Hydrodynamics from charged black branes

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\textbf{Abstract:} We extend the recent work on fluid-gravity correspondence to charged black-branes by determining the metric duals to arbitrary charged fluid configuration up to second order in the boundary derivative expansion. We also derive the energy-momentum tensor and the charge current for these configurations up to second order in the boundary derivative expansion. We find a new term in the charge current when there is a bulk Chern-Simons interaction thus resolving an earlier discrepancy between thermodynamics of charged rotating black holes and boundary hydrodynamics. We have also confirmed that all our expressions are covariant under boundary Weyl-transformations as expected.

\textbf{Keywords:} AdS/CFT, Hydrodynamics, Charge Black Holes.
1. Introduction

Modern theoretical physics provides mathematically precise descriptions of a bewildering variety of phenomena. The diversity of the phenomena studies by physicists encourages specialization and a consequent divergence in the field. We find it satisfying, however, that this push towards divergence is partially counterbalanced by periodic theoretical discoveries that unify - i.e. discover precise mathematical connections between - distinct fields of physics. In line with this tradition, recent string theory inspired studies of classical gravitational dynamics have found a precise mathematical connection between a long distance limit of the Einstein equations of gravity and the Navier Stokes equations of fluid dynamics. More specifically, it has recently been demonstrated that a class of long distance, regular, locally asymptotically $AdS_{d+1}$ solutions to Einstein’s equations with a negative cosmological constant is in one to one correspondence with solutions to the charge free Navier Stokes equations in $d$ dimensions [1–6, 8].

There exists a large literature in deriving linearise hydrodynamics from AdS/CFT. See([9] - [41]). There have been some recent work on hydrodynamics with higher derivative corrections [43, 44].
The connection between the equations of gravity and fluid dynamics, described above, was demonstrated essentially by use of the method of collective coordinates. The authors of [1,3–5,8] noted that there exists a $d$ parameter set of exact, asymptotically AdS$_{d+1}$ black brane solutions of the gravity equations parameterized by temperature and velocity. They then used the ‘Goldstone’ philosophy to promote temperatures and velocities to fields. The Navier Stokes equations turn out to be the effective ‘chiral Lagrangian equations’ of the temperature and velocity collective fields.

This initially surprising connection between gravity in $d + 1$ dimensions and fluid dynamics in $d$ dimensions is beautifully explained by the AdS/CFT correspondence. Recall that a particular large $N$ and strong coupling limit of that correspondence relates the dynamics of a classical gravitational theory (a two derivative theory of gravity interacting with other fields) on AdS$_{d+1}$ space to the dynamics of a strongly coupled conformal field theory in $d$ flat dimensions. Now the dynamics of a conformal field theory, at length scales long compared to an effective mean free path (more accurately an equilibration length scale) is expected to be well described by the Navier Stokes equations. Consequently, the connection between long wavelength solutions of gravity and the equations of fluid dynamics - directly derived in [1] - is a natural prediction of the AdS/CFT correspondence. Using the AdS/CFT correspondence, the stress tensor as a function of velocities and temperatures obtained above from gravity may be interpreted as the stress tensor fluid of the dual boundary field theory in its deconfined phase.

Now consider a conformal field theory that has a conserved charge $Q$ in addition to energy and momentum. This is especially an interesting extension of the hydrodynamics of the uncharged fluids since the hydrodynamics of many real fluids has a global conserved charge which is often just the number of particles that make up the fluid. The long distance dynamics of such a system is expected to be determined by the augmented Navier Stokes equations; $\nabla_\mu T^{\mu\nu} = 0$ together with $\nabla_\mu J^{\mu}_Q = 0$, where the stress tensor and charge current are now given as functions of the temperature, velocity and charge density, expanded to a given order in the derivative expansion. The bulk dual description of a field theory with a conserved charge always includes a Maxwell field propagating in the bulk. Consequently the AdS/CFT correspondence suggests asymptotically AdS long wavelength solutions of appropriate modifications of the the Einstein Maxwell equation are in one to one correspondence with solutions of the augmented Navier Stokes equations described above.

This expectation of the previous paragraph also fits well with the collective coordinate intuition described above. Recall that the Einstein Maxwell equations have a well known $d + 1$ dimensional set of charged black brane solutions, parameterized by the brane temperature, charge density and velocity. It seems plausible that the effective Goldstone equations, that arise from the promotion of these $d + 1$ dimensional parameters to fields, are simply the augmented Navier Stokes equations. In this paper we verify the expectations via a direct analysis of the relevant bulk equations. More concretely, we generalize the work out in [1] to set up a perturbative scheme to generate long wavelength solutions of the Einstein Maxwell equations plus a Chern Simons term (see below for more details) order by order in the derivative expansion. We also implement this expansion to second order, and thereby find explicit expressions for the stress tensor and charge current of our dual fluid to second order in the derivative expansion.

In this paper we work with the Einstein Maxwell equations augmented by a Chern Simon’s term. This is because the equations of IIB SUGRA on AdS$_5 \times S^5$ (which is conjectured to be dual to $\mathcal{N} = 4$ Yang Mills) with the restriction of equal charges for the three natural Cartans, admit a consistent
truncation to this system. Under this truncation, we get the following action:

$$S = \frac{1}{16\pi G_5} \int \sqrt{-g_5} \left[ R + 12 - F_{AB}F^{AB} + \frac{4\kappa}{3} \epsilon^{LABCD} A_L F_{AB} F_{CD} \right]$$

(1.1)

The value of the parameter $\kappa$ for $\mathcal{N} = 4$ Yang Mills is given by $\kappa = -1/(2\sqrt{3})$ - however, with a view to other potential applications we leave $\kappa$ as a free parameter in all the calculations below. Note in particular that our bulk Lagrangian reduces to the true Einstein Maxwell system at $\kappa = 0$.

Our expressions for the charge current and the stress tensor of the fluid are complicated, and are listed in detail in (). We would however to point out an important qualitative feature of our result. Already at first order, and at nonzero $\kappa$, the charge current includes a term proportional to $l^\alpha \equiv \epsilon^{\mu\nu\lambda\alpha} u_\mu \nabla_\nu u_\lambda$. The presence of this term in the current resolves an apparent mismatch between the predictions of fluid dynamics and the explicit form of charged rotating black holes in IIB supergravity reported in [42]. Note that despite the presence of the $\epsilon$ symbol, this term is parity symmetric, when accompanied by a flip in the charge of the brane, and so is CP symmetric in agreement with expectations of the CP symmetry of $\mathcal{N} = 4$ Yang Mills theory.

As we have explained above, the reduction of boundary field theory dynamics is expected to reduce to field theory dynamics only at long wavelength compared to an effective mean free path or equilibration length scale. All the gravitational constructions of this paper also work only in the same limit. It is consequently of interest to know the functional form of the equilibration length scale of our conformal fluid as a function of intensive fluid parameters.

In the case of $\mathcal{N} = 4$ Yang Mills, it follows from ‘t Hooft scaling and dimensional analysis that, at large $\lambda$, the effective equilibration length scale is given by $l_{mf} = f(\nu)/T$ where $\nu$ is the dimensionless chemical potential conjugate to the conserved charge of the theory. Explicit computation within gravity demonstrates that $f(\nu)$ is of unit order for generic values of $\nu$. Consequently, at generic values of $\nu$, all the considerations of this paper apply only when all fields vary at distances and times that are large compared to the local effective temperature. However, as was explained in detail in [42], the charged fluid we study in this paper has an upper bound on $\nu$ at $\nu_c$. At this special value of $\nu$, the black brane becomes extremal. $f(\nu)$ appears to have a simple zero about $\nu_c$. Consequently it appears to be possible to scale $\nu$ to $\nu_c$ and $T$ simultaneously to zero while keeping $l_{mf}$ (and all other thermodynamic densities) finite. Thus it would naively appear that the long distance field theory dynamics should be well described by fluid dynamics in this coordinated extremal limit. It turns out, however, that the bulk gravitational solutions described in this paper turn singular in the same limit. We are not completely sure how to interpret this fact. It would certainly be interesting to investigate this further.

**Note Added:** While this draft was in preparation, we became aware of a similar work by Haack et.al.(arXiv:0809.2488).

### 2. Notations and Conventions

In this section, we will establish the basic conventions and notations that we will use in the rest of the paper. We start with the five-dimensional action\(^2\)

$$S = \frac{1}{16\pi G_5} \int \sqrt{-g_5} \left[ R + 12 - F_{AB}F^{AB} + \frac{4\kappa}{3} \epsilon^{LABCD} A_L F_{AB} F_{CD} \right]$$

(2.1)

\(^2\)We use Latin letters $A, B \in \{r, v, x, y, z\}$ to denote the bulk indices and $\mu, \nu \in \{v, x, y, z\}$ to denote the boundary indices.
which is a consistent truncation of IIB SUGRA Lagrangian on $\text{AdS}_5 \times S^5$ background with a cosmological constant $\Lambda = -6$ and the Chern-Simons parameter $\kappa = -1/(2\sqrt{3})$ ([45] - [52]). However, for the sake of generality (and to keep track of the effects of the Chern-Simons term), we will work with an arbitrary value of $\kappa$ in the following.

The field equations corresponding to the above action are

$$
G_{AB} - 6g_{AB} + 2 \left[ F_{AC} F^C_B + \frac{1}{4} g_{AB} F_{CD} F^{CD} \right] = 0
$$

(2.2)

$$
\nabla_B F^{AB} + \kappa \epsilon^{ABCDE} F_{BC} F_{DE} = 0
$$

where $g_{AB}$ is the five-dimensional metric, $G_{AB}$ is the five dimensional Einstein tensor. These equations admit an AdS-Reissner-Nordström black-brane solution

$$
ds^2 = -2u_\mu dx^\mu dr - r^2 V(r, m, q) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu
$$

(2.3)

where $u_\mu dx^\mu = -dv; \quad V(r, m, q) \equiv 1 - \frac{m}{r^2} + \frac{q^2}{r^4}$

$$
P_{\mu\nu} \equiv \eta_{\mu\nu} + u_\mu u_\nu
$$

(2.4)

with $\eta_{\mu\nu} = \text{diag}(-+++)$ being the Minkowski-metric. Following the procedure elucidated in [1], we shall take this flat black-brane metric as our zeroth order metric/gauge field ansatz and promote the parameters $u_\mu, m$ and $q$ to slowly varying fields.

In the course of our calculations, we will find it convenient to use the following ‘rescaled’ variables

$$
\rho \equiv \frac{r}{R}; \quad M \equiv \frac{m}{R^4}; \quad Q \equiv \frac{q}{R^3}; \quad Q^2 = M - 1
$$

(2.5)

where $R$ is the radius of the outer horizon, i.e., the largest positive root of the equation $V = 0$. In terms of the rescaled variables, the outer and the inner horizon are given by

$$
\rho_+ \equiv 1 \quad \text{and} \quad \rho_- \equiv (Q^2 + 1/4)^{1/2} - 1/2
$$

and the extremality condition $\rho_+ = \rho_-$ corresponds to $(Q^2 = 2, M = 3)$. We shall assume the black-branes and the corresponding fluids to be non-extremal unless otherwise specified - this corresponds to the regime $0 < Q^2 < 2$ or $0 < M < 3$ which we will assume henceforth.

Using the flat black-brane solutions with slowly varying velocity, temperature and charge fields, our intention is to systematically determine the corrections to the metric and the gauge field in a derivative expansion. More precisely, we expand the metric and the gauge field in terms of derivatives of velocity, temperature and charge fields of the fluid as

$$
g_{AB} = g_{AB}^{(0)} + g_{AB}^{(1)} + g_{AB}^{(2)} + \ldots
$$

$$
A_M = A_M^{(0)} + A_M^{(1)} + A_M^{(2)} + \ldots
$$

(2.6)

where $g_{AB}^{(k)}$ and $A_M^{(k)}$ contain the k-th derivatives of the velocity, temperature and the charge fields with

$$
g_{AB}^{(0)} dx^A dx^B = -2u_\mu(x) dx^\mu dr - r^2 V(r, m(x), q(x)) u_\mu(x) u_\nu(x) dx^\mu dx^\nu + r^2 P_{\mu\nu}(x) dx^\mu dx^\nu
$$

$$
A_M^{(0)} dx^M = \frac{\sqrt{3} q(x)}{2r^2} u_\mu(x) dx^\mu
$$

(2.7)
In order to solve the Einstein-Maxwell-Chern-Simons system of equations, it is necessary to work in a particular gauge for the metric and the gauge fields. Following [1], we choose our gauge to be

\[ g_{rr} = 0; \quad g_{r\mu} \propto u_\mu; \quad A_r = 0; \quad Tr[(g^{(0)})^{-1} g^{(k)}] = 0 \quad (2.8) \]

Further, in order to relate the bulk dynamics to boundary hydrodynamics, it is useful to parameterise the fluid dynamics in the boundary in terms of a ‘fluid velocity’ \( u_\mu \). In case of relativistic fluids with conserved charges, there are two widely used conventions of how the fluid velocity should be defined. In this paper, we will work with the Landau frame velocity where the fluid velocity is defined with reference to the energy transport. In a more practical sense working in the Landau frame amounts to taking the unit time-like eigenvector of the energy-momentum tensor at a point to be the fluid velocity at that point.

Alternatively, one could work in the “Eckart frame” where the fluid velocity is defined with reference to the charge transport where the unit time-like vector along the charged current to be the definition of fluid velocity. Though the later is often the more natural convention in the context of charged fluids, we choose to use the Landau’s convention for the ease of comparison with the other literature. We will leave the conversion to the more natural Eckart frame to the future work.

In the next section, we will report in some detail the calculations relevant to finding the first order solution.

### 3. First Order Calculations

In this section, we present the computation of the metric and the gauge field up to first order in derivative expansion, the derivative being taken with respect to the boundary coordinates. We choose the boundary coordinates such that \( u_\mu = (1, 0, 0, 0) \) at \( x^\mu \). Since our procedure is ultra local therefore we intend to solve for the metric and the gauge field at first order about this special point \( x^\mu \). We shall then write the result thus obtained in a covariant form which will be valid for arbitrary choice of boundary coordinates.

In order to implement this procedure we require the zeroth order metric and gauge field expanded up to first order. For this we recall that the parameters \( m, q \) and the velocities \( (\beta_1) \) are functions of the boundary coordinates and therefore admit an expansion in terms of the boundary derivatives. These parameters expanded up to first order is given by,

\[ m = m_0 + x^\mu \partial_\mu m^{(0)} + \ldots \]
\[ q = q_0 + x^\mu \partial_\mu q^{(0)} + \ldots \]
\[ \beta_1 = x^\mu \partial_\mu \beta_1 + \ldots \]

(3.1)

The zeroth order metric expanded about \( x^\mu \) up to first order in boundary coordinates is given by

\[ ds^{(0)}_2 = 2 dv dr - r^2 V^{(0)}(r) dv^2 + r^2 dx_i dx^i - 2 x^\mu \partial_\mu \beta_1^{(0)} dx^i dr - 2 x^\mu \partial_\mu \beta_1^{(0)} r^2 (1 - V^{(0)}(r)) dx^i dv - \left( -\frac{x^\mu \partial_\mu m^{(0)}}{r^2} + \frac{2 m_0 x^\mu \partial_\mu q^{(0)}}{r^4} \right) dv^2, \]

(3.2)
where $m_0$ and $q_0$ are related to the mass and charge of the background blackbrane respectively and

$$V^{(0)} = 1 - \frac{m_0}{r^4} + \frac{q_0}{r^6}.$$ 

Similarly the zeroth order gauge fields expanded about $x^\mu$ up to first order is given by,

$$A = -\frac{\sqrt{3}}{2} \left[ \left( \frac{q_0 + x^\mu \partial_\mu q^{(0)}}{r^2} \right) dv - \frac{q_0}{r^3} x^\mu \partial_\mu \beta_i dx^i \right]$$ (3.3)

Since the background black brane metric preserves an $SO(3)$ symmetry, the Einstein-Maxwell equations separate into equations in scalar, transverse vector and the symmetric traceless transverse tensor sectors. This in turn allows us to solve separately for $SO(3)$ scalar, vector and symmetric traceless tensor components of the metric and the gauge field.

### 3.1 Scalars Of $SO(3)$

The scalar components of first order metric and gauge field perturbations ($g^{(1)}$ and $A^{(1)}$ respectively) are parameterized by the functions $h_1(r)$, $k_1(r)$ and $w_1(r)$ as follows,

$$\sum_i g^{(1)}_{ii}(r) = 3r^2 h_1(r),$$

$$g^{(1)}_{rr}(r) = \frac{k_1(r)}{r^2},$$

$$g^{(1)}_{vr}(r) = -\frac{3}{2} h_1(r),$$

$$A^{(1)}_v(r) = \frac{\sqrt{3} w_1(r)}{2 r^2}.$$ (3.4)

Note that $g^{(1)}_{ii}(r)$ and $g^{(1)}_{vr}(r)$ are related to each other by the gauge choice $Tr[(g^{(0)})^{-1}g^{(1)}] = 0$.

#### Constraint equations

We begin by finding the constraint equations that constrain various derivatives velocity, temperature and charge that appear in the first order scalar sector. The constraint equations are obtained by taking a dot of the Einstein and Maxwell equations with the vector dual to the one form $dr$. If we denote the Einstein and the Maxwell equations by $E_{AB} = 0$ and $M_{AB} = 0$, then there are three constraint relations.

Two of them come from Einstein equations. They are given by,

$$g^{rr} E_{rr} + g^{rv} E_{rv} = 0,$$ (3.5)

and,

$$g^{rr} E_{rr} + g^{vv} E_{vv} = 0.$$ (3.6)

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3We are referring to here to the $SO(3)$ rotational symmetry in the boundary spatial coordinates.

4Here $i$ runs over the boundary spatial coordinates, $v$ is the boundary time coordinate and $r$ is the radial coordinate in the bulk.
and the third constraint relation comes from Maxwell equations and is given by,
\[ g^{rr} M_r + g^{rv} M_v = 0 \tag{3.7} \]

Equation (3.5) reduces to
\[ \partial_v m^{(0)} = -\frac{4}{3} m_0 \partial_i \beta_i^{(0)} \tag{3.8} \]
which is same as the conservation of energy in the boundary at the first order in the derivative expansion, i.e., the above equation is identical to the constraint
\[ \partial_{\mu} T^{\mu \nu}_{(0)} = 0 \tag{3.9} \]
on the allowed boundary data.

The second constraint equation (3.6) in scalar sector implies a relation between \( h_1(r) \) and \( k_1(r) \).
\[ 2 \partial_i  \beta_i^{(0)} r^5 + 12 r^6 h_1(r) + 4 q_0 w_1(r) - m_0 r^3 h'_1(r) + 3 r^7 h'_1(r) - r^3 k'_1(r) - 2 q_0 r w'_1(r) = 0. \tag{3.10} \]

The constraint relation coming from Maxwell equation (See Eq. (3.7)) gives
\[ \partial_v q^{(0)} = -q_0 \partial_i \beta_i^{(0)} \tag{3.11} \]
This equation can be interpreted as the conservation of boundary current density at the first order in the derivative expansion.
\[ \partial_{\mu} J^\mu_{(0)} = 0. \tag{3.12} \]

We now proceed to find the scalar part of the metric dual to a fluid configuration which obeys the above constraints.

**Dynamical equations and the solutions**

Among the Einstein equations four are \( SO(3) \) scalars (namely the \( vv, rv, rr \) components and the trace over the boundary spatial part). Further the \( r \) and \( v \)-components of the Maxwell equations constitute two other equations in this sector. Two specific linear combination of the \( rr \) and \( vv \) components of the Einstein equations constitute the two constraint equations in (3.8). Further, a linear combination of the \( r \) and \( v \)-components of the Maxwell equations appear as a constraint equation in (3.11). Now among the six equations in the scalar sector we can use any three to solve for the unknown functions \( h_1(r), k_1(r) \) and \( w_1(r) \) and we must make sure that the solution satisfies the rest. The simplest two equations among these dynamical equations are,
\[ 5 h'_1(r) + rh''_1(r) = 0. \tag{3.13} \]
which comes from the \( rr \)-component of the Einstein equation and,
\[ 6 q_0 h'_1(r) + w'_1(r) - rw''_1(r) = 0. \tag{3.14} \]
which comes from the \( r \)-components of the Maxwell equation. We intend to use these dynamical equations (3.13), (3.14) along with one of the constraint equations in (3.8) to solve for the unknown functions \( h_1(r), k_1(r) \) and \( w_1(r) \).

Solving (3.13) we get,
\[ h_1(r) = \frac{C^1_{h_1}}{r^4} + C^2_{h_1}, \tag{3.15} \]
where $C_{h_1}^1$ and $C_{h_1}^2$ are constants to be determined. We can set $C_{h_1}^2$ to zero as it will lead to a non-normalizable mode of the metric. We then substitute the solution for $h_1(r)$ from (3.15) into (3.14) and solve the resultant equation for $w_1(r)$. The solution that we obtain is given by,

$$w_1(r) = C_{w_1}^1 r^2 + C_{w_1}^2 - q_0 \frac{C_{h_1}^1}{r^4}.$$  

(3.16)

Here again $C_{w_1}^1$, $C_{w_1}^2$ are constants to be determined. Again $C_{w_1}^1$ corresponds to a non-normalizable mode of the gauge field and therefore can be set to zero. Finally plugging in these solutions for $h_1(r)$ and $w_1(r)$ into one of the constraint equations in (3.8) and then solving the subsequent equation we obtain,

$$k_1(r) = \frac{2}{3} r^3 \partial_i \beta_i^{(0)} + C_{k_1(r)} - \frac{2q_0}{r} C_{w_1}^2 + \left( \frac{2q_0^2}{r^6} - \frac{m_0}{r^4} \right) C_{h_1}^1$$

(3.17)

Now the constants $C_{k_1}$ and $C_{w_1}^2$ may be absorbed into redefinitions of mass $(m_0)$ and charge $(q_0)$ respectively and hence may be set to zero. Further we can gauge away the constant $C_{h_1}^1$ by the following redefinition of the $r$ coordinate,

$$r \rightarrow r \left( 1 + \frac{C}{r^4} \right),$$

$C$ being a suitably chosen constant.

Thus we conclude that all the arbitrary constants in this sector can be set to zero and therefore our solutions may be summarized as,

$$h_1(r) = 0, \quad w_1(r) = 0, \quad k_1(r) = \frac{2}{3} r^3 \partial_i \beta_i^{(0)}.$$  

(3.18)

In terms of the first order metric and gauge field this result reduces to,

$$\sum_i g_{ii}^{(1)}(r) = 0,$$

$$g_{vv}^{(1)}(r) = \frac{2}{3} r \partial_i \beta_i^{(0)},$$

$$g_{vr}^{(1)}(r) = 0,$$

$$A_v^{(1)}(r) = 0.$$  

(3.19)

Now, we proceed to solving the equations in the vector sector.

### 3.2 Vectors Of $SO(3)$

The vector components of metric and gauge field $g^{(1)}$ and $A^{(1)}$ are parameterized by the functions $j_i^{(1)}(r)$ and $g_i^{(1)}(r)$ as follows,

$$g_{ii}^{(1)}(r) = \frac{m_0}{r^2} - \frac{q_0}{r^4} j_i^{(1)}(r)$$

$$A_i^{(1)}(r) = - \left( \frac{\sqrt{3}q_0}{2r^4} \right) j_i^{(1)}(r) + g_i^{(1)}(r)$$

(3.20)

Now we intend to solve for the functions $j_i^{(1)}(r)$ and $g_i^{(1)}(r)$. 

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Constraint equations

The constraint equations in the vector sector comes only from the Einstein equation. So there is only one constraint equation in this sector. It is given by

\[ g^{rr} E_{ri} + g^{rv} E_{vi} = 0 \]  \hspace{1cm} (3.21)

which implies

\[ \partial_i m^{(0)} = -4m_0 \partial_v \beta_i^{(0)}. \]  \hspace{1cm} (3.22)

These equations also follow from the conservation of boundary stress tensor at first order. We shall use this constraint equation to simplify the dynamical equations in the vector sector.

Dynamical equations and their solutions

In the vector sector we have two equations from Einstein equations (the \( ri \) and \( vi \)-components) and one from Maxwell equations (the \( i \)-component) \(^5\).

The dynamical equation obtained from the \( vi \)-component of the Einstein equations is given by,

\[ \left( q_0^2 - 3 m_0 r^2 \right) \frac{dg_i^{(1)}(r)}{dr} + 4 \sqrt{3} q_0 r^2 \frac{dg_i^{(1)}(r)}{dr} + (m_0 r^2 - q_0^2) r \frac{d^2 g_i^{(1)}(r)}{dr^2} = -3r^4 \partial_v \beta_i^{(0)}. \]  \hspace{1cm} (3.23)

Also the dynamical equation from the \( i \)-component of the Maxwell equation is given by,

\[
 r \left[ 2 \left( r^6 - m_0 r^2 + q_0^2 \right) \frac{d^2 j_i^{(1)}(r)}{dr^2} + (6r^7 + 2m_0r^3 - 6q_0^2) \frac{dj_i^{(1)}(r)}{dr} \right] \\
 - \sqrt{3} q_0 r \left( r^6 - m_0 r^2 + q_0^2 \right) \frac{d^2 j_i^{(1)}(r)}{dr^2} + \sqrt{3} q_0 \left( r^6 - 3m_0 r^2 + 5q_0^2 \right) \frac{dj_i^{(1)}(r)}{dr} \\
 = \sqrt{3} (q_0 \partial_v \beta_i^{(0)} + \partial_v q^{(0)}) r^3 - 24 q_0^2 \lambda r_i^{(0)},
\]

where \( l_i \) is defined as

\[ l_i \equiv \epsilon_{ijk} \partial_j \beta_k. \]  \hspace{1cm} (3.25)

Now in order to solve this coupled set of differential equations (3.23) and (3.24) we shall substitute \( g_i^{(1)}(r) \) obtained from (3.23) into (3.24) and solve the resultant equation for \( j_i^{(1)}(r) \). For any function \( j_i^{(1)}(r) \), using (3.23) \( g_i^{(1)}(r) \) may be expressed as,

\[ g_i^{(1)}(r) = (C_g)_i + \frac{1}{4 \sqrt{3} q_0} \left( - \partial_v \beta_i^{(0)} r^3 + 4m_0 j_i^{(1)}(r) - \frac{(m_0 r^2 - q_0^2) d^2 j_i^{(1)}(r)}{dr^2} \right). \]  \hspace{1cm} (3.26)

Here \((C_g)_i\) is an arbitrary constant. It corresponds to non normalizable mode of the gauge field and hence may be set to zero.

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\(^5\)Note that a linear combination of the \( ri \) and \( vi \)-components of the Einstein equation appear as the constraint equation in (3.22).
Substituting this expression for $g_i^{(1)}(r)$ into (3.24) we obtain the following differential equation for $j_i^{(1)}(r)$,

$$
(35q_0^2 + 5r^2 (r^4 - 6m_0) q_0^2 + 3m_0 r^4 (3r^4 + m_0)) \frac{dj_i^{(1)}(r)}{dr} r (-11q_0^2 - (5r^4 - 14m_0 r^2) q_0^{-2} m_0 r^4 (r^4 + m_0)) \frac{d^2 j_i^{(1)}(r)}{dr^2}
$$

$$
+ r^2 (q_0^2 - m_0 r^2) (r^6 - m_0 r^2 + 3 q_0^2) \frac{d^3 j_i^{(1)}(r)}{dr^3}
$$

$$
= \frac{1}{\sqrt{3}} \left( 6\sqrt{3} q_0 \partial_i q_0^{(0)} r^4 + 3 \sqrt{3} \partial_i \beta_i^{(0)} (5r^6 - m_0 r^2 - q_0^2) r^4 - 144 \frac{r}{3} l_i^{(0)} q_0^4 \right)
$$

The solution to this equation is given by,

$$
j_i^{(1)}(r) = (C_j^1)_i + \frac{1}{m_0} \frac{r \partial_i \beta_i^{(0)}}{r^2} \frac{6q_0 (q_0 \partial_i q_0^{(0)} + 3q_0 \partial_i \beta_i^{(0)})}{R_0^6 \left( \frac{m_0}{r^2} - \frac{q_0^2}{r^4} \right)} F_1(\frac{r}{R_0}, \frac{m_0}{R_0^2}),
$$

(3.28)

where again $(C_j^1)_i$ and $(C_j^2)_i$ are arbitrary constants. $(C_j^2)_i$ corresponds to a non-normalizable mode of the metric and so is set to zero. $(C_j^1)_i$ can be absorbed into a redefinition of the velocities and hence is also set to zero.

Here the function $F_1(\frac{r}{R_0}, \frac{m_0}{R_0})$ is given by$^6$,

$$
F_1(\rho, M) \equiv \frac{1}{3} \left( 1 - \frac{M}{\rho^4} + \frac{Q^{2}}{\rho^8} \right) \int_{\rho}^{\infty} dp \frac{1}{1 - \frac{M}{\rho^4} + \frac{Q^2}{\rho^8}} \left( 1 + \frac{1}{M} \right),
$$

(3.29)

where $Q^2 = M - 1$.

Substituting this result for $j_i^{(1)}(r)$ into (3.26) we obtain the following expression for $g_i^{(1)}(r)$,

$$
g_i^{(1)}(r) = \frac{1}{4 \sqrt{3} q_0 R_0^6 (R_0^6 + m_0 (r^2 - R_0^2))} \left[ 6q_0^2 \left( \beta \cdot q_0 \beta \right) \frac{R_0^8}{R_0^6} + q_0 (q_0 \partial_i q_0^{(0)} + 3q_0 \partial_i \beta_i^{(0)}) \left( (R_0^6 - m_0 R^2 + m_0 r^2) F_1^{(1,0)}(\rho, M) \right)^5 
$$

$$
+ 6F_1(\rho, M) R_0^3 \left( m_0 - R_0^6 \right) r^4 \right]
$$

(3.30)

where we use the notation $f^{(i,j)}(\alpha, \beta)$ to denote the partial derivative $\partial^{i+j} f/\partial \alpha^i \partial \beta^j$ of the function $f$.

Plugging back $j_i^{(1)}(r)$ and $g_i^{(1)}(r)$ back into (3.20) we conclude that the first order metric and gauge field in the vector sector is given by,

$$
j_v^{(1)}(r) = r \partial_i \beta_i^{(0)} + \frac{\sqrt{3} l_i^{(0)} q_0^4 \kappa}{m_0 r^4} + 6q_0 \partial_i q_0^{(0)} + 3q_0 \partial_i \beta_i^{(0)} \right) F_1(\frac{r}{R_0}, \frac{m_0}{R_0^2})
$$

$$
A_i^{(1)}(r) = -\frac{\sqrt{3} m_0 q_0 (q_0 \partial_i q_0^{(0)} + 3q_0 \partial_i \beta_i^{(0)} \right) F_1^{(1,0)}(\frac{r}{R_0}, \frac{m_0}{R_0^2})}{12m_0 q_0 r^2} R_0^3 r^7 + 18l_i q_0^4 \kappa.
$$

(3.31)

$^6$Although the expression for $F_1(\frac{r}{R_0}, \frac{m_0}{R_0^2})$ is very complicated but it satisfies some identities. One can use those identities and simplify the expressions.
3.3 Tensors Of $SO(3)$

The tensor components of the first order metric is parameterized by the function $\alpha^{(1)}_{ij}(r)$ such that,

$$g^{(1)}_{ij} = r^2 \alpha^{(1)}_{ij}.$$  \hspace{1cm} (3.32)

The gauge field does not have any tensor components therefore in this sector there is only one unknown function to be determined.

There are no constraint equations in this sector and the only dynamical equation is obtained from the $ij$-component of the Einstein equation. This equation is given by,

$$r \left( r^6 - m_0 r^2 + \frac{q_0^2}{2} \right) \frac{d^2 \alpha_{ij}(r)}{dr^2} - \left( -5 r^6 + m_0 r^2 + \frac{q_0^2}{2} \right) \frac{d \alpha_{ij}(r)}{dr} = -6 \sigma^{(0)}_{ij} r^4$$  \hspace{1cm} (3.33)

where $\sigma_{ij}$ is given by,

$$\sigma^{(0)}_{ij} = \frac{1}{2} \left( \partial_i \beta^{(0)}_j + \partial_j \beta^{(0)}_i \right) - \frac{1}{3} \partial_k \beta^{(0)}_k \delta_{ij}.$$  \hspace{1cm} (3.34)

The solution to equation (3.33) obtained by demanding regularity at the future event horizon and appropriate normalizability at infinity. The solution is given by,

$$\alpha^{(1)}_{ij} = \frac{2}{R_0} \sigma_{ij} F_2 \left( \frac{r}{R_0} \frac{m_0}{R_0^2} \right),$$  \hspace{1cm} (3.35)

where the function $F_2(\rho, M)$ is given by,

$$F_2(\rho, M) \equiv \int_{\rho}^{\infty} \frac{p \left( p^2 + p + 1 \right)}{(p + 1) (p^4 + p^2 - M + 1)} dp.$$  \hspace{1cm} (3.36)

with $M \equiv m/R^4$ as before.

Thus the tensor part of the first order metric is determined to be,

$$g^{(1)}_{ij} = \frac{2r^2}{R_0} \alpha^{(1)}_{ij} F_2 \left( \frac{r}{R_0} \frac{m_0}{R_0^2} \right).$$  \hspace{1cm} (3.37)

3.4 The global metric/gauge field at first order

In this subsection, we gather the results of our previous sections to write down the entire metric and the gauge field accurate up to first order in the derivative expansion.

We get the metric as

$$ds^2 = g_{AB} dx^A dx^B$$

$$= -2u_{\mu} dx^\mu dr - r^2 V u_{\mu} u_{\nu} dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu$$

$$- 2u_{\mu} dx^\mu \left[ u^\lambda \nabla_\lambda u_{\nu} - \frac{\nabla_\lambda u^\lambda}{3} u_{\nu} \right] dx^\nu + \frac{2v^2}{R} F_2(\rho, M) \sigma_{\mu\nu} dx^\mu dx^\nu$$

$$- 2u_{\mu} dx^\mu \left[ \frac{\sqrt{3} \kappa q^3}{mr^4} l_{\nu} + \frac{6q^2}{R^2} P_\nu^\lambda D_\lambda q F_1(\rho, M) \right] dx^\nu + \ldots$$  \hspace{1cm} (3.38)

$$A = \left[ \frac{\sqrt{3} q}{2r^2} u_{\mu} + \frac{3\kappa q^2}{mr^2} l_{\mu} + \frac{\sqrt{3} \kappa^5}{2R^8} P_\mu^\lambda D_\lambda q F_1^{(1,0)}(\rho, M) \right] dx^\mu + \ldots$$

where we have defined
\[
V \equiv 1 - \frac{m}{r^4} + \frac{q^2}{r_0^4}; \quad \mu^\nu = \epsilon^{\nu\lambda\sigma\mu} u_\lambda u_\sigma; \quad P_\mu^\lambda \mathcal{D}_\lambda q = P_\mu^\lambda \nabla_\lambda q + 3(u^\lambda \nabla_\lambda u_\mu)q \\
\rho \equiv \frac{r}{R}; \quad M \equiv \frac{m}{R^4}; \quad Q \equiv \frac{q}{R^3}; \quad Q^2 = M - 1
\]

and

\[
F_1(\rho, M) \equiv \frac{1}{3} \left(1 - \frac{M}{\rho^4} + \frac{Q^2}{\rho^6}\right) \int_\rho^\infty \frac{d\rho}{(1 - \frac{M}{\rho^4} + \frac{Q^2}{\rho^6})^2} \left(\frac{1}{p^8} - \frac{3}{4p^6} \left(1 + \frac{1}{M}\right)\right) \\
F_2(\rho, M) \equiv \int_\rho^\infty \frac{p(p^2 + p + 1)}{(p + 1)(p^4 + p^2 - M + 1)} d\rho
\]

### 3.5 The Stress Tensor and Charge Current at first order

In this section, we obtain the stress tensor and the charge current from the metric and the gauge field. The stress tensor can be obtained from the extrinsic curvature after subtraction of the appropriate counterterms. We get the first order stress tensor as

\[
T_{\mu\nu} = p(\eta_{\mu\nu} + 4u_\mu u_\nu) - 2\rho \sigma_{\mu\nu} + \ldots
\]

where the fluid pressure \(p\) and the viscosity \(\eta\) are given by the expressions

\[
p \equiv \frac{R^4}{16\pi G_5}; \quad \eta \equiv \frac{R^3}{16\pi G_5} = \frac{s}{4\pi}
\]

where \(s\) is the entropy density of the fluid obtained from the Bekenstein formula.

To obtain the charge current, we use

\[
J_\mu = \lim_{r \to \infty} \frac{r^2 A_\mu}{8\pi G_5} = n u_\mu - \mathcal{D} P_\mu^\nu \mathcal{D}_\nu n + \xi l_\mu + \ldots
\]

where the charge density \(n\), the diffusion constant \(\mathcal{D}\) and an additional transport coefficient \(\xi\) for the fluid under consideration are given by

\[
n \equiv \sqrt{\frac{3q}{16\pi G_5}}; \quad \mathcal{D} = \frac{1 + M}{4MR}; \quad \xi \equiv \frac{3\kappa q^2}{16\pi G_5 m}
\]

We note that when the bulk Chern-Simons coupling \(\kappa\) is non-zero, apart from the conventional diffusive transport, there is an additional non-dissipative contribution to the charge current which is proportional to the vorticity of the fluid. To the extent we know of, this is a hitherto unknown effect in the hydrodynamics which is exhibited by the conformal fluid made of \(\mathcal{N} = 4\) SYM matter. It would be interesting to find a direct boundary reasoning that would lead to the presence of such a term - however, as of yet, we do not have such an explanation and we hope to return to this issue in future.

The presence of such an effect was indirectly observed by the authors of [42] where they noted a discrepancy between the thermodynamics of charged rotating AdS blackholes and the fluid dynamical prediction with the third term in the charge current absent. We have verified that this discrepancy is resolved once we take into account the effect of the third term in the thermodynamics of the rotating \(\mathcal{N} = 4\) SYM fluid. In fact, one could go further and compare the first order metric that we have obtained with rotating blackhole metrics written in an appropriate gauge. We have done this comparison up to first order and we find that the metrics agree up to that order.
4. Second Order Hydrodynamics

In this section we will find out the metric, stress tensor and charge current at second order in derivative expansion. We will follow the same procedure as in [1] but in presence charge parameter $q$.

We will consider the following metric and gauge field perturbations in second order,

$$g_{\alpha\beta}^{(2)}dx^\alpha dx^\beta = -3h_2(r)dvdr + r^2h_2(r)dx^i dx^i + \frac{k_2(r)}{r^2}dv^2 + 12r^2 j_i^{(2)}dvdx^i + r^2\alpha_{ij}^{(2)}dx^i dx^j \quad (4.1)$$

and

$$A_i^{(2)} = \frac{\sqrt{3}}{2r^2}w_i^{(2)}(r)$$

$$A_i^{(2)} = \frac{\sqrt{3}}{2}r^5 g_i^{(2)}(r)dx^i. \quad (4.2)$$

Here we have used a little different parameterizations (from first order) for metric and gauge field perturbations in the vector sector. We found that the corresponding dynamical equations for $j_i^{(2)}(r)$ and $g_i^{(2)}(r)$ can be written in more tractable form (we will see later).

Like neutral black brane case, here also we will list all the source terms (second order in derivative expansion) which will appear on the right hand side of the constraint dynamical equations in scalar, vector and tensor sectors. These source terms are built out of second derivatives of $m$, $q$ and $\beta$ or square of first derivatives of these three fields. We can group these source terms according to their transformation properties under $SO(3)$ group. A complete list has been provided in table 4. $l_i$ and $\sigma_{ij}$ are given by,

$$l_i = \epsilon_{ijk}\partial_j\beta_k$$

$$\sigma_{ij} = \frac{1}{2}\left(\partial_i\beta_j + \partial_j\beta_k\right) - \frac{1}{3}\delta_{ij}\partial_k\beta_k. \quad (4.3)$$

In this table we have already employed the first order conservation relations i.e. equation 3.9 and 3.10. Using these two relations we have eliminated the first derivatives of $m$ and $q$. At second order in derivative expansion we have to use the relations,

$$\partial_\mu \partial_\nu T^{\mu\nu}_{(0)} = 0 \quad (4.4)$$

and

$$\partial_\lambda \partial_\mu J^{\mu}_{(0)} = 0. \quad (4.5)$$

These two equations imply some relations between the second order source terms listed in the table.

$$S1 = \frac{S3}{3} - \frac{8m}{3}ST1 + \frac{16m}{9}ST3 - \frac{2m}{3}ST4 + \frac{4m}{3}ST5$$

$$S2 = -\frac{1}{4m}S3 + 4ST1 + \frac{4}{2}ST4 - ST5$$

$$QS1 = q(-ST1 - S2 + ST3) - QS5$$

$$V1 = \frac{-40}{9}V4 - \frac{4}{9}V5 + \frac{56}{3}VT1 + \frac{4}{3}VT2 + \frac{8}{3}VT3$$

$$V2 = \frac{10}{9}V4 + \frac{1}{9}V5 - \frac{2}{3}VT1 + \frac{1}{6}VT2 - \frac{5}{3}VT3$$

\[ (4.6) \]
Table 1: An exhaustive list of two derivative terms in made up from the mass, charge and velocity fields. In order to present the results economically, we have dropped the superscript on the velocities $\beta_i$, charge $q$ and the mass $m$, leaving it implicit that these expressions are only valid at second order in the derivative expansion.

\[
V_3 = -\frac{1}{3} VT4 + VT5 \\
QV1 = q \left( \frac{10}{3} V4 + \frac{1}{2} (VT2 + 2VT1 + 2VT3) + \frac{1}{3} V5 \right) \\
\quad + QV2 + \frac{1}{2} \left( 2QV4 + QV3 + \frac{2}{3} QV2 \right) \\
T1 = -4m \left( T3 + \frac{1}{4} TT5 - 4TT1 + \frac{1}{3} TT4 + TT6 \right) \tag{4.7}
\]
With these relation between the source terms we will now solve the Einstein equations and Maxwell equations to find out the constraint and dynamical equations at second order in derivative expansion. We will do the analysis sector by sector.

4.1 Scalar Sector

Our parametrization for metric and gauge field are following,

\[ \sum_i g_{ii}^{(2)}(r) = 3r^2h_2(r), \]
\[ g_{vv}^{(2)}(r) = \frac{k_2(r)}{r^2}, \]
\[ g_{rr}^{(2)}(r) = -\frac{3}{2}h_2(r), \]
\[ A_v^{(2)}(r) = \frac{\sqrt{3}w_2(r)}{2r^2}. \]

**Scalar Constraint Equations**

As we have already explained, there are three constraint equations. First two come from Einstein equations (Eq. 3.5 and 3.5) and the third one comes from Maxwell equations (Eq. 3.7). The first constrain from Einstein equations gives,

\[ \partial_v m^{(1)} = \frac{2}{3}R_0^3 ST5 \]

Second constraint implies relation between \( k_2(r) \) and \( h_2(r) \). This constraint equation is given by,

\[ k'_2(r) = S_k \]

where \( S_k \) is given in appendix A.

The constraint relation coming from Maxwell equations is given by,

\[ \partial_v q^{(1)} = \frac{3q_0}{16m_0^2R_0} \left( R_0^4 + m_0 \right) S3 + \frac{(R_0^4 + m_0)}{4m_0R_0} QS2 - \frac{6\sqrt{3}q_0^2\kappa}{m_0} ST2 + \frac{\sqrt{3}(m_0 - 11R_0^4)}{8m_0R_0} QS5 - \frac{2\sqrt{3}q_0\kappa}{m_0} QS3 - \frac{q_0}{4m_0R_0^3} QS4 + \frac{9q_0}{4m_0R_0} ST1 \]

**Scalar Dynamical Equations**

Scalar Dynamical Equation is given by \((E_{rr} = 0)\),

\[ 5(h')_2(r) + r(h'')_2(r) = S_h. \]

The source term \( S_h \) is given in appendix A.

The second dynamical equation in this sector, which is \( M(r) = 0 \), is given by\(^7\),

\[ \frac{dw_2(r)}{dr} - r \frac{d^2w_2(r)}{dr^2} - 6q_0 \frac{dh_2(r)}{dr} = \frac{2}{\sqrt{3}r^6V^{(0)}(r)}S_M(r). \]

\(^7\)There is a relative negative sign in the r.h.s. with respect to first order equation and this is because of different parametrization.
The source term \( S_M(r) \) is given in appendix A.

The source terms have the same large \( r \) behavior as uncharged case because the charge dependent terms (leading) are more suppressed than that of charge independent terms. So one can follow the same procedure to find the solution for \( h_2(r) \) and \( k_2(r) \).

\[
h_2(r) = -\frac{1}{4r^2} S^\infty_h + \int_r^\infty \frac{dx}{x^3} \int_x^\infty dy y^4 (S_h(y) - \frac{1}{y^3} S^\infty_h)
\]

and

\[
k_2(r) = \frac{r^2}{2} S^\infty_k - \int_r^\infty dx (S_k(x) - x S^\infty_k)
\]

4.2 Vector Sector

Vector Sector Constraint Relation

The constraint relation coming from Einstein equations (3.21) is given by,

\[
\partial_i m^{(1)} = \frac{10 R_0^3}{9} V4_i + \frac{10 R_0^3}{9} V5_i + \frac{10 R_0^3}{3} VT1_i - \frac{5 R_0^3}{6} VT2_i
\]

\[
+ \frac{6 q_0 R_0}{m_0 - 3 R_0} QV4_i - \frac{(21 R_0^7 - 43 m_0 R_0^3)}{3 (m_0 - 3 R_0^3)} VT3_i .
\]

Vector Sector Dynamical Equations

There are two vector dynamical equations. The first equation which comes from Einstein equation is given by,

\[
3r \left( 5 r q_0^2 g_i^{(2)}(r) + r \frac{d g_i^{(2)}(r)}{dr} q_0^2 + 5 \frac{d j_i^{(2)}(r)}{dr} + r \frac{d^2 j_i^{(2)}(r)}{dr^2} \right) = S^{(1)}_v(r)
\]

where \( S^{(1)}_v \) is the source terms given in the appendix B. The second dynamical equation comes from Maxwell equation and is given by,

\[
5 \sqrt{3} (7 r^6 - 3 m_0 r^2 + q_0^2) g_i^{(2)}(r)
\]

\[
12 \sqrt{3} q_0 \frac{d j_i^{(2)}(r)}{dr} + \sqrt{3} r (13 r^6 - 9 m_0 r^2 + 7 q_0^2) \frac{d g_i^{(2)}(r)}{dr}
\]

\[
\sqrt{3} r^2 (r^6 - m_0 r^2 + q_0^2) \frac{d^2 g_i^{(2)}(r)}{dr^2} = -2 S^{(2)}_v(r)
\]

where \( S^{(2)}_v \) is other source terms given in the appendix B. From this above two equations one can replace \( g_i^{(2)}(r) \) and write a differential equation for \( j_i^{(2)}(r) \).

\[
3 \sqrt{3} (-35 r^5 + 15 m_0 r^2 + 7 q_0^2) \frac{d j_i^{(2)}(r)}{dr} - 3 \sqrt{3} r (13 r^6 - 9 m_0 r^2 + 7 q_0^2) \frac{d^2 j_i^{(2)}(r)}{dr^2}
\]

\[
-3 \sqrt{3} r^2 (r^6 - m_0 r^2 + q_0^2) \frac{d^3 j_i^{(2)}(r)}{dr^3} = - \left( 2 \sqrt{3} \left( m_0 - 3 r^4 \right) S^{(1)}_v(r) + 6 q_0 r S^{(2)}_v(r) - \sqrt{3} (r^6 - m_0 r^2 + q_0^2) \frac{d S^{(2)}_v(r)}{dr} \right)
\]
This equation can be written as,

\[
\frac{d}{dr} \left( -\frac{1}{8m_0 R_0} \frac{d}{dr} \left( r^7 (V(0))^2 \frac{d}{dr} \left( \frac{1}{V(0)} j_{\mu}^{(2)} \left( \frac{r}{R_0} \right) \right) \right) \right) \\
= -\frac{1}{24\sqrt{3}m_0 R_0^2} \left( 2\sqrt{3}r (m_0 - 3r^4) S_{\nu}^{(1)}(r) + 6q_0 r S_{\nu}^{(2)}(r) - \sqrt{3} (r^5 - m_0 r^2 + q_0^2) \frac{dS_{\nu}^{(1)}(r)}{dr} \right) \\
= S_{\text{source}} 
\]

(4.20)

### 4.3 Charge Current

To find the charge current density we need to solve the coupled dynamical equations (Eq.4.17 and Eq.4.18) in this sector and we have to consider the asymptotic limit. The charge current at second order in derivative expansion is given by,

\[
J_{\mu}^{(2)} = \lim_{r \to \infty} \frac{r^2 A_{\mu}^{(2)}}{8\pi G_5} . 
\]

(4.21)

Now all the computation can be done setting \( R_0 = 1 \). In the final solution the correct power of \( R_0 \) can be restored using Weyl covariance of the current. The solution to the Eq. 4.20 can be written as follows,

\[
J_{i}^{(2)}(r) = \frac{r^6 - M r^2 + Q^2}{\rho^6} \int_\rho^\infty f_i(x)dx 
\]

(4.22)

where, \( M, Q \) and \( \rho \) are given by, \( M = \frac{m_0}{R_0^4}, \ Q = \frac{q_0}{R_0^3}, \ \rho = \frac{r}{R_0} \). The function \( f_i(x) \) is given by,

\[
f_i(x) = \frac{x^5}{(x^6 - M x^2 + Q^2)^2} \left( M s_{(1)} x^4 - 8M s_{(2)} x - 4M s_{(3)} \log(x) + 8M \int_x^\infty y H(y) dy + C \right) 
\]

(4.23)

where \( H(y) \) is given by,

\[
H(y) = \int_y^\infty S_{\text{Norm}}(x)dx 
\]

(4.24)

and the constant \( C \) is given by,

\[
C = -s_{(1)}(2M - 3) + 2s_{(2)}(5M - 3) + s_{(3)}(M - 3) + B 
\]

(4.25)

where,

\[
B = -2(M + 3) \int_1^\infty S_{\text{Norm}}(x)dx + 4M \int_1^\infty x^2 S_{\text{Norm}}(x)dx . 
\]

(4.26)

We choose this constant \( C \) in such a way to cancel the divergence (in this case logarithmic) at the horizon \( R_0 = 1 \) for the function \( J_{i}^{(2)}(r) \). The function \( S_{\text{Norm}} \) is the normalized part of the source terms where we have subtracted out the asymptotic divergent parts.

\[
S_{\text{Norm}} = S_{\text{source}} - s_{(1)} r - \frac{s_{(2)}}{r^2} - \frac{s_{(3)}}{r^3} . 
\]

(4.27)

The solution to the gauge field perturbation is given by,

\[
\delta_{i}^{(2)}(\rho) = \sigma \left( \frac{r}{R_0}, \frac{m_0}{R_0^4}, \frac{q_0}{R_0^3} \right) = -\frac{1}{Q} \frac{dJ_{i}^{(2)}(\rho)}{d\rho} - \frac{1}{6Q\rho^3} s_{\nu}^{(1)} + \frac{1}{3Q\rho^3} \int_{\rho}^\infty x^3 S_{\nu_{\text{Norm}}}^{(1)}(x)dx . 
\]

(4.28)
where,

\[ S_{\text{v norm}}^{(1)} (x) = S_{\nu}^{(1)} \frac{8_{\nu}}{r^2}. \]  

(4.29)

The charge current can now be conveniently written down in a manifestly Weyl-covariant form. The charge current is given by,

\[ J_i^{(2)} = -\frac{1}{8\pi G_5} \sum_{i=1}^{6} C_i W_i \]  

(4.30)

where,

\[ C_1 = -R_0 \frac{3\sqrt{M-1}}{4M} \]
\[ C_2 = R_0 \frac{2\sqrt{M-1}(3(M-1)(2M-3)a^2 - M^2)}{3M^2} \]
\[ C_3 = \frac{1}{R_0^2} \left( \frac{B_1}{\sqrt{M-1}} + \frac{9}{16M\sqrt{M-1}} \right) \]
\[ C_4 = \frac{1}{R_0^2} \left( \frac{B_4}{\sqrt{M-1}} + \frac{1}{16} \left( -\frac{9}{M\sqrt{M-1}} + \frac{16}{M-3} + \frac{8}{M} + 4 \right) \right) \]
\[ C_5 = \frac{1}{R_0^2} \frac{B_5}{\sqrt{M-1}} \]
\[ C_6 = R_0 \frac{B_6}{\sqrt{M-1}} \]  

(4.31)

Here \( B_i \) is the coefficient of \( W_i \) in \( B \) (defined in (4.26)) and the Weyl-covariant vector combinations \( W_i \) are given by

\[ W_1 = \frac{5V_4}{9} + \frac{5V_5}{9} + \frac{5VT_1}{3} - \frac{5VT_2}{12} - \frac{11VT_3}{6} = P_{\mu}^{\nu} D_{\lambda} \sigma_{\nu}^{\lambda} \]
\[ W_2 = \frac{5V_4}{3} - \frac{V_5}{3} - VT_1 - \frac{VT_2}{4} + \frac{VT_3}{2} = P_{\mu}^{\nu} D_{\lambda} \omega_{\nu}^{\lambda} \]
\[ W_3 = \frac{QV_3}{2} + QV_4 + \frac{3VT_2}{2} + 3qVT_3 = -P_{\mu}^{\nu} u_{\lambda}^{\nu} D_{\lambda} q \]
\[ W_4 = QV_4 + 3qVT_3 = \sigma_{\mu}^{\lambda} D_{\lambda} q \]
\[ W_5 = QV_3 + 3qVT_2 = \omega_{\mu}^{\lambda} D_{\lambda} q \]
\[ W_6 = VT_5 = u_{\lambda}^{\nu} D_{\lambda} l_{\mu} \]  

(4.32)

where we have written the vectors in terms of the Weyl-covariant derivative introduced in [6].

4.4 Tensors Of \( SO(3) \)

We now consider the tensor modes at second order. Following the first order calculations we parametrize the traceless symmetric tensor components of the second order metric by the function \( \tau_{ij}(r) \) such that,

\[ g^{(2)}_{ij} = r^2 \tau_{ij}. \]  

(4.33)

The \( ij \)-component of the Einstein equation gives the dynamical equation for \( \tau_{ij} \) which is similar to (3.33). However the source term of the differential equation is modified in the second order. This equation is given by,
\( (r^6 - m_0 r^2 + q_0^2) r''_{ij}(r) - (-5r^6 + m_0 r^2 + q_0^2) r'_{ij}(r) = T_{ij}(r). \) (4.34)

where,

\[
T_{ij}(r) = C_1(r) \left( TT_{1,ij} + \frac{1}{3} TT_{4,ij} + TT_{3,ij} \right) + C_2(r) TT_{5,ij} + C_3(r) TT_{6,ij} + C_4(r) TT_{7,ij} \\
+ C_5(r) \left( QT_{3,ij} + 6q_0 QT_{4,ij} + 9q_0^2 TT_{1,ij} \right) + C_6(r) \left( QT_{1,ij} + 12q_0 TT_{1,ij} + 8QT_{4,ij} \right) \\
+ C_7(r) \left( QT_{2,ij} + 3q_0 TT_{2,ij} \right) + C_8(r) \left( 4TT_{2,ij} + 2T_{2,ij} - TT_{3,ij} \right). \
\] (4.35)

with the coefficient functions,

\[
C_1(r) = -\frac{(r^2 + R_0 r + R_0^2) (3r^3 + R_0^3) - m_0(r + R_0)}{(r + R_0)(r^4 + R_0^4 r^2 + R_0^4 - m_0)} \\
+ \frac{3r}{R_0^2} \left( F_2 \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) R_0^3 + 18 (m_0 - R_0^2) F_1 \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) R_0 + 6r (m_0 - R_0^4) F_{1(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) \right) \\
\]

\[
C_2(r) = \frac{1}{2} \left( \frac{144 \kappa^2 q_0^4}{m_0^2 r^6} + \frac{36 \kappa^2 q_0^4}{m_0^2 r^4} - \frac{36 \kappa^2 q_0^4}{m_0^2 r^2} + \frac{27 \kappa^2 q_0^4}{R_0^2 r^6} \right) \\
+ \frac{9r^2 F_{1(1,0)}^2 \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) q_0^2}{R_0^6} - \frac{2q_0^2 m_0}{r^6} - \frac{m_0}{r^4} - 1 \right) \\
\]

\[
C_3(r) = -\frac{1}{R_0^2(r + R_0)(r^4 + R_0^4 r^2 + R_0^4 - m_0)} \left( 2 (R_0^6 (m_0(r + R_0) - 2r^3 (r^2 + R_0 r + R_0^2)) \\
- 3r (r + R_0)(r^4 + R_0^4 r^2 + R_0^4 - m_0) \right) \left( F_2 \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) R_0^3 + 9 (R_0^4 - m_0) F_1 \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) R_0 \right) \right) \\
+ 3r (R_0^2 - m_0) F_{1(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) R_0 \right) \right) \\
\]

\[
C_4(r) = -\frac{2r^2 F_{1(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) r^2 + 3R_0 F_2 \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) r + R_0^2}{2R_0^2} \right) \right) \\
\]

\[
C_5(r) = -\frac{3r^2}{2R_0} \left( -5R_0 F_{1(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) - r F_{1(2,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) \right) \left( 2R_0^2 + r^2 (r^6 - R_0^6) \right) \right) \\
+ m_0 \left( R_0^2 - r^2 \right) \left( 5R_0 F_{1(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) + r F_{1(2,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) \right) \right) \\
- \frac{6r^2}{R_0^2 (3R_0^2 - m_0)} \left( 6 (2R_0^3 - 3m_0) F_1 \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) R_0^3 - 12m_0 (m_0 - R_0^4) F_{1(0,1)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) R_0^3 \right) \right) \\
+ r \left( R_0^2 - m_0 \right) \left( r F_{1(2,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) R_0^3 + 4m_0 F_{1(1,1)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) R_0^3 \right) \right) \\
+ 3r^2 \left( 3R_0^8 - 4m_0 R_0^4 + m_0^2 \right) F_{1(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) \right) \\
\] (4.36)
The solution to (4.34) which is regular at the outer horizon and normalizable at infinity is given by,

\[
\tau_{ij} = -\int_r^\infty d\xi \left( \frac{\xi}{q0^2 - m0\xi^2 + \xi^6} \right) \int_{R_0}^\xi d\zeta \, 2 \, \zeta \, T_{ij}(\zeta).
\]

(4.38)

For the computation of the second order stress tensor of the boundary fluid we require the large \( r \) behavior of \( \tau_{ij} \) only. Using (4.38) and the coefficient functions in (4.36) we obtain such an asymptotic expansion for \( \tau_{ij} \) which is given by,

\[
\tau_{ij} = \frac{1}{r^2} \left( -\frac{1}{4R_0^6} TT5_{ij} - \frac{1}{2} TT7_{ij} + \frac{1}{R_0^2} TT6_{ij} \right) + \frac{1}{r^4} \left( N_{TT5} TT5_{ij} + N_{TT7} TT7_{ij} + N_{TT6} TT6_{ij} + N_{T3} \left( TT1_{ij} + \frac{1}{3} TT4_{ij} + T3_{ij} \right) \right)
\]

(4.39)

\[+N_{QT2} \left( QT2_{ij} + 3q_0 TT2_{ij} \right) + N_{QT1} \left( QT1_{ij} + 12q_0 TT1_{ij} + 8QT4_{ij} \right)
\]

\[+N_{TT3} \left( 4TT2_{ij} + 2T2_{ij} - TT3_{ij} + N_{T3} \left( QT3_{ij} + 6q_0 QT4_{ij} + 9q_0^2 TT1_{ij} \right) \right) \].

where,

\[
N_{TT5} = -\frac{(m_0 - R_0^4)(21R_0^4 - 672(m_0 - R_0^4)\kappa^2 + 53m_0)}{224m_0R_0^6}
\]

\[
N_{TT7} = \frac{R_0^2}{4} \left( F_2(1, \frac{m_0}{R_0^4}) - \int_1^\infty d\rho \left( \rho^3 F_2^{(1,0)}(\rho, \frac{m_0}{R_0^4}) + \rho \right) \right)
\]

\[
N_{TT6} = \frac{1}{R_0^4} \left( 3 \int_1^\infty d\rho \left( \rho^2 F_2(\rho, \frac{m_0}{R_0^4} - \rho) \right) + \int_1^\infty d\rho \left( 2(1 + \rho^2 + 2\rho^4) - \frac{m_0}{R_0^4}(1 + \rho) \right) \right)
\]

\[+ \frac{9}{R_0^4} \left( 1 - \frac{m_0}{R_0^4} \right) F_1(1, \frac{m_0}{R_0^4}) - \frac{1}{2R_0^3} \]

\[
N_{T3} = \frac{1}{R_0^4} \left( \frac{3}{2} \int_1^\infty d\rho \left( \rho^2 F_2(\rho, \frac{m_0}{R_0^4} - \rho) \right) + \int_1^\infty \frac{\rho(1 + \rho + \rho^2)(2 + 3\rho^3) - \frac{m_0}{R_0^4}(1 + \rho)}{(1 + \rho)(\rho^4 + \rho^2 + 1 - \frac{m_0}{R_0^4})} \right)
\]

\[+ \left( \frac{3m_0}{56R_0^4} - \frac{3}{7R_0^3} + \frac{3R_0^4}{8m_0} \right) \].
$$N_{QT2} = \frac{\sqrt{3}}{2m_0 R_0} \left( m_0 - R_0^4 \right) \left( 24m_0 F_1\left(1, \frac{m_0}{R_0^4}\right) - R_0^4 \right) \kappa$$

$$N_{QT1} = \frac{3q_0}{R_0^3} F_1\left(1, \frac{m_0}{R_0^4}\right)$$

$$N_{TT3} = \frac{\sqrt{3} q_0^3 \kappa}{4m_0 R_0^3}$$

$$N_{QT3} = \frac{1}{2R_0^3} \int_r^\infty \int_r^\infty r C_5(r)$$

The leading term of this asymptotic expansion gives divergent contributions to the stress tensor which are cancelled by divergence arising from the expansion of $g^{(0)} + g^{(1)}$ up to second order. The subleading term in (4.39) yield the second order stress tensor.

### 4.5 Stress Tensor

The stress tensor is obtained by the following formula,

$$T^\mu_\nu = -2 \lim_{r \to \infty} r^4 \left( K^\mu_\nu - \delta^\mu_\nu \right),$$

where $K^\mu_\nu$ is the extrinsic curvature normal to the constant $r$ surface. If we evaluate this expression on our solution the second order stress tensor is obtained to be proportional to the coefficient of the $\frac{1}{r^4}$ term in (4.39) multiplied by a factor of $4$.

The stress tensor so obtained can be shown to be Weyl-covariant using the following relations

$$u^\lambda D_\lambda \sigma_{\mu\nu} = TT1 + \frac{1}{3} TT4 + TT3.$$  

$$4 \left( \omega^\mu_\lambda \omega^\lambda_\nu + \frac{1}{3} P^\mu_\nu \omega^{\alpha\beta} \omega_{\alpha\beta} \right) = TT5$$

$$\sigma^\mu_\lambda \sigma^\lambda_\nu + \frac{1}{3} P^\mu_\nu \sigma^{\alpha\beta} \sigma_{\alpha\beta} = TT6.$$  

$$-2 \left( \omega^\mu_\lambda \sigma^\lambda_\nu + \omega^\mu_\nu \sigma^{\lambda\mu} \right) = TT7$$

$$\frac{1}{2} \left( P^\alpha_\mu P^\beta_\nu + P^\alpha_\nu P^\beta_\mu - \frac{2}{3} P^{\alpha\beta} P_{\mu\nu} \right) D_\alpha q D_\beta q = QT1 + 6q QT4 + 9q^2 TT1$$

$$\frac{1}{2} \left( P^\alpha_\mu P^\beta_\nu + P^\alpha_\nu P^\beta_\mu - \frac{2}{3} P^{\alpha\beta} P_{\mu\nu} \right) D_\alpha q D_\beta q - 3q u^\lambda D_\lambda \sigma_{\mu\nu}$$

$$-3q \left( \omega^\mu_\lambda \omega^\lambda_\nu + \frac{1}{3} P^{\mu\nu} \omega^{\alpha\beta} \omega_{\alpha\beta} \right) = QT1 + 8q QT4 + 12q TT1$$

$$D_\mu l_\nu + D_\nu l_\mu = 4TT2 + 2TT - TT3.$$  

$$\frac{1}{2} \left( l_\mu D_\nu q + l_\nu D_\mu q \right) = QT2 + 3q TT2.$$  

These relations can be used to express the energy-momentum tensor in the manifestly Weyl-covariant way.

### 5. Discussion

In this paper, we have computed the metric dual to a fluid with a globally conserved charge and used that to find the energy-momentum tensor and the charge current in arbitrary fluid configurations to second order in the boundary derivative expansion.
Note that the corresponding construction of the bulk metric dual to an uncharged fluid flow was characterized by a great deal of universality. This universality had its origin in the fact that every two derivative theory of gravity (interacting with other fields) that admits AdS$_5$ as a solution, also admits a consistent truncation to the equations of Einstein gravity with a negative cosmological constant. This universality does not extend to Einstein Maxwell system. The possibility of extending the Einstein Maxwell system by a Chern Simon’s term of arbitrary coefficient, accounted for in this paper, is an illustration of the reduced universality of our calculations.

We have seen that a nonzero value for the coefficient of the Chern-Simons in the bulk leads to an interesting dual hydrodynamic effect (recall that this coefficient is indeed nonzero in strongly coupled $\mathcal{N} = 4$ Yang Mills). At first order in the derivative expansion we find that the charge current has a term proportional to $l^\alpha \equiv \epsilon^{\mu\nu\lambda\alpha} u_\mu \nabla_\nu u_\lambda$ in addition to the more familiar Fick type diffusive term. It would be interesting to reproduce this term from a computation directly in the boundary. In more general terms, it would be interesting to gain better intuition for effect induced by the bulk Chern-Simons term on boundary dynamics.

We have already remarked on the fact that the pseudo-tensor terms appearing in the stress tensor and the charge current solve an old puzzle raised in [42] regarding the fluid-gravity correspondence in large rotating AdS charged black holes. The discrepancy found by the authors of [42] between the fluid dynamical predictions and the known thermodynamics of rotating blackholes is basically resolved, once the qualitatively new effects due to the bulk Chern-Simons interaction on hydrodynamics is taken into account. We will reserve a deeper analysis of this issue along with a detailed comparison of our second order fluid dynamical metric and gauge field with charged black hole solutions to future work.

Though we have not yet worked out explicitly the position of the event horizon in our metric solutions, it is plausible that the analysis of [2] can be easily extended to the case of metrics with a global conserved charge(at least for the non-extremal case). This expectation, however has to be confirmed by an explicit computation. Further, it would be interesting to derive an entropy current for the charged fluid following the proposal outlined in [2].

In the bulk of our work we have refrained from commenting on the fluids near extremality. The preliminary analysis in [42] suggests that hydrodynamics would be a valid description at least for some class of extremal solutions. Unfortunately, we have not been able to shed more light on this issue in the present work - the perturbative metric and the gauge field we have obtained are ill-behaved at the horizon in the near-extremal limit. However, it is not clear what this behavior signifies. This is indeed one of the most pressing questions opened up by the present work and it would be interesting to understand the extremal limit of our solutions more clearly.

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A. Source Terms in Scalar Sector: Second Order

There are three source terms in scalar sector \( S_k(r) \), \( S_h(r) \) and \( S_M(r) \). They are quite complicated functions. Here we tried to put the explicit form of these source terms as much as possible.

\[
S_k = 12r^3h_2(r) + (3r^4 - 1)h_2(r) + \frac{4q_0w_2(r)}{r^3} - \frac{2q_0w_2'(r)}{r^2} - \frac{r^3}{6m_0R_0^8} (R_0 - A_1 + B_1 + C_1 + D_1 + E_1) \quad \text{(ST5)}
\]

\[
+ \frac{2}{3} \frac{r^3}{16m_0^2r^{10}} (ST4)
\]

\[
- \frac{2r}{9} \frac{r^3}{(ST3) + \frac{2}{3} \frac{r^3}{(ST5)}} \quad \text{(QST5)}
\]

\[
+ \frac{2}{3} \frac{3q_0}{rR_0} \left( 6R_0F_1 \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) + rF_1^{(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) \right) \quad \text{(QST2)}
\]

\[
- \frac{3\sqrt{3}q_0^3}{4m_0r^3R_0} \left( m_0 - r^4 \right) \kappa \left( 5R_0F_1^{(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) + rF_1^{(2,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) \right) \quad \text{(QST2)}
\]

\[
\frac{\sqrt{3}q_0^3}{4m_0r^3R_0} \left( m_0 - r^4 \right) \kappa \left( 5R_0F_1^{(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) + rF_1^{(2,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) \right) \quad \text{(QST3)}
\]

where,

\[
A_1 = \frac{q_0^2R_0 \left( r^2 + R_0r + R_0^2 \right)^2}{(r + R_0)^2 \left( r^4 + R_0^2r^2 + R_0^4 - m_0 \right)^2}
\]

\[
B_1 = - \frac{m_0r^2R_0 \left( r^2 + R_0r + R_0^2 \right)^2}{(r + R_0)^2 \left( r^4 + R_0^2r^2 + R_0^4 - m_0 \right)^2}
\]

\[
C_1 = \frac{r^6R_0 \left( r^2 + R_0r + R_0^2 \right)^2}{(r + R_0)^2 \left( r^4 + R_0^2r^2 + R_0^4 - m_0 \right)^2}
\]

\[
D_1 = \frac{4m_0 \left( r^2 + R_0r + R_0^2 \right) F_2 \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right)}{(r + R_0)^2 \left( r^4 + R_0^2r^2 + R_0^4 - m_0 \right)}
\]

\[
E_1 = - \frac{12r^4 \left( r^2 + R_0r + R_0^2 \right) F_2 \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right)}{(r + R_0)^2 \left( r^4 + R_0^2r^2 + R_0^4 - m_0 \right)}
\]

and the functions \( A_i(r) \) are complicated functions. Hence we are not giving the expressions here. But these functions have the correct asymptotic behavior. For example for large \( r \), \( \tilde{A}_1(r) \sim - \frac{1}{r} \).
The second source term is given by,

\[
S_h = \frac{2}{3} (A + B) \quad \text{ST}5
\]

\[
r^4 - \frac{3\sqrt{3}q_0^3\kappa^2}{m_0^2} \quad \text{ST}4
\]

\[
- \frac{9q_0^2r^7}{R_0^{18}} \left( 5R_0 F_1^{1(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) + r F_1^{(2,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) \right)^2 \quad \text{ST}1
\]

\[
+ \frac{12\sqrt{3}q_0^3\kappa \left( 5R_0 F_1^{1(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) + r F_1^{(2,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) \right)}{m_0 R_0^9} \quad \text{ST}2
\]

\[
+ \frac{4\sqrt{3}q_0^2\kappa \left( 5R_0 F_1^{1(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) + r F_1^{(2,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) \right)}{m_0 R_0^9} \quad \text{QS}3
\]

\[
- \frac{r^7}{R_0^{18}} \left( 5R_0 F_1^{1(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) + r F_1^{(2,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) \right)^2 \quad \text{QS}3
\]

\[
- \frac{6q_0 r^7}{R_0^{18}} \left( 5R_0 F_1^{1(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) + r F_1^{(2,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) \right)^2 \quad \text{QS}5
\]

(A.3)

where,

\[
A = \frac{r^3 \left( r^2 + R_0 r + R_0^2 \right)^2}{(r + R_0)^2 \left( r^4 + R_0^2 r^2 + R_0^4 - m_0 \right)^2} \quad \text{(A.4)}
\]

and

\[
B = \frac{2r \left( m_0 \left( 4r^3 + 8R_0 r^2 + 6R_0^2 r + 3R_0^3 \right) - 3R_0^3 \left( r^2 + R_0 r + R_0^2 \right)^2 \right) F_2 \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right)}{R_0 (r + R_0)^2 \left( r^4 + R_0^2 r^2 + R_0^4 - m_0 \right)^2} \quad \text{(A.5)}
\]
The source term $S_M(r)$ is given by,

\[
S_M(r) = \frac{3\sqrt{3}\sqrt{R_0^2 (m_0 - R_0^4)} \left( R_0^6 - 12r (m_0 - R_0^4) F1 \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) \right)}{8m_0 r^4 R_0^5} - \frac{\sqrt{3} \left( R_0^6 - 12r (m_0 - R_0^4) F1 \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) \right)}{2r^4 R_0^4} \text{QST2}
\]

\[
+ \frac{4\sqrt{3} \sqrt{R_0^2 (m_0 - R_0^4)} (-r^6 + R_0^6 + m_0 (r^2 - R_0^2)) F2 \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) F2^{(1.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right)}{r^2 R_0^6} \text{ST5}
\]

\[
+ \sqrt{3} (R_0^2 - r^2) \sqrt{R_0^2 (m_0 - R_0^4)} (r^4 + R_0^2 r^2 + R_0^4 - m_0)
\]

\[
(36k^2 R_0^2 - 24m_0 (r^2 + 3R_0^2) \kappa^2 R_0^6 + m_0 (r^4 + 24R_0^2 \kappa^2 r^2 + 36R_0^4 \kappa^2)) \frac{1}{2m_0 r^4 R_0^7} \text{ST4}
\]

\[
+ 9\sqrt{3} \sqrt{R_0^2 (m_0 - R_0^4)} ((m_0 - 3 R_0^4) R_0^6 + r (m_0 - R_0^4)) (24m_0 (m_0 - R_0^4) F1^{(0,1)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) R_0^5)
\]

\[
+ 3 F1 \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) \left( 13R_0^4 + 5m_0 \right) R_0^6 + 2r^2 (3R_0^4 - m_0) \left( (21R_0^2 + 21m_0 R_0^3 - 10m_0 r^2 R_0) F1^{(1.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) R_0^6 \right)
\]

\[
+ r (R_0^4 F1^{(1.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) \left( (11m_0 - 21R_0^2) R_0^6 + 15r^2 (r - R_0) (r + R_0) (r^4 + R_0^4 r^2 + R_0^4 - m_0) \right)
\]

\[
(3R_0^4 - m_0) F1^{(1.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) - r (3R_0^4 - m_0) (R_0^8 + 3r^2 (-r^6 + R_0^6 + m_0 (r - R_0) (r + R_0))
\]

\[
F1^{(1.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) F1^{(2.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) \right) \text{ST1}
\]

\[
- 9 \left( m_0 - R_0^4 \right) \kappa ((-5R_0^6 + 5m_0 R_0^2 - 4m_0 r^2) R_0^6 + 2r^2 (12R_0^2 - 24m_0 R_0^4 + r^2 (9r^4 + 11m_0)
\]

\[
R_0^6 + 12m_0 R_0^4 - m_0 r^2 (9r^4 + 11m_0) R_0^4) \left( 10m_0 r^4 (m_0 - r^4) \right) F1^{(1.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) R_0^6
\]

\[
+ 2r^5 ((3r^4 + m_0) R_0^6 - m_0 (3r^4 + m_0) R_0^6 + 2m_0 r^2 (m_0 - r^4)) F1^{(2.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) \text{ST2}
\]

\[
+ \sqrt{3} \sqrt{9(24m_0 (m_0 - R_0^4) F1^{(0.1)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) R_0^6 + 3 F1 \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) \left( R_0^4 + 9m_0 \right)
\]

\[
R_0^6 - 2r^2 (63 R_0^6 + m_0^2 (21 R_0^2 - 10r^2) + 6m_0 (5r^2 R_0^2 - 14 R_0^4)) F1^{(1.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) R_0^6
\]

\[
+ 2r^3 \left( -9R_0^6 - 6m_0 (r^2 - 2R_0^2) R_0^4 + m_0 (2r^2 - 3R_0^4) F1^{(2.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) R_0^4 \right)
\]

\[
+ r (m_0 - 3R_0^4) F1^{(2.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) R_0^8 + 15r^2 (3r^6 - R_0^6) R_0^4 + m_0^2 (r^2 - R_0^2)
\]

\[
m_0 (r^6 + 3 R_0^4 r^2 - 4 R_0^6) F1^{(1.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) R_0^8 + F1^{(1.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) \left( R_0^6 (11m_0 - 21R_0^2) \right)
\]

\[
- 3 (m_0 - R_0^4) \kappa ((-5R_0^6 + 5m_0 R_0^2 - 4m_0 r^2) R_0^6 + 2r^2 (3r^6 + 4 R_0^6) R_0^6
\]

\[
+ m_0^2 (10r^4 - 11R_0^2 r^2 + 12R_0^4) - m_0 (10r^8 + 9 R_0^6 r^2 - 11 R_0^6 r^2 + 24 R_0^4)) F1^{(1.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) R_0^6
\]

\[
+ 2r^5 (3r^4 R_0^2 + m_0^2 (2r^2 - R_0^2) + m_0 (2r^6 - 2R_0^2)) F1^{(2.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) \text{QST3}
\]

\[
+ \sqrt{3} \sqrt{5R_0^6 (m_0 - R_0^4) - 12r (m_0 - R_0^4) (24m_0 (m_0 - R_0^4) F1^{(0.1)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) R_0^6
\]

\[
+ r F1^{(1.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) (21R_0^2 - 11m_0 R_0^8 - 15r^2 (r - R_0) (r + R_0) (r^4 + R_0^2 r^2 + R_0^4 - m_0))}
\]
where,

\[
\begin{align*}
\text{func}(r) &= (3R_0^4 - m_0) F_1^{(1.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) R_0 + 2F_1 \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) (3r^2 (3R_0^4 - m_0) (21R_0^6 - 21m_0R_0^3 \\
&+ 10m_0r^2 R_0) F_1^{(1.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) r (3R_0^6 - 3m_0R_0^3 + 2m_0r^2) F_1^{(2.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) \\
&- R_0^3 (9R_0^6 + 11m_0) R_0 + r^2 (3R_0^4 - m_0) (R_0^6 + 3r^2 (-r^6 + R_0^6 + m_0(r - R_0)(r + R_0)) \\
&F_1^{(1.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) F_1^{(2.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) \right) \frac{1}{2r^4 R_0^5 (3R_0^4 - m_0)}
\end{align*}
\]  
\tag{A.7}
\]

B. Source Terms in Vector Sector: Second Order

The Source Terms

\[
\begin{align*}
S_v^{(1)}(r) &= - \frac{5 (4r^5 + 4R_0r^4 + 4R_0^2r^3 + 3R_0^3r^2 + 3R_0^4r + 3R_0^5 - 3m_0(r + R_0)) V4X}{9r^2(r + R_0)(r^4 + R_0^7r^2 + R_0^4 - m_0)} \\
&- \frac{(2r^5 + 2R_0r^4 + 2R_0^2r^3 - 3R_0^3r^2 - 3R_0^4r - 3R_0^5 + 3m_0(r + R_0)) V5X}{9r^2(r + R_0)(r^4 + R_0^7r^2 + R_0^4 - m_0)} \\
&- \frac{(2r^5 + 2R_0r^4 + 2R_0^2r^3 - 3R_0^3r^2 - 3R_0^4r + 3m_0r - 3R_0^5 + 3m_0R_0) VT1X}{3r^2(r + R_0)(r^4 + R_0^7r^2 + R_0^4 - m_0)} \\
&- \frac{q_0^3 (3r(r^2 + R_0r + R_0^2)^2 + m_0(r + 2R_0)) VT5X}{4\sqrt{3m_0r^2(r + R_0)^2} (r^4 + R_0^2r^2 + R_0^4 - m_0)^2} \\
&+ \frac{T1A + T1B + T2}{TD} VT3 \\
&+ \frac{TN1A + TN1B + TN2}{TD} QV4 \\
&- \frac{1}{2} VT2 \left( \frac{54F_1 \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) q_0^2}{R_0^7} + \frac{81F_1^{(1.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) q_0^2}{R_0^8} + \frac{9rF_1^{(2.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) q_0^2}{R_0^9} \right) \\
&- \frac{5r(r^2 + R_0r + R_0^2)}{6(r + R_0)(r^4 + R_0^2r^2 + R_0^4 - m_0)} \left( \frac{1}{r^2} \right) \\
&- \frac{3q_0^2 QV3X \left( 6F_1 \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) R_0^2 + r \left( 9R_0F_1^{(1.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) + rF_1^{(2.0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) \right) \right)}{2r R_0^7}
\end{align*}
\]  
\tag{B.1}
\]

where,

\[
\frac{T1A}{R_0^8} = 3 (r^2 + R_0r + R_0^2)^2 (8r^6 + 23R_0^3r^3 - 3R_0^6) R_0^4 \\
+ 3m_0^3 (r + R_0)^2 + m_0^2 (5r^6 + 10R_0r^5 - 26R_0^2r^4 - 19R_0^3r^3 - 21R_0^4r^2 - 30R_0^5r - 15R_0^6) \\
- m_0 (8r^{10} + 16R_0r^9 + 24R_0^2r^8 + 75R_0^3r^7 + 141R_0^4r^6 + 207R_0^5r^5) \\
- m_0 (109R_0^6r^4 + 32R_0^7r^3 - 45R_0^8r^2 - 42R_0^9r - 21R_0^{10})
\]  
\tag{B.2}
\]
The second source term is given by

\[
T1B = \frac{T1B}{-108q_0^2 r^4 R_0} = 9 \left( r^2 + R_0 r + R_0^2 \right)^2 \left( r^3 + 2 R_0^3 \right) R_0^4 + m_0^2 \left( 7 r^3 + 14 R_0 r^2 + 12 R_0^2 r + 6 R_0^3 \right) - 3 m_0 \left( r^7 + 2 R_0 r^6 + 3 R_0^2 r^5 + 4 R_0^3 r^4 + 12 R_0^4 r^3 + 20 R_0^5 r^2 + 16 R_0^6 r + 8 R_0^7 \right) F1 \left( \frac{r}{R_0}, \frac{m_0}{R_0^2} \right) \]  

(B.3)

\[
T2 = -108q_0^2 r^5 \left( r^3 + 2 R_0 r^2 + 2 R_0^2 r + R_0^3 \right) \]  

(B.4)

\[
TD = 6 r^2 R_0^8 \left( r^4 + R_0^2 r^2 + R_0^4 - m_0 \right)^2 \left( 3 R_0^4 - m_0 \right) \]  

(B.5)

\[
\frac{T1A}{3q_0 r} = R_0^{13} + 2 r \left( 9 \left( r^2 + R_0 r + R_0^2 \right)^2 \left( r^3 + 2 R_0^3 \right) R_0^4 + m_0^2 \left( 7 r^3 + 14 R_0 r^2 + 12 R_0^2 r + 6 R_0^3 \right) \right) - 2 r \left( 3 m_0 \left( r^7 + 2 R_0 r^6 + 3 R_0^2 r^5 + 4 R_0^3 r^4 + 12 R_0^4 r^3 + 20 R_0^5 r^2 + 16 R_0^6 r + 8 R_0^7 \right) \right) F1 \left( \frac{r}{R_0}, \frac{m_0}{R_0^4} \right) \]  

(B.6)

\[
T1B = 6 q_0 r^3 \left( r^3 + 2 R_0 r^2 + 2 R_0^2 r + R_0^3 \right) \]  

(B.7)

\[
TD = R_0^8 \left( r + R_0 \right)^2 \left( m_0 - 3 R_0^4 \right) \left( r^4 + R_0^2 r^2 + R_0^4 - m_0 \right)^2 \]  

(B.8)

The second source term is given by,

\[
S_{v}^{(2)} = \sqrt[3]{m_0 r^6} \left( \frac{r}{R_0^2} \left( m_0 - R_0^4 \right) \left( 24 \kappa^2 R_0^6 + m_0 \left( r^2 - 24 R_0^2 \kappa^2 \right) \right) \right) V5 + \sqrt[3]{m_0 r^6} \left( \frac{r}{R_0^2} \left( m_0 - R_0^4 \right) \left( 24 \kappa^2 R_0^6 + m_0 \left( r^2 - 24 R_0^2 \kappa^2 \right) \right) \right) VT1 + \frac{S1}{m_0 r^6} \left( r + R_0 \right) \left( r^4 + R_0^2 r^2 + R_0^4 - m_0 \right) VT5 - \frac{S2}{2 m_0 r^6 R_0^6} QV3 + \frac{S3}{2 m_0 r^6 R_0^6} VT2 - \frac{S4}{2 m_0 r^6 R_0^6} VT3 - \frac{S5}{r R_0^6 \left( r + R_0 \right) \left( r^4 + R_0^2 r^2 + R_0^4 - m_0 \right)} QV4 + \frac{S6}{\sqrt{3} \left( r + R_0 \right) \left( r^4 + R_0^2 r^2 + R_0^4 - m_0 \right)} V4 \]  

(B.10)
where,

\[ S1 = 3R_0 \left( r^2 R_0 \left( r (3r^5 + 3R_0 r^4 + 3R_0^2 r^3 + R_0^3 r^2 + R_0^4 r + R_0^5) - m_0 (3r^2 + 3R_0 r + 2R_0^2) \right) \right) \]
\[ - 8m_0(r + R_0) \left( r^4 + R_0^2 r^2 + R_0^4 - m_0 \right) S1 F2 \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) \]  
(B.11)

\[ S2 = \sqrt{3} \left( 24\kappa^2 R_0^6 - 24m_0\kappa^2 R_0^2 \right) R_0^6 + 6m_0 r^3 \left( R_0^4 - m_0 \right) F1 \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) R_0 \]
\[ + 6m_0 r^4 \left( m_0 - R_0^4 \right) F1^{(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) \]  
(B.12)

\[ S3 = \sqrt{3} \sqrt{R_0^2 \left( m_0 - R_0^4 \right)} \left( 5 \left( 24\kappa^2 R_0^6 - 24m_0\kappa^2 R_0^2 + m_0 r^2 \right) R_0^6 \right) \]
\[ - 36m_0 r^3 \left( m_0 - R_0^4 \right) F1 \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) R_0 \]
\[ + 36m_0 r^4 \left( m_0 - R_0^4 \right) F1^{(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) \]  
(B.13)

\[ S4 = \sqrt{3} \sqrt{R_0^2 \left( m_0 - R_0^4 \right)} \left( 6m_0 F1^{(2,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) \right) r^{11} + 30m_0 R_0^4 F1^{(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) r^7 \]
\[ + (6m_0 \left( r^2 + R_0 r + R_0^2 \right) \left( -R_0^9 + 12r \left( m_0 - R_0^4 \right) \right) F1 \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) R_0^4 \]
\[ + r^3 \left( r^2 + R_0^2 \right) \left( m_0 - R_0^4 \right) F1^{(2,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) \frac{1}{(r + R_0) \left( r^4 + R_0^2 r^2 + R_0^4 - m_0 \right)} \]
\[ + R_0^9 \left( 24\kappa^2 R_0^6 + m_0 \left( r^2 - 24R_0^2 \kappa^2 \right) \right) \]  
(B.14)

\[ S5 = \sqrt{3} \left( r(12R_0 \left( r^2 + R_0 r + R_0^2 \right) \left( m_0 - R_0^4 \right) F1 \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) \right) \]
\[ + r(r + R_0) \left( r^4 + R_0^2 r^2 + R_0^4 - m_0 \right) \left( 5R_0 F1^{(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) \right) \]
\[ + rF1^{(2,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) \]  
(B.15)

\[ S6 = 5\left( 6 \left( \sqrt{R_0^2 \left( m_0 - R_0^4 \right) - \sqrt{m_0 R_0^2 - R_0^6} \right) \right) R_0^2 \left( r^2 + R_0 r + R_0^2 \right) \left( m_0 - R_0^4 \right) F1 \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) \right) \]
\[ + r(r + R_0) \left( r^4 + R_0^2 r^2 + R_0^4 - m_0 \right) \left( 5R_0 F1^{(1,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) + rF1^{(2,0)} \left( \frac{r}{R_0}, \frac{m_0}{R_0} \right) \right) \]
\[ - (24r + R_0) \left( r^2 - R_0 r + R_0^2 \right) \left( r^2 + R_0 r + R_0^2 \right) \left( m_0 - R_0^4 \right) \kappa^2 R_0^6 \]
\[ + m_0^2 \left( r + R_0 \right) \sqrt{R_0^2 \left( m_0 - R_0^4 \right) \left( 24R_0^2 \kappa^2 - r^2 \right) \right) \]
\[ + m_0 \left( r + R_0 \right) \sqrt{R_0^2 \left( m_0 - R_0^4 \right) \left( 2r^2 + R_0 r + R_0^2 \right) \left( 7r^2 + R_0^3 \right) \right) \]
\[ - 24R_0^2 \left( r^2 + R_0^2 r^2 + 2R_0^4 \right) \kappa^2 - 6r^5 \left( r^2 + R_0 r + R_0^2 \right) \sqrt{m_0 R_0^2 - R_0^6} \]  
(B.16)
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