Bounds on the mass-to-radius ratio for non-compact field configurations

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Abstract

It is well known that a spherically symmetric compact star whose energy density decreases monotonically possesses an upper bound on its mass-to-radius ratio, \(2M/R \leq 8/9\). However, field configurations typically will not be compact. Here we investigate non-compact static configurations whose matter fields have a slow global spatial decay, bounded by a power law behavior. These matter distributions have no sharp boundaries. We derive an upper bound on the fundamental ratio \(\max_r \{2m(r)/r\}\) which is valid throughout the bulk. In its simplest form, the bound implies that in any region of spacetime in which the radial pressure increases, or alternatively decreases no faster than some power law \(r^{-(\gamma+4)}\), one has \(2m(r)/r \leq (2+2\gamma)/(3+2\gamma)\). (For \(\gamma \leq 0\) the bound degenerates to \(2m(r)/r \leq 2/3\).) In its general version, the bound is expressed in terms of two physical parameters: the spatial decaying rate of the matter fields, and the highest occurring ratio of the trace of the pressure tensor to the local energy density.

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1. Introduction

The spherically symmetric Schwarzschild black hole is characterized by a mass-to-radius ratio of \(2M/R = 1\). However, this solution of the Einstein equations has a central singularity, clothed by the black-hole horizon at \(R = 2M\). It was Schwarzschild who in 1916 asked the question: how large can \(2M/R\) possibly be for a regular field configuration?

This quantity is central for the determination of the spacetime geometry of an Einstein-matter solution, and it is also a measure for the strength of the gravitational interaction. In addition, a bound on \(2M/R\) has observational consequences, since it limits the observable redshift of the compact object (star).
For a compact configuration with a constant energy density and isotropic pressures, Schwarzschild obtained the bound [1]
\[ \frac{2M}{R} = \frac{8}{9}. \] (1)
Beyond this value the star would not be able to support its own gravitational field, and it would ultimately collapse to form a black hole. The existence of such an upper bound is intriguing because it occurs strictly before the appearance of an apparent horizon at \( \frac{2M}{R} = 1 \). Years later, Buchdahl [2] proved that the same bound holds for isotropic compact spheres in which the energy density decreases monotonically. The bound (1) is commonly known as the Buchdahl inequality, and is included in most text books on general relativity (see, e.g., [3, 4]).

Due to its astrophysical importance in determining the gravitational redshift factor of a compact star\(^1\), the Buchdahl bound has been investigated over the years by many researchers. It has been found that various bounds can be obtained on the mass-to-radius ratio, \( \frac{2M}{R} \), depending on the assumptions made on the structure of the static configuration and its matter content (see [6–19] and references therein).

Most of the former studies have considered compact bodies with sharp boundaries, for which the energy density has support in some finite region \([0, R]\), with the assumption that the radial pressure drops to zero at the surface of the compact spherical object, \( p(r = R) = 0 \). However, field configurations may typically be non-compact, characterized by energy densities and pressures that approach zero only asymptotically.

In this work we would like to study this regime, of non-compact slowly decaying matter distributions. Specifically, we shall analyze the strength of gravity for non-compact configurations in which the radial pressure increases locally, or alternatively decreases no faster than some inverse power law \( r^{-\alpha} \). In these cases the matter fields approach zero only asymptotically, and thus the corresponding configurations possess no sharp boundaries.

The rest of the paper is devoted to the investigation of how large the fundamental mass-to-radius ratio, \( \frac{2m(r)}{r} \), can be for these slowly decaying field configurations. The paper is organized as follows. In section 2 we formulate the Einstein-matter equations in a form which would be convenient for the analysis of the behavior of the radial pressure. In section 3 we obtain an upper bound on the quantity \( \max_r \left\{ \frac{2m(r)}{r} \right\} \) for the canonical case, in which the stress–energy tensor of the matter fields satisfies the commonly used energy conditions. The results are extended in section 4, where we consider a generalized version of the energy condition. We conclude in section 5 with a summary of the main results. We also discuss the physical implications of the analytically derived bound.

2. The Einstein-matter equations

The metric of a static spherically symmetric spacetime takes the following form in Schwarzschild coordinates [20]
\[ ds^2 = -e^{-2\delta} dt^2 + \mu^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \] (2)
where the metric functions \( \delta \) and \( \mu \) depend only on the Schwarzschild radius \( r \). Asymptotic flatness requires that as \( r \to \infty \),
\[ \mu(r) \to 1 \quad \text{and} \quad \delta(r) \to 0, \] (3)
and a regular center requires\(^2\)
\[ \mu(r) = 1 + O(r^2) \quad \text{and} \quad p(0) = p_T(0). \] (4)
\(^1\) The Buchdahl inequality may also have important implications for the study of trapped surfaces and the cosmic censorship conjecture (see [5]).
\(^2\) From equation (9) below one finds \( dp/dr = 2(p_T - p) \) as \( r \to 0 \) (where \( \mu \to 1 \)). Regularity of the pressure function therefore requires that \( p(0) = p_T(0) \).
Taking $T^r_r = -\rho$, $T^\theta_\theta = p_T$, and $T^\phi_\phi = p_T$, where $\rho, p$ and $p_T$ are identified as the energy density, radial pressure and tangential pressure, respectively [8], the Einstein equations $G^\mu_\nu = 8\pi T^\mu_\nu$ reads

$$
\mu' = -8\pi \tau \rho + (1 - \mu)/r, \\
\delta' = -4\pi r (\rho + p)/\mu,
$$

where the prime stands for differentiation with respect to $r$.

The mass $m(r)$ contained within a sphere of radius $r$ is given by

$$
m(r) = \int_0^r 4\pi r^2 \rho(r') \, dr'.
$$

Taking cognizance of the Einstein equation (5) one finds $\mu(r) = 1 - 2m(r)/r$. (We use gravitational units in which $G = c = 1$.)

The conservation equation, $T^\mu_\nu_{;\mu} = 0$, has only one nontrivial component [20]

$$
T^r_r_{;\mu} = 0.
$$

Substituting equations (5) and (6) in equation (8), one finds for the pressure gradient

$$
p'(r) = \frac{1}{2\mu r} [(3\mu - 1 - 8\pi r^2 p)(\rho + p) + 2\mu T - 8\mu p],
$$

where $T = -\rho + p + 2p_T$ is the trace of the energy–momentum tensor. Below we shall analyze the behavior of the function $P(r; \gamma) \equiv r^{\gamma+3} p(r)$, whose derivative is given by

$$
P'(r; \gamma) = \frac{r^{\gamma+3}}{2\mu} [(3\mu - 1 - 8\pi r^2 p)(\rho + p) + 2\mu T + 2\mu \gamma p] .
$$

When analyzing the coupled Einstein-matter system, one usually imposes some energy conditions on the matter fields. Two commonly used energy conditions are as follows:

- **The weak energy condition (WEC).** This means that the energy density, $\rho$, is positive semidefinite and that it bounds the pressures. In particular, one usually assumes $0 \leq p \leq \rho$.
- The trace of the pressure tensor plays a central role in determining the spacetime geometry of the equilibrium configuration. It is usually assumed to satisfy the relation $p + 2p_T \leq \rho$ (see [8] and references therein). This condition is likely to be satisfied by most realistic matter models [8], and in particular it holds for Vlasov matter [21, 22].

3. The canonical case

In this section we obtain upper bounds on the fundamental ratio $\max_r [2m(r)/r]$ for general Einstein-matter models which satisfy the canonical energy condition $p + 2p_T \leq \rho$ ($T \leq 0$). Taking cognizance of equation (10) together with the energy condition, one finds

$$
P'(r; \gamma) \leq \frac{r^{\gamma+3}}{2\mu} [(3\mu - 1)(\rho + p) + 2\mu \gamma p] .
$$

Next, we may use the inequalities $\gamma p \leq \gamma (\rho + p)$ for $\gamma \geq 0$, and $\gamma p \leq 0$ for $\gamma \leq 0$ in equation (11) and obtain

$$
P'(r; \gamma) \leq \frac{r^{\gamma+3}}{2\mu} [(3\mu - 1) + 2\mu \gamma \Theta(\gamma)](\rho + p),
$$

where $\Theta(x)$ is the Heaviside step function.
We consider non-compact field configurations for which any decline of the radial pressure, \( p(r) \), is bounded by some power law, \( r^{-(\gamma+4)} \). This implies that the combined pressure function, \( P(r) \), is a monotonic increasing function. Equation (12), therefore, yields a lower bound on \( \mu(r) \),

\[
\min_r \{\mu(r)\} \geq \frac{1}{3 + 2\gamma \Theta(\gamma)},
\]

which, in turn, implies an upper bound on the fundamental mass-to-radius ratio,

\[
\max_r \left\{ \frac{2m(r)}{r} \right\} \leq \frac{2 + 2\gamma}{3 + 2\gamma},
\]

for \( \gamma \geq 0 \), and

\[
\max_r \left\{ \frac{2m(r)}{r} \right\} \leq 2\gamma
\]

for \( \gamma \leq 0 \).

We point out that for \( \gamma \leq 3 \) (that is, any decrease of the radial pressure is bounded by \( r^{-7} \)), the newly derived upper bound is stronger than the canonical Buchdahl inequality, \( \max_r \{2m(r)/r\} \leq 8/9 \).

4. Generalized energy condition

Our results may be extended using a generalized energy condition of the form (see [17] and references therein)

\[
p + 2p_T \leq \Omega \rho,
\]

where \( \Omega \geq 0 \). This condition is very general. Indeed, a realistic matter model which satisfies the dominant energy condition (DEC) [23] would satisfy this inequality with \( \Omega = 3 \). We shall now consider two distinct cases:

Case I: \( \Omega \geq 1 \).—In this case one can substitute the inequalities \( T \leq (\Omega - 1)\rho \leq (\Omega - 1)(\rho + p) \) and \( \gamma p \leq \gamma(\rho + p)\Theta(\gamma) \) into equation (10) to obtain

\[
[(3\mu - 1) + 2\mu(\Omega - 1) + 2\mu\gamma \Theta(\gamma)](\rho + p) \geq 0.
\]

This sets lower bounds on the function \( \mu(r) \) which, in turn, yield an upper bound on the maximal mass-to-radius ratio of the configuration

\[
\max_r \left\{ \frac{2m(r)}{r} \right\} \leq \frac{2\Omega + 2\gamma}{1 + 2\Omega + 2\gamma},
\]

for \( \gamma \geq 0 \), and

\[
\max_r \left\{ \frac{2m(r)}{r} \right\} \leq \frac{2\Omega}{1 + 2\Omega},
\]

for \( \gamma \leq 0 \).

It is interesting to note that a self-gravitating configuration with zero radial pressure [24] which satisfies the energy condition \( 2p_T = \Omega \rho \) may saturate the upper bound (equation (19)).

\[\text{We note that for the total configuration's mass to be finite, the energy density (and as a consequence of the weak energy condition also the radial pressure) should approach zero faster than } r^{-1} \text{ asymptotically (see equation (7)). Thus, at the asymptotic region, } r \to \infty, \text{ one has } \gamma > -1.\]
Table 1. Upper bounds on the fundamental mass-to-radius ratio, $\max_r \{2m(r)/r\}$, for the various regimes of the parameter space. The radial pressure is assumed not to decrease faster than some inverse power law, $r^{-(\gamma+4)}$, and the matter fields satisfy the generalized energy condition $p + \frac{2}{\Omega} \rho \leq \frac{\rho}{\Omega}$.

| $\Omega$ | $\gamma$ | Bound |
|---------|---------|-------|
| $0 \leq \Omega \leq 1$ | $\gamma \leq 0$ | $\frac{1+\Omega+2\gamma}{2+\Omega+2\gamma}$ |
| $1 \leq \Omega$ | $\gamma \geq 0$ | $\frac{1+\Omega}{2+\Omega}$ |

Substituting these conditions into equation (10), one indeed finds $\mu = (1 + 2\Omega)^{-1}$, which implies $2m(r)/r = 2\Omega/(1 + 2\Omega)$.

Case II: $0 \leq \Omega \leq 1$.— In this case one may use the inequality $T \leq (\Omega - 1)\rho \leq (\Omega - 1)(\rho + p)/2$. Substituting this into equation (10), one obtains

$$[(3\mu - 1) + \mu(\Omega - 1) + 2\mu\gamma(\gamma)](\rho + p) \geq 0.$$  \hfill (20)

From here one finds the upper bounds

$$\max_r \left\{ \frac{2m(r)}{r} \right\} \leq \frac{1 + \Omega + 2\gamma}{2 + \Omega + 2\gamma},$$  \hfill (21)

for $\gamma \geq 0$, and

$$\max_r \left\{ \frac{2m(r)}{r} \right\} \leq \frac{1 + \Omega}{2 + \Omega},$$  \hfill (22)

for $\gamma \leq 0$.

5. Summary and conclusions

We have investigated the behavior of the fundamental mass-to-radius ratio, $2m(r)/r$, for non-compact static configurations in which the matter fields have a slow global spatial decay. Contrary to compact bodies studied in the past, the non-compact configurations studied here have no sharp boundaries, though their total mass is finite.

The physical importance of the non-compact field configurations studied here lies in the fact that, in physical situations, the pressure function and its derivatives are expected to be continuous analytic functions. On the other hand, a compact body has a non-continuous pressure gradient at its surface. Namely, $p'(r) < 0$ as $r \to R^-$, and $p'(r) = 0$ as $r \to R^+$, where $R$ is the radius of the compact body. It is, therefore, natural and highly important to analyze the behavior of static field configurations for which the matter content approaches zero only asymptotically. In these cases, which we have studied here, the radial pressure and its gradients are smooth analytic functions throughout the bulk.

We have shown that if the spatial decay of the radial pressure is not faster than some inverse power law, $r^{-(\gamma+4)}$, then the quantity $\max_r \{2m(r)/r\}$ is bounded from above by a simple relation which depends on the power index $\gamma$ and on the quantity $\Omega$ which bounds the ratio between the trace of the pressure tensor to the local energy density. The newly derived upper bound is summarized in table 1 for the various regimes of the parameter space.

We point out that the upper bound on $\max_r \{2m(r)/r\}$ becomes tighter monotonically as one decreases the value of the parameter $\Omega$. This parameter controls the strength of the pressures that prevent the static configuration from collapsing under its own gravity. This
result is in accord with common sense—the smaller the pressures are, the weaker gravity must be in order to allow the existence of static regular configurations.

Likewise, one finds that the upper bound on \( \max \{2m(r)/r\} \) becomes stronger as the value of \( \gamma \) decreases (for \( \gamma < 0 \) the bound becomes degenerate). The parameter \( \gamma \) is central in determining the pressure gradients. The smaller the value of \( \gamma \) is, the smaller the outward pressure gradients are, and this implies that the upper bound on the strength of gravity must also be tighter in order to avoid complete gravitational collapse.

Finally, it is worth emphasizing that had we considered only local behavior of the fields, our bound would still hold locally. That is, in any region of spacetime in which the radial pressure increases, or alternatively decreases no faster than some power law, the fundamental mass-to-radius ratio, \( 2m(r)/r \), conforms to the bound given by table 1.

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