A novel approach to the computation of one-loop three- and four-point functions. II - The complex mass case

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Abstract

This article is the second of a series of three presenting an alternative method to compute the one-loop scalar integrals. It extends the results of the first article to general complex masses. Let us remind the main features enjoyed by this method. It directly proceeds in terms of the quantities driving algebraic reduction methods. It applies to the four-point functions in the same way as to the three-point functions. Lastly, it extends to kinematics more general than the one of physical e.g. collider processes relevant at one loop.

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1 Introduction

This article is the second of a triptych. The first one [1] presents a method exploiting a Stokes-type identity to compute one-loop three- and four-point scalar integrals in the real mass case in a four dimension spacetime. The present article extends the results of [1] to the case of general complex masses. Let us sketch the motivation of this work. The automated calculation of loop multileg processes requires an efficient and numerically stable evaluation. This is particularly true for the calculation of two-loop three- and four-point functions in the general complex mass case which relies so far on multidimensional numerical integration through sector decomposition [2–6] which has a high computing cost. It would then be useful to perform part of the Feynman parameter integrations analytically to reduce the number of integrations to be performed numerically since the derivation of a fully analytic result remains beyond reach so far in the general mass case.

For the motivation and the general features of the method we refer to the introduction of [1]. However, let us notice an important difference with respect to the real mass case. In the latter, the imaginary part of the ratio of the kinematical $S$ matrix determinant over the Gram matrix one (or the various determinants of the pinched matrices formed from $S$ over their related Gram matrix determinants) was always positive and related to the Feynman prescription coming from the propagators. In the complex mass case, the signs of the imaginary part of these ratios depend on the kinematics and may be positive or negative. Despite this difference, the method, developed in [1], to perform analytical integration over the remaining parameters after the application of the Stokes-like identity can be applied in a systematic way for the various cases with slight adaptations. When expressed in terms of contour integrals, the different cases share a common structure supplemented by logarithmic terms which are case dependent.

This approach, initiated in [7], is based on the implementation of two-loop scalar $N$-point functions in four dimensions $^{(2)}I_N^4$ as double integrals whose integrands are building blocks $^{(1)}T_{N+1}^4(\rho, \xi)$ which are similar to “generalised” one-loop $(N+1)$-point Feynman-type integrals, in the form:

$$^{(2)}I_N^4 \sim \sum \int_0^1 d\rho \int_0^1 d\xi P(\rho, \xi) \; ^{(1)}T_{N+1}^4(\rho, \xi)$$

where $P(\rho, \xi)$ are some weighting functions. The $^{(1)}T_{N+1}^4(\rho, \xi)$ are “generalised” in the sense that the integration domain spanned by the Feynman parameters defining them is no longer the usual simplex $\{0 \leq z_j \leq 1, j = 1, \cdots, N + 1; \sum_{j=1}^{N+1} z_j = 1\}$. The generalised $(N+1)$-point integrals depend on two extra parameters $\rho$ and $\xi$ which span a larger kinematics than in the case of one-loop $(N+1)$-particle processes at colliders. As mentioned in [1] our method can be extended to compute $^{(1)}T_{N+1}^4(\rho, \xi)$ leaving only two numerical integrals, on $\rho$ and $\xi$, when evaluating two-loop $N$-point functions.

The method presented enjoys a couple of interesting features compared to the methods of [8] and [9–12]. Namely, it directly proceeds in terms of the algebraic quantities $\text{det}(S), \text{det}(G)$,
This novel method suffers from the same drawbacks as those mentioned in [1], namely, an inherent increase of the number of dilogarithms compared to the ’t Hooft-Veltman results or the Denner-Dittmaier ones. This point deserves further discussions but there exists ways to reduce this number.

The outline of this article follows closely the one of our preceding article [1]. We start by considering the three-point function $I_3^4$ with complex internal masses considered as a warm-up in sec. 2. After having reminded the necessary notations and definitions, we consider the two variants of the method presented in the real mass case, namely the “direct way” and the “indirect way”. The formulas for these two variants obtained in [1] still holds for the case of complex masses and so their derivations will not be reproduced in this article. Nevertheless, the equivalence between these two ways is more complicated to show and will be discussed in detail. We end this section by commenting on the apparent doubling of dilogarithms, already there in the real mass case. We then apply the “indirect way” to the four-point function with internal masses all complex in sec. 3. It results from this application eight formulas depending on the sign of the imaginary parts of the determinants of the $S$ matrix as well as its pinched ones. Various appendices gather a number of utilities: tools, proofs of steps, etc. we removed them from the main text to facilitate its reading but we consider them useful to supply. Accordingly, appendix A extends the companion appendix D of [1]. The so called “second type” integral is computed for the case where the complex numbers involved have a finite imaginary part. Appendix B is closely related to the appendix E of [1]. It adds to the latter the case where the parameters of the integrand are true complex numbers and also the cases where the integral has different bounds required for the treatment of complex masses. Appendix C widens the discussion started in the appendix F of [1] about the sign of the imaginary part of $\det(S)$ for general complex masses. Lastly, appendix D gives the conditions on the two complex numbers $A$ and $B$ for which one of the cuts of the logarithm $\ln(Az^2 + B)$ crosses the real segment $[0, 1]$ when $z$ spans the complex plane.

2 Warm-up: $I_3^4$

In the previous companion paper [1], we show how to compute the three point function using Stokes-type identity (cf. section 2 of ref. [1]) for real mass case. We want to extend these results for complex masses. To facilitate the reading, we recap the notations and some necessary definitions.

The usual Feynman integral representation of the three-point function in four dimensions $I_3^4$
Figure 1: The triangle picturing the one-loop three-point function.

is:

$$I^4_3 = - \int_0^1 \prod_{i=1}^3 dz_i \delta(1 - \sum_{i=1}^3 z_i) \left( - \frac{1}{2} Z^T \cdot S \cdot Z \right)^{-1}$$

(2.1)

Here $Z$ stands for a column 3-vector whose components are the $z_i$, $S$ is the $3 \times 3$ kinematic matrix associated to the diagram of fig. 1 encoding all the information on the kinematics associated to this diagram by:

$$S_{ij} = (q_i - q_j)^2 - m_i^2 - m_j^2$$

(2.2)

Each internal line with momentum $q_i$ stands for the propagator of a particle of mass $m_i$. Lastly, the superscript $^\text{T}$ stands for the matrix transpose. Note that in eq. (2.1) the infinitesimal prescription $-i \lambda$, there in [1], is overcome by the finite imaginary parts of the complex masses: it is irrelevant and we drop it in this article. Let us single out the subscript value $a$ ($a \in S_3 = \{1, 2, 3\}$) and write $z_a$ as $1 - \sum_{i \neq a} z_i$. We find:

$$- Z^T \cdot S \cdot Z = \sum_{i,j \in S_3 \setminus \{a\}} G_{ij}^{(a)} z_i z_j - 2 \sum_{j \in S_3 \setminus \{a\}} V_j^{(a)} z_j - C^{(a)}$$

(2.3)

where the $2 \times 2$ Gram matrix $G^{(a)}$ and the column 2-vector $V^{(a)}$ are defined by

$$G_{ij}^{(a)} = -(S_{ij} - S_{aj} - S_{ia} + S_{aa}), \quad i, j \neq a$$

$$V_j^{(a)} = S_{aj} - S_{aa}, \quad j \neq a$$

$$C^{(a)} = S_{aa}$$

(2.4)

Labelling $b$ and $c$ the two elements of $S_3 \setminus \{a\}$ with $b < c$, the polynomial (2.3) can be written as:

$$D^{(a)}(z_b, z_c) = X^{(a)T} \cdot G^{(a)} \cdot X^{(a)} - 2 V^{(a)T} \cdot X^{(a)} - C^{(a)}$$

$$X^{(a)} = \begin{bmatrix} z_b \\ z_c \end{bmatrix}$$

(2.5)

In ref. [1], we then applied once the Stokes-type identity presented in the appendix A of this reference to transform an integration over a Feynman parameter into a sum of integrals over
The derivation of this transformation is valid also in the complex mass case and will not be reproduced here, we refer the reader to ref. [1] for more details. At the end of this transformation, we could perform the integration over the half real line and got the result coined “direct way”:

\[
I_3^4 = -\sum_{i \in S_3} \frac{\bar{b}_i}{\det(G)} \int_0^1 \frac{dx}{D^{(i)(l)}(x) + \Delta_2} \left[ \ln \left(D^{(i)(l)}(x)\right) - \ln (-\Delta_2) \right] \tag{2.6}
\]

with \( l \equiv 1 + (i \text{ modulo } 3) \) and where\(^1\)

\[
\Delta_2 = \frac{\det(S)}{\det(G)} \tag{2.7}
\]

\[
\frac{\bar{b}_i}{\det(G)} = \left((G^{(a)})^{-1} \cdot V^{(a)}\right)_i \quad i \neq a \tag{2.8}
\]

\[
\frac{\bar{b}_a}{\det(G)} = 1 - \sum_{j \in S_3 \backslash \{a\}} \left((G^{(a)})^{-1} \cdot V^{(a)}\right)_j \tag{2.9}
\]

The second degree polynomial \( D^{(i)(l)}(x) \) is defined with the one-pinched \( S^{(i)} \) matrix as follows:

\[
D^{(i)(l)}(x) = G^{(i)(l)} x^2 - 2V^{(i)(l)} x - C^{(i)(l)} \tag{2.10}
\]

with

\[
G_{jk}^{(i)(l)} = -\left(S_{jk}^{(i)} - S_{jl}^{(i)} - S_{lk}^{(i)} + S_{ll}^{(i)}\right) \quad j, k \neq i, l \tag{2.11}
\]

\[
V_{j}^{(i)(l)} = S_{lj}^{(i)} - S_{ll}^{(i)} \quad j \neq i, l \tag{2.12}
\]

\[
C^{(i)(l)} = S_{ll}^{(i)} \tag{2.13}
\]

Note that in the case of the three-point function, since the set \( S_3 \) has only three elements, \( j \) has to be equal to \( k \) in eq. (2.11) so the matrix \( G^{(i)(l)} \) is a \( 1 \times 1 \) matrix and the vector \( V^{(i)(l)} \) has only one component, hence the notation used in eq. (2.10).

Unfortunately, in the case of the four-point function we did not succeed in proceeding as simply. We have therefore formulated an alternative to the “direct way”, henceforth coined “indirect”. In this formulation, the Stokes-type identity is applied twice and the three-point function is written as a sum over the coefficients \( \bar{b} \) and \( \bar{b}^{(i)} \) weighted by a two dimensional integral over the first quadrant\(^2\) (see ref. [1] for more details):

\[
I_3^4 = \sum_{i \in S_3} \sum_{j \in S_3 \backslash \{i\}} \frac{\bar{b}_i}{\det(G)} \frac{\bar{b}_j^{(i)}}{\det(G^{(i)})} L_3^4 \left(\Delta_2, \Delta_1^{(i)}, \bar{D}_{ij}\right) \tag{2.14}
\]

\(^1\)The different matrices \( G^{(a)} \) have the same determinant represented by \( \det(G) \).

\(^2\)As for the “direct way”, the derivation presented in [1] still holds for complex mass case.
with
\[
L_3^4 \left( \Delta_2, \Delta_1^{(i)}, \bar{D}_{ij} \right) = \int_0^{+\infty} \frac{d\xi}{(-\Delta_2 + \xi)} \times \int_0^{+\infty} \frac{d\rho}{(-\Delta_1^{(i)} + \xi + \rho^2)} (\bar{D}_{ij} + \xi + \rho^2)^{1/2}
\] (2.15)
and
\[
\frac{\bar{b}_j^{(i)}}{\det (G^{(i)})} = - \left(G^{(i)(l)}\right)^{-1} V^{(i)(l)}, \quad j \neq i, l \tag{2.16}
\]
\[
\frac{\bar{b}_i^{(i)}}{\det (G^{(i)})} = -1 + \left(G^{(i)(l)}\right)^{-1} V^{(i)(l)} \tag{2.17}
\]
\[
- \Delta_1^{(i)} = \frac{\det (S^{(i)})}{\det (G^{(i)})} \tag{2.18}
\]
\[
\bar{D}_{ij} = 2 m_i^2, \quad j \in S_3 \setminus \{i, l\} \tag{2.19}
\]

Some linear combinations of \(\Delta_2, \Delta_1^{(i)}\) and \(\bar{D}_{ij}\) are expressed in terms of the various determinants and \(\bar{b}\) coefficients (cf. the identities (2.42) and (2.43) of ref. [1]):
\[
\bar{D}_{ij} + \Delta_1^{(i)} = \frac{\bar{b}_j^{(i)} b_i^{(i)}}{\det (G^{(i)})} \tag{2.20}
\]
\[
\Delta_2 - \Delta_1^{(i)} = \frac{\bar{b}_i^2}{\det (G) \det (G^{(i)})} \tag{2.21}
\]

To show that the two ways “direct” and “indirect” are equivalent is more tricky in the complex mass case than in the real mass one. Let us discuss this point now. We have to distinguish according to the sign of \(\text{Im}(\Delta_1^{(i)})\) only. Indeed, after having performed the \(\rho\) integration, the \(\xi\) integration is always of the type
\[
\int_0^{+\infty} \frac{d\xi}{(\xi - \Delta_2) (\xi + A)} \tag{2.22}
\]
where \(A\) is a complex number. It has been shown in appendix D of [1] that the result of this integral does not depend on the sign of \(\text{Im}(\Delta_2)\), neither on the sign of \(\text{Im}(A)\). Furthermore, \(\bar{D}_{ij}\) which is equal to twice an internal mass squared has a negative imaginary part.

1) \(\text{Im}(\Delta_1^{(i)}) > 0\)

This case is a straightforward continuation of the real mass case. The result is readily given
by:

\[ L_3^4 \left( \Delta_2, \Delta_1^{(i)}, \tilde{D}_{ij} \right) = \int_0^1 \frac{dz}{(\tilde{D}_{ij} + \Delta_1^{(i)}) z^2 + \Delta_2 - \Delta_1^{(i)}} \left[ \ln \left( (\tilde{D}_{ij} + \Delta_1^{(i)}) z^2 - \Delta_1^{(i)} \right) - \ln (-\Delta_2) \right] \]  

(2.23)

(cf. eq. (2.37) of ref. [1]). Note that the apparent pole in the integrand is fake and the argument of the first logarithm never becomes real negative when \( z \) spans \( [0, 1] \).

2) \( \text{Im}(\Delta_1^{(i)}) < 0 \)

Let us come back to eq. (2.15). Instead of relying on eq. (A.5) of appendix A to get rid of the square-root, we have to use eq. (A.15) of this appendix. We are left with a \( \xi \) integration of the type:

\[ L_3^4 \left( \Delta_2, \Delta_1^{(i)}, \tilde{D}_{ij} \right) = \int_0^{+\infty} \frac{d\xi}{\xi - \Delta_2} \left[ i \int_0^{+\infty} \frac{dz}{\xi - (1 + z^2) \Delta_1^{(i)} - z^2 \tilde{D}_{ij}} \right. \\
- \left. \int_1^{+\infty} \frac{dz}{\xi - (1 - z^2) \Delta_1^{(i)} + z^2 \tilde{D}_{ij}} \right] 
\]

(2.24)

The \( \xi \) integration can be performed first, using eq. (D.4) of appendix D in [1] and we get:

\[ L_3^4 \left( \Delta_2, \Delta_1^{(i)}, \tilde{D}_{ij} \right) = -i \int_0^{+\infty} \frac{dz}{(\tilde{D}_{ij} + \Delta_1^{(i)}) z^2 - \Delta_2 + \Delta_1^{(i)}} \left[ \ln \left( (\tilde{D}_{ij} + \Delta_1^{(i)}) z^2 - \Delta_1^{(i)} \right) - \ln (-\Delta_2) \right] \\
- \int_1^{+\infty} \frac{dz}{(\tilde{D}_{ij} + \Delta_1^{(i)}) z^2 + \Delta_2 - \Delta_1^{(i)}} \left[ \ln \left( (\tilde{D}_{ij} + \Delta_1^{(i)}) z^2 - \Delta_1^{(i)} \right) - \ln (-\Delta_2) \right] 
\]

(2.25)

where each of the two integrals converges at \( \infty \), the apparent pole in the integrand of each term is fake again, and the arguments of logarithms never become real negative along the integration paths of none of the two integrals.

The two cases 1) vs. 2) disentangled above can be reunified by seeing eq. (2.25) as an analytic continuation in \( \Delta_1^{(i)} \) of eq. (2.23) which possibly requires a deformation of the contour \([0, 1]\) originally drawn along the real axis in eq. (2.23). This feature is discussed in appendix A on eqs. (A.4) and (A.14) vs. (A.15). It is interesting to formulate it on eq. (2.25) directly as follows. Let us alternatively rewrite eq. (2.25) as:

\[ L_3^4 \left( \Delta_2, \Delta_1^{(i)}, \tilde{D}_{ij} \right) = \left\{ \int_0^{+i\infty} + \int_{+\infty}^{1} \right\} \frac{dz}{(\tilde{D}_{ij} + \Delta_1^{(i)}) z^2 + \Delta_2 - \Delta_1^{(i)}} \\
\times \left[ \ln \left( (\tilde{D}_{ij} + \Delta_1^{(i)}) z^2 - \Delta_1^{(i)} \right) - \ln (-\Delta_2) \right] 
\]

(2.26)
The logarithm \( \ln((\tilde{D}_{ij} + \Delta_{ij}^{(i)}) z^2 - \Delta_{ij}^{(i)}) \) has two discontinuity cuts supported by one and the other branch of the hyperbola \( \{ \Im[(\tilde{D}_{ij} + \Delta_{ij}^{(i)}) z^2 - \Delta_{ij}^{(i)}] = 0 \} \) in the complex \( z \)-plane respectively. One of the two cuts\(^3\), let us label it \( C_{ij} \), lies in the right half \( z \)-plane \( \{ \Re(z) > 0 \} \). It originates at the point \( z_{ij} = [\Delta_{ij}^{(i)}/(\tilde{D}_{ij} + \Delta_{ij}^{(i)})]^{1/2} \) and slashes the right half plane away to \( \infty \) through the lower right quadrant \( \{ \Re(z) > 0, \Im(z) < 0 \} \). In case \( z_{ij} \) belongs to the upper right quadrant \( \{ \Re(z) > 0, \Im(z) > 0 \} \), this cut runs from \( z_{ij} \) away to \( \infty \) by crossing the real segment \([0, 1]\) at the value \( \Re(z) = [\Im(\Delta_{ij}^{(i)})/\Im(\tilde{D}_{ij} + \Delta_{ij}^{(i)})]^{1/2} \). The integration contour of the r.h.s. of eq. (2.26) can be closed by drawing an arc \((0, 1)_{i,j} \) between 0 and 1, the extra arc at \( \infty \) also involved by the Cauchy theorem to close the contour yields a vanishing contribution \( \sim \mathcal{O}(\ln R/R) \) where \( R \) is \( \|z\| \) on the contour at \( \infty \).

(i) If \( C_{ij} \) entirely belongs to the quadrant \( \{ \Re(z) > 0, \Im(z) < 0 \} \) this extra arc \((0, 1)_{i,j} \) can be taken along the real segment \([0, 1]\).

(ii) However if \( z_{ij} \) belongs to the quadrant \( \{ \Re(z) > 0, \Im(z) > 0 \} \), the extra arc \((0, 1)_{i,j} \) shall wrap the bit of \( C_{ij} \) inside the upper right quadrant from above as if \( C_{ij} \) were locally pushing up \((0, 1) \) away from the real segment \([0, 1]\) inside this quadrant as pictured on figure 2.

In either case, \( L_3^4(\Delta_2, \Delta_1^{(i)}, \tilde{D}_{ij}) \) can be represented also when \( \Im(\Delta_1^{(i)}) < 0 \) by an integral along the contour \((0, 1)_{i,j} \) whether along \([0, 1]\) in case (i) or deformed as described above in case (ii) according to the Cauchy theorem:

\[
L_3^4(\Delta_2, \Delta_1^{(i)}, \tilde{D}_{ij}) = \int_{(0,1)_{i,j}} \frac{dz}{(\tilde{D}_{ij} + \Delta_1^{(i)}) z^2 + \Delta_2 - \Delta_1^{(i)}} \times \left[ \ln \left( (\tilde{D}_{ij} + \Delta_1^{(i)}) z^2 - \Delta_1^{(i)} \right) - \ln (-\Delta_2) \right]
\]

(2.27)

which is the argued analytic continuation in \( \Delta_1^{(i)} \) of eq. (2.23). When the contour deformation is required, the split form (2.25) is more convenient from a computational point of view. However the alternative form (2.27) proves more convenient to extend to the complex mass case the recasting of the expression of \( I_3^4 \) obtained via the indirect way into the one obtained via the direct way.

Putting eq. (2.27) into eq. (2.15) results in a modification of eq. (2.44) of ref. [1] in the following form:

\[
I_3^4 = - \sum_{i \in S_3} \sum_{j \in S_3 \setminus \{i\}} \bar{b}_i b_j^{(i)} \int_{(0,1)_{i,j}} \frac{dz}{\det(G) b_j^{(i)} z^2 + \bar{b}_i} \times \left[ \ln \left( \frac{b_j^{(i)} z^2 + \det(\mathcal{S}^{(i)})}{\det(G^{(i)})} \right) - \ln \left( -\frac{\det(\mathcal{S})}{\det(G)} \right) \right]
\]

(2.28)

\(^3\)The other cut is the symmetric of \( C_{ij} \) under parity: located in the left half plane \( \{ \Re(z) < 0 \} \) it is irrelevant for our concern.
i.e. in eq.(2.28) each integral “from 0 to 1” is now understood in the sense of eq. (2.27) as an integral along a contour \( \widehat{(0,1)}_{i,j} \) specific to each \( i,j \). As we did in the real mass case, for each \( i \), we perform two operations: 1) the change of variable \( s = \bar{b}_j^{(i)} z \) in the integrals corresponding to the two values of \( j \in \mathcal{S}_3 \setminus \{i\} \), so that the integrands become identical in the two integrals; 2) the two integrals are joined end-to-end into a single one integrated along the contour \( \mathcal{L}_{k,l}^{(i)} \equiv -\bar{b}_k^{(i)} (0,1)_{i,k} \cup \bar{b}_l^{(i)} (0,1)_{i,l} \) in the complex \( s \)-plane. We again specify the two elements of \( \mathcal{S}_3 \setminus \{i\} \) to be \( k \equiv 1 + ((i + 1) \mod 3) \) and \( l \equiv 1 + (i \mod 3) \). In the real mass case, these two operations yield the following result:

\[
I_3^4 = \sum_{i \in \mathcal{S}_3} \bar{b}_i \int_{-\bar{b}_k^{(i)}}^{\bar{b}_l^{(i)}} \frac{ds}{s^2 \det (G) + \bar{b}_i^2} \left[ \ln \left( \frac{s^2 + \det (\mathcal{S}^{(i)})}{\det (G^{(i)})} - i \lambda \right) - \ln \left( -\frac{\det (\mathcal{S})}{\det (G)} - i \lambda \right) \right]
\]  

(2.29)

and the following change of variable

\[
s = -\bar{b}_k^{(i)} + \left( \bar{b}_k^{(i)} + \bar{b}_l^{(i)} \right) u = -\bar{b}_k^{(i)} - \det (G^{(i)}) u
\]

(2.30)

leads to the same formula obtained in the case of the “direct way”. For general complex masses the three points \( -\bar{b}_k^{(i)}, 0, \bar{b}_l^{(i)} \) are no longer aligned in general. Furthermore, either of
the paths $-b^{(i)}_k (0,1)_{i,k}$ and $b^{(i)}_l (0,1)_{i,l}$ (or both) may not be straight any more. Let us instead consider the extension of eq. (2.29) to the complex mass case: the integration contour in eq. (2.29) shall still be understood as the straight line stretched from $-b^{(i)}_k$ to $b^{(i)}_l$ and running parallel to the real axis cf. eq. (2.30). In this latter case, these two operations give:

$$I^4_3 = - \sum_{i \in S_3} b_i \int \tau^{(i)}_{k,l} \frac{ds}{s^2 \det (G) + b^2_i} \times \left[ \ln \left( \frac{s^2 + \det (S^{(i)})}{\det (G^{(i)})} \right) - \ln \left( \frac{\det (S)}{\det (G)} \right) \right]$$ (2.31)

Eq. (2.31) is the extension of eq. (2.45) of ref. [1]. Going from eq. (2.31) to the extension of eq. (2.29) which reads:

extension of r.h.s. (2.29)

$$= - \sum_{i \in S_3} b_i \int_{-\tau^{(i)}_{k,l}}^{\tau^{(i)}_{k,l}} \frac{ds}{s^2 \det (G) + b^2_i} \left[ \ln \left( \frac{s^2 + \det (S^{(i)})}{\det (G^{(i)})} \right) - \ln \left( \frac{\det (S)}{\det (G)} \right) \right]$$ (2.32)

thus involves a contour deformation by means of the (possibly distorted) “triangle” in the complex s-plane whose sides are $-b^{(i)}_k (0,1)_{i,k}$ and $b^{(i)}_l (0,1)_{i,l}$ (either or both being possibly non straight) and $[-b^{(i)}_k, b^{(i)}_l]$ (straight), as illustrated on figures 3 and 4.

Figure 3: Example of a contour deformation involving a triangle with one distorted side, for which no cut crosses the straight base $[-b^{(i)}_k, b^{(i)}_l]$.

This raises two issues. The first issue concerns the possible presence of poles in the integrand of eqs. (2.31), (2.32) inside the distorted “triangle”. Yet the poles are fake as their residues vanish by construction: this issue is thus irrelevant. The second issue concerns the respective location of the discontinuity cuts of the logarithm \(\ln[s^2 + \det (S^{(i)})/\det (G^{(i)})]\) w.r.t. the side $[-b^{(i)}_k, b^{(i)}_l]$ of the triangle, namely whether the side $[-b^{(i)}_k, b^{(i)}_l]$ crosses a discontinuity cut of \(\ln[s^2 + \det (S^{(i)})/\det (G^{(i)})]\) as illustrated by fig. 4. By means of the change of variable (2.30) this is equivalent to the issue whereby the real interval $[0,1]$ would cross a discontinuity cut of \(\ln[D^{(i)(i)}(u)]\) in the complex $u$-plane (cf. eq. (2.18) of ref. [1] for the
Figure 4: Hypothetical case of a contour deformation involving a triangle with one distorted side such that the contribution along the distorted side $-\vec{b}_{i,k}^{(i)}(0,1)$ wraps a cut which would cross the straight line $[-\vec{b}_{k}^{(i)}, \vec{b}_{l}^{(i)}]$: this would force one to distort the base $(-\vec{b}_{k}^{(i)}, \vec{b}_{l}^{(i)})$ of the triangle accordingly.

definition of $D^{(i)(l)}(u)$. In this respect we shall note that the imaginary part of $D^{(i)(l)}(u)$ is a convex combination of the form $\text{Im}(m_{k}^{2})u + \text{Im}(m_{l}^{2})(1 - u)$ where the imaginary parts of the masses $m_{k,l}^{2}$ have the same (negative) sign and $u$ spans $[0, 1]$, so that $\text{Im}(D^{(i)(l)}(u))$ keeps a constant (negative) sign on $[0, 1]$, which implies no cut crossing: the case at hand is the one pictured by fig. 3 whereas the case of fig. 4 does not occur. We can therefore safely deform $I_{k,l}^{(i)}$ into the straight line $[-\vec{b}_{k}^{(i)}, \vec{b}_{l}^{(i)}]$, map the latter onto $[0, 1]$ using eq. (2.30) and we finally recover the same expression as obtained according to the direct way with complex masses.

To finish this section let us comment about the proliferation of dilogarithms. To cover the case of general complex masses for the scalar three-point function, the integration contour has to be modified depending on the imaginary part of $\Delta_{1}^{(i)}$ (cf. eqs. (2.23) and (2.25)). But even if the contour is not on the real axis, it can be decomposed on a part along one of the half imaginary axes and another part on the real axis between $1$ and $+\infty$. As shown in appendix B, the contribution along the half imaginary axis gives only logarithms and the one on the real axis between $1$ and $+\infty$ yields the same combination of dilogarithms as an integration between $0$ and $1$ on the real axis (irrespectively of the fact that the cut of the integrand may cross the real axis between these bounds!). This is due to the fact that the integrand is even with respect to the integration variable and so, only the bound $1$ produces dilogarithms. To sum up, whatever the sign of the imaginary part of $\Delta_{1}^{(i)}$ is, the dilogarithms obtained after the last integration are the same to those of the real mass case. So the discussion of subsec. 2.3 of [1] about the number of dilogarithms versus [9] is still valid and leads to the same conclusion.
3 Leg up: $I_4^4$

Let us start this section by recapping the definitions and notations required for the extension to general complex masses. This section complements the section 3 of the companion article [1]. The usual integral representation of $I_4^4$ in terms of Feynman parameters is given by:

$$I_4^4 = \int_0^1 \prod_{i=1}^4 dz_i \delta(1 - \sum_{i=1}^4 z_i) \left(-\frac{1}{2} Z^T \cdot S \cdot Z\right)^{-2}$$  \hspace{1cm} (3.1)

where $Z$ is now a column 4-vector whose components are the $z_i$. Singling out arbitrarily the subscript value $a$ ($a \in S_4 = \{1, 2, 3, 4\}$), and writing $z_a$ as $1 - \sum_{j \neq a} z_j$, we find:

$$-Z^T \cdot S \cdot Z = \sum_{i,j \in S_4 \{a\}} G_{ij}^{(a)} z_i z_j - 2 \sum_{j \in S_4 \{a\}} V_j^{(a)} z_j - C^{(a)}$$  \hspace{1cm} (3.2)

where the $3 \times 3$ Gram matrix $G^{(a)}$ and the column 3-vector $V^{(a)}$ are defined by

$$G_{ij}^{(a)} = -(S_{ij} - S_{aj} - S_{ia} + S_{aa}), \quad i, j \neq a$$

$$V_j^{(a)} = S_{aj} - S_{aa}, \quad j \neq a$$

$$C^{(a)} = S_{aa}$$

Labelling $b$, $c$ and $d$ the three elements of $S_4 \{a\}$ with $b < c < d$, the polynomial (3.2) reads:

$$D^{(a)}(X) = X^T \cdot G^{(a)} \cdot X - 2V^{(a)T} \cdot X - C^{(a)}, \quad X = \begin{bmatrix} z_b \\ z_c \\ z_d \end{bmatrix}$$  \hspace{1cm} (3.4)

Again the dependencies on $G^{(a)}$, $V^{(a)}$ and $C^{(a)}$ will arise through quantities independent of the actual choice of $a$. In ref. [1], we applied three time the Stokes-type identity and traded

---

Footnote:

4Up to the fact that, in the case of general complex masses, the $S$ matrix elements entering into the arguments of the dilogarithms are complex numbers with a non vanishing imaginary part.
the three dimensional Feynman parameter integral over the simplex against a sum of three
dimensional integrals over the first octant of $\mathbb{R}^3$. The four-point function was written as a
sum over the coefficients $\overline{b}$, $\overline{b}^{(i)}$ and $\overline{b}^{(i,j)}$ weighted by a three dimensional integral over the
first octant (see ref. [1] for more details):

$$I_4^4 = \sum_{i \in S_4} \sum_{j \in S_4 \setminus \{i\}} \sum_{k \in S_4 \setminus \{i,j\}} \frac{\overline{b}_i}{\det (G)} \frac{\overline{b}_j^{(i)}}{\det (G^{(i)})} \frac{\overline{b}_k^{(i,j)}}{\det (G^{(i,j)})} L_4^4(\Delta_3, \Delta_2^{(i)}, \Delta_1^{(i,j)}, \vec{D}_{ijk})$$

(3.5)

with

$$L_4^4(\Delta_3, \Delta_2^{(i)}, \Delta_1^{(i,j)}, \vec{D}_{ijk}) = \kappa \int_0^{+\infty} \frac{d\xi}{(\xi^2 - \Delta_3)} \int_0^{+\infty} \frac{d\rho}{(\rho^2 + \xi^2 - \Delta_2^{(i)})} \int_0^{+\infty} \frac{d\sigma}{(\sigma^2 + \rho^2 + \xi^2 - \Delta_1^{(i,j)}) (\sigma^2 + \rho^2 + \xi^2 + \vec{D}_{ijk})^{1/2}}$$

$$\kappa = \frac{3}{16} B(2, 1/2) B(3/2, 1/2) B(1, 1/2)$$

(3.6)

The quantities $\Delta_3$, $\Delta_2^{(i)}$ and $\Delta_1^{(i,j)}$, involved in eq. (3.6), are expressed in terms of the
determinants of the $S$ matrix just as the one-pinched and two-pinched $S$ matrices and the
associated Gram matrices:

$$\Delta_3 = - \frac{\det (S)}{\det (G)}$$

(3.7)

$$\Delta_2^{(i)} = \frac{\det (S^{(i)})}{\det (G^{(i)})}$$

(3.8)

$$\Delta_1^{(i,j)} = - \frac{\det (S^{(i,j)})}{\det (G^{(i,j)})}$$

(3.9)

As for $\vec{D}_{ijk}$, it is proportional to an internal mass squared:

$$\vec{D}_{ijk} = 2 m_i^2 \quad l \in S_4 \setminus \{i, j, k\}$$

(3.10)

The coefficients $\overline{b}$, $\overline{b}^{(i)}$ and $\overline{b}^{(i,j)}$ can be built with respectively the Gram matrix $G^{(a)}$ and
the 3-vector $V^{(a)}$, the one-pinched Gram matrix $G^{(i)}$ and the 2-vector $V^{(i)}$ and the two-pinched Gram matrix$^5$ $G^{(i,j)}$ and the 1-vector $V^{(i,j)}$.$^6$

$$\frac{\overline{b}_j}{\det (G)} = - \left((G^{(a)})^{-1} \cdot V^{(a)}\right)_j, \quad j \in S_4 \setminus \{a\}$$

(3.11)

$$\frac{\overline{b}_a}{\det (G)} = - 1 + \sum_{j \in S_4 \setminus \{a\}} \left((G^{(a)})^{-1} \cdot V^{(a)}\right)_j$$

(3.12)

$^5$“$\det (G^{(i,j)})$” is merely a fancy notation to keep some unity in formulas, as $G^{(i,j)}$ reduces to one single scalar.

$^6$The coefficients $\overline{b}$ do not depend on the subscript which has been used to build the Gram matrix, it is
the same for the coefficients $\overline{b}^{(i)}$ and $\overline{b}^{(i,j)}$, cf. ref. [1].
\[
\frac{b_{ij}^{(i)}}{\det (G^{(i)})} = (G^{(i)(a)})^{-1} \cdot V^{(i)(a)}_j, \quad j \in S_4 \setminus \{i, a\} \quad (3.13)
\]
\[
\frac{\tilde{b}_{ij}^{(i,j)}}{\det (G^{(i,j)})} = 1 - \sum_{j \in S_4 \setminus \{i, a\}} (G^{(i)(a)})^{-1} \cdot V^{(i)(a)}_j \quad (3.14)
\]
\[
\frac{\tilde{b}_{ij}^{(i,j)}}{\det (G^{(i,j)})} = - (G^{(i,j)(a)})^{-1} V^{(i,j)(a)}_l, \quad l \in S_4 \setminus \{i, j, a\} \quad (3.15)
\]
\[
\frac{b_{ij}^{(i,j)}}{\det (G^{(i,j)})} = -1 + (G^{(i,j)(a)})^{-1} V^{(i,j)(a)} \quad (3.16)
\]

To finish the recap, we have introduced for convenience in ref. [1] the following quantities which will be used in the rest of this section:

\[
P_{ijk} = \tilde{D}_{ijk} + \Delta_{i,j}^{(ij)} \quad (3.17)
\]

\[
R_{ij} = \Delta_{2}^{(i)} - \Delta_{1}^{(i,j)} \quad (3.14)
\]

\[
Q_{i} = \Delta_{3} - \Delta_{2}^{(i)} \quad (3.13)
\]

\[
T = - \Delta_{3} \quad (3.15)
\]

3.1 Extension to the general complex mass case

We now extend the above results to the general complex mass case. Coming back to eq. (3.6), \(\Delta_{3}, \Delta_{2}^{(i)}, \Delta_{1}^{(i,j)}\) and \(\tilde{D}_{ijk}\) now assume finite i.e. non vanishing imaginary parts and the infinitesimal parameter \(\lambda\) specifying the Feynman contour prescription becomes irrelevant and can be put equal to zero. Whereas \(\text{Im}(\tilde{D}_{ijk})\) is always \(<0\) we have to distinguish between \(2^3 = 8\) cases according to the signs of \(\text{Im}(\Delta_{3}), \text{Im}(\Delta_{2}^{(i)}), \text{Im}(\Delta_{1}^{(i,j)})\).

1. (a) \(\text{Im}(\Delta_{3}) > 0, \text{Im}(\Delta_{2}^{(i)}) > 0, \text{Im}(\Delta_{1}^{(i,j)}) > 0\)

This case is a trivial extension of the real mass case. The expression of \(L_{4}^{4}(\Delta_{3}, \Delta_{2}^{(i)}, \Delta_{1}^{(i,j)}, \tilde{D}_{ijk})\) is provided by:

\[
L_{4}^{4}(\Delta_{3}, \Delta_{2}^{(i)}, \Delta_{1}^{(i,j)}, \tilde{D}_{ijk}) = - \int_{0}^{1} \frac{du}{u^2 P_{ijk} Q_{i} - R_{ij} T} \quad (3.18)
\]

\[
\times \ln \left( \frac{u^2 P_{ijk} + (R_{ij} + Q_{i} + T)}{u^2 (P_{ijk} + R_{ij} + Q_{i}) + T} \right) - \ln \left( \frac{Q_{i} + T}{u^2 Q_{i} + T} \right)
\]

which is the eq. (3.47) of ref. [1] with \(\lambda\) sets to 0. The result (3.18) is cast in a form such that the contributions of the two logarithms to the residue of the pole \(1/(u^2 P_{ijk} Q_{i} - R_{ij} T)\) cancel each other. This pole is fake, it is an artefact of partial fraction decomposition, cf. eq. (3.46) of ref. [1]. In each logarithm, the imaginary parts of the numerator and of the denominator of the argument have the same sign and this common sign is kept constant.
Logarithms of ratios can all be safely split into differences of logarithms, and the integration contour considered does not cross any discontinuity cut of any of the logarithms, so that eq. (3.18) takes the alternative form:

\[
L^4_4(\Delta_3, \Delta^{(i)}_2, \Delta^{[i,j]}_1, \tilde{D}_{ijk}) = - \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \times \left[ \ln \left( u^2 P_{ijk} + (R_{ij} + Q_i + T) \right) - \ln (Q_i + T) \right. \\
- \ln \left( u^2 (P_{ijk} + R_{ij} + Q_i) + T \right) + \ln \left( u^2 Q_i + T \right) \\
\left. - \ln \left( \frac{(R_{ij} + Q_i)}{Q_i}, Q_i + T \right) - \eta \left( \frac{P_{ijk} + R_{ij}}{P_{ijk}}, \frac{(R_{ij} + Q_i)}{Q_i} \right) \right] \\
\tag{3.19}
\]

On the alternative form (3.19) it is no longer manifest that the residue of the fake pole vanishes. Subtracting and adding the value taken at the pole by the split combination of logarithms leads to:

\[
L^4_4(\Delta_3, \Delta^{(i)}_2, \Delta^{[i,j]}_1, \tilde{D}_{ijk}) = - \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \times \left[ \ln \left( u^2 P_{ijk} + (R_{ij} + Q_i + T) \right) - \ln \left( \frac{(R_{ij} + Q_i)}{Q_i}, Q_i + T \right) \right. \\
- \ln \left( u^2 (P_{ijk} + R_{ij} + Q_i) + T \right) + \ln \left( u^2 Q_i + T \right) \\
+ \ln \left( \frac{(R_{ij} + Q_i)}{Q_i}, Q_i + T \right) - \eta \left( \frac{P_{ijk} + R_{ij}}{P_{ijk}}, \frac{(R_{ij} + Q_i)}{Q_i} \right) \\
\left. + \eta \left( \frac{(R_{ij} + Q_i)}{Q_i}, (Q_i + T) \right) - \eta \left( \frac{T}{P_{ijk}}, \frac{(R_{ij} + Q_i)}{Q_i} \right) \right] \\
\tag{3.20}
\]

Whereas the first three lines now manifestly vanish at the pole, the presence of the two extra \( \eta \) functions\(^7\) in the last line of eq. (3.20) might suggest that the pole residue no longer vanish. This paradox is solved as one realises that the splitting of the logarithms of ratios into differences of logarithms holds on the interval of integration but does not hold in general in the vicinity of the pole when the latter is remote from the integration contour. The splitting shall in general be supplemented by \( u \)-dependent \( \eta \) functions. These \( \eta \) functions vanish on the integration contour thus are not explicitly written in eq. (3.20). Yet these \( \eta \) functions take in general non vanishing values at the pole and these values combine into minus the last line of eq. (3.20). Let us note however that 1) if the pole happens to be close enough to - or even on - the segment \([0, 1]\), the last line of eq. (3.20) does vanish and the pole residue is manifestly zero indeed 2) if otherwise the pole is remote from the segment \([0, 1]\) the issue of subtraction of pole residue is irrelevant insofar as the fake pole generates no numerical instability whatsoever.

\(^7\)The \( \eta \) function is defined by eq. (E.6) in appendix B of [1]
For the seven other cases we follow the same strategy for step 4 as in the real mass case (cf. subsec. 3.4 of [1]). Two slight complications arise, though. One is induced when the variant (A.15) instead of (A.5) is at work for the integral (A.1) for \( \nu = 2 \), which now involves two integrals both ranging to \( \infty \), instead of one on \([0, 1]\) only. At substep 4a. of [1], when recasting the integral representation of \( M_1(\xi^2 + \rho^2) \) the extension itemises into 2 cases, depending on the sign of \( \text{Im}(\Delta_{ij}^1) > 0 \) w.r.t. the negative sign of \( \text{Im}(\widetilde{D}_{ijk}) \). Then at substep 4d. of [1] the extension itemises into \( 2 \times 2 = 4 \) subcases, depending of the relative signs of \( \text{Im}(\Delta_3) > 0 \) vs. \( \text{Im}(\Delta_3^i) \), and of \( \text{Im}(\Delta_3) > 0 \) vs. \( \text{Im}(\Delta_1^{ij}) \). The process of extension thus goes as follows:

1. if \( \text{Im}(\Delta_1^{ij}) > 0 \), at substep 4a. of [1] identity (A.5) is applied to the term involving \( \{\Delta_1^{ij}, \widetilde{D}_{ijk}\} \) which yields one term. Then at substep 4d. of [1],
   (a) if \( \text{Im}(\Delta_3) > 0 \) and \( \text{Im}(\Delta_2^{ij}) > 0 \) then
       identity (A.5) is applied to the term involving \( \{\Delta_3, \Delta_2^{ij}\} \)
       identity (A.5) is applied to the term involving \( \{\Delta_3, \Delta_1^{ij}\} \)
   (b) if \( \text{Im}(\Delta_3) > 0 \) and \( \text{Im}(\Delta_2^{ij}) < 0 \) then
       identity (A.15) is applied to the term involving \( \{\Delta_3, \Delta_2^{ij}\} \)
       identity (A.5) is applied to the term involving \( \{\Delta_3, \Delta_1^{ij}\} \)
   (c) if \( \text{Im}(\Delta_3) < 0 \) and \( \text{Im}(\Delta_2^{ij}) > 0 \) then
       identity (A.15) is applied to the term involving \( \{\Delta_3, \Delta_2^{ij}\} \)
       identity (A.15) is applied to the term involving \( \{\Delta_3, \Delta_1^{ij}\} \)
   (d) if \( \text{Im}(\Delta_3) < 0 \) and \( \text{Im}(\Delta_2^{ij}) < 0 \) then
       identity (A.5) is applied to the term involving \( \{\Delta_3, \Delta_2^{ij}\} \)
       identity (A.15) is applied to the term involving \( \{\Delta_3, \Delta_1^{ij}\} \);

2. else \( \text{Im}(\Delta_1^{ij}) < 0 \), thus at substep 4a. of [1] identity (A.15) is applied to the term involving \( \{\Delta_1^{ij}, \widetilde{D}_{ijk}\} \), which yields two terms in which the \( \sigma \) integration has been traded for a \( z \) integration ranging for one term from \( 0 \) to \( \infty \) and for the other between \( 1 \) to \( \infty \). Then at substep 4d. of [1], for the term having a \( z \) integration range between \( 1 \) to \( \infty \) the different splittings are the same as for the case \( \text{Im}(\Delta_1^{ij}) < 0 \), while for the other term
   (a) if \( \text{Im}(\Delta_3) > 0 \) and \( \text{Im}(\Delta_2^{ij}) > 0 \) then
       identity (A.5) is applied to the terms involving \( \{\Delta_3, \Delta_2^{ij}\} \)
       identity (A.15) is applied to the terms involving \( \{\Delta_3, \Delta_1^{ij}\} \)
   (b) if \( \text{Im}(\Delta_3) > 0 \) and \( \text{Im}(\Delta_2^{ij}) < 0 \) then
       identity (A.15) is applied to the terms involving \( \{\Delta_3, \Delta_2^{ij}\} \)
       identity (A.15) is applied to the terms involving \( \{\Delta_3, \Delta_1^{ij}\} \)
(c) if \( \text{Im}(\Delta_3) < 0 \) and \( \text{Im}(\Delta_2^{(i)}) > 0 \) then
identity (A.15) is applied to the terms involving \( \{\Delta_3, \Delta_2^{(i)}\} \)
identity (A.5) is applied to the terms involving \( \{\Delta_3, \Delta_1^{(i,j)}\} \)

(d) if \( \text{Im}(\Delta_3) < 0 \) and \( \text{Im}(\Delta_2^{(i)}) < 0 \) then
identity (A.5) is applied to the terms involving \( \{\Delta_3, \Delta_2^{(i)}\} \)
identity (A.5) is applied to the terms involving \( \{\Delta_3, \Delta_1^{(i,j)}\} \).

Another complication comes from the exchange of the orders of integrations over \( y \) and \( u \)
while going through the counterparts of eqs. (3.44) to (3.45) of ref. [1], whenever either of two
integrations (or both) is (are) not performed between 0 and 1 any more. A splitting into two
or more integrals may then be required. These two sources of complications thereby generate
both a proliferation and a diversification of integral contributions, resulting into as many final
forms as there are cases faced. Notwithstanding, further simplifications and rearrangements
lead to a somewhat common pattern, as will be described below. These complications let
aside, the extension of the derivation can be worked through without trouble and we quote
the results for each case, presented in the order in which they are met during the extension
process. As observed once the calculations have been done, the \( 2^3 \) cases all involve the same
three logarithms of second degree polynomials \( \ln(u^2 P_{ijk} + (R_{ij} + Q_i + T)) \), \( \ln(u^2 Q_i + T) \)
and \( \ln(u^2 (P_{ijk} + R_{ij} + Q_i) + T) \) integrated along contours stretched from 0 to 1, though
not necessarily along the real axis. Some of these contours may have to be deformed so as
to partly wrap cuts of the logarithms considered whenever some cut emerging from some
branch point at finite distance from the origin slashes across the real interval \([0, 1]\).

1.(b) \( \text{Im}(\Delta_3) > 0 \), \( \text{Im}(\Delta_2^{(i)}) < 0 \), \( \text{Im}(\Delta_1^{(i,j)}) > 0 \)

\[
L_1^{(i)}(\Delta_3, \Delta_2^{(i)}, \Delta_1^{(i,j)}, \bar{D}_{ijk})
= \int_0^\infty \frac{du}{u^2 P_{ijk} Q_i + R_{ij} T} \left[ \ln \left( \frac{-Q_i}{-u^2 Q_i + T} \right) - \ln \left( \frac{R_{ij}}{u^2 (P_{ijk} + R_{ij})} \right) \right] \]
+ \int_1^\infty \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( \frac{R_{ij}}{u^2 (P_{ijk} + R_{ij})} \right) - \ln \left( \frac{Q_i}{u^2 Q_i + T} \right) \right]
+ \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( \frac{u^2 P_{ijk} + R_{ij}}{u^2 (P_{ijk} + R_{ij})} \right) - \ln \left( \frac{u^2 P_{ijk} + (R_{ij} + Q_i + T)}{u^2 (P_{ijk} + R_{ij} + Q_i + T)} \right) \right]
+ \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( \frac{R_{ij}}{u^2 P_{ijk} + R_{ij}} \right) - \ln \left( \frac{Q_i}{Q_i + T} \right) \right]
\] (3.21)

In this case, we have \( \text{Im}(R_{ij}) < 0 \), \( \text{Im}(P_{ijk} + R_{ij}) < 0 \), \( \text{Im}(Q_i + T) > 0 \) and \( \text{Im}(Q_i) > 0 \).
Furthermore,

- \( u^2 Q_i - T = (1 + u^2) \Delta_3 - u^2 \Delta_2^{(i)} \)
  thus \( \text{Im}(u^2 Q_i - T) > 0 \) when \( u \in [0, \infty[ \),
\[ u^2 Q_i + T = (u^2 - 1) \Delta_3 - u^2 \Delta_2^{(i)} \]
thus \( \text{Im}(u^2 Q_i + T) > 0 \) when \( u \in [1, \infty[ \),

\[ u^2 P_{ijk} + R_{ij} = u^2 \tilde{D}_{ijk} - (1 - u^2) \Delta_1^{(i,j)} + \Delta_2^{(i)} \]
thus \( \text{Im}(u^2 P_{ijk} + R_{ij}) < 0 \) when \( u \in [0, 1] \),

\[ u^2 (P_{ijk} + R_{ij} + Q_i) + T = u^2 \tilde{D}_{ijk} - (1 - u^2) \Delta_3 \]
thus \( \text{Im}(u^2 (P_{ijk} + R_{ij} + Q_i) + T) < 0 \) when \( u \in [0, 1] \).

The result (3.21) is cast in a form such that, in each of the four integrals separately, the contributions of the logarithms to the residues of the fake pole \( 1/(u^2 P_{ijk} Q_i \pm R_{ij} T) \) cancel each other. This manifest and separate cancellation of residues is favoured at the expense of the economy of terms. Partial recombinations of integrals allow cancellations which reduce the number of terms. Let us showcase how the simplifications and rearrangements proceed on the case at hand. Similar handlings hold for the other cases listed further on, we will then only give the alternative form which they lead to in every other case.

In every logarithm in eq. (3.21), the imaginary parts of the numerator and of the denominator of the argument have the same sign which is kept constant over the integration interval considered. Logarithms of ratios can thus all be safely split into differences of logarithms in each integral, and in each integral the integration contour considered never crosses any discontinuity cut of any of the logarithms.

i) A first simplification occurs as the \( \ln(u^2 P_{ijk} + R_{ij}) \) terms cancel out among the last two integrals on \([0, 1]\) in eq. (3.21).

ii) The terms \( \ln(-u^2 Q_i + T) \) in the first integral and \( \ln(u^2 Q_i + T) \) in the second integral in eq. (3.21) can be combined into a single contour integral in the “south-east” quadrant \( \{ \text{Re}(u) > 0, \text{Im}(u) < 0 \} \) as follows. As detailed in appendix D, the cut of \( \ln(u^2 Q_i + T) \) in the right half plane \( \{ \text{Re}(u) > 0 \} \) emerges from \( \sqrt{-T/Q_i} \) and runs towards \( \infty \) across the “north-east” quadrant \( \{ \text{Re}(u) > 0, \text{Im}(u) > 0 \} \). On the other hand this cut does not extend towards \( \infty \) in the “south-east” quadrant. We can therefore make the change of variable \( v = -i u \) and rewrite

\[
\int_0^\infty \frac{du}{u^2 P_{ijk} Q_i + R_{ij} T} \ln(-u^2 Q_i + T) = \int_0^{-i\infty} \frac{dv}{u^2 P_{ijk} Q_i - R_{ij} T} \ln(v^2 Q_i + T)
\]

and concatenate\(^8\) the latter with minus the integral of the same integrand on \([1, +\infty[\) as the

---

\(^8\)A contribution “at \( \infty \)”, vanishing as \( O(R^{-1} \ln(R)) \) when the radius \( R \) of the added contour \( \rightarrow \infty \), is added.
single contour integral
\[
\left\{ \int_0^{-i\infty} + \int_{+\infty}^1 \right\} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \ln(u^2 Q_i + T)
\]
\[
= \int_{i\infty}^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \ln(u^2 Q_i + T) \tag{3.22}
\]

iii) In eq. (3.22) we then subtract and add to \(\ln(u^2 Q_i + T)\) its value at the pole \(\ln((P_{ijk} + R_{ij}) T/P_{ijk})\), so as to deform the integral
\[
\int_{i\infty}^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln(u^2 Q_i + T) - \ln\left( T \frac{(P_{ijk} + R_{ij})}{P_{ijk}} \right) \right] \tag{3.23}
\]
into an integral along a finite contour \((0,1)\) stretched from 0 to 1. The logarithm \(\ln(u^2 Q_i + T)\) has a cut in the half complex plane \(\{ \text{Re}(u) > 0 \}\) which extends towards infinity only through the “north-east” quadrant. Yet the branch point \(\sqrt{-T/Q_i}\), which the cut emerges from, may be located inside the “south-east” quadrant, so that the cut runs outside this quadrant crossing the segment \([0,1]\) to further slash the “north-east” quadrant. Whenever this occurs, \((0,1)\) shall differ from the straight line \([0,1]\). It shall instead wrap the arc of cut stretched between the branch point \(\sqrt{-T/Q_i}\) and the real axis, from below inside the “south-east” quadrant.

The left-over contribution of the forced counterterm \(\ln((P_{ijk} + R_{ij}) T/P_{ijk})\) can be rewritten:
\[
\int_{i\infty}^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \ln\left( T \frac{(P_{ijk} + R_{ij})}{P_{ijk}} \right) = \left\{ \int_{i\infty}^1 + \int_{0}^1 \right\} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \ln\left( T \frac{(P_{ijk} + R_{ij})}{P_{ijk}} \right) \tag{3.24}
\]
where the closed contour \(\Gamma_-\) encircles the “south-east” quadrant counterclockwise. It is also at work in step iv) next.

iv) We group contributions involving \(\ln(u^2 (P_{ijk} + R_{ij}))\) together with constant terms \(\ln(Q_i) - \ln(R_{ij})\) in eq. (3.21) in a similar way. Contributions from integrals on \([0,1]\) and \([1, +\infty]\) can be combined and cast in the form
\[
\int_0^{+\infty} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln(Q_i) - \ln(R_{ij}) + \ln(P_{ijk} + R_{ij}) + \ln(u^2) \right] \tag{3.25}
\]
whereas, with the change of variable \(v = -i u\), the left over contribution of the first integral of eq. (3.21) reads:
\[
\int_{-i\infty}^{0} \frac{dv}{v^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln(-Q_i) - \ln(R_{ij}) + \ln(-v^2 (P_{ijk} + R_{ij})) \right] \tag{3.26}
\]
We make use of the identity $\ln(z) = \ln(-z) + i\pi\text{sign}(\text{Im}(z))$ to write $\ln(-Q_i) = \ln(Q_i) - i\pi$, and, intending to combine eqs. (3.25) and (3.26) into a single integral on a closed contour encircling the “south-east” quadrant, we consider that in eq. (3.26) $v$ has an infinitesimal positive real part, so that $-v^2$ has an infinitesimal positive imaginary part. We can thus split $\ln(-v^2(P_{ijk} + R_{ij}))$ into $\ln(-v^2) + \ln(P_{ijk} + R_{ij})$ with $\ln(-v^2) = \ln(v^2) + i\pi$. As anticipated the contributions (3.25) and (3.26) are then combined into a single integral on the closed contour $\Gamma^-$ encircling the “south-east” quadrant counterclockwise$^9$:

$$\text{(3.25) + (3.26)} = \int_{\Gamma^-} \frac{du}{u^2P_{ijk}Q_i - R_{ij}T} \left[\ln(Q_i) - \ln(R_{ij}) + \ln(P_{ijk} + R_{ij}) + \ln(u^2)\right]$$

The term $\ln(u^2)$ in eq. (3.27) is then replaced by its residue value at the pole $\ln(R_{ij}T/(P_{ijk}Q_i))$.

v) In the contribution

$$\int_0^1 \frac{du}{u^2P_{ijk}Q_i - R_{ij}T} \left[\ln \left(u^2 P_{ijk} + (R_{ij} + Q_i + T)\right) - \ln \left(u^2(P_{ijk} + R_{ij} + Q_i + T)\right)\right]$$

we subtract and add the pole residue contribution so as to recast eq. (3.28) in the form:

$$\text{(3.28)} = \int_0^1 \frac{du}{u^2P_{ijk}Q_i - R_{ij}T} \left\{\ln \left(u^2 P_{ijk} + (R_{ij} + Q_i + T)\right) - \ln \left(\frac{Q_i + R_{ij}}{Q_i} (Q_i + T)\right)\right.\right.$$

$$- \ln \left(u^2 (P_{ijk} + R_{ij} + Q_i + T)\right) + \ln \left(\frac{T}{P_{ijk}} \frac{(P_{ijk} + R_{ij}) (Q_i + R_{ij})}{Q_i}\right)$$

$$+ \left[\ln \left(\frac{Q_i + R_{ij}}{Q_i} (Q_i + T)\right) - \ln \left(\frac{T}{P_{ijk}} \frac{(P_{ijk} + R_{ij}) (Q_i + R_{ij})}{Q_i}\right)\right]\left\} \right.$$

$^9$The counterclockwise orientation of the contour $\Gamma^-$ encircling the “south-east” quadrant is somewhat unusual. It is inherited from the construction of $\Gamma^-$ as the concatenation of the oriented contours $(0, -i\infty)$, $(+\infty, 1)$ and $(1, 0)$. Similarly, the contour $\Gamma^+$ encircling the “north-east” quadrant clockwise, constructed as the concatenation of the oriented contours $(0, +i\infty)$, $(+\infty, 1)$ and $(1, 0)$ is also used in subsequent cases. Yet this is all matter of presentation and readers preferring to handle contours with their favourite orientations can obviously modify the corresponding formulas by appropriate sign flips.
Putting steps i) to v) together, \( L_4^i(\Delta_3, \Delta_2^{(i)}, \Delta_1^{(i,j)}, \widetilde{D}_{ijk}) \) reads:

\[
L_4^i(\Delta_3, \Delta_2^{(i)}, \Delta_1^{(i,j)}, \widetilde{D}_{ijk}) = - \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( u^2 P_{ijk} + (R_{ij} + Q_i + T) \right) - \ln \left( \frac{Q_i + R_{ij}}{Q_i} (Q_i + T) \right) - \ln \left( u^2 (P_{ijk} + R_{ij} + Q_i + T) + \ln \left( \frac{T (P_{ijk} + R_{ij}) (Q_i + R_{ij})}{P_{ijk} Q_i} \right) \right) + \ln \left( \frac{(Q_i + R_{ij}) (Q_i + T)}{Q_i} \right) - \ln (Q_i + T) + \ln \left( \frac{T (P_{ijk} + R_{ij})}{P_{ijk} Q_i} \right) \right] + \int_{(0,1)} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( u^2 Q_i + T \right) - \ln \left( \frac{T (P_{ijk} + R_{ij})}{P_{ijk}} \right) \right] - \int_{\Gamma} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln (Q_i) - \ln (R_{ij}) + \ln (P_{ijk} + R_{ij}) + \ln \left( \frac{R_{ij} T}{P_{ijk} Q_i} \right) - \ln \left( \frac{T (P_{ijk} + R_{ij})}{P_{ijk}} \right) \right] \right) \]

(3.29)

vi) The combination of constant logarithms in the integral on \([0,1]\) in eq. (3.29) is the same as the one involved in case 1. (a), it therefore takes the same expression in terms of \( \eta \) functions as in the last line of eq. (3.20):

\[
\text{constant logs} = \eta \left( \frac{(Q_i + R_{ij})}{Q_i}, (Q_i + T) \right) - \eta \left( \frac{T (P_{ijk} + R_{ij})}{P_{ijk}}, \frac{(Q_i + R_{ij})}{Q_i} \right)
\]

vii) To put the combination of constant logarithms involved in the last integral in eq. (3.29) in a more compact form we split the logarithms as:

\[
\ln \left( \frac{R_{ij} T}{P_{ijk} Q_i} \right) = \ln \left( \frac{T}{P_{ijk}} \right) + \ln \left( \frac{R_{ij}}{Q_i} \right) + \eta \left( \frac{T}{P_{ijk}}, \frac{R_{ij}}{Q_i} \right)
\]

\[
\ln \left( \frac{R_{ij}}{Q_i} \right) = \ln (R_{ij}) - \ln (Q_i) + \eta \left( R_{ij}, \frac{1}{Q_i} \right)
\]

\[
\ln \left( \frac{T (P_{ijk} + R_{ij})}{P_{ijk}} \right) = \ln \left( \frac{T}{P_{ijk}} \right) + \ln (P_{ijk} + R_{ij}) + \eta \left( \frac{T}{P_{ijk}}, (P_{ijk} + R_{ij}) \right)
\]
so that the combination of five logarithms in the last integral combines into three $\eta$ functions:

$$
\ln \left( Q_i \right) - \ln \left( R_{ij} \right) + \ln \left( P_{ijk} + R_{ij} \right) + \ln \left( \frac{R_{ij} T}{P_{ijk} Q_i} \right) - \ln \left( \frac{T \left( P_{ijk} + R_{ij} \right)}{P_{ijk}} \right) = \eta \left( \frac{T}{P_{ijk}}, \frac{R_{ij}}{Q_i} \right) + \eta \left( R_{ij}, \frac{1}{Q_i} \right) - \eta \left( \frac{T}{P_{ijk}}, \left( P_{ijk} + R_{ij} \right) \right) 
$$

(3.30)

$L_4^i(\Delta_3, \Delta_2^{[i]}, \Delta_1^{[i,j]}, \tilde{D}_{ijk})$ finally reads:

$$
L_4^i(\Delta_3, \Delta_2^{[i]}, \Delta_1^{[i,j]}, \tilde{D}_{ijk}) = - \left\{ \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( u^2 P_{ijk} + \left( R_{ij} + Q_i + T \right) \right) - \ln \left( \frac{Q_i + R_{ij}}{Q_i} \left( Q_i + T \right) \right) \right. \\
- \ln \left( u^2 \left( P_{ijk} + R_{ij} + Q_i \right) + T \right) + \ln \left( \frac{T \left( P_{ijk} + R_{ij} \right)}{P_{ijk}} \frac{Q_i + R_{ij}}{Q_i} \right) \\
+ \eta \left( \frac{Q_i + R_{ij}}{Q_i}, (Q_i + T) \right) - \eta \left( \frac{T}{P_{ijk}}, \frac{Q_i + R_{ij}}{Q_i} \right) \right\} \\
+ \int_{(0,1)} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( u^2 Q_i + T \right) - \ln \left( \frac{T \left( P_{ijk} + R_{ij} \right)}{P_{ijk}} \right) \right] \\
- \int_{\Gamma_-} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \eta \left( \frac{T}{P_{ijk}}, \frac{R_{ij}}{Q_i} \right) + \eta \left( R_{ij}, \frac{1}{Q_i} \right) - \eta \left( \frac{T}{P_{ijk}}, \left( P_{ijk} + R_{ij} \right) \right) \right] \right\} 
$$

(3.31)

We thereby get an expression reminiscent of eq. (3.20) of case 1.(a), albeit modified in two ways. Firstly, the integral involving $\ln(u^2 Q_i + T)$ is performed along a contour $(0,1)$ stretched from 0 to 1 which however may differ from $[0, 1]$. The cut of $\ln(u^2 Q_i + T)$ indeed runs towards $\infty$ inside the "north-east" quadrant. Yet the branch point $\sqrt{-T/Q_i}$ which this cut emerges from may lie inside the "south-east" quadrant, in which case $(0, 1)$ shall wrap the branch point and arc of cut, from below inside this quadrant. Whereas the contents in terms of dilogarithms is unchanged, extra logarithmic contributions are generated along the wrapped cut; this feature is readily observed on $K_{1,\infty}^C$ functions in appendix B. Besides, the integral on $\Gamma_-$ provides an extra residue contribution involving a combination of $\eta$ functions. This contribution is non vanishing only if the pole $u = \sqrt{R_{ij} T/(P_{ijk} Q_i)}$ lies in the "south-east" quadrant i.e. if $\text{Im}(R_{ij} T/(P_{ijk} Q_i)) < 0$. 

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1.(c) \( \text{Im}(\Delta_3) < 0, \text{Im}(\Delta_2^{(i)}) > 0, \text{Im}(\Delta_1^{(i,j)}) > 0 \)

\[
L_4^4(\Delta_3, \Delta_2^{(i)}, \Delta_1^{(i,j)}, \tilde{D}_{ijk}) = \left\{ i \int_0^{+\infty} \frac{du}{u^2 P_{ijk} Q_i + R_{ij} T} \left[ \ln \left( \frac{R_{ij} + Q_i}{u^2 (P_{ijk} + R_{ij} + Q_i) - T} \right) - \ln \left( \frac{Q_i}{u^2 Q_i - T} \right) \right] \\
+ \int_1^{+\infty} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( \frac{R_{ij} + Q_i}{u^2 (P_{ijk} + R_{ij} + Q_i) + T} \right) - \ln \left( \frac{Q_i}{u^2 Q_i + T} \right) \right] \\
+ \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( \frac{R_{ij} + Q_i}{u^2 P_{ijk} + (R_{ij} + Q_i) + T} \right) - \ln \left( \frac{Q_i}{Q_i + T} \right) \right] \right\} 
\]

(3.32)

In this case, we have \( \text{Im}(R_{ij} + Q_i) < 0, \text{Im}(Q_i + T) < 0 \) and \( \text{Im}(Q_i) < 0 \). Furthermore,

- \( u^2 (P_{ijk} + R_{ij} + Q_i) - T = \tilde{D}_{ijk} u^2 + \Delta_3 (1 + u^2) \)
  thus \( \text{Im}(u^2 (P_{ijk} + R_{ij} + Q_i) - T) < 0 \) when \( u \in [0, +\infty[ \),

- \( u^2 Q_i - T = (u^2 + 1) \Delta_3 - u^2 \Delta_2^i \)
  thus \( \text{Im}(u^2 Q_i + T) < 0 \) when \( u \in [0, +\infty[ \),

- \( u^2 (P_{ijk} + R_{ij} + Q_i) + T = u^2 \tilde{D}_{ijk} + (u^2 - 1) \Delta_3 \)
  thus \( \text{Im}(u^2 (P_{ijk} + R_{ij} + Q_i) + T) < 0 \) when \( u \in [1, +\infty[ \),

- \( u^2 Q_i + T = (u^2 - 1) \Delta_3 - u^2 \Delta_2^i \)
  thus \( \text{Im}(u^2 Q_i + T) < 0 \) when \( u \in [1, +\infty[ \),

- \( u^2 P_{ijk} + R_{ij} + Q_i + T = u^2 \tilde{D}_{ijk} - (1 - u^2) \Delta_1^{(i,j)} \)
  thus \( \text{Im}(u^2 P_{ijk} + R_{ij} + Q_i + T) < 0 \) when \( u \in [0, 1] \).

Similar comments as in case 1.(b) hold regarding explicitly vanishing residues in each of the integrals, and further similar simplifications can be carried through exploiting the splittings of the logarithms and recombinations of integrals. We do not elaborate on their derivation.
again, we only quote the result and comment it:

\[ L_4^4(\Delta_3, \Delta_{i}^{[t]}, \Delta_{i,j}^{[t]}, \bar{D}_{ijk}) = - \left\{ \int_{(0,1)_1^+} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ - \ln \left( u^2 (P_{ijk} + R_{ij} + Q_i) + T \right) + \ln \left( T \frac{(P_{ijk} + R_{ij}) (R_{ij} + Q_i)}{P_{ijk} Q_i} \right) \right] \right. \]

\[ + \int_{(0,1)_2^+} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( u^2 Q_i + T \right) - \ln \left( T \frac{P_{ijk} + R_{ij}}{P_{ijk}} \right) \right] \]

\[ + \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( u^2 P_{ijk} + (R_{ij} + Q_i + T) \right) - \ln \left( \frac{R_{ij} + Q_i}{Q_i} \right) \left( Q_i + T \right) \right] - \eta \left( \frac{(R_{ij} + Q_i)}{Q_i}, \left( Q_i + T \right) \right) - \eta \left( T \frac{P_{ijk} + R_{ij}}{P_{ijk}}, \frac{(R_{ij} + Q_i)}{Q_i} \right) \]

\[ - \int_{\Gamma^+} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \eta \left( T \frac{(P_{ijk} + R_{ij})}{P_{ijk}}, \frac{(R_{ij} + Q_i)}{Q_i} \right) \}

\text{(3.33)}

Eq. (3.33) has a structure very similar to eq. (3.31). The respective cuts of \( \ln(u^2 Q_i + T) \) and \( \ln(u^2 (P_{ijk} + R_{ij} + Q_i) + T) \) both run towards \( \infty \) inside the “south-east” quadrant. Yet either or both branch points which each of these cuts emerge from may lie inside the “north-east” quadrant. Accordingly the contours \((0,1)_{1,2}^+\) on which the first two integrals are performed shall be deformations of \([0,1]\) so as to wrap the corresponding branch point and arc of cut from above inside the “north-east” quadrant. The two contours \((0,1)_{1,2}^+\) stretched from 0 to 1 may be chosen distinct from each other so as to best fit the respective cuts. The combination of two constant \( \eta \) terms in the integral on \([0,1]\) is the same as the one in eqs. (3.20) and (3.31). Lastly, and similarly to eq. (3.31) there is an extra “residue” contribution given by the integral of the pole factor on the closed contour \( \Gamma^+ \) encircling the “north-east” quadrant clockwise, weighted by a constant \( \eta \) term specific to the sign case 1.(c) at hand. The integral is non vanishing only if the pole \( \sqrt{R_{ij} T/(P_{ijk} Q_i)} \) lies inside the “north-east” quadrant.
1.(d) \( \text{Im}(\Delta_3) < 0, \text{Im}(\Delta_2^{(i)}) < 0, \text{Im}(\Delta_1^{(i,j)}) > 0 \)

\[
L_4^4(\Delta_3, \Delta_2^{(i)}, \Delta_1^{(i,j)}, \tilde{D}_{ijk}) = - \left\{ i \int_0^{+\infty} \frac{du}{u^2 P_{ijk} Q_i + R_{ij} T} \left[ \ln \left( \frac{R_{ij}}{u^2 (P_{ijk} + R_{ij})} \right) - \ln \left( \frac{R_{ij} + Q_i}{u^2 (P_{ijk} + R_{ij} + Q_i) - T} \right) \right] \right. \\
+ \int_1^{+\infty} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( \frac{R_{ij}}{u^2 (P_{ijk} + R_{ij})} \right) - \ln \left( \frac{R_{ij} + Q_i}{u^2 (P_{ijk} + R_{ij} + Q_i) + T} \right) \right] \\
+ \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( \frac{R_{ij}}{u^2 P_{ijk} + R_{ij}} \right) - \ln \left( \frac{R_{ij} + Q_i}{u^2 P_{ijk} + R_{ij} + Q_i + T} \right) \right] \\
+ \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( \frac{u^2 P_{ijk} + R_{ij}}{u^2 Q_i + T} \right) \right] \left\} \right.
\]

(3.34)

In this case we have \( \text{Im}(R_{ij}) < 0, \text{Im}(P_{ijk} + R_{ij}) < 0, \text{Im}(R_{ij} + Q_i) < 0, \text{Im}(Q_i + T) > 0 \). Furthermore,

- \( u^2 (P_{ijk} + R_{ij} + Q_i) - T = u^2 \tilde{D}_{ijk} + (1 + u^2) \Delta_3 \) 
  thus \( \text{Im}(u^2 (P_{ijk} + R_{ij} + Q_i) - T) < 0 \) when \( u \in [0, +\infty[ \),

- \( u^2 (P_{ijk} + R_{ij} + Q_i) + T = u^2 \tilde{D}_{ijk} + (u^2 - 1) \Delta_3 \) 
  thus \( \text{Im}(u^2 (P_{ijk} + R_{ij} + Q_i) + T) < 0 \) when \( u \in [1, +\infty[ \),

- \( u^2 P_{ijk} + R_{ij} = u^2 \tilde{D}_{ijk} - (1 - u^2) \Delta_1^{(i,j)} + \Delta_2^{(i)} \) 
  thus \( \text{Im}(u^2 P_{ijk} + R_{ij}) < 0 \) when \( u \in [0, 1[ \),

- \( u^2 Q_i + T = -(1 - u^2) \Delta_3 - u^2 \Delta_2^{(i)} \) 
  thus \( \text{Im}(u^2 Q_i + T) > 0 \) when \( u \in [0, 1[ \),

- \( u^2 P_{ijk} + R_{ij} + Q_i + T = u^2 \tilde{D}_{ijk} - (1 - u^2) \Delta_1^{(i,j)} \) 
  thus \( \text{Im}(u^2 P_{ijk} + R_{ij} + Q_i + T) < 0 \) when \( u \in [0, 1[ \).
The use of the same technics as in 1.(b) leads to the following alternative expression:

\[ L_4^4(\Delta_3, \Delta_2^{[i]}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk}) \]

\[ = - \left\{ \int_{(0,1)^+} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \right. \]

\[ \left. \left[ - \ln \left( u^2 (P_{ijk} + R_{ij} + Q_i) + T \right) + \ln \left( T \frac{(P_{ijk} + R_{ij}) (R_{ij} + Q_i)}{P_{ijk}} \right) \right] \right. \]

\[ + \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \]

\[ \left[ \ln \left( u^2 P_{ijk} + (R_{ij} + Q_i + T) \right) - \ln \left( \frac{(R_{ij} + Q_i)}{Q_i} (Q_i + T) \right) \right. \]

\[ + \ln \left( u^2 Q_i + T \right) - \ln \left( T \frac{(P_{ijk} + R_{ij})}{P_{ijk}} \right) \]

\[ + \eta \left( \frac{(R_{ij} + Q_i)}{Q_i}, (Q_i + T) \right) - \eta \left( T \frac{(P_{ijk} + R_{ij})}{P_{ijk}}, \frac{(R_{ij} + Q_i)}{Q_i} \right) \]

\[ \left. \right\} \]

\[ \left. \right\} \]

Again eq. (3.35) has a structure very similar to eqs. (3.31) and (3.33). The cut of \( \ln(u^2 (P_{ijk} + R_{ij} + Q_i) + T) \) runs towards \( \infty \) inside the “south-east” quadrant yet the branch point \( \sqrt{-T/(P_{ijk} + R_{ij} + Q_i)} \) which it emerges from may lie inside the “north-east” quadrant. Accordingly the contour \( \hat{\gamma}(0,1)^+ \) stretched from 0 to 1 may wrap the branch point and arc of cut from above inside the “north-east” quadrant.
2. (a) \( \text{Im}(\Delta_3) > 0, \text{Im}(\Delta_2^{(i)}) > 0, \text{Im}(\Delta_1^{(i,j)}) < 0 \)

\[
L_4^4(\Delta_3, \Delta_2^{(i)}, \Delta_1^{(i,j)}, \tilde{D}_{ijk}) = \left\{ i \int_0^{+\infty} \frac{du}{u^2 P_{ijk} Q_i + R_{ij} T} \left[ \ln \left( \frac{-u^2 P_{ijk} + R_{ij}}{-u^2 P_{ijk}} \right) - \ln \left( \frac{Q_i + T}{T} \right) \right] \\
+ \int_0^{+\infty} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( \frac{-R_{ij}}{-u^2 P_{ijk} + R_{ij}} \right) - \ln \left( \frac{Q_i + R_{ij}}{-u^2 P_{ijk} + R_{ij} + Q_i + T} \right) \right] \\
+ i \int_0^{+\infty} \frac{du}{u^2 P_{ijk} Q_i + R_{ij} T} \left[ \ln \left( \frac{R_{ij}}{-u^2 P_{ijk} + R_{ij}} \right) - \ln \left( \frac{Q_i + R_{ij}}{-u^2 P_{ijk} + R_{ij} + Q_i + T} \right) \right] \\
+ \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( \frac{u^2 (P_{ijk} + R_{ij} + Q_i + T)}{u^2 P_{ijk} + T} \right) - \ln \left( \frac{u^2 Q_i + T}{T} \right) \right] \\
+ \int_1^{+\infty} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( \frac{u^2 P_{ijk} + R_{ij} + Q_i + T}{u^2 P_{ijk} + T} \right) - \ln \left( \frac{Q_i + T}{T} \right) \right] \right\} \tag{3.36}
\]

In this case, we have \( \text{Im}(P_{ijk}) < 0, \text{Im}(R_{ij}) > 0, \text{Im}(R_{ij} + Q_i) > 0, \text{Im}(R_{ij} + Q_i + T) > 0, \text{Im}(Q_i + T) < 0 \) and \( \text{Im}(T) < 0 \). Furthermore,

- \( u^2 P_{ijk} - R_{ij} = u^2 \tilde{D}_{ijk} + (1 + u^2) \Delta_1^{(i,j)} - \Delta_2^{(i)} \)
  thus \( \text{Im}(u^2 P_{ijk} - R_{ij}) < 0 \) when \( u \in [0, +\infty[ \),

- \( u^2 (P_{ijk} + R_{ij} + Q_i) + T = u^2 \tilde{D}_{ijk} - (1 - u^2) \Delta_3 \)
  thus \( \text{Im}(u^2 (P_{ijk} + R_{ij} + Q_i + T)) < 0 \) when \( u \in [0, 1] \),

- \( u^2 P_{ijk} + T = u^2 (\tilde{D}_{ijk} + \Delta_1^{(i,j)}) - \Delta_3 \)
  thus \( \text{Im}(u^2 P_{ijk} + T) < 0 \) when \( u \in [0, +\infty[ \),

- \( -u^2 P_{ijk} + R_{ij} + Q_i + T = -[u^2 \tilde{D}_{ijk} + (u^2 + 1) \Delta_1^{(i,j)}] \)
  thus \( \text{Im}(-u^2 P_{ijk} + R_{ij} + Q_i + T) > 0 \) when \( u \in [0, +\infty[ \),

- \( u^2 Q_i + T = -(1 - u^2) \Delta_3 - u^2 \Delta_2^{(i)} \)
  thus \( \text{Im}(u^2 Q_i + T) < 0 \) when \( u \in [0, 1] \),

- \( u^2 P_{ijk} + R_{ij} + Q_i + T = u^2 \tilde{D}_{ijk} + (u^2 - 1) \Delta_1^{(i,j)} \)
  thus \( \text{Im}(u^2 P_{ijk} + R_{ij} + Q_i + T) < 0 \) when \( u \in [1, +\infty[ \).
The same tricks as in case 1.(b) lead to:

\[
L_4^4(\Delta_3, \Delta_2^{(i)}, \Delta_1^{(ij)}, \bar{D}_{ijk}) = - \left\{ \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \right. \\
\quad \left[ \ln (u^2 Q_i + T) - \ln \left( T \frac{(P_{ijk} + R_{ij})}{P_{ijk}} \right) \\
\quad \quad - \ln \left( u^2 (P_{ijk} + R_{ij} + Q_i) + T \right) + \ln \left( T \frac{(P_{ijk} + R_{ij})}{P_{ijk}} \frac{(Q_i + R_{ij})}{Q_i} \right) \\
\quad \quad + \eta \left( \frac{(R_{ij} + Q_i)}{Q_i}, (Q_i + T) \right) - \eta \left( T \frac{(P_{ijk} + R_{ij})}{P_{ijk}}, \frac{(R_{ij} + Q_i)}{Q_i} \right) \right] \\
\quad \left. + \int_{(0,1)^+} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( u^2 P_{ijk} + (R_{ij} + Q_i + T) \right) - \ln \left( \frac{(Q_i + R_{ij})}{Q_i} (Q_i + T) \right) \right] \\
\quad \left. + \int_{(0,1)^+} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \eta \left( (R_{ij} + Q_i)(Q_i + T), \frac{1}{Q_i} \right) - \eta \left( R_{ij} T, \frac{1}{Q_i} \right) \right] \right\} \tag{3.37}
\]

Again eq. (3.37) has a structure very similar to eqs. (3.31), (3.33) and (3.35). The cut of \( \ln(u^2 P_{ijk} + (R_{ij} + Q_i + T)) \) runs towards \( \infty \) inside the “south-east” quadrant, yet the branch point \( \sqrt{-(R_{ij} + Q_i + T)/P_{ijk}} \) may lie inside the “north-east” quadrant. In this case, the contour \( (0,1)^+ \) stretched from 0 to 1 shall wrap the branch point and the arc of cut located inside the “north-east” quadrant, from above inside this quadrant.
2. (b) $\text{Im}(\Delta_3) > 0$, $\text{Im}(\Delta_2^{(i)}) < 0$, $\text{Im}(\Delta_1^{(i,j)}) < 0$

$L_4^0(\Delta_3, \Delta_2^{(i)}, \Delta_1^{(i,j)}, \tilde{D}_{ijk})$

\[
= - \left\{ i \int_0^{+\infty} \frac{du}{u^2 P_{ijk} Q_i + R_{ij} T} \left[ \frac{\ln \left( \frac{Q_i u^2 - T}{T} \right)}{u^2} - \ln \left( \frac{P_{ijk} + R_{ij}}{P_{ijk}} \right) - \ln \left( \frac{Q_i + R_{ij}}{-u^2 P_{ijk} + (R_{ij} + Q_i + T)} \right) \right] 
- \int_0^{+\infty} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \frac{\ln \left( \frac{Q_i}{T} \right)}{u^2} - \ln \left( \frac{R_{ij} + Q_i}{-u^2 P_{ijk} - T} \right) \right] 
+ \int_1^{+\infty} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \frac{\ln \left( \frac{u^2 P_{ijk} + R_{ij}}{u^2 P_{ijk} + T} \right)}{u^2} - \ln \left( \frac{u^2 Q_i + T}{Q_i + T} \right) \right] 
+ \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( \frac{P_{ijk} + R_{ij}}{P_{ijk}} \right) - \ln \left( \frac{u^2 P_{ijk} + R_{ij} + Q_i + T}{u^2 P_{ijk} + T} \right) \right] \right\} 
\tag{3.38}
\]

We have here: $\text{Im}(P_{ijk}) < 0$, $\text{Im}(P_{ijk} + R_{ij}) < 0$, $\text{Im}(R_{ij} + Q_i) > 0$, $\text{Im}(Q_i) > 0$, $\text{Im}(Q_i + T) > 0$ and $\text{Im}(T) < 0$. Furthermore,

- $u^2 (P_{ijk} + R_{ij} + Q_i) + T = u^2 \tilde{D}_{ijk} - (1 - u^2) \Delta_3$
  thus $\text{Im}(u^2 (P_{ijk} + R_{ij} + Q_i) + T) < 0$ when $u \in [0, 1]$,

- $u^2 P_{ijk} + T = u^2 (\tilde{D}_{ijk} + \Delta_2^{(i)}) - \Delta_3$
  thus $\text{Im}(u^2 P_{ijk} + T) < 0$ when $u \in [0, +\infty[$,

- $u^2 Q_i + T = (u^2 - 1) \Delta_3 - u^2 \Delta_2^{(i)}$
  thus $\text{Im}(u^2 Q_i + T) > 0$ when $u \in [1, +\infty[$,

- $u^2 Q_i - T = (1 + u^2) \Delta_3 - u^2 \Delta_2^{(i)}$
  thus $\text{Im}(u^2 Q_i - T) > 0$ when $u \in [0, +\infty[$,

- $u^2 P_{ijk} + R_{ij} = u^2 \tilde{D}_{ijk} + (u^2 - 1) \Delta_1^{(i,j)} + \Delta_2^{(i)}$
  thus $\text{Im}(u^2 P_{ijk} + R_{ij}) < 0$ when $u \in [1, +\infty[$,

- $-u^2 P_{ijk} + (R_{ij} + Q_i + T) = -u^2 \tilde{D}_{ijk} - (1 + u^2) \Delta_1^{(i,j)}$
  thus $\text{Im}(-u^2 P_{ijk} + (R_{ij} + Q_i + T)) > 0$ when $u \in [0, +\infty[$,
• \( u^2 P_{ijk} + (R_{ij} + Q_i + T) = u^2 \tilde{D}_{ijk} + (u^2 - 1) \Delta^{(i,j)}_1 \)
thus \( \text{Im}(u^2 P_{ijk} + (R_{ij} + Q_i + T)) < 0 \) when \( u \in [1, +\infty[. \)

Using the same techniques as in previous cases yields:

\[
L_4^4(\Delta_3, \Delta_2^{(i)}, \Delta_1^{(i,j)}, \tilde{D}_{ijk})
= - \left\{ \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \right. \\
\left. \left[ - \ln \left( u^2 (P_{ijk} + R_{ij} + Q_i) + T \right) + \ln \left( T \frac{(P_{ijk} + R_{ij}) (Q_i + R_{ij})}{P_{ijk} Q_i} \right) \right. \\
\left. + \eta \left( \frac{(R_{ij} + Q_i)}{Q_i}, (Q_i + T) \right) - \eta \left( T \frac{(P_{ijk} + R_{ij})}{P_{ijk} Q_i}, \frac{(R_{ij} + Q_i)}{Q_i} \right) \right] \\
+ \int_{(0,1)^+} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( u^2 P_{ijk} + (R_{ij} + Q_i + T) \right) - \ln \left( \frac{(Q_i + R_{ij})}{Q_i} (Q_i + T) \right) \right] \\
+ \int_{(0,1)^-} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( u^2 Q_i + T \right) - \ln \left( T \frac{(P_{ijk} + R_{ij})}{P_{ijk}} \right) \right] \\
+ \int_{(1,-1)^+} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \eta \left( \frac{(R_{ij} + Q_i)}{Q_i}, (Q_i + T) \right) \\
+ \int_{(1,-1)^-} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \eta \left( T, \frac{(P_{ijk} + R_{ij})}{P_{ijk}} \right) \right\} \tag{3.39}
\]

Again eq. (3.39) has a structure very similar to eqs. (3.31), (3.33) and (3.35) and (3.37). The cut of \( \ln(u^2 P_{ijk} + (R_{ij} + Q_i + T)) \) runs towards \( \infty \) inside the “south-east” quadrant, yet the branch point which this cut originates from may lie in the “north-east” quadrant. Accordingly the contour \((0, 1)^+\) stretched from 0 to 1 shall wrap the branch point and finite arc of cut from above inside this quadrant. A mirror situation holds for the cut of \( \ln(u^2 Q_i + T) \) which runs towards \( \infty \) inside the north-east quadrant yet with the branch point possibly lying in the “south-east” quadrant. In the latter case the contour \((0, 1)^-\) stretched from 0 to 1 shall be wrap the branch point and finite arc of cut possibly located in the “south-east” quadrant, from below inside that quadrant.
\[ L^4_4(\Delta_3, \Delta_2^{(i)}, \Delta_1^{(i,j)}, \tilde{D}_{ijk}) \]
\[ = - \left\{ i \int_0^{+\infty} \frac{du}{u^2 P_{ijk} Q_i + R_{ij} T} \right. \]
\[ \ln \left( \frac{-u^2 P_{ijk} + R_{ij}}{-u^2 P_{ijk} + R_{ij}} \right) - \ln \left( \frac{-u^2 P_{ijk} + (R_{ij} + Q_i + T)}{-u^2 P_{ijk} + T} \right) \]
\[ + i \int_0^{+\infty} \frac{du}{u^2 P_{ijk} Q_i + R_{ij} T} \left[ \ln \left( \frac{-u^2 (P_{ijk} + R_{ij} + Q_i) + T}{-u^2 P_{ijk} + T} \right) - \ln \left( \frac{-u^2 Q_i + T}{T} \right) \right] \]
\[ + i \int_0^{+\infty} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( \frac{-R_{ij}}{-u^2 P_{ijk} + T} \right) - \ln \left( \frac{Q_i}{Q_i + T} \right) \right] \]
\[ - \int_0^{+\infty} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \left[ \ln \left( \frac{-u^2 (P_{ijk} + (R_{ij} + Q_i + T))}{u^2 P_{ijk} + (R_{ij} + Q_i + T)} \right) - \ln \left( \frac{-u^2 Q_i + T}{Q_i + T} \right) \right] \]
\[ \right\} \] \hspace{1cm} (3.40)

In this case, we have \( \text{Im}(P_{ijk}) < 0, \text{Im}(R_{ij}) > 0, \text{Im}(Q_i) < 0, \text{Im}(Q_i + T) < 0 \) and \( \text{Im}(T) > 0 \).

Furthermore,

- \( u^2 P_{ijk} - R_{ij} = u^2 \tilde{D}_{ijk} + (1 + u^2) \Delta_1^{(i,j)} - \Delta_2^{(i)} \) thus \( \text{Im}(u^2 P_{ijk} - R_{ij}) < 0 \) when \( u \in [0, +\infty[ \),

- \( u^2 (P_{ijk} + R_{ij} + Q_i) - T = \tilde{D}_{ijk} u^2 + \Delta_3 (1 + u^2) \) thus \( \text{Im}(u^2 (P_{ijk} + R_{ij} + Q_i) - T) < 0 \) when \( u \in [0, +\infty[ \),

- \( -u^2 P_{ijk} + T = -u^2 (\tilde{D}_{ijk} + \Delta_1^{(i,j)}) - \Delta_3 \) thus \( \text{Im}(-u^2 P_{ijk} + T) > 0 \) when \( u \in [0, +\infty[ \),

- \( -u^2 P_{ijk} + R_{ij} + Q_i + T = -u^2 \tilde{D}_{ijk} - (1 + u^2) \Delta_1^{(i,j)} \) thus \( \text{Im}(-u^2 P_{ijk} + R_{ij} + Q_i + T) > 0 \) when \( u \in [0, +\infty[ \),

- \( u^2 Q_i - T = (1 + u^2) \Delta_3 - u^2 \Delta_2^{(i)} \) thus \( \text{Im}(u^2 Q_i - T) < 0 \) when \( u \in [0, +\infty[ \),

- \( u^2 (P_{ijk} + R_{ij} + Q_i) + T = u^2 \tilde{D}_{ijk} + (u^2 - 1) \Delta_3 \) thus \( \text{Im}(u^2 (P_{ijk} + R_{ij} + Q_i) + T) < 0 \) when \( u \in [1, +\infty[ \),

- \( u^2 P_{ijk} + R_{ij} + Q_i + T = u^2 \tilde{D}_{ijk} + (u^2 - 1) \Delta_1^{(i,j)} \) thus \( \text{Im}(u^2 P_{ijk} + R_{ij} + Q_i + T) < 0 \) when \( u \in [1, +\infty[ \),
\( u^2 Q_i + T = (u^2 - 1) \Delta_3 - u^2 \Delta_2^{(i)} \)

thus \( \text{Im}(u^2 Q_i + T) < 0 \) when \( u \in [1, +\infty[. \)

The implementation of the technics used in case 1.(b) leads to:

\[
L_4^1(\Delta_3, \Delta_2^{(i)}, \Delta_1^{(i,j)}, \tilde{D}_{ijk})
\]

\[
= - \left\{ \int_{(0,1)}^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T}
\left[ \ln \left( u^2 P_{ijk} + (R_{ij} + Q_i + T) \right) - \ln \left( \frac{Q_i + R_{ij}}{Q_i} (Q_i + T) \right) \right]
+ \int_{(0,1)}^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T}
\left[ - \ln \left( u^2 (P_{ijk} + R_{ij} + Q_i) + T \right) + \ln \left( T \frac{P_{ijk} + R_{ij}}{P_{ijk}} \left( \frac{Q_i + R_{ij}}{Q_i} \right) \right) \right]
+ \int_{(0,1)}^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T}
\left[ \ln \left( u^2 Q_i + T \right) - \ln \left( T \frac{P_{ijk} + R_{ij}}{P_{ijk}} \right) \right]
+ \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T}
\left[ \ln \left( \frac{R_{ij} + Q_i}{Q_i}, (Q_i + T) \right) - \ln \left( T \frac{P_{ijk} + R_{ij}}{P_{ijk}} \right) \frac{(R_{ij} + Q_i)}{Q_i} \right]
+ \int_{\Gamma^+} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T}
\left[ \ln \left( \frac{R_{ij} + Q_i}{Q_i}, (Q_i + T) \right) - \ln \left( T \frac{P_{ijk} + R_{ij}}{P_{ijk}} \right) \frac{(R_{ij} + Q_i)}{Q_i} \right]
+ \int_{\Gamma^-} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T}
\left[ \eta \left( R_{ij} T, \frac{1}{P_{ijk} Q_i} \right) + \eta (-R_{ij}, -T) - \eta (P_{ijk}, Q_i) \right] \right\}
\]

Again eq. (3.41) has a structure very similar to eqs. (3.31), (3.33) and (3.35), (3.37) and (3.39). All three \( u \)-dependent logarithms have cuts running towards \( \infty \) in the south-east quadrant, yet the branch points which they respectively emerge from may be located inside the “north-east” quadrant. Accordingly the contours \( (0,1)^+_j, j = 1, 2, 3 \) are stretched from 0 to 1 and may wrap the branch points and arcs of cuts from above inside the “north-east” quadrant. These contours may be chosen distinct from each other so as to best fit the respective cuts.
2. (d) \( \text{Im}(\Delta_3) < 0, \text{Im}(\Delta_2^{(i)}) < 0, \text{Im}(\Delta_1^{(i,j)}) < 0 \)

\[
L_4^4(\Delta_3, \Delta_2^{(i)}, \Delta_1^{(i,j)}, \tilde{D}_{ijk})
= \left\{ \begin{array}{c}
\int_0^{+\infty} \frac{du}{u^2 P_{ijk} Q_i + R_{ij} T} \\
\phantom{+} \left[ \ln \left( \frac{P_{ijk} + R_{ij}}{P_{ijk}} \right) - \ln \left( \frac{-u^2 (P_{ijk} + R_{ij} + Q_i) + T}{-u^2 P_{ijk} + T} \right) \right]
\end{array} \right. \\
+ \int_0^{+\infty} \frac{du}{u^2 P_{ijk} Q_i + R_{ij} T} \\
\phantom{+} \left[ \ln \left( \frac{-u^2 P_{ijk} + (R_{ij} + Q_i + T)}{-u^2 P_{ijk} + T} \right) - \ln \left( \frac{Q_i + T}{T} \right) \right]
\end{array} \right. \\
+ \int_1^{+\infty} \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \\
\phantom{+} \left[ \ln \left( \frac{u^2 P_{ijk} + R_{ij}}{u^2 P_{ijk}} \right) - \ln \left( \frac{u^2 (P_{ijk} + R_{ij} + Q_i) + T}{u^2 P_{ijk} + (R_{ij} + Q_i + T)} \right) \right]
\end{array} \right. \\
+ \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} T} \\
\phantom{+} \left[ \ln \left( \frac{P_{ijk} + R_{ij}}{P_{ijk}} \right) - \ln \left( \frac{u^2 Q_i + T}{T} \right) \right] \right\} 
\tag{3.42}
\]

In this case, we have \( \text{Im}(P_{ijk}) < 0, \text{Im}(P_{ijk} + R_{ij}) < 0, \text{Im}(Q_i) > 0, \text{Im}(Q_i + T) > 0 \) and \( \text{Im}(T) > 0 \). Furthermore,

- \( u^2 (P_{ijk} + R_{ij} + Q_i) - T = u^2 \tilde{D}_{ijk} + (1 + u^2) \Delta_3 \),
  thus \( \text{Im}(u^2 (P_{ijk} + R_{ij} + Q_i) - T) < 0 \) when \( u \in [0, +\infty[ \),

- \( u^2 P_{ijk} - T = u^2 (\tilde{D}_{ijk} + \Delta_1^{(i,j)}) + \Delta_3 \),
  thus \( \text{Im}(u^2 P_{ijk} - T) < 0 \) when \( u \in [0, +\infty[ \),

- \( -u^2 P_{ijk} + (R_{ij} + Q_i + T) = -u^2 \tilde{D}_{ijk} - (1 + u^2) \Delta_1^{(i,j)} \),
  thus \( \text{Im}(-u^2 P_{ijk} + (R_{ij} + Q_i + T) > 0 \) when \( u \in [0, +\infty[ \),

- \( u^2 P_{ijk} + R_{ij} = u^2 \tilde{D}_{ijk} + (u^2 - 1) \Delta_1^{(i,j)} + \Delta_2^{(i)} \),
  thus \( \text{Im}(u^2 P_{ijk} + R_{ij}) < 0 \) when \( u \in [1, +\infty[ \),

- \( u^2 (P_{ijk} + R_{ij} + Q_i) + T = u^2 \tilde{D}_{ijk} + (u^2 - 1) \Delta_3 \),
  thus \( \text{Im}(u^2 (P_{ijk} + R_{ij} + Q_i) + T) < 0 \) when \( u \in [1, +\infty[ \),

- \( u^2 P_{ijk} + R_{ij} + Q_i + T = u^2 \tilde{D}_{ijk} + (u^2 - 1) \Delta_1^{(i,j)} \),
  thus \( \text{Im}(u^2 P_{ijk} + (R_{ij} + Q_i + T)) < 0 \) when \( u \in [1, +\infty[ \).
\[ u^2 Q_i + T = -(1 - u^2) \Delta_3 - u^2 \Delta_2^i \]

thus \( \text{Im}(u^2 Q_i + T) > 0 \) when \( u \in [0, 1] \).

After the use of the tricks developed in case 1.(b), the following alternative expression is obtained:

\[
L_4^i(\Delta_3, \Delta_2^{ij}, \Delta_1^{ij}, \tilde{D}_{ijk}) = - \left\{ \int_{(0,1)}^+ u^2 P_{ijk} Q_i - R_{ij} T \left[ \ln \left( u^2 P_{ijk} + (R_{ij} + Q_i + T) \right) - \ln \left( T \left( \frac{P_{ijk} + R_{ij}}{P_{ijk}} \right) \right) \right] \right. \\
+ \left. \int_{(0,1)}^+ u^2 P_{ijk} Q_i - R_{ij} T \left[ - \ln \left( u^2 (P_{ijk} + R_{ij} + Q_i) + T \right) + \ln \left( T \left( \frac{P_{ijk} + R_{ij}}{P_{ijk}} \right) \right) \right] \right\} (3.43)
\]

Eq. (3.43) shares the structure common to eqs. (3.31), (3.33), (3.35), (3.37), (3.39) and (3.41) as well. The cuts of \( \ln(u^2 P_{ijk} + (R_{ij} + Q_i + T)) \) and \( \ln(u^2 (P_{ijk} + R_{ij} + Q_i) + T) \) in the half plane \( \{ \text{Re}(u) > 0 \} \) both run towards \( \infty \) in the “south-east” quadrant”, whereas the contours \((0, 1)_{1,2}^+\) stretched from 0 to 1 shall wrap the branch points and cuts of \( \ln(u^2 P_{ijk} + (R_{ij} + Q_i + T)) \) and \( \ln(u^2 (P_{ijk} + R_{ij} + Q_i) + T) \) respectively, from above in the “north-east” quadrant in case the corresponding branch points lie in this quadrant; the two contours may be chosen distinct from each other so as to best fit the respective finite arcs of cuts partly slashing the “north-east” quadrant from the branch points.

### 3.2 Synthesis

As anticipated the number of integral contributions is profuse in a case-dependent way from (3.18) to (3.42). A common structure can however be achieved by means of case-dependant
contour deformations of the real interval $[0, 1]$ supplemented by extra pole residue contributions weighted by case-dependant combinations of $\eta$ functions. Can this common structure be a starting point to recombine terms further and reduce the number of contributions, as could be done for the three-point function in the general complex mass case treated according to the “indirect way”?

In the case of the three-point function case, we could first cast the integrals weighting the sum over the $\bar{b}_j^{(i)}$ as one-dimensional contour integrals of a common type along some case-dependent contour deformations of the interval $[0, 1]$ which was used in the real mass case. Then, after appropriate changes of variables absorbing the corresponding factor $\bar{b}_j^{(i)}$ in each of these contour integrals, we were able to concatenate these rescaled contour integrals into a single contour integral. Lastly, the compound contour of the latter was deformed in its turn into exactly the interval $[0, 1]$ involved in the real mass case. This resulted in a simplification which proved to coincide with the one coming out via the “direct way”. One may wonder whether the formal unification of the profuse diversity of expressions obtained for the four-point function with general complex masses could, at least partially, be exploited in a similar way following a similar programme. This quest appears much more complicated for the four-point function, all the more so as we already faced an issue in the reduction of the number of dilogarithms involved in the expression of the four-point function for the real mass case using the present approach, compared with ’t Hooft and Veltman’s approach. Nevertheless as already discussed in the end of sec. 2, the dilogarithms obtained after performing the last integration are the same for all the 8 cases and are similar to those of the real mass case. Here also, the discussion about the number of dilogarithms generated (cf. subsec. 3.5 of [1]) compared to ref. [9] still holds and the solutions which will be found to counteract this proliferation of dilogarithms in the real mass case will be able to apply without modifications.

4 Summary and outlook

In this article we presented an extension of the novel approach developed in a companion article (cf. [1]) for the computation of one-loop three- and four-point functions in the general complex mass case. The method naturally proceeds in terms of algebraic kinematical invariants involved in reduction algorithms and applies to general kinematics beyond the one relevant for one-loop collider processes, it thereby offers a potential application to the calculation of processes at two-loop using one-loop (generalised) $N$-point functions as building blocks. This novel approach enables a smooth extension to the complex mass case for the generalised one-loop building blocks expressed in terms of dilogarithms. Nevertheless, in the case of a two-loop computations, the analyticity of the one-loop integrand with respect of the two extra Feynman parameters has to be carefully studied. For sake of pedagogy, the method was exposed on “ordinary” three- and four-point functions in four dimensions in the real mass case in a companion article [1]. The complex mass case has been studied hereby. It can be extended in respect to the space-time dimension to tackle the infrared divergent case. Let us advertised it briefly.
In a third companion paper we extend the presented framework to the case where some vanishing internal masses cause infrared soft and/or collinear divergences. The method extends in a straightforward way, once a few intermediate steps and tools are appropriately adapted.

The question of the proliferation of dilogarithms in the expression of the four-point function computed in closed form with the present method comes up in the same terms as in the real mass case. It requires some extra work to be better apprehended, in order to counteract it. This issue will be addressed in a future article.

The last goal is to provide the generalised one-loop building blocks entering as integrands in the computation two-loop three- and four-point functions by means of an extra numerical double integration.

In memoriam

This work has been initiated by Prof. Shimizu after a visit to LAPTh. He explained us his idea about the numerical computation of scalar two loop three- and four-point functions, he shared his notes partly in English, partly in Japanese with us and he encouraged us to push this project forward. J.Ph. G. would like to thank Shimizu-sensei for giving him a taste of the Japanese culture and for his kindness.

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A General case for the second kind integral \( J(\nu) \)

This appendix extends the results of appendix D of ref. [1] concerning the second kind integral \( J(\nu) \) given by:

\[
J(\nu) = \int_0^{+\infty} \frac{d\xi}{(\xi^\nu + A) \sqrt{\xi^\nu + B}}
\]  

(A.1)

because new cases appear which were not covered in this reference. In what follows \( A \) and \( B \) are assumed dimensionless and complex valued, the signs of their real parts are unknown, and, contrary to the real mass case, the signs of their imaginary parts may or may not be the same. When no internal masses are vanishing it arises for \( \nu = 2 \) whereas infrared divergent cases regularised by dimensional continuation beyond \( n = 4 \) involve non integer \( \nu \). Anticipating our next paper on infrared divergent case, these various situations are treated all at once here, specifying \( \nu \) at will in the result. The integral need not be computed in closed form and shall instead be recast in an alternative, more handy form cleared from any
radical. So let us distinguish two cases according to the signs of the imaginary parts of $A$ and $B$.

1) $\text{Im}(A)$ and $\text{Im}(B)$ of the same sign

Whenever $\text{Im}(A)$ and $\text{Im}(B)$ have the same sign, the use of the celebrated Feynman “trick” is justified and leads to:

\[
\frac{1}{(\xi' + A) \sqrt{\xi' + B}} = \frac{1}{2} \int_0^1 \frac{dx \ x^{-\frac{1}{2}}}{(\xi' + (1 - x) A + x B)^{3/2}} \quad (A.2)
\]

$J(\nu)$ is readily rewritten as:

\[
J(\nu) = \frac{1}{2} \int_0^{+\infty} d\xi \int_0^1 \frac{dx \ x^{-\frac{1}{2}}}{(\xi' + (1 - x) A + x B)^{3/2}} \quad (A.3)
\]

Then the $\xi$ integration is performed first, using eq. (B.1) of appendix B of ref. [1]. Performing the change of variable $z = \sqrt{x}$ in the result obtained yields:

\[
J(\nu) = \frac{1}{\nu} B \left( \frac{3}{2} - \frac{1}{\nu}, \frac{1}{\nu} \right) \int_0^1 dz \ ( (1 - z^2) A + z^2 B )^{3/2 + 1/\nu} \quad (A.4)
\]

In particular for $\nu = 2$:

\[
J(2) = \int_0^1 \frac{dz}{(1 - z^2) A + z^2 B} \quad (A.5)
\]

2) $\text{Im}(A)$ and $\text{Im}(B)$ of opposite signs

This more annoying case can be met when the internal masses are complex. Naively reproducing the previous argument would again lead to eq. (A.4). However the derivation of the Feynman “trick” (A.2) assumes $\text{Im}(A)$ and $\text{Im}(B)$ to have the same sign (whenever the signs of their respective real values is undetermined, which is the case at hand): its use is illegitimate whenever $\text{Im}(A)$ and $\text{Im}(B)$ have opposite signs. We shall first recast the r.h.s. of eq. (A.1) so that the imaginary parts of both factors in the denominator of the integrand have the same sign:

\[
J(\nu) = -\int_0^{+\infty} \frac{d\xi}{(-\xi' - A) \sqrt{-\xi' + B}} \quad (A.6)
\]

Then we can apply the Feynman “trick” to eq. (A.6):

\[
J(\nu) = -\frac{1}{2} \int_0^{+\infty} d\xi \int_0^1 \frac{dx \ x^{-\frac{1}{2}}}{((2x - 1)\xi' - (1 - x) A + x B)^{3/2}} \quad (A.7)
\]

We again intend to perform the $\xi$ integration first, yet the task is a little more tricky than for (A.3). In order to use eq. (B.1) of ref. [1] we shall factor out a fractional power of $(2x - 1)$
which is not always positive when $x$ spans $[0, 1]$, so that some care is required. Introducing $S_B = \text{sign}(\text{Im}(B))$, $S'_x = \text{sign}(2x - 1)$ and an infinitesimal parameter $0 < \lambda \ll 1$ we have\(^{10}\):

\[
((2x - 1) \xi^\nu - (1 - x) A + x B)^{3/2} = (2x - 1 - i S B S'_x \lambda)^{3/2} \left[ \xi^\nu + \frac{x B - (1 - x) A}{2x - 1} \right]^{3/2}
\]

so that:

\[
J(\nu) = -\frac{1}{2} \int_0^1 \frac{dx}{x} \int_0^\infty \frac{d\xi}{(\xi^\nu + \frac{x B - (1 - x) A}{2x - 1})^{3/2}}
\]

(A.8)

The $\xi$ integration performed using eq. (B.1) of ref. [1] yields:

\[
J(\nu) = -\frac{1}{2 \nu} B \left( \frac{3}{2} - \frac{1}{\nu'}, \frac{1}{\nu} \right)
\]

\[
\times \int_0^1 \frac{dx}{x} \frac{1}{(2x - 1 - i S B S'_x \lambda)^{3/2}} \left( \frac{2x - 1}{(x - 1) A + x B} \right)^{3/2 - 1/\nu}
\]

(A.9)

Some care is required again to split the fraction raised to the non integer power $3/2 - 1/\nu$ into a fraction of powers:

\[
\left( \frac{2x - 1}{(x - 1) A + x B} \right)^{3/2 - 1/\nu} = \frac{(2x - 1 + i \lambda S B)^{3/2 - 1/\nu}}{(x - 1) A + x B)^{3/2 - 1/\nu}}
\]

(A.10)

Eq. (A.9) can be written as:

\[
J(\nu) = -\frac{1}{2 \nu} B \left( \frac{3}{2} - \frac{1}{\nu'}, \frac{1}{\nu} \right)
\]

\[
\times \int_0^1 \frac{dx}{x} \left( \frac{2x - 1 + i S B \lambda)^{3/2 - 1/\nu}}{(x - 1) A + x B)^{-3/2 + 1/\nu}}
\]

(A.11)

We now split the range of integration in $x$ in two parts : $0 \leq x \leq 1/2$ and $1/2 \leq x \leq 1$, so that in each sub-range, $2x - 1$ has a definite sign. $J(\nu)$ can be written as :

\[
J(\nu) = -\frac{1}{2 \nu} B \left( \frac{3}{2} - \frac{1}{\nu'}, \frac{1}{\nu} \right)
\]

\[
\times \left[ \int_0^{1/2} dx \int_0^{1/2} x^{-1/2} dx (2x - 1)^{-1/\nu} (B x + (x - 1) A)^{-3/2 + 1/\nu}}
\]

\[+ \int_{1/2}^1 dx \int_0^{1/2} x^{-1/2} (2x - 1)^{-1/\nu} (B x + (x - 1) A)^{-3/2 + 1/\nu}} \right]
\]

(A.12)

\(^{10}\)This comes from the splitting of $\ln(ab)$ with $a$ real not necessarily $> 0$ and $b$ is complex non real, for which [9]

\[
\ln(ab) = \ln(a - i \lambda S_b) + \ln(b), \quad S_b = \text{sign}(\text{Im}(b))
\]

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With the help of the Euler changes of variables \( \sqrt{x - 2x^2} = xt \) in the first integral and \( \sqrt{2x^2 - x} = xt \) in the second integral of eq. (A.12), we recast \( J(\nu) \) into:

\[
J(\nu) = -\frac{1}{\nu} B \left( \frac{3}{2} - \frac{1}{\nu}, \frac{1}{\nu} \right) \\
\times \left[ e^{-iS_B \pi/\nu} \int_0^{+\infty} dt \ t^{1-2/\nu} \ (B - (t^2 + 1) A)^{-3/2+1/\nu} \\
+ \int_0^1 dt \ t^{1-2/\nu} \ (B + (t^2 - 1) A)^{-3/2+1/\nu} \right]
\] (A.13)

Finally, we trade \( t \) for \( z = 1/t \) so that \( J(\nu) \) becomes:

\[
J(\nu) = -\frac{1}{\nu} B \left( \frac{3}{2} - \frac{1}{\nu}, \frac{1}{\nu} \right) \\
\times \left[ e^{-iS_B \pi/\nu} \int_0^{+\infty} dz \ (B z^2 - (1 + z^2) A)^{-3/2+1/\nu} \\
+ \int_0^{+\infty} dz \ (B z^2 + (1 - z^2) A)^{-3/2+1/\nu} \right]
\] (A.14)

In particular for \( \nu = 2 \), \( J(2) \) becomes:

\[
J(2) = \left[ i S_B \int_0^{+\infty} \frac{dz}{B z^2 - (1 + z^2) A} - \int_1^{+\infty} \frac{dz}{B z^2 + (1 - z^2) A} \right]
\] (A.15)

Note that the two integrals of the right hand size of eq. (A.14) are well defined because \( \text{Im}(B z^2 - (1 + z^2) A) \) and \( \text{Im}(B z^2 + (1 - z^2) A) \) never vanish in the respective \( z \) ranges of integration thus the branch cuts (poles for \( 3/2 - 1/\nu \) integer) of the integrands lie away from the integration ranges. We will elaborate a little more about their location in the complex \( z \) plane below.

The two cases 1) vs. 2) disentangled above can be reunified by seeing eq. (A.14) as an analytic continuation in \( A \) of eq. (A.4) which possibly requires a deformation of the contour \([0, 1]\) originally drawn along the real axis in eq. (A.4). The normalisation factor in \( J(\nu) \) is irrelevant in the following discussion, we drop it (apart the overall minus sign) to simplify the expressions.

\[
J(\nu) = -e^{-iS_B \pi/\nu} \int_0^{+\infty} dz \ (B z^2 - (1 + z^2) A)^{-3/2+1/\nu} \\
- \int_1^{+\infty} dz \ (B z^2 + (1 - z^2) A)^{-3/2+1/\nu}
\] (A.16)

\( J(\nu) \) can be alternatively written:

\[
J(\nu) = \left\{ \int_0^{-iS_B \infty} + \int_1^{+\infty} \right\} dz \ (B z^2 + (1 - z^2) A)^{-3/2+1/\nu}
\] (A.17)
The function \((B z^2 + (1 - z^2) A)^{-3/2+1/\nu}\) of the complex variable \(z\) has two discontinuity cuts supported respectively by either of the two branches of the hyperbola \(\{\text{Im}(B z^2 + (1 - z^2) A) = 0\}\). Let us label \(C_+\) the cut relevant\(^{11}\) for our concern. \(C_+\) lies in the right half \(z\)-plane \(\{\text{Re}(z) > 0\}\). It originates at the point \(z_+ = (A/(A - B))^{1/2}\) and slashes the right half plane through the quadrant \(\{\text{Re}(z) > 0, S_B \text{Im}(z) > 0\}\) away to \(\infty\). In case \(z_+\) belongs to the quadrant \(\{\text{Re}(z) > 0, S_B \text{Im}(z) < 0\}\) this cut crosses the real interval \([0, 1]\) at \(z_c = [\text{Im}(A)/\text{Im}(A - B)]^{1/2}\) (cf. appendix D for more details).

The integration contour in the r.h.s. of eq. (A.17) can be closed by drawing an arc \((0, 1)\) between 0 and 1 so as to reformulate \(J(\nu)\) as a contour integral along \((0, 1)\) according to the Cauchy theorem, the extra arc at \(\infty\) also involved by the Cauchy theorem to close the contour yields a vanishing contribution \(\sim O(1/R)\) where \(R \to +\infty\) is \(|z|\) on the contour at \(\infty\).

(i) If \(C_+\) entirely belongs to the quadrant \(\{\text{Re}(z) > 0, S_B \text{Im}(z) > 0\}\) this extra arc \((0, 1)\) can be taken along the real segment \([0, 1]\).

(ii) However if \(z_+\) belongs to the quadrant \(\{\text{Re}(z) > 0, S_B \text{Im}(z) < 0\}\) the extra arc \((0, 1)\) shall wrap the bit of \(C_+\) from slightly before \(z_c\) around \(z_+\) and back to slightly after \(z_c\) inside \(\{\text{Re}(z) > 0, S_B \text{Im}(z) < 0\}\) as if \(C_+\) were locally sinking the contour away from the real segment \([0, 1]\) inside this quadrant as pictured on figure 2.

In either case:

\[
J(\nu) = \int_{(0,1)} dz \ (B z^2 + (1 - z^2) A)^{-3/2+1/\nu}
\]  

(A.18)

is the argued analytic continuation in \(A\) of eq. (A.4).

**B Basic integrals in terms of dilogarithms and logarithms: \(K\)-type integrals**

This appendix comes in addition to appendix E of ref. [1]. The computations of the various \(N\)-point functions in closed form can be reduced to the calculation of integrals of simple types. The \(K\)-type is of the form

\[
K = \int_{b}^{a} du \ \frac{\ln(A u^2 + B) - \text{"subtracted term"}}{u^2 - u_0^2}
\]

where \(u_0^2 \neq -B/A\). In the case of complex masses, \(A\) is complex and the complex quantity \(B\) has a non vanishing imaginary part yet with \(\text{Im}(A u^2 + B)\) keeping a constant sign while

\(^{11}\)The other cut \(C_-\) is the symmetric of \(C_+\) under parity: located in the left half plane \(\{\text{Re}(z) < 0\}\) it is irrelevant here.
$u$ spans the real interval $[a, b]$. The general complex mass case involves the contour $[0, 1]$ as well as the two other contours $[0, +\infty[$ and $[1, +\infty[$.

In the complex mass case, the situation is less diverse than in the real mass case. The parameters $u_0^2$ in the $K$-type integrals are generically complex with a non infinitesimal imaginary part, in which case the poles in the integrands are well off the contour of integration thus the calculation can be formulated using either a vanishing or non vanishing “subtracted term” it does not matter. One may choose to use $K$-type integrals with a “subtracted term” equal to 0 so as to have the simplest possible expressions, or instead e.g. equal to $\ln(Au_0^2 + B)$ so as to involve similar building blocks as for the real mass case, cf. below: this thereby minimises the number of encoded functions in practical numerical implementations.

This appendix often makes use of the identity

$$\ln(z) = \ln(-z) + i \pi S(z) , \quad S(z) \equiv \text{sign}(\text{Im}(z)) \quad (B.1)$$

### B.1 $a = 0$, $b = 1$

The calculations of $N$-point functions are formulated so as to be expressed in terms of quantities of the form:

$$K_{0,1}^{C}(A, B, u_0^2) = \int_0^1 du \frac{\ln(Au^2 + B) - \ln(Au_0^2 + B)}{u^2 - u_0^2} \quad (B.2)$$

where $A$ and $B$ are now plain complex numbers yet with $\text{Im}(Au^2 + B)$ keeping a constant sign while $u$ spans the real interval $[0, 1]$. The logarithms may now be conveniently split as

$$\ln(Au^2 + B) = \ln(A) + \ln(u - \bar{u}) + \ln(u + \bar{u}) + \eta(A, -\bar{u}^2) \quad (B.3)$$
$$\ln(Au_0^2 + B) = \ln(A) + \ln(u_0 - \bar{u}) + \ln(u_0 + \bar{u})$$
$$+ \eta(u_0 - \bar{u}, u_0 + \bar{u}) + \eta(A, u_0^2 - \bar{u}^2) \quad (B.4)$$

where $\bar{u} = \sqrt{-B/A}$ and the function $\eta(z_1, z_2)$ is given by eq. (E.6) in appendix E of [1]. Substituting identities (B.3), (B.4) into eq. (B.2), and proceeding along the same line as for the real mass case, we get:

$$K_{0,1}^{C}(A, B, u_0^2) = \frac{1}{2u_0} \left\{ \mathcal{F}(u_0, \bar{u}) + [\eta(A, -\bar{u}^2) - \eta(A, u_0^2 - \bar{u}^2)$$
$$- \eta(u_0 - \bar{u}, u_0 + \bar{u})] \ln \left( \frac{u_0 - 1}{u_0 + 1} \right) \right\} \quad (B.5)$$

where $\mathcal{F}(y, z)$ is given by eq. (E.17) of ref. [1].
B.2 \( a = 1, b = +\infty \)

The computation of \( N \)-point functions in the complex mass case also involves integrals of the following kind:

\[
K_{1,\infty}^C(A, B, u_0^2) = \int_1^{\infty} du \frac{\ln(A u^2 + B) - \ln(A u_0^2 + B)}{u^2 - u_0^2} \tag{B.6}
\]

where \( A, B \) and \( u_0^2 \) are complex numbers such that \( \text{Im}(A u^2 + B) \) keeps a constant sign while \( u \) spans the range \([1, \infty[\) along the real axis. Logarithms can be split as in identities (B.3), (B.4) above, and the partial fraction decomposition of \( 1/(u^2 - u_0^2) \) proceeds as in the real mass case. We wish to conveniently handle the various terms resulting from the partial fraction decomposition separately. Yet the latter individually diverge logarithmically at large \( u \), although the integral in eq. (B.6) converges. We therefore introduce a regularisation procedure by means of “large \( u \)” cut-off \( \Lambda \). We then recombine individually divergent terms \( \propto \ln^2(\Lambda) \) and \( \propto \ln(\Lambda) \), so as to make them respectively cancel among each other explicitly. We then take the limit \( \Lambda \to +\infty \). With the above definitions of \( u_0 \) and \( \bar{u} \), the regularised splitting of \( K_{1,\infty}^C(A, B, u_0^2) \) reads:

\[
K_{1,\infty}^C(A, B, u_0^2) = \frac{1}{2 u_0} \left\{ \lim_{\Lambda \to +\infty} E(\Lambda) + \left[ \eta(A, 1 - \bar{u}^2) - \eta(A, u_0^2 - \bar{u}^2) \right. \right.
\]

\[
\left. - \eta(u_0 - \bar{u}, u_0 + \bar{u}) \right] [\ln (1 + u_0) - \ln (1 - u_0)] \right\} \tag{B.7}
\]

with

\[
E(\Lambda) = \int_1^{\Lambda} du \left[ \frac{1}{u - u_0} - \frac{1}{u + u_0} \right]
\]

\[
\times [\ln(u - \bar{u}) - \ln(u_0 - \bar{u}) + \ln(u + \bar{u}) - \ln(u_0 + \bar{u})] \tag{B.8}
\]

Introducing the quantity

\[
R^\Lambda(y, z) = \int_1^{\Lambda} du \frac{\ln(u - y) - \ln(z - y)}{u - z} \tag{B.9}
\]

\( E(\Lambda) \) reads in terms of \( R^\Lambda \):

\[
E(\Lambda) = R^\Lambda(u_0, \bar{u}) + R^\Lambda(u_0, -\bar{u}) - R^\Lambda(-u_0, \bar{u}) - R^\Lambda(-u_0, -\bar{u}) \]

\[
+ i \pi [S(u_0 + \bar{u}) + S(u_0 - \bar{u})] \ln \left( \frac{\Lambda + u_0}{1 + u_0} \right) \tag{B.10}
\]

The computation of \( R^\Lambda(y, z) \) proceeds along the same line as \( R'(y, z) \) in the appendix E of ref. [1] and we get:

\[
R^\Lambda(y, z) = \text{Li}_2 \left( \frac{1 - z}{y - z} \right) - \text{Li}_2 \left( \frac{\Lambda - z}{y - z} \right)
\]

\[
+ \eta \left( 1 - y, \frac{1}{z - y} \right) \ln \left( \frac{1 - z}{y - z} \right) - \eta \left( \Lambda - y, \frac{1}{z - y} \right) \ln \left( \frac{\Lambda - z}{y - z} \right) \tag{B.11}
\]
We use the identity relating \( \text{Li}_2(z) \) and \( \text{Li}_2(1/z) \) and we note that

\[
\eta \left( \Lambda - y, \frac{1}{z-y} \right) = \eta \left( 1 - \frac{y}{\Lambda}, \frac{1}{z-y} \right)
\]  
(B.12)

We rewrite eq. (B.11) as:

\[
R^\Lambda(y, z) = \text{Li}_2 \left( \frac{1-z}{y-z} \right) + \eta \left( 1 - y, \frac{1}{z-y} \right) \ln \left( \frac{1-z}{y-z} \right) + \frac{\pi^2}{6} - \text{Li}_2 \left( \frac{y-z}{\Lambda - z} \right) + \frac{1}{2} \left[ \ln(\Lambda) + \ln \left( \frac{1-z/\Lambda}{z-y} \right) \right]^2
\]

\[
- \eta \left( 1 - \frac{y}{\Lambda}, \frac{1}{z-y} \right) \left[ \ln(\Lambda) + \ln \left( \frac{1-z/\Lambda}{y-z} \right) \right]
\]  
(B.13)

For fixed \( y \) and \( z \), when \( \Lambda \) is large enough \( \eta(1-y/\Lambda, 1/(z-y)) = 0 \). Dropping all terms which vanish when \( \Lambda \to \infty \), eq. (B.13) can be rewritten as:

\[
R^\Lambda(y, z) = \text{Li}_2 \left( \frac{1-z}{y-z} \right) + \eta \left( 1 - y, \frac{1}{z-y} \right) \ln \left( \frac{1-z}{y-z} \right) + \frac{\pi^2}{6} + \frac{1}{2} \ln^2(z-y) - \ln(\Lambda) \ln(z-y) + \frac{1}{2} \ln^2(\Lambda)
\]  
(B.14)

Substituting eq. (B.14) into eq. (B.10), we get:

\[
E(\Lambda) = \text{Li}_2 \left( \frac{u_0 - 1}{u_0 - \bar{u}} \right) + \eta \left( 1 - \bar{u}, \frac{1}{u_0 - \bar{u}} \right) \ln \left( \frac{u_0 - 1}{u_0 - \bar{u}} \right)
\]

\[
+ \text{Li}_2 \left( \frac{u_0 - 1}{u_0 + \bar{u}} \right) + \eta \left( 1 + \bar{u}, \frac{1}{u_0 + \bar{u}} \right) \ln \left( \frac{u_0 - 1}{u_0 + \bar{u}} \right)
\]

\[
- \text{Li}_2 \left( \frac{u_0 + 1}{u_0 + \bar{u}} \right) - \eta \left( 1 - \bar{u}, \frac{-1}{u_0 + \bar{u}} \right) \ln \left( \frac{u_0 + 1}{u_0 + \bar{u}} \right)
\]

\[
- \text{Li}_2 \left( \frac{u_0 + 1}{u_0 - \bar{u}} \right) - \eta \left( 1 + \bar{u}, \frac{-1}{u_0 - \bar{u}} \right) \ln \left( \frac{u_0 + 1}{u_0 - \bar{u}} \right)
\]

\[
+ \frac{1}{2} \left( \ln^2(u_0 - \bar{u}) + \ln^2(u_0 + \bar{u}) - \ln^2(-u_0 - \bar{u}) - \ln^2(-u_0 + \bar{u}) \right)
\]

\[
- \ln(\Lambda) \left( \ln(u_0 - \bar{u}) + \ln(u_0 + \bar{u}) - \ln(-u_0 - \bar{u}) - \ln(-u_0 + \bar{u}) \right)
\]

\[
+ i \pi \left( S(u_0 + \bar{u}) + S(u_0 - \bar{u}) \right) \left[ \ln(\Lambda) - \ln(1 + u_0) \right]
\]  
(B.15)

Using eq. (B.1), the sums of logarithmic terms in eq. (B.15) can be expressed in terms of the sign function \( S \):

\[
\ln(u_0 - \bar{u}) + \ln(u_0 + \bar{u}) - \ln(-u_0 - \bar{u}) - \ln(-u_0 + \bar{u}) = i \pi \left( S(u_0 - \bar{u}) + S(u_0 + \bar{u}) \right)
\]  
(B.16)

\[
\ln^2(u_0 - \bar{u}) + \ln^2(u_0 + \bar{u}) - \ln^2(-u_0 - \bar{u}) - \ln^2(-u_0 + \bar{u})
\]

\[
= 2 \pi^2 + 2 i \pi \left[ S(u_0 - \bar{u}) \ln(u_0 - \bar{u}) + S(u_0 + \bar{u}) \ln(u_0 + \bar{u}) \right]
\]  
(B.17)
Substituting eqs. (B.16), (B.17) and (B.15) in (B.7), we get:

\[
K_{1,\infty}^C(A, B, u_0^2) = \frac{1}{2u_0} \left\{ \begin{array}{l}
\text{Li}_2 \left( \frac{u_0 - 1}{u_0 - \bar{u}} \right) + \eta \left( 1 - \bar{u}, \frac{1}{u_0 - \bar{u}} \right) \ln \left( \frac{u_0 - 1}{u_0 - \bar{u}} \right)
+ \text{Li}_2 \left( \frac{u_0 - 1}{u_0 + \bar{u}} \right) + \eta \left( 1 + \bar{u}, \frac{1}{u_0 + \bar{u}} \right) \ln \left( \frac{u_0 - 1}{u_0 + \bar{u}} \right)
- \text{Li}_2 \left( \frac{u_0 + 1}{u_0 + \bar{u}} \right) - \eta \left( 1 - \bar{u}, -\frac{1}{u_0 + \bar{u}} \right) \ln \left( \frac{u_0 + 1}{u_0 + \bar{u}} \right)
- \text{Li}_2 \left( \frac{u_0 + 1}{u_0 - \bar{u}} \right) - \eta \left( 1 + \bar{u}, -\frac{1}{u_0 - \bar{u}} \right) \ln \left( \frac{u_0 + 1}{u_0 - \bar{u}} \right)
+ i\pi S(u_0 - \bar{u}) \left[ \ln(u_0 - \bar{u}) - \ln(u_0 + 1) \right]
+ i\pi S(u_0 + \bar{u}) \left[ \ln(u_0 + \bar{u}) - \ln(u_0 + 1) \right] + \pi^2
+ \left[ \eta(A, 1 - \bar{u}^2) - \eta(A, u_0^2 - \bar{u}^2) - \eta(u_0 - \bar{u}, u_0 + \bar{u}) \right]
\times \left[ \ln \left( \frac{1 + u_0}{1 - u_0} \right) \right]
\end{array} \right\} \tag{B.18}
\]

The dilogarithms of eq. (B.18) are - up to an overall sign minus - the same than those appearing in the \( F \) function (c.f. eq. (E.17) of ref. [1]). We thus force the appearance of \( F \) by introducing the necessary extra \( \eta \) functions and, noting that \( \eta(1 + u_0, 1/(1 - u_0)) = 0 \) we rewrite eq.(B.18) as:

\[
K_{1,\infty}^C(A, B, u_0^2) = \frac{1}{2u_0} \left\{ -F(u_0, \bar{u}) + i\pi S(\bar{u}) \left[ \ln \left( \frac{u_0 + 1}{u_0 - \bar{u}} \right) - \ln \left( \frac{u_0 + 1}{u_0 + \bar{u}} \right) \right] + \pi^2
+ i\pi S(u_0 - \bar{u}) \eta \left( u_0 + 1, \frac{1}{u_0 - \bar{u}} \right)
+ i\pi S(u_0 + \bar{u}) \eta \left( u_0 + 1, \frac{1}{u_0 + \bar{u}} \right)
+ \left[ \eta(A, 1 - \bar{u}^2) - \eta(A, u_0^2 - \bar{u}^2) - \eta(u_0 - \bar{u}, u_0 + \bar{u}) \right]
\times \ln \left( \frac{1 + u_0}{1 - u_0} \right) \right\} \tag{B.19}
\]

Eq. (B.19) is obtained by noting that, for any two complex numbers \( a \) and \( b \):

\[
\eta(a, b) - \eta(-a, -b) = -i\pi \left[ S(a) + S(b) \right] \tag{B.20}
\]

Rearranging the logarithmic terms and noting that \( (S(u_0 + \bar{u}) - S(\bar{u})) \eta(u_0 + 1, 1/(u_0 + \bar{u})) \) as well as \( (S(u_0 - \bar{u}) + S(\bar{u})) \eta(u_0 + 1, 1/(u_0 - \bar{u})) \) always vanish, we end up with:

\[
K_{1,\infty}^C(A, B, u_0^2) = \frac{1}{2u_0} \left\{ -F(u_0, \bar{u}) + i\pi \left[ S(\bar{u}) \left[ \ln(u_0 + \bar{u}) - \ln(u_0 - \bar{u}) \right] - i\pi \right]
+ \left[ \eta(A, 1 - \bar{u}^2) - \eta(A, u_0^2 - \bar{u}^2) - \eta(u_0 - \bar{u}, u_0 + \bar{u}) \right]
\times \ln \left( \frac{1 + u_0}{1 - u_0} \right) \right\} \tag{B.21}
\]
B.3 \( a = 0, b = +\infty \)

The computation of \( N \)-point functions in the complex mass case also involves integrals of the following third kind:

\[
K_{0,\infty}^C(A, B, u_0^2) = \int_0^\infty du \frac{\ln(Au^2 + B) - \ln(Au_0^2 + B)}{u^2 - u_0^2}
\]  

(B.22)

where \( A, B \) and \( u_0^2 \) are complex numbers such that \( \text{Im}(Au^2 + B) \) keeps a constant sign while \( u \) spans the range \([0, \infty]\) along the real axis. Under the assumption made, \( K_{0,\infty}^C \) can be split as:

\[
K_{0,\infty}^C(A, B, u_0^2) = K_{0,1}^C(A, B, u_0^2) + K_{1,\infty}^C(A, B, u_0^2)
\]  

(B.23)

with the expressions of \( K_{0,1}^C(A, B, u_0^2) \) and \( K_{1,\infty}^C(A, B, u_0^2) \) computed above in eqs. (B.5) and (B.21) respectively. In the sum, the \( \mathcal{F} \) contribution drops out so that \( K_{0,\infty}^C(A, B, u_0^2) \) contains only logarithmic terms:

\[
K_{0,\infty}^C(A, B, u_0^2) = \frac{1}{2u_0} \left\{ i\pi \left[ S(\bar{u}) \left( \ln(u_0 + \bar{u}) - \ln(u_0 - \bar{u}) \right) - i\pi 
\right.ight.
\]

\[
+ \left\{ \eta(A, 1 - \bar{u}^2) - \eta(A, u_0^2 - \bar{u}^2) - \eta(u_0 - \bar{u}, u_0 + \bar{u}) \right\} S(u_0)
\]

\[
+ \left\{ \eta(A, 1 - \bar{u}^2) - \eta(A, 1 - \bar{u}^2) \right\} \ln \left( \frac{u_0 - 1}{u_0 + 1} \right) \right\} \right. \}
\]

(B.24)

By assumption the sign of \( \text{Im}(Au^2 + B) \) is constant when \( u \) spans the range \([0, +\infty]\), which means that \( S(A) = S(B) \). As \( S(B/A) = S(1 + B/A) \) then \( \eta(A, B/A) = \eta(A, 1 + B/A) \) or in terms of \( \bar{u} \) \( \eta(A, -\bar{u}^2) = \eta(A, 1 - \bar{u}^2) \), so the eq. (B.24) simplifies:

\[
K_{0,\infty}^C(A, B, u_0^2) = \frac{i\pi}{2u_0} \left[ S(\bar{u}) \left( \ln(u_0 + \bar{u}) - \ln(u_0 - \bar{u}) \right) - i\pi
\right.
\]

\[
+ \left\{ \eta(A, -\bar{u}^2) - \eta(A, u_0^2 - \bar{u}^2) - \eta(u_0 - \bar{u}, u_0 + \bar{u}) \right\} S(u_0)\right\}
\]

(B.25)

A comment is in order here. We used the “trick” (B.23) to obtain eq. (B.24) in an economical way. One shall be cautious that practical calculations, especially of four-point functions with general complex masses, involve \( K_{0,\infty}^C(A, B, u_0^2) \) and \( K_{1,\infty}^C(A', B', u_0'^2) \) where the arguments \( (A, B, u_0^2) \) differ from \( (A', B', u_0'^2) \) so that no cheap simplification can be made. A closer look reveals though that some pairs of \( K_{0,\infty}^C(A, B, u_0^2) \) and \( K_{1,\infty}^C(A', B', u_0'^2) \) may be combined using Cauchy’s theorem into analytic continuations of some \( K_{0,1}^C(A', B', u_0'^2) \) defined by contour integrals along some deformations \((0, 1)\) of the segment \([0, 1]\) designed to wrap the cuts of the logarithms \( \ln(A'\bar{u}^2 + B') \). In this respect see also the discussion at the end of appendix A.
C Prescription for the imaginary part of $\det(S)$: general complex mass case

This appendix extends the appendix F of ref. [1] for the complex mass case. Let us recap the result found in [1] (cf. eq. F.10)

$$\det [S + i\lambda E] = \det (S) + i\lambda B \det (S)$$

$$= \det (S) + i\lambda (-1)^{N-1} \det (G) \quad (C.1)$$

with $E_{ij} = 1$ for all $i, j = 1, \cdots, N$. It holds whether $\lambda$ is infinitesimal or finite: it thus tells the sign of $\text{Im}(\det (S))$ also for the particular complex mass case where the imaginary parts of all internal masses squared would be equal; however it is not enough to extract the sign of the imaginary part of $\det (S)$ in the general complex mass case. This general case is addressed below, and contains the one in ref. [1] as a particular subcase. Let us note, in this appendix, $Q_R = \text{Re}(Q)$ and $Q_I = \text{Im}(Q)$ for any complex number $Q$.

As shown in the appendix C (cf. eq. (C.9) of ref. [1]), the determinant of the $S$ matrix can be expressed in term of the determinant of the Gram matrix $G^{(N)}$ obtained by singling out the line and column $N$ of the $S$ matrix, as well as the matrix of cofactors of $G^{(N)} \text{Cof} [G^{(N)}]$:

$$\det (S) = (-1)^{N-1} \left\{ S_{NN} \det (G) + V^{(N)} \cdot \text{Cof} [G^{(N)}] \cdot V^{(N)} \right\} \quad (C.2)$$

The matrix $G^{(N)}$ is real, since all internal masses appearing in $S$ cancel among one another in the expression of $G^{(N)}$, all the imaginary parts are thus located in $S_{NN}$ and the $V^{(N)}$ and more precisely, the imaginary part of $\det (S)$ is linear in the latter and given by:

$$\text{Im} [\det (S)] = (-1)^{N-1} \left\{ S_{NNI} \det (G) \right. + 2 \left. V_R^{(N)} \cdot \text{Cof} [G^{(N)}] \cdot V_I^{(N)} \right\} \quad (C.3)$$

Notice that because of the definition of the vector $V^{(N)}$ (cf. eq. (2.4)), the components of its imaginary part are just the difference of the imaginary parts of two masses squared: $V_{ij}^{(N)} = -(m_{ij}^2 - m_{iN}^2)$ for $j = 1, \cdots, N - 1$. Furthermore, we have that $S_{NNI} = -2 m_I^2$. In the particular case where all masses squared have the same imaginary part $m_I$ which is negative, we have: $S_{NN} = -2m_I^2$, $V_I^{(N)} = 0$ and we recover the result (F.10) of [1]. However in the general complex mass case, the sign of $\text{Im} [\det (S)]$ is a more complicated function of the imaginary parts of the masses squared and of the Gram matrix whose sign depends on the kinematics.

According to appendix C of [1] the subtraction of the line and the column $N$ leads from the

\[12\] In particular when all masses squared are real, the Feynman contour prescription effectively provides a common infinitesimal Im part $-\lambda$. 

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$S$ matrix to the block matrix $\hat{S}^{(N)}$ written as:

$$\hat{S}^{(N)} = \begin{bmatrix}
- G^{(N)} & V^{(N)} \\
- V^{(N)\, T} & S_{NN}
\end{bmatrix}$$  \hspace{1cm} (C.4)

This matrix can be decomposed into its real and imaginary parts $\hat{S}_R^{(N)}$ and $\hat{S}_I^{(N)}$. Due to the fact that the Gram matrix $G^{(N)}$ is real, the blocks constituting the last two matrices are:

$$\hat{S}_R^{(N)} = \begin{bmatrix}
- G^{(N)} & V_R^{(N)} \\
- V_R^{(N)\, T} & S_{NNR}
\end{bmatrix}$$  \hspace{1cm} (C.5)

$$\hat{S}_I^{(N)} = \begin{bmatrix}
0 & V_I^{(N)} \\
- V_I^{(N)\, T} & S_{NNI}
\end{bmatrix}$$  \hspace{1cm} (C.6)

Let us note $b_{[R]\, j}, j = 1, \cdots, N$ the reduction coefficients solving the equation:

$$S_R \cdot b_{[R]} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$ \hspace{1cm} (C.7)

and $B_{[R]} = \sum_{j=1}^{N} b_{[R]\, j}$. With the help of eqs. (C.29), (C.30) and (C.31) in appendix C of [1], the solution of eq. (C.7) reads\(^\text{13}\):

$$B_{[R]} = (-1)^{N-1} \frac{\det(G)}{\det(S_R)}$$ \hspace{1cm} (C.8)

$$b_{[R]\, j} = (-1)^{N-1} \frac{1}{\det(S_R)} \left( \text{Cof} \left[ G^{(N)} \right] \cdot V_R^{(N)} \right)_{j}$$ \hspace{1cm} (C.9)

we wrote this solution in a way such that it is well behaved in the case where $\det(G) = 0$, cf. appendix C of [1]. Putting the expressions for $V_I^{(N)}$ and $S_{NNI}$ into eq. (C.3) and using the eqs. (C.8) and (C.9), we get an appealing form for the imaginary part of $\det(S)$:

$$\text{Im} [\det(S)] = -2 \det(S_R) \sum_{j=1}^{N} m_{1, j}^2 b_{[R] j}$$

$$= -2 \sum_{j=1}^{N} m_{1, j}^2 \overline{b}_{[R] j}$$ \hspace{1cm} (C.10)

\(^\text{13}\)The solution is given for non vanishing $\det(G)$, for special cases see ref. [1]. Let us remind that $\overline{b}_{[R]} = b_{[R]} \det(S_R)$.
When the \( m^2_{ij} \) are all equal eq. (C.10) reduces to \((-1)^{N-1} (-2 m_I^2) \det(G)\) whose sign is readily that obtained from \( \det(G) \). In our convention \( m_I^2 \) is negative and so the sign of the imaginary part of \( \det(S) \) in this case is the same as the one appearing in the real mass case as it should be (cf. appendix F of [1]). When the \( m^2_{ij} \) are unequal, the sign of eq. (C.10) is not explicit and may differ from the sign of \((-1)^{N-1} \det(G)\) depending on the kinematics, if the various \( b_{[R]} \) happen to have different signs.

Let us stress that the \( b_{[R]} \) are not the real parts of the \( b \), nor is \( \det(S_R) \) the real part of \( \det(S) \) in general, namely

\[
\text{Re}(\det(S)) - \det(S_R) \propto \left(V^{(N)}_I \cdot \text{Cof}[G^{(N)}] \cdot V^{(N)}_I \right)
\]

Yet, when \( V^{(N)}_I = 0 \) regardless of \( S_{NNI} \),

\[
b = \frac{\det(S_R)}{\det(S)} b_{[R]} ^{R}
\]

**D Location of cuts in the complex mass case**

Let us consider \( \ln(A z^2 + B) \) for two complex numbers \( A \) and \( B \) whose imaginary parts are non vanishing. We borrow from appendix B the function \( S(z) \equiv \text{sign}(\text{Im}(z)) \). We want to determine what are the conditions on \( A \) and \( B \) in order that one of the cuts of the logarithm crosses the real axis between 0 and 1. A necessary condition for that is given by the fact that the imaginary part of \( A z^2 + B \) changes its sign when \( z \) spans the real segment \([0, 1]\), which translates into \( S(A + B) \neq S(B) \). This last inequality implies that \( S(A) \neq S(B) \). For any complex \( Q \) let us note \( Q_R \equiv \text{Re}(Q) \) and \( Q_I \equiv \text{Im}(Q) \). With this new notation, the two conditions on the signs of the imaginary parts of \( A, B \) and \( A + B \) reads:

\[
0 < -\frac{B_I}{A_I} < 1 \quad \text{(D.1)}
\]

The logarithm considered has two cuts located where the two following conditions are simultaneously fulfilled:

\[
\begin{align*}
\text{Im}(A z^2 + B) &= 0 \quad \text{(D.2)} \\
\text{Re}(A z^2 + B) &\leq 0 \quad \text{(D.3)}
\end{align*}
\]

The branch points are located where the last inequality saturates i.e. at \( z_{\pm} = \pm (-B/A)^{1/2} \).

Eq. (D.2) reads:

\[
A_I (z^2)_R + A_R (z^2)_I + B_I = 0
\]

which is solved in \((z^2)_R\) parametrically in \((z^2)_I\) by:

\[
(z^2)_R = -\frac{1}{A_I} \left[ A_R (z^2)_I + B_I \right] \quad \text{(D.4)}
\]
Substituting eq. (D.4) into the l.h.s. of ineq. (D.3) we get:

\[ \text{l.h.s. (D.3)} = -S(A) \left| \frac{A}{A_I} \right|^2 [(z^2)_I - (z^2_+)_I] \]  

(D.5)

Since \((z^2)_I = 2z_R z_I\), the cut in the half complex plane \(\{z_R \geq 0\}\) slashes through \(\infty\) in the quadrant \(\{z_R > 0, S(A) z_I > 0\}\) whatever the sign of \(z_+\). This cut crosses the real segment \([0, 1]\) at the point \(z_c = (-B_I/A_I)^{1/2}\) if and only if the branch point \(z_+\) is not in the quadrant slashed through to \(\infty\) by the cut i.e. if and only if \(S(A)(z^2_+)_I < 0\), i.e. if and only if \(S(A)(z^2_+)_I < 0\). The latter condition reads explicitly:

\[ S(A) [B_I A_R - B_R A_I] < 0 \]  

(D.6)

So, the conditions necessary and sufficient for the cut in the half complex plane \(\{z_R \geq 0\}\) to cross the real segment \([0, 1]\) are given by eqs. (D.1) and (D.6).

Although the goal of this appendix has been reached, more details about the nature of the support of the cut and its parametrisation in term of \(z_I\) are given in the rest of this appendix.

Eq. (D.2) alternatively reads:

\[ A_I (z_R)^2 + 2 A_R z_I z_R + [B_I - A_I (z_I)^2] = 0 \]  

(D.7)

It is the Cartesian equation of a hyperbola since the coefficients of \((z_R)^2\) and \((z_I)^2\) are opposite. The two branches of hyperbola are symmetric to each other w.r.t. the origin as (D.7) is invariant under the parity transformation \(z_R \rightarrow -z_R, z_I \rightarrow -z_I\). Let us solve eq. (D.7) in \(z_R\) parametrically in \(z_I\). The discriminant \(\Delta'\) given by

\[ \Delta' = |A|^2 (z_I)^2 - B_I A_I > 0 \]

is manifestly > 0 for \(z_I\) spanning all \(\mathbb{R}\), so that the two roots are real for all \(z_I\) real. The product of these roots given by

\[ \Pi = -[(z_I)^2 + z^2_c] < 0 \]

indicates that the two roots have opposite signs. Let us label them \(z_{R\xi}, \xi = \pm\) as:

\[ z_{R\xi} = \frac{-S(A) A_R z_I + \xi \sqrt{\Delta'}}{|A_I|} \]  

(D.8)

and for all \(z_I\) real we have:

\[ z_{R-} < 0 < z_{R+} \]

\(^{14}\)We could have chosen just as well \((z^2)_I\) instead of \((z^2_+)_I\) in eq. (D.5) but, in the rest, we will consider only the cut in the half plane \(z_R > 0\).
We now focus on \( z_{R+} \). The variation of \( z_{R+} \) with \( z_I \) is captured by computing

\[
\frac{\partial}{\partial z_I} z_{R+} = \frac{1}{|A_I|} \left[ -S(A) A_R + \frac{|A|^2 z_I}{\sqrt{\Delta'}} \right]
\]  

(D.9)

whose sign is given by

\[
-S(A) A_R \sqrt{\Delta'} + |A|^2 z_I
\]  

(D.10)

Let us compute

\[
\mathcal{D} = -A_R^2 \Delta' + |A|^4 (z_I)^2
= A_I^2 |A|^2 \left[ (z_I)^2 - z_{I*}^2 \right]
\]  

(D.11)

with

\[
z_{I*} = \frac{|A_R|}{|A|} z_c
\]  

(D.12)

so that

\[
\mathcal{D} \text{ is } \left\{ \begin{array}{ll}
> 0 & \text{when } (z_I)^2 > z_{I*}^2 : z_I \text{ imposes its sign to } (\partial z_{R+}/\partial z_I) \\
< 0 & \text{when } (z_I)^2 < z_{I*}^2 : -S(A) A_R \text{ imposes its sign to } (\partial z_{R+}/\partial z_I)
\end{array} \right.
\]

Thus,

- if \(-S(A) A_R < 0\), \( z_{R+} \searrow \) for \( z_I < z_{I*} \), \( z_{R+} \) reaches its minimum \((z_{R+})_{min}\) at \( z_I = z_{I*} \), \( z_{R+} \nearrow \) for \( z_I > z_{I*} \);

- if \(-S(A) A_R > 0\), \( z_{R+} \searrow \) for \( z_I < -z_{I*} \), \( z_{R+} \) reaches its minimum \((z_{R+})_{min}\) at \( z_I = -z_{I*} \), \( z_{R+} \nearrow \) for \( z_I > -z_{I*} \).

As can be checked explicitly, the minimum \((z_{R+})_{min}\) takes the same analytic expression in both cases, and the latter is given by:

\[
(z_{R+})_{min} = \frac{|A_I|}{|A|} z_c
\]  

(D.13)

which is manifestly in \( ]0, z_c[ \) \( (z_c < 1 \) being also the value of \( z_{R+} \) when \( z_I = 0 \)).

When \( z_I \) is \(< 0\) and large,

\[
z_{R+} \sim -\frac{S(A) A_R + |A|}{|A_I|} z_I \to +\infty
\]

When \( z_I \) is \(> 0\) and large,

\[
z_{R+} \sim +\frac{-S(A) A_R + |A|}{|A_I|} z_I \to +\infty
\]

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Let us show that $z_I z_{R+}$ is a monotonously growing function of $z_I$. The derivative w.r.t. $z_I$ is given by
\[
\frac{\partial}{\partial z_I} (z_I z_{R+}) = \frac{1}{|A_I|} \left\{ -S(A) A_R z_I + \sqrt{\Delta'} + z_I \left[-S(A) A_R + \frac{|A|^2 z_I}{\sqrt{\Delta'}} \right] \right\}
\]
whose sign the same as $(\Delta' + |A|^2 z_I^2) - 2S(A) A_R z_I \sqrt{\Delta'}$, which is $> 0$ since
\[
(\Delta' + |A|^2 z_I^2)^2 - 4 A_R^2 z_I^2 \Delta' = A_I^2 \left[ 4 |A|^2 z_I^4 - 4B_I A_I z_I^2 + B_I^2 \right]
\]
\[
> 0 \quad q.e.d.
\]

Therefore, the r.h.s. of eq. (D.5) is $-S(A) \times$ a monotonously growing function of $z_I$ while $z_I$ spans all $R$. It vanishes once i.e. at the branch point $z_+$. Its sign is $S(A)$ when $z_I < z_+ I$ and $-S(A)$ when $z_I > z_+ I$. The cut corresponds to the arc such that this sign is “−”.

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