Eigenstate Entanglement Entropy in Random Quadratic Hamiltonians

Patrycja Lydżba,1,2 Marcos Rigol,3 and Lev Vidmar1,4

1Department of Theoretical Physics, J. Stefan Institute, SI-1000 Ljubljana, Slovenia
2Department of Theoretical Physics, Wrocław University of Science and Technology, 50-370 Wrocław, Poland
3Department of Physics, The Pennsylvania State University, University Park, Pennsylvania 16802, USA
4Department of Physics, Faculty of Mathematics and Physics, University of Ljubljana, SI-1000 Ljubljana, Slovenia

The eigenstate entanglement entropy has been recently shown to be a powerful tool to distinguish integrable from generic quantum-chaotic models. In integrable models, a unique feature of the average eigenstate entanglement entropy (over all Hamiltonian eigenstates) is that the volume-law coefficient depends on the subsystem fraction. Hence, it deviates from the maximal (subsystem fraction independent) value encountered in quantum-chaotic models. Using random matrix theory for quadratic Hamiltonians, we obtain a closed-form expression for the average eigenstate entanglement entropy as a function of the subsystem fraction. We test its correctness against numerical results for the quadratic Sachdev-Ye-Kitaev model. We also show that it describes the average entanglement entropy of eigenstates of the power-law random banded matrix model (in the delocalized regime), and that it is close but not the same as the result for quadratic models that exhibit localization in quasimomentum space.

Introduction. Entanglement, a genuine property of the quantum world, provides unique ways of characterizing quantum many-body systems [1–4]. Studies of entanglement indicators have contributed novel insights into properties of ground states [5, 6], quantum phase transitions [7, 8], information scrambling in nonequilibrium quantum dynamics [9, 10], and highly excited Hamiltonian eigenstates that comply with the eigenstate thermalization hypothesis (ETH) [11–13] (see Refs. [14–17] for reviews). Models that exhibit quantum chaos and eigenstate thermalization are usually referred to as quantum-chaotic models [15], and typical eigenstates in such models have been found to exhibit a maximal von-Neumann entanglement entropy [18–20].

Let |m⟩ be an eigenket of a lattice Hamiltonian with two states per site in one dimension (the Hamiltonians of interest in this work). To compute the von-Neumann entanglement entropy (in short, the entanglement entropy) of |m⟩, we bipartition the lattice with L sites into a subsystem A with L_A contiguous sites and the environment B with L − L_A sites, and trace out the environment sites to obtain the reduced density matrix of subsystem A, ρ_A = Tr_B{⟨m|m⟩}. The entanglement entropy is then obtained as S_m = −Tr{ln(ρ_A)}. For highly excited eigenstates of quantum-chaotic Hamiltonians, the leading term in S_m has been found to be proportional to L_A (for 1 ≪ L_A ≤ L/2) and consistent with the thermodynamic entropy at the corresponding energy [18–36]. Since the overwhelming majority of energy eigenstates in such systems is at “infinite temperature”, this means that typical eigenstates have

\[ S_m(L_A) \sim L_A \ln 2. \]  

(1)

This result matches the average entanglement entropy \( S \) of random pure states [37] (as well as the typical entropy [38]). The presence of conserved quantities other than the total energy, such as the particle number, only modifies subleading terms [19].

In sharp contrast to quantum-chaotic models, translationally invariant quadratic fermionic models (or models mappable to them) have been proved to exhibit a qualitatively different behavior of the average and typical entanglement entropy of their many-body eigenstates [39–41]. The leading term in the average was proved to be proportional to \( L_A \) (for 1 ≪ L_A ≤ L/2), but it depends on the subsystem fraction \( f = L_A/L \), and is smaller than the maximal value for \( f > 0 \). Qualitatively similar results were obtained numerically for free fermions in a superlattice in two dimensions [42], and for the translationally invariant interacting integrable spin-1/2 XXZ chain [43]. Actually, the results in the latter model were very close (potentially identical in the thermodynamic limit) to the ones for quadratic models in Refs. [39–41].

Integrable models are characterized by extensive numbers of nontrivial conserved quantities [44–46], and exhibit distinct properties such as absence of thermalization when taken far from equilibrium [47–50]. This is attributed to the fact that their eigenstates violate the ETH [13, 43, 50–54]. Lack of thermalization close to integrable points is robust enough as to be accessible in experiments with ultracold atoms [55–58], in which distributions of conserved quantities (rapidities) were recently measured [59]. It is then important to develop theoretical tools to distinguish integrable from quantum-chaotic models. The results in Refs. [39–43] show that the eigenstate entanglement entropy is one of such tools. In contrast to traditionally used spectral properties, it does not require finding and removing all symmetries of the model [15]. Another recently introduced eigenstate-based tool, the AGP norm, involves studying the response of many-body eigenstates to perturbations [60].

For the eigenstate entanglement entropy, a stepping stone missing for integrable models that is available for quantum-chaotic ones is a closed-form expression for the average entanglement entropy, like the one in Ref. [37], which could serve as a reference point to compare to re-
sults obtained for specific Hamiltonians. We provide such a stepping stone in this work.

The close-form expression is, for \( f \leq 1/2 \) [61],

\[
\bar{S}(L_A, f) = \left[ 1 - \frac{1 + f^{-1}(1 - f) \ln(1 - f)}{\ln 2} \right] L_A \ln 2 ,
\]

and we obtain it using random matrix theory (RMT) for quadratic Hamiltonians. We test Eq. (2) against numerical results for the quadratic Sachdev-Ye-Kitaev (SYK2) model, and provide evidence that it describes the average entanglement entropy of many-body eigenstates in the delocalized regime of the power-law random banded matrix (PLRBM) model. We also show that localization in quasimomentum space (e.g., because of translational invariance) results in (small) deviations from Eq. (2). This is in stark contrast to what happens in quantum-chaotic systems, in which the presence of translational invariance does not affect the leading term in the average entanglement entropy [19].

**Derivation of Eq. (2).** We consider quadratic fermionic Hamiltonians which, after diagonalization, can always be written as \( \hat{H} = \sum_{q=1}^{L} \varepsilon_q c_q^\dagger c_q \), where \( \varepsilon_q \) are the single-particle eigenenergies and \( \{|q\} = c_q^\dagger |\theta\rangle ; q = 1, \ldots, L \) are the single-particle energy eigenkets. Let the unitary transformation between the energy eigenkets and the position eigenkets \( \{|i\} = \hat{f}_i |\theta\rangle ; i = 1, \ldots, L \) be carried out by a matrix with elements \( v_{iq} \), so that \( \hat{f}_i = \sum_{q=1}^{L} v_{iq} c_q \).

The many-body eigenkets of \( \hat{H} \) can be written as \( \{|m\} = \prod_{\{q\}_m} c_{q_i}^\dagger |\theta\rangle ; m = 1, \ldots, 2^L \), where \( \{q\}_m \) represents the \( m \)-th set of occupied single-particle energy eigenkets. Introducing \( N_q = 2c_q^\dagger c_q - 1 \), for which \( N_q |m\rangle = N_q^m |m\rangle \) with \( N_q^m = 1 (-1) \) for occupied (empty) single-particle eigenkets in \( |m\rangle \), we can write the generalized one-body correlation matrix (in short, the correlation matrix) \( \mathcal{J}_m \) as

\[
(\mathcal{J}_m)_{ij} = 2 \langle m | \hat{f}_i^\dagger \hat{f}_j | m \rangle - \delta_{ij} = \sum_{q=1}^{L} N_q^m v_{iq}^* v_{jq} ,
\]

where \( i, j \leq L_A \), and we denote the eigenvalues as \( \{\lambda_j^m\} ; j = 1, \ldots, L_A \}. \) In Eq. (3) we used the orthonormality condition: \( \delta_{ij} = \sum_{q=1}^{L} v_{iq}^* v_{jq} \). Further on, we shorten the notation \( \mathcal{J}_m \rightarrow \mathcal{J} \) and \( \lambda_j^m \rightarrow \lambda_j \).

The entanglement entropy of many-body eigenket \( |m\rangle \) can then be computed as [6, 62],

\[
S_m = \sum_{j=1}^{L_A} \left( 1 + \frac{\lambda_j^m}{2} \ln \left[ \frac{1 + \lambda_j^m}{2} \right] + \frac{1 - \lambda_j^m}{2} \ln \left[ \frac{1 - \lambda_j^m}{2} \right] \right) ,
\]

and the average (over all eigenstates) entanglement entropy is defined as \( \bar{S} = 2^{-L} \sum_{m=1}^{L} S_m \).

In order to make analytic progress in the evaluation of \( \bar{S} \), one can write

\[
\bar{S} = L_A \ln 2 - \frac{\sum_{n=1}^{\infty} \text{Tr} \{ \mathcal{J}^{2n} \}}{2n(2n - 1)} .
\]

This series was proved to be convergent in Ref. [39]. The fact that only even powers of the correlation matrix \( \mathcal{J} \) appear in the series enabled the computation of upper and lower bounds in Refs. [39–41], e.g., truncating the sum yields an upper bound for \( \bar{S} \). In this work, however, we use Eq. (5) to obtain a closed-form expression for \( \bar{S} \).

Our central assumption is a random matrix theory (RMT) assumption for the single-particle energy eigenkets. We assume that \( v_{iq} = u_{iq}/\sqrt{L} \), where \( u_{iq} \) is a normally distributed complex variable with zero mean and unit variance. This is equivalent to assuming that the quadratic Hamiltonians are represented by random matrices drawn from the Gaussian unitary ensemble (GUE). We note that our (leading-term) results do not change if we assume \( u_{iq} \) to be a normally distributed real variable with the same mean and variance, or the Hamiltonians to be represented by matrices drawn from the Gaussian orthogonal ensemble (GOE), see Ref. [63].

Let us use our assumption to evaluate the first trace \( (n = 1) \) in the series in Eq. (5). Using Eq. (3), we get

\[
\sum_{i,j=1}^{L_A} \sum_{q_1, q_2 = 1}^{L} \sum_{m=1}^{L} N_{q_1}^m N_{q_2}^m u_{iq_1}^* u_{jq_2} u_{iq_2} u_{jq_1} = \frac{L_A^2 L_A}{L} \frac{L_A}{L} f ,
\]

where the average over all \( m \) is \( \sum_{m=1}^{L} u_{aq}^2 = L_A \) for \( a = i, j \), which yields

\[
\frac{\text{Tr} \{ \mathcal{J}^{2} \}}{L} = \frac{L_A^2}{L} = L_A f .
\]

Remarkably, this is exactly the universal result one gets for translationally invariant systems [39].

We have also computed the averages of traces for powers \( n = 2, 3, \) and \( 4 \) (see Ref. [63] for details),

\[
\begin{align*}
\sum_{i,j=1}^{L_A} \sum_{q_1, q_2 = 1}^{L} \sum_{m=1}^{L} N_{q_1}^m N_{q_2}^m u_{iq_1}^* u_{jq_2} u_{iq_2} u_{jq_1} & = \frac{L_A^2 L_A}{L} \frac{L_A^2}{L} f^2 , \\
\sum_{i,j=1}^{L_A} \sum_{q_1, q_2 = 1}^{L} \sum_{m=1}^{L} N_{q_1}^m N_{q_2}^m u_{iq_1}^* u_{jq_2} u_{iq_2} u_{jq_1} & = \frac{L_A^2 L_A}{L} \frac{L_A^2}{L} f^3 , \\
\sum_{i,j=1}^{L_A} \sum_{q_1, q_2 = 1}^{L} \sum_{m=1}^{L} N_{q_1}^m N_{q_2}^m u_{iq_1}^* u_{jq_2} u_{iq_2} u_{jq_1} & = \frac{L_A^2 L_A}{L} \frac{L_A^2}{L} f^4 + O(f^5) ,
\end{align*}
\]

Plugging these results in Eq. (5), we get

\[
\bar{S} = L_A \ln 2 - 2^{L - 1} \sum_{m=1}^{L} S_m .
\]

Equation (11) is an exact expansion in \( f \) up to \( O(f^5) \), since \( \frac{\text{Tr} \{ \mathcal{J}^{2m} \}}{L_A} \) is in general a polynomial that, in the thermodynamic limit \( (L \to \infty) \), contains only powers from \( f^n \) up to \( f^{2n-1} \) [39].
are normally distributed with zero mean and variance 2. Asymptotic results were obtained by averaging over 5 \times 10^5 many-body eigenstates, and 500 Hamiltonian realizations. For such matrices A, Eq. (13) is known as the Dirac fermion version of the SYK2 model (in short, the SYK2 model). Due to the consideration of particle-hole symmetry, the entanglement entropy of the SYK2 model does not change if one replaces the Dirac fermions with Majorana fermions [64]. In spite of large interest in entanglement properties of typical eigenstates of the SYK2 model [64, 65] (and the quantum-chaotic SYK4 model [66, 67]), a closed-form expression for the subsystem fraction dependence has remained elusive.

In Fig. 1(a), we show that the numerical evaluation of the average eigenstate entanglement entropy of the SYK2 model, for a chain with \( L = 100 \) sites, follows very closely the analytical prediction in Eq. (2). The small differences between the two are due to finite-size effects. In Fig. 1(b), finite-size scaling analyses for different values of \( f \) reveal that the numerical results converge to the analytic predictions with increasing system size.

We also studied the average eigenstate entanglement entropy in the so-called PLRBM model, which is a model that exhibits a delocalization-localization transition in one dimension [68]. In that model, the matrix elements \( A_{ij} \) are taken to be real numbers normally distributed with zero mean and an algebraically decaying variance \( \text{Var}(A_{ij}) = 1/(1+i-j)/\alpha^\beta \). We use open boundaries and focus on the delocalized regime by setting \( \alpha = 0.2 \) and \( \beta = 0.1 \) [68]. The results for \( L = 100 \), also shown in Fig. 1(a), closely follow our analytic prediction. Once again the deviations are due to finite-size effects, which are stronger than for the SYK2 model. In Fig. 1(c), finite-size scaling analyses for different values of \( f \) reveal that the numerical results for the PLRBM model converge to the predictions of Eq. (2) with increasing system size.

For both the SYK2 and PLRBM models, we find that the variances \( \sigma \) of the distributions of eigenstate entanglement entropies for \( f > 0 \) vanish in the thermodynamic limit, see Fig. 2. This means that the typical eigenstate entanglement entropy is the same as the average. Hence both terms can be used interchangeably. For both mod-
els, we find the scaling $\sigma \propto 1/L$ for large $L$. It is interesting to note that an identical scaling was found in Ref. [39] for translationally invariant free fermions in the limit $f \to 0$, for which it was shown that the distribution of eigenvalues of the correlation matrices $J$ is described by a Toeplitz Gaussian ensemble.

Deviations from Eq. (2). Many-body eigenstates of models that are localized in position space exhibit an area law entanglement entropy. Hence, the average and typical eigenstate entanglement entropy in those models is not described by Eq. (2). In what follows we discuss what happens for quadratic models that exhibit localization in quasimomentum space.

First, we consider spinless fermions with nearest-neighbor hoppings in a homogeneous lattice with either periodic (PBCs) or open (OBCs) boundary conditions,

$$\hat{H}_{\text{PBC/OBC}} = -\sum_{i=1}^{L-1} \left( \hat{f}^\dagger_i \hat{f}_{i+1} + \hat{f}^\dagger_{i+1} \hat{f}_i \right) - \eta \left( \hat{f}^\dagger_i \hat{f}_i + \hat{f}^\dagger_i \hat{f}^\dagger_i \right),$$

(14)

where $\eta = 1$ for $\hat{H}_{\text{PBC}}$ and $\eta = 0$ for $\hat{H}_{\text{OBC}}$. Several previous studies focused on the entanglement entropy of excited eigenstates in these and related models [39-42, 69-74]. The volume-law coefficient of the average entanglement entropy $\bar{S}_{\text{PBC}}/(L \ln 2)$ of $\hat{H}_{\text{PBC}}$ can be calculated numerically very accurately in the thermodynamic limit (the finite-size effects are exponentially small in $L$) [41]. Figure 3(a) depicts a finite-size scaling analysis of the difference between the latter and the volume-law coefficient in Eq. (2), $\bar{S}/(L \ln 2)$. This analysis shows that they are (slightly) different. Such a difference highlights that, contrary to quantum-chaotic systems in which localization in quasimomentum space does not affect the volume-law coefficient, in quadratic systems it does [75]. This is consistent with the analytic observation that the Macfarlane series of $\bar{S}$ and $\bar{S}_{\text{PBC}}$ (cf. Eq. (57) in Ref. [41]) differ starting with the term proportional to $f^2$. Using OBCs one can relax the condition of the single-particle energy eigenstates being quasimomentum eigenstates, while still keeping them localized in quasimomentum space. The results in Fig. 3(b) suggest that $\bar{S}_{\text{OBC}}/(L \ln 2) \to \bar{S}_{\text{PBC}}/(L \ln 2)$ in the thermodynamic limit, namely, that OBCs do not affect the volume law.

The second example is the quasiperiodic Aubry-André model [76]

$$\hat{H}_{\text{AA}} = \hat{H}_{\text{OBC}} + W \sum_{i=1}^{L} \cos(2\pi \sigma i + \phi) \hat{f}^\dagger_i \hat{f}_i,$$

(15)

whose eigenstate entanglement properties have also been studied in the past [77-79]. We set $\sigma = (\sqrt{5} - 1)/2$ to ensure incommensurability of the potential with the lattice periodicity, and $\phi$ is a constant. In the regime we are interested in, namely $W < 2$, the single-particle energy eigenstates are delocalized in position space and localized in quasimomentum space [76, 80].

We find that the subleading corrections to the volume-law coefficient of the average entanglement entropy $\bar{S}_{\text{AA}}/(L \ln 2)$ are much larger than in the homogeneous models. To subtract the effect of OBCs, we compute $(\bar{S}_{\text{OBC}} - \bar{S}_{\text{AA}})/(L \ln 2)$ and plot it vs $1/\sqrt{L}$ in Fig. 3(c). The results there suggest that the first subleading term in $\bar{S}_{\text{AA}}$ (for $f > 0$) is $\propto \sqrt{\ln L}$. We estimate the volume-law coefficient by finding the minimum of the quality $\delta$ of the linear fits in Fig. 3(c). The results in Fig. 3(d) strongly suggest that $\bar{S}_{\text{AA}}$ is not described by $\bar{S}$ in Eq. (2) (see the differences between the positions of the minima of $\delta$ and the vertical dashed lines). The leading volume-law term in $\bar{S}_{\text{AA}}$ is very close (possibly the same) as for $\bar{S}_{\text{PBC/OBC}}$. Summary. We report a closed-form expression for the average eigenstate entanglement entropy $S$ of random quadratic Hamiltonians [Eq. (2)], which can be seen as the RMT free-fermion counterpart to Page’s result [37]. We tested it against numerical results for the SYK2 model (and an analytical result for $f = 1/2$ [64]). We showed that $S$ describes the average and typical entanglement entropy in the PLRBM model. We also showed that, in contrast to what happens with quantum-chaotic systems, localization in quasimomentum space leads to (small) deviations from $\bar{S}$. Our results provide a stepping stone for studies of the average and typical entanglement
entropy of eigenstates of integrable models, whose structure has started to be unveiled as being much richer than the one of quantum-chaotic models [43, 60, 81].

ACKNOWLEDGMENTS

This work was supported by the Slovenian Research Agency (ARRS), Research core fundings Grants No. P1-0044 and No. J1-1696 (P.L. and L.V.) and by the National Science Foundation under Grant No. PHY-1707482 (M.R.).

[1] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Entanglement in many-body systems, Rev. Mod. Phys. 80, 517 (2008).
[2] I. Peschel and V. Eisler, Reduced density matrices and entanglement entropy in free lattice models, J. Phys. A 42, 504003 (2009).
[3] P. Calabrese and J. Cardy, Entanglement entropy and conformal field theory, J. Phys. A 42, 504005 (2009).
[4] J. Eisert, M. Cramer, and M. B. Plenio, Colloquium: Area laws for the entanglement entropy, Rev. Mod. Phys. 82, 277 (2010).
[5] K. Audenaert, J. Eisert, M. B. Plenio, and R. F. Werner, Entanglement properties of the harmonic chain, Phys. Rev. A 66, 042327 (2002).
[6] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Entanglement in quantum critical phenomena, Phys. Rev. Lett. 90, 227902 (2003).
[7] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Scaling of entanglement close to a quantum phase transition, Nature 416, 608 (2002).
[8] T. J. Osborne and M. A. Nielsen, Entanglement in a simple quantum phase transition, Phys. Rev. A 66, 032110 (2002).
[9] A. M. Kaufman, M. E. Tai, A. Lukin, M. Rispoli, R. Schittko, P. M. Preiss, and M. Greiner, Quantum thermalization through entanglement in an isolated many-body system, Science 355, 794 (2016).
[10] V. Alba and P. Calabrese, Entanglement and thermodynamics after a quantum quench in integrable systems, Proc. Natl. Acad. Sci. 114, 7947 (2017).
[11] J. M. Deutsch, Quantum statistical mechanics in a closed system, Phys. Rev. A 43, 2046 (1991).
[12] M. Srednicki, Chaos and quantum thermalization, Phys. Rev. E 50, 888 (1994).
[13] M. Rigol, V. Dunjko, and M. Olshanii, Thermalization and its mechanism for generic isolated quantum systems, Nature (London) 452, 854 (2008).
[14] J. Eisert, M. Friesdorf, and C. Gogolin, Quantum many-body systems out of equilibrium, Nat. Phys. 11, 124 (2015).
[15] L. D’Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics, Adv. Phys. 65, 239 (2016).
[16] J. M. Deutsch, Eigenstate thermalization hypothesis, Rep. Prog. Phys. 81, 082001 (2018).
[17] T. Mori, T. N. Ikeda, E. Kaminishi, and M. Ueda, Thermalization and prethermalization in isolated quantum systems: a theoretical overview, J. Phys. B 51, 112001 (2018).
[18] W. Beugeling, A. Andreanov, and M. Haque, Global characteristics of all eigenstates of local many-body Hamiltonians: participation ratio and entanglement entropy, J. Stat. Mech. (2015), P02002.
[19] L. Vidmar and M. Rigol, Entanglement Entropy of Eigenstates of Quantum Chaotic Hamiltonians, Phys. Rev. Lett. 119, 220603 (2017).
[20] J. R. Garrison and T. Grover, Does a single eigenstate encode the full Hamiltonian?, Phys. Rev. X 8, 021026 (2018).
[21] J. M. Deutsch, Thermodynamic entropy of a many-body energy eigenstate, New J. Phys. 12, 075021 (2010).
[22] L. F. Santos, A. Polkovnikov, and M. Rigol, Weak and strong typicality in quantum systems, Phys. Rev. E 86, 010102 (2012).
[23] J. M. Deutsch, H. Li, and A. Sharma, Microscopic origin of thermodynamic entropy in isolated systems, Phys. Rev. E 87, 042135 (2013).
[24] Z.-C. Yang, C. Chamon, A. Hamma, and E. R. Mucciolo, Two-component structure in the entanglement spectrum of highly excited states, Phys. Rev. Lett. 115, 267206 (2015).
[25] A. Dymarsky, N. Lashkari, and H. Liu, Subsystem eigenstate thermalization hypothesis, Phys. Rev. E 97, 012140 (2018).
[26] Y. O. Nakagawa, M. Watanabe, H. Fujita, and S. Sugiuara, Universality in volume-law entanglement of scrambled pure quantum states, Nat. Comm. 9, 1635 (2018).
[27] T.-C. Lu and T. Grover, Renyi entropy of chaotic eigenstates, Phys. Rev. E 99, 032111 (2019).
[28] Y. Huang, Universal eigenstate entanglement of chaotic local Hamiltonians, Nuc. Phys. B 938, 594 (2019).
[29] C. Murthy and M. Srednicki, Structure of chaotic eigenstates and their entanglement entropy, Phys. Rev. E 100, 022131 (2019).
[30] H. Wilming, M. Goihl, I. Roth, and J. Eisert, Entanglement-ergodic quantum systems equilibrate exponentially well, Phys. Rev. Lett. 123, 200604 (2019).
[31] Q. Miao and T. Barthel, Eigenstate entanglement: Crossover from the ground state to volume laws, arXiv:1905.07760.
[32] S. C. Morampudi, A. Chandran, and C. R. Laumann, Universal Entanglement of Typical States in Constrained Systems, Phys. Rev. Lett. 124, 050602 (2020).
[33] D. Faiez and D. Šafránek, How much entanglement can be created in a closed system, Phys. Rev. B 101, 060401 (2020).
[34] D. Faiez, D. Šafránek, J. M. Deutsch, and A. Aguirre, Typical and extreme entropies of long-lived isolated quantum systems, Phys. Rev. A 101, 052101 (2020).
[35] D. S. Bhakuni and A. Sharma, Entanglement and thermodynamic entropy in a clean many-body-localized system, J. Phys.: Cond. Mat. 32, 255603 (2020).

[36] K. Kaneko, E. Iyoza, and T. Sagawa, Characterizing complexity of many-body quantum dynamics by higher-order eigenstate thermalization, Phys. Rev. A 101, 042126 (2020).

[37] D. N. Page, Average entropy of a subsystem, Phys. Rev. Lett. 71, 1291 (1993).

[38] E. Bianchi and P. Donà, Typical entanglement entropy in the presence of a center: Page curve and its variance, Phys. Rev. D 100, 105010 (2019).

[39] L. Vidmar, L. Hackl, E. Bianchi, and M. Rigol, Entanglement Entropy of Eigenstates of Quadratic Fermionic Hamiltonians, Phys. Rev. Lett. 119, 020601 (2017).

[40] L. Vidmar, L. Hackl, E. Bianchi, and M. Rigol, Volume law and quantum criticality in the entanglement entropy of excited eigenstates of the quantum Ising model, Phys. Rev. Lett. 121, 220602 (2018).

[41] L. Hackl, L. Vidmar, M. Rigol, and E. Bianchi, Average eigenstate entanglement entropy of the XY chain in a transverse field and its universality for translationally invariant quadratic fermionic models, Phys. Rev. B 99, 075123 (2019).

[42] M. Storms and R. R. P. Singh, Entanglement in ground and excited states of gapped free-fermion systems and their relationship with Fermi surface and thermodynamic equilibrium properties, Phys. Rev. E 89, 012125 (2014).

[43] T. LeBlond, K. Mallayya, L. Vidmar, and M. Rigol, Entanglement and matrix elements of observables in interacting integrable systems, Phys. Rev. E 100, 062134 (2019).

[44] M. P. Grabowski and P. Mathieu, Structure of the conservation laws in quantum integrable spin chains with short range interactions, Ann. Phys. (N.Y.) 243, 299 (1995).

[45] M. Mierzejewski, P. Prelovšek, and T. Prosen, Identifying Local and Quasilocal Conserved Quantities in Integrable Systems, Phys. Rev. Lett. 114, 140601 (2015).

[46] E. Ilievski, M. Medenjak, T. Prosen, and L. Zadnik, Quasilocal charges in integrable lattice systems, J. Stat. Mech. (2016), 064008.

[47] M. Rigol, V. Dunjko, V. Vurovsky, and M. Olshanii, Relaxation in a Completely Integrable Many-Body Quantum System: An Ab Initio Study of the Dynamics of the Highly Excited States of 1D Lattice Hard-Core Bosons, Phys. Rev. Lett. 98, 050405 (2007).

[48] E. Ilievski, J. De Nardis, B. Wouters, J.-S. Caux, F. H. L. Essler, and T. Prosen, Complete Generalized Gibbs Ensembles in an Interacting Theory, Phys. Rev. Lett. 115, 157201 (2015).

[49] F. H. L. Essler and M. Fagotti, Quench dynamics and relaxation in isolated integrable quantum spin chains, J. Stat. Mech. (2016), 064002.

[50] L. Vidmar and M. Rigol, Generalized Gibbs ensemble in integrable lattice models, J. Stat. Mech. (2016), 064007.

[51] M. Rigol, Breakdown of Thermalization in Finite One-Dimensional Systems, Phys. Rev. Lett. 103, 100403 (2009).

[52] A. C. Cassidy, C. W. Clark, and M. Rigol, Generalized Thermalization in an Integrable Lattice System, Phys. Rev. Lett. 106, 140405 (2011).

[53] T. Yoshizawa, E. Iyoza, and T. Sagawa, Numerical large deviation analysis of the eigenstate thermalization hypothesis, Phys. Rev. Lett. 120, 200604 (2018).

[54] M. Mierzejewski and L. Vidmar, Quantitative Impact of Integrals of Motion on the Eigenstate Thermalization Hypothesis, Phys. Rev. Lett. 124, 040603 (2020).

[55] T. Kinoshita, T. Wenger, and S. D. Weiss, A quantum Newton’s cradle, Nature (London) 440, 900 (2006).

[56] T. Langen, S. Erne, R. Geiger, B. Rauer, T. Schweiger, M. Kuhnert, W. Rohringer, I. E. Mazets, T. Gasenzer, and J. Schmiedmayer, Experimental observation of a generalized Gibbs ensemble, Science 348, 207 (2015).

[57] T. Langen, T. Gasenzer, and J. Schmiedmayer, Prethermalization and universal dynamics in near-integrable quantum systems, J. Stat. Mech. (2016), 064009.

[58] Y. Tang, W. Kao, K.-Y. Li, S. Seo, K. Mallayya, M. Rigol, S. Gopalakrishnan, and B. L. Lev, Thermalization near integrability in a dipolar quantum Newton’s cradle, Phys. Rev. X 8, 021030 (2018).

[59] J. M. Wilson, N. Malvania, Y. Le, Y. Zhang, M. Rigol, and D. S. Weiss, Observation of dynamical renormalization as a sensitive probe for quantum chaos, arXiv:2004.05043.

[60] If \( f > 1/2 \), one needs to replace \( L_A \rightarrow L - L_A \) and \( f \rightarrow 1 - f \).

[61] I. Peschel, Calculation of reduced density matrices from correlation functions, J. Phys. A 36, L205 (2003).

[62] See Supplemental Material for details on the derivation of Eq. (2), for a discussion of GUE versus GOE random matrices, and for inverse participation ratios for the SYK2 model and the PLRB model.

[63] C. Liu, X. Chen, and L. Balents, Quantum entanglement of the Sachdev-Ye-Kitaev models, Phys. Rev. B 97, 245126 (2018).

[64] P. Zhang, C. Liu, and X. Chen, Subsystem Renyi Entropy of Thermal Ensembles for SYK-like models, arXiv:2003.09766.

[65] Y. Huang and Y. Gu, Eigenstate entanglement in the Sachdev-Ye-Kitaev model, Phys. Rev. D 100, 041901 (2019).

[66] P. Zhang, Entanglement Entropy and its Quench Dynamics for Pure States of the Sachdev-Ye-Kitaev model, arXiv:2004.05339.

[67] A. D. Mirlin, Y. V. Fyodorov, F.-M. Dittes, J. Quezada, and A. Polkovnikov, and D. Sels, Adiabatic eigenstate deformations as a sensitive probe for quantum chaos, arXiv:2003.05976.

[68] V. Alba, M. Fagotti, and P. Calabrese, Entanglement entropy of excited states, J. Stat. Mech. (2009), P10020.

[69] J. Möller, T. Barthel, U. Schollwöck, and V. Alba, Bound states and entanglement in the excited states of quantum spin chains, J. Stat. Mech. (2014), P10029.

[70] H.-H. Lai and K. Yang, Entanglement entropy scaling laws and eigenstate typicality in free fermion systems, Phys. Rev. B 91, 081110 (2015).

[71] S. Nandy, A. Sen, A. Das, and A. Dhar, Eigenstate Gibbs ensemble in integrable quantum systems, Phys. Rev. B 94, 245131 (2016).

[72] J. Rüdell and M. P. Müller, Generalized eigenstate typicality in translation-invariant quasifree fermionic models, Phys. Rev. B 97, 035129 (2018).

[73] A. Jafarizadeh and M. A. Rajabpour, Bipartite entanglement entropy of the excited states of free fermions and harmonic oscillators, Phys. Rev. B 100, 165135 (2019).
[75] In Ref. [63] we confirm that, as expected, the eigenkets of the SYK2 and the PLRBM models are delocalized in both position and quasimomentum space.

[76] S. Aubry and G. André, Analyticity breaking and Anderson localization in incommensurate lattices, Ann. Israel Phys. Soc. 3, 133 (1979).

[77] X. Li, J. H. Pixley, D.-L. Deng, S. Ganeshan, and S. Das Sarma, Quantum nonergodicity and fermion localization in a system with a single-particle mobility edge, Phys. Rev. B 93, 184204 (2016).

[78] N. Roy and A. Sharma, Study of counterintuitive transport properties in the Aubry-André-Harper model via entanglement entropy and persistent current, Phys. Rev. B 100, 195143 (2019).

[79] R. Modak and T. Nag, Many-body dynamics in long-range hopping models in the presence of correlated and uncorrelated disorder, Phys. Rev. Research 2, 012074 (2020).

[80] K. He, L. F. Santos, T. M. Wright, and M. Rigol, Single-particle and many-body analyses of a quasiperiodic integrable system after a quench, Phys. Rev. A 87, 063637 (2013).

[81] M. Brenes, J. Goold, and M. Rigol, Ballistic vs diffusive low-frequency scaling in the XXZ and a locally perturbed XXZ chain, arXiv:2005.12309.
S1. TRACES OF CORRELATION MATRICES

The main ingredient in the calculation of the average (over all eigenstates) entanglement entropy $S$ [cf. Eq. (5) in the main text] is the average of traces of powers of correlation matrices $\mathcal{J}$, given by the general expression

$$\overline{\text{Tr} \{ \mathcal{J}^{2n} \}} = \frac{1}{L^{2n}} \sum_{i_1,i_2,\ldots,i_{2n}}^{L_A} \sum_{q_1,q_2,\ldots,q_{2n}}^{L} N_{q_1}^{m} N_{q_2}^{m} \cdots N_{q_{2n}}^{m}
\times u_{i_1q_1}^{*} u_{i_2q_2}^{*} u_{i_3q_3}^{*} \cdots u_{i_{2n}q_{2n}}^{*} u_{i_1q_1},$$

(S1)

where $\{i_a\}$ are position indices running from 1 to $L_A$, and $\{q_a\}$ are single-particle energy eigenket indices running from 1 to $L$. The average $N_{q_1}^{m} N_{q_2}^{m} \cdots N_{q_{2n}}^{m}$ in Eq. (S1) is carried out over all eigenstates. The only nonzero contribution is $N_{q_1}^{m} N_{q_2}^{m} \cdots N_{q_{2n}}^{m} = 1$, which is obtained for configurations of $\{q_a\}$ with every $q_a$ belonging to some $r$-tuple of single-particle energy eigenstate indices ($r$ is an even integer). We denote an $r$-tuple as

$$(q_{a_1}, q_{a_2}, \ldots, q_{a_r}),$$

(S2)

which implies that $q_{a_1} = q_{a_2} = \ldots = q_{a_r}$, and every element from the tuple differs from every other single-particle energy eigenstate index $q_b$. For a fixed $n$ in Eq. (S1), the possible configurations of tuples that give nonzero contribution to $\overline{\text{Tr} \{ \mathcal{J}^{2n} \}}$ range from $n$ different 2-tuples to a single 2n-tuple.

We sort all $\{q_a\}$ on a unit circle, with $a = 1, 2, \ldots, 2n$. We consider an $r$-tuple to be connected if the indices $a_1, a_2, \ldots, a_r$ reside on $r$ consecutive sites on a unit circle. For a fixed $n$, we split the total contribution to $\overline{\text{Tr} \{ \mathcal{J}^{2n} \}}/L_A$ into contributions of different configurations of $r$-tuples. For example,

$$\chi_{r_1,r_2,r_3,\ldots}$$

(S3)

denotes the contribution to $\overline{\text{Tr} \{ \mathcal{J}^{2n} \}}/L_A$ that consists of an $r_1$-tuple, an $r_2$-tuple, an $r_3$-tuple, ..., with $r_1 + r_2 + r_3 + \ldots = 2n$. Moreover, $\tilde{r}_2$ in Eq. (S3) indicates that the $r_2$-tuple is connected.

The goal of our work is to derive the volume-law contribution to $S$ in the limit $1 \ll L_A \leq L/2$. This requires the calculation of the terms in $\chi_{r_1,r_2,r_3,\ldots}$ that are constant in the thermodynamic limit $L_A, L \to \infty$, while keeping $f$ fixed.

A. Case $n = 2$ in Eq. (S1)

For $n = 2$, nonzero contributions to $N_{q_1}^{m} N_{q_2}^{m} N_{q_3}^{m} N_{q_4}^{m}$ in Eq. (S1) stem from either configurations of two 2-tuples $\{(q_a, q_b), (q_c, q_d)\}$ or a single 4-tuple $\{(q_a, q_b, q_c, q_d)\}$. In what follows we consider each class of contributions separately.

1. Contribution from a single 4-tuple

The contribution from a single 4-tuple is denoted as $\chi_4$. In this case all single-particle energy eigenstate indices in Eq. (S1) are identical and we get

$$\chi_4 = \frac{1}{L_A L^4} \sum_{i_1,i_2,i_3,i_4} L \sum_{q_1} |u_{i_1q_1}|^2 |u_{i_2q_2}|^2 |u_{i_3q_3}|^2 |u_{i_4q_4}|^2.$$

(S4)

The latter expression can be rearranged by carrying out the sum over position indices first,

$$\chi_4 = \frac{1}{L_A^4} \sum_{q_1} \left( \sum_{i} |u_{iq_1}|^2 \right)^4.$$

(S5)

At this stage, we apply the random matrix theory (RMT) approximation for single-particle energy eigenstate coefficients,

$$\sum_{i=1}^{L_A} |u_{iq}|^2 = L_A,$$

(S6)

which simplifies Eq. (S5) to

$$\chi_4 = f^4.$$

(S7)

2. Contribution from two 2-tuples

There are three possible configurations of two 2-tuples: $\{(q_1, q_2), (q_3, q_4)\}$, $\{(q_2, q_3), (q_1, q_4)\}$ and $\{(q_1, q_3), (q_2, q_4)\}$. The contribution to Eq. (S1) from the first two is identical, we denote it as $\chi_{2,2}$, while the third contribution is denoted by $\chi_{2,4}$. We find $\chi_{2,2} \neq 0$ and $\chi_{2,4} = 0$. Below we sketch the derivation of $\chi_{2,2}$ in more detail.
For simplicity, we only consider \( q_1 = q_2 \) and \( q_3 = q_4 \) in Eq. (S1), which gives

\[
\chi_{\overline{2}, 2} = \frac{1}{L_A L^4} \sum_{i_1, i_3} \sum_{q_1 \neq q_3} u_{i_1 q_1}^* u_{i_2 q_1} u_{i_3 q_3}^* u_{i_4 q_3}
\times \left( \sum_{i_2 = 1}^{L_A} |u_{i_2 q_1}|^2 \right) \left( \sum_{i_4 = 1}^{L_A} |u_{i_4 q_3}|^2 \right).
\] (S8)

Using the RMT approximation, Eq. (S6), one can evaluate the sums over \( i_2 \) and \( i_4 \), which simplify Eq. (S8) to

\[
\chi_{\overline{2}, 2} = \frac{L_A}{L^2} \sum_{i_1, i_3} \sum_{q_1 \neq q_3} u_{i_1 q_1}^* u_{i_2 q_1} u_{i_3 q_3}^* u_{i_4 q_3}
- \chi_4.
\] (S9)

The single-particle energy eigenstate indices \( q_1 \) and \( q_3 \) in the first term on the r.h.s. of Eq. (S9) are independent, hence one needs to subtract \( \chi_4 \) (derived in Eq. (S7)) that includes the \( q_1 = q_3 \) contribution. We then use the orthogonality condition

\[
\sum_{q = 1}^{L} u_{i q}^* u_{i' q} = L \delta_{i i'}
\] (S10)

to arrive at the final expression

\[
\chi_{\overline{2}, 2} = \frac{L_A^2}{L^2} - \chi_4 = f^2 - f^3.
\] (S11)

3. Total contribution

To summarize, the volume-law contribution to \( \text{Tr} \{ \mathcal{F}^2 \} \) is a weighted sum of contributions derived in Sec. S1A,

\[
\frac{\text{Tr} \{ \mathcal{F}^2 \}}{L_A} = 2 \chi_{\overline{2}, 2} + \chi_4 = 2(f^2 - f^3) + f^3 = 2f^2 - f^3,
\] (S12)

which is Eq. (8) of the main text.

B. Case \( n = 3 \) in Eq. (S1)

At \( n = 3 \), nonzero contributions to single-particle energy eigenstate occupation averages \( N_{q_1}^N_{q_2} N_{q_3}^N_{q_4} N_{q_5}^N_{q_6} \) in Eq. (S1) stem from: (i) configurations of three 2-tuples \( \{ (q_a, q_b), (q_c, q_d), (q_e, q_f) \} \), (ii) configurations of a 4-tuple and a 2-tuple \( \{ (q_a, q_b, q_c, q_d), (q_e, q_f) \} \), and (iii) a single 6-tuple \( \{ (q_a, q_b, q_c, q_d, q_e, q_f) \} \). In what follows we consider each class of contributions separately.

1. Contribution from a single 6-tuple

The derivation of this contribution, denoted as \( \chi_6 \), is analogous to Eq. (S5). Using the RMT approximation, Eq. (S6), it gives

\[
\chi_6 = \frac{1}{L_A L^6} \sum_{q_1} \left( \sum_i |u_{i q_1}|^2 \right)^6 = f^5.
\] (S13)

Note that this type of contributions is ubiquitous for all \( n \) and yields \( \chi_{\overline{2n}} = f^{2n-1} \).

2. Contribution from a 4-tuple and a 2-tuple

There are fifteen possible configurations of a 4-tuple and a 2-tuple. Six of those configurations contain connected tuples only, and each of them provides an identical nonzero contribution \( \chi_{\overline{4}, 2} \neq 0 \). Contributions from all other configurations are zero. Below we sketch the derivation of \( \chi_{\overline{4}, 2} \).

For simplicity, we only consider the 2-tuple \( (q_1, q_2) \) and the 4-tuple \( (q_3, q_4, q_5, q_6) \). In this case we get

\[
\chi_{\overline{4}, 2} = \frac{1}{L_A L^6} \sum_{q_1 \neq q_3} \sum_{q_1, i_2} u_{i_1 q_1}^* u_{i_2 q_1} u_{i_3 q_3}^* u_{i_4 q_3}
\times \left( \sum_{i_2 = 1}^{L_A} |u_{i_2 q_1}|^2 \right) \left( \sum_{i_4 = 1}^{L_A} |u_{i_4 q_3}|^2 \right)^3.
\] (S14)

Using Eq. (S6) for the sums in the parentheses, we express \( \chi_{\overline{4}, 2} \) as

\[
\chi_{\overline{4}, 2} = \frac{L^3}{L_A} \sum_{q_1 \neq q_3} \sum_{i_2} u_{i_1 q_1}^* u_{i_2 q_1} u_{i_3 q_3}^* u_{i_4 q_3}
- \chi_6,
\] (S15)

which, after using the orthogonality condition in Eq. (S10) for the sums over \( q_1 \) and \( q_3 \), and the expression for \( \chi_6 \) in Eq. (S13), yields

\[
\chi_{\overline{4}, 2} = f^4 - f^5.
\] (S16)

3. Contribution from three 2-tuples

Here one needs to distinguish between cases with different degrees of connectivity of 2-tuples. We find that nonzero contributions stem from configurations of: (i) three and (ii) two connected 2-tuples, which we denote, respectively, as \( \chi_{\overline{2}, 2, 2} \) and \( \chi_{\overline{2}, 2, 2} \). Contributions from all other configurations are zero.

(i) There are two configurations of three connected 2-tuples, which yield identical contributions. For simplicity we only consider the configuration \( \{ (q_1, q_2), (q_3, q_4), (q_5, q_6) \} \). In this case,

\[
\chi_{\overline{2}, 2, 2} = \frac{1}{L_A L^6} \sum_{q_1 \neq q_3} \sum_{i_2, i_3} u_{i_1 q_1}^* u_{i_2 q_1} u_{i_3 q_3}^* u_{i_4 q_3}
\times \left( \sum_{i_2 = 1}^{L_A} |u_{i_2 q_1}|^2 \right) \left( \sum_{i_4 = 1}^{L_A} |u_{i_4 q_3}|^2 \right) \left( \sum_{i_6 = 1}^{L_A} |u_{i_6 q_5}|^2 \right).
\] (S17)
Using Eq. (S6) for the sums over \(i_2, i_4, \text{ and } i_6\), the expression is simplified to
\[
\chi_{2,2,2} = \frac{L_A^2}{L^6} \sum_{i_1, i_3, i_5} \left( \sum_{q_1=1}^L u_{i_1 q_1}^* u_{i_3 q_1} \right) \left( \sum_{q_3=1}^L u_{i_5 q_3}^* u_{i_5 q_3} \right) \times \left( \sum_{q_5=1}^L u_{i_5 q_5}^* u_{i_5 q_5} \right) - 3\chi_{4,2} - \chi_6. \tag{S18}
\]

Moreover, using the orthogonality condition in Eq. (S10) for the sums over \(q_1, q_3, \text{ and } q_5\), and results for \(\chi_{4,2}\) and \(\chi_6\) from Eqs. (S16) and (S13), we arrive at
\[
\chi_{2,2,2} = f^3 - 3f^4 + 2f^5. \tag{S19}
\]

(ii) There are three configurations of three 2-tuples, in which two of them are connected. They yield identical contributions. For simplicity we only consider the configuration \(\{(q_1, q_2), (q_3, q_6), (q_4, q_5)\}\). In this case,
\[
\chi_{2,2,2} = \frac{L_A}{L^6} \sum_{q_1, q_2, q_3, q_4} \sum_{q_5, q_6} u_{i_1 q_1}^* u_{i_3 q_1} u_{i_5 q_3} u_{i_5 q_3} \times \left( \sum_{i_2=1}^L \left| u_{i_2 q_1} \right|^2 \right) \left( \sum_{i_4=1}^L \left| u_{i_4 q_4} \right|^2 \right),
\]
which can be simplified using Eq. (S6) for the sums over \(i_2\) and \(i_5\) to
\[
\chi_{2,2,2} = \frac{L_A}{L^6} \sum_{i_1, i_3, i_4, i_6} \left( \sum_{q_1=1}^L u_{i_1 q_1}^* u_{i_3 q_1} \right) \left( \sum_{q_4=1}^L u_{i_4 q_4}^* u_{i_4 q_4} \right) \times \left( \sum_{q_3=1}^L u_{i_5 q_3}^* u_{i_5 q_3} u_{i_5 q_3} u_{i_5 q_3} \right) - 3\chi_{4,2} - \chi_{4,2} - \chi_6. \tag{S20}
\]

Note that by treating single-particle energy eigenstate indices \(q_1, q_3, \text{ and } q_4\) as independent on the r.h.s. of Eq. (S21) one needs to subtract the corresponding contributions from higher-order tuples, which in this case include contributions from both connected and disconnected tuples. The term \(\chi_{4,2}\) (i.e., the contribution from an unconnected 4-tuple and an unconnected 2-tuple) is zero. Then, using the orthogonality condition in Eq. (S10) for the sums over \(q_1\) and \(q_4\), we get
\[
\chi_{2,2,2} = \frac{L_A}{L^8} \sum_{q_1, q_4} \left( \sum_{i_1=1}^L \left| u_{i_1 q_1} \right|^2 \right) \left( \sum_{i_4=1}^L \left| u_{i_4 q_4} \right|^2 \right) - 2\chi_{4,2} - \chi_6. \tag{S22}
\]

Finally, applying Eq. (S6) for the sums over \(i_1\) and \(i_4\), as well as Eqs. (S13)-(S16) for results for \(\chi_6\) and \(\chi_{4,2}\), respectively, we arrive at
\[
\chi_{2,2,2} = f^3 - 2f^4 + f^5. \tag{S23}
\]

### 4. Total contribution

To summarize, the volume-law contribution to \(\overline{\text{Tr}}(\mathcal{J}^0)\) is a weighted sum of contributions derived in Sec. S1 B,
\[
\overline{\text{Tr}}(\mathcal{J}^0) = 2\chi_{2,2,2} + 3\chi_{2,2,2} + 6\chi_{4,2} + \chi_6
\]
\[
= 5f^3 - 6f^4 + 2f^5,
\]
which is Eq. (9) of the main text.

### C. Case \(n = 4\) in Eq. (S1)

The derivation becomes increasingly tedious with increasing \(n\). For \(n = 4\), we only compute the leading contribution, i.e., the contribution that is proportional to \(f^4\), which stems from configurations of four 2-tuples. In particular, the contribution \(\propto f^4\) is obtained by configurations of four 2-tuples with: (1) four connected 2-tuples, \(\chi_{2,2,2,2}\); (2) three connected 2-tuples, \(\chi_{2,2,2,2}\), and (3) two connected 2-tuples, \(\chi_{2,2,2,2}\). Below we derive each of those contributions separately.

#### 1. Four connected 2-tuples

There are two configurations of four connected 2-tuples. For simplicity we only consider the configuration \(\{(q_1, q_2), (q_3, q_4), (q_5, q_6), (q_7, q_8)\}\). The derivation of the leading term of \(\chi_{2,2,2,2}\) is then analogous to the derivation of \(\chi_{2,2,2}\) in Eq. (S17), and yields
\[
\chi_{2,2,2,2} = f^4 + \mathcal{O}(f^5). \tag{S25}
\]

#### 2. Three connected 2-tuples and one unconnected 2-tuple

There are eight configurations of four 2-tuples, in which exactly three of them are connected. All of them contain an identical leading term of \(\chi_{2,2,2,2}\). For simplicity we only consider the configuration \(\{(q_1, q_4), (q_2, q_3), (q_5, q_6), (q_7, q_8)\}\). In this case,
\[
\chi_{2,2,2,2} = \frac{1}{L^8} \sum_{q_1 < q_2 < q_3 < q_4} \sum_{i_1, i_2, i_4, i_5} \sum_{i_7} \left| u_{i_1 q_1} u_{i_2 q_1} u_{i_4 q_2} u_{i_4 q_2} u_{i_5 q_2} u_{i_5 q_2} u_{i_7 q_2} u_{i_7 q_2} \right|^2 \times \left( \sum_{i_3=1}^L \left| u_{i_3 q_2} \right|^2 \right) \left( \sum_{i_6=1}^L \left| u_{i_6 q_3} \right|^2 \right) \left( \sum_{i_8=1}^L \left| u_{i_8 q_4} \right|^2 \right). \tag{S26}
\]

Using Eq. (S6) for the sums over \(i_3, i_6, \text{ and } i_8\), followed by the orthogonality relation Eq. (S10) for the sums over...
For the GUE, we can rewrite this equation as
\[
\frac{1}{N} \sum_{i=1}^{N} |u_{iq}|^2 = \sum_{i=1}^{N} x_{iq}^2 + \sum_{i=1}^{N} y_{iq}^2 = 1 \tag{S33}
\]
and
\[
\frac{1}{N} \sum_{i=1}^{N} u_{iq}^* u_{ip} = \frac{1}{N} \sum_{i=1}^{N} x_{iq} x_{ip} + \frac{1}{N} \sum_{i=1}^{N} y_{iq} y_{ip} + \frac{1}{N} \sum_{i=1}^{N} x_{iq} y_{ip} - \frac{1}{N} \sum_{i=1}^{N} y_{iq} x_{ip} = \delta_{qp}, \tag{S34}
\]
which are valid in the thermodynamic limit (i.e., \(N \to \infty\)). Above, we used the fact that the mean of a product can be replaced by the product of means, i.e.,
\[
\frac{1}{N} \sum_{i=1}^{N} x_{iq} x_{ip} = \left( \frac{1}{N} \sum_{i=1}^{N} x_{iq} \right) \left( \frac{1}{N} \sum_{i=1}^{N} x_{ip} \right) \quad \text{for } p \neq q,
\]
\[
\frac{1}{N} \sum_{i=1}^{N} x_{iq} y_{ip} = \left( \frac{1}{N} \sum_{i=1}^{N} x_{iq} \right) \left( \frac{1}{N} \sum_{i=1}^{N} y_{ip} \right) \quad \text{for all } \{q, p\}. \tag{S35}
\]
where real parts \(x_{iq}\) can be exchanged for imaginary parts \(y_{iq}\) and vice versa. Note that the identities in Eqs. (S33) and (S34) are independent of whether the elements \(u_{iq}\) are real or complex. Such a difference in the nature of \(u_{iq}\) is only revealed when the mean values of higher powers of \(u_{iq}\) are considered. For example,
\[
\frac{1}{N} \sum_{i=1}^{N} |u_{iq}|^4 = \frac{1}{N} \sum_{i=1}^{N} x_{iq}^4 + \frac{1}{N} \sum_{i=1}^{N} y_{iq}^4 + \frac{2}{N} \sum_{i=1}^{N} x_{iq}^2 y_{iq}^2 \tag{S36}
\]
For the GUE, we can rewrite this equation as
\[
\frac{1}{N} \sum_{i=1}^{N} |u_{iq}|^4 = 3 \frac{1^2}{2} + 3 \frac{1^2}{2} + 2 \frac{1^2}{2} = 2. \tag{S37}
\]
whereas for the GOE we can rewrite it as

$$\frac{1}{N} \sum_{i=1}^{N} |u_{iq}|^4 = 3. \quad (S38)$$

However, such means of higher powers of $u_{iq}$ only contribute subleading terms to the average eigenstate entanglement entropy. For example, for $n = 1$,

$$\text{Tr}\{J^2\} = \frac{1}{L^2} \sum_{i,j=1}^{L_A} \sum_{q=1}^{L} |u_{iq}|^2 |u_{jq}|^2$$

$$= \frac{1}{L} \sum_{i \neq j} L_A \left( \frac{1}{L} \sum_{q=1}^{L} |u_{iq}|^2 \right) \left( \frac{1}{L} \sum_{q=1}^{L} |u_{jq}|^2 \right)$$

$$+ \frac{1}{L^2} \sum_{i=1}^{L_A} \sum_{q=1}^{L} |u_{iq}|^4 = L_A f + O(1). \quad (S39)$$

Hence, all the results reported in this work are independent of whether one deals with complex or real random matrices. We note that the Hamiltonian matrices for the SYK2 model are of the former type while the Hamiltonian matrices for the PLRBm model are of the latter type. Still, in the main text we showed that the average eigenstate entanglement entropies of both models are consistent with our close-form expression for $S$.

S3. INVERSE PARTICIPATION RATIOS

FIG. S1. Inverse participation ratio in (a) position space $\text{IPR}_n$, and (b) quasimomentum space $\text{IPR}_m$. The results are plotted vs the lattice size $L$ for the SYK2 (circles) and the PLRBm models. Lines are power-law fits in the interval $L \in [10^2, 10^3]$, which yield $\text{IPR}_n \propto L^{-1.00}$ and $\text{IPR}_m \propto L^{-1.00}$ for the SYK2 model, and $\text{IPR}_n \propto L^{-1.02}$ and $\text{IPR}_m \propto L^{-1.00}$ for the PLRBm model.

The SYK2 and the PLRBm (for the parameters considered in the main text) models are delocalized in position space and, in contrast to translationally invariant models, they are also delocalized in quasimomentum space. We illustrate this by calculating the inverse participation ratios (IPR) in position and quasimomentum space:

$$\text{IPR}_n = \frac{1}{L} \sum_{j=1}^{L} \sum_{i=1}^{L} |v_{ij}|^4, \quad \text{IPR}_m = \frac{1}{L} \sum_{j=1}^{L} \sum_{i=1}^{L} |w_{ij}|^4, \quad (S40)$$

where $v_{ij}$ ($w_{ij}$) stands for the projection of the $i$-th single-particle energy eigenket onto the $j$-th position (quasimomentum) eigenket.

Figure S1 shows that, since $\text{IPR}_n$ and $\text{IPR}_m$ vanish with increasing system size, the single-particle energy eigenkets are delocalized position and quasimomentum space, respectively.