EXOTIC ATOM FORMATION IN EXTERNAL ELECTRIC FIELDS

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Abstract
Validity of the coalescence model of exotic atom formation in the ion collisions is studied. An estimate is given for the rates of P state production in the electric fields of the colliding ions. Possibilities of quantum beat effects are indicated.

1 Introduction
We study possibilities of production of pairs composed of an elementary particle and its antiparticle in the atomic bound states. The particles may be produced in proton or ion collisions and the simplest cases of practical interest are the lightest pairs $e^+ e^-$ (positonium), $\pi^+ \pi^-$ (pionium) or $K^+ K^-$ (kaonium). These atoms are large systems when compared to the regions where the pairs are created. Thus, amplitudes for the formation of atoms in final states are proportional to the atomic wave functions at the origin and the pair creation amplitudes. This approximation known as the coalescence model yields rates of mesic atom formation high enough to allow experiments. Thus the search for pionium has been undertaken or contemplated at CELSIUS, CERN, COSY, Indiana UCL, [1,2,3,4,5].

The main physical motivation to study pionium (or kaonium) is a measurement of its lifetime. In the S states the latter may be expressed by the appropriate particle-antiparticle scattering length [6], a quantity of great theoretical interest. The main difficulty of these experiments is that the decay of the 1S state, due to strong interactions, is very fast (about $10^{-13}$ sec for pionium). In order to find the lifetime one has to compare the main $\pi^0 \pi^0$ decay mode with very rare two gamma decay mode. In principle, the atom’s life may be prolonged by orders of magnitude if it is formed in the P states that decay slowly to the ground state by the X-ray emission [7]. This would open new experimental possibilities and a new scale given by the calculable X-ray decay rate. However, a direct P wave pair production with the subsequent P state atom formation in a coalescent manner is very unlikely [6]. In this paper another mechanism for the P wave atom production is considered. We assume the pair to be produced at short ranges in an S wave. Subsequent formation of the P state atoms is achieved by a Stark mixing in the external electric field of the colliding particles. A related problem discussed here is the rate of destruction of the S state atoms formed via the coalescence mechanism and destroyed by the external electric field. The second question is relevant to the studies of atomic production in the proton-nucleus collisions at high energies [4].
The atom formation is discussed here as a problem of a pair created in an external static Coulomb field. In the full complication it becomes a formidable problem of three body Coulomb interactions in the continuum. Only part of it has been solved with satisfactory precision. In particular we consider only atoms moving with high velocities. In a kind of quasi-classical solution we find the rate of destruction of the 1S atoms in the external field and the rate of production of the P wave states. The former is significant the latter is rather small. An upper limit for these quantities is given with a help of completeness approximation. A system of successive approximations is shown to converge to this limit.

This model shows that the participation of different atomic states depends strongly on the final momentum of created pairs. If only a limited number of final states is allowed the atomic system displays effects of quantum beat. The atomic pair in the external, polarising, electric field forms quasi-stationary Stark type combinations. As a consequence the probability of production of each atomic state oscillates with respect to the final atom momentum. This phenomenon is rather unstable against the approximations. We believe it deserves more extended study as it may be related to the production of correlated $e^+ e^-$ pairs seen at GSI [8], in the heavy ion collisions.

2 The model of atomic formation

This section presents basic assumptions of this model. The equations that describe the atom formation are derived and simplified solutions are found. The latter allow to form an intuitive picture of the formation mechanism. An approximation scheme for the high energy region is derived.

In cases of practical interest the atomic Bohr radii, $B$, are much larger than radii of the interaction region where the pair is created. This leads to the expectation that the atom formation amplitude $A(E)$ is proportional to the wave function at the origin:

$$A(E) = F\psi(\vec{\rho} = 0, \vec{r} = 0)$$

(1)

where $F$ is an amplitude for the pair creation.

This paper is devoted entirely to the determination of the final state wave function $\psi$. The latter depends on two coordinates: $\vec{\rho}$ that describes the center of mass of the pair $\pi^+ \pi^-$, and $\vec{r}$ that is the relative coordinate in the center of mass. We assume the adiabatic situation i.e. fixed ions and fast mesons. Then the external polarization potential $V$ of colliding nuclei acting on the pair becomes:

$$V = Z\alpha\left(\frac{1}{|\vec{\rho} - \vec{r}|} - \frac{1}{|\vec{\rho} + \vec{r}|}\right)$$

(2)

where $Z$ is the total charge of the nuclei.
Let us start with the Lippman-Schwinger integral equation describing non-relativistic dynamical formation of atomic states of the pair \( \pi^+\pi^- \). We have,

\[
\psi = \psi_0 + \int GV \psi d^3\vec{r} d^3\vec{r}'
\]

and the boundary conditions, corresponding to the ingoing waves, are to be specified later. In this equation the "internal" interactions within the pair have been solved and described by the Green’s function \( G \). At first only the discrete terms of this Green’s function are taken into account.

\[
G \equiv G(\vec{r}, \vec{r}', \vec{\rho}, \vec{\rho}') \approx \sum_{nlm} | \varphi_{nlm} \rangle < \varphi_{nlm} | G(\vec{\rho}, \vec{\rho}', E - E_n),
\]

where \( E \) is the total energy of the pair. We are interested in the high kinetic energies (MeV region) and neglect the atomic binding energies \( E_n \) (KeV region). Then, \( E = \frac{p^2}{2m} \), where \( m \) is the particle mass. In addition, we assume that the pair is created with zero total angular momentum. Hence, the wave function can be written in the following form:

\[
\psi = \sum_{nlm} \frac{Y_{lm}(\vec{r}) Y_{l'm'}(\vec{\rho})}{\sqrt{2l + 1}} \psi_{nl}
\]

and the radial wave functions are assumed in the form

\[
\psi_{nl} = \varphi_{nl}(r) \phi_{nl}(\vec{\rho})
\]

\[
\phi_{nl} = j_l (p \rho) a_{nl}(\vec{\rho})
\]

where the unknown functions \( a_{nl}(\vec{\rho}) \) are to be found. The form of this expansion although general has been chosen for the high-energy region which allows simple approximate solutions for the functions \( a \). After eliminating the internal degrees of freedom by integration over \( r \) one obtains an infinite set of second order linear differential equations for \( a_{nl} \) that involve effective potentials in the "external" variable \( \vec{\rho} \)

\[
V_{nl'l'}(\vec{\rho}) = \sum_{\lambda=1,3...} \frac{< \varphi_{nl} | V_{\lambda} | \varphi_{n'l'} >}{\sqrt{(2l + 1)(2l' + 1)}} \Delta(l, l', \lambda)
\]

where

\[
\Delta(l, l', \lambda) = (2l + 1)(2l' + 1)(2\lambda + 1) \left( \begin{array}{ccc} l & l' & \lambda \\ 0 & 0 & 0 \end{array} \right)^2
\]

and \( \left( \begin{array}{ccc} l & l' & \lambda \\ 0 & 0 & 0 \end{array} \right)^2 \) is the Wigner symbol.

These potentials are related to the external Coulomb potential \( V \) by the multipole expansion:
V(\vec{r}, \vec{\varrho}) = \sum_{\lambda m} V_{\lambda m}(\vec{r}) Y_{\lambda m}(\vec{\varrho}) \tag{10}

with \( V_{\lambda} \) the radial part of the \( \lambda \)-pole polarisation potential given by

\[ V_{\lambda}(r, \varrho) = \frac{4M\alpha}{2\lambda + 1} \frac{x^{\lambda}}{x^{\lambda + 1}}, \ \lambda = 1, 3, ... \tag{11} \]

where \( x \) is \( \varrho \) or \( \frac{r}{\alpha} \).

The dominant dipole effective potentials are long-ranged of the \( \varrho^{-2} \) asymptotics. At distances close to the atomic radii these potentials fall down sharply and exhibit zeros of high order at the origin. Few examples are given in Appendix A. The long range of these potentials allows some vital approximations in the high energy region. We refer to Appendix B where the details are discussed in the language of integral equations while some numerical studies are performed in the next section. One effect of the range is some smoothness of the functions \( a \) and this allows for differential equations of the first order. Another special feature is related both to the long-range and the angular momentum carried by the effective potential. This allows for the expansion (6,7) where, one would otherwise expect also some Hankel functions corresponding to ingoing (outgoing) waves. The first order equation discussed in Appendix B is

\[ a'_{nl} = \sum_{n'l'} \frac{Z}{2p} V_{nln'l'} a_{n'l'} (-1)^l \tag{12} \]

where the length is expressed in the Bohr radius units. These equations for \( a_{nl}(\varrho) \) can be written in the following matrix form

\[ a' = Ua \tag{13} \]

or

\[ ia' = iUa \tag{14} \]

where \( U \) is an antisymmetric matrix and respectively \( iU \) is hermitian. The boundary conditions can be formulated via integral equation

\[ a = a_0 + \int_{\varrho}^{\infty} Ua \ d\varrho \tag{15} \]

which is the set of integral equations derived in Appendix B. For example for the incoming wave corresponding with the state \( 1S \)

\[ a_0 = (1, 0, 0, ...) \tag{16} \]

and the vector \( a \) represents following discrete states

\[ a = (1S, 2S, 2P, 3S, 3P, 3D, ...) \tag{17} \]
3 Illustrative example - two discrete states

To illustrate the formalism introduced above let us consider an idealised atomic system of only two: 2S and 2P states. It is easy to find, in this case, the exact as well as the approximate solutions for the integral or differential equations for the final atomic state of the pair. If the incident wave is formed with the 2S state one obtains from eq(12) or eq(15):

$$a_{20}(\varrho) = \cos\left(\frac{Z}{2p} \int_{\varrho}^{\infty} V_{2120}\right)$$ (18)

which generates the S wave amplitude of the pair with respect to the formation region. Now the 2S atom formation probability $pr(2S)$ is given by the square of the amplitude at the origin

$$pr(2S) = A \cos^2\left(\frac{Z}{2p} \int_{\varrho}^{\infty} V_{2120}\right)$$ (19)

where $A$ is a constant involving pair production amplitude and some phase space integrations, see 6 for more details.

In the same way one can solve for an amplitude $a_{20}$ in the case of the incident 2P state

$$a_{20}(\varrho) = \sin\left(\frac{Z}{2p} \int_{\varrho}^{\infty} V_{2120}\right)$$ (20)

and the probability of the 2P atomic state formation is

$$pr(2P) = A \sin^2\left(\frac{Z}{2p} \int_{\varrho}^{\infty} V_{2120}\right)$$ (21)

The total probability of the atom formation is then energy independent

$$pr = pr(2S) + pr(2P) = A$$ (22)

and equals in the high energy limit to the 2S state formation probability given by the coalescence model. In figs 1, 2 parts of the plots of $pr(2S)$ and $pr(2P)$ are presented for a special case of positronium formed in the strong field (Z=170) of heavy ions.
Let us notice several essential points.

1) The oscillations of the atomic production probability reflect quantum beating of this system. In the strong external electric field there are two diagonal combinations \( S(i)S(e) + P(i)P(e) \) and \( S(i)S(e) - P(i)P(e) \) where \( i \) denotes the internal atomic state and \( e \) the state of the external angular momentum.
of the atom. These two components beat in time and in space when the atom propagates through the polarising electric field.

2) This simple case may be solved exactly. We find that our high energy approximation works well for kinetic energies higher than the barrier inherent in the effective potential. In practice it covers the region of three last peaks in the formation probability.

3) The beating effects persist at kinetic energies higher than the atomic binding energies by orders of magnitude. The Coulomb potential and the dominant dipole polarising potential have no inherent scale parameter. This parameter is introduced via the atomic matrix elements (8) that determine the effective V. Thus the energy scale in fig 1 is determined by the dimensionless parameter Z/pB. It is much larger than the scale of the atomic binding energies.

The question to be answered is to what extent this beating of the Stark mixed atomic state persists in complete calculations. Such calculations should allow decays of the incident state to higher states and to the continuum. We study it by expanding the basis of the discrete levels allowed and by an approximate treatment of the continuum. The latter is discussed in the next section.

4 Approximate treatment of continuum

In appendix B we estimate the continuum part $\psi^c$ of the wave function $\psi$. This part is related to the continuous spectrum involved in the Coulomb Green function $G^c$ entering the Lippman-Shwinger eq(3). We argue that at high energies there exist an approximate relation

$$<\varphi_{nlm} | V G^c V | \varphi_{n'l'm'}> \approx <\varphi_{nlm} | V \delta(\vec{r} - \vec{r'})G^p V | \varphi_{n'l'm'}>$$

Then we expand the delta function into a system of eigenstates

$$\delta(\vec{r} - \vec{r'}) = \sum_{nlm} | \psi_{nlm} > < \psi_{nlm} | + \text{continuous part}$$

and neglect the continuous terms to obtain the following approximation

$$<\varphi_{nlm} | V G^c V | \varphi_{n'l'm'}> \approx <\varphi_{nlm} | V G^d V | \varphi_{n'l'm'}>$$

where $G^d$ is the discrete part of Coulomb Green’s function. In this way, we expand the continuous piece of the wave function $\psi^c$ into discrete states in the same way as it was done in eqs (5,6,7) with the $\psi^d$. The coefficients of this expansion are named $b$. Now defining a new vector $\tilde{a}$

$$\tilde{a} = (a, b)$$

we obtain the equation

$$\tilde{a'} = \tilde{U}\tilde{a}$$
where the new potential $\tilde{U}$ is defined as follows:

$$
\tilde{U} = \begin{pmatrix} U & U \\ U & 0 \end{pmatrix}
$$

(28)

The equation for $\tilde{a}$ is soluble with simple Runge-Kutta methods. This procedure doubles the dimension of the previously described set of differential equations. Its effect on the discrete part $\psi^d$ is equivalent to an optical potential contribution leading to the equation

$$
\psi^d = \psi^d_0 + G^d V \psi^d + G^d V G^d V \psi^d
$$

(29)

It turns out that contributions from $\psi^c$ at the origin vanish quickly with increasing energy and for intermediate energies stop playing an important role.

5 Calculations and Conclusions

This calculation describes formation of atomic states of a pair $e^+ e^-, \pi^+ \pi^-$ etc. in the electric field of colliding nuclei that create the pair. It is shown that the rate of creation of pairs in a given atomic state depends strongly on the final energy of the pair. This result depends on one scale parameter $\frac{2M_Z^2}{\alpha}$ so it can be rescaled for different masses of the pair, total atomic numbers of colliding nuclei $Z$ and energies. The probability to form the 1S state changes rather smoothly with energy from negligible values at low energies to the coalescence model result at high energies. A reliable calculation of the formation rates in higher atomic states requires a large number of atomic states in the basis of expansion. We have solved the system of differential equations up to $K=21$ states. Good convergence of probabilities with respect to $K$ have been obtained only for the first three states. Within higher shells some traces of the beating effects are still visible although these are rather unstable with respect to $K$. This question requires further studies and possibly better approximations as for large $K$ the effective potentials create numerical difficulties.

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6 Appendix A - Wave functions and potentials

The first three wave functions and effective potentials are given below. The atomic units (\(B = 1\)) are used. The radial wave functions:

\[
\varphi_{10} = 2e^{-r}
\]
\[
\varphi_{20} = \frac{(2 - r)}{2\sqrt{2}} e^{-\frac{r}{2}}
\]
\[
\varphi_{21} = \frac{r}{2\sqrt{6}} e^{-\frac{r}{2}}
\]

generate, under the dipole external field, the following transition potentials:

\[
V_{2110}(x) = (5.959 + (-5.959 - 17.88x - 26.82x^2 - 20.11x^3)e^{-3x}
\]
\[
V_{2120}(x) = (-24. + (24. + 48.x + 48.x^2 + 32.x^3 + 16.x^4)e^{-2x}
\]
\[
V_{3021}(x) = (4.33 + (-4.33 - 7.22x - 6.02x^2 - 3.17x^3 - 1.11x^4 - 3.71x^5)e^{-5x}
\]

All other transition potentials defined in section 1 can be found in explicit form as elementary functions.

7 Appendix B - High energy approximations

The wave function \(\psi\) can be divided into two parts - discrete and continuous:

\[
\psi = \psi^d + \psi^c
\]

This separation follows the same separation of the Coulomb Green function.

\[
G = G^d + G^c
\]

Where the first part contains the atomic and the second part the continuous internal states of the pair. Now Lippman-Schwinger equation is equivalent to coupled equations for the two parts of \(\psi\):

\[
\psi^d = \psi^d_0 + G^d V \psi^d + G^d V \psi^c
\]
\[
\psi^c = G^c V \psi^d + G^c V \psi^c
\]

In this calculation we are interested in atomic states, the continuum will be described in a simplified way similar to the optical potential method. Thus we
neglect the $G^cV\psi^c$ term that changes the continuum part of the propagator. Now

$$\psi^c = G^cV\psi^d$$

is substituted into equation for $\psi^c$ that generates the optical model equation

$$\psi^d = \psi^d_0 + G^dV\psi^d + G^dVG^cV\psi^d$$

For high energies, much larger than the atomic binding energies, one has an approximate relation for the matrix element

$$<\varphi_{nlm} | VG^cV | \varphi_{n'l'm'}> \approx <\varphi_{nlm} | VG_0V | \varphi_{n'l'm'}>$$

so in the above equations we interchange $G^c$ with $G_0$. The discrete part of propagator has the following form

$$G^d = G^d(\vec{r},\vec{g},\vec{r'},\vec{g'}) = \sum_{nlm} | \varphi_{nlm} > < \varphi_{nlm} | G_p(\vec{g},\vec{g'}) | E - E_n),$$

where for high energies $E$ we neglect the binding energies $E_n$. In a similar fashion for small relative momenta of the pair the Green function $G_0$ will be used in the following approximate form

$$G_0(\vec{r},\vec{g},\vec{r'},\vec{g'}) = \delta(\vec{r} - \vec{r'})G_p(\vec{g},\vec{g'})$$

The external potential changes the internal angular momentum. This is crucial for our problem and allows an approximation that we discuss now. Let us go over to the angular momentum representation and follow the expansion given in eqs(5) and (6). The propagator is also expanded into spherical harmonics

$$G_p(\vec{g},\vec{g'}) = \sum_{lm} g_l(\vec{g},\vec{g'}) Y^*_{lm}(\vec{g}) Y_{lm}(\vec{g'})$$

with

$$g_l(\vec{g},\vec{g'}) = -ipj_l(p\vartheta <) h_l(\vartheta >)$$

Now we can obtain the system of integral equations for the radial functions $\phi$ of eg(6)

$$\phi_{nl}(\vartheta) = j_{l_0}(p\vartheta)\delta_{n_0}\delta_{l_0} + Z \int g_l(\vartheta,\vartheta') \sum_{n'l'} V_{nln'l'}\phi_{nl}(\vartheta')$$

$$+ Z^2 \int g_l(\vartheta,\vartheta') \sum_{n'l'a} W_{nln'l'}^{\alpha}(\vartheta',\vartheta'') g_{\alpha}(\vartheta',\vartheta'') \phi_{n'l'}(\vartheta'')$$

The transition potential $V_{nln'l'}$ is defined in the following way:

$$V_{nln'l'}(\vartheta) = \sum_{\lambda=1,3...} \frac{<\varphi_{nl} | V_\lambda | \varphi_{n'l'}>} {\sqrt{(2l+1)(2l'+1)}} Delta(l, l', \lambda)$$
and \( W^{\alpha}_{nln' l'} \) is given by the formula

\[
W^{\alpha}_{nln' l'}(\varphi, \varphi') = \sum_{\lambda, \lambda'=1,3...} <\varphi | V_\lambda(\varphi, r) V_\lambda'(\varphi', r) | \varphi' > \frac{\Delta(l, \alpha, \lambda) \Delta(l', \alpha', \lambda')}{(2\alpha + 1) \sqrt{(2l + 1)(2l' + 1)}} (48)
\]

Iterations of these integral equations involve products of the Bessel and Neumann functions. These are integrated with the effective potentials \( V \) which are predominantly long-ranged. For high energies i.e. for high momenta \( p \) these functions oscillate rapidly and one obtains approximate relations for mean values:

\[
<j_l(x) j_{l'}(x) > \approx \frac{1}{2x^2} \cos((l - l') \frac{\pi}{2})
\]

\[
<n_l(x) n_{l'}(x) > \approx \frac{1}{2x^2} \cos((l - l') \frac{\pi}{2})
\]

\[
<j_l(x) n_{l'}(x) > \approx \frac{1}{2x^2} \sin((l - l') \frac{\pi}{2})
\]

where

\[ x = p \varphi \] (52)

and symbols \(< f(\varphi) >\) denote the mean value of an oscillating function \( f(\varphi) \) weighted by \( V \). We repeat again, that the success of these approximation is related to the long range of the dipole polarising potentials \( V \). When we write \( \varphi_{nl} \) in the standard form

\[ \varphi_{nl} = j_l(p \varphi) a_{nl}(\varphi) + h_l(p \varphi) b_{nl}(\varphi) \] (53)

and apply the expressions for mean values of Bessel functions we obtain

\[ a_{nl}(\varphi) = \delta_{nn_0} \delta_{ll_0} + \int_{0}^{\infty} \frac{1}{2p} \sin((l - l') \frac{\pi}{2}) \sum_{n' l'} V_{nln' l' n l} \varphi d\varphi' \]

\[ + \int_{0}^{\infty} \frac{1}{4p^2} \sin((\alpha - l) \frac{\pi}{2}) \sin((l' - \alpha) \frac{\pi}{2}) \sum_{n' l' \alpha} W^{\alpha}_{nln' l' l'}(\varphi', \varphi'') n_{l'}(\varphi'') \varphi d\varphi' d\varphi'' \] (54)

The equation for \( b_{nl} \) has the similar form:

\[ b_{nl}(\varphi) = \int_{0}^{\infty} \frac{1}{2p} \sin((l - l') \frac{\pi}{2}) \sum_{n' l'} V_{nln' l' n l} \varphi d\varphi' \]

\[ + \int_{0}^{\infty} \frac{1}{4p^2} \sin((\alpha - l) \frac{\pi}{2}) \sin((l' - \alpha) \frac{\pi}{2}) \sum_{n' l' \alpha} W^{\alpha}_{nln' l' l'}(\varphi', \varphi'') b_{n' l'}(\varphi'') \varphi d\varphi' d\varphi'' \] (55)

but because there is no initial term \( \delta_{nn_0} \delta_{ll_0} \) the solution of the above equation is trivial - i.e. \( b_{nl} = 0 \). The equation for \( a_{nl} \) is the gateway for the high energy approximation.