Vorticity, Helicity, Intrinsinc geometry for Navier-Stokes equations

Shizan Fang\(^1\ast\) Zhongmin Qian\(^2\dagger\)

\(^1\)I.M.B, Université de Bourgogne, BP 47870, 21078 Dijon, France
\(^2\)Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

October 14, 2019

Abstract

We will consider the Navier-Stokes equation on a Riemannian manifold \(M\) with Ricci tensor bounded below, the involved Laplacian operator is De Rham-Hodge Laplacian. The novelty of this work is to introduce a family of connections which are related to solutions of the Navier-Stokes equation, so that vorticity and helicity can be linked through the associated time-dependent Ricci tensor in intrinsic way in the case where \(\dim(M) = 3\).

MSC 2010: 35Q30, 58J65

Keywords: Vorticity, helicity, intrinsic Ricci tensor, De Rham-Hodge Laplacian, Navier-Stokes equations

1 Introduction

The Navier-Stokes equation in a domain of \(\mathbb{R}^n\) is a system of partial differential equations

\[
\partial_t u_t + (u_t \cdot \nabla) u_t - \nu \Delta u_t + \nabla p_t = 0, \quad \nabla \cdot u_t = 0, \quad u|_{t=0} = u_0,
\]

(1.1)

which describes the evolution of the velocity \(u_t\) and the pressure \(p_t\) of an incompressible viscous fluid with kinematic viscosity \(\nu > 0\). The model of periodic boundary conditions for (1.1) on a torus \(\mathbb{T}^n\) has been introduced to simplify mathematical considerations. In [14], Navier-Stokes equations on a compact Riemannian manifold \(M\) have been considered using the framework of the group of diffeomorphisms of \(M\) initiated by V. Arnold in [5]; where the Laplace operator involved in the text of [14] is de Rham-Hodge Laplacian \(\Box\), however, the authors said in the note added in proof that the convenient Laplace operator comes from deformation tensor.

In this article, we would like to explore the rich geometry coded in the Navier-Stokes equation on a manifold.

Let \(\nabla\) be the Levi-Civita connection on \(M\). For a vector field \(A\) on \(M\), the deformation tensor \(\text{Def}(A)\) is a symmetric tensor of type \((0, 2)\) defined by

\[
(\text{Def} A)(X, Y) = \frac{1}{2} \left( \langle \nabla_X A, Y \rangle + \langle \nabla_Y A, X \rangle \right), \quad X, Y \in \mathcal{X}(M),
\]

(1.2)

\(\ast\)Email: Shizan.Fang@u-bourgogne.fr.
\(\dagger\)Email: zhongmin.qian@maths.ox.ac.uk
where $\mathcal{X}(M)$ is the space of vector fields on $M$. Then $\text{Def} : TM \to S^2 T^* M$ maps a vector field to a symmetric tensor of type $(0, 2)$. Let $\text{Def}^* : S^2 T^* M \to TM$ be the adjoint operator. In [32] or in [36] (see page 493), the authors considered the following Laplacian

$$\Box = 2 \text{Def}^* \text{Def}.$$  

They considered the Navier-Stokes equation with viscosity described by $\Box$, namely

$$\partial_t u_t + \nabla u_t u_t + \nu \Box u_t = -\nabla p_t, \quad \text{div}(u_t) = 0, \quad u|_{t=0} = u_0,$$  

(1.4)

The reader may also refer to [33] in which the author considered the same equation as (1.4) on a complete Riemannian manifold with negative curvature. Variational principles in the class of incompressible Brownian martingales in the spirit of [5] were established recently in [10] [2] [3] [4] for the Navier-Stokes equation (1.4).

In this work, we will consider with a complete Riemannian manifold $M$ of dimension $n$, with Ricci curvature bounded from below. We are interested in the following Navier-Stokes equation on $M$ defined with the De Rham-Hodge Laplacian $\Box$,

$$\begin{cases} \partial_t u_t + \nabla u_t u_t + \nu \Box u_t = -\nabla p_t, \\ \text{div}(u_t) = 0, \quad u|_{t=0} = u_0, \end{cases}$$

(1.5)

where $u(x, t)$ denotes the velocity vector field at time $t$, and $p(x, t)$ models the pressure. If no confusion may arise, we will use $u_t$ (resp. $p_t$) to denote the vector field $u(\cdot, t)$ (resp. $p(\cdot, t)$) for each $t$.

There are a few works [26] [38] which support this choice of $\Box$. The probabilistic representation formulae behave better with Navier-Stokes equation (1.5) (see [11] [20] [19]). Our preference here for $\Box$ is motivated by its good geometric behavior and its deep links with Stochastic Analysis. See for example [6] [7] [8] [12] [13] [15] [18] [17] [22] [25] [27] [31] [34]. From the view of kinetic mechanics, the viscosity effect of a non-homogeneous fluid should be mathematically described by the Bochner Laplacian of the velocity vector field, where the metric tensor describes the local viscosity distribution. On the other hand, the de Rham-Hodge Laplacian operating on one forms is mathematically more appealing. By invoking de Rham-Hodge Laplacian in the model, according to the Bochner identity, one then produces a no-physical additional term which is however linear in the velocity. An additional linear term in the Navier-Stokes equation will not alter the fundamental difficulty, nor to alter the physics of the fluid flows, which justify the use of de Rham-Hodge Laplacian. There is also a good reason too to consider Navier-Stokes equations on manifolds, if one wants to model the global behavior of the pacific ocean climate for example.

Let’s first say a few words on the definition of $\Box$ on vector fields. There is a one-to-one correspondence between the space of vector fields $\mathcal{X}(M)$ and that of differential 1-forms $\Lambda^1(M)$. Given a vector field $A$ (resp. differential 1-form $\omega$), we shall denote by $\bar{A}$ (resp. $\omega^\sharp$) the corresponding differential 1-form (resp. vector field). To see more intuitively these correspondences, let’s explain on a local chart $U$: as usual, we denote by $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ the basis of the tangent space $T_x M$ and by $\{dx^1, \ldots, dx^n\}$ the dual basis of $T^*_x M$, called the co-tangent space at $x$, that is, $dx^i(\frac{\partial}{\partial x_j}) = \delta_{ij}$. The inner product in $T_x M$ as well as the one in the dual space $T^*_x M$ will be denoted by $\langle \cdot, \cdot \rangle$, while the duality between $T^*_x M$ and $T_x M$ will be denoted by $(\cdot, \cdot)$. Set $g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$. Let $u$ be a vector field on $M$, on $U$, $u = \sum_{i=1}^n u_i \frac{\partial}{\partial x_i}$,
then $\tilde{u}$ admits the expression

$$
\tilde{u} = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} g_{ij} u_j \right) dx^i.
$$

Let $g^{ij} = \langle dx^i, dx^j \rangle$. Then the matrix $(g^{ij})$ is the inverse matrix of $(g_{ij})$. For a differential

1-form $\omega = \sum_{j=1}^{n} \omega_j dx^j$, the associated vector field $\omega^\#$ has the expression

$$
\omega^\# = \sum_{i=1}^{n} \left( \sum_{\ell=1}^{n} g^{i\ell} \omega_\ell \right) \frac{\partial}{\partial x_i}.
$$

Concisely

$$(\omega, A) = (\omega^\#, A) = \langle \omega, \tilde{A} \rangle, \quad A \in \mathcal{X}(M), \; \omega \in \Lambda^1(M).$$

Now for $A \in \mathcal{X}(M)$, the De-Rham Hodge Laplacian $\Box A$ is defined by

$$
\Box A = (\Box \tilde{A})^\#, \quad \Box = dd^* + d^* d,
$$

where $d^*$ is adjoint operator of exterior derivative $d$. Then we have the following relation

$$
\int_M \langle \Box \omega, A \rangle dx = \int_M \langle \Box \omega, \tilde{A} \rangle dx = \int_M \langle \omega, \Box \tilde{A} \rangle dx = \int_M \langle \omega, \Box A \rangle dx
$$

where $dx$ denotes the Riemannian measure on $M$. The classical Bochner-Weitzenböck reads as

$$
\Box A = -\Delta A + \text{Ric}(A), \quad A \in \mathcal{X}(M),
$$

where Ric is the Ricci tensor on $M$ and $\Delta A = \text{Trace}(\nabla \nabla A)$, characterized by

$$
- \int_M \langle \Delta A, A \rangle dx = \int_M |\nabla A|^2 dx.
$$

Let $T : \mathcal{X}(M) \rightarrow \mathcal{X}(M)$ be a tensor of type $(1,1)$, and denote by $T^\# : \Lambda^1(M) \rightarrow \Lambda^1(M)$ its adjoint defined by

$$(T^\# \omega, A) = \langle \omega, T(A) \rangle, \quad A \in \mathcal{X}(M),$$

where we used notation $\Lambda^p(M)$ to denote the space of differential $p$-forms on $M$.

In the space of $\mathbb{R}^3$, the inner product between two vectors $u, v$ will be noted by $u \cdot v$. The vorticity $\xi_t$ of a velocity $u_t$ is a vector field defined as $\xi_t = \nabla \times u_t$. When $u_t$ is a solution to Navier-Stokes equation (1.1), then $\xi_t$ satisfies the following heat equation

$$
\frac{d\xi_t}{dt} + \nabla u_t \xi_t - \nu \Delta \xi_t = \nabla^{\#} \xi_t \cdot u_t
$$

where $\nabla^{\#} u_t$ is the symmetric part of $\nabla u_t$, such that $\nabla^{\#} u_t \cdot v = \text{Def} u_t(\xi_t, v)$ with Def introduced in (1.2). How to interpret the term $\nabla^{\#} \xi_t \cdot u_t$? From (1.9), a formal computation leads to

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\xi_t|^2 dx + \nu \int_{\mathbb{R}^3} |\nabla \xi_t|^2 dx = \int_{\mathbb{R}^3} \text{Def} (u_t)(\xi_t, \xi_t) dx.
$$
Since K. Itô introduced the tool of stochastic parallel translations along paths of Brownian motion on a Riemannian manifold, especially after the works by Eells, Elworthy, Malliavin and Bismut (see for example [31, 16, 8]), there are profound involvements of Stochastic Analysis in the study of linear second order partial differential equations and in Riemannian geometry [0, 31, 25, 29]. The purpose of this work is to geometrically explain the right hand side of (1.10). To this end, we will consider Navier-Stokes equation in a geometric framework in order that suitable geometric meaning could be found.

In what follows, we present the organisation of the paper and main results. In Section 2, first we follow more or less the exposition of [36]. To a solution $u_t$ to Navier-Stokes equation (1.5), we associate a differential 2-form $\tilde{\omega}_t$ which is the exterior derivative of $\tilde{u}_t$: a heat equation for $\tilde{\omega}_t$ will be established with involvement of $\nabla^s u_t$. When $M$ is of dimension 3, the Hodge star $\ast$ operator sends $\tilde{\omega}_t$ to a differential 1-form $\omega_t$. In flat case of $\mathbb{R}^3$, $\omega_t = \tilde{\nabla} \times u_t$. We call such $\omega_t$ the vorticity of $u_t$: a heat equation for $\omega_t$ is also obtained in Section 2. In second part of Section 2, the a priori evolution equation for $\omega_t$ is established. Using heat semi-group $e^{-t\Delta}$ on differential forms as well as Bismut formulae, the existence of weak solutions in the sense of Leray to Navier-Stokes equation (1.5) over any interval $[0, T]$ is proved under suitable hypothesis on boundedness of Ricci tensor: to our knowledge, these results are new while comparing to recent results obtained in [33]. In Section 3, we give an exposition of the involvement of Stochastic Analysis on Riemannian manifolds; stochastic differential equations on $M$, defining the Brownian motion with drift $u \in L^2([0, T], H^1(M))$ of divergence free is proved to be stochastic complete; then $\omega_t$ admits a probabilistic representation. By introducing a suitable metric compatible affine connection on $M$, a Brownian motion with drift $u$ on $M$ can be obtained by rolling without friction flat Brownian motion of $\mathbb{R}^n$ on $M$ with respect to it: it was a main idea in [31, 16], and well developed in [25]. So to a velocity $u_t$, we associate a metric compatible connection $\nabla^t$ on $M$, which admits the following global expression

$$\nabla^t_X Y = \nabla_X Y - \frac{2}{n-1} K_t(X,Y), \quad X, Y \in \mathcal{X}(M)$$

where $K_t(X,Y) = \langle Y, u_t \rangle X - \langle X, Y \rangle u_t$: it gives rise to a connection with torsion $T^t$ which is not of skew-symmetric. Section 4 is devoted to compute the associated intrinsic Ricci tensor $\text{Ric}^t$ which was first introduced by B. Driver in [12] as follows:

$$\text{Ric}^t(X) = \text{Ric}^t(X) + \sum_{i=1}^n (\nabla^t_{e_i} T^t)(X, e_i),$$

where $\text{Ric}^t$ is the Ricci tensor associated to $\nabla^t$ and $\{e_1, \ldots, e_n\}$ is an orthonormal basis at tangent spaces. The formula (1.10) has the following geometric counterpart for 3D Riemannian manifold $M$,

$$\frac{1}{2} \frac{d}{dt} \int_M |\omega_t|^2 \, dx + \nu \int_M |\nabla \omega_t|^2 \, dx = \frac{1}{2\nu} \int_M (\omega_t, u_t)^2 \, dx - \nu \int_M (\text{Ric}^t \ast \omega_t, \omega_t) \, dx. \quad (1.11)$$

As well as vorticity $\omega_t$ is not orthogonal to velocity $u_t$, a phenomenon of helicity $(\omega_t, u_t)$ will appear. Formula (1.11) says how helicity and intrinsic Ricci tensor fit into the evolution of vorticity in time and in space. Section 5 is devoted to interpretation of main results obtained in Section 4 in the framework of vector calculus. Finally in Section 6, we collect and prove technical results used previously.
2 Vorticity, Helicity and their evolution equations

Let \( u_t \) be a (smooth) solution to the Navier-Stokes equation on \( M \),
\[
\partial_t u_t + \nabla u_t \cdot u_t + \nu \Box u_t = -\nabla p_t, \quad \text{div}(u_t) = 0, \quad u_t|_{t=0} = u_0.
\] (2.1)

Transforming Equation (2.1) into differential forms, \( \tilde{u}_t \) satisfies
\[
\begin{cases}
\partial_t \tilde{u}_t + \nabla \tilde{u}_t \cdot \tilde{u}_t + \nu \Box \tilde{u}_t = -dp_t, \\
\ast \tilde{u}_t = 0, \quad \tilde{u}_t|_{t=0} = \tilde{u}_0.
\end{cases}
\] (2.2)

Let
\[
\omega_t = \ast \tilde{u}_t,
\] (2.3)
which is a differential 2-form. For vector fields \( X, v \) on \( M \), Lie derivative \( L \) satisfies the product rule, that is,
\[
L_v (\tilde{u}, X) = (L_v \tilde{u}, X) + (\tilde{u}, L_v X),
\]
where
\[
L_v (\tilde{u}, X) = (\nabla_v \tilde{u}, X) + (\tilde{u}, \nabla_v X).
\]

By taking \( v = u \), we get
\[
(L_u \tilde{u} - \nabla_u \tilde{u}, X) = (\tilde{u}, \nabla_u X - L_u X) = (\tilde{u}, \nabla_X u) = \langle u, \nabla_X u \rangle = \frac{1}{2} (|d|u|^2, X)
\]
which yields that
\[
L_u \tilde{u} - \nabla_u \tilde{u} = \frac{1}{2} d|u|^2.
\] (2.4)

By definition \( L_u = i_u d + di_u \) where \( i_u \) denotes the interior product by \( u \), so the exterior derivative \( d \) commutes with \( L_u \) since \( dL_u = L_ud = di_ud \), and therefore by using (2.4),
\[
d\nabla u \tilde{u} = dL_u \tilde{u} = L_u d\tilde{u}.
\]

It is obvious that \( \Box d = d\Box \). Then by acting \( d \) on the two sides of (2.2), we get
\[
\begin{cases}
\partial_t \omega_t + L_u \omega_t + \nu \Box \omega_t = 0, \\
\omega_t|_{t=0} = \omega_0.
\end{cases}
\] (2.5)

Remark 2.1. Since \( d^* \tilde{u} = 0 \), by definition (2.3), \( d^* \omega = d^* d\tilde{u} = \Box \tilde{u} \), and therefore, as \( \Box \) admits a spectral gap, \( \tilde{u} \) can be solved by
\[
\tilde{u} = \Box^{-1}(d^*\omega).
\]

It is sometimes more convenient to use covariant derivatives. To do this, let \( \beta \) be a differential \( p \)-form and \( T : \mathcal{X}(M) \to \mathcal{X}(M) \) be a tensor of type \((1, 1)\). Define for \( X_1, \ldots, X_p \),
\[
(\beta \circ T)(X_1, \ldots, X_p) = \beta(T(X_1), X_2, \ldots, X_p) + \ldots + \beta(X_1, \ldots, X_{p-1}, T(X_p)).
\] (2.6)

If \( \beta \) is a 2-form and \( T = \nabla u \), then for \( X, Y \in \mathcal{X}(M) \),
\[
(\beta \circ \nabla u)(X, Y) = \beta(\nabla_X u, Y) + \beta(X, \nabla_Y u).
\] (2.7)
In the same way as for proving (2.4), we have
\[(L_u\beta - \nabla v\beta)(X, Y) = \beta(\nabla_X v, Y) + \beta(X, \nabla_Y v) = (\beta \preceq \nabla v)(X, Y).\]
Now replacing $L_u\tilde{\omega}$ by $\nabla u\tilde{\omega} + \tilde{\omega} \preceq \nabla u$ in (2.5), we obtain the following form
\[
\begin{cases}
\partial_t \tilde{\omega} + \nabla u\tilde{\omega} + \nu \square \tilde{\omega} = -\tilde{\omega} \preceq \nabla u_t, \\
\tilde{\omega}|_{t=0} = \tilde{\omega}_0.
\end{cases}
\tag{2.8}
\]

**Proposition 2.2.** Let $\nabla^{sk} u$ be the skew-symmetric part of $\nabla u$, that is,
\[
\langle \nabla^{sk} u, X \otimes Y \rangle = \frac{1}{2} (\langle \nabla_X u, Y \rangle - \langle \nabla_Y u, X \rangle).
\]
Then $\tilde{\omega} \preceq \nabla^{sk} u = 0$.

**Proof.** Fix $x \in M$ and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_x M$. Then
\[
\nabla^{sk} u_X = \sum_{i,j=1}^n \langle \nabla^{sk} e_i u, e_j \rangle \langle X, e_i \rangle e_j = \sum_{i,j=1}^n d\tilde{u}(e_i, e_j) \langle X, e_i \rangle e_j = \sum_{j=1}^n \tilde{\omega}(X, e_j) e_j,
\]
so that
\[
\tilde{\omega}(\nabla^{sk} u_X, Y) = \sum_{j=1}^n \tilde{\omega}(X, e_j) \tilde{\omega}(e_j, Y) = \tilde{\omega}(\nabla^{sk} u_Y, X).
\]
Combining these relations and Definition (2.7), we have
\[
(\tilde{\omega} \preceq \nabla^{sk} u)(X, Y) = \tilde{\omega}(\nabla^{sk} u_Y, X) + \tilde{\omega}(X, \nabla^{sk} u_Y) = 0.
\]

Let $\nabla^s u$ be the symmetric part of $\nabla u$, that is
\[
\langle \nabla^s u, X \otimes Y \rangle = \frac{1}{2} (\langle \nabla_X u, Y \rangle + \langle \nabla_Y u, X \rangle).
\]
$\nabla^s u$ is called the rate of strain tensor in the literature on fluid dynamics. Therefore Equation (2.8) can be written in the following form:
\[
\begin{cases}
\partial_t \tilde{\omega} + \nabla u\tilde{\omega} + \nu \square \tilde{\omega} = -\tilde{\omega} \preceq \nabla^s u_t, \\
\tilde{\omega}|_{t=0} = \tilde{\omega}_0.
\end{cases}
\tag{2.9}
\]
In the case where $\dim(M) = 2$ or $3$, Equation (2.8) can be simplified using Hodge star operator $\ast$. Assume that $M$ is oriented and $\omega_n$ is the $n$-form of Riemannian volume, let $\omega = \ast \tilde{\omega}$, which is a $(n-2)$ form such that
\[
\tilde{\omega} \wedge \alpha = \langle \omega, \alpha \rangle \Lambda^{n-2} \omega_n, \quad \text{for any } \alpha \in \Lambda^{n-2}(M),
\]
or
\[
\beta \wedge \ast \tilde{\omega} = \langle \bar{\omega}, \beta \rangle \Lambda^2 \omega_n, \quad \text{for any } \beta \in \Lambda^2(M).
\]
Proposition 2.3. Let $\omega$ be a p-form on $M$ and $\text{div}(u) = 0$. Then $\nabla_u(*\omega) = *(\nabla_u \omega)$.

Proof. Let $\beta$ be a p-form. Then $\beta \wedge *\omega = \langle \beta, \omega \rangle \omega_n$. Taking the covariant derivative with respect to $u$, the left hand side gives

$$\nabla_u \beta \wedge (*\omega) + \beta \wedge \nabla_u (*\omega) = \langle \nabla_u \beta, \omega \rangle \omega_n + \beta \wedge \nabla_u (*\omega),$$

while the right hand side gives

$$\langle \nabla_u \beta, \omega \rangle \omega_n + \langle \beta, \nabla_u \omega \rangle \omega_n = \langle \nabla_u \beta, \omega \rangle \omega_n + \beta \wedge *\nabla_u \omega$$

as $\nabla_u \omega_n = 0$. Therefore $\beta \wedge \nabla_u (*\omega) = \beta \wedge (*\nabla_u \omega)$ holds for any p-form $\beta$, the result follows. $\Box$

Proposition 2.4. Assume $\dim(M) = 3$. Then

$$* (\omega \lrcorner \nabla^s u) = -(\ast \tilde{\omega}_t) \lrcorner \nabla^s u. \quad (2.10)$$

Proof. Fix $x \in M$; let $\{e_1, e_2, e_3\}$ be an orthonormal basis of $T_x M$, $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ be the dual basis of $T^*_x M$. Let $\{i_1, i_2, i_3\}$ be a direct permutation of $\{1, 2, 3\}$, and $\omega = \tilde{e}_{i_1} \wedge \tilde{e}_{i_2}$. Then

$$(\omega \lrcorner \nabla^s u)(X, Y) = \left( (\nabla^s_X u)_{i_1} X_{i_2} - Y_{i_1} (\nabla^s_X u)_{i_2} \right) + \left( (\nabla^s_Y u)_{i_2} X_{i_1} - X_{i_2} (\nabla^s_Y u)_{i_1} \right)$$

$$= \sum_{j=1}^3 \left[ (\nabla^s_{e_j} u)_{i_1} X_{i_2} - (\nabla^s_{e_j} u)_{i_2} X_{i_1} + (\nabla^s_{e_j} u)_{i_2} X_{i_1} Y_j - (\nabla^s_{e_j} u)_{i_1} X_{i_2} Y_j \right]$$

$$= \sum_{j=1}^3 (\nabla^s_{e_j} u)_{i_1} \left( X_{i_2} Y_j - X_{i_2} Y_j \right) + \sum_{j=1}^3 (\nabla^s_{e_j} u)_{i_2} \left( X_{i_1} Y_j - X_{i_2} Y_j \right).$$

It follows that

$$\omega \lrcorner \nabla^s u = \sum_{j=1}^3 (\nabla^s_{e_j} u)_{i_1} \tilde{e}_{i_2} \wedge \tilde{e}_{i_2} + \sum_{j=1}^3 (\nabla^s_{e_j} u)_{i_2} \tilde{e}_{i_1} \wedge \tilde{e}_{i_2}.$$

More precisely

$$\omega \lrcorner \nabla^s u = (\nabla^s_{e_1} u)_{i_1} \tilde{e}_{i_2} \wedge \tilde{e}_{i_2} + (\nabla^s_{e_2} u)_{i_2} \tilde{e}_{i_1} \wedge \tilde{e}_{i_2}$$

$$+ (\nabla^s_{e_3} u)_{i_3} \tilde{e}_{i_1} \wedge \tilde{e}_{i_2} + (\nabla^s_{e_3} u)_{i_2} \tilde{e}_{i_1} \wedge \tilde{e}_{i_3}.$$

Since $\sum_{j=1}^3 (\nabla^s_{e_j} u)_{i_j} = \text{Trace}(\nabla u) = \text{div}(u) = 0$, therefore finally we get

$$* (\omega \lrcorner \nabla^s u) = -(\nabla^s_{e_1} u)_{i_1} \tilde{e}_{i_2} - (\nabla^s_{e_2} u)_{i_2} \tilde{e}_{i_1} - (\nabla^s_{e_3} u)_{i_3} \tilde{e}_{i_1}. \quad (2.11)$$

On the other hand, $*\omega = \tilde{e}_{i_3}$, so that

$$(*\omega) \lrcorner (\nabla^s u)(X) = (*\omega)(\nabla^s_X u) = \sum_{j=1}^3 (\nabla^s_{e_j} u)_{i_3} X_j.$$

It follows that

$$(*\omega) \lrcorner (\nabla^s u) = (\nabla^s_{e_1} u)_{i_3} \tilde{e}_{i_1} + (\nabla^s_{e_2} u)_{i_3} \tilde{e}_{i_2} + (\nabla^s_{e_3} u)_{i_3} \tilde{e}_{i_3}. \quad (2.12)$$

Now combing (2.11), (2.12), and by symmetry of $\nabla^s u$, we get (2.10). $\Box$
Corollary 2.5. Let \( \dim(M) = 3 \) and \( \omega_t = \ast \tilde{\omega}_t \). Then
\[
\partial_t \omega_t + \nabla_y \omega_t + \nu \Box \omega_t = \omega_t \ast (\nabla^y u_t).
\]  
(2.13)

Proof. First note that \( \Box \ast = \ast \Box \) (see [40], p. 221), so (2.13) follows from Proposition 2.3 and Proposition 2.4.

Remark 2.6. Since \( \ast \ast = (-1)^{p(n-p)} \) on \( p \)-form, so for \( n = 3 \), \( \tilde{\omega}_t = \ast \omega_t \) and in the case where \( \Box \) admits a spectral gap, the following relation holds
\[
\tilde{u}_t = \Box^{-1} (d^\ast (\ast \omega_t)).
\]  
(2.14)

Proposition 2.7. In the smooth case, it holds
\[
\frac{1}{2} \frac{d}{dt} \int_M |u_t|^2 \, dx + \nu \int_M |\nabla u_t|^2 \, dx = -\nu \int_M \langle \text{Ric}, u_t \rangle \, dx.
\]  
(2.15)

Proof. Remark first that
\[
\int_M \langle \nabla u_t, u_t \rangle \, dx = \frac{1}{2} \int_M L_{u_t}|u_t|^2 \, dx = 0 \quad \text{and} \quad \int_M \langle \nabla p, u_t \rangle \, dx = 0.
\]
Then using equation (2.1), we get
\[
\frac{1}{2} \frac{d}{dt} \int_M |u_t|^2 \, dx + \nu \int_M \langle \Box u_t, u_t \rangle \, dx = 0.
\]
Now using Bochner-Weitzenb"ock formula (1.6) and (1.7) yields (2.15).

Proposition 2.8. Assume that there exists a constant \( \kappa \in \mathbb{R} \) such that
\[
\text{Ric} \geq -\kappa.
\]  
(2.16)

Then the following a priori estimate holds
\[
\frac{1}{2} ||u_t||^2_2 + \nu \int_0^t ||\nabla u_s||^2_2 \, ds \leq \frac{1}{2} ||u_0||^2_2 \exp(2\nu \kappa^+),
\]  
(2.17)

where \( \kappa^+ = \sup\{\kappa, 0\} \).

Proof. Using (2.16) and (2.15), we get inequality
\[
\frac{1}{2} \frac{d}{dt} \int_M |u_t|^2 \, dx + \nu \int_M |\nabla u_t|^2 \, dx \leq \nu \kappa \int_M |u_t|^2 \, dx \leq \nu \kappa^+ \int_M |u_t|^2 \, dx.
\]
Let \( \psi(t) = \frac{1}{2} ||u_t||^2_2 + \nu \int_0^t ||\nabla u_s||^2_2 \, ds \). Then \( \psi \) satisfies inequality
\[
\psi(t) \leq \frac{1}{2} ||u_0||^2_2 + 2\nu \kappa^+ \int_0^t \psi(s) \, ds.
\]
Gronwall lemma yields (2.17).

In what follows, we will establish the existence of weak solutions in Leray sense over any \( [0, T] \) and
\[
u \in L^2([0, T], H^1(M)) \cap L^\infty([0, T], L^2(M)).
\]
To this end, we will use the heat semi-group \( T_t = e^{-\nu t/2} \) to regularize vector fields. Let \( v \) be a continuous vector field on \( M \) with compact support and define \( T_t v = (T_t \tilde{v})^# \). Then \( T_t v \) solves the heat equation.
\[
\left( \frac{\partial}{\partial t} + \frac{1}{2} \Box \right) (T_t v) = 0.
\]
By ellipticity of \( \Box \) (see for example \[40\]), \((t, x) \to (T_t v)(x)\) is smooth. It was shown in \[21\] that
\[
div(T_t v) = T_t^M (\text{div}(v)),
\]
where \(T_t^M\) denotes heat semi-group on functions. Hence \(T_t\) preserves the space of divergence free vector fields. By \(\|\cdot\|\) in Section \[6\] it holds true that
\[
|T_t v| \leq e^{\epsilon n/2} T_t^M |v|.
\]
(2.18)
It follows that for \(1 \leq p \leq +\infty\), \(\|T_t v\|_p \leq e^{\epsilon n/2} \|v\|_p\), and for \(1 \leq p < +\infty\), \(T_t v \to v\) in \(L^p\).

Consider a family of smooth functions \(\varphi_\epsilon \in C_c^\infty(M)\) with compact support such that
\[
0 \leq \varphi_\epsilon \leq 1, \quad \varphi_\epsilon(x) = 1 \quad \text{for} \quad x \in B(x_M, 1/\epsilon) \quad \text{and} \quad \sup_{\epsilon > 0} \|\nabla \varphi_\epsilon\|_\infty < +\infty,
\]
(2.19)
where \(x_M\) is a fixed point of \(M\). For \(\epsilon > 0\), we define
\[
F_\epsilon(u) = -T_\epsilon P(\varphi_\epsilon \nabla T_\epsilon u(\varphi_\epsilon T_\epsilon u)) - \nu T_\epsilon \Box T_\epsilon u, \quad u \in L^2(M)
\]
where \(P\) is the orthogonal projection from \(L^2(M)\) to the subspace of vector fields of divergence free. We have
\[
\|T_\epsilon P(\varphi_\epsilon \nabla T_\epsilon u(\varphi_\epsilon T_\epsilon u))\|_2 \leq e^{\epsilon n/2} \|P(\varphi_\epsilon \nabla T_\epsilon u(\varphi_\epsilon T_\epsilon u))\|_2 \leq e^{\epsilon n/2} \|\nabla \varphi_\epsilon T_\epsilon u(\varphi_\epsilon T_\epsilon u)\|_2.
\]
Since \(\varphi_\epsilon\) is of compact support, we have
\[
\|\nabla \varphi_\epsilon T_\epsilon u(\varphi_\epsilon T_\epsilon u)\|_2 \leq \|\varphi_\epsilon T_\epsilon u\|_\infty \|\nabla(\varphi_\epsilon T_\epsilon u)\|_2.
\]
(2.20)
Again due to compact support of \(\varphi_\epsilon\), when \(n = 3\), by Sobolev’s embedding theorem, there is a constant \(\beta(\epsilon) > 0\) such that
\[
\|\varphi_\epsilon T_\epsilon u\|_\infty \leq \beta(\epsilon) \|\varphi_\epsilon T_\epsilon u\|_{H^2}.
\]
For the general case, it is sufficient to bound the uniform norm by the norm of \(H^m\) with \(m > \frac{n}{2}\).

**Proposition 2.9.** For any \(T > 0\), there are constants \(\beta_1, \beta_2\) such that
\[
\|\Box T_\epsilon u\|_2 \leq \frac{\beta_1}{\epsilon^2} \|u\|_2, \quad \|\nabla T_\epsilon u\|_2 \leq \frac{\beta_2}{\sqrt{\epsilon}}, \quad \epsilon > 0.
\]
(2.21)

**Proof.** We will restate, in Section \[6\] \(2.21\) with more precise coefficients dependent of curvatures of \(M\) and give a proof based on Bismut formulae obtained in \[18\] \[13\]. \(\square\)

By Proposition \[2.9\] there are constants \(\beta(\epsilon) > 0\), \(\tilde{\beta}(\epsilon) > 0\) such that
\[
\|\varphi_\epsilon T_\epsilon u\|_\infty \leq \beta(\epsilon) \|u\|_2, \quad \|T_\epsilon \Box T_\epsilon u\|_2 \leq \tilde{\beta}(\epsilon) \|u\|_2.
\]
(2.22)
Combining \[2.20\] and \[2.22\], there are two constants \(\beta_1(\epsilon) > 0\) and \(\beta_2(\epsilon) > 0\) such that
\[
\|F_\epsilon(u)\|_2 \leq \beta_1(\epsilon) \|u\|_2^2 + \beta_2(\epsilon) \|u\|_2.
\]
and $F_ε$ is locally Lipschitz. By theory of ordinary differential equation, there is a unique solution $u^ε_t$ to
\[
\frac{du^ε_t}{dt} = F_ε(u^ε_t), \quad u^ε_0 = u_0 \in L^2, \quad \text{div}(u^ε_t) = 0,
\] up to the explosion time $τ$.

**Theorem 2.10.** Assume that $||\text{Ric}||_{∞} < +∞$ and that $R_2$ is bounded below. Then for any $T > 0$, there is a weak solution $u \in L^2([0,T], H^1)$ to Navier-Stokes equation (2.21) such that
\[
\frac{1}{2}||u_t||^2_2 + ν \int_0^t ||\nabla u_s||^2_2 ds \leq \frac{1}{2}||u_0||^2_2 \exp(2νκ^+),
\]
where $κ$ is lower bound of $\text{Ric}$ and $R_2$ is the Weitzenböck curvature on 2-differential forms defined in (6.8).

**Proof.** Rewriting Equation (2.23) in the following explicit form, for $t < τ$,
\[
\frac{du^ε_t}{dt} + T_εP(φ_ε ∇T_εu^ε_t(φ_ε T_εu^ε_t)) + νT_ε□T_εu^ε_t = 0.
\]
Note that
\[
\int_M \langle T_εP(φ_ε ∇T_εu^ε_t(φ_ε T_εu^ε_t)), u^ε_t \rangle dx = \int_M \langle ∇T_εu^ε_t(φ_ε T_εu^ε_t), φ_ε T_εu^ε_t \rangle dx
\]
\[
= \int_M L_{T_εu^ε_t} |φ_ε T_εu^ε_t|^2 dx = 0,
\]
since $\text{div}(T_εu^ε_t) = 0$, and
\[
\int_M \langle T_ε□T_εu^ε_t, u^ε_t \rangle dx = \int_M |∇T_εu^ε_t|^2 dx + \int_M \langle \text{Ric}(T_εu^ε_t), T_εu^ε_t \rangle dx.
\]
Hence
\[
\frac{1}{2} \frac{d}{dt} \int_M |u^ε_t|^2 dx + ν \int_M |∇T_εu^ε_t|^2 dx = -ν \int_M \langle \text{Ric}(T_εu^ε_t), T_εu^ε_t \rangle dx
\]
\[
\leq -νκ \int_M |T_εu^ε_t|^2 dx,
\]
or in the form
\[
\frac{1}{2}||u^ε_t||^2_2 + ν \int_0^t ||∇T_εu^ε_s||^2_2 ds \leq \frac{1}{2}||u_0||^2_2 + ν\kappa^+ \int_0^t ||T_εu^ε_s||^2_2 ds. \tag{2.24}
\]
According to (2.18), above inequality implies that
\[
\frac{1}{2}||u^ε_t||^2_2 \leq \frac{1}{2}||u_0||^2_2 + ν\kappa^+ e^{ε\kappa^+} \int_0^t ||u^ε_s||^2_2 ds.
\]
Gronwall lemma implies that for $t < τ$
\[
\frac{1}{2}||u^ε_t||^2_2 \leq \frac{1}{2}||u_0||^2_2 \exp(tν\kappa^+ e^{ε\kappa^+}).
\]
It follows that $τ = +∞$. Now again by (2.18) and (2.24), we get
\[
\frac{1}{2}||T_εu^ε_t||^2_2 + νe^{ε\kappa^+} \int_0^t ||∇T_εu^ε_s||^2_2 ds \leq \frac{1}{2}e^{ε\kappa^+} ||u_0||^2_2 + ν\kappa^+ e^{ε\kappa^+} \int_0^t ||T_εu^ε_s||^2_2 ds.
\]
Gronwall lemma yields, for $\varepsilon \leq 1$, that

$$
\frac{1}{2}||T\varepsilon u_t^*||^2 + \nu \varepsilon \int_0^t ||\nabla T\varepsilon u_s^*||^2 ds \leq \frac{e^{\kappa^+}}{2}||u_0||^2 \exp(t \nu \kappa^+ e^+) .
$$

(2.25)

Let $T > 0$. By (2.25), the family $\{T\varepsilon u_t^*; \varepsilon \in (0, 1]\}$ is bounded in $L^2([0, T], H^1)$ as well as in $L^\infty([0, T], L^2)$. Then there is a sequence $\varepsilon_n$ and a $u \in L^2([0, T], H^1) \cap L^\infty([0, T], L^2)$ such that $T\varepsilon_n u_t^*$ converges weakly to $u$ in $L^2([0, T], H^1)$ and $*$-weakly in $L^\infty([0, T], L^2)$. Now standard arguments allow to prove that $u$ is a weak solution (2.1). The boundedness of Ric is needed while passing to the limit of the term $\int_M \langle \text{Ric}(T\varepsilon u_t^*), u_t \rangle \ dx$.

$$\square$$

Proposition 2.11. Let $\dim(M) = 3$. The vorticity $\omega_t$ satisfies a priori identity:

$$
\frac{1}{2} \frac{d}{dt} \int_M |\omega_t|^2 \ dx + \nu \int_M |\nabla \omega_t|^2 \ dx = -\nu \int_M \langle \text{Ric} \omega_t, \omega_t \rangle \ dx + \int_M \langle \omega_t \cdot \nabla^s u_t, \omega_t \rangle \ dx .
$$

(2.26)

Proof. Using Equation (2.13) and the same as proving (2.14) yields (2.26).

The term $H_t := \int_M \langle \omega_t, u_t \rangle \ dx$ is called helicity in theory of the fluid mechanics.

Proposition 2.12. Let $\dim(M) = 3$. Then

$$
\frac{d}{dt} \int_M \langle \omega_t, u_t \rangle \ dx = -\nu \int_M \langle d\omega_t, *\omega_t \rangle_{\Lambda^2} \ dx - \nu \int_M \langle \nabla^s, u_t \rangle \ dx - \nu \int_M \langle \omega_t, \text{Ric} u_t \rangle \ dx + \int_M \langle \omega_t, \nabla^s u_t \rangle \ dx .
$$

(2.27)

Proof. Using Equation (2.1) and Equation (2.13), we have

$$
\frac{d}{dt} \langle \omega_t, u_t \rangle = -(\nabla u_t \omega_t, u_t) - \mu(\nabla^s u_t, u_t) + \langle \omega_t \cdot \nabla^s u_t, u_t \rangle - \langle \omega_t, \nabla u_t \rangle - \nu(\omega_t, \nabla u_t) - \langle \omega_t, \nabla p \rangle .
$$

It is obvious that

$$
\int_M \left[ (\nabla u_t \omega_t, u_t) + \langle \omega_t, \nabla u_t \rangle \right] \ dx = \int_M L_{u_t}(\omega_t, u_t) \ dx = 0 .
$$

In addition, by (10, page 220), $d^* = (-1)^{(p+1)} * d*$ and $** = (-1)^{(n-p)}$ on $p$-forms. Then $d^* = \pm * d$, so that

$$
\int_M \langle \omega_t, dp \rangle \ dx = \int_M \langle *\tilde{\omega}_t, dp \rangle \ dx = \int_M d^*(*\tilde{\omega}_t)p \ dx = \pm \int_M *(d\tilde{\omega}_t)p \ dx = 0 .
$$

On one hand, using Hodge star operator,

$$
\int_M \langle \omega_t, \nabla u_t \rangle \ dx = \int_M \langle \omega_t, d^* du_t \rangle \ dx = \int_M \langle d\omega_t, \tilde{\omega}_t \rangle \ dx = \int_M \langle d\omega_t, *\omega_t \rangle \ dx .
$$

On the other hand, using Bochner-Weitzenböck formula,

$$
\int_M \langle \omega_t, \nabla u_t \rangle \ dx = \int_M \langle \nabla \omega_t, \nabla u_t \rangle \ dx + \int_M \langle \omega_t, \text{Ric} u_t \rangle \ dx .
$$
By putting these terms together we conclude that
\[
\frac{d}{dt} \int_M (\omega_t, u_t) \, dx = -\nu \int_M \langle d\omega_t, \ast \omega_t \rangle \, dx - \nu \int_M \langle \nabla \omega_t, \nabla u_t \rangle \, dx - \nu \int_M \langle \omega_t, \text{Ric} u_t \rangle \, dx + \int_M (\omega_t, \nabla^s u_t) \, dx,
\]
since \((\omega_t \ast \nabla^s u_t, u_t) = (\omega_t, \nabla^s u_t)\). We get \((2.27)\). 

3 Heat equations on differential forms

We will express solutions to equation \((2.13)\) by means of principal bundle of orthonormal frames \(O(M)\). An element \(r \in O(M)\) is an isometry from \(\mathbb{R}^n\) onto \(T_{\pi(r)}M\) where \(\pi : O(M) \to M\) is the canonical projection. More precisely, an element of \(O(M)\) is composed of \((x, r)\), where \(x = \pi(x, r)\) and \(r\) is an orthonormal frame at \(x\), that is, an isometry from \(\mathbb{R}^n\) onto \(T_x M\). For the sake of simplicity, we read \(r\) as \((\pi(r), r)\), but we sometimes have to distinguish them. The Levi-Civita connection on \(M\) gives rise to \(n\) canonical horizontal vector fields \(\{A_1, \ldots, A_n\}\) on \(O(M)\), which are such that \(d\pi(r) \cdot A_r = r \varepsilon_i\), where \(\{\varepsilon_1, \ldots, \varepsilon_n\}\) is the canonical basis of \(\mathbb{R}^n\). A vector field \(v\) on \(M\) can be lift to a horizontal vector field \(\bar{V}\) on \(O(M)\) such that \(d\pi(r) V_r = v_{\pi(r)}\). Let \(\omega\) be a differential 1-form. Following Malliavin \([31]\), we define
\[
F^i_\omega(r) = (\omega_{\pi(r)}, r \varepsilon_i) = (\pi^* \omega, A_i)_r, \quad i = 1, \ldots, n, \tag{3.1}
\]
where \(\pi^* \omega\) is the pull-back of \(\omega\) by \(\pi : O(M) \to M\). We have
\[
(L_{A_j} F^i_\omega)(r) = (\nabla_{r \varepsilon_j} \omega, r \varepsilon_i) = (\nabla \omega, r \varepsilon_j \otimes r \varepsilon_i), \tag{3.2}
\]
where the second duality makes sense in \(T_{\pi(r)} M \otimes T_{\pi(r)} M\). In fact, let \(t \to r(t) \in O(M)\) be the smooth curve such that \(r(0) = r, r'(0) = A_j(r)\). Let \(\xi_t = \pi(r(t))\). Then \(/ /_{t}^{-1} := r \circ r(t)^{-1}\) is the parallel translation from \(T_{\xi_t} M\) onto \(T_x M\) along \(\xi\) and
\[
F^i_\omega(r(t)) = (\omega_{\xi_t}, r(t) \varepsilon_i) = (/ /_{t}^{-1} \omega_{\xi_t}, r \varepsilon_i).
\]
Taking the derivative with respect to \(t\) at \(t = 0\) yields \((3.2)\). In the same way, we get \((L_{A_j} F^i_\omega)(r) = (\nabla_{r \varepsilon_j} \omega, r \varepsilon_j \otimes r \varepsilon_i)\). Therefore
\[
\Delta_{O(M)} F^i_\omega := \sum_{j=1}^n L^j_{A_j} F^i_\omega = (\Delta \omega, r \varepsilon_i) = F^i_{\Delta \omega}(r).
\]
Let \(U_t\) be the horizontal lift of \(u_t\) to \(O(M)\). Then \(U_t(r) = \sum_{j=1}^n \langle u_t(x), r \varepsilon_j \rangle A_j(r)\), where \(x = \pi(r)\) and according to \((3.2)\),
\[
(L_{U_t} F^i_\omega)(r) = \sum_{j=1}^n \langle u_t, r \varepsilon_j \rangle (L_{A_j} F^i_\omega)(r) = \langle \nabla u_t \omega, r \varepsilon_i \rangle = F^i_{\nabla u_t \omega}(r).
\]
Let \(\phi_t = \omega_t \ast \nabla^s u\); then
\[
F^i_{\phi_t}(r) = (\phi_t, r \varepsilon_i) = \omega_t(\nabla^s_{r \varepsilon_i} u_t) = \sum_{j=1}^n (\nabla^s_{r \varepsilon_i} u_t, r \varepsilon_j) (\omega_t, r \varepsilon_j) = \sum_{j=1}^n (\nabla^s_{r \varepsilon_i} u_t, r \varepsilon_j) F^j_{\omega_t}.
\]
Define 

$$K_{ij}(t,r) = \langle \nabla^*_r u_t(\pi(r)), r e_j \rangle$$

and 

$$K(t,r) = (K_{ij}(t,r)).$$

Then 

$$F_{\phi_t}(r) = K(t,r)F_{\omega_t}(r).$$

By applying Bochner-Weitzenböck formula (see (1.6)) to 1-form 

$$\omega,$$

we get the following heat equation defined on 

$$\Omega$$

$$\text{Ric}$$

Then for any 

$$\phi \in C_c(O(M))$$

we have 

$$F_{\phi_t} \rightarrow F_{\omega_t} \text{ as } t \to \infty.$$
It is known that out of $C_{x_0} \cup \{x_0\}$, $|\nabla_x d_M(x, x_0)| = 1$. Therefore out of $\pi^{-1}(C_{x_0} \cup \{x_0\})$,

$$|L_{V_1}d_M(\pi(\cdot), x_0)| \leq |V_1|.$$  

(3.7)

The lower bound of $\frac{1}{2}\Delta_O(M)\rho$ is more delicate. According to \[24\], page 90,

$$\frac{1}{2}\Delta_O(M)d_M(\pi(\cdot), x_0) \geq \frac{n-1}{2\rho} - \frac{1}{2}\sqrt{n(n-1)K_R}, \quad \text{quad} \pi(r) \in B(x_M, R) \setminus (C_{x_0} \cup \{x_0\}).$$  

(3.8)

where $K_R$ is the upper bound of sectional curvature on the big ball $B(x_M, R)$.

**Proposition 3.1.** Assume furthermore that

$$\rho(r_1) - \rho(r_0) = \beta_t + \int_0^t \left(\frac{1}{2}\Delta_O(M) + L_{V_s}\right)\rho(s) ds - \hat{L}_t, \quad t < \zeta(w, r_0).$$  

(3.10)

**Proof.** The proof will be given in Section 6. \qed

**Theorem 3.2.** Assume $\text{Ric} \geq -\kappa$ and (3.9) holds. Then for almost all $r_0$, $\zeta(w, r_0) = +\infty$ almost surely.

**Proof.** We have, by (3.10),

$$\rho(r_{t\wedge \zeta})^2 \leq \rho(r_0)^2 + t \wedge \zeta + 2 \int_0^{t \wedge \zeta} \rho(r_s) d\beta_s + 2 \int_0^{t \wedge \zeta} \rho(r_s)(L_s\rho)(r_s) ds,$$

where $L_s = \frac{1}{2}\Delta_O(M) + L_{V_s}$. Using (3.6) and (3.7), there is constants $C > 0$ such that

$$\mathbb{E}(\rho(r_{t\wedge \zeta})^2) \leq \rho(r_0)^2 + C \int_0^t \mathbb{E}\left(2\rho(r_s)(L_s\rho)(r_s) + 1\right) 1_{(s < \zeta)} ds$$

$$\leq \rho(r_0)^2 + 2C \int_0^t \mathbb{E}\left((1 + \rho(r_s))(1 + |V_s(r_s)|) 1_{(s < \zeta)}\right) ds.$$

Let $\mu$ be the probability measure on $O(M)$ defined in (6.6). Then

$$\int_{O(M)} \mathbb{E}(\rho(r_{t\wedge \zeta})^2) d\mu \leq \int_{O(M)} \rho(r_0)^2 d\mu + 2C \int_0^t \int_{O(M)} \mathbb{E}\left((1 + \rho(r_s))(1 + |V_s(r_s)|) 1_{(s < \zeta)}\right) d\mu ds$$

$$\leq \int_{O(M)} \rho(r_0)^2 d\mu + 4C \left(\int_0^t \int_{O(M)} \mathbb{E}\left((1 + \rho(r_{s\wedge \zeta})^2\right) d\mu ds\right)^{1/2} \times$$

$$\times \left(\int_0^t \int_{O(M)} \mathbb{E}\left((1 + |V_s(r_s)|)^2 1_{(s < \zeta)}\right) d\mu ds\right)^{1/2}.$$

Note that

$$\int_0^t \int_{O(M)} \mathbb{E}\left((1 + |V_s(r_s)|)^2 1_{(s < \zeta)}\right) d\mu ds \leq 2\left(T + \int_0^T \int_M |v_s(x)|^2 dx ds\right).$$
Set \( \psi(t) = \int_{O(M)} \mathbb{E}(\rho(r_{t \wedge \xi})^2) \, d\mu \) and
\[
C(T, v) = 4C\sqrt{2} \sqrt{T + ||v||^2_{L^2([0, T] \times M)}}. 
\] (3.11)

Remark that \( \sqrt{x} \leq 1 + \xi \) for \( \xi \geq 0 \), above two inequalities imply that
\[
\psi(t) \leq \left( \int_{O(M)} \rho(r_0)^2 \, d\mu + C(T, v) \right) + C(T, v) \int_0^t \psi(s) \, ds.
\]

The Gronwall lemma then yields
\[
\int_{O(M)} \mathbb{E}(\rho(r_{t \wedge \xi})^2) \, d\mu \leq \left( \int_{O(M)} \rho(r_0)^2 \, d\mu + C(T, v) \right) \exp(C(T, v)).
\]
The result follows.

Now we are going to obtain a probabilistic representation for solution to the heat equation [3.3]. To this end, set \( F(t, r) = F_{\omega_t}(r) \). Let \( T > 0 \) be fixed. Assume that \( u_t \) is a solution to \([2.1]\) such that \( (t, x) \to u_t(x) \) is continuous and for each \( t \geq 0 \), \( u_t \in C^{1+\alpha} \) with \( \alpha > 0 \). Consider the following SDE on \( O(M) \),
\[
\begin{aligned}
&dr_{s,t}(r, w) = \sqrt{2\nu} \sum_{i=1}^n A_i(r_{s,t}(r, w)) \circ dW^i_t - U_{T-t}(r_{s,t}(r, w)) \, dt, \quad s < t < T, \\
&r_{s,s}(r, w) = r.
\end{aligned}
\] (3.12)

Let \( v_t(x) = u_{T-t}(x) \). Then by Theorem [3.2] SDE \([3.12]\) is stochastic complete. Let \( Q_{s,t}(w) \) be solution to the resolvent equation
\[
\frac{d}{dt}Q_{s,t}(w) = Q_{s,t}(w)J_{T-t}(r_{s,t}(r, w)), \quad s < t < T, \quad Q_{s,s}(w) = Id
\] (3.13)
where
\[
J_t(r) = K(t, r) - \nu \ric_t. \quad (3.14)
\]

For the sake of simplicity, we denote \( r_{s,t} = rs_t(r, w) \). Applying Itô formula to \( Q_{s,t}F(T-t, r_{s,t}) \) for \( d_t \) with \( t \in (s, T) \), we have
\[
\begin{aligned}
d_t \left( Q_{s,t}F(T-t, r_{s,t}) \right) &= d_tQ_{s,t}F(T-t, r_{s,t}) + Q_{s,t}d_t \left( F(T-t, r_{s,t}) \right) \\
&= Q_{s,t}J_{T-t}(r_{s,t})F(T-t, r_{s,t}) + \sqrt{2\nu} Q_{s,t} \sum_{i=1}^n (L_{A_i}F)(T-t, r_{s,t}) \, dW^i_t \\
&+ Q_{s,t} \left( - (\partial_t F)(T-t, r_{s,t}) + \nu (\Delta_{O(M)}F)(T-t, r_{s,t}) - (L_{U_{T-t}}F)(T-t, r_{s,t}) \right) \, dt \\
&= \sqrt{2\nu} Q_{s,t} \sum_{i=1}^n (L_{A_i}F)(T-t, r_{s,t}) \, dW^i_t,
\end{aligned}
\]
where the last equality is due to Equation [3.3]. It follows that
\[
Q_{s,t}F(T-t, r_{s,t}) - F(T-s, r) = \sqrt{2\nu} \sum_{i=1}^n \int_s^t Q_{s,\tau} (L_{A_i}F)(T-\tau, r_{s,\tau}) \, dW^i_\tau.
\]
Taking expectation on the two sides gives $E\left(Q_{s,t} F(T-t,r_{s,t})\right) = F(T-s,r)$. Let $t = T$. Then $E\left(Q_{s,T} F(0,r_{s,T})\right) = F(T-s,r)$. Replacing $s$ by $T-t$, we get the following representation formula to (3.3):

$$F_{\omega t} = E\left(Q_{T-t,T} F_{\omega t}(r_{T-t,T})\right).$$

(3.15)

In what follows, we will explain how a vector field $v$ on $M$ gives rise to a metric compatible connection $\Gamma^v$. For a time-independent vector field $v$ on $M$, the diffusion processes \{${x_t, t \geq 0}$\} associated to the generator $\frac{1}{2}\Delta_M + v$ can be constructed in the following way:

$$dr_t = \sum_{i=1}^{n} A_i(r_t) \circ dW^i_t + V(r_t) dt$$

(3.16)

where $V$ is the horizontal lift of $v$ to $O(M)$, and let $x_t = \pi(r_t)$. We assume that the lift-time $\zeta = +\infty$ almost surely.

In Chapter V of [25], Ikeda and Watanabe introduced a metric compatible connection $\Gamma^v$ so that the diffusion process of generator $\frac{1}{2}\Delta_M + v$ can be constructed by rolling without friction Brownian motion on $\mathbb{R}^n$ with respect to the connection $\Gamma^v$. More precisely let \{${B_1,\ldots,B_n}$\} be the canonical horizontal vector fields on $O(M)$ with respect to $\Gamma^v$, consider SDE on $O(M)$:

$$dr_w(t) = \sum_{i=1}^{n} B_i(r_w(t)) \circ dW^i_t, \quad r_w(0) = r.$$  

Then the generator of diffusion process $t \to x_t(w) = \pi(r_w(t))$ is $\frac{1}{2}\Delta_M + v$. In fact, it holds

$$\frac{1}{2} \sum_{j=1}^{n} B_j^2(f \circ \pi) = \left(\left(\frac{1}{2}\Delta_M + v\right)f\right) \circ \pi.$$  

(3.17)

This connection $\Gamma^v$ was defined locally in [25]. On a local chart $U$, \{${\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}}$\} is a local basis of tangent spaces $T_x M$ with $x \in U$, and $v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x_i}$. Let $\Gamma_{ij}^{\delta k}$ be the Christoffel coefficients of Levi-Civita connection. According to ([25], p.271), the Christoffel coefficients $\Gamma_{ij}^{\delta k}$ of $\Gamma^v$ is defined by (see also [1]),

$$\Gamma_{ij}^{\delta k} = \Gamma_{ij}^{\delta k} - \frac{2}{n-1} \left(\delta_{kl} \sum_{\ell=1}^{n} g_{j\ell} v^{\ell} - g_{ij} v^{k}\right).$$  

(3.18)

**Proposition 3.3.** Let $\nabla^v$ be the covariant derivative with respect to the connection $\Gamma^v$, and $\nabla^0$ with respect to the Levi-Civita connection. Then for two vector fields $X,Y$ on $M$,

$$\nabla^v_X Y = \nabla^0_X Y - \frac{2}{n-1} K_v(X,Y),$$

(3.19)

where

$$K_v(X,Y) = \langle Y, v \rangle X - \langle X, Y \rangle v.$$  

(3.20)

**Proof.** We have, using (3.18),
\[ \nabla^v_k Y = \sum_{k=1}^n \left[ \sum_{i,j=1}^n X^i Y^j \Gamma^k_{ij} + \sum_{i=1}^n X^i \frac{\partial Y^k}{\partial x_i} \right] \frac{\partial}{\partial x_k} \]

\[ = \sum_{k=1}^n \left[ \sum_{i,j=1}^n X^i Y^j \Gamma^0_{ij} + \sum_{i=1}^n X^i \frac{\partial Y^k}{\partial x_i} \right] \frac{\partial}{\partial x_k} - \frac{2}{n-1} I_2, \]

where

\[ I_2 = \sum_{i,j,k=1}^n X^i Y^j \delta_{ki} \left( \frac{\partial}{\partial x_j}, v \right) \frac{\partial}{\partial x_k} - \sum_{i,j,k=1}^n X^i Y^j \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) v^k \frac{\partial}{\partial x_k}, \]

since \( \sum_{\ell=1}^n g_{j\ell} v^\ell = \left( \frac{\partial}{\partial x_j}, v \right) \). It is obvious to see that the first sum in \( I_2 \) is equal to \( \langle Y, v \rangle X \), while the second sum yields \( \langle X, Y \rangle v \). The relation (3.19) and (3.20) follow.

Having this explicit expression, we will compute the associated torsion tensor \( T^v \).

**Proposition 3.4.** \( T^v(X, Y) \) admits the expression:

\[ T^v(X, Y) = \frac{-2}{n-1} \left( \langle Y, v \rangle X - \langle X, v \rangle Y \right). \] (3.21)

Moreover, \( T^v \) is skew-symmetric (TSS), that is \( \langle T^v(X, Y), Z \rangle = -\langle T^v(Z, Y), X \rangle \) holds for all \( X, Y, Z \in \mathcal{X}(M) \) if and only if \( v = 0 \).

**Proof.** Using (3.19) and the fact \( \nabla^v_k Y - \nabla^0_k Y - [X, Y] = 0 \), we have

\[ T^v(X, Y) = -\frac{2}{n-1} \left( K_v(X, Y) - K_v(Y, X) \right) = -\frac{2}{n-1} \left( \langle Y, v \rangle X - \langle X, v \rangle Y \right), \]

that is nothing but (3.21). Now if for any \( X, Y, Z \in \mathcal{X}(M) \), \( \langle T^v(X, Y), Z \rangle + \langle T^v(Z, Y), X \rangle = 0 \), then this equality yields

\[ 2\langle Y, v \rangle \langle X, Z \rangle = \langle X, v \rangle \langle Y, Z \rangle + \langle Z, v \rangle \langle Y, X \rangle. \]

Taking \( Y = v \) and \( X = Z \) in above equality, we get

\[ |v|^2 |X|^2 = \langle X, v \rangle^2. \]

If \( v \neq 0 \), taking \( X \) orthogonal to \( v \) yields a contradiction.

**4 Intrinsic Ricci tensors for Navier-Stokes equations**

In what follows, we will denote Levi-Civita covariant derivative by \( \nabla^0 \). We first compute the Ricci tensor associated to the connection \( \nabla^v \).

**Proposition 4.1.** Let \( \text{Ric}^0 \) be the Ricci curvature associated to \( \nabla^0 \), and \( \text{Ric}^v \) to \( \nabla^v \). Then

\[ \text{Ric}^v(X) = \text{Ric}^0(X) - \frac{4(n-2)}{(n-1)^2} K_v(X, v) + \frac{2(n-2)}{n-1} \nabla^0_X v + \frac{2}{n-1} \text{div}(v) X. \] (4.1)
Proof. For the sake of simplicity, put $\nabla^v_Y Z = \nabla^0_Y Z + S(Y, Z)$, where $S$ is a $(1, 2)$ type tensor on $M$. Then
\[
\nabla^0_X \nabla^v_Y Z = \nabla^0_X \nabla^0_Y Z + S(X, \nabla^v_0 Y) = \nabla^0_X \left( \nabla^0_Y Z + S(Y, Z) \right) + S(X, \nabla^0_Y Z) = \nabla^0_X \nabla^0_Y Z + (\nabla^0_0 S)(Y, Z) + S(\nabla^0_X Y, Z) + S(Y, \nabla^0_X Z) + S(X, \nabla^0_Y Z).
\]
Changing role between $X$ and $Y$ yields
\[
\nabla^v_X \nabla^0_Y Z = \nabla^0_X \nabla^0_Y Z + S(X, \nabla^v_0 Y).
\]
Also
\[
\nabla^v_{[X, Y]} Z = \nabla^0_{[X, Y]} Z + S([X, Y], Z).
\]
Combining above equations, the curvature tensor
\[
R^v(X, Y) Z = \nabla^v_X \nabla^0_Y Z - \nabla^0_X \nabla^v_Y Z - \nabla^v_{[X, Y]} Z
\]
which admits the following expression
\[
R^0(X, Y) Z + (\nabla^0_X S)(Y, Z) - (\nabla^0_Y S)(X, Z) + S(\nabla^0_Y X, Z) - S(Y, S(X, Z)) + S(X, S(Y, Z)) - S([X, Y], Z).
\]
Let $x \in M$ and $\{e_1, \ldots, e_n\}$ an orthonormal basis of $T_x M$. Then $\text{Ric}^v(X) = \sum_{i=1}^n R^v(X, e_i) e_i$.

Note that $S(X, Y) = -\frac{2}{n - 1} K_v(X, Y)$. Put
\[
I_1 = \sum_{i=1}^n S(X, S(e_i, e_i)), \quad I_2 = \sum_{i=1}^n S(e_i, S(X, e_i)).
\]
\[
I_3 = \sum_{i=1}^n (\nabla^0_X S)(e_i, e_i), \quad I_4 = \sum_{i=1}^n (\nabla^0_Y S)(X, e_i).
\]
Then
\[
\text{Ric}^v(X) = \text{Ric}^0(X) + I_1 - I_2 + I_3 - I_4.
\]
By a completely elementary computation, we find
\[
I_1 = \frac{4}{(n - 1)^2} \sum_{i=1}^n K_v(X, K_v(e_i, e_i)) = -\frac{4(n - 1)}{(n - 1)^2} K_v(X, v)
\]
and
\[
I_2 = \frac{4}{(n - 1)^2} \sum_{i=1}^n K_v(e_i, K_v(X, e_i)) = -\frac{4}{(n - 1)^2} K_v(X, v).
\]
For two other terms,
\[
(\nabla^0_X S)(Y, Z) = -\frac{2}{n - 1} K_v^0_{X e}(Y, Z)
\]
and
\[
(\nabla^0_Y S)(X, Z) = -\frac{2}{n - 1} K_v^0_{Y e}(X, Z).
\]
Therefore

\[ I_3 = -\frac{2}{n - 1} \sum_{i=1}^{n} K_{\nabla_X^0 v}(e_i, e_i) = 2\nabla_X^0 v. \]

\[ I_4 = -\frac{2}{n - 1} \sum_{i=1}^{n} K_{\nabla_X^0 v}(X, e_i) = -\frac{2}{n - 1} \text{div}(v)X + \frac{2}{n - 1} \nabla_X^0 v. \]

Finally

\[ \text{Ric}^v(X) = \text{Ric}^0(X) - \frac{4(n - 2)}{(n - 1)^2} K_v(X, v) + \frac{2(n - 2)}{n - 1} \nabla_X^0 v + \frac{2}{n - 1} \text{div}(v)X \]

and the computations are complete. \(\square\)

Since the connection \(\nabla^v\) has torsion, we have to take account of torsion tensor into Ricci tensor in a suitable way. A Weitzenböck formula for a connection which is not of torsion skew-symmetric was established in \([17]\). Since the dual connection of \(\nabla^v\) is not metric, we prefer here avoid to use it. We will define the so-called \textit{Intrinsic Ricci tensor}, which was firstly introduced by B. Driver in \([12]\), in the framework of stochastic analysis on the path space of Riemannian manifolds (see also \([8, 22, 24, 30]\)). Such a connection was also used in \([1]\) to obtain an integration by parts formula for second order differential operators on Riemannian path spaces.

\textbf{Definition 4.2.} \textit{The intrinsic Ricci tensor is given by}

\[ \widehat{\text{Ric}}^v(X) = \text{Ric}^v(X) + \sum_{i=1}^{n} (\nabla_{e_i} T^v)(X, e_i). \]

(4.2)

where \((e_i)\) is a local orthonormal frame field of the tangent bundle.

\textbf{Theorem 4.3.} Assume that \(\dim(M) = 3\). Then \(\widehat{\text{Ric}}^v\) admits the following simple expression:

\[ \widehat{\text{Ric}}^v = \text{Ric}^0 + 2v \otimes v + 2\nabla^{0,s} v, \]

(4.3)

where \(\nabla^{0,s} v\) denotes the symmetric part of \(\nabla^0 v\).

\textbf{Proof.} By \([3,2,1]\),

\[ (\nabla_{e_i} T^v)(X, e_i) = -\frac{2}{n - 1} \left( (e_i, \nabla_{e_i} v)X - (X, \nabla_{e_i} v) e_i \right) \]

\[ = -\frac{2}{n - 1} \left( (e_i, \nabla_{e_i}^0 v)X - (X, \nabla_{e_i}^0 v) e_i \right) + J_i, \]

where

\[ J_i = \frac{4}{(n - 1)^2} \left( (e_i, K_v(e_i, v))X - (X, K_v(e_i, v)) e_i \right). \]

Then

\[ \sum_{i=1}^{n} J_i = \frac{4}{(n - 1)^2} \left( (n - 1)|v|^2 X - K_v(X, v) \right). \]

Therefore the sum \(\sum_{i=1}^{n} (\nabla_{e_i} T^v)(X, e_i)\) is equal to

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\[
\frac{-2}{n-1} \left( \text{div}(v) X - \sum_{i=1}^{n} (X, \nabla_{ei}^0 v) e_i \right) + \frac{4}{(n-1)^2} \left( (n-1)|v|^2 X - K_v(X, v) \right).
\]

When \( n = 3 \), the above formula yields that
\[
\sum_{i=1}^{3} (\nabla_{ei}^0 v)(X, e_i) = -\text{div}(v) X + \sum_{i=1}^{3} (X, \nabla_{ei}^0 v) e_i + 2|v|^2 X - K_v(X, v). \tag{4.4}
\]

On the other hand, by (4.1), for \( n = 3 \),
\[
\text{Ric}^v(X) = \text{Ric}^0(X) - K_v(X, v) + \nabla^0_X v + \text{div}(v) X. \tag{4.5}
\]

Note that
\[
\sum_{i=1}^{3} (X, \nabla_{ei}^0 v) e_i + \nabla^0_X v = \sum_{i=1}^{3} \left( (X, \nabla_{ei}^0 v) + (\nabla^0_X v, e_i) \right) e_i = 2\nabla^0_X v.
\]

According to this and summing up (4.4) and (4.5), we then obtain
\[
\hat{\text{Ric}}^v(X) = \text{Ric}^0(X) + 2|v|^2 X - 2K_v(X, v) + 2\nabla^0_X v.
\]

Now remarking that \( |v|^2 X - K_v(X, v) = \langle X, v \rangle v \), we deduce that
\[
\hat{\text{Ric}}^v(X) = \text{Ric}^0(X) + 2\langle X, v \rangle v + 2\nabla^0_X v
\]
for any vector field \( X \) and therefore (4.3) holds.

\[\Box\]

**Remark 4.4.** Consider the following SDE on \( O(M) \):
\[
dr_w(t) = \sqrt{2\nu} \sum_{i=1}^{n} B_i(r_w(t)) \circ dW^i_t, \quad r_w(0) = r,
\]
which has its infinitesimal generator
\[
\nu \sum_{i=1}^{n} L_{B_i}^2 (f \circ \pi) = \left( (\nu \Delta_M + 2\nu f) f \right) \circ \pi.
\]

According to Equation (3.12), we have to choose \( v = -\frac{1}{2\nu} u_t \). The term \( \text{Ric}^0 - \frac{1}{\nu} \nabla^0 u_t \) has already appeared in resolvent equation (3.13). In this case, we denote \( \text{Ric}^t \) instead of \( \text{Ric}^{-u_t/2\nu} \) and we have
\[
\text{Ric}^t = \text{Ric}^0 + \frac{1}{2\nu^2} u_t \otimes u_t - \frac{1}{\nu} \nabla^0 u_t.
\tag{4.6}
\]

**Proposition 4.5.** Assume that \( \dim(M) = 3 \). Then
(i) The following holds:
\[
\text{div}(\text{Ric}^t) = \text{div}(\text{Ric}^0) + \frac{1}{2\nu} \nabla u_t u_t - \frac{1}{\nu} \text{Ric}^0 u_t. \tag{4.7}
\]
(ii) Let \( \text{Scal}^t \) be the associated scalar curvature, that is \( \text{Scal}^t = \sum_{i=1}^{n} \langle \text{Ric}^t e_i, e_i \rangle \) for any orthonormal basis \( (e_i) \) of \( T_x M \). Then
\[
\text{Scal}^t = \text{Scal}^0 + \frac{1}{2\nu^2} |u_t|^2. \tag{4.8}
\]
Proof. (i) Since $\text{div}(u_t) = 0$, we have $\text{div}(u_t \otimes u_t) = \nabla u_t u_t$, and
\[ \nabla u_t = \nabla_s u_t + \nabla_s^k u_t. \]
We claim that
\[ \text{div}(\nabla_s^k u_t) = -\Box u_t. \]
In fact, let $X \in \mathcal{X}(M)$, we have
\[ \int_M \langle \text{div}(\nabla_s^k u_t), X \rangle \, dx = -\int_M \langle \nabla_s^k u_t, \nabla X \rangle \, dx = -\int_M \langle d\tilde{u}_t, dX \rangle \, dx = -\int_M \langle \Box \tilde{u}_t, X \rangle \, dx. \]
Therefore
\[ \text{div}(\nabla_s u_t) = \Delta u_t + \Box u_t = \text{Ric}^0 u_t. \]
The result (4.7) follows.

(ii) Concerning (4.8), by (4.6), it is enough to remark that
\[ \sum_{i=1}^n \langle \nabla^0_{e_i} u_t, e_i \rangle = \text{div}(u_t) = 0. \]

\[ \Box \]

\[ \text{Theorem 4.6.} \]
Let $\dim(M) = 3$, and $(u_t, \omega_t)$ be a regular solution to Equation (2.13). Then the following identity holds,
\[ \frac{1}{2} \frac{d}{dt} \int_M |\omega_t|^2 \, dx + \nu \int_M |\nabla^0 \omega_t|^2 \, dx = \frac{1}{2\nu} \int_M (\omega_t, u_t)^2 \, dx - \nu \int_M \langle \text{Ric}^\# \omega_t, \omega_t \rangle \, dx. \] (4.9)
where $\langle \text{Ric}^\# \omega_t, A \rangle = \langle \omega_t, \text{Ric} A \rangle$ for $A \in \mathcal{X}(M)$.

Proof. Using (4.6),
\[ \langle \text{Ric}^\# \omega_t, A \rangle = (\omega_t, \text{Ric}^0 A) + \frac{1}{2\nu} (\omega_t, u_t) \langle u_t, A \rangle - \frac{1}{\nu} (\omega_t, \nabla^0_{u_t} u_t). \]
Note that according to Definition (2.7), $(\omega_t, \nabla^0_{u_t} u_t) = (\omega_t \lhd \nabla^0_{u_t} u_t)(A)$. It follows that
\[ \text{Ric}^\# \omega_t = \text{Ric}^0 \omega_t + \frac{1}{2\nu} (\omega_t, u_t) \tilde{u}_t - \frac{1}{\nu} \omega_t \lhd \nabla^0_{u_t} u_t. \] (4.10)
We shall express the right hand side of (2.26) in term of $\text{Ric}^\#$. By (4.10),
\[ \langle \text{Ric}^\# \omega_t, \omega_t \rangle = \langle \text{Ric}^0 \omega_t, \omega_t \rangle + \frac{1}{2\nu} (\omega_t, u_t)^2 - \frac{1}{\nu} \langle \omega_t \lhd \nabla^0_{u_t} u_t, \omega_t \rangle. \]
Then
\[ -\nu \langle \text{Ric}^0 \omega_t, \omega_t \rangle + \langle \omega_t \lhd \nabla^0_{u_t} u_t, \omega_t \rangle = -\nu \langle \text{Ric}^\# \omega_t, \omega_t \rangle + \frac{1}{2\nu} (\omega_t, u_t)^2. \]
Substituting this term in the right hand side of (2.26), we get (4.9). \[ \Box \]
Remark 4.7. The term \((\omega_t, u_t)\) in the right hand side of (4.9) is called helical density, which involves explicitly in the evolution of vorticity in time and space.

Theorem 4.8. Let \(\dim(M) = 3\). Then

\[
\frac{d}{dt} \int_M (\omega_t, u_t) \, dx = -\nu \int_M (d\omega_t, \ast \omega_t) \Lambda^2 \, dx - \nu \int_M (\nabla \omega_t, \nabla u_t) \, dx \\
- \nu \int_M (\omega_t, \text{Ric}^t u_t) \, dx + \frac{1}{2\nu} \int_M (\omega_t, u_t) |u_t|^2 \, dx.
\]

(4.11)

Proof. By (5.1),

\[
\text{Ric}^t u_t = \text{Ric}^0 u_t + \frac{1}{2\nu} |u_t|^2 u_t - \frac{1}{\nu} \nabla_{u_t}^0 u_t.
\]

Hence

\[-\nu \text{Ric}^0 u_t + \nabla_{u_t}^0 u_t = -\nu \text{Ric}^t u_t + \frac{1}{2\nu} |u_t|^2 u_t.
\]

Substituting this term in the right hand of (2.27), we get (4.11). \(\square\)

5 Case of \(\mathbb{R}^3\)

We will specify results obtained in Section 4 on \(\mathbb{R}^n\). There are an ocean of publications on Navier-Stokes equations on \(\mathbb{R}^n\). We only refer to \[23, 28\] for nice expositions and to \[9\] for wellposedness of global solutions. We keep notations used in Section 2 for correspondences between vector fields and differential forms. In this case, \(\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}\) form an orthonormal basis at each tangent space of \(\mathbb{R}^3\), and \(\{dx, dy, dz\}\) an orthonormal basis at each co-tangent space. Let \(u\) be a vector field on \(\mathbb{R}^3\): \(u = u_1 \frac{\partial}{\partial x} + u_2 \frac{\partial}{\partial y} + u_3 \frac{\partial}{\partial z}\), then \(\tilde{u} = u_1 \, dx + u_2 \, dy + u_3 \, dz\) and

\[
\tilde{\omega} = d\tilde{u} = \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right) \, dz \wedge dx + \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right) \, dx \wedge dy + \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right) \, dy \wedge dz.
\]

Hodge star operator gives an isomorphism between \(\Lambda^2(\mathbb{R}^3)\) and \(\Lambda^1(\mathbb{R}^3)\), we have

\[
\omega = *\tilde{\omega} = \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right) \, dx \wedge dy + \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right) \, dx \wedge dz + \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right) \, dy \wedge dz.
\]

In this case \(\omega = \text{curl} \, u\), where \(\text{curl}(u)\) is the curl of \(u\), denoted sometimes by \(\nabla \times u\). We have the following relations

\[
\omega = \nabla \times u, \quad \nabla \times (\nabla \times u) = (d^* d\tilde{u})^\# = (d^* \tilde{\omega})^\#.
\]

(5.1)

By (5.1),

\[
\int_{\mathbb{R}^3} (d\omega_t, \ast \omega_t) \Lambda^2 \, dx = \int_{\mathbb{R}^3} (\omega_t, d^* \tilde{\omega}) \, dx = \int_{\mathbb{R}^3} \nabla \times (\nabla \times u) \cdot (\nabla \times u) \, dx.
\]

In what follows, we denote \(\xi_t = \nabla \times u_t\). In this flat case, the intrinsic Ricci tensor \(\text{Ric}^t\) defined in Formula (4.6) has expression

\[
\text{Ric}^t = \frac{1}{2\nu^2} u_t \otimes u_t - \frac{1}{\nu} \nabla^s u_t,
\]

(5.2)

where \(\nabla^s u_t\) is the rate of strains. Formula (4.9) becomes into the following form:
Let $\varepsilon < \varepsilon$. We claim that $x$.

In fact, if there exists $t > 0$ and set $t_q = t \wedge \sigma_q$. Then

This formula says that the variation of vorticity in time and in space can be explicitly measured by using helicity and the associated intrinsic Ricci tensor. Formula (4.11) has the form

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\xi_t|^2 \, dx + \nu \int_{\mathbb{R}^3} |\nabla \xi_t|^2 \, dx = \frac{1}{2\nu} \int_{\mathbb{R}^3} (\xi_t \cdot u_t)^2 \, dx - \nu \int_{\mathbb{R}^3} (\mathring{\text{Ric}}^i_{\xi_t}, \xi_t) \, dx. \tag{5.3}
\]

which shows how the helicity $\int_{\mathbb{R}^3} \xi_t \cdot u_t \, dx$ varies.

6 Appendix

6.1 Proof of Proposition 3.3

We first give a complete proof of Proposition 3.3 by following the proof of Theorem 3.5.1 in [24], and emphasize the steps we have to modify.

Proof. Let $i_x$ be the injectivity radius at $x$ and suppose that

\[
i_M = \inf \{i_x; \ x \in M\} > 0. \tag{6.1}\]

This means that the ball $B(x, i_M)$ does not meet the cut-locus $C_x$ of $x$. We prepare what we will need for proving (6.10).

Let $x \in B(x_0, i_M/2)^c$ which maybe is closed to or in $C_{x_0}$. Let $\gamma_x : [0, \eta(x)] \to M$ be a distance-minimizing geodesic connecting $x_0$ and $x$, parameterized by length. Then $\gamma_x(i_M/4) \notin C_x$ or $x \notin C_{\gamma_x(i_M/4)}$. Put $y = \gamma_x(i_M/4)$. Then $d_M(x_0, x) = d_M(x_0, y) + d_M(y, x)$. Since $C_y$ is closed, there is $\varepsilon_0 > 0$ such that

\[
B(x, \varepsilon_0) \cap C_y = \emptyset.
\]

We suppose that such $\varepsilon_0$ is valid for all $x$ (in fact, we will restrict ourselves in a compact set).

Let $\varepsilon < \varepsilon_0 \wedge \frac{i_M}{8}$, and define

\[
D_\varepsilon = \{ x \in M; \ d_M(x, C_{x_M}) < \varepsilon \}.
\]

We claim that

\[
D_\varepsilon \subset B(x_M, i_M/2)^c. \tag{6.2}
\]

In fact, if there exists $x \in D_\varepsilon$ such that $d_M(x, x_M) < i_M/2$; there is $z \in C_{x_M}$ such that $d_M(x, z) < \varepsilon$; then $d_M(x_M, z) \leq d_M(x_M, x) + d_M(x, z) < i_M$ which contradicts the definition of $i_M$. Let $\gamma_x$ be the geodesic considered above. Then $x \notin C_y$ with $y = \gamma_x(i_M/4)$.

Now introduce the stopping times $\sigma_q$ by $\sigma_0 = 0$ and

\[
\sigma_q = \inf \{ t > \sigma_{q-1}; \ d_M(\pi(r_t), \pi(r_{\sigma_{q-1}})) = \varepsilon \}.
\]

Let $t > 0$ and set $t_q = t \wedge \sigma_q$. Then
where $L$, page 95, so that $x_T = 0$.

(ii) Set $\pi(r_{t_q})$. If $x_{t_q-1} \in D_\varepsilon$, then by discussion at beginning, there is $y_{t_q-1}$ on a distance-minimizing geodesic $\gamma$ connecting $x_M$ and $x_{t_q-1}$ such that $d_M(x_M, y_{t_q-1}) = \frac{iM}{4}$ and $x_{t_q-1} \notin C_{y_{t_q-1}}$ and for $s \in [t_{q-1}, t_q]$,

$$d_M(\pi(r_s), x_{t_q-1}) \leq \varepsilon < \varepsilon_0.$$ 

Therefore $\pi(r_s) \notin C_{y_{t_q-1}}$. Let $\rho_q^*(r) = d_M(\pi(r), y_{t_q-1})$. Applying Itô formula to $\rho_q^*$, we have

$$\rho_q^*(r_{t_q}) - \rho_q^*(r_{t_q-1}) = \sum_{k=1}^{n} \int_{t_{q-1}}^{t_q} (L_A^k)\rho^*_q(r_s) dW_s^k + \int_{t_{q-1}}^{t_q} (L_s\rho^*_q)(r_s) ds.$$ 

On one hand

$$d_M(x_M, x_{t_q}) = d_M(x_M, y_{t_q-1}) + d_M(x_{t_q-1}, y_{t_q-1}) \quad \text{or} \quad \rho(r_{t_q-1}) = \frac{iM}{4} + \rho_q^*(r_{t_q-1}),$$

and on the other hand

$$d_M(x_M, x_{t_q}) \leq d_M(x_M, y_{t_q-1}) + d_M(x_{t_q-1}, y_{t_q-1}) \quad \text{or} \quad \rho(r_{t_q}) \leq \frac{iM}{4} + \rho_q^*(r_{t_q}).$$

It follows that

$$\rho(r_{t_q}) - \rho(r_{t_q-1}) \leq \rho_q^*(r_{t_q}) - \rho_q^*(r_{t_q-1}).$$

Therefore there exists $\hat{L}_q \geq 0$ such that

$$\rho(r_{t_q}) - \rho(r_{t_q-1}) = \rho_q^*(r_{t_q}) - \rho_q^*(r_{t_q-1}) - \hat{L}_q.$$ 

Define

$$\tau_R = \inf\{t > 0, \ d_M(x_M, \pi(r_t)) > R\}.$$ 

As did in [24], page 95, we get

$$\rho(r_{t\wedge\tau_R}) - \rho(r_0) = \beta_{t\wedge\tau_R} + \int_0^{t\wedge\tau_R} (L_s\rho)(r_s) ds - \hat{L}_q(t \wedge \tau_R) + R_\varepsilon(t \wedge \tau_R),$$

where

$$\hat{L}_q(t) = \sum_{q=1}^{+\infty} \hat{L}_q \mathbf{1}_{D_\varepsilon}(\pi(r_{t_q-1})).$$

which converges to $\hat{L}(t)$ as $\varepsilon \to 0$. The term $R_\varepsilon(t) = m_\varepsilon(t) + b_\varepsilon(t)$ with $m_\varepsilon(t)$ the same as in [24], page 95, so that

$$E(|m_\varepsilon(t)|^2) \leq 4 \int_0^t E(1_{D_\varepsilon}(\pi(r_s))) ds.$$

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Therefore for any compact subset \( K \subset B(x_M, R) \),
\[
\int_{\pi^{-1}(K)} \mathbb{E}((|m_x(t \wedge \tau_R)|)^2) \, dr_0 \leq 4 \int_0^t \int_{\pi^{-1}(K)} \mathbb{E}(1_{D_{2\varepsilon}}(\pi(r_{s \wedge \tau_R}))) \, dr_0 \, ds
\]
\[
\rightarrow 4 \int_0^t \int_{\pi^{-1}(K)} E(1_{C_{s, M}}(\pi(r_{s \wedge \tau_R}))) \, dr_0 \, ds \leq 4 \int_0^t \int_M 1_{C_{s, M}}(x) \, dx \, ds = 0.
\]
The term \( b_c(t) \) has to be modified such that
\[
b_c(t) = \sum_{q=1}^{+\infty} \int_{t_{q-1}}^{t_q} \left( L_s \rho_s^y(r_s) - L_s \rho(r_s) \right) \, ds 1_{D_s}(\pi(r_{t_{q-1}})).
\]
By (3.6) and (3.8), we have to control the term \( 1/\rho \). For \( x_{q-1} \in D_\varepsilon \) and for \( s \in [t_{q-1}, t_q) \),
\[
d_M(x_M, x_s) \geq d_M(x_M, x_{q-1}) - d_M(x_{q-1}, x_s) \geq \frac{i_M}{2} - \varepsilon \geq \frac{3i_M}{8},
\]
and
\[
d_M(y_{q-1}, x_s) \geq d_M(x_M, x_s) - d_M(x_M, y_{q-1}) \geq \frac{3i_M}{8} - \frac{i_M}{4} = \frac{i_M}{8}.
\]
Therefore, according to (3.7), since \( x_s = \pi(r_s) \in D_{2\varepsilon} \), there exists a constant \( \alpha > 0 \) such that
\[
\int_{t_{q-1}}^{t_q} \left| \left( L_s \rho_s^y(r_s) - L_s \rho(r_s) \right) \right| \, ds 1_{D_s}(\pi(r_{t_{q-1}})) \leq \alpha \int_{t_{q-1}}^{t_q} (1 + |V_s(r_s)|) 1_{D_{2\varepsilon}}(\pi(r_s)) \, ds.
\]
It follows that
\[
\mathbb{E}(b_c(t)) \leq \alpha \mathbb{E}\left( \int_0^t (1 + |V_s(r_s)|) 1_{D_{2\varepsilon}}(\pi(r_s)) \, ds \right). \tag{6.5}
\]
Again by hypothesis (2.10), there is a constant \( c_0 > 0 \) such that \( \text{vol}(B(x_0, \delta)) \leq e^{c_0 \delta} \), and therefore for a constant \( \lambda_0 > 0 \),
\[
C_M = \int_{O(M)} \exp(-\lambda_0 d_M^2(\pi(r_0), x_0)) \, dr_0 < +\infty.
\]
Define the probability measure \( d\mu \) on \( O(M) \) by
\[
d\mu(r_0) = \frac{1}{C_M} \exp(-\lambda_0 d_M^2(\pi(r_0), x_0)) \, dr_0. \tag{6.6}
\]
Now integrating with respect to \( d\mu(r_0) \), we get
\[
\int_0^t \int_{\pi^{-1}(K)} \mathbb{E}\left((1 + |V_s(r_s)|) 1_{D_{2\varepsilon}}(\pi(r_s)) 1_{(s \wedge \tau_R)}\right) \, d\mu(r_0) \, ds
\]
\[
\rightarrow \int_0^t \int_{\pi^{-1}(K)} \mathbb{E}\left((1 + |V_s(r_s)|) 1_{C_{s, M}}(\pi(r_s)) 1_{(s \wedge \tau_R)}\right) \, d\mu(r_0) \, ds
\]
\[
\leq \sqrt{T} \left( \int_0^t \int_M |v_s(x)|^2 1_{C_{s, M}}(x) \, dx \, ds \right)^{1/2} = 0,
\]
under the hypothesis (3.11). The proof of Proposition 3.3 is complete.  \( \square \)

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6.2 Bismut Formulae and Proof of Proposition 5.9

In this part, we will first present a nice derivative formula for heat semigroup \( T_t \) on differential \( p \)-forms obtained by Elworthy and Li in [18] and by Driver and Thalmaier in [13]. We keep notations introduced in Section 5. Let \( A_1, \ldots, A_n \) be the canonical horizontal vector fields on \( O(M) \). Consider the SDE on \( O(M) \)

\[
dr_t = \sum_{i=1}^n A_i(r_t) \circ dW_t^i, \quad r_{t=0} = r_0. \tag{6.7}
\]

Assume that the Ricci tensor is bounded below \( \text{Ric} \geq -\kappa \). Then SDE (6.7) is stochastic complete (see [24]). Set \( x_t = \pi(r_t) \) with \( x_0 = \pi(r_0) \). Then \( (x_t) \) is a semi-martingale on \( M \), with respect to which stochastic integral can be defined (see [7]). Then we can write

\[
dx_t = \pi(r_t) \circ dr_t = \sum_{i=1}^n d\pi(r_t) A_i(r_t) \circ dW_t^i = r_t \circ dW_t.
\]

Therefore \( W_t = \int_0^t r_s^{-1} \circ dx_s \), which is anti-development of \( \{x_t; t \geq 0\} \). Set

\[
B_t = r_0 W_t = \int_0^t \langle s^{-1} \circ dx_s, \rangle
\]

where \( \langle s \rangle = r_s \circ r_s^{-1} \) is Itô stochastic parallel translation along path \( \{x_t; t \geq 0\} \). Recall that Weitzenböck formula for \( p \)-differential forms reads as follows [25] [18]:

\[
\square = -\Delta + \mathcal{R}^\#_p, \tag{6.8}
\]

where \( \Delta \phi = \text{Trace}(\nabla \nabla \phi) \) for a \( p \)-form \( \phi \), and \( \mathcal{R}^\#_p : \Lambda^p(M) \to \Lambda^p(M) \) is a tensor, called Weitzenböck curvature. For \( p = 1 \), \( \mathcal{R}_1 = \text{Ric}^\# \) is Ricci tensor. As in [18], \( \mathcal{R}_p(x) \) is an endomorphism of \( p \)-vectors, that is, \( \mathcal{R}_p(x) : \Lambda^p T_x^* M \to \Lambda^p T_x M \). For \( r \in O(M) \), define \( \tilde{\mathcal{R}}_p(r) = r \circ \mathcal{R}_p(\pi(r)) \circ r^{-1} \), more precisely, for \( a_i, b_j \in \mathbb{R}^n \),

\[
\langle \tilde{\mathcal{R}}_p(r)(a_1 \wedge \cdots \wedge a_p), b_1 \wedge \cdots \wedge b_p \rangle = \langle \mathcal{R}_p(\pi(r))(ra_1 \wedge \cdots \wedge ra_p), rb_1 \wedge \cdots \wedge rb_p \rangle.
\]

Consider the heat equation on \( p \)-forms:

\[
\frac{d\phi_t}{dt} = -\frac{1}{2} \square \phi_t, \quad \phi_{t=0} = \phi_0. \tag{6.9}
\]

By definition \( T_t \phi_0 = \phi_t \). Consider the following resolvent equation on \( \Lambda^p \mathbb{R}^n \)

\[
\frac{d\tilde{Q}_t^p}{dt} = -\frac{1}{2} \tilde{\mathcal{R}}_p(r_t) \cdot \tilde{Q}_t^p, \quad \tilde{Q}_0^p = \text{Id}. \tag{6.10}
\]

Define \( Q_t^p : \Lambda^p(T_x^0 M) \to \Lambda^p(T_x^1 M) \) par \( Q_t^p V_0 = r_t \tilde{Q}_t^p(r_0^{-1}V_0) \). It is well-known (see [18]) that

\[
\langle T_t \phi(V_0) \rangle = \mathbb{E}(\langle \phi_{x_t}, V_t \rangle) = \mathbb{E}\left( \langle F_\phi(r_t), \tilde{Q}_t^p(r_0^{-1}V_0) \rangle \right), \tag{6.11}
\]

where \( F_\phi \) is defined in 3.11 if \( \phi \) is a differential 1-form, and \( F_\phi(r) \in \Lambda^p(\mathbb{R}^n) \) is such that

\[
\langle F_\phi(r), a_1 \wedge \cdots \wedge a_p \rangle = \langle \phi(\pi(r)), ra_1 \wedge \cdots \wedge ra_p \rangle \quad \text{where} \quad a_1, \ldots, a_p \in \mathbb{R}^n.
\]

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Proposition 6.1. Assume that
\[ R_p \geq -\kappa_p, \quad \kappa \in \mathbb{R}. \] (6.12)

Then
\[ |T_t \phi| \leq e^{\kappa_p t/2} |\phi|. \] (6.13)

Proof. Using (6.10) and (6.12), we have
\[ \frac{d}{dt} |\hat{Q}_t^p (r_0 V_0) |^2 = -\langle \hat{R}_p (r_t) Q_t^p (r_0 V_0), Q_t^p (r_0 V_0) \rangle \leq -\kappa_p |\hat{Q}_t^p (r_0 V_0) |^2. \]

The Gronwall lemma yields that
\[ |\hat{Q}_t^p (r_0 V_0) | \leq e^{\kappa_p t/2} |V_0|. \]

Since \(|F_\phi| = |\phi|\), (6.11) yields inequality (6.13).

For simplicity, for \( p = 1 \), we still denote \( \kappa \) instead of \( \kappa_1 \). In the case for 1-forms,
\[ |T_t \phi| \leq e^{\kappa t/2} T_t |\phi|. \] (6.14)

To our purpose, we only state the formula for 1-form established by Elworthy and Li; although it was stated for the case of compact Riemannian manifolds in [18], but it remains valid in non-compact cases as did by Driver and Thalmaier in [13], section 6.

Theorem 6.2. For 1-form \( \phi \) and a vector field \( v \),
\[ (\Box T_t \phi, v) = -\frac{4}{t^2} \mathbb{E} \left[ \left( \phi_{x_t}, Q_t^1 \int_{t/2}^t (Q_s^1)^{-1} dM_s(v) \right) \right] \] (6.15)

where \( dM_s(v) = dM^1_s(v) + dM^2_s(v) \) with
\[ dM^1_s(v) = \theta_{/s, dB_s} Q_s^2 \left( \int_{t/2}^t (Q_r^2)^{-1} (//r dB_r \wedge Q_r^1(v)) \right), \] (6.16)

where \( \theta \) is annihilation operator, and
\[ dM^2_s(v) = //s dB_s \left( \int_0^{t/2} (Q_r^1(v), //r dB_r) \right). \] (6.17)

Let \( \{ \varepsilon_1, \ldots, \varepsilon_n \} \) be the canonical basis of \( \mathbb{R}^n \) and set \( e_j = r_0 \varepsilon_j \). Then \( \{ e_1, \ldots, e_n \} \) is an orthonormal basis of \( T_{x_0} M \). By definition of \( \theta \), the term
\[ \left\langle \theta_{/s, dB_s} Q_s^2 \int_{0}^{t/2} (Q_r^2)^{-1} (//r dB_r \wedge Q_r^1(v)) \right\rangle, //s e_j \]

may be identified with the following
\[ \left\langle Q_s^2 \int_{0}^{t/2} (Q_r^2)^{-1} (//r dB_r \wedge Q_r^1(v)), //s dB_s \wedge //s e_j \right\rangle. \]

Hence
\[ dM^1_s(v) = \sum_{k,j=1}^n \left\langle Q_s^2 \int_{0}^{t/2} (Q_r^2)^{-1} (//r dB_r \wedge Q_r^1(v)), //s e_k \wedge //s e_j \right\rangle //s e_j dB_s^k, \]
and
\[ dM_s(v) = \sum_{k=1}^{n} \left( \int_0^{t/2} \langle Q_s^1(v), /s e_k dB^k_s \rangle \right) /s e_k dM_s^k. \]

Therefore \( dM_s(v) = \sum_{k=1}^{n} (a_k(s) + b_k(s)) dB^k_s \) with
\[ a_k(s) = \sum_{j=1}^{n} \left( Q_s^2 \int_0^{t/2} \left( Q^2_r \right)^{-1}(//r dB_r \wedge Q_s^1(v)) \right) /s e_k \wedge /s e_j. \]
and \( b_k(s) = \left( \int_0^{t/2} \langle Q_s^1(v), /r dB_r \rangle \right) /s e_k. \) It is obvious that \( \langle a_k(s), b_k(s) \rangle = 0. \)

**Lemma 6.3.** The quadratic variation \( dM_s(v) \cdot dM_s(v) \) of \( M_s(v) \) admits the following expression
\[ dM_s(v) \cdot dM_s(v) = 2 \left\| Q_s^2 \int_0^{t/2} \left( Q^2_r \right)^{-1}(//r dB_r \wedge Q_s^1(v)) \right\|_{\Lambda^2}^2 + \left\| \int_0^{t/2} \langle Q_s^1(v), /r dB_r \rangle \right\|^2. \]

**Theorem 6.4.** Assume that \( (6.12) \) holds for \( p = 1 \) and \( 2. \) Then for any differential 1-form \( \phi, \)
\[ ||T_r \phi||_2 \leq \frac{2}{t} e^{3\kappa t/2} \sqrt{2(n-1)e^{3\kappa t/2} + 1} ||\phi||_2, \quad t > 0. \] (6.18)

**Proof.** By Theorem \( [6.2], \)
\[ \left\| \Box T_r \phi \right\| \leq \frac{4}{t^2} \sqrt{E(\phi(x_1))} \left( E \left[ \left\| Q_t^1 \int_0^t \left( Q^2_r \right)^{-1} dM_s(v) \right\|^2 \right] \right)^{1/2} \]
\[ \leq \frac{4e^{\kappa t/2}}{t^2} \sqrt{E(\phi(x_1))} \left( E \left[ \left\| \int_0^t \left( Q^2_r \right)^{-1} dM_s(v) \right\|^2 \right] \right)^{1/2}. \] (6.19)

Note that \( (Q^p_r)^{-1} \) enjoys the same kind of equations as \( (6.10). \) Thus \( ||(Q^p_r)^{-1}|| \leq e^{\kappa t/2} \) under \( (6.12), \) so that
\[ E \left[ \int_0^t \left( Q^1_s \right)^{-1} dM_s(v) \right]^2 \leq E \left[ \int_0^t \sum_{k=1}^{n} (a_k(s) + b_k(s))^2 \right] \]
\[ \leq e^{\kappa t} E \left[ \int_0^t dM_s(v) \cdot dM_s(v) \right] = e^{\kappa t} (I_1(s) + I_2(s)), \]
where
\[ I_1(s) = E \left[ \int_0^t 2 \left\| Q_s^2 \int_0^{t/2} \left( Q^2_r \right)^{-1}(//r dB_r \wedge Q_s^1(v)) \right\|_{\Lambda^2}^2 ds \right] \]
\[ I_2(s) = E \left[ \int_0^t \left\| \int_0^{t/2} \langle Q_s^1(v), /r dB_r \rangle \right\|^2 ds \right]. \]

It is obvious that \( I_2(s) \leq \frac{t^2 e^{\kappa t/2}}{4} |v|^2 \) and
\[ I_1(s) \leq 2e^{\kappa t} \int_0^t E \left[ \left\| \left( Q^2_r \right)^{-1}(//r dB_r \wedge Q_s^1(v)) \right\|_{\Lambda^2}^2 \right] ds \] (6.20)
Since we have

\[(Q^2_r)^{-1}(\langle /r, dB_r \wedge Q^1_r(v) \rangle) = \sum_{k=1}^{n} (Q^2_r)^{-1}(\langle /r, e_k \wedge Q^1_r(v) \rangle) dB^k_r,\]

so that

\[
E \left[ \left\| \int_0^{t/2} (Q^2_r)^{-1}(\langle /r, dB_r \wedge Q^1_r(v) \rangle) \right\|^2 \right] = \sum_{k=1}^{n} \int_0^{t/2} \left\| (Q^2_r)^{-1}(\langle /r, e_k \wedge Q^1_r(v) \rangle) \right\|^2 \, dr
\]

\[
\leq \sum_{k=1}^{n} \int_0^{t/2} e^{\kappa r} \left\| /r, e_k \wedge Q^1_r(v) \right\|^2 \, dr.
\]

But

\[
\left\| /r, e_k \wedge Q^1_r(v) \right\|^2 = |Q^1_r(v)|^2 - \langle /r, e_k, Q^1_r(v) \rangle^2,
\]

we therefore have

\[
\sum_{k=1}^{n} \left\| /r, e_k \wedge Q^1_r(v) \right\|^2 = (n-1)|Q^1_r(v)|^2 \leq (n-1)e^{\kappa r}|v|^2.
\]

To simplify calculation, we note that \(e^{\kappa r} \leq e^{\kappa t/2} \) since \(r \in [0, t/2]\). Substituting these bounds first in (6.20), then together in (6.19), we finally get

\[
|\Box_T \phi| \leq \frac{2}{t} e^{3\kappa t/2} \sqrt{2(n-1)e^{3\kappa t/2} + 1 \sqrt{T^M}|\phi|^2},
\]

and the result (6.18) follows. \(\square\)

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