Geodesic Deviation in Sáez–Ballester Theory

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Abstract

We study the geodesic deviation (GD) equation in a generalized version of the Sáez–Ballester (SB) theory in arbitrary dimensions. We first establish a general formalism and then restrict to particular cases, where (i) the matter-energy distribution is that of a perfect fluid, and (ii) the spacetime geometry is described by a vanishing Weyl tensor. Furthermore, we consider the spatially flat FLRW universe as the background geometry. Based on this setup, we compute the GD equation as well as the convergence condition associated with fundamental observers and past directed null vector fields. Moreover, we extend that framework and extract the corresponding geodesic deviation in the modified Sáez–Ballester theory (MSBT), where the energy-momentum tensor and potential emerge strictly from the geometry of the extra dimensions. In order to examine our herein GD equations, we consider two novel cosmological models within the SB framework. Moreover, we discuss a few quintessential models and a suitable phantom dark energy scenario within the mentioned SB and MSBT frameworks. Noticing that our herein cosmological models can suitably include the present time of our Universe, we solve the GD equations analytically and/or numerically. By employing the correct energy conditions plus recent observational data, we consistently depict the behavior of the deviation vector $\eta(z)$ and the observer area distance $r_0(z)$ for our models. Concerning the Hubble constant problem, we specifically focus on the observational data reported by the Planck collaboration and the SH0ES collaboration to depict $\eta(z)$ and $r_0(z)$ for our herein phantom model. Subsequently, we contrast our results with those associated with the $\Lambda$CDM model. We argue that the MSBT can be considered as a fitting candidate for a proper description of the late evolution of the universe.

Keywords:
Sáez–Ballester theory, geodesic deviation; Mattig relation, focusing condition, extra dimensions, induced–matter theory, FLRW cosmology, quintessence, phantom dark energy, Hubble tension

1. Introduction

The literature referring to scalar–tensor theories applied to investigate problems in cosmology is vast, e.g., [1, 2, 3] and references therein. Sáez and Ballester may have been inspired by scalar–tensor theories and formulated a theory that is completely different with scalar–tensor theories in its construction and motivation [7]. More concretely, in the SB theory the scalar field

\footnote{1In Refs [4, 5, 6], once the Sáez–Ballester theory has been introduced, it was erroneously included in the class of scalar–tensor theories. It is important to emphasize that such statements has not affected the formulation and consequence of the theory.}
with a rather specific non–canonical kinetic term is added to the Einstein–Hilbert action. There was no scalar potential but a Lagrangian associated with ordinary matter was also considered. With this modification, the SB theory suggested a way to overcome the ‘missing matter problem’ in cosmology [8], which was actually the motivation with which the SB framework was originally proposed [7]. Since then, the SB theory has been appraised within classical cosmology in, e.g., [9, 10, 11, 12, 13, 14, 15, 16], whereas in [17] a quantization with the Wheeler–DeWitt equation was reported.

Notwithstanding the significant references associated with Sáez–Ballester (SB) theory in particular, regarding cosmological applications, it seems, with respect to the scalar–tensor theories, that it has been much less investigated. Although it is worth noting that SB theory is an attractive area of research due to the recent generalization [5].

The above-mentioned reasons have ingrained a robust motivation for our endeavor to explore and extract physical, testable consequences from SB cosmology [4, 5] yet on investigate the geodesic deviation (GD) construction in this paper. It is worthy noting that the GD equation has not been investigated within the SB theory.

The geodesic deviation (GD) equation is a pertinent tool to study properties of curved spacetimes [18, 19, 20, 21]. It has been extensively investigated within different gravitational theories, by means of various exact cosmological solutions (see, e.g., [22, 23, 24, 25, 26] and references therein). Therefore, we establish the GD formalism associated with the SB and MSBT alike [7, 5]. The latter is a new generalized version of the SB theory and is established by a dimensional reduction procedure upon the geometry of extra dimensions. The Lagrangian associated with the matter plus a scalar potential are present, with the number of dimensions assumed to be arbitrary but where, crucially, an effective energy–momentum tensor (EMT) and a potential are dictated from the geometry, instead of being added by ad hoc assumptions. As the particular case of MSBT, by considering a five-dimensional manifold (empty of ordinary matter), this yields an effective framework on a four-dimensional hypersurface, in which the usual right hand side of the field equations is explained solely in terms of the whole geometry. This effective framework is called the space–time matter theory or the induced-matter theory (IMT) [27, 28, 29, 30, 31] (see also [4, 5, 32, 33, 34, 35, 36, 37, 38, 39], as cosmological applications).

In this context, the main objectives of our paper are as follows: (i) To formulate the GD equation in a SB theory (either the original or extended settings) in arbitrary dimensions. (ii) To obtain new exact solutions in the context of a SB theory. This will allow us to apply the GD equation to pertinent cosmological case studies. In particular, we will consider the quintessential and a phantom dark energy scenarios and contrast them with results found for the \( \Lambda \)CDM model. (iii) To demonstrate that, although all the mentioned models yield similar behaviors for selected observables, as far as the MSBT setting is concerned, it still constitutes most satisfactorily a fair and realistic route to describe the evolution of the late current and late universe.

In the next section, after introducing an extended version of the SB theory in arbitrary dimensions, we investigate the corresponding GD equation. We first obtain the GD equation and then formulate it according to: (i) a line-element implying a vanishing Weyl tensor, (ii) a perfect fluid as the matter-energy sector, and (iii) a Friedmann–Lemaître–Robertson–Walker (FLRW) metric as the background geometry. Subsequently, we focus on the GD equation for fundamental observers and the null vector field past directed. In 2.2 by assuming a constant scalar field, we show that the formalism obtained in 2.1.2 reduces to that associated with GR in the presence of the cosmological constant in arbitrary dimensions. In 3 we obtain the energy conditions
(i.e., weak energy condition (WEC), null energy condition (NEC), strong energy condition (SEC) and dominant energy condition (DEC)) within the context of SB cosmology, and then study the GD equation associated with null vector field in the SB framework. In order to apply the formalism obtained in 2, we extract new exact cosmological solutions in the context of the SB theory in the absence of the ordinary matter. We show that these solutions can be applied to describe the accelerating late time epoch. Moreover, we investigate the GD equation associated with a phantom dark energy model and compare the results with the corresponding ones associated with the ΛCDM model. In 4, we review the MSBT framework, and then explore with similar detail as well the GD equation in this context. In 5, we present our conclusions.

2. GD equation in the context of the generalized SB theory in arbitrary dimensions

Let us retrieve the GD equation associated with the generalized SB theory in a D-dimensional spacetime and in the presence of a general scalar potential. The action associated with a D-dimensional SB theory, in analogy with the corresponding four-dimensional case \[ S(D) = \int d^Dx \sqrt{-g} \left[ R(D) - W \phi^n (\nabla_\alpha \phi)(\nabla_\beta \phi) - \frac{1}{2} g_{\alpha\beta} (\nabla_\gamma \phi)(\nabla^\gamma \phi) \right] - \frac{1}{2} g_{\mu\nu} V(\phi), \] (1) where \( g \) and \( R(D) \) stand for the determinant and Ricci scalar associated with the D-dimensional metric \( g_{\alpha\beta} \), respectively. Greek indices run from zero to \( D - 1 \) and \( \nabla \) denotes the covariant derivative on the D-dimensional spacetime. Throughout this work we use units where \( 8\pi G = c = 1 \) (where \( G \) and \( c \) are the Newton gravitational constant and the speed of light, respectively). Moreover, \( \phi \) is a dimensionless scalar field (which is hereafter designated as the SB scalar field), \( W \) and \( n \) are two dimensionless parameters of the model. The Lagrangian associated with the ordinary matter fields is denoted by \( L(D)_{\text{matt}} \), which is independent of the SB scalar field.

The equations of motion obtained from the action (1) are:

\[
G^{\mu\nu}_{(D)} = T^{\mu\nu}_{(D)} + W \phi^n \left[ (\nabla_\mu \phi)(\nabla_\nu \phi) - \frac{1}{2} g_{\mu\nu} (\nabla_\alpha \phi)(\nabla^\alpha \phi) \right] - \frac{1}{2} g_{\mu\nu} V(\phi),
\]

(2)

and

\[
2 \phi^n \nabla^2 \phi + n \phi^{n-1} (\nabla_\alpha \phi)(\nabla^\alpha \phi) - \frac{V_\phi}{W} = 0,
\]

(3)

where \( \nabla^2 \equiv \nabla_\alpha \nabla^\alpha \) and \( V_\phi \equiv \delta V/\delta \phi \). Here \( T^{\mu\nu}_{(D)} \) and \( G^{\mu\nu}_{(D)} \) denote the EMT (associated with the ordinary matter) and the Einstein tensor, respectively. Moreover, one can easily show

\[
\nabla_\mu T^{\mu\nu}_{(D)} = 0.
\]

(4)

Let us first obtain the general expression for the GD equation corresponding to the generalized SB theory (in the presence of the scalar potential) in \( D \) dimensions without choosing a line-element or any constraints on the EMT. Subsequently, we assume that the matter is a perfect fluid, which simplifies our expressions. We will then consider the spatially flat D-dimensional FLRW line-element as the background metric, and investigate the GD equation for different cases associated with the generalized SB framework.
Let $\gamma_1$ and $\gamma_2$ be two neighboring geodesic curves, both parameterized by $\zeta$. Consider $\mathbf{v}$ and $\mathbf{\eta}$ as the tangent vector to the curves and the connecting vector (which connects two points of $\gamma_1$ and $\gamma_2$ with the same value of the parameter $\zeta$), respectively. The GD of the curves is measured by $\mathbf{\eta}$. Assuming $\mathbf{v}$ and $\mathbf{\eta}$ as the coordinate basis vectors of a coordinate system, we have $[\mathbf{\eta}, \mathbf{v}] = 0$. Subsequently, using $\nabla_\mathbf{\eta} \mathbf{v} = 0$ (the curves have been assumed as geodesics) and the antisymmetry property for the Riemann tensor $R$, it is straightforward to show that

$$\nabla_\mathbf{v} \nabla_\mathbf{\eta} + R(\mathbf{\eta}, \mathbf{v}) \mathbf{v} = 0,$$

which is the GD equation. Equivalently, it can be rewritten as

$$\left(\frac{d^2 \eta^\alpha}{d\zeta^2}\right) = -R^\alpha_{\beta\gamma\delta} v^\beta \eta^\gamma v^\delta,$$

which implies that the measurements of the GD can determine completely the Riemann tensor. In our analysis, we assume that the tangent vector field $v^\alpha \equiv dx^\alpha(\zeta) d\zeta$ is normalized as $v^\alpha v^\alpha = \varepsilon$, where $\varepsilon = -1, 0, 1$ correspond to the timelike, null and spacelike geodesics, respectively. Moreover, as mentioned, $\mathbf{\eta}$ commutes with $\mathbf{v}$, i.e., $\eta_\alpha v^\alpha = \text{constant}$. Therefore, without loss of generality, we take $\eta_\alpha v^\alpha = 0$.

In $D$ dimensions (for $D \geq 3$), in the component form, we have:

$$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + \frac{1}{D-2} \left( g_{\alpha\gamma} R_{\beta\delta} - g_{\alpha\delta} R_{\gamma\beta} + g_{\beta\delta} R_{\alpha\gamma} - g_{\beta\gamma} R_{\alpha\delta} \right) + \frac{1}{(D-2)(D-3)} \left( g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma} \right),$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor. In order to obtain a more useful expression associated with the right hand side (r.h.s.) of equation (6), we do the following: (i) We only consider the background metrics whose Weyl tensor vanishes. (ii) We raise the first index of the Riemann tensor and then contract it with $v^\beta \eta^\gamma v^\delta$. (iii) In order to simplify, we compute explicitly the Ricci tensor and the Ricci curvature scalar. More concretely, equation (5) yields

$$R = -\frac{2 T}{(D-2)} + W^\phi(\nabla_\alpha \phi)(\nabla^\alpha \phi) + \left( \frac{D}{D-2} \right) V(\phi).$$

Replacing $R$ from (10) to (2) gives

$$R_{\mu\nu} = T_{\mu\nu} - \left( \frac{T}{D-2} \right) g_{\mu\nu} + W^\phi(\nabla_\mu \phi)(\nabla_\nu \phi) + \left( \frac{1}{D-2} \right) g_{\mu\nu} V(\phi).$$

$^2$From now on, we remove the upper index ($D$) from the quantities.
(iv) Substituting the Ricci tensor and Ricci scalar from relations (10) and (11) into the expression obtained from step (ii), we get

\[
R^l_{\rho\phi\beta} \eta^\rho v^\beta = \frac{1}{(D-2)} \left( \delta^l_\beta T_{\phi\beta} - \delta^l_\phi T_{\beta\phi} + g_{\phi\delta} T^l_\gamma - g_{\beta\gamma} T^l_\phi \right)
\]

\[ - \frac{2}{(D-1)} \left[ T + \frac{W}{2} \phi^\gamma (\nabla_\alpha \phi)(\nabla^\alpha \phi) - \frac{V(\phi)}{2} \right] \left( \delta^l_\gamma g_{\phi\beta} - \delta^l_\beta g_{\gamma\phi} \right) \]

\[ + \frac{W\phi^n}{(D-2)} \left[ \delta^l_\gamma (\nabla_\beta \phi)(\nabla^\beta \phi) - \delta^l_\beta (\nabla_\gamma \phi)(\nabla^\gamma \phi) + g_{\phi\beta}(\nabla_\gamma \phi)(\nabla^\gamma \phi) - g_{\beta\gamma}(\nabla_\phi \phi)(\nabla^\phi \phi) \right] v^\phi \eta^\gamma. \]

(v) We restrict ourselves to the special case where the EMT is taken as perfect fluid:

\[
T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu},
\]

where \( \rho \) and \( p \) denote the energy density and pressure of the fluid. The trace of (13) reads

\[
T = -\rho + (D - 1) p,
\]

where we have used \( u^\mu u_\mu = -1 \). Therefore, substituting \( T_{\mu\nu} \) and \( T \) from relations (13) and (14) into (12) as well as considering \( E = -\nu_\mu u^\mu, \eta_\mu u^\mu = 0 \), we obtain

\[
R^l_{\rho\phi\beta} \eta^\rho v^\beta = \frac{1}{(D-2)} \left( (\rho + p) E^2 + \frac{E}{(D-1)} \left[ 2\rho - W\phi^n (\nabla_\alpha \phi)(\nabla^\alpha \phi) + V(\phi) \right] \right) \eta^l
\]

\[ + \frac{W\phi^n}{(D-2)} \left[ \delta^l_\gamma (\nabla_\beta \phi)(\nabla^\beta \phi) - \delta^l_\beta (\nabla_\gamma \phi)(\nabla^\gamma \phi) + g_{\phi\beta}(\nabla_\gamma \phi)(\nabla^\gamma \phi) - g_{\beta\gamma}(\nabla_\phi \phi)(\nabla^\phi \phi) \right] v^\phi \eta^\gamma,
\]

where we have also used relations (7) and (8).

It is worth mentioning that all the above obtained equations are not only valid for the FLRW metric but also for all metrics whose Weyl tensor vanishes; although, we have restricted ourselves to the perfect fluid assumption. In the next subsection, we focus on a spatially flat FLRW metric as the background in D dimensions.

2.1. GD equation in the SB theory with a FLRW background

In the particular case where the background metric is the spatially flat FLRW, the D-dimensional spacetime is

\[
ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2_{D-2} \right),
\]

where \( k = -1, 0, 1 \), the scale factor is defined by \( a(t) \) and \( d\Omega^2_{D-2} = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \ldots + \sin^2 \theta_{D-3} d\theta_{D-2} \) for \( D \geq 3 \). Due to the spacetime symmetries, the components of the metric as well as the SB scalar field depend only on the cosmic time. Moreover, let us concentrate on the D-dimensional flat spacetime for which \( k = 0 \). Therefore, equation (15) reduces to

\[
R^l_{\rho\phi\beta} \eta^\rho v^\beta = \left( \frac{1}{D-2} \left( \rho + p \right) + \frac{2}{5} \frac{\rho}{(D-1)(D-2)} \right) \eta^l.
\]
In equation (17), we defined the effective energy density and effective pressure as
\[
\rho_{\text{eff}} \equiv \rho + \rho_\phi, \quad p_{\text{eff}} \equiv p + p_\phi,
\]
where
\[
\rho_\phi \equiv \frac{1}{2} \left[ W_{\phi} \dot{\phi}^2 + V(\phi) \right],
\]
(19)
\[
p_\phi \equiv \frac{1}{2} \left[ W_{\phi} \dot{\phi}^2 - V(\phi) \right].
\]
(20)
Indeed, the force term given by equation (17) is the generalized version of the one obtained in [20, 19].
Consequently, by substituting the force term (17) into equation (6), the GD equation associated with the SB framework with the FLRW background in \( D \) dimensions is
\[
d_2 \eta^{\lambda} = - \left\{ E^2 \left( \frac{\rho_{\text{eff}} + p_{\text{eff}}}{D - 2} \right) + 2 \varepsilon \left( \frac{\rho_{\text{eff}}}{(D - 1)(D - 2)} \right) \right\} \eta^{\lambda},
\]
(21)
which is the generalized version of the Pirani equation [19, 20]. Equation (21) implies that the spatial orientation of the connecting vector is not included in the GD equation. However, if we had not restricted ourselves to the isotropic symmetry, then the GD equation would have included not only the magnitude of the connecting vector along the geodesic but also its directional change; see for instance, [42].

2.1.1. GD equation for a fundamental observer

In this case, \( v^{\alpha} \) and the affine parameter \( \zeta \) can be replaced by the \( D \)-velocity of the fluid \( u^{\alpha} \) and the proper time \( t \), respectively. Moreover, letting the vector fields be normalized as \( E = 1 \), and considering temporal geodesics, i.e., \( \varepsilon = -1 \), equation (17) reduces to
\[
R_{\beta\gamma\delta}^{\lambda} u^{\beta} \eta^{\gamma} u^{\delta} = \left( \frac{(D - 3)\rho_{\text{eff}} + (D - 1)p_{\text{eff}}}{(D - 1)(D - 2)} \right) \eta^{\lambda}.
\]
(22)
Assuming the connecting vector to be \( \eta^{\lambda} = \theta e^{\lambda} \) (where the basis \( e^{\lambda} \) is propagated parallel to the \( D \)-velocity), then isotropy implies
\[
\frac{d e^{\lambda}}{dt} = 0,
\]
(23)
which leads to obtain
\[
\frac{d^2 \eta^{\lambda}}{dt^2} = \frac{d^2 \theta}{dt^2} e^{\lambda}.
\]
(24)
Consequently, the GD equation for this case is written as
\[
\frac{d^2 \theta}{dt^2} = - \left( \frac{(D - 3)p_{\alpha} + (D - 1)p_{\alpha}}{(D - 1)(D - 2)} \right) \theta,
\]
(25)
which is the Raychaudhuri equation associated with the SB theory (in \( D \) dimensions and in the presence of a scalar potential) when the universe is described by a spatially flat FLRW metric in
\( D \)-dimensions. (For a recent investigation of Raychaudhuri equation, see [43, 44].) Equation (25) can be applied to both comoving matter as well as non-comoving one, which, for the particular case where \( \phi = \text{constant} \) and \( D = 4 \), it has been investigated in [20]. Moreover, from equation (25), we see that focusing condition for all timelike geodesics is given by

\[
\frac{(D - 3)p_{ea} + (D - 1)p_{at}}{(D - 1)(D - 2)} > 0.
\]

(26)

In this study, let us merely consider the comoving matter where we set \( \vartheta = a(t) \). Therefore, equation (25) reduces to

\[
\ddot{a} = - \left[ \frac{(D - 3)p_{ea} + (D - 1)p_{at}}{(D - 1)(D - 2)} \right].
\]

(27)

We should note that the equation (27) can also be deduced from combining the field equations associated with the spatially flat FLRW metric in the context of the SB framework (including a scalar potential):

\[
\frac{(D - 1)(D - 2)}{2} H^2 = p_{ea},
\]

(28)

\[
(D - 2) \frac{\ddot{a}}{a} + \frac{(D - 2)(D - 3)}{2} H^2 = -p_{ea},
\]

(29)

\[
2\dot{\phi}^2 + 2(D - 1)H\dot{\phi}^2 + n\dot{\phi}^{D - 1} + V_{,\phi} = 0,
\]

(30)

where \( H \equiv \dot{a}/a \) is the Hubble parameter. Moreover, we have

\[
\dot{\rho} + (D - 1)H(\rho + p) = 0,
\]

(31)

\[
\dot{\rho}_\phi + (D - 1)H(\rho_\phi + p_\phi) = 0.
\]

(32)

The consistency of two different procedures for obtaining the Raychaudhuri equation indicates that all equations of herein model are correct.

2.1.2. GD equation for a past directed null vector field

We now extend our calculations for the past directed null vector fields. In this case, \( v^a = k^a \) and \( k_\alpha k^\alpha = 0 \). Therefore, the expressions associated with the force term according to equation (17) reduces to

\[
R_{\beta \gamma \eta \xi}^\lambda \eta^{\gamma} \xi^{\eta} = E^2 \left( \frac{\rho_{ea} + p_{at}}{D - 2} \right) \eta^{\lambda},
\]

(33)

which can be considered as the Ricci focusing in our herein SB framework. Using a parallelly propagated and aligned basis, i.e. admitting \( \frac{\partial k^\alpha}{\partial x^\alpha} = k^\alpha \nabla_a e^a = 0 \), and setting \( \eta^4 = \eta e^4 \), \( e_4 e^4 = 1 \), \( e_4 u^4 = e_4 k^4 = 0 \) \( \text{[20]} \), equation (21) reduces to

\[
\frac{d^2 \eta}{d\zeta^2} = -E^2 \left( \frac{\rho_{ea} + p_{at}}{D - 2} \right) \eta.
\]

(34)
From equation (34), we see that if the condition \( \rho + p + W \phi^a \phi^b > 0 \), is satisfied, all families of past-directed as well as future-directed null geodesics will experience focusing. In the particular case where \( W = 1 \) and \( n = 0 \) (the well-known cosmological model with a single scalar field minimally coupled to gravity), the above mentioned inequality reduces to \( \rho + p + \dot{\phi}^2 > 0 \) which is always satisfied for a special ordinary matter whose energy density and pressure are related as \( \rho + p = 0 \).

For our herein general case, it will be useful to transform equation (34) to the corresponding expression, which is written in terms of the redshift parameter \( z \). In this regard, we write

\[
\frac{d}{d\zeta} = \frac{dz}{d\zeta} \frac{d}{dz},
\]

which yields

\[
\frac{d^2}{d\zeta^2} = \left( \frac{dz}{d\zeta} \right)^{-2} \left[ \frac{d^2}{dz^2} - \left( \frac{dz}{d\zeta} \right)^{-1} \frac{d\zeta}{dz} \frac{d}{dz} \frac{d}{d\zeta} \right].
\]

Concerning the null geodesics, we can write

\[
1 + z = \frac{a_0}{a} = \frac{E}{E_0},
\]

where \( a_0 = 1 \) is the present value of the scale factor (throughout this paper, the index 0 denotes the value of the corresponding quantity at present time \( t_0 \)). Moreover, regarding the past directed case, using \( \frac{dt}{d\zeta} = E = E_0(1 + z) \) (note that for a past directed geodesic, while \( z \) increases, \( \zeta \) decreases) as well as (37), we obtain

\[
\frac{d\zeta}{dz} = \frac{1}{E_0 H(1 + z)^2}.
\]

Then, using

\[
\frac{dH}{dz} = \frac{d\zeta}{dz} \frac{dt}{d\zeta} \frac{dH}{dt} = -\frac{H}{H(1 + z)}
\]

we can show that

\[
\frac{d^2}{d\zeta^2} = \frac{1}{E_0 H^3 (1 + z)^3} \left( \frac{\ddot{a}}{a} - 3H^2 \right).
\]

Consequently, substituting \( d\zeta/dz \) and \( d^2\zeta/dz^2 \) respectively from (38) and (40) into (36) we get

\[
\frac{d^2 \eta}{d\zeta^2} = E_0^2 H^2 (1 + z)^4 \left[ \frac{d^2 \eta}{dz^2} + \left( \frac{3H^2 - \ddot{a}/a}{H^2 (1 + z)} \right) \frac{d\eta}{dz} \right].
\]

Using (27), (34), (37), and (41), we finally obtain the GD equation associated with the null vector fields past directed in terms of \( z \):

\[
\frac{d^2 \eta}{dz^2} + \frac{1}{1 + z} \left[ 3 + \frac{(D - 3)p_{\text{eff}} + (D - 1)p_{\text{eff}}}{(D - 1)(D - 2)H^2} \right] \frac{d\eta}{dz} + \frac{P_{\text{eff}} + P_{\text{eff}}}{(D - 2)(1 + z)^2 H^2} \eta = 0.
\]
Using (28), equation (42) can be written as
\[
\frac{d^2 \eta}{dz^2} + \frac{1}{1 + z} \left[ \frac{D + 3}{2} + \frac{p_{at}}{(D - 2) H^2} \right] \frac{d \eta}{dz} + \frac{1}{(1 + z)^2} \left[ \frac{D - 1}{2} + \frac{p_{at}}{(D - 2) H^2} \right] \eta = 0. \tag{43}
\]

Defining
\[
x \equiv 1 + z, \quad \eta \equiv \frac{\eta}{x}, \quad \ell(x) \equiv \frac{D + 3}{2} + \frac{p_{at}}{(D - 2) H^2},
\]
equation (43) can be written as
\[
\frac{d^2 y}{dx^2} + \left[ \frac{\ell(x) - 2}{x} \right] \frac{dy}{dx} = 0,
\tag{45}
\]
with a general solution
\[
\eta(z) = \frac{\eta_1}{1 + z} + \frac{\eta_2}{1 + z} \times \int dz \left\{ \exp \left[ \int dz' \left( \frac{D - 1}{2} + \frac{p_{at}}{(D - 2) H^2} \right) \right] \right\}, \tag{46}
\]
where \(\eta_1\) and \(\eta_2\) are constants of integration.

In Section 4, we will show that all of our herein calculations remain valid for the MSBT framework, for which the components of the induced EMT as well as the induced scalar potential are directly obtained from the corresponding equations without any ad hoc phenomenological assumptions.

### 2.2. GD equation in GR

Here we obtain the GD equation for null vector fields within a spatially flat FLRW background associated with a GR (in the presence of the cosmological constant, \(\Lambda\)) in arbitrary dimensions.

Let us first obtain the GD equation in the context of the MSBT framework by assuming that the perfect fluid has contributions from both dust and radiation:
\[
\rho = (D - 1) H_0^2 (1 + z)^{D-1} \left[ \Omega_m + \Omega_0 (1 + z) \right], \tag{47}
\]
\[
p = H_0^2 \Omega_0 (1 + z)^D, \tag{48}
\]
where \(\Omega_i \equiv \rho_i / [(D - 1) H^2] \) stands for the dimensionless cosmological density parameters; the indices \(m\) and \(r\) refer to the matter and radiation, respectively. Moreover, in the above equations \(p = p_r = \rho_r / (D - 1)\) for which the conservation law is assumed to be satisfied identically. For this case, equation (28) can be written as
\[
H^2 = \frac{2 H_0^2 (1 + z)^{D-1}}{(D - 2)} \left[ \Omega_m + \Omega_0 (1 + z) \right] + H_0^2 \Omega_{DE}, \tag{49}
\]
where
\[
\Omega_{DE} \equiv \frac{2 \rho_{DE}}{(D - 1)(D - 2) H_0^2}. \tag{50}
\]
Therefore, equation (52) reduces to

$$\frac{d^2 \eta}{dz^2} + \mathcal{P} \frac{d \eta}{dz} + Q \eta = 0,$$  \hspace{1cm} (51)

where \( \mathcal{P} = \mathcal{P}(H, dH/dz, z, D) \) and \( Q = Q(H, dH/dz, z, D) \) are given by

\[
\mathcal{P} \equiv \frac{3}{(1 + z)} + \frac{(D - 1)H_0^2(1 + z)^{D-1} [(D - 3)\Omega_{m0} + (D - 2)\Omega_{\phi0}(1 + z)] + [\Omega_{m0} + (1 + z)\rho_\phi + (D - 1)\rho_\phi]}{2(D - 1)H_0^2(1 + z)^{D-1} [\Omega_{m0} + \Omega_{\phi0}(1 + z)] + 2(1 + z)\rho_\phi},
\]

\[
Q \equiv \frac{(D - 1)}{2(1 + z)^2} \left\{ \frac{H_0^2(1 + z)^{D-1} [(D - 1)\Omega_{m0} + D\Omega_{\phi0}(1 + z)] + \rho_\phi + \rho_\phi}{(D - 1)H_0^2(1 + z)^{D-1} [\Omega_{m0} + \Omega_{\phi0}(1 + z)] + \rho_\phi} \right\}.
\]

It should be emphasized that, up to now, we have not restricted our attention to the GR limit. Moreover, admitting the conditions (53) still correspond to the generalized SB framework.

It is pertinent to note that the GR limit can be retrieved by assuming \( \phi = \) constant and \( V = 2\Lambda \). Concretely, relations (19) and (20) reduce to \( \rho_\phi = -\rho_\phi \equiv \Lambda \), where \( \Lambda \) is a constant. For this particular case, equation (21) then reduces to

$$\frac{D^2 \eta}{Dz^2} = -R_{\mu\nu}^\mu \eta^\nu \phi = -\left\{ E^2 \frac{\rho + \Lambda}{D - 2} + 2E \frac{\rho + \Lambda}{(D - 1)(D - 2)} \right\} \eta^1,$$  \hspace{1cm} (54)

which is a generalization of the Pirani equation (18) (19) (20). Moreover, admitting the conditions of the GR limit, equation (50) yields

$$\Omega_{DE} = \frac{2\Lambda}{(D - 1)(D - 2)H_0^2} \equiv \Omega_\Lambda = \text{constant}.$$  \hspace{1cm} (55)

Finally, from equation (51), we retrieve the GD equation for null vector fields in the context GR+\Lambda in arbitrary dimensions as

$$\frac{d^2 \eta}{dz^2} + \left\{ \frac{(1 + z)^{D-1} [(D + 3)\Omega_{m0} + (D + 4)\Omega_{\phi0}(1 + z)] + 2(D - 2)\Omega_\Lambda}{2(1 + z)^D [\Omega_{m0} + \Omega_{\phi0}(1 + z)] + (D - 2)(1 + z)\Omega_\Lambda} \right\} \frac{d \eta}{dz}
\]

$$+ \left\{ \frac{(D - 1)\Omega_{m0} + D(1 + z)\Omega_{\phi0}}{2(1 + z)^2 [\Omega_{m0} + (1 + z)\Omega_{\phi0}] + (D - 2)(1 + z)^{D-2}\Omega_\Lambda} \right\} \eta = 0,$$  \hspace{1cm} (56)

which is exactly the same equation obtained in (26), as expected.

In this paper, we will study the behavior of \( \eta(z) \) and the observer area distance, \( r_0(z) \), whose definition is:

$$r_0(z) = \sqrt{\left| \frac{dA_0(z)}{d\Omega_s} \right|} = \left| \frac{\eta(\zeta')}{\sqrt{d\Omega(\zeta')/d\zeta} \bigg|_{\zeta=0}} \right|^{10}.$$  \hspace{1cm} (57)
where $A_0$ is the area of the object and $\Omega_s$ stands for the solid angle. Note that to compute $r_0(z)$
we use $d/d\ell = E_0^{-1}(1 + z)^{-1}d/d\zeta = H(1 + z)d/dz$ and assume an initial condition as $\eta(z = 0) = 0$.

Equation (56) has been investigated in [26] for some cases. For later use, let us study another interesting case. Substituting $D = 4$ and $\Omega_{\Lambda_0} = 0$ in equations (47)-(49) and (56), we get

$$\rho(z) = 3H_0^2\Omega_m(1 + z)^3, \quad p = 0,$$

(58)

$$H(z) = H_0\left[\Omega_m(1 + z)^3 + \Omega_\Lambda\right],$$

(59)

$$\frac{d^2\eta}{dz^2} + \frac{3}{2} \left[\frac{7\Omega_m(1 + z)^3 + 4\Omega_\Lambda}{\Omega_m(1 + z)^3 + \Omega_\Lambda(z + 1)}\right] \frac{d\eta}{dz} + \frac{3}{2} \left[\frac{\Omega_m(1 + z)}{\Omega_m(1 + z)^3 + \Omega_\Lambda}\right] \eta = 0,$$

(60)

where $\Omega_\Lambda$ is given by (55). An exact solution for (60) is

$$\eta(z) = N \sum F_1\left(\frac{1}{3}, \frac{1}{2}; \frac{4}{3}; \frac{z}{\Omega_m} \Omega_\Lambda\right) + \frac{M}{\sqrt{\Omega_m}} \sum F_1\left(\frac{1}{3}, \frac{1}{2}; \frac{4}{3}; -\frac{\Omega_m}{\Omega_\Lambda}\right)z^{1/3},$$

(61)

where $N$ and $M$ are the integration constants, which carry the dimension of $\eta$, and $\sum F_1(a, b; c; z)$ is the hypergeometric function. Moreover, using the definition (57), one can show that

$$r_0(z) = \frac{1}{H_0\sqrt{\Omega_\Lambda}} \left[\sum F_1\left(\frac{1}{3}, \frac{1}{2}; \frac{4}{3}; \frac{z}{\Omega_m} \Omega_\Lambda\right) - \frac{2}{3} \sum F_1\left(\frac{1}{3}, \frac{1}{2}; \frac{4}{3}; \frac{\Omega_m}{\Omega_\Lambda}\right)\right].$$

(62)

For this case, using recent observational data, we will plot the behavior $\eta(z)$ and $r_0(z)$ in Sections 3 and 4.

3. GD equation in the SB theory for cosmological models

To apply the GD equation (42) we will first investigate the energy conditions associated with the SB theory and then consider some models based on the cosmological equations obtained in Section 2. Assuming that the SB scalar field dominates the dynamics during accelerating phase, in 3.2, we will obtain new cosmological exact solutions in the absence of ordinary matter. In 3.3, we will investigate the GD equation of a phantom dark energy model in the context of the SB theory, and compare the results with the corresponding ones of the $\Lambda$CDM model.

3.1. Energy conditions in the generalized SB theory

Applying the results found in [44, 46], one can show that the energy conditions in the generalized SB theory in arbitrary dimensions are

$$\text{NEC} : \quad \rho_{\text{tot}} \geq 0,$$

(63)

$$\text{WEC} : \quad \rho_{\text{tot}} \geq 0, \quad \text{and} \quad \rho_{\text{tot}} \geq 0,$$

(64)

$$\text{SEC} : \quad (D - 3)\rho_{\text{tot}} + (D - 1)p_{\text{tot}} \geq 0,$$

(65)

$$\text{DEC} : \quad \rho_{\text{tot}} \geq 0, \quad \text{and} \quad p_{\text{tot}} \geq 0.$$
Substituting $\rho_{\text{eff}}$ and $p_{\text{eff}}$ from (18) into the above conditions, we obtain

$$\text{NEC : } \rho + p + W\dot{\phi}^2 + \dot{\phi} \frac{\partial V}{\partial \phi} \geq 0,$$

(67)

$$\text{WEC : } \rho + \frac{1}{2} \left[ W\dot{\phi}^2 + V(\phi) \right] \geq 0, \quad \text{and} \quad \rho + p + W\dot{\phi}^2 \geq 0,$$

(68)

$$\text{SEC : } (D-3)\rho + (D-1)p - V(\phi) + (D-2)W\dot{\phi}^2 \geq 0, \quad \text{and} \quad \rho + p + W\dot{\phi}^2 \geq 0,$$

(69)

$$\text{DEC : } V(\phi) \geq 0, \quad \rho + \frac{1}{2} \left[ W\dot{\phi}^2 + V(\phi) \right] \geq 0, \quad \text{and} \quad \rho + p + W\dot{\phi}^2 \geq 0,$$

(70)

where we have used (19) and (20).

### 3.2. Cosmological exact solutions in vacuum

We will present two new exact solutions in the absence of ordinary matter in the context of the generalized SB theory. For a single scalar field in the absence of ordinary matter, using equations (28) and (32), we obtain

$$\dot{a} = \sqrt{W\dot{\phi}^2 + V(\phi)} \frac{(D-1)(D-2)}{2},$$

(71)

$$\frac{1}{W\dot{\phi}^2} \frac{d}{dr} \left[ W\dot{\phi}^2 + V(\phi) \right] = -2 \sqrt{\frac{D-1}{D-2}}.$$

(72)

#### 3.2.1. Solution I

Let us assume that the potential energy is a function of $\phi$ and $\dot{\phi}$, as

$$V(\phi) = \left( \frac{2}{\Gamma} - W \right) \phi^n \dot{\phi}^2,$$

(73)

where $\Gamma$ is a constant.

Substituting the potential from (73) into equation (72) leads to

$$\left( \frac{1}{\phi^n \dot{\phi}^2} \right) \frac{d}{dt} \left( \phi^n \dot{\phi}^2 \right) = -\kappa WT \sqrt{\frac{2}{\Gamma} \left( \frac{D-1}{D-2} \right)} \equiv A,$$

(74)

where we used $\sqrt{\phi^n \dot{\phi}^2} = \kappa \phi^2 \dot{\phi}$ with $\kappa = \pm 1$. It is straightforward to show that a solution of equation (74) is

$$\phi^2 \dot{\phi} = -\frac{2}{At},$$

(75)

where we have set the integration constant equal to zero.

---

[3] We will see that such an assumption (see also (95)) yields well-known potentials, which leads to a model that could account for the present epoch. More concretely, the choices (43) and (95) give, respectively, (79) and (101), which are the generalized versions of the exponential, power-law and Mexican-hat potentials.
Equation (75), implies
\[ \phi(t) = \begin{cases} [\phi_i + \left( \frac{2A}{2} \right) \ln \left( \frac{t}{t_i} \right) ]^{\frac{1}{2}}, & \text{for } n \neq -2, \\ \phi_i \left( \frac{t}{t_i} \right)^{-\frac{1}{2}}, & \text{for } n = -2, \end{cases} \]
(76)
where \( \phi_i \) is the value of the SB scalar field at \( t = t_i \).

From equations (71), (73) and (75), we get a power-law relation for the scale factor as
\[ a(t) = a_i \left( \frac{t}{t_i} \right)^\alpha, \]
(77)
where \( a_i \) is the value of the scale factor at \( t = t_i \). Moreover, employing relations (73) and (76), the potential can be obtained in terms of the cosmic time as well as SB scalar field:
\[ V(t) = \left[ \frac{4(2 - \mathcal{W}T)}{2} \right] \frac{1}{t^2}, \quad \forall n, \]
(78)
and
\[ V(\phi) = \begin{cases} V_i \exp \left[ -\left( \frac{2A}{2} \right) \left( \phi^{\frac{1}{2}} - \phi_i^{\frac{1}{2}} \right) \right], & \text{for } n \neq -2, \\ V_i \left( \frac{\phi}{\phi_i} \right)^4, & \text{for } n = -2, \end{cases} \]
(79)
where
\[ V_i = \left[ \frac{4(2 - \mathcal{W}T)}{2} \right] \frac{1}{t_i^2}, \quad \forall n. \]
(80)

Furthermore, for later use, let us also compute the component of the EMT associated with the scalar field. From (19), (20), (73) and (76), we obtain
\[ \rho_{\phi}(t) = \left( \frac{4}{2} \right) \frac{1}{t^2}, \quad \forall n, \]
(81)
and
\[ \rho_{\phi}(t) = \left[ \frac{4(\mathcal{W} - \Gamma)}{2} \right] \frac{1}{t^2}, \quad \forall n, \]
(82)
which satisfy the conservation law (32), as expected. Let us note that our solution associated with \( n \neq -2 \) is a generalized version of the Lucchin-Mataresse power-law solution [47]. More concretely, for the particular case where \( n = 0 \) and \( \mathcal{W} = 1 \), the action (1) reduces to the Einstein-Hilbert action including a single scalar field minimally coupled to gravity, and therefore our herein exact solution yields the D-dimensional Lucchin-Mataresse power-law solution, as expected. As we shown in Fig.1 when the SB scalar field grows the potential increases for \( \kappa = 1 \), while it decreases for \( \kappa = -1 \).
Let us investigate the GD equation for the solution I. Substituting $a(t)$, $\rho_0(t)$ and $p_0(t)$, respectively from relations (77), (81) and (82) into equation (42) and setting $\rho = 0 = p$, we get
\[
\frac{d^2\eta}{dz^2} + \frac{Q + 2}{1 + z} \frac{d\eta}{dz} + \frac{Q}{(1 + z)^2} \eta = 0,
\] (83)
where
\[
Q = \frac{(D - 1)WT}{2}.
\] (84)

It is straightforward to show that equation (83) yields an exact solution
\[
\eta(z) = C_1(z + 1)^{-\frac{1}{2}[|Q+1|+|Q-1|]} + C_2(z + 1)^{-\frac{1}{2}(|Q+1|-|Q-1|)},
\] (85)
where $C_1$ and $C_2$ are the constants of the integration carrying the dimension of $\eta$.

Using (57) and (85) the observer area distance is given by
\[
r_0(z) = \frac{(z + 1)^{-\frac{1}{2}[|Q+1|+|Q-1|]}[(z + 1)^{|Q-1|} - 1]}{H_0 |Q - 1|}.
\] (86)

To obtain the behavior of $\eta$ as a function of redshift parameter, we will apply the initial conditions $\eta(0) = 0$ and $d\eta(z)/dz |_{z=0} = 0.1$, leading to
\[
C_2 = -C_1 = \frac{0.1}{|Q - 1|}.
\] (87)

Demanding $\alpha > 1$, which corresponds to an accelerating scale factor, from equation (77), we obtain
\[
\frac{2}{(D - 1)WT} > 1.
\] (88)

Therefore, using (87), relations (85) can be rewritten as
\[
\eta(z) = \frac{H_0}{10} r_0(z) = \frac{(z + 1)^{-1} - (z + 1)^{-\frac{(D - 1)WT}{2}}}{5 [(D - 1)WT - 2]}.
\] (89)
Figure 2: The behavior of $\eta(z)$ (the left panel) and $r_0(z)$ (the right panel) associated with the null vector fields with four dimensional FLRW background for the $\Lambda$CDM (the black curves) and solution I (the solid and dashed blue curves). We used the units of $H_0^{-1}$, $8\pi G = 1$, and we have assumed $\eta(0) = 0$, $d\eta(z)/dz \bigg|_{z=0} = 0.1$, $H_0 = 67.4\text{km/s/Mpc}$. For plotting the dashed and the solid curves, we have assumed $\kappa W = 0.4$ ($q_0 \approx -0.4$) and $\kappa W = 0.3$ ($q_0 \approx -0.55$), respectively.

Setting $\rho = p = 0$, and substituting $\rho_\phi$ and $p_\phi$ from (81) and (82) into (63)-(66), we obtain

NEC: \[ W \geq 0, \quad (90) \]
WEC: \[ \Gamma \geq 0 \quad \text{and} \quad W \geq 0, \quad (91) \]
SEC: \[ -\frac{2}{\Gamma} + (D - 1)W \geq 0 \quad \text{and} \quad W \geq 0, \quad (92) \]
DEC: \[ \frac{2}{\Gamma} - W \geq 0, \quad \Gamma \geq 0 \quad \text{and} \quad W \geq 0, \quad (93) \]

where we have assumed $\phi'' > 0$.

Respecting the WEC and considering only an accelerating scale factor at present, equations (88) and (91) yield

\[ 0 \leq \Gamma W < \frac{2}{(D - 1)}, \quad (94) \]

Note that for any value of $\kappa W$ that satisfies (94), the NEC and DEC are also satisfied, whilst the SEC is violated.

Concretely, choosing the allowed values for $\kappa W$ enables us to obtain allowed values for $\Gamma$ (such that the inequality (94) is satisfied) in $D$ dimensions. Therefore, we get the corresponding values for $Q$ and can depict the behavior of $\eta(z)$. In figure 2, we show the behavior of $\eta(z)$ and $r_0(z)$ for the allowed values of the parameters associated with solution I and compare them with those of the $\Lambda$CDM model.

3.2.2. Solution II

In this case, we would assume the potential $V(\phi)$ to be

\[ V(\phi) = \frac{2}{\Gamma^2} \phi^2 \phi'^4 - \kappa W \phi'^2, \quad (95) \]

where $\Gamma > 0$ is an arbitrary constant. Substituting $V(\phi)$ from (95) into (72), we obtain

\[ \left( \frac{1}{\phi^2 \phi'^4} \right) \frac{d}{dt} (\phi'^2) = -\kappa W T \sqrt{2 \frac{D - 1}{(D - 2)}} \equiv B, \quad (96) \]
where $|\dot{\phi}^2| = \kappa \phi^2$ with $\kappa = \pm 1$.

The above equation for arbitrary $n$ has an exact solution with a complicated function. For simplicity of applying the GD equation for this case, let us focus on the particular case where $n = 0$ for which we get only one branch with $\kappa = +1$. Therefore, it is easy to show that equation (96) yields

$$\dot{\phi}(t) = \phi_i \exp \left[ \frac{B}{4} (t - t_i) \right],$$  
(97)

$$\phi(t) = \frac{4\phi_i}{B} \left( \exp \left[ \frac{B}{4} (t - t_i) \right] - 1 \right) + \phi_i,$$  
(98)

where $t_i$, $\phi_i$ and $\phi_i$ are integration constants such that $\dot{\phi}(t_i) = \phi_i$ and $\phi(t_i) = \phi_i$. Moreover, substituting $V(\phi)$ from (95) into (72), and then applying (97), we obtain

$$a(t) = a_i \exp \left\{ \frac{2\phi_i^2}{(D - 1)WT^2} \left[ 1 - \exp \left( \frac{B}{2} (t - t_i) \right) \right] \right\},$$  
(99)

where $a_i$ is the value of the scale factor at arbitrary time $t = t_i$.

Furthermore, substituting the scalar field from (98) into (95), we get

$$V(t) = \frac{2\phi_i^4}{T^2} \exp \left[ B(t - t_i) \right] - \phi_i^2 W \exp \left[ \frac{B}{2} (t - t_i) \right].$$  
(100)

Reemploying (98), we can obtain the scalar potential in terms of $\phi$:

$$V(\phi) = \frac{2\phi_i^4}{T^2} \left[ 1 + \frac{B}{4\phi_i} (\phi - \phi_i) \right]^4 - \phi_i^2 W \left[ 1 + \frac{B}{4\phi_i} (\phi - \phi_i) \right]^2.$$  
(101)

Substituting the scalar potential from (95) into relations (19) and (20), we obtain

$$\rho_{\phi}(t) = \frac{\phi_i^4}{T^2} \exp \left[ B(t - t_i) \right],$$  
(102)

$$p_{\phi}(t) = \omega W \phi_i^2 \exp \left[ \frac{B}{2} (t - t_i) \right] - \frac{\phi_i^4}{T^2} \exp \left[ B(t - t_i) \right],$$  
(103)

where we used (97). For this case, using relations (99), (102) and (103), it is straightforward to show that the conservation law (32) is satisfied identically. We should note that our herein formalism is the extended version of the Barrow–Burd–Lancaster–Madsen model, see [47] and references therein, which for the particular case where $W = 1$ and $n = 0$ reduces to their solution.

Let us now focus on the GD equation for the null vector fields associated with this case. For the latter use, let us first obtain an important relation. Assuming $t_i = t_0$, $a_0 = 1$ and using equations (37) and (99), we can easily show that

$$\exp \left[ \frac{B}{2} (t - t_0) \right] = 1 + \left( \frac{(D - 1)WT^2}{2\phi_0^2} \right) \ln(1 + z),$$  
(104)

by which we can express all the quantities in terms of the redshift parameter. Specifically, we
can easily show that the Hubble parameter and the deceleration parameter, 
\[ q \equiv -\ddot{a}/(aH^2), \]
are
\[ H(z) = \kappa W \Gamma \sqrt{\frac{D-1}{2(D-2)}} \left( \frac{2\phi_0^2}{(D-1)^2 W^2} + \ln(z+1) \right), \]  
(105)

\[ q(z) = -1 + \frac{1}{2} \frac{(D-1)W}{2\phi_0^2} + \frac{(D-1)^2 W^2 \Gamma^4 \ln(z+1)}{4\phi_0^4}. \]  
(106)

Assuming \( \rho = p = \rho \), substituting the scale factor from (99) and the components of the EMT (of the SB scalar field) from relations (102) and (103) into equation (42), and then using (104), we obtain
\[
\frac{d^2 \eta}{dz^2} + (\frac{\mathcal{P}}{1 + z}) \frac{d\eta}{dz} + \frac{Q}{(1 + z)^2} \eta = 0, 
\]  
(107)

where
\[
\mathcal{P} = \frac{8\phi_1^4 + 2(D-1)\phi_1^2 W T^2 + (D-1)^2 W^2 T^4 \ln(z+1)}{4\phi_1^4}, 
\]  
(108)
\[
Q = \frac{(D-1)^2 W^2}{2\phi_1^2 + (D-1)^2 W T^2 \ln(z+1)}. 
\]  
(109)

We should note that among the quantities obtained above, only \( q, \mathcal{P} \) and \( Q \) do not depend on \( \kappa \). Nevertheless, equation (105) indicates that the branch \( \kappa = 1 \) is the physical one for solution II.

It is seen that for the solution II, differently of the solution I, \( \mathcal{P} \) and \( Q \) are functions of the redshift parameter \( z \). Therefore, it is realistically impossible to obtain an exact solution for the differential equation (107) in a general case. In this respect, let us analyze this solution by using numerical methods. For this solution, assuming \( \phi > 0 \), we see that the NEC and WEC are satisfied provided that \( W \geq 0 \). Respecting the latter as well as admitting that our herein model might be suitable to describe the accelerating universe at late times, we will depict the behavior of \( \eta(z) \) and \( r_0(z) \) and compare them with those associated with the \( \Lambda \) CDM model, see, for instance, figure 3. It is seen that for small values of \( z \), the curves almost coincide.

Substituting \( V \) from (95) into (69), we find that the SEC is satisfied provided that
\[
\begin{cases} 
(D-1)^2 W T^2 \geq \frac{2\phi_1^2}{1 - \ln(z+1)}, & \text{for } z \leq e - 1, \\
(D-1)^2 W T^2 \leq \frac{2\phi_1^2}{1 - \ln(z+1)}, & \text{for } z > e - 1. 
\end{cases} 
\]  
(110)

On the other hand, demanding \( q < 0 \), from (106), we obtain
\[
\frac{-1 - \sqrt{1 + 4\phi_1^2 \ln(z+1)}}{\ln(z+1)} \leq (D-1)^2 W T^2 \leq \frac{-1 + \sqrt{1 + 4\phi_1^2 \ln(z+1)}}{\ln(z+1)}, \quad \text{for } z > 0. \]  
(111)

It is clear that the above determined regions for the quantity \( (D-1)^2 W T^2 \) given by inequalities (110) and (111) do not overlap for the corresponding values of \( z \). Therefore, by admitting \( q < 0 \), the SEC is violated for the solution II.
Figure 3: The behavior of $\eta(z)$ (the left panel) and $r_0(z)$ (the right panel) associated with the null vector fields with four dimensional FLRW background for the $\Lambda$CDM model (the black curves) and the solution II (the dashed and solid blue curves). We used the units of $H_0^{-1}, 8\pi G = 1$, and we have assumed $\eta(0) = 0, d\eta(z)/dz|_{z=0} = 1, H_0 = 67.4 km/s/Mpc$. For plotting the dashed curves (associated with a single scalar field minimally coupled to gravity, i.e., $W = 1$) and the solid blue curves (associated with a SB model with $W = 0.95$), it has been assumed $|\Gamma| = 0.8$ and $|\dot{\phi}| = 1.1$.

### 3.3. Phantom dark energy model

Among various cosmological models, the simplest dark energy model, i.e., $\Lambda$CDM model (the standard cosmological scenario), definitely can provide predictions, which are exquisitely well in agreement with the corresponding observational data. Notwithstanding, with the enhancement of the number and the accuracy of observations, it has been demonstrated that for some key cosmological parameters estimated by $\Lambda$CDM model, there is still conspicuous tension. Among them, the most obvious issue is estimating $H_0$ [48, 49, 50]. More concretely, applying Planck cosmic microwave background (CMB) and other cosmological observations based on the $\Lambda$CDM model yields $H_0 = 67.364 \pm 0.5 km/s/Mpc$ [51], which is much smaller than that found by local measurements, particularly, with that estimated by SH0ES collaboration by R20 team as $H_0 = 73.2 \pm 1.3 km/s/Mpc$ at %68 confidence level [50] (in tension at 4.2$\sigma$ with the Planck value in a $\Lambda$CDM scenario [48]).

The above mentioned strong discrepancy in estimating the Hubble constant has motivated scientific community to establish new physics beyond the concordance $\Lambda$CDM model to reconcile or alleviate the $H_0$ tension. For instance, one of the most important approaches has been introducing dynamical dark energy parameterizations scenarios for the late times, for a detailed study of the well-known models, see [48]. It has been demonstrated that most of these scenarios may solve the Hubble constant problem at the price of assuming a phantom-like dark energy equation of state [48, 49].

A toy model of a phantom energy component, for which $w_\phi < -1$ (where $w_\phi$ denotes the ratio of the pressure of the dark energy to its density), being compatible with the observational data, has been established by Caldwell [52]. A simple procedure to establish a phantom model is obtained by assuming the energy density and pressure [19] and [20] with negative kinetic term [53]. In this respect, in our herein model, we can also assume $W < 0$. In addition to the phantom dark energy, concerning taking the form of an ordinary matter associated with the present epoch, let us solve the field equations (28)-(32) in a particular case where only the non-relativistic matter fills the universe, i.e., we assume $p = 0$. Therefore, equation (31) yields

$$\rho = \rho_0 \left( \frac{a_0}{a} \right)^{D-1},$$

where the quantities with indices zero refer to their present values throughout.
As one of the main objectives of this paper is investigating the GD equation for specific cosmological models, thus let us abstain from solving the generalized complicated equations of motion associated with the herein phantom model in the context of the extended SB framework. Instead, we confine our attention to the most simplified phantom model established by taking \( W = -1, n = 0 \) and \( D = 4 \), which has been studied by applying different procedures [53] [54].

In [53], by assuming a nearly flat potential that satisfies slow-roll conditions, i.e.,

\[
\left( \frac{V_{,\phi}}{V} \right)^2 \ll 1 \quad \text{and} \quad \frac{V_{,\phi\phi}}{V} \ll 1, \quad (113)
\]

it has been shown that the equation of state associated with the scalar field, \( w_\phi \equiv \frac{p_\phi}{\rho_\phi} \), is obtained slightly less than \(-1\) at present [53]. However, in [54], by taking a reasonable assumption, analytic exact solutions have been obtained. In what follows, we focus on a phantom dark energy model investigated in [54], present more analysis of this model and finally investigate the GD equation for it.

It has been shown that for such a simple model (namely, assuming \( W = -1, n = 0 \) and \( D = 4 \)) the implicit symmetries in the corresponding equations lead us to take an appropriate ansatz as [54]

\[
\dot{\phi} = -\sigma H, \quad (114)
\]

where \( \sigma > 0 \) is a constant. From equation (114), we obtain the evolution of the scale factor as

\[
a = a_0 \exp \left( \frac{\phi - \phi_0}{\sigma} \right), \quad (115)
\]

or equivalently, we get

\[
\phi = \phi_0 + \sigma \ln(1 + z), \quad (116)
\]

where we set \( a = 1 \) at the present. Moreover, it is straightforward to show that [54]

\[
H^2(\phi) = K_1 \exp \left( \frac{3\phi}{\sigma} \right) + K_2 \exp(-\sigma\phi), \quad (117)
\]

\[
V(\phi) = -\sigma^2 K_1 \exp \left( \frac{3\phi}{\sigma} \right) + (6 + \sigma^2) K_2 \exp(-\sigma\phi), \quad (118)
\]

\[
\rho_\phi = -K_1 \sigma^2 \exp \left( \frac{3\phi}{\sigma} \right) + 3K_2 \exp(-\sigma\phi), \quad (119)
\]

\[
p_\phi = -(3 + \sigma^2) K_2 \exp(-\sigma\phi), \quad (120)
\]

where \( K_1 \) and \( K_2 \) are integration constants. Furthermore, from using equations (28), (119) and (120), we obtain the cosmological density parameter associated with the ordinary matter as

\[
\Omega_m = \frac{\rho}{3H^2} = \frac{K_1(3 + \sigma^2) \exp \left( \frac{3\phi}{\sigma} \right)}{3K_1 \exp \left( \frac{3\phi}{\sigma} \right) + 3K_2 \exp(-\sigma\phi)}, \quad (121)
\]

which, according to equation (28), is related to the density parameter associated with the phantom, \( \Omega_\phi \equiv \frac{\rho_\phi}{3H^2} \), as

\[
\Omega_m + \Omega_\phi = 1. \quad (122)
\]
It has been shown that the integration constants $K_1$ and $K_2$ are given by [54]

$$K_1 = \frac{3H_0^2 (1 - \Omega_{m0}) \text{Exp}(\frac{-3\sigma_0}{\sigma})}{3 + \sigma^2}, \quad (123)$$

$$K_2 = \frac{H_0^2 (3\Omega_{m0} + \sigma^2) \text{Exp}(\sigma\phi_0)}{3 + \sigma^2}. \quad (124)$$

For the later use for studying the GD equation [134], $H$ and $p_\phi$ should be expressed in terms of the redshift parameter $z$. Substituting $\phi$, $K_1$ and $K_2$, respectively, from [116], [123] and [124] into [117] and [120], it is easy to show that

$$H(z) = H_0 \left[ \frac{3\Omega_{m0}}{3 + \sigma^2} (1 + z)^3 + \left( \frac{3\Omega_{m0} + \sigma^2}{3 + \sigma^2} \right) (1 + z)^{-\sigma^2} \right]^{\frac{1}{2}}, \quad (125)$$

$$p_\phi(z) = -H_0^2 (3\Omega_{m0} + \sigma^2)(1 + z)^{-\sigma^2}. \quad (126)$$

Moreover, using equation (39), the deceleration parameter can be written as

$$q = -1 + (1 + z) \left( \frac{d \ln[H(z)]}{dz} \right), \quad (127)$$

Then, substituting $H(z)$ from [125] into [127], we obtain

$$q(z) = -3 + \sigma^2 \left( \frac{3\Omega_{m0}(1 + z)^{3 + \sigma^2}}{3\Omega_{m0} + \sigma^2} + 1 \right)^{-1} + \frac{1}{2}. \quad (128)$$

It is seen that the amount of the deceleration parameter for a specific $z$ depends on the values taken by $\Omega_{m0}$ and the free parameter $\sigma$. Concretely, from (128), we see that the accelerating phase began only recently after a transition obtained from equation $q(z) = 0$, which yields $z = z_\tau = f(\Omega_{m0}, \sigma)$.

Furthermore, using equations [115], [116] and [119]–[124], it is easy to show that $w_\phi$ can be written as

$$w_\phi = -\left( 1 + \frac{\sigma^2}{3\Omega_{m0}} \right) = \frac{(\sigma^2 + 3)(\sigma^2 + 3\Omega_{m0})}{3(\sigma^2 + 3\Omega_{m0}) + 3\sigma^2(\Omega_{m0} - 1)(z + 1)^{-\sigma^2}}. \quad (129)$$

In [54], some important features of the herein phantom dark energy model have been mentioned. Nevertheless, in what follows, in addition to the GD equation, let us further obtain a few novel interesting results. We should note that, instead of the observational data used in [54], let us focus on the considerations of [51], where we see ($H_0 = 67.364 \pm 0.5km/s/Mpc$, $\Omega_{m0} = 0.315 \pm 0.007, w_{\phi0} = -1.03 \pm 0.03$, $H_0 = 73.5.364 \pm 2.5km/s/Mpc, w_{\phi0} = -1.29^{+0.15}_{-0.12}$) and ($H_0 = 75.35 \pm 1.68km/s/Mpc, q_{0} = -1.08 \pm 0.29$), respectively. For instance, let us consider two examples: Assuming $\sigma = 0.22$ (1.06) and $\Omega_{m0} = 0.685$, the universe began acceleration very recently at redshifts about $z_\tau = 0.65$ (0.76). Moreover, we obtain $q_0 = -0.55 \pm (1.08)$ and $w_{\phi0} = -1.03 \pm (1.55)$. It is seen that the results of the first example, disregarding the value of $w_{\phi0}$, is in agreement with the $\Lambda$CDM model [51], while the second one (see the values in the parenthesis) is in agreement with the corresponding ones reported in [56].
Figure 4: The plot of $w_\phi(z)$ for small values of the redshift parameter for our herein phantom model. We considered a particular case of the SB theory with $W = -1$ and $n = 0$. We have assumed $8\pi G = 1 = c$, $\Omega_{\phi} = 0.685$, $\sigma = 0.23$ (the left panel) and $\sigma = 1.06$ (the right panel).

First, let us plot the evolution of $w_\phi$ as function of the redshift parameter, see figure 4. We see that by choosing various values for the free parameter $\sigma$, the model yields $w_{\phi0}$ such that it is in the range reported by the recent observational data.

Secondly, before investigating the GD equation, we would study the late time asymptotic behavior of some quantities as follows.

Substituting the integration constants from (123) and (124) into (118), and then using (116), we obtain

$$V = \frac{H_0^2(6 + \sigma^2)(3\Omega_{\phi0} + \sigma^2)(1 + z)^{-\sigma^2} + 3\sigma^2(\Omega_{\phi0} - 1)(1 + z)^3}{2(\sigma^2 + 3)}.$$ (130)

for which we get

$$\lim_{z \to 0} V = \frac{1}{2} H_0^2(\sigma^2 + 6\Omega_{\phi0}) = \text{constant}. \quad \text{(131)}$$

At late times, $V_\phi/V$ also asymptotically approaches to a constant:

$$\lim_{z \to 0} \frac{V_\phi}{V} = \sigma^2 \left[9(\Omega_{\phi0} - 1) - (6 + \sigma^2)(3 + \sigma^2\Omega_{\phi0})\right] / (3 + \sigma^2)^2(6\Omega_{\phi0} + \sigma^2). \quad \text{(132)}$$

Furthermore, using relations (43), we obtain

$$\lim_{z \to 0} w_\phi = \frac{\sigma^2}{3(\Omega_{\phi0} - 1)} - 1 \equiv w_{\phi0} = \text{constant}, \quad \text{(133)}$$

which, as $\Omega_{\phi0} < 1$, hence $w_{\phi0}$ will always be less than $-1$.

Let us now investigate the GD equation associated with this model. Concerning our herein model, setting $p = 0$ and $D = 4$, equation (43) reduces to

$$\frac{d^2\eta}{dz^2} + \frac{1}{2(1 + z)} \left(7 + \frac{p}{H^2}\right) \frac{d\eta}{dz} + \frac{1}{2(1 + z)^2} \left(3 + \frac{p}{H^2}\right) \eta = 0. \quad \text{(134)}$$
Using equations (125)-(128), one can show that

\[
\left( \frac{H^2}{p^0} \right)^{-1} = 2q(z) - 1.
\] 

Therefore, the GD equation (134) can be written as

\[
\frac{d^2 \eta}{dz^2} + \left[ 3 + q(z) \right] \frac{d \eta}{dz} + \left[ \frac{1 + q(z)}{(1+z)^2} \right] \eta = 0,
\]

where \( q(z) \) is given by (128). It seems that it is not possible to obtain analytic exact solutions for (136). In this regard, we will use a numerical approach to analyze it. Using recent observational data [51], in figure 5, we plot \( \eta \) and \( r_0 \) against the redshift parameter (see the blue solid curves). In addition to the observational data reported in [51], the blue dashed curves show the behavior of \( \eta(z) \) and \( r_0(z) \) by considering the value of the \( H_0 \) estimated by SH0ES collaboration, see for instance, [50]. In these figures, we have also compared the behavior of \( \eta \) and \( r_0 \) associated with herein phantom dark energy model with the corresponding case (i.e., assuming \( p = 0 \)) in the \( \Lambda \)CDM model presented in subsection 2.2. It is seen that the general behavior of \( \eta \) and \( r_0 \) are similar for all, as expected.

4. GD equation in the context of the MSBT

In this section, let us first review the MSBT in arbitrary dimensions [39, 5], and then investigate the GD equation in this framework.

In analogy to (2) and (3), their \((D+1)\)-dimensional counterpart field equations, in the absence of the scalar potential, are given by

\[
G^{(D+1)}_{ab} = \mathcal{W} \phi^{p} \left[ \nabla_a \phi (\nabla_b \phi) - \frac{1}{2} G_{ab} (\nabla_c \phi)(\nabla_c \phi) \right] + T_{ab}^{(D+1)}.
\]
where \( \nabla \) is the covariant derivative associated with \((D + 1)\)-dimensional spacetime (bulk) and \( \nabla^\prime \equiv \nabla_{\mu}^\prime \nabla_{\mu}^\prime \). We should note that the Lagrangian associated with the ordinary matter fields has also been taken nonzero in the bulk, i.e., \( L_m^{(D+1)} \neq 0 \). This choice was made in [5] with the purpose to establish a more generalized setting. Moreover, the tensors and quantities with index \((D + 1)\) and/or Latin indices (the Latin indices run from zero to \( D \)) are also associated with the \((D + 1)\)-dimensional spacetime (bulk).

Applying a specific reduction procedure, and considering [27, 30],

\[
dS^2 = g_{\mu\nu}(x^\mu)dx^\mu dx^\nu = g_{\mu\nu}(x^\mu, l)dx^\mu dx^\nu + \epsilon \psi^2 (x^\mu, l) dl^2,
\]

it has then been shown that the effective EMT as well as an induced scalar potential emerge intrinsically from the geometry of the extra dimension (for more detail, see [39, 5]). In (139), \( l \) denotes a non-compact coordinate along the extra dimension; the scalar field \( \psi \) depends on all coordinates and \( \epsilon = \pm 1 \). The hypersurface \( \Sigma_0 \) corresponding to \( l = l_0 = \text{constant is orthogonal to the} \ (D + 1)\)-dimensional unit vector

\[
n^a = \frac{\delta^a}{\psi}, \quad \text{where} \quad n_m n^m = \epsilon,
\]

along the extra dimension. Therefore, the induced metric \( g_{\mu\nu} \) on the hypersurface \( \Sigma_0 \) is given by

\[
dx^2 = g_{\mu\nu}(x^\mu, l_0)dx^\mu dx^\nu \equiv g_{\mu\nu}dx^\mu dx^\nu.
\]

Consequently, four sets of equations are retrieved (see [5] where more details can be found):

1. An equation for the scalar field \( \psi \):

\[
\frac{\nabla^2 \psi}{\psi} = -\frac{\epsilon}{2\dot{\psi}^2} \left[ g^{\mu\nu} \dot{g}_{\mu\nu} + \frac{1}{2} g^{\mu\nu} \dot{g}_{\mu\nu} \cdot \frac{g^{\mu\nu} \dot{g}_{\mu\nu}}{\psi} \right] - \frac{eW\dot{\phi}^2(\phi)}{\psi^2} + \left[ \frac{T^{(D+1)}_{\mu\nu}}{D - 1} - \frac{\epsilon T_{\mu\nu}^{(D+1)}}{\psi^2} \right],
\]

where \( \dot{A} \equiv \frac{\partial A}{\partial \mu} \).

2. The counterpart of the conservation law presented in IMT is given by

\[
G_{\alpha D}^{(D+1)} = \dot{R}_{\alpha D}^{(D+1)} = \psi P_{\alpha\beta} - T_{\alpha\beta}^{(D+1)} + W\dot{\phi}^2(\nabla_\alpha \phi),
\]

where

\[
P_{\alpha\beta} \equiv \frac{1}{2\dot{\psi}} \left( \dot{g}_{\alpha\beta} - g_{\alpha\beta} \dot{g}^{\mu\nu} \dot{g}_{\mu\nu} \right),
\]

3. The second pair of the field equations associated with the MSBT are also given by (2) and (3). However, in contrary to the conventional SB theory presented in section 2 in the MSBT framework the EMT as well as scalar potential are not added by phenomenological assumptions, but instead they are fully emerge from the geometry. Concretely, (i) the induced scalar potential \( V(\phi) \) is obtained from

\[
V_\phi \equiv \frac{-2W\dot{\phi}^2}{\dot{\psi}^2} \left( \psi (\nabla_\alpha \phi)(\nabla^\alpha \phi) + \frac{ne}{2} \left( \frac{\dot{\psi}^2}{\dot{\phi}^2} \right) \right) - \frac{\epsilon}{2} \left( \dot{\phi} + \dot{\phi} \left( \frac{g^{\mu\nu} \dot{g}_{\mu\nu}}{2\dot{\psi}} - \frac{\dot{\phi}}{\dot{\psi}} \right) \right).
\]
(ii) The induced EMT, $T^{(D+1)}_{\mu \nu}$, in (2) has four terms:

$$T^{(D+1)}_{\mu \nu} = E_{\mu \nu} + T^{[\text{IMT}]}_{\mu \nu} + T^{[\phi]}_{\mu \nu} + \frac{1}{2} g_{\mu \nu} V(\phi)$$  \hspace{1cm} (146)$$

where

- $E_{\mu \nu}$ represents the effective EMT induced from the $(D + 1)$-dimensional ordinary matter fields assumed in the bulk:

$$E_{\mu \nu} \equiv T^{(D+1)}_{\mu \nu} - g_{\mu \nu} \left[ \frac{T^{(D+1)}}{D - 1} - \frac{\epsilon T^{(D+1)}}{\psi^2} \right].$$ \hspace{1cm} (147)$$

- $T^{[\text{IMT}]}_{\mu \nu}$ is the same induced matter presented in the IMT [27]:

$$T^{[\text{IMT}]}_{\mu \nu} \equiv \frac{\nabla_\mu \nabla_\nu \psi}{\psi} - \frac{\epsilon}{2 \psi^2} \left\{ \psi g_{\mu \nu} - \frac{\epsilon}{2} \left[ g_{\mu \alpha} g^{\alpha \beta} s_{\beta \nu} + \left( g^{\alpha \beta} s_{\alpha \nu} \right)^2 \right] \right\}.$$ \hspace{1cm} (148)$$

- Another term of the induced EMT is:

$$T^{[\phi]}_{\mu \nu} \equiv \left[ \frac{\epsilon W}{2} \frac{\psi}{\phi} \right] g_{\mu \nu}.$$ \hspace{1cm} (149)$$

In summary, by considering the metric (139) and selecting a dimensional reduction procedure, the equations (137) and (138) associated with the $(D + 1)$-dimensional SB theory (in the absence of any potential and cosmological constant), are then reduced to the effective field equations (2), (3), (142) and (143) on the hypersurface. From the viewpoint of an observer on the hypersurface (who has no information concerning the reduction procedure as well as the existence of the extra dimension), (2) and (3) would be considered as the field equations for the SB theory (with a scalar potential) in $D$ dimensions, which can also be derived from the action (1) admitting

$$\sqrt{-g} \left( E_{ab} + T^{\text{SB}}_{ab} \right) \equiv 2 \delta \left( \sqrt{-g} L_\text{matter}^{(D)} \right)/\delta g^{ab}.$$ \hspace{1cm} (150)$$

In order to proceed our considerations in the context of the MSBT, let us mention an important remark: Equations (2) and (3) are the field equations that are valid not only for the conventional SB theory but also for the MSBT. However, concerning the former, both the EMT and the scalar potential should be chosen from phenomenological assumptions. Whilst for the latter, not only the EMT but also the scalar potential are thus extracted from the corresponding equations, namely, equations (146) and (145). More concretely, for the MSBT, we will employ the EMT as well as scalar potential directly dictated from the geometry.

4.1. GD equation for null vector field for cosmological models in the MSBT theory

In what follows, let us focus our attention on the GD equation in context of the MSBT. We should note that in order to obtain equation (12), we have merely used equations (2) and
without imposing any constraint. Therefore, it is also valid when we take MSBT as the underlying theory.

Nevertheless, it is worthy to stress that equation (15) has been deduced for the special case where we restricted ourselves to a perfect fluid. Therefore, this equation will be valid within the MSBT provided that the geometrically induced matter (on a $D$-dimensional hypersurface) to be also a perfect fluid. In this respect, let us choose the same assumptions used to derive the GD equation. More concretely, we consider a particular case of the metric (139):

$$dS^2 = ds^2 + \epsilon \psi^2(t) dl^2,$$

where the line-element associated with the hypersurface is given by (16). Moreover, we assume that there is no ordinary matter fields in the bulk. Therefore, from equations (147), we get $E_{\mu\nu} = 0$. By imposing the cylinder condition [30] (by which we must set the derivatives with respect to $l$ equal to zero), from equation (149), we obtain $T_{\mu\nu}^{(i)} = 0$. Regarding the assumptions mentioned above, equations (146), (142), (145) and (148) reduce to

$$T_{\mu\nu} = \frac{\nabla_{\mu} \nabla_{\nu} \psi}{\psi} + \frac{V(\phi)}{2} g_{\mu\nu}, \quad T = \frac{D V(\phi)}{2},$$

where

$$V_{,\phi} \equiv -2W^{\rho} \gamma_{,\phi} (\nabla_{\rho} \psi) (\nabla^{\rho} \phi), \quad \nabla^2 \psi = 0.$$  

(153)

Assuming $\phi = \phi(t)$ and substituting the components of the metric (16) into (146), the energy density $\rho$ and pressure $p$ of the induced matter is given by

$$\rho \equiv -T_{00} = \frac{\dot{\psi}}{2} - \frac{V(\phi)}{2},$$

$$p \equiv T_{ii} = -\frac{\dot{\psi}}{2} + \frac{V(\phi)}{2},$$

(154)

(155)

where $i = 1, 2, ..., (D - 1)$ (with no sum on $i$). Moreover, $V(\phi)$ in relations (154) and (155) should be obtained from solving the differential equation (153):

$$V_{,\phi} \mid_{\Sigma} = 2W \phi^2 \frac{\psi}{\dot{\psi}}.$$  

(156)

Hence, (155) implies that the pressure in all directions are equal (i.e., $p_1 = p_2 = \ldots = p_i = \dot{p}$), and consequently the induced matter on the $D$-dimensional hypersurface is a perfect fluid. The induced matter also obeys (13). So, we conclude that (15), (17) and (21) (which have been deduced in the SB framework) can also be applicable within the MSBT. However, let us emphasize once again that, contrary to the standard SB theory, herein $\rho$, $p$ and $V(\phi)$ have not put by hand, but instead they emerge from the geometry of the higher dimensions.

To study the GD equation in the context of the MSBT, let us first obtain exact solutions of our herein cosmological model (for more detail, see [5]). Equations (138) and (153), respectively, lead us to the following constants of motion

$$d^{D-1} \phi^2 \phi \dot{\phi} = c_1,$$

$$d^{D-1} \dot{\phi} = c_2.$$  

(157)

(158)
where \( c_1 \neq 0 \) and \( c_2 \neq 0 \) are constants of integration. Equations (157) and (158) imply

\[
\psi = \begin{cases} \psi_i \exp \left( \frac{2\beta}{n+2} \psi^{2+2} \right) & \text{for } n \neq -2, \\ \psi_i \phi^2 & \text{for } n = -2, \end{cases}
\]

and

\[
a = \begin{cases} a_i \exp \left( \frac{2\gamma(D)}{n+2} \phi^{2+2} \right) & \text{for } n \neq -2, \\ a_i \phi^2 & \text{for } n = -2, \end{cases}
\]

where to obtain (160), we have also used the Friedmann equation associated with the bulk in the absence of the ordinary matter [5]. Moreover, \( \psi_i \) and \( a_i \) are constants of integration, \( \beta \equiv \frac{c_2}{c_1} \) and \( \gamma(D) \) was defined as

\[
\gamma(D) \equiv \frac{1}{D-2} \left[ -\beta \pm \sqrt{\beta^2 + \left( \frac{D-2}{D-1} \right) W} \right].
\]

Replacing \( \psi \) and \( a \) from relations (159) and (160) into equation (157), we get

\[
\dot{\phi} \phi \psi^{2+2} = c_1 a_1 - D_i \psi_i \phi^2 \]

for \( n \neq -2 \) and

\[
\dot{\phi} \phi^{\gamma(D)} = \frac{c_1 a_1 - D_i \psi_i}{\psi_i} \phi^2
\]

for \( n = -2 \), where

\[
f(D) \equiv (D - 1) \gamma(D) + \beta.
\]

In order to obtain the unknowns of the model in terms of the cosmic time, we should first obtain \( \phi(t) \) by solving the above differential equations. However, whether or not \( f(D) \) is chosen to vanish, we obtain two classes of exact solutions.

4.1.1. GD equation for Exponential-law solution

In the particular case where \( f(D) = 0 \), the exact solutions corresponding to the equations (162) are given by [5]

\[
\phi(t) = \begin{cases} \left[ \frac{(n+2)(1-D)h(D)(t-t_i)}{2\beta} \right]^{\frac{1}{n+2}} & \text{for } n \neq -2, \\ \exp \left[ \frac{(1-D)h(D)(t-t_i)}{\beta} \right] & \text{for } n = -2, \end{cases}
\]

where \( t_i \) is an integration constant and

\[
h(D) \equiv \frac{c_1 \beta a_1^{1-D}}{(1-D)\psi_i}.
\]

From [5]:

\[
a(t) = a_i \exp[h(D)(t-t_i)], \quad \forall n,
\]

\[
\psi(t) = \psi_i \exp[(1-D)h(D)(t-t_i)], \quad \forall n.
\]
The induced potential on the hypersurface is:

\[ V(\phi) = \begin{cases} \frac{2V_0}{n+2} \phi^{\frac{n+2}{2}} & \text{for } n \neq -2, \\ V_0 \ln \left( \frac{\phi}{\phi_i} \right) & \text{for } n = -2, \end{cases} \]  

(168)

where \( \phi_i \) is an integration constant and

\[ V_0 \equiv 2\beta D(1-D)\tilde{h}^2(D). \]  

(169)

However, it has been shown that the following relations are independent of \( n \):

\[ \rho = (1-D)^2 \tilde{h}^2(D) [-Dh(D)(t - t_i) + 1], \]  

(170)

\[ p = (1-D) \tilde{h}^2(D) \times [D(1-D)h(D)(t - t_i) - 1], \]  

(171)

\[ \rho_\phi = \frac{D(1-D)\tilde{h}^2(D)}{2} \times [1 + 2 (1-D)h(D)(t - t_i)], \]  

(172)

\[ p_\phi = \frac{D(1-D)\tilde{h}^2(D)}{2} \times [1 - 2 (1-D)h(D)(t - t_i)]. \]  

(173)

Hence substituting the components of the induced matter and the matter associated with the SB scalar field into equation (42), the GD equation of the null vector fields past directed, in the context of the MSBT for \( f(D) = 0 \), is given by

\[ \frac{d^2\eta}{dz^2} + \frac{2}{1 + z} \frac{d\eta}{dz} = 0, \quad \forall n. \]  

(174)

Equation (174) yields an exact solution as

\[ \eta(z) = \frac{-C_1}{1 + z} + C_2, \quad \forall n, \]  

(175)

where \( C_1 \) and \( C_2 \) are integration constants, which have the same units of \( \eta \). Therefore, the observer area distance \( r_0(z) \) associated with this case is given by

\[ r_0(z) = \frac{z}{H_0(1 + z)}, \quad \forall n. \]  

(176)

4.1.2. GD equation for Power-law solution

For the case where \( f(D) \neq 0 \), the scale factor \( a(t) \) is given by a power-law form in terms of the cosmic time. Concretely, from equations (162), the SB scalar field is obtained

\[ \phi(t) = \begin{cases} \frac{n+2}{17f(D)} \ln \left[ \tilde{h}(D)(t - t_i) \right]^{\frac{1}{n+2}} & \text{for } n \neq -2, \\ \left[ \tilde{h}(D)(t - t_i) \right]^{\frac{1}{n-2}} & \text{for } n = -2, \end{cases} \]  

(177)

where

\[ \tilde{h}(D) \equiv \frac{c_1 f(D)}{a^{D-1} \psi_i} \]  

(178)
The scale factor $a$ and the scalar field $\psi$ are obtained in terms of the cosmic time $t$:

$$a(t) = a_i \left[ \hat{h}(D) (t-t_i) \right]^r, \quad \forall n,$$  \hspace{1cm} (179)

$$\psi(t) = \psi_i \left[ \hat{h}(D) (t-t_i) \right]^m, \quad \forall n.$$  \hspace{1cm} (180)

In equations (179) and (180), $r$ and $m$ were defined as

$$r \equiv \frac{\gamma}{f(D)}, \quad m \equiv \frac{\beta}{f(D)}, \quad \text{where} \quad m + (D-1)r = 1.$$  \hspace{1cm} (181)

The induced potential is given by \cite{5}

$$V(\phi) = \begin{cases} \frac{\tilde{V}_0}{\sqrt{D(D^2-1)}} \exp \left[ \frac{-2i(D)}{n+2} \phi^{n+2} \right] & \text{for} \quad n \neq -2, \\ -\frac{\tilde{V}_0}{\sqrt{D(D^2-1)}} \phi^{-2(D)} & \text{for} \quad n = -2, \end{cases}$$  \hspace{1cm} (182)

where $\tilde{V}_0$ is related to the other parameters of the model as

$$\tilde{V}_0 \equiv 2c^2 3^{\beta-2} W_0^{1-D} \eta^2.$$  \hspace{1cm} (183)

One can show that

$$\rho = -\frac{D(D-1)mr^2}{2(t-t_i)^2}, \quad p = -\frac{Dmr(1+m)}{2(t-t_i)^2},$$  \hspace{1cm} (184)

$$\rho_\phi = \frac{[(D-1)r]^2 [2m + (D-2)r]}{2(t-t_i)^2},$$  \hspace{1cm} (185)

$$p_\phi = \frac{[(D-1)r][2m + (D-1)r][2m + (D-2)r]}{2(t-t_i)^2},$$  \hspace{1cm} (186)

which are valid for all values of $n$.

Substituting $\rho$, $p$, $\rho_\phi$ and $p_\phi$ from relations (184)-(186) into \cite{12}, the GD equation for this case will be exactly the differential equation \cite{65} with an exact solution \cite{65}. Using \cite{67} for \cite{65}, one can show that the observer area distance is given by \cite{66}. However, for this case, it is important to note that $Q$ is given by

$$Q = Q(\beta, W, D) \equiv \frac{(D-1)}{W} \left[ W + \beta^2 \pm \beta \sqrt{\beta^2 + \frac{(D-2)}{(D-1)} W} \right] = \frac{1-m}{(D-1)}.$$  \hspace{1cm} (187)

It is worth to depict the behavior of the deviation vector as well as the observer area distance. For this aim, let us employ the following procedure.

In \cite{5}, it has been shown that for specific allowed ranges of the independent parameters of the model, i.e., either $\beta > 0, 2(D-1)\beta y < W < (D-1)\beta y < 0$ or $\beta < 0, 2(D-1)\beta y < W < (D-1)\beta y < 0$, it is feasible to obtain an accelerating scale factor which could be applicable for the present universe.

Let us express $Q$ in terms of the deceleration parameter (which reads for our herein power-law solution as $q = 1/(r-1)$): using equations (181) and (55), we obtain $Q = q + 1$. Moreover, we would use the same initial conditions used before, i.e., $\eta(0) = 0$ and $d\eta(z)/dz \big|_{z=0} = 0.1$, which leads to (67). We restrict our attention to the four-dimensional case for which we can use the recent observational data reported in \cite{51}. In figure 6, we plot the behavior of $\eta(z)$ and $r_0(z)$ for this case and compare them with those of the $\Lambda$CDM model.
5. Discussion and Conclusions

In this paper, we computed and investigated the general form for the GD equation in the framework of (i) an extended version of the conventional SB theory \[\text{(i.e., not only we have considered a general scalar potential, but also assumed an arbitrary number of spatial dimensions)}\] and (ii) the MSBT theory \[\text{[5]}.\] We have employed the ordinary matter as a perfect fluid and chosen the line-element such that the Weyl tensor vanishes. We focused on two particular case studies: fundamental observers and null vector fields. Subsequently, for the particular case where the SB scalar field takes constant values, the GD equation of the null vector fields reduces to the corresponding one in the \(\Lambda\)CDM model, as expected.

To apply the GD equation for the simplest case study, we assumed that the SB scalar field dominates the dynamics. In this regard, we have extracted two cosmological scenarios, in the form of new exact solutions. We have shown that, in a particular case, these models reduce to those retrieved in the context of GR where a scalar field is minimally coupled to gravity. In the particular case where \(n = 0, W = 1\) and \(D = 4\) (or using any other equivalent conditions, which can produce such a case), the Lucchin-Mataresse and Barrow–Burd–Lancaster–Madsen models are recovered from solutions I and II, respectively (please, see subsection 3.2). We have also used a specific form of a perfect fluid which is described by non-interacting dust and radiation.

We study a phantom dark energy model in the context of an SB theory. Assuming this setting is applicable for the late time accelerating universe, we used the GD equation for small values of the redshift parameter. We have employed the energy conditions and recent observational data to find the allowed values of the corresponding parameters of the model. Such a procedure assisted us to depict the evolution of the deviation vector as well as the observer area distance against the redshift parameter. As it is well-known that phantom dark energy models are able to alleviate the \(H_0\) tension, we have therefore considered two sets of the observational data, which have been reported by the Planck collaboration and by the SH0ES collaboration, to plot the behavior of \(\eta(z)\) and \(r_0(z)\). We have also compared the behavior of these quantities with those plotted according to the \(\Lambda\)CDM model. Our endeavors have shown that the general behavior of these quantities are similar for all models.

We also studied the GD equation in the MSBT \[\text{[5].}\] In this respect, we have shown that all the formalism in the context of the generalized SB theory could also be applied for the MSBT. Subsequently, we have retrieved the corresponding cosmological exact solutions within the MSBT.
framework, namely within a spatially flat FLRW background. We have investigated the GD equation for a null vector field past directed, specifically for those mentioned cosmological solutions, and plotted the behavior of $\eta(z)$ and $r_0(z)$.

We have shown that the behaviour of the plotted observables (i.e., $\eta(z)$ and $r_0(z)$ for small values of the redshift parameter), either appraising them quantitative or qualitative associated with the new cosmological solutions extracted in the generalized SB theory and the MSBT (which have been also contrasted with either ΛCDM model or a phantom dark energy model), are all similar. However, it is important to note that only the latter could be considered as fundamental. In contrast to the other cosmological settings investigated in this paper, the EMT as well as the scalar potential present in the MSBT are not added by ad hoc assumptions to the action, but instead, they emerge strictly from dimensional reduction from the geometry, including the extra spatial dimensions [5]. Let us emphasize that the analysis associated with the GD equation in the MSBT i.e., the null vector fields appraisal, is fully consistent with the current observational data.

Let us close this section with the following comments.

- We should note that, for the sake of generality, all of our calculations have been done in arbitrary dimensions. Although, as a toy model, it is easy to plot the figures for any values of $D \geq 3$, using the observational data we have examined our herein model only for the cases with $D = 4$.

- For a general case, it may not possible to consider transformations by which the action (1) proceeds to a corresponding case with a canonical kinetic term. Notwithstanding, because of the importance of this point, let us assume a particular case such that the coupling function only takes positive values, i.e., $W(\phi) \equiv J(\phi) > 0$. In this case, defining a canonical scalar field as $d\tilde{\phi} = \sqrt{J(\phi)}d\phi$, the gravitational sector of the SB model (1) becomes [57]

$$S^{(n)} = \int d^Dx \sqrt{-g} \left[ R^{(n)} - g^{\alpha\beta} (\nabla_\alpha \tilde{\phi}) (\nabla_\beta \tilde{\phi}) - U(\tilde{\phi}) \right].$$

(188)

Note that the canonical potential and $V(\phi)$ are related as $U(\tilde{\phi}) = V(\phi)$. It is straightforward to show that the coupling function $J(\phi)$ can be expressed in terms of the potentials, such that the action (188) is rewritten as

$$S^{(n)} = \int d^Dx \sqrt{-g} \left[ R^{(n)} - \left( V - \frac{dU^{-1}(V(\phi))}{dV(\phi)} \right)^2 g^{\alpha\beta} (\nabla_\alpha \phi) (\nabla_\beta \phi) - V(\phi) \right],$$

(189)

where $U^{-1}$ is the inverse function of $U$. It should also be noted that actions (188) and (189) are equivalent and they determine the same predictions [57]. However, it seems that for any non-canonical model with specified coupling function (see e.g., the SB model with $J(\phi) = W(\phi)^n > 0$), it is important to note that, using the above transformation for getting the canonical kinetic term, restricted us to take a special canonical potential, see (189). From what we pointed out above, we find that our discussions associated with the GD equation in the SB context, in particular cases, can also be applied for the gravitational models whose actions possess a canonical kinetic term. We emphasize that, to the best of our knowledge, the GD equation associated with the latter case has not yet been investigated.

- Furthermore, we should note that it is not easy to find transformations by which the field equations of our generalized SB theory (for general values of $W$ and $n$) can transform to the corresponding ones of the Brans-Dicke theory. However, the GD equations (21), (25)
and (42) bear close resemblance to those obtained in the context of the Brans-Dicke theory. More concretely, letting

$$\rho_{\phi} \rightarrow \frac{1}{\varphi}(\rho + \rho_{\phi}), \quad p_{\phi} \rightarrow \frac{1}{\varphi}(p + p_{\phi}),$$

(190)

where $\rho_{\phi}$ and $p_{\phi}$ stand for the energy density and pressure associated with the BD scalar field $\varphi$, respectively:

$$\rho_{\varphi} \equiv \frac{\omega}{2} \varphi^2 + \frac{V(\varphi)}{2} - (D - 1)H\varphi,$$

(191)

$$p_{\varphi} \equiv \frac{\omega}{2} \varphi^2 - \frac{V(\varphi)}{2} + \alpha + (D - 2)H\varphi,$$

(192)

(where $\omega$ is the BD coupling parameter), then equations (21), (25) and (42) transform to the corresponding ones obtained in the context of the BD theory, for more details we refer the reader to [26].

- It is important to note that equation (136) is valid (as the GD equation associated with the past directed null vector field corresponding to the spatially flat FLRW metric) not only for the phantom dark energy model, but also for any cosmological model investigated in the context of the generalized SB theory in arbitrary dimensions. Such a significant consequence can be easily shown from using equations (27), (28), (31), (32), (42) and the definition of the deceleration parameter. However, we should emphasize that the $q(z)$ is, obviously, a model dependent quantity.

- One of the biggest shortcomings of GR is predicting existence of singularities, which can be indicated by singularity theorems, see, for instance, [58, 59]. The Raychaudhuri equation has been employed as one of the important ingredients to prove such theorems. A congruence singularity, whether or not could be considered as a curvature singularity, is caused by focusing of congruence, by which, together with a few additional reasonable conditions on a spacetime, the singularities emerge. In GR, the convergence condition $R_{\mu\nu}u^\mu u^\nu \geq 0$ (which leads to geodesic focusing from an attractive gravity) is retrieved from the SEC. As the field equations associated with alternative theories to GR are different, therefore, even if the SEC is satisfied, it is possible that the convergence condition is violated [43, 44]. Let us focus on our herein model. For the case established in part (iii) of Section 2, from using equation (11), we obtain

$$R_{\mu\nu}u^\mu u^\nu = \left[ T_{\mu\nu} - \left( \frac{T}{D-2} \right) g_{\mu\nu} + W\phi^2 (\nabla_{\mu} \phi)(\nabla_{\nu} \phi) + \left( \frac{1}{D-2} \right) g_{\mu\nu} V(\phi) \right] u^\mu u^\nu.$$

(193)

For the case of perfect fluid (which was discussed in part (v) of Section 2), equation (193) for the geodesic congruences with timelike and null vector fields reduces to

$$R_{\mu\nu}u^\mu u^\nu = \begin{cases} \frac{1}{D-2} \left[ (D - 3)p + (D - 1)\rho \right] + W\phi^2 \dot{\phi}^2 - \frac{V(\phi)}{D-2} & \text{(timelike)}, \\ \frac{1}{D-2} \left( \rho + p + W\phi^2 \dot{\phi}^2 \right) & \text{(null)}, \end{cases}$$

(194)

which can also be read from equations (25) and (34). Obviously, without considering a specific exact solution, we cannot proceed discussion. In this regard, it is straightforward
to determine the overall signature of $R_{\mu\nu u^\mu u^\nu}$ for our exact solutions obtained in Sections 3 and 4. Such an investigation to study the violation of convergence condition may constrain the parameters of the model.

- Finally, it is worth noting that further investigation is required to obtain concrete constraints on the SB coupling parameter $W$, so to be consistent with current observational data. Such a procedure is not in the scope of this paper and might be presented in our future investigations.

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References

[1] V. Faraoni, *Cosmology in Scalar Tensor Gravity* (Kluwer Academic Publishers, Netherlands, 2004).
[2] Israel Quiros, *Int.J.Mod.Phys.D* 28, 1930012 (2019).
[3] Tsutomu Kobayashi, *Rep. Prog. Phys.* 82, 086901 (2019).
[4] S. M. M. Rasouli and P. V. Moniz, Class. Quantum Grav. 35 35, 025004 (2018).
[5] S. M. M. Rasouli, R. Pacheco, M. Sakellariadou and P. V. Moniz *Physics of the Dark Universe* 27, 100446 (2020).
[6] S. M. M. Rasouli, *Universe* 8, 165 (2022).
[7] D. Sáez and V. J. Ballester, *Phys. Lett.* 113A, 9, 1986.
[8] S. Capozziello, L. Consiglio, M. De Laurentis, G. De Rosa and C. Di Donato, “The missing matter problem: from the dark matter search to alternative hypotheses” [arXiv:1110.5026v2].
[9] Luis O. Pimentel, *Astrophysics and Space Science* 132, 387, 1987.
[10] T. Singh and A. K. Agrawal, *Astrophysics and Space Science* 182, 289, 1991.
[11] C. P. Singh and Shri RAM, *Astrophysics and Space Science* 284, 1199, 2003.
[12] G. Mohanty, R.R. Sahoo and K.L. Mahanta, *Astrophys Space Sci.* 312, 321, 2007.
[13] R.L. Naidu, B. Satyanarayana and D.R.K. Reddy, *Int. J. Theor. Phys.* 51, 1997, 2012.
[14] V.U.M. Rao, G. Sreedevi Kumari and D. Neelima, *Astrophys Space Sci.* 337, 499, 2012.
[15] A. K. Yadav, *Research in Astron. Astrophys.* 13, 772, 2013.
[16] V.U.M. Rao, D.C. PapaRao and D.R.K. Reddy, *Astrophys Space Sci.* 357, 164, 2015.
[17] M. Sabido, J. Socorro and L. Arturo Ureña López, Classical and quantum cosmology of the Sáez-Ballester theory, *Fizika B* 19 177, 2010; arXiv: 0904.0422.
[18] J. L. Synge, *Ann. Math.* 35, 705 (1934).
[19] F. A. E. Pirani, *Acta Phys. Polon.* 15, 389 (1956).
[20] George F.R. Ellis and Henk van Elst, “Deviation of geodesics in FLRW spacetime geometries”, arXiv: gs/99709060.
[21] F. A. E. Pirani, *Phys. Rev.* 105, 1089 (1957).
[22] A. Guzmazo, L. Castañeda and Juan M. Tejeiro, *Gen. Relativ. Gravit.* 43, 2713 (2011).
[23] Ines G. Salako, M. J. S. Houndjo and Abdul Jawad, *Int. J. of Mod. Phys. D* 25, 1650076 (2016).
[24] S. M. M. Rasouli, A. F. Bahrehbakhsh, S. Jalalzadeh and M. Farhoudi, *EPL* 87, 40006 (2009).
[25] F. Darabi, M. Mousavi and K. Atazadeh, *Phys. Rev. D* 91 084023 (2015).
[26] S. M. M. Rasouli and F. Shoja Physics of the Dark Universe 32, 100781 (2021).
[27] P.S. Wesson, *Space-Time-Matter: Modern Kaluza-Klein Theory* (World Scientific, Singapore, 1999).
[28] P.S. Wesson, *Five-Dimensional Physics* (World Scientific, Singapore, 2006).
[29] J.M. Overduin and P.S. Wesson, *Phys. Rep.* 283, 303 (1997).
[30] J. Lidsey, C. Romero, R. Tavakol and S. Rippl, *Class. Quant. Grav.* 14, 865 (1997).
[32] J.E.M. Aguilar, C. Romero and A. Barros, Gen. Rel. Grav. 40, 117 (2008).
[33] N. Doroud, S.M. M. Rasouli and S. Jalalzadeh, Gen. Rel. Grav. 41, 2637 (2009).
[34] S.M. M. Rasouli and S. Jalalzadeh, Ann. Phys. (Berlin) 19, 276 (2010).
[35] S.M. M. Rasouli, M. Farhoudi and H.R. Sepangi, Class. Quant. Grav. 28, 155004 (2011).
[36] S.M. M. Rasouli, M. Farhoudi and P. V. Moniz, Classical Quantum Gravity 31, 115002 (2014).
[37] S.M. M. Rasouli, Springer Proc. in Math. Statist. 60 371 (2014).
[38] S. M. M. Rasouli and P. Moniz, Class. Quant. Grav. 33, 035006 (2016).
[39] S.M. M. Rasouli and P. Moniz, Class. Quant. Grav. 36, 075010 (2019).
[40] Øyvind Grøn and Sibjorn Hervik, Einstein’s General Theory of Relativity: With Modern Application in Cosmology, (Springer, New York 2007).
[41] Ray D Inverno, Introducing Einstein’s Relativity, (Oxford University Press, Cambridge, 1992).
[42] D.L. Caceres, L. Castaneda, J. M. Tejeiro, “Geodesic deviation equation in Bianchi cosmologies”, J. Phys. Conf. Ser. 229, 012076 (2010) [arXiv:0912.4220v1].
[43] Daniel J. Burger, Nathan Moynihan, Saurya Das, S. Shahjul Haque, and Bret Underwood Phys. Rev. D 98, 02400 (2018).
[44] Shibendu Gupta Choudhury, Ananda Dasgupta and Narayan Banerjee International Journal of Geometric Methods in Modern Physics 18, 2150115 (2021).
[45] M. Sharif and Saira Waheed, Advances High Energy Phys. 2013, 253985 (2013).
[46] Hideki Maeda, Crisitán Martinez, “Energy conditions in arbitrary dimensions”, Progress of Theoretical and Experimental Physics 2020, 4 (2020), [arXiv:1810.02487]
[47] A. A. Garcia and S. Carlip, Physics Letters B 645, 101 (2007).
[48] Eleonora Di Valentinoa et al, Class. Quantum Grav. 38, 153001 (2021).
[49] Eleonora Di Valentineto al, “Cosmology Intertwined II: The Hubble Constant Tension”, [arXiv:2008.11284 [astro-ph.CO]].
[50] A. G. Riess, S. Casertano, W. Yuan, J. B. Bowers, L. Macri, J. C. Zinn and D. Scolnic, Astrophys. J. 908 L6, (2021).
[51] Planck Collaboration, N. Aghanim, et al., Planck 2018 results VI: “Cosmological parameters”, Astron. Astrophys. 641, A6 (2020), [arXiv:1807.06209]
[52] R. R. Caldwell, Physics Letters B 545 23 (2002).
[53] Robert J. Scherrer and A. A. Sen, Phys. Rev. D 78 067303 (2008).
[54] Wang Wen-Fu, Shui Zheng-Wei and Tang Bin, Chin. Phys. B 19, 119801 (2010).
[55] Eleonora Di Valentinoa, Alessandro Melchiorrib and Joseph Silka, Physics Letters B 761, 242 (2016).
[56] David Camarena and Valerio Marra, Phys. Rev. Research 2,013028 (2020).
[57] Zhu Yi and Zong-Hong Zhu, “Inflationary attractors from a non-canonical kinetic term”, [arXiv:2106.10303]
[58] R. Penrose, Phys. Rev. Lett. 14, 57 (1965).
[59] S. W. Hawking and R. Penrose, Proc. Roy. Soc. Lond. A314, 529 (1970).