LOCAL HOLOMORPHIC ISOMETRIES OF A MODIFIED PROJECTIVE SPACE INTO A STANDARD PROJECTIVE SPACE; RATIONAL CONFORMAL FACTORS

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ABSTRACT. We consider local modifications \( \omega_n + f^*\omega_d \) of the Fubini-Study metric (with associated \((1,1)\)-form \( \omega_n \)) on an open subset \( \Omega \subset \mathbb{C}P^n \) induced by a local holomorphic mapping \( f: \Omega \rightarrow \mathbb{P}^d \). Our main result is that there are "gaps" in potential dimensions \( m \) such that the modification can be obtained as \( \mu h^*\omega_m \) for some local holomorphic mapping \( h: \Omega \rightarrow \mathbb{C}P^m \). We also consider the case of rational conformal factors.

1. Introduction

We shall consider complex projective space \( \mathbb{CP}^n \) equipped with the standard Fubini-Study metric, and we shall denote by \( \omega_n \) its associated \((1,1)\)-form. In affine coordinates \( z = (z_1, \ldots, z_n) \) in an affine chart \( \mathbb{C}^n \subset \mathbb{CP}^n \), we have

\[
\omega_n = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left( 1 + \sum_{i=1}^{n} |z_i|^2 \right) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left( 1 + \|z\|^2 \right).
\]

Let \( \Omega \subset \mathbb{C}P^n \) be an open subset and \( F: \Omega \rightarrow \mathbb{C}P^d \) a holomorphic mapping. For a nonnegative real number \( \lambda \), the \( \lambda \)-modification of the Fubini-Study metric (in \( \Omega \)) induced by this mapping is the metric whose associated \((1,1)\)-form is given by \( \omega_{n,F,\lambda} = \omega_n + \lambda F^*\omega_p \). The considerations in this paper are local, so we shall assume that \( \Omega \) and \( F(\Omega) \) are contained in affine charts of \( \mathbb{C}P^n \) and \( \mathbb{C}P^d \), respectively; thus, if we express \( F \) in affine coordinates, \( F(z) = [1 : f(z)] \) with \( f(z) = (f_1(z), \ldots, f_n(z)) \), then \( \omega_{n,f,\lambda} = \omega_{n,F,\lambda} \) is given by

\[
\omega_{n,f,\lambda} := \omega_n + \lambda \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left( 1 + \sum_{i=1}^{n} |f_i(z)|^2 \right) = \omega_n + \lambda \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left( 1 + \|f\|^2 \right).
\]

We shall further assume that there is a positive integer \( m \), a positive real number \( \mu \), and a holomorphic mapping \( h: \Omega \subset \mathbb{C}^n \rightarrow \mathbb{C}^m \subset \mathbb{C}P^m \) such that \( \omega_{n,f,\lambda} = \mu h^*\omega_m \); i.e.,

\[
\omega_n = \mu h^*\omega_m - \lambda f^*\omega_d.
\]

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This situation, but in a more general setting where the source is a general simply connected Kähler manifold and the target is a product of projective spaces, was considered in the recent paper [10] by X. Huang, and Y. Yuan. They show that strong rigidity properties hold under suitable number theoretic conditions on the conformal factors $\mu$ and $\lambda$. In the restrictive setting considered here, their result would state that if there are no positive rational numbers $s, t$ such that $s \lambda = t \mu$, then $f$ and $h$ extend as global holomorphic immersions $f: \mathbb{CP}^n \to \mathbb{CP}^d$, $h: \mathbb{CP}^n \to \mathbb{CP}^m$, and furthermore, $f$ and $h$ are both conformal isometries (i.e., $f^* \omega_d = a \omega_n$, $h^* \omega_m = b \omega_n$) with integral conformal factors $a, b$ such that $1 = a \mu - b \lambda$. The reader is referred to [10] for a discussion of the relevance and general context of this problem.

In this paper, we shall consider in some sense the opposite case, where the conformal factors $\lambda$ and $\mu$ are rational numbers, in which case the number theoretic condition in [10] of course fails. In this case, the rigidity properties established in [10] also fail, as is pointed out in that paper: The mappings $f$ and $h$ do not extend as global mappings and they are not conformal isometries, in general. However, there are range estimates that hold for the rank of $h$, in general depending on the dimension $d$ of the modification as well as the conformal factors $\mu$ and $\lambda$. In the special case where the conformal factor $\lambda$ equals one, there are ”gaps” in the range of possible ranks of $h$ such that the integers in these gaps cannot occur as the rank of an $h$ satisfying (3) for any $f$ or integral $\mu$. (This phenomenon is akin to the codimensional gaps that are predicted by the Huang-Ji-Yin Gap Conjecture [9] for CR mappings between spheres. Indeed, the underlying reasons are similar, in both cases boiling down to rank properties of certain sums of squares; see Section 2.)

We shall say that a mapping $G: \Omega \to \mathbb{CP}^N$ is minimally embedded if the image $G(\Omega)$ is not contained in a proper projective plane. Since projective space equipped with the Fubini-Study metric is a homogeneous space, there is no loss of generality in assuming that $0 \in \Omega$ and that $f(0) = h(0) = 0$. In this case, $g$ being minimally embedded is equivalent to the components of $g = (g_1, \ldots, g_N)$, in affine coordinates near 0, being linearly independent. We first state our result in the special case where the conformal factors $\lambda$ and $\mu$ are both one:

**Theorem 1.1.** Let $\Omega \subset \mathbb{C}^n \subset \mathbb{CP}^n$ be a connected open set, $f: \Omega \to \mathbb{C}^d \subset \mathbb{CP}^d$ a minimally embedded holomorphic mapping, and $\omega_{n,f} = \omega_n + f^* \omega_d$ the 1-modification of the Fubini-Study metric $\omega_n$ induced by $f$. Then, there is a minimally embedded holomorphic mapping $h: \Omega \to \mathbb{C}^m \subset \mathbb{CP}^m$, unique up to multiplication by a unitary $m \times m$ matrix, such that $\omega_{n,f} = h^* \omega_m$ or, equivalently,

$$\omega_n = h^* \omega_m - f^* \omega_d,$$

(4)
and the dimension $m$ satisfies the following:

(i) If $d \leq n$, then

\[
 n + \sum_{l=0}^{d-1} (n-l) = n(d+1) - \frac{d(d-1)}{2} \leq m \leq dn + n + d = n(d+1) + d.
\]

(ii) If $d \geq n$, then $m \geq \max(n(n+3)/2, d).

Remark 1.2. Observe that, for fixed $n$, the function $d \mapsto n(d+1) - \frac{d(d-1)}{2}$ is strictly increasing in $d$ for $1 \leq d \leq n$. The existence of the mapping $h$ is trivial in this case (see the proof of Theorem 1.1 below), and the uniqueness is a consequence of a well known lemma by D’Angelo [4]. The main point of the theorem is the range estimates in (i) on the dimension $m$ for low dimensional ($d \leq n$) modifications. We note that there are "gaps" in the range of possible dimensions $m$ that can occur as target dimensions for $h$, regardless of the modification (i.e., regardless of the modifying mapping $f$ and dimension $d$). For instance, if $d = 1$, then $2n \leq m \leq 2n+1$. If $d = 2$, then $3n-1 \leq m \leq 3n+2$, etc. As $d$ grows towards $n$, these possible ranges of $m$ will initially be disjoint, but the "gaps" between them will shrink until eventually (when $d \sim \sqrt{2n}$) they disappear. The gap intervals of dimensions $m$ for which no minimally embedded $h: \Omega \to \mathbb{C}^m \subset \mathbb{CP}^m$ is isometric to a 1-modification induced by any $f$ go as follows (until they disappear):

\[
(0, 2n), (2n+1, 3n-1), (3n+2, 4n-3), \ldots,
\]

with the $d$th gap being given by $(n(d+1)+d, n(d+2)-d(d+1)/2)$, which as the reader can readily verify becomes empty when $d$ is sufficiently large, $d \sim \sqrt{2n}$ as mentioned above. The estimate provided in (ii) is of less interest. It will become very poor as the dimension $d$ of the modification grows; for generic choices of $f$ the dimension $m$ will grow on the order of the right hand side of (5). We formulate the gap result as a corollary:

Corollary 1.3. Let $\Omega \subset \mathbb{C}^n \subset \mathbb{CP}^n$ be a connected open set, $h: \Omega \to \mathbb{C}^m \subset \mathbb{CP}^m$ a minimally embedded holomorphic mapping such that

\[
 n(k+1) + k < m < n(k+2) - \frac{k(k+1)}{2},
\]

for some $k$. Then, $h^*\omega_m$ is not the 1-modification $w_{n,f} = w_n + f^*\omega_d$ for any $f: \Omega \to \mathbb{C}^d \subset \mathbb{CP}^d$.

In order to formulate a result with general, rational conformal factors $\lambda$ and $\mu$, we need to introduce some terminology and notation. Let $\phi = (\phi_1, \ldots, \phi_a)$ and $\psi = (\psi_1, \ldots, \psi_b)$ be holomorphic mappings $\Omega \to \mathbb{C}^a \subset \mathbb{CP}^a$ and $\Omega \to \mathbb{C}^b \subset \mathbb{CP}^b$, respectively. The tensor
product $\phi \otimes \psi$ is defined to be the mapping $\Omega \to \mathbb{C}^{ab} \subset \mathbb{C}P^{ab}$ whose components are $\phi_i \psi_j$ as $i$ and $j$ run over the sets $\{1, \ldots, a\}$ and $\{1, \ldots, b\}$, respectively, in some predetermined ordering of pairs $(i, j)$. The notation $\phi^{\otimes k}$ denotes the tensor product of $\phi$ with itself $k$ times. The rank of a holomorphic mapping $\phi: \Omega \to \mathbb{C}^a \subset \mathbb{C}P^a$ is the smallest integer $r$ such that the image $\phi(\Omega)$ is contained in an affine complex plane (or, equivalently, projective plane if considered as a mapping into $\mathbb{C}P^a$) of dimension $r$; in particular, $\phi$ is minimally embedded in $\mathbb{C}P^a$ if and only if the rank is $a$.

**Theorem 1.4.** Let $\Omega \subset \mathbb{C}^n \subset \mathbb{C}P^n$ be a connected open set, $f: \Omega \to \mathbb{C}^d \subset \mathbb{C}P^d$ a minimally embedded holomorphic mapping, and $a, b, c$ positive integers without a common prime factor. Let $\omega_{n,f,c/b} = \omega_n + (c/b)f^*\omega_d$ be the $c/b$-modification of the Fubini-Study metric $\omega_n$ induced by $f$. Assume that there is a minimally embedded holomorphic mapping $h: \Omega \to \mathbb{C}^m \subset \mathbb{C}P^m$ such that $\omega_{n,f,c/b} = (a/b)h^*\omega_m$ or, equivalently,

$$
\omega_n = \frac{a}{b}h^*\omega_m - \frac{c}{b}f^*\omega_d.
$$

If $(1, f)^{\otimes c}$ has rank $e + 1$, then

$$
\text{cd} \leq e \leq \sum_{k=1}^c \binom{d+k-1}{k}
$$

and the following hold:

(i) If $e \leq n$ and $b = 1$, then the following two inequalities hold:

$$
\begin{cases}
n(e + 1) - \frac{e(e - 1)}{2} \leq \sum_{k=1}^a \binom{m+k-1}{k} \\
am \leq n(e + 1) + e.
\end{cases}
$$

(ii) If $e \geq n$ or $b \geq 2$, then

$$
\frac{n(n+3)}{2} \leq \sum_{k=1}^a \binom{m+k-1}{k}.
$$

We note that if $a = 1$ in Theorem 1.4, then the estimates on $m$ are of the same type as those of Theorem 1.1 with the rank $e$ playing the role of the dimension $d$ in the latter theorem. If $a \geq 2$, then the existence of the mapping $h$ has to be assumed, as it may not exist in general. We also note that in this case the sum on the right in the first inequality of (10) and on the right in (11) is a polynomial of degree $a$ in $m$ with positive coefficients. Consequently, the lower bound on $m$ provided by these two estimates will be roughly on the order of the $a$th root of the left hand sides. In the case (i) in Theorem 1.4, this means (as the reader can verify) that the possible intervals of $m$ provided by (10) will in general not be disjoint for different values of $e \geq cd$, as is the case in Theorem
Thus, the gaps in possible values of $m$, regardless of the modification, that exist when the conformal factors are both one, cannot be predicted (although they may still exist) for general rational conformal factors by Theorem 1.4, except for the first gap that exists for sufficiently small values of $a$ (compared to the dimension $n$): For fixed $c \geq 1$, we have $e \geq c$ and hence (10) and (11) imply that

$$n(c + 1) - \frac{c(c - 1)}{2} \leq \sum_{k=1}^{a} \binom{m + k - 1}{k}.$$ 

(If $c \geq n$, we replace the left hand side by $n(n + 3)/2$, but let us assume here that $c \leq n$.) Observe that if $m = 1$, then the right hand side equals $a$. It follows, regardless of the modification, that if $a < n(c + 1) - c(c - 1)/2$, then $m \geq 2$. Similarly, if $m = 2$, then the right hand side equals $(a + 1)(a + 2)/2 - 1$ and, hence, if $(a + 1)(a + 2)/2 - 1 < n(c + 1) - c(c - 1)/2$, then $m \geq 3$, etc. In general, Theorem 1.4 should be regarded as estimates on $m$ for a given rank $e$, but estimates that do not depend on the modifying mapping $f$ itself. As above, the main point is the estimates for low ranks $e \leq n$. We should point out, however, that the gap phenomenon described in Corollary 1.3 holds for the possible ranks of $(1, h)^{\otimes a}$. This follows directly from the proofs of Theorems 1.1 and 1.4.

**Remark 1.5.** If $a \geq 2$, then the lower bounds provided by Theorem 1.4 are at best $m \geq n$, and in general worse than this. To see this, observe that in the case $a \geq 2$, if

$$\frac{n(n + 3)}{2} \leq m + \frac{m(m + 1)}{2},$$

then the inequalities for lower bounds on $m$ in (10) and (11) hold. Thus, any lower bound $m \geq A$ that follows from Theorem 1.4 is implied by the lower bound that follows from (12), and the reader can readily verify that this bound is precisely $m \geq n$.

If $f$ in Theorem 1.4 is assumed to be a rational mapping, then one can show (using Huang’s Lemma; see below) that in fact $m \geq n$. Thus, in view of Remark 1.5 for rational mappings, the lower bounds in Theorem 1.4 do not yield any new information beyond $m \geq n$. It is also possible, in this case, that estimates that are linear in $m$ could be proved, if further progress is made on the SOS problem (see Section 2) in the general situation. The case where $f$ is rational is discussed in Section 4.

Standard arguments will reduce the proofs of Theorems 1.1 and 1.4 to statements about ranks of certain sums of squares. (A reader unfamiliar with these standard arguments might want to read Section 3 before reading Section 2.) The results concerning the latter that are needed for the proofs are stated and proved in Section 2. The proofs of Theorems 1.1 and 1.4 are then given in Section 3. A discussion of the case where the modifying
map is rational is conducted in Section 4. Some examples are also given in the latter
section.

The connection between results concerning sums of squares and isometric embedding
problems has also been explored in, e.g., [2], [3].

2. Ranks of Sums of Squares

In this section, we shall consider some Sums of Squares (SOS) problems that arise in
the context of holomorphic mappings between projective spaces with (modified) Fubini-
Study metrics. The (standard) connection will be made in Section 3.

2.1. Setup and Basics. We shall first consider the following Sums of Squares (SOS)
Equation in $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$:

$$
\left( \sum_{j=1}^{n} |z_j|^2 \right) a(z, \bar{z}) = \sum_{k=1}^{m} |h_k(z)|^2,
$$

where $a(z, \bar{z})$ is a real-analytic, Hermitian function; $h = (h_1, \ldots, h_m)$ is a local (germ at 0
of a) holomorphic mapping $(\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ whose components are linearly independent
over $\mathbb{C}$ (or, equivalently, whose image is not contained in a proper subspace of $\mathbb{C}^m$). For
brevity, we shall also use the notation

$$
\|z\|^2 := \sum_{j=1}^{n} |z_j|^2, \quad \|h\|^2 = \|h(z)\|^2 := \sum_{k=1}^{m} |h_k(z)|^2,
$$

the dimension of the complex space whose Euclidian norm is used will be clear from the
context. Thus, equation (13) can then be written

$$
\|z\|^2 a(z, \bar{z}) = \|h\|^2.
$$

If the Hermitian function $a(z, \bar{z})$ is a polynomial, then it can then be written as a differ-
ce of finite squared norms:

$$
a(z, \bar{z}) = \sum_{i=1}^{p} |f_i(z)|^2 - \sum_{j=1}^{q} |g_j(z)|^2 = \|f\|^2 - \|g\|^2,
$$

where $f = (f_1, \ldots, f_p)$, $g = (g_1, \ldots, g_q)$ are polynomial mappings. If $a$ is real-analytic but
not polynomial, then a similar decomposition can be achieved with (in general, infinite
dimensional) Hilbert space valued $f$ and $g$. In what follows, we shall assume that $a$ can
be decomposed as a difference of finite squared norms (as in (15)), which is always the
case if $a$ is polynomial. When the components of $f$ and $g$ are linearly independent (as
can always be achieved), then the pair $(p, q)$ is called the rank of $a$. If $q = 0$ (meaning
that (15) can be achieved with $g \equiv 0$), then $a$ is said to be a (finite) SOS and we will
simply refer to \( p \) (rather than \((p,0)\)) as its rank; thus, e.g., \( |h|^2 \) above is a finite SOS of rank \( m \). A fundamental problem of general interest (and whose solution would have direct implications for the Huang-Ji-Yin Conjecture in CR geometry mentioned in the introduction) can be described as follows:

**SOS Problem.** Let \( a(z, \bar{z}) \) be a Hermitian real-analytic function in neighborhood of 0 in \( \mathbb{C}^n \) and assume that \( |z|^2 a(z, \bar{z}) \) is a finite SOS, i.e., there exists a holomorphic mapping \( h = (h_1, \ldots, h_m) \) satisfying (14). Relate the possible values of the rank \( m \) of the SOS \( |h|^2 \) to the rank \((p,q)\) of \( a \) and the dimension \( n \).

The only general result known, to the best of the author’s knowledge, about the SOS Problem is Huang’s Lemma \[8\], which states that if \( a(z, \bar{z}) \) is not identically zero, then the rank \( m \geq n \). In this paper, we shall only consider the SOS problem in the special case where \( a(z, \bar{z}) \) itself is an SOS.

In what follows, we shall also use the following notation: Let \( F = (F_1, \ldots, F_a) \) and \( G = (G_1, \ldots, G_b) \) be local holomorphic mappings \( (\mathbb{C}^n, 0) \to \mathbb{C}^a \) and \( (\mathbb{C}^n, 0) \to \mathbb{C}^b \), respectively. Then, \( F \oplus G \) denotes the mapping \( (\mathbb{C}^n, 0) \to \mathbb{C}^{a+b} \) given by \( F \oplus G := (F, G) \), and \( F \otimes G \) the mapping \( (\mathbb{C}^n, 0) \to \mathbb{C}^{ab} \) whose components are \( F_i G_j \) as \( i \) and \( j \) run over the sets \( \{1, \ldots, a\} \) and \( \{1, \ldots, b\} \), respectively, in some predetermined ordering of pairs \((i,j)\). We observe immediately that

\[
\|F \oplus G\|^2 = \|F\|^2 + \|G\|^2; \quad \|F \otimes G\|^2 = \|F\|^2 \|G\|^2.
\]

We shall use the notation \( V_F \subset \mathbb{C}\{z\} \cong \mathcal{O}_n \) for the vector space over \( \mathbb{C} \) spanned by the components of \( F \). Clearly, the rank of the SOS \( \|F\|^2 \) equals the dimension of \( V_F \).

2.2. The SOS problem when \( a(z, \bar{z}) \) is an SOS. The following result concerning the case when \( a \) is a bihomogeneous Hermitian polynomial SOS was proved in \[7\] (Proposition 3) using an estimate by Macauley on the growth of the Hilbert function of a homogeneous polynomial ideal:

**Proposition 2.1.** \(\[7\]\) Let \( A(Z, \bar{Z}) \) be a bihomogeneous Hermitian polynomial in \( Z = (Z_0, Z_1, \ldots, Z_n) \) and \( \bar{Z} \), and assume that \( A(Z, \bar{Z}) \) is an SOS of rank \( p \), i.e.,

\[
A(Z, \bar{Z}) = \sum_{i=1}^{p} |F_i(Z)|^2,
\]

where \( F_1(Z), \ldots, F_p(Z) \) are linearly independent homogeneous polynomials. If \( p \leq n + 1 \), then the rank \( R \) of the SOS \( |Z|^2 A(Z, \bar{Z}) \) satisfies

\[
\sum_{i=0}^{p-1} (n + 1 - l) = (n + 1)p - \frac{p(p - 1)}{2} \leq R \leq p(n + 1),
\]
and if \( p \geq n + 1 \), then \( R \geq (n + 1)(n + 2)/2 \).

**Remark 2.2.** We note that both the lower and upper bound in can be achieved for each \( p \leq n \). It is easy to see that the lower bound is achieved with, e.g., \( F_i(Z) = Z_i \) for \( i = 1, \ldots, p \). The upper bound is achieved for ”generic” choices of \( F_i(Z) \).

A straightforward argument using homogenization of polynomials yields the following:

**Theorem 2.3.** Let \( a(z, \bar{z}) \) be a Hermitian real-analytic function near 0 in \( \mathbb{C}^n \), and assume that \( a(z, \bar{z}) \) is a finite SOS of rank \( p \), i.e.,

\[
a(z, \bar{z}) = \sum_{i=1}^{p} |f_i(z)|^2;
\]

where \( f_1(z), \ldots, f_p(z) \) are linearly independent holomorphic functions near 0. If \( p \leq n \), then the rank \( r \) of the SOS \( \|z\|^2 a(z, \bar{z}) \) satisfies

\[
\sum_{l=0}^{p-1} (n - l) = np - \frac{p(p - 1)}{2} \leq r \leq pn,
\]

and if \( p \geq n \), then \( r \geq \max(n(n + 1)/2, p) \).

**Proof.** Let us first assume that \( a(z, \bar{z}) \) is a polynomial of bidegree \( (d, d) \) and that (18) can be achieved with linearly independent polynomials \( f_i(z) \) of degree at most \( d \). Let us introduce homogeneous coordinates \( Z = (Z_0, Z_1, \ldots, Z_n) = (Z_0, \bar{Z}) \) and define homogeneous polynomials of degree \( d \) by

\[
F_i(Z) := Z_0^d f_i(Z/Z_0), \quad i = 1, \ldots, p.
\]

Clearly, the \( F_i \) are linearly independent since the \( f_i \) are. It follows that the bihomogeneous Hermitian polynomial

\[
A(Z, \bar{Z}) := |Z_0|^{2d} a \left( \frac{\bar{Z}}{Z_0}, \frac{Z_0}{\bar{Z}} \right)
\]

then satisfies

\[
A(Z, \bar{Z}) = \sum_{i=1}^{p} |F_i(Z)|^2,
\]

and has rank \( p \). Let us first assume that \( p \leq n \). By Proposition 2.1, the rank \( R \) of the SOS \( \|Z\|^2 A(Z, \bar{Z}) \) satisfies \( R \geq (n + 1)p - p(p - 1)/2 \). Since

\[
\|Z\|^2 A(Z, \bar{Z}) = (\|Z_0\|^2 + \|Z\|^2)A(Z, \bar{Z}) = \sum_{i=1}^{p} |Z_0|^2 |F_i(Z)|^2 + \|\bar{Z}\|^2 A(Z, \bar{Z})
\]

and \( \|\bar{Z}\|^2 A(Z, \bar{Z}) \) is also an SOS, it is then clear that the rank \( r \) of \( \|\bar{Z}\|^2 A(Z, \bar{Z}) \) must satisfy

\[
r \geq (n + 1)p - p(p - 1)/2 - p = np - p(p - 1)/2,
\]

and

\[
r \geq \max(n(n + 1)/2, p).
\]
In other words, there are linearly independent homogeneous polynomials $H_i(Z)$ of degree $d+1$ such that

\[ r \geq (n+1)n - n(n-1)/2 - n = n(n+1)/2. \]

(22)

In other words, there are linearly independent homogeneous polynomials $H_i(Z)$ of degree $d+1$ such that

\[ \|\bar{Z}\|^2 A(Z, \bar{Z}) = \sum_{i=1}^{r} |H_i(Z)|^2, \]

where $r$ satisfies the lower bound (21). Noting that $f_i(z) = F_i(1, z)$ and $a(z, \bar{z}) = A((1, z), (\bar{1}, z))$, we conclude by substituting $Z = (1, z)$ in (23) that

\[ \|z\|^2 a(z, \bar{z}) = \sum_{i=1}^{r} |h_i(z)|^2, \]

where $h_i(z) = H_i(1, z)$. The $h_i$ are linearly independent since the $H_i$ are, and therefore the rank of $\|z\|^2 a(z, \bar{z})$ equals $r$, where $r$ satisfies (21). Clearly, by construction we have $r \leq pn$ (since $pn$ is the total number of terms obtained when the product $\|z\|^2 a(z, \bar{z})$ is multiplied out), proving (19) when $p \leq n$. If $p \geq n$, then the rank $r$ will be greater than or equal to the corresponding rank obtained when $a(z, \bar{z})$ is replaced by the sum on the right in (18) truncated after $n$ terms, i.e., $r \geq n(n+1)/2$. The fact that $r \geq p$ is trivial. This establishes the statement of Theorem 2.3 in the polynomial case.

Next, let $a(z, \bar{z})$ be a Hermitian real-analytic function near 0 in $\mathbb{C}^n$ satisfying (18), where the $f_i$ are linearly independent holomorphic functions near 0, and let

\[ \|z\|^2 a(z, \bar{z}) = \sum_{i=1}^{r} |h_i(z)|^2, \]

be a SOS decomposition of $\|z\|^2 a(z, \bar{z})$ with the $h_i(z)$ being linearly independent holomorphic functions near 0. If we truncate the Taylor series of the $f_i(z)$ at degree $d$ and those of the $h_i(z)$ at degree $d+1$, then we obtain a Hermitian polynomial $a^d(z, \bar{z})$ of bidegree $(d, d)$ such that

\[ a^d(z, \bar{z}) = \sum_{i=1}^{p} |f^d_i(z)|^2, \quad \|z\|^2 a^d(z, \bar{z}) = \sum_{i=1}^{p} |h^{d+1}_i(z)|^2 \]

where the $f^d_i(z)$ and $h^{d+1}_i(z)$ denote the truncated Taylor polynomials of $f_i(z)$ and $h_i(z)$ at degrees $d$ and $d+1$, respectively. Since the sets of holomorphic functions $f_1, \ldots, f_p$ and $h_1, \ldots, h_r$ both are linearly independent, it is clear that the sets of polynomials $f^d_1, \ldots, f^d_p$ and $h^{d+1}_1, \ldots, h^{d+1}_r$ both are linearly independent for $d$ sufficiently large. In other words, for $d$ sufficiently large, the Hermitian polynomial $a^d(z, \bar{z})$ is an SOS of rank $p$ and $\|z\|^2 a^d(z, \bar{z})$ is an SOS of rank $r$. The conclusion of Theorem 2.3 now follows from the corresponding
statement in the polynomial case, already established above. This concludes the proof of Theorem 2.3.

We are now ready to state and prove a "nonhomogeneous" version of Theorem 2.3 that will be used in the proofs of Theorems 1.1 and 1.4 below.

**Theorem 2.4.** Let $f_1(z), \ldots, f_p(z)$ be linearly independent local holomorphic functions vanishing at 0 in $\mathbb{C}^n$ such that

$$
(1 + |z|^2)(1 + \sum_{i=1}^{p} |f_i(z)|^2) = 1 + \sum_{i=1}^{r} |h_i(z)|^2,
$$

where $h_1(z), \ldots, h_r(z)$ are linearly independent local holomorphic functions near 0. If $p \leq n$, then the rank $r$ of the SOS $\|h\|^2 = \sum_{i=1}^{r} |h_i(z)|^2$ satisfies

$$
n + \sum_{l=0}^{p-1} (n - l) = n(p + 1) - \frac{p(p - 1)}{2} \leq r \leq pn + n + p = n(p + 1) + p,
$$

and if $p \geq n$, then $r \geq n(n + 3)/2$.

**Remark 2.5.** Both the upper and lower bound in (28) can be achieved by examples similar to those in Remark 2.2.

**Proof.** If we write $f = (f_1, \ldots, f_p)$ and $h = (h_1, \ldots, h_r)$, then (27) can be written

$$
(1 + \|z\|^2)(1 + \|f\|^2) = 1 + \|h\|^2,
$$

which when multiplied out is equivalent to

$$
\|z\|^2 + \|f\|^2 + \|z\|^2 \|f\|^2 = \|h\|^2.
$$

It is clear that the vector spaces $V_z$ and $V_{f \otimes z}$ only intersect at 0 (since the Taylor series of the $z_if_i(z)$ have no constant or linear terms by the assumption that $f_i(0) = 0$) and therefore the rank of the SOS $\|z\|^2 + \|z\|^2 \|f\|^2 (= \dim_{\mathbb{C}} V_z \oplus V_{f \otimes z})$ is $\geq$ the rank of $\|z\|^2 \|f\|^2$ plus $n$. It then follows immediately from Theorem 2.3 that if $p \leq n$, then

$$
r \geq np - \frac{p(p - 1)}{2} + n = n(p + 1) - \frac{p(p - 1)}{2},
$$

which is the lower bound in (28). The lower bound $r \geq n(n + 3)/2$ when $p \geq n$ follows from Theorem 2.3 in the same way. The upper bound in (28) is obtained directly by counting the number of squares on the left in (29). This completes the proof of Theorem 2.4. □
2.3. Another Sums of Squares Problem. For the proof of Theorem 1.4 we shall also need the following result concerning the rank of powers of \((1 + \| f \|^2)^t\):

**Proposition 2.6.** Let \(f_1(z), \ldots, f_p(z)\) be linearly independent local holomorphic functions vanishing at 0 in \(\mathbb{C}^n\). Let \(t \in \mathbb{Z}_+\) and express \((1 + \sum_{i=1}^p |f_i(z)|^2)^t\) as follows:

\[
(1 + \sum_{i=1}^p |f_i(z)|^2)^t = 1 + \sum_{i=1}^r |h_i(z)|^2,
\]

where \(h_1(z), \ldots, h_r(z)\) are linearly independent local holomorphic functions near 0. Then the rank \(r\) of the SOS \(\| h \|^2 = \sum_{i=1}^r |h_i(z)|^2\) satisfies

\[
(31) \quad t p \leq r \leq \sum_{k=1}^t \binom{p + k - 1}{k}.
\]

**Remark 2.7.** The lower bound in (31) is realized by taking \(f_i(z) = z_i^1\), and the upper bound can be realized by choosing the \(f_i\) to be “suitably spaced” monomials. For instance, to realize the upper bound if \(p \leq n\), we can simply take \(f_i(z) = z_i\); if \(p > n\), then we can take the first \(n\) \(f_i\)'s of this form, and then take subsequent ones to be monomials with an increment in the degrees so that the degrees of monomials up to order \(t\) in \(f_1, \ldots, f_k\) is lower than the degree of the monomials \(f_{k+1}, \ldots, f_p\).

**Proof.** Observe that the rank \(r\) is the dimension of the complex vector space \(V_F \subset \mathbb{C}\{z\}\) spanned by the collection \(F\) of all monomials \(f^\alpha := f_1^{\alpha_1} \ldots f_p^{\alpha_p}\) with \(1 \leq |\alpha| \leq t\). The upper bound in (31) is easily seen to hold. The number on the right in (31) is the number of distinct monomials of degree \(\leq t\) in \(p\) variables, and the rank \(r\) can clearly not exceed this. To prove the lower bound, we proceed as follows. First, a moment's reflection will convince the reader that, by iteratively replacing the \(k\)th generator \(f_k(z)\) with a suitable linear combination \(f_k(z) - \sum_{i=1}^{k-1} c_i f_i(z)\) if necessary, we may assume that the generators are of the form \(f_i(z) = q_i(z) + O(|z|^{s_i+1})\) where the \(q_i(z)\) are homogeneous polynomials of degree \(s_i\) such that \(q_1, \ldots, q_p\) are linearly independent. We shall need the following lemma, in which the notation \(\mathbb{C}[w]_{\leq t}\) is used for the space of polynomials in \(w = (w_1, \ldots, w_q)\) of degree \(\leq t\).

**Lemma 2.8.** For any positive integers \(t\) and \(n\), there are positive integers \(a_1, \ldots, a_n\) such that the algebra homomorphism \(\mathbb{C}[z] \to \mathbb{C}[\zeta]\) induced by the map

\[
p(z) \mapsto p(\zeta^{a_1}, \ldots, \zeta^{a_n})
\]

is injective when restricted as a linear map \(\mathbb{C}[z]_{\leq t} \to \mathbb{C}[\zeta]_{\leq \max(a_1, \ldots, a_n)}\).
Proof of Lemma 2.8. It suffices to prove the lemma for \( n = 2 \), since this result can then be repeated iteratively to ”collapse” two variables to one until the desired map \( \mathbb{C}[z]_{\leq t} \to \mathbb{C}[\zeta]_{\leq t_{\max}(a_1, \ldots, a_n)} \) is obtained. Thus, assume \( n = 2 \). Choose \( a_1 \neq a_2 \) to be prime numbers \( \geq t + 1 \). Then, the induced homomorphism maps the monomial \( z^\alpha := z_1^{\alpha_1}z_2^{\alpha_2} \) to \( \zeta^{a_1\alpha_1+a_2\alpha_2} \). Suppose, in order to reach a contradiction, that the linear map \( \mathbb{C}[z]_{\leq t} \to \mathbb{C}[\zeta]_{\leq t_{\max}(a_1, a_2)} \) is not injective. Then, there must be \( |\alpha| \leq t \) and \( |\beta| \leq t \) such that \( z^\alpha \) and \( z^\beta \) get mapped to the same monomial \( \zeta^k \); i.e., \( a_1\alpha_1 + a_2\alpha_2 = a_1\beta_1 + a_2\beta_2 \) or, equivalently, \( a_1(\alpha_1 - \beta_1) = a_2(\beta_2 - \alpha_2) \). This is clearly a contradiction since \( a_1 \neq a_2 \) are prime numbers \( \geq t + 1 \) and \( |\alpha_1 - \beta_1|, |\beta_2 - \alpha_2| \) are both \( \leq t \). This completes the proof. □

We now return to the proof of Proposition 2.6. By Lemma 2.8, we can find positive integers \( a_1, \ldots, a_n \) such that the one-variable polynomials \( f_i(\zeta) = f_i(\zeta^{a_1}, \ldots, \zeta^{a_n}) \), for \( 1 = 1, 2, \ldots, p \), are linearly independent. The rank \( r \) in Proposition 2.6 will be \( \geq \) the dimension \( \tilde{r} \) of the complex vector space \( V_{\tilde{F}} \subset \mathbb{C}\{z\} \) spanned by the collection \( \tilde{F} \) of all monomials \( \tilde{f}^\alpha := \tilde{f}_{\alpha_1} \cdots \tilde{f}_{\alpha_p}^p \) with \( 1 \leq |\alpha| \leq t \). Again, by iteratively replacing the \( k \)th generator \( \tilde{f}_k(\zeta) \) with a suitable linear combination \( \tilde{f}_k(\zeta) = \sum_{i=1}^{k-1} c_i \tilde{f}_i(\zeta) \) and then renumbering if necessary, we may assume that \( \tilde{f}_1(\zeta) = b_1\zeta^{a_1} + O(\zeta^{s_1+1}) \) where \( b_1 \neq 0 \) and \( 1 \leq s_1 < \ldots < s_p \). As a final reduction, we note that the dimension \( \tilde{r} \) is \( \geq \) the dimension \( r' \) of the complex vector space \( V' \subset \mathbb{C}\{z\} \) spanned by the collection \( M \) of all monomials

\[
\zeta^{\alpha_1s_1 + \ldots + \alpha_ps_p}, \quad 1 \leq |\alpha| \leq t.
\]

To finish the proof, we shall show that \( r' \) satisfies the lower bound in (31). We shall prove this by induction on \( p \). Clearly, if \( p = 1 \), then \( r' = t = tp \). Next, assume that the lower bound \( r' \geq tp \) has been proved for \( p < p_0 \). Then, with \( p = p_0 \), we obtain at least \( t(p - 1) \) distinct monomials \( \zeta^q \) with \( q = \alpha_1s_1 + \ldots + \alpha_{p-1}s_{p-1} \) such that \( |\alpha| = |(\alpha_1, \ldots, \alpha_{p-1}, 0)| \leq t \). We must prove that we obtain at least \( t \) new distinct monomials \( \zeta^q \) with \( q = \alpha_1s_1 + \ldots + \alpha_{p-1}s_{p-1} + \alpha.ps_p \) where \( \alpha.p \geq 1 \) and \( |\alpha| \leq t \). We claim that every \( q = ks_p - (m - k)s_p \), for \( k = 0, \ldots, t - 1 \), is such that it cannot be obtained as \( q = \alpha_1s_1 + \ldots + \alpha_{p-1}s_{p-1} \) with \( |\alpha| = |(\alpha_1, \ldots, \alpha_{p-1}, 0)| \leq m \). To see this, note that since \( 1 \leq s_1 < \ldots < s_p \) we have \( \alpha_1s_1 + \ldots + \alpha_{p-1}s_{p-1} \leq ts_{p-1} \). Moreover, we have

\[
k s_p - (t - k)s_p - ts_{p-1} = (t - k)(s_p - s_{p-1}) > 0, \quad k = 0, \ldots, t - 1,
\]

which proves the claim, and shows that \( r' \geq (t - 1)p + t = tp \) also for \( p = p_0 \). This completes the proof of the proposition. □

3. Proof of Theorems 1.1 and 1.4

Proof of Theorem 1.1. Let \( f : \Omega \to \mathbb{C}^d \subset \mathbb{C}P^d \) and \( h : \Omega \to \mathbb{C}^m \subset \mathbb{C}P^m \) be minimally embedded holomorphic mappings satisfying (4). Since projective space equipped with
the Fubini-Study metric is a homogeneous space, we may assume that $0 \in \Omega$ and $f(0) = 0$, $h(0) = 0$. By (1) and (2), we conclude that $\log(1 + \|z\|^2) + \log(1 + \|f\|^2) = \log(1 + \|h\|^2) + \Phi$, where $\Phi$ is a real-valued polyharmonic function, i.e., $\Phi(z, \bar{z}) = \phi(z) + \overline{\phi(z)}$ near 0 for some local holomorphic function $\phi(z)$. By comparing the Taylor series of the log-terms (no constant terms and no pure terms in $z$ or $\bar{z}$) with that of $\Phi$, we conclude that $\Phi \equiv 0$, and hence

$$\log(1 + \|z\|^2) + \log(1 + \|f\|^2) = \log(1 + \|h\|^2).$$

By exponentiating this, we obtain the SOS identity

$$\tag{32} (1 + \|z\|^2)(1 + \|f\|^2) = (1 + \|h\|^2).$$

Conversely, if $f$ and $h$ satisfy this SOS identity, then (1) holds. It follows that if $f$ is a minimally embedded holomorphic mapping, then there exists a mapping $h$ satisfying (32) (simply carry out the multiplication on the left), and elementary linear algebra shows that we may assume (after replacing the $h_i$ obtained by multiplying out the left hand side of (32) by a suitable basis for the vector space spanned by these) that $h$ is also minimally embedded. If there are two minimally embedded $h$ and $\tilde{h}$ that both satisfy (32), then $\|h\|^2 = \|\tilde{h}\|^2$ and hence, by a lemma of D’Angelo [4], it follows that $h = U\tilde{h}$ for some unitary $m \times m$ matrix $U$. To finish the proof of Theorem 1.1 we recall that $f$ and $h$, with $f(0) = 0$ and $h(0) = 0$, are minimally embedded precisely when their components are linearly independent. Thus, the estimates in (i) and (ii) follow immediately from Theorem 2.4. □

**Proof of Theorem 1.4.** As in the proof of Theorem 1.1 above, we may assume that $0 \in \Omega$ and $f(0) = 0$, $h(0) = 0$. The same argument as in that proof also shows that the mappings $f$, $h$ satisfy

$$\tag{33} (1 + \|z\|^2)^b(1 + \|f\|^2)^c = (1 + \|h\|^2)^a.$$ 

As noted in Section 2 we have $(1 + \|f\|^2)^c = \|(1, f)^{\otimes c}\|^2$. Thus, if the rank of $(1, f)^{\otimes c}$ is $e + 1$, then we can find a minimally embedded holomorphic mapping $g: \Omega \to \mathbb{C}^e$, with $g(0) = 0$, such that

$$\tag{34} (1 + \|f\|^2)^c = \|(1, f)^{\otimes c}\|^2 = (1 + \|g\|^2).$$

The estimate (9) for $e$ follows from Proposition 2.6. We may now rewrite (33) as follows:

$$\tag{35} (1 + \|z\|^2)^b(1 + \|g\|^2) = (1 + \|h\|^2)^a.$$ 

The estimates in (i) and (ii) in the case $b = 1$ now follow immediately by combining the estimates in Theorem 2.4 and Proposition 2.6. If $b \geq 2$, we observe that

$$ (1 + \|z\|^2)^b(1 + \|g\|^2) = (1 + \|z\|^2)^{b-1} ((1 + \|z\|^2)(1 + \|g\|^2))$$
and \((1 + \|z\|^2)(1 + \|g\|^2)\) has rank at least \(n\) by Theorem 2.4. The estimate in (ii) now follows by again combining the estimates in Theorem 2.4 and Proposition 2.6. This completes the proof of Theorem 1.4.

\[\Box\]

4. The case where the mapping \(f\) is rational; Examples

Let us examine closer the case where \(f : \Omega \to \mathbb{C}P^d\) is a rational mapping in Theorem 1.4. Clearly, by the uniqueness property of the mapping \(h\) (up to linear transformations as explained in the proof of Theorem 1.1), it follows that \(h : \Omega \to \mathbb{C}P^m\) is also rational. Let us choose homogeneous coordinates \(Z = [Z_0 : Z_1 : \ldots : Z_n]\) in \(\mathbb{C}P^n\). The Fubini-Study metric then has the associated \((1, 1)\)-form

\[
\omega_n = \sqrt{-1} \frac{1}{2\pi} \partial \bar{\partial} \log \left( \sum_{i=0}^{n} |Z_i|^2 \right) = \sqrt{-1} \frac{1}{2\pi} \partial \bar{\partial} \log \|Z\|^2.
\]

We shall denote the rational mappings \(\mathbb{C}P^n \to \mathbb{C}P^d\) and \(\mathbb{C}P^n \to \mathbb{C}P^m\) corresponding to \(f\) and \(h\) by \(F\) and \(H\), respectively. Thus, in homogeneous coordinates on the target projective spaces, we have

\[
F(Z) = [F_0(Z) : \ldots : F_d(Z)] \quad \text{and} \quad H(Z) = [H_0(Z) : \ldots : H_m(Z)],
\]

where \(F_i(Z)\) and \(H_j(Z)\) are homogeneous polynomials, and the assumptions in Theorem 1.4 are equivalent to the identity

\[
(\|Z\|^2)^b (\|F\|^2)^c = (\|H\|^2)^a.
\]

Let us proceed under the assumption that \(a \geq 2\). By complexifying (37), i.e. replacing \(\bar{Z}\) by an independent complex variable \(\chi\) and using the notation \(\bar{\phi}(\chi) := \phi(\bar{\chi})\), we obtain

\[
\left( \sum_{j=0}^{n} Z_j \chi_j \right)^b \left( \sum_{i=0}^{d} F_i(Z) \bar{F}_i(\chi) \right)^c = \left( \sum_{k=0}^{m} H_k(Z) \bar{H}_k(\chi) \right)^a.
\]

Since the polynomial \(\sum_{j=0}^{n} Z_j \chi_j\) is irreducible, the identity (38) implies that \(\sum_{j=0}^{n} Z_j \chi_j\) divides \(\sum_{k=0}^{m} H_k(Z) \bar{H}_k(\chi)\), and we conclude that there exists a (homogeneous) polynomial \(R(Z, \chi)\) such that

\[
\left( \sum_{j=0}^{n} Z_j \chi_j \right) R(Z, \chi) = \sum_{k=0}^{m} H_k(Z) \bar{H}_k(\chi),
\]

or

\[
\|Z\|^2 R(Z, \bar{Z}) = \|H\|^2.
\]

Substituting this in (37), we obtain

\[
(\|Z\|^2)^b (\|F\|^2)^c = (\|Z\|^2)^a R(Z, Z)^a.
\]
From (41), we can conclude that the Hermitian polynomial $R(z, \bar{z})$ belongs to various "positivity classes" introduced by D'Angelo and Varolin (see [6]; see also [5]). For instance, if $b \geq a$, then it follows that $R(Z, \bar{Z})^a$ is an SOS; however, $R(z, \bar{z})$ may not be an SOS itself in general. Thus, in order to use (41) instead of (37) to estimate the dimension $m$, we would need to solve the SOS Problem in Section 2 in its general form. Nevertheless, Huang’s Lemma (described in Section 2 above) applied to (40) implies that $m \geq n$, which is a stronger lower bound in the first gap than that provided by Theorem 1.4 for general, not necessarily rational mappings (see the discussion following the statement of Theorem 1.4 and Remark 1.5). When $R(z, \bar{z})$ is not an SOS, then the modification of $\omega_n$ obtained by applying $(\sqrt{-1}/2\pi)\bar{\partial}\partial$ to log of the left side of (40) arises from the pullback of a mapping into a projective space with an indefinite Fubini-Study type "metric" (as in [1]).

We should remark that for local holomorphic mappings $f$ and $h$, reducing the identity

$$(1 + \|z\|^2)^b(1 + \|f\|^2)^c = (1 + \|h\|^2)^a$$

(42)

is not useful, unless $Q(z, \bar{z})$ itself is an SOS. Any mapping $h$ has a local analytic identity of this form with

$$1 + Q(z, \bar{z}) = \frac{1 + \|h\|^2}{1 + \|z\|^2}.$$ 

We conclude this section by giving an example (taken from [6] and [5]) illustrating that even in the rational (polynomial) case, the situation in Theorem 1.4 cannot be reduced to a situation covered by Theorem 1.1.

**Example 4.1.** Let $n = 1$ and

$$R_\lambda(z, \bar{z}) := (1 + |z|^2)^4 - \lambda|z|^4 = 1 + 4|z|^2 + (6 - \lambda)|z|^4 + 4|z|^6 + |z|^8. \quad (43)$$

We observe that $R_\lambda(z, \bar{z}) = 1 + Q_\lambda(z, \bar{z})$ where $Q_\lambda(z, \bar{z})$ is an SOS if and only if $\lambda \leq 6$. A straightforward calculation shows that

$$R_\lambda(z, \bar{z}) = 1 + 5|z|^2 + (10 - \lambda)|z|^4 + (10 - \lambda)|z|^6 + 5|z|^8 + |z|^10, \quad (44)$$

which is of the form $1 + \|h\|^2$ if and only $\lambda \leq 10$. Another calculation shows that

$$R_\lambda(z, \bar{z})^2 = 1 + 1 + 8|z|^2 + 2(14 - \lambda)|z|^4 + 8(7 - \lambda)|z|^6 + (6 - \lambda)^2 + 34|z|^8 + 8(7 - \lambda)|z|^10 + 2(14 - \lambda)|z|^12 + 9|z|^14 + |z|^16, \quad (45)$$
which is of the form $1 + \|f\|^2$ if and only $\lambda \leq 7$. (According to [6] and [5], there exists $k \geq 2$ such that $R_\lambda(z, \bar{z})^k = 1 + \text{an SOS}$ if and only if $\lambda < 8$.) Thus, if we choose $\lambda = 7$ and denote $R_7$ by $R$, then we have

$$
(46) \quad (1 + |z|^2)R(z, \bar{z}) = 1 + \|h\|^2,
$$

with $h = (h_1, \ldots, h_m)$ and $m = 5$, but $R(z, \bar{z})$ is not an SOS. However, if we square (46) and use (45), then we obtain

$$
(47) \quad (1 + |z|^2)^2(1 + \|f\|^2) = (1 + \|h\|^2)^2,
$$

with $h = (h_1, \ldots, h_m)$ and $m = 5$, $f = (f_1, \ldots, f_d)$ and $d = 6$. Thus, this is an example where the conditions in Theorem 1.4 are satisfied (with $a = b = 2$, $c = 1$, $e = d = 6$), but which cannot be reduced to a situation covered by Theorem 1.1. Of course, in this example the lower bound for $m$ (which is 5) provided by Theorem 1.4 (ii) (or the one provided by Huang’s Lemma as above) is a poor estimate, as it reduces to $m \geq 1$ (which is a trivial bound) when $n = 1$. We do note, however, that $m < d = e$ in contrast to the situation in Theorem 1.1.

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