AN EXTENSION OF PROPERTIES OF SYMMETRIC GROUP TO MONOIDS AND A PRETORSION THEORY IN THE CATEGORY OF MAPPINGS

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Abstract. Several elementary properties of the symmetric group $S_n$ extend in a nice way to the full transformation monoid $M_n$ of all maps of the set $X := \{1, 2, 3, \ldots, n\}$ into itself. The group $S_n$ turns out to be in some sense the torsion part of the monoid $M_n$. More precisely, there is a pretorsion theory in the category of all maps $f: X \to X$, $X$ an arbitrary finite non-empty set, in which bijections are exactly the torsion objects.

1. Introduction

In all this paper, $n \geq 1$ denotes a fixed integer and $X$ is the set $\{1, 2, 3, \ldots, n\}$. We teach every year to our first year students that:

(1) Every permutation can be written as a product of disjoint cycles, in a unique way up to the order of the factors. (We prove this associating a graph to every permutation, so that the decomposition as a product of pairwise disjoint cycles of the permutation follows from the partition of the graph into its connected components).
(2) Disjoint cycles permute.
(3) Every permutation can be written as a product of transpositions.
(4) Let $S_n$ be the symmetric group, i.e., the group of all permutations of $X$. There is a group morphism $\text{sgn}: S_n \to \{1, -1\}$. For every permutation $f \in S_n$, the number $\text{sgn}(f)$ is called the sign of the permutation $f$.
(5) As a consequence, the group $S_n$ of permutations has a normal subgroup $A_n$, the alternating subgroup, which is a subgroup of index 2 of $S_n$ when $n \geq 2$. It follows that, for $n \geq 2$, $S_n$ is the semidirect product of $A_n$ and any subgroup of $S_n$ generated by a trasposition.

In this paper, we develop the naive idea of extending the five results above from permutations $X \to X$ to arbitrary mappings $X \to X$. That is, we generalize the five results above from the group $S_n$ to the monoid $M_n$ of all mappings $X \to X$. The operation on $M_n$ is the composition of mappings. The group $S_n$ has order $n!$, while the monoid $M_n$ has $n^n$ elements. We find that:

(1) Every mapping $f: X \to X$ can be written as a product of “forests on a cycle”, in a unique way up to the order of the factors. (This corresponds to the partition into connected components of an undirected graph $G^n_f$ associated to the...
mapping \( f \)). The decomposition of \( f \) as a product of disjoint forests on a cycle corresponds exactly to the decomposition of a permutation \( \sigma \) as a product of disjoint cycles.

(2) Any two disjoint forests on a cycle permute.

(3) Any forest on a cycle can be written as a product of moves and transpositions. A move is a mapping \( X \to X \) that fixes all elements of \( X \) except for one. It is an idempotent mapping \( X \to X \).

(4) Let \( M_n \) be the monoid of all mappings \( X \to X \). Then \( M_n \) is the disjoint union of its groups of units \( S_n \) and the completely prime two-sided ideal \( I_n \) of \( M_n \) consisting of all non-injective mappings \( X \to X \). As a consequence, there is a monoid morphism \( \text{sgn} : M_n \to \{0, 1, -1\} \) of the monoid \( M_n \) into the multiplicative monoid \( \{0, 1, -1\} \). The mappings \( f \in M_n \) with \( \text{sgn}(f) = 1 \) are exactly those in the alternating group \( A_n \), the mappings \( f \) with \( \text{sgn}(f) = -1 \) are those in the coset \( S_n \setminus A_n \) of \( S_n \), and the mappings \( f \in M_n \) with \( \text{sgn}(f) = 0 \) are those in the two-sided ideal \( I_n \).

(5) The submonoid \( I_n \) of \( M_n \) is generated by the set of all moves in \( M_n \). The submonoid \( I'_n := I_n \cup \{1_{M_n}\} \) generated by the set of all moves is such that \( M_n = S_n \cup I'_n \) and \( S_n \cap I'_n = \{1_{M_n}\} \). There is a canonical epimorphism of the semidirect product \( I'_n \rtimes S_n \) of \( I'_n \) and \( S_n \) onto \( M_n \).

In the second part of the paper, we consider the category \( \mathcal{M} \) whose objects \((X, f)\) are all finite nonempty sets \( X \) with a mapping \( f : X \to X \). We show that in \( \mathcal{M} \) there is a very nice and interesting pretorsion theory \((\mathcal{C}, \mathcal{F})\) in the sense of [1]. The pretorsion class \( \mathcal{C} \) consists of all objects \((X, f)\) of \( \mathcal{M} \) with \( f \) a bijection, and \( \mathcal{F} \) consists of all objects \((X, f)\) of \( \mathcal{M} \) for which the graph \( G^f_i \) associated to \( f \) is a forest (equivalently, of all objects \((X, f)\) of \( \mathcal{M} \) with \( f^n = f^{n+1} \)).

In this paper, all undirected graphs are simple and don’t have loops, while directed graphs are simple, but can have loops.

### 2. A description of mappings \( X \to X \).

Recall that, in all the paper, \( X := \{1, 2, 3, \ldots, n\} \) for some positive integer \( n \).

**Proposition 2.1.** Let \( f : X \to X \) be a mapping. Then there exist an integer \( m \geq 0 \) and a partition \( \{A_i \mid i = 0, 1, 2, \ldots, m\} \) of \( X \) such that \( f(A_0) = A_0 \) and \( f(A_i) \subseteq A_{i-1} \) for every \( i = 1, 2, \ldots, m \).

**Proof.** Consider the descending chain \( X \supseteq f(X) \supseteq f^2(X) \supseteq \ldots \) of subsets of \( X \). This descending chain is stationary, hence there exists a least index \( t \geq 0 \) with \( f(f^t(X)) = f^t(X) \). Set \( A_0 := f^t(X) \), so \( f(A_0) = A_0 \). Now define an ascending chain of subsets \( B_i \) of \( X \), \( i \geq 0 \), setting \( B_0 := A_0 \) and \( B_i := f^{-1}(B_{i-1}) \) for every \( i > 0 \). Then \( B_i \supseteq B_{i-1} \) for every \( i > 0 \) (Induction on \( i > 0 \)). For \( i = 1 \): \( f(A_0) = A_0 \), so that \( A_0 \subseteq f^{-1}(A_0) \), i.e., \( B_0 \subseteq B_1 \). Suppose the inclusion true for \( i \), i.e., \( B_i \supseteq B_{i-1} \). Then \( B_{i+1} = f^{-1}(B_i) \supseteq f^{-1}(B_{i-1}) = B_i \). Also, \( B_i \supseteq f^{-1}(X) \) for every \( i = 0, 1, 2, \ldots, t \), as can be easily seen by induction on \( i \). Hence \( B_t = X \).

Thus we have an ascending chain \( B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_m = X \) of subsets of \( X \). Let \( m \geq 0 \) be the smallest integer \( \geq 0 \) with \( B_m = X \). Then the chain \( B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_m = X \) is strictly ascending. Set \( A_i := B_i \setminus B_{i-1} \) for every \( i = 1, 2, \ldots, m \), so that \( \{A_i \mid i = 0, 1, 2, \ldots, m\} \) is a partition of \( X \).

It remains to prove that \( f(A_i) \subseteq A_{i-1} \) for every \( i = 1, 2, \ldots, m \). We have that \( B_i = f^{-1}(B_{i-1}) \) for every \( i > 0 \). Thus, if \( x \in A_i \), then \( x \in B_i \) and \( x \notin B_{i-1} \), so \( x \in
\[
f^{-1}(B_{i-1}) \text{ and } (i \geq 2) \ x \notin f^{-1}(B_{i-2}). \text{ Therefore } f(x) \in B_{i-1} \setminus B_{i-2} = A_{i-1}, \text{ as desired.}
\]

For any finite set \( X \), a mapping \( X \to X \) is injective, if and only if it is surjective, if and only if it is bijection. Therefore the role of \( A_0 \) in the previous proposition is completely different from the role of the other blocks \( A_1, \ldots, A_m \) of the partition: the restriction of \( f \) to \( A_0 \) is a permutation of \( A_0 \), while, for \( i > 0 \), \( f \) maps \( A_i \)
into \( A_{i-1} \).

3. The directed graph and the undirected graph associated to a mapping \( X \to X \).

Recall that all undirected graphs in this paper don’t have multiple edges and loops, while directed graphs don’t have multiple edges, but can have loops.

Given any mapping \( f : X \to X \), it is possible to associate to \( f \) a directed graph \( G^d_f = (X, E^d_f) \), called the graph of the function \( f \), having \( X \) as a set of vertices and \( E^d_f := \{ (i, f(i)) \mid i \in X \} \) as a set of arrows. Hence \( G^d_f \) has \( n \) vertices and \( n \) arrows, one arrow from \( i \) to \( f(i) \) for every \( i \in X \). In the directed graph \( G^d_f \) every vertex has outdegree 1. In Graph Theory, a direct graph in which every vertex has outdegree 1 is sometimes called a directed pseudoforest. The corresponding \( m \), defined as in Proposition 2.1, will be called the height of the pseudoforest.

Similarly, it is possible to associate to \( f \) an undirected graph \( G^u_f = (X, E^u_f) \) having \( X \) as a set of vertices and \( E^u_f := \{ \{i, f(i)\} \mid i \in X, \ f(i) \neq i \} \) as a set of (undirected) edges. The graph \( G^u_f \) also has \( n \) vertices, but \( \leq n \) edges. This occurs because of the fixed points, that is, the elements \( i \in X \) with \( f(i) = i \), and because of traspositions, i.e., the pairs \( i, j \) of distinct elements of \( X \) with \( f(i) = j \) and \( f(j) = i \).

We will now describe the connected components of the graph \( G^u_f \). Fix a vertex \( x_0 \in X \). We will determine the connected component of \( x_0 \). The connected component of \( x_0 \) in \( G^u_f \) consists of all vertices \( x \in X \) for which there exists a path from \( x_0 \) to \( x \) in \( G^u_f \). In our graph \( G^u_f \), the edges are all of the form \( \{x, f(x)\} \), provided \( x \neq f(x) \). Thus the vertices at a distance \( \leq 1 \) from \( x_0 \) are exactly those in the set \( \{x_0, f(x_0)\} \cup f^{-1}(x_0) \). Hence the connected component of \( x_0 \) is the closure of \( \{x_0\} \) with respect to taking images and inverse images via \( f \). Starting from the fixed vertex \( x_0 \in X \), we can define a sequence of vertices \( x_0, x_1, x_2, \ldots \) in \( X \) with \( x_{t+1} = f(x_t) \) for every \( t \geq 0 \). Since \( X \) is finite, there exists a smallest index \( t_0 \geq 0 \) with \( x_{t_0} = x_t \) for some index \( t > t_0 \). Let \( t_1 \) be the smallest index \( t > t_0 \) with \( x_{t_0} = x_t \). Then, in the graph \( G^u_f \), there is a simple (=no repetitions of vertices and edges) directed cycle \( x_{t_0}, x_{t_0+1}, \ldots, x_{t_1} \). Now recursively define an ascending chain of subsets \( B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots \) of \( X \) setting \( B_0 := \{x_{t_0}, x_{t_0+1}, \ldots, x_{t_1-1}\} \) (the set of vertices on the cycle, which can also be a cycle of length 1) and \( B_{t+1} := f^{-1}(B_t) \). Since the chain of submodules is ascending and the set \( X \) is finite, the chain is necessarily stationary, so that there exists an index \( m \) with \( B_{m+1} = B_m \). Equivalently, \( B_m = f^{-1}(B_m) \). Moreover, \( f(B_i) \subseteq B_{i-1} \) for every \( i \geq 1 \). There is no edge between any vertex in \( X \setminus B_m \) and any vertex in \( B_m \). Also, any two vertices in \( B_m \) are connected by a path in \( G^u_f \). The vertex \( x_0 \) is in \( B_m \), because \( f^m(x_0) = x_{t_0} \in B_0 \), so that \( x_0 \in (f^m)^{-1}(B_0) \subseteq B_{t_0} \subseteq B_m \). This proves that \( B_m \) is the connected component of \( G^u_f \) containing \( x_0 \).
The full subgraph of $G_f^d = (X, E_f^d)$ whose set of vertices is $B_m$ is a connected directed pseudoforest. We will call a connected directed pseudoforest a *forest on a cycle*. Its typical form is like in Figure 1.

We will say that a function $f: X \to X$ is a *forest on a cycle* if there are a subset $A$ of $X$ and an element $x_0 \in A$ such that: (1) for every $x \in A$ there exists an integer $t \geq 0$ with either $f^t(x_0) = x$ or $f^t(x) = x_0$, and (2) $f(y) = y$ for every $y \in X \setminus A$. Hence a bijection $f: X \to X$ is a forest on a cycle if and only if it is a cycle.

We say that two functions $f, g: X \to X$ are *disjoint* if, for every $x \in X$, either $f(x) = x$ or $g(x) = x$ (or both). Similarly to the fact that any two disjoint cycles commute, any two disjoint functions commute. In particular, any two disjoint forests on a cycle commute.

Since every undirected graph can be decomposed in a unique way into its connected components, in the same way as we see that every permutation can be written as a product of disjoint cycles in a unique way up to the order of the factors, we similarly get that:

**Theorem 3.1.** Every mapping $f: X \to X$ can be written as a product of disjoint forests on a cycle, in a unique way up to the order of the factors.

This corresponds to the decomposition into connected components of the undirected graph $G_f^u$.

### 4. Transpositions and moves.

Every cycle, hence every permutation, is a product of transpositions. Let’s see the analog for any mapping $f: X \to X$. 

\[ B_4 \setminus B_3 \]
\[ B_3 \setminus B_2 \]
\[ B_2 \setminus B_1 \]
\[ B_1 \setminus B_0 \]
\[ B_0 \]

**Figure 1.** A forest on a cycle, that is, a connected directed pseudoforest.
A move is a mapping \( f : X \to X \) such that there exists \( x_0 \in X \) with \( f(x_0) \neq x_0 \) and \( f(x) = x \) for every \( x \in X \), \( x \neq x_0 \). We will denote the move that maps \( x_0 \) to \( x_1 \) and fixes all the elements \( x \in X \), \( x \neq x_0 \), by \( m(x_0, x_1) \).

If \( f : X \to X \) is any mapping, and \( G^n_f \) is its associated pseudoforest, we will call length of \( f \) the number of edges of \( G^n_f \). Hence mappings of length 1 are transpositions and moves.

**Theorem 4.1.** Let \( f : X \to X \) be a mapping, \( n = |X| \) and \( p = |f^n(X)| = |A_0| \). Then \( f \) can be written as a product

\[
f = m_1 m_2 \ldots m_{n−p} t_1 t_2 \ldots t_s,
\]

where:

1. \( m_1, m_2, \ldots, m_{n−p} \) are \( n − p \) moves, one for each element of \( X \setminus A_0 \), which fix all the elements of \( A_0 = f^n(X) \), and
2. \( t_1, t_2, \ldots, t_s \) are \( s \geq 0 \) transpositions of elements of \( A_0 \).

**Proof.** Let \( \{ A_i \mid i = 0, 1, 2, \ldots, m \} \) be the partition of \( X \) as in Proposition 2.1. Then \( A_0 \) is invariant for \( f \), and the restriction \( f^n_{A_0} : A_0 \to A_0 \) is a bijection. Extending this bijection to a bijection \( \sigma : X \to X \) that is the identity on \( X \setminus A_0 \), we get an element \( \sigma \in S_n \). Write \( \sigma \) as a product \( \sigma = t_1 \circ t_2 \circ \cdots \circ t_s \) of \( s \geq 0 \) transpositions of elements of \( A_0 \). For every \( i = 1, 2, \ldots, m \), let \( x_{i1}, x_{i2}, \ldots, x_{in_i} \) be the elements of \( A_i \), where \( n_i \) is the cardinality of \( A_i \). Then the required decomposition of \( f \) is

\[
f = m(x_{11}, f(x_{11})) \circ m(x_{21}, f(x_{21})) \circ \cdots \circ m(x_{mn}, f(x_{mn})) \circ \circ m(x_{m−11}, f(x_{m−11})) \circ \cdots \circ m(x_{m−1n−1}, f(x_{m−1n−1})) \circ \cdots \circ m(x_{1n_1}, f(x_{1n_1})) \circ t_1 \circ t_2 \circ \cdots \circ t_s.
\]

\( \square \)

**Remark 4.2.** (1) In the statement of Theorem 4.1 the two composite mappings \( m := m_1 m_2 \ldots m_{n−p} \) and \( \sigma := t_1 t_2 \ldots t_s \) are completely determined by \( f \). So \( f = m \sigma \), where \( m \) is a product of moves and \( \sigma \in S_n \) is a permutation. Notice that, in general, a decomposition \( f = g \tau \) of a mapping \( f \) as a product \( g \) of moves and a product \( \tau \) of transpositions is not unique. For instance, \( m(1, 2) \circ (1 2) = m(1, 2) \). Hence the additional conditions “the moves, one for each element of \( X \setminus A_0 \), fix all the elements of \( A_0 \), and the transpositions \( t_1, t_2, \ldots, t_s \) act on elements of \( A_0 \)” in the statement of Theorem 4.1 are necessary.

(2) If we have two product decompositions \( f = m \sigma \) and \( f' = m' \sigma' \) like in the statement of Theorem 4.1 then \( f f' = (m \sigma)(m' \sigma') = (m \sigma m') \sigma' \), and \( m \sigma m' \sigma^{-1} \) is a product of moves, because the conjugate \( \sigma m(i, j) \sigma^{-1} \) of any move \( m(i, j) \) is the move \( m(\sigma(i), \sigma(j)) \), as is easily verified. But, in the decomposition \( f f' = (m \sigma m' \sigma^{-1})(\sigma \sigma') \), the additional conditions “the moves, one for each element of \( X \setminus A_0 \), fix all the elements of \( A_0 \), and the transpositions act on elements of \( A_0 \)” in the statement of Theorem 4.1 do not hold.

5. **Idempotent mappings, products of moves, the sign sgn:** \( M_n \to \{0, 1, −1\} \), semidirect products.

Any move is an idempotent mapping, that is, a mapping \( f : X \to X \) such that \( f^2 = f \) (where \( f^2 = f \circ f \)). Let us determine the structure of idempotent mappings. Let \( f : X \to X \) be an idempotent mapping and \( \{ A_i \mid i = 0, 1, 2, \ldots, m \} \) the corresponding partition of \( X \) according to Proposition 2.1. Then \( f(A_i) \subseteq A_{i−1} \).
A mapping \( f : X \to X \) is idempotent if and only if its graph is a forest of height at most 1, if and only if there is a subset \( A_0 \) of \( X \) such that \( f(X \setminus A_0) \subseteq A_0 \) and the restriction of \( f \) to \( A_0 \) is the identity of \( A_0 \).

As a consequence, we have that in the product representation of Theorem 4.1 of an idempotent mapping \( f : X \to X \) as a product \( f = m_1 m_2 \ldots m_{n-2} \) of moves (no transpositions are necessary), the moves \( m_i \) commute pairwise.

Let \( I_n \) be the subset of \( M_n \) consisting of all non-injective mappings \( X \to X \) (equivalently, all non-surjective mappings \( X \to X \)). Clearly, for every \( f, g \in M_n \), \( fg \in I_n \) if and only if either \( f \in I_n \) or \( g \in I_n \), that is, \( I_n \) is a completely prime two-sided ideal of the monoid \( M_n \). By [4], every mapping in \( I_n \) is a product of finitely many idempotent mappings. Since every idempotent mapping is a product of moves (Theorem 4.1), it follows that the set of all moves in \( M_n \) is a set of generators of the subsemigroup \( I_n \) of \( M_n \).

Remark 5.2. The fact that \( I_n = M_n \setminus S_n \) is the subsemigroup \( M_n \) generated by the set of all moves, implies that all elements \( f \in I_n \) are product of moves, and all the other elements \( f \in S_n = M_n \setminus I_n \) are product of transpositions.

Now the monoid \( M_n \) is the disjoint union of its group of units \( S_n = U(M_n) \) and the completely prime two-sided ideal \( I_n \). As a consequence, it is possible to define the sign of any mapping \( f : X \to X \), which will be one of the three integers 1, −1 and 0. There is a monoid morphism \( \text{sgn} : M_n \to \{0, 1, -1\} \) of the monoid \( M_n \) into the multiplicative monoid \( \{0, 1, -1\} \). It is defined, for every \( f \in M_n \), by

\[
\text{sgn}(f) = \begin{cases} 
1 & \text{if } f \in A_n, \\
-1 & \text{if } f \in S_n \setminus A_n, \\
0 & \text{if } f \in I_n.
\end{cases}
\]

Hence, for \( n \geq 2 \), there is an equivalence relation \( \sim \) on the monoid \( M_n \), compatible with the operation of \( M_n \), whose equivalence classes are the three classes \( A_n \), \( S_n \setminus A_n \) and \( I_n \), respectively.

For the symmetric group \( S_n \), the sign \( \text{sgn} : S_n \to \{1, -1\} \) is a surjective group morphism when \( n \geq 2 \), so that \( S_n \) has a normal subgroup \( A_n \), the alternating subgroup, which is a subgroup of index 2 of \( S_n \). Fix any transposition in \( S_n \), for

![Figure 2. An idempotent mapping: a forest of height at most 1.](image)
instance the transposition \((1\ 2)\). Then there is an endomorphism of \(S_n\) that maps all even permutations of \(S_n\) to the identity of \(X\) and maps all odd permutations of \(S_n\) to the transposition \((1\ 2)\). This endomorphism of \(S_n\) is an idempotent endomorphism of \(S_n\). For groups, the existence of such an idempotent endomorphism is sufficient to have a splitting as a semidirect product: if \(G\) is any group and \(\varphi: G \to G\) is any idempotent endomorphism of \(G\), then \(G\) is the semidirect product of the kernel and the image of \(\varphi\). Therefore, for \(n \geq 2\), we have that \(S_n = A_n \rtimes \langle (1\ 2) \rangle\), i.e., \(S_n\) splits as a semidirect product of its normal subgroups \(A_n\) and the subgroup \(\langle (1\ 2) \rangle\) of order 2. The group \(\langle (1\ 2) \rangle\) acts on \(A_n\) via conjugation.

Let us see how this generalizes to the monoid \(M_n\). The completely prime ideal \(I_n\) of \(M_n\) is a subsemigroup without identity of \(M_n\). Let \(I_n'\) be the submonoid of \(M_n\) obtained from \(I_n\) adjoining to it the identity of \(M_n\), i.e., \(I_n' := I_n \cup \{1_{M_n}\}\). Then \(I_n'\) is the submonoid of \(M_n\) generated by the set of all moves, \(M_n = S_n \cup I_n'\) and \(S_n' \cap I_n' = \{1_{M_n}\}\).

There is an action of \(S_n\) on the normal submonoid \(I_n'\) of \(M_n\), i.e., a group homomorphism \(S_n \to \text{Aut}(I_n')\) that maps any \(\tau \in S_n\) to the inner automorphism \(g \in I_n' \mapsto \tau g \tau^{-1}\) of the monoid \(I_n'\). Hence it is possible to construct the semidirect product \(I_n' \rtimes S_n\). There is a canonical surjective monoid homomorphism

\[
\psi: I_n' \rtimes S_n \to M_n, \quad \psi: (g, \tau) \mapsto g \tau,
\]

because multiplication in the semidirect product is such that

\[
(g, \tau)(g', \tau') = (g \tau g'^{-1} \tau', \tau').
\]

If we consider the inverse image via \(\psi\) of the elements of \(M_n\), we find that the inverse image of an element \(f \in M_n\) is \(\varphi^{-1}(f) = \{(g, \tau) \mid g \in I_n', \tau \in S_n, g \tau = f\}\), which has cardinality 1 if \(f \in S_n\) and has cardinality \(n!\) if \(f \in M_n \setminus S_n\).

Notice that there is no idempotent endomorphism \(\varphi\) of \(M_n\) whose image is \(S_n\). This is because a monoid morphism maps idempotents to idempotents. All moves are idempotents of \(M_n\), and the only idempotent in \(S_n\) is the identity. Thus we would have that \(\varphi\) should map all moves of \(M_n\) to the identity of \(S_n\) and fix the elements of \(S_n\). The equality \(m(1, 2) \circ (1, 2) = m(1, 2)\) seen in Section 4 shows that this is not possible.

The monoid \(I_n' \rtimes S_n\) contains as a submonoid \(I_n' \rtimes A_n\), on which the group \(\langle (1\ 2) \rangle\) acts via conjugation. Hence it is possible to construct the semidirect product \((I_n' \rtimes A_n) \rtimes \langle (1\ 2) \rangle\), and there is a canonical epimorphism \(p: (I_n' \rtimes A_n) \rtimes \langle (1\ 2) \rangle \to M_n\). The composite morphism \(\text{sgn} \circ p: (I_n' \rtimes A_n) \times \langle (1\ 2) \rangle \to \{1, -1, 0\}\) is the product \((\iota \times \alpha) \times \tau\) of the three morphisms:

1. \(\iota: I_n' \to \{1, -1, 0\}\) that maps all elements of \(I_n\) to 0 and the identity of \(I_n'\) to 1.
2. \(\alpha: A_n \to \{1, -1, 0\}\) that maps all elements of \(A_n\) to 1. And
3. the injective monoid morphism \(\tau: \langle (1\ 2) \rangle \to \{1, -1, 0\}\) that maps the transposition \((1\ 2)\) to \(-1\) and the identity of \(\langle (1\ 2) \rangle\) to 1:

\[
(I_n' \rtimes A_n) \times \langle (1\ 2) \rangle \xrightarrow{p} M_n \xrightarrow{\text{sgn}} \{1, -1, 0\}.
\]
6. Forests, cycles, and trivial mappings.

Proposition 6.1. The following conditions are equivalent for a mapping $f: X \to X$ with $|X| = n$:

(a) The graph of $f$ is a forest.

(b) $f^n = f^{n+1}$.

Proof. The graph of $f$ is a forest if and only if every connected component of the graph of $f$ is a tree, if and only if, for every $x_0 \in X$, the connected component of $x_0$ has no cycles. In the notation of Section 3, this means that $t_1 = t_0 + 1$ for every vertex $x_0$ in $X$. Hence, equivalently, the graph of $f$ is a forest if and only for every $x \in X$ there is a $t_x \geq 0$ such that $f^{t_x}(x) = f^{t_x+1}(x)$. Without loss of generality, we can suppose $t_x \leq n$ because $|X| = n$. Thus the graph of $f$ is a forest if and only $f^n(x) = f^{n+1}(x)$ for every $x \in X$, that is, $f^n = f^{n+1}$. □

Proposition 6.2. The following conditions are equivalent for a mapping $f: X \to X$ with $|X| = n$:

(a) $f$ is a bijection.

(b) $f^n$ is the identity $\iota_X : X \to X$.

Proof. If $f$ is a bijection, it is in the group of units $S_n$ of the monoid $M_n$, and $S_n$ has order $n!$, so that $f^n$ is the identity $\iota_X : X \to X$ of the group $S_n$.

Conversely, if $n = 1$, then $f$ is certainly a bijection $X \to X$. If $n > 1$, then $n! > 1$, so that $f^n = \iota_X$ implies that $f^{n-1}$ is the inverse of $f$. Thus $f$ is a bijection. □

Let $\mathcal{M}$ be the category whose objects are all pairs $(X, f)$, where $X = \{1, 2, 3, \ldots, n\}$ for some $n \geq 1$ and $f: X \to X$ is a mapping. Hence $\mathcal{M}$ will be a small category with countably many objects. A morphism $g: (X, f) \to (X', f')$ in $\mathcal{M}$ is any mapping $g: X \to X'$ for which the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X' \\
\downarrow f & & \downarrow f' \\
X & \xrightarrow{g} & X'
\end{array}
\]

commutes.

Remark 6.3. Our category $\mathcal{M}$ can also be seen from the point of view of Universal Algebra. It is equivalent to the category (variety) of all finite algebras $(X, f)$ with one unary operation $f$ and no axioms. The morphisms in the category $\mathcal{M}$ are exactly the homomorphisms in the sense of Universal Algebra. The product decomposition of $f$ as a product of disjoint forests on a cycle corresponds to the coproduct decomposition in this category $\mathcal{M}$ as a coproduct of indecomposable algebras. A congruence on $(X, f)$, in the sense of Universal Algebra, is an equivalence relation $\sim$ on the set $X$ such that, for all $x, y \in X$, $x \sim y$ implies $f(x) \sim f(y)$.

Now let $\mathcal{C}$ be the full subcategory of $\mathcal{M}$ whose objects are the pairs $(X, f)$ with $f: X \to X$ a bijection. Let $\mathcal{F}$ be the full subcategory of $\mathcal{M}$
whose objects are the pairs \((X, f)\) where \(f\) is a mapping whose graph is a forest. (Here, \(\mathcal{C}\) stands for cycles and \(\mathcal{F}\) stands for forests, or torsion-free objects, as we will see.) Clearly, an object of \(\mathcal{M}\) is an object both in \(\mathcal{C}\) and in \(\mathcal{F}\) if and only if it is of the form \((X, \iota_X)\), where \(\iota_X : X \to X\) is the identity mapping. We will call these objects \((X, \iota_X)\) the trivial objects of \(\mathcal{M}\). Let \(\text{Triv}\) be the full subcategory of \(\mathcal{M}\) whose objects are all trivial objects \((X, \iota_X)\).

Call a morphism \(g : (X, f) \to (X', f')\) in \(\mathcal{M}\) trivial if it factors through a trivial object. That is, if there exists a trivial object \((Y, \iota_Y)\) and morphisms \(h : (X, f) \to (Y, \iota_Y)\) and \(\ell : (Y, \iota_Y) \to (X', f')\) in \(\mathcal{M}\) such that \(g = \ell h\).

**Lemma 6.4.** Let \(g : (X, f) \to (X', f')\) be a morphism in \(\mathcal{M}\). Then \(g\) is a trivial morphism in \(\mathcal{M}\) if and only if \(g\) is constant on the connected components of \(G^u_f\) and the image of \(g\) consists of elements of \(X'\) fixed by \(f'\).

**Proof.** Let \(g\) be a trivial morphism in \(\mathcal{M}\), so that there is a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow{f} & \quad & \downarrow{\ell} \\
X' & \xrightarrow{f'} & Y \\
\end{array}
\]

where \(g = \ell h\). In order to show that \(g\) is constant on the connected components of \(G^u_f\), it suffices to prove that if \(x \in X\), then \(g(x) = g(f(x))\).

Now the commutativity of the square on the left of diagram (2) yields that \(h(x) = h(f(x))\), so that \(g(x) = \ell(h(x)) = \ell(h(f(x))) = g(f(x))\). In order to prove that the image of \(g\) consists of elements of \(X'\) fixed by \(f'\), we must show that, for every \(x \in X\), \(g(x) = f'(g(x))\). The commutativity of the square on the right of diagram (2) tells us that \(\ell = f' \circ \ell\). Hence, for every \(x \in X\), \(f'(g(x)) = f'\ell(h(x))) = \ell(h(x))) = g(x)\).

Conversely, let \(g : X \to X'\) be a mapping that is constant on the connected components of \(G^u_f\) and whose image consists of elements of \(X'\) fixed by \(f'\). For every \(x \in X\), the vertices \(x\) and \(f(x)\) are in the same connected component of \(G^u_f\), so that \(g(x) = g(f(x))\). Since the image of \(g\) consists of elements of \(X'\) fixed by \(f'\), we get that \(f'(g(x)) = g(x)\). This proves that \(g = gf\) and \(f'g = g\). Now \(g\) factors as \(g = \varepsilon \circ g|^{g(X)}\), where \(\varepsilon : g(X) \to X'\) is the inclusion and \(g|^{g(X)} : X \to g(X)\) is the corestriction of \(g\) to \(g(X)\). Thus the equalities \(g = gf\) and \(f'g = g\) can be rewritten as \(\varepsilon g|^{g(X)} = \varepsilon g|^{g(X)} f\) and \(f'\varepsilon g|^{g(X)} = \varepsilon g|^{g(X)} f\) and \(f'\varepsilon = \varepsilon\) because \(\varepsilon\) is injective and \(g|^{g(X)}\) is surjective. Thus the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g|^{g(X)}} & g(X) \\
\downarrow{f} & \quad & \downarrow{\varepsilon} \\
X & \xrightarrow{g|^{g(X)}} & g(X) \\
\end{array}
\]

is commutative. Hence \(\varepsilon\) and \(g|^{g(X)}\) are morphisms in \(\mathcal{M}\), and \(g = \varepsilon \circ g|^{g(X)}\) is a factorization of \(g\) through the trivial object \((g(X), \iota_{g(X)})\). \(\square\)
We conclude this section with a lemma that will be often useful in the sequel.

**Lemma 6.5.** Let \( g: (X, f) \to (X', f') \) be a morphism in \( \mathcal{M} \). Then:

(a) The image of an arrow in \( G^d_f \) is an arrow in \( G^d_{f'} \).

(b) The image of a directed path (a directed cycle) in \( G^d_f \) is a directed path (a directed cycle) in \( G^d_{f'} \).

(c) The image of a connected component of \( G^u_f \) is contained in a connected component of \( G^u_{f'} \).

**Proof.** (a) We must show that for an arbitrary arrow \( x \to f(x) \) of \( G^d_f \), the arrow \( g(x) \to g(f(x)) \) is in \( G^d_{f'} \), i.e., that \( g(f(x)) = f'(g(x)) \), which holds because of the commutativity of digram \([\text{1}]\). Statements (b) and (c) follow immediately from (a). \( \square \)

**Corollary 6.6.** If \((X, f)\) and \((X', f')\) are objects of \( \mathcal{M} \), where \( f \) is a bijection and the graph of \( f' \) is a forest, then every morphism \( g: (X, f) \to (X', f') \) is trivial.

**Proof.** By Lemma [6.3, we must show that if \( C \) is a connected component of \( G^d_f \), then \( g \) is constant on \( C \) and \( g(C) \) consists of an element of \( X' \) fixed by \( f' \). Since \( f \) is a bijection, the connected component \( C \) of \( G^d_f \), that is, a cycle, corresponds to an oriented cycle in \( G^d_{f'} \). By Lemma [6.5(b), the image of a directed cycle of \( G^d_f \) via \( g \) is a directed cycle in \( G^d_{f'} \). But the directed cycles in \( G^d_{f'} \) are only the trivial cycles of length 1 based on the elements of \( X' \) fixed by \( f' \). This concludes the proof. \( \square \)

**7. Prekernels and Precokernels**

Let \( f: X \to X' \) be a morphism in \( \mathcal{M} \). We say that a morphism \( k: K \to X \) in \( \mathcal{M} \) is a **prekernel** of \( f \) if the following properties hold:

(a) \( fk \) is a trivial morphism.

(b) Whenever \( \ell: Y \to X \) is a morphism in \( \mathcal{M} \) and \( f\ell \) is trivial, then there exists a unique morphism \( \ell': Y \to K \) in \( \mathcal{M} \) such that \( \ell = k\ell' \).

A **precokernel** of \( f \) is a morphism \( p: X' \to P \) such that:

(a) \( pf \) is a trivial morphism.

(b) Whenever \( h: X' \to Y \) is a morphism such that \( hf \) is trivial, then there exists a unique morphism \( h': P \to Y \) with \( h = h'p \).

Not all morphisms in \( \mathcal{M} \) have a prekernel. For instance, let \( f: X \to X \) be any fixed-point-free permutation of \( X \), for example any cycle of length \( n \). Then, for every trivial object \((Y, \iota_Y)\) of \( \mathcal{M} \), there are no morphisms \((Y, \iota_Y) \to (X, f)\) in \( \mathcal{M} \). Hence there is no trivial morphism \((Z, h) \to (X, f)\) in \( \mathcal{M} \), for any object \((Z, h)\) of \( \mathcal{M} \). Therefore for every object \((W, \ell)\) of \( \mathcal{M} \), all morphisms \((W, \ell) \to (X, f)\) have no prekernel.

Now let \((X, f)\) be an object of \( \mathcal{M} \). Let \( \sim \) be the congruence on the universal algebra \((X, f)\) generated by the subset \( A := \{ (f^n(x), f^{n+1}(x)) \mid x \in X \} \) of \( X \times X \). That is, \( \sim \) is the intersection of all the congruences of the universal algebra \((X, f)\), viewed as subsets of \( X \times X \), that contain the subset \( A \).
Lemma 7.1. The following conditions are equivalent for any two elements $x_1, x_2 \in X$:

(a) $x_1 \sim x_2$.
(b) there exist non-negative integers $t_1, t_2$ such that $x_1 = f^{t_1}(x_2)$ and $x_2 = f^{t_2}(x_1)$.
(c) $x_1$ and $x_2$ are on an oriented cycle (possibly of length 1) of $G_i^d$.

Proof. (a) $\Rightarrow$ (b). Let $\sim'$ be the relation on $X$ defined, for every $x_1, x_2 \in X$, by $x_1 \sim' x_2$ if there exist non-negative integers $t_1, t_2$ such that $x_1 = f^{t_1}(x_2)$ and $x_2 = f^{t_2}(x_1)$. We want to show that $\sim \subseteq \sim'$. Now $\sim'$ is reflexive (take $t_1 = t_2 = 0$), and clearly symmetric. In order to prove that it is transitive, suppose $x_1 \sim' x_2$ and $x_2 \sim' x_3$ $(x_1, x_2, x_3 \in X)$. There exist integers $t_1, t_2, t_3, t_4 \geq 0$ such that $x_1 = f^{t_1}(x_2)$, $x_2 = f^{t_2}(x_3)$ and $x_3 = f^{t_3}(x_2)$. Then $x_1 = f^{t_1}(x_2) = f^{t_1 + t_3}(x_3)$ and $x_3 = f^{t_3}(x_2) = f^{t_4 + t_2}(x_1)$. Hence $x_1 \sim' x_3$. This proves that $\sim'$ is an equivalence relation on the set $X$. Moreover, if $x_1 \sim' x_2$, then $x_1 = f^{t_1}(x_2)$ and $x_2 = f^{t_2}(x_1)$ for suitable $t_1, t_2 \geq 0$, so $f(x_1) = f^{t_1}(f(x_2))$ and $f(x_2) = f^{t_2}(f(x_1))$. Thus $f(x_1) \sim' f(x_2)$. This shows that $\sim'$ is a congruence for the universal algebra $(X, f)$. Now $A \subseteq \sim'$, because if $(f^n(x), f^{n+1}(x))$ is an arbitrary element of $A$ $(x \in X)$, then, in the notation of Section $n$, $f^n(x) = f^{t_1}(x)$, so $f^n(x) = f^{n+k}(x)$ for some $k > 0$. Thus, for $t_1 := k - 1$, we have that $f^n(x) = f^{n+k}(x) = f^{k-1}(f^{n+1}(x)) = f^{t_1}(f^{n+1}(x))$ and, for $t_2 := 1$, $f^{n+1}(x) = f^{t_2}(f^n(x))$. This proves that $f^n(x) \sim' f^{n+1}(x)$. We have thus shown that $A \subseteq \sim'$. As $\sim$ is generated by $A$, we obtain $\sim \subseteq \sim'$, that is, (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a). We must show that $\sim' \subseteq \sim$. Suppose $x_1 \sim' x_2$ $(x_1, x_2 \in X)$. If $x_1 = x_2$, then $x_1 \sim x_2$. Therefore we can suppose $x_1 \neq x_2$. Thus there exist $t_1, t_2 \geq 0$ such that $x_1 = f^{t_1}(x_2)$, $x_2 = f^{t_2}(x_1)$ and $t_1 + t_2 > 0$. Hence $x_1 = f^{t_1 + t_2}(x_1)$. Consider the subset $\{x_1, f(x_1), f^2(x_1), \ldots, f^{t_1+t_2-1}(x_1)\}$ of $X$. Then $f$ permutes cyclically the elements of this subset, so that $f^n$ permutes the elements of this subset

$$\{x_1, f(x_1), f^2(x_1), \ldots, f^{t_1+t_2-1}(x_1)\}
= \{f^n(x_1), f^{n+1}(x_1), f^{n+2}(x_1), \ldots, f^{n+t_1+t_2-1}(x_1)\}.$$\

But $(f^n(x_1), f^{n+1}(x_1)) \in A$, $(f^{n+1}(x_1), f^{n+2}(x_1)) \in A$, and so on, so that $f^n(x_1) \sim f^{n+1}(x_1)$, $f^{n+1}(x_1) \sim f^{n+2}(x_1)$, \ldots, $f^{n+t_1+t_2-1}(x_1) \sim f^{n+t_1+t_2-2}(x_1)$. Thus all the elements of the subset

$$\{f^n(x_1), f^{n+1}(x_1), f^{n+2}(x_1), \ldots, f^{n+t_1+t_2-2}(x_1)\}$$\

are equivalent modulo $\sim$. But $x_1$ and $x_2 = f^{t_2}(x_1)$ are two of those elements, so $x_1 \sim x_2$.

(b) $\Leftrightarrow$ (c) is trivial. \qed

Clearly, the canonical projection $\pi: X \to X/\sim$ is the coequalizer of the two endomorphisms $f^n$ and $f^{n+1}$ of $(X, f)$. Also, since $\sim$ is a congruence on the unary universal algebra $(X, f)$, $f$ induces a mapping $\overline{f}: X/\sim \to X/\sim$. 

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Thus \((X/\sim, \mathcal{F})\) is a universal algebra and the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & X/\sim \\
\downarrow{f} & & \downarrow{\mathcal{F}} \\
X & \xrightarrow{\pi} & X/\sim
\end{array}
\]

commutes. Notice that \(X/\sim\) is a forest, because \(\mathcal{F}^n = \mathcal{F}^{n+1}\), so that \(\mathcal{F}\) always has fixed points (the roots of the trees in the forest \(G^d_f\)).

As an example, we have drawn the graph \(G^d_f\) of a mapping \(f: X \to X\) in Figure 3. In this example, \(X\) has \(n = 27\) elements. In Figure 4, we have represented the partition of the set \(X\) of vertices of the graph \(G^d_f\) in equivalence classes modulo \(\sim\). One equivalence class has 6 elements, another one has 4 elements, and all the other classes consist of one element. Hence there are 19 equivalence classes. Figure 5 represents the graph of \((X/\sim, \mathcal{F})\), which is a graph with 19 vertices.

Let us prove that the morphism \(\pi: X \to X/\sim\) has a prekernel, which is the embedding \(\varepsilon: (A_0, f|_{A_0}) \to (X, f)\). Here \(A_0\) is the subset of \(X\) as defined in Proposition 2.1. Hence the restriction \(f|_{A_0}: A_0 \to A_0\) of \(f\) to \(A_0\) is a bijection. Clearly, \(\varepsilon\) is a morphism in \(\mathcal{M}\), \(\pi\varepsilon: A_0 \to X/\sim\) sends the cycles of \(A_0\) to the roots of the tree \(X/\sim\), hence \(\pi\varepsilon\) is a trivial morphism by Lemma 6.4. In order to prove property (2) in the definition of prekernel, fix a morphism \(\ell: (Y, g) \to (X, f)\) in \(\mathcal{M}\) with \(\pi\ell\) trivial. We
have a commutative diagram

\[
\begin{array}{ccc}
Y & \overset{\ell}{\longrightarrow} & X \\
\downarrow^{g} & & \downarrow^{f} \\
Y & \overset{\ell}{\longrightarrow} & X \left/ \sim \right. \\
\end{array}
\]

Thus, for any connected component \( C_Y \) of the graph \( G^n_G \) of \( (Y, g) \), we have that \( \ell(C_Y) \) is contained in a cycle of \( X \). It follows that \( \ell(Y) \subseteq A_0 \). Let \( \ell' := \ell|_{A_0} : Y \to A_0 \) be the corestriction of \( \ell \), obtained by restricting the codomain \( X \) of \( \ell \) to \( A_0 \). Then \( \ell = \varepsilon \ell' \). As far as the uniqueness of such an \( \ell' \) is concerned, suppose that we also have another morphism \( \ell'' : Y \to A_0 \)

\[\text{Figure 4. The partition of the set } X \text{ of vertices in equivalence classes modulo } \sim.\]

\[\text{Figure 5. The quotient set } X/\sim.\]
with $\ell = \varepsilon \ell''$. Then $\ell' = \varepsilon \ell''$, so $\ell' = \ell''$ because $\varepsilon$ is an injective mapping. This proves that the embedding $\varepsilon: (A_0, f|_{A_0}) \hookrightarrow (X, f)$ is the prekernel of the canonical projection $\pi: X \to X/\sim$.

Conversely, we will now show that the canonical projection $\pi: X \to X/\sim$ is the precokernel of the embedding $\varepsilon: (A_0, f|_{A_0}) \hookrightarrow (X, f)$. It is sufficient to prove property (2) in the definition of precokernel. Hence suppose we have a commutative diagram

$$
\begin{array}{ccc}
A_0 & \xrightarrow{\varepsilon} & X \\
\downarrow f|_{A_0} & & \downarrow f \\
A_0 & \xrightarrow{\varepsilon} & X
\end{array}
\begin{array}{ccc}
\downarrow \alpha & & \downarrow \alpha \\
Z & & Z
\end{array}
\begin{array}{ccc}
\downarrow g & & \downarrow g \\
X & & X
\end{array}
\begin{array}{ccc}
\downarrow \pi & & \downarrow \pi \\
X/\sim & & X/\sim
\end{array}
$$

with $\alpha\varepsilon$ trivial. Then the image $\alpha(C_f)$ of any cycle $C_f$ of $f$ is a fixed point of $(Z, g)$, that is, is one of the roots of a tree in the graph $G^u_g$ of $(Z, g)$. But if $x, y \in X$ and $x \sim y$, then $\alpha(x) = \alpha(y)$. Hence $\alpha$ factors through $\pi$:

$$
\begin{array}{ccc}
X & \xrightarrow{\alpha} & Z \\
\downarrow \pi & & \downarrow \pi \\
X/\sim & & X/\sim
\end{array}
$$

The uniqueness of $\alpha'$ follows from the surjectivity of $\pi$.

We will recall the exact definition of pretorsion theory in Section 9, but the next result shows that the pair $(\mathcal{C}, \mathcal{F})$ is a pretorsion theory in $\mathcal{M}$.

**Theorem 7.2.** In the notation above, we have that:

(a) For every object $(X, f)$ of $\mathcal{M}$, there are morphisms

$$(3) \quad (A_0, f|_{A_0}) \xrightarrow{\varepsilon} (X, f) \xrightarrow{\varepsilon'} (X/\sim, \overline{f})$$

in the category $\mathcal{M}$ such that $\varepsilon$ is a prekernel of $\pi$ and $\pi$ is a precokernel of $\varepsilon$.

(b) $\text{Hom}_{\mathcal{M}}(C, F) = \text{Triv}_{\mathcal{M}}(C, F)$ for every $C \in \mathcal{C}$, $F \in \mathcal{F}$.

(c) If $X$ is an object of $\mathcal{M}$ and $\text{Hom}_{\mathcal{M}}(X, F) = \text{Triv}_{\mathcal{M}}(X, F)$ for every $F \in \mathcal{F}$, then $X$ is an object of $\mathcal{C}$.

(d) If $X$ is an object of $\mathcal{M}$ and $\text{Hom}_{\mathcal{M}}(C, X) = \text{Triv}_{\mathcal{M}}(C, X)$ for every $C \in \mathcal{C}$, then $X$ is an object of $\mathcal{F}$.

**Proof.** (a) was proved in the discussion above, before the statement of the theorem.

(b) is Corollary 6.6.

(c) Let $X$ be an object of $\mathcal{M}$ with $\text{Hom}_{\mathcal{M}}(X, F) = \text{Triv}_{\mathcal{M}}(X, F)$ for every $F \in \mathcal{F}$. In (3), we have that the morphism $\pi$ is trivial because $X/\sim$ is in $\mathcal{F}$. Hence $\pi$ sends any connected component of $G^u_j$ to a root of the forest $(X/\sim, \overline{f})$. The inverse images via $\pi$ of the roots of the forest $(X/\sim, \overline{f})$ are the cycles of $(X, f)$. Hence any connected component of $G^u_j$ is contained in a cycle of $G^u_j$. Thus any connected component of $G^v_j$ is a cycle of $G^v_j$. Therefore $f$ is a bijection.

(d) Finally, let $X$ be an object of $\mathcal{M}$. Assume that $\text{Hom}_{\mathcal{M}}(C, X) = \text{Triv}_{\mathcal{M}}(C, X)$ for every $C \in \mathcal{C}$. Suppose that $X$ has a directed cycle, so that...
the cycle is contained in $A_0$ necessarily. We have the embedding $\varepsilon: A_0 \to X$, where $(A_0, f|_{A_0})$ is an object of $C$. By the hypothesis $\text{Hom}_{\mathcal{M}}(A_0, X) = \text{Triv}_{\mathcal{M}}(A_0, X)$, we have that $\varepsilon$ is trivial, so that the inclusion $\varepsilon$ maps every cycle in $A_0$ to a root of $X$. Thus every cycle consists of a single point, so that in $A_0$ there are no cycles of length $> 0$. Hence in $G_J^n$ there are no cycles. This proves that $X$ is an object of $\mathcal{F}$. \hfill \Box

8. Some noteworthy functors from the category $\mathcal{M}$ and adjoint functors of the embeddings.

In this section, we are going to consider some remarkable functors.

We call \textit{preradical} of $\mathcal{M}$ any functor $R: \mathcal{M} \to \mathcal{M}$ such that, for each object $(X, f)$ of $\mathcal{M}$, $R(X, f)$ is a subalgebra of $(X, f)$ and, for every morphism $g: (X, f) \to (X', f')$, $R(g): R(X, f) \to R(X', f')$ is the restriction $g|_{R(X, f)}$ of $g$. Thus a preradical is a subfunctor of the identity functor on $\mathcal{M}$. We say that a preradical $R$ is \textit{idempotent} if $R \circ R = R$.

Dually, we call \textit{precoradical} of $\mathcal{M}$ any functor $C: \mathcal{M} \to \mathcal{M}$ such that, for each object $(X, f)$ of $\mathcal{M}$, $C(X, f)$ is a quotient algebra $(X/\equiv, \overline{f})$ of $(X, f)$ for some congruence $\equiv$ on the algebra $(X, f)$ and, for every morphism $g: (X, f) \to (X', f')$, $C(g): C(X, f) \to C(X', f')$ is induced by $g$. Hence a precoradical is a quotient functor of the identity functor on $\mathcal{M}$. We say that a precoradical $C$ is \textit{idempotent} if, for every object $(X, f)$ of $\mathcal{M}$, $C(\pi): C(X, f) \to C(C(X, f))$ is an isomorphism. Here $\pi: (X, f) \to C(X, f) = (X/\equiv, \overline{f})$ is the canonical projection.

The first noteworthy functor we study in this section is an idempotent functor $C: \mathcal{M} \to \mathcal{M}$ such that $C(X, f) = (X/\sim, \overline{f}) \in \text{Ob}(\mathcal{F})$ for every object $(X, f)$ of $\mathcal{M}$. The notation is an in Lemma \ref{7.1}.

**Proposition 8.1.** There is an idempotent precoradical $C: \mathcal{M} \to \mathcal{M}$ such that $C(X, f) = (X/\sim, \overline{f}) \in \text{Ob}(\mathcal{F})$ for every object $(X, f)$ of $\mathcal{M}$.

**Proof.** In order to prove that $C: \mathcal{M} \to \mathcal{M}$ is a precoradical, we must show that every morphism $g: (X, f) \to (X', f')$ induces a morphism $\overline{g}: (X/\sim, \overline{f}) \to (X'/\sim', \overline{f'})$. By Lemma \ref{6.5}(b), $g$ maps any oriented cycle of $G_J^n$ to an oriented cycle of $G_J^{n'}$. From Lemma \ref{7.1} it follows that, for every $x_1, x_2 \in X$, $x_1 \sim x_2$ implies $g(x_1) \sim' g(x_2)$. Thus the kernel $\sim_{\pi'g}$ of $\pi'g$ contains the kernel $\sim_{\pi}$ of $\pi$. We have the following diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & X/\sim \\
\downarrow{g} & & \downarrow{\equiv_{\pi}} \\
X' & \xrightarrow{\pi'} & X'/\sim'
\end{array}
\]

Hence there exists a unique mapping $\overline{g}: X/\sim \to X'/\sim'$ that makes the diagram commute. It is now clear that $C$ is a precoradical.

We have already seen in Section \ref{6} that $(X/\sim, \overline{f})$ is an object of $\mathcal{F}$ for every object $(X, f)$ of $\mathcal{M}$. Moreover, for every $(F, h)$ in $\mathcal{F}$ the congruence $\sim$ on $F$ is the equality on $f$. It follows that the precoradical $C$ is idempotent. \hfill \Box
Proposition 8.2. There is an idempotent preradical \( R: \mathcal{M} \to \mathcal{M} \) such that \( R(X,f) = (A_0,f|_{A_0}) \in \text{Ob}(\mathcal{C}) \) for every object \((X,f)\) of \(\mathcal{M}\). Here \(A_0 = f^n(X)\) is the set of all vertices of \(G^d_f\) on some oriented cycle.

Proof. Like for the previous proposition, to prove that \( R: \mathcal{M} \to \mathcal{M} \) is a preradical, it suffices to show that every morphism \( g: (X,f) \to (X,f') \) maps \(A_0,f\) to \(A_0,f'\). By Lemma 6.5, \( g \) maps any cycle of \(G^d_f\) to a cycle of \(G^d_{f'}\). Therefore \( g(A_0,f) \subseteq A_0,f' \). The rest is clear. \(\square\)

Notice that the canonical mapping \((X,f) \to C(X,f)\) has a prekernel, which is \(R(X,f)\) (Theorem 7.2). The prekernel of the canonical embedding \(R(X,f) \hookrightarrow (X,f)\) does not exist in general.

A third remarkable functor \(W: \mathcal{M} \to \mathcal{C}\) is the following. Let \((X,f)\) be an object of \(\mathcal{M}\) and we write \(f = m\sigma\), where \(m\) is a product of moves (of the vertices on the forest) and \(\sigma\) is a permutation (of the vertices on the cycles), like in Theorem 11.3. That is, \(m\) is a product of moves of elements of \(X \setminus A_0\) to elements of \(X\) and \(\sigma\) permutes the elements of \(A_0\). Then \(f^n(x) \in A_0\) for every \(x \in X\), so that \(\sigma^{−(n)}(f^n(x)) \in A_0\).

Lemma 8.3. The mapping \(w: (X,f) \to (A_0,\sigma)\), defined by

\[w(x) = \sigma^{−(n)}(f^n(x)) \quad \text{for every } x \in X,\]

is a morphism in \(\mathcal{M}\).

Proof. We must prove that \(\sigma w = wf\). Now \(f^n(x) \in A_0\) for every \(x \in X\), and \(f\) and \(\sigma\) coincide on \(A_0\), so that \(ff^n = \sigma f^n\). Therefore \(\sigma w = \sigma \sigma^{−(n)} f^n = \sigma^{−(n)} \sigma f^n = \sigma^{−(n)} \sigma f^n = \sigma^{−(n)} f^n f = w f\). \(\square\)

Notice that \(w\) fixes the points of \(A_0\), i.e., the points on the cycles, and winds up to the trees of the forest around the cycles. The roots of the trees are fixed by \(w\). The permutation \(\sigma\) is the restriction of \(f\) to \(A_0\).

We are ready to study the existence of right and left adjoints of the embeddings \(\mathcal{C} \hookrightarrow \mathcal{M}\) and \(\mathcal{F} \hookrightarrow \mathcal{M}\).

Proposition 8.4. The category \(\mathcal{C}\) is a reflective and coreflective subcategory of \(\mathcal{M}\), that is, the embedding \(\mathcal{C} \hookrightarrow \mathcal{M}\) has both a left adjoint and a right adjoint. The left adjoint and right adjoint is the functor restriction \(R: \mathcal{M} \to \mathcal{C}\) that maps the object \((X,f)\) to its restriction \((A_0,\sigma)\).

Proof. In order to show that \(R: \mathcal{M} \to \mathcal{C}\) is the left adjoint of the embedding \(\mathcal{C} \hookrightarrow \mathcal{M}\), with unit the morphism \(w: (X,f) \to (A_0,\sigma)\), it suffices to show that for every object \((X,f)\) of \(\mathcal{M}\), every object \((C,\tau)\) of \(\mathcal{C}\) and every morphism \(\varphi: (X,f) \to (C,\tau)\), there exists a unique morphism \(\tilde{\varphi}: (A_0,\sigma) \to (C,\tau)\) such that \(\tilde{\varphi}w = \varphi\). Let \(\tilde{\varphi}: (A_0,\sigma) \to (C,\tau)\) be the restriction to \(A_0\) of \(\varphi: (X,f) \to (C,\tau)\). Then

\[\tilde{\varphi}w = \tilde{\varphi} \sigma^{−(n)} f^n = \tau^{−(n)} \tilde{\varphi} f^n = \tau^{−(n)} \tilde{\varphi} f^n = \tau^{−(n)} f^n \tilde{\varphi} = \varphi.\]

The uniqueness of \(\tilde{\varphi}\) follows from the surjectivity of \(w\).

We will now prove that \(R\) is the right adjoint of the embedding \(\mathcal{C} \hookrightarrow \mathcal{M}\). The counit of the adjunction is the inclusion morphism \(\varepsilon: (A_0,\sigma) \to (X,f)\). It suffices to show that, for every object \((X,f)\) of \(\mathcal{M}\), every object \((C,\tau)\)
of $C$ and every morphism $\varphi: (C, \tau) \to (X, f)$, there exists a unique morphism $\varphi': (C, \tau) \to (A_0, \sigma)$ with $\varepsilon \varphi' = \varphi$. Now the morphism $\varphi: (C, \tau) \to (X, f)$ maps cycles to cycles, so that $\varphi(C) \subseteq A_0$. Hence the corestriction $\varphi': (C, \tau) \to (A_0, \sigma)$ of $\varphi: (C, \tau) \to (X, f)$ obtained by restricting the codomain to $A_0$ has the property that $\varepsilon \varphi' = \varphi$. The uniqueness of $\varphi'$ follows from the injectivity of $\varepsilon$. \qed

**Proposition 8.5.** The category $\mathcal{F}$ is a reflective subcategory of $\mathcal{M}$ which is not coreflecting, that is, the embedding $\mathcal{F} \hookrightarrow \mathcal{M}$ has a left adjoint, but not a right adjoint. The left adjoint is the preradical $C$, viewed as a functor of $\mathcal{M}$ to $\mathcal{F}$.

**Proof.** To show that $C: \mathcal{M} \to \mathcal{F}$ is the left adjoint of the embedding $\mathcal{F} \hookrightarrow \mathcal{M}$, with unit the canonical projection $\pi: (X, f) \to (X/\sim, \bar{f})$, it suffices to prove that for every object $(X, f)$ of $\mathcal{M}$, every object $(F, g)$ of $\mathcal{F}$ and every morphism $\varphi: (X, f) \to (F, g)$, there exists a unique morphism $\bar{\varphi}: (X/\sim, \bar{f}) \to (F, g)$ such that $\bar{\varphi} \pi = \varphi$. This is equivalent to proving that $\varphi: (X, f) \to (F, g)$ induces a morphism $\bar{\varphi}: (X/\sim, \bar{f}) \to (F, g)$, that is, that the congruence $\sim$ is contained in the kernel $\sim_{\varphi}$ of $\varphi$. Now $\sim$, corestriction of $f^n$ and $f^{n+1}$, is the congruence generated by the arrows on the cycles of $f$, so that it suffices to prove that $\varphi$ corequalizes $f^n$ and $f^{n+1}$, i.e., that $\varphi f^n = \varphi f^{n+1}$. But $\varphi f^n = g^n \varphi = g^{n+1} \varphi = \varphi f^{n+1}$, as desired.

In order to prove that the embedding $\mathcal{F} \hookrightarrow \mathcal{M}$ does not have a right adjoint, it suffices to show that there exist objects $(X, f)$ in $\mathcal{M}$ for which $\text{Hom}_\mathcal{M}(\mathcal{F}, (X, f)) = \emptyset$. To this end, it suffices to take as $(X, f)$ any object with $f$ fixed-point-free, for instance a cycle permuting all elements of $X$ ($|X| > 1$). Then every cycle of length zero of any $F$ in $\mathcal{F}$ (that is, any fixed point, which exists in all $F \in \mathcal{F}$), should be mapped to a fixed point of $X$, which does not exist. \qed

9. A PRETORSION THEORY

The second half of this paper, from Section 8 on, has been deeply influenced by the article [1]. In that paper, the category $\text{Preord}$ of preordered sets $(A, \rho)$ is studied. Here $A$ is any non-empty set, and $\rho$ is a preorder on $A$, that is, a reflexive and transitive relation on $A$. The morphisms $g: (A, \rho) \to (A', \rho')$ in the category $\text{Preord}$ are the mappings $g$ of $A$ into $A'$ such that $a \rho b$ implies $g(a) \rho' g(b)$ for all $a, b \in A$. In $\text{Preord}$, there is a pretorsion theory $(\text{Equiv}, \text{ParOrd})$, where $\text{Equiv}$ is the class of all objects $(A, \rho)$ of $\text{Preord}$ with $\rho$ an equivalence relation on $A$ and $\text{ParOrd}$ is the class of all objects $(A, \rho)$ of $\text{Preord}$ with $\rho$ a partial order on $A$.

Torsion theories in general categories are studied in the papers [2], [3] and [5]. Since these articles are rather technical while our setting is extremely simple, the first author and Carmelo Finocchiaro introduced in [1] a much easier notion of pretorsion theory. We are very grateful to Marino Gran, Marco Grandis and Sandra Mantovani for some suggestions.

Our setting is the following. Fix an arbitrary category $\mathcal{C}$ and two non-empty classes $\mathcal{T}, \mathcal{F}$ of objects of $\mathcal{C}$, both closed under isomorphism. Set $\mathcal{Z} := \mathcal{T} \cap \mathcal{F}$. For every pair $A, B$ of objects of $\mathcal{C}$, $\text{Triv}_{\mathcal{Z}}(A, B)$ indicates
the set of all morphisms in \( \mathcal{C} \) that factors through an object of \( Z \). The morphisms in \( \text{Triv}_Z(A, B) \) are called \( Z \)-trivial.

Let \( f: A \to A' \) be a morphism in \( \mathcal{C} \). A \( Z \)-prekernel of \( f \) is a morphism \( k: X \to A \) in \( \mathcal{C} \) such that:

(a) \( fk \) is a \( Z \)-trivial morphism.

(b) Whenever \( \ell: Y \to A \) is a morphism in \( \mathcal{C} \) and \( f \ell \) is \( Z \)-trivial, then there exists a unique morphism \( \ell': Y \to X \) in \( \mathcal{C} \) such that \( \ell = k \ell' \).

The \( Z \)-prekernel of \( f: A \to A' \) turns out to be a subobject of \( A \) unique up to isomorphism, when it exists.

Dually, for the \( Z \)-precokernel of \( f \), which is a morphism \( A' \to X \). If \( f: A \to B \), \( g: B \to C \) are morphisms in \( \mathcal{C} \), we say that

\[
A \xrightarrow{f} B \xrightarrow{g} C
\]

is a short \( Z \)-preexact sequence in \( \mathcal{C} \) if \( f \) is a \( Z \)-prekernel of \( g \) and \( g \) is a \( Z \)-prekernel of \( f \).

We say that a pair \( (T, F) \) is a pretorsion theory for \( \mathcal{C} \), where \( T, F \) are classes of objects of \( \mathcal{C} \) closed under isomorphism and \( Z : = T \cap F \), if it satisfies the following two conditions:

1. For every object \( B \) of \( \mathcal{C} \) there is a short \( Z \)-preexact sequence

\[
A \xrightarrow{f} B \xrightarrow{g} C
\]

with \( A \in T \) and \( C \in F \).

2. \( \text{Hom}_C(T, F) = \text{Triv}_Z(T, F) \) for every pair of objects \( T \in T, F \in F \).

There is a clear overlapping between this definition of pretorsion theory and the theory developed in [9], in a case concerning the category \( \mathcal{C} = \text{Preord} \) of all non-empty preordered set was studied. The pretorsion theory was the pair \( (\text{Equiv}, \text{ParOrd}) \) of sets endowed with an equivalence relation and sets endowed with a partial order, respectively. In this paper, we have seen that \( (\mathcal{C}, F) \) is a pretorsion theory in \( \mathcal{M} \) (a) and (b) in Theorem 7.2.

There is a functor \( U: \mathcal{M} \to \text{Preord} \), which is a canonical embedding. It associates to any object \( (X, f) \) of \( \mathcal{M} \) the preordered set \( (X, \rho_f) \), where \( \rho_f \) is defined, for every \( x, y \in X \), setting \( x \rho_f y \) if \( x = f^t(y) \) for some integer \( t \geq 0 \). Any morphism \( g: (X, f) \to (X', f') \) in \( \mathcal{M} \), is a morphism \( g: (X, \rho_f) \to (X', \rho_{f'}) \) in \( \text{Preord} \), because if \( x, y \in X \) and \( x \rho_f y \), then \( x = f^t(y) \) for some integer \( t \geq 0 \). The commutativity of diagram (1) yields that \( g(x) = g f^t(y) = f'^t g(y) \), so \( g(x) \rho_{f'} g(y) \). Thus we have a functor \( U: \mathcal{M} \to \text{Preord} \), which is clearly faithful. Hence \( \mathcal{M} \) is isomorphic to a subcategory of \( \text{Preord} \).

**Proposition 9.1.** The following equalities hold:

(a) \( U(\mathcal{C}) = U(\text{Ob}(\mathcal{M})) \cap \text{Equiv} \).

(b) \( U(\mathcal{F}) = U(\text{Ob}(\mathcal{M})) \cap \text{ParOrd} \).

**Proof.** The proof of the inclusion \( U(\mathcal{C}) \subseteq U(\text{Ob}(\mathcal{M})) \cap \text{Equiv} \) in (a) is easy. Conversely, assume \( (X, f) \in \text{Ob}(\mathcal{M}) \) and \( (X, \rho_f) \in \text{Equiv} \). For every \( x \in X \), we have that \( f(x) \rho_f x \). As \( \rho_f \) is an equivalence relation on \( X \), we get that \( x \rho_f f(x) \). Thus there exists \( t \geq 0 \) such that \( x = f^{t+1}(x) \). It
follows that \( x \in f(X) \). Therefore the mapping \( f \) is onto, i.e., a bijection, so \((X,f) \in \mathcal{C}\).

For (b), suppose \((X,f) \in \mathcal{F}\), i.e., that \(G_f^d\) is a forest. In order to prove that \(\rho_f\) is a partial order on \(X\), assume \(x,y \in X\), \(x \rho_f y\) and \(y \rho_f x\). Then there are \(t,u \geq 0\) such that \(x = f^t(y)\) and \(y = f^u(x)\). Then \(x = f^{t+u}(x)\).

Since \(G_f^d\) is a forest, i.e., has no cycle of length \(> 1\), it follows that \(t + u = 0\), so \(t = u = 0\), and \(x = y\). For the opposite inclusion, assume \((X,f) \in \text{Ob}(\mathcal{M})\) and \((X,\rho_f) \in \text{ParOrd}\). For every \(x \in X\), we have that \(f^{n+1}(x) \rho_f f^n(x)\). Now \(G_f^d\) is a forest on a cycle, and \(f^n(x)\) is on the cycle, so that there exists \(v > 0\) such that \(f^n(x) = f^{n+v}(x)\). Hence \(f^n(x) = f^{n+v}(x)\rho_f f^{n+1}(x)\). Therefore \(f^n(x) \rho_f f^{n+1}(x)\) and \(f^{n+1}(x) \rho_f f^n(x)\), which imply \(f^{n+1}(x) = f^n(x)\) because \(\rho_f\) is a partial order. Thus \(f^n = f^{n+1}\), so \((X,f) \in \mathcal{F}\).

The functor \(U\) is not full. For example, consider the permutations \(f = (1\ 2\ 3)\) and \(g = (1\ 2)\) of \(X = \{1,2,3\}\). Then \((X,f)\) is an object of \(\mathcal{M}\), but \(g\) is not an endomorphism of \((X,f)\) in \(\mathcal{M}\), because \(fg \neq gf\). Nevertheless \(\rho_f\) is the trivial preorder on \(X\), in which any two elements of \(X\) are in the relation \(\rho_f\), so that \(g\) is an endomorphism of \((X,\rho_f)\) in the category \(\text{Preord}\).

### 10. The stable category \(\mathcal{M}\)

Following [1], it is now natural to introduce a congruence \(R\) on the category \(\mathcal{M}\) that identifies all trivial morphisms between two objects of \(\mathcal{M}\). Then it is possible to construct the quotient category \(\mathcal{M}/R\) and call it the stable category of \(\mathcal{M}\). But we will see that such a category \(\mathcal{M}/R\) and the category \(\mathcal{M}\) have a terminal object, but don’t have initial objects. So our prekernels and precokernels in \(\mathcal{M}\) do not correspond to kernels and cokernels in the stable category \(\mathcal{M}/R\).

For every pair of objects \((X,f),(X',f')\) of \(\mathcal{M}\), there is an equivalence relation \(R_{X,X'}\) on the set \(\text{Hom}_\mathcal{M}((X,f),(X',f'))\) defined, for every \(g,h\), \((X,f) \to (X',f')\), by \(g R_{X,X'} h\) if, for every connected component \(C\) of \(G_f^d\), either

1. \(g(x) = h(x)\) for every \(x \in C\), or
2. \(|g(C)| = 1\), \(|h(C)| = 1\), \(f'(g(C)) = g(C)\) and \(f'(h(C)) = h(C)\).

Here we view the connected components \(C\) as subsets of \(X\), and \(|A|\) denotes the cardinality of the set \(A\).

In order to view that \(R_{X,X'}\) is an equivalence relation, reflexivity and symmetry are clear. Transitivity is a little more tricky. Suppose \(g,h,\ell\): \((X,f) \to (X',f')\), \(g R_{X,X'} h\) and \(h R_{X,X'} \ell\). For every connected component \(C\) of \(G_f^d\), we have four possible cases.

**First case:** Condition (1) holds for both the pairs \((g,h)\) and \((h,\ell)\). That is, \(g(x) = h(x) = \ell(x)\) for every \(x \in C\). Then \(g(x) = \ell(x)\) for every \(x \in C\), and we have Condition (1) for the pair \((g,\ell)\) as well.

**Second case:** Condition (1) holds for the pair \((g,h)\) and Condition (2) holds for the pair \((h,\ell)\). That is, \(g(x) = h(x)\) for every \(x \in C\), \(|h(C)| = 1\), \(|\ell(C)| = 1\), \(f'(h(C)) = h(C)\) and \(f'(\ell(C)) = \ell(C)\). Then \(|g(C)| = 1\) and \(f'(g(C)) = g(C)\), so that Condition (2) holds for the pair \((g,\ell)\) as well.
Third case: Condition (2) holds for the pair \((g, h)\) and Condition (1) holds for the pair \((h, \ell)\). This is similar to the second case.

Fourth case: Condition (2) holds for both the pairs \((g, h)\) and \((h, \ell)\).

Then, trivially, Condition (2) holds for the pair \((g, \ell)\) as well.

Now we want to prove that the assignment \((X, X') \mapsto R_{X, X'}\) is a congruence on the category \(\mathcal{M}\) in the sense of \([6, \text{page 51}]\). Let \(g, h : (X, f) \to (X', f')\) be morphisms with \(g R_{X, X'} h\), and

\[ m : (Y, g) \to (X, f), \quad \ell : (X', f') \to (X'', f'') \]

be arbitrary morphisms in \(\mathcal{M}\). We must show that \(gm R_{Y, X'} hm\) and \(\ell g R_{X, X''} \ell h\). This is also straightforward, as several of the previous verifications, and we omit it here.

Therefore it is possible to construct the stable category \(\mathcal{M} := \mathcal{M}/R\) like in \([\Pi]\). The difference with \([\Pi]\) is that now the stable category \(\mathcal{M}\) does not have a zero object, as we will show in the next paragraph, so that it is not possible to construct kernels and cokernels in \(\mathcal{M}\) in the classical sense, hence prekernels and precokernels in \(\mathcal{M}\) do not correspond to kernels and cokernels in \(\mathcal{M}\).

The categories \(\mathcal{M}\) and \(\mathcal{M}\) have terminal objects (the singleton \(\{1\}, (1)\) with the identity morphism). To see that \(\mathcal{M}\) and \(\mathcal{M}\) don't have zero objects, notice that there are objects of \(\mathcal{M}\), for instance the cycle \(\{1, 2, 3\}, (1 2 3)\), for which

\[ \text{Hom}_{\mathcal{M}}((\{1\}, (1)), (\{1, 2, 3\}, (1 2 3))) \]

is the empty set. Hence \(\text{Hom}_{\mathcal{M}}((\{1\}, (1)), (\{1, 2, 3\}, (1 2 3)))\) is the empty-set, which is not possible in a pointed category.

REFERENCES

[1] A. Facchini and C. A. Finocchiaro, Pretorsion theories, stable category and preordered sets, to appear, 2019.
[2] M. Grandis and G. Janelidze From torsion theories to closure operators and factorization systems, to appear, 2019.
[3] M. Grandis, G. Janelidze and L. Márki, Non-pointed exactness, radicals, closure operators, J. Aust. Math. Soc. 94 (2013), no. 3, 348–361.
[4] J. M. Howie, The subsemigroup generated by the idempotents of a full transformation semigroup, J. London Math. Soc. 41 (1966), 707–716.
[5] G. Janelidze and W. Tholen, Characterization of torsion theories in general categories, in “Categories in algebra, geometry and mathematical physics”, A. Davydov, M. Batanin, M. Johnson, S. Lack and A. Neeman Eds., Contemp. Math. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 249–256.
[6] S. MacLane, “Categories for the Working Mathematician”, 2nd edn., Springer-Verlag, 1998.
[7] V. M. Usenko, On semidirect products of monoids, Ukrain. Mat. Zh. 34 (2) (1982), 185–189, 268. English translation in Ukrainian Math. J. 34 (2) (1982), 151–155. Available in https://link.springer.com/content/pdf/10.1007/BF01091519.pdf