A new matrix modified Korteweg-de Vries equation: Riemann-Hilbert approach and exact solutions

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Abstract

A new matrix modified Korteweg-de Vries (mmKdV) equation with a $p \times q$ complex-valued potential matrix function is first studied via Riemann-Hilbert approach, which can be reduced to the well-known coupled modified Korteweg-de Vries equations by selecting special potential matrix. Starting from the special analysis for the Lax pair of this equation, we successfully establish a Riemann-Hilbert problem of the equation. By introducing the special conditions of irreguarity and reflectionless case, some interesting exact solutions, including the $N$-soliton solution formula, of the mmKdV equation are derived through solving the corresponding Riemann-Hilbert problem. Moreover, due to the special symmetry of special potential matrices and the $N$-soliton solution formula, we make further efforts to classify the original exact solutions to obtain some other interesting solutions which are all displayed graphically. It is interesting that the local structures and dynamic behaviors of soliton solutions, breather-type solutions and bell-type soliton solutions are all analyzed via taking different types of potential matrices.

Keywords: Matrix modified Korteweg-de Vries equation, Riemann-Hilbert approach, Exact solutions, Multi-soliton solutions, Soliton classification.

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1. Introduction

Soliton theory is one of the important research directions in the field of nonlinear science in the world today. Solitons of nonlinear differential equations play an important role in revealing some important physical laws such as fluid mechanics, plasma physics, nonlinear optics, classical field theory, and quantum theory. Based on this fact, many domestic and foreign scholars have devoted themselves to the study of solitons. In recent decades, a number of effective methods have been produced such as Hirota bilinear method \([1]\), Darboux and Bäcklund transformation \([2]\), inverse scattering transformation \([3, 4]\). Apart from them, the Riemann-Hilbert(RH) approach is also a very effective method which can not only solve the soliton solutions of a series of nonlinear evolution equations \([6–21]\), but also study integrable systems with non-zero boundaries \([22–26]\), the asymptoticity of integrable system solutions \([27–31]\), certain important properties of orthogonal polynomials \([32]\), etc. Because of the superiority of RH approach, many scholars have done a lot of work about the exact solutions of single equation and partial coupled equations with it. However, there are very few work about the solutions of matrix-type equation via the RH approach. Therefore, the main purpose of our work is to study the RH problem and exact solutions with their properties of a new matrix modified Korteweg-de Vries(mmKdV) equation in this work.

In this work, we focus on Riemann-Hilbert problem and exact solutions with their propagation behaviors for the mmKdV equation \([33]\)

\[
Q_t + Q_{xxx} - 3\epsilon(Q, Q^\dagger Q + QQ^\dagger Q_x) = 0, \quad \epsilon = \pm 1, \quad (1.1)
\]

where \(Q\) is a \(p \times q\) complex-valued matrix function of variation \(x\) and \(t\). If \(Q\) is restricted to be a real matrix, Eq. (1.1) is the same as the equation presented in \([34]\). When \(Q\) is taken as some special forms, Eq. (1.1) can be reduced to the coupled modified Korteweg-de Vries(cmKdV) equations

\[
\frac{\partial \nu_i}{\partial t} - 6 \left( \sum_{j=0}^{M-1} \epsilon_j \nu_j^2 \right) \frac{\partial \nu_i}{\partial x} + \frac{\partial^3 \nu_i}{\partial x^3} = 0, \quad \epsilon_j = \pm 1, \quad i = 0, 1, \ldots, M - 1. \quad (1.2)
\]

Moreover, as far as known, there are already two different methods to complete the reduction to cmKdV in \([33, 35]\). Here is a brief introduction to the reduced method in \([33]\). We define

\[
Q^{(1)} = \mu_0 v_0 + i\nu_1, \quad R^{(1)} = \epsilon_1(\mu_0 v_0 - i\nu_1), \quad (1.3)
\]

\[
Q^{(m+1)} = \begin{bmatrix}
Q^{(m)} & -\epsilon_{2m+1}(\mu_{2m} v_{2m} + i\nu_{2m})I_{2m-1} \\
-(\mu_{2m} v_{2m} - i\nu_{2m})I_{2m-1} & -R^{(m)}
\end{bmatrix}, \quad (1.4)
\]
where \( I_{2^{m-1}} \) is the \( 2^{m-1} \times 2^{m-1} \) identity matrix, \( \epsilon = \pm 1 \), and \( \mu_{2m} \) satisfies
\[
\mu_{2m}^2 = \frac{\epsilon_{2m}}{\epsilon_{2m+1}} = \epsilon_{2m}\epsilon_{2m+1}.
\]

Substituting \( Q^{(m)} \) and \( R^{(m)} \) for \( Q \) and \( R \) into Eq. (1.4), we can obtain Eq. (1.2) \((M = 2m)\).

In addition, the conservation laws and Hamilton structure of the mmKdV equation have been studied carefully by Tsuchida and Wadati in [33]. Starting from a special class \( p = q = n \), the Lax representation for Eq. (1.2) can be written as
\[
\Phi_1 \left( \begin{array}{c}
\Phi_1 \\
\Phi_2
\end{array} \right) = \left[ \begin{array}{cc}
-i\zeta I & Q \\
R & i\zeta I
\end{array} \right] \left( \begin{array}{c}
\Phi_1 \\
\Phi_2
\end{array} \right),
\]
\[
\Phi_1 \left( \begin{array}{c}
\Phi_1 \\
\Phi_2
\end{array} \right) = \left[ \begin{array}{cc}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{array} \right] \left( \begin{array}{c}
\Phi_1 \\
\Phi_2
\end{array} \right),
\]
where every element is a \( n \times n \) matrix. We define \( \Gamma = \Phi_2\Phi_1^{-1} \), with the aid of Lax pair and the compatibility condition of it, some special relationship can be derived by
\[
\{\text{tr}(Q\Gamma)\}_l = \{\text{some function of } \Gamma, Q, R \text{ and } \zeta\}_x,
\]
\[
2iQ\Gamma = -QR + Q(Q^{-1}Q\Gamma)_x + (Q\Gamma)^2,
\]
where \( \text{tr}(Q\Gamma) \) is the generating form of conserved densities. By expanding \( Q\Gamma \) with respect to the spectral parameter \( \zeta \) as \( Q\Gamma = \sum_{l=1}^{\infty} \frac{1}{(2\zeta)^l}F_l \). A recursion formula can be obtained as follows
\[
F_{l+1} = -\delta_{l,0}QR + Q(Q^{-1}F_l)_x + \sum_{k=1}^{l} F_kF_{l-k}, \quad l = 0, 1, \ldots ,
\]
where \( \text{tr}(F_l) \) is a conserved density for any positive integer \( l \). Moreover, on the basis of conserved density, the Hamilton structure and the Possion bracket of mmKdV equation can be obtained as follows
\[
H = \text{tr} \int [iF_4] \, dx = \text{tr} \int \left\{ -iQR_{xxx} + i\frac{3}{2}QR(QR_x - Q_xR) \right\} \, dx,
\]
\[
\{Q(x) \circ Q(y)\} = \{R(x) \circ Q(y)\} = 0,
\]
\[
\{Q(x) \circ R(y)\} = i\delta(x - y)\Pi,
\]
where \( \{X \otimes Y\}_{kl}^{ij} = \{X_{ij}, Y_{kl}\} \), and \( \Pi \) denotes the \( n^2 \times n^2 \) permutation matrix. The complete integrability can be proved by a classical \( r \)-matrix [33]. The exact solutions of \( \text{mmKdV} \) equation as \( p = q = n \) and \( \epsilon = -1 \) have been studied by using the classical inverse scattering method [33]. Different from previous work about \( \text{mmKdV} \) equation, our work is to perfect the exact solutions of \( \text{mmKdV} \) equation for any positive integers \( p, q \) and \( \epsilon = \pm 1 \) via RH approach, and obtain some other interesting and meaningful phenomenon by analyzing the special properties of potential matrix which is important to understand \( \text{mmKdV} \) equation more thoroughly.

The outline of this work is as follows. In Section 2, we perform the spectral analysis of Lax pair, and analyze the symmetry and analyticity of scattering matrix. In Section 3, on the basis of the results of the last section, the RH problem is formulated. In Section 4, we complete the spatiotemporal evolution of the scattered data. In Section 5, we obtain general form of solutions by solving the RH problem. Finally, some specific forms of potential matrix are considered. According to the special properties unique to the particular forms, we can further classify soliton solutions based on the original soliton solutions. Then, we obtain some other interesting solutions such as \( N \)-soliton solutions, breather-type soliton solutions and bell-type soliton solutions. The conclusions are discussed in the last section.

2. Spectral Analysis

2.1. Lax pair and eigenfunction

The equivalent form of Lax pair of Eq. (1.1) can be written as

\[ \begin{cases} \Phi_x = M\Phi, & M = -i\zeta \sigma + U, \\ \Phi_t = N\Phi, & N = -4i\zeta^3\sigma + 4\zeta^2U - 2i\zeta(U^2 + U_x)\sigma + 2U^3 - U_{xx} + U_xU - UU_x, \end{cases} \]

(2.1)

with

\[ \sigma = \begin{bmatrix} I_1 & 0 \\ 0 & -I_2 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & Q \\ R & 0 \end{bmatrix}, \]

(2.2)

where \( \Phi = \Phi(x, t; \zeta) \) is a \((p + q)\)-component vector, \( Q \) is a \( p \times q \) matrix, \( R \) is a \( q \times p \) matrix, and \( I_1 \) and \( I_2 \) are \( p \times p \) and \( q \times q \) identity matrix, respectively. The potential matrices satisfy \( R = \epsilon Q^\dagger \) and \( \epsilon = \pm 1 \). In addition, Eq. (1.1) can be derived via the compatibility condition of Eq. (2.1).

For the convenience of discussion, we can rewrite Eq. (2.1) as

\[ \begin{cases} \Phi_x + i\zeta \sigma\Phi = U\Phi, \\ \Phi_t + 4i\zeta^3 \sigma\Phi = V\Phi, \end{cases} \]

(2.3)
where
\[ V = 4\zeta^2 U + i\zeta(2U_xU + 2U_y)\sigma + 2U^3 - U_{xx} + U_xU - UU_x. \] (2.4)

According to Eq. (2.3), when \(|x| \to \infty\),
\[ \Phi \propto \exp(-i\zeta \sigma x - 4i\zeta \sigma t). \] (2.5)

Let \( \Psi = \Phi \exp(i\zeta \sigma x + 4i\zeta \sigma t) \), then \( \Psi \) satisfy:
\[ \Psi_x + i\zeta [\sigma, \Psi] = U\Psi, \] (2.6)
\[ \Psi_1 + 4i\zeta^3 [\sigma, \Psi] = V\Psi, \] (2.7)
where \([\sigma, \Psi] = \sigma \Psi - \Psi \sigma\) is the commutator. Based on Eqs. (2.6) and (2.7), we can get the formula
\[ d(e^{i(\zeta + 4\zeta^3) \sigma t})\Psi = e^{i(\zeta + 4\zeta^3) \sigma t}(Udx + Vdt)\Psi. \] (2.8)

Now we begin to consider the spectral analysis of Lax pair (2.6) and (2.7), for which we merely focus on the spectral problem (2.6), because the analysis will take place at a fixed time, and the \( t \)-dependence will be suppressed. As for (2.6), we can write its two matrix Jost solutions as a collection of columns, that is
\[ \Psi_1 = ([\Psi_1], [\Psi_1]_2, \ldots, [\Psi_1]_p, [\Psi_2]_{p+1}, \ldots, [\Psi_2]_{p+q}), \]
\[ \Psi_2 = ([\Psi_2], [\Psi_2]_2, \ldots, [\Psi_2]_p, [\Psi_1]_{p+1}, \ldots, [\Psi_1]_{p+q}), \] (2.9)

obeying the asymptotic conditions
\[ \Psi_1 \to \mathbb{I}, \quad x \to -\infty, \]
\[ \Psi_2 \to \mathbb{I}, \quad x \to +\infty. \] (2.10)

Here \( \mathbb{I} \) is a \((p + q) \times (p + q)\) identity matrix. \( \Psi_1 \) and \( \Psi_2 \) are uniquely determined by the integral equations of Volterra type
\[ \Psi_1 = \mathbb{I} + \int_{-\infty}^{x} e^{-i\zeta(\sigma(x-y))U(y)}\Psi_1(y, \zeta)e^{i\zeta(\sigma(x-y))}dy, \] (2.11)
\[ \Psi_2 = \mathbb{I} - \int_{x}^{\infty} e^{-i\zeta(\sigma(x-y))U(y)}\Psi_2(y, \zeta)e^{i\zeta(\sigma(x-y))}dy. \]

By direct computation, we can get
\[ e^{-i\zeta(\sigma(x-y))U(y)e^{i\zeta(\sigma(x-y))}} = \begin{pmatrix} 0 & e^{-i\zeta(\sigma(x-y))I_1}Qe^{-i\zeta(\sigma(x-y))I_1} \\ e^{i\zeta(\sigma(x-y))I_1}R & 0 \end{pmatrix}, \] (2.12)

By direct analysis, we can see that
\[ [\Psi_1], [\Psi_1]_2, \ldots, [\Psi_1]_p, [\Psi_2]_{p+1}, [\Psi_2]_{p+2}, \ldots, [\Psi_2]_{p+q} \] (2.13)
are analytic for $\zeta \in \mathbb{C}^+$ and continuous for $\zeta \in \mathbb{C}^+ \cup \mathbb{R}$, and
\[
[\Psi_2^1, [\Psi_2^2]_2, \ldots, [\Psi_2^p]_p, [\Psi_1^1]_{p+1}, [\Psi_1^2]_{p+2}, \ldots, [\Psi_1^q]_{p+q} \quad (2.14)
\]
are analytic for $\zeta \in \mathbb{C}^-$ and continuous for $\zeta \in \mathbb{C}^- \cup \mathbb{R}$, where $\mathbb{C}^+$ and $\mathbb{C}^-$ respectively the upper and lower half $\zeta$-plane. It is indicated owing to the Abels identity and $tr(U) = 0$ that the determinants of $\Psi_1$ and $\Psi_2$ are independent of the variable $x$. Through evaluating $\det(\Psi_1)$ at $x = -\infty$ and $\det(\Psi_2)$ at $x = +\infty$, we know
\[
\det(\Psi_1) = \det(\Psi_2) = 1, \quad \zeta \in \mathbb{R}. \quad (2.15)
\]

2.2. Symmetry and analyticity of scattering matrix

Since $\Psi_1 E$ and $\Psi_2 E$ are matrix solutions of the spectral problem (12), where $E = e^{-i(\zeta + \zeta^*)^*}$, therefore, $\Psi_1 E$ and $\Psi_2 E$ are linear dependent, namely,
\[
\Psi_1 E = \Psi_2 E S(\zeta), \quad \zeta \in \mathbb{R}, \quad (2.16)
\]
where $S(\zeta)$ is a $(p + q) \times (p + q)$ matrix, i.e., $S(\zeta) = (S_{ij})_{(p+q)\times(p+q)}$. By taking determinant at both ends of Eq. (2.16), it is obvious that
\[
\det(S) = 1. \quad (2.17)
\]

In order to construct the RH problem, we need to consider the inverse of $\Psi_1$ and $\Psi_2$, and we partition the inverse matrices of $\Psi_1$ and $\Psi_2$ into rows, that is
\[
\Psi_1^{-1} = \begin{pmatrix}
[\Psi_1^{-1}]^1_1 \\
[\Psi_1^{-1}]^2_1 \\
\vdots \\
[\Psi_1^{-1}]^{p+q}_1
\end{pmatrix}, \quad \Psi_2^{-1} = \begin{pmatrix}
[\Psi_2^{-1}]^1_1 \\
[\Psi_2^{-1}]^2_1 \\
\vdots \\
[\Psi_2^{-1}]^{p+q}_1
\end{pmatrix}, \quad (2.18)
\]
in which each $[\Psi_1^{-1}]^{l}_1$ and $[\Psi_2^{-1}]^{l}_1$ denote the $l$-th row of the matrices $\Psi_1^{-1}$ and $\Psi_2^{-1}$. At the same time, $\Psi_1^{-1}$ and $\Psi_2^{-1}$ meet the equation
\[
T_x + i\zeta[\sigma, T] = -\gamma U. \quad (2.19)
\]
With the aid of Eq. (2.19), we can see that
\[
[\Psi_1^{-1}]_1, [\Psi_1^{-1}]_2, \ldots, [\Psi_1^{-1}]_p, [\Psi_2^{-1}]_{p+1}, \ldots, [\Psi_2^{-1}]_{p+q} \quad (2.20)
\]
are analytic in $\zeta \in \mathbb{C}^-$, while
\[
[\Psi_2^{-1}]_1, [\Psi_2^{-1}]_2, \ldots, [\Psi_2^{-1}]_p, [\Psi_1^{-1}]_{p+1}, \ldots, [\Psi_1^{-1}]_{p+q} \quad (2.21)
\]
are analytic in $\zeta \in \mathbb{C}^+$. Moreover, from Eq. (2.16), we can get that
\[
E^{-1}\Psi_1^{-1} = R(\zeta)E^{-1}\Psi_2^{-1},
\] (2.22)
where $R(\zeta) = (R_{i,j})_{(p+q)\times(p+q)} = S^{-1}(\zeta)$, we can call it as inverse scattering matrix.
After finding the scattering matrix $S(\zeta)$ and the inverse scattering matrix $R(\zeta)$, we give the following theorem to describe the analyticity of the elements for the two matrices.

**Theorem 2.1.** Assume that the scattering matrix $S(\zeta)$ and the inverse scattering matrix $R(\zeta)$ are divided into the following forms
\[
\begin{pmatrix}
S_1 & S_2 \\
S_3 & S_4
\end{pmatrix} = \begin{pmatrix}
R_1 & R_2 \\
R_3 & R_4
\end{pmatrix},
\] (2.23)
where $S_1$ and $R_1$ are p×p matrices, $S_4$ and $R_4$ are q×q matrices, $S_2$ and $R_2$ are p×q matrices, and $S_3$ and $R_3$ are q×p matrices. Then the elements of $S_1$ and $R_1$ can be analytic extension to $\zeta \in \mathbb{C}^+$; the elements of $S_4$ and $R_1$ can be extended analytically to $\zeta \in \mathbb{C}^-$; the elements of $S_2, S_3, R_2$ and $R_3$ are not analytic in $\zeta \in \mathbb{C}^+$ and $\zeta \in \mathbb{C}^-$; the elements of $S_2$ and $S_3$ are continuous in $\zeta \in \mathbb{R}$; the elements of $R_2$ and $R_3$ can not be extended analytically to $\zeta \in \mathbb{R}$.

**Proof.** From Eq. (2.16), we can get
\[
E^{-1}\Psi_1^{-1}E = S(\zeta).
\] (2.24)

More explicitly, we have
\[
\Psi_2^{-1}\Psi_1 =
\begin{bmatrix}
[\Psi_2^{-1}]_1[\Psi_1]_1 & \ldots & [\Psi_2^{-1}]_1[\Psi_1]_p & [\Psi_2^{-1}]_1[\Psi_1]_{p+1} & \ldots & [\Psi_2^{-1}]_1[\Psi_1]_{p+q} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
[\Psi_2^{-1}]_p[\Psi_1]_1 & \ldots & [\Psi_2^{-1}]_p[\Psi_1]_p & [\Psi_2^{-1}]_p[\Psi_1]_{p+1} & \ldots & [\Psi_2^{-1}]_p[\Psi_1]_{p+q} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
[\Psi_2^{-1}]_{p+1}[\Psi_1]_1 & \ldots & [\Psi_2^{-1}]_{p+1}[\Psi_1]_p & [\Psi_2^{-1}]_{p+1}[\Psi_1]_{p+1} & \ldots & [\Psi_2^{-1}]_{p+1}[\Psi_1]_{p+q} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
[\Psi_2^{-1}]_{p+q}[\Psi_1]_1 & \ldots & [\Psi_2^{-1}]_{p+q}[\Psi_1]_p & [\Psi_2^{-1}]_{p+q}[\Psi_1]_{p+1} & \ldots & [\Psi_2^{-1}]_{p+q}[\Psi_1]_{p+q}
\end{bmatrix}
\] (2.25)

With the aid of the analyticity of the columns of $\Psi_1$, the rows of $\Psi_2$ and the matrix $E$, we can proof the analyticity of the elements of $S(\zeta)$. Similarly, the analyticity of the elements of $R(\zeta)$ can be obtained. Thus, we complete the proof. □

In addition, the potential matrix $U$ has the symmetry as follows:
\[
U = \begin{bmatrix} 0 & Q \\ R & 0 \end{bmatrix}, \quad R = \epsilon Q^3, \quad \epsilon = \pm 1.
\] (2.26)
Case 1: $\epsilon = -1$.

From $R = -Q^\dagger$, we can derive that $U^\dagger = -U$, and then obtain

$$\Psi_i^\dagger(\zeta^*) = \Psi_i^{-1}(\zeta), \quad i = 1, 2.$$  \hfill (2.27)

According to Eq. (2.16), we have

$$S^\dagger(\zeta^*) = S^{-1}(\zeta) = R(\zeta),$$  \hfill (2.28)

which gives the following relationships

$$s_{i,j}(\zeta^*) = r_{i,j}(\zeta), \quad 1 \leq i \leq p, \quad 1 \leq j \leq p, \quad \zeta \in \mathbb{C}^*,$$

$$s_{i,j}(\zeta) = r_{i,j}(\zeta), \quad 1 \leq i \leq p, \quad p \leq j \leq p + q, \quad \zeta \in \mathbb{R},$$

$$s_{i,j}(\zeta) = r_{i,j}(\zeta), \quad p \leq i \leq p + q, \quad 1 \leq j \leq p, \quad \zeta \in \mathbb{R},$$

$$s_{i,j}(\zeta^*) = r_{i,j}(\zeta), \quad p + 1 \leq i \leq p + q, \quad p + 1 \leq j \leq p + q, \quad \zeta \in \mathbb{C}^-.$$

With the aid of Eq. (2.27), we have

$$\Gamma_1^\dagger(\zeta^*) = \Gamma_2(\zeta), \quad \zeta \in \mathbb{C}^-.$$  \hfill (2.30)

Case 2: $\epsilon = 1$.

From $R = Q^\dagger$, we can see that $U^\dagger = -\sigma U\sigma$, and then obtain

$$\Psi_i^\dagger(\zeta^*) = \sigma^i \Psi_i^{-1}(\zeta)\sigma, \quad i = 1, 2.$$  \hfill (2.31)

According to Eq. (2.16), we have

$$S^\dagger(\zeta^*) = \sigma R(\zeta)\sigma.$$  \hfill (2.32)

With the aid of Eq. (2.31), we have

$$\Gamma_1^\dagger(\zeta^*) = \sigma \Gamma_2(\zeta)\sigma, \quad \zeta \in \mathbb{C}^-.$$  \hfill (2.33)

In a word, when

$$U = \begin{bmatrix} 0 & Q \\ R & 0 \end{bmatrix}, \quad R = \epsilon Q^\dagger, \quad \epsilon = \pm 1,$$  \hfill (2.34)

we have

$$\det(\Gamma_1(\zeta^*)) = (\det(\Gamma_2(\zeta)))^*, \quad \zeta \in \mathbb{C}^-.$$  \hfill (2.35)

It is shown that if $\zeta$ is the zero of $\det(\Gamma_1)$, then $\zeta^*$ is one zero of $\det(\Gamma_2)$, which reveals the relationship between the zeros of $\det(\Gamma_1)$ and $\det(\Gamma_2)$.
3. Riemann-Hilbert Problem

A RH problem desired for the mmKdV equation involves two matrix functions: one is analytic in $\mathbb{C}^+$, and the other is analytic in $\mathbb{C}^-$. Define the first matrix function

$$\Gamma_1(x, \zeta) = ([\Psi_1^1], [\Psi_1^2], \ldots, [\Psi_1^p], [\Psi_2^1], [\Psi_2^2], \ldots, [\Psi_2^p]),$$  \hspace{1cm} (3.1)

which is an analytic function of $\zeta$ in $\mathbb{C}^+$. Next, we study the very large $\zeta$ asymptotic behavior of $\Gamma_1(x, \zeta)$, which has the asymptotic expansion

$$\Gamma_1 = \Gamma_1^{(0)} + \frac{\Gamma_1^{(1)}}{\zeta} + \frac{\Gamma_1^{(2)}}{\zeta^2} + O\left(\frac{1}{\zeta^3}\right), \quad \zeta \to \infty.$$  \hspace{1cm} (3.2)

Inserting Eq. (3.2) into Eq. (2.6), and comparing the coefficients of $\zeta$ directly bring about

$$O(1) : \Gamma_1^{(0)} + i[\sigma, \Gamma_1^{(1)}] = U \Gamma_1^{(0)},$$  
$$O(\zeta) : i[\sigma, \Gamma_1^{(0)}] = 0,$$  
$$O(\zeta^2) : i[\sigma, \Gamma_1^{(1)}] = U \Gamma_1^{(0)},$$  \hspace{1cm} (3.3)

from which we have $\Gamma_1^{(0)} = I$, namely,

$$\Gamma_1 \to I, \quad \zeta \in \mathbb{C}^+ \to \infty.$$  \hspace{1cm} (3.4)

We can introduce the matrix function $\Gamma_2$, which is analytic in $\zeta \in \mathbb{C}^-$ in terms of

$$\Gamma_2(x, \zeta) = \begin{pmatrix} [\Psi_1^{-1}]_1 \\ \vdots \\ [\Psi_1^{-1}]_p \\ [\Psi_2^{-1}]_{p+1} \\ \vdots \\ [\Psi_2^{-1}]_{p+q} \end{pmatrix}(x, \zeta).$$  \hspace{1cm} (3.5)

Similarly, $\Gamma_2$ satisfies the following condition:

$$\Gamma_2 \to I, \quad \zeta \in \mathbb{C}^- \to \infty.$$  \hspace{1cm} (3.6)

In addition, we define some auxiliary matrices

$$H_i = \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0),$$  \hspace{1cm} (3.7)

then

$$\Gamma_1 = \Psi_1 H_1 + \cdots + \Psi_1 H_p + \Psi_2 H_{p+1} + \cdots + \Psi_2 H_{p+q},$$  
$$\Gamma_2 = H_1 \Psi_1^{-1} + \cdots + H_p \Psi_1^{-1} + H_{p+1} \Psi_2^{-1} + \cdots + H_{p+q} \Psi_2^{-1}.$$  \hspace{1cm} (3.8)
Before we give the normal RH Problem, we need to do some calculations:
\[
\Gamma_2(\zeta)\Gamma_1(\zeta) = (H_1\Psi_1^{-1} + \cdots + H_p\Psi_p^{-1} + H_{p+1}\Psi_{p+1}^{-1} + \cdots + H_{p+q}\Psi_{p+q}^{-1})
\times (\Psi_1H_1 + \cdots + \Psi_1H_p + \Psi_2H_{p+1} + \cdots + \Psi_2H_{p+q}).
\]

Note that
\[
H_iH_j = \begin{cases} H_i, & i = j, \quad 1 \leq i, j \leq p + q, \\ 0, & i \neq j, \quad 1 \leq i, j \leq p + q, \end{cases}
\]

\[
\Psi_1^{-1}\Psi_2 = ER(\zeta)E^{-1} =
\begin{bmatrix}
    r_{1,1} & \cdots & r_{1,p} & r_{1,p+1}e^{-2i(\zeta + 4\zeta^i)} & \cdots & r_{1,p+q}e^{-2i(\zeta + 4\zeta^i)} \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    r_{p,1} & \cdots & r_{p,p} & r_{p,p+1}e^{-2i(\zeta + 4\zeta^i)} & \cdots & r_{p,p+q}e^{-2i(\zeta + 4\zeta^i)} \\
    r_{p+1,1}e^{2i(\zeta + 4\zeta^i)} & \cdots & r_{p+1,p}e^{2i(\zeta + 4\zeta^i)} & r_{p+1,p+1} & \cdots & r_{p+1,p+q} \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    r_{p+q,1}e^{2i(\zeta + 4\zeta^i)} & \cdots & r_{p+q,p}e^{2i(\zeta + 4\zeta^i)} & r_{p+q,p+1} & \cdots & r_{p+q,p+q} \\
\end{bmatrix}
\]

and
\[
\Psi_2^{-1}\Psi_1 = ES(\zeta)E^{-1} =
\begin{bmatrix}
    s_{1,1} & \cdots & s_{1,p} & s_{1,p+1}e^{-2i(\zeta + 4\zeta^i)} & \cdots & s_{1,p+q}e^{-2i(\zeta + 4\zeta^i)} \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    s_{p,1} & \cdots & s_{p,p} & s_{p,p+1}e^{-2i(\zeta + 4\zeta^i)} & \cdots & s_{p,p+q}e^{-2i(\zeta + 4\zeta^i)} \\
    s_{p+1,1}e^{2i(\zeta + 4\zeta^i)} & \cdots & s_{p+1,p}e^{2i(\zeta + 4\zeta^i)} & s_{p+1,p+1} & \cdots & s_{p+1,p+q} \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    s_{p+q,1}e^{2i(\zeta + 4\zeta^i)} & \cdots & s_{p+q,p}e^{2i(\zeta + 4\zeta^i)} & s_{p+q,p+1} & \cdots & s_{p+q,p+q} \\
\end{bmatrix}
\]

we can obtain
\[
\Gamma_2\Gamma_1 = J(x, \zeta)
\]
\[
\begin{bmatrix}
    1 & \cdots & 0 & r_{1,p+1}e^{-2i(\zeta + 4\zeta^i)} & \cdots & r_{1,p+q}e^{-2i(\zeta + 4\zeta^i)} \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    0 & \cdots & 1 & r_{p,p+1}e^{-2i(\zeta + 4\zeta^i)} & \cdots & r_{p,p+q}e^{-2i(\zeta + 4\zeta^i)} \\
    s_{p+1,1}e^{2i(\zeta + 4\zeta^i)} & \cdots & s_{p+1,p}e^{2i(\zeta + 4\zeta^i)} & 1 & \cdots & 0 \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    s_{p+q,1}e^{2i(\zeta + 4\zeta^i)} & \cdots & s_{p+q,p}e^{2i(\zeta + 4\zeta^i)} & 0 & \cdots & 1 \\
\end{bmatrix}
\]

We denote \(\Gamma_1\) and \(\Gamma_2\) as \(\Gamma^+\) and \(\Gamma^-\), based on which a RH problem can be set up as follows:
- $\Gamma^+$ are analytic in $\mathbb{C}^+ = \mathbb{C} \setminus \Sigma$ and $\mathbb{C}^- = \mathbb{C} \setminus \Sigma$, respectively, where $\Sigma$ represent the directed path along the positive direction of $\text{Im}\zeta = 0$.

- jump condition
  \[
  \Gamma^-\Gamma^+ = J(x, \zeta), \quad \zeta \in \mathbb{R}
  \]  
  where $J(x, \zeta)$ is from Eq. (3.13).

- normalization condition
  \[
  \Gamma^+ \rightarrow I, \quad \text{as} \quad \zeta \rightarrow \infty,
  \]
  \[
  \Gamma^- \rightarrow I, \quad \text{as} \quad \zeta \rightarrow \infty.
  \]  

4. Time evolution of the scattering data

In the previous section, we have obtained the zeros of $\det(\Gamma_1)$ or $\det(\Gamma_2)$. The zero and non-zero vectors $\vartheta_j, \widehat{\vartheta}_j$ form complete scattering data, which satisfy

\[
\Gamma_1(\zeta_j)\vartheta_j = 0, \\
\widehat{\vartheta}_j\Gamma_2(\zeta_j^*) = 0,
\]  
where $\vartheta_j$ and $\widehat{\vartheta}_j$ represent row vector and column vector, respectively.

Next, we begin to analyze the law of time and space evolution. First, we derive the Eq. (4.1) from $x$, $\Gamma_1, x\vartheta_j + \Gamma_1\vartheta_j, x = 0$, $\Gamma_1, t\vartheta_j + \Gamma_1\vartheta_j, t = 0$.

Since

\[
\Gamma_{1,x} = \Psi_{1,x}H_1 + \cdots + \Psi_{1,x}H_p + \Psi_{2,x}H_{p+1} + \cdots + \Psi_{2,x}H_{p+q},
\]  
and

\[
\Psi_x = -i\zeta[\sigma, \Psi] + U\Psi,
\]  
we can obtain

\[
\Gamma_{1,x} = -i\zeta[\sigma, \Gamma_1] + U\Gamma_1.
\]  
Following the similar process as $\Gamma_{1,x}$, we get

\[
\Gamma_{1,t} = -4i\zeta^3[\sigma, \Gamma_1] + U\Gamma_1.
\]  
Substituting Eqs. (4.5) and (4.6) into Eq. (4.2), and note that $\Gamma_1\vartheta_j = 0$, we have

\[
\vartheta_{j,x} + i\zeta_j[\sigma, \vartheta_j] = 0, \\
\vartheta_{j,t} + 4i\zeta_j^3[\sigma, \vartheta_j] = 0.
\]
By solving the above vector differential equations, we have
\[ \vartheta_j = e^{-i(\zeta_j + 4\zeta_j^3)t}\vartheta_{j,0}, \quad (4.8) \]
where \( \vartheta_{j,0} \) is a constant vector.

Moreover, for \( \epsilon = -1 \), we have \( \Psi_i^{(1)}(\zeta) = \Psi_i^{-1}(\zeta)(i = 1, 2) \), then
\[ \Gamma_i^{(1)}(\zeta) = \Gamma_i(\zeta), \quad \zeta \in \mathbb{C}^- . \quad (4.9) \]

From Eq. (4.1) and Eq. (4.9), we know
\[ \hat{\vartheta}_j = \vartheta_j^{\dagger}, \quad 1 \leq j \leq N, \quad (4.10) \]
where \( N \) is the number of zeros of \( \det(\Gamma_1(\zeta)) \). Therefore,
\[ \hat{\vartheta}_j = \vartheta_j^{\dagger}e^{i(\sigma \zeta_j + 4\zeta_j^3)t}\sigma. \quad (4.11) \]

For \( \epsilon = 1 \), we have \( \Psi_i^{(1)}(\zeta) = \sigma \Psi_i^{-1}(\zeta) \sigma, (i = 1, 2) \), then
\[ \Gamma_i^{(1)}(\zeta) = \sigma \Gamma_i(\zeta) \sigma, \quad \zeta \in \mathbb{C}^- . \quad (4.12) \]

From Eq. (4.1) and Eq. (4.12), we know
\[ \hat{\vartheta}_j = \vartheta_j^{\dagger} \sigma, \quad 1 \leq j \leq N, \quad (4.13) \]
where \( N \) is the number of zeros of \( \det(\Gamma_1(\zeta)) \). Therefore,
\[ \hat{\vartheta}_j = \vartheta_j^{\dagger}e^{i(\sigma \zeta_j + 4\zeta_j^3)t}\sigma, \quad 1 \leq j \leq N . \quad (4.14) \]

Thus, we complete the analysis of the law of time and space evolution.

5. Exact solutions of mmKdV equation

In fact, \( \Gamma_1 \) has the asymptotic expansion as follows
\[ \Gamma_1(\zeta) = I + \frac{P^{(1)}(\zeta)}{\zeta} + \frac{P^{(2)}(\zeta)}{\zeta^2} + O(\frac{1}{\zeta^3}), \quad \zeta \to \infty . \quad (5.1) \]

Substituting the expression (5.1) into Eq. (2.6) yields
\[ i[\sigma, \Gamma^{(1)}_1] = U. \quad (5.2) \]

Comparing with the elements of the matrices of Eq. (5.2), we can finally recover the potential function.
We define
\[
Q = \begin{bmatrix}
Q_{1,1} & Q_{1,2} & \cdots & Q_{1,q} \\
\vdots & \vdots & \ddots & \vdots \\
Q_{p,1} & Q_{p,2} & \cdots & Q_{p,q}
\end{bmatrix},
\]
where
\[
Q_{i,j} = 2i(F^{(1)}_{i,j}), \quad 1 \leq i, j \leq p.
\]

For the above RH problem, under the reflectionless case ($S_3=0$, in Eq. (2.23)), the discrete spectrum corresponds to multiple soliton solutions. In addition, the solution where we obtain the explicit expression of the exact solutions of the mmKdV equation by
\[
\text{if we define}
\]

Then we obtain the solutions of the mmKdV equation as follows:
\[
Q_{m,n} = 2ie \sum_{k=1}^{N} \sum_{j=1}^{N} \alpha_{m,k} \alpha_{n,j}^* e^{\theta_k - \theta_j} (M^{-1})_{k,j}, \quad \epsilon = \pm 1.
\]

If we define
\[
F_{i,j} = \begin{vmatrix}
0 & \alpha_{i,1} e^{\theta_1} & \alpha_{i,2} e^{\theta_2} & \cdots & \alpha_{i,N} e^{\theta_N} \\
-e\alpha_{j,p,1} e^{-\theta_1} & M_{1,1} & M_{1,2} & \cdots & M_{1,N} \\
-e\alpha_{j,p,2} e^{-\theta_2} & M_{2,1} & M_{2,2} & \cdots & M_{2,N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-e\alpha_{j,p,N} e^{-\theta_N} & M_{N,1} & M_{N,2} & \cdots & M_{N,N}
\end{vmatrix}, \quad \epsilon = \pm 1,
\]
we obtain the explicit expression of the exact solutions of the mmKdV equation by
\[
Q_{i,j} = 2i \frac{\det(F_{i,j})}{\det(M)}, \quad 1 \leq i \leq p, 1 \leq j \leq q.
\]
6. Special examples

Next, let’s consider the following special cases:

6.1. $Q$ is taken as $2 \times 2$ form

At first, we consider the following form of potential matrix

$$Q = \begin{bmatrix} u_1(x, t) & u_2(x, t) \\ -u_2(x, t) & u_1(x, t) \end{bmatrix},$$

(6.1)

where $R = Q^\dagger$ and $p = q = 2$. It is a direct calculation to verify that the two-component mKdV equations of system (1.2) can be obtained by substituting Eq. (6.1) into Eq. (1.1), which is also studied in [35].

6.1.1. A variety of Rational Solutions and Physical Visions

Taking $N = 1$ in Eq. (5.8), once-iterated solutions are as follows

$$\begin{align*}
  u_1(x, t) &= 2i \frac{\alpha_{1,1} \alpha_{3,1}^*(\zeta_1 - \zeta_1^*) e^{\theta_1 - \theta_1^*}}{(|\alpha_{1,1}|^2 + |\alpha_{2,1}|^2) e^{i \theta_1 - i \theta_1^*} - (|\alpha_{3,1}|^2 + |\alpha_{4,1}|^2) e^{-i \theta_1 + i \theta_1^*}}, \\
  u_2(x, t) &= 2i \frac{\alpha_{1,1} \alpha_{4,1}^*(\zeta_1 - \zeta_1^*) e^{\theta_1 - \theta_1^*}}{(|\alpha_{1,1}|^2 + |\alpha_{2,1}|^2) e^{i \theta_1 - i \theta_1^*} - (|\alpha_{3,1}|^2 + |\alpha_{4,1}|^2) e^{-i \theta_1 + i \theta_1^*}},
\end{align*}$$

(6.2)

where

$$\zeta_1 = a_1 + ib_1, \quad \theta_1 = -i(\zeta_1 x + 4\zeta_1^3 t).$$

(6.3)

If we set $\alpha_{3,1} = \frac{\sqrt{3}}{2}$, $\alpha_{4,1} = \frac{1}{2}$ and $|\alpha_{1,1}|^2 = |\alpha_{2,1}|^2 = \frac{1}{4} e^{2\zeta}$, then Eq. (6.2) can be simplified as follows

$$\begin{align*}
  u_1(x, t) &= -\sqrt{3}\alpha_{1,1} b_1 e^{\theta_1 - \theta_1^* - \xi} \cosh(\theta_1 + \theta_1^* + \xi), \\
  u_2(x, t) &= -\alpha_{1,1} b_1 e^{\theta_1 - \theta_1^* - \xi} \cosh(\theta_1 + \theta_1^* + \xi).
\end{align*}$$

(6.4)

The localized structures and dynamic behaviors of once-iterated solutions are plotted in Fig. 1. It is noteworthy that although the waves corresponding to the solutions propagate in a fixed direction and show strong periodicity, the values are infinity in certain points, i.e. the solutions in Eq. (6.4) are singular which are different from usual soliton solutions.
In what follows, if we take \( N = 2 \), the twice-iterated solutions can be expressed by

\[
\begin{align*}
    u_1(x, t) &= \frac{2i}{M_{11}M_{22} - M_{12}M_{21}} (\alpha_{1,1}\alpha_{3,1}e^{\theta_{1} - \theta_{2}}M_{22} - \alpha_{1,1}\alpha_{3,2}e^{\theta_{1} - \theta_{2}}M_{12} - \alpha_{1,2}\alpha_{3,1}e^{\theta_{1} - \theta_{2}}M_{21} + \alpha_{1,2}\alpha_{3,2}e^{\theta_{1} - \theta_{2}}M_{11}), \\
    u_2(x, t) &= \frac{2i}{M_{11}M_{22} - M_{12}M_{21}} (\alpha_{1,1}\alpha_{4,1}e^{\theta_{1} - \theta_{2}}M_{22} - \alpha_{1,1}\alpha_{4,2}e^{\theta_{1} - \theta_{2}}M_{12} - \alpha_{1,2}\alpha_{4,1}e^{\theta_{1} - \theta_{2}}M_{21} + \alpha_{1,2}\alpha_{4,2}e^{\theta_{1} - \theta_{2}}M_{11}),
\end{align*}
\]

(6.5)

where

\[
\begin{align*}
    M_{11} &= ((|\alpha_{1,1}|^2 + |\alpha_{2,1}|^2)e^{\theta_{1} + \theta_{1}} - (|\alpha_{3,1}|^2 + |\alpha_{4,1}|^2)e^{-\theta_{1} - \theta_{1}}) \\
    M_{12} &= (\alpha_{1,1}^*\alpha_{2,1} + \alpha_{1,2}^*\alpha_{2,2})e^{\theta_{1} + \theta_{1}} - (\alpha_{3,1}^*\alpha_{3,2} + \alpha_{4,1}^*\alpha_{4,2})e^{-\theta_{1} - \theta_{1}}), \\
    M_{21} &= (\alpha_{1,2}^*\alpha_{1,1} + \alpha_{2,2}^*\alpha_{2,1})e^{\theta_{1} + \theta_{1}} - (\alpha_{3,2}^*\alpha_{3,1} + \alpha_{4,2}^*\alpha_{4,1})e^{-\theta_{1} - \theta_{1}}), \\
    M_{22} &= ((|\alpha_{1,2}|^2 + |\alpha_{2,2}|^2)e^{\theta_{1} + \theta_{1}} - (|\alpha_{3,2}|^2 + |\alpha_{4,2}|^2)e^{-\theta_{1} - \theta_{1}}) \\
    \end{align*}
\]

(6.6)

with \( \zeta_1 = \alpha_1 + i\beta_1, \zeta_2 = \alpha_2 + i\beta_2, \theta_1 = -i(\zeta_1 x + 4\zeta_1^2 t) \) and \( \theta_2 = -i(\zeta_2 x + 4\zeta_2^2 t) \).

If we select the parameters as \( \alpha_{3,1} = \alpha_{3,2} = \frac{i}{2}, \alpha_{4,1} = \alpha_{4,2} = -\frac{i}{2}, \alpha_{1,1} = \alpha_{1,2}, \alpha_{2,1} = \alpha_{2,2} \) and \( |\alpha_{1,1}|^2 = |\alpha_{2,1}|^2 = \frac{1}{8}e^{2\xi} \), then the explicit solutions can be expressed as...
follow

\[
\begin{align*}
  u_1(x,t) &= \sqrt{2i} \frac{(\alpha_1 e^{\theta_1 - \theta_*} M_{2,2} - \alpha_1 e^{\theta_1 - \theta_*} M_{1,1} - \alpha_1 e^{\theta_1 - \theta_*} M_{2,1} + \alpha_1 e^{\theta_1 - \theta_*} M_{1,1})}{M_{11} M_{22} - M_{12} M_{21}}, \\
  u_2(x,t) &= -\sqrt{2i} \frac{(\alpha_1 e^{\theta_1 - \theta_*} M_{2,2} - \alpha_1 e^{\theta_1 - \theta_*} M_{1,1} - \alpha_1 e^{\theta_1 - \theta_*} M_{2,1} + \alpha_1 e^{\theta_1 - \theta_*} M_{1,1})}{M_{11} M_{22} - M_{12} M_{21}},
\end{align*}
\]

(6.7)

where

\[
\begin{align*}
  M_{11} &= \frac{i}{b_1} e^{\xi} \sinh(\theta_1 + \theta_* + \xi), \\
  M_{12} &= \frac{2e^{\xi}}{a_2 - a_1 + i(b_1 + b_2)} \sinh(\theta_* + \theta_2 + \xi), \\
  M_{21} &= \frac{2e^{\xi}}{a_1 - a_2 + i(b_1 + b_2)} \sinh(\theta_* + \theta_1 + \xi), \\
  M_{22} &= \frac{i}{b_2} e^{\xi} \sinh(\theta_2 + \theta_* + \xi).
\end{align*}
\]

(6.8)

The localized structures and dynamic behaviors of twice-iterated solutions are plotted in Fig. 2. It can be seen from Fig. 2 that with the special parameters, the solution as a whole is wavy and shows a significant attenuation trend as time goes on.

![Figure 2. Twice-iterated solutions to Eq. (6.7) with parameters $\xi = 0.3$, $a_1 = -0.3$, $b_1 = 0.001$, $a_2 = 0.4$, $b_2 = 0.002$, $\alpha_{1,1} = \alpha_{1,2} = \frac{\sqrt{2}}{4} e^{\xi}$. (a)(d): the structures of the twice-iterated solutions, (b)(e): the density plot, (c)(f): the wave propagation of the twice-iterated solutions.](image-url)
6.2. *Q is taken as 4×4 form*

In this subsection, we consider the special case for potential matrix as follows

\[ Q = \begin{bmatrix}
    u_1(x, t) & u_2(x, t) & u_3(x, t) & 0 \\
    -u_2(x, t) & u_1(x, t) & 0 & u_3(x, t) \\
    -u_3(x, t) & 0 & u_1(x, t) & -u_2(x, t) \\
    0 & -u_3(x, t) & u_2(x, t) & u_1(x, t)
\end{bmatrix}, \quad (6.9) \]

Here \( R = -Q^\dagger, \ p = q = 4 \) and \( Q \) is a real-valued matrix. It is a direct calculation to verify that the three-component modified KdV equations of system (1.2) can be obtained by substituting Eq. (6.9) into Eq. (1.1), which is also studied in [35].

Note that, besides \( R = -Q^\dagger \), another symmetry \( Q^* = Q \) should be considered. Based on above symmetry, we can easily get that

\[ \Psi^*_i(-\zeta^*) = \Psi^*_i(\zeta), \quad i = 1, 2. \quad (6.10) \]

Moreover,

\[ S^*(-\zeta^*) = S(\zeta), \quad (6.11) \]

equivalently,

\[
\begin{cases}
    s_{i,j}(-\zeta^*) = s_{i,j}(\zeta), & 1 \leq i \leq p, \quad 1 \leq j \leq p, \\
    s_{i,j}(-\zeta) = s_{i,j}(\zeta), & 1 \leq i \leq p, \quad p \leq j \leq p + q, \\
    s_{i,j}(\zeta) = s_{i,j}(\zeta), & p \leq i \leq p + q, \quad 1 \leq j \leq p, \\
    s_{i,j}(-\zeta^*) = s_{i,j}(\zeta), & p + 1 \leq i \leq p + q, \quad p + 1 \leq j \leq p + q,
\end{cases}
\]

Further, we obtain

\[ \det(\Gamma_1(-\zeta^*))^* = \det(\Gamma_1(\zeta)), \quad \zeta \in \mathbb{C}^+. \quad (6.13) \]

Since \( \det(\Gamma_1(-\zeta^*))^* = \det(\Gamma_1(\zeta)), \ \zeta \in \mathbb{C}^+ \) and \( \det(\Gamma_1(\zeta^*)) = (\det(\Gamma_2(\zeta)))^*, \ \zeta \in \mathbb{C}^- \), we can obtain the relationship of zeros of \( \Gamma_1(\zeta) \) and \( \Gamma_2(\zeta) \). Based on these results, we consider the following three cases:

**Case 1:** \( \det(\Gamma_1(\zeta)) \) has 2\( \Delta_1 \) simple zeros \( \zeta_j (1 \leq j \leq 2\Delta_1) \) in \( \mathbb{C}^+ \), and \( \zeta_j = -\zeta^*_j - \Delta_1, \ (\Delta_1 + 1 \leq j \leq 2\Delta_1) \).

Considering the Eq. (4.1) and Eq. (6.10), we have

\[
\begin{cases}
    \hat{\theta}_j = \theta^*_j, & 1 \leq j \leq 2\Delta_1, \\
    \theta_j = \theta^*_{j-\Delta_1}, & \Delta_1 + 1 \leq j \leq 2\Delta_1.
\end{cases}
\]

(6.14)

From Eqs. (4.8) and (6.14), we can obtain

\[
\theta_j = \begin{cases}
    e^{\rho^* \sigma^*} \theta^*_{j,0}, & 1 \leq j \leq \Delta_1, \\
    e^{\rho^*_{-\Delta_1} \sigma^*} \theta^*_{j-\Delta_1,0}, & \Delta_1 + 1 \leq j \leq 2\Delta_1.
\end{cases}
\]

(6.15)
\[
\hat{\theta}_j = \begin{cases} 
\theta_j, & 1 \leq j \leq \Delta_1, \\
\theta_j^\dagger, & \Delta_1 + 1 \leq j \leq 2\Delta_1.
\end{cases}
\quad (6.16)
\]

Moreover, the solutions of RH problem in Eq. (5.5) can be rewritten as follows:
\[
\begin{align*}
\Gamma_1(\zeta) &= \mathbb{I} - \sum_{k=1}^{2\Delta_1} \sum_{j=1}^{2\Delta_1} \frac{\partial_k \theta_j (M^{-1}) k_j}{\zeta - \zeta_j^*}, \\
\Gamma_2(\zeta) &= \mathbb{I} + \sum_{k=1}^{2\Delta_1} \sum_{j=1}^{2\Delta_1} \frac{\partial_k \hat{\theta}_j (M^{-1}) k_j}{\zeta - \zeta_k}.
\end{align*}
\quad (6.17)
\]

**Case 2:** \(\det(\Gamma_1(\zeta))\) has \(\Delta_2\) simple zeros \(\zeta_j (1 \leq j \leq \Delta_2)\) in \(\mathbb{C}^+\), and each \(\zeta_j\) is pure imaginary.

Similarly, \(\hat{\theta}_j\) and \(\hat{\theta}_j\) satisfy:
\[
\begin{align*}
\hat{\theta}_j &= e^{\theta_j} \hat{\theta}_j, & \quad 1 \leq j \leq \Delta_2, \\
\hat{\theta}_j &= \hat{\theta}_j^\dagger, & \quad 1 \leq j \leq \Delta_2.
\end{align*}
\quad (6.18)
\]

Furthermore,
\[
\begin{align*}
\theta_j &= e^{\theta_j} \theta_j, & \quad 1 \leq j \leq \Delta_2, \\
\hat{\theta}_j &= \hat{\theta}_j^\dagger, & \quad 1 \leq j \leq \Delta_2.
\end{align*}
\quad (6.19)
\]

Moreover, the solutions of RH problem in Eq. (5.5) can be rewritten as follows
\[
\begin{align*}
\Gamma_1(\zeta) &= \mathbb{I} - \sum_{k=1}^{\Delta_1} \sum_{j=1}^{\Delta_1} \frac{\partial_k \theta_j (M^{-1}) k_j}{\zeta - \zeta_j^*}, \\
\Gamma_2(\zeta) &= \mathbb{I} + \sum_{k=1}^{\Delta_2} \sum_{j=1}^{\Delta_2} \frac{\partial_k \hat{\theta}_j (M^{-1}) k_j}{\zeta - \zeta_k}.
\end{align*}
\quad (6.20)
\]

**Case 3:** \(\det(\Gamma_1(\zeta))\) has \(2\Delta_1 + \Delta_2\) simple zeros in \(\mathbb{C}^+\), where the first \(2\Delta_1\) zeros satisfy \(\zeta_{\Delta_1 + j} = -\zeta_j^*, 1 \leq j \leq \Delta_1\), and \(\zeta_j, (2\Delta_1 + 1 \leq j \leq 2\Delta_1 + \Delta_2)\), are pure imaginary.

Similarly, \(\theta_j\) and \(\hat{\theta}_j\) satisfy
\[
\begin{align*}
\theta_j &= e^{\theta_j} \theta_j, & \quad 1 \leq j \leq 2\Delta_1 + \Delta_2, \\
\hat{\theta}_j &= \hat{\theta}_j^\dagger, & \quad 1 \leq j \leq 2\Delta_1 + \Delta_2, \\
\hat{\theta}_j &= \hat{\theta}_j^\dagger, & \quad 1 \leq j \leq \Delta_1 + \Delta_2.
\end{align*}
\quad (6.21)
\]

Furthermore,
\[
\theta_j = \begin{cases} 
\theta_j, & 1 \leq j \leq \Delta_1, \\
\theta_j^\dagger, & \Delta_1 + 1 \leq j \leq 2\Delta_1, \\
\theta_j^\dagger, & \Delta_1 + 1 \leq j \leq 2\Delta_1 + \Delta_2.
\end{cases}
\quad (6.22)
\]
\[
\hat{\theta}_j = \begin{cases} 
\theta_{j,0} e^{\theta_j \sigma}, & 1 \leq j \leq \Delta_1, \\
\theta_{j,\Delta_1} e^{\theta_j \sigma}, & \Delta_1 + 1 \leq j \leq 2\Delta_1, \\
\theta_{j,2\Delta_1} e^{\theta_j \sigma}, & 2\Delta_1 + 1 \leq j \leq 2\Delta_1 + \Delta_2.
\end{cases}
\] (6.23)

Moreover, the solutions of RH problem in Eq. (5.5) can be rewritten as follows:

\[
\begin{align*}
\Gamma_1(\zeta) &= I - \sum_{k=1}^{2\Delta_1 + \Delta_2} \sum_{j=1}^{2\Delta_1 + \Delta_2} \frac{\hat{\theta}_k \hat{\theta}_j (M^{-1})_{k,j}}{\zeta - \zeta_j}, \\
\Gamma_2(\zeta) &= I + \sum_{k=1}^{2\Delta_1 + \Delta_2} \sum_{j=1}^{2\Delta_1 + \Delta_2} \frac{\hat{\theta}_k \hat{\theta}_j (M^{-1})_{k,j}}{\zeta - \zeta_j}.
\end{align*}
\] (6.24)

6.2.1. Multi-soliton solutions

For case 1, we suppose that

\[
\theta_j = (\alpha_{1,j} e^{\theta_j}, \ldots, \alpha_{4,j} e^{\theta_j}, \alpha_{5,j} e^{-\theta_j}, \ldots, \alpha_{8,j} e^{-\theta_j})^T, \quad 1 \leq j \leq \Delta_1,
\] (6.25)

naturally,

\[
\theta_{\Delta_1+j} = (\alpha_{1,j}^* e^{\theta_j}, \ldots, \alpha_{4,j}^* e^{\theta_j}, \alpha_{5,j}^* e^{-\theta_j}, \ldots, \alpha_{8,j}^* e^{-\theta_j})^T, \quad 1 \leq j \leq \Delta_1,
\] (6.26)

then \(\Delta_1\)-breather solution are as follows

\[
u_k = 2 \det(F_{k+4}) \det(M),
\] (6.27)

where

\[
F_{k+4} = \begin{bmatrix} 0 & \beta_k^T \\ \beta_{k+4} & M \end{bmatrix}, \quad 1 \leq k \leq 3,
\] (6.28)

\[
M = \begin{bmatrix} M_{1,1} & \ldots & M_{1,\Delta_1} & M_{1,\Delta_1+1} & \ldots & M_{1,2\Delta_1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
M_{\Delta_1,1} & \ldots & M_{\Delta_1,\Delta_1} & M_{\Delta_1,\Delta_1+1} & \ldots & M_{\Delta_1,2\Delta_1} \\
M_{\Delta_1+1,1} & \ldots & M_{\Delta_1+1,\Delta_1} & M_{\Delta_1+1,\Delta_1+1} & \ldots & M_{\Delta_1+1,2\Delta_1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
M_{2\Delta_1,1} & \ldots & M_{2\Delta_1,\Delta_1} & M_{2\Delta_1,\Delta_1+1} & \ldots & M_{2\Delta_1,2\Delta_1} \end{bmatrix},
\] (6.29)

with

\[
\beta_1 = (\alpha_{1,1} e^{\theta_1}, \ldots, \alpha_{1,\Delta_1} e^{\theta_1}, \alpha_{1,\Delta_1+1} e^{\theta_1}, \ldots, \alpha_{1,2\Delta_1} e^{\theta_1})^T = (\alpha_{1,1} e^{\theta_1}, \ldots, \alpha_{1,\Delta_1} e^{\theta_1}, \alpha_{1,\Delta_1} e^{\theta_1})^T,
\] (6.30)

\[
\beta_{k+4} = (\alpha_{k+4,1} e^{-\theta_1}, \ldots, \alpha_{k+4,\Delta_1} e^{-\theta_1}, \alpha_{k+4,\Delta_1+1} e^{-\theta_1}, \ldots, \alpha_{k+4,2\Delta_1} e^{-\theta_1})^T = (\alpha_{k+4,1} e^{-\theta_1}, \ldots, \alpha_{k+4,\Delta_1} e^{-\theta_1}, \alpha_{k+4,\Delta_1} e^{-\theta_1})^T.
\]
For case 2, we set
\[ \theta_j = (\alpha_1 e^{\theta_1}, \ldots, \alpha_4 e^{\theta_1}, \alpha_5 e^{-\theta_1}, \ldots, \alpha_8 e^{-\theta_1})^T, \quad 1 \leq j \leq \Delta_2, \tag{6.31} \]
then we can obtain \( \Delta_2 \)-bell solutions
\[ u_k = 2i \frac{\text{det}(F_{k+4})}{\text{det}(M)}, \tag{6.32} \]
where
\[ F_{k+4} = \begin{bmatrix} 0 & \beta_1^T \\ \beta_{k+4} & M \end{bmatrix}, \quad 1 \leq k \leq 3, \tag{6.33} \]
\[ M = \begin{bmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,\Delta_2} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,\Delta_2} \\ \vdots & \vdots & \ddots & \vdots \\ M_{\Delta_1,1} & M_{\Delta_1,2} & \cdots & M_{\Delta_1,\Delta_2} \end{bmatrix}, \tag{6.34} \]
\[ \beta_1 = (\alpha_1 e^{\theta_1}, \alpha_1 e^{\theta_1}, \ldots, \alpha_{\Delta_1} e^{\theta_{\Delta_1}})^T, \]
\[ \beta_{k+4} = (\alpha_{\Delta_1+1} e^{-\theta_{\Delta_1}}, \alpha_{\Delta_1+2} e^{-\theta_{\Delta_1}}, \ldots, \alpha_{\Delta_1+\Delta_2} e^{-\theta_{\Delta_1}})^T. \tag{6.35} \]

For case 3, combining the results of case 1 and case 2, we can easily obtain the \( 2\Delta_1 + \Delta_2 \) soliton solutions.

6.2.2. A variety of Rational Solutions and Physical Visions

For case 1, if we take \( \Delta_1 = 1 \), we can obtain single-breather solutions as follows
\[ u_1(x, t) = 2i \begin{vmatrix} 0 & \alpha_{1,1} e^{\theta_1} & \alpha_{1,1}^* e^{\theta_1} \\ \alpha_{5,1} e^{-\theta_1} & M_{1,1} & M_{1,2} \\ \alpha_{5,1} e^{\theta_1} & M_{2,1} & M_{2,2} \end{vmatrix}, \tag{6.36} \]
\[ u_2(x, t) = 2i \begin{vmatrix} 0 & \alpha_{1,1} e^{\theta_1} & \alpha_{1,1}^* e^{\theta_1} \\ \alpha_{6,1} e^{-\theta_1} & M_{1,1} & M_{1,2} \\ \alpha_{6,1} e^{\theta_1} & M_{2,1} & M_{2,2} \end{vmatrix}, \tag{6.36} \]
\[ u_3(x, t) = 2i \begin{vmatrix} 0 & \alpha_{1,1} e^{\theta_1} & \alpha_{1,1}^* e^{\theta_1} \\ \alpha_{7,1} e^{-\theta_1} & M_{1,1} & M_{1,2} \\ \alpha_{7,1} e^{\theta_1} & M_{2,1} & M_{2,2} \end{vmatrix}. \]
where

\[
\begin{align*}
M_{11} &= \frac{(|a_{1,1}|^2 + |a_{2,1}|^2 + |a_{3,1}|^2 + |a_{4,1}|^2)e^{\eta_1 + \theta_1}}{\zeta_1 - \zeta_1^*} + \frac{(|a_{5,1}|^2 + |a_{6,1}|^2 + |a_{7,1}|^2 + |a_{8,1}|^2)e^{-\eta_1 - \theta_1}}{\zeta_1 - \zeta_1^*}, \\
M_{12} &= \frac{(a_{5,1}^* a_{1,1}^* + a_{2,1}^* a_{3,1}^* + a_{3,1}^* a_{5,1}^* + a_{4,1}^* a_{4,1}^*)e^{\eta_1 + \theta_1}}{\zeta_2 - \zeta_1^*} + \frac{(a_{5,1}^* a_{6,1}^* + a_{7,1}^* a_{7,1}^* + a_{8,1}^* a_{8,1}^*)e^{-\eta_1 - \theta_1}}{\zeta_2 - \zeta_1^*}, \\
M_{21} &= \frac{(a_{1,1} a_{1,1} + a_{2,1} a_{2,1} + a_{3,1} a_{3,1} + a_{4,1} a_{4,1})e^{\eta_1 + \theta_1}}{\zeta_1 - \zeta_2^*} + \frac{(a_{5,1} a_{5,1} + a_{6,1} a_{6,1} + a_{7,1} a_{7,1} + a_{8,1} a_{8,1})e^{-\eta_1 - \theta_1}}{\zeta_1 - \zeta_2^*}, \\
M_{22} &= \frac{|a_{1,1}|^2 + |a_{2,1}|^2 + |a_{3,1}|^2 + |a_{4,1}|^2)e^{\eta_1 + \theta_1}}{\zeta_2 - \zeta_2^*} + \frac{|a_{5,1}|^2 + |a_{6,1}|^2 + |a_{7,1}|^2 + |a_{8,1}|^2)e^{-\eta_1 - \theta_1}}{\zeta_2 - \zeta_2^*}.
\end{align*}
\]

(6.37)

with \(\zeta_1 = a_1 + ib_1(b_1 \neq 0, b_1 > 0), \theta_1 = -i(\zeta_1 x + 4\zeta_1^3 t)\) and \(\zeta_2 = -\zeta_1^*\). In addition, the localized structures and dynamic behaviors of single-breather solutions are shown in Fig. 3.

For case 2, if we take the \(\Delta_2 = 1\), we can obtain the single-breather soliton solution as follows

\[
\begin{align*}
&u_1(x, t) = \frac{-2i a_{1,1} a_{5,1}^* (\zeta_1 - \zeta_1^*)e^{\eta_1 - \theta_1}}{\gamma_1 e^{\eta_1 + \theta_1} + \gamma_2 e^{-\eta_1 - \theta_1}}, \\
&u_2(x, t) = \frac{-2i a_{1,1} a_{6,1}^* (\zeta_1 - \zeta_1^*)e^{\eta_1 - \theta_1}}{\gamma_1 e^{\eta_1 + \theta_1} + \gamma_2 e^{-\eta_1 - \theta_1}}, \\
&u_3(x, t) = \frac{-2i a_{1,1} a_{7,1}^* (\zeta_1 - \zeta_1^*)e^{\eta_1 - \theta_1}}{\gamma_1 e^{\eta_1 + \theta_1} + \gamma_2 e^{-\eta_1 - \theta_1}},
\end{align*}
\]

(6.38)

where

\[
\begin{align*}
\gamma_1 &= |a_{1,1}|^2 + |a_{2,1}|^2 + |a_{3,1}|^2 + |a_{4,1}|^2, \\
\gamma_2 &= |a_{5,1}|^2 + |a_{6,1}|^2 + |a_{7,1}|^2 + |a_{8,1}|^2, \\
\zeta_1 &= ib_1(b_1 > 0), \theta_1 = -i(\zeta_1 x + 4\zeta_1^3 t).
\end{align*}
\]

(6.39)
From the 3D plot and density plot in Fig. 3, we can see that the waves corresponds to the solutions which travels left along the $x$-axis and their amplitude increases and decreases periodically. Moreover, it can be seen from the dynamic behaviors that the waveforms when time is negative are almost symmetrical with the waveform when time is positive, but they are not symmetrical about the central axis of themselves. At $t = 0$, the waveforms are symmetrical about the central axis.

6.3. $Q$ is taken as a special 1×3 form

In this subsection, we consider following special case for matrix $Q$

$$Q = (u_1(x, t), u_2(x, t), u_3(x, t))^T, \quad (6.40)$$

where $p = 1, q = 3$ and $R = -Q^T$. 

Figure 3. (Color online) Single-breather solutions to Eq. (6.36) with the parameters $a_1 = 0.2, b_1 = 0.2, \alpha_{1,1} = \alpha_{3,1} = 0.1, \alpha_{2,1} = \alpha_{5,1} = 0.2, \alpha_{4,1} = 0.05, \alpha_{7,1} = 0.12, \alpha_{8,1} = 0.13$. (a)(d)(g): the structures of the single-breather solutions, (b)(e)(h): the density plot , (c)(f)(i): the wave propagation of the single-breather solutions.
6.3.1. A variety of Rational Solutions and Physical Visions

If we set \( N = 1, 2, 3, \) \( p = 1, \) and \( q = 3 \) in Eq. \((5.8)\), we can easily get the one-soliton solutions, two-soliton solutions and three-soliton solutions, respectively,

\[
\begin{align*}
    u_1(x, t) &= -2i \sum_{k=1}^{N} \sum_{j=1}^{N} \alpha_{1,k}^* \alpha_{2,j} e^{i \alpha_1 - \theta_j} (M^{-1})_{k,j}, \quad N = 1, 2, 3, \\
    u_2(x, t) &= -2i \sum_{k=1}^{N} \sum_{j=1}^{N} \alpha_{1,k}^* \alpha_{3,j} e^{i \alpha_1 - \theta_j} (M^{-1})_{k,j}, \quad N = 1, 2, 3, \\
    u_3(x, t) &= -2i \sum_{k=1}^{N} \sum_{j=1}^{N} \alpha_{1,k}^* \alpha_{4,j} e^{i \alpha_1 - \theta_j} (M^{-1})_{k,j}, \quad N = 1, 2, 3,
\end{align*}
\]

where \( M_{k,j} = \frac{\alpha_{1,k} \alpha_{1,j} e^{i \theta_j} + \alpha_{1,k} \alpha_{2,j} e^{i \theta_j} + \alpha_{3,k} \alpha_{3,j} e^{i \theta_j} + \alpha_{4,k} \alpha_{4,j} e^{i \theta_j}}{\zeta_j - \zeta_k}. \)

\[\text{(a)(d)(g): the structures of the one-soliton solutions, (b)(e)(h): the density plot, (c)(f)(i): the wave propagation of the one-soliton solutions.}\]
Fig. 4 shows the localized structures and dynamic propagation behaviors of one-soliton solutions by choosing appropriate parameters. As shown in Fig. 4, we can see that three one-soliton solutions have the same direction of propagation, at the same time, they have similar properties in amplitude, peak and so forth.

![Figures](images)

**Figure 5.** (Color online) Two-soliton solutions to Eq. (6.41) with the parameters $a_1 = b_1 = 0.25$, $a_2 = b_2 = 0.5$, $\alpha_{1,1} = 0.4$, $\alpha_{2,1} = 0.5$, $\alpha_{3,1} = 0.6$, $\alpha_{4,1} = 0.7$, $\alpha_{1,2} = 0.5$, $\alpha_{2,2} = 0.25$, $\alpha_{3,2} = 0.55$, $\alpha_{4,2} = 0.8$. (a)(d)(g): the structures of the two-soliton solutions, (b)(e)(h): the density plot, (c)(f)(i): the wave propagation of the two-soliton solutions.

Fig. 5 presents the localized structures and dynamic propagation behaviors of two-soliton solutions by choosing appropriate parameters. Through careful observation of $u_1$ in Figs. 5(a) – 5(c), we can see that the propagation direction of lower left soliton is unexchanged before and after two solitons collide with each other, but its energy has been improved, while another soliton in the upper right has exchanged its position and decreased its energy. Similar results for $u_2$ and $u_3$ can be obtained by observing Figs. 5(d) – 5(f) and Figs. 5(g) – 5(i), respectively.
Fig. 6 displays the localized structures and dynamic propagation behaviors of three-soliton solutions by choosing appropriate parameters. Via analyzing the figures carefully, we can obtain the information about the energy, propagation direction and position of three solitons. For example, as for $u_1$, before three solitons collide with each other, two of the three solitons travel in nearly parallel directions and the third soliton travel in another different direction. After collision, the propagation directions of all three solitons are unexchanged, but their positions have shifted and their energies have been increased or decreased.

**Figure 6.** (Color online) Three-soliton solutions to Eq. (6.41) with the parameters $a_1 = b_1 = 0.25$, $a_2 = b_2 = 0.5$, $a_3 = b_3 = 0.3$, $\alpha_{1,1} = 0.25$, $\alpha_{2,1} = 0.45$, $\alpha_{3,1} = 0.75$, $\alpha_{4,1} = 0.6$, $\alpha_{1,2} = 1.9$, $\alpha_{2,2} = 1.6$, $\alpha_{3,2} = 1.3$, $\alpha_{4,2} = 1.0$, $\alpha_{1,3} = 2.5$, $\alpha_{2,3} = 2.9$, $\alpha_{3,3} = 2.3$, $\alpha_{4,3} = 2.1$. (a)(d)(g): the structures of the three-soliton solutions, (b)(e)(h): the density plot, (c)(f)(i): the wave propagation of the three-soliton solutions.
6.4. $Q$ is taken as a special $6 \times 1$ form

In last subsection, we take $Q$ as special case as follows

$$Q = (u_1, u_1^*, u_2, u_2^*, u_3, u_3^*)^T,$$  \hspace{1cm} (6.42)

Then

$$U = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & u_1 \\
0 & 0 & 0 & 0 & 0 & 0 & u_1^* \\
0 & 0 & 0 & 0 & 0 & 0 & u_2 \\
0 & 0 & 0 & 0 & 0 & 0 & u_2^* \\
- u_1^* & - u_1 & - u_2^* & - u_2 & - u_3^* & - u_3 & 0
\end{bmatrix},$$  \hspace{1cm} (6.43)

where $p = 6, q = 1$. Specially, besides $U = -U^\dagger$, we can derive another symmetry for $U$,

$$U^* = \Lambda U \Lambda,$$  \hspace{1cm} (6.44)

where

$$\Lambda = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}.$$  \hspace{1cm} (6.45)

With the aid of Eq. (6.44), we can derive

$$\Lambda \Psi_1^*(-\xi^*) \Lambda = \Psi_1(\xi), \hspace{0.5cm} \Lambda \Psi_2^*(-\xi^*) \Lambda = \Psi_2(\xi),$$  \hspace{1cm} (6.46)

which leads to

$$\Lambda S^*(-\xi^*) \Lambda = S(\xi).$$  \hspace{1cm} (6.47)

Moreover, Eq. (6.46) also yield a property,

$$\Lambda \Gamma_1^*(-\xi^*) \Lambda = \Gamma_1(\xi),$$  \hspace{1cm} (6.48)

which indicates that if $\xi_j$ is one zero of $\text{det}(\Gamma_1)$, $-\xi_j^*$ is also one zero of $\text{det}(\Gamma_1)$. Therefore, we consider the zeros of $\text{det}(\Gamma_1)$ in the following three cases:

- Suppose that $\text{det}(\Gamma_1)$ has $2N_1$ simple zeros satisfying $\text{Re}(\xi_j) \neq 0$, $\xi_j = -\xi_{j-N_1}^*$, $N_1 + 1 \leq j \leq 2N_1$, which are all in $\mathbb{C}^*$. 

• Suppose that det($\Gamma_1$) has $N_2$ simple zeros $\zeta_j$ which are all pure imaginary in $\mathbb{C}^+$;

• Suppose that det($\Gamma_1$) has $2N_1 + N_2$ simple zeros $\zeta_j$, where the first $2N_1$ zeros satisfy Re($\zeta_j$) $\neq$ 0 and $\zeta_j = -\zeta_{j-N_1}^*$, $N_1 + 1 \leq j \leq 2N_1$, the last $N_2$ zeros are pure imaginary, and all zeros $\zeta_j (1 \leq j \leq 2N_1 + N_2)$ are all in $\mathbb{C}^+$.

Similar with the last subsection, we can derive the similar results. First, suppose that the discrete scattering data consists of $\{\zeta_j, \hat{\zeta}_j, \hat{\theta}_j, \hat{\theta}_j\}$ satisfy

$$\Gamma_1(\hat{\zeta}_j)\hat{\theta}_j = 0, \quad (6.49)$$

$$\hat{\theta}_j\Gamma_2(\hat{\zeta}_j) = 0. \quad (6.50)$$

For case 1, we can obtain

$$\hat{\theta}_j = \theta_j^\dagger, \quad 1 \leq j \leq 2N_1,$$

$$\hat{\theta}_j = \Lambda \theta_{j-N_1}^*, \quad N_1 + 1 \leq j \leq 2N_1. \quad (6.51)$$

More specifically,

$$\theta_j = \begin{cases} 
\frac{e^{i\phi_j^\sigma}}{\Lambda e^{i\phi_{j-N_1}^\sigma}} \theta_{j-N_1}, & 1 \leq j \leq N_1, \\
\frac{\theta_{N_1} e^{i\phi_{j-N_1}^\sigma}}{\theta_{j-N_1} e^{i\phi_{j-N_1}^\sigma} \Lambda}, & N_1 + 1 \leq j \leq 2N_1.
\end{cases} \quad (6.52)$$

$$\hat{\theta}_j = \begin{cases} 
\frac{\theta_{j-N_1} e^{i\phi_{j-N_1}^\sigma}}{\theta_{j-N_1} e^{i\phi_{j-N_1}^\sigma} \Lambda}, & 1 \leq j \leq N_1, \\
\frac{\theta_{N_1} e^{i\phi_{j-N_1}^\sigma}}{\theta_{j-N_1} e^{i\phi_{j-N_1}^\sigma} \Lambda}, & N_1 + 1 \leq j \leq 2N_1.
\end{cases} \quad (6.53)$$

Moreover, the solutions of RH problem in Eq. (5.5) can be rewritten as follows,

$$\begin{cases} 
\Gamma_1(\zeta) = \mathbb{I} - \sum_{k=1}^{2N_1} \sum_{j=1}^{2N_1} \frac{\bar{\theta}_k \bar{\theta}_j (M^{-1})_{k,j}}{\zeta - \zeta_j}, \\
\Gamma_2(\zeta) = \mathbb{I} + \sum_{k=1}^{2N_1} \sum_{j=1}^{2N_1} \frac{\bar{\theta}_k \bar{\theta}_j (M^{-1})_{k,j}}{\zeta - \zeta_k}.
\end{cases} \quad (6.54)$$

then we have

$$\Gamma_1^{(1)}(\zeta) = \sum_{k=1}^{2N_1} \sum_{j=1}^{2N_1} \bar{\theta}_k \bar{\theta}_j (M^{-1})_{k,j}. \quad (6.55)$$

finally, the solutions are as follows

$$\begin{cases} 
u_1(x, t) = -2i(\Gamma_1^{(1)})_{1,7}, \\
u_2(x, t) = -2i(\Gamma_1^{(1)})_{3,7}, \\
u_3(x, t) = -2i(\Gamma_1^{(1)})_{5,7}.
\end{cases} \quad (6.56)$$
Similarly, for case 2, we have the following process

\[
\begin{align*}
\vartheta_j &= e^{i\sigma_j} \vartheta_j^o, & 1 \leq j \leq N_1, \\
\hat{\vartheta}_j &= \vartheta_j^o e^{i\sigma_j}, & 1 \leq j \leq N_2. 
\end{align*}
\]

(6.57)

Moreover, the solutions of RH problem in Eq. (5.5) can be rewritten as follows

\[
\begin{align*}
\Gamma_1(\zeta) &= I - \sum_{k=1}^{N_2} \sum_{j=1}^{N_1} \frac{\vartheta_k \vartheta_j^o (M^{-1})_{k,j}}{\zeta - \zeta_j}, \\
\Gamma_2(\zeta) &= I + \sum_{k=1}^{N_2} \sum_{j=1}^{N_1} \frac{\vartheta_k \vartheta_j^o (M^{-1})_{k,j}}{\zeta - \zeta_k}.
\end{align*}
\]

(6.58)

then we have

\[
\Gamma_1^{(1)}(\zeta) = \sum_{k=1}^{N_2} \sum_{j=1}^{N_1} \vartheta_k \vartheta_j^o (M^{-1})_{k,j}.
\]

(6.59)

Finally, the solutions are as follows

\[
\begin{align*}
u_1(x, t) &= -2i(\Gamma_1^{(1)})_{1,7}, \\
u_2(x, t) &= -2i(\Gamma_1^{(1)})_{3,7}, \\
u_3(x, t) &= -2i(\Gamma_1^{(1)})_{3,7}.
\end{align*}
\]

(6.60)

Equally, for case 3, we have the following process:

\[
\vartheta_j = \begin{cases} 
\vartheta_j^o e^{i\sigma_j} & 1 \leq j \leq N_1, \\
e^{i\sigma_j} \vartheta_j^o & N_1 + 1 \leq j \leq 2N_1, \\
\vartheta_j^o e^{i\sigma_j} & 2N_1 + 1 \leq j \leq 2N_1 + N_2,
\end{cases}
\]

(6.61)

\[
\hat{\vartheta}_j = \begin{cases} 
\vartheta_j^o e^{i\sigma_j} & 1 \leq j \leq N_1, \\
\vartheta_j^o e^{i\sigma_j} & N_1 + 1 \leq j \leq 2N_1, \\
\vartheta_j^o e^{i\sigma_j} & 2N_1 + 1 \leq j \leq 2N_1 + N_2.
\end{cases}
\]

(6.62)

Moreover, the solutions of RH problem in Eq. (5.5) can be rewritten as follows

\[
\begin{align*}
\Gamma_1(\zeta) &= I - \sum_{k=1}^{2N_1+N_2} \sum_{j=1}^{N_1+N_2} \frac{\vartheta_k \vartheta_j^o (M^{-1})_{k,j}}{\zeta - \zeta_j}, \\
\Gamma_2(\zeta) &= I + \sum_{k=1}^{2N_1+N_2} \sum_{j=1}^{N_1+N_2} \frac{\vartheta_k \vartheta_j^o (M^{-1})_{k,j}}{\zeta - \zeta_k}.
\end{align*}
\]

(6.63)

then we have

\[
\Gamma_1^{(1)}(\zeta) = \sum_{k=1}^{2N_1+N_2} \sum_{j=1}^{N_1+N_2} \vartheta_k \vartheta_j^o (M^{-1})_{k,j}.
\]

(6.64)
Finally, the solutions are as follows:

\[
\begin{align*}
  u_1(x, t) &= -2i(\Gamma_1^{(1)})_{1,7}, \\
  u_2(x, t) &= -2i(\Gamma_1^{(1)})_{3,7}, \\
  u_3(x, t) &= -2i(\Gamma_1^{(1)})_{5,7}.
\end{align*}
\]  

(6.65)

6.4.1. Multi-soliton solutions

For case 1, we set \( \tilde{\theta}_{1,0} = (\alpha_{1, j}, \alpha_{2, j}, \alpha_{3, j}, \alpha_{4, j}, \alpha_{5, j}, \alpha_{6, j})^T \), then

\[
\begin{align*}
  u_1(x, t) &= 2i \sum_{k=1}^{N_1} \sum_{j=1}^{N_1} \alpha_{1,k} e^{\theta_{0,j} - \theta_{0} - \theta_{1}} (M^{-1})_{k,j} + 2i \sum_{k=1}^{N_1} \sum_{j=N_1+1}^{2N_1} \alpha_{1,k} e^{\theta_{0,j} - \theta_{0} - \theta_{1}} (M^{-1})_{k,j} + 2i \sum_{k=N_1+1}^{2N_1} \sum_{j=1}^{N_1} \alpha_{2,k-N_1} e^{\theta_{0,j} - \theta_{0} - \theta_{1}} (M^{-1})_{k,j} + 2i \sum_{k=N_1+1}^{2N_1} \sum_{j=N_1+1}^{2N_1} \alpha_{2,k-N_1} e^{\theta_{0,j} - \theta_{0} - \theta_{1}} (M^{-1})_{k,j} + 2i \sum_{k=1}^{N_1} \sum_{j=1}^{N_1} \alpha_{3,k} e^{\theta_{0,j} - \theta_{0} - \theta_{1}} (M^{-1})_{k,j} + 2i \sum_{k=1}^{N_1} \sum_{j=N_1+1}^{2N_1} \alpha_{3,k} e^{\theta_{0,j} - \theta_{0} - \theta_{1}} (M^{-1})_{k,j} + 2i \sum_{k=N_1+1}^{2N_1} \sum_{j=1}^{N_1} \alpha_{4,k-N_1} e^{\theta_{0,j} - \theta_{0} - \theta_{1}} (M^{-1})_{k,j} + 2i \sum_{k=N_1+1}^{2N_1} \sum_{j=N_1+1}^{2N_1} \alpha_{4,k-N_1} e^{\theta_{0,j} - \theta_{0} - \theta_{1}} (M^{-1})_{k,j} + 2i \sum_{k=1}^{N_1} \sum_{j=1}^{N_1} \alpha_{5,k} e^{\theta_{0,j} - \theta_{0} - \theta_{1}} (M^{-1})_{k,j} + 2i \sum_{k=1}^{N_1} \sum_{j=N_1+1}^{2N_1} \alpha_{5,k} e^{\theta_{0,j} - \theta_{0} - \theta_{1}} (M^{-1})_{k,j} + 2i \sum_{k=N_1+1}^{2N_1} \sum_{j=1}^{N_1} \alpha_{6,k-N_1} e^{\theta_{0,j} - \theta_{0} - \theta_{1}} (M^{-1})_{k,j} + 2i \sum_{k=N_1+1}^{2N_1} \sum_{j=N_1+1}^{2N_1} \alpha_{6,k-N_1} e^{\theta_{0,j} - \theta_{0} - \theta_{1}} (M^{-1})_{k,j}.
\end{align*}
\]  

(6.66)

where \( M \) is an \( 2N_1 \times 2N_1 \) matrix. For simplicity, we define

\[
\begin{align*}
  M_{k,j}^{(1)} &= \alpha_{1,k}^{*} \alpha_{1,j} + \alpha_{2,k}^{*} \alpha_{2,j} + \cdots + \alpha_{6,k}^{*} \alpha_{6,j}, \\
  M_{k,j}^{(2)} &= \alpha_{1,k}^{*} \alpha_{2,j-N_1} + \alpha_{2,k}^{*} \alpha_{1,j-N_1} + \cdots + \alpha_{6,k}^{*} \alpha_{5,j-N_1}, \\
  M_{k,j}^{(3)} &= \alpha_{2,k-N_1} \alpha_{1,j} + \alpha_{1,k-N_1} \alpha_{2,j} + \cdots + \alpha_{5,k-N_1} \alpha_{6,j}, \\
  M_{k,j}^{(4)} &= \alpha_{1,k-N_1} \alpha_{1,j-N_1} + \alpha_{2,k-N_1} \alpha_{2,j-N_1} + \cdots + \alpha_{6,k-N_1} \alpha_{6,j-N_1}.
\end{align*}
\]  

(6.67)
then

\[
\frac{M_{k,j}^{(1)} e^{\theta_k \theta_j} + e^{-\theta_k \theta_j}}{\zeta_j - \zeta_k}, \quad 1 \leq k, j \leq N_1
\]

\[
\frac{M_{k,j}^{(2)} e^{\theta_k \theta_j} + e^{-\theta_k \theta_j}}{\zeta_j - \zeta_k}, \quad 1 \leq k \leq N_1, N_1 + 1 \leq j \leq 2N_1
\]

\[
\frac{M_{k,j}^{(3)} e^{\theta_k \theta_j} + e^{-\theta_k \theta_j}}{\zeta_j - \zeta_k}, \quad N_1 + 1 \leq k \leq 2N_1, 1 \leq j \leq N_1.
\]

(6.68)

N_1 + 1 \leq k, j \leq 2N_1.

For case 2, we set \( \theta_{j,0} = (\alpha_{1,j}, \alpha_{2,j}, \alpha_{3,j}, \alpha_{4,j}, \alpha_{5,j}, \alpha_{6,j})^T \), then

\[
\begin{align*}
\{ \quad u_1(x, t) & = 2i \sum_{k=1}^{N_1} \sum_{j=1}^{N_2} a_{1,k} e^{\theta_k - \theta_j} (M^{-1})_{k,j}, \\
\{ \quad u_2(x, t) & = 2i \sum_{k=1}^{N_1} \sum_{j=1}^{N_2} a_{3,k} e^{\theta_k - \theta_j} (M^{-1})_{k,j}, \\
\{ \quad u_3(x, t) & = 2i \sum_{k=1}^{N_1} \sum_{j=1}^{N_2} a_{5,k} e^{\theta_k - \theta_j} (M^{-1})_{k,j},
\end{align*}
\]

(6.69)

where

\[
m_{k,j} = \frac{M_{k,j}^{(1)} e^{\theta_k \theta_j} + e^{-\theta_k \theta_j}}{\zeta_j - \zeta_k}.
\]

(6.70)

For case 3, combine the results of case 1 and case 2 reasonably, we can easily obtain the final soliton solutions.

6.4.2. A variety of Rational Solutions and Physical Visions

For case 1, we usually are interested in the simple situation in \( N_1 = 1 \),

\[
\begin{align*}
\{ \quad u_1(x, t) & = 2ia_{1,1} e^{\theta_1 - \theta_1} (M^{-1})_{1,1} + 2ia_{1,1} e^{\theta_1 - \theta_1} (M^{-1})_{1,2} \quad + 2ia_{2,1} e^{\theta_1 - \theta_1} (M^{-1})_{2,1} + 2ia_{2,1} e^{\theta_1 - \theta_1} (M^{-1})_{2,2}, \\
\{ \quad u_2(x, t) & = 2ia_{3,1} e^{\theta_1 - \theta_1} (M^{-1})_{1,1} + 2ia_{3,1} e^{\theta_1 - \theta_1} (M^{-1})_{1,2} \quad + 2ia_{4,1} e^{\theta_1 - \theta_1} (M^{-1})_{2,1} + 2ia_{4,1} e^{\theta_1 - \theta_1} (M^{-1})_{2,2},
\end{align*}
\]

(6.71)

where \( \xi_1 = a_1 + ib_1(a_1 \neq 0, b_1 > 0) \), and \( \theta_1 = -i(\xi_1 x + 4\xi_1^3 t) \). If we take the parameters as \( \alpha_{2,1} = \alpha_{1,1}^*, \alpha_{4,1} = \alpha_{3,1}^*, \alpha_{6,1} = \alpha_{5,1}^* \), then the breather-type soliton solutions are
obtained

\[
\begin{align*}
&u_1(x,t) = -2 \sqrt{a_1 a_{11}} b_1 \frac{a_1 \cosh X_1 \cos Y_1 + b_1 \sinh X_1 \sin Y_1}{\sqrt{|a_{11}|^2 + |\alpha_{31}|^2 + |\alpha_{51}|^2}} a_1^2 \cosh^2 X_1 + b_1^2 \sin^2 Y_1, \\
&u_2(x,t) = -2 \sqrt{a_1 a_{31}} a_{11} b_1 \frac{a_1 \cosh X_1 \cos Y_1 + b_1 \sinh X_1 \sin Y_1}{\sqrt{|a_{11}|^2 + |\alpha_{31}|^2 + |\alpha_{51}|^2}} a_1^2 \cosh^2 X_1 + b_1^2 \sin^2 Y_1, \\
&u_3(x,t) = -2 \sqrt{a_1 a_{51}} a_{11} b_1 \frac{a_1 \cosh X_1 \cos Y_1 + b_1 \sinh X_1 \sin Y_1}{\sqrt{|a_{11}|^2 + |\alpha_{31}|^2 + |\alpha_{51}|^2}} a_1^2 \cosh^2 X_1 + b_1^2 \sin^2 Y_1, \\
\end{align*}
\]

(6.72)

where \(X_1 = 2b_1[x+4(3a_1^2-b_1^2)t] - \ln \sqrt{|a_{11}|^2 + |\alpha_{31}|^2 + |\alpha_{51}|^2}, Y_1 = 2a_1[x+4(a_1^2-3b_1^2)t].\)

The breather-type soliton solutions are plotted in Fig. 7.

**Figure 7.** (Color online) Single-breather solutions to Eq. (6.71) with the parameters \(a_{11} = 0.2 + 0.3i, \alpha_{31} = 0.2 + 0.4i, \alpha_{51} = 0.2 + 0.5i, a_1 = 0.2, b_1 = 0.3.\) (a)(d)(g): the structures of the single-breather solutions, (b)(e)(h): the density plot, (c)(f)(i): the wave propagation of the single-breather solutions.
For case 2, we usually are interested in the simple situation in \( N_2 = 1 \),

\[
\begin{align*}
    u_1(x, t) &= 2i \frac{\alpha_{1,1} e^{\theta_1 - \theta_i} (\xi_1 - \zeta_1^*)}{M_{1,1} e^{\theta_1^* + \theta_i} + e^{-\theta_1 - \theta_i}}, \\
    u_2(x, t) &= 2i \frac{\alpha_{3,1} e^{\theta_1 - \theta_i} (\xi_1 - \zeta_1^*)}{M_{1,1} e^{\theta_1^* + \theta_i} + e^{-\theta_1 - \theta_i}}, \\
    u_3(x, t) &= 2i \frac{\alpha_{5,1} e^{\theta_1 - \theta_i} (\xi_1 - \zeta_1^*)}{M_{1,1} e^{\theta_1^* + \theta_i} + e^{-\theta_1 - \theta_i}},
\end{align*}
\]

(6.73)

where \( \xi_1 = ib_1(b_1 > 0) \), and \( \theta_i = -i(\xi_1 x + 4\zeta_1^2 t) \). If we set \( \alpha_{2,1} = \alpha_{4,1}^* = \alpha_{3,1}^* \), and \( \alpha_{6,1} = \alpha_{5,1}^* \), then Eq. (6.73) can be converted to the following form

\[
\begin{align*}
    u_1(x, t) &= \frac{-\sqrt{2} \alpha_{1,1} b_1}{\sqrt{|\alpha_{1,1}|^2 + |\alpha_{3,1}|^2 + |\alpha_{5,1}|^2}} \text{sech} \left( 2b_1 x - 8b_1^2 t + \ln \left( 2(|\alpha_{1,1}|^2 + |\alpha_{3,1}|^2 + |\alpha_{5,1}|^2) \right) \right), \\
    u_2(x, t) &= \frac{-\sqrt{2} \alpha_{3,1} b_1}{\sqrt{|\alpha_{1,1}|^2 + |\alpha_{3,1}|^2 + |\alpha_{5,1}|^2}} \text{sech} \left( 2b_1 x - 8b_1^2 t + \ln \left( 2(|\alpha_{1,1}|^2 + |\alpha_{3,1}|^2 + |\alpha_{5,1}|^2) \right) \right), \\
    u_3(x, t) &= \frac{-\sqrt{2} \alpha_{5,1} b_1}{\sqrt{|\alpha_{1,1}|^2 + |\alpha_{3,1}|^2 + |\alpha_{5,1}|^2}} \text{sech} \left( 2b_1 x - 8b_1^2 t + \ln \left( 2(|\alpha_{1,1}|^2 + |\alpha_{3,1}|^2 + |\alpha_{5,1}|^2) \right) \right).
\end{align*}
\]

(6.74)

In Fig. 8, we display the localized structures and dynamic behaviors of one-bell solutions vividly.

(a)  (b)  (c)

(d)  (e)  (f)
Figure 8. (Color online) One-bell solutions to Eq. (6.74) with the parameters $\alpha_{1,1} = 0.2 + 0.3i$, $\alpha_{3,1} = 0.2 + 0.4i$, $\alpha_{5,1} = 0.2 + 0.5i$, $a_1 = 0$, $b_1 = 0.3$. (a)(d)(g): the structures of the one-bell solutions, (b)(e)(h): the density plot, (c)(f)(i): the wave propagation of the one-bell solutions.

When taking $N_2 = 2$, two-bell soliton solutions are as follows

\[
\begin{align*}
\begin{cases}
  u_1(x, t) &= 2i\alpha_{1,1}e^{\theta_{1,1}-\theta_{2,1}}(M^{-1})_{1,1} + 2i\alpha_{1,1}e^{\theta_{1,1}-\theta_{2,1}}(M^{-1})_{1,2} \\
  &\quad + 2i\alpha_{1,2}e^{\theta_{1,2}-\theta_{2,2}}(M^{-1})_{2,1} + 2i\alpha_{1,2}e^{\theta_{1,2}-\theta_{2,2}}(M^{-1})_{2,2}, \\
  u_2(x, t) &= 2i\alpha_{3,1}e^{\theta_{3,1}-\theta_{4,1}}(M^{-1})_{1,1} + 2i\alpha_{3,1}e^{\theta_{3,1}-\theta_{4,1}}(M^{-1})_{1,2} \\
  &\quad + 2i\alpha_{3,2}e^{\theta_{3,2}-\theta_{4,2}}(M^{-1})_{2,1} + 2i\alpha_{3,2}e^{\theta_{3,2}-\theta_{4,2}}(M^{-1})_{2,2}, \\
  u_3(x, t) &= 2i\alpha_{5,1}e^{\theta_{5,1}-\theta_{6,1}}(M^{-1})_{1,1} + 2i\alpha_{5,1}e^{\theta_{5,1}-\theta_{6,1}}(M^{-1})_{1,2} \\
  &\quad + 2i\alpha_{5,2}e^{\theta_{5,2}-\theta_{6,2}}(M^{-1})_{2,1} + 2i\alpha_{5,2}e^{\theta_{5,2}-\theta_{6,2}}(M^{-1})_{2,2}.
\end{cases}
\end{align*}
\tag{6.75}
\]
Figure 9. (Color online) Two-bell solutions to Eq. (6.75) with the parameters $\alpha_{1,1} = \alpha_{1,2} = \alpha_{3,1} = \alpha_{3,2} = \alpha_{5,1} = \alpha_{5,2} = 0.2, \alpha_{2,1} = \alpha_{2,2} = \alpha_{4,1} = \alpha_{4,2} = \alpha_{6,1} = \alpha_{6,2} = 0.3, b_1 = 0.3, b_2 = 0.7$. (a)(d)(g): the structures of the two-bell solutions, (b)(e)(h): the density plot, (c)(f)(i): the wave propagation of the two-bell solutions.

In addition, the localized structures and dynamic propagation behaviors of two-bell soliton solutions are displayed in Fig. 9, from which we can see that after two solitons collide with each other, their propagation direction has been misaligned, but the energies of them are basically unchanged.

6.5. Summary

From these special cases, we can see that the exact solutions of mmKdV equation we obtain are suitable for matrices of any order whether $p = q, p > q$ or $p < q$. More importantly, in the process of solving specific equations, the special properties of the potential matrices will help us further classify soliton solutions to find other meaningful solutions such as breather-type solutions and bell-type solutions, which can help us understand the meaning of solutions more comprehensively.

7. Conclusions

As we all know, many scholars have done a lot of works on the mKdV-type equation or equations, for example, Zhang et al. have studied two-component cmKdV equations by Darboux transformation [35], and Liu et al. have discussed initial–boundary problems for the vector modified Korteweg–de Vries (vmKdV) equation via Fokas unified transform method [36]. However, many scholars pay less attention to mmKdV equation with a $p \times q$ complex-valued potential matrix function in recent years. Actually, vmKdV equations, cmKdV equations and so on are only the sub cases of the mmKdV equation. In this work, we study the exact solutions with their propagation behaviors for the mmKdV equation which can solve the solutions of not only previous vmKdV and cmKdV equations, but also other generalized mKdV-type equation or equations.
The main propose of this work is to study the multi-soliton solutions of mmKdV equation with the aid of RH approach. Firstly, we performed the spectral analysis of Lax pair in order to structure the RH problem. Secondly, we obtain general solution form of mmKdV equation. Next, we choose certain special cases of potential matrix $Q$ including $2 \times 2$, $4 \times 4$, $1 \times 3$ and $6 \times 1$ form. By analyzing the special characteristics of potential matrices, some interested solutions can be derived on the basis of original exact solutions. At the same time, the localized structures and dynamic behaviors of above interesting solutions are displayed vividly.

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