THE TRANSFER IS FUNCTORIAL

JOHN R. KLEIN AND CARY MALKIEWICH

Abstract. We prove that the Becker-Gottlieb transfer is functorial up to homotopy, for all fibrations with finitely dominated fibers. This resolves a lingering foundational question about the transfer, which was originally defined in the late 1970s in order to simplify the proof of the Adams conjecture. Our approach is a direct generalization of the functoriality argument for smooth fiber bundles with compact manifold fibers. It leads to a “multiplicative” description of the transfer, different from the standard presentation as the trace of a diagonal map.

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1. Introduction

A transfer is a construction that takes a map $X \to Y$ to some “wrong way” map from $Y$ to $X$. The prototype of a transfer was probably first defined by Schur in 1902 as a construction in group theory [S]. A half a
century later, the transfer idea was imported into the field of algebraic topology as a wrong way map in cohomology.

In 1972, Roush defined a transfer for finite covering spaces $X \to Y$ as a stable wrong way map $\Sigma^\infty Y_+ \to \Sigma^\infty X_+$ [R]. In 1975, this idea was expanded on by Becker and Gottlieb to define transfers for smooth fiber bundles with compact fiber [BG1]. This was used to give a “remarkably simple proof”\(^1\) of the Adams conjecture. In 1976, Becker and Gottlieb produced a purely homotopy theoretic construction of the transfer using fiberwise $S$-duality, thereby extending the definition to the class of Hurewicz fibrations with finitely dominated fibers [BG2]. For a history of such constructions and their applications, see [BG3].

The transfer is most elementary to describe for a smooth fiber bundle $p: E \to B$ with compact fibers and compact base $B$. We first observe that $p$ admits a fiberwise smooth embedding $E \subset B \times \mathbb{R}^n$ for $n$ suitably large. Let $\tau$ and $\nu$ be the vertical tangent and vertical normal bundles of $p$, respectively. The Pontryagin-Thom construction gives a map $\Sigma^n (B_+) \to E^\nu$, where $E^\nu$ denotes the Thom space of $\nu$. Composing this with the inclusion of the zero section of the tangent bundle, we get a map of spaces

$$\Sigma^n (B_+) \to E^\nu \subset E^{\nu \oplus \tau} = \Sigma^n (E_+).$$

This defines a stable map $p^!: \Sigma^\infty B_+ \to \Sigma^\infty E_+$, which is the transfer. It is well-defined up to contractible choice as $n$ tends to $\infty$.

As the Pontryagin-Thom construction is functorial on compositions of embeddings, it is straightforward to deduce that for any composition of smooth fiber bundles with compact fibers

$$X \xrightarrow{p} Y \xrightarrow{q} Z$$

the transfer is functorial up to homotopy, $(q \circ p)^! \simeq p^! \circ q^!$ (cf. §3).

Given that the transfer is defined for a larger class of fibrations, it is reasonable to ask how much more generally functoriality holds. Lewis, May and Steinberger give a functoriality statement in the context of fiber bundles with a compact Lie group $G$ as the structure group where the fibers have the structure finite $G$-CW complex [LMS, th. IV.7.1]. In [D, thm. 8.7] functoriality is proven for “parametrized Euclidean neighborhood retracts,” a condition with strong finiteness assumptions.

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\(^1\)In the words of Peter May [M].
Essentially, the objects Dold considers are $C^0$-submersions with compact base and ENR fibers (this in particular includes the case of fiber bundles with ENR fibers).

In the general case of (Hurewicz) fibrations, the question of functoriality remains open, even though four decades have transpired since the homotopy-theoretic transfer was defined.

The purpose of this paper is to settle this question in the affirmative.

**Theorem A.** Suppose $p: X \to Y$ and $q: Y \to Z$ are fibrations with finitely dominated fibers. Then the transfer $(q \circ p)^!$ coincides with the composite of transfers $p^! \circ q^!$ in the stable homotopy category.

**Remark 1.1.** The existence of the transfer always assumes a finiteness condition on the fibers. In the algebraic topology community, the most common condition is that of homotopy finiteness, i.e., spaces which are homotopy equivalent to a finite CW complex. However, for applications to manifold theory, it is more useful to work with the larger class of fibrations having finitely dominated fibers. Recall that a space is finitely dominated if it is a retract up to homotopy of a finite complex.

**Remark 1.2.** The proof of Theorem A employs the dualizing spectrum of [K]. Our argument starts with a very explicit proof of functoriality in the smooth fiber bundle case, and carefully builds a homotopy-theoretic analog of each of the terms. The result is a framework for the homotopy-theoretic transfer which is more “multiplicative” than the usual construction as the trace of a diagonal map [DP], [MS, ch. 15]. Our point of view seems to be the most natural for the question of functoriality. It would not be surprising if it had other applications as well.

There are two other approaches which come to mind in trying to establish the functoriality of the transfer. The first of these is inductive and runs as follows: for simplicity, consider the special case in which $Z = \ast$ is a point and $Y$ as being built up from a point by cell attachments. Assume $Y = Y_0 \cup D^j$ and functoriality holds for $X_0 \to Y_0 \to \ast$,

where $X_0 = Y_0 \times_Y X$ is the fiber product. Let $U = D^j \times_Y X$. Then, $X = X_0 \cup U$ and as $U \to D^j \to \ast$ is trivially functorial, we appeal to the additivity of the transfer to conclude that the transfer is functorial for $X \to Y \to Z$. Unfortunately, there are technical problems with
this argument since additivity of the transfer is known to hold only up to homotopy and there is therefore a homotopy gluing problem to deal with. One way around this might be to prove a stronger statement that additivity of the transfer holds in the $A_\infty$-sense. Unfortunately, we are not aware of how to prove this.

The other approach which comes to mind arises from considerations of Casson and Gottlieb [CG]: we attempt to thicken the fibrations to smooth fiber bundles with compact manifold fibers and then appeal to functoriality in the smooth case. The problem with this approach is that the existence of such thickenings is generally obstructed.\footnote{The obstruction lies in the (spectrum) cohomology of the base with coefficients twisted by the Whitehead spectrum of the fibers; this is the Converse Riemann-Roch Theorem of [DWW].} To kill the obstruction one has to take the cartesian product of the total space with a circle, and taking such a product is known to trivialize the transfer.

Remark 1.3. Theorem A is a vast generalization of the well-known statement that the Euler characteristic is multiplicative for fibrations of finitely dominated spaces.

To see that, we take $Z = *$ to be a point. In this case denote the transfer of the map $Y \to *$ by $\chi_Y : S \to \Sigma^\infty Y_+$. The map $\chi_Y$ represents the Euler characteristic of $Y$ in the sense that composing it with the map $\Sigma^\infty Y_+ \to S$ that collapses $Y$ to a point yields a stable map $S \to S$ whose degree coincides with the Euler characteristic of $Y$. In what follows we assume without loss in generality that $Y$ is connected.

In this case, Theorem A shows that $\chi_X \simeq p^!(\chi_Y)$, for a fibration $p : X \to Y$ with finitely dominated fibers. Applying the map $p_+^! : \Sigma^\infty X_+ \to \Sigma^\infty Y_+$ to both sides yields the equation $p_+^!(\chi_X) = p_+^! \circ p^!(\chi_Y)$. But one of the basic properties of the transfer is that $p_+^! \circ p^!: \Sigma^\infty Y_+ \to \Sigma^\infty Y_+$ is given by multiplication by the Euler-characteristic of the fiber $F$ of $p$ on singular homology. The multiplicativity of the Euler characteristic immediately follows.

Outline. §2 is about language. In §3 we exhibit functoriality in the case of smooth fiber bundles. §4 gives a definition of the transfer which involves viewing a space as a module over its Spanier-Whitehead dual and conversely viewing the Spanier-Whitehead dual as a comodule over the space; our definition coincides with the one of Becker and Gottlieb in [BG2]. In §5 we reduce the problem of checking functoriality to the
case when \( q \) is either 1-connected or a finite-sheeted covering space. In §6 we deal with the case when \( q \) is 1-connected and in §7 we handle the case when \( q \) is a finite-sheeted cover. §§6-7 form the technical heart of the paper and contain large diagrams. We have chosen to include these diagrams in the interest of transparency, so that an interested reader may check them directly. This is particularly important because there have been previous unsuccessful attempts by other authors to prove Theorem A.

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2. Notation and conventions

The spaces in this paper are taken in the compactly generated sense.

Most of our work takes place in the stable homotopy category, so it does not matter which model of spectra we use, so long as they form a symmetric monoidal model category. All of our constructions are derived, unless otherwise noted. We assume throughout that every topological group \( G \) has a nondegenerate basepoint and has the homotopy type of a CW complex.

When we must pick representatives for maps in the stable category, we will use the category of orthogonal spectra from [MMSS], although this choice is not essential for our arguments to work. We let \( S \) denote the sphere spectrum. We will frequently suppress the symbol \( \Sigma^\infty \) for suspension spectrum, so \( E_+ \) will almost always mean \( \Sigma^\infty(E_+) \).

When we mention the category of \( G \)-spectra, we always mean the most naïve theory possible. That is, the objects are spectra with an action of the topological group \( G \), and the weak equivalences are determined by forgetting the \( G \)-action.

When we define point-set maps between mapping spectra \( F(X,Y) \), we will insist that \( X \) is cofibrant up to homotopy equivalence, but we will not take a fibrant replacement of the target \( Y \). This will not get us into trouble, because \( X \) will always be a suspension spectrum of some finitely dominated space. Similarly, when we consider a point-set model for the spectrum of \( G \)-fixed maps \( F^G(X,Y) \), \( X \) will be a suspension spectrum of a \( G \)-finitely dominated space, and so we will
not need to make $Y$ fibrant. This convention will simplify some of the more complicated proofs.

3. THE FIBER BUNDLE CASE

As motivation for our argument, we consider the case in which $p : E \to B$ is a submersion of compact smooth manifolds and $q : B \to \ast$ is the trivial bundle over a point. Let $\tau^f$ denote the tangent bundle along the fibers of $p$. Then the Whitney sum $p^* \tau_B \oplus \tau^f$ is identified with $\tau_E$. Let $D_+(p)$ denote the $S$-dual of the map $p_+ : E_+ \to B_+$; this is a Pontryagin-Thom collapse map $B^{-\tau_B} \to E^{-\tau_E}$ by Atiyah duality. Similarly, we have the Thomification map $p^{-\tau_B} : E^{-p^* \tau_B} \to B^{-\tau_B}$, which dualizes to a collapse map $D(p^{-\tau_B}) : B_+ \to E^{-\tau^f}$. We also have the unit $1_B : S \to B^{-\tau_B}$ (the map representing the fundamental class), which is just the $S$-dual of the counit $B_+ \to S$. Denote the transfer of $q$ by $\chi_B : S \to B_+$; this is the composite $(\oplus \tau_B) \circ 1_B$.

The key observation is that every subdiagram of following diagram of Thom spectra commutes:

\[ \begin{array}{ccc}
\text{S} & \downarrow 1_B & \text{B}^{-\tau_B} \\
1_E & D_+ p_+ & \oplus \tau^f \\
B_+ & \oplus p^* \tau_B & E_+ \\
\oplus \tau_B & E^{-\tau^f} & \oplus \tau^f \\
\end{array} \]

The transfer $p'$ of $p$ is the bottom horizontal composite. Hence

\[ p'(\chi_B) = p'((\oplus \tau_B) \circ 1_B) = (p^* \tau_B) \circ \oplus \tau^f \circ D_+ p \circ 1_B = (p^* \tau_B) \circ \oplus \tau^f \circ 1_E = (\oplus \tau_E) \circ 1_E = \chi_E. \]

One can then pass to the case of a smooth fiber bundle $q : Y \to Z$ with fiber $B$ by performing this construction for each point of $Z$ in a continuous fashion.

The idea in what follows is to construct an analogue of the above diagram in the case of a general fibration of finitely dominated spaces.
To illustrate further, we subdivide diagram (1) as follows.

(2)

\[
\begin{array}{c}
\mathbb{S} \\
\downarrow^{\text{col}} \\
B^{-\tau_B} \xrightarrow{\text{col}} E^{-\tau_E} \xrightarrow{\Delta_M} (E \times_B E)^{-\tau_E} \xrightarrow{\nabla_M} E^{-\tau_B} \\
\downarrow^{\Delta_B} \\
B^{-\tau_B} \wedge B_+ \xrightarrow{\text{id} \wedge \text{col}} B^{-\tau_B} \wedge E^{-\tau_f} \xrightarrow{\Delta_B} E^{-\tau_E} \wedge E_+ \\
\downarrow^{\nabla_B} \\
B_+ \xrightarrow{\text{col}} E^{-\tau_f} \xrightarrow{\Delta_M} (E \times_B E)^{-\tau_f} \xrightarrow{\nabla_M} E_+ \\
\end{array}
\]

These maps all have explicit geometric interpretations. The maps labeled “col” are Pontryagin-Thom collapses. The maps labeled $\Delta$ are the evident diagonal maps or duplication maps. The map $\nabla_B$ is a scanning map, which collapses away all pairs of points whose images in $B$ are too far apart. On nearby points, $\nabla_B$ thinks of the difference between the two points as a tangent vector in $B$, and uses that tangent vector to cancel out the $-\tau_B$. The scanning maps $\nabla_E$, $\nabla_M$, and $\tilde{\nabla}_B$ are defined similarly.

It is a pleasurable exercise to check that diagram (2) is a subdivision of diagram (1) in which all the regions commute, up to homotopies with reasonable geometric interpretations. Therefore the Becker-Gottlieb transfer is functorial, for smooth fiber bundles. Our proof in the next few sections will generalize this approach to the case of fibrations with finitely dominated fibers.

4. Homotopy theoretic definition of the transfer

In this section we recall the definition of the Becker-Gottlieb transfer. We describe a “multiplicative” factorization of the transfer that corresponds to the geometric factorization from the previous section.

**Definition 4.1.** For an unbased space $X$, we write $D_+X$ for the function spectrum $F(X_+, \mathbb{S})$. When $X$ is finitely dominated, this gives the Spanier-Whitehead dual to $X_+$.

Of course, $D_+X$ is a ring spectrum, with multiplication $D(\delta_X)$ and unit 1 coming from the dualization of the diagonal $\delta_X : X \to X \times X$ and
collapse $X \to \ast$. When $X$ is finitely dominated, there are evaluation and coevaluation maps in the homotopy category

$$e_X : D_+X \wedge X_+ \to S \quad c_X : S \to D_+X \wedge X_+$$

which give the duality between $X_+$ and $D_+X$. The evaluation map is easy to define, but coevaluation is slightly subtle. Since $X$ is dualizable, we allow ourselves to use the isomorphism in the homotopy category

$$D_+X \wedge Y \cong F(X_+, Y)$$

Under this isomorphism, the coevaluation map is the lift of the identity map $S \to F(X_+, X_+)$. More generally, the smash product of coevaluation with the identity of $Y$ is the lift of the map

$$Y \to F(X_+, X_+ \wedge Y)$$

which smashes $Y$ with the identity map of $X$.

We define two maps $\nabla_X$ and $\Delta_X$ by the composites

$$\begin{array}{ccc}
D_+X & \xrightarrow{\Delta_X} & D_+X \wedge X_+ \\
\downarrow{\text{id} \wedge c_X} & & \downarrow{\text{id} \wedge \delta_X} \\
D_+X \wedge D_+X \wedge X_+ & \xrightarrow{\nabla_X} & D_+X \wedge X_+ \wedge X_+
\end{array}$$

We use $\delta_X$ to refer to the diagonal map of spaces $X \to X \times X$, to distinguish it from the map of spectra $\Delta_X$ we are defining above. We observe that $\nabla_X$ gives $X_+$ the structure of a $D_+X$-module, and dually, $\Delta_X$ gives $D_+X$ the structure of a $X_+$-comodule. In fact, $D_+X \wedge X_+$ is self-dual and the map $\Delta_X$ is just the dual of the map $\nabla_X$.

If $X$ has an action by a topological group $G$, then all the maps above are $G$-equivariant. To be more precise, we always take $G$ to act on a mapping spectrum by conjugation, so that the action on $D_+X$ is given by $(gf)(x) = f(g^{-1}x)$. We also take the diagonal $G$-action on the smash product of two $G$-spectra. We emphasize that the equivariant version of the coevaluation map $c_X$ must be the lift of the identity $S \to F(X_+, X_+)$, in the homotopy category of $G$-spectra.

Given a fibration $p : E \to B$, with $B$ connected and based, we can functorially identify $B$ with $BG$ for a suitable topological group $G$. (For example, we can take $G$ to be the geometric realization of the Kan loop group of $SB$ where the latter is the simplicial total singular complex of $B$). Hence, without loss in generality, we take $B = BG$. 


Define a $G$-space
\[ F := \lim_{\leftarrow} (EG \to B \leftarrow E) \]
where $F$ is given the $G$-action induced by the inclusion $F \subset EG \times E$ in which $G$ acts on the first factor. Then $F$ is a $G$-equivariant model for the homotopy fiber of $p$, and $E$ is identified with the Borel construction $EG \times_G F$. We adopt the convention that $G$ acts on both $F$ and $EG$ on the left, and $\times_G$ divides out the diagonal $G$-action.

**Definition 4.2.** For any fibration $p: E \to B$ with $B$ path connected, the pretransfer $\chi_F: S \to F_+$ is the stable $G$-equivariant composition
\[ S \xrightarrow{1_F} D_+ F \xrightarrow{\Delta F} D_+ F \wedge F_+ \xrightarrow{\Sigma F} F_+. \]
The transfer $p^!: B_+ \to E_+$ is obtained from the pretransfer by taking the homotopy orbits:
\[ p^!: B_+ \simeq EG_+ \wedge_G S \xrightarrow{(\chi_F)_G} EG_+ \wedge_G F_+ \simeq E_. \]
When $B$ is not path connected, we perform this operation on each component of $B$ and add the resulting maps together.

It is easy to observe that our pretransfer gives the same equivariant map as the composition
\[ S \xrightarrow{e_+} D_+ F \wedge F_+ \xrightarrow{id \wedge \Delta F} D_+ F \wedge F_+ \wedge F_+ \xrightarrow{e \wedge id} F_+. \]
Therefore our definition agrees with the usual one. This definition preserves weak equivalences, in the sense that weakly equivalent fibrations are sent to equivalent maps in the stable homotopy category. Therefore we may assume throughout that $F$ has been made cofibrant as a space with a left $G$-action.

5. **Reduction to special cases**

We begin the proof of Theorem A by reducing to two special cases, which we handle in the next two sections.

Let $X \xrightarrow{p} Y \xrightarrow{q} Z$ be fibrations with finitely dominated fibers $F$ and $B$, respectively. Our goal is to show that the transfer for $q \circ p$ agrees with the composite of the transfers for $q$ and for $p$, in the stable homotopy category. By standard additivity properties of the transfer, it suffices to consider the case where both $Z$ and $Y$ are path-connected.
Consider the diagram of pullbacks

\[
\begin{array}{ccc}
F & \rightarrow & * \\
\downarrow & & \downarrow \\
E & \rightarrow & B \rightarrow * \\
\downarrow & & \downarrow \\
X & \rightarrow & Y \rightarrow Z
\end{array}
\]

Let \( G \to P \) be a homomorphism of topological groups, each homotopy equivalent to a CW complex, whose classifying spaces are equivalent to \( Y \to Z \). Without loss of generality, the homomorphism is a fibration (one can for instance use the mapping-path construction on \( G \)), and we let \( H \) be the kernel. We remark that \( G/H \) has the homotopy type of a cell complex, and so \( G \) has the homotopy type of a free \( H \)-cell complex.

Under our assumptions, \( \pi_0 G \) must have finite index in \( \pi_0 P \cong \pi_1 Z \). Therefore it corresponds to a finite-sheeted cover \( \tilde{Z} \to Z \), through which \( q \) factors. By an easy diagram-chase, if we can prove functoriality in the special cases where \( q \) is either 1-connected or a finite-sheeted covering space, the general case follows.

6. The case of \( q \) a 1-connected fibration

In this case we may assume that \( G \to P \) is surjective. We have a short exact sequence of nonabelian topological groups

\[
1 \to H \to G \to P \to 1
\]

We replace \( q : Y \to Z \) by \( BG \to BP \). Then we pull back \( p : X \to Y \) to \( BG \), and replace it by the equivalent fibration \( EG \times_G F \to BG \).

It suffices to prove functoriality for our new fibrations, whose fibers may be arranged in the homotopy-commuting diagram of homotopy pullback squares

\[
\begin{array}{ccc}
F & \rightarrow & * \\
\downarrow & & \downarrow \\
EG \times_H F & \rightarrow & EG \times_H * \rightarrow * \\
\downarrow & & \downarrow \\
EG \times_G F & \rightarrow & BG \quad \rightarrow \quad BP
\end{array}
\]
We recall that $H$ is a normal subgroup of $G$, and both $F$ and $BH = EG \times_H *$ are finitely dominated spaces. To set up the rest of our argument, we recall the notion of the dualizing spectrum from [K].

**Definition 6.1.** The dualizing spectrum $D_H$ is defined as the derived homotopy fixed point spectrum

$$S[H]^{hH} = F^H(EG_+, S[H]),$$

The $H$ superscript denotes maps which are $H$-equivariant. As before, $EG$ has a free left $G$-action, and $H$ acts on $S[H] = \Sigma^{\infty}H_+$ by left translation. We make $D_H$ a $G$-spectrum by the action

$$(gf)(s) := gf(g^{-1}s)g^{-1}$$

For any $G$-spectrum $E$, one has a $G/H$-equivariant norm map

$$D_H \wedge_{hH} E \to E^{hH},$$

given explicitly by a composite of assembly, cancellation, and the collapse of $EG$ to a point:

$$F^H(EG_+, S[H]) \wedge_H (EG_+ \wedge E) \cong F^H(EG_+, S[H] \wedge_H (EG_+ \wedge E))$$

$$\cong F^H(EG_+, EG_+ \wedge E)$$

$$\cong F^H(EG_+, E)$$

In the top-right mapping space, the maps are equivariant with respect to the left $H$-action acting only on $S[H]$ by left translation. The $\wedge_H$ is then formed from the right translation action, and the remaining action on $EG_+ \wedge E$. Under these conventions, the norm map is a map of $G/H$-spectra, and it is an equivalence because $BH$ is finitely dominated ([K], Theorem D).

Next we define the homotopy theoretic analog of the maps $\Delta_B$ and $\nabla_B$ from §3. We continue to assume that $BH$ is finitely dominated and $H \leq G$ is a normal subgroup.

**Definition 6.2.** For any $G$-spectrum $X$, define two natural $G/H$-equivariant maps in the homotopy category

$$X^{hH} \xrightarrow{\mu} S^{hH} \wedge X_{hH} \xrightarrow{\mu} X_{hH}$$
by the zig-zags
\[
F^H(EG_+, X) \xrightarrow{\sim} F^H(EG_+, EG_+ \land EG_+ \land X) \\
\xrightarrow{\sim} F^H(EG_+, EG_+ \land EG_+ \land H X) \\
\xrightarrow{\sim} F^H(EG_+, EG_+) \land EG_+ \land H X \\
\xrightarrow{\mu} EG_+ \land H X
\]
The map \(\mu\) is the action of self-maps of \(EG\) on \(EG_+ \land H X\). The \(G\)-actions are the usual conjugation actions.

Remark 6.3. The map \(\sim\) is only defined when \(BH\) is finitely dominated, while \(\mu\) has no such restriction. Although \(\sim\) needs to be defined as a zig-zag, if we model \(X^{hH}\) by \(\mathcal{D}_{H \land hH} X\), then the zig-zag is no longer necessary.

Proposition 6.4. When \(X = S\), the maps \(\sim\) and \(\mu\) coincide in the \(G/H\)-equivariant homotopy category with \(\Delta_{BH}\) and \(\nabla_{BH}\), respectively.

Proof. We begin by comparing \(\sim\) with the definition of \(\Delta_{BH}\), from §4. Since coevaluation is given by the unit \(u\) of an adjunction, we can write \(\Delta_{BH}\) as the composite
\[
F(BH_+, \ast_+) \xrightarrow{\land id_{BH}} F(BH_+ \land BH_+, BH_+) \xrightarrow{- \circ \delta_{BH}} F(BH_+, BH_+) \\
\xrightarrow{\sim} F(BH_+, F(BH_+, \ast_+) \land BH_+)
\]
The two maps we wish to compare expand to
\[
F^H(EG_+, EG_+ \land EG_+) \xrightarrow{\sim} F^H(EG_+, EG_+ \land BH_+) \xrightarrow{- \circ \delta_{BH}} F^H(EG_+, EG_+) \land BH_+ \\
F^H(EG_+, \ast_+) \xrightarrow{- \circ \delta_{BH}} F(BH_+, BH_+) \xrightarrow{\sim} F^H(EG_+, \ast_+) \land BH_+
\]
The unmarked equivalences all collapse \(EG\) to a point, except the one in the bottom-right, which is the assembly equivalence. (We use this as our preferred identification between \(F(BH_+, BH_+)^\) and \(D_+ BH \land BH_+\) throughout this section.) The right-hand rectangle easily commutes,
while the left can be simplified and subdivided as

\[(3)\]

\[
\begin{array}{c}
F_H(EG_+, EG_+) \quad /H \\
\sim \\
\sim \\
F_H(EG_+, BH_+) \quad /H \\
\end{array}
\]

The only nontrivial part of (3) is the triangle on the left, which commutes because all maps are isomorphisms in the homotopy category, and the composition that starts and ends in the lower-left corner is the identity as a strict map of spectra.

To prove that \( \mu \) coincides in the homotopy category with \( \nabla_{BH} \), we prove the commutativity of the upper-left triangle in this diagram:

\[
\begin{array}{c}
F_H(EG_+, EG_+) \quad /H \\
\sim \\
\sim \\
F_H(EG_+ \wedge BH_+, BH_+) \\
\end{array}
\]

But the lower-right triangle commutes by a straightforward diagram chase, and the outer square is the adjoint of diagram (3).

We conclude that the maps \( /H \) and \( \mu \) are natural variants of \( \Delta_B \) and \( \nabla_B \) for spectra parametrized over \( B \simeq BH \). In fact, when \( B \) is a compact manifold, they agree with the fiberwise variants of \( \Delta_B \) and \( \nabla_B \) that we alluded to at the end of \( \S3 \). We omit the proof since we will not use this directly.

This motivates us to draw the following homotopy theoretic analog of the geometric diagram from \( \S3 \). Rather than proving the commutativity of this diagram in a parametrized fashion over \( Z \simeq BP \), we will show that it commutes in the homotopy category of spectra with an action
of the group $P = G/H$. Here $B = EG \times_H *$ and $E = EG \times_H F$.

(4)

$$S \downarrow \downarrow \quad \downarrow_{1_B} \quad \downarrow_{1_E} \quad \downarrow_{(1_E)^hP} \quad \downarrow_{(1_E)^{hH}}$$

$$S^{hH} \quad \downarrow_{/H} \quad \downarrow_{/H} \quad \downarrow_{/H} \quad \downarrow_{/H}$$

$$\downarrow_{/H} \quad \downarrow_{/H} \quad \downarrow_{/H} \quad \downarrow_{/H}$$

$$S^{hH} \quad \downarrow_{\mu} \quad \downarrow_{\mu} \quad \downarrow_{\mu} \quad \downarrow_{\mu}$$

$$S^{hH} \wedge S^{hH} \quad \downarrow_{(1_E)^{hP}} \quad \downarrow_{(1_E)^{hH}} \quad \downarrow_{(1_E)^{hP}} \quad \downarrow_{(1_E)^{hH}}$$

All of these maps have been defined as $P$-equivariant maps, except for the map $\sim_{/H}$, which we will define when we check that the diagram commutes.

**Proposition 6.5.** The diagram (4) commutes in the homotopy category of $P$-equivariant spectra.

Proposition 6.5 will immediately imply Theorem A in the 1-connected case, because we may take the homotopy $P$-orbits of diagram (4) and use the natural equivalence

$$EP_+ \wedge_P (EG_+ \wedge_H X) \sim \ast_+ \wedge_P (EG_+ \wedge_H X) \cong EG_+ \wedge_G X$$

to get the commuting triangle

$$S^{hP} \quad \downarrow_{(1_E)^{hP}} \quad \downarrow_{(1_E)^{hH}} \quad \downarrow_{(1_E)^{hP}} \quad \downarrow_{(1_E)^{hH}}$$

$$\downarrow_{/H} \quad \downarrow_{/H} \quad \downarrow_{/H} \quad \downarrow_{/H}$$

$$\downarrow_{\mu} \quad \downarrow_{\mu} \quad \downarrow_{\mu} \quad \downarrow_{\mu}$$

$$S^{hP} \quad \downarrow_{(1_E)^{hG}} \quad \downarrow_{(1_E)^{hG}} \quad \downarrow_{(1_E)^{hG}} \quad \downarrow_{(1_E)^{hG}}$$

The bottom composite is by definition the transfer map $p^! : Y_+ \to X_+$.

The diagonal composite is the transfer $(q \circ p)^! : Z_+ \to X_+$, so long as we are careful to make sure that $\Delta_E$ and $\nabla_E$ have the correct equivariant definitions. The left-hand composite agrees with the transfer $q^! : Z_+ \to Y_+$ using Proposition 6.4.
Proof. (of Proposition 6.5) In diagram (4), the units easily commute, and naturality of $\mathcal{L}/H$ and $\mu$ take care of both the left-hand rectangle and the large square on the right. We are left with a small triangle and a trapezoid. To check these commute in the homotopy category, we expand out our definitions of $\Delta(-)$ and $\nabla(-)$ as equivariant maps, and reduce to checking that various strict maps of spectra strictly commute.

By our definition of $\Delta(-)$ as an equivariant map, the small triangle expands to

$$
\begin{align*}
(D_+ F)^{hH} \xrightarrow{(c_F)^{hH}} (D_+ F \wedge D_+ F \wedge F_+)^{hH} \xrightarrow{(\circ \delta_F)^{hH}} (D_+ F \wedge F_+)^{hH} \\
D_+ E \wedge D_+ E \wedge E_+ \xrightarrow{\delta_E} D_+ E \wedge E_+
\end{align*}
$$

We get to pick the map $\widetilde{\mathcal{L}/H}$. We define it just as $\mathcal{L}/H$ was defined earlier, the only difference being that the source in the mapping spectrum is $EG_+ \wedge F_+$ instead of $EG_+$. 

---
The triangle (5) then expands further as

\[ (6) \]

\[ F^H(EG_+ \land F_+ \land F_+ \land EG_+) \xrightarrow{\land id_F} F^H(EG_+ \land F_+ \land F_+ \land EG_+) \xrightarrow{-\delta_F} F^H(EG_+ \land F_+, EG_+ \land F_+) \xrightarrow{\sim} \]

\[ \sim \]

\[ F^H(EG_+ \land F \land EG_+, EG_+) \xrightarrow{\delta_{EG}} \sim \]

\[ F^H(EG_+ \land F_+ \land EG_+ \land F_+, EG_+ \land F_+) \xrightarrow{\delta_{EG \times F}} \]

\[ F(E_+ \land H F_+ \land EG_+ \land H F_+, EG_+ \land H F_+) \xrightarrow{\delta_{EG \times H F}} F(E_+ \land H F_+, EG_+ \land H F_+) \]

The left-hand triangle commutes in the homotopy category by the same argument as in Proposition 6.4 above. The rest of the regions commute by a straightforward diagram chase.
It remains to check the trapezoid-shaped region on the right-hand side of (4). By our definition of $\nabla_{(-)}$ as an equivariant map, this expands as

(7) \[
(D + F \wedge F^+)^hH \xrightarrow{(\delta F)^{hH}} (D + F \wedge F^+ \wedge F^+)^hH \xrightarrow{(e_F)^{hH}} (F^+)^{hH}
\]

\[
\begin{array}{c}
D + E \wedge E^+ \xrightarrow{\delta_E} D + E \wedge E^+ \wedge E^+ \\
\mu \downarrow \downarrow \downarrow
\end{array}
\]

We define the new map $\tilde{\beta}$ so that the left rectangle of (7) commutes:

\[
\begin{array}{c}
F^H(EG +, EG + \wedge F^+) \xrightarrow{\delta_{EG \times F}} F^H(EG +, EG + \wedge F + \wedge EG + \wedge F +) \\
\tilde{\beta} \downarrow \downarrow \downarrow
\end{array}
\]

\[
\begin{array}{c}
F^H(EG +, F +, EG + \wedge H F +) \xrightarrow{\delta_{EG \times H F}} F^H(EG +, F +, EG + \wedge H F + \wedge EG + \wedge H F +) \\
\tilde{\beta} \downarrow \downarrow \downarrow
\end{array}
\]

For the final trapezoid of (7), it seems essential to use the dualizing spectrum $\mathcal{D}_H$ from Definition 6.1, so that both $\beta$ and $\mu$ can be written as strict maps and not zig-zags. The maps $\beta$ and $\tilde{\beta}$ can be defined on the source of the norm map $\eta$ just as easily as they can be defined on the target, for instance:

\[
\begin{array}{c}
F^H(EG +, EG + \wedge F^+) \xleftarrow{\eta} F^H(EG +, S[H]) \wedge_H (EG + \wedge EG + \wedge F^+) \\
\eta \downarrow \downarrow \downarrow
\end{array}
\]

\[
\begin{array}{c}
F^H(EG +, EG + \wedge H F^+) \xleftarrow{\eta} F^H(EG +, S[H]) \wedge_H EG + \wedge EG + \wedge H F^+
\end{array}
\]

To make everything derived, we also assume that $\mathcal{D}_H = F^H(EG, S[H])$ has been made cofibrant, and we define maps out of the cofibrant approximation by defining them on $\mathcal{D}_H$ itself. This transforms the final
trapezoid of (7) into the following diagram. We suppress the + decorations to save space.

(8)
\[
\begin{array}{c}
F^H(EG, S[H]) \wedge_H (EG \wedge DF \wedge EG \wedge F^{(1)} \wedge F^{(2)}) \\
\sim \delta_{EG} \\
F^H(EG, S[H]) \wedge_H (EG \wedge DF \wedge EG \wedge F^{(1)} \wedge EG \wedge F^{(2)}) \\
\xrightarrow{i_H} \xrightarrow{e} F^H(EG, S[H]) \wedge_H (EG \wedge EG \wedge EG \wedge F^{(2)}) \\
F^H(EG, S[H]) \wedge_H (EG \wedge DF) \wedge EG \wedge_H F^{(1)} \wedge EG \wedge_H F^{(2)} \\
\xrightarrow{\eta} F^H(EG, S[H]) \wedge_H (EG \wedge EG \wedge EG \wedge F^{(2)}) \\
F^H(EG, DF \wedge EG \wedge_H F^{(2)}) \wedge EG \wedge_H F^{(1)} \\
\sim \eta \\
F^H(EG, EG) \wedge EG \wedge_H F^{(2)} \\
\xrightarrow{\mu} F^H(EG \wedge F, EG \wedge_H F^{(2)}) \wedge EG \wedge_H F^{(1)} \\
\xrightarrow{e} \xrightarrow{\mu} EG \wedge_H F^{(2)}
\end{array}
\]

We have used the labels $F^{(1)}$ and $F^{(2)}$ to denote the same space $F$, emphasizing that the two copies play different roles. One checks that both routes from the top of the diagram to the bottom commute as strict maps of spectra, and so the parallelogram in (8) commutes in the homotopy category. This finishes the proof of Proposition 6.5. □

7. The Case of $q$ a Finite Covering Space

In this case, instead of $P = G/H$, the homomorphism $G \to P$ is an inclusion of connected components with finite index $n$. The fiber of the covering map $q : Y \to Z$ is the finite $P$-set $P/G$ with cardinality $n$. The space $F$ still has a $G$ action, and $E = P \times_G F$, which is non-canonically equivalent to $n$ disjoint copies of $F$. The diagram of
homotopy pullbacks from §5 may be rewritten as

\[
\begin{array}{c}
\downarrow & & \\
F & \rightarrow & \star \\
\downarrow & & \\
P \times_G F & \rightarrow & P/G & \rightarrow & \star \\
\downarrow & & \\
EP \times_G F & \rightarrow & EP \times_G * & \rightarrow & BP
\end{array}
\]

Since \( B = P/G \) consists of \( n \) points, the composite \( \nabla_B \circ \Delta_B \) is an equivalence, so our usual form for the diagram can be made simpler. We will prove commutativity of the diagram of \( P \)-equivariant maps (9)

\[
\begin{array}{c}
S \\
\downarrow^{1_B} & \downarrow^{1_E} & \\
F^G(P_+, S) & \rightarrow & F^G(P_+, D_+ F) \\
\downarrow \sim & \downarrow \sim & \downarrow \Delta_E \\
P_+ \wedge_G S & \rightarrow & P_+ \wedge_G D_+ F & \rightarrow & (P_+ \wedge_G F_+) \wedge (P_+ \wedge_G F_+) & \rightarrow & P_+ \wedge_G F_+
\end{array}
\]

We explain a few conventions for this section. It is desirable to have \( P \times_G F \) formed from the left \( G \)-action on \( F \) and the right \( G \)-action on \( P \), so we also adopt this convention for \( EP \times_G F \) as well, and assume that \( P \) acts on \( EP \) on the right. To accommodate an isomorphism of \( P \)-spectra

\[
F^G(P_+, D_+ F) \cong F^G(P_+ \wedge G, F_+) \cong F(P_+ \wedge G, F_+, *)
\]

we give the \( P \) in the source a left \( G \)-action by multiplication on the right by the inverse. If \( X \) is any \( G \)-spectrum, we let \( F^G(P_+, X) \) refer to maps that send this left \( G \)-action on \( P \) to the left \( G \)-action on \( X \):

\[
f(pg^{-1}) = gf(p)
\]

Then \( F^G(P_+, X) \) has a left \( P \)-action given by

\[
(pf)(p') = f(p^{-1}p')
\]

It is then straightforward to verify the isomorphism (10).
We freely use the following variant of the Wirthmüller isomorphism. For any $G$-spectrum $X$, define a map
\[ P_+ \wedge_G X \xrightarrow{\sim} F^G(P_+, X) \]
sending $p, x$ to a map which vanishes outside the left coset of $G$ containing $p$, and on that left coset it sends $pg^{-1}$ to $gx$. This defines a $P$-equivariant map, which is nonequivariantly the inclusion of a finite wedge into a finite product, and is therefore an equivalence. At one point, we will also use a refinement of this equivalence
\[ P_+ \wedge_G F(X, Y) \xrightarrow{\sim} F^G(P_+ \wedge X, Y) \]
which sends $p, f$ to a map that takes $pg^{-1}, x$ to $gf(g^{-1}x)$.

**Proposition 7.1.** The diagram (9) commutes in the homotopy category of $P$-equivariant spectra.

Again, we have set this up so that when we take homotopy $P$-orbits, the left-hand column gives $q'$, the bottom row gives $p'$, and the diagonal gives $(q \circ p)'$. Therefore the proof of this proposition will conclude the proof of Theorem A.

**Proof.** Throughout this proof, $p$ and $q$ will refer to elements of $P$, not the fibrations $X \to Y \to Z$. As before, only the trapezoid in the middle and the triangle in the lower-right of (9) pose any difficulty. We expand the trapezoid as (11) and the triangle as (12) on the next two pages. Let $G^2 = G \times G$. Then when we write spectra such as
\[ P_+^2 \wedge_{G^2} (X \wedge Y) \quad \text{or} \quad P_+^2 \wedge_{G^2} F(X, Y) \]
we adopt the convention that the first copy of $P$ acts only on $X$, and the second copy of $P$ acts only on $Y$. Each map labeled $|\Delta_P$ duplicates the $P$ coordinate. The maps labeled $|\Delta_P$ restrict from $P_+^2 \wedge_{G^2} (X \wedge Y)$ to $P_+ \wedge_G (X \wedge Y)$, by collapsing to the basepoint each summand in which the pair of elements in $P$ are not identical, up to the right action of $G^2$. The equivalences are all assembly maps and Wirthmüller isomorphisms.

In diagram (11), the three regions making up the big square in the top-right commute easily. Of the remaining regions, from left to right, in the first both branches are given by the formula
\[ p, f \mapsto (pg^{-1}, x) \mapsto p, f(g^{-1}x), g^{-1}x) \]
where the map on the right vanishes on $q, x$ when $q$ is in a different left $G$-coset from $p$. The function $f$ is a function from $F_+$ into some level
of the sphere spectrum, and the notation $f(x), x$ refers to the a point in the same level of the sphere spectrum smashed with $F_+$. The next region has both branches given by

$$p, pg^{-1}, q, f_1, f_2, x \mapsto p, q, (y \mapsto f_1(y), f_2(gy), x)$$

Here we mean that if the first two coordinates in the source are in different $G$-cosets then the map vanishes, and otherwise it is given by this formula. The last region has both branches given by

$$p, q, f, x \mapsto (pg^{-1}, y \mapsto q, f(g^{-1}y), x)$$

Again, the map we get at the end is nonzero only on the coset of $p$.

In diagram (12), all regions commute easily, except for the right-hand region, where both branches have the formula

$$p, f, x_1, x_2 \mapsto f(x_1), p, x_2$$

This finishes the proof. □
\[(12)\]

\[
P \land_G (D_+ F \land F_+) \xrightarrow{\delta_F} P_+ \land_G (D_+ F \land F_+^{(1)} \land F_+^{(2)}) \xrightarrow{\epsilon_1} P_+ \land_G F_+^{(2)}
\]

\[
P_+ \land_G D_+ F \land P_+ \land_G F_+ \xrightarrow{\delta_{P \times_G F}^F} P_+ \land_G D_+ F \land P_+ \land_G F_+^{(1)} \land P_+ \land_G F_+^{(2)}
\]

\[
P_+ \land_G D_+ F \land P_+ \land_G F_+ \xrightarrow{\sim} P_+ \land_G F_+^{(2)}
\]

\[
P_+ \land_G D_+ F \land P_+ \land_G F_+ \xrightarrow{\sim} P_+ \land_G F_+^{(2)}
\]

\[
P_+ \land_G D_+ F \land P_+ \land_G F_+ \xrightarrow{\sim} P_+ \land_G F_+^{(2)}
\]

\[
P_+ \land_G D_+ F \land P_+ \land_G F_+ \xrightarrow{\sim} P_+ \land_G F_+^{(2)}
\]

\[
P_+ \land_G D_+ F \land P_+ \land_G F_+ \xrightarrow{\sim} P_+ \land_G F_+^{(2)}
\]

\[
P_+ \land_G D_+ F \land P_+ \land_G F_+ \xrightarrow{\sim} P_+ \land_G F_+^{(2)}
\]
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Department of Mathematics, Wayne State University, Detroit, MI 48202

E-mail address: klein@math.wayne.edu

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801

E-mail address: cmalkiew@illinois.edu