(\(\zeta, \delta(\mu)\))-closed sets in strong generalized topological spaces

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Abstract: This paper deals with the concepts of \(\zeta\delta(\mu)\)-sets and \((\zeta, \delta(\mu))\)-closed sets in a strong generalized topological space and investigate properties of several low separation axioms of strong generalized topologies constructed by the families of these sets. Some properties of \((\zeta, \delta(\mu))\)-\(R_0\) and \((\zeta, \delta(\mu))\)-\(R_1\) strong generalized topological spaces will be given. Finally, several characterizations of weakly \((\zeta, \delta(\mu))\)-continuous functions are discussed.

Subjects: Analysis - Mathematics; Pure Mathematics; Foundations & Theorems

Keywords: \((\zeta, \delta(\mu))\)-closed set; \((\zeta, \delta(\mu))\)-open set; \((\zeta, \delta(\mu))\)-\(R_0\) space; \((\zeta, \delta(\mu))\)-\(R_1\) space; weakly \((\zeta, \delta(\mu))\)-continuous function

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1. Introduction

General topology is important in many fields of applied sciences as well as branches of mathematics. The theory of generalized topology, which was founded by Császár (Császár, 1997), is one of the most important development of general topology in recent years. Especially, the author defined some basic operators on generalized topological spaces. Noiri and Roy (Noiri & Roy, 2011) introduced a new kind of sets called generalized \(\mu\)-closed sets in a topological space by using the concept of generalized open sets introduced by Császár. In 2007, Noiri (Noiri, 2007) introduced a new set called \(mg\)-closed which is defined on a family of sets satisfying some minimal conditions and obtained several basic properties of \(mg\)-closed sets. Moreover, the present author (Noiri, 2008) introduced and studied the notion of \(mg\)-closed sets defined in a set with two minimal structures. Ekici (Ekici, 2012) introduced the notion of generalized hyperconnected spaces and investigated various characterizations of generalized hyperconnected spaces and preservation theorem. In (Ekici, 2011), the present author introduced and studied the concept of generalized submaximal spaces. Ekici and Roy (Ekici & Roy, 2011) introduced new types of sets called \(\land\)-sets and \(\lor\)-sets and investigated some of their fundamental properties. Roy and Ekici (Roy & Ekici, 2011) introduced and studied \(\land\)-\(\mu\)-open sets and \(\lor\)-\(\mu\)-closed sets via \(\mu\)-open and \(\mu\)-closed sets in generalized topological spaces. Shanin (Shanin, 1943) introduced the notion of \(R_0\) topological spaces. Davis (Davis, 1961) introduced the notion of a separation axiom called \(R_1\). These...
notions are further investigated by Naimpally (Naimpally, 1967), Mur德shwar and Naimpally (Mur德shwar & Naimpally, 1966), Dube (Due, 1982) and Dorsett (Dorsett, 1978b). As natural generalizations of the separation axioms $R_0$ and $R_1$, the concepts of semi-$R_0$ and semi-$R_1$ were introduced and studied by Maheshwari and Prasad (Maheshwari & Prasad, 1975) and Dorsett (Dorsett, 1978a). Caldas et al. (Caldas, Jafari, & Noiri, 2004) introduced and studied two new weak separation axioms called $\Lambda_\mu - R_0$ and $\Lambda_\mu - R_1$ by the concepts of $(\Lambda, \theta)$-closure operators and $(\Lambda, \theta)$-open sets. Cammaroto and Noiri (Cammaroto & Noiri, 2005) have defined a weak separation axiom $m-R_0$ in $m$-spaces which are equivalent to generalized topological spaces due to Lugojan (Lugojan, 1982). Recently, Noiri (Noiri, 2006) introduced the notion of $m-R_1$ spaces and investigated several characterizations of $m-R_0$ and $m-R_1$ spaces. Roy (Roy, 2010) introduced the concepts of generalized $R_0$ and $R_1$ topological spaces by using closure operators defined on a generalized topological space and investigated some properties of generalized $R_0$ and $R_1$ topological spaces.

Continuity is basic concept for the study in topological spaces. The concept of weak continuity due to Levine (Levine, 1963) is one of the most important weak forms of continuity in topological spaces. Rose (Rose, 1984) has introduced the notion of subweakly continuous functions and investigated the relationships between subweak continuity and weak continuity. Popa and Stan (Popa & Stan, 1973) introduced and studied the notion of weakly quasi-continuous functions. Weak quasi-continuity is implied by both quasi-continuity and weak continuity which are independent of each other. Janković (Janković, 1985) introduced the concept of almost weakly continuous functions. It is shown in (Popa & Noiri, 1992) that almost weak continuity is equivalent to quasi precontinuity due to Paul and Bhattacharyya (Paul & Bhattacharyya, 1992). Noiri (Noiri, 1987) introduced the notion of weakly $\alpha$-continuous functions. Several characterizations of weakly $\alpha$-continuous functions are studied in (Noiri, 1987), (Rose, 1990) and (Sen & Bhattacharyya, 1993). In (Popa & Noiri, 1994), the present authors introduced and studied weakly $\beta$-continuous functions. Ekici et al. (Ekici, Jafari, Caldas, & Noiri, 2008) established a new class of functions called weakly $\lambda$-continuous functions which is weaker than $\lambda$-continuous functions and investigated some fundamental properties of weakly $\lambda$-continuous functions. Popa and Noiri (Popa & Noiri, 2002a) introduced the notion of weakly $(\tau_m)$-continuous functions as functions from a topological space into a set satisfying some minimal conditions and investigated several characterizations of such functions. Moreover, the present authors (Popa & Noiri, 2002b) introduced the concept of weakly $M$-continuous functions as functions from a set satisfying some minimal conditions into a set satisfying some minimal conditions and investigated some characterizations of weakly $M$-continuous functions. Min (Min, 2009) introduced the notions of weakly $(\mu, \mu')$-continuous functions and weakly $(\psi, \psi')$-continuous functions on generalized topological spaces and generalized neighbourhood systems, respectively, and investigated several characterizations for such functions and the relationships between weak $(\mu, \mu')$-continuity and weak $(\psi, \psi')$-continuity.

In this paper, we define $\zeta(\delta(\mu))$-sets, $(\zeta, \delta(\mu))$-closed sets in a strong generalized topological $(X, \mu)$ and introduce the concepts of the $(\zeta, \delta(\mu))$-closure and $(\zeta, \delta(\mu))$-open sets by utilizing $\delta(\mu)$-open sets and $\delta(\mu)$-closure operators. In Section 3, we obtain fundamental properties of $(\zeta, \delta(\mu))$-closed sets. In Section 4, we investigate properties of several low separation axioms of strong generalized topologies constructed by the concepts of $(\zeta, \delta(\mu))$-closure operators and $(\zeta, \delta(\mu))$-open sets. In the last section, we present the notion of weakly $(\zeta, \delta(\mu))$-continuous functions and investigate some characterizations of such functions.

2. Preliminaries

Let $X$ be a non-empty set and $P(X)$ the power set of $X$. We call a class $\mu \subseteq P(X)$ a generalized topology (briefly, GT) on $X$ if $\emptyset \in \mu$ and an arbitrary union of elements of $\mu$ belongs to $\mu$ (Császár, 2002). A set $X$ with a generalized topology $\mu$ on it is said to be a generalized topological space (briefly, GTS) and is denoted by $(X, \mu)$. For a generalized topological space $(X, \mu)$, the elements of $\mu$ are called $\mu$-open sets and the complements of $\mu$-open sets are called $\mu$-closed sets. Let $\mu$ be a generalized topology on $X$. Observe that $X \in \mu$ must not hold; if all the same $X \in \mu$, then we say that the generalized topology $\mu$ is strong (Császár, 2004). In general, let $M_{\mu}$ denote the union of all elements of $\mu$; of course, $M_{\mu} \in \mu$ and $M_{\mu} = X$ if $\mu$ is a
strong generalized topology. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all $\mu$-closed sets containing $A$ and by $i_\mu(A)$ the union of all $\mu$-open sets contained in $A$ (Császár, 2005). Moreover, $i_\mu(X - A) = X - c_\mu(A)$. According to (Császár, 2008), for $A \subseteq X$ and $x \in X$, we have $x \in c_\mu(A)$ if $x \in M \in \mu$ implies $M \cap A \neq \emptyset$. Consider a generalized topology $\mu$ on $X$. Let us define $\delta(\mu) = \delta \subseteq \mathcal{P}(X)$ by $A \in \delta(\mu)$ iff $A \subseteq X$ and, if $x \in A$, then there is a $\mu$-closed set $Q$ such that $x \in i_\mu(Q) \subseteq A$ (Császár, 2008).

**Proposition 2.1.** (Császár, 2008) Let $(X, \mu)$ be a generalized topological space. Then $\delta(\mu)$ is a generalized topology on $X$.

A subset $A$ of a generalized topological space $(X, \mu)$ is said to be $\mu$-open (Császár, 2008) (resp. $\mu$-closed) if $A = i_\mu(c_\mu(A))$ (resp. $A = c_\mu(i_\mu(A))$).

**Theorem 2.2.** (Császár, 2008) Let $(X, \mu)$ be a generalized topological space. Then, the elements of $\delta(\mu)$ coincide with the union of all $\mu$-open sets.

**Theorem 2.3.** (Császár, 2008) Let $(X, \mu)$ be a generalized topological space and $x \in X$. Then $x \in c_{\delta(\mu)}(A)$ if and only if $R \cap \emptyset \subseteq A$ for every $\mu$-open set $R$ containing $x$.

**Proposition 2.4.** (Császár, 2008) Let $A$ be a subset of a generalized topological space $(X, \mu)$. Then $A$ is $\delta(\mu)$-closed if and only if $A = c_{\delta(\mu)}(A)$.

A subset $A$ of a generalized topological space $(X, \mu)$ is called $\delta(\mu)$-open if the complement of $A$ is $\delta(\mu)$-closed. The family of all $\delta(\mu)$-closed sets in a generalized topological space $(X, \mu)$ is denoted by $\delta(\mu)\mathcal{C}$.

**Theorem 2.5.** (Min, 2010) For a subset $A$ of a generalized topological space $(X, \mu)$, the following properties hold:

1. $A \subseteq c_{\delta(\mu)}(A)$.
2. $c_\mu(V) = c_{\delta(\mu)}(V)$ for every $\mu$-open set $V$.
3. $c_{\delta(\mu)}[c_{\delta(\mu)}(A)] = c_{\delta(\mu)}(A)$.
4. $c_{\delta(\mu)}(A)$ is $\mu$-closed.

**Theorem 2.6.** (Min, 2010) For a subset $A$ of a generalized topological space $(X, \mu)$, the following properties hold:

1. $i_{\delta(\mu)}(A) = A$ iff $A$ is $\delta(\mu)$-open.
2. $i_{\delta(\mu)}(A)$ is $\mu$-open.
3. $i_{\delta(\mu)}[i_{\delta(\mu)}(A)] = i_{\delta(\mu)}(A)$.
4. $i_{\delta(\mu)}(i_{\delta(\mu)}(A)) = i_{\delta(\mu)}(A)$.

**Theorem 2.7.** (Min, 2010) For a subset $A$ of a generalized topological space $(X, \mu)$, the following properties hold:

1. $x \in i_{\delta(\mu)}(A)$ iff there exists a $\mu$-open set $R$ containing $x$ such that $R \subseteq A$.
2. $i_{\delta(\mu)}(A) = X - c_{\delta(\mu)}(X - A)$.
(3) \( \zeta_{\delta(\mu)}(A) = X - i_{\delta(\mu)}(X - A) \).

3. \( (\zeta, \delta(\mu)) \)-closed sets

In this section, we introduce the notion of \( (\zeta, \delta(\mu)) \)-closed sets. Moreover, several interesting properties of \( (\zeta, \delta(\mu)) \)-closed sets are investigated.

Definition 3.1. Let \( A \) be a subset of a strong generalized topological space \( (X, \mu) \). A subset \( \zeta_{\delta(\mu)}(A) \) is defined as follows: \( \zeta_{\delta(\mu)}(A) = \cap \{ U \in \delta(\mu) | A \subseteq U \} \).

Lemma 3.2. For subsets \( A, B \) and \( C_\alpha (\alpha \in \mathbb{V}) \) of a strong generalized topological space \( (X, \mu) \), the following properties hold:

(1) \( A \subseteq \zeta_{\delta(\mu)}(A) \).

(2) \( \zeta_{\delta(\mu)}(\zeta_{\delta(\mu)}(A)) = \zeta_{\delta(\mu)}(A) \).

(3) If \( A \subseteq B \), then \( \zeta_{\delta(\mu)}(A) \subseteq \zeta_{\delta(\mu)}(B) \).

(4) \( \zeta_{\delta(\mu)}(\cap \{ C_\alpha | \alpha \in \mathbb{V} \}) \subseteq \cap \{ \zeta_{\delta(\mu)}(C_\alpha) | \alpha \in \mathbb{V} \} \).

(5) \( \zeta_{\delta(\mu)}(\cup \{ C_\alpha | \alpha \in \mathbb{V} \}) = \cup \{ \zeta_{\delta(\mu)}(C_\alpha) | \alpha \in \mathbb{V} \} \).

Proof. (1) This is obvious from the definition.

(2) By (1), we have \( \zeta_{\delta(\mu)}(\zeta_{\delta(\mu)}(A)) \supseteq \zeta_{\delta(\mu)}(A) \). Suppose that \( x \notin \zeta_{\delta(\mu)}(A) \). Then there exists \( U \in \delta(\mu) \) such that \( A \subseteq U \) and \( x \notin U \). Since \( A \subseteq \zeta_{\delta(\mu)}(A) \subseteq U \), we have \( x \notin \zeta_{\delta(\mu)}(\zeta_{\delta(\mu)}(A)) \) and hence \( \zeta_{\delta(\mu)}(\zeta_{\delta(\mu)}(A)) \subseteq \zeta_{\delta(\mu)}(A) \).

(3) Suppose that \( x \notin \zeta_{\delta(\mu)}(B) \). Then there exists \( U \in \delta(\mu) \) such that \( B \subseteq U \) and \( x \notin U \). Since \( A \subseteq B \), we have \( x \notin \zeta_{\delta(\mu)}(A) \) and hence \( \zeta_{\delta(\mu)}(A) \subseteq \zeta_{\delta(\mu)}(B) \).

(4) Suppose that \( x \notin \cap \{ \zeta_{\delta(\mu)}(C_\alpha) | \alpha \in \mathbb{V} \} \). There exists \( a_0 \in \mathbb{V} \) such that \( x \notin \zeta_{\delta(\mu)}(C_{a_0}) \) and there exists a \( \delta(\mu) \)-open set \( U \) such that \( x \notin U \) and \( C_{a_0} \subseteq U \). Since \( \cap_{a \in \mathbb{V}} C_a \subseteq C_{a_0} \), we have \( x \notin \zeta_{\delta(\mu)}(\cap \{ C_\alpha | \alpha \in \mathbb{V} \}) \) and hence, \( \zeta_{\delta(\mu)}(\cap \{ C_\alpha | \alpha \in \mathbb{V} \}) \subseteq \cap \{ \zeta_{\delta(\mu)}(C_\alpha) | \alpha \in \mathbb{V} \} \).

(5) Since \( C_a \subseteq \cup_{a \in \mathbb{V}} C_a \), by (3) we have \( \zeta_{\delta(\mu)}(C_a) \subseteq \zeta_{\delta(\mu)}(\cup_{a \in \mathbb{V}} C_a) \) and \( \cup_{a \in \mathbb{V}} \zeta_{\delta(\mu)}(C_a) \subseteq \zeta_{\delta(\mu)}(\cup_{a \in \mathbb{V}} C_a) \). On the other hand, suppose that \( x \notin \cup_{a \in \mathbb{V}} \zeta_{\delta(\mu)}(C_a) \).

Then \( x \notin \zeta_{\delta(\mu)}(C_a) \) for each \( a \in \mathbb{V} \) and hence there exists \( U_a \in \delta(\mu) \) such that \( C_a \subseteq U_a \) and \( x \notin U_a \) for each \( a \in \mathbb{V} \). Therefore, we have \( \cup_{a \in \mathbb{V}} C_a \subseteq \cup_{a \in \mathbb{V}} U_a \) and \( \cup_{a \in \mathbb{V}} U_a \) is a \( \delta(\mu) \)-open set not containing \( x \). Thus, \( x \notin \zeta_{\delta(\mu)}(\cup_{a \in \mathbb{V}} C_a) \). This implies that \( \cup_{a \in \mathbb{V}} \zeta_{\delta(\mu)}(C_a) \supseteq \zeta_{\delta(\mu)}(\cup_{a \in \mathbb{V}} C_a) \). Consequently, we obtain \( \zeta_{\delta(\mu)}(\{ C_\alpha | \alpha \in \mathbb{V} \}) \subseteq \cup \{ \zeta_{\delta(\mu)}(C_\alpha) | \alpha \in \mathbb{V} \} \).

Definition 3.3. A subset \( A \) of a strong generalized topological space \( (X, \mu) \) is called a \( \zeta_{\delta(\mu)} \)-set if \( A = \zeta_{\delta(\mu)}(A) \). The family of all \( \zeta_{\delta(\mu)} \)-sets of \( (X, \mu) \) is denoted by \( \zeta_{\delta(\mu)}(X, \mu) \).

Lemma 3.4. For subsets \( A \) and \( B_\alpha (\alpha \in \mathbb{V}) \) of a strong generalized topological space \( (X, \mu) \), the following properties hold:

(1) \( \zeta_{\delta(\mu)}(A) \) is a \( \zeta_{\delta(\mu)} \)-set.

(2) If \( A \) is \( \delta(\mu) \)-open, then \( A \) is a \( \zeta_{\delta(\mu)} \)-set.
(3) If \( B_\alpha \) is a \( \zeta_{(\mu)} \)-set for each \( \alpha \in V \), then \( \cap_{\alpha \in V} B_\alpha \) is a \( \zeta_{(\mu)} \)-set.

(4) If \( B_\alpha \) is a \( \zeta_{(\mu)} \)-set for each \( \alpha \in V \), then \( \cup_{\alpha \in V} B_\alpha \) is a \( \zeta_{(\mu)} \)-set.

Proof. (1) and (2) are obvious.

(3) Let \( B_\alpha \in \zeta_{(\mu)}(X, \mu) \) for each \( \alpha \in V \). By Lemma 3.2(4), we have
\[
\cap_{\alpha \in V} B_\alpha = \cap_{\alpha \in V} \zeta_{(\mu)}(B_\alpha) \supseteq \zeta_{(\mu)}(\cap_{\alpha \in V} B_\alpha) \supseteq \cap_{\alpha \in V} B_\alpha.
\]
Thus, \( \cap_{\alpha \in V} B_\alpha \in \zeta_{(\mu)}(X, \mu) \) and hence, \( \cap_{\alpha \in V} B_\alpha \in \zeta_{(\mu)}(X, \mu) \).

(4) Let \( B_\alpha \in \zeta_{(\mu)}(X, \mu) \) for each \( \alpha \in V \). By Lemma 3.2(5), we have
\[
\cup_{\alpha \in V} B_\alpha = \cup_{\alpha \in V} \zeta_{(\mu)}(B_\alpha) = \zeta_{(\mu)}(\cup_{\alpha \in V} B_\alpha) \supseteq \cup_{\alpha \in V} B_\alpha.
\]
Therefore, we obtain \( \cup_{\alpha \in V} B_\alpha = \zeta_{(\mu)}(\cup_{\alpha \in V} B_\alpha) \) and so \( \cup_{\alpha \in V} B_\alpha \in \zeta_{(\mu)}(X, \mu) \).

Definition 3.5. Let \( A \) be a subset of a strong generalized topological space \( (X, \mu) \). A subset \( \zeta^{(\mu)}(A) \) is defined as follows: \( \zeta^{(\mu)}(A) = \cup \{ F \in \delta(\mu) | C(F) \subseteq A \} \).

Definition 3.6. A subset \( A \) of a strong generalized topological space \( (X, \mu) \) is called a \( \zeta^{(\mu)} \)-set if \( A = \zeta^{(\mu)}(A) \). The family of all \( \zeta^{(\mu)} \)-sets in a strong generalized topological space \( (X, \mu) \) is denoted by \( \zeta^{(\mu)}(X, \mu) \).

Lemma 3.7. For subsets \( A, B \) and \( C_\alpha(\alpha \in V) \) of a strong generalized topological space \( (X, \mu) \), the following properties hold:

(1) \( \zeta^{(\mu)}(A) \subseteq A \).

(2) If \( A \subseteq B \), then \( \zeta^{(\mu)}(A) \subseteq \zeta^{(\mu)}(B) \).

(3) If \( A \) is \( \delta(\mu) \)-closed, then \( \zeta^{(\mu)}(A) = A \).

(4) \( \zeta^{(\mu)}(\cap \{ C_\alpha(\alpha \in V) \}) = \cap \{ \zeta^{(\mu)}(C_\alpha(\alpha \in V)) \} \).

(5) \( \cup \{ \zeta^{(\mu)}(C_\alpha(\alpha \in V)) \} \subseteq \zeta^{(\mu)}(\cup \{ C_\alpha(\alpha \in V) \}) \).

(6) \( \zeta_{(\mu)}(X - A) = X - \zeta^{(\mu)}(A) \) and \( \zeta^{(\mu)}(X - A) = X - \zeta_{(\mu)}(A) \).

Lemma 3.8. For subsets \( A \) and \( B_\alpha(\alpha \in V) \) of a strong generalized topological space \( (X, \mu) \), the following properties hold:

(1) \( \zeta^{(\mu)}(A) \) is a \( \zeta^{(\mu)} \)-set.

(2) If \( A \) is \( \delta(\mu) \)-closed, then \( A \) is \( \zeta^{(\mu)} \)-closed.

(3) If \( B_\alpha \) is a \( \zeta^{(\mu)} \)-set for each \( \alpha \in V \), then \( \cap_{\alpha \in V} B_\alpha \) is a \( \zeta^{(\mu)} \)-set.

(4) If \( B_\alpha \) is a \( \zeta^{(\mu)} \)-set for each \( \alpha \in V \), then \( \cup_{\alpha \in V} B_\alpha \) is a \( \zeta^{(\mu)} \)-set.

Definition 3.9. A subset \( A \) of a strong generalized topological space \( (X, \mu) \) is called \( (\zeta, \delta(\mu)) \)-closed if \( A = T \cap F \), where \( T \) is a \( \zeta_{(\mu)} \)-set and \( F \) is a \( \delta(\mu) \)-closed set. The family of all \( (\zeta, \delta(\mu)) \)-closed sets in a strong generalized topological space \( (X, \mu) \) is denoted by \((\zeta, \delta(\mu))C\).
Theorem 3.10. For a subset $A$ of a strong generalized topological space $(X, \mu)$, the following properties are equivalent:

1. $A$ is $(\zeta, \delta(\mu))$-closed.

2. $A = T \cap c_{\delta(\mu)}(A)$, where $T$ is a $\zeta_{\delta(\mu)}$-set.

3. $A = \zeta_{\delta(\mu)}(A) \cap c_{\delta(\mu)}(A)$.

Proof. (1) $\Rightarrow$ (2): Let $A = T \cap F$, where $T$ is a $\zeta_{\delta(\mu)}$-set and $F$ is a $\delta(\mu)$-closed set. Since $A \subseteq F$, we have $c_{\delta(\mu)}(A) \subseteq F$ and $A = T \cap F \supseteq T \cap c_{\delta(\mu)}(A) \supseteq A$. Therefore, we obtain $A = T \cap c_{\delta(\mu)}(A)$.

(2) $\Rightarrow$ (3): Let $A = T \cap c_{\delta(\mu)}(A)$, where $T$ is a $\zeta_{\delta(\mu)}$-set. Since $A \subseteq T$, we have $\zeta_{\delta(\mu)}(A) \subseteq \zeta_{\delta(\mu)}(T) = T$ and hence, $A \subseteq \zeta_{\delta(\mu)}(A) \cap c_{\delta(\mu)}(A) \subseteq T \cap c_{\delta(\mu)}(A) = A$. Consequently, we obtain $A = \zeta_{\delta(\mu)}(A) \cap c_{\delta(\mu)}(A)$.

(3) $\Rightarrow$ (1): Since $\zeta_{\delta(\mu)}(A)$ is a $\zeta_{\delta(\mu)}$-set, $c_{\delta(\mu)}(A)$ is $\delta(\mu)$-closed and $A = \zeta_{\delta(\mu)}(A) \cap c_{\delta(\mu)}(A)$.

Lemma 3.11. Every $\zeta_{\delta(\mu)}$-closed set is $(\zeta, \delta(\mu))$-closed.

Definition 3.12. A subset $A$ of a strong generalized topological space $(X, \mu)$ is called $(\zeta, \delta(\mu))$-open if the complement of $A$ is $(\zeta, \delta(\mu))$-closed. The family of all $(\zeta, \delta(\mu))$-open sets in a strong generalized topological space $(X, \mu)$ is denoted by $(\zeta, \delta(\mu))\text{O}$.

Proposition 3.13. Let $A_{\alpha}(\alpha \in \mathbb{V})$ be a subset of a strong generalized topological space $(X, \mu)$.

1. If $A_{\alpha}$ is $(\zeta, \delta(\mu))$-closed for each $\alpha \in \mathbb{V}$, then $\cap \{A_{\alpha}| \alpha \in \mathbb{V}\}$ is $(\zeta, \delta(\mu))$-closed.

2. If $A_{\alpha}$ is $(\zeta, \delta(\mu))$-open for each $\alpha \in \mathbb{V}$, then $\cup \{A_{\alpha}| \alpha \in \mathbb{V}\}$ is $(\zeta, \delta(\mu))$-open.

Proof. (1) Suppose that $A_{\alpha}$ is $(\zeta, \delta(\mu))$-closed for each $\alpha \in \mathbb{V}$. Then, for each $\alpha$, there exist a $\zeta_{\delta(\mu)}$-set $T_{\alpha}$ and a $\delta(\mu)$-closed set $F_{\alpha}$ such that $A_{\alpha} = T_{\alpha} \cap F_{\alpha}$. Then, we have $\cap_{\alpha \in \mathbb{V}} A_{\alpha} = \cap_{\alpha \in \mathbb{V}} (T_{\alpha} \cap F_{\alpha}) = (\cap_{\alpha \in \mathbb{V}} T_{\alpha}) \cap (\cap_{\alpha \in \mathbb{V}} F_{\alpha})$. By Lemma 3.4, $\cap_{\alpha \in \mathbb{V}} T_{\alpha}$ is a $\delta(\mu)$-set and $\cap_{\alpha \in \mathbb{V}} F_{\alpha}$ is a $\delta(\mu)$-closed set. This shows that $\cap_{\alpha \in \mathbb{V}} A_{\alpha}$ is $(\zeta, \delta(\mu))$-closed.

(2) Let $A_{\alpha}$ is $(\zeta, \delta(\mu))$-open for each $\alpha \in \mathbb{V}$. Then $X - A_{\alpha}$ is $(\zeta, \delta(\mu))$-closed for each $\alpha \in \mathbb{V}$. By (1), we have $X - \cup_{\alpha \in \mathbb{V}} A_{\alpha} = \cap_{\alpha \in \mathbb{V}} (X - A_{\alpha})$ is $(\zeta, \delta(\mu))$-closed. Therefore, $\cup_{\alpha \in \mathbb{V}} A_{\alpha}$ is $(\zeta, \delta(\mu))$-open.

Theorem 3.14. For a subset $A$ of a strong generalized topological space $(X, \mu)$, the followings are equivalent:

1. $A$ is $(\zeta, \delta(\mu))$-open.

2. $A = T \cup U$, where $T$ is a $\zeta_{\delta(\mu)}$-set and $U$ is a $\delta(\mu)$-open set.

3. $A = T \cup \zeta_{\delta(\mu)}(A)$, where $T$ is a $\zeta_{\delta(\mu)}$-set.

4. $A = \zeta_{\delta(\mu)}(A) \cup \zeta_{\delta(\mu)}(A)$.

Proof. (1) $\Rightarrow$ (2): Suppose that $A$ is a $(\zeta, \delta(\mu))$-open set. Then $X - A$ is $(\zeta, \delta(\mu))$-closed and $X - A = C \cap F$, where $C$ is a $\zeta_{\delta(\mu)}$-set and $F$ is a $\delta(\mu)$-closed set. Hence, we have $A = (X - C) \cup (X - F)$, where $X - C$ is a $\zeta_{\delta(\mu)}$-set and $X - F$ is a $\delta(\mu)$-open set.
(2) ⇒ (3): Let \( A = T \cup U \), where \( T \) is a \( \zeta^{\delta(\mu)} \)-set and \( U \) is a \( \delta(\mu) \)-open set. Since \( U \subseteq A \) and \( U \) is \( \delta(\mu) \)-open, \( U \subseteq i_{\delta(\mu)}(A) \) and hence, \( A = T \cup U \subseteq T \cup i_{\delta(\mu)}(A) \subseteq A \). Therefore, we obtain \( A = T \cup i_{\delta(\mu)}(A) \).

(3) ⇒ (4): Let \( A = T \cup i_{\delta(\mu)}(A) \), where \( T \) is a \( \zeta^{\delta(\mu)} \)-set. Since \( T \subseteq A \), we have \( \zeta^{\delta(\mu)}(A) \supseteq \zeta^{\delta(\mu)}(T) \) and hence, \( A \supseteq \zeta^{\delta(\mu)}(A) \cup i_{\delta(\mu)}(A) \supseteq \zeta^{\delta(\mu)}(T) \cup i_{\delta(\mu)}(A) = T \cup i_{\delta(\mu)}(A) = A \). Consequently, we obtain \( A = \zeta^{\delta(\mu)}(A) \cup i_{\delta(\mu)}(A) \).

(4) ⇒ (1): Let \( A = \zeta^{\delta(\mu)}(A) \cup i_{\delta(\mu)}(A) \). Then, we have

\[
X - A = (X - \zeta^{\delta(\mu)}(A)) \cap (X - i_{\delta(\mu)}(A)) = \zeta_{\delta(\mu)}(X - A) \cap c_{\delta(\mu)}(X - A).
\]

By Lemma 3.4, \( \zeta_{\delta(\mu)}(X - A) \) is a \( \zeta_{\delta(\mu)} \)-set and \( c_{\delta(\mu)}(X - A) \) is a \( \delta(\mu) \)-closed set. Therefore, \( X - A \) is a \( (\zeta, \delta(\mu)) \)-closed set and so \( A \) is \( (\zeta, \delta(\mu)) \)-open.

**Definition 3.15.** Let \( A \) be a subset of a strong generalized topological space \( (X, \mu) \). A point \( x \in X \) is called a \( (\zeta, \delta(\mu)) \)-cluster point of \( A \) if for every \( (\zeta, \delta(\mu)) \)-open set \( U \) of \( (X, \mu) \) containing \( x \), we have \( A \cap U \neq \emptyset \). The set of all \( (\zeta, \delta(\mu)) \)-cluster points is called the \( (\zeta, \delta(\mu)) \)-closure of \( A \) and is denoted by \( c_{(\zeta, \delta(\mu))}(A) \).

**Lemma 3.16.** Let \( A \) and \( B \) be subsets of a strong generalized topological space \( (X, \mu) \). For the \( (\zeta, \delta(\mu)) \)-closure, the following properties hold:

1. \( A \subseteq c_{(\zeta, \delta(\mu))}(A) \) and \( c_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(A)) = c_{(\zeta, \delta(\mu))}(A) \).
2. \( c_{(\zeta, \delta(\mu))}(A) = \cap \{ F | A \subseteq F \text{ and } F \text{ is } (\zeta, \delta(\mu)) \text{-closed} \} \).
3. If \( A \subseteq B \), then \( c_{(\zeta, \delta(\mu))}(A) \subseteq c_{(\zeta, \delta(\mu))}(B) \).
4. \( A \) is \( (\zeta, \delta(\mu)) \)-closed if and only if \( A = c_{(\zeta, \delta(\mu))}(A) \).
5. \( c_{(\zeta, \delta(\mu))}(A) \) is \( (\zeta, \delta(\mu)) \)-closed.

**Lemma 3.17.** For a subset \( A \) of a strong generalized topological space \( (X, \mu) \), the following properties hold:

1. If \( A \) is \( \delta(\mu) \)-closed, then \( A \) is \( (\zeta, \delta(\mu)) \)-closed.
2. If \( A \) is \( (\zeta, \delta(\mu)) \)-closed, then \( A = \zeta_{\delta(\mu)}(A) \cap c_{(\zeta, \delta(\mu))}(A) \).

**Proof.** (1) It is sufficient to observe that \( A = X \cap A \), where the whole set \( X \) is a \( \zeta_{\delta(\mu)} \)-set.

(2) Let \( A \) be \( (\zeta, \delta(\mu)) \)-closed, then there exists a \( \zeta_{\delta(\mu)} \)-set \( T \) and a \( \delta(\mu) \)-closed set \( C \) such that \( A = T \cap C \). Since \( A \subseteq T \) and \( A \subseteq C \), we have \( A \subseteq \zeta_{\delta(\mu)}(A) \subseteq \zeta_{\delta(\mu)}(T) = T \) and \( A \subseteq c_{(\zeta, \delta(\mu))}(A) \subseteq c_{(\zeta, \delta(\mu))}(C) = C \). This implies that \( A \subseteq \zeta_{\delta(\mu)}(A) \cap c_{(\zeta, \delta(\mu))}(A) \subseteq T \cap C = A \).

Consequently, we obtain \( A = \zeta_{\delta(\mu)}(A) \cap c_{(\zeta, \delta(\mu))}(A) \).

**Definition 3.18.** Let \( A \) be a subset of a strong generalized topological space \( (X, \mu) \). The union of all \( (\zeta, \delta(\mu)) \)-open sets contained in \( A \) is called the \( (\zeta, \delta(\mu)) \)-interior of \( A \) and is denoted by \( i_{(\zeta, \delta(\mu))}(A) \).

**Lemma 3.19.** Let \( A \) and \( B \) be subsets of a strong generalized topological space \( (X, \mu) \). For the \( (\zeta, \delta(\mu)) \)-interior, the following properties hold:
(1) $i_{(\zeta,\delta)(\mu)}(A) \subseteq A$ and $i_{(\zeta,\delta)(\mu)}(i_{(\zeta,\delta)(\mu)}(A)) = i_{(\zeta,\delta)(\mu)}(A)$.

(2) If $A \subseteq B$, then $i_{(\zeta,\delta)(\mu)}(A) \subseteq i_{(\zeta,\delta)(\mu)}(B)$.

(3) $i_{(\zeta,\delta)(\mu)}(A) = \cup\{G|G \subseteq A \text{ and } G \text{ is } (\zeta,\delta(\mu))-\text{open}\}$.

(4) $i_{(\zeta,\delta)(\mu)}(A)$ is $(\zeta,\delta(\mu))-\text{open}$.

(5) $A$ is $(\zeta,\delta(\mu))-\text{open}$ if and only if $i_{(\zeta,\delta)(\mu)}(A) = A$.

(6) $c_{(\zeta,\delta)(\mu)}(X - A) = X - i_{(\zeta,\delta)(\mu)}(A)$.

**Definition 3.20.** A subset $A$ of a strong generalized topological space $(X,\mu)$ is said to be generalized $(\zeta,\delta(\mu))-\text{closed}$ (briefly $g-(\zeta,\delta(\mu))-\text{closed}$) set if $c_{(\zeta,\delta)(\mu)}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in (\zeta,\delta(\mu))O$.

**Definition 3.21.** A strong generalized topological space $(X,\mu)$ is said to be $(\zeta,\delta(\mu))-\text{symmetric}$ if for any $x$ and $y$ in $X$, $x \in c_{(\zeta,\delta)(\mu)}(\{y\})$ implies $y \in c_{(\zeta,\delta)(\mu)}(\{x\})$.

**Theorem 3.22.** A strong generalized topological space $(X,\mu)$ is $(\zeta,\delta(\mu))-\text{symmetric}$ if and only if $\{x\}$ is $g-(\zeta,\delta(\mu))-\text{closed}$ for each $x \in X$.

**Proof.** Assume that $x \in c_{(\zeta,\delta)(\mu)}(\{y\})$, but $y \not\in c_{(\zeta,\delta)(\mu)}(\{x\})$. This implies that the complement of $c_{(\zeta,\delta)(\mu)}(\{x\})$ contains $y$. Therefore, the set $\{y\}$ is a subset of the complement of $c_{(\zeta,\delta)(\mu)}(\{x\})$. This implies that $c_{(\zeta,\delta)(\mu)}(\{y\})$ is a subset of the complement of $c_{(\zeta,\delta)(\mu)}(\{x\})$. Now, the complement of $c_{(\zeta,\delta)(\mu)}(\{x\})$ contains $x$ which is a contradiction.

Conversely, suppose that $\{x\} \subseteq V \in (\zeta,\delta(\mu))O$, but $c_{(\zeta,\delta)(\mu)}(\{x\})$ is not a subset of $V$. This means that $c_{(\zeta,\delta)(\mu)}(\{x\})$ and the complement of $V$ are not disjoint. Let $y$ belongs to their intersection. Now, we have $x \in c_{(\zeta,\delta)(\mu)}(\{y\})$ which is a subset of the complement of $V$ and $x \notin V$. This is a contradiction.

**Theorem 3.23.** A subset $A$ of a strong generalized topological space $(X,\mu)$ is $g-(\zeta,\delta(\mu))-\text{closed}$ if and only if $c_{(\zeta,\delta)(\mu)}(A) - A$ contains no non-empty $(\zeta,\delta(\mu))-\text{closed}$ set.

**Proof.** Let $F$ be a $(\zeta,\delta(\mu))-\text{closed}$ subset of $c_{(\zeta,\delta)(\mu)}(A) - A$. Now, $A \subseteq X - F$ and since $A$ is $g-(\zeta,\delta(\mu))-\text{closed}$, we have $c_{(\zeta,\delta)(\mu)}(A) \subseteq X - F$ and $F \subseteq X - c_{(\zeta,\delta)(\mu)}(A)$. Thus, $F \subseteq c_{(\zeta,\delta)(\mu)}(A) \cap (X - c_{(\zeta,\delta)(\mu)}(A)) = \emptyset$ and $F$ is empty.

Conversely, suppose that $A \subseteq U$ and $U$ is $(\zeta,\delta(\mu))-\text{open}$. If $c_{(\zeta,\delta)(\mu)}(A) \not\subseteq U$, then $c_{(\zeta,\delta)(\mu)}(A) \cap (X - U)$ is a non-empty $(\zeta,\delta(\mu))-\text{closed}$ subset of $c_{(\zeta,\delta)(\mu)}(A) - A$.

**Corollary 3.24.** Let $A$ be a $g-(\zeta,\delta(\mu))-\text{closed}$ subset of a strong generalized topological space $(X,\mu)$. Then $A$ is $(\zeta,\delta(\mu))-\text{closed}$ if and only if $c_{(\zeta,\delta)(\mu)}(A) - A$ is $(\zeta,\delta(\mu))-\text{closed}$.

**Proof.** If $A$ is $(\zeta,\delta(\mu))-\text{closed}$, then $c_{(\zeta,\delta)(\mu)}(A) - A = \emptyset$.

Conversely, suppose that $c_{(\zeta,\delta)(\mu)}(A) - A$ is $(\zeta,\delta(\mu))-\text{closed}$. But $A$ is $g-(\zeta,\delta(\mu))-\text{closed}$ and $c_{(\zeta,\delta)(\mu)}(A) - A$ is a $(\zeta,\delta(\mu))-\text{closed}$ subset of itself. By Theorem 3.23, $c_{(\zeta,\delta)(\mu)}(A) - A = \emptyset$ and hence $c_{(\zeta,\delta)(\mu)}(A) = A$.

**Proposition 3.25.** For a subset $A$ of a strong generalized topological space $(X,\mu)$, the following properties hold:
(1) If $A$ is $(ζ, δ(μ))$-closed, then $A$ is $g(ζ, δ(μ))$-closed.

(2) If $A$ is $g(ζ, δ(μ))$-closed and $(ζ, δ(μ))$-open, then $A$ is $(ζ, δ(μ))$-closed.

(3) If $A$ is $g(ζ, δ(μ))$-closed and $A \subseteq B \subseteq c_{(ζ, δ(μ))}(A)$, then $B$ is $g(ζ, δ(μ))$-closed.

Proof. (1) Let $A$ be $(ζ, δ(μ))$-closed and $A \subseteq U \in (ζ, δ(μ))O$. Then, by Lemma 3.16 $A = c_{(ζ, δ(μ))}(A) \subseteq U$ and hence, $A$ is $g(ζ, δ(μ))$-closed.

(2) Let $A$ be $g(ζ, δ(μ))$-closed and $(ζ, δ(μ))$-open. Then $c_{(ζ, δ(μ))}(A) \subseteq A$ and so $A$ is $(ζ, δ(μ))$-closed.

(3) Let $B \subseteq U$ and $U \in (ζ, δ(μ))O$. Then $A \subseteq U$ and $A$ is $g(ζ, δ(μ))$-closed, we have $c_{(ζ, δ(μ))}(A) \subseteq U$. By Lemma 3.16, $c_{(ζ, δ(μ))}(A) = c_{(ζ, δ(μ))}(B)$ and hence, $c_{(ζ, δ(μ))}(B) \subseteq U$. Consequently, we obtain $B$ is $g(ζ, δ(μ))$-closed.

Definition 3.26. Let $A$ be a subset of a strong generalized topological space $(X, μ)$. The $(ζ, δ(μ))$-frontier of $A$, $Fr_{(ζ, δ(μ))}(A)$, is defined as follows:

$$Fr_{(ζ, δ(μ))}(A) = c_{(ζ, δ(μ))}(A) \cap c_{(ζ, δ(μ))}(X - A).$$

Proposition 3.27. Let $A$ be a subset of a strong generalized topological space $(X, μ)$. If $A$ is $g(ζ, δ(μ))$-closed and $A \subseteq V \in (ζ, δ(μ))O$, then $Fr_{(ζ, δ(μ))}(V) \subseteq i_{(ζ, δ(μ))}(X - A)$.

Proof. Let $A$ be $g(ζ, δ(μ))$-closed and $A \subseteq V \in (ζ, δ(μ))O$. Then $c_{(ζ, δ(μ))}(A) \subseteq V$. Suppose that $x \in Fr_{(ζ, δ(μ))}(V)$. Since $V \in (ζ, δ(μ))O$, we have $Fr_{(ζ, δ(μ))}(V) = c_{(ζ, δ(μ))}(V) - V$.

Therefore, $x \notin V$ and $x \notin c_{(ζ, δ(μ))}(A)$. This shows that $x \in i_{(ζ, δ(μ))}(X - A)$ and hence, $Fr_{(ζ, δ(μ))}(V) \subseteq i_{(ζ, δ(μ))}(X - A)$.

Proposition 3.28. Let $(X, μ)$ be a strong generalized topological space. For each $x \in X$, either $\{x\}$ is $(ζ, δ(μ))$-closed or $\{x\}$ is $g(ζ, δ(μ))$-open.

Proof. Suppose that $\{x\}$ is not $(ζ, δ(μ))$-closed. Then $X - \{x\}$ is not $(ζ, δ(μ))$-open and the only $(ζ, δ(μ))$-open set containing $X - \{x\}$ is $X$ itself. Therefore, $c_{(ζ, δ(μ))}(X - \{x\}) \subseteq X$ and hence, $X - \{x\}$ is $g(ζ, δ(μ))$-closed. Thus, $\{x\}$ is $g(ζ, δ(μ))$-open.

Theorem 3.29. A subset $A$ of a strong generalized topological space $(X, μ)$ is $g(ζ, δ(μ))$-open if and only if $F \subseteq i_{(ζ, δ(μ))}(A)$ whenever $F \subseteq A$ and $F$ is $(ζ, δ(μ))$-closed.

Proof. Suppose that $A$ is $g(ζ, δ(μ))$-open. Let $F \subseteq A$ and $F$ is $(ζ, δ(μ))$-closed. Then $X - A \subseteq X - F \in (ζ, δ(μ))O$ and $X - A$ is $g(ζ, δ(μ))$-closed. Therefore, we obtain

$$X - i_{(ζ, δ(μ))}(A) = c_{(ζ, δ(μ))}(X - A) \subseteq X - F$$

and so $F \subseteq i_{(ζ, δ(μ))}(A)$.

Conversely, let $X - A \subseteq U$ and $U \in (ζ, δ(μ))O$. Then $X - U \subseteq A$ and $X - U$ is $(ζ, δ(μ))$-closed. By the hypothesis, we have $X - U \subseteq i_{(ζ, δ(μ))}(A)$ and hence, $c_{(ζ, δ(μ))}(X - A) = X - i_{(ζ, δ(μ))}(A) \subseteq U$. Therefore, $X - A$ is $g(ζ, δ(μ))$-closed and $A$ is $g(ζ, δ(μ))$-open.

Corollary 3.30. For a subsets $A, B$ of a strong generalized topological space $(X, μ)$, the following properties hold:

(1) If $A$ is $(ζ, δ(μ))$-open, then $A$ is $g(ζ, δ(μ))$-open.
(2) If $A$ is $g_{(\zeta, \delta(\mu))}$-open and $(\zeta, \delta(\mu))$-closed, then $A$ is $(\zeta, \delta(\mu))$-open.

(3) If $A$ is $g_{(\zeta, \delta(\mu))}$-open and $i_{(\zeta, \delta(\mu))}(A) \subseteq B \subseteq A$, then $B$ is $g_{(\zeta, \delta(\mu))}$-open.

Proof. This follows from Proposition 3.25.

Lemma 3.31. Let $A$ be a subset of a strong generalized topological space $(X, \mu)$ and $G \in (\zeta, \delta(\mu))O$. If $A \cap G = \emptyset$, then $c_{(\zeta, \delta(\mu))}(A) \cap G = \emptyset$.

Theorem 3.32. For a subset $A$ of a strong generalized topological space $(X, \mu)$, the following properties are equivalent:

(1) $A$ is $g_{(\zeta, \delta(\mu))}$-closed.

(2) $c_{(\zeta, \delta(\mu))}(A) - A$ contains no non-empty $(\zeta, \delta(\mu))$-closed set.

(3) $c_{(\zeta, \delta(\mu))}(A) - A$ is $g_{(\zeta, \delta(\mu))}$-open.

Proof. (1) $\Rightarrow$ (2) : Let $F \subseteq c_{(\zeta, \delta(\mu))}(A) - A$ and $F$ be $(\zeta, \delta(\mu))$-closed. By (2), we have $F = \emptyset$. Then $F \subseteq i_{(\zeta, \delta(\mu))}c_{(\zeta, \delta(\mu))}(A) - A$. It follows from Theorem 3.29 that $c_{(\zeta, \delta(\mu))}(A) - A$ is $g_{(\zeta, \delta(\mu))}$-open.

(2) $\Rightarrow$ (3) : Let $F \subseteq c_{(\zeta, \delta(\mu))}(A) - A$ and $F$ be $(\zeta, \delta(\mu))$-closed. Since $c_{(\zeta, \delta(\mu))}(A) - A = U \subseteq i_{(\zeta, \delta(\mu))}c_{(\zeta, \delta(\mu))}(A) - A$, we have $c_{(\zeta, \delta(\mu))}(A) - U \subseteq \emptyset$. Therefore, we have $c_{(\zeta, \delta(\mu))}(A) \subseteq U$ and hence, $A$ is $g_{(\zeta, \delta(\mu))}$-closed. Then, the proof of $i_{(\zeta, \delta(\mu))}c_{(\zeta, \delta(\mu))}(A) - A = \emptyset$ is given as follows. Suppose that $i_{(\zeta, \delta(\mu))}c_{(\zeta, \delta(\mu))}(A) - A \neq \emptyset$.

There exists $x \in i_{(\zeta, \delta(\mu))}c_{(\zeta, \delta(\mu))}(A) - A$. Then, there exists $G \in (\zeta, \delta(\mu))O$ such that $x \in G \subseteq c_{(\zeta, \delta(\mu))}(A) - A$. Since $G \subseteq X - A$, we have $G \cap A = \emptyset$ and hence, $G \subseteq X - c_{(\zeta, \delta(\mu))}(A)$. Therefore, we obtain $G \subseteq X - c_{(\zeta, \delta(\mu))}(A) \cap c_{(\zeta, \delta(\mu))}(A) = \emptyset$. This is a contradiction.

Theorem 3.33. A subset $A$ of a strong generalized topological space $(X, \mu)$ is $g_{(\zeta, \delta(\mu))}$-closed if and only if $F \cap c_{(\zeta, \delta(\mu))}(A) = \emptyset$ whenever $A \cap F = \emptyset$ and $F$ is $(\zeta, \delta(\mu))$-closed.

Proof. Suppose that $A$ is $g_{(\zeta, \delta(\mu))}$-closed. Let $A \cap F = \emptyset$ and $F$ be $(\zeta, \delta(\mu))$-closed. Then $c_{(\zeta, \delta(\mu))}(A) \subseteq X - F$. Therefore, we have $F \cap c_{(\zeta, \delta(\mu))}(A) = \emptyset$.

Conversely, let $A \subseteq U$ and $U \in (\zeta, \delta(\mu))O$. Then $A \cap (X - U) = \emptyset$ and $X - U$ is $(\zeta, \delta(\mu))$-closed. By the hypothesis, $(X - U) \cap c_{(\zeta, \delta(\mu))}(A) = \emptyset$ and hence, $c_{(\zeta, \delta(\mu))}(A) \subseteq U$. Consequently, we obtain $A$ is $g_{(\zeta, \delta(\mu))}$-closed.

Theorem 3.34. A subset $A$ of a strong generalized topological space $(X, \mu)$ is $g_{(\zeta, \delta(\mu))}$-closed if and only if $A \cap \{x\} \neq \emptyset$ for every $x \in c_{(\zeta, \delta(\mu))}(A)$.

Proof. Let $A$ be a $g_{(\zeta, \delta(\mu))}$-closed set. Assume that $A \cap \{x\} = \emptyset$ for some $x \in c_{(\zeta, \delta(\mu))}(A)$. By Lemma 3.16, $c_{(\zeta, \delta(\mu))}(\{x\})$ is $(\zeta, \delta(\mu))$-closed and hence, $A \subseteq X - c_{(\zeta, \delta(\mu))}(\{x\}) \in (\zeta, \delta(\mu))O$. Since $A$ is $g_{(\zeta, \delta(\mu))}$-closed, $c_{(\zeta, \delta(\mu))}(A) \subseteq X - c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq X - \{x\}$.

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This contradicts that $x \in c_{(\zeta, \delta)(\mu)}(A)$.

Conversely, suppose that $A$ is not $g-(\zeta, \delta(\mu))-\text{closed}$, then $\emptyset \neq c_{(\zeta, \delta)(\mu)}(A) - U$ for some $U \in (\zeta, \delta(\mu))O$ containing $A$. There exists $x \in c_{(\zeta, \delta)(\mu)}(A) - U$. Since $x \notin U$, by Lemma 3.31 $U \cap c_{(\zeta, \delta)(\mu)}(\{x\}) = \emptyset$ and hence, $A \cap c_{(\zeta, \delta)(\mu)}(\{x\}) \subseteq U \cap c_{(\zeta, \delta)(\mu)}(\{x\}) = \emptyset$.

This shows that $A \cap c_{(\zeta, \delta)(\mu)}(\{x\}) = \emptyset$ for some $x \in c_{(\zeta, \delta)(\mu)}(A)$.

**Corollary 3.35.** For a subset $A$ of a strong generalized topological space $(X, \mu)$, the following properties are equivalent:

1. $A$ is $g-(\zeta, \delta(\mu))-\text{open}$.
2. $A - i_{(\zeta, \delta)(\mu)}(A)$ contains no non-empty $(\zeta, \delta(\mu))-\text{closed}$ set.
3. $A - i_{(\zeta, \delta)(\mu)}(A)$ is $g-(\zeta, \delta(\mu))-\text{open}$.
4. $(X - A) \cap c_{(\zeta, \delta)(\mu)}(\{x\}) \neq \emptyset$ for every $x \in A - i_{(\zeta, \delta)(\mu)}(A)$.

**Proof.** This follows from Theorems 3.32 and 3.34.

**Proposition 3.36.** For a strong generalized topological space $(X, \mu)$, the following properties are equivalent:

1. For every $(\zeta, \delta(\mu))-\text{open}$ set $U$, $c_{(\zeta, \delta)(\mu)}(U) \subseteq U$.
2. Every subset of $X$ is $g-(\zeta, \delta(\mu))-\text{closed}$.

**Proof.** (1) $\Rightarrow$ (2) : Let $A$ be any subset of $X$ and $A \subseteq U \in (\zeta, \delta(\mu))O$. By (1), $c_{(\zeta, \delta)(\mu)}(U) \subseteq U$ and so $c_{(\zeta, \delta)(\mu)}(A) \subseteq c_{(\zeta, \delta)(\mu)}(U) \subseteq U$. Therefore, $A$ is $g-(\zeta, \delta(\mu))-\text{closed}$.

(2) $\Rightarrow$ (1) : Let $U \in (\zeta, \delta(\mu))O$. By (2), we have $U$ is $g-(\zeta, \delta(\mu))-\text{closed}$ and hence, $c_{(\zeta, \delta)(\mu)}(U) \subseteq U$.

**Theorem 3.37.** A subset $A$ of a strong generalized topological space $(X, \mu)$ is $g-(\zeta, \delta(\mu))-\text{open}$ if and only if $U = X$ whenever $U$ is $(\zeta, \delta(\mu))-\text{open}$ and $(X - A) \cup i_{(\zeta, \delta)(\mu)}(A) \subseteq U$.

**Proof.** Suppose that $A$ is $g-(\zeta, \delta(\mu))-\text{open}$ and $U \in (\zeta, \delta(\mu))O$ such that

$$(X - A) \cup i_{(\zeta, \delta)(\mu)}(A) \subseteq U.$$ 

Then $X - U \subseteq c_{(\zeta, \delta)(\mu)}(X - A) - (X - A)$. Since $X - A$ is $g-(\zeta, \delta(\mu))-\text{closed}$ and $X - U$ is $(\zeta, \delta(\mu))-\text{closed}$, by Theorem 3.23 $X - U = \emptyset$ and hence, $X = U$.

Conversely, suppose that $F \subseteq A$ and $F$ is $(\zeta, \delta(\mu))-\text{closed}$. By Lemma 3.19, we have $(X - A) \cup i_{(\zeta, \delta)(\mu)}(A) \subseteq (X - F) \cup i_{(\zeta, \delta)(\mu)}(A) \subseteq c_{(\zeta, \delta)(\mu)}O(X, \mu)$. By the hypothesis, $X = (X - F) \cup i_{(\zeta, \delta)(\mu)}(A)$ and hence,

$$F = F \cap ((X - F) \cup i_{(\zeta, \delta)(\mu)}(A)) = F \cap i_{(\zeta, \delta)(\mu)}(A) \subseteq i_{(\zeta, \delta)(\mu)}(A).$$

It follows from Theorem 3.29 that $A$ is $g-(\zeta, \delta(\mu))-\text{open}$.

**Proposition 3.38.** Let $A$ be a subset of a strong generalized topological space $(X, \mu)$. If $A$ is $g-(\zeta, \delta(\mu))-\text{open}$ and $i_{(\zeta, \delta)(\mu)}(A) \subseteq B \subseteq A$, then $B$ is $g-(\zeta, \delta(\mu))-\text{open}$.
Proof. We have $X - A \subseteq X - B \subseteq X - i_{c(\xi, \delta(\mu))}(A) = c_{c(\xi, \delta(\mu))}(X - A)$. Since $X - A$ is $g-(\xi, \delta(\mu))$-closed, it follows from Proposition 3.25(3) that $X - B$ is $g-(\xi, \delta(\mu))$-closed and hence, $B$ is $g-(\xi, \delta(\mu))$-open.

Definition 3.39. A subset $A$ of a strong generalized topological space $(X, \mu)$ is said to be locally $(\xi, \delta(\mu))$-closed if $A = U \cap F$, where $U \in (\xi, \delta(\mu))O$ and $F$ is $(\xi, \delta(\mu))$-closed.

Theorem 3.40. For a subset $A$ of a strong generalized topological space $(X, \mu)$, the following properties are equivalent:

1. $A$ is locally $(\xi, \delta(\mu))$-closed.
2. $A = U \cap c_{c(\xi, \delta(\mu))}(A)$ for some $U \in (\xi, \delta(\mu))O$.
3. $c_{c(\xi, \delta(\mu))}(A) - A$ is $(\xi, \delta(\mu))$-closed.
4. $A \cup (X - i_{c(\xi, \delta(\mu))}(A)) \in (\xi, \delta(\mu))O$.
5. $A \subseteq i_{c(\xi, \delta(\mu))}(A \cup (X - c_{c(\xi, \delta(\mu))}(A)))$.

Proof. (1) $\Rightarrow$ (2): Suppose that $A = U \cap F$, where $U \in (\xi, \delta(\mu))O$ and $F$ is $(\xi, \delta(\mu))$-closed. Since $A \subseteq F$, we have $c_{c(\xi, \delta(\mu))}(A) \subseteq c_{c(\xi, \delta(\mu))}(F) = F$. Since $A \subseteq U$, $A \subseteq U \cap c_{c(\xi, \delta(\mu))}(A) \subseteq U \cap F = A$. Consequently, we obtain $A = U \cap c_{c(\xi, \delta(\mu))}(A)$ for some $U \in (\xi, \delta(\mu))O$.

(2) $\Rightarrow$ (3): Suppose that $A = U \cap c_{c(\xi, \delta(\mu))}(A)$ for some $U \in (\xi, \delta(\mu))O$. Then, we have $c_{c(\xi, \delta(\mu))}(A) - A = (X - [U \cap c_{c(\xi, \delta(\mu))}(A)]) \cap c_{c(\xi, \delta(\mu))}(A) = (X - U) \cap c_{c(\xi, \delta(\mu))}(A)$.

Since $(X - U) \cap c_{c(\xi, \delta(\mu))}(A)$ is $(\xi, \delta(\mu))$-closed and hence, $c_{c(\xi, \delta(\mu))}(A) - A$ is $(\xi, \delta(\mu))$-closed.

(3) $\Rightarrow$ (4): Since $X - [c_{c(\xi, \delta(\mu))}(A) - A] = [X - c_{c(\xi, \delta(\mu))}(A)] \cup A$ and by (3), we obtain $A \cup [X - c_{c(\xi, \delta(\mu))}(A)] \in (\xi, \delta(\mu))O$.

(4) $\Rightarrow$ (5): By (4), $A \subseteq A \cup [X - c_{c(\xi, \delta(\mu))}(A)] = i_{c(\xi, \delta(\mu))}(A \cup (X - c_{c(\xi, \delta(\mu))}(A)))$.

(5) $\Rightarrow$ (1): We put $U = i_{c(\xi, \delta(\mu))}(A \cup (X - c_{c(\xi, \delta(\mu))}(A)))$. Then $U \in (\xi, \delta(\mu))O$ and hence,

$A = A \cap U \subseteq U \cap c_{c(\xi, \delta(\mu))}(A)$

$\subseteq [A \cup [X - c_{c(\xi, \delta(\mu))}(A)] \cap c_{c(\xi, \delta(\mu))}(A)$

$= A \cap c_{c(\xi, \delta(\mu))}(A) = A$.

Therefore, we obtain $A = U \cap c_{c(\xi, \delta(\mu))}(A)$, where $U \in (\xi, \delta(\mu))O$ and $c_{c(\xi, \delta(\mu))}(A)$ is $(\xi, \delta(\mu))$-closed. This shows that $A$ is locally $(\xi, \delta(\mu))$-closed.

Theorem 3.41. A subset $A$ of a strong generalized topological space $(X, \mu)$ is $(\xi, \delta(\mu))$-closed if and only if $A$ is locally $(\xi, \delta(\mu))$-closed and $g-(\xi, \delta(\mu))$-closed.

Proof. Let $A$ be $(\xi, \delta(\mu))$-closed. By Proposition 3.25(1), $A$ is $g-(\xi, \delta(\mu))$-closed. Since $X \in (\xi, \delta(\mu))O$ and $A \subseteq X \cap A$, we have $A$ is locally $(\xi, \delta(\mu))$-closed.

Conversely, suppose that $A$ is locally $(\xi, \delta(\mu))$-closed and $g-(\xi, \delta(\mu))$-closed. Since $A$ is locally $(\xi, \delta(\mu))$-closed, by Theorem 3.40 we have $A \subseteq i_{c(\xi, \delta(\mu))}(A \cup (X - c_{c(\xi, \delta(\mu))}(A)))$. 

By Lemma 3.19, \( i_{(\zeta, \delta(\mu))}(A \cup (X - c_{(\zeta, \delta(\mu))}(A))) \in (\zeta, \delta(\mu))\mathcal{O} \) and \( A \) is \( g-(\zeta, \delta(\mu))\)-closed. Therefore, we have
\[
c_{(\zeta, \delta(\mu))}(A) \subseteq i_{(\zeta, \delta(\mu))}(A \cup (X - c_{(\zeta, \delta(\mu))}(A))) \subseteq A \cup (X - c_{(\zeta, \delta(\mu))}(A))
\]
and hence, \( c_{(\zeta, \delta(\mu))}(A) \subseteq A \). Thus, \( c_{(\zeta, \delta(\mu))}(A) = A \) and by Lemma 3.16, we obtain \( A \) is \((\zeta, \delta(\mu))\)-closed.

**Definition 3.42.** A subset \( A \) of a strong generalized topological space \((X, \mu)\) is said to be:

(i) \( s(\zeta, \delta(\mu))\)-open if \( A \subseteq c_{(\zeta, \delta(\mu))}[i_{(\zeta, \delta(\mu))}(A)] \);

(ii) \( p(\zeta, \delta(\mu))\)-open if \( A \subseteq i_{(\zeta, \delta(\mu))}[c_{(\zeta, \delta(\mu))}(A)] \);

(iii) \( a(\zeta, \delta(\mu))\)-open if \( A \subseteq i_{(\zeta, \delta(\mu))}[c_{(\zeta, \delta(\mu))}\{i_{(\zeta, \delta(\mu))}(A)] \};

(iv) \( \beta(\zeta, \delta(\mu))\)-open if \( A \subseteq c_{(\zeta, \delta(\mu))}[i_{(\zeta, \delta(\mu))}\{c_{(\zeta, \delta(\mu))}(A)] \}.

The complement of a \( s(\zeta, \delta(\mu))\)-open (resp. \( p(\zeta, \delta(\mu))\)-open, \( a(\zeta, \delta(\mu))\)-open, \( \beta(\zeta, \delta(\mu))\)-open) set is said to be \( s(\zeta, \delta(\mu))\)-closed (resp. \( p(\zeta, \delta(\mu))\)-closed, \( a(\zeta, \delta(\mu))\)-closed, \( \beta(\zeta, \delta(\mu))\)-closed).

The family of all \( s(\zeta, \delta(\mu))\)-open (resp. \( p(\zeta, \delta(\mu))\)-open, \( a(\zeta, \delta(\mu))\)-open, \( \beta(\zeta, \delta(\mu))\)-open) sets in a strong generalized topological space \((X, \mu)\) is denoted by \( s(\zeta, \delta(\mu))\mathcal{O} \) (resp. \( p(\zeta, \delta(\mu))\mathcal{O}, a(\zeta, \delta(\mu))\mathcal{O}, \beta(\zeta, \delta(\mu))\mathcal{O} \)).

**Proposition 3.43.** For a strong generalized topological space \((X, \mu)\), the following properties hold:

1. \( (\zeta, \delta(\mu))\mathcal{O} \subseteq a(\zeta, \delta(\mu))\mathcal{O} \subseteq s(\zeta, \delta(\mu))\mathcal{O} \subseteq \beta(\zeta, \delta(\mu))\mathcal{O} \).

2. \( a(\zeta, \delta(\mu))\mathcal{O} \subseteq p(\zeta, \delta(\mu))\mathcal{O} \subseteq \beta(\zeta, \delta(\mu))\mathcal{O} \).

3. \( a(\zeta, \delta(\mu))\mathcal{O} = s(\zeta, \delta(\mu))\mathcal{O} \cap p(\zeta, \delta(\mu))\mathcal{O} \).

**Proof.** (1) Let \( V \) is a \( (\zeta, \delta(\mu))\)-open set. Then, we have
\[
V = i_{(\zeta, \delta(\mu))}(V) \subseteq i_{(\zeta, \delta(\mu))}[c_{(\zeta, \delta(\mu))}\{i_{(\zeta, \delta(\mu))}(V)]
\]
\[
\subseteq c_{(\zeta, \delta(\mu))}[i_{(\zeta, \delta(\mu))}]\{V)]
\]
\[
\subseteq c_{(\zeta, \delta(\mu))}[i_{(\zeta, \delta(\mu))}\{c_{(\zeta, \delta(\mu))}(V)]
\]
and so \( (\zeta, \delta(\mu))\mathcal{O} \subseteq a(\zeta, \delta(\mu))\mathcal{O} \subseteq s(\zeta, \delta(\mu))\mathcal{O} \subseteq \beta(\zeta, \delta(\mu))\mathcal{O} \).

(2) Let \( V \) is a \( a(\zeta, \delta(\mu))\)-open set. Then, we have
\[
V \subseteq i_{(\zeta, \delta(\mu))}\{c_{(\zeta, \delta(\mu))}(V)] \subseteq c_{(\zeta, \delta(\mu))}[i_{(\zeta, \delta(\mu))}\{c_{(\zeta, \delta(\mu))}(V)]
\]
and hence, \( a(\zeta, \delta(\mu))\mathcal{O} \subseteq p(\zeta, \delta(\mu))\mathcal{O} \subseteq \beta(\zeta, \delta(\mu))\mathcal{O} \).

(3) By (1) and (2), we obtain \( a(\zeta, \delta(\mu))\mathcal{O} \subseteq s(\zeta, \delta(\mu))\mathcal{O} \cap p(\zeta, \delta(\mu))\mathcal{O} \). Let \( V \in s(\zeta, \delta(\mu))\mathcal{O} \cap p(\zeta, \delta(\mu))\mathcal{O} \). Then, we have \( V \in s(\zeta, \delta(\mu))\mathcal{O} \) and \( V \in p(\zeta, \delta(\mu))\mathcal{O} \). Therefore, \( V \subseteq c_{(\zeta, \delta(\mu))}\{i_{(\zeta, \delta(\mu))}(V)] \) and \( V \subseteq i_{(\zeta, \delta(\mu))}\{c_{(\zeta, \delta(\mu))}(V)] \). Hence, \( V \subseteq i_{(\zeta, \delta(\mu))}\{c_{(\zeta, \delta(\mu))}(V)] \subseteq i_{(\zeta, \delta(\mu))}\{c_{(\zeta, \delta(\mu))}(V)] [i_{(\zeta, \delta(\mu))}(V)] \) and so \( V \in a(\zeta, \delta(\mu))\mathcal{O} \). Therefore, \( s(\zeta, \delta(\mu))\mathcal{O} \cap p(\zeta, \delta(\mu))\mathcal{O} \subseteq a(\zeta, \delta(\mu))\mathcal{O} \). Consequently, we obtain \( a(\zeta, \delta(\mu))\mathcal{O} = s(\zeta, \delta(\mu))\mathcal{O} \cap p(\zeta, \delta(\mu))\mathcal{O} \).
Definition 3.44. A subset $A$ of a strong generalized topological space $(X, \mu)$ is said to be $r(\zeta, \delta(\mu))$-open set if $A = i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(A))$. The complement of a $r(\zeta, \delta(\mu))$-open set is said to be $r(\zeta, \delta(\mu))$-closed.

The family of all $r(\zeta, \delta(\mu))$-open (resp. $r(\zeta, \delta(\mu))$-closed) sets in a strong generalized topological space $(X, \mu)$ is denoted by $r(\zeta, \delta(\mu))$ (resp. $r(\zeta, \delta(\mu))$).

Definition 3.45. A subset $D$ of a strong generalized topological space $(X, \mu)$ is said to be $(\zeta, \delta(\mu))$-dense if $c_{(\zeta, \delta(\mu))}(D) = X$. $D$ is said to be $(\zeta, \delta(\mu))$-codense if $X - D$ is $(\zeta, \delta(\mu))$-dense.

Proposition 3.46. For a subset $D$ of a strong generalized topological space $(X, \mu)$, the following properties are equivalent:

1. $D$ is $(\zeta, \delta(\mu))$-dense.
2. If $F$ is any $(\zeta, \delta(\mu))$-closed set and $D \subseteq F$, then $F = X$.
3. Each non-empty $(\zeta, \delta(\mu))$-open set contains an element of $D$.
4. The complement of $D$ has empty $(\zeta, \delta(\mu))$-interior.

Proof. (1) $\Rightarrow$ (2) : Let $F$ be a $(\zeta, \delta(\mu))$-closed set such that $D \subseteq F$. Then $X = c_{(\zeta, \delta(\mu))}(D) \subseteq c_{(\zeta, \delta(\mu))}(F) = F$.

(2) $\Rightarrow$ (3) : Let $U$ be non-empty $(\zeta, \delta(\mu))$-open set such that $U \cap D = \emptyset$; then $D \subseteq X - U \neq X$, which contradicts (2), since $X - U$ is $(\zeta, \delta(\mu))$-closed.

(3) $\Rightarrow$ (4) : Suppose that $i_{(\zeta, \delta(\mu))}(X - D) \neq \emptyset$; since $i_{(\zeta, \delta(\mu))}(X - D)$ is a $(\zeta, \delta(\mu))$-open set such that $i_{(\zeta, \delta(\mu))}(X - D) \subseteq X - D$, we have $i_{(\zeta, \delta(\mu))}(X - D)$ contains no point of $D$.

(4) $\Rightarrow$ (1) : $i_{(\zeta, \delta(\mu))}(X - D) = X - c_{(\zeta, \delta(\mu))}(D) = \emptyset$ so that $c_{(\zeta, \delta(\mu))}(D) = X$.

Remark 3.47. Let $D$ be a subset of a strong generalized topological space $(X, \mu)$. If $D$ is $(\zeta, \delta(\mu))$-dense, then $D$ is $p(\zeta, \delta(\mu))$-open.

Proposition 3.48. Let $A$ be a subset of a strong generalized topological space $(X, \mu)$. If $A$ is $p(\zeta, \delta(\mu))$-open, then $A$ is the intersection of a $r(\zeta, \delta(\mu))$-open set and a $(\zeta, \delta(\mu))$-dense set.

Proof. Suppose that $A$ is $p(\zeta, \delta(\mu))$-open. Then, we have $A \subseteq i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(A))$ and $A = (A \cup |X - c_{(\zeta, \delta(\mu))}(A)) \cap i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(A))$. Put $C = i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(A))$ and $D = A \cup |X - c_{(\zeta, \delta(\mu))}(A))$. Then $C$ is $r(\zeta, \delta(\mu))$-open, also $c_{(\zeta, \delta(\mu))}(A) \subseteq c_{(\zeta, \delta(\mu))}(D)$ since $A \subseteq D$ and $X - c_{(\zeta, \delta(\mu))}(A) \subseteq D \subseteq c_{(\zeta, \delta(\mu))}(D)$. Thus, we have $c_{(\zeta, \delta(\mu))}(D) = X$.

Definition 3.49. A strong generalized topological space $(X, \mu)$ is said to be $(\zeta, \delta(\mu))$-submaximal if each $(\zeta, \delta(\mu))$-dense set of $X$ is $(\zeta, \delta(\mu))$-open.

Proposition 3.50. Let $(X, \mu)$ be a strong generalized topological space. If each $p(\zeta, \delta(\mu))$-open set is $s(\zeta, \delta(\mu))$-open and each $a(\zeta, \delta(\mu))$-open set is $(\zeta, \delta(\mu))$-open, then $(X, \mu)$ is $(\zeta, \delta(\mu))$-submaximal.

Proof. Let $D$ be a $(\zeta, \delta(\mu))$-dense set of $X$. Since $c_{(\zeta, \delta(\mu))}(D) = X$, then $D$ is a $(\zeta, \delta(\mu))$-open set. This implies that $D$ is a $s(\zeta, \delta(\mu))$-open set. Since any set is $s(\zeta, \delta(\mu))$-open if and only if it is $s(\zeta, \delta(\mu))$-open and $p(\zeta, \delta(\mu))$-open, then $D$ is an $a(\zeta, \delta(\mu))$-open set. Hence, since each $a(\zeta, \delta(\mu))$-open set is $(\zeta, \delta(\mu))$-open, we have $D$ is $(\zeta, \delta(\mu))$-open. Thus, $(X, \mu)$ is $(\zeta, \delta(\mu))$-submaximal.
Proposition 3.51. Let \((X, \mu)\) be a strong generalized topological space. If each \(p(\zeta, \delta(\mu))\)-open set is \((\zeta, \delta(\mu))\)-open, then \((X, \mu)\) is \((\zeta, \delta(\mu))\)-submaximal.

Proof. Suppose that each \(p(\zeta, \delta(\mu))\)-open set is \((\zeta, \delta(\mu))\)-open. It follows that every \(p(\zeta, \delta(\mu))\)-open set is \(s(\zeta, \delta(\mu))\)-open. Since each \(a(\zeta, \delta(\mu))\)-open set is \(p(\zeta, \delta(\mu))\)-open, then each \(a(\zeta, \delta(\mu))\)-open set is \((\zeta, \delta(\mu))\)-open. Thus, by Proposition 3.50, \((X, \mu)\) is \((\zeta, \delta(\mu))\)-submaximal.

Proposition 3.52. For a strong generalized topological space \((X, \mu)\), the following properties are equivalent:

1. \((X, \mu)\) is \((\zeta, \delta(\mu))\)-submaximal.
2. Each \((\zeta, \delta(\mu))\)-codense set \(C\) of \(X\) is \((\zeta, \delta(\mu))\)-closed.

Theorem 3.53. For a strong generalized topological space \((X, \mu)\), the following properties are equivalent:

1. \((X, \mu)\) is \((\zeta, \delta(\mu))\)-submaximal.
2. Each subset of \(X\) is locally \((\zeta, \delta(\mu))\)-closed set.
3. Each \((\zeta, \delta(\mu))\)-dense set of \(X\) is the intersection of a \((\zeta, \delta(\mu))\)-closed set and a \((\zeta, \delta(\mu))\)-open set.

Proof. (1) \(\Rightarrow\) (2): Suppose that \((X, \mu)\) is \((\zeta, \delta(\mu))\)-submaximal. Let \(A\) be any subset of \(X\). Then, we have \(c_{(\zeta, \delta(\mu))}(X - [c_{(\zeta, \delta(\mu))}(A) - A]) = c_{(\zeta, \delta(\mu))}(A \cup [X - c_{(\zeta, \delta(\mu))}(A)]) = X\) and hence, \(X - [c_{(\zeta, \delta(\mu))}(A) - A]\) is a \((\zeta, \delta(\mu))\)-dense set. By (1), \(X - [c_{(\zeta, \delta(\mu))}(A) - A]\) is a \((\zeta, \delta(\mu))\)-open set. Thus, we have \(X - [c_{(\zeta, \delta(\mu))}(A) - A] = A \cup [X - c_{(\zeta, \delta(\mu))}(A)]\) is \((\zeta, \delta(\mu))\)-open. Consequently, \(A = (A \cup (X - c_{(\zeta, \delta(\mu))}(A)]) \cap c_{(\zeta, \delta(\mu))}(A)\) is a locally \((\zeta, \delta(\mu))\)-closed set.

(2) \(\Rightarrow\) (3): This is obvious.

(3) \(\Rightarrow\) (1): Let \(D\) be a \((\zeta, \delta(\mu))\)-dense set. By (3), there exist a \((\zeta, \delta(\mu))\)-open set \(U\) and a \((\zeta, \delta(\mu))\)-closed set \(F\) such that \(D = U \cap F\). Since \(D \subseteq F\) and \(D\) is a \((\zeta, \delta(\mu))\)-dense set, we have \(i_{(\zeta, \delta(\mu))}(F) \supseteq i_{(\zeta, \delta(\mu))}(D)\) = \(i_{(\zeta, \delta(\mu))}(X) = X\). This implies that \(F = X\) and \(D = U\) is \((\zeta, \delta(\mu))\)-open. Hence, \((X, \mu)\) is \((\zeta, \delta(\mu))\)-submaximal.

Theorem 3.54. For a strong generalized topological space \((X, \mu)\), the following properties are equivalent:

1. \((X, \mu)\) is \((\zeta, \delta(\mu))\)-submaximal.
2. Each \((\zeta, \delta(\mu))\)-codense set of \(X\) is the union of a \((\zeta, \delta(\mu))\)-open set and a \((\zeta, \delta(\mu))\)-closed set.

Proof. This is an immediate consequence of Theorem 3.53.

Definition 3.55. A strong generalized topological space \((X, \mu)\) is called \((\zeta, \delta(\mu))\)-hyperconnected if every non-empty \((\zeta, \delta(\mu))\)-open set is \((\zeta, \delta(\mu))\)-dense.

Theorem 3.56. For a strong generalized topological space \((X, \mu)\), the following properties are equivalent:

1. \((X, \mu)\) is \((\zeta, \delta(\mu))\)-hyperconnected.
(2) $V$ is $(\zeta, \delta(\mu))$-dense for every non-empty set $V \in p(\zeta, \delta(\mu))O$.

(3) $V \cup i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(V)) = X$ for every non-empty set $V \in p(\zeta, \delta(\mu))O$.

(4) $V \cup i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(i_{(\zeta, \delta(\mu))}(V))) = X$ for every non-empty set $V \in s(\zeta, \delta(\mu))O$.

(5) $V \cup c_{(\zeta, \delta(\mu))}(i_{(\zeta, \delta(\mu))}(V)) = X$ for every non-empty set $V \in s(\zeta, \delta(\mu))O$.

Proof. (1) $\Rightarrow$ (2): Let $V$ be any non-empty $p(\zeta, \delta(\mu))$-open set. Then, we have $i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(V)) \neq \emptyset$ and $c_{(\zeta, \delta(\mu))}(V) = c_{(\zeta, \delta(\mu))}(i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(V))) = X$.

(2) $\Rightarrow$ (3): Let $V$ be any non-empty $p(\zeta, \delta(\mu))$-open set. By (2), $c_{(\zeta, \delta(\mu))}(V) = X$ and so $V \cup i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(V)) = V \cup i_{(\zeta, \delta(\mu))}(V) = V \cup X = X$.

(3) $\Rightarrow$ (4): Let $V$ be any non-empty $s(\zeta, \delta(\mu))$-open set. Then, we have $c_{(\zeta, \delta(\mu))}(V) = c_{(\zeta, \delta(\mu))}(i_{(\zeta, \delta(\mu))}(V))$.

By (3), we obtain $X = V \cup i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(V)) = V \cup i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(i_{(\zeta, \delta(\mu))}(V)))$.

(4) $\Rightarrow$ (5): Let $V$ be any non-empty $s(\zeta, \delta(\mu))$-open set. By (4), we have $X = V \cup i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(i_{(\zeta, \delta(\mu))}(V))) \subseteq V \cup c_{(\zeta, \delta(\mu))}(i_{(\zeta, \delta(\mu))}(V))$.

Consequently, we obtain $V \cup c_{(\zeta, \delta(\mu))}(i_{(\zeta, \delta(\mu))}(V)) = X$.

(5) $\Rightarrow$ (1): Let $V$ be any non-empty $(\zeta, \delta(\mu))$-open set. Then $V$ is $s(\zeta, \delta(\mu))$-open. By (5), we have $V \cup c_{(\zeta, \delta(\mu))}(i_{(\zeta, \delta(\mu))}(V)) = X$ and hence, $c_{(\zeta, \delta(\mu))}(V) = X$. Therefore, we obtain $(X, \mu)$ is $(\zeta, \delta(\mu))$-hyperconnected.

**Theorem 3.57.** For a strong generalized topological space $(X, \mu)$, the following properties are equivalent:

(1) $(X, \mu)$ is $(\zeta, \delta(\mu))$-hyperconnected.

(2) $V$ is $(\zeta, \delta(\mu))$-dense for every non-empty set $V \in p(\zeta, \delta(\mu))O$.

(3) $V \cup i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(V)) = X$ for every non-empty set $V \in p(\zeta, \delta(\mu))O$.

(4) $V \cup c_{(\zeta, \delta(\mu))}(i_{(\zeta, \delta(\mu))}(V)) = X$ for every non-empty set $V \in s(\zeta, \delta(\mu))O$.

Proof. It is similar to that of Theorem 3.56.

**Theorem 3.58.** For a strong generalized topological space $(X, \mu)$, the following properties are equivalent:

(1) $(X, \mu)$ is $(\zeta, \delta(\mu))$-hyperconnected.

(2) $V$ is $(\zeta, \delta(\mu))$-dense for every non-empty set $V \in s(\zeta, \delta(\mu))O$.

(3) $V \cup i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(V)) = X$ for every non-empty set $V \in s(\zeta, \delta(\mu))O$.

Proof. It is similar to that of Theorem 3.56.
Definition 3.59. A strong generalized topological space \((X, \mu)\) is called \((\zeta, \delta(\mu))\)-extremally disconnected if \(c_{(\zeta,\delta(\mu))}(V)\) is \((\zeta, \delta(\mu))\)-open in \((X, \mu)\) for every \((\zeta, \delta(\mu))\)-open set \(V\).

Theorem 3.60. A strong generalized topological space \((X, \mu)\) is \((\zeta, \delta(\mu))\)-extremally disconnected if and only if \(c_{(\zeta,\delta(\mu))}(U) \cap c_{(\zeta,\delta(\mu))}(V) = \emptyset\) for every \((\zeta, \delta(\mu))\)-open sets \(U\) and \(V\) such that \(U \cap V = \emptyset\).

Proof. Suppose that \(U\) and \(V\) are \((\zeta, \delta(\mu))\)-open sets such that \(U \cap V = \emptyset\). By Lemma 3.31, we obtain \(c_{(\zeta,\delta(\mu))}(U) \cap V = \emptyset\) and \(c_{(\zeta,\delta(\mu))}(U) \cap c_{(\zeta,\delta(\mu))}(V) = \emptyset\).

Conversely, let \(U\) be any \((\zeta, \delta(\mu))\)-open set. Then \(X - \mu\) is \((\zeta, \delta(\mu))\)-closed and hence, \(i_{(\zeta,\delta(\mu))}(X - \mu)\) is \((\zeta, \delta(\mu))\)-open such that \(U \cap i_{(\zeta,\delta(\mu))}(X - \mu) = \emptyset\).

By the hypothesis, we have \(c_{(\zeta,\delta(\mu))}(U) \cap c_{(\zeta,\delta(\mu))}(V) = \emptyset\) which implies that \(c_{(\zeta,\delta(\mu))}(U) \cap (X - i_{(\zeta,\delta(\mu))}(i_{(\zeta,\delta(\mu))}(U))) = \emptyset\).

Therefore, \(c_{(\zeta,\delta(\mu))}(U) \subseteq i_{(\zeta,\delta(\mu))}(c_{(\zeta,\delta(\mu))}(U))\) and hence, \(c_{(\zeta,\delta(\mu))}(U) = i_{(\zeta,\delta(\mu))}(c_{(\zeta,\delta(\mu))}(U))\). This shows that \(c_{(\zeta,\delta(\mu))}(U)\) is \((\zeta, \delta(\mu))\)-open. Consequently, we obtain \((X, \mu)\) is \((\zeta, \delta(\mu))\)-extremally disconnected.

Lemma 3.61. Let \(A\) be a subset of a strong generalized topological space \((X, \mu)\). If \(U\) is \((\zeta, \delta(\mu))\)-open, then \(U \cap c_{(\zeta,\delta(\mu))}(A) \subseteq c_{(\zeta,\delta(\mu))}(U \cap A)\).

Theorem 3.62. For a strong generalized topological space \((X, \mu)\), the following properties are equivalent:

1. \((X, \mu)\) is \((\zeta, \delta(\mu))\)-extremally disconnected.
2. \(c_{(\zeta,\delta(\mu))}(U) \cap c_{(\zeta,\delta(\mu))}(V) = c_{(\zeta,\delta(\mu))}(U \cap V)\) for every \((\zeta, \delta(\mu))\)-open sets \(U\) and \(V\).
3. \(i_{(\zeta,\delta(\mu))}(E) \cup i_{(\zeta,\delta(\mu))}(F) = i_{(\zeta,\delta(\mu))}(E \cup F)\) for every \((\zeta, \delta(\mu))\)-closed sets \(E\) and \(F\).

Proof. (1) \(\Rightarrow\) (2) : Let \(U\) and \(V\) be \((\zeta, \delta(\mu))\)-open sets. Then by (1), we have \(c_{(\zeta,\delta(\mu))}(U)\) and \(c_{(\zeta,\delta(\mu))}(V)\) are \((\zeta, \delta(\mu))\)-open sets. By Lemma 3.61,
\[
c_{(\zeta,\delta(\mu))}(U) \cap c_{(\zeta,\delta(\mu))}(V) \subseteq c_{(\zeta,\delta(\mu))}(U \cap c_{(\zeta,\delta(\mu))}(V))
\]
\[
\subseteq c_{(\zeta,\delta(\mu))}(c_{(\zeta,\delta(\mu))}(U \cap V))
\]
\[
= c_{(\zeta,\delta(\mu))}(U \cap V).
\]
Consequently, we obtain \(c_{(\zeta,\delta(\mu))}(U) \cap c_{(\zeta,\delta(\mu))}(V) = c_{(\zeta,\delta(\mu))}(U \cap V)\).

(2) \(\Rightarrow\) (3) : Let \(E\) and \(F\) be \((\zeta, \delta(\mu))\)-closed sets. Then \(X - E\) and \(X - F\) are \((\zeta, \delta(\mu))\)-open. By (2) and Lemma 3.19, we obtain
\[
i_{(\zeta,\delta(\mu))}(E) \cup i_{(\zeta,\delta(\mu))}(F) = X - (X - i_{(\zeta,\delta(\mu))}(E) \cup i_{(\zeta,\delta(\mu))}(F))
\]
\[
= X - (X - i_{(\zeta,\delta(\mu))}(E)) \cap (X - i_{(\zeta,\delta(\mu))}(F))
\]
\[
= X - [c_{(\zeta,\delta(\mu))}(X - E) \cap c_{(\zeta,\delta(\mu))}(X - F)]
\]
\[
= X - c_{(\zeta,\delta(\mu))}[(X - E) \cap (X - F)]
\]
\[
= i_{(\zeta,\delta(\mu))}(X - [(X - E) \cap (X - F)])
\]
\[ \delta \in V \mu \text{ and } r \text{ is } c F \delta / C_0. \] Since \( \delta \) and \( V \) is \( V \delta \), the following properties are obtained and hence, \( r \text{ is } \delta \mu \overline{U} \text{ is } c \). Therefore, \( \delta / \delta \mu \text{ is } \delta / \delta \mu \), and only if \( r \text{ is } \delta / \delta \mu \) extremally disconnected.

\textbf{Theorem 3.63.} For a strong generalized topological space \((X, \mu)\), the following properties are equivalent:

1. \((X, \mu)\) is \( (\zeta, \delta(\mu))\)-extremally disconnected.
2. \( c_{(\zeta, \delta(\mu))}(U) \cap c_{(\zeta, \delta(\mu))}(V) \) for every \( (\zeta, \delta(\mu))\)-open sets \( U \) and \( V \).
3. \( c_{(\zeta, \delta(\mu))}(U) \cap c_{(\zeta, \delta(\mu))}(V) = \emptyset \) for every \( (\zeta, \delta(\mu))\)-open sets \( U \) and \( V \) such that \( U \cap V = \emptyset \).

\textbf{Proof.} This follows from Theorems 3.60 and Theorem 3.62.

\textbf{Theorem 3.64.} A strong generalized topological space \((X, \mu)\) is \( (\zeta, \delta(\mu))\)-extremally disconnected if and only if \( \delta(\zeta, \delta(\mu)) O = r(\zeta, \delta(\mu)) C \).

\textbf{Proof.} Suppose that \((X, \mu)\) is \( (\zeta, \delta(\mu))\)-extremally disconnected. Let \( V \in r(\zeta, \delta(\mu)) O. \) Then, we have \( V = i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(V)) \). Since \((X, \mu)\) is \( (\zeta, \delta(\mu))\)-extremally disconnected, 
\[ c_{(\zeta, \delta(\mu))}(i_{(\zeta, \delta(\mu))}(V)) = i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(V)) \]
\[ = i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(i_{(\zeta, \delta(\mu))}(V))) \]
\[ = i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(V)) = V \]
and so \( V \in r(\zeta, \delta(\mu)) C \). Therefore, we obtain \( r(\zeta, \delta(\mu)) O \subseteq r(\zeta, \delta(\mu)) C \). On the other hand, let \( V \in r(\zeta, \delta(\mu)) C \). Then, we have \( V = i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(V)) \). Since \((X, \mu)\) is \( (\zeta, \delta(\mu))\)-extremally disconnected, 
\[ i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(V)) = i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(i_{(\zeta, \delta(\mu))}(V))) \]
\[ = i_{(\zeta, \delta(\mu))}(c_{(\zeta, \delta(\mu))}(V)) \]
and hence, \( V \in r(\zeta, \delta(\mu)) O \). Therefore, \( r(\zeta, \delta(\mu)) C \subseteq r(\zeta, \delta(\mu)) O \). Consequently, we obtain \( r(\zeta, \delta(\mu)) O = r(\zeta, \delta(\mu)) C \).

Conversely, suppose that \( r(\zeta, \delta(\mu)) O = r(\zeta, \delta(\mu)) C \). Let \( V \) be any \( (\zeta, \delta(\mu))\)-open set. Then, we have \( c_{(\zeta, \delta(\mu))}(i_{(\zeta, \delta(\mu))}(V)) \in r(\zeta, \delta(\mu)) C \) and so \( c_{(\zeta, \delta(\mu))}(i_{(\zeta, \delta(\mu))}(V)) \in r(\zeta, \delta(\mu)) O \). Therefore, we obtain...
For a strong generalized topological space $c_\delta(\mu,\nu)$, the following properties are equivalent:

1. $(X,\mu)$ is $(\zeta,\delta(\mu))$-extremely disconnected.
2. For each $U \in s(\zeta,\delta(\mu))\Omega$, $c_{(\zeta,\delta(\mu))}(U) \in (\zeta,\delta(\mu))\Omega$.
3. For each $U, V \in s(\zeta,\delta(\mu))\Omega$, $c_{(\zeta,\delta(\mu))}(U \cap V) = c_{(\zeta,\delta(\mu))}(U) \cap c_{(\zeta,\delta(\mu))}(V)$.
4. For each $U, V \in (\zeta,\delta(\mu))\Omega$, $c_{(\zeta,\delta(\mu))}(U \cap V) = c_{(\zeta,\delta(\mu))}(U) \cap c_{(\zeta,\delta(\mu))}(V)$.

Proof. (1) $\Rightarrow$ (2): Let $U \in s(\zeta,\delta(\mu))\Omega$. Then, we have $U \subseteq c_{(\zeta,\delta(\mu))}[i_{(\zeta,\delta(\mu))}(U)]$. Since $(X,\mu)$ is $(\zeta,\delta(\mu))$-extremely disconnected,

$$c_{(\zeta,\delta(\mu))}(U) \subseteq c_{(\zeta,\delta(\mu))}[i_{(\zeta,\delta(\mu))}(U)] = i_{(\zeta,\delta(\mu))}[c_{(\zeta,\delta(\mu))}[i_{(\zeta,\delta(\mu))}(U)]] \subseteq i_{(\zeta,\delta(\mu))}[c_{(\zeta,\delta(\mu))}(U)].$$

Consequently, we obtain $c_{(\zeta,\delta(\mu))}(U) = i_{(\zeta,\delta(\mu))}[c_{(\zeta,\delta(\mu))}(U)]$. Therefore, $c_{(\zeta,\delta(\mu))}(U) \in (\zeta,\delta(\mu))\Omega$.

(2) $\Rightarrow$ (3): Let $U, V \in s(\zeta,\delta(\mu))\Omega$. By (2), we have $c_{(\zeta,\delta(\mu))}(U), c_{(\zeta,\delta(\mu))}(V) \in (\zeta,\delta(\mu))\Omega$ and hence,

$$c_{(\zeta,\delta(\mu))}(U \cap V) = c_{(\zeta,\delta(\mu))}[c_{(\zeta,\delta(\mu))}(U)] \cap c_{(\zeta,\delta(\mu))}(V) \subseteq c_{(\zeta,\delta(\mu))}[c_{(\zeta,\delta(\mu))}(U) \cap c_{(\zeta,\delta(\mu))}(V)] \subseteq c_{(\zeta,\delta(\mu))}[c_{(\zeta,\delta(\mu))}[i_{(\zeta,\delta(\mu))}(U)]] \cap c_{(\zeta,\delta(\mu))}(V) = c_{(\zeta,\delta(\mu))}[c_{(\zeta,\delta(\mu))}[i_{(\zeta,\delta(\mu))}(U) \cap c_{(\zeta,\delta(\mu))}(V)] = c_{(\zeta,\delta(\mu))}[c_{(\zeta,\delta(\mu))}(U) \cap c_{(\zeta,\delta(\mu))}(V)] \subseteq c_{(\zeta,\delta(\mu))}[i_{(\zeta,\delta(\mu))}(U) \cap c_{(\zeta,\delta(\mu))}(V)] \subseteq c_{(\zeta,\delta(\mu))}[c_{(\zeta,\delta(\mu))}(U) \cap V]\subseteq c_{(\zeta,\delta(\mu))}[i_{(\zeta,\delta(\mu))}(U) \cap V] = c_{(\zeta,\delta(\mu))}[i_{(\zeta,\delta(\mu))}(U) \cap V] \subseteq c_{(\zeta,\delta(\mu))}(U \cap V)$.

Therefore, we obtain $c_{(\zeta,\delta(\mu))}(U \cap V) = c_{(\zeta,\delta(\mu))}(U) \cap c_{(\zeta,\delta(\mu))}(V)$.

(3) $\Rightarrow$ (4): This is obvious since every $(\zeta,\delta(\mu))$-open set is $s(\zeta,\delta(\mu))$-open.
(4) $\Rightarrow$ (1): The proof is obvious from Theorem 3.62.

**Definition 3.66.** A strong generalized topological space $(X, \mu)$ is called a $\delta(\mu)$-$R_0$ space if for each $\delta(\mu)$-open set $U$ and each $x \in U$, $c_{\delta(\mu)}(\{x\}) \subseteq U$.

**Definition 3.67.** A strong generalized topological space $(X, \mu)$ is said to be:

(i) $\delta(\mu)$-$T_0$ if for any distinct pair of points in $X$, there exists a $\delta(\mu)$-open set containing one of the points but not the other.

(ii) $\delta(\mu)$-$T_1$ if for any distinct pair of points $x$ and $y$ in $X$, there exist a $\delta(\mu)$-open set $U$ containing $x$ but not $y$ and a $\delta(\mu)$-open set $V$ containing $y$ but not $x$.

**Theorem 3.68.** Let $(X, \mu)$ be a $\delta(\mu)$-$R_0$ strong generalized topological space. A singleton $\{x\}$ is $(\zeta, \delta(\mu))$-closed if and only if for each $x \in X$, this shows that $\delta(\mu)$-$T_0$.

**Proof.** Suppose that $\{x\}$ is $(\zeta, \delta(\mu))$-closed. Then by Theorem 3.10, $\{x\} = \zeta_{\delta(\mu)}(\{x\}) \cap c_{\delta(\mu)}(\{x\})$.

For any $\delta(\mu)$-open set $U$ containing $x$, $c_{\delta(\mu)}(\{x\}) \subseteq U$ and hence, $c_{\delta(\mu)}(\{x\}) \subseteq \zeta_{\delta(\mu)}(\{x\})$. Therefore, we have $\{x\} = \zeta_{\delta(\mu)}(\{x\}) \cap c_{\delta(\mu)}(\{x\}) \supseteq c_{\delta(\mu)}(\{x\})$. This shows that $\{x\}$ is $\delta(\mu)$-closed.

Conversely, suppose that $\{x\}$ is $\delta(\mu)$-closed. Since $\{x\} \subseteq \zeta_{\delta(\mu)}(\{x\})$, we have $\zeta_{\delta(\mu)}(\{x\}) \cap c_{\delta(\mu)}(\{x\}) = \zeta_{\delta(\mu)}(\{x\}) \cap \{x\} = \{x\}$. This shows that $\{x\}$ is $(\zeta, \delta(\mu))$-closed.

**Theorem 3.69.** A strong generalized topological space $(X, \mu)$ is $\delta(\mu)$-$T_0$ if and only if for each $x \in X$, the singleton $\{x\}$ is $(\zeta, \delta(\mu))$-closed.

**Proof.** Suppose that $(X, \mu)$ is $\delta(\mu)$-$T_0$. For each $x \in X$, it is obvious that $\{x\} \subseteq \zeta_{\delta(\mu)}(\{x\}) \cap c_{\delta(\mu)}(\{x\})$.

If $y \neq x$, (i) there exists a $\delta(\mu)$-open set $V_y$ such that $y \notin V_y$ and $x \in V_y$ or (ii) there exists a $\delta(\mu)$-open set $V_y$ such that $x \notin V_y$ and $y \in V_y$. In case of (i), $y \notin \zeta_{\delta(\mu)}(\{x\})$ and $y \notin c_{\delta(\mu)}(\{x\})$. This shows that $\{x\} \supseteq \zeta_{\delta(\mu)}(\{x\}) \cap c_{\delta(\mu)}(\{x\})$. In case (ii), $y \notin \zeta_{\delta(\mu)}(\{x\})$ and $y \notin c_{\delta(\mu)}(\{x\})$. This shows that $\{x\} \subseteq \zeta_{\delta(\mu)}(\{x\}) \cap c_{\delta(\mu)}(\{x\})$. Consequently, we obtain $\{x\} = \zeta_{\delta(\mu)}(\{x\}) \cap c_{\delta(\mu)}(\{x\})$.

Conversely, suppose that $(X, \mu)$ is not $\delta(\mu)$-$T_0$. There exist two distinct points $x, y$ such that (i) $y \in V_x$ for every $\delta(\mu)$-open set $V_x$ containing $x$ and (ii) $x \in V_y$ for every $\delta(\mu)$-open set $V_y$ containing $y$. From (i) and (ii), we obtain $y \in \zeta_{\delta(\mu)}(\{x\})$ and $x \in c_{\delta(\mu)}(\{y\})$, respectively. Therefore, we have $y \in \zeta_{\delta(\mu)}(\{x\}) \cap c_{\delta(\mu)}(\{x\})$. By Theorem 3.10, $\{x\} = \zeta_{\delta(\mu)}(\{x\}) \cap c_{\delta(\mu)}(\{x\})$ since $\{x\}$ is $(\delta, \zeta(\mu))$-closed. This is contrary to $x \neq y$.

**Corollary 3.70.** Let $(X, \mu)$ be a $\delta(\mu)$-$R_0$ strong generalized topological space. Then $(X, \mu)$ is $\delta(\mu)$-$T_0$ if for each $x \in X$, the singleton $\{x\}$ is $\delta(\mu)$-closed.

**Proof.** It is an immediate consequence of Theorem 3.68 and Theorem 3.69.

**Theorem 3.71.** A strong generalized topological space $(X, \mu)$ is $\delta(\mu)$-$T_1$ if and only if for each $x \in X$, the singleton $\{x\}$ is a $\zeta_{\delta(\mu)}$-set.

**Proof.** Suppose that $y \in \zeta_{\delta(\mu)}(\{x\})$ for some point $y$ distinct from $x$. Then, we have $y \in \cap \{V_x | x \in V_x; V_x \in \delta(\mu)\}$ and $y \in V_x$ for every $\delta(\mu)$-open set $V_x$ containing $x$. This contradicts that $(X, \mu)$ is $\delta(\mu)$-$T_1$. 

Conversely, suppose that \( \{x\} \) is a \( \xi(\mu) \)-set for each \( x \in X \). Let \( x \) and \( y \) be any distinct points. Then \( y \not\in \xi(\mu)(\{x\}) \) and there exists a \( \delta(\mu) \)-open set \( V_x \) such that \( x \in V_x \) and \( y \notin V_x \). Similarly, \( x \not\in \xi(\mu)(\{y\}) \) and there exists a \( \delta(\mu) \)-open set \( V_y \) such that \( y \in V_y \) and \( x \notin V_y \). This shows that \( (X, \mu) \) is \( \delta(\mu) \)-T1.

4. \( (\zeta, \delta(\mu)) \)-open sets and associated separation axioms

In this section, we introduce and investigate several new low separation axioms by utilizing the notion of \( (\zeta, \delta(\mu)) \)-open sets.

Definition 4.1. Let \( A \) be a subset of a strong generalized topological space \( (X, \mu) \). A subset \( \xi(\zeta, \mu(\mu)) \) is defined as follows: \( \xi(\zeta, \mu(\mu)) = \{ U \in (\zeta, \delta(\mu)) : A \subseteq U \} \).

Lemma 4.2. For subsets \( A, B \) of a strong generalized topological space \( (X, \mu) \), the following properties hold:

1. \( A \subseteq \xi(\zeta, \mu(\mu))(A) \)
2. If \( A \subseteq B \), then \( \xi(\zeta, \mu(\mu))(A) \subseteq \xi(\zeta, \mu(\mu))(B) \).
3. \( \xi(\zeta, \mu(\mu))(\xi(\zeta, \mu(\mu))(A)) = \xi(\zeta, \mu(\mu))(A) \).
4. If \( A \) is \( (\zeta, \delta(\mu)) \)-open, then \( \xi(\zeta, \mu(\mu))(A) = A \).

Lemma 4.3. Let \( (X, \mu) \) be a strong generalized topological space and \( x, y \in X \). Then, \( y \in \xi(\zeta, \mu(\mu))(\{x\}) \) if and only if \( x \in c(\zeta, \mu(\mu))(\{y\}) \).

Proof. Let \( y \notin \xi(\zeta, \mu(\mu))(\{x\}) \). Then, there exists a \( (\zeta, \delta(\mu)) \)-open set \( V \) containing \( x \) such that \( y \notin V \). Hence, \( x \notin c(\zeta, \mu(\mu))(\{y\}) \). The converse is similarly shown.

A subset \( N \) of a strong generalized topological space \( (X, \mu) \) is said to be \( (\zeta, \delta(\mu)) \)-neighbourhood of a point \( x \in X \) if there exists an \( (\zeta, \delta(\mu)) \)-open set \( U \) such that \( x \in U \subseteq N \).

Lemma 4.4. A subset of a strong generalized topological space \( (X, \mu) \) is \( (\zeta, \delta(\mu)) \)-open in \( (X, \mu) \) if and only if it is an \( (\zeta, \delta(\mu)) \)-neighbourhood of each of its points.

Definition 4.5. Let \( (X, \mu) \) be a strong generalized topological space and \( x \in X \). A subset \( \{x\} \) \( (\zeta, \mu(\mu)) \) is defined as follows: \( \{x\} = \xi(\zeta, \mu(\mu))(\{x\}) \).

Theorem 4.6. For a strong generalized topological space \( (X, \mu) \), the following properties hold:

1. \( \xi(\zeta, \mu(\mu))(A) = \{ x \in X | A \cap c(\zeta, \mu(\mu))(\{x\}) \neq \emptyset \} \) for each subset \( A \) of \( X \).
2. For each \( x \in X \), \( \xi(\zeta, \mu(\mu))(\{x\}) = \xi(\zeta, \mu(\mu))(\{x\}) \).
3. For each \( x \in X \), \( c(\zeta, \mu(\mu))(\{x\}) = c(\zeta, \mu(\mu))(\{x\}) \).
4. If \( U \) is \( (\zeta, \delta(\mu)) \)-open in \( (X, \mu) \) and \( x \in U \), then \( \{x\} \subseteq U \).
5. If \( F \) is \( (\zeta, \delta(\mu)) \)-closed in \( (X, \mu) \) and \( x \in F \), then \( \{x\} \subseteq F \).

Proof. (1) Suppose that \( A \cap c(\zeta, \mu(\mu))(\{x\}) \neq \emptyset \). Then \( x \notin X - c(\zeta, \mu(\mu))(\{x\}) \) which is a \( (\zeta, \delta(\mu)) \)-open set containing \( A \). Therefore, \( x \notin \xi(\zeta, \mu(\mu))(A) \). Consequently, we have \( \xi(\zeta, \mu(\mu))(A) \subseteq \{ x \in X | A \cap c(\zeta, \mu(\mu))(\{x\}) \neq \emptyset \} \).
Next, let \( x \in X \) such that \( A \cap c(\zeta, \mu(\mu))(\{x\}) \neq \emptyset \) and suppose that \( x \notin \xi(\zeta, \mu(\mu))(A) \). Then, there exists a...
(ζ, δ(μ))-open set U containing A and \(X \notin U\). Let \(y \in A \cap C(ξ,μ)(\{x\})\). Hence, \(U\) is a \((ξ, δ(μ))\)-neighbourhood of \(y\) which does not contain \(x\). By this contradiction \(y \in C(ξ,μ)(\{x\})\).

(2) Let \(x \in X\), then we have \(\{x\} \subseteq C(ξ,μ)(\{x\}) \cap C(ξ,μ)(\{y\}) = (X)_{C(ξ,μ)}\). By Lemma 4.2, we obtain \(ξ(\{x\}) \subseteq ζ(\{x\}) \subseteq ζ(\{y\})\). Next, we show the opposite implication. Suppose that \(y \notin C(ξ,μ)(\{x\})\). Then, there exists a \((ξ, δ(μ))\)-open set \(V\) such that \(x \in V\) and \(y \notin V\). Since \(X = ζ(\{x\}) \subseteq ζ(\{y\})\), we have \(\{x\} \subseteq ζ(\{y\})\). Consequently, we obtain \(ζ(\{x\}) \subseteq ζ(\{y\})\) and hence, \(C(ξ,μ)(\{x\}) = C(ξ,μ)(\{y\})\).

(3) By the definition of \((X)_{C(ξ,μ)}\), we have \(\{x\} \subseteq (X)_{C(ξ,μ)}\) and \(C(ξ,μ)(\{x\}) \subseteq C(ξ,μ)(\{y\})\) by Lemma 3.16. On the other hand, we have \((X)_{C(ξ,μ)} \subseteq C(ξ,μ)(\{x\})\) and \(C(ξ,μ)(\{x\}) \subseteq C(ξ,μ)(\{y\})\). Therefore, we obtain \(C(ξ,μ)(\{x\}) = C(ξ,μ)(\{y\})\).

(4) Since \(x \in U\) and \(U\) is a \((ξ, δ(μ))\)-open set, we have \(ζ(\{y\}) \subseteq U\). Hence, \((X)_{C(ξ,μ)} \subseteq U\).

(5) Since \(x \in F\) and \(F\) is a \((ξ, δ(μ))\)-closed set, we have \(X = F \subseteq C(ξ,μ)(\{x\}) \cap C(ξ,μ)(\{y\}) = (X)_{C(ξ,μ)}\).

Lemma 4.7. The following properties are equivalent for any points \(x\) and \(y\) in a strong generalized topological space \((X, μ):\)

(1) \(ζ(\{x\}) \neq ζ(\{y\})\).

(2) \(C(ξ,μ)(\{x\}) \neq C(ξ,μ)(\{y\})\).

Proof. (1) \(\Rightarrow\) (2): Suppose that \(ζ(\{x\}) \neq ζ(\{y\})\). Then there exists a point \(z \in X\) such that \(z \in ζ(\{x\})\) and \(z \notin ζ(\{y\})\) or \(z \notin ζ(\{x\})\) and \(z \notin ζ(\{y\})\). We prove only the first case being the second analogous. From \(z \in ζ(\{x\})\) it follows that \(\{z\} \cap C(ξ,μ)(\{y\}) = \emptyset\) which implies \(x \in C(ξ,μ)(\{z\})\). By \(z \notin ζ(\{y\})\), we have \(\{y\} \cap C(ξ,μ)(\{z\}) = \emptyset\). Since \(x \in C(ξ,μ)(\{z\})\), \(C(ξ,μ)(\{x\}) \subseteq C(ξ,μ)(\{z\})\) and \(\{y\} \cap C(ξ,μ)(\{z\}) = \emptyset\). Therefore, it follows that \(C(ξ,μ)(\{x\}) \neq C(ξ,μ)(\{y\})\).

(2) \(\Rightarrow\) (1): Suppose that \(C(ξ,μ)(\{x\}) \neq C(ξ,μ)(\{y\})\). Then, there exists a point \(z \in X\) such that \(z \in C(ξ,μ)(\{x\})\) and \(z \notin C(ξ,μ)(\{y\})\) or \(z \in C(ξ,μ)(\{y\})\) and \(z \notin C(ξ,μ)(\{x\})\). We prove only the first case being the second analogous.

It follows that there exists a \((ξ, δ(μ))\)-open set containing \(z\) and \(x\) but not \(y\). This means that \(y \notin ζ(\{x\})\) and thus, \(ζ(\{x\}) = ζ(\{y\})\).

Lemma 4.8. Let \((X, μ)\) be a strong generalized topological space and \(x, y \in X\). Then, the following properties hold:

(1) \(y \in ζ(\{x\})\) if and only if \(x \in C(ξ,μ)(\{y\})\).

(2) \(ζ(\{x\}) = ζ(\{y\})\) if and only if \(C(ξ,μ)(\{x\}) = C(ξ,μ)(\{y\})\).

Proof. (1) Let \(x \notin C(ξ,μ)(\{y\})\). Then, there exists \(U \in (ξ, δ(μ))O\) such that \(x \in U\) and \(y \notin U\). Thus, \(y \notin ζ(\{x\})\). The converse is similarly shown.
Suppose that $V$, $Y/C_22$, $X$ and $F_2$. For any distinct points $x$ and $y$ of $X$ and $r_2$, $\delta_2$, $\delta_2$, $\zeta$ such that $x \in \zeta(C_{\delta(\mu)}(\{x\}))$, by (1), $y \in \zeta(C_{\delta(\mu)}(\{y\}))$. By Lemma 3.16, we have

$$c_{\zeta(C_{\delta(\mu)})(\{y\})} \subseteq c_{\zeta(C_{\delta(\mu)})(\{x\})}.$$ Similarly, we have $c_{\zeta(C_{\delta(\mu)})(\{x\})} \subseteq c_{\zeta(C_{\delta(\mu)})(\{y\})}$ and hence, $c_{\zeta(C_{\delta(\mu)})(\{x\})} = c_{\zeta(C_{\delta(\mu)})(\{y\})}$. On the other hand, suppose that $c_{\zeta(C_{\delta(\mu)})(\{x\})} = c_{\zeta(C_{\delta(\mu)})(\{y\})}$. Since $x \in \zeta(C_{\delta(\mu)})(\{x\})$, we have $x \in c_{\zeta(C_{\delta(\mu)})(\{y\})}$ and by (1), $y \in \zeta(C_{\delta(\mu)})(\{x\})$. By Lemma 4.2, $\zeta(C_{\delta(\mu)})(\{y\}) \subseteq \zeta(C_{\delta(\mu)})(\{x\}) = \zeta(C_{\delta(\mu)})(\{x\})$. Similarly, we have $\zeta(C_{\delta(\mu)})(\{x\}) \subseteq \zeta(C_{\delta(\mu)})(\{y\})$ and hence, $\zeta(C_{\delta(\mu)})(\{x\}) = \zeta(C_{\delta(\mu)})(\{y\})$.

**Definition 4.9.** A strong generalized topological space $(X, \mu)$ is called a $(\zeta, \delta(\mu))$-$R_0$ space if every $(\zeta, \delta(\mu))$-open set contains the $(\zeta, \delta(\mu))$-closure of each of its singletons.

**Theorem 4.10.** For a strong generalized topological space $(X, \mu)$, the following properties are equivalent:

1. $(X, \mu)$ is a $(\zeta, \delta(\mu))$-$R_0$ space.
2. For any $F \in (\zeta, \delta(\mu))C$, $x \notin F$ implies $F \subseteq U$ and $x \notin U$ for some $U \in (\zeta, \delta(\mu))O$.
3. For any $F \in (\zeta, \delta(\mu))C$, $x \notin F$ implies $F \cap c_{\zeta(C_{\delta(\mu)})(\{x\})} = \emptyset$.
4. For any distinct points $x$ and $y$ of $X$, either $c_{\zeta(C_{\delta(\mu)})(\{x\})} = c_{\zeta(C_{\delta(\mu)})(\{y\})}$ or $c_{\zeta(C_{\delta(\mu)})(\{x\})} \cap c_{\zeta(C_{\delta(\mu)})(\{y\})} = \emptyset$.

**Proof.** (1) $\Rightarrow$ (2) : Let $F \in (\zeta, \delta(\mu))C$ and $x \notin F$. Then by (1), $c_{\zeta(C_{\delta(\mu)})(\{x\})} \subseteq X - F$. Put $U = X - c_{\zeta(C_{\delta(\mu)})(\{x\})}$, then $U \in (\zeta, \delta(\mu))O$, $F \subseteq U$ and $x \in U$.

(2) $\Rightarrow$ (3) : Let $F \in (\zeta, \delta(\mu))C$ and $x \notin F$. There exists $U \in (\zeta, \delta(\mu))O$ such that $F \subseteq U$ and $x \notin U$. Since $U \in (\zeta, \delta(\mu))O$, we have $U \cap c_{\zeta(C_{\delta(\mu)})(\{x\})} = \emptyset$ and

$$F \cap c_{\zeta(C_{\delta(\mu)})(\{x\})} = \emptyset.$$  

(3) $\Rightarrow$ (4) : Suppose that $c_{\zeta(C_{\delta(\mu)})(\{x\})} \neq c_{\zeta(C_{\delta(\mu)})(\{y\})}$ for distinct points $x, y \in X$. There exists $z \in c_{\zeta(C_{\delta(\mu)})(\{x\})}$ such that $z \notin c_{\zeta(C_{\delta(\mu)})(\{y\})}$ or $z \in c_{\zeta(C_{\delta(\mu)})(\{y\})}$ such that $z \notin c_{\zeta(C_{\delta(\mu)})(\{x\})}$. There exists $V \in (\zeta, \delta(\mu))O$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin c_{\zeta(C_{\delta(\mu)})(\{y\})}$. By (3), we obtain $c_{\zeta(C_{\delta(\mu)})(\{x\})} \cap c_{\zeta(C_{\delta(\mu)})(\{y\})} = \emptyset$. The proof for the other case is similar.

(4) $\Rightarrow$ (1) : Let $V \in (\zeta, \delta(\mu))O$ and $x \in V$. For each $y \notin V$, $y \neq x$ and $x \notin c_{\zeta(C_{\delta(\mu)})(\{y\})}$. This shows that $c_{\zeta(C_{\delta(\mu)})(\{x\})} \neq c_{\zeta(C_{\delta(\mu)})(\{y\})}$. By (4), we have $c_{\zeta(C_{\delta(\mu)})(\{x\})} \cap c_{\zeta(C_{\delta(\mu)})(\{y\})} = \emptyset$ for each $y \in X - V$ and hence, $c_{\zeta(C_{\delta(\mu)})(\{x\})} \cap \bigcup_{y \notin V} c_{\zeta(C_{\delta(\mu)})(\{y\})} = \emptyset$. On the other hand, since $V \in (\zeta, \delta(\mu))O$ and $y \in X - V$, we have $c_{\zeta(C_{\delta(\mu)})(\{y\})} \subseteq X - V$ and so $X - V = \bigcup_{y \notin V} c_{\zeta(C_{\delta(\mu)})(\{y\})}$. Consequently, we obtain $(X - V) \cap c_{\zeta(C_{\delta(\mu)})(\{x\})} = \emptyset$ and $c_{\zeta(C_{\delta(\mu)})(\{x\})} \subseteq V$. This shows that $(X, \mu)$ is a $(\zeta, \delta(\mu))$-$R_0$ space.

**Corollary 4.11.** A strong generalized topological space $(X, \mu)$ is a $(\zeta, \delta(\mu))$-$R_0$ space if and only if for any $x, y \in X$, $c_{\zeta(C_{\delta(\mu)})(\{x\})} \neq c_{\zeta(C_{\delta(\mu)})(\{y\})}$ implies $c_{\zeta(C_{\delta(\mu)})(\{x\})} \cap c_{\zeta(C_{\delta(\mu)})(\{y\})} = \emptyset$.

**Proof.** Suppose that $(X, \mu)$ is a $(\zeta, \delta(\mu))$-$R_0$ space. Let $x, y \in X$ such that $c_{\zeta(C_{\delta(\mu)})(\{x\})} \neq c_{\zeta(C_{\delta(\mu)})(\{y\})}$. Then by (4), $c_{\zeta(C_{\delta(\mu)})(\{x\})} \cap c_{\zeta(C_{\delta(\mu)})(\{y\})} = \emptyset$. On the other hand, since $V \in (\zeta, \delta(\mu))O$ and $y \in X - V$, we have $c_{\zeta(C_{\delta(\mu)})(\{y\})} \subseteq X - V$ and so $X - V = \bigcup_{y \notin V} c_{\zeta(C_{\delta(\mu)})(\{y\})}$. Consequently, we obtain $(X - V) \cap c_{\zeta(C_{\delta(\mu)})(\{x\})} = \emptyset$ and $c_{\zeta(C_{\delta(\mu)})(\{x\})} \subseteq V$. This shows that $(X, \mu)$ is a $(\zeta, \delta(\mu))$-$R_0$ space.
Then, there exists \( z \in c(\zeta, \delta(\mu)) \{ \{ x \} \} \) such that \( z \not\in c(\zeta, \delta(\mu)) \{ \{ y \} \} \) or \( z \in c(\zeta, \delta(\mu)) \{ \{ y \} \} \) such that \( z \not\in c(\zeta, \delta(\mu)) \{ \{ x \} \} \). There exists \( V \in (\zeta, \delta(\mu))O \) such that \( y \not\in V \) and \( z \in V \); hence \( x \in V \). Therefore, we have \( x \not\in c(\zeta, \delta(\mu)) \{ \{ y \} \} \). Thus,

\[
x \in X - c(\zeta, \delta(\mu)) \{ \{ y \} \} \in (\zeta, \delta(\mu))O,
\]

which implies \( c(\zeta, \delta(\mu)) \{ \{ x \} \} \subseteq X - c(\zeta, \delta(\mu)) \{ \{ y \} \} \) and \( c(\zeta, \delta(\mu)) \{ \{ x \} \} \cap c(\zeta, \delta(\mu)) \{ \{ y \} \} = \emptyset \). The proof for otherwise is similar.

Conversely, let \( V \in (\zeta, \delta(\mu))O \) and \( x \in V \). Now, we will show that \( c(\zeta, \delta(\mu)) \{ \{ x \} \} \subseteq V \). Let \( y \notin V \), i.e., \( y \in X - V \). Then, we have \( x \not= y \) and \( x \not\in c(\zeta, \delta(\mu)) \{ \{ y \} \} \). This shows that \( c(\zeta, \delta(\mu)) \{ \{ x \} \} \neq c(\zeta, \delta(\mu)) \{ \{ y \} \} \). By the hypothesis, \( c(\zeta, \delta(\mu)) \{ \{ x \} \} \cap c(\zeta, \delta(\mu)) \{ \{ y \} \} = \emptyset \)

and \( y \not\in c(\zeta, \delta(\mu)) \{ \{ x \} \} \). This implies that \( c(\zeta, \delta(\mu)) \{ \{ x \} \} \subseteq V \). Consequently, we obtain \( (X, \mu) \in (\zeta, \delta(\mu))R_0 \).

**Theorem 4.12.** A strong generalized topological space \( (X, \mu) \) is an \( (\zeta, \delta(\mu)) \)-\( R_0 \) space if and only if for any \( x, y \in X \), \( \zeta(\zeta, \delta(\mu))(\{ x \}) \neq \zeta(\zeta, \delta(\mu))(\{ y \}) \) implies \( \zeta(\zeta, \delta(\mu))(\{ x \}) \cap \zeta(\zeta, \delta(\mu))(\{ y \}) = \emptyset \).

**Proof.** Suppose that \( (X, \mu) \) is an \( (\zeta, \delta(\mu)) \)-\( R_0 \) space. Thus by Lemma 4.7, for any points \( x, y \in X \) if \( \zeta(\zeta, \delta(\mu))(\{ x \}) \neq \zeta(\zeta, \delta(\mu))(\{ y \}) \), then \( c(\zeta(\zeta, \delta(\mu)) \{ \{ x \} \}) \neq c(\zeta(\zeta, \delta(\mu)) \{ \{ y \} \}) \). Now, we prove that \( c(\zeta(\zeta, \delta(\mu)) \{ \{ x \} \}) \cap c(\zeta(\zeta, \delta(\mu)) \{ \{ y \} \}) = \emptyset \). Assume that \( z \in c(\zeta(\zeta, \delta(\mu)) \{ \{ x \} \}) \cap c(\zeta(\zeta, \delta(\mu)) \{ \{ y \} \}) \).

By \( z \in c(\zeta(\zeta, \delta(\mu)) \{ \{ x \} \}) \), it follows that \( x \in c(\zeta(\zeta, \delta(\mu)) \{ \{ x \} \}) \). Since \( x \in c(\zeta(\zeta, \delta(\mu)) \{ \{ x \} \}) \), by Corollary 4.11, \( c(\zeta(\zeta, \delta(\mu)) \{ \{ x \} \}) = c(\zeta(\zeta, \delta(\mu)) \{ \{ x \} \}) \). Similarly, we have \( c(\zeta(\zeta, \delta(\mu)) \{ \{ y \} \}) = c(\zeta(\zeta, \delta(\mu)) \{ \{ y \} \}) \). This is a contradiction. Consequently, we obtain

\( \zeta(\zeta, \delta(\mu)) \{ \{ x \} \} \cap \zeta(\zeta, \delta(\mu)) \{ \{ y \} \} = \emptyset \).

Conversely, let \( (X, \mu) \) be a strong generalized topological space such that for any points \( x, y \in X \), \( \zeta(\zeta, \delta(\mu)) \{ \{ x \} \} \neq \zeta(\zeta, \delta(\mu)) \{ \{ y \} \} \) implies \( \zeta(\zeta, \delta(\mu)) \{ \{ x \} \} \cap \zeta(\zeta, \delta(\mu)) \{ \{ y \} \} = \emptyset \). If \( c(\zeta(\zeta, \delta(\mu)) \{ \{ x \} \}) \neq c(\zeta(\zeta, \delta(\mu)) \{ \{ y \} \}) \), then by Lemma 4.7, \( \zeta(\zeta, \delta(\mu)) \{ \{ x \} \} \neq \zeta(\zeta, \delta(\mu)) \{ \{ y \} \} \). Therefore, \( \zeta(\zeta, \delta(\mu)) \{ \{ x \} \} \cap \zeta(\zeta, \delta(\mu)) \{ \{ y \} \} = \emptyset \) which implies \( c(\zeta(\zeta, \delta(\mu)) \{ \{ x \} \}) \cap c(\zeta(\zeta, \delta(\mu)) \{ \{ y \} \}) = \emptyset \).

Because \( z \in c(\zeta(\zeta, \delta(\mu)) \{ \{ x \} \}) \) implies that \( x \in c(\zeta(\zeta, \delta(\mu)) \{ \{ x \} \}) \) by Lemma 4.8 and therefore \( \zeta(\zeta, \delta(\mu)) \{ \{ x \} \} \cap \zeta(\zeta, \delta(\mu)) \{ \{ z \} \} = \emptyset \). By the hypothesis, we have \( \zeta(\zeta, \delta(\mu)) \{ \{ x \} \} = \zeta(\zeta, \delta(\mu)) \{ \{ z \} \} \). Then \( z \in c(\zeta(\zeta, \delta(\mu)) \{ \{ x \} \}) \cap c(\zeta(\zeta, \delta(\mu)) \{ \{ y \} \}) \) would imply that \( c(\zeta(\zeta, \delta(\mu)) \{ \{ x \} \}) = c(\zeta(\zeta, \delta(\mu)) \{ \{ z \} \}) = c(\zeta(\zeta, \delta(\mu)) \{ \{ y \} \}) \).

This is a contradiction. Therefore, \( c(\zeta(\zeta, \delta(\mu)) \{ \{ x \} \} \cap c(\zeta(\zeta, \delta(\mu)) \{ \{ y \} \} = \emptyset \) and by Corollary 4.11, \( (X, \mu) \) is a \( (\zeta, \delta(\mu)) \)-\( R_0 \) space.

**Theorem 4.13.** For a strong generalized topological space \( (X, \mu) \), the following properties are equivalent:

1. \( (X, \mu) \in (\zeta, \delta(\mu))R_0 \).
2. For any non-empty set \( A \) and \( G \in (\zeta, \delta(\mu))O \) such that \( A \cap G \neq \emptyset \), there exists \( F \in (\zeta, \delta(\mu))C \) such that \( A \cap F \neq \emptyset \) and \( F \subseteq G \).
3. For any \( G \in (\zeta, \delta(\mu))O \), \( G = \bigcup \{ F \in (\zeta, \delta(\mu))C \mid F \subseteq G \} \).
4. For any \( F \in (\zeta, \delta(\mu))C \), \( F \subseteq \bigcap \{ G \in (\zeta, \delta(\mu))O \mid F \subseteq G \} \).
5. For any \( x \in X \), \( c(\zeta(\zeta, \delta(\mu)) \{ \{ x \} \} \subseteq (\zeta, \delta(\mu))\{ \{ x \} \} \).
Proof. (1) ⇒ (2) : Let A be any non-empty set of X and G ∈ (ζ, δ(μ))O such that A ∩ G ≠ ∅. There exists x ∈ A ∩ G. Since x ∈ G, we have c_{ζ,δ(μ)}(\{x\}) ⊆ G. Put F = c_{ζ,δ(μ)}(\{x\}), then F ∈ (ζ, δ(μ))C, F ⊆ G and A ∩ F ≠ ∅.

(2) ⇒ (3) : Let G ∈ (ζ, δ(μ))O, then G ⊇ ∪\{F ∈ (ζ, δ(μ))C|F ⊆ G\}. Let x be any point of G. By (2), there exists F ∈ (ζ, δ(μ))C such that x ∈ F and F ⊆ G. Therefore, we have x ∈ F ⊆ ∪\{F ∈ (ζ, δ(μ))C|F ⊆ G\} and hence, G ⊇ ∪\{F ∈ (ζ, δ(μ))C|F ⊆ G\}.

Consequently, we obtain G = ∪\{F ∈ (ζ, δ(μ))C|F ⊆ G\}.

(3) ⇒ (4) : This is obvious.

(4) ⇒ (5) : Let x be any point of X and y ∉ ζ_{ζ,δ(μ)}(\{x\}). There exists V ∈ (ζ, δ(μ))O such that x ∈ V and y ∉ V; hence, V ∩ ζ_{ζ,δ(μ)}(\{y\}) = ∅. By (4), \{G ∈ (ζ, δ(μ))O|ζ_{ζ,δ(μ)}(\{y\}) ⊆ G\} \cap V = ∅ and there exists G ∈ (ζ, δ(μ))O such that X ∉ G and c_{ζ,δ(μ)}(\{y\}) ⊆ G. Therefore, G ∩ c_{ζ,δ(μ)}(\{x\}) = ∅ and hence, G contains y. This shows that \{x\} is a \{ζ, δ(μ)\}-R₀ space.

Corollary 4.14. For a strong generalized topological space (X, μ), the following properties are equivalent:

(1) (X, μ) is \{ζ, δ(μ)\}-R₀.

(2) c_{ζ,δ(μ)}(\{x\}) = ζ_{ζ,δ(μ)}(\{x\}) for all x ∈ X.

Proof. (1) ⇒ (2) : Suppose that (X, μ) is a \{ζ, δ(μ)\}-R₀ space. By Theorem 4.13, c_{ζ,δ(μ)}(\{x\}) ⊆ ζ_{ζ,δ(μ)}(\{x\}) for each x ∈ X. Let y ∈ ζ_{ζ,δ(μ)}(\{x\}), then c_{ζ,δ(μ)}(\{x\}) ∩ c_{ζ,δ(μ)}(\{y\}) ≠ ∅. By Corollary 4.11, we have c_{ζ,δ(μ)}(\{x\}) = c_{ζ,δ(μ)}(\{y\}). Therefore, y ∈ c_{ζ,δ(μ)}(\{x\}) and so ζ_{ζ,δ(μ)}(\{x\}) ⊆ c_{ζ,δ(μ)}(\{x\}). Consequently, we obtain c_{ζ,δ(μ)}(\{x\}) = ζ_{ζ,δ(μ)}(\{x\}).

(2) ⇒ (1) : By Theorem 4.13.

Corollary 4.15. Let (X, μ) be a \{ζ, δ(μ)\}-R₀ strong generalized topological space and x ∈ X. If ζ_{ζ,δ(μ)}(\{x\}) = \{x\}, then c_{ζ,δ(μ)}(\{x\}) = \{x\}.

Proof. This is a consequence of Corollary 4.14.

Theorem 4.16. For a strong generalized topological space (X, μ), the following properties are equivalent:

(1) (X, μ) is \{ζ, δ(μ)\}-R₀.

(2) x ∈ c_{ζ,δ(μ)}(\{y\}) if and only if y ∈ c_{ζ,δ(μ)}(\{x\}).

Proof. (1) ⇒ (2) : Suppose that (X, μ) is \{ζ, δ(μ)\}-R₀. Let x ∈ c_{ζ,δ(μ)}(\{y\}) and U be any (ζ, δ(μ))-open set such that y ∈ U. Therefore, ζ_{ζ,δ(μ)}(\{y\}) ⊆ U. Since x ∈ c_{ζ,δ(μ)}(\{y\}) and (X, μ) is \{ζ, δ(μ)\}-R₀, by Corollary 4.14, x ∈ ζ_{ζ,δ(μ)}(\{y\}) ⊆ U. Therefore, every (ζ, δ(μ))-open set containing y contains x. Hence, y ∈ c_{ζ,δ(μ)}(\{x\}).
(2) ⇒ (1): Let $U$ be any $(\zeta, \delta(\mu))$-open set and $x \in U$. If $y \notin U$, then $x \notin c_{(\zeta, \delta(\mu))}(\{y\})$ and hence, $y \notin c_{(\zeta, \delta(\mu))}(\{x\})$. This implies that $c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq U$. Hence, $(X, \mu)$ is a $(\zeta, \delta(\mu))$- $R_0$ space.

**Theorem 4.17.** For a strong generalized topological space $(X, \mu)$, the following properties are equivalent:

(1) $(X, \mu)$ is $(\zeta, \delta(\mu))$-$R_0$.

(2) $(X)_{(\zeta, \delta(\mu))} = c_{(\zeta, \delta(\mu))}(\{x\})$ for each $x \in X$.

(3) $(X)_{(\zeta, \delta(\mu))}$ is $(\zeta, \delta(\mu))$-closed for each $x \in X$.

**Proof.** (1) ⇒ (2): By Corollary 4.14, $c_{(\zeta, \delta(\mu))}(\{x\}) = \xi_{(\zeta, \delta(\mu))}(\{x\})$ for each $x \in X$. Hence, $c_{(\zeta, \delta(\mu))}(\{x\}) = c_{(\zeta, \delta(\mu))}(\{x\}) \cap \xi_{(\zeta, \delta(\mu))}(\{x\}) = (X)_{(\zeta, \delta(\mu))}$.

(2) ⇒ (1): Since $c_{(\zeta, \delta(\mu))}(\{x\}) = (X)_{(\zeta, \delta(\mu))}$ for each $x \in X$, we have $c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq \xi_{(\zeta, \delta(\mu))}(\{x\})$. By Theorem 4.13, $(X, \mu)$ is $(\zeta, \delta(\mu))$-$R_0$.

(2) ⇔ (3): This is a consequence of Theorem 4.6.

**Theorem 4.18.** For a strong generalized topological space $(X, \mu)$, the following properties are equivalent:

(1) $(X, \mu)$ is $(\zeta, \delta(\mu))$-$R_0$.

(2) For each non-empty set $A$ of $X$ and each $U \in (\zeta, \delta(\mu))O$ such that $A \cap U \neq \emptyset$, there exists a $(\zeta, \delta(\mu))$-closed set $F$ such that $A \cap F \neq \emptyset$ and $F \subseteq U$.

(3) $F = c_{(\zeta, \delta(\mu))}(F)$ for any $(\zeta, \delta(\mu))$-closed set $F$.

(4) $c_{(\zeta, \delta(\mu))}(\{x\}) = \xi_{(\zeta, \delta(\mu))}(\{x\})$ for each $x \in X$.

**Proof.** (1) ⇒ (2): By Theorem 4.13.

(2) ⇒ (3): Let $F$ be any $(\zeta, \delta(\mu))$-closed set. By Lemma 4.2, we have $F \subseteq \xi_{(\zeta, \delta(\mu))}(F)$. Next, we show $F \supseteq c_{(\zeta, \delta(\mu))}(F)$. Let $x \notin F$. Then $x \in X - F \in (\zeta, \delta(\mu))O$ and by (2), there exists a $(\zeta, \delta(\mu))$-closed set $K$ such that $x \in K$ and $K \subseteq X - F$. Now, put $U = X - K$. Then $F \subseteq U \in (\zeta, \delta(\mu))O$ and $x \notin U$. Therefore, $x \notin c_{(\zeta, \delta(\mu))}(F)$. This shows that $F = c_{(\zeta, \delta(\mu))}(F)$.

(3) ⇒ (4): Let $x \in X$ and $y \notin c_{(\zeta, \delta(\mu))}(\{x\})$. There exists a $(\zeta, \delta(\mu))$-open set $U$ such that $x \in U$ and $y \notin U$. Hence, $c_{(\zeta, \delta(\mu))}(\{y\}) \cap U = \emptyset$. By (3), $c_{(\zeta, \delta(\mu))}(\{y\}) \cap U = \emptyset$. Since $x \notin c_{(\zeta, \delta(\mu))}(\{y\})$, there exists a $(\zeta, \delta(\mu))$-open set $G$ such that $c_{(\zeta, \delta(\mu))}(\{y\}) \subseteq G$ and $x \notin G$. This implies that $c_{(\zeta, \delta(\mu))}(\{x\}) \cap G = \emptyset$.

Since $y \in G$, we have $y \notin c_{(\zeta, \delta(\mu))}(\{y\})$. Therefore, $c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq \xi_{(\zeta, \delta(\mu))}(\{x\})$. Moreover, $c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq \xi_{(\zeta, \delta(\mu))}(\{x\}) \subseteq \xi_{(\zeta, \delta(\mu))}(\{x\}) = c_{(\zeta, \delta(\mu))}(\{x\})$. Consequently, we obtain $c_{(\zeta, \delta(\mu))}(\{x\}) = \xi_{(\zeta, \delta(\mu))}(\{x\})$.

(4) ⇒ (1): By Corollary 4.14.

**Definition 4.19.** Let $(X, \mu)$ be a strong generalized topological space, $x \in X$ and $(x_a)_{a \in \mathbb{V}}$ be a net in $(X, \mu)$. A net $(x_a)_{a \in \mathbb{V}}$ is called $(\zeta, \delta(\mu))$-converges to $x$, if for each $(\zeta, \delta(\mu))$-open set $U$ containing $x$, there exists $a_0 \in \mathbb{V}$ such that $x_a \in U$ for each $a \geq a_0$. 

\[ (2) \implies (1): \text{Let } U \text{ be any } (\zeta, \delta(\mu))\text{-open set and } x \in U. \text{ If } y \notin U, \text{ then } x \notin c_{(\zeta, \delta(\mu))}(\{y\}) \text{ and hence, } y \notin c_{(\zeta, \delta(\mu))}(\{x\}). \text{ This implies that } c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq U. \text{ Hence, } (X, \mu) \text{ is a } (\zeta, \delta(\mu))\text{- } R_0 \text{ space.} \]
Lemma 4.20. Let \((X, \mu)\) be a strong generalized topological space and let \(x\) and \(y\) be any two points of \(X\) such that every net in \((X, \mu)\) \((\zeta, \delta(\mu))\)-converging to \(y\) \((\zeta, \delta(\mu))\)-converges to \(x\). Then \(x \in c_{(\zeta, \delta(\mu))}(\{y\})\).

Proof. Suppose that \(x_n = y\) for each \(n \in \mathbb{N}\). Then \(\{x_n\}_{n \in \mathbb{N}}\) is a net in \(c_{(\zeta, \delta(\mu))}(\{y\})\). By the fact that \(\{x_n\}_{n \in \mathbb{N}}\) converges to \(y\), \(\{x_n\}_{n \in \mathbb{N}}\) converges to \(x\) and this means that \(x \in c_{(\zeta, \delta(\mu))}(\{y\})\).

Theorem 4.21. For a strong generalized topological space \((X, \mu)\), the following properties are equivalent:

1. \((X, \mu)\) is a \((\zeta, \delta(\mu))\)-R\(_0\) space.

2. If \(x, y \in X\) then \(y \in c_{(\zeta, \delta(\mu))}(\{x\})\) if and only if every net in \((X, \mu)\) \((\zeta, \delta(\mu))\)-converging to \(y\) \((\zeta, \delta(\mu))\)-converges to \(x\).

Proof. (1) \(\Rightarrow\) (2) : Let \(x, y \in X\) such that \(y \in c_{(\zeta, \delta(\mu))}(\{x\})\). Suppose that \(\{x_n\}_{n \in \mathbb{N}}\) is a net in \((X, \mu)\) such that \(\{x_n\}_{n \in \mathbb{N}}\) \((\zeta, \delta(\mu))\)-converges to \(y\). Since \(y \in c_{(\zeta, \delta(\mu))}(\{x\})\), by Theorem 4.16, we have \(c_{(\zeta, \delta(\mu))}(\{x\}) = c_{(\zeta, \delta(\mu))}(\{y\})\). Thus, \(x \in c_{(\zeta, \delta(\mu))}(\{y\})\). This means that \(\{x_n\}_{n \in \mathbb{N}}\) \((\zeta, \delta(\mu))\)-converging to \(y\).

Conversely, let \(x, y \in X\) such that every net in \((X, \mu)\) \((\zeta, \delta(\mu))\)-converging to \(y\) \((\zeta, \delta(\mu))\)-converges to \(x\). Then \(x \in c_{(\zeta, \delta(\mu))}(\{y\})\) by Lemma 4.20. By Theorem 4.16, we have \(c_{(\zeta, \delta(\mu))}(\{x\}) = c_{(\zeta, \delta(\mu))}(\{y\})\). Hence, \(y \in c_{(\zeta, \delta(\mu))}(\{x\})\).

Definition 4.22. A strong generalized topological space \((X, \mu)\) is called a \((\zeta, \delta(\mu))\)-R\(_1\) space if for any points \(x, y \in X\) with \(c_{(\zeta, \delta(\mu))}(\{x\}) \neq c_{(\zeta, \delta(\mu))}(\{y\})\), there exist disjoint \((\zeta, \delta(\mu))\)-open sets \(U\) and \(V\) such that \(c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq U\) and \(c_{(\zeta, \delta(\mu))}(\{y\}) \subseteq V\).

Proposition 4.23. If \((X, \mu)\) is a \((\zeta, \delta(\mu))\)-R\(_1\) space, then \((X, \mu)\) is \((\zeta, \delta(\mu))\)-R\(_0\).

Proof. Let \(U\) be any \((\zeta, \delta(\mu))\)-open set and let \(x \in U\). If \(y \notin U\), then since \(x \notin c_{(\zeta, \delta(\mu))}(\{y\})\), \(c_{(\zeta, \delta(\mu))}(\{x\}) \neq c_{(\zeta, \delta(\mu))}(\{y\})\) and there exists a \((\zeta, \delta(\mu))\)-open set \(V\) such that \(c_{(\zeta, \delta(\mu))}(\{y\}) \subseteq V\) and hence, \(x \notin V\), which implies \(y \notin c_{(\zeta, \delta(\mu))}(\{x\})\). Thus, \(c_{(\zeta, \delta(\mu))}(\{x\}) \cup \{y\} \subseteq U\). Therefore, \((X, \mu)\) is \((\zeta, \delta(\mu))\)-R\(_0\).

Theorem 4.24. For a strong generalized topological space \((X, \mu)\), the following properties are equivalent:

1. \((X, \mu)\) is a \((\zeta, \delta(\mu))\)-R\(_1\) space.

2. For each \(x, y \in X\) one of the following hold:
   a) For any \((\zeta, \delta(\mu))\)-open set \(U\), \(x \in U\) if and only if \(y \in U\).
   b) There exist disjoint \((\zeta, \delta(\mu))\)-open sets \(U\) and \(V\) such that \(x \in U\) and \(y \in V\).

3. For each \(x, y \in X\) such that \(c_{(\zeta, \delta(\mu))}(\{x\}) \neq c_{(\zeta, \delta(\mu))}(\{y\})\), there exist \((\zeta, \delta(\mu))\)-closed sets \(F_x\) and \(F_y\) such that \(x \in F_x\), \(y \notin F_x\), \(y \in F_y\), \(x \notin F_y\) and \(X = F_x \cup F_y\).

Proof. (1) \(\Rightarrow\) (2) : Let \(x, y \in X\). Then (a) \(c_{(\zeta, \delta(\mu))}(\{x\}) = c_{(\zeta, \delta(\mu))}(\{y\})\) or (b) \(c_{(\zeta, \delta(\mu))}(\{x\}) \neq c_{(\zeta, \delta(\mu))}(\{y\})\). If \(c_{(\zeta, \delta(\mu))}(\{x\}) = c_{(\zeta, \delta(\mu))}(\{y\})\) and \(U\) is any \((\zeta, \delta(\mu))\)-open set, then by Proposition 4.23, \(x \in U\) implies
\( y \in c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq U \) and also \( y \in U \) implies \( x \in c_{(\zeta,\delta(\mu))}(\{y\}) \subseteq U \). If \( c_{(\zeta,\delta(\mu))}(\{x\}) \neq c_{(\zeta,\delta(\mu))}(\{y\}) \), then by (1) there exist disjoint \((\zeta, \delta(\mu))\)-open sets \( U \) and \( V \) such that \( x \in c_{(\zeta,\delta(\mu))}(\{x\}) \subseteq U \) and \( y \in c_{(\zeta,\delta(\mu))}(\{y\}) \subseteq V \).

(2) \( \Rightarrow \) (3): Let \( x, y \in X \) such that \( c_{(\zeta, \delta(\mu))}(\{x\}) \neq c_{(\zeta, \delta(\mu))}(\{y\}) \). Then, we have \( x \notin c_{(\zeta, \delta(\mu))}(\{y\}) \) or \( y \notin c_{(\zeta, \delta(\mu))}(\{x\}) \), say \( x \notin c_{(\zeta, \delta(\mu))}(\{y\}) \). Then, there exists a \((\zeta, \delta(\mu))\)-open set \( G \) such that \( x \in G \) and \( y \notin G \). This shows that (b) holds. There exist disjoint \((\zeta, \delta(\mu))\)-open sets \( U \) and \( V \) such that \( x \in U \) and \( y \in V \). Put \( F_x = X - V \) and \( F_y = X - U \). Then \( F_x \) and \( F_y \) are \((\zeta, \delta(\mu))\)-closed sets such that \( x \in F_x, y \notin F_x, y \notin F_y, x \notin F_y \) and \( X = F_x \cup F_y \).

(3) \( \Rightarrow \) (1): First, we shall show that \((X, \mu)\) is a \((\zeta, \delta(\mu))\)-\( R_0 \) space. Let \( U \) be any \((\zeta, \delta(\mu))\)-open set and \( x \in U \). Suppose that \( y \notin U \). Then \( U \cap c_{(\zeta, \delta(\mu))}(\{y\}) = \emptyset \) and hence, \( c_{(\zeta, \delta(\mu))}(\{x\}) \neq c_{(\zeta, \delta(\mu))}(\{y\}) \) since \( x \in U \). By (3), there exist \((\zeta, \delta(\mu))\)-closed sets \( F_x \) and \( F_y \) such that \( x \in F_x, y \notin F_x, y \notin F_y, x \notin F_y \) and \( X = F_x \cup F_y \). Then \( y \in X - F_x, x \notin X - F_x \) and \( X - F_x \) is \((\zeta, \delta(\mu))\)-open. Therefore, we have \( y \notin c_{(\zeta, \delta(\mu))}(\{x\}) \) and we obtain \( c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq U \). This shows that \((X, \mu)\) is a \((\zeta, \delta(\mu))\)-\( R_0 \) space.

Next, we show that \((X, \mu)\) is \((\zeta, \delta(\mu))\)-\( R_1 \). Let \( x, y \in X \) such that \( c_{(\zeta, \delta(\mu))}(\{x\}) \neq c_{(\zeta, \delta(\mu))}(\{y\}) \). By (3), there exist \((\zeta, \delta(\mu))\)-closed sets \( F_x \) and \( F_y \) such that \( x \in F_x, y \notin F_x, y \notin F_y, x \notin F_y \) and \( X = F_x \cup F_y \). Now put \( U = X - F_y \) and \( V = X - F_x \). Then \( x \in U \), \( y \in V \) and \( U \cap V = \emptyset \). Since every \((\zeta, \delta(\mu))\)-\( R_1 \) space is \((\zeta, \delta(\mu))\)-\( R_0 \), \( c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq U \) and \( c_{(\zeta, \delta(\mu))}(\{y\}) \subseteq V \). Therefore, \((X, \mu)\) is a \((\zeta, \delta(\mu))\)-\( R_1 \) space.

**Theorem 4.25.** A strong generalized topological space \((X, \mu)\) is \((\zeta, \delta(\mu))\)-\( R_1 \) if and only if for every pair of points \( x \) and \( y \) of \( X \) such that \( c_{(\zeta, \delta(\mu))}(\{x\}) \neq c_{(\zeta, \delta(\mu))}(\{y\}) \), there exist a \((\zeta, \delta(\mu))\)-open set \( U \) and a \((\zeta, \delta(\mu))\)-open set \( V \) such that \( x \in U \), \( y \in V \) and \( U \cap V = \emptyset \).

**Proof.** Suppose that \((X, \mu)\) is a \((\zeta, \delta(\mu))\)-\( R_1 \) space. Let \( x, y \) be points of \( X \) such that \( c_{(\zeta, \delta(\mu))}(\{x\}) \neq c_{(\zeta, \delta(\mu))}(\{y\}) \). Then, there exist disjoint \((\zeta, \delta(\mu))\)-open sets \( U \) and \( V \) such that \( x \in c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq U \) and \( y \in c_{(\zeta, \delta(\mu))}(\{y\}) \subseteq V \).

Conversely, suppose that there exist a \((\zeta, \delta(\mu))\)-open set \( U \) and a \((\zeta, \delta(\mu))\)-open set \( V \) such that \( x \in U \) and \( y \in V \) and \( U \cap V = \emptyset \). Since every \((\zeta, \delta(\mu))\)-\( R_1 \) space is \((\zeta, \delta(\mu))\)-\( R_0 \), \( c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq U \) and \( c_{(\zeta, \delta(\mu))}(\{y\}) \subseteq V \). Hence, the claim.

**Definition 4.26.** Let \( A \) be a subset of a strong generalized topological space \((X, \mu)\). The \( \theta(\zeta, \delta(\mu)) \)-closure of \( A \), \( c_{\theta(\zeta, \delta(\mu))}(A) \), is defined as follows:

\[
c_{\theta(\zeta, \delta(\mu))}(A) = \{ x \in X | A \cap c_{\theta(\zeta, \delta(\mu))}(U) \neq \emptyset \text{ for each } U \in (\zeta, \delta(\mu)) O \text{ containing } x \}.
\]

A subset \( A \) of a strong generalized topological space \((X, \mu)\) is called \( \theta(\zeta, \delta(\mu)) \)-closed if \( A = c_{\theta(\zeta, \delta(\mu))}(A) \). The complement of a \( \theta(\zeta, \delta(\mu)) \)-closed set is said to be \( \theta(\zeta, \delta(\mu)) \)-open.

In Theorem 4.17, we obtain that a strong generalized topological space \((X, \mu)\) is \((\zeta, \delta(\mu))\)-\( R_0 \) if and only if \( c_{\zeta(\delta(\mu))}(\{x\}) = c_{\zeta(\delta(\mu))}(\{y\}) \) for each \( x \in X \). For a \((\zeta, \delta(\mu))\)-\( R_1 \) space, we have the following theorem.

**Theorem 4.27.** A strong generalized topological space \((X, \mu)\) is \((\zeta, \delta(\mu))\)-\( R_1 \) if and only if \( c_{\zeta(\delta(\mu))}(\{x\}) = c_{\zeta(\delta(\mu))}(\{y\}) \) for each \( x \in X \).

**Proof.** Let \((X, \mu)\) be \((\zeta, \delta(\mu))\)-\( R_2 \), then by Proposition 4.23, it is \((\zeta, \delta(\mu))\)-\( R_0 \) and by Theorem 4.17, \( c_{\zeta(\delta(\mu))}(\{x\}) \subseteq c_{\zeta(\delta(\mu))}(\{y\}) \) for each \( x \in X \). Therefore, \( c_{\zeta(\delta(\mu))}(\{x\}) \subseteq c_{\zeta(\delta(\mu))}(\{y\}) \) for each \( x \in X \). In order to show the opposite inclusion, suppose that \( y \notin c_{\zeta(\delta(\mu))}(\{x\}) \). Then \( c_{\zeta(\delta(\mu))}(\{x\}) \neq c_{\zeta(\delta(\mu))}(\{y\}) \). Since \((X, \mu)\) is \((\zeta, \delta(\mu))\)-\( R_1 \), by Theorem 4.17, \( c_{\zeta(\delta(\mu))}(\{x\}) \neq c_{\zeta(\delta(\mu))}(\{y\}) \). Since \((X, \mu)\) is \((\zeta, \delta(\mu))\)-\( R_1 \), there exist disjoint \( U, V \in (\zeta, \delta(\mu)) O \) such that \( c_{\zeta(\delta(\mu))}(\{x\}) \subseteq U \) and \( c_{\zeta(\delta(\mu))}(\{y\}) \subseteq V \). Since
\{x\} \cap c_{(\zeta, \delta(\mu))}(x) \subseteq \bigcap c_{(\zeta, \delta(\mu))}(\{x\})$, we have $y \notin c_{(\zeta, \delta(\mu))}(\{x\})$. Consequently, we obtain $c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq (\{x\})_{(\zeta, \delta(\mu))}$ and hence, $c_{(\zeta, \delta(\mu))}(\{x\}) = \langle x \rangle_{(\zeta, \delta(\mu))}$.

Conversely, suppose that $c_{(\zeta, \delta(\mu))}(\{x\}) = \langle x \rangle_{(\zeta, \delta(\mu))}$ for each $x \in X$. Then $\langle x \rangle_{(\zeta, \delta(\mu))} = c_{(\zeta, \delta(\mu))}(\{x\}) \supseteq c_{(\zeta, \delta(\mu))}(\{x\}) \supseteq \langle x \rangle_{(\zeta, \delta(\mu))}$ and $\langle x \rangle_{(\zeta, \delta(\mu))} = c_{(\zeta, \delta(\mu))}(\{x\})$

for each $x \in X$. By Theorem 4.17, $(X, \mu)$ is $(\zeta, \delta(\mu))$-$R_0$. Suppose that

$c_{(\zeta, \delta(\mu))}(\{x\}) \neq c_{(\zeta, \delta(\mu))}(\{y\})$.

Then by Corollary 4.11, $c_{(\zeta, \delta(\mu))}(\{x\}) \cap c_{(\zeta, \delta(\mu))}(\{y\}) = \emptyset$ and by Theorem 4.17, we have $\langle x \rangle_{(\zeta, \delta(\mu))} \cap \langle y \rangle_{(\zeta, \delta(\mu))} = \emptyset$. Therefore, $c_{(\zeta, \delta(\mu))}(\{x\}) \cap c_{(\zeta, \delta(\mu))}(\{y\}) = \emptyset$. Since $y \notin c_{(\zeta, \delta(\mu))}(\{x\})$, there exists $U \in (\zeta, \delta(\mu))O$ such that $y \in U \subseteq c_{(\zeta, \delta(\mu))}(U) \subseteq X - \{x\}$. Let $V = X - c_{(\zeta, \delta(\mu))}(U)$, then $x \in V \in (\zeta, \delta(\mu))O$. Since $(X, \mu)$ is $(\zeta, \delta(\mu))$-$R_0$, we obtain $c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq V, c_{(\zeta, \delta(\mu))}(\{y\}) \subseteq U$ and $U \cap V = \emptyset$. This shows that $(X, \mu)$ is $(\zeta, \delta(\mu))$-$R_1$.

**Theorem 4.28.** A strong generalized topological space $(X, \mu)$ is $(\zeta, \delta(\mu))$-$R_1$ if and only if $c_{(\zeta, \delta(\mu))}(\{x\}) = c_{(\zeta, \delta(\mu))}(\{y\})$ for each $x \in X$.

**Proof.** Let $(X, \mu)$ be $(\zeta, \delta(\mu))$-$R_1$. By Theorem 4.27, we have $c_{(\zeta, \delta(\mu))}(\{x\}) \supseteq \langle x \rangle_{(\zeta, \delta(\mu))} = c_{(\zeta, \delta(\mu))}(\{x\}) \supseteq c_{(\zeta, \delta(\mu))}(\{y\})$ and hence, $c_{(\zeta, \delta(\mu))}(\{x\}) = c_{(\zeta, \delta(\mu))}(\{y\})$ for each $x \in X$.

Conversely, suppose that $c_{(\zeta, \delta(\mu))}(\{x\}) = c_{(\zeta, \delta(\mu))}(\{y\})$ for each $x \in X$. First, we show that $(X, \mu)$ is $(\zeta, \delta(\mu))$-$R_0$. Let $U \in (\zeta, \delta(\mu))O$ and $x \in U$. If $y \notin U$, then $U \cap c_{(\zeta, \delta(\mu))}(\{y\}) = U \cap c_{(\zeta, \delta(\mu))}(\{y\}) = \emptyset$. Hence, $x \notin c_{(\zeta, \delta(\mu))}(\{y\})$. There exists $V \in (\zeta, \delta(\mu))O$ such that $x \in V$ and $y \notin c_{(\zeta, \delta(\mu))}(V)$. Since $c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq c_{(\zeta, \delta(\mu))}(V), y \notin c_{(\zeta, \delta(\mu))}(\{x\})$. This shows that $c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq U$ and $(X, \mu)$ is $(\zeta, \delta(\mu))$-$R_0$. By Theorem 4.17, $(X, \mu) = c_{(\zeta, \delta(\mu))}(\{x\}) = c_{(\zeta, \delta(\mu))}(\{x\})$ for each $x \in X$ and by Theorem 4.27, we obtain $(X, \mu)$ is $(\zeta, \delta(\mu))$-$R_2$.

**Definition 4.29.** A strong generalized topological space $(X, \mu)$ is said to be:

(i) $(\zeta, \delta(\mu))$-$T_0$ if for any distinct pair of points in $X$, there exists a $(\zeta, \delta(\mu))$-open set containing one of the points but not the other.

(ii) $(\zeta, \delta(\mu))$-$T_1$ if for any distinct pair of points $x$ and $y$ in $X$, there exist a $(\zeta, \delta(\mu))$-open set $U$ containing $x$ but not $y$ and a $(\zeta, \delta(\mu))$-open set $V$ containing $y$ but not $x$.

(iii) $(\zeta, \delta(\mu))$-$T_2$ if for any distinct pair of points $x$ and $y$ in $X$, there exist $(\zeta, \delta(\mu))$-open sets $U$ and $V$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

**Theorem 4.30.** A strong generalized topological space $(X, \mu)$ is $(\zeta, \delta(\mu))$-$T_0$ if and only if for each pair of distinct points $x$ and $y$ of $X$, $c_{(\zeta, \delta(\mu))}(\{x\}) \neq c_{(\zeta, \delta(\mu))}(\{y\})$.

**Proof.** Suppose that $x, y \in X, x \neq y$ and $c_{(\zeta, \delta(\mu))}(\{x\}) \neq c_{(\zeta, \delta(\mu))}(\{y\})$. Let $z$ be a point of $X$ such that $z \in c_{(\zeta, \delta(\mu))}(\{x\})$ but $z \notin c_{(\zeta, \delta(\mu))}(\{y\})$. We claim that $x \notin c_{(\zeta, \delta(\mu))}(\{y\})$. For, if $x \in c_{(\zeta, \delta(\mu))}(\{y\})$, then $c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq c_{(\zeta, \delta(\mu))}(\{y\})$. And this contradicts the fact that $z \notin c_{(\zeta, \delta(\mu))}(\{y\})$. Consequently, $x$ belongs to the $(\zeta, \delta(\mu))$-open set $X - c_{(\zeta, \delta(\mu))}(\{x\})$ to which $y$ does not belong.

Conversely, let $(X, \mu)$ be a $(\zeta, \delta(\mu))$-$T_0$ space and $x, y$ be any two distinct points of $X$. There exists a $(\zeta, \delta(\mu))$-open set $V$ containing $x$ or $y$, say $x$ but not $y$. Then $X - V$ is a $(\zeta, \delta(\mu))$-closed set which does not contain $x$ but contains $y$. Since $c_{(\zeta, \delta(\mu))}(\{y\})$ is the smallest $(\zeta, \delta(\mu))$-closed set containing $y$, $c_{(\zeta, \delta(\mu))}(\{y\}) \subseteq X - V$ and so $x \notin c_{(\zeta, \delta(\mu))}(\{y\})$. Consequently, we obtain $c_{(\zeta, \delta(\mu))}(\{x\}) \neq c_{(\zeta, \delta(\mu))}(\{y\})$. 


Theorem 4.31. A strong generalized topological space $(X, \mu)$ is $(\zeta, \delta(\mu))$-$T_1$ if and only if the singletons are $(\zeta, \delta(\mu))$-closed sets.

Proof. Suppose that $(X, \mu)$ is $(\zeta, \delta(\mu))$-$T_1$ and $x$ be any point of $X$. Let $y \in X - \{x\}$. Then $x \neq y$ and so there exists a $(\zeta, \delta(\mu))$-open set $V_x$ such that $y \in V_x$ but $x \notin V_x$. Consequently, $y \in V_x \subseteq X - \{x\}$, i.e., $X - \{x\} = \cup \{V_x | y \in X - \{x\}\}$ which is $(\zeta, \delta(\mu))$-open.

Conversely, suppose that $\{z\}$ is $(\zeta, \delta(\mu))$-closed for every $z \in X$. Let $x, y \in X$ such that $x \neq y$. Now, $x \neq y$ implies $y \in X - \{x\}$. Hence, $X - \{x\}$ is a $(\zeta, \delta(\mu))$-open set containing $y$ but not $x$. Similarly, $X - \{y\}$ is a $(\zeta, \delta(\mu))$-open set containing $x$ but not $y$. Therefore, $(X, \mu)$ is a $(\zeta, \delta(\mu))$-$T_1$ space.

Theorem 4.32. For a strong generalized topological space $(X, \mu)$, the following properties are equivalent:

1. $(X, \mu)$ is $(\zeta, \delta(\mu))$-$T_1$.

2. For any $x \in X$, $\{x\}$ is $(\zeta, \delta(\mu))$-closed.

3. $(X, \mu)$ is $(\zeta, \delta(\mu))$-$R_0$ and $(\zeta, \delta(\mu))$-$T_0$.

Proof. (1) $\Rightarrow$ (2): Let $x$ be any point of $X$. Let $y$ be any point of $X$ such that $y \neq x$. There exists a $(\zeta, \delta(\mu))$-open set $V$ such that $y \in V$ and $x \notin V$. This implies that $y \notin c_{(\zeta, \delta(\mu))}(\{x\})$. Consequently, we obtain $c_{(\zeta, \delta(\mu))}(\{x\}) = \{x\}$ and hence, $\{x\}$ is $(\zeta, \delta(\mu))$-closed.

(2) $\Rightarrow$ (3): The proof is obvious.

(3) $\Rightarrow$ (1): Let $x$ and $y$ be any distinct points of $X$. Since $(X, \mu)$ is a $(\zeta, \delta(\mu))$-$T_1$ space, there exists a $(\zeta, \delta(\mu))$-open set $V$ such that either $x \in V$ and $y \notin V$ or $x \notin V$ and $y \in V$. In case $x \in V$ and $y \notin V$, we have $x \in c_{(\zeta, \delta(\mu))}(\{x\}) \subseteq V$ and hence, $y \in X - V \subseteq X - c_{(\zeta, \delta(\mu))}(\{x\})$. Since the proof of the other is quite similar, $(X, \mu)$ is a $(\zeta, \delta(\mu))$-$T_1$ space.

Theorem 4.33. For a strong generalized topological space $(X, \mu)$, the following properties are equivalent:

1. $(X, \mu)$ is $(\zeta, \delta(\mu))$-$T_2$.

2. $(X, \mu)$ is $(\zeta, \delta(\mu))$-$R_1$ and $(\zeta, \delta(\mu))$-$T_1$.

3. $(X, \mu)$ is $(\zeta, \delta(\mu))$-$R_1$ and $(\zeta, \delta(\mu))$-$T_0$.

Proof. (1) $\Rightarrow$ (2): Since $(X, \mu)$ is $(\zeta, \delta(\mu))$-$T_2$, $(X, \mu)$ is $(\zeta, \delta(\mu))$-$T_1$. Let $x$ and $y$ be any points of $X$ such that $c_{(\zeta, \delta(\mu))}(\{x\}) \neq c_{(\zeta, \delta(\mu))}(\{y\})$. Then, by Theorem 4.32, $\{x\} = c_{(\zeta, \delta(\mu))}(\{x\}) \neq c_{(\zeta, \delta(\mu))}(\{y\}) = \{y\}$ and there exist disjoint $(\zeta, \delta(\mu))$-open sets $U$ and $V$ such that $c_{(\zeta, \delta(\mu))}(\{x\}) = \{x\} \subseteq U$ and $c_{(\zeta, \delta(\mu))}(\{y\}) = \{y\} \subseteq V$. This shows that $(X, \mu)$ is a $(\zeta, \delta(\mu))$-$R_1$ space.

(2) $\Rightarrow$ (3): The proof is obvious.

(3) $\Rightarrow$ (1): Let $(X, \mu)$ be $(\zeta, \delta(\mu))$-$R_1$ and $(\zeta, \delta(\mu))$-$T_0$. By Proposition 4.23 and Theorem 4.32, $(X, \mu)$ is a $(\zeta, \delta(\mu))$-$T_1$ space and every singleton is $(\zeta, \delta(\mu))$-closed. Let $x$ and $y$ be distinct points of $X$. Then, we have $c_{(\zeta, \delta(\mu))}(\{x\}) = \{x\} \neq \{y\} = c_{(\zeta, \delta(\mu))}(\{y\})$ and there exist disjoint $(\zeta, \delta(\mu))$-open sets $U$ and $V$ such that $x \in U$ and $y \in V$. This shows that $(X, \mu)$ is a $(\zeta, \delta(\mu))$-$T_2$ space.
Proposition 4.34. If a strong generalized topological space \((X, \mu)\) is \((\zeta, \delta(\mu))\)-symmetric, then \((X, \mu)\) is \((\zeta, \delta(\mu))\)-symmetric.

Proof. In a \((\zeta, \delta(\mu))\)-space singleton sets are \((\zeta, \delta(\mu))\)-closed by Theorem 4.31 and hence, \(g-(\zeta, \delta(\mu))\)-closed by Proposition 3.25. By Theorem 3.22, the space is \((\zeta, \delta(\mu))\)-symmetric.

Theorem 4.35. For a strong generalized topological space \((X, \mu)\), the following properties are equivalent:

1. \((X, \mu)\) is \((\zeta, \delta(\mu))\)-symmetric and \((\zeta, \delta(\mu))\)-\(T_1\).

2. \((X, \mu)\) is \((\zeta, \delta(\mu))\)-\(T_1\).

Proof. (1) \(\Rightarrow\) (2): Suppose that \((X, \mu)\) is \((\zeta, \delta(\mu))\)-symmetric and \((\zeta, \delta(\mu))\)-\(T_0\). Let \(x, y \in X\) such that \(x \neq y\) and by \((\zeta, \delta(\mu))\)-\(T_0\), we may assume that \(x \in U \subseteq X - \{y\}\) for some \(U \subseteq (\zeta, \delta(\mu))\)O. Then \(x \notin \epsilon_{\zeta(\delta(\mu))}(\{y\})\). Therefore, we have \(y \notin \epsilon_{\zeta(\delta(\mu))}(\{x\})\). There exists a \((\zeta, \delta(\mu))\)-open set \(V\) such that \(y \in V \subseteq X - \{x\}\). Consequently, we obtain \((X, \mu)\) is a \((\zeta, \delta(\mu))\)-\(T_1\) space.

(2) \(\Rightarrow\) (1): Suppose that \((X, \mu)\) is \((\zeta, \delta(\mu))\)-\(T_1\). Since \((X, \mu)\) is \((\zeta, \delta(\mu))\)-\(T_1\), \((X, \mu)\) is \((\zeta, \delta(\mu))\)-\(T_0\) and by Proposition 4.34, we have \((X, \mu)\) is \((\zeta, \delta(\mu))\)-symmetric.

Definition 4.36. A subset \(A\) of a strong generalized topological space \((X, \mu)\) is called \(\zeta_{\zeta(\delta(\mu))}\)-set if \(A = \epsilon_{\zeta(\delta(\mu))}(A)\). The family of all \(\zeta_{\zeta(\delta(\mu))}\)-sets of \((X, \mu)\) is denoted by \(\epsilon_{\zeta(\delta(\mu))}(X, \mu)\).

Definition 4.37. A subset \(A\) of a strong generalized topological space \((X, \mu)\) is called a generalized \(\zeta_{\zeta(\delta(\mu))}\)-set (briefly \(g-\zeta_{\zeta(\delta(\mu))}\)-set) if \(\epsilon_{\zeta(\delta(\mu))}(A) \subseteq F\) whenever \(A \subseteq F\) and \(F\) is \((\zeta, \delta(\mu))\)-closed.

Definition 4.38. A strong generalized topological space \((X, \mu)\) is called a \((\zeta, \delta(\mu))\)-\(T_2\)-space if every \(g-(\zeta, \delta(\mu))\)-closed set of \((X, \mu)\) is \((\zeta, \delta(\mu))\)-closed.

Lemma 4.39. For a strong generalized topological space \((X, \mu)\), the following properties hold:

1. For each \(x \in X\), the singleton \(\{x\}\) is \((\zeta, \delta(\mu))\)-closed or \(X - \{x\}\) is \(g-(\zeta, \delta(\mu))\)-closed.

2. For each \(x \in X\), the singleton \(\{x\}\) is \((\zeta, \delta(\mu))\)-open or \(X - \{x\}\) is a \(g-\zeta_{\zeta(\delta(\mu))}\)-set.

Proof. (1) Let \(x \in X\) and the singleton \(\{x\}\) be not \((\zeta, \delta(\mu))\)-closed. Then \(X - \{x\}\) is not \((\zeta, \delta(\mu))\)-open and \(X\) is the only \((\zeta, \delta(\mu))\)-open set which contains \(X - \{x\}\) and hence, \(X - \{x\}\) is \(g-(\zeta, \delta(\mu))\)-closed.

(2) Let \(x \in X\) and the singleton \(\{x\}\) be not \((\zeta, \delta(\mu))\)-open. Then \(X - \{x\}\) is not \((\zeta, \delta(\mu))\)-closed and the only \((\zeta, \delta(\mu))\)-closed set which contains \(X - \{x\}\) is \(X\) and hence, \(X - \{x\}\) is a \(g-\zeta_{\zeta(\delta(\mu))}\)-set.

Theorem 4.40. For a strong generalized topological space \((X, \mu)\), the following properties are equivalent:

1. \((X, \mu)\) is a \((\zeta, \delta(\mu))\)-\(T_2\)-space.

2. For each \(x \in X\), the singleton \(\{x\}\) is \((\zeta, \delta(\mu))\)-open or \((\zeta, \delta(\mu))\)-closed.

3. Every \(g-\zeta_{\zeta(\delta(\mu))}\)-set is a \(\zeta_{\zeta(\delta(\mu))}\)-set.
Proof. (1) ⇒ (2) : By Lemma 4.39, for each \( x \in X \), the singleton \( \{ x \} \) is \((\zeta, \delta(\mu))\)-closed or \( X - \{ x \} \) is \( g-(\zeta, \delta(\mu))\)-closed. Since \((X, \mu)\) is a \((\zeta, \delta(\mu))\)-\(T_2\)-space, \( X - \{ x \} \) is \((\zeta, \delta(\mu))\)-closed and hence, \( \{ x \} \) is \((\zeta, \delta(\mu))\)-open in the latter case. Therefore, the singleton \( \{ x \} \) is \((\zeta, \delta(\mu))\)-open or \((\zeta, \delta(\mu))\)-closed.

(2) ⇒ (3) : Suppose that there exists a \( g-(\zeta, \delta(\mu))\)-set \( A \) which is not a \((\zeta, \delta(\mu))\)-set. There exists \( x \in \zeta(A) \) such that \( x \notin A \). In case the singleton \( \{ x \} \) is \((\zeta, \delta(\mu))\)-open, \( A \subseteq X - \{ x \} \) and \( X - \{ x \} \) is \((\zeta, \delta(\mu))\)-closed. Since \( A \) is a \( g-(\zeta, \delta(\mu))\)-set, \( \zeta(A) \subseteq X - \{ x \} \). This is a contradiction. In case the singleton \( \{ x \} \) is \((\zeta, \delta(\mu))\)-closed, \( A \subseteq X - \{ x \} \) and \( X - \{ x \} \) is \((\zeta, \delta(\mu))\)-open. By Lemma 4.2, \( \zeta(A) \subseteq \zeta(X - \{ x \}) = X - \{ x \} \). This is a contradiction. Therefore, every \( g-(\zeta, \delta(\mu))\)-set is a \((\zeta, \delta(\mu))\)-set.

(3) ⇒ (1) : Suppose that \((X, \mu)\) is not a \((\zeta, \delta(\mu))\)-\(T_2\)-space. Then, there exists a \( g-(\zeta, \delta(\mu))\)-closed set \( A \) which is not \((\zeta, \delta(\mu))\)-closed. Since \( A \) is not \((\zeta, \delta(\mu))\)-closed, there exists a point \( x \in c_{\zeta}(A) \) such that \( x \notin A \). By Lemma 4.39, the singleton \( \{ x \} \) is \((\zeta, \delta(\mu))\)-open or \( X - \{ x \} \) is a \((\zeta, \delta(\mu))\)-set. (a) In case \( \{ x \} \) is \((\zeta, \delta(\mu))\)-open, since \( x \in c_{\zeta}(A) \), \( \{ x \} \cap A \neq \emptyset \) and \( x \in A \). This is a contradiction. (b) In case \( X - \{ x \} \) is a \((\zeta, \delta(\mu))\)-set, \( x \) is not \((\zeta, \delta(\mu))\)-closed, \( X - \{ x \} \) is not \((\zeta, \delta(\mu))\)-open and \( \zeta(X - \{ x \}) = X \). Hence, \( X - \{ x \} \) is not a \((\zeta, \delta(\mu))\)-set. This contradicts (3). If \( x \) is \((\zeta, \delta(\mu))\)-closed, \( A \subseteq X - \{ x \} \subseteq (\zeta, \delta(\mu))O \) and \( A \) is \( g-(\zeta, \delta(\mu))\)-closed. Hence, we have \( c_{\zeta}(A) \subseteq X - \{ x \} \). This contradicts that \( x \in c_{\zeta}(A) \). Therefore, \((X, \mu)\) is a \((\zeta, \delta(\mu))\)-\(T_2\)-space.

Definition 4.41. A strong generalized topological space \((X, \mu)\) is said to be \((\zeta, \delta(\mu))\)-regular if for each \((\zeta, \delta(\mu))\)-closed set \( F \) not containing \( x \), there exist disjoint \((\zeta, \delta(\mu))\)-open sets \( U \) and \( V \) such that \( x \in U \) and \( F \subseteq V \).

Theorem 4.42. For a strong generalized topological space \((X, \mu)\), the following properties are equivalent:

1. \((X, \mu)\) is \((\zeta, \delta(\mu))\)-regular.
2. For each \( x \in X \) and each \( U \in (\zeta, \delta(\mu))O \) such that \( x \in U \), there exists \( V \in (\zeta, \delta(\mu))O \) such that \( x \in V \subseteq c_{\zeta}(V) \subseteq U \).
3. For each \((\zeta, \delta(\mu))\)-closed set \( F \), \( \cap(c_{\zeta}(V) \mid V \subseteq \zeta(F)) = F \).
4. For each subset \( A \) of \( X \) and each \( U \in (\zeta, \delta(\mu))O \) such that \( A \cap U \neq \emptyset \), there exists \( V \in (\zeta, \delta(\mu))O \) such that \( A \cap V \neq \emptyset \) and \( c_{\zeta}(V) \cap V \subseteq U \).
5. For each non-empty subset \( A \) of \( X \) and each \((\zeta, \delta(\mu))\)-closed set \( F \) such that \( A \cap F = \emptyset \), there exist \( V, W \in (\zeta, \delta(\mu))O \) such that \( A \cap V \neq \emptyset \), \( F \subseteq W \) and \( V \cap W = \emptyset \).
6. For each \((\zeta, \delta(\mu))\)-closed set \( F \) and \( x \notin F \), there exist \( U \in (\zeta, \delta(\mu))O \) and a \( g-(\zeta, \delta(\mu))\)-open set \( V \) such that \( x \in U \), \( F \subseteq V \) and \( U \cap V = \emptyset \).
7. For each subset \( A \) of \( X \) and each \((\zeta, \delta(\mu))\)-closed set \( F \) such that \( A \cap F = \emptyset \), there exist \( U \in (\zeta, \delta(\mu))O \) and a \( g-(\zeta, \delta(\mu))\)-open set \( V \) such that \( A \cap U \neq \emptyset \), \( F \subseteq V \) and \( U \cap V = \emptyset \).

Proof. (1) ⇒ (2) : Let \( G \in (\zeta, \delta(\mu))O \) and \( x \in G \). Then \( x \in X - G \), there exist disjoint \( U, V \in (\zeta, \delta(\mu))O \) such that \( X - G \subseteq U \) and \( x \in V \). Thus, \( V \subseteq X - U \) and so \( x \in V \subseteq c_{\zeta}(V) \subseteq X - U \subseteq G \).

(2) ⇒ (3) : For any \( F \in (\zeta, \delta(\mu))O \), we always have
\[ F \subseteq \cap(c_{\zeta}(V) \mid V \subseteq (\zeta, \delta(\mu))O) \].
On the other hand, let \( F \in (\zeta, \delta(\mu))C \) such that \( x \in X - F \). Then by \( (2) \), there exists \( U \in (\zeta, \delta(\mu))O \) such that \( x \in U \subseteq c_{(\zeta, \delta(\mu))}(U) \subseteq X - F \). Therefore, \( F \subseteq X - c_{(\zeta, \delta(\mu))}(U) = V \in (\zeta, \delta(\mu))O \) and \( U \cap V = \emptyset \). Then \( x \notin c_{(\zeta, \delta(\mu))}(V) \). Thus, \( F \supseteq \cap c_{(\zeta, \delta(\mu))}(V) \subseteq V \in (\zeta, \delta(\mu))O \).

\[ (3) \Rightarrow (4) \text{: Let } A \text{ be a subset of } X \text{ and } U \in (\zeta, \delta(\mu))O \text{ such that } A \cap U \neq \emptyset \text{. Let } x \in A \cap U \text{. Then } x \in X - U \text{. Hence by } (3) \text{, there exists } W \in (\zeta, \delta(\mu))O \text{ such that } X - U \subseteq W \text{ and } x \notin c_{(\zeta, \delta(\mu))}(W) \text{. Put } V = X - c_{(\zeta, \delta(\mu))}(W) \text{ which is a } (\zeta, \delta(\mu))\text{-open set containing } x \text{ and } A \cap V \neq \emptyset \text{. Now, } V \subseteq X - W \text{ and hence, } c_{(\zeta, \delta(\mu))}(V) \subseteq X - W \subseteq U. \]

\[ (4) \Rightarrow (5) \text{: Let } A \text{ be a non-empty subset of } X \text{ and } F \text{ be a } (\zeta, \delta(\mu))\text{-closed set such that } A \cap F = \emptyset \text{. Then } X - F \in (\zeta, \delta(\mu))O \text{ such that } A \cap (X - F) \neq \emptyset \text{ and by } (4) \text{, there exists } V \in (\zeta, \delta(\mu))O \text{ such that } A \cap V \neq \emptyset \text{ and } c_{(\zeta, \delta(\mu))}(V) \subseteq X - F \text{. Put } W = X - c_{(\zeta, \delta(\mu))}(V) \text{ then } W \text{ is a } (\zeta, \delta(\mu))\text{-open set such that } F \subseteq W \text{ and } W \cap V = \emptyset. \]

\[ (5) \Rightarrow (1) \text{: Let } F \text{ be any } (\zeta, \delta(\mu))\text{-closed set not containing } x \text{. Then } F \cap \{x\} = \emptyset \text{. Thus by } (5) \text{, there exists } V, W \in (\zeta, \delta(\mu))O \text{ such that } x \in V, F \subseteq W \text{ and } V \cap W = \emptyset. \]

\[ (1) \Rightarrow (6) \text{: The proof is obvious.} \]

\[ (6) \Rightarrow (7) \text{: Let } A \text{ be a subset of } X \text{ and } F \text{ be a } (\zeta, \delta(\mu))\text{-closed set such that } A \cap F = \emptyset \text{. Then, for } x \in A \text{, } x \notin F \text{ and hence by } (6) \text{, there exist } U \in (\zeta, \delta(\mu))O \text{ and a } g-(\zeta, \delta(\mu))\text{-open set } V \text{ such that } x \in U \text{, } F \subseteq V \text{ and } U \cap V = \emptyset \text{. Consequently, we obtain } A \cap U \neq \emptyset, F \subseteq V \text{ and } U \cap V = \emptyset \text{.} \]

\[ (7) \Rightarrow (1) \text{: Let } F \text{ be any } (\zeta, \delta(\mu))\text{-closed set such that } x \notin F \text{. Since } \{x\} \cap F = \emptyset \text{, by } (7) \text{ there exist } U \in (\zeta, \delta(\mu))O \text{ and a } g-(\zeta, \delta(\mu))\text{-open set } W \text{ such that } x \in U \text{, } F \subseteq W \text{ and } U \cap W = \emptyset \text{. Since } W \text{ is } g-(\zeta, \delta(\mu))\text{-open, by Theorem 3.29, we have } F \subseteq c_{(\zeta, \delta(\mu))}(W) = V \in (\zeta, \delta(\mu))O \text{ and hence, } U \cap V = \emptyset. \]

**Definition 4.43.** A strong generalized topological space \((X, \mu)\) is said to be \((\zeta, \delta(\mu))\)-normal if for any pair of disjoint \((\zeta, \delta(\mu))\)-closed sets \(F \) and \(H \), there exist disjoint \((\zeta, \delta(\mu))\)-open sets \(U \) and \(V \) such that \( F \subseteq U \) and \( H \subseteq V \).

**Theorem 4.44** For a strong generalized topological space \((X, \mu)\), the following properties are equivalent:

1. \((X, \mu)\) is \((\zeta, \delta(\mu))\)-normal.
2. For every pair of \((\zeta, \delta(\mu))\)-open sets \(U \) and \(V \) whose union is \(X \), there exist \((\zeta, \delta(\mu))\)-closed sets \(F \) and \(H \) such that \( F \subseteq U \), \( H \subseteq V \) and \( F \cup H = X \).
3. For every \((\zeta, \delta(\mu))\)-closed set \(F \) and every \((\zeta, \delta(\mu))\)-open set \(G \) containing \(F \), there exists a \((\zeta, \delta(\mu))\)-open set \(U \) such that \( F \subseteq U \subseteq c_{(\zeta, \delta(\mu))}(U) \subseteq G \).
4. For every pair of disjoint \((\zeta, \delta(\mu))\)-closed sets \(F \) and \(H \), there exist \((\zeta, \delta(\mu))\)-open sets \(U \) and \(V \) such that \( F \subseteq U \), \( H \subseteq V \) and \( c_{(\zeta, \delta(\mu))}(U) \cap c_{(\zeta, \delta(\mu))}(V) = \emptyset \).

**Proof.** \( (1) \Rightarrow (2) \): Let \( U \) and \( V \) be a pair of \((\zeta, \delta(\mu))\)-open sets in a \((\zeta, \delta(\mu))\)-normal space \((X, \mu)\) such that \( X = U \cup V \). Then \( X - U \) and \( X - V \) are disjoint \((\zeta, \delta(\mu))\)-closed sets. Since \((X, \mu)\) is \((\zeta, \delta(\mu))\)-normal, there exist disjoint \((\zeta, \delta(\mu))\)-open sets \(G \) and \(W \) such that \( X - U \subseteq G \) and \( X - V \subseteq W \). Put \( F = X - G \) and \( H = X - W \). Then \( F \) and \( H \) are \((\zeta, \delta(\mu))\)-closed sets such that \( F \subseteq U \), \( H \subseteq V \) and \( F \cup H = X \).

\( (2) \Rightarrow (3) \): Let \( F \) be a \((\zeta, \delta(\mu))\)-closed set and \( G \) be a \((\zeta, \delta(\mu))\)-open set containing \(F \). Then \( X - F \) and \( G \) are \((\zeta, \delta(\mu))\)-open sets whose union is \(X \). Then by \( (2) \), there exist \((\zeta, \delta(\mu))\)-closed sets \(M \) and \(N \) such that \( M \subseteq X - F \), \( N \subseteq G \) and \( M \cup N = X \). Then \( F \subseteq X - M \), \( X - G \subseteq X - N \) and \( (X - M) \cap (X - N) = \emptyset \).
Put $U = X - M$ and $V = X - N$. Then $U$ and $V$ are disjoint $(\zeta, \delta(\mu))$-open sets such that $F \subseteq U \subseteq X - V \subseteq G$. As $X - V$ is a $(\zeta, \delta(\mu))$-closed set, we have $c_{(\zeta, \delta(\mu))}(U) \subseteq X - V$ and $F \subseteq U \subseteq c_{(\zeta, \delta(\mu))}(U) \subseteq G$.

$(3) \implies (4)$: Let $F$ and $H$ be two disjoint $(\zeta, \delta(\mu))$-closed sets. Then $F \subseteq X - H$ and $X - H$ is $(\zeta, \delta(\mu))$-open. By $(3)$, there exists a $(\zeta, \delta(\mu))$-open set $U$ such that $F \subseteq U \subseteq c_{(\zeta, \delta(\mu))}(U) \subseteq X - H$. Again, since $X - H$ is a $(\zeta, \delta(\mu))$-open set containing the $(\zeta, \delta(\mu))$-closed set $c_{(\zeta, \delta(\mu))}(U)$, there exists a $(\zeta, \delta(\mu))$-open set $W$ such that $F \subseteq U \subseteq c_{(\zeta, \delta(\mu))}(U) \subseteq W \subseteq c_{(\zeta, \delta(\mu))}(W) \subseteq X - H$. Put $V = X - c_{(\zeta, \delta(\mu))}(W)$. Then $V$ is $(\zeta, \delta(\mu))$-open and $H \subseteq V$. Since $X - c_{(\zeta, \delta(\mu))}(W) \subseteq X - W$, we have $V \subseteq X - W$ and $c_{(\zeta, \delta(\mu))}(V) \subseteq c_{(\zeta, \delta(\mu))}(X - W) = X - W$. Therefore, $c_{(\zeta, \delta(\mu))}(U) \cap c_{(\zeta, \delta(\mu))}(V) = \emptyset$.

$(4) \implies (1)$: The proof is obvious.

**Theorem 4.45.** For a strong generalized topological space $(X, \mu)$, the following properties are equivalent:

1. $(X, \mu)$ is $(\zeta, \delta(\mu))$-normal.
2. For every pair of disjoint $(\zeta, \delta(\mu))$-closed sets $F$ and $H$, there exist disjoint $g$-$(\zeta, \delta(\mu))$-open sets $U$ and $V$ such that $F \subseteq U$ and $H \subseteq V$.
3. For each $(\zeta, \delta(\mu))$-closed set $F$ and each $(\zeta, \delta(\mu))$-open set $G$ containing $F$, there exists a $g$-$(\zeta, \delta(\mu))$-open set $U$ such that $F \subseteq U \subseteq c_{(\zeta, \delta(\mu))}(U) \subseteq G$.
4. For each $(\zeta, \delta(\mu))$-closed set $F$ and each $g$-$(\zeta, \delta(\mu))$-open set $G$ containing $F$, there exists a $(\zeta, \delta(\mu))$-open set $U$ such that $F \subseteq U \subseteq c_{(\zeta, \delta(\mu))}(U) \subseteq G$.
5. For each $(\zeta, \delta(\mu))$-closed set $F$ and each $g$-$(\zeta, \delta(\mu))$-open set $G$ containing $F$, there exists a $g$-$(\zeta, \delta(\mu))$-open set $U$ such that $F \subseteq U \subseteq c_{(\zeta, \delta(\mu))}(U) \subseteq G$.
6. For each $g$-$(\zeta, \delta(\mu))$-closed set $F$ and each $(\zeta, \delta(\mu))$-open set $G$ containing $F$, there exists a $(\zeta, \delta(\mu))$-open set $U$ such that $c_{(\zeta, \delta(\mu))}(F) \subseteq U \subseteq c_{(\zeta, \delta(\mu))}(U) \subseteq G$.
7. For each $g$-$(\zeta, \delta(\mu))$-closed set $F$ and each $(\zeta, \delta(\mu))$-open set $G$ containing $F$, there exists a $g$-$(\zeta, \delta(\mu))$-open set $U$ such that $c_{(\zeta, \delta(\mu))}(F) \subseteq U \subseteq c_{(\zeta, \delta(\mu))}(U) \subseteq G$.

**Proof.**

$(1) \implies (2)$: The proof is obvious.

$(2) \implies (3)$: Let $F$ be a $(\zeta, \delta(\mu))$-closed set and $G$ be a $(\zeta, \delta(\mu))$-open set containing $F$. Then $F$ and $X - G$ are two disjoint $(\zeta, \delta(\mu))$-closed sets. Hence by $(2)$, there exist disjoint $g$-$(\zeta, \delta(\mu))$-open sets $U$ and $V$ such that $F \subseteq U$ and $X - G \subseteq V$. Since $V$ is $g$-$(\zeta, \delta(\mu))$-open and $X - G$ is $(\zeta, \delta(\mu))$-closed, by Theorem 3.29, $X - G \subseteq i_{(\zeta, \delta(\mu))}(V)$. Since $U \cap V = \emptyset$, we have $c_{(\zeta, \delta(\mu))}(U) \subseteq c_{(\zeta, \delta(\mu))}(X - V) = X - i_{(\zeta, \delta(\mu))}(V) \subseteq G$. Thus, $F \subseteq U \subseteq c_{(\zeta, \delta(\mu))}(U) \subseteq G$.

$(3) \implies (1)$: Let $F$ and $H$ be two disjoint $(\zeta, \delta(\mu))$-closed sets. Then $X - H$ is a $(\zeta, \delta(\mu))$-open set containing $F$. Thus by $(3)$, there exists a $g$-$(\zeta, \delta(\mu))$-open set $U$ such that $F \subseteq U \subseteq c_{(\zeta, \delta(\mu))}(U) \subseteq X - H$. Therefore, $H \subseteq X - c_{(\zeta, \delta(\mu))}(U)$. Since $F$ is a $(\zeta, \delta(\mu))$-closed set and $U$ is a $g$-$(\zeta, \delta(\mu))$-open set, by Theorem 3.29, we have $F \subseteq i_{(\zeta, \delta(\mu))}(U)$. Consequently, we obtain $(X - c_{(\zeta, \delta(\mu))}(U)) \cap i_{(\zeta, \delta(\mu))}(U) = \emptyset$. This shows that $(X, \mu)$ is $(\zeta, \delta(\mu))$-normal.

$(4) \implies (5) \implies (2)$: This is obvious.

$(6) \implies (7) \implies (3)$: This is obvious.
(3) ⇒ (5): Let \( F \) be a \((\zeta, \delta(\mu))-\)closed set and \( G \) be a \(g-\)(\(\zeta, \delta(\mu))-\)open set containing \( F \). Since \( G \) is \(g-\)(\(\zeta, \delta(\mu))-\)open and \( F \) is \((\zeta, \delta(\mu))-\)closed, by Theorem 3.29, \( F \subseteq i_{(\zeta, \delta(\mu))}(G) \). Thus by (3), there exists a \(g-\)(\(\zeta, \delta(\mu))-\)open set \( U \) such that \( F \subseteq U \subseteq c_{(\zeta, \delta(\mu))}(U) \subseteq i_{(\zeta, \delta(\mu))}(G) \).

(5) ⇒ (6): Let \( F \) be a \(g-\)(\(\zeta, \delta(\mu))-\)closed set and \( G \) be a \((\zeta, \delta(\mu))-\)open set containing \( F \). Then \( c_{(\zeta, \delta(\mu))}(F) \subseteq X \). Since \( G \) is \(g-\)(\(\zeta, \delta(\mu))-\)open, there exists a \(g-\)(\(\zeta, \delta(\mu))-\)open set \( U \) such that \( c_{\delta(\mu)}(F) \subseteq U \subseteq c_{(\zeta, \delta(\mu))}(U) \subseteq G \). Since \( U \) is \(g-\)(\(\zeta, \delta(\mu))-\)open and \( i_{(\zeta, \delta(\mu))}(F) \) is \((\zeta, \delta(\mu))-\)closed, by Theorem 3.29, we have \( c_{(\zeta, \delta(\mu))}(F) \subseteq i_{(\zeta, \delta(\mu))}(U) \). Put \( V = i_{(\zeta, \delta(\mu))}(U) \). Then, we have \( V \) is \((\zeta, \delta(\mu))-\)open and \( c_{(\zeta, \delta(\mu))}(F) \subseteq V \subseteq c_{(\zeta, \delta(\mu))}(U) \subseteq i_{(\zeta, \delta(\mu))}(G) \).

(6) ⇒ (4): Let \( F \) be a \((\zeta, \delta(\mu))-\)closed set and \( G \) be a \(g-\)(\(\zeta, \delta(\mu))-\)open set containing \( F \). Then by Theorem 3.29, we have \( F \subseteq i_{(\zeta, \delta(\mu))}(G) \). Since \( F \) is \(g-\)(\(\zeta, \delta(\mu))-\)closed and \( i_{(\zeta, \delta(\mu))}(G) \) is \((\zeta, \delta(\mu))-\)open, by (6) there exists a \((\zeta, \delta(\mu))-\)open set \( U \) such that \( F \subseteq c_{(\zeta, \delta(\mu))}(F) \subseteq U \subseteq c_{(\zeta, \delta(\mu))}(U) \subseteq i_{(\zeta, \delta(\mu))}(G) \).

5. Characterizations of weakly \((\zeta, \delta(\mu))-\)continuous functions

We begin this section by introducing the notion of weakly \((\zeta, \delta(\mu))-\)continuous functions.

Definition 5.1. A function \( f : (X, \mu) \to (Y, \mu') \) is said to be weakly \((\zeta, \delta(\mu))-\)continuous at a point \( x \in X \) if for each \((\zeta, \delta(\mu))-\)open set \( V \) containing \( f(x) \), there exists a \((\zeta, \delta(\mu))-\)open set \( U \) containing \( x \) such that \( f(U) \subseteq c_{(\zeta, \delta(\mu))}(V) \). A function \( f : (X, \mu) \to (Y, \mu') \) is said to be weakly \((\zeta, \delta(\mu))-\)continuous if it has this property at each point \( x \in X \).

Theorem 5.2. A function \( f : (X, \mu) \to (Y, \mu') \) is weakly \((\zeta, \delta(\mu))-\)continuous at \( x \in X \) if and only if for each \((\zeta, \delta(\mu))-\)open set \( V \) containing \( f(x) \), \( x \in i_{(\zeta, \delta(\mu))}(f^{-1}(c_{(\zeta, \delta(\mu))}(V))) \).

Proof. Let \( V \) be any \((\zeta, \delta(\mu))-\)open set containing \( f(x) \). Then, there exists a \((\zeta, \delta(\mu))-\)open set \( U \) containing \( x \) such that \( f(U) \subseteq c_{(\zeta, \delta(\mu))}(V) \). Then, we have \( x \in U \subseteq f^{-1}(c_{(\zeta, \delta(\mu))}(V)) \) and hence, \( x \in i_{(\zeta, \delta(\mu))}(f^{-1}(c_{(\zeta, \delta(\mu))}(V))) \).

Conversely, let \( V \) be any \((\zeta, \delta(\mu))-\)open set containing \( f(x) \). Then, by the hypothesis, we have \( x \in i_{(\zeta, \delta(\mu))}(f^{-1}(c_{(\zeta, \delta(\mu))}(V))) \). There exists a \((\zeta, \delta(\mu))-\)open set \( U \) such that \( x \in U \) and \( U \subseteq f^{-1}(c_{(\zeta, \delta(\mu))}(V)) \) ; hence \( f(U) \subseteq c_{(\zeta, \delta(\mu))}(V) \). This shows that \( f \) is weakly \((\zeta, \delta(\mu))-\)continuous at \( x \in X \).

Theorem 5.3. A function \( f : (X, \mu) \to (Y, \mu') \) is weakly \((\zeta, \delta(\mu))-\)continuous if and only if \( f^{-1}(V) \subseteq i_{(\zeta, \delta(\mu))}(f^{-1}(c_{(\zeta, \delta(\mu))}(V))) \) for every \((\zeta, \delta(\mu))-\)open set \( V \) of \( Y \).

Proof. Let \( V \) be any \((\zeta, \delta(\mu))-\)open set of \( Y \) and \( x \in f^{-1}(V) \). Then \( f(x) \in V \). Since \( f \) is weakly \((\zeta, \delta(\mu))-\)continuous at \( x \), by Theorem 5.2, we have \( x \in i_{(\zeta, \delta(\mu))}(f^{-1}(c_{(\zeta, \delta(\mu))}(V))) \) and hence, \( f^{-1}(V) \subseteq i_{(\zeta, \delta(\mu))}(f^{-1}(c_{(\zeta, \delta(\mu))}(V))) \).

Conversely, let \( x \in X \) and \( V \) be any \((\zeta, \delta(\mu))-\)open set of \( Y \) containing \( f(x) \). Then, we have \( x \in f^{-1}(V) \subseteq i_{(\zeta, \delta(\mu))}(f^{-1}(c_{(\zeta, \delta(\mu))}(V))) \) and by Theorem 5.2, \( f \) is weakly \((\zeta, \delta(\mu))-\)continuous.

The following theorems give some characterizations of weakly \((\zeta, \delta(\mu))-\)continuous functions.

Theorem 5.4. For a function \( f : (X, \mu) \to (Y, \mu') \), the following properties are equivalent:

1. \( f \) is weakly \((\zeta, \delta(\mu))-\)continuous.

2. \( f^{-1}(U) \subseteq i_{(\zeta, \delta(\mu))}(f^{-1}(c_{(\zeta, \delta(\mu))}(U))) \) for every \((\zeta, \delta(\mu))-\)open set \( U \) of \( Y \).
(3) \( c_{(\zeta,\delta')} (f^{-1}(i_{(\zeta,\delta')} (F))) \subseteq f^{-1}(F) \) for every \((\zeta,\delta')\)-closed set \( F \) of \( Y \).

(4) \( c_{(\zeta,\delta')} (f^{-1}(i_{(\zeta,\delta')} (C_{(\zeta,\delta')} (A)))) \subseteq f^{-1}(c_{(\zeta,\delta')} (A)) \) for every subset \( A \) of \( Y \).

(5) \( f^{-1}(i_{(\zeta,\delta')} (A)) \subseteq i_{(\zeta,\delta')} (f^{-1}(c_{(\zeta,\delta')} ([i_{(\zeta,\delta')} (A)]))) \) for every subset \( A \) of \( Y \).

(6) \( c_{(\zeta,\delta')} (f^{-1}(U)) \subseteq f^{-1}(c_{(\zeta,\delta')} (U)) \) for every \((\zeta,\delta')\)-open set \( U \) of \( Y \).

**Proof.** (1) \( \Rightarrow \) (2) : This is obvious from Theorem 5.3.

(2) \( \Rightarrow \) (3) : Let \( F \) be any \((\zeta,\delta')\)-closed set of \( Y \). Then \( Y - F \) is \((\zeta,\delta')\)-open and by (2), we have

\[
X - f^{-1}(F) = f^{-1}(Y - F)
\]

\[
\subseteq i_{(\zeta,\delta')} (f^{-1}(c_{(\zeta,\delta')} (Y - F)))
\]

\[
= i_{(\zeta,\delta')} (f^{-1}(Y - i_{(\zeta,\delta')} (F)))
\]

\[
= X - c_{(\zeta,\delta')} (f^{-1}(i_{(\zeta,\delta')} (F))).
\]

Consequently, we obtain \( c_{(\zeta,\delta')} (f^{-1}(i_{(\zeta,\delta')} (F))) \subseteq f^{-1}(F) \).

(3) \( \Rightarrow \) (4) : Let \( A \) be any subset of \( Y \). Then \( c_{(\zeta,\delta')} (A) \) is \((\zeta,\delta')\)-closed, by (3), we have

\[
\]

(4) \( \Rightarrow \) (5) : Let \( A \) be any subset of \( Y \). By (4), we have

\[
f^{-1}(i_{(\zeta,\delta')} (A)) = X - f^{-1}(c_{(\zeta,\delta')} (Y - A))
\]

\[
\subseteq X - c_{(\zeta,\delta')} (f^{-1}(i_{(\zeta,\delta')} (C_{(\zeta,\delta')} (A))))
\]

\[
= i_{(\zeta,\delta')} (f^{-1}(c_{(\zeta,\delta')} ([i_{(\zeta,\delta')} (A)]))).
\]

Thus, we get the result.

(5) \( \Rightarrow \) (6) : Let \( U \) be any \((\zeta,\delta')\)-open set of \( Y \). Suppose that \( x \notin f^{-1}(c_{(\zeta,\delta')} (U)) \).

Then, we have \( f(x) \notin c_{(\zeta,\delta')} (U) \) and there exists a \((\zeta,\delta')\)-open set \( V \) containing \( f(x) \) such that \( U \cap V = \emptyset \) and hence, \( U \cap c_{(\zeta,\delta')} (V) = \emptyset \). By (5), \( x \in f^{-1}(V) \subseteq i_{(\zeta,\delta')} (f^{-1}(c_{(\zeta,\delta')} (V))). \) There exists a \((\zeta,\delta')\)-open set \( W \) containing \( x \) such that \( x \in W \subseteq f^{-1}(c_{(\zeta,\delta')} (V)). \) Since \( U \cap c_{(\zeta,\delta')} (V) = \emptyset \) and \( f(W) \subseteq c_{(\zeta,\delta')} (V), \) we have \( W \cap f^{-1}(U) = \emptyset. \) This implies that \( x \notin c_{(\zeta,\delta')} (f^{-1}(U)). \) Therefore, \( c_{(\zeta,\delta')} (f^{-1}(U)) \subseteq f^{-1}(c_{(\zeta,\delta')} (U)). \)

(6) \( \Rightarrow \) (1) : Let \( x \in X \) and \( U \) be any \((\zeta,\delta')\)-open set containing \( f(x) \). Since \( U = i_{(\zeta,\delta')} (U) \subseteq i_{(\zeta,\delta')} (c_{(\zeta,\delta')} (U)) \) and by (6),

\[
x \in f^{-1}(U) \subseteq f^{-1}(i_{(\zeta,\delta')} (c_{(\zeta,\delta')} (U)))
\]

\[
= X - f^{-1}(c_{(\zeta,\delta')} (Y - c_{(\zeta,\delta')} (U))]
\]

\[
\subseteq X - c_{(\zeta,\delta')} (f^{-1}(Y - c_{(\zeta,\delta')} (U))]
\]

\[
= i_{(\zeta,\delta')} (f^{-1}(c_{(\zeta,\delta')} (U))).
\]

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So there exists a \((\zeta, \delta(\mu))\)-open set \(V\) containing \(x\) such that \(V \subseteq f^{-1}(c_{(\zeta,\mu')}(U))\). This shows that \(f\) is weakly \((\zeta, \delta(\mu))\)-continuous.

**Theorem 5.5.** For a function \(f : (X, \mu) \rightarrow (Y, \mu')\), the following properties are equivalent:

1. \(f\) is weakly \((\zeta, \delta(\mu))\)-continuous.
2. \(c_{(\zeta,\mu')}(f^{-1}(i_{(\zeta,\mu')}(F))) \subseteq f^{-1}(F)\) for every \(r(\zeta, \delta(\mu'))\)-closed set \(F\) of \(Y\).
3. \(c_{(\zeta,\mu')}(f^{-1}(i_{(\zeta,\mu')}(c_{(\zeta,\mu')}(U)))) \subseteq f^{-1}(c_{(\zeta,\mu')}(U))\) for every \(\beta(\zeta, \delta(\mu'))\)-open set \(U\) of \(Y\).
4. \(c_{(\zeta,\mu')}(f^{-1}(i_{(\zeta,\mu')}(c_{(\zeta,\mu')}(U)))) \subseteq f^{-1}(c_{(\zeta,\mu')}(U))\) for every \(s(\zeta, \delta(\mu'))\)-open set \(U\) of \(Y\).

**Proof.** (1) \(\Rightarrow\) (2) : Let \(F\) be any \(r(\zeta, \delta(\mu'))\)-closed set of \(Y\). Then, we have \(i_{(\zeta,\mu')}(F)\) is \((\zeta, \delta(\mu'))\)-open, by Theorem 5.4(6), \(c_{(\zeta,\mu')}(f^{-1}(i_{(\zeta,\mu')}(F))) \subseteq f^{-1}(c_{(\zeta,\mu')}(i_{(\zeta,\mu')}(F)))\).

Since \(F\) is \(r(\zeta, \delta(\mu'))\)-closed, we have

\[c_{(\zeta,\mu')}(f^{-1}(i_{(\zeta,\mu')}(F))) \subseteq f^{-1}(c_{(\zeta,\mu')}(i_{(\zeta,\mu')}(F))) = f^{-1}(F).\]

(2) \(\Rightarrow\) (3) : Let \(U\) be any \(\beta(\zeta, \delta(\mu'))\)-open set. Then, we have

\[c_{(\zeta,\mu')}(U) \subseteq c_{(\zeta,\mu')}(i_{(\zeta,\mu')}(c_{(\zeta,\mu')}(U))) \subseteq c_{(\zeta,\mu')}(U)\]

and hence, \(c_{(\zeta,\mu')}(U)\) is \(r(\zeta, \delta(\mu'))\)-closed. By (2), it follows that

\[c_{(\zeta,\mu')}(f^{-1}(i_{(\zeta,\mu')}(c_{(\zeta,\mu')}(U)))) \subseteq f^{-1}(c_{(\zeta,\mu')}(U)).\]

(3) \(\Rightarrow\) (4) : The proof is obvious.

(4) \(\Rightarrow\) (1) : Let \(U\) be any \((\zeta, \delta(\mu'))\)-open set of \(Y\). By (4), we have

\[c_{(\zeta,\mu')}(f^{-1}(U)) \subseteq c_{(\zeta,\mu')}(f^{-1}(i_{(\zeta,\mu')}(c_{(\zeta,\mu')}(U)))) \subseteq f^{-1}(c_{(\zeta,\mu')}(U)).\]

Hence, by Theorem 5.4(6), \(f\) is weakly \((\zeta, \delta(\mu))\)-continuous.

**Theorem 5.6.** For a function \(f : (X, \mu) \rightarrow (Y, \mu')\), the following properties are equivalent:

1. \(f\) is weakly \((\zeta, \delta(\mu))\)-continuous.
2. \(c_{(\zeta,\mu')}(f^{-1}(i_{(\zeta,\mu')}(c_{(\zeta,\mu')}(U)))) \subseteq f^{-1}(c_{(\zeta,\mu')}(U))\) for every \(p(\zeta, \delta(\mu'))\)-open set \(U\) of \(Y\).
3. \(c_{(\zeta,\mu')}(f^{-1}(U)) \subseteq f^{-1}(c_{(\zeta,\mu')}(U))\) for every \(p(\zeta, \delta(\mu'))\)-open set \(U\) of \(Y\).
4. \(f^{-1}(U) \subseteq i_{(\zeta,\mu')}(f^{-1}(c_{(\zeta,\mu')}(U)))\) for every \(p(\zeta, \delta(\mu'))\)-open set \(U\) of \(Y\).

**Proof.** (1) \(\Rightarrow\) (2) : Let \(U\) be any \(p(\zeta, \delta(\mu'))\)-open set of \(Y\). Then, we have \(c_{(\zeta,\mu')}(U) = c_{(\zeta,\mu')}(i_{(\zeta,\mu')}(c_{(\zeta,\mu')}(U)))\) and so \(c_{(\zeta,\mu')}(U)\) is \(r(\zeta, \delta(\mu'))\)-closed. By Theorem 5.5(2), it follows that

\[c_{(\zeta,\mu')}(f^{-1}(i_{(\zeta,\mu')}(c_{(\zeta,\mu')}(U)))) \subseteq f^{-1}(c_{(\zeta,\mu')}(U)).\]
\[(2) \Rightarrow (3) : \text{Let } U \text{ be any } p(\zeta, \delta(\mu'))-\text{open set of } Y. \text{ Then, we have} \]
\[U \subseteq \overline{\{i(\zeta,\mu')|c(\zeta,\mu')(U)\}} \text{ and by (2),} \]
\[c(\zeta,\mu')(f^{-1}(U)) \subseteq c(\zeta,\mu')(f^{-1}(\overline{\{i(\zeta,\mu')|c(\zeta,\mu')(U)\}})) \]
\[\subseteq f^{-1}(c(\zeta,\mu')(U)). \]
\[(3) \Rightarrow (4) : \text{Let } U \text{ be any } p(\zeta, \delta(\mu'))-\text{open set of } Y. \text{ By (3), it follows that} \]
\[f^{-1}(U) \subseteq f^{-1}(\overline{\{i(\zeta,\mu')|c(\zeta,\mu')(U)\}}) \]
\[= X - f^{-1}(c(\zeta,\mu')(Y - c(\zeta,\mu')(U))) \]
\[\subseteq X - c(\zeta,\mu')(f^{-1}[Y - c(\zeta,\mu')(U)]) \]
\[= i(\zeta,\mu')(f^{-1}(c(\zeta,\mu')(U))). \]
\[(4) \Rightarrow (1) : \text{Since every } (\zeta, \delta(\mu'))-\text{open set is } p(\zeta, \delta(\mu'))-\text{open, by (4) and Theorem 5.4(2), it follows that} \]
\[f \text{ is weakly } (\zeta, \delta(\mu))-\text{continuous.} \]

**Theorem 5.7.** For a function \(f:(X, \mu) \to (Y, \mu')\), the following properties are equivalent:

1. \(f\) is weakly \((\zeta, \delta(\mu))-\text{continuous.}\)
2. \(c(\zeta,\mu')(f^{-1}(\overline{\{i(\zeta,\mu')|c(\zeta,\mu')(A)\}})) \subseteq f^{-1}(c(\zeta,\mu'(A)))\) for every subset \(A\) of \(Y.\)
3. \(c(\zeta,\mu')(f^{-1}(\overline{\{i(\zeta,\mu')|c(\zeta,\mu')(F)\}})) \subseteq f^{-1}(F)\) for every \(r(\zeta, \delta(\mu'))-\text{closed set } F.\)
4. \(c(\zeta,\mu')(f^{-1}(U)) \subseteq f^{-1}(c(\zeta,\mu'(U)))\) for every \((\zeta, \delta(\mu'))-\text{open set } U.\)
5. \(f^{-1}(U) \subseteq c(\zeta,\mu')(f^{-1}(c(\zeta,\mu'(U))))\) for every \((\zeta, \delta(\mu'))-\text{open set } U.\)
6. \(c(\zeta,\mu')(f^{-1}(U)) \subseteq f^{-1}(c(\zeta,\mu'(U)))\) for every \(p(\zeta, \delta(\mu'))-\text{open set } U.\)
7. \(f^{-1}(U) \subseteq c(\zeta,\mu')(f^{-1}(c(\zeta,\mu'(U))))\) for every \(p(\zeta, \delta(\mu))-\text{open set } U.\)

**Proof.** \(1) \Rightarrow (2) : \text{Let } A \text{ be any subset of } Y \text{ and } x \notin f^{-1}(c(\zeta,\mu')(A)). \text{ Then, we have} \]
\[f(x) \notin c(\zeta,\mu')(A) \text{ and there exists a } (\zeta, \delta(\mu'))-\text{open set } U \text{ containing } f(x) \text{ such that} \]
\[U \cap A = \emptyset. \text{ This implies that} \)
\[c(\zeta,\mu')(U) \cap c(\zeta,\mu')(A) = \emptyset. \text{ Since } f \text{ is weakly } (\zeta, \delta(\mu'))-\text{continuous, there exists a} \]
\[(\zeta, \delta(\mu'))-\text{open set } W \text{ containing } x \text{ such that} \]
\[f(W) \subseteq c(\zeta,\mu')(U). \text{ Then}\]
\[W \cap f^{-1}(\overline{\{i(\zeta,\mu')|c(\zeta,\mu')(A)\}}) = \emptyset \text{ and hence, } x \notin c(\zeta,\mu')(f^{-1}(\overline{\{i(\zeta,\mu')|c(\zeta,\mu')(A)\}})). \text{ Consequently, we obtain}\]
\[c(\zeta,\mu')(f^{-1}(\overline{\{i(\zeta,\mu')|c(\zeta,\mu')(A)\}})) \subseteq f^{-1}(c(\zeta,\mu'(A))).\]

\(2) \Rightarrow (3) : \text{Let } F \text{ be any } r(\zeta, \delta(\mu'))-\text{closed set of } Y. \text{ By (2), we have} \]
\[c(\zeta,\mu')(\overline{\{i(\zeta,\mu')|c(\zeta,\mu')(F)\}})) = c(\zeta,\mu')(f^{-1}(\overline{\{i(\zeta,\mu')|c(\zeta,\mu')(F)\}})) \]
\[\subseteq f^{-1}(c(\zeta,\mu')(F)) = f^{-1}(F). \]

\(3) \Rightarrow (4) : \text{Let } U \text{ be any } (\zeta, \delta(\mu'))-\text{open set of } Y. \text{ Since } c(\zeta,\mu')(U) \text{ is } r(\zeta, \delta(\mu'))-\text{closed and by (3), we have} \]
\[c(\zeta,\mu')(f^{-1}(U)) \subseteq c(\zeta,\mu')(f^{-1}(\overline{\{i(\zeta,\mu')|c(\zeta,\mu')(U)\}})) \subseteq f^{-1}(c(\zeta,\mu'(U))).\]
Let $U$ be any $(\zeta, \delta(\mu'))$-open set of $Y$. Since $Y - c_{(\zeta, \delta(\mu'))}(U)$ is $(\zeta, \delta(\mu'))$-open and by (4), we have $X - i_{(\zeta, \delta(\mu'))}(f^{-1}(c_{(\zeta, \delta(\mu'))}(U))) = c_{(\zeta, \delta(\mu'))}(f^{-1}(Y - c_{(\zeta, \delta(\mu'))}(U)))$

$$\subseteq f^{-1}(c_{(\zeta, \delta(\mu'))}(Y - c_{(\zeta, \delta(\mu'))}(U)))$$

$$= f^{-1}(Y - i_{(\zeta, \delta(\mu'))}[c_{(\zeta, \delta(\mu'))}(U)])$$

$$= X - f^{-1}(i_{(\zeta, \delta(\mu'))}[c_{(\zeta, \delta(\mu'))}(U)])$$

$$\subseteq X - f^{-1}(U).$$

Consequently, we obtain $f^{-1}(U) \subseteq i_{(\zeta, \delta(\mu'))}(f^{-1}(c_{(\zeta, \delta(\mu'))}(U)))$.

(5) $\Rightarrow$ (1): Let $x \in X$ and $U$ be any $(\zeta, \delta(\mu'))$-open set containing $f(x)$. Then, we have $x \in f^{-1}(U) \subseteq i_{(\zeta, \delta(\mu'))}(f^{-1}(c_{(\zeta, \delta(\mu'))}(U)))$. Put $W = i_{(\zeta, \delta(\mu'))}(f^{-1}(c_{(\zeta, \delta(\mu'))}(U)))$. Thus, $f(W) \subseteq c_{(\zeta, \delta(\mu'))}(U)$ and hence, $f$ is weakly $(\zeta, \delta(\mu))$-continuous.

(1) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (1): This is a consequence of Theorem 5.6.

**Definition 5.8.** A strong generalized topological space $(X, \mu)$ is called $(\zeta, \delta(\mu))$-connected if $X$ cannot be written as a disjoint union of two non-empty $(\zeta, \delta(\mu))$-open sets.

**Proposition 5.9.** For a strong generalized topological space $(X, \mu)$, the following properties are equivalent:

(1) $(X, \mu)$ is $(\zeta, \delta(\mu))$-connected.

(2) The only subsets of $X$, which are both $(\zeta, \delta(\mu))$-open and $(\zeta, \delta(\mu))$-closed are $\emptyset$ and $X$.

**Definition 5.10.** A strong generalized topological space $(X, \mu)$ is said to be $(\zeta, \delta(\mu))$-Urysohn if for each distinct points $x, y \in X$, there exist $U, V \in (\zeta, \delta(\mu))O$ containing $x$ and $y$, respectively, such that $c_{(\zeta, \delta(\mu))}(U) \cap c_{(\zeta, \delta(\mu))}(V) = \emptyset$.

**Proposition 5.11.** If $f : (X, \mu) \rightarrow (Y, \mu')$ is a weakly $(\zeta, \delta(\mu))$-continuous injection and $(Y, \mu')$ is $(\zeta, \delta(\mu'))$-Urysohn, then $(X, \mu)$ is $(\zeta, \delta(\mu))$-T₂.

**Proof.** Let $x, y$ be distinct points of $X$. Then $f(x) \neq f(y)$. Since $(Y, \mu')$ is $(\zeta, \delta(\mu'))$-Urysohn, there exist $U, V \in (\zeta, \delta(\mu'))O$ containing $f(x)$ and $f(y)$, respectively, such that $c_{(\zeta, \delta(\mu'))}(U) \cap c_{(\zeta, \delta(\mu'))}(V) = \emptyset$. Since $f$ is weakly $(\zeta, \delta)$-continuous, there exist $G, W \in (\zeta, \delta(\mu))O$ containing $x$ and $y$, respectively, such that $f(G) \subseteq c_{(\zeta, \delta(\mu'))}(U)$ and $f(W) \subseteq c_{(\zeta, \delta(\mu'))}(V)$. Therefore, $G \cap W = \emptyset$. Consequently, we obtain $(X, \mu)$ is $(\zeta, \delta(\mu))$-T₂.

**Proposition 5.12.** Let $f : (X, \mu) \rightarrow (Y, \mu')$ be a weakly $(\zeta, \delta(\mu))$-continuous surjection. If $(X, \mu)$ is $(\zeta, \delta(\mu))$-connected, then $(Y, \mu')$ is $(\zeta, \delta(\mu))$-connected.

**Proof.** Suppose that $(Y, \mu')$ is not $(\zeta, \delta(\mu'))$-connected. Then, there exist non-empty $(\zeta, \delta(\mu'))$-open sets $V_1, V_2$ such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = Y$. By Theorem 5.3, we have $f^{-1}(V_i) \subseteq i_{(\zeta, \delta(\mu'))}(f^{-1}(c_{(\zeta, \delta(\mu'))}(V_i)))$ for $i = 1, 2$. Since $V_i$ is $(\zeta, \delta(\mu'))$-closed in $(Y, \mu')$ for each $i = 1, 2$. Therefore, we obtain $f^{-1}(V_i) \subseteq i_{(\zeta, \delta(\mu'))}(f^{-1}(V_i))$ and hence by Lemma 3.19, $f^{-1}(V_i)$ is $(\zeta, \delta(\mu))$-open for $i = 1, 2$. Moreover, $X$ is union of non-empty disjoint sets $f^{-1}(V_1)$ and $f^{-1}(V_2)$. This implies that $(X, \mu)$ is not $(\zeta, \delta(\mu))$-connected. This is contrary to the hypothesis that $(X, \mu)$ is $(\zeta, \delta(\mu))$-connected. Therefore, $(Y, \mu')$ is $(\zeta, \delta(\mu'))$-connected.
Definition 5.13. A subset $K$ of a strong generalized topological space $(X, \mu)$ is said to be $S_{(\zeta, \delta(\mu))}$-closed (resp. $(\zeta, \delta(\mu))$-compact) relative to $(X, \mu)$ if for any cover $\{V_i| i \in I\}$ of $K$ by $(\zeta, \delta(\mu))$-open sets of $X$, there exists a finite subset $I_0$ of $I$ such that $K \subseteq \cup \{c_{(\zeta, \delta(\mu))}(V_i)| i \in I_0\}$ (resp. $K \subseteq \cup V_i| i \in I_0\}$). If $X$ is $S_{(\zeta, \delta(\mu))}$-closed (resp. $(\zeta, \delta(\mu))$-compact) relative to $(X, \mu)$, then $(X, \mu)$ is said to be $S_{(\zeta, \delta(\mu))}$-closed (resp. $(\zeta, \delta(\mu))$-compact).

Proposition 5.14. If $f: (X, \mu) \rightarrow (Y, \mu')$ is weakly $(\zeta, \delta(\mu))$-continuous and $K$ is $(\zeta, \delta(\mu))$-compact, then $f(K)$ is $S_{(\zeta, \delta(\mu'))}$-closed relative to $(Y, \mu')$.

Proof. Let $\{V_i| i \in I\}$ be any cover of $f(K)$ by $(\zeta, \delta(\mu'))$-open sets of $Y$. For each $x \in K$, there exists $i(x) \in I$ such that $f(x) \in V_{i(x)}$. Since $f$ is weakly $(\zeta, \delta(\mu))$-continuous, there exists an $(\zeta, \delta(\mu))$-open set $U(x)$ containing $x$ such that $f(U(x)) \subseteq c_{(\zeta, \delta(\mu))}(V_{i(x)})$. The family $\{U(x)| x \in K\}$ is a cover of $K$ by $(\zeta, \delta(\mu))$-open sets of $(X, \mu)$. Since $K$ is $(\zeta, \delta(\mu))$-compact relative to $(X, \mu)$, there exists a finite number of points, say, $x_1, x_2, ..., x_n$ in $K$ such that $K \subseteq \cup\{U(x_k)| x_k \in K, 1 \leq k \leq n\}$.

Therefore, we obtain

$f(K) \subseteq \cup\{f(U(x_k))| x_k \in K, 1 \leq k \leq n\}$

$\subseteq \cup\{c_{(\zeta, \delta(\mu'))}(V_{i(x_k)})| x_k \in K, 1 \leq k \leq n\}$.

This shows that $f(K)$ is $S_{(\zeta, \delta(\mu'))}$-closed relative to $(Y, \mu')$.

Corollary 5.15. If $f: (X, \mu) \rightarrow (Y, \mu')$ is a weakly $(\zeta, \delta(\mu))$-continuous surjection and $(X, \mu)$ is $(\zeta, \delta(\mu))$-compact, then $(Y, \mu')$ is $S_{(\zeta, \delta(\mu'))}$-closed.

Theorem 5.16. The set of all points $x \in X$ at which a function $f: (X, \mu) \rightarrow (Y, \mu')$ is not weakly $(\zeta, \delta(\mu))$-continuous is identical with the union of the $(\zeta, \delta(\mu))$-frontiers of the inverse images of the $(\zeta, \delta(\mu))$-closures of $(\zeta, \delta(\mu))$-open sets containing $f(x)$.

Proof. Suppose that $f$ is not weakly $(\zeta, \delta(\mu))$-continuous at $x \in X$. There exists a $(\zeta, \delta(\mu'))$-open set $V$ containing $f(x)$ such that $f(U)$ is not contained in $c_{(\zeta, \delta(\mu))}(V)$ for every $(\zeta, \delta(\mu))$-open set $U$ containing $x$. Then $U \cap (X - f^{-1}(c_{(\zeta, \delta(\mu))}(V))) \neq \emptyset$ for every $(\zeta, \delta(\mu))$-open set $U$ containing $x$ and $x \in c_{(\zeta, \delta(\mu))}(X - f^{-1}(c_{(\zeta, \delta(\mu))}(V)))$. On the other hand, we have $x \in f^{-1}(V) \subseteq c_{(\zeta, \delta(\mu))}(f^{-1}(c_{(\zeta, \delta(\mu))}(V)))$ and hence, $x \in Fr_{(\zeta, \delta(\mu))}(f^{-1}(c_{(\zeta, \delta(\mu))}(V)))$.

Conversely, suppose that $f$ is weakly $(\zeta, \delta(\mu))$-continuous at $x \in X$ and let $V$ be any $(\zeta, \delta(\mu))$-open set containing $f(x)$. Then by Theorem 5.2, we have $x \in i_{(\zeta, \delta(\mu))}(f^{-1}(c_{(\zeta, \delta(\mu))}(V)))$. Therefore, $x \not\in Fr_{(\zeta, \delta(\mu))}(f^{-1}(c_{(\zeta, \delta(\mu))}(V)))$ for each $(\zeta, \delta(\mu'))$-open set $V$ of $Y$ containing $f(x)$. This completes the proof.

Proposition 5.17. If $f: (X, \mu) \rightarrow (Y, \mu')$ is weakly $(\zeta, \delta(\mu))$-continuous and $(Y, \mu')$ is $(\zeta, \delta(\mu'))$-T2, then $f$ has $(\zeta, \delta(\mu))$-closed point inverses.

Proof. Let $y \in Y$. We show that $f^{-1}(y) = \{x \in X| f(x) = y\}$ is $(\zeta, \delta(\mu))$-closed, or equivalently $G = \{x \in X| f(x) = y\}$ is $(\zeta, \delta(\mu))$-open. Let $x \in G$. Since $f(x) = y$ and $(Y, \mu')$ is $(\zeta, \delta(\mu'))$-T2, there exist disjoint $(\zeta, \delta(\mu))$-open sets $U, V$ such that $f(x) \in U$ and $y \in V$. Since $U \cap V = \emptyset$, by Lemma 3.31, we have $c_{(\zeta, \delta(\mu))}(U) \cap V = \emptyset$. Thus, $y \not\in c_{(\zeta, \delta(\mu))}(U)$. Since $f$ is weakly $(\zeta, \delta(\mu))$-continuous, there exists a $(\zeta, \delta(\mu))$-open set $W$ containing $x$ such that $f(W) \subseteq c_{(\zeta, \delta(\mu))}(U)$. Now, suppose that $W$ is not contained in $G$. Then, there exists a point $w \in W$ such that $f(w) = y$. Since $f(W) \subseteq c_{(\zeta, \delta(\mu))}(U)$, we have $y = f(w) \in c_{(\zeta, \delta(\mu))}(U)$. This is a contradiction. Therefore, $W \subseteq G$ and so $G$ is $(\zeta, \delta(\mu))$-neighbourhood of $x$. By Lemma 4.4, we obtain $G$ is a $(\zeta, \delta(\mu))$-open set.

Proposition 5.18. Let $(X, \mu)$ be a strong generalized topological space. If for each pair of distinct points $x_1$ and $x_2$ in $X$, there exists a function $f$ of $(X, \mu)$ into $(Y, \mu')$ such that
In general, we have $\delta U_1 \cap \delta U_2 = \emptyset$. Thus, there exists $U_1 \subseteq \delta U_2$ containing $x_i$ such that $f(U_1) \subseteq \delta (\delta U_2)(V_i)$. Hence, we get $U_1 \cap U_2 = \emptyset$. Therefore, $(X, \mu)$ is $(\zeta, \delta(\mu))$-T-2.

**Corollary 5.19.** If $f : (X, \mu) \to (Y, \mu')$ is a weakly $(\zeta, \delta(\mu))$-continuous injection and $(Y, \mu')$ is $(\zeta, \delta(\mu'))$-Urysohn, then $(X, \mu)$ is $(\zeta, \delta(\mu))$-T-2.

**Lemma 5.20.** Let $A$ be a subset of a strong generalized topological space $(X, \mu)$. Then $x \in c_{(\zeta, \delta(\mu))}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \subseteq \delta(\delta U)(A)$ containing $x$.

**Proof.** Suppose that there exists $U \subseteq \delta(\delta U)(A)$ containing $x$ such that $U \cap A = \emptyset$. Then $A \subseteq X - U$ and hence, $c_{(\zeta, \delta(\mu))}(A) \subseteq X - U$. Since $x \in U$, we have $x \notin c_{(\zeta, \delta(\mu))}(A)$.

Conversely, suppose that $x \notin c_{(\zeta, \delta(\mu))}(A)$. There exists a $(\zeta, \delta(\mu))$-closed set $F$ of $X$ such that $X - F \subseteq \delta(\delta U)(A)$, $A \subseteq F$ and $x \notin F$. Thus, there exists $F \subseteq \delta(\delta U)(A)$ containing $x$ such that $(X - F) \cap A = \emptyset$.

**Lemma 5.21.** For a subset $A$ of a strong generalized topological space $(X, \mu)$, the following properties hold:

1. If $A$ is $(\zeta, \delta(\mu))$-open, then $c_{(\zeta, \delta(\mu))}(A) = c_{\delta(\delta U)}(A)$.
2. $c_{(\zeta, \delta(\mu))}(A)$ is $(\zeta, \delta(\mu))$-closed for every subset $A$ of $X$.

**Proof.** (1) In general, we have $c_{(\zeta, \delta(\mu))}(A) \subseteq c_{\delta(\delta U)(\mu)}(A)$. Suppose that $x \notin c_{\delta(\delta U)(\mu)}(A)$. Then by Lemma 5.20, there exists $U \subseteq \delta(\delta U)(A)$ containing $x$ such that $U \cap A = \emptyset$; hence $A \cap c_{\delta(\delta U)(\mu)}(U) = \emptyset$ since $A$ is $(\zeta, \delta(\mu))$-open. This shows that $x \notin c_{\delta(\delta U)(\mu)}(A)$. Consequently, we obtain $c_{(\zeta, \delta(\mu))}(A) = c_{\delta(\delta U)(\mu)}(A)$.

(2) Let $x \in X - c_{\delta(\delta U)(\mu)}(A)$. Then, we have $x \notin c_{\delta(\delta U)(\mu)}(A)$. There exists $U_x \subseteq \delta(\delta U)(\mu)$ containing $x$ such that $A \cap c_{\delta(\delta U)(\mu)}(U_x) = \emptyset$. Then $U_x \subseteq c_{\delta(\delta U)(\mu)}(A) = \emptyset$ and hence, $x \in U_x \subseteq X - c_{\delta(\delta U)(\mu)}(A)$. Therefore, $X - c_{\delta(\delta U)(\mu)}(A) = \bigcup_{x \in X - c_{\delta(\delta U)(\mu)}(A)} U_x$. This shows that $c_{\delta(\delta U)(\mu)}(A)$ is $(\zeta, \delta(\mu))$-closed.

**Theorem 5.22.** For a function $f : (X, \mu) \to (Y, \mu')$, the following properties are equivalent:

1. $f$ is weakly $(\zeta, \delta(\mu))$-continuous.
2. $f(c_{(\zeta, \delta(\mu))}(A)) \subseteq c_{\delta(\delta U)(\mu)}(f(A))$ for every subset $A$ of $X$.
3. $c_{(\zeta, \delta(\mu))}(f^{-1}(B)) \subseteq f^{-1}(c_{\delta(\delta U)(\mu)}(B))$ for every subset $B$ of $Y$.
4. $c_{(\zeta, \delta(\mu))}(f^{-1}(V)) \subseteq f^{-1}(c_{\delta(\delta U)(\mu)}(V))$ for every $(\zeta, \delta(\mu'))$-open set $V$ of $Y$.

**Proof.** (1) $\Rightarrow$ (2) : Let $A$ be any subset of $X$. Suppose that $x \in c_{(\zeta, \delta(\mu))}(A)$ and $V$ be any $(\zeta, \delta(\mu'))$-open set containing $f(x)$. Since $f$ is weakly $(\zeta, \delta(\mu))$-continuous, there exists a...
(ζ, δ(μ))-open set U containing x such that \( f(U) \subseteq c_{(ζ, δ(μ))}(V) \). Since \( x \in c_{(ζ, δ(μ))}(A) \), we have \( U \cap A = \emptyset \). It follows that \( \emptyset \neq f(U) \cap f(A) \subseteq c_{(ζ, δ(μ))}(V) \cap f(A) \).

Therefore, \( c_{(ζ, δ(μ))}(V) \cap f(A) \neq \emptyset \) and hence, \( f(x) \in c_{(ζ, δ(μ))}(f(A)) \).

(2) \( \Rightarrow \) (3) : Let \( B \) be any subset of \( Y \). By (2), we have

\[
f(c_{(ζ, δ(μ))}(f^{-1}(B))) \subseteq c_{(ζ, δ(μ'))}(f(f^{-1}(B))) \subseteq c_{(ζ, δ(μ))}(B)
\]

and so \( c_{(ζ, δ(μ))}(f^{-1}(B)) \subseteq f^{-1}(c_{(ζ, δ(μ))}(B)) \).

(3) \( \Rightarrow \) (4) : Let \( V \) be any \( (ζ, δ(μ')) \)-open set of \( Y \). By Lemma 5.21, \( c_{(ζ, δ(μ'))}(V) = c_{(ζ, δ(μ))}(V) \). Thus, the proof is obvious.

(4) \( \Rightarrow \) (1) : Let \( V \) be any \( (ζ, δ(μ')) \)-open set containing \( f(x) \). Since \( V \cap (Y - c_{(ζ, δ(μ))}(V)) = \emptyset \) we have \( f(x) \notin c_{(ζ, δ(μ'))}(Y - c_{(ζ, δ(μ))}(V)) \) and hence, \( x \notin f^{-1}(c_{(ζ, δ(μ'))}(Y - c_{(ζ, δ(μ))}(V))) \).

Since \( Y - c_{(ζ, δ(μ'))}(V) = i_{(ζ, δ(μ'))}(Y - V) \in (ζ, δ(μ'))O \) and by (4),
\[
x \notin c_{(ζ, δ(μ'))}(f^{-1}(Y - c_{(ζ, δ(μ))}(V)))
\]

There exists a \( (ζ, δ(μ)) \)-open set \( U \) containing \( x \) such that
\[
U \cap f^{-1}(Y - c_{(ζ, δ(μ))}(V)) = \emptyset;
\]
hence \( f(U) \subseteq c_{(ζ, δ(μ))}(V) \). Therefore, \( f \) is weakly \( (ζ, δ(μ)) \)-continuous.

**Lemma 5.23.** Let \((X, μ)\) be a \( (ζ, δ(μ)) \)-regular space. Then, the following properties hold:

1. \( c_{(ζ, δ(μ))}(A) = c_{(ζ, δ(μ))}(A) \) for every subset \( A \) of \( X \).
2. Every \( (ζ, δ(μ)) \)-open set is \( θ(ζ, δ(μ)) \)-open.

**Proof.** (1) In general, we have \( c_{(ζ, δ(μ))}(A) \subseteq c_{(ζ, δ(μ))}(A) \) for every subset \( A \) of \( X \). Next, we show that \( c_{(ζ, δ(μ))}(A) \subseteq c_{(ζ, δ(μ))}(A) \). Let \( x \in c_{(ζ, δ(μ))}(A) \) and \( U \) be any \( (ζ, δ(μ)) \)-open set containing \( x \). Then by Theorem 4.42, there exists a \( (ζ, δ(μ')) \)-open set \( V \) such that \( x \in V \subseteq c_{(ζ, δ(μ))}(V) \subseteq U \). Since \( x \in c_{(ζ, δ(μ))}(A) \), it follows that \( A \cap c_{(ζ, δ(μ))}(V) \neq \emptyset \) and hence \( U \cap A \neq \emptyset \). By Lemma 5.20, we have \( x \in c_{(ζ, δ(μ))}(A) \). Thus, \( c_{(ζ, δ(μ))}(A) \subseteq c_{(ζ, δ(μ))}(A) \). Consequently, we obtain \( c_{(ζ, δ(μ))}(A) = c_{(ζ, δ(μ))}(A) \).

(2) Let \( V \in (ζ, δ(μ))O(X, μ) \). By (1), we have \( X - V = c_{(ζ, δ(μ))}(X - V) = c_{(ζ, δ(μ))}(X - V) \) and so \( X - V \) is \( θ(ζ, δ(μ)) \)-closed. Therefore, \( V \) is \( θ(ζ, δ(μ)) \)-open.

**Theorem 5.24.** Let \((X, μ)\) be a \( (ζ, δ(μ)) \)-regular space. Then, for a function \( f : (X, μ) \rightarrow (Y, μ') \), the following properties are equivalent:

1. \( f^{-1}(c_{(ζ, δ(μ))}(B)) \) is \( θ(ζ, δ(μ)) \)-closed in \( X \) for every subset \( B \) of \( Y \).
2. \( f \) is weakly \( (ζ, δ(μ)) \)-continuous.
3. \( f^{-1}(F) \) is \( (ζ, δ(μ)) \)-closed in \( X \) for every \( θ(ζ, δ(μ)) \)-closed set \( F \) of \( Y \).
4. \( f^{-1}(V) \) is \( (ζ, δ(μ)) \)-open in \( X \) for every \( θ(ζ, δ(μ)) \)-open set \( V \) of \( Y \).
Proof. \((1) \Rightarrow (2): \) Let \(B\) be any subset of \(Y\). Then, we have 
\[ c_{(\zeta, \delta(\mu'))}(f^{-1}(B)) \subseteq c_{(\zeta, \delta(\mu'))}(f^{-1}(c_{(\zeta, \delta(\mu'))}(B))) = f^{-1}(c_{(\zeta, \delta(\mu'))}(B)). \]
Therefore, by Theorem 5.22, \(f\) is weakly \((\zeta, \delta(\mu'))\)-continuous.

\((2) \Rightarrow (3): \) Let \(F\) be any \(\theta(\zeta, \delta(\mu'))\)-closed set of \(Y\). By Theorem 5.22, we have 
\[ c_{(\zeta, \delta(\mu'))}(f^{-1}(F)) \subseteq f^{-1}(c_{(\zeta, \delta(\mu'))}(F)) = f^{-1}(F). \]
Therefore, \(f^{-1}(F)\) is \((\zeta, \delta(\mu'))\)-closed.

\((3) \Rightarrow (4): \) The proof is obvious.

\((4) \Rightarrow (1): \) Let \(B\) be any subset of \(Y\). By Lemma 5.21, we have \(c_{(\zeta, \delta(\mu'))}(B)\) is \((\zeta, \delta(\mu'))\)-closed and so 
\[ Y - c_{(\zeta, \delta(\mu'))}(B) \text{ is } (\zeta, \delta(\mu'))-open. \]
Therefore, by Lemma 5.23, \(Y - c_{(\zeta, \delta(\mu'))}(B)\) is \((\zeta, \delta(\mu'))\)-open and by (4), 
\[ X - f^{-1}(c_{(\zeta, \delta(\mu'))}(B)) = f^{-1}(Y - c_{(\zeta, \delta(\mu'))}(B)) \text{ is } (\zeta, \delta(\mu'))-open. \]
Consequently, we obtain \(f^{-1}(c_{(\zeta, \delta(\mu'))}(B))\) is \((\zeta, \delta(\mu'))\)-closed.

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