ON LOG K-STABILITY FOR ASYMPTOTICALLY LOG FANO VARIETIES

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Abstract. The notion of asymptotically log Fano varieties was given by Cheltsov and Rubinstein. We show that, if an asymptotically log Fano variety \((X, D)\) satisfies that \(D\) is irreducible and \(-K_X - D\) is big, then \(X\) does not admit Kähler-Einstein edge metrics with angle \(2\pi\beta\) along \(D\) for any sufficiently small positive rational number \(\beta\). This gives an affirmative answer to a conjecture of Cheltsov and Rubinstein.

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1. Introduction

The purpose of this article is to give a simple necessary criterion for log K-stability of \(((X, D), -K_X - (1-\beta)\beta)\) with cone angle \(2\pi\beta\) in the sense of [OS11], where \(X\) is projective log terminal and \(D\) is a reduced Weil divisor with \(-K_X - (1-\beta)\beta)\) ample. The motivation comes from a recent preprint of Cheltsov and Rubinstein [CR15], who treated the case that the dimension of \(X\) is equal to two. In this article, we show the following result.

**Theorem 1.1** (=Theorem 3.9). Let \(X\) be a normal projective variety which is log terminal, \(D\) be a nonzero reduced Weil divisor on \(X\) which is \(\mathbb{Q}\)-Cartier, and \(0 \leq \beta \leq 1\) be a rational number. Assume that the pair \((X, (1-\beta)\beta)\) is dlt, \(-K_X - (1-\beta)\beta)\) is ample, and \(((X, D), -K_X - (1-\beta)\beta)\) is ample. Then \(X\) does not admit Kähler-Einstein edge metrics with angle \(2\pi\beta\) along \(D\) for any sufficiently small positive rational number \(\beta\).
$(1 - \beta)D$ is log $K$-stable (resp. log $K$-semistable) with cone angle $2\pi\beta$. Then we have $\eta_\beta(D) > 0$ (resp. $\geq 0$), where

$$\eta_\beta(D) := \beta \cdot \text{vol}_X(-K_X - (1 - \beta)D) - \int_0^\infty \text{vol}_X(-K_X - (1 - \beta + x)D)dx.$$  

Note that $\text{vol}_X$ is the volume function (see [Laz04]).

Theorem 1.1 immediately gives the following corollary.

**Corollary 1.2** (see Corollary 3.11). Let $X$ be a smooth projective variety and $D$ be a nonzero reduced simple normal crossing divisor on $X$. Assume that $-K_X - (1 - \beta)D$ is ample for any $0 < \beta \ll 1$ and the divisor $-K_X - D$ is big. Then $((X, D), -K_X - (1 - \beta)D)$ is not log $K$-semistable with cone angle $2\pi\beta$ for any $0 < \beta \ll 1$ with $\beta \in \mathbb{Q}$. In particular, $X$ does not admit Kähler-Einstein edge metrics with angle $2\pi\beta$ along $D$ for any $0 < \beta \ll 1$ with $\beta \in \mathbb{Q}$.

Corollary 1.2 gives an affirmative answer for a conjecture of Cheltsov and Rubinstein for asymptotically log Fano varieties [CR13] with irreducible boundaries in any dimension. Although the following corollary is a special case of Corollary 1.2, we state the assertion for the readers’ convenience.

**Corollary 1.3** (see [CR13, Conjecture 1.11 (i)]). Let $(X, D)$ be an asymptotically log Fano variety with $D$ irreducible, that is, $X$ is a smooth projective variety and $D$ is a smooth irreducible divisor on $X$ such that $-K_X - (1 - \beta)D$ is ample for any $0 < \beta \ll 1$. If the divisor $-K_X - D$ is big, then $X$ does not admit Kähler-Einstein edge metrics with angle $2\pi\beta$ along $D$ for any $0 < \beta \ll 1$ with $\beta \in \mathbb{Q}$.

**Remark 1.4.** In [CR15, Theorem 1.6] (see also [CR15, Conjecture 1.5]), Cheltsov and Rubinstein proved Corollary 1.3 in dimension two by using a construction of flops on the deformation to the normal cone.

The strategy for the proof of Theorem 1.1 is essentially same as the strategy in [Fuj15]. We consider a kind of “log-version” of divisorial stability along $D$ in the sense of [Fuj15]. We construct a specific log semi test configuration from certain section ring (see Remark 3.3) and calculate its log Donaldson-Futaki invariant explicitly by using the theory of “geography of models” (see Theorem 3.4).

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A variety stands for a reduced, irreducible, separated and of finite type scheme over the complex number field \( \mathbb{C} \). For the theory of minimal model program, we refer the readers to \[KM98\]. For any Weil divisor \( E \) on a normal variety \( X \), the divisorial sheaf on \( X \) is denoted by \( \mathcal{O}_X(E) \). More precisely, the section \( \Gamma(U, \mathcal{O}_X(E)) \) on any open subscheme \( U \subset X \) is given by the following:

\[
\{ f \in \mathbb{C}(X) \mid \text{div}(f)|_U + E|_U \geq 0 \},
\]
where \( \mathbb{C}(X) \) is the function field of \( X \).

For varieties \( X_1 \) and \( X_2 \), let \( p_i: X_1 \times X_2 \to X_i \) \((i = 1, 2)\) be the projection morphisms.

2. Log K-stability

We recall the definition of log K-stability.

**Definition 2.1** (see [OS11]). Let \( X \) be an \( n \)-dimensional normal projective variety, \( L \) be an ample line bundle on \( X \), and \( D \) be a reduced Weil divisor on \( X \).

1. A coherent ideal sheaf \( \mathcal{I} \subset \mathcal{O}_{X \times \mathbb{A}^1} \) is said to be a flag ideal if \( \mathcal{I} \) is of the form

\[
\mathcal{I} = I_M + I_{M-1}t^1 + \cdots + I_1t^{M-1} + (t^{M}) \subset \mathcal{O}_{X \times \mathbb{A}^1},
\]
where \( I_M \subset \cdots \subset I_1 \subset \mathcal{O}_X \) is a sequence of coherent ideal sheaves of \( X \).

2. Let \( m \in \mathbb{Z}_{>0} \), and let \( \mathcal{I} \subset \mathcal{O}_{X \times \mathbb{A}^1} \) be a flag ideal. A log semi test configuration \( ((\mathcal{X}, \mathcal{D}), \mathcal{M})/\mathbb{A}^1 \) of \( ((X, D), L^\otimes m) \) obtained by \( \mathcal{I} \) is given from the following data:

- \( \Pi: \mathcal{X} \to X \times \mathbb{A}^1 \) is the blowing up along \( \mathcal{I} \), \( \mathcal{D} \subset \mathcal{X} \) is given by the blowing up of \( D \times \mathbb{A}^1 \) along \( \mathcal{I}|_{D \times \mathbb{A}^1} \), and \( E \subset \mathcal{X} \) is the Cartier divisor defined by \( \mathcal{O}_\mathcal{X}(-E) = \mathcal{I} \cdot \mathcal{O}_\mathcal{X} \),
- \( \mathcal{M} \) is the line bundle on \( \mathcal{X} \) defined by \( \mathcal{M} := \Pi^*p_1^*L^\otimes m \otimes \mathcal{O}_\mathcal{X}(-E) \),

such that we require the following:

- \( \mathcal{I} \) is not of the form \((t^M)\), and
- \( \mathcal{M} \) is semiample over \( \mathbb{A}^1 \).

3. Assume that \( ((\mathcal{X}, \mathcal{D}), \mathcal{M})/\mathbb{A}^1 \) is a log semi test configuration of \( ((X, D), L^\otimes m) \) obtained by \( \mathcal{I} \). Then the multiplicative group \( \mathbb{G}_m \) naturally acts on \( (\mathcal{X}, \mathcal{M}) \) and \( (\mathcal{D}, \mathcal{M}|_D) \). For \( k \in \mathbb{Z}_{>0} \), let \( w(k) \) be the total weight of \( \mathbb{G}_m \)-action on \( H^0(\mathcal{X}_0, \mathcal{M}^\otimes k|_{\mathcal{X}_0}) \) and \( \tilde{w}(k) \) be the total weight of \( \mathbb{G}_m \)-action on \( H^0(\mathcal{D}_0, \mathcal{M}^\otimes k|_{\mathcal{D}_0}) \),
where $X_0 \subset X$ and $D_0 \subset D$ are the scheme-theoretic fibers at $0 \in \mathbb{A}^1$, respectively. It is known that, for $k \gg 0$, $w(k)$ (resp. $\tilde{w}(k)$) is a polynomial function of degree at most $n+1$ (resp. $n$). For $k \gg 0$, we set
\[
\chi(X, L^\otimes k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}),
\]
\[
\chi(D, L|_D^\otimes k) = a_0 k^{n-1} + O(k^{n-2}),
\]
\[
w(k) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}),
\]
\[
\tilde{w}(k) = \tilde{b}_0 k^n + O(k^{n-1}).
\]

For any $\beta \in [0, 1]$, we set the \textit{log Donaldson-Futaki invariant}
\[
\text{DF}_\beta((\mathcal{X}, \mathcal{D}), \mathcal{M}) \text{ with cone angle } 2\pi \beta
\]
as
\[
\text{DF}_\beta((\mathcal{X}, \mathcal{D}), \mathcal{M}) := 2(b_0 a_1 - b_1 a_0) + (1 - \beta)(a_0 \tilde{b}_0 - b_0 \tilde{a}_0).
\]

(4) Let $\beta \in [0, 1]$. $((X, D), L)$ is said to be \textit{log K-stable} (resp. \textit{log K-semistable}) with cone angle $2\pi \beta$ if $\text{DF}_\beta((\mathcal{X}, \mathcal{D}), \mathcal{M}) > 0$ (resp. $\geq 0$) holds for any $m \in \mathbb{Z}_{>0}$, for any flag ideal $\mathcal{I}$, and for any log semi test configuration $((\mathcal{X}, \mathcal{D}), \mathcal{M})/\mathbb{A}^1$ of $((X, D), L^\otimes m)$ obtained by $\mathcal{I}$. For an ample $\mathbb{Q}$-divisor $A$ on $X$, $((X, D), A)$ is said to be \textit{log K-stable} (resp. \textit{log K-semistable}) with cone angle $2\pi \beta$ if $((X, D), \mathcal{O}_X(aA))$ is so for some $a \in \mathbb{Z}_{>0}$ with $aA$ Cartier (this definition does not depend on the choice of $a$).

The following theorem is important.

**Theorem 2.2** (see [Ber12] [CR15] [OS11]). \textit{Let $X$ be a smooth projective variety, $D$ be a reduced simple normal crossing divisor on $X$, and let $\beta \in [0, 1] \cap \mathbb{Q}$. Assume that $-K_X - (1 - \beta)D$ is ample and $X$ admits Kähler-Einstein edge metrics with angle $2\pi \beta$ along $D$. Then $((X, D), -K_X - (1 - \beta)D)$ is log K-semistable with cone angle $2\pi \beta$.}

3. **Construcing log semi test configurations**

In this section, from a pair $(X, D)$, we construct a specific log semi test configuration via $D$. The construction is essentially in the same way as in [Fuj15 §3]. We fix the following condition:

**Assumption 3.1.** \textit{Let $X$ be an $n$-dimensional normal projective variety which is log terminal, $D$ is a nonzero reduced Weil divisor on $X$ which is $\mathbb{Q}$-Cartier, and $\beta \in [0, 1] \cap \mathbb{Q}$. Assume that the pair $(X, (1 - \beta)D)$ is dlt and $-K_X - (1 - \beta)D$ is ample.}

**Definition 3.2.** Under Assumption 3.1, we set
\[
\tau(D) := \sup \{ \tau \in \mathbb{R}_{>0} \mid -K_X - \tau D \text{ big} \},
\]
\[
\tau_\beta(D) := \sup \{ \tau \in \mathbb{R}_{>0} \mid -K_X - (1 - \beta + \tau)D \text{ big} \}.
\]
It is obvious that $\tau_\beta(D) = \tau(D) - (1 - \beta)$. We remark that $\tau(D) > 1$ holds if and only if the divisor $-K_X - D$ is big.

**Remark 3.3.** By [BCHM10, Corollary 1.1.9], the $\mathbb{C}$-algebra

$$\bigoplus_{k \in \mathbb{Z}_{\geq 0}} H^0(X, \mathcal{O}_X(\lfloor k(-K_X - (1 - \beta)D) - jD \rfloor))$$

is finitely generated, where $\lfloor k(-K_X - (1 - \beta)D) - jD \rfloor$ is the biggest $\mathbb{Z}$-divisor which is contained by $k(-K_X - (1 - \beta)D) - jD$. We note that $H^0(X, \mathcal{O}_X(\lfloor k(-K_X - (1 - \beta)D) - jD \rfloor)) = 0$ if $j > k\tau_\beta(D)$. Thus, there exists $r \in \mathbb{Z}_{>0}$ such that

- $L_\beta := r(-K_X - (1 - \beta)D)$ is Cartier, and
- the $\mathbb{C}$-algebra

$$\bigoplus_{k \in \mathbb{Z}_{\geq 0}} H^0(X, \mathcal{O}_X(kL_\beta - jD))$$

is generated by

$$\bigoplus_{j \in [0, r\tau_\beta(D)] \cap \mathbb{Z}} H^0(X, \mathcal{O}_X(L_\beta - jD)).$$

From now on, we fix such $r$ (and $L_\beta$).

**Theorem 3.4 ([KKL12, Theorem 4.2]).** Under Assumption 3.1, there exist

- a sequence of rational numbers
  
  $$0 = \tau_0 < \tau_1 < \cdots < \tau_m = \tau_\beta(D),$$

- normal projective varieties $X_1, \ldots, X_m$ such that $X_1 = X$, and

- mutually distinct birational contraction maps $\phi_i: X \dashrightarrow X_i$ with $\phi_1 = \text{id}_X$ ($1 \leq i \leq m$)

such that the following hold:

- for any $x \in [\tau_{i-1}, \tau_i]$, $\phi_i$ is a semiample model (see [KKL12, Definition 2.3]) of $-K_X - (1 - \beta + x)D$, and

- if $x \in (\tau_{i-1}, \tau_i)$, then $\phi_i$ is the ample model (see [KKL12, Definition 2.3]) of $-K_X - (1 - \beta + x)D$.

**Proof.** By [BCHM10, Corollary 1.4.3], there exists a projective birational morphism $\sigma: \tilde{X} \to X$ such that $\sigma$ is an isomorphism in codimension one and $\tilde{X}$ is $\mathbb{Q}$-factorial. Let $\tilde{D}$ be the strict transform of $D$ on $\tilde{X}$. A semiample model (resp. the ample model) of $-K_{\tilde{X}} - (1 - \beta + x)\tilde{D}$
is a semiample model (resp. the ample model) of $-K_X - (1 - \beta + x)D$. Moreover, the $\mathbb{C}$-algebra
\[
\bigoplus_{k \in \mathbb{Z}_{\geq 0}} \bigoplus_{j \in \mathbb{Z}_{\geq 0}} H^0(X, \mathcal{O}_X([k(-K_X - (1 - \beta)D) - jD]))
\]
is equal to the $\mathbb{C}$-algebra
\[
\bigoplus_{k \in \mathbb{Z}_{\geq 0}} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}([k(-K_{\tilde{X}} - (1 - \beta)\tilde{D}) - j\tilde{D}])).
\]
Thus we can apply \cite[Theorem 4.2]{KKL12}.

We construct a log semi test configuration of \(((X, D), \mathcal{O}_X(L_\beta))\) from $D$. For any $j \in [0, r\tau_\beta(D)] \cap \mathbb{Z}$, we set
\[
I_j := \text{Image}(H^0(X, \mathcal{O}_X(L_\beta - jD)) \otimes_\mathbb{C} \mathcal{O}_X(-L_\beta) \to \mathcal{O}_X),
\]
where the homomorphism is the evaluation. Note that, for any $j \in [0, r\tau_\beta(D)] \cap \mathbb{Z}$ and
\[
0 \subset I_{r\tau_\beta(D)} \subset \cdots \subset I_1 \subset I_0 = \mathcal{O}_X
\]
hold. For $k \in \mathbb{Z}_{>0}$ and $j \in [0, kr\tau_\beta(D)] \cap \mathbb{Z}$, we define
\[
J_{(k,j)} := \sum_{j_1 + \cdots + j_k = j, \ j_1, \ldots, j_k \in [0, r\tau_\beta(D)] \cap \mathbb{Z}} I_{j_1} \cdots I_{j_k} \subset \mathcal{O}_X.
\]

**Lemma 3.5 (see \cite[Lemma 3.3]{Fuj15}).** The $J_{(k,j)} \subset \mathcal{O}_X$ is equal to
\[
\text{Image}(H^0(X, \mathcal{O}_X(kL_\beta - jD)) \otimes_\mathbb{C} \mathcal{O}_X(-kL_\beta) \to \mathcal{O}_X).
\]
In particular, we have
\[
H^0(X, \mathcal{O}_X(kL_\beta - jD)) = H^0(X, \mathcal{O}_X(kL_\beta) \cdot J_{(k,j)}).
\]

**Proof.** Set
\[
V_{k,j} := H^0(X, \mathcal{O}_X(kL_\beta - jD))
\]
for simplicity. We remark that, by the choice of $r \in \mathbb{Z}_{>0}$, the homomorphism
\[
\bigoplus_{j_1 + \cdots + j_k = j, \ j_1, \ldots, j_k \in [0, r\tau_\beta(D)] \cap \mathbb{Z}} V_{1,j_1} \otimes_\mathbb{C} \cdots \otimes_\mathbb{C} V_{1,j_k} \to V_{k,j}
\]
is surjective. For any $1 \leq i \leq k$, the ideal sheaf $I_{j_i}$ is nothing but
\[
\text{Image}(V_{1,j_i} \otimes_\mathbb{C} \mathcal{O}_X(-L_\beta) \to \mathcal{O}_X).
\]
Thus the assertion follows. \qed
Set the flag ideal $\mathcal{I}$ such that 
\[ \mathcal{I} := I_{\tau_{\beta}(D)} + I_{\tau_{\beta}(D)-1} + \cdots + I_1 t^{\tau_{\beta}(D)-1} + (t^{\tau_{\beta}(D)}) \subset \mathcal{O}_{X \times \mathbb{A}^1}. \]
For any $k \in \mathbb{Z}_{>0}$, we have 
\[ \mathcal{I}^k = J_{(k, \tau_{\beta}(D))} + J_{(k, \tau_{\beta}(D)-1)} + \cdots + J_{(k, 1)} t^{k \tau_{\beta}(D)-1} + (t^{k \tau_{\beta}(D)}) \]
by the construction of $J_{(k, j)}$. Let $\Pi: \mathcal{X} \to X \times \mathbb{A}^1$ be the blowing up along $\mathcal{I}$ and let $E \subset \mathcal{X}$ be the Cartier divisor given by the equation $O_X(-E) = E \cdot O_X$. Set $\mathcal{L}_{\beta} := \Pi^* p^*_1 O_X(k \beta) \otimes O_X(-E)$. Let $\mathcal{D} \to D \times \mathbb{A}^1$ be the blowing up along $\mathcal{I}|_{D \times \mathbb{A}^1}$. We note that $\mathcal{I}|_{D \times \mathbb{A}^1} = (t^{\tau_{\beta}(D)}) \subset \mathcal{O}_{D \times \mathbb{A}^1}$ since $I_j \subset O_X(-jD) \subset O_X(-D)$ for any $j > 0$. In particular, $\mathcal{D} \simeq D \times \mathbb{A}^1$ holds.

**Lemma 3.6** (see [Fuj15], Lemma 3.4). $((\mathcal{X}, \mathcal{D}), \mathcal{L}_{\beta})/\mathbb{A}^1$ is a log semi test configuration of $((X, D), L_{\beta})$.

**Proof.** Set $\alpha := p_2 \circ \Pi: \mathcal{X} \to \mathbb{A}^1$. It is enough to check that $\mathcal{L}_{\beta}$ is $\alpha$-semiample. By Lemma 3.3 the homomorphism
\[ H^0(X, O_X(k \beta) \cdot J_{(k, j)}) \otimes C \cdot O_X \to O_X(k \beta) \cdot J_{(k, j)} \]
is surjective for any $k \in \mathbb{Z}_{>0}$ and $j \in [0, \tau_{\beta}(D)] \cap \mathbb{Z}$. Thus
\[ H^0(X \times \mathbb{A}^1, p_1^* O_X(k \beta) \cdot T^k) \otimes C[t] \cdot O_{X \times \mathbb{A}^1} \to p_1^* O_X(k \beta) \cdot T^k \]
is surjective for any $k \in \mathbb{Z}_{>0}$. From [Laz04], Lemma 5.4.24, we have
\[ \alpha^* \alpha_* \mathcal{L}_{\beta}^\otimes k \simeq \Pi^* (p_2)^*(p_2)_*(p_1^* O_X(k \beta) \cdot T^k) \]
\[ = \Pi^* \left( H^0(X \times \mathbb{A}^1, p_1^* O_X(k \beta) \cdot T^k) \otimes C[t] \cdot O_{X \times \mathbb{A}^1} \right) \]
\[ \to \Pi^* \left( p_1^* O_X(k \beta) \cdot T^k \right) \]
\[ \to \Pi^* p_1^* O_X(k \beta) \otimes O_X(-kE) = \mathcal{L}_{\beta}^\otimes k \]
for $k \gg 0$. 

**Definition 3.7.** We say the log semi test configuration $((\mathcal{X}, \mathcal{D}), \mathcal{L}_{\beta})/\mathbb{A}^1$ the basic log semi test configuration of $((X, D), O_X(L_{\beta}))$ via $D$.

Now we calculate the log Donaldson-Futaki invariant of the basic log semi test configuration $((\mathcal{X}, \mathcal{D}), \mathcal{L}_{\beta})/\mathbb{A}^1$ of $((X, D), O_X(L_{\beta}))$ via $D$. By the asymptotic Riemann-Roch theorem, we have
\[ a_0 = \frac{(L_{\beta}^n)}{n!}, \quad a_1 = \frac{(L_{\beta}^{n-1} \cdot -K_X)}{2 \cdot (n-1)!}, \quad a_0 = \frac{(L_{\beta}^{n-1} \cdot D)}{(n-1)!}. \]
(We follow the notation in Definition 2.1) By [Odk13] §3,
\[ w(k) = -\dim \left( \frac{H^0(X \times \mathbb{A}^1, p_1^* O_X(k \beta))}{H^0(X \times \mathbb{A}^1, p_1^* O_X(k \beta) \cdot T^k)} \right) \]
\[ = -k \tau_{\beta}(D) \cdot h^0(X, O_X(k \beta)) + v(k), \]
where

\[ v(k) := \sum_{j=1}^{kr\tau_\beta(D)} h^0(X, \mathcal{O}_X(kL_\beta - jD)). \]

By the same argument,

\[ \tilde{w}(k) = -\dim \left( \frac{H^0(D \times \mathbb{A}^1, p_1^* \mathcal{O}_X(kL_\beta)|_D)}{H^0(D \times \mathbb{A}^1, p_1^* \mathcal{O}_X(kL_\beta)|_D \cdot (t^{kr\tau_\beta(D)})} \right) = -kr\tau_\beta(D) \cdot h^0(D, \mathcal{O}_X(kL_\beta)|_D). \]

Thus

\[ \tilde{b}_0 = -r\tau_\beta(D) \left( \frac{L_{\beta}^{n-1} \cdot D}{(n-1)!} \right). \]

We set \( v(k) = v_0 k^{n+1} + v_1 k^n + O(k^{n-1}) \). We calculate the values \( v_0 \) and \( v_1 \). Let \( L_{\beta,i} \) and \( D_i \) be the divisors on \( X_i \) which are the push-forwards of \( L_\beta \) and \( D \), respectively. For \( k \gg 0 \) sufficiently divisible, by [KKL12, Remark 2.4 (i)] and [Fuj15, Proposition 4.1], \( v(k) \) is equal to

\[
\sum_{i=1}^{m} \sum_{j=kr\tau_{i-1}+1}^{kr\tau_i} h^0(X_i, \mathcal{O}_{X_i}(kL_{\beta,i} - jD_i)) \\
= \sum_{i=1}^{m} \left( \frac{(kr)^{n+1}}{n!} \int_{\tau_{i-1}}^{\tau_i} ( (1/r)L_{\beta,i} - xD_i)^n \right) dx \\
- \frac{(kr)^n}{2 \cdot (n-1)!} \int_{\tau_{i-1}}^{\tau_i} ( (1/r)L_{\beta,i} - xD_i)^{n-1} \cdot (K_{X_i} + D_i)) dx \right) \\
+ O(k^{n-1}).
\]

This implies that

\[
v_0 = \frac{r^{n+1}}{n!} \sum_{i=1}^{m} \int_{\tau_{i-1}}^{\tau_i} ((1/r)L_{\beta,i} - xD_i)^n \right) dx,
\]

\[
v_1 = \frac{-r^n}{2 \cdot (n-1)!} \sum_{i=1}^{m} \int_{\tau_{i-1}}^{\tau_i} ((1/r)L_{\beta,i} - xD_i)^{n-1} \cdot (K_{X_i} + D_i)) dx.
\]

Thus we have

\[
DF_\beta((\mathcal{X}, \mathcal{D}), L_\beta) \\
= 2(v_0a_1 - v_1a_0) + (1 - \beta)(a_0\tilde{b}_0 - (v_0 - r\tau_\beta(D)a_0)\tilde{a}_0) \\
= \frac{n \cdot r^n(L_{\beta}^n)}{(n!)^2} \sum_{i=1}^{m} \int_{\tau_{i-1}}^{\tau_i} (\beta - x) \left( (K_{X_i} - (1 - \beta + x)D_i)^n \cdot D_i \right) dx.
\]
Lemma 3.8 (cf. [Fuj15, Theorem 5.2]). We have

$$\eta_\beta(D) = n \sum_{i=1}^{m} \int_{\tau_{i-1}}^{\tau_i} (\beta - x) \left( (-K_{X_i} - (1 - \beta + x)D_i)^{n-1} \cdot D_i \right) dx.$$  

Proof. By [KKL12, Remark 2.4 (i)], we have

$$\text{vol}_X(-K_X - (1 - \beta + x)D) = ((-K_{X_i} - (1 - \beta + x)D_i)^n)$$

for any \( x \in [\tau_{i-1}, \tau_i] \). From partial integration, we have

$$n \sum_{i=1}^{m} \int_{\tau_{i-1}}^{\tau_i} (\beta - x) \left( (-K_{X_i} - (1 - \beta + x)D_i)^{n-1} \cdot D_i \right) dx$$

$$= \sum_{i=1}^{m} \left( \left[ (x - \beta) \text{vol}_X(-K_X - (1 - \beta + x)D) \right]_{\tau_{i-1}}^{\tau_i} \right.$$  

$$- \int_{\tau_{i-1}}^{\tau_i} \text{vol}_X(-K_X - (1 - \beta + x)D) dx \left) = \eta_\beta(D). \right.$$  

We remark that \( \text{vol}_X(-K_X - (1 - \beta + x)D) = 0 \) if \( x \geq \tau_\beta(D) \). \( \square \)

Therefore we have obtained the following.

Theorem 3.9. Under Assumption 3.1, assume that \((X,D), -K_X - (1 - \beta)D)\) is log K-stable (resp. log K-semistable) with cone angle \(2\pi \beta\). Then \(\eta_\beta(D) > 0\) (resp. \(\geq 0\)) holds.

Remark 3.10. If \(\beta = 1\), then the value \(\eta_1(D)\) is noting but the value \(\eta(D)\) in [Fuj15, Definition 1.1].

Corollary 3.11. Let \(X\) be a normal projective variety which is log terminal, \(D\) is a nonzero reduced Weil divisor on \(X\) which is \(\mathbb{Q}\)-Cartier. Assume that \((X,(1 - \beta)D)\) is klt and \(-K_X - (1 - \beta)D\) is ample for any \(0 < \beta \ll 1\). Moreover, we assume that \(-K_X - D\) is big. Then for any \(0 < \beta \ll 1\) rational number, \((X,D), -K_X - (1 - \beta)D\) is not log K-semistable with cone angle \(2\pi \beta\).

Proof. We have \(\eta_\beta(D) = \eta_+(\beta) - \eta_-\), where

$$\eta_+(\beta) := \beta \cdot \text{vol}_X(-K_X - (1 - \beta)D) - \int_{1-\beta}^{1} \text{vol}_X(-K_X - xD) dx,$$

$$\eta_- := \int_1^{\infty} \text{vol}_X(-K_X - xD) dx.$$  

If \(-K_X - D\) is big, then \(\eta_- > 0\). On the other hand, \(\lim_{\beta \to 0} \eta_+(\beta)\) is equal to zero. Thus the assertion follows from Theorem 3.9. \( \square \)

Corollary 1.2 is immediately obtained from Theorem 2.2 and Corollary 3.11.
4. Examples

We see some examples.

**Example 4.1.** Let $X$ be an $n$-dimensional Fano manifold, and let $D$ be a smooth divisor on $X$ with $-K_X \sim_Q lD$ for some $l \in [1, n+1] \cap \mathbb{Q}$. Then $-K_X - (1 - \beta)D$ is ample for any $\beta \in (0, 1]$. In this case, we have

$$
\eta_\beta(D) = \frac{n}{n+1} \text{vol}_X(-K_X - (1 - \beta)D) \left( \beta - \frac{l-1}{n} \right).
$$

If $\beta < (l-1)/n$, then $\eta_\beta(D) < 0$ holds.

**Remark 4.2.** In Example 4.1 if $D \sim -K_X$ (i.e., $l = 1$), then $\eta_\beta(D) > 0$ for any $\beta \in (0, 1]$. Thus our argument does not give any destabilizing information in this case. Compare with [CR15, §6].

**Example 4.3.** Let $Y := \mathbb{P}^1(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, $C$ be a section of the $\mathbb{P}^1$-bundle $Y/\mathbb{P}^1$ with $(C^2) = 1$, $\pi: X \to Y$ be the blowing up along $p \in Y$ with $p \in C$, $E$ be the exceptional divisor of $\pi$, and set $D := \pi_*^{-1}C$. Then $-K_X - (1 - \beta)D$ is ample for any $\beta \in (0, 1]$. (The pair $(X, D)$ is nothing but [CR13] (I8B.1).) In this case, $\tau_1 = \beta$, $X_2 = Y$, $\tau_2 = \tau_\beta(D) = 1 + \beta$ and

$$
\text{vol}_X(-K_X - xD) = \begin{cases} 
-4x + 7 & \text{if } x \in [0, 1], \\
x^2 - 6x + 8 & \text{if } x \in [1, 2].
\end{cases}
$$

Thus $\eta_\beta(D) = 2(\beta^2 - 2/3)$. For example, if we set $\beta := 1/2$, then $\eta_{1/2}(D) < 0$ holds. In this case ($\beta = 1/2$), we can check that $r := 2$ satisfies the condition in Remark 3.3 and the corresponding flag ideal $\mathcal{I}$ is of the form

$$
\mathcal{I} = \mathcal{O}_X(-3D - 2E) + \mathcal{O}_X(-2D - E)t^1 + \mathcal{O}_X(-D)t^2 + (t^3).
$$

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