A Study of System Reliability Analysis Using Linear Programming

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Abstract
The main drawback of using the linear programming (LP) bounds method for computing bounds on the reliability of a general system is that the size of the LP problem increases exponentially with the number of components. Although a multi-scale approach has been proposed to deal with a system with a large number of components, the selection of the subsystems in a multi-scale approach could still be difficult or impossible for a general system. In order to overcome the main drawback of the LP bounds method, the relaxed linear programming (RLP) bounds method was developed for a pure series system and pure parallel system; it provides results comparable to that of the LP bounds method. This paper extends the applicability of linear programming by presenting an approach to handle a general system by decomposing the entire system into subsystems based on failure modes. The proposed approach relies on individual component state probabilities and joint probabilities of the states of a small number of components, and it can provide the bounds for the failure probability of large systems, especially when other methods are not applicable. This paper also presents a strategy for decreasing the number of constraints in the RLP bounds method.

Keywords: system reliability analysis; linear programming; LP bounds method; RLP bounds method; failure modes

1. Introduction
The linear programming (LP) bounds method was proposed for computing bounds for the failure probability of a general system based on the joint probabilities of the states of \( k \) components (when \( k = 1 \), these joint probabilities become the individual component state probabilities) (Song and Der Kiureghian, 2003). However, there is a critical drawback in the LP bounds method: the size of the LP problem, which is determined by the number of design variables and the number of constraints, increases exponentially with the number of components. The size issue – both the number of design variables and the number of constraints – can be a hindrance in the application of the LP bounds method to a large system.

Based on the LP bounds method, Der Kiureghian and Song (2008) proposed a multi-scale approach, whereby the system is decomposed into subsystems and a hierarchy of analysis is performed by considering each subsystem or set of subsystems "separately". This decomposition facilitates the determination of the system reliability by the LP bounds method, whereby the large LP problems for the entire system are replaced by several LP problems of much smaller size. The multi-scale approach has two disadvantages. First, the size of any subsystem will still be limited by the restriction on the LP bounds method. Second, even though guidelines for the effective selection of the subsystems have been enumerated, the selection might still be difficult or impossible for a general system, rendering the multi-scale approach difficult to apply.

Chang and Mori (2013) developed the relaxed linear programming (RLP) bounds method as an efficient reliability tool for a system with a large number of components. The RLP bounds method can overcome the main drawback of the LP bounds method while providing results comparable to those of the LP bounds method. The RLP bounds method employs the universal generating function (UGF) (Levitin, 2005) to reduce the number of design variables from \( 2^n \) in the LP bounds method to \( n^2 - n + 2 \), where \( n \) is the number of components. The number of constraints can also be reduced substantially. Although the RLP bounds method is more efficient and can solve problems involving larger systems than the LP bounds method, the method can still be improved in the following two ways. First, the RLP bounds method can be used to estimate the bounds for a pure series system and pure parallel system; however, it cannot handle a general system consisting of both series and parallel subsystems. Second, the rule for decreasing the
number of constraints is not quite clear; the number of constraints may still be enormously large for a large system. This paper presents an approach that combines the RLP bounds method and the LP bounds method to handle a general system by decomposing the entire system into subsystems based on failure modes. Authors also present a strategy for decreasing the number of constraints in the RLP bounds method. The accuracy and efficiency of the proposed approach are investigated using numerical examples.

2. Review of Existing Methods
2.1 Linear Programming Bounds Method

Considering a system with \( n \) two-state components, Hailperin (1965) divided the sample space of component states into \( 2^n \) mutually exclusive and collectively exhaustive (MECE) events, each consisting of a distinct intersection of the failure events \( F_i \) and their complements \( F_i^c \) (functional events), \( i = 1, 2, ..., n \). Authors call them the basic MECE events and denote them by \( e_r \), \( r = 1, 2, ..., 2^n \). It should be noted that only the joint failure probability of \( k \) components such as \( P(F_i) \) and \( P(F_i \cap F_j) \), \( i, j = 1, 2, ..., n \), and \( i \neq j \), is known, but any probability of the basic MECE event \( e_r \), \( P_{m_r} = P(e_r) \), is not known in advance.

Because of the mutual exclusivity of the basic MECE events, the probability of any union of these events is the sum of the corresponding probabilities. In particular, the probability of any failure event \( F_i \) is the sum of the probabilities of the basic MECE events that constitute the event \( F_i \). Any joint failure probability is given as the sum of the basic MECE events that constitute the intersection events. More generally, authors write

\[
P(F_i) = P_i = \sum_{m_r \subset F_i} P_{m_r},
\]

\[
P(F_i \cap F_j) = P_{ij} = \sum_{m_r \subset F_i \cap F_j} P_{m_r}
\]

(1)

(2)

By the basic axioms of probability, the above probabilities \( P = \{P_{m_1}, P_{m_2}, ..., P_{m_{2^n}}\} \) are subject to the following linear constraints:

\[
\sum_{m_r \subset F_i} P_{m_r} = 1
\]

(3)

\[
P_{m_r} \geq 0; \quad r = 1, 2, ..., 2^n
\]

The lower bound and the upper bound of the system failure probability are obtained as the minimum and the maximum of the objective function of the LP problem, respectively. The formulation of the LP problem appropriate for this analysis has the following form:

\[
\begin{align*}
\text{minimize (maximize)} & \quad C^T P_m \\
\text{subject to} & \quad A_1 P_m = B \\
& \quad A_2 P_m \geq B
\end{align*}
\]

(4)

where \( P_m = \{P_{m_1}, P_{m_2}, ..., P_{m_{2^n}}\} \) is the vector of design variables and represents the probabilities of the basic MECE events; \( C \) is a matrix that relates the system failure event to the component failure events; \( C^T P_m \) is the linear objective function; and \( A_1 \) and \( A_2 \) and \( B \) are the coefficient matrices and vectors, respectively, which represent the information given in terms of joint failure probabilities of \( k \) components. \( A_1 \) and \( B \) are obtained from Eq. (1). \( A_2 \) and \( B \) are also obtained from Eq. (1), but when one has information of the form \( P(F_i) \geq x \) or \( P(F_i) \leq x \) rather than \( P(F_i) = x \). Additional linear constraints are imposed by the axioms of probability (Eqs. (2) and (3)) (Song and Der Kiureghian, 2003). The above probabilities \((B_i \text{ and } B_j)\) associated with the related matrices \((A_i \text{ and } A_j)\) consist of the constraints of the LP problem.

For a system with \( n \) two-state components, the number of design variables, \( n_d, \) is \( 2^n \). When the complete set of joint failure probabilities of each combination of \( k \) components, i.e., the joint failure probabilities of all combinations up to each combination of \( k \) components, is available, the total number of constraints of the LP bounds method, \( n_c \), can be expressed as

\[
n_c = 2^n + 1 + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{k}
\]

(5)

Obviously, the main drawback of the LP bounds method is that the number of design variables, \( n_d, \) grows exponentially with the number of components. The number of constraints is also intractably large for a large system. For example, suppose authors have an LP solver that can handle \( 2^{18} = 262144 \) design variables and all the constraints up to the complete set of joint failure probabilities of each combination of 3 components. Then, in theory the limitation of the number of components in the LP bounds method is 18.

2.2 Multi-Scale Approach

Based on the LP bounds method, a multi-scale approach has been proposed to deal with a system with a larger number of components. In this proposed approach, the system is decomposed into subsystems as shown in Fig.1. (Der Kiureghian and Song, 2008). Subsystem (super-component) \( [i] \) \( (i = 1, 2, ..., n_s) \), where \( n_s \) is the number of super-components) consists of a group of components; no components can belong to more than one subsystem. The bounds of failure probability of subsystem \([i], P_{[i]}\), can be obtained by the LP bounds method. In addition, the bounds on the joint failure probability of subsystems such as \( P_{[i,j]} \) (the joint failure probability of subsystems \( i \) and \( j \)) can be obtained by the LP bounds method. Then, based on the failure probabilities of the subsystems, bounds for the entire system can be obtained by the LP bounds method.

However, there are many cases in which numerous components are common in different failure modes.
In addition, there are many cases for which the number of components in a subsystem is beyond the limitations of the LP bounds method. In such cases, the multi-scale approach is difficult or impossible to apply.

2.3 Relaxed Linear Programming Bounds Method

Consider a system consisting of \( n \) components, and suppose that component \( i \) has two possible states, for \( i = 1, 2, ..., n \). After some basic algebra, a conceptually simplified universal generating function (UGF) (Levitin, 2005) of the system can be expressed as (Chang and Mori, 2013)

\[
U(z) = p_1 z^0 + p_2 z^1 + ... + p_n z^{n-1} + p_n z^n + \sum_{i=1}^n p_{n+1} z^{i+1} + \sum_{i=1}^n p_{n+2} z^{i+2} + \sum_{i=1}^n p_{n+3} z^{i+3} + ... + \sum_{i=1}^n p_{n+k} z^{i+k} + ...
\]

where \( z' \) is the \( z \)-transform of \( x \) (Grimmett and Stirzaker, 1992, Ross, 2000), the \( 0 \) of \( z^0 \) encodes the subset of the state in which no component survives, and \( p_i \) is the probability corresponding to the state encoded by the \( 0 \) of \( z^i \); the \( x_i \) of \( z^i \) encodes the subset of the state in which only component \( i \) survives, and \( p_{n+1} \) is the probability corresponding to the state encoded by the \( x_i \) of \( z^i \); the \( 2x_i \) of \( z^i \) encodes the portion of subsets of states for which only two components including component \( i \) survive, and \( p_{n+2} \) is the probability corresponding to the state encoded by the \( 2x_i \) of \( z^i \); the union of all states encoded by the \( 2x_i \) of \( z^i \), \( i = 1, 2, ..., n \), is the subset of the states in which only two components survive, the sum of the associated probabilities is the probability corresponding to the states in which only two components survive; the \( nx \) of \( z^n \) encodes the subset of the state in which all components survive, and \( P_{n+2} \) is the probability corresponding to the state encoded by the \( nx \) of \( z^n \), and so on.

Note that these events, like the one encoded by \( z^{2n} \), are not the basic MECE events; however, the union of the events \( z^{ni} \) (\( i = 1, 2, ..., n \)), denoted as \( E_i \), \( j = 1, 2, ..., n-1 \), is mutually exclusive with \( z^*, z^m \), and \( E \) (\( k \neq j \)). In addition, \( z^0 \), \( z^n \), and \( E \) are collectively exhaustive.

According to the basic axioms of probability, the above probabilities \( p_r \), \( r = 1, 2, ..., n^2-n+2 \), are subject to the following linear constraints:

\[
\sum_{r=1}^{n^2-n+2} p_r = 1 \tag{7}
\]

\[
p_r \geq 0; \quad r = 1, 2, ..., n^2-n+2 \tag{8}
\]

Similar to Eq. (4), the probabilities \( p_r \), \( r = 1, 2, ..., n^2-n+2 \), in the UGF of the system serve as the design variables in the LP, and the number of design variables is \( n = n^2-n+2 \).

One can find \( k \) equalities when the complete set of joint failure probabilities of each \( k \) component is available. Because the number of constraints based on the axioms of probability is \( n^2-n+3 \) (Eqs. (7) and (8)), the total number of constraints of the relaxed linear programming (RLP) bounds method is given by

\[
n_k = (n^2-n+3) + 2 \left[ \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{k} \right] + k \tag{9}
\]

Suppose authors have the same LP solver described in Section 2.1. Then in theory the limitation of the number of components in the RLP bounds method is 512. Although the RLP bounds method is more efficient than the LP bounds method, it can solve problems involving only a pure series system or a pure parallel system. The rule for decreasing the number of constraints is not quite clear either.

3. Strategy for Decreasing the Number of Constraints in the Relaxed Linear Programming Bounds Method

When some of the joint failure probabilities of \( k \) components are identical, the inequality constraints based on these joint failure probabilities have the same effect in the calculation of the RLP bounds method. For such a case, one can sum up each side of these constraints respectively, and thereby reduce the number of constraints without losing any accuracy. When some of the joint failure probabilities of \( k \) components are not identical but close to each other in value, one can still sum up each side of the inequality constraints with only a slight loss of accuracy.

Based on such observations, in this paper, authors propose dividing the range of the joint failure probabilities of \( k \) components (i.e., from their minimum value to the maximum value) into \( t_k \cdot n \) (\( t_k = 1, 2, ... \)) intervals evenly using a log scale, and to sum up the inequality constraints of the joint failure probabilities of \( k \) components in the same interval. By using this procedure, the total number of constraints
can be reduced from that given by Eq. (9) to

\[ n_c = (n^2 - n + 3) + 2 \sum_{i=1}^{k} t_i \cdot n + k \]  

(10)

Clearly, the efficiency increases rapidly with an increase in the value of \( n \).

For example, the UGF of a system with 3 two-state components can be expressed as

\[ U(z) = p_1 z^0 + p_2 z^1 + p_3 z^2 + p_4 z^3 + p_5 z^4 + p_6 z^5 + p_7 z^6 + p_8 z^7 + p_9 z^8 \]  

(11)

When \( k = 1 \), each of the component failure probabilities can be expressed as two inequalities in the RLP bounds method. For component 1 the two inequalities can be expressed as

\[ P(F_1) < p_1 + p_3 + p_4 \]
\[ > p_1 + p_3 + p_5 + p_6 + p_7 \]  

(12)

If \( P(F_1) \) and \( P(F_2) \) are in the same interval, \( P(F_1) \) and \( P(F_2) \) can be replaced by \( P(F_1) + P(F_2) \), and the four constraints relative to \( P(F_1) \) and \( P(F_2) \) can be decreased to two constraints relative to \( P(F_1) + P(F_2) \). Thus the number of inequality constraints in the RLP bounds method for \( k = 1 \) may be reduced from 6 to 4.

The selection of \( t_i \) depends on the dispersion degree of the joint failure probabilities of \( k \) components. A large dispersion requires a correspondingly large \( t_i \). The accuracy of the RLP bounds method using this procedure is investigated in numerical example 1.

4. An Approach to System Reliability Analysis Using Linear Programming

4.1 Outline

The RLP bounds method can be applied to a single series or parallel system, but it is not applicable to a general system that consists of both series subsystems and parallel subsystems. In this section, authors propose an approach to extend the applicability of the combination of the RLP bounds method and the LP bounds method to handle a general system by decomposing the entire system into subsystems based on failure modes.

A system with multi-failure modes can be modeled as a series system if it fails whenever any of its critical failure modes occur. Many methods have been introduced and used to determine the critical failure modes of a system (Rashedi and Moses, 1986, Song and Der Kiureghian, 2005). In such a system each critical failure mode or so-called cut set can be considered as a "component".

Each critical failure mode is also a system (subsystem) itself. The bounds on its failure probability can be computed by the RLP bounds method if it is a series or parallel system, and the bounds on its joint failure probability can also be computed by the RLP bounds method. These bounds estimated by the RLP bounds method are then used as constraints in solving the LP problem in order to estimate the failure probability of the entire system.

The bounds for the entire system could be wide using the RLP bounds method when the information for many of the probabilities is given by inequalities. Therefore, the LP bounds method is preferred for determining the bounds for the entire system.

\[ \text{Each component can belong to more than one subsystem} \]

![Diagram of the Proposed Approach](Image)

Fig.2. Diagram of the Proposed Approach

When the number of failure modes is large, a group of failure modes is treated as a subsystem. The grouping is made based on the degree of correlation; failure modes having a strong correlation can be combined into a group. The diagram of the proposed approach is shown in Fig. 2., in which \( P(f_i) \) is the probability of the occurrence of subsystem \( i \) \((i = 1, 2, ..., n_s)\) where \( n_s \) is the number of subsystems, and \( n_i \) is the number of failure modes), and \( P(f_{i,j}) \) is the probability of the occurrence of subsystem \( i \) and \( j \), and so on.

For a system with a very large number of failure modes, subsystems are considered again, i.e., a group of subsystems can be grouped together as a single subsystem. Also, if a subsystem is not a pure series system or a pure parallel system, this proposed method is applied to estimate the bounds of its failure probability again.

4.2 Limitation of the Size of the Proposed Approach

Since the bounds of the entire system and subsystems are obtained by using the LP bounds method and the RLP bounds method, the application of the proposed approach is limited by the size of the LP problem of the entire system and of each subsystem.

Suppose authors have the same LP solver described in Section 2.1. Then, in theory the bounds for the system failure probability can be obtained using the proposed approach when \( n_s \leq 18 \), \( k_s \leq 3 \), \( n_m \leq 512 \), and \( k_m \leq 3 \), where \( n_s \) denotes the number of subsystems, \( k_s \) denotes the number of failure modes up to which is considered in the joint failure probability, \( n_m \) denotes the number of components in subsystem \( m \), and \( k_m \) denotes the...
number of components up to which is considered in the joint failure probability in subsystem $m_i$.

4.3 A System Subjected to a Common Source of Hazard

Consider a system subjected to a common source of hazard such as an earthquake, and assume that there are several critical failure modes. The correlation among failure modes of the system could be strong because of the common source of hazard. When the proposed approach is applied, a system with strong correlation among failure modes usually requires values of $k_i$ and $k_m$ larger than those for a system with weaker correlations in order to achieve comparable accuracy. Such strong correlation among failure modes can be avoided by considering the conditional failure probability of the system (Karamchandani and Cornell, 1991).

Suppose that $X$ is a random variable causing the statistical dependence among failure modes, which authors call a common source random variable (CSRV). Let $f_i(x)$ be the probability density function (PDF) of $X$. Then, using the theorem of total probability and applying Gaussian integration, authors may estimate the system failure probability $P(F_i)$ by

$$P(F_i) = \int_{x_{\text{min}}}^{x_{\text{max}}} P(F_i | X = x) f_i(x) \, dx$$

\begin{align}
&\approx \int_{x_{\text{min}}}^{x_{\text{max}}} P(F_i | X = x) f_i(x) \, dx \\
&\approx \sum_{j} w_j P(F_i | X = x) f_i(x_j)
\end{align}

(13)

where $[x_{\text{min}}, x_{\text{max}}]$ is the range of $x$ used for Gaussian integration, $P(F_i | X = x)$ is the conditional system failure probability given that $X = x$. $N$ is the number of points in the Gaussian integration, the $x_j$ ($i = 1, 2, ..., N$) are abscissas, and $w_j$ is the corresponding weight factor for $x_j$.

The values of $x_{\text{min}}$ and $x_{\text{max}}$ can be determined based on the expected order of accuracy of $P(F_i)$. Since $P(F_i | X = x)$ is a monotonically increasing function, the center of gravity of the product $P(F_i | X = x)f_i(x)$ shifts to the right of the mean of $X$. In Eq. (13), because the area under this product to the left of $x_{\text{min}}$ and to the right of $x_{\text{max}}$ is neglected, the range should be wide enough so that the neglected area does not cause a significant error in the estimate. One might consider $P(X \geq x_{\text{min}}) = 10^{-\beta_2}$ and $P(X \geq x_{\text{max}}) = 10^{-\beta_3}$, where the expected order of accuracy is $10^{-\beta}$. For example, if $P(F_i) \approx 10^{-5}$, $x_{\text{min}}$ and $x_{\text{max}}$ was determined so that $P(X \leq x_{\text{min}}) \approx 10^{-2}$ and $P(X \geq x_{\text{max}}) \approx 10^{-6}$.

The number $N$ used in the Gaussian integration is considered acceptable if the differences between the lower bounds and the upper bounds of system failure probability by two Gaussian integrations with $N - 1$ and $N$ are negligible.

4.4 Advantages of the Proposed Approach

The main advantage of our proposed approach over the multi-scale approach is that each component of the system can belong to more than one subsystem in the proposed approach because the subsystems are based on the failure modes. In the multi-scale approach, though, any component of the system can belong to only one subsystem because it is not possible to define the system failure event (the union of failure modes) when one component belongs to more than one subsystem. There are many cases in which many components are common in different failure modes. In such cases, the multi-scale approach is difficult or impossible to apply, while our proposed approach is applicable. In addition, an advantage of the RLP bounds method over the LP bounds method is that a subsystem with a much larger number of components can be handled, while the limitation on the number of components in each subsystem remains the same as that of the RLP bounds method.

There are two advantages of the proposed approach over the Monte Carlo (MC) simulation. One is that the MC simulation can be impractical when the failure probability is small, whereas our proposed approach is unaffected by the magnitude of the failure probability. Another advantage of our approach over the MC simulation is that the latter is not applicable when information on the probabilities is incomplete, such as when information on the statistical characteristics of one component or on the correlation between one component and another is missing; our proposed approach is still applicable under such circumstances.

5. Numerical Examples

5.1 Example 1: Truss with Seven Components

Consider the truss shown in Fig. 3. (Song and Der Kiureghian, 2003, Chang and Mori, 2013), to see the effect of the reduction of the number of constraints in the RLP bounds method by considering the strategy proposed in Section 3. Let $S$ denote the load acting on the truss and let $X_i$, $i = 1, 2, ..., 7$, denote either the tensile strength of a component in tension or the compressive strength of a component in compression. Here, the buckling failure mode is neglected. Based on the distribution of internal forces shown in Fig. 3., the failure states of the individual components are $F_i = \{X_i \leq S/2\sqrt{3}\}$ for $i = 1$ and 2, and $F_i = \{X_i \leq S/\sqrt{3}\}$, for $i = 3, 4, ..., 7$. Suppose the load has the deterministic value $S = 100$ and the component strengths $X_i$, $i = 1, 2, ..., 7$, are normally distributed random variables with $X_i$ and $X_j$ having means of 100 and standard deviations of 20, and $X_i - X_j$, having means of 200 and standard deviations of 40. Under these conditions, the components have equal failure probabilities, $P_i = 1.88x10^{-4}$, $i = 1, 2, 3, 4, 5, 6, 7$.

Assume that the $X_i$’s have a Dunnet and Sobel (1955) class correlation matrix, which is specified as $ho_{ij} = k_{ij}$ for $i \neq j$ and $\rho_{ii} = 1$. Suppose that $r_1 = 0.90$, $r_2 = 0.96$, $r_3 = 0.91$, $r_4 = 0.95$, $r_5 = 0.92$, $r_6 = 0.94$, and $r_7 = 0.93$. The joint probabilities of two components and three components, $P_{ij}$ and $P_{i}$, respectively, have been computed for this example (Chang and Mori, 2013). In Tables 1. and 2., “LP” denotes the LP bounds method,
"RLP" denotes the RLP bounds method considering all equalities and inequalities, "SRLP" denotes the RLP bounds method with the strategy proposed in Section 3 when \( t = 1 \), and \( k_m \) is the number of components up to which is considered in the joint failure probability. The failure probabilities and the number of equality and inequality constraints for each method are shown in Tables 1. and 2.

In Table 1., the bounds estimated by "SRLP", "RLP", and "LP", are identical when \( k_m = 1 \). When \( k_m = 2 \) and \( k_m = 3 \), the bounds of "SRLP" and "RLP" are found to be somewhat wider than those of "LP". The bounds of "SRLP" are also slightly wider than those of "RLP"; however, the difference between "SRLP" and "RLP" is negligible, and the number of constraints in "SRLP" is smaller than that of "RLP". The CPU times for all cases are found to be less than 0.5s, and the CPU times using "SRLP" is the shortest among them. Note that the CPU times for the LP algorithm are determined by the number of design variables and constraints.

5.2 Example 2: Rigid-plastic Structure as a General System

As a general system example, consider an ideally elastic-plastic cantilever beam of moment capacity \( M \), propped by an ideally rigid-brittle bar of strength \( T \) as shown in Fig.4. (Song and Der Kiureghian, 2003). A random load, \( X \), acts at the midpoint of the beam. Ignoring shear failure, there are three possible paths, or cut sets, for the failure of the system as follows:

- Cut set 1: The bar fractures (failure event \( F_1 \)), followed by the formation of a hinge at the fixed end of the cantilever beam (failure event \( F_2 \)).

Thus, the system failure event can be expressed as

\[
F_s = F_1 F_2 \cup F_1 F_3 \cup F_2 F_3
\]

After the structural analysis of the cantilever beam, one finds that the component failure event is defined as follows:

\[
\begin{align*}
F_1 &= \{T - 5X / 16 \leq 0\}, \quad F_2 = \{M - l \cdot X \leq 0\} \\
F_3 &= \{M - 3l \cdot X / 8 \leq 0\}, \quad F_4 = \{M - l \cdot X / 3 \leq 0\} \\
F_5 &= \{M + 2l \cdot T - l \cdot X \leq 0\}
\end{align*}
\]

For the sake of simplicity, authors assume in this example that \( l \) is deterministic and equals 5. In addition, \( M, T, \) and \( X \) are assumed to be uncorrelated normal random variables with means and standard deviations as shown in Table 3. Then the limit-state functions defining component events are jointly normal. The component failure probabilities and the joint failure probabilities of the components can be computed from the standard normal cumulative distribution function (CDF) and the joint normal CDF, respectively.

The bounds and CPU times estimated by our proposed approach and the LP bounds method are shown in Tables 4. and 5., respectively. Their \( k_m \) denotes the number of components considered.
to calculate the bounds of the subsystem failure probability. By MC simulation with $10^7$ simulations, the system failure probability is estimated as $7.75 \times 10^{-3}$.

Comparing the results in Table 4., one can find that the bounds determined by the proposed approach are rather wide, but identical to those resulting from the LP bounds method when $k_m = 1$. When $k_m = 2$ and $k_f = 1$, the bounds estimated by the proposed approach are found to be acceptable. The difference between the bounds is negligible, particularly when $k_m = 2$ and $k_f = 2$. The CPU times of the proposed approach, shown in Table 5., are greater than those of the LP bounds method in this example; however, the proposed approach can handle a much larger system efficiently, while the LP bounds method cannot, as shown in the next section.

5.3 Example 3: Seismic Reliability of a Substation Power Network

Consider the four substation power network system shown in Fig.5., which is designed based on the hypothetical substation example in Ostrom (2004) and Brown (2002). Equipment items of the system of the substation power network are shown in Table 6.

Assume that the substation power network is located in an earthquake-prone region. Let $A$ denote the bedrock peak ground acceleration (PGA) in the region of the substations, and let $S_i$ denote a factor representing the local site response of equipment item $i$, such that $A \cdot S_i$ is the actual peak acceleration experienced by the $i^{th}$ equipment item. Assume that $A$ is a lognormal random variable with mean 0.15 g (in units of gravity acceleration, g) and a coefficient of variation (c.o.v.) equal to 0.5. Assume also that the $S_i, i=1, ..., n$, are lognormal random variables independent of each other and also independent of $A$, with means 1.0 and c.o.v.'s 0.2. Let $R_i$ denote the capacity of the $i^{th}$ equipment item with respect to the base acceleration in units of $g$, and assume that $R_i$ are lognormally distributed with statistics as shown in Table 6. Assume that the tie breakers have an equal correlation coefficient of 0.5, and that the other equipment items have an equal correlation coefficient of 0.3. The capacities of equipment items in different categories are assumed to be statistically independent of each other.

The ability of the substation power network to supply power from the input line of Substation 1 (Input) to the output line of Substation 4 (Output) is assumed to be the performance criterion. For this criterion, there are 54 components in the system of the substation power network as shown in Fig.5. For example, (1 ~ 2) DS means that there are components 1 and 2, and that they are a Disconnect Switch.

The failure events of the individual equipment items are formulated as $F_i = [\ln R_i - \ln A - \ln S_i \leq 0], i = 1, 2, ..., 54$. Let $V_i = \ln R_i - \ln A - \ln S_i$. Because $R_i, A$, and $S_i$ are lognormally distributed, $V_i$ is normally distributed.

| Equipment Items          | Mean | c.o.v. |
|--------------------------|------|--------|
| Disconnect Switch (DS)   | 0.7g | 0.3    |
| Circuit Breaker (CB)     | 0.6g | 0.3    |
| Tie Breaker (TB)         | 1.0g | 0.3    |
| Power Transformer (PT)   | 1.5g | 0.5    |

Table 6. The Mean and Coefficient of Variation of Equipment Capacities

Fig.5. Four Substation Power Network
Table 7. Bounds Calculated by the Proposed Approach

| Bounds       | Case 1 | Case 2 |
|--------------|--------|--------|
| \(k_i = 1\)  | 2.90-37.88 | 2.90-37.88 | 2.90-43.71 |
| \(k_i = 2\)  | 2.90-9.55  | 2.92-9.54  | 2.90-10.13 |
| \(k_i = 3\)  | 2.90-7.27  | 3.53-7.19  | 2.90-7.38 |

Table 8. CPU Times (Seconds) of the Proposed Approach

| CPU Time     | Case 1 | Case 2 |
|--------------|--------|--------|
| \(k_i = 1\)  | 8.9    | 8.9    | 7.7    |
| \(k_i = 2\)  | 9.0    | 84.2   | 9.9    |
| \(k_i = 3\)  | 20.8   | 630.7  | 20.8   |

The system has 22 cut sets which are expressed using component identification numbers shown in Fig.5. as follows: \(F_{n_1} = \{1, 2\}, F_{n_2} = \{3, 4, 5, 6, 7, 8\}, F_{n_3} = \{3, 4, 13\}, F_{n_4} = \{5, 6, 7, 8, 9, 10, 11, 12\}, F_{n_5} = \{9, 10, 11, 12, 13\}, F_{n_6} = \{14, 15\}, F_{n_7} = \{16, 17, 18, 19, 20, 21\}, F_{n_8} = \{16, 17, 26\}, F_{n_9} = \{18, 19, 20, 21, 22, 23, 24, 25\}, F_{n_{10}} = \{22, 23, 24, 25, 26\}, F_{n_{11}} = \{27, 28\}, F_{n_{12}} = \{29, 30, 31, 32, 33, 34\}, F_{n_{13}} = \{29, 30, 39\}, F_{n_{14}} = \{31, 32, 33, 34, 35, 36, 37, 38\}, F_{n_{15}} = \{35, 36, 37, 38, 39\}, F_{n_{16}} = \{40, 41\}, F_{n_{17}} = \{42, 43, 44, 45, 46, 47\}, F_{n_{18}} = \{42, 43, 52\}, F_{n_{19}} = \{44, 45, 46, 47, 48, 49, 50, 51\}, F_{n_{20}} = \{48, 49, 50, 51, 52\}, F_{n_{21}} = \{53\}, F_{n_{22}} = \{54\}. Because the number of components is large and the relationships among the components are complex in this system, the LP bounds method is not applicable and a multi-scale approach is also difficult to apply.

For the proposed approach, the failure probability can be obtained from Eq. (13) by replacing \(X\) by \(A\). The failure events of the individual equipment items are formulated as \(F_i = \{n R_i - \ln a - \ln S_i \leq 0 \mid A = a\}, i = 1, 2, ..., 54\), where \(a\) is the value of \(A\). Then, \(V_i = \ln R_i - \ln a - \ln S_i\) and the failure probabilities information is given by the product of the conditional marginal (PCM) method (Pandey, 1998, Yuan and Pandey, 2006). Since authors want to obtain an accuracy of approximately \(10^{-3}\), \(w_m\) and \(w_{ma}\) are determined to be 0.045g and 1.2g, respectively, so that \(P(A \leq w_m) \approx 10^{-3}\) and \(P(A \geq w_{ma}) \approx 10^{-5}\). The number \(N\) in the Gaussian integration is set equal to 11. In this example, because the number of failure modes is larger than 18, the set of failure modes is divided into 7 subsystems as follows: \((F_{n_1}, F_{n_2}, F_{n_3}, F_{n_4}, F_{n_5}, F_{n_6}, F_{n_7}), (F_{n_8}, F_{n_9}, F_{n_{10}}, F_{n_{11}}, F_{n_{12}}, F_{n_{13}}, F_{n_{14}}), (F_{n_{15}}, F_{n_{16}}, F_{n_{17}}, F_{n_{18}}, F_{n_{19}}, F_{n_{20}}, F_{n_{21}}), (F_{n_{22}}, F_{n_{23}}, F_{n_{24}}, F_{n_{25}}, F_{n_{26}}, F_{n_{27}}, F_{n_{28}}), (F_{n_{29}}, F_{n_{30}}, F_{n_{31}}, F_{n_{32}}, F_{n_{33}}, F_{n_{34}}, F_{n_{35}}), (F_{n_{36}}, F_{n_{37}}, F_{n_{38}}, F_{n_{39}}, F_{n_{40}}, F_{n_{41}}, F_{n_{42}}), (F_{n_{43}}, F_{n_{44}}, F_{n_{45}}, F_{n_{46}}, F_{n_{47}}, F_{n_{48}}, F_{n_{49}}), (F_{n_{50}}, F_{n_{51}}, F_{n_{52}}, F_{n_{53}}, F_{n_{54}})\).

The failure probabilities and the CPU times of our proposed approach are shown in Tables 7. and 8., respectively, as Case 1. Using MC simulations with 10⁷ simulations, the system failure probability is estimated as 4.57×10⁻² and the corresponding CPU time is 39.7s.

Furthermore, suppose that the information for the statistics of component 52 is missing; in this case, the probabilities involving this equipment item are not available. With the proposed approach, the relative constraints are removed. The failure probabilities and the CPU times of the proposed approach are also shown in Tables 7. and 8., respectively, as Case 2.

Note that with incomplete probability information, MC simulations cannot be performed.

From Tables 7. and 8., one finds that the accuracy of the proposed approach is acceptable when \(k_m \geq 2\) and \(k_r \geq 1\), and that narrower bounds can be obtained by increasing \(k_m\) and \(k_r\).

6. Conclusion

A system reliability analysis approach using linear programming is developed in this paper to provide an efficient reliability tool for a general system with a large number of components. Our method is based on the information from a few probabilities, and can provide bounds for the failure probability of a large system. The proposed approach provides results comparable to those of the LP bounds method when the LP bounds method is applicable. The efficiency and the accuracy of our approach for a large system with a small number of failure modes are demonstrated through a concrete example.

This paper also presents a strategy for decreasing the number of constraints in the RLP bounds method in such a way as to cause little or no loss of accuracy in the result.

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