HIGH RANK ELLIPTIC CURVES INDUCED BY RATIONAL DIOPHANTINE TRIPLES

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Dedicated to the memory of our friend and coauthor Julián Aguirre

Abstract. A rational Diophantine triple is a set of three nonzero rational \(a, b, c\) with the property that \(ab + 1, ac + 1, bc + 1\) are perfect squares. We say that the elliptic curve \(y^2 = (ax + 1)(bx + 1)(cx + 1)\) is induced by the triple \(\{a, b, c\}\). In this paper, we describe a new method for construction of elliptic curves over \(\mathbb{Q}\) with reasonably high rank based on a parametrization of rational Diophantine triples. In particular, we construct an elliptic curve induced by a rational Diophantine triple with rank equal to 12, and an infinite family of such curves with rank \(\geq 7\), which are both the current records for that kind of curves.

1. Introduction

A set \(\{a_1, a_2, \ldots, a_m\}\) of \(m\) distinct nonzero rationals is called a rational Diophantine \(m\)-tuple if \(a_ia_j + 1\) is a perfect square for all \(1 \leq i < j \leq m\). The first rational Diophantine quadruple \(\{\frac{1}{16}, \frac{33}{16}, \frac{12}{16}, \frac{105}{16}\}\) was found by Diophantus, while the first Diophantine quadruple in integers \(\{1, 3, 8, 120\}\) was found by Fermat. In 1969, Baker and Davenport ([2]) proved that Fermat’s set cannot be extended to a Diophantine quintuple in integers. It was proved in [6] that there does not exist a Diophantine sextuple in integers and there are only finitely many Diophantine quintuples in integers. Recently, He, Togbé and Ziegler proved that there are no Diophantine quintuples in integers ([21]). Euler proved that there are infinitely many rational Diophantine quintuples. In particular, he extended Fermat’s quadruple by the fifth positive rational number \(\frac{777480}{8288641}\). In 2019, Stoll ([28]) proved that extension of Fermat’s set

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to a rational quintuple with the same property is unique. The first example of a rational Diophantine sextuple, the set \( \{ \frac{11}{122}, \frac{35}{122}, \frac{1312}{27}, \frac{1180873}{16} \} \), was found by Gibbs ([20]), while Dujella, Kazalicki, Mikić and Szikszai ([11]) recently proved that there are infinitely many rational Diophantine sextuples (see also [10, 12, 13]). For an overview of results on Diophantine \( m \)-tuples and its generalizations see [8].

Let \( \{ a, b, c \} \) be a rational Diophantine triple. Then there exist nonnegative rationals \( r, s, t \) such that \( ab + 1 = r^2, ac + 1 = s^2 \) and \( bc + 1 = t^2 \). In order to extend the triple \( \{ a, b, c \} \) to a quadruple, we have to solve the system of equations

\[
\begin{align*}
ax + 1 &= \Box, \\
bx + 1 &= \Box, \\
cx + 1 &= \Box.
\end{align*}
\]

We assign the following elliptic curve to the system (1.1):

\[
E : y^2 = (ax + 1)(bx + 1)(cx + 1).
\]

We say that the elliptic curve \( E \) is induced by the rational Diophantine triple \( \{ a, b, c \} \).

Elliptic curves induced by rational Diophantine triples were used for the first time in the construction of elliptic curves with relatively large rank in [5] (let us mention that in [22] all \( S \)-integral points on some elliptic curves associated with the quintuple \( \{ 1, \frac{33}{16}, \frac{105}{16}, 20, 1140 \} \) were computed, which was a motivation for considering connections between elliptic curves and Diophantine \( m \)-tuples). By using subtriples of certain rational Diophantine quintuples, elliptic curves with rank 7 over \( \mathbb{Q} \) and rank 4 over \( \mathbb{Q}(t) \) were constructed in [5]. That result was improved in [7] where several examples of curves with rank 9 were found by considering subtriples of the following generalization of Fermat’s quadruple: \( \{ k - 1, k + 1, 4k, 16k^3 - 4k \} \). These results were further improved in our joint paper with Julián Aguirre ([1]), where we constructed an elliptic curve with rank 11 over \( \mathbb{Q} \) (induced by the triple \( \{ \frac{795625}{3128544}, \frac{-22247424}{7951245}, \frac{9750161189120}{290125899} \} \)) and rank 5 over \( \mathbb{Q}(t) \). The construction was based on subtriples of quadruples of the form \( \{ a, a(k + 1)^2 - 2k, a(2k + 1)^2 - 8k - 4, ak^2 - 2k - 2 \} \). We used similar method in [16] and constructed several new elliptic curves with rank 11 over \( \mathbb{Q} \) and rank 6 over \( \mathbb{Q}(t) \) (see also [17]).

Note that in all mentioned results the elliptic curves have torsion group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). The application of elliptic curves induced by rational Diophantine triples in construction of high rank curves appears to be even more fruitful in the case of larger torsion groups. Such curves were used in [15, 17] for finding elliptic curves with the largest known rank over \( \mathbb{Q} \) (rank 9; induced by the triples \( \{-301273, 550614, 535707232\} \) and \( \{-301273, 550614, 535707232\} \)) and \( \mathbb{Q}(t) \) (rank 4) with torsion group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \). This construction uses triples of the form \( \{ a, -\frac{1}{a}, c \} \) which induce elliptic curves with points of order 4. It is shown in [16] that the elliptic curve with largest known rank over \( \mathbb{Q} \)
(rank 6; originally found by Elkies in 2006) with torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ is induced by the triple \( \{ 31269599, -23721120, 1461969791 \} \).

Furthermore, it was shown in [7] that every elliptic curve with torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ is induced by a Diophantine triple (see also [3, 14]). In particular, the triple \( \{ 408, 145, -145, 408, -145439, 59160 \} \) induces the curve with torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ and rank 3 over $\mathbb{Q}$, found by Connell and Dujella in 2000, what is the largest known rank for curves with that torsion group.

Although in the case of torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the record ranks over $\mathbb{Q}$ (rank 15) and $\mathbb{Q}(t)$ (rank 7) were discovered by Elkies ([18, 19]) with different methods, we believe that it is still interesting question to investigate how large can be the rank of elliptic curves induced by rational Diophantine triples. In this paper, we construct an elliptic curve induced by a rational Diophantine triples with rank equal to 12, and an infinite family of such curves with rank $\geq 7$, which both improve previous results of the type.

### 2. Construction of an elliptic curve with rank 12

By the coordinate transformation $x \mapsto \frac{a}{abc}$, $y \mapsto \frac{b}{abc}$, applied to the curve $E$, we obtain the equivalent curve

\[ E' : \quad y^2 = (x + ab)(x + ac)(x + bc). \]

The curve $E'$ has three 2-rational points $A = [-bc, 0]$, $B = [-ac, 0]$, $C = [-ab, 0]$, and other two rational points $P = [0, abc]$ and $S = [1, rst]$, where $ab + 1 = r^2$, $ac + 1 = s^2$, $bc + 1 = t^2$. We may expect that in general the points $P$ and $S$ will be independent points of infinite order, so that the rank of $E'$ will be at least 2.

To increase the rank, we will use the parametrization of rational Diophantine triples due to Lasić ([23]) (see also [13]):

\[
\begin{align*}
    a &= \frac{2t_1(1 + t_1t_2(1 + t_2t_3))}{(-1 + t_1t_2t_3)(1 + t_1t_2t_3)} \\
    b &= \frac{2t_2(1 + t_2t_3(1 + t_3t_1))}{(-1 + t_1t_2t_3)(1 + t_1t_2t_3)} \\
    c &= \frac{2t_3(1 + t_3t_1(1 + t_1t_2))}{(-1 + t_1t_2t_3)(1 + t_1t_2t_3)}
\end{align*}
\]

We have noted that the rank jumps if $t_3(t_3 - t_2)$ is a perfect square (and, cyclicly, if $t_1(t_1 - t_3)$ is a perfect square or if $t_2(t_2 - t_1)$ is a perfect square). Indeed, if we insert

\[
x = \frac{4(t_3^2t_4 - t_3 + t_2)(t_3^2t_4^2 + 1 + t_3t_1)(t_2t_3 + t_2t_3^2t_1 + 1)}{t_3(-1 + t_1t_2t_3)^2(1 + t_1t_2t_3)^2}
\]
(note that \(x + ab = \frac{b(c-b)}{t_2t_3}\) into the equation (2.1), we obtain
\[
y^2 = 64(1 + t_3t_1)^2(t_1t_2t_3 - t_2 - t_2^2t_3 + t_3)^2(t_2t_3 + t_2t_3^2t_1 + 1)^2(1 + t_2t_3)^2
\times (t_3t_2^2t_1 + 1 + t_3t_1)^2(t_3 - t_2)t_3^{-3}(-1 + t_1t_2t_3)^{-6}(1 + t_1t_2t_3)^{-6},
\]
which leads to the condition that \(t_3(t_3 - t_2)\) is a perfect square.

Thus, if we find a triple \((t_1, t_2, t_3)\) of rationals such that
\[
(2.2) \quad t_3(t_3 - t_2), \quad t_1(t_1 - t_3), \quad t_2(t_2 - t_1)
\]
are all perfect squares, we may expect that our curve will have rank \(\geq 5\) (since we started with rank \(\geq 2\)).

One way to satisfy conditions (2.2) is through so called almost perfect cuboids. Indeed, if we put
\[
t_3 = s_3^2, \quad t_1 = -s_1^2, \quad t_2 = s_2^2, \quad s_3^2 - s_2^2 = s_4^2,
\]
then we have
\[
(2.3) \quad s_1^2 + s_2^2 = 1, \quad s_2^2 + s_4^2 = 1, \quad s_1^2 + s_2^2 + s_4^2 = 1
\]
Thus we get an almost perfect cuboid (only one diagonal is not an integer).

In [29], one can find a parametric solution of (2.3):
\[
s_1 = 2(m^2 + m)(m^2 - 1)(m^2 + 1 + 4m), \quad s_2 = 4(m^2 + m + 1)(2m + 1)(m^2 - 1)(2m + m^2),
\]
\[
s_4 = (2m + 1)(2m + m^2)(3m^2 + 2m + 1)(m^2 + 2m + 3),
\]
which gives
\[
t_1 = -4(m^2 + m + 1)^2(m^2 - 1)^4(m^2 + 1 + 4m)^2,
\]
\[
t_2 = 16(m^2 + m + 1)^2(2m + 1)^2(m^2 - 1)^2(2m + m^2)^2,
\]
\[
t_3 = m^2(2m + 1)^2(m + 2)^2(5m^2 + 8m + 5)^2(m^2 + 1)^2.
\]

We now present another approach which yields a two-parametric solution, more appropriate for numerical experiments for finding specializations with higher rank. We satisfy the first two conditions by putting
\[
t_3(t_3 - t_2) = (t_3 + u)^2, \quad t_1(t_1 - t_3) = (t_1 + v)^2
\]
and we get
\[
t_2 = -\frac{u(2t_3 + u)}{t_3}, \quad t_3 = -\frac{v(2t_1 + v)}{t_1}.
\]
By inserting this into the third condition \(t_2(t_2 - t_1) = 1\) we get
\[
(2.4) \quad (8uv^2 - 2u^2v)t_1^3 + (-8u^3v + 15u^2v^2 + u^4 + 8uv^3)t_2^3
\]
\[
+ (-4u^3v^2 + 2u^4 + 16u^2v^3)t_1 + 4v^4u^2 = 1
\]
The equation (2.4) can be viewed as an elliptic curve over $\mathbb{Q}(u,v)$, with an obvious point $P = [0, 2u^2v^2]$. By taking the point $2P$, we obtain

$$t_1 = \frac{v^2(-v + 16u)}{8u(-4v + u)},$$

which gives

$$a = \frac{v^2(-v + 16u)(16u^2 - 64u^2 - v^4 + 16u^3 - 4v^5u + v^4u^2)}{u(2 + v)(4 - 2v + v^2)(v - 2)(v^2 + 2v + 4)(2u - v)(2u + v)(-4v + u)},$$

$$b = \frac{16u(-4v + u)v(4v - 64u + 16uv^2 - 4uv^2 - v^5 + 4u^2v^3)}{(2 + v)(4 - 2v + v^2)(v - 2)(v^2 + 2v + 4)(2u - v)(2u + v)(-4v + u)},$$

$$c = \frac{4(256uv - 64u^2 - 16v^4 + 64uv^2 + v^6 - 16v^5u)(2u - v)(2u + v)}{u(2 + v)(4 - 2v + v^2)(v - 2)(v^2 + 2v + 4)(-v + 16u)(-4v + u)}. $$

This gives the elliptic curve with rank $\geq 5$ over $\mathbb{Q}(u,v)$. Indeed, if we write the curve in the form $y^2 = x^3 + Ax^2 + Bx$, where

$$A = v(256v^{13} - 32v^{15} + v^{17} + 140288v^9u^2 + 741888v^7u^4 - 4096v^{10}u - 1167360v^8u^3 - 21258240v^6u^5 - 7936v^{12}u + 664832v^{10}u^3 + 11440128v^8u^5 + 32192v^{11}u) - 2785824v^5u^4 - 32380416v^7u^6 + 28747776v^5u^6 + 6463488v^9u^7 + 71860224v^7u^4 - 2205960u^8v^5 + 1536v^{14}u - 24192v^{13}u^2 - 22528v^{12}u^3 + 591360v^{11}u^4 - 3244800u^9v^{10} - 12848328v^8u^8 - 12979200v^6u^7 + 7816v^{15}u^2 - 36160v^{14}u^3 - 8616v^{13}u^4 + 100992v^{12}u^5 - 128v^{16}u - 2023776v^{11}u^6 + 4v^{18}u - 449v^{17}u^2 + 7824v^{16}u^3 - 3136v^{15}u^4 + 2860032v^{10}u^7 + 70176v^{14}u^5 + 112296v^3u^6 + 946176v^7u^8 - 2785824v^9u^8 - 332160v^{12}u^7 + 128188416v^2u^9 - 37027840v^4u^9 - 1441792v^{10}u^5 + 2659328v^5u^9 + 46368v^{11}u^8 - 6193512v^{10}u^5 + 515072u^{10}u^7 - 291840v^9u^{10} + 16818240v^9u^6 - 29425664v^{10}u^9 + 32014336u^{10}u^3 + 140288u^9u^{10} - 2097152u^{11}v^2 + 1572864u^{11}v^4 - 507904u^{11}v^6 - 16384u^{11}v^8 + 65536u^{12}v^5 + 65536u^{12}v - 131072u^{12}v^3 + 1048576u^{11}),$$

$$B = 4(8vu^2 - 8u^2 + 16vu - v^2u + v^2 + 2v^3)(8uv^2 + 8u^2 - 16vu - v^2u - v^2 + 2v^3) \times (-16v^2 + 64u^2 + v^4 - 16u^2u)(4v - 64u + 16u^2u - 4uv^2 - v^5 + 4u^2v^3) \times (2v^2 - 16u^2 + 2uv + 8u^2 - 4v^2 - v^3)(16uvu - 4u^2 - v^4 + 4u^2v^2) \times (16v^2 - 64u^2 - v^4 + 16u^3u - 4v^5u + v^4u^2) \times (2v^2 + 16u^2 - 2uv + 8u^2 + v^2 - v^3)(-v + 16u)^2(-4v + u)^2u^2v^3.$
then five independent points of infinite order are

\[
P = [-4(4v - 64u + 16v^2u - 4vu^2 - v^5 + 4v^3u^2)(-v + u)^2(-v + 16u)^2 \\
\times (16v^2 - 64u^2 - v^4 + 16v^3u - 4v^5u + v^4u^2)u^2v^3, \\
8(64v^2u^2 - 64u^2 - 16v^5u + 256vu + v^6 - 16v^4)(4v - 64u + 16v^2u - 4vu^2 - v^5 + 4v^3u^2) \\
\times (16v^2 - 64u^2 - v^4 + 16v^3u - 4v^5u + v^4u^2)(2u - v)^2(2u + v)^2(-v + u)^2 \times (-v - 16u)^2u^2v^3],
\]

\[
R = [4(16vu - 4v^2 - v^4 + 4v^2u^2)(16v^2 - 64u^2 - v^4 + 16v^3u - 4v^5u + v^4u^2) \\
\times (8vu^2 - 8u^2 + 16vu - v^2u + v^2 + 2v^3)(8vu^2 + 8u^2 - 16vu - v^2u - v^2 + 2v^3) \\
\times (-v + 16u)(-4v + u)vu, \\
4(8vu^2 - 8u^2 + 16vu - v^2u + v^2 + 2v^3)(16v^2 - 64u^2 - v^4 + 16v^3u - 4v^5u + v^4u^2) \\
\times (8vu^2 + 8u^2 + 32vu - 16v^3u - 4v^5u - v^7)(16vu - 4u^2 - v^4 + 4v^5u^2) \\
\times (8vu^2 + 8u^2 - 16vu - v^2u - v^2 + 2v^3)(8vu^2 - 8u^2 - 32vu - 16v^3u + 4v^5u^2 - v^3) \\
\times (2u + v)(2u - v)v^2(-v + 16u)(-4v + u)u],
\]

\[
T_1 = [16(16vu - 4v^2 - v^4 + 4v^2u^2)(2vu^2 - 16u^2 + 2vu + 8v^3u - 4v^2 - v^3) \\
\times (2vu^2 + 16u^2 - 2vu + 8v^3u + 4v^2 - v^3)(4v - 64u + 16v^3u - 4vu^2 - v^5 + 4v^3u^2) \\
\times (-v + 16u)(-4v + u)u, \\
8(16vu - 4u^2 - v^4 + 4v^2u^2)(2vu^2 - 16u^2 + 2vu + 8v^3u - 4v^2 - v^3) \\
\times (2vu^2 + 16u^2 - 2vu + 8v^3u + 4v^2 - v^3)(-v + 16u - 4v^2u + v^2u^2) \\
\times (v^6 - 16v^5u + 256vu - 64u^2 - 16v^4 + 4v^2u^2)(8u^2 - vu + 2v^2) \\
\times (4v - 64u + 16v^3u - 4vu^2 - v^5 + 4v^3u^2)(-v + 16u)(-4v + u)u]u/v^2,
\]

\[
T_2 = [-4(8vu^2 - 8u^2 + 16vu - v^2u + v^2 + 2v^3)(-16v^2 + 64v^2u - v^4 + 16v^3u) \\
\times (16vu - 4v^2 - v^4 + 4v^2u^2)(8vu^2 + 8u^2 - 16vu - v^2u - v^2 + 2v^3) \\
\times (16v^2 - 64u^2 - v^4 + 16v^3u - 4v^5u + v^4u^2)(-4v + u)u/v^2, \\
4(8vu^2 - 8u^2 + 16vu - v^2u + v^2 + 2v^3)(8vu^2 + 8u^2 - 16vu - v^2u - v^2 + 2v^3) \\
\times (16vu - 4v^2 - v^4 + 4v^2u^2)(-16v^2 + 64v^2u - v^4 + 16v^3u)(2u + v)(2u - v) \\
\times (8vu^2 - 16vu - v^2)(16vu - 64v^3u + 4v^5u + 256vu - 64u^2) \\
\times (16v^2 - 64u^2 - v^4 + 16v^3u - 4v^5u + v^4u^2)(-4v + u)u/v^3],
\]

\[
T_3 = [(4v - 64u + 16v^3u - 4vu^2 - v^5 + 4v^3u^2)(-16v^2 + 64v^2u + v^4 - 16v^3u) \\
\times (16v^2 - 64u^2 - v^4 + 16v^3u - 4v^5u + v^4u^2)(2u^2 + 8vu - v^2)^2(-v + 16u), \\
2(-16v^2 + 64v^2u + v^4 - 16v^3u)(8vu^2 - 8u^2 - 32vu - 16v^2u + 4v^2 - v^3) \\
\times (8vu^2 + 8u^2 + 32vu - 16v^2u + 4v^5u + v^4u^2)(2u^2 + 8vu - v^2) \\
\times (16v^2 - 64u^2 - v^4 + 16v^3u - 4v^5u + v^4u^2)(2u - v)(2u + v)(-v + 16u)].
\]
Here the point $P$ corresponds to $[0, abc]$ on $y^2 = (x + ab)(x + ac)(x + bc)$, the point $R$ satisfies $2R = S$, where $S$ corresponds to $[1, rst]$ on $y^2 = (x + ab)(x + ac)(x + bc)$, the point $T_1$ corresponds to the condition $t_3(t_3 - t_2) = 0$, the point $T_2$ corresponds to the condition $t_1(t_1 - t_3) = 0$ while the point $T_3$ corresponds to the condition $t_2(t_2 - t_1) = 0$. Since the specialization map in a homomorphism, it suffices to find a specialization $(u_0, v_0)$ for which the points $P$, $R$, $T_1$, $T_2$ and $T_3$ are independent points of infinite order on $y^2 = x^3 + Ax^2 + Bx$. We checked that this is the case for $(u_0, v_0) = (2, 1)$, since the points $[170605, 39532697], [302665, -66247363], [795565, -637321303], [-447095, -24260803], [8673115/4, -25165674989/8]$ are independent on $y^2 = x^3 + 21361758597x^2 - 28803989016278714304x$.

Now we search for specializations $(u, v)$ with higher rank, in particular with rank 11 and 12. We use a sieving methods similar to those used, e.g., in [1, 16]. We searched for curves with relatively large Mestre-Nagao sum

$$S(N, E) = \sum_{p=2}^{N} -a_p + 2 \log p,$$

where $a_p = a_p(E) = p+1- \#E(\mathbb{F}_p)$, since it is experimentally known ([24, 25]) that we may expect that high rank curves have large $S(N, E)$, and large Selmer rank (as implemented in mwrank with option -a). In search for rank 12 curves we also use the condition that the root-number is equal to 1 (conjecturally this implies that rank is even). We searched also in some restricted subfamilies, including e.g. $u = v$. We implemented the sieving algorithm in Pari ([26]). For the curves which pass our searching conditions, we calculate the rank by Cremona’s program mwrank ([4]).

We find curves with rank 11 for the following parameters:

$$(u, v) = \begin{cases} (11, 5), (-145, 29), (136, 68), (16, 4), (473, 43), (89, 89), \\ (-62, 93), (71, 142), (224, 7), (1032, 172), (-1501, 158), \\ (1358, 194), (-2072, 148), (454, 227), (77, 77), (163, 163), \\ (1007, 53), (1819, 107), (481, 37), (173, 173), (137, 137) \end{cases}$$

The details (minimal Weierstrass equation, torsion points and independent points of infinite order) are given in [9]. Let us mention that the curve corresponding to $(u, v) = (-62, 93)$, i.e.

$$\{a, b, c\} = \begin{cases} 21409906185, 31580198976, 10309975195, \\ 74591676404, 18647919101, 18647919101 \end{cases}$$
with the minimal Weierstrass equation
\[ y^2 + xy = x^3 - x^2 - 21252276640652798739707819217x + 938627524108684110053910801619511357084941, \]

has the minimal discriminant among all known curves with rank 11 and torsion group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

Finally, we found a curve with rank 12 for \((u, v) = (-95, 50)\), i.e.
\[ \{a, b, c\} = \{6125241375, 11907531272, 5535371271425\}, \]

with the minimal Weierstrass equation
\[ y^2 + xy + y = x^3 - x^2 - 14444917075285913568560891846409119571268950880x + 55992158377962542124284683584939561762456224290170437461555851482041439747, \]

the torsion points
\[
\begin{align*}
O, & \quad [910954389920845836020349, -455477194960422918010175], \\
[55448727291190824028230629/4, 5448727291190824028230625/8], & \quad [-451227432876860171037309, -225613716438430085518655], \\
[951514410733369555670349, 21667652092127680529970331439049825], & \quad [15358084490095818094207095, 140142144080498380363069785533616999513], \\
[448008187422056021535383, 73569216148613399817347986859758945], & \quad [-1206000615871044278678751, -74021042560921761514326945233545375], \\
[192562292438693523617091, -911556889640548767064630159456313855], & \quad [10508879668527356682921249, 33851800053181168926568362825476385625], \\
[951514410733369555670349, 21667652092127680529970331439049825], & \quad [7355680099955426271741581/81, -605705671933225602690651446390633849125/729].
\end{align*}
\]

and 12 independent points of infinite order
\[
\begin{align*}
P_1 &= [158850932500649600134809, 578334775816714524616276221704042845], \\
P_2 &= [351104017200784386392209, 30989796694495116194624198332593845], \\
P_3 &= [-427722660290928813983135, -1048576645526111528109185629948786727], \\
P_4 &= [954500781939375762742909, 225326008863345220543071618783370945], \\
P_5 &= [423679598259676591990099, 154829810959547852593332987635966145], \\
P_6 &= [15358084490095818094207095, 140142144080498380363069785533616999513], \\
P_7 &= [448008187422056021535383, 73569216148613399817347986859758945], \\
P_8 &= [-1206000615871044278678751, -74021042560921761514326945233545375], \\
P_9 &= [-192562292438693523617091, -911556889640548767064630159456313855], \\
P_{10} &= [10508879668527356682921249, 33851800053181168926568362825476385625], \\
P_{11} &= [951514410733369555670349, 21667652092127680529970331439049825], \\
P_{12} &= [-7355680099955426271741581/81, -605705671933225602690651446390633849125/729].
\end{align*}
\]

Let us also mention a minor, somewhat related result: for \( t_1 = 44/27, t_2 = 17/2, t_3 = 33/4 \), i.e. \( a = 815848/164547, b = 1012544/1860017, c = 320069/201173 \), we get the elliptic curve
\[ y^2 = x^3 + x^2 - 19393636086496946772176x + 29453641253718130506136229522416740 \]

with rank 10, which is the curve with smallest known conductor among curves with rank 10 and torsion group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). It is obtained by brute-force
search (not in parametric families) within triples $t_1, t_2, t_3$ with small numerators and denominators.

3. Infinite families of elliptic curves with rank $\geq 7$

The construction of the two-parametric family of curves with rank $\geq 5$ from the previous section is related with the construction from our joint paper with Julián Aguirre ([1]). In [1], we constructed a two-parametric family of curves with rank $\geq 4$ over $\mathbb{Q}(m, n)$, and by choosing $n = 7/3$ we obtained a family with rank $\geq 5$ over $\mathbb{Q}(m)$. It can be checked that by taking $m = \frac{-20(4u^2 - 1)}{9u(u+4)}$ we obtain the same family as the family obtained from our new two-parametric family by specializing $v = -1$.

It is shown in [16] that inserting $n = 7/3$ already in the family from [1] with rank $\geq 3$ over $\mathbb{Q}(a, n)$, gives a simple family with rank $\geq 4$ over $\mathbb{Q}(a)$, which is very suitable for constructing subfamilies with higher rank. That family is

$$y^2 = x^3 + A(a)x^2 + B(a)x,$$

where

$$A(a) = -2(-51200 + 109440a + 38880a^2 + 55404a^3 + 6561a^4),$$

$$B(a) = 243a^2(20 + 3a)(-4 + 9a)(16 + 9a)(80 + 9a)(320 + 81a^2),$$

and the $x$-coordinates of four independent points of infinite order are

$$x_1 = 81a^2(-4 + 9a)(80 + 9a),$$

$$x_2 = 27a(20 + 3a)(-4 + 9a)(80 + 9a),$$

$$x_3 = \frac{1}{441}(-4 + 9a)(80 + 9a)(160 + 171a)^2,$$

$$x_4 = 3(20 + 3a)(-4 + 9a)(320 + 81a^2).$$

There are several substitutions which give subfamilies with rank $\geq 6$:

$$a = \frac{-2(-27 + 13w_1^2)(-13 + 27w_1^2)}{9(9 + 178w_1^2 + 9w_1^4)},$$

$$a = \frac{64(831744 - 40128w_2 + 4288w_2^2 - 44w_2^3 + w_2^4)}{9(-1520 + 88w_2 + w_2^2)(-2736 - 264w_2 + 5w_2^2)},$$

$$a = \frac{10732176 + 628992w_3 + 19192w_3^2 + 576w_3^3 + 9w_3^4}{36w_3(27 + w_3)(364 + 9w_3)},$$

$$a = \frac{5(-10 + 6w_4 + w_4^2)(-18 - 18w_4 + 5w_4^2)}{9(12 - 2w_4 + w_4^2)(3 - w_4 + w_4^2)},$$

$$a = \frac{5(584820 + 135432w_5 - 18288w_5^2 + 396w_5^3 + 5w_5^4)}{9(684 - 66w_5 + w_5^2)(171 - 33w_5 + w_5^2)}. $$
The first four substitutions were already given in [16], while the fifth substitution is new.

In order to find infinite families with rank $\geq 7$, we try to find intersections of these five families with rank $\geq 6$. We compare their $j$-invariants by factorizing their difference and seeking for the factors which correspond to curves with genus 1.

If we compare the second and third substitution, we find two suitable factors, which give the following conditions:

\begin{align}
& w_2^2 w_3^2 + 72 w_2^2 w_3 + 88 w_2 w_3^2 + 1820 w_2^2 - 1520 w_3^2 \\
& - 96096 w_2 - 65664 w_3 - 995904 = 0, \tag{3.2}
\end{align}

\begin{align}
& 5 w_2^2 w_3^2 + 216 w_2^2 w_3 - 264 w_2 w_3^2 + 3276 w_2^2 - 2736 w_3^2 \\
& + 288288 w_2 - 196992 w_3 - 497952 = 0. \tag{3.3}
\end{align}

Both conditions lead to

\begin{equation}
54 w_4^4 + 2736 w_3^3 + 66592 w_3^2 + 2987712 w_3 + 64393056 = 0. \tag{3.4}
\end{equation}

This quartic is birationally equivalent to the elliptic curve

$$ y^2 = x^3 + x^2 - 28174550x + 45644288448 $$

with rank equal to 3, hence the elliptic curve, and also the quartic, have infinitely many rational solutions. Many of them produce curves with rank $= 7$, e.g. $w_3 = -234, -30, -18, 26, 42, 94, -\frac{202}{3}, -\frac{182}{3}, -\frac{14}{3}$.

Consider the four points given by (3.1) and additional two points corresponding to the second and third substitutions. The second substitution gives the curve

\begin{equation}
 y^2 = x^3 + a_{62} x^2 + b_{62} x, \tag{3.5}
\end{equation}

where

\begin{align*}
a_{62} &= 79573 w_1^{16} + 2281840 w_1^{15} - 791687936 w_1^{14} - 34844285696 w_1^{13} \\
&+ 3065917324288 w_1^{12} + 556971294060544 w_1^{11} - 64165839736733696 w_1^{10} \\
&+ 336021145234263552 w_1^9 - 130403990149389221888 w_1^8 \\
&+ 3064512846261648359424 w_1^7 - 53369552205989831245824 w_1^6 \\
&+ 42249006919046891516732 w_1^5 + 2120995723090424777146368 w_1^4 \\
&- 21983951517250398896259072 w_1^3 - 455536370311599498486349824 w_1^2 \\
&+ 1197427029434259336824094720 w_1 + 38082411231292796255084740608,
\end{align*}
The third condition gives the curve

\[ b_{62} = -5184(w_2^4 - 44w_2^3 + 4288w_2^2 - 40128w_2 + 831744)^2 \]
\[ \times (w_2^4 + 352w_2^3 - 50720w_2^2 + 321024w_2 + 831744) \]
\[ \times (3w_2^4 + 352w_2^3 + 15328w_2^2 - 642048w_2 + 5822208) \]
\[ \times (7w_2^4 - 704w_2^3 + 15328w_2^2 + 321024w_2 + 2495232) \]
\[ \times (7w_2^4 - 176w_2^3 + 11680w_2^2 - 160512w_2 + 5822208) \]
\[ \times (7w_2^4 + 352w_2^3 - 61664w_2^2 + 321024w_2 + 5822208) \]
\[ \times (59w_2^4 + 3344w_2^3 - 572128w_2^2 + 3049728w_2 + 49072896), \]

and six independent points of infinite order with \( x \)-coordinates:

\[ x_{21} = -576(w_2^4 - 44w_2^3 + 4288w_2^2 - 40128w_2 + 831744)^2 \]
\[ \times (7w_2^4 - 176w_2^3 + 11680w_2^2 - 160512w_2 + 5822208) \]
\[ \times (7w_2^4 + 352w_2^3 - 61664w_2^2 + 321024w_2 + 5822208), \]
\[ x_{22} = 36(w_2^4 - 44w_2^3 + 4288w_2^2 - 40128w_2 + 831744) \]
\[ \times (7w_2^4 - 176w_2^3 + 11680w_2^2 - 160512w_2 + 5822208) \]
\[ \times (7w_2^4 + 352w_2^3 - 61664w_2^2 + 321024w_2 + 5822208) \]
\[ \times (59w_2^4 + 3344w_2^3 - 572128w_2^2 + 3049728w_2 + 49072896), \]
\[ x_{23} = -16/49(7w_2^4 - 176w_2^3 + 11680w_2^2 - 160512w_2 + 5822208) \]
\[ \times (7w_2^4 + 352w_2^3 - 61664w_2^2 + 321024w_2 + 5822208) \]
\[ \times (13w_2^4 - 2552w_2^3 + 330784w_2^2 - 2327424w_2 + 10812672)^2, \]
\[ x_{24} = -27/4(3w_2^4 + 352w_2^3 + 15328w_2^2 - 642048w_2 + 5822208) \]
\[ \times (7w_2^4 - 704w_2^3 + 15328w_2^2 + 321024w_2 + 2495232) \]
\[ \times (7w_2^4 - 176w_2^3 + 11680w_2^2 - 160512w_2 + 5822208) \]
\[ \times (59w_2^4 + 3344w_2^3 - 572128w_2^2 + 3049728w_2 + 49072896), \]
\[ x_{25} = -108(w_2^2 - 912)^2(w_2^4 + 352w_2^3 - 50720w_2^2 + 321024w_2 + 831744) \]
\[ \times (7w_2^4 + 352w_2^3 - 61664w_2^2 + 321024w_2 + 5822208) \]
\[ \times (59w_2^4 + 3344w_2^3 - 572128w_2^2 + 3049728w_2 + 49072896), \]
\[ x_{26} = 324(w_2^2 - 912)^2(w_2^4 + 352w_2^3 - 50720w_2^2 + 321024w_2 + 831744) \]
\[ \times (7w_2^4 - 176w_2^3 + 11680w_2^2 - 160512w_2 + 5822208) \]
\[ \times (7w_2^4 + 352w_2^3 - 61664w_2^2 + 321024w_2 + 5822208). \]

The third condition gives the curve

\[ y^2 = x^3 + a_{63}x^2 + b_{63}x, \]
where
\[
a_{03} = -13122w_3^{16} - 7348320w_3^{15} - 1570137696w_3^{14} - 206172584064w_3^{13} \\
- 19541430237312w_3^{12} - 1402008391816704w_3^{11} - 77606011598363136w_3^{10} \\
- 3410103604914358272w_3^9 - 123219415654113963008w_3^8 \\
- 3723833136566769233024w_3^7 - 9254375014630498607104w_3^6 \\
- 1852654232153731017572352w_3^5 - 27787335201034030779236352w_3^4 \\
- 32014307055930939026382848w_3^3 - 2662401630093588063697895424w_3^2 \\
- 136065032272957111027839631360w_3 - 26532681293226636504287281152,
\]

\[
b_{03} = 81(w_3^4 + 72w_3^3 + 8504w_3^2 + 550368w_3 + 10732176) \\
\times (3w_3^4 + 144w_3^3 + 3160w_3^2 + 157248w_3 + 3577392) \\
\times (3w_3^4 + 1152w_3^3 + 71144w_3^2 + 1257984w_3 + 3577392) \\
\times (9w_3^4 + 504w_3^3 + 8504w_3^2 + 78624w_3 + 1192464) \\
\times (9w_3^4 + 576w_3^3 + 19192w_3^2 + 628992w_3 + 10732176)^2, \\
\]

and six independent points of infinite order with \(x\)-coordinates:
\[
x_{31} = 9(3w_3^4 + 144w_3^3 + 3160w_3^2 + 157248w_3 + 3577392) \\
\times (3w_3^4 + 1152w_3^3 + 71144w_3^2 + 1257984w_3 + 3577392) \\
\times (9w_3^4 + 576w_3^3 + 19192w_3^2 + 628992w_3 + 10732176), \\
\]
\[
x_{32} = 9(3w_3^4 + 144w_3^3 + 3160w_3^2 + 157248w_3 + 3577392) \\
\times (3w_3^4 + 1152w_3^3 + 71144w_3^2 + 1257984w_3 + 3577392) \\
\times (9w_3^4 + 576w_3^3 + 19192w_3^2 + 628992w_3 + 10732176) \\
\times (9w_3^4 + 2736w_3^3 + 164872w_3^2 + 2987712w_3 + 10732176), \\
\]
\[
x_{33} = 1/49(3w_3^4 + 144w_3^3 + 3160w_3^2 + 157248w_3 + 3577392) \\
\times (3w_3^4 + 1152w_3^3 + 71144w_3^2 + 1257984w_3 + 3577392) \\
\times (171w_3^4 + 16704w_3^3 + 753128w_3^2 + 18240768w_3 + 203911344)^2, \\
\]
\[
x_{34} = 27(w_3^4 + 72w_3^3 + 8504w_3^2 + 550368w_3 + 10732176) \\
\times (3w_3^4 + 144w_3^3 + 3160w_3^2 + 157248w_3 + 3577392) \\
\times (9w_3^4 + 504w_3^3 + 8504w_3^2 + 78624w_3 + 1192464) \\
\times (9w_3^4 + 2736w_3^3 + 164872w_3^2 + 2987712w_3 + 10732176), \\
\]
\[
x_{35} = 27(w_3^4 - 1092)^2(3w_3^4 + 1152w_3^3 + 71144w_3^2 + 1257984w_3 + 3577392) \\
\times (9w_3^4 + 1152w_3^3 + 58040w_3^2 + 1257984w_3 + 10732176) \\
\times (9w_3^4 + 2736w_3^3 + 164872w_3^2 + 2987712w_3 + 10732176),
\]
the curves (3.5) and (3.6) are isomorphic, where the isomorphism is given by $w \mapsto \frac{3}{2} x^2 + \frac{1}{2} x + 1$.

We also give $y$-coordinates of the points corresponding to $x_{21}$ on (3.5) and $x_{31}$ on (3.6):

$$y_{21} = 5760(7w_2^4 + 352w_2^3 - 61664w_2^2 + 321024w_2 + 5822208)$$

$$\times (7w_2^4 - 176w_2^3 + 11680w_2^2 - 160512w_2 + 5822208)$$

$$\times (w_2^4 + 88w_2 - 1520)(5w_2^4 - 264w_2 - 2736)^2$$

$$\times (w_2^4 - 44w_2^3 + 4288w_2^2 - 40128w_2 + 831744)^2,$$

$$y_{31} = 92160w_2^2(3w_2^4 + 1152w_2^3 + 71144w_2^2 + 1257984w_3 + 3577392)$$

$$\times (9w_2^4 + 576w_2^3 + 19192w_2^2 + 628992w_3 + 10732176)^2$$

$$\times (3w_2^4 + 144w_2^3 + 3160w_2^2 + 157248w_3 + 3577392)(9w_3 + 364)^2(w_3 + 27)^2.$$
(3.2) or (3.3). Thus we proved that there are indeed infinitely many elliptic curves induced by rational Diophantine triples with rank \( \geq 7 \).

Analogous result can be obtained by considering the second and fifth substitution for the parameter \( a \). Here the conditions are

\[
9w_2^2w_5^2 - 4w_2w_5^2 - 198w_2^2 + 528w_5^2 + 1368w_2 - 8208w_5 = 0, \\
11w_2^2w_5^2 - 171w_2^2w_5 - 76w_2w_5^2 + 25992w_2 + 155952w_5 - 3430944 = 0,
\]

and they lead to the quartic

\[
w_5^4 - 1188w_5^3 + 43920w_5^2 - 406296w_5 + 116964 = 0,
\]

which is equivalent to the elliptic curve

\[y^2 = x^3 - x^2 - 124056x - 10126800\]

with rank equal to 1, so we again have infinitely many rational solutions. These solutions give seven points on the curve (3.5). By taking the specialization \((w_2, w_5) = \left( \frac{6392}{69}, \frac{6392}{69} \right)\) we can check that these seven points are indeed independent, and by Silverman’s specialization theorem we conclude that we obtained another infinite family of curves with rank \( \geq 7 \).

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