Cosmological Einstein-Yang-Mills equations *†

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Abstract

We use a systematic construction method for invariant connections on homogeneous
spaces to find the Einstein-$SU(n)$-Yang-Mills equations for Friedmann-Robertson-Walker
and locally rotationally symmetric homogeneous cosmologies. These connections de-
pend on the choice of a homomorphism from the isotropy group into the gauge group.
We consider here the cases of the gauge group $SU(n)$ and $SO(n)$ where these homo-
morphisms correspond to unitary or orthogonal representations of the isotropy group.
For some of the simpler cases the full system of the evolution equations are derived, for
others we only determine the number of dynamical variables that remain after some
mild fixing of the gauge.

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1. Introduction

There has been extensive work on Einstein-Yang-Mills (EYM) cosmological models in the last decade. This work was partly motivated by the successes of inflationary models driven by scalar fields in solving flatness and (to large extent) horizon problems in cosmology. The interest in inflationary models driven by fields other than scalar fields is a consequence of less attractive features of the former [1]. Mini-superspace EYM cosmology is a natural extension of a non-perturbative treatment of self-gravitating scalar fields. It has been realized that despite a large phase space associated with seemingly redundant extra gauge degrees of freedom, there already exists a systematic mathematical method (based on Wang’s theorem [2]) for the construction of invariant connections over homogeneous spaces in the same spirit as that of Kaluza-Klein theories. As will be seen in section 2 such invariant connections are related to the representation theory of compact Lie algebras. For some of the most easily constructed cases of $SO(n)$-YM fields the solutions were obtained on closed Friedman-Robertson-Walker (FRW) cosmologies [3],[4],[5]. $SU(2)$-YM fields on open FRW cosmologies have also been of some interest [6],[7]. In this particular representation there is one degree of freedom associated with the YM fields.

Conformal invariance of YM field equations (due to the fact that they are zero-rest-mass fields) results for the homogeneous and isotropic case in a decoupling of the gravitational and YM degrees of freedom. The energy momentum tensor is that of a radiation perfect fluid and the geometry is that of a Tolman universe.

Despite the fact that it is known that the construction of invariant YM connections could be generalized — at least in principle — to other compact gauge groups and cosmological models with compact and non-compact spatial sections, a systematic attempt to study models based on more complicated representations in FRW and anisotropic homogeneous cosmologies has not been conducted.

In the present article we derive the EYM equations for $SU(n)$-FRW and $SU(n)$ locally rotationally symmetric (LRS) cosmologies.

Section 2 is an exposition of a general but rather explicit construction of the Riemann and YM curvatures based on the theory of connections invariant under symmetry groups that act transitively on the base manifold. It turns out that the resulting purely algebraic Yang-Mills
equations do not require any explicit choice of gauge. Such space-time homogeneous models are not considered to be realistic physically and we make no attempt in this paper to find any exact solutions.

In section 3 we derive the EYM equations for spatially homogeneous cosmological models. The result is a system of ordinary differential equations where again the YM gauge needs to be fixed only mildly, for example, by setting the temporal component of the potential to zero. The spatially homogeneous and isotropic models are discussed in section 4. Although the space-time geometry is completely determined independently of the YM fields, the latter satisfy in general some complicated coupled system of evolution equations. We derive here a few general facts for arbitrary gauge groups and some more explicit equations corresponding to different possible YM fields for the gauge groups $SU(n)$ and $SO(n)$.

Finally, in section 5 we consider, in a unified way, all LRS cosmological models with a $SU(n)$ Yang-Mills source. In such models, after solving for the constraints, there are $2(n-1)$ degrees of freedom associated with the YM fields. Here we just concentrate on what we consider the simplest YM-connections that contain a 'magnetic' part and derive the full evolution equations of the EYM-system. An analysis of the solutions of this quite complicated system is beyond the scope of this paper. Even the system for homogeneous YM fields in two-dimensional flat space is known to be non-integrable. A dynamical system analysis of LRS Bianchi I models with $SU(2)$-YM fields was given in Ref. [8].

2. Einstein-Yang-Mills equations on homogeneous space-time

Following the conventions of Ref. [9] we let $(M,g)$ be a connected pseudo-Riemannian manifold with its Levi-Civita connection, and $K$ its isometry group. Its (left) action,

\[ \bar{\psi} : K \times M \rightarrow M : (a, x) \mapsto \bar{\psi}_a x \]

is transitive and effective on $M$. Fixing a point $x_0 \in M$ (to be called the origin) the isotropy subgroup $K_0$ of $K$ is defined by

\[ K_0 := \{ a \in K \mid \bar{\psi}_a x_0 = x_0 \} \]

and all isotropy subgroups for different points of $M$ are conjugate. The manifold $M$ is diffeomorphic to the set of left cosets of $K$ with respect to $K_0$, $M \cong K/K_0$, and there
is a one-to-one correspondence between $K$-invariant pseudo-Riemannian metrics on $M$ and $\text{ad}_{K_0}$-invariant non-degenerate symmetric bilinear forms $\tilde{g}$ on the quotient space $\mathfrak{k}/\mathfrak{k}_0$ of the corresponding Lie algebras.

We wish to describe Yang-Mills connections that have as many symmetries as the metric of space-time and therefore assume that the full isometry group also acts by principal bundle automorphisms

$$\tilde{\psi} : K \times P \to P$$

on the principal bundle $P$ that project onto isometries on $M$ thus satisfying

$$\pi \circ \tilde{\psi} = \tilde{\psi} \circ \pi \quad \text{and} \quad \tilde{\psi}_a \circ R_g = R_g \circ \tilde{\psi}_a \quad \forall \ a \in K \ \forall \ g \in G$$

where $\pi$ is the projection, $G$ the structure (gauge) group, and $R$ the right action of $P$. If the gauge potential is invariant under this action, i.e. if the connection form $\tilde{\omega}$ on $P$ is invariant, $\tilde{\psi}_a^* \tilde{\omega} = \tilde{\omega}$ for all $a \in K$, then so is the curvature form $\tilde{\Omega}$, $\tilde{\psi}_a^* \tilde{\Omega} = \tilde{\Omega}$. It follows that

$$\mathcal{L}_X \tilde{\omega} = 0 \quad \text{and} \quad \mathcal{L}_X \tilde{\Omega} = 0 \quad \forall \ X \in \mathfrak{k}.$$  

where $X$ is the infinitesimal generator of the action $\tilde{\psi}$ on $P$ corresponding to $X \in \mathfrak{k}$.

Now it is known (see, for example, Ref. [10]) that equivalence classes of such $\tilde{\psi}$-invariant principal bundles $P$ over $M$ are in one-to-one correspondence with conjugacy classes of homomorphisms $\lambda : K_0 \to G$. Here $\lambda$ and $\tilde{\psi}$ are related by

$$\tilde{\psi}_a(u_0) = R_{\lambda(a)}u_0 \quad \forall \ a \in K_0$$

where $u_0$ is any fixed element of $\pi^{-1}(x_0)$.

Moreover, Wang’s theorem (Ref. [2],also see Ref. [9]) states that (for fixed $\lambda$) the set of $\tilde{\psi}$-invariant connections on $P$ is in one-to-one correspondence with the set of linear maps $\Lambda : \mathfrak{k} \to \mathfrak{g}$ that satisfy

$$\Lambda(X) = \lambda(X) \quad (X \in \mathfrak{k}_0),$$

$$\Lambda \circ \text{ad}_k = \text{ad}_{\lambda(k)} \circ \Lambda \quad (k \in K_0)$$

(7)
(where \( \lambda \) now also denotes the induced Lie algebra homomorphism), and the invariant connection and curvature on \( P \) are then given by

\[
<\bar{X}, \bar{\omega}> = \Lambda(X) \quad (X \in \mathfrak{k}),
\]

\[
<\bar{X} \wedge \bar{Y}/\bar{\Omega}> = [\Lambda(X), \Lambda(Y)] - \Lambda([X, Y]) \quad (X, Y \in \mathfrak{k}).
\]

The symbol \( \overset{\circ}{\circ} \) indicates that these equations only hold at the fixed point \( u_0 \in P \). The second equation of \((7)\) becomes infinitesimally

\[
\Lambda([X, Y]) = [\lambda(X), \Lambda(Y)] \quad \forall X \in \mathfrak{k}_0, \forall Y \in \mathfrak{k}.
\]

For practical calculations we need to introduce a basis \( \{e_\alpha | \alpha = 1 \ldots m\} \) of the Lie algebra \( \mathfrak{k} \) such that the corresponding generators \( \{e_a | a = 1 \ldots n\} \) span the tangent space \( T_{x_0} M \) at \( x_0 \) while the \( \{e_\Gamma | \Gamma = n+1 \ldots m\} \) span the Lie subalgebra \( \mathfrak{k}_0 \). Thus, if the structure constants \( c^{\lambda}_{\alpha \beta} \) are introduced by

\[
[e_\alpha, e_\beta] = c^{\lambda}_{\alpha \beta} e_\lambda,
\]

then

\[
c^{a}_{\Gamma \Delta} = 0.
\]

The infinitesimal generators \( \bar{e}_a \) on \( M \) corresponding to \( e_a \) form a frame field in a neighborhood of \( x_0 \), but this will, in general, only be global on \( M \) if \( M \) admits a simply transitive isometry subgroup and is thus a group manifold. Let \( \{\bar{\theta}^a\} \) be the local 1-form field dual to \( \{e_a\} \).

A pseudo-Riemannian metric \( g \) on \( M \) can now be written in the form

\[
g = g_{ab} \bar{\theta}^a \otimes \bar{\theta}^b,
\]

where the components \( \check{g}_{ab} := g_{ab}(x_0) \) satisfy

\[
\check{g}_{\Gamma(a} c^{\rho}_{b)\Gamma} = 0,
\]
because of the ad\(_{K_0}\)-invariance. The coefficients of the Levi-Civita connection and the curvature tensor with respect to this frame at \(x_0\) are then given by

\[
\Gamma^a_{bc} \overset{\circ}{=} -\frac{1}{2}\epsilon^a_{bc} + g^{ar}c^s_{r(b}g_{c)s} \tag{15}
\]

\[
R^a_{bcd} \overset{\circ}{=} \Gamma^a_{rc}\Gamma^r_{bd} - \Gamma^a_{rd}\Gamma^r_{bc} - c^r_{cd}\Gamma^a_{br} + c^a_{0\Sigma}c^\Sigma_{cd} \tag{16}
\]

From here on the symbol \(\overset{\circ}{=}\) denotes equality at \(x_0\) only. Note that neither the components of \(g\), \(\Gamma\), nor \(R\) are constant on \(M\), in general.

Equation (15) is easily derived from the equation \(\mathcal{L}_{\bar{e}_a}g = 0\) stating that the \(\bar{e}_a\) are infinitesimal isometries, and that assumptions that the linear connection is metric and symmetric.

Equation (16) is obtained most conveniently from Wang’s theorem applied to the bundle of pseudo-orthogonal frames over \(M\). Here, however, the principal bundle and the connection on it are already fixed as well as the action of \(K\) on the bundle, which is the natural lift of the action on \(M\). Thus (8) fixes the Wang map together with the requirement of zero torsion and (9) then leads to (16) (cf. Ref. [9], Ch. X). In a systematic study of EYM-systems from a Kaluza-Klein perspective in Ref. [11], the Riemann tensor for metrics on homogeneous spaces is also calculated in a very explicit form in terms of the structure constants of the symmetry group by another method which leads to a different but equivalent expression.

The gauge fields being invariant under a transitive symmetry group are also determined by their values at just one point of \(M\) which we take to be the origin \(x_0\). Their derivatives that occur in the Yang-Mills equations can be computed using again Wang’s theorem so that the field equations are reduced to a purely algebraic form. Let \(\sigma\) be a local section of \(P\), thus satisfying \(\pi \circ \sigma = \text{id}_M\), and introduce the local gauge potential \(A\) and the gauge field \(F\) by

\[
A = \sigma^*\tilde{\omega}, \quad F = \sigma^*\tilde{\Omega}. \tag{17}
\]

Then we have the following lemma:

**Lemma 1.** Under the above assumptions the Lie derivative of the gauge curvature \(F\) at \(x_0 \in M\) can be written in the form

\[
\mathcal{L}_X F \overset{\circ}{=} [\Lambda(X), F] - [\langle \bar{X}, A \rangle, F]. \tag{18}
\]
Proof. Since $\bar{X} = \pi_* \circ \sigma_* \bar{X} = \pi_* \bar{X}$ the vector field $\hat{X} = \sigma_* \bar{X} - \bar{X}$ is vertical on $P$. Now $\mathcal{L}_\hat{X} F = \mathcal{L}_\hat{X} \sigma^* \tilde{\Omega} = \sigma^* \mathcal{L}_{\sigma_* \bar{X}} \tilde{\Omega} = \sigma^* (\mathcal{L}_{\bar{X} + \hat{X}} \tilde{\Omega}) = \sigma^* \mathcal{L}_\hat{X} \tilde{\Omega}$ in view of (3). But

$$\mathcal{L}_\hat{X} \tilde{\Omega} = \iota_{\hat{X}} d\tilde{\Omega} + d\iota_{\hat{X}} \tilde{\Omega} = -\iota_{\hat{X}} [\hat{\omega} \wedge \tilde{\Omega}] = -[\hat{X}, \hat{\omega}] + [\hat{\omega} \wedge \iota_{\hat{X}} \tilde{\Omega}] = -[\hat{X}, \hat{\omega}] \quad (19)$$

in view of the Bianchi identities, $d\tilde{\Omega} + [\hat{\omega} \wedge \tilde{\Omega}] = 0$, and the fact that $\iota_Z \tilde{\Omega} = 0$ for any vertical vector field $Z$. Pulling back (19) to $M$ by $\sigma$,  

$$\mathcal{L}_{\bar{X}} F = \sigma^* \mathcal{L}_\hat{X} \tilde{\Omega} = -\sigma^* [\hat{X}, \hat{\omega}] = \iota_{\hat{X}} A - \Lambda(\bar{X}) \quad (8).$$

We choose now for the vector field $\bar{X}$ the local space-time frame vectors $\bar{e}_a$ and let

$$A = A_b \bar{\theta}^b, \quad F = \frac{1}{2} F_{ab} \bar{\theta}^a \wedge \bar{\theta}^b. \quad (20)$$

Then, introducing the (space-time) covariant derivatives $F_{ab/c} = (\mathcal{L}_{\bar{e}_c} F)_{ab} + 2 F_{ra} \Gamma^r_{bc}$, together with (18), we have

$$F_{ab/c} = [\Lambda_c - A_c, F_{ab}] + 2 F_{ra} \Gamma^r_{bc}, \quad (21)$$

where $\Lambda_c := \Lambda(e_c)$.

Since the gauge-covariant derivative of $F$ is defined by

$$D_\alpha F_{\beta\gamma} = \nabla_\alpha F_{\beta\gamma} + [A_\alpha, F_{\beta\gamma}], \quad (22)$$

we now find, interestingly, that the Yang-Mills equations, $D^\lambda F_{\lambda\alpha} = 0$ can be written in these frame components without involving the gauge potentials,

$$[\Lambda^r, F_{ra}] + \Gamma^\ell_{ar} F_{\ell r} + F_{at} \Gamma^\ell_{rs} g^{rs} \Lambda^\ell = 0. \quad (23)$$

In view of (3), the frame components $F_{ab}$ of the Yang-Mills field are given by

$$F_{ab} = [\Lambda_a, \Lambda_b] - c_{ab} \Lambda_r - c_{ab} \Lambda_\Sigma. \quad (24)$$

Einstein’s equations are also easily formulated in these frame components,

$$R_{ab} = \kappa T_{ab}. \quad (25)$$
where $\kappa = 8\pi$ (Newton’s constant), the velocity of light is set to unity,
\[ T_{ab} = X_{ab} - \frac{1}{4} X_r^{gr} g_{ab}, \quad X_{ab} := \langle F_a^r, F_b^r \rangle, \] (26)
and $\langle, \rangle$ represents a bi-invariant scalar product on the gauge group Lie algebra $\mathfrak{g}$. The stress energy tensor $T_{ab}$ has zero trace, and the Ricci tensor components are obtained from (14).

All these equations hold only at the origin $x_0 \in M$ and they form a complicated algebraic system. For a given isometry group $K$ of space-time and a chosen basis of $\mathfrak{k}$ the structure constants can be considered fixed. The homomorphism $\lambda$ can be chosen arbitrarily and then fixed. Possible choices are found by considering the subgroups of the gauge group $G$ onto which there are homomorphisms from the isotropy group $K_0$, in particular, imbeddings of $K_0$ in $G$. This classification is discussed (for semisimple $K_0$ and semisimple $G$) in Ref. [12], Ref. [13]. After the choice of a particular homomorphism, equations (7) or, infinitesimally, (14), i.e.
\[ [\Lambda_a, \lambda_f] + c^r_{af} \Lambda_r = -c^\Sigma_{af} \lambda_\Sigma, \] (27)
must be solved for $\Lambda$ that is then substituted into (23), (24) and into Einstein’s equations (25).

In the (most important) case of a reductive homogeneous space $c^\Sigma_{af} = 0$ and (27) is a homogeneous linear system. Then $\Lambda$ can also be regarded as an intertwining operator between two linear representations of the isotropy group $K_0$ in the following way. We have $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{m}$ as a vector space and the map $\Lambda$ in (7) is fully determined by the linear map $\bar{\Lambda} : \mathfrak{m} \to \mathfrak{g}$ that satisfies
\[ \bar{\Lambda} \circ \phi = \psi \circ \bar{\lambda}, \] (28)
where $\phi : K_0 \times \mathfrak{m} \to \mathfrak{m} : (a, X) \mapsto \text{ad}_a X$ and $\psi : K_0 \times \mathfrak{g} \to \mathfrak{g} : (a, Z) \mapsto \text{ad}_{\lambda(a)} Z$. Then $\bar{\Lambda}$ is an intertwining operator between these representations of $K_0$, namely the adjoint representation $\phi$ on $\mathfrak{m}$ and the representation $\psi$ on $\mathfrak{g}$.

Also, the $g_{ab}$ are arbitrary, subject to (14). But not all choices need lead to nonisometric space-times. One can reduce the number of free parameters by bringing $g_{ab}$ into a canonical form using basis transformations by automorphisms of $K$ that leave the subgroup $K_0$ invariant.
3. EYM equations in spatially homogeneous cosmological models

Let \((M, g)\) now be an \(n + 1\)-dimensional space-time manifold with an isometry group \(K\) whose orbits are \(n\)-dimensional space-like hypersurfaces so that \(M = \Sigma \times \mathbb{R}\) with \(K\) acting transitively on \(\Sigma\) and \(K_0\) the isotropy subgroup at \(x_0 \in \Sigma\). We choose to describe the metric by a coordinate time \(t\) and a frame field \(\{\bar{e}_a\}\) of Killing vector fields on \(\Sigma\),

\[
g = -dt \otimes dt + g_{ab} \bar{\theta}^a \otimes \bar{\theta}^b. \tag{29}
\]

Assume also that the \(\bar{e}_\Gamma\) \((\Gamma = n + 1 \ldots m)\) vanish at a fixed point \(x_0 \in \Sigma\). It then follows that the \(\Sigma_t\)-coordinate components of the frame vectors \(\bar{e}_a\) do not depend on the time \(t\), so that

\[
[\partial_t, \bar{e}_\alpha] = 0 \quad \forall \alpha = 1 \ldots m. \tag{30}
\]

The connection and curvature components with respect to the local space-time frame field \(\{\bar{e}_0 = \partial_t, \bar{e}_a\}\) can then be calculated in the standard fashion. If

\[
K_{ab} = \frac{1}{2} \dot{g}_{ab} \tag{31}
\]

is the extrinsic curvature of the hypersurfaces and a dot denotes the time derivative, we have for the Ricci tensor components

\[
R_{00} \overset{\circ}{=} -g^{rs} \ddot{K}^{s}_r + K^r_s \dot{K}^s_r, \tag{32}
\]

\[
R_{0b} \overset{\circ}{=} K^r_b c^s_r + K^r_s c^s_b, \tag{33}
\]

\[
R_{ab} \overset{\circ}{=} \ddot{K}_{ab} + K^r_a K^{br}_b - 2K_{ar} \dot{K}^r_b + \ddot{R}_{ab}. \tag{34}
\]

Here \(\ddot{R}\) is the Ricci tensor on \(\Sigma\) and is given, according to \((\Sigma)\), by

\[
\dddot{R}_{ab} \overset{\circ}{=} \Gamma^r_{ab} \Gamma^s_r - \Gamma^r_{sb} \Gamma^s_{ar} - \Gamma^s_{ar} c^r_{sb} + c_{a\Sigma} c_{r\Sigma}. \tag{35}
\]

The \(g_{ab}\) and \(K_{ab}\) depend on \(t\), the \(c^a_{\beta\gamma}\) are constant (on \(\{x_0\} \times \mathbb{R}\)) and the \(\Gamma^a_{bc}\) are still given by \((\Sigma)\).

The calculation of the Yang-Mills equations for a gauge connection invariant under a symmetry group with orbits on surfaces of constant \(t\) is analogous to the one on spherically
symmetric static space-times and is done as first outlined in Ref. [10] (see also Ref. [14]). Locally one can introduce a gauge potential $A = A_0 dt + A$ where $A$ is the potential of a ($t$-dependent) invariant connection on $\Sigma$ and $A_0$ is a $g$-valued scalar, invariant under $Ad_{\lambda(K_0)}$. In practice (unless there are incompatible boundary conditions in the time evolution) $A_0$ can be gauged away. This is because a time-dependent gauge transformation to achieve such a result needs to satisfy an ordinary differential equation on the gauge group that can always be solved, at least locally in $t$.

In terms of the space-time co-frame $\{\bar{\theta}^0 = dt, \bar{\theta}^a\}$ we now write for the Yang-Mills field

$$ F = E_a dt \wedge \bar{\theta}^a + \frac{1}{2} B_{ab} \bar{\theta}^a \wedge \bar{\theta}^b. $$

Then the Lie derivative of $F$ in the time direction is

$$ \mathcal{L}_\partial F = \dot{E}_a dt \wedge \bar{\theta}^a + \frac{1}{2} \dot{B}_{ab} \bar{\theta}^a \wedge \bar{\theta}^b $$

and those along $\Sigma$ are still given by (18). Just as in section 2 we can then compute the frame components of the covariant derivatives and find

$$ F_{ab/c} = [\Lambda_c - A_c, B_{ab}] + 2B_{r[a} \Gamma^r_{bc]} + 2E_{[a} K_{b]c}, $$

$$ F_{0b/c} = [\Lambda_c - A_c, E_b] - E_r \Gamma^r_{bc} + B_{br} K^r_c, $$

$$ F_{ab/0} = \dot{B}_{ab} + 2B_{r[a} K^r_b], $$

$$ F_{0b/0} = \dot{E}_b - E_r K^r_c. $$

The Yang-Mills equations thus become

$$ [E^r, \Lambda_r] - c^r_r E^s \overset{\circ}{=} 0, $$

$$ \dot{E}_a + [A_0, E_a] + K^r_a E_a - 2K^r_a E_r - [B_{ar}, \Lambda^r] + B^s_a c^r_{rs} - \frac{1}{2} g_{pq} c^r_{pq} B_{pq} \overset{\circ}{=} 0, $$

where

$$ B_{ab} = [\Lambda_a, \Lambda_b] - c^r_{ab} \Lambda_r - c^r_{ab} \Lambda \Sigma, $$

$$ E_a = \partial_t \Lambda_a + [A_0, \Lambda_a], $$

and we may choose the gauge such that $A_0 = 0.$
For the stress-energy tensor components we find (if we now restrict to \( n = 3 \))
\[
T_{00} = \frac{1}{2} (E^2 + B^2),
\]
\[
T_{0a} = \epsilon_a{}^{rs} < E_r, B_s >,
\]
\[
T_{ab} = - < E_a, E_b > - < B_a, B_b > + \frac{1}{2} (E^2 + B^2) g_{ab}
\]
where \( B_a := \frac{1}{2} \epsilon_a{}^{rs} B_{rs}, E^2 := < E_r, E_r >, \) and \( B^2 := < B_r, B_r >. \) Einstein’s equations (25) can now be brought into the form
\[
\Sigma \mathcal{R} + (K_r^r)^2 - K^{rs} K_{rs} = \kappa (E^2 + B^2),
\]
\[
K_r^r \epsilon_{sr} + K_s^r \epsilon_{ar} = \kappa \epsilon_a{}^{rs} < E_r, B_s >,
\]
\[
\dot{K}_{ab} - 2K_{ar} K_b^r + K_r^r K_{ab} + \Sigma \mathcal{R}_{ab} = \kappa T_{ab}.
\]

If we choose the gauge such that \( A_0 = 0 \) then, after a basis of the symmetry Lie algebra \( \mathfrak{k} \) and the homomorphism \( \lambda : K_0 \to G \) are chosen and a point \( x_0 \in \Sigma \) is fixed, we have as dynamical variables the functions \( g_{ab}(t) \), subject to (14), and the \( \mathfrak{g} \)-valued functions \( \Lambda_a(t) \), subject to (27). Equations (49) and (50) can be considered the Hamiltonian and the momentum constraints, respectively. They restrict somewhat the choice of initial values for an initial time but will afterwards be preserved by the time evolution. This follows as a special case from the general analysis of the Cauchy problem in EYM theory.

Only a time-independent basis transformation in \( \mathfrak{k} \) by automorphisms leaving \( \mathfrak{k}_0 \) invariant can now be used to possibly eliminate some variables. The algebraic problem of finding the possible homomorphisms \( \lambda \) and solving for \( \Lambda \) is similar to the one mentioned in section 2 but a little simpler. The isotropy group \( K_0 \) is now a subgroup of \( SO(3) \) and thus compact so that the homogeneous space is reductive. Moreover, on the three-dimensional space-like space sections the isotropy group can only be either \( SO(3) \) or \( U(1) \) (or trivial). We will consider in the following sections some of these cases that can be handled without recourse to the more advanced techniques of the theory of Lie algebra representations.

4. Isotropic cosmological models

The isotropy subgroup \( K_0 \) of a space-time transitive isometry group must be a subgroup of the Lorentz group and a classification of all homomorphisms of such a subgroup into

\[\]
any compact gauge group $G$ is a nontrivial algebraic problem. For a cosmological model with three-dimensional homogeneous space sections the situation is much simpler, since $K_0$ must be a subgroup of $SO(3)$ which leaves only $SO(3)$, $U(1)$ or the trivial subgroup. In this section we consider the “physically isotropic” models where $K_0$ is $SO(3)$. There are still many possible conjugacy classes of homomorphisms $\lambda$ and a complete classification for arbitrary compact groups $G$ may not be known. We will here mainly consider the case when $G$ is either $SU(n)$ or a real orthogonal group.

When $SO(3)$ is the isotropy group of an isometric action on the three-dimensional manifold $\Sigma$ the $(\Sigma, \tilde{g})$ must be of constant curvature $k$ and its isometry group $K$ is $SO(4)$, $E(3)$ or $SO(3,1)$, respectively, depending on whether $k$ is positive, zero or negative. The Lie algebra has a basis $\{e_i, f_i\}$ ($i = 1 \ldots 3$) with commutators

$$[e_i, e_j] = k \epsilon_{ij}^r f_r,$$
$$[e_i, f_j] = \epsilon_{ij}^r e_r,$$  \hspace{1em} (52)
$$[f_i, f_j] = \epsilon_{ij}^r f_r,$$  \hspace{1em} (53)

where the $f_i$ span the Lie algebra of the isotropy group. We can choose $k$ to be $\pm 1$ or 0 and the $\epsilon_{ij}^r$ in this section now refers to the Euclidean metric in $\mathbb{R}^3$.

The geometry of these isotropic models is then already determined, namely the one of the well known Friedman-Robertson-Walker space-times. We have in the terminology of section 8

$$g_{ab} = a(t)\delta_{ab}, \quad K_{ab} = \frac{1}{2}\dot{a}\delta_{ab}, \quad \Sigma R_{ab} = 2k\delta_{ab}$$  \hspace{1em} (55)

where the bar was dropped and $\{\theta^i\}$ is the co-frame dual to $\{e_i\}$. In terms of the *conformal time* $\tau$ the metric is

$$g = R(\tau)^2(-d\tau^2 + \delta_{ab}\theta^a \otimes \theta^b)$$  \hspace{1em} (56)

so that $a = R^2$ and $\dot{\phi} = d\phi/dt = R^{-1}d\phi/d\tau = \dot{\phi}$ for any function $\phi$. The stress tensor, being isotropic, is of the form

$$T_{ab} = pg_{ab}$$  \hspace{1em} (57)
where $p$ is the pressure and, since the source will be a zero-rest-mass Yang-Mills field, the mass-energy density is $\mu = 3p$. Einstein’s equations are now equivalent to

$$\ddot{a} = -2k \quad \text{and} \quad \kappa p = \frac{1}{4} a^{-2} \dot{a}^2 + ka$$

or, in terms of the conformal time,

$$R'' + kR = 0 \quad \text{and} \quad \kappa p = R^{-4} R^2 + kR^{-2} = (\text{const.}) R^{-4}.$$  \hspace{1cm} (59)

The complete time evolution of the geometry and thus the stress-energy tensor is therefore easily obtained explicitly. It remains to formulate the equations for the Yang-Mills field. If we use again the notation $\Lambda_i = \Lambda(e_i)$ and now $\lambda_i = \lambda(f_i)$ then equations (27) become

$$[\lambda_i, \Lambda_j] = \epsilon_{ij}^r \Lambda_r.$$  \hspace{1cm} (60)

They represent a system of linear equations for the $\Lambda_i$ once the $\lambda_i$, i.e. the homomorphism is chosen. We have from (44) and (45)

$$E_i = \dot{\Lambda}_i = R^{-1} \Lambda'_i \quad \text{and} \quad B_i = R^{-1} \left( \frac{1}{2} \epsilon_{ij}^{rs}[\Lambda_r, \Lambda_s] - k\lambda_i \right)$$

for the Yang-Mills fields (where the indices on $\Lambda$ and $\lambda$ are raised and lowered with respect to $\delta_{ij}$) so that

$$E^2 = R^{-4} \delta^{rs} <\Lambda'_r, \Lambda'_s>$$

$$B^2 = \frac{1}{2} R^{-4} \left( <[\Lambda_r, \Lambda_s], [\Lambda^r, \Lambda^s]> - 4k <\Lambda_r, \Lambda^r > + 2k^2 <\lambda_r, \lambda^r > \right)$$

The YM field equations become

$$\Lambda''_i - 2k \Lambda_i - [[\Lambda_i, \Lambda_r], \Lambda^r] = 0$$

$$[\Lambda'_r, \Lambda^r] = 0.$$  \hspace{1cm} (65)

From (49), (50) and (57) we have, moreover,

$$\epsilon_i^{rs} <\lambda_r, \Lambda'_s> = 0$$

$$<E_i, E_j> + <B_i, B_j> = 2p g_{ij}.$$  \hspace{1cm} (67)
To derive these expressions we have used, whenever convenient, (60) as well as the invariance of the inner product $<,>$ on $\mathfrak{g}$.

We can go a little further before we need to specify the gauge group $G$, but the specific structure of the isotropy group and its action on $\Sigma$ incorporated in equations (60) are essential. Equations (60) are a system of linear equations for the ($\mathfrak{g}$-valued) $\Lambda_i$. Let \( \{\Lambda^K_i, K = 0, \ldots, r - 1\} \) be a basis of the solution space where $\Lambda^K_0 = \lambda_i$ since $\lambda_i$ is always a solution and is nonzero except if $\lambda$ is the trivial homomorphism.

**Lemma 2.** The basis vectors \( \{\Lambda^K_i, K = 0, \ldots, r - 1\} \) of the solution space of Wang’s conditions (60) satisfy the following relations

\[
\epsilon_{ij}^{rs}[\Lambda^K_i, \Lambda^L_j] = \gamma_{rs}^{KL} \Lambda^K_i
\]

\[
[\Lambda^K_i, \Lambda^K_j] = \frac{1}{2} \epsilon_{ij}^{rs} \gamma_{rs}^{KL} \Lambda^K_i
\]

\[
\gamma_{KL} = \gamma_{KL}
\]

\[
L^{KL} := \delta^{rs}[\Lambda^K_i, \Lambda^L_j] = -L^{LK}
\]

\[
<\Lambda^K_i, \Lambda^K_j> = \alpha^{KL} \delta_{ij} \quad \text{with} \quad \alpha^{KL} = \alpha^{LK}
\]

\[
\gamma_{KL} \alpha^{SM} = \gamma_{KS} \alpha^{LM}
\]

\[
\gamma^0_L = 2\delta^K_L \quad \text{and} \quad L^0_K = 0
\]

**Proof.** To prove (68) let $L_i$ and $M_j$ be solutions of (60) and $N_i = \epsilon_{ij}^{rs}[L_r, M_s]$. Then we can show that $[\lambda_i, N_j] = \epsilon_{ij}^{rs}N_r$ by a simple calculation using the Jacobi identity and the identities satisfied by the Levi-Civita symbol $\epsilon_{ijk}$. Thus $N_i$ is also a solution of (60). (However, the full solution space need not be a Lie subalgebra of $\mathfrak{g}$, in general.)

Equations (69) and (71) follow immediately from the antisymmetry of the Lie bracket and (70) is a consequence of either (68) or (62).

To prove (72) we let $\alpha^{KL}_{ij} := <\Lambda^K_i, \Lambda^L_j>$ and use (60) and the invariance of the scalar product $<,>$,

\[
\epsilon_{ij}^{r} \alpha^{KL}_{rk} = \epsilon_{ij}^{r} \Lambda^K_r, \Lambda^L_k = <[\lambda_i, \Lambda^K_j], \Lambda^L_k> = -<\Lambda^K_j, [\lambda_i, \Lambda^L_k]>
\]

\[
= -\epsilon_{ij}^{r} \alpha^{KL}_{rk}
\]

from which the result easily follows.

Finally, (73) follows directly from the invariance of the scalar product and (74) is an immediate consequence of (60) since $\Lambda^0_i = \lambda_i$. □
The only time-dependent quantities are now the amplitudes $\Phi_K(\tau)$ which satisfy the Yang-Mills equations in the form

$$L^{KL}\Phi'_K\Phi_L = 0,$$

$$\Phi''_K - 2k\Phi_K + \frac{1}{2}\gamma^M_{KL}\gamma^P_Q\Phi_L\Phi_P\Phi_Q = 0.$$  \hfill (75)

Here $(L^{KL})$, defined in (71), is an array of skewsymmetric matrices one for each dimension of the Lie algebra $\mathfrak{g}$. From (61) we have

$$E_i = R^{-1}\Phi'_K\Lambda^K_i \quad \text{and} \quad B_i = R^{-1}\left(\frac{1}{2}\gamma^M_{KL}\Phi_K\Phi_L - k\delta^0_M\right)\Lambda^M_i$$  \hfill (77)

and, in view of (72), Einstein’s equations (66) and (67) reduce to (59) and the following expression for the mass-energy density

$$\mu = \frac{1}{2}(E^2 + B^2)$$  \hfill (78)

where now

$$E^2 = 3R^{-4}\alpha^{KL}\Phi'_K\Phi'_L,$$

$$B^2 = 3R^{-4}\left(k^2\alpha^{00} - k\alpha^{0M}\gamma^M_{KL}\Phi_K\Phi_L + \frac{1}{4}\gamma^R_{KL}\gamma^{RS}_{PQ}\Phi_K\Phi_L\Phi_P\Phi_Q\right).$$  \hfill (79)

(Using the relations of Lemma 2 it can be verified that $\mu R^4$ is constant as it should be.)

The quantities $\alpha^{KL}$, $\gamma^M_{KL}$ and $L^{KL}$ depend only on the Lie algebra $\mathfrak{g}$ and the homomorphism $\lambda : \mathfrak{su}(2) \to \mathfrak{g}$. Hence, to find all possible isotropic EYM equations one has to find all $\mathfrak{su}(2)$ subalgebras of $\mathfrak{g}$ (up to inner isomorphism), thus choosing the homomorphism $\lambda$ (see Ref. [13] and then solve the equation (28) for the intertwining operator $\bar{\Lambda} = (\Lambda^K_i)$). This can be done in a systematic way using a Cartan-Weyl basis of $\mathfrak{g}$ by the methods given in Ref. [12].

Here we will only consider those examples that can be dealt with in a more elementary way, without involving the theory of Lie algebra root systems.

We know that all (connected) compact gauge groups can be imbedded as subgroups of $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ (in fact in $SO(n)$) for some $n$. Moreover, all finite-dimensional complex (real) representations of $SU(2)$ are equivalent to unitary (real orthogonal) ones and decompose orthogonally into irreducible parts. Thus at least for the unitary and the real
orthogonal groups we can determine the possible homomorphisms directly from the well known representation theory. If, for example, \( \tilde{\lambda} \) is a \( n \times n \)-unitary representation of \( K_0 \), i.e. \( \tilde{\lambda} : a \mapsto U_a \forall a \in K_0 \) where \( U_a \) is a unitary matrix, then \( \lambda : a \mapsto (\det U_a)^{-1/n} U_a \) is a homomorphism into \( SU(n) \). Moreover, it is easily seen that equivalent representations define conjugate homomorphisms and that, in fact, conjugacy classes of homomorphisms of \( K_0 \) into \( SU(n) \) are in one-to-one correspondence with equivalence classes of \( n \)-dimensional unitary representations of \( K_0 \). Similarly, any real \( n \)-dimensional orthogonal representation of \( K_0 \) immediately defines a homomorphism into \( SO(n) \).

If now \( K_0 = SU(2) \) then any \( n \)-dimensional unitary (or real orthogonal) representation is a direct sum of irreducible unitary (real orthogonal) representations, i.e. any homomorphism \( \lambda : SU(2) \to SU(n) \) is conjugate to one that maps into block matrices

\[
\lambda(a) = \begin{pmatrix} D_{k_1}(a) \\ \vdots \\ D_{k_r}(a) \end{pmatrix}
\]

where each \( D_{k_i} \) is an irreducible \( k_i \)-dimensional representation and where \( k_1 + \cdots + k_r = n \). As is well known, the Lie algebra representation corresponding to an \( n \)-dimensional irreducible representation can be written as follows. If \( \{\tau_1, \tau_2, \tau_3\} \) is the standard basis of \( su(2) \) in terms of anti-Hermitian matrices and \( \lambda_k = \lambda(\tau_k) \) are the images in \( su(n) \) then the latter can be represented by the matrices

\[
(\lambda_+)_\ell m = \sqrt{m(n-m)} \delta_{\ell,m+1}, \quad \lambda_- = \lambda_+^H \\
\lambda_1 = -\frac{i}{2}(\lambda_+ + \lambda_-), \quad \lambda_2 = -\frac{1}{2}(\lambda_+ - \lambda_-), \quad (\lambda_3)_\ell m = -i(\frac{n+1}{2} - m) \delta_{\ell m}
\]

Consider first a homomorphism class from \( SU(2) \) to \( SU(n) \), that arises from an irreducible unitary representation in \( \mathbb{C}^n \). Then the \( \lambda_i \) in (80) can be chosen as the matrices (82) and the system (81) can be explicitly solved (this also follows from more general results of representation theory) for the \( \Lambda_j \) that can now be taken to be \( (n \times n) \) skew-Hermitian matrices. It follows that

\[
\Lambda_i = \Phi \lambda_i
\]
i.e. the solution space is one-dimensional. In this case the YM-potential is thus determined
by a single function $\Phi(\tau)$. For a simple Lie algebra like $\mathfrak{su}(n)$ the invariant product $<,>$
must be a multiple of the Killing form,

$$<X, Y> = -c_n \kappa(X, Y)$$  \hspace{1cm} (85)

for some constant $c_n > 0$ which we will choose to be 1. It follows from (68) and (69) that
$\gamma^0_0 = 2$ and $\alpha^0_0 = n^2(n^2 - 1)/6$ so that the Yang-Mills equations become

$$\Phi'' - 2(k - \Phi^2)\Phi = 0$$  \hspace{1cm} (86)

whence

$$\frac{d\Phi}{\sqrt{c^2 - (\Phi^2 - k)^2}} = d\tau$$  \hspace{1cm} (87)

where the constant $c^2 = 4\mu R^4/(n^2(n^2 - 1))$. Thus $\Phi(\tau)$ is periodic in the cosmological
time $\tau$ and can be expressed in terms of an inverse elliptic integral. It is easily seen that
the “electric” and “magnetic” contributions to the energy density $\mu$ oscillate in the time $\tau$.
These equations (for $G = SU(2)$) have previously been derived and analyzed by Gal’tsov
and Volkov [6].

If the homomorphism class is not induced by an irreducible representation the gauge field
may be more complicated. However, since the evolution of the geometry of space-time is
already determined only the evolution of the gauge fields can be affected. Table I shows the
dimensions $d$ of the solution space of (60) for the some homomorphisms $\lambda : \mathfrak{su}(2) \rightarrow \mathfrak{su}(n)$.
Here $1 \oplus 2$, for example, means that $\lambda$ is obtained from a representation in $\mathbb{C}^3$ that decomposes
into a (trivial) one-dimensional one and an irreducible two-dimensional one. In these cases,
according to (70), the YM field depends on $d$ independent amplitudes $\Phi_K(\tau)$ that each satisfy
a second order equation. However, at least for $n \leq 6$, the $c$ constraint conditions (73) (which
are not linearly independent in general) simply imply that many of the $\Phi_K$ are proportional
to each other so that the remaining number $n_{eq}$ of second order equations that must be
solved is much smaller.
To give one example, for an $SU(5)$-theory with the homomorphism $\lambda$ corresponding to a representation of the type $1 \oplus 1 \oplus 3$ we find the Yang-Mills equations

\begin{align*}
\Phi'' + 2\Phi(\Phi^2 + 3\Psi^2 - k) &= 0 \quad \text{(88)} \\
\Psi'' + 2\Psi(3\Phi^2 + \Psi^2 - k) &= 0 \quad \text{(89)}
\end{align*}

and

\begin{align*}
E^2 &= 20R^{-4} \left( \Phi'^2 + \Psi'^2 \right) \quad \text{and} \quad B^2 = 20R^{-4} \left[ (\Phi^2 + \Psi^2 - k)^2 + 4\Phi^2\Psi^2 \right]. \quad \text{(90)}
\end{align*}

The contribution of the electric and the magnetic part to the mass-energy density changes in time similarly as in the ‘irreducible’ case, but the gauge fields now ‘rotate’ in the Lie algebra in more dimensions.

If the gauge group is $SO(n)$ we can similarly classify the $\lambda$ by considering all $n$-dimensional real orthogonal representations of $\mathfrak{su}(2)$. These decompose into irreducible blocks of dimensions $2k + 1$ or $4k$ for integer $k$, but not $2k + 2$ (see, e.g. Ref. [16]). It does not seem to be simple to write down formulae for these representations for arbitrary $n$ as in (82) and (83). But there exists an algorithm to construct them explicitly. First note that an irreducible complex representation of $\mathfrak{su}(2)$ leaves invariant a bilinear form $\beta$ on $\mathbb{C}^n$. For the choice of $\lambda$ in (82),(83) we find that $\beta_{k\ell} = (-1)^k \delta_{\ell,n+1-k}$ which is symmetric for odd $n$ and skew for even $n$.

Thus if $n$ is odd then $\lambda$ is of real type, i.e. the representation is unitarily equivalent to one by real orthogonal matrices. In fact,

\begin{align*}
\tilde{\lambda}_k &= U^H \lambda_k U \quad \text{where} \quad U^H U = \text{id} \quad \text{and} \quad U^T \beta U = \text{id} \quad \text{(91)}
\end{align*}

are the generators of the orthogonal representation. The matrices $U$ can be easily computed by diagonalizing $\beta$ by congruence. For a $\lambda : \mathfrak{su}(2) \to \mathfrak{so}(2k + 1)$ that corresponds to an irreducible representation it now follows easily from the complex case that the solutions of (60) are again of the form (84) and the single time dependent amplitude $\Phi$ satisfies (86).

For $n = 4k$ the explicit irreducible representations are obtained via the Lie algebra homomorphism

\begin{align*}
\rho : \mathfrak{gl}(\ell, \mathbb{C}) \to \mathfrak{gl}(2\ell, \mathbb{R}) : A = A_1 + iA_2 \mapsto \tilde{A} = \begin{pmatrix}
A_1 & -A_2 \\
A_2 & A_1
\end{pmatrix} \quad \text{(92)}
\end{align*}
which maps $\mathfrak{su}(\ell)$ into $\mathfrak{so}(2\ell)$. For $\ell = 2k$ the image of the matrices $\lambda_k$ generate an irreducible $4k$-dimensional real orthogonal representation of $\mathfrak{su}(2)$. Again, it can be verified explicitly that (60) has only the solutions (84) and that the only amplitude satisfies (86).

The remaining equivalence classes of homomorphisms $\lambda$ into $\mathfrak{so}(n)$ can now be obtained from reducible orthogonal representations in the same way as those for $\mathfrak{su}(n)$. Some examples are tabulated in Table 2. The corresponding equations and expressions for $E^2$ and $B^2$ are very similar to (88), (89), and (90).

5. Locally rotationally symmetric cosmological models

Spatially homogeneous cosmological models with $K_0 = U(1)$ have been extensively studied and are known as locally rotationally symmetric (LRS) models. Our construction of four-dimensional isometry groups of LRS models is along the lines with Ref. [17]. It is known that if $K_0$ is compact, then there exists a reductive decomposition of $\mathfrak{k}$ (i.e. there is a subspace $\mathfrak{m}$ such that $\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{m}$ and $[\mathfrak{k}_0, \mathfrak{m}] \subset \mathfrak{m}$ and $\mathfrak{k}_0 \cap \mathfrak{m} = 0$). The choice of such a reductive decomposition is not unique. As it will be seen shortly, a judicious choice of a reductive decomposition, greatly simplifies the EYM equations. It is interesting to note that for all Bianchi cosmologies except Bianchi III, there is a reductive decomposition in which $\mathfrak{m}$ is a Lie subalgebra (such a decomposition for BIII would require $SU(1,1)$ to be solvable which contradicts the simplicity of $SU(1,1)$). In a suitable basis $e_1, \cdots, e_4$ such that $e_1, e_2, e_3$ span $\mathfrak{m}$ and $e_4$ span $\mathfrak{k}_0$,

$$-c_{14}^2 = c_{24}^1 = 1, \ c_{34}^a = 0 \quad (a = 1, 2, 3). \quad (93)$$

The Ad($K_0$)-invariance of the metric expressed via (14) then restricts the space-metric to the form $\text{diag}(f^2, f^2, f^2\sigma^2)$ where $f$ and $\sigma$ are functions of $t$. Given an invariant basis on a homogeneous space, one can start from this metric and, after integrating the Killing equations, find out which spatially homogeneous space-times admit the action of a four-dimensional isotropy group (cf. Table 3 and Ref. [18]). Kramer et al. [17] have classified all such space-times with two integers $\ell$ and $k$ (Bianchi V (BV) does not fall into this category and is treated separately). All homogeneous spaces which have the same four-dimensional isometry group, belong to group manifolds (Bianchi cosmologies). Such group manifolds
correspond to different three-dimensional subgroups of the isometry group which act simply transitively on the hypersurfaces of homogeneity. Kramer et al.’s classification (section 11.1) is based on the metric

\[ g = f^2(2C^{-2}(dx^2 + dy^2) + \frac{1}{2}\sigma^2dz^2 - \ell\sigma^2C^{-1}(ydx - xdy)dz + \ell^2\sigma^2C^{-2}(ydx - xdy)^2], \]

where \( C := 1 + 1/2k(x^2 + y^2) \) \hspace{1cm} (94)

or, for Bianchi V,

\[ g = f^2(e^{2\gamma}(dy^2 + dx^2) + \sigma^2dz^2]. \]

These metrics all have (generically) four-dimensional isometry groups. We must now select a frame field of Killing vectors in such a way as to let \( e_4 \) generate the isotropy group and the structure constants to satisfy (93). The following choice achieves this.

\[
\begin{align*}
e_1 &= -\frac{k}{\sqrt{2}}xy\partial_y - \frac{1}{\sqrt{2}}(1 + K)\partial_x + \sqrt{2}\ell y\partial_z, \\
e_2 &= \frac{k}{\sqrt{2}}xy\partial_x + \frac{1}{\sqrt{2}}(1 - K)\partial_y + \sqrt{2}\ell x\partial_z, \\
e_3 &= -2\partial_z, \\
e_4 &= x\partial_y - y\partial_x,
\end{align*}
\]

where \( K := (k/2)(x^2 - y^2) \) and the entries of the right column are the Killing vector fields of BV. The above Killing vector fields and non-vanishing structure constants

\[
c^3_{12} = \ell, \quad c^4_{12} = k \quad \text{or} \quad c^1_{13} = c^2_{23} = -1 \quad \text{for BV},
\]

(96)
determine the isometry group, embeddings of the isotropy group in the isometry group up to conjugacy class, and identify the three-dimensional homogeneous spaces that admit an action of a four-dimensional isometry group. Here \( \Sigma \) is simply connected. It is known that the number of degrees of freedom in mini-superspace models depends on the choice of topology [19].

Our aim is to construct the invariant \( SU(n) \)-YM connections for homogeneous spaces listed in the above table. In doing so, we have to find all the conjugacy classes of homomorphisms \( \lambda : U(1) \rightarrow SU(n). \)
Such conjugacy classes of homomorphisms are well understood for spherically symmetric solutions of the EYM equations (cf. Ref. [14]). These classes of homomorphisms are basically of the same form as (81). However, since the irreducible representations of $U(1)$ are one-dimensional, $D_k$ have only one entry. Therefore if $U(1) = \{ z \in \mathbb{C} : |z| = 1 \}$, then

$$\lambda : z \mapsto \text{diag}(z^{j_1}, \cdots, z^{j_n}) \quad (\sum_{i=1}^{n} j_i = 0, \ j_i = \text{an integer}) \quad (97)$$

is clearly a homomorphism of $U(1)$ into $SU(n)$. The set of integers $j_p \ (p = 1, \cdots, n)$ such that $j_p \geq j_q$ for $p < q$, yields all conjugacy classes of homomorphisms $\lambda : U(1) \to SU(n)$. Denoting $\mathcal{D} := (i/2)\text{diag}(j_1, \cdots, j_n)$ we have

$$\Lambda[e_4, e_i] = [\lambda(e_4), \Lambda_i] = [\mathcal{D}, \Lambda_i] \implies c_{i,r}^{\Lambda} \Lambda_r = [\mathcal{D}, \Lambda_i] \quad (98)$$

in which $\Lambda_i$ are traceless antihermitian matrices as in section 4. These equations and (93) give

$$\Lambda_2 = -[\mathcal{D}, \Lambda_1], \quad \Lambda_1 = [\mathcal{D}, \Lambda_2], \quad [\mathcal{D}, \Lambda_3] = 0, \quad (99)$$

which in turn yield

$$(\Lambda_l)_{pq}[4 - (j_p - j_q)^2] = 0, \quad l = (1, 2). \quad (100)$$

The solution to the above equations is

$$\Lambda_1 = i/2(\Lambda_+ - \Lambda_-), \quad \Lambda_2 = -1/2(\Lambda_+ + \Lambda_-), \quad \Lambda_+ = -(\Lambda_-)^H \quad (101)$$

where $j_p \geq j_q$ for $p < q$ and therefore $\Lambda_+(\Lambda_-)$ is a strictly upper (lower) triangular matrix. Moreover, $(\Lambda_+)^{pq} \neq 0$ only if $j_p = j_q + 2$. The general solution of the above equations is in the root space corresponding to $\mathcal{D} \subset$ (the Cartan subalgebra of $su(n)$) and in principle could be obtained for any compact group. However, such a general treatment is out of the scope of the present paper (cf. Ref. [20]). Some interesting special cases to consider are the following:

(a) $j_p = 0, \ \forall \ p \in \{1, \cdots, n\}$, (trivial homomorphism) requires $\Lambda_1 = \Lambda_2 = 0$ and $\Lambda_3$ is completely undetermined.
(b) If \( |j_p - j_q| \neq 2 \) \( \forall p, q \in \{1, \ldots, n\} \) then \( \Lambda_1 = \Lambda_2 = 0 \) and \( \Lambda_3 \) is a diagonal traceless anti-Hermitian matrix. In this case the gauge group reduces to its maximal torus (i.e. \( U(1) \otimes \cdots \otimes U(1) \subset SU(n) \)).

(c) If \( j_p = j_{p+1} + 2, \forall p \in \{1, \ldots, n - 1\} \) \( \Rightarrow \mathcal{D} = (i/2)\text{diag}(n - 1, n - 3, \ldots, -n + 1) \). Then (99) and (100) respectively imply that \( \Lambda_3 \) is an anti-hermitian traceless diagonal matrix and \( (\Lambda_+)_{p,p+1} = -(\Lambda^H)_{p+1,p} \) are the only non-vanishing entries of \( \Lambda_\pm \).

In (b) the EYM equations for \( SU(2) \)-YM fields reduce to that of axially symmetric electromagnetic fields and one can show that (a) and (b) are gauge equivalent [3]. We consider (c) the simplest non-Abelian YM field in which the entries of \( \mathcal{D} \) correspond to the magnetic quantum numbers in the \( n \)-dimensional unitary representation of \( SU(2) \). Up to a gauge transformation, this representation yields the only possible non-abelian connection for \( SU(2) \)-YM fields. Therefore we derive the EYM equations for this particular example starting with

\[
(\Lambda_+)_{p,p+1} = \omega_p e^{\gamma_p}, \quad p \in \{1, \ldots, n - 1\}
\]

\[
\Lambda_3 = i \text{ diag} (\alpha_1, \ldots, \alpha_p - \alpha_{p-1}, \ldots, -\alpha_{n-1}).
\]  

(102)

The YM constraints (42) in terms of these variables are as follows

\[
\omega_p^2 \dot{\gamma}_p + 2\dot{\alpha}_p \sigma^{-2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.
\]  

(103)

Terms in the upper (lower) part of the braces refer to the ‘general’ (BV) case. The YM dynamical equations (43) consist of

\[
\ddot{\omega}_p + (f^{-1} \dot{f} + \sigma^{-1} \dot{\sigma}) \dot{\omega}_p + f^{-2} \omega_p \left( \sigma^{-2} \dot{\alpha}_p^2 + \frac{1}{2} \dot{\omega}_p - f^2 \dot{\gamma}_p^2 - \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right) = 0,
\]

\[
\ddot{\gamma}_p + (2\dot{\omega}_p \sigma^{-1} + f^{-1} \dot{f} + \sigma^{-1} \dot{\sigma}) \dot{\gamma}_p + 2(f \sigma)^{-2} \ddot{\alpha}_p \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0,
\]  

(104)

\[
\ddot{\alpha}_p + (f^{-1} \dot{f} - \sigma^{-1} \dot{\sigma}) \dot{\alpha}_p + f^{-2} \dot{\alpha}_p \sigma^2 - \frac{1}{2} \sigma^2 f^{-2} [W_p + p(n - p)k] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0
\]
and Einstein equations \([49-51]\) are, respectively,

\[
3f^{-2}j^2 + 2f^{-1}j\hat{\sigma}^{-1}\hat{\sigma} + f^{-2} \begin{cases} 
   k - \frac{1}{4}\ell^2\sigma^2 \\
   -3\sigma^{-2}
\end{cases} = \kappa f^{-2}(T_1 + T_2),
\]

\[
\dot{j} + 2f^{-1}j^2 + j\hat{\sigma}^{-1}\hat{\sigma} + f^{-1} \begin{cases} 
   k - \frac{1}{2}\ell^2\sigma^2 \\
   -2\sigma^{-2}
\end{cases} = \kappa f^{-1}T_1,
\]

\[
\dot{\sigma} + 3f^{-1}j\dot{\sigma} - f^{-2}(k\sigma - \ell^2\sigma^3) \begin{cases} 
   1 \\
   0
\end{cases} = \kappa \sigma f^{-2}(T_2 - 2T_1).
\]

with the only non-trivial momentum constraint given by

\[
\sigma^{-1}\dot{\sigma} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \kappa nf^{-2} \left( \sum_p \hat{\alpha}_p \hat{\gamma}_p \omega_p^2 - \sum_p \omega_p \omega_p \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).
\]

Here we have used the abbreviations

\[
\begin{aligned}
\hat{\alpha}_p &:= 2\alpha_p - \alpha_{p-1} - \alpha_{p+1}, \\
W_p &:= \omega_p^2 - \begin{cases} 
   2\ell\alpha_p \\
   0
\end{cases}, \\
\bar{W}_p &:= 2W_p - W_{p-1} - W_{p+1} + \begin{cases} 
   4k \\
   0
\end{cases},
\end{aligned}
\]

and

\[
\begin{aligned}
T_1 &:= n \left[ \sigma^{-2} \sum_p \hat{\alpha}_p \hat{\alpha}_p + \frac{1}{4}f^{-2} \left( \sum_p \bar{W}_p W_p + (1/3)n(n^2 - 1)k^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \right], \\
T_2 &:= n \sum_p \left[ \hat{\omega}_p^2 + \omega_p^2 \hat{\gamma}_p^2 + \omega_p^2(f\sigma)^{-2} \left( \hat{\alpha}_p^2 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right].
\end{aligned}
\]

and it is understood that all subscripted quantities are zero when the index is outside the range \(\{1, \ldots, n-1\}\).

At this point, we do not intend to give a complete analysis of the above system of differential equations. However, a few points are in order. For the general case, if \(\omega_p \neq 0 \forall p, \hat{\gamma}_p = 0\) and the first equation in (103), the Hamiltonian constraint, is the only constraint of the system. The dynamical evolution is expected to preserve the constraint \(\dot{H} = 0\). Indeed, as a check
on the consistency of the above equations, one can show, for example for $G = SU(2)$, that
$$\dot{H} = -(6\dot{f}/f + 2\dot{\sigma}/\sigma)H.$$ One observes that there are $2(n-1)$ degrees of freedom associated with YM fields. Such an explicit integration is very complicated for the Bianchi V case, but as mentioned at the end of section we would expect the constraints to be conserved in view of the general consistency of the Cauchy problem.

The above system is the set of $SU(n)$-EYM equations for the particular homomorphism from $U(1)$ to $SU(n)$ chosen above for all spatially homogeneous cosmologies with isotropy group $U(1)$. These equations are mildly gauge dependent ($A_0$ was set to 0). Nevertheless, the gauge-invariant quantities like the various components of the energy-momentum tensor, are easily expressible in terms of $\alpha_p, \gamma_p,$ and $\omega_p$. We plan to pursue a more detailed analysis of these equations for $SU(2)$-YM fields.

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Table 1: This table gives for different homomorphisms $\lambda : \mathfrak{su}(2) \to \mathfrak{su}(n)$ the number $d$ of dimensions of the solution space of (60), the number $c$ of nonzero constraint conditions (75) and the number $n_{eq}$ of independent amplitudes that satisfy second order equations in time. (Trivial homomorphisms and those arising from irreducible representations are not included.)

| $n$ | $\lambda$ | $d$ | $c$ | $n_{eq}$ |
|-----|-----------|-----|-----|---------|
| 3   | $1 \oplus 2$ | 1   | 0   | 1       |
| 4   | $1 \oplus 1 \oplus 2$ | 1   | 0   | 1       |
|     | $1 \oplus 3$ | 3   | 1   | 2       |
|     | $2 \oplus 2$ | 4   | 3   | 2       |
| 5   | $1 \oplus 1 \oplus 1 \oplus 2$ | 1   | 0   | 1       |
|     | $1 \oplus 1 \oplus 3$ | 5   | 6   | 2       |
|     | $1 \oplus 4$ | 1   | 0   | 1       |
|     | $1 \oplus 2 \oplus 2$ | 4   | 4   | 2       |
|     | $2 \oplus 3$ | 2   | 0   | 2       |
| 6   | $1 \oplus 1 \oplus 1 \oplus 1 \oplus 2$ | 1   | 0   | 1       |
|     | $1 \oplus 1 \oplus 1 \oplus 3$ | 7   | 11  | 2       |
|     | $1 \oplus 1 \oplus 4$ | 1   | 0   | 1       |
|     | $1 \oplus 1 \oplus 2 \oplus 2$ | 4   | 4   | 2       |
|     | $1 \oplus 2 \oplus 3$ | 4   | 3   | 3       |
|     | $1 \oplus 5$ | 1   | 0   | 1       |
|     | $2 \oplus 2 \oplus 2$ | 9   | 11  | 2       |
|     | $2 \oplus 4$ | 4   | 4   | 3       |
|     | $3 \oplus 3$ | 4   | 5   | 2       |
Table 2: Values $d$, $c$ and $n_{eq}$ for the equivalence classes of homomorphisms $\lambda : \mathfrak{su}(2) \to \mathfrak{so}(n)$ for small $n$. Trivial homomorphisms and those arising from irreducible representations are not included. The question marks indicate cases where the constraint equations do not simply imply that some amplitudes are proportional to others.

| $n$ | $\lambda$       | $d$ | $c$ | $n_{eq}$ |
|-----|------------------|-----|-----|----------|
| 4   | $1 \oplus 3$     | 2   | 0   | 2        |
| 5   | $1 \oplus 1 \oplus 3$ | 3   | 1   | 2        |
|     | $1 \oplus 4$     | 1   | 0   | 1        |
| 6   | $1 \oplus 1 \oplus 1 \oplus 3$ | 4   | 3   | 2        |
|     | $1 \oplus 1 \oplus 4$ | 1   | 0   | 1        |
|     | $1 \oplus 5$     | 1   | 0   | 1        |
|     | $3 \oplus 3$     | 3   | 1   | 2        |
| 7   | $1 \oplus 1 \oplus 1 \oplus 1 \oplus 3$ | 5   | 6   | 2        |
|     | $1 \oplus 1 \oplus 1 \oplus 4$ | 1   | 0   | 1        |
|     | $1 \oplus 1 \oplus 5$ | 1   | 0   | 1        |
|     | $1 \oplus 3 \oplus 3$ | 5   | 1   | ?        |
|     | $3 \oplus 4$     | 2   | 0   | 2        |
| 8   | $1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 3$ | 6   | 10  | 2        |
|     | $1 \oplus 1 \oplus 1 \oplus 1 \oplus 4$ | 1   | 0   | 1        |
|     | $1 \oplus 1 \oplus 1 \oplus 5$ | 1   | 0   | 1        |
|     | $1 \oplus 1 \oplus 3 \oplus 3$ | 7   | 2   | ?        |
|     | $1 \oplus 7$     | 1   | 0   | 1        |
|     | $3 \oplus 5$     | 3   | 0   | 3        |
|     | $4 \oplus 4$     | 6   | 17  | 2        |
Table 3: The three-homogeneous cosmologies with a four-dimensional isometry group. \( WH \) refers to Weyl-Heisenberg group.

| Class | Homogeneous cosmology | Isometry group        | \( l \) | \( k \) |
|-------|-----------------------|-----------------------|--------|--------|
| A     | BI                    | \( E(2) \otimes U(1) \) | 0      | 0      |
| A     | BVII\(_0\)           |                       |        |        |
| B     | BV                    | \( BVII_h \otimes U(1) \) | -      | -      |
| B     | BVII\(_h\)           |                       |        |        |
| B     | BIII                  | \( SU(1,1) \otimes U(1) \) | 0      | -1     |
| A     | BVIII                 |                       | 1      | -1     |
| A     | BII                   | \( WH \otimes U(1) \)  | 1      | 0      |
| A     | BIX                   | \( SU(2) \otimes U(1) \) | 1      | 1      |
| -     | Kantowski-Sachs       | \( SU(2) \otimes R \) | 0      | 1      |