SDIFF(2) KP HIERARCHY

KANEHISA TAKASAKI\textsuperscript{1} and TAKASHI TAKEBE\textsuperscript{2}
\textsuperscript{1} Institute of Mathematics, Yoshida College, Kyoto University
Yoshida-Nihonmatsu-cho, Sakyo-ku, Kyoto 606, Japan
\textsuperscript{2} Department of Mathematics, Faculty of Science, University of Tokyo
Hongo, Bunkyo-ku, Tokyo 606, Japan

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ABSTRACT
An analogue of the KP hierarchy, the SDiff(2) KP hierarchy, related to the group of
area-preserving diffeomorphisms on a cylinder is proposed. An improved Lax formalism
of the KP hierarchy is shown to give a prototype of this new hierarchy. Two important
potentials, $S$ and $\tau$, are introduced. The latter is a counterpart of the tau function
of the ordinary KP hierarchy. A Riemann-Hilbert problem relative to the group of area-
diffeomorphisms gives a twistor theoretical description (nonlinear graviton construction)
of general solutions. A special family of solutions related to topological minimal models
are identified in the framework of the Riemann-Hilbert problem. Further, infinitesimal
symmetries of the hierarchy are constructed. At the level of the tau function, these
symmetries obey anomalous commutation relations, hence leads to a central extension
of the algebra of infinitesimal area-preserving diffeomorphisms (or of the associated Poisson
algebra).

1. Introduction
This article presents, with technical details, our recent attempt to construct an
analogue of the KP hierarchy with a different Lie algebraic structure. The ordinary
KP hierarchy is known to be characterized by the algebra gl($\infty$) of infinite matrices
\cite{1}\cite{2}. Our idea is to replace this algebra by the Lie algebra of Hamiltonian vector
fields (or the Poisson algebra) associated with area-preserving diffeomorphisms on
a surface (in this case, a cylinder $S^1 \times R^1$). We refer to such groups, in general, of
area-preserving diffeomorphisms as “SDiff(2)” somewhat symbolically; “2” means
that we deal with a two-dimensional manifold on which to consider diffeomorphisms.
Our goal is to verify that a number of remarkable properties of the ordinary KP
hierarchy persist in this SDiff(2) version.

Our attempt is primarily motivated by recent work of Krichever \cite{3} on the notion
of “dispersionless Lax equations” and its application to “topological minimal mod-
Kanehisa Takasaki and Takashi Takebe

Krichever’s dispersionless Lax equations are a kind of “quasi-classical” version of ordinary Lax equations for the KP and generalized KdV equations. A main characteristic is that the commutator on the right hand side of Lax equations in the ordinary sense is now replaced by a Poisson bracket. Despite of this difference, Krichever pointed out that the notion of the tau function can be extended to these equations (at least for a class of special solutions). This strongly suggests that dispersionless Lax equations can be treated more systematically along the line of the Kyoto group [1][2].

Actually, we were first led to this issue from the study of a different nonlinear equation. This equation (an SDiff(2) version of Toda field theory) was first discovered as a 3-d reduction of the 4-d self-dual vacuum Einstein equation by Boyer and Finley [6] and studied in more detail by Gegenberg and Das [7]. Since this is a continuous limit of the ordinary Toda chain, methods of nonlinear integrable systems have been applied by Golenisheva-Kutuzova and Reiman [8] and by Savilev and his coworkers [9][10]. Bakas [11] and Park [12] studied the same equation in the context of extended conformal symmetries (\(w_\infty\) algebras). Park presented a remarkable observation on hidden symmetries of this equation. Meanwhile, twistor people [13][14][15][16] arrived at the same equation from a different direction (“minitwistor theory”). Inspired by these observations, we introduced the notion of the “SDiff(2) Toda hierarchy” and attempted to apply the approach of the Kyoto group to this new nonlinear system [17]. In the course of that study, we encountered the work of Krichever and noticed that his dispersionless Lax equations are very similar to our SDiff(2) Toda hierarchy. Therefore we decided to deal with these two cases in a parallel way, incorporating in particular the twistor theoretical point of view into Krichever’s dispersionless Lax equations. This program indeed has turned out to be very useful.

We should further mention that a prototype of such a twistor theory of the SDiff(2) KP hierarchy can also be found in Orlov’s work [18] on the KP hierarchy. Actually, we learned Orlov’s work through a paper by Awada and Sin [19], who applied it to \(d = 1\) string theory. An essence of Orlov’s idea is to enlarge the usual Lax formalism of the KP hierarchy with a single Lax operator \(L\) by adding another Lax operator \(M\). \(L\) and \(M\) give a “canonical conjugate pair” as \([L, M] = 1\). Our twistor theoretical approach to Krichever’s dispersionless Lax equations is based upon a similar pair \(\mathcal{L}\) and \(\mathcal{M}\) that are now functions rather than operators, and give a “classical” canonical conjugate pair for a Poisson bracket as \(\{\mathcal{L}, \mathcal{M}\} = 1\). This provides us with a nice dictionary between the ordinary and SDiff(2) KP hierarchies.

We first review Orlov’s idea in Section 2. The SDiff(2) KP hierarchy is introduced in Section 3 along with one of our main technical device, a Kähler-like 2-form. Section 4 is devoted to the notion of the \(S\) function. This is a key object in Krichever’s approach. We do not actually need the \(S\) function because the \((\mathcal{L}, \mathcal{M})\)-pair plays essentially the same role. It turns out that the \(S\) function is a counterpart of the logarithm of a wave function (a solution of the linear system) for
the KP hierarchy. The notion of tau function is introduced in Section 5. A twistor theoretical description (which is a kind of Riemann-Hilbert problem in the SDiff(2) group) of general solutions is treated in Section 6. This idea is applied to special solutions related to topological minimal models (Section 7) and to the construction of infinitesimal symmetries on the space of solutions (Section 8). In the final stage, infinitesimal symmetries are extended to the tau function and exhibit, at that level, anomalous commutation relations. A central extension of the SDiff(2) algebra thus naturally emerges. In Section 9, we give a few concluding remarks.

2. \((L,M)\)-pair in KP hierarchy

The KP hierarchy describes a set of isospectral deformations of a first order pseudo differential operator

\[
L = \partial + \sum_{i=1}^{\infty} u_{i+1}(t)\partial^{-i},
\]

(2.1)

where \(t = (t_1, t_2, \ldots)\) are deformation parameters and

\[
\partial = \partial/\partial x, \quad x = t_1.
\]

(2.2)

The deformation equations can be written in the so called Lax form as

\[
\frac{\partial L}{\partial t_n} = [B_n, L], \quad B_n = (L^n)_{\geq 0},
\]

(2.3)

where \((\quad)_{\geq 0}\) stands for the differential operator part (i.e., nonnegative powers of \(\partial\)). They give, formally, the Frobenius integrability conditions of the linear system

\[
\frac{\partial \psi(t, \lambda)}{\partial t_n} = B_n \psi(t, \lambda), \quad \lambda \psi(t, \lambda) = L \psi(t, \lambda).
\]

(2.4)

The system of Lax equations has a zero-curvature representation of the Zakharov-Shabat form,

\[
\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + [B_m, B_n] = 0.
\]

(2.5)

It is also known that a zero-th order pseudo-differential operator

\[
W = 1 + w_1 \partial^{-1} + w_2 \partial^{-2} + \ldots
\]

(2.6)

exists and converts the above equations into the Sato form

\[
\frac{\partial W}{\partial t_n} = B_n W - W \partial^n, \quad L = W \partial W^{-1}.
\]

(2.7)

Accordingly, the linear system has a solution of the form

\[
\psi(t, \lambda) = W \exp(\sum_{n=1}^{\infty} t_n \lambda^n).
\]

(2.8)
It is this pseudo-differential operator $W$ that encodes the KP hierarchy into a dynamical system on an infinite dimensional Grassmannian manifold [1][2][20]. In that respect, one should consider the Lax operator rather a secondary object. Actually, the Lax operator $L$ cannot reproduce $W$ uniquely.

Orlov’s idea [18] is to introduce the pseudo-differential operator

$$M = \text{def } W(\sum_{n=1}^{\infty} nt_n \partial^{n-1})W^{-1} = WxW^{-1} + \sum_{n=2}^{\infty} nt_n L^{n-1} \tag{2.9}$$

as a second Lax operator. In fact, $M$ satisfies Lax equations

$$\frac{\partial M}{\partial \ell_n} = [B_n, M] \tag{2.10}$$

and the linear equation

$$\frac{\partial \psi(t, \lambda)}{\partial \lambda} = M\psi(t, \lambda). \tag{2.11}$$

Further, $L$ and $M$ obey the canonical commutation relation

$$[L, M] = 1. \tag{2.12}$$

With this extended Lax formalism, Orlov pointed out the existence of an infinite number of infinitesimal symmetries of the KP hierarchy. These symmetries can be identified with those originating in the geometry of the infinite dimensional Grassmannian manifold [1][2]. One may thus reorganize the KP hierarchy in terms of the $(L, M)$-pair.

The linear equations for $\psi(t, \lambda)$ yield several interesting relations. Note first that Eq. (2.1) has always an inversion formula:

$$\vartheta = L + \sum_{i=1}^{\infty} q_{i+1} L^{-i}. \tag{2.13}$$

This is a general fact independent of the KP hierarchy. Meanwhile, because of the linear equation for $L$, one has

$$\partial \psi(t, \lambda) = (\lambda + \sum_{i=1}^{\infty} q_{i+1} \lambda^{-i})\psi(t, \lambda), \tag{2.14}$$

hence

$$\frac{\partial \log \psi(t, \lambda)}{\partial x} = \lambda + \sum_{i=1}^{\infty} q_{i+1} \lambda^{-i}. \tag{2.15}$$

Similarly, one can expand $M$ and $B_n$ in powers of $L$ as

$$M = \sum_{n=1}^{\infty} nt_n L^{n-1} + \sum_{i=1}^{\infty} v_{i+1} L^{-i-1},$$

$$B_n = L^n + \sum_{i=1}^{\infty} q_{n,i+1} L^{-i}. \tag{2.16}$$
Then from the linear equations,
\[
\frac{\partial \log \psi(t, \lambda)}{\partial \lambda} = \sum_{n=1}^{\infty} nt_n\lambda^{n-1} + \sum_{i=1}^{\infty} v_{i+1}\lambda^{-i-1},
\]
\[
\frac{\partial \log \psi(t, \lambda)}{\partial t_n} = \lambda^n + \sum_{i=1}^{\infty} q_{n,i+1}\lambda^{-i}.
\]
(2.17)

If one expands \( \log \psi(t, \lambda) \) as
\[
\log \psi(t, \lambda) = \sum_{n=1}^{\infty} t_n\lambda^n + \sum_{i=1}^{\infty} S_{i+1}\lambda^{-i},
\]
(2.18)
the previous Laurent coefficients can be written
\[
v_{i+1} = -iS_{i+1}, \quad q_{i+1} = \frac{\partial S_{i+1}}{\partial x}, \quad q_{n,i+1} = \frac{\partial S_{i+1}}{\partial t_n}.
\]
(2.19)

The coefficients \( q_{i+1} \) are related to conservation laws of the KP hierarchy [21] [22] [23]. The other coefficients, too, will have a similar interpretation. We shall find that all the above relations have some counterpart in the SDiff(2) KP hierarchy.

3. SDiff(2) KP hierarchy and Kähler-like 2-form
Krichever’s proposal is to consider a “quasi-classical” version of the KP hierarchy replacing \( \partial \) by \( \lambda \) and commutators by Poisson brackets:
\[
[\ , \ ] \rightarrow \{ \ , \ \},
\]
\[
[\partial, x] = 1 \rightarrow \{\lambda, x\} = 1.
\]
(3.1)
Pseudo-differential operators will then be replaced by Laurent series of \( \lambda \). According to this prescription, it would be natural to consider the following system of equations as an analogue of Orlov’s improved Lax formalism of the KP hierarchy.
\[
\frac{\partial \mathcal{L}}{\partial t_n} = \{B_n, \mathcal{L}\}, \quad \frac{\partial \mathcal{M}}{\partial t_n} = \{B_n, \mathcal{M}\},
\]
\[
\{\mathcal{L}, \mathcal{M}\} = 1,
\]
(3.2)
where \( \mathcal{L} \) and \( \mathcal{M} \) are Laurent series in \( \lambda \),
\[
\mathcal{L} \overset{\text{def}}{=} \lambda + \sum_{i=1}^{\infty} u_{i+1}\lambda^{-i},
\]
\[
\mathcal{M} \overset{\text{def}}{=} \sum_{n=1}^{\infty} nt_n\mathcal{L}^{n-1} + \sum_{i=1}^{\infty} v_{i+1}\mathcal{L}^{-i-1},
\]
(3.3)
and the $B$'s are given by the polynomial part (i.e., nonnegative powers of $\lambda$) of powers of $L$,

$$B_n \overset{\text{def}}{=} (L^n)_{\geq 0}. \quad (3.4)$$

(The projection $(\quad)_{\geq 0}$ now means the polynomial part of Laurent series of $\lambda$.)

Finally, $\{\quad\}$ stands for the Poisson bracket

$$\{F, G\} \overset{\text{def}}{=} \frac{\partial F}{\partial \lambda} \frac{\partial G}{\partial x} - \frac{\partial G}{\partial \lambda} \frac{\partial F}{\partial x}. \quad (3.5)$$

We call this system the SDiff(2) KP hierarchy. “SDiff(2)” now refers to the structure of a Poisson algebra given by the above Poisson bracket, which corresponds to the group of area-preserving diffeomorphisms on a cylinder $S^1 \times \mathbb{R}^1$.

A number of characteristics of the KP hierarchy, indeed, persist in this hierarchy. For example, one can prove the following fact with purely algebraic manipulation (as proven for the ordinary KP hierarchy [2]).

**Proposition 1.** The Lax equations for $L$ are equivalent to the “zero-curvature equations”

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + \{B_m, B_n\} = 0 \quad (3.6)$$

and to its “dual” form

$$\frac{\partial B^-}{\partial t_n} - \frac{\partial B^-}{\partial t_m} - \{B^-_m, B^-_n\} = 0, \quad (3.7)$$

where

$$B^-_n \overset{\text{def}}{=} L^n - B_n = (L^n)_{\leq -1} \quad (3.8)$$

and $(\quad)_{\leq -1}$ stands for the negative power part of Laurent series of $\lambda$.

A crucial difference, however, is that there is no direct counterpart of the $W$ operator. The successful understandings of the KP hierarchy [1][2][20] are all based upon the use of $W$. We need something new in place of $W$ for the study of the SDiff(2) version.

We already know equations of the above type in the analysis of the self-dual vacuum Einstein equation and hyper-Kähler geometry [24]. Actually, they are slightly different in the sense that the spectral variable $\lambda$ therein is merely a parameter. In the above setting, one has to treat $\lambda$ as a true variable that enters into the definition of the Poisson bracket along with $x$. Apart from this difference, both situations are almost the same. In the study of the vacuum Einstein equation (as well as its hyper-Kähler version), a Kähler-like 2-form and associated “Darboux coordinates” play a central role. We now show that the SDiff(2) KP hierarchy has a similar structure. Let $\omega$ be a 2-form given by

$$\omega \overset{\text{def}}{=} \sum_{n=1}^{\infty} dB_n \wedge dt_n = d\lambda \wedge dx + \sum_{n=2}^{\infty} dB_n \wedge dt_n, \quad (3.9)$$
where “\( \mathcal{D} \) now stands for total differentiation in both \( t \) and \( \lambda \). From the definition, obviously, \( \omega \) is a closed form,
\[
d\omega = 0. \tag{3.10}
\]

The zero-curvature equations for \( B_n \) can be cast into a compact form as
\[
\omega \wedge \omega = 0. \tag{3.11}
\]

These two relations ensure the existence of “Darboux coordinates” \( \mathcal{P} \) and \( \mathcal{Q} \) (functions of \( t \) and \( \lambda \)) such that
\[
\omega = d\mathcal{P} \wedge d\mathcal{Q}. \tag{3.12}
\]

(In the case of the self-dual vacuum Einstein equation, \( \lambda \) is considered a constant under the total differentiation.) Actually, \( \mathcal{L} \) and \( \mathcal{M} \) give (and are characterized as) such a pair of functions:

**Proposition 2.** The \( SDiff(2) \) KP hierarchy is equivalent to the exterior differential equation
\[
\omega = d\mathcal{L} \wedge d\mathcal{M}. \tag{3.13}
\]

**Proof:** We only show the derivation of the Lax system from the exterior differential equation; the converse can be checked by simply tracing back the following reasoning. Expanding both sides of the exterior differential equation as linear combination of \( d\lambda \wedge dt_n \) and \( dt_m \wedge dt_n \) give rise to an infinite set of partial differential equations. From coefficients of \( d\lambda \wedge dx \) \((x = t_1)\), one has
\[
1 = \frac{\partial \mathcal{L}}{\partial \lambda} \frac{\partial \mathcal{M}}{\partial x} - \frac{\partial \mathcal{M}}{\partial \lambda} \frac{\partial \mathcal{L}}{\partial x} = \{\mathcal{L}, \mathcal{M}\}. \tag{3.14}
\]

Similarly, from coefficients of \( d\lambda \wedge dt_n \) and \( dx \wedge dt_n \), respectively,
\[
\begin{align*}
\frac{\partial B_n}{\partial \lambda} &= \frac{\partial \mathcal{L}}{\partial \lambda} \frac{\partial \mathcal{M}}{\partial t_n} - \frac{\partial \mathcal{M}}{\partial \lambda} \frac{\partial \mathcal{L}}{\partial t_n}, \\
\frac{\partial B_n}{\partial x} &= \frac{\partial \mathcal{L}}{\partial x} \frac{\partial \mathcal{M}}{\partial t_n} - \frac{\partial \mathcal{M}}{\partial x} \frac{\partial \mathcal{L}}{\partial t_n}. \tag{3.15}
\end{align*}
\]

One can easily solve these relations for \( \partial \mathcal{L}/\partial t_n \) and \( \partial \mathcal{M}/\partial t_n \) because the coefficient matrix is unimodular, \( \{\mathcal{L}, \mathcal{M}\} = 1 \), as we have just deduced above. The result is:
\[
\begin{align*}
\frac{\partial \mathcal{L}}{\partial t_n} &= \frac{\partial B_n}{\partial \lambda} \frac{\partial \mathcal{L}}{\partial x} - \frac{\partial B_n}{\partial x} \frac{\partial \mathcal{L}}{\partial \lambda} = \{B_n, \mathcal{L}\}, \\
\frac{\partial \mathcal{M}}{\partial t_n} &= \frac{\partial B_n}{\partial \lambda} \frac{\partial \mathcal{M}}{\partial x} - \frac{\partial B_n}{\partial x} \frac{\partial \mathcal{M}}{\partial \lambda} = \{B_n, \mathcal{M}\}. \tag{3.16}
\end{align*}
\]

This completes the proof. □
4. $S$ function

Note that the fundamental relation

$$\omega = d\mathcal{L} \wedge d\mathcal{M}$$  \hspace{1cm} (4.1)

can be rewritten

$$d\left(\mathcal{M}d\mathcal{L} + \sum_{n=1}^{\infty} B_n dt_n\right) = 0.$$ \hspace{1cm} (4.2)

This implies the existence of a function $S$ such that

$$dS = \mathcal{M}d\mathcal{L} + \sum_{n=1}^{\infty} B_n dt_n,$$ \hspace{1cm} (4.3)

or, equivalently,

$$\mathcal{M} = \left(\frac{\partial S}{\partial \mathcal{L}}\right)_{t \text{ fixed}},$$ \hspace{1cm} (4.4)

$$B_n = \left(\frac{\partial S}{\partial t_n}\right)_{\mathcal{L}, t_m (m \neq n) \text{ fixed}}.$$ \hspace{1cm} (4.5)

This potential $S$ is introduced by Krichever [3]. In his formulation, $\mathcal{M}$ is implicit in various calculations and the $S$ function plays rather a central role.

Actually, we have the following explicit realization of the $S$ function in terms of the Laurent coefficients $v_{i+1}$ of $\mathcal{M}$.

**Proposition 3.** $S$ is given by

$$S = \sum_{i=1}^{\infty} t_i \mathcal{L}^i + \sum_{i=1}^{\infty} S_{i+1} \mathcal{L}^{-i}, \quad S_{i+1} = -\frac{1}{i} v_{i+1}.$$ \hspace{1cm} (4.6)

In particular, $B_n$ can be written explicitly in terms of the Laurent coefficients $S_{i+1}$ as

$$B_n = \mathcal{L}^n + \sum_{i=1}^{\infty} \frac{\partial S_{i+1}}{\partial t_n} \mathcal{L}^{-i}.$$ \hspace{1cm} (4.7)

It is amusing to compare this result with the KP hierarchy (Section 2). One will immediately find that the $S$ function plays the same role as the logarithm of $\psi$. Note, in particular, that Eq. (4.7) for $n = 1$ becomes

$$\lambda = \mathcal{L} + \sum_{i=1}^{\infty} \frac{\partial S_{i+1}}{\partial x} \mathcal{L}^{-i},$$ \hspace{1cm} (4.8)

hence gives an explicit inversion formula

$$\lambda = \mathcal{L} + \sum_{i=1}^{\infty} q_{i+1} \mathcal{L}^{-i}$$ \hspace{1cm} (4.9)
of the Laurent expansion of $L$ in $\lambda$. These coefficients $q_{i+1}$ are known in the theory of singularities as “flat coordinates” [25]. It is thus natural that the notion of flat coordinates plays an important role in topological minimal models [4][5] (see Section 7).

Let us give a proof of the above result. To this end, we use the notion of formal residue of 1-forms:

$$\text{res} \sum a_n \lambda^n d\lambda = a_{-1}. \tag{4.10}$$

and the following basic properties without proofs.

**Lemma A.** For any (formal) Laurent series $F$ and $G$ of $\lambda$,

$$\text{res} d\lambda F = 0, \tag{4.11}$$
$$\text{res} F d\lambda G = -\text{res} G d\lambda F, \tag{4.12}$$
$$\text{res} F d\lambda G = \text{res} (F_{\geq 0}) d\lambda (G_{\leq -1}) + \text{res} (F_{\leq -1}) d\lambda (G_{\geq 0}), \tag{4.13}$$

where “$d\lambda$” stands for total differentiation with respect to $\lambda$.

**Lemma B.** For any integer $n$,

$$\text{res} L^n d\lambda L = \delta_{n,-1}. \tag{4.14}$$

Bearing these observations in mind, we first prove:

**Proposition 4.** The $t$-derivatives of $v_{i+1}$ are given by

$$\frac{\partial v_{i+1}}{\partial t_n} = \text{res} \left[ \left( \frac{\partial M}{\partial L} \right)_{t,v \text{ fixed}} \frac{\partial L}{\partial t_n} + nL^{n-1} + \sum_{i=1}^{\infty} \frac{\partial v_{i+1}}{\partial t_n} L^{-i-1} \right] d\lambda L. \tag{4.15}$$

**Proof:** By the chain rule of differentiation,

$$\frac{\partial M}{\partial t_n} = \left( \frac{\partial M}{\partial L} \right)_{t,v \text{ fixed}} \frac{\partial L}{\partial t_n} + nL^{n-1} + \sum_{i=1}^{\infty} \frac{\partial v_{i+1}}{\partial t_n} L^{-i-1}, \tag{4.16}$$

where “$t, v$ fixed” means that $M$ is differentiated with respect to $L$ while the Laurent coefficients $t_n$ and $v_i$ being fixed. In other words,

$$\left( \frac{\partial M}{\partial L} \right)_{t,v \text{ fixed}} = \sum_{n=1}^{\infty} n(n-1)t_n L^{n-2} - \sum_{i=1}^{\infty} (i+1)v_{i+1} L^{-i-2},$$

though one does not need this explicit expansion. Therefore, using Lemma B as well, one has

$$\frac{\partial v_{i+1}}{\partial t_n} = \text{res} \left[ \left( \frac{\partial M}{\partial L} \right)_{t,v \text{ fixed}} \frac{\partial L}{\partial t_n} \right] d\lambda L. \tag{4.17}$$
The “[…]” part times $d_{\lambda}L$ in the last formula can be calculated as:

$$
\left[\ldots\right] d_{\lambda}L = \frac{\partial M}{\partial t_n} d_{\lambda}L - \frac{\partial L}{\partial t_n} d_{\lambda}M = \{B_n, M\} d_{\lambda}L - \{B_n, L\} d_{\lambda}M
$$

$$
= \left( \frac{\partial B_n}{\partial \lambda} \frac{\partial M}{\partial x} - \frac{\partial B_n}{\partial x} \frac{\partial M}{\partial \lambda} \right) \frac{\partial L}{\partial \lambda} d\lambda - \left( \frac{\partial B_n}{\partial \lambda} \frac{\partial L}{\partial x} - \frac{\partial B_n}{\partial x} \frac{\partial L}{\partial \lambda} \right) \frac{\partial M}{\partial \lambda} d\lambda
$$

$$
= \left( \frac{\partial L}{\partial \lambda} \frac{\partial M}{\partial x} - \frac{\partial L}{\partial x} \frac{\partial M}{\partial \lambda} \right) \frac{\partial B_n}{\partial \lambda} d\lambda = d_{\lambda}B_n.
$$

(4.18)

(We have also used the canonical Poisson relation of $L$ and $M$.)

**Proof of Proposition 3:** Eq. (4.4) is obviously satisfied because of the construction. One has to prove Eq. (4.5) or, equivalently, Eq. (4.7). Use the previous lemmas to continue the right hand side of Eq. (4.15) as

$$
\frac{\partial v_{i+1}}{\partial t_n} = -\text{res } B_n d_{\lambda}(L^i) = -i \text{ res } B_n L^{i-1} d_{\lambda}L.
$$

(4.19)

From these relations for $i \geq 1$ (and Lemma B), $B_n$ turns out to have a Laurent expansion as

$$
B_n = \sum_{m \geq 0} b_{nm} L^m - \sum_{i=1}^{\infty} \frac{1}{i} \frac{\partial v_{i+1}}{\partial t_n} L^{-i-1}
$$

(4.20)

with yet undetermined coefficients $b_{nm}$. The $(\ )_{\geq 0}$ part of both hand sides then gives rise to such a relation as

$$
B_n = \sum_{m \geq 0} b_{nm} B_m,
$$

(4.21)

however $B_m$’s should be linearly independent polynomials of $\lambda$, hence

$$
b_{nm} = \delta_{nm}.
$$

(4.22)

Since $S_{i+1} = -v_{i+1}/i$, the last expression of $B_n$ gives Eq. (4.7).

We conclude this section with a comment on the missing coefficients $u_1$ and $v_1$ in the definition of the hierarchy. We have excluded these terms because they are absent in the ordinary KP hierarchy. Actually, we could have included them in $L$ and $M$, but it turns out that they can be absorbed into redefinition of $\lambda$ and $M$. This is due to the following fact.

**Proposition 5.** Suppose that $u_1$ and $v_1$ are inserted as

$$
L = \lambda + \sum_{i=0}^{\infty} u_{i+1} \lambda^{-i}, \quad M = \sum_{n=1}^{\infty} n t_n \lambda^{n-1} + \sum_{i=0}^{\infty} v_{i+1} \lambda^{-i-1}
$$

(4.23)
and the same Lax equations (including the \( n = 1 \) case) and the canonical Poisson relation are satisfied. Then \( u_1 \) and \( v_1 \) become constants.

**Proof:** Consider first the Lax equation for \( \mathcal{L} \). The right hand side can be evaluated as

\[
\{ \mathcal{B}_n, \mathcal{L} \} = -\{(\mathcal{L}^n)_{\leq -1}, \mathcal{L} \} = \{O(\lambda^{-1}), \lambda + u_1 + O(\lambda^{-1})\} = O(\lambda^{-1}).
\]

(4.24)

hence from the \( \lambda^0 \)-part,

\[
\frac{\partial u_1}{\partial t_n} = 0.
\]

(4.25)

To prove that \( v_1 \) is a constant, we note that Eq. (4.15) is also valid for \( i = 0 \). Accordingly,

\[
\frac{\partial v_1}{\partial t_n} = \text{res } d\lambda \mathcal{B}_n = 0.
\]

(4.26)

This completes the proof. \( \blacksquare \)

Consequently, \( u_1 \) and \( v_1 \) can be absorbed by redefinition of \( \lambda \) and \( \mathcal{M} \) as

\[
\lambda + u_1 \rightarrow \lambda,
\]

\[
\mathcal{M} - v_1 \mathcal{L}^{-1} \rightarrow \mathcal{M}.
\]

(4.27)

With this prescription, one may always assume

\[
u_1 = 0, \quad v_1 = 0
\]

(4.28)

without losing generality. Later, however, we shall have to relax the second condition because the nonlinear graviton construction (see Section 6) can, in general, generate a nonzero \( v_1 \) term.

5. Tau function

We define the tau function \( \tau \) of the SDiff(2) KP hierarchy by the equations

\[
\frac{\partial \log \tau}{\partial t_n} = v_{n+1}, \quad n = 1, 2, \ldots
\]

(5.1)

This is due to the following basic fact.

**Proposition 6.** The functions \( v_{n+1} \) on the right hand side satisfy the integrability condition

\[
\frac{\partial v_{m+1}}{\partial t_n} = \frac{\partial v_{n+1}}{\partial t_m}, \quad m, n = 1, 2, \ldots
\]

(5.2)

**Proof:** From Eq. (4.15) and Lemma A,

\[
\frac{\partial v_{m+1}}{\partial t_n} = \text{res } (\mathcal{L}^n)_{\leq -1} d\lambda (\mathcal{L}^m)_{\geq 0}.
\]

(5.3)
Taking difference after interchanging $m \leftrightarrow n$ and using Lemmas A and B, one has

$$
\frac{\partial v_{m+1}}{\partial t_n} - \frac{\partial v_{n+1}}{\partial t_m} = \text{res} \left[ (L^m)_{\leq -1} d\lambda (L^n)_{\geq 0} \right] - \text{res} \left[ (L^n)_{\leq -1} d\lambda (L^m)_{\geq 0} \right]
= \text{res} \left[ (L^m)_{\leq -1} d\lambda (L^n)_{\geq 0} \right] + \text{res} \left[ (L^m)_{\geq 0} d\lambda (L^n)_{\leq -1} \right]
= \text{res} \left[ L^m d\lambda L^n \right]
= n\delta_{m+n,0},
$$

(5.4)

which vanishes because $m$ and $n$ are now positive integers. □

The tau function contains all information of the hierarchy. In fact, we can reproduce $u_i$ and $v_i$ (hence $L$ and $M$) from the tau function. This can be seen as follows. First, from the construction, the $v$'s can be written

$$
v_{n+1} = \frac{\partial \log \tau}{\partial t_n}.
$$

(5.5)

To obtain a similar expression for the $u$'s, recall Eqs. (4.6)-(4.9). From these relations, one can see that $q_{n+1}$ can be written

$$
q_{n+1} = -\frac{1}{n} \frac{\partial v_{n+1}}{\partial x} = -\frac{1}{n} \frac{\partial^2 \log \tau}{\partial t_n \partial x}.
$$

(5.6)

This means that $u_{n+1}$ is a differential polynomial of $\log \tau$, because the $u_{n+1}$'s and the $q_{n+1}$'s are connected by an invertible polynomial relation. Actually, they are linked in a more explicit form by residue formulas:

$$
u_{n+1} = -\frac{1}{n} \text{res } \lambda^n d\lambda L,
q_{n+1} = -\frac{1}{n} \text{res } L^n d\lambda.
$$

(5.7)

Thus the SDiff(2) KP hierarchy can be, in principle, rewritten as a system of differential equations for the tau function $\tau$. In the case of the ordinary KP hierarchy [1][2], this leads to the celebrated Hirota bilinear equations. No similar expression has been discovered for the SDiff(2) version.

It is instructive to compare our definition of the tau function with the case of the ordinary KP hierarchy. To distinguish between the ordinary KP hierarchy and the SDiff(2) version, we now put superscript $^{KP}$ for the ordinary KP hierarchy as $\tau^{KP}$, $u_n^{KP}$, $v_n^{KP}$, etc., whereas $\tau$, $u_n$, $v_n$, etc. stand for their SDiff(2) counterparts.

In the KP hierarchy, $\psi = \psi(t, \lambda)$ is linked with the tau function as [2]

$$
\log \psi = \sum_{n=1}^{\infty} t_n \lambda^n + \log \frac{\tau^{KP}(t_1 - \frac{1}{N}, t_2 - \frac{1}{2N}, \ldots)}{\tau^{KP}(t_1, t_2, \ldots)}
= \sum_{n=1}^{\infty} t_n \lambda^n + \left[ \exp \left( -\sum_{n=1}^{\infty} \frac{1}{n \lambda^n} \frac{\partial}{\partial t_n} \right) - 1 \right] \log \tau^{KP}(t)
= \sum_{n=1}^{\infty} t_n \lambda^n + \sum_{N=1}^{\infty} \frac{1}{N!} \left( -\sum_{n=1}^{\infty} \frac{1}{n \lambda^n} \frac{\partial}{\partial t_n} \right)^N \log \tau^{KP}(t).
$$

(5.8)
We have seen in the previous section that the $S$ function of the SDiff(2) KP hierarchy should correspond to $\log \psi$ of the KP hierarchy. In fact, we now have

$$S = \sum_{n=1}^{\infty} t_n L^n + \sum_{n=1}^{\infty} S_{n+1} L^{-n}$$

$$= \sum_{n=1}^{\infty} t_n L^n - \sum_{n=1}^{\infty} \frac{1}{n L^n} \frac{\partial}{\partial t_n} \log \tau(t),$$

(5.9)

which is very similar to the above formula except that

- only the $N = 1$ term is retained, and
- $\lambda$ is replaced by $L$.

6. Riemann-Hilbert problem in SDiff(2) group

The nonlinear graviton construction of Penrose [26] generates, in principle, all (both local and global) solutions of the self-dual vacuum Einstein equation and its hyper-Kähler version [27]. This method can be extended to the SDiff(2) KP hierarchy. A key step is to solve the functional equation

$$f(L, M) \leq -1 = 0,$$
$$g(L, M) \leq -1 = 0,$$

(6.1)

where $f = f(\lambda, x)$ and $g = g(\lambda, x)$ are arbitrary holomorphic functions defined in a neighborhood of $\lambda = \infty$ except at $\lambda = \infty$ itself, and required to satisfy the canonical Poisson relation

$$\{ f(\lambda, x), g(\lambda, x) \} = 1.$$  

(6.2)

Thus the pair $(f, g)$ may be thought of as an area-preserving diffeomorphism. (For simplicity, we do not specify the domain where $x$ is supposed to take values. That depends on the situation.) One may rewrite the functional equations as

$$f(L, M) = \hat{L},$$
$$g(L, M) = \hat{M},$$

(6.3)

where $\hat{L}$ and $\hat{M}$ are another set of unknown functions and required to have Laurent expansion in $\lambda$ with only nonnegative powers,

$$\hat{L} = \hat{L}_{\geq 0}, \quad \hat{M} = \hat{M}_{\geq 0}.$$  

(6.4)

In the latter expression, the functional equations look more like a “Riemann-Hilbert problem” as we know for the case of the self-dual vacuum Einstein equation and hyper-Kähler geometry [26][27].

We now assume that the above “Riemann-Hilbert problem” has a unique solution with $L$ and $M$ in the form given in the definition of the SDiff(2) KP hierarchy. This is indeed ensured, as Penrose observed in the case of the self-dual vacuum Einstein equation, if $(f, g)$ is sufficiently close to the trivial one $(f, g) = (\lambda, x)$. 
Proposition 7. Such a solution \((\mathcal{L}, \mathcal{M})\) of the functional equations gives a solution of the \(SDiff(2)\) KP hierarchy.

Proof: We first derive the canonical Poisson relation. By differentiating Eqs. (6.3) with respect to \(\lambda\) and \(x\),

\[
\begin{pmatrix}
\frac{\partial f(\mathcal{L}, \mathcal{M})}{\partial \mathcal{L}} & \frac{\partial f(\mathcal{L}, \mathcal{M})}{\partial \mathcal{M}} \\
\frac{\partial g(\mathcal{L}, \mathcal{M})}{\partial \mathcal{L}} & \frac{\partial g(\mathcal{L}, \mathcal{M})}{\partial \mathcal{M}}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \mathcal{L}}{\partial \lambda} & \frac{\partial \mathcal{L}}{\partial x} \\
\frac{\partial \mathcal{M}}{\partial \lambda} & \frac{\partial \mathcal{M}}{\partial x}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial \hat{\mathcal{L}}}{\partial \lambda} & \frac{\partial \hat{\mathcal{L}}}{\partial x} \\
\frac{\partial \hat{\mathcal{M}}}{\partial \lambda} & \frac{\partial \hat{\mathcal{M}}}{\partial x}
\end{pmatrix}.
\tag{6.5}
\]

Since the first matrix on the left hand side is unimodular because of Eq. (6.2), the determinants of both hand sides give

\[
\{\mathcal{L}, \mathcal{M}\} = \{\hat{\mathcal{L}}, \hat{\mathcal{M}}\}.
\tag{6.6}
\]

Following the idea of the proof of Proposition 4, one can calculate the left hand side as

\[
\{\mathcal{L}, \mathcal{M}\} = \frac{\partial \mathcal{L}}{\partial \lambda} \frac{\partial \mathcal{M}}{\partial x} - \frac{\partial \mathcal{M}}{\partial \lambda} \frac{\partial \mathcal{L}}{\partial x}
= \frac{\partial \mathcal{L}}{\partial \lambda} \left( \frac{\partial \mathcal{M}}{\partial \mathcal{L}} \right)_{t,v \text{ fixed}} \frac{\partial \mathcal{L}}{\partial x} + 1 + \sum_{i=1}^{\infty} \frac{\partial v_{i+1}}{\partial x} \mathcal{L}^{-i}
- \frac{\partial \mathcal{L}}{\partial x} \left( \frac{\partial \mathcal{M}}{\partial \mathcal{L}} \right)_{t,v \text{ fixed}} \frac{\partial \mathcal{L}}{\partial \lambda}.
\]

but terms containing \((\partial \mathcal{M}/\partial \mathcal{L})_{t,v \text{ fixed}}\) in the last line cancel, hence

\[
\{\mathcal{L}, \mathcal{M}\} = 1 + (\text{negative powers of } \lambda).
\tag{6.7}
\]

Meanwhile, Laurent expansion of \(\{\hat{\mathcal{L}}, \hat{\mathcal{M}}\}\) contains only nonnegative powers of \(\lambda\). Therefore strictly negative (as well as) powers of \(\lambda\) in the last line should be absent, thus

\[
\{\mathcal{L}, \mathcal{M}\} = \{\hat{\mathcal{L}}, \hat{\mathcal{M}}\} = 1.
\tag{6.8}
\]

This gives the canonical Poisson commutation that we have sought for. We now show that the Lax equations for \(\mathcal{L}\) and \(\mathcal{M}\) are indeed satisfied. Differentiating Eqs. (6.3) now with respect to \(t_n\) gives

\[
\begin{pmatrix}
\frac{\partial f(\mathcal{L}, \mathcal{M})}{\partial \mathcal{L}} & \frac{\partial f(\mathcal{L}, \mathcal{M})}{\partial \mathcal{M}} \\
\frac{\partial g(\mathcal{L}, \mathcal{M})}{\partial \mathcal{L}} & \frac{\partial g(\mathcal{L}, \mathcal{M})}{\partial \mathcal{M}}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \mathcal{L}}{\partial \lambda} & \frac{\partial \mathcal{L}}{\partial t_n} \\
\frac{\partial \mathcal{M}}{\partial \lambda} & \frac{\partial \mathcal{M}}{\partial t_n}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial \hat{\mathcal{L}}}{\partial \lambda} & \frac{\partial \hat{\mathcal{L}}}{\partial t_n} \\
\frac{\partial \hat{\mathcal{M}}}{\partial \lambda} & \frac{\partial \hat{\mathcal{M}}}{\partial t_n}
\end{pmatrix}.
\tag{6.9}
\]

Combining Eqs. (6.5) and (6.9), one can eliminate the derivative matrix of \((f, g)\) by \((\mathcal{L}, \mathcal{M})\) and obtain the matrix relation

\[
\begin{pmatrix}
\frac{\partial \mathcal{L}}{\partial \lambda} & \frac{\partial \mathcal{L}}{\partial x} \\
\frac{\partial \mathcal{M}}{\partial \lambda} & \frac{\partial \mathcal{M}}{\partial x}
\end{pmatrix}^{-1}
\begin{pmatrix}
\frac{\partial \mathcal{L}}{\partial \lambda} & \frac{\partial \mathcal{L}}{\partial t_n} \\
\frac{\partial \mathcal{M}}{\partial \lambda} & \frac{\partial \mathcal{M}}{\partial t_n}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial \hat{\mathcal{L}}}{\partial \lambda} & \frac{\partial \hat{\mathcal{L}}}{\partial t_n} \\
\frac{\partial \hat{\mathcal{M}}}{\partial \lambda} & \frac{\partial \hat{\mathcal{M}}}{\partial t_n}
\end{pmatrix}.
\tag{6.10}
\]
Since the two $2 \times 2$ matrices on both sides are unimodular because of Eq. (6.8), the inverse can also be written explicitly. In components, thus, the above matrix relation gives:

$$\frac{\partial M}{\partial x} \frac{\partial L}{\partial t_n} - \frac{\partial L}{\partial x} \frac{\partial M}{\partial t_n} = \frac{\partial \hat{M}}{\partial x} \frac{\partial \hat{L}}{\partial t_n} - \frac{\partial \hat{L}}{\partial x} \frac{\partial \hat{M}}{\partial t_n},$$

$$\frac{\partial M}{\partial \lambda} \frac{\partial L}{\partial t_n} - \frac{\partial L}{\partial \lambda} \frac{\partial M}{\partial t_n} = \frac{\partial \hat{M}}{\partial \lambda} \frac{\partial \hat{L}}{\partial t_n} - \frac{\partial \hat{L}}{\partial \lambda} \frac{\partial \hat{M}}{\partial t_n}. \quad (6.11)$$

The left hand side of Eqs. (6.11) can be calculated just as we have done above for derivatives in $(x, \lambda)$. For the first equation of (6.11),

$$\frac{\partial M}{\partial x} \frac{\partial L}{\partial t_n} - \frac{\partial L}{\partial x} \frac{\partial M}{\partial t_n} = \left[ \left( \frac{\partial M}{\partial L} \right)_{t,v \ fixed} \frac{\partial L}{\partial x} + 1 + \sum_{i=1}^{\infty} \frac{\partial v_{i+1}}{\partial x} \frac{L^i}{L} \right] \frac{\partial L}{\partial t_n}$$

$$- \frac{\partial L}{\partial x} \left[ \left( \frac{\partial M}{\partial L} \right)_{t,v \ fixed} \frac{\partial L}{\partial t_n} n L^{n-1} + \sum_{i=1}^{\infty} \frac{\partial v_{i+1}}{\partial t_n} \frac{L^{-i}}{L} \right], \quad (6.12)$$

and terms containing $(\partial M/\partial L)_{t,v \ fixed}$ cancel. The rest can be easily evaluated. Thus,

$$\frac{\partial M}{\partial x} \frac{\partial L}{\partial t_n} - \frac{\partial L}{\partial x} \frac{\partial M}{\partial t_n} = -\partial \left( L^n \right)_{\geq 0} + (\text{negative powers of } \lambda). \quad (6.13)$$

Similarly,

$$\frac{\partial M}{\partial \lambda} \frac{\partial L}{\partial t_n} - \frac{\partial L}{\partial \lambda} \frac{\partial M}{\partial t_n} = -\partial \left( L^n \right)_{\geq 0} + (\text{negative powers of } \lambda). \quad (6.14)$$

The right hand side of Eqs. (6.11), meanwhile, have Laurent expansion with only nonnegative powers of $\lambda$. Therefore only nonnegative powers of $\lambda$ should survive, thus

$$\frac{\partial M}{\partial x} \frac{\partial L}{\partial t_n} - \frac{\partial L}{\partial x} \frac{\partial M}{\partial t_n} = -\partial \left( L^n \right)_{\geq 0} = -\frac{\partial B_n}{\partial x},$$

$$\frac{\partial M}{\partial \lambda} \frac{\partial L}{\partial t_n} - \frac{\partial L}{\partial \lambda} \frac{\partial M}{\partial t_n} = -\partial \left( L^n \right)_{\geq 0} = -\frac{\partial B_n}{\partial \lambda}. \quad (6.15)$$

These equations can be readily solved (again due to the unimodular property, (6.8), of the coefficient matrix):

$$\frac{\partial L}{\partial t_n} = -\frac{\partial L}{\partial \lambda} \frac{\partial B_n}{\partial x} + \frac{\partial M}{\partial x} \frac{\partial B_n}{\partial \lambda} = \{B_n, L\},$$

$$\frac{\partial M}{\partial t_n} = -\frac{\partial M}{\partial \lambda} \frac{\partial B_n}{\partial x} + \frac{\partial M}{\partial x} \frac{\partial B_n}{\partial \lambda} = \{B_n, M\}. \quad (6.16)$$

One can thus derive the Lax equations as well. □
We thus anyway have a solution scheme for general solutions of the SDiff(2) KP hierarchy. Unfortunately, finding an explicit form of the solution of the functional equations is a very hard problem at this moment. The case of the self-dual vacuum Einstein equation and its hyper-Kähler version is slightly simpler because \( \lambda \) therein is just a parameter unlike the present setting; nevertheless, explicitly solvable cases are very limited [27] [28][29]. Very few is known for the SDiff(2) version. We now turn to this issue.

7. Reductions and special solutions

7.1. Krichever’s dispersionless Lax equations. In Krichever’s original formulation [3], “dispersionless Lax equations” are Lax equations

\[
\frac{\partial P}{\partial t_n} = \{ B_n, P \}
\]  

of a polynomial

\[
P = \lambda^N + p_2\lambda^{N-2} + \cdots + p_N
\]

rather than an infinite Laurent series like the \( \mathcal{L} \). The \( B_n \)’s are now given by

\[
B_n = (P^{n/N})_{\geq 0}.
\]

Obviously, the \( N \)-th root

\[
\mathcal{L} = P^{1/N}
\]

of \( \mathcal{L} \) satisfies the Lax equations of the SDiff(2) KP hierarchy, and conversely, the above situation is characterized by the condition

\[
(\mathcal{L}_N)^{\leq -1} = 0.
\]

In the case of the ordinary KP hierarchy [1][2], this amounts a reduction of generalized KdV type (the KdV equation for \( N = 2 \), the Boussinesq for \( N = 3 \), etc.). After the terminology therein, one may call this reduction “\( N \)-reduction.”

Eqs. (7.1) give rise to a system of evolution equations for \( p_2, \ldots, p_N \),

\[
\frac{\partial p_i}{\partial t_n} = \sum_{j=2}^{N} f_{ijN}(p_2, \ldots, p_N) \frac{\partial p_j}{\partial x},
\]

where \( f_{ijN}(p_2, \ldots, p_N) \) are polynomials of \( p_2, \ldots, p_N \). These equations fall into a family of equations of “hydrodynamic type” studied by Krichever, Novikov and Tsarev [30][31][32]. Krichever [3] pointed out a link of the above dispersionless Lax equations with topological minimal models [4][5]. Our goal in this section is to give an interpretation of his observation in our framework.
7.2. Hodograph transformation method for $N = 2$. According to Tsarev [31], the traditional method of “hodograph transformation” can be extended to these equations. We now illustrate this method in the $N = 2$ case.

It is convenient to write $\mathcal{P}$ as

$$\mathcal{P} = \lambda^2 + 2p, \quad p = p_2/2. \quad (7.7)$$

In this case, only “odd” flows associated with $(t_3, t_5, \ldots)$ are nontrivial. For “even” flows, we have $B_{2n} = \mathcal{L}^{2n}$ for $n = 1, 2, \ldots$, therefore

$$\frac{\partial \mathcal{L}}{\partial t_{2n}} = \{\mathcal{L}^{2n}, \mathcal{L}\} = 0, \quad (7.8)$$

$$\frac{\partial \mathcal{M}}{\partial t_{2n}} = \{\mathcal{L}^{2n}, \mathcal{M}\} = 2n\mathcal{L}^{2n-1}, \quad (7.9)$$

which shows that $u_i$ and $v_i$ are independent of $t_{2n}$. To describe odd flows, we introduce a set of polynomials of $p$:

$$r_n(p) = \text{res} \mathcal{P}^{n+1/2} d\lambda. \quad (7.10)$$

They correspond to the Gelfand-Dikii resolvent functionals of the KdV equation [33]. In the present setting, the fractional powers of $\mathcal{P}$ can be calculated by means of the binomial expansion of $\mathcal{P}^{n+1/2} = (\lambda^2 + 2p)^{n+1/2}$ as:

$$r_n(p) = \left( \begin{array}{c} n + 1/2 \\ n + 1 \end{array} \right) (2p)^n = \frac{(2n + 1)!!}{(n + 1)!} p^n. \quad (7.11)$$

They obey a set of recursion relations just as the Gelfand-Dikii resolvent functionals:

$$\frac{\partial r_n(p)}{\partial p} = (2n + 1)r_{n-1}(p). \quad (7.12)$$

Now the right hand side of the Lax equations for odd flows can be written

$$\{(\mathcal{P}^{n+1/2})_{\geq 0}, \mathcal{P}\} = -\{(\mathcal{P}^{n+1/2})_{\leq -1}, \mathcal{P}\} = 2\frac{r_n(p)}{\partial x} + O(\lambda^{-1}). \quad (7.13)$$

Actually, the left hand side of Eq. (7.13) should be a polynomial of $\lambda$, hence the remainder term $O(\lambda^{-1})$ on the right hand side must be absent. Thus the Lax equations can be reduced to evolution equations of $p$:

$$\frac{\partial p}{\partial t_{2n+1}} = \frac{\partial r_n(p)}{\partial x} = (2n + 1)r_{n-1}(p)\frac{\partial p}{\partial x}. \quad (7.14)$$

The $t_3$-flow gives the “dispersionless KdV equation”

$$\frac{\partial p}{\partial t_3} = \frac{\partial}{\partial x} \left( \frac{3}{2} p^2 \right) = 3p\frac{\partial p}{\partial x}. \quad (7.15)$$

The method of “generalized hodograph transformations” [31] gives a general solution $p = p(x, t_3, t_5, \ldots)$ of Eqs. (7.15) as an implicit function:

$$x + \sum_{n=1}^{\infty} (2n + 1)r_{n-1}(p)t_{2n+1} = \phi(p), \quad (7.16)$$

where $\phi$ is an arbitrary function of one variable. Hence solving this (transcendental) equation, one obtains a solution that depends on an arbitrary function. Note that if $\phi$ has a Taylor expansion at the origin, this arbitrary data can be absorbed into shift of the time variables.
7.3. Special solution for general $N$. We construct a special solution for a general value of $N$ by the method of Section 6. Let the $(f,g)$-pair be given by
\[ f(\lambda, x) = \lambda^N / N, \quad g(\lambda, x) = x \lambda^{1-N}. \] (7.17)

It is convenient to use
\[ P = \defeq \mathcal{L}^N / N, \quad Q = \defeq \mathcal{L}^{1-N} \mathcal{M} \] (7.18)
rather than $\mathcal{L}$ and $\mathcal{M}$. This is essentially a change of canonical variables:
\[ \omega = d\mathcal{L} \land d\mathcal{M} = dP \land dQ. \] (7.19)

The Riemann-Hilbert problem for $\mathcal{L}$ and $\mathcal{M}$ is thus converted to solving the equations
\[ (P)_{\leq -1} = 0, \quad (Q)_{\leq -1} = 0 \] (7.20)
for $P$ and $Q$. The first equation simply means that $P$ is a polynomial in $\lambda$:
\[ P = \lambda^N / N + p_2 \lambda^{N-2} + \cdots + p_N. \] (7.21)

[We have slightly modified the parameterization in (7.2).] In view of (7.18), $Q$ is required to be a Laurent series of $\mathcal{L}$ of the following form.
\[ Q = \sum_{n=1}^{\infty} nt_n \mathcal{L}^{n-N} + \sum_{i=0}^{\infty} v_{i+1} \mathcal{L}^{-i-N} \] (7.22)

(Actually, we shall see that the $v_1$-term disappears.) The second equation of (7.20) then becomes
\[ \sum_{n=1}^{N-1} nt_n \mathcal{L}^{n-N} + \sum_{i=0}^{\infty} v_{i+1} \mathcal{L}^{-i-N} + \sum_{k=N+1}^{\infty} kt_k (\mathcal{L}^{k-N})_{\leq -1} = 0. \] (7.23)

Now multiply this equation with $\mathcal{L}^{N-n-1} d\lambda \mathcal{L}$ or $\mathcal{L}^{N+i-1} d\lambda \mathcal{L}$ and take the residue. In the first and second sums of (7.23), only a single term survives after this manipulation (recall Lemma B of Section 4). In the third sum, the $k = N$ term disappears because $(\mathcal{L}^{k-N})_{\leq -1} = 0$ for $k = N$, but the other terms give nontrivial contribution in general. Thus we have
\[ nt_n + \sum_{k=N+1}^{\infty} kt_k \text{ res } [(\mathcal{L}^{k-N})_{\leq -1} \mathcal{L}^{N-n-1} d\lambda \mathcal{L}] = 0 \] (7.24)
for $n = 1, \ldots, N-1$, and
\[ v_{i+1} + \sum_{k=N+1}^{\infty} kt_k \text{ res } [(\mathcal{L}^{k-N})_{\leq -1} \mathcal{L}^{N+i-1} d\lambda \mathcal{L}] = 0 \] (7.25)
for $i = 0, 1, \ldots$. We now argue that
- Eqs. (7.24) may be thought of as a generalized hodograph transformation that determines the coefficients of $P$ as implicit functions,
- Eqs. (7.25) give the Laurent coefficients $v_{i+1}$ of $Q$, and
- Krichever’s formula for the tau function and “dispersionless analogues of Virasoro constraints” [3] can be reproduced in the present setting.
7.4. Structure of Eqs. (7.24) and (7.25). We first consider Eqs. (7.24). A basic tool is the residue identities

\[
\text{res} [(L_{n})_{\leq -1}d_{\lambda}L] = \frac{1}{m} \text{res} [(L_{n})_{\leq -1}d_{\lambda}(L_{m})] \\
= \frac{1}{m} \text{res} [(L_{n})_{\leq -1}d_{\lambda}B_{m}] \\
= \frac{1}{m} \text{res} [L_{n}d_{\lambda}B_{m}] \tag{7.26}
\]

that we have used in Sections 4 and 5 [see (4.19) and (5.4)]. Note that these residues give polynomials of \( p_{2}, \ldots, p_{N} \). In particular [see (5.7)],

\[
\text{res} [(L_{n})_{\leq -1}L\lambda d\lambda L] = \frac{1}{N - n} \text{res} [(L^{N-n})_{\leq -1}d_{\lambda}L] \\
= -q_{N-n+1} \tag{7.27}
\]

The last identity gives us a key to understand the role of flat coordinates [25] in topological minimal models [4][5]. To see this, restrict part of time variables to special values:

\[
t_{N+1} = \frac{1}{N + 1}, \quad t_{N+2} = t_{N+3} = \cdots = 0. \tag{7.28}
\]

Eqs. (7.24) then takes a very simple form:

\[
nt_{n} - q_{N-n+1} = 0 \quad (n = 1, \ldots, N - 1). \tag{7.29}
\]

These are the fundamental relation in topological minimal models that links time variables (which are identified with “coupling constants” [4][5]) of flow equations with flat coordinates. The same relation can also be found in Krichever’s interpretation.

Let us continue the analysis of Eqs. (7.24). To see how the unknown functions \( p_{2}, \ldots, p_{N} \) are to be determined, we have to know the structure of the \( q_{n}'s \) as polynomials of the \( p_{n}'s \). The theory of flat coordinates [25] provides very detailed information on this issue. For the moment, however, the following is sufficient.

**Lemma C.** The invertible polynomial relation connecting \( p_{2}, \ldots, p_{N} \) and \( q_{2}, \ldots, q_{N} \) can be written

\[
q_{n} = -p_{n} + \text{(polynomials of } p_{2}, \ldots, p_{N}) \tag{7.30}
\]

for \( n = 2, \ldots, N \).

**Proof:** The \( i \)-th power of \( L \), in general, can be expanded in powers of \( \lambda \) as:

\[
L^{i} = \lambda^{i} + iu_{2}\lambda^{i-2} + iu_{3}\lambda^{i-3} + \cdots \\
+ (iu_{j} + \text{(polynomial of } u_{2}, \ldots, u_{j-1})\lambda^{i-j} + \cdots,
\]

where \( u_{j} \) are constants.
therefore, letting \( i = N \) and \( j = n \), we have
\[
Np_{n+1} = Nu_{n+1} + (\text{polynomial of } u_2, \ldots, u_n).
\]

Meanwhile, by the same expansion with \( i = n \) and \( j = n+1 \),
\[
q_{n+1} = -\frac{1}{n} \ \text{res } \mathcal{L}^n d\lambda = -u_{n+1} + (\text{polynomial of } u_2, \ldots, u_n).
\]

From these relations, one can deduce (7.30). \( \Box \)

Consequently, Eqs. (7.24) takes such a form as:
\[
\begin{align*}
t_1 - (N + 1)t_{N+1}q_N + \sum_{k=N+2}^{\infty} kt_k \ \text{res } [\cdots] &= 0, \\
2t_2 - (N + 1)t_{N+1}q_{N-1} + \sum_{k=N+2}^{\infty} kt_k \ \text{res } [\cdots] &= 0, \\
\cdots \cdots \\
(N - 1)t_{N-1} - (N + 1)t_{N+1}q_2 + \sum_{k=N+2}^{\infty} kt_k \ \text{res } [\cdots] &= 0, \\
\end{align*}
\]
and the residues \( \text{res } [\cdots] \) are polynomials of \( p_2, \ldots, p_N \) (or, equivalently, of \( q_2, \ldots, q_N \)). If \( t_{N+1} \neq 0 \), one can rewrite these equations as:
\[
\begin{align*}
q_N &= \frac{t_1}{(N + 1)t_{N+1}} + \sum_{k=N+2}^{\infty} \frac{kt_k \ \text{res } [\cdots]}{(N + 1)t_{N+1}}, \\
q_{N-1} &= \frac{2t_2}{(N + 1)t_{N+1}} + \sum_{k=N+2}^{\infty} \frac{kt_k \ \text{res } [\cdots]}{(N + 1)t_{N+1}}, \\
\cdots \cdots \\
q_2 &= \frac{(N - 1)t_{N-1}}{(N + 1)t_{N+1}} + \sum_{k=N+2}^{\infty} \frac{kt_k \ \text{res } [\cdots]}{(N + 1)t_{N+1}}.
\end{align*}
\]

One can now use an ordinary implicit function theorem to ensure the existence of functions \( q_2, \ldots, q_N \) (hence \( p_2, \ldots, p_N \)) that satisfy these equations. Note that these functions depend only on the ratios \( t_n/t_{N+1} \) \( (n \neq N + 1) \) rather than \( t_n \) themselves.

Thus:

**Proposition 8.** Eqs. (7.24) has a solution that consists of homogeneous functions \( p_2, \ldots, p_N \) of degree zero. These functions are defined in a domain where \( t_{N+1} \neq 0 \) and \( t_n/t_{N+1} \ (n \neq N + 1) \) are small.

Having such a solution of Eqs. (7.24), one can readily solve Eqs. (7.25) as:
\[
v_{i+1} = -\sum_{k=N+1}^{\infty} kt_k \ \text{res } [(\mathcal{L}^{k-N})_{\leq -1} \mathcal{L}^{N+i-1} d\lambda \mathcal{L}].
\]

The residues on the right hand side are polynomial functions of \( p_2, \ldots, p_N \), hence homogenous functions of degree zero. Therefore:
Proposition 9. A set of functions $v_{i+1}$ defined by (7.32) are homogeneous functions of degree one and give a solution of Eqs. (7.25). Further, $v_1 = 0$.

Proof of last part: From (9.32) [and recalling (7.26)], we have

$$v_1 = - \sum_{k=N+1}^{\infty} k t_k \text{res} [(\mathcal{L}^{k-N})^{\leq -1} \mathcal{L}^{N-1} d_{\lambda} \mathcal{L}]$$

$$= - \sum_{k=N+1}^{\infty} k t_k \text{res} [(\mathcal{L}^N)^{\leq -1} d_{\lambda} B_{k-N}].$$

The $N$-th power $\mathcal{L}^N$, however, is a polynomial of $\lambda$ because of the construction [see (7.21)]. Therefore $(\mathcal{L}^N)^{\leq -1} = 0$, and the last residues should vanish. This means $v_1 = 0$. \(\square\)

7.5. Tau function and nonlinear constraints. Recall that the tau function is defined by

$$\frac{\partial \log \tau}{\partial t_n} = v_{n+1} \quad (n = 1, 2, \ldots). \quad (7.33)$$

Since the right hand side of these equations are homogeneous functions of degree one (Proposition 9), $\log \tau$ becomes a homogeneous function of degree two (plus an integration constant). Consequently,

$$\sum_{n=1}^{\infty} t_n \frac{\partial \log \tau}{\partial t_n} = 2 \log \tau. \quad (7.34)$$

Applying the operator $\sum_{n=1}^{\infty} t_n \partial / \partial t_n$ once again, we have

$$\sum_{n,m=1}^{\infty} t_m t_n \frac{\partial^2 \log \tau}{\partial t_m \partial t_n} = 2 \log \tau. \quad (7.35)$$

These relations show an explicit form of the tau function:

Proposition 10. The tau function, up to an integration constant, is given by

$$\log \tau = \frac{1}{2} \sum_{n=1}^{\infty} t_n v_{n+1} + \text{const.} \quad (7.36)$$

or, equivalently, by

$$\log \tau = \frac{1}{2} \sum_{n,m=1}^{\infty} t_m t_n \frac{\partial v_{n+1}}{\partial t_m} + \text{const.} \quad (7.37)$$

The last formula, (7.37), has another expression. To see this, recall that (Section 4)

$$B_m = \mathcal{L}^m - \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial v_{n+1}}{\partial t_m} \mathcal{L}^{-n}. \quad (7.38)$$
By virtue of this identity, (7.37) can be rewritten
\[ \log \tau = \frac{1}{2} \text{res} \left[ \sum_{n=1}^{\infty} t_n \mathcal{L}^n \right] d_{\lambda} \left( \sum_{n=1}^{\infty} t_m \mathcal{B}_m \right). \] (7.39)

This exactly reproduces Krichever's formula of the tau function [3].

Krichever’s dispersionless analogues of Virasoro constraints [3], too, can be deduced as follows.

**Proposition 11.** The tau function satisfies the constraints
\[ \sum_{k=N+1}^{\infty} k t_k \frac{\partial \log \tau}{\partial t_{k-N}} + \frac{1}{2} \sum_{i=1}^{N-1} i(N-i) t_i t_{N-i} = 0, \] (7.40)
\[ \sum_{k=1}^{\infty} k t_k \frac{\partial \log \tau}{\partial t_k} = 0, \] (7.41)
\[ \sum_{k=1}^{\infty} k t_k \frac{\partial \log \tau}{\partial t_{k+mN}} + \frac{1}{2} \sum_{i=1}^{mN-1} \frac{\partial \log \tau}{\partial t_i} \frac{\partial \log \tau}{\partial t_{mN-i}} = 0, \] (7.42)
where \( m \) ranges over \( m = 1, 2, \ldots \).

**Proof:** We first derive (7.40) from Eqs. (7.24) and (7.25). Recall, again, the identity (see Section 5)
\[ \text{res} \left[ \mathcal{L}^n d_{\lambda} \mathcal{B}_m \right] = \frac{\partial v_{N+1}}{\partial t_m} = \frac{\partial v_{m+1}}{\partial t_n} = \frac{\partial^2 \log \tau}{\partial t_m \partial t_n}. \] (7.43)

Because of these identities, Eqs. (7.24) and (7.25) become differential equations for \( \log \tau \):
\[ (N-n) n t_n + \sum_{k=N+1}^{\infty} k t_k \frac{\partial^2 \log \tau}{\partial t_{N-n} \partial t_{k-N}} = 0, \] (7.44)
\[ (N+i) \frac{\partial \log \tau}{\partial t_i} + \sum_{k=N+1}^{\infty} k t_k \frac{\partial^2 \log \tau}{\partial t_{N+i} \partial t_{k-N}} = 0, \] (7.45)
where the indices \( n \) and \( i \) range over \( n = 1, \ldots, N-1 \) and \( i = 1, 2, \ldots \). Eq. (7.45), after replacing \( i \to i-N \), can be rewritten
\[ \frac{\partial}{\partial t_i} \left( \sum_{k=N+1}^{\infty} k t_k \frac{\partial \log \tau}{\partial t_{k-N}} \right) = 0, \] (7.46)
which means that \( \sum_{k=N+1}^{\infty} k t_k \frac{\partial \log \tau}{\partial t_{k-N}} \) does not depend on \( t_{N+1}, t_{N+2}, \ldots \). Eqs. (7.44) can be rewritten, by similar calculations, as
\[ \frac{\partial}{\partial t_{N-n}} \left( \sum_{k=N+1}^{\infty} k t_k \frac{\partial \log \tau}{\partial t_{k-N}} + \frac{1}{2} \sum_{i=1}^{N-1} i(N-i) t_i t_{N-i} \right) = 0. \] (7.47)
These relations show that
\[ \sum_{k=N+1}^{\infty} k t_k \partial \log \tau_{k-N} + \frac{1}{2} \sum_{i=1}^{N-1} i(N-i)t_i t_{N-i} = \text{const.} \]

The left hand side of this equality, however, is a homogeneous function of degree two, hence the constant on the right hand side has to vanish. One can thus derive (7.40). To derive (7.41) and (7.42), note that Eqs. (7.24) and (7.25) are simply a restatement of the second equation of (7.20). Actually, \( \mathcal{P} \) and \( \mathcal{Q} \) obeying (7.20) also satisfy an infinite number of similar equations such as
\[ (\mathcal{P}^{m+1} \mathcal{Q})_{\leq -1} = 0 \quad (m = 0, 1, 2, \ldots), \quad (7.48) \]
and each of these equations give rise to a set of relations like Eqs. (7.24) and (7.25). Starting from these relations, one can deduce (7.41) and (7.42) in the same way as we have derived (7.40). This completes the proof. \( \square \)

The above proof shows that there are actually more constraints that our tau function are satisfying. Besides (7.48), in fact, we have
\[ (\mathcal{P}^{m+1} \mathcal{Q}^n)_{\leq -1} = 0 \quad (m \geq -1, n \geq 2), \quad (7.49) \]
and accordingly a corresponding set of nonlinear constraints. This observation reminds us of the “W constraints” in the \( d < 1 \) string theory [34][35] and “twisted \( W_\infty \) constraints” in a \( d = 1 \) version [19]. We shall show a more systematic interpretation of these constraints in Section 8, exploiting SDiff(2) (= \( w_\infty \)) symmetries of our hierarchy.

7.6. Deformations. The previous solution has a family of deformations generated by the \( (f, g) \)-pair
\[ f(\lambda, x) = \lambda^N / N, \quad g(\lambda, x) = \lambda^{1-N} x + h(\lambda), \quad (7.50) \]
where \( h(\lambda) \) is an arbitrary function with Laurent expansion
\[ h(\lambda) = \sum_{k=-\infty}^{\infty} h_k \lambda^k. \quad (7.51) \]
The \( (\mathcal{P}, \mathcal{Q}) \)-pair is now given by
\[ \mathcal{P} = \mathcal{L}^N / N, \quad \mathcal{Q} = \mathcal{L}^{1-N} \mathcal{M} + h(\mathcal{L}). \quad (7.52) \]
Eqs. (7.24) and (7.25) are also deformed:
\[ n t_n + h_{n-N} + \sum_{k=N+1}^{\infty} (kt_k + h_{k-N}) \text{res} \left[ (\mathcal{L}^{k-N})_{\leq -1} \mathcal{L}^{N-n-1} d_\lambda \mathcal{L} \right] = 0, \quad (7.53) \]
\[ v_{i+1} + h_{-i-N} + \sum_{k=N+1}^{\infty} (kt_k + h_{k-N}) \text{res} \left[ (\mathcal{L}^{k-N})_{\leq -1} \mathcal{L}^{N+i-1} d_\lambda \mathcal{L} \right] = 0. \quad (7.54) \]
Previous calculations can be mostly carried over to this case, except that the $v_1$-term now does not vanish in general,

$$v_1 = -h_{-N}. \quad (7.55)$$

(This is, however, a redundant degree of freedom as we have mentioned in Section 4.) If $N = 2$, Eqs. (7.53) reduces to a single equation:

$$x + h_{-1} + \sum_{k=N+1}^{\infty} (kt_k + h_{k-2}) \text{res} [(L^{k-1})_{\leq -1} d\lambda L] = 0. \quad (7.56)$$

The residues in the last part can be written

$$\text{res} [(L^{k-1})_{\leq -1} d\lambda L] = \text{res} [L^{k-2} d\lambda] = \begin{cases} r_{n-1}(p) & \text{if } k = 2n + 1, \\ 0 & \text{if } k = 2n. \end{cases} \quad (7.57)$$

Thus Eq. (7.56) is essentially the same “hodograph” relation as Eq. (7.16). The arbitrary function $\phi(p)$ is connected with the Riemann-Hilbert data $h(\lambda)$ as:

$$\phi(p) = -\text{res} h(L)d\lambda. \quad (7.58)$$

The tau function of the deformed solutions can be readily determined, because the presence of $h(\lambda)$ simply affects as shifting $t_n$’s and $v_{i+1}$’s by constants:

**Proposition 12.** Let $\tau_0 = \tau_0(t)$ be the tau function of the undeformed solution ($h = 0$). The tau function $\tau_h = \tau_h(t)$ of the deformed solution is then given by

$$\tau_h(t) = \text{const.} \exp \left( -\sum_{n=1}^{\infty} h_{-n-N} t_n \right) \times \tau_0(t_1 + h_{1-N}, t_2 + h_{2-N}/2, \ldots, t_k + h_{k-N}/k, \ldots). \quad (7.59)$$

In the context of topological minimal models, insertion of $h(\lambda)$ amounts to “perturbations” of the model [4][5]. Of course, as we have seen above, these deformations are actually absorbed into the time variables $t_n$ of the hierarchy. In the same context, Eqs. (7.24) and (7.53) may be interpreted as “Landau-Ginzburg equations,” i.e., “topological” analogues of $d < 1$ string equations [36]. Unlike the latter, these topological analogues contain no derivatives of unknown functions. One can therefore resort to the implicit function theorem to ensure the existence of solutions.
8. SDiff(2) symmetries

8.1. Symmetries in terms of \((\mathcal{L}, \mathcal{M})\). The method of construction of symmetries is based upon the same principle as the case of the self-dual vacuum Einstein equations \[37\][38]. We have established a correspondence between \((\mathcal{L}, \mathcal{M})\) and \((f, g)\). The SDiff(2) group structure in the data \((f, g)\) gives rise to transformations of a solution to another by, say, the right action of a given SDiff(2) group element. To find their infinitesimal form, we consider a one-parameter family of transformations \((\mathcal{L}, \mathcal{M}) \rightarrow (\mathcal{L}(\epsilon), \mathcal{M}(\epsilon))\) generated by the right action

\[
(f, g) \longrightarrow (f, g) \circ \exp(-\epsilon \{F, \cdot\}),
\]

where \(\{F, \cdot\}\) is a Hamiltonian vector field,

\[
\{F, \cdot\} = \frac{\partial F}{\partial \lambda} \frac{\partial}{\partial x} - \frac{\partial F}{\partial x} \frac{\partial}{\partial \lambda},
\]

and \(F(\lambda, x)\) is assumed to have the same analyticity properties as \(f(\lambda, x)\) and \(g(\lambda, x)\). We then calculate the transformed pair \((\mathcal{L}(\epsilon), \mathcal{M}(\epsilon))\) to the first order of \(\epsilon\):

\[
\mathcal{L}(\epsilon) = \mathcal{L} + \epsilon \delta \mathcal{L} + O(\epsilon^2),
\]

\[
\mathcal{M}(\epsilon) = \mathcal{M} + \epsilon \delta \mathcal{M} + O(\epsilon^2).
\]

The coefficients \(\delta \mathcal{L}\) and \(\delta \mathcal{M}\) define a linear operator \(\delta = \delta_F\) that represents an infinitesimal symmetry of the SDiff(2) KP hierarchy. By definition, \(\delta_F\) acts on any function of \(\mathcal{L}\) and \(\mathcal{M}\) as an abstract derivation,

\[
\delta_F G(\mathcal{L}, \mathcal{M}) = \frac{\partial G}{\partial \mathcal{L}} \delta_F \mathcal{L} + \frac{\partial G}{\partial \mathcal{M}} \delta_F \mathcal{M}
\]

whereas leaves invariant the independent variables of the hierarchy,

\[
\delta_F t_n = \delta_F x = \delta_F \lambda = 0.
\]

(This is a formal way to understand infinitesimal symmetries of differential equations in the language of differential algebras \[39\].)

For the self-dual vacuum Einstein equation, infinitesimal symmetries thus constructed have an explicit and compact expression \[38\]; this is also the case for the present situation. Since precise calculations are only required to the first order of \(\epsilon\), one can write the Riemann-Hilbert problem for \(\mathcal{L}(\epsilon)\) and \(\mathcal{M}(\epsilon)\) as:

\[
\begin{align*}
\left. f \left( \lambda + \epsilon F_x(\lambda, x) + O(\epsilon^2), x - \epsilon F_x(\lambda, x) + O(\epsilon^2) \right) \right|_{\lambda \rightarrow \mathcal{L}(\epsilon), x \rightarrow \mathcal{M}(\epsilon)} &= \mathcal{L} + \epsilon \delta \mathcal{L} + O(\epsilon^2), \\
\left. g \left( \lambda + \epsilon F_x(\lambda, x) + O(\epsilon^2), x - \epsilon F_x(\lambda, x) + O(\epsilon^2) \right) \right|_{\lambda \rightarrow \mathcal{L}(\epsilon), x \rightarrow \mathcal{M}(\epsilon)} &= \mathcal{M} + \epsilon \delta \mathcal{M} + O(\epsilon^2),
\end{align*}
\]
where \( F_\lambda(\lambda,x) \equiv \partial F(\lambda,x)/\partial \lambda \), etc. as usual; note that the right hand side are also affected by the change of the data \((f,g)\). From the \( \epsilon \)-terms, one obtains a set of equations that should determine \( \delta_f, \delta_g \) also affected by the change of the data \((f,g)\). In a matrix form, these equations can be written

\[
\begin{pmatrix}
\frac{\partial f(\mathcal{L},\mathcal{M})}{\partial \mathcal{L}} & \frac{\partial f(\mathcal{L},\mathcal{M})}{\partial \mathcal{M}} \\
\frac{\partial F(\mathcal{L},\mathcal{M})}{\partial \mathcal{L}} & \frac{\partial F(\mathcal{L},\mathcal{M})}{\partial \mathcal{M}}
\end{pmatrix}
\begin{pmatrix}
\delta \mathcal{L} + \frac{\partial F(\mathcal{L},\mathcal{M})}{\partial \mathcal{M}} \\
\delta \mathcal{M} - \frac{\partial F(\mathcal{L},\mathcal{M})}{\partial \mathcal{L}}
\end{pmatrix}
= \begin{pmatrix}
\delta \hat{\mathcal{L}} \\
\delta \hat{\mathcal{M}}
\end{pmatrix}.
\] (8.7)

Now the situation is very similar to the proof of Proposition 7, except that \( \partial \mathcal{L}/\partial t_n \) etc. therein are replaced as:

\[
\begin{align*}
\frac{\partial \mathcal{L}}{\partial t_n} & \to \delta \mathcal{L} + \frac{\partial F(\mathcal{L},\mathcal{M})}{\partial \mathcal{M}}, & \frac{\partial \hat{\mathcal{L}}}{\partial t_n} & \to \delta \hat{\mathcal{L}}, \\
\frac{\partial \mathcal{M}}{\partial t_n} & \to \delta \mathcal{M} - \frac{\partial F(\mathcal{L},\mathcal{M})}{\partial \mathcal{L}}, & \frac{\partial \hat{\mathcal{M}}}{\partial t_n} & \to \delta \hat{\mathcal{M}}.
\end{align*}
\] (8.8)

After eliminating the derivative matrix of \((f,g)\) by \((\mathcal{L},\mathcal{M})\), one has

\[
\begin{align*}
\frac{\partial \mathcal{M}}{\partial x} & \left( \delta \mathcal{L} + \frac{\partial F(\mathcal{L},\mathcal{M})}{\partial \mathcal{M}} \right) - \frac{\partial \mathcal{L}}{\partial \mathcal{M}} \delta \mathcal{M} - \frac{\partial F(\mathcal{L},\mathcal{M})}{\partial \mathcal{L}} = \frac{\partial \hat{\mathcal{M}}}{\partial x} \delta \hat{\mathcal{L}} - \frac{\partial \hat{\mathcal{L}}}{\partial x} \delta \hat{\mathcal{M}}, \\
\frac{\partial \mathcal{M}}{\partial \lambda} & \left( \delta \mathcal{L} + \frac{\partial F(\mathcal{L},\mathcal{M})}{\partial \mathcal{M}} \right) - \frac{\partial \mathcal{L}}{\partial \mathcal{M}} \delta \mathcal{M} - \frac{\partial F(\mathcal{L},\mathcal{M})}{\partial \mathcal{L}} = \frac{\partial \hat{\mathcal{M}}}{\partial \lambda} \delta \hat{\mathcal{L}} - \frac{\partial \hat{\mathcal{L}}}{\partial \lambda} \delta \hat{\mathcal{M}}.
\end{align*}
\] (8.9)

or, in a more compact form,

\[
\begin{align*}
\frac{\partial \mathcal{M}}{\partial x} \delta \mathcal{L} - \frac{\partial \mathcal{L}}{\partial x} \delta \mathcal{M} - \frac{\partial F(\mathcal{L},\mathcal{M})}{\partial \mathcal{M}} = \frac{\partial \hat{\mathcal{M}}}{\partial x} \delta \hat{\mathcal{L}} - \frac{\partial \hat{\mathcal{L}}}{\partial x} \delta \hat{\mathcal{M}}, \\
\frac{\partial \mathcal{M}}{\partial \lambda} \delta \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \lambda} \delta \mathcal{M} - \frac{\partial F(\mathcal{L},\mathcal{M})}{\partial \mathcal{M}} = \frac{\partial \hat{\mathcal{M}}}{\partial \lambda} \delta \hat{\mathcal{L}} - \frac{\partial \hat{\mathcal{L}}}{\partial \lambda} \delta \hat{\mathcal{M}}.
\end{align*}
\] (8.10)

The \((\ )_{\leq -1}\)-part of the last equations give

\[
\begin{align*}
\frac{\partial \mathcal{M}}{\partial x} \delta \mathcal{L} - \frac{\partial \mathcal{L}}{\partial x} \delta \mathcal{M} - \frac{\partial F(\mathcal{L},\mathcal{M})}{\partial \mathcal{M}}_{\leq -1} = 0, \\
\frac{\partial \mathcal{M}}{\partial \lambda} \delta \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \lambda} \delta \mathcal{M} - \frac{\partial F(\mathcal{L},\mathcal{M})}{\partial \mathcal{M}}_{\leq -1} = 0.
\end{align*}
\] (8.11)

which can easily be solved with respect to \( \delta \mathcal{L} \) and \( \delta \mathcal{M} \) because the coefficient matrix is unimodular. One thus arrives at the following result.

**Proposition 13.** The infinitesimal symmetries \( \delta_F \mathcal{L} \) and \( \delta_F \mathcal{M} \) are given by

\[
\delta_F \mathcal{L} = \{ F(\mathcal{L},\mathcal{M})_{\leq -1}, \mathcal{L} \},
\]

\[
\delta_F \mathcal{M} = \{ F(\mathcal{L},\mathcal{M})_{\leq -1}, \mathcal{M} \}.
\] (8.12)

8.2. Symmetries in terms of \( v_{i+1} \). Symmetries of the self-dual vacuum Einstein equation are further extended to potentials called Plebanski’s “key functions” [38]. For several reasons, it is \( v_2 \) that should correspond to the key functions. For comparison, we now show the action of \( \delta_F \) on the Laurent coefficients \( v_i \). The formula for \( \delta_F v_2 \) is indeed reminiscent of a similar formula for the key functions.
**Proposition 14.** For $i = 1, 2, \ldots$, $\delta_F v_{i+1}$ are given by

$$\delta_F v_{i+1} = - \, \text{res} \, F(\mathcal{L}, \mathcal{M}) d_\lambda B_i.$$  \hspace{1cm} (8.13)

In particular,

$$\delta_F v_2 = - \, \text{res} \, F(\mathcal{L}, \mathcal{M}) d\lambda.$$ \hspace{1cm} (8.14)

**Remark.** The above formula is actually valid for $i = 0$ and gives

$$\delta_F v_1 = 0$$ \hspace{1cm} (8.15)

if one retains the $v_1$-term in the definition of the hierarchy (Section 4). This means that the above infinitesimal symmetries leave invariant $v_1$. Nevertheless we have seen in the previous section that a nonzero $v_1$-term can be generated in the course of solving a Riemann-Hilbert problem. This apparent discrepancy is due to the special form of the one-parameter family of deformations, (8.1), for which we have assumed the existence of a single-valued generating function $F(\lambda, x)$.

**Proof of Proposition 14:** The essence is the same as the proof of Proposition 4. Since $\delta_F$ is a derivation like $\partial / \partial t_n$, the chain rule applies:

$$\delta_F \mathcal{M} = \left( \frac{\partial \mathcal{M}}{\partial \mathcal{L}} \right)_{t, v \text{ fixed}} \delta_F \mathcal{L} + \sum_{i=1}^{\infty} (\delta_F v_{i+1}) \mathcal{L}^{-i},$$ \hspace{1cm} (8.16)

and from the $\mathcal{L}^{-i}$-term, one has

$$\delta_F v_{i+1} = \text{res} \, \mathcal{L}^i \left[ \delta_F \mathcal{M} - \left( \frac{\partial \mathcal{M}}{\partial \mathcal{L}} \right)_{t, v \text{ fixed}} \delta_F \mathcal{L} \right] d_\lambda \mathcal{L}
= \text{res} \, \mathcal{L}^i [\delta_F \mathcal{M} d_\lambda \mathcal{L} - \delta_F \mathcal{L} d_\lambda \mathcal{M}]
= \text{res} \, \mathcal{L}^i [\{ F(\mathcal{L}, \mathcal{M}) \leq -1, \mathcal{M} \} d_\lambda \mathcal{L} - \{ F(\mathcal{L}, \mathcal{M}) \leq -1, \mathcal{L} \} d_\lambda \mathcal{M}]
= \text{res} \, \mathcal{L}^i d_\lambda F(\mathcal{L}, \mathcal{M}) \leq -1,$$ \hspace{1cm} (8.17)

and finally, due to the properties of formal residues (Lemmas A and B),

$$= \text{res} \, \mathcal{B}_i d_\lambda F(\mathcal{L}, \mathcal{M})
= - \, \text{res} \, F(\mathcal{L}, \mathcal{M}) d_\lambda B_i.$$

This proves (8.13). ❑

**8.3. Symmetries extended to tau function.** Since our definition of the tau function is always accompanied with an integration constant, symmetries at the level of $\mathcal{L}$ and $\mathcal{M}$ do not automatically extend to the tau function. Besides, if such an extension exists, it is not ensured whether the extension has a simple form. In fact, the following result shows that such an extension does exist with a very simple expression.
Proposition 10. The infinitesimal symmetries $\delta F$ of the $(\mathcal{L}, \mathcal{M})$-pair can be consistently extended to the tau function by defining

$$\delta F \log \tau = - \text{res} F^x(\mathcal{L}, \mathcal{M}) d\lambda \mathcal{L},$$

(8.18)

where $F^x(\lambda, x)$ is a primitive function of $F(\lambda, x)$ normalized as

$$F^x(\lambda, x) \overset{\text{def}}{=} \int_0^x F(\lambda, y) dy.$$ 

(8.19)

“Consistency” means that the following relation is satisfied.

$$\frac{\partial}{\partial t} \delta F \log \tau = \delta F \frac{\partial \log \tau}{\partial t}.$$ 

(8.20)

Proof: The right hand side of the consistency relation has been calculated (Proposition 13):

$$\delta F \frac{\partial \log \tau}{\partial t} = \delta F v_{n+1} = - \text{res} F(\mathcal{L}, \mathcal{M}) d\lambda \mathcal{B}_n.$$ 

(8.21)

To calculate the other side, we introduce a set of functions $F_i = F_i(t, v)$ of $t$ and $v = (v_2, v_3, \ldots)$ by the Laurent expansion

$$F^x(\mathcal{L}, \mathcal{M}) = F^x(\mathcal{L}, \sum_{n=1}^{\infty} n t_n \mathcal{L}^{n-1} + \sum_{i=1}^{\infty} v_{i+1} \mathcal{L}^{-i-1}) = \sum_{i=-\infty}^{\infty} F_i \mathcal{L}^i$$

(8.22)

with respect to $\mathcal{L}$. The $t$-dependence of $F_i$ comes only from the $t$’s and $v$’s included in $\mathcal{M}$ of $F^x(\mathcal{L}, \mathcal{M})$; in this definition, $\mathcal{L}$ simply plays the role of an independent parameter. Therefore, to calculate the $t$-derivatives of $F_i$, one may temporally consider $\mathcal{L}$ as a constant, and differentiate both hand sides of the above relation.

Thus one has

$$\sum_{i=-\infty}^{\infty} \frac{\partial F_i}{\partial t_n} \mathcal{L}^i = \frac{\partial F^x(\mathcal{L}, \mathcal{M})}{\partial \mathcal{M}} \left( \frac{\partial \mathcal{M}}{\partial t_n} \right)_{\mathcal{L}, t_m (m \neq n) \text{ fixed}}$$

$$= F(\mathcal{L}, \mathcal{M}) \left( \frac{\partial \mathcal{M}}{\partial t_n} \right)_{\mathcal{L}, t_m (m \neq n) \text{ fixed}}.$$ 

(8.23)

Now one can apply the method of proof of Proposition 4 once again. First, the last line can be continued as:

$$= F(\mathcal{L}, \mathcal{M}) \left[ \frac{\partial \mathcal{M}}{\partial t_n} - \left( \frac{\partial \mathcal{M}}{\partial \mathcal{L}} \right)_{t, v \text{ fixed}} \frac{\partial \mathcal{L}}{\partial t_n} \right].$$

(8.23)

Then from the $\mathcal{L}^{-1}$-term,

$$\frac{\partial}{\partial t_n} \delta F \log \tau = - \frac{\partial \mathcal{F}_1}{\partial t_n}$$

$$= - \text{res} F(\mathcal{L}, \mathcal{M}) \left[ \frac{\partial \mathcal{M}}{\partial t_n} - \left( \frac{\partial \mathcal{M}}{\partial \mathcal{L}} \right)_{t, v \text{ fixed}} \frac{\partial \mathcal{L}}{\partial t_n} \right] d\lambda \mathcal{L}$$
and this can be calculated just as in the proof of Proposition 4:

\[ - \text{res } F(L, M) \left[ \frac{\partial M}{\partial t_n} d\lambda L - \frac{\partial L}{\partial t_n} d\lambda M \right] \]

\[ = - \text{res } F(L, M) d\lambda B_n. \tag{8.24} \]

Eqs. (8.21) and (8.24) give the identical results, and this is exactly the consistency relation. \( \square \)

8.4. Commutation relations of symmetries. Our final task is to calculate the commutation relations of these infinitesimal symmetries.

**Proposition 16.** For any two generating functions \( F_1 = F_1(\lambda, x) \) and \( F_2 = F_2(\lambda, x) \), the infinitesimal symmetries \( \delta_{F_1} \) and \( \delta_{F_2} \) obey the commutation relations

\[ [\delta_{F_1}, \delta_{F_2}] \log \tau = \delta_{\{F_1, F_2\}} \log \tau + c(F_1, F_2) \tag{8.25} \]

for the tau function and

\[ [\delta_{F_1}, \delta_{F_2}] K = \delta_{\{F_1, F_2\}} K \tag{8.26} \]

for \( K = L, M \), where

\[ c(F_1, F_2) = \text{def } \text{res } F_1(\lambda, 0) dF_2(\lambda, 0). \tag{8.27} \]

**Proof:** (8.26) is an immediate consequence of (8.25) and the consistency in the sense of Proposition 14. We shall only prove (8.25). Without loss of generality, we may assume that

\[ F_1 = f_1(\lambda)x^j, \quad F_2 = f_2(\lambda)x^k \tag{8.28} \]

where \( j \) and \( k \) are nonnegative integers. Accordingly,

\[ F_1^x = \frac{f_1(\lambda)x^{j+1}}{j+1}, \quad F_2^x = \frac{f_2(\lambda)x^{k+1}}{k+1}, \]

\[ \{F_1, F_2\}^x = \begin{cases} (kf_1 f_2' - jf_1 f_2')x^{j+k} & \text{if } j + k > 0, \\ 0 & \text{if } j + k = 0. \end{cases} \tag{8.29} \]

Let us examine the action of the commutator. From the construction,

\[ [\delta_{F_1}, \delta_{F_2}] \log \tau = -\delta_{F_1} \text{ res } F_2^x(L, M)d\lambda L + \delta_{F_2} \text{ res } F_1^x(L, M)d\lambda L. \tag{8.30} \]

Note that the situation is the same as the proof of the previous proposition; one has to calculate a derivative of a formal residue. The only difference is that we now have a more abstract derivation \( \delta_F \) rather than \( \partial/\partial t_n \). For the first term on the right hand side, thus,

\[ \delta_{F_1} \text{ res } F_2^x(L, M)d\lambda L = \text{ res } F_2(L, M)[\delta_{F_1}Md\lambda L - \delta_{F_1}Ld\lambda M] \]
and by the method of proof of Proposition 4, again,

\[ = \text{res } F_2(\mathcal{L}, \mathcal{M})d_\lambda F_1(\mathcal{L}, \mathcal{M})_{\leq -1} \]
\[ = \text{res } F_2(\mathcal{L}, \mathcal{M})_{\geq 0}d_\lambda F_1(\mathcal{L}, \mathcal{M})_{\leq -1}. \quad (8.31) \]

Similarly,

\[ \delta F_2 \text{ res } F_1^\ast(\mathcal{L}, \mathcal{M})d_\lambda \mathcal{L} = \text{res } F_1(\mathcal{L}, \mathcal{M})_{\geq 0}d_\lambda F_2(\mathcal{L}, \mathcal{M})_{\leq -1}. \quad (8.32) \]

From (8.30)-(8.32),

\[ [\delta_{F_1}, \delta_{F_2}] \log \tau = \text{res } F_1(\mathcal{L})d_\lambda f_2(\mathcal{L}) \]
\[ = \text{res } f_1(\lambda)d_\lambda f_2(\lambda) = c(F_1, F_2) \quad (8.34) \]

whereas

\[ \delta_{(F_1, F_2)} \log \tau = 0, \quad (8.35) \]
hence (8.25) is satisfied for this case. Meanwhile, if \( j + k > 0 \),

\[ [\delta_{F_1}, \delta_{F_2}] \log \tau = \text{res } f_1(\mathcal{L})\mathcal{M}^j[f_2'(\mathcal{L})\mathcal{M}^k d_\lambda \mathcal{L} + k f_2(\mathcal{L})\mathcal{M}^{k-1} d_\lambda \mathcal{M}] \]
\[ = \text{res } f_1(\mathcal{L})f_2'(\mathcal{L})\mathcal{M}^{j+k}d_\lambda \mathcal{L} + f_1(\mathcal{L})f_2(\mathcal{L})d_\lambda (\mathcal{M}^{j+k}) \]
\[ = \text{res } f_1(\mathcal{L})f_2'(\mathcal{L})\mathcal{M}^{j+k}d_\lambda \mathcal{L} - \frac{k}{j + k}f_1(\mathcal{L})f_2(\mathcal{L})d_\lambda (f_1(\mathcal{L})f_2(\mathcal{L})) \]
\[ = - \text{res } \left[ \frac{k}{j + k}f_1'(\mathcal{L})f_2(\mathcal{L}) - \frac{j}{j + k}f_1(\mathcal{L})f_2'(\mathcal{L}) \right] \mathcal{M}^{j+k}d_\lambda \mathcal{L} \]
\[ = \delta_{(F_1, F_2)} \log \tau. \quad (8.36) \]

Since \( c(F_1, F_2) = 0 \), this shows that (8.25) is satisfied for this case as well. This completes the proof. \( \square \)

We have thus observed that the infinitesimal symmetries at the level of the tau function exhibit anomalous commutation relations. The anomalous term, \( c(F_1, F_2) \), is a cocycle of the SDiff(2) algebra on a cylinder, hence gives rise to a central extension. This result, too, advocates that our definition of the tau function is an appropriate one as an analogue of the tau function of the ordinary KP hierarchy. One should note that anomalous commutation relations of the SDiff(2) version are limited to the “spin-1” sector (i.e., \( \delta_F \) with \( F = F(\lambda) \)) of the SDiff(2) algebra. This
is in contrast with the case of the ordinary KP hierarchy; anomalous commutation relations therein take place in all sector of the $gl(\infty)$ algebra [2] or of the $\hat{W}_\infty$ algebra [19].

Cocycles of SDiff(2) algebras on various surfaces are classified by physicists [40][41][42][43]. According to their analysis, there are $2g$ linearly independent cocycles on a genus $g$ surface. Since a cylinder $S^1 \times R^1$ may be thought of as a genus $g = 1/2$ surface, the space of nontrivial cocycle should be one-dimensional. Our cocycle gives a realization of such a cocycle.

### 8.5. SDiff(2) constraints of topological minimal models.

We have seen in Section 7 that the tau function $\tau = \tau_0(t)$ of the undeformed ($h = 0$) solution constructed therein satisfies an infinite set of nonlinear constraints. Actually, these constraints have a very simple interpretation in terms of the SDiff(2) symmetries as follows. Let us start from the basic relations

$$\left( L^{(m-1)N+n(1-N)}M^n \right)_{\leq -1} = \text{const.} \quad \left( P^{n-1}Q^n \right)_{\leq -1} = 0 \quad (8.37)$$

for $m \geq -1$ and $n \geq 0$. Each of these relations is equivalent to the requirement that the equations

$$\text{res} \left[ L^{(m-1)N+n(1-N)}M^n d\lambda B_i \right] = 0 \quad (8.38)$$

be satisfied for $i \geq 1$, because the $B_i$’s ($i \geq 1$) form a basis of the vector subspace of polynomials of $\lambda$ in the space of Laurent series. By Proposition 14, one can rewrite Eq. (8.38) in terms of SDiff(2) symmetries as:

$$\delta_\lambda \left( L^{(m-1)N+n(1-N)}M^n \right) v_{i-1} = 0 \quad (8.39)$$

Further, since $v_{i+1} = \partial \log \tau / \partial t_i$ and the infinitesimal symmetries commute with $\partial / \partial t_i$ in the sense of Proposition 15, Eq. (8.39) becomes

$$\frac{\partial}{\partial t_i} \delta_\lambda \left( L^{(m-1)N+n(1-N)}M^n \right) \log \tau = 0 \quad (8.40)$$

Consequently,

$$\delta_\lambda \left( L^{(m-1)N+n(1-N)}M^n \right) \log \tau = \text{const.} \quad (8.41)$$

The left hand side of this equation can be written in a residue form (Proposition 15) as:

$$\delta_\lambda \left( L^{(m-1)N+n(1-N)}M^n \right) \log \tau = \frac{1}{n+1} \text{res} \left[ L^{(m-1)N+n(1-N)}M^{n+1} d\lambda \mathcal{L} \right] \quad (8.42)$$

and this is a homogeneous function of degree $n + 1$ ($\neq 0$). Hence the constant on the right hand side of (8.41) has to vanish. To summarize:
Proposition 17. The tau function $\tau = \tau_0(t)$ of Proposition 10 satisfies the constraints

$$\delta_{\lambda(m-1)N+n(1-N)} x^n \log \tau = -\frac{1}{n+1} \text{res} \left[ \mathcal{L}^{(m-1)N+n(1-N)} \mathcal{M}^{n+1} d\lambda \mathcal{L} \right] = 0 \quad (8.43)$$

for $m \geq -1$ and $n \geq 0$.

Eqs. (8.43) give a compact expression of the constraints mentioned in Section 7. If $n = 0$, one obtains the obvious relations

$$v_{(m-1)N} = \frac{\partial \log \tau}{\partial t_{(m-1)N}} = 0, \quad (8.44)$$

which are satisfied by any solution of the $N$-reduced hierarchy. If $n = 1$, Eqs. (8.42) are nothing but Krichever’s dispersionless analogue of Virasoro constraints. The others for $n \geq 2$ give higher constraints equivalent to (7.49).

9. Conclusion

Our SDiff(2) KP hierarchy is, in many aspects, very similar to the ordinary KP hierarchy. Their differences should be ultimately due to the difference of the underlying Lie algebras, i.e., the SDiff(2) algebra and the $\text{gl}(\infty)$ algebra. The $S$ function and the $\tau$ function both have counterparts in the KP hierarchy. Infinitesimal symmetries are constructed and shown to exhibit anomalous commutation relations at the level of the tau function. These show a remarkable similarity between the two distinct hierarchies.

Besides the similarity with the KP hierarchy, the SDiff(2) KP hierarchy shares a number of characteristics with the self-dual vacuum Einstein equations and its 3-d reductions [6][7]. The Riemann-Hilbert problem in the SDiff(2) group is a key to connect this hierarchy with the minitwistor theory [13][14][15][16].

This double nature of the SDiff(2) KP hierarchy can also be seen in its Toda version [17]. We therefore expect these SDiff(2) hierarchies to play the role of a bridge that connects two distinct families of nonlinear integrable systems, i.e., soliton equations (most of which live in two dimensions) and self-duality equations (which live in four dimensions). In this respect, an intriguing problem will be to pursue Orlov’s approach to the KP hierarchy [18][19] as a “noncommutative minitwistor theory.” If this turns out to be successful, the next step would be naturally to construct a noncommutative analogue of full twistor theory in four dimensions that should reproduce the twistor theory of the self-dual vacuum Einstein equation [26] as a “quasi-classical” limit. We do not know what a corresponding nonlinear “integrable” system looks like.

Our knowledge on special solutions of the SDiff(2) KP hierarchy is very limited. As Krichever [3] pointed out in his framework of dispersionless Lax equations, topological minimal models [4][5] give such special solutions. We have presented a characterization of these solutions in the language of the Riemann-Hilbert problem,
and found a set of nonlinear constraints satisfied by the corresponding tau function. These constraints include Krichever’s “dispersionless analogues of Virasoro constraints” [3], hence may be called “SDiff(2) (or \(w_{\infty}\)) constraints.”

An important problem still left open is to find a geometric structure like the infinite dimensional Grassmannian manifold \([1][2][20]\). A useful expression of the tau function will be obtained from such a geometric structure. For the SDiff(2) Toda equation, Saveliev and his collaborators [10] indeed presented an expression of solutions that seems to provide a hint to this problem. This issue should also be related to some quantum field theory like the free fermion theory emerging in the KP hierarchy [2]. In view of the relation to topological minimal models, an underlying field theory will be a kind of topological field theory in a generalized sense.

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Note added

After completing this work, we are informed of papers by Kodama and Gibbons [44]. They deal with the same hierarchy (and a Toda version) and present remarkable results on special solutions. We thank Takahiro Shiota for this information.

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