Generalized coherent and intelligent states for exact solvable quantum systems

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Abstract

The so-called Gazeau-Klauder and Perelomov coherent states are introduced for an arbitrary quantum system. We give also the general framework to construct the generalized intelligent states which minimize the Robertson-Schrödinger uncertainty relation. As illustration, the Pöschl-Teller potentials of trigonometric type will be chosen. We show the advantage of the analytical representations of Gazeau-Klauder and Perelomov coherent states in obtaining the generalized intelligent states in analytical way.
1 Introduction

Coherent states, known as the closest states to classical ones, play an important role in many different contexts of theoretical and experimental physics, especially quantum optics [1, 2, 3]. Schrödinger first discovered the coherent states for the harmonic oscillator potential in 1926 [4] and much work has been done since then on their properties and applications [5, 6]. The coherent states have also been found in systems with the Lie group symmetry [7, 8]. Recently, coherent states have been found in special Hamiltonians [9]. These coherent states are called minimum uncertainty coherent states. In coherent states the standard deviation of $X$ (coordinate) and $P$ (momentum) are equal and their product is minimum over states. There are also quantum states where, through we have minimum uncertainty for the standard deviation of coordinate and momentum, they are not equal any more; those states are called squeezed states. These states are as important as coherent ones their generation play an important role in many different branch of physics.

There exist three definitions of coherent states. The first one defines the usual coherent states as eigenstates of the annihilation operator $a^-$ for each individual oscillator mode of the electromagnetic field

$$a^- |z\rangle = z |z\rangle. \quad (1)$$

Here $[a^-, a^+] = 1 ((a^-)^\dagger = a^+)$ and $z$ is a complex constant with conjugate $\bar{z}$. The resulting unit normalized states $|z\rangle$ are given by

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle, \quad (2)$$

where $|n\rangle$ is an element of the Fock space $\mathcal{H} \equiv \{|n\rangle, n \geq 0\}$. A second definition of coherent states for oscillators assumes the existence of a unitary "displacement" operator $D(z)$ defined as

$$D(z) = \exp \left( za^+ - \bar{z} a^- \right). \quad (3)$$

The coherent states parametrized by $z$ are given by the action of $D(z)$ on the ground state $|0\rangle$. The unitarity of $D(z)$ ensures the correct normalization of $|z\rangle$. The Baker-Campbell-Hausdorff relation (BCH)

$$e^A e^B = e^{A+B + \frac{1}{2} [A, B]}, \quad (4)$$

valid only for any two operators $A$ and $B$ that both commute with the commutator $[A, B]$, implies the equivalence of this definition with the one above.

A third definition is based on the uncertainty relation, with the position $X$ and momentum $P$ given, as usual, by

$$X = \frac{1}{\sqrt{2}} (a^- + a^+), \quad P = \frac{i}{\sqrt{2}} (a^+ - a^-). \quad (5)$$
The coherent states defined above have the minimum-uncertainty value $2\Delta X \Delta P = 1$ and maintain this relation in time (temporal stability of coherent states). Coherent states have two important properties. First, they are not orthogonal to each other. Second, they provide a resolution of the identity, i.e., they form an over complete set states.

A central goal of this article is to extend the above three definitions for an arbitrary quantum system (exactly solvable) and comparing the equivalence between them. Note that an attempt in this sense was considered by Nieto et al [9] concluding that the three definition are generally inequivalents. Our analysis is different from the Nieto et al ones for several reasons which will be clear in the sequel of this paper.

The method we adopt is an extension of the group-theoretical approach to coherent states which generalizes the displacement operator definition. We call the obtained coherent states: coherent states of Perelomov type. The latter will be compared with Gazeau-Klauder coherent states constructed using the approach adopted by Barut-Girardello [10, 11] (see also the references 12, 13 and 14) for an arbitrary quantum system. To extend to third definition, we solve the eigenvalue equation of states minimizing the Robertson-Schrödinger uncertainty relation which extend the Heisenberg one. These states are called Generalized Intelligent States (GIS) [15, 16]. We show that the set of GIS includes the Gazeau-Klauder coherent states in a particular situation.

This paper is organized as follows: Creation and annihilation operators for an arbitrary quantum system (exactly solvable) are introduced in section 2. These operators are used to define Gazeau-Klauder coherent states in section 3. Section 4 is devoted to give a general algorithm leading to the Perelomov coherent states. States minimizing the Robertson-Schrödinger uncertainty relation are constructed in section 5. The results of sections 3, 4 and 5 are applied to a quantum system evolving in Pöschl-Teller potentials. In particular, using the analytical representations of Gazeau-Klauder coherent states and Perelomov ones, we give the generalized intelligent states under analytical forms (section 6). The last section concerns a summary of the main results of this work.

2 Creation and annihilation operators for an arbitrary quantum system

We start with general consideration on the creation and annihilation operators from the factorization of a given Hamiltonian admitting a non-degenerate discrete infinite energy spectrum. Let us assume that the Hamiltonian $H$ of a quantum system admits infinite spectrum of energy $\{E_n, n = 0, 1, 2,...\}$ such that the fundamental energy $E_0 = 0$ and the others are in increasing order, i.e.,

$$E_0 = 0 < E_1 < E_2 ... < E_{n-1} < E_n < ...$$

For such a system, we known that the fundamental state $\psi_0(x)$ and the potential $V(x)$ are closely related so that the factorization is possible. Indeed, the time
independent Schrödinger equation for $\psi_0(x)$ reads

$$H\psi_0(x) = \left( -\frac{1}{2} \frac{d^2}{dx^2} + V(x) \right) \psi_0(x) = 0,$$

(7)

and we have

$$V(x) = \frac{1}{2} \frac{\psi''_0(x)}{\psi_0(x)},$$

(8)

where the prime means the derivation with respect to $x$.

The usual factorization of $H$ is then given by

$$H = A^+ A^-$$

(9)

with

$$A^+ = \frac{1}{\sqrt{2}} \left( -\frac{d}{dx} + W(x) \right), \quad A^- = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + W(x) \right),$$

(10)

where the superpotential $W(x)$ satisfies the Riccati equation

$$V(x) = \frac{1}{2} \left( W^2(x) - W'(x) \right).$$

(11)

It is clear, from equations (8) and (11), that $W(x)$ takes the form

$$W(x) = -\frac{\psi'_0(x)}{\psi_0(x)}.$$

(12)

From equation (10), we have

$$[A^-, A^+] = W'(x),$$

(13)

which generalizes the usual one for the harmonic oscillator ($W(x) = x$). The operators $A^+$ and $A^-$ are not the creation and annihilation operators of $H$. Then, we are interested now in identifying the operators creating and annihilating the quantum states of the system under consideration. The key ingredients in constructing them is to define the operator $H_+ = A^- A^+$ obtained from $H = H_- = A^+ A^-$ by reversing the order of $A^-$ and $A^+$. The operator $H_+$ is in fact an Hamiltonian corresponding to a new potential $V_+(x)$.

$$H_+ = -\frac{1}{2} \frac{d^2}{dx^2} + V_+(x), \quad V_+(x) = \frac{1}{2} \left( W^2(x) + W'(x) \right).$$

(14)

The potentials $V_-(x) = V(x)$ and $V_+(x)$ are known as supersymmetric partner potentials and $H_- \equiv H$ and $H_+$ are isospectrals ($H_+$ is also exactly solvable). Indeed, the Schrödinger equation for $H_-$

$$H_- |\psi_n\rangle = E_n |\psi_n\rangle,$$

(15)

implies

$$H_+ \left( A^- |\psi_n\rangle \right) = E_n \left( A^- |\psi_n\rangle \right).$$

(16)

Similarly, the Schrödinger equation for $H_+$.
\( H_+ |\theta_n\rangle = e_n |\theta_n\rangle, \)

implies

\[ H_- \left( A^+ |\theta_n\rangle \right) = e_n \left( A^+ |\theta_n\rangle \right), \]

where \( e_n \) are the eigenvalues and \(|\theta_n\rangle\) are eigenstates of \( H_+ \). From the latter equations and the fact that \( E_0 = 0 \), it is clear that the energies and eigenstates of \( H_- \) and \( H_+ \) are related by

\[ e_n = E_{n+1}, \]

\[ A^- |\psi_{n+1}\rangle = \sqrt{E_{n+1}} e^{i(E_{n+1}-E_n)\alpha} |\theta_n\rangle, \]  

\[ A^+ |\theta_n\rangle = \sqrt{e_n} e^{-i(E_{n+1}-E_n)\alpha} |\psi_{n+1}\rangle, \]

where \( \alpha \in \mathbb{R} \). Notice that if the eigenstates \(|\psi_{n+1}\rangle (|\theta_n\rangle)\) of \( H_- (H_+) \) is normalized, then the wavefunctions \(|\theta_n\rangle (|\psi_{n+1}\rangle)\) in equations (19) and (20) is also normalized. Further, the operator \( A^- (A^+) \) converts an eigenfunction of \( H_- (H_+) \) into an eigenfunction of \( H_+ (H_-) \) with the same energy. Thus, the operators \( A^- \) and \( A^+ \) connect the states \(|\psi_n\rangle\) and \(|\theta_n\rangle\) and can not be considered as creation and annihilation operators for \( H \equiv H_- \). To define the ladder operators for the quantum system described by \( H \), we consider the unitary transformation \( U \) connecting the basis \( \{|\psi_n\rangle\} \) and \( \{|\theta_n\rangle\} \) as follows

\[ |\theta_n\rangle = U |\psi_n\rangle \]

with

\[ UU^+ = U^+U = I. \]

The explicit structure of the unitary operator \( U \) is given by

\[ U = \sum_{n,m} U_{nm} |\psi_n\rangle \langle \psi_m|, \]

where the elements \( U_{nm} \) are evaluated by

\[ U_{nm} = \langle \psi_n |\theta_m\rangle = \int \psi_n^*(x) \theta_m(x) \, dx. \]

Note that in the harmonic oscillator case \( U = I \).

At this stage, we can introduce the creation and annihilation operators of \( H \) by

\[ a^+ = A^+ U, \quad a^- = U^+ A^- . \]

The actions of the operators \( a^+ \) and \( a^- \) on the states \( \{|\psi_n\rangle\} \) are given by

\[ a^+ |\psi_n\rangle = \sqrt{E_{n+1}} e^{-i(E_{n+1}-E_n)\alpha} |\psi_{n+1}\rangle, \]

\[ a^- |\psi_n\rangle = \sqrt{E_n} e^{i(E_n-E_{n-1})\alpha} |\psi_{n-1}\rangle. \]
Note that \( a^+ a^- = A^+ A^- = H \). It is easy to show that
\[
|\psi_n\rangle = \frac{(a^+)^n}{\sqrt{E(n)}} e^{iE(n)\alpha} |\psi_0\rangle, \quad n > 0, \tag{28}
\]
where we have defined
\[
E(n) = E_1 E_2 ... E_n \tag{29}
\]
and for \( n = 0 \), \( E(0) = 1 \).

The exponential factor appearing in all these expressions produces only a phase factor and will be significant for the temporal stability of the coherent states we will construct in the following. From the equations (26) and (27), we have also
\[
[a^-, a^+] |\psi_n\rangle = (E_{n+1} - E_n) |\psi_n\rangle. \tag{30}
\]

Let us now introduce the operator \( N \) such that
\[
N |\psi_n\rangle = n |\psi_n\rangle, \tag{31}
\]
which in general (for an arbitrary quantum system) different from the product \( a^+ a^- (= H) \). We can see that it satisfies the following properties
\[
a^- N = (N + 1) a^- , \quad a^+ (N + 1) = Na^+. \tag{32}
\]

We are then able to define an operator \( G \) such that
\[
[a^-, a^+] = G(N), \tag{33}
\]
which acts in the states \( |\psi_n\rangle \) as
\[
G(N) |\psi_n\rangle = (E_{n+1} - E_n) |\psi_n\rangle. \tag{34}
\]

The operator \( G \) is hermitian.

\section{3 \ Gazeau-Klauder coherent states}

\subsection{3.1 \ Eigenstates of annihilation operator}

The Gazeau-Klauder coherent states are eigenstates of the annihilation operator of the system under consideration. For the system governed by the Hamiltonian \( H \) (\( = A^+ A^- = a^+ a^- \)), such states are labelled by \( |z, \alpha\rangle \), \( z \in \mathbb{C} \) and \( \alpha \in \mathbb{R} \) (\( \alpha \) is the parameter entering in the eqs (26) and (27)), and they assumed to be the solution of the eigenvalue equation
\[
a^- |z, \alpha\rangle = z |z, \alpha\rangle. \tag{35}
\]

To have their explicit form we decompose it in the basis \( \{|\psi_n\rangle\} \) such that
\[
|z, \alpha\rangle = \sum_{n=0}^{+\infty} a_n |\psi_n\rangle \tag{36}
\]
and insert this equation in (35). Using equation (27), we find

\[ a_n = \frac{z^n}{\sqrt{E(n)}} e^{-iE_n \alpha} a_0, \quad n > 0, \tag{37} \]

with \( E(n) \) is given by (29). For \( n = 0 \), the states \( |\psi_0\rangle \) is an eigenstate of \( a^- \) with eigenvalue 0. Finally, the coherent states \( |z, \alpha\rangle \) take the form

\[ |z, \alpha\rangle = a_0 \sum_{n=0}^{+\infty} \frac{z^n}{\sqrt{E(n)}} e^{-iE_n \alpha} |\psi_n\rangle. \tag{38} \]

The constant \( a_0 \) will be fixed by imposing the normalization to unity. We get

\[ |a_0|^{-2} = \sum_{n=0}^{+\infty} \frac{|z|^{2n}}{E(n)}. \tag{39} \]

The coherent states (38) are continuous in \( z \in \mathbb{C} \) and \( \alpha \in \mathbb{R} \). Moreover, the presence of the phase factor in the definitions equations (26) and (27) of the \( a^- \) and \( a^+ \) actions leads to temporal stability of the coherent states. Indeed, we have

\[ e^{iHt} |z, \alpha\rangle = |z, \alpha + t\rangle. \tag{40} \]

The analysis of completeness (in fact, the overcompleteness) require to compute the identity resolution, that is

\[ \int |z, \alpha\rangle \langle z, \alpha| d\mu(z) = I_H. \tag{41} \]

Note that the integral is over the disk \( \{z \in \mathbb{C}, |z| < \mathcal{R}\} \), where the radius of convergence \( \mathcal{R} \) is

\[ \mathcal{R} = \lim_{n \to \infty} \sqrt{n} \sqrt{E(n)} \tag{42} \]

and the measure \( d\mu(z) \) has to be determined. To determine it, we suppose that \( d\mu(z) \) depends only on \( |z| \) (isotropy condition), we take

\[ d\mu(z) = [a_0]^{-2} h(r^2) r dr d\varphi \quad ; \quad z = re^{i\varphi}. \tag{43} \]

Hence, the identity resolution can be written in the following form

\[ I_H = \sum_{n=0}^{+\infty} |\psi_n\rangle \langle \psi_n| \left[ \frac{\pi}{E(n)} \int_0^{\mathcal{R}^2} h(u) u^n du \right]. \tag{44} \]

The last equation is satisfied when we have

\[ \int_0^{\mathcal{R}^2} h(u) u^n du = \frac{E(n)}{\pi}. \tag{45} \]

It is clear that the identity resolution is then equivalent to the determination of the function \( h(u) \) satisfying the equation (45). For \( \mathcal{R} \to \infty \), the function \( h(u) \) is the inverse Mellin transform of \( \pi^{-1}E(s-1) \)

\[ h(u) = \frac{1}{2\pi^2 i} \int_{c-i\infty}^{c+i\infty} E(s-1)u^{-s} ds; \quad c \in \mathbb{R}. \tag{46} \]
Note that explicit computation of the function $h(u)$ requires the knowledge of the spectrum of the quantum mechanical system under consideration.

Using the equation (35), one can obtain the mean value of the Hamiltonian $H$ in the states $|z, \alpha\rangle$

$$\langle z, \alpha | H | z, \alpha \rangle = |z|^2. \quad (47)$$

This relation is known as the action identity.

Finally, we remark that the coherent states $|z, \alpha\rangle$ can be written as an operator $U(z)$ acting in the ground state $|\psi_0\rangle$

$$U(z) = a_0 \exp \left( z \frac{N}{g(N)} a^+ \right) \quad (48)$$

such that we have

$$|z, \alpha\rangle = U(z) |\psi_0\rangle. \quad (49)$$

In (48), $g(N) \equiv H = a^+ a^-$. The operator $U(z)$ is not unitary and cannot be interpreted as the displacement operator in the Perelomov’s sense.

A final comment can be made in connection with the work of Gazeau and Klauder [11]. In fact, the coherent states (38) satisfy all the requirements (continuity, temporal stability, identity resolution, action identity) given in their approach but they are more general since we are working with $z \in \mathbb{C}$ and $\alpha \in \mathbb{R}$. They are eigenstates of the annihilation operator $a^-$. Additional properties of this set of states will be considered in section (5).

### 3.2 Fock-Bargmann representation

It is well known that the Fock-Bargmann representation enable one to find simple solutions of a number of problems, exploiting the theory of analytical entire functions. In this subsection, generalizing the pioneering work of Bargmann [17] for the usual harmonic oscillator, we give the Bargmann representation of an arbitrary quantum mechanical system. We recall that in the Fock-Bargmann representation for the standard harmonic oscillator, the creation operator $a^+$ is the multiplication by $z$ while the annihilation operator $a^-$ is the differentiation with respect to $z$.

For an arbitrary quantum system, we define the Fock-Bargmann space as a space of functions which are holomorphic on a ring $D$ of the complex plane. The scalar product is written with an integral of the form

$$\langle f | g \rangle = \int \overline{f(z)} g(z) d\mu(z), \quad (50)$$

where $d\mu(z)$ is the measure defined above (see Eq (43)). Let $|f\rangle$ be an arbitrary quantum state of the system under study

$$|f\rangle = \sum_{n=0}^{+\infty} f_n |\psi_n\rangle, \quad \text{with} \quad \sum_{n=0}^{+\infty} |f_n|^2 < \infty. \quad (51)$$

Any state $|f\rangle$ is represented, in the Fock-Bargmann representation, as a function of the complex variable $z$ (using the so-called coherent states associated with the
quantum system under consideration)

\[ f(z) \equiv \langle \bar{z}, \alpha | f \rangle = \sum_{n=0}^{+\infty} \frac{z^n}{\sqrt{E(n)}} e^{iE_n f_n}. \]  \hfill (52)

In particular, to the vectors \(|\psi_n\rangle\) there correspond the monomials

\[ \langle \bar{z}, \alpha | \psi_n \rangle = \frac{z^n}{\sqrt{E(n)}} e^{iE_n \alpha}. \]  \hfill (53)

Using the equations (52) and (53), we can prove the following result: In the Fock-Bargmann representation, we realize the annihilation operator \(a^-\) by

\[ a^- = z^{-1} g(z \frac{d}{dz}), \]  \hfill (54)

the creation operator \(a^+\)

\[ a^+ = z, \]  \hfill (55)

and the operator number by

\[ N = z \frac{d}{dz}. \]  \hfill (56)

The Fock-Bargmann representation exists if we have a measure such that

\[ \int |z, \alpha \rangle \langle z, \alpha| d\mu(z) = I_H. \]  \hfill (57)

The existence of the measure, discussed previously for the so-called Gazeau-Klauder coherent states, ensures that the scalar product takes the form (50). We note that in the case where

\[ g(z \frac{d}{dz}) = z \frac{d}{dz}, \]  \hfill i.e. \[ g(N) = N. \]  \hfill (58)

we recover the well-known Fock-Bargmann representation of the harmonic oscillator. The Fock-Bargmann realization discussed here will be the main tool to construct the generalized intelligent states (see section 6).

### 4 Coherent states of Perelomov’s type

In view of the second definition of coherent states for the standard harmonic oscillator (group-theoretical approach), we define, for an arbitrary quantum system, the states

\[ |z, \alpha\rangle = \exp \left( za^+ - \bar{z}a^- \right) |\psi_0\rangle, \quad \text{for} \quad z \in \mathbb{C}, \]  \hfill (59)

which we call of Perelomov’s type. We have to compute the action of the displacement operator

\[ D(z) = \exp \left( za^+ - \bar{z}a^- \right) \]  \hfill (60)

on the ground state \(|\psi_0\rangle\) of the quantum system under study. We will give the result of this action in a closed form. An illustration is treated for the Pöschl-Teller and square-well potentials (in section 6).
Using the actions of the annihilation and creation operators on the Hilbert space \( \{ |\psi_n\rangle, n = 0, 1, 2, ... \} \) (eqs (26) and (27)), one can, after more or less complicated computations, show that the states \( |z, \alpha\rangle \) can be written as follows

\[
|z, \alpha\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{F_n(|z|)}} e^{-iE_n \alpha} |\psi_n\rangle. \tag{61}
\]

The quantities \( F_n(|z|) \) satisfy

\[
F_n(|z|) E(n) (c_n (|z|))^2 = 1, \tag{62}
\]

where the coefficients \( c_n (|z|) \) are given by

\[
c_n (|z|) = \sum_{j=0}^{\infty} \frac{(-|z|^2)^j}{(n+2j)!} \left( \sum_{i_1=1}^{n+1} E_{i_1} \sum_{i_2=1}^{i_1+1} E_{i_2} ... \sum_{i_j=1}^{i_{j-1}+1} E_{i_j} \right). \tag{63}
\]

Setting

\[
\pi(n+1,j) = \sum_{i_1=1}^{n+1} E_{i_1} \sum_{i_2=1}^{i_1+1} E_{i_2} ... \sum_{i_j=1}^{i_{j-1}+1} E_{i_j} \quad \text{and} \quad \pi(n+1,0) = 1, \tag{64}
\]

one can verify that the \( \pi \)'s satisfy the following relation:

\[
\frac{\pi(n+1,j) - \pi(n,j)}{E_{n+1}} = \pi(n+2,j-1). \tag{65}
\]

Using this recurrence formula, one can show that the coefficients \( c_n (|z| = r) \) satisfy the following differential equation

\[
\frac{dc_n (r)}{dr} = \frac{1}{r} c_{n-1} (r) - \frac{n}{r} c_n (r) - E_{n+1} c_{n+1} (r) r. \tag{66}
\]

Hence, solving this equation, we can obtain explicitly the coherent states \( |z, \alpha\rangle \) of Perelomov’s type. Of course, to solve this equation for an arbitrary quantum system is, in general, not an easy task. However, solutions in some particular (and interesting physical system) will be given in section 6. Here, as a first illustration of the approach leading to coherent states of Perelomov’s type, we give the standard harmonic oscillator coherent states using the above considerations. In this case we show that (61) coincides with (2). For the harmonic oscillator \( E_n = n \) and \( E(n) = n! \).

To solve the equation (66), we set

\[
c_n (r) = \frac{1}{n!} \sum_{m=0}^{\infty} a_m r^m. \tag{67}
\]

Substituting this expression in (66), we get the coefficients \( a_m \),

\[
a_{2p} = \frac{(-1)^p}{2^p p!} a_0 \quad \text{and} \quad a_{2p+1} = 0, \tag{68}
\]

where \( a_0 = 1 \) because \( c_0 (r = 0) = 1 \). Finally, we have

\[
F_n (|z|) = n! \exp \left( |z|^2 \right) \tag{69}
\]

and

\[
|z, \alpha\rangle = \exp \left( -\frac{|z|^2}{2} \right) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} e^{-i\alpha n} |n\rangle. \tag{70}
\]

We recover a well known result.
5 Generalized intelligent states

These states minimize the Robertson-Schrödinger uncertainty relation [18, 19], and
generalize the Gazeau-Klauder coherent states.

Using the creation $a^+$ and annihilation $a^-$ operators, we introduce the hermitian
operators

$$X = \frac{1}{\sqrt{2}} (a^+ + a^-), \quad P = \frac{i}{\sqrt{2}} (a^+ - a^-),$$

which satisfy the commutation relation

$$[X, P] = iG (N) \equiv iG.$$  \hspace{1cm} (72)

The operator $G (N)$, defined by (34), is not necessarily a multiple of the unit operator (for
an arbitrary quantum system). It is well known that for two hermitian operators $X$ and $P$
satisfying the noncanonical commutation relation (72), the variances $(\Delta X)^2$ and $(\Delta P)^2$
satisfy the Robertson-Schrödinger uncertainty relation

$$(\Delta X)^2 (\Delta P)^2 \geq \frac{1}{4} \left( \langle G \rangle^2 + \langle F \rangle^2 \right),$$

where the operator $F$ is defined by

$$F = \{ X - \langle X \rangle, P - \langle P \rangle \}$$  \hspace{1cm} (74)

or by

$$F = i \left[ (2a^- - \langle a^- \rangle) \langle a^- \rangle + (-2a^+ + \langle a^+ \rangle) \langle a^+ \rangle - a^- a^+ - a^+ a^- \right]$$

in terms of the operators $a^-$ and $a^+$. The symbol $\{,\}$ in (74) stands for the
anti-commutator. When there is a correlation between $X$ and $P$, i.e. $\langle F \rangle \neq 0$, the
relation (73) is a generalization of the usual one (the Heisenberg uncertainty condition)

$$(\Delta X)^2 (\Delta P)^2 \geq \frac{1}{4} \langle G \rangle^2.$$ \hspace{1cm} (76)

The special form (76) is identical with the general form (73) if $X$ and $P$ are
uncorrelated, i.e., $\langle F \rangle = 0$. The general uncertainty relation (73) is better suited
to determine the lower bound on the product of variances in the measurement of
observables corresponding to the noncanonical operators. The so-called generalized
intelligent states are obtained when the equality in the Robertson-Schrödinger
uncertainty relation is realized [20]. The inequality in (73) becomes equality for
the states satisfying the equation (see also, [20 – 23])

$$(X + i\lambda P) |\psi\rangle = z\sqrt{2} |\psi\rangle, \quad \lambda, z \in \mathbb{C}.$$  \hspace{1cm} (77)

As a consequence, we have the following relations

$$(\Delta X)^2 = |\lambda| \Delta, \quad (\Delta P)^2 = \frac{1}{|\lambda|} \Delta,$$  \hspace{1cm} (78)
with
\[ \Delta = \frac{1}{2} \sqrt{\langle G \rangle^2 + \langle F \rangle^2}. \] (79)

The average values \( \langle G \rangle \) and \( \langle F \rangle \), in the states satisfying the eigenvalue equation (77), can be expressed in terms of the variances as follows:
\[ \langle G \rangle = 2 \text{Re}(\lambda) \langle \Delta P \rangle^2, \quad \langle F \rangle = 2 \text{Im}(\lambda) \langle \Delta P \rangle^2. \] (80)

It is clear, from (78), that if \( |\lambda| = 1 \) we have
\[ (\Delta X)^2 = (\Delta P)^2. \] (81)

We call the states satisfying (81) with \( |\lambda| = 1 \), the generalized coherent states. For \( |\lambda| \neq 1 \), the states are called generalized squeezed states.

Using Eq. (77), one can obtain some general relations for the average values and dispersions of \( X \) and \( P \) in the states which minimize the Robertson-Schrödinger uncertainty relation (73). We have
\[ (\Delta X)^2 = \frac{1}{2} \text{Re}(\lambda) \langle G \rangle + \text{Im}(\lambda) \langle F \rangle, \] (82)
\[ (\Delta P)^2 = \frac{1}{2 |\lambda|^2} \left( \text{Re}(\lambda) \langle G \rangle + \text{Im}(\lambda) \langle F \rangle \right), \] (83)
\[ \text{Im}(\lambda) \langle G \rangle = \text{Re}(\lambda) \langle F \rangle. \] (84)

In order to give a complete classification of the so-called generalized intelligent states for an arbitrary quantum system, we have to solve the eigenvalue equation (77). Such computation was considered previously by the authors in [15, 16]. The states minimizing the Robertson-Schrödinger uncertainty relation are given by
\[ |\psi\rangle \equiv |z, \lambda, \alpha\rangle = \sum_{n=0}^{+\infty} d_n |\psi_n\rangle, \quad d_n \equiv d_n(z, \alpha, \lambda). \] (85)

For the case where \( \lambda \neq -1 \), the coefficients \( d_n \) are given by the following expression
\[ d_n = d_0 \frac{(2z)^n}{(1 + \lambda)^n \sqrt{E(n)}} \left[ \sum_{h=0(1)}^{[\frac{n}{2}]} (-1)^h \left( \frac{1 - \lambda^2}{(2z)^2} \right)^h \Delta(n, h) \right] e^{-i\alpha E_n}, \] (86)
where the symbol \( \left[ \frac{n}{2} \right] \) stands for the integer part of \( \frac{n}{2} \) and the function \( \Delta(n, h) \) is defined by
\[ \Delta(n, h) = \sum_{j_1=1}^{n-(2h-1)} E_{j_1} \left[ \sum_{j_2=j_1+2}^{n-(2h-3)} E_{j_2} \left[ \sum_{j_{h-1}+2}^{n-1} E_{j_{h-1}} \right] \right] \ldots \] (87)

We note that the case \( \lambda = -1 \), leading to the unnormalized solution, is not of interest.

The states \( |z, \lambda, \alpha\rangle \) can be also given as the action of some operator on the ground state \( |\psi_0\rangle \) of \( H \). A more or less complicated manipulation give the following result:
\[ |z, \lambda, \alpha\rangle = U(\lambda, z) |\psi_0\rangle, \] (88)
where
\[ \U (\lambda, z) = d_0 \sum_{n=0}^{\infty} \left( \frac{2z}{\lambda+1} a^+ + \frac{\lambda-1}{\lambda+1} \frac{1}{g(N)} (a^+)^2 \right)^n. \] (89)

Note that the states \(|z, \lambda, \alpha\rangle\) are stable temporally. As a first illustration of this construction, we can obtain the generalized intelligent states for the standard harmonic oscillator \((g(N) = N)\). We have (up to normalization constant)
\[ |z, \lambda, \alpha\rangle = \exp \left[ \left( \frac{\lambda-1}{\lambda+1} \right) \frac{(a^+)^2}{2} \right] \exp \left[ \left( \frac{2z}{\lambda+1} a^+ \right) \right] |0\rangle, \] (90)

where \(|0\rangle\) is the ground states for the harmonic oscillator.

The Gazeau-Klauder coherent states correspond to the situation \(\lambda = 1\). In this case, the coefficients \(d_n\) are given by
\[ d_n = d_0 \frac{z^n}{\sqrt{E(n)}} e^{-i\alpha E_n}, \] (91)

and the coherent states \(|z, \lambda = 1, \alpha\rangle\) coincide with Gazeau-Klauder ones \(|z, \alpha\rangle\) given by eq (38). The normalization factor \(d_0\) is given by eq (39). The states \(|z, \lambda = 1, \alpha\rangle \equiv |z, \alpha\rangle\) minimize the Heisenberg uncertainty relation (76) and are eigenvectors of the annihilation operator \(a^-\). We have
\[ (\Delta X)^2 = (\Delta P)^2 = \frac{1}{2} \langle G \rangle, \] (92)

where
\[ \langle G \rangle = d_0^2 \sum_{n=0}^{+\infty} \frac{|z|^{2n}}{E(n)} E_{n+1} - |z|^2 \quad \text{and} \quad \langle F \rangle = 0. \] (93)

The latter equation traduce the fact that there is no correlation between \(X\) and \(P\). For the harmonic oscillator, it is easy to see that \(\langle G \rangle = 1\) and \(2(\Delta X)^2 = 2(\Delta P)^2 = 1\).

As we mentioned above, the coherent states minimizing Robertson-Schrödinger uncertainty relation correspond to the case \(|\lambda| = 1\). The case \(\lambda = 1\) correspond the Gazeau-Klauder coherent states and \(\lambda = -1\) is not allowed by our construction. Setting \(\lambda = e^{i\theta} (\theta \neq k\pi; k \in \mathbb{N})\), the states \(|z, \lambda, \alpha\rangle\) are coherent and dispersions are given by
\[ (\Delta X)^2 = (\Delta P)^2 = \frac{1}{2 |\cos \theta|} \langle G \rangle. \] (94)

The main value of the operator \(F\) is nonvanishing (vanish only in the Gazeau-Klauder coherent states, i.e., \(\lambda = 1\)) and it is given by
\[ \langle F \rangle = \tan \theta \langle G \rangle. \] (95)

From the latter equation, we conclude that the presence of the correlation \((\langle F \rangle \neq 0)\) does not forbid the system to be prepared in a coherent states. This result is true for any quantum system. The properties of the states \(|z, \lambda, \alpha\rangle\) turned out to be sensitive about the spectral properties of the commutator \([a^-, a^+] = G(N)\).
To close this section, we note that the minimization of the Robertson-Schrödinger uncertainty relation leads to more general expressions of coherent states associated to an arbitrary quantum system. The Gazeau-Klauder coherent states (\(\lambda = 1\)) (eigenvectors of the annihilation operator) constitute a particular case of such coherent states class (\(|\lambda| = 1\)).

6 Application: Pöschl-Teller potentials

We start by recalling the eigenvalues and eigenstates of infinite square well and Pöschl-Teller potentials [24] (see also [25] and references therein). We consider the Hamiltonian

\[
H = -\frac{d^2}{dx^2} + V_{\kappa,\kappa'}(x)
\]

(96)

describing a particle on the line, and submitted to the potential

\[
V_{\kappa,\kappa'}(x) = \begin{cases} 
\frac{1}{4a^2} \left[ \kappa(\kappa-1) \sin^2 \left( \frac{x}{2a} \right) + \kappa'(\kappa'-1) \cos^2 \left( \frac{x}{2a} \right) \right] , & 0 < x < \pi a \\
\infty & x \leq 0, \quad x \geq \pi a
\end{cases}
\]

(97)

for \(\kappa > 1\) and \(\kappa' > 1\). It is well known that the Pöschl-Teller potentials interpolate between the harmonic oscillator and infinite square well. The infinite square well takes place in the limit \(\kappa = \kappa' = 1\).

The Hamiltonian \(H\) can be written in the factorized form

\[
H = A^+_{\kappa,\kappa'} A^-_{\kappa,\kappa'},
\]

(98)

where the operators \(A^+_{\kappa,\kappa'}\) and \(A^-_{\kappa,\kappa'}\) are given by

\[
A^\pm_{\kappa,\kappa'} = \mp \frac{d}{dx} + W_{\kappa,\kappa'}(x)
\]

(99)

in terms of the superpotentials \(W_{\kappa,\kappa'}(x)\)

\[
W_{\kappa,\kappa'}(x) = \frac{1}{2a} \left[ \kappa \cotg \left( \frac{x}{2a} \right) - \kappa' \tang \left( \frac{x}{2a} \right) \right].
\]

(100)

The eigenvectors are given by

\[
\psi_n(x) = \left[ c_n(\kappa, \kappa') \right]^{-\frac{1}{2}} \left( \cos \frac{x}{2a} \right)^\kappa \left( \sin \frac{x}{2a} \right)^{\kappa'} P_n^{(\kappa-\frac{1}{2}, \kappa'-\frac{1}{2})} \left( \cos \frac{x}{a} \right)
\]

(101)

with \(c_n(\kappa, \kappa')\) are the normalization constant which takes the form

\[
c_n(\kappa, \kappa') = a \frac{\Gamma(n + \kappa + \frac{1}{2})\Gamma(n + \kappa' + \frac{1}{2})}{\Gamma(n + 1)\Gamma(n + \kappa + \kappa')\Gamma(2n + \kappa + \kappa')}
\]

(102)

and \(P_n^{(\alpha,\beta)}\)'s stands for the Jacobi polynomials.

The eigenvalues of \(H\) are given by

\[
H |\psi_n\rangle = n(n + \kappa + \kappa') |\psi_n\rangle.
\]

(103)
To find the annihilation and creation operators for the Pöschl-Teller system, we follow the strategy given in section 2. So, we denote $H$ by $H^-$ and $V_{\kappa,\kappa'}(x)$ by $V^+_{\kappa,\kappa'}(x)$ the Hamiltonian $H_+ = A^+_{\kappa,\kappa'}A^-_{\kappa,\kappa'}$ (supersymmetric partner of $H \equiv H^-$)

$$H_+ = -\frac{1}{2} \frac{d^2}{dx^2} + V^+_{\kappa,\kappa'}(x),$$

(104)

describes a quantum system trapped in the potentials

$$V_{\kappa,\kappa'}(x) = \begin{cases} \frac{\kappa(\kappa-1)}{8a^2} + \frac{\kappa'(\kappa'-1)}{\cos^2(x/a)} - \frac{(\kappa + \kappa')^2}{8a^2}, & 0 < x < \pi a \\ 0, & x \leq 0 \\ 0, & x \geq \pi a. \end{cases}$$

(105)

The eigenstates of $H_+$ are given by

$$\theta_n(x) = \left[ c_n(\kappa + 1, \kappa' + 1) \right]^{-\frac{1}{2}} \left( \cos \frac{x}{2a} \right)^{\kappa + 1} \left( \sin \frac{x}{2a} \right)^{\kappa' + 1} P_{n}^{(\kappa + \frac{1}{2}, \kappa' + \frac{1}{2})} \left( \cos \frac{x}{a} \right),$$

(106)

where the $c_n(\kappa, \kappa')$ are defined by (102).

The eigenvalues are $\epsilon_n = (n + 1)(n + \kappa + \kappa' + 1)$. Using the operators $A^-_{\kappa,\kappa'}$ and $A^+_{\kappa,\kappa'}$ and the unitary transformation $U$ connecting $\psi_n(x)$ and $\theta_n(x)$ (see section 2), we define the creation and annihilation operators by

$$a^+_{\kappa,\kappa'} = A^+_{\kappa,\kappa'}U \quad \text{and} \quad a^-_{\kappa,\kappa'} = U^+ A^-_{\kappa,\kappa'}.$$

(107)

The creation and annihilation operators $a^+_{\kappa,\kappa'}$ and $a^-_{\kappa,\kappa'}$ act on $|\psi_n\rangle$ as follows

$$a^+_{\kappa,\kappa'} |\psi_n\rangle = \sqrt{(n + 1)(n + 1 + \kappa + \kappa')} e^{-i\alpha(n+1+\kappa+\kappa')} |\psi_{n+1}\rangle,$$

$$a^-_{\kappa,\kappa'} |\psi_n\rangle = \sqrt{n(n + \kappa + \kappa')} e^{i\alpha(2n+1+\kappa+\kappa')} |\psi_{n-1}\rangle,$$

(108)

and satisfy the following commutation relation

$$[a^-_{\kappa,\kappa'}, a^+_{\kappa,\kappa'}] = G_{\kappa,\kappa'}(N),$$

(109)

where

$$G_{\kappa,\kappa'}(N) \equiv G(N) = 2N + (1 + \kappa + \kappa').$$

(110)

We note that $N \neq a^+_{\kappa,\kappa'}a^-_{\kappa,\kappa'} = H$.

### 6.1 Gazeau-Klauder coherent states

Using the result of section (3), the so-called Gazeau-Klauder coherent states (eigenstates of the annihilation operator $a^-_{\kappa,\kappa'}$) reads as

$$|z, \alpha\rangle = \mathcal{N}(|z|) \sum_{n=0}^{+\infty} \frac{z^n e^{-i\alpha(n+\kappa+\kappa')}}{\sqrt{n!(n+\kappa+\kappa'+1)}} |\psi_n\rangle,$$

(111)
with $\mathcal{N}(|z|)$ the normalization constant which takes the form

$$[\mathcal{N}(|z|)]^2 = \frac{|z|^{\kappa + \kappa'}}{I_{\kappa + \kappa'}(2|z|)},$$

(112)

where $I_{\kappa + \kappa'}(2|z|)$ is the modified Bessel function of the first kind.

The identity resolution is given explicitly by

$$\int |z,\alpha\rangle \langle z,\alpha| \, d\mu(z) = I_H,$$

(113)

where the measure can be computed by the inverse Mellin transform [26]

$$d\mu(z) = \frac{2}{\pi} I_{\kappa + \kappa'}(2r) K_{\kappa + \kappa'}(2r) \, rdrd\phi, \quad z = r \exp(i\phi).$$

(114)

The Gazeau-Klauder coherent states of the infinite square well are obtained from the Pöschl-Teller ones simply by putting $\kappa + \kappa' = 2$.

The Gazeau-Klauder coherent states form an overcomplete family of states (resolving the unity by integration with respect to the measure given by (114)), and provide a representation of any state $|f\rangle$ by an entire function

$$f(z,\alpha) = \sqrt{\frac{I_{\kappa + \kappa'}(2|z|)}{|z|^{\kappa + \kappa'}}} \langle z,\alpha|f\rangle$$

$$= \sum_{n=0}^{+\infty} \langle \psi_n|f\rangle \frac{z^n e^{in(\kappa + \kappa')}}{\Gamma(n+1) \Gamma(n + \kappa + \kappa' + 1)}.$$ 

(115)

In particular, the analytic functions corresponding to the vectors $|\psi_n\rangle$ are

$$\mathcal{F}_n(z,\alpha) = \frac{z^n e^{in(\kappa + \kappa')}}{\sqrt{\Gamma(n+1) \Gamma(n + \kappa + \kappa' + 1)}}.$$ 

(116)

Using the Fock-Bargmann representation discussed in the subsection (3.2), the creation and annihilation operators, for quantum system evolving in Pöschl-Teller (or in the infinite square well) potentials, are realized by

$$a_{\kappa,\kappa'}^+ = z, \quad a_{\kappa,\kappa'}^- = z \frac{d^2}{dz^2} + (\kappa + \kappa' + 1) \frac{d}{dz},$$

(117)

and the operator $G_{\kappa,\kappa'}(N)$, in this representation, acts as

$$G = 2z \frac{d}{dz} + (\kappa + \kappa' + 1).$$

(118)

In fact, one can verify that

$$a_{\kappa,\kappa'}^+ \mathcal{F}_n(z,\alpha) = \sqrt{(n+1)(n+1+\kappa+\kappa')}e^{-i\alpha(\kappa+\kappa')} \mathcal{F}_{n+1}(z,\alpha),$$

(119)

$$a_{\kappa,\kappa'}^- \mathcal{F}_n(z,\alpha) = \sqrt{n(n+\kappa+\kappa')}e^{i\alpha(2n+1+\kappa+\kappa')} \mathcal{F}_{n-1}(z,\alpha),$$

(120)

$$G_{\kappa,\kappa'}(N) \mathcal{F}_n(z,\alpha) = (2n+1+\kappa+\kappa') \mathcal{F}_n(z,\alpha).$$

(121)

This realization will be useful, as we will see, to construct the Pöschl-Teller generalized intelligent states which minimize the Robertson-Schrödinger uncertainty relation.
6.2 Pöschl-Teller coherent states of Perelomov’s type

In section (4), we defined coherent states of Perelomov’s type for an arbitrary quantum system. The expressions of these states are given by infinite series (more or less complicated). As a first illustration, we discussed the harmonic oscillator system. Here, we construct the Pöschl-Teller coherent states à la Perelomov. In this order, we have to solve the differential equation (66) for the Pöschl-Teller potentials \( E_n = n(n + \kappa + \kappa') \). In this case, the solutions are

\[
c_n(r) = \frac{1}{n!} r^n \beta_{\frac{1}{2}(\kappa+\kappa' + 1), \frac{1}{2}(\kappa+\kappa' + 1)} (\cosh(2r)),
\]

because the Jacobi functions \( \beta \) satisfy the following differential equation [27]

\[
\frac{d}{dr} \beta_{m,n-l}(\cosh (2r)) = n \beta_{m,n-1-l}(\cosh (2r)) - (n - 2l) \beta_{m,n+1-l}(\cosh (2r)), \tag{123}
\]

where \( l = -\frac{1}{2}(\kappa + \kappa' + 1) \) and \( m \) is a free integer parameter which will be fixed after. These functions play an important role in the representation theory of the \( QU(2) \) group of unimodular quasi-unitary matrices.

The differential equation (123) admits several solutions. However, an admissible solution is obtained by noting that \( D(z = 0) = 1 \). Using the definition of the Jacobi functions [27], the unique solution, compatible with the condition \( D(z = 0) = 1 \), is given by

\[
c_n(r) = \frac{1}{n!} (\cosh(r))^{-(\kappa + \kappa' + 1)} \left( \frac{\tanh r}{r} \right)^n. \tag{125}
\]

The coherent states of Perelomov’s type take the form

\[
|z, \alpha \rangle = \left( 1 - \tanh^2 |z| \right)^{\frac{1}{2}(\kappa + \kappa' + 1)} \sum_{n=0}^{+\infty} \left( \frac{z \tanh |z|}{|z|} \right)^n \times \left[ \frac{\Gamma \left(n + \kappa + \kappa' + 1\right)}{\Gamma \left(n + 1\right) \Gamma \left(\kappa + \kappa' + 1\right)} \right]^{\frac{1}{2}} e^{-i \alpha n (\kappa + \kappa')} |\psi_n \rangle. \tag{126}
\]

Finally, setting \( \zeta = \frac{z \tanh |z|}{|z|} \), we obtain

\[
|\zeta, \alpha \rangle = \left( 1 - |\zeta|^2 \right)^{\frac{1}{2}(\kappa + \kappa' + 1)} \sum_{n=0}^{+\infty} \zeta^n \left[ \frac{\Gamma \left(n + \kappa + \kappa' + 1\right)}{\Gamma \left(n + 1\right) \Gamma \left(\kappa + \kappa' + 1\right)} \right]^{\frac{1}{2}} e^{-i \alpha n (\kappa + \kappa')} |\psi_n \rangle. \tag{127}
\]

We note that the parameter \( \zeta \) belongs to the unit disk \( D = \{ \zeta \in \mathbb{C}, \ |\zeta| < 1 \} \).

The states are stable temporally. Indeed

\[
e^{-iHt} |\zeta, \alpha \rangle = |\zeta, \alpha + t \rangle. \tag{128}
\]
The identity resolution is given by

\[ \int |\zeta, \alpha\rangle \langle \zeta, \alpha| d\mu(\zeta) = I_H, \]  

(129)

where the measure is

\[ d\mu(\zeta) = \frac{\kappa + \kappa'}{\pi} \frac{d^2\zeta}{(1 - |\zeta|^2)^2}. \]  

(130)

There are two main consequence arising from the former result. First, we can express any coherent state \(|\zeta', \alpha\rangle\) in terms of the others

\[ |\zeta', \alpha\rangle = \int |\zeta, \alpha\rangle \langle \zeta, \alpha| \zeta', \alpha\rangle d\mu(\zeta). \]  

(131)

The kernel \(\langle \zeta, \alpha| \zeta', \alpha\rangle\) is easy to evaluate from (127)

\[ \langle \zeta, \alpha| \zeta', \alpha\rangle = \sqrt{\left(1 - |\zeta|^2\right)^{(\kappa + \kappa' + 1)}} \left(1 - |\zeta'|^2\right)^{(\kappa + \kappa' + 1)} \sum_{n=0}^{+\infty} \zeta^n \zeta'^n \times \frac{\Gamma(n + \kappa + \kappa' + 1)}{\Gamma(n + 1) \Gamma(\kappa + \kappa' + 1)} e^{-i(\alpha' - \alpha)n(\kappa + \kappa' + 1)}. \]  

(132)

The coherent states are normalized (\(\langle \zeta, \alpha| \zeta, \alpha\rangle = 1\)), but they are not orthogonal to each other.

Second, an arbitrary element state of the Hilbert space \(H\), let us call it \(|f\rangle\), can be written in terms of the coherent states

\[ |f\rangle = \int (1 - |\zeta|^2)^{\frac{1}{2}(\kappa + \kappa' + 1)} f(\zeta, \alpha) |\zeta, \alpha\rangle d\mu(\zeta), \]  

(133)

where the analytic function

\[ f(\zeta, \alpha) = (1 - |\zeta|^2)^{-\frac{1}{2}(\kappa + \kappa' + 1)} \langle \zeta, \alpha| f \rangle \]

\[ = \sum_{n=0}^{+\infty} \zeta^n \left[ \frac{\Gamma(n + \kappa + \kappa' + 1)}{\Gamma(n + 1) \Gamma(\kappa + \kappa' + 1)} \right]^\frac{1}{2} e^{i\alpha n(\kappa + \kappa' + 1)} \langle \psi_n| f \rangle \]  

(134)

determines in a complete way the state \(|f\rangle \in H\). The state \(|\psi_n\rangle\) is represented by the function

\[ F'_n(\zeta, \alpha) = \zeta^n \left[ \frac{\Gamma(n + \kappa + \kappa' + 1)}{\Gamma(n + 1) \Gamma(\kappa + \kappa' + 1)} \right]^\frac{1}{2} e^{i\alpha n(\kappa + \kappa' + 1)}. \]  

(135)

The creation \(a^+_{\kappa, \kappa'}\), annihilation \(a^-_{\kappa, \kappa'}\), and \(G_{\kappa, \kappa'}(N)\) operators act in the Hilbert space of analytic functions \(f(\zeta, \alpha)\) as a first-order differential operators:

\[ a_{\kappa, \kappa'}^+ = \zeta^2 \frac{d}{d\zeta} + (\kappa + \kappa' + 1)\zeta, \quad a_{\kappa, \kappa'}^- = \frac{d}{d\zeta}, \quad G_{\kappa, \kappa'}(N) \equiv G = 2\zeta \frac{d}{d\zeta} + (\kappa + \kappa' + 1). \]  

(136)
One can verify that
\[
a^+_{\kappa,\kappa'} F_n' (\zeta, \alpha) = \sqrt{(n + 1)(n + 1 + \kappa + \kappa')} e^{-\alpha (2n + 1 + \kappa + \kappa')} F_n' (\zeta, \alpha),
\]
\[
a^-_{\kappa,\kappa'} F_n' (\zeta, \alpha) = \sqrt{n(n + \kappa + \kappa')} e^{\alpha (2n - 1 + \kappa + \kappa')} F_n' (\zeta, \alpha),
\]
\[
G_{\kappa,\kappa'} (N) F_n' (\zeta, \alpha) = (2n + 1 + \kappa + \kappa') F_n' (\zeta, \alpha).
\]

The analytic representation of the Gazeau-Klauder coherent states and the analytical realization of the Perelomov ones in the unit disk are related through a Laplace transform. Indeed one can verify easily that
\[
F_n' (\zeta, \alpha) = \frac{\zeta^{-(\kappa + \kappa' + 1)}}{\Gamma(\kappa + \kappa' + 1)} \int_0^{+\infty} z^{\kappa + \kappa'} F_n (z, \alpha) e^{-z \zeta} dz,
\]
which means that the function \(F_n' (\zeta, \alpha)\) is the Laplace transform of \(z^{\kappa + \kappa'} F_n (z, \alpha)\).

A similar result was obtained in [28] showing that the representation in the unit disk and Barut-Girardello one, based on the \(su(1, 1)\) coherent states, are related through a Laplace transform.

### 6.3 Pöschl-Teller generalized intelligent states

The generalized intelligent states can be determined by using two analytic representation, one based on the so-called Gazeau-Klauder coherent states (section 3) and the other one on the Perelomov’s coherent states (section 4).

#### 6.3.1 The Gazeau-Klauder analytic representation

We introduce the analytic function
\[
\Phi_{(z', \lambda, \alpha)} (z) = \sqrt{\frac{\Gamma(\kappa + \kappa') (2 \lvert z \rvert)}{\lvert z \rvert^{\kappa + \kappa'}}} \langle \zeta, \alpha \rvert z', \lambda, \alpha \rangle
\]
by mean of which one convert the eigenvalues equation
\[
\left[ (1 + \lambda) a^-_{\kappa,\kappa'} + (1 - \lambda) a^+_{\kappa,\kappa'} \right] \lvert z', \lambda, \alpha \rangle = 2z' \lvert z', \lambda, \alpha \rangle
\]
into the second-order linear homogeneous differential equation
\[
\left[ (1 + \lambda) \left( z \frac{d^2}{dz^2} + (\kappa + \kappa' + 1) \frac{d}{dz} \right) + (1 - \lambda) z - 2z' \right] \Phi_{(z', \lambda)} (z) = 0.
\]

We first consider the general case \(\lambda \neq \pm 1\). Setting
\[
\Phi_{(z', \lambda)} (z) = \exp \left( \pm \sqrt{\frac{\lambda - 1}{\lambda + 1}} \right) F_{(z', \lambda)} (z),
\]
The equation can be transformed in the Kummer equation
\[
\left[ Z \frac{d^2}{dZ^2} + (\kappa + \kappa' + 1 - Z) \frac{d}{dZ} - \left( \frac{\kappa + \kappa' + 1}{2} \mp \frac{z'}{\sqrt{\lambda^2 - 1}} \right) \right] F_{(z', \lambda)} (z) = 0.
\]
where $Z = \pm 2 \sqrt{\frac{\lambda - 1}{\lambda + 1}} z$

Then the solutions of the equation (143) are given by

$$
\Phi_{(\zeta', \lambda)}(z) = \exp \left( \pm \sqrt{\frac{\lambda - 1}{\lambda + 1}} z \right) \, {}_1F_1 \left( \frac{\kappa + \kappa' + 1}{2} \pm \frac{z'}{\sqrt{\lambda^2 - 1}}, \kappa + \kappa' + 1; \mp 2 \sqrt{\frac{\lambda - 1}{\lambda + 1}} z \right)
$$

or

$$
\Phi_{(\zeta', \lambda)}(z) = \exp \left( \pm \sqrt{\frac{\lambda - 1}{\lambda + 1}} z \right) z^{-(\kappa + \kappa')} \, {}_1F_1 \left( \frac{1 - (\kappa + \kappa')}{2} \pm \frac{z'}{\sqrt{\lambda^2 - 1}}, 1 - (\kappa + \kappa'); \mp 2 \sqrt{\frac{\lambda - 1}{\lambda + 1}} z \right)
$$

The first solution (146) is always analytic, but the solution (147) is not (Remember that $\kappa > 1$ and $\kappa' > 1$). The upper and lower signs in equation (146) are equivalent, because the confluent hypergeometric function ${}_1F_1(\alpha, \gamma; z)$ can be written in two equivalents forms which are related by Kummer’s transformation

$$
{}_1F_1(\alpha, \gamma; z) = e^z \, {}_1F_1(\gamma - \alpha, \gamma, -z).
$$

Using the properties of this hypergeometric functions, we conclude that the squeezing parameter $\lambda$ obeys to the condition

$$
\sqrt{\frac{\lambda - 1}{\lambda + 1}} < 1 \iff \text{Re}(\lambda) > 0,
$$

which exactly the restriction on $\lambda$ imposed by the positivity of the commutator $[a_{-\kappa, \kappa'之外}; a_{+\kappa, \kappa'之外}] = G_{\kappa, \kappa'}(N)$ (see equations (109) and (110)).

We consider now the degenerate cases $\lambda = -1$ and $\lambda = 1$. For the $\lambda = -1$ the equation (143) does not have any normalized analytic solution (the operator $a_{+\kappa, \kappa'}$ does not have any eigenstate). For $\lambda = 1$, using the power series of ${}_1F_1(a, b; z)$, we get

$$
\Phi_{(\zeta', \lambda=1)}(z) = {}_0F_1(\kappa + \kappa' + 1; z z').
$$

The result (150) coincides with the solution (111) (up to normalization constant) for $\lambda = 1$, and we recover the Pöschl-Teller coherent states defined as the $a_{-\kappa, \kappa'}$ eigenstates.

### 6.3.2 The Perelomov coherent state basis and analytic representation in the unit disk

To solve the eigenvalues equation (142), using the analytic representation of Perelomov coherent states in the unit disk, we introduce the analytic function

$$
\Phi_{(\zeta', \lambda)}(\zeta) = \left(1 - |\zeta|^2\right)^{-\frac{1}{2}(\kappa + \kappa' + 1)} \langle \zeta', \alpha | \zeta, \lambda, \alpha \rangle.
$$

Equation (142) is then converted to the following differential equation

$$
\left[ (1 - \lambda)\zeta^2 + (1 + \lambda) \frac{d}{d\zeta} + (1 - \lambda)(\kappa' + \kappa + 1)\zeta - 2\zeta' \right] \Phi_{(\zeta', \lambda)}(\zeta) = 0.
$$
Admissible values of $\lambda$ and $\zeta'$ are determined by the requirements that the functions $\Phi_{(\zeta', \lambda)}(\zeta)$ must be analytic in the unit disk. We consider the general case. The solution of Eq. (152) is

$$\Phi_{(\zeta', \lambda)}(\zeta) = \mathcal{N}^{-\frac{1}{2}} \prod_{l=\pm 1} \left( 1 + t \left( \frac{\lambda - 1}{\lambda + 1} \right)^{1/2} \zeta \right)^{\pm \frac{1}{2}(\kappa + \kappa' + 1) + 1/\sqrt{\lambda - 1}},$$

(153)

where $\mathcal{N}$ is a normalization constant. The condition of analyticity requires

$$\left| \frac{\lambda - 1}{\lambda + 1} \right| < 1 \iff \text{Re}\lambda > 0.$$  

(154)

If $\text{Re}\lambda < 0$, the function $\Phi_{(\zeta', \lambda)}(\zeta)$ cannot be analytic in the unit disk.

The decomposition of the generalized intelligent states $|\zeta', \lambda, \alpha\rangle$ over the Hilbert orthonormal basis $\{|\psi_n\rangle\}$ can be obtained by expanding the function $\Phi_{(\zeta', \lambda)}(\zeta)$ into a power series in $\zeta$. This can be done using the following relations

$$\left( 1 + \left( \frac{\lambda - 1}{\lambda + 1} \right)^{1/2} \zeta \right)^{\alpha_+} \left( 1 - \left( \frac{\lambda - 1}{\lambda + 1} \right)^{1/2} \zeta \right)^{\alpha_-} = \sum_{n=0}^{+\infty} \zeta^n \left( 2 \sqrt{\frac{\lambda - 1}{\lambda + 1}} \right)^n P_n^{(\alpha_+ - n, \alpha_- - n)}(0),$$

(155)

where

$$\alpha_\pm = -\frac{1}{2}(\kappa + \kappa' + 1) \pm \frac{\zeta'}{\sqrt{\lambda - 1}}.$$  

(156)

Then, the function $\Phi_{(\zeta', \lambda)}(\zeta)$ can be expanded in terms of the Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$. Using the relation between the hypergeometric function and Jacobi polynomials [27], one can show

$$|\zeta', \lambda, \alpha\rangle = \mathcal{N}^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \frac{\Gamma(\kappa + \kappa' + 1)}{n! \Gamma(\kappa + \kappa' + 1 + n)} \left[ \frac{\Gamma(\alpha_+ + 1)}{\Gamma(\alpha_+ - n + 1 + n)} \right]^{1/2} \left\{ 2 \sqrt{\frac{\lambda - 1}{\lambda + 1}} \right\}^n \binom{\alpha_+ - n}{\alpha_- - n} |\psi_n\rangle,$$

(157)

or

$$|\zeta', \lambda, \alpha\rangle = \mathcal{N}^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \frac{n! \Gamma(\kappa + \kappa' + 1)}{\Gamma(\kappa + \kappa' + 1 + n)} \left[ \frac{\Gamma(\alpha_+ + 1)}{\Gamma(\alpha_+ - n + 1 + n)} \right]^{1/2} \left\{ 2 \sqrt{\frac{\lambda - 1}{\lambda + 1}} \right\}^n P_n^{(\alpha_+ - n, \alpha_- - n)}(0)e^{-i\alpha E_n} |\psi_n\rangle.$$  

(158)

The generalized intelligent states $\Phi_{(\zeta', \lambda)}(\zeta)$ and $\Phi_{(z', \lambda)}(z)$ are related through a Laplace transform. In fact, equation (152) can be written as

$$\left[ \left( 1 + \lambda \zeta^2 + (1 - \lambda) \right) \frac{d}{d\zeta} - \frac{(1 - \lambda)(\kappa + \kappa' + 1)}{\zeta} + 2\zeta' \right] \Phi_{(\zeta', \lambda)} \left( \frac{1}{\zeta} \right) = 0.$$  

(159)

Using

$$\Phi_{(\zeta', \lambda)} \left( \frac{1}{\zeta} \right) = \frac{\zeta^{-(\kappa + \kappa' + 1)}}{\sqrt{\Gamma(\kappa + \kappa' + 1)}} \int_0^{+\infty} z^{\kappa + \kappa'} \Phi_{(\zeta', \lambda)}(z)e^{-\frac{z}{\zeta}} dz.$$  

(160)
It is easy to see that the eigenvalues equation (159) becomes

\[
(1 + \lambda) \left( z \frac{d^2}{dz^2} + (\kappa + \kappa' + 1) \frac{d}{dz} \right) + (1 - \lambda) z - 2\zeta' \Phi_{(\zeta', \lambda)}(z) = 0, \tag{161}
\]

which coincides with (143) ones ($\zeta' = z'$) that gives the generalized intelligent states (146).

7 Summary

In this work, we have explicitly constructed the Gazeau-Klauder and Perelomov coherent states for an arbitrary quantum system. As an application, of this construction, we considered the system trapped in the Pöschl-Teller potentials type. We shown that the analytical representations of Gazeau-Klauder and Perelomov coherent states (which are related through a Laplace transform) enables us to compute the generalized intelligent states for the Pöschl-Teller potentials. Finally, it should be interesting to investigate further applications of the results obtained on this work. Indeed, it is interest, in our opinion, to construct the coherent states and generalized intelligent states for the Shape invariant potentials [29]. This matter will be considered in a forthcoming work.

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