A LEVEL-ONE REPRESENTATION OF THE QUANTUM AFFINE SUPERALGEBRA $U_q(\hat{sl}(M+1|N+1))$

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Abstract. A level-one representation of the quantum affine superalgebra $U_q(\hat{sl}(M+1|N+1))$ and vertex operators associated with the fundamental representations are constructed in terms of free bosonic fields. Character formulas of level-one irreducible highest weight modules of $U_q(\hat{sl}(2|1))$ are conjectured.

1. Introduction

In the recent studies of the mathematical physics and solvable systems [DFJMN] (see also references in [JM]), affine quantum algebras [D1,J] played an important role. In these works, techniques of the representation theory, infinite dimensional highest weight modules and corresponding vertex operators etc. have proved to be very powerful to study low-dimensional systems. Therefore, it may be natural to expect that representation theories of the affine Lie superalgebras or their quantum analogues will greatly help our future studies (as for applications to the number theory see [KWk1]). The defining relations of the quantum affine superalgebras are obtained by Yamane [Y]. In that paper, the Drinfeld realization is also studied [D2]. The representation theories of the Lie superalgebras are much more complicated than non-super cases and have reach structures [K1,K2,K3,FSS,KWn,KWk1]. Hence, to obtain concrete representation spaces is desirable. The aim of this article is to construct a level-one representation of $U_q(\hat{sl}(M+1|N+1))$ by bosonizing the Drinfeld generators basing on the free boson representation of $\hat{sl}(M+1|N+1)$ obtained in [BCMN] and study character formulas for highest weight modules.

This paper is organized as follows. In this section we review the definition of the quantum affine superalgebra $U_q(\hat{sl}(M+1|N+1))$ and construct level-zero representations. In Section 2, we study the bosonization of $U_q(\hat{sl}(M+1|N+1))$ and using that, character formulas for level-one irreducible highest weight modules of $U_q(\hat{sl}(M+1|N+1))$ are conjectured. Section 3 is devoted to the study of the bosonization of the vertex operators.
1.1 Quantum affine super algebra $U_q(\widehat{\mathfrak{sl}}(M+1|N+1))$. We will study the quantum affine superalgebra $U_q(\widehat{\mathfrak{sl}}(M+1|N+1))$ for $M, N = 0, 1, \cdots$ and we will restrict our analysis to $M \neq N$. The Cartan matrix of the affine Lie superalgebra $\widehat{\mathfrak{sl}}(M+1|N+1)$ is

\[
(a_{ij}) = \begin{pmatrix}
0 & -1 & 1 \\
-1 & 2 & -1 \\
1 & -2 & 1 \\
\end{pmatrix}
\]

(0 ≤ i, j ≤ M+N+1),

where the diagonal part is $(a_{ii}) = (0, 2, \cdots, 2, 0, -2, \cdots, -2)$. Let us introduce orthonormal basis $\{\varepsilon'_i|i=1, \cdots, M+N+2\}$ with the bilinear form $(\varepsilon'_i|\varepsilon'_j) = \nu_i\delta_{i,j}$, where $\nu_i = 1$ for $i = 1, \cdots, M+N+1$ and $\nu_j = -1$ for $i = M+2, \cdots, M+N+2$. Define $\varepsilon_i = \varepsilon'_i - \nu_i \sum_{j=1}^{M+N+2} \varepsilon'_j/(M-N)$. The classical simple roots are defined by $\alpha_i = \nu_i \varepsilon'_i - \nu_{i+1} \varepsilon'_{i+1}$ and the classical weights are $\Lambda_i = \sum_{j=1}^{i} \varepsilon_j$ for $i = 1, \cdots, M+N+1$. Introduce the affine weight $\Lambda_0$ and the null root $\delta$ having $(\Lambda_0|\varepsilon'_i) = (\delta|\varepsilon'_i) = 0$ for $i = 1, \cdots, M+N+2$ and $(\Lambda_0|\Lambda_0) = (\delta|\delta) = 0, (\Lambda_0|\delta) = 1$. The other affine weights and affine roots are given by $\Lambda_i = \Lambda_{i} + \Lambda_0$ and $\alpha_i = \alpha_i$ for $i = 1, \cdots, M+N+1$ and $\alpha_0 = \delta - \sum_{i=1}^{M+N+1} \alpha_i$. Let $P = \oplus_{i=0}^{M+N+1} \mathbb{Z} \Lambda_i \oplus \mathbb{Z} \delta$ and $P^* = \oplus_{i=0}^{M+N+1} \mathbb{Z} h_i \oplus \mathbb{Z} d$ be the affine weight lattice and its dual lattice, respectively.

The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}(M+1|N+1)) [Y]$ is a $q$-analogue of the universal enveloping algebra of $\widehat{\mathfrak{sl}}(M+1|N+1)$ generated by the Chevalley generators, $e_i, f_i, t_i^{\pm 1}$ over the base field $\mathbb{Q}(q)$. The $\mathbb{Z}_2$-grading $| \cdot : U_q(\widehat{\mathfrak{sl}}(M+1|N+1)) \rightarrow \mathbb{Z}_2$ of the generators are: $|e_0| = |f_0| = |e_{M+1}| = |f_{M+1}| = 1$ and zero otherwise. The relations among these generators are

\[ t_i t_j = t_j t_i, \]
\[ t_i e_j t_i^{-1} = q^{a_{ij}} e_j, \]
\[ t_i f_j t_i^{-1} = q^{-a_{ij}} f_j, \]
\[ [e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}}, \]

\[
\begin{align*}
|e_j, [e_j, e_i]_{q^{-1}}|_q = 0 \\
[f_j, [f_j, f_i]_{q^{-1}}]_{q} = 0
\end{align*}
\]

for $|a_{ij}| = 1, i \neq 0, M+1$ \hspace{1cm} (1.6)

\[
\begin{align*}
|e_l, [e_k, e_l, e_m]_{q^{-1}}|_q = 0 \\
[f_l, [f_k, f_l, f_m]_{q^{-1}}]_{q} = 0
\end{align*}
\]

for $(k, l, m) = (M+2, M+1, M), (1, 0, M+N+1), \hspace{1cm} (1.7)$
A Level-One Representation of the Quantum Affine Superalgebra $U_q(\widehat{\mathfrak{sl}(M+1|N+1)})$

where we have used the notations $[X,Y]\xi = XY - (-1)^{|X||Y|}YX$. Here and hereafter, we write $[X,Y]_i$ as $[X,Y]$ for simplicity. If $M = 0$ or $N = 0$, we have extra fifth order Serre relations. As for the explicit forms of the extra Serre relations, we will refer the reader to ref. [Y].

The quantum affine super algebra $U_q(\widehat{\mathfrak{sl}(M+1|N+1)})$ can be endowed with the graded Hopf algebra structure. We take the following coproduct

$$
\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta(t_i^{\pm 1}) = t_i^{\pm 1} \otimes t_i^{\pm 1}, \quad (1.8)
$$

and the antipode

$$
a(e_i) = -t_i^{-1}e_i, \quad a(f_i) = -f_it_i, \quad a(t_i^{\pm 1}) = t_i^{\mp 1}. \quad (1.9)
$$

The coproduct is an algebra automorphism $\Delta(xy) = \Delta(x)\Delta(y)$ and the antipode is a graded algebra anti-autoorphism $a(xy) = (-1)^{\alpha(x)\beta(y)}a(y)a(x)$ for $x, y \in U_q(\widehat{\mathfrak{sl}(M+1|N+1)})$.

Let $V$ and $W$ be graded representations of $U_q(\widehat{\mathfrak{sl}(M+1|N+1)})$. Hereafter, the $\mathbb{Z}_2$-grading on representation space will also be denoted by $\cdot$. The graded action of $U_q(\widehat{\mathfrak{sl}(M+1|N+1)}) \otimes U_q(\widehat{\mathfrak{sl}(M+1|N+1)})$ on the tensor representation $V \otimes W$ is defined by $x \otimes y \cdot v \otimes w = (-1)^{|y||v|}xv \otimes yw$ for $x, y \in U_q(\widehat{\mathfrak{sl}(M+1|N+1)})$ and $v \in V, w \in W$.

In order to construct the bosonic representation of $U_q(\widehat{\mathfrak{sl}(M+1|N+1)})$, we give another realization of $U_q(\widehat{\mathfrak{sl}(M+1|N+1)})$ using the Drinfeld basis [Y]: $\{X_{m}^{+}, h_{n}^{i}, (K^{i})^{\pm 1}, \gamma^{\pm 1/2} | i = 1, \ldots, M+N+1, m \in \mathbb{Z}, n \in \mathbb{Z} \neq 0\}$. The $\mathbb{Z}_2$-grading of the Drinfeld generators are: $|X_{m}^{+}\gamma^{M+1}| = 1$ for $m \in \mathbb{Z}$ and zero otherwise. The relations are

$$
\gamma \text{ is central,} \quad (1.10)
$$

$$
[K^{i}, h_{m}^{j}] = 0, \quad [h_{m}^{i}, h_{n}^{j}] = \delta_{m+n,0} \frac{[a_{ij}m](\gamma^{m} - \gamma^{-m})}{m(q - q^{-1})}, \quad (1.11)
$$

$$
K^{i}X_{m}^{\pm j} = q^{\pm a_{ij}}X_{m}^{\pm j}K^{i}, \quad (1.12)
$$

$$
[h_{m}^{i}, X_{n}^{\pm j}] = \pm \frac{[a_{ij}m]}{m} \gamma^{\pm |m|/2}X_{n+m}^{\pm j}, \quad (1.13)
$$

$$
[X_{n}^{+,i}, X_{n}^{-,j}] = \delta_{i,j} \frac{1}{q - q^{-1}}(\gamma^{(m-n)/2}\psi_{m+n}^{+,i} - \gamma^{-(m-n)/2}\psi_{m+n}^{-,j}), \quad (1.14)
$$

$$
[X_{n}^{+\pm i}, X_{n}^{\pm j}] = 0 \quad \text{for} \quad a_{ij} = 0, \quad (1.15)
$$

$$
[X_{n+1}^{\pm i}, X_{n}^{\pm j}]_{q^{a_{ij}}} + [X_{n}^{\pm j}, X_{n+1}^{\pm i}]_{q^{a_{ij}}} = 0, \quad \text{for} \quad a_{ij} \neq 0 \quad (1.16)
$$

$$
\text{Sym}_{i,m}[X_{i}^{\pm i}, [X_{m}^{\pm i}, X_{n}^{\pm j}]_{q^{-1}}]_{q} = 0, \quad \text{for} \quad a_{ij} = 0, i \neq M+1 \quad (1.17)
$$

$$
\text{Sym}_{k,m}[X_{k}^{\pm, M+1}, [X_{i}^{\pm, M+2}, [X_{m}^{\pm, M+1}, X_{n}^{\pm, M}]]_{q^{-1}}]_{q} = 0, \quad (1.18)
$$

where

$$
\sum_{m \in \mathbb{Z}} \psi_{m}^{\pm i}z^{-m} = (K^{i})^{\pm 1}\exp \left(\pm(q - q^{-1})\sum_{m>0}h_{\pm n}^{i}z^{\mp n}\right), \quad (1.19)
$$
and the symbol Sym$_{k,l}$ means symmetrization with respect to $k$ and $l$. We used the standard notation $[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$. If $M = 0$ or $N = 0$, we have extra fifth order Serre relations for the Drinfeld generators. As for the explicit forms, we will refer the reader to the ref. [Y].

The Chevalley generators are obtained by the formulas:

\begin{align}
t_i &= K_i, \quad e_i = X_{i,i}^+, \quad f_i = X_{i,i}^- \quad \text{for } i = 1, \ldots, M + N + 1 \\
t_0 &= \gamma K_1^{-1} \cdots K_{M+N+1}^{-1} \\
e_0 &= (-1)^{N+1} [X_0^{-M+N+1} \cdots [X_0^{-M+2}, [X_0^{-M+1} \cdots, [X_0^{-2}, X_1^{-1}]_{q-1} \cdots]_{q-1}]_q]_q \\
f_0 &= K_1 \cdots K_{M+N+1} [\cdots [[X_{-1}^+, X_0^{+2}q]_{q} \cdots X_0^{+M+1}q, X_0^{+M+2}q_{q-1} \cdots, X_0^{+M+N+1}q]_{q-1}^{-1}
\end{align}

### 1.2 level-zero representations of $U_q(\mathfrak{sl}(M+1|N+1))$

Let $E_{i,j}$ be the $(M + N + 2) \times (M + N + 2)$ matrix whose $(i,j)$-element is unity and zero elsewhere, set $v_i = t^i (0, \ldots, 1, 0, \ldots, 0, \ldots, 0)$ for $i = 1, \ldots, M + N + 2$. We will adopt the $\mathbb{Z}_2$-grading to the basis by $|v_i| = (\nu_i + 1)/2$. For the sake of simplicity, we will not study another possibility $|v_i| = (-\nu_i + 1)/2$ in this article. The $M + N + 2$ dimensional level-zero representation $V_z$ of $U_q(\mathfrak{sl}(M+1|N+1))$ with basis $\{v_i \otimes z^n | i, 1, \ldots, M + N + 2, n \in \mathbb{Z}\}$ is defined by

\begin{align}
e_i &= E_{i,i+1}, \quad f_i = \nu_i E_{i+1,i}, \quad t_i = q^{\nu_i} E_{i,i+1}, \quad t_i = q^{-\nu_i} E_{i+1,i+1}, \\
e_0 &= -z E_{M+N+2,1}, \quad f_0 = z^{-1} E_{1,M+N+2}, \quad t_0 = q^{-E_{1,1} - E_{M+N+2,1}} - E_{M+N+2,1} + E_{M+N+2,1}
\end{align}

for $i = 1, \ldots, M + N + 1$. Let $V^*_z$ be the dual space of $V_z$ with basis $\{v^*_i \otimes z^n | i, 1, \ldots, M + N + 2, n \in \mathbb{Z}\}$ such that $\langle v^*_i \otimes z^m, v_j \otimes z^n \rangle = \delta_{i,j} \delta_{m,n}$. We also regard $v^*_i$ as the vector $v_i = t^i (0, \ldots, 1, 0, \ldots, 0, \ldots, 0)$. The $U_q(\mathfrak{sl}(M+1|N+1))$-module structure is given by $\langle xv, w \rangle = \langle v, (\nu_1 - 1)_{|x|} a(x)w \rangle$ for $v \in V^*_z, w \in V_z$ and we call the module as $V^*_z$. The representation is:

\begin{align}
e_i &= -\nu_i \nu_{i+1} q^{-\nu_i} E_{i+1,i+1}, \quad f_i = -\nu_i q^{\nu_i} E_{i+1,i}, \quad t_i = q^{-\nu_i} E_{i+1,i+1} q^{\nu_i} E_{i+1,i+1}, \\
e_0 &= qz E_{1,M+N+2}, \quad f_0 = q^{-z} E_{M+N+2,1}, \quad t_0 = q^{E_{1,1} + E_{M+N+2,1}} - E_{M+N+2,1} + E_{M+N+2,1}
\end{align}

The Drinfeld generators on $V_z$ are represented by

\begin{align}
h^i_m &= \frac{[m]}{m} (q^{\nu_i} z)^m \nu_1 q^{-\nu_i} E_{i,i} - \nu_{i+1} q^{\nu_i+1} E_{i+1,i+1}, \\
X^+_{m,i} &= (q^{\nu_i} z)^m E_{i,i+1}, \quad X^-_{m,i} = \nu_i (q^{\nu_i} z)^m E_{i+1,i},
\end{align}

and on $V^*_z$

\begin{align}
h^i_m &= -\frac{[m]}{m} (q^{-\nu_i} z)^m \nu_1 q^{\nu_i} E_{i,i} - \nu_{i+1} q^{-\nu_i+1} E_{i+1,i+1}, \\
X^+_{m,i} &= -\nu_i \nu_{i+1} q^{-\nu_i} (q^{-\nu_i} z)^m E_{i+1,i}, \quad X^-_{m,i} = -\nu_i q^{\nu_i} (q^{-\nu_i} z)^m E_{i+1,i},
\end{align}

where $\mu_i = \sum_{k=1}^i \nu_k$. 
2. A level-one representation of $U_q(\hat{\mathfrak{sl}}(M + 1|N + 1))$

2.1 Free boson realization. Now, we will study the free boson realization of $U_q(\hat{\mathfrak{sl}}(M + 1|N + 1))$ which gives us a level-one representation. It is well known that the representations of the non-super affine algebras are constructed in terms of bosonic fields at level-one [FK]. Basing on this realization, Frenkel and Jing constructed the free boson representation of the quantum affine algebras at level-one [FJ]. We will show that this kind of bosonization can be extended to the affine superalgebras of $A$-type. Our representation can be regarded as a $q$-deformation of the free field realization of $\hat{\mathfrak{sl}}(M + 1|N + 1)$ studied by Bouwknegt et.al. [BCMN]. The structure of the deformation is essentially the same as that of Frenkel and Jing’s except for the deformation of the $\beta, \gamma$ ghost-system. To this ghost-system, however, the technique for bosonizing the $\beta, \gamma$ ghost which was discussed in the papers on deformed Wakimoto realization of $U_q(\hat{\mathfrak{sl}}(N))$ (see [AOS] and references therein) is applicable.

Let us introduce the bosonic oscillators $\{a^i_n, b^i_n, c^i_n, Q_{a^i}, Q_{b^i}, Q_{c^i} \mid n \in \mathbb{Z}, i = 1, \cdots, M + 1, j = 1, \cdots, N + 1\}$ satisfying the commutation relations

\[
[a^i_m, a^j_n] = \delta_{i,j} \delta_{m+n,0} \frac{[m]^2}{m}, \quad [a^i_0, Q_{a^j}] = \delta_{i,j}, \tag{2.1}
\]

\[
[b^i_m, b^j_n] = -\delta_{i,j} \delta_{m+n,0} \frac{[m]^2}{m}, \quad [b^i_0, Q_{b^j}] = -\delta_{i,j}, \tag{2.2}
\]

\[
[c^i_m, c^j_n] = \delta_{i,j} \delta_{m+n,0} \frac{[m]^2}{m}, \quad [c^i_0, Q_{c^j}] = \delta_{i,j}. \tag{2.3}
\]

The remaining commutators vanish.

Define the generating functions for the Drinfeld basis by $X^{i, \pm}(z) = \sum_{m \in \mathbb{Z}} X^{i, \pm}_{m} z^{-m-1}$, and introduce $h^i_0$ by setting $K^i = q^{h^i_0}$. Define $Q_{h^i} = Q_{a^i} - Q_{a^{i+1}}$ for $i = 1, \cdots, M$, $Q_{h^M+1} = Q_{a^{M+1}} + Q_{b^1}$ and $Q_{h^{M+1+j}} = -Q_{b^j} + Q_{b^{j+1}}$ for $j = 1, \cdots, N$. Let us introduce the notation

\[
h^i(z; \beta) = -\sum_{n \neq 0} \frac{h^i_n}{[n]} q^{-|n|} z^{-n} + Q_{h^i} + h^i_0 \ln z, \tag{2.4}
\]

for the Drinfeld generators $h^i_n$, $Q_{h^i}$ and $\beta \in \mathbb{R}$. In this article, we will adopt this notation for other bosonic fields, for example, the boson field $c^j(z; \beta)$ should be defined in the same way. We introduce the $q$-differential operator defined by

\[
\partial_z f(z) = \frac{f(qz) - f(q^{-1}z)}{(q-q^{-1})z}. \tag{2.5}
\]

Now we state the result of the bosonization.

**Proposition 2.1.** The Drinfeld generators at level-one are realized by the free
boson fields as
\[
\begin{align*}
\gamma &= q, \\
h^i_m &= a^i_m q^{-|m|/2} - a^{i+1}_m q^{|m|/2}, \\
h^{M+1}_m &= e^{M+1}_m q^{-|m|/2} + b^1_m q^{-|m|/2}, \\
h^{M+1+j}_m &= -b^j_m q^{|m|/2} + b^{j+1}_m q^{-|m|/2}, \\
X^{+,i}(z) &= e^{H^i(z:1/2)} : e^{i\pi a^i_0}, \\
X^{+,M+1}(z) &= e^{h^{M+1}(z;1/2)} e^{c^i(z;0)} : \prod_{i=1}^M e^{-i\pi a^i_0}, \\
X^{+,M+1+j}(z) &= e^{h^{M+1+j}(z;1/2)} [1, \partial_z e^{-c^i(z;0)}] e^{c^{j+1}(z;0)} : , \\
X^{-,i}(z) &= -e^{-h^i(z;-1/2)} : e^{-i\pi a^i_0}, \\
X^{-,M+1}(z) &= e^{-h^{M+1}(z;-1/2)} [1, \partial_z e^{-c^i(z;0)}] : \prod_{i=1}^M e^{i\pi a^i_0}, \\
X^{-,M+1+j}(z) &= -e^{-h^{M+1+j}(z;-1/2)} e^{c^{j+1}(z;0)} [1, \partial_z e^{-c^i(z;0)}] : ,
\end{align*}
\]

for \( m \in \mathbb{Z}_{\neq 0}, i = 1, \ldots, M \) and \( j = 1, \ldots, N \). The usual normal ordering is denoted by : \( \cdots : \).

We can check the commutation relations by studying the operator products among the bosonized generators.

2.2 Highest weight \( U_q(\hat{\mathfrak{sl}}(2|1))\)-modules. To exploit special features of the quantum affine superalgebras, we study the simplest example \( U_q(\hat{\mathfrak{sl}}(2|1)) \) of the level-one representation obtained in the last subsection. We begin by defining the Fock module. The vacuum vector \( |0\rangle \) is defined by \( a^i_n |0\rangle = b_n |0\rangle = c_n |0\rangle = 0 \) for \( n \geq 0 \), and the vector carrying the weight \( (\lambda_1, \lambda_2, \lambda_b, \lambda_c) \in \mathbb{C}^4 \) by
\[
|\lambda_1, \lambda_2, \lambda_b, \lambda_c\rangle = e^{\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_b Q_b + \lambda_c Q_c} |0\rangle.
\]

The Fock module \( \mathcal{F}_{\lambda_1, \lambda_2, \lambda_b, \lambda_c} \) is generated by acting creation operators \( h^i_m = a^i_0 q^{m^2} - a^i_m q^{-m^2}, h^2_m = a^2_0 q^{m^2} + b_m q^{m^2} \) and \( c_n \) \( (n < 0) \) on \( |\lambda_1, \lambda_2, \lambda_b, \lambda_c\rangle \).

To obtain highest weight vectors of \( U_q(\hat{\mathfrak{sl}}(2|1)) \), we impose the conditions:
\[
\begin{align*}
e_i |\lambda_1, \lambda_2, \lambda_b, \lambda_c\rangle &= 0, \\
h_i |\lambda_1, \lambda_2, \lambda_b, \lambda_c\rangle &= \lambda^i |\lambda_1, \lambda_2, \lambda_b, \lambda_c\rangle, \quad \text{for } i = 0, 1, 2,
\end{align*}
\]

Solving these equations, we obtain the following classification:

1. \( (\lambda_1, \lambda_2, \lambda_b, \lambda_c) = (\beta, \beta, -\alpha, -\alpha) \), where \( \alpha \) and \( \beta \) are arbitrary. The weight of this vector is \( (\lambda^0, \lambda^1, \lambda^2) = (1 - \alpha, 0, \alpha) \). Thus we have the identification: \(|(1 - \alpha) \Lambda_0 + \alpha \Lambda_2\rangle = |\beta, \beta, -\alpha, -\alpha\rangle\).

2. \( (\lambda_1, \lambda_2, \lambda_b, \lambda_c) = (\beta + 1, \beta, 0) \), where \( \beta \) is arbitrary. The weight is \( (\lambda^0, \lambda^1, \lambda^2) = (0, 1, 0) \). We have \(|\Lambda_1\rangle = |\beta + 1, \beta, \beta, 0\rangle\).

3. \( (\lambda_1, \lambda_2, \lambda_b, \lambda_c) = (\beta + 1, \beta + 1, \beta, 0) \), where \( \beta \) is arbitrary. The weight is \( (\lambda^0, \lambda^1, \lambda^2) = (0, 0, 1) \) and we have \(|\Lambda_2\rangle = |\beta + 1, \beta + 1, \beta, 0\rangle\).
According to this classification, let us introduce the following spaces

\[ F_{(\alpha;\beta)} = \bigoplus_{i,j \in \mathbb{Z}} F_{\beta+i,\beta-i+j,\beta-\alpha+j,-\alpha+j}, \]

(2.17)

\[ F_{(1,0);\beta} = \bigoplus_{i,j \in \mathbb{Z}} F_{\beta+1+i,\beta-i+j,\beta+j,j}, \quad F_{((0,1);\beta)} = \bigoplus_{i,j \in \mathbb{Z}} F_{\beta+1+i,\beta-i+j,\beta+j,j}. \]

It is not difficult to see that the bosonized actions of \( U_q(\widehat{\mathfrak{sl}}(2|1)) \) on these spaces are closed i.e. \( U_q(\widehat{\mathfrak{sl}}(2|1)) F_{(\kappa,\gamma)} = F_{(\kappa,\gamma)} \) where \( \kappa = (1,0), (0,1) \). These spaces are not irreducible in general. It is convenient to introduce a pair of Fermionic fields \( \eta(z) = \sum_{n \in \mathbb{Z}} \eta_n z^{-n-1} = e^{e(z;0)} \) and \( \xi(z) = \sum_{n \in \mathbb{Z}} \xi_n z^{-n} = e^{-e(z)} \) to obtain the irreducible subspaces of these \( U_q(\widehat{\mathfrak{sl}}(2|1)) \)-modules. The mode expansion of \( \eta(z) \), \( \xi(z) \) is well defined on \( F_{(\alpha;\beta)} \) for \( \alpha \in \mathbb{Z} \) and on \( F_{(1,0);\beta} \), \( F_{((0,1);\beta)} \) and the relations are \( \{\xi_r,\xi_s\} = \{\eta_r,\eta_s\} = 0, \{\xi_r,\eta_s\} = \delta_{r+s,0} \). In these cases, we have the direct sum decompositions \( F_{(\kappa,\gamma)} = \eta_0 \xi_0 F_{(\kappa,\gamma)} \oplus \xi_0 \eta_0 F_{(\kappa,\gamma)} \). As usual, we call \( \eta_0 \xi_0 F_{(\kappa,\gamma)} \) as \( \text{Ker} \eta_0 F_{(\kappa,\gamma)} \) and \( F_{(\kappa,\gamma)}/\eta_0 \xi_0 F_{(\kappa,\gamma)} \) as \( \text{Coker} \eta_0 F_{(\kappa,\gamma)} \). Since \( \eta_0 \) commutes (or anti-commutes) with every element of \( U_q(\widehat{\mathfrak{sl}}(2|1)) \), we can regard \( \text{Ker} \eta_0 F_{(\kappa,\gamma)} \) and \( \text{Coker} \eta_0 F_{(\kappa,\gamma)} \) as \( U_q(\widehat{\mathfrak{sl}}(2|1)) \)-modules.

From now on, we study the character formulas of these \( U_q(\widehat{\mathfrak{sl}}(2|1)) \)-modules we have constructed in the bosonic Fock space. The character of a space \( F \) is defined by

\[ \text{ch}_F(q,x,y) = \text{tr}_F q^{-d} x^{h_0^1} y^{h_0^2}, \]

(2.18)

where the grading operator \( d \) is [BCMN]

\[ d = -\sum_{n \geq 1} \left( \frac{r}{[n]} \right)^2 \{a_-^n a_1 + a_-^n a_n^2 - b_n b_n + c_-^n c_n - (a_-^n + a_n^2 + b_n)(a_1^1 + a_n^2 + b_n)\}

- \frac{1}{2} \{(a_0^1)^2 + (a_2^1)^2 - (b_0)^2 + c_0(c_0 + 1) - (a_0^1 + a_2^2 + b_0)^2\}, \]

(2.19)

and \( h_0^1 = a_0^1 - a_0^2, h_0^2 = a_2^0 + b_0 \).

(I) Character of \( F_{(\alpha;\beta)} \) for \( \alpha \not\in \mathbb{Z} \). Since \( \eta_0 \) is not defined on this module, it is expected that \( F_{(\alpha;\beta)} \) is irreducible highest weight \( U_q(\widehat{\mathfrak{sl}}(2|1)) \)-module. Thus, we conjecture the following.

**Conjecture 2.1.** We have the identification of the highest weight \( U_q(\widehat{\mathfrak{sl}}(2|1)) \)-modules:

\[ F_{(\alpha;\beta)} \cong V((1 - \alpha)\Lambda_0 + \alpha \Lambda_2) \quad \text{for} \ \alpha \not\in \mathbb{Z} \ \text{and arbitrary} \ \beta, \]

(2.20)

where \( V(\lambda) \) denotes the irreducible highest weight \( U_q(\widehat{\mathfrak{sl}}(2|1)) \)-modules with the highest weight weight \( \lambda \).

**Proposition 2.2.** We obtain the character of \( F_{(\alpha;\beta)} \) as

\[ \text{ch}_{F_{(\alpha;\beta)}}(q,x,y) = \frac{q^{-\frac{1}{2}(\alpha+1)}}{\prod_{n=1}^\infty (1 - q^n)^3} \sum_{s,t \in \mathbb{Z}} q^{\frac{1}{2}(2s^2 - 2st + t^2)} x^a s^{-t} y^a s. \]

(2.21)

(II) Character of \( \text{Ker}\eta_0 \) and \( \text{Coker}\eta_0 \). Next, let us consider the character of the modules on which \( \eta_0 \) is well defined.
Proposition 2.3. The character of $\text{Ker} \eta_0(\alpha; \beta)$ for $\alpha \in \mathbb{Z}$ is obtained as

$$
\text{chKer} \eta_0(\alpha, \beta)(q, x, y) = \frac{q^{-\frac{1}{2}\alpha(\alpha+1)}}{\prod_{n=1}^{\infty}(1-q^n)^3} \left( \sum_{s, t, \ell \in \mathbb{Z}} (-1)^{l+1} q^{\frac{l(l+1)}{2} + l(\alpha-t) + \frac{1}{2}(2s^2 - 2st + t^2) x} 2s - ty^\alpha - s \right) \right).
$$

The character of $\text{Coker} \eta_0(\alpha, \beta)$ ($\alpha \in \mathbb{Z}$) is

$$
\text{chCoker} \eta_0(\alpha, \beta)(q, x, y) = \frac{q^{-\frac{1}{2}\alpha(\alpha+1)}}{\prod_{n=1}^{\infty}(1-q^n)^3} \left( \sum_{s, t, \ell \in \mathbb{Z}} (-1)^{l+1} q^{\frac{l(l+1)}{2} - l(\alpha-t) + \frac{1}{2}(2s^2 - 2st + t^2) x} 2s - ty^\alpha - s \right) \right).
$$

These formulas are obtained by inserting the projectors $\eta_0 \xi_0$ and $\xi_0 \eta_0$ to the trace of the Fock space. As for the details of this technique, see [BCMN]. We also have the following formulas.

Proposition 2.4. We have the equalities $\text{chCoker} \eta_0((0,1); \beta)(q, x, y) = \text{chCoker} \eta_0((1); \beta)(q, x, y)$, and

$$
\text{chCoker} \eta_0((1,0); \beta)(q, x, y) = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)^3} \left( \sum_{s, t, \ell \in \mathbb{Z}} (-1)^{l+1} q^{\frac{l(l+1)}{2} - l(\alpha-t) + \frac{1}{2}(2s^2 - 2st + t^2) x} 2s - ty^\alpha - s \right) \right).
$$

Since we have the conditions $\eta_0 | \beta, \beta, \beta - \alpha, -\alpha \rangle \neq 0$ for $\alpha = 0, 1, \cdots$ and $\eta_0 | \beta, \beta, \beta - \alpha, -\alpha \rangle = 0$ for $\alpha = -1, -2, \cdots$, the modules $\text{Coker} \eta_0(\alpha; \beta)$ ($\alpha \in \mathbb{Z}$), $\text{Coker} \eta_0((0,1); \beta)$, $\text{Coker} \eta_0((1,0); \beta)$ and $\text{Ker} \eta_0(\alpha; \beta)$ ($\alpha = -1, -2, \cdots$) are highest weight $U_q(\hat{sl}(2|1))$-modules. It is expected that these modules are also irreducible with respect to the action of $U_q(\hat{sl}(2|1))$.

Conjecture 2.2. We have the following identifications of the highest weight $U_q(\hat{sl}(2|1))$-modules:

$$
V((1-\alpha)\Lambda_0 + \alpha \Lambda_2) \cong \text{Coker} \eta_0(\alpha; \beta) \text{ for } \alpha = 0, 1, \cdots,
$$

$$
\cong \text{Ker} \eta_0(\alpha; \beta) \text{ for } \alpha = -1, -2, \cdots,
$$

and $V(\Lambda_1) \cong \text{Coker} \eta_0((1,0); \beta)$, $V(\Lambda_2) \cong \text{Coker} \eta_0((0,1); \beta)$ for arbitrary $\beta$.

We have checked the validity of Conjecture 2.1 and Conjecture 2.2 for $V(\Lambda_0), V(\Lambda_1), V(\Lambda_2)$ up to certain degrees by comparing our results with those of Kac and Wakimoto [KWk2].
3. Vertex Operators

3.1 Vertex operators for $U_q(\widehat{\mathfrak{sl}}(M+1|N+1))$. In this section, we study free boson realization of vertex operators for $U_q(\widehat{\mathfrak{sl}}(M+1|N+1))$. Let $V(\lambda)$ be the highest weight $U_q(\widehat{\mathfrak{sl}}(M+1|N+1))$-module with the highest weight $\lambda$. The $\mathbb{Z}_2$-gradation of $V(\lambda)$ is also denoted by $|\cdot|$. The vertex operators $\Phi^\mu_V(z), \Phi^\nu_V(z), \Psi^\mu_V(z), \Psi^\nu_V(z)$ are defined as the following intertwiners of $U_q(\widehat{\mathfrak{sl}}(M+1|N+1))$-modules if they exist:

\[
\begin{align*}
\Phi^\mu_V(z) & : V(\lambda) \rightarrow V(\mu) \otimes V_z, \quad \Phi^\nu_V(z) : V(\lambda) \rightarrow V(\mu) \otimes V_z, \\
\Psi^\mu_V(z) & : V(\lambda) \rightarrow V_z \otimes V(\mu), \quad \Psi^\nu_V(z) : V(\lambda) \rightarrow V_z \otimes V(\mu),
\end{align*}
\]

for $\forall x \in U_q(\widehat{\mathfrak{sl}}(M+1|N+1))$ together with the gradation $|\Phi^\mu_V(z)| = |\Phi^\nu_V(z)| = |\Psi^\mu_V(z)| = |\Psi^\nu_V(z)| = 0$. We expand the vertex operators as

\[
\begin{align*}
\Phi^\mu_V(z) &= \sum_{l=1}^{M+N+2} \Phi^\mu_{Vl}(z) \otimes v_l, \quad \Phi^\nu_V(z) = \sum_{l=1}^{M+N+2} v_l \otimes \Phi^\nu_{Vl}(z), \\
\Psi^\mu_V(z) &= \sum_{l=1}^{M+N+2} v_l \otimes \Psi^\mu_{Vl}(z), \quad \Psi^\nu_V(z) = \sum_{l=1}^{M+N+2} \Psi^\nu_{Vl}(z) \otimes v_l,
\end{align*}
\]

We define the graded action of these expanded operators by $\Phi^\mu_{Vl}(z)|u\rangle = \sum_{l=1}^{M+N+2} \Phi^\mu_{Vl}(z)|u\rangle \otimes v_l(-1)^{|u||v|} |u\rangle$, $\Phi^\nu_{Vl}(z)|u\rangle = \sum_{l=1}^{M+N+2} v_l \otimes \Phi^\nu_{Vl}(z)|u\rangle$, and $\Psi^\mu_{Vl}(z)|u\rangle = \sum_{l=1}^{M+N+2} v_l \otimes \Psi^\mu_{Vl}(z)|u\rangle$, $\Psi^\nu_{Vl}(z)|u\rangle = \sum_{l=1}^{M+N+2} \Psi^\nu_{Vl}(z) \otimes v_l$, for $|u\rangle \in V(\lambda)$.

Let us introduce the following combinations of the Drinfeld operators:

\[
\begin{align*}
h_{ij}^m &= \sum_{j=1}^{M+N+1} \frac{[\alpha_{ij}m][\beta_{ij}m]}{[(M+N)m]m} h_{ij}^m, \quad Q_{ij}^m = \sum_{j=1}^{M+N+1} \frac{\alpha_{ij} \beta_{ij}}{M-N} Q_{h_{ij}}, \quad h_{ij} = \sum_{j=1}^{M+N+1} \frac{\alpha_{ij} \beta_{ij}}{M-N} h_{ij}, \\
Q_{ij} &= \sum_{j=1}^{M+N+1} \frac{\alpha_{ij} \beta_{ij}}{M-N} Q_{h_{ij}}, \quad h_{ij} = \sum_{j=1}^{M+N+1} \frac{\alpha_{ij} \beta_{ij}}{M-N} h_{ij}.
\end{align*}
\]

where $h_{ij}^m$ is defined by $K^j = q^{h_{ij}^m}$ and

\[
\begin{align*}
\alpha_{ij} &= \begin{cases} 
\min(i,j) & \text{if } \min(i,j) \leq M+1, \\
2(M+1) - \min(i,j) & \text{if } \min(i,j) > M+1,
\end{cases} \\
\beta_{ij} &= \begin{cases} 
M-N - \max(i,j) & \text{if } \max(i,j) \leq M+1, \\
- M-N - 2 + \max(i,j) & \text{if } \max(i,j) > M+1.
\end{cases}
\end{align*}
\]

Note that using these notations, we have the inverse of the Cartan matrix as $(a_{ij})^{-1} = \alpha_{ij} \beta_{ij}/(M-N).$ We obtain the relations $[h_{ij}^m, h_{kl}^n] = \delta_{ij,kl} \delta_{m+n, \beta_{ij}m} / m$, $[h_{ij}^m, h_{ij}^p] = \delta_{m+n, \beta_{ij}m} / m$, and $[h_{ij}^m, Q_{ij}] = \delta_{ij,kl} [h_{ij}^m, Q_{ij}^m] = a_{ij}^{-1}$.

Define the operators $\phi_l(z), \phi_l^*(z), \psi_l(z), \psi_l^*(z)$ ($i = 1, \ldots, M+N+2$) iter-
atively by

\[
\phi_{M+N+2}(z) = e^{-h_i(z)+1}(q^{M-N+1}z^{-1/2}+c^{N+1}(q^{M-N+1}z^{0}) : \prod_{i=1}^{M+1} e^{i\pi \frac{h_i(z)}{M-N}a_0} :,
\]

\[
\nu_i\phi_i(z) = [\phi_{i+1}(z), f_i]_{q^{\nu_i+1}},
\]

\[
\phi_i(z) = e^{H_i(z)} : \prod_{i=1}^{M+1} e^{i\pi \frac{1}{M-N}a_0},
\]

\[-\nu_i q^{\nu_i}\phi_{i+1}(z) = [\phi_i(z), f_i]_{q^{\nu_i+1}},
\]

\[
\psi_i(z) = e^{-H_i(z)} : \prod_{i=1}^{M+1} e^{i\pi \frac{1}{M-N}a_0},
\]

\[
\psi_{i+1}(z) = [\psi_i(z), e_i]_{q^{\nu_i+1}},
\]

\[
\psi_{M+N+2}(z) = e^{H_i(z)} : \prod_{i=1}^{M+1} e^{i\pi \frac{1}{M-N}a_0},
\]

\[-\nu_i \nu_{i+1} q^{\nu_i}\psi_{i+1}(z) = [\psi_{i+1}(z), e_i]_{q^{\nu_i+1}},
\]

where \( h_i(z) \) is defined in the same manner as \( h_i(z; \beta) \). The gradations are given by \( |\phi_i(z)| = |\phi_i(z)| = |\psi_i(z)| = |\psi_i(z)| = \frac{\nu_i+1}{2} \). Define the operators \( \phi(z), \phi^*(z), \psi(z) \) and \( \psi^*(z) \) by \( \phi(z) = \sum_{i=1}^{M+N+2} \phi_i(z)\cdot v_i, \phi^*(z) = \sum_{i=1}^{M+N+2} \phi_i^*(z)\cdot v_i^*, \psi(z) = \sum_{i=1}^{M+N+2} v_i\cdot \psi_i(z) \) and \( \psi^*(z) = \sum_{i=1}^{M+N+2} v_i^\ast\cdot \psi_i^*(z) \) respectively. Then we have the following result.

**Proposition 3.1.** The operators \( \phi(z), \phi^*(z), \psi(z) \) and \( \psi^*(z) \) satisfy the same commutation relations as \( \Phi^\mu_V(z), \Phi^\mu_{V^*}(z), \Psi^\mu_{V}(z) \) and \( \Psi^\mu_{V^*}(z) \) respectively have, respectively.

To prove the proposition, the equations \([\psi_i(z), e_i]_{q}, e_1]_{q^{-1}} = 0, [\psi_i(z), e_i] = 0 \) \((i \neq 1) \), \((qz - q^{-1}x)\psi(z)X^{+1}(x) = (z-x)X^{+1}(x)\psi(z) \) and similar formulas for \( \psi^\ast_{M+N+2}, \Phi^\ast_{M+N+2} \) and \( \phi_1^* \) are helpful.

Remark. These operators can almost be determined by the method used for the level-one bosonization of the vertex operators of \( U_q(\tilde{\mathfrak{sl}}(N)) \) [JMMN,Ko]. Namely, we obtained those by studying the commutation relations between the vertex operators and some of the Drinfeld basis. Relevant explicit coproduct formulas for the Drinfeld basis can be obtained in the same way as Chari and Pressley’s [CP]. We have the bosonic fields \( c \)'s whose contribution to the vertex operators cannot be determined by studying the commutation relations with \( h_m \), because they do not contain \( c \)'s. However, the following two information enables us to find the unique solutions as above: \( i \) the vertex operators of \( q \rightarrow 1 \) limit, \( ii \) the commutation relations with \( X^{+1}(z) \) (or \( X^{-1}(z) \)) for type I (or type II).

### 3.2 \( U_q(\tilde{\mathfrak{sl}}(2|1)) \) case

We study the action of the bosonized vertex operators of \( U_q(\tilde{\mathfrak{sl}}(2|1)) \) on the Fock space defined in Subsection 2.2. Using the bosonic representations of the vertex operators, we have the homomorphisms of \( U_q(\tilde{\mathfrak{sl}}(2|1))-
A LEVEL-ONE REPRESENTATION OF THE QUANTUM AFFINE SUPERALGEBRA $U_q(\hat{\mathfrak{sl}}(M + 1|N + 1))$

modules:

\[
\begin{align*}
\phi(z) : & \quad \begin{cases}
\mathcal{F}(\alpha;\beta) \to \mathcal{F}(\alpha-1;\beta+1) \otimes V_z, \\
\mathcal{F}(1,0;\beta) \to \mathcal{F}(0;\beta+1) \otimes V_z, \\
\mathcal{F}(0,1;\beta) \to \mathcal{F}(1;\beta+1) \otimes V_z, \\
\mathcal{F}(0;\beta) \to \mathcal{F}(1;0;\beta+1) \otimes V_z,
\end{cases} \\
\psi(z) : & \quad \begin{cases}
\mathcal{F}(\alpha;\beta) \to \mathcal{F}(\alpha-1;\beta+1) \otimes V_z, \\
\mathcal{F}(1,0;\beta) \to \mathcal{F}(0;\beta+1) \otimes V_z, \\
\mathcal{F}(0,1;\beta) \to \mathcal{F}(1;\beta+1) \otimes V_z, \\
\mathcal{F}(2;0;\beta) \to \mathcal{F}(1;0;\beta+1) \otimes V_z,
\end{cases}
\end{align*}
\]

\[
\phi^*(z) : \quad \begin{cases}
\mathcal{F}(\alpha;\beta) \to \mathcal{F}(\alpha+1;\beta-1) \otimes V_z^{s\alpha}, \\
\mathcal{F}(1,0;\beta) \to \mathcal{F}(2;\beta-1) \otimes V_z^{s\alpha}, \\
\mathcal{F}(0,1;\beta) \to \mathcal{F}(3;\beta-1) \otimes V_z^{s\alpha}, \\
\mathcal{F}(1,1;\beta) \to \mathcal{F}(0,1;\beta-1) \otimes V_z^{s\alpha}, \\
\mathcal{F}(0;\beta) \to \mathcal{F}(1;0;\beta-1) \otimes V_z^{s\alpha},
\end{cases}
\]

\[
\psi^*(z) : \quad \begin{cases}
\mathcal{F}(\alpha;\beta) \to \mathcal{F}(\alpha+1;\beta-1) \otimes V_z^{s\alpha}, \\
\mathcal{F}(1,0;\beta) \to \mathcal{F}(2;\beta-1) \otimes V_z^{s\alpha}, \\
\mathcal{F}(0,1;\beta) \to \mathcal{F}(3;\beta-1) \otimes V_z^{s\alpha}, \\
\mathcal{F}(1,1;\beta) \to \mathcal{F}(0,1;\beta-1) \otimes V_z^{s\alpha}, \\
\mathcal{F}(0;\beta) \to \mathcal{F}(1;0;\beta-1) \otimes V_z^{s\alpha},
\end{cases}
\]

Next let us consider the vertex operators which intertwine highest weight $U_q(\hat{\mathfrak{sl}}(2|1))$-modules by using the above results. It is easy to see that the vertex operators also commute (or anti-commute) with $\eta_0$. Noting this property, the above homomorphisms and Conjecture 2.1, 2.2, we can study the conditions of existence for the vertex operators which intertwine the irreducible highest weight modules.

**Conjecture 3.1.** The following vertex operators associated with the level-one irreducible highest weight modules exist:

\[
\begin{align*}
\Phi^\lambda_{\alpha-1} : V(\lambda) & \longrightarrow V(\lambda_{\alpha-1}) \otimes V_z, \\
\Phi^\lambda_{\Lambda_1} : V(\Lambda_1) & \longrightarrow V(\Lambda_0) \otimes V_z, \\
\Phi^\lambda_{\Lambda_2} : V(\Lambda_2) & \longrightarrow V(\Lambda_1) \otimes V_z, \\
\Phi^\lambda_{\Lambda_0+1} : V(\lambda) & \longrightarrow V(\lambda_{\alpha+1}) \otimes V_z^{s\alpha}, \\
\Phi^\lambda_{\Lambda_0} : V(\Lambda_0) & \longrightarrow V(\Lambda_1) \otimes V_z^{s\alpha}, \\
\Phi^\lambda_{\Lambda_1} : V(\Lambda_1) & \longrightarrow V(\Lambda_2) \otimes V_z^{s\alpha},
\end{align*}
\]

where $\lambda_\alpha = (1-\alpha)\Lambda_0 + \alpha\Lambda_2$ for $\alpha \in \mathbb{R}$.

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