The fixed point iteration of positive concave mappings converges geometrically if a fixed point exists

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Abstract—We prove that the fixed point iteration of arbitrary positive concave mappings with nonempty fixed point set converges geometrically for any starting point. We also show that positivity is crucial for this result to hold, and the concept of (nonlinear) spectral radius of asymptotic mappings provides us with information about the convergence factor. As a practical implication of the results shown here, we rigorously explain why some power control and load estimation algorithms in wireless networks, which are particular instances of the fixed point iteration, have empirically shown good convergence speed, even though the algorithms are derived by considering a more general class of mappings (namely, standard interference mappings) for which its usage with the fixed point iteration can result in sublinear convergence.

Index Terms—Positive concave mappings, geometric convergence, fixed point analysis, nonlinear Perron-Frobenius theory.

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I. INTRODUCTION

Many problems in economy [1], wireless communications [2]–[10], and machine learning [11], to cite a few fields, use the so-called standard interference mappings [2] in their formulations, and we note that these mappings are also known as (positive) order-preserving (or monotone) strictly subhomogeneous mappings in the mathematical literature [12]–[14]. If we restrict the attention to the wireless domain, problems such as load estimation [3], [4], [10], [15]–[17], power control [5]–[7], [9], and signal-to-interference-noise ratio (SINR) feasibility studies [8], [18] often involve standard interference mappings. From a mathematical perspective, these problems have in common that the objective is to answer whether a standard interference mapping has a fixed point, and, if so, whether the fixed point, which is unique if it exists [2], can be obtained with simple algorithms.

Regarding the first question, the existence of a fixed point, recently the study in [4] has used the concept of spectral radius of asymptotic mappings to obtain a necessary and sufficient condition for an arbitrary (possibly nonlinear) standard interference mapping to have a fixed point (see Fact 4). This condition is often easy to verify in practice, and it generalizes and unifies many mathematical tools for feasibility studies in the wireless literature. If a standard interference mapping has a fixed point, then its usage with the fixed point iteration produces a sequence that converges to the fixed point of the mapping [2], thus we have an answer to the second question posed above; namely, how to obtain the fixed point.

Given the simplicity and widespread use of the fixed point iteration of standard interference mappings in the wireless domain, understanding and improving its convergence speed have been the focus of many studies [3], [19]–[21] [5, Ch. 5.3]. From a theoretical perspective, an important but somewhat disappointing result in this direction has been established in [20, Example 2], where the authors have shown that the fixed point iteration can converge sublinearly. However, this bad performance does not usually manifest in practice. A possible explanation is that very frequently the standard interference mappings have additional structure that is not explored to prove convergence of the fixed point iteration. In particular, positive concave mappings are common in applications [2]–[6], [9], [10], [21], and we recall that these mappings are a subclass of standard interference mappings [3, Proposition 1]. Nevertheless, to the best our knowledge, to date it is unknown whether this structure is sufficient to guarantee geometric convergence except in special cases.

Geometric convergence has been established if the fixed point iteration uses a concave mapping constructed by taking the coordinate-wise minimum of finitely many positive affine mappings [21], provided that a fixed point exists. Nevertheless, extending the analysis in that study to arbitrary positive concave mappings does not seem trivial, and we emphasize that concave mappings without the particular structure considered in [21] are very common in the wireless domain [3], [4], [9], [15]–[17], [22].

If a positive concave mapping has a fixed point, the study in [19] has shown that, under a mild assumption, the convergence of the fixed point iteration cannot be better than geometric, and, for specific starting points, the convergence factor is bounded by the spectral radius of the asymptotic mapping associated with the concave mapping. Nonetheless, those results do not rule out the possibility of sublinear convergence. In this study, we give a complete answer to this question. In more
detail, our main contributions can be summarized as follows:

1) We prove that the fixed point iteration of an arbitrary (continuous) positive concave mapping with an arbitrary starting point is guaranteed to converge geometrically if the mapping has a fixed point (Proposition 3). We also construct an example illustrating that this property does not necessarily hold if we drop the assumption of positivity (Example 3).

2) We establish useful connections between the convergence factor of the fixed point iteration in Thompson’s metric space and normed vector spaces (Proposition 2).

3) We show that the concept of (nonlinear) spectral radius of the asymptotic mapping associated with a continuous positive concave mapping provides us with useful information about the convergence factor for any starting point of the fixed point iteration (Proposition 4).

4) We illustrate the main implications of the results derived here in the problem of load estimation in wireless networks (Sect. IV).

To keep this study self-contained, we list in Appendix A technical definitions and facts that are required in proofs but not for understanding the main contributions of this study. Existing results that are crucial for setting the stage for the main contributions are listed in the next section.

II. Preliminaries

We introduce the following necessary notation on the nonnegative cone

$$\mathbb{R}^k_+ = \{ (x_1, \ldots, x_k) \in \mathbb{R}^k : x_i \geq 0 \text{ for } 1 \leq i \leq k \}.$$ 

Let $x, y \in \mathbb{R}^k_+$; then, $x \leq y$ denotes the partial ordering induced by the nonnegative cone, i.e., $x \leq y \iff y - x \in \mathbb{R}^k_+$.

Similarly, $x < y$ means that $y - x \in \mathbb{R}^k_+$ and $x \neq y$, while $x \ll y$ means that $y - x \in \mathbb{R}^k_+$, where int$(\mathbb{R}^k_+)$ denotes the interior of the nonnegative cone $\mathbb{R}^k_+$. Let $x \in \mathbb{R}^k_+$. We call $U \subseteq \mathbb{R}^k_+$ a neighbourhood of $x$ if there exists an open set $V \subseteq \mathbb{R}^k_+$ such that $x \in V \subseteq U$. We denote by $x[i] \in \mathbb{R}$ for $i \in \{1,2,\ldots, k\}$ the $i$-th coefficient of a vector $x \in \mathbb{R}^k$, and by $[X]_{i,j} \in \mathbb{R}$ the element of a matrix $X \in \mathbb{R}^{m \times n}$ at the $i$-th row and $j$-th column with $i \in \{1,2,\ldots, m\}$ and $j \in \{1,2,\ldots, n\}$.

Given a mapping $f : X \to Y$ for two sets $X$ and $Y$ satisfying $Y \subseteq \mathbb{R}^k_+$, we denote by

$$\text{Fix}(f) = \{ x \in X \mid f(x) = x \}$$

the set of fixed points of $f$. Moreover, for a metric space $(X,d_1)$, we note that a sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ converges to $x \in X$ if $\forall \epsilon > 0 \exists n_0 \in \mathbb{N}$ such that $\forall n > n_0$ one has that $d_1(x_n,x) < \epsilon$. We also note that a norm $\| \cdot \|$ of a normed vector space $(\mathbb{R}^k_+, \| \cdot \|)$ induces a metric on $X$ by $d_2(x,y) \overset{df}{=} \| x - y \|$ for $x, y \in X$. In such a case, we say that $d_2$ is induced by a norm $\| \cdot \|$.

We now proceed to list definitions and facts that are crucial for understanding the main results in this study.

**Definition 1.** A mapping $f : \mathbb{R}^k_+ \to \mathbb{R}^k_+$ is said to be a standard interference mapping (SI mapping) if it is

1) monotonic

$$\forall \ x, y \in \mathbb{R}^k_+ \quad x \leq y \implies f(x) \leq f(y), \text{ and }$$

2) scalable

$$\forall \ x \in \mathbb{R}^k_+ \quad \forall \lambda > 1 \quad f(\lambda x) \ll \lambda f(x).$$

**Fact 1** ([23]). Let $f : \mathbb{R}_+^k \to \mathbb{R}_+^k$ be an SI mapping. Then $f(\mathbb{R}_+^k) \subseteq \text{int}(\mathbb{R}_+^k)$.

**Definition 2.** Let $f : \mathbb{R}_+^k \to \text{int}(\mathbb{R}_+^k)$ be concave w.r.t. cone order:

$$\forall \ x, y \in \mathbb{R}_+^k \quad \forall t \in (0,1) \quad f(tx + (1-t)y) \geq tf(x) + (1-t)f(y).$$

Then $f$ is called a positive concave (PC) mapping.

We note the following fact on SI mappings.

**Fact 2** ([3], [4]). Let $f : \mathbb{R}_+^k \to \text{int}(\mathbb{R}_+^k)$ be a PC mapping. Then $f$ is an SI mapping.

**Definition 3.** We define Thompson’s metric by

$$d_T : \text{int}(\mathbb{R}_+^k) \times \text{int}(\mathbb{R}_+^k) \to [0, \infty)$$

$$(x,y) \mapsto \ln(\max\{M(x,y), M(y,x)\}),$$

where $M(x,y) \overset{df}{=} \inf\{\beta > 0 \mid x \leq \beta y\}$ and analogously for $M(y,x)$.

**Remark 1.** We note that the metric space $(\text{int}(\mathbb{R}_+^k), d_T)$ is isometric (and, hence, also homeomorphic) to $(\mathbb{R}_+^k, \| \cdot \|_\infty)$, with an isometry given by the componentwise natural logarithm; namely

$$L : \text{int}(\mathbb{R}_+^k) \to \mathbb{R}_+^k : x \mapsto (\ln x[1], \ln x[2], \ldots, \ln x[k]),$$

where $x = (x[1], x[2], \ldots, x[k]) \in \text{int}(\mathbb{R}_+^k)$ [12, Proposition 2.2.1]. This fact implies that $(\text{int}(\mathbb{R}_+^k), d_T)$ inherits all topological and metric properties of $(\mathbb{R}_+^k, \| \cdot \|_\infty)$. In particular, $(\text{int}(\mathbb{R}_+^k), d_T)$ is also a complete metric space. Furthermore, if $U \subseteq \text{int}(\mathbb{R}_+^k)$ is a closed, bounded, or compact set in $(\text{int}(\mathbb{R}_+^k), d_T)$, then $L(U)$ is a closed, bounded, or compact set in $(\mathbb{R}_+^k, \| \cdot \|_\infty)$, and by the norm equivalence in $\mathbb{R}_+^k$, is a closed, bounded, or compact set in $(\mathbb{R}_+^k, \| \cdot \|)$ for any norm $\| \cdot \|$.

**Definition 4.** Let $f : \text{int}(\mathbb{R}_+^k) \to \text{int}(\mathbb{R}_+^k)$. We say that $f$ is a para-contraction (is strictly nonexpansive) w.r.t. Thompson’s metric on $(\text{int}(\mathbb{R}_+^k))$ if

$$\forall \ x, y \in \text{int}(\mathbb{R}_+^k) \quad x \neq y \implies d_T(f(x), f(y)) < d_T(x, y).$$

Moreover, we say that $f$ is a $c$-Lipschitz contraction w.r.t. Thompson’s metric on $(\text{int}(\mathbb{R}_+^k))$ if

$$\exists c \in [0,1) \quad \forall \ x, y \in \text{int}(\mathbb{R}_+^k) \quad d_T(f(x), f(y)) \leq c d_T(x, y).$$

We call $c$ in (7) a contraction factor of $f$. Finally, we say that $f$ is a local $c$-Lipschitz contraction w.r.t. Thompson’s metric on a compact set $U \subseteq \text{int}(\mathbb{R}_+^k)$ if

$$\exists c \in [0,1) \quad \forall \ x, y \in U \quad d_T(f(x), f(y)) \leq c d_T(x, y).$$
We call \( c \) in (8) a local contraction factor of \( f \) on \( U \).

**Definition 5.** Let \( f : \text{int}(\mathbb{R}^k_+) \to \text{int}(\mathbb{R}^k_+) \) and let

\[
x_{n+1} \overset{\Delta}{=} f(x_n) \quad \text{with} \quad x_1 \in \mathbb{R}^k_+ , \quad n \in \mathbb{N}.
\]

We call the scheme in (9) a point iteration of \( f \).

**Definition 6.** Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence in \( \mathbb{R}^k_+ \). We say that \( (x_n)_{n \in \mathbb{N}} \) converges geometrically to \( \bar{x} \in \mathbb{R}^k_+ \) with a factor \( c \in [0,1) \) and a constant \( \gamma > 0 \) if

\[
\exists c \in [0,1) \quad \exists \gamma > 0 \quad \forall n \in \mathbb{N} \quad \|x_n - \bar{x}\| \leq \gamma c^n.
\]

Similarly, we say that \( (x_n)_{n \in \mathbb{N}} \) converges linearly to \( \bar{x} \in \mathbb{R}^k_+ \) with a factor \( c \in [0,1) \) if

\[
\exists c \in [0,1) \quad \forall n \in \mathbb{N} \quad \|x_{n+1} - \bar{x}\| \leq c \|x_n - \bar{x}\|.
\]

We verify that, if \( (x_n)_{n \in \mathbb{N}} \) converges linearly to \( \bar{x} \in \mathbb{R}^k_+ \) with a factor \( c \in [0,1) \), then it converges geometrically to \( \bar{x} \in \mathbb{R}^k_+ \) with the same factor \( c \).

**Remark 2.** Let \( f : \text{int}(\mathbb{R}^k_+) \to \text{int}(\mathbb{R}^k_+) \) be a \( c \)-Lipschitz contraction w.r.t. Thompson’s metric on \( \text{int}(\mathbb{R}^k_+) \). Then, from the Banach fixed point theorem, there exists a unique fixed point \( x^* \in \text{int}(\mathbb{R}^k_+) \) of \( f \) such that the fixed point iteration in (9) converges both linearly and geometrically to \( x^* \) with a (contraction) factor \( c \), and, in the latter case, with \( \gamma = d_T(x_2, x_1)(1-c)^{-1} \).

**Fact 3.** [12, Lemma 2.1.7] Let \( f : \mathbb{R}^k_+ \to \text{int}(\mathbb{R}^k_+) \) be an SI mapping and let \( f_{|\text{int}(\mathbb{R}^k_+)} \) be the restriction of \( f \) to \( \text{int}(\mathbb{R}^k_+) \). Then \( f_{|\text{int}(\mathbb{R}^k_+)} \) is a para-contraction w.r.t. Thompson’s metric on \( \text{int}(\mathbb{R}^k_+) \).

**Remark 3.** Let \( f : \mathbb{R}^k_+ \to \text{int}(\mathbb{R}^k_+) \) be an SI mapping. We note that

- Fact 3 does not imply that \( f \) has a fixed point in \( \text{int}(\mathbb{R}^k_+) \).
  
- Moreover, even if \( f \) has a fixed point \( x^* \in \text{int}(\mathbb{R}^k_+) \), Fact 3 guarantees neither linear nor geometric convergence of the sequence constructed via (9).

However, a solution is provided to the first problem for continuous SI mappings in Fact 4 below, and the second problem is going to be dealt with and solved for continuous PC mappings in Section III.

**Definition 7.** [4] Let \( f : \mathbb{R}^k_+ \to \text{int}(\mathbb{R}^k_+) \) be a continuous SI mapping. The asymptotic mapping associated with \( f \) is the continuous mapping defined by

\[
f_\infty : \mathbb{R}^k_+ \to \mathbb{R}^k_+ : x \mapsto \lim_{p \to \infty} \frac{1}{p} f(px).
\]

We note that the above limit always exists in any normed vector space.

The asymptotic mapping associated with a continuous SI mapping can be used to define its (nonlinear) spectral radius as follows.

**Definition 8.** The (nonlinear) spectral radius of a continuous SI mapping \( f : \mathbb{R}^k_+ \to \text{int}(\mathbb{R}^k_+) \) is given by the largest eigenvalue of the corresponding asymptotic mapping [4], [13], [25]:

\[
\rho(f_\infty) = \max \{ \lambda \in \mathbb{R}_+ \mid \exists x \in \mathbb{R}^k_+ \setminus \{0\} \text{ s.t. } f_\infty(x) = \lambda x \in \mathbb{R}_+ \}.
\]

The existence of such an eigenvalue is established in [13].

We have the following fact as a solution to the first problem posed in Remark 3.

**Fact 4.** If \( f : \mathbb{R}^k_+ \to \text{int}(\mathbb{R}^k_+) \) is a continuous SI mapping, then \( f \) has a fixed point if and only if \( \rho(f_\infty) < 1 \) [4]. Furthermore, if a fixed point exists, then it is unique [2].

III. CONVERGENCE OF THE FIXED POINT ITERATION WITH CONTINUOUS PC MAPPINGS

As discussed in the Introduction, many algorithms in science and engineering (and, in particular, many power control and load estimation algorithms in wireless networks) are particular instances of the fixed point iteration \( x_{n+1} = f(x_n) \), \( x_1 \in \mathbb{R}^k_+ \), \( n \in \mathbb{N} \) in (9), where \( f : \mathbb{R}^k_+ \to \text{int}(\mathbb{R}^k_+) \) is a continuous PC mapping. Therefore, a natural question when using these algorithms is whether a fixed point exists, and this question can be answered by computing the spectral radius \( \rho(f_\infty) \) of the asymptotic mapping \( f_\infty \) (Fact 4) because positive concave mappings are particular instances of SI mappings (Fact 2). We also recall that there exists a simple iterative method to compute \( \rho(f_\infty) \) [4, Remark 2]. In some applications involving possibly nonlinear mappings \( f \), such as those in neural networks [11] and wireless networks (see Sect. IV and the references therein), \( \rho(f_\infty) \) is a linear function, in which case \( \rho(f_\infty) \) reduces to the spectral radius of a matrix.

If a PC mapping \( f \) has a fixed point, which is known to be unique (Fact 4), then the next question is whether the fixed point iteration converges to the fixed point. As shown in Fact 6 in the Appendix, the fixed point iteration converges in norm to the fixed point for any starting point in the cone \( \mathbb{R}^k_+ \), but that result does not provide us with any indication of the convergence speed. Against this background, the main objective of this section is analyze in detail the convergence of the fixed point iteration of arbitrary continuous PC mappings.

In more detail, we start by proving that PC mappings \( f : \mathbb{R}^k_+ \to \text{int}(\mathbb{R}^k_+) \) with \( \text{Fix}(f) \neq \emptyset \) are local contractions w.r.t. Thompson’s metric \( d_T \) in (4) in a neighborhood of the fixed point \( x^* \in \text{Fix}(f) \). Therefore, since \( (\text{int}(\mathbb{R}^k_+), d_T) \) is a complete metric space, and the fixed point iteration also converges to \( x^* \) in this metric space, then standard arguments based on the Banach fixed point theorem guarantees geometric convergence in \( (\text{int}(\mathbb{R}^k_+), d_T) \). These results are the main contributions of Sect. III-A.

However, in engineering applications such as those in the wireless domain, we are typically not interested in geometric...
convergence in the metric space \((\text{int}(\mathbb{R}^k_+), \, d_T)\), but in a normed vector space \((\mathbb{R}^k_+, \, \| \cdot \|)\) (the choice of the norm is irrelevant because of the equivalence of norms in finite dimensional spaces). Unfortunately, the homeomorphism in (5) between \((\text{int}(\mathbb{R}^k_+), \, d_T)\) and \((\mathbb{R}^k_+, \, \| \cdot \|_\infty)\) is nonlinear, so it may be unclear whether the results in Sect. III-A imply geometric convergence in a normed vector space. This problem is addressed in Sect. III-B, where we prove that geometric convergence of the fixed point iteration in \((\text{int}(\mathbb{R}^k_+), \, d_T)\) with a factor \(c \in [0, 1)\) implies geometric convergence in \((\mathbb{R}^k_+, \, \| \cdot \|)\) with the same factor \(c\).

Since the contraction factor \(c\) gives useful information about the convergence speed of the fixed point iteration [see (10)], in Sect. III-C we show that the concept of (nonlinear) spectral radius is useful not only for establishing the existence of a fixed point, but also for bounding \(c\).

### A. Local \(c\)-Lipschitz contractivity

Before delving into the convergence of the fixed point iteration of PC mappings, we first derive properties required for a mapping to be a contraction w.r.t. Thompson’s metric. We begin with the following lemma, which is an extension of [12, Lemma 2.1.7] to \(c \in [0, 1)\).

**Lemma 1.** Let \(f: \text{int}(\mathbb{R}^k_+) \to \text{int}(\mathbb{R}^k_+)\) be monotonic. Then \(f\) is a \(c\)-Lipschitz contraction w.r.t. Thompson’s metric if and only if

\[
\exists c \in [0, 1), \text{ s.t. } \forall x \in \text{int}(\mathbb{R}^k_+) \quad \forall \lambda > 1 \\
\quad f(\lambda x) \leq \lambda^c f(x). \tag{13}
\]

**Proof.** Let \(x, y \in \text{int}(\mathbb{R}^k_+)\) such that \(x \leq y\). Then \(d_T(x, y) = \ln \lambda\) for some \(\lambda \geq 1\). If \(\lambda = 1\), then \(d_T(x, y) = 0\), and thus \(x = y\), in which case \(f\) is trivially contractive. Therefore, we assume below that \(\lambda > 1\). From the definition of \(d_T\), one has that \(x \leq \lambda y\) and \(y \leq \lambda x\). From monotonicity of \(f\) we also obtain that \(f(x) \leq f(\lambda y)\) and \(f(y) \leq f(\lambda x)\), and from (13) one has that \(f(\lambda y) \leq \lambda^c f(y)\) and \(f(\lambda x) \leq \lambda^c f(x)\). As a result,

\[
f(x) \leq f(\lambda y) \leq \lambda^c f(y) \quad \text{and} \quad f(y) \leq f(\lambda x) \leq \lambda^c f(x).
\]

Thus,

\[
\beta_1 \triangleq \inf \{ \beta > 0 \mid f(x) \leq \beta f(y) \} \leq \lambda^c \quad \text{and} \quad \\
\beta_2 \triangleq \inf \{ \beta > 0 \mid f(y) \leq \beta f(x) \} \leq \lambda^c,
\]

and we verify that \(\max\{\beta_1, \beta_2\} \leq \lambda^c\). We now conclude that

\[
d_T(f(x), f(y)) = \ln(\max\{\beta_1, \beta_2\}) \leq \ln \lambda^c = c \ln \lambda = c d_T(x, y).
\]

Conversely, assume that \(f\) is a \(c\)-Lipschitz contraction w.r.t. Thompson’s metric, and let \(x \in \text{int}(\mathbb{R}^k_+)\) and \(\lambda > 1\) be as above and chosen arbitrarily. We note that \(d_T(x, \lambda x) = \ln \lambda\), and thus

\[
d_T(f(x), f(\lambda x)) \leq c d_T(x, \lambda x) = c \ln \lambda = \ln \lambda^c.
\]

Let \(\beta_3 \triangleq \inf \{ \beta > 0 \mid f(x) \leq \beta f(\lambda x) \} \) and \(\beta_4 \triangleq \inf \{ \beta > 0 \mid f(\lambda x) \leq \beta f(x) \} \).

From monotonicity of \(f\) we have that \(f(x) \leq f(\lambda x)\) because \(\lambda > 1\), thus \(\beta_3 \leq \beta_4\). Therefore,

\[
d_T(f(x), f(\lambda x)) = \ln \beta_4 \leq \ln \lambda^c. \tag{14}
\]

From the definition of \(\beta_4\), we obtain that \(f(\lambda x) \leq \beta_4 f(x)\), and from (14) we obtain in particular that \(\beta_4 \leq \lambda^c\), and thus we also have \(\beta_4 f(x) \leq \lambda^c f(x)\), and the proof of the desired result \(f(\lambda x) \leq \lambda^c f(x)\) is now complete. ■

**Remark 4.** We note that the mappings satisfying (13) are known as \(c\)-concave mappings in the mathematical literature, where they are considered in the context of partially ordered subsets (such as cones) of generic Banach spaces; see, for example, [26]–[28].

Continuous PC mappings (and, in particular, affine mappings restricted to \(\mathbb{R}^k_+\)) do not satisfy the properties in Lemma 1 in general, even if the mappings have a fixed point. Therefore, these mappings are not global contractions in the metric space \((\text{int}(\mathbb{R}^k_+), \, d_T)\), and thus Lemma 1 cannot be directly used with the Banach fixed point theorem to deduce geometric convergence of the fixed point iteration with PC mappings. Nevertheless, the following corollary shows that a weaker condition than (13) is sufficient for \(f\) to be a local \(c\)-Lipschitz contraction w.r.t. Thompson’s metric on a compact set \(U \subset \text{int}(\mathbb{R}^k_+)\), and in Proposition 1 we show that this property is satisfied by PC mappings.

**Corollary 1.** Let \(U \subset \text{int}(\mathbb{R}^k_+)\) be a nonempty compact set such that \(d_T \triangleq \max_{x,y \in U} d_T(x, y) = \ln \lambda_0\) with \(\lambda_0 > 1\). Let \(f: \text{int}(\mathbb{R}^k_+) \to \text{int}(\mathbb{R}^k_+)\) be monotonic. Then \(f\) is a \(c\)-Lipschitz contraction w.r.t. Thompson’s metric on \(U\) if

\[
\exists c \in [0, 1), \text{ s.t. } \forall x \in U \quad \forall \lambda \in (1, \lambda_0] \quad f(\lambda x) \leq \lambda^c f(x). \tag{15}
\]

**Proof.** Let \(x, y \in U\) be such that \(x \leq y\). Then \(d_T(x, y) = \ln \lambda\) for some \(\lambda \geq 1\). If \(\lambda = 1\), then \(d_T(x, y) = 0\), and thus \(x = y\), in which case \(f\) is trivially contractive on \(U\). Therefore, we assume below that \(\lambda > 1\), in which case \(d_T(x, y) = \ln \lambda\) for some \(1 < \lambda \leq \lambda_0\). The remaining part of the proof now follows from the corresponding part of the proof of Lemma 1. ■

Thanks to Lemma 1 and Corollary 1, we are able to prove the following proposition on local \(c\)-Lipschitz contractivity of continuous PC mappings.

**Proposition 1.** Let \(f: \mathbb{R}^k_+ \to \text{int}(\mathbb{R}^k_+)\) be a continuous and concave (w.r.t. to cone order) mapping (i.e., a continuous PC mapping), and let \(U \subset \text{int}(\mathbb{R}^k_+)\) be a nonempty compact set. Then \(f\) is a local \(c\)-Lipschitz contraction w.r.t. Thompson’s metric on \(U\).

**Proof.** From concavity of \(f\), one has in particular that, for \(t \in (0, 1)\) and \(x \in \mathbb{R}^k_+\):

\[
f(tx) = f(tx + (1-t)0) \geq tf(x) + (1-t)f(0).
\]
Furthermore,
\[ f(x) = f(t(x/t)) \geq tf(x/t) + (1 - t)f(0). \]

Let \( \lambda \overset{!}{=} t^{-1} > 1. \) Then, the above inequality can be rewritten as
\[ f(x) \geq \lambda^{-1} f(\lambda x) + (1 - \lambda^{-1}) f(0), \]
thus
\[ \lambda f(x) \geq f(\lambda x) + (\lambda - 1)f(0), \]
and consequently
\[ f(\lambda x) \leq \lambda f(x) + (1 - \lambda)f(0), \quad x \in \mathbb{R}^k. \] (16)

By our assumptions, \( f \) is continuous, and \( U \subset \text{int}(\mathbb{R}^k_+) \) is closed and bounded, hence \( f(U) \) is also closed and bounded. In particular, there exists \( \mu > 0 \) such that \( f(0) \geq \mu f(x) \) for all \( x \in U \). We note that \( \mu \in (0, 1] \) because concavity of \( f \) on \( \mathbb{R}^k_+ \) implies its monotonicity on \( \mathbb{R}^k_+ \) [3] (see also Fact 2).

Noticing that \( 1 - \lambda < 0 \), we obtain from (16) that
\[ f(\lambda x) \leq \lambda f(x) + (1 - \lambda)f(0), \quad x \in U. \]
Thus,
\[ f(\lambda x) \leq \lambda \left( 1 + \frac{1 - \lambda}{\lambda} \mu \right) f(x) = \lambda \left( 1 - \frac{(1 - \lambda)\mu}{\lambda} \right) f(x) = \lambda f(\lambda x) = \lambda f(x), \quad x \in U, \] (17)
where \( \eta \overset{!}{=} (1 - \mu)\lambda \mu > 0 \), because \( \lambda > 1 \) and \( \mu \in (0, 1] \). Therefore, using the change of base rule of logarithms, we note that \( \log_\lambda(\eta f(x)) = \frac{\log(\eta f(x))}{\log(\lambda)} \), and hence \( \eta f(x) = \lambda^{-\frac{\log(\eta f(x))}{\log(\lambda)}} \).

Thus, we obtain from (17) that
\[ f(\lambda x) \leq \lambda^{\frac{\log(\eta f(x))}{\log(\lambda)}} f(x) = \lambda^{c(\lambda)} f(x), \quad x \in U, \]
where \( c(\lambda) = \frac{\log(\eta f(x))}{\log(\lambda)} = \frac{\log((1 - \mu)\lambda \mu)}{\log(\lambda)}. \) We now show that (15) in Corollary 1 holds for a given \( \mu \in (0, 1] \).

To this end, we note that \( c'(\lambda) = \frac{(1 - \mu)\lambda +(1 - \mu)\lambda}{(1 - \mu)\lambda + \mu} > 0 \) for \( \mu \in (0, 1) \) and \( c'(\lambda) = 0 \) for \( \mu = 1 \), in which case \( c(\lambda) = 0 \) for all \( \lambda > 1 \). Therefore, for \( \mu \in (0, 1) \), \( c(\lambda) \) is an increasing mapping of \( \lambda > 1 \), and we have \( \lim_{\lambda \to \infty} c(\lambda) = 1 \), whereas the minimal value of \( c(\lambda) \) is obtained for \( \lim_{\lambda \to 1^+} c(\lambda) = 1 - \mu \). Therefore, \( c \) in Corollary 1 can be selected as \( c = c(\lambda_0) < 1 \) for \( \mu \in (0, 1) \), where \( \max_{x,y \in U} d_T(x,y) = \log_\lambda \), as in this case
\[ f(\lambda x) \leq \lambda^{c(\lambda)} f(x) \leq c^\lambda f(x), \quad x \in U. \] (18)

Moreover, if \( \mu = 1 \), then \( c \) in Corollary 1 can be selected as \( c = 0 \). We conclude that \( f \) satisfies inequality (15) in Corollary 1, implying that \( f \) is a local \( c \)-Lipschitz contraction w.r.t. Thompson’s metric on \( U \).

We note the following remark on Proposition 1.

Remark 5. If \( f \) has a fixed point \( x^* \in \text{int}(\mathbb{R}^k_+) \), then \( U \) can be selected as a compact neighbourhood of \( x^* \). In this case, Proposition 1 provides a local contraction factor for \( f \) on \( U \). We also note that the convergence of a fixed point iteration of \( f \) on \( U \) is linear in this case.

Example 1. Let \( k = 1 \) and define \( f_1(x) = (1/2)x + 1/2 \) for \( x \in \mathbb{R}_+ \). Let \( U = [1/2, 3/2] \). Then, \((U, d_T)\) is a complete metric space with \( f_1(U) \subseteq U \), and, in Proposition 1, we may set \( \mu = 1/3 \) because \( f_1(0) = 1/2 \) and \( f_1(3/2) = 5/4 \). Moreover, we have \( \lambda_0 = 3 \), thus \( c \overset{!}{=} c(\lambda_0) \approx 0.771 \), which is a local contraction factor of \( f_1 \) on \( U \). Thus, by the Banach fixed point theorem, \( f_1 \) admits a unique fixed point over \( U \), which is \( x^* = 1 \) in this case.

Example 2. Let \( k = 1 \) and define \( f_2(x) = x + 1 \) for \( x \in \mathbb{R}_+ \). As above, let \( U = [1/2, 3/2] \), so that \((U, d_T)\) is a complete metric space. Then, in Proposition 1, we may choose \( \mu = 1/3 \) because \( f_2(0) = 1 \) and \( f_2(3/2) = 5/2 \), and, as above, we have \( \lambda_0 = 3 \), thus \( c \overset{!}{=} c(\lambda_0) \approx 0.771 \), which is a local contraction factor of \( f_2 \) on \( U \). We note that we have the same values of \( \mu, \lambda_0 \), and \( c \) as in the preceding example. However, the Banach fixed point theorem cannot be applied in this case because \( f_2(U) \not\subseteq U \). Indeed, it is seen that \( f_2 \) does not have a fixed point over \( U \).

B. Geometric convergence of fixed point iteration in normed vector spaces

Having established linear, and, hence, geometric convergence of the fixed point iteration with continuous PC mappings in the metric space \((\text{int}(\mathbb{R}^k_+), d_T)\), we now proceed to prove that we also have geometric convergence in any normed vector space \((\mathbb{R}^k, \|\cdot\|)\), and the convergence factors in \((\text{int}(\mathbb{R}^k_+), d_T)\) and \((\mathbb{R}^k, \|\cdot\|)\) are the same. To this end, we need the next result.

Proposition 2. Let \( f : \text{int}(\mathbb{R}^k_+) \to \text{int}(\mathbb{R}^k_+) \) be a \( c \)-Lipschitz contraction w.r.t. Thompson’s metric and let \( x^* \in \text{int}(\mathbb{R}^k_+) \) be the unique fixed point of \( f \). Then, the fixed point iteration of \( f \) with \( x_1 \in \text{int}(\mathbb{R}^k_+) \) converges geometrically to \( x^* \) with a factor \( c \in (0, 1) \) w.r.t. any metric induced by a norm in \( \mathbb{R}^k \).

Proof. We divide the proof into two parts. We first show that the sequence constructed via (9) converges geometrically in \((\text{int}(\mathbb{R}^k_+), d_T)\), and then we proceed to prove geometric convergence in normed vector spaces.

1) Thompson’s metric: Let \( x_1 \in \text{int}(\mathbb{R}^k_+) \) and let \( (x_n)_{n \in \mathbb{N}} \) be the sequence defined through (9). We note first that, if \( x_1 = x^* \), then \( (x_n)_{n \in \mathbb{N}} \) trivially converges to \( x^* \) - and, in particular, geometrically with \( c = 0 \) and an arbitrary \( \gamma > 0 \). Thus, we assume below that \( x_1 \neq x^* \). In such a case, we note that \((\text{int}(\mathbb{R}^k_+), d_T)\) is a complete metric space (see Remark 1), so \((x_n)_{n \in \mathbb{N}} \) is bounded and converges (linearly, and hence, geometrically) to \( x^* \) w.r.t. Thompson’s metric \( d_T \).

1a) Metric induced by a norm: We note that \((\text{int}(\mathbb{R}^k_+), d_T)\) is isometric with \((\mathbb{R}^k, \|\cdot\|)\) (see Remark 1), so the sequence \((x_n)_{n \in \mathbb{N}} \) is also bounded in \((\mathbb{R}^k, \|\cdot\|)\), and, by the equivalence of norms in \( \mathbb{R}^k \), it is bounded in \((\mathbb{R}^k, \|\cdot\|)\) for any norm \( \|\cdot\| \). Let \( b > 0 \) be such that \( \forall n \in \mathbb{N} \|x_n\| \leq b \) for a given norm \( \|\cdot\| \). We note that we also have \( \|x^*\| \leq b \) because \((x_n)_{n \in \mathbb{N}} \) converges to \( x^* \). Now, let \( \delta > 0 \) be the normality constant of \( \mathbb{R}^k_+ \) for this norm (see Definition 9 in the Appendix). Then, using the fact that \( \forall n \in \mathbb{N} d_T(x_n, x^*) \leq c^{n-1} d_T(x_1, x^*) \) (because \( f \) is a...
c-Lipschitz contraction w.r.t. Thompson’s metric), we may use Fact 5 in the Appendix A to deduce that

\[
\forall n \in \mathbb{N} \quad \|x_n - x^*\| \leq b(1 + 2\delta)(e^{d_f(x_n, x^*)} - 1) \\
\leq b(1 + 2\delta)(e^{-\epsilon d_f(x_1, x^*)} - 1) \\
= b(1 + 2\delta)(e^{-\epsilon d_f(x_1, x^*)}(1 - \frac{1}{e^{-\epsilon d_f(x_1, x^*)}})) .
\]

(19)

Then, noticing that \(\forall n \in \mathbb{N}\) one has \(e^{\epsilon c - 1} \in [0, 1]\), we obtain
\[e^{\epsilon c - 1} d_f(x_1, x^*) \leq e^{d_f(x_1, x^*)},\]
and, consequently,
\[
\forall n \in \mathbb{N} \quad \|x_n - x^*\| \leq s \left(1 - \frac{1}{e^{\epsilon c - 1} d_f(x_1, x^*)}\right),
\]
where \(s \triangleq b(1 + 2\delta)e^{d_f(x_1, x^*)}.\) Using the well-known bound \(1 - 1/v \leq \ln v\) for \(v > 0\), we obtain
\[
\forall n \in \mathbb{N} \quad \|x_n - x^*\| \leq s \ln e^{\epsilon c - 1} d_f(x_1, x^*) = s e^{\epsilon c - 1} d_f(x_1, x^*) = \gamma e^n,
\]
where \(\gamma \triangleq s e^{\epsilon c - 1} d_f(x_1, x^*)\), and the proof is complete. \(\blacksquare\)

We now have all ingredients to establish geometric convergence of the fixed point iteration with metric mappings:

**Proposition 3.** Let \(f : \mathbb{R}^k_+ \rightarrow \text{int}(\mathbb{R}^k_+)\) be a continuous and concave (w.r.t. to cone order) mapping (i.e., a continuous PC mapping) with a fixed point \(x^* \in \text{int}(\mathbb{R}^k_+)\). Then, the fixed point iteration of \(f\) with \(x_1 \in \text{int}(\mathbb{R}^k_+)\) converges geometrically to \(x^*\) with a factor \(c \in [0, 1)\) w.r.t. any metric induced by a norm in \(\mathbb{R}^k\).

**Proof.** We first note that the fixed point \(x^*\) of \(f\) satisfies \(x^* \geq f(x^*)\), thus \(f\) is feasible in the sense of [2]. Therefore, by Fact 6 in the Appendix A, the fixed point iteration of \(f\) with \(x_1 \in \text{int}(\mathbb{R}^k_+)\) produces a sequence \((x_n)\) converging to \(x^*\) w.r.t. any norm on \(\mathbb{R}^k_+\). Let \(U \subset \text{int}(\mathbb{R}^k_+)\) be a compact neighbourhood of \(x^*\). Then, we can find \(n_0 \in \mathbb{N}\) such that \(\forall n > n_0\) one has \(x_n \in U\). Thus, let \((y_n)_{n \in \mathbb{N}} \subset U\) be a sequence defined as \(y_n = x_{n + n_0}\) for \(n \in \mathbb{N}\). We note that, if \(y_1 = x^*\), then \((y_n)_{n \in \mathbb{N}}\) trivially converges to \(x^*\). In particular, geometrically with \(c = 0\) and an arbitrary \(\gamma_y > 0\). We also note that \(f\) is a local c-Lipschitz contraction on \(U\) in view of Proposition 1. Moreover, using the isometry provided in (5) in Remark 1, we conclude that \((U, d_T)\) is a complete metric space isometric to the complete metric subspace of \((\mathbb{R}^k, \|\cdot\|_{\infty})\) [29]. Therefore, as \((y_n)_{n \in \mathbb{N}} \subset U\) is a fixed point iteration of \(f\) converging to \(x^*\) in a complete metric space \((U, d_T)\), and \(f\) is a local c-Lipschitz contraction on \(U\), we conclude that \((y_n)_{n \in \mathbb{N}}\) is obviously bounded and that it converges linearly (and hence, also geometrically) to \(x^*\) w.r.t. Thompson’s metric \(d_T\). Owing to this fact, we may reproduce now Part b) of the proof of Proposition 2, with the sequence \((x_n)_{n \in \mathbb{N}} \subset \text{int}(\mathbb{R}^k_+)\) considered therein replaced with the sequence \((y_n)_{n \in \mathbb{N}} \subset U\), and the complete metric space \((\text{int}(\mathbb{R}^k_+), d_T)\) considered therein replaced with the complete metric space \((U, d_T)\), to conclude that \((y_n)_{n \in \mathbb{N}}\) converges geometrically to \(x^*\) with a factor \(c \in [0, 1)\) w.r.t. any metric induced by a norm in \(\mathbb{R}^k\), with a constant \(\gamma_y > 0\) determined through (19)-(21). To show that the sequence \((x_n)_{n \in \mathbb{N}}\) also has this property, let us define \(\eta \triangleq \max_{i, j \in \{1, 2, \ldots, n_0\}} \|x_i - y_i\|\), and assume that \(c = 0\). Then, \(f\) is constant on \(U\), hence \(\forall n \in \mathbb{N}\) one has \(y_n = x_{n_0 + n} = x^*\). Then, taking into consideration that for any \(v \in (0, 1)\), we obtain

\[
\forall n \in \{1, \ldots, n_0\} \quad \|x_n - x^*\| \leq \eta = \left(\frac{\eta}{\ln v}\right) v^n \leq \left(\frac{\eta}{\ln v^{n_0}}\right) v^n,
\]

and we conclude that the sequence \((x_n)_{n \in \mathbb{N}}\) converges geometrically to \(x^*\) with a factor \(v \in (0, 1)\) w.r.t. any metric induced by a norm in \(\mathbb{R}^k\), with a constant \(\eta/v^{n_0}\). Assume now that \(c \in (0, 1)\). Then,

\[
\forall n \in \{1, \ldots, n_0\} \quad \|x_n - x^*\| \leq \eta + \gamma_y e^n = \left(\frac{\eta}{e^{n_0}} + \gamma_y\right) e^n \leq \eta e^{n_0}/(\eta + \gamma_y) e^n = \gamma x_1 c^n,
\]

where \(\gamma x_1 \triangleq \eta e^{n_0} + \gamma_y\). Moreover,

\[
\forall n > n_0 \quad \|x_n - x^*\| = \|y_n - x^*\| \leq \gamma c^n - n_0 \leq \gamma x_2 c^n,
\]

where \(\gamma x_2 \triangleq c^{-n_0} \gamma_y\). Therefore, letting \(\gamma_x \triangleq \max\{\gamma x_1, \gamma x_2\}\), we have found \(c \in (0, 1)\) and \(\gamma_x > 0\) such that \(\forall n \in \mathbb{N}\) one has \(\|x_n - x^*\| \leq \gamma_x c^n\).

An interesting question is whether the fixed point iteration is guaranteed to converge geometrically (in a normed vector space) with a more general class of concave mappings than those considered in Proposition 3. The next example shows that this special property of PC mappings does not necessarily hold even if we simply replace the assumption of positivity by nonnegativity.

**Example 3.** In [20], the authors consider the mapping
\[
f : \mathbb{R}_+ \rightarrow \text{int}(\mathbb{R}) : x \mapsto 4/(1 + e^{2 - x}),
\]
and they show that \(f\) is an SI mapping with the property that the fixed point iteration of \(f\) converges sublinearly to its unique fixed point \(x^* = 2\). In particular, the fixed point iteration of \(f\) satisfies neither (10) nor (11) for \(x = x^*\) and \(x_{n+1} = f(x_n)\) with \(x_1 \in \mathbb{R}_+, n \in \mathbb{N}\). This example motivates us to consider the function \(g : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) such that \(g(x) = x\) for \(x \in [0, 2]\) and \(g(x) = f(x)\) for \(x > 2\). Namely, let
\[
g(x) = \begin{cases} 
\frac{x}{4} & \text{for } 0 \leq x \leq 2 \\
\frac{4 - x}{1 + e^{2 - x}} & \text{for } x > 2.
\end{cases}
\]

(22)

Then, through algebraic manipulations, the following expressions of the first and second derivatives of \(g\) can be obtained:
\[
g'(x) = \begin{cases} 
\frac{1}{4e^{2 + x}} & \text{for } 0 \leq x \leq 2 \\
\frac{4e^{2 + x}}{(e^{2 + x})^2} & \text{for } x > 2,
\end{cases}
\]

and
\[
g''(x) = \begin{cases} 
\frac{0}{(e^{2 + x})^3} & \text{for } 0 \leq x \leq 2 \\
\frac{4e^{2 + x}(e^{2 - x})}{(e^{2 + x})^3} & \text{for } x > 2.
\end{cases}
\]

In particular, we observe that both \(g'\) and \(g''\) are continuous on \(\mathbb{R}_+\) and that \(g''(x) \leq 0\) for \(x \in \mathbb{R}_+\), implying that \(g\) is continuous and concave on \(\mathbb{R}_+\). Therefore, since \(g(x) = f(x)\) for \(x > 2\), we obtain a continuous and concave function, for which the fixed point iteration converges sublinearly to
$x^* = 2$ for $x_1 > 2$ irrespective of the selection of a compact neighbourhood of $x^* = 2$, e.g., on the interval $U = [1, x_1]$. Fig. 1 illustrates the convergence of the fixed point iteration starting at $x_1 = 4$, and we note that the ratio $|x_{n+1} - x^*|/|x_n - x^*|$ converges to the value one, which is the expected behavior of sequences that converge sublinearly. To highlight further the slow convergence of the algorithm, we also remark that the error $|x_n - x^*|$ remains greater than $10^{-3}$ with $n = 10^5$ iterations in this experiment. Thus, at first sight, this example invalidates in particular the assertion of Proposition 3 on geometric convergence of the fixed point iteration of $g$. However, we note that $g(0) = 0$, so none of the assertions in Propositions 1-3 apply to $g$. Therefore, this example underlines the importance of positivity of the mappings under consideration for the results in the above propositions to hold.

Example 4. By adding a small constant $\epsilon > 0$ to the function $g$ in (22), we obtain a continuous positive concave mapping, and thus we re-establish geometric convergence of the fixed point iteration as an implication of Proposition 3.

This fact is illustrated in Fig. 2, where we set $\epsilon = 10^{-3}$. The result in this figure indicates linear convergence of the fixed point iteration, and we recall that linear convergence implies geometric convergence. In this experiment, the error $|x_n - x^*|$ is smaller than $10^{-3}$ already with $n = 400$ iterations, and with around 2,000 iterations the error is reaching the machine precision of the computer used for the simulation.

We finish this section by showing that existing results in the literature emerge as corollaries of Proposition 3, with the added benefit that assumptions used in some studies can be dropped.

Example 5. The studies in [20], [21] prove geometric convergence of the fixed point iteration in a power control problem with a PC mapping given by:

$$f : \mathbb{R}^k \to \text{int}(\mathbb{R}^k_+) : x \mapsto \min_{r \in \{1, \ldots, L\}} \{M_r x + n_r \},$$

where $L \in \mathbb{N}$, $M_r \in \mathbb{R}^{k \times k}_+$, and $n_r \in \text{int}(\mathbb{R}^k_+)$ are given problem parameters for every $r \in \{1, \ldots, L\}$. In (23), $\min\{\cdot\}$ denotes the coordinate-wise min operator of a set in $\mathbb{R}^k_+$. In particular, in [20] geometric convergence of the fixed point iteration of the above mapping has been established under
an additional assumption on the matrices \( (M_i)_{i \in \{1, \ldots, L\}} \). In contrast, the result in Proposition 3 does not require any assumptions except for the existence of a fixed point, which, as shown in Fact 4, can be easily verified by computing spectral radius of \( \rho(f(x)) \) with the numerical techniques described in [4, Remark 2].

C. Spectral radius and convergence rate

It follows from (10) that the convergence factor \( c \) gives an indication on the number of iterations required to obtain a good approximation of the fixed point of a PC mapping. The results derived so far does not directly provide us with computationally simple means of obtaining information about \( c \). This limitation is addressed in this section. In particular, we note that Proposition 4 proves that, under mild assumptions, the spectral radius of a PC mapping is a lower bound on \( c \), and we recall that simple algorithms for computing the spectral radius are available in the literature [4].

Proposition 4. Let \( f : \mathbb{R}^+_n \rightarrow \text{int}(\mathbb{R}^+_n) \) be a continuous and concave (w.r.t. to cone order) mapping (i.e., a continuous PC mapping) with a fixed point \( x^* \in \text{int}(\mathbb{R}^+_n) \). Let \( U \subset \text{int}(\mathbb{R}^+_n) \) be a compact neighbourhood of \( x^* \). Furthermore, let \( c \in [0, 1] \) be a local contraction factor of \( f \) on \( U \), and denote by \( \rho \equiv \rho_f(\infty) \) the spectral radius of the asymptotic mapping \( f_\infty \) of \( f \). Further assume that there exists a positive eigenvector associated with \( \rho \); i.e., there exists \( \v \geq 0 \) such that \( f_\infty(v) = \rho v \). Then \( c \geq \rho \).

Proof. The result is trivial if \( \rho = 0 \), so we focus on the case \( \rho > 0 \). Let \( x^* \in U \) be the unique fixed point of \( f \) (see Fact 4) and let \( (x_n)_{n \in \mathbb{N}} \) be constructed via the fixed point iteration of \( f \) with \( x_1 \in U \) such that \( x_1 \leq x^* \). Then, there exists \( \v > 0 \) and \( v \geq 0 \) such that \( \|x_n - x^* - v\| \leq (\rho c)^n \|v\| \). Therefore, \( c \leq \rho \), and the proof is complete.

Remark 6. The study in [30] discusses simple techniques to verify the existence of a positive positive eigenvector of the asymptotic mapping \( f_\infty \) associated with a continuous PC mapping \( f \), which is an assumption required in Proposition 4.

Remark 7. Let \( f : \mathbb{R}^+_n \rightarrow \text{int}(\mathbb{R}^+_n) \) be a continuous and concave (w.r.t. to cone order) mapping (i.e., a continuous PC mapping) with a fixed point \( x^* \in \text{int}(\mathbb{R}^+_n) \). Let \( U \subset \text{int}(\mathbb{R}^+_n) \) be a compact neighbourhood of \( x^* \). Then, it is important to highlight that the meaning of \( c \in [0, 1] \) is consistent among

- Proposition 1, where it is used to denote the local contraction factor of \( f \) on \( U \),
- Proposition 4, where \( c \) is used in the sense of Proposition 1,
- Proposition 3, where \( c \) is used in the sense of Propositions 1 and 4, and it is also recognized as a factor of geometric convergence of the fixed point iteration of \( f \) to the unique fixed point.

In particular, the proof of Proposition 1 provides a method to determine the value of \( c \) as \( c \geq c(\lambda_0) = \frac{\ln(1 - \mu/\lambda_0)}{\ln(1 - \mu)} \), with \( \lambda_0 = e^{\max_{x, y \in U} \|f(x) - f(y)\|} \in (1, \infty) \) and \( \mu \in (0, 1] \) such that \( f(0) \geq \mu f(x) \) for all \( x \in U \). Therefore, it is seen that, if \( f \) is increasing slowly in the sense that \( x^* = f(x^*) \) is not much larger than \( f(0) \), and the neighbourhood \( U \) of \( x^* \) is sufficiently small, then the value of \( c \) can be fixed to a value close to 1. Similarly, the smaller the neighbourhood \( U \) of \( x^* \) is considered, the closer the value of \( \lambda_0 \) to 1, with \( \lim_{\lambda \rightarrow 1} c(\lambda) = 1 - \mu \). Thus, we obtain a useful insight into the relationship between \( f \) and \( U \), and we also obtain an achievable lower bound for \( c \) alternative to the one provided in Proposition 4, where \( c \) is bounded below by the spectral radius of the asymptotic mapping \( f_\infty \) of \( f \).

IV. Numerical example

We illustrate the results in the previous section with the problem of load estimation in wireless networks [3], [4], [9], [10], [15]–[17], [19], [31]. In more detail, the network being simulated uses the orthogonal frequency division multiple access (OFDMA) technology, and we consider the same downlink scenario (i.e., data transmission is from base stations to users) used in the simulations in [19]. To pose the problem, we use the following definitions:

- \( B = 2 \cdot 10^5 \) Hz - bandwidth of each resource block;
- \( \forall j \in \mathcal{N} \) \( d[j] = 10^6 \) bits/s - traffic requested by the \( j \)th user;
- \( g[i, j] > 0 \) - pathloss between base station \( i \) and user \( j \);
- \( K = 25 \) - number of resource blocks in the OFDMA system;
- \( \mathcal{M} = \{1, \ldots, K := 25\} \) - set of base stations;
- \( \mathcal{N} \) - set of users;
- \( N_i \neq \emptyset \) - set of users connected to base station \( i \);
- \( \forall i \in \mathcal{M} \) \( p[i] = 1.6 \) W - transmit power per resource block of base station \( i \);
- \( \sigma^2 = 6.2 \cdot 10^{-18} \) W - noise power per resource block.

In the simulation, the pathloss \( g[i, j] \) between base station \( i \) and user \( j \) is computed using the standard Hata model for urban scenarios with the height of the antennas of the users and the base stations set to, respectively, 1.5m and 30m. The \( k = 25 \) base stations are uniformly distributed in a square of dimensions 2km \( \times \) 2km, and 400 users are distributed uniformly at random in a square of dimensions 2.5km \( \times \) 2.5km, which is concentric to the square where base stations are placed. Users are assigned to base stations with the lowest pathloss.

Given the above problem parameters, the objective is to determine the fraction of resource blocks used at each base...
station to serve the traffic demand of the users. This fraction, which is the so-called load, is denoted by the vector \( x = [x[1], \ldots, x[k]] \in \mathbb{R}^k \). In real systems the load \( x[i] \) at base station \( i \in \mathcal{M} \) cannot exceed the value one, but in network planning and optimization tools values greater than one are allowed because we gain useful information about the unserved demand [16]. As shown in [3], [4], [9], [15]–[17], [19], [31], the load is the fixed point of the positive concave mapping defined by

\[
f : \mathbb{R}^k_+ \to \text{int}(\mathbb{R}^k_+) : x \mapsto [t_1(x), \ldots, t_k(x)],
\]

where, for all \( i \in \mathcal{M} \),

\[
t_i : \mathbb{R}^k_+ \to \text{int}(\mathbb{R}_+)
\]

\[
x \mapsto \sum_{j \in \mathcal{N}_i} d[j] \frac{K B \log_2 \left( 1 + \frac{\sum_{n \in \mathcal{M}\setminus \{i\}} \frac{1}{\sigma^2} x[n] p[n] g[n, j] + \sigma^2}{n} \right)}{K B g[i, j] \ln(2)}.
\]

The asymptotic mapping associated with \( f \) is given by [4]

\[
f_\infty : \mathbb{R}^k_+ \to \mathbb{R}^k_+ : x \mapsto \text{diag}(p)^{-1} M \text{diag}(p)x,
\]

where \( \text{diag}(p) \) is a diagonal matrix with diagonal elements given by \( \forall i \in \mathcal{M} \quad [\text{diag}(p)]_{ii} = p[i], \) and \( M \) is the matrix constructed as follows:

\[
[M]_{i,n} = \begin{cases} 0, & \text{if } i = 1 \\ \ln(2) d_i g[n, j] / K B g[i, j], & \text{otherwise.} \end{cases}
\]

Note that the asymptotic mapping \( f_\infty \) is linear, so \( \rho(f_\infty) = \rho(\text{diag}(p)^{-1} M \text{diag}(p)) = \rho(M) \) is simply the spectral radius \( \rho(M) = \rho(\text{diag}(p)^{-1} M \text{diag}(p)) \) of the matrix \( M \), or, equivalently, of the matrix \( \text{diag}(p)^{-1} M \text{diag}(p) \), where the spectral radius of matrices should be understood in the conventional sense in linear algebra. It now follows from Fact 4 that

\[
\text{Fix}(f) \neq \emptyset \Leftrightarrow \rho(M) < 1,
\]

which is a result first obtained in [10].

For the simulated scenario, we have \( \rho(M) \approx 0.92 \), so the fixed point exists, and it has been computed by constructing the sequence \( (x_n)_{n\in\mathbb{N}} \) via the fixed point iteration \( x_{n+1} = f(x_n) \), where \( x_1 = f(0) \). In Fig. 3 we show the convergence of the fixed point iterations and also the lower bound derived in [19, Proposition 5(iii)]. In Fig. 4 we show the ratio \( \|x_{n+1} - x^*\|_2 / \|x_n - x^*\|_2 \) as a mapping of the number of iterations. In these figures, \( \|\cdot\|_2 \) denotes the standard Euclidean norm.

In light of Proposition 4, \( \rho(M) \) is a lower bound for the local contraction factor around the fixed point \( x^* \) of \( f \), and this property is indeed consistent with the result in Figs. 3 and 4. More specifically, note that the bound has a slightly greater slope than the error \( \|x_n - x^*\|_2 \) for the iterations shown in Fig. 3. This property is expected because the bound as a mapping of the iteration index \( n \) takes the form \( y[n] = C_1 \rho(M)^n \) for some \( C_1 > 0 \), whereas the error takes the form \( \|x_{n+1} - x^*\|_2 \leq C_2 e^n \) for some \( C_2 > 0 \) and some \( c \in [\rho(M), 1) \). Furthermore, the ratio in Fig. 4 is converging to a value \( \mu \) within the interval \((0, 1)\), which provides us with an indication of the type of convergence described in Proposition 4 with \( \mu \geq \rho(M) \) [32]. Note, however, that the results in that proposition are valid for any sequence produced via the fixed point iteration, and this simulation shows the convergence of sequence produced by starting the iterations with a particular vector.

**APPENDIX A**

**Definitions and Known Results**

**Definition 9.** ([12, p. 45]) Let \( \|\cdot\| \) be any norm on \( \mathbb{R}^k \). We call \( \delta > 0 \) the normality constant of the cone \( \mathbb{R}^k_+ \) for the norm \( \|\cdot\| \) if

\[
\delta \triangleq \inf \{ q \in \mathbb{R}_+ : \forall (x, y) \in C \quad \|x\| \leq q \|y\| \},
\]

where \( C \triangleq \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^k_+ \mid x \leq y\} \).
Fact 5 ([12, Lemma 2.5.1.]). Let $K$ be a cone in a normed space $(V,\|\cdot\|)$. Let $\delta > 0$ be the normality constant of $K$. For each $x, y \in K$ with $\|x\| \leq b$ and $\|y\| \leq b$, we have that
$$\|x - y\| \leq b(1 + 2\delta)(e^{2\delta}\|x - y\| - 1).$$

Fact 6 ([2, Theorem 2]). Let $f : \mathbb{R}_+^k \to \text{int}(\mathbb{R}_+^k)$ be a continuous $SI$ mapping satisfying the feasibility condition $f(x) \leq x$ for a vector $x \in \mathbb{R}_+^k$. Then, for any $x \in \text{int}(\mathbb{R}_+^k)$, the fixed point iteration of $f$ produces a sequence $(x_n)_{n \in \mathbb{N}}$ converging in norm to the unique vector $x^* \in \text{Fix}(f)$, i.e.,
$$\lim_{n \to \infty} \|x_n - x^*\| = 0$$
for an arbitrary norm $\|\cdot\|$.

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\footnote{For example, $V = \mathbb{R}^n$ with $K = \mathbb{R}_+^k$.}