SMOOTHING PROPERTIES OF THE DISCRETE FRACTIONAL MAXIMAL OPERATOR ON BESOV AND TRIEBEL–LIZORKIN SPACES

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Abstract. Motivated by the results of Korry and Kinnunen and Saksman, we study the behaviour of the discrete fractional maximal operator on fractional Hajlasz spaces, Hajlasz–Besov and Hajlasz–Triebel–Lizorkin spaces on metric measure spaces. We show that the discrete fractional maximal operator maps these spaces to the spaces of the same type with higher smoothness. Our results extend and unify aforementioned results. We present our results in general setting, but they are new already in the Euclidean case.

1. Introduction

Maximal functions are standard tools in harmonic analysis. They are usually used to estimate absolute size, but recently there has been interest in studying their regularity properties, see [1], [2], [3], [11], [12], [13], [15], [17], [18], [19], [21], [22], [23], [25], [26], [28]. A starting point was [17], where Kinnunen observed that the Hardy-Littlewood maximal operator is bounded on $W^{1,p}(\mathbb{R}^n)$ for $1 < p \leq \infty$. In [22] and [23] Korry extended this result by showing that the maximal operator preserves also fractional Sobolev spaces as well as Besov and Triebel–Lizorkin spaces. Another kind of extension was given in [20], where Kinnunen and Saksman showed that the fractional maximal operator $M_\alpha$, defined by

$$M_\alpha u(x) = \sup_{r>0} \frac{r^\alpha}{|B(x,r)|} \int_{B(x,r)} |u(y)| dy,$$

is bounded from $W^{1,p}(\mathbb{R}^n)$ to $W^{1,p^*}(\mathbb{R}^n)$, where $p^* = np/(n - \alpha p)$, and from $L^p(\mathbb{R}^n)$ to $W^{1,q}(\mathbb{R}^n)$, where $q = np/(n - (\alpha - 1)p)$ and $W^{1,q}(\mathbb{R}^n)$ is the homogeneous Sobolev space. These results indicate that $M_\alpha$ has similar smoothing properties as the Riesz potential.

It is natural to ask whether these results can be seen as special cases of the behaviour of the fractional maximal operator on Besov and Triebel–Lizorkin spaces. In this paper we show that this is indeed the case, and that all these results can be obtained by the same rather simple method. Instead of the
standard fractional maximal operator, we consider its variant, the so-called
discrete fractional maximal operator $M^\ast$. This allows us to present our results
in a setting of doubling metric measure spaces. In this generality, the standard
fractional maximal operator behaves quite badly. Indeed, one can construct
spaces, where the fractional maximal function of a Lipschitz function fails to be
continuous, see [3] and [15]. Since $M^\ast$ and $M^\alpha$ are comparable, for practical
purposes it does not matter which one we choose. The discrete fractional
maximal operator was introduced in [18] and further studied in [21], [1] and
[13].

Among the many possible definitions of Besov and Triebel–Lizorkin spa-
tes, the most suitable for our purposes is the one based on Hajlasz type pointwise
inequalities. This approach, introduced by Koskela, Yang and Zhou in [24],
provides a new point of view to the classical Besov and Triebel–Lizorkin spaces.
On the other hand, it allows these spaces to be defined in the setting of metric
measure spaces.

By employing this definition, we can prove very general results using only
simple ”telescoping” arguments and Poincaré type inequalities. As special
cases, we obtain versions of the results of Kinnunen and Saksman as well as
those of Korry, see Remark 5.4 and Theorems 4.5 and 4.6. We prove our results
in doubling metric measure spaces but they are new even in Euclidean spaces.
Our main results (Theorems 4.3 and 4.4) imply that if $\alpha \geq 0$ and $0 < s + \alpha < 1$,
then $M^\ast$ is bounded from $\dot{F}^{s,p,q}(\mathbb{R}^n)$ to $\dot{F}^{s+\alpha,p,q}(\mathbb{R}^n)$ for
$n/(n+s) < p, q < \infty$ and
from $\dot{B}^{s,p,q}(\mathbb{R}^n)$ to $\dot{B}^{s+\alpha,p,q}(\mathbb{R}^n)$ for $n/(n+s) < p < \infty, 0 < q < \infty$, see Section 4
for the definition of Triebel–Lizorkin and Besov spaces.

2. Preliminaries and notation

We assume that $X = (X, d, \mu)$ is a metric measure space equipped with a
metric $d$ and a Borel regular outer measure $\mu$, which satisfies $0 < \mu(U) < \infty$
whenever $U$ is nonempty, open and bounded. We assume that the measure is
doubling, that is, there exists a fixed constant $c_d > 0$, called the doubling
constant, such that
\begin{equation}
\mu(B(x, 2r)) \leq c_d \mu(B(x, r))
\end{equation}
for every ball $B(x, r) = \{y \in X : d(y, x) < r\}$.

The doubling condition implies that
\begin{equation}
\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq C \left(\frac{r}{R}\right)^Q
\end{equation}
for every $0 < r \leq R$ and $y \in B(x, R)$ for some $C$ and $Q > 1$ that only depend
on $c_d$. In fact, we may take $Q = \log_2 c_d$.

For the boundedness of the fractional maximal operator in $L^p$, we have to
assume, in Theorems 2.1 and 3.3 (b), that the measure $\mu$ satisfies the lower bound condition
\begin{equation}
\mu(B(x, r)) \geq c_l r^Q
\end{equation}
with some constant $c_l > 0$ for all $x \in X$ and $r > 0$.

Throughout the paper, $C$ will denote a positive constant whose value is not necessarily the same at each occurrence.

**The fractional maximal function.** Let $\alpha \geq 0$. The fractional maximal function of a locally integrable function $u$ is

\begin{equation}
M_{\alpha} u(x) = \sup_{r>0} r^{\alpha} \int_{B(x,r)} |u| \, d\mu,
\end{equation}

where $u_B = \int_{B} u \, d\mu = \frac{1}{\mu(B)} \int_{B} u \, d\mu$ is the integral average of $u$ over $B$. For $\alpha = 0$, we have the usual Hardy-Littlewood maximal function

$$
M u(x) = \sup_{r>0} \int_{B(x,r)} |u| \, d\mu.
$$

The following Sobolev type inequality for the fractional maximal operator follows easily from the boundedness of the Hardy-Littlewood maximal operator in $L^p$, for the proof, see [4], [6] or [15].

**Theorem 2.1.** Assume that the measure lower bound condition holds. If $p > 1$ and $0 < \alpha < Q/p$, then there is a constant $C > 0$, depending only on the doubling constant, constant in the measure lower bound, $p$ and $\alpha$, such that

$$
\| M_{\alpha} u \|_{L^{p^*}(X)} \leq C \| u \|_{L^p(X)},
$$

for every $u \in L^p(X)$ with $p^* = Qp/(Q - \alpha p)$.

**Remark 2.2.** If $u$ is only locally integrable, then $M_{\alpha} u$ may well be identically infinite. However, if $M_{\alpha} u(x_0) < \infty$ for some $x_0 \in X$, then $M_{\alpha} u(x) < \infty$ for almost every $x$. This follows from the estimate

$$
r^{\alpha} \int_{B(x,r)} |u| \, d\mu \leq \frac{\mu(B(x_0, r + d(x, x_0)))}{\mu(B(x, r))} M_{\alpha} u(x_0)
$$

combined with the doubling condition and the fact that

$$
\lim_{r \to 0} r^{\alpha} \int_{B(x,r)} |u| \, d\mu < \infty,
$$

whenever $x$ is a Lebesgue point of $u$.

**The discrete fractional maximal function.** We begin the construction of the discrete maximal function with a covering of the space. Let $r > 0$. Since the measure is doubling, there are balls $B(x_i, r)$, $i = 1, 2, \ldots$, such that

$$
X = \bigcup_{i=1}^{\infty} B(x_i, r) \quad \text{and} \quad \sum_{i=1}^{\infty} \chi_{B(x_i, 6r)} \leq N < \infty,
$$

where $\chi_{B(x_i, 6r)}$ is the characteristic function of the ball $B(x_i, 6r)$. This means that the dilated balls $B(x_i, 6r)$, $i = 1, 2, \ldots$, are of bounded overlap. The constant $N$ depends only on the doubling constant and, in particular, it is independent of $r$. 
Then we construct a partition of unity subordinate to the covering $B(x_i, r)$, $i = 1, 2, \ldots$, of $X$. Indeed, there is a family of functions $\varphi_i$, $i = 1, 2, \ldots$, such that $0 \leq \varphi_i \leq 1$, $\varphi_i = 0$ in $X \setminus B(x_i, 6r)$, $\varphi_i \geq \nu$ in $B(x_i, 3r)$, $\varphi_i$ is Lipschitz with constant $L/r$ with $\nu$ and $L$ depending only on the doubling constant, and

$$\sum_{i=1}^{\infty} \varphi_i(x) = 1$$

for every $x \in X$.

The discrete convolution of a locally integrable function $u$ at the scale $3r$ is

$$u_r(x) = \sum_{i=1}^{\infty} \varphi_i(x) u_{B(x_i, 3r)}$$

for every $x \in X$, and we write $u_r^\alpha = r^\alpha u_r$.

Let $r_j$, $j = 1, 2, \ldots$ be an enumeration of the positive rationals and let balls $B(x_i, r_j)$, $i = 1, 2, \ldots$ be a covering of $X$ as above. The discrete fractional maximal function of $u$ in $X$ is

$$\mathcal{M}_u^\alpha u(x) = \sup_j |u|_{r_j, \alpha}^\alpha$$

for every $x \in X$. For $\alpha = 0$, we obtain the Hardy-Littlewood type discrete maximal function $M^*$ studied in [18], [21] and [1]. The discrete fractional maximal function is easily seen to be comparable to the standard fractional maximal function, see [13].

3. Fractional Hajlasz spaces

Let $u$ be a measurable function and let $s \geq 0$. A nonnegative measurable function $g$ is an $s$-Hajlasz gradient of $u$ if there exists $E \subset X$ with $\mu(E) = 0$ such that for all $x, y \in X \setminus E$,

$$|u(x) - u(y)| \leq d(x, y)^s (g(x) + g(y)).$$

The collection of all $s$-Hajlasz gradients of $u$ is denoted by $\mathcal{D}^s(u)$. A homogeneous Hajlasz space $M^{s,p}(X)$ consists of measurable functions $u$ such that

$$\|u\|_{M^{s,p}(X)} = \inf_{g \in \mathcal{D}^s(u)} \|g\|_{L^p(X)}$$

is finite. The Hajlasz space $M^{s,p}(X)$ is $M^{s,p}(X) \cap L^p(X)$ equipped with the norm

$$\|u\|_{M^{s,p}(X)} = \|u\|_{L^p(X)} + \|u\|_{M^{s,p}(X)}.$$

The space $M^{1,p}(X)$, a counterpart of a Sobolev space in metric measure space, was introduced in [9], see also [10]. The fractional spaces $M^{s,p}(X)$ were introduced in [30] and studied for example in [16] and [14]. Notice that $M^{0,p}(X) = L^p(X)$.

The pointwise definition of the Hajlasz spaces implies the validity of Sobolev-Poincaré type inequalities without the assumption that the space admits any weak Poincaré inequality.
Lemma 3.1. Let $s \in [0, \infty)$ and let $p \in (0, Q/s)$. There exists a constant $C$ such that for all measurable functions $u$ with $g \in D^s(u)$, all $x \in X$ and $r > 0$,

\begin{equation}
\inf_{c \in \mathbb{R}} \left( \int_{B(x,r)} |u(y) - c|^p \, d\mu(y) \right)^{1/p} \leq Cr^s \left( \int_{B(x,2r)} g^p \, d\mu \right)^{1/p},
\end{equation}

where $p^*(s) = Qp/(Q - sp)$.

Moreover, if $p \geq Q/(Q + s)$ and $g \in D^s(u) \cap L^p(X)$, then (3.2) implies that $u$ is locally integrable and that

\begin{equation}
\int_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq Cr^s \left( \int_{B(x,2r)} g^p \, d\mu \right)^{1/p}.
\end{equation}

For the case $s = 1$, see [9] and [10].

In the next theorem, we use the following simple result. If $u_i, i \in \mathbb{N}$, are measurable functions with a common $s$-Hajlasz gradient $g$ and $u = \sup_i u_i$ is finite almost everywhere, then $g$ is an $s$-Hajlasz gradient of $u$.

**Theorem 3.2.** Assume that $M^* u \neq \infty$. Let $t \geq Q/(Q + s)$ and let $g$ be an $s$-Hajlasz gradient of $u$.

a) If $0 < s + \alpha \leq 1$, then there exists a constant $C > 0$ such that

\[ \tilde{g} = C \left( M g^t \right)^{1/t} \]

is an $(s + \alpha)$-Hajlasz gradient of $M^* u$.

b) If $s + \alpha > 1$, then there exists a constant $C > 0$ such that

\[ \tilde{g} = C \left( M_{(s+\alpha-1)} g^t \right)^{1/t} \]

is a $1$-Hajlasz gradient of $M^* u$.

**Proof.** We begin by proving the claims for $u^*_r$. Let $r > 0$, let $g \in D^s(u)$ and let $x, y \in X$.

Assume first that $r \geq d(x, y)$. Let $I_{xy}$ be a set of indices $i$ for which $x$ or $y$ belongs to $B(x_i, 6r)$. Then, for each $i \in I_{xy}$, $B(x_i, 3r) \subset B(x, 10r) \subset B(x_i, 17r)$. This together with the doubling condition, the properties of the functions $\varphi_i$, the fact that there are bounded number of indices in $I_{xy}$ and Poincaré inequality (3.3) implies that

\begin{equation}
|u_r^*(x) - u_r^*(y)| \leq r^\alpha \sum_{i \in I_{xy}} |\varphi_i(x) - \varphi_i(y)| |u_{B(x_i, 3r)} - u_{B(x_i, 10r)}| \\
\leq Cr^{\alpha-1} d(x, y) \int_{B(x, 10r)} |u - u_{B(x, 3r)}| \, d\mu \\
\leq Cr^{\alpha-1} d(x, y) \int_{B(x, 10r)} |u - u_{B(x, 10r)}| \, d\mu \\
\leq Cr^{s+\alpha-1} d(x, y) \left( \int_{B(x, 20r)} g^t \, d\mu \right)^{1/t}.
\end{equation}
If $0 < s + \alpha \leq 1$, then by (3.4) and the assumption $r \geq d(x,y)$, we have that

$$|u^\alpha_r(x) - u^\alpha_r(y)| \leq Cd(x,y)^{s+\alpha}(\mathcal{M}g^t(x))^{1/t}.$$  

If $s + \alpha > 1$, then by (3.4),

$$|u^\alpha_r(x) - u^\alpha_r(y)| \leq Cd(x,y)(\mathcal{M}_{t(s+\alpha-1)}g^t(x))^{1/t}.$$  

This shows that Hajłasz gradient inequality (3.1) with desired exponent holds when $r \geq d(x,y)$.

Assume then that $r < d(x,y)$. Let $R = d(x,y)$. Then $B(y,r) \subset B(x,2R)$ and

$$|u^\alpha_r(x) - u^\alpha_r(y)| \leq r^\alpha \left( \sum_{i \in I_x} \varphi_i(x)|u_{B(x,3r)}(x) - u_{B(x,9R)}(x)| \right.$$  
$$+ \sum_{i \in I_y} \varphi_i(y)|u_{B(x,3r)}(x) - u_{B(x,9R)}(x)|),$$  

where $I_x$ is a set of indices $i$ for which $x$ belongs to $B(x_i,6r)$ and $I_y$ the corresponding set for $y$. Let $k \in \mathbb{N}$ be the smallest integer such that $2^k r \geq R$.

Assume first that $0 < s + \alpha \leq 1$. If $i \in I_x$, then

$$|u_{B(x,3r)}(x) - u_{B(x,9R)}(x)| \leq |u_{B(x,3r)}(x) - u_{B(x,9r)}(x)| + \sum_{i=1}^{k} |u_{B(x,2^i9r)}(x) - u_{B(x,2^{i-1}9r)}(x)|$$  
$$+ |u_{B(x,2^k9r)}(x) - u_{B(x,9R)}(x)|.$$  

By the doubling condition and Poincaré inequality (3.3), we have

$$r^\alpha |u_{B(x,3r)}(x) - u_{B(x,9r)}(x)| \leq C r^\alpha \int_{B(x,9r)} |u - u_{B(x,9r)}| \, d\mu$$  
$$\leq C r^{s+\alpha} \left( \int_{B(x,18r)} g^t \, d\mu \right)^{1/t}$$  
$$\leq CR^{s+\alpha}(\mathcal{M}g^t(x))^{1/t},$$  

and, by the doubling condition, Poincaré inequality (3.3), the fact that $r \leq 2^i 9r$ for all $i$, and the selection of $k$,

$$r^\alpha \sum_{i=1}^{k} |u_{B(x,2^{i}9r)}(x) - u_{B(x,2^{i-1}9r)}(x)| \leq C r^\alpha \sum_{i=1}^{k} \int_{B(x,2^{i}9r)} |u - u_{B(x,2^{i}9r)}| \, d\mu$$  
$$\leq C \sum_{i=1}^{k} (2^{i}9r)^{s+\alpha} \left( \int_{B(x,2^{i+1}9r)} g^t \, d\mu \right)^{1/t}$$  
$$\leq CR^{s+\alpha}(\mathcal{M}g^t(x))^{1/t}.$$  

Similarly we obtain that
\[ r^\alpha |u_{B(x,2^k9r)} - u_{B(x,9r)}| \leq CR^{s+\alpha} \left( \int_{B(x,36R)} g^t \, d\mu \right)^{1/t} \]
\[ \leq CR^{s+\alpha} \left( \mathcal{M} g^t(x) \right)^{1/t}. \]

If \( i \in I_y \), we use balls \( B(y, 2^kr) \) instead of balls \( B(x, 2^kr) \) in (3.9). Estimates corresponding (3.7) and (3.8) are as above (\( x \) replaced by \( y \)) and, corresponding to (3.9),
\[ r^\alpha |u_{B(y,2^k9r)} - u_{B(y,9r)}| \leq CR^{s+\alpha} \left( \int_{B(x,38R)} g^t \, d\mu \right)^{1/t} \]
\[ \leq CR^{s+\alpha} \left( \mathcal{M} g^t(x) \right)^{1/t}. \]

Now, by (3.5)-(3.10) and the fact \( R = d(x, y) \), we have
\[ r^\alpha |u_{B(x,3r)} - u_{B(x,3r)}| \leq C d(x, y)^{s+\alpha} \left( \left( \mathcal{M} g^t(x) \right)^{1/t} + \mathcal{M} g^t(y) \right)^{1/t}. \]

If \( s + \alpha > 1 \), then similar estimates as above show that if \( i \in I_x \cup I_y \), then (3.11)
\[ r^\alpha |u_{B(x,3r)} - u_{B(x,3r)}| \leq C d(x, y)^{s+\alpha} \left( \left( \mathcal{M} g^t(x) \right)^{1/t} + \mathcal{M} g^t(y) \right)^{1/t}. \]

These estimates together with (3.5) and the fact that there are bounded number of indices in \( I_x \) and \( I_y \) imply that Hajlasz gradient inequality (3.1) with desired exponent holds when \( r < d(x, y) \).

The claim for \( u^\alpha \) follows from the estimates above and for \( \mathcal{M}^\alpha u \) from the discussion before the theorem.

**Theorem 3.3.** Let \( Q/(Q+s) < p < \infty \).

a) If \( 0 < s + \alpha \leq 1 \), there exists a constant \( C > 0 \), such that
\[ \| \mathcal{M}^\alpha u \|_{\dot{M}^{s+\alpha,p}(X)} \leq C \| u \|_{\dot{M}^{s,p}(X)} \]
for all \( u \in \dot{M}^{s,p}(X) \) with \( \mathcal{M}^\alpha u \not\equiv \infty \).

b) If \( 1 < s + \alpha \leq 1 + Q/p \) and the measure lower bound condition holds, there exists a constant \( C > 0 \) such that
\[ \| \mathcal{M}^\alpha u \|_{\dot{M}^{1,q}(X)} \leq C \| u \|_{\dot{M}^{s,p}(X)}, \]
where \( q = Qp/(Q-(s+\alpha-1)p) \), for all \( u \in \dot{M}^{s,p}(X) \) with \( \mathcal{M}^\alpha u \not\equiv \infty \).

**Proof.** a) Let \( Q/(Q+s) \leq t < p \). By Theorem 3.2 the function \( C(\mathcal{M} g^t)^{1/t} \) is an \((s + \alpha)\)-gradient of \( \mathcal{M}^\alpha u \). Since \( g \in L^p(X) \), the claim follows from the boundedness of the Hardy–Littlewood maximal operator in \( L^p(X) \) for \( q > 1 \).

b) Let \( Q/(Q+s) \leq t < p \). By Theorem 3.2 the function \( (\mathcal{M} g^t)^{1/t} \) is a 1-gradient of \( \mathcal{M}^\alpha u \). Since \( g \in L^p(X) \), the claim follows from Theorem 2.1. \( \square \)
Remark 3.4. In the cases $s = 0$ and $s = 1$ of Theorem 3.3b), we obtain counterparts of the results of Kinnunen and Saksman.

Remark 3.5. As a special case of Theorems 3.2 and 3.3 we obtain boundedness results for the discrete maximal operator $M^*$ in $M^{s,p}(X)$. If $0 < s \leq 1$, then $\tilde{g} = C(M g^t)^{1/t}$ is an $s$-Hajlasz gradient of $M^* u$ for all $t \geq Q/(Q + s)$ and

$$\|M^* u\|_{M^{s,p}(X)} \leq C\|u\|_{M^{s,p}(X)}$$

for all $u \in M^{s,p}(X)$, $p > Q/(Q + s)$.

If $1 < s \leq 1 + Q/p$, then $\tilde{g} = C(M_{t(s-1)} g^t)^{1/t}$ is a 1-Hajlasz gradient of $M^* u$ for all $t \geq Q/(Q + s)$ and

$$\|M^* u\|_{M^{s,p}(X)} \leq C\|u\|_{M^{s,p}(X)},$$

where $q = Qp/(Q - (s - 1)p)$, for all $u \in M^{s,p}(X)$.

Moreover, when $s = 1$, we obtain boundedness results for the discrete maximal operator $M^*$ in (homogeneous) Hajlasz spaces $M^{1,p}(X)$, proved earlier for $M^{1,p}(X)$ in [18] and [21].

4. Hajlasz–Besov and Hajlasz–Triebel–Lizorkin spaces

Let $u$ be a measurable function and let $s \in (0, \infty)$. Following [24], we say that a sequence of nonnegative measurable functions $(g_k)_{k \in \mathbb{Z}}$ is a fractional $s$-Hajlasz gradient of $u$ if there exists $E \subset X$ with $\mu(E) = 0$ such that

$$|u(x) - u(y)| \leq d(x,y)^s (g_k(x) + g_k(y))$$

for all $k \in \mathbb{Z}$ and all $x, y \in X \setminus E$ satisfying $2^{-k-1} \leq d(x,y) < 2^{-k}$. The collection of all fractional $s$-Hajlasz gradients of $u$ is denoted by $D^s(u)$.

For $p \in (0, \infty)$, $q \in (0, \infty]$ and a sequence $(f_k)_{k \in \mathbb{Z}}$ of measurable functions, we write

$$\| (f_k)_{k \in \mathbb{Z}} \|_{L^p(X, iv)} = \inf \{ \| (f_k)_{k \in \mathbb{Z}} \|_{L^p(X)} \}$$

and

$$\| (f_k)_{k \in \mathbb{Z}} \|_{iv(L^p(X))} = \inf \{ \| f_k \|_{L^p(X)} \} \|_{iv},$$

where $\| (f_k) \|_{iv} = (\sum_{k \in \mathbb{Z}} |f_k|^q)^{1/q}$ if $0 < q < \infty$ and $\| (f_k) \|_{iv \infty} = \sup_{k \in \mathbb{Z}} |f_k|$.

The homogeneous Hajlasz–Triebel–Lizorkin space $\dot{M}^{s}_{p,q}(X)$ consists of measurable functions $u$ such that

$$\|u\|_{\dot{M}^{s}_{p,q}(X)} = \inf_{(g_k)_{k \in \mathbb{Z}}} \| (g_k) \|_{L^p(X, iv)}$$

is finite. The Hajlasz–Triebel–Lizorkin space $M^{s}_{p,q}(X)$ is $\dot{M}^{s}_{p,q}(X) \cap L^p(X)$ equipped with the norm

$$\|u\|_{M^{s}_{p,q}(X)} = \|u\|_{L^p(X)} + \|u\|_{\dot{M}^{s}_{p,q}(X)}.$$

The homogeneous Hajlasz–Besov space $\dot{N}^{s}_{p,q}(X)$ consists of measurable functions $u$ such that

$$\|u\|_{\dot{N}^{s}_{p,q}(X)} = \inf_{(g_k)_{k \in \mathbb{Z}}} \| (g_k) \|_{iv(L^p(X))}$$
is finite and the Hajłasz–Besov space $N^s_{p,q}(X)$ is $N^s_{p,q}(X) \cap L^{p}(X)$ equipped with the norm

$$\|u\|_{N^s_{p,q}(X)} = \|u\|_{L^{p}(X)} + \|u\|_{N^s_{p,q}(X)}.$$ 

Notice that $M^{s}_{p,\infty}(X)$ is the homogeneous fractional Hajłasz space $M^{s,p}(X)$, for the simple proof, see [24] Prop. 2.1. The homogeneous Hajłasz–Triebel–Lizorkin space $M^{s,q,q}_{p,q}(\mathbb{R}^{n})$ coincides with the classical homogeneous Triebel–Lizorkin space $\hat{F}_{p,q}^{s}(\mathbb{R}^{n})$ for $s \in (0, 1)$, $p \in (n/(n + s), \infty)$ and $q \in (n/(n + s), \infty]$. Similarly, $\hat{N}_{p,q}^{s}(\mathbb{R}^{n})$ coincides with the classical homogeneous Besov space $\hat{B}_{p,q}^{s}(\mathbb{R}^{n})$ for $s \in (0, 1)$, $p \in (n/(n + s), \infty)$ and $q \in (0, \infty]$ by [24] Thm 1.2. For the definitions of $F_{p,q}^{s}(\mathbb{R}^{n})$ and $B_{p,q}^{s}(\mathbb{R}^{n})$, see [29].

If $X$ supports a (weak) $(1, p)$-Poincaré inequality with $p \in (1, \infty)$, then for all $q \in (0, \infty)$, the spaces $M_{p,q}^{s}(X)$ and $\hat{N}_{p,q}^{s}(X)$ are trivial, that is, they contain only constant functions, see [7, Thm 4.1].

**Lemma 4.1** ([7]). Let $s \in (0, \infty)$ and $p \in (0, Q/s)$. Then for every $\varepsilon, \varepsilon' \in (0, s)$ with $\varepsilon < \varepsilon'$ there exists a constant $C > 0$ such that for all measurable functions $u$ with $(g_{j}) \in \mathcal{D}^{s}(u)$, $x \in X$ and $k \in \mathbb{Z}$,

$$\inf_{c \in \mathbb{R}} \left( \int_{B(x, 2^{-k})} |u(y) - c|^{p^{*}(\varepsilon)} \, d\mu(y) \right)^{1/p^{*}(\varepsilon)} \leq C 2^{-k\varepsilon'} \sum_{j \geq k-2} 2^{-j(s-\varepsilon')} \left( \int_{B(x, 2^{-k+1})} g_{j}^{p} \, d\mu \right)^{1/p},$$

where $p^{*}(\varepsilon) = Qp/(Q - \varepsilon p)$.

If $p \geq Q/(Q + \varepsilon)$, then (4.1) implies that

$$\int_{B(x, 2^{-k})} |u - u_{B(x, 2^{-k})}| \, d\mu \leq C 2^{-k\varepsilon'} \sum_{j \geq k-2} 2^{-j(s-\varepsilon')} \left( \int_{B(x, 2^{-k+1})} g_{j}^{p} \, d\mu \right)^{1/p}.$$ 

We are now ready to state and prove our main results. Theorem 4.2 below gives a formula for an $(s + \alpha)$-Hajłasz gradient of $\mathcal{M}^{s}_{\alpha}$ in terms of an $s$-Hajłasz gradient of $u$. This easily implies the desired boundedness results for $\mathcal{M}^{s}_{\alpha}$ in homogeneous Hajłasz–Besov and Hajłasz–Triebel–Lizorkin spaces. For related results concerning Riesz potentials in the metric setting, see [31].

**Theorem 4.2.** Assume that $\mathcal{M}^{s}_{\alpha} u \neq \infty$ and that $(g_{k}) \in \mathcal{D}^{s}(u)$. Let $0 < s + \alpha < 1$, $0 < \delta < 1 - s - \alpha$, $0 < \varepsilon < \varepsilon' < s$ and $t \geq Q/(Q + \varepsilon)$. Then there is a constant $C > 0$, indepent of $u$ and $(g_{k})$, such that $(Cg_{k})$, where

$$\tilde{g}_{k} = \sum_{j=-\infty}^{k} 2^{(j-k)\delta} (\mathcal{M} g_{j}^{1/\epsilon})^{1/t} + \sum_{j=k-7}^{\infty} 2^{(k-j)(s-\epsilon')} (\mathcal{M} g_{j}^{1/\epsilon})^{1/t},$$

is a fractional $(s + \alpha)$-Hajłasz gradient of $\mathcal{M}^{s}_{\alpha} u$. 

Proof. Let $k \in \mathbb{Z}$ and let $x, y \in X$ such that $2^{-k-1} \leq d(x, y) < 2^{-k}$. We will show that

$$|u_r^k(x) - u_r^k(y)| \leq C d(x, y)^{s+\alpha}(\tilde{g}_k(x) + \tilde{g}_k(y)),$$

where $C$ is independent of $r$ and $k$.

Assume first that $d(x, y) > r$. Then

$$|u_r(x) - u_r(y)| \leq |u_r(x) - u_B(x, 2^{-k+4})| + |u_r(y) - u_B(x, 2^{-k+4})|$$

$$\leq \sum_{i \in I_x} \varphi_i(x)|u_B(x, 3r) - u_B(x, 2^{-k+4})|$$

$$+ \sum_{i \in I_y} \varphi_i(y)|u_B(x, 3r) - u_B(x, 2^{-k+4})|,$$

where $I_x$ is a set of indices $i$ for which $x$ belongs to $B(x, 6r)$ and $I_y$ the corresponding set for $y$. Let $m \in \mathbb{Z}$ be such that $2^{-m-1} < 9r \leq 2^{-m}$. Since $r < d(x, y) < 2^{-k}$, it follows that $m \geq k - 4$. If $i \in I_x$, we obtain

$$|u_B(x, 3r) - u_B(x, 2^{-k+4})| \leq |u_B(x, 3r) - u_B(x, 2^{-m})| + \sum_{l=k-4}^{m-1} |u_B(x, 2^{-l}) - u_B(x, 2^{-l-1})|$$

$$\leq C \sum_{l=k-4}^{m} \int_{B(x, 2^{-l})} |u - u_B(x, 2^{-l})| \, d\mu$$

and hence Poincaré inequality (1.2) implies that

$$|u_B(x, 3r) - u_B(x, 2^{-k+4})| \leq C \sum_{l=k-4}^{\infty} 2^{-l} \sum_{j=l-2}^{\infty} 2^{-j(s-\varepsilon')} \left(\mathcal{M} g_j^l(x)\right)^{1/t}$$

$$= C \sum_{j=k-6}^{\infty} 2^{-j(s-\varepsilon')} \left(\mathcal{M} g_j^l(x)\right)^{1/t} \sum_{l=k-4}^{j+2} 2^{-l}$$

$$\leq C 2^{-k\varepsilon'} \sum_{j=k-6}^{\infty} 2^{-j(s-\varepsilon')} \left(\mathcal{M} g_j^l(x)\right)^{1/t}$$

$$= C 2^{-ks} \sum_{j=k-6}^{\infty} 2^{(k-j)(s-\varepsilon')} \left(\mathcal{M} g_j^l(x)\right)^{1/t}$$

$$\leq C 2^{-ks} \tilde{g}_k(x).$$

Similarly, if $i \in I_y$, then

$$|u_B(x, 3r) - u_B(x, 2^{-k+4})| \leq |u_B(x, 3r) - u_B(y, 2^{-m})| + \sum_{l=k-4}^{m-1} |u_B(y, 2^{-l}) - u_B(y, 2^{-l-1})|$$

$$+ |u_B(y, 2^{-k+5}) - u_B(x, 2^{-k+4})|$$

$$\leq C \sum_{l=k-5}^{m} \int_{B(y, 2^{-l})} |u - u_B(y, 2^{-l})| \, d\mu,$$
which implies that
\[ |u_{B(x_i,3r)} - u_{B(x,2^{-k+4})}| \leq C2^{-k\alpha} \hat{g}_k(y). \]

It follows that
\[ |u^\alpha_r(x) - u^\alpha_r(y)| \leq Cr^\alpha 2^{-k\alpha}(\hat{g}_k(x) + \hat{g}_k(y)) \leq Cd(x,y)^{\alpha+\delta}(\hat{g}_k(x) + \hat{g}_k(y)). \]

Suppose then that \( d(x,y) \leq r \). Let \( I_{xy} \) be a set of indices \( i \) for which \( x \) or \( y \) belongs to \( B(x_i,6r) \). Let \( l \) be such that \( 2^{-l-1} < 10r \leq 2^{-l} \). Using the doubling condition, the properties of the functions \( \varphi_i \), the fact that there are bounded number of indices in \( I_{xy} \) and Poincaré inequality \( (4.2) \), we have that
\[
|u^\alpha_r(x) - u^\alpha_r(y)| \leq r^\alpha \sum_{i=1}^{\infty} |\varphi_i(x) - \varphi_i(y)||u_{B(x_i,3r)} - u_{B(x,2^{-l})}|
\]
(4.4)
\[
\leq Cd(x,y)r^{\alpha-1-2\epsilon t} \sum_{j=-1}^{\infty} 2^{-j(s-\epsilon t)} (\mathcal{M} g_j^f(x))^{1/t}.
\]

Using the assumptions \( 0 < \delta < 1 - \alpha - s \), \( r \geq d(x,y) \) and \( d(x,y) < 2^{-k} \), we have that
\[
d(x,y)^{\alpha-1-2\epsilon t} \leq Cd(x,y)^{s+\alpha+\delta-1} 2^{(s-\epsilon t + \delta)} \leq Cd(x,y)^{s+\alpha+\delta} 2^{(s-\epsilon t + \delta)}
\]
\[
\leq Cd(x,y)^{s+\alpha+2(s-\epsilon t + \delta)}.
\]

This together with \( (4.4) \) implies that
\[
|u^\alpha_r(x) - u^\alpha_r(y)| \leq Cd(x,y)^{s+\alpha} \sum_{j=-2}^{\infty} 2^{(s-\epsilon t + \delta)(j-\delta)} (\mathcal{M} g_j^f(x))^{1/t}.
\]

By splitting the sum in two parts and using the estimates \( l \leq j+2 \) and \( l \leq k \), we obtain
\[
\sum_{j=-2}^{\infty} 2^{(s-\epsilon t + \delta)(j-\delta)} (\mathcal{M} g_j^f(x))^{1/t}
\]
\[
= \sum_{j=-2}^{k-1} 2^{(s-\epsilon t + \delta)(j-\delta)} (\mathcal{M} g_j^f(x))^{1/t} + \sum_{j=k}^{\infty} 2^{(s-\epsilon t + \delta)(j-\delta)} (\mathcal{M} g_j^f(x))^{1/t}
\]
\[
\leq C \left( \sum_{j=-\infty}^{k-1} 2^{(s-\epsilon t + \delta)(j-\delta)} (\mathcal{M} g_j^f(x))^{1/t} + \sum_{j=k}^{\infty} 2^{(s-\epsilon t + \delta)(j-\delta)} (\mathcal{M} g_j^f(x))^{1/t} \right),
\]

which implies the claim for \( u^\alpha_r \). The claim for \( \mathcal{M} \) follows similarly as in the proof of Theorem 4.3. \( \square \)

**Theorem 4.3.** Let \( 0 < s + \alpha < 1 \) and \( Q/(Q+s) < p, q < \infty \). Then there exists a constant \( C > 0 \) such that
\[
\| \mathcal{M}^* u \|_{\dot{B}^{s+\alpha}_{p,q}(X)} \leq C\| u \|_{\dot{B}^s_{p,q}(X)}
\]
for all \( u \in \dot{B}^s_{p,q}(X) \) with \( \mathcal{M}^* u \neq \infty \).
Proof. Let \( \delta = \frac{1}{2}(1 - (s + \alpha)) \), \( \varepsilon = \frac{1}{2} \max\{s, s + \frac{Q - r}{r}\} \), \( \varepsilon' = \frac{1}{2}(\varepsilon + s) \), where \( r = \min\{p, q\} \), and let \( t = Q/(Q + \varepsilon) \). Then \( 0 < \varepsilon < \varepsilon' < s \) and \( Q/(Q + s) < t < \min\{p, q\} \). By Theorem 4.2, \((Cg_k)\) defined by (4.3) is a fractional \((s + \alpha)\)-Hajlasz gradient of \( M^*_\alpha u \).

It suffices to show that \((\tilde{g}_k) \in L^p(X, l^q)\). We estimate the \( L^p(X, l^q) \) norm of

\[
\left( \sum_{j = -\infty}^{k} 2^{(j-k)\delta} (Mg_j^t)^{1/t} \right)_{k \in \mathbb{Z}},
\]

the other part can be estimated similarly. If \( q \geq 1 \), we have, by the Hölder inequality, that

\[
\sum_{k \in \mathbb{Z}} \left( \sum_{j = -\infty}^{k} 2^{(j-k)\delta} (Mg_j^t)^{1/t} \right)^q \leq C \sum_{k \in \mathbb{Z}} \sum_{j = -\infty}^{k} 2^{(j-k)\delta} (Mg_j^t)^{q/t} \leq C \sum_{j \in \mathbb{Z}} (Mg_j^t)^{q/t} \sum_{k \in \mathbb{Z}} 2^{(j-k)\delta} \leq C \sum_{j \in \mathbb{Z}} (Mg_j^t)^{q/t}.
\]

If \( q < 1 \), we obtain the same estimate by using the elementary inequality \((\sum_j a_j)^q \leq \sum_j a_j^q\) for \( a_j \geq 0 \).

By the Fefferman–Stein vector valued maximal function theorem from [5] (for a metric space version, see for example [27] or [8]), we obtain now the desired estimate

\[
\left\| \left( \sum_{j = -\infty}^{k} 2^{(j-k)\delta} (Mg_j^t)^{1/t} \right) \right\|_{L^p(X, l^q)} \leq C \left\| (Mg_k^t) \right\|_{L^{p/t}(X, l^{q/t})}^{1/t} \leq C \left\| (g_k) \right\|_{L^{p/t}(X, l^{q/t})} = C \left\| (g_k) \right\|_{L^p(X, l^q)}.
\]

\[\square\]

Theorem 4.4. Let \( 0 < s + \alpha < 1 \), \( Q/(Q + s) < p < \infty \) and \( 0 < q < \infty \). Then there exists a constant \( C > 0 \) such that

\[
\left\| M^*_\alpha u \right\|_{N^*_p,q(X)} \leq C \left\| u \right\|_{N^*_p,q(X)}
\]

for all \( u \in N^*_p,q(X) \) with \( M^*_\alpha u \neq \infty \).

Proof. Let \( \delta = \frac{1}{2}(1 - (s + \alpha)) \), \( \varepsilon = \frac{1}{2} \max\{s, s + \frac{Q - p}{p}\} \), \( \varepsilon' = \frac{1}{2}(\varepsilon + s) \), and let \( t = Q/(Q + \varepsilon) \). Then \( 0 < \varepsilon < \varepsilon' < s \) and \( Q/(Q + s) < t < p \). Then \((Cg_k)\) defined by (4.3) is a fractional \((s + \alpha)\)-Hajlasz gradient of \( M^*_\alpha u \) by Theorem 1.2.
It suffices to show that $\| (\tilde{g}_k) \|_{L^q(L^p(X))} \leq C \| (g_k) \|_{L^q(L^p(X))}$. By the Hardy–Littlewood maximal theorem,

$$\left\| \sum_{j=-\infty}^{k} 2^{(j-k)\delta} (M g_j^t)^{1/t} \right\|_{L^p(X)} \leq \sum_{j=-\infty}^{k} 2^{(j-k)\delta} \| (M g_j^t)^{1/t} \|_{L^p(X)} \leq \sum_{j=-\infty}^{k} 2^{(j-k)\delta} \| g_j \|_{L^p(X)}.$$ 

If $q \geq 1$, we have by the Hölder inequality,

$$\sum_{k \in \mathbb{Z}} \left( \sum_{j=-\infty}^{k} 2^{(j-k)\delta} \| g_j \|_{L^p(X)} \right)^q \leq C \sum_{k \in \mathbb{Z}} \sum_{j=-\infty}^{k} 2^{(j-k)\delta} \| g_j \|_{L^p(X)} \leq C \sum_{j \in \mathbb{Z}} \| g_j \|_{L^p(X)}^q \sum_{k=j}^{\infty} 2^{(j-k)\delta} \leq C \sum_{j \in \mathbb{Z}} \| g_j \|_{L^p(X)}^q.$$ 

If $q < 1$, we use the inequality $(\sum a_j)^q \leq \sum a_j^q$ instead of the Hölder inequality. The second part of $(\tilde{g}_k)$ can be estimated similarly.

Theorems 4.3, 4.4 and the Hardy–Littlewood maximal theorem imply the following results for the discrete maximal operator.

**Theorem 4.5.** Let $0 < s < 1$.

a) If $Q/(Q+s) < p, q < \infty$, then there exist a constant $C > 0$ such that

$$\| \mathcal{M}^* u \|_{M^s_{p,q}(X)} \leq C \| u \|_{M^s_{p,q}(X)},$$

whenever $u \in M^s_{p,q}(X)$ and $\mathcal{M}^* u \not\equiv \infty$.

b) If $1 < p, q < \infty$, then there exist a constant $C > 0$ such that

$$\| \mathcal{M}^* u \|_{M^s_{p,q}(X)} \leq C \| u \|_{M^s_{p,q}(X)},$$

for all $u \in M^s_{p,q}(X)$.

**Theorem 4.6.** Let $0 < s < 1$.

a) If $Q/(Q+s) < p < \infty$ and $0 < q < \infty$, there exist a constant $C > 0$ such that

$$\| \mathcal{M}^* u \|_{N^s_{p,q}(X)} \leq C \| u \|_{N^s_{p,q}(X)},$$

for all $u \in N^s_{p,q}(X)$ with $\mathcal{M}^* u \not\equiv \infty$.

b) If $1 < p < \infty$ and $0 < q < \infty$, there exist a constant $C > 0$ such that

$$\| \mathcal{M}^* u \|_{N^s_{p,q}(X)} \leq C \| u \|_{N^s_{p,q}(X)},$$

for all $u \in N^s_{p,q}(X)$. 
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