The isolated Heisenberg magnet as a quantum time crystal

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We demonstrate analytically and numerically that the paradigmatic model of quantum magnetism, the Heisenberg XXZ spin chain, does not equilibrate. It constitutes an example of persistent non-stationarity in a quantum many-body system that does not rely on external driving or coupling to an environment. We trace this phenomenon to the existence of extensive dynamical symmetries. We discuss how the ensuing persistent oscillations that seemingly violate one of the most fundamental laws of physics could be observed experimentally.

Introduction. Isolated systems consisting of many interacting particles are generally assumed to relax to a stationary equilibrium state whose macroscopic properties are described by the laws of statistical physics. This has been confirmed by a large amount of recent theoretical1–17 and experimental18–24 work, particularly focusing on the Heisenberg spin chain in quench setups.25–49 On the other hand time crystals describe a phase of mater which never relaxes to stationarity and breaks the continuous time-translation symmetry (TTS) in analogy with the continuous space translation symmetry breaking in ordinary crystals. Historically the research into quantum time crystals was instigated by an intriguing possibility that a system at zero temperature could exhibit perpetual motion33, however this has subsequently been disputed,50–52 leading to possible generalisations to finite temperature, and systems far from thermal equilibrium. Despite the large amount of work on Floquet (breaking the discrete time translation symmetry) or dissipation induced time crystal53–57 such behavior was believed to be impossible to realize in isolated many-body systems.51–55,58 These typically relax to stationary states depending only on few parameters, such as energy and particle number.59–62

Despite numerous studies on relaxation in many-body quantum systems, there have been no results on spontaneous time translation symmetry breaking in locally strongly interacting Hamiltonian systems close to equilibrium. Previous examples of TTS breaking include models with a well defined mean field limit61,62, and the case of spin precession. In strongly correlated systems one a priori expects relaxation to stationarity63–65. The absence of such results might be expected in the light of the no-go theorem66, however there is a crucial defining property underlying its derivation, which we relax. It assumes that the system should exhibit long range spatial correlations, which are not relevant to time-translation symmetry breaking, or non-stationarity.

In this letter we show that systems can indeed fail to relax and relate this type of behavior to extensive dynamical symmetries that are local in space and have a periodic dependence on time. We find that, as a consequence of the presence of dynamical symmetries, a system can fail to equilibrate after a quantum quench and is instead described by a time-dependent statistical ensemble. Non-stationarity also shows up on the level of dynamical response functions describing the behavior near equilibrium and the stability of the equilibrium state to small perturbations.

We demonstrate the effects of local dynamical symmetries for the one-dimensional Heisenberg spin chain and study its stability under integrability breaking perturbations. This paradigmatic model is used to describe many experimentally relevant situations including organic compounds,53 various materials,54 cold atom implementation,55 and quantum dots.56

Quantum time crystals: We introduce two related definitions of spontaneous time translation symmetry breaking motivated by analogy with previous literature - one will be oscillations in the autocorrelation function at equilibrium, and the other is motivated by relaxation following quenches.

Watanabe and Oshikawa57 defined quantum time crystals as interacting systems exhibiting persistent oscillations. These can be probed by the auto-correlation function \( f(t) = \frac{1}{V} \langle f(t) \rangle \), where \( V \) is the volume of the system and \( f \) is an extensive observable.67 They consider this auto-correlation function as a perturbation to equilibrium and show that it is time independent at zero and finite temperature for local Hamiltonians (see67 for non-local results). Their definition does not capture all physically measurable persistent oscillations because the time dependent (connected) part of the function \( f(t) \) vanishes in the absence of long range correlations in the large volume limit at all times \( t \). We instead consider connected auto-correlation functions that are initially normalized \( f(t) = \frac{1}{O} \langle O(t)O \rangle \). This probes TTS breaking of measurable local operators if the equilibrium ensemble is perturbed by the extensive operator \( O \), even in the absence of long range correlations. Following such a perturbation quantum time crystals will never reach stationarity. In contrast quantum many-body scarred models55,60 exhibit oscillations only for special set of initial states and are expected to relax to stationarity close to equilibrium.

While there are different ways of identifying the many-body nature of the phenomenon, the definition we use
here is that single-body observables relax to stationarity, while some of the many-body observables oscillate persistently. This way oscillations of the higher point correlation functions cannot be attributed to single-body oscillations.

Recently, most of the work on quantum time-crystals has focused on the discrete time translation symmetry breaking in Floquet systems. In this case TTS breaking is studied in an out-of-equilibrium quench setup. In this setup the system is prepared in a generic pure state $|\psi\rangle$ and then allowed to evolve under the action of a Hamiltonian. The system is identified as a discrete quantum time crystal if the dynamics breaks the discrete time symmetry of the driving period $T$ with a subharmonic response $\langle \psi | o(t + nT) | \psi \rangle = \langle \psi | o(t) | \psi \rangle$, for some integer $n > 1$, and $\langle \psi | o(t + t_1) | \psi \rangle \neq \langle \psi | o(t) | \psi \rangle$ for $t_1 < nT$. We make the analogous identification for continuous time evolution by requiring that $\langle \psi | o(t + T) | \psi \rangle = \langle \psi | o(t) | \psi \rangle$ and $\langle \psi | o(t + t_1) | \psi \rangle \neq \langle \psi | o(t) | \psi \rangle$, for $t_1 < T$ and for some $T$.

**Extensive dynamical symmetries:** An important insight into the phenomenon of equilibration is provided by the eigenstate thermalization hypothesis in generic systems or generalized eigenstate thermalization hypothesis (GETH) in integrable systems. It states that off-diagonal elements of local observables in the eigenbasis of a local Hamiltonian vanish exponentially in the thermodynamic limit, and that their expectation values in a given eigenstate are smooth functions of conserved quantities. Assuming the validity of the GETH the system is expected to locally relax to the maximal entropy, or generalized Gibbs ensemble (GGE) $\rho_{\text{GGE}} = \exp(-\sum_j \mu_j X_j - \mu_Y Y - \mu_Y Y^\dagger)$, which take the form of time dependent generalized Gibbs ensembles ($t\text{GGE}$) for the intermediate times,

$$\rho_{\text{tGGE}} = \exp(-\sum_j \mu_j X_j - \mu_Y Y - \mu_Y Y^\dagger), \quad (3)$$

where the conserved quantities $[H, X_j] = 0$ as usual. The values of the chemical potentials $\mu_j, \mu_Y$ can be fixed in the following way. The maximum entropy non-stationary ensemble which correctly reproduces the initial value of $X_k$, and the dynamics of dynamical symmetries $Y_k$ is obtained by requiring for an initial state $|\psi\rangle$

$$\langle \psi | Y(t) | \psi \rangle = \text{tr} (Y(t) \rho_{\text{GGE}}),$$

$$\langle \psi | X_k | \psi \rangle = \text{tr} (X_k \rho_{\text{GGE}}), \quad (4)$$

**Dynamical response functions in thermal equilibrium:** Here we focus on the response of the system in equilibrium. In the large time limit a local observable $O(t)$ can be represented as a linear combination of conserved quantities and dynamical symmetries

$$O(t) = \lim_{t \to \infty} \alpha_O^Y \exp(i\omega t) Y + h.c. + \sum_j \alpha_j^O X_j, \quad (5)$$

for calculating the dynamical susceptibilities $\langle O_1(t) O_2 \rangle$. This equality is valid only in the hydrodynamic level and in the long-time limit. We restrict the discussion to the case with $\langle O_1 \rangle = \langle O_2 \rangle = 0$, which can be relaxed by considering connected correlation functions. In general the coefficients $\alpha_O^Y$ and $\alpha_j^O$ depend not only on the observable $O$ but also on the thermal ensemble ($\bullet$). If these coefficients vanish, the observable is not expected to oscillate.

**Heisenberg model and extensive dynamical symmetries:**
We will consider the anisotropic Heisenberg Hamiltonian

$$H = J \left[ \sum \limits_j s_j^x s_{j+1}^x + s_j^y s_{j+1}^y + \Delta s_j^z s_{j+1}^z + \alpha \left( \sum \limits_j s_j^z s_{j+2}^z + s_j^y s_{j+2}^y + \Delta s_j^z s_{j+2}^z \right) \right] + h s_j^z.$$

as a paradigm example of a system breaking TTS. In [6] we introduced the spin $-\frac{1}{2}$ operators $s_j^{x,y,z}$, anisotropy $\Delta$, hopping amplitude $J$, the magnetic field $h$, and the integrability breaking parameter $\alpha$, which is set to 0 except when otherwise specified. One of the crucial aspects of the Heisenberg model, which has a paramount effect on physical properties are its extensive conservation laws [78–81]. Their known effects range from the absence of thermalization to the ideal (ballistic) energy and spin conductivity at any temperature. Despite the absence of thermalization, the Heisenberg model has in recent years served as a testbed for studying equilibration properties of strongly interacting systems [25–28]. In what follows we will show that in the easy plane regime $-1 < \Delta < 1$, it does not, in general, reach equilibrium if $h \neq 0$.

This can be seen as a consequence of the existence of semi-cyclic quantities $Y_\Delta(\phi)$, which were introduced in [29] (see also [30]), with $\phi$ parametrising the infinite set of quantities $Y_\Delta(\phi)$ at $\Delta$. We will suppress the dependence on $\phi$ for sake of simplicity. While these quantities commute with the Hamiltonian [31] in the absence of the field $h = 0$, the $Y_\Delta$ operators do not commute with the total magnetization. Due to them having a surplus of exactly $m$ spin operators they rather satisfy $[\sum \limits_j s_j^z, Y_\Delta] = m Y_\Delta$ for the anisotropy parameter $\Delta = \cos(\phi/m)$, with $n = 2N$, $m \in 2N + 1$. This leads to the $Y_\Delta$ operators becoming dynamical symmetries of $H$ for non-zero $h$ in the sense of [32]. Interestingly, this also means that the frequency of their oscillations $\omega = \hbar m$ is a discontinuous function of $\Delta$ (see [33] for details). We emphasize that the set of dynamical symmetries $Y_\Delta$ are different for each value of $m$. This means the operators exhibiting persistent oscillations change with $\Delta$. Likewise, the operators $Y_\Delta^\dagger$ have a surplus of $m$ operators $s^{-}$.

For the sake of simplicity we will now mainly focus on the oscillations of the transverse correlation function $s_j^x s_{j+2}^y$ at $\Delta = -\frac{1}{2}$. Physically, such observables correspond to correlations of the three-site measurement statistics - the average measured value of each individual spin relaxes according to standard statistical physics, but the measured values will be such that on average their product oscillates in time. Alternatively, they may be thought of as oscillations of the higher moments of the $m$-site quantum fluctuations (e.g. $(s_j^x + s_j^y + s_j^z)^3(t)$). Note that the fact that quantities responsible for oscillations $Y$ do not exist at the non-interacting point $\Delta = 0$ solidifies the argument that the oscillations are a genuine many-body phenomenon.

In order to obtain the dynamics of temporal correla-
perturbation strengths (see also §64). the integrability breaking term \( \alpha \) does not pertain to any dynamical symmetry (see (4) and §5). We emphasize that only certain (infinite) set of local observables will have overlap with the \( Y_\Delta \)s at a given value of \( \Delta \) which can be deduced from their form in §65.

**Results:** Using the hydrodynamical projection [5] and the known form of charges, we can calculate the asymptotic value of the autocorrelation function \( C(t) = \frac{\langle O(t)O \rangle}{\langle O^2 \rangle} \), for the observable \( O_3 = \sum_j s^+_j s^+_{j+1} s^-_{j+2} + s^-_j s^+_{j+1} s^-_{j+2} \), and the infinite temperature ensemble \( \langle \bullet \rangle = \frac{\langle \operatorname{tr}(\bullet) \rangle}{\langle \operatorname{tr}(1) \rangle} \). \( C(t) \) is given by

\[
\frac{1}{N} \left( \frac{27\sqrt{3} - 8}{\pi} \right) \cos(3ht) \tag{32}
\]

asymptotic time correlation functions is more involved, which can be engineered with \( N \)-particle quantum systems. [7], which can be engineered [62]. Preparation and measurement of auto-correlation functions is available through quantum gas microscopes for cold atom systems [65]. For experiments an important discovery is that oscillations can be observed for a quench from the ferromagnetic initial state \((\uparrow | \downarrow )\), which can be engineered [62]. Preparation and measurement of auto-correlation functions is more involved, but can be achieved through the use of ancilla qubits in Rydberg atoms [62]. Our results could potentially also have far-reaching applications in quantum metrology [62] due to the sensitivity of the dynamical symmetries \( Y_\Delta \) to the anisotropy. In cold atom simulations this can be directly related to the strength of the external magnetic field used to achieve Feshbach resonance of the spin-spin interaction [62]. In this regard, an important observation is that the amplitude seems to be less affected by integrability breaking, than by a change of the anisotropy. We have tested this prediction by fitting the long-time parts of the quench protocol for any initial state of the form \( | \phi \rangle \). A particular choice of \( \alpha \) was made due to the non-zero overlap with the \( Y_\Delta \) quantities of the Heisenberg XXZ spin chain [79] in the sense of [5]. We emphasize that only certain (infinite) set of local observables will have overlap with the \( Y_\Delta \)s at a given value of \( \Delta \) which can be deduced from their form in §65.

**Conclusion:** Numerous questions remain open. While we have addressed the question of stability to perturba-

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-1 < \Delta < 1 \quad \text{which can be parametrised as} \quad \Delta = \cos\left(\frac{\pi n}{m}\right), \quad \text{with} \quad n \in 2\mathbb{N}, \quad m \in 2\mathbb{N} + 1. \quad \text{The infinite set of these operators (parametrised by} \phi \text{) consist of an infinite sum of an infinite number of different local operators [62]. The observables with a finite overlap with these operators will oscillate at a frequency \( \omega = \hbar m \) (see Fig. 2 for some examples). The observables become 'less' local with increasing \( m \), the smallest one being supported on \( m \) consecutive sites.

**Experimental realization:** Due to the demonstrated stability, we expect that in current quantum cold-atom simulations of the XXZ spin chain, such as the ones in I. Bloch's group [88], the lattice depth can be sufficiently tuned to make the dynamics fast enough compared to integrability-breaking effects to observe oscillations.
tions from a practical perspective, stability to all orders remains an open problem, related to the long standing question of the existence of the KAM theorem in systems with infinitely many degrees of freedom. That being said, the crucial ingredient for oscillations is not integrability itself, but rather local or extensive quantities satisfying the relation \( \rho_{\text{GGE}} \). Importantly, we were able to identify similar quantities in topological models\(^{11}\), conformal field theories\(^{57}\), 2D cold atom systems\(^{58}\), and approximately in mean-field models\(^{59,60}\), and certain localized systems\(^{61}\). The gimmers of similar dynamical symmetries have also been identified in locally constrained models exhibiting quantum many-body scars, preventing the systems from relaxing for certain special initial conditions\(^{65,66}\). Indeed, a possible relation to integrability has also been drawn\(^{53}\).

Some questions remain also from the standpoint of integrable systems. Here we only focused on the lowest frequency of oscillations at a given \( \Delta \), while in general the state \( \rho_{\text{GGE}} \) should support a complete harmonic spectrum \( \omega = km \), for \( k \in \mathbb{N} \). Furthermore, due to the non-commutativity of the conserved quantities and dynamical symmetries, subtleties might arise in obtaining the correct form of \( \rho_{\text{GGE}} \). The answers to these questions should be attainable by extending the thermodynamic Bethe ansatz description\(^{62}\), to include additional quantities \( Y \). The existence of these quantities implies that the standard GGE description is in fact incomplete even in the absence of persistent oscillations (\( h = 0 \)). We found that microscopic nearest-neighbor correlation functions do not relax to stationarity, whereas the single-particle function do. Thus, another exciting question is whether the dynamical symmetries, and absence of many-body equilibration has a counterpart in the realm of classical physics. Otherwise, the phenomenon would constitute one of the first many-body quantum effects that can be observed in macroscopic systems on large space and time scales solely due to the extensive many-body nature of the \( Y \) operators.

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