On the Convergence of Perturbed Distributed Asynchronous Stochastic Gradient Descent to Second Order Stationary Points in Non-convex Optimization

Lifu Wang  
Bo Shen  
Department of Electronic and Information Engineering, Beijing Jiaotong University, Beijing, China  
Key Laboratory of Communication and Information Systems, Beijing Municipal Commission of Education  
Beijing Jiaotong University, Beijing, China  
Ning Zhao  
Department of Electronic and Information Engineering, Beijing Jiaotong University, Beijing, China  
State Key Lab of Rail Traffic Control and Safety, Beijing Jiaotong University, Beijing, China

Abstract

In this paper, the second order convergence of non-convex optimization in asynchronous stochastic gradient descent (ASGD) algorithm is studied systematically. We investigate the behavior of ASGD near and away from saddle points and show that, different from general stochastic gradient descent (SGD), ASGD may return back after escaping the saddle points, yet after staying near a saddle point for a long enough time ($O(T)$), ASGD will finally go away from strictly saddle points. An inequality is given to describe the process of ASGD to escape from saddle points. We show the exponential instability of the perturbed gradient dynamics near the strictly saddle points and use a novel Razumikhin-Lyapunov method to give a more detailed estimation about how the time delay parameter $T$ influence the speed to escape. In particular, we consider the optimization of smooth nonconvex functions, and propose a perturbed asynchronous stochastic gradient descent algorithm with guarantee of convergence to second order stationary points with high probability in $O(1/\epsilon^4)$ iterations. To the best of our knowledge, this is the first work on the second order convergence of asynchronous algorithm.

Keywords: non-convex optimization, asynchronous algorithms, saddle points escaping, machine learning, stochastic gradient descent

1. Introduction

Since the pioneering work of Alexnet Krizhevsky et al. (2012) in 2012, deep learning has become the mainstream in machine learning. One of the most major concern, however, is that with more and more training data and larger and larger model capacity, the training of a large model becomes very challenging. Distributed training is now a popular approach to overcome this difficulty and accelerate the training process. Recently in large-scale distributed deep learning system, the asynchronous parallel algorithms has attracted a lot of attention Recht et al. (2011); Lian et al. (2015); Li et al. (2014); Yun et al. (2014), because of its advantage of reducing the system overhead largely. In asynchronous algorithms, all works will be carried out in parallel, with synchronization being performed independently. Asynchronous versions of various algorithms have been shown to be very successful in speeding up the optimization process. Meanwhile, it is very important yet not easy to understand the success of the asynchronous algorithms. In the early paper of various versions of asynchronous SGD Recht et al. (2011); Li et al. (2014); Zhang et al. (2015), the convergence of
asynchronous stochastic gradient descent was proved only for convex optimization problems. The convergence property for non-convex optimization was studied in Lian et al. (2015), in which the authors revealed how the synchronization time delay $T$ can influence the convergence of asynchronous SGD.

Another challenge in deep learning is the nonconvexity. Different from the past models in machine learning, deep learning models are generally non-convex, which make it very hard to be analyzed theoretically, since it can be NP-hard in the worst case. However, in practice, SGD is considered to be a very good optimizer in deep learning. To give an answer, many works have been carried out to investigate the theoretic properties of SGD(or Stochastic Gradient Langevin Dynamics(SGLD)) and there are two conclusions have been widely accepted. First, it is very hard for SGD to escape deep local minima(Lemma 15 in Ge et al. (2015), but it is easy to escape the shallow ones, which is proved by Zhang et al. (2017). Second, gradient descent algorithm can only converge to local minima but not saddle points Lee et al. (2016). However, it has been shown that general gradient descent algorithm can take exponential time to escape saddle points Du et al. (2017), but stochastic gradient descent with noise will escape saddle points in polynomial time Ge et al. (2015); Jin et al. (2017). Even for non-convex problem, local minima can be good enough in many practical problems, such as learning multi-layer neural networks Kawaguchi (2016), matrix completion, matrix sensing, robust PCA Ge et al. (2016), Burer-Monteiro style low rank optimization Zhu et al. (2018) and over-parametrization neural network Du and Lee (2018). In these problems, under good conditions, the loss functions can be strictly saddle and there is only one local minimum, such that the global minimum is the only second-order stationarity point($\nabla f = 0$, $\nabla^2 f > 0$) and SGD will succeed.

Saddle points escaping is one of the core issues in non-convex optimization. After the pioneering work of Ge Rong in Ge et al. (2015), there have been a lot of researches to investigate this problem. In the work of Lee et al. (2016), stable manifolds in dynamical system theory is used to show that all the strictly saddle points are unstable. The work of Ge et al. (2015) and Jin et al. (2017) are virtually to illustrate the exponential instability of gradient dynamics near the strictly saddle points. In fact, the process of escaping from saddle points is described by Lyapunov’s first theorem, which is to consider the linearization of the dynamic systems near saddle points. However, if we consider asynchronous algorithms, stability problems will differ.

Based on the understanding of saddle points escaping properties for the synchronous version of SGD, two questions arise naturally.
1) What the asynchronous algorithm behaves near strictly saddle points?
2) How the time delay parameter $T$ in the asynchronous SGD affects escaping?

These two problems are studied in this paper. We propose an estimation to illustrate the influencing mode of parameter $T$ on the speed to escape. The mechanism of the asynchronous SGD algorithm to decrease the function value near and away from strictly saddle points is illustrated. As a result, we give the convergence rates in the ASGD process.

1.1 Our Contribution

In this paper, the second order convergence properties of asynchronous stochastic gradient descent is studied systematically. We prove that the perturbed asynchronous SGD algorithm will converge to a second order stationary points with a high probability. To the best of our knowledge, this is the
first work on the theoretical guarantees on strictly saddle points escaping problem for asynchronous algorithms.

Our main technical contributions are listed below:

- The first theoretical guarantees for asynchronous SGD algorithms to escape strictly saddle points in polynomial time is given and the relationship between learning rate $\eta$ and time delay parameter $T$ such that the algorithm can be guaranteed to converge to a second order stationary point is studied.

  Further, the influencing mechanism of the delay bound $T$ on the process of saddle points escaping is investigated, which can explain the performance differences between asynchronous and synchronous algorithms.

- By using Lyapunov-Razumikhin method, the growth rate of time-delay linear system is studied, and the influencing mode of parameter $T$ on the speed of saddle points escaping is explored.

- Different from the synchronous case, it is possible that ASGD will return back to the saddle points even though it has escaped from saddle points. Yet in this work it is shown that since we can prove the first order convergence of the algorithm, after a "calm down" process, ASGD will finally go away from the strictly saddle points.

1.2 Related Works

Asynchronous SGD algorithm is firstly proposed in Agarwal and Duchi (2011), and a lock-free version Hogwild is proposed in Recht et al. (2011). ASGD is used in Google to train deep learning network effectively in Dean et al. (2012). The convergence is only proved for convex cases in Agarwal and Duchi (2011); Recht et al. (2011). Non-convex cases are studied in Lian et al. (2015), De Sa et al. (2015).

In the non-convex cases, the main technical obstacle is that there is no guarantee for the function value decreasing with updating. In Lian et al. (2015), the authors use a trick to estimate $\|\nabla f(x_k) - \frac{1}{n} \sum_{m=1}^{M} \nabla f(x_k - \tau_m)\|^2$ term and in De Sa et al. (2015) propose a new functional instead of the function value to show that the new functional will keep decreasing. This method is generalize to the unbounded delay cases in Hannah and Yin (2018); Zhang et al. (2018). All these work are limited to the first order convergence.

The saddle escaping problem is firstly studied in Ge et al. (2015). They show the dynamics near a saddle point can be approximated by SGD of quadratic loss function and escaping strictly saddle points is easy by adding an isotropy enough noise. More detailed studies are given in Jin et al. (2017) for perturbed gradient descent and Jin et al. (2019a) for SGD. Stable manifold in dynamical system is used in Lee et al. (2016) to show gradient descent will always finally reach a local minimum with probability almost 1 if we use random initialization. However, the work in Du et al. (2017) points out that if we don’t add any noise, it is possible that gradient descent will take exponential time to escape strictly saddle points.

Saddle points escaping is closely related to the instability of dynamical system. The gradient descent dynamic can be studied using linearization to show strictly saddle points are unstable, which is in fact the first theorem of Lyapunov. For the time delay system, the stability has been studied in many work Zhou (2018); Gu (1999); Han (2005); Bugong Xu (1994);
Kharitonov and Zhabko (2003). Yet there are only a few articles about the instability of time delay system, e.g. Haddock and Zhao (1996); Haddock and Ko (1995); Sedova (2010); Hale (1965); Raffoul (2013). The basic tools for analyzing time delay system are Lyapunov-Krasovskii functional and Lyapunov-Razumikhin method. These works didn’t study the exponential instability except Raffoul (2013) and the criterion in Raffoul (2013) requires $T \sim \frac{1}{\gamma}$, where $\gamma$ is the eigenvalue of the matrix in the linear equation. In this work, we can reduce to the case that all the coefficients are positive, so that we can give a much stronger theorem by Razumikhin type theorem.

1.3 Structure of This Paper

The remainder of this paper is organized as follows: We present the main results for this paper in Section 2. Some preliminary theorems are listed in Section 3. In Section 4, we prove the main theorem in this paper. We present a simple experiment to illustrate the effect of PASGD to escape the saddle points in Section 5. The influence of parameter $T$ on the speed of convergence is discussed in Section 6. Finally, we conclude our work in Section 7. The omitted proof of theorems in Section 4 are in the Appendix.

2. Main Results

In this paper, we consider using asynchronous stochastic gradient descent to solve

$$\min_{x \in \mathbb{R}^d} f(x)$$

(1)

$f(x)$ is smooth and can be non-convex.

Considering a network with star-shaped topology, the center of the star is the master machine, which maintains the parameters of neural network. Other node machines will compute stochastic gradients, sending gradients to the master and update the parameters. All the node machines work independently and simultaneously.

The major difference between ASGD and general SGD is that, because of asynchronous updating, some stochastic gradients sent by the node machines might be $g(x_{t-\tau, i})$, which is computed from some early value of parameters instead of the current gradient $g(x_t)$.

**Algorithm 1: Perturbed Asynchronous Stochastic Gradient Descent**

**Input:** Initial parameters $x_0$, learning rate $\eta$, perturbation radius $r$.

**At the master machine:**

At time $t$, wait till receiving $M$ stochastic gradients $G(x_{t-\tau, i}, \theta_{t,i})$ from node machines.

$x_{t+1} = x_t - \eta (\sum_{i=1}^{M} G(x_{t-\tau, i}, \theta_{t,i}) + M \zeta_t), \quad \zeta_t \sim N(0, (r^2/d)I)$;

**At node machines:**

Exchange information with the master machine and update the parameters.

Random select samples, compute stochastic gradient $G(x_{t-\tau, i}, \theta_{t,i})$ and send to the master machine.

The process is shown in algorithm 1.

Some assumptions needed in this work are listed below:

**Assumption 1** Function $f(x)$ should be $L$ smooth:

$$\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\| \quad \forall x, y$$

(2)
Assumption 2 Function $f(x)$ should be $\rho$-Hessian Lipschitz:

$$||\nabla^2 f(x) - \nabla^2 f(y)|| \leq \rho ||x - y|| \ \forall x, y$$ (3)

Assumption 3 Stochastic gradient $g(x, \theta)$ should be $s$-norm-subGaussian:

$$\mathbb{E}g(x, \theta) = \nabla f(x), \ P(||g(x, \theta) - \nabla f(x)|| \geq t) \leq 2\exp(-t^2/(2s^2))$$ (4)

The main result of this paper is the following theorem.

Theorem 1 Given any $\epsilon_1, \epsilon_2, \delta$, for a smooth function $f(x)$, and stochastic gradient $g$ satisfying above assumptions, and all $t, i, \tau_{i,t} \leq T$, we run perturbed asynchronous stochastic gradient with parameter $r \sim O(\sqrt{ds}), \eta \sim O(\frac{1}{e^{\max(T+1, \sqrt{d})}})$. Then with probability at least $1 - \delta$, asynchronous stochastic gradient will reach points with $||\nabla f(x)||^2 \leq \epsilon_1^2$ and $\lambda_{\min}(\nabla^2 f(x)) \geq -\sqrt{\rho \epsilon_2}$ at last ones in $O(\frac{1}{\epsilon^2} \frac{d}{\epsilon^2})$ iterations.

Remark 2.1 In our case, we should keep $r \sim O(\sqrt{ds}), \eta \sim O(\frac{1}{\sqrt{d}})$, yet using the assumption that stochastic gradient function $g(x, \cdot)$ is Lipschitz (such as the deep learning case), we can prove that it is enough if $r \sim O(s), \eta \sim O(1/L)$, and show the convergence in $O(\frac{1}{\epsilon^2})$ iterations. However, this is not the main intention of this paper, so we omit it.

Remark 2.2 The noise added in the algorithm is not necessarily Gaussian. The only thing we need is to guarantee $||q_p(t) + q_{\theta_{\varphi}}(t)||$ in (45) can be large enough. This can be satisfied if the noise in stochastic gradient is isotropic enough.

3. Preliminaries

In this section, some concentration theorems used in this paper are listed below.

Definition 1 A sequence of random vectors $X_1, X_2, \ldots X_n \in \mathbb{R}^d$ with filtrations $F_i = \sigma(X_1, X_2 \ldots X_i)$ such that

$$\mathbb{E}[X_i | F_{i-1}] = 0, \mathbb{E}[e^{r||X_i||} | F_{i-1}] \leq e^{4r^2 \sigma^2}, \sigma_i \in F_{i-1}$$ is called zero-mean nSG($\sigma_i$) sequence.

For a zero-mean nSG($\sigma_i$) sequence, we have two important lemmas from Jin et al. (2019b).

Lemma 2 (Hoeffding type inequality for norm-subGaussian, lemma 6 in Jin et al. (2019b)) With probability at last $1 - 2(d + 1)e^{-1}$:

$$||\sum_{i=1}^{n} X_i || \leq c \sqrt{\sum_{i=1}^{n} \sigma_i^2}$$

The proof is based on lemma 4 in Jin et al. (2019b). Even if $\mathbb{E}Y^{2p+1} \neq 0$, we can still prove that

$$\mathbb{E}e^\theta Y = I + \sum_{p} \frac{\theta^p \mathbb{E}Y^{2p}}{(2p)!} + Z$$

where $Z$ is the $2p + 1$ parts with $trZ = 0$.

And we can still prove that $\mathbb{E}e^\theta Y \leq e^{\epsilon^2 \sigma^2} I$ where $\epsilon \geq 4$, so that we can use this to prove lemma 6 in Jin et al. (2019b). This is from the fact that $\mathbb{E}||X|| \leq e^{4r^2 \sigma^2}$.

Using the same way, it is easy to prove the square sum theorem:
Lemma 3 (Lemma 29 in Jin et al. (2019a)) For a zero-mean \( nSG(\sigma) \) sequence \( X_i \) with \( \sigma_i = \sigma \), we have with probability at least \( 1 - e^{-\lambda} \):

\[
\sum_i ||X_i||^2 \leq c\sigma^2(n + 1)
\]

The following Azuma inequality is needed in the proof the main theorem.

Lemma 4 (Azuma-Hoeffding inequality) Let \( \sum_i Y_i \) be a sub-martingale and \( |Y_i| \leq c \), then we have

\[
P\left( N \sum_i Y_i \leq \mathbb{E}\{ \sum_i Y_i \} - \lambda \right) \leq e^{-\lambda^2/(2c^2N)}
\]

The proof can be found in Chung and Lu (2006).

4. Proof of the Main Theorem

4.1 Notation

In algorithm 1, the updating rule is

\[
x_{t+1} = x_t - \eta \left( \sum_{i=1}^M G(x_{t-\tau_i}, \theta_{t,i}) + M\zeta_t \right),
\]

where \( \zeta_t = N(0, (r^2/d)I) \).

Let \( \xi_{t,i} = G(x_{t-\tau_i}, \theta_{t,i}) - \nabla f(x_{t-\tau_i}) \). We denote \( \xi_{t,m} = \xi_{t,m} + \xi_2 \). Since \( \xi_{t,m} \) and \( \xi_2 \) are independent when \( t_1 = t_2 \), \( \sum_i \zeta_{t,i} \) is \( M^2\sigma^2 = M^2s^2 + M^2r^2 \) sub-Gaussian.

We set:

\[
\eta = \frac{1}{wML}, \quad r = c_1 \sqrt{ds}, \quad f = (T + 1)M\eta \sqrt{\rho e_2}, \quad T_{max} = T + \frac{uef}{M\eta \sqrt{\rho e_2}}, \quad F = 50c\sigma^2\eta MLT, \quad F_2 = 2r^2, \quad S = \eta M \sqrt{15cT_{max} \sigma}, \quad c = 4
\]

\[
b = \log(2(d + 1)) + \log 6, \quad C = 12\sqrt{8bc}, \quad p = \frac{1}{1 + C}, \quad c_1 = 6c \sqrt{\log(2(d + 1)) + \log 3}
\]

\( u \) should be large enough such that

\[
2^u \geq 12 \sqrt{30du + 30dM\eta LT} \frac{\sigma}{r}
\]

and

\[
\frac{uef}{M\eta \sqrt{\rho e_2}} \geq \log 48
\]

then \( u \sim O(\sqrt{d}) \).

For \( X \in \mathbb{R} \), \( P(|X| > t) \leq 2exp(-\frac{t^2}{2\sigma^2}) \), then \( \mathbb{E}\{|e^{||X||}|\} \leq e^{4\sigma^2s^2} \), so we see the parameter \( c \) in concentration inequality are all not larger than 4.

We assume \( w \sim O(\sqrt{d}) \) is large enough such that the following conditions are satisfied:

(a) \( \eta^2(\frac{3M}{4} + L^2MT^2\eta) - \frac{r}{2M} < 0 \)

(b) \( (2T_{max})4cL^2M^2\eta^2T^2 \leq 1 \)

(c) \( \eta MT_{max} \rho S \leq p \)

Note that \( T_{max} > T + 1 \). From (b) we have

\[
2L^2\eta^2M^2T^3 \leq 1
\]
4.2 Sketch of the Proof

Asynchronous SGD algorithm is quite different from the general(synchronous) version of SGD. One of the most important differences is that ASGD even cannot guarantee to decrease the function value in the sense of average. In fact, we have

$$f(x_{k+1}) - f(x_k) \leq \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2} ||x_{k+1} - x_k||^2$$

$$= -\langle \nabla f(x_k), \eta \sum_{m=1}^{M} \nabla f(x_{k-\tau_m}) + \eta \sum_{m=1}^{M} \zeta_{k,m} \rangle + \frac{\eta^2 L}{2} \sum_{m=1}^{M} ||\nabla f(x_{k-\tau_m}) + \sum_{m=1}^{M} \zeta_{k,m}||^2$$

(10)

However, when we add some assumptions such that the time delay $T$ and learning rate $\eta$ is small enough, it has been shown in Lian et al. (2015) that after large enough number of steps, we have

$$\mathbb{E}\{f(x_{K+1})\} - f(x_0) \leq \sum_{k=0}^{K} -C \eta M ||\nabla f(x_k)||^2 + \text{terms about variance of SGD}$$

(11)

Using this equation, we find that if $||\nabla f(x_k)||$ is large enough, the function will keep decreasing, so that we can prove the first order convergence of ASGD, as in Theorem 5 in section 4.3.

One of the differences between the method in this paper and the standard strategy to prove Jin et al. (2017, 2019a) is that, even if the sequence in the updating dynamics has escaped from the saddle points, it may also go back, due to the fact that the function value may not decrease. Thus we turn to study the delay-free gradient $\nabla f(x_k)$ rather than the delay gradient $\sum_{i=0}^{M} \nabla f(x_{k-\tau_i})$ used in updating. Then we prove Lemma 8 which shows that if $\sum_{k=-2T}^{-1} ||\nabla f(x_k)||^2$ is small enough and $||x(t) - x(0)||$ is large, $\sum_{k=-T}^{T} ||\nabla f(x_k)||^2$ will be large to decrease the function value. In Appendix D, we show the exponential instability of ASGD near the strictly saddle points, so that we can prove main theorem in section 4.3. The basic tool is Azuma’s inequality for submartingale.

4.3 Descent Lemma

In order to prove the convergence, it is necessary to show that AGSD(Asynchronous Stochastic Gradient Descent) can decrease the function value when the gradient is large. Different from the proof of general SGD, this is not trivial. The first order convergence is proved in Lian et al. (2015), which is about the expectation of loss function. We will give a probability version of the first order convergence.

**Theorem 5** Suppose $\eta^2 (\frac{3L}{4} + L^2 MT^2 \eta) - \frac{\eta}{2M} < 0$, with probability $1 - 2e^{-1}$, we have

$$f(x_{l_0 + \tau + 1}) - f(x_{l_0}) \leq \frac{3M \eta}{8} ||\nabla f(x_{l_0})||^2$$

$$+ \epsilon \eta \sigma^2 + \left( \frac{3 \eta^2 L}{2} + L^2 T^2 M^2 \eta^3 \right) M^2 c \sigma^2 (\tau + 1 + 1)$$

$$+ L^2 TM^3 \sum_{k=l_0-T}^{l_0-1} T ||\nabla f(x_{l_0-\tau_m})||^2$$

(12)
Following the method in Lian et al. (2015), we can prove

**Lemma 6** For any \( k \) we have:

\[
    f(x_{k+1}) - f(x_k) \leq -\frac{M \eta}{2} \|\nabla f(x_k)\|^2 + \left(\frac{3\eta^2L}{4} - \frac{\eta}{2M}\right) \|\sum_{m=1}^{M} \nabla f(x_{k-t_m})\|^2 \\
    + L^2TM \eta \sum_{j=k-T}^{k-1} \eta^2 \|\sum_{m=1}^{M} \nabla f(x_{j-t_j})\|^2 \\
    + \frac{3\eta^2L}{2} \|\sum_{m=1}^{M} \xi_{k,m}\|^2 + L^2TM \eta \sum_{j=k-T}^{k-1} \eta^2 \|\sum_{m=1}^{M} \xi_{j,m}\|^2
\]

(13)

Terms with delay is hard to estimated. However if \( \eta^2 \left( \frac{3L^2}{4} + L^2MT^2 \eta \right) - \frac{\eta}{2M} < 0, \)
\( \sum_{k=0}^{k-T} \eta^2 \|\sum_{m=1}^{M} \nabla f(x_{k-t_m})\|^2 \) will counterbalance \( \sum_{j=k-T}^{k-1} \eta^2 \|\sum_{m=1}^{M} \nabla f(x_{j-t_j})\|^2 \), then the theorem can be deduced by direct calculation.

### 4.4 Escaping Saddle Points

In order to show ASGD can escape from saddle points, we will prove the following theorem.

**Theorem 7** Given a point \( x_k \), let \( H = \nabla^2 f(x_k) \), and \( e_1 \) be the minimum eigendirection of \( H, \gamma = -\lambda_{\min}(H) \geq \sqrt{\theta_2} \) and \( \sum_{t=k-2T}^{k-1} \|\nabla f(x_t)\|^2 \leq F \). We have

\[
P\left( \sum_{t=k}^{k+T_{max}-1} \|\nabla f(x(t))\|^2 \geq F_2 \right) \geq 1/24
\]

(14)

In order to prove it, we can turn to show the distance from the saddle point will be large after several iterations. This is due to the following lemma:

**Lemma 8** Localization lemma for SGD

\[
    \sum_{k=t_0}^{t-1+t_0} (1 + 2L^2 \eta^2 M^2 T^3) \|\nabla f(x_k)\|^2 \geq \frac{\|x_{t_0} + t - x_{t_0}\|^2 - 3\eta^3 \|\sum_{t=t_0}^{t_0+t-1} \xi_{k,m}\|^2}{3\eta^2 M^2 t} \\
    - \sum_{k=t_0-2T}^{t_0-1} 2L^2 M^2 \eta^2 T^3 \|\nabla f(x_k)\|^2 \\
    - \sum_{k=t_0}^{t-1+t_0} \sum_{j=k-T}^{k-1} 2L^2 \eta^2 T \|\sum_{m=1}^{M} \xi_{j,m}\|^2
\]

(15)

**Remark 4.1** In the \( T = 0 \) case, the \( \sum_{i=0}^{t} \|\nabla f(x_i)\|^2 \sim \|x(t) - x(0)/t, \) so that if \( \|x(t) - x(0)\| \)

is large, \( \|\nabla f(x(t))\|^2 \) will be large, and the saddle escaping theorem in Jin et al. (2017).Jin et al. (2019a) can be in fact used for any point \( x \) with \( \lambda_{\min}\nabla^2 f(x) < 0, \) the condition \( \|\nabla f(x)\| \) is small is useless. In our case, if \( \sum_{k=t_0-2T}^{t_0-1} 2L^2 \eta^2 M^2 T^3 \|\sum_{m=1}^{M} \nabla f(x_k)\|^2 \) is large, the localization theorem will be invalid, but if the updating sequence stays at saddle point for a long time \( O(T) \), it may not return after escaping from the strictly saddle points.

A direct corollary is the following lemma:
Lemma 9 \textit{Considering a sequence }\{x(k+t)\} \textit{as a run of algorithm 1 and }\sum_{t=k-2T}^{k+T_{\text{max}}-1} ||\nabla f(x(t))||^2 \leq F, \textit{we have}

\[ P\left( \sum_{t=k}^{k+T_{\text{max}}-1} ||\nabla f(x(t))||^2 \geq F_2, \text{ or } \forall t \leq T_{\text{max}}, ||x(k+t)-x(k)||^2 \leq S^2 \right) \geq 1 - 1/24 \quad (16) \]

Using these lemmas, in order to prove ASGD can decrease the function value, we can turn to show \( ||x(t)-x(0)|| > S \). This can be proved by analyzing the exponential instability of ASGD dynamics near the strictly saddle points. We can prove that:

\textbf{Theorem 10} \textit{With probability at least }1/6, \textit{max}_{0 \leq t \leq T_{\text{max}}} (||x_1(k+t)-x_1(k)||, ||x_2(k+t)-x_2(k)||) \geq S \textit{or } ||x(k)|| \geq \frac{\beta(k)M\eta r}{6\sqrt{d}}, \textit{and for all }k \geq T, q = M\eta\sqrt{\rho}\epsilon e^{-\frac{1}{f}}, \textit{we have }\beta^2(k) \geq T + \frac{(1+q)^{k-T}}{2q}.

Then we are ready to prove the main theorem in this subsection.

\textbf{Proof of Theorem 7:}

From Lemma 9, we need to show \textit{max}_{t \leq T_{\text{max}}} ||x(k+t) - x(k)|| \geq S. Using theorem 10, for coupling sequences \(\{x_1(t)\}, \{x_2(t)\}\)

\[ \max(||x_1(k+t) - x(k)||, ||x_2(k+t) - x(k)||) \geq \frac{1}{2} ||x(t)|| \quad (17) \]

Let \(q = M\eta\sqrt{\rho}\epsilon e^{-\frac{1}{f}}\). We have \(\beta^2(k) \geq T + \frac{(1+q)^{k-T}}{2q}\). Assuming \(u\) is large enough to satisfy Eq. (7), we have

\[ \frac{\beta(T_{\text{max}})M\eta r}{6\sqrt{d}} \geq \frac{(1+q)^{T_{\text{max}}-T}M\eta r}{6\sqrt{2qd}} \geq 2\eta^{M}\sqrt{\rho}\epsilon e^{-\frac{1}{f}} \geq 2\eta M\sqrt{15cT_{\text{max}}\sigma} = 2S \]

Thus Theorem 10 shows that, with probability at least \(1/6\)

\[ \max_{t \leq T_{\text{max}}} (||x_1(k+t) - x(k)||, ||x_2(k+t) - x(k)||) \geq S \]

so that we have

\[ P(\max_{t \leq T_{\text{max}}} ||x_1(k+t) - x(k)|| \geq S) \geq \frac{1}{2} P(\max_{t \leq T_{\text{max}}} (||x_1(k+t) - x(k)||, ||x_2(k+t) - x(k)||) \geq S) \geq 1/12 \quad (18) \]

Using Lemma 9, we have

\[ P\left( \sum_{t=k}^{k+T_{\text{max}}-1} ||\nabla f(x_1(t))||^2 \geq F_2 \right) \geq 1/24 \quad (19) \]

\textbf{4.5 Proof of the main theorem}

To prove the main Theorem 1 we need a new definition.
\textbf{Definition 2} \ Let the total number of iterations be

$$K = \max \{100T \frac{f(x_0) − f(x_*)}{M\eta F}, 100t \frac{f(x_0) − f(x_*)}{M\eta F_2}\}$$

We divide $K$ into $\lceil K/2T \rceil$ blocks $S_k = \{i|k2T \leq i < (k+1)2T\}$.

Blocks $S_k$ satisfying $\sum_{i \in S_k} \|\nabla f(x_i)\|^2 \geq F$ are called first kind blocks. Let $F_k = \max_{i \in S_k} + 1$, the right after iteration of $S_k$. $S_k$ with $\sum_{i \in S_k} \|\nabla f(x_i)\|^2 < F, \lambda_{\min}(\nabla^2 f(x_{F_k})) \leq -\sqrt{\rho \varepsilon_2}$ are called second kind blocks. Blocks are of third kind, if $\sum_{i \in S_k} \|\nabla f(x_i)\|^2 < F$ and $\lambda_{\min}(\nabla^2 f(x_{F_k})) > -\sqrt{\rho \varepsilon_2}$.

\textbf{Lemma 11} \ With probability at last $1 - 2e^{-t}$:

1. There are at most $\lceil K/8T \rceil$ first kind blocks.
2. There are at most $\lceil K/8T \rceil$ second kind blocks.
so that at least $\lfloor K/4T \rfloor$ blocks are of third kind.

\textbf{Proof} \ We follow the proof of theorem 5 in Jin et al. (2019a).

Using Theorem 5, if there are more than $\lceil K/8T \rceil$ first kind blocks, with probability $1 - 2e^{-t}$

$$f(x_{K+1}) - f(x_0) \leq \sum_{k=0}^{K} -\frac{3M\eta}{8} \|\nabla f(x_k)\|^2$$

$$+ c\eta \sigma^2 t + \left(\frac{3\eta^2L}{2} + L^2T^2M\eta^3\right) M^2 c\sigma^2 (K + 1 + t)$$

$$\leq \sum_{k=0}^{K} \sum_{i \in S_k} -\frac{3M\eta}{8} \|\nabla f(x_k)\|^2$$

$$+ c\eta \sigma^2 t + \left(\frac{3\eta^2L}{2} + L^2T^2M\eta^3\right) M^2 c\sigma^2 (K + 1 + t)$$

$$\leq -\frac{K}{8T} \frac{3M\eta}{8} F$$

$$+ c\eta \sigma^2 t + \left(\frac{3\eta^2L}{2} + L^2T^2M\eta^3\right) M^2 c\sigma^2 (K + 1 + t)$$

$$= -\frac{K}{8T} \frac{3M\eta}{8} (F - \left[\frac{32\eta M L T}{3} + \frac{64L^2 T^3 \eta^2 M^2}{3}\right] c\sigma^2)$$

$$+ c\eta \sigma^2 t + \left(\frac{3\eta^2L}{2} + L^2T^2M\eta^3\right) M^2 c\sigma^2 (1 + t)$$

$$\leq -\frac{K}{8T} \frac{3M\eta}{8} (F - 40c\sigma^2 \eta M L T)$$

$$+ c\eta \sigma^2 t + \left(\frac{3\eta^2L}{2} + L^2T^2M\eta^3\right) M^2 c\sigma^2 (1 + t)$$

$$\leq -\frac{K}{8T} \frac{3M\eta}{8} \frac{1}{5} F$$

$$+ c\eta \sigma^2 t + \left(\frac{3\eta^2L}{2} + L^2T^2M\eta^3\right) M^2 c\sigma^2 (1 + t)$$

Since $K = 100T \frac{f(x_0) - f(x_*)}{M\eta F}$, and $t$ is large enough, it can not be achieved.
As for 2, let $z_i$ be the stopping time such that
\[ z_1 = \inf \{ j | S_j \text{ is of second kind} \} \]
\[ z_i = \inf \{ j | T_{\max}/2T \leq j - z_{i-1} \text{ and } S_j \text{ is of second kind} \} \quad (21) \]

Let $N = \max \{ i | 2Tz_i + T_{\max} \leq K \}$. We have
\[
f(x_{K+1}) - f(x_0) \leq \sum_{k=0}^{K} - \frac{3M\eta}{8}||\nabla f(x_k)||^2 + c\eta \sigma^2 i \\
+ \frac{3\eta^2 L}{2} M^2 c \sigma^2 (K + 1 + t) + L^2 T^2 M\eta^3 M^2 c \sigma^2 (K + 1 + T + t) \\
\leq c\eta \sigma^2 i + \frac{3\eta^2 L}{2} M^2 c \sigma^2 (K + 1 + t) + L^2 T^2 M\eta^3 M^2 c \sigma^2 (K + 1 + T + t) \\
+ \sum_{i}^{N} F_{i}^{T} + T_{\max}^{-1} - \frac{3M\eta}{8} ||\nabla f(x_k)||^2 \quad (22)\]

Let
\[ X_i = \sum_{k=F_{i-1}}^{F_{i} + T_{\max}^{-1}} ||\nabla f(x_k)||^2 \]
\[ \sum X_i \text{ is a submartingale and the last term of Eq.}(22) \text{ is } -\sum X_i. \]

Note that $P(X_i \geq F_{2}) \geq 1/24$. Let $Y_i$ be a random variable, such that $Y_i = X_i$ if $X_i \leq F_2$ else $Y_i = F_2$. Then we have a bounded submartingale $0 \leq Y_i \leq X_i$, so that we can use Azuma’s inequality:
\[ P(\sum_{i}^{N} X_i \geq \mathbb{E}(\sum_{i} Y_i) - \lambda) \geq P(\sum_{i}^{N} Y_i \geq \mathbb{E}(\sum_{i} Y_i) - \lambda) \geq 1 - 2e^{-\frac{\lambda^2}{2F_{2}N}} \quad (23) \]

If it easy to see $\mathbb{E}(\sum_{i}^{N} Y_i) \geq \frac{1}{24} NF_{2}$. We have
\[ P(\sum_{i}^{N} Y_i \geq \frac{1}{24} NF_{2} - \sqrt{2NF_{2} \sqrt{t}}) \geq 1 - e^{-t} \quad (24) \]

If there are more then $K/8T$ second kind blocks, we have $N \geq K/4T_{\max}$.
\[ \frac{1}{24} N - \sqrt{2N \sqrt{t}} \geq \frac{1}{48} N \]

With probability at least $1 - 2e^{-t}$
\[
f(x_{K+1}) - f(x_0) \leq c\eta \sigma^2 i + \frac{3\eta^2 L}{2} M^2 c \sigma^2 (K + 1 + t) + L^2 T^2 M\eta^3 M^2 c \sigma^2 (K + 1 + T + t) \\
- \frac{3M\eta}{8} NF_{2} \quad (25) \]

If $N \geq K/4T_{\max}$, and $K = 100T_{\max} \frac{f(x_0) - f(x_c)}{M\eta F_{2}}$, it can not be achieved.
Now we are ready to prove the main theorem.

**Proof of Theorem 1:**

The above lemma shows that with high probability at last $K/2$ iterations are of the third kind \( \sum_{i=k-2T}^{k-1} \| \nabla f(x_i) \|^2 < F \) and \( \lambda_{\min}(\nabla^2 f(x_k)) > -\sqrt{pF_2} \).

Let \( x_k = x_{F_j} \), the right after iteration of block \( S_k \). We will show \( x_k \) is a second order stationary point with high probability.

\[
\| \nabla f(x_k) \|^2 \leq 2\| \nabla f(x_{k-1}) \|^2 + 2\| \nabla f(x_k) - \nabla f(x_{k-1}) \|^2 \\
\leq 2\| \nabla f(x_{k-1}) \|^2 + 2L^2\| x_k - x_{k-1} \|^2 \\
\leq 2\| \nabla f(x_{k-1}) \|^2 + 4L^2\eta^2\| \sum_{m=1}^{M} \nabla f(x_{k-1-\tau_{1,m}}) \|^2 + 4L^2\eta^2\| \xi_k \|^2 \\
\leq 2\| \nabla f(x_{k-1}) \|^2 + 4L^2\eta^2\| M\nabla f(x_{k-1}) \|^2 + 4L^2\eta^2\| \sum_{m=1}^{M} \nabla f(x_{k-1-\tau_{1,m}}) \|^2 + 4L^2\eta^2\| \sum_{m=1}^{M} \xi_{j,m} \|^2
\]

(26)

From

\[
\| M\nabla f(x_k) - \sum_{m=1}^{M} \nabla f(x_{k-\tau_{1,m}}) \|^2 \\
\leq M^22L^2[ \sum_{j=k-T}^{k-1} \eta^2\| \sum_{m=1}^{M} \xi_{j,m} \|^2 + \| \sum_{m=1}^{M} \nabla f(x_{j-\tau_{1,m}}) \|^2 ] \\
\leq M^22L^2\eta^2\| \sum_{j=k-T}^{k-1} \sum_{m=1}^{M} \xi_{j,m} \|^2 + M^22L^2\eta^2T^2 + M^22L^2\eta^2T \sum_{j=k-2T}^{k-1} \sum_{m=1}^{M} \xi_{j,m} \|^2
\]

(27)

We have

\[
\| \nabla f(x_k) \|^2 \leq (2 + 4L^2\eta^2M^2(1 + 2L^2M^2\eta^2T^2)) \sum_{i=k-2T}^{k-1} \| \nabla f(x_i) \|^2 + 4L^2\eta^2\| \sum_{m=1}^{M} \xi_{j,m} \|^2 + 4L^2\eta^2M^22L^2\eta^2T \sum_{j=k-T}^{k-1} \sum_{m=1}^{M} \xi_{j,m} \|^2
\]

(28)

\[
\| \nabla f(x_k) \|^2 \leq (2 + 4L^2\eta^2M^2(1 + 2L^2M^2\eta^2T^2))F + 4L^2\eta^2\| \sum_{m=1}^{M} \xi_{j,m} \|^2 + 4L^2\eta^2M^22L^2\eta^2T \sum_{j=k-T}^{k-1} \sum_{m=1}^{M} \xi_{j,m} \|^2
\]

so with probability \( 1 - \epsilon^{-1/\eta} \), \( \| \nabla f(x_k) \|^2 \leq 2F + F = 3F \), the third kind block corresponds to a second order stationary point.

We can sum up all the \( F_k \). Let \( B \leq \frac{K}{2T} \) be the number of third kind block. We have

\[
\frac{1}{B} \sum_{F_k} \| \nabla f(x_k) \|^2 \leq (2 + 4L^2\eta^2M^2(1 + 2L^2M^2\eta^2T^2))F + 4L^2\eta^2\| \sum_{m=1}^{M} \xi_{j,m} \|^2 + 4L^2\eta^2M^22L^2\eta^2T \sum_{j=k-T}^{k-1} \sum_{m=1}^{M} \xi_{j,m} \|^2
\]

(29)
with probability at last $1 - e^{-B/\eta} \geq 1 - e^{-t}$, we have

$$\min_{F_k} \left\| \nabla f(x_k) \right\|^2 \leq \frac{1}{N} \sum_{F_k} \left\| \nabla f(x_k) \right\|^2 \leq 2F + F = 3F$$

Using Lemma 11, with probability at last $1 - 3e^\iota$, PASGD reach a second order stationary point with

$$\left\| \nabla f(x) \right\|^2 \leq 3F$$

and

$$\lambda_{\min}(\nabla^2 f(x)) > -\sqrt{\rho \varepsilon}$$

Given any $\varepsilon_1, \varepsilon_2, \delta$ in Theorem 1, there is a large enough $w, t$ such that $\eta$ is small enough, $3F \leq \varepsilon^2$, and

$$1 - 3e^{-1} \geq 1 - \delta$$

Our theorem follows.

5. Numerical Results

We use a simple example to illustrate the effect of saddle points escaping.

Consider a nonconvex objective function

$$f(x) = \frac{1}{2} x^T A x + \frac{1}{4} \|x\|^4, \quad \|x\| \leq B$$

(30)

Let $B^2 \geq \|A\|$, then $\nabla f(x)$ is $2B^2$-Lipschitz and $6B$-Hessian Lipschitz. We have $\nabla^2 f(x)|_{x=0} = A$. Supposing $\lambda_{\min}(A) < 0, x = 0$ is a strictly saddle point. Note that all the nonconvex functions near a strictly saddle points have the same local structure as this case. We set

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

$M = 5, \eta = 0.001/M, r = 0.05, T = 10$

In the simulation, the time delay $\tau_{t,i} \leq T$ of every node is random, and we randomly initialize near the saddle point $x = 0$. The convergence speed is shown in Figure 1. It is shown that perturbed asynchronous gradient algorithm can escape from the strictly saddle points faster.

6. Discussion

The behavior of ASGD near a saddle point is completely described by Lemma 8 and Theorem 10.

From Lemma 8, we have

$$\sum_{k=0}^{t-1} \left\| \nabla f(x_k) \right\|^2 \geq (1 + 2L^2 \eta^2 M^2 T^3)^{-1} \left\{ \sum_{i=0}^{t-1} \left\| x_{i+1} - x_i \right\|^2 - \sum_{i=0}^{t-1} \sum_{j=0}^{i-1} \sum_{m} \zeta_{i,j,m} \right\}$$

(31)
When $\eta \sim \frac{1}{M L(T + 1)}$, $L^2 \eta^2 M^2 T^3$ can be large. If $\sum_{k=0}^{T-1} ||\nabla f(x_k)||^2$ is large, i.e. the block before the saddle point is of the first kind, in the worst case, $\sum_k ||\nabla f(x_k)||^2$ can be still very small although $\max_{t \leq T_{\text{max}}} ||x_{k+t} - x_k|| \geq S$. This is due to there is no guarantee that ASGD can decrease the function value, so it is possible that the algorithm will finally return to a point near the saddle point. However, if $||\nabla f(x_k)||^2$ is small for a long enough time ($O(T)$), we have $\sum_{k=0}^{T-1} ||\nabla f(x_k)||^2 \leq F$, then $\max_{t \leq T_{\text{max}}} ||x_{k+t} - x_k|| \geq S \Rightarrow \sum_{k=0}^{T_{\text{max}}} ||\nabla f(x_k)||^2 \geq F_2$ from Lemma 8 and 9. Thus ASGD will finally escape.

In our experiment, we find when $\eta$ is large (e.g. larger than $0.02/M$), the escaping time will be sensitive to the initialization. This is from our above analysis that $-\sum_{k=0}^{T-1} 2L^2 M^2 \eta^2 T^3 ||\nabla f(x_k)||^2$ will heavily influence the decline of function value.

The influence on the growth rate of $||x_{k+t} - x_k||$ is described by Theorem 10 and corollary 21. ASGD with time delay $T$ will take $e_f = e^{(T+1)M\eta T}$ times as long to make $||x_{k+t} - x_k|| > S$ in the worst case. When $\eta \sim \frac{1}{M L(T + 1)}$, $e_f$ will not be very large.

**7. Conclusion**

In this paper, we study the theoretical properties of Perturbed Asynchronous Stochastic Gradient Descent (PASGD) algorithm and give the first theoretical guarantees of convergence to second order stationary points. The main contribution of this work is to give a new analysis of asynchronous algorithm. We show the exponential instability of asynchronous updating system by Razumikhin-Lyapunov method from the control theory. Then we give an explicit expression of how the asyn-
chronous algorithm behave near and far away from strictly saddle points and local minimum, and how time delay parameter $T$ in asynchronous process influences the escaping behaviors.

Acknowledgments

This research was funded by the National Key Research and Development Program of China, grant number 2018YFC0831300, and the Fundamental Research Funds for the Central Universities, grant number No.2018YJS003.

Appendix A. Proof of the theorems in section 4.3

Proof of Lemma 6:

This lemma is a transformation of Eq.(30) in Lian et al. (2015). The following equations are based on the proof in Lian et al. (2015).

$$f(x_{k+1}) - f(x_k) \leq \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2} \|x_{k+1} - x_k\|^2$$  

$$= - \left( \nabla f(x_k), \eta \sum_{m=1}^{M} \nabla f(x_{k-\tau,m}) + \eta \sum_{m=1}^{M} \zeta_{k,m} \right)$$

$$+ \frac{\eta^2 L}{2} \left\| \sum_{m=1}^{M} \nabla f(x_{k-\tau,m}) + \sum_{m=1}^{M} \zeta_{k,m} \right\|^2$$

$$= - \left( \nabla f(x_k), \eta \sum_{m=1}^{M} \nabla f(x_{k-\tau,m}) \right) + \left( \nabla f(x_k), \sum_{m=1}^{M} \zeta_{k,m} \right)$$

$$+ \frac{\eta^2 L}{2} \left\| \sum_{m=1}^{M} \nabla f(x_{k-\tau,m}) + \sum_{m=1}^{M} \zeta_{k,m} \right\|^2$$

$$\leq - \frac{M \eta}{2} \left( \|\nabla f(x_k)\|^2 + \frac{1}{M} \sum_{m=1}^{M} \|\nabla f(x_{k-\tau,m})\|^2 \right)$$

$$- \|\nabla f(x_k) - \frac{1}{M} \sum_{m=1}^{M} \nabla f(x_{k-\tau,m})\|^2 - \eta \left( \nabla f(x_k), \sum_{m=1}^{M} \zeta_{k,m} \right)$$

$$+ \frac{\eta^2 L}{2} \left\| \sum_{m=1}^{M} \nabla f(x_{k-\tau,m}) + \sum_{m=1}^{M} \zeta_{k,m} \right\|^2$$

$$= - \frac{M \eta}{2} \left( \|\nabla f(x_k)\|^2 + \frac{1}{M} \sum_{m=1}^{M} \|\nabla f(x_{k-\tau,m})\|^2 - \|\nabla f(x_k) - \frac{1}{M} \sum_{m=1}^{M} \nabla f(x_{k-\tau,m})\|^2 \right)$$

$$- \eta \left( \nabla f(x_k), \sum_{m=1}^{M} \zeta_{k,m} \right) + \frac{\eta^2 L}{2} \left( \frac{3}{2} \left\| \sum_{m=1}^{M} \nabla f(x_{k-\tau,m}) \right\|^2 + 3 \left\| \sum_{m=1}^{M} \zeta_{k,m} \right\|^2 \right)$$

$$\leq - \frac{M \eta}{2} \left( \|\nabla f(x_k)\|^2 + \frac{1}{M} \sum_{m=1}^{M} \|\nabla f(x_{k-\tau,m})\|^2 \right)$$
In (1) we use the fact $\langle a, b \rangle = \frac{1}{2}(||a||^2 + ||b||^2 - ||a - b||^2)$, and (2) is from the estimation of $T_1$ in Lian et al. (2015).

From this lemma, we can observe that, different from the general SGD, since the gradients used in asynchronous SGD are not those in the current time, we can’t guarantee the function value will decrease in every step. However, it can be proved that the overall trend of the function value is still decreasing.

**Proof of theorem 5:**

\[
f(x_{t_0+\tau+1}) - f(x_{t_0}) = \sum_{k=t_0}^{t_0+\tau} f(x_{k+1}) - f(x_k)
\]

\[
\leq \sum_{k=t_0}^{t_0+\tau} - \frac{M\eta}{2} ||\nabla f(x_k)||^2 + \left( \frac{3\eta^2 L}{4} - \frac{\eta}{2M} \right) ||\sum_{m=1}^{M} \nabla f(x_{k-\tau,m})||^2
\]

\[
+ L^2 TM\eta \sum_{j=k-T}^{k-1} \eta^2 ||\sum_{m=1}^{M} \nabla f(x_{j-\tau,m})||^2
\]

\[
- \eta \left( \nabla f(x_k), \sum_{m=1}^{M} \zeta_{k,m} \right) + \frac{3\eta^2 L}{2} ||\sum_{m=1}^{M} \zeta_{k,m}||^2 + L^2 TM\eta \sum_{j=k-T}^{k-1} \eta^2 ||\sum_{m=1}^{M} \zeta_{j,m}||^2
\]

\[
= \sum_{k=t_0}^{t_0+\tau} - \frac{M\eta}{2} ||\nabla f(x_k)||^2 + \sum_{k=t_0}^{t_0+\tau} \left( \frac{3\eta^2 L}{4} - \frac{\eta}{2M} \right) ||\sum_{m=1}^{M} \nabla f(x_{k-\tau,m})||^2
\]
\[
+ L^2 TM \eta \sum_{j=k-T}^{k-1} \eta^2 \| \sum_{m=1}^M \nabla f(x_{j-\tau_m}) \|^2
- \sum_{k=t_0}^{t_0+\tau} \eta \langle \nabla f(x_k), \sum_{m=1}^M \zeta_{k,m} \rangle + \frac{3 \eta^2 L}{2} \| \sum_{m=1}^M \zeta_{k,m} \|^2 + L^2 TM \eta \sum_{j=k-T}^{k-1} \eta^2 \| \sum_{m=1}^M \zeta_{j,m} \|^2
\leq \sum_{k=t_0}^{t_0+\tau} - \frac{M \eta}{2} \| \nabla f(x_k) \|^2
+ \sum_{k=t_0}^{t_0+\tau} \eta \left( \frac{3L}{4} + L^2 MT^2 \eta \right) - \frac{\eta}{2M} \| \sum_{m=1}^M \nabla f(x_{k-\tau_m}) \|^2
\]

In order to estimate \( T_2 \), we can use lemmas in Jin et al. (2019a). Let \( \zeta_k = \frac{1}{M} \sum_{m=1}^M \zeta_{k,m} \). With probability \( 1 - e^{-t} \), we have

\[
- \sum_{k=t_0}^{t_0+\tau} \eta \langle M \nabla f(x_k), \zeta_k \rangle \leq \frac{\eta M}{8} \sum_{k=t_0}^{t_0+\tau} \| \nabla f(x_k) \|^2 + c \eta \sigma^2 I \tag{34}
\]

This is from Lemma 30 in Jin et al. (2019a).

With probability \( 1 - e^{-t} \),

\[
\sum_{k=t_0}^{t_0+\tau} \frac{3 \eta^2 L}{2} \| \sum_{m=1}^M \zeta_{k,m} \|^2 + \sum_{k=t_0-\tau}^{t_0+\tau} L^2 TM \eta^3 \| \sum_{m=1}^M \zeta_{j,m} \|^2
\leq \frac{3 \eta^2 L}{2} M^2 c \sigma^2 (\tau + 1 + t) + L^2 T^2 M^2 \eta^3 M^2 c \sigma^2 (\tau + 1 + T + t) \tag{35}
\]

We have, with probability \( 1 - 2e^{-t} \),

\[
T_1 \leq \frac{\eta M^{t_0+\tau}}{8} \sum_{k=t_0}^{t_0+\tau} \| \nabla f(x_k) \|^2 + c \eta \sigma^2 I + \frac{3 \eta^2 L}{2} M^2 c \sigma^2 (\tau + 1 + t) + L^2 T^2 M^2 \eta^3 M^2 c \sigma^2 (\tau + 1 + T + t)
\]

Note that \( \eta^2 (\frac{3L}{4} - L^2 MT^2 \eta) - \frac{\eta}{2M} < 0 \), with probability \( 1 - 2e^{-t} \),

\[
f(x_{t_0+\tau+1}) - f(x_{t_0}) \leq \sum_{k=t_0}^{t_0+\tau} - \frac{3 \eta M}{8} \| \nabla f(x_k) \|^2 + c \eta \sigma^2 I
+ \frac{3 \eta^2 L}{2} M^2 c \sigma^2 (\tau + 1 + t) + L^2 T^2 M^2 \eta^3 M^2 c \sigma^2 (\tau + 1 + T + t)
\]

\[
+ L^2 TM \eta^3 \sum_{k=t_0-\tau}^{t_0+\tau} T \| \sum_{m=1}^M \nabla f(x_{j-\tau_m}) \|^2 \tag{36}
\]
Theorem 5 follows.

Appendix B. Proof of Localization theorems in section 4.4

Proof of Lemma 8:

$$||x_{t_0+t} - x_{t_0}||^2 - 3\eta^2 ||\sum_{m=0}^{t-1} \zeta_{i,m}||^2$$

$$= \eta^2 ||\sum_{k=t_0}^{t-1+t_0} M \nabla f(x_{k-\tau_m}) + \sum_{m=0}^{t-1} \zeta_{i,m}||^2 - 3\eta^2 ||\sum_{m=0}^{t-1} \zeta_{i,m}||^2$$

$$\leq 3\eta^2 \sum_{k=t_0}^{t-1} ||M \nabla f(x_k)||^2$$

(a) $$\leq 3\eta^2 \sum_{k=t_0}^{t-1+t_0} M^2 t \||\nabla f(x_k)||^2$$

$$+ 3\eta^2 t \sum_{k=t_0}^{t-1+t_0} M^2 2L^2 \eta^2 T \||\sum_{j=k-T}^{k-1} \sum_{m=1}^{M} \zeta_{j,m}||^2$$

$$+ 3\eta^2 \sum_{k=t_0}^{t-1+t_0} M^2 2L^2 \eta^2 T \||\sum_{m=1}^{M} \nabla f(x_{j-\tau_m})||^2$$

In (a), we use the estimation for $T_1$ in the previous section.
Proof of Lemma 9:

Supposing there is a $\tau \leq T_{\text{max}}$, such that $\|x(k + \tau) - x(k)\|^2 \geq S^2$, with probability at last $1 - 2e^{T-T_{\text{max}}}$, we have

\[
T_{\text{max}}^{-1+t_0} \sum_{k=t_0}^{t_0+1} (1 + 2L^2\eta^2M^2T^3)\|\nabla f(x_k)\|^2 \geq \sum_{k=t_0}^{t_0+1} (1 + 2L^2\eta^2M^2T^3)\|\nabla f(x_k)\|^2 \tag{39}
\]

\[
\geq \frac{\|x_{t_0} + \tau - 1 - x_{t_0}\|^2 - 3\eta^2 \sum_{m=0}^{t_0+\tau-1} \zeta_{i,m}^2}{3\eta^2M^2T_{\text{max}}} 
- \sum_{k=t_0}^{t_0-2T} 2L^2\eta^2T^3 \|\nabla f(x_k)\|^2 
- \sum_{k=t_0}^{t_0-2T} \sum_{j=k-T}^{k-1} 2L^2\eta^2T \|\sum_{m=1}^{M} \zeta_{j,m}\|^2
\]

\[
\geq \frac{S^2 - 3\eta^2M^2c\sigma^22T_{\text{max}}}{3\eta^2M^2T_{\text{max}}} - 2L^2\eta^2T^3M^2F 
- c\sigma^22T_{\text{max}}2L^2M^2\eta^2T^2r^2 
\geq 3cr^2 - 2L^2\eta^2M^2T^3(50c\sigma^2\eta MLT) - cr^2 
= 2cr^2 - 2L^3\eta^3M^3T^4100cr^2 
\geq cr^2 
\geq 4r^2
\]

Using (9), $2L^2\eta^2M^2T^3 \leq 1$. We have

\[
\sum_{k=t_0}^{T_{\text{max}}-1+t_0} \|\nabla f(x_k)\|^2 \geq 2r^2 = F_2 \tag{40}
\]

In (a) we use with probability at last $1 - e^{T-T_{\text{max}}}$

\[
\sum_{k=t_0}^{T_{\text{max}}-1+t_0} \sum_{j=k-T}^{k-1} 2L^2\eta^2T \|\sum_{m=1}^{M} \zeta_{j,m}\|^2 \leq c\sigma^22T_{\text{max}}2L^2M^2\eta^2T^2 \leq r^2
\]

and with probability at last $1 - e^{-T_{\text{max}}}$, we have

\[
3\eta^2\sum_{m}^{t_0+\tau-1} \sum_{i=t_0}^{M} \zeta_{i,m}\|^2 \leq 3\eta^2M^2c\sigma^2(T_{\text{max}} + T_{\text{max}})
\]

From (8), $2e^{T-T_{\text{max}}} \leq 1/24$. Our claim follows. ■

Appendix C. Proof of theorem 10

In order to analyze the $x(t)$ under ASGD updating rules, as in Jin et al. (2017), the standard proof strategy to consider two sequences $\{x_1(t)\}$ and $\{x_2(t)\}$ as two separate runs ASGD starting from
\(x(k)\) (for all \(t \leq k, x_1(t) = x_2(t)\)). They are coupled, such that for the Gaussian noise \(\xi_1(t)\) and \(\xi_2(t)\) in algorithm 1: \(\xi_1^T \xi_1 = -\xi_2^T \xi_2\), and the components at any direction perpendicular to \(e_1\) of \(\xi_1\) and \(\xi_2\) are equal. Given coupling sequence \(\{x_1(k + t)\}\) and \(\{x_2(k + t)\}\), let \(x(t) = x_1(k + t) - x_2(k + t)\). We have

\[
x(k) = x(k - 1) + \eta \left( \sum_{m=1}^{M} \nabla f(x_1(k - \tau_{k,m})) + \xi_{1,k,m} - \sum_{m=1}^{M} \nabla f(x_2(k - \tau_{k,m})) - \xi_{2,k,m} \right)
\]

We have \(\nabla f(x_1) - \nabla f(x_2) = \int_0^1 \nabla^2 f(tx_1 + (1 - t)x_2) (x_1 - x_2) dt = \int_0^1 \nabla^2 f(tx_1 + (1 - t)x_2) dt - \nabla^2 f(x_0))(x_1 - x_2)\). Let \(H = \nabla^2 f(x_0), \Delta_{x_1,x_2} = \int_0^1 \nabla^2 f(tx_1 + (1 - t)x_2) dt - \nabla^2 f(x_0)\). We have

\[
x(k) = x(k - 1) + \eta \left[ \sum_{m=1}^{M} (H + \Delta_{x_1(k - \tau_{k,m}),x_2(k - \tau_{k,m})})x(k - \tau_{k,m}) + \xi_{1,k,m} - \xi_{2,k,m} \right]
\]

Note that since we want to estimate the probability of event

\[
\{ \max_{t \leq T_{\max}} (||x_1(k + t) - x_1(k)||^2, ||x_2(k + t) - x_2(k)||^2) \geq S^2 \text{ or } ||x(t)|| \geq 2S \}
\]

It is enough to consider a random variable \(x'\) such that \(x'(t)|E - x(t)|E = 0\) where \(E\) is the event \(\{\forall t \leq T_{\max}: \max_{t \leq T_{\max}} (||x_1(k + t) - x_1(k)||^2, ||x_2(k + t) - x_2(k)||^2) \leq S^2\}\). This is due to the fact that

\[
P(\forall t \leq T_{\max} \max(||x_1(k + t) - x_1(k)||^2, ||x_2(k + t) - x_2(k)||^2) \leq S^2 \text{ or } ||x(t)|| < 2S) = P(\forall t \leq T_{\max} \max(||x_1(k + t) - x_1(k)||^2, ||x_2(k + t) - x_2(k)||^2) \leq S^2 \text{ or } ||x'(t)|| < 2S)
\]

Then we can turn to consider \(x'\), such that

\[
x'(k) = x'(k - 1) + \eta \left[ \sum_{m=1}^{M} (H + \Delta'_{x_1(k - \tau_{k,m}),x_2(k - \tau_{k,m})})x(k - \tau_{k,m}) + \xi_{1,k,m} - \xi_{2,k,m} \right]
\]

if \(\max(||x_1(t) - x(t)||^2, ||x_2(t) - x(t)||^2) \leq S^2\)

\[
\Delta'(t) = \Delta
\]

else

\[
\Delta'(t) = \rho S
\]

Then \(\Delta'(x_1'(k - \tau_{k,m}), x_2'(k - \tau_{k,m})) \leq \rho S\). In order to simplify symbols, we denote \(x = x'\).

To show that \(||x(T_{\max})|| \geq 2S\), firstly note that from Eq.(43), we can show there is a polynomial function \(f(t_0, t, y)\) such that \(x(k) = q_p(k) + q_h(k) + q_{sg}(k)\)

\[
q_p(k) = M \eta \sum_{i=0}^{k-1} f(i, k, H) \xi_i
\]

\[
q_h(k) = \eta \sum_{m=0}^{k-1} \sum_{i=0}^{k-1} f(i, k, H) \Delta(i - \tau_{i,m}) x(i - \tau_{i,m})
\]

\[
q_{sg}(k) = \eta \sum_{m=0}^{k-1} \sum_{i=0}^{k-1} f(i, k, H) \xi_{i,m}
\]

20
\( f(t_0, t, H) \) is the solution (fundamental solution) of the following linear equation

\[
    x(k) = x(k-1) - \eta [\sum_{m=1}^{m} H x(k - \tau_{k,m})]
\]

\[
    x(t_0) = I \\
    x(n) = 0 \text{ for all } n < t_0
\]  

(46)

This is an easy inference for linear time-varying systems. And if the minimum eigenvalue of \( H \) is \( \gamma \), it is easy to show \( ||f(t_0, t, H)||_2 \leq f(t_0, t, H) \).

**Lemma 12** Let \( f(t_0, t) = f(t_0, t, \gamma) \), \( \beta^2(k) = \sum_{i=0}^{k} f^2(i, k) \) we have

1. \( f(t_0, t_1)f(t_1, t_2) \leq f(t_0, t_2) \)
2. \( f(t_1, t_2) \geq f(t_1, t_2 - 1) \)
3. \( f(k, t)\beta(k) = \sqrt{\sum_{j=0}^{k-1} f^2(k, t)f^2(j, k)} \leq \sqrt{\sum_{j=0}^{k-1} f^2(j, t)} \leq \beta(t) \)
4. \( f(k, t + 1) \geq (1 + M\eta \gamma e^{-\eta \gamma_{T+1}}M) f(k, t) \text{ if } t - k \geq T. \)

**Proof** All inequalities are trivial, except for the last one. The last inequality is from theorem 21 we give in the next subsection.

Now we can estimate \( q_b \) term.

\[
    q_b(t + 1) = \eta \sum_{m=0}^{t} \sum_{n=0}^{t} f(n, t + 1, H) \Delta(n - \tau_{n,m}) x(n - \tau_{n,m})
\]

(47)

We want to give an estimation for \( \mathbb{E}\{e^{q_b(t)}\} \) then use Chernoff bound.

As in Jin et al. (2019b), we construct a matrix \( Y \)

\[
    Y = \begin{bmatrix} 0 & X^T \\ X & 0 \end{bmatrix}
\]

\( X \)

**Theorem 13** We have

\[
    \mathbb{E}e^{\theta Y_p(t)} \leq e^{\theta^2 \beta^2(i)M^2 \eta^2 \gamma^2 d/\eta} I \\
    \mathbb{E}e^{\theta Y_{\psi}(t)} \leq e^{\theta^2 \beta^2(i)M^2 \eta^2 \gamma^2 d/\eta} I \\
    \mathbb{E}tr\{e^{\theta Y_p(t)} + \theta Y_{\psi}(t)\} \leq e^{\theta^2 \beta^2(i)M^2 \eta^2 \gamma^2 d/\eta} (d + 1)
\]

(48)

This is from the fact that \( q_p \) and \( q_{\psi} \) are sub-Gaussian. The last one is from the following lemma:

**Lemma 14** Let \( Y_i \) the random matrix such that \( \mathbb{E}\{Y_i\} = 0 \) and \( \mathbb{E}tr\{e^{\theta Y_i}\} \leq e^{\theta^2 \sigma^2 (d + 1)} \), then we have \( \mathbb{E}tr\{e^{\theta \Sigma Y_i}\} \leq e^{\theta^2 (\Sigma \sigma^2)(d + 1)} \)
Proof
For any semi-positive definite matrix $A_i$ and $\sum_i a_i = 1$, $a_i \geq 0$, we have
\[
tr\prod_{i=1}^M A_i^{a_i} \leq \sum_i a_i tr A_i
\]
However it is impossible to use this inequality directly. When $Y_i$ and $Y_j$ are not commutative if $i \neq j$, we have $e^{\sum_i Y_i} \neq \prod_i^n e^{Y_i}$ even $tr\{e^{\sum_i Y_i}\} \neq tr\{\prod_i^n e^{Y_i}\}$. In the case $n = 2$, we have Golden-Thompson inequality Golden (1965) $tr\{e^{Y_1+Y_2}\} \leq tr\{e^{Y_1}e^{Y_2}\}$. However it is false when $n = 3$, which is studied by Lieb in Lieb (1973). Fortunately, for $n > 3$, we have Sutter-Berta-Tomamichel inequality Sutter et al. (2017):

Let $||\cdot||$ be the trace norm, and $H_k$ be Hermitian matrix. We have
\[
\log ||\exp(\sum_k^n H_k)|| \leq \int \log ||\prod_k^n \exp((1+it)H_k)|| d\beta(t)
\]
where $\beta$ is a probability measure.
For the right hand side, we have
\[
||\prod_k^n \exp((1+it)H_k)|| \leq \sum_i \sigma_i (\prod_k^n \exp((1+it)H_k) \leq \sum_i \sigma_i (\exp(H_1))\sigma_i (\exp(H_2))...\sigma_i (\exp(H_n))
\]
where $\sigma_i$ is the ith singular value.
If all $H_k$ are semi-positive definite, $\lambda_i = \sigma_i$, using the elementary inequality that
\[
\sum_i \lambda_i^{\alpha_i} (\exp(H_1))\lambda_i^{\alpha_i} (\exp(H_2))...\lambda_i^{\alpha_i} (\exp(H_n)) \leq \sum_i (\sum_k \alpha_k \lambda_i (\exp(H_k)))
\]
where $\sum_i \alpha_i = 1$, we have
\[
||\prod_k^n \exp((1+it)H_k)|| \leq tr\{\sum_k \alpha_k \exp(H_k)\}
\]
so that
\[
tr\{\exp(\sum_k^n \alpha_k H_k)\} \leq \sum_k \alpha_k tr\{\exp(H_k)\}
\]
Using lemma 14 and $\beta(t)2r/\sqrt{d} + \beta(t)\sqrt{2s} \leq \beta(t)2\sqrt{2r/d}$, the last equation follows.

Lemma 15 Let $Y_h$ be the $Y$ matrix constructed by $q_{h,t}(t)$, $Y(t)$ be the $Y$ matrix constructed by $x(t)$
\[
\mathbb{E}tr\{e^{\theta Y_h(t)}\} \leq e^{\theta^2(\sum_{i=1}^p \beta_i)^2 M^2 \eta^2 r^2/d} (d+1)
\]
\[
\mathbb{E}tr\{e^{\theta Y(t)}\} \leq e^{\theta^2(1+\sum_{i=1}^p \beta_i)^2 M^2 \eta^2 r^2/d} (d+1)
\]
(52)
Proof

We use mathematical induction.

For \( t = 0 \), the first inequality is obviously true. For the second one

\[
x(0) = q_p(0) + q_h(0) + q_{ch}(0) = q_p(0) + q_{ch}(0)
\]

so from theorem in the next section, we have

\[
\mathbb{E}Tr\{e^{\theta X(0)}\} \leq e^{e^{\theta^2 \beta^2(0)M^2 \eta^2 r^2/d}(d + 1)}
\]

Then supposing the lemma is true for all \( \tau \leq t \), we consider \( t + 1 \).

\[
\mathbb{E}Tr\{e^{\theta Y(i+1)}\} = \mathbb{E}Tr\{e^{\theta(\eta \sum_{i=1}^{t+1} f(i,H)\Delta(i-\tau,\eta)Y(i-\tau,m))}\}
\]

so we have

\[
\mathbb{E}Tr\{e^{\theta Y(t+1)}\} = \mathbb{E}Tr\{e^{\theta(\eta \sum_{i=1}^{t+1} f(i+1+1,H)\Delta(i-\tau,\eta)Y(i-\tau,m))}\}
\]

(1) is from lemma 14  \( e^X = 1 + X + \frac{X^2}{2} + \ldots \) \( ||f(i, t+1, H)|| \leq f(i, t + 1), \Delta(i-\tau, \eta) \leq \rho S \) and \( \eta MT_{max} \rho S \leq p \).

Using Chernoff bound in the proof of Corollary 7 in Jin et al. (2019b), we have

**corollary 16** For any \( t > 0 \)

\[
P(||q_b(k)|| \leq \frac{\sqrt{c} \beta(k) M \eta 2\sqrt{2} \sigma}{C \sqrt{d}} \sqrt{t}) \geq 1 - 2(d + 1)e^{-t}
\]

where \( \frac{1}{t} = \sum_{i=1}^{t} p^i = \frac{p}{1-p} \).

We select \( t = b = \log(2(d + 1)) + \log 6 \), then \( \frac{\sqrt{c} \beta(k) M \eta 2\sqrt{2} \sigma}{C \sqrt{d}} \sqrt{t} = \frac{1}{12} \), \( P(||q_b(k)|| \leq \frac{\beta(k) M \eta r}{2\sqrt{d}}) \geq \frac{5}{6} \)

**Remark 1** Note that this estimation is much stronger than Lemma 22 in Jin et al. (2019a), because we estimate \( q_b \) directly, instead of estimating at every step.

**Lemma 17** For all \( k \):

\[
P(||q_p(k)|| \geq \frac{\beta(k) M 2 \eta r}{3\sqrt{d}}) \geq \frac{2}{3}
\]

\[
P(||q_{ch}(k)|| \leq \frac{\beta(k) M \eta r}{3\sqrt{d}}) \geq \frac{2}{3}
\]

\[
P(||q_p(k) + q_{ch}(k)|| \geq \frac{\beta(k) M \eta r}{3\sqrt{d}}) \geq \frac{1}{3}
\]
Proof Since $q_p$ is Gaussian, $P(|X| \leq \lambda \sigma) \leq 2\lambda / \sqrt{2\pi} \leq \lambda$ for all normal random variable $X$. Let $\lambda = \frac{1}{3}$, $q_p \geq \frac{\beta(k)M\eta r}{3\sqrt{d}}$ with probability $2/3$.

As for the second one,

$$P(||q_{sg}(k)|| \leq c\beta(k)M\eta 2\sqrt{T}) \leq 1 - 2(d + 1)e^{-t}$$

let

$$t = \log 2(d + 1) + \log 3$$

$$c\beta(k)M2\eta s\sqrt{t} = \beta(k)M2\eta r \frac{c\sqrt{\log 2(d + 1) + \log 3}}{c_1 \sqrt{d}} \leq \frac{\beta(k)M2\eta r}{6\sqrt{d}}$$

so that the last inequality follows.

Now, we are ready to prove Theorem 10:

$$P(||q_p(k) + q_{sg}(k)|| \geq \frac{\beta(k)M\eta r}{3\sqrt{d}}) \geq \frac{1}{2},$$

so that with probability $1/6$, $||x(k)|| \geq ||q_p(k) + q_{sg}(k)||$$ - ||q_h(k)|| \geq \frac{\beta(k)M\eta r}{6\sqrt{d}}$

Appendix D. The Growth Rate of Polynomial $f(t_1, t_2)$

In this section, we will prove the last property of polynomial $f(t_1, t_2)$ in lemma 12. Firstly, in the synchronous case, the delay $T = 0$. We know Lyapunov’s First Theorem.

Lemma 18 Let $A$ to be a symmetric matrix, with maximum eigenvalue $\gamma > 0$. Suppose the updating rules of $x$ is

$$x(n + 1) = x(n) + Ax(n)$$

(55)

Then $x(n)$ is exponential unstable in the neighborhood of zero.

This can be proved by choosing a Lyapunov function. We consider $V(n) = x(n)^T Px(n)$, where $P$ is the Projection matrix to the subspace of the maximum eigenvalue. We can show that $V(n + 1) = (1 + \gamma)^2 V(n)$.

This method can be generalized to the asynchronous(time-delay) systems. There are many works on the stability of time-delay system by considering Lyapunov functional Kharitonov and Zhabko (2003); Gu (1999); Han (2005). Constructing a Lyapunov functional is generally difficult. One way to avoid this is to use Razumikhin-type theorem Bugong Xu (1994); Zhou (2018) and a stochastic version of Razumikhin theorems is proved in Mao (1999). There are few works on the instability of time-delay system. Haddock and Zhao (1996) used Razumikhin-type theorems to study the instability, and the work in Raffoul (2013) constructed a Lyapunov functional, then it was shown that when the delay is small enough, the system is exponential unstable.

D.1 A Rough Estimation

Here we give a much easier analysis for the linear time-delay system without using Lyapunov functional.
Lemma 19 Let \( A \) to be a symmetric matrix

\[
x(n + 1) = x(n) + \sum_{i}^{m} Ax(n - \tau_{n,i})
\]

\[
x(0) = I, x(t) = 0 \text{ for all } t < 0
\]

with \( 0 \leq \tau \leq T \), the largest eigenvalue of \( A \) is \( \gamma \). Let \( P \) be the projection matrix to the eigenvalues \( \gamma \). \( V(n) = x(n)^T P x(n) \). If \( m\gamma - m^{2}\gamma^{2} T^{2} = q > 0 \), we have \( V(n + 1) \geq (1 + q) V(n) \) for \( n \geq T \) and \( V(n + 1) \geq V(n) \) for \( n < T \).

Let \( P \) be the projection matrix of \( A \) to the subspace of maximum eigenvalue and \( V(n) = x(n)^T P x(n) \). We have

\[
V(n + 1) = x(n)^T P x(n) + 2x(n)^T P \left[ \sum_{i}^{m} Ax(n - \tau_{n,i}) \right]
\]

\[
+ \left[ \sum_{i}^{m} Ax(n - \tau_{n,i}) \right] ^{T} P \left[ \sum_{i}^{m} Ax(n - \tau_{n,i}) \right]
\]

\[
\geq V(n) + 2x(n)^T P \sum_{i}^{m} Ax(n - \tau_{n,i})
\]

Let \( i \) in the set \( \{1, 2, 3...m'(n)\} \) such that \( x(n - \tau_{n,i}) \neq 0 \) and if \( n \geq T, m'(n) = m \). For simplicity, we use \( m \) to represent \( m'(n) \). Using the fact \( \langle a, b \rangle = \frac{1}{2} (||a||^{2} + ||b||^{2} - ||a - b||^{2}) \), we have

\[
V(n + 1) = V(n) + mx(n)^T P Ax(n) + m \sum_{i}^{m} x(n - \tau_{n,i})^{T} P A \sum_{i}^{m} x(n - \tau_{n,i})
\]

\[
- m[x(n) - \frac{1}{m} \sum_{i}^{m} x(n - \tau_{n,i})] ^{T} P A[x(n) - \frac{1}{m} \sum_{i}^{m} x(n - \tau_{n,i})]
\]

\[
\geq V(n) + m\gamma V(n) + m \sum_{i}^{m} x(n - \tau_{n,i})^{T} P A \sum_{i}^{m} x(n - \tau_{n,i})
\]

\[
- m[x(n) - \frac{1}{m} \sum_{i}^{m} x(n - \tau_{n,i})] ^{T} P A[x(n) - \frac{1}{m} \sum_{i}^{m} x(n - \tau_{n,i})]
\]

\[
\geq V(n) + m\gamma V(n) + m \sum_{i}^{m} x(n - \tau_{n,i})^{T} P A \sum_{i}^{m} x(n - \tau_{n,i})
\]

\[
- m \sum_{t = n - \tau_{n,i}}^{n - 1} \sum_{i}^{m} Ax(t - \tau_{n,i}) ^{T} P A [\sum_{t = n - \tau_{n,i}}^{n - 1} \sum_{i}^{m} Ax(t - \tau_{n,i})]
\]

\[
\geq V(n) + m\gamma V(n) + m \sum_{i}^{m} x(n - \tau_{n,i})^{T} P \sum_{i}^{m} x(n - \tau_{n,i})
\]

\[
- m\gamma^{2} \sum_{t = n - \tau_{n,i}}^{n - 1} \sum_{i}^{m} (x(t - \tau_{n,i})) ^{T} P [\sum_{i}^{m} x(t - \tau_{n,i})]
\]

\[
\geq V(n) + m\gamma V(n)
\]

\[
+ m \gamma \sum_{i}^{m} x(n - \tau_{n,i})^{T} P \sum_{i}^{m} x(n - \tau_{n,i})
\]

\[
\geq V(n) + m\gamma V(n) + m \sum_{i}^{m} x(n - \tau_{n,i})^{T} P \sum_{i}^{m} x(n - \tau_{n,i})
\]

\[
\geq V(n) + m\gamma V(n)
\]

\[
+ m \gamma \sum_{i}^{m} x(n - \tau_{n,i})^{T} P \sum_{i}^{m} x(n - \tau_{n,i})
\]

25
\[-m^2\gamma^3 T \sum_{t=n-T}^{n-1} \sum_{i} V(t-\tau_{i,t})\]

Note that from Eq.(56), since $\gamma > 0$, $||P_x(n)||$ will keep increasing. $V(n) \geq V(n-\tau)$ for all $\tau \geq 0$. If $m\gamma - m^3\gamma^3 T^2 = q > 0$, we have $V(n+1) \geq (1+q)V(n)$, $||P_x(n+1)|| \geq \sqrt{1+q}||P_x(n)||$ if $n > T$.

D.2 Razumikhin-Lyapunov Method

$q = m\gamma - m^3\gamma^3 T^2$, even when $T = 0$, $q \leq m\gamma$. But we know that $V(n+1) = (1+2m\gamma+m^2\gamma^2)V(n)$, so that $1+q$ is a very rough estimation. Here, using Razumikhin technique, we give a new theorem to get a better estimation and it can go beyond $T \sim \frac{1}{T}$ cases ($m\gamma - m^3\gamma^3 T^2 > 0$), This theorem is inspired by the proof in Mao (1996, 1999).

**Theorem 20** (Razumikhin unboundness theorem for discrete system) For a discrete system, $V(n,x)$ is a positive value function $V(n,x)$. Let $\Omega$ be the space of discrete function $\phi$ from $\{-T,..,0,1,2,..\}$ to $\mathbb{R}$ and $\phi$ is a solution of discrete system equation. Suppose the following two conditions are satisfied

\[(a)V(t+1, \phi(t+1)) \geq q_m V(t, \phi(t)) \quad (\text{Bounded difference condition.})\]

\[(b)\text{If } V(t-\tau, \phi(t-\tau)) \geq (1+q)^{-T} \frac{q_m}{1+q} V(t, \phi(t)) \forall 0 \leq \tau \leq T\]

\[\text{then } V(t+1, \phi(t+1)) \geq (1+q) V(t, \phi(t)) \quad (\text{Razumikhin condition.})\]

Then for a $\phi \in \Omega$ such that for all $-T \leq t \leq 0$, $V(t) \geq pV(0)$ with $0 < p \leq 1$, we have $V(t) \geq (1+q)^t pV(0)$ for all $t > 0$.

**Proof**

Let $B(n) = (1+q)^{-n}V(n)$, in order to prove our theorem, we only need to show $B(n)$ have a lower bound.

$B(0) = V(0) \geq pV(0) \triangleq p'$. Assuming there is a $t > 0$ such that $B(t) = (1+q)^{-t}V(t) < p'$, select the minimum one as $t$, such that $B(k) \geq p'$ for all $k < t$, and $B(t) < p'$. Note that $V(t) \geq q_m V(t-1)$ so that $B(t) \geq p' \frac{q_m}{1+q}$. Then for all $k$ satisfying $t-T \leq k \leq t$

\[V(k) = (1+q)^k B(k) \geq (1+q)^k p' \frac{q_m}{1+q} = (1+q)^k -t (1+q)^{k-t} p' \frac{q_m}{1+q}\]

\[\leq (1+q)^k (1+q)^{t} \frac{q_m}{1+q} B(t) \geq (1+q)^{-T} \frac{q_m}{1+q} V(t)\]

So that we have $V(t+1) \geq (1+q) V(t)$, $B(t+1) \geq B(t) \geq p' \frac{q_m}{1+q}$. If $B(t+1) \geq p'$, $V(t+2) \geq q_m V(n+1)$, so that $B(t+2) \geq B(t+1) \geq p' \frac{q_m}{1+q}$. If $B(t+1) < p'$, $V(t+1-\tau) \geq (1+q)^{-T} \frac{q_m}{1+q} V(t+1)$, from the condition in (59), $B(t+2) \geq B(t+1) \geq p' \frac{q_m}{1+q}$. This process can continue, such that $B(t) \geq p' \frac{q_m}{1+q}$ for any $t$. Our claim follows. \[\square\]

Using Theorem 20 to (56), we set $V(n,x) = ||P_x(n)||$. Supposing $x(0) = I, x(-t) = 0$ for all $t > 0$, $||P_x(t)|| = e_T^1 x(t)$ and $q_m = 1$. We have the following corollary:
**Corollary 21** Let \( f(k,t) \) be the polynomial in lemma 12, we have \( f(k,t+1) \geq (1+q)f(k,t) \) if \( t-k \geq T \), where \( q = M\eta \gamma e^{-(T+1)M\eta \gamma} \).

**Proof**

Condition (59) has the form

\[
1 + M\eta \gamma (1+q)^{-T-1} \geq 1 + q
\]

which is equal to

\[
q(1+q)^{T+1} \leq M\eta \gamma
\]

(61)

(62)

It is easy to see that for all \( T > 0 \), since \( M\gamma > 0 \), there is a \( q > 0 \) satisfying Razumikhin condition (59), so that the system is exponential unstable.

In the case the time delay is \( O(1/M\eta \gamma) \), we can easily estimate the value of \( q \). Let \( T + 1 = \frac{T}{M\eta \gamma} \).

\[
(1+q)^{T+1} = (1+q)^{\frac{T}{M\eta \gamma}} = (1+q)^{\frac{1}{2}} \frac{M\eta \gamma}{2}\gamma
\]

Since \( 0 < q \leq M\eta \gamma \)

\[
(1+q)^{T+1} = (1+q)^{\frac{1}{2}} \frac{M\eta \gamma}{2}\gamma \leq e^{\frac{M\eta \gamma}{2}\gamma} \leq e^{f}
\]

so that \( q \geq e^{-fM\eta \gamma}, \|Px(n+1)\| \geq (1+e^{-fM\eta \gamma})\|Px(n)\| \) if \( n > T \).

References

Alekh Agarwal and John C Duchi. Distributed delayed stochastic optimization. In J. Shawe-Taylor, R. S. Zemel, P. L. Bartlett, F. Pereira, and K. Q. Weinberger, editors, *Advances in Neural Information Processing Systems 24*, pages 873–881. Curran Associates, Inc., 2011.

Yongqing Liu Bugong Xu. An improved Razumikhin-type theorem and its applications. *IEEE Transactions on Automatic Control*, 39(3):429–430, 1994.

Fan R K Chung and Lincoln Lu. Concentration inequalities and martingale inequalities: A survey. *Internet Mathematics*, 3(1):79–127, 2006.

Christopher M De Sa, Ce Zhang, Kunle Olukotun, Christopher R, and Christopher R. Taming the wild: A unified analysis of hogwild-style algorithms. In C. Cortes, N. D. Lawrence, D. D. Lee, M. Sugiyama, and R. Garnett, editors, *Advances in Neural Information Processing Systems 28*, pages 2674–2682. Curran Associates, Inc., 2015.

Jeffrey Dean, Greg Corrado, Rajat Monga, Kai Chen, Matthieu Devin, Mark Mao, Marc Aurelio Ranzato, Andrew Senior, Paul Tucker, Ke Yang, Quoc V. Le, and Andrew Y. Ng. Large scale distributed deep networks. In F. Pereira, C. J. C. Burges, L. Bottou, and K. Q. Weinberger, editors, *Advances in Neural Information Processing Systems 25*, pages 1223–1231. Curran Associates, Inc., 2012.

Simon Du and Jason Lee. On the power of over-parametrization in neural networks with quadratic activation. In Jennifer Dy and Andreas Krause, editors, *Proceedings of the 35th International Conference on Machine Learning*, volume 80 of *Proceedings of Machine Learning Research*, pages 1329–1338, Stockholmsm?sson, Stockholm Sweden, 10–15 Jul 2018. PMLR.

27
Simon S Du, Chi Jin, Jason D Lee, Michael I Jordan, Aarti Singh, and Barnabas Poczos. Gradient
descent can take exponential time to escape saddle points. In I. Guyon, U. V. Luxburg, S. Bengio,
H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, editors, Advances in Neural Information
Processing Systems 30, pages 1067–1077. Curran Associates, Inc., 2017.

Rong Ge, Furong Huang, Chi Jin, and Yang Yuan. Escaping from saddle points — online stochastic
gradient for tensor decomposition. In Peter Grunwald, Elad Hazan, and Satyen Kale, editors,
Proceedings of The 28th Conference on Learning Theory, volume 40 of Proceedings of Machine
Learning Research, pages 797–842, Paris, France, 03–06 Jul 2015. PMLR.

Rong Ge, Jason D Lee, and Tengyu Ma. Matrix completion has no spurious local minimum. In
D. D. Lee, M. Sugiyama, U. V. Luxburg, I. Guyon, and R. Garnett, editors, Advances in Neural
Information Processing Systems 29, pages 2973–2981. Curran Associates, Inc., 2016.

Sidney Golden. Lower bounds for the helmholtz function. Phys Rev, 137(4B):B1127–B1128, 1965.

Keqin Gu. Discretized lyapunov functional for uncertain systems with multiple time-delay. International Journal of Control, 72(16):1436–1445, 1999.

John R Haddock and Younhee Ko. Liapunov-razumikhin functions and an instability theorem for
autonomous functional differential equations with finite delay. Rocky Mountain Journal of Mathematics, 25(1):261–267, 1995.

John R Haddock and Jiaxiang Zhao. Instability for autonomous and periodic functional differential
equations with finite delay. Funkcialaj Ekvacioj, 39(3):553–570, 1996.

Jack K Hale. Sufficient conditions for stability and instability of autonomous functional-differential
equations. Journal of Differential Equations, 1(4):452–482, 1965.

Qing Long Han. On stability of linear neutral systems with mixed time delays: A discretized
lyapunov functional approach. Automatica, 41(7):1209–1218, 2005.

Robert Hannah and Wotao Yin. On unbounded delays in asynchronous parallel fixed-point algo-
rithms. Journal of Scientific Computing, 76(1):299–326, 2018.

Chi Jin, Rong Ge, Praneeth Netrapalli, Sham M Kakade, and Michael I Jordan. How to escape
saddle points efficiently. international conference on machine learning, pages 1724–1732, 2017.

Chi Jin, Praneeth Netrapalli, Rong Ge, Sham M Kakade, and Michael I Jordan. Stochastic gradient
descent escapes saddle points efficiently. arXiv: Learning, 2019a.

Chi Jin, Praneeth Netrapalli, Rong Ge, Sham M Kakade, and Michael I Jordan. A short note on
concentration inequalities for random vectors with subgaussian norm. arXiv: Probability, 2019b.

Kenji Kawaguchi. Deep learning without poor local minima. In D. D. Lee, M. Sugiyama, U. V.
Luxburg, I. Guyon, and R. Garnett, editors, Advances in Neural Information Processing Systems
29, pages 586–594. Curran Associates, Inc., 2016.

V. L. Kharitonov and A.P. Zhabko. Lyapunov krasovskii approach to the robust stability analysis of
time-delay systems. Automatica, 39(1):15–20, 2003.
Alex Krizhevsky, Ilya Sutskever, and Geoffrey E Hinton. ImageNet classification with deep convolutional neural networks. *neural information processing systems*, 141(5):1097–1105, 2012.

Jason D. Lee, Max Simchowitz, Michael I. Jordan, and Benjamin Recht. Gradient descent only converges to minimizers. In Vitaly Feldman, Alexander Rakhlin, and Ohad Shamir, editors, *29th Annual Conference on Learning Theory*, volume 49 of *Proceedings of Machine Learning Research*, pages 1246–1257, Columbia University, New York, New York, USA, 23–26 Jun 2016. PMLR.

Mu Li, David G Andersen, Alexander J Smola, and Kai Yu. Communication efficient distributed machine learning with the parameter server. In Z. Ghahramani, M. Welling, C. Cortes, N. D. Lawrence, and K. Q. Weinberger, editors, *Advances in Neural Information Processing Systems 27*, pages 19–27. Curran Associates, Inc., 2014.

Xiangru Lian, Yijun Huang, Yuncheng Li, and Liu Ji. Asynchronous parallel stochastic gradient for nonconvex optimization. *neural information processing systems*, 2015.

Elliott H Lieb. Convex trace functions and the wigner-yanase-dyson conjecture. *Advances in Mathematics*, 11(3):267–288, 1973.

Xuerong Mao. Razumikhin-type theorems on exponential stability of stochastic functional differential equations. *Stochastic Processes and their Applications*, 65(2):233–250, 1996.

Xuerong Mao. Razumikhin-type theorems on exponential stability of neutral stochastic differential equations. *Chinese Science Bulletin*, 44(24):2225–2228, 1999.

Youssef Raffoul. Inequalities that lead to exponential stability and instability in delay difference equations. *Journal of Inequalities in Pure & Applied Mathematics*, 10(10), 2013.

Benjamin Recht, Christopher Re, Stephen J Wright, and Feng Niu. Hogwild: A lock-free approach to parallelizing stochastic gradient descent. *neural information processing systems*, pages 693–701, 2011.

N.O. Sedova. Lyapunovrazumikhin pairs in the instability problem for infinite delay equations. *Nonlinear Analysis: Theory, Methods & Applications*, 73(7):2324 – 2333, 2010. ISSN 0362-546X. doi: https://doi.org/10.1016/j.na.2010.06.027.

David Sutter, Mario Berta, and Marco Tomamichel. Multivariate trace inequalities. *Communications in Mathematical Physics*, 352(1):1–22, 2017.

Hyokun Yun, Hsiangfu Yu, Choji Hsieh, S V N Vishwanathan, and Inderjit S Dhillon. Nomad: non-locking, stochastic multi-machine algorithm for asynchronous and decentralized matrix completion. *Very large data bases*, 7(11):975–986, 2014.

Sixin Zhang, Anna E Choromanska, and Yann LeCun. Deep learning with elastic averaging sgd. In C. Cortes, N. D. Lawrence, D. D. Lee, M. Sugiyama, and R. Garnett, editors, *Advances in Neural Information Processing Systems 28*, pages 685–693. Curran Associates, Inc., 2015.

Xin Zhang, Jia Liu, and Zhengyuan Zhu. Taming convergence for asynchronous stochastic gradient descent with unbounded delay in non-convex learning. *arXiv: Learning*, 2018.
Yuchen Zhang, Percy Liang, and Moses Charikar. A hitting time analysis of stochastic gradient
langevin dynamics. *Conference on Learning Theory*, 2017.

Bin Zhou. Improved razumikhin and krasovskii approaches for discrete-time time-varying time-
delay systems. *Automatica*, 91:256–269, 2018.

Zhihui Zhu, Qiuwei Li, Gongguo Tang, and Michael B. Wakin. Global optimality in low-rank
matrix optimization. *IEEE Transactions on Signal Processing*, PP(99):1–1, 2018.