MAXIMAL IDEALS IN COMMUTATIVE BANACH ALGEBRAS

H. G. DALES

Abstract. We show that each maximal ideal in a commutative Banach algebra has codimension 1.

1.

Let $A$ be a commutative algebra (over the complex field, $\mathbb{C}$). An ideal $M$ in $A$ is maximal if $M \neq A$ and if $M = I$ or $I = A$ for each ideal $I$ in $A$ with $M \subset I$.

We wonder if every maximal ideal in a commutative Banach algebra is of codimension 1 in $A$?

This is not resolved in [1]; I cannot see a comment on the point in other texts. Here we show that the above is indeed the case.

Let $A$ be an algebra. Then $A^{[2]} = \{ab : a, b \in A\}$ and $A^2$ is the linear span of $A^{[2]}$, so that $A^2$ is an ideal in $A$.

Let $A$ be a commutative algebra without an identity. We denote by $A^\sharp$ the unital algebra formed by adjoining an identity, $e$, to $A$, so that $A$ is a maximal ideal in $A^\sharp$; in the case where $A$ is a Banach algebra, $A^\sharp$ is also a Banach algebra.

In the case where $A$ is a topological algebra, a maximal ideal is either closed or dense in $A$.

First suppose that $M$ is a maximal modular ideal in a commutative algebra $A$. Then $A/M$ is a field containing $\mathbb{C}$. In the case where $A$ is a commutative Banach algebra, $M$ is necessarily closed, and so $A/M$ is a Banach algebra. By the Gel’fand–Mazur theorem $A/M \cong \mathbb{C}$, and $M$ is the kernel of a continuous character. In particular, $M$ has codimension 1. In fact, the Gel’fand–Mazur theorem applies to locally convex $F$-algebras [1, Theorem 2.2.42], and so the previous comment also applies to this class of topological algebras.

Now suppose that $A$ is a commutative, unital Fréchet algebra. Then a maximal ideal (which is necessarily modular) is not necessarily either closed or of codimension 1, as the following example, which essentially repeats [1, Proposition 4.10.27], shows.

Example 1.1. Let $O(\mathbb{C})$ denote the set of entire functions on $\mathbb{C}$. This is a commutative, unital algebra for the pointwise algebraic operations, and it is a Fréchet algebra with respect to the topology of uniform convergence on
compact subsets of \( \mathbb{C} \). It is standard that each maximal modular ideal \( M \) of codimension 1 in \( O(\mathbb{C}) \) is closed and is such that there exists \( z \in \mathbb{C} \) such that

\[
M = M_z := \{ f \in O(\mathbb{C}) : f(z) = 0 \}.
\]

Now let \( I \) be the set of functions \( f \in O(\mathbb{C}) \) such that \( f(n) = 0 \) for each sufficiently large \( n \in \mathbb{N} \). Clearly \( I \) is an ideal in \( O(\mathbb{C}) \), and it is easy to see that \( I \) is dense in \( O(\mathbb{C}) \). Since \( O(\mathbb{C}) \) has an identity, \( I \) is contained in a maximal modular ideal, say \( M \), of \( O(\mathbb{C}) \). The ideal \( M \) is dense in \( O(\mathbb{C}) \). Clearly, \( M \) is not of the form \( M_z \) for any \( z \in \mathbb{C} \), and so \( M \) does not have codimension 1 in \( O(\mathbb{C}) \); the quotient \( A/M \) is a ‘large field’.

\[ \square \]

2.

Now suppose that \( A \) is a commutative algebra, and that \( M \) is a maximal ideal in \( A \). Set \( I = A^2 + M \), so that \( I \) is an ideal in \( A \) containing \( M \). Thus either \( A^2 \subset M \) or \( A^2 + M = A \).

Consider the case in which \( A^2 \subset M \), so that \( A/M \) is a commutative algebra with zero product. Let \( E \) be a subspace of codimension 1 of \( A \) with \( E \supset M \). Then \( E \) is an ideal in \( A \), and so \( E = M \). Thus it is indeed the case that \( M \) has codimension 1. (This remark is essentially Exercise 6(e) in \cite[Chapter 1]{2}.)

By considering an infinite-dimensional Banach space with zero product, we see that there are examples of maximal ideals of codimension 1 that are closed and that are dense in a commutative Banach algebra \( A \).

It remains to consider the case where \( A^2 + M = A \).

3.

We now show that the latter case does not occur in a commutative Banach algebra \( A \).

**Proposition 3.1.** Let \( A \) be a commutative Banach algebra. Suppose that \( I \) is an ideal in \( A \) such that \( I \) is dense in \( A \) and \( A^2 + I = A \). Then \( I \) is not maximal.

**Proof.** Necessarily \( A \) does not have an identity. The norm in \( A^2 \) is denoted by \( \| \cdot \| \).

For each \( n \in \mathbb{N} \), we denote by \( A_n \) the set of elements \( a \in A \) such that there exist \( b_1, \ldots, b_n, c_1, \ldots, c_n \in A \) and \( x \in I \) with

\[
a = b_1 c_1 + \cdots + b_n c_n + x.
\]

Choose an element \( a_0 \in A \setminus I \) such that \( a_0 \) has a representation of the form \( (1) \) and such that \( n \) is the minimum natural number with this property. Define

\[
J = A^2 a_0 + I.
\]
Then $J$ is an ideal in $A$ with $a_0 \in J$, and $J \supseteq I$. We shall show that $J \neq A$, and hence that $I$ is not a maximal ideal in $A$.

Assume towards a contradiction that $J = A$. Then $b_1 \in J$, and so there exists $d_1 \in A^\sharp$ such that

$$b_1 - d_1 a_0 \in I,$$

say $\|d_1\| = m$. Choose $d_2 \in A$ such that $c_1 - d_2 \in I$ and $\|d_2\| < 1/m$; this is possible because $I$ is dense in $A$.

Now we have

$$b_1 - d_1 (b_1 d_2 + b_2 c_2 + \cdots + b_n c_n) \in I,$$

and so

$$b_1 (e - d_1 d_2) \in d_1 (b_2 c_2 + \cdots + b_n c_n) + I.$$

However $\|d_1 d_2\| \leq \|d_1\| \|d_2\| < 1$, and so the element $e - d_1 d_2$ is invertible in $A^\sharp$, say with inverse $d_3$. Thus

$$a_0 \in c_1 d_1 d_3 (b_2 c_2 + \cdots + b_n c_n) + (b_2 c_2 + \cdots + b_n c_n) + I = (e + c_1 d_1 d_3) (b_2 c_2 + \cdots + b_n c_n) + I.$$

In the case where $n > 1$, this shows that $a_0 \in A_{n-1}$, a contradiction of the minimality of the choice of $n$. In the case where $n = 1$, the argument shows that $b_1 (e - d_1 d_2) \in I$ and hence that $b_1 \in I$, and then that $a_0 = b_1 c_1 \in I$, a contradiction of the fact that $a_0 \notin I$. Thus $J \subset A$. □

There is a closely-related algebraic result that is surely known and in some text.

**Proposition 3.2.** Let $R$ be a commutative, radical algebra. Suppose that $I$ is an ideal in $R$ such that $R^2 + I = A$. Then $I$ is not maximal.

**Proof.** The element $c_1 d_1$ in the above proof (with $R$ for $A$) is such that $e - c_1 d_1$ is invertible in $R^\sharp$ because $R$ is radical, and so the argument of the above proof gives the result. □

**Theorem 3.3.** Let $A$ be a commutative Banach algebra. Then every maximal ideal $M$ in $A$ has codimension 1 in $A$. Further, either $A/M \cong \mathbb{C}$ or $A^2 \subset M$.

**Proof.** Let $M$ be a maximal ideal in $A$. We have noted that $M$ is either closed or dense in $A$. We have also noted that either $A^2 \subset M$ or $A^2 + M = A$, and that in the former case $M$ does have codimension 1 in $A$.

Thus we may suppose that $A^2 + M = A$. By Proposition 3.1 it cannot be that $M$ is dense in $A$, and so $M$ is closed. Thus $A/M$ has dimension 1, and so $M$ has codimension 1 in $A$. Further, $A/M \cong \mathbb{C}$. □

**Corollary 3.4.** Let $R$ be a commutative, radical Banach algebra such that $R^2 = R$. Then $R$ has no maximal ideals.
Proof. Assume that $M$ is a maximal ideal in $R$. Then it is not the case that $R/M \cong \mathbb{C}$ because $R$ is radical, and it is not the case that $R^2 \subset M$ because $R^2 = R$. In either case, we have a contradiction of Theorem 3.3. Thus $R$ has no maximal ideals. \qed

There are many examples of commutative, radical Banach algebras such that $R^2 = R$. Each commutative, radical Banach algebra with a bounded approximate identity has this property. For example, this is the case for the Volterra algebra $\mathcal{V}$, which is the space $L^1[0,1]$ with convolution product $\star$ given by
\[
(f \star g)(t) = \int_0^1 f(t-s)g(s) \, ds \quad (t \in [0,1])
\]
for $f, g \in \mathcal{V}$. There are also examples which are integral domains.

The following example shows that there are commutative, radical Banach algebras $R$ such that $R^2 = R$, but such that $R$ does have a maximal ideal, necessarily of codimension 1.

Example 3.5. Let $\mathbb{I} = [0,1]$, and define
\[
R = \{ f \in C(\mathbb{I}) : f(0) = 0 \} ,
\]
taken with the uniform norm $| \cdot |_\mathbb{I}$ and the above truncated convolution product. Then $R$ is a commutative, radical Banach algebra.

Set $I = \{ f \in R : \lim_{t \to 0^+} f(t)/t = 0 \}$, so that $I$ is a linear subspace of $R$; in fact, $I$ is an ideal in $(R, \star)$.

Take $f, g \in R$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that
\[
|g(s)|_\mathbb{I} < \varepsilon \quad (0 \leq s \leq \delta).
\]
Thus, for $0 \leq t \leq \delta$, we have
\[
|\int_0^t (f \star g)(s) \, ds| < \varepsilon \int_0^t |g(s)| \, ds < \varepsilon t |f|_\mathbb{I},
\]
and so $f \star g \in I$. Hence $R^2 \subset I \subset M$, and so $M$ is a maximal ideal (of codimension 1) in $R$. \qed

References

[1] H. G. Dales, *Banach algebras and automatic continuity*, London Mathematical Society Monographs, Volume 24, Clarendon Press, Oxford, 2000.

[2] T. W. Gamelin, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, New Jersey, 1969.

Department of Mathematics and Statistics
University of Lancaster
Lancaster LA1 4YF
United Kingdom
g.dales@lancaster.ac.uk