A proof of the Kazdan-Warner identity via the Minkowski spacetime

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Abstract
Any 2-dim Riemannian manifold with spherical topology can be embedded isometrically into a lightcone of the Minkowski spacetime. We apply this fact to give a proof of the Kazdan-Warner identity.

1 The Kazdan-Warner identity

$(S^2, g_{S^2})$ is the sphere of radius 1 at the origin in the 3-dim Euclidean space $\mathbb{E}^3$. $(S^2, g_{S^2})$ has the constant Gauss curvature 1. $\{x_1, x_2, x_3\}$ is the rectangular coordinate system of $\mathbb{E}^3$. We denote $\nabla$ the Levi-Civita connection on $(S^2, g_{S^2})$. Then the vector fields $\nabla x_1, \nabla x_2, \nabla x_3$ are conformal Killing vector fields on $(S^2, g_{S^2})$, i.e. the diffeomorphisms generated by them are conformal.

$g$ is another Riemannian metric on the sphere $S^2$. By the uniformization theorem, there exists a function $f$ on $S^2$ such that the conformal metric $e^{-2f}g$ has the constant Gauss curvature 1. Hence there is a diffeomorphism $\psi : S^2 \to S^2$ such that $\psi^*g = e^{2f}\psi g_{S^2}$.

In the following of this note, we assume that $g$ is conformal to the standard metric $g_{S^2}$ with the conformal factor $e^{2f}$. Let $K_g$ be the Gauss curvature of $(S^2, g)$. Then we have the following from [2]:

The Kazdan-Warner identity.

\[
\int_{S^2} \langle \nabla^g K_g, \nabla x_i \rangle_{g_{S^2}} dvol_g = 0. \tag{1.1}
\]

We denote $\tilde{\nabla}$ the Levi-Civita connection on $(S^2, g)$. Then we can rewrite the Kazdan-Warner identity as following

\[
\int_{S^2} \langle \tilde{\nabla} K_g, \nabla x_i \rangle_g dvol_g = 0. \tag{1.2}
\]

Since $\nabla x_i$ is a conformal Killing vector field for $g_{S^2}$, it is also a conformal Killing vector field for $g$. Actually we have that for any conformal Killing vector field $X$ on $(S^2, g)$,

\[
\int_{S^2} \langle \tilde{\nabla} K_g, X \rangle_g dvol_g = 0. \tag{1.3}
\]
We will give a proof of (1.3), thus the Kazdan-Warner identity follows.

2 The Minkowski spacetime

\((\mathbb{M}, \eta)\) is the 4-dim Minkowski spacetime. \(\{x_0, x_1, x_2, x_3\}\) is the rectangular coordinate system of \(\mathbb{M}\). The metric \(\eta = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2\).

\(t, r\) are two functions on \(\mathbb{M}\): \(t = x_0\) and \(r = \sqrt{x_1^2 + x_2^2 + x_3^2}\). We define the optical functions \(u, v\) by \(u = \frac{1}{2}(t - r)\) and \(v = \frac{1}{2}(t + r)\). So \(t = u + v\) and \(r = v - u\).

Let \(C_u\) be the level set of \(u\), which is the future lightcone at the point \((u, 0, 0, 0)\). Let \(C_v\) be the level set of \(v\), which is the past lightcone at the point \((v, 0, 0, 0)\). Let \(S_{u,v}\) be the intersection of \(C_u\) and \(C_v\). \(S_{u,v}\) is a sphere of radius \(r = v - u\).

We define a map \(\phi: \mathbb{M}\ \setminus \{r = 0\} \rightarrow \mathbb{S}^2\) by \(\phi: x = (x_0, x_1, x_2, x_3) \mapsto \left(\frac{x_1}{r}, \frac{x_2}{r}, \frac{x_3}{r}\right)\). \(\mathbb{R}^2_{v > u}\) is the open half-plane \(\{(v, u) \in \mathbb{R}^2 \mid v > u\}\). Then we have another coordinate system \(\Phi\) on \(\mathbb{M}\ \setminus \{r = 0\}\) given by

\[
\Phi: \quad \mathbb{M}\ \setminus \{r = 0\} \rightarrow \mathbb{R}^2_{v > u} \times \mathbb{S}^2,
\]

\[
x = (x_0, x_1, x_2, x_3) \mapsto \left((v, u), \phi(x) = \left(\frac{x_1}{r}, \frac{x_2}{r}, \frac{x_3}{r}\right)\right). \tag{2.1}
\]

Let \(\partial_t\) be the vector field \(\partial_0\) and \(\partial_r\) be the vector field \(\frac{x_0}{r} \partial_1 + \frac{x_2}{r} \partial_2 + \frac{x_3}{r} \partial_3\). In the coordinate system \(\Phi\), the coordinate vector fields \(\partial_v = \partial_t + \partial_r\) and \(\partial_u = \partial_t - \partial_r\).

\(\partial_u\) and \(\partial_v\) are both null vector fields, i.e. \(\eta(\partial_u, \partial_u) = \eta(\partial_v, \partial_v) = 0\). The inner product of \(\partial_u\) and \(\partial_v\) is \(-2\), i.e. \(\eta(\partial_u, \partial_v) = -2\). We see that \(\partial_u\) and \(\partial_v\) are orthonormal to the tangent space of any \(S_{u,v}\). So in the coordinate system \(\Phi\), the metric \(\eta = -2(du \otimes dv + dv \otimes du) + r^2 g_{\mathbb{S}^2}\).

In particular, when we restrict the coordinate system \(\Phi\) on the lightcone \(C_0\ \setminus \{o\}\), we get a coordinate system of \(C_0\ \setminus \{o\}\):

\[
\Phi|_{C_0\ \setminus \{o\}}: \quad C_0\ \setminus \{o\} \rightarrow \mathbb{R}^+ \times \mathbb{S}^2, \quad x \in C_0\ \setminus \{o\} \mapsto (v, \phi(x)). \tag{2.2}
\]

The induced metric \(\eta|_{C_0\ \setminus \{o\}} = v^2 g_{\mathbb{S}^2}\) is degenerated.

3 The isometric embedding of \((\mathbb{S}^2, g)\) into a lightcone of the Minkowski spacetime

Via the coordinate system \(\Phi|_{C_0\ \setminus \{o\}}\), we can represent any closed spacelike surface in \(C_0\ \setminus \{o\}\) as a graph of a function on \(\mathbb{S}^2\). \(S\) is a closed spacelike surface in \(C_0\ \setminus \{o\}\), there exists a unique function \(h\) on \(\mathbb{S}^2\) such that

\[
\Phi|_{C_0\ \setminus \{o\}}(S) = \left\{(e^{h(\theta)} \theta, \theta) \in \mathbb{R}^+ \times \mathbb{S}^2 \mid \theta \in \mathbb{S}^2\right\}. \tag{3.1}
\]
Conversely for any function $h$ on $S^2$, the map
\[ \psi_h : S^2 \to \mathbb{R}^+ \times S^2, \quad \theta \in S^2 \mapsto (e^{h(\theta)}, \theta) \] (3.2)
is an embedding of $S^2$ into $C_0\setminus\{0\}$. Its image
\[ S_h = \psi_h(S^2) = \left\{ (e^{h(\theta)}, \theta) \in \mathbb{R}^+ \times S^2 \mid \theta \in S^2 \right\} \] (3.3)
is a closed spacelike surface in $C_0\setminus\{0\}$. Moreover, $\psi_h^* (\eta|_{S_h}) = e^{2h} g_{S^2}$ since $\eta|_{C_0\setminus\{0\}} = v^2 g_{S^2}$.

Hence, we can embed $(S^2, g = e^{2f} g_{S^2})$ isometrically into $C_0\setminus\{0\}$ as above by taking $h = f$.

4 The geometry of a spacelike surface in the Minkowski spacetime

Let $S$ be a orientable spacelike surface in the Minkowski spacetime $(\mathbb{M}, \eta)$. $TS$ is the tangent bundle and $NS$ is the normal bundle of $S$ in $(\mathbb{M}, \eta)$.

Since $S$ is spacelike, $\eta|_{TS}$ is positive definite and $\eta|_{NS}$ is of signature $(1,1)$. Hence we can choose a null frame \{L, \underline{L}\} of $NS$ such that $L$ and \underline{L} are both future-directed null vector fields and their inner product is $-2$, i.e.

\[ \eta(L, L) = \eta(L, \underline{L}) = 0, \quad \eta(L, \underline{L}) = -2. \] (4.1)

Such a choice isn’t unique, since for any positive function $a$ on $S$, the frame \{aL, a^{-1} \underline{L}\} also satisfies the above conditions.

In the following, we fix such a null frame \{L, \underline{L}\}. Then we can choose an orthonormal frame \{e_1, e_2\} of $TS$ at least locally such that \{e_1, e_2, e_3 = \underline{L}, e_4 = L\} is positive oriented in $\mathbb{M}$. Choose the orientation of $S$ to be the orientation of \{e_1, e_2\}. Let $A, B = 1, 2$.

The intrinsic geometry of $S$ is given by the induced metric $\eta|_S$. Let $\nabla$ be the Levi-Civita connection on $(S, \eta|_S)$ and $d$ be the exterior derivative on $S$. Let $\epsilon$ be the volume form on $(S, \eta|_S)$. We define the intrinsic differential operators $\text{curl}, d\text{iv}$ on $S$ by

\[ \text{curl}\omega = e^{AB} \nabla_A \omega_B, \quad \text{for any 1-form } \omega \text{ on } S; \] (4.2)
\[ (d\text{iv} T)_A = \nabla^B T_{BA}, \quad \text{for any symmetric 2-tensor field } T \text{ on } S. \] (4.3)

The extrinsic geometry of $S$ in $(\mathbb{M}, \eta)$ is given by the null second fundamental forms $\chi, \underline{\chi}$ and the torsion $\zeta$ defined as following:

\[ \chi(X, Y) = \eta(\nabla_X L, Y), \quad \text{for any } X, Y \in TS; \] (4.4)
\[ \underline{\chi}(X, Y) = \eta(\nabla_X \underline{L}, Y), \quad \text{for any } X, Y \in TS; \] (4.5)
\[ \zeta(X) = \frac{1}{2} \eta(\nabla_X L, \underline{L}), \quad \text{for any } X \in TS. \] (4.6)
\( \chi, \chi \) are covariant symmetric 2-tensor fields and \( \zeta \) is a 1-form on \( S \). Let \( \text{tr}\chi \) and \( \text{tr}\chi \) be the traces of \( \chi \) and \( \chi \) on \( S \), i.e. the contractions with \( \eta|_S^{-1} \) on \( TS \). Let \( \hat{\chi} \) and \( \hat{\chi} \) be the tracefree parts of \( \chi \) and \( \chi \) on \( S \).

In analogy with the Gauss equation and Codazzi equation for a surface in 3-dim Euclidean spacetime, we have the following equations:

**The Gauss equations**

\[
K + \frac{1}{4} \text{tr}\chi \text{tr}\chi - \frac{1}{2} (\hat{\chi}, \hat{\chi}) \eta|_S = 0, \quad (4.7)
\]

\[
\text{curl}\zeta + \frac{1}{2} \hat{\chi} \wedge \hat{\chi} = 0; \quad (4.8)
\]

**The Codazzi equations**

\[
d\hat{\chi} - \frac{1}{2} d \text{tr}\chi + \hat{\chi} \cdot \zeta - \frac{1}{2} \text{tr}\chi \zeta = 0, \quad (4.9)
\]

\[
d\hat{\chi} - \frac{1}{2} d \text{tr}\chi - \hat{\chi} \cdot \zeta + \frac{1}{2} \text{tr}\chi \zeta = 0; \quad (4.10)
\]

where

\[
(\hat{\chi}, \hat{\chi}) \eta|_S = (\eta|_S^{-1})^{AC}(\eta|_S^{-1})^{BD} \hat{\chi}^{\hat{A}\hat{B}} \hat{\chi}_{\hat{C}\hat{D}}, \quad \hat{\chi} \wedge \hat{\chi} = \epsilon^{AB} \hat{\chi}_A \hat{\chi}_B, \quad (4.11)
\]

and

\[
(\hat{\chi} \cdot \zeta)_A = \hat{\chi}_A \hat{\chi}_B, \quad (\hat{\chi} \cdot \zeta)_A = \hat{\chi}_A \hat{\chi}_B. \quad (4.12)
\]

One can find the proofs of these equations in [1].

We can apply these equations to \( S \) in the lightcone \( C_0 \setminus \{0\} \). We see \( \hat{\zeta} \) along \( S \) is a null vector field in \( NS \). So we choose \( L \) to be the null vector field \( \hat{\zeta} \) over \( S \). Then we can find \( L \) in \( NS \) such that \( \{L, L\} \) is a null frame of \( NS \).

Since the metric \( \eta|_{C_0 \setminus \{0\}} = \nu^2 g_{52} \), we see that the induced metric on \( S \) deforms conformally when we deform \( S \) in the direction of \( \hat{\zeta} \). This means that the seconded fundamental form \( \chi \) of \( S \) is a multiple of the induced metric \( \eta|_S \). Hence \( \hat{\chi} = 0 \). So on \( S \), the equations are simpler:

\[
K + \frac{1}{4} \text{tr}\chi \text{tr}\chi = 0, \quad (4.13)
\]

\[
\text{curl}\zeta = 0, \quad (4.14)
\]

\[
d\text{tr}\chi + \text{tr}\chi \zeta = 0, \quad (4.15)
\]

\[
d\hat{\chi} - \frac{1}{2} d \text{tr}\chi - \hat{\chi} \cdot \zeta + \frac{1}{2} \text{tr}\chi \zeta = 0. \quad (4.16)
\]
5 The proof of the Kazdan-Warner identity

$S$ is a closed surface in $C_0 \setminus \{a\}$. Let $X$ be a conformal Killling vector field on $(S, \eta|_S)$. The deformation tensor field $^{(X)}\pi = L_X \eta|_S$ of $X$ is a multiple of $\eta|_S$. Then $^{(X)}\pi = ^{(X)}\Omega \cdot \eta|_S$ where $^{(X)}\Omega = \frac{1}{2} \delta \hat{\nu} X$. We define the operator sym by

$$(\text{sym} T)_{AB} = T_{AB} + T_{BA},$$

for any covariant 2-tensor field $T$ on $S$. Then

$$\text{sym} (L_X \eta|_S) = ^{(X)}\pi = \Omega^X \cdot \eta|_S.$$ 

Then

$$\int_S \langle \nabla K, X \rangle \eta|_S \, d\text{vol}|_S \overset{(1.13)}{=} \int_S -\frac{1}{4} \langle \nabla (\text{tr} \chi), X \rangle \eta|_S \, d\text{vol}|_S$$

$$= \int_S -\frac{1}{4} \langle \text{tr} \chi \nabla \text{tr} \chi + \text{tr} \chi \nabla \text{tr} \chi, X \rangle \eta|_S \, d\text{vol}|_S$$

$$\overset{(1.15)}{=} \int_S -\frac{1}{2} \{ \text{tr} \chi \delta \hat{\nu} \hat{\chi} \cdot X - \hat{\chi} (\text{tr} \chi, X) \} \, d\text{vol}|_S$$

$$\overset{(1.15)}{=} \int_S -\frac{1}{4} \{ \text{tr} \chi \hat{\chi} \cdot \nabla \text{tr} \chi + \hat{\chi} (\nabla \text{tr} \chi + \text{tr} \chi \cdot X) \} \, d\text{vol}|_S$$

$$\overset{(1.15)}{=} \int_S \frac{1}{4} \text{tr} \chi \langle \hat{\chi}, \nabla \text{tr} \chi \rangle \eta|_S \, d\text{vol}|_S$$

$$= \int_S \frac{1}{4} \text{tr} \chi \langle \hat{\chi}, \text{sym} (\nabla X) \rangle \eta|_S \, d\text{vol}|_S$$

$$= \int_S \frac{1}{4} \text{tr} \chi \langle \hat{\chi}, \Omega X \cdot \eta|_S \rangle \eta|_S \, d\text{vol}|_S$$

$$= 0.$$ 

The last equality follows from that $\hat{\chi}$ is tracefree.

Together with the constructions in section 3, we prove (1.3).

6 The gauge transformations on the normal bundle of a
spacelike surface in the Minkowski spacetime

Recall that in the section 4 we introduced the normal bundle $NS$ of a oriented spacelike surface $S$ in the Minkowski spacetime $(\mathbb{M}, \eta)$. Since that for any $p \in S$, the normal space $N_p S$ endowed with the induced metric $\eta|_{N_p S}$ is isometric to the 2-dim Minkowski spacetime, we have a 1-dim non-compact abelian group of isometries for $(N_p S, \eta|_{N_p S})$. The group of isometries on $(N_p S, \eta|_{N_p S})$ is just the group of Lorentz rotations of the 2-dim Minkowski spacetime. We can explicitly write down the isometries via the null frame $\{ L, L \}$. Any positive number $a \in \mathbb{R}_{>0}$, we have the mapping $L_a : N_p S \rightarrow N_p S$ defined by

$$L_a : \quad L_p \rightarrow a L_p, \quad L_p \rightarrow a^{-1} L_p.$$ 

The group structure is simply given by $L_a \circ L_b = L_{ab}$ for any $a, b \in \mathbb{R}_{>0}$.
So the normal bundle $(NS, \eta|_{NS})$ is a vector bundle on $S$ with the group action of $(\mathbb{R}_{>0}, \cdot)$. Moreover, the null frame bundle of $(NS, \eta|_{NS})$ is a principal $(\mathbb{R}_{>0}, \cdot)$-bundle. This principal bundle is actually trivial since we can find a global section, which is a global null frame. The parallel transport on the normal bundle $NS$ defines a principal connection on the null frame bundle. We see that the torsion $\zeta$ of a null frame $\{L, L\}$ is actually the connection 1-form for this principal connection. Assume now that $a$ is a positive function over $S$, then $\{aL, a^{-1}L\}$ is another null frame of $(NS, \eta|_{NS})$. Let us denote $\zeta_a$ being the torsion for $\{aL, a^{-1}L\}$. Direct calculation by the definition of the torsion (4.6) shows that

$$\zeta_a = \zeta - a^{-1}\partial a = \zeta - \partial \log a, \quad (6.2)$$

which is just the transformation formula for the connection form. However $\text{curl} \zeta_a = \text{curl} \zeta$ keeps invariant, because it is actually the curvature of this connection. In particular, we can choose a positive function $a$ such that

$$\text{div} \zeta_a = \text{div} \zeta - \Delta \log a = 0. \quad (6.3)$$

Now Assume that the oriented spacelike surface $S$ is contained in the lightcone $C_0 \setminus \{o\}$. We take the null frame $\{L, L\}$ of $(NS_h, \eta|_{NS_h})$ such that its torsion $\zeta$ satisfies $\text{div} \zeta = 0$. Associated with this null frame $\{L, L\}$, we have the Gauss equations and Codazzi equations. Since $\chi = 0$ still holds, we have $\text{curl} \zeta = 0$. Then the equations

$$\text{div} \zeta = 0, \quad \text{curl} \zeta = 0, \quad (6.4)$$

imply that $\zeta = 0$. Then we have

$$K + \frac{1}{4} \text{tr} \chi \text{tr} \chi = 0, \quad (6.5)$$

$$\partial \text{tr} \chi = 0, \quad (6.6)$$

$$\text{div} \chi - \frac{1}{2} \partial \text{tr} \chi = 0. \quad (6.7)$$

Hence $\text{tr} \chi$ is a constant function over $S$ and we can assume $\text{tr} \chi \equiv 1$ since we can always achieve this by modifying the null frame by a positive constant. So we get that

$$\partial K = -\frac{1}{2} \text{div} \chi. \quad (6.8)$$

We consider the following operator $\text{div}$ taking a 2-covariant symmetric, traceless tensor $\xi$ into the 1-form $\text{div} \xi$. The $L^2$-adjoint of $\text{div}$ is the operator taking a 1-form $f$ into the 2-covariant symmetric, traceless tensor $-\frac{1}{2} \mathcal{L}_f \eta|_S$, where $\mathcal{L}_f \eta|_S$ is the traceless part of the
Lie derivative of $\eta|_S$ with respect to the vector field $f^\sharp$. This can be shown as the following:

$$
\int_S \langle \nabla f, \xi \rangle \, d\nu|_S = \int_S \langle \xi, -\nabla f \rangle \, d\nu|_S
$$

$$
= \int_S \langle \xi, -\frac{1}{2} \text{sym } \nabla f \rangle \, d\nu|_S
$$

$$
= \int_S \langle \xi, -\frac{1}{2} \mathcal{L}_{f|_S} \eta \rangle \, d\nu|_S
$$

$$
= \int_S \langle \xi, -\frac{1}{2} \mathcal{L}_{f|_S} \eta \rangle \, d\nu|_S.
$$

The kernel of the $L^2$-adjoint of $\nabla \nu$ consists of the 1-form $f$ such that $f^\sharp$ is a conformal Killing vector field. Since the range of $\nabla \nu$ is $L^2$ orthogonal to the kernel of its $L^2$-adjoint, then the identity (1.3) follows from (6.8).

References

[1] D. Christodoulou, *The Formation of Black Holes in General Relativity*. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2009.

[2] J. L. Kazdan, F. W. Warner, Curvature Functions for Compact 2-Manifolds, *Ann. of Math.* 99 (1974), 14–47.