New criteria for self-adjointness and its application to
Dirac-Maxwell Hamiltonian

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Abstract

We present a new theorem concerning a sufficient condition for a symmetric operator acting in
a complex Hilbert space to be essentially self-adjoint. By applying the theorem, we prove that the
Dirac Maxwell Hamiltonian, which describes a quantum system of a Dirac particle and a radiation
field minimally interacting with each other, is essentially self-adjoint. Our theorem covers the case
where the Dirac particle is in the Coulomb type potential.

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1 Introduction

One of the most important mathematical studies of quantum systems is to prove the self-adjointness
of the Hamiltonian. A self-adjoint Hamiltonian generates a unique time evolution operator, while
symmetric but not self-adjoint Hamiltonians may generate no natural time evolution or may generate
a lot of different dynamics, because they have, in general, no self-adjoint extensions or infinitely
many ones. Moreover, the “probability interpretation” in quantum theory crucially depends upon
the existence of a spectral measure supported on the real line, which belongs only to self-adjoint
operators. In these viewpoints, proving the self-adjointness of a Hamiltonian is not just a problem
on a mathematical technicality but also of physical importance, and therefore developing general
mathematical theorems for the self-adjointness would contribute both to mathematics and to physics.

The Dirac-Maxwell model is expected to describe a quantum system consisting of a Dirac particle
and a radiation field with the minimal interaction. Informal perturbation method shows that this model
derives the Klein-Nishina formula for the cross section of the Compton scattering of an electron and a
photon, which agrees with the experimental results very well [7]. Hence, it is strongly suggested that
the Dirac-Maxwell model describes a class of natural phenomena where the quantized radiation field
plays an essential role. The mathematically rigorous study of this model was initiated by Arai in Ref.
[2], and several mathematical aspects of the model was analyzed so far (see, e.g., [3], [4], [5], [9], and
[11]). The Hamiltonian of this model has a certain singularity coming from the fact that the free part
Hamiltonian is not bounded below, which is quite special among Hamiltonians of realistic quantum
systems. The essential self-adjointness of the Dirac-Maxwell Hamiltonian has been analyzed in Refs. [2] and [11], but, to our best knowledge, the proof of the essential self-adjointness in the case where the Dirac particle lies in the Coulomb type potential, is still missing, although this is one of the most important situations in physics. The main goal of the present paper is to give a proof of it.

The present paper can be regarded as a sequel to Ref. [6]. In Ref. [6], the authors developed a general theory on the existence of solutions of initial value problems for the Schrödinger and Heisenberg equations generated by a linear operator $H$ in some Hilbert space $\mathcal{H}$. One of the merits of the general theory established there is that it is applicable to the case where $H$ is not symmetric or even not normal, but it will also help us to attack mathematical problems of quantum field theories with a usual symmetric Hamiltonian. In this paper, we develop a new theorem for a symmetric operator to be essentially self-adjoint, making a good use of materials obtained in Ref. [6]. Assumptions we employ here seem to be compatible with a large class of Hamiltonians of mathematical quantum field models with the interaction being linear in bosonic field operators. In fact, the Dirac-Maxwell Hamiltonian with the Coulomb type potential fulfills our assumptions and turns out to be essentially self-adjoint on its natural domain, in spite of the fact that it is not semi-bounded.

2 General Theorem on essential Self-adjointness

In this section, we develop a general strategy for proving the essential self-adjointness, by using the results obtained in Ref. [6]. Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. For a linear operator $T$ in $\mathcal{H}$, we denote, in general, its domain and range by $\text{D}(T)$ and $\text{R}(T)$, respectively. We also denote the adjoint of $T$ by $T^*$ and the closure by $\overline{T}$, if these exist. For a self-adjoint operator $T$, $E_T(\cdot)$ denotes the spectral measure of $T$.

Let $H_0$ be a self-adjoint operator in $\mathcal{H}$, and $H_1$ be a symmetric one in $\mathcal{H}$. Suppose that $\text{D}(H_0) \cap \text{D}(H_1)$ is dense in $\mathcal{H}$. The main object we consider in the present section is the symmetric operator

$$H := H_0 + H_1.$$  \hspace{1cm} (2.1)

Assumptions we employ here are as follows [6] :

**Assumption 2.1.** There exists an operator $A$ in $\mathcal{H}$ satisfying the following conditions:

(I) $A$ is self-adjoint and non-negative.

(II) $A$ and $H_0$ are strongly commuting.

(III) $H_1$ is $A^{1/2}$- bounded, where $A^{1/2}$ defined through operational calculus.

(IV) There exists a constant $b > 0$ such that, for all $L \geq 0$, $\xi \in R(E_A([0, L]))$ implies $H_1\xi \in R(E_A([0, L + b]))$.

For the notational simplicity, we put

$$D := \bigcup_{L \geq 0} R(E_A([0, L])), \hspace{1cm} (2.2)$$

and $D' := D(H_0) \cap D$. Note that $D$ is a dense subspace in $\mathcal{H}$, because $A$ is self-adjoint by Assumption 2.1(I), and $D'$ is also dense if Assumption 2.1(I) and (II) are valid [6].

Our goal of the present section is to prove the following theorem :

**Theorem 2.1.** Suppose that Assumption 2.1 holds.

(i) Exactly one of the following (a) or (b) holds.

(a) $H$ has no self-adjoint extension.

...
(b) $H$ is essentially self-adjoint.

(ii) If $D'$ is a core of $H$, then $H$ is essentially self-adjoint on $D'$.

To prove Theorem 2.1 we first recall the result obtained in Ref. [6]. Define for $t \in \mathbb{R}$,

$$H_1(t) := e^{iH_0t} H_1 e^{-iH_0t}.$$ 

All we need here are summarized as the following lemma:

**Lemma 2.1.** Suppose that Assumption 2.1 holds. Then, for each $t, t' \in \mathbb{R}$ and $\xi \in D$, the series

$$U(t, t') \xi := \xi + (-i) \int_t^{t'} d\tau_1 H_1(\tau_1) \xi + (-i)^2 \int_t^{t'} \int_{\tau_1}^{\tau_2} d\tau_2 H_1(\tau_1) H_1(\tau_2) \xi + \cdots$$

(2.3)

converges absolutely, where each of integrals is taken in the sense of strong integral. Furthermore, the following (i) and (ii) hold.

(i) The operator $U(t, t')$ has a unitary extension $\overline{U(t, t')}$ uniquely.

(ii) For each $\xi \in D'$, put $\xi(t) := e^{-iH_0t} \overline{U(t, 0)} \xi$. Then, $\xi(t) \in D(H)$ for all $t \in \mathbb{R}$, and the vector valued function $t \mapsto \xi(t)$ is strongly differentiable. Moreover, it is a solution of the initial value problem for the Schrödinger equation:

$$\frac{d}{dt} \xi(t) = -iH \xi(t), \quad \xi(0) = \xi,$$

(2.4)

where the derivative in $t$ is taken in the strong sense (this applies in what follows unless otherwise stated).

To go further, we need a little bit more lemmas.

The following fact is well known (see, e.g., Ref. [10]):

**Lemma 2.2.** Let $T$ be a symmetric operator in $H$. If $T$ has a unique self-adjoint extension, then $T$ is essentially self-adjoint.

**Lemma 2.3.** Let $T$ be a symmetric operator in $H$. If there exists a dense subspace $V$ such that for any $\xi \in V$ the initial value problem

$$\frac{d}{dt} \xi(t) = -iT \xi(t), \quad \xi(0) = \xi,$$

(2.5)

has a solution $\mathbb{R} \ni t \mapsto \xi(t) \in D(T)$, then, exactly one of the following (a) or (b) holds.

(a) $T$ has no self-adjoint extension.

(b) $T$ is essentially self-adjoint.

**Proof.** It is sufficient to prove that, if there exists a self-adjoint extension, then $T$ is essentially self-adjoint. Suppose that $T$ has a self-adjoint extension $\overline{T}$. Then, for each $\eta \in D(\overline{T})$ and $\xi \in V$, we have

$$\frac{d}{dt} \left\langle \eta, e^{it\overline{T}} \xi(t) \right\rangle = \left\langle -i\overline{T} e^{-it\overline{\eta}} \eta, \xi(t) \right\rangle + \left\langle e^{-it\overline{T}} \eta, -iT \xi(t) \right\rangle.$$  

(2.6)

Since $\xi(t)$ belongs to $D(T)$, the first term on the right hand side of (2.6) is equal to $\left\langle -ie^{-it\overline{T}} \eta, T \xi(t) \right\rangle$. Hence

$$\frac{d}{dt} \left\langle \eta, e^{it\overline{T}} \xi(t) \right\rangle = 0$$

(2.7)
for all \( t \in \mathbb{R} \). Thus, we have
\[
\langle \eta, e^{it\hat{T}} \xi(t) \rangle = \langle \eta, \xi(0) \rangle = \langle \eta, \xi \rangle, \quad t \in \mathbb{R}.
\] (2.8)

Since \( \eta \in D(\hat{T}) \) is arbitrary, \( \xi(t) = e^{-it\hat{T}} \xi(t \in \mathbb{R}) \) for all \( \xi \in V \). This implies that, if \( T \) has another self-adjoint extension \( \hat{T}' \), then \( e^{-it\hat{T}} = e^{-it\hat{T}'} \) \( (t \in \mathbb{R}) \). Hence, \( \hat{T} = \hat{T}' \) by Stone’s theorem. This means that the self-adjoint extension of \( T \) is unique. Thus, \( T \) is essentially self-adjoint by Lemma 2.2. \( \square \)

The next lemma is related to Stone’s theorem and the proof can be found in e.g., Ref. [8, p. 267].

**Lemma 2.4.** Let \( T \) be a symmetric operator in the Hilbert space \( \mathcal{H} \). If for any \( \xi \in D(T) \) the initial value problem
\[
\frac{d}{dt}\xi(t) = -iT\xi(t), \quad \xi(0) = \xi,
\] (2.9)
has a solution \( \mathbb{R} \ni t \mapsto \xi(t) \in D(T) \), then \( T \) is self-adjoint.

**Proof of Theorem 2.1.** We first prove (i). From Lemma 2.1, we find that for all \( \xi \in D(T) \), there is a solution \( \xi(t) = e^{-it\hat{T}} U(t,0) \xi \in D(T) \) of the initial value problem generated by \( T \),
\[
\frac{d}{dt}\xi(t) = -iH\xi(t), \quad \xi(0) = \xi.
\] (2.10)
Since \( D(T) \) is dense, the assertion follows from Lemma 2.3.

Next, we prove (ii). Let \( H' \) be a restriction of \( H \) to \( D' \). Then \( H' \) is a symmetric operator, and \( \overline{H'} = \overline{H} \), because \( D' \) is a core of \( H \). From Lemma 2.1 we conclude that for all \( \xi \in D' = D(H') \) there is a solution \( \xi(t) = e^{-it\hat{H}} U(t,0) \xi \in D(H) \) of the initial value problem generated by \( H \),
\[
\frac{d}{dt}\xi(t) = -iH\xi(t) = -i\overline{H'}\xi(t), \quad \xi(0) = \xi
\] (2.11)
From Lemma 2.4 it follows that \( \overline{H'} \) is self-adjoint, but this immediately implies that \( H \) is essentially self-adjoint, because \( \overline{H'} = \overline{H} \). \( \square \)

## 3 Dirac-Maxwell Hamiltonian

In this section, we introduce the Dirac-Maxwell Hamiltonian and prove its essential self-adjointness under a suitable condition. The Dirac-Maxwell Hamiltonian describes a quantum system consisting of a Dirac particle under a potential \( V \) and a radiation field minimally interacting with each other. We will use the unit system in which the speed of light and \( \hbar \), the Planck constant divided by \( 2\pi \), are set to be unity.

Firstly, we consider the Dirac particle sector. Let us denote the mass and the charge of the Dirac particle by \( M > 0 \) and \( q \in \mathbb{R} \), respectively. The Hilbert space of state vectors for the Dirac particle is taken to be
\[
\mathcal{H}_D := L^2(\mathbb{R}^3; \mathbb{C}^4),
\] (3.1)
the \( \mathbb{C}^4 \)-valued square integrable functions on \( \mathbb{R}^3 = \{ x = (x^1, x^2, x^3) | x^j \in \mathbb{R}, j = 1, 2, 3 \} \). The vector space \( \mathbb{R}^3 \) here represents the position space of the Dirac particle. We sometimes omit the subscript \( x \)
and just write $\mathbb{R}^3$ instead of $\mathbb{R}^3$ when no confusion may occur. The target space $\mathbb{C}^4$ realizes a representation of the four dimensional Clifford algebra accompanied by the four dimensional Minkowski vector space. The generators $\{\gamma^\mu\}_{\mu=0,1,2,3}$ satisfy the anti-commutation relations
\[
\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3,
\] (3.2)
where the Minkowski metric tensor $(g_{\mu\nu})$ is given by
\[
(g_{\mu\nu}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\] (3.3)
and $g^{\mu\nu}$ denotes the $\mu\nu$-component of the inverse matrix of the above $(g_{\mu\nu})$ (numerically the same as $g_{\mu\nu}$). We assume $\gamma^0$ to be Hermitian and $\gamma^j$'s ($j = 1, 2, 3$) be anti-Hermitian. We use the notations following Dirac:
\[
\beta := \gamma^0, \quad \alpha^j := \gamma^0 \gamma^j, \quad j = 1, 2, 3.
\] (3.4)
(3.5)
Then, $\alpha^j$'s and $\beta$ satisfy the anti-commutation relations
\[
\{\alpha^i, \alpha^j\} = 2\delta^{ij}, \quad i, j = 1, 2, 3,
\] (3.6)
\[
\{\alpha^j, \beta\} = 0, \quad \beta^2 = 1, \quad j = 1, 2, 3,
\] (3.7)
where $\delta^{ij}$ is the Kronecker delta. The momentum operator of the Dirac particle is given by
\[
p := (p_1, p_2, p_3) := (-iD_1, -iD_2, -iD_3)
\] (3.8)
with $D_j$ being the generalized partial differential operator on $L^2(\mathbb{R}^3; \mathbb{C}^4)$ with respect to the variable $x^j$, the $j$-th component of $x = (x^1, x^2, x^3) \in \mathbb{R}^3$. We write in short
\[
\alpha \cdot p := \sum_{j=1}^3 \alpha^j p_j.
\]
The potential is represented by a $4 \times 4$ Hermitian matrix-valued function $V$ on $\mathbb{R}^3$ with each matrix components being Borel measurable. Note that the function $V$ naturally defines a multiplication operator acting in $\mathcal{H}_D$. We denote it by the same symbol $V$. The Hamiltonian of the Dirac particle under the influence of this potential $V$ is then given by the Dirac operator
\[
H_D(V) := \alpha \cdot p + M\beta + V
\] (3.9)
acting in $\mathcal{H}_D$, with the domain $D(\mathcal{H}_D(V)) := H^1(\mathbb{R}^3; \mathbb{C}^4) \cap D(V)$, where $H^1(\mathbb{R}^3)$ denotes the $\mathbb{C}^4$ valued Sobolev space of order one. Let $C$ be the conjugation operator in $\mathcal{H}_D$ defined by
\[
(Cf)(x) = f(x)^*, \quad f \in \mathcal{H}_D, \quad x \in \mathbb{R}^3,
\]
where $\ast$ means the usual complex conjugation. By Pauli's lemma \cite{12}, there is a $4 \times 4$ matrix $U$ satisfying
\[
U^2 = 1, \quad UC = CU,
\]
\[
U^{-1} \alpha^j U = \alpha^j, \quad j = 1, 2, 3, \quad U^{-1} \beta U = -\beta,
\] (3.10)
(3.11)
where for a matrix $A$, $\overline{A}$ denotes its complex-conjugated matrix and 1 the identity matrix. We assume that the potential $V$ satisfies the following conditions:
Assumption 3.1.  
(I) Each matrix component of $V$ belongs to $L^2_{\text{loc}}(\mathbb{R}^3) := \left\{ f : \mathbb{R}^3 \to \mathbb{C} \mid \text{Borel measurable and } \int_{|x| \leq R} |f(x)|^2 < \infty \text{ for all } R > 0 \right\}$.

(II) $V$ is Charge-Parity (CP) invariant in the following sense:

$$U^{-1}V(x)U = V(-x)^*, \quad \text{a.e. } x \in \mathbb{R}^3.$$  \hfill (3.12)

(III) $H_D(V)$ is essentially self-adjoint.

Hereafter, we denote the closure of $H_D(V)$, which is self-adjoint by Assumption 3.1, by the same symbol. The important remark is that the Coulomb type potential

$$V(x) = -\frac{Zq^2}{|x|}$$  \hfill (3.13)

satisfies Assumption 3.1 provided that $Zq^2 < 1/2$, or more concretely, $Z \leq 68$ if we put $q = e$, the elementary charge [12].

Secondly, we introduce the free radiation field Hamiltonian in the Coulomb gauge. We adopt as the one-photon Hilbert space

$$\mathcal{H}_{\text{ph}} := L^2(\mathbb{R}^3; \mathbb{C}^2).$$  \hfill (3.14)

The above $\mathbb{R}^3 := (k = (k^1, k^2, k^3) \mid k^j \in \mathbb{R}, \ j = 1, 2, 3)$ represents the momentum space of photons, and the the target space $\mathbb{C}^2$ represents the degrees of freedom coming from the polarization of photons. We often omit the subscript $k$ in $\mathbb{R}^3_k$, and just denote it by $\mathbb{R}^3$, when there is no danger of confusion.

The Hilbert space for the quantized radiation field in the Coulomb gauge is given by

$$\mathcal{T}_{\text{ph}} := \bigoplus_{n=0}^{\infty} \bigotimes_s \mathcal{H}_{\text{ph}} = \left\{ \Psi = (\Psi(n))_{n=0}^{\infty} \mid \Psi(n) \in \bigotimes_s \mathcal{H}_{\text{ph}}, \|\Psi\|^2 := \sum_{n=0}^{\infty} \|\Psi(n)\|^2 < \infty \right\},$$  \hfill (3.15)

the Boson Fock space over $\mathcal{H}_{\text{ph}}$, where $\bigotimes_s^n$ denotes the $n$-fold symmetric tensor product with the convention $\bigotimes_0^0 \mathcal{H}_{\text{ph}} := \mathbb{C}$. Let $\omega(k) := |k|, \ k \in \mathbb{R}^3$, the energy of a photon with momentum $k \in \mathbb{R}^3$. The multiplication operator by the $2 \times 2$ matrix-valued function

$$k \mapsto \begin{pmatrix} \omega(k) & 0 \\ 0 & \omega(k) \end{pmatrix}$$  \hfill (3.16)

acting in $\mathcal{H}_{\text{ph}}$ is self-adjoint, and we also denote it by the same symbol $\omega$. This operator $\omega$ is a one-photon Hamiltonian in $\mathcal{H}_{\text{ph}}$, and the free Hamiltonian (kinetic term) of the quantum radiation field is given by its second quantization

$$H_{\text{rad}} := d\Gamma_{\text{b}}(\omega) := \bigoplus_{n=0}^{\infty} \left( \bigotimes_{j=1}^{n} I \otimes \cdots \otimes I \otimes I^{\text{th}} \otimes \omega I \otimes I \otimes \cdots \otimes I \right) \uparrow \bigotimes D(\omega).$$  \hfill (3.17)

The operator $H_{\text{rad}}$ is self-adjoint.

Thirdly, we will introduce the point-like quantum radiation field $A(x)$ at $x \in \mathbb{R}^3$ with an ultraviolet (UV) cut-off in the Coulomb gauge. This is defined in terms of photon polarization vectors $\{e_r\}_{r=1,2}$ and an ultraviolet cut-off function $\chi \in L^2(\mathbb{R}^3)$ as follows. Photon polarization vectors are $\mathbb{R}^3_k$-valued measurable functions $e_r(x) = (e_r^1(x), e_r^2(x), e_r^3(x)) \ (r = 1, 2)$ on $\mathbb{R}^3_k$ such that, for all $k \in M_0 := \mathbb{R}^3 \setminus \{(0, 0, k^3) \mid k^3 \in \mathbb{R}\}$,

$$e_r(x) \cdot e_{r'}(x) = \delta_{rr'}, \quad e_r(x) \cdot k = 0, \quad r, r' = 1, 2,$$  \hfill (3.18)
acting in direct integral with the base space (self-adjoint operator since the mapping of state vectors for the coupled system, which is taken to be the Hilbert space of We remark that this Hilbert space can be identified as given by Note that such vector valued functions can be chosen so that they are continuous on $M_0$. An ultraviolet cut-off function $\chi \in L^2(\mathbb{R}^3)$ is a real valued function on $\mathbb{R}^3$ satisfying
\[
\frac{\bar{\chi}}{\sqrt{\omega}} \in L^2(\mathbb{R}^3),
\] (3.19)
where $\bar{\chi}$ denotes the Fourier transform of $\chi$. Let us denote by $a(F) (F \in \mathcal{H}_{ph})$ the annihilation operator on $\mathcal{F}_{ph}$, and $\phi(F)$ by the Segal field operator
\[
\phi(F) := \frac{a(F) + a(F)^*}{\sqrt{2}}.
\] (3.20)
It is well known that $\phi(F)$ is self-adjoint. For each $f \in L^2(\mathbb{R}^3)$, we define
\[
a^{(1)}(f) := a(f, 0), \quad a^{(2)}(f) := a(0, f).
\] (3.21)
Then, the point-like quantized radiation field $A(x) := (A^1(x), A^2(x), A^3(x))$ with the UV cut-off $\chi$ is given by
\[
A^j(x) := \phi(g^j_x), \quad j = 1, 2, 3,
\] (3.22)
with
\[
g^j_x(k) := \left( \frac{\bar{\chi}(k)e^{ij}\bar{\chi}(k)e^{-ikx}}{\sqrt{\omega(k)}}, \frac{\bar{\chi}(k)e^{ij}\bar{\chi}(k)e^{-ikx}}{\sqrt{\omega(k)}} \right) \in \mathbb{C}^2.
\] (3.23)
Forthly, we introduce the interaction Hamiltonian and the total Hamiltonian in the Hilbert space of state vectors for the coupled system, which is taken to be
\[
\mathcal{F}_{DM} := \mathcal{H}_{D} \otimes \mathcal{F}_{rad}.
\] (3.24)
We remark that this Hilbert space can be identified as
\[
\mathcal{F}_{DM} = L^2(\mathbb{R}^3, \mathbb{R}) = \int_{\mathbb{R}^3} \omega \mathcal{F}_{rad},
\] (3.25)
the Hilbert space of $\mathcal{F}_{rad}$-valued Lebesgue square integrable functions on $\mathbb{R}^3$ (the constant fibre direct integral with the base space $(\mathbb{R}^3, dx)$ and fibre $\mathcal{F}_{rad}$). We freely use this identification. Now, since the mapping $x \mapsto g^j_x$ from $\mathbb{R}^3$ to $\mathcal{H}_{ph}$ is strongly continuous, we can define a decomposable self-adjoint operator $A^j$ by
\[
A^j := \int_{\mathbb{R}^3} \omega \mathcal{F}_{rad}.
\] (3.26)
acting in $\int_{\mathbb{R}^3} \omega \mathcal{F}_{rad}$. We have now arrived at the position to define the minimal interaction Hamiltonian $H_1$, between the Dirac particle and the quantized radiation field with the UV cutoff $\chi$. It is given by
\[
H_1 := -q \mathbf{a} \cdot \mathbf{A} = -q \sum_{j=1}^3 a^j A^j.
\] (3.27)
The total Hamiltonian of the coupled system is then given by

\[ H_{DM}(V) := H_0 + H_1, \quad H_0 := H_D(V) + H_{\text{rad}}. \] (3.28)

This is called the **Dirac-Maxwell Hamiltonian**. The essential self-adjointness of \( H_{DM}(V) \) is discussed in [2]. However, when the potential \( V \) is of the Coulomb type (3.13), the essential self-adjointness of \( H_{DM}(V) \) remains to be proved.

The rest of the present paper is devoted to prove

**Theorem 3.1.** Suppose that the potential \( V \) satisfies Assumption [3.7] and that the Fourier transformation of the UV cut-off, which we denote by \( \hat{\chi} \), is real-valued. Then, \( H_{DM}(V) \) is essentially self-adjoint.

We emphasize here again that Theorem 3.1 certainly covers the Coulomb potential case (3.13), if \( Zq^2 < 1/2 \).

4 Proof of Theorem 3.1

First, we recall the important result obtained in Ref. [2].

**Lemma 4.1.** Suppose that Assumption [3.7] is valid and \( \hat{\chi} \) is real-valued. Then, \( H_{DM}(V) \) has a self-adjoint extension.

**Proof.** See Ref. [2], Theorem 1.2. □

Let \( N_b \) be the photon number operator which is defined by

\[ N_b := 1 \otimes d\Gamma_b(1), \] (4.1)

acting in \( \mathcal{H}_D \otimes \mathcal{F}_{\text{rad}} \). Note that the operator \( N_b \) can be identified with the following decomposable operator in the sense of (3.25):

\[ N_b = \int_{\mathbb{R}^3} d\mathbf{x} \, d\Gamma_b(1). \] (4.2)

**Lemma 4.2.** The Dirac Maxwell Hamiltonian \( H_{DM}(V) \) fulfills Assumption [2.7] where the photon number operator \( N_b \) plays a role of \( A \) in it. Namely,

(i) \( N_b \) is self-adjoint and non-negative.

(ii) \( N_b \) and \( H_0 \) are strongly commuting.

(iii) \( H_1 \) is \( N_b^{1/2} \)-bounded.

(iv) If \( \Psi \in E_{N_b}([0, L]) \), then \( H_1 \Psi \in E_{N_b}([0, L + 1]) \).

**Proof.** The assertion (i) and (ii) are well known.

We prove (iii). It is well known [1] that for all \( \Psi \in D(d\Gamma_b(1)^{1/2}) \) and \( F \in \mathcal{H}_{\text{ph}} \), \( \Psi \) belongs to \( D(a(F) \cap D(a(F)^*) \) and the estimates

\[ \|a(F)\Psi\| \leq \|F\| \|d\Gamma_b(1)^{1/2}\Psi\|, \quad \|a(F)^*\Psi\| \leq \|F\| \|d\Gamma_b(1)^{1/2}\Psi\| + \|F\| \|\Psi\| \] (4.3)

are valid. Therefore, we find that, for all \( \Psi \in D(d\Gamma_b(1)^{1/2}), x \in \mathbb{R}^3 \) and \( j = 1, 2, 3 \), \( \Psi \) is in \( D(A^j(x)) \) and that

\[ \|A^j(x)\Psi\| \leq \sqrt{2} \|g_{x,j} \| \|d\Gamma_b(1)^{1/2}\Psi\| + \frac{1}{\sqrt{2}} \|g_{x,j} \| \|\Psi\|. \] (4.4)
Take arbitrary $\Psi \in D(N_{b}^{1/2})$ as a subspace of $\mathcal{F}_{DM}$. Then, for almost every $x \in \mathbb{R}^3$ and $j = 1, 2, 3$, we have $\Psi(x) \in D(A^{j}(x))$ and

$$\|A^{j}(x)\Psi(x)\| \leq \sqrt{2}\|g_{x}^{j}\| \|d\Gamma_{b}(1)^{1/2}\Psi(x)\| + \frac{1}{\sqrt{2}}\|g_{x}^{j}\| \|\Psi(x)\|. \quad (4.5)$$

From the elementary inequality

$$(a + b)^2 \leq 2a^2 + 2b^2,$$

and the fact that $\|g_{x}^{j}\| = \|g_{0}^{j}\|$, it follows that

$$\|A^{j}(x)\Psi(x)\|^2 \leq 4\|g_{0}^{j}\|^2 \|d\Gamma_{b}(1)^{1/2}\Psi(x)\|^2 + \|g_{0}^{j}\|^2 \|\Psi(x)\|^2. \quad (4.6)$$

By integrating both sides of (4.6) with respect to $x$ on $\mathbb{R}^3$, one obtains $\Psi \in D(A^{j})$ and

$$\|A^{j}\Psi\|^2 = \int_{\mathbb{R}^3} dx \|A^{j}(x)\Psi(x)\|^2 \leq 4\|g_{0}^{j}\|^2 \int_{\mathbb{R}^3} dx \|d\Gamma_{b}(1)^{1/2}\Psi(x)\|^2 + \|g_{0}^{j}\|^2 \int_{\mathbb{R}^3} dx \|\Psi(x)\|^2 = 4\|g_{0}^{j}\|^2 \|N_{b}^{1/2}\Psi\|^2 + \|g_{0}^{j}\|^2 \|\Psi\|^2. \quad (4.7)$$

Hence, we obtain

$$\|A^{j}\Psi\| \leq 2\|g_{0}^{j}\| \|N_{b}^{1/2}\Psi\| + \|g_{0}^{j}\| \|\Psi\|. \quad (4.8)$$

Thus, one obtains from (4.8) the estimate for all $\Psi \in D(N_{b}^{1/2})$,

$$\|H_{1}\Psi\| = |q| \sum_{j=1}^{3} \|\alpha^{j}A^{j}\Psi\| \leq |q| \sum_{j=1}^{3} \|\alpha^{j}\| \|A^{j}\Psi\| \leq 2|q| \sum_{j=1}^{3} \|\alpha^{j}\| \|g_{0}^{j}\| \|N_{b}^{1/2}\Psi\| + |q| \sum_{j=1}^{3} \|\alpha^{j}\| \|g_{0}^{j}\| \|\Psi\|, \quad (4.9)$$

which proves (iii).

We prove (iv). Let us introduce the closed subspace $\mathcal{F}_{N} \subset \mathcal{F}_{DM}$ for $N \in \mathbb{N} \cup \{0\}$ by

$$\mathcal{F}_{N} := \mathcal{H}_{D} \otimes \left(\bigoplus_{n=0}^{N} \mathcal{H}_{ph}\right). \quad (4.10)$$

As is well known, the spectrum of $N_{b}$ is equal to the discrete set $\mathbb{N} \cup \{0\}$, and the eigenspace belonging to an eigenvalue $n \in \mathbb{N} \cup \{0\}$ is $\mathcal{H}_{D} \otimes (\otimes_{s=0}^{n} \mathcal{H}_{ph})$. Hence, for $L \geq 0$, $E_{N_{b}}((0, L))$ is the orthogonal projection onto $\mathcal{F}_{[L]}$ with $[L]$ denoting the integer satisfying $L - 1 < [L] \leq L$, and we have

$$R(E_{N_{b}}((0, L))) = \mathcal{F}_{[L]}. \quad (4.11)$$

Since the interaction Hamiltonian $H_{1}$ creates at most one photon, $H_{1}$ maps $\mathcal{F}_{N}$ into $\mathcal{F}_{N+1}$, and we find that $\Psi \in R(E_{N_{b}}((0, L)))$ implies $\Psi \in R(E_{N_{b}}((0, [L] + 1)))$ and $R(E_{N_{b}}((0, L + 1)))$. This proves (iv). \hspace{1cm} \Box

**Proof of Theorem 3.7** From Lemma 4.2 and Theorem 2.1, we find that if there exists at least one self-adjoint extension of $H_{DM}(V)$, then $H_{DM}(V)$ is essentially self-adjoint. But from Lemma 4.1, $H_{DM}(V)$ indeed has a self-adjoint extension. This completes the proof. \hspace{1cm} \Box
We remark that our proof presented here is also applicable to similar particle-field Hamiltonians. For instance, we can prove that the Dirac-Klein-Gordon Hamiltonian $H_{DKG}(V)$ acting in the Hilbert space $L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathcal{F}_b(L^2(\mathbb{R}^3))$, given by

$$H_{DKG}(V) = H_D(V) + 1 \otimes d\Gamma_b(\omega) + \lambda \int_{\mathbb{R}^3} d\mathbf{x} \beta \phi(x_j),$$

with

$$x_j(x)(k) := \frac{\hat{\chi}(k)}{\sqrt{\omega(k)}} e^{-ikx}, \quad \chi \in L^2(\mathbb{R})$$

and $\lambda \in \mathbb{R}$, is essentially self-adjoint as long as the above assumptions are satisfied. This Hamiltonian describes a quantum system of a Dirac particle under the potential $V$ interacting with a neutral scalar field.

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