BUMP CONDITIONS FOR GENERAL ITERATED COMMUTATORS WITH APPLICATIONS TO COMPACTNESS

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ABSTRACT. We prove new sufficient bump conditions for general iterated commutators of Calderón-Zygmund operators and fractional integral operators. As an application of our results we derive two weight compactness theorems for higher order iterated commutators with a CMO function.

1. Introduction

Let $L$ be a linear integral operator and consider the commutator of $L$ and a function $b$ is given by

$$[b, L]f(x) = b(x)Lf(x) - L(bf)(x).$$

In this paper we consider the general iterated commutator with symbol $b = (b_1, \ldots, b_m)$

$$L_b f(x) = [b_m \ldots [b_1, L] \ldots] f(x)$$

when $L$ is a Calderón-Zygmund operator $T$ or fractional integral operator $I_\alpha$. The action

$$(b_1, \ldots, b_m, f) \mapsto L_b f$$

defines an $(m + 1)$-linear operator. For this reason, $L_b$ is sometimes referred to as a multilinear commutator; not to be confused with a commutator of a multilinear operator. In the case of a Calderón-Zygmund operator with kernel $K(x,y)$ it is formally given by

$$T_b f(x) = \int_{\mathbb{R}^n} \prod_{j=1}^m (b_j(x) - b_j(y)) K(x,y) f(y) dy.$$

These general iterated commutators were first introduced by Pérez and Trujillo-González [20], who proved endpoint estimates and weighted inequalities. When $L$ is the fractional integral operator $I_\alpha$

$$I_{\alpha,b} f(x) = \int_{\mathbb{R}^n} \prod_{j=1}^m (b_j(x) - b_j(y)) \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Later, Gorosito, Pradolini, and Salinas [7] proved similar results for $I_{\alpha,b}$. The operators $L_{(b,\ldots,b)} = L^m_b$ are called the $m$-th iterated commutators and have been studied by a number of mathematicians.

It is well known that the $A_p$ condition

$$[w]_{A_p} = \sup_Q \left( \frac{1}{|Q|} \int_Q w \right) \left( \int_Q w^{1-p'} \right)^{p-1} < \infty$$
is sufficient for singular integral operators to be bounded on $L^p(w)$. However Muckenhoupt and Wheeden in [17] showed that the correlating two-weight $A_p$ condition
\[
\sup_Q \left( \frac{1}{Q} \right)^{\frac{1}{p}} \left( \frac{1}{Q} \right)^{\frac{1}{q'}} = \sup_Q \|u^{\frac{1}{p}}\|_{L^p(Q)} \|v^{-\frac{1}{q'}}\|_{L^{q'}(Q)} < \infty
\]
is almost never sufficient for the $L^p(v) \to L^p(u)$ two weight boundedness of operators (see [3] for more details). To ameliorate this, the stronger $A_p$-bump condition was introduced:
\[
\sup_Q \|u^{\frac{1}{p}}\|_{X,Q} \|v^{-\frac{1}{q'}}\|_{Y,Q} < \infty,
\]
where $X$ and $Y$ are slightly larger norms than $L^p$ and $L^{q'}$ respectively. To state these results, we recall some background material on Young functions and Orlicz spaces. A function $A : [0, \infty) \to [0, \infty)$ is called a Young function if it is increasing, convex, $A(0) = 0$ and $A(t)/t \to \infty$ as $t \to \infty$. Given a Young function $A$, there exists another Young function $\bar{A}$ which we call the associate function of $A$, and the two functions satisfy
\[
A^{-1}(t)\bar{A}^{-1}(t) \approx t.
\]
The Orlicz average with respect to $A$ of $f$ over $Q$ is given by
\[
\|f\|_{A,Q} = \inf \left\{ \lambda > 0 : \frac{1}{Q} A \left( \frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.
\]
The quantity $\| \cdot \|_{A,Q}$ is indeed a norm. Also note that if $A(t) = t^p$, $p \geq 1$, then
\[
\|f\|_{A,Q} = \|f\|_{L^p,Q} = |Q|^{-\frac{1}{p}} \|f\chi_Q\|_{L^p}
\]
is the usual $L^p$ average. When $A(t) = t^p \log(e + t)^{\alpha}$ we will write
\[
\|f\|_{A,Q} = \|f\|_{L^p(\log L)^{\alpha},Q}.
\]
For $B(t) = e^t - 1$ we will write
\[
\|f\|_{B,Q} = \|f\|_{\exp L,Q}.
\]
The $B_p$ integrability condition was introduced by Pérez in [18] as a way to quantify the bump conditions. Given $1 < p < \infty$, we say that a Young function belongs to $B_p$ if
\[
\int_1^{\infty} \frac{A(t)}{t^p} \frac{dt}{t} < \infty.
\]
Typical Young functions that satisfy the $B_p$ condition are $A(t) = t^{p-\delta}$ and $A(t) = t^p \log(e + t)^{-1-\delta}$ for some $\delta > 0$. Pérez showed that the condition
\[
\sup_Q \left( \frac{1}{Q} \right)^{\frac{1}{p}} \|v^{-\frac{1}{q'}}\|_{B,Q} < \infty
\]
for some $\bar{B} \in B_p$ is sufficient for the Hardy-Littlewood maximal function
\[
Mf(x) = \sup_{Q \ni x} \frac{1}{Q} \int_Q |f(y)| dy,
\]
to be bounded from $L^p(v)$ to $L^p(u)$. Pérez also showed that the condition

$$\sup_{Q} |Q|^{\frac{n}{p} + \frac{1}{q} - \frac{1}{p}}\|u^\frac{1}{q}\|_{A,Q}\|v^{-\frac{1}{p}}\|_{B,Q} < \infty$$

(1.2)

where $\tilde{A} \in B_{q'}$ and $\tilde{B} \in B_p$ is sufficient for $I_{\alpha} : L^p(v) \to L^q(u)$ when $1 < p \leq q < \infty$. In the strict off-diagonal case, $p < q$, Cruz-Uribe and the second author improved (1.4) in two ways. First, the $B_{p,q}$ condition is introduced for a Young function $A$

$$\int_1^\infty A(t)\frac{\tilde{t}^\frac{q}{p}}{t^n}\frac{dt}{t} < \infty.\quad (1.3)$$

The $B_{p,q}$ is a weaker condition on the Young function in the sense that $B_p \subseteq B_{p,q}$ when $p < q$ (of course $B_p = B_{p,p}$). Second, they separated the bumps and showed that

$$\sup_{Q} |Q|^{\frac{n}{p} + \frac{1}{q} - \frac{1}{p}}\left(\int_u^1 u^\frac{1}{q}\right)^\frac{1}{p}\|u^\frac{1}{q}\|_{A,Q} + \sup_{Q} |Q|^{\frac{n}{p} + \frac{1}{q} - \frac{1}{p}}\|u^\frac{1}{q}\|_{B,Q}\left(\int_Q v^{1-p'}\right)^\frac{1}{p'} < \infty$$

for some $\tilde{A} \in B_{q',q'}$ and $\tilde{B} \in B_p$ is sufficient for $I_{\alpha} : L^p(v) \to L^q(u)$. Interestingly, these separated bumps are not know to be sufficient in the case $p = q$ for $I_{\alpha}$. For Calderón-Zygmund operators Lerner [13] showed that

$$\sup_{Q} \|u^\frac{1}{q}\|_{A,Q}\|v^{-\frac{1}{p}}\|_{B,Q} < \infty$$

(1.4)

where $\tilde{A} \in B_{q'}$ and $\tilde{B} \in B_p$ is sufficient for $T$ and for the maximal truncation operator $T^\sharp$ to be bounded from $L^p(v)$ to $L^p(u)$.

For commutators, the story is more complicated. Cruz-Uribe and the second author [4] began the study of bump conditions for commutators with $BMO$ functions (see Section. In [4] it shown that the condition

$$\sup_{Q} \|u^\frac{1}{q}\|_{L^p(\log L)^{2p-1+\delta},Q}\|v^{-\frac{1}{p}}\|_{L^{p'}(\log L)^{2p'-1+\delta},Q} < \infty$$

(1.5)

for some $\delta > 0$ is sufficient for the inequality

$$|[b,T]f|_{L^p(u)} \leq C\|b\|_{BMO}\|f\|_{L^p(v)}.$$

For fractional integrals

$$\sup_{Q} |Q|^{\frac{n}{p} + \frac{1}{q} - \frac{1}{p}}\left(\int v^{\frac{1}{q}}\right)^\frac{1}{p}\|u^\frac{1}{q}\|_{L^p(\log L)^{(m+1)e^\delta},Q}\|v^{-\frac{1}{p}}\|_{L^{p'}((\log L)^{(m+1)e^\delta}Q, Q} < \infty$$

(1.6)

for $\delta > 0$ is sufficient for

$$\|(I_{\alpha})^m f|_{L^p(u)} \leq C\|b\|_{BMO}^m\|f\|_{L^p(v)}.$$
[6] found a way to combine the oscillation class with the weighted class. In particular they showed that

$$\text{sup}_Q \| (b - b_Q)^m u_{\frac{1}{p}} \|_{A,Q} \| v_{\frac{1}{p}} \|_{B,Q} + \text{sup}_Q \| u_{\frac{1}{p}} \|_{A,Q} \| (b - b_Q)^m v_{\frac{1}{p}} \|_{B,Q} < \infty$$

for some $\tilde{A} \in B_{p'}$ and $\tilde{B} \in B_p$ is sufficient for

$$\| T^m b f \|_{L^p(v)} \leq C \| f \|_{L^p(v)}.$$  

The last author and Wu [23] extended this theory to the fractional integral case, showing that

$$\text{sup}_Q |Q|^{\frac{\alpha}{m} + \frac{1}{p} - \frac{1}{q}} \| (b - b_Q)^m u_{\frac{1}{p}} \|_{A,Q} \| v_{\frac{1}{p}} \|_{B,Q}$$

$$+ \text{sup}_Q |Q|^{\frac{\alpha}{m} + \frac{1}{p} - \frac{1}{q}} \| u_{\frac{1}{p}} \|_{A,Q} \| (b - b_Q)^m v_{\frac{1}{p}} \|_{B,Q} < \infty$$

is sufficient for $(I_a)^m_b : L^p(v) \to L^q(u)$ when $\tilde{A} \in B_{p'}$ and $\tilde{B} \in B_{p,q}$. The oscillation condition (1.7) is related to Bloom’s BMO, in the sense that it is a weighted BMO assumption. However, it does not require any strong assumptions on the individual weights $u$ and $v$. Moreover, it is important because the condition (1.7) implies (1.6) when $b \in BMO$ and condition (1.7) can be used to study $b$ in different oscillation classes, for example $b^k \in BMO$ for $k \in \mathbb{N}$ (see [6]).

In this article we extend the oscillation bump conditions (1.7) and (1.8) to the operators $T_b$ and $I_{a,b}$. This is not done to generalize for the sake of generalizing, but rather as a necessary tool to study the compactness of the higher order iterated commutators. Our main theorem for the general iterated commutator is the following.

**Theorem 1.** Let $1 < p < \infty$, $b = (b_1, \ldots, b_m) \in L^1_{loc}(\mathbb{R}^n)^m$, and $T$ be a Calderón-Zygmund operator. If $(u, v)$ are weights such that

$$K = \sum_{\tau \subseteq \{1, \ldots, m\}} \text{sup}_Q \| \prod_{i \in \tau} (b_i - (b_i)_Q) u_{\frac{1}{p}} \|_{A,Q} \| \prod_{i \in \tau} (b_i - (b_i)_Q) v_{\frac{1}{p}} \|_{B,Q} < \infty$$

for $\tilde{A} \in B_{p'}$ and $\tilde{B} \in B_p$, then

$$\| T_b f \|_{L^p(v)} \lesssim K \| f \|_{L^p(v)}.$$  

For the fractional integral operator we work in the lower triangular case $p \leq q$.

**Theorem 2.** Let $1 < p \leq q < \infty$, $0 < \alpha < n$, and $b = (b_1, \ldots, b_m) \in L^1_{loc}(\mathbb{R}^n)^m$. If $(u, v)$ are weights such that

$$K_\alpha = \sum_{\tau \subseteq \{1, \ldots, m\}} \text{sup}_Q |Q|^{\frac{\alpha}{m} + \frac{1}{p} - \frac{1}{q}} \| \prod_{i \in \tau} (b_i - (b_i)_Q) u_{\frac{1}{p}} \|_{A,Q} \| \prod_{i \in \tau} (b_i - (b_i)_Q) v_{\frac{1}{p}} \|_{B,Q} < \infty$$

for any $p \leq s \leq q$ such that $\tilde{A} \in B_{q', s'}$ and $\tilde{B} \in B_{p, s}$, then

$$\| I_{a,b} f \|_{L^p(v)} \lesssim K_\alpha f \|_{L^p(v)}.$$
Remark 3. When $p < q$ we have a scale of conditions corresponding to $s$ with $p \leq s \leq q$. When $s = p$ we have $A \in B_{q', p'}$ and $B \in B_p$ and when $s = q$ we have $A \in B_q$ and $B \in B_{p, q}$. This scale of conditions was first found by Tran [21].

When we turn to the bump conditions on the individual weights we have the following Corollaries.

Theorem 4. Let $1 < p < \infty$, $b = (b_1, \ldots, b_m) \in BMO^m$, and $T$ be a Calderón-Zygmund operator. If $(u, v)$ are weights such that

$$\hat{K} = \sup_Q \|u^{\frac{3}{2}} \|_{L^p(\log L)^{(m+1)p-1+\delta}, Q} \|v^{-\frac{1}{2}} \|_{L^{p'}(\log L)^{(m+1)p'-1+\delta}, Q} < \infty$$

for some $\delta > 0$, then

$$\|Tbf\|_{L^p(u)} \lesssim \hat{K} \prod_{j=1}^m \|b_j\|_{BMO} \|f\|_{L^p(v)}.$$

Theorem 5. Let $1 < p \leq q < \infty$, $0 < \alpha < n$, and $b = (b_1, \ldots, b_m) \in BMO^m$. If $(u, v)$ are weights such that

$$\hat{K}_\alpha = \sup_Q |Q|^{\frac{\alpha}{\alpha+n}} \|u^{\frac{1}{\alpha+n}} \|_{L^\alpha(\log L)^{(m+\frac{1}{n})q+\delta}, Q} \|v^{-\frac{1}{\alpha+n}} \|_{L^{p'}(\log L)^{(m+\frac{1}{n})p'+\delta}, Q} < \infty$$

for any $p \leq s \leq q$ and $\delta > 0$ then

$$\|I_{\alpha, bf}\|_{L^p(u)} \lesssim \hat{K}_\alpha \prod_{j=1}^m \|b_j\|_{BMO} \|f\|_{L^p(v)}.$$

Remark 6. For $p \leq s \leq q$, the powers on logarithmic terms satisfy $(m+\frac{1}{n})q + \delta \leq (m+1)q - 1 + \delta$ and $(m+\frac{1}{n})p' + \delta \leq (m+1)p' - 1 + \delta$ which were the previous powers by Li [15]. However, when $p = q$ the condition collapses to

$$\sup_Q |Q|^{\frac{\alpha}{\alpha+n}} \|u^{\frac{1}{\alpha+n}} \|_{L^\alpha(\log L)^{(m+1)p+\delta}, Q} \|v^{-\frac{1}{\alpha+n}} \|_{L^{p'}(\log L)^{(m+1)p'+\delta}, Q} < \infty.$$

The main application our results is to prove compactness of higher order iterated commutators. Recall that a linear operator $T : X \to Y$ between two Banach spaces is compact if $T(B_X)$ has compact closure in $Y$ ($B_X$ being the the unit ball in $X$). In [22] Uchiyama showed that restricting $b$ to a subset $CMO \subseteq BMO$ gives us sufficient conditions for a Calderón-Zygmund operator to satisfy $[b, T] : L^p \to L^p$ being a compact operator (see [16] for more context on recent work regarding compactness with respect to $T$). The first and second authors in [16] established conditions on a two-weight system $(u, v)$ such that $b \in CMO$ is sufficient to establish that $[b, T] : L^p(v) \to L^q(u)$ is a compact operator. When studying the compactness of the operator $[b, T] = T^1_b$ the linearity of the map $b \mapsto [b, T]$ is important and does not readily transfer to the operator $T^{m}_b$. However, by analyzing the operator $T_b$ we are able to avoid this obstacle. As an application to the general iterated norm inequalities we prove in section 3, we extend this result to iterated commutators:
Theorem 7. Let $1 < p < \infty$, $b \in \text{CMO}(\mathbb{R}^n)$, and $T$ a Calderón-Zygmund operator. If $(u, v)$ are a pair of weights satisfying
\[
\sup_Q \| u^{\frac{1}{p}} \|_{L^p(\log L)^{(m+1)p-1+\delta},Q} \| v^{-\frac{1}{p}} \|_{L^{p'}(\log L)^{(m+1)p'-1+\delta},Q} < \infty
\]
for some $\delta > 0$, then $T^m_b$ is a compact operator from $L^p(v)$ to $L^p(u)$ for all natural numbers $m$.

In the one weight setting it is shown in [24] that $b \in \text{CMO}$ characterizes compactness of the operator $[b, I_\alpha] : L^p(w^p) \to L^q(w^q)$. However this requires the assumption that $w \in A_{p,q}$, which implies that $w^p \in A_p$ and $w^q \in A_q$. We will prove general iterated commutator norm inequalities in section 3 using similar sparse domination arguments to the ones used for Calderón-Zygmund operators. We then prove bump conditions for iterated commutators of the Riesz potential to be compact operators.

Theorem 8. Let $0 < \alpha < n$, $1 < p \leq q < \infty$, $b \in \text{CMO}(\mathbb{R}^n)$. If $(u, v)$ are a pair of weights satisfying
\[
(1.9) \quad \sup_Q |Q|^{\frac{1}{s} + \frac{1}{s'}} \| u^{\frac{1}{s}} \|_{L^s((\log L)^{(m+1)s-1+\delta}),Q} \| v^{-\frac{1}{s'}} \|_{L^{s'}((\log L)^{(m+1)s'-1+\delta}),Q} < \infty
\]
for any $p \leq s \leq q$ and $\delta > 0$, then $(I_\alpha)_b^m : L^p(v) \to L^q(u)$ is a compact operator for all natural numbers $m$.

2. Preliminaries

Given a measurable function $b$ and a cube $Q$, by $b_Q$ we denote the average value of $b$ on $Q$:
\[
b_Q = \frac{1}{|Q|} \int_Q b(x) dx.
\]
We say $b$ is of bounded mean oscillation, denoted $\text{BMO}$, if
\[
\| b \|_{\text{BMO}} = \sup_Q \int_Q |b(x) - b_Q| dx < \infty.
\]
This quantity fails to be a norm since any constant function $c$ is such that $\| c \|_{\text{BMO}} = 0$. We remedy this by considering the space $\text{BMO}$ modulo constants. Define the space $\text{CMO}$ as the closure of $C^\infty_0(\mathbb{R}^n)$ in the $\text{BMO}$ norm $\| \cdot \|_{\text{BMO}}$. Recall that $\text{BMO}$ functions satisfy the John-Nirenberg inequality,
\[
\int_Q \exp \left( \frac{c|b(x) - b_Q|}{\| b \|_{\text{BMO}}} \right) dx \leq C
\]
for some constants $c, C > 0$. In terms of Orlicz norms the John-Nirenberg inequality implies
\[
\| b \|_{\text{BMO}} \approx \sup_Q \| b - b_Q \|_{\exp L,Q}.
\]

The following characterizations follow the work by Cruz-Uribe, Martell, and Pérez in [3]. When working with a pair of weights $(u, v)$ we think of non-negative, locally integrable functions such that $u$ is positive on a set of positive measure and $v$ is positive almost everywhere.
We need the following multilinear version Hölder inequality for Orlicz spaces which was proven in [20].

**Lemma 9.** Let $A_1, \ldots, A_n, C$ be non-negative, continuous, strictly increasing functions on $[0, \infty)$ that satisfy

$$(A_1^{-1} \cdots A_n^{-1})(t) \leq C^{-1}(t) \text{ for all } t \geq 0.$$ 

Also assume $C$ is a Young function. If $Q$ is a cube and $f_1, \ldots, f_n$ are measurable functions, then

$$\|f_1 \cdots f_n\|_{C,Q} \leq n\|f_1\|_{A_1,Q} \cdots \|f_n\|_{A_n,Q}.$$ 

Given a Young function $A$, define the maximal operator associated with $A$ by

$$M_A f(x) = \sup_{Q \ni x} \|f\|_{A,Q}.$$ 

Pérez showed in [19] that the $B_p$ integrability condition, (1.1), characterizes the $L^p(\mathbb{R}^n)$ boundedness of $M_A$, namely, $A \in B_p$ if, and only if, $M_A : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. We will use condition introduced in [5] similar to the $B_p$ condition but better suited to our fractional bump conditions. If we define the fractional Orlicz maximal operator

$$M_{\beta,A} f(x) = \sup_{Q \ni x} |Q|^{\beta/n} \|f\|_{A,Q}.$$ 

If $\frac{\beta}{n} = \frac{1}{p} - \frac{1}{q}$ the authors in [5] prove that the $B_{p,q}$ condition on $A$, (1.3), is sufficient for the boundedness $M_{\beta,A} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$.

Recall that $T$ is a Calderón-Zygmund operator (CZO) on $\mathbb{R}^n$ if $T$ is bounded on $L^2(\mathbb{R}^n)$ and it admits the following representation

$$T f(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \quad f \in L^\infty_c(\mathbb{R}^n), x \notin \text{supp } f.$$ 

The kernel $K(x,y)$ defined on $\{(x,y) : x \neq y\}$ satisfies the size condition:

$$|K(x,y)| \leq \frac{C}{|x-y|^n}, \quad x \neq y \tag{2.1}$$

and the smoothness condition

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq \frac{C|x - x'|}{|x-y|^{n+1}}, \tag{2.2}$$

for all $|x - y| > 2|x - x'|$. We will also need the maximal truncation operator which is given by

$$T^\# f(x) = \sup_{\eta > 0} \left| \int_{|x-y| > \eta} K(x,y)f(y)dy \right|.$$ 

In [18], it is shown that if the pair of weights satisfies the condition

$$\sup_Q \left( \int_Q u \right)^{\frac{1}{p}} \|v^{1/p}\|_{L^{p'}(\log L)^{p'-1+\delta,q}} < \infty, \tag{2.3}$$
for some $\delta > 0$ then $M : L^p(v) \to L^p(u)$. For CZOs bump conditions on both weights are needed. Lerner [13] showed that the condition

$$\sup_Q \|u^\frac{1}{p}\|_{L^p(\log L)^{p-1+\delta},Q} \|v^{-\frac{1}{p'}}\|_{L^{p'}(\log L)^{p'-1+\delta},Q} < \infty$$

for some $\delta > 0$ is sufficient for the boundedness of $T$ and $T^\sharp$ from $L^p(v)$ to $L^p(u)$.

Define the fractional integral operator by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n.$$ 

As mentioned in the introduction, [18], Pérez proved that if $1 < p \leq q < \infty$ and $(u, v)$ satisfy

$$\sup_Q |Q|^{\frac{1}{n} + \frac{1}{q} - \frac{1}{p}} \|u^\frac{1}{q}\|_{L^q(\log L)^{q-1+\delta},Q} \|v^{-\frac{1}{p'}}\|_{L^{p'}(\log L)^{p'-1+\delta},Q} < \infty$$

for $\delta > 0$, then $I_\alpha : L^p(v) \to L^q(u)$. However commutators of singular integrals are more singular than their associated operators as is seen in the different conditions needed for sharp norm inequalities. In [15], Li showed that given (1.9), then $I_{\alpha,b}^m : L^p(v) \to L^q(u)$. This is sharp in the sense that it is not true for $\delta = 0$ (see [4]).

This inequality is crucial for a reduction we make in our compactness arguments. Note how the $m = 1$ case reflects the higher singularity of the commutator from the higher power of the logarithm in our Young functions. Also note that the condition given in (1.9) is sufficient for $I_\alpha$ and $M_\alpha$ to be bounded from $L^p(v)$ to $L^q(u)$, where $M_\alpha$ is the maximal operator associated with the Riesz potential,

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{\frac{1}{n} - \frac{\alpha}{n}}} \int_Q |f(y)| dy.$$ 

To prove our compactness results will use a weighted version of the Kolmogorov-Riesz theorem due to Guo and Zhao [8].

**Lemma 10.** Let $1 \leq p < \infty$, and let $u$ be a weight. If $\mathcal{F} \subseteq L^p(u)$ satisfies the following conditions:

(a) $\mathcal{F}$ is uniformly bounded,

$$\sup_{f \in \mathcal{F}} \|f\|_{L^p(u)} \lesssim 1;$$

(b) $\mathcal{F}$ uniformly vanishes at infinity:

$$\lim_{R \to \infty} \sup_{f \in \mathcal{F}} \|f \chi_{\mathbb{R}^n \setminus B(0,R)}\|_{L^p(u)} = 0;$$

(c) $\mathcal{F}$ is uniformly equicontinuous:

$$\lim_{h \to 0} \sup_{f \in \mathcal{F}} \|f(\cdot + h) - f(\cdot)\|_{L^p(u)} = 0,$$

then the family $\mathcal{F}$ has compact closure in $L^p(u)$.
3. Sparse Domination

Recall that a dyadic grid $D$ is a collection of half-open cubes $\prod_{i=1}^{n}[a_i, b_i)$ such that:

(a) Each cube has side-length $2^k$ for some integer $k$. If $D_k$ is the collection of cubes $Q \in D$ of side-length $2^k$, then $D_k$ partitions $\mathbb{R}^n$.

(b) for all $x \in \mathbb{R}^n$, there is a unique cube in each family $D_k$ containing it.

(c) given any two distinct cubes in $D$, they are either disjoint or one is contained in the other.

(d) for each cube $Q \in D_k$, there is a unique cube in $Q_{k+1}$ containing it; this cube is denoted $\hat{Q}$ and is called the dyadic parent of $Q$.

(e) if $Q \in D_k$ then there are $2^n$ cubes in $D_{k-1}$ contained in $Q$.

Given a cube $Q$ then $D(Q)$ is the set of all dyadic cubes with respect to $Q$, i.e. the cubes obtained by repeatedly subdividing $Q$ and its descendants into $2^n$ congruent cubes. An important sub-family of dyadic grids are sparse families. A family $S$ of cubes in $\mathbb{R}^n$ is sparse if there exists $0 < \alpha < 1$ such that for all $Q \in S$ there is a measurable set $E_Q \subseteq Q$ such that $|E_Q| \geq \alpha |Q|$ and the collection $\{E_Q\}_{Q \in S}$ is pairwise disjoint.

We wish to establish strong norm inequalities in the two-weight setting for both $T_b$ and $I_{\alpha, b}$ by way of sparse domination. Given a dyadic grid $D$ and sparse family $S \subseteq D$ define the sparse operator

$$T_{S, b}^{0, \tau} f(x) = \sum_{Q \in S} |Q|^{\frac{\alpha}{n}} \left( \prod_{i \in \tau} |b_i(x) - (b_i)_Q| \int_{Q \cap \tau} \prod_{l \in \tau} |b_l(y) - (b_l)_Q| f(y) dy \right) \chi_Q(x)$$

where $0 \leq \alpha < n$ and $\tau \subseteq \{1, \ldots, m\}$. When $\alpha = 0$ we simply write $T_{S, b}^{0, \tau} = T_{S, b}^{\tau}$.

We now show that our iterated commutators can be bounded by a linear combination of the above sparse operators. For Calderón-Zygmund operators we have the following lemma, which is a simplified version of Proposition 2.1 in [11]:

**Lemma 11.** Let $T$ be a Calderón-Zygmund operator and $\tau_m = \{1, \ldots, m\}$. Given $b = (b_1, \ldots, b_m)$ with $b_i \in L^1_{\text{loc}}(\mathbb{R}^n)$, there exists $C = C(n, T)$ such that for any $f \in L^\infty_{\text{loc}}(\mathbb{R}^n)$, there exists $3^n$ sparse families $S_j$ of dyadic cubes such that

$$|T_b f(x)| \leq C \sum_{j=1}^{3^n} \sum_{\tau \subseteq \tau_m} T_{S_j, b}^{\tau} f(x).$$

For $I_{\alpha, b}$ there does not seem to be known pointwise sparse domination formulas, unless $b_1 = b_2 = \cdots = b_m$ (see [1]), so we will prove one. First we need the following lemmas.

**Lemma 12.** (cf. [12]) Given a dyadic lattice $D$, there exist $3^n$ dyadic lattices $D_1, \ldots, D_{3^n}$ such that

$$\{3Q : Q \in D\} = \bigcup_{j=1}^{3^n} D_j$$
and for each cube $Q \in \mathcal{D}$ we can find a cube $R_Q$ in each $\mathcal{D}_j$ such that $Q \subseteq R_Q$ and $3l_Q = l_{R_Q}$.

Letting $\mathcal{D}$ be a dyadic lattice, we note that for any cube $Q \subseteq \mathbb{R}^n$ we can always find a cube $Q' \in \mathcal{D}$ such that $l_Q/2 < l_{Q'} \leq l_Q$ and $Q \subset 3Q'$. By the above lemma, for some $j \in \{1, \ldots, 3^n\}$, it is easy to see that $3Q' = P \in \mathcal{D}_j$. Hence, for each cube $Q \subseteq \mathbb{R}^n$, we can find a cube $P \in \mathcal{D}_j$ that satisfies $Q \subset P$ and $l_P \leq 3l_Q$.

For a cube $Q_0 \subset \mathbb{R}^n$, define the grand maximal truncated operator $M_{I_{\alpha}}$ and local grand maximal truncated operator $M_{I_{\alpha},Q_0}$ by

$$M_{I_{\alpha},f}(x) = \sup_{Q \ni x} \text{ess sup } |I_{\alpha}(f \chi_{\mathbb{R}^n \setminus 3Q})(\xi)|,$$

$$M_{I_{\alpha},Q_0}f(x) = \sup_{x \in Q \subset Q_0} \text{ess sup } |I_{\alpha}(f \chi_{\mathbb{R}^n \setminus 3Q})(\xi)|,$$

respectively.

**Lemma 13.** (cf. [1]) Let $0 < \alpha < n$. Let $Q_0 \subset \mathbb{R}^n$ be a cube. The following pointwise estimates holds:

1. For a.e. $x \in Q_0$, $|I_{\alpha}(f \chi_{Q_0})(x)| \leq M_{I_{\alpha},Q_0}f(x)$;

2. $M_{I_{\alpha}}$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{n/(n-\alpha)}(\mathbb{R}^n)$.

**Lemma 14.** Let $0 < \alpha < n$, $I_{\alpha}$ be fractional integral operators and $\tau_m = \{1, \ldots, m\}$. Given $b(x) = (b_1(x), \ldots, b_m(x))$ with $b_i \in L^1_{\text{loc}}(\mathbb{R}^n)$, there exists a constant $C = C(n, \alpha)$ so that for any $f \in L^\infty_{\text{loc}}(\mathbb{R}^n)$, there exists $3^n$ sparse families $S_j$ of dyadic cubes such that

$$|I_{\alpha,b}f(x)| \leq C \sum_{j=1}^{3^n} \sum_{\tau \subseteq \{1, \ldots, m\}} T^{\alpha,\tau}_{S_j,b}(f)(x).$$

**Proof.** As we previously noted there are $3^n$ dyadic lattices such that for any cube $Q \subset \mathbb{R}^n$, there is a cube $R_Q \in \mathcal{D}_j$ for some $j$, for which $3Q \subset R_Q$ and $|R_Q| \leq 9^n|Q|$.

Following a similar scheme as in [12], it reduces to show that for any cube $Q_0 \subset \mathbb{R}^n$, there is a $1/2$-sparse family $S \subset \mathcal{D}(Q_0)$ such that for a.e. $x \in Q_0$

$$|I_{\alpha,b}(f \chi_{3Q_0})(x)| \leq C \sum_{Q \in S} \left[ \sum_{\tau \subseteq \tau_m} \left( \prod_{i \in \tau} |b_i(x) - (b_i)_{R_Q}| \left( \prod_{k \in \tau_m \setminus \tau} |b_k - (b_k)_{R_Q,Q_0}| \right)^{\alpha/3Q_0} \right] \times |3Q_0|^\alpha \chi_{Q_0}(x).$$

To prove (3.1), it suffices to prove the following recursive estimate: there is a disjoint family of cubes $P_j \in \mathcal{D}(Q_0)$ such that $\sum_j |P_j| \leq |Q_0|/2$ and for a.e. $x \in Q_0$,

$$|I_{\alpha,b}(f \chi_{3Q_0})(x)| \leq C \left[ \sum_{\tau \subseteq \tau_m} \left( \prod_{i \in \tau} |b_i(x) - (b_i)_{R_Q_Q_0}| \left( \prod_{k \in \tau_m \setminus \tau} |b_k - (b_k)_{R_Q_Q_0}| \right)^{\alpha/3Q_0} \right]$$
\[ \times |3Q_0|^{\alpha/n} \chi_{Q_0}(x) + \sum_j |I_{\alpha,b}(f \chi_{3P_j})(x)| \chi_{P_j}(x). \]

Indeed, iterating (3.2), we get (3.1) with \( \mathcal{S} = \{ P_j^0 \} \), where \( \{ P_j^0 \} = \{ Q_0 \} \), \( \{ P_j^1 \} = \{ P_j \} \) and \( \{ P_j^k \} \) are the cubes obtained at the \( k \)-th stage of this iterative process.

For any mutually disjoint cubes \( P_j \in \mathcal{D}(Q_0) \), (3.2) follows from the below estimate (3.3)

\[ |I_{\alpha,b}(f \chi_{3Q_0})| \chi_{Q_0 \cup P_j} + \sum_j |I_{\alpha,b}(f \chi_{3Q_0}) - I_{\alpha,b}(f \chi_{3P_j})| \chi_{P_j} \]

\[ \leq C \left[ \sum_{\tau \subseteq \tau_m} \left( \prod_{i \in \tau} |b_i(x) - (b_i)_{R_{Q_0}}| \right) \left( \prod_{k \in \tau_m \setminus \tau} |b_k - (b_k)_{R_{Q_0}}||f| \right) \right] |3Q_0|^{\alpha/n} \chi_{Q_0}(x). \]

Now we are in the position to prove (3.3). Since \( I_{\alpha,b-c} = I_{\alpha,b} \) for any vector constant \( c \), the left side of inequality (3.3) is bounded by

\[ \leq \sum_{\tau \subseteq \tau_m} \prod_{i \in \tau} |b_i - (b_i)_{R_{Q_0}}| \left| I_{\alpha} \left( \prod_{k \in \tau_m \setminus \tau} (b_k - (b_k)_{R_{Q_0}}) f \right) \chi_{Q_0 \cup P_j} \right| \]

\[ + \sum_j \sum_{\tau \subseteq \tau_m} \prod_{i \in \tau} |b_i - (b_i)_{R_{Q_0}}| \left| I_{\alpha} \left( \prod_{k \in \tau_m \setminus \tau} (b_k - (b_k)_{R_{Q_0}}) f \chi_{3Q_0 \setminus 3P_j} \right) \right| \chi_{P_j}. \]

Define \( E = \cup_{\tau \subseteq \tau_m} E_\tau \), where

\[ E_\tau = \left\{ x \in Q_0 : M_{I_{\alpha,Q_0}} \left( \prod_{k \in \tau_m \setminus \tau} (b_k - (b_k)_{R_{Q_0}}) f \right)(x) \right\}. \]

Note that

\[ M_{I_{\alpha,Q_0}} g \leq M_{I_{\alpha}}(g \chi_{3Q_0}). \]

Then for each \( \tau \), by Lemma 13, we have

\[ |E_\tau| \leq \left( \frac{c_{n,R} \int_{3Q_0} \left| \prod_{k \in \tau_m \setminus \tau} (b_k - (b_k)_{R_{Q_0}}) f \right|}{C|3Q_0|^{\alpha/n} \left( \prod_{k \in \tau_m \setminus \tau} |b_k - (b_k)_{R_{Q_0}}||f| \right)_{3Q_0}} \right)^{\frac{n}{n-\alpha}} \]

\[ = 3^n \left( \frac{c_{n,R} C}{3^n} \right)^{\frac{n}{n-\alpha}} |Q_0|. \]

Choose \( C \) big enough such that \( |E| \leq \sum_{\tau \subseteq \tau_m} |E_\tau| \leq |Q_0|/2^{n+2} \).

Applying the Calderón-Zygmund to \( \chi_E \) on \( Q_0 \) at height \( h = 1/2^{n+1} \), we get mutually disjoint cubes \( P_j \in \mathcal{D}(Q_0) \) such that for each \( j \),

\[ 2^{-n-1}|P_j| \leq |P_j \cap E| \leq 2^{-1}|P_j|, \quad |E \cup P_j| = 0, \]

which further implies that

\[ \sum_j |P_j| < |Q_0|/2, \quad P_j \cap E^c \neq \emptyset. \]
We return to prove (3.3). For \( x \in Q_0 \setminus \bigcup_j P_j \), then \( x \notin E_\tau \) yields that
\[
\sum_{\tau \leq \tau_m} \prod_{i \in \tau} \left| b_i - (b_i)_{RQ_0} \right| I_\alpha \left( \prod_{k \in \tau_m \setminus \tau} \left| b_k - (b_k)_{RQ_0} \right| f \right) \chi_{Q_0 \setminus \bigcup_j P_j} \\
\leq \sum_{\tau \leq \tau_m} \prod_{i \in \tau} \left| b_i - (b_i)_{RQ_0} \right| M_{I_\alpha, Q_0} \left( \prod_{k \in \tau_m \setminus \tau} \left| b_k - (b_k)_{RQ_0} \right| f \right) \\
\leq C \sum_{\tau \leq \tau_m} \prod_{i \in \tau} \left| b_i - (b_i)_{RQ_0} \right| \left| \frac{3Q_0}{|a/n|} \right| \left( \prod_{k \in \tau_m \setminus \tau} \left| b_k - (b_k)_{RQ_0} \right| |f| \right)_{3Q_0}.
\]
On the other hand, fix some \( j \), using \( P_j \cap E^c \neq \emptyset \), we deduce that
\[
M_{I_\alpha, Q_0} \left( \prod_{k \in \tau_m \setminus \tau} \left| b_k - (b_k)_{RQ_0} \right| f \right) \leq C \left| \frac{3Q_0}{|a/n|} \right| \left( \prod_{k \in \tau_m \setminus \tau} \left| b_k - (b_k)_{RQ_0} \right| |f| \right)_{3Q_0}
\]
It follows that
\[
\sum \sum_{\tau \leq \tau_m} \prod_{i \in \tau} \left| b_i - (b_i)_{RQ_0} \right| I_\alpha \left( \prod_{k \in \tau_m \setminus \tau} \left| b_k - (b_k)_{RQ_0} \right| f \chi_{3Q_0 \setminus 3P_j} \right) \chi_{P_j} \\
\leq C \sum_{\tau \leq \tau_m} \prod_{i \in \tau} \left| b_i - (b_i)_{RQ_0} \right| \left| \frac{3Q_0}{|a/n|} \right| \left( \prod_{k \in \tau_m \setminus \tau} \left| b_k - (b_k)_{RQ_0} \right| |f| \right)_{3Q_0}.
\]
Hence, (3.4), (3.5) and (3.6) show (3.3). This completes the proof. \( \square \)

To prove Theorems 1 and 2 it suffices to bound \( T^{\alpha, \tau}_{S, b} \) for an arbitrary sparse family \( S \). We have the following norm inequality for a general sparse operator.

**Theorem 15.** Let \( 1 < p \leq q < \infty \), \( b(x) = (b_1(x), \ldots, b_m(x)) \) with \( b_i \in L_{1, \infty}^1(\mathbb{R}^n) \). Let \( 0 \leq \alpha < n \), \( \tau \subseteq \{1, \ldots, m\} \), and \( S \) be a sparse family of dyadic cubes. Assume that \( A \) and \( B \) are Young functions that satisfy \( \hat{A} = B_{q', s'} \), \( B = B_{p, s} \) for some \( p \leq s \leq q \). If \( u \) and \( v \) are a pair of weights such that
\[
K_{\alpha, \tau} = \sup_{Q \in S} \frac{|Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}}}{\prod_{i \in \tau} (b_i(x) - (b_i)_Q) u_i^\frac{1}{q}} \left\| \prod_{l \in \tau^c} (b_l(y) - (b_l)_Q) v_l^{-\frac{1}{p}} \right\|_{B, Q},
\]
then \( \| T^{\alpha, \tau}_{S, b} f \|_{L^p(u)} \lesssim K_{\alpha, \tau} \| f \|_{L^p(v)} \).

**Proof.** By duality it suffices to show that
\[
\int_{\mathbb{R}^n} T^{\alpha, \tau}_{S, b} f(x) g(x) u(x)^{\frac{1}{q}} dx
\]
is bounded by \( \| f \|_{L^p(v)} \), given \( \| g \|_{L^{q'}} = 1 \). Let \( p \leq s \leq q \). Using the generalized Hölder inequality for Young functions and letting \( \frac{\beta}{\alpha} = \frac{1}{q} - \frac{1}{s} = \frac{1}{q} - \frac{1}{q'} \) and \( \frac{\gamma}{\alpha} = \frac{1}{p} - \frac{1}{s} \) we have
\[
\left| \int_{\mathbb{R}^n} T^{\alpha, \tau}_{S, b} f(x) g(x) u(x)^{\frac{1}{q}} dx \right| \leq \sum_{Q \in S} |Q|^\frac{\alpha}{n} + 1 \left\| \prod_{i \in \tau} (b_i - (b_i)_Q) u_i^\frac{1}{q} \right\|_{A, Q} \| g \|_{A, Q}
\]
\[ \prod_{l \in \tau} (b_l - (b_l)_Q) v^{-\frac{1}{p}} \|f v^{\frac{1}{p}}\|_{B,Q} \]
\[ \leq K_{\alpha,\tau} \sum_{Q \in S} |Q|^{\frac{1}{p} - \frac{1}{2} + \frac{1}{q}} g_{A,Q} \|f v^{\frac{1}{p}}\|_{B,Q} \]
\[ = K_{\alpha,\tau} \sum_{Q \in S} |Q|^{\frac{1}{p} - \frac{1}{2} + \frac{1}{q}} g_{A,Q} \|f v^{\frac{1}{p}}\|_{B,Q} |Q| \]
\[ \lesssim K_{\alpha,\tau} \sum_{Q \in S} |Q|^{\frac{1}{p}} g_{A,Q} |Q|^{\frac{1}{p} - \frac{1}{2} + \frac{1}{q}} \|g v^{\frac{1}{p}}\|_{B,Q} \]
\[ \leq K_{\alpha,\tau} \sum_{Q \in S} |Q|^{\frac{1}{p} + \frac{1}{q}} g_{A,Q} |Q|^{\frac{1}{p} - \frac{1}{2} + \frac{1}{q}} \|g v^{\frac{1}{p}}\|_{B,Q} \]
\[ \leq K_{\alpha,\tau} \int_{\mathbb{R}^n} M_{\beta,\tilde{A}}(g)(x) M_{\gamma,\tilde{B}}(f v^{\frac{1}{p}})(x) dx \]
\[ \leq K_{\alpha,\tau} \|M_{\beta,\tilde{A}}(g)\|_{L^{p'}} \|M_{\gamma,\tilde{B}}(f v^{\frac{1}{p}})\|_{L^s} \]
\[ \leq K_{\alpha,\tau} \|M_{\beta,\tilde{A}}\|_{L^{p'} \rightarrow L^s} \|M_{\gamma,\tilde{B}}\|_{L^p \rightarrow L^s} \|f\|_{L^p(v)} \]
\]

Using Theorem 15, and Lemmas 11 and 14, we have the desired norm inequality for \( I_{a,b} \), and with the restrictions \( \alpha = 0 \) and \( p = q \), we also attain our norm inequality for \( T_b \).

Finally, we end with the proof of Theorem 4 and note that the proof of Theorem 5 is similar.

**Proof of Theorem 4.** We show that with certain \( \tilde{A} \in B_{p'} \) and \( \tilde{B} \in B_p \) and \( b \in BMO^m \) then the conditions of Theorem 1 are satisfied. Let \( \tau \subseteq \{1, \ldots, m\} \) with \( |\tau| = j \) and let \( U(t) = t^p \log(e + t)^{(m + \frac{1}{p})p + \delta} \) for some \( \delta > 0 \) and \( \phi(t) = e^t - 1 \). We choose the following function \( A \) so that \( \tilde{A} \in B_{p'} \):

\[ \tilde{A}(t) = \frac{t^{p'}}{(\log(e + t)(1+\epsilon))} \]

for some \( \epsilon > 0 \) to be determined. Then we have

\[ A^{-1}(t) \approx \frac{t^{\frac{1}{p}}}{\log(e + t)^{1+\epsilon}} \]
\[ U^{-1}(t) \approx \frac{t^{\frac{1}{p}}}{\log(e + t)^{m+ \frac{1}{p} + \frac{1}{p} + \delta}} \]
\[ \phi^{-1}(t) \approx \log(e + t) \]

We want to show \( (\phi^{-1}(t))^j U^{-1}(t) \lesssim A^{-1}(t) \). So

\[ (\phi^{-1}(t))^j U^{-1}(t) \approx \frac{t^{\frac{1}{p}}}{\log(e + t)^{\frac{1}{p} + m + j + \frac{1}{p}}} \]
and have our desired result:

\begin{align*}
\left\| \left( \prod_{i \in \tau} b_i - (b_i)_{Q} \right) \right\|_{A,Q} & \leq \left( \prod_{i \in \tau} \|b_i\|_{BMO} \right) \left\| u^\frac{1}{p} \right\|_{L^p(\log L)^{(m+\frac{1}{p})p+\delta}} \\quad \text{and} \\
\left\| \left( \prod_{i \in \tau^c} (b_i - (b_i)_{Q}) \right) \right\|_{B,Q} & \leq \left( \prod_{i \in \tau^c} \|b_i\|_{BMO} \right) \left\| v^{-\frac{1}{p}} \right\|_{L^{p'}((\log L)^{(m+\frac{1}{p})p+\delta}}} \\quad \text{with this we may use Theorem 1 and have our desired result:}
\end{align*}

\[ \| T_b f \|_{L^p(v)} \]

\[ \leq \sum_{\tau \subseteq \{1, \ldots, m\}} \sup_Q \left\| \left( \prod_{i \in \tau} b_i - (b_i)_{Q} \right) \right\|_{A,Q} \left\| \left( \prod_{i \in \tau^c} (b_i - (b_i)_{Q}) \right) \right\|_{B,Q} \\quad \text{by letting } \epsilon \text{ be small enough so that } \frac{\epsilon}{p} \leq \frac{\delta}{p}. \]

Choosing \( \epsilon > 0 \) such that \( \frac{\epsilon}{p} \leq \frac{\delta}{p} \) satisfies the conditions of Theorem 9. By a nearly identical argument with \( V(t) = t^{p'} \log(e + t)^{(m+\frac{1}{p})p+\delta} \), we see that

\[ (\phi^{-1}(t))^{m-j}V^{-1}(t) \lesssim B^{-1}(t), \]

by letting \( \epsilon \) be small enough so that \( \frac{\epsilon}{p} \leq \frac{\delta}{p} \). With the exponential integrability of \( BMO \) functions we have

\[ \left\| \left( \prod_{i \in \tau} (b_i - (b_i)_{Q}) \right) u^\frac{1}{p} \right\|_{A,Q} \]

\[ \leq \left( \prod_{i \in \tau} \|b_i\|_{BMO} \right) \left\| u^\frac{1}{p} \right\|_{L^p((\log L)^{(m+\frac{1}{p})p+\delta}} \]

and

\[ \left\| \left( \prod_{i \in \tau^c} (b_i - (b_i)_{Q}) \right) v^{-\frac{1}{p}} \right\|_{B,Q} \leq \left( \prod_{i \in \tau^c} \|b_i\|_{BMO} \right) \left\| v^{-\frac{1}{p}} \right\|_{L^{p'}((\log L)^{(m+\frac{1}{p})p+\delta}}} \]

\[ \leq \left( \prod_{i=1}^{m} \|b_i\|_{BMO} \right) \left\| f \right\|_{L^p(v)}. \]

4. Proof of Theorems 7 and 8

In this section we prove our compactness results, Theorems 7 and 8.

Proof of Theorem 7. Consider the unit ball in \( L^p(v) \),

\[ B_{L^p(v)} := \{ f \in L^p(v) : \left\| f \right\|_{L^p(v)} \leq 1 \}. \]

We need to show that the set

\[ \mathcal{F} = T^m_b(B_{L^p(v)}) \]

satisfies (a)-(c) of Lemma 10. We will make some reductions to prove it. From the definition of \( CMO(\mathbb{R}^n) \), for any \( \epsilon > 0 \), there is a function \( b_\epsilon \in C^\infty_c(\mathbb{R}^n) \) such that

\[ \| b - b_\epsilon \|_{BMO(\mathbb{R}^n)} < \epsilon. \]

Observe that

\[ (b(x) - b(y))^m - (b_\epsilon(x) - b_\epsilon(y))^m \]

\[ \leq \frac{t^{\frac{1}{p}}}{\log(e + t)^{\frac{1}{p}}}. \]
\[= (b - b_\varepsilon)(x) - (b - b_\varepsilon)(y) \sum_{i+j=m-1} (b(x) - b(y))^i(b_\varepsilon(x) - b_\varepsilon(y))^j.\]

This, together with Theorems 4 and 5, allows us to deduce that
\[
\left\| T_b^m f - T_{b_\varepsilon}^m f \right\|_{L^p(\mu)} \\
\leq \sum_{i=0}^{m-1} \left\| T_{b_i}^m f \right\|_{L^p(\mu)} \\
\lesssim \sum_{i=0}^{m-1} \| b - b_\varepsilon \|_{BMO} \| b_\varepsilon \|_{BMO} \| b_\varepsilon \|_{BMO}^{m-1-i} \lesssim \| b \|_{BMO}^{m-1},
\]
where
\[
b_0 := (b - b_\varepsilon, b, \ldots, b), \quad b_{m-1} := (b - b_\varepsilon, b, \ldots, b)
\]
and
\[
b_i := (b - b_\varepsilon, b, \ldots, b, b_\varepsilon, \ldots, b_\varepsilon), \quad i \in \tau_{m-2}.
\]

Given that the space of compact operators \(K(L^p(v), L^p(u))\) is a closed subset of the space of bounded operators \(B(L^p(v), L^p(u))\), then \(T_b^m f \to T_{b_\varepsilon}^m f\) in the operator topology as \(b_\varepsilon \to b\) in \(BMO\). Hence, it suffices to show
\[
\mathcal{F} = T_b^m (B_{L^p(v)}), \quad b \in C_c^\infty(\mathbb{R}^n)
\]
satisfies (a)-(c) of Lemma 10.

It will also be useful to employ a smooth truncation of our operators, an idea that goes back to Krantz and Li [10] (see also [2]). Given \(\eta > 0\), we let \(K_\eta(x, y)\) be a smooth truncation of the \(K(x, y)\) such that

(a) \(K_\eta(x, y) = 0\) if \(|x - y| \leq \eta\);

(b) \(K_\eta(x, y) = K(x, y)\) if \(|x - y| > 2\eta\);

(c) \(K_\eta\) satisfies the same size and regularity estimates as \(K\), namely

\[
|K_\eta(x, y)| \leq \frac{C}{|x - y|^n} \quad \text{and} \quad |\nabla K_\eta(x, y)| \leq \frac{C}{|x - y|^{n+1}}.
\]

Let \(T_\eta\) be the operator associated with our truncated kernel \(K_\eta\). Note that
\[
|T_\eta f(x)| \leq C(M f(x) + T^\varepsilon f(x))
\]
Thus \(T_\eta\) is a bounded operator from \(L^p(v)\) to \(L^p(u)\). Let \(T_b^{m-\eta}\) be the iterated commutators associated with our truncated operator, we have
\[
|T_b^{m-\eta} f(x) - T_b^m f(x)| \lesssim \int_{|x-y| \leq 2\eta} |b(x) - b(y)|^m |f(y)| |x-y|^n \, dy \\
\lesssim \|\nabla b\|_{L^\infty} \sum_{k=0}^\infty (\eta^{-k})^{m-n} \int_{\eta^{2-k} < |x-y| \leq \eta^{2-k+1}} |f(y)| \, dy
\]
\[
\leq \eta^m \sum_{k=0}^{\infty} 2^{-km} (\eta 2^{-k})^{-n} \int_{|x-y| \leq \eta 2^{-k+1}} |f(y)| dy
\]
\[
\lesssim \eta^m M f(x).
\]

With this we have
\[
\|T^{m,\eta}_b f - T^{m}_b f\|_{L^p(u)} \lesssim \eta^m \|M f\|_{L^p(u)} \lesssim \eta^m,
\]
which gives us that \(T^{m,\eta}_b \to T^{m}_b\) as \(\eta \to 0\) in the operator topology. Therefore we must show that \(F = T^{m,\eta}_b(B_{L^p(v)})\) satisfies (a)-(c) of Lemma 10.

By (4.2) it is easy to show that \(L^{m,\eta}_b\) is bounded from \(L^p(v)\) to \(L^p(u)\) and
\[
\|T^{m,\eta}_b\|_{L^p(u)} \leq C \|f\|_{L^p(v)} \leq C,
\]
thus \(F\) satisfies (a) of Lemma 10. To satisfy (b) from Lemma 10 let
\[
A > \max\{1, 2 \sup \{|y| : y \in \text{supp } b\}\}
\]
and let \(Q\) be a cube containing \(\text{supp } b\). Then for \(|x| > A\) and \(y \in \text{supp } b\) we have \(|x-y| > |x|/2\) and
\[
|T^{m,\eta}_b f(x)| \leq \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|^m}{|x-y|^n} |f(y)| dy
\]
\[
\leq \frac{C \|b\|_{L^\infty}^m}{|x|^n} \int_{\text{supp } b} |f(y)| dy
\]
\[
\leq \frac{C \|b\|_{L^\infty} \|f\|_{L^p(v)} \left( \int_Q v^{-\frac{1}{p'}} \right)^\frac{1}{p'}}{|x|^n}
\]
\[
\leq \frac{C_{b,v}}{|x|^n},
\]
where \(C_{b,v}\) depends on \(b\) and \(v\), but not \(f\). To see this, note that
\[
\left( \int_Q v^{-\frac{1}{p'}} \right)^\frac{1}{p'} \leq C |Q|^{\frac{1}{p'}} \|v^{-\frac{1}{p'}}\|_{L^{p'}(\log L)^{2p'-1+\delta}} < \infty.
\]

Thus
\[
(4.3) \quad \int_{|x| > A} |T^{m,\eta}_b f(x)|^p u(x) dx \leq C_{b,v}^p \int_{|x| > A} \frac{u(x)}{|x|^{np}} dx.
\]

We must show that the right-hand side of (4.3) is finite. Noting that \(M : L^p(v) \to L^p(u)\) and
\[
M(\chi_{[-1,1]^n})(x) \geq \frac{C}{(|x|+1)^n},
\]
then
\[
\int_{\mathbb{R}^n} \frac{u(x)}{(|x|+1)^{np}} dx \leq C \int_{\mathbb{R}^n} M(\chi_{[-1,1]^n})(x)^p u(x) dx < \infty.
\]

Thus we have
\[
\lim_{|A| \to \infty} \left( \int_{|x| > A} |T^{m,\eta}_b f(x)|^p u(x) dx \right)^{1/p} = 0
\]
and so $\mathcal{F}$ satisfies (b) of Lemma 10.

Finally, we check (c) of Lemma 10 holds. Take $h \in \mathbb{R}^n$ such that $|h| \leq \eta/2$. It is direct that

$$
T_b^{m,\eta}f(x + h) - T_b^{m,\eta}f(x)
= \int_{\mathbb{R}^n} (b(x + h) - b(y))^m(K^\eta(x + h, y) - K^\eta(x, y))f(y)dy
+ \int_{\mathbb{R}^n} [(b(x + h) - b(y))^m - (b(x) - b(y))^m]K^\eta(x, y)f(y)dy
:= Af(x, h) + Bf(x, h).
$$

The kernels $K^\eta(x + h, y)$ and $K^\eta(x, y)$ both vanish when $|x - y| < \eta/2$ and by (4.1) we have

$$
Af(x, h) \lesssim \|b\|_{L^\infty}^m \int_{|x-y| \geq \eta/2} \frac{|h|}{|x-y|^{n+1}}|f(y)|dy
\leq \|b\|_{L^\infty}^m \sum_{k=0}^{\infty} \int_{2^{k-1}\eta \leq |x-y| < 2^k \eta} \frac{|h|}{|x-y|^{n+1}}|f(y)|dy
\lesssim \frac{|h|}{\eta} \|b\|_{L^\infty}^m \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{1}{(\eta 2^{k-1})^{n-\alpha}} \int_{|x-y| \leq \eta 2^k} |f(y)|dy
\lesssim \frac{|h|}{\eta} \|b\|_{L^\infty}^m Mf(x).
$$

This gives us

$$
(4.5) \quad \sup_{f \in B_{L^p(\nu)}} \|Af\|_{L^p(\nu)} \leq C_{b,\delta}|h|\|Mf\|_{L^p(\nu)} \leq C_{b,\delta}|h|,
$$

which will go to zero uniformly as $h \to 0$.

For $Bf$ we use the difference of powers formula to write:

$$
(b(x + h) - b(y))^m - (b(x) - b(y))^m
= (b(x + h) - b(x) + b(x) - b(y))^m - (b(x) - b(y))^m
= \sum_{k=1}^{m} \binom{m}{k} (b(x + h) - b(x))^k(b(x) - b(y))^{m-k}
= \sum_{k=1}^{m} \binom{m}{k} (b(x + h) - b(x))^k \sum_{j=0}^{m-k} \binom{m-k}{j} b(x)^j b(y)^{m-k-j}.
$$

We now break of the regions of integration for $B$ as follows

$$
Bf = \int_{|x-y| > 2\eta} + \int_{\eta \leq |x-y| < 2\eta} =: B_1f + B_2f
$$
We have the following estimates for $B_1 f$:

$$|B_1 f(x, h)| \leq \sum_{k=1}^{m} \binom{m}{k} |b(x + h) - b(x)|^k \sum_{j=0}^{m-k} \binom{m-k}{j} |b(x)|^j$$

$$\times |\int_{|x-y|>2\eta} K(x, y) b(y)^{m-k-j} f(y) \, dy|$$

$$\leq |h| \|\nabla b\|_{L^\infty} \sum_{k=1}^{m} \binom{m}{k} |b(x + h) - b(x)|^{k-1}$$

$$\times \sum_{j=0}^{m-k} \binom{m-k}{j} |b(x)|^j T^*_q(b^{m-k-j} f)(x)$$

$$\leq C_b |h| \sum_{k=1}^{m} \sum_{j=0}^{m-k} T^*_q(b^{m-k-j} f)(x).$$

Hence we have

$$\|B_1 f\|_{L^p(u)} \leq C_b |h| \sum_{k=1}^{m} \sum_{j=0}^{m-k} \|T^*_q(b^{m-k-j} f)\|_{L^p(u)}$$

$$\leq C_b |h| \sum_{k=1}^{m} \sum_{j=0}^{m-k} \|b^{m-k-j} f\|_{L^p(v)} \leq C_b |h| \|f\|_{L^p(v)}.$$

For $B_2 f$ we have

$$|B_2 f(x, h)| \lesssim |h| \|\nabla b\|_{L^\infty} \|b\|_{L^\infty}^{m-1} \int_{\eta<|x-y| \leq 2\eta} \frac{|f(y)|}{|x-y|^n} \, dy$$

$$\lesssim C_b |h| \frac{1}{(2\eta)^n} \int_{|x-y| \leq 2\eta} |f(y)| \, dy$$

$$\leq C_b |h| Mf(x)$$

which gives us

$$\|B_2 f\|_{L^q(u)} \leq C_b |h| \|f\|_{L^p(v)}.$$

This completes the equicontinuity argument and so by Lemma 10 proves compactness. \qed

**Proof of Theorem 8.** The proof for $(I_\alpha)^m_b$ is similar to that of Theorem 7 and we only sketch the details. However, there are some simplifications that we point out. Again, it suffices to show that

$$\mathcal{F} = (I_\alpha)^m_B(L^p(v)) \subseteq L^q(u).$$
satisfies (a)-(c) of Lemma 10. By the same reductions we may assume \( b \in C_c^\infty(\mathbb{R}^n) \) and the truncated operator for \( \eta > 0 \)

\[
I_\alpha^n f(x) = \int_{|x-y|>\eta} K_\alpha^n(x, y) f(y) \, dy
\]

where

\[
K_\alpha^n(x, y) = \begin{cases} 
|x-y|^{\alpha-n}, & |x-y| \geq 2\eta \\
0, & |x-y| \leq \eta
\end{cases}
\]

and

\[
|K_\alpha^n(x, y)| \leq \frac{1}{|x-y|^{n-\alpha}} \quad \text{and} \quad |\nabla K_\alpha^n(x, y)| \leq \frac{C}{|x-y|^{n-\alpha+1}}.
\]

Notice that

\[
|I_\alpha^n f(x)| \leq I_\alpha(|f|)(x)
\]

and thus is bounded \( I_\alpha^n : L^p(v) \to L^q(u) \) when the weights satisfy condition (1.9). Moreover consider the truncated commutator

\[
(I_\alpha^n)_b^m f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^m K_\alpha^n(x, y) f(y) \, dy.
\]

Then we have

\[
|(I_\alpha^n)_b^m f(x) - (I_\alpha)_b^m f(x)| \lesssim \eta^m M_\alpha f(x),
\]

and this implies that \( (I_\alpha^n)_b^m \) is a bounded operator from \( L^p(v) \) to \( L^q(u) \) and

\[
\| (I_\alpha^n)_b^m f - (I_\alpha)_b^m f \|_{L^q(u)} \lesssim \eta^m \| M_\alpha f \|_{L^p(v)} \lesssim \eta^m
\]

which gives us that \( (I_\alpha^n)_b^m \to (I_\alpha)_b^m \) as \( \eta \to 0 \) in the operator topology. Therefore we must show that \( \mathcal{F} = (I_\alpha^n)_b^m (B_{L^p(v)}) \) satisfies (a)-(c) of Lemma 10. Since \( (I_\alpha^n)_b^m \) is a bounded operator we have that \( \mathcal{F} \) satisfies (a) of Lemma 10. The proof of (b) from Lemma 10 is completely analogous: for \( A \) and let \( Q \) be a cube containing \( \text{supp} \, b \) we have

\[
|(I_\alpha^n)_b^m f(x)| \leq \frac{C_{b,v}}{|x|^{n-\alpha}}
\]

and

\[
\int_{|x|>A} |(I_\alpha^n)_b^m f(x)|^q u(x) \, dx \leq C_{b,v}^q \int_{|x|>A} \frac{u(x)}{|x|^{(n-\alpha)q}} \, dx.
\]

Noting that \( M_\alpha : L^p(v) \to L^q(u) \) and

\[
M_\alpha(\chi_{[-1,1]^n})(x) \geq \frac{C}{(|x| + 1)^{n-\alpha}},
\]

yields

\[
\lim_{A \to \infty} \left( \int_{|x|>A} |(I_\alpha^n)_b^m f(x)|^q u(x) \, dx \right)^{1/q} = 0
\]

and so \( \mathcal{F} \) satisfies (b) of Lemma 10.

To check (c) of Lemma 10 we take a similar approach. For \( h \in \mathbb{R}^n \) such that \( |h| \leq \frac{\eta}{2} \), we have

\[
(I_\alpha^n)_b^m f(x+h) - (I_\alpha^n)_b^m f(x)
\]
\[
= \int_{\mathbb{R}^n} (b(x + h) - b(y))^m (K_\alpha^n(x + h, y) - K_\alpha^n(x, y)) f(y) dy
\]
\[
+ \int_{\mathbb{R}^n} [(b(x + h) - b(y))^m - (b(x) - b(y))^m] K_\alpha^n(x, y) f(y) dy
\]
\[=: Af(x, h) + B f(x, h).
\]
One can see that \(K_\alpha^n(x + h, y)\) and \(K_\alpha^n(x, y)\) both vanish when \(|x - y| < \frac{\eta}{2}\) and by the smoothness condition on \(K_\alpha^n\) we have
\[
Af(x, h) \lesssim \|b\|_{L^\infty}^m \int_{|x-y| \geq \frac{\eta}{2}} \frac{|h|}{|x-y|^{n-\alpha+1}} |f(y)| dy
\]
\[
\leq \|b\|_{L^\infty}^m \sum_{k=0}^{\infty} \int_{2^{k-1}\eta \leq |x-y| < 2^k\eta} \frac{|h|}{|x-y|^{n-\alpha+1}} |f(y)| dy
\]
\[
\lesssim \frac{|h|}{\eta} \|b\|_{L^\infty}^m \sum_{k=0}^{\infty} \frac{1}{2^k (\eta 2^{k-1})^{n-\alpha}} \int_{|x-y| \leq 2^k\eta} |f(y)| dy
\]
\[
\lesssim \frac{|h|}{\eta} \|b\|_{L^\infty}^m M_\alpha f(x).
\]
This gives us
\[
\sup_{f \in B_{L^p(\nu)}} \|Af\|_{L^q(u)} \leq C |h| \|M_\alpha f\|_{L^p(u)} \leq C |h|,
\]
which will go to zero uniformly as \(h \to 0\).

For \(B f\) we use a slightly different power formula to write:
\[
(b(x + h) - b(y))^m - (b(x) - b(y))^m
\]
\[= (b(x + h) - b(x)) \left( \sum_{k=0}^{m-1} (b(x) - b(y))^k (b(x) - b(y))^{m-1-k} \right).
\]
We now break of the regions of integration for \(B\) as follows
\[
B f = \int_{|x-y| > 2\eta} + \int_{|x-y| \leq 2\eta} := B_1 f + B_2 f
\]
We have the following estimates for \(B_1 f\):
\[
|B_1 f(x, h)| \lesssim |h| \|\nabla b\|_{L^\infty} \|b\|_{L^\infty}^{m-1} \int_{|x-y| > 2\eta} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \leq C_b |h| I_\alpha f(x),
\]
which gives
\[
\|B_1 f\|_{L^q(u)} \lesssim \|f\|_{L^p(\nu)}.
\]
For \(B_2 f\) we have
\[
|B_2 f(x, h)| \lesssim |h| \|\nabla b\|_{L^\infty} \|b\|_{L^\infty}^{m-1} \int_{|x-y| \leq 2\eta} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy
\]
\[
\lesssim C_b |h| \frac{1}{(2\eta)^{n-\alpha}} \int_{|x-y| \leq 2\eta} |f(y)| dy
\]
\[ \leq C_0|h|\mu f(x) \]

which yields,

\[ \|B_2f\|_{L^q(u)} \leq C|h|\|f\|_{L^p(v)}. \]

This completes the equicontinuity argument and so by Lemma 10 proves compactness. \qed

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