Dynamic Modeling and Attitude Control System of Underactuated Fuel Slosh with Flexible Appendages

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Abstract: This paper investigates the dynamics and attitude control of the pitching motion of an axisymmetric liquid-filled satellite with flexible appendages under the conditions of zero gravity and constant thrust (fixed axial acceleration). Using Lagrange method, the dynamics model of the flexible liquid-filled satellite is established. The flexible appendages are considered as Euler–Bernoulli beams, and the sloshing liquid is modeled as a spring-mass model. Considering the motion of the satellite in the orbit plane and pitching attitude motion, a sliding mode controller is designed by decoupling the satellite dynamics equation into a typical underactuated system. The influence of the vibration of flexible appendages and the slosh of liquid fuel on the attitude control of the spacecraft is considered. The results show that the designed controller can effectively suppress the liquid sloshing and the vibration of the flexible appendages in such a manner as to achieve the desired result and control target.

1. Introduction  
With today's large and complex spacecraft, modern spacecraft contain large flexible solar panels and significant amounts of liquid propellant for on-orbit mission necessity. Liquid sloshing and flexible vibration are the most probable causing unexpected coupling effects on the dynamics of spacecraft orbit and attitude. In particular, when the natural frequencies of prevailing fluid slosh modes and flexible vibration are very close, the attitude control during the spacecraft proceed attitude will be a very tricky problem. Controlling the vibration while the sloshing and describing the flexible–fluid–spacecraft coupling dynamics have made critical problems of the technologies. The ordinarily appended flexible members on a modern spacecraft can be assorted into two types, beam-type appendages, and plate-type appendages. The dynamic modeling and the attitude control of spacecraft with beam-type and plate-type flexible appendages were studied in [1-3]. For a kind of flexible spacecraft, it was derived control schemes through Lyapunov stability attitude errors for the flexible vibration induced in the maneuver [4,5].

Liquid slosh in moving containers becomes an attractive new field of study, essentially in modern spacecraft, which tend to carry large amounts of liquid fuel. Although a liquid in the spacecraft has many types of motion when the spacecraft suffers disturbance [6,7], there are generally two principal slosh modes: lateral sloshing mode motivated by spacecraft translation and rotary sloshing mode. In addition, there is possibly a rigid motion of liquid regard to the spacecraft when undergoes large angle attitude motion [8]. Anyway, incorporate the liquid sloshing dynamics into the overall spacecraft stability analysis and control design is always complicated and tricky work to describe. Equivalent
mechanical models are particularly useful when designing a control system or creating a model for stability or performance analysis which commonly includes the pendulum model and the spring–mass model which considered in this paper.

2. Dynamic modeling of flexible liquid-filled satellite
As shown in Fig.1, consider a spacecraft consisting of a central rigid body and two symmetrically mounted flexible panels. The central rigid body is an axisymmetric structure with a liquid fuel tank inside. The panel is approximated by a uniform flexible cantilever beam. The bending deformation of the flexible cantilever beam has two forms: symmetric mode and antisymmetric mode. In this paper, the antisymmetric model is adopted. Since the motion of two flexible sailboards is anti-symmetric, only the modal coordinates of one of the boards are taken into consideration when establishing the equation of motion of flexible accessories. The body coordinate system of the satellite is established with the satellite's center of mass as the origin. In figure 1 the axis $X_o$ and $Z_o$ axis is a pair of coordinate axes of the orbital coordinate system in the pitching plane. The flexible liquid-filled satellite has a constant thrust $F$ acting through the center of mass of the satellite. The velocity component of the spacecraft along the longitudinal axis of the body ($OZ_b$ axis of the body coordinate system) is $V_x$, and the velocity component along the horizontal axis of the body ($OZ_b$ axis of the body coordinate system) is $V_z$.

![Figure 1. Flexible liquid-filled satellite model](image)

The sloshing displacement of the liquid is equivalent to the sloshing displacement of the block along the horizontal axis $\eta$. The spacecraft's Angle $\theta$ of rotation around Y-axis is influenced by the transverse thrust $f$ and the rotational moment acting on the center of mass $M$. The constant thrust in the system is $F$ and the mass of the rigid body spacecraft is $m$. The fuel mass in fuel tank $m_f$ where the non-sloshing fuel mass is $m_0$. The distance to the spacecraft center of mass is $h_0$ while the mass of sloshing liquid is $m_1$. The distance to the spacecraft's center of mass is $h_1$ and the moment of inertia of the rigid part of the spacecraft relative to the center of mass of the spacecraft is $I$. The rotational inertia of the fuel relative to the spacecraft's centroid is $I_0$ and $I_1$ respectively. The distance between the connecting point of the flexible appendage and spacecraft rigid body to spacecraft centroid is $a$ and the distance between the spacecraft mass center and the geometric center of the fuel tank is $b$. Finally, the elastic coefficient of the spring in the spring mass block model is $k$.

The equations of motion for flexible liquid-filled spacecraft can be obtained as follows

\[
(m + m_f + 2m_\rho)a_x + m_1b\dot{\theta}^2 + m_1(\eta\ddot{\theta} + 2\dot{\eta}\dot{\theta}) = F
\]

(1)

\[
(m + m_f + m_\rho)a_z + m_1b\ddot{\theta} + m_1(\dot{\eta} - \eta\dot{\theta}^2) = f
\]

(2)
\[ l \ddot{\theta} + m_1 ( \eta a_x + 2 \eta \dot{\theta} + (b - h_1) \dot{\eta}) + m_f a_x + 2 B_{rot} \ddot{\eta} = M + b f \]  
\[ m_1 ( \ddot{\eta} + a_x + (b - h_1) \dot{\theta} - \eta \dot{\theta}^2) + k \eta + c \dot{\eta} = 0 \]  
\[ \dot{q}_i + \omega_i^2 q_i + B_{trans} a_x + B_{rot} \dot{\theta} = 0 \]  

Where, \( \ddot{i} = \ddot{i} + 2 \rho A \int_a^{a+1} x^2 \, dz \).

During the spacecraft motion, when the pitch angle of spacecraft changes slowly and the liquid sloshing amplitude is small, the influence on the spacecraft axial acceleration \( a_x \) can be regarded as a constant, so that \( a_x \) can be simplified as
\[ a_x = \dot{v}_x + \dot{\theta} v_x = \frac{M + b f - l \dot{\theta} - m_1 ( \dot{v}_x + \dot{\theta} v_x)}{m_f b} - 2 m_1 ( \eta \dot{\theta} + (b - h_1) \dot{\eta}) - 2 B_{rot} \ddot{\eta}. \]  

3. Attitude controller design of flexible liquid-filled satellite

The control objective of this section is to design a sliding mode controller so that the transverse acceleration \( v_x \) and attitude angle \( \theta \) of the spacecraft can reach the equilibrium point, and at the same time to suppress the vibration of the flexible appendage and small sloshing of liquid fuel, so that the spacecraft can be asymptotically stable.

Equation (3) in the system dynamics equation is rewritten as
\[ a_x = \dot{v}_x + \dot{\theta} v_x = \frac{M + b f - l \dot{\theta} - m_1 ( \dot{v}_x + \dot{\theta} v_x)}{m_f b} - 2 m_1 ( \eta \dot{\theta} + (b - h_1) \dot{\eta}) - 2 B_{rot} \ddot{\eta}. \]  

In order to control the transverse velocity reaching the equilibrium point, the control input \( M \) can be designed as
\[ M = l \dot{\theta} - b f + m_1 ( \dot{v}_x + \dot{\theta} v_x) + 2 m_1 ( \eta \ddot{\theta} + (b - h_1) \dot{\eta}) + m_f b (-k_1 v_x - \dot{\theta} v_x) + 2 B_{rot} \ddot{\eta}. \]  

Substituting equation (8) into equation (7), can get
\[ \ddot{v}_x = -k_1 v_x. \]  

According to the above formula, as long as the appropriate normal number \( k_1 \) is selected, \( v_x \) can be asymptotically stable. Therefore, \( v_x \) and \( \dot{v}_x \) can be treated as external variables during the design of the sliding mode controller.

3.1 Model decoupling

It can be obtained from the equation (4) that
\[ \ddot{\eta} = \eta \dot{\theta}^2 - a_x - \frac{k}{m_1} \eta - \frac{c}{m_1} \dot{\eta} - (b - h_1) \dot{\theta}. \]  

Substitute equation (10) into equation (2) to get
\[ \ddot{\theta} = \frac{k \eta + c \dot{\eta} - (m_T - m_1) a_x}{M_1} + \frac{f}{M_1} = f_1 (\eta, \dot{\eta}, \dot{\theta}) + \frac{1}{M_1} u. \]  

Where \( m_T = m + m_f + 2 m_p, M_1 = m_f b - m_1 b + m_1 h_1, f_1 = \frac{k \eta + c \dot{\eta} - (m_f - m_p) a_x}{m_1}, u = f. \)

Substitute equation (11) into equation (10) to get
\[ \ddot{\eta} = \eta \dot{\theta}^2 + \left( M_2 k - \frac{k}{m_1} \right) \eta + \left( M_2 c - \frac{c}{m_1} \right) \dot{\eta} \]  
\[ -[1 + M_2 (m_T - m_1)] a_x + M_2 u = f_2 (\eta, \dot{\eta}, \dot{\theta}) + M_2 u \]  

Where, \( M_2 = \frac{(h_1 - b)}{M_1}, \ f_2 = \eta \dot{\theta}^2 + \left( M_2 k - \frac{k}{m_1} \right) \eta + \left( M_2 c - \frac{c}{m_1} \right) \dot{\eta} - [1 + M_2 (m_T - m_1)] a_x. \)
Due to the existence of axial acceleration $a_x$, it can be observed from equation (5) that when the system is in equilibrium, the vibration displacement of the flexible panel does not tend to 0, but tend to a constant value. In order to facilitate analysis and design, it is desirable to avoid the second derivative of the attitude angle $\theta$ in the equation of the flexible vibration [9]. The vibration equation is

$$p_i = -\omega_i^2 p_i + \omega_i^2 B_{rott_1}$$

(13)

Where,

$$p_i = q_i + B_{rott_1} + \frac{B_{rott_1} a_x}{\omega_i^2}.$$  

(14)

Decoupling algorithm, let

$$\begin{aligned}
    x_1 &= \theta - \frac{1}{M_1 M_2} \eta,
    x_2 &= \dot{\theta} - \frac{1}{M_1 M_2} \eta,
    x_3 &= p_1,
    x_4 &= \dot{p}_1,
    x_5 &= p_2,
    x_6 &= \dot{p}_2,
    x_7 &= \theta,
    x_8 &= \dot{\theta},
\end{aligned}$$  

(15)

3.2 Sliding mode controller design

Define the sliding surface as

$$s = \sum_{i=1}^{8} c_i x_i.$$  

(16)

Where, $c_1, c_2, ..., c_8$ is an undetermined real number which is obtained by Hurwitz stability analysis of the sliding surface equation below

$$\dot{s} = c_1 x_2 + c_2 \alpha + c_3 x_4 + c_4 \xi + c_5 x_6 + c_6 \phi + c_7 x_8 + c_8 f_1 + \frac{u}{M_1}.$$  

(17)

Where, $\alpha = (f_1 - \frac{1}{M_1 M_2} f_2)$, $\xi = (\omega_1^2 x_3 + \omega_1^2 B_{rot1} x_7)$, $\phi = (-\omega_2^2 x_5 + \omega_2^2 B_{rot2} x_7)$.

Let $\dot{s} = 0$ the equivalent control law

$$u_{eq} = -\left(\frac{M_1}{c_8}\right) [c_1 x_2 + c_2 \alpha + c_3 x_4 + c_4 \xi + c_5 x_6 + c_6 \phi + c_7 x_8 + c_8].$$  

(18)

Chosen the exponential approach law:

$$\dot{s} = -\frac{\beta sgn(s) + K s}{c_8 sgn(s)} , \beta > 0, K > 0.$$  

Where, $\beta$ and $K$ are positive numbers and $sgn(s)$ is a symbolic function as fallow:

$$sgn(s) = \begin{cases} 
1, & s > 0, \\
0, & s = 0, \\
-1, & s < 0. 
\end{cases}$$

In the exponential approach law, in order to ensure that the chattering can be reduced while rapidly approaching the sliding surface, we should increase $K$ and decrease $\beta$ at the same time. Therefore, the design of the sliding mode control as follows
\[ u = u_{eq} + u_{sw} = -\left(\frac{M_1}{c_8}\right) \left[ c_1x_2 + c_2\alpha + c_3x_4 + c_4\xi + c_5x_6 + c_6\varphi + c_7x_8 + c_8f_1 \\
+ \frac{\beta sgn(s) + K(s)}{c_8sgn(s)} \right]. \]  \tag{19}

Take Lyapunov function as
\[ V = \frac{1}{2} s^2. \]  \tag{20}

Substituting the control law (19) into the \( s \) of equation (17), the Lyapunov function is derived as:
\[ \dot{V} = s \ddot{s} = -\beta|s| - Ks^2 < 0. \]  Where, \( \beta \) and \( K > 0 \).

It can be seen from the above analysis that the sliding mode surface \( s \) will eventually approach 0, i.e.
\[ s = \sum_{i=1}^{7} c_i x_i + c_8 x_8 = 0. \]

Next, it is necessary to prove that when \( s = 0 \), \( x_1 \to 0 \) can be established by the selection of the sliding surface parameters \( c_1, c_2, \ldots, c_8 \).

Let \( y_1 = x_1, y_2 = x_2, \ldots, y_7 = x_7 \), then \( \dot{\eta} = M_1M_2(y_7 - y_1), \ddot{\eta} = M_1M_2 \left( -\sum_{i=1}^{7} \frac{c_i x_i}{c_8} - y_2 \right) \), then the system equations become
\[
\begin{align*}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= \bar{A} y_1 + \bar{B} y_2 + \bar{C} y_3 + \bar{D} y_4 + \bar{E} y_5 + \bar{F} y_6 + \bar{G} y_7 + \eta + \frac{\nu \eta}{M_1M_2}, \\
\dot{y}_3 &= y_4, \\
\dot{y}_4 &= -\omega_1^2 y_3 + \omega_1^2 Brot_1 y_7, \\
\dot{y}_5 &= y_6, \\
\dot{y}_6 &= -\omega^2 y_5 + \omega^2 Brot_2 y_7, \\
\dot{y}_7 &= -\frac{\varphi y_1}{c_8} - \frac{\varphi y_2}{c_8} - \frac{\varphi y_3}{c_8} - \frac{\varphi y_4}{c_8} - \frac{\varphi y_5}{c_8} - \frac{\varphi y_6}{c_8} - \frac{\varphi y_7}{c_8},
\end{align*}
\]

where,
\[
\begin{align*}
\bar{A} &= -\frac{k}{m_1} - \frac{cc_1}{c_8 m_1} + \frac{v_x c_1}{c_8 M_1 M_2}, \\
\bar{B} &= -\frac{c}{m_1} - \frac{cc_2}{c_8 m_1} + \frac{v_x c_2}{c_8 M_1 M_2}, \\
\bar{C} &= -\frac{cc_3}{c_8 m_1} + \frac{v_x c_3}{c_8 M_1 M_2}, \\
\bar{D} &= -\frac{cc_4}{c_8 m_1} + \frac{v_x c_4}{c_8 M_1 M_2}, \\
\bar{E} &= -\frac{cc_5}{c_8 m_1} + \frac{v_x c_5}{c_8 M_1 M_2}, \\
\bar{F} &= -\frac{cc_6}{c_8 m_1} + \frac{v_x c_6}{c_8 M_1 M_2}, \\
\bar{G} &= \frac{k}{m_1} - \frac{cc_7}{c_8 m_1} + \frac{v_x c_7}{c_8 M_1 M_2}.
\end{align*}
\]
\[ o(y) = (y_1 - y_7)(-\sum_{i=1}^{7} c_i y_i)^2 \]

Organize the equation (21) into the following form
\[
\dot{y} = Ay + o(y) + \frac{\dot{v}_z}{M_2}.
\]

Where,
\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\bar{A} & \bar{B} & \bar{C} & \bar{D} & \bar{E} & \bar{F} & \bar{G} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\omega_2^2 & 0 & \omega_2^2 B_{rot1} \\
0 & 0 & 0 & 0 & 0 & 0 & \omega_2^2 B_{rot2} \\
-c_1 & -c_2 & -c_3 & -c_4 & -c_5 & -c_6 & -c_7 \\
c_8 & c_8 & c_8 & c_8 & c_8 & c_8 & c_8 \\
\end{bmatrix}
\]

Through pole configuration, the system \( \dot{y} = Ay \) is globally asymptotically stable and \( A \) needs to satisfy Hurwitz stability. It is necessary to ensure that all eigenvalues of \( A \) are negative in real part while the real part of \( |A - \lambda I| = 0 \) is negative which make it possible to design the parameters \( c_1, c_2, ..., c_8 \) of the sliding surface.

When \( A \) satisfies Hurwitz stability, Then the linear principal part of the system \( \dot{y} = Ay \) is stable. The system \( \dot{y} = Ay + o(y) \) satisfies local asymptotic stability after considering high order small amount \( o(y) \). Since the \( \dot{v}_z \) controlled by formula (8) finally tends to zero, it can be regarded as a disturbance tending to zero, it can be proved by the counter-evidence method that the state quantity of the system in equation (22) eventually tends to zero.

4. Simulation results
For the sliding mode controller designed in this paper, the digital simulation method is used to verify its control effect. The satellite parameters in the system dynamics equation (1) - (5) are \( m = 600 \text{kg} \), \( m_0 = 150 \text{kg} \), \( m_1 = 100 \text{kg} \), \( l = 720 \text{kg.m}^2 \), \( I_0 = 70 \text{kg.m}^2 \), \( J_1 = 34 \text{kg.m}^2 \), \( a = 1 \text{m} \), \( b = 0.6 \text{m} \), \( \rho = 300 \text{kg/m}^3 \), \( EI = 80 \text{N.m} \), \( A = 7.2968 \times 10^{-4} \text{m}^2 \), \( t = 10 \text{m} \), \( F = 500 \text{N} \), \( \kappa = 500 \text{N/m} \), \( c = 30 \text{N/s} \).

The controller parameters designed in formula (8) designed as: \( k_1 = 1 \) The \( c_1 = 0.02,c_2 = 0.01,c_3 = 0.4 \), \( c_5 = 2 \).The parameters of the sliding mode controller of formula are: \( c_6 = 4 \), \( c_7 = 20 \), \( \beta = 0.0005 \), \( K = 0.3 \), \( \Delta = 0.1 \). The initial state value of the system: \( v_x(0) = 200 \text{m/s} \), \( v_y(0) = 20 \text{m/s} \), \( \eta(0) = 0.1 \text{m} \), \( \eta(0) = 0 \text{m/s} \), \( \theta(0) = 5^\circ \), \( \dot{\theta}(0) = 0^\circ/s \), \( q_1(0) = 0.1 \text{m} \), \( q_2(0) = 0 \text{m} \), \( \dot{q}_2(0) = 0 \text{m/s} \).The simulation results are shown in Fig. 2-6.

![Figure 2. Change of spacecraft attitude state](image-url)
Figure 3. Sloshing displacement state and its acceleration

Figure 4. First-order and second-order vibration modes of flexible appendages

Figure 5. Variation of spacecraft centroid velocity
As can be seen from Fig. 5, the velocity $v_z$ quickly reaches the equilibrium point with less oscillation. However, due to the coupling effect of the flexible appendage vibration, the sliding surface fluctuates more and the control quantity fluctuates more intensely, which results in the attitude angle of the spacecraft, the sloshing displacement of the liquid and the first and second order vibration flutter of the flexible appendage. In the flexible liquid-filled satellite of this paper, the satellite is stabilized in the equilibrium position in about 160 seconds under the attitude angle and sloshing displacement. The sloshing displacement of liquid fuel in the spacecraft reaches the equilibrium position in 180 seconds. The first-order vibration of the flexible accessory reaches the equilibrium position in about 180 seconds, while the second-order vibration mode reaches the equilibrium position in about 100 seconds. Due to the existence of axial acceleration $a_x$, the first-order vibration of flexible accessories is stable at -1.44 meters, and the second-order vibration is stable at -0.02 meters, which is in line with expectations. However, the control amount $f$ and rotational moment $M$ are too large at the beginning of control. Although the required lateral thrust reduced to about 2000N in about 20 seconds and the required rotational torque decreases to 1000n.m in 30 seconds as the system gradually stabilizes, but still does not meet the actual output control quantity. At the same time, the maximum amplitude of liquid fuel sloshing reaches about 0.7 meters, which exceeds the maximum radius of the liquid fuel tank.

5. Conclusions
In this paper, the influence of the vibration of flexible appendages and sloshing of liquid fuel on attitude control of spacecraft is considered. After decoupling the spacecraft dynamics equation into a typical underactuated system, a sliding mode controller is designed. The simulation results show that the designed controller can achieve the control targets and effectively suppress the liquid sloshing and the vibration of flexible accessories. However, due to the coupling effect of the vibration of flexible accessories, the vibration of the control quantity and the controlled state quantity in the control process is quite severe, which requires further research on a more suitable control algorithm.

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