Aspects of proper differential sequences of ordinary differential equations

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Abstract:
We define a proper differential sequence of ordinary differential equations and introduce a method to derive an alternative sequence of integrals for such a sequence. We describe some general properties which are illustrated by several examples.

1 Introduction

In a recent paper M Euler and the present authors reported a symmetry analysis and Painlevé analysis of two sequences of ordinary differential equations, namely a Riccati sequence and an Ermakov-Pinney sequence [5]. Andriopoulos and Leach [1] used the singularity analysis and explicit solution of the Riccati Sequence as a vehicle to demonstrate some specific results which can arise during the course of the singularity analysis. Subsequently Andriopoulos et al [2] made a detailed study of the symmetry and singularity properties of the Riccati Sequence.

The aim of the present paper is to define the proper differential sequence and discuss its integrability. We also introduce an alternative sequence where the equations of the higher order members in the sequence do not increase in order, but are fixed by the first equation in the sequence. This approach can help to integrate the sequence and provides in some cases a direct route to the first integrals of the equations in the differential sequence.

The paper is organised as follows: In Section 2 we give definitions regarding proper differential sequences, their Lie symmetry algebra and their integrability. In Section 3 we introduce a method to derive an alternative sequence and provide several examples to illustrate the concept of compatible and completely compatible sequences. In an Appendix we provide details of the Lie point symmetry analysis of some of the sequences discussed in Section 3.

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2 General description

Consider the variables $x$ and $u$, where $u = u(x)$ with $u_x = du/dx$, $u_{xx} = d^2u/dx^2$ and the additional notation $u_{nx} = d^n u/dx^n$. We now consider a differential sequence of $m$ equations,

$$\{E_1, E_2, \ldots, E_m\}, \quad (2.1)$$

in the following form:

$$E_1 := F(u, u_x, u_{xx}, \ldots, u_{nx}) = 0$$

$$E_2 := R^k[u] F(u, u_x, u_{xx}, \ldots, u_{nx}) = 0$$

$$E_3 := (R^k[u])^2 F(u, u_x, u_{xx}, \ldots, u_{nx}) = 0 \quad (2.2)$$

$$\vdots$$

$$E_m := (R^k[u])^{m-1} F(u, u_x, u_{xx}, \ldots, u_{nx}) = 0,$$

where $R^k[u]$ is a $k$th-order integrodifferential operator of the form

$$R^k[u] = G_k D^k_x + G_{k-1} D^{k-1}_x + \cdots + G_0 + Q D^{-1}_x \circ J. \quad (2.3)$$

The adjoint of $R^k[u]$ has the form

$$(R^k)^*[u] = \sum_{i=0}^{k} (-1)^i D^i_x \circ G_i - J D^{-1}_x \circ Q. \quad (2.4)$$

We term $E_1$ the seed equation of the differential sequence $\{E_i\}$. Note that the second equation,

$$E_2 := R^k[u] F(u, u_x, u_{xx}, \ldots, u_{nx}) = 0,$$ \quad (2.5)

is of order $n + k$, the third equation,

$$E_3 := (R^k)^2[u] F(u, u_x, u_{xx}, \ldots, u_{nx}) = 0,$$ \quad (2.6)

is of order $n + 2k$ and the $m$th equation $E_m$ is of order $n + (m - 1)k$.

Let $L_{E_i}[u]$ denote the linear operator

$$L_{E_i}[u] = \frac{\partial E_i}{\partial u} + \frac{\partial E_i}{\partial u_x} D_x + \frac{\partial E_i}{\partial u_{xx}} D^2_x + \cdots + \frac{\partial E_i}{\partial u_{nx}} D^m_x \quad (2.7)$$

and $L^*_i[u]$ the adjoint to $L_{E_i}[u]$, namely

$$L^*_i[u] = \frac{\partial E_i}{\partial u} - D_x \circ \frac{\partial E_i}{\partial u_x} + D^2_x \circ \frac{\partial E_i}{\partial u_{xx}} + \cdots + (-1)^q D^q_x \circ \frac{\partial E_i}{\partial u_{nx}}. \quad (2.8)$$
We denote by \( Z^i(E_i) \) the vertical symmetry generator of the equation \( E_i \) in the sequence (2.2), namely
\[
Z^i(E_i) = Q(x, u, u_x, u_{xx}, u_{3x}, \ldots, u_{jx}) \partial_u
\]
where the necessary and sufficient invariance condition for equation \( E_i \) is
\[
L_{E_i}Q \bigg|_{E_i=0} = 0.
\]
(2.10)

Note that (2.9) includes the point symmetry generators
\[
\Gamma_i = \xi(x, t, u) \partial_x + \eta(x, t, u) \partial_u
\]
(2.11)
with symmetry characteristic \( Q(x, u, u_x) = \xi(x, u)u_x - \eta(x, u) \) and equivalent vertical form
\[
Z^i = [\xi(x, u)u_x - \eta(x, u)] \partial_u.
\]
(2.12)

**Definition 1:** The sequence (2.2) admits a \( p \)-dimensional Lie symmetry algebra, \( \mathcal{L} \), spanned by the linearly independent symmetry generators
\[
\{ Z_1^i(E_i), Z_2^i(E_i), \ldots, Z_p^i(E_i) \}
\]
if each equation in the sequence (2.2), \( \{ E_1, E_2, \ldots, E_m \} \), admits a \( p \)-dimensional Lie symmetry algebra, \( \mathcal{L}' \), isomorphic to \( \mathcal{L} \).

**Definition 2:** \( J = J(x, u, u_x, u_{xx}, \ldots) \) is an integrating factor for the differential sequence (2.2) if \( J \) is an integrating factor for each equation in the sequence.

**Definition 3:** The operator \( R^{[k]}[u] \) of the form (2.3) is defined as a \( k \)-th-order recursion operator of the differential sequence (2.2) under the following conditions:
\[
\left[ L_{E_i}[u], R^{[k]}[u] \right] = 0, \quad i = 1, 2, \ldots, m,
\]
(2.14a)
\[
(R^{[k]})^*[u]J_k = \alpha J_l \quad \forall \quad k, l = 1, 2, \ldots, p,
\]
(2.14b)
where \( \alpha \) is a nonzero constant, \( i = 1, 2, \ldots, m \) and \( p \) is the total number of integrating factors, \( J_l \), valid for all members of the sequence. For some values of \( l \), \( J_l \) may be zero.

**Definition 4:** A proper differential sequence of ordinary differential equations is a differential sequence which admits at least one recursion operator of the form (2.3).

**Definition 5:** An integrable differential sequence is defined as a proper differential sequence of ordinary differential equations for which each equation in the sequence is integrable.
Remark: By an integrable ordinary differential equation of nth order, we mean an equation which admits a solution, \(u = \phi(x; c_1, \ldots, c_n)\), where \(c_j, j = 1, n\) are independent arbitrary constants. In less strict sense we require that the nth-order equation admits \(n - 1\) functionally independent first integrals such that the general solution can be expressed as a quadrature. Integrability of a nonlinear ordinary differential equation can also be expressed in terms of its singularity structure in the complex domain. This is known as the Painlevé Property and requires that the solutions possess only movable poles as singularities (see for example [4] for some recent reviews on the Painlevé Property). It can be of interest to study the Painlevé Property of proper differential sequences, but this falls outside the scope of the current paper. We refer the to the papers by Andriopoulos et al [1, 2] in which singularity analysis is used to study the integrability of a Riccati sequence.

Let
\[E_i := u_{qx} - f_i(x, u, u_x, \ldots, u_{(q-1)x}) = 0,\] (2.15)

where
\[q = n + (m - 1)k.\]

We introduce the following total derivative operator
\[D_{E_i} = D_x\bigg|_{E_i=0} = \frac{\partial}{\partial x} + \sum_{j=0}^{q-1} u_{jx} \frac{\partial}{\partial u_{(j-1)x}} + f_i(x, u, u_x, \ldots, u_{(q-1)x}) \frac{\partial}{\partial u_{(q-1)x}}.\] (2.16)

**Proposition 1:** \(J_s\) is an integrating factor for the sequence (2.2) if and only if the following conditions are satisfied:
\[L_{E_i[u]}^* J_s(x, u, u_x, \ldots)\bigg|_{E_i=0} = 0, \quad i = 1, 2, \ldots, m,\] (2.17a)
\[\frac{\partial J_s}{\partial u_{(q-2r)x}} + \sum_{j=1}^{2r-1} (-1)^{j-1} \frac{\partial}{\partial u_{(q-1)x}} \left\{ D_{E_i}^{j-1} \left( \frac{\partial f_i}{\partial u_{(j+q-2r)x}} J_s \right) \right\} \]
\[+ \frac{\partial}{\partial u_{(q-1)x}} \left( D_{E_i}^{2r-1} J_s \right) = 0, \quad s = 1, 2, \ldots, p, \quad r = 1, 2, \ldots, \left\lfloor \frac{q}{2} \right\rfloor.\] (2.17b)

Here \(\left\lfloor \frac{q}{2} \right\rfloor\) is the largest natural number less than or equal to the number \(\frac{q}{2}\), \(i = 1, 2, \ldots, m\), and \(p\) is the total number of integrating factors, \(J_s\), valid for all members of the sequence, i.e. \(s = 1, 2, \ldots, p\).

Remark: The derivation of the necessary and sufficient conditions for integrating factors of single ordinary differential equations of order \(n\) are derived in the book of Bluman and Anco [3]. Proposition 1 is a natural extension of this result to proper differential sequences of ordinary differential equations. Note that condition (2.17a) ensures that each \(J_s\) is an...
adjoint symmetry for each equation in the sequence and (2.17b) ensures that this adjoint symmetry is an integrating factor for each member of the sequence.

Example 1: We consider the seed equation

\[ u_{xx} + u_x^2 = 0. \]  

(2.18)

with the differential operator

\[ R[u] = D_x + u_x. \]  

(2.19)

This gives a proper differential sequence

\[ R^j[u] \left( u_{xx} + u_x^2 \right) = 0 \]  

(2.20)

with zeroth-order integrating factors

\[ J_1(x, u) = e^u, \quad J_2(x, u) = xe^u. \]  

(2.21)

Here

\[ R^*[u]e^u = 0, \quad R^*[u](xe^u) = -e^u. \]  

(2.22)

In the next section we show that the sequence (2.20) is an integrable sequence of ordinary differential equations and discuss its properties.

3 An alternative description

It is of interest to construct an alternative sequence,

\[ \{ \tilde{E}_1, \tilde{E}_2, \ldots, \tilde{E}_m \}, \]  

(3.1)

to (2.2), ie \{E_1, E_2, \ldots, E_m\}, namely one in which the order of the differential equations in the sequence (3.1) does not increase but is fixed by the seed equation \( \tilde{E}_1 \). For the same seed equation, \( E_1 = \tilde{E}_1 \), the two sequences (2.1) and (3.1) should then be compatible or completely compatible.

Definition 6: Two equations, \( E_j \) and \( \tilde{E}_j \) from the sequences (2.2) and (3.1) respectively, are called compatible if the equations admit at least one common solution. The two equations are called completely compatible if the general solution of \( \tilde{E}_j \) gives the general solution for \( E_j \). Two sequences of \( m \) equations, \{E_1, E_2, \ldots, E_m\} and \{\tilde{E}_1, \tilde{E}_2, \ldots, \tilde{E}_m\} with the same seed equation \( E_1 = \tilde{E}_1 \), is called compatible if each equation in the sequence admits at least one common solution between the corresponding members in the two sequences. The sequences are called completely compatible if the general solution of \( \tilde{E}_j \) provides the general solution for \( E_j \) for all members of the sequence, ie for all \( j = 1, 2, \ldots, m \). The sequence (3.1) is termed an alternative sequence to (2.2) if the two sequences are at least compatible.
Since the order of the equations in an alternative sequence \((3.1)\) does not increase, the equations that make up an alternative sequence should define integrals for the equations in the proper differential sequence \((2.1)\) to guarantee compatibility of its solutions. We introduce the following

**Proposition 2:** Consider a proper differential sequence \(\{E_1, E_2, \ldots, E_m\}\) with recursion operator \(R^{[k]}[u]\). An alternative sequence, \(\{\tilde{E}_1, \tilde{E}_2, \ldots, \tilde{E}_m\}\), of the form

\[
\tilde{E}_1 := F(u, u_x u_{xx}, \ldots, u_{n_x}) = 0 \tag{3.2a}
\]

\[
\tilde{E}_{j+1} := F(u, u_x u_{xx}, \ldots, u_{n_x}) = Q_j(x, u, u_x, \ldots, \omega^1, \omega^2, \ldots, \omega^\ell; c_1, c_2, \ldots, c_s) \tag{3.2b}
\]

\(j = 1, 2, \ldots, m - 1,\)

is compatible with the proper differential sequence \(\{E_1, E_2, \ldots, E_m\}\) with \(E_1 = \tilde{E}_1\) if

\[
R^{[k]}Q_1 = 0 \tag{3.3a}
\]

\[
R^{[k]}Q_i = Q_{i-1}, \quad i = 2, 3, \ldots, m. \tag{3.3b}
\]

Here \(\omega^1, \omega^2, \ldots, \omega^\ell\) are nonlocal coordinates defined by

\[
\frac{d\omega^1}{dx} = g_1(u), \tag{3.4a}
\]

\[
\frac{d\omega^2}{dx} = g_2(\omega^1), \quad \frac{d\omega^3}{dx} = g_3(\omega^2), \quad \ldots, \quad \frac{d\omega^\ell}{dx} = g_\ell(\omega^{\ell-1}) \tag{3.4b}
\]

for some differentiable functions \(g_k\).

Below we discuss several examples of proper differential sequences which are compatible or completely compatible with their alternative sequences. The examples are sufficiently simple to demonstrate the method of construction and to investigate the properties of the sequences. A classification is not at all attempted here, but will be addressed in future works. For our examples we consider the following four seed equations which are part of the list of second-order linearisable evolution equations in \((1 + 1)\) dimensions reported in [6]:

\[
u_{xx} + u_x^2 = 0
\]

\[
u_{xx} + h(u)u_x^2 = 0
\]

\[
u_{xx} + \lambda u_x - \frac{h'(u)}{h(u)}u_x^2 + h(u) = 0
\]

\[
u_{xx} + uu_x = 0
\]

for arbitrary differentiable functions \(h\), where prime denotes the derivative with respect to \(u\).
**Example 2:** Firstly we consider the proper differential sequence, \( \text{(2.20)} \), already introduced in Example 1, where
\[
R[u] = D_x + u_x.
\]

The proper differential sequence is
\[
\begin{align*}
E_1 &:= F(u, u_x u_{xx}) = u_{xx} + u_x^2 = 0 \\ 
E_2 &:= R[u]F(u, u_x u_{xx}) = u_{3x} + 3u_x u_{xx} + u_x^3 = 0 \\ 
E_3 &:= R^2[u]F(u, u_x u_{xx}) = u_{4x} + 4u_x u_{3x} + 3u_{xx}^2 + 6u_x^2 u_{xx} + u_x^4 = 0 \\
& \vdots \\
E_m &:= R^{m-1}[u]F(u, u_x u_{xx}) = u_{(m+1)x} + \cdots = 0.
\end{align*}
\]

We apply Proposition 2 to calculate the alternative sequence with seed equation (3.5a). The second member in the alternative sequence is
\[
E_1 := F(u, u_x u_{xx}) = u_{xx} + u_x^2 = 0
\]
under the condition
\[
R[u]Q_1(x, u, u_x, \ldots) = 0.
\]

Condition (3.7) is of the form
\[
D_x(Q_1) = -u_x Q_1
\]
with general solution
\[
Q_1(u, c_1) = c_1 e^{-u},
\]
where \( c_1 \) is an arbitrary constant of integration. Thus the second member in the alternative sequence is
\[
u_{xx} + u_x^2 = c_1 e^{-u}.
\]
The third member, \( \text{(3.5c)} \), becomes
\[
u_{xx} + u_x^2 = Q_2(x, u, u_x, \ldots)
\]
in the alternative sequence under the condition
\[
R[u]Q_2(x, u; c_1, c_2) = Q_1(u; c_1),
\]
which admits the general solution
\[
Q_2(x, u; c_1, c_2) = c_1 x e^{-u} + c_2 e^{-u}
\]
with $c_2$ another constant of integration. Thus the third member in the alternative sequence is
\[ u_{xx} + u_x^2 = e^{-u} (c_1 x + c_2) \] (3.14)
which can be presented in the form
\[ u_{xx} + u_x^2 = e^{-u} D_x^{-1} c_1. \] (3.15)

The next member in the alternative sequence is
\[ u_{xx} + u_x^2 = e^{-u} \left( \frac{1}{2} c_1 x^2 + c_2 x + c_3 \right) \equiv e^{-u} D_x^{-2} c_1. \] (3.16)

The functions $Q_k$ in $E_{k+1}$ are
\[ Q_k = e^{-u} q(x) \text{ with } q^{(k)}(x) = 0 \iff q(x) = \sum_{j=1}^{k} \frac{c_j}{(k-j)!} x^{k-j}. \] (3.17)

We thus conclude that the alternative sequence to the differential sequence (3.5a) - (3.5d) has the form
\[ \tilde{E}_1 := u_{xx} + u_x^2 = 0 \] (3.18a)
\[ \tilde{E}_j := u_{xx} + u_x^2 = e^{-u} D_x^{-(j-2)} c_1, \quad j = 2, 3, \ldots, m, \] (3.18b)

where $D_x^{-q}$ are $q \in \mathbb{N}$ compositions of the integral operator $D_x^{-1}$ and $D_x^0 := 1$. In explicit form the alternative sequence is
\[ \tilde{E}_1 := u_{xx} + u_x^2 = 0 \] (3.19a)
\[ \tilde{E}_2 := u_{xx} + u_x^2 = Q_1 \text{ with } Q_1 = e^{-u} c_1 \] (3.19b)
\[ \tilde{E}_3 := u_{xx} + u_x^2 = Q_2 \text{ with } Q_2 = e^{-u} (c_1 x + c_2) \] (3.19c)
\[ \tilde{E}_4 := u_{xx} + u_x^2 = Q_3 \text{ with } Q_3 = e^{-u} \left( \frac{1}{2} c_1 x^2 + c_2 x + c_3 \right) \] (3.19d)
\[ \vdots \] (3.19e)
\[ \tilde{E}_m := u_{xx} + u_x^2 = Q_{m-1} \text{ with } Q_{m-1} = e^{-u} \left( \sum_{j=1}^{m-1} \frac{c_j}{(m-j-1)!} x^{m-j-1} \right). \] (3.19f)

It is easy to establish that the proper differential sequence (3.5a) - (3.5d) is an integrable differential sequence since each member of the sequence is linearisable by the change for variables
\[ w(X) = u_x e^u, \quad X = x. \] (3.20)
Moreover the alternative sequence \((3.19a) - (3.19f)\) is linearisable by the change of variables

\[ w(X) = e^u, \quad X = x. \quad (3.21) \]

To establish the compatibility or complete compatibility of the two sequences \((3.5a) - (3.5d)\) and \((3.19a) - (3.19f)\) we take a closer look at the corresponding members.

- Compare the members \(E_2\) and \(\tilde{E}_2\):
  A first integral for \(E_2\) is given by \(\tilde{E}_2\), namely
  \[ c_1 = e^u \left( u_{xx} + u_x^2 \right). \quad (3.22) \]
  Therefore the general solution of \(\tilde{E}_2\) gives the general solution of \(E_2\) with \(c_1\) as one of the constants of integration for \(E_2\). Hence the two equations, \(E_2\) and \(\tilde{E}_2\), are completely compatible.

- Compare the members \(E_3\) and \(\tilde{E}_3\):
  A second integral for \(E_3\) is given by \(\tilde{E}_3\), namely
  \[ c_1 x + c_2 = e^u \left( u_{xx} + u_x^2 \right). \quad (3.23) \]
  Therefore the general solution of \(\tilde{E}_3\) gives the general solution of \(E_3\) (with \(c_1\) and \(c_2\) as two of the constants of integration for \(E_3\)) and the two equations \(E_3\) and \(\tilde{E}_3\) are completely compatible. A similar argument follows for all equations in the proper differential sequence \((3.5a) - (3.5d)\).

We conclude that the two sequences \((3.5a) - (3.5d)\) and \((3.19a) - (3.19f)\) are completely compatible.

Another interesting property of the sequences \((3.5a) - (3.5d)\) and \((3.19a) - (3.19f)\) is that the symmetry characteristics, \(\eta_j\), of the solution symmetries,

\[ \Gamma^s_j = \eta_j(x, u)\partial_u, \quad (3.24) \]

for the equations in \((3.5a) - (3.5d)\) are given by the functions \(Q_1, Q_2, \ldots\) in the alternative sequence \((3.19a) - (3.19f)\). In particular

The symmetry characteristic, \(\eta_j\), for the solution symmetry of \(E_j\) in \((3.5a) - (3.5d)\) is given by \(Q_{j+1}\) of the equation \(\tilde{E}_{j+2}\) in \((3.19a) - (3.19f)\) for all \(j = 1, 2, \ldots, m\).

For example \(E_1 := u_{xx} + u_x^2 = 0\) admits the solution symmetry

\[ \Gamma^s_1 = Q_2\partial_u, \quad (3.25) \]

where \(Q_2 = e^{-u}(c_1 x + c_2)\) corresponds to the right-hand expression in \(\tilde{E}_3\).

Note that the complete set of all point symmetries for the proper differential sequence \((3.5a) - (3.5d)\) is the following:
For $E_1$ the complete set of Lie point symmetries are
\[
\{e^{-u}\partial_u, xe^{-u}\partial_u, \partial_x, x\partial_x + x\partial_u, x^2\partial_x + x\partial_u, e^u\partial_x, xe^u\partial_x + e^u\partial_u\} \tag{3.26}
\]
For $E_k$, $k = 2, 3, \ldots, m$, the complete set of Lie point symmetries are
\[
\{q(x)e^{-u}\partial_u, \partial_u, \partial_x, x\partial_x + \frac{1}{2}(n-1)\partial_u, x^2\partial_x + (n-1)x\partial_u\}, \tag{3.27}
\]
where $n$ is the order of the differential equation, $E_k$, and
\[
q^{(n)}(x) = 0. \tag{3.28}
\]
The Lie point symmetry properties of the alternative sequence $(3.19a) – (3.19f)$ are discussed in the Appendix.

**Example 3:** Consider the seed equation
\[
u_{xx} + h(u)u_x^2 = 0 \tag{3.29}
\]
with recursion operator
\[
R[u] = D_x + h(u)u_x. \tag{3.30}
\]
This defines the proper differential sequence of the form
\[
E_1 := u_{xx} + h(u)u_x^2 = 0 \tag{3.31a}
\]
\[
E_{j+1} := R^j[u] (u_{xx} + h(u)u_x^2) = 0, \quad j = 1, 2, \ldots, m - 1. \tag{3.31b}
\]
We apply Proposition 2 and calculate the functions $Q_j$ in the same manner as in Example 2. This leads to the following alternative sequence
\[
\tilde{E}_1 := u_{xx} + u_x^2 = 0 \tag{3.32a}
\]
\[
\tilde{E}_{j+1} := u_{xx} + u_x^2 = \exp \left[ - \int h(u)du \right] D_x^{-(j-1)c_1}, \quad j = 1, 2, \ldots, m - 1. \tag{3.32b}
\]
It is easy to show that the sequence $(3.31a) – (3.31b)$ and its alternative sequence $(3.32a) – (3.32b)$ are completely compatible and that $(3.31a) – (3.31b)$ is an integrable sequence. The linearisation and Lie point symmetries of $(3.32a) – (3.32b)$ are discussed in the Appendix.

**Example 4:** Consider the seed equation
\[
u_{xx} + \lambda u_x - \frac{h'(u)}{h(u)}u_x^2 + h(u) = 0 \tag{3.33}
\]
with the recursion operator
\[
R[u] = D_x - \frac{h'(u)}{h(u)}u_x. \tag{3.34}
\]
This gives the proper differential sequence

\[ E_1 := u_{xx} + \lambda u_x - \frac{h'(u)}{h(u)} u_x^2 + h(u) = 0 \]  

(3.35a)

\[ E_{j+1} := R^j[u] \left( u_{xx} + \lambda u_x - \frac{h'(u)}{h(u)} u_x^2 + h(u) \right) = 0, \quad j = 1, 2, \ldots, m - 1, \]  

(3.35b)

with its alternative sequence

\[ \tilde{E}_1 := u_{xx} + \lambda u_x - \frac{h'(u)}{h(u)} u_x^2 + h(u) = 0 \]  

(3.36a)

\[ \tilde{E}_{j+1} := u_{xx} + \lambda u_x - \frac{h'(u)}{h(u)} u_x^2 + h(u) = h(u) D_x^{(j-1)} c_1, \quad j = 1, 2, \ldots, m - 1, \]  

(3.36b)

where \( h \) is an arbitrary differentiable function. Just like the sequences in Example 2 and Example 3 the sequences (3.35a) – (3.35b) and (3.36a) – (3.36b) are completely compatible and (3.35a) – (3.35b) is an integrable sequence. Details are given in the Appendix.

**Example 5:** To the Burgers equation

\[ u_{xx} + uu_x = u_t \]  

(3.37)

one can (with standard symmetry reduction following the \( t \)-translation invariance) associate

\[ u_{xx} + uu_x = 0 \]  

(3.38)

which shares the same integrodifferential recursion operator,

\[ R[u] = D_x + \frac{1}{2} u + \frac{1}{2} u_x D_x^{-1} \circ 1. \]  

(3.39)

The proper differential sequence, which we name the *Burgers Sequence*, is

\[ E_1 := u_{xx} + uu_x = 0 \]  

(3.40a)

\[ E_{j+1} := R^j[u] (u_{xx} + uu_x) = 0, \quad j = 1, 2, \ldots, m. \]  

(3.40b)

We now construct an alternative Burger’s Sequence following Proposition 2. The solution of \( R[u] Q_1 = 0 \) is

\[ Q_1 = \left( -2A \exp \left[ -\frac{1}{2} \int u \, dx \right] + 2B \exp \left[ -\frac{1}{2} \int u \, dx \right] \int \exp \left[ \frac{1}{2} \int u \, dx \right] \, dx \right) \]  

(3.41)

where \( A \) and \( B \) are constants of integration. Equation (3.41) is an integrodifferential equation. It can be rendered as an ordinary differential equation by defining

\[ w = \int \exp \left[ \frac{1}{2} \int u \, dx \right] \, dx \]  

(3.42)
so that
\[ u_{xx} + uu_x = \left( -2A \exp \left[ -\frac{1}{2} \int u \, dx \right] + 2B \exp \left[ -\frac{1}{2} \int u \, dx \right] \int \exp \left[ \frac{1}{2} \int u \, dx \right] \, dx \right)_x \] (3.43)
becomes
\[ \frac{w_{4x}}{w_x} = \frac{w_{xx}w_{3x}}{w_x^2} = A w_{xx} + B \left( 1 - \frac{ww_{xx}}{w_x^2} \right). \] (3.44)

In a similar fashion the equation \( R[u]Q_2 = Q_1 \) has the solution
\[ Q_2 = \left\{ 2C \exp \left[ -\frac{1}{2} \int u \, dx \right] \int \exp \left[ \frac{1}{2} \int u \, dx \right] \, dx - 2Ax \exp \left[ -\frac{1}{2} \int u \, dx \right] \right. \\
+ 2B \exp \left[ -\frac{1}{2} \int u \, dx \right] \int \left( \int \exp \left[ \frac{1}{2} \int u \, dx \right] \, dx \right) \, dx \right\}, \] (3.45)
where \( C \) is also a constant of integration, and the integrodifferential equation is
\[ u_{xx} + uu_x = \left\{ 2C \exp \left[ -\frac{1}{2} \int u \, dx \right] \int \exp \left[ \frac{1}{2} \int u \, dx \right] \, dx - 2Ax \exp \left[ -\frac{1}{2} \int u \, dx \right] \right. \\
+ 2B \exp \left[ -\frac{1}{2} \int u \, dx \right] \int \left( \int \exp \left[ \frac{1}{2} \int u \, dx \right] \, dx \right) \, dx \right\}. \] (3.46)
The corresponding higher-order ordinary differential equation is
\[ \frac{w_{5x}}{w_{xx}} = w_{xx}w_{3x} = C \left( 1 - \frac{w_{xx}w_{3x}}{w_{xx}^2} \right) + A \left( \frac{xw_{3x}}{w_{xx}^2} - \frac{1}{w_{xx}} \right) + B \left( \frac{w_x}{w_{xx}} - \frac{ww_{xx}}{w_{xx}^2} \right), \] (3.47)
where now
\[ w = \int \left( \int \exp \left[ \frac{1}{2} \int u \, dx \right] \, dx \right) \, dx \] (3.48)
or equivalently
\[ u = 2 \frac{w_{3x}}{w_{xx}}. \] (3.49)

Evidently one could continue in like fashion. It contrast to Examples 2, 3 and 4 in which the recursion operator did not contain an inverse derivative to obtain an ordinary differential equation one must redefine the dependent variable as in (3.42) and (3.47) (and the obvious extension for higher elements of the sequence). We note that the terms on the left sides of (3.44) and (3.46) have the same form apart from the increase in the order of each derivative. Consequently, if we wish to have a differential sequence based upon the differential equation (3.38) and its recursion operator, (3.39), of \( m \) elements in terms of differential equations, all differential equations belonging to the sequence must be written in terms of a differential equation of order \( m + 2 \). The alternative is an integrodifferential equation of increasing nonlocality.
We thus conclude that the first three terms in the alternative sequence take the following forms

\[
\tilde{E}_1(w) := \frac{w_{5x} - w_{3x}w_{4x}}{w_{xx}^2} = 0
\]

\[\Leftrightarrow \left( \frac{w_{4x}}{w_{xx}} \right)_x = 0 \] \hspace{1cm} (3.50a)

\[\Leftrightarrow w_{4x} = k_1w_{2x} \] \hspace{1cm} (3.50b)

\[
\tilde{E}_2(w) := \frac{w_{5x} - w_{3x}w_{4x}}{w_{xx}^2} = A\frac{w_{3x}}{w_{xx}^2} + B \left( 1 - \frac{w_x w_{3x}}{w_{xx}^2} \right)
\]

\[\Leftrightarrow \left( \frac{w_{4x}}{w_{xx}} \right)_x = -\left( \frac{A}{w_{xx}} \right)_x + B \left( \frac{w_x}{w_{xx}} \right)_x \] \hspace{1cm} (3.50c)

\[\Leftrightarrow w_{4x} = a_1w_{xx} + Bw_x - A \] \hspace{1cm} (3.50d)

\[
\tilde{E}_3(w) := \frac{w_{5x} - w_{3x}w_{4x}}{w_{xx}^2} = C \left( 1 - \frac{w_x w_{3x}}{w_{xx}^2} \right) + A \left( \frac{xw_{3x}}{w_{xx}^2} - \frac{1}{w_{xx}} \right)
\]

\[+ B \left( \frac{w_x}{w_{xx}} - \frac{w w_{3x}}{w_{xx}^2} \right) \] \hspace{1cm} (3.50e)

\[\Leftrightarrow \left( \frac{w_{4x}}{w_{xx}} \right)_x = C \left( \frac{w_x}{w_{xx}} \right)_x - A \left( \frac{x}{w_{xx}} \right)_x + B \left( \frac{w}{w_{xx}} \right)_x \] \hspace{1cm} (3.50f)

\[\Leftrightarrow w_{4x} = a_2w_{xx} + Cw_x + Bw - Ax. \] \hspace{1cm} (3.50g)

However, such a differential sequence should not be confused with the normal type of differential sequence since, as the value of \( m \) increases, both the left side of the equations and the recursion operator must be redefined.

In order to establish compatibility of the first three members of the two sequences, (3.40) – (3.40) and (3.50) – (3.50), need to be written in the same variable \( w \), that is, we need to apply the transformation (3.49) and transform the first three members of the differential sequence (3.40) – (3.40) in order to write the equations in terms of the variable \( w \). We obtain the following:

\[
E_1(w) := \frac{w_{5x} - w_{3x}w_{4x}}{w_{xx}^2} = 0 \Leftrightarrow \left( \frac{w_{4x}}{w_{xx}} \right)_x = 0 \Leftrightarrow w_{3x} = k_1w_x + k_{11} \] \hspace{1cm} (3.51a)

\[
E_2(w) := \frac{w_{6x} - w_{3x}w_{5x}}{w_{xx}^2} = 0 \Leftrightarrow \left( \frac{w_{5x}}{w_{xx}} \right)_x = 0 \Leftrightarrow w_{4x} = k_2w_x + k_{21} \] \hspace{1cm} (3.51b)

\[
E_3(w) := \frac{w_{7x} - w_{3x}w_{6x}}{w_{xx}^2} = 0 \Leftrightarrow \left( \frac{w_{6x}}{w_{xx}} \right)_x = 0 \Leftrightarrow w_{5x} = k_3w_x + k_{31} \] \hspace{1cm} (3.51c)
All \( k \)s are constants of integration.

Comparing (3.50f) with (3.51b) and (3.50i) with (3.51c) it is clear that these two sequences are compatible but not completely compatible. In particular the expression

\[
\w_4 = B \w_x - A,
\]

which is (3.50f) with \( a_1 = 0 \), is a second integral for \( E_2 \), namely equation (3.51b). However, the same second integral for (3.51c) follows from (3.50i), where the two additional parameters \( a_2 \) and \( C \) have to be zero for compatibility. Therefore the higher members of the alternative sequence do not provide additional parameters for the integration of the differential sequence (3.40a) – (3.40b) and we conclude that the two sequences only share special solutions. The proper differential sequence, (3.40a) – (3.40b), is, however, integrable since every member of the sequence can be linearised. The same is of course true for the alternate sequence (3.50a), (3.50d) and (3.50g).

Unlike the Examples 2, 3 and 4 there is no preservation of Lie point symmetries in the alternate sequence. In the cases of (3.50a), (3.50d) and (3.50g) we obtain

\[
\begin{align*}
\Gamma_1 &= \partial_x, \quad \Gamma_2 = \partial_w, \quad \Gamma_3 = x \partial_x, \quad \Gamma_4 = x \partial_w, \quad \Gamma_5 = w \partial_w \\
\Gamma_1 &= \partial_x, \quad \Gamma_2 = \partial_w, \quad \Gamma_6 = (Ax - Bw) \partial_w \\
\Gamma_7 &= B \partial_x + A \partial_w, \quad \Gamma_8 = (ABx - B^2 w - AC) \partial_w,
\end{align*}
\]

respectively.

Above we only looked at the first three members of the sequence. We end this example with the following statements about the full sequence (3.40a) – (3.40b).

The \( n \)th element of the alternative Burgers Differential Sequence written in the integro-differential form

\[
u_{xx} + uu_x = \exp \left[ -\frac{1}{2} \int udx \right] \left( \sum_{i=1}^{n-1} B_i D_x^{-i} \exp \left[ \frac{1}{2} \int udx \right] \right)
\]

is linearised to

\[
W_{(n+1)x} = B_{n-2} + B_{n-1} W,
\]

where \( W = D_x^{-(n-1)} \exp \left[ \frac{1}{2} \int udx \right] \).

Remark: This procedure for the linearisation of (3.54) is a natural generalisation of the Cole-Hopf transformation which can also be derived via the \( x \)-generalised hodograph transformation for evolution equations [6].

The \( n \)th element of the Burgers Differential Sequence (3.40a) – (3.40b), ie

\[
R^{n-1}[u] (u_{xx} + uu_x) = 0,
\]

(3.56)
where $R[u] = D_x + \frac{1}{2}u + \frac{1}{2}u_x D_x^{-1}$, is linearised to

$$v_{(n+1)} = \Omega^{n+1}_n v,$$

where $u = 2v_x/v$ and $\Omega$ are arbitrary constants.

**Remark:** Here we make use of the relationship between the elements of the Burgers Differential Sequence (3.40a) – (3.40b) and the sequence in Example 2.

### 4 Discussion

Although no general Theorem has been provided to investigate the integrability of a proper differential sequence, the paper gives definitions of these objects and suggests some roots for the investigations illustrated by several examples. In particular, the foregoing examples strongly suggest that the alternative sequence and Proposition 2, which addresses the compatibility/complete compatibility of sequences, provides a useful root to the integrals of a proper differential sequence. In this sense, the current paper should be appreciated as a starting point for the investigations of proper differential sequences rather than a concluding paper on this subject.

We deliberately concentrate on simple examples, namely proper differential sequences for which the general solution can be derived via linearisations of the equations in the sequences, in order to gain an understanding of the properties. We recall that equations of the sequence in the last example, the Burger’s Sequence of Example 5, are linearisable by a Cole-Hopf type transformation, whereas all other sequences are examples of equations linearisable by point transformations. The point-linearisable sequences have beautiful properties in view of the Lie symmetry structure of the sequence and the usefulness of the alternative sequence for the construction of the complete set of first integrals of the proper differential sequences. For the Burgers’ Sequence we have to introduce nonlocal variables for the general solution of the operator equation

$$R[u]Q_1 = 0$$

which then results in a higher order alternative sequence in terms of local variables. This example clearly suggests that further investigations are necessary in order to handle such cases, namely when nonlocal variables come into play. We suspect that nonlocal symmetries and nonlocal integrating factors will play an important role for this investigation.

### A Appendix

The sequences discussed in Examples 2, 3 and 4 are integrable sequences and their alternative sequences preserve the maximal Lie symmetry algebra of point symmetries as given by the corresponding seed equations. A detailed Lie point symmetry analysis is the aim of this Appendix.

We determine the Lie point symmetries of the general equations in the systems

1. $u_{xx} + u_x^2 = 0$ (A.1a)
2. $u_{xx} + u_x^2 = e^{-u} D_x^{(j-1)} c_1, \quad j = 1, m,$ (A.1b)
\[ u_{xx} + u_x^2 = 0 \] (A.2a)

\[ u_{xx} + u_x^2 = \exp \left[ - \int h(u) du \right] D_x^{(j-1)c_1}, \quad j = 1, m, \] (A.2b)

and

\[ u_{xx} + \lambda u_x - \frac{h'(u)}{h(u)} u_x^2 + h(u) = 0 \] (A.3a)

\[ u_{xx} + \lambda u_x - \frac{h'(u)}{h(u)} u_x^2 + h(u) = h(u) D_x^{(j-1)c_1}. \] (A.3b)

It is evident that (A.1a) – (A.1b) is subsumed into (A.2a) – (A.2b).

Equation (A.2b) belongs to the class of equations

\[ u_{xx} + u_x^2 = \exp \left[ - \int h(u) du \right] f(x) \] (A.4)

and (A.3b) to the class

\[ u_{xx} + \lambda u_x - \frac{h'(u)}{h(u)} u_x^2 + h(u) = h(u) f(x), \] (A.5)

where \( f(x) \) is at least \( C^1 \). This is more than adequately covers the polynomials of the original equations.

If in (A.4) we make the change of dependent variable,

\[ w = \int \exp \left[ \int h(u) du \right] du, \] (A.6)

(A.4) becomes

\[ w_{xx} = f(x). \] (A.7)

In like manner the change of dependent variable,

\[ w = \int \frac{du}{h(u)}. \] (A.8)

converts (A.5) to

\[ w_{xx} + \lambda w_x = f(x) - 1. \] (A.9)

Since the transformation in the dependent variable is a point transformation in both cases, the Lie point symmetries of (A.7) and (A.9), which are reasonably easy to calculate, lead directly to the Lie point symmetries of (A.4) and (A.5). Since (A.7) and (A.9) are linear
second-order ordinary differential equations, they each possess eight Lie point symmetries with the algebra \( \text{sl}(3, R) \).

The coefficient functions for a Lie point symmetry of (A.7),

\[
\Gamma = \xi(x, w) \partial_x + \eta(x, w) \partial_w,
\]

have the forms

\[
\begin{align*}
\xi &= a(x) + b(x)w \\
\eta &= b_x(x)w^2 + c(x)w + d(x),
\end{align*}
\]

in which the functions of \( x \) satisfy

\[
\begin{align*}
b_{xx} &= 0 \\
c_{xx} &= b f_x \\
a_{xx} &= b + 2c_x \\
d_{xx} &= a f_x + (2a_x - c)f.
\end{align*}
\]

We solve the equations in (A.12a) – (A.12d) in turn to obtain

\[
\begin{align*}
b &= B_0 + B_1 x \\
c &= C_0 + C_1 x + \int \int (b f_x) dx \, dx \\
a &= A_0 + A_1 x + \int \int (b + 2c_x) \, dx \, dx \\
d &= D_0 + D_1 x + \int \int [a f_x + (2a_x - c)f] \, dx \, dx.
\end{align*}
\]

We did not give the explicit formulæ for the integrals for general \( f(x) \) as they are not informative. The important thing to note is that there are eight Lie point symmetries.

The Lie point symmetries of (A.9) have the same dependence upon \( w \) as given in (A.11a). Now the equations to be satisfied by the functions of \( x \) are

\[
\begin{align*}
b_{xx} - \lambda b_x &= 0 \\
c_{xx} + \lambda c_x &= b f_x + 2\lambda b(f - 1) \\
a_{xx} - \lambda a_x &= 2c_x - 3b(f - 1) \\
d_{xx} + \lambda d_x &= a f_x + (2a_x - c)(f - 1)
\end{align*}
\]

which can easily be solved and again provides eight arbitrary constants and hence eight Lie point symmetries.

The explicit Lie point symmetries of the original equations, (A.2b) and (A.3b), require both the inversion of the transformations (A.6) and (A.8) and the specification of the function \( f(x) \). To maintain a modicum of simplicity we take (A.1b), for which \( u = \log w \),
and $f(x) = c_3 + c_2 x + \frac{1}{2} c_1 x^2$, ie we list the Lie point symmetries of the equation which is completely compatible with the fourth element of the differential sequence (2.20).

$$\Gamma_1 = e^{-u} \partial_u$$
$$\Gamma_2 = x e^{-u} \partial_u$$
$$\Gamma_3 = 6 \partial_x + (3 c_2 x^2 + c_1 x^3) e^{-u} \partial_u$$
$$\Gamma_4 = \left[ 24 - (12 c_3 x^2 + 4 c_2 x^3 + c_1 x^4) e^{-u} \right] \partial_u$$
$$\Gamma_5 = 6 x \partial_x + (6 c_3 x^2 + 3 c_2 x^3 + c_1 x^4) e^{-u} \partial_u$$
$$\Gamma_6 = 24 x^2 \partial_x + \left[ 24 x + (12 c_3 x^3 + 8 c_2 x^4 + 3 c_1 x^5) e^{-u} \right] \partial_u$$
$$\Gamma_7 = (144 e^u - 216 c_3 x^2 - 24 c_2 x^3 - 6 c_1 x^4) \partial_x + \left[ 72 c_2 x^2 + 24 c_1 x^3 - (144 c_3^2 x^3 + 108 c_3 c_2 x^4 + 12 c_2^2 x^5 + 36 c_3 c_1 x^5 + 7 c_2 c_1 x^6 + c_1^2 x^7) e^{-u} \right] \partial_u$$
$$\Gamma_8 = (576 e^u x - 288 c_3 x^3 - 96 c_2 x^4 - 24 c_1 x^5) \partial_x + \left[ 576 e^u + 96 c_2 x^3 + 48 c_1 x^4 - (144 c_3^2 x^3 + 144 c_3 c_2 x^4 + 32 c_2^2 x^5 + 48 c_3 c_1 x^6 + 20 c_2 c_1 x^7 + 3 c_1^2 x^8) e^{-u} \right] \partial_u.$$

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