Resonant frequencies and spatial correlations in frustrated arrays of Josephson type nonlinear oscillators

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Abstract
We present a theoretical study of resonant frequencies and spatial correlations of Josephson phases in frustrated arrays of Josephson junctions. Two types of one-dimensional arrays, namely, the diamond and sawtooth chains, are discussed in detail. For these arrays in the linear regime the Josephson phase dynamics is characterized by multiband dispersion relation $\omega(k)$, and the lowest band becomes completely flat at a critical value of frustration, $f = f_c$.

In a strongly nonlinear regime such critical value of frustration determines the crossover from non-frustrated ($0 < f < f_c$) to frustrated ($f_c < f < 1$) regimes. The crossover is characterized by the thermodynamic spatial correlation functions of phases on vertices, $\langle \cos(p(\phi_i - \phi_j)) \rangle$, displaying the transition from long- to short-range spatial correlations. We find that higher-order correlations functions, e.g. $p = 2$ and $p = 3$, restore the long-range behavior deeply in the frustrated regime, $f \approx 1$. Monte-Carlo simulations of the thermodynamics of frustrated arrays of Josephson junctions are in good agreement with analytical results. We also outline the extension of our results to the case of kagome lattice, prototypical 2D frustrated lattice, and other higher dimensional lattices.

Keywords: Josephson junction array, flatbands, dispersion relation, spatial correlation function, frustrated networks

(Some figures may appear in colour only in the online journal)
1. Introduction

Great attention has been devoted to theoretical and experimental study of dynamics of various systems of interacting nonlinear oscillators. Interesting effects, e.g. collective (synchronized) behavior [1, 2], electromagnetic field induced dynamic metastable states [3], solitons and breathers [4, 5], just to name a few, were predicted and observed in diverse solid state systems.

In the linear regime the dynamics of complex networks of interacting oscillators is characterized by a multiband dispersion relation \( \omega_m(k) \), where \( k \) is the wave vector of the extended linear excitations and \( m \) is the band index. The dispersion relation can be probed by a resonant response of a system to a small external time-dependent perturbation [3, 6].

In [7] it was predicted that the spectrum of electronic excitations of the Lieb lattice contains a flatband. Lately flatbands were identified theoretically in various one- and two-dimensional lattices [8–12] and several methods to engineer flatbands were suggested [13–18]. Flat bands have been observed experimentally in magnetic and high-\( T_c \) superconducting materials [19, 20], and they have been implemented in photon lattices [21], exciton–polariton condensates [22], and arrays of superconducting Josephson junctions [23].

Dynamics of electrons, photons or phonons on lattices supporting flatbands shows interesting physical properties: magnetic phase transitions [8], localization in absence of disorder—appearance of compactons: eigenstates strictly localized on few sites, destructive interference of electromagnetic waves propagating in such lattices. In presence of disorder or non-linearity the flatband spectrum leads to solitons with non-exponential tails [24], Fano resonances in the scattering of electromagnetic waves on nonlinearities [13], and topological effects, e.g. preservation of topological flatbands in applied magnetic field [25].

The case of Josephson junctions is of special interest since the current technology allows one to engineer arbitrary one- and two-dimensional lattices of coupled Josephson junctions. Furthermore, external magnetic fields allow one to change the ground state and the dispersion relation \( \omega(k) \) of linear oscillations in these systems. E.g. it was shown theoretically that a simple diamond chain of identical Josephson junctions exhibits a classical (quantum) phase transition at specific strengths of the applied magnetic field [10, 26, 27]. This phenomenon was dubbed 4\( e \)-condensation at variance with the usual 2\( e \)-condensation occurring in simple Josephson junction arrays (lattices). Some evidences in support of such a 4\( e \)-condensation have been reported in [23].

In this paper we present a systematic study of dynamic and thermodynamic properties of frustrated Josephson junction arrays. The standard way to introduce frustration in the Josephson junction lattices that is used in the majority of works, is to apply an external magnetic field perpendicular to the lattice cells [10, 26, 27]. In this case the frustration is naturally determined as the ratio of an external magnetic flux per cell \( \Phi \) to the flux quantum \( \Phi_0 \), i.e. \( f_m = 2\pi \Phi / \Phi_0 \). We are using a different definition because frustration arises due to the Josephson couplings having alternating signs in a single lattice cell in our model. In the simplest case the Josephson couplings in a single cell are chosen as follows: a single Josephson coupling (in dimensionless units) is equal \( \alpha \) that varies from 1 up to \( -1 \), and the rest of Josephson couplings are set to one. In this case the frustration quantified by a frustration parameter \( f = (1 - \alpha) / 2 \), and the properties of the arrays, namely, resonant frequencies and ground states depend strongly on the value of \( f \). We emphasize here that there is no direct mapping between two definitions of the frustration parameter, \( f \) and \( f_m \), except the case of \( \alpha = -1, f = 1 \), which maps exactly on the \( f_m = 1/2 \) case discussed in [10, 26]: our Hamiltonian for the diamond chain becomes identical to that of Douçot and Vidal [10].

In particular thanks to the new way of introducing frustration, we find a transition between non-frustrated and frustrated regimes characterized by the critical value, \( f = f_c \), and study in...
detail the frustrated regime, \( f_c < f < 1 \). We show that the lowest band in the linear spectrum \( \omega(k) \) turns flat at \( f = f_c \), and this can be considered as the precursor of the crossover. The most spectacular difference between the non-frustrated and frustrated regimes is in the properties of the ground states. The ground state in non-frustrated regime is unique, with all Josephson phases equal to zero. In contrast, the ground state in frustrated regime is macroscopically degenerate and the Josephson phases can take two different sets of values in each cell of the array.

The important advantages of this approach are threefold: first, the frustrated regime is characterized by macroscopic degeneracy of the ground state, that persists for a finite range of frustrations, unlike the previous works \([10, 26, 27]\) where the degeneracy was observed for a single value of \( f \). Second, the transition between frustrated and non-frustrated regimes is intrinsically related to the appearance of a flat band at zero frequency in the spectrum of linear modes. Third, the Josephson junctions arrays (lattices) with Josephson couplings of alternating signs can be viewed as generic frustrated XY Hamiltonians with nearest-neighbor ferromagnetic/antiferromagnetic couplings similarly to a few other implementations \([28, 29, 30]\).

Experimental realizations of such arrays requires Josephson couplings of different signs. Such Josephson couplings are provided by the so-called \( \pi \)-Josephson junctions that can be fabricated on basis of superconductor-ferromagnet-superconductor junctions \([31]\), different facets of grain boundaries of high temperature superconductors \([32, 33]\), Josephson junctions between two-bands superconductors \([34, 35]\), or various single- or multi-junctions SQUIDS in externally applied magnetic field \([30, 36, 37]\).

The paper is organized as follows: in section 2 we introduce the models of frustrated arrays of Josephson junctions and define their Lagrangians, partition functions and dynamic equations. In this paper we focus on two examples of one-dimensional frustrated arrays: diamond and sawtooth chains. We also present briefly the extension of our analysis to the case of 2D frustrated kagome lattice. In section 3 we analyze the linearized dynamics of the frustrated arrays, derive the dispersion relation \( \omega(k) \) and study its dependence on the frustration strength \( f \). Section 4 is devoted to the derivation of the thermodynamic spatial correlation functions as functions of temperature and frustration, and detailed discussion of the crossover between frustrated and non-frustrated regimes. Section 5 presents numerical support for the analytical results of the previous sections by the Monte-Carlo simulations. Section 6 provides discussion and conclusions.

### 2. Model, Lagrangian, partition function and dynamic equations

Several lattices of coupled Josephson junctions: the one-dimensional diamond and sawtooth chains, and the two-dimensional kagome lattice, are shown in figure 1. Every vertex \( i \) of the network is characterized by a time-dependent phase, \( \varphi_i(t) \). Nearest neighbor vertices are binded by Josephson type non-linear oscillators, and therefore, the Lagrangian of network is as follows:

\[
L\{\varphi_i, \dot{\varphi}_i\} = E_J \left[ \sum_i \frac{\dot{\varphi}_i^2}{2\omega_p^2} + \sum_{\langle ij \rangle} \alpha_{ij} \cos(\varphi_i - \varphi_j) \right],
\]

where \( \langle ij \rangle \) are the two nearest neighbor vertices coupled by a Josephson junction, \( E_J \) and \( \omega_p \) are the Josephson coupling strength and the plasma frequency, respectively. \( \alpha_{ij} \) are the relative Josephson coupling strength of the \( ij \)-bond. The set of dynamic equations is then written as:
Thermodynamic properties are given by the partition function $Z$ that can be expressed through the path integral in the imaginary time-representation:

$$Z = \int D[\phi_n(\tau)] \exp \left[ \frac{1}{\hbar} \int_0^{\hbar/(\hbar k_B T)} L\{\phi_n, \dot{\phi}_n, i\tau\} \, d\tau \right].$$

(3)

A natural way to quantitatively characterize the long/short spatial correlations of the Josephson phases in the network of Josephson junctions is the set of correlation functions [38]:

Figure 1. Schematic plots of frustrated networks/lattices of Josephson junctions (indicated by crosses for the 1D cases): the diamond (top) and sawtooth (middle) chains, 2D kagome lattice (bottom). The phases $\phi$ of the vertices and the Josephson couplings $\alpha_{ij}$ in a single cell are shown.
The correlation function $C_{p}(n)$ allows one to distinguish different ground states: ordered and disordered. The correlation functions $C_{p}(n)$ for $p > 1$ are fingerprints of particular arrangements of Josephson phases inside of the cells.

Next we use the generic equations (1)–(4) to analyze the dynamic and thermodynamic properties of the frustrated arrays of coupled Josephson junctions. The frustrated arrays are characterized by specific distribution of Josephson coupling strengths, $\alpha_{ij}$, which display alternating signs of the couplings in every unit cell and destabilize the uniform (ferromagnetic) ground state.

### 3. Linear regime: dispersion relation $\omega(k)$

In this section we study the stability of the uniform (ferromagnetic) ground state on the three lattices displayed in figure 1 by analyzing the spectrum of linear excitations around the ground state [39]. As we will see the uniform solution generically becomes unstable for high enough frustration, and the linear spectrum acquires negative modes. Also unlike the more conventional transitions, where a single or few modes destabilize the ground state, in the models considered, the entire lower band turns flat, indicating the onset of massive degeneracy.

#### 3.1. Diamond chain

The state of the diamond chain of Josephson junctions is described by three phases per unit cell, $\varphi_{n} = \{\varphi_{0,n}, \varphi_{+n}, \varphi_{-n}\}$ (see the top part of figure 1). The distribution of Josephson coupling strength in a single cell is chosen as $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $\alpha_4 = \alpha$ (the couplings of a single rhombus are labelled clockwise). The precise choice of $\alpha_4$ is not important, and it can be swapped with any other $\alpha_i$ as implied by the symmetry of the diamond chain. The parameter $\alpha$ can take values from 1 to $-1$. As we mentioned in the Introduction the frustration parameter $f$ is defined as $f = (1 - \alpha)/2$ varying from 0 (non-frustrated arrays) to 1 (maximal frustration).

We linearize equation (2) around the uniform (ferromagnetic) solution, i.e. we write $\varphi_{\pm0,n} = \phi + \theta_{\pm0,n}$ and neglect the quadratic and higher order terms of $\theta_{\pm0,n}$. Then the dynamic equation (2) become

$$
\begin{align*}
\frac{1}{\omega^2_p} \tilde{\varphi}_{+n} &= \theta_{0,n} + \theta_{0,(n+1)} - 2\theta_{+,n}, \\
\frac{1}{\omega^2_p} \tilde{\varphi}_{-n} &= (1 - 2f)\theta_{0,n} + \theta_{0,(n+1)} - 2(1 - f)\theta_{-,n}, \\
\frac{1}{\omega^2_p} \tilde{\theta}_{0,n} &= \theta_{+,n} + \theta_{+,n-1} + (1 - 2f)\theta_{-,n} + \theta_{-,n-1} - 2(2 - f)\theta_{0,n}.
\end{align*}
$$

(5)
Using periodicity of the system we apply the Fourier transform with respect to space and time and obtain the dispersion relations \( \omega_k^p \equiv \omega^D(k) \) as solutions of the following transcendent equation (\( \alpha = 1 - 2f \)):

\[
\begin{align*}
-(\omega_k^D)^2 \omega_p^2 + 3 + \alpha = & \frac{4 \cos^2 \left( \frac{\theta}{2} \right)}{2 - (\omega_k^D/\omega_p)^2} + \frac{(\alpha - 1)^2 + 4\alpha \cos^2 \left( \frac{\theta}{2} \right)}{-(\omega_k^D/\omega_p)^2 + 1 + \alpha}.
\end{align*}
\]

(6)

This equation has three solutions \( \omega^D(k) \), representing the three bands which are shown in figure 2 for different values of the frustration parameter \( f \). One can see that flatbands occur for two particular values of the frustration parameter, \( f = 0 \) and \( f = 2/3 \). In the latter case the flatband is the lowest band of the spectrum, indicating the phase transition and the change of the ground state as the frustration parameter goes to the region \( f > f_{\text{c}}^D = 2/3 \).

### 3.2. Sawtooth chain

Similarly to the previous subsection, here, we analyze the dynamics of the sawtooth chain of Josephson oscillators (see the bottom part of figure 1). The sawtooth chain is described by two phases per unit cell, \( \varphi_n = \{ \varphi_{0,n}, \varphi_{+n} \} \). The Josephson couplings strengths in a single cell (as shown in figure 1) is fixed to \( \alpha_1 = \alpha_2 = 1 \) and \( \alpha_3 = \alpha \) (the couplings of a single triangle are labeled clockwise). Unlike the diamond chain, the choice of \( \alpha_3 \) differs from \( \alpha_{1,2} \) since swapping \( \alpha_3 \) with \( \alpha_{1,2} \) gives a different Hamiltonian. We will discuss this other choice in section 6. The frustration parameter \( f = (1 - \alpha)/2 \).

By expanding the Josephson phases around the uniform groundstate: \( \varphi_{+0,n} = \phi + \theta_{+0,n} \), and neglecting the quadratic and high-order terms of \( \theta_{+0,n} \), we arrive at the following linearized dynamical equation (2) for the sawtooth chain

\[
\begin{align*}
\frac{1}{\omega_p^2} \dot{\theta}_{+n} = & \theta_{0,n} + \theta_{0,(n+1)} - 2\theta_{+n}, \\
\frac{1}{\omega_p^2} \dot{\theta}_{0,n} = & \theta_{+n} + \theta_{+(n-1)} + (1 - 2f)\theta_{0,n-1} \\
& + (1 - 2f)\theta_{0,(n+1)} - 2(1 - 2f)\theta_{0,n}.
\end{align*}
\]

(7)

The dispersion relation \( \omega^{\text{ST}}(k) \) for time and space periodic solution reads

\[
\omega^{\text{ST}}(k) = \omega_p \left\{ 2 + 2\alpha \sin^2 \frac{k}{2} \pm \sqrt{4\alpha^2 \sin^4 \frac{k}{2} + 4\cos^2 \frac{k}{2}} \right\}^{1/2}.
\]

(8)

There are two solutions for every \( k \) corresponding to 2 bands, shown in figure 3 for several values of the frustration parameter \( f \). The flatband occurs for two particular values of the frustration parameter, \( f = 0.25 \) and \( f = 0.75 \). Again, in the latter case the flatband is the lowest band of the spectrum, and it marks the change of the ground state as the frustration parameter goes to the region \( f > f_c^{\text{ST}} = 3/4 \).

### 3.3. Kagome lattice

The analysis of 1D Josephson junction arrays extends straightforwardly to higher dimensions. We consider here the case of frustrated 2D kagome lattice (see, figure 1), where all couplings are set to 1 except the horizontal links, where the couplings are set equal to \( \alpha \). Similar to the
diamond chain and unlike the sawtooth chain, the choice of the links with $\alpha$ coupling is not important as long as the links form lines. The frustration parameter $f = (1 - \alpha)/2$.

Indeed, writing the set of dynamic equations for phases on the three sublattices, $\varphi_{0,ij}, \varphi_{ij}^\pm$ (indicated by black, red and green circles respectively in figure 1), we obtain the dispersion relation $\omega_k \equiv \omega(k_x, k_y)$:

**Figure 2.** The dispersion relation $\omega^D(k)$ for the diamond chain of Josephson junctions for several values of the frustration parameter: $f = 0$ (black lines), $f = 0.55$ (blue line), $f = 0.65$ (red line). Flat bands occur for $f = 0$ and $f = 2/3$. The flatband at $f_c = 2/3$ is the lowest one and marks the transition. Beyond $f_c = 2/3$ the uniform state is not a ground state, marked by the appearance of the negative (unstable modes).

**Figure 3.** The dispersion relation $\omega(k)$ for the sawtooth chain of Josephson junctions for several values of the frustration parameter: $f = 0$ (black lines), $f = 0.25$ (blue lines), $f = 0.5$ (green lines), $f = 0.75$ (red lines). The flatbands occurring at $f = 0.25$ and $f = 0.75$ (the lowest one) are shown.
\[
\left[ -\frac{\omega^2}{\omega_p} + 4 \right] \left[ -\frac{\omega^2}{\omega_p} + 2\alpha + 2 \right]^2 - 4\alpha^2 \cos^2 \frac{k_x + k_y}{2}
\]
\[= 4 \left( \cos \frac{k_x}{2} + \cos \frac{k_y}{2} \right) \left( -\frac{\omega^2}{\omega_p} + 2\alpha + 2 \right)
\]
\[+ 16\alpha \cos \frac{k_x + k_y}{2} \cos \frac{k_x}{2} \cos \frac{k_y}{2}. \tag{9}
\]

Three bands are obtained as the roots of the above polynomial for every pair of values \(k_x, k_y\). The lowest band becomes completely flat at the critical value of frustration \(f = 3/4\) (\(\alpha = -1/2\)): \(\omega_{KGL}(k_x, k_y) = 0\). This is illustrated in figure 4. Like the 1D case this indicates the instability of the ferromagnetic state and marks the transition to a different (non-ferromagnetic) degenerate ground state [40].

4. Thermodynamics of frustrated arrays of Josephson junctions: spatial correlation functions

In the classical regime the thermodynamic properties of the arrays of interacting Josephson junctions are described by the partition function

\[
Z = \int D\{\varphi_i\} \exp \left[ -\frac{U}{k_B T} \right], \tag{10}
\]

where the potential energy \(U\{\varphi_n\}\) depends on the set of vertex phases. The correlation functions of interest are expressed as

\[
C_p(n) = \frac{1}{Z} \int D\{\varphi_i\} \exp \left\{ -\frac{U}{k_B T} + ip[\varphi_{0,0} - \varphi_{0,m+n}] \right\}. \tag{11}
\]

Note that due to the global symmetry \(\varphi_n \rightarrow -\varphi_n\) respected by the energy \(U\) the above expression is real.

Below we discuss spatial dependencies of the correlation functions \(C_p(n)\) for several values of \(p\) and compare the behavior of such correlation functions with the \(C_1(n)\).

4.1. Frustrated diamond chain

In a particular case of diamond chain the potential energy is written as

\[
U\{\varphi_n\} = -E_J \sum_n \cos[\varphi_{+n} - \varphi_{0,n}] + \cos[\varphi_{+n} - \varphi_{0,n+1}]
\]
\[+ \cos[\varphi_{-n} - \varphi_{0,n+1}] + (1 - 2f) \cos[\varphi_{-n} - \varphi_{0,n}]. \tag{12}
\]

The correlation functions \(C_p(n)\) (11) can be computed exactly for the chain by introducing new variables: \(\varphi_{+n} - \varphi_{0,n} = s_{3n}, \varphi_{0,n+1} - \varphi_{+n} = s_{2n}, \varphi_{-n} - \varphi_{0,n+1} = s_{3n}\) and \(\varphi_{0,n} - \varphi_{-n} = s_{4n}\). New variables satisfy a simple constraint [10], \(s_{1n} + s_{2n} + s_{3n} + s_{4n} = 0\). Integrating over \(s_{3n}\) and \(s_{1n} - s_{2n}\) the spatial correlation functions evaluate to

\[
C_p^D(n) = \left\{ \frac{F_p^D}{F_p^0} \right\}^n \quad \alpha = 1 - 2f,
\]

\[
F_p^D = \int_0^{2\pi} d\varphi \exp \left[ \frac{2K \cos u}{2} \right] I_0 \left[ K \sqrt{1 + \alpha^2 + 2\alpha \cos u} \right], \tag{13}
\]

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where \( K = E_j/(k_B T) \) and \( I_0(z) \) is the modified Bessel function \([41]\). For high temperatures \( k_B T \gg E_j \) the lowest band, painted in black, becomes completely flat, marking instability of the uniform ground state towards new, highly degenerate ground state.

For \( K \ll 1 \) one obtains the expected strong suppression of correlations

\[
C^D_p(n) \simeq \eta_p(f) \left( \frac{E_j}{k_B T} \right)^{2m} n, \quad (14)
\]

where \( \eta_p(f) \) is a smooth function of order one that depends on both \( p \) and \( f \). For \( p = 1 \) one has explicitly \( \eta_1(f) = 1 - f \).

The situation is less obvious in the low temperature regime as \( k_B T \ll E_j \). The spatial decay of correlation functions strongly depends on the value of the frustration parameter \( f \). Indeed, for \( 0 < f < 2/3 \) the value \( u = 0 \) gives a most important contribution to the integrals over \( u \) in the equation (13), and therefore, the spatial correlation function displays long-range correlations as

\[
C^D_p(n) = \exp \left[ -\frac{(1 - f)k_B T}{(2 - 3f)E_j} p^2 n \right], \quad 0 < f < 2/3. \quad (15)
\]

Thus, one can see that in this regime the correlation length \( \xi_p \) defined as \( 1/\xi_p = -\ln[C_p(n)]/n \), increases as \( \xi_p \simeq 1/T \) with the decrease of temperature \( T \).

However in the highly frustrated regime as \( f_c^D = 2/3 < f < 1 \), the values \( u = \pm u_0 = \pm 2 \arccos(\sqrt{f}/[2(2f - 1)]) \) give the most important contribution to the integrals over \( u \) in equation (13), and we obtain a crossover to a regime of short-range correlations as

\[
C^D_p(n) = \exp \left[ -\frac{\beta_p(f)k_B T}{E_j} p^2 n \right] \{\cos(\mu u_0)\}^n, \quad (16)
\]

where \( \beta_p(f) \) is a smooth function of \( p \) of order one.

The correlation functions, \( C^D_p(n) \), computed for several values of temperature and frustration are shown in figure 5. As one can see the critical frustration \( f_c^D = 2/3 \) that
determines the crossover from long-range to short-range correlations, coincides with the value of $f$ where the lowest band in the spectrum $\omega(k)$ of linear excitations turns flat. This corresponds to a transition from unique ground state $u = 0$, to a highly degenerate ground state characterized by two possible values of $u = \pm u_0$ that can be chosen independently for every unit cell. Explicitly the single cell phases for two degenerate states are written as $\varphi_{0,n} = -u_0/2$; $\varphi_{+,n} = 0$; $\varphi_{0,n+1} = u_0/2$; $\varphi_{-,n} = 0$ and the mirror state $\varphi_{0,n} = +u_0/2$; $\varphi_{+,n} = 0$; $\varphi_{0,n+1} = -u_0/2$; $\varphi_{-,n} = 0$.

However, as we turn to higher order correlation functions with $p > 1$ we find that e.g. for frustration $f \approx 0.5$ the value of $u \approx \pi/2$ and the correlation function $C_2(n)$ displays a long-range behavior with sign alternation on adjacent cells. Since the correlation functions $C_p(n)$ characterize the thermodynamic properties of Cooper pairs with the charge, $2pe$, such phenomenon,
namely, a short-range behavior of the correlation function $C_1(n)$ and a long-range behavior of the correlation function $C_2(n)$ was coined as the 2e-4e transition in [10, 26, 27].

4.2. Frustrated sawtooth chain

For the sawtooth chain of Josephson junctions the potential energy $U$ reads

$$U\{\varphi_n\} = -E_J \sum_n \cos[\varphi_{+n} - \varphi_{0,n}] + \cos[\varphi_{+n} - \varphi_{0,n+1}] + (1 - 2f) \cos[\varphi_{0,n} - \varphi_{0,n+1}], \quad (17)$$

The correlation functions are computed similarly to the case of the diamond chain: the path integrals in the equations (10) and (11) are evaluated by introducing new variables $\varphi_{+n} - \varphi_{0,n} = s_{1n}$, $\varphi_{0,n+1} - \varphi_{+n} = s_{2n}$ and $\varphi_{0,n} - \varphi_{0,n+1} = s_{3n}$. The spatial correlation functions are expressed as ratios $C_{ST}^p(n) = (F_{ST}^p / F_{ST}^0)^n$, where

$$F_{ST}^p = \int_0^{2\pi} du e^{iu} I_0\left[\frac{2K \cos \frac{u}{2}}{2}\right] e^{-K(1-2f) \cos u}, \quad (18)$$

and $K$ and $I_0$ are the same as in the diamond chain case. For high temperatures $k_B T \gg E_J$ ($K \ll 1$) the correlation function $C_{ST}^p(n) \simeq [E_J / k_B T]^{2n}$ shows fast exponential decay. In the low temperature regime as $k_B T \ll E_J$ ($K \gg 1$) the spatial decay of correlation functions strongly depends on frustration $f$ just like in the diamond chain case. Indeed, for $0 < f < 0.75$ the value $u \simeq 0$ gives the most important contribution to the integrals over $u$ (18), and the spatial correlation function displays long-range correlations as

$$C_{ST}^p(n) = \exp \left[ - \frac{k_B T}{(3 - 4f)E_J} p^2 n \right]. \quad (19)$$
In the frustration regime \( f_{ST}^c = 0.75 < f < 1 \) the values \( u = \pm u_0 = \pm 2 \arccos[1/(4f - 2)] \) give the most important contribution to the integrals over \( u \) in (18), and we find a crossover to a regime of short-range correlations

\[
C_{ST}^p(n) = \exp \left[ -\frac{k_B T}{E_J} p^n \right] \{\cos(pu_0)\}^n.
\]  

(20)

The correlation functions, \( C_{p=1,3}^{ST}(n) \), computed for low temperature and for several values of frustration are shown in figure 6. As for the diamond chain, the critical frustration \( f_{ST}^c = 3/4 \) marks the crossover from long-range to short-range correlations and coincides with the value of \( f \) where the lowest band in the spectrum \( \omega(k) \) of linear excitations becomes flat. This indicates the transition from unique ground state \( u = 0 \), to the highly degenerate ground state characterized by independent choice of any of the two values of \( u = \pm u_0 \) in every unit cell. In this case the single cell phases for two degenerate states are written as \( \varphi_{0,n} = -u_0/2; \ \varphi_{\pm,n} = 0; \ \varphi_{0,n+1} = u_0/2 \).
and the mirror state- $\varphi_{0,R} = +u_0/2$; $\varphi_{+,R} = 0$; $\varphi_{0,R+1} = -u_0/2$; $\varphi_{-,R} = 0$. However, as we turn to higher order correlation functions with $p > 1$ we obtain that e.g. for $f \approx 1$ the value of $u \approx 2\pi/3$ and therefore the correlation function $C_3(n)$ displays a long-range behavior. Notice here, that diamond chain $C_3(n)$ displays oscillating behavior with $n$ at $f = 1$ while the sawtooth model $C_3(n)$ does not oscillate with $n$. This difference—sign alternation—does not seem to be related to the difference in geometry of the two models: diamond chain is bipartite while sawtooth chain is not. One can modify the diamond chain model by adding a coupling between $\varphi_{n,\pm}$ and $\varphi_{n,-\pm}$, that breaks the chiral symmetry of the Hamiltonian, nevertheless the $C_3(n)$ still shows oscillations as a function of $n$.

5. Spatial correlation functions: Monte-Carlo results

In order to check the analytical predictions we performed detailed classical Monte-Carlo simulations using the Hamiltonians (12) and (17), treating the vertex phases $\varphi_{x,y}$ as classical XY spins $\vec{s}_{x,y} = (\cos \varphi_{x,y}, \sin \varphi_{x,y})$. We used systems with $N = 900$ (diamond chain) and $N = 600$
(sawtooth chain) sites, each corresponding to 300 unit cells. A typical run consisted of a simulated annealing where the system was heated up from $k_B T = 0.01 E_J$ to $10 E_J$ in 100 steps. A set of correlation functions $C_p(n)$ with $p = 1 \ldots 8$ was computed during the annealing.

The Monte-Carlo results fully support the analytical predictions. Indeed, for both the diamond and sawtooth chains the spatial correlations are clearly ferromagnetic for low frustration $f < 2/3$ for diamond and $f < 3/4$ for sawtooth chains), i.e. the long-range spatial correlations develop in $C_1(n)$ as the temperature is lowered to zero. This behavior is shown in figure 7. Once the frustration $f$ is large enough, both the diamond and the sawtooth models show spatial correlations in $C_1(n)$ that drop to zero beyond few nearest unit cells (see figure 8). This observation is quantified with the help of the correlation length $\xi_1$ extracted from the fitting $C_1(n) \propto \exp(-n/\xi_1)$ (see (15) and (19)). As expected $\xi_1$ grows with decreasing temperature for low frustration. For large frustrations exceeding the critical value, $f > f_{c(DST)}$, the length $\xi_1$ is of order one, i.e. of a few unit cell, for all temperatures.

Higher order correlation functions display long-range ordering for specific values of frustration $f$ and $p$. The origin of this ordering is the $(m/p)\pi$ ($m$ are arbitrary integers) values of the Josephson phases in the ground state for these values of $f$. Indeed, the Monte-Carlo simulation for the diamond chain with the largest frustration $f = 1$, shows that the long-range correlations are restored in $C_2(n)$ for low temperatures. This result is displayed in figure 9 (similar behavior is observed for the sawtooth chain (not shown)). While $C_1(n)$ decays to zero beyond 2 unit cells, the $C_2(n)$ shows long-range oscillatory behavior with sign alternation. This is in good agreement with the analytical results (see figure 5). Similarly, for the sawtooth chain, $C_1(n)$ vanishes beyond few unit cells at $f = 1$, while $C_3(n)$ displays perfect ordering (not shown). The $C_2(n)$ correlation function displays the same behavior as $C_1(n)$.

The long-range correlations developing at $f = 1$ in the diamond and sawtooth chains can be quantified by correlation lengths $\xi_2$ and $\xi_3$ ($C_p(n) \propto \exp(-n/\xi_p)$) respectively as shown in figure 11. For small frustration $f < f_{c(DST)}$, as the ferromagnetic ordering is realized at $T = 0$, the correlation length $\xi_1$ is of order one, i.e. of a few unit cell, for all temperatures.

![Figure 9. The spatial correlation functions for the diamond chain, $C_2^D(n)$ for $p = 1$ (red line) and $p = 2$ (blue line) and $f = 1$ at $k_B T = 0.01 E_J$. For the convenience of the visualization $C_1(n)$ has been shifted down by 1.0 and the lower x axis reversed. The correlation function with $p = 1$ drops to zero beyond the second unit cell, while the $p = 2$ correlation function shows long-range features. The fluctuations at large distances are due to stronger fluctuations in $d = 1$ systems.](image-url)
the $C_p$ for all $p$ become long-ranged ones. Corresponding correlation lengths, $\xi_p$, increase with decreasing temperature. The growth is diminishing as the critical frustration $f_c^{D(ST)}$ value is approached and spatial correlations vanish. Upon entering the highly frustrated regime $f \geq f_c^{D(ST)}$ and approaching $f = 1$ the lengths again start to show growth with decreasing temperature and a long-range ordering establishes at $f = 1, T = 0$. Similar effect is observed in higher order correlation functions for several other values of $f > f_c$ for both models.
We verified the validity of our analysis for the kagome frustrated lattice (figure 1) by carrying out numerical energy minimization to identify the ground states. We repeated the minimization for several system sizes, up to \( N = 1024 \) spins. These simulations confirmed the stability of the ferromagnetic ground state up to \( f = 3/4 \) (figure 12(a), \( f = 0.7 \)), that is replaced by highly degenerate ground states as \( f > 3/4 \) (figure 12(b), \( f = 1 \)).
6. Discussion and conclusions

We have studied analytically and using Monte-Carlo simulations the static and dynamic properties of frustrated arrays of Josephson junction classical nonlinear oscillators. Such periodic arrays are characterized by Josephson coupling strengths in every cell that can have different signs. In the models considered we took all the couplings to be equal to 1 except a subset of links set to $\alpha$. Our results are robust with respect to (at least) small changes of the ferromagnetic couplings. We have considered two one-dimensional models of such frustrated arrays, namely, the diamond and sawtooth chains, and extended our results to the case of 2D kagome lattice (see figure 1).

In the linear regime the dynamics of these arrays is determined by a multi-band dispersion relation $\omega(k)$ (see figures 2 and 3). Such spectrum can be experimentally accessed through the analysis of the resonant response of the arrays to an applied small amplitude electromagnetic wave. We find that the spin-wave spectrum of frustrated arrays has a flatband for particular values of frustration $f$. Moreover at the critical value of frustration $f_c$ the lowest band becomes flat, and the transition between non-frustrated and frustrated nonlinear regimes occurs at the

![Figure 12. The ground states of frustrated kagome lattice for two values of frustration $f$ obtained by energy minimization: (a) the ferromagnetic ground state for low frustration, $f = 0.7$; (b) one of the many highly degenerate ground states for large frustration, $f = 1$.](image)
same value of frustration. We identified the critical values of frustration $f_{c}^{D(ST)}$ for both diamond and sawtooth chains.

The transition is reflected in the temperature and frustration dependencies of spatial correlation functions of different orders, $C_p(n)$. In particularly, we obtain that the Josephson junction diamond and sawtooth chains display the ferromagnetic order for low frustration, $f < f_c$, as all Josephson phases (the phase differences between nearest-neighbor vertices) are equal to zero. The ferromagnetic ground state is characterized by the exponential decay of the spatial correlation function $C_p(n)$, where the correlation length $\xi_1$ increases with the decreasing temperature. These results are presented in the top part of figures 5 (analytical results) and 7 (Monte-Carlo simulations).

At $f = f_c$ the transition to the frustrated regime occurs. It is characterized by the appearance of the massive degeneracy of the ground state, where any of the two different configurations of the phases, that depend on the frustration value $f$, can be chosen independently for every cell. This results into a drastic suppression of spatial correlations in the correlation function $C_p(n)$ (see the bottom part of figures 5 (bottom) and 6 (analytic results), and 8 (Monte-Carlo simulations)). The dependence of correlation length $\xi_1$ on temperature and frustration is shown as a density plot in figure 10.

Long-range correlations are recovered in higher-order correlation functions $C_p(n)$ with $p > 1$ deep in the frustration regime. Indeed, in the limit $f \rightarrow 1$ the long-range correlations have been identified for $C_p^D(n)$ for the diamond chain (see figures 5 (bottom), 9 and 11 (top)) and $C_p^{ST}(n)$ for the sawtooth chain (see figures 6 and 11 (bottom)). Furthermore, we found that the $C_p^D(n)$ changes sign in every cell. The long-range spatial correlations of high-order correlation functions indicate the presence of $2ne$-condensation ($n$ is larger than 1) in the frustrated arrays of Josephson junctions similarly to the model studied in [10] where the long-range correlations have been obtained in $C_j(n)$. It is worth noting, that the frustration strength $f = 1$ for which $C_p(n)$ with $p = 2$ and $p = 3$ show long-range order for the diamond and sawtooth chains respectively, is not unique. There are other values of frustration $f$ where higher order correlation functions $C_p(n)$ display long-range behavior at low temperatures. A few examples for the diamond chain are $f = 0.75$ with $p = 3$, $f \approx 0.71$ with $p = 4$, $f = 0.81$ with $p = 5$, $f \approx 0.85$ with $p = 7$ and $f \approx 0.78$ with $p = 8$. Interestingly there are no oscillations in $C_3(n)$ for $f = 0.81$.

Sawtooth chain admits two irreducible representations for the Josephson coupling distribution (see figure 1 and section 3): $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = \alpha$ (the bottom link of the triangle) and $\alpha_1 = \alpha$, $\alpha_2 = \alpha_3 = 1$. The first case was discussed above. In the second case one finds that the critical value of the frustration $f_{c}^{ST}$ does not change but both the spectrum of linear modes and the high-order correlations functions $C_p(n)$ with $p > 1$ vary in respect to the first case. In particular, we obtain that the correlation function $C_3(n)$ displays large oscillations for the value of frustration $f = 1$.

It is instructive to compare these results to the previously studied models of Josephson arrays with magnetic field induced frustration and to stress an important novel behavior present in our models: the model elaborated in [10] shows a massively degenerate ground state only for a single value of the frustration, corresponding to the half flux quantum per plaquette (as well as the higher dimensional models analyzed in [42, 43, 44]). Frustration does persist around this massive degeneracy point—the uniform ground state is unstable—, but the ground state is unique. Our models have highly degenerate ground states for a finite range of values of frustration $f_c < f < 1$. This remarkable feature was previously observed in a diamond chain of two-band superconductors [34], that is similar to our diamond chain model, however this degeneracy and its consequences were not explored by the authors of [34].
The geometry of the diamond and sawtooth chains studied strongly suggests that similar results hold for other corner sharing chains and lattices. The flatband construction of [15] relying on repetition of mini-arrays appears particularly promising. Our initial analysis of the frustrated kagome lattice have shown a peculiar transition from the ferromagnetic to highly degenerated ground state beyond some critical value of frustration (see figure 12), and corresponding appearance of a flat band at critical $f_c$ (see equation (9)). We expect that similar degeneracies might be present in models on other 2D and 3D lattices: dice [42, 43], honeycomb [44, 45].

At low temperatures quantum fluctuations start to play a crucial role. In the non-frustrating regime such quantum fluctuations result in the continuous quantum phase transition which is equivalent to an order-disorder phase transition occurring in the classical two-dimensional XY model [46]. In the frustrating regime a complex phase diagram is anticipated because of an intriguing interplay of the quantum tunneling induced transitions between classically degenerate ground states and excitation of quantum vortex (antivortex) pairs penetration in frustrated arrays of Josephson junctions. We will address the quantum-mechanical regime of such systems elsewhere.

Finally, we would like to note that our Hamiltonian can be viewed as generic frustrated XY Hamiltonians, not connected to any Josephson junction arrays. Our results suggests that breaking sublattice symmetry might be a promising route to discover novel frustrated phases, with highly degenerate ground states. So far, massive degeneracies in the ground state were limited to isolated points, or required inclusion of longer range interactions, like the 2nd or 3rd nearest neighbours.

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