HERMITE–HADAMARD TYPE INEQUALITIES FOR
WRIGHT-CONVEX FUNCTIONS OF SEVERAL VARIABLES

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ABSTRACT. We present Hermite–Hadamard type inequalities for Wright-convex, strongly convex and strongly Wright-convex functions of several variables defined on simplices.

1. INTRODUCTION

One of the most classical inequalities in the theory of convex functions is the Hermite–Hadamard inequality. It states that if \( f : [a, b] \to \mathbb{R} \) is convex then

\[
\frac{f(a) + f(b)}{2} \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]

It plays an important role in convex analysis, so in the literature one can find its various generalizations and applications. For example, an exhausting study of this inequality is given in the book [2].

Recall that a function \( f : D \to \mathbb{R} \), where \( D \subset \mathbb{R}^n \) is a convex set, is called Wright-convex (W-convex for short), if

\[
f(tx + (1-t)y) + f((1-t)x + ty) \leq f(x) + f(y)
\]

for any \( x, y \in D \) and \( t \in [0,1] \). Trivially we can see that any convex function is necessarily W-convex and any W-convex function is Jensen-convex (i.e. it fulfills the above inequality with \( t = \frac{1}{2} \)). However, these inclusions are proper. It is evident, if one knows the famous Ng’s representation (cf. [8]). It states that any W-convex function defined on an open and convex set \( D \subset \mathbb{R}^n \) is the sum of an additive function \( a : \mathbb{R}^n \to \mathbb{R} \) and a convex function \( g : D \to \mathbb{R} \). Therefore, if \( f : \mathbb{R} \to \mathbb{R} \) is W-convex, then either \( f \) is continuous (and then convex), or the graph of \( f \) is a dense subset of a plane. Hence by putting \( f(x) = |a(x)| \), where \( a : \mathbb{R} \to \mathbb{R} \) is a discontinuous additive function, we obtain a Jensen-convex function, which is not W-convex. Of course, the function \( a \) is a W-convex function, which is not convex.

It was natural to generalize the Hermite–Hadamard inequality to the functions of several variables. In the case of simplices, for the first time it was done by Neuman [7] (see also [1], [3] and [13] for the functions defined on simplices and [2], [9] for more general domains). Recently Olbryś [10] obtained the following inequality of Hermite–Hadamard type: if \( f : \mathcal{I} \to \mathbb{R} \) (where \( \mathcal{I} \subset \mathbb{R} \) is an open

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interval) is W-convex, then
\[
2f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \left(f(x) + f(a+b-x)\right) \, dx \leq f(a) + f(b)
\]
for any \(a, b \in \mathcal{I}\). Because the note \([10]\) is actually unpublished, let us mention that (1) is an immediate consequence of the Hermite–Hadamard inequality. It is enough to apply it to the convex (due to Ng’s representation of \(f\)) function \([a, b] \ni x \mapsto f(x) + f(a+b-x)\).

Motivated by this beautiful Olbryš’s result we present in this paper its multivariate counterparts. We also give some related inequalities for strongly convex and strongly W-convex functions of several variables.

2. Definitions and Basic Properties

Let \(v_0, \ldots, v_n \in \mathbb{R}^n\) be affine independent and let \(S = \text{conv}\{v_0, \ldots, v_n\}\) be a simplex with vertices \(v_0, \ldots, v_n\). Denote by \(|S|\) its volume and by \(b\) its barycenter, i.e.
\[
b = \frac{1}{n} \sum_{i=1}^n v_i.
\]

Any element \(x \in S\) is uniquely represented by a convex combination of the vertices:
\[
x = \sum_{i=0}^n t_i v_i,
\]
where the coefficients \(t_i \geq 0, i = 0, \ldots, n\), with \(t_0 + \cdots + t_n = 1\), are called the barycentric coordinates of \(x\). Moreover, any \(x \in \mathbb{R}^n\) has the above (unique) representation with real scalars summing up to 1.

Denote by \(C\) the set of all cyclic permutations of \(\{0, \ldots, n\}\). Any \(\sigma \in C\) generates an affine transformation \(\sigma : \mathbb{R}^n \to \mathbb{R}^n\) in the manner
\[
\sigma\left(\sum_{i=0}^n t_i v_i\right) = \sum_{i=0}^n t_{\sigma(i)} v_i.
\]

From now on we identify \(\sigma \in C\) with the affine map \(\sigma\) given as above. For \(\sigma \in C\) and for any function \(f : S \to \mathbb{R}\) we define the function \(f_\sigma : S \to S\) by
\[
f_\sigma(x) = f(\sigma(x)).
\]
Next we introduce the symmetrization \(F\) of a function \(f\) as follows:
\[
F(x) = \sum_{\sigma \in C} f_\sigma(x), \quad x \in S.
\]
It is easy to observe that \(F\) is symmetric with respect to the barycenter, which means that \(F(\sigma(x)) = F(x)\) for any \(\sigma \in C\).

In our article we use the Hermite–Hadamard inequality on simplices, which was firstly given by Neuman \([7]\), then reproved by Guessab and Schmeisser \([3]\), Bessenyei \([1]\) and the second author \([13, \text{Corollary 3}]\):

**Theorem 1.** If \(f : S \to \mathbb{R}\) is convex then
\[
f(b) \leq \frac{1}{|S|} \int_S f(x) \, dx \leq \frac{1}{n+1} \sum_{i=0}^n f(v_i).
\]
To prove the Hermite-Hadamard type inequality for W-convex functions, we need two lemmas. The first of them could be found in [12, Lemma 2.2].

**Lemma 2.** If \( g : S \to R \) is convex then so is \( g_\sigma \).

**Lemma 3.** If \( a : \mathbb{R}^n \to \mathbb{R} \) is additive, then its symmetrization \( A = \sum_{\sigma \in C} a_\sigma \) is constant.

**Proof.** Any \( x \in \mathbb{R}^n \) may be written as \( x = \sum_{i=0}^n t_i v_i \) with real scalars \( t_0, \ldots, t_n \) (possibly not all positive) summing up to 1. Then by additivity of \( a \) and affinity of \( \sigma \) we get

\[
A(x) = \sum_{\sigma \in C} a_\sigma(x) = \sum_{\sigma \in C} a(\sigma(x)) = a\left( \sum_{\sigma \in C} \sigma\left( \sum_{i=0}^n t_i v_i \right) \right) = a\left( \sum_{i=0}^n \sigma(v_i) \right) = a\left( \sum_{i=0}^n v_i \right),
\]

which is a constant. \( \square \)

3. HERMITE–HADAMARD TYPE INEQUALITY FOR W-CONVEX FUNCTIONS

**Theorem 4.** Let \( D \subset \mathbb{R}^n \) be an open and convex set and \( S \subset D \) be a simplex. If \( f : D \to \mathbb{R} \) is W-convex, then its symmetrization \( F \) is convex on \( S \).

**Proof.** Because \( f \) is W-convex, then \( f = a + g \) for some additive function \( a : \mathbb{R}^n \to \mathbb{R} \) and a convex function \( g : D \to \mathbb{R} \). The function \( A \) (symmetrization of \( a \), cf. (2)) is constant by Lemma 3, while the function \( G \) (symmetrization of \( g \) on \( S \)), is convex by Lemma 2. Thus \( F \) is convex on \( S \). \( \square \)

**Theorem 5.** Let \( D \subset \mathbb{R}^n \) be an open and convex set and \( S \subset D \) be a simplex with vertices \( v_0, \ldots, v_n \) and barycenter \( b \). If \( f : D \to \mathbb{R} \) is W-convex, then

\[
(n + 1)f(b) \leq \frac{1}{|S|} \int_S \left( \sum_{\sigma \in C} f_\sigma(x) \right) dx \leq \sum_{i=0}^n f(v_i).
\]

**Proof.** Let \( F = \sum_{\sigma \in C} f_\sigma \) be the symmetrization of \( f \) on \( S \). By the previous theorem \( F \) is convex on \( S \). Using the Hermite–Hadamard inequality (cf. Theorem 1) we arrive at

\[
F(b) \leq \frac{1}{|S|} \int_S F(x) dx \leq \frac{1}{n + 1} \sum_{i=0}^n F(v_i).
\]

Observe that \( \sigma(b) = b \). Therefore

\[
F(b) = \sum_{\sigma \in C} f_\sigma(b) = \sum_{\sigma \in C} f(\sigma(b)) = (n + 1)f(b),
\]

(4)
while

\[(5) \quad \sum_{i=0}^{n} F(v_i) = \sum_{i=0}^{n} \sum_{\sigma \in C} f_\sigma(v_i) = \sum_{i=0}^{n} (f(v_0) + \cdots + f(v_n))
\]

\[= (n + 1) \sum_{i=0}^{n} f(v_i)\]

and the proof is finished. □

**Remark 6.** For \(n = 1\) and \(S = [a, b]\) we obtain immediately the result due to Olbryś [10] given by (1).

Observe that in the Hermite–Hadamard inequality (3) one fact was essential: the integral mean value \(T[f] = \frac{1}{|S|} \int_S f(x) \, dx\) is a positive linear operator. This motivated the second author to prove the operator version of the inequality (3) (cf. [13, Theorem 2]). Our next theorem offers an extension of Theorem 5 to positive linear operators going in this direction.

**Theorem 7.** Let \(D \subset \mathbb{R}^n\) be an open and convex set and \(S \subset D\) be a simplex with vertices \(v_0, \ldots, v_n\) and barycenter \(b\). Let \(T\) be positive linear functional defined (at least) on a linear subspace of all functions mapping \(S\) into \(\mathbb{R}\) generated by a cone of convex functions. Assume that

\[T(\pi_i) = \frac{1}{|S|} \int_S \pi_i(x) \, dx, \quad i = 1, \ldots, n,\]

where \(\pi_i\) is the projection onto the \(i\)-th axis and \(T(1) = 1\). If \(f : D \to \mathbb{R}\) is \(W\)-convex and \(F\) is the symmetrization of \(f\) on \(S\), then

\[(6) \quad F(b) \leq T[F] \leq \frac{1}{n + 1} \sum_{i=0}^{n} F(v_i).\]

**Proof.** Take an arbitrary affine function \(\varphi : \mathbb{R}^n \to \mathbb{R}\). It has a form

\[\varphi(x) = \sum_{i=0}^{n} \alpha_i x_i + \beta = \sum_{i=0}^{n} \alpha_i \pi_i(x) + \beta, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n\]

for some scalars \(\alpha_0, \ldots, \alpha_n, \beta\). The linearity yields

\[T[\varphi] = T\left[\sum_{i=0}^{n} \alpha_i \pi_i + \beta\right] = \sum_{i=0}^{n} \alpha_i T[\pi_i] + \beta T[1] = \sum_{i=0}^{n} \frac{\alpha_i}{|S|} \int_S \pi_i(x) \, dx + \beta \]

\[= \frac{1}{|S|} \int_S \left(\sum_{i=0}^{n} \alpha_i \pi_i(x) + \beta\right) \, dx = \frac{1}{|S|} \int_S \varphi(x) \, dx.\]

Therefore \(T\) meets the assumptions of [13, Theorem 2]. Hence, by convexity of \(F\), the inequality (6) holds. □

Of course, taking in the above theorem \(T[f] = \frac{1}{|S|} \int_S f(x) \, dx\), we obtain immediately the Theorem 5.
4. HERMITE–HADAMARD TYPE INEQUALITY FOR STRONGLY CONVEX FUNCTIONS

Let \( D \subset \mathbb{R}^n \) be a convex set and \( c > 0 \). The function \( f : D \to \mathbb{R} \) is called strongly convex with modulus \( c \), if

\[
f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) - ct(1 - t)\|x - y\|^2
\]

for all \( x, y \in D \) and \( t \in [0, 1] \). Strongly convex functions were introduced by Polyak [11] (see also [5] for some interesting remarks on this class of functions). Let us only mention that a strongly convex function is necessarily convex, but the converse does not hold (for instance, affine functions are not strongly convex).

Below we present the multivariate counterpart of a result due to Merentes and Nikodem [5].

**Theorem 8.** If \( f : S \to \mathbb{R} \) is strongly convex with modulus \( c \), then

\[
f(b) + c \left( \frac{1}{|S|} \int_S \|x\|^2 \, dx - \|b\|^2 \right) \leq \frac{1}{|S|} \int_S f(x) \, dx \leq f(b) - c \|b\|^2 \leq \frac{1}{n + 1} \sum_{i=0}^{n} f(v_i) + c \left( \frac{1}{|S|} \int_S \|x\|^2 \, dx - \frac{1}{n + 1} \sum_{i=0}^{n} \|v_i\|^2 \right).
\]

**Proof.** We take a function \( g : S \to \mathbb{R} \) of the form \( g = f - c\|\cdot\|^2 \). Since \( f \) is strongly convex with modulus \( c \), then \( g \) is convex (for a quick reference see [4] or [5]). Therefore \( g \) satisfies the Hermite–Hadamard inequality (3):

\[
g(b) \leq \frac{1}{|S|} \int_S g(x) \, dx \leq \frac{1}{n + 1} \sum_{i=0}^{n} g(v_i).
\]

Then we arrive at

\[
f(b) - c\|b\|^2 \leq \frac{1}{|S|} \int_S f(x) \, dx - \frac{c}{|S|} \int_S \|x\|^2 \, dx \leq \frac{1}{n + 1} \sum_{i=0}^{n} f(v_i) - c \frac{1}{n + 1} \sum_{i=0}^{n} \|v_i\|^2.
\]

and our result follows by adding the term \( c \frac{1}{|S|} \int_S \|x\|^2 \, dx \) to both sides of the above inequality. \( \square \)

Denote by \( S_1 \) the unit simplex in \( \mathbb{R}^n \), i.e. the simplex with vertices \( e_0 = (0, 0, \ldots, 0) \), \( e_1 = (1, 0, \ldots, 0) \), \ldots , \( e_n = (0, \ldots, 0, 1) \). Then

\[
b = \frac{1}{n + 1} \sum_{i=0}^{n} e_i.
\]

It is well-known that \( |S| = \frac{1}{n!} \). The second author noticed in [13] (proof of Corollary 8) that

\[
\int_{S_1} \pi^2_i(x) \, dx = \frac{2}{(n + 2)!}, \quad i = 1, \ldots, n.
\]

For strongly convex functions defined on the unit simplex \( S_1 \), Theorem 8 together with (7) gives us
Corollary 9. If $f : S_1 \to \mathbb{R}$ is strongly convex with modulus $c$, then
\[
f(b) + \frac{cn^2}{(n + 1)^2(n + 2)} \leq n! \int_{S_1} f(x) \, dx \leq \frac{1}{n + 1} \sum_{i=0}^{n} f(e_i) - \frac{cn^2}{(n + 1)(n + 2)}.
\]

Remark 10. For $n = 1$ we obtain the inequality
\[
f\left(\frac{1}{2}\right) + \frac{c}{12} \leq \int_{0}^{1} f(x) \, dx \leq \frac{f(0) + f(1)}{2} - \frac{c}{6},
\]
which corresponds with the result of Merentes and Nikodem presented in \cite[Theorem 6]{6} (the full version of their inequality could be derived from Theorem 8 by setting $n = 1$ and an arbitrary compact interval $[a, b]$ in the role of $S$).

5. HERMITE–HADAMARD TYPE INEQUALITY FOR STRONGLY W-CONVEX FUNCTIONS

Let $D \subset \mathbb{R}^n$ be a convex set and $c > 0$. The function $f : D \to \mathbb{R}$ is called strongly W-convex with modulus $c$, if
\[
f(tx + (1 - t)y) + f((1 - t)x + ty) \leq f(x) + f(y) - 2ct(1 - t)||x - y||^2
\]
for all $x, y \in D$ and $t \in [0, 1]$. Such functions were introduced by Merentes, Nikodem and Rivas in \cite{6}. We present below a counterpart of Theorem 4 for strongly W-convex functions.

Theorem 11. Let $D \subset \mathbb{R}^n$ be an open and convex set and $S \subset D$ be a simplex. If $f : D \to \mathbb{R}$ is strongly W-convex with modulus $c$, then its symmetrization $F$ is strongly convex on $S$ with modulus $(n + 1)c$. In particular, $F$ is integrable on $S$.

Proof. Since $f$ is strongly W-convex with modulus $c$, there exists a W-convex function $h : D \to \mathbb{R}$ such that $f(x) = h(x) + c||x||^2$, $x \in D$ (cf. \cite[Corollary 5]{6}). Take an arbitrary vector $x \in S$. If $\sigma \in C$, then $||\sigma(x)|| = ||x||$, $x \in S$, whence $f_\sigma(x) = h_\sigma(x) + c||x||^2$. Therefore
\[
F(x) = \sum_{\sigma \in C} f_\sigma(x) = \sum_{\sigma \in C} h_\sigma(x) + \sum_{\sigma \in C} c||x||^2 = \sum_{\sigma \in C} h_\sigma(x) + (n + 1)c||x||^2.
\]
Theorem 4 yields that a function $\sum_{\sigma \in C} h_\sigma$ is convex on $S$. Then $F$ is strongly convex on $S$ with modulus $(n + 1)c$ (cf. \cite{4} or \cite{5}) and the proof is finished. \hfill \Box

Corollary 12. Let $D \subset \mathbb{R}^n$ be an open and convex set and $S \subset D$ be a simplex with vertices $v_0, \ldots, v_n$ and barycenter $b$. If $f : D \to \mathbb{R}$ is strongly W-convex with modulus $c$, then
\[
f(b) + c\left(\frac{1}{|S|}\int_{S} ||x||^2 \, dx - ||b||^2\right) \leq \frac{1}{(n + 1)|S|} \int_{S} \left(\sum_{\sigma \in C} f_\sigma(x) \, dx\right)
\]
\[
\leq \frac{1}{n + 1} \sum_{i=0}^{n} f(v_i) + c\left(\frac{1}{|S|}\int_{S} ||x||^2 \, dx - \frac{1}{n + 1} \sum_{i=0}^{n} ||v_i||^2\right).
\]
Proof. Since the symmetrization $F$ of the function $f$ is strongly convex on $S$ with modulus $(n + 1)c$, then by virtue of Theorem 8

$$F(b) + (n + 1)c \left( \frac{1}{|S|} \int_S \|x\|^2 \, dx - \|b\|^2 \right) \leq \frac{1}{|S|} \int_S F(x) \, dx$$

$$\leq \frac{1}{n + 1} \sum_{i=0}^{n} F(v_i) + (n + 1)c \left( \frac{1}{|S|} \int_S \|x\|^2 \, dx - \frac{1}{n + 1} \sum_{i=0}^{n} \|v_i\|^2 \right).$$

We have by (4), (5)

$$F(b) = (n + 1)f(b) \quad \text{and} \quad \sum_{i=0}^{n} F(v_i) = (n + 1) \sum_{i=0}^{n} f(v_i),$$

from which the Corollary follows. \qed

For strongly W-convex functions on the unit simplex $S_1$, Corollary 12 together with (7) gives us

**Corollary 13.** Let $D \subset \mathbb{R}^n$ be an open and convex such that $S_1 \subset D$. If $f : D \to \mathbb{R}$ is strongly W-convex with modulus $c$, then

$$f(b) + \frac{cn^2}{(n + 1)^2(n + 2)} \leq \frac{n!}{(n + 1)} \int_{S_1} \left( \sum_{\sigma \in C} f_{\sigma}(x) \right) \, dx$$

$$\leq \frac{1}{n + 1} \sum_{i=0}^{n} f(v_i) - \frac{cn^2}{(n + 1)(n + 2)}.$$

**Remark 14.** For $n = 1$ we obtain the inequality

$$f\left(\frac{1}{2}\right) + \frac{c}{12} \leq \frac{1}{2} \int_0^1 (f(x) + f(1 - x)) \, dx \leq \frac{f(0) + f(1)}{2} - \frac{c}{6},$$

(compare with Remark 10). The similar inequality for an arbitrary compact interval $[a, b]$ could be derived directly from Corollary 12.

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