Reduced-Order and Full-Order Interval Observers Design for Discrete-Time System

WEIJIE REN¹, RENYANG YOU¹, WEI YU², AND SHENGHUI GUO¹

¹College of Electronics and Information Engineering, Suzhou University of Science and Technology, Suzhou 215009, China
²School of Automation, Foshan University, Foshan 528225, China

Corresponding author: Shenghui Guo (shguo@usts.edu.cn)

This work was supported in part by the National Natural Science Foundation of China under Grant 61703296, Grant 61733015, and Grant 51875380, and in part by the Research Foundation of Suzhou University of Science and Technology under Grant XKZ2018004.

ABSTRACT In this paper, the design of reduced-order and full-order interval observers for discrete-time system is investigated. First, the original system is transformed into a descriptor system with the ability to estimate the unknown noise of sensors. Next, a reduced-order interval observer design method for descriptor system is presented. The state matrix of the system is obtained by linear transformation, and a Sylvester equation is constructed to find the appropriate observer gain matrix. Then, an existing design method of full-order interval observer is introduced in this paper. Finally, a simulation example is given to demonstrate that the reduced-order interval observer has more accurate interval estimation results than the full-order interval observer.

INDEX TERMS Reduced-order interval observer, interval estimation, descriptor system, sensor noise, discrete-time system.

I. INTRODUCTION

State feedback plays a significant role in practical system control and fault diagnosis, but some important system states can not be obtained directly due to the limitations of measurement means, so the study of state observers attracts many scholars. And due to the unavoidable noises and disturbances in the system, the researches of robust observers have become a hot topic [1]–[5], such as high order sliding mode observer [3], unknown input observer [4]. However, the design methods proposed in the above literature require noises or disturbances to meet many restrictive conditions, for example, the observer matching condition, which is very difficult to meet for practical systems. As a result, the applications of these methods are limited.

In literature [6], the validity of approximate interval estimation is proved by theoretical research, and the concept of interval observers is first proposed in [7], then interval observers become a new method to estimate states. Compared with the traditional asymptotic observers, which make point estimation of the system states, interval observers can provide the upper and lower boundaries ($x_k^+$ and $x_k^-$) of the estimation value, so that real states of the system are always in the estimation interval ($k \geq 0, x_k \in [x_k^-, x_k^+]$) [8]. Compared with the traditional asymptotic observers, the design procedures of interval observers are much more convenient [9]. In addition, the constraints of input and output disturbances is loose, only the upper and lower boundaries of the disturbances need to be known so that the interval observer can complete an accurate interval estimation. Moreover, interval estimation contains more information about system states than traditional point estimation. Therefore, as good supplements to asymptotic observers, interval observers have been widely studied by scholars [8]–[15], and have been applied in many practical systems [16]–[18].

Descriptor systems, also known as differential-algebraic systems, are represented by differential and algebraic equations. In comparison with the standard state space representation, descriptor systems can describe a wider range of practical system models. Therefore, the researches of descriptor systems have already achieved many important results. As we know, there are many researches on interval estimation of descriptor systems based on interval observers [19]–[25]. In [19], a method of interval observer design is proposed for a class of descriptor systems which have time delays. Considering uncertain nonlinear descriptor systems are of great significance in many practical applications, [20] assumes the uncertainties are bounded and manages to make interval
estimations for uncertain input and output. And [21] takes into account a special situation that \((\Gamma, A, C)\) may not always be detectable. On account of this, a new Luenberger-like interval observer is developed for descriptor systems and achieves good results.

However, all of the above methods are based on full-order interval observers, and there are few researches on the designs of reduced-order interval observers [26]–[28]. It is worth mentioning that the structure of reduced-order interval observers is simpler than that of full-order interval observers. Also, the amount of data to be processed is much less. Therefore, reduced-order interval observers do have certain advantages in the area of interval estimation. To be specific, [27] adopts a reduced-order interval observer method to estimate the states of an induction machine system which is time-variant and gets excellent interval estimation results.

Based on discussions above, a design method of reduced-order interval observer is presented, and the estimation results are compared with a full-order method. The main contributions of this paper are summarized as follows: (1) The reduced-order interval observer designed in this paper achieves accurate state interval estimations for a complex system, and the simulation results show it is better than the existing full-order interval observer design method. (2) By constructing a descriptor system, the reduced-order interval observer can estimate the unknown sensor noise effectively.

The rest of this paper is organized as follows: Some preliminaries and problem statements are formulated in Section II. Main results are given in Section III. The simulation example and a conclusion of the whole paper are presented in Section IV and Section V, respectively.

Here, we give the definitions of the symbols used in this paper. \(\mathbb{R}\) is the set of all real numbers, \(\mathbb{C}\) is the set of all complex numbers, and \(\mathbb{R}^n\) is a \(n\) dimensional set with positive elements. The \(A > B\) or \(A < B\) (\(A\) and \(B\) denote matrices or vectors) in the paper should be considered elementwise. If the spectral radius of a matrix \(A \in \mathbb{R}^{n \times n}\) is less than one, we call it a Schur matrix. And if all the elements in the matrix \(A \in \mathbb{R}^{n \times n}\) are less than zero, we call it a non-negative matrix. For a matrix \(A \in \mathbb{R}^{m \times n}\), we give a definition to \(A^+ = \text{max}\{0, A\}\) and \(A^- = \text{max}\{0, -A\}\), so the relation of them is \(A = A^+ - A^-\).

II. PRELIMINARIES AND PROBLEM FORMULATION

Consider a class of discrete-time systems as follows

\[
\begin{align*}
    x_{k+1} &= Ax_k + Bu_k + D\eta_k \\
    y_k &= Cx_k + Fw_k
\end{align*}
\]

(1)

where \(x_k \in \mathbb{R}^n\), \(u_k \in \mathbb{R}^m\) and \(y_k \in \mathbb{R}^p\) are the state vector, control input vector and output vector of the system, respectively. \(\eta_k \in \mathbb{R}^q\) and \(w_k \in \mathbb{R}^s\) denote input and sensor noises. \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times s}\) and \(F \in \mathbb{R}^{p \times s}\), which are all constant matrices. Moreover, \(D\) and \(F\) are matrices with full column rank. In addition, it is assumed that the initial state vector of the system satisfies the interval bounded condition \(x_0 \in [x_0^+, x_0^-]\). \(x_0^+\) and \(x_0^-\) represent the upper and lower boundaries of the state vector respectively. In addition, the system input noise \(\eta_k\) and initial sensor noise \(w_k\) are all bounded, that is to say, they meet \(\eta^- \leq \eta_k \leq \eta^+\) and \(w^- \leq w_k \leq w^+\). We assume that the system dimension satisfies \(p \geq q + s\).

In order to estimate the sensor noise, an augmented state vector of the system is constructed. We can get the following new system state vector

\[
\bar{x}_k = [x_k^T \ w_k^T]^T \in \mathbb{R}^{n+s}
\]

And the system state space representation after transforming is given as

\[
\begin{align*}
    E\bar{x}_{k+1} &= \bar{A}\bar{x}_k + \bar{B}u_k + \bar{D}\eta_k \\
    y_k &= \bar{C}\bar{x}_k
\end{align*}
\]

(2)

The original system (1) can be written as

\[
\begin{align*}
    E\tilde{x}_{k+1} &= \tilde{A}\tilde{x}_k + \tilde{B}u_k + \tilde{D}\eta_k \\
    y_k &= \tilde{C}\tilde{x}_k
\end{align*}
\]

(2)

In this descriptor system, \(E, \tilde{A} \in \mathbb{R}^{(n+s) \times (n+s)}, \tilde{B} \in \mathbb{R}^{(n+s) \times m}, \tilde{C} \in \mathbb{R}^{p \times (n+s)}\) and \(\tilde{D} \in \mathbb{R}^{p \times (n+s)}\). \(E\) is singular, i.e. \(\text{rank}(E) < n+s\). In addition, we do not need to know the upper and lower boundaries of the sensor noise in this case. It is easy to see (2) satisfies

\[
\text{rank}\begin{bmatrix}
    E & 0 \\
    C & F
\end{bmatrix} = \text{rank}\begin{bmatrix}
    I_n & 0 \\
    0 & 0_s
\end{bmatrix} = n+s
\]

(3)

Assuming that the original system (1) is observable, the following rank condition can be obtained

\[
\text{rank}\begin{bmatrix}
    zI - A & 0 \\
    C & F
\end{bmatrix} = n+s, \quad \forall z \in \mathbb{C}, \ |z| \geq 1
\]

(4)

According to (3), there exists a row full rank matrix which makes

\[
\begin{align*}
    [H_1 \ H_2] &\begin{bmatrix}
        E \\
        C
    \end{bmatrix} \\
    &\begin{bmatrix}
        I_n & 0 \\
        0 & 0_s
    \end{bmatrix} \\
    &\begin{bmatrix}
        C & F
    \end{bmatrix} \\
    &H_1 \begin{bmatrix}
        I_n & 0 \\
        0 & 0_s
    \end{bmatrix} + H_2 \begin{bmatrix}
        C & F
    \end{bmatrix}
\end{align*}
\]

(5)
In (5), \( H_1 \in \mathbb{R}^{(n+s)\times(n+s)} \) and \( H_2 \in \mathbb{R}^{(n+s)\times p} \) are the parameter matrices that need to be designed and they should satisfy the following constraint equation

\[
H_1E + H_2\bar{C} = I_{n+s} \tag{6}
\]

Remark 1: In the process of constructing the descriptor system from the original one, only the corresponding matrix transformation is used, without using any assumptions and inferences. Therefore, the constructed descriptor system is completely equivalent to the original. Because of this, system (2) is observable as we assume system (1) is. If there is an interval observer of the descriptor system (2), we can estimate the augmented state vector, and then the upper and lower boundaries \( (\bar{w}_k^x, \underline{w}_k^x) \) of the sensor noise in the original system can be obtained.

Lemma 1 [22]: For the following matrix equation

\[
MN = U
\]

where \( M \in \mathbb{R}^{a\times b}, N \in \mathbb{R}^{b\times c} \) and \( U \in \mathbb{R}^{a\times c} \), if \( \text{rank}(N) = c \), the general solution of \( M \) is

\[
M = N^\dagger + (I_b - NN^\dagger)
\]

\( S \in \mathbb{R}^{a\times b} \) is an arbitrary matrix, and \( N^\dagger \) denotes the pseudo-inverse of the matrix \( N \).

Lemma 2 [9]: Assume that the state vector \( x_k \in \mathbb{R}^n \) satisfies \( x_k^+ \leq x_k \leq x_k^- \) and \( x_k^+ \in \mathbb{R}^n \) are vectors which are already known, then we have \( \Theta^+x_k^- \leq \Theta x_k \leq \Theta^+x_k^- \Theta^-x_k^+ \), where \( \Theta \in \mathbb{R}^{m\times n} \) is an arbitrary constant matrix.

Lemma 3 [13]: In the discrete-time system \( x_{k+1} = Ax_k + d_k \), \( x_k \in \mathbb{R}^n \) and \( x_0 \in \mathbb{R}^n \) are the state vector and the initial state vector of the system, respectively. Assume that \( A \in \mathbb{R}^{n\times n} \) is a non-negative matrix, and \( d_k \in \mathbb{R}^n \) for \( \forall k \geq 0 \). If the condition \( x_0 \geq 0 \) is satisfied, then the discrete-time system \( x_{k+1} = Ax_k + d_k \) has a non-negative solution \( x_k \geq 0 \) for \( \forall k \geq 0 \).

III. MAIN RESULTS

A. REDUCED-ORDER INTERVAL OBSERVER DESIGN

On the basis of Lemma 1, the solution of the row full rank matrix in (5) is

\[
\begin{bmatrix} H_1 & H_2 \end{bmatrix} = \begin{bmatrix} E^\dagger & \Upsilon \end{bmatrix} \begin{bmatrix} I_{n+s+p} & -E \end{bmatrix} \begin{bmatrix} E^\dagger \end{bmatrix} \tag{7}
\]

where \( \Upsilon \in \mathbb{R}^{(n+s)\times(n+s+p)} \) is an arbitrary matrix. We can get the solutions of \( H_1, H_2 \), respectively

\[
\begin{align*}
H_1 &= \Lambda^\dagger\alpha_1 + \Upsilon \Xi \alpha_1 \\
H_2 &= \Lambda^\dagger\alpha_2 + \Upsilon \Xi \alpha_2
\end{align*}
\]

In (7), \( \Lambda = \begin{bmatrix} E^\dagger \end{bmatrix} \), \( \Xi = I_{n+s+p} - \begin{bmatrix} E \end{bmatrix} \begin{bmatrix} E^\dagger \end{bmatrix} \), \( \alpha_1 = \begin{bmatrix} I_{n+s} \noalign{\medskip} \sigma \end{bmatrix} \), \( \alpha_2 = \begin{bmatrix} 0_{(n+s)p} & I_p \end{bmatrix} \), \( \Lambda \in \mathbb{R}^{(n+s+p)\times(n+s)}, \Xi \in \mathbb{R}^{(n+s+p)\times(n+s+p)}, \sigma \in \mathbb{R}^{(n+s+p)\times(n+s)}, \) and \( \alpha_2 \in \mathbb{R}^{(n+s+p)p} \).

Using \( H_1, H_2 \) designed in (7), if \( H_1 \) is full column rank, then the descriptor system (2) can be transformed into the following form

\[
\begin{align*}
H_1E\tilde{x}_{k+1} &= H_1\tilde{A}\tilde{x}_k + H_1\tilde{B}u_k + H_1\tilde{D}\eta_k \\
H_2\tilde{C}\tilde{x}_{k+1} &= H_2y_{k+1}
\end{align*}
\]

According to (8) and constraint equation (6) we can get

\[
\begin{align*}
\tilde{x}_{k+1} &= (H_1E + H_2\tilde{C})^{-1}(H_1\tilde{A}\tilde{x}_k + H_1\tilde{B}u_k + H_1\tilde{D}\eta_k + H_2y_{k+1}) \\
y_k &= \tilde{C}\tilde{x}_k
\end{align*}
\]

The descriptor system (2) becomes

\[
\begin{align*}
\tilde{x}_{k+1} &= H_1\tilde{A}\tilde{x}_k + H_1\tilde{B}u_k + H_1\tilde{D}\eta_k + H_2y_{k+1} \\
y_k &= \tilde{C}\tilde{x}_k
\end{align*}
\]

Let \( z_k = \tilde{x}_k - H_2y_k \), a transformed state equation can be obtained

\[
\begin{align*}
z_{k+1} &= H_1\tilde{A}z_k + H_1\tilde{A}H_2y_k + H_1\tilde{B}u_k + H_1\tilde{D}\eta_k \\
\tilde{C}z_k &= (I - \tilde{C}H_2)z_k
\end{align*}
\]

Selecting a matrix \( T = [CT MT]^T \), where \( M \in \mathbb{R}^{(n+s-p)\times(n+s)} \) is an arbitrary matrix which make \( T \) is non-singular. We define \( T^{-1}z_k = \theta_k \), and substitute it into (10)

\[
\begin{align*}
T\theta_{k+1} &= H_1\tilde{A}T\theta_k + H_1\tilde{A}H_2y_k + H_1\tilde{B}u_k + H_1\tilde{D}\eta_k \\
(I - \tilde{C}H_2)y_k &= (\tilde{C}T)(T^{-1}z_k) = [I_p 0] \theta_k
\end{align*}
\]

Write \( \theta_k \) as the form of \( \theta_k = \begin{bmatrix} \theta_{1,k} \\ \theta_{2,k} \end{bmatrix} \), we can get

\[
\begin{bmatrix} I_p & 0 \end{bmatrix} \begin{bmatrix} \theta_{1,k} \\ \theta_{2,k} \end{bmatrix} = \theta_{1,k} \text{, so } (11) \text{ can be expressed by (12)}
\]

\[
\begin{align*}
\theta_{k+1} &= T^{-1}H_1\tilde{A}T\theta_k + T^{-1}H_1\tilde{A}H_2y_k + T^{-1}H_1\tilde{B}u_k + T^{-1}H_1\tilde{D}\eta_k \\
\theta_{1,k} &= (I - \tilde{C}H_2)y_k
\end{align*}
\]

We can regard the corresponding parameter matrices in (12) as a whole

\[
\begin{bmatrix} \theta_{k+1} = A\theta_k + \varphi y_k + Bu_k + \Delta\eta_k \\ \theta_{1,k} = My_k \end{bmatrix}
\]

Then, write the parameter matrices in (13) as the form of block matrices

\[
A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \Delta = \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}
\]

where \( A_1 \in \mathbb{R}^{p\times p}, A_2 \in \mathbb{R}^{p\times((n+s)-p)}, A_3 \in \mathbb{R}^{((n+s)-p)\times p}, A_4 \in \mathbb{R}^{((n+s)-p)\times((n+s)-p)}, \) \( \varphi_1 \in \mathbb{R}^{p\times p}, \varphi_2 \in \mathbb{R}^{((n+s)-p)\times p}, B_1 \in \mathbb{R}^{p\times p}, B_2 \in \mathbb{R}^{((n+s)-p)\times p}, \Delta_1 \in \mathbb{R}^{p\times p}, \) and \( \Delta_2 \in \mathbb{R}^{((n+s)-p)\times p} \).

The results are presented as follows

\[
\begin{align*}
\begin{bmatrix} \theta_{1,k+1} \\ \theta_{2,k+1} \end{bmatrix} &= \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} \theta_{1,k} \\ \theta_{2,k} \end{bmatrix} + \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} y_k + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_k + \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} \eta_k
\end{align*}
\]
Here, we define \( \theta \). In the light of (16), the relationships among \( L \), \( \xi_1, k \), \( \xi_2, k \), \( \theta_1, k \), and \( \theta_2, k \) can be shown as follows

\[
\begin{align*}
\xi_1, k &= \theta_1, k \nonumber \\
\xi_2, k &= \theta_2, k - L \theta_1, k
\end{align*}
\]

Substitute (17) into (15)

\[
\begin{align*}
\xi_1, k &= \theta_1, k + L \theta_1, k \nonumber \\
\xi_2, k &= \theta_2, k + L \theta_1, k - L \theta_1, k
\end{align*}
\]

Based on (13) and (16), \( \theta_2, k \) can be replaced by \( \theta_2, k = L \theta_1, k + \xi_2, k = LM \eta_1 + \xi_2, k \), then we bring it into (18)

\[
\begin{align*}
\xi_2, k+1 &= (A_4 - LA_2) (LM \eta_1 + \xi_2, k) \\
&+ [(A_3 - LA_1) M + (\theta_2 - L \theta_1)] y_k \\
&+ B_2 u_k - LB_1 u_k + \Delta_2 \eta_1 - L \Delta_1 \eta_1 \\
\xi_1, k &= \theta_1, k + L \theta_1, k
\end{align*}
\]

Remark 2: Based on (19), the reduced-order interval observer can be designed directly. What we need to do is to find out the appropriate observer gain matrix, so that the observer can meet the design conditions of asymptotic stability. In the designs of traditional asymptotic observers, for example, [24] takes an LMI method, making \( (A_4 - LA_2) \) a Schur matrix. But when designing interval observers, it is very difficult to find the appropriate matrix \( L \) directly because \( (A_4 - LA_2) \) is required to be Schur and non-negative at the same time. In literature [29], it uses a time-varying transformation. However, the calculation process is very complicated and the solution is usually hard to find. Here, a simpler linear transformation method is given as follows [13].

Define \( \tau_k = W \xi_2, k \), where \( W \in \mathbb{R}^{(n+s)-p \times (n+s)-p} \) is a nonsingular matrix, (19) becomes the following form

\[
\begin{align*}
\tau_{k+1} &= W (A_4 - LA_2) W^{-1} \tau_k + W (\theta_2 - L \theta_1) y_k \\
&+ W [(A_3 - LA_1) M + (A_4 - LA_2) L M] y_k \\
&+ W B_2 u_k - W L B_1 u_k + W \Delta_2 \eta_1 - W L \Delta_1 \eta_1 \\
&= P \tau_k + W (\theta_2 - L \theta_1) y_k \\
&+ W [(A_3 - LA_1) M + (A_4 - LA_2) L M] y_k \\
&+ W B_2 u_k - W L B_1 u_k + W \Delta_2 \eta_1 - W L \Delta_1 \eta_1 \quad \text{(20)}
\end{align*}
\]

In (20), \( P = W (A_4 - LA_2) W^{-1} \) is Schur and non-negative matrix. The reduced-order interval observer designed on the basis of (20) is presented in (21)

\[
\begin{align*}
\tau_{k+1} &= P \tau_k + (W \Delta_2 - W L \Delta_1)^{-1} \eta_k \\
&= - (W \Delta_2 - W L \Delta_1)^{-1} \eta_k \\
&+ W [(A_3 - LA_1) M + (A_4 - LA_2) L M] y_k \\
&+ W (\theta_2 - L \theta_1) y_k + W B_2 u_k - W L B_1 u_k \\
&= P \tau_k + (W \Delta_2 - W L \Delta_1)^{-1} \eta_k \\
&+ W [(A_3 - LA_1) M + (A_4 - LA_2) L M] y_k \\
&+ W (\theta_2 - L \theta_1) y_k + W B_2 u_k - W L B_1 u_k \quad \text{(21)}
\end{align*}
\]

Theorem 1: If there exists a nonsingular matrix \( W \in \mathbb{R}^{(n+s)-p \times (n+s)-p} \) and an observer gain matrix \( L \in \mathbb{R}^{(n+s)-p \times p} \) which can guarantee \( P = W (A_4 - LA_2) W^{-1} \) is a Schur and non-negative matrix, and define \( \tau_0^+ = W^+ \xi^+_{2,0} - W^- \xi^-_{2,0} \), \( \tau_0^- = W^+ \xi^+_{2,0} - W^- \xi^-_{2,0} \), then (21) is a reduced-order interval observer of (20). That is to say, the states of (20) meet \( \tau_k^- \leq \tau_k \leq \tau_k^+ \) for \( V_k \geq 0 \).

Proof: The upper estimation error is defined as \( \tilde{\tau}_k^+ = \tau_k^+ - \tau_k \), so we have

\[
\begin{align*}
\tilde{\tau}_{k+1} &= \tilde{\tau}_{k+1}^- + \tau_{k+1} \\
&= P \tilde{\tau}_k + (W \Delta_2 - W L \Delta_1)^{-1} \eta_k \\
&= - (W \Delta_2 - W L \Delta_1)^{-1} \eta_k \\
&+ W [(A_3 - LA_1) M + (A_4 - LA_2) L M] y_k \\
&+ W (\theta_2 - L \theta_1) y_k + W B_2 u_k - W L B_1 u_k \\
&= P \tilde{\tau}_k + (W \Delta_2 - W L \Delta_1)^{-1} \eta_k \\
&+ W [(A_3 - LA_1) M + (A_4 - LA_2) L M] y_k \\
&+ W (\theta_2 - L \theta_1) y_k + W B_2 u_k - W L B_1 u_k \quad \text{(21)}
\end{align*}
\]

The system after transformation is presented as below:

\[
\begin{align*}
\begin{cases}
\xi_2, k+1 &= (A_4 + LA_2) \xi_2, k + (\theta_2 - L \theta_1) y_k \\
&+ [(A_3 - LA_1) M + (A_4 - LA_2) L M] y_k \\
&+ B_2 u_k - LB_1 u_k + \Delta_2 \eta_1 - L \Delta_1 \eta_1 \\
\xi_1, k &= \theta_1, k + L \theta_1, k
\end{cases}
\end{align*}
\]

Based on Lemma 2, we know that \( (W \Delta_2 - W L \Delta_1)^{-1} \eta_k \leq (W \Delta_2 - W L \Delta_1)^{-1} \eta_k^- - (W \Delta_2 - W L \Delta_1)^{-1} \eta_k^+ \leq 0 \). So \( (W \Delta_2 - W L \Delta_1)^{-1} \eta_k^- \leq (W \Delta_2 - W L \Delta_1)^{-1} \eta_k^+ \geq 0 \). We also have \( \xi_{2,0}^+ = \tau_0^+ - \tau_0 = W^+ \xi_{2,0}^+ - W^- \xi_{2,0}^- - W^\xi_{2,0} \geq 0 \) and \( P \) is a Schur and non-negative matrix.
By using Lemma 3 we get $\bar{\tau}_k^+ = \tau_k^+ - \tau_k \geq 0$. That is $\tau_k \leq \tau_k^+$ for $\forall k \geq 0$.

Similarly, we define the lower estimation error as $\bar{\tau}_k^- = \tau_k - \tau_k^-$, then we have

$$\bar{\tau}_{k+1} = \tau_{k+1} - \tau_{k+1}^- = P\bar{x}_k - W(-L_1)u_k$$

and $$(W_2 - W_1)\bar{x}_k = (W_2 - W_1)\eta_k^- = \bar{\tau}_k^- = \tau_k - \tau_k^+ \geq 0.$$ According to the definition, $\bar{\tau}_0 = \tau_0 - \tau_0^- = W_2\xi_0 - (W_2^+\xi_2^+ - W_2^-\xi_2^-) \geq 0$, and $P$ is a Schur and non-negative matrix, using again Lemma 3, we have $\bar{\tau}_k = \tau_k - \tau_k^+ \geq 0$. Like the proof result we get above, for $\forall k \geq 0$, $\tau_k^+ \leq \bar{\tau}_k$.

In summary, the estimation results of the interval observer always satisfy $\tau_k^+ \leq \tau_k \leq \tau_k^+$ for $\forall k \geq 0$. Theorem 1 is proved.

In order to find the appropriate observer gain matrix $L$ and nonsingular matrix $W$, we construct a Sylvester equation from $P = W(A_4 - LA_2)W^{-1}$ to calculate.

First, we give the general form of Sylvester equation

$$AX + XB = -C$$

From $P = W(A_4 - LA_2)W^{-1}$, we can get an equation with the same form of (22)

$$(P)W + W(-A_4) = -(QA_2)$$

In (23), $Q \in \mathbb{R}^{(n+3)p \times p}$ is an arbitrary matrix, and we let $Q = WL$. When $(P \otimes I_{(n+3)p} + I_{3})$ is a nonsingular matrix, we can obtain a unique solution of $L$ and $W(\otimes)$ is Kronecker product.

### B. FULL-ORDER INTERVAL OBSERVER DESIGN

In this section, we choose the design method of a Luenberger-like interval observer in [21]. Although it considers the sensor noise, it does not estimate the noise value further. As a result, it is necessary for the sensor noise to meet the limit condition of a bounded interval $[W, \bar{W}]$, i.e. the noise belongs to an already known interval. In this paper, the method of constructing the augmented state vector does not need the above condition, so it is simpler in design procedures.

Next, we briefly describe the design method of a full-order observer.

On the basis of (9), we let $z_k = \bar{x}_k - H_2y_k$, then the new state equation is as follows

$$\begin{cases} z_{k+1} = H_1\bar{z}_k + H_1\bar{A}H_2y_k + H_1\tilde{B}u_k + H_1\tilde{D}\eta_k \\
y_k = \tilde{C}(z_k + H_2y_k) \end{cases}$$

There exists a matrix $V$ that is nonsingular, define $\phi_k = Vz_k$, (24) can be written as (25) by using linear transformation

$$\begin{cases} \phi_{k+1} = VH_1\tilde{A}V^{-1}\phi_k + VH_1\tilde{A}H_2y_k + VH_1\tilde{B}u_k + VH_1\tilde{D}\eta_k \\
y_k = \tilde{C}(V^{-1}\phi_k + H_2y_k) \end{cases}$$

(26) is the full-order interval observer of system (25)

$$\begin{cases} \phi_{k+1}^+ = S\phi_k^+ + V((H_1\tilde{A} - L\tilde{C})H_2 + L)z_k + VH_1\tilde{B}u_k \\
\phi_{k+1}^- = S\phi_k^- + V((H_1\tilde{A} - L\tilde{C})H_2 + L)z_k + VH_1\tilde{B}u_k \end{cases}$$

where $S = V(H_1\tilde{A} - L\tilde{C})V^{-1}$ is a Schur and non-negative matrix. One can refer to [21] for more information about the proof process.

### IV. SIMULATION STUDIES

In this section, we use a simulation example to demonstrate the proposed reduced-order interval observer has better performances than the full-order interval observer.

**Example:** Consider the descriptor system we discussed in (2) with parameter matrices as follows

$$\begin{bmatrix} 0.9841 & 0 & 0.0419 & 0 & 0 \\ 0 & 0.9888 & 0 & 0.0294 & 0 \\ 0 & 0 & 0.9581 & 0 & 0 \\ 0 & 0 & 0 & 0.9706 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tilde{A}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tilde{B} = \begin{bmatrix} 0.0796 & 0 \\ 0 & 0.0593 \\ 0.0351 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\tilde{D} = \begin{bmatrix} 0.3 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \tilde{C} = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0.4 \\ 0 & 0.5 & 0 & 0 & 0.4 \\ 0.3 & 0.5 & 0.5 & 0 & -0.4 \end{bmatrix}.$$
It is easy to see $\eta_k$ is in an interval with the upper and lower boundaries $\eta_k^+ = 0.05$ and $\eta_k^- = -0.05$. The selected initial conditions of the system are presented next

$\begin{align*}
x_0 &= \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}^T \\
x_0^+ &= \begin{bmatrix} 0.3 & 0.3 & 0.3 & 0.3 \end{bmatrix}^T \\
x_0^- &= -\begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}^T
\end{align*}$

From the results, it is extremely clear to see that the reduced-order interval observer designed in this paper has more accurate estimation results of system states than the full-order one in [21].

According to (7), we can compute the solutions of $H_1$ and $H_2$

$H_1 = \begin{bmatrix} 0.6923 & 0 & -0.1538 & 0 & 0 \\
0 & 0.8 & 0 & 0 & 0 \\
-0.1538 & 0 & 0.9231 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-0.0962 & 0 & 0.5769 & 0 & 0.5 \end{bmatrix}$

$H_2 = \begin{bmatrix} 0.3077 & 0 & 0.3077 \\
0 & 0.4 & 0 \\
0.1538 & 0 & 0.1538 \\
0 & 0 & 0 \\
1.3462 & 0 & -1.1538 \end{bmatrix}$
Results of the reduced-order interval observer we proposed are presented first.

\[ T = \begin{bmatrix} 2 & 0 & 0 & 0 & -0.8 \\ 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 1.6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]

And we select a Schur and non-negative matrix \( P \) and an arbitrary matrix \( Q \)

\[ P = \begin{bmatrix} 0.966 & 0 & 0.933 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \]

Using (23), a Sylvester equation is solved to get the linear transformation matrix \( W \) and the observer gain matrix \( L \)

\[ W = \begin{bmatrix} 2.5565 & -1.1524 \\ 0 & 1.3777 \end{bmatrix}, \quad L = \begin{bmatrix} 0.7183 & 0.3912 & 0.3272 \\ 0.7258 & 0 & 0.7258 \end{bmatrix} \]

In the following, results of the full-order interval observer are given. Similarly, a Schur and non-negative matrix \( S \) and an arbitrary matrix \( Q \) are chosen as

\[ S = \begin{bmatrix} 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0.9638 & 0 \\ 0 & 0 & 0 & 0 & 0.9315 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \]

So, we can compute \( V \) and the observer gain matrix \( L \) by using the method in [21]. The equation can be derived, as shown at the top of this page.

The simulation results are shown in Fig.1-Fig.5.

V. CONCLUSION

In this paper, a design method of reduced-order interval observers is proposed for discrete-time systems. By treating sensor noise as an augmented state, the original system is transformed into an equivalent descriptor system which is not affected by the noise. An observer is designed based on the linear transformation, and the appropriate gain matrix can be easily obtained by solving Sylvester equation. Through the simulation example, the observer designed in this paper is compared with an existing full-order interval observer. The results show that the proposed method in this paper is more effective and accurate.

REFERENCES

[1] D.-J. Lee, Y. Park, and Y.-S. Park, “Robust \( H_{\infty} \) sliding mode descriptor observer for fault and output disturbance estimation of uncertain systems,” IEEE Trans. Autom. Control, vol. 57, no. 11, pp. 2928–2934, Nov. 2012.

[2] F. Zhu and J. Yang, “Fault detection and isolation design for uncertain nonlinear systems based on full-order, reduced-order and high-order high-gain sliding-mode observers,” Int. J. Control, vol. 86, no. 10, pp. 1800–1812, Oct. 2013.

[3] K. Kalisi, J. Lian, S. Hui, and S. H. Zak, “Sliding-mode observers for systems with unknown inputs: A high-gain approach,” Automatica, vol. 46, no. 2, pp. 347–353, Feb. 2010.

[4] B. A. Charandabi and H. J. Marquez, “A novel approach to unknown input filter design for discrete-time linear systems,” Automatica, vol. 50, no. 11, pp. 2835–2839, Nov. 2014.

[5] S. Guo, F. Zhu, and L. Xu, “Unknown input observer design for Takagi-Sugeno fuzzy stochastic system,” Int. J. Control, Autom. Syst., vol. 13, no. 4, pp. 1003–1009, Aug. 2015.

[6] A. Agresti and B. A. Coull, “Approximate is better than ‘exact’ for interval estimation of binomial proportions,” Amer. Statist., vol. 52, no. 2, pp. 119–126, 1998.

[7] J. L. Gouzé, A. Rapaport, and M. Z. Hadj-Sadok, “Interval observers for uncertain biological systems,” Ecol. Model., vol. 133, nos. 1–2, pp. 45–56, Aug. 2000.

[8] F. Mazenc and O. Bernard, “Interval observers for linear time-invariant systems with disturbances,” Automatica, vol. 47, no. 1, pp. 140–147, Jan. 2011.

[9] D. Efimov, T. Raii, S. Chebotarev, and A. Zolghadri, “Interval state observer for nonlinear time varying systems,” Automatica, vol. 49, no. 1, pp. 200–205, Jan. 2013.

[10] L. Meyer, D. Ichalal, and V. Vigneron, “Interval observer for LPV systems with unknown inputs,” IET Control Theory Appl., vol. 12, no. 5, pp. 649–660, Mar. 2018.

[11] D. Guçük-Deirigny, T. Raii, and A. Zolghadri, “A note on interval observer design for unknown input estimation,” Int. J. Control, vol. 89, no. 1, pp. 25–37, Jan. 2016.

[12] Y. Wang, D. M. Bevly, and R. Rajamani, “Interval observer design for LPV systems with parametric uncertainty,” Automatica, vol. 60, pp. 79–85, Oct. 2015.

[13] T. Raii, D. Efimov, and A. Zolghadri, “Interval state estimation for a class of nonlinear systems,” IEEE Trans. Autom. Control, vol. 57, no. 1, pp. 260–265, Jan. 2012.
[14] D. Efimov, W. Perruquetti, T. Raissi, and A. Zolghadri, “Interval observers for time-varying discrete-time systems,” IEEE Trans. Autom. Control, vol. 58, no. 12, pp. 3218–3224, Dec. 2013.

[15] F. Mazenc, T. N. Dinh, and S. I. Niculescu, “Interval observers for discrete-time systems,” Int. J. Robust Nonlinear Control, vol. 24, no. 17, pp. 2867–2890, Nov. 2014.

[16] X. Zhang, F. Zhu, and S. Guo, “Actuator fault detection for uncertain systems based on the combination of the interval observer and asymptotical reduced-order observer,” Int. J. Control, pp. 1–9, May 2019, doi: 10.1080/00207179.2019.1620329.

[17] S.-A. Raka and C. Combastel, “Fault detection based on robust adaptive thresholds: A dynamic interval approach,” Ann. Rev. Control, vol. 37, no. 1, pp. 119–128, Apr. 2013.

[18] D. Efimov, L. Fridman, T. Raissi, and R. Seydou, “Interval estimation for LPV systems applying high order sliding mode techniques,” Automatica, vol. 48, no. 9, pp. 2365–2371, Sep. 2012.

[19] D. Efimov, A. Polyakov, and J.-P. Richard, “Interval observer design for estimation and control of time-delay descriptor systems,” Eur. J. Control, vol. 23, pp. 26–35, May 2015.

[20] G. Zheng, D. Efimov, F. J. Bejarano, W. Perruquetti, and H. Wang, “Interval observer for a class of uncertain nonlinear singular systems,” Automatica, vol. 71, pp. 159–168, Sep. 2016.

[21] S. Guo, B. Jiang, F. Zhu, and Z. Wang, “Luenberger-like interval observer design for discrete-time descriptor linear system,” Syst. Control Lett., vol. 126, pp. 21–27, Apr. 2019.

[22] Z. Wang, M. Rodrigues, D. Theilliol, and Y. Shen, “Actuator fault estimation observer design for discrete-time linear parameter-varying descriptor systems,” Int. J. Adapt. Control Signal Process., vol. 29, no. 2, pp. 242–258, Feb. 2015.

[23] S. H. Guo, F. L. Zhu, and B. Jiang, “Reduced-order switched UIO design for switched discrete-time descriptor systems,” Nonlinear Anal., Hybrid Syst., vol. 30, pp. 240–255, Nov. 2018.

[24] Z. Wang, Y. Shen, X. Zhang, and Q. Wang, “Observer design for discrete-time descriptor systems: An LMI approach,” Syst. Control Lett., vol. 61, no. 6, pp. 683–687, Jun. 2012.

[25] S. Guo and F. Zhu, “Reduced-order observer design for discrete-time descriptor systems with unknown inputs: An linear matrix inequality approach,” J. Dyn. Syst., Meas., Control, vol. 137, no. 8, p. 84503, Aug. 2015.

[26] D. Efimov, W. Perruquetti, and J.-P. Richard, “On reduced-order interval observers for time-delay systems,” in Proc. Eur. Control Conf. (ECC), Jul. 2013, pp. 2116–2121.

[27] S. Krebs, C. Schnurr, M. Pfeifer, J. Weigold, and S. Hohmann, “Reduced-order hybrid interval observer for verified state estimation of an induction machine,” Control Eng. Pract., vol. 57, pp. 157–168, Dec. 2016.

[28] M. Pourasghar, V. Puig, C. Ocampo-Martinez, and Q. Zhang, “Reduced-order interval-observer design for dynamic systems with time-invariant uncertainty,” IFAC-PapersOnLine, vol. 50, no. 1, pp. 6271–6276, Jul. 2017.

[29] Z. Wang, C.-C. Lim, and Y. Shen, “Interval observer design for uncertain discrete-time linear systems,” Syst. Control Lett., vol. 116, pp. 41–46, Jun. 2018.

WEIJIE REN was born in 1998. He is currently pursuing the bachelor’s degree with the College of Electronics and Information Engineering, Suzhou University of Science and Technology, China. His research interests include model-based fault detection and observer design.

RENYANG YOU was born in 1994. He received the B.S. degree in applied physics from the Nanjing University of Posts and Telecommunications, Nanjing, China, in 2017. He is currently pursuing the master’s degree with the College of Electronics and Information Engineering, Suzhou University of Science and Technology, China. His research interests include model-based fault detection and multi-agent systems.

WEI YU received the M.S. and Ph.D. degrees from the School of Automation Science and Engineering, South China University of Technology, Guangzhou, China, in 2009 and 2014, respectively. From 2014 to 2016, he held a postdoctoral position at the Guangdong University of Technology, Guangzhou. He is currently a Lecturer with the School of Automation, Foshan University, China. His research interests include fractional calculus, system identification, and fault diagnosis.

SHENGHUI GUO received the Ph.D. degree in control theory and control engineering from Tongji University, China, in 2016. He was a Postdoctoral Fellow of the College of Automation Engineering, Nanjing University of Aeronautics and Astronautics, China. He is currently an Associate Professor with the College of Electronics and Information Engineering, Suzhou University of Science and Technology, China. His research interests include observer design, model-based fault detection, and fault-tolerant control.