Enumeration of extended irreducible binary Goppa codes

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Abstract

The family of Goppa codes is one of the most interesting subclasses of linear codes. As the McEliece cryptosystem often chooses a random Goppa code as its key, knowledge of the number of inequivalent Goppa codes for fixed parameters may facilitate in the evaluation of the security of such a cryptosystem. In this paper we present a new approach to give an upper bound on the number of inequivalent extended irreducible binary Goppa codes. To be more specific, let \( n > 3 \) be an odd prime number and \( q = 2^n \); let \( r \geq 3 \) be a positive integer satisfying \( \gcd(r, n) = 1 \) and \( \gcd(r, q(q^2 - 1)) = 1 \). We obtain an upper bound for the number of inequivalent extended irreducible binary Goppa codes of length \( q + 1 \) and degree \( r \).

MSC: 94B50.

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1 Introduction

The family of Goppa codes is one of the most interesting subclasses of linear codes, for example, see [1, 2, 5, 6]. This family of codes also contains long codes that have good parameters. Goppa codes have attracted considerable attention in cryptography; the McEliece cryptosystem and the Niederreiter cryptosystem are examples of public-key cryptosystems that use Goppa codes, for example, see [11, 15].

One of the reasons why Goppa codes receive interest from cryptographers may be that Goppa codes have few invariants and the number of inequivalent codes grow exponentially with the length and dimension of the code, which makes it possible to resist to any structural attack. More specifically, in the McEliece cryptosystem a random Goppa code is often chosen as a key. When we give the assessment of the security of this cryptosystem against the enumerative attack, it is important for us to know the number of Goppa codes for any given set of parameters. An enumerative attack in the McEliece cryptosystem is to find all Goppa codes for a given set of parameters and to test their equivalences with the public codes [11]. Thus one of the key issues for the McEliece cryptosystem is the enumeration of inequivalent Goppa codes for a given set of parameters.

There has been tremendous interest in developing the enumeration of inequivalent Goppa codes. Making use of the invariant property under the group of transformations, Moreno [16] classified cubic and quartic irreducible Goppa codes and obtained that there are four inequivalent quartic Goppa codes of length 33; in addition, Moreno proved that there is only one inequivalent extended irreducible binary Goppa code with any length and degree 3. Ryan studied irreducible Goppa codes [23, 25]. Ryan and Fitzpatrick [26] obtained an upper bound for the number of irreducible Goppa codes of length \( q^n \) over \( \mathbb{F}_q \). Ryan [24] made a great improvement on giving a much tighter upper bound (compared to the previous work [23]) on the number of inequivalent extended irreducible binary quartic Goppa codes of length \( 2^n + 1 \), where \( n > 3 \) is a prime number. Following that line of research, an upper bound on the number of inequivalent extended irreducible binary Goppa codes of degree \( 2^m \) and length \( 2^n + 1 \) was given in [18], where \( n \) is an odd prime and \( m > 1 \) is a positive integer. Magamba and Ryan [13] obtained an upper bound on the number of inequivalent extended irreducible \( q \)-ary Goppa codes of degree \( r \) and length

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is the projective semi-linear group and 
the set of all monic irreducible polynomials of degree 
do not use the Cauchy-Frobenius Theorem to obtain our main result ; we introduce an action of 
PΓL on 
S to count the number of orbits of PΓL on 
PGL
It allows us to convert the problem of counting the number of orbits of PΓL on 
2
S
of the advantages of considering the action of PΓL on 
PGL
q
n
+ 1, where 
q
= p', 
n and 
r > 2 are both prime numbers. In [17], an upper bound on the number of inequivalent extended irreducible binary Goppa codes of degree 2p and length 2n + 1 was produced, where 
n and 
p are two distinct odd primes such that 
p
does not divide 2n ± 1. Very recently, Huang and Yue [17] obtained an upper bound on the number of extended irreducible binary Goppa codes of degree 6 and length 2n + 1, where 
n > 3 is a prime number.

In this paper, we explore further the ideas in [17] and [24] to give an upper bound on the number of inequivalent extended irreducible binary Goppa codes of length 2n + 1 and degree 
r, where 
n > 3 is an odd prime number and 
r ≥ 3 be a positive integer satisfying gcd(r, n) = 1 and gcd (r, q(q 2 − 1)) = 1. By virtue of a result in [24], the number of inequivalent extended irreducible binary Goppa codes of length 2n + 1 and degree 
r is less than or equal to the number of orbits of PTL on 
S, where PTL = PGL × Gal
is the projective semi-linear group and 
S is a subset of 
F
2n+1. With the help of this result, one only needs to count the number of orbits of PTL on 
S. The papers [7], [13], [17], [18] and [24] used the Cauchy-Frobenius Theorem to calculate the number of orbits of PTL on 
S. Distinguishing from this approach, we do not use the Cauchy-Frobenius Theorem to obtain our main result; we introduce an action of PTL on the set of all monic irreducible polynomials of degree 
r over 
F
q, say 
I
r (see Lemma 3.1). We then show that the number of orbits of PTL on 
S is equal to the number of orbits of PTL on 
I
r (see Lemma 3.3). It allows us to convert the problem of counting the number of orbits of PTL on 
S to that of counting the number of orbits of PTL on 
I
r. We then use a basic strategy to count the number of orbits of PTL on 
I
r (see Lemma 3.4): the number of orbits of PTL on 
I
r is equal to the number of orbits of Gal on 
PGL\I
r, where 
PGL is the projective linear group and 
PGL\I
r is the set of orbits of PGL on 
I
r. One of the advantages of considering the action of PTL on 
I
r rather than that on 
S is that the orbits of 
⟨σ
r⟩ on 
PGL\I
r are trivial to see, where 
σ is the Frobenius generator of Gal.

This paper is organized as follows. In Section 2, we review some definitions and basic results about extended irreducible Goppa codes, some matrix groups and group actions. In Section 3, we find a formula for the number of orbits of PTL on the set 
I
r, which naturally gives an upper bound for the number of inequivalent extended irreducible Goppa codes of length 2n + 1 and degree 
r, where 
n and 
r satisfy certain conditions; we also give two small examples to illustrative our main result. We conclude this paper with remarks and some possible future works in Section 4.

2 Preliminaries

Starting from this section till the end of this paper, we assume that 
n > 3 is an odd prime number and 
q
= 2
n
; let 
r ≥ 3 be a positive integer satisfying gcd(r, n) = 1 and gcd (r, q(q 2 − 1)) = 1. Let 
F
q be the finite field with 
q elements and let 
F
q∗ = 
F
q \ {0} be the multiplicative group of the finite field 
F
q. Suppose 
x is an indeterminate over the finite field 
F
q and let 
F
q[x] be the polynomial ring in variable 
x with coefficients in 
F
q. As usual, for a polynomial 
f(x) ∈ 
F
q[x] (or simply denoted by 
f), deg 
f is the degree of 
f; for a finite set 
X, let 
|X| denote the number of elements of 
X.

We start by recalling the notion of irreducible binary Goppa codes of length 
q. For the general definition and more detail information about Goppa codes, readers may refer to [10] or [12].

2.1 Extended irreducible Goppa codes

Definition 2.1. Let 
g(x) be a polynomial in 
F
q[x] of degree 
r, and let 
L = 
F
q = {α0, α1, · · · , α
q−1} such that 
L ∩ {zeros of 
g(x)} = ∅. The binary Goppa code 
Γ (L, 
g) of length 
q and degree 
r is defined as

\[
\Gamma (L, g) = \left\{ c = (c0, c1, · · · , c\ q−1) \in \mathbb{F}_q^q \mid \sum_{i=0}^{q-1} \frac{c_i}{x - \alpha_i} \equiv 0 \pmod{\ g(x)} \right\}.
\]

The polynomial 
g(x) is called the Goppa polynomial. When 
g(x) is irreducible, 
Γ (L, 
g) is called an irreducible binary Goppa code of degree 
r.

The Goppa code of length 
q can be extended to a code of length 
q + 1 by appending a coordinate in the set 
L = \mathbb{F}_q. In this paper, we mainly consider extended irreducible Goppa codes. The definition of extended irreducible Goppa codes of length 
q + 1 and degree 
r is given below.

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Definition 2.2. For a given monic irreducible polynomial \( g(x) \) of degree \( r \), let \( \Gamma(L, g) \) be an irreducible binary Goppa code of length \( q \) as given above. The extended Goppa code \( \overline{\Gamma(L, g)} \) of length \( q + 1 \) is defined as

\[
\overline{\Gamma(L, g)} = \left\{ (c_0, c_1, \cdots, c_q) \in \mathbb{F}_2^{q+1} \mid (c_0, c_1, \cdots, c_{q-1}) \in \Gamma(L, g) \text{ and } \sum_{i=0}^{q} c_i = 0 \right\}.
\]

Chen [3] showed that the irreducible binary Goppa code \( \Gamma(L, g) \) is completely determined by any root of the Goppa polynomial \( g(x) \); more precisely, if \( \alpha \) is a root of \( g(x) \) in some extension field over \( \mathbb{F}_q \), then

\[
H(\alpha) = \left( \frac{1}{\alpha_0 - \alpha_0}, \frac{1}{\alpha - \alpha_1}, \cdots, \frac{1}{\alpha - \alpha_{q-1}} \right)
\]

can be served as a parity check matrix for \( \Gamma(L, g) \). As such, let \( C(\alpha) \) denote the code \( \Gamma(L, g) \) and let \( \overline{C(\alpha)} \) denote the code \( \overline{\Gamma(L, g)} \). Therefore, every extended irreducible binary Goppa code of length \( q + 1 \) and degree \( r \) can be described as \( \overline{C(\alpha)} \) for some \( \alpha \in \mathbb{F}_{q^r} \).

2.2 Equivalent extended irreducible Goppa codes

The primary purpose of this paper is to give an upper bound for the number of (inequivalent) extended irreducible binary Goppa codes of length \( q + 1 \) and degree \( r \). This problem can be reduced to that of counting the number of orbits of the projective semi-linear group action on some set (see [1], [7] or [24]). To state this result clearly, we need the notions of group actions and some matrix groups. In the following we collect the matrix groups that we will use later and fix the notations.

1. The general linear group of size \( 2 \times 2 \) over \( \mathbb{F}_q \)

\[
\text{GL} = \text{GL}_2(\mathbb{F}_q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{F}_q, ad - bc \neq 0 \right\}.
\]

2. The affine general linear group of size \( 2 \times 2 \) over \( \mathbb{F}_q \)

\[
\text{AGL} = \text{AGL}_2(\mathbb{F}_q) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_{q^*}, b \in \mathbb{F}_q \right\}.
\]

3. The projective general linear group of size \( 2 \times 2 \) over \( \mathbb{F}_q \)

\[
\text{PGL} = \text{PGL}_2(\mathbb{F}_q) = \text{GL}/\mathcal{Z},
\]

where \( \mathcal{Z} \) is the center of \( \text{GL} \) consisting of the multiples of the identity matrix by elements of \( \mathbb{F}_{q^*} \).

4. The projective semi-linear group

\[
\text{PTL} = \text{PTL}_2(\mathbb{F}_q) = \text{PGL} \rtimes \text{Gal} = \left\{ A\sigma^i \mid A \in \text{PGL}, 0 \leq i \leq rn - 1 \right\},
\]

where \( \text{Gal} = \text{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_2) = \text{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2) = \langle \sigma \rangle \) is the Galois group of order \( rn \) generated by \( \sigma \) (\( \sigma \) sends each \( \alpha \in \mathbb{F}_{q^r} \) to \( \alpha^2 \)). The operation \( " \cdot " \) in \( \text{PTL} \) is defined as follows:

\[
A\sigma^i \cdot B\sigma^j = A\sigma^i(B)\sigma^{i+j}, \quad 0 \leq i, j \leq rn - 1,
\]

where \( \sigma(B) = \left( \begin{array}{cc} \sigma^t & \sigma^u \\ \sigma^w & \sigma^v \end{array} \right) \) for \( B = \left( \begin{array}{cc} t & u \\ v & w \end{array} \right) \in \text{PGL} \) (\( \sigma^a \) means \( \sigma^a a = a^{2^a} \) for \( a \in \mathbb{F}_{q^r} \)). It is clear that \( E_2 \sigma^n \) is the identity element of \( \text{PTL} \), where \( E_2 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \) is the identity matrix.

Now it is the turn of group actions (for example, see [22]). Let \( S = \mathcal{S}(r, n) \) denote the set of elements in \( \mathbb{F}_{q^r} \) of degree \( r \) over \( \mathbb{F}_q \); in other words,

\[
S = \left\{ \alpha \in \mathbb{F}_{q^r} \mid \text{there exists a monic irreducible polynomial of degree } r \text{ over } \mathbb{F}_q \text{ satisfying } f(\alpha) = 0 \right\}.
\]

It is known that \( \text{PGL} \) and \( \text{PTL} \) can act on the set \( S \) in the following ways (see [7] or [24]):
The action of the projective general linear group on $S$:

$$\text{PGL} \times S \to S \quad (A, \alpha) \mapsto A\alpha = \frac{a\alpha + b}{ca + d}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PGL}$.

The action of the projective semi-linear group on $S$:

$$\text{PΓL} \times S \to S \quad (A\sigma^i, \alpha) \mapsto A\sigma^i\alpha = \frac{a\sigma^i(\alpha) + b}{c\sigma^i(\alpha) + d} = \frac{a\alpha^2 + b}{ca^2 + d}$$

We are ready to recall a sufficient condition which guarantees two extended irreducible Goppa codes to be equivalent; thus, in particular, it gives an upper bound for the number of inequivalent codes in

$$\{ \overline{C(\alpha)} \mid \alpha \in S \},$$

see [1], [7] or [24].

**Lemma 2.3.** Let $\alpha \in S$ and $\beta \in S$. If $\alpha, \beta$ lie in the same PΓL-orbit: $\alpha = A\sigma^i\beta$ for some $A\sigma^i \in \text{PΓL}$, then the extended Goppa code $\overline{C(\alpha)}$ is (permutation) equivalent to the extended Goppa code $\overline{C(\beta)}$. In particular, the number of inequivalent extended irreducible binary Goppa codes of length $q + 1$ and degree $r$ is less than or equal to the number of orbits of PΓL on $S$.

With the help of Lemma 2.3 we only need to count the number of orbits of PΓL on $S$.

### 3 An upper bound for the number of extended Goppa codes

The papers [7], [13], [17], [18] and [24] used the Cauchy-Frobenius Theorem to calculate the number of orbits of PΓL on $S$. Here we introduce another group action: The group PΓL can act on the set of all monic irreducible polynomials of degree $r$ over $\mathbb{F}_q$. Let $\mathcal{I}_r$ be the set of all monic irreducible polynomials of degree $r$ over $\mathbb{F}_q$. We will show that the number of orbits of PΓL on $S$ is equal to the number of orbits of PΓL on $\mathcal{I}_r$.

#### 3.1 The action of PΓL on $\mathcal{I}_r$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PGL}, \alpha \in \mathbb{F}_q$ and $f(x) = a_0 + a_1x + \cdots + a_rx^r \in \mathbb{F}_q[x]$, where $r \geq 1, a_r \neq 0$. We make the following definitions:

$$f(x)^* = \frac{1}{a_r}f(x), \quad A\alpha = \frac{a\alpha + b}{ca + d}$$

$$Af = (-cx + a)^rf(A^{-1}x) = (-cx + a)^rf\left(\frac{dx - b}{-cx + a}\right),$$

$$\sigma^i f = \sigma^i(f(x)) = \sigma^i a_0 + \sigma^i a_1x + \cdots + \sigma^i a_r x^r.$$

Now we show that the group PΓL acts on the set $\mathcal{I}_r$, as stated below.

**Lemma 3.1.** With the notation given above, we have a group action PΓL on the set $\mathcal{I}_r$ defined by

$$\text{PΓL} \times \mathcal{I}_r \to \mathcal{I}_r \quad (A\sigma^i, f) \mapsto (A\sigma^i)(f) = \left(\frac{A(\sigma^i f)}{A(\sigma^i) f}\right)^*.$$
Proof. Verification of the group action conditions is routine. We first introduce the strategy: Let \( X_r \) denote the set of all irreducible polynomials of degree \( r \) over \( \mathbb{F}_q \). There is an equivalence relation defined on \( X_r \): \( f \sim g \) if and only if there exists \( \lambda \in \mathbb{F}_q^* \) such that \( f = \lambda g \). We claim that

\[
\text{PGL} \times X_r \rightarrow X_r \\
(A\sigma^i, f) \mapsto (A\sigma^i)(f) = A(\sigma^i f)
\]

defines an action of PGL on \( X_r \). Once this claim is established, one can easily show that if \( f \sim g \) then \( A\sigma^i f \sim A\sigma^i g \); this implies that PGL acts on the equivalence classes \( X_r/\sim \). Since every equivalence class contains a unique monic polynomial, it is trivial to see that PGL acts on \( \mathbb{F}_q \), in the way stated in the lemma. Therefore, it is enough to verify that PGL acts on \( X_r \).

We will give a detailed proof by carrying out the following steps, although it is somewhat tedious.

Step 1. \( \deg(A(\sigma^i f)) = \deg(f) \) for any \( f \in X_r \).

Suppose that \( f(x) = f_0 + f_1 x + \cdots + f_{r-1} x^{r-1} + f_r x^r \), where \( f_r \neq 0 \). Then

\[
A(\sigma^i f(x)) = (\sigma^i f)(\frac{dx - b}{-cx + a}) \cdot (-cx + a)^r
\]

\[
= \sigma^i f_r (dx - b)^r + \sigma^i f_{n-1}(dx - b)^{r-1}(-cx + a) + \cdots + \sigma^i f_0(-cx + a)^r
\]

\[
x^r \left[ \sigma^i f_r d^r + \sigma^i f_{r-1}(-c)d^{r-1} + \cdots + \sigma^i f_0(-c)^r \right] + \cdots.
\]

If \( c \neq 0 \), then

\[
\sigma^i f_r d^r + \sigma^i f_{r-1}(-c)d^{r-1} + \cdots + \sigma^i f_0(-c)^r
\]

\[
= (-c)^r \left[ \sigma^i f_0 + \sigma^i f_1 \left( -\frac{d}{c} \right) + \cdots + \sigma^i f_{r-1} \left( -\frac{d}{c} \right)^{r-1} + \sigma^i f_r \left( -\frac{d}{c} \right)^r \right]
\]

\[
= (-c)^r (\sigma^i f) \left( -\frac{d}{c} \right).
\]

Noting that \( c, d \in \mathbb{F}_q \) and \( \sigma^i f \) is irreducible over \( \mathbb{F}_q \), we obtain \((-c)^r (\sigma^i f) \left( -\frac{d}{c} \right) \neq 0\).

If \( c = 0 \), then \( A \in \text{PGL} \), which yields \( d \neq 0 \), and we have \( \sigma^i f_r d^r \neq 0 \), i.e., the leading coefficient of \( x^r \) in \((A\sigma^i)f\) is nonzero.

In conclusion, we have \( \deg(A(\sigma^i f)) = \deg(f) \) for any \( f(x) \in X_r \).

Step 2. \( A(\sigma^i f) \in X_r \) for any \( f \in X_r \).

It is enough to show that \( A(\sigma^i f) \) is irreducible over \( \mathbb{F}_q \). To this end, we use the following result (see [22 Proposition 4.13]): Let \( k \) be a field and let \( p(x) \in k[x] \) have no repeated roots. If \( E/k \) is a splitting field of \( p(x) \), then \( p(x) \) is irreducible if and only if the Galois group of \( E \) over \( k \), denoted by \( \text{Gal}(E/k) \), acts transitively on the roots of \( p(x) \).

Suppose that \( \alpha, \alpha^q, \ldots, \alpha^{q^{r-1}} \) are all the distinct roots of \( f(x) \). Then \( A(\sigma^i \alpha), A(\sigma^i \alpha^q), \ldots, A(\sigma^i \alpha^{q^{r-1}}) \) are all the roots of \( A(\sigma^i f) \). Since for \( j = 0, 1, 2, \ldots, r - 1 \),

\[
A(\sigma^i \alpha^q) = \frac{a \sigma^i \alpha^q + b}{a \sigma^i \alpha^q + d} = \left( \frac{a \sigma^i \alpha + b}{a \sigma^i \alpha + d} \right)^q = (A(\sigma^i \alpha))^q,
\]

\( A(\sigma^j \alpha), A(\sigma^i \alpha^q), \ldots, A(\sigma^i \alpha^{q^{r-1}}) \) are distinct; that is to say \( A(\sigma^i f) \) has no repeated roots. On the other hand, \( A(\sigma^i f) \in \mathbb{F}_q[x] \) and the splitting field of \( A(\sigma^i f) \) is \( \mathbb{F}_q \). Hence \( \mathbb{F}_q \) is the splitting field of \( A(\sigma^i f) \).

Clearly, \( \text{Gal}(\mathbb{F}_q/\mathbb{F}_q) = (\sigma^n) \). Let \( \tau = \sigma^n \) and then all the distinct roots of \( A(\sigma^i f) \) are

\[
\beta, \tau(\beta), \tau^2(\beta), \ldots, \tau^{r-1}(\beta).
\]
Thus $\text{Gal}(\mathbb{F}_q / \mathbb{F}_q)$ acts transitively on the roots of $A(\sigma^i f)$. Hence according to [22] Proposition 4.13 $A(\sigma^i f)$ is irreducible over $\mathbb{F}_q$.

Step 3. Clearly, $(E_2(\sigma^0 f)) f = E_2(\sigma^0 f) = f$, where $E_2 \sigma^0$ is the identity element of the group PΓL.

Step 4. We are left to check that

$$(A\sigma^i)[(B\sigma^j)f] = [(A\sigma^i) \cdot (B\sigma^j)] f,$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PGL}$, $B = \begin{pmatrix} t & u \\ v & w \end{pmatrix} \in \text{PGL}$. On the one hand,

$$(A\sigma^i)[(B\sigma^j)f] = (A\sigma^i) \left[(-vx + t)^r(\sigma^i f) \left(\frac{wx - u}{-vx + t} \right) \right]$$

$$= (-cx + a)^r(\sigma^i)^r(\sigma^j)^r(\sigma^i)^r(\sigma^j)^r(\sigma^j)^r$$

$$= \left[ -\sigma^i v(dx - b) + \sigma^j(t(-cx + a)) \right](\sigma^i)^r(\sigma^j)^r$$

$$= \left[ \sigma^i v(dx - b) - \sigma^j u(-cx + a) \right](\sigma^i)^r(\sigma^j)^r.$$

On the other hand, since

$$(A\sigma^i) \cdot (B\sigma^j) = A\sigma^i(B)\sigma^i + j$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sigma^i t & \sigma^i u \\ \sigma^i w & \sigma^i u \end{pmatrix} \sigma^i + j$$

$$= \begin{pmatrix} a\sigma^i t + ba^i & a\sigma^i u + be^i \\ ca^i t + da^i & ca^i u + da^i \end{pmatrix} \sigma^i + j$$

and

$$(A\sigma^i(B))^r = \begin{pmatrix} ca^i t + da^i & ca^i u + da^i \end{pmatrix}$$

we have

$$[(A\sigma^i) \cdot (B\sigma^j)] f = \left( \begin{pmatrix} a\sigma^i t + ba^i & a\sigma^i u + be^i \\ ca^i t + da^i & ca^i u + da^i \end{pmatrix} \sigma^i + j \right) f$$

$$= \left( (-ca^i t + da^i v) x + \sigma^i t + ba^i v \right)^r(\sigma^i)^j$$

Then we obtain $(A\sigma^i)[(B\sigma^j)f] = [(A\sigma^i) \cdot (B\sigma^j)] f$, as wanted.

**Remark 3.2.** Many authors have studied the action of PGL on $I_r$, focusing on the characterization and number of $A$-invariants where $A \in \text{PGL}$ (for example, see [4], [19], [20], [21], [27]). The paper [14] considered an action of PTL on $I_r$, and our definition of PTL on $I_r$ is different from that of [14].

### 3.2 The orbits of PΓL on $I_r$

In this subsection we analyze the orbits of PTL on $I_r$. We first introduce some notations. For a general group $G$ acting on a set $X$, let $G(x)$ denote the orbit containing $x \in X$, namely $G(x) = \{gx | g \in G\}$; let $\text{Stab}_G(x)$ be the stabilizer of the point $x \in X$ in $G$, namely $\text{Stab}_G(x) = \{g \in G | gx = x\}$. For example, $\text{PGL}(\alpha) = \{A\alpha | A \in \text{PGL}\}$ denotes the orbit of $\alpha \in S$ under the action of PGL on $S$, and $\text{PTL}(f) = \{(A\sigma^i)(f) | A \in \text{PGL}, 0 \leq i \leq rn - 1\}$ denotes the orbit of $f \in I_r$, under the action of PTL on $I_r$.

The next result reveals that the problem of counting the number of orbits of PTL on $S$ can be completely converted to the problem of counting the number of orbits of PTL on $I_r$.

**Lemma 3.3.** The number of orbits of PTL on $S$ is equal to the number of orbits of PTL on $I_r$. 

Proof. Let $\Gamma \setminus S$ be the set of orbits of $\Gamma$ on $S$ and let $\Gamma \setminus I_r$ be the set of orbits of $\Gamma$ on $I_r$. To prove $|\Gamma \setminus S| = |\Gamma \setminus I_r|$, it suffices to show that there is a bijection between $\Gamma \setminus S$ and $\Gamma \setminus I_r$. Define a map $\varphi$ as follows:

$$\varphi: \Gamma \setminus I_r \to \Gamma \setminus S$$

$$\Gamma(f) \mapsto \varphi(\Gamma(f)),$$

where

$$\varphi(\Gamma(f)) = \{ \alpha \mid \text{there exists a polynomial } g(x) \in \Gamma(f) \text{ such that } g(\alpha) = 0 \}.$$

We will show that $\varphi$ is a bijection by carrying out the following steps.

(1) $\varphi(\Gamma(f)) \in \Gamma \setminus S$. Let $f(\alpha) = 0$, i.e., $\alpha$ is a root of $f(x)$. Then $(A\sigma^i f)(A(\sigma^i(\alpha))) = 0$, i.e., $A(\sigma^i(\alpha))$ is a root of $(A\sigma^i(f(x)))$. Assume that $\alpha, \alpha^q, \cdots, \alpha^{q^{r-1}}$ are all the distinct roots of $f(x)$. Based on the above fact we have that

$$A(\sigma^i(\alpha)), A(\sigma^i(\alpha^q)), \cdots, A(\sigma^i(\alpha^{q^{r-1}}))$$

are all the distinct roots of the polynomial $(A\sigma^i(f(x)))$. Noting that

$$A(\sigma^i(\alpha^j)) = (A(\sigma^i\alpha))^q^j = A(\sigma^{i+nj}\alpha), \quad j = 0, 1, \cdots, r-1,$$

we obtain that

$$\varphi(\Gamma(f)) = \{ (A(\sigma^i\alpha))^q^j \mid A \in GL, 0 \leq i \leq rn - 1, 0 \leq j \leq r - 1 \}$$

$$= \{ A(\sigma^{i+nj}\alpha) \mid A \in GL, 0 \leq i \leq rn - 1, 0 \leq j \leq r - 1 \}$$

$$= \{ A(\sigma^i\alpha) \mid A \in GL, 0 \leq i \leq rn - 1 \}$$

$$= \Gamma(\alpha),$$

which shows that $\varphi(\Gamma(f)) \in \Gamma \setminus S$.

(2) $\varphi$ is injective. According to the definition of $\varphi$ it is obvious that $\varphi$ is injective.

(3) $\varphi$ is surjective. Given an orbit $\Lambda = \Gamma(\alpha)$ in $\Gamma \setminus S$, i.e., $\alpha \in S$ and

$$\Lambda = \{ A(\sigma^i\alpha) \mid A \in PGL, 0 \leq i \leq rn - 1 \}.$$

Suppose that $f(x)$ is an irreducible polynomial of degree $r$ over $\mathbb{F}_q$ with $f(\alpha) = 0$, then $\varphi(\Gamma(f)) = \Lambda$. Thus $\varphi$ is surjective. \hfill $\Box$

By Lemma 3.3 our ultimate aim is to find the number of orbits of $\Gamma = PGL \times Gal$ on the set $I_r$. For this purpose, we will repeatedly use the following fact to achieve this goal (for example, see [8, Pages 35-36]):

**Lemma 3.4.** Let $G$ be a finite group acting on a finite set $X$ and let $N$ be a normal subgroup of $G$. It is clear that $N$ naturally acts on $X$. Suppose the $N$-orbits are denoted by $N\setminus X = \{ N(x) \mid x \in X \}$. Then the factor group $G/N$ acts on $N\setminus X$ and the number of orbits of $G$ on $X$ is equal to the number of orbits of $G/N$ on $N\setminus X$.

As $PGL$ is a normal subgroup of $\Gamma$, by virtue of Lemma 3.4 we first count the number of orbits of $PGL$ on the set $I_r$. The next result shows that if $\gcd(r, q(q^2 - 1)) = 1$, then the size of each orbit of $PGL$ on $I_r$ is equal to $q(q^2 - 1)$; in other words, $\text{Stab}_{PGL}(f) = \{ E_2 \}$ for any $f \in I_r$, where $E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix.

**Lemma 3.5.** Let $r$ be a positive integer satisfying $r \geq 3$ and $\gcd(r, q(q^2 - 1)) = 1$. Let $f(x) \in I_r$ and

$$\text{Stab}_{PGL}(f) = \{ A \in PGL \mid Af = f \}.$$

Then $\text{Stab}_{PGL}(f) = \{ E_2 \}$.
Proof. Fix an irreducible polynomial $f(x)$ of degree $r$ over $\mathbb{F}_q$, i.e., $f(x) \in \mathcal{I}_r$. Suppose $A \in \text{Stab}_{\text{PGL}}(f)$ and the order of $A \in \text{PGL}$ is equal to $\ell$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. In the following we want to prove that $A$ must be the identity matrix, i.e., $A = E_2$. Let $\alpha$ be a root of $f(x)$. Then $A\alpha$ is a root of $Af$. Thus $A\alpha$ is also a root of $f(x)$. Hence there exists a positive $s$ satisfying $0 \leq s \leq r - 1$ such that $A\alpha = \alpha^s$. Since $A^\ell = E_2$, we have $A^s = E_2\alpha = \alpha$, therefore $\alpha^s = \alpha$. This shows that $\mathbb{F}_q = \mathbb{F}_q(\alpha) \subseteq \mathbb{F}_q(\alpha^{1/r})$, which gives that $r$ is a divisor of $s\ell$.

Clearly, $\ell$ is a divisor of $q(q^2 - 1)$. Since $\gcd(r, q(q^2 - 1)) = 1$, we obtain that $\gcd(r, \ell) = 1$. Hence $r$ is a divisor of $s$. We know that $\alpha^s = \alpha$, and we get $\alpha^{s\ell} = \alpha$; it follows from $A\alpha = \alpha^s$ that $A\alpha = \alpha$. Substitute $A$ to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and it leads to

\[
\frac{aa + b}{ca + d} = \alpha,
\]

that is to say $ca^2 + (d + a)\alpha + b = 0$. By our assumption $r \geq 3$, the above equality tells us that $c = b = 0$ and $a = d$, i.e., $A = E_2$. This concludes the proof.

Lemma 3.5 shows that the size of every orbit $\text{PGL}(f)$ of $\text{PGL}$ on $\mathcal{I}_r$ is equal to

\[
|\text{PGL}(f)| = |\text{PGL} : \text{Stab}_{\text{PGL}}(f)| = |\text{PGL}| = q(q^2 - 1).
\]

Let $\text{PGL}\backslash\mathcal{I}_r$ be the set of all orbits of $\text{PGL}$ on $\mathcal{I}_r$. It follows from Lemma 3.5 and the enumerative formula for the size of $\mathcal{I}_r$ (see [9, Theorem 3.25]) that

\[
|\text{PGL}\backslash\mathcal{I}_r| = \frac{|\mathcal{I}_r|}{q(q^2 - 1)} = \frac{\sum d | r \mu(d)q^{r/d}}{rq(q^2 - 1)},
\]

(3.1)

where $\mu$ is the Möbius function.

### 3.3 The orbits of $\text{Gal}$ on $\text{PGL}\backslash\mathcal{I}_r$

In order to get the number of orbits of $\text{PGL}$ on $\mathcal{I}_r$, by Lemmas 3.3 and 3.5 we have to count the number of orbits of $\text{Gal}$ on $\text{PGL}\backslash\mathcal{I}_r$. Recall that the Galois group $\text{Gal} = \text{Gal}(\mathbb{F}_q'/\mathbb{F}_2) = \text{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2) = \langle \sigma \rangle$ is the cyclic group of order $rn$ generated by $\sigma$. The action of $\text{Gal}$ on $\text{PGL}\backslash\mathcal{I}_r$ is given by

\[
\text{Gal} \times \text{PGL}\backslash\mathcal{I}_r \rightarrow \text{PGL}\backslash\mathcal{I}_r
\]

\[
(\sigma, \text{PGL}(f)) \rightarrow \sigma(\text{PGL}(f)) = \text{PGL}(\sigma^r f).
\]

Recall also that $n \geq 3$ is a prime number, $q = 2^n$ and $r$ is a positive integer satisfying $\gcd(r, n) = 1$ and $\gcd(r, q(q^2 - 1)) = 1$. Since $\gcd(r, n) = 1$, $\text{Gal} = \langle \sigma \rangle$ has the following decomposition into direct products:

\[
\text{Gal} = \langle \sigma^r \rangle \times \langle \sigma^n \rangle.
\]

In order to count the number of orbits of $\text{Gal}$ on $\text{PGL}\backslash\mathcal{I}_r$, using Lemma 3.4 again we first consider the action of $\langle \sigma^n \rangle$ on $\text{PGL}\backslash\mathcal{I}_r$ (i.e., $\langle \sigma^n \rangle$ is certainly a normal subgroup of $\text{Gal}$). Note that the action of $\langle \sigma^n \rangle$ on $\text{PGL}\backslash\mathcal{I}_r$ is given by

\[
\langle \sigma^n \rangle \times \text{PGL}\backslash\mathcal{I}_r \rightarrow \text{PGL}\backslash\mathcal{I}_r
\]

\[
(\sigma^n, \text{PGL}(f)) \rightarrow \sigma^{ni}(\text{PGL}(f)) = \text{PGL}(\sigma^{ni} f).
\]

Observe that $\sigma^{n} a = a^{2^n} = \sigma^q = a$ for any $a \in \mathbb{F}_q$, which gives

\[
\text{PGL}(\sigma^{ni} f) = \text{PGL}(f) \quad \text{for any } f \in \mathcal{I}_r.
\]

That is to say that $\langle \sigma^n \rangle$ fixes each $\text{PGL}(f)$ in $\text{PGL}\backslash\mathcal{I}_r$; in other words, the set of orbits of $\langle \sigma^n \rangle$ on $\text{PGL}\backslash\mathcal{I}_r$ remains $\text{PGL}\backslash\mathcal{I}_r$. By Lemma 3.4, the number of orbits of $\text{Gal}$ on $\text{PGL}\backslash\mathcal{I}_r$ is equal to the number of orbits of $\langle \sigma^r \rangle$ on $\text{PGL}\backslash\mathcal{I}_r$. Since $\langle \sigma^r \rangle$ is of prime order $n$, the size of every orbit of $\langle \sigma^r \rangle$ on $\text{PGL}\backslash\mathcal{I}_r$ is equal to $1$ or $n$. Thus it is enough to determine the number of orbits of $\langle \sigma^r \rangle$ on $\text{PGL}\backslash\mathcal{I}_r$ with size $1$. 

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Lemma 3.6. Let notation be the same as before. Assume that $r \geq 3$ is a positive integer satisfying $\gcd(r, q(q^2 - 1)) = 1$. Let $f \in \mathcal{I}_r$ and let $\alpha$ be a root of $f(x)$. Then $\PGL^r(f) = \PGL(f)$ if and only if $\PGL^r(\alpha) = \PGL(\alpha)$.

Proof. Suppose $\PGL^r(f) = \PGL(f)$. Then there exists $A \in \PGL$ such that $A(\sigma f) = f$. This implies that $A(\sigma^r \alpha)$ is also a root of $f(x)$. Thus there is an integer $s \geq 0$ satisfying

$$A(\sigma^r \alpha) = \alpha^{q^s}. \tag{3.2}$$

Simple algebraic calculations in $\PGL^2$ show that

$$(A\sigma)^n = A\sigma^r(A\sigma^{2r}(A) \cdots \sigma^{r(n-1)}(A)\sigma^n = A\sigma^r(A)\sigma^{2r}(A) \cdots \sigma^{r(n-1)}(A)\sigma^0,$$

where $\sigma$ be the set of all orbits of $\AGL$ on $\mathcal{I}_r$. The affine general linear group $\AGL$ can be viewed naturally as a subgroup of $\PGL$. Hence, the group $\AGL$ acts on the set $\mathcal{I}_r$. Then $A\sigma f$ is also a root of $f(x)$, and we obtain $\PGL^r(\sigma f) = \PGL(f)$.

Conversely, suppose that $\PGL^r(\sigma \alpha) = \PGL(\alpha)$. Then there is a matrix $A \in \PGL$ such that $A(\sigma^r \alpha) = \alpha$. Note that $A(\sigma^r \alpha)$ is a root of $A(\sigma^r f)$, and then we obtain $\PGL^r(\sigma^r f) = \PGL(f)$. We are done. \hfill \qed

To count the number of $\PGL(f) \in \PGL \mathcal{I}_r$ that are fixed by $\langle \sigma^r \rangle$, by virtue of \cite{24} we need to use the affine general linear group $\AGL$. The affine general linear group $\AGL$ can be viewed naturally as a subgroup of $\PGL$. Hence, the group $\AGL$ acts on the set $\mathcal{S}$ naturally. Let

$$\AGL \mathcal{S} = \{ \AGL(\alpha) \mid \alpha \in \mathcal{S} \}$$

be the set of all orbits of $\AGL$ on $\mathcal{S}$. Then the cyclic group $\langle \sigma^r \rangle$ acts on $\AGL \mathcal{S}$ in the following way:

$$\langle \sigma^r \rangle \times \AGL \mathcal{S} \to \AGL \mathcal{S}, \quad (\sigma^r, \AGL(\alpha)) \mapsto \sigma^r(\AGL(\alpha)) = \AGL(\sigma^r \alpha). \tag{3.3}$$

It is not hard to verify that this is indeed a group action. We now turn to consider the orbit $\PGL(\alpha)$ where $\alpha \in \mathcal{S}$. There is an action of $\AGL$ on $\PGL(\alpha)$:

$$\AGL \times \PGL(\alpha) \to \PGL(\alpha)$$

$$(C, A\alpha) \mapsto C A\alpha.$$

Therefore, $\PGL(\alpha)$ is the disjoint union of $\AGL$-orbits. Indeed, one can easily check that there are exactly $q + 1$ right cosets of $\AGL$ in $\PGL$ and

$$t_0 = E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad t_\gamma = \begin{pmatrix} 0 & 1 \\ 1 & \gamma \end{pmatrix} \quad \text{for any } \gamma \in \mathbb{F}_q^*$$

consists of a right coset representative of $\AGL$ in $\PGL$. The coset decomposition

$$\PGL = \AGL t_0 \bigcup \AGL t_1 \bigcup \bigcup_{\gamma \in \mathbb{F}_q^*} \AGL t_\gamma$$

gives rise to the orbit decomposition of $\PGL(\alpha)$ into $\AGL$-orbits

$$\PGL(\alpha) = \AGL(t_0 \alpha) \bigcup \AGL(t_1 \alpha) \bigcup \bigcup_{\gamma \in \mathbb{F}_q^*} \AGL(t_\gamma \alpha).$$

We have arrived at the following result, which has been appeared previously in \cite{7} and \cite{24}.

Lemma 3.7. Let $\alpha \in \mathcal{S}$. Then

$$\PGL(\alpha) = \bigcup_{\gamma \in \mathbb{F}_q^*} \AGL \left( \frac{1}{\alpha + \gamma} \right) \bigcup \AGL(\alpha),$$

is a partition of $\PGL(\alpha)$ into $\AGL$-orbits.
Lemma 3.7 implies that
\[
PGL(\sigma^r(\alpha)) = \bigcup_{\gamma \in \mathbb{F}_q} AGL\left(\frac{1}{\sigma^r(\alpha) + \gamma}\right) \bigcup AGL(\sigma^r(\alpha))
\]
\[
= \bigcup_{\gamma \in \mathbb{F}_q} AGL\left(\frac{1}{\sigma^r(\alpha) + \sigma^r(\gamma)}\right) \bigcup AGL(\sigma^r(\alpha))
\]
\[
= \bigcup_{\gamma \in \mathbb{F}_q} AGL\left(\frac{1}{\alpha + \gamma}\right) \bigcup AGL(\sigma^r(\alpha)).
\]
Suppose now that PGL(\alpha) is fixed by the cyclic group \langle \sigma^r \rangle, i.e., PGL(\sigma^r \alpha) = PGL(\alpha). In this case, the cyclic group \langle \sigma^r \rangle acts on the set of AGL-orbits
\[
AGL \backslash PGL(\alpha) = \left\{ AGL(\alpha), AGL\left(\frac{1}{\alpha + \gamma}\right) \big| \gamma \in \mathbb{F}_q \right\}
\]
in the way given in Lemma 3.3. The next result has been appeared in [7] and [21].

**Lemma 3.8.** Let \( n > 3 \) be a prime number. If PGL(\sigma^r \alpha) = PGL(\alpha), then there exists a fixed point of \langle \sigma^r \rangle on AGL \backslash PGL(\alpha). In other words, either AGL(\sigma^r \alpha) = AGL(\alpha) or AGL(\sigma^r(\frac{1}{\alpha + \gamma})) = AGL(\frac{1}{\alpha + \gamma})\) for some \( \gamma \in \mathbb{F}_q \).

**Proof.** Note that the following properties hold: (1) \( |AGL \backslash PGL(\alpha)| = q + 1 \); (2) \( n \) does not divide \( q + 1 \); (3) the size of each orbit of \langle \sigma^r \rangle on AGL \backslash PGL(\alpha) is either 1 or \( n \). We get the required result. \( \square \)

The following result is crucial to our enumeration.

**Lemma 3.9.** Let notation be the same as before. Let \( f \in \mathbb{Z} \). Then PGL(\sigma^r f) = PGL(f) if and only if there is a polynomial \( g(x) \in PGL(f) \) such that \( g(x) \) divides \( x^{2^r} + x \).

**Proof.** To complete the proof, we mainly modify the arguments in [21] Lemma 3.1 and Theorem 3.2]. Let \( \alpha \) be a root of \( f(x) \). Lemma 3.6 says that PGL(\sigma^r f) = PGL(f) if and only if PGL(\sigma^r \alpha) = PGL(\alpha).

Suppose that PGL(\sigma^r \alpha) = PGL(\alpha), then according to Lemma 3.8 we need to consider two cases:

(1) AGL(\sigma^r \alpha) = AGL(\alpha). In this case there are \( \theta \neq 0, \tau e \mathbb{F}_q \) such that \( \alpha^{2^r} = \theta \alpha + \tau \). Let \( \rho \) be a primitive element of \( \mathbb{F}_q \), i.e., \( \mathbb{F}_q^* = \langle \rho \rangle \) and \( \rho \) is of order \( q - 1 = 2^n - 1 \). Since \( (2^r - 1, 2^n - 1) = 1 \), \( \mathbb{F}_q^* = \langle \rho^{2^r - 1} \rangle \). Assume that \( \theta = \rho^{2^r - 1} \kappa \), where \( \kappa \) is a positive integer. Then there exists an element \( \mu = \rho^{-\kappa} \in AGL(\alpha) \) for each \( \nu \in \mathbb{F}_q \) and
\[
(\mu \alpha + \nu)^{2^r} = \mu^{2^r} \alpha^{2^r} + \nu^{2^r}
\]
\[
= \mu^{2^r} (\theta \alpha + \tau) + \nu^{2^r}
\]
\[
= \mu^{2^r} - \theta (\mu \alpha + \nu) + \mu^{2^r - 1} \theta \nu + \mu^{2^r} \tau + v^{2^r}
\]
\[
= (\mu \alpha + \nu) + (\nu + \mu^{2^r} \tau + v^{2^r}).
\]
Write \( \beta = \mu \alpha + \nu \) and \( \xi = \nu + \mu^{2^r} \tau + v^{2^r} \). Then \( \beta \in AGL(\alpha) \), \( \xi \in \mathbb{F}_q \) and \( \beta^{2^r} = \beta + \xi \). This gives \( \xi = \beta^{2^r} + \beta \), which yields
\[
\xi + \xi^{2^r} + \cdots + \xi^{2^r(n - 1)} = 0.
\]
It is known that \( 0, 1, 2, \cdots, n - 1 \) is a complete set of residues modulo \( n \). From \( \gcd(r, n) = 1 \) we have that \( 0, r, 2r, \cdots, (n - 1)r \) is also a complete set of residues modulo \( n \). By \( \xi^{2^r} = \xi \) we have
\[
\xi + \xi^2 + \cdots + \xi^{2^r(n - 1)} = 0.
\]
This is equivalent to saying that (see [9] Definition 2.22) \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\xi) = 0 \). Hence there exists \( \omega_0 \in \mathbb{F}_q \) such that \( \xi = \omega_0^2 + \omega_0 \) (see [9] Theorem 2.25]). Therefore, there is an element \( \omega \in \mathbb{F}_q \) such that \( \xi = \omega^{2^r} + \omega \), because one can easily see that (using the fact \( \gcd(r, n) = 1 \))
\[
\{\eta^2 + \eta \mid \eta \in \mathbb{F}_q\} = \{\eta^{2^r} + \eta \mid \eta \in \mathbb{F}_q\}.
\]
It follows that
\[ \beta^{2^r} = \beta + \xi = \beta + \omega^{2^r} + \omega, \]
which yields
\[ (\beta + \omega)^{2^r} + (\beta + \omega) = 0; \]
that is to say, \( \beta + \omega \in \text{AGL}(\alpha) \) is a root of \( x^{2^r} + x \). Let \( g(x) \) be the minimal polynomial of \( \beta + \omega \) over \( \mathbb{F}_q \). Then there is \( g(x) \in \text{PGL}(f) \) such that \( g(x) \) divides \( x^{2^r} + x \).

(2) \( \text{AGL}(\sigma^r(\alpha)) = \text{AGL}(\frac{\alpha}{\alpha^{2^r}}) \). Using arguments essentially the same as those in case (1), we conclude that there exists \( g(x) \in \text{PGL}(f) \) such that \( g(x) \) divides \( x^{2^r} + x \).

Conversely, suppose that there is a polynomial \( g(x) \in \text{PGL}(f) \) such that \( g(x) \) divides \( x^{2^r} + x \). Then \( f = D(g) \), where \( D \in \text{PGL} \). We then have \( \text{PGL}(f) = \text{PGL}(g) \). In addition, from \( f = D(g) \) we can get that \( \sigma^r(f) = \sigma^r(D(g)) = \sigma^r(D)\sigma^r(g) \). Therefore \( \text{PGL}(\sigma^r f) = \text{PGL}(\sigma^r g) \). Let \( \zeta \) be a root of \( g(x) \). Since \( g(x) \) divides \( x^{2^r} + x \), we obtain \( \zeta^{2^r} = \zeta \), i.e., \( \sigma^r(\zeta) = \zeta \), which implies that \( \text{PGL}(\zeta) = \text{PGL}(\sigma^r(\zeta)) \).

Using Lemma 3.6 one has \( \text{PGL}(g) = \text{PGL}(\sigma^r g) \). It follows that \( \text{PGL}(\sigma^r f) = \text{PGL}(f) \). The proof is complete. \( \square \)

Now we are ready to determine the number of orbits of \( (\sigma^r) \) on \( \text{PGL} \setminus \mathcal{I}_r \) with size 1. Since \( r \geq 3 \), we have that \( f(x) \in \mathcal{I}_r \) divides \( x^{2^r} + x \) if and only if \( f(x) \in \mathcal{I}_r \) divides \( x^{2^r-1} - 1 \). Let \( \text{ord}(f) \) denote the order of the polynomial \( f \) (see [9] Definition 3.2). It follows from [9] Lemma 3.6 that \( f(x) \) divides \( x^{2^r} + x \) if and only if \( \text{ord}(f) \) divides \( 2^r - 1 \). The set \( E(r, q) \) is defined by
\[
E(r, q) = \left\{ e \mid e > 1 \text{ is an integer dividing } 2^r - 1 \text{ but } e \text{ does not divide } q^d - 1 \text{ for any } 1 \leq d < r \right\}. \tag{3.4}
\]
Then according to [9] Theorem 3.5, the number of polynomials \( f(x) \in \mathcal{I}_r \) such that \( f(x) \) divides \( x^{2^r} + x \) is equal to
\[
\left| \left\{ f(x) \in \mathcal{I}_r \middle| f(x) \text{ divides } x^{2^r} + x \right\} \right| = \sum_{e \in E(r, q)} \frac{\phi(e)}{r},
\]
where \( \phi \) is the Euler’s function.

In the following we provide another characterization about the number of polynomials \( f(x) \in \mathcal{I}_r \) such that \( f(x) \) divides \( x^{2^r} + x \).

**Lemma 3.10.** With the notation as above. Then
\[
\left| \left\{ f(x) \in \mathcal{I}_r \middle| f(x) \text{ divides } x^{2^r} + x \right\} \right| = \frac{1}{r} \sum_{d \mid r} (2^d - 1) \mu(d),
\]
where \( \mu \) is the Möbius function.

**Proof.** Let \( d \) be a positive integer. Suppose that \( \Omega_d(x) = \prod_{f(x) \in \mathcal{S}_d(x)} f(x) \),
where \( \mathcal{S}_d(x) = \left\{ f(x) \in \mathcal{I}_d \middle| f(x) \text{ divides } x^{2^d-1} - 1 \right\} \).

Note that \( \gcd(2^r - 1, q) = \gcd(2^r - 1, 2^n) = 1 \) and \( \gcd(r, n) = 1 \) and we get that \( \text{ord}_{2^r-1}(q) = r \), which shows that \( d \mid r \). Then
\[
x^{2^r-1} - 1 = \prod_{d \mid r} \Omega_d(x).
\]
So
\[
2^r - 1 = \sum_{d \mid r} d \left| \mathcal{S}_d(x) \right|.
\]
In virtue of the M"obius inversion formula we obtain that
\[ r | S_r(x) | = \frac{1}{r} \sum_{d|r} (2^\frac{r}{d} - 1) \mu(d). \]

Hence we have
\[ \left| \left\{ f(x) \in \mathcal{I}_r \mid f(x) \text{ divides } x^{2^r} + x \right\} \right| = |S_r(x)| = \frac{1}{r} \sum_{d|r} (2^\frac{r}{d} - 1) \mu(d), \]

where \( \mu \) is the M"obius function.

We have seen that the number of monic irreducible polynomials of degree \( r \) over \( \mathbb{F}_q \) that divide \( x^{2^r} + x \) is equal to
\[ \frac{1}{r} \sum_{e \in E(r,q)} \phi(e) \text{ or } \frac{1}{r} \sum_{d|r} (2^\frac{r}{d} - 1) \mu(d). \]

The following result reveals that if \( \text{PGL}(f) \) contains a polynomial that divides \( x^{2^r} + x \), then \( \text{PGL}(f) \) contains exactly 6 such polynomials.

**Lemma 3.11.** Suppose that \( f(x) \in \mathcal{I}_r \) such that \( f(x) \) divides \( x^{2^r} + x \). Then
\[ \left| \left\{ h(x) \mid h(x) \in \text{PGL}(f), h(x) \text{ divides } x^{2^r} + x \right\} \right| = 6. \]

**Proof.** For simplifying notations, let \( \Delta = \left\{ h(x) \mid h(x) \in \text{PGL}(f), h(x) \text{ divides } x^{2^r} + x \right\} \). Let \( \alpha \) be a root of \( f(x) \), which gives \( \alpha^{2^r} = \alpha \) since \( f(x) \) divides \( x^{2^r} + x \). Observe that
\[ |\Delta| = \left| \left\{ h(x) \mid h(x) \in \text{PGL}(f), h(x) \text{ divides } x^{2^r} + x \right\} \right| = \left| \left\{ A f(x) \mid A f(x) \text{ divides } x^{2^r} + x \right\} \right| = \left| \left\{ A \in \text{PGL} \mid (A\alpha)^{2^r} + A\alpha = 0 \right\} \right|. \]

Assume that \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Then
\[ (A\alpha)^{2^r} + A\alpha = 0 \iff \begin{pmatrix} a\alpha + b \\ c\alpha + d \end{pmatrix} \text{ divides } x^{2^r} + x \]
\[ \iff \begin{cases} (a\alpha + b)^{2^r} + (c\alpha + d)^{2^r} = 0 \\ a^{2^r}\alpha^{2^r} + b^{2^r} + (c\alpha + d)^{2^r} = 0 \\ \frac{a^{2^r}\alpha^{2^r} + b^{2^r}}{c^{2^r} + d^{2^r}} + \frac{a\alpha + b}{c\alpha + d} = 0 \\ (ca^{2^r} + bc^{2^r})\alpha^{2^r} + (da^{2^r} + bc^{2^r} + ad^{2^r} + cb^{2^r})\alpha + (bd^{2^r} + db^{2^r}) = 0 \end{cases}. \]

Case 1: \( a \neq 0, c = 0 \). Since \( A \) is invertible, \( d \neq 0 \). From the second equality we have \( a = d \). If \( b \neq 0 \), then \( b = d \). Hence, there are two cases:

\[ b = c = 0, \quad a = d \neq 0; \quad c = 0, \quad a = b = d \neq 0. \]

Therefore in this case
\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \]
Case 2: \( c \neq 0, \ a = 0 \). Since \( A \) is invertible, \( b \neq 0 \) and \( c \neq 0 \). By the second equality we obtain \( b = c \). If \( d \neq 0 \), then \( b = d \). Hence, there are two cases:

\[
b = c \neq 0, \ a = d = 0; \ a = 0, \ b = c = d \neq 0.
\]

Therefore

\[
A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.
\]

Case 3: \( c \neq 0, \ a \neq 0 \). From the first equality we get \( a = c \). We consider three subcases separately.

Subcase 3.1: \( b = 0, \ d \neq 0 \). From the second equality we get \( a = d \).

Subcase 3.2: \( b \neq 0, \ d = 0 \). From the second equality we get \( b = c \).

Subcase 3.3: \( b \neq 0, \ d \neq 0 \). From the last equality we get \( b = d \). However, the determinate of \( A \) is \( ad - bc = 0 \). This is impossible.

Hence, there are two cases:

\[
b = 0, \ a = c = d \neq 0; \ d = 0, \ a = b = c \neq 0.
\]

Therefore in this case \( A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \) or \( \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \).

In conclusion, we have \( |\Delta| = 6 \).

### 3.4 Our main result and two illustrative examples

Collecting all the results that we have established, we arrive at the following result, which gives an upper bound for the number of inequivalent extended irreducible binary Goppa codes of length \( 2^n + 1 \) and degree \( r \).

**Theorem 3.12.** We assume that \( n > 3 \) is an odd prime number and \( q = 2^n \); let \( r \geq 3 \) be a positive integer satisfying \( \gcd(r, n) = 1 \) and \( \gcd(r, q(q^2 - 1)) = 1 \). The number of inequivalent extended irreducible binary Goppa codes of length \( q + 1 \) and degree \( r \) is at most

\[
n - 1 \sum_{d \mid r} \mu(d)(2^\frac{r}{d} - 1) + \frac{1}{rnq(q^2 - 1)} \sum_{d \mid r} \mu(d)q^\frac{r}{d},
\]

where \( \mu \) is the Möbius function.

**Proof.** By Lemma 2.3, the number of inequivalent extended irreducible binary Goppa codes of length \( q + 1 \) and degree \( r \) is less than or equal to the number of orbits of PGL on \( S \). By Lemma 3.3, let \( s \) be the number of orbits of PGL on \( I_r \). Using Lemmas 3.9 and 3.11, we have

\[
\frac{1}{6r} \sum_{d \mid r} \mu(d)(2^\frac{r}{d} - 1) + n \left( s - \frac{1}{6r} \sum_{d \mid r} \mu(d)(2^\frac{r}{d} - 1) \right) = |\text{PGL}\setminus I_r| = \frac{|I_r|}{q(q^2 - 1)},
\]

from which we obtain

\[
s = \frac{n - 1}{6rn} \sum_{d \mid r} \mu(d)(2^\frac{r}{d} - 1) + \frac{|I_r|}{nq(q^2 - 1)}.
\]

Substituting \( |I_r| = \frac{1}{2} \sum_{d \mid r} \mu(d)q^{r/d} \) into the above equation, we obtain the desired result. We are done.

We give two small examples to illustrate Theorem 3.12.

**Example 3.13.** Take \( n = 5 \) and \( r = 7 \) in Theorem 3.12. This gives \( q = 2^n = 32 \) and thus \( q(q^2 - 1) = 32(32^2 - 1) = 31 \cdot 32 \cdot 33 = 32736 \). It is readily seen that \( \gcd(r, n) = \gcd(7, 5) = 1 \) and \( \gcd(r, q(q^2 - 1)) = \gcd(7, 32(32^2 - 1)) = 1 \), namely, the conditions listed in Theorem 3.12 are satisfied.
Table 1: Upper bounds on the number of inequivalent extended irreducible binary Goppa codes of length 129 and degree $r$

| Degree | Upper bound |
|--------|-------------|
| $r = 5$ | 469         |
| $r = 11$ | 935870030557051 |
| $r = 13$ | 12974326183623782445 |
| $r = 17$ | 2663294067654074513871726265 |
| $r = 19$ | 390422089513449508526138877707059 |

Then Theorem 3.12 says that the number of inequivalent extended irreducible binary Goppa codes of length 33 and degree 7 is at most

$$n - 1 \sum_{d | r} \mu(d)(2^d - 1) + \frac{1}{rnq(q^2 - 1)} \sum_{d | r} \mu(d)q^d$$

$$\begin{align*}
&= 4 \cdot 7 \cdot 5 \sum_{d | r} \mu(d)(2^d - 1) + \frac{1}{7 \cdot 5 \cdot 32(32^2 - 1)} \sum_{d | r} \mu(d)32^d \\
&= 12 + \frac{1}{7 \cdot 5 \cdot 32(32^2 - 1)}(32^7 - 32) \\
&= 29991.
\end{align*}$$

Example 3.14. Take $n = 7$ and $q = 2^n = 2^7 = 128$ and thus $q(q^2 - 1) = 128(128^2 - 1) = 2^7 \cdot 3 \cdot 43 \cdot 127$. Assume that $\gcd(r, n) = \gcd(r, 7) = 1$ and $\gcd(r, q(q^2 - 1)) = \gcd(r, 2^7 \cdot 3 \cdot 43 \cdot 127) = 1$. Then according to Theorem 3.12 the number of inequivalent extended irreducible binary Goppa codes of length 129 and degree $r$ is at most

$$n - 1 \sum_{d | r} \mu(d)(2^d - 1) + \frac{1}{rnq(q^2 - 1)} \sum_{d | r} \mu(d)q^d$$

$$\begin{align*}
&= 1 \frac{1}{7r} \sum_{d | r} \mu(d)(2^d - 1) + \frac{1}{r \cdot 7 \cdot 2^7 \cdot 3 \cdot 43 \cdot 127} \sum_{d | r} \mu(d)128^d. \\
\end{align*}$$

In the following we provide the values of the upper bounds on the number of inequivalent extended irreducible binary Goppa codes of length 129 and some possible degrees $r$, which are listed in Table I.

4 Concluding remarks and future work

It is known that the number of inequivalent extended irreducible binary Goppa codes of length $2^n + 1$ and degree $r$ is less than or equal to the number of orbits of PTL on $S$. In this paper, we present a new approach to get an upper bound for the number of inequivalent extended irreducible binary Goppa codes by introducing a group action of PTL on $I_r$, the set of all monic irreducible polynomials of degree $r$ over $\mathbb{F}_q$. We show that the number of orbits of PTL on $S$ is equal to the number of orbits of PTL on $I_r$. There are some advantages of considering the action of PTL on $I_r$ which permits us to find a formula for the number of orbits of PTL on $S$. Therefore, we obtain an upper bound for the number of inequivalent extended irreducible binary Goppa codes of length $2^n + 1$ and degree $r$, where $n > 3$ and $r$ satisfy certain conditions.

A possible direction for future work is to find tight upper bounds for the number of inequivalent extended irreducible binary Goppa codes in more cases. It also would be interesting to find the exact value of extended irreducible binary Goppa codes.

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