PERSISTENCE AND EXTINCTION OF A STOCHASTIC SIS EPIDEMIC MODEL WITH REGIME SWITCHING AND LÉVY JUMPS

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Abstract. This paper is devoted to a stochastic regime-switching susceptible-infected-susceptible epidemic model with nonlinear incidence rate and Lévy jumps. A threshold $\lambda$ in terms of the invariant measure, different from the usual basic reproduction number, is obtained to completely determine the extinction and prevalence of the disease: if $\lambda > 0$, the disease is persistent and there is a stationary distribution; if $\lambda < 0$, the disease goes to extinction and the susceptible population converges weakly to a boundary distribution. Moreover, some numerical simulations are performed to illustrate our theoretical results. It is very interesting to notice that random fluctuations (including the white noise and Lévy noise) acting the infected individuals can prevent the outbreak of disease, that the disease of a regime-switching model may have the opportunity to persist eventually even if it is extinct in one regime, and that the prevalence of the disease can also be controlled by reducing the value of transmission rate of disease.

1. Introduction. Medical research has shown that some diseases do not have permanent immunity and infected individuals become susceptible individuals immediately again after recovery. This type of diseases can be modelled appropriately by susceptible-infected-susceptible (SIS) epidemic models of the form

$$
\begin{align*}
\frac{dS(t)}{dt} &= [\Lambda - \beta g(S(t), I(t)) + \gamma I(t) - \mu S(t)]dt, \\
\frac{dI(t)}{dt} &= [\beta g(S(t), I(t)) - (\mu + \gamma + \alpha)I(t)]dt,
\end{align*}
$$

where $S(t)$ and $I(t)$ denote the numbers of susceptible and infected individuals at $t$, respectively, $\Lambda > 0$ and $\mu > 0$ are the recruitment rate and natural death rate of the population, respectively, $\alpha \geq 0$ is the disease-related death rate of $I$, and $\gamma \geq 0$ is the recovery rate. The transmission of the infection is described by a nonlinear incidence $\beta g(S(t), I(t))$, where $\beta \geq 0$ represents the disease transmission coefficient, $g: [0, \infty) \times [0, \infty) \to [0, \infty)$ is Lipschitz continuous and satisfies the linear growth condition. Throughout this paper, we always assume that $\lim_{S \to 0} g(S, I)/S$ exists for

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all $I \geq 0$, $g'(S, 0) \triangleq \lim_{t \to 0} g(S, I)/I > 0$ for all $S > 0$, and $g(S, I)/I$ is monotone nonincreasing with respect to $I \in (0, \infty)$ for $S \in (0, \infty)$. Obviously, the function $g$ includes some special incidence rates [10, 11, 15, 17, 25, 26, 40, 41], such as bilinear incidence rate $g(S, I) = SI$, saturation incidence rate $g(S, I) = SI/(1 + mI)$ with a positive constant $m$ denoting the half-saturation constant, Holling type II incidence rate $g(S, I) = SI/(aS + bI + c)$ with positive constants $a$, $b$, and $c$, and some other kinds of incidence rates like $g(S, I) = e^{-mI}SI$ and $g(S, I) = SI/(1 + mI^\theta)$ with positive constants $m$ and $\theta$ (see, for example [8, 30, 36]). In [8], model (1) with bilinear incidence rate always has a globally asymptotically stable disease-free equilibrium $E^0$ and so the disease disappears if the basic reproduction number $R_0 = \frac{\Lambda g}{\mu + \alpha + \lambda} \leq 1$. When $R_0 > 1$, however, $E^0$ loses the stability and a globally asymptotically stable endemic equilibrium $E^*$ appears, which means that the disease is persistent. As we known, species are inevitably subject to various environmental random factors, and hence stochastic noises are introduced into deterministic models to reveal the influence of environmental randomness on the dynamics of epidemic models [6, 13, 14, 18, 30, 32, 39].

Population dynamics may also suffer sudden environmental perturbations, such as hurricanes, harvesting, epidemics, earthquakes and so on, which cause jumps in population but cannot be described by white noise. Situ [27] and Bao et al. [2, 4] proposed that stochastic perturbation of jumps can be described by Lévy jumps. In addition, parameters in population models may be affected by abrupt changes, for example, some species may have much different growth rates in sultry summer and in freezing winter. These phenomena can be modeled by regime switching in a finite state space ([9, 28, 31]). So, to make the model more practical, we introduce a stationary Poisson point process as the driven process and incorporate regime switching in a Markov chain on a finite state space $\{1, 2, \ldots, n\}$ with $1 \leq n < \infty$ satisfying

$$\mathbb{P}\{r(t + \delta) = j | r(t) = i\} = \begin{cases} q_{ij}\delta + o(\delta) & \text{if } i \neq j, \\ 1 + q_{ii}\delta + o(\delta) & \text{if } i = j \end{cases}$$

for a sufficiently small $\delta > 0$. By the Markov chain theory, $r(t)$ is ergodic and has a unique stationary distribution $\theta_i$, $i = 1, 2, \ldots, n$. $S(t^-)$ and $I(t^-)$ are the left limits of $S(t)$ and $I(t)$, respectively. For $i = 1, 2$, $c_i(r(t), u)$ reflects the birth rate (when it is positive) or death rate (when it is negative) caused by the jumps in some degree. $N(dt, du)$ is a Poisson counting measure with the stationary compensator $\vartheta(du)dt$, defined on a measurable subset $\mathcal{Y}$ of $(0, \infty)$ with $\vartheta(\mathcal{Y}) < \infty$, $\vartheta$ is the intensity
measure of $N$. Define $\widetilde{N}(dt, du) := N(dt, du) - \vartheta(du)dt$ is the compensated random measure. Throughout this paper, we assume that

(i): $B_1, B_2, r(t)$ and $N$ are independent;

(ii): $c_1, c_2 : \mathbb{S} \times \mathbb{Y} \to \mathbb{R}$ are continuous and bounded with respect to $\vartheta$, and $c_i(r(t), u) > -1$ for $i = 1, 2, u \in \mathbb{Y}$;

(iii): there is $K > 0$ such that

$$\int_{\mathbb{Y}} |\ln(1 + c_k(i, u))|^2 \vartheta(du) < K \quad \text{for all} \quad i \in \mathbb{S}, \quad k = 1, 2;$$

(iv): the $Q$-matrix $Q = (q_{ij})_{i,j \in \mathbb{S}}$ of the Markov chain $r(t)$ is conservative and irreducible, i.e., $q_{ii} = -q_i = -\sum_{j \neq i} q_{ij} < 0$ for $i \in \mathbb{S}$.

The three-component process $(S(t), I(t), r(t))$ defined by (2) and (3) is a regime-switching diffusion process, consisting of a continuous part $(S(t), I(t))$ and a random switching device $r(t)$. Corresponding to the process $(S(t), I(t), r(t))$, there is a family of diffusion processes $(S_i(t), I_i(t)), i \in \mathbb{S}$ defined by

$$\begin{align*}
\text{d}S_i(t) &= [\Lambda(i) - \beta(i)g(S_i(t), I_i(t)) + \gamma(i)I_i(t) - \mu(i)S_i(t)]dt \\
&\quad + \alpha(i)S_i(t)dB_1(t) + \int_{\mathbb{Y}} c_1(i, u)S(t^-)\widetilde{N}(dt, du), \\
\text{d}I_i(t) &= [\beta(i)g(S_i(t), I_i(t)) - (\mu(i) + \gamma(i) + \alpha(i))I_i(t)]dt \\
&\quad + b(i)I_i(t)dB_2(t) + \int_{\mathbb{Y}} c_2(i, u)I(t^-)\widetilde{N}(dt, du).
\end{align*}
$$

(4)

In other words, $(S_i(t), I_i(t)), i \in \mathbb{S}$ are the diffusion processes associated with $(S(t), I(t), r(t))$ in each fixed environment, while the solution process $(S(t), I(t))$ of (2) can be regarded as the result of solution processes $(S_i(t), I_i(t))$ of (4) switching from one to the other by the law of the Markov chain. What we are interested in is the extinction and prevalence of the disease in such a regime-switching model as (2). Thus, it is natural to connect this behavior with its behavior in each fixed environment. As we shall see in the numerical simulation section (Section 5), however, this connection is rather complicated.

Usually, the so-called basic reproduction number is adopted to investigate the dynamics of epidemic models [30, 37, 39]. For example, Lin and Zhao [20] considered the exponential ergodicity of (2) with a standard incidence $SI/(S + I)$ by verifying a Foster-Lyapunov condition. Zhou et al. [37] considered the following model, which is obviously a special case of (2) in a fixed environment,

$$\begin{align*}
\text{d}S(t) &= [\Lambda - \beta S(t)I(t) + \gamma I(t) - \mu S(t)]dt + aS(t)dB_1(t) \\
&\quad + \int_{\mathbb{Y}} c_1(u)S(t^-)\widetilde{N}(dt, du), \\
\text{d}I(t) &= [\beta S(t)I(t) - (\mu + \gamma + \alpha)I(t)]dt + bI(t)dB_2(t) \\
&\quad + \int_{\mathbb{Y}} c_2(u)I(t^-)\widetilde{N}(dt, du).
\end{align*}
$$

(5)

Zhou et al. [37] found that dynamics of model (5) is determined by a threshold $\tilde{R}_0$ given by

$$\tilde{R}_0 = \frac{1}{\mu + \alpha + \gamma} \left[ \frac{\Lambda \beta}{\mu} - \frac{b^2}{2} + \int_{\mathbb{Y}} (\ln(1 + c_2(u)) - c_2(u))\vartheta(du) \right].$$

(6)
To be more specific, Zhou et al. [37] found that if $\tilde{R}_0 < 1$, the disease of (5) vanishes exponentially with probability one, and that if $\tilde{R}_0 > 1$, every solution to model (5) tends to a point in time average and hence the disease is persistent.

In this paper, we want to seek a different threshold from (and perhaps much better than) the basic reproduction number to determine the existence of stationary distribution and the convergence of the population in system (2). For this purpose, we first obtain some estimates of solutions in probability by using some standard inequalities. To obtain the bound of the total population in system (2), we have to estimate the stochastic terms. However, the presence of regime-switching and Lévy jumps makes it difficult to estimate the stochastic terms [19]. We shall employ Chebyshev’s inequality and comparison principle to overcome this difficulty. In order to determine whether the disease goes to extinction or not, we first pay attention to the equation on the boundary (that is, $I(t) = 0$), which has a stationary distribution. We define a threshold $\lambda$ in terms of the invariant measure. In fact, as $I(t)$ is small for a long enough time, $\lambda$ is the Lyapunov exponent of $I(t)$. On the one hand, instead of adopting a Lyapunov function (see, for example [20]), we define a stopping time to help us to show that the disease is persistent and system (2) has a stationary distribution when $\lambda > 0$. On the other hand, we shall see that the disease converges to 0 and the susceptible population converges to the boundary distribution when $\lambda < 0$. Furthermore, we shall see the persistence condition of the disease (i.e., $\lambda > 0$) is much better than all the relevant results obtained in [30, 37, 39] (for more details, see Corollaries 2 and 3. Nevertheless, a main barrier in our analysis is that the monotonicity doesn’t work here, which makes us have to resort to the ergodicity.

Apparently, our model (2) is more accurate since the population of the species might be affected by kinds of environmental random factors, for example, white noise, color noise, jumps, etc. The previous threshold, for example, introduced in [13, 37, 39], depends on the upper bound of the total population and parameters of the system under investigation. Note that the susceptible population is persistent and plays an important part in system (2). By comparison, our threshold $\lambda$ is well-defined to determine the dynamics of system (2) because it relies on the parameters $(b, c_2, \mu, \gamma, \alpha)$ and the boundary distribution of the susceptible population. Furthermore, we not only prove the persistence and extinction of the disease, but also give the existence of the stationary distribution and the weak convergence of the susceptible population. Most importantly, the stochastic SIS epidemic model (2) is more general than models in literature, our approach is applicable to a variety of other stochastic differential equations, and our main results improve some relevant results in [13, 20, 30, 37, 39].

The paper is organized as follows. In Section 2, we review some basic concepts and results and give some estimates of the solution of (2). Section 3 is devoted to the persistence of the disease and the existence of stationary distribution of system (2). Section 4 is devoted to the extinction of the disease and the weak convergence of susceptible population. In Section 5, we perform some numerical simulations to illustrate our results. Finally, in Section 6 we summarize our contribution and also present some perspectives of this paper.

2. Preliminaries. In this section, we first recall some basic results on the existence of the solutions of (2). Denote that $z = (x, y) \triangleq (S(0), I(0))$ is the initial value in $\mathbb{R}_+^2$, and $(S_z(t), I_z(t))$ the solution to (2) with initial value $z$, where $\mathbb{R}_+^2 \triangleq \{(x, y) : \ldots \}$.
It follows that
\[\hat{a} := \min_{i \in S} a(i), \quad \hat{\beta} := \max_{i \in S} \beta(i), \quad \hat{\delta}_j := \min_{j \in S} c_j(i, u), \quad \hat{\gamma}_j := \max_{j \in S} c_j(i, u), \quad j = 1, 2,\]
and the other parameters \(\hat{\Lambda}, \hat{\Phi}, \hat{\beta}, \ldots\) are defined analogously.

Brownian motion and jump processes can suppress the explosion (see for example [23, 34]). The global existence of positive solutions of system (2) can be established by using the Lyapunov analysis method similar to [22]. Namely, we obtain the following result.

**Lemma 2.1.** For any initial value \((S(0), I(0), r(0)) \in \mathbb{R}_+^3 \times S\), system (2) has a unique global positive solution \((S(t), I(t), r(t)) \in \mathbb{R}_+^3 \times S\) almost surely for all \(t \geq 0\).

By applying Chebyshev’s inequality and comparison principle, we can obtain the upper and lower bound of the solution of (2) in probability.

**Lemma 2.2.** For \(p \geq 1\), there exists \(N_1 > 0\) such that \(E((S_t + I_t)^p) \leq N_1\) for all \(t \geq 0\).

**Proof.** Using a similar method as [37], we have
\[
d(S_t + I_t) = ([\Lambda(r(t)) - \mu(r(t))S_t - \mu(r(t))I_t - \alpha(r(t))I_t]dt + a(r(t))S_t dB_1(t) + b(r(t))I_t dB_2(t)
+ \int_{\mathbb{Y}} c_1(r(t), u)S_t N(dt, du) + \int_{\mathbb{Y}} c_2(r(t), u)I_t N(dt, du).
\]
It is easy to see that \(d \mathbb{E}(S_t + I_t) \leq [\hat{\Lambda} - \hat{\mu} \mathbb{E}(S_t + I_t)]dt\). By comparison principle, it yields that
\[
\mathbb{E}(S_t + I_t) \leq \frac{\hat{\Lambda}}{\hat{\mu}} + \left(x + y - \frac{\hat{\Lambda}}{\hat{\mu}}\right)e^{-\hat{\mu}t} \quad \text{for all } t \geq 0.
\]
Define \(X_t = S_t + I_t\) and \(V(X) = X_p, p > 1\). Itô’s formula yields that
\[
dV(X) \leq LV dt + pX^{p-1}[a(r(t))S dB_1 + b(r(t))I dB_2]
+ \int_{\mathbb{Y}} X^p[(1 + c_1(r(t), u) \vee c_2(r(t), u))^p - 1]N(dt, du),
\]
where
\[
LV \leq pX^{p-1}(\Lambda(r(t)) - \mu(r(t))X - \alpha(r(t))I) + \frac{p(p-1)}{2}X^{p-2}(a^2(r(t))S^2 + b^2(r(t))I^2)
+ \int_{\mathbb{Y}} X^p[(1 + c_1(r(t), u) \vee c_2(r(t), u))^p - 1 - pc_1(r(t), u) \wedge c_2(r(t), u)] \vartheta(du)
\leq pX^{p-2} \left\{ \Lambda(r(t))X - \left[ \mu(r(t)) - \frac{p-1}{2}(a^2(r(t)) \vee b^2(r(t))) \right] X^2 \right\}.
\]
Choose \(p > 1\) such that
\[
A = \hat{\mu} - \frac{p-1}{2}(\hat{a}^2 \vee \hat{b}^2) - \frac{1}{p} \int_{\mathbb{Y}} [(1 + c_1(u) \vee c_2(u))^p - 1 - pc_1(u) \wedge c_2(u)] \vartheta(du) > 0.
\]
It follows that
\[
dV(X) \leq pX^{p-2}(\hat{\Lambda}X - AX^2)dt + pX^{p-1}[\hat{a}S dB_1 + \hat{b}I dB_2]
+ \int_{\mathbb{Y}} X^p[(1 + c_1(u) \vee c_1(u))^p - 1]N(dt, du),
\]
and hence that for $0 < k < A_p$,
\[
de^{kt}V(X) \leq L[e^{kt}V(X)]dt + e^{kt}pX^{p-1}[\dot{a}SdB_1 + \dot{b}IdB_2] \\
+ e^{kt} \int_\gamma X^p[(1 + \tilde{c}_1(u) \vee \tilde{c}_1(u))^p - 1]\tilde{N}(dt, du).
\]
Integrating the above inequality from 0 to $t$ yields that
\[
\int_0^t de^{kt}V(X) \leq \int_0^t [ke^{ks}V(u(s)) + e^{ks}LV(X)]ds \\
+ \int_0^t e^{ks}[\dot{a}S(s)dB_1(s) + \dot{b}I(s)dB_2(s)] \\
+ \int_0^t e^{ks} \int_\gamma X^p[(1 + \tilde{c}_1(u) \vee \tilde{c}_1(u))^p - 1]\tilde{N}(ds, du)
\]
and hence that
\[
Ee^{kt}V(X) \leq V(X(0)) + E \left\{ \int_0^t [ke^{ks}V(u(s)) + e^{ks}LV(X)]ds \right\}.
\]
Therefore,
\[
ke^{kt}V(X) + e^{kt}LV(X) \leq ke^{kt}X^p + pe^{kt}X^{p-2}[-AX^2 + \dot{\Lambda}X] \\
=pe^{kt}X^{p-2} [\frac{k}{p}X^2 - AX^2 + \dot{\Lambda}X] \\
=pe^{kt}X^{p-2} \left[ - \left( A - \frac{k}{p} \right) X^2 + \dot{\Lambda}X \right] \leq pe^{kt}B,
\]
where $B = \sup_{X \in \mathbb{R}^+} \left\{ X^{p-2} \left[ - \left( A - \frac{k}{p} \right) X^2 + \dot{\Lambda}X \right] + 1 \right\}$. It follows that
\[
Ee^{kt}V(X) \leq V(X(0)) + E \int_0^t pe^{ks}Bds \leq V(X(0)) + \frac{pB}{k}e^{kt}
\]
and hence that
\[
\lim_{t \to \infty} E(X^p) \leq \frac{pB}{k} \quad \text{a.s.}
\]
Thus, there exists a constant $N_1 > 0$ such that $E(X^p) \leq N_1$ for all $t \geq 0$. The proof is completed.

The following is the exponential martingale inequality with jumps (see, for example, [1, Theorem 5.2.9]).

**Lemma 2.3.** Assume that $G : [0, \infty) \to \mathbb{R}$ and $F : [0, \infty) \times \gamma \to \mathbb{R}$ are both predictable $\mathcal{F}_t$-adapted processes such that for any $T > 0$
\[
\int_0^T |G(t)|^2 dt < \infty \quad \text{and} \quad \int_0^T \int_\gamma |F(t, u)|^2 \vartheta(du) dt \leq \infty \quad \text{a.s.}
\]
Then for any constants $p, q > 0$,
\[
P\left\{ \sup_{0 \leq t \leq T} \left[ \int_0^t G(s)dB(s) - \frac{p}{2} \int_0^t |G(s)|^2 ds + \int_0^t \int_\gamma F(s, u)\tilde{N}(ds, du) \\
- \frac{1}{p} \int_0^t \int_\gamma (e^{pF(s, u)} - 1 - pF(s, u))\vartheta(du)ds \right] > q \right\} \leq e^{-pq}.
\]
Lemma 2.4. For any $z \in \mathbb{R}_+^2$, there are positive constants $N^0$, $\overline{H}$, $T$ and $\widetilde{H}$ such that

$$
\begin{align*}
\mathbb{P} \left\{ S_z(t) + I_z(t) \geq N^0 \text{ for all } t \geq 0 \right\} &\geq 1 - \varepsilon, \\
\mathbb{P} \left\{ S_z(t) \geq \overline{H} \text{ for all } t \geq 0 \right\} &\geq 1 - \varepsilon, \\
\mathbb{P} \left\{ I_z(t) \geq \widetilde{H} \text{ for all } t \in [0, T] \right\} &\geq 1 - \varepsilon.
\end{align*}
$$

(7)

Furthermore, for all $z \in \mathbb{R}_+^2$,

$$
\lim_{t \to \infty} \frac{1}{t} \int_0^t (S_z(u) + I_z(u))du \leq N_0 := \frac{\Lambda}{\mu} \text{ a.s.}
$$

Proof. Note that

$$
\begin{align*}
\mathbb{E} \left[ \int_0^T a(r(t))S_z(t)dB_1(t) + \int_0^T \int_{\mathcal{Y}} c_1(r(t), u)S(t^-)\tilde{N}(dt, du) \right]^2 \\
\leq \int_0^T \left[ \tilde{a}^2 E S_z^2(t) + \int_{\mathcal{Y}} \tilde{c}_1^2(u)\vartheta(u) E S^2(t^-) \right] dt \\
\leq \left( \tilde{a}^2 + \int_{\mathcal{Y}} \tilde{c}_1^2(u)\vartheta(u) \right) N_1 T,
\end{align*}
$$

then by Chebyshev’s inequality, we have $\mathbb{P} \{ \Omega_1 \} \geq 1 - \varepsilon$ and $\mathbb{P} (\Omega_1^c) \geq 1 - \varepsilon$, where

$$
\begin{align*}
\Omega_1 &= \left\{ \int_0^T a(r(t))S_z(t)dB_1(t) + \int_0^T \int_{\mathcal{Y}} c_1(r(t), u)S_z(t^-)\tilde{N}(dt, du) \leq \frac{\widetilde{M}}{\varepsilon} \sqrt{T} \right\}, \\
\Omega_1^c &= \left\{ \int_0^T b(r(t))I_z(t)dB_2(t) + \int_0^T \int_{\mathcal{Y}} c_2(r(t), u)I_z(t^-)\tilde{N}(dt, du) \leq \frac{\widetilde{M}}{\varepsilon} \sqrt{T} \right\},
\end{align*}
$$

where $\widetilde{M} > 0$ is a constant. Using the comparison principle again, for $\omega \in \Omega_1 \cap \Omega_1^c$, we have

$$
S_z(t) + I_z(t) \geq \frac{\Lambda}{\mu + \alpha} + \left( z - \frac{\Lambda}{\mu + \alpha} \right) e^{-\left(\mu + \alpha\right)t} - \frac{\widetilde{M}}{\varepsilon} \int_0^t s^{-\frac{1}{2}} e^{\left(\beta M_1 + \mu\right)(s-t)} ds.
$$

(8)

By using the similar method, we have

$$
\lim_{t \to \infty} \frac{1}{t} \int_0^t (S_z(u) + I_z(u))du \leq \frac{\Lambda}{\mu}.
$$

By virtue of the continuity of $g(S, I)$ with respect to $(S, I)$, the existence of $\lim_{S \to 0} g(S, I)/S$, and the upper boundedness of $(S_z(t), I_z(t))$, there exists $M_1 > 0$ such that $g(S_z, I_z) \leq M_1 S_z$ for all $z \in \mathbb{R}_+^2$. Thus, for $\omega \in \Omega_1$ and $z \in \mathbb{R}_+^2$, we have

$$
dS_z(t) \geq \left[ \frac{\Lambda}{\beta M_1 + \mu} - \frac{\widetilde{M}}{2\varepsilon} t^{\frac{1}{2}} \right] dt - \frac{\widetilde{M}}{2\varepsilon} t^{-\frac{1}{2}} dt
$$

for all $t \geq 0$. Applying the comparison principle yields that for all $t \geq 0$,

$$
S_z(t) \geq \frac{\Lambda}{\beta M_1 + \mu} + \left( x - \frac{\Lambda}{\beta M_1 + \mu} \right) e^{-\left(\beta M_1 + \mu\right)t} - \frac{\widetilde{M}}{2\varepsilon} \int_0^t s^{-\frac{1}{2}} e^{\left(\beta M_1 + \mu\right)(s-t)} ds.
$$

(9)
As for \( I_z(t) \), for any \( k > 0 \) and \( t \geq 0 \), by Itô’s formula, we have

\[
dI_z^{-k}(t) = -kI_z^{-k}(t) \left[ \frac{\beta(r(t)g(S_z(t),I_z(t))}{I_z(t)} - (\mu(r(t)) + \gamma(r(t)) + \alpha(r(t)))}{2} \right] dt
+ \int_{\gamma} I_z^{-k}(t) \left[(1 + c_2(r(t), u))^{-k} - 1 + kc_2(r(t), u)) \right] \tilde{\vartheta}(du) dt
- kb(r(t))I_z^{-k}(t) dB_2(t) + \int_{\gamma} I_z^{-k}(t) \left[(1 + c_2(r(t), u))^{-k} - 1 \right] \tilde{N}(dt, du).
\]

We can find \( k > 0 \) such that

\[
\Psi := k(\bar{\mu} + \bar{\gamma} + \bar{\alpha}) + k(k + 1) \bar{b}^2 + \int_{\gamma} \left[(1 + \bar{c}_2(u))^{-k} - 1 + k\bar{c}_2(u)) \right] \tilde{\vartheta}(du),
\]

and hence that

\[
d\mathbb{E}I_z^{-k}(t) \leq \Psi \mathbb{E}I_z^{-k}(t) dt.
\]

Then for \( t \in [0, T] \), there exists a constant \( M_2 > 0 \) such that

\[
\mathbb{E}[I_z^{-k}(t)] \leq y^{-k}e^{\Psi t} \leq y^{-k}M_2.
\]

Using Chebyshev’s inequality, for all \( t \in [0, T] \), we have

\[
P \left\{ I_z(t) \geq \tilde{H} \right\} = 1 - P \left\{ \tilde{H}^{-k} \geq I_z^{-k}(t) \right\} \geq 1 - \tilde{H}^k \mathbb{E}[I_z^{-k}(t)] \geq 1 - \tilde{H}^ky^{-k}M_2,
\]

then choosing \( \tilde{H} \) such that \( \tilde{H}^ky^{-k}M_2 < \epsilon \), we obtain that

\[
P \left\{ I_z(t) \geq \tilde{H} \text{ for all } t \in [0, T] \right\} \geq 1 - \epsilon.
\]

This completes the proof. \( \square \)

In model (2), all newborns are born as susceptible population and after a period the infectious individual becomes susceptible again. In order to take a deep study of dynamical behaviors of susceptible population, we consider the following boundary equation. When \( I(t) = 0 \), we have

\[
d\varphi(t) = [\Lambda(r(t)) - \mu(r(t))\varphi(t)] dt + a(r(t))\varphi(t) dB_1(t)
+ \int_{\gamma} c_1(r(t), u)\varphi(t^-) \tilde{N}(dt, du).
\]

**Lemma 2.5.** For any initial values \( x > 0 \), equation (10) is asymptotically stable in distribution. Moreover, there is a unique invariant measure \( \pi^* \) for the process \((\varphi, r(t))\) in equation (10).

**Proof.** It follows from (10) that

\[
\frac{d\mathbb{E}\varphi(t)}{dt} = \Lambda(r(t)) - \mu(r(t))\mathbb{E}\varphi(t) \leq (\tilde{\Lambda} - \bar{\mu}\mathbb{E}\varphi(t)) \leq \tilde{\Lambda}.
\]

By comparison principle, we have

\[
\mathbb{E}\varphi(t) \leq \frac{\tilde{\Lambda}}{\bar{\mu}} + \left( x - \frac{\tilde{\Lambda}}{\bar{\mu}} \right) e^{-\bar{\mu}t} \leq N_3,
\]

where \( N_3 \) is a positive constant. Thus, \( \mathbb{E}\varphi(t) \) is a uniformly continuous function on \([0, \infty)\).
Lemma 2.6. Assume

\[ d|\varphi_1(t) - \varphi_2(t)| = -\mu(r(t))E|\varphi_1(t) - \varphi_2(t)|dt. \]

Integrating and taking expectations of the above equation yields that

\[ \mathbb{E}[\varphi_1(t) - \varphi_2(t)] \leq |x_1 - x_2| - \bar{\mu} \int_0^t \mathbb{E}[|\varphi_1(s) - \varphi_2(s)|]ds, \]

which implies that

\[ \int_0^\infty \mathbb{E}[\varphi_1(t) - \varphi_2(t)]dt < \frac{|x_1 - x_2|}{\bar{\mu}} < \infty. \]

Consequently, (12) together with Barbalat’s conclusion in [5] implies that

\[ \lim_{t \to \infty} \mathbb{E}[\varphi_1(t) - \varphi_2(t)] = 0, \]

and hence that different solutions of (10) squeeze together in the sense of expectation. It follows from (11) and (13) that (10) is asymptotically stable in distribution, which means that the transition probability of \((\varphi(t), r(t))\) converges weakly to a probability measure \(\pi^*\) as \(t \to \infty\). That completes the proof. \(\square\)

Using the similar method of [33, Theorem 4.4], we have the following ergodicity result.

Lemma 2.7. Let \(f: \mathbb{R} \times S \to \mathbb{R}\) be a bounded measurable function. Then

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t f(\varphi_s(s), r(s))ds = \sum_{i \in S} \int_0^\infty f(\phi, i)\pi^*(d\phi, i) \quad \text{a.s.,} \]

where \(\varphi_s\) is the solution to (10) starting at \(x\).

Using the similar method in [37], we have the following result.

Lemma 2.7. Let \((S(t), I(t))\) be the solution of model (2) with initial value \((S(0), I(0)) \in \mathbb{R}_+^2\), then

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t S(s)dB_1(s) = 0, \]

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t I(s)dB_2(s) = 0, \quad \text{a.s.,} \]

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t \int_{\mathbb{Y}} c_1(r(s), u)S(s^-)N(ds, du) = 0, \]

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t \int_{\mathbb{Y}} c_2(r(s), u)I(s^-)N(ds, du) = 0, \quad \text{a.s.} \]

It follows from Lemma 2.7 and (10) that

\[ \frac{\varphi_s(t) - x}{t} = \frac{1}{t} \int_0^t \left\{ [\Lambda(r(s)) - \mu(r(s))\varphi(s)] ds \right. \]

\[ + a(r(s))\varphi(s)dB_1(s) + \int_{\mathbb{Y}} c_1(r(s), u)\varphi(s^-)N(ds, du) \left. \right\}, \]

which together with \(\liminf_{t \to \infty} \frac{\varphi_s(t) - x}{t} = 0\) and the large number theorem for martingales yields that

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t \varphi_s(u)du = \sum_{i \in S} \theta_i \Lambda(i) \quad \text{a.s.,} \]

\[ \sum_{i \in S} \theta_i \mu(i). \]
where \( \theta_i \) \((i \in \mathbb{S})\) is the unique stationary distribution of \( r(t) \).

**Theorem 2.8.** The susceptible population in system (2) is persistent in mean with probability one.

**Proof.** In view of Lemma 2.4, for any \( z \in \mathbb{R}^2 \), we have

\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t S_z(u) du \leq N_0 \quad \text{a.s.}
\]

Integrating the first equation of system (2) yields that for any \( t \geq 0 \),

\[
\frac{S_z(t) - x}{t} = \frac{1}{t} \int_0^t \left[ \Lambda(r(s)) - \beta(r(s))g(S_z(s), I_z(s)) - \gamma(r(s))I_z(u) + \mu(r(s))S_z(s) \right] ds
\]

\[
+ \frac{1}{t} \int_0^t \left[ a(r(s))S_z(s)dB_1(s) + \int_Y c_1(r(s), u)S_z(s^-)\tilde{N}(ds, du) \right]
\]

\[
\geq \hat{\Lambda} - \frac{1}{t} \int_0^t \left[ (\hat{\beta}M_1 + \hat{\mu})S_z(s) \right] ds
\]

\[
+ \frac{1}{t} \int_0^t \left[ a(r(s))S_z(s)dB_1(s) + \int_Y c_1(r(s), u)S_z(s^-)\tilde{N}(ds, du) \right].
\]

Note that \( \lim_{t \to \infty} \frac{S_z(t) - x}{t} = 0 \). Then by Lemma 2.7, we have

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t S_z(u) du \geq \frac{\hat{\Lambda}}{\beta M_1 + \mu} \quad \text{a.s.}
\]

This completes the proof. \( \square \)

3. **Persistence of disease.** In this section, we shall investigate when the disease is persistent and whether there is a stationary distribution of system (2). To this end, we shall investigate the Lyapunov exponent \( \lambda \) given by

\[
\lambda \triangleq \sum_{i \in \mathbb{S}} \theta_i \Upsilon(i) + \sum_{i \in \mathbb{S}} \int_0^\infty \beta(i)g'(\phi, 0)\pi^*(d\phi, i), \quad (14)
\]

where

\[
\Upsilon(i) = -\left( \mu(i) + \gamma(i) + \alpha(i) + \frac{b^2(i)}{2} \right) + \int_Y (\ln(1 + c_2(i, u)) - c_2(i, u))\vartheta(d\mu).
\]

**Lemma 3.1.** For any \( T, C > 1, \varepsilon > 0 \) and \( \sigma > 0 \), there exists a \( \delta = \delta(T, \varepsilon, \sigma) > 0 \) such that

\[
\mathbb{P}\{\tau_z^c \geq T\} \geq 1 - \varepsilon \quad \text{for all } z \in [C^{-1}, C] \times (0, \sigma],
\]

where the stopping time is defined as \( \tau_z^c = \inf\{t \geq 0 : I_z(t) \geq \sigma\} \).

**Proof.** It follows from the exponential martingale inequality that \( \mathbb{P}(\Omega_2) \geq 1 - \varepsilon \), where

\[
\Omega_2 = \left\{ \int_0^t b(r(s))dB_2(s) + \int_0^t \int_Y \ln(1 + c_2(r(s), u))\tilde{N}(ds, du) \right.
\]

\[
< \frac{1}{2} \int_0^t \left[ b^2(r(s)) + \int_Y (c_2(r(s), u) - \ln(1 + c_2(r(s), u)))\vartheta(du) \right] ds
\]

\[
+ \ln \frac{1}{\varepsilon} \quad \text{for all } t \geq 0 \right\}.
\]
By applying Itô’s formula, we have

\[
\frac{\ln I_z(T)}{T} - \frac{\ln y}{T} = \frac{1}{T} \int_0^T \left[ \beta(r(t)) \frac{g(S_z(t), I_z(t))}{I_z(t)} \right. \\
- \left( \mu(r(t)) + \gamma(r(t)) + \alpha(r(t)) + \frac{b^2(r(t))}{2} \right) dt + \frac{1}{T} \int_0^T b(r(t)) dB_2(t) \tag{15}
\]

+ \frac{1}{T} \int_0^T \int_Y [\ln(1 + c_2(r(t), u) - c_2(r(t), u)] \vartheta(du) dt

+ \frac{1}{T} \int_0^T \int_Y \ln(1 + c_2(r(t), u)) \tilde{N}(dt, du).

Let \( M_3 \triangleq \max\{g'(x, 0) : 0 \leq x \leq N_1\} \) and there exists \( M_4 \geq M_3 > 0 \) such that \( \beta(i)M_4 > \mu(i) + \gamma(i) + \alpha(i), i \in \mathbb{S} \). When \( \omega \in \Omega_2 \) we have

\[
\ln I_z(t) \leq \ln y + \ln \frac{1}{\xi}
\]

+ \int_0^t \left[ \beta(r(s)) \frac{g(S_z(s), I_z(s))}{I_z(s)} - (\mu(r(s)) + \gamma(r(s)) + \alpha(r(s))) \right] ds

\leq \ln y + \ln \frac{1}{\xi} + \sum_{i \in \mathbb{S}} [\beta(i)M_4 - (\mu(i) + \gamma(i) + \alpha(i))] t.

Let \( \delta = \sigma \xi \sum_{i \in \mathbb{S}} (\mu(i) + \gamma(i) + \alpha(i) - \beta(i)M_4) T \). Then \( I_z(t) < \sigma \) for all \( t < T, \omega \in \Omega_2 \) if \( y \leq \delta \).

Lemma 3.2. For any \( C, T > 1, \varepsilon, \nu > 0 \), there exists a \( \sigma > 0 \) such that for all \( z \in [C^{-1}, C] \times (0, \sigma] \),

\[ \mathbb{P}\{ |\varphi_x(t) - S_z(t)| < \nu \ \text{for all} \ t \in [0, T \wedge \tau^z_\nu] \} \geq 1 - \varepsilon. \]

Proof. In view of Lemma 2.4, we have

\[ \mathbb{P}\{ \mathbb{E}(\varphi_x(t) \vee S_z(t)) \leq N_1 \ \text{for all} \ t \leq \tau^z_\nu \} \geq 1 - \frac{\varepsilon}{2} \ \text{for all} \ z \in [C^{-1}, C] \times (0, \delta]. \]

Applying Itô’s formula, we have

\[
|\varphi_x(s) - S_z(s)| \leq \int_0^s \mu(r(u))|\varphi_x(u) - S_z(u)| du + \int_0^s \beta(r(u))g(S_z(u), I_z(u)) du
\]

+ \int_0^s \gamma(r(u))I_z(u) du + \int_0^s a(r(u))(\varphi_x(u) - S_z(u)) dB_1(u) + \int_0^s \int_Y c_1(r(v), u)(\varphi(v^-) - S(v^-)) \tilde{N}(dv, du)
\].
and hence
\[
\mathbb{E} \sup_{s \leq t} [\varphi_x(s \wedge \xi_z) - S_z(s \wedge \xi_z)]^2 \\
\leq 16 \mathbb{E} \left( \int_0^{s \wedge \xi_z} \mu(r(u))|\varphi_x(u) - S_z(u)|^2 \right) \\
+ 16 \mathbb{E} \left( \int_0^{s \wedge \xi_z} (\beta(r(u))g(S_z(u), I_z(u)) + \gamma I_z(u))^2 \right) \\
+ 16 \mathbb{E} \sup_{s \leq t} \left| \int_0^{s \wedge \xi_z} a(r(u))(\varphi_x(u) - S_z(u)) dB_1(u) \right|^2 \\
+ 16 \mathbb{E} \sup_{s \leq t} \left| \int_0^{s \wedge \xi_z} c_1(r(v), u)(\varphi(v^-) - S(v^-)) \tilde{N}(dv, du) \right|^2,
\]
where \(\xi_z \triangleq \tau_z^\sigma \wedge \inf \{u \geq 0 : (\varphi_x(u) \vee S_z(u)) \geq N_1\}\). By using Burkholder-Davis-Gundy inequality and Hölder’s inequality, for all \(t \leq T\), we have
\[
\mathbb{E} \sup_{s \leq t} [\varphi_x(s \wedge \xi_z) - S_z(s \wedge \xi_z)]^2 \\
\leq 16m_1\mu^2 \int_0^{s \wedge \xi_z} (\varphi_x(u) - S_z(u))^2 \, du \\
+ 16m_2(\tilde{\beta}M_3 + \tilde{\gamma})^2 \sigma^2 T^2 \\
+ 16m_3a^2 \mathbb{E} \int_0^{t \wedge \xi_z} (\varphi_x(u) - S_z(u))^2 \, du \\
+ 16 \mathbb{E} \sup_{s \leq t} \int_0^{t \wedge \xi_z} c_1(v)^2 \vartheta(dv) \int_0^{s \wedge \xi_z} |(\varphi(u^-) - S(u^-))|^2 \, du \\
\leq \overline{m} \left( \sigma^2 + \mathbb{E} \int_0^{t \wedge \xi_z} (\varphi_x(u) - S_z(u))^2 \, du \right) \\
\leq \overline{m} \left( \sigma^2 + \int_0^t \mathbb{E} \sup_{s \leq u} (\varphi_x(s \wedge \xi_z) - S_z(s \wedge \xi_z))^2 \, du \right)
\]
for some \(\overline{m} = \overline{m}(M_3, T) > 0\), where \(m_i, i = 1, 2, 3, 4\) are positive constants. Applying Grönwall’s inequality, we have
\[
\mathbb{E} \sup_{s \leq T} [\varphi_x(s \wedge \xi_z) - S_z(s \wedge \xi_z)]^2 \leq \overline{m} \sigma^2 e^{\overline{m} T}.
\]
From Chebyshev’s inequality, it follows that when \(\sigma\) is sufficiently small,
\[
\mathbb{P} \left\{ \sup_{s \leq T} [\varphi_x(s \wedge \xi_z) - S_z(s \wedge \xi_z)]^2 \geq \nu^2 \right\} \leq \frac{\overline{m} \sigma^2 e^{\overline{m} T}}{\nu^2} < \frac{\varepsilon}{2},
\]
which together with
\[
\mathbb{P}\{s \wedge \xi_z = s \wedge \tau_z^\sigma \text{ for all } s \in [0, T]\} \geq \mathbb{P} \left\{ \sup_{s \leq T} \{\varphi_x(s) \wedge S_z(s)\} \leq N_1 \right\} \geq 1 - \frac{\varepsilon}{2}
\]
yields the conclusion of this lemma. This completes the proof. \(\Box\)
Lemma 3.3. Assume that $\lambda > 0$, then for any $\varepsilon > 0$, $C > 1$, there exist $T = T(\varepsilon, C) > 0$ and $\delta_0 = \delta_0(\varepsilon, C)$ such that $\mathbb{P}(\Omega_z) > 1 - 3\varepsilon$ for all $z \in [C^{-1}, C] \times (0, \delta_0]$, where $\Omega_z = \{\ln \xi_+(T) - \ln y \geq \frac{1}{2}T\}$.

Proof. From the definition of $\eta_1$, we have

$$\sum_{i \in S} \theta_i T(i) + \sum_{i \in S} \int_0^\infty \beta(i)(g'(\phi, 0) - \eta_1) \pi^*(d\phi, i) \geq \frac{3}{4} \lambda$$

for sufficiently small $\eta_1$. By ergodicity, there exists $T(\varepsilon, C) > 0$ such that

$$\mathbb{P}\left\{ \frac{1}{T} \int_0^T [\beta(r(t))(g'(\varphi_C(t), 0) - \eta_1) + \Upsilon(r(t))] dt \geq \frac{1}{2} \lambda \right\} \geq 1 - \varepsilon.$$ 

By the uniqueness of solution, $\varphi_x(t) \leq \varphi_C(t)$ a.s. for all $x \in [C^{-1}, C]$ of $\mathbb{R}$ and hence that $\mathbb{P}(\Omega_3) \geq 1 - \varepsilon$, where

$$\Omega_3 = \left\{ \int_0^T [\beta(r(t))(g'(\varphi_x(t), 0) - \eta_1) + \Upsilon(r(t))] dt \geq \frac{1}{2} \lambda T \right\}.$$

By virtue of Lemma 3.2, we can choose $\sigma, \nu > 0$ such that

$$|g(S_z(t), I_z(t)) - g'(S_z(t), 0)I_z(t)| < \varepsilon \quad \text{for } t \in [0, T \wedge \tau^\pi_z]$$

and $\mathbb{P}(\Omega_4) \geq 1 - \varepsilon$ and that for all $\omega \in \Omega_4$,

$$\frac{|g(S_z(t), I_z(t))|}{I_z(t)} - g'(\varphi_x(t), 0) \leq \frac{|g(S_z(t), I_z(t)) - g'(S_z(t), 0)|}{I_z(t)} + |g'(S_z(t), 0) - g'(\varphi_x(t), 0)| < \eta_1,$$

where $\Omega_4 = \{||\varphi_x(t) - S_z(t)|| < \nu \text{ for all } t \in [0, T \wedge \tau^\pi_z]\}$. Note that

$$\mathbb{E}\left[ \int_0^T b(r(t))dB_2(t) + \int_0^T \int_Y \ln(1 + c_2(r(t), u)) \tilde{N}(dt, du) \right]^2$$

$$\leq \int_0^T (b^2 + \int_0^T \int_Y |\ln(1 + c_2(r(t), u))|^2 \vartheta(du)dt$$

$$\leq (b^2 + K)T,$$

then applying Chebyshev’s inequality, we have

$$\mathbb{P}\left\{ \left| \int_0^T b(r(t))dB_2(t) + \int_0^T \int_Y \ln(1 + c_2(r(t), u)) \tilde{N}(dt, du) \right| \leq \frac{M}{\varepsilon} \sqrt{T} \right\} \geq 1 - \varepsilon,$$

where $M = M(b, c, M_3) > 0$ is a constant. Taking $T > \frac{16M}{\varepsilon 4T}$, we have $\mathbb{P}(\Omega_5) \geq 1 - \varepsilon$, where

$$\Omega_5 = \left\{ \left| \int_0^T b(r(t))dB_2(t) + \int_0^T \int_Y \ln(1 + c_2(r(t), u)) \tilde{N}(dt, du) \right| \leq \frac{\lambda}{4} T \right\}.$$
For \( z \in [C^{-1}, C] \times (0, \delta_0) \) and \( w \in \hat{\Omega}_z = \bigcap_{i=3}^{5} \Omega_i \),
\[
\ln I_z(T) - \ln y \\
\geq \int_0^T \left[ \beta(r(t)) \frac{g(S_z(t), I_z(t))}{I_z(t)} - \left( \mu(r(t)) + \gamma(r(t)) + \alpha(r(t)) + \frac{b^2(r(t))}{2} \right) \right] \, dt \\
+ \int_0^T \left[ \ln(1 + c_2(r(t), u)) - c_2(r(t), u) \right] \delta(du) \, dt \\
- \int_0^T (b(r(t))dB_2(t) + \int_0^T \ln(1 + c_2(r(t), u)) \tilde{N}(dt, du) \\
\geq \int_0^T [\beta(r(t))(g'(\varphi_z(t), 0) - \varepsilon) + Y(r(t))] \, dt - \frac{\lambda}{4} T \\
\geq \frac{\lambda}{4} T.
\]
Consequently, we obtain \( \mathbb{P}(\hat{\Omega}_z) = \mathbb{P}(\bigcap_{i=3}^{5} \Omega_i) \geq 1 - 3\varepsilon \), which completes the proof.  \( \square \)

**Theorem 3.4.** Assume that \( \lambda > 0 \), then the disease in system (2) is persistent.

**Proof.** Using the similar method to that in [24] and noting Lemmas 2.4 and 3.3, we see that for any \( \varepsilon > 0 \), there exist \( T = T(\varepsilon) \) and \( \delta_1(\varepsilon) > 0 \) such that
\[
\limsup_{n \to \infty} \sum_{k=0}^{n-1} \mathbb{P}\{I_z(kT) \leq \delta_1\} \leq \varepsilon \quad \text{for all } z \in \mathbb{R}_+^2, \tag{17}
\]
which together with Lemma 2.4 implies that there exists a compact set \( G \subset \mathbb{R}_+ \) such that
\[
\limsup_{n \to \infty} \sum_{k=0}^{n-1} \mathbb{P}\{I_z(kT) \in G\} \geq 1 - 2\varepsilon \quad \text{for all } z \in \mathbb{R}_+^2.
\]
By virtue of Lemma 2.4, there exists \( N_2 > 1 \) such that
\[
\mathbb{P}\{N_2^{-1} \leq I_z(t) \leq N_2\} \geq 1 - \varepsilon
\]
for all \( y \in G, t \leq T \). By the Markov property, for all \( t \leq T \)
\[
\mathbb{P}\{N_2^{-1} \leq I_z(t)(kT + t) \leq N_2\} \geq (1 - \varepsilon)\mathbb{P}\{I_z(kT) \in G\}.
\]
As a result, for any \( z \in \mathbb{R}^2_+ \),
\[
\liminf_{n \to \infty} \frac{1}{nT} \int_0^{nT} \mathbb{P}\{N_2^{-1} \leq I_z(t) \leq N_2\} \, dt \geq (1 - 2\varepsilon)(1 - \varepsilon) \geq (1 - 3\varepsilon),
\]
which implies that
\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{P}\{N_2^{-1} \leq I_z(u) \leq N_2\} \, du \geq 1 - 3\varepsilon.
\]
This completes the proof.  \( \square \)

By virtue of Lemma 2.4 and [7, Theorem 4.14], \((S_z(t), I_z(t), r(t))\) has a stationary distribution. Then we have the following result.

**Theorem 3.5.** Assume that \( \lambda > 0 \), then the process \((S(t), I(t), r(t))\) of system (2) has a unique stationary distribution.
### 4. Extinction of Disease

In this section, we discuss the extinction of disease and the convergence of the susceptible population.

**Lemma 4.1.** Assume that $\lambda < 0$. For any $C > 1$, $\varepsilon, \nu > 0$, there exists $\tilde{\delta} > 0$ such that

$$
P \left( \left\{ \lim_{t \to \infty} \frac{\ln I_z(t)}{t} = \lambda \right\} \cap \{ |\varphi_x(t) - S_z(t)| \leq \nu \text{ for all } t \geq 0 \} \right) \geq 1 - 3\varepsilon$$

for all $z \in [C^{-1}, C] \times [0, \tilde{\delta}]$.

**Proof.** Since

$$
\sum_{i \in S} \theta_i \int \ln (1 + c_2(i, u)) - c_2(i, u) \, \vartheta(du) - \sum_{i \in S} \int_0^\infty \beta(i) g'(\phi, 0) \pi^*(d\phi, i)
$$

$$
= -\lambda - \sum_{i \in S} \theta_i \left( \mu(i) + \gamma(i) + \alpha(i) + \frac{b^2(i)}{2} \right) < \infty,
$$

then we can find $\eta_2 > 0$ such that

$$
-\sum_{i \in S} \theta_i \vartheta(0) - \sum_{i \in S} \int_0^\infty \beta(i) (g'(\phi, 0) + \eta_2) \pi^*(d\phi, i) \geq -\frac{3}{4} \lambda.
$$

Similar to the method in the proof of Lemma 3.3, there exists $T_1 = T_1(\varepsilon, C)$ such that for all $t \geq T_1$, $P(\Omega_6) \geq 1 - \varepsilon$, where

$$
\Omega_6 = \left\{ \frac{1}{t} \int_0^t [\beta(r(s))g'(\varphi_x(s), 0) + \eta_2] + \vartheta(r(s)) \, ds \leq \frac{1}{2} \lambda \right\}.
$$

By the continuity of $g(\cdot, \cdot)$, there are constants $\nu > 0$ and $\sigma > 0$ such that

$$
\frac{g'(r, 0)}{y} - \frac{g(p, q)}{y} < \eta_2 \quad \text{whenever} \quad |p - r| < \nu \quad \text{and} \quad 0 < |q| < \sigma.
$$

Let $\vartheta_z = \inf \{ t > 0 : |\varphi_x(t) - S_z(t)| \geq \nu \}$, then there exists $\tilde{\sigma} \in (0, \sigma)$ such that

$$
\left| g'(\varphi_x(t), 0) - \frac{g(S_z(t), I_z(t))}{I_z(t)} \right| < \eta_2
$$

for all $t \in [0, \vartheta_z \wedge \tau_x^z]$. Hence when $\omega \in \Omega_6 \cap \{ \vartheta_z \geq T_1 \}$,

$$
\frac{1}{t} \int_0^t [\beta(r(s))g'(S_z(s), 0) + \vartheta(r(s))] \, ds \leq \frac{1}{2} \lambda
$$

for all $t \in [T_1, \vartheta_z \wedge \tau_x^z]$.

Similar to the method in the proof of the Lemma 3.3, taking $T_1 > \frac{16\sigma^2}{\lambda \varphi z_x^2}$, we have

$$
P(\Omega_7) \geq 1 - \varepsilon,
$$

where

$$
\Omega_7 = \left\{ \int_0^T b(r(t)) \, dB_2(t) + \int_0^T \int \ln (1 + c_2(r(t), u)) \, N(du) \right\} \leq \frac{|\lambda|}{4} T_1.
$$

Applying (18) and (19) into (15), then for $\omega \in \Omega_6 \cap \Omega_7 \cap \{ \vartheta_z \wedge \tau_x^z \geq T_1 \}$, we obtain

$$
\ln I_z(t) \leq \ln y - \frac{|\lambda|}{4} t
$$

for all $t \in [T_1, \vartheta_z \wedge \tau_x^z]$. By virtue of Lemmas 3.1 and 3.2, we can find sufficiently small $\tilde{\delta} = \delta(\varepsilon, H)$ such that

$$
\ln \tilde{\delta} - \frac{|\lambda|}{4} T_1 < \ln \tilde{\sigma}
$$

(21)
and \( \mathbb{P}(\Omega_8) \geq 1 - \varepsilon \) for all \( z \in [C^{-1}, C] \times (0, \delta] \), where \( \Omega_8 = \{ \zeta^* \triangleq \theta^* \wedge \tau_z^* \geq T_1 \} \). Consequently, we obtain \( \mathbb{P}(\Omega_8) \geq 1 - 3\varepsilon \), where \( \Omega_8 = \cap_{i=0}^8 \Omega_i \). It follows from the definition of \( \zeta^* \) that for \( \omega \in \Omega_8 \),

\[
|\varphi_x(t \wedge \zeta^*) - S_z(t \wedge \zeta^*)| \leq \nu.
\]

Then in \( \Omega_8 \), \( t \wedge \zeta^* < \theta^* \) for all \( t \geq T_1 \), which implies that \( \Omega_8 \subset \{ \zeta^* \leq \theta^* \} \). Since \( \zeta^* = \theta^* \wedge \tau_z^* \), we have \( \Omega_8 \subset \{ \tau_z^* \leq \theta^* \} \). As a result, for all \( z \in [C^{-1}, C] \times (0, \delta] \) and \( \omega \in \Omega_8 \), it follows from (20) and (21) that

\[
\ln I_z(t \wedge \tau_z^*) \leq \ln \frac{|\lambda|}{4} (t \wedge \tau_z^*) < \ln \sigma \text{ for all } t \geq T_1.
\]

This implies that \( t \wedge \tau_z^* < \tau_z^* \) for all \( t \geq T_1 \), \( z \in [C^{-1}, C] \times (0, \delta] \) and \( \omega \in \Omega_8 \), and hence that \( \tau_z^* = \theta_z^* = \infty \) for any \( z \in [C^{-1}, C] \times (0, \delta] \) and \( \omega \in \Omega_8 \). If \( \omega \in \Omega_8 \), by ergodicity and strong law of large numbers for martingales, we have

\[
\limsup_{t \to \infty} \frac{1}{t} \ln |I_z(t)| \leq \lambda \leq \limsup_{t \to \infty} \frac{1}{t} \int_0^t \left| \frac{g(S_z(s), I_z(s))}{I_z(s)} - g'(\varphi_x(s), 0) \right| ds \leq c_0 \varepsilon,
\]

where \( c_0 \) is a positive constant. Thus, for any \( \nu > 0 \) and \( z \in [C^{-1}, C] \times (0, \delta] \), we obtain

\[
\mathbb{P} \left\{ \limsup_{t \to \infty} \frac{\ln I_z(t)}{t} = \lambda \right\} = \mathbb{P} \left\{ \limsup_{t \to \infty} |\varphi_x(t) - S_z(t)| \leq \nu \text{ for all } t \geq 0 \right\} \geq \mathbb{P}(\Omega_8) \geq 1 - 3\varepsilon.
\]

The proof is completed. \( \square \)

**Theorem 4.2.** Assume that \( \lambda < 0 \), then the disease in system (2) is extinct exponentially and the distribution of \( (S(t), r(t)) \) converges weakly to \( \pi^* \).

**Proof.** By virtue of Lemma 4.1, we know that \( I(t) \) is not recurrent in \( \mathbb{R}_+ \). Note that the diffusion in (2) is nondegenerate, then \( I(t) \) is transient, and hence for any \( t \geq 0 \), there exists \( \hat{\delta} > 0 \) such that

\[
\mathbb{E}(\tau_z^*) < \infty \quad \text{for all } y \in (\hat{\delta}, \infty),
\]

where \( \tau_z^* \triangleq \inf\{ t \geq 0 : I_z(t) \in (0, \hat{\delta}) \} \). It follows from Lemma 4.1 that

\[
\mathbb{P} \left\{ \limsup_{t \to \infty} \frac{\ln I_z(t)}{t} = \lambda \text{ for all } t \geq 0 \right\} \geq 1 - 2\varepsilon \quad \text{for all } y > 0,
\]

which implies that as \( t \to \infty \), \( I_z(t) \to 0 \) a.s. that is, the disease in system (2) is extinct. In what follows, we prove that the susceptible population weakly converges to \( \pi^* \).

In view of Theorem 2.8, by using similar method to (17), for any \( \epsilon > 0 \), there exist \( T_2 = T_2(\epsilon) \) and \( \delta_2(\epsilon) > 0 \) such that

\[
\limsup_{n \to \infty} \sum_{k=0}^{n-1} \mathbb{P}\{S_z(kT_2) \leq \delta_2 \} \leq \epsilon \quad \text{for all } z \in \mathbb{R}_+^2.
\]

Assume \( h(\cdot) : \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R} \) is a Lipschitz function with constant \( K_h \) such that \( |h(x_1, i)| \leq K_h \) and \( |h(x_1, i) - h(x_2, i)| \leq K_h |x_1 - x_2| \) for any \( i \in \mathbb{S}, x_1, x_2 \in \mathbb{R}_+ \). By the ergodicity theorem, for every function \( h(\cdot) : \mathbb{R} \to \mathbb{R} \) satisfying that \( \sum_{i \in \mathbb{S}} \int_0^\infty |h(\phi, i)| \pi^*(d\phi, i) < \infty \), we have

\[
\mathbb{P} \left\{ \lim_{t \to \infty} \frac{1}{T} \int_0^T h(\varphi_x(t), r(t))dt = \sum_{i \in \mathbb{S}} \int_0^\infty h(\phi, i) \pi^*(d\phi, i) \right\} = 1 \quad \text{for all } x > 0.
\]
In order to prove the weak convergence of the distribution of \((S_z(t), r(t))\) to the measure \(\pi^*\), we only need to prove that
\[
\lim_{t \to \infty} \mathbb{E} h(S_z(t), r(t)) = h^* \triangleq \sum_{i \in S} \int_0^\infty h(\phi, i) \pi^*(d\phi, i) \quad \text{for all } z \in \mathbb{R}^2_+.
\]

Since \(|\mathbb{E} h(S_z(t), r(t)) - h^*| \leq |\mathbb{E} h(\phi_z(t), r(t)) - h^*| + |\mathbb{E} h(S_z(t), r(t)) - \mathbb{E} h(\phi_z(t), r(t))|\) and \(\phi_z(t)\) converges weakly to \(\pi^*\), we have
\[
\limsup_{t \to \infty} |\mathbb{E} h(S_z(t), r(t)) - h^*| \leq K_h \nu \limsup_{t \to \infty} \mathbb{P}\{|\phi_z(t) - S_z(t)| \leq \nu\}
+ 2K_h \limsup_{t \to \infty} \mathbb{P}\{|\phi_z(t) - S_z(t)| \geq \nu\}.
\] (24)

By the Markov property,
\[
|\mathbb{E} h(S_z(t + i_0 T_2), r(t)) - h^*| \\
\leq \int_{\mathbb{R}^2_+} |\mathbb{E} h(S_z(t), r(t)) - h^*| \mathbb{P}\{S_z(i_0 T_2) \in dz\} \\
\leq \int_U |\mathbb{E} h(S_z(t), r(t)) - h^*| \mathbb{P}\{S_z(i_0 T_2) \in dz\} + 2K_h \mathbb{P}\{S(i_0 T_2) \notin U_1\}.
\] (25)

Applying (22), (24) and Fatou’s lemma into (25), we have
\[
\limsup_{t \to \infty} |\mathbb{E} h(S_z(t + i_0 T_2), r(t)) - h^*| \leq K_h \nu + K_h \epsilon + 6K_h \epsilon.
\]

The proof is completed due to the arbitrariness of \(\epsilon\) and \(\nu\). \(\square\)

**Corollary 1.** Assume that \(\lambda < 0\) then \(\lim_{t \to \infty} I_z(t) = 0\) a.s., and hence that
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t S_z(u) du = \lim_{t \to \infty} \frac{1}{t} \int_0^t \phi_z(u) du = \frac{\sum_{i \in S} \theta_i A(i)}{\sum_{i \in S} \theta_i \mu(i)} \text{ a.s.}
\]

**Remark 1.** In view of Theorem 4.2 and the expression (14) of the threshold \(\lambda\), we see that the white noise and Lévy jumps acting on \(I(t)\) in (2) have a positive effect on controlling the outbreak of the disease. To be more specific, if \(\max\{\{b(i) : i \in S\}\) (respectively, \(\max\{\{c_2(i, u) : i \in S, u \in Y\}\) is close to 0, then white noise (respectively, Lévy jumps) has a little effect on controlling the outbreak of the disease; if \(\min\{\{|b(i) : i \in S\}\) (respectively, \(\min\{\{c_2(i, u) : i \in S, u \in Y\}\) are more away from 0, then white noise (respectively, Lévy jumps) has a strong effect on controlling the disease. Furthermore, in a regime-switching model, the disease might be extinct eventually, even if it is persistent in one regime.

5. **Application to system** (5). Note that system (5) is a special case of system (2). In this section, we apply the previous theoretical results to system (5). By using the similar method of Theorem 2.8 we have the following result.

**Corollary 2.** The susceptible population in system (5) is persistent in mean with probability one. Moreover,
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t S_z(u) du \leq \limsup_{t \to \infty} \frac{1}{t} \int_0^t \phi_z(u) du = \frac{\Lambda}{\mu} \text{ a.s.}
\]
In what follows, we apply Theorems 3.4 and 3.5 to system (5). The threshold $\lambda$ associated with (5) is given as follows:

$$
\lambda = - \left( \mu + \gamma + \alpha + \frac{b_2}{2} \right) + \int_Y (\ln(1 + c_2(u)) - c_2(u))\vartheta(du) + \beta \int_0^\infty \phi \pi^*(d\phi)
$$

$$=(\mu + \gamma + \alpha)(\tilde{R}_0 - 1) + \beta \int_0^\infty \phi \pi^*(d\phi) - \frac{\Lambda \beta}{\mu},$$

where $\tilde{R}_0$ is given in (6). It follows from the ergodicity theorem and Corollary 2 that

$$\int_0^\infty \phi \pi^*(d\phi) = \limsup_{t \to \infty} \frac{1}{t} \int_0^t \varphi_x(u)du = \frac{\Lambda}{\mu} \text{ a.s.}$$

and hence that

$$\lambda = (\mu + \gamma + \alpha)(\tilde{R}_0 - 1). \quad (26)$$

Thus, the following result is an immediate consequence of Theorems 3.4 and 3.5.

**Corollary 3.**

(i): If $\tilde{R}_0 > 1$ then the disease in (5) is persistent and system (5) has a stationary distribution;

(ii): If $\tilde{R}_0 < 1$ then the disease in system (5) is extinct.

Obviously, this corollary is consistent with the results obtain by Zhou et al. [37].

6. **Numerical simulations.** In this section, we shall give an example and perform some numerical simulations to illustrate our results in Sections 3 and 4. Throughout this section, we take $g(S, I) = SI/(S + I)$, $Y = \{1, 2, 3\}$, $\vartheta(Y) = 2$, $\vartheta(\{1\}) = 0.5$, $\vartheta(\{2\}) = 1$, $\vartheta(\{3\}) = 0.5$, $S = \{1, 2\}$, and

$$Q = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$**
switching from one to the other due to the law of the Markov chain. First, we take
\[ \beta(1) = 0.8, \quad c_2(1, 1) = c_2(1, 3) = -0.1, \quad c_2(1, 2) = -0.2, \]
\[ \beta(2) = 0.55, \quad c_2(2, 1) = c_2(2, 3) = -0.2, \quad c_2(2, 2) = -0.3. \] (29)
The corresponding deterministic systems of (27) and (28) have a unique globally
asymptotically stable endemic equilibrium \((S^*, I^*) = (\frac{9}{5}, \frac{2}{5})\) and \((S^*, I^*) = (\frac{16}{7}, \frac{8}{45})\),
respectively (for more details, see for example [30]). From Figure 1, we see that in
regime 1, the solution of (2) oscillates around the deterministic endemic equilibrium
\(S^*\) and the disease \(I\) is extinction in both regime 1 and regime 2. In this case, white noise and
\(\lambda\) and the disease \((\text{regime 1, the solution of (2) oscillates around the deterministic endemic equilibrium}
respectively (for more details, see for example [30]). From Figure 1, we see that in
regime 1, the solution of (2) with parameters (29) is persistent and the curves of
\(S\) population to (2) oscillates around the deterministic endemic equilibrium
\(\phi\) the process
is larger than that of model (2) without jumps. It follows from Lemma 2.5 that
\(\lambda\) has a negative effect on the model (2). To find out the effect of intensity of jumps on dynamical system (2), we shall
decrease the intensity \(c_2\). First set
\[ \beta(1) = 0.8, \quad c_2(2, 1) = c_2(2, 3) = -0.2, \quad c_2(2, 2) = -0.3, \]
\[ \beta(2) = 0.55, \quad c_2(1, 1) = c_2(1, 3) = -0.3, \quad c_2(1, 2) = -0.4 \] (30)
and take other parameter values the same to those of (27) and (28). We see from
Figures 4 and 5 that the disease \(I(t)\) of model (2) with jumps is extinct in both
regime 1 and regime 2. In this case, \(\lambda \approx -0.0166\), and hence it follows from Theorem
4.2 that the disease \(I(t)\) of the regime-switching model (2) with jumps is extinct
eventually and \(S\) converges to the unique stationary distribution of (10) (see Figures
6 and 7). Comparing with the parameters (29), we see that Lévy noise can suppress
disease outbreak.

From example (30) we can conclude that the disease \(I(t)\) of the regime-switching
model (2) with jumps is extinct eventually if the disease \(I(t)\) is extinct in both
regime 1 and regime 2. Note that the disease \(I(t)\) of the regime-switching model (2)
may have the opportunity to persist eventually even if it is extinct in one regime.
It is interesting to see whether the disease \(I(t)\) of the regime-switching model (2)
with jumps dies out eventually if it is extinct in one regime but is persistent in the
other. In fact, this is possible. For example, we fix the following parameters
\[ \beta(1) = 0.8, \quad c_2(1, 1) = c_2(1, 3) = -0.1, \quad c_2(1, 2) = -0.2, \]
\[ \beta(2) = 0.55, \quad c_2(2, 1) = c_2(2, 3) = -0.5, \quad c_2(2, 2) = -0.6 \] (31)
and take other parameter values the same to those of (27) and (28). Figure 8
shows that in regime 1, the susceptible and infected individuals fluctuate around
the deterministic steady-state values, while in regime 2, the disease \(I(t)\) of (2) with
jumps is extinct. In this case, \( \lambda \approx -0.0671 \), and hence Theorem 4.2 says that the disease \( I(t) \) of the regime-switching model (2) with jumps is extinct eventually and \( S(t) \) converges to the unique stationary distribution of (10) (see Figures 10 and 11).

In what follows we shall investigate the influence of the disease transmission coefficient \( \beta(i) \), \( i = 1, 2 \). We take
\[
\beta(1) = 0.8, \quad c_2(1, 1) = c_2(1, 3) = -0.1, \quad c_2(1, 2) = -0.2, \\
\beta(2) = 0.65, \quad c_2(2, 1) = c_2(2, 3) = -0.2, \quad c_2(2, 2) = -0.3. \tag{32}
\]
Direct calculation yields that \( \lambda \approx 0.1094 \). This, together with Theorems 3.4 and 3.5, implies that the system (2) with jumps is persistent and has a stationary distribution (see Figure 14). In fact, we see from Figures 12 and 13 that in regime 1, regime 2 and regime-switching case, the infected and susceptible individuals are persistent. Furthermore, the curve of the susceptible population fluctuates around the deterministic steady-state value and the infected population of (2) with jumps is less than the deterministic steady-state value.

Now, we decrease \( \beta(2) \) but keep the other parameters unchanged:
\[
\beta(1) = 0.8, \quad c_2(1, 1) = c_2(1, 3) = -0.1, \quad c_2(1, 2) = -0.2, \\
\beta(2) = 0.45, \quad c_2(2, 1) = c_2(2, 3) = -0.2, \quad c_2(2, 2) = -0.3. \tag{33}
\]
In this case, we see that in regime 2 the disease \( I(t) \) of the system (2) is extinct, but in regime 1 and regime-switching case, the disease \( I(t) \) is persistent eventually (see Figures 15 and 16). An easy calculation yields that \( \lambda \approx 0.0427 \). This, together with Theorems 3.4 and 3.5, implies that system (2) is persistent and has a stationary distribution. This theoretical result is illustrated by Figure 17. Finally, we further decrease \( \beta(1) \) but keep the other parameters unchanged:
\[
\beta(1) = 0.6, \quad c_2(1, 1) = c_2(1, 3) = -0.1, \quad c_2(1, 2) = -0.2, \\
\beta(2) = 0.45, \quad c_2(2, 1) = c_2(2, 3) = -0.2, \quad c_2(2, 2) = -0.3. \tag{34}
\]
From Figures 18 and 19, we see that the curve of susceptible population fluctuates the deterministic steady-state value \( S^* = 2 \) and the disease is extinct eventually in regime 1, regime 2 and regime-switching case. Easy calculation yields that \( \lambda \approx -0.0573 \), which implies that the disease \( I(t) \) is extinct and \( S(t) \) converges to the boundary distribution of (10) according to Theorem 4.2 (see Figures 20 and 21). Comparing (33) with the parameters (33) and (34), we see that the prevalence of the disease can be controlled by means of appropriate protection measures to reduce the value of transmission rate \( \beta(i) \) (\( i \in S \)) of disease when susceptible individuals get in touch with infected individuals.

7. Conclusions and discussions. In this paper, we introduce a more accurate threshold which is sufficient and almost necessary condition of persistence and extinction of the disease in a class of stochastic regime-switching SIS epidemic models with Lévy jumps and nonlinear incidence rate, which include the standard incidence, Beddington-DeAngelis incidence, and nonlinear incidence of the form \( f(S)h(I) \). Moreover, we take several different perturbations (white noise, Lévy jumps and regime-switching) into consideration and our model (2) is more general because many stochastic SIS models can be regarded as special cases of our model (for example, [12, 20, 21, 37]). In particular, we see that the oscillating amplitude of the solution curve of model (2) with jumps is larger than that of model (2) without jumps. In view of the expression (14) of the threshold \( \lambda \), we can draw the following five conclusions:
(i): The white noise acting on $I(t)$ play a positive role in controlling the spread of the disease;
(ii): The Lévy noise acting on $I(t)$ play a positive role in controlling the spread of the disease;
(iii): Even if the disease is persistent in one regime, it will have the opportunity to die out eventually;
(iv): The white noise acting on the death rate of $S(t)$ has little influence on the asymptotic behavior of the susceptible individuals in model (2);
(v): The prevalence of the disease can be controlled reducing the value of transmission rates of disease when susceptible individuals contact with infected individuals.

Therefore, we may determine treatment strategies and forecast epidemic dynamics by making use of the expression (14) of the threshold $\lambda$.  

Figure 1. Trajectories of solutions to model (2) with parameters (29) in regimes 1 and 2.

Figure 2. Trajectories of susceptible and infected populations of model (2) with parameters (29).
Figure 3. The joint density distribution of $(S, I)$ of model (2) with parameters (29). (a) The case without jumps; (b) The case with jumps.

Figure 4. Trajectories of solutions to model (2) with parameters (30) in regimes 1 and 2.
We introduce the following stochastic susceptible-infected-recovered (SIR) epidemic model

\[
\begin{align*}
    dS(t) &= [\Lambda(r(t)) - \beta(r(t))g(S(t), I(t)) - \mu(r(t))S(t)]dt \\
        &+ a(r(t))S(t)dB_1(t) + \int_{\mathbb{V}} c_1(r(t), u)S(t^-)\tilde{N}(dt, du), \\
    dI(t) &= [\beta(r(t))g(S(t), I(t)) - (\mu(r(t)) + \gamma(r(t)) + \alpha(r(t)))I(t)]dt \\
        &+ b(r(t))I(t)dB_2(t) + \int_{\mathbb{V}} c_2(r(t), u)I(t^-)\tilde{N}(dt, du), \\
    dR(t) &= [-\mu(r(t))R(t) + \gamma(r(t))I(t)]dt + \sigma(r(t))R(t)dB_3(t) \\
        &+ \int_{\mathbb{V}} c_3(r(t), u)R(t^-)\tilde{N}(dt, du),
\end{align*}
\]

(35)

where \( R(t) \) represents the number of recovered individuals, \( \sigma \geq 0, r(t), N \) and \( B_i(t), i = 1, 2, 3 \) are mutually independent Brownian motions, and all the other
Figure 7. The joint density distribution of \((S, I)\) of model (2) with parameters (30). (a) The case without jumps; (b) The case with jumps.

Figure 8. Trajectories of solutions to model (2) with parameters (31) in regimes 1 and 2.
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Figure 9. Trajectories of susceptible and infected populations of model (2) with parameters (31).

Figure 10. The weak convergence of $S$ to the stationary solution $\varphi$ of (10) with parameters (31).
**Figure 11.** The joint density distribution of \((S, I)\) of model (2) with parameters (31). (a) The case without jumps; (b) The case with jumps.

**Figure 12.** Trajectories of solutions to model (2) with parameters (32) in regimes 1 and 2.
Figure 13. Trajectories of susceptible and infected populations of model (2) with parameters (32).

Figure 14. The joint density distribution of \((S, I)\) of model (2) with parameters (32). (a) The case without jumps; (b) The case with jumps.
parameters have the same meaning as those in (2). Note that the dynamics of recovered individuals have no influence on the disease transmission dynamics and we can omit to discuss the third equation. Thus, our theoretical results can also be applied to the stochastic susceptible-infected-recovered (SIR) epidemic model of the form (35). There are a number of results about persistence and extinction of the disease in the SIR model (35). For the bilinear incidence $g(S, I) = SI$, Zhang and Wang [34] studied the asymptotical behavior of (35) without regime-switching; Guo [16] obtained a unique global positive solution and investigated the asymptotical behavior of the stochastic SIR model (35).

Comparing with [16, 34], we weaken the restrictions in the assumptions and improve their results to some degree. By choosing appropriate Lyapunov functions, Zhou and Zhang [38] investigated the extinction and persistence of the disease of
Figure 17. The joint density distribution of \((S, I)\) of model (2) with parameters (33). (a) The case without jumps; (b) The case with jumps.

the following system

\[
\begin{aligned}
\text{d}S(t) &= [\lambda - \beta S(t)I(t) - \mu S(t)]\text{d}t \\
&\quad + aS(t)\text{d}B_1(t) + \int_Y c_1(u)S(t^-)\tilde{N}(dt, du), \\
\text{d}I(t) &= [\beta S(t)I(t) - (\mu + \gamma + \alpha)I(t)]\text{d}t \\
&\quad + bI(t)\text{d}B_2(t) + \int_Y c_2(u)I(t^-)\tilde{N}(dt, du), \\
\text{d}R(t) &= [-\mu R(t) + \gamma I(t)]\text{d}t \\
&\quad + \sigma R(t)\text{d}B_3(t) + \int_Y c_3(u)R(t^-)\tilde{N}(dt, du).
\end{aligned}
\] (36)

Zhou and Zhang [38] concluded that if \(\bar{R}_0 < 1\) then the disease of system (36) ultimately vanishes from the population, and that if \(\bar{R}_0 > 1\) then the disease of system (36) persists in the population, where \(\bar{R}_0\) is given in (6). It is easy to see
that the relevant results obtained in [38] are completely covered by Corollaries 2 and 3.

Furthermore, it would be very interesting to see whether our analysis and methods in this paper are applicable to a SIRS epidemic model with regime-switching and Lévy jumps because in this case the system cannot be de-coupled any more (see [29, 35]). Recently, Bao and Shao [3] established a criterion to judge extinction of the infectious individuals for a state-dependent regime-switching SIRS model driven by two independent Brownian motions. It is interesting to give some sufficient conditions on persistence and extinction (not only extinction) of the infectious individuals for a SIRS epidemic model with state-dependent regime-switching and Lévy jumps. We look forward to investigating these problems in the future.

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Figure 19. Trajectories of susceptible and infected populations of model (2) with parameters (34) in regimes 1 and 2.

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Figure 20. The weak convergence of $S$ to the stationary solution $\varphi$ of (10) with parameters (34).

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