A nonperturbative determination of $C_A$

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We describe a non-perturbative determination of $c_A$ using correlators containing the axial-vector and pseudoscalar currents at zero and non-zero momentum. We apply the method of Bhattacharya et al to extract $c_A$ from the requirement that the ratio of appropriate correlators for the PCAC relation becomes independent of time in the excited state region. We find that the result depends strongly on the order of the derivatives used in the PCAC relation. We also find that, using the lowest order derivatives, we cannot get a consistent value of $c_A$ between zero and non-zero momentum cases. The $c_A$ values that we obtain as we improve the derivatives are consistent and decrease in magnitude heading towards the perturbative result.

1. Introduction

The use of Symanzik-improved lattice actions and matrix elements is widespread. However, with each improvement term added the corresponding coefficient must be determined to enable discretisation effects to be reduced to the desired level. Considering the light hadron spectrum and matrix elements, the relevant $O(a)$ improvement coefficients are, for the most part, only known to 1-loop in perturbation theory. A nonperturbative determination of these coefficients is desirable and may be simpler than performing higher loop perturbative calculations.

The ALPHA collaboration, using Schrödinger functional techniques, have calculated several $O(a)$ improvement coefficients nonperturbatively. In most cases agreement is found with 1-loop (tadpole-improved) perturbation theory, or the discrepancy is consistent with estimates of the size of omitted higher orders in $a$ (albeit in some cases assuming a slow convergence of the perturbative series). However, for the $O(a)$ improvement of the axial-vector current

$$A_\mu \rightarrow A'_\mu = A_\mu + c_A \partial_\mu P + O(a^2)$$

where $A_\mu = \bar{\psi} \gamma_\mu \gamma_5 \psi$ and $P = \bar{\psi} \gamma_5 \psi$, $c_A$ is found to be many times larger than the 1-loop perturbative value even at reasonably high $\beta$, as shown in figure 1, taken from reference [1].

$c_A$ appears in the expression for the pion decay constant, $f_\pi$,

$$f_\pi = (1 + b_A) Z_A (f_\pi^{(0)} + c_A a f_\pi^{(1)} + O(a^2))$$

and the bare quark mass (from the PCAC relation)

$$2m = \frac{\partial_\mu <A_\mu P> + c_A a \partial_\mu^2 <PP>} {2 <PP>} + O(a^2)$$

Choosing the perturbative or nonperturbative value for $c_A$ leads to significantly different values for these quantities at finite $\beta$. Bhattacharya et al. proposed an alternative method for determining $c_A$, and other $O(a)$ coefficients in reference [2]. Here, we apply their method, but with higher statistics than reported in reference [2]. In

![Figure 1. $C_A$ extracted by the ALPHA collaboration. The circles indicate the results of reference [1], the dotted line the results of 1-loop perturbation theory and the crosses the results from 1-loop tadpole-improved perturbation theory.](image-url)
addition, we extract $c_A$ at finite, as well as zero, momentum.

2. $C_A$ from the PCAC relation

The PCAC relation in euclidean space
\[
<\partial_\mu A_\mu(x)P>=2m<P(x)P>
\] (4)
should hold for all $x$ on the lattice up to discretisation terms. For simplicity we assign
\[
r(t) = <\partial_\mu A_\mu(t)P> <P(t)P>
\] (5)
\[
s(t) = <\partial_\mu^2 P(t)P> <P(t)P>
\] (6)
Thus,
\[
2m = r(t) + O(a)
\] (7)
\[
2m = r(t) + c_A s(t) + O(a^2).
\] (8)
Equation 8 only holds if the $O(a)$-improvement term (the clover term) is included in the light quark action and the coefficient, $c_{SW}$, is determined nonperturbatively. The method of Bhat- tacharya et. al. is to determine $c_A$ by minimizing the dependence of $r(t) + c_A s(t)$ on $t$ and, hence, reduce the discretisation errors in this quantity. Using this method, we investigate the effect of improving $\partial_\mu$ on the determination of $c_A$; discretisation errors in the lattice derivatives may be the dominant source of lattice spacing errors in the bare quark mass, and hence, the value obtained for $c_A$ may be artificially high.

Normally, the symmetric lattice derivatives
\[
\partial_\mu \rightarrow \Delta^{(+-)}_\mu = \frac{1}{2}(\delta_{\vec{x},\vec{x}+\mu} - \delta_{\vec{x},\vec{x}-\mu})
\] (9)
\[
\partial_\mu^2 \rightarrow \Delta^{(2)}_\mu = \delta_{\vec{x},\vec{x}+\mu} - 2\delta_{\vec{x},\vec{x}} + \delta_{\vec{x},\vec{x}-\mu}
\] are used, which contain $O(a^2)$ errors. We will consider the effect of using
\[
\tilde{\Delta}^{(+-)}_\mu = \Delta^{(+-)}_\mu - \frac{1}{6} \Delta^{+}\Delta^{(-)} \Delta^{-}
\] (10)
\[
\tilde{\Delta}^{(2)}_\mu = \Delta^{(2)}_\mu - \frac{1}{12} [\Delta^+\Delta^-]^2
\] which are correct up to $O(a^4)$. One can continue to correct to $O(a^6)$, we denote these derivatives $\tilde{\Delta}^{(+-)}_\mu$ and $\tilde{\Delta}^{(2)}_\mu$. As an additional constraint on $c_A$, we investigate whether Lorentz invariance is restored to sufficient accuracy i.e. that consistent values of $m$ are obtained at zero and finite momentum.

3. Estimating $C_A$ using excited states

The configurations and light quark propagators were provide by the UKQCD collaboration. The simulation parameters are given in table 1. For more details see reference 3. Note that the $C_{SW}$ values correspond to those determined by the ALPHA collaboration.

Figure 2 shows the ratio of correlators, $r(t)$, for the data set at $\beta = 6.0$ for correlators local at the source and sink (LL) and those fuzzed at the source and local at the sink (FL). The $O(a^2)$ lattice derivatives, equation 8, were used. Towards the center of the lattice $r(t)$ tends to a

| $\beta$ | Volume | $n_{confs}$ | $C_{SW}$ | $\kappa$ |
|---------|--------|-------------|----------|---------|
| 6.0     | $16^3 \times 48$ | 496          | 1.77     | 0.13344 |
| 6.2     | $24^3 \times 48$  | 214          | 1.61     | 0.13460 |

Table 1
Simulation details.
constant, $2m \sim 0.1096$. However, in the LL case, close to the origin there are significant deviations from this value due to discretisation effects. The much wider plateau in $r(t)$ for the FL correlators (where the fuzzing has been chosen to more or less eliminate excited state contributions) show that the discretisation effects in $r(t)$ LL are associated with excited states. We can determine $c_A$ by adding the term $c_A s(t)$ to reduce the discretisation effects at small $t$. To do this we perform a correlated fit to $r(t)$ using $r(t) = 2m - c_A s(t)$, to extract $2m$ and $c_A$. The values of $c_A$ obtained are detailed in the next section.

To see how well the discretisation effects are reduced we plug these values back into $r(t) + c_A s(t)$ for LL correlators and compare with the unimproved case. At $\beta = 6.0$ we find that the plateau in $2m$ can be extended from $t = 11$ to $t = 6$ when $c_A = -0.10$ and $O(a^2)$ lattice derivatives are used. However, as shown in figure 3, if one improves $\partial_\mu$, the discretisation effects can be removed by the same amount but lower values of $c_A$ are required. Clearly the value of $c_A$ and the form of the derivatives significantly affects the value of $m$ extracted. Furthermore, we find that consistent values of $m$ at zero and finite momentum can only be obtained if $O(a^4)$ or $O(a^3)$ derivatives are used. Figures 4 and 5 show the difference in the
bare quark mass at zero and finite momentum for standard and improved derivatives.

4. Results for $C_A$

Results are shown in figures 6 and 7. The main points are:

- A reliable estimate for $c_A$ is indicated by a stable value with $t_{\text{min}}$. At $\beta = 6.0$, the statistical errors grow rapidly for $t_{\text{min}} > 8$, although the results are reasonably stable for $t_{\text{min}} = 6 - 8$. The situation is similar at $\beta = 6.2$.

- Clearly $c_A$ decreases when the temporal derivatives are improved. The decrease is less going from derivatives correct to $O(a^4)$ to those correct to $O(a^6)$, as expected.

- $c_A$ obtained using improved derivatives is below the value obtained by the ALPHA collaboration. For the $O(a^6)$-correct derivatives $c_A$ is close to the 1-loop tadpole-improved value (although the perturbation theory has not been performed with improved derivatives). In principle, our results can differ with those of reference [1] by $O(\Lambda_{\text{QCD}}/a)$ around $\kappa_{\text{strange}}$.

- The results can be compared to those obtained by Bhattacharya et. al. [2]. They obtain $c_A = -0.02(2)(2)$ at $\beta = 6.0$ and $-0.033(4)(3)$ at $\beta = 6.2$ for $\kappa_c$ with a significantly smaller number of configurations. Our results are at finite $\kappa$ around $\kappa_{\text{strange}}$. A preliminary study indicates that the light quark mass dependence of $c_A$ is small.

5. Conclusions

We have applied the method of reference [3] to extract the $O(a)$ improvement coefficient, $c_A$. We find that the accuracy and reliability of this method is limited by the small range of timeslices over which a stable value of $c_A$ is found. Nevertheless, we have shown that the value of $c_A$ obtained depends significantly on the discretisation chosen for the lattice derivatives appearing in the PCAC relation. $c_A$ reduces as the lattice derivatives are improved, and lies between the value obtained by the ALPHA collaboration and the 1-loop tadpole-improved value. Furthermore, consistent values for the bare quark mass at zero and finite momentum can only be obtained for derivatives correct to $O(a^4)$ and above.

These results suggest a similar study should be performed within the Schrödinger functional approach. In the future, we plan to investigate the quark mass dependence of $c_A$ and check the scaling properties of the renormalised light quark mass and $f_\pi$ obtained using our values of $c_A$.

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Figure 7. The results for $c_A$ at $\beta = 6.2$. The figure is labelled in the same way as figure 6.

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