Solar System Tests of Higher-Dimensional Gravity

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ABSTRACT

The classical tests of general relativity — light deflection, time delay and perihelion shift — are applied, along with the geodetic precession test, to the five-dimensional extension of the theory known as Kaluza-Klein gravity, using an analogue of the four-dimensional Schwarzschild metric. The perihelion advance and geodetic precession calculations are generalized for the first time to situations in which the components of momentum and spin along the extra coordinate do not vanish. Existing data on light-bending around the Sun using long-baseline radio interferometry, ranging to Mars using the Viking lander, and the perihelion precession of Mercury all constrain a small parameter $b$ associated with the extra part of the metric to be less than $|b| < 0.07$ in the solar system. An order-of-magnitude increase in sensitivity is possible from perihelion precession, if better limits on solar oblateness become available. Measurement of geodetic precession by the Gravity Probe B satellite will improve this significantly, probing values of $b$ with an accuracy of one part in $10^4$ or more.

Subject headings: gravitation — relativity — solar system: general
1. Introduction

There is now a substantial literature on the higher-dimensional extension of Einstein’s general theory of relativity known as Kaluza-Klein gravity (Overduin and Wesson 1997, Wesson 1999). There are several ways to test the theory, with perhaps the most straightforward involving the motion of test particles in the field of a static, spherically-symmetric mass like the Sun or the Earth. Birkhoff’s theorem in the usual sense does not hold in higher dimensions (Bromnikov and Melnikov 1997, Schmidt 1997), so some question arises in identifying the appropriate metric to use for this problem. In the five-dimensional (5D) case (with one extra coordinate $y \equiv x^4$), most attention has focused on the soliton metric (Gross and Perry 1983, Sorkin 1983, Davidson and Owen 1985), which satisfies the 5D vacuum field equations, reduces to the standard four-dimensional (4D) Schwarzschild solution on hypersurfaces $y = \text{const}$, and contains no explicit $y$-dependence. The assumption of a vacuum in 5D is consistent with the spirit of Kaluza’s idea, that 4D matter and gauge fields appear as a manifestation of pure geometry in the higher-dimensional world. The soliton metric has been generalized in various ways to incorporate time-dependence (Liu et al. 1993), $y$-dependence (Billyard and Wesson 1996) and electric charge (Liu and Wesson 1997), among other things (eg, Wesson and Liu 1998); see for review Overduin and Wesson (1997). We confine ourselves here to the original (two-parameter) soliton metric.

The motion of test bodies in the gravitational field of the soliton can be studied using the familiar classical tests of general relativity (gravitational redshift, light deflection, perihelion advance and time delay), along with the geodetic precession test. Work done so far along these lines (Lim, Overduin and Wesson 1995; Kalligas, Wesson and Everitt 1995, hereafter “KWE”) has demonstrated the existence of small but potentially measurable departures from the standard 4D Einstein predictions. In the present paper, we extend these earlier calculations in several ways, clarifying the physical meaning of the light deflection and time delay results for massless test particles and presenting new generalizations of the perihelion shift and geodetic precession formulas for massive ones. We take special care to compare our results to the latest experimental data in each case, obtaining new numerical constraints on the small parameter $b$ associated with the extra part of the soliton metric.

2. The Soliton Metric

In what follows, lowercase Greek indices $\mu, \nu \ldots$ will be taken to run over $0, 1, 2, 3$ as usual, while capital Latin indices $A, B, C \ldots$ run over all five coordinates $(0, 1, 2, 3, 4)$. Units are such that $G = c = 1$
except where stated otherwise. It is important to distinguish between the 4D line element \( (ds) \) and its 5D counterpart \( (dS) \), the two being related by

\[
dS^2 = ds^2 + g_{44} dy^2. \tag{1}
\]

To interpret an expression containing \( d/dS \) physically, one can always make the conversion

\[
\frac{d}{dS} = \frac{ds}{dS} \frac{d}{ds} = \sqrt{1 - g_{44} \left( \frac{dy}{dS} \right)^2} \frac{d}{ds}. \tag{2}
\]

We emphasize in particular that \( d/dS \neq d/ds \) if \( dy/dS \neq 0 \).

The soliton metric may be written (following Gross and Perry 1983 but switching to nonisotropic form, and defining \( a \equiv 1/\alpha \), \( b \equiv \beta/\alpha \) and \( M \equiv 2m \))

\[
dS^2 = A^a dt^2 - A^{-a-b} dr^2 - A^{1-a-b} \times r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) - A^b dy^2, \tag{3}
\]

where \( A(r) \equiv 1 - 2M/r \), \( M \) is a parameter related to the mass of the object at the center of the geometry, and the constants \( a, b \) satisfy a consistency relation

\[
a^2 + ab + b^2 = 1, \tag{4}
\]

so that any two of \( M, a, b \) may be taken as independent metric parameters. We will treat \( b \) as the primary free parameter of the theory in what follows, noting that the 4D Schwarzschild metric is recovered (on hypersurfaces \( y = \text{const} \)) in the limit \( b \to 0 \) (and \( a \to +1 \)). In general, larger values of \( |b| \) will give rise to increasing departures from Einstein’s theory, subject to the upper bound \( |b| \leq 2/\sqrt{3} \approx 1.15 \) imposed by equation (4). Possible theoretical expectations for this parameter in the solar system and elsewhere are discussed further in §8.

3. Equation of Motion

We proceed now with the analysis of experimental constraints. The Lagrangian for a test particle in the field described by the metric (3) is

\[
\mathcal{L} = \left[ A^a \dot{t}^2 - A^{-a-b} \dot{r}^2 - A^{1-a-b} \times r^2 \left( \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right) - A^b \dot{y}^2 \right]^{1/2}, \tag{5}
\]

where \( A(r) \equiv 1 - 2M/r \), \( M \) is a parameter related to the mass of the object at the center of the geometry, and the constants \( a, b \) satisfy a consistency relation

\[
a^2 + ab + b^2 = 1, \tag{4}
\]

so that any two of \( M, a, b \) may be taken as independent metric parameters. We will treat \( b \) as the primary free parameter of the theory in what follows, noting that the 4D Schwarzschild metric is recovered (on hypersurfaces \( y = \text{const} \)) in the limit \( b \to 0 \) (and \( a \to +1 \)). In general, larger values of \( |b| \) will give rise to increasing departures from Einstein’s theory, subject to the upper bound \( |b| \leq 2/\sqrt{3} \approx 1.15 \) imposed by equation (4). Possible theoretical expectations for this parameter in the solar system and elsewhere are discussed further in §8.
where the overdot represents differentiation with respect to an affine parameter $\lambda$ along the geodesics.

The Euler-Lagrange equations read

$$\frac{d}{d\lambda} \left( \frac{\partial L}{\partial \dot{x}^C} \right) - \frac{\partial L}{\partial x^C} = 0.$$  \hfill (6)

We confine ourselves to orbits with $\theta = \pi/2$ and $\dot{\theta} = 0$, so that $L$ becomes

$$L = \left( A^a \dot{t}^2 - A^{-a-b} r^2 - A^{1-a-b} r^2 \dot{\phi}^2 - A^b \dot{y}^2 \right)^{1/2}. \hfill (7)$$

We can identify three constants of motion

$$\ell \equiv \frac{1}{L} A^a \dot{t} = A^a \frac{dt}{dS},$$
$$h \equiv \frac{1}{L} A^{-a-b} r^2 \dot{\phi} = A^{1-a-b} r^2 \frac{d\phi}{dS},$$
$$k \equiv \frac{1}{L} A^b \dot{y} = A^b \frac{dy}{dS}, \hfill (8)$$

where we have used the relation $L = dS/d\lambda$. From these equations we find that

$$\left( \frac{dr}{d\phi} \right)^2 + Ar^2 - \left( \frac{\ell^2}{h^2} A^{2-2a-b} - \frac{k^2}{h^2} A^{2-a-2b} \right) - \frac{1}{h^2} A^{2-a-b} r^4 = 0. \hfill (9)$$

The derivation here differs slightly from that of KWE, where $L \equiv (dS/d\lambda)^2$. Although the two approaches are physically equivalent, we have found that results are obtained more simply if the three constants of motion $\ell, h, k$ are defined in terms of $d/dS$ rather than $d/d\lambda$ (or $d/ds$).

### 4. Light deflection

Experimental upper limits on possible violations of local Lorentz invariance are extremely tight (Will 1993), so that we are justified in assuming that photons follow 4D null geodesics, $ds = 0$. The situation is not so clear with regard to the 5D line element. However, it is economical to follow KWE and suppose that \textit{all particles follow ND null geodesics in N-dimensional gravity, whether massive or not}.\footnote{This assumption is supported by various lines of argument. In one version of 5D gravity, for example, the fifth coordinate $y$ is related to \textit{rest mass} $m$ (Wesson 1984), so that one has $dS^2 = ds^2 + g_{44}(G/c^2)^2 dm^2$. If all particles move on 5D null geodesics, then $ds^2 = -g_{44}(G/c^2)^2 dm^2$. It then follows that $ds = 0$ for photons.} Proceeding on...
this assumption, and substituting $ds = dS = 0$ into equation (1), we get $dy = 0$ also, so that $\ell, h \to \infty$ and $k$ is undefined. The ratios $\ell/h$ and $k/h$ are however well-behaved, and read

$$\frac{\ell}{h} = r^{-2} A^{2a+b-1} \frac{dt}{d\phi} = \text{finite},$$

$$\frac{k}{h} = r^{-2} A^{a+2b-1} \frac{dy}{d\phi} = 0. \quad (10)$$

For self-consistency, therefore, the terms in $k/h$ can be dropped from KWE equations (7),(8),(11) and (12). Equation (8) of that paper, in particular, reduces to

$$\left(\frac{du}{d\phi}\right)^2 + A u^2 - \frac{\ell^2}{h^2} A^{2-2a-b} = 0, \quad (11)$$

and the definition of the parameter $p$, KWE equation (12), becomes just

$$p \equiv -(2 - 2a - b) \frac{\ell^2}{h^2}. \quad (12)$$

The photon’s trajectory is deflected by an angle

$$\delta \phi = \omega = \frac{4M}{r_o} + 2Mpr_o, \quad (13)$$

which agrees with KWE equation (18.1). At the closest approach to the central body, we have $u = u_o = 1/r_o$ and $du/d\phi = 0$, so that equation (11) gives

$$\frac{\ell^2}{h^2} = A_o^{2a+b-1} u_o^2 = \frac{1}{r_o^2} + O(\varepsilon), \quad (14)$$

where $\varepsilon \equiv M$ is a small parameter. Putting equations (12) and (14) into equation (13), we find for the final light deflection angle

$$\delta \phi = (4a + 2b) \frac{M}{r_o} + O(\varepsilon^2), \quad (15)$$

as in KWE equation (18.2), where however it is presented as a special case $k = 0$. We see here that equation (15) is in fact entirely general for light deflection, and does not depend on any choice of $k$, which is in any case undefined when $ds = dS = 0.$

which have $m = \text{const} = 0$. For massive particles with $ds \neq 0$, one expects variations in rest mass $m$, which are however below currently detectable levels, owing to the small size of the dimension-transposing constant $G/c^2$ (Overduin and Wesson 1997, 1998). Recent work on incorporating non-relativistic quantum theory into higher-dimensional gravity also strongly suggests that all test particles travel on ND null geodesics in the classical limit (Seahra 2000).
To obtain experimental constraints from the light deflection result, let us express equation (15) for the Sun in terms of the deviation $\Delta_{LD}$ from the general relativity prediction $\delta \phi_{GR}$, as follows

$$\delta \phi = \delta \phi_{GR}(1 + \Delta_{LD}) ,$$

(16)

where (to first order in $\varepsilon$):

$$\delta \phi_{GR} \equiv 4M_\odot/r_o ,$$

$$\Delta_{LD} \equiv a + b/2 - 1 .$$

(17)

Using the consistency relation (4) we find

$$a = -b/2 \pm (1 - 3b^2/4)^{1/2} .$$

(18)

Theoretical and numerical work indicates that $|b| \ll 1$ in the solar system (§8), and our experimental limits bear this out. The negative roots of equation (18) may also be ignored, as they are inconsistent with the limiting Schwarzschild case, and also imply the possibility of negative gravitational and/or inertial soliton mass (Gross and Perry 1983, Lim et al. 1995; Overduin and Wesson 1997). We therefore take

$$a = 1 - b/2 - 3b^2/8 + O(b^4) ,$$

(19)

in the solar system, whereupon equation (17) gives

$$\Delta_{LD} = -3b^2/8 + O(b^4) .$$

(20)

The best available constraints on $\Delta_{LD}$ come from long-baseline radio interferometry, which implies that $|\Delta_{LD}| \leq 0.0017$ (Robertson et al. 1991, Lebach et al. 1995). We therefore infer an upper limit

$$|b| \leq 0.07 ,$$

(21)

for the Sun. This could potentially be tightened by more than an order of magnitude using a proposed astrometric optical interferometer sensitive to departures from Einstein’s theory of as little as $|\Delta_{LD}| \leq 10^{-5}$ (Reasenberg and Shapiro 1986).

It is important to bear in mind, however, that the parameter $b$ characterizing the soliton metric (3) is not a universal constant of nature like $G$ or $c$, but may in principle vary from soliton to soliton. Kaluza-Klein gravity as an alternative to 4D general relativity is therefore best constrained by the application of two or more tests to the same system. With this in mind we can use a recent measurement of light deflection by Jupiter, for which $|\Delta_{LD}| \leq 0.17$ (Treuhaft and Lowe 1991), to obtain

$$|b| \leq 0.7 ,$$

(22)
for that planet. It has also been proposed to measure light deflection by the Earth using the Hipparcos satellite, with an estimated precision of 12% (Gould 1993). Such a test would be sensitive to values of $|b| \leq 0.6$ for the Earth. The Gravity Probe B satellite should also be able to detect this effect by means of its guide star telescope, though with a somewhat lower precision (Adler 2000).

5. Time delay

The arguments in the previous section regarding the parameter $k$ also apply to the time delay (or radar ranging) test, and circular photon orbits as well. That is, terms in $k/h$ and $k/\ell$ may be dropped from KWE equations (20-24) for radar ranging, and KWE equations (28-30) for circular orbits. The final results given there, however — equations (25) and (31) respectively — are correct. In fact, they hold not only for the special case $k = 0$, but quite generally.

In particular, the excess round-trip time delay $\Delta \tau$ for signals emitted from Earth (at distance $r_e$ from the Sun) which graze the Sun (at nearest distance $r_o$) and bounce off another planet (at $r_p$) may be calculated by setting $k/\ell=0$ in KWE equation (24) to obtain

$$\Delta \tau = \Delta \tau_{GR}(1 + \Delta_{TD}),$$

(23)

where (to first order in $\varepsilon$)

$$\Delta \tau_{GR} \equiv 4M_\odot \left[ \ln \left( \frac{r_p + \sqrt{r_p^2 - r_o^2}}{r_o} \right) + \ln \left( \frac{r_o + \sqrt{r_e^2 - r_o^2}}{r_o} \right) \right],$$

$$\Delta_{TD} \equiv a + b/2 - 1$$

$$= -3b^2/8 + O(b^4).$$

(24)

We note that departures from 4D general relativity for time delay have exactly the same form as they do for light deflection.

The best experimental constraint on time delay so far has come from the Viking lander on Mars, and gives $|\Delta_{TD}| \leq 0.002$ (Reasenberg et al. 1979). This leads immediately to the upper bound

$$|b| \leq 0.07,$$

(25)

for the Sun, exactly the same as the limit obtained in the case of light deflection using long-baseline interferometry.
Keeping in mind that values of $b$ can differ from soliton to soliton, however, it is possible that different physical setups could provide new information. For instance, one could attempt to measure $b$ for the Earth by sending grazing signals from an orbiting satellite past our planet and bouncing them off the Moon; retroreflectors left there by Apollo astronauts are routinely used for lunar laser ranging (Williams et al. 1996). Substituting $M_e$ for $M_\odot$ and replacing $r_e, r_p$ and $r_o$ with the appropriate distances, we find an expected excess time delay of order 400 ps using a satellite in geostationary orbit. This is well above the currently available resolution of $\sim 50$ ps (Samain et al. 1998). The feasibility of such a proposal would likely be limited by the weakness of the reflected signal. Better results might be obtained by active ranging between two orbiting satellites, or by statistical analysis of ranging data between two such satellites and an Earth station (the latter would however require excellent atmospheric modelling).

In the same vein, one might attempt to measure $b$ for the Moon by grazing it with signals from the Earth and bouncing them off the Viking lander on Mars. This might be done when Mars is at nearest approach (on the same side of the Sun as the Earth) to minimize signal contamination from the competing effect of the Sun. Substituting $M_m$ for $M_\odot$ in equation (23), however, and replacing $r_e, r_p$ and $r_o$ with the appropriate distances, we find that the Moon’s excess time delay (of order 10 ps) would be so short as to make this a daunting task at present.

6. Perihelion advance

We now switch our attention to massive test particles. In terms of a new variable $u \equiv 1/r$, equation (4) becomes

$$
\left( \frac{du}{d\phi} \right)^2 + Au^2 - \left( \frac{\ell^2}{h^2} A^{2-a-b} - \frac{k^2}{h^2} A^{2-a-2b} \right) - \frac{1}{h^2} A^{2-a-b} = 0 .
$$

(26)

Differentiating with respect to $\phi$ (and letting primes denote $d/d\phi$), we find that noncircular orbits ($u' \neq 0$) are governed by the following differential equation

$$
u'' + (1 + \gamma \epsilon) u = B + \epsilon B^{-1} u^2 + O(\epsilon^2)
$$

(27)

where five new quantities have been introduced

$$
\gamma \equiv -\frac{f}{3d} , \quad \epsilon \equiv 3MB , \quad B \equiv \frac{M d}{h^2} ,
$$

$$
d \equiv (2-a-b) - \ell^2(2-2a-b)
$$
$$+ k^2 (2 - a - 2b),$$

$$f \equiv 2 (2 - a - b)(-1 + a + b)$$

$$+ 2\ell^2 (-2 + 2a + b)(-1 + 2a + b)$$

$$+ 2k^2 (2 - a - 2b)(-1 + a + 2b).$$  (28)

These expressions agree with KWE equations (32-36). (We have however chosen to relabel their $e$ as $f$, for reasons that will become clear shortly.)

The solution of the differential equation (27) is

$$u = \frac{1}{r} = B + \left( 1 - \frac{\gamma}{2} \right) C \cos \left\{ \left[ 1 - \epsilon \left( 1 - \frac{\gamma}{2} \right) \right] \phi \right\}$$

$$+ \epsilon (1 - \gamma) B + \epsilon \frac{C^2}{2B} \left( 1 - \frac{\gamma}{2} \right)^2$$

$$- \epsilon \frac{C^2}{6B} \left( 1 - \frac{\gamma}{2} \right)^2 \cos 2\phi + O(\epsilon^2),$$  (29)

where $C$ is an integration constant. [This result differs slightly from KWE equation (37), where the factors of $(1 - \gamma/2)^2$ were omitted.] Equation (29) can be written in a physically more transparent form by introducing two new quantities $e$ and $\omega$ via

$$\left( 1 - \frac{\gamma}{2} \right) C \equiv B e, \quad 1 - \epsilon \left( 1 - \frac{\gamma}{2} \right) \equiv \omega.$$  (30)

With these definitions, we find that

$$u = \frac{1}{r} = B (1 + e \cos \omega \phi) + \frac{1}{2} \epsilon B^2\left[ -e^2 \cos 2\omega \phi \right.$$

$$+ 6 \left( 1 - \gamma + \frac{1}{2} e^2 \right)] + O(\epsilon^2),$$  (31)

where $\epsilon \equiv M$ is a small parameter as before, and

$$\omega = 1 - 3\epsilon B \left( 1 + \frac{f}{6d} \right) + O(\epsilon^2).$$  (32)

The first term on the right-hand side of equation (31) is of order $\epsilon^0$, and shows explicitly the elliptical shape of the orbit. This is then modified by the second term, of order $\epsilon^1$. Note that $e$ is just the eccentricity of the ellipse. The angular shift between two successive perihelia is given by

$$\delta \phi = \phi - 2\pi = \frac{6\pi M^2d}{h^2} \left( 1 + \frac{f}{6d} \right) + O(\epsilon^2),$$  (33)

in agreement with the final result (38.1) of KWE. It should be emphasized that the angular momentum $h$ is not in general the same quantity in 5D as it is in 4D. In particular, putting equations (2) and (3) into
the second of equations (33), we find

\[ h = A^{1-a-b} \frac{d\phi}{dS} = A^{1-a-b} \frac{d\phi}{ds} \sqrt{1 + A^{-b}k^2} \]

\[ = h_{(4D)} \sqrt{1 + A^{-b}k^2}. \]  \hspace{1cm} (34)

If \( k \neq 0 \), therefore, it follows that \( h \neq h_{(4D)} \).

To eliminate \( h \) from equation (33), let us consider the points along the orbit where \( r \) takes its minimum value \( r_- \) and maximum value \( r_+ \) respectively. From inspection of equation (31) we see that

\[ r_- = B^{-1}(1 + e)^{-1} + O(\varepsilon) \] at \( \omega \Phi = 0 \) and

\[ r_+ = B^{-1}(1 - e)^{-1} + O(\varepsilon) \] at \( \omega \Phi = \pi \). The semimajor axis \( a_o \) of the ellipse is then

\[ a_o \equiv \frac{1}{2} \left( r_- + r_+ \right) = \frac{1}{B(1 - e^2)} + O(\varepsilon), \]  \hspace{1cm} (35)

so that

\[ B \equiv \frac{Md}{h^2} = \frac{1}{a_o(1 - e^2)} + O(\varepsilon), \]  \hspace{1cm} (36)

or

\[ h^2 = \varepsilon(1 - e^2) a_o d + O(\varepsilon^2). \]  \hspace{1cm} (37)

Substituting equation (36) into equation (33), we find

\[ \delta \phi = \frac{6\pi M}{a_o(1 - e^2)} \left[ 1 + \frac{f}{6d} \right] + O(\varepsilon^2). \]  \hspace{1cm} (38)

Only one term in this result remains physically obscure, and that is the ratio \( f/d \). This is given in terms of \( \ell \) and \( k \) by the definitions (28). The latter two constants are related by equation (26) as follows

\[ \ell^2 = h^2 \left[ \left( \frac{du}{d\phi} \right)^2 + A u^2 \right] A^{2a+b-2} + k^2 A^{a-b} + A^a. \]  \hspace{1cm} (39)

Since \( h^2 \) is of order \( \varepsilon^1 \) by equation (37), while \( u \) and \( u' \) are of order \( \varepsilon^0 \) by equation (31), it follows from equation (39) that \( \ell^2 = 1 + k^2 + O(\varepsilon) \). Using the definitions (28), we therefore obtain

\[ \frac{f}{6d} = -1 + a + \frac{2b}{3} + \frac{k^2 b(a-b)/3}{a + k^2(a-b)} + O(\varepsilon), \]  \hspace{1cm} (40)

so that the final perihelion precession angle (38) becomes

\[ \delta \phi = \frac{6\pi M}{a_o(1 - e^2)} \left[ a + \frac{2b}{3} + \frac{k^2 (a-b)b/3}{a + k^2(a-b)} \right] + O(\varepsilon^2). \]  \hspace{1cm} (41)

This represents the generalization of KWE equation (38.2) to cases in which \( k \neq 0 \) (and eccentricity \( e \neq 0 \)). In the special case \( b = 0 \) (and \( a = +1 \), for which the metric (8) reduces to Schwarzschild form on
hypersurfaces \( y = \text{const} \), it is interesting to note that one recovers the standard 4D general relativity result, regardless of the value of \( k \). In this limit, therefore, the perihelion shift test is insensitive to the momentum of the test body along the extra coordinate. And in general, one must choose a soliton with \( b \neq 0 \) in order to distinguish experimentally between test particles with different values of \( k \).

As usual, let us parametrize our result in terms of the departure from 4D general relativity so that

\[
\delta \phi = \delta \phi_{GR}(1 + \Delta_{\nu \nu}) ,
\]

where (to first order in \( \varepsilon \)):

\[
\delta \phi_{GR} \equiv \frac{6 \pi M}{a_0(1-e^2)} ,
\]

\[
\Delta_{\nu \nu} \equiv a + \frac{2}{3} b + \frac{k^2(a-b)b/3}{a+k^2(a-b)} - 1 .
\]

(43)

Theoretical work indicates that \( k \), which is a measure of momentum along the fifth dimension, is related to the charge-to-mass ratio of the test body (Wesson and Liu 1997). For a planet such as Mercury, we may take \( k = 0 \). Putting equation (19) into equation (43), we therefore have

\[
\Delta_{\nu \nu} = \frac{b}{6} - 3b^2/8 + O(b^4) .
\]

(44)

Perihelion precession is thus a potentially more sensitive probe of higher-dimensional gravity than either light deflection or time delay, in that it depends on the first, as well second order in \( b \).

Unfortunately, however, this increased sensitivity is offset in the case of Mercury’s orbit about the Sun by uncertainty in the solar oblateness. The latter introduces a new term \( \xi J_2 \) inside the brackets on the right-hand side of equation (42), where \( \xi \equiv R_\odot^2/2M_\odot a_0(1-e^2) \) and \( J_2 \) is the solar quadrupole moment (Campbell et al. 1983). Dividing through by the orbital period \( T \), we may therefore write for the rate of perihelion advance (to order \( b^3 \))

\[
\Delta \omega \equiv \frac{\delta \phi}{T} = \Delta \omega_{GR}(1 + \xi J_2 + b/6 - 3b^2/8) ,
\]

(45)

where \( \Delta \omega_{GR} \equiv \delta \phi_{GR}/T = 42.98 \) arcsec/century. The observed value of Mercury’s perihelion precession rate is quite close to this value, \( \Delta \omega = 43.11 \pm 0.21 \) arcsec per century (Shapiro, Counselman and King 1976). Experimental data on \( J_2 \) is a good deal more controversial and has ranged over two orders of magnitude, from a maximum value of \( (23.7 \pm 2.3) \times 10^{-6} \) (Dicke and Goldenberg 1967) to a minimum of \( (0.17 \pm 0.02) \times 10^{-6} \) (Duvall et al. 1984). One straightforward least-squares fit to a number of published measurements leads to intermediate value of \( J_2 = 5.0 \times 10^{-6} \), which however implies a general relativistic
precession rate more than two standard deviations away from that observed (Campbell et al. 1983). Such a discrepancy could be explained in the context of higher-dimensional gravity by modelling the Sun as a soliton with $b = -0.062$. This is just consistent with the constraint $|b| \leq 0.07$ from light deflection (§4) and time delay (§5), which is intriguing since these tests probe somewhat independent aspects of relativistic gravity. Improved experimental data relating to any of the three tests would be of great interest.

Conservative limits on $b$ from perihelion precession may be obtained by quoting the results of a recent review in which all available data (to 1997) have been combined to give a weighted mean value for the solar oblateness of $J_2 = (3.64 \pm 2.84) \times 10^{-6}$ (Rozelet and Rösch 1997). Using this uncertainty range, together with that in the observed value of $\Delta \omega$ for Mercury’s orbit, we find that

$$b = -0.03 \pm 0.07,$$

(46)

for the Sun. This is consistent with the bounds obtained from light deflection and time delay. Sensitivity of the perihelion precession test to the value of $b$ could be improved by an order of magnitude if better data on $J_2$ were to become available; the proposed ASTROD mission, for example, might measure this parameter to an accuracy of $5 \times 10^{-8}$ (Ni 1998).

7. **Geodetic effect**

We now move on to consider *spinning* massive test particles with velocity 5-vectors $u^C \equiv dx^C/dS$ and spin 5-vectors $S^C$. The motion of these objects is governed by three central equations; namely, the geodesic equation

$$\frac{d^2x^C}{dS^2} + \hat{\Gamma}^C_{AB} u^A u^B = 0,$$

(47)

the parallel transport equation

$$\frac{dS^C}{dS} + \hat{\Gamma}^C_{AB} S^A u^B = 0,$$

(48)

and the orthogonality condition

$$u^C S_C = 0.$$

(49)

Here $\hat{\Gamma}^C_{AB}$ refers to the 5D Christoffel symbol for the metric (3). This is defined in exactly the same manner as the usual 4D Christoffel symbol, with indices running over five values instead of four (see KWE, Appendix A1 for details).

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2There are some minor typographical errors in this appendix, which we note briefly here. The factors of $(1 - 2M)/r$ in equations (A2.2), (A2.6) and (A2.7) should read $1 - 2M/r$. The same thing applies to
In order to simplify the problem, we follow KWE in assuming that the test particle moves in a circular orbit with $\theta = \pi/2$, $r = r_o$ and $\dot{\theta} = \dot{r} = 0$. Its velocity $u^C$ may then be expressed as follows in terms of the constants of motion $\ell, h$ and $k$, as given by equations (8)

$$u^C \equiv \frac{dx^C}{dS} = (\ell A^{-a}, 0, 0, h r_o^{-2} A^{a+b-1}, k A^{-b}) \ .$$

(50)

From the metric (3), we have

$$1 = A^a \left( u^0 \right)^2 - A^{1-a-b} r_o^2 \left( u^3 \right)^2 - A^b \left( u^4 \right)^2 ,$$

(51)

which, with equation (50), implies

$$\ell^2 - h^2 r_o^{-2} A^{2a+b-1} - k^2 A^{a-b} - A^a = 0 \ .$$

(52)

It may be shown that the motion of the test body as given by equations (50) and (52) is geodesic in the sense of equation (47).

We now propose to generalize the treatment of KWE by leaving the extra component $S^4$ of spin unrestricted, rather than setting it to zero. In fact, writing explicitly $S^C \equiv (S^0, S^1, S^2, S^3, S^4)$, we find that the orthogonality condition (49) imposes the following restriction on the spin components

$$\ell S^0 - h S^3 - k S^4 = 0 ,$$

(53)

so that $S^4$ will not vanish in general, if the parameter $k$ is well-defined.

We now proceed to solve the parallel transport equation (48), taking one value of the index $C$ at a time. To begin with, the $C = 2$ component gives

$$S^2 = \frac{H_2}{r_o} = \text{const} \ , \ H_2 = \text{const} \ .$$

(54)

(Note that, due to our choice of coordinates, $S^0, S^1$ and $S^4$ are dimensionless while $S^2$ and $S^3$ have units of inverse length.) Defining a new function $g = g(S)$ of the 5D proper time, we may write without loss of generality

$$S^0 \equiv H_0 g , \ H_0 = \text{const} \ .$$

(55)

equations (57) and (58) in the main body of KWE. Also, the exponents $-(1/2)$ and $1/2$ in equations (57) and (58) should be switched, in agreement with equations (A2.7) and (A2.2) respectively. These discrepancies do not affect any of the other equations or conclusions reported in KWE, and do not appear in the new reference book on Kaluza-Klein gravity by Wesson (1999).
The $C=0$ component of equation (48) then reads

$$S^1 = -\frac{H_0}{a\ell M} r_o^2 A^{a+1} \frac{dg}{dS}. \quad (56)$$

The $C=4$ component, meanwhile, takes the form

$$S^4 = H_4 g + K_4,$$

$$H_4 = \frac{b k}{a \ell} H_0 A^{a-b} = \text{const},$$

$$K_4 = \text{const}, \quad (57)$$

where we have used equation (56). In a similar way, the $C=3$ component of equation (48) gives

$$S^3 = \frac{H_3}{r_o} g + K_3,$$

$$H_3 = \frac{h H_0}{a \ell M} A^{2a+b-1} \left[ 1 - (1 + a + b) \frac{M}{r_o} \right] = \text{const},$$

$$K_3 = \text{const}. \quad (58)$$

We now solve the $C=1$ component of equation (48), assuming for simplicity that $K_3 = K_4 = 0$. Using equations (50), (55), (57) and (58), we find

$$\frac{dS^1}{dS} = -\frac{H_0}{a \ell r_o^2} r_o^2 A^{a+1} \left\{ a^2 \ell^2 - b^2 k^2 A^{a-b} - \frac{M^2}{h^2} \left[ 1 - (1 + a + b) \frac{M}{r_o} \right]^2 A^{2a+b-1} \right\} g. \quad (59)$$

Differentiating equation (56) with respect to $S$, meanwhile, gives

$$\frac{dS^1}{dS} = -\frac{H_0}{a \ell M} r_o^2 A^{a+1} \frac{d^2 g}{dS^2}. \quad (60)$$

Equating these two expressions, we obtain

$$\frac{d^2 g}{dS^2} = -\Omega^2 g, \quad (61)$$

where

$$\Omega^2 \equiv \frac{h^2}{r_o^2} A^{b-2} \left\{ 1 - (1 + a + b) \frac{M}{r_o} \right\} \left[ a^2 \ell^2 - b^2 k^2 A^{a-b} \right] - \frac{M^2}{h^2} \left( a^2 \ell^2 - b^2 k^2 A^{a-b} \right) \right\}. \quad (62)$$
The general solution of equation (61) is
\[ g(S) = K_1 \sin(\Omega S) + K_2 \cos(\Omega S) . \]
We choose \( K_1 = 1 \) and \( K_2 = 0 \) for simplicity. The spin components are then given by
\[
S^0 = H_0 \sin(\Omega S), \quad S^1 = H_1 \cos(\Omega S), \\
S^2 = \frac{H_2}{r_o}, \quad S^3 = \frac{H_3}{r_o} \sin(\Omega S), \\
S^4 = H_4 \sin(\Omega S),
\]
where \( H_1 \) and \( H_2 \) are arbitrary constants and
\[
H_0 = -\frac{\alpha \ell M}{r_o^2 \Omega} A^{\alpha-1} H_1, \\
H_3 = -\frac{\hbar}{r_o^2 \Omega} A^{\alpha+b-2} \left[ 1 - (1 + a + b) \frac{M}{r_o} \right] H_1, \\
H_4 = -\frac{\beta k M}{r_o^2 \Omega} A^{-b-1} H_1 .
\] (64)
The spatial part of \( S^C \) is thus seen to rotate in the plane of the orbit with angular speed \( \Omega \). Substituting these results into equation (53) yields
\[
\alpha \ell^2 - \frac{\hbar^2}{M r_o} A^{2 \alpha+b-1} \left[ 1 - (1 + a + b) \frac{M}{r_o} \right] \\
- \beta k^2 A^{\alpha-b} = 0 .
\] (65)
Solving simultaneously with equation (62), we obtain for the constants of motion
\[
\ell^2 = A^a \left( 1 + k^2 A^{-b} + \frac{M}{r_o} \left[ \frac{a + (a-b)k^2 A^{-b}}{1 - (1 + 2a + b)M/r_o} \right] \right) , \\
\hbar^2 = M r_o A^{1-a-b} \left[ \frac{a + (a-b)k^2 A^{-b}}{1 - (1 + 2a + b)M/r_o} \right] .
\] (66)
These expressions can be written in terms of a small parameter \( \varepsilon \equiv M \) as usual
\[
\ell^2 = (1 + k^2) \left[ 1 - \left( a - \frac{\beta k^2}{1 + k^2} \right) \frac{M}{r_o} \right] + O(\varepsilon^2) , \\
\hbar^2 = M r_o \left[ a + (a-b)k^2 \right] \left[ 1 + \left( 4a + 3b - 1 \right) \right] \\
+ \frac{2\beta(a-b)k^2}{a + (a-b)k^2} \frac{M}{r_o} + O(\varepsilon^3) .
\] (67)
With the aid of equation (62), we then find for the angular speed of the spin vector
\[
\Omega = \sqrt{\frac{(a + (a-b)k^2)M}{r_o^2} \left[ 1 + \frac{M}{2r_o} \left[ 3(1-a-b) \right. \right. \\
\left. + \frac{b(a-b)k^2}{a + (a-b)k^2} \right] + O(\varepsilon^2) \right]} .
\] (68)
This quantity is not the same as the test body’s orbital angular speed, which is given in terms of the 5D proper time $dS$ as

$$\omega \equiv \frac{d\phi}{dS} = hr_o^{-2}A^{a+b-1}$$

$$= \sqrt{\frac{M}{r_o^3}} A^{(a+b-1)/2} \sqrt{\frac{a + (a - b)k^2 A^{-b}}{1 - (1 + 2a + b)M/r_o}},$$

where we have used equations (8) and (66). In terms of $\varepsilon$

$$\omega = \sqrt{\frac{a + (a - b)k^2 M}{r_o^3}} \left\{ 1 + \frac{M}{r_o} \left( \frac{3 - b}{2} \right. \right.$$  

$$+ \frac{b(a - b)k^2}{a + (a - b)k^2} \left. \right\} + O(\varepsilon^2).$$

(70)

It is precisely the excess of $\Omega$ over $\omega$ that gives rise to the geodetic effect.

Suppose the spin vector $S^C$ is initially oriented in the radial direction; i.e., $H_2 = 0$ at $S = 0$. During one orbit, the test body’s angular displacement $\phi$ goes from 0 to $2\pi$, so that $\delta S = 2\pi/\omega$. In the same period, $S^3$ goes from its initial value of zero at $S = 0$ to its final value at proper time $S$. To first order in $\varepsilon$, the spin vector has advanced through an angle

$$\delta \phi = \frac{r_o[S^3(S) - S^3(0)]}{S^1(0)} + O(\varepsilon^2),$$

$$= 2\pi \frac{H_3}{H_1} \left( \frac{\Omega}{\omega} - 1 \right) + O(\varepsilon^2),$$

$$= -2\pi \left( \frac{\Omega}{\omega} - 1 \right) + O(\varepsilon^2),$$

(71)

where we have used equations (4), (7) and (8). Combining equations (8) and (7), we find that

$$\frac{\Omega}{\omega} - 1 = -\frac{3M}{2r_o} \left[ a + \frac{2}{3} b + \frac{k^2(a - b)/3}{a + (a - b)k^2} \right] + O(\varepsilon^2),$$

(72)

so that the geodetic precession angle can finally be expressed as follows in terms of its deviation from the prediction of 4D general relativity

$$\delta \phi = \delta \phi_{\text{GR}} (1 + \Delta_{\text{GP}}),$$

(73)

where (to first order in $\varepsilon$)

$$\delta \phi_{\text{GR}} \equiv 3\pi M/r_o,$$

$$\Delta_{\text{GP}} \equiv a + \frac{2}{3} b + \frac{k^2(a - b)b/3}{a + k^2(a - b)} - 1.$$
This represents the generalization of KWE equation (66) to cases in which \( S^4 \neq 0 \). Deviations from 4D general relativity have exactly the same form for geodetic precession as they do for perihelion precession. Taking \( k = 0 \) and using equation (19), as in §6, we find that

\[
\Delta_{\text{GP}} = b/6 - 3b^2/8 + O(b^4).
\]

Like the perihelion shift, geodetic precession depends on \( b \) to first as well as second order, and is thus a potentially more sensitive probe of the theory than either light deflection or time delay.

The Gravity Probe B satellite, currently scheduled for launch in early 2001, has been designed to measure deviations from 4D general relativity with a precision of better than \(|\Delta_{\text{GP}}| \leq 2.5 \times 10^{-5}\) (Buchman et al. 1996). Using equation (75), we find that this corresponds to a sensitivity to values as small as

\[
|b| \leq 1 \times 10^{-4},
\]

or better for the Earth — a constraint some five hundred times stronger than any other solar system bound obtained to date, and five thousand times stronger than the only other Earth-based test (light deflection using Hipparcos; §4).

We conclude this section by noting that a complementary analysis of geodetic precession has been carried out for a static, spherically-symmetric 5D metric different from that given by equation (3), one in which the fifth dimension is flat (Mashhoon, Liu and Wesson 1994; Mashhoon, Wesson and Liu 1998). The inclusion of spin is of special importance in this case since the classical tests (based on the equations of motion) alone cannot discriminate between 4D and 5D effects. The geodetic precession rate has been computed, and differs from the 4D Einstein value in the weak-field, low velocity limit (Liu and Wesson 1996). A preliminary interpretation of the discrepancy indicates, however, that it is likely to be somewhat below the threshold of detection by Gravity Probe B (Overduin and Wesson 1998).

### 8. Discussion

Having obtained upper limits on \(|b|\) of order 0.07 (and possibly \(10^{-4}\)) from experiment, we consider here the range of values that might be expected for this parameter on theoretical grounds. These turn out to be small (perhaps of order \(10^{-8}\) to \(10^{-2}\)) in the solar system, but could be larger (of order 0.1) in larger systems such as clusters of galaxies.

These estimates are based on the fact that the soliton’s effective 4D mass is not concentrated at a point,
like that of a black hole, but has instead a finite (though sharply peaked) density profile whose steepness depends on the metric parameters (Liu and Wesson 1992, Wesson and Ponce de Leon 1994). Quoting the latter authors, but replacing their metric parameters \( \tilde{a}, \epsilon, k \) (due to Davidson and Owen 1985) with our \( M, a, b \) via 
\[
M \equiv \frac{2}{\tilde{a}}, \quad a \equiv \epsilon k, \quad b \equiv -\epsilon,
\]
we find for the density of the soliton
\[
8\pi \rho(r) = \frac{-ab M^2/r^4}{1 - (M/2r)^2} \left( \frac{1 - M/2r}{1 + M/2r} \right)^{2(a+b)}.
\]
(77)
Pressure is given by \( p = \rho/3 \), so that the matter described by equation (77) could be radiation-like, or composed of ultrarelativistic particles such as neutrinos. Total gravitational mass (as deduced from the asymptotic form of the metric) is \( M_g = aM \), so it is clear that \( b \) must be negative for positive density.

Numerical analysis further reveals that the mass of the soliton is increasingly concentrated at small \( r \) as \( |b| \) approaches zero, and that the 4D Schwarzschild limit \( (b = 0) \) can in fact be viewed as a maximally compressed soliton (Wesson and Ponce de Leon 1994). Physically, this means that solar system bodies, which (viewed as solitons) are essentially point masses, are likely to be associated with very small values of \( |b| \).

To attach some numbers to these qualitative remarks, we make use of equation (19) and consider the weak-field \( (r \gg M/2) \), small-\( b \) limit, in which
\[
\rho(r) \approx -b GM_g^2/(8\pi c^2 r^4),
\]
(78)
where we have reverted to physical units. Equation (78) allows us to associate ranges of \( b \)-values with solitons of mass \( M_g \), if the density \( \rho \) can be estimated at some radius \( r \). It has, for instance, been suggested (e.g., Freese 1986, Gould 1992) that relativistic hot dark matter in the form of massive neutrinos could be trapped inside the Earth. Krauss et al. (1986) have derived one possible density profile for such particles, assuming that equilibrium is established between those undergoing capture, annihilation, and escape from the Earth’s gravitational potential. We do not attempt to fit our equation (78) to this profile at all radii, but merely take the predicted neutrino density at the Earth’s surface as illustrative. From Fig. 2 of Krauss et al. (1986), the expected escape rate for 10 GeV neutrinos is \( 2 \times 10^{16} \text{ s}^{-1} \), which translates into a density at the Earth’s surface of \( \rho(R_\oplus) = 3 \times 10^{-20} \text{ kg m}^{-3} \) (about 50 times the canonical local halo dark matter density of \( 5 \times 10^{-22} \text{ kg m}^{-3} \)). If we suppose that this is rather associated with solitonic matter making up some fraction \( \zeta \) of the Earth’s total mass \( (M_g = \zeta M_\oplus) \), then equation (78) gives \( b = -4 \times 10^{-14} \zeta^{-2} \).

For dark matter of this kind to be significant, \( b \) must be small for solar system bodies. With \( \zeta \sim 10^{-3} \), for example, we have \( b \sim -4 \times 10^{-8} \), while \( \zeta \sim 10^{-6} \) would correspond to \( b \sim -0.04 \). These numbers are consistent with the experimental limits obtained in §§3 - 7 above. It may be possible to constrain
the theory more tightly by looking at violations of the weak equivalence principle by solar system bodies (Overduin 2000).

On larger scales, systems such as galaxies and clusters of galaxies are suspected by many to harbor significant amounts of relativistic hot dark matter. We take here as an example a recent numerical simulation (Kofman et al. 1996) in which light (2.3 eV) neutrinos make up 20% (by mass) of a cluster whose total mass \( M_T = 6 \times 10^{14} M_\odot \). Fig. 3 of this paper shows a typical neutrino density of \( \rho(r) \approx 200 \rho_c \) at \( r = 0.03 \) Mpc, where \( \rho_c = 2 \times 10^{-26} h_0^2 \) kg m\(^{-3}\) is the critical density. If this were instead attributed to solitonic dark matter of total mass \( M_g = \zeta M_T \), then the latter would have \( b = -0.01 \zeta^{-2} \) by equation (78), where we have taken \( h_0 = 0.65 \). If all the hot dark matter were solitonic (\( \zeta = 0.2 \)), then \( |b| \) could be as large as 0.3. These values are illustrative only, since density profiles of hot dark matter in clusters are likely somewhat shallower than that indicated by equation (78). Nevertheless they establish that values of \( |b| \) in galaxy clusters might in principle be significantly larger than those in the solar system, and this encourages us to speculate that stronger tests of higher-dimensional gravity might be carried out using the excellent observational data now available on gravitational lensing by these objects.

9. Conclusions

We have re-examined the classical tests of general relativity, as well as the geodetic precession test, when Einstein’s theory is extended from four to five dimensions. The physical meaning of previous calculations for light deflection and time delay have been clarified physically, and the restriction of zero momentum and/or spin along the extra coordinate that characterized the earlier calculations of perihelion shift and geodetic precession has been lifted.

Our results show that Kaluza-Klein gravity remains consistent with experiment. The free parameter of the theory, however, is increasingly constrained to small values. Thus, data on light deflection, radar ranging to Mars and the perihelion precession of Mercury all imply a value of \( |b| \leq 0.07 \) for the Sun. Improved data on solar oblateness should improve the sensitivity of the perihelion precession bound by as much as an order of magnitude. And the upcoming launch of Gravity Probe B will allow us to measure values of \( |b| \) for the Earth with an accuracy of one part in \( 10^4 \) or better.

\(^1\) Density profiles with \( \rho \propto r^{-4} \) at large \( r \) have however been discussed in other contexts, such as elliptical galaxies (Jaffe 1983, de Zeeuw 1985, Hernquist 1990).
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REFERENCES

Adler, R. J. 2000, Gravity Probe B preprint, Stanford University

Billyard, A. and Wesson, P. S. 1996, Phys. Rev., D55, 731

Bronnikov, K. A. and Melnikov, V. N. 1995, Gen. Rel. Grav., 27, 465

Buchman, S. et al. 1996, in proc. 7th Marcel Grossmann Meeting on General Relativity, ed. R. T. Jantzen and G. M. Keiser (Singapore, World Scientific), p. 1533

Campbell, L., McDow, J. C., Moffat, J. W. and Vincent, D. 1983, Nature, 305, 508

Davidson, A. and Owen, D. A. 1985, Phys. Lett., B155, 247

de Zeeuw, T. 1985, Mon. Not. R. Astron. Soc., 216, 273

Dicke, R. H. and Goldenberg, H. M. 1967, Phys. Rev. Lett., 18, 313

Duvall, T. L., Dziembowski, W. A., Goode, P. R., Gough, D. O., Harvey, J. W. and Leibacher, J. W. 1984, Nature, 310, 22

Freese, K. 1986, Phys. Lett., 167B, 295

Gould, A. 1992, ApJ, 388, 338

Gould, A. 1993, ApJ, 414, L37

Gross, D. J. and Perry, M. J. 1983, Nucl. Phys., B226, 29

Hernquist, L. 1990, ApJ, 356, 359

Jaffe, W. 1983, Mon. Not. R. Astron. Soc., 202, 995

Kalligas, D., Wesson, P. S. and Everitt, C. W. F. 1995, ApJ, 439, 548

Kofman, L., Klypin, A., Pogosyan, D. and Henry, J. P. 1996, ApJ, 470, 102

Kolb, E. W. and Turner, M. S. 1990, The Early Universe (Reading, Addison-Wesley), p. 441

Krauss, L. M., Srednicki, M. and Wilczek, F. 1986, Phys. Rev., D33, 2079
Lebach, D. E. et al. 1995, Phys. Rev. Lett., 75, 1439

Lim, P. H., Overduin, J. M. and Wesson, P. S. 1995, J. Math. Phys., 36, 6907

Liu, H. and Wesson, P. S. 1992, J. Math. Phys., 33, 3888

Liu, H. and Wesson, P. S. 1996, Class. Quant. Grav., 13, 2311

Liu, H. and Wesson, P. S. 1997, Class. Quant. Grav., 14, 1651

Liu, H., Wesson, P. S. and Ponce de Leon, J. 1992, J. Math. Phys., 34, 4070

Mashhoon, B., Liu, H. and Wesson, P. S. 1994, Phys. Lett., B331, 305

Mashhoon, B., Wesson, P. and Liu, H. 1998, Gen. Rel. Grav., 30, 555

Ni, W.-T. 1998, J. Jpn. Soc. Microgravity Appl., Vol. 15, Suppl. II, 66

Overduin, J. M. 2000, Phys. Rev. D., submitted

Overduin, J. M. and Wesson, P. S. 1997, Phys. Rep., 283, 303

Overduin, J. M. and Wesson, P. S. 1998, in Current Topics in Mathematical Cosmology, ed. M. Rainer and H.-J. Schmidt (Singapore, World Scientific), p. 293

Reasenberg, R. D. and Shapiro, I. I. 1986, in Relativity in Celestial Mechanics and Astrometry, ed. J. Kovalevsky and V. A. Brumberg (Dordrecht, Reidel), p. 383

Reasenberg, R. D. et al. 1979, ApJ, 234, L219

Robertson, D. S., Carter, W. E. and Dillinger, W. H. 1991, Nature, 349, 768

Rozelot, J. P. and Rösch, J. 1997, Solar Phys., 172, 11

Samain, E. et al. 1998, Astron. Astrophys. Suppl. Ser., 130, 235

Schmidt, H.-J. 1997, Grav. Cosm., 3, 185

Seahra, S. S. 2000, Phys. Rev. D., submitted

Shapiro, I. I., Counselman, C. C. and King, R. W. 1976, Phys. Rev. Lett., 36, 555
Sorkin, R. D. 1983, Phys. Rev. Lett., 51, 87

Treuhaft, R. N. and Lowe, S. T. 1991, AJ, 102, 1879

Wesson, P. S. 1984, Gen. Rel. Grav., 16, 193

Wesson, P. S. 1999, Space, Time and Matter (Singapore, World Scientific)

Wesson, P. S. and Liu, H. 1997, Int. J. Theor. Phys., 36, 1865

Wesson, P. S. and Liu, H. 1998, Phys. Lett., B432, 266

Wesson, P. S. and Ponce de Leon, J. 1994, Class. Quant. Grav., 11, 1341

Will, C. M. 1993, Theory and experiment in gravitational physics (Cambridge University Press, Cambridge), §2

Williams, J. G., Newhall, X. X. and Dickey, J. O. 1996, Phys. Rev., D53, 6730