Constraints on fourth order generalized f(R) gravity

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Abstract
A fourth order generalized f(R) gravity theory (FOG) is considered with the Einstein-Hilbert action 
\[ R + aR^2 + bR_{\mu\nu}R^{\mu\nu}, \]
\( R_{\mu\nu} \) being Ricci's tensor and \( R \) the curvature scalar. The field equations are applied to spherical bodies where Newtonian gravity is a good approximation. The result is that for \( 0 \leq a \sim -b << R^2 \), \( R \) being the body radius, the gravitational field outside the body contains two Yukawas, one attractive and the other one repulsive, in addition to the Newtonian term. For \( a \sim -b >> R^2 \) the gravitational field near the body is zero but at distances greater than \( \sqrt{a} \sim \sqrt{-b} \) the field is practically Newtonian. From the comparison with laboratory experiments I conclude that \( \sqrt{a} \) and \( \sqrt{-b} \) should be smaller than a few millimeters, which excludes any relevant effect of FOG on stars, galaxies or cosmology.

1 Fourth order generalized f(R) gravity and quantum vacuum

In recent years a great effort has been devoted to extended gravity theories which modify general relativity. The main motivation was the search for physical explanations to the observed accelerated expansion of the universe and other astrophysical observations, like the flat rotation curves in galaxies [1]. It is plausible to derive the extended theory from a generalized Einstein-Hilbert action

\[
S = \frac{1}{2k} \int d^4x\sqrt{-g}(R + F) + S_{mat},
\]

(1)

1
where $F$ should be a function of the scalars which may be obtained by combining the Riemann tensor, $R_{\mu\nu\lambda\sigma}$, and its derivatives, with the metric tensor, $g_{\mu\nu}$. The theory derived from the particular choice $F(R)$, where $R$ is the Ricci scalar, has been extensively explored under the name of f(R)-gravity\cite{2}. But it is possible to consider more general forms of $F$, e. g. depending on the Ricci tensor in addition to the curvature scalar. The most simple extension of general relativity seems to be fourth order gravity (FOG), where the functional $F$ has the form

$$F = aR^2 + bR_{\mu\nu}R^{\mu\nu}, \quad (2)$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ the Ricci tensor, $R$ the curvature scalar and $k$ is $8\pi$ times Newton’s constant, $a$ and $b$ being two constant parameters with dimensions of length squared\cite{3}, \cite{4}.

Another approach to FOG comes from the assumption that the energy of the quantum vacuum is not zero in curved spacetime\cite{5}, \cite{6}. This fact may be taken into account by adding a new stress-energy tensor, $T_{\mu\nu}^{\text{vac}}$, to the matter one, $T_{\mu\nu}^{\text{mat}}$, so that Einstein’s equation becomes

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = k \left( T_{\mu\nu}^{\text{mat}} + T_{\mu\nu}^{\text{vac}} \right), \quad (3)$$

Eq.(3) may be derived from the action (1), the tensor $T_{\mu\nu}^{\text{vac}}$ coming from the functional $F$. In not too strong gravitational fields a plausible form of $F$ is given by eq.(2)\cite{3}. In summary, we see that FOG may be seen as either an extension of general relativity or a quantum vacuum effect. In the latter case the tensor $T_{\mu\nu}^{\text{vac}}$ appears on the right side of eq.(3), in the former it would appear on the left. But in both cases it gives rise to the same physical theory. In this paper I will use a language corresponding to the quantum vacuum assumption.

The modifications derived from $F$ should be small in weak gravitational fields, where GR is valid, but might be relevant in more strong fields like those existing in compact stars or the early universe. In order to study that possibility we should find the range or the parameters $a$ and $b$ compatible with known data, which is the purpose of this paper.

### 2 Field equations

The tensor field equation derived from the functional (2) may be taken from the literature\cite{4}. I shall write it in terms of the Einstein tensor, $G_{\mu\nu}$, rather
than the Ricci tensor, $R_{\mu\nu}$, and in a form that looks like the standard Einstein equation of general relativity eq.(3). That is

$$G_{\mu\nu} = kT_{\mu\nu}, \quad T_{\mu\nu} = T_{\mu\nu}^{\text{mat}} + T_{\mu\nu}^{\text{vac}},$$

$$kT_{\mu\nu}^{\text{vac}} \equiv -(2a + b) \left[ \nabla_\mu \nabla_\nu G - g_{\mu\nu} \Box G \right] - 2(a + b) \left[ -GG_{\mu\nu} + \frac{1}{4}g_{\mu\nu}G^2 \right] - b \left[ 2G_\mu^\sigma G_{\sigma\nu} - \frac{1}{2}g_{\mu\nu}G_\lambda^\sigma G^{\lambda\sigma} - \nabla_\sigma \nabla_\nu G_\mu^\sigma - \nabla_\sigma \nabla_\mu G_\nu^\sigma + \Box G_{\mu\nu} \right], \quad (4)$$

with obvious notation.

We are interested in static problems of spherical symmetry and will use the standard metric

$$ds^2 = - \exp (\beta(r)) \, dt^2 + \exp (\alpha(r)) \, dr^2 + r^2 d\Omega^2. \quad (5)$$

Thus $G_{\mu\nu}(r)$ and $T_{\mu\nu}^{\text{mat}}(r)$ have 3 independent components each, so that including $\alpha(r)$ and $\beta(r)$ there are 8 unknown functions of $r$. On the other hand there are 8 equations, namely 3 eqs.(4), 3 more equations giving the independent components of $G_{\mu\nu}$ in terms of $\alpha$ and $\beta$ and 2 equations of state relating the 3 independent components of $T_{\mu\nu}^{\text{mat}}$. I shall assume local isotropy for matter, so that one of the latter will be the equality $T_{11}^{\text{mat}} = T_{22}^{\text{mat}} (= T_{33}^{\text{mat}}$ in spherical symmetry.) In principle the remaining 7 coupled nonlinear equations may be solved exactly by numerical methods.

Before proceeding, a note about the signs convention is in order. As is well known different authors use different signs in the definition of the relevant quantities. Here I shall make a choice which essentially agrees with the one of Ref.[2]. It may be summarized as follows

$$g_{00} = - \exp \beta, \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = kT_{\mu\nu}, \quad T_0^0 = -\rho. \quad (6)$$

After that I shall write the three independent components of eq.(4) using the notation

$$T_0^0 = -\rho, T_1^1 = p, T_2^2 = q, T_\mu^\mu = T = p + 2q - \rho,$$

$$(T_{\text{mat}})_0^0 = -\rho_{\text{mat}}, (T_{\text{mat}})_1^1 = (T_{\text{mat}})_2^2 = (T_{\text{mat}})_3^3 = p_{\text{mat}}. \quad (7)$$

In the following I will name $\rho, p$ and $q$ the total density, radial pressure and transverse pressure respectively, whilst $\rho_{\text{mat}}$ and $p_{\text{mat}}$ will be named matter.
density and pressure respectively (remember that we assume local isotropy for matter, that is equality of radial and transverse matter pressures.) The differences $\rho - \rho_{mat}, p - p_{mat}$ and $q - p_{mat}$ will be named vacuum density, radial pressure and transverse pressure, respectively.

After some algebra I get for the components of the tensor eq.(4) as follows

$$-\rho_{mat} = -\rho + (2a + b)e^{-\alpha}\left[ -\frac{d^2T}{dr^2} - \left( \frac{2}{r} - \frac{1}{2}\alpha' \right) \frac{dT}{dr} \right]$$

$$+ (a + b)k\left( \frac{r}{2}T^2 + 2T \rho \right) + b \left[ -\Delta \rho + 2k\rho^2 - \frac{1}{2}k \left[ \rho^2 + p^2 + 2q^2 \right] \right]$$

$$+ b \exp(-\alpha) \left[ -\frac{2\beta'}{r} (p + q) + \left( \frac{1}{2} \alpha' \beta' - \beta'' \right) (\rho + p) \right], \quad (8)$$

$$p_{mat} = p - (2a + b)e^{-\alpha}\left( \frac{2}{r} + \frac{1}{2}\beta' \right) \frac{dT}{dr} + (a + b)k\left( \frac{1}{2}T^2 - 2T \rho \right)$$

$$+ b \left[ \Delta \rho + 2k\rho^2 - \frac{1}{2}k \left[ \rho^2 + p^2 + 2q^2 \right] \right]$$

$$+ b \exp(-\alpha) \left[ \left( \frac{2\alpha'}{r} + \frac{4}{r^2} \right) (q - p) + \left( -\frac{1}{2} \alpha' \beta' + \beta'' \right) (\rho + p) \right], \quad (9)$$

$$p_{mat} = q - (2a + b)e^{-\alpha}\left[ \frac{d^2T}{dr^2} + \left( \frac{1}{r} + \frac{1}{2}\beta' - \frac{1}{2}\alpha' \right) \frac{dT}{dr} \right]$$

$$+ (a + b)k\left( \frac{1}{2}T^2 - 2T q \right) + b \left[ \Delta q + 2kq^2 - \frac{1}{2}k \left[ \rho^2 + p^2 + 2q^2 \right] \right]$$

$$+ b \exp(-\alpha) \left[ \left( -\frac{\alpha'}{r} - \frac{2}{r^2} \right) (q - p) - \frac{\beta'}{r} (\rho + q) \right]. \quad (10)$$

Addition of these 3 equations gives the trace equation, that is

$$T_{mat} \equiv 3p_{mat} - \rho_{mat} = T - (6a + 2b) \Delta T, \quad (11)$$

where $\Delta$ is the Laplacean operator in curved space-time, that is

$$\Delta \equiv \exp(-\alpha)\left[ \frac{d^2}{dr^2} + \left( \frac{2}{r} + \frac{1}{2}\beta' - \frac{1}{2}\alpha' \right) \frac{d}{dr} \right]. \quad (12)$$
The quantities $G^{\nu}_{\mu}$ are related to the metric coefficients $\alpha$ and $\beta$ and their derivatives, hence to $\rho, p$ and $q$, that is

$$\exp(-\alpha) = 1 - \frac{2m}{r}, \quad \frac{\alpha'}{2} = \frac{m - 4\pi \rho r^3}{r^2 - 2mr}, \quad \beta' = \frac{2m + 4\pi r^3p}{r^2 - 2mr},$$

$$\beta'' = \frac{8\pi r^2(r \rho + rp + 3p')}{r^2 - 2mr} - \frac{4(m + 4\pi r^3p)(r - m - 4\pi r^3p)}{(r^2 - 2mr)^2}$$

where I have used units $k = 8\pi, c = 1$ and the radial derivative of $\alpha$ ($\beta'$) is labelled $\alpha'$ ($\beta''$). The mass parameter $m$ is defined by

$$m = \int_0^r 4\pi x^2 \rho(x) dx.$$  \hspace{1cm} (14)

The condition that Einstein tensor, $G_{\mu\nu}$, is divergence free leads to the hydrostatic equilibrium equation, that is

$$\frac{dp}{dr} = \frac{2(q - p)}{r} - \frac{1}{2} \beta' (\rho + p).$$  \hspace{1cm} (15)

### 3 Terrestrial constraints on fourth order gravity (FOG)

The theory derived from $F = 0$ in the action (1), that is general relativity, is known to give good agreement with observations for a wide range of intensities of the gravitational field (that is curvature of spacetime.) As a consequence the corrections due to finite, nonzero, values of the parameters $a$ and $b$ should be below the uncertainties in the data in that domain. In a previous paper [3] I considered the problem, but there are additional constraints not taken into account there, which makes necessary a more detailed study.

In order to study the constraints on $a$ and $b$ derived from terrestrial and solar system observations I shall start solving eqs.(8) to (15) for spherical bodies - like the Sun, the Earth or a laboratory sphere of metal - where Newtonian gravity is a fairly good approximation. Corrections to Newtonian gravity coming from FOG have been obtained via the Newtonian approximation of the field eqs.(8) to (15) [4], [5]. Here I shall calculate the gravitational field by solving directly the field equations, with appropriate approximations.
For the Earth the parameters $\alpha$ and $\beta$ of the metric (??) are very close to unity and terms like $GG^\nu_\mu$ or $g^\nu_\mu G^2$ are smaller than the main term, $G^\nu_\mu$, by about

$$ak\rho_{mat}/c^2 \sim bk\rho_{mat}/c^2 \lesssim 10^{-26},$$

for $a \sim b \lesssim 1$ m$^2$. In addition the matter pressure, $p_{mat}$, is negligible in comparison with matter density, $\rho_{mat}$. As a consequence eqs. to (10) may be approximated by the following

$$b \left[ \frac{d^2 \rho}{dr^2} + \frac{2d\rho}{rdr} \right] + (2a + b) \left[ \frac{d^2 T}{dr^2} + \frac{2dT}{rdr} \right] + \rho = \rho_{mat},$$

(16)

$$-b \left[ \frac{d^2 p}{dr^2} + \frac{2dp}{rdr} + \frac{4}{r^2} (q - p) \right] + (2a + b) \frac{2dT}{rdr} - p = 0,$$

(17)

$$-b \left[ \frac{d^2 q}{dr^2} + \frac{2dq}{rdr} - \frac{2}{r^2} (q - p) \right] + (2a + b) \left[ \frac{d^2 T}{dr^2} + \frac{1dT}{rdr} \right] - q = 0.$$

(18)

Eqs. to (18) together with eqs. to (15), plus the equation of state, are a system of coupled differential equations whose solution is involved. However a great simplification is possible if we assume that the corrections due to finite values of the parameters $a$ and $b$ would modify but slightly the function $\rho_{mat}(r)$ with respect to the results obtained from a Newtonian treatment. Thus we may take $\rho_{mat}(r)$ as given, which decouples eqs. to (18) from the remaining ones. Still the three eqs. to (18) are coupled amongst themselves, but from them it is possible to get two decoupled ones. In fact if I add eq. (16), eq. (17) plus twice eq. (18) I get the trace eq. (11), which may be rewritten

$$(6a + 2b) \left[ \frac{d^2 T}{dr^2} + \frac{2dT}{rdr} \right] - T = -T_{mat} \simeq \rho_{mat}.$$ 

(19)

Subtracting this minus three times eq. (16) I obtain

$$b \left[ \frac{d^2(T + 3\rho)}{dr^2} + \frac{2d(T + 3\rho)}{rdr} \right] + T + 3\rho = 2\rho_{mat}.$$ 

(20)

Now the general solution of the trace eq. (19) is trivial and I get

$$T(r) = \frac{1}{\sqrt{6a + 2b}} \int_0^r \sinh \left( \frac{r - z}{\sqrt{6a + 2b}} \right) z \rho_{mat}(z) dz$$

$$+ \frac{A}{r} \exp \left( \frac{r}{\sqrt{6a + 2b}} \right) + \frac{B}{r} \exp \left( -\frac{r}{\sqrt{6a + 2b}} \right).$$ 

(21)
The integration constants $A$ and $B$ may be got from the boundary conditions, that is the function $\rho (r)$ should be finite at the origin and go to zero at infinity, which gives

$$A = -B = -\frac{1}{2\sqrt{6a + 2b}} \int_0^R \exp \left( -\frac{z}{\sqrt{6a + 2b}} \right) z\rho_{\text{mat}} (z) \, dz,$$

(22)

where $R$ is the radius of the body, that is, the radius of the matter distribution. Note that a part of the total mass of the body lies in the region without matter, associated to the vacuum density. Eq. (20) may be solved by a method similar to the one used for eq.(19), which gives the function $\rho (r)$. Once we know the function $T(r)$ the solutions of eqs. (17) and (18) are straightforward.

I am interested in the functions $T(r), \rho (r), p(r)$ and $q(r)$ outside the body (i.e. $r > R$, where $\rho_{\text{mat}} = 0$) and for the particular case where $\rho_{\text{mat}}$ is a constant inside the body (which is a good approximation for the Earth or a metallic sphere). Thus the $z$ integrals in eqs. (21) and (22) may be performed analytically, and similarly in the solution of eq. (20). In order to simplify the notation I shall introduce the following dimensionless variables

$$x \equiv \frac{r}{\sqrt{6a + 2b}}, \quad X \equiv \frac{R}{\sqrt{6a + 2b}}, \quad y \equiv \frac{r}{\sqrt{-b}}, \quad Y \equiv \frac{R}{\sqrt{-b}},$$

(23)

and the mass parameters

$$M_x \equiv \frac{3M}{2X^3} \left[ X - 1 + (X + 1) \exp (-2X) \right], \quad M_y \equiv \frac{3M}{2Y^3} \left[ Y - 1 + (Y + 1) \exp (-2Y) \right].$$

(24)

Thus I get for $r > R$

$$T(r) = -\frac{M_x}{4\pi (6a + 2b) r} \exp (X - x),$$

(25)

$$\rho (r) = \frac{1}{12\pi r} \left[ (6a + 2b)^{-1} M_x \exp (X - x) + 2 |b|^{-1} M_y \exp (Y - y) \right].$$

(26)

Hence it follows

$$p + 2q = T + \rho = \frac{1}{6\pi r} \left[ |b|^{-1} M_y \exp (Y - y) - (6a + 2b)^{-1} M_x \exp (X - x) \right].$$

(27)
In order to get $p$ and $q$ separately we shall solve the equation resulting from the subtraction of eq. (18) minus eq. (17) in the region $r > R$, that is

$$
|b| \left[ \frac{d^2 h}{dr^2} + \frac{2}{r} \frac{dh}{dr} - \frac{6h}{r^2} \right] - h = (2a + b) \left( \frac{1}{r} \frac{dT}{dr} - \frac{d^2 T}{dr^2} \right) - \frac{6h}{r^2} - h = (2a + b) M_x 4\pi (6a + 2b)^{5/2} \left( 1 + \frac{3}{x} + \frac{3}{x^2} \right) \exp (X - x),
$$

where eq. (25) has been taken into account. Now I use the ansatz

$$
h = A \left( 1 + \frac{3}{x} + \frac{3}{x^2} \right) \exp (X - x) + B \left( 1 + \frac{3}{y} + \frac{3}{y^2} \right) \exp (Y - y),$$

which inserted in the left side of eq. (28) gives

$$A(\frac{|b|}{6a + 2b} - 1) \left( 1 + \frac{3}{x} + \frac{3}{x^2} \right) \exp (X - x).$$

This leads to

$$A(\frac{|b|}{6a + 2b} - 1) = \frac{(2a + b) M_x}{6a + 2b} \Rightarrow A = -\frac{M_x}{12\pi},$$

whence I get

$$h = -\frac{M_x}{12\pi} \left( 1 + \frac{3}{x} + \frac{3}{x^2} \right) \exp (X - x) + B \left( 1 + \frac{3}{y} + \frac{3}{y^2} \right) \exp (Y - y).$$

Combining this with eq. (27) I obtain

$$p = \frac{1}{6\pi r} \left[ |b|^{-1} M_y \exp (Y - y) - (6a + 2b)^{-1} M_x \exp (X - x) \right]
\quad + \frac{M_x}{6\pi} \left( 1 + \frac{3}{x} + \frac{3}{x^2} \right) \exp (X - x) - 2B \left( 1 + \frac{3}{y} + \frac{3}{y^2} \right) \exp (Y - y).$$

The integration constant $B$ is fixed by the condition that eq. (15) is fulfilled with the approximation of neglecting the last term. That is

$$\frac{dp}{dr} = \frac{2(q - p)}{r} \iff \frac{d}{dr} (r^3 p) = r^2 (2q + p).$$

Finally this leads to
\[ p(r) = \frac{M_x}{6\pi r^3} [x + 1] \exp (X - x) - \frac{M_y}{6\pi r^3} [y + 1] \exp (Y - y). \]  

(29)

All results obtained up to now for the solution of the differential eqs. (16) to (20) have assumed that both \( 6a + 2b \) and \( -b \) are positive. If one or both quantities were negative we would obtain sinus or cosinus functions rather than exponentials, which would clearly violate empirical facts. Thus the parameters are constrained to the range

\[ 0 \leq -b \leq 3a. \]

(30)

The remaining bounds are derived in the following.

The vacuum density eq. (26) gives rise to a mass distribution which I calculate as follows. Firstly I show that the total mass of the body is the same as the mass of matter, that is the total vacuum mass is zero. This follows trivially if we integrate the two sides of eq. (16) after multiplication times the volume element. In fact we obtain

\[
\int_0^\infty \nabla^2 \rho 4\pi r^2 dr = 4\pi \int_0^\infty r \frac{d^2}{dr^2} (rp) dr = -4\pi \int_0^\infty \frac{d}{dr} (rp) dr = 0,
\]

where I have performed an integration by parts and taken into account that \( rp \to 0 \) for \( r \to \infty \). A similar result holds true in the integral of \( \nabla^2 T \). Thus the mass associated to the total density \( \rho \) equals the mass associated to the matter density \( \rho_{mat} \). In our language we may say that the total mass of the quantum vacuum (associated to \( T^{vac}_{\mu\nu} \), see eq. (21)) is zero. Nevertheless the vacuum density changes the spatial distribution of the total mass. In fact, we may obtain the (vacuum) mass distribution outside the body by integrating eq. (26) between \( R \) and \( r \). I get

\[
M_{ext}(r) = 4\pi \int_R^r r^2 \rho dr
\]

\[
= \frac{M_x}{3} [X + 1 - (x + 1) \exp (X - x)] + \frac{2M_y}{3} [Y + 1 - (y + 1) \exp (Y - y)].
\]

Hence it follows that the total mass enclosed in a sphere of radius \( r \) is (see eqs. (23))

\[
M(r) = M - \frac{M_x}{3} (x + 1) \exp (X - x) - \frac{2M_y}{3} (y + 1) \exp (Y - y),
\]

(31)
where the condition $M(\infty) = M$ has been taken into account. Remember that $M(\infty) - M$ corresponds to the vacuum mass, which is zero as shown above. The effect is dramatic for a spherical body with a radius $R \ll \sqrt{a}, \sqrt{|b|}$. In this case $X, Y \ll 1$, $M_x \simeq M_y \simeq M$ which implies $M(r) \simeq 0$, that is the total mass in the interior of the body is zero. That is the vacuum mass (negative) cancels the matter mass (positive) in the interior of the body.

Now we may calculate the gravitational field, $g$, near the Earth surface, which we should identify with $-1/2$ times the quantity $\beta'$ defined in (13), where we may neglect $m << r$ in the denominator. Thus from eqs.(26) and (29) I get, to lowest order in $\sqrt{6a + 2b}$ and $\sqrt{|b|}$,

$$g = -\frac{G}{r^2} \left[ M(r) + 4\pi r^3 p(r) \right]$$

$$\simeq -\frac{GM}{r^2} - \frac{GM_x}{3r^2} (x + 1) \exp(X - x) + \frac{4GM_y}{3r^2} (y + 1) \exp(Y - y) \tag{32}$$

where $G$ is Newton constant. We see that when $\sqrt{6a + 2b}, \sqrt{|b|} \ll R$ eq.(32) corresponds to Newtonian gravity plus a small correction consisting of two Yukawa-type terms, one of them attractive, the other one repulsive. Although these results have been derived for bodies with constant density they are valid for any celestial body because the parameters $M_x$ and $M_y$ depend only on a small region near the body’s surface where the density is effectively a constant.

In sharp contrast for a small sphere where $\sqrt{6a + 2b}, \sqrt{|b|} \gg R$ and $M_x \simeq M_y \simeq M$ (see eqs.(24)) we obtain

$$|g| \simeq \frac{GM}{r^2} \left| 1 + \frac{1}{3} \exp(X - x) - \frac{4}{3} \exp(Y - y) \right|$$

$$\simeq \frac{GM}{3r^2} (r - R) \left| \frac{1}{\sqrt{6a + 2b}} - \frac{4}{\sqrt{|b|}} \right| \ll \frac{GM}{r^2}$$

for $r$ not much larger than both $\sqrt{6a + 2b}$ and $\sqrt{|b|}$. Now performed laboratory experiments have shown that the field $g$ agrees fairly well with Newtonian predictions for bodies greater than a few millimeters\[7\]. Thus our calculation shows that these experiments exclude values of the parameters $a$ and $|b|$ greater than a fraction of squared centimeter.
4 Conclusions

I conclude that $\sqrt{a}$ and $\sqrt{-b}$ should be smaller than one centimeter, which probably excludes any relevant effect of FOG on stars, galaxies or cosmology.

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