INTRODUCTION.
We examine the Cauchy problem for General Relativity as the time history of the two fundamental forms of the geometry of a space-like hypersurface, its metric $\bar{g}$ and its extrinsic curvature $K$. By using a 3+1 decomposition of the Ricci tensor, we split the Einstein equations into “constraints”, equations containing only $\bar{g}$, $K$ and their space derivatives, and evolution equations giving the time derivatives of $\bar{g}$ and $K$ in terms of space derivatives of these quantities and also of the lapse and shift, i.e., the choice of the time lines. The constraints can be posed and solved as an elliptic system by known methods. However, the equations of evolution in time for $\bar{g}$ and $K$ are not, despite their form as canonical hamiltonian equations, a mathematically well posed system; and they do not manifest directly the propagation of physical gravitational effects along the light cone. These equations, of course, contain gauge effects and cannot, therefore, yield directly a physical wave equation, that is, a hyperbolic system with suitable characteristics. (For a review of the relevant geometry, see, for example.)

In this paper we give two different methods for obtaining a nonlinear wave equation for the evolution of the extrinsic curvature $K$. The evolution of $\bar{g}$ is just its dragging by $K$ along the axis orthogonal to the time slices.

The first method corresponds to a choice of time slicing by fixation of the mean extrinsic curvature of the space slices. It leads to a mixed elliptic and hyperbolic system for which we prove local in time, global in space, existence theorems in the cases of compact or asymptotically euclidean space slices.

The second method, which relies on a harmonic time slicing condition, leads to equations of motion equivalent to a first order symmetric hyperbolic system with only physically appropriate characteristics. We construct this system explicitly. Among the propagated quantities is, in effect, the Riemann curvature. The space coordinates and the shift are arbitrary; and, in this sense, the system is gauge invariant.

Our gauge invariant non linear hyperbolic system, being exact and always on the physical light cone, is well suited for use in a number of problems which now confront gravity theorists. These include large scale computations of astrophysically significant processes (such as black hole collisions) that require efficient stable numerical integration, extraction of gravitational radiation with arbitrarily high accuracy from Cauchy data, gauge invariant perturbation and approximation methods, and posing boundary conditions compatibly with the causal structure of space time.

n+1 DECOMPOSITION, ARBITRARY SHIFT.
The $n+1$ decomposition of the curvature of a lorentzian manifold and the resulting decomposition of Einstein equations into constraints and evolution equations have been given in the case of zero shift by Lichnerowicz (1939) and in the case of an arbitrary shift by Choquet (Foures)- Bruhat (1956). Arnowitt, Deser and Misner (1962) have obtained this decomposition through a hamiltonian formalism and have deduced from it interesting physical consequences, for example, the definition of the gravitational mass.

We recall this straightforward decomposition, using a coframe with time axis orthogonal to the space slices and introducing the notation of spatial Lie derivative like, for example,
We consider on a manifold $V = M \times \mathbb{R}$ a pseudo Riemannian metric of Lorentzian signature which reads

$$ds^2 \equiv g_{\alpha \beta} \theta^\alpha \theta^\beta \equiv -N^2 (\theta^0)^2 + g_{ij} \theta^i \theta^j$$

with $(t \in \mathbb{R}, x^i, i = 1, \ldots, n$ are local coordinates on $M)$

$$\theta^0 \equiv dt \quad , \quad \theta^i \equiv dx^i + \beta^i dt$$

With this choice of coframe, the dual frame has a time axis orthogonal to the space slices $M_t \equiv M \times \{ t \}$ while the space axis are tangent to them. The Pfaff derivatives $\partial_\alpha$ with respect to $\theta^\alpha$ are

$$\partial_0 \equiv \frac{\partial}{\partial t} - \beta^i \partial_i \quad , \quad \partial_i \equiv \frac{\partial}{\partial x^i}$$

The structure constants of our coframe are such that

$$d\theta^\alpha = -\frac{1}{2} c^\alpha_{\beta \gamma} \theta^\beta \wedge \theta^\gamma$$

Hence

$$c^0_{\alpha \beta} = 0 \quad , \quad c^i_{jk} = 0 \quad , \quad c^i_{j0} = -c^0_{ij} = -\partial_j \beta^i$$

We denote by $\bar{\Gamma}^i_{jk}$ the Christoffel symbols of the (time dependent) space metric $\bar{g} \equiv g_{ij} dx^i dx^j$, and by $\omega^\lambda_{\alpha \beta}$ the coefficients of the Riemannian connection of $g$ in the coframe $\theta^\alpha$. We have

$$\omega^i_{jk} = \bar{\Gamma}^i_{jk}$$

$$\omega^0_{00} = N^{-1} \partial_0 N \quad , \quad \omega^0_{0i} = \omega^0_{i0} = N^{-1} \partial_i N$$

$$\omega^0_{ij} = \frac{1}{2} N^{-2} (\partial_0 g_{ij} - g_{hi} (\partial_j \beta^h)) \equiv \frac{1}{2} N^{-2} (\frac{\partial}{\partial t} g_{ij} - L_{\beta} g_{ij})$$

where we define the symmetrization by $f_{(ij)} \equiv f_{ij} + f_{ji}$ (no factor of $1/2$).

We define for any $t$ dependent space tensor $T$ another such tensor $\hat{\partial}_0 T$ of the same type by setting

$$\hat{\partial}_0 \equiv \frac{\partial}{\partial t} - L_{\beta}$$

where $L_{\beta}$ is the Lie derivative on $M_t$ with respect to $\beta$.

The extrinsic curvature of the space slices is given by

$$K_{ij} \equiv -N \omega^0_{ij} \equiv -\frac{1}{2} N^{-1} \hat{\partial}_0 g_{ij}$$

and we find

$$\omega^k_{0i} = -N K^k_i \quad , \quad \omega^k_{00} = -N K^k_i + \partial_i \beta^k$$

**Curvature tensor.**

The components of the curvature tensor of $g$ in the frame $\theta^\alpha$ are found to be, using the general formulas given in Choquet-Bruhat and DeWitt-Morette I, p. 306, (with an opposite sign)

$$R^j_{i \ kl} = \bar{R}^j_{i \ kl} - K^j_K K_{li} + K^j_l K_{ki}$$

$$R^i_{0 \ kl} = N^{-1} (\bar{\nabla}_K K_{li} - \bar{\nabla}_l K_{ki})$$

$$R^0_{i \ j0} = -(N^{-1} \hat{\partial}_0 K_{ij} + K_{im} K^m_j + N^{-1} \bar{\nabla}_j \partial_i N )$$

**Ricci tensor.**

From these formulas result the following ones for the Ricci curvature, where we have denoted by $H$ the mean extrinsic curvature of the space slices, i.e., we have set
\[ H \equiv K_h^\alpha, \]
\[ R_{ij} \equiv -N^{-1}\partial_0 K_{ij} + HK_{ij} - 2K_{im}K^m_j - N\partial_j N + R_{ij} \]
\[ R_{0ij} \equiv N(-\nabla_j K^\alpha_k + \partial_j H) \]
\[ R_{00} \equiv N(\nabla^i \partial_i N - NK_{ij}K^{ij} + \partial_0 H) \]

**SECOND ORDER EQUATION FOR K.**

In the formula above \( R_{ij} \) is, like the right hand side giving its decomposition, a time dependent space tensor, the projection on space of the Ricci tensor of the space time metric. We compute its \( \partial_0 \) derivative. First we compute \( \partial_0 R_{ij} \). The infinitesimal variation of the Ricci curvature corresponding to an infinitesimal \( \delta \hat{g} \) variation of the metric is

\[ \delta R_{ij} = \frac{1}{2}\{\nabla^h \nabla_{(i} \delta g_{j)h} - \nabla_h \nabla^h \delta g_{ij} - \nabla_j \partial_i (g^{hk} \delta g_{hk})\} \]

This expression applies to \((\partial/\partial t)\hat{R}_{ij}\) with \( \delta g_{ij} = (\partial/\partial t)g_{ij} \) and to \( \mathcal{L}_\beta \hat{R}_{ij} \) with \( \delta g_{ij} = \mathcal{L}_\beta g_{ij} \). Therefore, using the relation between \( \partial_0 g_{ij} \) and \( K_{ij} \), we obtain

\[ \partial_0 R_{ij} \equiv -\nabla^h \nabla_{(i} (NK_{j)h}) + \nabla_h \nabla^h (NK_{ij}) + \nabla_j \partial_i (NH) \]
\[ \equiv -\nabla_{(i} (N\nabla^h (K_{j)h}) - \nabla_{(i} (K_{j}^k h) \partial_k N - 2N\hat{R}_{ijm}K^m_k - N\hat{R}_{m(i}K^m_{j)} \]
\[ + \nabla_h \nabla^h (NK_{ij}) + \nabla_j \partial_i (NH). \]

We now use the expressions for \( R_{0ij} \) and \( R_{ij} \) to obtain the identity

\[ \Omega_{ij} \equiv \partial_0 R_{ij} - \nabla_{(i} (R_{j)0} \equiv \]
\[ -\partial_0 (N^{-1}\partial_0 K_{ij}) + \hat{\partial}_0 (HK_{ij} - 2K_{im}K^m_j) - \partial_0 (N^{-1}\nabla_j \partial_i N) - N\nabla_i \partial_j H \]
\[ -\nabla_{(i} (K_{j)h} \partial^h N) - 2N\hat{R}_{ijm}K^m_h - N\hat{R}_{m(i}K^m_{j)} + \nabla_h \nabla^h (NK_{ij}) + H\nabla_j \partial_i N \]

This identity shows that for a solution of the Einstein equations

\[(E) \quad R_{\alpha\beta} = \rho_{\alpha\beta} \]

the extrinsic curvature \( K \) satisfies a second order differential system which is quasidiagonal with principal part the wave operator, except for the terms \( \nabla_i \partial_j H \). The other unknowns \( \hat{g} \) and \( N \) appear at second order except for the term \( \hat{\partial}_0 \nabla_j \partial_i N \). We will use these facts in the following paragraphs to obtain geometrical well posed systems for the Einstein equations.

**I. MIXED HYPERBOLIC AND ELLIPTIC SYSTEM FOR \( \hat{g}, K, N \) WHEN H IS GIVEN.**

A procedure to reduce the second order equation for \( K \) obtained above to a quasi diagonal system with principal part the wave operator is to replace in the term \( \nabla_j \partial_i H \) the mean curvature \( H \) by an apriori given function \( h \): this is a gauge condition on the space slices. It was used by Christodoulou and Klainerman\(^7\), with \( h = 0 \), in the asymptotically euclidean case. With this replacement the equations \( \Omega_{ij} = \Theta_{ij} \) become, when \( N \) is known, a quasi diagonal second order system for \( K \) with principal part the wave operator, namely:

\[ \square K_{ij} = P_{ij} + \Theta_{ij} \]
with
\[ \Box K_{ij} \equiv -\hat{\partial}_0(N^{-1}\hat{\partial}_0 K_{ij}) + \nabla^h \nabla_h(NK_{ij}) \]
where \( P_{ij} \) depends only on \( K \) and its first derivatives, on \( \bar{g}, N \) and \( \hat{\partial}_0 N \) together with their space derivatives of order \( \leq 2 \), and is given by:
\[
P_{ij} \equiv \hat{\partial}_0(-HK_{ij} + 2g^{hm} K_{ih} K_{jm}) + \hat{\partial}_0(N^{-1}\nabla_j \partial_i N)
+ \nabla_i(K_{j}h, h) + 2NK_{ij}K^m + NR_{m(i}K_{j)m} - H\nabla_j \partial_i N + N\nabla_j \partial_i h
\]

while \( \Theta_{ij} \), zero in vacuum, is:
\[ \Theta_{ij} \equiv \hat{\partial}_0 \rho_{ij} - \nabla_i(\rho_{j}0) \]

When \( \beta, N \) and the sources \( \rho \) are known, the above equation together with
\[
(\rho') \quad \hat{\partial}_0 g_{ij} = -2NK_{ij}
\]
constitute a third order quasi diagonal system for \( \bar{g} \), hyperbolic if \( N^2 > 0 \) and \( \bar{g} \) is properly riemannian.

On the other hand the equation \( R^0_0 = \rho^0_0 \) together with \( H = h \) imply the equation
\[
\nabla^i \partial_i N - (K_{ij}K^{ij} - \rho^0_0)N = -\hat{\partial}_0 h
\]

This equation is an elliptic equation for \( N \) when \( \bar{g}, K \) and \( \rho \) are known.

Note that for energy sources satisfying the energy condition we have \(-\rho^0_0 \geq 0\) as well as \( K.K \equiv K_{ij}K^{ij} \geq 0 \), an important property for the solution of the elliptic equation. The mixed hyperbolic elliptic system that we have constructed will determine the unknowns \( N \) and \( \bar{g} \) in a neighbourhood of \( M \) in \( M \times \mathbb{R} \) when the shift \( \beta \) is chosen.

LOCAL EXISTENCE THEOREM(case of given \( H \)).

We consider the vacuum case. The system for \( \bar{g}, K \)
\[
(1) \quad \hat{\partial}_0 g_{ij} = -2NK_{ij}
(2) \quad \Box K_{ij} = P_{ij}
\]
is equivalent to a third order hyperbolic system. It requires for its solution only local Sobolev spaces (cf. Leray), but the lapse \( N \) is now determined by an elliptic equation on each \( M_t \) which must be solved globally:
\[
(3) \quad \nabla^i \partial_i N - K.K N = -\hat{\partial}_0 h
\]

We will consider the two cases of a compact and of an asymptotically euclidean \( M \). In both cases we will use an iteration scheme solving alternatively the elliptic equation and a linear hyperbolic system deduced in an obvious manner from (1), (2).

We specify the results in the physical case of space dimensions \( n = 3 \). There are problems in extending them to other dimensions in the asymptotically euclidean case.

Case of a compact \( M \).

In the case of a compact \( M \) a good choice in order to have a positive lapse is to take \( \hat{\partial}_0 h = f > 0 \). Standard elliptic theory gives the following lemma, where Sobolev spaces \( H_s \) are relative to some smooth fixed metric \( e \) on \( M \), \( I \) is an interval of \( \mathbb{R} \) and the shift \( \beta \) is taken smooth on \( M \times I \).
Lemma 1. a) If $\tilde{g}(t) \in H_3, K(t) \in H_2$, with $K.K \neq 0$, and $f \in H_2$ the lapse equation has a unique solution $N(t) \in H_4$. By the maximum principle this solution is such that $N > 0$ on $M$ if $f \geq 0$ on $M$ and $f \neq 0$.
b) If moreover $\tilde{g} \in \cap C^{3-k}(I, H_k)$, while $f, K \in \cap C^{2-k}(I, H_k)$
then
$$N \in \cap C^{2-k}(I, H_{2+k}).$$

Remark. We have $K.K \geq \frac{1}{4}H^2$ hence $K.K \neq 0$ if $H \neq 0$; we will prove later that the equality $H = h$ holds for a solution of the coupled hyperbolic and elliptic system if it holds initially, but we cannot use this property in the iteration scheme. The condition $K.K \neq 0$ must be deduced from such an hypothesis on the initial data and the continuity properties possessed by solutions of the hyperbolic system.

We now consider the linear system for $\tilde{g}(n+1), K(n+1)$

$$\begin{align*}
(1_n) & \quad \hat{\partial}_t g_{ij}(n+1) = -2N(n)K_{ij}(n), \\
(2_n) & \quad \Box_n K_{ij}(n+1) = P_{ij}(n)
\end{align*}$$

obtained by replacing in $\Box$ the metric $\tilde{g}$ by a metric $\bar{g}(n)$ and in $P_{ij}$ also $K$ by $K(n)$ and $N$ by $N(n)$.

**Lemma 2. Hypothesis.** Given:

1) a) $N(n) \geq N_0 > 0$, $N(n) \in \cap C^{2-k}(I, H_{2+k}),$

b) $h \in \cap C^{3-k}(I, H_k), I \equiv [0, T],$

c) $\tilde{g}(n) \in \cap C^{3-k}(I, H_k)$ and $K(n) \in \cap C^{2-k}(I, H_k).$

2) Cauchy data such that

$$g_{ij}(n+1)(0, .) = \gamma_{ij} \in H_3, \quad K_{ij}(n+1)(0, .) = k_{ij} \in H_2,$$

where the metric $\gamma$ is uniformly equivalent to the given metric $e$ and $k, k \neq 0$ on $M$.

The data $\hat{\partial}_t K_{ij}(n+1)(0, .) = \hat{k}_{ij}$, determined by the equation $R_{ij}(0, .) = 0$, belong to $H_1$.

**Conclusion.** The Cauchy problem for the system $(1_n), (2_n)$ has a unique solution

$$\tilde{g}(n+1) \in \cap C^{3-k}(I, H_k), \quad K(n+1) \in \cap C^{2-k}(I, H_k).$$

Moreover there exists an interval $I_1 = [0, T_1], 0 < T_1 \leq T$, and a number $\epsilon > 0$ depending only on the norms of the given quantities in their respective spaces such that $\tilde{g}(n+1)$ is uniformly equivalent to $\gamma$ on $M \times I_1$ and

$$\| K(n+1).K(n+1) \|_{C^0(M \times I_1)} \geq \epsilon.$$

**Proof.** The system is equivalent to a third order hyperbolic system, and the results can be proved by standard methods.

**Remark.** When $\tilde{g}(n), K(n), N(n)$ satisfy the indicated hypothesis the equation $(2_n)$ is a wave equation for $K(n+1)$ whose coefficients satisfy the required hypothesis for the existence of a solution in

$$\cap C^{2-k}(I, H_k).$$

The integration of equation $(1_n)$ determines $0 \leq k \leq 2$.
then $\bar{g}(n+1)$, but the general theory of first order equations is not sufficient to assert its required regularity. One must use the fact that $\bar{g}(n+1)$ satisfies a third order hyperbolic equation to prove that result.

We remark also that the regularity hypothesis made on $h$ is stronger than the regularity we prove on $K$.

**Theorem.** Let $(M,e)$ be a smooth compact riemannian manifold. Let be given on $M \times I$, with $I \equiv [0,T]$, a pure space smooth vector field $\beta$ and a function $h$ such that

$$h \in \cap_{2 \leq k \leq 3} C^{3-k}(I, H_k), \quad \partial_0 h \geq 0, \quad \partial_0 h \neq 0.$$

There exists an interval $J \equiv [0, \ell]$ such that the system (1), (2), (3) has one and only one solution on $M \times J$

$$\bar{g} \in \cap_{1 \leq k \leq 3} C^{3-k}(J, H_k), \quad K \in \cap_{1 \leq k \leq 2} C^{2-k}(J, H_k), \quad N \in \cap_{0 \leq k \leq 2} C^{2-k}(J, H_{2+k}).$$

with $N > 0$ and $\bar{g}$ uniformly equivalent to $e$, taking the initial data

$$g_{ij}(0,.) = \gamma_{ij} \in H_3, \quad K_{ij}(0,.) = k_{ij} \in H_2$$

if $\gamma$ is a properly riemannian metric uniformly equivalent to $e$ and $k.k \neq 0$.

**Proof.** One determines $N(0)$ on $M \times I$ by the equation on each $M_t$

$$\triangle \gamma N(0) - k.k N(0) = -\partial_0 h(0)$$

and then proceeds to the iterations, with $N(n)$ determined from $\bar{g}(n), K(n)$.

One uses the elliptic estimates for $N(n)$ and the energy estimates of first and second order for $K(n)$ and $\bar{g}(n)$ to prove the boundedness of the iterates in the indicated norms, as well as to show that there exists $I_1 = [0, \ell_1] \subset I$ such that, on $M \times I_1$, $\bar{g}(n)$ is uniformly equivalent to $e$ and $\|K(n), K(n)\| \geq \epsilon > 0$.

The elliptic estimates and the first order energy estimates applied to the difference of two successive iterates show that there exists $J \equiv [0, \ell]$ such that the sequence converges strongly but in a weaker norm

$$(C^0(J, H_2) \cap C^1(J, H_1)) \times (C^0(J, H_1) \cap C^1(J, H_0)) \times (C^0(J, H_3) \cap C^1(J, H_2)).$$

Since the sequence is bounded in $L^\infty(J, H_3) \times L^\infty(J, H_2) \times L^\infty(J, H_4)$, the dual of a Banach space, there exists a subsequence which converges weakly in that space, and the limit found from the strong convergence belongs therefore to that space. An analogous argument applied to the time derivatives shows the existence of a subsequence whose first [respectively second] time derivatives converge weakly in $L^\infty(J \times H_2) \times L^\infty(J \times H_1)$ [respectively in $L^\infty(J, H_1) \times L^\infty(J \times H_0)$], and the limit found belongs therefore to these spaces. It can be proved using properties of strong and weak convergence that the limit satisfies the non linear system (1), (2), (3). It can also be proved that this solution is in fact in the space given in the theorem (cf. a proof for second order systems by Hughes, Kato and Marsden10 and for symmetric hyperbolic first order systems by Majda11).

**Case of $(M,e)$ euclidean at infinity.**

We now consider the case of a manifold $M$ which is the union of a compact set and a finite number of disjoint sets (“ends”) diffeomorphic to the exterior of a ball in $\mathbb{R}^3$; $M$ is endowed with a smooth properly riemannian metric $e$ which reduces on each end to the euclidean metric.
We will look for an asymptotically minkowskian solution of (1), (2), (3) on M×I, i.e. a solution belonging to some weighted Sobolev space \(H_{s,\delta}\) on each space slice \(M_t\). We recall that \(H_{s,\delta}\) is the completion of \(C_0^\infty\) in the norm
\[
\|f\|_{H_{s,\delta}} \equiv \left\{ \int_M \sum_{0 \leq k \leq s} \sigma^{2k+2\delta} |D^k f|^2 \mu(e) \right\}^{1/2}
\]
where \(\sigma^2 \equiv 1+d^2\), \(d\) the distance in the metric \(e\) to some fixed point in \(M\).

The theory of elliptic equations on an asymptotically euclidean manifold in \(H_{s,\delta}\) spaces (Choquet-Bruhat and Christodolou\(^{12}\)) gives the following lemma.

**Lemma 1.** a) If \(\bar{g}(t) - e \in H_{3,-1}, K(t) \in H_{2,0}\) and \(f = \partial_0 h \in H_{2,1}\) the lapse equation has a unique solution with \(1 - N(t) \in H_{4,-1}\). By the maximum principle this solution is such that \(N > 0\) on \(M\) if \(\partial_0 h \geq 0\) on \(M\).

b) If moreover:
\[
\bar{g} - e \in \bigcap_1^{\infty} C^{3-k}(I, H_{k,-1}), \quad K \in \bigcap_0^2 C^{2-k}(I, H_{k,0}),
\]
then
\[
1 - N \in \bigcap_0^2 C^{2-k}(I, H_{2+k,-1}).
\]

**Proof.** a) The multiplication properties of weighted Sobolev spaces show that with the hypothesis made \((K,K')(t) \in H_{2,1}\). Since \(-\frac{3}{2} < -1 < -\frac{1}{2}\) the equation
\[
\bar{\nabla}^i \partial_i u - K.K u = -\partial_0 h + K.K , \quad N = 1 + u
\]
has one and only one solution \(u \in H_{4,-1}\); one deduces the property \(N > 0\) from the maximum principle and the fact that \(N\) tends to 1 at infinity.

b) The first time derivative of \(u\) can be proven to be in \(H_{3,\delta}\), but again with \(-\frac{3}{2} < \delta < -\frac{1}{2}\), hence in \(H_{3,-1}\) (not in \(H_{3,0}\)), and analogously for the second time derivative with 3 replaced by 2.

**Lemma 2. Hypothesis.** We give on \(M\times I, I \equiv [0,T]\):

a) \(\beta \in \bigcap_1^{\infty} C^{k-3}(I, H_{k,-1}), \quad h \in \bigcap_2^3 C^{k-3}(I, H_{k,1})
\]
b) \(N(n) \geq N_0 > 0, N(n) \in \bigcap_0^2 C^{2-k}(I, H_{2+k,-1})
\]
c) \(\bar{g}(n) - e \in \bigcap_1^{\infty} C^{3-k}(I, H_{k,-1}), K(n) \in \bigcap_0^{\infty} C^{2-k}(I, H_{k,0}),
\]
with moreover \(\partial_0 K_n \in \bigcap_1^{\infty} C^{1-k}(I, H_{k,1})
\]
d) The Cauchy data (independent of \(n\)) such that
\[
g_{ij}(n+1)(0,\cdot) = \gamma_{ij}, \quad \gamma - e \in H_{3,-1}, \quad K_{ij}(n+1)(0,\cdot) = k_{ij} \in H_{2,0},
\]
and the metric \(\gamma\) is uniformly equivalent to the given metric \(e\).

The data \(\partial_0 K_{ij}(n+1)(0,\cdot) = \bar{k}_{ij}\), determined by the equation \(R_{ij}(0,\cdot) = 0\), belong to \(H_{1,1}\).

**Conclusion.** The Cauchy problem for the system \((1_n), (2_n)\) has a unique solution such that
The first weighted energy estimate needed to prove the lemma is obtained for\( \bar{g}(n+1) - e \in C^3-I, H_{k,-1}, K(n+1) \in C^2-I, H_{k,0} \)\( 1 \leq k \leq 3 \)\( 0 \leq k \leq 2 \)\( \partial_t K(n+1) \in C^1-I, H_{k,1} \)\( 0 \leq k \leq 1 \)

There exists an interval \( I_1 = [0, T_1] \), \( 0 < T_1 \leq T \), and a number \( \epsilon > 0 \) depending only on the norms of the given quantities in their respective spaces such that \( \bar{g}(n+1) \) is uniformly equivalent to \( \gamma \) on \( M \times I_1 \).

**Proof.** The first weighted energy estimate needed to prove the lemma is obtained for \( \bar{D}K(n+1) \) (denoted \( K \) to simplify the writing) and \( \partial_t K \) by multiplying its wave equation by \( \sigma^2 \nabla_0 K^{ij} \), after checking that under the hypothesis made on the \( n \)th iterates we have \( \sigma P_{ij}(n) \in C^0(I, L^2) \). The estimates for \( \partial_t \bar{D}K \) and \( \bar{D}^2K \) are obtained after derivation of the equation in the space directions. We obtain their boundedness in \( H_{0,2} \) because \( \sigma^2 \bar{D}P_{ij} \in C^0(I, L^2) \). The estimate for \( \partial_t^2 \bar{K} \) is obtained only in \( H_{0,1} \) because only \( \sigma \partial_t P_{ij} \in C^0(I, L^2) \), due to the term in \( \bar{D}^2 \partial_t^2 N \).

The estimate for \( K \) can be obtained by integration in \( t \) from the estimate of \( \partial_t K \): it implies \( K - k \in C^0(I, H_{0,1}) \), hence \( K \in C^0(I, H_{0,0}) \) if it is so of \( k \).

The property \( \partial_t \bar{g}(n+1) \in C^0(I, H_{2,0}) \), \( \partial^2_{ttt} \bar{g}(n+1) \in C^0(I, H_{1,1}) \), \( \partial^3_{ttt} \bar{g}(n+1) \in C^0(I, H_{0,1}) \) results from the equation \( (1_n) \). By integration with respect to \( t \) we obtain \( \bar{g}(n+1) - \gamma \in C^0(H_{2,0}) \), hence \( \bar{g}(n+1) - e \in C^0(I, H_{2,-1}) \) by the hypothesis on \( \gamma \). To prove that it belongs to \( C^0(I, H_{3,-1}) \) one must use the third order equation for \( \bar{g}(n+1) \) deduced from \( (1_n), (2_n) \).

The convergence of the iterates in a small enough time interval is proved analogously as in the case of a compact \( M \) and gives the following theorem.

**Theorem.** The system \( (1), (2), (3) \) with Cauchy data \( \gamma \) and \( k \) on the manifold \( (M, e) \), euclidean at infinity with given \( \beta \) and \( h \) on \( M \times [0, T] \) satisfying the hypothesis spelled out in lemmas 1 and 2, has one and only one solution \( (\bar{g}, K, N) \) on \( M \times J \equiv [0, \ell] \) a sufficiently small subinterval of \( I \), if the Cauchy data are such that \( \gamma - e \in H_{3,-1} \) and is uniformly equivalent to \( e \), \( k \in H_{2,0}, \bar{k} \in H_{1,1} \). The solution belongs to the functional spaces indicated for the iterates in the lemmas and is such that \( N \geq 0 \) and \( \bar{g} \) is uniformly equivalent to \( \gamma \).

**Remark.** The estimates show in fact that \( \bar{g}_t - \gamma \) is such that for each \( t \) we have

\[
\bar{g}_t - \gamma \in H_{3,0},
\]

i.e., it has a stronger asymptotic fall off than \( \gamma - e \) or \( \bar{g}_t - e \); this property is related to the A.D.M.\(^3\) theorem of mass conservation.

**SOLUTION OF THE FULL EINSTEIN EQUATIONS.**

We suppose that we have solved the reduced equations, that is

\[
\Omega_{ij} + N \nabla_i \partial_j (H - h) = \Theta_{ij}
\]

and

\[
R_{00} - N \partial_0 (H - h) = \rho_{00}
\]

We will show that the solution obtained satisfies the original Einstein equations if the source \( \rho \) is such that the associated stress energy tensor satisfies the conservation laws
\[ \nabla_\alpha T^{\alpha\beta} = 0 \]

where \((8\pi G = c = 1)\)

\[ T_{\alpha\beta} \equiv \rho_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \rho \]

and the initial data \(\gamma\) and \(k\) satisfy the constraints:

\[ S_0^0 = T_0^0, \quad S_i^0 = T_i^0, \quad \text{for} \quad t = 0 \]

with moreover

\[ \text{tr} k \equiv H \big|_{t=0} = h \big|_{t=0} \]

while the initial values of \(\check{\partial}_0 K\) are chosen so that

\[ R_{ij} \big|_{t=0} = \rho_{ij} \big|_{t=0}, \]

i.e., the Einstein equations are satisfied initially.

The proof follows with \(A_{\alpha\beta} \equiv R_{\alpha\beta} - \rho_{\alpha\beta}, H - h \equiv X\). We will use the Bianchi identities \(\nabla_\alpha S^{\alpha\beta} \equiv 0\) where

\[ S_{\alpha\beta} = T_{\alpha\beta}, \quad S_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \]

For any covariant symmetric space time 2-tensor \(A\) we find by using the values of the \(\omega\)'s that we have the identities

\[ \nabla_0 A_{ij} \equiv \check{\partial}_0 A_{ij} + NK_{(i} A_{j)h} - N^{-1} \partial_{(i} N A_{j)0} \]

\[ \nabla_i A_{j0} \equiv \check{\nabla}_i A_{j0} + N^{-1} K_{ij} A_{00} - N^{-1} A_{0j} \partial_i N + N A_{hj} K_i^h \]

\[ \nabla_i \partial_j X \equiv \check{\nabla}_i \partial_j X + N^{-1} K_{ij} \partial_0 X \]

A straightforward computation using these identities shows that the reduced equations can also be written:

\[ \nabla_0 A_{ij} - \nabla_{(i} A_{j)0} + N \nabla_i \partial_j X + K_{ij} \partial_0 X = 0 \]

and

\[ A_{00} - N \partial_0 X = 0 \]

The Bianchi identities together with the conservation laws give the conservation equations:

\[ \nabla_\alpha \Sigma^{\alpha\beta} = 0, \quad \text{with} \quad \Sigma^{\alpha\beta} \equiv S^{\alpha\beta} - T^{\alpha\beta} \]

In our frame we have the identities

\[ \Sigma^{00} \equiv \frac{1}{2} (A^{00} + N^{-2} g^{ij} A_{ij}) \]

\[ \Sigma^{ij} \equiv A^{ij} + \frac{1}{2} g^{ij} (N^2 A^{00} - g^{hk} A_{hk}) \]

The conservation equation with \(\beta = 0\) implies therefore

\[ \frac{1}{2} (\nabla_0 A^{00} + N^{-2} g^{ij} \nabla_0 A_{ij}) + \nabla_i A^{i0} = 0. \]

Hence, when the reduced equations are satisfied (recall that \(\nabla_0 N = 0\)),

\[ -N^{-2} \nabla_0 \partial_0 X + g^{ij} \nabla_i \partial_j X + N^{-1} H \partial_0 X = 0. \]
This wave equation for $X$ implies $X = 0$ on $M \times I$ if $X$ and $\partial_0 X$ vanish initially. We have then also $A_{00} = 0$

We can now proceed in a manner analogous to \(^6\). We differentiate the conservation equation with $\beta = j$ and use the Ricci identity to obtain

$$\nabla_0 \nabla_0 A_{0j} + \nabla_i \nabla_0 A_{ij} + R_{0i}^j \lambda A^{\lambda j} - R_{0\lambda} A^{j \lambda} - \frac{1}{2} \nabla^j \nabla_0 A_{ij}^i + \frac{1}{2} N^2 \nabla^j \nabla_0 A^{i00} = 0.$$  

Using the reduced equations now with $A_{00} = X = 0$, we find

$$(\nabla_0 \nabla_0 - N^2 \nabla_i \nabla^i) A^{j0} - N^2 (\nabla_i \nabla^j A^{i0} - \nabla^j \nabla_i A^{i0}) + \frac{1}{2} N^2 \nabla^j \nabla_0 A^{i00} \sim 0$$

where $\sim 0$ means modulo linear terms in $A^{i\lambda}$. Using again the Ricci identity and the conservation law with $\beta = 0$, we find

$$-\square A^{j0} \equiv (N^2 \nabla_0 \nabla_0 - \nabla_i \nabla^i) A^{j0} \sim 0$$

where $\sim 0$ denotes terms linear in $A^{i\lambda}$ and its derivatives of order $\leq 1$. We deduce then from the first reduced equation:

$$\square \nabla_0 A_{ij} \sim 0$$

where $\sim 0$ is modulo linear terms in $A^{i\lambda}$ and their derivatives of order $\leq 2$. We have obtained for $A_{i\lambda}$ a strictly hyperbolic system which is linear and homogeneous. The uniqueness theorem of Leray\(^8\) for solutions of such systems shows that we have $A_{i\lambda} = 0$ on $M \times I$ if the corresponding initial data are zero.

The set of initial data in our problem are the values on $M_0$ of $X$, $\partial_0 X$, $A_{i0}$, $\partial_0 A_{i0}$, $A_{ij}$, $\partial_0 A_{ij}$, $\partial_{00} A_{ij}$. These values can be proved to be all zero for a solution of the reduced equations if $(H - h) \mid_t = 0$ and the Einstein equations are satisfied on $M_0$. Our proof is complete.

II. HARMONIC TIME, ALGEBRAIC GAUGE

HYPERBOLIC SYSTEM FOR $\bar{g}$, $K$, $N$.

We shall eliminate at the same time the third derivatives of $N$ and the second derivatives of $H$ in the second order equation for $K$ by a particular choice of $N$ which we now explain (cf. this elimination in the case of zero shift in Choquet Bruhat and Ruggeri\(^6\)). We compute

$$\hat{\partial}_0 \nabla_j \partial_i N \equiv (\frac{\partial}{\partial t} - \mathcal{L}_\beta) \nabla_j \partial_i N$$

We find at once

$$\frac{\partial}{\partial t} \nabla_j \partial_i N \equiv \nabla_j \partial_i \frac{\partial N}{\partial t} - \frac{1}{2} g^{kl} (\nabla_k (\frac{\partial}{\partial t} g_{lj}) - \nabla_l (\frac{\partial}{\partial t} g_{jk}) \partial_k N$$

and we find an analogous formula when the operator $\partial/\partial t$ is replaced by $\mathcal{L}_\beta$ by using the property

$$\partial_i \mathcal{L}_\beta N \equiv \mathcal{L}_\beta \partial_i N \equiv 0$$

which leads to

$$\mathcal{L}_\beta \nabla_j \partial_i N - \nabla_j \partial_i \mathcal{L}_\beta N \equiv -\partial_h N (\nabla_j \nabla_i h - \beta^l \hat{R}_{ij}^l h) \equiv -\frac{1}{2} g^{kl} (\nabla_l \mathcal{L}_\beta g_{jk}) \partial_k N$$

Finally

$$\hat{\partial}_0 \nabla_j \partial_i N \equiv \nabla_j \partial_i \partial_0 N + \frac{1}{2} (\nabla_l K_{j}^l - \nabla_l K_{ij}) \partial_l N.$$
The third order terms in $N$ and second order terms in $H$ in the second order equation for $K$ can therefore be written under the following form

$$C_{ij} \equiv -N^{-1}\nabla_j \partial_i (\partial_0 N + N^2 H)$$

We satisfy the condition $C_{ij} = 0$ by imposing on $N$ the differential equation

$$(N') \partial_0 N + N^2 H = 0$$

The second order equation for $K$ reads then as a wave type non linear system:

$$(K') N \Box K_{ij} = NQ_{ij} + \Theta_{ij}$$

where we now set

$$\hat{\Box} K_{ij} \equiv N^{-2} \hat{\partial}_0 \hat{\partial}_0 K_{ij} + \nabla^h \nabla_h K_{ij}$$

$$N Q_{ij} \equiv -K_{ij} \partial_0 H + 2g^{hm} K_{m(i} \hat{\partial}_0 K_{j)h} + 4N g^{ht} g^{mk} K_{lk} K_{im} K_{jh}$$

$$+ (2 \nabla_{(i} K_{j)l}) \partial^l N - 2HN^{-1} \partial_i N \partial_j N - 2 \partial(i N \partial_j H)$$

$$-3 \partial_h N \nabla^h K_{ij} - K_{ij} \nabla^h N - N^{-1} K_{ij} \partial_h N \partial_i N + N^{-1} K_{(i} \partial_{j)} N \partial_h N$$

$$+ (\nabla_{(i} \partial^h N) K_{j)h} + 2N \tilde{\nabla}^{hijm} K^m_{lij} + N \tilde{R}^{m(i} K^l_{j)h} - 2H \nabla_j \partial_i N$$

$$\Theta_{ij} \equiv \hat{\partial}_0 \rho_{ij} - \tilde{\nabla}_{(i} \rho_{j)}^0$$

The equation $(N')$ expresses that the time coordinate is harmonic, namely:

$$\nabla^\alpha \partial_\alpha x^0 \equiv N^2 \omega^0_0 - g^{ij} \omega^0_{ij} = 0.$$
The preceding results can be extended without major change to dimensions greater than 4. We will show that in dimension 4 a solution of the vacuum Einstein equations, together with the harmonic time gauge condition, satisfies a first order symmetric system, hyperbolic if $\bar{g}$ is properly riemannian and $N^2 > 0$. Such a system could be useful to establish a priori estimates relevant to global problems. It is of great importance for numerical computations, because symmetric hyperbolic systems occur in many areas of mathematical physics, in particular in fluid dynamics, and codes have been developed to study such systems.

We have obtained for the unknowns $\bar{g}$, $K$, $N$ the equations

\begin{align*}
(1) & \quad \hat{\partial}_0 g_{ij} = -2NK_{ij} \\
(2) & \quad \partial_0 N = -N^2 H \\
(3) & \quad \Box K_{ij} \equiv Q_{ij}
\end{align*}

To obtain a first order system we take as additional unknowns:

$\hat{\partial}_0 K_{ij} = M_{0ij}$, $\nabla_h K_{ij} = M_{hij}$, $\partial_i N = N_i$, $\hat{\partial}_0 \partial_i N = N_{0i}$, $\nabla_j \partial_i N = N_{ji}$.

We take as equation (3′)

\begin{equation}
(3') \quad \hat{\partial}_0 K_{ij} = M_{0ij}
\end{equation}

The equation (3) gives

\begin{equation}
(4) \quad \hat{\partial}_0 M_{0ij} - N^2 \nabla^h M_{hij} = -N^2 Q_{ij}
\end{equation}

In three space dimensions the riemann tensor is a linear function of the Ricci tensor:

$$
\bar{R}_{lijm} \equiv g_{lj} \bar{R}_{im} + g_{im} \bar{R}_{jl} - g_{ij} \bar{R}_{lm} - g_{lm} \bar{R}_{ij} - \frac{1}{2} (g_{lj} g_{im} - g_{ij} g_{lm}) \bar{R}
$$

Using the equation $R_{ij} = 0$ we have

$$
\bar{R}_{ij} = N^{-1} M_{0ij} - HK_{ij} + 2K_{ih} K^h_j + N^{-1} N_{ji}
$$

We then can write $N^2 Q_{ij}$ as a polynomial in the unknowns, $N^{-1}$ and $g^{ij}$.

We prove the following lemma for instance by using the commutativity $\hat{\partial}_0 \partial_i = \partial_i \hat{\partial}_0$ on components of tensors and the value of $\hat{\partial}_0 \Gamma^h_{ij}$ in terms of $K$:

**Lemma.** For an arbitrary covariant vector $u_i$ we have

$$
\hat{\partial}_0 \nabla_h u_i = \nabla_h \hat{\partial}_0 u_i + u_l \{ \nabla_h (NK^l_i) + \nabla_i (NK^l_h) - \nabla^l (NK_{ih}) \}
$$

and an analogous formula for tensors with additional terms for each index.

Using this lemma we see that $M_{0ij}$ and $M_{hij}$ must satisfy the equations
\( (5) \hat{\partial}_0 M_{hij} - \bar{\nabla}_h M_{0ij} = NK_{l(j}(M_{hi}^{l} + M_{ih}^{l} - M_{ih}^{l}) + K_{l(j}(K_{h}^{l}N_{h} + N_{i})K_{h}^{l} - K_{i}hN^{l}). \)

On the other hand by definition

\( (6) \hat{\partial}_0 N_{i} = N_{0i} \)

while \( N_{hi} \) and \( N_{0i} \) must satisfy

\( (7) \hat{\partial}_0 N_{hi} - \bar{\nabla}_h N_{0i} = NN_{i}(M_{hi}^{l} + M_{ih}^{l} - M_{ih}^{l}) + N_{i}(K_{h}^{l}N_{h} + K_{h}^{l}N_{i} - N^{l}K_{ih}). \)

Now we deduce the value of \( \hat{\partial}_0 \hat{\partial}_0 N_{0i} \equiv \hat{\partial}_0 N_{0i} \) from (2) and from the Einstein equation

\[ R_{0}^{0} \equiv -\{N^{-1}\bar{\nabla}^{h}\bar{\nabla}_{h}N - K_{ij}K^{ij} + N^{-1}\partial_{0}H\} = 0 \]

Indeed these two equations imply that \( N \) satisfies the following inhomogeneous wave equation

\[ \partial_{0}\partial_{0}N - N^{2}\bar{\nabla}^{h}\bar{\nabla}_{h}N = -N^{3}K_{ij}K^{ij} + 2N^{3}H^{2} \]

Hence by differentiation, use of the Ricci formula and the definitions of \( N_{hi} \) and \( N_{0i} \), we find

\( (8) \hat{\partial}_0 N_{0i} - N^{2}\bar{\nabla}^{h}N_{hi} = -R_{i}^{h}N_{h}N^{2} + 2NN_{i}N_{h}^{h} - 2N^{3}K_{hl}M_{i}^{hl} \)

\[ -3N^{2}N_{i}K_{hl}K^{hl} + 4N^{3}HM_{i}^{h}h + 6N^{2}H^{2}N_{i}. \]

We use again the equation \( R_{ij} = 0 \) to replace \( R_{ij} \) by its value in terms of the unknowns.

We have obtained a first order system (equations 1, 2, 3', 4, 5, 6, 7, 8) in all the unknowns. The right hand sides are polynomial in the unknowns, \( g^{ij} \) and \( N^{-1} \). They do not depend on their derivatives. The left hand sides are linear operators on the all the unknowns. Their coefficients depend on these unknowns, and not on their derivatives except for the derivatives of \( \bar{g} \). In order to obtain a covariant quasilinear first order system for all the unknowns we can, for instance, introduce on \( M \) an a priori given metric \( e \) which may depend on \( t \) but is such that

\( \hat{\partial}_0 e_{ij} = 0. \)

We denote by \( D \) the covariant derivative in the metric \( e \). We introduce the additional unknown \( G_{hij} \equiv D_{h}g_{ij} \) and deduce from (1) the equation:

\( (9) \hat{\partial}_0 G_{hij} = -2\{N_{h}K_{ij} + N(M_{hij} + S_{h(i}^{m}K_{j)m})\} \)

where the tensor \( S \), the difference of the connections of \( \bar{g} \) and \( e \), is given by
\[ S_{ij}^m = \frac{1}{2} g^{mh} (G_{(ij)h} - G_{hij}) \]

We have now obtained a quasi linear first order system for all the unknowns. Its characteristic matrix, obtained by replacing in the principal matrix the operator \( \partial \) by a covariant vector \( \xi \), is constituted of blocks around the diagonal, some reduced to one element \( \xi_0 \), and some \( 4 \times 4 \) matrices with determinant \( \xi_0^2 - N^2 \xi_i \xi_i \). The characteristics are the light cone and the time axis. On the other hand the system can be symmetrized by multiplication with a matrix constituted of blocks around the diagonal equal to one element, 1, or the matrix \( (g^{ij}) \). In other words the system is a first order symmetric hyperbolic system, with domain of dependence determined by the light cone. Known local existence theorems apply to such a system.

Remark. It is possible to obtain a system whose right hand sides are polynomial in the unknowns and \( g^{ij} \) by making a slightly different choice of these unknowns, in particular by introducing \( a_i \equiv N^{-1} \partial_i N \) (cf.\textsuperscript{13,14}).

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Bibliography

1. A. Lichnerowicz, “Problèmes Globaux en Mecanique Relativiste”, Hermann 1939.
2. Y. (Foures)-Bruhat, J. Rat. Mechanics and Anal. 5 p.951-966, 1956.
3. A. Arnowitt, S. Deser and C. Misner in “Gravitation: An Introduction to Current Research”, L. Witten ed. Wiley 1962.
4. A. Fisher and J. Marsden, J. Math. Phys. 13 p.546-568, 1972.
5. Y. Choquet-Bruhat and J. W. York, “The Cauchy Problem” in General Relativity and Gravitation, A. Held ed. Plenum 1980.
6. Y. Choquet-Bruhat and T. Ruggeri, Comm. Math. Phys. 89, p.269-275, 1983.
7. D. Christodoulou and S. Klainerman, “The Global Nonlinear Stability of the Minkowski Space ” Princeton 1992.
8. J. Leray, “Hyperbolic Differential Equations” I. A. S. Princeton 1952.
9. J. W. York, “Kinematics and Dynamics of General Relativity” in Sources of Gravitational Radiation, L. Smarr ed., Cambridge 1979.
10. T. Hughes, T. Kato and J. Marsden, Arch. Rat. Mech. Anal. 63 p.273-274, 1976.
11. A. Majda, “Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables”, Springer 1984.
12. Y. Choquet-Bruhat and D. Christodoulou, Acta Mathematica 146 p.129-150, 1981.
13. Y. Choquet-Bruhat and J. W. York, C.R. Acad. Sci. Paris 321 , p. 1089-1095, 1995.
14. A. Abrahams, A. Anderson, Y. Choquet-Bruhat and J. W. York, Phys. Rev. Letters 75, p.3377-3381, 1995.

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