On Shallow Packings and Tusnády’s Problem

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Abstract

Tusnády’s problem asks to bound the discrepancy of points and axis-parallel boxes in $\mathbb{R}^d$. Algorithmic bounds on Tusnády’s problem use a canonical decomposition of Matoušek for the system of points and axis-parallel boxes, together with other techniques like partial coloring and/or random-walk based methods. We use the notion of shallow cell complexity and the shallow packing lemma, together with the chaining technique, to obtain an improved decomposition of the set system. Coupled with an algorithmic technique of Bansal and Garg for discrepancy minimization, which we also slightly extend, this yields improved algorithmic bounds on Tusnády’s problem.

For $d \geq 5$, our bound matches the lower bound of $\Omega(\log^{d-1} n)$ given by Matoušek, Nikolov and Talwar [IMRN, 2020] — settling Tusnády’s problem, up to constant factors. For $d = 2, 3, 4$, we obtain improved algorithmic bounds of $O(\log^{7/4} n)$, $O(\log^{5/2} n)$ and $O(\log^{13/4} n)$ respectively, which match or improve upon the non-constructive bounds of Nikolov for $d \geq 3$.

Further, we also give improved bounds for the discrepancy of set systems of points and polytopes in $\mathbb{R}^d$ generated via translations of a fixed set of hyperplanes.

As an application, we also get a bound for the geometric discrepancy of anchored boxes in $\mathbb{R}^d$ with respect to an arbitrary measure, matching the upper bound for the Lebesgue measure, which improves on a result of Aistleitner, Bilyk, and Nikolov [MC and QMC methods, Springer, Proc. Math. Stat., 2018] for $d \geq 4$.

1 Introduction

Given a set system $(X, \mathcal{R})$, consisting of a ground set $X$ and a family $\mathcal{R} \subset 2^X$ of subsets of $X$, its combinatorial discrepancy $\text{disc}(X, \mathcal{R})$ is given by

$$\text{disc}(X, \mathcal{R}) = \min_{x \in \{-1, 1\}^n} \max_{S \in \mathcal{R}} \left| \sum_{i \in S} x(i) \right|.$$ 

Thus, the discrepancy of a set system quantifies the minimum necessary imbalance that must occur, in at least one of the subsets in the family $\mathcal{R}$, when the ground set $X$ is bi-partitioned. More generally, given an $m \times n$ matrix $M$ with entries $(a_{ij})_{i \in [m], j \in [n]}$, such that $|a_{ij}| \leq 1$ for all $i, j$, the $\ell_\infty$-discrepancy of $M$ is defined to be

$$\text{disc}(M) = \min_{x \in \{-1, 1\}^n} \|Mx\|_\infty.$$ 

The above definition gives the discrepancy of a set system $(X, \mathcal{R})$ by taking $M$ to be the adjacency matrix of $(X, \mathcal{R})$. The vector $x \in \{-1, 1\}^n$ is the coloring of the columns of $M$ (or the elements of $X$), as it corresponds to the bi-partitioning of $X$.

Discrepancy theory initially arose in several areas of mathematical analysis including the number-theoretic work of Erdős, van der Corput and others [43, 42, 57], and since the 80s has found several connections to other areas of mathematics, as well as many applications in computer science and related areas [14, 15, 30].

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Hereditary Discrepancy

Though the discrepancy of a matrix (or a set system) seems to be a fundamental property and has several applications, it can be prone to change sharply under slight perturbation. That is, a set system with very low (or zero) discrepancy can have sub-systems which have high discrepancy. A related notion therefore is that of hereditary discrepancy, which is more robust with respect to deletions in the set system. Formally, the hereditary discrepancy of a set system \((X, \mathcal{S})\) is defined to be 
\[
\text{herdisc}(X, \mathcal{S}) := \max_{Y \subseteq X} \text{disc}(Y, \mathcal{S}|_Y),
\]
where \(\mathcal{S}|_Y := \{ S \cap Y \mid S \in \mathcal{S} \}\) is the projection of \((X, \mathcal{S})\) on to \(Y\). Thus, herdisc\((X, \mathcal{S})\) simultaneously bounds the discrepancy of every sub-system of \((X, \mathcal{S})\).

In terms of matrices, herdisc\((M)\) is the family of subsets of \(X\) which arise as intersections of \(X\) with the family of all possible \(d\)-dimensional axis-parallel boxes. In Tusnády’s problem the goal is to compute a bound on the discrepancy of any set system formed by points and axis-parallel boxes in \(\mathbb{R}^d\). Let \(\text{disc}(n, \mathcal{B}_d)\), \(\text{herdisc}(n, \mathcal{B}_d)\) be the maximum discrepancy and hereditary discrepancy respectively, of any system of \(n\) points and axis-parallel boxes in \(\mathbb{R}^d\).

Tusnády’s Problem

Tusnády’s problem is one of the most well-known problems in the discrepancy theory of geometric set systems. Given a universe \(X\) of points in \(\mathbb{R}^d\), the set system \((X, \mathcal{B}_d)\) generated by axis-parallel boxes is the family of subsets of \(X\) which arise as intersections of \(X\) with the family of all possible \(d\)-dimensional axis-parallel boxes. In Tusnády’s problem the goal is to compute a bound on the discrepancy of any set system having \(n\) points and \(n\) sets. Since then, several more results using further ideas from different areas of mathematics, especially convex geometry and random walks, have been developed.

Previous Results

Combinatorial discrepancy began to be studied intensively in the 80’s following the work of Beck and Fiala [6]. A significant breakthrough was achieved by Spencer [40], who showed that the discrepancy of a system with \(n\) elements and \(n\) sets, is at most \(6\sqrt{n}\). We refer the reader to the books [30] [15] [16] for many more interesting results in the area. In the last decade or so, there have been several breakthroughs in combinatorial discrepancy theory beginning with Bansal’s [2] algorithmization of Spencer’s result. Bansal’s remarkable breakthrough used ideas from random walks, martingale concentration bounds, as well as semidefinite optimization to obtain an algorithmic bound of \(O(\sqrt{n})\) for a set system having \(n\) points and \(n\) sets. Since then, several more results using further ideas from different areas of mathematics, especially convex geometry and random walks, have been developed [26] [3] [1] [38] [35].

In 1980 Tusnády asked whether the discrepancy of points and axis-parallel rectangles in the plane was bounded by a constant. Beck [7] answered this negatively, and also showed an upper bound of \(O(\log^4 n)\). Successive improvements by Beck [8], Bohus [11], Srinivasan [41] and Matoušek [30] resulted in an upper bound of \(O(\log^{d+1/2} n \sqrt{\log \log n})\), which was slightly improved to \(O(\log^{d+1/2} n)\) by Larsen [24]. These remained the best known bounds until Bansal and Garg [4] proved a constructive upper bound of \(O(\log^d n)\), which was non-constructively improved to \(O(\log^{d-1/2} n)\) by Nikolov [35], which is the current best upper bound. The current best lower bound is \(\Omega(\log^{d-1} n)\) by Matoušek, Nikolov and Talwar [31]. This was obtained using the notion of \(\gamma_2\)-factorization norm from Banach space theory, and improved upon a long-standing earlier bound where the exponent of the logarithm was roughly \((d - 1)/2\). Their proof is for the hereditary discrepancy, i.e. they show that
\[
\text{disc}(n, \mathcal{B}_d) \geq \text{herdisc}(n, \mathcal{B}_d) \geq \Omega(\log^{d-1} n).
\]

Observe that the first inequality holds because the set systems \((X, \mathcal{B}_d)\) are defined by taking intersections with objects in \(\mathcal{B}_d\). Suppose \(\text{disc}(n, \mathcal{B}_d) < \text{herdisc}(n, \mathcal{B}_d)\), then there exists a set system \((X, \mathcal{B}_d)\) with \(\text{disc}(X, \mathcal{B}_d) = \text{disc}(n, \mathcal{B}_d)\), and a subset \(Y \subset X\), such that \(\text{disc}(Y, \mathcal{B}_d) > \text{disc}(X, \mathcal{B}_d)\). But then the system \((Y, (\mathcal{B}_d)|_Y)\) has fewer elements than \(X\) and
discrepancy greater than \( \text{disc}(n, B_d) \). Adding \( n - |Y| \) points to \( Y \), without including the new points in any member of \((B_d)_Y\), gives a set system with \( n \) elements and discrepancy > \( \text{disc}(n, B_d) \), which contradicts the fact that \( \text{disc}(n, B_d) \) is the maximum discrepancy of any \( n \)-point set system of points and axis-parallel boxes.

**Discrepancy of Polytopes in \( \mathbb{R}^d \)** Let \( L \) be a fixed set of \( k \) hyperplanes, and let \( POL(L) \) denote the set of all polytopes of the form \( \cap_{i=1}^k P_i \), where \( P_i \)s are halfspaces generated from the hyperplanes in \( L \) via translations. Given a set \( X \) of \( n \) points in \( \mathbb{R}^d \), let \( (X, POL(L)) \) denote the set system formed by taking intersections of \( X \) with all possible polytopes in \( POL(L) \). Matoušek [28] studied the problem of determining the maximum discrepancy of \( n \)-point set systems generated by \( POL(L) \), and proved an upper bound of \( O(\log^{d+1/2} n / \sqrt{\log \log n}) \). This improved to \( O(\log^d n) \) in [4]. Nikolov [35] gave a further improved bound of \( O(\log^{d-1/2} n) \), though non-constructively. The best lower bound known is \( \Omega(\log^{d-1} n) \) given by Nikolov [35] for the case of the set system formed by points and translations and homothetic transformations of a generic polytope.

**Geometric Discrepancy** Much of the early impetus to discrepancy theory came from the area of geometric discrepancy. Here the input is a family \( F \) of Borel sets in \( \mathbb{R}^d \), and the goal is to find an \( n \)-point set which minimizes the maximum deviation of the empirical measure from the Lebesgue measure – or more generally, a given measure \( \mu_d \) – within the unit hypercube \([0,1]^d\). That is, the (normalized) geometric discrepancy of \( n \) points \( x_1, \ldots, x_n \in \mathbb{R}^d \) with respect to the family \( F \) is given by:

\[
G_{\mu_d}(x_1, \ldots, x_n; F) := \sup_{F \in F} \left| \frac{|P \cap F|}{n} - \mu_d(F \cap [0,1]^d) \right|.
\]

Let \( G_{\mu_d}(n, F) \) denote the infimum of \( G(x_1, \ldots, x_n; F) \) over all \( n \)-point sets in \( \mathbb{R}^d \). For the case when \( F \equiv B_d \) and \( \mu_d \) is the Lebesgue measure (which we’ll denote by omitting the subscript \( \mu_d \)), the problem has been investigated since at least the early 1950’s. A fundamental result of Schmidt [39] shows that \( G(n, B_2) = \Theta(n^{-1} \log n) \). In higher dimensions however, there is a significant gap between the lower and the upper bounds. The seminal result of Roth [36] showing that \( G(n, B_d) = \Omega_d(n^{-1} \log^{(d-1)/2} n) \), gives the order of the exponent of the lower bound, as a function of \( d \), although the current best lower bound is that of Bilyk, Lacey and Vagharshakyan [10], who improve upon Roth’s exponent by a small additive function of \( d \), which goes to zero as \( d \to \infty \). The upper bound of \( O(n^{-1} \log^{d-1} n) \) has several proofs, e.g. using the Halton-Hammersley construction [21, 22]. Closing the gap between the upper and lower bounds for \( G(n, B_d) \) has been referred to as the “Great Open Problem” (in geometric discrepancy) by Beck and Chen [9]. Recently, Nikolov [35] and Aistleitner, Bilyk and Nikolov [11] investigated the problem of minimizing the geometric discrepancy of axis-parallel boxes in \( \mathbb{R}^d \), with respect to an arbitrary given measure. Using a well-known transference principle, which basically states that the geometric discrepancy is at most the combinatorial discrepancy (up to constant factors), they showed that there exists an \( n \)-point set in \( \mathbb{R}^d \), such that the geometric discrepancy with respect to a given measure is essentially \( O(n^{-1} (\log n)^{d-1/2}) \).

**Set Systems with Bounded VC dimension; Packing and Chaining** Though Spencer’s result [10] optimally bounds the discrepancy of a general set system (up to constants), it can be far from optimal for specific classes of systems - as for example in the case of Tusnády’s problem. One such class is that for set systems with bounded VC dimension. The optimal upper bound for the discrepancy of such systems by Matoušek [29], is via a decomposition of the set system using the *Packing Lemma* of Haussler [23], coupled with a *chaining* technique, attributed to Kolmogorov. Haussler’s packing lemma, which bounds the maximum size of a family of sets with a given minimum pairwise symmetric difference, was a seminal breakthrough and has been applied in a wide variety of problems in machine learning.
Constructive Discrepancy Minimization: The Bansal-Garg Algorithm The question of finding a low-discrepancy ±1-coloring has been of interest to computer scientists and mathematicians since the early 1980’s. As mentioned earlier, there have been several algorithmic proofs of discrepancy bounds in the past decade, employing a variety of techniques, from semidefinite duality theory, Gaussian random walks and martingale concentration, to ideas in convex geometry and functional analysis. In this paper, we shall focus on one such algorithm – that of Bansal and Garg [4], which seems most suited for proving our bounds. Bansal and Garg, building on a previous work with Dadush [3], proved a general framework for discrepancy minimization of set systems, and used it to improve discrepancy bounds for several problems, including Tusnády’s problem, which they showed has an $O(\log^d n)$ discrepancy. Using the canonical decomposition of Matoušek [28] for the set system of points and axis-parallel boxes (with a slight fine-tuning of the parameters), they showed that a family of protected sets could be chosen at each time step in a way that any axis-parallel box $B$ in $\mathcal{B}_d$ would be not protected with respect to at most $O(\log^{2d-2} n)$ points at any step of the algorithm, and further this set of points would change at most $O(\log n)$ times during the algorithm execution. Thus the number of unprotected points in $B$ would be at most $O(\log^{2d-1} n)$. This bound, together with the above-mentioned $\sqrt{\log n}$-factor for the union bound over all distinct axis-parallel boxes, gives their bound on the discrepancy.

Our Contribution

In this article, we prove constructive upper bounds on Tusnády’s problem. Roughly, our bound can be stated as $O\left(\log^{\max\{d-1,(3d+1)/4\}} n\right)$.

**Theorem 1.1** (Tusnády’s Problem). Let $(X, \mathcal{B}_d)$ be a set system generated by points and axis-parallel boxes in $\mathbb{R}^d$. Then there exists an efficient algorithm that finds a $\{-1,1\}$ coloring of the points in $X$, having discrepancy bounded by

$$\text{disc}(X, \mathcal{B}_d) = O\left(\log^t n\right),$$

where $t = \max\{d-1,(3d+1)/4\}$, and the constant in the $O$-notation depends on $d$.

For small values of $d$, the bounds obtained using Theorem 1.1 can be combined with the lower bound of [4], to get the following corollary.

**Corollary 1.2.** The set system generated by points and axis parallel boxes in $\mathbb{R}^d$ has the following constructive bounds on the discrepancy.

| $d$ | Previous Constructive Bound | Previous Best Bound | New Bound | Lower Bound |
|-----|-----------------------------|---------------------|-----------|------------|
| 2   | $O(\log^2 n)$              | $O(\log^{3/2} n)$  | $O(\log^{7/4} n)$ | $\Omega(\log n)$ |
| 3   | $O(\log^3 n)$              | $O(\log^{5/2} n)$  | $O(\log^{5/2} n)$ | $\Omega(\log^2 n)$ |
| 4   | $O(\log^4 n)$              | $O(\log^{7/2} n)$  | $O(\log^{13/4} n)$ | $\Omega(\log^3 n)$ |
| $\geq 5$ | $O(\log^d n)$            | $O(\log^{d-1/2} n)$ | $O(\log^{d-1} n)$ | $\Omega(\log^{d-1} n)$ |

Thus, for $d \geq 5$, the discrepancy of the set system formed by points and axis-parallel boxes in $\mathbb{R}^d$ is $\Theta\left(\log^{d-1} n\right)$, and there exists an efficient algorithm which can generate a $\{-1,1\}$-coloring with the optimal discrepancy.

1though it may very well be possible to obtain similar bounds by more recent algorithmic frameworks.
Remark Ignoring constant factors dependent on \(d\), for \(d=2\), the bound in Theorem 1.1 improves on the Bansal-Garg constructive bound by an asymptotic factor of \(\log^{1/4} n\), though it is worse than Nikolov’s non-constructive bound of \(\log^{3/2} n\). For \(d \geq 3\), we match or improve upon Nikolov’s bound of \(\log^{d-1/2} n\). For \(d=4\), we get a new bound that is smaller than Nikolov’s by a factor of \(\log^{1/4} n\), though still \(\log^{1/4} n\) off the optimal. For \(d \geq 5\) we get the optimal bound of \(\log^{d-1} n\), which settles Tusnády’s problem.

As a consequence of Corollary 1.2, we also get corresponding bounds on the maximum hereditary discrepancy \(\text{herdisc}(n, B_d)\) of a set system with \(n\)-points and axis-parallel boxes in \(\mathbb{R}^d\), matching the lower bound for \(d \geq 5\).

Corollary 1.3 (Hereditary Discrepancy). For any \(n \in \mathbb{Z}^+\) and \(d \geq 2\),

\[
\text{herdisc}(n, B_d) = O(\log^t n),
\]

where \(t = \max\{d - 1, (3d + 1)/4\}\).

Proof. Fix \(d \geq 2\), and let \(C = C_d > 0\) be the constant in the bound in Theorem 1.1. Suppose for each \(n \in \mathbb{Z}^+\), there exists a set system \((X, B_d) = (X_n, B_d)\) such that the hereditary discrepancy of \((X, B_d)\) is greater than \(C_d \log^t n\). Then by the definition of hereditary discrepancy, it follows that there exists a subset \(Y \subset X\), such that \(\text{disc}(Y, (B_d)_Y) > C_d \log^t n\). However this would mean that the set system \((Y, (B_d)_Y)\) has fewer points than \(n\) while having discrepancy more than \(C_d \log^t n\), which would contradict Theorem 1.1. \(\square\)

Our techniques also translate to the discrepancy of polytopes. Let \(L\) be a fixed set of \(k\) hyperplanes in \(\mathbb{R}^d\), and \(\text{POL}(L)\) be the set of polytopes of the form \(\bigcap_{i=1}^k P_i\), where \(P_i\) are halfspaced generated via translations of the hyperplanes in \(L\). We thus obtain the following.

Theorem 1.4 (Discrepancy of Polytopes). Given a set \(X\) of \(n\) points in \(\mathbb{R}^d\), there exists an efficient algorithm that finds a \([-1, 1]^n\)-coloring of the points in \(X\), such that the discrepancy of any polytopes in \(\text{POL}(L)\) is \(O(\log^{7/4} n), O(\log^{5/2} n), \text{and } O(\log^{13/4} n)\) for \(d = 2, 3, 4\) respectively, and \(O(\log^{d-1} n)\) for \(d \geq 5\), with the constant in the \(O\)-notation depending only on \(d\) and \(k\).

As a simple consequence of Theorem 1.1 together with the transference principle of [1], we also get an improvement in the upper bound for the star discrepancy (i.e. the geometric discrepancy of the family of axis-parallel boxes with one vertex at the origin) of point sets in \(\mathbb{R}^d\) for an arbitrary measure, matching the known upper bound for the uniform measure on \([0, 1]^d\).

Theorem 1.5 (Geometric Discrepancy of Boxes). Given any \(d \geq 1\), \(n \geq 2\), and a normalized Borel measure \(\mu_d\) in \([0, 1]^d\), there exists a set \(\{x_1, \ldots, x_n\} \subset \mathbb{R}^d\) of \(n\) points, such that the geometric discrepancy \(G_{\mu_d}(x_1, \ldots, x_n; B_d)\) satisfies

\[
G_{\mu_d}(x_1, \ldots, x_n; B_d) \leq O\left(\frac{\log^t n}{n}\right),
\]

where \(t := \max\{d - 1, (3d + 1)/4\}\), and the constant in the \(O\)-notation depends on \(d\).

The proof of the above theorem is on the same lines as the proof in [1], where we plug in the bound on the combinatorial discrepancy \(\text{disc}(n, B_d)\) from Theorem 1.1. Thus the proof follows directly from Theorem 1.1 and the transference principle of Aistleitner et al. [1] Theorem 3], after verifying that the condition of [1] Theorem 3] is satisfied since the family \(B_d\) has bounded VC dimension and therefore forms a uniform Glivenko-Cantelli class.
QMC Methods and the Gates of Hell Problem

For \( d \geq 4 \), Theorem \( 1.5 \) improves on the result of Aistleitner, Bilyk and Nikolov \([1]\), and as in their case, has an application in the quasi Monte Carlo method in Numerical Integration. A version of the Koksma-Hlawka inequality, for general measures, proved by Götz \([20]\), shows that given an in the quasi Monte Carlo method in Numerical Integration. A version of the Koksma-

Thus Theorem \( 1.5 \) reduces the minimum approximation error for \( d \)-variate numerical integrals over arbitrary measures on \([0,1]^d\).

In \([1\), Introduction], the authors pose an open problem which they call the Gates of Hell Problem. This problem asks if there exists a measure on \([0,1]^d\) which is harder to approximate by finite atomic measures with equal weights, than the Lebesgue measure. Theorem \( 1.5 \) together with the known upper bound of \( O(n^{-1}\log^{d-1} n) \) for the Lebesgue measure, gives some evidence towards a negative answer to this problem, i.e. seems to indicate that the Lebesgue measure is the hardest measure to approximate. Of course it could be that both upper bounds are off the mark, so we are far from a complete resolution of this problem.

Overview of the Proof of Theorem \( 1.1 \)

We briefly summarize the ideas in the proof of Theorem \( 1.1 \). Broadly, the plan is to build upon a well-known partition tree-like decomposition (sometimes referred as a dyadic decomposition) of the set system formed by points and axis-parallel boxes in \( \mathbb{R}^d \), by conducting a finer analysis of some of the leaf-level sets, using the notion of shallow cell complexity and the Shallow Packing Lemma. Then, the new decomposition is used in a constructive discrepancy minimization algorithm, such as the Bansal-Garg algorithm (with a slight extension of its analysis).

Several previous bounds, such as by Matoušek \([28]\), and Bansal and Garg \([4]\), were based on a decomposition of the set system of points and axis-parallel boxes, which can be thought of as initially constructing a binary partition tree on the point set according to the \( x_1 \)-coordinates of the points, followed by further partitioning the subset corresponding to each node of this tree, along the \( x_2 \)-axis, and so on recursively up to the \( x_d \)-axis. Thus we get a tree of partition trees, having \( d \) levels, which has \( x_i \)-leaf nodes corresponding to each coordinate axis \( x_i \), \( i \in [d] \). Each \( x_i \)-leaf corresponds to a subset of points of \( X \). Using this decomposition, each axis-parallel box in the set system can be expressed as the disjoint union of canonical boxes, which correspond to nodes in the recursive tree structure, and \( x_i \)-error sets, which correspond to intersections of boxes with \( x_i \)-level canonical boxes.

It is easy to see that a typical axis-parallel box \( B \in \mathcal{B}_d \) intersects with \( O(\log^{d-1} n) \) \( x_i \)-leaf level boxes, each of which has size \( O(\log^{d-1} n) \). Therefore the number of points in the \( x_i \)-error sets of \( B \) is \( O(\log^{d+1-2} n) \). Thus the main contribution to the unprotected sets of \( B \) comes from \( x_d \)-error sets.

Our technique is to consider the family of all possible \( x_d \)-error sets, and use the Shallow Packing Lemma (Mustafa \([32]\) to obtain a finer decomposition. We show that the only possible subsets of points which can arise as \( x_d \)-error sets, correspond to taking the set difference of a pair of almost-fixed boxes. An almost-fixed box \( A \) is an axis-parallel box contained in an \( x_d \)-leaf \( L \), whose which shares the \( x_1, \ldots, x_{d-1} \)-boundaries of \( L \), together with one \( x_d \)-boundary of \( L \). Thus the only variable parameter in an almost-fixed box is choosing its other \( x_d \)-boundary, which clearly has complexity linear in \( |L| \). As we formally prove in Section \([4]\), this system has small shallow cell complexity \( \psi(n,k) = O(k) \). This allows us to use the Shallow Packing Lemma, together with the chaining technique of Kolmogorov,
to obtain a further decomposition with smaller canonical sets and leftover error sets of size \( O(\log^{d-1} n) \), i.e. \( \ll \log^{d-1} n \). Finally we run the Bansal-Garg algorithm on the set system with this decomposition (this requires an extension of their analysis, discussed in Section 2), which has significantly fewer corrupted points inside a given box, thus giving an improved constructive bound on the discrepancy.

Outline

The rest of this paper is organized as follows. In Section 2, we revisit the Bansal-Garg algorithm and state and prove a slight extension of their analysis. In Section 3, we revise the decomposition of Matoušek. In Section 4, we review the concepts of shallow cell complexity and the shallow packing lemma, together with an application of the chaining method, and put everything together to complete the proof of Theorem 1.1.

2 The Bansal-Garg Algorithm and its Extension

In this section, we shall describe the Bansal-Garg algorithmic framework for discrepancy minimization, and state and prove a slight extension of their main theorem. We shall show that under the Bansal-Garg algorithm, the discrepancy of weighted linear combinations of rows also has an almost-subgaussian distribution.

2.1 Overview of the Bansal-Garg Algorithm

As mentioned the Introduction, the problem of discrepancy minimization takes as input an \( m \times n \) matrix \( M \) with entries \( |a_{ij}| \leq 1 \), and seeks to obtain a coloring \( x \in \{-1, 1\}^n \) which minimizes \( \|Mx\|_\infty \). The regime of interest for us is when \( m \leq n^{c_0} \) is polynomial in \( n \) (i.e. \( c_0 \) is a constant independent of \( m, n \)). In essence, the Bansal-Garg algorithm works by relaxing the domain of the coloring from the vertices of the unit hypercube in \( \mathbb{R}^n \), to its entire interior i.e. \([-1,1]^n\), and simulating a time-discretized Brownian motion in this domain, where at each time step, the random increment vector is constructed in a way that its covariance matrix satisfies a certain matrix semi-definiteness condition.

Let \( J := [m] \times 2^{[n]} \) denote the set of ordered pairs of rows and subsets of columns of \( M \). Given a pair \((j,S) \in J\) define the vector \( a_{j,S} \in \mathbb{R}^n \) to be

\[
(a_{j,S})_i = \begin{cases} a_{ji}, & i \in S, \\ 0, & i \not\in S, \end{cases}
\]

In one step, the algorithm can change color only for those columns of \( M \) which are not close to \( \pm 1 \), i.e., only for the alive columns, where a column \( i \in [n] \) is alive at time \( t \) if \( |x_i(t)| < 1 - 1/n \), and frozen otherwise. Let \( N(t) := \{i \in [n] : x_i(t-1) < 1 - 1/n\} \) denote the set of alive columns before the \( t \)-th step. Let \( A(t) \subset N(t) \) denote the active columns of \( M \) whose color is to be updated at time \( t \). The algorithm initially assigns the coloring \( x(0) = 0 \in \mathbb{R}^n \). At the \( t \)-th time step, the coloring is updated from \( x(t-1) \) to \( x(t) \), by adding a suitably scaled random update vector:

\[
x(t) = x(t-1) + \gamma U(t) g(t).
\]

Thus the update vector is \( \gamma U(t) g(t) \), where \( U(t) \) is an \( n \times n \) matrix, \( g(t) \in \{-1, 1\}^n \) is a random vector obtained by choosing each coordinate to be \( \pm 1 \) independently with equal probability, and \( \gamma = n^{-\Omega(1)} \) is a suitably small scaling factor. The existence and constructibility of the matrix \( U(t) \) follows from the Universal Vector Coloring theorem below, which was proved in [4]. To describe the theorem, we introduce some notation first.
Protection and Corruption  A key feature of the Bansal-Garg algorithm is the notion of protection and corruption, which we now explain. Given a fractional coloring \( x \in [-1,1]^n \) and a pair \((j,S) \in J\), define the signed discrepancy \( \text{disc}(x,j,S) \) as \( \text{disc}(x,j,S) := \langle a_{j,S}, x \rangle \), i.e. \( \sum_{i \in S} a_{ji} x_i \). A zero-constraint at time \( t \) is a vector \( u \in \mathbb{R}^n \) such that the change in the signed discrepancy of the update vector at time \( t \), i.e. \( \langle u, r(t) \rangle \) is constrained to be zero. The set of zero constraints at time \( t \) is specified before the update at step \( t \) occurs, and is denoted by \( Y(t) \), having cardinality \( l := |Y(t)| \).

Given a column \( k \in [n] \) of \( M \), a pair \((j,S) \in J\), and a time step \( t \), \( k \) is protected at time \( t \) with respect to \((j,S)\), if and only if there exist zero-constraints \( u_1, \ldots, u_s \in Y(t) \), such that for each \( i \in [s] \), and each \( i' \notin S \), \( (u_i)_{i'} = 0 \), and \( \sum_{i=1}^{s} (u_i)_{k} = a_{jk} \). \( k \) is corrupted with respect to \((j,S)\) if there existed any time \( t \) at which \( k \) was not protected with respect to \((j,S)\). Let \( C_{j,S} \) denote the set of corrupted elements for the row \( j \) restricted to the columns in \( S \), after the \( T \)-th step of the Bansal-Garg algorithm.

The Universal Vector Coloring Theorem can be now stated as follows.

**Theorem 2.1** (Theorem 6, Universal Vector Coloring [4]). Given an \( m \times n \) matrix \( M \) with entries \(|a_{ji}| \leq 1\), a set \( Y(t) \) of zero-constraints with \( |Y(t)| = \delta n \), where \( \delta \in (0,1) \), and \( \beta \in (0,1-\delta) \), there exists an efficient algorithm to obtain a \( n \times n \) matrix \( U \) with columns \( u_1, \ldots, u_n \in \mathbb{R}^n \), such that

1. For each vector \( v_k \in Y(t) \), \( \sum_{i \in [n]} (v_k)_i u_i = 0 \).
2. For any vector \( w \in \mathbb{R}^n \), \( \|\sum_i w(i) u_i\| \leq \frac{1}{\beta} \sum_i w(i)^2 \|u_i\|^2 \).
3. For each \( i \in [n] \), \( \|u_i\| \leq 1 \) and \( \sum_i \|u_i\|^2 \geq (1-\delta-\beta)n \).

In order to get a \( \pm 1 \) coloring, we need to ensure that for our choice of \( T \), with high probability, none of the elements of the ground set \( X \) are alive after \( T \) steps, and that the rounding off error is small. This is ensured by showing that with high probability, for each \( |x_i(T)| \in (1-1/n,1) \), and can therefore be easily rounded off without incurring any significant error in the discrepancy. This essentially follows from Theorem 2.1(iii), together with elementary bounds on the step variance of the random walk, and is shown in lemmas 17-19 in [4], which we state below for reference.

**Lemma 2.2** ([4](Lemma 17)). For each time step \( t \), \( \sum_i \|u_i(t)\|^2 \geq (1-\delta-\beta)|A(t)| \), where \( u_i(t) \) is the \( i \)-th column of the matrix \( U(t) \) in [1].

**Lemma 2.3** ([4](Lemma 19)). After \( T = \frac{6 n \log n}{(1-\delta-\beta)^2} \) time steps, there are no alive variables left with probability at least \( 1-o(1) \).

Subgaussianity Factor of Discrepancy  The main technical result in the analysis of Bansal and Garg was that for any given pair \((j,S) \in J\), the final signed discrepancy resulting from their algorithm has a nearly subgaussian tail, with subgaussianity factor at most \( C_{j,S} \).

**Theorem 2.4** (Bansal-Garg [4]). There is a constant \( c > 0 \) such that given \( \delta \in (0,1) \), an \( m \times n \) matrix \( M \) with entries \(|a_{ij}| \leq 1\), and a set \( Y(t) \) of user-determined zero-constraints such that for all \( 0 \leq t \leq T \), \( |Y(t)| \leq \delta n \), then for any \( j \in [m] \), \( S \subset [n] \) and any \( \lambda > 0 \), the coloring \( x \in \{-1,1\}^n \) returned by the Bansal-Garg algorithm satisfies

\[
\mathbb{P} \left[ \text{disc}(x,j,S) \geq c \lambda \left( \lambda + \left( \sum_{i \in C_{j,S}} a_{ji}^2 \right)^{1/2} \right) \right] \leq 2 \cdot \exp \left( -\lambda^2/2 \right).
\]
2.2 Extension to Linear Combinations of Pairs

The concentration bound of Bansal and Garg (Theorem 2.4) applies to any given vector \( a_{j,S} \), corresponding to a pair \((j,S) \in J\). Consider a generalization, where we have a linear combination \( \sum_{i} \rho_{i} a_{j_{i},S_{i}} \) of such vectors with a given expansion

\[
\sum_{i} \rho_{i} a_{j_{i},S_{i}} = \sum_{k=0}^{k_{\text{max}}} \lambda_{k}(t) a_{j_{k}(t),S_{k}(t)},
\]

where \( \lambda_{k}(t), j_{k}(t), S_{k}(t) \) are allowed to change during the execution of the algorithm, while satisfying \( \| \), and \( k_{\text{max}} \) is polynomial in \( n,m \). In this setting it is not clear a priori that a similarly subgaussian tail bound still exists. However, another crucial observation of Bansal and Garg was that for their algorithm the signed discrepancy \( \text{disc}(x(t),j,S) \) is essentially a supermartingale with respect to the squared \( \ell_{2} \) norm of the corrupted columns in \( S \). Therefore, we can hope that if for each \( t \geq 0 \), the linear combination for the \( t \)-th step is determined before the step occurs, then a similar result might still hold.

The following theorem shows that this is indeed the case. Let \( \mathbf{w} := \sum_{i} \rho_{i} a_{j_{i},S_{i}} \) be a weighted linear combination of vectors corresponding to pairs in \( J \), whose discrepancy at time \( t \) is given by the expansion

\[
\text{disc}_{\mathbf{w}}(t) := \sum_{i} \rho_{i} \text{disc}(x(t), j_{i}, S_{i}) = \sum_{k=0}^{k_{\text{max}}} \lambda_{k}(t) \text{disc}(x(t), j_{k}(t), S_{k}(t)),
\]

where \( k_{\text{max}} \leq n^{c'} \), the coefficients \( \lambda_{k}(t) \) and the pairs \( (j_{k}(t), S_{k}(t)) \) are a function of \( M \) and \( x(t-1) \). Assume that the coefficients \( \lambda_{k}(t) \) are in \([-1,1]\).

Let \( K := \bigcup_{t,k} (\lambda_{k}(t), C_{j_{k}(t),S_{k}(t)}) \) be the set of pairs \((\lambda_{k}(t), C_{j_{k}(t),S_{k}(t)})\) corresponding to terms \( \lambda_{k}(t) \text{disc}(x(t), j_{k}(t), S_{k}(t)) \), which appear in the expansion of \( \text{disc}_{\mathbf{w}}(t) \) at some time \( t \) during the course of the algorithm. Define \( B := \sum_{K} \lambda_{k}^{2}(t) \left( \sum_{i\in C_{j_{k}(t),S_{k}(t)}} a_{j_{k}(t),i}^{2} \right) \).

**Theorem 2.5.** Given \( c' > 0, \delta \in (0,1) \), an \( m \times n \) matrix \( M \) with entries \( |a_{ij}| \leq 1 \), a set \( Y(t) \) of user-determined zero-constraints satisfying \( |Y(t)| \leq \delta n \) for each \( 0 \leq t \leq T \) and an arbitrary linear combination of vectors \( \mathbf{w} := \sum_{i} \rho_{i} a_{j_{i},S_{i}} \) with a linear expansion \( \mathbf{w} = \sum_{k=0}^{k_{\text{max}}} \lambda_{k}(t) a_{j_{k}(t),S_{k}(t)} \), as a function of the state, at time \( t \), of the Bansal-Garg algorithm run on \( M \), where the coefficients \( \lambda_{k}(t) \in [-1,1] \) and \( k_{\text{max}} \leq n^{c'} \), there is a constant \( c > 0 \) such that the coloring \( x \in \{-1,1\}^{n} \) returned by the algorithm satisfies

\[
\mathbb{P} \left[ |\text{disc}_{\mathbf{w}}(T)| \geq c\lambda \left( \lambda + B^{1/2} \right) \right] \leq 2 \cdot \exp\left(-\lambda^{2}/2\right).
\]

**Remark** The above theorem shows for instance that the Bansal-Garg algorithm generates an output that is subgaussian for any vector in \( \mathbb{R}^{m} \): given a vector \( v = \sum_{i=1}^{m} v_{i} \tilde{e}_{i} \in \mathbb{R}^{m} \), where \( \tilde{e}_{i}, 1 \leq i \leq m \) are the unit vectors along the coordinate axes in \( \mathbb{R}^{m} \), define \( \mathbf{w} = \sum_{i=1}^{m} \rho_{i} a_{i,S_{i}} \), where \( S_{i} = \{ i \} \), and \( \rho_{i} = \lambda_{i} = \frac{v_{i}}{\sum_{i=1}^{m} (v_{i}/a_{i})^{2}}^{1/2} \). Applying Theorem 2.5 now gives a subgaussian distribution for \( \text{disc}_{\mathbf{w}}(t) \).

**Proof of Theorem 2.5.** We’ll closely follow the proof of Bansal and Garg [1], [Theorem 1], which involves the following lemmas. By the symmetry of \( \text{disc}_{\mathbf{w}}(T) \) about the origin, it suffices to focus only on upper-bounding the upper tail, i.e. \( \mathbb{P} \left[ \text{disc}_{\mathbf{w}}(T) \geq c\lambda \left( \lambda + B^{1/2} \right) \right] \).

**Lemma 2.6 ( [1], Theorem 21, Theorem 1).** Given \( b \in (0,1), B \geq 0 \) and sequences of random variables \((U_{i}, V_{i})_{i=0}^{T} \) with \( U_{0} = V_{0} = 0 \) and \( V_{i} \in [0,B] \) for all \( 0 \leq t \leq T \), suppose that for any \( a \in (0,1/3) \), \( Y_{t} := U_{t} - abV_{t} \) satisfies the following for all \( t \in [T] \):

\[
(i) \quad |Y_{t} - Y_{t-1}| \leq 1, \quad \text{and}
\]

9
(ii) $\mathbb{E}_{t-1} [Y_t - Y_{t-1}] \leq -a \mathbb{E}_{t-1} [(Y_t - Y_{t-1})^2]$. 

Then for any $\lambda \geq 0$,

$$
\mathbb{P} \left[ U_T \geq c\lambda (\lambda + B^{1/2}) \right] \leq \exp \left( -\lambda^2 / 2 \right),
$$

for any $c \geq \max\{\sqrt{b}, 5b / 2\}$. 

Given a pair $(j', S') \in J$ and a time $t$, let $C_{j', S'}(t)$ denote the set of columns in $[n]$ that are not protected for $(j', S')$ at some time $t' \leq t$, and $\Delta C_{j', S'}(t)$ be the columns that are not protected for $(j', S')$ at time $t$. For a column $i \in [n]$, let $t_i$ be the last time step when $i$ is protected for $(j', S')$. Define the energy of $(j', S')$ at time $t$ to be $V_{j', S'}(t) := \sum_{i \in C_{j', S'}(t)} a_{ji}^2 |x(t)_i^2 - x(t_i)_i^2|$. Let $\text{disc}_{j', S'}(t) := \text{disc}(x(t), j', S')$. Then the following statements can be easily seen from the definitions of the respective terms.

(i) $\text{disc}_{j', S'}(t) - \text{disc}_{j', S'}(t-1) = \gamma \langle g_t, \sum_{i \in S'} a_{ji}^2 u_i(t) \rangle = \gamma \langle g_t, \sum_{i \in \Delta C_{j', S'}(t)} a_{ji}^2 u_i(t) \rangle$.

(ii) $V_{j', S'}(t) - V_{j', S'}(t-1) = \gamma^2 \sum_{i \in C_{j', S'}(t)} a_{ji}^2 \langle g_t, u_i(t) \rangle^2 + 2 \gamma \langle g_t, \sum_{i \in C_{j', S'}(t)} a_{ji}^2 x_i(t - 1) u_i(t) \rangle$.

(iii) $\mathbb{E}_{t-1} [\text{disc}_{j', S'}(t) - \text{disc}_{j', S'}(t-1)] = 0$.

(iv) $\mathbb{E}_{t-1} [V_{j', S'}(t) - V_{j', S'}(t-1)] = \gamma^2 \sum_{i \in C_{j', S'}(t)} a_{ji}^2 \| u_i(t) \|^2$.

Next, let $Y_{j', S'}(t) = \text{disc}_{j', S'}(t) - abV_{j', S'}(t)$, where $a, b \geq 0$ such that $ab \leq 1 / 5$, are parameters to be set later. Consider the given linear combination of pairs, at time $t$: $\sum_k \lambda_k(t) \text{disc}_{jk(t), Sk(t)}(t)$. The corresponding energy terms are given by $\sum_k \lambda_k^2(t) V_{jk(t), Sk(t)}(t)$. To simplify notation, we’ll denote $\lambda_k(t), j_k(t), S_k(t)$ by $\lambda, j_k, S_k$ respectively.

Define $Y_w(t) := \sum_k (\lambda_k \text{disc}_{jk, Sk}(t) - ab\lambda_k^2 V_{jk, Sk}(t))$, and let $\Delta Y_w(t) := Y_w(t) - Y_w(t - 1)$. We have

$$
\mathbb{E}_{t-1} [\Delta Y_w(t)] = -ab \sum_k \lambda_k^2 V_{jk, Sk}(t).
$$

$$
\mathbb{E}_{t-1} [((\Delta Y_w(t))^2)] = \gamma^2 \left\| \sum_k \left( \lambda_k \sum_{i \in \Delta C_{jk, Sk}(t)} a_{ji}^2 u_i(t) - 2a\lambda_k^2 \sum_{i \in C_{jk, Sk}(t)} a_{ji}^2 x_i(t - 1) u_i(t) \right) \right\|^2 + O(\gamma^3 n^{2c^3 + 4})
$$

$$
= \gamma^2 \left\| \sum_k \left( \sum_{i \in \Delta C_{jk, Sk}(t)} \lambda_k a_{ji}^2 (1 - 2a\lambda_k x_i(t) - 1) u_i(t) \right) \right\|^2 + O(\gamma^3 n^{2c^3 + 4})
$$

$$
\leq \frac{b\gamma^2}{2} \sum_k \left( \sum_{i \in \Delta C_{jk, Sk}(t)} \lambda_k^2 a_{ji}^4 (1 - 2ab\lambda_k x_i(t) - 1)^2 \| u_i(t) \|^2 \right)
$$

$$
+ 4a^2 b^2 \sum_{i \in \Delta C_{jk, Sk}(t)} \lambda_k^4 a_{ji}^4 x_i^2(t - 1) \| u_i(t) \|^2 + O(\gamma^3 n^{2c^3 + 4})
$$

$$
\leq b\gamma^2 \sum_k \sum_{i \in \Delta C_{jk, Sk}(t)} \lambda_k^2 a_{ji}^2 \| u_i(t) \|^2 + O(\gamma^3 n^{2c^3 + 4}),
$$

(3)
where in step (3) we applied Theorem 2.1 on the matrix $M$, with $\beta = 2/b$, $\delta = \delta$, and zero-constraints $Y(t)$, and in the final step (4) we used that $(1 - 2ab\lambda_kx_2^2(t-1))^2 \leq 2$, since $\lambda_k$ and $x_2^2(t-1)$ are each at most 1 and $ab \leq 1/5$. Ignoring the $O(\gamma^3n^{2\epsilon^4})$ term, since it can contribute at most $O(T^\gamma n^{2\epsilon^4}) = o(1)$ to the total variance, we get that $\mathbb{E}_{t-1}[(\Delta Y_w(t))^2] \leq b_k \lambda_k^2 V_{jk,Sk}(t)$. Therefore,

$$\mathbb{E}_{t-1}[\Delta Y_w(t)] = -a \left( b_k \sum_{k} \lambda_k^2 V_{jk,Sk}(t) \right) \leq -a \mathbb{E}_{t-1}[(\Delta Y_w(t))^2].$$

We can now apply Lemma 2.6 with $U_t = \text{disc}_w(t)$, $V_t = \sum_k \lambda_k^2 V_{jk,Sk}(t)$, and $K$ and $B$ as defined in the premise of the theorem, to complete the proof.

Using the above theorem, one can prove a general bound on the discrepancy of points and axis-parallel boxes, for a more general class of decompositions involving set unions and differences.

**Lemma 2.7.** For the set system $(X, B_d)$ of $n$ points and axis-parallel boxes in $\mathbb{R}^d$, suppose that there exists

(a) a collection $T = T_X$ of at most $n/8$ subsets of $X$, together with

(b) another family $D$ of at most $n^{2d\epsilon'}$ sets,

with each member of the collections $T$ and $D$ being a subset of some $B \in B_d$, such that each axis-parallel box in $B_d$ can be expressed as a combination of set unions and differences, consisting of (i) a disjoint union of sets in $T_X$, together with (ii) at most $n^{\epsilon'}$ sets from $D$, whose union contains at most $k$ points, then the discrepancy of $(X, B_d)$ is at most

$$\text{disc}(X, B_d) = O\left(\sqrt{k} \cdot \log n\right).$$

**Proof of Lemma 2.7.** We’ll proceed as in the proof of Theorem 2 in [1], plugging in our general assumption on the existence of a decomposition of the boxes in $B_d$. Briefly, in the proof in [1], the idea is to consider the incidence matrix $M$ of the set system $(X, B_d)$, and apply Theorem 2.4 ensuring that the set of zero-constraints always satisfies the conditions of the theorem such that the number of corrupted elements in a given axis-parallel box throughout the running of the algorithm, is at most $k^2 \log n$. The matrix $M$ for us will be the same, i.e. the incidence matrix of $(X, B_d)$.

The main difference now is that we no longer have a decomposition for the sets in our set system. Instead, each set in $B_d$ is generated by a Boolean combination involving unions and differences of sets from the families $T$ and $D$. Note that since each set $D \in T \cup D$ is a subset of some box in $B_d$, the indicator vector of $D$ can be expressed as a subset of columns of some row in $M$. Therefore, we can apply Theorem 2.5 instead of Theorem 2.4 with $\lambda_k = 1$ for set unions and $\lambda_k = -1$ for set differences. We choose $\delta = 1/4$.

Let us verify that the set $Y(t)$ of zero-constraints satisfies $|Y(t)| \leq \delta n$ throughout the course of the algorithm. Recall the relevant definitions from Section 2.1. For all $0 \leq t \leq T$, we set $A(t) = N(t)$. The constraints in $Y(t)$ are now fixed as follows. At a given time step $t$, let $m_t$ be such that $m_t^{n/4m_t} \leq |N(t)| \leq m_t^{n/4m_t}$, and let $t_1 \leq t$ be the first time at which the event $|N(t)| \leq n^{n/4m_t} \delta n$ occurred. Set $Y(t)$ to be the set of indicator vectors of the subsets in $T_{N(t)}$. This can be done, since we have that

$$|Y(t)| = |T_{N(t)}| \leq \frac{|N(t_1)|}{8} \leq \frac{n}{8} \cdot \frac{n}{2m_t} = \frac{n}{4} \cdot \frac{n}{2m_t+1} \leq \frac{|N(t)|}{4},$$

where $m_t$ is such that $m_t^{n/4m_t} \leq |N(t)| \leq m_t^{n/4m_t}$. If
Next, we need to bound the parameter $B$ in Theorem 2.3. Fix a box $B_1 \in B_d$. By the premise of the lemma, at any time step $t$, $B_1$ is a Boolean combination of set unions and differences involving sets from $\mathcal{T}$ and at most $k$ elements from $\mathcal{D}$. Since the sets in $\mathcal{T}$ are protected, the corrupted elements in the expansion $\sum_k \lambda_k(t) a_{ji}(t), S_k(t)$, come only from the sets in $\mathcal{D}$. Since $\lambda_k = \pm 1$ and $a_{ji} \in \{0, 1\}$, the contribution to $B$ of all such elements is simply the number of elements in the expansion, which is at most $k$ by the assumption in the lemma. Since $Y(t)$ changes at most $O(\log n)$ times during the execution of the algorithm, we get that the parameter $B$ is at most $O(k \log n)$.

Taking $\lambda = \sqrt{4d \log n}$, we get that the discrepancy of the box $B_1$ is at most $O\left(\sqrt{k \lambda \log n}\right)$ with probability at least $1 - 1/n^{2d}$. Taking a union bound over all $O(n^{2d})$ possible distinct boxes $B \in B_d$ gives us the statement of the lemma. \hfill \square

3 Points and Axis-Parallel Boxes

In this section we shall see how each box arising in a set system of points and axis-parallel boxes in $\mathbb{R}^d$, can be decomposed into a collection of canonical intervals, and study some properties of the decomposition.

3.1 Canonical Intervals

The idea is to construct a recursive tree structure of canonical intervals over the coordinate axes. We first build a binary tree $T$ of intervals over the $x_1$ axis. For each node $v$ in $T$, we'll construct a tree $T_v$ of $x_2$-intervals. For each node $v' \in T_v$, there’ll be a tree $T_{v,v'}$ of $x_3$ intervals, and so on. The size of the leaf-level intervals of a tree on $x_i$ will be determined by the parameters $l_i$, which are fixed as follows. For $i \in [d] \setminus \{d - 1\}$, let us fix $l_i = 32(d + \log n)^{d-1}$ and let $l_{d-1} := 32(d + \log n)^{d-2}$.

The $x_1$-interval tree of $X$ can now be constructed as follows. Let $p_1, \ldots, p_n$ be the points of $X$ ordered according to increasing values of their $x_1$-coordinates. The root node or level zero canonical interval along the $x_1$-axis is $[p_1, p_n]$, and corresponds to the subset of points having indices in $[1, n]$. For $i = 0, 1, \ldots, \lfloor \log(n/l_1) \rfloor$, given the collection of nodes at level $i$, the canonical intervals at level $i + 1$ are created by splitting each level-$i$ interval $[a, b]$ into the sub-intervals $[a, \lfloor \frac{a+b}{2} \rfloor]$ and $[\lfloor \frac{a+b}{2} \rfloor + 1, b]$, which are set to be the children of the node $[a, b]$. The points corresponding to an interval $I$ are the points having their $x_1$-coordinates in $I$. The leaf-level canonical $x_1$-intervals, or $x_1$-leaves, are the intervals having at most $l_1$ points in them.

Next, for each canonical $x_1$-interval $I$ at level $i_1$, construct an $x_2$-interval tree $T_i$ for the points in the interval $I$ by partitioning them according to increasing values of their $x_2$-coordinates. $T_i$ has root at level $i_1$, and $x_2$-leaves with at most $l_2$ points in them. Continuing in this way for each coordinate $j \in [d - 1]$, for each canonical $(x_1, \ldots, x_j)$-interval at levels $(i_1, \ldots, i_j)$, construct an $x_{j+1}$-interval tree $T_{i_1, \ldots, i_j}$ with root at level $i_j$ and $x_{j+1}$-leaves with at most $l_{j+1}$ points.

Decomposition. Now we shall see how any box $B \in B_d$ can be expressed as the disjoint union of products of canonical intervals together with at most $O(\log^{i-1} n)$ $x_i$-error sets, for $i = 1, \ldots, d$, which are intersections of $x_i$-leaves with $B$.

Indeed, given $B \in B_d$, $B$ can be written as the product of intervals $\prod_{i=1}^d [a_i, b_i] = [a_1, b_1] \times B$, where $B \subset \mathbb{R}^{d-1}$. Let $I_1$ be the minimum-level canonical $x_1$-interval, having exactly one end-point $c_1 \in [a_1, b_1]$. Split $B$ into the boxes $B_1 := [a_1, c_1] \times B$ and $B_2 := [c_1, b_1] \times B$. We recursively split each box $B_1$, $B_2$ in this manner, stopping a branch of the recursion whenever we obtain a canonical $x_1$-interval completely contained in $[a_1, b_1]$. (If $|B \cap X| < l_1$, it may be that $[a_1, b_1]$ is contained in a single canonical $x_1$-leaf, and in this
case we would stop before making any split). We thus obtain, in at most $O(\log n)$ levels of recursion, a decomposition of $B$ along the $x_1$-coordinate into intervals $J_1, \ldots, J_t$, all of which, with the possible exception of $J_1$ and $J_t$ are canonical intervals along the $x_1$-axis. Let $C_{B,x_1}$ denote the set of boxes corresponding to the canonical $x_1$-intervals $J_2, \ldots, J_{t-1}$. Call the collection of boxes $L_{B,x_1}$ corresponding to the intervals $J_1$ and $J_t$ as $x_1$-error sets. (If $[a_1, b_1]$ lies in a single $x_1$-leaf, then $L_{B,x_1} := \{B\}$ and $C_{B,x_1} = \emptyset$).

Next, we split each box in $C_{B,x_1}$ along the $x_2$-coordinate, to obtain a set of $O(\log n)$ disjoint boxes. Let $C_{B,x_2}$ denote the union of these sets of boxes. Observe that each $x_1$-canonical box in $C_{B,x_1}$ contributes $O(\log n)$ boxes to $C_{B,x_2}$, and at most 2 boxes that have canonical intervals in the $x_1$-coordinate, but non-canonical along the $x_2$-coordinate. Call these $x_2$-error sets, and let the collection of $x_2$-error sets be denoted by $L_{B,x_2}$. It follows that $|C_{B,x_2}| = O(\log^2 n)$, and $|L_{B,x_2}| = O(\log n)$.

In this manner for $j = 1, \ldots, d - 1$, we recursively split each collection of boxes $C_{B,x_j}$ along the coordinate axes $x_{j+1}$. obtaining the families of boxes $C_{B,x_{j+1}}$ and $L_{B,x_{j+1}}$, with $|C_{B,x_{j+1}}| = O(\log^{j+1} n)$ and $|L_{B,x_{j+1}}| = O(\log^j n)$. Finally we get a collection $C_{B,x_d}$ of at most $O(\log^d n)$ boxes, which are our desired canonical boxes. We note the following prop-
erties of the boxes in $C_{B,x_d}$.

**Properties of $C_{B,x_d}$**

(i) Each box in $C_{B,x_d}$ arises uniquely as a product of canonical intervals.

(ii) Each box $S \in C_{B,x_d}$ lies completely in $B$, since $S$ is a product of canonical intervals which are contained in in the corresponding intervals of $B$.

Similarly, for each $i = 1, \ldots, d$, the collection $L_{B,x_i}$ satisfies the following properties.

**Properties of $L_{B,x_i}$, $i \in [d]$**

(i) $|L_{B,x_i}| = O\left(\frac{n}{\log^{d-1} n}\right)$.

(ii) For each $i \in [d]$, each $x_i$-error set in $L_{B,x_i}$ is a box which is a product of canonical intervals along the $x_1, \ldots, x_{i-1}$-axes, and not necessarily canonical intervals in the coordinates $x_i, \ldots, x_d$.

**Number of Canonical Boxes and Elementary Error Sets**

Let $\mathcal{C} = \bigcup_{B \in \mathcal{B}_d} C_{B,x_d}$ denote the collection of all canonical boxes in $\mathcal{B}_d$, and let $\mathcal{L}_i = \bigcup_{B \in \mathcal{B}_d} L_{B,x_i}$ denote the collection of all $x_i$-error sets, $i \in [d]$. The size of the canonical collection $\mathcal{C}$ can be easily bounded as follows. Observe that for any fixed choice of levels $(i_1, \ldots, i_{d-1})$ the total number of canonical $x_d$-leaves is at most

$$\frac{n}{2^{i_1}} \cdot \frac{2^{i_2}}{2^{i_3}} \cdots \frac{2^{i_{d-1}}}{l_d} = \frac{n}{l_d}.$$  

Since each such $x_d$-interval tree is a binary tree, the number of canonical boxes is the number of nodes of the tree, which is at most twice the number of leaves, i.e. $\frac{2n}{l_d}$. Now choosing the first $d-1$ levels of the $x_d$-interval tree in $\prod_{i=1}^{d-1} (d + \log(n/l_i)) = O\left(\log^{d-1} n\right)$ ways, gives us that the total number of canonical boxes is at most $\frac{n}{l_d} \cdot (d + \log n)^{d-1} \leq n/16$.

From the above discussion, we can conclude the following general properties about boxes in $\mathcal{B}_d$, implicit in Matoušek’s decomposition.

**Lemma 3.1.** Given any set $X$ of $n$ points in $\mathbb{R}^d$, there exists (i) a collection $\mathcal{C}$ of canonical axis-parallel boxes in $\mathcal{B}_d$, and (ii) for each $j = 1, \ldots, d$, a collection of $x_j$-error sets $\mathcal{L}_j$, such that any axis-parallel box $B \in \mathcal{B}_d$ can be expressed as the disjoint union of boxes in $\mathcal{C}$, together with a set of at most $O\left(\log^{i-1} n\right)$ boxes from $\mathcal{L}_i$. Further, the boxes in $\mathcal{L}_j$, satisfy the following properties.

(i) Each error set is a box contained in an $x_j$-leaf, for $i \in [d]$.

(ii) For any error set $B$ contained in an $x_j$-leaf node $v$, for $1 \leq j \leq i-1$, the $x_j$-boundaries of $B$ coincide with the $x_j$-boundaries of $v$.

(iii) For any given $B \in \mathcal{B}_d$ and $j \in [d]$, $|L_{B,x_j}| = O\left(\log^{i-1} n\right)$.

(iv) $|C| \leq \frac{n}{16}$.

**4 Shallow Packings and Matoušek’s Decomposition**

Let us describe the main ideas in the proof, in somewhat greater detail. In the decomposition of Matoušek, it can be observed that the dominant contribution to the unprotected set of any axis-parallel box comes from a collection of $O\left(\log^{d-1} n\right)$ $x_d$-error sets (see definition in Section 3). Briefly, each such set is a subset of the points represented by a leaf-level node of an $x_d$-interval tree. Since an $x_d$-leaf has at most $O\left(\log^{d-1} n\right)$ points of $X$, this gives a
dominant contribution of $O(\log^{2d-2} n)$ points to the unprotected set of any axis-parallel box.

Our main new idea is to consider the set system formed by the points and all $x_d$-error sets. We show that this set system has a small shallow cell complexity by observing that each $x_d$-error set can be expressed as the set difference of two almost-fixed boxes (see Definition 4.4). These are axis-parallel boxes with $2d-1$ of their sides anchored to canonical intervals, and the last side can be chosen in at most $O(\log^{d-1} n)$ ways.

We then use the Shallow Packing Lemma, together with a chaining-based decomposition, to express each range in this set system as a combination of set unions and differences of some protected sets, together with $O(1)$ new unprotected sets of size $O(\log^{d-1} n)$). Finally, plugging this decomposition into the Bansal-Garg technique and using Theorem 2.5, gives the new bounds.

In this section therefore, we’ll first introduce some necessary background - the notions of shallow cell complexity and shallow packings and the shallow packing lemma of [32], and then use these ideas to extend the decomposition of set systems of points and axis-parallel boxes. Finally, we’ll use the new decomposition to prove Theorem 1.1.

4.1 Projections, Shallow Cell Complexity and Shallow Packings

The projection of a set system $(X, R)$ on to a subset $Y \subset X$ of the ground set is denoted by $R|_Y := \{ R \cap Y \mid R \in R \}$.

The notion of shallow cell complexity has found many applications in Computational Geometry, including improved bounds on $\varepsilon$-nets and related structures (see e.g. [34]).

Definition 4.1. A set system $(X, R)$ has shallow-cell complexity $\psi(\cdot, \cdot)$ if for any $Y \subseteq X$, the number of subsets in $R|_Y$ of size $l$ is at most $|Y| \cdot \psi(|Y|, l)$.

A central lemma for set systems of bounded shallow cell complexity, is the Shallow Packing Lemma of Mustafa [32]. To state the lemma, first we need to define shallow packings.

Definition 4.2 (Shallow Packing). Let $(X, R)$ be a set system, and $\delta$ and $k$ be positive integers. A subset of ranges $P \subseteq R$ is a $k$-shallow $\delta$-packing or $(k, \delta)$-packing, if any pair of ranges in $P$ have symmetric difference greater than $\delta$, and for all $R \in P$ we have $|R \cap X| \leq k$.

Now follows the Shallow Packing Lemma of Mustafa [32], which gives optimal bounds for shallow packings of set systems, in terms of their shallow cell complexity.

Theorem 4.3 (Shallow Packing Lemma, Mustafa [32]). Let $(X, R)$ be a set system with $|X| = n$ and shallow cell complexity $\varphi_R$. If the VC dimension of $(X, R)$ is at most $d_0$, and $(X, R)$ is a $k$-shallow $\delta$-packing then

$$\frac{24d_0 \delta}{\delta} \cdot \varphi_R \left( \frac{4d_0 \delta}{\delta}, \frac{12d_0 k}{\delta} \right) \leq C_{d_0} \frac{n}{\delta} \psi \left( \frac{n}{\delta}, \frac{k}{\delta}, \delta \right),$$

where $C_{d_0}$ is a constant independent of $n$ and $\delta$.

Now we come to the new decomposition of the set system, which extends Matousek’s decomposition.
4.2 Shallow Packings of Elementary Error Sets

Definition 4.4. Let $v$ be an $x_d$-leaf in the $x_d$-interval tree $T_{i_1,...,i_{d-1}}$, and $B_v$ be the corresponding canonical box. An almost-fixed box is a box $B \in B_d$ containing a subset of points in $B_v$, such that for $j = 1, \ldots, d-1$, the $x_j$-boundaries of $B$ coincide with the $x_j$ boundaries of $B_v$, and one $x_d$-boundary of $B$ coincides with an $x_d$-boundary of $B_v$.

We first observe a simple fact about almost-fixed boxes.

Claim 4.5. For a given $k \leq l_d$, and a given $x_d$-leaf node in a fixed $x_d$-interval tree, $v \in T_{i_1,...,i_{d-1}}$, there are $2k$ almost-fixed boxes having at most $k$ points.

Proof. Indeed, the points in the canonical box $B_v$ corresponding to $v$ can be ordered according to increasing $x_d$ coordinate values. Now, one of the two $x_d$-boundaries of $B_v$ can be chosen in 2 ways. Accordingly, the point in $B_v$ with the $k$-th maximum or $k$-th minimum value of the $x_d$-coordinate can be chosen to determine uniquely an almost-fixed box with $k$ points and boundaries given by $v$. Therefore the number of almost-fixed boxes corresponding to $v$ having at most $k$ points is $2k$.

The following claim indicates the need for almost-fixed boxes in our analysis.

Claim 4.6. Each $x_d$-error set is the difference set of a pair of almost-fixed boxes.

Proof. Consider a box $B \in B_d$. Let $B = \prod_{i=1}^d [a_i, b_i]$ be the decomposition of $B$ as a product of intervals. Recall the construction of the set $L_{B,x_d}$ in Section 3. Let $S \in L_{B,x_d}$ be an $x_d$-error set contained in $B$. Then $S$ is contained in a canonical $x_d$-leaf $L$. By Lemma 3.1(ii), for each $j = 1, \ldots, d-1$, both the $x_j$-boundaries of $S$ coincide with the $x_j$-boundaries of $L$. If both $x_d$-boundaries of $S$ coincide with the $x_d$-boundaries of $L$, then $S = L$. Otherwise, if exactly one $x_d$-boundary of $S$ is canonical, then by the definition of almost-fixed boxes, $S$ is an almost-fixed box, say $B_S$, and we are done by taking the pair consisting of $B_S$ and the empty box (i.e. $\emptyset$). Otherwise, $S$ is the difference of two almost-fixed boxes: take any of the $x_d$-boundaries of $L$, say $r$, and consider the pair of almost-fixed boxes generated by $r$ and each of the $x_d$-boundaries of $L$ separately.

Consider the set system $(X, L)$ formed by the points of $X$ and all possible almost-fixed boxes generated by the $x_d$-leaf nodes of the tree $T(X)$. Our central observation is that $(X, L)$ has shallow cell complexity $\psi(n, k) = O(k)$. This is fairly straightforward to observe, however we give a proof below for completeness.

Claim 4.7. The set system $(X, L)$ has shallow cell complexity $\psi(n, k) = O(k)$.

Proof. Note that for any subset $Y \subseteq X$, the projected range space $L|_Y$ is generated by taking intersections with ranges generated by the tree $T_X$, and not by generating a new interval tree on $Y$. Consider such a subset $Y \in \binom{X}{m}$. Any almost-fixed box in $L$ is given by choosing an $x_d$-tree $T_d$ at levels $(i_1, \ldots, i_{d-1})$, then choosing a leaf-node $v \in T_d$, and finally choosing a $k$-shallow almost-fixed box in $v$. The set of levels $(i_1, \ldots, i_{d-1})$ can be chosen in $\prod_{i=1}^{d-1} \lceil \log(n/l_i) \rceil = O(\log^{d-1} n)$ ways, followed by choosing a leaf node in $T_d$ in $\frac{m^d}{d^d} = O\left(\frac{m^d}{\log^{d-1} n}\right)$ ways (since the $x_d$-leaves for each choice of levels $(i_1, \ldots, i_{d-1})$, are pairwise disjoint and each such $x_d$-leaf contains at most $l_d = \log^{d-1} n$ points), and finally choosing one of $2k$ possible $k$-shallow ranges, in $2k$ ways. Thus the maximum number of possible $k$-shallow ranges in $L|_Y$ is $O(mk)$. Now applying the definition of shallow cell complexity, we get that the shallow cell complexity of $(X, L)$ is $O(k)$.

Now applying the Shallow Packing Lemma gives that the size of any $(l_d, \delta)$-packing is at most $O\left(\frac{m^d}{\delta^d}\right)$. Next, we use the chaining technique to decompose the set system $(X, L)$.
Given a range $S \in L$, each range $S$ can be decomposed into a collection of sets $S = A_S \cup B_S \cup (\bigcup_{F \in F_S} F)$, such that $|A_S|, |B_S| = O(\sqrt{d})$, and $F_S$ is a collection of sets, such that the total number of sets in the collections $F_S$, over all $S \in L$, satisfies $\sum_{S \in L} |F_S| \leq n/16$.

**Proof.** Denote $k := (\log(l_d C_d))/2$, where $C_d$ is the constant in the Shallow Packing Lemma 4.3. For each $i \in k, k + 1, \ldots, 2k$, let $P_i$ be a maximal $(l_d, \delta_i)$-packing, where $\delta_i = (12 \cdot 2^i)$. By the Shallow Packing Lemma 4.3, the number of ranges in $P_i$ is at most

$$|P_i| = O\left(\frac{n}{\delta_i} \cdot \left(\frac{n}{\delta_i} \cdot \frac{l_d}{\delta_i}\right)^{\frac{1}{\delta_i}}\right) = O\left(\frac{n l_d}{\delta_i}\right) = O\left(\frac{n l_d}{12^2 \cdot 2^i}\right).$$

Given a range $S \in L$, let $P_k \in P_k$ be the nearest neighbour of $S$ in $P_k$, $P_{k+1}$ be the nearest neighbour of $P_k$ in $P_{k+1}$, and so on, until $P_{2k} \in P_{2k}$. The set $S$ satisfies

$$S = (P_k \cup (S \setminus P_k)) \setminus (P_k \setminus S).$$

Since $P_k$ is the nearest neighbour of $S$, we have that $|S \setminus P_k|, |P_k \setminus S| \leq \delta_k$. Expanding further, we get that

$$S = (\ldots ((P_{2k} \cup (P_{2k-1} \setminus P_{2k})) \cup (P_{2k} \setminus P_{2k-1})) \cup \ldots \cup (S \setminus P_k)) \setminus (P_k \setminus S).$$

Since for each $i = k, \ldots, 2k$, $P_i$ is the unique nearest neighbour of $P_{i-1}$ in $P_i$, the sets $(P_i \setminus P_{i-1}), (P_{i-1} \setminus P_i)$ satisfy

$$|P_i \setminus P_{i-1}|, |P_{i-1} \setminus P_i| \leq \delta_i.$$ 

In particular, $|S \setminus P_k|, |P_k \setminus S| \leq \delta_k = 12 \cdot 2^k = 12 \sqrt{C_d d} = O\left(\log(d-1)/2 \cdot n\right)$. Now set $F_S$ to be the collection of the sets $P_i \setminus P_{i-1}, P_{i-1} \setminus P_i$, for $i = k + 1, \ldots, 2k$, and $A_S, B_S$ to be $S \setminus P_k$ and $P_k \setminus S$ respectively.

It remains to verify that $A_S, B_S$ and $F_S$ satisfy the conditions of the lemma. Observe that, for each $i = k + 1, \ldots, 2k$, the number of pairs $(P_i, P_{i-1})$ is at most $|P_{i-1}|$, since each
pair \((P_{i-1}, P_i)\) can be mapped to \(P_{i-1} \in \mathcal{P}_{i-1}\), as the nearest neighbour \(P_i \in \mathcal{P}_i\) is unique for a given \(P_{i-1}\). Hence, the number of sets \(P_i \setminus P_{i-1}, P_{i-1} \setminus P_i\), over all \(i = 2, \ldots, k\), is at most
\[
2 \sum_{j=k}^{2k} |\mathcal{P}_j| \leq 2C_d \sum_{j=k}^{2k} \frac{n d}{12^{2i} 2^j}
\]
\[
\leq 2C_d n \sum_{j=0}^{k} \frac{l_d}{12^{2i} 2^{k+2j}}
\]
\[
\leq 2C_d n \sum_{j=0}^{k} \frac{l_d}{12^{2i} C_d 2^{2j}} \leq 4n/12^2 \leq n/16.
\]
Therefore by the above calculation it follows that
\[
\sum_{S \in \mathcal{L}} |\mathcal{F}_S| \leq 2 \cdot \sum_{j=k}^{2k} |\mathcal{P}_j| \leq n/16.
\]
\[\square\]

From Lemma 4.8 and the canonical decomposition in Lemma 3.1 we can conclude the following.

**Lemma 4.9.** Given a set system \((X, \mathcal{B}_d)\) of points and axis-parallel boxes in \(\mathbb{R}^d\), there exists a collection \(\mathcal{T}\) of at most \(n/8\) subsets of \(X\), such that each box \(B \in \mathcal{B}_d\) can be decomposed into a Boolean combination of unions and differences of sets from \(\mathcal{T}\), together with \(O(\log^{2d-4} n) + O(\log^{3(d-1)/2} n)\) other points.

**Proof.** The proof follows from Lemma 3.1 and Lemma 4.8. Take \(\mathcal{T} = \mathcal{C} \cup (\cup_{S \in \mathcal{L}} \mathcal{F}_S)\). We have
\[
|\mathcal{T}| \leq |\mathcal{C}| + \sum_{S \in \mathcal{L}} |\mathcal{F}_S| \leq \frac{n}{16} + \frac{n}{16}.
\]
To bound the number of other points, observe that given any box \(B \in \mathcal{B}_d\), from Lemma 3.1 for \(i = 1, \ldots, d\), there is a collection \(\mathcal{L}_{B, x_i}\) of \(O(\log^{i-1} n)\) \(x_i\)-error sets contained in \(B\), whose disjoint union with some sets in \(\mathcal{T}\) is equal to \(B\). Now applying Lemma 4.8 to each \(x_i\)-error set \(S \in \mathcal{L}_{B, x_i}\), we obtain sets \(|A_S|, |B_S|\) whose size is at most \(O(\sqrt{l_d}) = O(\log^{(d-1)/2} n)\). This gives that the total number of points in \(\{A_S \cup B_S : S \in \mathcal{L}_{B, x_i}\}\) is at most \(|\mathcal{L}_{B, x_i}| \cdot O(\log^{(d-1)/2} n)\), which is \(O(\log^{3(d-1)/2} n)\), since \(|\mathcal{L}_{B, x_i}| = O(\log^{d-1} n)\). We still need to bound the points in the other \(x_i\)-error sets, for \(i \neq d\): \(\cup_{i \in [d-1]} \cup_{S \in \mathcal{L}_{B, x_i}} S\). This is simply the total number of all such \(x_i\)-error sets, times the maximum number of points in each such set:
\[
\left| \bigcup_{i \in [d-1]} \bigcup_{S \in \mathcal{L}_{B, x_i}} S \right| \leq \sum_{i=1}^{d-2} |\mathcal{L}_{B, x_i}| \times O(\log^{d-1} n) + |\mathcal{L}_{B, x_{d-1}}| \times O(\log^{d-2} n)
\]
\[
= \sum_{i=1}^{d-2} O(\log^{i-1} n) \times O(\log^{d-1} n) + O(\log^{d-2} n) \times O(\log^{d-2} n)
\]
\[
= O(\log^{2d-4} n).
\]
This completes the proof of the lemma. \[\square\]

We can now plug the above results into Lemma 2.7.

**Theorem 4.1.** The premises of Lemma 2.7 now apply directly, using Lemma 4.9.
(i) For \( d = 2 \), we get that the number of extra points in the decomposition in Lemma 4.9 is at most \( O(1) + O(\log^{3/2} n) \), which is \( O(\log^{3/2} n) \). Now applying Lemma 2.7 gives us a constructive discrepancy bound of \( O(\log^{(3/2)/2+1} n) = O(\log^{7/4} n) \).

(ii) For \( d = 3 \), the number of extra points in Lemma 4.9 is \( O(\log^2 n) + O(\log^3 n) = O(\log^3 n) \), which gives us a discrepancy bound of \( O(\log^{2+1} n) = O(\log^{5/2} n) \), matching Nikolov’s bound.

(iii) For \( d = 4 \), the number of extra points in Lemma 4.9 is \( O(\log^4 n) + O(\log^{9/2} n) = O(\log^{9/2} n) \), which gives us a discrepancy bound of \( O(\log^{3.25} n) \).

(iv) For \( d \geq 5 \), the number of extra points in Lemma 4.9 is \( O(\log^{2d-4} n) + O(\log^{3(d-1)/2} n) = O(\log^{2(d-4)} n) \), which gives us a discrepancy bound of \( O(\log^{d-1} n) \). Observe that this bound is optimal up to constants, due to the lower bound of \( \Omega \left( \log^{d-1} n \right) \) of Matoušek, Nikolov and Talwar [31].

Remark It may be instructive to observe the reason behind choosing the leaf-size \( l_{d-1} \) to be \( \Theta(\log^{d-2} n) \), while \( l_i = \Theta(\log^{d-1} n) \) for all other \( i \), since this choice of \( l_{d-1} \) seems to be “just right” to get the discrepancy bound of \( O(\log^{d-1} n) \) for \( d \geq 5 \). Clearly, \( l_{d-1} \) cannot be \( o(\log^{d-2} n) \), since this would increase the number of zero-constraints corresponding to nodes of \( x_{d-1} \)-trees to \( \gg n \). (Recall that there are \( \Omega(n/l_{d-1}) \) \( x_{d-1} \)-leaves for each choice of the \( x_1, \ldots, x_{d-2} \)-levels, and each such leaf gives a zero-constraint. Thus there are \( \Omega((n/l_{d-1}) \cdot \log^{d-2} n) \) zero-constraints corresponding to \( x_{d-1} \)-leaves.)

Further, decreasing \( l_{d-1} \) from \( \log^{d-1} n \) to \( \log^{d-2} n \) does not have any ill-effects - the canonical boxes corresponding to \( x_{d-1} \)-intervals smaller than \( \log^{d-1} n \), have \( x_d \)-trees with just the root node.

4.3 Polytopes in \( \mathbb{R}^d \)

Our proof techniques for Theorem 1.1 easily extend to the case of polytopes, in the manner described in [30, 4], and we give only a brief outline here. Firstly, we have an analogue of the canonical construction for axis-parallel boxes, to polytopes: For each pair of hyperplanes in \( L \), we construct canonical parallelopipeds in the same manner as in Section 3. The idea is to align translates of one hyperplane parallel to \( x_d = 0 \), and have the translates of the other hyperplane form canonical parallelopipeds. Note that in the construction in Section 3 and in obtaining the new decomposition of Lemma 4.9 in the proof of Theorem 1.1, we did not use the fact that the coordinate axes are orthogonal to each other. Thus, for \( k = 2 \), our techniques readily translate to the discrepancy of polytopes. For larger \( k \), we apply the same construction for each pair of hyperplanes in \( L \). Sacrificing some polylogarithmic factors in \( k \) (which result from the blow-up of \( \binom{k}{2} \) in the number of zero-constraints, we get the same bounds as for Tusnády’s problem.

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A Appendix

Proof of Lemma 2.6. Fix $a \in (0, 0.5)$. We can check that the sequence $Y_t := U_t - abV_t$, $t \geq 0$, satisfies the premises of Theorem 21 [4] with $\alpha := a$. Applying the theorem, we get that for all $\lambda \geq 0$,

$$
\mathbb{P}[Y_T \geq \lambda] \leq \exp(-a\lambda). \quad (5)
$$

Let $c \geq 0$ be a parameter which will be set later. Therefore,

$$
\mathbb{P}
\left[
U_T \geq c(\lambda + B^{1/2})
\right]
\leq \mathbb{P}
\left[
Y_T \geq c(\lambda + B^{1/2}) - abB
\right]
\leq \exp\left(-ac(\lambda + B^{1/2}) + a^2bB\right) \quad (7)
$$

For $\lambda \leq B^{1/2}/5c$, choosing $a = c(\lambda + B^{1/2})/2bB$, gives the right-hand side of the last inequality above, to be

$$
\exp\left(-\frac{c^2\lambda^2}{2bB}(\lambda + B^{1/2})^2 + \frac{c^2\lambda^2B}{4bB^2}(\lambda + B^{1/2})^2\right),
$$

which completes the proof of Lemma 2.6.
which is at most
\[
\exp \left(-\frac{c^2 \lambda^2}{4bB} (\lambda + B^{1/2})^2 \right) = \exp \left(-\frac{c^2 \lambda^2}{4bB} (\lambda^2 + B + 2 \lambda B^{1/2}) \right) \\
\leq \exp \left(-\frac{c^2 \lambda^2}{4bB} (\lambda^2 + 1 + 2 \lambda B^{-1/2})/4b \right) \\
\leq \exp \left(-\frac{c^2 \lambda^2}{2b} \right) \leq \exp \left(-\lambda^2/2 \right),
\]
for \(c^2 \geq b\). Here to get the last line above, we used that \(\lambda^2/B \leq 1/25c^2 \leq 1/2\) and \(2\lambda/B^{1/2} \leq 2/(5c) \leq 1/2\), since \(c \geq 4/5\). For \(\lambda \geq B^{1/2}/5c\), choosing \(a = 1/(5b)\) gives:
\[
\mathbb{P} \left[U_T \geq c\lambda(\lambda + B^{1/2}) \right] \leq \mathbb{P} \left[Y_T \geq c\lambda(\lambda + B^{1/2}) - abB \right] \\
\leq \exp \left(-ac\lambda(\lambda + B^{1/2}) + a^2 b B \right) \\
\leq \exp \left(-\frac{1}{5b} \left(c\lambda^2 + c\lambda B^{1/2} - B/5 \right) \right) \leq \exp \left(-\frac{c\lambda^2}{5b} \right),
\]
since \(c\lambda \geq B^{1/2}/5\). The last bound above is at most \(\exp \left(-\lambda^2/2 \right)\) for \(c \geq 5b/2\).

Thus, in both the above cases, we get that \(\mathbb{P} \left[U_T \geq c\lambda(\lambda + B^{1/2}) \right] \leq \exp \left(-\lambda^2/2 \right)\), which completes the proof of the lemma. \(\square\)