Loops for Hot QCD

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In this talk we review the status concerning vacuum integral s needed in perturbative expansions of QCD at non-zero temperature. We will focus on the differences as compared to familiar zero-temperature techniques, and provide a list of known basic master integrals.

1. Introduction

Weak-coupling expansions in finite temperature field theory can be organized in a way that is very similar to the better-known zero-temperature case. In particular, for observables describing the physics of thermally equilibrated systems, the so-called imaginary time formalism provides a very close analogy.

For a given system that is coupled to an external heat-bath with temperature $T$, the relevant observables have to be averaged with the density matrix, $\langle O \rangle = \text{Tr}(O e^{-H/T}) / \text{Tr}(e^{-H/T})$, where $H$ denotes the Hamiltonian of the system. In the case of interacting quantum fields $\phi$ in thermal equilibrium, one can lift this to a functional integral representation of the averages,

$$\langle O[\phi] \rangle = \frac{\int D\phi O[\phi] e^{-\int \frac{1}{T} d\tau f d^3x L_E[\phi]}}{\int D\phi e^{-\int \frac{1}{T} d\tau f d^3x L_E[\phi]}} ,$$

with Euclidean action and where the trace is enforced by constraining the path integral to fields that are periodic in imaginary time $\tau$:

$$\phi(\tau + 1/T, \vec{x}) = \pm \phi(\tau, \vec{x}) ,$$

where the upper/lower sign refers to bosonic/fermionic fields.

Finite-temperature Feynman rules hence differ from zero-temperature ones by that fact, due to the compact support in imaginary time $\tau$, the integration measure includes a discrete sum

$$\int \frac{dp_0}{2\pi} \rightarrow T \sum_{p_0}$$

over the Matsubara frequencies $p_0 = 2\pi n T$ for bosons and $p_0 = 2\pi T(n + \frac{1}{2})$ for fermions, respectively, where $n \in \mathbb{Z}$. As the presence of a preferred direction – the rest frame of the heat-bath – has broken the 4d Lorentz symmetry, one works with four-momenta $P = (p_0, \vec{p})$ and uses Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$, such that $P^2 = p_0^2 + \vec{p}^2$.

In this short note, we will assume the reader be familiar with standard methods of zero-temperature perturbation theory, such as integration by parts (IBP) \cite{1} and its systematic algorithmic use \cite{2}, and then point out the main differences that occur in a thermal system. For illustration purposes, we will concentrate on the most basic class of sum-integrals, vacuum (or bubble) integrals, which are needed to compute e.g. thermodynamic observables that directly relate to vacuum diagrams, like the pressure (see \cite{3} and references therein). These vacuum integrals are at the same time needed as building blocks for higher-point Greens functions, after expanding in external momenta, see e.g. \cite{5}. A collection of analytically known integrals that are relevant in these contexts will be presented.

2. Bosonic master integrals

We work in dimensional regularization, and for convenience take the integral measure to be

$$\sum_P \equiv T \sum_{n=-\infty}^{\infty} (4\pi T^2)^n \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}} .$$

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2.1. One-loop bosonic integrals

The basic bosonic 1-loop tadpole integral is

\[ \mathcal{I}_n^m = \oint \frac{(p_0)^m}{P^n} \]

\[ = \frac{2\pi^2 T^4}{(2\pi T)^{2n-m}} \frac{4^\epsilon \Gamma(n-\frac{d}{2}+\epsilon)}{\Gamma(\frac{d}{2}) \Gamma(n)} \times \]

\[ \times \zeta(2n-m-3+2\epsilon) \, . \tag{4} \]

It obeys the recursion relation \( \mathcal{I}_{n+2}^{m+2} = \frac{2n-3+2\epsilon}{2n} \mathcal{I}_n^m \), as can either be derived from integration by parts (IBP) in the integral over 3-momentum, or confirmed directly from the above explicit solution. Note that, in contrast to the zero-temperature case, already at 1-loop level one finds in principle infinitely many master integrals. In practice, however, only a small number of them appear in any practical computation.

1-loop master integrals that are needed for the pressure are

\[ \mathcal{I}_1^0 = \frac{T^2}{4\pi^2} \frac{4^\epsilon \Gamma(\frac{d}{2}+\epsilon)}{\Gamma(\frac{d}{2})} \zeta(-1+2\epsilon) \frac{1}{1-2\epsilon} \]

\[ \approx \frac{T^2}{2} \left[ 1 + \epsilon(2 - \gamma_E + 2Z_1 + \ldots) \right] , \]

\[ \mathcal{I}_2^0 = \frac{1}{(4\pi^2)^2} \frac{4^\epsilon \Gamma(\frac{d}{2}+\epsilon)}{\Gamma(\frac{d}{2})} 2\epsilon \zeta(1+2\epsilon) \]

\[ \approx \frac{1}{(4\pi^2)^2} \frac{1}{\epsilon} \left[ 1 + \epsilon \gamma_E + \ldots \right] , \]

\[ \mathcal{I}_3^0 = \frac{2\zeta(3)}{(4\pi^2)^2} \frac{4^\epsilon \Gamma(\frac{d}{2}+\epsilon)}{\Gamma(\frac{d}{2})} \frac{\zeta(3+2\epsilon)}{\zeta(3)} (1+2\epsilon) \]

\[ \approx \frac{2\zeta(3)}{(4\pi^2)^2} \left[ 1 + \epsilon \left( 2 - \gamma_E + 2 \zeta(3) + \ldots \right) \right] , \]

where \( Z_1 = \frac{\zeta'(-1)}{\zeta(-1)} \).

2.2. Two-loop bosonic integrals

At 2-loop order, there is a major simplification: the basic sunset-type master vacuum integral (cf. Fig. 2) vanishes identically,

\[ S = \oint P Q R \frac{1}{(P^2)} = 0 , \tag{5} \]

as can be proven by IBP. In fact, using that the integral of a total derivative vanishes in dimensional regularization,

\[ 0 = \oint P Q R \frac{1}{(P^2)} \partial_P f_1 \frac{1}{(P^2)(P^2)} , \tag{6} \]

and choosing (here, \( d = 4 - 2\epsilon \) denotes the spacetime dimension)

\[ f_1 = (d-3)(p_i + q_i) + \frac{2}{Q^2} (p_0 + q_0)(p_0 q_i - q_0 p_i) , \tag{7} \]

after working out the derivatives, using the shift \( Q \rightarrow P - Q \), and exploiting symmetry of the sum-integral under \( P \leftrightarrow Q \), one gets

\[ 0 = (d-3)(d-4) S , \]

which completes the proof of Eq. (5).

2.3. Three-loop bosonic integrals

At the 3-loop level, a number of master integrals have been encountered in the computation of the pressure. For a classification, let us adopt a naming scheme advocated in [3],

\[ \mathcal{M}_{ij} = \oint P Q R \frac{1}{(P^2)^i Q^2 R^2} \times \frac{1}{(P - Q)^2 (Q - R)^2 (R - P)^2} . \tag{8} \]

Note that \( \mathcal{M} \) is symmetric in its two indices. The known cases (cf. Fig. 4) include a basketball-type and a spectacles-type integral

\[ \mathcal{M}_{100} \approx \frac{T^4}{(4\pi^2)^2} \frac{1}{24 \epsilon} \left[ 1 + \epsilon b_{11} + \epsilon^2 b_{12} + \ldots \right] . \tag{9} \]
$M_{-22} \approx \frac{T^4}{(4\pi)^4} \frac{11}{216} [1 + \epsilon m_{11} + \ldots]$ , \hspace{1cm} (10)

as well as the modified basketball-type integral that is needed at higher orders \cite{4},

$M_{00} = \sum_{PQR} \frac{1}{(Q^2)^2} \frac{1}{R^2 (P - Q)^2 (R - P)^2}$

$\approx \frac{T^4}{(4\pi)^4} \frac{1}{8 \epsilon^2} [1 + \epsilon b_{21} + \epsilon^2 b_{22} + \ldots]$ . \hspace{1cm} (11)

The expansion coefficients read \cite{6,4}

$b_{11} = \frac{\gamma}{2} - 3 \gamma_e + 8 Z_1 - 2 Z_3$ , \hspace{1cm} (12)

$b_{21} = \frac{\gamma}{2} + \gamma_e + 2 Z_1$ , \hspace{1cm} (13)

$b_{22} = 48.796$.. , \hspace{1cm} (14)

$m_{11} = \frac{\gamma}{2} - \frac{\gamma}{2} \gamma_e + \frac{\gamma}{2} Z_1 - \frac{\gamma}{2} Z_3$ , \hspace{1cm} (15)

while $b_{12}$ has not been computed yet.

Other integrals of the class Eq. (8) can be reduced by standard methods, like shifts of integration momenta exploiting the symmetry of the integrand, such as

$M_{-11} = -\frac{1}{2} M_{00} + 2 T_4^0 S$ , \hspace{1cm} (16)

or, in a graphical representation

\[ \begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{diagram.png}} \\
\text{where the notation is like in Fig. 1}
\end{array} \]

2.4. Four-loop bosonic integrals

At the 4-loop level, the classification and evaluation of sum-integrals has not been tackled in a systematic way yet. There is however some pioneering work exploring scalar theory at this order \cite{1}. The major simplification (as compared to QCD) is that the reduction step is trivial, and there is only one genuine 4-loop sum-integral to compute. Hence, the only 4-loop master integral that is known presently is (cf. Fig. 1)

$T = \sum_{PQRS} \frac{1}{P^2 (P + S)^2 (Q + S)^2 R^2 (R + S)^2}$

$\approx \frac{T^4}{(4\pi)^4} \frac{1}{16 \epsilon^2} [1 + \epsilon t_{11} + \epsilon^2 t_{12} + \ldots]$ , \hspace{1cm} (17)

with coefficients

$t_{11} = \frac{44}{3} - 4 \gamma_e + 12 Z_1 - 4 Z_3 - 3 \zeta(2)$ , \hspace{1cm} (18)

$t_{12} = 2 b_{12} + 25.7055.. - 3 \zeta(2)(28.9250..)$ . \hspace{1cm} (19)

These coefficients have been calculated in \cite{4}, profiting from the particular structure of the integral Eq. (17), which contains three insertions of 1-loop bubble-type integrals, and can hence be tackled in complete analogy to the one in Eq. (9), as pioneered in \cite{6}. The numerical values given above result from simple numerical integrations and are known to more than 10 digit accuracy.

Note that in the physics calculation where this integral was needed, it occurred in combination with the one in Eq. (9) such that the number $b_{12}$ cancels in the final result.

3. Fermionic master integrals

Let us denote fermionic four-momenta by braces, $\{P\}$, to indicate that their $p_0 = 2\pi T(n + \frac{1}{2})$ with $n \in \mathbb{Z}$.

There is an important class of relations between fermionic and bosonic sum-integrals, which can be derived by partitioning the Matsubara sums

$\sum_{n \in \mathbb{Z}} \sum_{n \text{ even}} + \sum_{n \text{ odd}}$ \hspace{1cm} (20)

and then rescaling spatial integration momenta on the left-hand-side as $p_i \rightarrow \frac{1}{2} p_i$ (see, e.g., \cite{7}).

3.1. One- and two-loop fermionic integrals

The relations obtained as just described reduce all fermionic 1- and 2-loop tadpoles to the respective bosonic case,

$\tilde{T}_n^m = \sum_{\{P\}} \frac{(p_0)^m}{(P^2)^n}$

$= (2^{2n-m-3+2\epsilon} - 1) T_n^m$ , \hspace{1cm} (21)

$\tilde{S} = \sum_{\{P\}Q} \frac{1}{P^2 Q^2 (P - Q)^2}$

$= \frac{1}{3} (2^{4\epsilon} - 1) S = 0$ , \hspace{1cm} (22)

where in the last step Eq. (5) was used.

3.2. Three-loop fermionic integrals

To classify the master integrals at the 3-loop level, let us again adopt the notation of \cite{8}.

$N_{ij} = \frac{1}{(P - Q)^2 ((Q - R)^2)^r (R - P)^2}$ , \hspace{1cm} (23)
the basis of master integrals, such that it suffices to include one of them into the basis of master integrals, 

\[ \tilde{\mathcal{M}}_{ij} \approx \frac{T^4}{(4\pi)^2} \left( \frac{1}{108 \epsilon} \right) [1 + \epsilon b_{31} + ...] \]  

and 

\[ \tilde{\mathcal{N}}_{-22} \approx \frac{T^4}{(4\pi)^2} \left( \frac{1}{1728 \epsilon} \right) [1 + \epsilon n_{21} + ...] \]  

\[ \tilde{\mathcal{M}}_{1-1} \approx \frac{T^4}{(4\pi)^2} \left( \frac{1}{192 \epsilon} \right) [1 + \epsilon m_{21} + ...] \]  

\[ \tilde{\mathcal{M}}_{-22} \approx \frac{T^4}{(4\pi)^2} \left( \frac{1}{1728 \epsilon} \right) [1 + \epsilon m_{31} + ...] \] 

where \[ \text{Eq. (25)} \] 

Note that, while \( \mathcal{N} \) is still symmetric in its two indices, \( \mathcal{M} \) is not.

It turns out that, by the same partitioning trick as above, the two possible fermionic 3-loop basketball-type vacuum integrals are related, such that it suffices to include one of them into the basis of master integrals, 

\[ \tilde{\mathcal{M}}_{00} = \frac{1}{6} (2^{9\epsilon - 1} - 1) \mathcal{M}_{00} - \frac{1}{6} \mathcal{N}_{00} \]  

\[ \mathcal{N}_{00} \approx \frac{T^4}{(4\pi)^2} \left( \frac{1}{108 \epsilon} \right) [1 + \epsilon b_{31} + ...] \]  

where \[ \text{Eq. (26)} \] 

Further 3-loop masters (see also Fig. 2) are 

\[ \tilde{\mathcal{N}}_{-22} \approx \frac{T^4}{(4\pi)^2} \left( \frac{1}{108 \epsilon} \right) [1 + \epsilon n_{21} + ...] \]  

\[ \tilde{\mathcal{M}}_{1-1} \approx \frac{T^4}{(4\pi)^2} \left( \frac{1}{192 \epsilon} \right) [1 + \epsilon m_{21} + ...] \]  

\[ \tilde{\mathcal{M}}_{-22} \approx \frac{T^4}{(4\pi)^2} \left( \frac{1}{1728 \epsilon} \right) [1 + \epsilon m_{31} + ...] \] 

with \[ \text{Eq. (29)} \] 

\[ n_{21} = \frac{3}{8} \gamma_E - \frac{\pi^2}{60} \ln 2 + 5 Z_1 - \frac{1}{2} Z_3 \]  

\[ m_{21} = \frac{36}{60} \gamma_E + \frac{\pi^2}{72} \ln 2 - 4 Z_1 + 4 Z_3 \]  

\[ m_{31} = \frac{36}{90} - \frac{36}{90} \gamma_E - \frac{9}{20} \ln 2 + \frac{136}{90} Z_1 - \frac{19}{90} Z_3 \]

In close analogy to the bosonic case Eq. (10), others get reduced by applying a shift of integration variables and exploiting the integral’s symmetry, 

\[ \mathcal{N}_{-11} = \frac{1}{2} \mathcal{N}_{00} + 2 \tilde{T}_1^0 \tilde{S} \]  

\[ \tilde{\mathcal{M}}_{-11} = \frac{1}{2} \tilde{\mathcal{M}}_{00} + \tilde{T}_1^0 \tilde{S} + \tilde{T}_1^0 \mathcal{S} \]  

3.3. Four-loop fermionic integrals

Not a single genuine fermionic 4-loop sum-integral has been computed to date. However, at this level, a theoretically important contribution is expected to be made for the QCD pressure [10], adding the last missing perturbative piece of a physically complete leading-order description, which includes (known) non-perturbative coefficients. Since, naturally, an analytic treatment of diagrams containing quark loops seems to be a good starting point (due to the simpler structure of fermionic vertices and propagators), progress with this type of sum-integrals can be expected in the near future, before the gluonic sector (cf. Sec. 2.4) will be classified.

To obtain a first useful relation, one can apply the strategy of Eq. (23) to the bosonic integral of Eq. (17), to get a linear relation between the four distinct fermionic integrals of the same topology, as depicted in Fig. 3

4. Chemical potential

In systems which allow for an excess of, say, particles over anti-particles, one introduces further parameters – the chemical potentials \( \mu \) – as...
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Lagrange multipliers of the particle number operators into the Lagrangian. The effect is to change four-momentum to be used in the free propagators of the respective particle species as

$$p_0 \rightarrow p_0 - i\mu,$$ \hspace{1cm} (36)

such that all sum-integrals pick up a dependence on the parameter(s) \( \mu \). In the case of QCD, to accommodate for a non-zero baryon number density, chemical potentials have to be introduced for all flavors of quarks.

In fact, contributions to the QCD pressure have been calculated including the effects of a quark chemical potential, requiring a generalization of the sum-integrals discussed in Sec. 3 to include the \( \mu \)-dependence. This was done through 3-loop order in [10], where all required expansions are shown. While for the 1-loop fermionic tadpole an analytic result can simply be given in terms of the generalized Zeta function \( \zeta(x, z) \) (cf. Eq. (10) below), the 3-loop generalizations \( \tilde{N}_{ij}(\mu) \) and \( \tilde{M}_{ij}(\mu) \) are somewhat more complicated to evaluate, and their expansions turn out to contain derivatives \( \partial_{\mu} \zeta(x, z) \) [10].

One notable structural change as compared to the case \( \mu = 0 \) is that certain relations that relied on symmetries of the integrand no longer hold true. In particular, there are now non-vanishing \( \mu \)-dependent 2-loop sum-integrals of sunset-type, such as (cf. Eq. (22))

$$\tilde{S}(\mu) \approx -\frac{\mu^2}{(4\pi)^4} \frac{4}{\epsilon} [1 + \epsilon s_{11}(\mu) + ...],$$ \hspace{1cm} (37)

with [10, 11]

$$s_{11}(\mu) = 2 - 2\gamma_\text{E} + \frac{4}{1 - 2z} [\zeta'(0, z) - \zeta'(0, 1 - z)] = 2 - 2\gamma_\text{E} + \frac{4}{2z - 1} \ln \frac{\Gamma(1 - z)}{\Gamma(z)},$$ \hspace{1cm} (38)

where \( z = \frac{1}{2} - i\mu/(4\pi T) \) and \( \zeta'(x, z) \equiv \partial_x \zeta(x, z) \). Note that the result Eq. (37) is compatible with Eq. (22) in the limit \( \mu \rightarrow 0 \), as it has to be.

We will not list the further known integrals here, but finally briefly discuss a different setting, in which related sum-integrals appear.

5. \( q \)-integrals

In some cases, one might be interested in integrals which (for \( \mu = 0 \)) interpolate smoothly between the bosonic and fermionic cases, using

$$p_0 = 2\pi T(n + q), \quad n \in \mathbb{Z} \quad \text{and} \quad q \in \mathbb{R}. \quad (39)$$

Below, four-momenta with such \( q \)-dependent components will be denoted by brackets, \([P] \). Hence, all sum-integrals pick up a dependence on the parameter \( q \), and, since the integers \( n \) are summed, are periodic with respect to a unit change in \( q \), such that it is sufficient to determine them on the interval \( q \in [0, 1) \). Note that \( q = 0 \) corresponds to the bosonic integrals discussed in Sec. 2 while \( q = \frac{1}{2} \) corresponds to the fermionic ones discussed in Sec. 3.

Notably, the need for such \( q \)-dependent sum-integrals arises in attempts to analyze the behavior of the so-called spatial ‘t Hooft loop [12], which can be taken as an order parameter for the deconfinement phase transition in pure Yang-Mills gauge theory, and requires the minimization of a certain effective action with respect to the parameter \( q \) [13].

To show a concrete example, the basic bosonic 1-loop tadpole can now be represented as

$$I_n^m(q) = \sum_{[P]} \frac{(p_0)^m}{(P^2)^n} = \frac{\pi^2 T^4}{(2\pi T)^{2n-m}} \frac{4^n (n - \frac{1}{2} + \epsilon)}{\Gamma(\frac{1}{2})^n \Gamma(n)} \times \left[ \left(\frac{1}{\Gamma(2n-m)} \right)^m \right] \left[ (2n-m-3+2\epsilon, 1-q) + \zeta(2n-m-3+2\epsilon, q) \right],$$ \hspace{1cm} (40)

where \( \zeta(x, q) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+q)^x} \) is the generalized Zeta function, with \( \zeta(x, 1) = \zeta(x) \) and \( \zeta(x, \frac{1}{2}) = (2^x - 1)\zeta(x) \) relating \( I_n^m(q) \) to Eq. (4) and Eq. (24).

We will not discuss higher loop orders here, although some results are also available in the literature [13], while others (notably up to the 3-loop level) can in principle be deduced from [10, 11] by analytically continuing the \( q \)-dependence to complex values \( q \rightarrow z = \frac{1}{2} - i\mu/(2\pi T) \), such as the \( q \)-dependent 2-loop sunset-type sum-integral

$$S(q) = \frac{T^2}{(4\pi)^2} \frac{(2q - 1)^2}{4\epsilon} [1 + \epsilon s_{11}(q) + ...],$$
\[ s_{11}(q) = 2 - 2 \gamma_E + \frac{4}{2q - 1} \ln \frac{\Gamma(1 - q)}{\Gamma(q)}, \]

as obtained by explicit calculation, or by analytic continuation from Eq. (37). It vanishes as \( q \to \frac{1}{2} \), as it should (cf. Eq. (23)).

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