Resurgence and Lefschetz thimble in 3d $\mathcal{N} = 2$ supersymmetric Chern-Simons matter theories

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We study a certain class of supersymmetric (SUSY) observables in 3d $\mathcal{N} = 2$ SUSY Chern-Simons (CS) matter theories and investigate how their exact results are related to the perturbative series with respect to coupling constants given by inverse CS levels. We show that the observables have nontrivial resurgent structures by expressing the exact results as a full transseries consisting of perturbative and non-perturbative parts. As real mass parameters are varied, we encounter Stokes phenomena at an infinite number of points, where the perturbative series becomes non-Borel-summable due to singularities on the positive real axis of the Borel plane. We also investigate the Stokes phenomena when the phase of the coupling constant is varied. For these cases, we find that the Borel ambiguities in the perturbative sector are canceled by those in nonperturbative sectors and end up with an unambiguous result which agrees with the exact result even on the Stokes lines. We also decompose the Coulomb branch localization formula, which is an integral representation for the exact results, into Lefschetz thimble contributions and study how they are related to the resurgent transseries. We interpret the non-perturbative effects appearing in the transseries as contributions of complexified SUSY solutions which formally satisfy the SUSY conditions but are not on the original path integral contour.
# Contents

I. Introduction

II. $\mathcal{N} = 3$ Chern-Simons SQED
   A. Exact results as resurgent transseries
   B. Thimble decomposition
      1. Analytical results for small $g$
      2. Numerical results for finite $g$ and comparison with resurgent transseries
   C. Stokes phenomena in terms of $\arg(g)$
      1. Resurgent transseries
      2. Thimble decomposition
   D. “Mirror” description

III. $\mathcal{N} = 3$ $SU(2)$ Chern-Simons SQCD
   A. Exact results as resurgent transseries
   B. Thimble decomposition

IV. Generalization
   A. General rank-1 $\mathcal{N} = 2$ Chern-Simons matter theory
      1. Exact results as resurgent transseries
      2. Thimble decomposition
   B. Other observables
      Supersymmetric Wilson loop
      Bremsstrahlung function in SCFT on $\mathbb{R}^3$
      Two-point function of $U(1)$ flavor symmetry currents in SCFT on $\mathbb{R}^3$
      Partition function and Wilson loop on Squashed $S^3$
      Two point function of stress tensor in SCFT on $\mathbb{R}^3$

V. Path integral interpretation of the non-perturbative effects

VI. Summary and Discussion

Acknowledgments

A. Supersymmetric actions in 3D $\mathcal{N} = 2$ theory on $S^3
1. $\mathcal{N} = 2$ vector multiplet
2. $\mathcal{N} = 2$ chiral multiplet

B. Details on computation of perturbative series
   1. $\mathcal{N} = 3$ CS SQED
   2. $\mathcal{N} = 3$ SU(2) CS SQCD

C. Brief review of the thimble analysis

References
I. INTRODUCTION

Perturbative series in quantum field theory (QFT) is usually divergent \[1\]. One of standard procedures to take resummation of divergent series is Borel resummation. Given a perturbative series

\[
F_{\text{pert}}(g) = \sum_{\ell=0}^{\infty} c_\ell g^{a+\ell},
\]

(I.1)

Borel resummation of \(F_{\text{pert}}(g)\) along the direction \(\varphi\) is defined by

\[
S_\varphi F_{\text{pert}}(g) = \int_0^{\infty} e^{i\varphi t} e^{-\frac{t}{\pi}B_{\text{pert}}(t)},
\]

(I.2)

where \(B_{\text{pert}}(t)\) is the analytic continuation of the formal Borel transformation \(\sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} t^{a+\ell-1}\) and \(\varphi\) is usually taken as \(\varphi = \text{arg}(g)\). It is known (or expected) that \(B_{\text{pert}}(t)\) in typical QFT has singularities along the positive real axis \(\mathbb{R}_+\) in complex \(t\)-plane called the Borel plane. Some of the famous examples are quantum mechanics with degenerate classical vacua and asymptotically free field theories \[2\]. In this situation, the integral \(I.2\) with \(\varphi = 0\) is ill-defined and hence we have to deform the integration contour or equivalently complexify the parameter \(g\) to avoid the singularities. Consequently, the integral becomes ambiguous depending on the way of avoiding the singularities. In resurgence theory \[3\], which is often useful in such situations, one considers the following ansatz called a “transseries” for the exact result of the physical quantity

\[
F(g) = C_0 \sum_{\ell} c_\ell^{(0)} g^\ell + \sum_{I \in \text{saddles}} C_I e^{-\frac{S_I}{g}} \sum_{\ell} c_\ell^{(I)} g^\ell,
\]

(I.3)

where \(I\) labels nonperturbative saddle points and \(S_I\) are the actions at the saddle points. \(C_I\) denotes a transseries parameter which can jump at certain values of parameters called “Stokes lines”. It is expected that the ambiguities of perturbative Borel resummation are canceled by those of the nonperturbative saddles and one can obtain an unambiguous answer which is equivalent to the exact result. Typically a divergent perturbative series and non-perturbative contributions are related with each other via the cancellation of the ambiguities. Such a significant relation, called a “resurgent relation”, enables us to reconstruct non-perturbative terms from divergent perturbative series and vice versa \[3\]–\[11\].

Resurgence theory has a long history in quantum mechanics and differential equations. There have been various applications in a variety of physical systems including quantum mechanics (QM) \[12\]–\[27\], hydrodynamics \[28\], non-critical \[29\] and topological string theory \[30\]–\[31\] as well as QFT \[90\]. There are various types of applications to QFT such as in weak coupling expansions,
strong coupling expansions [34], 1/N-expansions [31], large-\(N_f\) expansions [35] and expansions by geometric parameters of space [36]. In this paper we make further progress in understanding the applications of resurgence theory to the weak coupling expansions in QFT with Lagrangians. The weak coupling expansion of QFT in the context of resurgence theory has been studied in 2D quantum field theories [37–46], 3D pure Chern-Simons theory [47, 48], 4D non-SUSY QFT [49–51] and supersymmetric (SUSY) gauge theories in various dimensions [52–58]. In all the known examples with sufficient data, observables have resurgent structures with respect to the coupling parameter and unambiguous transseries expressions, which agree with exact results. However it is currently unclear which observables/theories have resurgent structures. In other words, we do not know when one obtains an unambiguous answer by the resurgence procedure and when the answer obtained in this manner agrees with the exact result. If we can identify such a class, then we can obtain “semi-classical decoding” [59] of exact results or conversely, may use the resurgent structure to define QFT for this class.

In general, it is much harder to study the resummation problem in QFT than in quantum mechanics since Schrödinger equations are not available and we have to confront the saddle-point analysis of path integrals “seriously”. According to the recent progress in understanding the resurgent structure of QM from the path integral viewpoint [15], what we have to do is as follows:

- Find all critical points including complex saddles.
- See which critical points contribute in terms of Lefschetz thimble decompositions.
- Study perturbative expansions around contributing critical points.

We know that the first step is already technically hard in typical QFT and the second step is harder than the first step. Indeed there are only few known examples of physical quantities satisfying the following ideal conditions:

1. physical quantities in \(d\)-dimensional QFT \((d \geq 2)\),

2. quantities for which mathematically well-defined descriptions for their exact results are known [91],

3. quantities with the non-trivial resurgent structure.

To the best of our knowledge, the only examples satisfying all these three conditions are 2d pure YM theory [87] and pure CS theory [47, 92]. Main reasons for the difficulties to find such examples
are that the condition 2 is not satisfied in most cases at present and it is too complicated to check whether or not they satisfy the condition 3. Although exactly solvable quantities trivially satisfy the condition 2, they often do not satisfy the condition 3. Namely, they usually have truncated, convergent or Borel summable weak-coupling perturbative series, which has the trivial resurgent structure and gives an unambiguous result. A certain class of models becomes solvable in the large-$N$ limit but perturbative series with respect to the 't Hooft coupling in large-$N$ QFTs is typically convergent \[91\] \[.\] In some supersymmetric gauge theories, we have non-renormalizable theorems which imply that some observables are tree-level or 1-loop exact. The prepotentials of 4d $\mathcal{N} = 2$ theories receive an infinite number of instanton corrections but its perturbative series in each sector is truncated \[62\]. One of more non-trivial examples is a class of SUSY observables in 4d $\mathcal{N} = 2$ theories which also receive instanton corrections and have an asymptotic perturbative series in every sector, but all the perturbative series are Borel summable and hence unambiguous \[52, 54, 58\].

In this paper we propose an infinite number of examples satisfying all the above conditions 1, 2 and 3. The examples are a certain class of supersymmetric observables in 3d $\mathcal{N} = 2$ SUSY Chern-Simons (CS) theories coupled to matters, which appear in a broad context of theoretical physics such as AdS/CFT, M-theory, duality, higher spin gauge theory, condensed matter physics and so on. A typical quantity of this class is the partition function on $S^3$. Although the partition function is originally defined by the infinite-dimensional path integral, it is known that the partition function of 3d $\mathcal{N} = 2$ theory on $S^3$ has a finite-dimensional integral representation obtained by the SUSY localization method \[69\] whose dimension is a rank of gauge group \[94\]

$$Z_{S^3}(g, m) = \int_{\mathbb{R}^N} d^N \sigma\ e^{-S[\sigma]}, \quad (I.4)$$

where $g$ is a coupling constant proportional to the inverse of CS level $k$, $N$ is rank of gauge group and $\sigma$ is a Coulomb branch parameter. The integrand is uniquely determined by specifying the gauge group, the representation of matters, $U(1)_R$ charges, CS levels, FI parameters and real masses \[95\]. Since this is just a finite-dimensional integral, it obviously satisfies the condition 2. Furthermore, we will discuss that it is a resurgent function of $g$ and has non-trivial resurgent structures.

Another motivation of this paper comes from mysterious results in the same setup previously found by one of the present authors \[56\]. First, the work \[56\] found an explicit finite-dimensional integral representation of perturbative Borel transformation for the $S^3$ partition functions in 3d $\mathcal{N} = 2$ SUSY Chern-Simons matter theory \[96\]. Second, Borel summability along $\mathbb{R}_+$ on the Borel
plane depends on matter contents and values of real masses. Third, the exact result is always the
same as the Borel resummation along half imaginary axis:
\[ Z_{S^3} = \int_0^{-i\infty} dt \ e^{-\frac{i}{\hbar} BZ(t)} \quad (k > 0). \]  
(1.5)

Technically these results were obtained by rewriting the exact result and we did not have ap-
propriate interpretations for them. To obtain more precise understanding of these results, we
decompose the integration path of the Coulomb branch localization formula (1.4) into a sum of
Lefschetz thimbles (steepest descent contours) which has been recently applied in a variety of
contexts such as analytic continuation of path integral, real time path integral, black hole infor-
mation problem, cosmology, the sign problems in Monte Carlo simulation, and of course resurgence
theory. The advantage to use Lefschetz thimbles in our problem is that one can systematically express the exact result as a sum over contributions from critical points. In particular, we can determine which critical points contribute to the integral by
looking at the intersections of dual thimbles (steepest ascent contours) and the original integral
contour even if the critical points are not on the original integration contour. As we will see, in
our setups, the intersection numbers depend on the values of the real masses and precisely describe
the step-function behavior of the transseries parameter. We will discuss how the Lefschetz thimble
decomposition is related to the resurgent transseries.

We explicitly demonstrate the above arguments based on partition functions of a certain class
of rank-1 3d \( \mathcal{N} = 2 \) CS matter theories on \( S^3 \). Let us briefly summarize our results in the simplest
nontrivial theory: the \( \mathcal{N} = 3 \) CS SQED which is \( \mathcal{N} = 3 \ U(1) \) CS theory coupled to a charge-1
hyper multiplet with a real mass \( m \). This model can be regarded as a special case of the 3d \( \mathcal{N} = 2 \)
theories. The exact result for the sphere partition function of this theory is simply given by
\[ Z = \int_0^{\infty} d\sigma \ \frac{e^{ik\pi g^2}}{2 \cosh \frac{\sigma - m}{2}}. \]  
(1.6)

It has been shown that this expression is regarded as the Borel resummation along the direction
\( \varphi = -\pi/2 \):
\[ Z = \int_0^{-i\infty} dt \ e^{-\frac{i}{\hbar} BZ(t)}, \]  
(1.7)

where the Borel transformation \( BZ(t) \) will be explicitly given in (II.3) later. By changing the
integration contour, we can also write this as the Borel resummation along \( \mathbb{R}_+ \) plus residues in the
4th quadrant of Borel plane:
\[ Z = \int_0^{\infty} dt \ e^{-\frac{i}{\hbar} BZ(t)} + \sum_{\text{poles} \in 4\text{th quadrant}} \text{Res}_{t=pole} \left[ e^{-\frac{i}{\hbar} BZ(t)} \right], \]  
(1.8)
where the second term generates non-perturbative corrections. The most important point here is that distribution of the poles depends on the mass $m$. We will discuss that the number of the poles in the 4th quadrant is $|n|$ when $(2n - 1)\pi < m < (2n + 1)\pi$ and find that $Z$ has the following transseries expression

$$Z = \sum_q c_q^{(0)}(m) g^q + \sum_{n=1}^{\infty} \theta(m - (2n - 1)\pi) c_q^{\frac{4}{3}(m+(2n-1)\pi)^2} \sum_q c_q^{(n)}(m) g^q,$$

(1.9)

where $\theta(x)$ is step function and the perturbative coefficients $c_q^{(n)}(m)$ will be given in (II.10). The second term consists of exponentially suppressed corrections which are identified as the non-perturbative contributions. We will show that the transseries has a nontrivial resurgent structure and hence gives the unambiguous answer in agreement with the exact result. We will also decompose the Coulomb branch localization formula (I.4) in terms of Lefschetz thimbles and discuss relations between the transseries expression and the thimble decomposition. We will first find critical points around the origin and singularities of the integrand, which are interpreted as perturbative and non-perturbative critical points respectively. It will be shown that the value of real mass $m$ determines which thimbles associated with the nonperturbative critical points are contributing while the perturbative thimble always contributes to the partition function. However, it will be also shown that the correspondence between each thimble integral and each of the building blocks of the transseries is complicated for finite $g$. We will argue that one building block of the transseries can be given by the multiple thimble integrals. For example, a sum of the perturbative thimble integral and one of the nonperturbative thimble integrals coincides with the perturbative Borel resummation along $\mathbb{R}^+$ in a certain region of the real mass $m$.

We also discuss path integral interpretation of the non-perturbative contributions appearing in the resurgent transseries. Recently, one of the present authors has found complexified supersymmetric solutions in general 3d $\mathcal{N}=2$ SUSY field theory on $S^3$ which formally satisfy SUSY conditions but are not on the original path integral contour, and then proposed that these solutions correspond to the singularities of the Borel transformation of the perturbative series (Borel singularities) in 3d $\mathcal{N}=2$ SUSY Chern-Simons matter theory [56]. We discuss possible interpretation of the nonperturbative effects in terms of the complexified SUSY solutions.

This paper is organized as follows: In Sec. II we first obtain the full transseries expression of the partition function in the $\mathcal{N}=3$ CS SQED. Next we discuss the thimble decomposition of the partition function expressed as the integral with respect to the Coulomb branch parameter, with emphasis on the Stokes phenomena at the special values of real mass. In Sec. III we obtain the full transseries of partition function in $SU(2)$ vector multiplet with the Chern-Simons term coupled
with hyper multiplets (CS SQCD), where we discuss the thimble decomposition and the Stokes phenomena. In Sec. [IV] we discuss generalization to more generic theories and other observables. In Sec. [V] we propose an interpretation on the relation between the complex saddles of the Coulomb branch parameter and the complex SUSY solutions of the CS SQED and CS SQCD. Sec. [VI] is devoted to summary and discussion.

II. \( \mathcal{N} = 3 \) CHERN-SIMONS SQED

In this section, we study the \( S^3 \) partition function of 3D \( \mathcal{N} = 3 \) \( U(1)_k \) CS theory with \( N_f \) charge +1 hyper multiplets, which we call \( \mathcal{N} = 3 \) CS SQED \([98]\). In the 3D \( \mathcal{N} = 2 \) language, this theory consists of an \( \mathcal{N} = 2 \) vector multiplet, an adjoint chiral multiplet with \( U(1)_R \) charge 1 and \( N_f \) pairs of charge +1 and −1 chiral multiplets with \( U(1)_R \) charge 1/2 \([99]\). We also turn on real masses \( m_a (a = 1, 2, \ldots, N_f) \) associated with the \( U(N_f) \) flavor transformation of the hyper multiplets \([100]\).

Applying the SUSY localization \([69]\) to the present theory, the partition function is expressed as \([70, 71]\)

\[
Z = \int_{\sigma} \frac{d\sigma}{\prod_{a=1}^{N_f} 2 \cosh \frac{\sigma - m_a}{2}},
\]

where \( \sigma \) is the Coulomb branch parameter given by constant configuration of the adjoint scalar in 3d \( \mathcal{N} = 2 \) vector multiplet \([101]\).

In Sec. [II.A] we show that the exact partition function obtained by the localization technique with respect to the Coulomb branch parameter can be written as a full transseries with non-perturbative exponential contributions. In Sec. [II.B] we argue the thimble decomposition of the integral with respect to the Coulomb branch parameter. In both cases, we discuss the Stokes phenomena at the special real masses.

A. Exact results as resurgent transseries

Let us take \( k > 0 \) and \( m_a \geq 0 \) for simplicity \([102]\). By changing the variables as \( g = \frac{4\pi}{k} \) and \( \sigma = \sqrt{it} \), we rewrite the partition function as

\[
Z = \int_{0}^{-i\infty} dt \ e^{-\frac{1}{2}t \mathcal{B}Z(t)},
\]
where

$$BZ(t) = \frac{i}{4\sqrt{it}} \left[ \prod_{a=1}^{N_f} \frac{1}{\cosh \sqrt{it-m_a}} + \prod_{a=1}^{N_f} \frac{1}{\cosh \sqrt{it+m_a}} \right].$$  \hspace{1cm} (II.3)$$

Note that this expression is similar to Borel resummation \(I.2\) along \(\varphi = -\pi/2\). Indeed it has been proved in \(55\) that the function \(BZ(t)\) is rigorously the same as the Borel transformation of the perturbative series of \(Z\). This shows that the exact result is equivalent to the Borel resummation along \(-i\mathbb{R}_+\). The Borel transformation has simple poles at

$$t^*_n = -i [m_a \pm (2n_a - 1)\pi i]^2,$$  \hspace{1cm} (II.4)$$

with \(n_a \in \mathbb{N}\) for each of flavors. We can easily see that \(\arg(t)\) of the poles depends on the values of the real masses as depicted in Fig. \(1\) for \(N_f = 1\). In particular, with \(m_a = (2n_a - 1)\pi\), we have Borel singularities on the real axis

$$t^*_n \big|_{m_a=(2n_a-1)\pi} = \pm 2(2n_a - 1)^2 \pi^2,$$  \hspace{1cm} (II.5)$$

which leads to non-Borel-summability of the perturbative series along \(\mathbb{R}_+\). This means that \(m_a = (2n_a - 1)\pi\) is the Stokes line, where the Stokes phenomena occur. We depict the Borel singularities for \(N_f = 1\) with \(m = m_a = 0\) and \(n = n_a\) in Fig. \(1\). As we turn on the real mass, the degenerate singularities (double poles) on the positive imaginary axis for \(m_a = 0\) get lifted and move to positive and negative real directions. When the real mass goes beyond \(m = (2n-1)\pi\), a singularity crosses the positive real axis and come into the fourth quadrant from the first quadrant.

Since the exact result is given by the integral along \(-i\mathbb{R}_+\) in the Borel resummation, by use of Cauchy integration theorem, the exact result turns out to be composed of the Borel resummation along \(\mathbb{R}_+\) (perturbative part) and the residue of all the singularities in the fourth quadrant of the Borel plane (non-perturbative part):

$$Z = \int_0^\infty dt \ e^{-\frac{t}{2}} BZ(t) + \sum_{\text{poles \in \text{4th quadrant}}} \text{Res}_{t=t_{\text{pole}}} \left[ e^{-\frac{t}{2}} BZ(t) \right],$$  \hspace{1cm} (II.6)$$

where \(\text{Res}_{z=z_0} [f(z)]\) denotes residue of \(f(z)\) at \(z = z_0\). The number of the singularities in this region is \(|n_a|\) for the real mass \((2n_a - 1)\pi < m_a < (2n_a + 1)\pi\) \((n_a \in \mathbb{N}^0)\) for each of the flavors. This is also a correct statement even for negative \(m_a\) with \(n_a \in \mathbb{Z}\). When the real mass \(m_a\) crosses \((2n_a + 1)\pi\), we start to receive a contribution from another Borel singularity which leads to ambiguity of the perturbative Borel resummation at \(m_a = (2n_a + 1)\pi\) as we have discussed above. This is how the Stokes phenomena emerge in the present example. For the degenerate mass \(m = m_a\) for all flavors, the singularities are also degenerate, where the order of their poles is \(N_f\).
FIG. 1: Borel singularities (red crosses) are depicted for \( N_f = 1 \) \( \mathcal{N} = 2 \) CS SQED with the real masses \( m = 0, \pi/2, \pi, 3\pi, 4\pi \). Among these choices, \( m = \pi, 3\pi \) are Stokes lines.

To show these results explicitly, first let us focus on \( N_f = 1 \). For \( (2n-1)\pi < m < (2n+1)\pi \), the second term in (II.6) is given by

\[
\sum_{\ell=1}^{n} (-1)^{\ell-1} 2\pi e^{\frac{4}{\beta}[m+(2\ell-1)\pi]^{2}} \text{ for } (2n-1)\pi < m < (2n+1)\pi.
\] (II.7)

Note that it vanishes for \( n = 0 \), where we just have the perturbative part. By use of the step function \( \theta(x) \), the partition function \( Z \) is also written as

\[
Z = Z_{pt} + \sum_{n=1}^{\infty} Z_{np}^{(n)},
\] (II.8)

\[
Z_{pt} = \int_{0}^{\infty} dt \, e^{-\frac{t}{\beta}} BZ(t), \quad Z_{np}^{(n)} = \theta (m - (2n-1)\pi) 2\pi (-1)^{n-1} e^{\frac{4}{\beta}[m+(2n-1)\pi]^{2}}.
\] (II.9)

Note that this decomposition is well defined for almost all values of \( m \) in the sense that it is
apparently ambiguous on the Stokes lines as we will discuss later. Here \( Z_{pt} \) is the perturbative contribution while \( Z_{np}^{(n)} \) is the nonperturbative contribution. By expanding the perturbative part \( Z_{pt} \) with respect to \( t \) and looking into coefficients, we obtain the asymptotic form of the perturbative contribution as

\[
Z_{pt} = \frac{\sqrt{i g}}{2} \sum_{q=0}^{\infty} \frac{\Gamma(q+1/2)}{\Gamma(q+1)} \partial_q \left( \frac{1}{\cosh \frac{\sqrt{it} - m}{2}} + \frac{1}{\cosh \frac{\sqrt{it} + m}{2}} \right) \bigg|_{t=0}^q g^q
\]

where \( E_n \) is the Euler number \[104\]. It is notable that this asymptotic series is Borel-summable along \( \mathbb{R}_+ \) for \( m \neq (2n - 1)\pi \) while it is not for \( m = (2n - 1)\pi \). However, even for \( m \neq (2n - 1)\pi \), the Borel resummation of the perturbative series along \( \mathbb{R}_+ \) does not give an exact result for \( m > \pi \).

These are consistent with the argument on the Stokes phenomena mentioned above. Now we are ready to write down the full transseries expansion of \( Z \):

\[
Z = C_0 \sum_{q=0}^{\infty} c_q^{(0)} g^{q-1/2} + \sum_{n=1}^{\infty} C_n e^{-\frac{S_n}{g}} \sum_{q=0}^{\infty} c_q^{(n)} g^q.
\]

(II.11)

Comparing this with the above data, we identify the above parameters with

\[
C_0 = 1, \quad c_q^{(0)} = \frac{i^{q+1} \Gamma(q+1/2)}{2 \Gamma(2q+1)} \sum_{a=0}^{\infty} \frac{E_{2(q+a)} \Gamma(2q+1)}{2^2(q+a) \Gamma(2q+1)} m^{2a},
\]

\[
C_n = \theta \left( m - (2n - 1)\pi \right), \quad S_n = -i [m + (2n - 1)\pi] \left( \frac{1}{g} \right)^2, \quad c_q^{(n)} = 2\pi (-1)^{n-1} \delta q 0. \quad (II.12)
\]

For \( m = (2n-1)\pi \), we need to take a handle with care. This is because \( Z_{pt} \) and \( Z_{np}^{(n)} \) are ambiguous due to the non-Borel summability along \( \mathbb{R}_+ \) and the step function behavior of the transseries parameter \( C_n \), respectively, while the other non-perturbative corrections are unambiguous at this point. Their ambiguities are indeed canceled as follows. In the context of resurgence theory, the Borel ambiguity is usually estimated by the difference of the lateral Borel resummations as

\[
(S_{0+} - S_{0-}) Z(g, m).
\]

(II.13)

Instead let us estimate the ambiguities of perturbative and nonperturbative contributions by

\[
Z_{pt}(g, m = (2n - 1)\pi + 0_+) - Z_{pt}(g, m = (2n - 1)\pi + 0_-),
\]

and

\[
Z_{np}^{(n)}(g, m = (2n - 1)\pi + 0_+) - Z_{np}^{(n)}(g, m = (2n - 1)\pi + 0_-).
\]
Noting
\[ Z_{pt}(m = (2n - 1)\pi + 0_\pm) = P \int_0^\infty dt \ e^{-\frac{t}{\pi}BZ(t)} \mp \frac{1}{2} \text{Res}_{t=t_n^*} [e^{-\frac{t}{\pi}BZ(t)}], \] (II.14)
the Borel ambiguity in the perturbative sector is
\[ Z_{pt}(m = (2n - 1)\pi + 0_+) - Z_{pt}(m = (2n - 1)\pi + 0_-) = -\text{Res}_{t=t_n^*} [e^{-\frac{t}{\pi}BZ(t)}], \] (II.15)
while the non-perturbative ones are
\[ Z_{np}(m = (2n - 1)\pi + 0_+) - Z_{np}(m = (2n - 1)\pi + 0_-) = \begin{cases} 0 & \text{for } \ell \neq n \\ \text{Res}_{t=t_n^*} [e^{-\frac{t}{\pi}BZ(t)}] & \text{for } \ell = n \end{cases}. \] (II.16)
\[ \text{Res}_{t=t_n^*} [...] \text{ stands for the residue at the singularity on the positive real axis denoted as } t_n^*. \] Thus the ambiguities are canceled and the whole transseries (II.11) gives the unambiguous result which is equivalent to the exact result. We note that the importance of the Borel singularities at the first and fourth quadrants on the perturbative Borel plane has been stressed in [31] in the context of $1/N$-expansion of 3d $\mathcal{N} = 6$ superconformal field theory as their residues give exponentially suppressed corrections. There, it is argued that these singularities correspond to nonperturbative contributions, thus they should be taken into account even if the perturbative series is Borel-summable. The present case is one of the examples consistent with this argument.

We also show the results for a generic number of flavors $N_f \geq 1$ with degenerate mass $m_a = m$:
\[ Z_{pt} = \frac{\sqrt{\pi}i}{2} \sum_{\{q_a\}=0}^\infty \sum_{\{l_a\}=0}^\infty \frac{\Gamma(q + 1/2)}{2^{2(q+l)}} \left[ \prod_{a=1}^{N_f} \frac{E_{2(q_a+l_a)}}{\Gamma(2q_a+1)\Gamma(2l_a+1)} m_a^{2l_a} \right] (ig)^q, \] (II.17)
\[ Z_{np} = \frac{\pi i}{2^{N_f-1} \Gamma(N_f)} \sum_{\ell=1}^n \lim_{z \to z_\ell^*} \frac{\partial^{N_f-1}}{\partial z^{N_f-1}} \left[ \frac{(z - z_\ell^*)^{N_f} \left( \frac{z^2}{2m} \right)^{N_f} e^{-\frac{z}{\sqrt{2}}} \right], \] for $(2n - 1)\pi < m < (2n + 1)\pi$.
(II.18)
where $\bar{q} = \sum_{a=1}^{N_f} q_a$, $\bar{l} = \sum_{a=1}^{N_f} l_a$ and $z_\ell^* = m + (2\ell - 1)\pi i$. We note that $Z_{np} = 0$ for $n = 0$.

We again emphasize that the order of poles of Borel singularities is $N_f$ when the masses of the flavors are degenerate. For these cases with the degenerate mass, we still have the exact result as the full transseries, where the Stokes phenomena occur at the special values of the real mass $m = (2n - 1)\pi$. This is regarded as the resurgent structure beyond the argument with the standard “resurgent function” with simple poles or branch cuts [3].

We end this subsection by a comment on uniqueness of the decomposition of the exact partition function into perturbative and nonperturbative parts in Eq. (II.9). We have defined perturbative
part as the Borel resummation that is obtained by integration of the Borel transform along \( \mathbb{R}_+ \), and have decomposed the exact result into the perturbative and nonperturbative parts. Provided we have a perturbative series for a certain quantity, its Borel resummation just gives one of analytic functions, whose asymptotic expansion becomes the perturbative series. Thus the Borel resummation is not a unique definition of the perturbative contribution. In addition, provided we have an exact result of the quantity, its decomposition into perturbative and nonperturbative parts is not unique. This point will be discussed again when we study the thimble decomposition of the exact result in the next subsection.

### B. Thimble decomposition

Here we decompose the Coulomb branch localization formula (II.1) into Lefschetz thimbles (steepest descents) and compare the result with the transseries expression in the previous subsection. A brief review on the Lefschetz thimble decomposition is given in Appendix. Here we concentrate on the \( N_f = 1 \) case for simplicity. Generalization to multi-flavors is straightforward.

First we rewrite (II.1) as

\[
Z = \int_{-\infty}^{\infty} d\sigma \ e^{-S[\sigma]},
\]

(II.19)

where

\[
S[\sigma] = -\frac{i}{g} \sigma^2 - \log \frac{1}{2 \cosh \frac{z - m}{2}}.
\]

(II.20)

We regard \( S[\sigma] \) as “action” of the Coulomb branch parameter \( \sigma \) and extend \( \sigma \in \mathbb{R} \) to a complex value \( z \in \mathbb{C} \) since saddle points and the associated Lefschetz thimbles are complex-valued in general. The saddle points \( z_c \) are obtained from the saddle-point equation,

\[
\frac{\partial S[z]}{\partial z} \bigg|_{z=z_c} = -\frac{2i}{g} z_c + \frac{1}{2} \tanh \frac{z_c - m}{2} = 0.
\]

(II.21)

Let us label the saddle points by \( z_f \). Note that although we have the infinitely many saddle points \( \{z_f\} \), each saddle point may or may not contribute to the integral (II.19). This is determined by looking at saddle points passed by the steepest descent contours obtained by deforming the original contour without changing the value of the integral. In general, this depends on the original integral contour, the parameters \( (g, m) \) and properties of the (dual) Lefschetz thimbles as explained below.

The Lefschetz thimble or the steepest descent contour \( J_f \) associated with the saddle point \( z_f \) is
obtained by solving the differential equation called the flow equation,
\[
\frac{dz}{ds}_{J_I} = \frac{\partial S[z]}{\partial z} = \frac{2i}{g} \bar{z} + \frac{1}{2} \tanh \frac{\bar{z} - m}{2},
\]  
(I.22)

with the initial condition
\[
\lim_{s \to -\infty} z(s) = z^c_I,
\]
(I.23)

with \(s\) being the flow parameter. Using the flow equation, we can easily prove
\[
\frac{d \text{Re} S[z(s)]}{ds}_{J_I} \geq 0 \quad \text{and} \quad \frac{d \text{Im} S[z(s)]}{ds}_{J_I} = 0,
\]
(I.24)

which indicate that integrals along Lefschetz thimbles are rapidly convergent and non-oscillating.

We can express the original contour \(C_R\) as the linear combination of the thimbles
\[
C_R = \sum_{I \in \text{saddles}} n_I J_I.
\]
(I.25)

When \(n_I\) is nonzero, the saddle point \(z^c_I\) and its associated thimble contribute to the integral while we have no contributions from saddle points with \(n_I = 0\). It is known that the expansion coefficient \(n_I\) is an integer because \(n_I\) is the same as an intersection number between the original contour \(C_R\) and the dual thimble (steepest ascent contour) \(K_I\) associated with \(z^c_I\) defined by
\[
\frac{dz}{ds}_{K_I} = \frac{\partial S[z]}{\partial z} \quad \text{with} \quad \lim_{s \to +\infty} z(s) = z^c_I.
\]
(I.26)

In general \(n_I\) depends on \((g, m)\) but its dependence is not continuous since \(n_I\) is integer. Typically \(n_I\) is a constant or a step function and the latter case leads us to a Stokes phenomenon.

Let us analyze the structures of the Lefschetz thimbles in the present example. The saddle point equation (I.21) implies that the critical points \(z^c\) are complicated functions of \((g, m)\) and it is hard to compute them and their thimbles analytically. Therefore we exhibit the critical points and solve the associated flow equations numerically for finite \(g\). Before showing the numerical results, we discuss the weak coupling limit analytically to get an intuitive understanding on the thimble structures.

1. Analytical results for small \(g\)

In the weak-coupling limit \(g \to 0\) we can ignore the second term in (I.21) and the critical points are determined by
\[
z^c(g, m) \cosh \frac{z^c(g, m) - m}{2} = 0 \quad \text{for} \quad g \to 0,
\]
(I.27)
FIG. 2: Locations of critical points and their actions as functions of $g$ for $m = \pi/2$, which are computed numerically. [Left Top] $\text{Re}z_{pt}^c$ (black solid) and $\text{Im}z_{pt}^c$ (red solid). [Right Top] $\text{Re}z_2^c$ (black solid) and $\text{Im}z_2^c$ (red solid) compared with $\text{Re}z_2^* = m$ (black dotted) and $\text{Im}z_2^* = 3\pi$ (red dotted). [Left Bottom] $\text{Re}S[z_{pt}^c]$ (black solid) and $\text{Im}S[z_{pt}^c]$ (red solid) compared with $\text{Re}S[0] = \log 2 \cosh \frac{m}{g}$ (black dotted) and $\text{Im}S[0] = 0$ (red dotted). [Right Bottom] $\text{Re}S[z_2^c]$ (black solid) and $\text{Im}S[z_2^c]$ (red solid) compared with $\text{Re}\left[\frac{1}{g}(m + 3\pi i)^2\right]$ (black dotted) and $\text{Im}\left[\frac{1}{g}(m + 3\pi i)^2\right]$ (red dotted).

in which we obtain $z^c(0, m) = 0, m + (2\ell - 1)i\pi$ with $\ell \in \mathbb{Z}$. Therefore we have an infinite number of critical points approaching these values in $g \to 0$. Let us denote as $z_{pt}^c$ and $z_\ell^c$ the critical points satisfying

$$\lim_{g \to 0} z_{pt}^c(g, m) = 0, \quad \lim_{g \to 0} z_\ell^c(g, m) = m + (2\ell - 1)i\pi. \quad (\text{II.28})$$

As $g$ increases, the critical points go away from (II.28) as shown in the top panels of Fig. 2. The critical points $z_{pt}^c$ and $z_\ell^c$ approximately correspond to the saddle points for the perturbative and nonperturbative contributions respectively since, for $g \to 0$, the action at $z_{pt}^c$ behaves as $\mathcal{O}(1)$ while the one at $z_\ell^c$ behaves as $-\frac{1}{g}(m + (2\ell - 1)i\pi)^2 + \mathcal{O}(1)$ as illustrated in the bottom panels of Fig. 2. This behavior precisely matches with the exponent of the nonperturbative corrections appearing in our resurgent transseries (II.9). Moreover $\text{Im}S$ of $z_{pt}^c(0, m)$ and $z_\ell^c(0, m)$ coincide at the special values: $m = \pm(2\ell - 1)\pi$. This is expected from the fact that the Stokes phenomena of the transseries (II.9) occurs at $m = (2\ell - 1)\pi$. 
We can easily compute the thimble flowing from \( z_{pt}^c \) in the \( g \to 0 \) limit. The flow equation for \( g \to 0 \) is given by
\[
\frac{dz}{ds} = \frac{2i}{g} \bar{z} \\
\text{as} \quad g \to 0,
\]
which is solved by
\[
\lim_{g \to 0} z_{pt}(g, m; s) = \epsilon \exp \left( \frac{2}{g} s + \frac{\pi i}{4} \right),
\]
with a parameter \( \epsilon \in \mathbb{R} \) for the initial condition. Note that this thimble corresponds to the integration \( Z_{pt} \) in (II.9), or the perturbative contribution. For the non-perturbative one \( z_{\ell}^c \), it is hard to analytically solve the flow equation globally even in the \( g \to 0 \) limit.

Note that \( z_{\ell}^c \) for \( g \to 0 \) given by (II.28) is precisely the same as the location of the poles of the integrand \( e^{-S[x]} \), which are zeroes of \( \cosh \frac{z^c}{2} \) in the denominator and given by
\[
z_{\ell}^c(m) = m + (2 \ell - 1) \pi i \quad \text{with} \quad \ell \in \mathbb{Z}.
\]
This always happens when we study the following type of integral:
\[
\int dx \frac{dx}{f(x)} e^{-\frac{1}{g} h(x)},
\]
where \( f(x) \) is a function without poles but may have zeroes. The critical points for this integrand are determined by
\[
\frac{\partial h}{\partial x} + g \frac{1}{f} \frac{\partial f}{\partial x} = 0.
\]
By examining the limiting behavior of critical points as \( g \to 0 \), we find that at least one of critical points inevitably goes to each zero of \( f(x) \) in the limit. In summary, the asymptotic values of the critical points in the \( g \to 0 \) limit satisfy
\[
\frac{\partial h(x)}{\partial x} \cdot f(x) = 0.
\]
This fact has important implications for structures of (dual) thimbles. Since the actions at the poles are \(-\infty\), dual thimbles can end on the poles while thimbles cannot pass through the poles. In other words, the poles play a role of source of the dual thimble. Therefore, noting that the critical points for finite \( g \) are located near the poles, the dual thimble associated with one of the critical points goes from the pole to another region with \( \text{Re}S \to -\infty \) via the critical point. On the other hand, the thimble associated with the same critical point connects two regions with \( \text{Re}S \to +\infty \) via the critical point but circumvents the poles. As we will see below, the thimble integrals associated with the critical points near the poles are equivalent to their residues.
Next we take into account a small $g$ correction by taking the ansatz $z^c(g, m) = z^{c,0}(m) + g z^{c,1}(m) + \mathcal{O}(g^2)$ to see explicitly what would be going on for nonzero $g$. Matching $\mathcal{O}(1)$ terms in (II.21) gives

$$z^c_{\text{pt}}(g, m) = \frac{ig}{4} \tanh \frac{m}{2} + \mathcal{O}(g^2), \quad z^c_{\ell}(g, m) = z^*_\ell + \frac{g}{2i z^*_\ell} + \mathcal{O}(g^2),$$  

which have the actions

$$S[z^c_{\text{pt}}(g, m)] = \log \left(2 \cosh \frac{m}{2}\right) + \mathcal{O}(g),$$

$$S[z^c_{\ell}(g, m)] = -\frac{i}{g} z^{*2}_\ell + \log g + \left(-1 + \log \frac{(-1)^{\ell-1}}{2z^*_\ell} \right) + \mathcal{O}(g).$$  

From these actions, a necessary condition for having Stokes phenomenon is

$$0 = \Im S[z^c_{\ell}(g, m)] - \Im S[z^c_{\text{pt}}(g, m)]$$

$$= -\frac{1}{g} \Re \left[ z^{*2}_\ell \right] + \arg \frac{(-1)^{\ell-1}}{z^*_\ell} + \mathcal{O}(g).$$

Note that this condition is not satisfied by $m = (2\ell - 1)\pi$, which was a solution in the $g \to 0$ limit. This implies that the Stokes phenomenon in the thimble decomposition for nonzero $g$ occurs at a different point $m = \tilde{m}_\ell(g)$ from those of the transseries and they coincide in the weak coupling limit:

$$\lim_{g \to 0} \tilde{m}_\ell(g) = (2\ell - 1)\pi.$$  

Consequently for finite $g$ we need to distinguish Stokes phenomena in the sense of thimble decomposition and in the sense of transseries. This happens in general when coupling is not multiplicative to the whole action or when we include operators (with no or different coupling dependence) as a part of the effective action. We will readily see this effect by a numerical analysis for finite $g$ performed below and discuss relation to the resurgent transseries.

2. Numerical results for finite $g$ and comparison with resurgent transseries

Now let us turn to the finite $g$ case. As we already illustrated in Fig. 2, $z^c_{\text{pt}}$ and $z^c_{\ell}$ are distinct from $z = 0$ and $z = m + (2\ell - 1)\pi i$ respectively and their actions receive finite $g$ corrections. We have numerically solved the flow equation and obtained the thimbles and dual thimbles for the saddle points for finite $g$, where we figure out the structure of thimble decomposition for several choices of the real mass $m$ as follows. Figs. 3 and 5 summarize the thimble structure for $g = 4\pi/100 \approx 0.126$ ($k = 100$) and $g = 4\pi \approx 12.56$ ($k = 1$) with $m = 2\pi, 3\pi, 4\pi$ in complexified $\sigma$ plane ($z$...
FIG. 3: Thimble structure of partition function of $\mathcal{N} = 2$ CS SQED with $N_f = 1$ hyper multiplet for $k = 100$ ($g = 4\pi/k \approx 0.126$) on Re-$z$-Im-$z$ plane. The green points and red crosses stand for critical points and singularities, respectively. The green points (saddle points) are hidden by the red crosses (singularities) except at the origin. The red dotted lines stand for dual thimbles with nonzero intersection numbers, whereas the blue dashed lines for corresponding thimbles. The arrows represent flow lines for increasing flow parameter $s$.

FIG. 4: Zoom-up of Fig. 3(a) around the singularity $z = z_1^* = m + \pi i$. The thimble integral associated with $z_1^*$ is equivalent to the residue around $z = z_1^*$.

For smaller $g$, we see in Fig. 3 that the nonperturbative saddle points (green points) and the singularities (red crosses) are almost degenerate, while they are slightly more separated in Fig. 5 for larger $g$. We term a saddle point near the origin as a “perturbative” one and others as “nonperturbative” ones. As we will see later, this naming gets precisely appropriate only for the $g \to 0$ limit.

We first discuss the case with small $g$ in Fig. 3, which can be seen as an approximate example of the $g \to 0$ limit. The main results in Fig. 3 are summarized as follows: For $m = 2\pi$, two of the saddle points contribute to the partition function: a thimble associated with the perturbative
saddle $\tilde{s}^c_{pt}$ near the origin and another one associated with $z_1^c$ near $z = m + \pi i$. By Cauchy's theorem, the integral along $J_{pt}$ is equivalent to the one along $e^{\pi i} \mathbb{R}$, namely $Z_{pt}$, while the integral along $J_0$ (the first nonperturbative thimble) corresponds to the residue at $z = z_1^*$ (see Fig. 4):

$$Z(g,m) = \int_{J_{pt} + J_1} dz e^{-S[z]}, \quad (II.39)$$

$$\int_{J_{pt}} dz e^{-S[z]} = Z_{pt}(g,m), \quad \int_{J_1} dz e^{-S[z]} = \text{Res}_{z=z_1^*} \left[ e^{-S[z]} \right] \quad \text{at} \quad m = 2\pi. \quad (II.40)$$

For $m = 3\pi$, there are two important changes. First, the dual thimble associated with $z_2^c$ (another nonperturbative saddle) intersects the real axis. Second, the thimble associated with $z_2^c$ seems to pass $z_2^c$. More precisely, this does not pass $z_2^c$ in a rigorous sense but almost passes $z_2^c$. These facts imply that Stokes phenomena in the sense of the thimble decomposition occur at a certain point $m = \tilde{m}(g)$ which is slightly below $m = 3\pi$ as expected from the subleading small-$g$ correction.

$$Z(g,m) = \int_{J_{pt} + J_1 + J_2} dz e^{-S[z]}, \quad (II.41)$$

$$\int_{J_{pt}} dz e^{-S[z]} = Z_{pt}(g,m + 0_+), \quad \int_{J_{1,2}} dz e^{-S[z]} = \text{Res}_{z=z_{1,2}^*} \left[ e^{-S[z]} \right] \quad \text{at} \quad m = 3\pi. \quad (II.42)$$

For $m = 4\pi$, three of saddle points contribute to the partition function: a thimble associated with the perturbative saddle (near the origin) and two thimbles associated with the nonperturbative saddles.

$$Z(g,m) = \int_{J_{pt} + J_1 + J_2} dz e^{-S[z]}, \quad (II.43)$$

$$\int_{J_{pt}} dz e^{-S[z]} = Z_{pt}(g,m), \quad \int_{J_{1,2}} dz e^{-S[z]} = \text{Res}_{z=z_{1,2}^*} \left[ e^{-S[z]} \right] \quad \text{at} \quad m = 4\pi. \quad (II.44)$$

For larger $g$, the differences from the weak coupling case are more explicit as illustrated in Fig. 5. First the critical points $z_{pt}^c$ and $z_c^c$ are clearly separated from the origin and singularities respectively. The thimble structures at $m = 2\pi$, $3\pi$ and $4\pi$ are the same as the ones of $g \approx 0.126$ but the value of $\tilde{m}_2(g)$ clearly deviates from $m = 3\pi$. We here give a short explanation on the thimble structure before the detailed discussion: For $\pi < m < \tilde{m}_2$ the perturbative contribution $Z_{pt}$ is only composed of the thimble associated with the perturbative saddle point near the origin, while $Z_{pt}$ gets composed of the perturbative thimble and one more thimble associated with the nonperturbative saddle for $\tilde{m}_2 < m < 3\pi$. This nonperturbative thimble comes to contribute as the "genuine" nonperturbative contribution at the Stokes line $m = 3\pi$. In Fig. 6 we plot $\tilde{m}_2(g)$ as a function of $g$. We immediately see that $\tilde{m}_2(g)$ deviates from $3\pi$ for strong coupling.

We have analyzed the cases for generic values of $(g,m)$ and summarized the thimble structures related to $z_{pt}^c$ and $z_c^c$ in Fig. 7 which is the schematic expanded version of Fig. 5. In the figure, we...
FIG. 5: Thimble structure of partition function of $\mathcal{N} = 2$ CS SQED with $N_f = 1$ hyper multiplet for $k = 1$ ($g \approx 12.6$) on Re $z$-Im $z$ plane. The green points and red crosses stand for critical points and singularities, respectively. The red dotted lines stand for dual thimbles with nonzero intersection numbers, whereas the blue dashed lines for corresponding thimbles. The arrows represent the flow lines.

FIG. 6: The red dotted line denotes $\tilde{m}_2(g)$ as a function of $g$. The black dotted line denotes $3\pi$.

only show two saddle points $z_{\text{pt}}^1$, $z_{\text{pt}}^2$ and one singularity $z_{*}^2$ to discuss the Stokes phenomena just around $m = 3\pi$. We manifest their associated thimbles $\mathcal{J}_{\text{pt}}$, $\mathcal{J}_2$ and dual thimbles $\mathcal{K}_{\text{pt}}$, $\mathcal{K}_2$. We now look into the intersection of the dual thimbles with “Full contour”: $\mathbb{R}$ and “Perturbative contour”: $e^{\pi i} \mathbb{R}$. The full contour stands for the integration contour giving the exact partition function $Z$ while the perturbative contour is the one giving the perturbative part $Z_{\text{pt}}$ of the transseries, which is the Borel resummation along $\mathbb{R}_+$. The results of Fig. 7 is summarized as follows:

- For $m < \tilde{m}_2$, the dual thimble $\mathcal{K}_{\text{pt}}$ intersects with both the full and perturbative contours. It indicates that the perturbative thimble $\mathcal{J}_{\text{pt}}$ yields the perturbative contribution $Z_{\text{pt}}$ in the
FIG. 7: Schematic expanded figures for Fig. 5 around $m \approx 3\pi$. We only show two saddle points $z_{pt}^c$ and $z_2^c$ (green points) to discuss the Stokes phenomena around $m = 3\pi$. We also exhibit the associated thimbles $J_{pt}$, $J_2$ (blue dashed lines) and dual thimbles $K_{pt}$, $K_2$ (red dotted lines). One can figure out the Stokes lines by looking into the intersection of the dual thimbles with “Full contour” (black bold line) and “Pert. contour” (black solid line).

full transseries of the partition function:

$$Z(g, m) = \int_{J_{pt} + J_i} dze^{-S[z]}, \quad \int_{J_{pt}} dze^{-S[z]} = Z_{pt}(g, m), \quad \text{for } \pi < m < \tilde{m}_2. \quad (II.45)$$

- For $m = \tilde{m}_2$, the two saddle points $z_{pt}^c$ and $z_2^c$ are connected by the thimble $J_{pt}$ and the dual thimble $K_2$. This indicates that our thimble decomposition has the Stokes phenomenon and is apparently ambiguous at $m = \tilde{m}_2$.

- For $\tilde{m}_2 < m < 3\pi$, the dual thimble $K_2$ intersects with both the full and perturbative contours, which means that the nonperturbative thimble $J_2$ contributes, but just as part of
the perturbative contribution \( Z_{pt} \):

\[
\int_{\mathcal{J}_{pt,1}} dz e^{-S[z]} = Z_{pt}(g, m), \quad \text{for } \tilde{m}_2 < m < 3\pi.
\] (II.46)

Therefore we can express the exact result in this regime as

\[
Z(g, m) = \int_{\mathcal{J}_{pt,1}+\mathcal{J}_2} dz e^{-S[z]} = Z_{pt}(g, m) + \text{Res}_{z=z_1^*} \left[ e^{-S[z]} \right], \quad \text{for } \tilde{m}_2 < m < 3\pi,
\] (II.47)

which agrees the transseries representation.

- At \( m = 3\pi \), the integral along the perturbative contour is ill-defined due to the pole \( z_2^* \) but the integral along \( \mathcal{J}_2 \) is still related to \( Z_{pt} \) as the ambiguous part:

\[
\int_{\mathcal{J}_{pt}} dz e^{-S[z]} = Z_{pt}(g, m + 0_+) = \mathcal{P} \int_0^\infty dt e^{-\frac{t}{2}} BZ(t) - \frac{1}{2} \text{Res}_{z=z_2^*} \left[ e^{-S[z]} \right], \quad \text{at } m = 3\pi.
\] (II.48)

The Lefschetz thimble decomposition of the exact result is well-defined at this point and the exact result is expressed as

\[
Z(g, m) = \int_{\mathcal{J}_{pt,1}+\mathcal{J}_2} dz e^{-S[z]} = Z_{pt}(g, m + 0_+) + \text{Res}_{z=z_1} \left[ e^{-S[z]} \right] + \text{Res}_{z=z_2} \left[ e^{-S[z]} \right]
\] at \( m = 3\pi \),

(II.49)

which is equivalent to the transseries expression at \( m = 3\pi + 0_+ \). Note that using

\[
Z_{pt}(g, m + 0_-) = \mathcal{P} \int_0^\infty dt e^{-\frac{t}{2}} BZ(t) + \frac{1}{2} \text{Res}_{z=z_2^*} \left[ e^{-S[z]} \right],
\]

we can also write the exact result as

\[
Z(g, m) = Z_{pt}(g, m + 0_-) + \text{Res}_{z=z_1^*} \left[ e^{-S[z]} \right] \quad \text{at } m = 3\pi,
\] (II.50)

which is the transseries representation at \( m = 3\pi + 0_- \). This is what we expect from the resurgence analysis. Namely we have manifested that the transseries has the Stokes phenomena at \( m = 3\pi \) and the well-defined thimble decomposition of the exact result at \( m = 3\pi \) coincides with the unambiguous answer obtained by the resurgence:

\[
Z(g, m) = \mathcal{P} \int_0^\infty dt e^{-\frac{t}{2}} BZ(t) + \text{Res}_{z=z_1^*} \left[ e^{-S[z]} \right] + \frac{1}{2} \text{Res}_{z=z_2^*} \left[ e^{-S[z]} \right] \quad \text{at } m = 3\pi.
\] (II.51)

- For \( m > 3\pi \), let us take \( m \) to be smaller than the next Stokes line to keep that another Stokes phenomena with \( z_{\ell \geq 3}^* \) does not matter, namely \( 3\pi < m < \tilde{m}_3(g) \). In this regime the
dual thimble $K_2$ does not intersect with the perturbative contour while it still intersects with the full contour. It indicates that the nonperturbative thimble $J_2$ comes to contribute as the nonperturbative contribution, not as part of the perturbative contribution:

$$\int_{J_{pt}} dz e^{-S[z]} = Z_{pt}(g, m), \quad \text{for } 3\pi < m < \tilde{m}_2(g), \quad \text{(II.52)}$$

which leads us to

$$Z(g, m) = \int_{J_{pt} + J_1 + J_2} dz e^{-S[z]} = Z_{pt}(g, m) + \text{Res}_{z=z_1}^1 [e^{-S[z]}] + \text{Res}_{z=z_2}^2 [e^{-S[z]}]$$

for $3\pi < m < \tilde{m}_3(g).$ \quad \text{(II.53)}

If we further increase $m$, then we encounter Stokes phenomenon with other critical points in similar ways. We conclude that the Lefschetz thimble decomposition for any $m$ is

$$Z(g, m) = \int_{J_{pt}} dz e^{-S[z]} + \sum_{\ell=1}^{\infty} \theta(m - \tilde{m}_\ell(g)) \int_{J_\ell} dz e^{-S[z]}, \quad \text{(II.54)}$$

This shows that we have the decomposition

$$C_\mathbb{R} = n_{pt} J_{pt} + \sum_{\ell} n_\ell J_\ell, \quad \text{(II.55)}$$

with the intersection numbers

$$n_{pt} = 1, \quad n_\ell = \theta(m - \tilde{m}_\ell(g)). \quad \text{(II.56)}$$

The thimble integral along $J_\ell$ is equivalent to the residue of the Borel singularities, which is the nonperturbative exponential part other than the step function in $Z_{np}^n$ of the transseries \quad \text{(II.9)}. $Z_{pt}(g, m)$ is related to the thimble integrals in a complicated way due to the intersection number between $e^{\frac{\pi}{4}} \mathbb{R}$ and the dual thimbles. In terms of the Boxcar function $\Pi_{a,b}(x)$

$$\Pi_{a,b}(x) = \theta(x-a) - \theta(x-b) = \begin{cases} 
0 & \text{for } x < a \\
1 & \text{for } a < x < b \\
0 & \text{for } x > b 
\end{cases}, \quad \text{(II.57)}$$

$Z_{pt}(g, m)$ is decomposed as

$$Z_{pt}(g, m) = \int_{J_{pt}} dz e^{-S[z]} + \sum_{\ell=1}^{\infty} \Pi_{\tilde{m}_\ell(g), (2\ell-1)\pi}(m) \int_{J_\ell} dz e^{-S[z]}, \quad \text{(II.58)}$$
This decomposition is ambiguous at \( m = \tilde{m}_\ell(g) \) and \( (2\ell - 1)\pi \). At \( m = (2\ell - 1)\pi \), the integral along \( \mathcal{J}_{pt} \) is related to \( Z_{pt} \) by

\[
\int_{\mathcal{J}_{pt}} dz e^{-S[z]} = Z_{pt}(g, m + 0_+) = P \int_0^\infty dt \ e^{-\frac{1}{\theta} BZ(t)} - \frac{1}{2} \text{Res}_{z=z^*_\ell} [e^{-S[z]}], \quad \text{at } m = (2\ell - 1)\pi.
\]

(II.59)

Thus, at \( m = (2\ell - 1)\pi \), we can rewrite the exact result as

\[
Z(g, m) = Z_{pt}(g, m + 0_+) + \sum_{\ell'=1}^\ell \text{Res}_{z=z^*_{\ell'}} [e^{-S[z]}] = Z_{pt}(g, m + 0_-) + \sum_{\ell'=1}^{\ell-1} \text{Res}_{z=z^*_{\ell'}} [e^{-S[z]}] = P \int_0^\infty dt \ e^{-\frac{1}{\theta} BZ(t)} + \frac{1}{2} \text{Res}_{z=z^*_\ell} [e^{-S[z]}] \quad \text{at } m = (2\ell - 1)\pi,
\]

(II.60)

which is the same as the unambiguous answer obtained in the resurgent transseries. We can easily derive the resurgent transseries from the thimble decomposition by considering small-\( g \) expansion of the expression (II.54). Noting \( \tilde{m}_\ell(g) = (2\ell - 1)\pi + O(g) \), we can replace \( \theta(m - \tilde{m}_\ell(g)) \) by \( \theta(m - (2\ell - 1)\pi) \) and arrive at

\[
Z(g, m) = Z_{pt}(g, m) + \sum_{\ell=1}^\infty \theta(m - (2\ell - 1)\pi) \text{Res}_{z=z^*_\ell} [e^{-S[z]}], \quad \text{which is nothing but the resurgent transseries representation.}
\]

(II.61)

Now we comment on the definition of the perturbative contribution. As we mentioned in the end of the previous subsection, the definition of the perturbative contribution based on the Borel resummation is just one of definitions. In our work, we define the perturbative part as the Borel resummation of the perturbative series and decompose the exact result into the perturbative and nonperturbative parts. We may be able to propose another feasible definition of the perturbative contribution: the thimble integral associated with the perturbative saddle \( z^c_{pt} \) is regarded as the perturbative contribution while the nonperturbative contributions are defined as the thimble integral associated with the nonperturbative saddles \( z^c_{\ell} \). In this alternative definition, the Stokes phenomenon of thimble decomposition at \( m = m^* \) becomes a Stokes phenomenon of transseries while \( m = (2n - 1)\pi \) is no longer a Stokes line. We emphasize that the two definitions get equivalent in the \( g \to 0 \) limit.

Finally we mention a technical subtlety of \( \text{Im}S \) for different thimbles. As well-known, a necessary condition for having a Stokes phenomenon between two thimbles \( \mathcal{J} \) and \( \mathcal{J}' \) is to have the same imaginary part of action: \( \text{Im}S|_{\mathcal{J}} = \text{Im}S|_{\mathcal{J}'} \). However, we have to be careful in evaluating \( \text{Im}S \) when the action has branch cuts as noted in [68]. For our case, we have infinitely many logarithmic
branch cuts extended from the poles of the integrand, which generate ambiguities in specifying “log 1” = 2πiZ. Thus, the necessary condition for the Stokes phenomenon can be modified as

$$\text{Im} S|_{\mathcal{J}_p} = \text{Im} S|_{\mathcal{J}_2} + 2\pi n', \quad n' \in \mathbb{Z},$$

(II.62)

and one can determine $n'$ of each thimble by looking into the Stokes phenomena in details. For example, we present $\frac{\text{Im}(S[z_{\text{cpt}}'] - S[z_c])}{2\pi}$ as a function of $m/\pi$ for $g = 10$ in Fig. 8. We take the notation “log 1” = 0 in computing Im$S$. Here, the Stokes line in the thimble decomposition is given as $\tilde{m}_2(g = 10) \simeq 2.355\pi$. For this value of $m$, we have $\frac{\text{Im}(S[z_{\text{cpt}}'] - S[z_c])}{2\pi} \approx -1$, which means $\text{Im} S|_{\mathcal{J}_{\text{pt}}} = \text{Im} S|_{\mathcal{J}_2} - 2\pi$ at $m = \tilde{m}_2(g)$. This is a clear example where we need to take care of the branch cuts to consider thimble decompositions.

### C. Stokes phenomena in terms of arg($g$)

So far we have discussed the Stokes phenomena and the resurgence structure by changing the real mass parameter while we have fixed the coupling $g$ to be real positive. It would be also interesting to change arg($g$) with fixed $m$ in the integral (II.1) as in the usual analyses of the resurgence theory. Note that it is unclear whether or not (II.1) for complex $g$ can be interpreted as $S^3$ partition function of the theory with complex $g$ except arg($g$) = 0, $\pi$, since the localization procedure requires gauge invariance naively. In order to see the relation in a more precise manner, we need to perform analogue of the reference [63] for 3d $\mathcal{N} = 2$ CS matter theory but we do not discuss this in the present work. This subsection is motivated by technical comparison with the standard resurgence analyses. We take $N_f = 1$ for simplicity in this subsection.
Let us take complex \( g \) in the integral (II.1). In order to keep the integral finite, we restrict ourselves to

\[-\pi \leq \arg(g) \leq 0 \quad (0 \leq \arg(k) \leq \pi).\]

(II.63)

Repeating the argument of [55], we can easily show that the exact result for nonzero \( \arg(g) \) can be still written as

\[ Z = \int_{0}^{-\infty} dt \ e^{-\frac{t}{2}BZ(t)} , \]

(II.64)

where the Borel transformation \( BZ(t) \) is given by (II.13). The main difference from the \( \arg(g) = 0 \) case is that the “standard direction” of the Borel resummation is \( \varphi = \arg(g) \) rather than \( \varphi = 0 \), which is equivalent to \( \arg(\sigma) = \frac{\pi}{4} + \frac{\arg(g)}{2} \) in the language of \( \sigma \). Therefore considering a contour integral along the fan \( \mathcal{F} \) connecting \(-i\mathbb{R}_+\) and \( e^{i\arg(g)}\mathbb{R}_+ \) (see Fig. 9), we find

\[ Z = \int_{0}^{e^{i\arg(g)}\infty} dt \ e^{-\frac{t}{2}BZ(t)} + \sum_{\text{poles} \in \mathcal{F}} \text{Res} \left[ e^{-\frac{t}{2}BZ(t)} \right] , \]

(II.65)

which is the extension of (II.6) to general \( \arg(g) \). As in the \( \arg(g) = 0 \) case, we identify the first and second terms with perturbative and non-perturbative contributions respectively. On the Borel plane, the non-perturbative corrections are given by the residues around the Borel singularities \( t_n^* = -i[m + (2n - 1)\pi i]^2 \) satisfying \( \varphi = \arg(g) > \arg(t_n^*) \). As changing \( \arg(g) \) from 0 to \(-\pi\), the Stokes line rotates clockwise but the locations of the singularities are unchanged since they are independent of \( g \) with fixed \( m \). Then, except for the Stokes lines with respect to \( \arg(g) \), we can
write the partition function as

\[ Z = Z_{\text{pt}} + \sum_n Z_{\text{np}}^{(n)}, \]

\[ Z_{\text{pt}} = \int_0^{e^{i\arg(g)\infty}} dt \ e^{-\frac{t}{g}\mathcal{B}Z(t)}, \quad Z_{\text{np}}^{(n)} = \theta(\arg(g) - \arg(t_n^*)) 2\pi(-1)^{n-1} e^{\frac{t}{g}|m+(2n-1)\pi i|^2}, \quad (\text{II.66)} \]

where

\[ \arg(t_n^*) = -\frac{\pi}{2} + 2\arctan\left(\frac{2n-1}{m}\pi\right) = -\arctan\left(\frac{m^2-(2n-1)^2\pi^2}{2m(2n-1)\pi}\right). \quad (\text{II.67}) \]

Note that the only differences from the \( \arg(g) = 0 \) case are the change of the contour of the perturbative Borel resummation and the step function in \( Z_{\text{np}}^{(n)} \). Namely the perturbative series in every sector is unchanged and only the transseries parameter is changed. We emphasize that we are changing \( \arg(g) \) rather than \( m \). This is why the variable in the step function is not \( m \) but \( \arg(g) \). We can see from \( (\text{II.66}) \) that for \(-\pi/2 < \arg(g) \leq 0\), the total partition function has the non-perturbative part coming only from the Borel singularities \( t_n^* = -i|m+(2n-1)\pi i|^2 \) with the positive \( n \) and the fan \( \mathcal{F} \) becomes narrower for smaller \( \arg(g) \) (larger \( |\arg(g)| \)) in this regime.

In particular, for \( \arg(g) = -\pi/2 \), the fan coincides with \(-i\mathbb{R}_+\) and the exact result has only the perturbative part. For \(-\pi \leq \arg(g) < -\pi/2\), the partition function receives non-perturbative corrections from \( n \in \mathbb{Z}_{\leq 0} \).

For \( \arg(g) = \arg(t_n^*) \), \( Z_{\text{pt}} \) and \( Z_{\text{np}}^{(n)} \) are apparently ambiguous since the integral in \( Z_{\text{pt}} \) hits the singularity at \( t = t_n^* \) and the step function in \( Z_{\text{np}}^{(n)} \) is ambiguous. The ambiguities are indeed canceled as in Sec. IIIA. Let us estimate the Borel ambiguity by

\[ (\mathcal{S}_{\arg(t_n^*)+0^+} - \mathcal{S}_{\arg(t_n^*)+0^-}) Z(g,m), \quad (\text{II.68}) \]

as usual. Noting

\[ Z_{\text{pt}}(g,m)|_{\arg(g)=\arg(t_n^*)+0^\pm} = P \int_0^{e^{i\arg(g)\infty}} dt \ e^{-\frac{t}{g}\mathcal{B}Z(t)} \mp \frac{1}{2}\text{Res}_{t=t_n^*} \left[ e^{-\frac{t}{g}\mathcal{B}Z(t)} \right], \quad (\text{II.69}) \]

the Borel ambiguity in the perturbative sector is

\[ Z_{\text{pt}}(g,m)|_{\arg(g)=\arg(t_n^*)+0^+} - Z_{\text{pt}}(g,m)|_{\arg(g)=\arg(t_n^*)+0^-} = -\text{Res}_{t=t_n^*} \left[ e^{-\frac{t}{g}\mathcal{B}Z(t)} \right], \quad (\text{II.70}) \]

while the non-perturbative ones are

\[ Z_{\text{np}}^{(\ell)}|_{\arg(g)=\arg(t_n^*)+0^+} - Z_{\text{np}}^{(\ell)}|_{\arg(g)=\arg(t_n^*)+0^-} = \begin{cases} 0 & \text{for } \ell \neq n, \\ +\text{Res}_{t=t_n^*} \left[ e^{-\frac{t}{g}\mathcal{B}Z(t)} \right] & \text{for } \ell = n, \end{cases} \quad (\text{II.71}) \]

Therefore the ambiguities are canceled and the whole transseries gives the unambiguous answer, which agrees with the exact result.
2. Thimble decomposition

Let us decompose the exact result into Lefschetz thimble contributions. First, we discuss a small-$g$ regime analytically. As with the $\arg(g) = 0$ case, the critical points up to $O(g^2)$ are given by (II.35)

$$z_{\text{pt}}^{c}(g, m) = \frac{ig}{4} \tanh \frac{m}{2} + O(g^2), \quad z_{\ell}^{c}(g, m) = z_{\ell}^{*} + g \frac{g}{2i z_{\ell}^{*}} + O(g^2),$$

which approach the origin and the positions of poles of the integrand in the $g \to 0$ limit respectively.

The perturbative thimble in the $g \to 0$ limit is given by (II.72)

$$\lim_{g \to 0} z_{\text{pt}}(g, m; s) = \epsilon \exp \left[ \frac{2s}{|g|} + i \left( \frac{\pi}{4} - \frac{\arg(g)}{2} \right) \right], \quad (II.72)$$

with a parameter $\epsilon \in \mathbb{R}$ for the initial condition. The actions at the critical points are still given by (II.36), but the necessary condition for having Stokes phenomenon is slightly modified as (II.73)

$$0 = -\Re \left[ \frac{z_{\ell}^{*2}}{g} \right] + \arg \left( \frac{-1}{\ell} \frac{g}{z_{\ell}^{*}} \right) + O(g), \quad (II.73)$$

or equivalently (II.74)

$$0 = -\left| \frac{z_{\ell}^{*2}}{g} \right| \cos \left( 2\arg(z_{\ell}^{*}) - \arg(g) \right) + \arg \left( \frac{-1}{\ell} \frac{g}{z_{\ell}^{*}} \right) + O(g). \quad (II.74)$$

In the $|g| \to 0$ limit, one of the solutions of this condition is $\arg(g) = -\pi/2 + 2\arg(z_{\ell}^{*}) = \arg(t_{\ell}^{*})$. This is consistent with the Stokes phenomena of the transseries (II.66) at $\arg(g) = \arg(t_{\ell}^{*})$, which we encountered above. Note that $\arg(g) = \arg(t_{\ell}^{*})$ is no longer solution of (II.74) for nonzero $|g|$. This indicates that for nonzero $|g|$, we have the Stokes phenomena of the thimble decomposition at a different point $\arg(g) = \arg(\tilde{g}_{n})(|g|, m)$ which approaches $\arg(t_{\ell}^{*})$ in the weak coupling limit:

$$\lim_{|g| \to 0} \arg(\tilde{g}_{n})(|g|, m) = \arg(t_{\ell}^{*}), \quad (II.75)$$

which is the counter part of $\tilde{m}_{n}(g)$ in the case of $\arg(g) = 0$ with varying $m$. It is worth to note that the Stokes line in the $g$-plane is curved rather than straight for given $m$ since $\arg(\tilde{g}_{n})$ depends also on $|g|$.

In Fig. 11, we show numerical plots of the thimble structures for $|g| = \frac{4\pi}{100}$, $m = 2\pi$ with varying $\arg(g)$. Since $|g|$ is small, we expect that Stokes phenomena occur around the Stokes lines of the transseries, namely $\arg(g) = \arg(t_{\ell}^{*})$. One can check this expectation by looking at Fig. 11(b) with $\arg(g) = \arg(t_{1}^{*})$ and (d) with $\arg(g) = \arg(t_{0}^{*})$ at which the transseries has the Stokes phenomena. We easily see from these figures that the perturbative thimbles approximately pass the two critical
\[ \arg(g) = \frac{\arg(t^*_1)}{2} \simeq -0.32 \]

\[ \arg(g) = \arg(t^*_1) \simeq -0.64 \]

\[ \arg(g) = -\pi/2 \simeq -1.57 \]

\[ \arg(g) = \arg(t^*_0) \simeq -2.50 \]

\[ \arg(g) = 1.2 \cdot \arg(t^*_0) \simeq -3.00 \]

**FIG. 10:** Thimble structures for \(|g| = \frac{4\pi}{100}\), \(m = 2\pi\) with varying \(\arg(g)\). Values of \(\arg(t^*_1)\) and \(\arg(t^*_0)\) for \(m = 2\pi\) are \(-\arctan\frac{3}{4}\) and \(-\frac{\pi}{2} - 2\arctan\frac{1}{2}\) respectively.

points. Furthermore Fig. 10 (a), (c) and (e) show that the number of contributing critical points is changed when we cross \(\arg(g) \simeq \arg(\tilde{g}_1^*)\) and \(\arg(g) \simeq \arg(\tilde{g}_0^*)\). In summary, for \(\pi < m < 3\pi\) and \(|g| \ll 1\), we have the following pictures:

- For \(0 \geq \arg(g) > \arg(\tilde{g}_1) \simeq \arg(t^*_1)\), we have contributions from \(z^c_{\text{pt}}\) and \(z^c_1\).

- For \(\arg(\tilde{g}_1) > \arg(g) > \arg(\tilde{g}_0) \simeq \arg(t^*_0)\), only the perturbative critical point \(z^c_{\text{pt}}\) contributes. Especially, the perturbative Lefschetz thimble for \(\arg(g) = -\pi/2\) is almost the same as the original integral contour.

- For \(\arg(t^*_1) > \arg(g) \geq -\pi\), we have contributions from \(z^c_{\text{pt}}\) and \(z^c_0\).

As \(|g|\) increases, \(\arg(\tilde{g}_n)\) becomes typically further from \(\arg(t^*_n)\). In other regimes of \(m\), the number of \(\arg(t^*_n)\)'s satisfying \(0 \geq \arg(t^*_n) \geq -\pi\) is different which determines the number of times we encounter the Stokes phenomena.
D. “Mirror” description

The CS SQED has another description, which is connected to the original description by 3d mirror symmetry \[73\]. The $S^3$ partition function has a different integral representation but turns out to take the same value. In this subsection we briefly study thimble structures of the mirror integral. To derive the mirror description, it is convenient to use the Fourier transformation \[74\]:

$$\frac{1}{2 \cosh \frac{z}{2}} = \frac{1}{2 \pi} \int dp \frac{e^{\frac{i}{2\pi}px}}{2 \cosh \frac{p}{2}}, \quad (\text{II.76})$$

which leads us to

$$Z = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} d\tilde{\sigma} \frac{e^{\frac{i}{4\pi k} \sigma^2 + \frac{i}{2\pi}(\sigma - m)\tilde{\sigma}}}{2 \cosh \frac{\tilde{\sigma}}{2}} = \sqrt{\frac{i}{k}} \int_{-\infty}^{\infty} d\tilde{\sigma} \frac{e^{-\frac{i}{4\pi k} \tilde{\sigma}^2 - \frac{i}{2\pi}m\tilde{\sigma}}}{2 \cosh \frac{\tilde{\sigma}}{2}}. \quad (\text{II.77})$$

This is formally the same as the Coulomb branch localization formula for the $S^3$ partition function of $U(1)$ Chern-Simons theory coupled to charge-1 hyper multiplet with level $-1/k$ and FI-parameter $-m/2\pi$.

Let us perform thimble decomposition in this integral representation.

$$\tilde{Z} = \int_{-\infty}^{\infty} d\tilde{\sigma} \ e^{-S_{\text{mirror}}[\tilde{\sigma}]} \quad (\text{II.78})$$

where

$$S_{\text{mirror}}[\tilde{\sigma}] = \frac{ig}{16\pi^2} \tilde{z}^2 + \frac{i}{2\pi}m\tilde{z} + \log \left(2 \cosh \frac{\tilde{z}}{2}\right). \quad (\text{II.79})$$

Note that the action does become large for $g \to 0$ since weak coupling in the original theory corresponds to strong coupling in the mirror theory and vice versa. Therefore it is much easier to analyze Lefschetz thimble for $g \to \infty$. In this limit, the saddle point $\tilde{z}_c^c(g,m)$ is approximately determined by

$$\tilde{z}_c^c(g,m) \cosh \frac{\tilde{z}_c^c(g,m)}{2} = 0 \quad \text{for } g \to \infty, \quad (\text{II.80})$$

which leads us to $\lim_{g \to \infty} \tilde{z}_c^c(g,m) = 0$, $(2\ell - 1)\pi i$ with $\ell \in \mathbb{Z}$. We denote as $\tilde{z}_c^c_{\text{origin}}$ and $\tilde{z}_c^c_{\ell}$ the critical points satisfying

$$\lim_{g \to \infty} \tilde{z}_c^c_{\text{origin}}(g,m) = 0, \quad \lim_{g \to \infty} \tilde{z}_c^c_{\ell}(g,m) = (2\ell - 1)\pi i. \quad (\text{II.81})$$

Note their roles in the transseries are unclear just from this information in contrast to $z_{pt}^c$ and $z_{\ell}^c$ in the original theory. In other words, we do not have one-to-one correspondences between the critical points in the original and mirror theories although their final results are the same. In the large-$g$
expansion, $\tilde{z}_c^{\text{origin}}(g, m)$ and $\tilde{z}_c^{\epsilon}$ correspond to perturbative and non-perturbative critical points of $1/g$-expansion. We can easily solve the thimble associated with $\tilde{z}_c^{\text{origin}}$ in the $g \to \infty$ limit by

$$\lim_{g \to \infty} \tilde{z}_c^{\text{origin}}(g, m; s) = \epsilon \exp \left( \frac{g}{16\pi^2} s - \frac{\pi i}{4} \right). \quad (\text{II.82})$$

For the other critical points, it is hard to solve the flow equation analytically as in the original theory.

In Fig. 11 we present numerical plots for the thimble structures in the mirror theory. We take $(g, m)$ as parameters in Fig. 11 (a)-(c) as in Fig. 5. For $(g, m) = (4\pi, 2\pi)$, contributing critical points are $\tilde{z}_c^{\text{origin}}$ and $\tilde{z}_0^c$, and the thimble integral associated with $\tilde{z}_0^c$ is equivalent to the residue around the pole, which is here denoted as $\tilde{z} = \tilde{z}_0^c$. For $m = 3\pi$ and $4\pi$ with $g = 4\pi$, another critical point $\tilde{z}_1^c$ also contributes and the thimble associated with $\tilde{z}_1^c$ passes between $\tilde{z}_1^c$ and $\tilde{z}_2^c$ in contrast to the $m = 2\pi$ case. We have more complicated structures for smaller $g$: Fig. 11 (d) shows that the contributing critical points are $\tilde{z}_c^{\text{origin}}$, $\tilde{z}_0^c$ and $\tilde{z}_2^c$ for $(g, m) = (2\pi, 2\pi)$. For this case, the thimble integral associated with $\tilde{z}_0^c$ is equivalent to the residues around the two poles $\tilde{z} = \tilde{z}_0^c$.
and $\tilde{z}_{-1}$. Similarly, for $(g, m) = (4\pi/3, 2\pi)$, we have contributions from $\tilde{z}^c_{\text{origin}}$, $\tilde{z}^c_0$ and $\tilde{z}^c_{-3}$, and the thimble integral associated with $\tilde{z}^c_0$ is the same as the residues around the three poles $\tilde{z} = \tilde{z}^*_0$, $\tilde{z}^*_1$ and $\tilde{z}^*_2$. These results clearly show that the thimble decomposition in the mirror theory has the Stokes phenomena. While the sum of the thimble integrals over the contributing critical points is the same as the exact result by construction, we have not found precise understanding on a connection between the thimble structure and the resurgent structure in the mirror theory. It would be interesting to study this problem in more details in the future.

### III. $\mathcal{N} = 3$ SU(2) Chern-Simons SQCD

We next investigate the $S^3$ partition function of the 3D $\mathcal{N} = 3$ SU(2)$_k$ CS theory with $N_f$ fundamental hyper multiplets and real masses $m_a$, which we call CS SQCD [109]. We rewrite the exact partition function obtained by the Coulomb branch localization into the full transseries with nonperturbative exponential contributions. We also discuss the thimble decomposition and the Stokes phenomena in a manner parallel to the case of CS SQED in the previous section.

#### A. Exact results as resurgent transseries

The partition function of the $\mathcal{N} = 3$ SU(2) CS SQCD is given by [110]

$$Z = \int_{-\infty}^{\infty} d\sigma \, e^{ik \frac{2\pi i}{\sigma^2}} \left( \frac{2 \sinh \sigma}{2 \cosh \frac{\sigma-m_a}{2}} \cdot \frac{2 \cosh \frac{\sigma+m_a}{2}}{2 \cosh \frac{\sigma-m_a}{2}} \right).$$

(III.1)

We again focus on $k > 0$ and $m_a > 0$ mainly. Taking $g = \frac{2\pi}{k}$ and $\sigma = \sqrt{it}$, we rewrite the partition function as

$$Z = \int_{0}^{-i\infty} dt \, e^{-\frac{t}{i}BZ(t)},$$

(III.2)

where $BZ(t)$ is the Borel transformation of the perturbative series of the CS SQCD [55]:

$$BZ(t) = \frac{i \left( \frac{2 \sinh \sqrt{it}}{\sqrt{it}} \right)^2}{\sqrt{it} \prod_{a=1}^{N_f} 2 \cosh \frac{\sqrt{it}+m_a}{2} \cdot 2 \cosh \frac{\sqrt{it}-m_a}{2}}.$$  

(III.3)

This Borel transformation has simple poles at $t = -i [m_a \pm (2n_a - 1)\pi i]^2$ with $n_a \in \mathbb{N}$. With $m_a = (2n_a - 1)\pi$, we have Borel singularities at positive real axis as $t|_{m_a=(2n_a-1)\pi} = \pm 2(2n_a - 1)^2 \pi^2$, leading to non-Borel-summability of the perturbative series, thus $m_a = (2n_a - 1)\pi$ is a Stokes line.

As with the case of CS SQED, the exact result (III.2) is decomposed into the Borel resummation along $\mathbb{R}_+$ (perturbative part) and the residue of all the singularities in the fourth quadrant of the
The nonperturbative part $Z$ in the fourth quadrant of the Borel plane. We below show the results of $S$ to the contribution with the action $\mathcal{S}$ for non-degenerate real masses, separately. For $m_0 = (2n_a + 1)\pi$, leading to ambiguity of the perturbative Borel resummation, that is the Stokes phenomenon. It is also notable that, for the degenerate masses $m = m_a$, the singularities are degenerate, where the order of their poles gets equivalent to $N_f$.

We first focus on $N_f = 2$ for simplicity. By expanding $Z_{pt}$ with respect to $t$ and extracting coefficients, we obtain an asymptotic series of the perturbative part as

$$Z_{pt} = \frac{\sqrt{\pi g}}{8} \sum_{(s_b) = 0}^{\infty} \sum_{(s_b) = 0}^{\infty} \sum_{(s_b) = 0}^{\infty} 2^{-2q} \left( \prod_{b=1}^{2} \frac{1}{\Gamma(2s_b + 2)} \right) \left( \prod_{b=1}^{4} \frac{E_{2q_b}}{\Gamma(2q_b - l_b + 1)\Gamma(l_b + 1)} \right) \times \Gamma(q - l/2 + s/2 + 3/2) \cdot (ig)^{\bar{q} - l/2 + s + 1} m_1^{l_1} m_2^{l_2} \delta_{l \bmod 2, 0}. \quad (III.6)$$

with $q = \sum_{b=1}^{4} q_b$, $l = \sum_{b=1}^{4} l_b$, and $s = s_1 + s_2$. This perturbative series is Borel-summable along $\mathbb{R}_+$ for $m_0 \neq (2n_a - 1)\pi$. However, even if $m_0 \neq (2n_a - 1)\pi$, the Borel resummation of the perturbative series does not give an exact result for $m_0 > \pi$ as in the CS SQED case. The nonperturbative part $Z_{np}$ can be calculated by the residues of the Borel singularities in the fourth quadrant of the Borel plane. We below show the results of $Z_{np}$ for non-degenerate and degenerate real masses, separately. For $m_1 \neq m_2$ with $(2n_1 - 1)\pi < m_1 < (2n_1 + 1)\pi$ and $(2n_2 - 1)\pi < m_2 < (2n_2 + 1)\pi$, the nonperturbative part is given by

$$Z_{np} = 4\pi i \left[ \sum_{\ell_1=1}^{m_1} \frac{\frac{e^{\frac{i}{2}[m_1+(2\ell_1-1)\pi i]^2}}{\cosh m_1 - \cosh m_2} \sinh m_1} + \sum_{\ell_2=1}^{m_2} \frac{\frac{e^{\frac{i}{2}[m_2+(2\ell_2-1)\pi i]^2}}{\cosh m_2 - \cosh m_1} \sinh m_2} \right]. \quad (III.7)$$

For $m = m_1 = m_2$ with $(2n - 1)\pi < m_1 < (2n + 1)\pi$, it is obtained as

$$Z_{np} = 4\pi i \sum_{\ell=1}^{n} \frac{e^{\frac{i}{2}[m+(2\ell-1)\pi i]^2}}{\cosh m} \left( \frac{2i[m+(2\ell-1)\pi i]}{g} + \frac{1}{\tanh m} \right). \quad (III.8)$$

In these expressions of the full transseries expansion, each of the nonperturbative parts corresponds to the contribution with the action $S = -i[m_a+(2\ell_a-1)\pi i]^2/g$, which is consistent with the position of the singularities in the Borel transform (III.3).
In the case of general $N_f$ with degenerate mass, $Z_{pt}$ and $Z_{np}$ are given by

$$Z_{pt} = \frac{\sqrt{ig}}{2^{2N_f-1}} \sum_{\{s_b\}=0}^{\infty} \sum_{\{t_b\}=0}^{\infty} 2^{-2q} \left( \prod_{b=1}^{2N_f} \frac{1}{\Gamma(2s_b + 2)} \right) \left( \prod_{b=1}^{2N_f} \frac{E_{2q_b}}{\Gamma(2q_b - l_b + 1)\Gamma(l_b + 1)} \right) \times \Gamma(q - \ell/2 + s/2 + 3/2) \cdot (ig)^{q - \ell/2 + s + 1} m^{\ell/2} \delta_{\ell mod 2, 0},$$

(III.9)

$$Z_{np} = \frac{\pi i}{2^{2N_f-3}\Gamma(N_f)} \sum_{\ell=0}^{n-1} \lim_{z \to \ell} \frac{\partial^{N_f-1}}{\partial z^{N_f-1}} \left( \frac{(z - z^*)^{N_f} \sinh^2 z \cdot e^{iz/g}}{(\cosh \frac{z-m}{2} \cosh \frac{z+m}{2})^{N_f}} \right),$$

(III.10)

with $\tilde{q} = \sum_{b=1}^{2N_f} q_b$, $\tilde{\ell} = \sum_{b=1}^{2N_f} \ell_b$, $\tilde{s} = \sum_{b=1}^{2} s_b$ and $z^* = m + (2\ell - 1)\pi i$. Introducing the step function, the nonperturbative part for general $(N_f, m_a)$ is expressed as

$$Z_{np} = \sum_{a=1}^{N_f} \sum_{\ell_a=1}^{\infty} \theta(m_a - (2\ell_a - 1)\pi) \text{Res}_{t=-i|m_a+(2\ell_a-1)\pi i|^2} \left[ e^{\frac{i\pi}{2} BZ(t)} \right].$$

(III.11)

As in the CS SQED cases, the transseries expression is apparently ambiguous for $m_a = (2n_a - 1)\pi$ ($n_a \in \mathbb{N}$) due to the Borel ambiguities and step function behaviors of the transseries parameters. The ambiguity in the perturbative part is estimated by

$$Z_{pt}(\{m_b\})|_{m_a=(2n_a-1)\pi+0_+} - Z_{pt}(\{m_b\})|_{m_a=(2n_a-1)\pi+0_-} = -\text{Res}_{t=-i|m_a+(2n_a-1)\pi|^2} \left[ e^{\frac{i\pi}{2} BZ(t)} \right],$$

(III.12)

while the non-perturbative ambiguity is

$$Z_{np}(\{m_b\})|_{m_a=(2n_a-1)\pi+0_+} - Z_{np}(\{m_b\})|_{m_a=(2n_a-1)\pi+0_-} = +\text{Res}_{t=-i|m_a+(2n_a-1)\pi|^2} \left[ e^{\frac{i\pi}{2} BZ(t)} \right].$$

(III.13)

It is clear that these ambiguities are canceled and we obtain the unambiguous result equivalent to the exact result.

### B. Thimble decomposition

The effective action of the present example with respect to $\sigma$ reads as

$$S[\sigma] = -\frac{i\sigma^2}{g} - \log \frac{4 \sinh^2 \sigma}{\prod_{a=1}^{N_f} 2 \cosh \frac{z-a}{2} \cdot 2 \cosh \frac{z+a}{2}}.$$  

(III.14)

We consider the complexification $\sigma \to z$ and study thimble structures in a parallel manner to the CS SQED case. First the saddle point $z^c$ is determined by

$$\frac{\partial S[z]}{\partial z} \bigg|_{z=z^c} = -\frac{2i}{g} z^c - 2 \coth z^c + \frac{1}{2} \sum_{a=1}^{N_f} \sum_{\pm} \tanh z^c \pm \frac{m_a}{2} = 0.$$  

(III.15)
As in the $U(1)$ case, we can analytically find the saddle points in the $g \to 0$ limit:

$$
\begin{align*}
&e^c \sinh^2 e^c \prod_{a=1}^{N_f} \cosh \frac{e^c - m_a}{2} \cosh \frac{e^c + m_a}{2} = 0 \quad \text{for } g \to 0.
\end{align*}
$$

(III.16)

Note that while the third factor comes from the poles of the integrand, which we had also in the CS SQED case, the second factor comes from the zeroes, which were absent in the $U(1)$ case. The zeroes of the integrand are given by

$$
\sinh^2 z_{\text{zero}} = 0 \quad \Rightarrow \quad z_{\text{zero}} = n\pi i, \quad n \in \mathbb{Z}.
$$

(III.17)

The zeroes add qualitatively new features to the thimble structure because they can be end points of Lefschetz thimbles and thimbles may terminate at finite $z$ ($= z_{\text{zero}}$) in contrast to the CS SQED. This always happens when we analyze the following type of integral

$$
\int dx \frac{P(x)}{Q(x)} e^{-\frac{1}{g}h(x)},
$$

(III.18)

where $P(x)$ and $Q(x)$ are functions without poles. Saddle points of this integral in the $g \to 0$ limit are given by

$$
\frac{\partial h(x)}{\partial x} P(x)Q(x) = 0,
$$

(III.19)

which indicates that the poles and zeroes coincide with the saddle points in the $g \to 0$ limit. In the present example with $g \to 0$, the perturbative part of the transseries, which is the Borel resummation along $\mathbb{R}_+$, corresponds to a sum of two thimble integrals associated with two saddle points around $z = 0$ \[111\] as we will see soon. Another important feature comes from the fact that the action \[111\] is invariant under the $\mathbb{Z}_2$ transformation $z \to -z$. This symmetry forces the singularities and saddle points to be located symmetrically in the complex $z$-plane. These facts imply that each of the contributions in the transseries \[111\] is composed of a pair of two thimble integrals associated with two saddle points even in $g \to 0$ limit.

Now we present some samples of numerical results. Fig. \[12\] depicts the thimble structure for $N_f = 1$ in CS SQCD with $g \approx 0.126$ ($k = 100$) and $m = 2\pi, 3\pi, 4\pi$, which can be regarded as approximate cases of the weak-coupling limit. We term two saddle points near the origin as “perturbative” ones and others as “nonperturbative” ones. Note that the red crosses are almost overlapped with the green circles since the nonperturbative saddle points (green circles) and the singularities (red crosses) are almost degenerate for small $g$. Each pair of saddle points constituting one sector of the transseries is located in a $\mathbb{Z}_2$ symmetrical manner. For $m = 2\pi$, four thimbles (two pairs) contribute to the partition function: two thimbles associated with the perturbative
FIG. 12: Thimble structures of the SU(2) CS SQCD with \( N_f = 1 \) and \( g \approx 0.126 \) \((k = 100)\). The green points, red crosses, and blue square stand for critical points, poles and zeroes of the integrand, respectively. The red dotted lines denote dual thimbles having nonzero intersection numbers with the original integral contour \( \mathbb{R} \) while the blue dashed lines for the corresponding thimbles. The arrows represent flow lines for increasing flow parameter \( s \).

saddles near the origin and the other two thimbles associated with the nonperturbative saddles around \( z = \pm(m + \pi i) \). For \( m = 3\pi \), the perturbative thimbles almost pass the saddles around \( z = \pm(m + 3\pi i) \). This reflects the fact that \( m = 3\pi \) corresponds to the Stokes line of the transseries and the result starts to receive contributions from the two thimbles associated with the saddles around \( z = \pm(m + 3\pi i) \) as the nonperturbative effects. Note that, in this limit, the Stokes lines of transseries and thimble decomposition almost coincides. For \( m = 4\pi \), the six thimbles (three pairs) contribute to the partition function: the two thimbles associated with the perturbative saddles and four thimbles associated with the nonperturbative saddles around \( z = \pm(m + \pi i) \) and \( \pm(m + 3\pi i) \).

Fig. 13 shows the \( N_f = 2 \) case. In this case, we depict the thimble structures for \( m_1 = 2\pi, 3\pi, 4\pi \) with \( g \approx 0.126 \) and \( m_2 = 4\pi \) fixed, where the nonperturbative saddles (green points) and singularities (red crosses) are almost degenerate again. For \( m_1 = 2\pi \), eight thimbles (four pairs) contribute to the partition function: two thimbles associated with the perturbative saddles (near the origin) and the other six thimbles associated with the nonperturbative saddles around \( z = \pm(m_2 + \pi i), \pm(m_2 + 3\pi i) \) and \( \pm(m_1 + \pi i) \). For \( m_1 = 3\pi \), two more thimbles (one pair) associated with the nonperturbative saddles come in as the nonperturbative contributions since this parameter is the Stokes line of the transseries. For \( m_1 = 4\pi \), the poles get degenerate and therefore nonperturbative saddles also become degenerate, where the eight nonperturbative saddles around \( z = \pm(m_{1,2} + \pi i) \) and \( \pm(m_{1,2} + 3\pi i) \) are merged into the four degenerate saddles. We end
FIG. 13: Thimble structure of two-flavor $SU(2)$ CS theory with $g \approx 0.126$ ($k = 100$) and $m_2 = 4\pi$. The green points, red crosses, and blue square stand for critical points, singularities, and zero-points, respectively. The red dotted lines stand for dual thimbles with nonzero intersection numbers, and the blue dashed lines for the corresponding thimbles. The arrows represent the flow lines.

up with two thimbles (one pair) associated with the perturbative saddles near the origin and four thimbles (two pairs) associated with the nonperturbative degenerate saddles.

In the $g \to 0$ limit, each of the thimble integrals associated with nonperturbative saddle points is equivalent to each of the residues of the Borel singularities. Thus, when the real mass crosses the Stokes line $m_a \approx (2n_a - 1)\pi$, the saddle point around the pole starts to contribute to the partition function as the nonperturbative effect. As in the case of the CS SQED, for finite $g$, the Stokes phenomena of the thimble decomposition occur at different points $\tilde{m}$’s from those of the transseries which approach the same points in the weak coupling limit $g \to 0$. Let us consider the $N_f = 1$ case for simplicity. The perturbative contribution $Z_{pt}$ is only composed of a pair of the thimbles associated with the perturbative saddle points near the origin for $(2n - 1)\pi < m < \tilde{m}$ ($n \geq 1$). However, it gets composed of these perturbative thimbles and two more thimbles associated with nonperturbative saddles for $\tilde{m} < m < (2n+1)\pi$. The role played by these nonperturbative thimbles changes at $m = (2n+1)\pi$, where they come to contribute to the partition function as the “genuine” nonperturbative contribution. As in the $U(1)$ case, if we are not on the Stokes lines, the result based on the thimble decomposition is in exact agreement with that of the resurgent transseries without subtleties. On the Stokes lines, they are apparently ambiguous and we need to take limits from opposite sides as in the $U(1)$ case. Up to these subtleties, they are equivalent for any $(g, \{m_a\})$. 
IV. GENERALIZATION

So far we have analyzed the sphere partition functions of the $\mathcal{N} = 3$ theories for simplicity. In this section we generalize these analyses to more general theories and other observables.

A. General rank-1 $\mathcal{N} = 2$ Chern-Simons matter theory

Let us consider general rank-1 $\mathcal{N} = 2$ Chern-Simons matter theory, which is $U(1)$ theory coupled to charge-$q_a$ chiral multiplets with R-charge $\Delta_a$ and real mass $m_a$, or $SU(2)$ theory coupled to isospin-$j_a$ chiral multiplets with R-charge $\Delta_a$ and real mass $m_a$. The localization formula for the sphere partition function is given by

$$Z = \int_{-\infty}^{\infty} d\sigma e^{\frac{i}{g^2} \sigma^2} Z_{1\text{-loop}}(\sigma)$$ (IV.1)

where $g = 1/\pi k$ for $U(1)$ and $g = 1/2\pi k$ for $SU(2)$. $Z_{1\text{loop}}$ is given by

$$Z_{1\text{loop}}(\sigma) = \begin{cases} \prod_{a=1}^{N_f} s_1(q_a \sigma + m_a - i(1 - \Delta_a)) & \text{for } U(1) \\ \prod_{a=1}^{N_f} \prod_{\Delta_a=-2\Delta_a}^{2\Delta_a} s_1(q_a \cdot \sigma + m_a - i(1 - \Delta_a)) & \text{for } SU(2) \end{cases}$$ (IV.2)

where $\prod$ denotes $\prod$ with step 2 and $s_1(x)$ is given by

$$s_1(x) = \prod_{n=1}^{\infty} \left( \frac{n - ix}{n + ix} \right)^n.$$ (IV.3)

The most important difference from the $\mathcal{N} = 3$ theories is that each matter contribution has both zeroes and poles, whose degrees are not necessarily one.

1. Exact results as resurgent transseries

We can extend the analyzes in Sec. [III A] and [III A] straightforwardly. Taking $\sigma = \sqrt{it}$ again leads us to

$$Z(g, \{m_a\}) = \int_{0}^{-i\infty} dt e^{-\frac{t}{2g}} BZ(t),$$ (IV.4)

where $BZ(t)$ is the perturbative Borel transformation

$$BZ(t) = \frac{i}{\sqrt{it}} \sum_{\pm} Z_{1\text{loop}}(\pm \sqrt{it}).$$ (IV.5)
Note that the $a$-th chiral multiplet gives poles of $Z_{1\text{loop}}(z)$ with degree $\ell_a$ at
\[ z_{a,\ell}^* = -m_a - i(1 - \Delta_a + \ell_a), \quad (\text{IV.6}) \]
which gives Borel singularities at
\[ t_{a,\ell}^* = -\frac{i}{q_a^2}(m_a + i(1 - \Delta_a + \ell_a))^2 \quad \text{with} \quad \ell_a \in \mathbb{Z}_+. \quad (\text{IV.7}) \]
Changing the integral contour as in Fig. 9 we decompose the exact result into the perturbative and nonperturbative parts:
\[ Z(g, \{m_a\}) = \int_0^\infty dt \ e^{-\frac{t}{g}BZ(t)} + \sum_{\text{poles } \in 4\text{th quadrant}} \text{Res}_{t=t_{a,\ell}^*} \left[ e^{-\frac{t}{g}BZ(t)} \right]. \quad (\text{IV.8}) \]
Noting that the poles start to come into 4th quadrant when $m_a = 1 - \Delta_a + \ell_a$, we can write the partition function as
\[ Z(g, \{m_a\}) = Z_{\text{pt}} + \sum_{a=1}^{N_f} \sum_{\ell_a=1}^{\infty} Z_{\text{np}}^{(a,\ell_a)}, \quad (\text{IV.9}) \]
where
\[ Z_{\text{pt}} = \int_0^\infty dt \ e^{-\frac{t}{g}BZ(t)}, \]
\[ Z_{\text{np}}^{(a,\ell)} = \theta(m_a - (1 - \Delta_a + \ell_a)) \text{Res}_{t=t_{a,\ell}^*} \left[ e^{-\frac{t}{g}BZ(t)} \right]. \quad (\text{IV.10}) \]
As in the previous cases, this decomposition is apparently ambiguous for $m_a = 1 - \Delta_a + \ell_a$ because of the Borel ambiguities and step function behavior of the transseries parameter. Indeed the Borel ambiguity in the perturbative sector is
\[ Z_{\text{pt}}(\{m_b\})|_{m_a=1-\Delta_a+\ell_a+0_+} - Z_{\text{pt}}(\{m_b\})|_{m_a=1-\Delta_a+\ell_a+0_-} = -\text{Res}_{t=t_{a,\ell}^*} \left[ e^{-\frac{t}{g}BZ(t)} \right], \quad (\text{IV.11}) \]
while the non-perturbative ones are
\[ Z_{\text{np}}^{(b,\ell)}(\{m_b\})|_{m_a=1-\Delta_a+\ell_a+0_+} - Z_{\text{np}}^{(b,\ell)}(\{m_b\})|_{m_a=1-\Delta_a+\ell_a+0_-} \]
\[ = \begin{cases} 0 & \text{for } b \neq a, \text{ or } \ell \neq \ell_a \\ +\text{Res}_{t=t_{a,\ell}^*} \left[ e^{-\frac{t}{g}BZ(t)} \right] & \text{for } b = a \text{ and } \ell = \ell_a. \end{cases} \quad (\text{IV.12}) \]
Thus the ambiguities are canceled and we find the unambiguous answer consistent with the exact result.
We can also find the resurgent structure in the situation with fixing \( m \) and varying \( \arg(g) \) as in Sec. II C. By use of similar arguments, the exact result is decomposed as

\[
Z = Z_{\text{pt}} + \sum_{a=1}^{N_f} \sum_{\ell_a=1}^{\infty} Z^{(a,\ell_a)}_{\text{np}},
\]

\[
Z_{\text{pt}} = \int_0^{\arg(g)\infty} dt \ e^{-\frac{t}{g} B Z(t)}, \quad Z^{(a,\ell)}_{\text{np}} = \theta \left( \arg(g) - \arg(t^*_{a,\ell}) \right) \text{Res}_{t=t^*_{a,\ell}} \left[ e^{-\frac{t}{g} B Z(t)} \right]. \quad (IV.13)
\]

Although this decomposition apparently has ambiguities for \( \arg(g) = \arg(t^*_{a,\ell}) \) estimated by

\[
(S_{\arg(t^*_{a,\ell})+0^+} - S_{\arg(t^*_{a,\ell})+0^-}) Z(g, \{m_b\}), \quad (IV.14)
\]

they are precisely canceled and the transseries leads us to the unambiguous answer.

2. Thimble decomposition

We discuss thimble decomposition of the integral

\[
Z = \int_{-\infty}^{\infty} d\sigma \ e^{-S[\sigma]}, \quad S[z] = -\frac{i z^2}{g} - \log Z_{\text{loop}}(z). \quad (IV.15)
\]

Saddle point equation under this action is given by

\[
0 = \begin{cases}
\frac{2i z}{g} + \pi i \sum_{a=1}^{N_f} \frac{q_a z + m_a - i(1-\Delta_a)}{\tanh(\pi(q_a z + m_a - i(1-\Delta_a)))} & \text{for } U(1) \\
\frac{2iz}{g} - \frac{2\pi}{\tanh(\pi z)} + \pi i \sum_{a=1}^{N_f} \sum_{j_a = 1}^{2j_a} \frac{q_a z + m_a - i(1-\Delta_a)}{\tanh(\pi(q_a z + m_a - i(1-\Delta_a)))} & \text{for } SU(2)
\end{cases}, \quad (IV.16)
\]

where we have used the identity \([7.1]\)

\[
\frac{\partial \log s_1(z)}{\partial z} = \frac{\pi i z}{\tanh(\pi z)}. \quad (IV.17)
\]

We can analytically solve this equation in weak coupling limit as in the previous cases. For \( g \to 0 \) the saddle points are approximately determined by

\[
0 = \begin{cases}
z^c \prod_{a=1}^{N_f} \frac{\sinh(\pi(q_a z^c + m_a - i(1-\Delta_a)))}{q_a z^c + m_a - i(1-\Delta_a)} & \text{for } U(1) \\
\left(z^c \sinh(\pi z^c) \prod_{a=1}^{N_f} \frac{\sinh(\pi(q_a z^c + m_a - i(1-\Delta_a)))}{q_a z^c + m_a - i(1-\Delta_a)} \right) & \text{for } SU(2)
\end{cases}, \quad (g \to 0), \quad (IV.18)
\]

whose solutions are \( z = 0 \), zeros and poles of the integrand in (IV.1) as expected. Note that these general cases have much more critical points than the \( N = 3 \) cases since each \( N = 2 \) chiral multiplet gives an infinite number of zeroes as well as poles. Let \( z^c_{\text{pt}} \) denoting the critical point satisfying

\[
\lim_{g \to 0} z^c_{\text{pt}} = 0, \quad (IV.19)
\]
then we can easily compute the Lefschetz thimble associated with $z_{pt}^c$ in the weak coupling limit:

$$\lim_{g \to 0} z_{pt}(s) = \epsilon \exp \left( \frac{2}{g} s + \frac{\pi i}{4} \right). \quad (IV.20)$$

As in the $\mathcal{N} = 3$ cases, it is hard to find critical points analytically for nonzero $g$. In addition, numerical analysis is also inapplicable without specifying theories. Therefore, we here provides expected thimble structures for general case based on its resurgent structure in the last subsubsection and the examples of the thimble structures. For weak coupling, there are critical points around $z = 0$, the zeroes and poles of the integrand. We identify $z_{pt}^c$ as a “perturbative critical point” and the ones around the poles as “nonperturbative critical points”. There are two possibilities of the behavior of the perturbative thimble $z_{pt}(s)$ for finite $g$: it would run between $e^{\pm \frac{\pi i}{4} \infty}$ as in (IV.20) or it would terminate at a zero of the integrand. For the latter case, another critical point around the zero contributes and its thimble runs from the zero to $e^{\pm \frac{\pi i}{4} \infty}$ so that the thimble combined with the perturbative thimble $z_{pt}(s)$ gets equivalent to (IV.20) as in Fig. 12 for the $\mathcal{N} = 3 \, SU(2)$ SQCD case. It is also expected that a critical point around the pole $z = z_{a,\ell}^*$ starts to contribute around $m_a = 1 - \Delta_a + \ell$. Then, there are again two possibilities: the thimble integral associated with this critical point would be equivalent to residue around $z = z_{a,\ell}^*$ or would terminate at a zero. In the latter case, a combination of thimble integrals of the critical points around the pole and the zero gets equivalent to the residue. It is left for the future work to check these expectations explicitly.

B. Other observables

So far we have considered only the partition function on a round sphere. In this subsection, we discuss extension of our argument to other observables.

*Supersymmetric Wilson loop*

Let us start with the Wilson loop

$$W_R(C) = \text{tr} R P \exp \left[ \oint_C ds (i A_\mu \dot{x}^\mu + \sigma |\dot{x}|) \right], \quad (IV.21)$$

It is known that this operator preserves two supercharges if the contour $C$ is the great circle of $S^3$. Hence, we can compute an expectation value of the SUSY Wilson loop by localization:

$$\langle W_R(C) \rangle = \langle \text{tr} e^{\sigma} \rangle_{M.M.}, \quad (IV.22)$$
where $\langle \cdots \rangle_{M.M.}$ denotes an expectation value in the integral \[(I.4)\]. Note that the difference from the sphere partition function is just insertion of entire function of $\sigma$. Therefore we can repeat the analyses in the previous sections straightforwardly. Namely, the SUSY Wilson loop has the same Borel singularities as the sphere partition function and their resurgent structures are the same although there are differences in some details such as values of perturbative coefficients and residues around the poles. The insertion of the Wilson loop changes saddle point equation of the integral and hence thimble structures as well. However, since the difference is negligible in the weak coupling limit, the Wilson loop should not affect the relation between transseries and thimble decomposition, which we have seen in the sphere partition functions.

**Bremsstrahlung function in SCFT on $\mathbb{R}^3$**

If we restrict ourselves to superconformal case, we can also compute Bremsstrahlung function $B$ on $\mathbb{R}^3$ by localization which determines an energy radiated by accelerating quarks with small velocities as $E = 2\pi B \int dt v^2$. It was conjectured in [75] that the Bremsstrahlung function in 3d $\mathcal{N} = 2$ superconformal theory is given by

$$B(g) = \frac{1}{4\pi^2} \frac{\partial}{\partial b} \log \langle \text{tr} e^{ib\sigma} \rangle_{M.M.} \bigg|_{b=1}.$$  \[(IV.23)\]

As in the Wilson loop, the net effect is just insertion of the entire function and hence we basically arrive at the same conclusion as the Wilson loop. However, note that we cannot turn on real masses for this case since we are considering superconformal case. In other words, we can formally turn on real masses at the level of the integral \[(I.4)\] but its physical interpretation is unclear. Nevertheless, it is notable that the RHS of \[(IV.23)\] with nonzero $m$ shares the common resurgent structures with the sphere partition function and Wilson loop.

**Two-point function of $U(1)$ flavor symmetry currents in SCFT on $\mathbb{R}^3$**

We can also compute two-point function of the $U(1)$ flavor symmetry current $j^a_\mu$ for superconformal cases by localization. It is known that the two-point function is fixed by the 3d conformal symmetry as

$$\langle j^a_\mu(x) j^b_\nu(0) \rangle = \tau_{ab} \left( \delta^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu \right) \frac{1}{x^2} + \frac{i\kappa_{ab}}{2\pi} \epsilon^{\mu\nu\rho} \partial_\rho \delta^{(3)}(x),$$  \[(IV.24)\]

where $\tau_{ab}(g)$ and $\kappa_{ab}(g)$ are coefficients depending on couplings. The work [76] showed that these coefficients are generated by the sphere partition function with real mass $\{m_a\}$ associated with the
\[ U(1) \text{ symmetries:} \]
\[
\tau_{ab}(g) = -\frac{2}{\pi^2} \text{Re} \left[ \frac{1}{Z_{S^3}(g,0)} \frac{\partial^2 Z_{S^3}(g,m)}{\partial m_a \partial m_b} \right] \{m_a=0\},
\]
\[
\kappa_{ab}(g) = \frac{1}{2\pi} \text{Im} \left[ \frac{1}{Z_{S^3}(g,0)} \frac{\partial^2 Z_{S^3}(g,m)}{\partial m_a \partial m_b} \right] \{m_a=0\}. \quad (IV.25)
\]

The derivatives by the real masses do not change locations of singularities while their degrees are changed. This difference, however, does not lead to qualitative change on the resurgent structure and the thimble structures for weak coupling.

**Partition function and Wilson loop on Squashed S³**

Let us consider partition function on squashed sphere \( S^3_b \) with the squashing parameter \( b \), which has a simple relation to supersymmetric Renyi entropy [77]. The difference from the round sphere partition function in localization formula is just the one-loop determinant [78]:
\[
Z_{1\text{loop}}(\sigma) = \prod_{\alpha \in \text{root}^+} \frac{4 \sinh (\pi b \alpha \cdot \sigma) \sinh (\pi b^{-1} \alpha \cdot \sigma)}{\prod_{n=1}^{N_f} \prod_{\rho \in \text{R}_a} s_b(\rho_a \cdot \sigma - \frac{iQ}{2}(1 - \Delta_a))}, \quad (IV.26)
\]
where \( Q = b + b^{-1} \) and
\[
s_b(z) = \prod_{n_1=0}^{\infty} \prod_{n_2=0}^{\infty} \frac{n_1 b + n_2 b^{-1} + Q/2 - iz}{n_1 b + n_2 b^{-1} + Q/2 + iz}. \quad (IV.27)
\]

Note that the round sphere case corresponds to \( b = 1 \). It was shown in [56] that we can obtain the perturbative Borel transform for general \( b \) in a parallel way to the \( b = 1 \) case and rewrite the \( S^3_b \) partition function as the Borel resummation along \( \varphi = -\pi/2 \). Therefore the change of the 1-loop determinant \( (IV.26) \) affects Borel singularities. Two important differences for us are

- Borel singularities associated with each of chiral multiplets become simple poles and are labeled by two integers.
- Locations of the singularities depend on \( b \).

Even for this case, we can still write the partition function as
\[
Z_{S^3_b}(g, \{m_a\}) = \int_0^\infty dt \ e^{-\frac{t}{2} BZ(t)} + \sum_{\text{poles} \in \text{4th quadrant}} \text{Res}_{t=pole} \left[ e^{-\frac{t}{2} BZ_{S^3_b}(t)} \right], \quad (IV.28)
\]
which is a valid expression except for the Stokes lines. We regard the first and second terms as perturbative and nonperturbative parts respectively. As in the previous cases, we need to
take care of the singularities in the fourth quadrant. A short calculation shows that the Borel singularities come on \( \mathbb{R}_+ \) for \( m_a = \frac{n_1 + \Delta}{2} b + \frac{n_2 + \Delta}{2} b^{-1} \) with \( n_1, n_2 \in \mathbb{Z}_{\geq 0} \) for \( b \in \mathbb{R}_+ \) \[113\]. For this case, the decomposition \[ IV.28 \] is apparently ambiguous but the ambiguities are canceled between perturbative and nonperturbative parts.

We can also put the supersymmetric Wilson loop on a squashed sphere constructed in \[79\]. Localization formula for the Wilson loop is insertion of \( \text{tr}_{\mathbb{R}^2} e^{\sigma} \) or \( \text{tr}_{\mathbb{R}^2} e^{b^{\pm} \sigma} \) to the localization formula of the partition function. Therefore the Wilson loop gives only minor differences such as values of perturbative coefficients and details on thimble structures for nonzero \( g \).

Two point function of stress tensor in SCFT on \( \mathbb{R}^3 \)

For superconformal case, we can also compute a two-point function of the normalized stress tensor at separate points, whose expression is determined by conformal symmetry as

\[
\langle T_{\mu\nu}(x)T_{\rho\sigma}(0) \rangle = \frac{c_T}{64} (P_{\mu\rho} P_{\nu\sigma} + P_{\nu\rho} P_{\mu\sigma} - P_{\mu\nu} P_{\rho\sigma}) \frac{1}{16\pi^2 x^2},
\]

(IV.29)

where \( P_{\mu\nu} = \delta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu \) \[114\]. The coefficient \( c_T(g) \) is generated by \( Z_{S^3} \) as \[80\]

\[
c_T(g) = \frac{32}{\pi^2} \text{Re} \left[ \frac{1}{Z_{S^3}(g)} \frac{\partial^2 Z_{S^3}(g)}{\partial b^2} \right]_{b=1}.
\]

(IV.30)

Although the derivative with respect to \( b \) changes degrees of the singularities, this does not change the resurgent structures so much as the two-point function of the flavor symmetry currents.

V. PATH INTEGRAL INTERPRETATION OF THE NON-PERTURBATIVE EFFECTS

In this section we discuss possible interpretations of the non-perturbative effects appearing in the transseries from the path-integral viewpoint. It is technically obvious that the non-perturbative effects come from the Borel singularities or equivalently the poles of the integrand of the Coulomb branch localization formula. In \[56\], one of the present authors has proposed that the Borel singularities correspond to complexified SUSY solutions (CSS) which satisfy SUSY conditions but are not on the original path-integral contour. The CSS have been constructed for generic 3d \( \mathcal{N} = 2 \) SUSY theory with Lagrangian and \( U(1)_R \) symmetry put on a sphere. For theories with CS terms, it has been shown that classical actions of the CSS are precisely the same as the exponents of the residues around Borel singularities \[56\], which give the non-perturbative corrections appearing in the transseries we have discussed in the present work. In more detail,
the work \[56\] discussed that there exist two types of CSS in general: one has a bosonic parameter while the other has a fermionic one, which are referred to as bosonic and fermionic complexified supersymmetric solutions, respectively. Then it has been proposed that if there are \(n_B\) bosonic and \(n_F\) fermionic solutions with the action \(S = S_c/g\), then the Borel transformation includes the following factor

\[
BZ(t) \supset \prod_{\text{CSS}} \frac{1}{(t - S_c)^{n_B - n_F}}. \tag{V.1}
\]

For example, in the \(\mathcal{N} = 3\) \(U(1)\) theory discussed in Sec. II, there are \(\ell\) bosonic and \((\ell - 1)\) fermionic solutions with the action

\[
S_\ell = -\frac{i}{g} \left[ m + (2\ell - 1)\pi i \right]^2, \tag{V.2}
\]

which are precisely the exponentials in the transseries. Thus it is reasonable to conjecture that the non-perturbative effects appearing in the transseries correspond to the complexified SUSY solutions. In this interpretation, the Borel ambiguity in the perturbative sector is canceled by ambiguities in the nonperturbative CSS contributions and the total unambiguous answer obtained in this procedure agrees with the exact result.

However, there are three subtleties in this interpretation. First, SUSY solutions are not necessarily saddle points on the curved space contrary to the flat space. Indeed the CSS constructed in \[56\] are saddle points of 3d \(\mathcal{N} = 2\) SYM coupled to matters but when we turn on either CS or FI terms, the CSS do not satisfy saddle-point equations while SUSY conditions are still satisfied. We emphasize that this does not contradict Lipatov’s argument \[84\] which states that Borel singularities correspond to saddle points of the theory; In the localization procedure, we analyze the following type of path integral

\[
Z(g) = \int \mathcal{D}\Phi \ e^{-\frac{S}{g} - t_{\text{def}} Q V}, \tag{V.3}
\]

where \(S\) is the original action, \(Q\) is supercharge and \(V\) is a fermionic functional. The result is independent of the deformation parameter \(t_{\text{def}}\), which is usually taken to be \(t_{\text{def}} \to 0\) so that the saddle point analysis becomes exact. In 3d \(\mathcal{N} = 2\) theories, actions of the \(\mathcal{N} = 2\) SYM theory and chiral multiplets can be written in \(Q\)-exact forms while the CS and FI terms are \(Q\)-closed but not \(Q\)-exact. In Coulomb branch localization, we regard the SYM and matter actions as the deformation term, and the CS/FI terms as “operators” technically. Therefore the CSS are saddle point of the deformation term but may not be for the whole action. Now let us extend the Lipatov’s
argument to the integral \((V.3)\). We can extract \(n\)-th order perturbative coefficient by

\[
\frac{1}{2\pi i} \int \frac{dg}{g^{n+1}} Z(g) = \frac{1}{2\pi i} \int dg \int D\Phi \exp \left[ -\frac{1}{g} S[\Phi] - t_{\text{def}} QV - (n + 1) \ln g \right],
\]

which is independent of \(t_{\text{def}}\). For large-\(n\), the integral is dominated by the conditions

\[
\frac{\delta}{\delta \Phi} \left( \frac{S}{g} + t_{\text{def}} QV \right) \bigg|_{\Phi = \Phi_*} = 0, \quad -\frac{1}{g_*} S[\Phi_*] + \frac{n + 1}{g_*} = 0,
\]

which leads us to the Borel singularity at \(t = S[\Phi_*]\). Now we use the extra property of the integral \((V.3)\), namely independence of \(t_{\text{def}}\). When \(t_{\text{def}}\) is very large, the first condition approximately becomes

\[
\frac{\delta}{\delta \Phi} QV \bigg|_{\Phi = \Phi_*} = 0,
\]

which is nothing but the condition giving a localization locus. Therefore, Borel singularities correspond to the localization loci, where the positions of the Borel singularities are given by the actions of the original theory evaluated on the localization loci. This is similar to the case of the perturbative series for some operators in field theory in the sense that operators slightly affect saddle point equations and the original action in localization procedure can be technically regarded as an operator.

Second, to verify our conjecture, we have to check the following two facts: (1) The Stokes phenomena regarding the nonperturbative contributions we have shown should be identified as jumps of intersection numbers between the original path integral contour and dual Lefschetz thimbles associated with the CSS. (2) The perturbative series in the nonperturbative sector of the transseries should agree with perturbative series around the CSS. Especially the perturbative series should terminate at the one-loop order. We may be able to check this statement in future works.

Third, to our knowledge, most of analyses of SUSY localization in the literature have not preformed serious saddle-point analysis including complex saddles and therefore there is possibility that we are missing contributions from complex saddles. In particular, in the Coulomb branch localization formula for 3d \(\mathcal{N} = 2\) theory on \(S^3\), we have picked up only real SUSY solutions which are Coulomb branch solutions, but we currently know the existence of the CSS, which may or may not contribute. Although we think that this possibility is very unlikely since the localization formula has passed many nontrivial tests such as dualities, AdS/CFT, \(F\)-theorem and so on, we have to verify at least that the known localization formula is exact by using Lefschetz thimble analysis.
VI. SUMMARY AND DISCUSSION

We summarize the results obtained in this paper as follows:

(i) We have expressed the exact results for the SUSY observables in 3d $\mathcal{N} = 2$ Chern-Simons matter theories, which are also seen as $\mathcal{N} = 3$ theories, as the full resurgent transseries composed of the perturbative and nonperturbative sectors. The nonperturbative sectors are given by the residues around the Borel singularities in the fourth quadrant. The transseries is also understood from the viewpoint of Lefschetz thimble associated with saddle points of the effective action with respect to Coulomb branch parameter.

(ii) We have found that, when the real masses cross the special values, some of Borel singularities get on the real positive axis and come to contribute to the partition function as nonperturbative contributions. It leads to Stokes phenomena, where the perturbative Borel resummation becomes ambiguous. For example, in the $\mathcal{N} = 3$ $U(1)$ CS theory with $N_f = 1$ for the mass $(2n-1)\pi < m < (2n+1)\pi$, the $|n|$ Borel singularities contribute to the partition function, and one more singularity comes to contribute at $m = (2n+1)\pi$. In the $g \to 0$ limit, we can rephrase this in the language of the thimble decomposition that the perturbative thimble and the $|n|$ thimbles associated with the nonperturbative saddles contribute to the partition function in this regime.

(iii) We have shown that the relation between each of the thimble integrals in the thimble decomposition and each of building blocks of the transseries do not necessarily have one-to-one correspondence for finite $g$. Each building block of the transseries can be expressed as the multiple thimble integrals in general. For example, we have shown that a sum of the thimble integrals associated with the “perturbative saddle” and one of the “nonperturbative saddles” gives the perturbative Borel resummation along $\mathbb{R}_+$ for $\tilde{m} < m < (2n+1)\pi$, where $\tilde{m}$ is a certain value smaller than $(2n+1)\pi$.

(iv) We have proposed path integral interpretations of the nonperturbative contributions appearing in the transseries. We interpret the nonperturbative effects as the complexified SUSY solutions constructed in [56], up to the three subtleties discussed in Sec. V. The contributions from the complex SUSY solutions should be shown to yield the nonperturbative exponential contributions in the full transseries of the partition function by calculating their one-loop or quasi-zero-mode integrals (thimble integral). We leave this task for future works, which will test our interpretation.

(v) Based on our results, one may expect that, even if a perturbative series of a physical quantity is Borel-summable along $\varphi = \arg(g)$ (e.g. $\varphi = 0$ for real positive $g$) and its resurgent structure is trivial, one could obtain its exact result by including the residues of “some” of perturbative Borel
singularities on the right-half Borel plane which correspond to the nonperturbative contributions. In other words, one may be able to obtain the exact result by deforming a contour in the Borel resummation. However, in general, we cannot know which Borel singularities contribute to the exact result only from the perturbative series and we may need to perform the thimble decomposition of the path integral. In the examples of this paper, we have easily found that the Borel singularities in the fourth quadrant are relevant while those in the first quadrant are not since we have rewritten the integral representation for the exact result directly in terms of the Borel resummation. Note that, even if we do not know this representation of the exact result, the Lefschetz-thimble analysis enables us to derive the correct contour as we have explicitly demonstrated.

We conclude this paper with discussing possible future studies. It is known that, in the Coulomb branch localization formula, picking up poles of the one-loop determinant gives rise to Higgs branch representation of the partition function which includes a product of vortex and anti-vortex partition functions for some theories \cite{81, 85}. Since we know that the poles correspond to the bosonic CSS, it is natural to expect that the CSS are closely related to the Higgs branch representation. It would be illuminating to make this expectation more precise.

It is interesting to see whether the resurgent structures become simplified for higher SUSY theories such as 3d $\mathcal{N} = 4$ CS matter theories. For example, it is known that sphere partition function of the 3d $\mathcal{N} = 6$ $U(2) \times U(2)$ ABJM theory (without mass) has a Borel summable series along $\mathbb{R}_+ \ [52]$. This implies simplifications of the resurgent structure for the theories with higher SUSY.

Finally, we make a comment on the paper \cite{36}, which discusses the resurgent structure for expansions by the geometric parameter $q = e^h$ in 3d $\mathcal{N} = 2$ theories on $D^2 \times_q S^1$. In that analysis, critical points in the localization formula for their partition functions are determined by the twisted effective potentials of 2d $\mathcal{N} = (2, 2)$ theories with infinite KK towers or equivalently so-called Bethe vacua. Although the authors of \cite{36} consider the expansion and the space that differ from ours, we expect that some aspects in their problem are also of importance in our problem since $D^2 \times_q S^1$ is the building block of 3d manifolds including spheres \cite{81, 85}. It would be nice to see connections between the analysis in \cite{36} and ours. Perhaps the detailed analysis of the case for the squashed sphere with $b \to 0$ may shed some lights on this question.
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Appendix A: Supersymmetric actions in 3D $\mathcal{N} = 2$ theory on $S^3$

In this appendix we write down supersymmetric actions in 3D $\mathcal{N} = 2$ theory on $S^3$ known in literature.

1. $\mathcal{N} = 2$ vector multiplet

The 3D $\mathcal{N} = 2$ vector multiplet is dimensional reduction of 4D $\mathcal{N} = 1$ vector multiplet and consists of gauge field $A_\mu$, adjoint scalar $\sigma$, auxiliary field $D$ and gaugino $(\bar{\lambda}, \lambda)$. The 3D $\mathcal{N} = 2$ SYM has the following action

$$S_{\text{YM}} = \frac{1}{g_{\text{YM}}^2} \int_{S^3} d^3x \sqrt{g} \text{Tr} \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \sigma D^\mu \sigma + \frac{1}{2} \left( D + \frac{\sigma}{R_{S^3}} \right)^2 \right] + \frac{i}{2} \bar{\lambda} \gamma^\mu D_\mu \lambda + \frac{i}{2} \bar{\lambda} [\sigma, \lambda] - \frac{1}{4R_{S^3}} \bar{\lambda} \lambda, \quad (A.1)$$

while the SUSY CS term is given by

$$S_{\text{CS}} = \frac{ik}{4\pi} \int_{S^3} d^3x \sqrt{g} \text{Tr} \left[ e^{\mu\nu\rho} \left( A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho \right) - \bar{\lambda} \lambda + 2D\sigma \right], \quad (A.2)$$

If gauge group includes $U(1)$, we can add the FI term

$$S_{\text{FI}} = -\frac{i\zeta}{2\pi R_{S^3}} \int_{S^3} d^3x \sqrt{g} \text{Tr} \left( D - \frac{\sigma}{R_{S^3}} \right). \quad (A.3)$$
2. $\mathcal{N} = 2$ chiral multiplet

The 3D $\mathcal{N} = 2$ chiral multiplet is dimensional reduction of 4D $\mathcal{N} = 1$ chiral multiplet and consists of scalars ($\phi, \bar{\phi}$), auxiliary field ($F, \bar{F}$) and fermions ($\psi, \bar{\psi}$). The SUSY action of the chiral multiplet without superpotential is given by

$$S_{\text{chiral}} = \int_{S^3} d^3x \sqrt{g} \left( D_\mu \bar{\phi} D^\mu \phi + \bar{\phi} \sigma^2 \phi + \frac{i(2\Delta - 1)}{R_{S^3}} \bar{\phi} \sigma \phi + \frac{\Delta(2 - \Delta)}{R^2_{S^3}} \phi \bar{\phi} + i\bar{\phi} D\phi + \bar{F} F \right.$$

$$-i\bar{\psi} \gamma^\mu D_\mu \psi + i\bar{\psi} \sigma \psi - \frac{2\Delta - 1}{2R_{S^3}} \bar{\psi} \psi + i\bar{\psi} \lambda \phi - i\bar{\phi} \lambda \psi \right), \quad (A.4)$$

where $\Delta$ is the $U(1)_R$ charge.

Appendix B: Details on computation of perturbative series

In this appendix we compute the perturbative coefficients in the standard way while we have derived the same results in main text by Taylor expanding the Borel transformations.

1. $\mathcal{N} = 3$ CS SQED

In terms of Euler number and the binomial theorem, we rewrite the hyper multiplet contribution as

$$\frac{1}{2 \cosh \frac{\sigma - m}{2}} = \frac{1}{2} \sum_{q=0}^{\infty} \sum_{a=0}^{2q} \frac{E_{2q}}{2^{2q} \Gamma(2q - a + 1) \Gamma(2a + 1)} \sigma^{2q-a} m^a. \quad (B.1)$$

Then the perturbative part of the partition function for $N_f = 1$ is given by

$$Z_{pt} = \frac{1}{2} \sum_{q=0}^{\infty} \sum_{a=0}^{q} \frac{E_{2q}}{2^{2q} \Gamma(2q - 2a + 1) \Gamma(2a + 1)} m^{2a} \int_{-\infty}^{\infty} d\sigma \ \sigma^{2q-2a} e^{\frac{ig}{2} \sigma^2}. \quad (B.2)$$

Using

$$\int_{-\infty}^{\infty} dx \ x^{2n} e^{\frac{ig}{2} x^2} = \Gamma \left( n + \frac{1}{2} \right) (ig)^{\frac{2n+1}{2}}, \quad (B.3)$$

we find

$$Z_{pt} = \frac{\sqrt{ig}}{2} \sum_{q=0}^{\infty} \sum_{a=0}^{q} \frac{E_{2q} \Gamma \left( q - a + \frac{1}{2} \right)}{2^{2q} \Gamma(2q - 2a + 1) \Gamma(2a + 1)} m^{2a} \ (ig)^{q-a}$$

$$= \frac{\sqrt{ig}}{2} \sum_{q=0}^{\infty} \sum_{a=0}^{\infty} \frac{E_{2q+2a} \Gamma \left( q + \frac{1}{2} \right)}{2^{2q+2a} \Gamma(2q + 1) \Gamma(2a + 1)} m^{2a} (ig)^q. \quad (B.4)$$
For general $N_f$, applying (B.1) to the contribution from each hyper multiplet ($\frac{1}{2 \cosh \frac{x-m_a}{2}}$), we obtain

$$Z_{pt} = \frac{\sqrt{ig}}{2} \sum_{q_a=0}^{\infty} \sum_{l_a=0}^{\infty} \frac{\Gamma(q+1/2)}{2^{2(q+l)}} \left[ \prod_{a=1}^{N_f} \frac{E_{2(q_a+l_a)}}{\Gamma(2q_a+1)\Gamma(2l_a+1)} m_a^{2l_a} \right] (ig)^q,$$  \hspace{1cm} (B.5)

with $q = \sum_{a=1}^{N_f} q_a$ and $l = \sum_{a=1}^{N_f} l_a$.

### 2. $N = 3$ SU(2) CS SQCD

The only difference from SQED is the presence of $(2 \sinh \sigma)^2$. We first consider $N_f = 2$ case. Using Taylor expansion of this factor and applying (B.1) to each \hspace{1cm} \frac{1}{2 \cosh \frac{x-m_a}{2}} \hspace{1cm}, we find

$$Z_{pt} = \frac{\sqrt{ig}}{8} \sum_{s_b=0}^{\infty} \sum_{q_b=0}^{\infty} \sum_{p_b=0}^{\infty} \sum_{l_b=0}^{\infty} \sum_{k_b=0}^{\infty} 2^{-2q_b} \Gamma(Q-L/2+s/2) (\sqrt{i}x)^{Q-L/2+s+1} \delta_{L \mod 2,0}$$

$$\times \prod_{b=1}^{2} \left( \frac{2q_b}{l_b} \right) \left( \frac{2p_b}{k_b} \right) \frac{E_{2q_b} E_{2p_b}}{\Gamma(2s_b+2)\Gamma(2q_b+1)\Gamma(2p_b+1)} m_b^{l_b+k_b}$$

$$= \frac{\sqrt{ig}}{8} \sum_{s_b=0}^{\infty} \sum_{q_b=0}^{\infty} \sum_{p_b=0}^{\infty} \sum_{l_b=0}^{\infty} \sum_{k_b=0}^{\infty} 2^{-2\tilde{Q}} \Gamma(\tilde{Q}-L/2+s+1) (\sqrt{i}x)^{\tilde{Q}-L/2+s+1} \delta_{\tilde{L} \mod 2,0}$$

$$\times \prod_{b=1}^{2} \frac{E_{2q_b} E_{2p_b}}{\Gamma(2s_b+2)\Gamma(2q_b+1)\Gamma(2p_b+1)\Gamma(l_b+1)\Gamma(k_b+1)} m_b^{l_b+k_b},$$  \hspace{1cm} (B.6)

with $\tilde{Q} = q_1 + q_2 + p_1 + p_2$, $\tilde{L} = l_1 + l_2 + k_1 + k_2$, and $\tilde{s} = s_1 + s_2$.

For general $N_f$ it is expressed as

$$Z_{pt} = \frac{\sqrt{ig}}{2^{2N_f-1}} \sum_{s_b=0}^{\infty} \sum_{q_b=0}^{\infty} \sum_{p_b=0}^{\infty} \sum_{l_b=0}^{\infty} \sum_{k_b=0}^{\infty} 2^{-2\tilde{Q}} \Gamma(\tilde{Q}-L/2+s+1) (\sqrt{i}x)^{\tilde{Q}-L/2+s+1} \delta_{\tilde{L} \mod 2,0}$$

$$\times \prod_{b=1}^{N_f} \frac{1}{\Gamma(2s_b+2)} \prod_{b=1}^{N_f} \frac{E_{2q_b} E_{2p_b}}{\Gamma(2q_b-l_b+1)\Gamma(2p_b-k_b+1)\Gamma(l_b+1)\Gamma(k_b+1)} m_b^{l_b+k_b},$$  \hspace{1cm} (B.7)

with $\tilde{Q} = \sum_{b=1}^{N_f} (q_b + p_b)$, $\tilde{L} = \sum_{b=1}^{N_f} (l_b + k_b)$, and $\tilde{s} = \sum_{b=1}^{2} s_b$. 
Appendix C: Brief review of the thimble analysis

In the thimble analysis, we firstly extend the real variable \( x \in \mathbb{R} \) to the complex one \( z \in \mathbb{C} \), and then, we obtain the steepest descent given by

\[
\frac{dz(s)}{ds} = F(z(s)), \quad F(z) = \frac{\partial S[z]}{\partial z},
\]

where \( s \) is a real flow parameter and \( \overline{F} \) denotes the complex conjugation of \( F \). The critical points \( z^c_{\sigma} \) are obtained by solving \( F(z^c_{\sigma}) = 0 \). The thimble \( J_{\sigma} \) associated with the critical point \( \sigma \) is determined as a particular flow with the initial condition given by

\[
\lim_{s \to -\infty} z_{\sigma}(s) = z^c_{\sigma},
\]

whereas the dual thimble \( K_{\sigma} \) is defined by a flow with the condition \( \lim_{s \to +\infty} z_{\sigma}(s) = z^c_{\sigma} \). One can easily find that

\[
\frac{d\text{Re}S[z(s)]}{ds} \geq 0 \quad \text{and} \quad \frac{d\text{Im}S[z(s)]}{ds} = 0.
\]

The original integration contour \( C_{\mathbb{R}} \) can be reproduced by a linear combination of the thimbles as

\[
C_{\mathbb{R}} = \sum_{\sigma \in \Sigma} n_{\sigma} J_{\sigma},
\]

where \( \Sigma \) is a set of the critical points and \( n_{\sigma} \) is an integer, called the intersection number. The intersection number is determined by each of dual-thimbles \( K_{\sigma} \) so as to have the same homology class as the real contour: \( n_{\sigma} = \pm 1 \) if the dual-thimble has an intersection with the real contour, \( n_{\sigma} = 0 \) otherwise.

By choosing particular values of parameters, one might encounter the Stokes phenomenon, which is defined as

\[
\lim_{s \to +\infty} z_{\sigma_1}(s) = z^c_{\sigma_2}, \quad (\sigma_1 \neq \sigma_2),
\]

and (C.4) becomes ill-defined. Even if the Stokes phenomenon occurs, one can avoid the phenomenon by introducing a sufficiently small complex phase to a parameter. This fact implies that there is an ambiguity regarding choices of the modified contours to avoid the Stokes phenomenon.

The complexified configuration space generally has not only critical points but also other objects such as singularities(sources) and zero-points(sinks) defined as

\[
\text{Re}S[z] = \begin{cases} 
-\infty & \text{for singularities} \\
+\infty & \text{for zero-points}
\end{cases}
\]

(C.6)
These points have the role of end-points of the thimbles.

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See also reviews on math side [32] and physics side [33].

This does not necessarily mean that closed expressions for the exact results are explicitly known. For example, localization method typically provides finite dimensional integral representations for the exact results but we likely do not know how to perform the integrals analytically for gauge theories with multiple finite ranks. In this situation, we know the mathematically well-defined descriptions for the exact results but do not know their final closed expressions.

If we count so-called Cheshire cat resurgence [23], then the $S^2$ partition function of the 2d $\mathcal{N} = (2, 2)$ $\mathbb{CP}^N$ model (and correlators generated by it) also provides the example [51], where an expansion parameter is inverse of FI-parameter.
When we have IR renormalons, this would not be true.

Partition function on $S^3$ (more generally odd-dimensional sphere) is physical in the following sense: First, there is no IR divergence since sphere is compact. Second, $\log|Z|$ has power-law UV divergence but does not have log-divergence in odd dimensions. Therefore $O(1)$ part of $\log|Z|$ cannot be changed by counter terms and is physical, though there are counter terms to shift phase of $Z$ to some extent.

A real mass is given by a constant background of the flavor vector multiplet. More precisely a class of theory considered in [56] is theory with well-defined sphere partition functions though ill-defined cases is also interesting [72].

Decomposition of localization formula by Lefschetz thimble has been considered in [86] in the context of Witten index of SUSY QM.

By “$G_k$ CS theory”, we mean CS theory with gauge group $G$ and CS level $k$.

The adjoint chiral multiplet with $U(1)_R$ charge 1 is technically irrelevant because this contributes to the integrand by 1.

In 3d $\mathcal{N} = 2$ language, we have $U(N_f) \times U(N_f)$ flavor symmetry whose first one rotates the charge +1 chiral with $N_f$ representation while second one rotates the charge $-1$ chirals with $\bar{N}_f$ representation. If we denote real masses associated with these two $U(N_f)$ symmetries by $m^f_a$ and $\bar{m}^f_a$ respectively, then we are taking $m_a = m^f_a = \bar{m}^f_a$ which corresponds to so-called vector real mass. Since we are considering $U(1)$ gauge theory, the diagonal part of the vector real mass $m$ is absorbed by shifting $\sigma \to \sigma + m$ but this absorption gives FI-term with the coefficient $km/2\pi$ and $U(1)$-flavor CS term with level $k$ because of the $U(1)$ gauge CS term. Thus one can also say that this setup is $U(1)_k$ CS theory with FI-parameter and vector real masses associated with $SU(N_f)$ flavor symmetry.

We are assuming $\arg(g) = 0$.

Although the results could include small numerical errors, the main arguments in the following are not affected by the details.

More precisely, except $k \in \mathbb{Z}$.

We assume that $m > 0$ as with the case $\arg(g) = 0$.

In 3D $\mathcal{N} = 2$ language, this theory consists of $SU(2)_k$ vector multiplet, adjoint chiral multiplet with $U(1)_R$-charge 1 and $N_f$ pairs of fundamental chiral multiplets with $U(1)_R$-charge 1/2 and the real
masses.

[110] In 3d $\mathcal{N} = 2$ language, we have $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_A$ global symmetry where $U(1)_B$ is the baryon symmetry. The diagonal part of $m_a$ corresponds to the real mass associated with $U(1)_B$ while we are turning off the one associated with $U(1)_A$.

[111] The fact that the number of the saddles around $z = 0$ is two can be analytically checked by considering the expansion $z^c = 0 + g^{1/2} z^{c_1} + \mathcal{O}(g^{3/2})$ in (III.15), which leads us to $z^{1,c} = \pm e^{\frac{\pi i}{4}}$.

[112] We have rescaled $\sigma$ as $\sigma \to 2\pi \sigma$ as well as $m_a$.

[113] If $b$ is complex, this condition becomes $m_a - \frac{n_1 + \Delta}{2} \text{Im} b - \frac{n_1 + \Delta}{2} \text{Im} b^{-1} = \frac{n_1 + \Delta}{2} \text{Re} b + \frac{n_1 + \Delta}{2} \text{Re} b^{-1}$.

[114] We take a normalization such that $c_T = 1$ for single free real scalar.