \tau} + \Im \left[ \frac{e^{i\omega c \tau}}{\pi} \int_0^\infty dy e^{-y \tau} (F_I(\omega_c + i y) - F_{II}(\omega_c + i y)) \right] \right), \quad (51)$$

with $K^{-1} = z_0 - \pi G'(z_0) + i \pi r'(z_0)$.

As seen from this formula, the pole contribution given by the first term describes the exponential decay as mentioned already. The rest of contribution gives the power law decay at very late times; $\propto t^{-\alpha - 1}$ [3]. The power $-\alpha - 1$ is related to the threshold behavior of the response weight, $r(\omega) \propto (\omega - \omega_c)^\alpha$. This continuous integral also describes the correct early time behavior.

Emergence of the power law term in the quantum Brownian motion has been noted in some specific models, but we find this behavior as a general property in the presence of the non-local dissipation. We would like to stress that the analytic structure of the cut $\omega$ plane is very important to derive the power law decay. For instance, if one takes the Ohmic form of the response weight given by [4]

$$r^{(4)}(\omega) = \frac{4c \Omega \gamma \omega}{(\omega^2 - \Omega^2 + \frac{\gamma^2}{4})^2 + \Omega^2 \gamma^2}, \quad (52)$$

then one does not obtain the power law decay. The point is that this function, consisting of pole terms alone, does not have the branch point singularity needed for the power law behavior.

Quadratic quantities averaged over the initial environment ensemble are computed as [3]

$$\langle q^2(t) \rangle = \int_{\omega_c}^{\infty} d\omega \langle 2n_i(\omega) + 1 \rangle_{env} r(\omega) |h(\omega, t)|^2$$

$$+ g^2(t) \langle p_i^2 \rangle + \dot{g}^2(t) \langle q_i^2 \rangle + g(t) \dot{g}(t) \langle p_i q_i + q_i p_i \rangle,$$

$$\langle p^2(t) \rangle = \int_{\omega_c}^{\infty} d\omega \langle 2n_i(\omega) + 1 \rangle_{env} r(\omega) |k(\omega, t)|^2$$

$$+ \dot{g}^2(t) \langle p_i^2 \rangle + \ddot{g}^2(t) \langle q_i^2 \rangle + \ddot{g}(t) \dot{g}(t) \langle p_i q_i + q_i p_i \rangle,$$  

(53)
\[
\frac{1}{2} \langle q(t) p(t) + p(t) q(t) \rangle = \int_{\omega_c}^{\infty} d\omega \langle 2n_i(\omega) + 1 \rangle_{\text{env}} r(\omega) h(\omega, t) k^*(\omega, t) \\
+ g(t) g(\frac{p^2}{\hbar}) + \dot{g}(t) \dot{g}(t) \langle q^2 \rangle + \langle \dot{g}^2(t) + g(t) \dot{g}(t) \rangle \frac{1}{2} \langle p_i q_i + q_i p_i \rangle. \tag{55}
\]

Here we assumed both \( \langle b_i(\omega) \rangle = \langle b_i^\dagger(\omega) \rangle = 0 \) and \( \langle b_i(\omega) b_j(\omega') \rangle = \langle b_i^\dagger(\omega) b_j^\dagger(\omega') \rangle = 0 \).

Equivalent expressions are obtained using identities:

\[
\int_{\omega_c}^{\infty} d\omega \coth\left(\frac{\beta \omega}{2}\right) r(\omega) |h(\omega, t)|^2 = 2 \int_0^t d\tau \int_0^\tau ds \ g(t - \tau) \alpha_R(\tau - s) g(t - s), \tag{56}
\]

\[
\int_{\omega_c}^{\infty} d\omega \coth\left(\frac{\beta \omega}{2}\right) r(\omega) |k(\omega, t)|^2 = 2 \int_0^t d\tau \int_0^\tau ds \ \dot{g}(t - \tau) \alpha_R(\tau - s) \dot{g}(t - s), \tag{57}
\]

\[
\alpha_R(\tau) = \int_{\omega_c}^{\infty} d\omega \langle 2n_i(\omega) + 1 \rangle \frac{\cos(\omega \tau)}{2\omega}. \tag{58}
\]

Let us turn to the late time behavior. At asymptotically late times the initial value dependence disappears, as mentioned above, leading to

\[
q(t) \to -\int_{\omega_c}^{\infty} d\omega \sqrt{r(\omega)} \left( F^*(\omega - i0^+) e^{-i\omega t} b_i(\omega) + \text{(h.c.)} \right), \tag{59}
\]

\[
p(t) \to i \int_{\omega_c}^{\infty} d\omega \omega \sqrt{r(\omega)} \left( F^*(\omega - i0^+) e^{-i\omega t} b_i(\omega) - \text{(h.c.)} \right), \tag{60}
\]

since \( h(\omega, \infty) = F(\omega - i0^+) \). The system evolution is thus governed by the initial distribution of the environment variables, \( b_i(\omega), b_i^\dagger(\omega) \), with the following probability functions;

\[
r(\omega)|h(\omega, \infty)|^2 = H(\omega) = \frac{|\langle \langle q | \bar{Q}(\omega) \rangle \rangle|^2}{2\omega}, \quad \text{for } q(t), \tag{61}
\]

\[
r(\omega)|k(\omega, \infty)|^2 = \omega^2 H(\omega) = \frac{\omega}{2} \ |\langle \langle q | \bar{Q}(\omega) \rangle \rangle|^2, \quad \text{for } p(t). \tag{62}
\]

Note that the overlap probability \( |\langle \langle q | \bar{Q}(\omega) \rangle \rangle|^2 \) has a universal character, being determined by general properties of the system and the environment alone, irrespective of their particular initial states. The overlap probability between the system \( q \) variable and the diagonal \( \bar{Q}(\omega) \) respects the unitarity relation,

\[
\int_{\omega_c}^{\infty} d\omega |\langle \langle q | \bar{Q}(\omega) \rangle \rangle|^2 = \int_{\omega_c}^{\infty} d\omega 2\omega H(\omega) = 1. \tag{63}
\]

Asymptotic values of averaged quadratic quantities are

\[
\langle q^2(\infty) \rangle = \int_{\omega_c}^{\infty} d\omega \langle 2n_i(\omega) + 1 \rangle_{\text{env}} H(\omega), \tag{64}
\]

\[
\langle p^2(\infty) \rangle = \int_{\omega_c}^{\infty} d\omega \langle 2n_i(\omega) + 1 \rangle_{\text{env}} \omega^2 H(\omega). \tag{65}
\]
Taking the thermal bath with \( \langle 2n_i(\omega) + 1 \rangle_{\text{env}} = \coth\left(\frac{\beta \omega}{2}\right) \), along with the pole approximation, this gives the well known results:

\[
\langle q^2(\infty) \rangle \approx \frac{T}{\bar{\omega}^2}, \quad \langle p^2(\infty) \rangle \approx T.
\] (66)

Let us discuss more closely the asymptotic form of the occupation number in thermal bath,

\[
n(\infty) \equiv \frac{1}{2} \left[ \frac{p^2(\infty)}{\bar{\omega}} + \bar{\omega} q^2(\infty) \right] - \frac{1}{2} = \frac{1}{2} \int_{\omega_c}^{\infty} d\omega \coth\left(\frac{\beta \omega}{2}\right) \left(\bar{\omega} + \frac{\omega^2}{\bar{\omega}}\right) H(\omega) - \frac{1}{2}.
\] (67)

The temperature dependent part of the occupation number is then

\[
n^\beta = \int_{\omega_c}^{\infty} d\omega \frac{1}{e^{\beta \omega} - 1} \left(\bar{\omega} + \frac{\omega^2}{\bar{\omega}}\right) H(\omega).
\] (68)

When the pole term dominates, or equivalently one approximates \( H(\omega) \) by the Breit-Wigner function, \( n^\beta \) has the factor of Boltzmann suppression, \( e^{-\bar{\omega}/T} \) at low temperatures. But the pole approximation is not good at low temperatures. Indeed, let us examine a typical example by taking the form of \( r(\omega) = c (\omega - \omega_c)\alpha \), in the range of \( \omega_c < \omega < \Omega (\Omega \gg \omega_c) \). We assume \( 0 < \alpha < 1 \) and \( \bar{\omega} \gg \text{Max} (\omega_c, T) \). The result is

\[
n^\beta \approx \frac{c}{\bar{\omega}^3} \Gamma(\alpha + 1) e^{-\beta \omega_c} T^{\alpha + 1}.
\] (69)

\( \Gamma \) is the Euler’s gamma function.

This shows that instead of the exponential suppression at low temperatures, what is left in medium after the decay has a power-law behavior of temperature dependence \((\propto T^{\alpha + 1})\). An implication of this behavior to the unstable particle decay will be discussed in the next section. It means that the remnant fraction in thermal medium does not suffer from the Boltzmann suppression factor at temperatures even much lower than the mass of unstable particle.

Related time evolution is as follows. The corresponding behavior of \( g(t) \) computed from the continuous part of \( H(\omega) \) integration gives

\[
g(t) \approx -\frac{2c}{\bar{\omega}^4} \Gamma(\alpha + 1) \frac{\cos(\omega_c t + \frac{\pi \alpha}{2})}{t^{\alpha + 1}}.
\] (70)

One can estimate the transient time \( t_* \) from the exponential period to the power period by equating the two formulas of \( g(t) \) in their respective ranges, to obtain

\[
t_* \approx \frac{1}{\gamma} \ln \left( \frac{\bar{\omega}^3}{2c \Gamma(\alpha + 1) \gamma^{\alpha + 1}} \right),
\] (71)
with $\gamma = -\Im z_0$ the decay rate. For a very small $c$ the factor inside the logarithm becomes large ($\propto c^{-2a-3}$), and by the time $t_*$ the initial population has decreased like
\[ e^{-2\gamma t_*} \propto c^{4a+6}. \]

(72)

It may thus be claimed that the power law behavior is difficult to observe. But we shall later discuss that this may not be so in cosmology.

We shall refer to our original paper [3] on detailed discussion of correlation functions. But let me mention the asymptotic form. For both $t_1$ and $t_2$ in the asymptotic late time region,
\[
\langle q(t_1)q(t_2) \rangle \rightarrow \int_{\omega_c}^{\infty} d\omega H(\omega) \cos(\omega(t_1 - t_2)) \coth(\frac{\beta \omega}{2}),
\]
\[
\langle p(t_1)p(t_2) \rangle \rightarrow \int_{\omega_c}^{\infty} d\omega \omega^2 H(\omega) \cos(\omega(t_1 - t_2)) \coth(\frac{\beta \omega}{2}).
\]

(73)

(74)

The other correlator vanishes: $\langle q(t_1)p(t_2) + p(t_2)q(t_1) \rangle \rightarrow 0$.

Finally let us mention how some of our fundamental formulas are related to quantities in complete thermal equilibrium. The limiting values of the analytic function $G(z)$ as $z \rightarrow \text{real}$ are related to the well-known real-time thermal Green’s function [10], when one takes the initial environment in thermal state. For a single oscillator,
\[
G^R(\omega) \equiv \overline{G}(\omega + i\epsilon), \quad G^A(\omega) \equiv \overline{G}(\omega - i\epsilon),
\]
\[
\frac{1}{1 - e^{-\beta \omega}} G^R(\omega) + \frac{1}{1 - e^{\beta \omega}} G^A(\omega) = i \int_{-\infty}^{\infty} d\tau e^{i\omega \tau} \text{tr} \left( \rho_\beta T \left[ Q(\tau) Q(0) \right] \right).
\]

(75)

(76)

The operator method is thus very powerful. On the other hand, it is difficult to extract quantum statistical nature of the subsystem state in this approach. For this purpose, the path integral approach is useful, to which we shall now turn.

IIB Path integral method

The basic idea [3] is that one is interested in the behavior of the $q-$system alone and traces out the environment variable altogether in the path integral formula. In the influence functional approach by Feynman and Vernon one directly deals with the probability instead of the amplitude. This way one can compute the reduced density matrix that describes the state of the small system incorporating effects of interacting
environment. We define the influence functional by convoluting with the initial state of the environment. To do so we assume for technical reasons that initially we may take an environment state uncorrelated with the system. The influence functional is thus obtained after integrating out the environment variables:

\[
F[q(\tau), q'(\tau)] \equiv \int \mathcal{D}q(\tau) \int \mathcal{D}q'(\tau) \int dq_i \int dQ_i \int dq_f \int dQ_f \delta(Q_f - Q'_f) \\
\cdot K(q(\tau), Q(\tau)) \, K^*(q'(\tau), Q'(\tau)) \, \rho_i(Q_i, Q'_i),
\]

(77)

\[
K(q(\tau), Q(\tau)) = \exp \left( iS_0[Q] + iS_{\text{int}}[q, Q] \right),
\]

(78)

\[
S_0[Q] + S_{\text{int}}[q, Q] = \int_0^t \, d\tau \, ( L_Q[Q] + L_{\text{int}}[q, Q] ) .
\]

(79)

The influence functional is a functional of the entire path of the system \(q(\tau)\) and its conjugate path \(q'(\tau)\).

\[
\rho_i(Q_i, Q'_i) = \sum_n w_n \psi_n^*(Q_i') \psi_n(Q_i), \quad (0 \leq w_n \leq 1)
\]

(80)

is the initial density matrix of the environment, which can be any mixture of pure quantum states with probability \(w_n\). What deserves to be stressed is that one does not observe the final state of the environment, hence integration with respect to the final values of \(Q_f = Q'_f\) is performed here.

Once the influence functional is known, one may compute the transition probability and any physical quantities of the \(q\)–system by convoluting dynamics of the system under study. For instance, the transition probability is given, with introduction of the density matrix \(\rho^{(R)}\), by

\[
\rho^{(R)} = \int \mathcal{D}q(\tau) \int \mathcal{D}q'(\tau) \int dq_i \int dQ_i \psi_i^*(q_i) \psi_i(q_i) \, F[q(\tau), q'(\tau)] \, \exp \left( iS_q[q] - iS_{\text{int}}[q] \right),
\]

(82)

where \(\psi_i,q\)'s are wave functions of the initial and the final \(q\)–states, and \(S_q[q]\) is the action of the \(q\)–system.

The form of the influence functional is dictated by general principles such as probability conservation and causality. Feynman and Vernon found a closed quadratic form consistent with these,

\[
F[q(\tau), q'(\tau)] = \exp \left[ - \int_0^t d\tau \int_0^\tau ds \left( \xi(\tau) \alpha_R(\tau - s) \xi(s) + i \xi(\tau) \alpha_f(\tau - s) X(s) \right) \right],
\]

(83)

with \(\xi(\tau) = q(\tau) - q'(\tau), \quad X(\tau) = q(\tau) + q'(\tau)\).

(84)
Thus two real functions $\alpha_i(\tau)$ are all we need to characterize the system-environment interaction. These are defined here in the range of $\tau \geq 0$. The fact that $\alpha_i$ depends on the difference of time variables, $\tau - s$, is due to the assumed stationarity of the environment. The Feyman-Vernon formula is valid for general $L_Q[q]$ and $L_q[Q]$, not limited to the harmonic oscillator model if the interaction $L_{int}[q, Q]$ is bilinear.

The correlation kernels appear in the influence functional as a form of the nonlocal interaction and they are the dissipation $\alpha_I$ and the noise $\alpha_R$. The dissipation kernel $\alpha_I$ thus computed agrees with the one defined in eq.(10) \[1\]. Let us now specialize to the case of the oscillator bath of temperature $T = 1/\beta$, which is described for a single oscillator of frequency $\omega$ by

$$\rho_\beta(Q, Q') = \left(\frac{\omega}{\pi \coth(\beta\omega/2)}\right)^{1/2} \cdot \exp\left[-\frac{\omega}{2 \sinh(\beta\omega)} \left((Q^2 + Q'^2) \cosh(\beta\omega) - 2QQ'\right)\right].$$ \hspace{1cm} (85)

The noise kernel, along with the dissipation kernel, are then given by

$$\alpha_R(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} d\omega \coth(\beta\omega/2) r(\omega) e^{-i\omega\tau}, \hspace{1cm} (86)$$

$$\alpha_I(\tau) = -\frac{i}{2} \int_{-\infty}^{\infty} d\omega r(\omega) e^{-i\omega\tau}. \hspace{1cm} (87)$$

Combined together, it gives the real-time thermal Green’s function:

$$\alpha(\tau) \equiv \alpha_R(\tau) + i\alpha_I(\tau) = \sum_k c_k^2 \text{tr} \left( \rho_\beta T \left[ Q(\omega_k, \tau) Q(\omega_k, 0) \right] \right), \hspace{1cm} (88)$$

$$\alpha(\omega) \equiv \int_{-\infty}^{\infty} d\tau \alpha(\tau) e^{i\omega\tau} = i \sum_k c_k^2 \left( \frac{1}{\omega^2 - \omega_k^2 + i\epsilon} - \frac{2\pi i}{e^{\beta\omega_k} - 1} \delta(\omega^2 - \omega_k^2) \right). \hspace{1cm} (89)$$

As noted already, these are given in terms of the response weight $r(\omega)$, and are governed by the analytic function $\mathcal{T}(z)$.

For the system dynamics we further assume a single harmonic oscillator of frequency $\omega_0$. In the path integral approach integration over the sum variable $X(\tau)$ is trivial in this case, since both the local part and the nonlocal action above are linear in this variable:

$$\frac{1}{2} \int_0^t \left( \dot{\xi}(\tau) \dot{X}(\tau) - \omega_0^2 \xi(\tau)X(\tau) \right) - \int_0^t d\tau \int_0^\tau ds \left( \xi(\tau)\alpha_R(\tau - s)\xi(s) + i \xi(\tau)\alpha_I(\tau - s)X(s) \right). \hspace{1cm} (90)$$
Thus result of the path integration of the system variable $X(\tau)$ gives the classical integro-differential equation for $\xi(\tau)$:

$$\frac{d^2 \xi}{d\tau^2} + \omega_0^2 \xi(\tau) + 2 \int_\tau^t ds \xi(s) \alpha_I(s - \tau) = 0. \quad (91)$$

The end result of the $\xi$ path integral then contains an integral of the form,

$$- \int_0^t d\tau \int_0^\tau ds \xi(\tau) \alpha_R(\tau - s) \xi(s), \quad (92)$$

using the classical solution $\xi(\tau)$ with specified boundary conditions, $\xi(0) = \xi_i$, $\xi(t) = \xi_f$.

In the local approximation often used the dissipation kernel is taken to have the form of

$$\alpha_I(\tau) = \delta \omega^2 \delta(\tau) + \eta \delta'(\tau), \quad (93)$$

with $\delta \omega^2$ representing the frequency shift and the $\eta$ term the local friction. This choice enables one to solve the $\xi$ equation (91) by elementary means. On the other hand, the noise kernel is usually given by the response weight of the form,

$$r(\omega) = \frac{\eta}{\pi} \omega f\left(\frac{\omega}{\Omega}\right), \quad (94)$$

with $f(x)$ some cutoff function and $\Omega$ a high frequency cutoff. The cutoff is needed to tame the high frequency integral of $\alpha_R(t)$. The simplest cutoff function $f(x) = \theta(1 - x)$ gives an approximate form of $\alpha_I(t)$ with the friction $\eta$ and

$$\delta \omega^2 \approx - \frac{2}{\pi} \eta \Omega. \quad (95)$$

At high temperatures this approximation reduces to the well known classical form of the fluctuation,

$$\alpha_R(\tau) = \frac{\eta T}{\pi} \delta(\tau). \quad (96)$$

At low temperatures, however, the use of the cutoff function only in the noise kernel while retaining the local form of the dissipation makes validity of this approximation dubious. Our path integral approach \[2\] does not make this local approximation, and instead uses exact solutions of the classical equation (91). This equation has been used in other approaches \[4\], too.
The rest of deduction uses the Laplace transform, and we shall be brief, leaving technical details to our original paper [2]. Solution of the integro-differential equation is, using \( g(\tau) \) defined by eq. (40), given as

\[
\xi(\tau) = \xi_i \frac{g(t-\tau)}{g(t)} + \xi_f \left( \dot{g}(t-\tau) - \frac{g(t-\tau)\dot{g}(t)}{g(t)} \right),
\]

(97)

with the dot denoting derivative. Both \( g(t) \) and \( \dot{g}(t) \) can be shown to satisfy the integro-differential equation of the form \( x = g \) or \( \dot{g} \),

\[
\frac{d^2 x}{dt^2} + \omega_0^2 x + 2 \int_0^t d\tau \alpha f(t-\tau) x(\tau) = 0.
\]

(98)

The reduced density matrix of the quantum system at any time is obtained from the action written in terms of the boundary values, \( S_{cl}(\xi_f, X_f; \xi_i, X_i) \), by convoluting with the initial density matrix of the thermal environment. This action is computed as

\[
i S_{cl} = -\frac{U}{2} \xi_f^2 - \frac{V}{2} \xi_i^2 - W \xi_i \xi_f + \frac{i}{2} X_f \dot{\xi}_f - \frac{i}{2} X_i \dot{\xi}_i,
\]

(99)

\[
U = 2 \int_0^t d\tau \int_0^\tau ds z(\tau) \alpha_R(\tau - s) z(s),
\]

(100)

\[
V = 2 \int_0^t d\tau \int_0^\tau ds y(\tau) \alpha_R(\tau - s) y(s),
\]

(101)

\[
W = \int_0^t d\tau \int_0^\tau ds \left( y(\tau) z(s) + y(s) z(\tau) \right) \alpha_R(\tau - s),
\]

(102)

\[
y(\tau) = \frac{g(t-\tau)}{g(t)},
\]

(103)

\[
z(\tau) = \dot{g}(t-\tau) - g(t-\tau) \frac{\dot{g}(t)}{g(t)},
\]

(104)

\[
\dot{\xi}(\tau) = -\xi_i \frac{\dot{g}(t-\tau)}{g(t)} - \xi_f \left( \ddot{g}(t-\tau) - \frac{\dot{g}(t-\tau)\ddot{g}(t)}{g(t)} \right).
\]

(105)

The master equation for the reduced density matrix, which holds for any initial state of the system, may be derived [11] from this effective action:

\[
i \frac{\partial \rho}{\partial t} = -2 \frac{\partial^2 \rho}{\partial \xi \partial X} + \frac{\Omega^2(t)}{2} \xi X \rho - iC(t) \xi \frac{\partial \rho}{\partial \xi} + 2i\xi^2 D_{pp}(t) \rho + 4D_{xp}(t) \xi \frac{\partial \rho}{\partial X},
\]

(106)

where some new quantities are defined as

\[
D_{pp}(t) = \frac{1}{2} \left( g \frac{\ddot{g}}{g} + \frac{\dot{g}}{g} \frac{\dot{g}}{g} - \frac{g^2 \ddot{g} + \dot{g}}{g} \frac{\dot{g}}{g} + \frac{g^2 \ddot{g} + \dot{g}}{g} \right) (W - \hat{U} - \hat{W}),
\]

(107)

\[
D_{xp}(t) = U - gW + \frac{g^2 \ddot{g} - 2g \dot{g} + g^3}{g \ddot{g} - \frac{g^2}{2}} W.
\]

(108)
The same master equation was derived by Hu, Paz, and Zhang \cite{6}, and their derivation was later simplified by ref \cite{12}. These works however do not give exact solutions like ours. Moreover, our derivation is different, in that we deduce the master equation from the exact solution.

We shall discuss the question of the positivity, using the master equation. This argument is based on that evolution of a pure quantum state necessarily requires \cite{13}

\[
\left( \frac{d}{dt} \text{tr} \rho^2 \right)_{t=0} \leq 0 . \tag{109}
\]

If this inequality is violated, the initial pure state with tr $\rho^2 = 1$ would evolve to a state with tr $\rho^2 > 1$. But with tr $\rho = 1$, this means that the state at $t \sim 0$ has a negative diagonal element of the density matrix. Using the general equation for $\frac{d}{dt} \text{tr} \rho^2$ derived from the master equation, and the initial behavior,

\[
C(t) \sim -\frac{t^3}{3} \left( g^{(3)}(0)^2 - g^{(5)}(0) \right), \tag{110}
\]

\[
D_{pp}(t) \sim -\frac{U}{4} \sim -\frac{t}{8} \mathcal{I}[1] < 0, \tag{111}
\]

\[
D_{xp}(t) \sim U \sim \frac{t^2}{4} \mathcal{I}[1], \tag{112}
\]

one concludes that

\[
\left( \frac{d}{dt} \text{tr} \rho^2 \right)_{t=0} \sim -\mathcal{I}[1] (\Delta q)^2 t < 0 . \tag{113}
\]

We introduced the following notation:

\[
\mathcal{I}[f(\omega)] \equiv \int_{\omega_c}^{\infty} d\omega \ coth \left( \frac{\beta \omega}{2} \right) r(\omega) f(\omega) . \tag{114}
\]

In this sense the correct approach satisfies the positivity constraint at this level. A more direct and general proof of the positivity will be given elsewhere \cite{14}.

Comparison to the pole approximation, or the local friction approximation, at $t \sim 0$ would be instructive. Here as the initial behavior one has

\[
C(t) = 2\gamma , \quad D_{pp}(t) = O[t] , \quad D_{xp}(t) = O[t^2] . \tag{115}
\]

Hence at $t \sim 0$

\[
\frac{d}{dt} \text{tr} \rho^2 \sim 2\gamma \text{tr} \rho^2 . \tag{116}
\]

This clearly violates the positivity constraint (109).
For further discussion we take as the initial state a product of thermal states, a system of temperature $T_0 = 1/\beta_0$ and an environment of temperature $T = 1/\beta$.

We may take $T_0 = T$ when we apply to the decay process of excited level initially in thermal equilibrium. On the other hand, in the limit of $T_0 \to 0$ it describes the ground state of the system harmonic oscillator.

After a series of straightforward Gaussian integration we find the reduced density matrix as a function of $X$ and $\xi$, of the form,

$$\rho^{(R)}(X_f, \xi_f) = 2 \sqrt{\frac{A}{\pi}} \exp\left(-AX_f^2 - B\xi_f^2 + iC X_f \xi_f\right), \quad (117)$$

$$A = \frac{1}{8I_1}, \quad B = \frac{1}{2} \left( I_3 - \frac{I_2^2}{I_1} \right), \quad C = \frac{I_2}{2I_1}, \quad (118)$$

$$I_1 = \mathcal{I}[|h(\omega, t)|^2] + \frac{1}{2\omega_0} \coth\left(\frac{\beta_0\omega_0}{2}\right) (\dot{\gamma}^2 + \omega_0^2 g^2), \quad (119)$$

$$I_2 = \Re\mathcal{I}[h(\omega, t)k^*(\omega, t)] + \frac{1}{2\omega_0} \coth\left(\frac{\beta_0\omega_0}{2}\right) \dot{\gamma} (\ddot{\gamma} + \omega_0^2 g), \quad (120)$$

$$I_3 = \mathcal{I}[|k(\omega, t)|^2] + \frac{1}{2\omega_0} \coth\left(\frac{\beta_0\omega_0}{2}\right) (\ddot{\gamma}^2 + \omega_0^2 \ddot{g}), \quad (121)$$

$\omega_0$ is a reference frequency taken as that of the initial system state, and equated here to the initial bare frequency. If one so desires, either the renormalized $\omega_R$ or the pole $\bar{\omega}$ may be taken as another choice. But we imagine the situation a small system was added to a large environment at some time, its mutual interaction being absent prior to the initial time. In this circumstance it is appropriate to take $\omega_0$ as the reference frequency. Since dependence on the initial state dies away quickly as time passes, the choice of the initial reference is not crucial for determining the behavior of states at late times. Both of $h(\omega, t)$ and $k(\omega, t)$ are already defined in the preceding subsection. The density matrix $\rho^{(R)}$ from which any physical quantity at time $t$ can be computed has explicitly been given by the discontinuity, $H(\omega)$ or $r(\omega)$.

The diagonal part of the reduced density matrix with $q = q'$, namely with $\xi = 0$, is

$$\rho^{(R)}(X_f = 2q, 0) = 2 \sqrt{\frac{A}{\pi}} \exp\left(-4Aq^2\right). \quad (122)$$

The width of the Gaussian peak of $\approx 1/\sqrt{A}$ is a measure of how the system behaves. For instance, if the width increases with time, the system is more excited than originally prepared, while if it decreases, it is more deexcited.
The basic quantities that appear in the reduced density matrix are related to expectation values of the coordinate and the momentum operators at the same moment by

\[
\langle q^2 \rangle = \frac{1}{8A} = I_1, \quad (123)
\]

\[
\langle p^2 \rangle = 2B + \frac{C^2}{2A} = I_3, \quad (124)
\]

\[
\frac{1}{2} \langle qp + pq \rangle = \frac{C}{4A} = I_2. \quad (125)
\]

Thus one may write the density matrix as

\[
\rho^{(R)}(x_f, \xi_f) = \sqrt{\frac{1}{2\pi \langle q^2 \rangle}} \exp\left[-\frac{1}{8\langle q^2 \rangle} X_f^2 - \left(\frac{\langle p^2 \rangle}{2} - \frac{\langle qp + pq \rangle^2}{8\langle q^2 \rangle}\right) \xi_f^2 + i \frac{\langle qp + pq \rangle}{4\langle q^2 \rangle} X_f \xi_f \right]. \quad (126)
\]

The reduced density matrix is thus characterized by expectation values of quadratic operators, just as in the case of pure Gaussian system without the environmental effect.

It is sometimes useful to transform the density matrix in the configuration space to the Wigner function \( f_W(x, p) \),

\[
f_W(x, p) \equiv \int_{-\infty}^{\infty} d\xi \rho^{(R)}(2x, \xi) e^{-i\xi x}, \quad (127)
\]

\[
= \sqrt{\frac{4A}{B}} \exp\left[-4Ax^2 - \frac{(p - 2C)x^2}{4B}\right]. \quad (128)
\]

The Wigner function is expected to give the probability distribution in the phase space \((x, p)\) when the semi-classical picture is valid. Expectation value of the number operator, namely the occupation number, in terms of the reference frequency, equated to the pole location \(\bar{\omega}\) here, is calculated most easily from the Wigner function:

\[
\langle n \rangle \equiv \langle -\frac{1}{2\bar{\omega}} \frac{d^2}{dq^2} + \frac{\bar{\omega}}{2} q^2 - \frac{1}{2} \rangle = \frac{B}{\bar{\omega}} + \frac{C^2}{4\bar{\omega}A} + \frac{\bar{\omega}}{16A} - \frac{1}{2}
\]

\[
= \frac{1}{2\bar{\omega}} \left( I_3 + \bar{\omega}^2 I_1 \right) - \frac{1}{2}. \quad (129)
\]

It consists, except the trivial \(\frac{1}{2}\), of two terms, the term \(\bar{\omega}/(16A)\) from the Gaussian width of the diagonal density matrix element and the rest from the kinetic term \(-\frac{d^2}{dq^2}\). This formula of course agrees with that of the previous derivation in the operator method. We shall discuss the rate of particle production in more detail when we turn to the specific application, namely the case of periodic frequency \(\omega^2(\tau)\).
An important measure to explore the behavior of the quantum state under the environment action is the effective entropy of the subsystem under study. Even if the entire system of the subsystem plus the environment is in a pure quantum state, the subsystem can have a nonvanishing entropy, since one cannot measure the environment and traces out its freedom. The reduced density matrix derived in the path integral approach is especially suited to calculation of the subsystem entropy $\rho$. We define as usual the entropy of the system as (henceforth we omit the index $R$ for the reduced density matrix)

$$S = - \text{tr} \rho \ln \rho,$$

(130)

where the trace operation is performed on the system variable $q$. A useful device to compute the logarithmic matrix is to convert it to a power series according to the formula like

$$S = - \text{tr} \left( \frac{d}{ds} \frac{1}{\Gamma(3 - s)} \rho^3 \int_0^\infty du u^{2-s} e^{-\rho u} \right)_{s=1}. \quad (131)$$

The arbitrary power $\text{tr} \rho^n$ is calculable as

$$\text{tr} \rho^n = \int \left( \prod_i dq_i \right) \rho(q_1 + q_2, q_1 - q_2) \cdots \rho(q_n + q_1, q_n - q_1)$$

$$= \frac{(4A)^{n/2}}{(\sqrt{B} + \sqrt{A})^n - (\sqrt{B} - \sqrt{A})^n}. \quad (132)$$

A more efficient way to compute the entropy is first to observe, with the presence of the power series expansion, independence of the factor $C$ in the entropy formula. This justifies neglect of the $C$ term (by setting $C = 0$) in the density matrix (117) for computation of the entropy, and only for this purpose. The next step is to identify this density matrix with $C = 0$ as an equivalent harmonic oscillator system of frequency $\tilde{\omega}$ under a thermal bath of temperature $1/\tilde{\beta}$:

$$\sqrt{\frac{\tilde{\omega}}{\pi \coth(\tilde{\beta}/2)}} \exp \left( - \frac{\tilde{\omega}}{4} \coth(\tilde{\beta}/2) \xi_f^2 - \frac{\tilde{\omega}}{4} \tanh(\tilde{\beta}/2) X_f^2 \right). \quad (133)$$

The equivalence is possible only for $B > A$ and is established by the parameter relation,

$$\tilde{\omega} = 4 \sqrt{AB}, \quad \tanh \frac{\tilde{\beta}\tilde{\omega}}{2} = \sqrt{\frac{A}{B}}. \quad (134)$$
With this identification the equivalent temperature and the entropy are given by

\[
\tilde{T} = \frac{4 \sqrt{AB}}{\ln \left( (\sqrt{B} + \sqrt{A})/(\sqrt{B} - \sqrt{A}) \right)},
\]

\[
S = \frac{\tilde{\omega} \beta}{e^{\tilde{\omega} \beta} - 1} - \ln(1 - e^{-\tilde{\omega}}) = \frac{1}{2} (x - 1) \ln \frac{x + 1}{x - 1} + \ln \frac{x + 1}{2},
\]

\[
x = \sqrt{\frac{B}{A}} = 2 \sqrt{I_1 I_3 - I_2^2} = \sqrt{4 \langle q^2 \rangle \langle p^2 \rangle - \langle qp + pq \rangle^2}.
\]

The entropy \( S \) is a monotonic function of the single variable \( x = \sqrt{B/A} \). Its limiting values are

\[
S \rightarrow -\frac{1}{2} (x - 1) \ln \frac{x - 1}{2}, \quad \text{as } x \rightarrow 1^+,
\]

\[
\rightarrow \ln \frac{x}{2}, \quad \text{as } x \rightarrow \infty.
\]

It is interesting to note that the entropy initially increases irrespective of the relation between the two temperatures of the subsystem and the environment, \( T_0 \) and \( T \). This can be proved by first writing down the time evolution equation for the quantity \( x^2 = 4 \langle q^2 \rangle \langle p^2 \rangle - \langle qp + pq \rangle^2 \):

\[
\frac{dx^2}{dt} = -2C(t) x^2 - 16D_{pp}(t) \langle q^2 \rangle - 8D_{xp}(t) \langle qp + pq \rangle.
\]

As \( t \rightarrow 0^+ \), the dominant term in the right hand side is the \( D_{pp}(t) \) term. Thus at \( t \sim 0 \)

\[
\frac{dx^2}{dt} \sim 2I[1] t \langle q^2 \rangle > 0.
\]

This proves the entropy increase.

When \( A \ll B \) holds, considerable simplification follows:

\[
\tilde{\omega} = 4\sqrt{AB},
\]

\[
\tilde{T} \sim 2B,
\]

\[
S \sim \frac{1}{2} \ln \frac{B}{4A} + 1.
\]

When \( B = A \), or equivalently \( \langle p^2 \rangle \langle q^2 \rangle = \langle qp^2 q \rangle \), the density matrix is factorizable into a product of \( \psi(q) \psi^*(q') \), since \( \text{tr} \rho^n = 1 \). This is a pure quantum state, having the entropy \( S = 0 \).
When the initial subsystem temperature $T_0 = 1/\beta_0$ is nonvanishing, the entropy is

$$S_i = \frac{\beta_0 \omega_0}{e^{\beta_0 \omega_0} - 1} - \ln(1 - e^{-\beta_0 \omega_0}),$$

(145)

initially. For $T_0 \gg \omega_0$, $S_i \approx \ln \frac{T_0}{\omega_0}$. It ends finally with

$$S_f = \ln \frac{\bar{T}}{\bar{\omega}},$$

(146)

for $B \gg A$. As will be made clear shortly, this is equal to $\ln \frac{T}{\omega}$ in the high temperature limit. This variation, from $\ln(T_0/\omega_0)$ to $\ln(T/\bar{\omega})$, is a result one would naively expect.

Existence of the equivalent harmonic oscillator basis for $B \geq A$ actually implies much more than a convenient means of the entropy computation. We have found the basis of the states that diagonalize the density matrix at any instant of time: it is the Fock basis $|n\rangle$ of the equivalent harmonic oscillator of frequency $\bar{\omega}$ with diagonal elements:

$$\rho_n = 2 \sinh\left(\frac{\bar{\beta} \bar{\omega}}{2}\right) e^{-\frac{\bar{\beta} \bar{\omega}}{2}} \left(\frac{\sqrt{B} - \sqrt{A}}{2\sqrt{A}}\right)^n.$$

(147)

Since the parameters, $A$ and $B$, are time dependent, the diagonalization of the density matrix is instantaneous. One thus thinks of the subsystem temperature varying with time as

$$\bar{T} = \frac{1}{2 \langle q^2(t) \rangle} \frac{x(t)}{\ln \frac{x(t) + 1}{x(t) - 1}},$$

(148)

however with frequency also changing as

$$\bar{\omega} = \frac{x(t)}{2 \langle q^2(t) \rangle}.$$

(149)

To explore a more detailed nature of the final state, let us work out the frequency distribution of populated harmonic oscillator levels. Suppose that one asks the probability of finding the system in the unit energy interval around $\omega = \frac{p^2}{2} + \omega_0^2 q^2$. We get the energy distribution function $f(\omega)$ [1] from the Wigner function:

$$f(\omega) \equiv \int \frac{dq dp}{2\pi} f_W(q,p) \delta(\omega - \frac{p^2}{2} - \omega_0^2 \frac{q^2}{2})$$

$$= \frac{2}{\omega_0 \sqrt{B}} \exp\left( -\omega \frac{4A}{\omega_0} + \frac{4C^2 + \omega_0^2}{4\omega_0 B} \right) I_0\left(\frac{\omega}{\omega_0} z\right),$$

(150)

$$z = \left( \frac{4A}{\omega_0} + \frac{4C^2 + \omega_0^2}{4\omega_0 B} \right)^2 - 4 \frac{A}{B} \right)^{1/2}. $$

(151)
Here

$$I_0(w) = \frac{1}{\pi} \int_0^\pi d\theta e^{-w \cos \theta}$$

(153)

is the modified Bessel function, that behaves as

$$I_0(w) \to \frac{1}{\sqrt{2\pi w}} e^w \ (\text{as } w \to \infty), \quad \to 1 \ (\text{as } w \to 0).$$

(154)

$\omega_0$ is again the reference frequency not to be confused with the original subsystem frequency.

This distribution differs from the thermal one and has some interesting features. At $\omega \to \infty$, it decreases as

$$f(\omega) \to \frac{2}{\sqrt{2\pi \omega_0 \omega z}} \sqrt{\frac{A}{B}} e^{-\omega D},$$

(155)

$$D = \frac{1}{\omega_0} \left[ \frac{4A}{\omega_0} + \frac{4C^2 + \omega_0^2}{4\omega_0 B} - \left( \frac{4A}{\omega_0} + \frac{4C^2 + \omega_0^2}{4\omega_0 B} \right)^2 - \frac{4A}{B} \right]^{1/2},$$

(156)

assuming $z \neq 0$. The high frequency tail of this distribution decreases more rapidly than that of the Boltzmann distribution, $e^{-\beta \omega}$. On the other hand, at $\omega = 0$

$$f(0) = \frac{2}{\omega_0} \sqrt{\frac{A}{B}}.$$

(157)

The average energy of the distribution is computed as

$$\langle \omega \rangle = \int_0^\infty d\omega \omega f(\omega) = \omega_0 B \left( \frac{4A}{\omega_0} + \frac{4C^2 + \omega_0^2}{4\omega_0 B} \right).$$

(158)

This average satisfies the relation,

$$\langle \omega \rangle = \omega_0 \left( \langle n \rangle + \frac{1}{2} \right),$$

(159)

when $\bar{\omega}$ is replaced by $\omega_0$ in eq.(129).

The exceptional case of $z = 0$ with an exact exponential form is actually important, since it is realized in thermal equilibrium. This is possible only when $C = 0$, and $\sqrt{AB} = \omega_0/4$, giving

$$f(\omega) = \frac{e^{-\omega/T}}{T},$$

(160)

$$\langle \omega \rangle = T.$$

(161)
The normalized spectrum shape written in terms of a scaled energy, \( x = \omega / \langle \omega \rangle \), is characterized by a single parameter \( \delta \),

\[
f_0(x; \delta) = \frac{1}{\delta} e^{-x/\delta^2} I_0 \left( \frac{\sqrt{1 - \delta^2}}{\delta^2} x \right)
\]

\[
= \frac{1}{\pi \delta} e^{-x/\delta^2} \int_0^{\pi} d\theta \exp \left[ -x \frac{\sqrt{1 - \delta^2}}{\delta^2} \cos \theta \right],
\]

with which the energy distribution is given by

\[
f(\omega) = \frac{1}{\langle \omega \rangle} f_0 \left( \frac{\omega}{\langle \omega \rangle}; \delta \right),
\]

\[
\delta = 2 \sqrt{\frac{A}{B}} \left( \frac{4A}{\omega_0^2} + \frac{4C^2 + \omega_0^2}{4\omega_0 B} \right)^{-1} = \omega_0 \sqrt{\frac{4 \langle q^2 \rangle \langle p^2 \rangle - \langle qp + pq \rangle^2}{\langle p^2 \rangle + \omega_0^2 \langle q^2 \rangle}} = \frac{x}{2\langle n \rangle + 1}.
\]

This function has the following limiting behavior:

\[
f_0(x; \delta) \to e^{-x} \text{ as } \delta \to 1,
\]

\[
f_0(x; \delta) \to \frac{1}{\sqrt{2\pi x}} e^{-x/2} \text{ as } \delta \to 0.
\]

The first moment \( \bar{x} \) happens to be the same in the two limits of \( \delta = 1 \) and \( \delta = 0 \). The main difference of these two limiting distributions is in the low \( x \) part. We call \( \delta \) the spectral shape parameter.

The asymptotic late time behavior of the reduced density matrix is determined by

\[
h(\omega, \infty) = \int_0^{\infty} d\tau \ g(\tau) e^{-i\omega \tau},
\]

which is equal to the boundary value, \( F(\omega - i0^+) \), thus giving the already known relations, \( r(\omega)|h(\omega, \infty)|^2 = H(\omega) \), \( r(\omega)|k(\omega, \infty)|^2 = \omega^2 H(\omega) \). Thus, at asymptotic late times

\[
A \to \frac{1}{8} \left( \int_{\omega_c}^{\infty} d\omega \ \coth \left( \frac{\beta \omega}{2} \right) H(\omega) \right)^{-1},
\]

\[
B \to \frac{1}{2} \int_{\omega_c}^{\infty} d\omega \ \coth \left( \frac{\beta \omega}{2} \right) \omega^2 H(\omega),
\]

\[
C \to 0.
\]

Equivalently,

\[
\langle q^2 \rangle \to \int_{\omega_c}^{\infty} d\omega \ \coth \left( \frac{\beta \omega}{2} \right) H(\omega)
\]

\[
\langle p^2 \rangle \to \int_{\omega_c}^{\infty} d\omega \ \coth \left( \frac{\beta \omega}{2} \right) \omega^2 H(\omega).
\]

These agree with those in the operator approach. In these computations the analytic structure and only that is important.
In the high temperature limit
\[ \langle q^2 \rangle \sim \frac{T}{\omega_R^2}, \quad \langle p^2 \rangle \sim T. \]  
(173)

The renormalized frequency \( \omega_R \) is defined in the preceding subsection, eq.(28) and is given by
\[ \omega^2_R = \left( \int_{\omega_c}^{\infty} d\omega \frac{2H(\omega)}{\omega} \right)^{-1}. \]  
(174)

In this contribution the continuous integral during the period of the power law decay is numerically subdominant, suppressed by the factor \( 1/T^2 \) relative to the one from the exponential period. On the other hand, at low temperatures the contribution from the threshold region, \( \omega \approx \omega_c \), cannot be ignored, giving the dominant contribution to the power law period, as noted in the preceding subsection.

It may be useful to write crude formulas for asymptotic values in the Breit-Wigner approximation, using \( 2\omega H(\omega) \approx \delta(\omega - \bar{\omega}) \):
\[ \langle q^2(\infty) \rangle \approx \frac{1}{2\bar{\omega}} \coth(\frac{\beta \bar{\omega}}{2}), \quad \langle p^2(\infty) \rangle \approx \frac{\bar{\omega}}{2} \coth(\frac{\beta \bar{\omega}}{2}), \quad \tilde{\omega}(\infty) \approx \bar{\omega}, \quad \tilde{T}(\infty) \approx T, \quad x(\infty) \approx \coth(\frac{\beta \bar{\omega}}{2}), \quad \delta(\infty) \approx \frac{2\bar{\omega}\omega_0}{\bar{\omega}^2 + \omega_0^2}. \]  
(175)

These are not precise, but they are nevertheless instructive to understand precise formulas for these quantities. \( \delta(\infty) < 1 \) for \( \omega_0 \neq \bar{\omega} \). But if one takes the reference frequency equal to \( \bar{\omega} \), one gets \( \delta(\infty) = 1 \), in complete agreement with thermal equilibrium.

In closing this section, we would like to stress that the two approaches we described, the operator and the path integral, are both useful and complement each other. In both approaches the maximal use of the extended analyticity has been instrumental to completely clarify the details of the linear open system. The model of the linear open system is thus analytically solvable.

### III Particle Decay in Thermal Medium

Let us apply these considerations to the decay of unstable particle; \( \varphi \rightarrow \chi + \chi \). We assume that the decay product \( \chi \) is a part of thermal components that make up the environment. The parent particle \( \varphi \) of mass \( M \) may or may not be in thermal
equilibrium with the rest of medium: since we focus on the late time behavior, the initial state dependence may be ignored.

Let me first explain how the field theory may be incorporated in the framework so far discussed. The point is that the infinitely many environment variable $Q_a$ can actually be a composite field of more fundamental fields. We consider as the system variable $q_k$ a Fourier component of some scalar field $\varphi(x)$ in a fully relativistic field theory. As a simplest field theory model that may describe the decay process, take a Yukawa type of interaction with a new bose field $\chi(x)$;

$$L_{\text{int}} = \int_V d^3x \mathcal{L}_{\text{int}}, \quad \mathcal{L}_{\text{int}} = -\frac{\mu}{2} \varphi \chi^2,$$

with $\mu$ a parameter of mass dimension. $V$ is the normalization volume. By identifying the Fourier-mode with

$$q_k e^{i \vec{k} \cdot \vec{x}} = \frac{1}{\sqrt{2\omega_k V}} (a_k + a_k^\dagger) e^{i \vec{k} \cdot \vec{x}},$$

we deduce the environment variable $Q_k(\omega)$ coupled to it,

$$\int d\omega c(\omega) Q_k(\omega) = \frac{\mu}{2} \int_V d^3x \chi^2(x) e^{-i \vec{k} \cdot \vec{x}}.$$

The continuous label $\omega$ of the environment variable $Q_k(\omega)$ is thus identified to the internal configuration of two body states with a given total momentum $\vec{k}$. This field theory model was introduced in ref [1]. Henceforth we regard the field $\chi$ as a fundamental variable of the environment, assumed to be in thermal equilibrium.

The correlation function $\alpha_k(\tau)$, or more conveniently its Fourier transform $\alpha_k(\omega)$, can be computed as in the real-time formalism at finite temperatures. To lowest order in $\mu^2$

$$i\alpha_k(\tau - s) = i \left( \frac{\mu}{2} \right)^2 \int_V d^3x e^{i \vec{k} \cdot \vec{x}} \text{tr} \left( T [\chi^2(\vec{x},\tau) \chi^2(\vec{0},s)] \rho_\beta \right)$$

is calculable in terms of the $\chi-$propagator in the momentum space,

$$i \left( \frac{1}{\omega^2 - \vec{k}^2 - m_\chi^2 + i\epsilon} - \frac{2\pi i}{e^\beta \omega_k - 1} \delta(\omega^2 - \vec{k}^2 - m_\chi^2) \right).$$

We omit the suffix $\chi$ for the $\chi-$mass $m_\chi$ in what follows.

As in the $T = 0$ field theory it is easiest to first compute the discontinuity instead of the full diagramatic contribution. Physically the most transparent way to compute the imaginary part of the self-energy diagram, $\Im \Pi(\omega)$, is to use the
analytically continued expression [15] from the imaginary-time formalism. The analytically continued imaginary part, or more precisely the discontinuity, directly gives the response weight via
\[ r(\omega) = 2\Im \Pi(\omega) = 2\omega \Gamma(\omega), \quad \Im \Pi(\omega) \equiv \frac{1}{2i} (\Pi(\omega - i\epsilon) - \Pi(\omega + i\epsilon)), \] (182)
where \( \Gamma(\omega) \) is interpreted as a decay rate in thermal medium.

The result for the \( \chi \) loop diagram at finite temperatures is well known [15], and physically consists of two parts; the process \( \varphi \leftrightarrow \chi + \chi \) in the region of \( \omega > \sqrt{k^2 + 4m^2} \) and the other process \( \varphi + \chi \leftrightarrow \chi \) (forbidden when all particles are on the mass shell, but allowed in thermal environment) in \( 0 < \omega < k \), where \( m \) is the daughter \( \chi \) mass. The response weight \( r(\omega) \) does not vanish for \( |\omega| > \sqrt{k^2 + 4m^2} \) and \( |\omega| < |k| \) from the kinematics of the decay and the inverse decay of particles off the mass shell, with the constraint of the momentum conservation. Thus a gap of the spectrum exists in \( k < |\omega| < \sqrt{k^2 + 4m^2} \). The finite non-vanishing mass of the daughter particle \( (m \neq 0) \) is important for the existence of the gap and for associated physical consequences that follow.

Let us explain some details of the calculation of the response weight in the sub-threshold region of \( |\omega| < k \). One loop contribution from the imaginary-time formalism is given by [15]
\[ \Im \Pi(\omega) = \frac{\mu^2}{16\pi k} \int_{-\omega_-}^{\infty} dE \left( n(E) - n(E + \omega) \right), \] (183)
where \( \omega_{\pm} = \frac{\omega}{2} \pm \frac{k}{2} \sqrt{1 - \frac{4m^2}{\omega^2 - k^2}}, \) (184)
and the Planck distribution function of \( T = 1/\beta \). Since
\[ n(E) - n(E + \omega) = n(E)(1 + n(E + \omega)) - n(E + \omega)(1 + n(E)), \] (186)
the imaginary part \( \{183\} \) for \( |\omega| < k \) is a sum of the two contributions, \( \chi + \varphi \rightarrow \chi \) and its inverse process that is allowed to occur in thermal medium. Note that \( \varphi \) can be off the mass shell: \( \omega^2 - k^2 \neq \varphi \) mass\(^2\). The factor \( 1 + n \) represents the effect of stimulated boson emission.
The result of computation is now summarized. For \( \omega > \sqrt{k^2 + 4m^2} \) the response weight is

\[
\frac{\mu^2}{16\pi} \left( \sqrt{1 - \frac{4m^2}{\omega^2 - k^2}} + \frac{2}{k\beta} \ln \frac{1 - e^{-\beta\omega^+}}{1 - e^{-\beta|\omega^-|}} \right). \tag{187}
\]

Note that \( r(\omega) \rightarrow \) a constant (= \(2 \times \) decay rate in the rest frame of \( \varphi \)), hence the frequency renormalization or the subtracted form of \( \omega \) integral is necessary. For \( 0 < \omega < k \) only the second term in the bracket of Eq.(187) contributes.

A useful, and adequate approximation we exploit for subsequent estimate is the weak coupling scheme with correct threshold and asymptotic behaviors incorporated:

\[
F(z) = \frac{1}{-z^2 + \bar{\omega}^2 - i \pi r(z)}. \tag{188}
\]

In this approximation we replaced the \( \omega \) dependent real part of the self-energy \( \Pi(\omega) \) by a constant, hence by the constant pole location \( \bar{\omega} \).

A quantity of physical interest is the fraction of remaining particles given by the occupation number at asymptotic late times,

\[
n_k = \frac{1}{2} \left( \frac{p_k^2}{\omega_k} + \omega_k q_k^2 \right) - \frac{1}{2} \approx \frac{B_k}{\omega_k} + \frac{\omega_k}{16A_k} - \frac{1}{2}, \tag{189}
\]

for each \( \vec{k} \) mode (\( \omega_k = \sqrt{\vec{k}^2 + M^2} \)). The temperature dependent part of this quantity is

\[
n_k^\beta = \int_0^\infty d\omega \frac{1}{e^{\beta\omega} - 1} \left( \omega_k + \frac{\omega^2}{\omega_k} \right) H(\omega, k). \tag{190}
\]

One must sum over momentum \( \vec{k} \) to obtain the number density of remnants.

We shall limit our discussion here to the decay that occurs when the parent \( \varphi \) becomes non-relativistic,

\[
\omega_k \sim M + \frac{\vec{k}^2}{2M} \gg T. \tag{191}
\]

This condition is relevant in interesting cosmological problems of the neutron decay at the time of nucleosynthesis and GUT X boson decay at baryogenesis [16].

Computation of the temperature dependent part of the occupation number \( n_k^\beta \) may proceed by deforming the contour of \( \omega \) integration, in the same way as in the discussion of \( g(\tau) \) in the preceding section. There are then two types of contribution: one is the pole term that gives the usual Boltzmann suppressed contribution of \( e^{-\beta\omega_k} \).

When mode-summed, it gives the number density,

\[
\int \frac{d^3k}{(2\pi)^3} e^{-\beta(M+k^2/2M)} = \left( \frac{MT}{2\pi} \right)^{3/2} e^{-M/T}. \tag{192}
\]
This is the familiar Boltzmann suppressed formula.

The second one is contribution from the continuous complex path that gives the power law behavior of temperature dependence. A part of this contribution in the region $\omega > \sqrt{k^2 + 4m^2}$ is analytically calculable by using $r(\omega) = \frac{\mu^2}{16\pi} + O[m^2]$, valid for a small daughter mass $m$. It is

$$
\frac{n}{T^3} \approx \frac{1}{1440} \frac{\mu^2 T^4}{M^3}.
$$

This calculation however ignores complicated logarithmic factors in $r(\omega)$ of eq.(187).

We numerically computed all terms including the logarithmic factor in $r(\omega)$ along with $O[m^2]$ corrections. It turns out that the total contribution is ten times larger than the analytic result above: in the $m \to 0$ limit,

$$
n \approx 2.4 \times 10^{-2} \frac{\mu^2 T^4}{M^3}.
$$

The main part of this large contribution comes from $|\omega| < k$. With a dimensionless constant introduced by $\mu = gM$, this gives, relative to the photon number density ($= \frac{2\zeta(3)}{\pi^2} T^3$),

$$
\frac{n}{T^3} \approx 2 \times 10^{-12} \left( \frac{g}{G_F m_N^2} \right)^2 \frac{T}{M}.
$$

We wrote here the numerical value using $G_F$ the weak interaction constant of mass dimensions $-2$ ($G_F m_N^2 \approx 10^{-5}$), as if it were relevant to the neutron decay.

One may estimate the equal time temperature $T_{eq}$ at which the power contribution becomes equal to the Boltzmann suppressed number density, to give

$$
\frac{T_{eq}}{M} \approx \frac{1}{33}, \quad \frac{n}{T_{eq}^3} \approx 7 \times 10^{-14},
$$

taking as an example $\mu = 10^{-5} M$, the weak interaction strength. This number is in an interesting range to affect nucleosynthesis, but we should keep in mind that we did not work out the relevant three-body decay, $n \to p + e + \bar{\nu}_e$.

The physical interpretation of the pole term is a conventional one in terms of the remnant created by the inverse decay $\chi + \chi \to \varphi$, with all relevant particles on the mass shell, hence suppressed kinematically by the Boltzmann factor $e^{-M/T}$. On the other hand, the contribution from the continuous contour can only be interpreted as remnant particles far off the mass shell that may exist in thermal equilibrium.
The usual kinetic approach such as the Boltzmann-like equation is based on the rates computed from S-matrix elements on the mass shell and gives the Boltzmann suppressed abundance in equilibrium for $M \gg T$. Our fully quantum mechanical approach yields a different result.

We note that the local friction approximation is equivalent to the pole model (with identification of $\omega_0^2 + \delta \omega^2 = (\Re z_0)^2$, $\eta = -2\Im z_0$) that ignores the continuum integral around the threshold. Hence the pole model, or the local friction approximation, fails to correctly describe the off-shell remnant.

We shall mention another application of immediate interest in cosmology; the heavy $X$ boson decay for GUT baryogenesis. It has been argued [16] that there exists a severe mass bound of order,

$$m_X > O[\alpha_X m_{pl}] \approx 10^{16}\text{GeV}, \quad (197)$$

to block the inverse process of the $X$ boson decay so that generation of the baryon asymmetry proceeds with sufficient abundance of parent $X$ particles. The usual estimate of the mass bound mentioned above is however based on the on-shell Boltzmann equation. More appropriate formula in this estimate is our remnant number density,

$$n_X \approx O[2 \times 10^{-2}] g^2_X T^4 \frac{T^4}{m_X}. \quad (198)$$

(In a more realistic estimate one should consider the $X$ boson decay into quarks and leptons. But for an order of magnitude estimate difference in statistics is not important.) With the GUT coupling of $g^2_X/4\pi = 1/40$, the equal temperature is roughly

$$T_{eq} \approx \frac{M}{2}. \quad (199)$$

Thus, already at temperature of about half of the $X$ mass the Boltzmann suppressed formula is replaced by the power formula. The kinematical condition for baryogenesis must be reconsidered in view of our off-shell formula.

**IV Parametric Resonant Particle Production by Coherent Field Oscillation**

We now switch to an entirely different dynamical system under dissipative environment. Coherent field oscillation often appears in modern cosmology. Two main
examples worthy of explicit mention are inflaton oscillation that gives rise to the hot big band from the null state after inflation, and the moduli field oscillation that often appears in supergravity theories.

We consider the case in which a quantum system field \( \varphi \) is coupled to a coherent field oscillation \( \xi(t) \propto \cos(m_\xi t) \). The coherent field may be viewed as an aggregate of zero-momentum particles with some kind of precise coherence. \( m_\xi \) is thus the mass of these bosons. Here this field oscillation is regarded as given, hence we do not discuss field damping due to particle production of the \( \varphi \)-field. Our main concern here is in the effect of thermal bath on the \( \varphi \)-particle production caused by \( \xi \)-oscillation.

Under this circumstance the harmonic oscillator variable is identified to \( q_k = \frac{1}{\sqrt{2\omega_k}} (a_k + a_k^\dagger) \), where the creation and the annihilation operators are those of Fourier \( \vec{k} \)-mode \( (\propto e^{i\vec{k} \cdot \vec{x}}) \) of the system field \( \varphi(x) \). For the oscillator-system interaction we take the quartic coupling given by

\[
\frac{1}{2} g^2 \xi^2 \varphi^2, \tag{200}
\]

primarily because this case has been analyzed in considerable detail without taking into account the environment effect. The \( \vec{k} \)-mode variable \( q_k \) then obeys the evolution equation,

\[
\left( \frac{d^2}{d\tau^2} + \omega_k^2(\tau) \right) q_k(\tau) = 0 , \quad \omega_k^2(\tau) = \vec{k}^2 + m_\varphi^2 + g^2\xi_0^2 \cos^2(m_\xi \tau) . \tag{201}
\]

\( \xi_0 \) is the amplitude of oscillation. The standard form of this type of equation is called the Mathieu equation and is usually written in dimensionless units:

\[
\left( \frac{d^2}{dz^2} + h + 2\theta \cos(2z) \right) q_k(z) = 0 , \tag{202}
\]

\[
z = m_\xi \tau , \quad h = \frac{\vec{k}^2 + m_\varphi^2}{m_\xi^2} + 2\theta , \quad \theta = \frac{g^2\xi_0^2}{4m_\xi^2} . \tag{203}
\]

It is well known that this quantum system exhibits instability in infinitely many band regions \([17]\) of the parameters of \( (|\vec{k}|, g\xi_0) \), or \( (h, \theta) \). It has been realized that this instability gives rise to particle production \([16]\). The most important recent development is that for large amplitude oscillation particle production and associated field decay is greatly expedited, typically ending in a time scale of \( O[10^{−100}] \) times \( 1/m_\xi \) \([18], [19], [20], [21], [22]\). However, in all initial investigations so far no systematic estimate of the environmental effect has been attempted. This is precisely
what we wish to do in the present section. Throughout this section we assume that the time scale of cosmic expansion is much larger than the oscillation time scale so that the cosmological expansion may be ignored. This is usually valid at the explosive stage of particle production.

This dynamical problem belongs to the more general class of dynamics of time dependent harmonic oscillator. We shall briefly mention how much of the general formalism for the simple harmonic oscillator is modified in this case. There are two time dependences in this system: the nonlocal effect of the correlation $\alpha(\tau - s)$ introduced by the environmental interaction, and the time dependent frequency $\omega(\tau)$. The most important cases of applications can be analyzed using the approximate localized friction for $\alpha_{l}(\tau - s)$, since we are primarily interested in the late time behavior. In this case one can explicitly perform the path integral in terms of a classical solution for $q_k(\tau)$.

The notion of the localized friction arises by focusing on the late time behavior of $\alpha_{l}(\tau)$. By the late time here we mean the asymptotic time that is much larger than any time scale of excitation in the environment; $t \gg 1/\Omega$ (maximal excitation frequency) $\approx 1/\Omega$. The limiting form is then given by

$$\alpha_{l}(\tau) = -\int_{0}^{\infty} d\omega \ r(\omega) \sin(\omega \tau) \to \eta \delta'(\tau) + \delta \omega^2 \delta(\tau), \quad (204)$$

$$\eta = -2 \int_{0}^{\infty} d\tau \ \tau \alpha_{l}(\tau) = \pi \ r'(0), \quad \text{for } r(\omega = \infty) = 0, \quad (205)$$

$$\alpha_{R}(\tau) \to \frac{\eta}{\pi} \int_{0}^{\infty} d\omega \omega \ f(\frac{\omega}{\Omega}) \coth(\frac{\beta \omega}{2}) \cos(\omega \tau). \quad (206)$$

The essence of this approximation is the low frequency truncation to the response weight,

$$r(\omega) = \omega \ r'(0) = \frac{\eta}{\pi} \omega. \quad (207)$$

Although the dissipation given by $\alpha_{l}(\tau)$ is local by the time scale larger than $1/\Omega$ the cutoff scale, the damping time scale of the noise kernel $\alpha_{R}(\tau)$ may differ. It can be shown that at high temperatures of $T \geq \Omega$ the locality of the correlation holds for $\tau \geq 1/\Omega$, giving

$$\alpha_{R}(\tau) \sim \frac{\eta T}{\pi} \delta(\tau). \quad (208)$$

On the other hand, at low temperatures of $T \leq \Omega$ the local approximation may be inadequate and a high frequency cutoff is needed. As already discussed, we replace in this case the response weight by multiplying some cutoff function $f(x)$, for instance the simple frequency cutoff at $\omega = \Omega$ with $f(x) = \theta(1 - x)$, as is often practiced.
With $\alpha_I(\tau) \propto \delta'(\tau)$, the exponent factor in the path integral becomes, including the mass renormalization effect,

\[-i \frac{1}{2} \int_0^t d\tau X(\tau) \left( \frac{d^2}{d\tau^2} + \omega_R^2(\tau) \right) \xi(\tau) + \frac{i}{2} \eta \int_0^t d\tau X(\tau) \dot{\xi}(\tau)\]
\[+ \frac{i}{2} \left[ X\dot{\xi} \right]_0^t - \frac{i}{2} \eta \xi_0 X \xi \text{sign}(\tau) - \int_0^t d\tau \int_0^\tau ds \xi(\tau) \alpha_R(\tau - s) \xi(s) . \] 

(209)

A remarkable feature of this formula is that a part of the action excluding the $\alpha_R$ term is a local integral described in terms of a renormalized frequency $\omega_R(\tau)$ and the friction term $\propto \eta$. This much is enough to considerably simplify calculation of the transition amplitude of the $q$–system.

Path integral over the system variable $q$ is standard: as in the more general case the equation for the classical path of the difference path $\xi(\tau)$, which is obtained by functional differentiation with respect to $X(\tau)$, is

\[\left( \frac{d^2}{d\tau^2} + \omega_R^2(\tau) - \eta \frac{d}{d\tau} \right) \xi(\tau) = 0 . \] 

(210)

This equation is best analyzed by introducing a new $y(\tau)$,

\[y(\tau) = e^{\eta \tau / 2} \xi(\tau) , \]
\[\left( \frac{d^2}{d\tau^2} + \omega_R^2(\tau) - \frac{\eta^2}{4} \right) y(\tau) = 0 . \] 

(211)

(212)

The technique of the Laplace transform, which was very useful for the simple harmonic oscillator in the previous sections, does not work here due to the time dependence $\omega_R^2(\tau)$. This is another main reason we have to resort to the local friction approximation.

Limiting our analysis to modes within the instability band of the modified Mathieu equation, we note that two linearly independent solutions are either growing or decaying according to

\[y(\tau) = e^{\lambda m_\xi \tau} P(\tau) , \quad \text{or} \quad e^{-\lambda m_\xi \tau} R(\tau) , \] 

(213)

where $\lambda > 0$ is the dimensionless growth rate and both $P(\tau)$ and $R(\tau)$ are periodic with period of $2\pi/m_\xi$. There are two independent solutions for $y(t)$, $u(t)$ and $v(t)$ satisfying the boundary condition, $u(0) = 0$, $v(t) = 0$. These are linear combinations of the growing and the decaying solutions. With the normalization appropriately given, leading asymptotic behaviors of these are

\[u(t) \sim e^{\lambda m_\xi t} \hat{P}(t) , \quad \hat{v}(t) = -\frac{\dot{u}(0)v(0)}{u(t)} \sim e^{-\lambda m_\xi t} \hat{R}(t) , \] 

(214)

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with both $\bar{P}(t)$ and $\bar{R}(t)$ bounded functions.

Note that the growth rate $\lambda$ here is related to the solution of the modified Mathieu equation, eq.(212), and not of the original one. Thus we should keep in mind that the growth rate $\lambda$ does depend on the friction $\eta$: $\lambda = \lambda(\eta)$. Suppose for instance that the renormalized $\varphi$–mass vanishes: $m_\varphi^2 + \delta \omega^2 = 0$. The $\eta$ term in eq.(212) then has an effect of lowering the band level, since the $h$ parameter in the standard Mathieu equation is modified to

$$\bar{h} = h - \frac{\eta^2}{4m_\xi^2} = \frac{\bar{k}^2}{m_\xi^2} + 2\theta - \frac{\eta^2}{4m_\xi^2},$$

(215)

in this case. Although it is in general difficult to estimate the influence of the friction $\eta$ on the growth rate $\lambda(\eta)$, the general trend is obvious: the lowering effect of the band level tends to increase the growth rate. In the extreme case of a very large $\eta$ the $h$ parameter becomes negative, and the parameter is clearly in the instability region. In perturbation theory we later work with, the leading $\lambda$ term is however independent of $\eta$.

The exponential growth of the original mode function $q_k(\tau)$ is possible for $2\lambda(\eta) m_\xi > \eta$. We observe two competitive factors for the growth: the rate of the parametric amplification $2\lambda(\eta) m_\xi$ against the friction $\eta$. As will be shown in subsequent computations of physical quantities, the friction, when it is small, does act to diminish the parametric particle production, but does not wipe out the parametric effect. Thus, instead of the blocking factor, it is more appropriate to view the friction as the inverse time scale for the system to be driven towards thermalization. Hence if the friction is small enough, the parametric amplification never loses against the friction.

In subsequent discussion we only present analytic computation of time evolution of physical quantities and leave results of numerical calculation to our paper [1]. In analytic estimate we concentrate on the asymptotic late time behavior for which $\tau \gg \max \left( \frac{1}{\eta}, \frac{1}{(\lambda m_\xi)} \right)$. With the asymptotic behavior of the classical solution, $u(\tau)$ and $v(\tau)$, the three quantities that appear in the reduced density matrix (117) of the $\varphi$–system are

$$A \rightarrow \frac{1}{8K} \left( \frac{\ddot{u}(0)}{u(t)} \right)^2,$$

$$B \rightarrow \frac{A}{2} - \frac{C^2}{2K},$$

(216),

(217)
\[ C \to \frac{C \cdot \dot{u}(0)}{2K \cdot u(t)} - \frac{1}{2} \left( \frac{\eta}{2} - \frac{\dot{u}(t)}{u(t)} \right). \quad (218) \]

The quantities \( K, A, C \) are defined by
\[
K = B + e^{-\eta t} \frac{\omega_0}{2} \coth \left( \frac{\beta_0 \omega_0}{2} \right) \left( 1 + \frac{1}{\omega_0^2} \left( \frac{\eta}{2} + \frac{\dot{v}(0)}{v(0)} \right)^2 \right), \quad (219)
\]
\[
(A, B, C) = \int_0^\infty d\omega \coth \left( \frac{\beta \omega}{2} \right) r(\omega) (a, b, c)(\omega), \quad (220)
\]
\[
a(\omega) = \left| \int_0^t ds \frac{u(s)}{u(t)} e^{i\omega s-\frac{\eta}{2}(t-s)} \right|^2, \quad (221)
\]
\[
b(\omega) = \left| \int_0^t ds \frac{v(s)}{v(0)} e^{i\omega s-\frac{\eta}{2}(t-s)} \right|^2, \quad (222)
\]
\[
c(\omega) = \Re \left( \int_0^t ds \frac{u(s)}{u(t)} e^{i\omega s-\frac{\eta}{2}(t-s)} \cdot \int_0^t ds \frac{v(s)}{v(0)} e^{-i\omega s-\frac{\eta}{2}(t-s)} \right). \quad (223)
\]

They have asymptotic behaviors:
\[
A = O[1], \quad C \leq O[\text{Max. } (e^{-\lambda m \xi t}, e^{-\eta t/2})], \quad K = O[\text{Max. } (e^{-2\lambda m \xi t}, e^{-\eta t})]. \quad (224)
\]

Thus, we conclude that \( A, B, C \) behave asymptotically as
\[
\mathcal{A} = O[\text{Min. } (1, e^{-2\lambda m \xi t+\eta t})], \quad \mathcal{B} = O[1], \quad \mathcal{C} = O[1]. \quad (225)
\]

We omitted dimensional scales of these quantities such as \( \omega_0 \) and \( \eta \), since these differ from case to case, depending on the parameter range of \( T, \omega_0, m_\xi, \eta \). For the reference frequency we take \( \omega_0^2 = \omega_R^2(0) \) : the initial value of the frequency.

The formulas so far are valid for any response weight \( r(\omega) \). We now specialize to the environment system that may be described by the approximate form of the localized friction, \( r(\omega) = \frac{\eta}{\pi} \omega \times \text{ (high } \omega \text{-cutoff)}. \) In particular, we are very much interested in the high and low temperature limits of various quantities. The high temperature here effectively means that \( T \gg \text{Max. } (\omega_0, m_\xi, \eta, \Omega) \).

Here we shall write only some typical results for the average energy, the entropy, and the distribution function: for asymptotic late times we find at high temperatures
\[
\langle \omega \rangle \approx \omega_0 \langle n \rangle \approx \frac{\eta \omega_0^2 T}{2 \lambda m_\xi} e^{-\eta t} \left( \frac{u(t)}{\dot{u}(0)} \right)^2 \left( 1 + \frac{1}{\omega_0^2} \left( \frac{\eta}{2} - \frac{\dot{u}(t)}{u(t)} \right)^2 \right), \quad (226)
\]
\[
S \approx \ln \left( \frac{\eta T}{\lambda m_\xi} \left[ \frac{u(t)}{u(0)} \right]^{e^{-\eta t/2}} \right), \quad (227)
\]
\[
f(\omega) \approx \frac{1}{\sqrt{2\pi \langle \omega \rangle}} \frac{e^{-\omega/(2\langle \omega \rangle)}}{\sqrt{\omega}}. \quad (228)
\]

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Both the number $\langle n \rangle$ and the average energy $\langle \omega \rangle$ increase exponentially with time, the rate $2\lambda m_\xi - \eta$ being somewhat diminished from the value taken without environment. Note also that a rather non-trivial enhancement factor $\propto T$ appears in the prefactor.

Despite of the suppression of the growth rate from $2\lambda m_\xi$ to $2\lambda m_\xi - \eta$, the environment interaction does not erase the parametric effect if

$$2\lambda(\eta) m_\xi > \eta.$$  \hfill (229)

Let us look into the meaning of this condition that guarantees the exponential rate of particle production in thermal bath. Since $\eta$ is the relaxation rate of $\varphi$–field disturbance towards thermalization, the condition simply implies that the thermal relaxation never catches up the parametric amplification for a sufficiently small friction. At the same time the condition implies that there exists a critical strength of the friction $\eta$ above which the parametric effect does not occur. The equation $2\lambda(\eta) m_\xi = \eta$ thus gives the critical strength of the friction constant $\eta_c$. We wish to stress that although the suppression of the growth rate might have been anticipated from a naive consideration \[^{18}\], it was not clear in the past how one can verify the parametric amplification itself, starting from quantum mechanical principles. In our view the significance of the friction $\eta$ is in its role as the relaxation rather than destruction of the coherence. Hence if the relaxation time scale is larger than the amplification time scale, the parametric effect wins over the decoherence due to the environment.

We carried out detailed numerical analysis in \[^{1}\]. It supports analytical estimates. We only quote our main conclusions from that work:

- For the first time it was verified that the parametric amplification can occur in medium, using the influence functional method. The average energy and the total number of population exponentially increases with time, if the growth rate of the parametric amplification in medium is larger than the friction $\eta$, interpreted to be a measure of the relaxation rate towards thermalization of the system variable. The critical friction for the exponential growth has also been given, above which the parametric effect does not occur.

- The resulting energy distribution deviates from the thermal Boltzmann distribution, having a long tail of the high energy component characterized by
an exponentially increasing average energy and an exponentially decreasing \( \delta \) parameter (spectrum shape parameter).

- Late time behavior of the average energy, the entropy, and the energy spectrum is insensitive to the environment temperature if time is displaced in comparison at different temperatures. This is true, even including the \( T = 0 \) vacuum.

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References

[1] M. Hotta, I. Joichi, Sh. Matsumoto, and M. Yoshimura, Quantum System under Periodic Perturbation: Effect of Environment, TU/96/506 and hep-ph/9608374.

[2] I. Joichi, Sh. Matsumoto, and M. Yoshimura, Quantum Dissipation and Decay in Medium, TU/96/510 and hep-th/9609223.

[3] I. Joichi, Sh. Matsumoto, and M. Yoshimura, Quantum Dissipation in Open Harmonic Systems: Operator Solution, TU/96/514 and cond-mat/9612235.

[4] P. Ullersma, Physica 32, 27(1966) and references therein; K. Lindenberg and B.J. West, Phys. Rev. A30, 568(1984); G.W. Ford, J.T. Lewis, and R.F. O’Connell, Phys. Rev. A37, 4419(1988) and references therein.

[5] R.P. Feynman and F.L. Vernon, Ann. Phys. (N.Y.) 24, 118(1963); R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integral, McGraw Hill, New York (1965).
[6] A.O. Caldeira and A.J. Leggett, *Physica* **121A**, 587(1983); *Ann. Phys. (N.Y.)* **149**, 374(1983);
W.G. Unruh and W.H. Zurek, *Phys. Rev. D* **40**, 1071(1989);
B.L. Hu, J.P. Paz, and Y. Zhang, *Phys. Rev. D* **45**, 2843(1992).

[7] For a recent review from the path integral point, see H. Grabert, P. Schramm, and G-L. Ingold, *Phys. Rep.* **168**, 115(1988).

[8] G.W. Ford, J.T. Lewis, and R.F. O’Connell, *Phys. Rev. A* **37**, 4419(1988). This work writes down the quantum Langevin equation corresponding to our model. However, these authors do not discuss solutions of this model.

[9] The contour distortion used here is the one in M.L. Goldberger and K.M. Watson, *Collision Theory*, John Wiley and Sons, New York, (1964), Chap.8, in which detailed account of the decay of prepared states in quantum mechanics is given.

[10] For a review, see A.L. Fetter and J.D. Walecka, *Quantum Theory of Many-Particle System*, McGraw Hill, New York (1971).

[11] The master equation in our model does not belong to the Lindblad’s form, given by G. Lindblad, *Commun.Math.Phys.* **48**, 119(1976). The Lindblad’s form is suggested only as a sufficient condition to guarantee the positivity of the density matrix. As we stated, but not proved here, our bilinear model correctly treated without the local approximation, fulfills the positivity. Apparently the same master equation as Lindblad’s was used by T. Banks, L. Susskind, and M.E. Peskin, *Nucl. Phys. B* **244**, 125(1984), in their discussion of violation of quantum mechanics related to the black hole evaporation.

[12] J.J. Halliwell and T. Yu, *Phys. Rev. D* **53**, 2012(1996).

[13] V. Ambegaokar, Ber. Bunsenges. Phys. Chem. **95**, 400(1991) and references therein.

[14] I. Joichi, Sh. Matsumoto, and M. Yoshimura, in preparation.

[15] H.A. Weldon, *Phys. Rev. D* **28**, 2007(1983).
[16] For a recent review, M. Yoshimura, *Baryogenesis and thermal history after inflation*, hep-th/9605246, and *Journal of the Korean Physical Society* **29**, S236(1996).

[17] For a review of physical aspects, L. Landau and E. Lifschitz, *Mechanics* (Pergamon, Oxford, 1960), p80; for a review of mathematical aspects, E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, (MacGraw-Hill, New York, 1955).

[18] A.D. Dolgov and D.P. Kirilova, *Sov. J. Nucl. Phys.* **51**, 172(1990); J.H. Traschen and R.H. Brandenberger, *Phys. Rev.* **D42**, 2491(1990).

[19] L. Kofman, A. Linde, and A.A. Starobinsky, *Phys. Rev. Lett.* **73**, 3195(1994).

[20] D. Boyanovsky, M. D’Attanasio, H.J. de Vega, and R. Holman, *Phys. Rev.* **D52**, 6809(1996) and references therein.

[21] M. Yoshimura, *Prog. Theor. Phys.* **94**, 873(1995); M. Yoshimura, *Decay Rate of Coherent Field Oscillation*, hep-ph/9603350 and to appear in the Proceedings of the Symposium on *Frontiers in Quantum Field Theory*, (World Scientific, Singapore, 1996).

[22] H. Fujisaki, K. Kumekawa, M. Yamaguchi, and M. Yoshimura, *Phys. Rev.* **D53**, 6805(1996); H. Fujisaki, K. Kumekawa, M. Yamaguchi, and M. Yoshimura, *Phys. Rev.* **D54**, 2494(1996).