LEGENDRIAN SUBMANIFOLDS FROM BOHR–SOMMERFELD COVERS OF MONOTONE LAGRANGIAN TORI

GEORGIOS DIMITROGLOU RIZELL AND ROMAN GOLOVKO

Abstract. By a result due to Ziltener, there exist no closed embedded Bohr–Sommerfeld Lagrangians inside $\mathbb{C}P^n$ for the prequantisation bundle whose total space is the standard contact sphere. On the other hand, any embedded monotone Lagrangian torus has a canonical nontrivial cover which is a Bohr–Sommerfeld immersion. We draw the front projections for the corresponding Legendrian lifts inside a contact Darboux ball of the threefold covers of both the two-dimensional Clifford and Chekanov tori (the former is the Legendrian link of the Harvey–Lawson special Lagrangian cone), and compute the associated Chekanov–Eliashberg algebras. Although these Legendrians are not loose, we show that they both admit exact Lagrangian cobordisms to the loose Legendrian sphere; they hence admit exact Lagrangian caps in the symplectisation, which are non-regular Lagrangian cobordisms. Along the way, we also compute bilinearised Legendrian contact homology of a general Legendrian surface in the standard contact vector space when all Reeb chords are of positive degree, as well as the augmentation variety in the case of tori.

1. Introduction

Prequantisation $S^1$-bundles $\pi: E \to M$ form an important class of contact manifolds $(E^{2n+1}, \alpha)$ that have been well studied from many different points-of-view. The contact form $\alpha$ is a connection one-form for the $S^1$-bundle, and the curvature is a symplectic two-form $\omega \in \Omega^2(M)$ on $M^{2n}$. For that reason, there is a close relationship between the symplectic geometry of $(M, \omega)$ and the contact geometry of $(E, \ker \alpha)$. For instance, every Legendrian immersion inside $E$ projects to a so-called Bohr–Sommerfeld Lagrangian immersion inside $(M, \omega)$. Our goal here is to study certain embedded Legendrians $\Lambda \subset (E, \ker \alpha)$ whose projection $\pi(\Lambda) \subset M$ again has an embedded image (but which is possibly multiply covered). In fact, we are mainly interested in the case $M = \mathbb{C}P^n$ and the line-bundle $\mathcal{O}(-1)$, which produces the standard round contact sphere $(E, \alpha) = (S^{2n+1}, \alpha_{st})$. The general connection between Bohr–Sommerfeld Lagrangian immersions in the projective plane and Legendrians in the standard contact sphere was recently studied in [3] by Bldrige–McCarthy–Vela-Vick.

If the image of the Legendrian is embedded, then it necessarily is a nontrivial multiple cover. Namely, as was shown by Ziltener:

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Theorem 1.1 \((\[51\])\). Any closed Legendrian submanifold inside the round contact \((S^{2n+1}, \alpha_{st})\) admits a Reeb chord of length \((0, \pi/2]\) (i.e. at most half the minimal period of a Reeb orbit). In particular, there exists no closed Bohr–Sommerfeld Lagrangian embeddings inside the standard symplectic \(\mathbb{C}P^n\) for the prequantisation bundle \(\mathcal{O}(-1)\).

On the other hand, there exists plenty of Bohr–Sommerfeld immersions that are multiply covered Lagrangian embeddings. Here we are interested mainly in the case of monotone tori. By the very recent result of Vianna \([49]\) there are infinitely many different monotone Lagrangian tori inside \(\mathbb{C}P^2\) up to Hamiltonian isotopy. They all admit Bohr–Sommerfeld covers by the following result shown in Section 3.

Theorem 1.2. Any monotone Lagrangian torus inside \(\mathbb{C}P^n\) has a canonically defined \(n+1\)-fold cover which is a Bohr–Sommerfeld immersion for the prequantisation bundle \(\mathcal{O}(-1)\). Moreover, the corresponding Legendrian lift is embedded inside \((S^{2n+1}, \xi_{st})\), has vanishing Maslov class, and only Reeb chords of positive degree. (Here we mean the degree in the Bott sense as explained in Section 3.2.)

In Section 8 we conjecture that the canonical Bohr–Sommerfeld covers of Vianna’s different Lagrangian tori have lifts that remain different up to Legendrian isotopy. Vianna’s infinite family of tori exhibit intricate Floer homological properties. As objects in the Fukaya category they constitute the infinite charts on the cluster variety that is the mirror to \(\mathbb{C}P^2\) (in the sense of homological mirror symmetry); see recent work of Pascaleff–Tonkonog \([44]\).

One should expect that this structure has a rich counterpart for also the Legendrian lifts of their Bohr–Sommerfeld covers. This paper is the starting point for such an investigation, since we compute the Legendrian invariants for the first two tori that appear in the family. For the first torus in the family (the well-known Clifford torus) we recover Nadler’s computation \([42]\) which exhibits the mirror to the one-dimensional pair of pants.

Then we turn our eye to some particular examples of Bohr–Sommerfeld covers of embedded Lagrangians inside \(\mathbb{C}P^2\). Since \(S^{2n+1} \setminus \{pt\}\) is contactomorphic to a Darboux ball \((\mathbb{R}^{2n+1}, dz - ydx)\) by \([35\] Proposition 2.1.8\), they can all be described by their associated front projections to the \((z,x)\)-plane. Recall that the front-projection recovers the Legendrian embedding. However, since the aforementioned contactomorphism is not strict, this identification of a Darboux ball is not so easy to work with. For that reason, we instead take a different path, and produce an explicit contact isotopy of the Legendrian into a small contact-form preserving Darboux ball with respect to the prequantisation contact form \(\alpha\).

The advantage of placing the Legendrians inside a Darboux ball is that it makes their Legendrian contact homology as defined by Eliashberg–Givental–Hofer and Chekanov \([32, 12]\) computable by using Ekholm’s theory of gradient flow trees \([21]\). Legendrian contact homology is a Legendrian invariant which is a homology of a differential graded algebra (DGA for short) called the Chekanov–Eliashberg algebra; in the setting considered here the precise construction was carried out in \([25]\) by Ekholm–Etnyre–Sullivan.

Notably we study the Legendrian link of the Harvey–Lawson cone from \([38\] Example III.3.A\]. This is a conical special Lagrangian inside \(\mathbb{C}^3\), whose intersection with the standard contact sphere \(S^5\) is a Legendrian torus whose projection to \(\mathbb{C}P^2\) is a threefold cover of
the monotone Clifford torus. This example was also studied in [3, Example 4.1] from the perspective of contact geometry. After a Legendrian isotopy into a small contact Darboux ball, we obtain the Legendrian $\Lambda_{Cl}$ whose front projection is shown in Figure 1; it is the symmetric figure-8 curve with two horizontal cusps rotated along its $z$-axis of symmetry. The Chekanov–Eliashberg algebra of $\Lambda_{Cl}$ was computed by the first author in [16]; see Section 5.2.

A different front associated to this Legendrian was described by Treumann–Zaslow in [48, Section 3.1], who placed it as a type of satellite around the standard Legendrian sphere. The front inside the standard neighbourhood $J^1S^2$ of the standard Legendrian two-sphere then has a caustic which is the tetrahedron cubic graph on $S^2$. Our front projection is simpler, at least in the sense that the number of Reeb chords for a small and generic perturbation is minimal for our representative.

The second example that we study is the Legendrian lift of the threefold canonical Bohr–Sommerfeld cover of the monotone Chekanov torus inside $CP^2$ [10]. The front projection of a representative $\Lambda_{Ch}$ placed inside a small contact Darboux ball is shown in Figure 4 in Section 4.2. Again the front is symmetric with respect to rotation around the $z$-axis. We compute its Chekanov–Eliashberg algebra and deduce that it is not Legendrian isotopic to $\Lambda_{Cl}$; recall that Chekanov–Schlenk [13] proved that the underlying Lagrangian torus also is not Hamiltonian isotopic to the Clifford torus inside $CP^2$. To conclude:

**Theorem 1.3** (Theorem 4.4). The threefold canonical Bohr–Sommerfeld covers of the Clifford and Chekanov tori in $CP^2$ has embedded Legendrian lifts to the prequantisation space $S^5 \rightarrow CP^2$ that are not Legendrian isotopic. Moreover, both tori have vanishing Maslov classes and are subloose but not loose.

We call a Legendrian subloose if there exists an exact Lagrangian cobordism to a loose Legendrian; see Definition 2.13 below in Section 6. Subloose but not loose Legendrian submanifolds first appeared in the work of the first author [16]. They play an important role in understanding the geography of Legendrian submanifolds. More precisely, the Chekanov–Eliashberg algebra of such submanifolds has an unusual behaviour: it is acyclic with field coefficients, but is linearisable with the so-called Novikov coefficients. There are plenty of subloose Legendrians that can be obtained from the examples in [16] by taking cusp-connect sums with e.g. appropriate families of Legendrian spheres. However, note that almost nothing is known when it comes to what augmentation varieties are realisable by such
examples (in fact this is also the case for more general Legendrian tori). In view of Vianna’s infinite family of monotone tori, it seems like there is a very rich family of augmentation varieties for Legendrian tori inside the standard contact sphere.

Since the Legendrians $\Lambda_{Cl}$ and $\Lambda_{Ch}$ both satisfy a symmetry of their fronts with respect to rotation around the $z$-axis, they have Lagrangian projections that are well-behaved with respect to the standard Lefschetz fibration $\mathbb{C}^2 \rightarrow \mathbb{C}$ given by $(z_1, z_2) \mapsto z_1 \cdot z_2$. (One must first translate the Legendrians to ensure that the axis of the $S^1$-symmetry is precisely $\{x_1 = x_2 = 0\}$.) Namely, they project to generic immersions of a closed curve inside $\mathbb{C}^*$; see the curves $\gamma_{Ch}$ and $\gamma_{Cl}$ in the bottom right of Figures 6 and 7, respectively.

We also consider the lift of the canonical twofold cover of the monotone product torus $S^1 \times S^1$ inside the monotone quadric surface $(\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{FS} \oplus \omega_{FS})$. It turns out that its Legendrian lift is Legendrian isotopic to the conormal lift of the unknot inside $S^3$; see Theorem 4.1. Recall that the conormal lift of any submanifold of $\mathbb{R}^n \subset S^n$ has a front projection associated to its representation inside the jet-space $(J^1 \mathbb{R}^n, dz - ydx) \cong UT^* \mathbb{R}^n \subset (UT^* S^n, pdq)$ which can be recovered from the geometric properties of the smooth submanifold itself.

It should be possible to apply the same strategy to all of the infinitely many monotone Lagrangian tori produced by Vianna [49], even though the explicit isotopy into a Darboux ball is more complicated. However, once this has been found, note that the Legendrian contact homology is possible to compute for Legendrians inside a Darboux ball by using Ekholm’s theory of gradient flow trees [21]. In this manner interesting enumerative invariants for these Legendrians are thus possible to compute (at least in theory). In particular, in Section 8.1 we formulate a conjecture relating the augmentation polynomial and the superpotential. Since the underlying Lagrangian live in pairwise different Hamiltonian isotopy classes, as detected by their different superpotentials, we expect that two different monotone tori also have different Legendrian lifts. We expect that this family of subloose Legendrian tori exhibits interesting properties reflecting the invariants of Vianna’s family. For instance, their augmentations varieties should correspond to certain threefold covers of the zero loci of the superpotentials of the tori in Vianna’s family. In general, not much is known, what varieties are possible to obtain as augmentation varieties of Legendrian surfaces in $\mathbb{R}^5$. Here we compute the invariants for the two first tori in the family. The Legendrian torus corresponding to the Clifford torus has augmentation variety being the one-dimensional pair of pants. This is in accordance with the verification of the mirror of the pair of pants done by Nadler in [42]. We will study these questions further in future investigations.

Along the way, we also provide the following reasonably general computational result:

**Theorem 1.4 (Theorem 5.6).** Let $\varepsilon_0, \varepsilon_1: (\mathcal{A}, \partial) \rightarrow (\mathcal{F}, 0)$ be two graded augmentations for a Legendrian oriented genus $g \geq 0$ surface $\Lambda \subset (\mathbb{R}^5, \xi_{st})$ of vanishing Maslov class and with all Reeb chords in positive degrees. Then

- when $\varepsilon_0 = \varepsilon_1$:
  \[
  LCH^{\varepsilon_0, \varepsilon_1}_k(\Lambda) = \begin{cases} 
  \mathcal{F}, & k = 2, \\
  \mathcal{F}^9, & k = 1, \\
  0, & k \neq 1, 2,
  \end{cases}
  \]
when \(0 \leq g \leq 1\) and \(\varepsilon_0 \neq \varepsilon_1\): \(LCH_k^{\varepsilon_0,\varepsilon_1}(\Lambda) = 0\) for all \(k\), is satisfied for the bilinearised Legendrian contact homology groups.

(In other words, in the case \(g = 1\), the \(LCH_*\) groups for augmentations behave as the \(\text{Ext}\)-groups for skyscraper sheaves on an algebraic curve under the above assumptions.) In addition, we obtain restrictions on the variety of augmentations of a Legendrian torus that satisfies the assumptions of Theorem 5.6.

**Theorem 1.5 (Theorem 5.8).** Let \(\Lambda \subset (\mathbb{R}^5, \xi_{\text{st}})\) be a Legendrian torus of vanishing Maslov class and with all Reeb chords in positive degrees. Its augmentation variety over \(\mathbb{C}\) is then either empty, or of complex dimension equal to one.

The restriction on the degree of Reeb chords given by Theorem 1.2 should imply that the two results above are applicable to the Legendrian lift of the canonical Bohr–Sommerfeld cover of an arbitrary monotone Lagrangian torus in \(\mathbb{C}P^2\).

Then in Section 6 we construct an infinite family of non-regular exact Lagrangian caps in the symplectisation of the contact Euclidean space of all odd dimensions \(\geq 5\). This provides a partial negative answer to the question of Eliashberg–Ganatra–Lazarev [31, 30]; see Question 6.2.

**Theorem 1.6 (Theorem 6.3).** For any \(g > 0\) and \(k_1, \ldots, k_r \geq 0\) there exists infinitely many different Legendrian isotopy classes of subloose Legendrian embeddings \(\Lambda \subset (\mathbb{R}^{2k_1+\ldots+2k_r+5}, \xi_{\text{st}})\) of the manifold \(S^{k_1} \times \ldots \times S^{k_r} \times \Sigma_g\), where \(\Sigma_g\) denotes the surface of genus \(g\), which:

- have vanishing Maslov classes;
- have Chekanov–Eliashberg algebras with Novikov coefficients \(R = \mathbb{C}[H_1(\Lambda)]\) that admit (0-graded) augmentations; and
- admit orientable exact Lagrangian caps inside the symplectisation with vanishing Maslov classes.

In particular, none of these Legendrian surfaces are loose.

In addition, we provide a refinement of the question of Eliashberg–Ganatra–Lazarev, see Question 6.5.

Finally, in Section 7 we investigate the “standard” Legendrian disc from the point of view of the standard Lefschetz fibration \(\mathbb{C}^n \to \mathbb{C}\). We conclude that:

**Theorem 1.7.** The Legendrian disc inside \((S^{2n+1}, \alpha_{\text{st}})\) given by the upper hemisphere of the standard Legendrian sphere, i.e.
\[
\Lambda_0 := \{x_{n+1} \geq 0\} \cap \mathbb{R}^{n+1} \cap S^{2n+1} \subset (S^{2n+1}, \alpha_{\text{st}}),
\]

admits a loose chart contained entirely in the complement of a neighbourhood of its boundary.

Note that \((S^{2n+1} \setminus \{x_{n+1} = 0\}, \xi_{\text{st}})\) is the local model of the trivial contact open book structure with page a critical Weinstein handle, i.e. the contact structure on \(\partial(D^2 \times DT^*B^n)\), with \(B^n\) the open ball, induced by the standard product Weinstein structure \((D^2 \times DT^*B^n, r^2d\theta \oplus pdq)\). We refer to [36] for the definitions of an open book in the settings of contact geometry. The Legendrian \(\Lambda_0\) can be identified with the Legendrian \(\{1\} \times 0_{B^n} \subset S^1 \times DT^*B^n\) that is
contained entirely in a page of this open book, where it moreover intersects the Lagrangian cocore $DT_0^*B^n$ of the handle transversely in a single point. The above theorem thus in particular reproves the following criterion for looseness established in [6, Proposition 2.9] by Casals–Murphy: inside a trivial open book $\partial(W \times D^2)$ with page being a Weinstein manifold $(W, d\eta)$, any Legendrian which is contained entirely inside a page and which intersects some cocore disc transversely in a single point is loose (i.e. a Legendrian that coincide with a closed regular Lagrangian inside a page $(W, d\eta)$).

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2. Background

2.1. Prequantisation spaces and Bohr–Sommerfeld Lagrangians. A prequantisation space is a contact manifold $(E, \alpha)$ equipped with a contact form $\alpha$ whose Reeb flow defines a free $S^1$-action. It follows that $(M = E/S^1, \omega = d\alpha)$ naturally is a symplectic manifold via symplectic reduction. Phrased differently, the $S^1$-bundle $\pi: E \to M$ has a connection 1-form $\alpha$ with a curvature 2-form given by $\omega$. We fix the convention that the lengths of the corresponding simple Reeb orbits on $(E, \alpha)$ all are equal to $\pi$, i.e. the fibre is canonically identified with $S^1 = \mathbb{R}/\pi\mathbb{Z}$ when parametrised by the Reeb flow. In this case the latter symplectic form then satisfies

$$\omega = \pi \cdot c_1(E) \in \pi \cdot \text{im}(H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R}))$$

where $c_1(E)$ denotes the first Chern class of the $S^1$-bundle.

Definition 2.1. A closed Lagrangian $L \subset (M, \omega)$ satisfies the Bohr–Sommerfeld condition for the (prequantisation) $S^1$-bundle $E \to M$ if the symplectic action class of $L$ satisfies

$$\omega \in \text{im}(\pi \cdot \text{im}(H^2(M, L, \mathbb{Z}) \to H^2(M, L, \mathbb{R})))$$

and the symplectic form is cohomologous to the multiple $[\omega] = \pi \cdot c_1(E)$ of the first Chern class of the $S^1$-bundle $E \to M$.

In the following, when it is clear to which prequantisation bundle we are referring, we will simply say that a Lagrangian immersion is Bohr–Sommerfeld.

The following lemma is standard.
Lemma 2.2. Any closed immersed Legendrian submanifold $\Lambda \subset (E, \alpha)$ projects to a Bohr–Sommerfeld Lagrangian immersion $\pi(\Lambda) \subset (M, \omega)$. If $M$ is simply connected, then the converse is true as well: any Bohr–Sommerfeld immersion lifts to a Legendrian immersion which is uniquely determined up to a global application of the Reeb flow.

Proof. The prequantisation bundle is flat when pulled back under a Lagrangian immersion. The Bohr–Sommerfeld condition together with the simple connectivity implies that the monodromy of the parallel transport moreover is trivial along any loop in the Lagrangian. It is thus possible to construct the lift as a parallel section along the immersion. \qed

The Legendrian lift is in fact an embedding given that a condition on the double points of the Lagrangian immersion is met; this can always be made to hold after a small generic perturbation through Bohr–Sommerfeld Lagrangian immersions.

For simplicity we will in the following assume that $\pi_1(M) = 0$. In this case, any symplectic form satisfying the above cohomological condition can conversely be seen to give rise to a line bundle with the structure of a prequantisation space.

The prequantisation spaces that we will study here are the following.

- The standard round contact sphere $S^{2n-1} \subset \mathbb{C}^n$ equipped with the coordinates $(z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n)$ and the contact form $\alpha_{st} := \frac{1}{2} \sum_{i=1}^n (x_idy_i - y_idx_i)$.

The corresponding symplectic reduction is the quotient under the Hopf map, which gives $M = \mathbb{C}P^{n-1}$ equipped with the Fubini–Study Kähler form $\omega_{FS}$ for which the symplectic area of a line has been normalised to $\int_A \omega_{FS} = \pi$. This is also the length of a minimal periodic Reeb orbit for the above contact form.

- The unit cotangent bundle $(UT^*S^n, pdq)$ of the round $n$-sphere of radius $1/2$ has a completely periodic Reeb flow with minimal length of a periodic orbit equal to $\pi$. The symplectic reduction gives $UT^*S^n$ the structure of a prequantisation space over the monotone projective quadric $(n-1)$-fold. In particular, when $n = 3$, this unit cotangent bundle is a prequantisation space over the monotone quadric surface $(\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{FS} \oplus \omega_{FS})$.

The contact structures of the prequantisation spaces in the above examples both have vanishing first Chern classes $c_1(E, \alpha) \in H^2(E)$. This is of course obvious for the sphere, while it is also a general fact for all unit cotangent bundles. For completeness we also recall the following standard result; since the above examples are prequantisation spaces over monotone symplectic manifolds, it can also be used to deduce the vanishing of $c_1(E, \alpha)$.

Lemma 2.3. Suppose that $c_1(M, \omega)$ vanishes on any $A \in \pi_2(M)$ with $\int_A \omega = 0$. Then $c_1(E, \alpha)$ vanishes on $\pi_2(E)$.
Remark 2.4. Even more can be said: prequantisation spaces over simply connected monotone symplectic manifolds have torsion first Chern class, as follows from the argument of [14, Lemma 7.3].

Here follows basic examples of Lagrangian Bohr–Sommerfeld immersions inside the projective space and quadric.

Example 2.5. (1) The anti-diagonal \( \{(z, \bar{z}) \in \mathbb{C}P^1 \times \mathbb{C}P^1 \} \) is an embedded Lagrangian sphere and it is thus automatically Bohr–Sommerfeld. In general, every affine complex quadric is symplectomorphic to \( (T^*S^n, d(pdq)) \) and thus it contains an embedded Lagrangian sphere as well (i.e. the zero section). The Legendrian lift can be readily seen to be Legendrian isotopic to a unit cotangent fibre \( UT^*_\text{pt}S^n \).

(2) The standard Legendrian sphere \( \Lambda_{\text{std}} \subset (\mathbb{S}^{2n+1}, \alpha_{\text{st}}) \) is the intersection of the sphere with the real part \( \mathbb{R}\mathbb{C}^{n+1} \). This Legendrian is the lift of the twofold cover of \( \mathbb{R}P^n \subset \mathbb{C}P^n \), which thus is a Bohr–Sommerfeld immersion. It can be readily seen to be Legendrian isotopic to a representative inside a contact Darboux ball which has a rotationally symmetric front projection given by the “flying saucer” with precisely one Reeb chord, and singularities consisting of a spherical cusp edge. The Lagrangian projection of this representative is the so-called (exact Lagrangian) Whitney immersion of a sphere with a single transverse double point.

2.2. Lagrangians and Legendrians from Lefschetz fibrations. It will be convenient for us to describe Lagrangians inside \((\mathbb{C}P^n, \omega_{\text{FS}})\), as well as Legendrian lifts to the prequantisation space \((\mathbb{S}^{2n+1}, \alpha_{\text{st}})\), via their images under the standard Lefschetz fibration \( \mathbb{C}^n \to \mathbb{C} \)

\[
(z_1, \ldots, z_n) \mapsto z_1^2 + \ldots + z_n^2
\]

in an affine chart of the symplectic base \( \mathbb{C}P^n \). To consider Lagrangians in relation to a Lefschetz fibration has turned out to be a useful perspective, which goes back to the work [34] by Eliashberg–Polterovich. We call a Lagrangian which projects to a curve under a symplectic fibration compatible with the fibration.

We proceed to give details in the case \( n = 2 \); the general case needed in Section 7 is treated analogously in the end of this subsection.

In dimension \( n = 2 \) we choose coordinates so that the Lefschetz fibration becomes \( (z_1, z_2) \mapsto z_1 \cdot z_2 \). In order to fix notation, we endow \( \mathbb{C}^2 \) with the standard linear symplectic form \( \omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 \). For any \( r > 0 \) and primitive \( a = (a_1, a_2) \in \mathbb{Z}^2 \) we construct the three-dimensional hypersurface

\[
\Sigma_{a,r} := \{z(a_1, a_2); z \in \mathbb{C}^*, \} \cap D^4_r \subset \mathbb{C}^2 \setminus \{0\}.
\]

This hypersurface is smooth and foliated by holomorphic lines. In addition it is foliated by the Lagrangian product tori \( a_1S^1_s \times a_2S^1_s \) for \( s \in \left(0, \frac{r}{\|a\|}\right] \).

The symplectic fibration \( (z_1, z_2) \mapsto z_1^{a_1} \cdot z_2^{a_2} \) with a singularity at the origin restricts to a surjective and smooth \( S^1 \)-fibration

\[
f_a : \Sigma_{a,r} \to D^2_r \setminus \{0\}
\]
of the radius
\[ \tilde{r} = a_1^{a_2} a_2^{a_1} (r/\|a\|)^{a_1+a_2}. \]

The following lemma allows us to easily understand the Lagrangians inside \( \Sigma_{a,r} \) via the fibration \( f_a \).

**Lemma 2.6.** The characteristic distribution \( \ker(\omega|_{T\Sigma_{a,r}}) \) is spanned by the infinitesimal generator of the \( \mathbb{R} \)-action
\[ (z_1, z_2) \mapsto (e^{ia_2 t} z_1, e^{-ia_1 t} z_2), \]
with \( t \in \mathbb{R} \), i.e. it is tangent to the fibres of \( f_a \).

In particular, by dimensional reasons, any (two-dimensional) Lagrangian immersion that is contained inside \( \Sigma_{a,r} \) projects to a smooth immersed curve inside \( D_r^2 \setminus \{0\} \) under \( f_a \) and vice versa: since the fibres of \( f_a \) are tangent to the characteristic distribution, the preimage of any curve inside the base \( D_r^2 \setminus \{0\} \) under the same map is a Lagrangian immersion of \( \Sigma_{a,r} \).

In the following we will be interested in \( \Sigma_{a,r} \) with \( a = (1,1) \), in which case we will drop the subscript \( a \). To be able to study the symplectic action properties of such Lagrangians, the following simple computation is useful.

**Lemma 2.7.** Denote by \( \omega_0|_{T\Sigma_r} \) the pullback to \( \Sigma_r \subset \mathbb{C}^2 \) of the standard symplectic form
\[ \omega_0 = d\left( \frac{r_1^2}{2} d\theta_1 + \frac{r_2^2}{2} d\theta_2 \right) \]
on \( \mathbb{C}^2 \). Then \( \omega_0|_{T\Sigma_r} = f^* d(r/2 \, d\theta) \), where \( d(r/2 \, d\theta) \) is a symplectic form on \( D_r^2 \setminus \{0\} \) of total area \( \pi \tilde{r} \).

It will be necessary to consider a certain compactification of the symplectic hypersurfaces \( \Sigma_r \), but rather of certain compactifications. For any primitive vector \( n = (n_1, n_2) \in \mathbb{Z}^2 \) for which \( n_1 a_1 + n_2 a_2 \neq 0 \) we now construct \( \Sigma_n^{a,r} \) from \( \Sigma_{a,r} \) by taking the quotient of its Lagrangian boundary
\[ \partial\Sigma_{a,r} = a_1 S_r^{1/\|a\|} \times a_2 S_r^{1/\|a\|} \]
under the action of the closed subgroup
\[ e^t n \subset \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \cong a_1 S_r^{1/\|a\|} \times a_2 S_r^{1/\|a\|}. \]
The compactification \( \Sigma_n^{a,r} \) can explicitly seen to again be the the hypersurface of a symplectic manifold.

**Example 2.8.** The projective plane \( \mathbb{C}P^2 \) with the Fubini–Study symplectic form can be obtained by performing symplectic reduction to the boundary \( \partial B^4 = S^3 \) of the standard symplectic ball \( (B^4, \omega_0) \). The symplectic reduction is the quotient under the Hopf fibration, which when restricted to \( \Sigma_{(1,1),1} \) produces the space \( \Sigma_{(1,1),1} \).

The image of the torus \( \partial\Sigma_{a,r} \rightarrow \Sigma_n^{a,r} \) inside the above compactification is an embedded closed curve \( S_n^{a,r} \subset \Sigma_n^{a,r} \). It has a neighbourhood given as a disc normal-bundle \( D^2 \times S \) with
the following explicit description. Choose \( c = (c_1, c_2) \in \mathbb{Z}^2 \) for which \( n_1 c_2 - n_2 c_1 = 1 \) (here we use that \( n \) is primitive). Then compactify the family

\[
\left\{(a_1 \cdot s \cdot e^{i(n_1 \varphi + c_1 \theta)}, a_2 \cdot s \cdot e^{i(n_2 \varphi + c_2 \theta)}); \quad \frac{r}{\|a\|} - \epsilon \leq s \leq \frac{r}{\|a\|} \right\} \subset \Sigma_{a,r}
\]

of symplectic annuli; each annulus are parametrised by \((r, \varphi)\) while the family is parametrised by \( \theta \in [0, 2\pi) \), which collapses each boundary components \( \{s = r/\|a\|\} \) to a point. Hence, the above family of annuli becomes naturally identified with a solid torus \( D^2_x \times S^n_{a,r} \) inside the quotient foliated by the symplectic discs \( D^2_x \times \{pt\} \). The symplectic condition means that the characteristic distribution of \( \Sigma_{a,r} \) is transverse to these disc leaves.

**Lemma 2.9.** The characteristic distribution on the solid torus \( D^2_x \times S^n_{a,r} \) integrates to a symplectic monodromy of any \( D^2_x \times \{pt\} \) given by the rotation

\[
\varphi \mapsto \varphi + \frac{2\pi}{a_1 n_1 + a_2 n_2}
\]

with respect to the above coordinates, where \( a_1 n_1 + a_2 n_2 \) is the intersection number between the two curves \( t \cdot (a_2, -a_1) \) (i.e. an integral curve of the characteristic distribution) and \( t \cdot n \) on the torus \( \mathbb{R}^2/\mathbb{Z}^2 \).

Any Lagrangian submanifold contained inside \( \Sigma^n_{a,r} \) must be tangent to the characteristic distribution. The above lemma thus in particular implies that

**Lemma 2.10.** Any properly immersed Lagrangian contained inside \( B^2_x \times S^n_{a,r} \) without boundary must intersect each \( B^2_x \times \{pt\} \) in a curve which is invariant under the rotation

\[
\varphi \mapsto \varphi + \frac{2\pi}{a_1 n_1 + a_2 n_2}
\]

with respect to the above coordinates. Conversely, any such submanifold is Lagrangian.

Our main technique will be to present Lagrangians inside \( \Sigma^{(1,1)} \subset \mathbb{C}P^2 \) that are disjoint from the divisor \( \ell_\infty \) at infinity as the curves being their images under \( f \). Regular homotopies through Bohr–Sommerfeld Lagrangians will then be constructed by families of such curves, while making careful area configurations considerations using Lemma 2.7. In order to describe a Lagrangian regular homotopy that traverses the line at infinity \( \ell_\infty \cap \Sigma^{(1,1)} \) the projection \( f \) cannot be used; we instead use the presentation of the Lagrangian in the normal bundle over the divisor given by Lemma 2.10.

In the above case we will write \( \Sigma := \Sigma^{(1,1)} \setminus \Sigma^{(1,1)} \), \( \Sigma := \Sigma^{(1,1)} \setminus \Sigma^{(1,1)} \), and \( f := f^{(1,1)} : \Sigma \to D^2 / \Sigma \) for the restriction of the Lefschetz fibration.

In the case \( n > 2 \), the subset \( \Sigma \subset \mathbb{C}P^n \) that we are interested in is the quotient of the real \((n + 1)\)-dimensional submanifold

\[
\Sigma := \{z(x_1, \ldots, x_n); \quad z \in S^1 \subset \mathbb{C}^*, x_i \in \mathbb{R}\} \cap D^{2n}
\]

under the symplectic reduction \( D^n \to \mathbb{C}P^n \) along the boundary \( S^{2n-1} = \partial D^{2n} \). Again the characteristic distribution are the tangencies to the \( S^{n-1}\)-fibres of the symplectic Lefschetz
Figure 2. The standard momentum polytope for $\mathbb{C}P^2$, the fibre over the point $(\pi/3, \pi/3)$ (shown in blue) is the Clifford torus while the subset over the line $\{u = v\}$ (shown in green) is symplectomorphic to $\Sigma^n_{(1,1),1}$ with $n = (1, 1)$.

fibration $z_1^2 + \ldots + z_n^2$ restricted to $\Sigma$. Observe that there is a similar description in the case $n = 2$ as well which is equivalent to the previous one after a suitable coordinate change.

2.3. The Chekanov–Eliashberg algebra. Here we give a brief recollection of the Chekanov–Eliashberg algebra in the contact manifold $(\mathbb{C}^n \times \mathbb{R}, dz - ydx)$ as developed in [25]. More precisely, the algebraic formalism from [26] will be used (called the “fully noncommutative Legendrian DGA”) where the Novikov coefficients do not commute with the Reeb chord generators; also c.f. [9]. For tori, this DGA computes partially wrapped Floer homology where the Legendrian is used as a stop [29].

Let $\Lambda \subset \mathbb{C}^n \times \mathbb{R}$ be a closed Legendrian submanifold with a set $Q$ of Reeb chords, which are assumed to be finite. Denote by $R := \mathbb{F}[H_1(\Lambda)]$ the group ring of $H_1(\Lambda)$ over the field $\mathbb{F}$; there is also a weaker version of the invariant where we take $R := \mathbb{F}$. We can thus consider the free graded $R$-bimodule $A$ generated by $Q$, where the grading is induced by the Conley–Zehnder index as in [25]. The underlying unital algebra of the Chekanov–Eliashberg DGA in our setting is the tensor ring

$$A(\Lambda) := \bigoplus_{k \geq 0} A^{\otimes R_k}$$

where

$$A^{\otimes R_0} = R,$$

$$A^{\otimes R_k} := \underbrace{A \otimes_R \cdots \otimes_R A}_k, \quad k \geq 1,$$

all are $R$-bimodules. The contributions from the homotopy classes of the pseudoholomorphic discs in the definition of the differential $\partial$ is then determined by auxiliary choices of capping paths from each of the two endpoints of every Reeb chord to a fixed based point $\star \in \Lambda$. We refer to [26, 9] for more details.
Recall that an **augmentation** is a unital DGA-morphism

\[ \varepsilon : \mathcal{A}(\Lambda) \rightarrow \mathbb{F} \]

which thus satisfies \( \varepsilon \circ \partial = 0 \). Here we will only consider **graded augmentations** which by definition vanish on all generators in nonzero degrees. Observe that an augmentation restricts to a unital algebra map

\[ \varepsilon : \mathbb{F}[H_1(\Lambda)] \rightarrow \mathbb{F}, \]

which can be identified with a local system in \( \mathbb{F} \) when \( \pi_1(\Lambda) \) is abelian. Following [43] we define the **augmentation variety** to be the Zariski closure of the set of points in \( \text{Sp}(\mathbb{F}[H_1(\Lambda)]) \), i.e. unital \( \mathbb{F} \)-algebra maps \( \mathbb{F}[H_1(\Lambda)] \rightarrow \mathbb{C} \), which extend to an augmentation via the canonical inclusion \( \mathbb{F}[H_1(\Lambda)] \subset \mathcal{A}(\Lambda) \).

Given a pair of augmentations, Bourgeois and Chantraine defined the **bilinearised Legendrian contact homology** in [5], which is a chain complex with underlying vector space \( Q^\mathbb{F} \). In the setting of the fully noncommutative Legendrian DGA we refer to [9] for more details.

### 2.4. Lagrangian cobordisms and subloose Legendrian submanifolds.

The class of loose Legendrian submanifolds of dimension two and more was introduced by Murphy in [41]. By definition a Legendrian is loose if one can find a so-called **loose chart**, which is a contact Darboux ball where the Legendrian is in a particular position. In the aforementioned article an h-principle was established for this class of Legendrians, which implies that their classification up to Legendrian isotopy is determined by their formal Legendrian isotopy classes.

The Chekanov–Eliashberg algebra of a loose Legendrian can be computed, in some suitable standard model, to be acyclic. This is equivalent to the unit being a boundary. Note that an acyclic DGA admits no augmentations.

We now describe a loose chart that is well-behaved with respect to the restricted Lefschetz fibration \( f : \Sigma \rightarrow \mathring{D}^2_{1/2} \) given in Section 2.2. Consider a Legendrian inside \((S^{2n+1}, \alpha_{\text{st}})\) whose Lagrangian projection in \( \mathbb{C}P^n \) has the following description. Inside some \( f^{-1}(U) \subset \Sigma \) for an open domain \( U \subset \mathring{D}^2_{1/2} \) with smooth boundary, the projection of \( \Lambda \cap f^{-1}(U) \) under \( f \) is a single arc \( \gamma \) intersecting \( \partial U \) transversely in two points, where this arc moreover has a single transverse self-intersection; see Figure 3. Let \( A \subset U \setminus \gamma \) be the connected component which is bounded in \( \mathbb{C} \setminus \gamma \) while \( B \subset U \setminus \gamma \) is the connected component which is adjacent to \( A \) only at the double-point of \( \gamma \); again c.f. Figure 3.

**Lemma 2.11.** If the \((1/2)dr \wedge d\theta\)-area of \( A \) is strictly smaller than that of \( B \), then we can assume that \( A \) is arbitrarily small after a compactly supported Legendrian isotopy supported inside \( f^{-1}(U) \subset \Sigma \). In this case, \( \Lambda \) is moreover loose.

**Proof.** The compactly supported Legendrian isotopy is readily constructed explicitly as the lifts of \( f^{-1}(\gamma_t) \) for a suitable deformation \( \gamma_t \) of the curve \( \gamma_0 = \gamma \).

After a suitable such isotopy one can then show that there exists a neighbourhood of the form

\[ (D_\varepsilon T^*S^{n-1} \times [-\varepsilon, \varepsilon]^3, -pdq - ydx + dz) \hookrightarrow (S^{2n+1}, \alpha_{\text{st}}) \]
Figure 3. A loose chart described by using the standard Lefschetz fibration, in the sense that \( f(\Lambda) = \gamma \) is an arc with a single self-intersection. The area of the region \( A \) must be greater than the area of the region \( B \).

in which the Legendrian coincides with \( 0_{S^n} \times \Lambda_{\text{stab}} \), for a stabilised Legendrian arc \( \Lambda_{\text{stab}} \subset ([-\epsilon, \epsilon]^3, dz-ydx) \). I.e. the Legendrian coincides with the \( S^{n-1} \)-spun of a stabilised Legendrian arc in some neighbourhood, which implies looseness \([19]\).

Definition 2.12. Given two closed Legendrian submanifolds \( \Lambda_- \) and \( \Lambda_+ \) of a contact manifold \((M, \alpha)\). An **exact Lagrangian cobordism from \( \Lambda_- \) to \( \Lambda_+ \)** is a properly embedded submanifold \( L \subset (\mathbb{R} \times M, d(e^t \alpha)) \) in the symplectisation such that for some \( T > 0 \)

(i) \( L \cap (-\infty, -T) \times M = (-\infty, -T) \times \Lambda_- \) and \( L \cap (T, +\infty) \times M = (T, +\infty) \times \Lambda_+ \),

(ii) \( L \cap [-T, T] \times M \) is compact.

(iii) there is a function \( f_L \in C^\infty(L) \) such that

- \( e^t \alpha|_{TL} = df_L \),
- \( f_L|(-\infty, -T) \times \Lambda_- \), \( f_L|(T, +\infty) \times \Lambda_+ \) are constant functions.

We call \( (T, +\infty) \times \Lambda_+ \) and \( (-\infty, -T) \times \Lambda_- \) the **positive end** and the **negative end** of \( L \), respectively.

Recall the fact established in [22] that an exact Lagrangian cobordism from \( \Lambda_- \) to \( \Lambda_+ \) induced a unital DGA morphism

\[ \Phi: (\mathcal{A}(\Lambda_+), \partial_+) \to (\mathcal{A}(\Lambda_-), \partial_-) \]

with the following caveat: the Novikov “coefficients” from the cobordism must be used in the definition of both Chekanov–Eliashberg algebras. In particular, note that an augmentation of \( \Lambda_- \) can be **pulled back** to an augmentation of \( \Lambda_+ \) (again under the above caveat concerning the coefficients used). This means that an exact Lagrangian **filling** of \( \Lambda \), i.e. an exact cobordism with \( \Lambda_- = \emptyset \) and \( \Lambda_+ = \Lambda \), induces an augmentation of \( \mathcal{A}(\Lambda) \) with coefficients in \( R = \mathbb{F} \).
Definition 2.13. A Legendrian submanifold \( \Lambda \) is said to be \textbf{subloose} if there exists an exact Lagrangian cobordism from \( \Lambda \) to a loose Legendrian submanifold.

Remark 2.14. Observe that according to the definition of a subloose Legendrian, every loose Legendrian submanifold is subloose (taking a trivial exact Lagrangian cobordism \( \mathbb{R} \times \Lambda \) of a loose Legendrian \( \Lambda \), we see that \( \Lambda \) is subloose).

Proposition 2.15. Any subloose Legendrian has an acyclic Chekanov–Eliashberg algebra without Novikov coefficients, i.e. when we take \( R = \mathbb{F} \). In particular, it admits no exact Lagrangian filling.

Proof. The exact Lagrangian cobordism induces a DGA morphism from the trivial ring to the homology of the Chekanov–Eliashberg algebra of the subloose Legendrian. (Here it is important that we do not use Novikov coefficients.) The result then follows from the algebraic fact: if a given ring admits a unital map from the trivial ring (i.e. for which \( 1 = 0 \)) then the former ring is trivial as well. \( \square \)

In Section 6 we show that the Legendrian tori \( \Lambda_{Cl} \) and \( \Lambda_{Ch} \) both are subloose. This, however, does not mean that their Chekanov–Eliashberg algebras are uninteresting. As computed in Section 5 they even admit augmentations when Novikov coefficients are used. (While they a priori must have acyclic DGAs without Novikov coefficients by Proposition 2.15)

3. Canonical Bohr–Sommerfeld covers

In this section we present some fairly general considerations on how to obtain canonically defined Bohr–Sommerfeld covers from rational Lagrangian embeddings, and how to compute the Maslov classes of their lifts. Theorem 1.2 then follows as a special case of this. Here we assume that \( (E, \alpha) \) is a given prequantisation bundle on \( (M, \omega) \); when talking about Bohr–Sommerfeld Lagrangians it is with respect to this choice of bundle.

3.1. Bohr–Sommerfeld covers. Any Lagrangian submanifold whose symplectic action class satisfies

\[
[\omega] \in \frac{\pi}{k} \cdot \text{im}(H^2(M, L, \mathbb{Z}) \rightarrow H^2(M, L, \mathbb{R}))
\]

for some \( k = 1, 2, 3, \ldots \), admits a canonical \( k \)-fold cover which is a Bohr–Sommerfeld Lagrangian immersion. Here we describe the construction.

For simplicity we now assume that \( \pi_1(L) = H_1(L, \mathbb{Z}) \) is abelian and that \( H_1(M, \mathbb{Z}) = 0 \). The symplectic action class \( \sigma: H_2(M, L, \mathbb{Z}) \rightarrow \mathbb{R} \) descends to a well-defined morphism

\[
\bar{\sigma}: H_1(L, \mathbb{Z}) \rightarrow \left( \frac{\pi}{k} \cdot \mathbb{Z} \right) / (\pi \cdot \mathbb{Z}) \cong \mathbb{Z}_k,
\]

which we assume is surjective.

Consider the subgroup \( \ker \bar{\sigma} \subset H_1(L, \mathbb{Z}) \) and the corresponding \( k \)-fold cover \( \tilde{L} \rightarrow L \subset M \), which can be considered as a Lagrangian immersion \( \tilde{L} \looparrowright (M, \omega) \).
Definition 3.1. In the case when \( k = 1, 2, \ldots \) is chosen to be minimal with the property that Equation (3.1) is satisfied, then we will call \( \tilde{L} \) the canonical Bohr–Sommerfeld cover of \( L \).

From Lemma 2.2 together with the main theorem of [3] (in order to ensure embeddedness), we deduce that:

**Proposition 3.2.** Assume that \( L \) is a closed embedded Lagrangian submanifold of \( (M, \omega) \) that satisfies Equation (3.1). Its canonical \( k \)-fold Bohr–Sommerfeld cover \( \tilde{L} \) then admits an embedded Legendrian lift \( \Lambda \) of \( (E, \alpha) \).

### 3.2. Computing the Maslov class and degrees

In order to compute the Maslov class of a Legendrian lift produced by Proposition 3.2 above we use the following general result. Here we assume that the Chern class of the contact manifold \( (E, \alpha) \) vanishes. This is the case for both the contact sphere and unit cotangent bundles. Further we assume that \( \pi_1(M) = 0 = \pi_1(E) \), which again is the case for the contact spheres as well as for unit cotangent bundles of spheres of dimension at least three. Now consider a Bohr–Sommerfeld immersion \( \iota: L \hookrightarrow (M, \omega) \) together with a Legendrian lift \( \Lambda \subset (E, \alpha) \).

Recall the definition of the Bott version of the Conley–Zehnder index \( \mu(\gamma) \) of a periodic Reeb orbit from [15] (c.f. Remark 5.4 in the same) and [4], which also is called the “Maslov index.” Its significance is as follows: the integer \( \mu(\gamma) \) of a periodic Reeb orbit in \( \gamma \) of \( (E, \alpha) \) which is non-degenerate in the Bott sense is equal to the expected dimension of the moduli space of pseudoholomorphic planes inside the symplectisation \( (\mathbb{R} \times E, d(e^t\alpha)) \) that are asymptotic to the given orbit, for an almost complex structure which is cylindrical with respect to \( \alpha \).

**Example 3.3.** When \( \gamma \) is the periodic Reeb orbit of minimal period \( \pi \) inside \( (S^{2n+1}, \alpha_{st}) \) and \( UT^*S^n \) we compute \( \mu(\gamma) = 2n \) and \( \mu(\gamma) = 2(n-1) \), respectively. For the latter computation, recall that \( UT^*S^n \) is the prequantisation of a projective conic of complex dimension \( n-1 \), and that the dimension of the aforementioned moduli space of planes coincides with the dimension of lines inside \( CP^{n-1} \) that are contained inside an affine conic of dimension \( 2n \) and which are asymptotic to a fixed point at the projective conic at infinity.

**Proposition 3.4.** Let \( A \in H_2(M, \iota(L)) \) be a relative cycle with boundary in class \( \iota_*(\eta) \) for some \( \eta \in H_1(L) \). The Maslov classes of \( L \) and its Legendrian lift \( \Lambda \subset (E, \alpha) \) are related by the formula

\[
\mu_M(A) = \mu_E(\eta) + \frac{\mu(\gamma)}{\pi} \int_A \omega
\]

where \( \mu(\gamma) \) denotes the Bott version of the Conley–Zehnder index of a Reeb orbit of \( \alpha \) of smallest period (this period is \( \pi \) with our convention).

**Proof.** We start by noting that both the determinant bundle of \( TM \) and the line bundle \( E \) become trivial over the chain \( A \).

The restriction to the boundary of \( A \) of the Legendrian lift of \( L \) is a section that has winding number \( \omega = \frac{1}{\pi} \int_A \omega \) proportional to the curvature in this trivialisation of \( E \). In
order to lift the chain $A$ to $E$ with boundary on $\Lambda$ we must thus first remove a number $w$ of points on the chain $A$, and make it into a chain with a number $w$ of additional boundary component on Reeb orbits of minimal period on $(E, \alpha)$ parametrised by the backwards Reeb flow.

The statement finally follows from the fact that the trivialisations of along the Reeb orbit induced by a frame of $TM$ at the point beings its projection, and the trivialisation induced by a capping disc inside $E$ have determinant lines that differ by precisely by the winding number $\mu(\gamma)$. □

Finally we compute the degree of the Reeb chords on the above Legendrian lift. Since the prequantisation space is foliated by Reeb chords for the prequantisation form, this is of course a very degenerate situation. In addition, since the projection of the Legendrian has an embedded image, all Reeb chords on the Legendrian $\Lambda$ automatically come in $\dim \Lambda$-dimensional Bott families. By the (Bott) version of the degree of such a Reeb chord we mean the expected dimension of a pseudoholomorphic half-plane inside the symplectisation $(\mathbb{R} \times E, d(e^t\alpha))$ that has boundary on $\mathbb{R} \times \Lambda$ and which is asymptotic to a fixed orbit inside the Bott family, where the almost complex structure is cylindrical.

**Proposition 3.5.** Denote by $\Lambda \subset (E, \alpha)$ a Legendrian lift of the canonical $k$-fold Bohr–Sommerfeld cover of a monotone Lagrangian embedding in $(M^{2n}, \omega)$. The degree of a Reeb chord on $\Lambda$ which has length $l\pi/k$, $l \in \mathbb{N}$, is equal to $-2 + \mu + 1$, where $\mu$ is the Maslov index of a disc $D$ inside $M$ with boundary on $L$ and of symplectic area $\int_D \omega = l\pi/k$.

**Proof.** Assume the existence of a disc $D$ with the given symplectic area $l\pi/k$. Fix a point $pt \in L$ through which $\partial D$ is passing, and now consider $D$ as a half-plane (i.e. a punctured disc) with boundary on $L$. By curvature considerations it thus follows that there is a trivialisation of $E$ along the half-plane in which $\Lambda$ is a constant section over the boundary, and such that the constant section over the puncture parametrises the given Reeb chord of length $l\pi/k$ over the point $pt$.

In other words, the disc $D$ lifts to a half-plane with boundary on $\mathbb{R} \times \Lambda$ inside the symplectisation, such that it has a positive puncture asymptotic to the Reeb chord on $\Lambda$ of length $l\pi/k$ under consideration. It thus remains to compute the expected dimension of the pseudoholomorphic half-planes in this class, with the asymptotic constraint to the same Reeb chord.

Recall that $-2 + \mu$ is the expected dimensions of pseudoholomorphic discs in $M$ in the class under consideration with boundary on $L$ and a single boundary-point constraint at $pt \in L$; see e.g. [2]. The expected dimension of the lift of the disc to the symplectisation can be seen to be equal to $-2 + \mu + 1$, where the additional degree of freedom is due to the presence of the symmetry by translations in the symplectisation. To that end, one must again use the properties of the trivialisation of $E$ along $D$ as established in the first paragraph of the proof. □
4. Some concrete Lagrangian tori and their lifts

It is now time to restrict attention to some particular examples of Legendrian tori produced by Proposition 3.2 as lifts of canonical Bohr–Sommerfeld covers.

4.1. Lifts of monotone tori in the quadric surface. The quadric surface is cut out by the polynomial equation \( \{ z_1^2 + z_2^2 + z_3^2 = 1 \} \subset \mathbb{C}P^3 \). We endow this variety with the restriction of the Fubini–Study symplectic form, which descends to the standard monotone symplectic form \((\{ z_1^2 + z_2^2 + z_3^2 = 1 \}, \omega_{\text{FS}}) \) under a suitable identification. The affine part of this variety \( \{ z_1^2 + z_2^2 + z_3^2 = 1 \} \subset \mathbb{C}^3 \) is symplectomorphic to an open subset \((DT^*S^2, d(pdq))\) of the cotangent bundle. We consider the prequantisation space \( E \) which is the unit cotangent bundle \((E, \alpha) = (UT^*S^3, pdq)\) of the round three-sphere of radius \( 1/2 \). This bundle can be identified with the \( S^1 \)-bundle corresponding to the complex line-bundle \( O(-1, -1) \) over the conic.

Recall that a Lagrangian is said to be monotone if its symplectic action class is a non-negative multiple of its Maslov class. Any monotone Lagrangian torus inside \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) thus has a symplectic action class which lives inside \( \pi_2 \cdot \text{im}(H^2(\mathbb{C}P^1 \times \mathbb{C}P^1, L, \mathbb{Z}) \to H^2(\mathbb{C}P^1 \times \mathbb{C}P^1, L, \mathbb{R})) \).

In fact, the symplectic action class cannot be an integer multiple of \( \pi \) by the result [15, Theorem 1.21] by Cieliebak–Mohnke.

The most basic example is the monotone product torus \( S^1 \times S^1 \subset \mathbb{C}P^1 \times \mathbb{C}P^1 \), which we now proceed to direct our attention to. (This torus also goes under the name of “the Clifford torus,” but in order to avoid confusion we will not call it by that.)

Recall that given a knot \( K \) in \( S^3 \), one can define the conormal lift of it in the following way. First one takes \( L_K \) defined by

\[
L_K = \{ (q, p) \mid q \in K, \langle p, v \rangle = 0 \text{ for all } v \in T_q K \} \subset T^*S^3.
\]

It is a standard exercise to check that \( L_K \) is a Lagrangian submanifold of \( T^*S^3 \). Then one takes the unit cotangent bundle \( UT^*S^3 \) of unit covectors of \( S^3 \) with respect to some metric. The conormal lift \( \Lambda_K \) is given by \( \Lambda_K = L_K \cap UT^*S^3 \), and it is a Legendrian torus in \( UT^*S^3 \).

**Theorem 4.1.** Under the identification of \( E \) with \((UT^*S^3, pdq)\) the Legendrian lift of the twice covered monotone product torus is Legendrian isotopic to the conormal lift of the unknot.

**Proof.** The monotone product torus \( S^1 \times S^1 \subset \mathbb{C}P^1 \times \mathbb{C}P^1 \) given as the product of equators is invariant under the standard Hamiltonian \( \mathbb{T}^2 \)-action on the conic (which is toric). One can readily check that this \( \mathbb{T}^2 \)-action lifts to strict contactomorphisms of the prequantisation space, and that the Legendrian lift of the canonical Bohr–Sommerfeld cover is invariant under the latter \( \mathbb{T}^2 \)-action.

From the point of view of \( UT^*S^3 \), this \( \mathbb{T}^2 \)-action is induced by lifting the restriction to \( S^3 \) of the standard toric \( \mathbb{T}^2 \)-action on \( \mathbb{C}^2 \) to the cotangent bundle. (Observe that this \( \mathbb{T}^2 \)-action
is by isometries on the round three-sphere.) We refer to e.g. [1, Section 6.1] for a careful treatment of this contact toric structure on $UT^*S^3$. It follows that any Legendrian which is invariant under this $T^2$-action must thus live above a (possibly degenerate) torus in the base of the form $S^1_a \times S^1_b \subset S^3$ with $a^2 + b^2 = 1$. In other words, the Legendrian lift of the Bohr–Sommerfeld cover as considered is the conormal lift of one of these tori.

Finally, since the one-sided conormal lifts of all these tori can be seen to be Legendrian isotopic to the conormal lift of an unknot $S^1 \times \{0\} \subset S^3$ by a suitable application of the Reeb flow induced by the round metric, the result now follows. □

Remark 4.2. Alternatively, the result can be proven following the same recipe as in the proof of Theorem 4.4 below in Section 4.3. There the Legendrian isotopy is obtained by describing a homotopy of Bohr–Sommerfeld Lagrangian immersions via their images under a Lefschetz fibration. In this case one then uses the Lefschetz fibration which on the affine part $\{z_1^2 + z_2^2 + z_3^2 = 1\} \subset \mathbb{C}^3$ of the conic is equal to the restriction of $(z_1, z_2, z_3) \mapsto z_3$. In fact, the present situation is significantly simpler compared to the situation in Theorem 4.4 due to the fact that this defines a Lefschetz fibration on the full conic, and not just its affine part. In order to identify the conormal lift of the unknot, we refer to [26], where it is shown that a suitable representative has a Lagrangian projection inside the conic that lives above an immersed figure-8 curve that encircles both critical values in the Lefschetz fibration. The proof below instead uses a more direct approach.

There are infinitely many monotone Lagrangian tori inside the quadric surface by a result [50] due to Vianna. We do not know if any other of these tori have a lift that is isotopic to the conormal lift of a smooth knot.

In any case, the above Legendrian lift of the product torus is not subloose since the conormal lift of the unknot admits an exact Lagrangian filling inside $(T^*S^3, d(pdq))$; e.g. the Lagrangian conormal of the knot. In fact, since the canonical Bohr–Sommerfeld covers of monotone Lagrangians in the conic are two-fold covers, it is not difficult to show the following.

**Proposition 4.3.** Any monotone Lagrangian torus $L \subset (\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{FS} \oplus \omega_{FS})$ has a two-fold canonical Bohr–Sommerfeld cover whose Legendrian lift admits a monotone Lagrangian filling inside the complex line bundle $O(-1, -1)$ which is contained in the preimage of $L$ under the bundle projection.

We believe that these monotone Lagrangian fillings also are obstructions to being subloose.

4.2. Lifts of monotone tori in the projective plane. A monotone Lagrangian torus $L$ inside the projective plane has a symplectic action class which lives inside

$$\frac{\pi}{3} \cdot \text{im}(H^2(\mathbb{C}P^2, L, \mathbb{Z}) \to H^2(\mathbb{C}P^2, L, \mathbb{R})).$$

By Theorem 4.11 this symplectic action class is not an integer multiple of $\pi$. In other words, an appropriate three-fold cover of any monotone Lagrangian torus is Bohr–Sommerfeld for the prequantisation space

$$S^1 \to S^5 \to \mathbb{C}P^2.$$
The first two well-known examples of monotone Lagrangian tori are the Clifford and the Chekanov torus [11, 13]. In terms of the fibration
\[ f: \Sigma \to \hat{D}_{1/2}^2 \]
from Section 2.2, i.e. the restriction of the Lefschetz fibration \((z_1, z_2) \mapsto z_1 \cdot z_2\) to the subset \(\Sigma \subset \mathbb{C}P^2\), these two tori have the following descriptions.

- The **Clifford torus** is the preimage of an embedded closed curve in \((\hat{D}_{1/2}^2, (1/2)d(r\,d\theta))\) of symplectic area \(\pi/3\) which *encircles* the origin; while
- The **Chekanov torus** is the preimage of an embedded closed curve in \((\hat{D}_{1/2}^2, (1/2)d(r\,d\theta))\) of symplectic area \(\pi/3\) which does *not* encircle the origin.

See Lemma 2.7 for more details concerning the area considerations.

The Legendrian lift of the Bohr–Sommerfeld cover of the Clifford torus is already well studied, since it is the link of the singularity of the Harvey–Lawson cone [38]. Its contact topology was been studied by Nadler [42], Treumann–Zaslow [48], as well as Baldrige–McCarthy–Vela-Vick [3]. By Theorem 4.4 below, this torus is, in addition, Legendrian isotopic to the “knotted” Legendrian torus considered in [16] by the first author.

**Theorem 4.4.** The threefold Bohr–Sommerfeld coverings of the Clifford and Chekanov tori in \(\mathbb{C}P^2\) have Legendrian lifts in the prequantisation space \(S^5 \to \mathbb{C}P^2\) that are Legendrian isotopic into a Darboux ball with front projections as shown in Figures 7 and 4.

**4.3. Proof of Theorem 4.4.** Both the Clifford and Chekanov tori live inside the subset \(\overline{\Sigma} \subset \mathbb{C}P^2\) described in Section 2.2, and they project to simple closed curves under the Lefschetz fibration \(f: \Sigma \to \hat{D}_{1/2}^2\) described in the same; the former encircles the origin, while the latter does not. The sought Legendrian isotopies will be constructed inside the prequantisation bundle above the same subset \(\overline{\Sigma} \subset \mathbb{C}P^2\); we describe the corresponding Lagrangian projection, which thus is a Lagrangian regular homotopy of Bohr–Sommerfeld immersions inside the same. While doing this one must of course also take precaution so that no self-intersections arise in the Legendrian lifts.

Recall that any immersed Lagrangian torus inside \(\Sigma\) projects to an immersed closed curve inside \(\hat{D}_{1/2}^2\) under the fibration \(f\) by Lemma 2.6, and that conversely preimages of such curves
are Lagrangian immersions. The sought regular homotopies are described in Figures 6 and 7 respectively, by regular homotopies of immersed curves inside $\hat{D}_{1/2}^2$, with one caveat: at one moment we must let our Lagrangian immersion pass through the intersection $S_{(1,1),1}$ of the line at infinity of $\mathbb{C}P^2$ and $\Sigma$. In that region $f$ is no longer suitable for describing the Lagrangian, and we instead use Lemma 2.10. We proceed with some more details.

We start on the top left in either of Figures 6 or 7. Here we see the threefold canonical Bohr–Sommerfeld covers of the respective monotone Lagrangian tori that have been generically perturbed inside the subset $\Sigma$ through Lagrangian immersions.

Going from the top right projection to the bottom left projection we must let the Legendrian pass over the line at infinity of $\mathbb{C}P^2$, i.e. the Lagrangian intersects $S_{(1,1),1}$ of the line at infinity of $\mathbb{C}P^2$ and $\Sigma$. The fact that the Lefschetz fibration $(z_1, z_2) \mapsto z_1 \cdot z_2$ defined on $\mathbb{C}^2$ obtains a singular fibre (a twofold branched cover of the line at infinity) when extended to $\mathbb{C}P^2$ makes this move slightly complicated. Instead of using $f$ we therefore pass to the description given in Lemma 2.10 of a neighbourhood of the intersection $S_{(1,1),1}$ of the line at infinity and $\Sigma$; see Figure 5.

**Figure 5.** Top left: the image under $f: \Sigma \to \hat{D}_{1/2}^2$ of a Lagrangian cylinder inside $\Sigma$. Top right: the cylinder after a Lagrangian regular homotopy that slides it over the embedded circle $S_{(1,1),1} \subset \Sigma$ (which lives over the boundary of $\hat{D}_{1/2}^2$ as shown in red). The bottom pictures show the respective intersection of the Lagrangian in a fibre of the normal bundle $D_x^2 \times S_{(1,1),1} \to S_{(1,1),1}$ inside $\Sigma$. 
Figure 6. The sequence of curves depicts the projection to $\dot{D}_{1/2}^2$ of a homotopy of Bohr–Sommerfeld immersions contained inside $\Sigma \subset \mathbb{CP}^2$ under the fibration $f: \Sigma \to \dot{D}_{1/2}^2$. The numbers denote approximate symplectic areas of the regions with respect to the symplectic form from Lemma 2.7, for which $\dot{D}_{1/2}^2$ has total area $\pi/2$.

In the end we produce the Lagrangian projection living over the immersed curves $\gamma_{\text{Ch}}$ and $\gamma_{\text{Cl}}$ shown on the bottom right in Figures 6 and 7 respectively. Note that these curves bound signed symplectic area equal to zero for the symplectic form on $\dot{D}_{1/2}^2$ from Lemma 2.7. Shrinking these curves are thus obviously a regular homotopy through exact, and hence Bohr–Sommerfeld, Lagrangian immersions inside $\mathbb{C}^2$. Shrinking these curves sufficiently, the corresponding Legendrians end up inside a small contact Darboux ball and have front projections as shown in Figures 1 and 4 respectively. \qed
Figure 7. A homotopy of Bohr–Sommerfeld immersions of tori inside $\Sigma$ which starts at the canonical threefold Bohr–Sommerfeld cover of the Chekanov torus and ends at the Lagrangian projection of $\Lambda_{Ch}$. The curves depict the projections under the fibration $f: \Sigma \to \mathbb{D}^2_{1/2}$.

5. Computations of the Chekanov–Eliashberg algebras

For Legendrian surfaces inside the standard contact vector space $(\mathbb{R}^5, dz - ydx)$ the full Chekanov–Eliashberg algebra as constructed in [25] can readily be computed by using Ekholm’s theory of gradient flow-trees [21]. Here we compute this invariant for the Legendrian tori $\Lambda_{Cl}$ and $\Lambda_{Ch}$ when considered as Legendrian submanifolds inside a strict contact Darboux ball. (The DGA inside the full $(\mathbb{S}^5, \alpha_{st})$ is of a more complicated form with infinitely many generators, but up to quasi-isomorphism we should get the same result; c.f. [18].)

Remark 5.1. The Chekanov–Eliashberg algebra computed in a contact Darboux ball is a priori a Legendrian invariant of the Legendrian also when considered inside the entire contact sphere. Namely, any Legendrian isotopy can be assumed to miss a generic point after a small
perturbation. By [35, Proposition 2.1.8] it may thus be assumed to actually be confined to the contact Darboux ball itself.

We have found representatives for the Legendrians $\Lambda_{\text{Cl}}$ and $\Lambda_{\text{Ch}}$ that live in a Darboux ball $(\mathbb{C}^2 \times \mathbb{R}, dz - ydx)$ and which moreover satisfy a rotational symmetry around the $z$-axis. This is equivalent to the fact that they live inside $\Sigma \subset \mathbb{C}^2$ as considered in Section 2.2 and can thus be described by their projections under $f: \Sigma \to \tilde{D}_{1/2}$. Their projections are the closed immersed curves $\gamma_{\text{Cl}}$ and $\gamma_{\text{Ch}} \subset \mathbb{C}^*$ shown at the bottom right in Figures 6 and 7 respectively. This symmetry will significantly facilitate the computations of their Chekanov–Eliashberg algebras.

Remark 5.2. In the case when the origin is contained in the unbounded component of $\mathbb{C} \setminus \gamma$, then a Legendrian satisfying the above symmetry is the so-called $S^1$-spin of the Legendrian knot with Lagrangian projection $\gamma \subset \mathbb{C}$; see [28] for the definition, where Ekholm–Kálmán moreover give a description of the full Chekanov–Eliashberg algebra of such a Legendrian torus entirely in terms of the Chekanov–Eliashberg algebra of the knot alone. In practice, this means that we can compute the differential of the torus simply by counting polygons on $\gamma$ in $\mathbb{C}$.

The case considered here is not the $S^1$-spin but rather the so-called symmetric $S^1$-spin of a Legendrian knot. Computing the full DGA in this case is slightly harder, but it can be done e.g. using the cellular DGA by Rutherford–Sullivan [47, 46] as done by Li [40, Proposition 6.2] in a particular case. Here we only do a partial computation of the DGA of $\Lambda_{\text{Ch}}$; we postpone the computation of the full DGA to future work.

5.1. Generators and grading. Let $\gamma \subset \mathbb{C}^*$ be a generic immersion of a closed curve which is the image of a Legendrian torus $\Lambda \subset (\mathbb{C}^2 \times \mathbb{R}, dz - ydx)$ under the above standard Lefschetz fibration. The Legendrian condition translates to an exactness property of this curve (i.e. that it bounds zero symplectic area); c.f. Lemma 2.7 for the relevant area form for that purpose.

The Chekanov–Eliashberg algebra of $\Lambda$ is the unital noncommutative DGA $(\mathcal{A}(\Lambda), \partial)$ freely generated by the Reeb chords of a generic perturbation of $\Lambda$ (in order to make all Reeb chords transverse) over the commutative group ring $\mathbb{F}[H_1(\Lambda)]$.

A choice of basis of $H_1(\Lambda)$: We begin by pin-pointing a $\mathbb{Z}$-basis $\langle \mu, \lambda \rangle = H_1(\Lambda)$. This gives an identification $\mathbb{F}[H_1(\Lambda)] \cong \mathbb{F}[\mu^{\pm 1}, \lambda^{\pm 1}]$ with the ring of Laurent polynomials in two variables.

First we take $\mu \in H_1(\Lambda)$ that is represented by (a suitably orientated) $S^1$-fibre of $\Lambda$ under the Lefschetz fibration. Such a curve can be taken to lie e.g. parallel to one of the circular cusp edges in the front projection of $\Lambda$. Then we take $\lambda$ to be represented by a simple closed curve inside $\Lambda$ that projects to the curve in $\mathbb{C}$ (covered precisely once). Note that this choice of class is only well-defined up to an addition of $k\mu, k \in \mathbb{Z}$. Any such curve becomes a curve in the front projection of the Legendrian which traverses all the cusp edges and front-cone singularities. The ambiguity of the choice of $\lambda$ can be seen also here: lifting such a curve on the front to the Legendrian itself, the homotopy class depends precisely on a choice made at each cone-point (where the lift is ambiguous).
The Reeb chords: Closed Legendrians that satisfy the $S^1$-symmetry considered here never have transverse Reeb chords; the preimage of a double point in the base $C$ of the Lefschetz fibration is always a circle of Reeb chords. In the case when the intersection point of the projection is a transverse double point, the circle of Reeb chords is still generic in the Bott sense. For a generic perturbation constructed by a Morse function on the Bott manifold (i.e. $S^1$) having precisely two critical points, each Bott manifold gives rise to precisely two Reeb chords $x$ and $\hat{x}$ corresponding to the minimum and maximum of the Morse function, respectively. The grading of these generators are then $|\hat{x}| = |x| + 1$ and $|x| = CZ(x) - 1$, where $CZ(x)$ moreover coincides with the Bott version of the Conley–Zehnder index for the corresponding $S^1$-family of Reeb chords. For a small generic perturbation we thus get precisely two Reeb chords $x$ and $\hat{x}$ for any double point of $\gamma$. Moreover, the gradings of the Novikov parameters are given by their respective Maslov indices.

The following result allows us to compute the gradings entirely in terms of data on the projection $\gamma \subset \mathbb{C}$.

Lemma 5.3. (1) The Maslov index of $\mu$ vanishes, while the Maslov index of $\lambda$ is equal to twice the tangent winding number $\tau \in \mathbb{Z}$ of the closed curve $\gamma$.

(2) The Conley–Zehnder index $CZ(x)$ is equal to the Conley–Zehnder index of the corresponding transverse self-intersection of $\gamma \subset \mathbb{C}$ as defined for immersed curves in [12].

Proof. (1): The Maslov index of $\mu$ vanishes by an explicit computation e.g. using the formula in [24, Lemma 3.4]. To that end, observe that $\mu$ can be represented by an embedded path which is invariant under the $S^1$-symmetry and hence which traverses no cusp-edge in the front projection of the Legendrian. For the computation of the Maslov index of $\lambda$ we argue as follows.

It is clear that the Maslov index of $\lambda$ is computed by the formula $2\tau + cw$ for some universal constant $c \in \mathbb{Z}$, where $w$ is the total winding number of $\gamma$ around the origin. Indeed, this formula holds for the Lagrangian $f^{-1}(s + \gamma)$ over the translation $s + \gamma$, $s > 0$, of the curve to the right halfplane (and thus in particular satisfying $w = 0$). To show that the same formula also holds for the initial Lagrangian $f^{-1}(\gamma)$, it suffices to compare the difference in Maslov index between the Lagrangians living over a vertical arc just to the right and just to the left of the origin, respectively. Since we claim that the contribution to the Maslov class is the same for both curves in this dimension, we conclude that $c = 0$ as sought. (One can e.g. use the computation [24, Lemma 3.4] in terms of the front projection; the Lagrangian above the arc to the left is the front cone, whose generic resolution was described in [26, Section 3.1] and [16] while the Lagrangian above the arc to the right is the $S^1$-spun of a single cusp edge; an arc from the top to the bottom thus in particular traverses precisely one cusp edge in downwards direction in either case.)

(2): In view of the considerations of the behaviour of the Maslov index in the previous paragraph, this is the same as the computation of the degrees of the Reeb chords on the $S^1$-spun of a Legendrian knot in terms of the degrees of the Reeb chords of the knot itself; see [28].
Figure 8. The blue curve depicts the caustic of the front projection of a generic resolution of the front cone that involves four swallow-tail singularities. We also depict the two possibilities for a rigid partial gradient flow-tree that involves a single front-cone; the tree on the left has a single vertex being (an edge-vertex at the cusp edge), while the tree on the right has three vertices (one three-valent vertex of type Y and two edges-vertices at two cusp-edges).

5.2. **Computations for** $\Lambda_{\text{Cl}}$. The Chekanov–Eliashberg algebra $(\mathcal{A}(\Lambda_{\text{Cl}}), \partial)$ of $\Lambda_{\text{Cl}} \subset (\mathbb{R}^5, \alpha_{st})$ was computed by the first author in [16]. Here we redo this computation in the present algebraic setting (in which the Novikov coefficients $\mathbb{C}[H_1(\Lambda_{\text{Cl}})]$ do not commute with the Reeb chord generators).

The $S^1$-symmetric version of $\Lambda_{\text{Cl}}$ has a single $S^1$-Bott manifold of Reeb chords that correspond to the unique double point of the curve shown at bottom right in Figure 6. A small generic perturbation of the $S^1$-symmetric front of $\Lambda_{\text{Cl}}$ produces two generic chords $a$ and $\hat{a}$, where we compute $|\hat{a}| = |a| + 1$ and $|a| = 1$.

**Theorem 5.4.** For the Lie group spin structure and suitable choices of capping paths and basis $\langle \mu, \lambda \rangle = H_1(\Lambda)$ we have

$$\partial a = \partial \hat{a} = 1 + \lambda(1 + \mu),$$

$$\partial \hat{a} = a - \mu a \mu^{-1}.$$  

In particular, the augmentation variety of $(\mathcal{A}(\Lambda_{\text{Cl}}), \partial)$ is equal to the one-dimensional complex pair of pants

$$\text{Sp}(\mathbb{C}[\mu^\pm 1, \lambda^\pm 1]/(1 + \lambda(1 + \mu)))$$

and for two augmentations $\varepsilon_0, \varepsilon_1 \colon H_0(\mathcal{A}) \to \mathbb{C}$ the bilinearised Legendrian contact homology group satisfies

$$LCH^*_{\varepsilon_0, \varepsilon_1}(\Lambda_{\text{Cl}}) = \begin{cases} H_{s-1}(S^1; \mathbb{C}), & \varepsilon_0 = \varepsilon_1, \\ 0, & \varepsilon_0 \neq \varepsilon_1. \end{cases}$$

**Proof.** In [16] the differential

$$\partial a = 1 + \lambda(1 + \mu)$$
was computed for some suitable (but unspecified) choice of spin structure (and choice of sign of the generator "a"), where the terms are in bijection with the rigid pseudoholomorphic discs with exactly one puncture. The term "1" corresponds to the rightmost immersed teardrop bounded by \( \gamma_{C1} \) (c.f. Figure 5), i.e. which does not contain the origin. The capping paths of \( a \) are chosen to be the two curves in \( \Lambda \) that correspond to the edge of this flow-tree, and the basepoint is thus located precisely at the cusp-edge. The remaining two terms both correspond to the immersed teardrop to the left bounded by \( \gamma_{C1} \), i.e. that pass through the origin. After generically resolving the front-cone singularity as described in [26] or [16], these flow-trees flow towards the cone-point where they then proceed in either of the two manners shown in Figure 8.

In order to show that the signs of all three discs are the same for the Lie group spin structure we argue as follows.

First, we claim that the discs counted by \( \partial a \) which appears from the resolution shown on the left in Figure 8 and the disc that corresponds to the term "1" both are counted with the same sign (and hence +1 without loss of generality). The claim follows by the computation of DGA for the \( S^1 \)-spun of the standard unknot with a single Reeb chord, since the two aforementioned discs can be identified with the two discs that contribute to \( \partial a' = 1 - \lambda \) for the unique generator \( a' \) in degree one and the spin structure that extends over the solid torus in which \( \lambda \) is a longitude (this is different from the Lie group spin structure along \( \lambda \)). To see the sign in the case of the \( S^1 \)-spun of the unknot, note that the latter spin structure extends to the standard exact Lagrangian filling of the aforementioned \( S^1 \)-spun by a solid torus in the symplectisation, in which the two discs become the endpoints of a one-dimensional moduli space of discs with boundary on the filling.

Second, we claim that the disc counted by \( \partial a \) that corresponds to the term \( \lambda \mu \) also is counted with the same sign as the previous discs. If not we would have \( \partial a = 1 + \lambda(1 + \mu) \) for the spin structure that extends to the exact Lagrangian cobordism to the loose unknot as constructed in Section 6 in particular see Figure 12. (Note that this spin structure differs from the Lie group spin structure along the cycle \( \mu \).) This, however, would mean that \( \partial a = 0 \) with coefficients in \( \mathbb{Z}_3 \), thus contradicting the fact that the loose two-sphere has no augmentation with any choice of coefficients.

In the setting of the “fully noncommutative Legendrian DGA” considered here we also compute

\[
\partial \hat{a} = a - \mu a \mu^{-1},
\]

where the two terms correspond to the two flow-lines in the Bott \( S^1 \)-manifold of Reeb chords that appear after the generic perturbation. (The signs of the two terms can be determined by \( \partial^2 \hat{a} = 0 \), after a choice of sign of the generator \( \hat{a} \).) Here we have taken the capping paths of \( \hat{a} \) to follow the \( S^1 \)-manifolds of Reeb chords (before the Morse perturbation) to the corresponding endpoint of the chord \( a \), concatenated with the corresponding capping path for the chord \( a \).

For the computation of the bilinearised Legendrian contact homology, we can either allude to Theorem 5.6 below or perform the following computation by hand.
For the explicit computation, we first note that the bilinearised differential satisfies
\[ \partial^{\varepsilon_0,\varepsilon_1}(\hat{a}) = (1 - \varepsilon_0(\mu)\varepsilon_1(\mu)^{-1})a. \]

Finally, two augmentations are the same if and only if they take the same value on the variable \( \mu \in \mathbb{C}[H_1(\Lambda_{Ch})] \), as follows from the relation \( \lambda^{-1} = -(1 + \mu) \). □

In other words, the homology groups \( LCH^* \) behave like the Ext groups for different skyscraper sheaves on the augmentation variety. This should be compared to Nadler’s computation [42] based upon the technique of sheaves with microsupport on the Lagrangian cone on \( \Lambda_{Cl} \subset \partial B^4 \).

5.3. Computations for \( \Lambda_{Ch} \). Here we compute the homology of the Chekanov–Eliashberg algebra \( (\mathcal{A}(\Lambda_{Ch}), \partial) \) of \( \Lambda_{Ch} \subset (\mathbb{R}^5, \alpha_{st}) \) in degree zero. This is sufficient for computing its augmentation variety, and to distinguish it from the Legendrian torus \( \Lambda_{Cl} \) up to Legendrian isotopy. The computation of the full DGA is postponed to future studies. For the bilinearised Legendrian contact homologies, we refer to the general computation in Section 5.4.

The \( S^1 \)-symmetric version of \( \Lambda_{Cl} \) has the three \( S^1 \)-Bott manifolds of Reeb chords corresponding to the double point shown in the bottom right of Figure 6. The Bott indices are \( |a| = 1, |c| = 3, \) and \( |b| = 2, \) for the rightmost, middle, and leftmost double points, respectively. A small generic perturbation thus gives us a Legendrian with six Reeb chords \( a, \hat{a}, c, \hat{c}, b, \hat{b} \), with
\[ |a| = |\hat{a}| - 1 = 1, \quad |c| = |\hat{c}| - 1 = 3, \quad \text{and} \quad |b| = |\hat{b}| - 1 = 2. \]

Theorem 5.5. For the Lie group spin structure and suitable choices of capping paths and basis \( \langle \mu, \lambda \rangle = H_1(\Lambda) \) we have
\[ \partial a = 1 + \lambda(1 + \mu)^2, \quad \partial \hat{a} = a - \mu \alpha \mu^{-1}, \]
and in particular \( H_0(\mathcal{A}, \partial) = \mathbb{C}[\mu^{\pm 1}, \lambda^{\pm 1}]/(1 + \lambda(1 + \mu)^2) \).

Proof. The version of the Legendrian that satisfies the \( S^1 \)-symmetry shown in Figure 4 is degenerate for two different reasons: first, the Reeb chords come in \( S^1 \)-Bott manifolds and, second, it has two double-cone singularities in its front projection both located inside the axis of the rotational symmetry.

We will compute the rigid gradient flow-trees [21] after a generic perturbation which, by the same work, computes the differential for a suitable choice of almost complex structure. Since we only care about such flow-trees with a single positive puncture at the chord \( a \) of degree \( |a| = 1 \), these flow-trees will have no additional (negative) punctures.

If we keep the \( S^1 \)-Bott family of Reeb chords, then this is the same as a rigid gradient flow-tree with a positive puncture at the Bott family of chords itself, but with a fixed asymptotic constraint.

We start with the flow-trees that do not enter the region near the axis of symmetry. Here there is single rigid gradient flow-tree with a single positive puncture at \( a \), which is as shown in Figure 9. This flow-tree has precisely two vertices (one puncture at \( a \) and one so-called edge-vertex at the cusp-edge) and one edge. We take the capping paths of \( a \) to coincide
with the two arcs on \( \Lambda \) that correspond to the edge of the flow-tree; the basepoint is hence located precisely at the cusp-edge. With this choice, together with an appropriate choice of sign for the generator, we obviously get a disc that contributes to a term
\[
\partial(a) = \ldots + (\pm 1) + \ldots
\]
of the differential.

We then consider gradient flow-trees which enter the region near the axis of symmetry. By action reasons, and using the fact that the positive puncture is unique, it is not possible for such a gradient flow-tree to have an edge that involves two sheets corresponding to different cones. (The difference in \( z \)-coordinate for two such sheets is always greater than the length of \( a \).) In other words, the only edges that we need to consider near the cone-region in this case are those which are associated to precisely one of the two cones. Such edges are given simply by a gradient flow towards the cone-point, and the vertex at the cone-point will be called a cone-vertex. See Figure 10 for an example of a partial flow tree with precisely two such vertices.

In the perturbed Legendrian which resolves the cone-point to a generic front, the rigid partial gradient flow-trees with cone-vertices in the sense described above can be resolved to honest rigid gradient flow-trees by the following process. First use the perturbations of the Legendrian described in [26] and [16] in order to resolve the front-cone singularities to a generic front. In this perturbation, every cone-vertex has precisely the two possibilities of a completion to an honest gradient flow tree that are shown in Figure 8. Conversely, every rigid gradient flow tree corresponds to a unique partial rigid flow-tree after reintroducing the degenerate front-cone.

In view of the above, we are able to postpone the perturbation that makes the front generic near the cone points, and simply start by finding the rigid partial gradient flow-trees that have cone-vertices and a puncture at a fixed point on the Bott manifold of Reeb chords \( a \). There is precisely one such partial flow-tree; see Figure 10. By considering all possible completions of the latter partial flow-tree after the perturbation of the front cone, we thus find the remaining four rigid gradient flow-trees that contribute to the four terms
\[
\partial a = \ldots + (\pm) \lambda \pm \lambda \mu \pm \lambda \mu \pm \lambda \mu^2 + \ldots
\]
of the differential.

We have now found all five rigid flow-trees that contribute to \( \partial a \). As in the proof of Theorem 5.4 we can show that the disc contributing to the term “1” and the disc whose resolutions at both cone-points is as shown on the left in Figure 8 are counted with the same signs. A neck-stretching argument near the cone regions then show that the local sign contribution from the two resolutions shown in Figure 8 are the same at both cone points, from which we conclude that
\[
\partial a = 1 + \lambda(1 \pm \mu)^2
\]
(as opposed to \( \partial a = 1 + \lambda(1 - \mu)(1 + \mu) \)). As in the proof of Theorem 5.4 the fact that \( \Lambda_{Ch} \) is subloose then pin-points the final sign in the last factor.

The computation of \( \partial \hat{a} \) is the same as for \( A(\Lambda_{Ch}) \) in the proof of Theorem 5.4. \( \square \)
5.4. **A general computation of bilinearised LCH.** Even if we do not compute the full DGA of $\Lambda_{\text{Ch}}$ we can still compute its bilinearised Legendrian contact homologies by a general result that we establish. Assume for now that we have a general Legendrian embedding of an oriented genus $g \geq 0$ surface $\Lambda \subset (\mathbb{R}^5, \xi_{\text{st}})$. Further assume that $\Lambda$ has vanishing Maslov class and only Reeb chords of positive degree. The duality long exact sequence \cite{27} by Ekholm–Etnyre–Sabloff and its generalisation \cite{5} by Bourgeois–Chantraine to the setting of bilinearised contact homology can then be used to compute all different bilinearised contact homologies.

**Theorem 5.6.** Let $\varepsilon_0, \varepsilon_1 : (\mathcal{A}, \partial) \to (\mathcal{F}, 0)$ be two graded augmentations for a Legendrian oriented genus $g \geq 0$ surface $\Lambda \subset (\mathbb{R}^5, \xi_{\text{st}})$ of vanishing Maslov class and with all Reeb chords in positive degrees. Then
• when $\varepsilon_0 = \varepsilon_1$:

$$LCH_{k}^{\varepsilon_0, \varepsilon_1}(\Lambda) = \begin{cases} 
\mathbb{F}, & k = 2, \\
\mathbb{F}^g, & k = 1, \\
0, & k \neq 1, 2,
\end{cases}$$

• when $0 \leq g \leq 1$ and $\varepsilon_0 \neq \varepsilon_1$: $LCH_{k}^{\varepsilon_0, \varepsilon_1}(\Lambda) = 0$ for all $k$,

is satisfied for the bilinearised Legendrian contact homology groups.

Remark 5.7. In the present situation two different augmentations are always inequivalent in the sense of [5], due to the fact that there are no Reeb chords of negative degree.

Proof. Let $\Sigma_g$ be an oriented genus $g \geq 0$ surface. First, assume that $0 \leq g \leq 1$. Denote by $H_k(\Sigma_g; \varepsilon_0 \otimes \varepsilon_1^{-1})$ the Morse homology of $\Sigma_g$ with coefficients in $\mathbb{F}$ which is induced by the local system which takes the value $\varepsilon_0(\alpha)\varepsilon_1(\alpha)^{-1} \in \mathbb{F}$ on the class $\alpha \in H_1(\Sigma_g; \mathbb{F})$. Recall the standard fact that this homology vanishes in all degrees unless the latter local system is trivial, i.e. unless $\varepsilon_0 = \varepsilon_1$, while in the latter case we simply get the standard singular homology $H_k(\Sigma_g; \mathbb{F})$ of the surface. (Of course when $g = 0$ all local systems are trivial.)

The duality long-exact sequence [5, Theorem 1.5] applied in this situation is equal to

$$\cdots \rightarrow H_{k+1}(\Sigma_g; \varepsilon_0 \otimes \varepsilon_1^{-1}) \rightarrow LCH_{1-k}^{\varepsilon_0, \varepsilon_1}(\Lambda) \rightarrow LCH_{k}^{\varepsilon_0, \varepsilon_1}(\Lambda) \rightarrow H_k(\Sigma_g; \varepsilon_0 \otimes \varepsilon_1^{-1}) \rightarrow \cdots$$

The assumption on the gradings of the Reeb chords of $\Lambda$ immediately implies the vanishing $LCH_{k}^{\varepsilon_0, \varepsilon_1}(\Lambda) = 0$ whenever $k \leq 0$ and that $LCH_{1-k}^{\varepsilon_0, \varepsilon_1}(\Lambda) = 0$ whenever $k \geq 1$. Combined with the vanishing of the homology of the oriented genus $0 \leq g \leq 1$ surface with nontrivial local systems, the conclusion now follows in the case when $\varepsilon_0 \neq \varepsilon_1$. (Recall that two different augmentations automatically induce different local systems by our degree assumptions.)

What remains is the case when $\varepsilon_0 = \varepsilon_1$. Now we consider all $g \geq 0$. The immediate conclusion of the above vanishing result is that

$$LCH_{k}^{\varepsilon_0, \varepsilon_0}(\Lambda) = 0 \text{ when } k \neq 1, 2, \quad (5.1)$$

and hence we can write part of the duality long exact sequence as

$$0 = LCH_{-1}^{\varepsilon_0, \varepsilon_0}(\Lambda) \rightarrow LCH_{2}^{\varepsilon_0, \varepsilon_0}(\Lambda) \rightarrow H_2(\Sigma_g; \mathbb{F}) \rightarrow LCH_{0}^{\varepsilon_0, \varepsilon_0}(\Lambda) = 0,$$

which implies that

$$LCH_{2}^{\varepsilon_0, \varepsilon_0}(\Lambda) = \mathbb{F}.$$ 

Recall that

$$\dim LCH_{*}^{\varepsilon_0, \varepsilon_0}(\Lambda) = \dim LCH_{*}^{\varepsilon_0, \varepsilon_0}(\Lambda), \quad (5.2)$$

is satisfied in general.

We now claim that $\dim LCH_{1}^{\varepsilon_0, \varepsilon_0}(\Lambda) = g$. In order to show it, consider the following part of the duality long exact sequence, where zero terms appear because of Equations 5.1 and 5.2

$$0 \rightarrow LCH_{1}^{\varepsilon_0, \varepsilon_0}(\Lambda) \rightarrow H_1(\Sigma_g; \mathbb{F}) \rightarrow LCH_{0}^{\varepsilon_0, \varepsilon_0}(\Lambda) \rightarrow 0. \quad (5.3)$$
Using Equation 5.2 and the fact that $H_1(\Sigma_g; F) \cong \mathbb{F}^{2g}$, we can rewrite long exact sequence 5.3 as
\begin{equation}
0 \to LCH_{1}^{\varepsilon_0, \varepsilon_0}(\Lambda) \to \mathbb{F}^{2g} \to LCH_{1}^{\varepsilon_0, \varepsilon_0}(\Lambda) \to 0.
\end{equation}
Since all terms in long exact sequence 5.4 are $\mathbb{F}$-vector spaces, we see that $LCH_{1}^{\varepsilon_0, \varepsilon_0}(\Lambda) \cong \mathbb{F}^g$. This finishes the proof. □

In addition, it is also possible to obtain restrictions on the variety of augmentations of a Legendrian torus satisfying the assumptions of Theorem 5.6.

**Theorem 5.8.** Let $\Lambda \subset (\mathbb{R}^5, \xi_{st})$ be a Legendrian torus of vanishing Maslov class and with all Reeb chords in positive degrees. Its augmentation variety over $\mathbb{C}$ is then either empty, or of complex dimension equal to one.

**Proof.** First we argue that the augmentation variety is not all of $\text{Sp}(\mathbb{C}[H_1(\Lambda)]) \cong (\mathbb{C}^*)^2$. It suffices to show that there exists a Reeb chord of degree one with a nontrivial boundary (whose image necessarily is an element of $\mathbb{C}[H_1(\Lambda)]$). We prove that there exists such a nonvanishing boundary with $\mathbb{Z}_2$-coefficients, which simplifies the considerations essentially, and implies the claim.

Argue by contradiction and assume that all such boundaries vanish. Thus the canonical unital DGA morphism $\varepsilon_0: (A, \partial) \to \mathbb{Z}_2$ is clearly an augmentation. Consider the map
\[ LCH_{1}^{\varepsilon_0, \varepsilon_0}(\Lambda) \xrightarrow{\rho} H_1(\mathbb{T}^2; \mathbb{Z}_2) \]
that arises in the duality long exact sequence [27], which is defined by the choice of an auxiliary generic Morse function and Riemannian metric on $\Lambda$. By [27] Theorem 3.6(3) and Section 3.3.3] the value $\langle \rho(c), s \rangle$ for a Reeb chord $c \in LCH_{1}^{\varepsilon_0, \varepsilon_0}(\Lambda)$ and a critical point $s \in H_1(\mathbb{T}^2; \mathbb{Z}_2)$ of index one is given as the count of generalised pseudoholomorphic discs which consist of:

- a pseudoholomorphic polygon in $\mathbb{R}^4$ with boundary on the Lagrangian projection $\Pi_{\text{Lag}}(\Lambda) \subset \mathbb{R}^4$ and a single positive puncture at the double point that corresponds to $c$, together with
- a negative gradient flow-line in $\Lambda$ that connects the boundary of the polygon to $s$, such that the configuration moreover is rigid. Under suitable identifications, this count can be seen to be equal to the gradient $\nabla \partial(c) \in \mathbb{C}[\lambda^\pm, \mu^\pm]$ of the Laurent polynomial $\partial(c)$ evaluated at $(1, 1) \in \mathbb{Z}_2^2$. The assumption that $\partial(c)$ vanishes for all Reeb chords $c$ of degree one is now seen to contradict the fact that $\rho$ is an inclusion of a one-dimensional $\mathbb{Z}_2$-vector space, as was shown in the proof of Theorem 5.6 above.

Now consider an augmentation $\varepsilon_0: (A, \partial) \to \mathbb{C}$. We will show that there exists a one-dimensional component of the augmentation variety that passes through the corresponding point $\varepsilon_0 \in \text{Sp}(\mathbb{C}[H_1(\Lambda)])$. Choose a basis $\langle a_0, a_1, \ldots, a_m \rangle$ of the degree-one subspace of the complex $(LCC_{s_0, \varepsilon_0}^{*}(\Lambda), \partial^{s_0, \varepsilon_0})$, which automatically consists of cycles, together with a basis $\langle b_1, \ldots, b_n \rangle$ of the degree-two subspace, such that $\partial^{s_0, \varepsilon_0}(b_i) = a_i$ moreover is satisfied for $i \geq 1$. This is possible by Theorem 5.9 above.
On the level of the DGA, after making the Novikov generators \( \mu, \lambda \) commute with the Reeb chord generators, we thus conclude that

\[
\partial b_i = \sum_{j=0}^{m} P_j^i(\mu, \lambda)a_j,
\]

where \( P_j^i \in \mathbb{C}[\mu^{\pm 1}, \lambda^{\pm 1}] \) for \( i = 1, \ldots, m; \ j = 0, 1, \ldots, m \) are Laurent polynomials that satisfy \( \varepsilon_0(P_j^i) = \delta_j^i \).

Then consider the localisation

\[
\mathbb{C}[\mu^{\pm 1}, \lambda^{\pm 1}] \to \mathbb{C}[\mu^{\pm 1}, \lambda^{\pm 1}][(P_1^1 \cdots P_m^m)^{-1}],
\]

and the induced DGA \((\widetilde{A}, \widetilde{\partial})\) with coefficients in \( \mathbb{C}[\mu^{\pm 1}, \lambda^{\pm 1}][(P_1^1 \cdots P_m^m)^{-1}] \) together with the canonical unital DGA morphism \((A, \partial) \to (\widetilde{A}, \widetilde{\partial})\). Since none of the polynomials \( P_i^i \) vanish at the point corresponding to \( \varepsilon_0 \), the augmentation \( \varepsilon_0 \) descends to an augmentation \( \widetilde{\varepsilon}_0: (\widetilde{A}, \widetilde{\partial}) \to \mathbb{C} \). In other words, we have

\[
(5.5) \quad \widetilde{\partial} b_i = \sum_{j=0}^{m} P_j^i a_j, \quad i \in [1, m],
\]

where \( P_i^i \in \mathbb{C}[\mu^{\pm 1}, \lambda^{\pm 1}][(P_1^1 \cdots P_m^m)^{-1}] \) is a unit and \( \widetilde{\varepsilon}_0(P_j^i) = 0 \) if \( i \neq j \).

By induction we assume that we can perform a \( \mathbb{C}[\mu^{\pm 1}, \lambda^{\pm 1}][(P_1^1 \cdots P_m^m)^{-1}] \)-linear change of coordinates of the \( \mathbb{C}[\mu^{\pm 1}, \lambda^{\pm 1}][(P_1^1 \cdots P_m^m)^{-1}] \)-submodule generated by \( \langle a_1, \ldots, a_m \rangle \) after which, in additional to Equality (5.5), also \( \partial a_i = 0 \) is satisfied for the (possibly empty) set of values \( i \in [m_0 + 1, m] \) and some \( m_0 \in [1, m] \). Since \( \widetilde{\partial}^2 b_{m_0} = 0 \) is satisfied, we can express

\[
\widetilde{\partial} a_{m_0} = -(P_{m_0}^{m_0})^{-1} \sum_{j < m_0} P_j^{m_0} \widetilde{\partial} a_j, \quad i = 1, \ldots, m.
\]

Hence we can make \( \widetilde{\partial} a_{m_0} = 0 \) satisfied as well after the change of coordinates

\[
a_{m_0} \mapsto a_{m_0} + (P_{m_0}^{m_0})^{-1} \sum_{j < m_0} P_j^{m_0} a_j.
\]

By induction we may thus readily assume that \( \widetilde{\partial} a_i = 0 \) is satisfied for all \( i \in [1, m] \) in addition to Equality (5.5).

In conclusion, we have shown that the augmentation variety is cut out by the single function \( \widetilde{\partial} a_0 \in \mathbb{C}[\mu^{\pm 1}, \lambda^{\pm 1}][(P_1^1 \cdots P_m^m)^{-1}] \) inside the principal open subset

\[
\{P_1^1 \cdots P_m^m \neq 0\} \subset \text{Sp}(\mathbb{C}[\mu^{\pm 1}, \lambda^{\pm 1}]),
\]

which implies the claim. □
6. Sublooseness and existence of exact caps

There exists an elementary exact Lagrangian cobordism from both $\Lambda_{Cl}, \Lambda_{Ch} \subset (S^5, \xi_{st})$ to the loose sphere. This elementary cobordism arises as the handle attachment of a single handle as constructed in [17].

We recall this construction. The critical version of Legendrian surgery can be performed given that we have a Legendrian disc $D \subset (S^5, \xi_{st})$ that intersects a Legendrian $\Lambda_-$ cleanly precisely along its boundary. Given this data, Legendrian ambient surgery produces an embedded Legendrian $\Lambda_+$ which is topologically obtained from $\Lambda_-$ by surgery along the embedded codimension one sphere $\partial D \subset \Lambda_-$. There is also an associated exact Lagrangian handle attachment cobordism that goes from $\Lambda_-$ (at the negative end) to $\Lambda_+$ (at the positive end). Note that the handle attachment cobordism can be assumed to be a trivial cylinder outside of an arbitrarily small neighbourhood of $\mathbb{R} \times D \subset (\mathbb{R} \times S^5, d(e^\alpha_{st}))$ in the symplectisation.

The ambient surgery can be described using the Lefschetz fibration $f$ from Section 2.2. On the left in Figure 11 is the local model for the Legendrian $\Lambda_-$ together with the Legendrian surgery disc, while the effect of the surgery is shown on the right side. In terms of the front projection, the Legendrian $\Lambda_-$ is just a double-cone with the surgery disc $D$ living above the cone-point, while the result $\Lambda_+$ of the surgery resolves the cone-point to two smooth sheets that intersects transversely along an embedded $S^1$; see Figure 12.

![Figure 11. The local model of the (critical) Legendrian ambient surgery described using the Lefschetz fibration $f$ from Section 2.2. To the left: the Legendrian $\Lambda_-$ before the surgery together with the Legendrian surgery disc $D$. To the right: the effect of the surgery $\Lambda_+$, which is a Legendrian with one additional Reeb chord (which in this case projects to the origin).]
The lift of the Clifford torus: In this case both the surgery disc and the result of the Legendrian ambient surgery are shown in Figure 12. The result is clearly the loose two-sphere, since it coincides with an $S^1$-spun of a stabilised arc in some neighbourhood as in the proof of Lemma 2.11 (alternatively, one can apply Lemma 2.11 directly to the Lefschetz fibration).

The lift of the Chekanov torus: In this case we take the surgery disc to live above the lower of the two cone-points in the front projection Figure 4. The result of the Legendrian ambient surgery is then shown on the left side of Figure 13. After an $S^1$-spun of a Reidemeister-II move (this preserves the Legendrian isotopy class) we see that the result of the surgery indeed is the loose two-sphere as well; in the Legendrian on the right in Figure 13 one can clearly see the $S^1$-spun of a stabilised arc as described in the proof of Lemma 2.11 (alternatively, one can apply Lemma 2.11 directly to the Lefschetz fibration).

Since it is well-known that the loose sphere does not admit any exact Lagrangian filling, neither does any of the two former Legendrian tori $\Lambda_{Cl}$ or $\Lambda_{Ch}$. In addition, the loose sphere was shown to admit an exact Lagrangian cap in [23]. By concatenation of exact Lagrangian cobordisms, it follows that both tori $\Lambda_{Cl}$ and $\Lambda_{Ch}$ also admit exact Lagrangian caps.
6.1. Regularity of exact Lagrangian cobordisms. The class of regular Lagrangians in a Weinstein manifold was first introduced by Eliashberg–Ganatra–Lazarev in [31] and was then studied in [31, 30, 39].

We start with the definition of a regular exact Lagrangian cobordism.

Definition 6.1. Given a Weinstein cobordism \((W, \omega, X, \phi)\), where \(\omega\) denotes a symplectic form, \(X\) is an expanding Liouville vector field for \((W, \omega)\), \(\phi: W \to \mathbb{R}\) is defining for \(W\) and Lyapunov for \(X\), an exact Lagrangian cobordism \(L \subset W\) is called regular if \((W, \omega, X, \phi)\) can be deformed to a Weinstein cobordism \((W, \omega', X', \phi')\) through Weinstein structures for which \(L\) is Lagrangian and \(X'\) is tangent to \(L\).

The following question about regularity of exact Lagrangian cobordisms has been asked by Eliashberg–Ganatra–Lazarev in [31], and by Eliashberg in [30].

Question 6.2 ([31, 30]). Given a Weinstein cobordism \((W, \omega, X, \phi)\) such that \(\partial_- W \neq \emptyset\) and an exact Lagrangian cobordism \(L \subset (W, \omega, X, \phi)\) such that \(\partial_- L \neq \emptyset\) is not loose, is \(L\) regular?

Since an exact Lagrangian cap inside a symplectization never is regular by [31, Lemma 2.4], the following theorem provides a partial negative answer to the above Question 6.2.

Theorem 6.3. For any \(g > 0\) and \(k_1, \ldots, k_r \geq 0\) there exists infinitely many different Legendrian isotopy classes of subloose Legendrian embeddings \(\Lambda \subset (\mathbb{R}^{2k_1 + \ldots + 2k_r + 5}, \xi_{st})\) of the manifold \(S^{k_1} \times \ldots \times S^{k_r} \times \Sigma_g\), where \(\Sigma_g\) denotes the surface of genus \(g\), which:
- have vanishing Maslov classes;
- have Chekanov–Eliashberg algebras with Novikov coefficients \(R = \mathbb{C}[H_1(\Lambda)]\) that admits (0-graded) augmentations; and
- admit orientable exact Lagrangian caps inside the symplectisation with vanishing Maslov classes.

In particular, none of these Legendrian surfaces are loose.

Remark 6.4. It can further be shown that there are infinitely many diffeomorphism types of orientable exact Lagrangian caps for the above subloose Legendrians; see [23].

In view of the above we would like to make the following reasonable refinement of Question 6.2.

Question 6.5. Let \(L\) be an exact Lagrangian cobordism in a Weinstein cobordism \((W, \omega, X, \phi)\) such that \(\partial_- L \neq \emptyset\) is not subloose. Is \(L\) regular?

6.2. Proof of Theorem 6.3.

6.2.1. The case of surfaces. We start by considering the subloose surface of genus \(g > 0\) obtained by taking the cusp-connect sum of \(g\) number of copies of \(\Lambda_{Cl}\); for the definition of cusp-connect sum see [17]. These Legendrian surfaces were also considered in [16]. Denote the resulting Legendrian embedding by \(\Sigma_g \subset (\mathbb{R}^5, \xi)\). Observe that this Legendrian surface satisfies the statements of the theorem. What then remains is to make modifications to yield an infinite set of Legendrian up to Legendrian isotopy.
For any \( N > 0 \) we then produce a Legendrian 2-sphere \( \Lambda_N \) which admits an exact spin Lagrangian filling \( L_N \) of vanishing Maslov class such that
\[
\dim \bigoplus_i H_i(L_N; \mathbb{C}) \geq N
\]
is satisfied. For instance, \( \Lambda_N \) can be taken to be the cusp connected sum \( N \) copies of the Legendrian sphere from [8, Example 9.4.2]. By Seidel’s isomorphism (see e.g. [7, 8]) it follows that the augmentation \( \varepsilon_{L_N} \) induced by this filling has a Legendrian contact cohomology
\[
\bigoplus_i L\text{CH}^i_{\varepsilon_{L_N}}(\Lambda_N) \cong \bigoplus_i H_i(L_N; \mathbb{C})
\]
with complex coefficients whose total rank is at least \( N \) as well.

By the long exact sequence described in [8, Theorem 1.1] it follows that there exists an augmentation \( \varepsilon \) of the cusp-connected sum \( \Lambda \subset (\mathbb{R}^5, \xi_{st}) \) of \( \Lambda_N \) and \( \Sigma_g \) for which the linearised Legendrian contact cohomology \( L\text{CH}^*_{\varepsilon}(\Lambda) \) with coefficients in \( \mathbb{C} \) has total rank equal to at least \( N - 1 \).

Recall that the set of isomorphism classes of linearised Legendrian contact cohomologies for all different augmentations is a Legendrian isotopy invariant [12]. Further, the dimension of the linearised contact cohomology for any augmentation has an a priori upper bound given by the number of Reeb chords for any given generic Legendrian representative. It is thus clear that we can construct infinitely many Legendrian embeddings \( \Lambda \) in the above manner by choosing \( N > 0 \) larger and larger.

6.2.2. The existence of caps. Since we saw above that \( \Lambda_{C_1} \) admits an orientable exact Lagrangian cobordism to the loose sphere, the same is true for \( \Lambda \) as well. (We simply have to do a \( g \) number of suitable exact Lagrangian handle attachments, corresponding to Legendrian ambient surgeries.) The existence of exact Lagrangian caps for the loose sphere was established in [33] by Eliashberg–Murphy; also see the work [23] by Ekholm–Eliashberg–Murphy–Smith. The sought cap of \( \Lambda \) is then given by the concatenation of the exact Lagrangian cobordism to the loose sphere and the exact Lagrangian cap of the loose sphere.

6.2.3. The higher dimensional case. We apply the spherical front-spinning construction by the second author [37] to the above Legendrian surfaces and exact Lagrangian cobordisms to produce the subloose embeddings and their exact Lagrangian caps.

In order to deduce the existence of augmentations with Novikov coefficients we use the partial computation [20, Theorem 4.1] of the DGA of the \( S^m \)-spun of a Legendrian.

In order to deduce that the spuns remain in different Legendrian isotopy classes we use Künneth’s formula [7, Theorem 2.5], by which
\[
L\text{CH}^*_{\varepsilon_{\Sigma_{S^m}L_N}}(\Sigma_{S^m} \Lambda_N) \cong L\text{CH}^*_{\varepsilon_{L_N}}(\Lambda_N) \otimes H_*(S^m).
\]
Then we iterate this procedure and compute \( L\text{CH}^*_{\varepsilon_{\Sigma_{S^{k_1}} \cdots S^{k_l}L_N}}(\Sigma_{S^{k_1}} \cdots S^{k_l} \Lambda_N) \). Note that this isomorphism holds also with coefficients in \( \mathbb{C} \) given that the filling is spin. In addition, observe that [37, Proposition 1.1] implies that the spherical front spin of the cusp-connected sum is the cusp-connected sum of the spherical front spuns of the components. Using this and
the long exact sequence described in [8, Theorem 1.1], similarly to the 2-dimensional case, we observe that there exists infinitely many different Legendrian isotopy classes of subloose Legendrian embeddings $\Lambda \subset (\mathbb{R}^{2k_1+\ldots+2k_r+5}, \xi_{st})$ of the manifold $S^{k_1} \times \ldots \times S^{k_r} \times \Sigma_g$. □

7. Looseness of the standard Legendrian disc

![Figure 14. A regular homotopy through Lagrangian discs described in terms of the fibration $f: \Sigma \to \tilde{D}^2_{1/2}$.](image)

The standard Legendrian disc has a Lagrangian projection to $\mathbb{C}P^n$ that can be identified with the real part inside an affine chart $\mathbb{C}^n$. This is the vanishing thimble for the standard Lefschetz fibration on $\mathbb{C}^n$, and hence we identify it with the Lagrangian which lives above the positive $x$-axis inside $\tilde{D}^2_{1/2}$ under the fibration $f: \Sigma \to \tilde{D}^2_{1/2}$.

We must produce a Legendrian isotopy which is fixed near the boundary of the aforementioned Legendrian to exhibit a loose chart. To do this we follow the same strategy as in the proof of Theorem 1.4 given in Section 1.3. The sequence of moves is shown in Figure 14 where going from the top right curve to the bottom left curve requires that we pass over
the line at infinity. Note that the Legendrian lifts do not intersect the boundary of the disc, which is fixed, even though the Lagrangian projections clearly do intersect. In order to see this it suffices to compute the symplectic action at the points that intersect the boundary, and to check that it is not an integer multiple of \( \pi \).

In the bottom left of the figure resulting from the homotopy, we can indeed easily see the loose chart.

8. Further directions

Here we present some expectations that we hope to show in future work.

8.1. Relations between the augmentation variety and the superpotential. Recall that there exists precisely three pseudoholomorphic Maslov-two disc families in \( \mathbb{C}P^2 \) with boundary on the Clifford torus. Similarly, the augmentation variety of the Legendrian \( \Lambda_{Cl} \) is the zero-locus of the Laurent polynomial \( \partial a = 1 + \lambda(1 + \mu) \), which also is given by a count of precisely three pseudoholomorphic discs. In general we expect that we should be able to compute the augmentation variety in the Bott degenerate situation of \( (E, \alpha) \), where the embedded Legendrian lives above the embedded Lagrangian torus, and in this manner obtain a relation between the augmentation variety and the count of such pseudoholomorphic Maslov-two discs. Recall that the count of the latter discs define the so-called superpotential of the Lagrangian, which also is a Laurent polynomial.

In the case of the monotone Clifford and the Chekanov torus \( L_{Cl}, L_{Ch} \subset (\mathbb{C}P^2, \omega_{FS}) \) such a correspondence can indeed be confirmed by means of a hands-on comparison. Recall that the superpotentials of the two tori are given by

\[
\mathcal{P}_{L_{Cl}}(u, v) = u(1 + v) + 1/u^2v \in u \cdot \mathbb{C}[u^{\pm 3}, v^{\pm 1}],
\]

\[
\mathcal{P}_{L_{Ch}}(u, v) = u + (1 + v)^2/u^2v \in u \cdot \mathbb{C}[u^{\pm 3}, v^{\pm 1}],
\]

for suitable spin structures, and relative a suitable basis of \( \langle u, v \rangle = H_1(\mathbb{T}^2) \) where \( u \) and \( v \) correspond to generators of Maslov index equal to two and zero modulo six, respectively; see [2].

The precise relation between the augmentation polynomial and the superpotential is given by the following conjecture. Assume that

\[
H_0(\mathcal{A}(\Lambda), \partial) = \mathbb{C}[\mu^{\pm 1}, \lambda^{\pm 1}]/\langle \text{Aug}_\Lambda(\mu, \lambda) \rangle
\]

where \( \text{Aug}_\Lambda(\mu, \lambda) \) is the so-called augmentation polynomial (which under the assumptions is uniquely determined up to the multiplication by a unit).

**Conjecture 8.1.** For an embedded monotone Lagrangian two-torus \( L \subset (\mathbb{C}P^2, \omega_{FS}) \) the superpotential \( \mathcal{P}_L(u, v) \) can be recovered (up to the multiplication by a unit) from the augmentation polynomial \( \text{Aug}_\Lambda(\mu, \lambda) \) of the Legendrian lift \( \Lambda \) of its canonical threefold Bohr–Sommerfeld cover from the equality

\[
\mathcal{P}_L(u, v) = \text{Aug}_\Lambda(u^3, v) \in u \cdot \mathbb{C}[u^{\pm 3}, v^{\pm 1}]
\]

of ideals, given that we use suitable choices of bases of \( H_1(L) \) and \( H_1(\Lambda) \), as well as capping paths, and that \( \text{Aug}_\Lambda(\mu, \lambda) \) has been normalised appropriately.
Remark 8.2. The basis of $H_1(L) = \langle u, v \rangle$ should be chosen so that $u$ and $v$ are mapped to primitive classes that are the boundaries of disc of Maslov index two and zero, respectively. The capping paths in $\Lambda$ must then be chosen accordingly.

Since Vianna’s infinite family of monotone Lagrangian tori in $[49]$ can be distinguished by computations of their superpotentials, Conjecture 8.1 would imply that also their Legendrian lifts all live in different Legendrian isotopy classes.

8.2. **Sublooseness of monotone tori in the projective plane.** As described in Proposition 4.3, the Legendrian lift of any canonical twofold Bohr–Sommerfeld cover of a monotone Lagrangian torus inside $(\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{FS} \oplus \omega_{FS})$ can be seen to admit a monotone filling inside full complex line-bundle. Even if monotone fillability is weaker than exact fillability, in the present setting they should still allow us to conclude the existence of an augmentation with coefficients in $\mathbb{F}$.

Contrary to this, similarly as in the case of the Clifford and Chekanov torus, we believe that the canonical threefold Bohr–Sommerfeld cover of any monotone Lagrangian torus inside $(\mathbb{C}P^2, \omega_{FS})$ always has a subloose Legendrian lift.

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**Department of Mathematics, Uppsala University, Box 480, SE-751 06, Uppsala, Sweden**

*E-mail address: georgios.dimitroglou@math.uu.se*

**Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 18000 Praha 8, Czech Republic**

*E-mail address: golovko@karlin.mff.cuni.cz*