Dynamics of Entanglement Wedge Cross Section from Conformal Field Theories

Yuya Kusuki\(^1\) and Kotaro Tamaoka\(^1\)

\(^1\)Center for Gravitational Physics, Yukawa Institute for Theoretical Physics (YITP), Kyoto University, Kitashirakawa Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan.

(Dated: July 17, 2019)

We derive dynamics of the entanglement wedge cross section directly from the two-dimensional holographic CFTs with a local operator quench. This derivation is based on the reflected entropy, a correlation measure for mixed states. We further compare these results with the mutual information and ones for RCFTs. Our results directly suggest the classical correlation also plays an important role in the subregion/subregion duality even for dynamical setup. Besides a local operator quench, we study the reflected entropy in a heavy state and provide improved bulk interpretation. We checked the above results also hold for the odd entanglement entropy, which is another measure for mixed states related to the entanglement wedge cross section.

I. INTRODUCTION AND SUMMARY

The non-equilibrium dynamics in a given strongly coupled system attracts a lot of attention in the physics community. One useful tool to capture this dynamical process is the entanglement entropy (EE), which is defined by

\[ S(A) = -\text{tr}\rho_A \log \rho_A, \quad (1) \]

where \( \rho_A \) is a reduced density matrix for a subsystem \( A \), obtained by tracing out its complement \( A^c \). This quantity measures entanglement between subsystem \( A \) and its complement \( A^c \) if a pure state describes the entire system. The EE also plays a significant role in quantum gravity via the AdS/CFT correspondence\([1–4]\). In particular, we expect that the dynamics of the entanglement in the certain \( d \)-dimensional system is related to the dynamics of the spacetime in \( d+1 \)-dimensional asymptotically AdS spacetime.

If one considers mixed states \( \rho_{AB} \) on a system \( AB \equiv A \cup B \) and wishes to measure the correlation between \( A \) and \( B \), however, we have many measures for mixed states in the literature and no unique choice as opposed to the EE for pure states. Therefore, from both conceptual and practical viewpoints, we should use the one(s) which have a clear meaning in the setup under consideration.

In this Letter, we will focus on the reflected entropy \( S_R \)[5] which has a sharp (conjectured) interpretation in the context of AdS/CFT. We expect that

\[ S_R(A : B) = 2E_W(A : B) \quad (2) \]

where \( E_W \) is area of the minimal cross section of the entanglement wedge\([6, 7]\) dual to the reduced density matrix\([8–10]\). (See also \([11–27]\) for further developments in this direction.) We will give the definition of the reflected entropy in the next section. This bulk object, called entanglement wedge cross section (EWCS), is a natural generalization of the minimal surfaces. In particular, if \( B = A^c \) and \( \rho_{AB} \) is a pure state, \( E_W(A : B) \) reduces to the area of the minimal surfaces associated with the \( S(A) (= S(A^c)) \). In the same way, \( S_R(A : B) \) reduces to the \( 2S(A) \) for pure states. One main motivation of the present Letter is to understand how the reflected entropy describes the dynamics of correlation in various setups. It will provide us a key to understanding which kind of correlation is important in the subregion/subregion duality.

Let us summarize the results of the present Letter. First, we have studied the time evolution of the reflected entropy by a local operator quench and see a perfect agreement with the EWCS for a falling particle geometry\([28]\). Comparing with the mutual information, our results directly suggest that classical correlations are also important even in the dynamical process of subregion/subregion duality. Second, we study the reflected entropy for heavy states. Interestingly, we can see the phase transition of the EWCS originally discussed in \([6, 7]\). Our analysis leads improvement of the bulk dual of the heavy state: we should take into account the end of the world brane wrapping around the blackhole horizon. Note that this is also the case for the usual EE first discussed in \([29]\).

Third, we study the local operator quench for rational conformal field theories (RCFTs) and see the agreement with the quasi-particle picture. We should stress that the above analysis also holds for the odd entanglement entropy\([23]\), which is another generalization of the EE for mixed states. These results can be achieved by using the fusion kernel approach in two-dimensional CFT\([30–32]\). We will report the detail of technical parts (for both CFT and gravity) in our upcoming paper\([33]\).

II. REFLECTED ENTROPY

Here we review the definition of the reflected entropy. We consider the following mixed state,

\[ \rho_{AB} = \sum_n p_n \rho_{AB}^{(n)}, \quad (3) \]

where each \( \rho_{AB}^{(n)} \) represents a pure state as

\[ \rho_{AB}^{(n)} = \sum_{i,j} \sqrt{t_{ij}^{(n)}} |i_n\rangle_A \langle i_n|_A |j_n\rangle_B \langle j_n|_B, \quad (4) \]

where \( |i_n\rangle_A \in \mathcal{H}_A, |j_n\rangle_B \in \mathcal{H}_B \) and \( t_{ij}^{(n)} \) is a positive number such that \( \sum_i t_{ij}^{(n)} = 1 \). The real number \( p_n \) is the corresponding probability associated with its appearance in the ensemble.
For this mixed state, we can provide the simplest purification as

\[ |\sqrt{\rho_{AB}}\rangle = \sum_{i,j,n} \sqrt{p_{ij}} |i_n\rangle_A |j_n\rangle_B |i_j\rangle_{A^*} |j_i\rangle_{B^*}, \]

where \(|i_n\rangle_{A^*} \in \mathcal{H}_A^*\) and \(|i_j\rangle_{B^*} \in \mathcal{H}_B^*\) are just copies of \(\mathcal{H}_A\) and \(\mathcal{H}_B\). Then, the reflected entropy is defined by

\[ S_R(A : B) \equiv -\text{tr}_{AA^*} \log \rho_{AA^*}, \]

where \(\rho_{AA^*}\) is the reduced density matrix of \(\rho_{AB^*BB^*}\) after tracing over \(\mathcal{H}_B \otimes \mathcal{H}_B^*\).

### III. SETUP

Our interest in this Letter is to study a local operator quench state \([5] [55]\), which is created by acting a local operator \(O(x)\) on the vacuum in a given CFT at \(t = 0\),

\[ |\Psi(t)\rangle = \sqrt{N} e^{-\epsilon H - i\epsilon H} O(x) |0\rangle, \]

where \(x\) represents the position of insertion of the operator, \(\epsilon\) is a UV regularization of the local operator and \(N\) is a normalization factor so that \(\langle \Psi(t) | \Psi(t) \rangle = 1\).

The reflected entropy can be evaluated in the path integral formalism \([5]\). For example, the Renyi reflected entropy in the vacuum can be computed by a path integral on \(m \times n\) copies as shown in FIG. 1. Here, we would view this manifold as a correlator with twist operators as in the lower of FIG. 1, where

\[ \langle \sigma_{g_A}(u_1) \sigma_{g_A}^{-1}(v_1) \sigma_{g_B}(u_2) \sigma_{g_B}^{-1}(v_2) \rangle_{\text{CFT}^m \otimes \text{CFT}^n}. \]

we define the twist operators \(\sigma_{g_A}\) and \(\sigma_{g_B}\). Here, we focus on the following mixed state,

\[ \rho_{AB} = \text{tr}_{A^*B^*} |\Psi(t)\rangle \langle \Psi(t)|, \]

where \(|\Psi(t)\rangle\) is a time-dependent pure state as \(|\Psi(t)\rangle = \sqrt{N} e^{-\epsilon H - i\epsilon H} O(0) |0\rangle\). Then, in a similar manner to the method in \([34]\), the replica partition function in this state can be obtained by a correlator as

\[ \frac{1}{1 - n} \log \frac{\langle \sigma_{g_A}(u_1) \sigma_{g_A}^{-1}(v_1) O^{mn}(w_1, w_2) \rangle}{\langle \sigma_{g_B}(u_2) \sigma_{g_B}^{-1}(v_2) O^{mn}(w_1, w_2) \rangle} \]

where we abbreviate \(V(z, \bar{z}) \equiv V(z)\) if \(z \in \mathbb{R}\) and the operators \(O\) are inserted at

\[ w_1 = t + i\epsilon, \quad \bar{w}_1 = -t + i\epsilon, \quad w_2 = t - i\epsilon, \quad \bar{w}_2 = -t - i\epsilon. \]

The twist operator \(\sigma_{g_m}\) is just the usual twist operator \(\sigma_m\) based on the \(m\)-cyclic permutation group, which has the conformal dimension \(h_{\sigma_m} = \frac{c}{24} \left( m - \frac{1}{m} \right) \equiv h_m\). To avoid unnecessary technicalities, we do not show the precise definition of the twist operators \(\sigma_{g_A}\) and \(\sigma_{g_B}\) (which can be found in \([3]\)) because in this Letter, we only use the scaling dimension of the twist operators,

\[ h_{\sigma_{g_A}} = h_{\sigma_{g_B}} = h_{\sigma_{g_A}^{-1}} = h_{\sigma_{g_B}^{-1}} = \frac{c}{24} \left( m - \frac{1}{m} \right) = nh_m, \]

\[ h_{\sigma_{g_A}^{-1} g_B} = h_{\sigma_{g_B}^{-1} g_A} = \frac{c}{12} \left( n - \frac{1}{n} \right) = 2h_n. \]

Here \(O^{\otimes N} \equiv O \otimes O \otimes \cdots \otimes O\) is an abbreviation of the operator on \(N\) copies of CFT (CFT\(^N\)). We will take \(n, m \to 1\) limit so that the \([9]\) reduces to the original reflected entropy. The denominator in \([9]\) corresponds to the entanglement entropy after a local quench.

### IV. HOLOGRAPHIC CFT

As a concrete example, we consider the setup described in FIG. 2. Namely, we set our subregion \(A = \{ u_1, v_1 \}, B = \{ u_2, v_2 \}\) and assume \(0 < \epsilon \ll t < u_2 < v_2 < -u_1 < v_2\).
Growth of Reflected Entropy vs. Mutual Information

\[
\Delta S_R(A : B) = S_R(A : B)[O] - S_R(A : B)[\bar{O}],
\]
\[
\Delta I(A : B) = I(A : B)[O] - I(A : B)[\bar{O}],
\]
which measure a growth of correlations after a local quench. In fact, they behave very similarly, but interestingly, we find the following inequalities for the mutual information and reflected entropy,

\[
\begin{align*}
\Delta S_R(A : B) &\geq \Delta I(A : B), \quad \text{if } t \notin [-v_1, -u_1], \\
\Delta S_R(A : B) &\leq \Delta I(A : B), \quad \text{if } t \in [-v_1, -u_1].
\end{align*}
\]

It implies that the reflected entropy measure the dynamics of the correlations in a quite different way from the mutual information. And this inequalities might be a key to understanding what correlations are measured by reflected entropy from the physical point of view. Possibly, it might be interpreted in the following. The growth in \( t \in [-v_1, -u_1] \) is strongly caused by the quantum correlations, on the other hand, it would be expected that in \( t \notin [-v_1, -u_1] \), the excitation changes both quantum correlations and classical correlations in a similar manner. The point is that in the holographic CFT, the mutual information probes quantum correlations more purely than the reflected entropy. Therefore, the quantum correlations in \( t \in [-v_1, -u_1] \) (compared with the classical correlations result in the large growth of the mutual information, thus we obtain \( \Delta S_R(A : B) \leq \Delta I(A : B) \), while in \( t \notin [-v_1, -u_1] \), the change of the quantum correlations are not larger than the classical correlations enough to satisfy \( \Delta S_R(A : B) \leq \Delta I(A : B) \). This implies that the classical correlation also plays an important role in the subregion/subregion duality (even for dynamical setup).

It would be worth mentioning that in the nontrivial time region \( t \in [u_2, v_2] \), there are two phases as shown in the figure. The remarkable features in each phase is as follows:

\begin{itemize}
  \item \( t \in [u_2, -v_1] \cup [-u_1, v_2] \)
    
    The reflected entropy is independent of the conformal dimension \( h_O \) and does not include high energy scale (the UV cutoff parameter \( \epsilon \)).
  \item \( t \in [-v_1, -u_1] \)
    
    The reflected entropy depends on the conformal dimension \( h_O \) and includes high energy scale.
\end{itemize}

It means that when the left or right moving excitation enters one interval, the excitation affects the reflected entropy but its effect is not so strong. On the other hand, if both left and right moving excitations enter two intervals, then the reflected entropy becomes much larger than that for the vacuum. This strong effect comes from the entanglement between two intervals, which is created by the excitation. However, we do not have any clear explanation of the small effect found in

\[
\sigma_R \approx 1 - \frac{\sinh \pi \bar{\gamma}}{\bar{\gamma}} \approx 1 - \frac{\sinh \pi \gamma}{\gamma}.
\]
We can also obtain one for left panel of the same figure. As we will see later, this small effect does not appear in RCFTs.

Since our analysis in CFT is consistent with the entanglement wedge cross section, we can relate the above discussion to original conjecture, the holographic entanglement of purification (EoP) $E_P(A : B)$\[^{6,7}\]. In particular, the EoP is more sensitive to the classical correlation than the reflected entropy, thus the importance of classical correlation becomes more remarkable. (For example, we have the lower bound of EoP for any states $E_P(A : B) \geq I(A : B)/2$, whereas we have the stronger lower bound for separable states $E_P(A : B) \geq I(A : B)$\[^{36}\].)

\section{V. HEAVY STATE}

We consider a CFT on a circle with length $L$. Then, the reflected entropy for a heavy state can be obtained from

$$\frac{1}{1 - u} \log \left( \frac{O_{mn} \sigma_g(u_1) \sigma_A^{-1}(u_1)}{(O_m \sigma_{g_m}(u_1) \sigma_{g_m^{-1}}(v_1)} \right)_{CFT} \frac{\sigma_{g_B}(u_2) \sigma_{g_B^{-1}}(v_2)}{\sigma_{g_m}(u_2) \sigma_{g_m^{-1}}(v_2)} \left( O_{mn} \right)_{CFT}.$$  \hspace{1cm} (18)

Here, this correlator is defined on a cylinder. This can be mapped to the plane $(\zeta, \bar{\zeta})$ by

$$z = e^{\frac{2\pi i}{L}} \zeta, \quad \bar{z} = e^{-\frac{2\pi i}{L}} \bar{\zeta}.$$  \hspace{1cm} (19)

For a sufficiently large subsystem, we have obtained

$$S_R(A : B) = \frac{c}{6} \log \left( \coth \frac{\pi \gamma (u_2 - v_1)}{2L} \right) + \frac{c}{6} \log \left( \coth \frac{\pi \gamma (u_2 - v_1)}{2L} \right). \hspace{1cm} (20)$$

This result perfectly matches the entanglement wedge cross section in the BTZ metric\[^{6}\], namely the cross section described in the right panel of the FIG. 4. It means that the thermalization in the large $\epsilon$ limit \[^{37–41}\] can also be found in the reflected entropy. (For a sufficiently small subsystem, we can also obtain one for left panel of the same figure.)

Our result also answers the interesting question, what is the bulk dual of our quench state. We show that the surface ends at the horizon of the black hole. This can be explained by considering the horizon as an end of the world brane\[^{42–44}\]. In this case, the surface can end at the horizon even if we consider a pure state black hole. We have to mention that this idea should be also applied to the EE in a heavy state because the reflected entropy\[^{20}\] should reproduces the double of the EE in the pure state limit. Note that the pure state limit of the\[^{20}\] does not match the result in\[^{29}\]. This is because their derivation implicitly assumes that the change of the dominant channel (i.e., the transition shown in FIG. 4) does not happen. However, the result under such an assumption contradicts the pure state limit, and basically there is no reason to remove the possibility of the transition even in the EE. It is also important to note that this disconnected phase can never dominate at the $\epsilon \to 0$ limit in the previous section.

\section{VI. RCFT}

It is very interesting to compare our result to the dynamics of the reflected entropy in other CFTs, especially RCFTs.

If we consider the setup $(0 < \epsilon \ll u_2 < -v_1 < -u_1 < v_2$ and $O$ is acted on $x = 0$ at $t = 0$.) for example, we obtain

$$\Delta S_R(A : B)[O] = \begin{cases} 0, & \text{if } t < -v_1, \\ 2 \log d_O, & \text{if } -v_1 < t < -u_1, \\ 0, & \text{if } -u_1 < t, \end{cases} \hspace{1cm} (21)$$

where $d_O$ is a constant, so-called quantum dimension, which is re-expressed in terms of the modular $S$ matrix as \[^{45,46}\]

$$d_O = \frac{S_{00}}{S_{00}}. \hspace{1cm} (22)$$

One can find two significant differences from FIG. 5.

- The small effect in $t \in [u_2, -v_1] \cup [-u_1, v_2]$ does not appear in RCFTs, unlike the holographic CFT.
- The holographic CFT shows the logarithmic growth in $t \in [-v_1, -u_1]$, on the other hand, the growth of RCFT approaches a finite constant.

It would be interesting to note that this growth pattern\[^{21}\] is exactly the same as that of the mutual information, which is quite natural for RCFTs because the quasi particle picture can be applied in any time region.

\section{VII. DISCUSSION}

One can reproduce the above results from the odd entanglement entropy\[^{23}\], which is defined by

$$S_o(A : B) \equiv \lim_{n_o \to 1} \frac{1}{1 - n_o} \left[ \text{tr} \left( \rho_A^{n_o} \rho_B^{n_o} \right) - 1 \right]. \hspace{1cm} (23)$$
where $\rho_{AB}$ is a reduced density matrix for subsystems $A$ and $B$, obtained by tracing out its complement. The limit $n_o \to 1$ is the analytic continuation of an odd integer and $T_B$ is the partial transposition with respect to the subsystem $B$. The odd EE for holographic CFT is expected to have the following relation,

$$S_o(A : B) - S(AB) = E_W(A : B). \quad (24)$$

Indeed, one can replace $S_R(A : B)$ with $2(S_o(A : B) - S(AB))$ for the above results (In fact, this is also the case for RCFTs). This coincidence can happen because we are considering large $c$ limit and/or Regge limit which give us quite universal consequences. In more general parameter regimes, these two quantities should behave differently. It is very interesting to study further such regimes.

Interestingly, we can also discuss the non-perturbative quantum correction of the reflected entropy (and odd entropy) by using the Virasoro conformal block before taking the large $c$ expansion. We have observed that this correction is always negative. This is quite natural because the discontinuous transition between connected and disconnected entanglement wedge should become smooth once we go to the finite $c$ regime. We will describe the detail of this point in the upcoming paper[33].

Finally, there are several interesting future directions which can be accomplished in a similar manner. For example, it would be interesting to understand a relation to negativity [20], to study dynamics in other irrational CFTs [47], [48], to investigate information spreading by using the reflected entropy [49], and evaluate the Renyi reflected entropy, in particular, its replica transition [30] [32] [50].

**ACKNOWLEDGMENTS**

We thank Souvik Dutta, Thomas Hartman, Jonah Kudler-Flam, Masamichi Miyaji, Masahiro Nozaki, Tokiro Numasawa, Tadashi Takayanagi and Koji Umemoto for fruitful discussions and comments. YK is supported by the JSPS fellowship. KT is supported by JSPS Grant-in-Aid for Scientific Research (A) No.16H02182 and Simons Foundation through the “It from Qubit” collaboration. We are very grateful to “Quantum Information and String Theory 2019” and “Strings 2019” where the final part of this work has been completed.

[1] J. M. Maldacena, Int. J. Theor. Phys. 38, 1113 (1999), [Adv. Theor. Math. Phys.2,231(1998)], hep-th/9711200.
[2] S. Ryu and T. Takayanagi, JHEP 08, 045 (2006), hep-th/0605073.
[3] S. Ryu and T. Takayanagi, Journal of High Energy Physics 2006, 045 (2006).
[4] V. E. Hubeny, M. Rangamani, and T. Takayanagi, JHEP 07, 062 (2007), 0705.0016.
[5] S. Dutta and T. Faulkner (2019), 1905.00577.
[6] T. Takayanagi and K. Umemoto, Nature Phys. 14, 573 (2018), 1708.09393.
[7] P. Nguyen, T. Devakul, M. G. Halbasch, M. P. Zaletel, and B. Swingle, JHEP 01, 098 (2018), 1709.07424.
[8] B. Czech, J. L. Karczmarek, F. Nogueira, and M. Van Raamsdonk, Class. Quant. Grav. 29, 155009 (2012), 1204.1330.
[9] A. C. Wall, Class. Quant. Grav. 31, 225007 (2014), 1211.3494.
[10] M. Headrick, V. E. Hubeny, A. Lawrence, and M. Rangamani, JHEP 12, 162 (2014), 1408.6300.
[11] N. Bao, G. Penington, J. Sorce, and A. C. Wall (2018), 1812.01171.
[12] K. Umemoto and Y. Zhou, JHEP 10, 152 (2018), 1805.02625.
[13] H. Hirai, K. Tamaoka, and T. Yokoya, PTEP 2018, 063B03 (2018), 1803.10539.
[14] N. Bao and I. F. Halpern, Phys. Rev. D99, 046010 (2019), 1805.00476.
[15] R. Espindola, A. Guijosa, and J. F. Pedraza, Eur. Phys. J. C78, 646 (2018), 1804.05855.
[16] N. Bao and I. F. Halpern, JHEP 03, 006 (2018), 1710.07643.
[17] W.-Z. Guo (2019), 1901.00330.
[18] N. Bao, A. Chatwin-Davies, and G. N. Remmen, JHEP 02, 110 (2019), 1811.01983.
[19] R.-Q. Yang, C.-Y. Zhang, and W.-M. Li, JHEP 01, 114 (2019), 1810.00420.
[20] J. Kudler-Flam and S. Ryu, Phys. Rev. D99, 106014 (2019), 1808.00446.
[21] K. Babaei Veln, M. R. Mohammadi Mozaffar, and M. H. Vahidian, JHEP 05, 200 (2019), 1903.08490.
[22] P. Caputa, M. Miyaji, T. Takayanagi, and K. Umemoto, Phys. Rev. Lett. 122, 111601 (2019), 1812.05268.
[23] K. Tamaoka, Phys. Rev. Lett. 122, 141601 (2019), 1809.09109.
[24] W.-Z. Guo (2019), 1904.12124.
[25] N. Bao, A. Chatwin-Davies, J. Pollack, and G. N. Remmen (2019), 1905.04317.
[26] J. Harper and M. Headrick (????), 1906.05970.
[27] J. Kudler-Flam, M. Nozaki, S. Ryu, and M. T. Tan (????), 1906.07639.
[28] M. Nozaki, T. Numasawa, and T. Takayanagi, JHEP 05, 080 (2013), 1302.5703.
[29] C. T. Asplund, A. Bernamonti, F. Galli, and T. Hartman, JHEP 02, 171 (2015), 1410.1392.

**FIG. 5.** The growth of reflected entropy in holographic CFT (blue) and Ising model (yellow). $\Delta S_B$ means the difference between the excited state and the vacuum state. Here $(\nu_1, \nu_1, \nu_2) = (\nu_{10}, -1, 3, 10)$, $\epsilon = 10^{-3}$ and we divide them by $\frac{1}{6}$. We choose $\gamma = 2$ in holographic CFT and $O = \sigma$ in Ising model. Each blue dot shows a transition of itself or its first derivative.
[30] Y. Kusuki (2018), 1810.01335.
[31] S. Collier, Y. Gobeil, H. Maxfield, and E. Perlmutter (2018), 1811.05710.
[32] Y. Kusuki and M. Miyaji (2019), 1905.02191.
[33] Y. Kusuki and K. Tamaoka (2019), in preparation.
[34] M. Nozaki, T. Numasawa, and T. Takayanagi, Physical review letters 112, 111602 (2014).
[35] M. Nozaki, JHEP 10, 147 (2014), 1405.5875.
[36] B. M. Terhal, M. Horodecki, D. W. Leung, and D. P. DiVincenzo, Journal of Mathematical Physics 43, 4286 (2002).
[37] A. L. Fitzpatrick, J. Kaplan, and M. T. Walters, JHEP 11, 200 (2015), 1501.05315.
[38] N. Lashkari, A. Dymarsky, and H. Liu, JHEP 03, 070 (2018), 1710.10458.
[39] Y. Hikida, Y. Kusuki, and T. Takayanagi (2018), 1804.09658.
[40] A. Romero-Bermúdez, P. Sabella-Garnier, and K. Schalm (2018), 1804.08899.
[41] E. M. Brehm, D. Das, and S. Datta (2018), 1804.07924.
[42] T. Hartman and J. Maldacena, JHEP 05, 014 (2013), 1303.1080.
[43] A. Almheiri, A. Mousatov, and M. Shyani (2018), 1803.04434.
[44] S. Cooper, M. Rozali, B. Swingle, M. Van Raamsdonk, C. Waddell, and D. Wakeham (2018), 1810.10601.
[45] T. Numasawa, JHEP 12, 061 (2016), 1610.06181.
[46] S. He, T. Numasawa, T. Takayanagi, and K. Watanabe, Physical Review D 90, 041701 (2014).
[47] P. Caputa, Y. Kusuki, T. Takayanagi, and K. Watanabe, J. Phys. A50, 244001 (2017), 1701.03110.
[48] P. Caputa, Y. Kusuki, T. Takayanagi, and K. Watanabe, Phys. Rev. D96, 046020 (2017), 1703.09939.
[49] C. T. Asplund, A. Bernamonti, F. Galli, and T. Hartman, JHEP 09, 110 (2015), 1506.03772.
[50] Y. Kusuki and T. Takayanagi, JHEP 01, 115 (2018), 1711.09913.