Comparison of various models of particle multiplicity distributions using a general form of the grand canonical partition function

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Various phenomenological models of particle multiplicity distributions are discussed using a general form of the grand canonical partition function. These phenomenological models include a wide range of varied processes such as coherent emission or Poisson processes, chaotic emission resulting in a negative binomial distribution, combinations of coherent and chaotic processes called signal/noise distributions, and models based on field emission from Lorentzian line shapes leading to Lorentz/Catalan distributions. These specific cases can be written as special cases of a more general distribution. Using this grand canonical approach moments and cumulants, combinants, hierarchical structure, void scaling relations, KNO scaling features, clan variables and branching laws associated with stochastic or ancestral variables are discussed. It is shown that just looking at the mean and fluctuation of data is not enough to distinguish these distributions or the underlying mechanism. A generalization of the Poisson transform of a distribution and the Poissonian decomposition of it into a compound or sequential process is also given.

I. INTRODUCTION

Pion multiplicity distribution and their associated fluctuation and correlations have been studied at the CERN SPS in the past, are being investigated at RHIC presently, and in the future will be studied at the LHC. Because of the very large number of pions and other produced particles in very high energy collisions, event-by-event studies can be carried out and are important tools for many reasons. One current reason for studying them resides in the hope that anomalous fluctuations will remain from a transition of a quark-gluon phase to hadronic phase, thus offering one of the signals of the formation of the QG phase. Several models predict large fluctuations such as the disoriented chiral condensate and in density fluctuations from droplets arising in a first order phase transition. A well known procedure for studying correlations uses the Bose-Einstein symmetries associated with pions in a Hanbury Brown-Twiss analysis. Such an analysis gives information about the space time history of the collision through measurements of source parameters. If the density of pions becomes large, Bose-Einstein correlation may also lead to a strongly emitting system which has been called a pion laser. The pion laser model has been recently solved analytically by T. Csorgo and J. Zimanyi. The importance of Bose-Einstein correlations has also been illustrated in the observation of a condensation of atoms in a harmonic oscillator or laser trap. Previous interest in pionic distributions have centered around the possibility of intermittency behavior and fractal structure based on parallels with turbulent flow in fluids. A distribution widely used to discuss such features has been the negative binomial (NB) distribution with its associated clan structure and KNO scaling feature. KNO scaling properties have been interpreted in terms of a phase transition associated with a Feynman-Wilson gas. Various other issues associated with pions include evidence for thermalization, critical point fluctuations, fluctuations from a first order phase transition, charge particle ratios and question of chemical equilibrium, the behavior of fluctuations in net charge in a QG plasma for transition.

For lower energy heavy ion collisions, multifragmentation of nuclei takes place. The fragment distribution can also be described statistically by considering all the possible partition of A nucleons into smaller clusters. This study gives a tool for the description of nuclear multifragmentation distributions, critical exponent, intermittency, and chaotic behavior of nuclear multifragmentation. The same model can describe pion distribution. This possibility arises in our approach which is based on Feynman path integral methods where symmetrization of bosons or anti-symmetrization of fermions leads to a cycle class decomposition of the permutations associated with these symmetries. The correspondance comes from the identification of clusters of size k and the cycles of length k in a permutation as discussed below.

In next section a summary of the generalized statistical model will be given. Various models of particle multiplicity distribution will then be developed. Moreover, we derive a generalized model which can further be reduced to a...
geometric, negative binomial, and Lorentz/Catalan model. These various models used in describing pion and particle multiplicity distributions then are examined with the generalized model in Sect. III. In Sect. IV, we further compare various statistical properties between different models within the generalized model.

II. GENERALIZED PROBABILITY DISTRIBUTION

Consider a system composed of \( N \) different types of species or objects which could be the fragments in a fragmentation or in a cycle class description. Any event of such a system can be associated with a vector \( \vec{n} = \{ n_k \} = (n_1, n_2, \ldots, n_k, \ldots, n_N) \) or \( 1^{n_1}2^{n_2}3^{n_3} \ldots k^{n_k} \ldots N^{n_N} \) where the non-negative integer \( n_k \) is the number of individuals of species \( k \). For example \( n_k \) can be the number of clusters of size \( k \) or the number of cycles of length \( k \) in a given permutation of \( n \) particles. The later is important for Bose-Einstein and Fermi-Dirac statistics. A general block picture of \( \vec{n} \) is shown in Fig. 1a. Fig. 1b shows how the various partition can be developed as an evolution from successively smaller systems. The multiplicity, i.e., the total number of individuals is then

\[
M = \sum_{k=1}^{N} n_k
\]

The number of species \( N \) can be infinity in general.

Various probability distributions related with this system can be developed by assigning an appropriate weight \( x_k \) to each individual of type \( k \) species. A weight \( W(\vec{x}, \vec{n}) \) is then given to each event \( \vec{n} \) and the type of weight that will be considered has the structure;

\[
W(\vec{x}, \vec{n}) = N \prod_{k=1}^{N} \left[ \frac{x_k^{n_k}}{n_k!} \right]
\]

The last equation holds due to the form of Eq.(2) of the weight \( W(\vec{x}, \vec{n}) \), i.e., the factor \( x_k^{n_k}/n_k! \) is the \( n_k \)-th order expansion term of \( e^{x_k} \).

Introducing other quantities \( \alpha_k \) to each individual entity or group of type \( k \), \( \vec{\alpha} = \{ \alpha_k \} = (\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots, \alpha_N) \), we can define a canonical partition function \( Z_A(\vec{x}) \) with a fixed \( A \) as
\[ A = \sum_{k=1}^{N} \alpha_k n_k = \hat{\alpha} \cdot \hat{n} \]  
\[ Z_A(\vec{x}) = \sum_{\vec{n}_A} W(\vec{x}, \vec{n}) = \sum_{\vec{n}_A} \prod_{k=1}^{N} \left[ \frac{z^{n_k}}{n_k!} \right] \]  
with \( Z(\vec{x}) = \sum_A Z_A(\vec{x}) \). Here \( \sum_{\vec{n}_A} \) is the summation over all events with a fixed value of \( A \), i.e., over a canonical ensemble of a fixed \( A \), and the \( \sum_{\vec{n}_A} \) is a summation over all the possible values of \( A \); it becomes \( \sum_{A=0}^{\infty} \) for the case of \( \alpha_k = k \) with positive integer \( k \). The case \( \alpha_k = k \) is encountered in fragmentation problems and permutations problems. If we take \( x_k \propto z^{\alpha_k} \), then the canonical partition function \( Z_A(\vec{x}) \) is the \( z^A \) dependent term of the grand canonical partition function \( Z(\vec{x}) \). The physics of the canonical ensemble depends on the choice of the quantity \( \alpha_k \). There is always at least one event having \( A = 0 \), i.e., the event where all \( n_k \)'s are zero, \( \hat{n} = \vec{0} \). Thus \( Z_0(\vec{x}) = 1 \) if all \( \alpha_k \) are non zero positive since then there are no other possible events having \( A = 0 \). Due to the form of the weight \( W(\vec{x}, \vec{n}) \) given by Eq. (6), the canonical partition function \( Z_A(\vec{x}) \) satisfies a recurrence relation \[ Z_A(\vec{x}) = \frac{1}{A} \sum_k \alpha_k x_k Z_A - \alpha_k(\vec{x}) \]  
This relation is nothing but the constraint Eq. (4) in terms of the mean \( < n_k >_A \) using Eq. (8). For non-zero positive \( \alpha_k \), there is no case having \( A < 0 \), i.e., \( Z_A = 0 \) for \( A < 0 \) and thus the \( Z_A \) can be obtained by the recurrence relation of Eq. (4) starting from \( Z_0(\vec{x}) = 1 \).

To study the statistical properties of a system where the number of individuals is independent of their species type, we can choose \( \alpha_k = 1 \). For fragmentation, all the clusters are treated the same independent of their size \( k \) or internal structure by choosing \( \alpha_k = 1 \). In pion distribution, \( \pi^+ \), \( \pi^0 \), and \( \pi^- \) are treated the same if we choose \( \alpha_k = 1 \) independent of their charge. By choosing \( \alpha_k = 1 \), we can study jet distributions without considering any further process of hadronization. Then \( A = \sum_k n_k = M \) and the canonical ensemble is a set of systems having the same number of individuals \( A = M \) independent of their internal structure and the canonical partition function \( Z_A(\vec{x}) = Z_M(\vec{x}) \) is the expansion of grand canonical partition function \( Z(\vec{x}) \) in terms of \( z \) with a power of multiplicity \( M \) and \( z^M = z^{\sum n_k} \) counts the total multiplicity of individuals in the system. As we will see later, this uniform treatment of all the species independent to their internal structure leads to a Poisson distribution.

If we take \( \alpha_k = k \) with the \( k \) to be a positive integer representing the number of constituent particles (such as nucleons in a nuclear fragment or cycle length of a permutation), then \( A = \sum_k k n_k \) is the total number of the constituent particles in the system and \( z^A \) term in \( Z(\vec{x}) \) is the canonical partition function \( Z_A \) with the total of \( A \) constituent particles. For this case the weight variable \( z \) is the variable counting the number of constituent particles, i.e., the fugacity of the constituent with \( z = e^\mu \) and with \( \mu \) the chemical potential of the constituent particle. This case is related to the nuclear multifragmentation model in Refs. \[23,24\] and various models of pion distribution discussed below. For \( \alpha_k = k \), and if we also take \( x_k = x \) then all the clusters are treated the same independent of the size \( \alpha_k = k \). The weight \( W \propto x^M \) is then the same for all partitions having the same multiplicity except for the Gibb’s factor. Thus \( x \) counts the multiplicity with \( x \) being a fugacity of a cluster. If we choose \( x_k = z^k \) then the weight \( W(\vec{x}, \vec{n}) \) is the same for all partitions having same \( A \). All the nucleons are treated the same independent of the cluster it belongs to thus counting the number of constituent nucleons with \( z \) being a fugacity of the constituent. If we choose \( x_k = z^{\alpha_k} \) for a general value of \( \alpha_k \) then the weight \( W(\vec{x}, \vec{n}) \) is the same for all partitions having same \( A \). All the nucleons per unit \( \alpha_k \) are treated the same independent of the cluster it belongs to thus counting \( A \) as the corresponding total quantity of \( \alpha_k \) with \( z \) being a fugacity of unit \( \alpha_k \).

On the other hand, if we choose \( \alpha_k \) to be the energy \( \epsilon_k \) of species \( k \) or \( k \)-th level, then \( A = \sum_k \alpha_k n_k = \sum_k \epsilon_k n_k = E \) is the total energy of the system and \( Z_A(\vec{x}) \) becomes the partition function of a canonical ensemble with a fixed total energy \( E \). If we choose \( \alpha_k \) to be the charge \( q_k \) of species \( k \), then \( A = \sum_k q_k n_k = Q \) is the total charge of the system. Most discussions in this section applies to a general value of \( \alpha_k \). However we will concentrate more on the choice of \( \alpha_k = k \) here for simplicity in notation.

**A. Probability distribution**

In a canonical ensemble of fixed \( A \), we can define a probability distribution of a specific partition \( \vec{n} \) as

\[ P_A(\vec{x}, \vec{n}) = \frac{W(\vec{x}, \vec{n})}{Z_A(\vec{x})} \]
With this probability, various mean values, fluctuations and correlations of the number of species \( n_k \) can be evaluated as a ratio of canonical partition functions for two different values of \( A \) such as \[30\]

\[
\left( \frac{n_k!}{(n_k-m)! (n_j-l)!} \right)_A = \sum_{\tilde{n}_A} \frac{n_k!}{(n_k-m)!} \frac{n_j!}{(n_j-l)!} P_A(\tilde{x}, \tilde{n}) = x_k^m x_j^n \frac{Z_A - m \alpha_k - l \alpha_j(\tilde{x})}{Z_A(\tilde{x})}
\]

(8)

Thus we have \(< n_k >_A = x_k \frac{Z_A + \alpha_k(\tilde{x})}{Z_A(\tilde{x})}\), which shows that the mean number of species \( k \) in a canonical ensemble with fixed \( A \) is proportional to the weight \( x_k \) assigned to the species and the ratio of canonical partition function with different \( A \). The recurrence relation of Eq. (6) then follows simply using the fact that \( A \) is determined, various statistical quantities can be evaluated. Also the grand canonical partition function can also be looked at as a grand canonical partition function with the weight \( z = 1 \), where the variable \( z \) counts \( A \) explicitly and \( Z(\tilde{x}, 1) = Z(\tilde{x}) \). If we consider \( P_A(\tilde{x}, z) = P_A(\tilde{x}) z^A \) then \( z \) has two roles; one as a weight which is assigned the same to each constituent and another as a generating parameter of the probability \( P_A(\tilde{x}) \). For the case that \( Z_0(\tilde{x}) = 1 \), the void probability \( P_0(\tilde{x}) \) is the inverse of the grand canonical partition function, i.e.,

\[
P_0(\tilde{x}) = \frac{Z_0(\tilde{x})}{Z(\tilde{x})} = Z^{-1}(\tilde{x})
\]

(11)

Inversely the canonical partition function \( Z_A(\tilde{x}) \) is the probability \( P_A \) normalized or rescaled by the void probability \( P_0 \), i.e.,

\[
Z_A(\tilde{x}) = \frac{1}{\Gamma(A+1)} \left[ \left( \frac{d}{dz} \right)^A Z(\tilde{x}, z) \right]_{z=0} = \frac{P_A(\tilde{x})}{P_0(\tilde{x})}
\]

(12)

Another type of generating function of \( P_A \) that is frequently used may be defined as

\[
G(\tilde{x}, u) = \sum_A P_A(\tilde{x})(1-u)^A = \frac{1}{Z(\tilde{x})} \sum_A Z_A(\tilde{x})(1-u)^A = \frac{Z(\tilde{x}, z=1-u)}{Z(\tilde{x})} = \exp \left[ \sum_k x_k [(1-u)^{\alpha_k} - 1] \right]
\]

(13)

\[
P_A(\tilde{x}) = \frac{1}{\Gamma(A+1)} \left[ \left( - \frac{d}{du} \right)^A G(\tilde{x}, u) \right]_{u=1}
\]

(14)

We see that \( G(\tilde{x}, 0) = 1, G(\tilde{x}, 1) = P_0(\tilde{x}) = Z_0(\tilde{x})/Z(\tilde{x}) \) and \( G(\tilde{x}, 1-z) = Z(\tilde{x}, z)/Z(\tilde{x}) \). Once the probability \( P_A(\tilde{x}) \) is determined, various statistical quantities can be evaluated. Also the grand canonical partition function \( Z(\tilde{x}, z) \) is given once we know the thermodynamic grand potential

\[
\Omega(\tilde{x}, z) = - \ln Z(\tilde{x}, z) = - \sum_k x_k z^\alpha_k.
\]

(15)

This result can be used to study the statistical properties of a system. Moreover various moments and cumulants, mean values and fluctuations may be obtained using the generating function \[23, 24\].
B. Moments and cumulants; combinants and hierachical structure

This subsection gives general expression for various quantities that will be used later when we discuss specific models. Since the probability of a specific event \( n \) in a grand canonical ensemble is given by

\[
P(\vec{x}, \vec{n}) = \frac{W(\vec{x}, \vec{n})}{Z(\vec{x})}
\]  

the mean of a quantity \( F \) in a grand canonical ensemble is related to the mean of \( F \) in a canonical ensemble as

\[
<F> = \sum_{\vec{n}} F P(\vec{x}, \vec{n}) = \frac{1}{Z(\vec{x})} \sum_{\vec{n}} F W(\vec{x}, \vec{n}) = \sum_{A} \frac{Z_A(\vec{x})}{Z(\vec{x})} \sum_{\vec{n}_A} F W(\vec{x}, \vec{n})
\]

\[
= \sum_{A} P_A(\vec{x}) \frac{1}{Z_A(\vec{x})} \sum_{\vec{n}_A} F P_A(\vec{x}, \vec{n}) = \sum_{A} P_A(\vec{x}) <F>_A
\]

Thus once we obtain \( Z_A(\vec{x}) \) using the recurrence relation of Eq.(11), then using the mean in canonical ensemble given by Eq.(8), we can obtain mean in grand canonical ensemble. Since, in a grand canonical ensemble, \( A \) can be infinity, i.e., \( 0 \leq A < \infty \) with \( Z_A(\vec{x}) = 0 \) for \( A < 0 \) if all \( \alpha_k \geq 0 \) and \( -\infty < A < \infty \) if \( \alpha_k \) can be negative, we have

\[
Z(\vec{x}) = \sum_{A} Z_A(\vec{x}) = \sum_{A} Z_{A-m \alpha_k - \alpha_k}(\vec{x})
\]

for integer \( m \) and \( l \). Using this fact, we can see easily that

\[
<n_k> = \sum_{\vec{n}} n_k P(\vec{x}, \vec{n}) = \sum_{A} P_A(\vec{x}) <n_k>_A = \sum_{A} P_A(\vec{x}) x_k \frac{Z_{A-\alpha_k}(\vec{x})}{Z_A(\vec{x})}
\]

\[
= x_k \sum_{A} \frac{Z_{A-\alpha_k}(\vec{x})}{Z(\vec{x})} = x_k
\]

\[
<M> = \sum_{\vec{n}} \left( \sum_{k} n_k \right) P(\vec{x}, \vec{n}) = \sum_{k} x_k
\]

This result shows that the weight factor \( x_k \) in this model is the mean number \(<n_k>\) in a grand canonical ensemble. The \( m \)-th power moment of \( A \) and its factorial moments are given simply by

\[
<A^m>(\vec{x}) = \sum_{\vec{n}} \left( \sum_{k=1}^{N} \alpha_k n_k \right)^m P(\vec{x}, \vec{n}) = \frac{1}{Z(\vec{x})} \left[ \left( \frac{z}{dz} \right)^m Z(\vec{x}, z) \right]_{z=1}
\]

\[
\langle \frac{\Gamma(A+1)}{\Gamma(A-m+1)} \rangle(\vec{x}) = \sum_{\vec{n}} \frac{\Gamma(A+1)}{\Gamma(A-m+1)} P(\vec{x}, \vec{n}) = \frac{1}{Z(\vec{x})} \left[ \left( \frac{d}{dz} \right)^m Z(\vec{x}, z) \right]_{z=1}
\]

Similarly the \( m \)-th cumulants , which is the power moments of \( \alpha_k \), and the factorial cumulants are

\[
<\alpha^m>(\vec{x}) = \left( \sum_{k=1}^{N} \alpha_k^m n_k \right) = \left[ \left( \frac{d}{dz} \right)^m \ln Z(\vec{x}, z) \right]_{z=1} = \sum_{k=1}^{\infty} \alpha_k^m x_k
\]

\[
f_m(\xi) = \left( \sum_{k=1}^{N} \frac{\Gamma(\alpha_k+1)}{\Gamma(\alpha_k-m+1)} n_k \right) = \left[ \left( \frac{d}{dz} \right)^m \ln Z(\vec{x}, z) \right]_{z=1}
\]

\[
= \left[ \left( \frac{d}{du} \right)^m \ln G(\vec{x}, u) \right]_{u=0} = \sum_{k=1}^{N} \frac{\Gamma(\alpha_k+1)}{\Gamma(\alpha_k-m+1)} x_k
\]

We can see easily that, for the power moments of \( \alpha_k \),

\[
<\alpha^0> = \sum_{k=1}^{N} x_k = f_0 = <M>
\]

(25)
Thus, from the factorial structure of $f$ we also have

$$<\alpha > = \sum_{k=1}^{N} \alpha_k x_k = f_1 = < A > \tag{26}$$

$$<\alpha^2 > = \sum_{k=1}^{N} \alpha_k^2 x_k = f_2 + f_1 = < (A- < A >)^2 > = < A^2 > - < A >^2 = \sigma^2 \tag{27}$$

$$<\alpha^3 > = \sum_{k=1}^{N} \alpha_k^3 x_k = f_3 + 3f_2 + f_1 = < (A- < A >)^3 > \tag{28}$$

The power moments of $\alpha_k$ are directly related to the power moments of $A$ measured from the mean $< A >$, i.e., the cumulants $<\alpha^m >$ are same with the central moments of $A$. This simple relation does not hold for $m \geq 4$ but we can evaluate them starting from $<\alpha^3 >$ using the recurrence relation

$$<\alpha^{m+1} > (\bar{x}, z) = \left( z \frac{d}{dz} \right)^{m+1} \ln Z(\bar{x}, z) = \left( z \frac{d}{dz} \right) <\alpha^m > (\bar{x}, z) \tag{29}$$

Similarly the $m$-th factorial cumulants $f_m$, which is the factorial moments of $\alpha_k$, can be found using recurrence relation

$$\frac{f_{m+1}(\bar{x}, z)}{f_m(\bar{x}, z)} = z \left( \frac{d}{dz} \right) \ln f_m(\bar{x}, z) - m \tag{30}$$

starting from

$$f_0(\bar{x}, z) = < M > (\bar{x}, z) = \ln Z(\bar{x}, z) = -\Omega(\bar{x}, z) = \sum_{k=1}^{N} x_k z^{\alpha_k} \tag{31}$$

$$f_1(\bar{x}, z) = < A > (\bar{x}, z) = \sum_{k=1}^{N} \alpha_k x_k z^{\alpha_k} \tag{32}$$

The reduced factorial cumulants $\kappa_m$ defined in Ref. [30] corresponds to the factorial cumulants $f_m$ normalized with mean number $< A > = A$ as

$$\kappa_m(\bar{x}, z) = \frac{f_m(\bar{x}, z)}{A^m} \tag{33}$$

Thus with $\kappa_1 = 1$ and $\kappa_0 = f_0 = < M >$.

A Taylor expansion of $(p+q)^a$ with a real number $a$, w.r.t. $p$ for $q \neq 0$, is

$$ (p+q)^a = \sum_{n=0}^{\infty} \frac{p^n}{n!} \left( \frac{d}{dp} \right)^n (p+q)^a = \sum_{n=0}^{\infty} \frac{\Gamma(a+1)}{n! \Gamma(a-n+1)} q^{-n} p^n = \sum_{n=0}^{\infty} \binom{a}{n} p^n q^{-n} \tag{34} $$

Thus, from the factorial structure of $f_m$ and from Eqs.(14) and (22),

$$ G(\bar{x}, u) = \frac{Z(\bar{x}, z = 1-u)}{Z(\bar{x})} = \exp \left[ \sum_{m=1}^{\infty} \frac{(-u)^m}{m!} f_m(\bar{x}) \right] = \exp \left[ \sum_{m=1}^{\infty} \frac{(-u\bar{A})^m}{m!} \kappa_m(\bar{x}) \right] \tag{35}$$

Since $P_0(\bar{x}) = G(\bar{x}, 1)$ the probability $P_A$ can be obtained by

$$ P_A(\bar{x}) = \frac{1}{\Gamma(A+1)} \left[ \left( \frac{d}{du} \right)^A G(\bar{x}, u) \right]_{u=1} = \frac{(-\bar{A})^A}{\Gamma(A+1)} \left[ \left( \frac{d}{dA} \right)^A P_0(\bar{x}) \right] \tag{36} $$

when $\kappa_m$ is independent of the mean $< A > = \bar{A}$ as used in Ref. [30]. Using the power moment $<\alpha^m >$ of Eq.(23), we also have

$$ G(\bar{x}, u = 1 - e^{-\lambda}) = \frac{Z(\bar{x}, z = e^{-\lambda})}{Z(\bar{x})} = \exp \left[ \sum_{m=1}^{\infty} \frac{(\lambda)^m}{m!} <\alpha^m > (\bar{x}) \right] \tag{37} $$
Since $f_0 = \ln Z = \langle \alpha^0 \rangle$,

$$Z(\vec{x}, z) = G(\vec{x}, u = 1 - z)Z(\vec{x}) = G(\vec{x}, u = 1 - z)e^{f_0(\vec{x})}$$

$$= \exp \left[ \sum_{m=0}^{\infty} \frac{(z - 1)^m}{m!} f_m(\vec{x}) \right] = \exp \left[ \sum_{m=0}^{\infty} \frac{(\ln z)^m}{m!} < \alpha^m > (\vec{x}) \right]$$  \hspace{1cm} (38)

The generating function $Z(\vec{x}, z)$ differs from the generating function $G(\vec{x}, u)$ only by an extra term of $m = 0$ in their exponent.

The relation between the $x_k$’s and the $Z$ shows that the $x_k$’s are also the combinatorial cumulants of $Z(\vec{x}, z)$, which will be discussed below. The generating function $Z(\vec{x}, z)$ is usually fixed. In multiparticle production the $Z$’s are also the combinatorial cumulants of Ref. \cite{31}. In turn the combinatorial cumulants $f_m$ are the $m$-th order factorial moments of $\alpha_k$ of Eq.\cite{24}. Thus

$$f_m = m! \sum_{k=m}^{\infty} \frac{k}{m} x_k$$  \hspace{1cm} (40)

for $\alpha_k = k$. The normalized factorial cumulants, i.e., the reduced cumulant, is

$$\kappa_m = \frac{f_m}{\bar{A}^m} = (m - 1)! \kappa_{m-1}$$  \hspace{1cm} (41)

for a negative binomial (NB) distribution. This result of Eq.\cite{41} shows that $\kappa_m$ for NB has an hierarchical structure of a distribution at the reduced cumulant level which was realized for the NB distribution in Ref. \cite{31}. This result will be generalized later.

Also using the above power moments and factorial moments we can study voids and void scaling relation, hierarchical structure, combinator and cumulant properties which will be discussed below.

C. Multi-fragmentation versus multiparticle production

Since our approach was first used to discuss multifragmentation and then later extended to include multiparticle production, we briefly mention some of difference between multifragmentation and multiparticle production.

In nuclear multifragmentation, $\alpha_k = k$ is the number of nucleons in a fragment and $n_k$ is the number of fragments of size $k$. The mean values of the total number of fragments $M = \sum_k n_k$ and the total number of nucleons $A = \sum_k k n_k$ are

$$< M > = \sum_k < n_k > = \sum_k x_k = f_0 = \ln Z$$  \hspace{1cm} (42)

$$< A > = \sum_k k < n_k > = \sum_k k x_k = f_1$$  \hspace{1cm} (43)

In nuclear multifragmentation, the total number of nucleons $A$ is usually fixed. In multiparticle production the $A$ is the total number of produced particles and is not fixed. Thus in multifragmentation we consider canonical ensemble instead of a grand canonical ensemble. The mean multiplicity $< M >_A = \sum_k < n_k >_A$ and $< n_k >_A = x_k z A^{-k} Z_A(\vec{x})$ in a canonical ensemble are related with grand canonical ensemble as

$$< M > = \sum_{A=0}^{\infty} < M >_A P_A$$

$$< n_k > = \sum_{A=0}^{\infty} < n_k >_A P_A = x_k$$  \hspace{1cm} (45)

The $x_k = < n_k >$ in a grand canonical ensemble. But the weight factor $x_k$ is different from the mean multiplicity of cluster size $k$ in a canonical ensemble $< n_k >_A$. From Eq.\cite{6}, $< n_k >_A$ is the weight $x_k$ multiplied by the ratio of $Z_{A-k}$ and $Z_A$, i.e., $< n_k >_A = x_k Z_{A-k}/Z_A$. In multifragmentation studies \cite{23,24}, we have used $x_k = x_k^A / k$ with the same weight $x$ for each cluster and the same weight $z$ for each nucleon. This choice of $x_k$ gives $Z_A(x, z) = \frac{z^A \Gamma(x + A)}{\Gamma(x)}$ and thus $Z_{A-k}/Z_A \rightarrow z^{-k}$ as $A \rightarrow \infty$ where the $z = e^{\mu}$ with $\mu$ the chemical potential. To determine the weight $x_k$ experimentally, we should use a grand canonical ensemble, i.e., we need to consider various system with different values of $A$.  

7
Van Hove and Giovannini have introduced clan variables $N_c$ and $n_c$ to describe a general class of probability distributions, with most discussions of these variables centering around the negative binomial distribution [13]. These variables are defined as

$$N_c = < M > = \ln Z = f_0, \quad n_c = < A > / N_c = f_1 / f_0$$

(46)

where the mean number of clans is $N_c$ and the $n_c$ is the mean number of members per clan. The $Z$ is the grand canonical generating function and thus $N_c = \sum x_k$ where $x_k$ is the cycle class weight distribution $\bar{x}$. The $< A > = \sum k < n_k >$ is the mean number of total members (particles). The $n_k$ here is the number of clans of size $k$ having $k$ members.

The clan variable $N_c$ is also related to the void probability $P_0 = Z_0 / Z = 1 / Z = e^{-N_c}$; thus $N_c = - \ln P_0 = f_0$. An important function in void analysis is $\chi = - \ln P_0 / < A > = N_c / < A > = 1 / n_c$. Thus the void parameters, void probability $P_0$ and void function $\nu$ [30], are equivalent to the generalized clan parameters $N_c$ and $n_c$ with the equivalence given by:

$$f_0(\bar{x}) = \ln Z(\bar{x}) = - \ln P_0(\bar{x}) = N_c$$

(47)

$$\chi(\bar{x}) = \frac{f_0(\bar{x})}{< A >} = \nu = n_c^{-1}$$

(48)

The $-\chi$ is the normalized grand potential $\Omega = -f_0$ for the mean $< A >$.

Void analysis looks for scaling properties associated with $\chi$; specifically, $\chi$ is a function of the combination $< A > \xi$ where $\xi$ is the coefficient of $< A >^2$ in the fluctuation $\sigma^2 = < A > + \xi < A >^2$. Since $f_2 = < \alpha^2 > = < A > + \xi < A >^2$, the variance of $A$ in a grand canonical ensemble becomes

$$\sigma^2 = < A^2 > = < A > + f_2 = < A > + \xi < A >^2 = < A > [1 + \xi(\bar{x}) < A >]$$

(49)

with $\xi = \kappa_2$, i.e., the normalized factorial cumulant. Since the variance for a Poissonian distribution is the same as the mean, $\sigma^2 = < A >$, the $\xi = 0$; thus, the parameter $\xi < A >$ represents a degree of departure from Poissonian fluctuation normalized by mean $< A >$ of the distribution. A well known non-Poissonian example is a NB distribution which has $\xi = \frac{1}{\bar{x}}$ and this becomes Plank distribution with $x = 1$. Using the recurrence relation Eq.(30) for $f_m$, we can show that

$$\xi(\bar{x}, z) < A > (\bar{x}) = \chi^{-1}(\bar{x}, z) - z \left( \frac{d}{dz} \right) \ln \chi(\bar{x}, z) - 1 = \frac{1 - z \left( \frac{d}{dz} \right) \chi(\bar{x}, z)}{\chi(\bar{x}, z)} - 1$$

(50)

$$\kappa_3(\bar{x}, z) < A >^3 = \frac{f_3(\bar{x}, z)}{< A >^3} = \frac{\xi(\bar{x}, z)}{< A >} \left( z \left( \frac{d}{dz} \right) \ln \left[ \xi(\bar{x}, z) < A >^2 \right] - 2 \xi(\bar{x}, z) < A > \right)$$

(51)

A NB distribution has $\chi = \ln(1 + \xi < A >) / (\xi < A >)$ while the Lorentz/Catalan (LC) distribution discussed in Ref. [32] and below has $\chi = (\sqrt{2x} < A > + 1 - 1) / (\xi < A >)$. We will study in Sect. [IV.A] $\chi$ vs $\xi < A >$, i.e., the void or clan variable vs the fluctuation for various choices of $x_k$ summarized in Table I.

E. Ancestral or evolutionary variables

The LC model was shown to be a useful model for discussing an underlying splitting dynamics when ancestral or evolutionary variables $p$ and $\beta$ were introduced into $x$ and $z$ as discussed in Ref. [32]. Percolation or splitting dynamics with a branching probability $p$ and survival probability $(1 - p)$ has a hierarchical topology as shown in Fig. 2. Weighting each diagram by $x_k = \beta C_k p^{k-1}(1 - p)^k$, the evolutionary or ancestral variables are related to the clan variables $N_c = < M >$ and $n_c = < A > / N_c$. By taking $C_k$ to be the number of diagrams of size $k$ shown in Fig. 2, the evolutionary dynamics is just that of the LC model. Then with $\beta$ set equal to 1, $x_1 = (1 - p)$, $x_2 = p(1 - p)$, $x_3 = 2p^2(1 - p)^3$, $x_4 = 5p^3(1 - p)^4$, etc. The interpretation of this set of $x_k$’s reads as follows: $x_1$ has 1 surviving line without a branch ($p^0(1 - p)^1$) and one diagram ($C_1 = 1$), $x_2$ has 1 branch ($p^1$) leading to 2 surviving lines ($((1 - p)^2$) and one diagram ($C_2 = 1$), $x_3$ has 2 branch points ($p^2$) leading 3 surviving lines ($((1 - p)^3$) and two diagrams ($C_3 = 2$), $x_4$ has 3 branch points ($p^3$), 4 surviving lines ($((1 - p)^4$) and 5 diagrams ($C_4 = 5$), etc. In these evolutionary/ancestral
variables the \( f_0 \) which determines \( Z \) is \( f_0 = \sum x_k = \beta \) for all \( p \leq 1/2 \). For \( p \geq 1/2 \), \( f_0 = \sum x_k \) is no longer a constant and is \( f_0 = \sum x_k = \beta(1 - p)/p \).

Since the clan variables are \( N_c = < M > = f_0 \) and \( n_c = < A > /N_c = f_1 / f_0 \), then, for the LC model with evolutionary or ancestral variables,

\[
N_c = < M > = \beta \frac{1 - |1 - 2p|}{2p} \tag{52}
\]

\[
n_c = < A > /N_c = \frac{2p(1 - p)}{|1 - 2p(1 - 1 - 2p)|} \tag{53}
\]

\[
< A > = \beta \frac{(1 - p)}{|1 - 2p|} \tag{54}
\]

\[
p = \frac{1}{2} \left[ 1 \mp \frac{1}{2n_c - 1} \right] \tag{55}
\]

\[
\beta = N_c \frac{n_c - (1 + 1)/2}{n_c - 1} \tag{56}
\]

These can be reduced to \( N_c = \beta \), \( n_c = (1 - p)/(1 - 2p) \), and \( 2p = 1 - 1/(2n_c - 1) \) for \( p \leq 1/2 \) while \( N_c = \beta(1 - p)/p \), \( n_c = p/(2p - 1) \), and \( 2p = 1 + 1/(2n_c - 1) \) for \( p > 1/2 \). Since the branching probability \( p \) varies in \( 0 \leq p \leq 1 \), the clan variable \( n_c \) has \( n_c \geq 1 \) with \( n_c = 1 \) at \( p = 0 \) and \( 1 \) and \( n_c = \infty \) at \( p = 1/2 \). The behavior of \( x_k \) above \( p = 1/2 \) will be discussed in Sec. \( \text{III D} \) where a \( x_0 = \phi_{\infty} \) will be introduced. The LC model thus connects the clan variable \( n_c \) to the probability \( p \) of branching in the evolutionary or ancestral picture of Fig. 3 or in a percolation model. For Poisson processes \( p = 0 \) (no branching), \( x_k = \beta \delta_{k1} \) (only unit cycles and no BE correlations) and \( n_c = 1 \) (one member in each clan in average).

\[\text{FIG. 2. Evolutionary lines of descent in a hierarchical topology. Each branch increases the cycle length with probability } p, \text{ survival } 1 - p. \text{ The probability distribution evolves from Poisson to chaotic. For clusters each branch generates a bigger cluster.}\]

F. Various distributions with Gauss hypergeometric series

In describing either nuclear multifragmentation or multiparticle distributions, \( \alpha_k \) is taken to be a positive integer \( k \). In a nuclear fragment distribution, an initial number of \( A \) nucleons are clustered into \( n_k \) small nuclei of \( k \) nucleons. For multiparticle distributions, \( k \) can be related to the length of the cycle in a cycle class representation of permutations for Bosonic or Fermionic symmetry. Then \( A \) is the total number of produced particles of a given type such as pions and \( n_k \) is the number of cycle classes of cycle length \( k \) for a permutation of \( A \) identical particles. These cases have no situation with \( k = 0 \). If we consider a formation of \( M \) jets which are followed by a pion creation process, then \( k \) may be related to the number of pions from a jet. For this case, \( k = 0 \) can also be included as a jet having no pion.

Once we identify an appropriate \( x_k \) for a physical system, then we may use our general model to study the statistical behavior of the system. The various models used in pion distribution can be related to our general model with \( \alpha_k = k \) by choosing \( x_k \) as a term in a Gauss hypergeometric series \( F(a, b; c; z) \);

\[
F(a, b; c; z) = \sum_{m=0}^{\infty} \frac{[a]_m [b]_m z^m}{[c]_m m!} \tag{57}
\]

\[
[a]_m = \frac{\Gamma(a + m)}{\Gamma(a)} = \frac{\Gamma(a + n + m - n)}{\Gamma(a + n)} = [a]_n [a + n]_m - n \tag{58}
\]

Usefull values of \([a]_n\) are \([1/2]_n = (2n)!/(n!)^2 \), \([1]_n = n! \), \([2]_n = (n + 1)! \). Considering only positive \( k \), we choose
\[ x_k = x \frac{[a]_{k-1} [b]_k}{[c]_{k-1}} \frac{z^k}{(k-1)!} = x \frac{\Gamma(a + k - 1)}{\Gamma(a)} \frac{\Gamma(c + k)}{\Gamma(c + k + 1)} \frac{\Gamma(b + k - 1)}{\Gamma(b)} \frac{z^k}{(k-1)!} \]

(59)

For this case, the thermodynamic grand potential or the generating function is

\[ f_0(\bar{x}) = f_0(x, z) = \log Z(x, z) = -\Omega(x, z) = \sum_{k=1}^{\infty} x_k = xzF(a, b; c; z) \]

(60)

If we allow jets without a pion, then we may allow \( k = 0 \) also. For such a case,

\[ x_k = x \frac{[a]_{k-1} [b]_k}{[c]_{k-1}} \frac{z^k}{(k-1)!} = x \frac{\Gamma(a + k)}{\Gamma(a)} \frac{\Gamma(c + k + 1)}{\Gamma(c + k)} \frac{\Gamma(b + k)}{\Gamma(b)} \frac{z^k}{k!} \]

(61)

\[ f_0(\bar{x}) = f_0(x, z) = \log Z(x, z) = -\Omega(x, z) = \sum_{k=0}^{\infty} x_k = xF(a, b; c; z) \]

(62)

We can see the only difference of the generating functions between above two cases is the extra factor \( z \) for the grand potential. We will mostly concentrate on the first case, i.e., \( k \neq 0 \).

Using Eq. (24) or the recurrence relation Eq. (30) and Eqs. (57) – (58),

\[ f_m(x, z) = -z^m \frac{d^m}{dz^m} \log Z(x, z) = z^m \left( \frac{d^m}{dz^m} xzF(a, b; c; z) \right) \]

\[ = \frac{z^m [a]_m [b]_m [c]_m}{[c]_m} \]

(63)

The second order normalized factorial cumulant \( \xi = k_2 \) and the void variable \( \chi \) are then

\[ \xi(x, z) < A > = \frac{f_2(x, z)}{f_1(x, z)} = \frac{f_2(x, z)}{f_1(x, z)} = \frac{\Gamma(a + 2, b + 2; c + 2; z)}{\Gamma(a + 1, b + 1; c + 1; z)} \]

(64)

\[ \chi(x, z) < A > = \frac{f_0(x, z)}{f_1(x, z)} = \frac{f_0(x, z)}{f_1(x, z)} = \frac{\Gamma(a, b; c; z)}{\Gamma(a + 1, b + 1; c + 1; z)} \]

(65)

For some values of \( a, b, \) and \( c, \) the hypergeometric function become a simple function:

\[ F(a, b; c; z) = (1 - z)^{-a} \]

\[ F(a, b; a; z) = (1 - z)^{-b} \]

\[ F(a, 1; 2; z) = \frac{1 - (1 - z)^{1-a}}{z(1-a)} \]

\[ F(1, 1; 2; z) = \lim_{a \to 1} F(a, 1; 2; z) = -\frac{\ln(1 - z)}{z} \]

(66)

Various models of pion distributions can be related with these functions as listed in Table I.

| Model          | \( x_k \) | \( f_0(\bar{x}) = \ln Z \) | \( a \) | \( b \) | \( c \) |
|---------------|----------|--------------------------|-------|-------|-------|
| Poisson (P)   | \( N \delta_{k,1} \) or \( x \) for \( k = 1, 2, \ldots, N \) | \( \frac{N_x}{1-z} \) | \( a \) | 1     | \( a \) |
| Geometric (Geo)| \( xz^k \) | \( \frac{N_x}{1-z} \) | \( a \) | 1     | \( a \) |

TABLE I. Various models with specific choice of \( \alpha_k = k \) and \( x_k \) in hypergeometric series \( F(a, b; c; z) \) of Eq.(74). Here \( k = 0 \) is not included, and thus \( f_0 = \ln Z = \sum_{k=1}^{\infty} x_k = xzF(a, b; c; z) \). Here \( 1 \leq k \leq N \) with \( N \to \infty \) except for Poisson which has a finite \( N_x \).
Negative Binomial (NB) \[ \frac{1}{k} x^k \left( y + \frac{1}{k} \right)^z \] Signal/Noise (SN) \[ \frac{1}{z} e^{-x \ln(1 - z)} \] Hypergeometric (HG) \[ \frac{x}{k} \left[ \frac{1}{z} - 1 \right]^x \] \[ -x \ln(1 - z) \] \[ 1 \] \[ \frac{1}{2} \] \[ 1 \] \[ a \] \[ \frac{1}{2} \] \[ 1 \]

Lorentz/Catalan (LC) \[ 2^{2(k-1)} \left( \frac{2(k-1)}{k-1} \right) xz \] \[ 2x(1 - (z - 2))^{1/2} \] \[ a \] \[ \frac{1}{2} \] \[ 1 \] \[ \frac{1}{2} \] \[ 1 \] \[ \frac{1}{2} \]

Random Walk-1d (RW1D) \[ 2^{2(k-1)} \left( \frac{2(k-1)}{k-1} \right) xz \] \[ xz(1 - z)^{-1/2} \] \[ \frac{1}{2} \] \[ 1 \] \[ \frac{1}{2} \] \[ 1 \] \[ \frac{1}{2} \] \[ 1 \]

Random Walk-2d (RW2D) \[ \left[ 2^{2(k-1)} \left( \frac{2(k-1)}{k-1} \right) \right]^2 xz^k \] \[ xzF(\frac{1}{2}, \frac{1}{2}; 1; z) \] \[ \frac{1}{2} \] \[ \frac{1}{2} \] \[ 1 \]

Generalized RW1D (GRW1D) \[ \frac{1}{(k-1)x} xz^k \] \[ xz(1 - z)^{-a} \] \[ a \] \[ b \] \[ b \]

Generalized RW2D (GRW2D) \[ \left[ \frac{1}{(k-1)x} \right]^2 xz^k \] \[ xzF(a, a; 1; z) \] \[ a \] \[ a \] \[ 1 \]

### TABLE II. Factorial cumulants for various choices of \( x_k \) of Table I

| Model   | \( f_0 = \log Z \) | \( f_1 = \langle A \rangle \) | \( z^{-2} f_2 \) | \( z^{-m} f_m \) |
|---------|-------------------|-------------------------------|------------------|-------------------|
| P       | \( N_x = A \)     | \( A \)                       | \( 0 \)          | \( 0 \)          |
| Geo     | \( x \frac{x}{1 - z} \) | \( x \frac{x}{1 - z} \) | \( x \frac{y}{(1 - z)^{1/2}} \) | \( x \frac{y}{(1 - z)^{m/2}} \) |
| NB      | \( -x \ln(1 - z) \) | \( y \frac{x}{(1 - z)^{1/2}} \) | \( y \frac{x}{(1 - z)^{1/2}} \) | \( y \frac{m}{(1 - z)^{m/2}} \) |
| SN      | \( \frac{x}{1 - z} = x \ln(1 - z) \) | \( x \frac{x}{1 - z} \) | \( x \frac{y}{(1 - z)^{1/2}} \) | \( x \frac{y}{(1 - z)^{m/2}} \) |
| LC      | \( 2x(1 - (z - 1)^{1/2}) \) | \( x \frac{x}{1 - z} \) | \( x \frac{y}{(1 - z)^{1/2}} \) | \( x \frac{y}{(1 - z)^{m/2}} \) |
| HG      | \( \frac{x}{1 - z} = x \ln(1 - z) \) | \( x \frac{x}{1 - z} \) | \( x \frac{y}{(1 - z)^{1/2}} \) | \( x \frac{y}{(1 - z)^{m/2}} \) |
| RW1D    | \( x \frac{x}{1 - z} \) | \( x \frac{x}{1 - z} \) | \( x \frac{y}{(1 - z)^{1/2}} \) | \( x \frac{y}{(1 - z)^{m/2}} \) |
| GRW1D   | \( x \frac{x}{1 - z} \) | \( x \frac{x}{1 - z} \) | \( x \frac{y}{(1 - z)^{1/2}} \) | \( x \frac{y}{(1 - z)^{m/2}} \) |
More detail discussions and related physical systems of these distributions will be given and discussed in the next section. Brief discussion about the weight $x_k$ of each model follows. A Poisson (P) distribution is generated when there is monomers only, i.e., $x_k = Nx_0$ which gives $f_0 = Nx$. A Poisson can also be generated if all the clusters or cycles are treated the same with the same weight of $x_k = x$ independent of their size $k$ for $1 \leq k \leq N$ which also gives $f_0 = Nx$. The second case can also be viewed as weighting each constituents by $x^{1/k}$ so that $x_k = (x^{1/k})^k = x$. The weight $x$ counts the multiplicity $M$ and the multiplicity of clusters (or of monomers for $x_k = Nx_0$) has the Poisson distribution. All the other distributions considered in Table I have several factors in the weight $x_k$. One factor in $x_k$ is $z^k$ which is a $k$ dependent geometric term and comes from assigning the same weight $z$ to each constituents independent of the cluster or cycle classes it belongs to. Another factor of weight is independent of $k$ such as the $x$ in Table I which comes from assigning the same weight $x$ to each cluster or the cycle class as a whole independent of its internal structure. These two factors, $x z^k$, are multiplied by a $k$ dependent or independent prefactor. A geometric (Geo) distribution follows when there is no other weight factor beside $x$ and $z$, i.e., no $k$ dependent prefactor so that $x_k = x z^k$. The Geo with $z = 1$ for a finite $N$ is the same as the Poisson distribution; both have $f_0 = Nx$. The negative binomial (NB) which appears frequently in various studies has a weight factor assigned to a cluster or cycle class given by $x z F(a, b; z)$ where $F(a, b; z) = z^b / \Gamma(a)$ is a confluent hypergeometric function with weight $x$. Its associated standard normal distribution is given by

$$
Z(x, z) = e^{f_0} = e^{x z F(a, 1; 2; z)} = \exp \left[ \frac{x}{(a - 1)} \left( \frac{1}{(1 - z)(a - 1)} - 1 \right) \right] \quad (68)
$$

is shown in Table I. From Table I, we have $\sum_{k=0}^{\infty} x F(a, b; k; z) = x z F(a, b; z)$ as a consequence of the invariance of $z^k$ when $k$ is a positive integer. Note that $x F(a, b; z) = 1$ when $z = 0$. The Geo model is given by

$$
x_k = x z^k \frac{(a)_k}{k!} = x z^k \frac{\Gamma(a + k)}{k! \Gamma(a)} \quad (67)
$$

and the NB model has $f_0 = x z F(1, 1; 2; z)$. The Geometric model $x_k = y z^k$ has $f_0 = y z F(1, 1; 2; z)$ while the SN model is a combination of the geometric plus NB cases. These functions are special cases of $f_0 = x z F(a, 1; 2; z)$ of the HGa model. The generalized random walk in 1-dimension (GRW1D) has $f_0 = x z F(a, b; z)$ and the generalized RW in 2-dimension (GRW2D) has $f_0 = x z F(a, a; 1; z)$. The factorial cumulants $f_n$ for these models are summarized in Table I. The cases with $c = 2$ have a canonical partition function $Z_n$ which can be written in terms of confluent hypergeometric functions $U(u, v; w)$ and standard factor $z^n / n!$. 

G. Generalized model of Hypergeometric (HGa)

We consider in more detail the hypergeometric model with $b = 1$ and $c = 2$ (HGa) here since it includes the NB, Geo, and LC models as special cases. This generalized model is related with the hypergeometric function with $b = 1$ and $c = 2$ with an arbitrary value of $a$, i.e., $F(a, 1; 2; z)$, and has the weight of

$$
x_k = x z^k \frac{(a)_{k-1}}{k!} = x z^k \frac{\Gamma(a + k)}{k! \Gamma(a)} \quad (67)
$$

Its associated grand canonical partition function

$$
Z(x, z) = e^{f_0} = e^{x z F(a, 1; 2; z)} = \exp \left[ \frac{x}{(a - 1)} \left( \frac{1}{(1 - z)(a - 1)} - 1 \right) \right] \quad (68)
$$

is shown in Table I. From Table I, we have
\[ f_m(x, z) = [a]_{m-1} \frac{xz^m}{(1 - z)^{a+m-1}} = x \frac{\Gamma(a + m - 1)}{\Gamma(a)} \left( \frac{z^m}{1 - z} \right) \] (69)

\[ \kappa_m(x, z) = \frac{[a]_{m-1}}{x^{m-1}} \frac{1 - (1 - z)^{(a-1)(m-1)}}{\Gamma(a)} = \frac{\Gamma(a + m - 1)}{\Gamma(a)} \left( \frac{(1 - z)^{(a-1)}}{x} \right)^{m-1} \] (70)

The normalized factorial cumulant, i.e., the reduced cumulant \( \kappa_m \) shows the hierarchical structure of HG\( a \) at the reduced cumulant level with \( A_m = a^{-(m-1)} \Gamma(a + m - 1) / \Gamma(a) \) where \( \kappa_m \) is related to \( \kappa_2 \). This property was realized for the NB distribution in Ref. [30] which is obtained for Eq.(70) with \( a = 1 \) giving \( A_m = (m - 1)! \). The result of Eq.(70) is a generalization of the NB result.

Some moments for HG\( a \) are

\[ < A >(x, z) = f_1(z, x) = \frac{xz}{(1 - z)^a} \] (71)

\[ \chi(x, z) = \frac{f_0(x, z)}{f_1(x, z)} = 1 \frac{1 - (1 - z)^{a-1}}{(1 - a) \left( \frac{1 - z}{z} \right)^{a-1}} \] (72)

\[ \xi(x, z) = \kappa_2(x, z) = \frac{f_2(x, z)}{f_1(x, z)} = \frac{a}{x} \left( \frac{1 - z}{1 - z} \right)^{(a-1)} \] (73)

Since these relations give

\[ \xi(x, z) < A >(x, z) = \kappa_2(x, z) < A >(x, z) = \frac{f_2(x, z)}{f_1(x, z)} = \frac{a}{x - (1 - z)} \] (74)

the void parameters can be obtained in terms of the normalized fluctuation \( \xi \) and the mean number \( < A > = \bar{A} \) by

\[ z(\bar{A}, \xi) = \frac{x \bar{A}}{a + \xi A} = \frac{f_2(\bar{A})}{f_2 + aA} \frac{f_2}{f_2 + aA} \] (75)

\[ x(\bar{A}, \xi) = \frac{\bar{A}}{z(1 - z)^a} = \frac{a}{\bar{A}} \left( \frac{a}{a + \xi A} \right)^{(a-1)} = \frac{aA^2}{f_2} \left( \frac{aA^2}{a + f_2} \right)^{-a-1} \] (76)

\[ f_0(\bar{A}, \xi) = \log Z(\bar{A}, \xi) = \frac{x}{a - 1} \left[ \left( 1 + \frac{\xi A}{a} \right)^{a-1} - 1 \right] = \frac{x}{a - 1} \left[ \left( 1 + \frac{f_2}{a \bar{A}} \right)^{a-1} - 1 \right] \] (77)

\[ \chi(\bar{A}, \xi) = \frac{f_0(\bar{A}, \xi)}{\bar{A}} = 1 \frac{a}{(1 - a) \xi A} \left[ 1 + \frac{\xi A}{a} \right]^{a-1} - 1 = \frac{1}{(a - 1) f_2} \left[ 1 + \frac{f_2}{a \bar{A}} \right]^{a-1} - 1 \] (78)

\[ \kappa_m(\bar{A}, \xi) = \frac{f_m(\bar{A}, \xi)}{A^m} = \frac{\Gamma(a + m - 1)}{\Gamma(a)} \left( \frac{\xi}{a} \right)^{m-1} \] (79)

for a given mean value of \( < A > = \bar{A} \) and the fluctuation \( \xi \) or \( \xi \bar{A} \) or \( f_2 \).

Table 3 shows that the generalized HG model becomes the Lorentz/Catalan (LC) model with \( a = 1/2 \), the negative binomial (NB) model with \( a = 1 \), and geometric (Geo) distribution with \( a = 2 \). However the NB should be considered as a \( a \to 1 \) limit of HG\( a \);

\[ \lim_{a \to 1} f_0 = -x \log(1 - z) \] (80)

\[ \lim_{a \to 1} Z = (1 - z)^{-x} \] (81)

\[ \lim_{a \to 1} f_0(\bar{A}, \xi) = x \ln (1 + \xi \bar{A}) = x \ln (1 + f_2 / \bar{A}) \] (82)

\[ \lim_{a \to 1} \chi(\bar{A}, \xi) = \frac{1}{\xi \bar{A}} \log (1 + \xi \bar{A}) \] (83)

Thus a NB distribution has \( \chi = \ln(1 + \xi \bar{A}) / (\xi \bar{A}) \) with \( x = 1 / \xi \) while the LC distribution has \( \chi = (\sqrt{1 + 2 \xi \bar{A}} - 1) / (\xi \bar{A}) \) with \( x = \sqrt{1 + 2 \xi \bar{A} / (2 \xi)} \) and a Geo distribution has \( \chi = [1 - (1 + \xi \bar{A} / 2)^{-1}] / (\xi \bar{A}) \) with \( 1 / x = \xi (1 + \xi \bar{A} / 2) / 2 \) as limiting expressions of a more general \( \chi \) given by Eq.(78) for HG\( a \).
III. PION PROBABILITY DISTRIBUTION

In this section we discuss various features of the pion multiplicity distribution and its associated fluctuations. In the introduction we noted the importance of such studies. The present work is a continuation of the approach developed in Ref. [35] which is based on Feynman path integral methods [33]. These methods lead to a cycle class decomposition of any permutation, with permutations appearing when Bose-Einstein or Fermi-Dirac symmetries are included into the density matrix. Any permutation of a particle has associated with it a vector \( \vec{\text{x}} \) of any permutation, with permutations appearing when Bose-Einstein or Fermi-Dirac symmetries are included into Ref. [35] which is based on Feynman path integral methods [33]. These methods lead to a cycle class decomposition introduction we noted the importance of such studies. The present work is a continuation of the approach developed

\[ < n > = \sum_k k x_k \]

\[ \sigma^2 = < n^2 > - < n >^2 = < n > + \xi < n >^2 = \sum_k k^2 x_k \]  

(84)

from Eqs. (20) and (27).

A. Poisson distribution and cycle of length 2 correlations

A widely used distribution which also is a basis for comparison is the Poisson distribution which is obtained for the \( x_k \)’s by either having unit cycles only, i.e., \( x_k = x_1 \delta_{k1} \) or uniformly distributed cycles independent of their cycle length which is equivalent to the weight \( \langle \text{x}/N \rangle^{1/k} \) of each pion in a cycle class with size \( k \) (see Table I). For this case \( n = M \) and the cycle class multiplicity \( M \) has a Poisson distribution. Then \( < n > = x_1, \sigma^2 = < n >, \) and \( P_n \) is

\[ P_n = < n >^n e^{-< n >}/n! \]  

(85)

with \( Z_n = x_1^n/n! \) and \( Z = e^{x_1} \). The probability of no events, the void probability distribution has \( n = 0 \) and \( P_0 = e^{-< n >} \). Poisson distributions appear in coherent state emission and are the Maxwell-Boltzmann limit of Bose-Einstein and Fermi-Dirac distribution in statistical physics. Fermi-Dirac and Bose-Einstein statistics lead to departures from Poisson results. The voids parameters are \( \xi = 0 \) and \( \chi = 1 \) for this distribution.

The first correction to Poisson comes from cycles of length 2. Then from Eq. (84) \( < n > = x_1 + 2x_2 \) and \( \sigma^2 = x_1 + 4x_2 \) \( \neq < n > \). The \( P_n \) distribution can be generated for \( \vec{x} = (x_1, x_2, 0, \cdots) \) using the recurrence relation Eq. (8) to obtain \( Z_n \) and \( Z(\vec{x}) = \exp[x_1 + x_2] \). The \( Z_n \) is simply

\[ Z_n = \frac{1}{n} (x_1 Z_{n-1} + 2x_2 Z_{n-2}) \]

(86)

The \( Z(\vec{x}) = \sum_n Z_n \) becomes the Hermit polynomial \( e^{2xz-z^2} = \sum_n \frac{1}{n!} H_n(x) z^n \) when \( x_1 = 2xz \) and \( x_2 = -z^2 \).

A system of one type only gives a Poisson distribution and a system of two or more types exhibits non-Poissonian behavior. Departures from Poisson statistics have been noted in level densities where random matrix theory gives the Wigner distribution which is not a Poisson distribution. The next subsection discusses a simple distribution which is the negative binomial and it can have large non-Poissonian fluctuations.

B. Negative binomial (NB) distribution

A distribution frequently discussed for situation that depart from Poisson statistics is the negative binomial (NB) distribution. The cycle class vector \( \vec{x} \) which generates the NB distribution is \( x_k = x^k/k \), logarithmic distribution, and \( x_k \) contains cycle lengths \( k \) of all sizes. For the choice (see Tables 1 and 2); \( Z = (1 - z)^{-x}, Z_n = \frac{n}{n} \Gamma(x + n)/\Gamma(x), \)

\( < n > = xz/(1 - z), \sigma^2 = xz/(1 - z)^2 \) \( \neq < n > (1 + < n > / x) \). The \( P_n \) can be written in the well-known form

\[ P_n = \frac{x^n}{n!} (1 - z)^x \frac{\Gamma(x + n)}{\Gamma(x)} = \left( n + x - 1 \right) \left( \frac{1}{1 + \frac{< n >}{x}} \right)^x \left( \frac{< n > / x}{1 + < n > / x} \right)^n \]  

(87)
The $x = 1$ limit is also referred to as a Planck distribution since the fluctuation is $\sigma^2 < n > (1+ < n >)$ and has the characteristic Planck behavior. Departures from Poisson statistics arise from chaotic sources as discussed in quantum optics. Intermittency and its association with chaos have also been studied using the negative binomial distribution [10]. The voids parameters are $\xi = \frac{1}{x}$ and $\chi = \frac{\ln(1+\xi < n >)}{\xi < n >}$ for this distribution.

Ref. [11] gives several sources for the origin of the negative binomial distribution. These sources include sequential processes that arise from a composite of logarithmic and Poisson distributions, self-similar cascade processes and connections with Cantor sets and fractal structure, generalization of the Plank distribution, solutions to stochastic differential equations. Becattini et al [36] have shown that the NB distribution arises from decaying resonances. The $\alpha$-model of Ref. [10], which is a self-similar random cascading process, leads to NB like behavior. The stochastics aspects of the NB distribution have been discussed by R. Hwa [37]. Hegyi [38] has discussed the NB distribution in Ref. [11] gives several sources for the origin of the negative binomial distribution. These sources include sequential processes that arise from a composite of logarithmic and Poisson distributions, self-similar cascade processes and connections with Cantor sets and fractal structure, generalization of the Plank distribution, solutions to stochastic differential equations. Becattini et al [36] have shown that the NB distribution arises from decaying resonances. The $\alpha$-model of Ref. [10], which is a self-similar random cascading process, leads to NB like behavior. The stochastics aspects of the NB distribution have been discussed by R. Hwa [37]. Hegyi [38] has discussed the NB distribution in

C. Signal/noise (SN) model; Coherent and chaotic emission

The signal/noise model (SN in Table I and II) offers a continuous connection between a Poisson distribution ($x = 0$ with $y \to 0$ and $z \to 1$) keeping $< n > = yz/(1-z)^2$ finite) and the Planck or Bose-Einstein (BE) ($x = 1$ and $y = 0$) or more generally a negative binomial distribution (arbitrary $x$ with $y = 0$). Since a coherent state is a Poisson emitter and a Planck or NB distribution comes from a chaotic state, the SN model interpolates between coherent and chaotic emission. The SN model is discussed in Ref. [39] and its connection to the cycle class picture developed here was initially shown in Ref. [32]. Here, further developments of it are discussed. To begin with the cycle weights have a geometric piece

$$x = \frac{1}{S} L_n^{-1} (y) = \frac{(N/x)^n}{(1+N/x)^{n+1}} \exp \left[ -\frac{S}{1+N/x} \right] L_n^{-1}(\frac{-S}{N/x(1+N/x)})$$

(88)

with $x = 1$ being the Glauber-Lach model and $L_n^a$ is an associated Laguerre function. When $N \to 0$ the Poisson limit is realized with $x_k = S\delta_{k,1}$ and when $S \to 0$, the distribution goes into a NB distribution. In terms of $S$ and $N$ the mean

$$< n > = \frac{yz}{(1-z)^2} + \frac{xz}{1-z} = S + N$$

(89)

while the fluctuation

$$\sigma^2 = < n > + \xi < n >^2 = \frac{yz}{(1-z)^2} + \frac{xz}{1-z} + \frac{xz^2}{(1-z)^2} + \frac{2yz^2}{(1-z)^3}$$

$$= S + N + \frac{N^2}{x} + \frac{2SN}{x}$$

(90)

The void parameters are

$$\xi = \frac{f_2}{f_1} = \frac{N(2S+N)}{x(S+N)^2}$$

(91)

$$\chi = \frac{f_0}{f_1} = \frac{x \ln(1+N/x) + S/(1+N/x)}{S + N}$$

(92)

for this distribution. The SN model has important application to quantum optics and, in particular, to photon counts from lasers [10]. Biyajima [11] has suggested using it for particle multiplicity distribution as does Ref. [39]. When the noise level $N \to 0$, $\xi \to 0$ and $\chi \to 1$. When the signal level $S \to 0$, $\xi \to 1/x$ and $\chi \to \frac{2}{x} \ln(1+N/x)$.

D. Lorentz/Catalan (LC) distribution; underlying splitting probability

Probability distribution associated from field emission from line shapes which are Lorentzian have appeared in quantum optics. Its cycle class description was developed in Ref. [32] and is $x_k = x C_k z^k / 2^k(k-1)$ with $C_k = \binom{2(k-1)}{k-1} / k$
a shifted Catalan number, distributed with NB with $x = 1/2$ (see Eq.(108) in Sect. V C). The partition function is $Z(x, z) = \exp(2x[1 - (1 - z)^{1/2}])$ and the probability distribution associated with it is

$$P_n(x, z) = e^{-2x(1 - \sqrt{1 - z})} (2x)^{2n} \frac{U(n, 2n; 4x) e^{2n}}{n!}$$

with $U(u, v; w)$ a confluent hypergeometric function. The void parameters are $\xi = \frac{2x}{2x \sqrt{1 - z}}$ and $\chi = \frac{2x}{2x \sqrt{1 - z}} \left[ \sqrt{1 + 2x} < n > - 1 \right]$ with $z = 2\xi < n > / (1 + 2\xi < n >)$ for this distribution.

The results of quantum optics, in the notation of Ref. [40], can be obtained [32] when $x = T\Omega / 2$, $z = 2W \gamma / \Omega^2$, $\Omega^2 = \gamma^2 + 2W^2 \gamma$. The $W$ is an integral of the Lorentzian line shape $\Gamma(\omega) = b / [(\omega - \omega_0)^2 + \gamma^2]$, $T$ is the time, and $2x\sqrt{1 - z} = \gamma T$.

An interesting feature of the model arises when evolutionary or ancestral variables are introduced which contain a branching probability $p$, survival probability $(1 - p)$ as discussed in Sect. I E. Percolation or splitting dynamics with evolutionary parameters $p$ and $\beta$ become equivalent to the LC model with clan variables $N_c = \beta$ and $n_c = \frac{(1 - p)}{(1 - 2p)}$ for $p < 1/2$. Specifically $x = \beta / 4p$ and $z = 4p(1 - p)$ giving $x_k = \beta C_p k^{1 - 1 - p}$. In these evolutionary/ancestral variables the $f_0 = \sum x_k = 2x(1 - \sqrt{1 - z})$ which determines $Z$ has the interesting property $f_0 = \sum x_k = \beta$ for all $p < 1/2$. For $p \geq 1/2$, $f_0 = \sum x_k$ is no longer a constant and is $f_0 = \sum x_k = \beta(1 - p)/p$. To keep $f_0 = \sum x_k$ a constant without changing $< n > = \sum_k k < n_k > = \beta (\frac{1 - p}{1 - 2p})$, a $x_0 = \phi_\infty$ was introduced in Ref. [32]. The $\phi_\infty = 0$ for $p \leq 1/2$ and $\phi_\infty = \beta (2p - 1)/p$ for $p \geq 1/2$. For $p \geq 1/2$ there is a finite probability that the splitting will go on forever. In percolation above a certain p an infinite cluster is formed and $\phi_\infty$ is similar to the strength of the infinite cluster. Moreover the sudden appearance of $\phi_\infty$ is similar to the behavior of an order parameter in a phase transition. The appearance of $\phi_\infty$ can also be interpreted as the sudden appearance of a jet without pion ($k = 0$), when $x_0 = \phi_\infty$.

A connection of the LC model can also be made with a Ginzberg-Landau theory of phase transitions and a Feynman-Wilson Gas. These connections are discussed in Ref. 31.

E. Thermal models

Thermal emission of pions based on statistical mechanics and equilibrium ideas have been popular descriptions of pions coming from relativistic heavy ion collisions. For thermal models [33], the $x_k = (V T^3 / 2\pi^2)(mk/T)^2 K_2(km/T)/k$ for a cycle length $k$ or a cluster of size $k$ with $K_2$ a Mac Donald function. For low temperatures, $x_k = (V/\lambda_T^2) e^{-km/T}/k^{5/2}$ and the Boltzmann factor in mass, $e^{-km/T}$, suppresses large fluctuations. In the high temperature limit and/or zero pion mass limit $x_k = (V/\pi^2) T^3/k^4$. The $x_k$ can be used to generate the pion probability distribution $P_k$. The thermal models can be combined with hydrodynamic descriptions and an application was given [33] to 158 A GeV Pb+Pb data measured by the CERN/NA44 and CERN/NA49 collaborations. The results of Ref. 33 showed a Gaussian distribution with a width about 20 % larger than the Poisson result. For Table II of Ref. 33 has $x_1 = 260$, $x_2 = 9.957$, $x_3 = 1.997$, $x_4 = 0.163$, · · ·. The $x_k$’s have a rather sharp fall off with $k$ and two terms $x_1$ and $x_2$ give a reasonably good description of the Gaussian. A Poisson emitter with large $<n>$ has also a Gaussian distribution with a width such that $\sigma^2 = <n>$. For the first two $x_k$’s just given $x_1 = 260$, $x_2 = 10$, the $\sum x_k = 270$, $< n > = \sum kx_k = 280$ and $\sigma^2 = 300$.

F. Fisher exponent $\tau$ and other distributions

Various forms for $x_k$ encountered for the cycle class weights have a form $x_k = xz^k/k^\tau$. The NB has $\tau = 1$, the geometric has $\tau = 0$ and the SN has a combination. The LC model has $\tau = 3/2$ for large $k$ using Stirling approximation. In clusters yields $\tau$ is called Fisher critical exponent and it determines the droplet yields $<n_k>$ around a critical point of a first order liquid-gas phase transition. It has been studied extensively in medium energy heavy ion collisions [42]. Here we discuss its importance in particle multiplicity distributions. A power law of $<n_k>$ is the power law of $x_k$ according to Eq.(14) in grand canonical ensemble and Eq.(6) in canonical ensemble with finite size effect through $Z_k$. The $\tau$ determines the grand canonical partition function $Z(\vec{x}) = \exp[\sum_k x_k] = \exp[x \sum_k z^k/k^\tau]$. Bose-Einstein condensation of atoms in a box of sides $L$ of dimensions $d$ have $x = L^d/\lambda_T^2$ with $\lambda_T = h/(2\pi m k_B T)^{1/2}$ and have $\tau = 1 + d/2$. Feynman used random walk arguments, the closing of a cycle parallels a closed random walk, to discuss his choice of $x_k$ in his discussion of a superfluid phase transition in liquid equilibrium. We have therefore consider two other case for $x_k$ which asymptotically have $\tau = 1/2$ and $\tau = 1$. These are called random walk (RW) in 1D and 2D respectively. The RW1D has an $x_k$ which is just $k$ times the $x_k$ of the LC model. The RW2D has the same $\tau$ as the NB distribution in the asymptotic limit of large $k$. 

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A simple emitting source model for pions was introduced by Pratt [7] and solved analytically by Cs"org"o and Zim"any [8]. The model can be used to study the role of Bose-Einstein symmetrization effects on pionic distributions. An interesting property of this model is that a Poisson emitter of pions with strength \( \eta \) can become a pion “laser”. A manifestation of this property is a large enhancement of pions in a zero momentum mode. For this model the

\[
x_k = \left( \frac{\eta}{\eta_c} \right)^k \left( 1 - \left( \gamma - \frac{1}{2} \right) \frac{k}{\eta_c} \right)^3 \left( 1 - \frac{\gamma}{\gamma_c} \right)^{3/2},
\]

with 

\[
\gamma = \frac{1}{2} \left( 1 + y \pm \sqrt{1 + y} \right), \quad y = \frac{2R^2mT}{1 - 2mT}, \quad \eta_c = \frac{\gamma_c^3}{2}.
\]

For very large \( k \),

\[
x_k = \left( \frac{\eta}{\eta_c} \right)^k / k^4
\]

and for small \( k \) and typical values of \( R \) and \( T \) the

\[
x_k = \left( \frac{R^2mT}{1 - 2mT} \right)^{3/2} / k^4.
\]

The \( 1/k^4 \) dependence also appears in thermal models with zero mass pions.

IV. COMPARISON OF PROBABILITY DISTRIBUTIONS WITHIN HGA

The hypergeometric case (HGa) which we studied more in detail in Sect. II G included various cases of Table I: Geo, NB, LC which are distinguished by one parameter \( a \) of the hypergeometric model. Thus we use HGa to compare various models for pion distribution and other particle distributions.

A. Voids and void scaling relations

Void analysis looks for scaling properties associated with \( \chi = f_0 / < n > \); specifically, \( \chi \) is a function of the combination \( \xi < n > = f_2 / < n > \) where \( \xi = \kappa_2 \) is the coefficient of \( < n >^2 \) in the fluctuation \( \sigma^2 = < n^2 > - < n >^2 = < n > + \xi < n >^2 \). For example, a NB distribution has \( \chi = \ln(1 + \xi < n > ) / (\xi < n > ) \) while the LC distribution has \( \chi = (\sqrt{2\xi} < n > + \xi - 1) / (\xi < n > ) \). These two cases are limiting expressions of a more general \( \chi \) given by

\[
\chi = \frac{(1 + \xi < n > / a)^{1-a} - 1}{(1-a)\xi < n > / a}.
\]

of the HGa model. The NB \( \chi \) is obtained from the limit \( a \to 1 \) while the LC \( \chi \) follows for \( a = 1/2 \).

We show the void variable \( \chi \) as a function of \( \xi < n > \) in Fig. 3. This shows that we can vary \( a \) to fit data. Ref. 30 claims the NB distribution fits reasonably well the void distribution for single jet events in \( e^+e^- \) annihilation but Fig. 3 shows all of the curves might fit such data up to \( \xi < n > \sim 1 \) since the various curves are reasonably close up to this \( \xi < n > \). Thus further investigations of this data is required to distinguish various models. Higher moments or cumulants might need to be compared for this purpose. Within HGa, \( \kappa_3(\bar{n}, \xi, a) = \left( \frac{a+1}{a} \right) \xi^2 \) for a given value of \( < n > = \bar{n} \) and \( \xi \) according to Eq.(79). Thus \( A_3 = \kappa_3 / \xi^2 = (a+1)/a = 3, 2, 3/2, 4/3, 5/4 \) for \( a = 1/2, 1, 2, 3, 4 \) independent of \( \xi \). The \( \kappa_3 \) may help in distinguishing various models for the data with a small value of \( \xi \), i.e., the region of \( \xi < n > < 1 \) in Fig. 3. The differences of \( A_m \) between models with different value of \( a \) becomes larger as the order \( m \) of the reduced factorial cumulant becomes higher.
FIG. 3. $\chi$ vs $\xi < n >$ for $a = 1/2$ (solid line; LC), 1 (dash; NB), 2 (dash-dot; Geo), 3 (dot), 4 (dash-dot-dot-dot). For Poisson distribution $\xi = 0$ and $\chi = 1$.

B. Probability distribution

Once we know the generating function $Z(\vec{x})$ or $f_0(\vec{x})$ we may study the probability distribution $P_n$ using Eq.(9) or the recurrence relation Eq.(6):

$$P_n(x, z) = \frac{Z_n(x, z)}{Z(x, z)} = \frac{1}{Z(x, z)} \frac{n!}{n!} \left[ \left( \frac{d}{dz} \right)^n Z(x, z) \right]_{z=0}$$

for $n = \sum_k k n_k$. For NB which is a special case of HGa in the $a \to 1$ limit, $x_k = x \frac{k}{\bar{n}}$,

$$Z^{NB}(x, z) = (1 - z)^{-x} = P_0^{-1}(x, z)$$

$$P_n^{NB}(x, z) = \frac{1}{Z} \frac{n!}{n!} \frac{\Gamma(x + n)}{\Gamma(x)} = (1 - z)^{-x} \frac{z^n}{n!} \frac{\Gamma(x + n)}{\Gamma(x)}$$

The $P_n$ for various cases of Table I are shown in Fig.4 and Fig.5 with the same mean value $<n> = \bar{n}$ and the same fluctuation $\xi = \kappa_2 = f_2/\bar{n}^2$ for fixed $<n> = 10$. Fig.6 shows KNO plots of $<n> P_n$ versus $n/ <n>$ for fixed $<n> = 10$ and 20. Fig.7 shows that the various models considered here have almost the same distribution for small fluctuation ($\xi = 0.01$) and in this case they are very similar to a Poisson’s distribution. For $\xi = 0.05$ the models are similar to each other except for larger $n/\bar{n}$ even though they are different from a Poisson distribution. For larger fluctuations such as $\xi = 0.5$, the models have very different forms even if they have the same mean value and fluctuation. Fig.8 shows that the probability distribution of these models differ in their form for fluctuations larger than $\xi \approx 0.2$. 

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FIG. 4. $P_n$ for fixed $<n> = 10$ for $\alpha = 1/2$ (solid line), 1 (dash), 2 (dash-dot), 3 (dot), 4 (dash-dot-dot-dot) and for $\xi = 0.01$, 0.05, and 0.5 in log scale on the left and linear scale on the right. For $\xi = 0.01$ all distributions become very close to Poisson (thin solid curve) already. ($P_0$ becomes large for large $\xi$.)
FIG. 5. \( P_n \) for fixed \( \langle n \rangle = 10 \) and for \( a = 1/2, 1, 2, 3, 4 \). The value \( \xi = f_2/\langle n \rangle^2 \) are shown in each figure. The various choices of \( a \) for each curve are given in the figure caption of Figs. 3 and 4.
FIG. 6. KNO plot of $\langle n \rangle P_n$ for fixed $\langle n \rangle = 10$ and 20 for $a = 1/2, 1, 2, 3, 4$. The KNO behavior for $\bar{n} \to \infty$ can also be studied for the models listed in Table 1. According to KNO scaling, for distributions with a large mean value, the distribution $\langle n \rangle P_n$ becomes model independent in the new variable $n/\bar{n}$, i.e., variable scaled by mean value. In general, any distribution becomes Gaussian for large mean $\langle n \rangle$ according to the central limit theorem; specifically

$$P_n(\bar{n}, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(n-\bar{n})^2}{2\sigma^2}\right]$$

$$= \left(1 + \frac{1}{\xi n}\right)^{-1/2} \frac{1}{\bar{n}\sqrt{2\pi\xi}} \exp \left[-\frac{1}{2\xi} \left(\frac{n}{\bar{n}} - 1\right)^2 \left(1 + \frac{1}{\xi n}\right)^{-1}\right]$$

(98)

with the mean $\langle n \rangle = \bar{n}$ and the standard deviation $\sigma$ which is related to the reduced factorial cumulant $\xi$ by $\sigma^2 = \langle n^2 \rangle - \langle n \rangle^2 = \bar{n} + \xi \bar{n}^2$. This means that the KNO scaling follows when the fluctuation is given by $\sigma^2 = \xi \bar{n}^2$ with a constant $\xi$ or when $\xi \bar{n} \gg 1$. Thus to compare KNO scaling properties of $\bar{n}P_n$ for different models, the fluctuation of these models should have the same value of $\xi = \xi_2/\bar{n}^2$. For a small $\xi$, the Poisson component of the fluctuation $\sigma^2 = \bar{n} + \xi \bar{n}^2$ becomes dominant and thus the KNO scaling behavior is broken. For large $\xi$ the Poisson component is negligible and KNO scaling is realized, i.e.,

$$\bar{n}P_n(\bar{n}, \xi) = \frac{1}{\sqrt{2\pi\xi}} \exp \left[-\frac{1}{2\xi} \left(\frac{n}{\bar{n}} - 1\right)^2\right]$$

(99)

The KNO plot of Fig. 6 shows that the various distributions have no KNO scaling property for small fluctuation ($\xi = 0.1$) but show a KNO scaling property for large fluctuation ($\xi = 0.5$). The effect of mean on $\bar{n}P_n$ is larger than the difference between different model for $\xi = 0.1$ while the effect of mean on $\bar{n}P_n$ is much smaller than the difference between different models exhibiting KNO scaling behavior for $\xi = 0.5$. Ref. [39] shows the total charge distribution in hadronic collisions. From their fits we can extract the corresponding fluctuation which are $\xi = 0.05 \sim 0.5$ and $\bar{n} = 6 \sim 13$. This means that the KNO scaling behavior is marginal for these data, i.e., just fitting the distribution $\bar{n}P_n$ of the data does not show clear evidence for KNO scaling. We need to evaluate the explicit values of the mean number $\bar{n}$ and the fluctuation $\xi$ to check the KNO behavior of this data; the value of $\xi \bar{n}$ should be large enough to show KNO scaling behavior as can be seen from Eqs. (86) and (89).
A Poisson distribution plays a very important role in physics. As already noted, in statistical physics, Maxwell-Boltzmann statistics leads to Poisson probabilities. Other distributions are compared to the Poisson distribution which acts as a benchmark for comparison. The distributions considered in this paper can have large non-Poissonian fluctuations. The purpose of this section is to show how they can be rewritten as a compound process or sequential process involving one aspect that has a Poisson character. As an example the negative binomial distribution can be obtained from a compound Poisson-logarithmic distribution as discussed in Ref. [39]. Here, we extended this result to include the other distributions and we also show that the final distribution can be obtained from compounding it with another distribution, such as the negative binomial. In general, the underlying picture for a sequential process involves a two step procedure in which the observed particles arise from the production of “clusters” with the subsequent decay of each cluster producing its distribution of particles. The final distribution is obtained by compounding the probability distribution of the clusters with another distribution coming from each cluster and suming over clusters.

The grand partition function or the generating function \( Z = e^{\beta \Omega} \) for any distribution can be represented as a Poisson distribution whose mean value is the void variable \( f_0 \) or the grand potential \( \Omega = -f_0 \). On the other hand and in general we can rewrite the void variable as

\[
 f_0(\bar{x}) = \ln Z(\bar{x}) = \sum_k x_k = N \sum_k \mathcal{P}_k(\bar{x})
\]

where \( \mathcal{N} = f_0 = \sum_k x_k \) and

\[
 \mathcal{P}_k(\bar{x}) = \frac{x_k}{\mathcal{N}}
\]

The \( \mathcal{P}_k(\bar{x}) \) can be connected to its generating function \( G(\bar{x}, u) \):
\[ G(\bar{x}, u) = \sum_k (1 - u)^k P_k(\bar{x}) = \frac{1}{N} \sum_k x_k (1 - u)^k \] (105)

Thus the generating function \( G(\bar{x}, u) \) of \( P_n \) can be expanded in terms of \( P_k \) as

\[
G(\bar{x}, u) = \sum_n (1 - u)^n P_n(\bar{x}) = \frac{1}{Z(\bar{x})} \sum_n Z_n(\bar{x})(1 - u)^n = \frac{e^{N G(\bar{x}, u)}}{e^N} 
\]

\[
= \sum_{M=0}^{\infty} \frac{1}{e^N} \frac{[NG]^M}{M!} = \sum_M P'_M(\mathcal{N}) \left[ \sum_j (1 - u)^j P_j \right]^M 
\]

\[
= \sum_M P'_M(\mathcal{N}) \sum_{\vec{n}_M} \prod_k [(1 - u)^k P_k]^n_k 
\] (106)

where \( P'_M(\mathcal{N}) = e^{-\mathcal{N} \frac{M}{M!}} \) is the Poisson distribution with the mean of \( \mathcal{N} = f_0 \). Here the sum over \( \vec{n}_M \) is the sum over partitions \( \vec{n} \) with a fixed \( M = \sum_k n_k \). Thus we have

\[
P_n(\bar{x}) = \sum_M P'_M(\mathcal{N}) \sum_{\vec{n}_M} \prod_k P_k^{n_k}(\bar{x}) 
\] (107)

with \( M = \sum_k n_k \) and \( n = \sum_k k n_k \). Any distribution obtained from a generating function of the form of \( Z(\bar{x}) = e^{f_0} = \exp(\sum_k x_k) \) can therefore be decomposed as a compound Poisson’s distribution with some other distribution \( P_k = x_k/f_0 \) obtained from the weight \( x_k \).

The sequential nature of a process is explicitly shown on Eq.(107). The observed particle multiplicity distribution arises from a two step process in which \( M = \sum_k n_k \) clusters are first distributed according to a Poisson distribution. This is then sequentially followed by breaking each of the \( n_k \) clusters of type \( k \) into \( k \) particles with probability \( P_k = x_k/\mathcal{N} \) and with \( n = \sum_k k n_k \). The probability associated with a given \( M \) and \( \vec{n} \) with \( \vec{x} \) is \( P_M(\vec{x}, \vec{n}) = P'_M(\mathcal{N}) \prod_k P_k^{n_k} = P_M(\mathcal{N}) \prod_k (x_k/\mathcal{N})^{n_k} \).

As an illustration we consider the LC model with \( x_k = x C_k z^k/k! \) for LC or \( a = 1/2 \) for NB. The weight \( x_k \) has the structure of the probability \( P_k(x, z) \) of NB distribution given by Eq.(107), i.e.,

\[
x_k = x \frac{z^k \Gamma(a + k - 1)}{k! \Gamma(a)} = \frac{x}{a - 1} (1 - z)^{a - 1} P_{\text{NB}}^k(a - 1, z) 
\] (108)

thus \( P_k = [1 - (1 - z)^{a - 1}]^{-1} P_{\text{NB}}^k \) for HGa. Therefore the HGa \( P_n \) distribution is a compound Poisson-NB distribution. This may interpreted as a sequential process in which clusters with a Poisson cluster distribution \( P_k \) breakup into particles with a particle distribution \( P_k \) given by a NB distribution. For the various models considered here with their \( x_k \) listed in Table II, the corresponding distribution \( P_k \) and the normalization factor \( \mathcal{N} = f_0 \) are listed in Table III. We can further see that HGa can be looked as a compound Poisson-Poisson-Logarithmic distribution, i.e., a distribution having three sequential steps; Poissonian breakup into clusters \( \rightarrow \) Poissonian breakup of each cluster \( \rightarrow \) logarithmic breakup of each of them.

| Model | Weight \( \mathcal{N} \) | Distribution \( P_k \) | Comments on \( P_k \) |
|-------|-----------------|-----------------|----------------------|
| P     | \( < n > \) | \( \frac{x}{a - 1} \) | Monomer only or Uniform for \( \mathcal{N} \) species |
| Geo   | \( x \frac{1}{1 - z} \) | \( (1 - z)^{k-1} \) | Uniform with constituents |
| NB    | \( x \ln \left(\frac{1}{1 - z}\right) \) | \( \frac{x^k}{k! \ln \left(\frac{1}{1 - z}\right)} \) | logarithmic with constituents |
| LC    | \( 2\pi \left[1 - \sqrt{1 - z^2}\right] \) | \( \frac{1}{(1 - z)^{1/2}} \) | NB with constituents with \( P_0 = 0 \) |
| HGa   | \( \frac{x}{1 - (1 - z)^{1-\alpha}} \) | \( e^{-z \Gamma\left(\frac{k}{2}\right)} \) | NB with constituents without \( k = 0 \) |

The sequential nature of a process is explicitly shown on Eq.(107). The observed particle multiplicity distribution arises from a two step process in which \( M = \sum_k n_k \) clusters are first distributed according to a Poisson distribution. This is then sequentially followed by breaking each of the \( n_k \) clusters of type \( k \) into \( k \) particles with probability \( P_k = x_k/\mathcal{N} \) and with \( n = \sum_k k n_k \). The probability associated with a given \( M \) and \( \vec{n} \) with \( \vec{x} \) is \( P_M(\vec{x}, \vec{n}) = P'_M(\mathcal{N}) \prod_k P_k^{n_k} = P_M(\mathcal{N}) \prod_k (x_k/\mathcal{N})^{n_k} \).

As an illustration we consider the LC model with \( x_k = x C_k z^k/k! \) for LC or \( a = 1/2 \) for NB. The weight \( x_k \) has the structure of the probability \( P_k(x, z) \) of NB distribution given by Eq.(107), i.e.,

\[
x_k = x \frac{z^k \Gamma(a + k - 1)}{k! \Gamma(a)} = \frac{x}{a - 1} (1 - z)^{a - 1} P_{\text{NB}}^k(a - 1, z) 
\] (108)

thus \( P_k = [1 - (1 - z)^{a - 1}]^{-1} P_{\text{NB}}^k \) for HGa. Therefore the HGa \( P_n \) distribution is a compound Poisson-NB distribution. This may interpreted as a sequential process in which clusters with a Poisson cluster distribution \( P_k \) breakup into particles with a particle distribution \( P_k \) given by a NB distribution. For the various models considered here with their \( x_k \) listed in Table II, the corresponding distribution \( P_k \) and the normalization factor \( \mathcal{N} = f_0 \) are listed in Table III. We can further see that HGa can be looked as a compound Poisson-Poisson-Logarithmic distribution, i.e., a distribution having three sequential steps; Poissonian breakup into clusters \( \rightarrow \) Poissonian breakup of each cluster \( \rightarrow \) logarithmic breakup of each of them.
Because of a unique role played by the Poisson distribution and the form $e^{f_0} = \exp[\sum_k x_k]$ of the generating function, the cluster distribution $P_G$ is usually taken to be a Poisson. However, as noted before, other divisions are possible. Using the same approach used above for $P_G = P^G_M$, we can expand any distribution using a NB instead of Poisson, i.e., with $P_G = P^NB_M$, using the form of the generating function $Z(x, z) = (1 - z)^{-x}$ for the NB. Replacing $z$ by $N(z)$, the normalization factor of a new distribution, we have

$$Z(x, z) = [1 - N(z)]^{-x} = [1 - N(z)G(u = 0)]^{-x}$$

(109)

The $G(u)$ is

$$G(u) = \frac{N([1 - u]z)}{N(z)} = \sum_k (1 - u)^k P_k$$

while the $G(u)$ is

$$G(u) = \sum_n (1 - u)^n P_n = \frac{Z(x, [1 - u]z)}{Z(x, z)} = \frac{[1 - N G(u)]^{-x}}{[1 - N]^{-x}}$$

(111)

Thus we have

$$G(u) = \sum_{M=0}^{\infty} (1 - N)^x \frac{N^M}{M!} \frac{\Gamma(x + M)}{\Gamma(x)}[G(u)]^M = \sum_{M=0}^{\infty} P^NB_M(x, N)[G(u)]^M$$

$$= \sum_{M=0}^{\infty} P^NB_M \sum_{j=0}^{\infty} (1 - u)^j P_j \left(\sum_{k=0}^{\infty} \prod_{\{n_k\}_M} [1 - u]^k P_k\right)^{n_k}$$

$$= \sum_{M=0}^{\infty} P^NB_M(x, N) \sum_{\{n_k\}_M} \prod_k P_k^{n_k}(\bar{x})$$

(112)

The result of Eq. (113) shows that the distribution $P_n$ can be written as a compound probability distribution of a negative binomial $P^NB_M$ with another probability $P_k$ distribution generated from $G(u)$. For the case of $N(z) = e^z$, which may be considered as the fugacity $e^z = e^\mu$ for a particle with chemical potential $\mu = z$, $G$ can further be decomposed as

$$G(u) = \frac{N([1 - u]z)}{N(z)} = \frac{e^{(1-u)z}}{e^z} = \sum_{k=0}^{\infty} (1 - u)^k P^P_k(z)$$

(114)

i.e., $P_k$ for this case is Poisson $P^P_k(z)$. If $N = 1 - e^{f_0/x}$, then $P_k = P^P_k(f_0/x)$ without $k = 0$ and $Z(x, z) = [1 - (1 - e^{f_0/x})]^x = e^{f_0}$. For $f_0 = \sum_k x_k$ given in Table 1, the $P_k$ becomes the same probability with $x_k$ replaced by $x_k/x$. As an example the HGa with $Z(x, z) = \exp \left(\frac{x}{x-a}[1 - (1 - z)^{1-a}]\right)$ can be decomposed as a sequential process consisting of a NB distribution of clusters $Z(x, N = 1 - e^{f_0/x})$ which is then followed by a breakup of clusters distributed with a HGa distribution with $Z(x = 1, z)$ but without voids, i.e., with $P_0 = 0$. Thus this decomposition separates the parameter $x$ assigned to cluster from other parameters.

**D. Poisson transformation and other transformation**

Compound distribution such as those considered in the previous section can also be understood using the fact that the Laplace transform of $G(u)$ is related to the Poisson transform for the probability distribution $P_n$ [39];

$$G(u\bar{\bar{n}}) = \int_0^\infty dyf(y)e^{-uy\bar{\bar{n}}} = \int_0^\infty dyf(y)G^P(-uy\bar{\bar{n}})$$

(115)

where $G^P(\bar{\bar{n}}) = e^{-\bar{\bar{n}}}e^{(1-u)\bar{\bar{n}}}$ is the generating function for Poisson’s distribution and
\[
\begin{align*}
\int_0^\infty df(y) &= 1 \\
\int_0^\infty dyf(y) &= 1 \\
\int_0^\infty dyg^m f(y) &= \tilde{n}^{-m} \left(\frac{n!}{(n-m)!}\right)
\end{align*}
\]

(116)

For a Poisson distribution,

\[
\begin{align*}
\mathcal{L}\{f(y)\} &= \delta(y-1) \\
\mathcal{P}_n(\tilde{n}) &= \frac{\langle \tilde{n} \rangle^n}{n!} e^{-\tilde{n}} \\
\mathcal{G}(u) &= \sum_{n=0}^\infty (1-u)^n \mathcal{P}_n(\tilde{n}) = \sum_{n=0}^\infty \frac{(1-u)\tilde{n}^n}{n!} e^{-\tilde{n}} = e^{-u\tilde{n}}
\end{align*}
\]

(117, 118, 119)

By a Laplace transform or Poisson transform,

\[
\begin{align*}
\mathcal{G}(u) &= \sum_{n=0}^\infty (1-u)^n \mathcal{P}_n(\tilde{n}) = \sum_{n=0}^\infty \frac{(1-u)\tilde{n}^n}{n!} e^{-\tilde{n}} = \sum_{n=0}^\infty (1-u)^n \int_0^\infty dyf(y) e^{-uy\tilde{n}} = \sum_{n=0}^\infty (1-u)^n \int_0^\infty dyf(y) G^P(yu\tilde{n}) \\
\mathcal{P}_n(\tilde{n}) &= \int_0^\infty dyf(y) \frac{(yu\tilde{n})^n}{n!} e^{-yu\tilde{n}} = \int_0^\infty dyf(y) \mathcal{P}_n(yu\tilde{n})
\end{align*}
\]

(120, 121)

where \(\mathcal{P}_n(y\tilde{n})\) is the Poissonian probability with a mean of \(y\tilde{n}\). The Poisson transform of \(f(y)\) is \(\mathcal{P}_n(\tilde{n})\) and the Laplace transform of \(f(y)\) is \(\mathcal{G}(u)\). Thus \(f(y)\) is an inverse Poisson transform of \(\mathcal{P}_n(\tilde{n})\) or an inverse Laplace transform of \(\mathcal{G}(u)\). However the probability \(\mathcal{P}_n(\tilde{n})\) which is a Poisson transform of \(f(y)\) may also be considered as a Laplace transform of \(f(y) \frac{(yu\tilde{n})^n}{n!}\) instead of \(f(y)\) itself. The probability \(\mathcal{P}_n(\tilde{n})\) is a superposition of a Poisson distribution \(\mathcal{P}^P_n(y\tilde{n})\) with weight of \(f(y)\). Thus we may interpret the \(\mathcal{P}_n(\tilde{n})\) with a mean number \(\tilde{n}\) as the probability in an ensemble of mixed systems with various values of the energy or temperature which is distributed with weight \(f(y)\). Each of the system with a fixed energy or temperature breakups to give a Poisson distribution \(\mathcal{P}^P_n(y\tilde{n})\) with a scaled mean number \(y\tilde{n}\).

Since the \(G(z, u) = Z((1-u)z)/Z(z)\), we have \(G(z, u = 1) = Z_0(z)/Z(z) = Z^{-1}(z)\) for the case of \(Z_0 = 1\) and thus

\[
\mathcal{L}\{f(y)\} = \int_0^\infty dyf(y) e^{-uy\tilde{n}} = \int_0^\infty dyf(y) \left[ \frac{1}{Z(y\tilde{n})} \right]
\]

(122)

Thus \(f(y)\) is also an inverse Laplace transform of \(1/Z(z)\) with \(u = 1\), i.e., \(1/Z^P(y\tilde{n}) = e^{-y\tilde{n}}\). In general, the Laplace transform and its inversion are

\[
\begin{align*}
\mathcal{F}(s) &= \int_0^\infty e^{-st} f(t) dt \\
\mathcal{F}^{-1}(s) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds
\end{align*}
\]

(123)

Some examples are

| \(F(s)\) | \(f(t)\) | constraint and comments |
|----------------|----------------|-------------------------|
| \(\frac{1}{(s+a)^n}\) | \(\frac{t^{n-1}e^{-at}}{(n-1)!}\) | \(n = 1, 2, 3, \cdots\) NB |
| \(e^{-k\sqrt{t}}\) | \(\frac{k}{2\sqrt{\pi t}} \exp \left(\frac{-k^2}{4t}\right)\) | \(k > 0\) LC |

(124)

For NB with \(Z^{NB}(x, z) = (1-z)^{-x}\) and \(\tilde{n} = \frac{xz}{(1-z)}\),

\[
\begin{align*}
G(x, z, u) &= \frac{(1-z)^x}{(1-(1-u)z)^x} = \left(1 + u \frac{z}{1-z}\right)^{-x} = \frac{x^x}{(u\tilde{n} + x)^x} \\
Z^{-1}(x, z) &= (1-z)^x = \frac{x^x}{(x + \tilde{n})^x} = G(x, z, 1)
\end{align*}
\]

(125, 126)

25
Thus $P_{NB}$ is the Poisson transform of

$$f(y) = \frac{x^y y^{x-1} e^{-xy}}{(x-1)!} = x^y (xy)^{x-1} e^{-xy} \quad (x-1)!$$

which is the inverse Laplace transform of $n^n (s + a)^{-n}$ with $s = \bar{n}$ and $a = x$ and $n = x$ from Eq. (124) as shown in Ref. [39]. With $x = 1$, the NB becomes Bose-Einstein (BE) distribution with $f(y) = e^{-y}$. For LC with $Z^{LC}(x, z) = e^{2x(1 - \sqrt{1 - z})}$ and $\bar{n} = \frac{xz}{\sqrt{1 - z}}$.

$$G(x, z, u) = \exp \left[ 2x \sqrt{1 - z} \left( 1 - \frac{1 + u}{x \sqrt{1 - z} \sqrt{1 - z}} \right) \right]$$

$$= e^{2x \sqrt{1 - z}} \exp \left[ -2 \sqrt{x \sqrt{1 - z} \sqrt{1 - z} + u \bar{n}} \right] \quad (128)$$

$$Z^{-1}(x, z) = \exp [-2x(1 - \sqrt{1 - z})] = \exp \left[ 2x \sqrt{1 - z} \left( 1 - \frac{1}{\sqrt{1 - z}} \right) \right]$$

$$= e^{2x \sqrt{1 - z}} \exp \left[ -2 \sqrt{x \sqrt{1 - z} \sqrt{1 - z} + \bar{n}} \right] = G(x, z, 1) \quad (129)$$

Thus $P^{LC}$ is the Poisson transform of

$$f(y) = e^{2x \sqrt{1 - z} \sqrt{x \sqrt{1 - z}} \sqrt{1 - z}} \exp \left[ -x \sqrt{1 - z} - y x \sqrt{1 - z} \right]$$

which is the inverse Laplace transform of $e^{-k x \sqrt{z} e^{2x \sqrt{1 - z}}}$ with $s = \bar{n} + x \sqrt{1 - z}$ and $k = 2 \sqrt{x \sqrt{1 - z}}$ from Eq. (124).

Similarly we can also decompose any distribution as a superposition of NB distribution instead of superposition of Poisson distribution by replacing $G^{P}(y\bar{n}) = e^{-y\bar{n}}$ in Laplace transform by the generating function $G^{NB}(x, z, u) = (1 + u \frac{x}{z})^{-z} = \left( \frac{x}{x + u\bar{n}} \right)^z$ of NB distribution. That is using Mellin transform $F(s) = \int_0^\infty f(y) y^{s-1} dy$, $G^{NB}(u\bar{n}, x) = \sum_{n=0}^{\infty} (1 - u)^n P_n^{NB}(\bar{n}, x) = \frac{x^x (u\bar{n} + x)^{(-x+1)-1}}{(x + u\bar{n})^x}$

$$G(u\bar{n}, x) = \sum_{n=0}^{\infty} (1 - u)^n P_n(\bar{n}, x) = \int_0^{\infty} f(y) x^y (uy\bar{n})^{(-x+1)-1} dy'$$

$$G(\bar{n}, x) = \int_0^{\infty} f(y + x/\bar{n}) x^{y/\bar{n}} G^{NB}(uy\bar{n}, x) dy$$

$$P_n(\bar{n}, x) = \int_0^{\infty} f(y + x/\bar{n}) P_n^{NB}(y\bar{n}, x) dy$$

with $s = 1 - x$. Thus a NB transform for the probability $P_n$ may be defined as a shifted Mellin transform for the generator $G(u)$. Further investigation of these transformation properties may give interesting features of various distributions. As a special case of Mellin transform $(s-1)! = \int_0^\infty e^{-t} t^{s-1} dt$.

V. CONCLUSION

Event-by-event studies from ultrarelativistic heavy ion collisions such as done at RHIC are being used to study the details of particle multiplicity distributions as, for example, those associated with pions. Such studies not only give information about the mean number of particles produced, but also information about fluctuations and higher order moments of the probability distribution which are important tools for studying the underlying processes and mechanisms that operate. They are also useful in distinguishing various phenomenological models. Issues associated with fluctuations play an important role in many areas of physics and departures from Poisson statistics are of current
interest. One purpose of this paper was an investigation of various models of particle multiplicity distributions that can be used in event-by-event analysis. This study was done using a generalized model based on a hypergeometric series (HGa) and uses a grand canonical ensemble as its basic framework. This framework has its origin in a Feynman path integral approach and involves a cycle class decomposition of the permutation symmetries that originate from the underlying wavefunctions associated with the produced particles. Many of the existing distributions used in particle phenomenology are shown to be special cases of this more general hypergeometric model. Several quantities are shown to be quite general; such as hierarchical scaling relations (Eq. (13)), initially discussed in terms of a particular distribution such as the negative binomial distribution. Various models and associated distribution can be developed in a unified way. These include the Poisson distribution coming from coherent emission, chaotic emission producing a negative binomial distribution, combinations of coherent and chaotic process leading to signal/noise distributions and field emission from Lorentzian line shapes producing the Lorentz/Catalan distribution which are all shown to be special cases of the HGa model. The HGa model and its associated special cases are used to explore a wide variety of phenomena. These include; linked pair approximations leading to hierarchical scaling relations on the reduced cumulant level, generalized void scaling relations, clan variable descriptions and their connections with stochastic variables associated with branching processes, KNO scaling behavior, enhanced non-Poissonian fluctuations.

Our results show that even though various distributions have the same mean and fluctuation the distribution itself or the underlying mechanism could be very different. Thus to find the correct distribution and underlying mechanism from the data more informations than just the mean and its fluctuation are necessary. The model used here, HGa, has three parameters, \( x \), \( z \), and \( a \). Thus with a given value of \( a \), there is no extra controlling parameter for a given mean and fluctuation. All the higher moments and cumulants are then determined by the model without any further controlling parameter. Further studies of general models having more parameters are needed where the higher moments can be used as extra conditions in comparing various distributions.

We compared various pionic distribution within a generalized hypergeometric model (HGa) which is a special case of much more general distribution in grand canonical ensemble. In our model used here for a grand canonical ensemble, the weight factor \( x_k \) for each species is the same as the mean number \( < n_k > \) of the species in the grand canonical ensemble. Thus we may determine the weight \( x_k \) through experimental data. This result also shows that any power law behavior in \( < n_k > \) is directly related to the power law of \( x_k \).

It is explicitly shown that KNO scaling works only for large fluctuations for all the distribution related to the HGa model. Comparison within the HGa model also shows that just comparing void variables \( \chi \) and \( \xi \) is not enough to distinguish different models that describing pion data. Thus new parameters have to be found which are quite different between different models. Beside mean values and fluctuations higher order reduced factorial cumulants need also to be evaluated.

In this paper we have also generalized the Poisson transformation and the compound distribution that arises from sequential process. Specifically, the underlying sequential picture involves a two step process where the final distribution arises from the production of clusters followed by a subsequent decay of the clusters. For the HGa model, the final distribution is obtained from compounding a Poisson distribution of clusters with a NB distribution coming from the decay of each of the clusters. The HGa may arise through a three step sequential process of Poisson-Poisson-Logarithmic compound distribution. It is also shown that the HGa can arise from a two step sequential process of a NB distribution followed by a new HGa with a different mean value.

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