Nonuniform sampling and approximation in Sobolev space from perturbation of the framelet system

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Abstract The Sobolev space \( H^s(\mathbb{R}^d) \), where \( s > d/2 \), is an important function space that has many applications in various areas of research. Attributed to the inertia of a measurement instrument, it is desirable in sampling theory to recover a function by its nonuniform sampling. In the present paper, based on dual framelet systems for the Sobolev space pair \( (H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d)) \), where \( d/2 < s < \varsigma \), we investigate the problem of constructing the approximations to all the functions in \( H^s(\mathbb{R}^d) \) by nonuniform sampling. We first establish the convergence rate of the framelet series in \( (H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d)) \), and then construct the framelet approximation operator that acts on the entire space \( H^s(\mathbb{R}^d) \). We examine the stability property for the framelet approximation operator with respect to the perturbations of shift parameters, and obtain an estimate bound for the perturbation error. Our result shows that under the condition \( d/2 < s < \varsigma \), the approximation operator is robust to shift perturbations. Motivated by Hamm (2015)’s work on nonuniform sampling and approximation in the Sobolev space, we do not require the perturbation sequence to be in \( \ell^2(\mathbb{Z}^d) \). Our results allow us to establish the approximation for every function in \( H^s(\mathbb{R}^d) \) by nonuniform sampling. In particular, the approximation error is robust to the jittering of the samples.

Keywords Sobolev space, framelet series, truncation error, perturbation error, nonuniform sampling and approximation

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1 Introduction

Sampling is a fundamental tool for the conversion between an analogue signal and its digital form (A/D). The most classical sampling theory is the Whittaker-Kotelinkov-Shannon (WKS) sampling theorem [43], which states that a bandlimited signal can be perfectly reconstructed if it is sampled at a rate greater than its Nyquist frequency. The WKS sampling theorem holds only for bandlimited signals. In order to extend the sampling theorem to non-bandlimited signals, researchers have established various sampling theorems for many other function spaces. Such examples include the sampling theory for shift-invariant subspaces [1–3, 12, 31, 32, 49, 51, 52], reproducing kernel subspaces of \( L^2(\mathbb{R}^d) \) [9, 20, 28, 41] and subspaces from the generalized sinc functions [8].
1.1 Goal of the applicable scope and sampling flexibility

For any $\varsigma \in \mathbb{R}$, the Sobolev space $H^\varsigma(\mathbb{R}^d)$ is defined as

$$H^\varsigma(\mathbb{R}^d) := \left\{ f : \int_{\mathbb{R}^d} \hat{f}(\xi)^2 (1 + \|\xi\|^2)^\varsigma d\xi < \infty \right\}, \quad (1.1)$$

where $\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-ix\cdot \xi} dx$ is the Fourier transform of $f$. For $\varsigma > d/2$, by the similar analysis in [47, Chapter 9.1] one can check that the functions in $H^\varsigma(\mathbb{R}^d)$ are continuous. From now on it is assumed that $\varsigma > d/2$. The function theory in $H^\varsigma(\mathbb{R}^d)$ is important for many problems, including the boundedness of the Fourier multiplier operator [7, 18], the viscous shallow water system [39, 54], the PDE or the ODE [26, 42], and signal analysis [47]. Moreover, it is easy to check that many important function spaces such as the bandlimited function space [43], and the shift-invariant subspace [1–3, 10, 11, 27, 48] (in which the generator is continuous) are contained in $H^\varsigma(\mathbb{R}^d)$ for some appropriate $\varsigma > d/2$. However, in general it is not easy to determine whether a function in $H^\varsigma(\mathbb{R}^d)$ belongs to a desired subspace or not. Therefore it is practically useful to establish some recovery methods for the entire space $H^\varsigma(\mathbb{R}^d)$. Besides the aspect of the applicable scope, the samples we acquire may also be well jittered and thus usually nonuniform [45, 46, 48]. Therefore the goal of this paper is to establish a sampling theory for the entire space $H^\varsigma(\mathbb{R}^d)$. This will allow us to construct the approximations to all the functions in $H^\varsigma(\mathbb{R}^d)$, which admit nonuniform sampling points. To the best of our knowledge, this has not been examined in the literature. Our goal will be achieved in Theorem 4.1 by the theory of dual framelets in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$, with $d/2 < s < \varsigma$, which was introduced by Han and Shen [27]. In what follows we introduce some necessary terminologies for framelets in the Sobolev spaces. More details can be found in [27] and Han’s continuing work [23–25] on dual framelets in distribution spaces.

1.2 Preliminary terminologies for dual framelets in dual Sobolev spaces

By (1.1), $H^\varsigma(\mathbb{R}^d)$ is equipped with the inner product $\langle \cdot, \cdot \rangle_{H^\varsigma(\mathbb{R}^d)}$ defined by

$$\langle f, g \rangle_{H^\varsigma(\mathbb{R}^d)} := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}(1 + \|\xi\|^2)^\varsigma d\xi, \quad \forall f, g \in H^\varsigma(\mathbb{R}^d), \quad (1.2)$$

where $\overline{\cdot}$ is the complex conjugate. Naturally, the deduced norm is defined by

$$\|f\|_{H^\varsigma(\mathbb{R}^d)} := \frac{1}{(2\pi)^{d/2}} \left( \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 (1 + \|\xi\|^2)^\varsigma d\xi \right)^{1/2}, \quad \forall f \in H^\varsigma(\mathbb{R}^d).$$

It is easy to check that the functional $\langle \cdot, \cdot \rangle : (H^\varsigma(\mathbb{R}^d), H^{-\varsigma}(\mathbb{R}^d)) \to \mathbb{C}$ defined by

$$\langle f, g \rangle := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)} d\xi, \quad \forall f \in H^\varsigma(\mathbb{R}^d), \quad g \in H^{-\varsigma}(\mathbb{R}^d)$$

can be bounded by

$$|\langle f, g \rangle| \leq \|f\|_{H^\varsigma(\mathbb{R}^d)}\|g\|_{H^{-\varsigma}(\mathbb{R}^d)}.$$  

Clearly, $H^{\varsigma_1}(\mathbb{R}^d) \supseteq H^{\varsigma_2}(\mathbb{R}^d)$ if and only if $\varsigma_1 \leq \varsigma_2$. Moreover, $H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ and correspondingly $\|\cdot\|_{H^0(\mathbb{R}^d)} = \|\cdot\|_{L^2}$. For any two functions $f, g : \mathbb{R}^d \to \mathbb{C}$ and $\mu \in \mathbb{R}$, define their bracket product $[f, g]_\mu$ by

$$[f, g]_\mu(\xi) := \sum_{k \in \mathbb{Z}^d} f(\xi + 2k\pi)g(\xi + 2k\pi)(1 + \|\xi + 2k\pi\|^2)^\mu, \quad (1.3)$$

whenever the above series converge. Readers can refer to Han’s method [21,25] for estimating the bracket product.

A $d \times d$ integer matrix $M$ is referred to as a dilation matrix if all its eigenvalues are strictly larger than 1 in modulus. Throughout this paper, we are interested in the case where $M$ is isotropic. Specifically, $M$ is similar to $\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$ with

$$|\lambda_1| = |\lambda_2| = \cdots = |\lambda_d| = m := |\det M|^{1/d}.$$
Suppose that $\phi \in H^s(\mathbb{R}^d)$ is an $M$-refinable function given via the $M$-refinement equation

$$\hat{\phi}(M^t \cdot) = \hat{\varphi}(\cdot) \hat{\phi}(\cdot).$$

(1.4)

Here, $\hat{\varphi}(\xi) := \sum_{k \in \mathbb{Z}^d} a[k] e^{ik \cdot \xi}$, $\xi \in \mathbb{R}^d$ is referred to as the mask symbol of $\phi$, and $\{\psi^\ell\}_{\ell=1}^L$ is a set of wavelet functions defined by

$$\hat{\psi}^\ell(M^t \cdot) = \hat{\varphi}(\cdot) \hat{\psi}^\ell(\cdot),$$

(1.5)

where the $2\pi \mathbb{Z}^d$-periodic trigonometric polynomial $\hat{\varphi}(\cdot)$ is the mask symbol of $\psi^\ell$. Now a wavelet system $X^s(\phi; \psi^1, \ldots, \psi^L)$ in $H^s(\mathbb{R}^d)$ is defined as

$$X^s(\phi; \psi^1, \ldots, \psi^L) := \{\phi_{0,k} : k \in \mathbb{Z}^d\} \cup \{\psi^\ell_{j,k} : k \in \mathbb{Z}^d, j \in \mathbb{N}_0, \ell = 1, \ldots, L\},$$

(1.6)

where $\phi_{0,k} = \hat{\phi}(\cdot - k)$, $\psi^\ell_{j,k} = m^{j/(d-2)-s} \hat{\psi}^\ell(M^j \cdot - k)$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If there exist two positive constants $C_1$ and $C_2$ such that for every $f \in H^s(\mathbb{R}^d)$,

$$C_1 \|f\|_{H^s(\mathbb{R}^d)}^2 \leq \sum_{k \in \mathbb{Z}^d} |\langle f, \phi_{0,k} \rangle_{H^s(\mathbb{R}^d)}|^2 + \sum_{\ell=1}^L \sum_{j \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}^d} |\langle f, \psi^\ell_{j,k} \rangle_{H^s(\mathbb{R}^d)}|^2 \leq C_2 \|f\|_{H^s(\mathbb{R}^d)}^2,$$

(1.7)

then we say that $X^s(\phi; \psi^1, \ldots, \psi^L)$ is an $M$-framelet system in $H^s(\mathbb{R}^d)$. If there exists another $M$-framelet system $X^{-s}(\phi; \tilde{\psi}^1, \ldots, \tilde{\psi}^L)$ in $H^{-s}(\mathbb{R}^d)$ such that for any $f \in H^s(\mathbb{R}^d)$ and $g \in H^{-s}(\mathbb{R}^d)$,

$$\langle f, g \rangle = \sum_{k \in \mathbb{Z}^d} \langle f, \phi_{0,k} \rangle \langle f, \phi_{0,k} \rangle + \sum_{\ell=1}^L \sum_{j \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}^d} \langle f, \psi^\ell_{j,k} \rangle \langle f, \psi^\ell_{j,k} \rangle,$$

(1.8)

then we say that $X^s(\phi; \psi^1, \ldots, \psi^L)$ and $X^{-s}(\phi; \tilde{\psi}^1, \ldots, \tilde{\psi}^L)$ form a pair of dual $M$-framelet systems in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$. For any function $f \in H^s(\mathbb{R}^d)$, it follows from (1.8) that

$$f = \sum_{k \in \mathbb{Z}^d} \langle f, \phi_{0,k} \rangle \phi_{0,k} + \sum_{\ell=1}^L \sum_{j \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}^d} \langle f, \psi^\ell_{j,k} \rangle \psi^\ell_{j,k}.$$

(1.9)

**Remark 1.1.**  (i) The framelets in $L^2(\mathbb{R}^d)$ must have at least one vanishing moment such that the framelet series converge unconditionally (see [11, 15, 27]). However when $s > 0$, the vanishing moment of $\psi^\ell$ is not necessary for the convergence of the series in (1.9), $\ell = 1, \ldots, L$. This is the most significant difference between the framelets in $L^2(\mathbb{R}^d)$ and those in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$. For more details about the conditions for the convergence of framelet series in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$, readers can refer to [25, 27].

(ii) Our goal is to construct the approximations to all the functions in $H^s(\mathbb{R}^d)$ ($s > d/2$). The construction scheme is sketched as follows. We first choose $d/2 < s < \varsigma$, and design special dual framelet systems $X^s(\phi; \psi^1, \ldots, \psi^L)$ and $X^{-s}(\phi; \tilde{\psi}^1, \ldots, \tilde{\psi}^L)$ in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$. Recall that the target $f \in H^s(\mathbb{R}^d) \subseteq H^s(\mathbb{R}^d)$. Then we shall use (1.9) to establish the approximation to $f$. The reason for $s > \varsigma$ is postponed to Remark 2.2.  

**1.3 Main results and structure**

The main results of the present paper are stated in Theorems 2.3, 3.1 and 4.1. As assumed in Remark 1.1(ii), the target $f \in H^s(\mathbb{R}^d)$, and $X^s(\phi; \psi^1, \ldots, \psi^L)$ and $X^{-s}(\phi; \tilde{\psi}^1, \ldots, \tilde{\psi}^L)$ are the dual framelet systems in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ where $d/2 < s < \varsigma$. It will be clear in Theorem 4.1 that the truncation version $S^N_\phi f$ of the series in (1.9) with respect to the scale $j$, defined by

$$S^N_\phi f := \sum_{k \in \mathbb{Z}^d} \langle f, \phi_{0,k} \rangle \phi_{0,k} + \sum_{\ell=1}^L \sum_{j=0}^{N-1} \sum_{k \in \mathbb{Z}^d} \langle f, \psi^\ell_{j,k} \rangle \psi^\ell_{j,k},$$

(1.10)
is crucial for establishing the sampling and approximation. Naturally, the first problem is how to estimate the approximation error \( \| (I - S^N_\phi) f \|_{H^s} \) for any \( f \in H^s(\mathbb{R}^d) \), where \( I \) is the identity operator. The answer to this problem will be given in Theorem 2.3. It should be noted that the estimation of the approximation error established in this paper holds for any \( f \in H^s(\mathbb{R}^d) \). In [36], the error estimation was established for a class of target functions. For the one-dimensional case, the error was estimated in the sense of Sobolev seminorm by [25, Corollary 4.7.3]. Theorem 2.3 is not the trivial generalization of [36] and [25, Corollary 4.7.3]. More details of comparison will be given in Comparison 2.5.

We next turn to the perturbation of \( S^N_\phi f \). It will be clear in (3.6) and (4.4) that the sampling nonuniformity is substantially derived from the perturbation of shift parameter \( k \) of the system

\[
\{ \phi^s_N,k \}_{k \in \mathbb{Z}^d} \subseteq H^{-s}(\mathbb{R}^d),
\]

where \( \phi^s_N,k = m^{N(d/2+s)}(M^N \cdot k) \). Thus, in order to construct the approximation by nonuniform sampling, we need to estimate the perturbation error of \( \| (I - \tilde{S}^N_\phi) f \|_{L^2} \), where \( \varepsilon = \{ \varepsilon_k \}_{k \in \mathbb{Z}^d} \) is the perturbation sequence of the shift sequence \( \{ k \}_{k \in \mathbb{Z}^d} \). Theorem 3.1 establishes such an error estimation. Motivated by Hamm’s recent work [19] on the nonuniform sampling-based approximation, the perturbation sequence in the present paper is not required to sit in \( \ell^2(\mathbb{Z}^d) \) (the square summable sequence space). Since the perturbation sequence is not necessarily in \( \ell^2(\mathbb{Z}^d) \), the error cannot be estimated by the brute force estimation but by using some crucial techniques in Subsection 3.1. More details about the techniques will be summarized in Subsection 5.3.

The rest of this paper is organized as follows. In Section 2, we shall establish an error estimation of \( \| (I - S^N_\phi) f \|_{H^s(\mathbb{R}^d)} \) for any \( f \in H^s(\mathbb{R}^d) \). Theorem 2.3 is not the trivial generalization of \([36]\) and \([25, \text{Corollary } 4.7.3]\), our main approximation results in Theorems 2.3, 3.1 and 4.1, and the estimation techniques are not trivial generalizations of the results available in the literature. In Section 5, we make detailed comparisons between the main results and the estimation techniques of this paper with the existing ones in the literature. Two simulation examples are presented in Section 6 to demonstrate the approximation efficiency.

## 2 Framelet approximation system in the Sobolev space

In Lemma 2.1 we will estimate the convergence rate of the coefficient sequence \( \{ (f, \psi^j_{k,x}) \}_{j,k} \) in (1.9) with respect to the scale \( j \). Based on Lemma 2.1 we will establish an estimation for \( \| (I - S^N_\phi) f \|_{H^s(\mathbb{R}^d)} \) in Theorem 2.3, where \( f \in H^s(\mathbb{R}^d) \) with \( \varsigma > s > d/2 \). The following notations and definitions are needed for our discussion.

For any \( \alpha := (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d_0 \) and \( x := (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \), define \( x_\alpha := \prod_{k=1}^d x_k^{\alpha_k} \). For a function \( f : \mathbb{R}^d \to \mathbb{C} \), its \( \alpha \)-th partial derivative \( \frac{\partial^\alpha}{\partial x^\alpha} f \) is defined as

\[
\frac{\partial^\alpha}{\partial x^\alpha} f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} f.
\]

We say that a function \( f : \mathbb{R}^d \to \mathbb{C} \) has \( \kappa + 1 \) (\( \in \mathbb{N} \)) vanishing moments if

\[
\frac{\partial^\alpha}{\partial x^\alpha} f(0) = 0
\]

for any \( \alpha \in \mathbb{N}^d_0 \) such that \( \| \alpha \|_1 \leq \kappa \). The characteristic function of the set \( E \subseteq \mathbb{R}^d \) is denoted by \( \chi_E \). Motivated by [25, Theorem 4.6.5], we establish the convergence rate of the wavelet series in \( H^s(\mathbb{R}^d) \) in the following lemma.
Lemma 2.1. Suppose that $\tilde{\phi} \in H^{-\varsigma}(\mathbb{R}^d)$ is $M$-refinable such that

$$\|\tilde{\phi} \ast \tilde{\phi}\|_{L^\infty(\mathbb{T}^d)} < \infty,$$

where $\varsigma > 0$ and $T = (-\pi, \pi]$. Moreover, $\tilde{\phi}$ belongs to $H^{-s}(\mathbb{R}^d)$ with $0 < s < \varsigma$. A wavelet function $\tilde{\psi}$ given by $\tilde{\psi}(MT^j \cdot) = b(M^j \cdot)\hat{\phi}(\cdot)$ has $\kappa + 1$ vanishing moments, where $b$ is a $2\pi \mathbb{Z}^d$-periodic trigonometric polynomial, $\kappa \in \mathbb{N}_0$ and $\kappa + 1 > \varsigma$, i.e., there exists $g_1 \left(g_1 > 0\right)$ such that

$$|\tilde{\psi}(\xi)| \leq g_1 \|\xi\|_{2}^{\kappa + 1} \text{ for a.e. } \xi \in \mathbb{T}^d.$$

Then there exists a positive constant $H_{\tilde{\psi}}(\varsigma, s)$ such that for any $f \in H^s(\mathbb{R}^d)$,

$$\sum_{j=N}^{\infty} \sum_{k \in \mathbb{Z}^d} |\langle f, \tilde{\psi}_{j,k} \rangle|^2 \leq H_{\tilde{\psi}}(s, s) \|f\|_{H^s(\mathbb{R}^d)}^2 \|M^{-2N(s-s)}(\cdot)\|_{L^2(\mathbb{T}^d)}^2.$$

Proof. Through the direct calculation we have

$$\sum_{k \in \mathbb{Z}^d} |\langle f, \tilde{\psi}_{j,k} \rangle|^2 = \frac{m^d}{2(2\pi)^d} \int_{(-\pi, \pi)^d} m^2 \|f\|^2 \|M^{-j}(\cdot)\|_{L^2(\mathbb{T}^d)}^2 \left|\mathcal{F}(f)(\xi)\right|^2 d\xi$$

$$\leq \frac{m^d}{2d-1\pi^d} \int_{(-\pi, \pi)^d} m^2 \|f\|^2 \|M^{-j}(\xi)\|_{L^2(\mathbb{T}^d)}^2 \left|\mathcal{F}(f)(\xi)\right|^2 d\xi$$

$$+ \frac{m^d}{2d-1\pi^d} \int_{(-\pi, \pi)^d} m^2 \sum_{0 \neq k \in \mathbb{Z}^d} \left|\mathcal{F}(f)(\xi + 2\pi k)\right|^2 \|\mathcal{F}(f)(\xi + 2\pi k)\|_{L^2(\mathbb{T}^d)}^2 d\xi$$

$$=: I_{1,j} + I_{2,j},$$

where the Cauchy-Schwarz inequality is used in the inequality,

$$I_{1,j} := \frac{1}{2d-1\pi^d} \int_{\mathbb{R}^d} \left|\mathcal{F}(\tilde{\psi}(\cdot))\right|^2 (1 + \|\xi\|_2^2)^s (1 + \|\xi\|_2^{-\varsigma}) m^2 \|\mathcal{F}(\tilde{\psi}(\cdot))\|^2 \|\mathcal{F}(f)(\xi)\|^2 \chi_{\Lambda_j}(\xi) d\xi$$

and

$$I_{2,j} := \frac{\|\tilde{\psi} \ast \tilde{\phi}\|_{L^\infty(\mathbb{T}^d)}^2}{2d-1\pi^d} \int_{\mathbb{R}^d} \left|\mathcal{F}(\tilde{\psi}(\cdot))\right|^2 (1 + \|\xi\|_2^2)^s \chi_{\mathbb{R}^d \setminus \Lambda_j}(\xi) d\xi.$$

with $\Lambda_j := (MT)^j \mathbb{T}^d$.

Define

$$B_{1,N}(\xi) := \left(1 + \|\xi\|_2^2\right)^{-\varsigma} \sum_{j=N}^{\infty} m^2 \|\mathcal{F}(f)(\xi)\|^2 \chi_{\Lambda_j}(\xi)$$

and

$$B_{2,N}(\xi) := \left(1 + \|\xi\|_2^2\right)^{-\varsigma} \sum_{j=N}^{\infty} m^2 \|\mathcal{F}(f)(\xi)\|^2 \chi_{\mathbb{R}^d \setminus \Lambda_j}(\xi).$$

It follows from (2.2) that

$$\sum_{k \in \mathbb{Z}^d} |\langle f, \tilde{\psi}_{j,k} \rangle|^2 \leq 2 \max\left\{1, \|\tilde{\psi} \ast \tilde{\phi}\|_{L^\infty(\mathbb{T}^d)} \right\} \|f\|_{H^s(\mathbb{R}^d)} \max\{\|B_{1,N}\|_{L^\infty(\mathbb{R}^d)}, \|B_{2,N}\|_{L^\infty(\mathbb{R}^d)}\}. \quad (2.3)$$

In what follows we estimate $\|B_{1,N}\|_{L^\infty(\mathbb{R}^d)}$ and $\|B_{2,N}\|_{L^\infty(\mathbb{R}^d)}$. Clearly if $\xi \in \Lambda_j$ then $\|\xi\|_2 \leq \sqrt{\pi} m^j$. Consequently, if $0 < \|\xi\|_2 \leq \sqrt{\pi} m^j$ then we have

$$B_{1,N}(\xi) = \left(1 + \|\xi\|_2^2\right)^{-\varsigma} \sum_{j=N}^{\infty} m^2 \|\mathcal{F}(f)(\xi)\|^2 \chi_{\Lambda_j}(\xi)$$
\[
\leq g_1^2(1 + \|\xi\|_2)^{2\kappa} \|\xi\|_2^{2(\kappa+1)} \sum_{j=N}^{\infty} m^{-2(\kappa + 1-s)j} \\
\leq g_1^2(\sqrt{d\pi})^{2(\kappa + 1-\varsigma)} \frac{m^{-2(\varsigma-s)N}}{1 - m^{-2(\kappa+1-s)}}.
\]

where we use \(\kappa + 1 \geq \varsigma > s\) in the last inequality. Next, we estimate \(B_{1,N}(\xi)\) when \(\|\xi\|_2 > \sqrt{d\pi}m^N\). By the above analysis, if \(\xi \in \Lambda_j\) then

\[
j \geq J_\xi := \max \left\{ 0, \left\lfloor \log_m \frac{\|\xi\|_2}{\sqrt{d\pi}} \right\rfloor \right\},
\]

where \(\lfloor x \rfloor\) is the smallest integer that is larger than \(x\). Therefore, whenever \(\|\xi\|_2 > \sqrt{d\pi}m^N\) we have

\[
B_{1,N}(\xi) = (1 + \|\xi\|_2)^{-\varsigma} \sum_{j=N}^{\infty} m^{2js} \sim \psi((M^{-T})^j \xi)^2 \chi_{\Lambda_j}(\xi)
\]

\[
= (1 + \|\xi\|_2)^{-\varsigma} \sum_{j=N}^{\infty} m^{2js} \sim \psi((M^{-T})^j \xi)^2 \chi_{\Lambda_j}(\xi)
\]

\[
\leq g_1^2(1 + \|\xi\|_2)^{2\kappa} \|\xi\|_2^{2(\kappa+1)} \sum_{j=N}^{\infty} m^{-2(\kappa + 1-s)j} \\
\leq g_1^2 \|\xi\|_2^{2(\kappa+1-\varsigma)} \sum_{j=N}^{\infty} m^{-2(\kappa + 1-s)j} \\
= g_1^2 \|\xi\|_2^{2\kappa} \|\xi\|_2^{2(\kappa+1)} \frac{m^{-2(\kappa+1-s)J_\xi}}{1 - m^{-2(\kappa+1-s)}} \\
\leq g_1^2 \|\xi\|_2^{2\kappa} \|\xi\|_2^{2(\kappa+1)} \frac{m^{-2(\kappa+1-s)J_\xi}}{1 - m^{-2(\kappa+1-s)} \log_m \frac{\|\xi\|_2}{\sqrt{d\pi}}} \\
= g_1^2 (\sqrt{d\pi})^{2(\kappa + 1-\varsigma)} \frac{\|\xi\|_2^{2(\kappa+1)} \log_m \frac{\|\xi\|_2}{\sqrt{d\pi}}}{1 - m^{-2(\kappa+1-s)}} \\
\leq g_1^2 \frac{1}{\sqrt{d\pi}}^{2(\kappa - \varsigma)} \log_m m^{J_\xi} \|\xi\|_2^{2(\kappa + 1-\varsigma)} \frac{m^{-2(\kappa+1-s)N}}{1 - m^{-2(\kappa+1-s)}} \\
= g_1^2 \frac{1}{\sqrt{d\pi}}^{2(\kappa - \varsigma)} \frac{m^{-2(\kappa+1-s)N}}{1 - m^{-2(\kappa+1-s)}}
\]

where \(J_\xi \geq N\) is used in the first inequality. Then it follows from (2.4) and (2.5) that

\[
\|B_{1,N}\|_{L^\infty(\mathbb{R}^d)} \leq g_1^2 \frac{1}{\sqrt{d\pi}}^{2(\kappa - \varsigma)} \frac{m^{-2(\kappa+1-s)N}}{1 - m^{-2(\kappa+1-s)}}.
\]

We next estimate \(\|B_{2,N}\|_{L^\infty(\mathbb{R}^d)}\). Denote the sphere \(\{\xi \in \mathbb{R}^d : \|\xi\|_2 \leq r\}\) by \(S_d(r)\). Clearly,

\[
(M^T)^j \mathbb{T}^d = S_d(m^j \pi).
\]

As mentioned previously we can prove that \(\xi \notin \mathbb{R}^d \setminus (M^T)^j \mathbb{T}^d\) if \(j \geq \widehat{J}_\xi\), where \(\widehat{J}_\xi := \max \{0, \left\lfloor \log_m \frac{\|\xi\|_2}{\pi} \right\rfloor \}\). Consequently, \(B_{2,N}(\xi) = 0\) when \(\|\xi\|_2 < \pi m^N\). When \(\|\xi\|_2 \geq \pi m^N\), \(\|B_{2,N}(\xi)\|\) is estimated as follows:

\[
B_{2,N}(\xi) = (1 + \|\xi\|_2)^{-\varsigma} \sum_{j=N}^{\widehat{J}_\xi - 1} m^{2js} (1 + \|M^{-T})^{-j} \xi\|_2)^2 \chi_{\mathbb{R}^d \setminus \Lambda_j}(\xi)
\]

\[
\leq 2^\varsigma (1 + \|\xi\|_2)^{-\varsigma} \sum_{j=N}^{\widehat{J}_\xi - 1} m^{2js} (1 + m^{-2j\varsigma}) \|\xi\|_2^{2\varsigma}.
\]
\[
\begin{align*}
    \frac{2^s m^{2N_s} m^{2s(\tilde{\ell}-N)}}{\|\xi\|_2^{2s} m^{2s} - 1} + 2^s \frac{\|\xi\|_2^{2s}}{(1 + \|\xi\|_2^s)^{s}} \sum_{j=N}^{\tilde{\ell}-1} m^{-2j(s-s)} \\
    \leq \frac{2^s m^{2s\tilde{\ell}}}{\|\xi\|_2^{2s} m^{2s} - 1} + 2^{s+1} \frac{\|\xi\|_2^{2s}}{(1 + \|\xi\|_2^s)^{s}} m^{-2N(s-s)} \\
    \leq \frac{2^s m^{2s\tilde{\ell}}}{\|\xi\|_2^{2s} m^{2s} - 1} + 2^{s+1} \frac{m^{-2N(s-s)}}{1 - m^{-2(s-s)}} \\
    \leq \left( \frac{2^s m^{2s}}{\pi^{2s}} \frac{1}{m^{2s} - 1} + \frac{2^{s+1}}{1 - m^{-2(s-s)}} \right) m^{-2N(s-s)}, \quad (2.7)
\end{align*}
\]

where we use

\[(1 + |x|^2)^s \leq (2 \max\{1, |x|^2\})^s \leq 2^s (1 + |x|^2)\]

and

\[\tilde{\ell} \leq 1 + \log_m \frac{\|\xi\|_2}{\pi}\]

in the first and last inequalities, respectively. Define

\[H_{\tilde{\psi}}(\varsigma, s) := 2 \max\{1, \|\tilde{\psi} - \varsigma\|_{L^\infty(\mathbb{R}^d)} \} \times \max\left\{ \frac{g^2(\sqrt{\pi})^{2(\varsigma+1)}}{1 - m^{-2(\varsigma+1-s)}}, \left( \frac{2^s m^{2s}}{\pi^{2s}} \frac{1}{m^{2s} - 1} + \frac{2^{s+1}}{1 - m^{-2(s-s)}} \right) \right\} < \infty.\]

Now the proof can be concluded by (2.3), (2.6) and (2.7).

\[\square\]

**Remark 2.2.** The condition \(\varsigma - s > 0\) guarantees that the series in (2.5) and (2.7) converge.

Based on Lemma 2.1, we estimate the approximation error \(\|I - S^N_\phi f\|_{H^s(\mathbb{R}^d)}\) in the following theorem, where \(S^N_\phi\) is defined in (1.10).

**Theorem 2.3.** Suppose that \(X^s(\phi; \psi_1, \psi_2, \ldots, \psi_L)\) and \(X^{-s}(\tilde{\phi}, \tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_L)\) form a pair of dual M-framelet systems for \((H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))\). Moreover, \(\phi \in H^s(\mathbb{R}^d)\) and \(\tilde{\psi}\) has \(\kappa + 1\) vanishing moments, where \(0 < s < \varsigma < \kappa + 1, \kappa \in \mathbb{N}_0\) and \(\ell = 1, 2, \ldots, L\). Then there exists a positive constant \(C(s, \varsigma)\) such that for any \(f \in H^s(\mathbb{R}^d)\),

\[\|I - S^N_\phi f\|_{H^s(\mathbb{R}^d)} \leq C(s, \varsigma) \|f\|_{H^s(\mathbb{R}^d)} m^{-N(s-s)/2}. \quad (2.8)\]

**Proof.** Suppose that \(\psi\) and \(\tilde{\psi}\) are defined by \(\tilde{\psi}(M^T \cdot) = \tilde{\psi}(\cdot)\phi(\cdot)\) and \(\tilde{\psi}(M^T \cdot) = \tilde{\psi}(\cdot)\phi(\cdot)\), respectively. Denote by \(\ell^2(\mathbb{Z}^d \times \mathbb{N}_0 \times \mathbb{Z}^d \times \mathbb{L})\) the space of square summable sequences supported on \(\mathbb{Z}^d \times \mathbb{N}_0 \times \mathbb{Z}^d \times \mathbb{L}\). Let

\[P : H^s(\mathbb{R}^d) \rightarrow \ell^2(\mathbb{Z}^d \times \mathbb{N}_0 \times \mathbb{Z}^d \times \mathbb{L})\]

be the analysis operator of \(X^s(\phi; \psi_1, \psi_2, \ldots, \psi_L)\), i.e., for any \(g \in H^s(\mathbb{R}^d)\) the mapping \(P\) is defined as

\[P(g) := \{ (g, \phi_{0,n})_{H^s(\mathbb{R}^d),} (g, \psi_{j,k}^{\ell,s})_{H^s(\mathbb{R}^d)} : n, k \in \mathbb{Z}^d, j \in \mathbb{N}_0, \ell = 1, \ldots, L \}.\]

By (1.7), \(P\) is a bounded operator from \(H^s(\mathbb{R}^d)\) to \(\ell^2(\mathbb{Z}^d \times \mathbb{N}_0 \times \mathbb{Z}^d \times \mathbb{L})\). Then

\[\|P g\|_2 \leq \|P\| \|g\|_{H^s(\mathbb{R}^d)}. \quad (2.9)\]

By the isomorphic map \(\theta_s : H^s(\mathbb{R}^d) \rightarrow H^{-s}(\mathbb{R}^d)\),

\[\theta_s g(\xi) = \tilde{g}(\xi)(1 + \|\xi\|_2^s)^s, \quad \forall g \in H^s(\mathbb{R}^d),\]

it is easy to prove that (1.7) holds if and only if

\[C_1 \|\tilde{g}\|^2_{H^{-s}(\mathbb{R}^d)} \leq \sum_{k \in \mathbb{Z}^d} |\langle \tilde{g}, \phi_{0,k} \rangle|^2 + \sum_{\ell=1}^L \sum_{j \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}^d} |\langle \tilde{g}, \psi_{j,k}^{\ell,s} \rangle|^2 \leq C_2 \|\tilde{g}\|^2_{H^{-s}(\mathbb{R}^d)} \quad (2.10)\]
for any $\tilde{g} \in H^{-s}(\mathbb{R}^d)$. By (2.10) and [37, Theorem 2.1] we have
\[ \|P\| \leq h(s, \varsigma), \quad (2.11) \]
where
\[ h(s, \varsigma) = \left( L\left[1 + \frac{m^d}{(2\pi)^d} \left( \frac{m^2(s+\varsigma)^2 \varsigma}{m^{2(s-\varsigma)} - 1} + \frac{2\varsigma}{1 - m^{-2\varsigma}} \right) \right] \max_{1 \leq l \leq L} \left\{ \|\hat{d}^l\|_{L^\infty(\mathbb{T}^d)} \right\} \right)^{1/2}. \]

Next, we compute $P^*$, the adjoint operator of $P$. For any $g \in H^s(\mathbb{R}^d)$ and $c \in L^2(\mathbb{Z}^d \times \mathbb{N}_0 \times \mathbb{Z}^d \times L)$ such that its elements are $c_k$ and $c_{j,k,l,-s}$, we have
\[ \langle P^* c, g \rangle_{H^s(\mathbb{R}^d)} = \langle c, Pg \rangle_{\ell^2} = \sum_{k \in \mathbb{Z}^d} c_k \langle g, \phi_{0,k} \rangle_{H^s(\mathbb{R}^d)} + \sum_{\ell=1}^L \sum_{j \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}^d} c_{j,k,l,-s} \langle g, \psi_{j,k}^{\ell,s} \rangle_{H^s(\mathbb{R}^d)}. \]

Therefore,
\[ P^* c = \sum_{k \in \mathbb{Z}^d} c_k \phi_{0,k} + \sum_{\ell=1}^L \sum_{j \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}^d} c_{j,k,l,-s} \psi_{j,k}^{\ell,s}. \]

From $\|P^*\| = \|P\|$, we arrive at
\[ \|P^*(c)\|_{H^s(\mathbb{R}^d)} \leq \|P\|\|c\|_{\ell^2}. \quad (2.12) \]

For any $f \in H^s(\mathbb{R}^d)$, it follows from (2.12), (2.11) and (2.1) that
\[ \left\| \sum_{\ell=1}^L \sum_{j=N}^\infty \sum_{k \in \mathbb{Z}^d} \langle f, \psi_{j,k}^{\ell,-s} \rangle \langle \psi_{j,k}^{\ell,s} \rangle \right\|_{H^s(\mathbb{R}^d)} \leq \|P\| \left( \sum_{\ell=1}^L \sum_{j=N}^\infty \sum_{k \in \mathbb{Z}^d} |\langle f, \psi_{j,k}^{\ell,s} \rangle|^2 \right)^{1/2} \leq h(s, \varsigma) \sqrt{H(s, \varsigma)} m^{-N(s-\varsigma)/2} \|f\|_{H^s(\mathbb{R}^d)}, \quad (2.13) \]

where $H(\varsigma, s) := \sum_{\ell=1}^L H(\psi_{\ell}(\varsigma, s)$ with $H(\psi_{\ell}(\varsigma, s)$ being given by Lemma 2.1 and (2.1). Now we define
\[ C(s, \varsigma) := h(s, \varsigma) \sqrt{H(\varsigma, s)} \]
to conclude the proof.

**Remark 2.4.** For a target function $f \in H^s(\mathbb{R}^d)$, the approximation error $O(m^{-N(s-\varsigma)/2})$ in (2.8) holds provided that $f$ also lies in $H^s(\mathbb{R}^d)$ for some $\varsigma > s$. As mentioned in Remark 1.1(ii), our aim in this paper is to construct the approximation to all the functions in $H^s(\mathbb{R}^d)$ for any $\varsigma > d/2$. In fact, the above requirement $\varsigma > s$ will not bring any negative impact on our aim. To make this point, we sketch the procedures for constructing the approximation in Theorem 2.3. For any $f \in H^s(\mathbb{R}^d)$, we choose $d/2 < s < \varsigma$ and construct a pair of dual $M$-framelet systems $X^s(\phi; \psi^1, \psi^2, \ldots, \psi^L)$ and $X^{-s}(\phi; \tilde{\psi}^1, \tilde{\psi}^2, \ldots, \tilde{\psi}^L)$ of $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$. Then $f$ can be approximated by $S^N_\phi f$ in (2.8).

**Comparison 2.5.** (i) There are some bounds for the error $\| (I - S^N_\phi) f \|_{L^2}$ in the literature such as in [34,36]. When $f$ belongs to the Schwartz class $S(\mathbb{R}^d)$, the above error was estimated in [34, Theorem 16]. For the target $f$ satisfying
\[ |\tilde{f}(\xi)| \leq C(1 + \|\xi\|_2)^{-\frac{d+\alpha}{2}} \quad \text{for every } \xi \in \mathbb{R}^d, \quad \alpha > 0, \quad (2.14) \]
the error was estimated in [36]. Clearly, there are many functions sitting in $L^2(\mathbb{R}^d) \backslash S(\mathbb{R}^d)$ or in $H^s(\mathbb{R}^d) \backslash S(\mathbb{R}^d)$. Moreover, there are also many functions in $H^s(\mathbb{R}^d)$ not satisfying (2.14). For example, the following class of generalized sinc functions (see [8,38]), given by the Fourier transform
\[ \hat{f}(\xi) = \sum_{n \in \mathbb{Z}} e^{i\alpha n} \chi_{[\,n,n+\alpha]}(\xi), \quad (2.15) \]
where \( \lambda_n > 0 \) and \( 0 < \epsilon_n < \min \{ e^{-2\lambda_n}, 1 \} \), does not satisfy (2.14) for any \( \alpha \) and \( C_0 \). Contrary to [34,36], Theorem 2.3 holds for the entire space \( H^c(\mathbb{R}^d) \) with \( \zeta > 0 \). Therefore, Theorem 2.3 is not the trivial generalization of [34,36].

(ii) For the case of \( d = 1 \), the Sobolev seminorm

\[
|\{1 - S^N_\phi \}f|_{W^2_2(\mathbb{R})} := \left( \int_{\mathbb{R}} |\hat{f}(\xi) - \tilde{S}^N_\phi \hat{f}(\xi)|^2 |\xi|^2 d\xi \right)^{1/2}
\]

was estimated in [25, Corollary 4.7.3]. Clearly, \( |f|_{W^2_2} \leq 2\pi \|f\|_{H^c} \) but the two norms are not equivalent. Therefore for \( d = 1 \), Theorem 2.3 is not the trivial generalization of [25, Corollary 4.7.3].

3 Approximation by the shift-perturbed system in \( H^s(\mathbb{R}^d) \) when the perturbation sequence is not necessarily in \( \ell^\alpha(\mathbb{Z}^d) \)

The complete set of representatives of distinctive cosets of the quotient group \( \{(MT)^{-1}\mathbb{Z}^d\}/\mathbb{Z}^d \) is denoted by \( \Gamma_{M^r} := \{ \gamma_0, \ldots, \gamma_{m^d-1} \} \) with \( \gamma_0 = 0 \). Recall that the mixed extension principle (MEP) is an efficient algorithm (see [15,16,27]) for designing dual framelet systems. In this section, we will use MEP to design dual systems, we will construct the approximations to the functions in \( H^c(\mathbb{R}^d) \) with \( \zeta > s > d/2 \). Since the dual systems are derived from MEP, the mask symbols \( \{ \tilde{b}^1, \ldots, \tilde{b}^L \} \) of \( \{ \psi^1, \ldots, \psi^L \} \), and \( \{ \tilde{b}^1, \ldots, \tilde{b}^L \} \) of \( \{ \tilde{\psi}^1, \ldots, \tilde{\psi}^L \} \) satisfy

\[
\begin{align*}
\sum_{l=1}^L \tilde{b}^l \tilde{b}^l + \tilde{a} \tilde{a} &= 1, \\
\sum_{l=1}^L \tilde{b}^l \tilde{b}^l + \tilde{a} \tilde{a} &= 0, \quad \forall \ j \in \{1, \ldots, m^d - 1\},
\end{align*}
\]

where \( \tilde{a} \) and \( \tilde{a} \) are the mask symbols of \( \phi \) and \( \tilde{\phi} \), respectively. It follows from (3.1) that

\[
S^N_\phi f = \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\phi}^\perp_{N,k} \rangle \phi^\perp_{N,k},
\]

where

\[
\phi^\perp_{N,k} := m^{N(d/2-s)} \phi(M^N \cdot -k) \quad \text{and} \quad \tilde{\phi}^\perp_{N,k} := m^{N(d/2+s)} \tilde{\phi}(M^N \cdot -k),
\]

i.e., \( S^N_\phi \) can be reexpressed by the system \( \{ \tilde{\phi}^\perp_{N,k}, \phi^\perp_{N,k} \}_{k \in \mathbb{Z}^d} \). By (2.8), when the scale \( N \) is sufficiently large, \( f \) can be well approximated by using the inner products \( \langle f, \phi^\perp_{N,k} \rangle, k \in \mathbb{Z}^d \). In what follows we introduce the perturbed version of \( S^N_\phi f \). Motivated by Hamm [19], suppose that the perturbation sequence

\[
\varepsilon := \{ \varepsilon_k : k \in \mathbb{Z}^d \} \subseteq \mathbb{R}^d
\]

satisfies

\[
\left( \sum_{k \in \mathbb{Z}^d} \|\varepsilon_k - \lambda\|^2_2 \right)^{1/\alpha} < \infty
\]

for some \( \lambda \in \mathbb{R}^d \) and \( \alpha > 0 \). Clearly, if \( \lambda \neq 0 \) then \( \varepsilon \) does not lie in \( \ell^{\alpha}(\mathbb{Z}^d) \) defined by

\[
\ell^{\alpha}(\mathbb{Z}^d) := \left\{ \{x_k\}_{k \in \mathbb{Z}^d} \subseteq \mathbb{R}^d : \|\{x_k\}\|_{\ell^{\alpha}(\mathbb{Z}^d)} = \left( \sum_{k \in \mathbb{Z}^d} \|x_k\|^2_2 \right)^{1/\alpha} < \infty \right\}.
\]
Now define the perturbed version $S^N_{\phi, \varepsilon}$ of $S^N_{\phi} : H^s(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by
\[
S^N_{\phi, \varepsilon} f = \sum_{k \in \mathbb{Z}^d} \langle f, m^{N(d/2+s)}_{\phi}(M^N \cdot -k - \varepsilon_k)\rangle \phi_{N,k}, \quad \forall f \in H^s(\mathbb{R}^d).
\] (3.6)

Our main task in the present section is to establish the approximation error $\|(I - S^N_{\phi, \varepsilon})f\|_{L^2}$ for any function $f \in H^s(\mathbb{R}^d)$ with $\zeta > s > d/2$. The main result is stated in the following theorem.

**Theorem 3.1.** Suppose that $\zeta > d/2$ and $\phi \in H^s(\mathbb{R}^d)$ is $M$-refinable. Choose $d/2 < s < \zeta$ and construct an $M$-refinable function $\tilde{\phi} \in H^{-s}(\mathbb{R}^d)$. Moreover, suppose that the sequence $\varepsilon = \{\varepsilon_k\}_{k \in \mathbb{Z}^d}$ satisfies (3.4) for some $\lambda \in \mathbb{R}^d$ and $0 < \alpha < \min\{2s - d, 2\}$, and
\[
N \geq \frac{2s + 2 - \alpha}{2 - \alpha} \log_d d
\]
is arbitrary. Then there exists a constant $C_3(s, \zeta, \alpha, d) > 0$ such that for any $f \in H^s(\mathbb{R}^d)$,
\[
\|(I - S^N_{\phi, \varepsilon})f\|_{L^2} \leq \|(I - S^N_{\phi})f\|_{L^2} + C_3(s, \zeta, d, \alpha)\|f\|_{H^s(\mathbb{R}^d)} m^{-N\zeta} \|\lambda\|_2^2 + \|\varepsilon - \lambda\|_{\text{max}},
\] (3.7)
where
\[
\zeta = \min \left\{ 1, \zeta - s, \frac{4s + (\alpha - 2)d}{2s - \alpha + 2} + d \right\}.
\] (3.8)
and
\[
\|\varepsilon - \lambda\|_{\text{max}} = \max \{ \|\varepsilon_k - \lambda\|_{\ell^2(\mathbb{Z}^d)}, \|\varepsilon_k - \lambda\|_{\ell^2(\mathbb{Z}^d)}^{\alpha/2} \}.
\]

**Proof.** The proof is given in Subsection 3.2. \(\square\)

### 3.1 Auxiliary results for proving Theorem 3.1

In this subsection we present some auxiliary results which will be helpful for proving Theorem 3.1.

**Lemma 3.2.** Let $J \geq \log_d d$ and $s > d/2$. Then
\[
\sum_{j \in \mathbb{Z}^d, \|j\|_2 \geq m^J} \|j\|_2^{-2s} \leq \hat{C}(s,d) m^{-J(2s-d)},
\] (3.9)
where
\[
\hat{C}(s,d) := 2^{d-1} d^{-s} \left[ \left( \frac{1}{2s-d} + 1 \right) \sum_{n=1}^{d-1} \left( \frac{d-1}{n} \right) \prod_{l=1}^{n} \frac{1}{2s-l} + \frac{1}{2s-1} + 1 \right].
\]

**Proof.** The proof is given in Appendix A. \(\square\)

As mentioned previously the perturbation sequence $\varepsilon$ satisfying (3.4) does not necessarily sit in $\ell^s(\mathbb{Z}^d)$. When it sits in $\ell^s(\mathbb{Z}^d)$, we establish the approximation error $\|(I - S^N_{\phi, \varepsilon})f\|_{L^2}$ in the following lemma, which will be used in the proof of Theorem 3.1.

**Lemma 3.3.** Suppose that both $\phi \in H^s(\mathbb{R}^d)$ and $\tilde{\phi} \in H^{-s}(\mathbb{R}^d)$ are $M$-refinable, where $s > d/2$ and $\|\tilde{\phi}\|_{L^\infty(\mathbb{R}^d)} < \infty$. Moreover, suppose that $\varepsilon_k := \{\varepsilon_k\}_{k \in \mathbb{Z}^d}$ lies in $\ell^s(\mathbb{Z}^d)$ with $0 < \alpha < \min\{2s - d, 2\}$. Then there exists a positive constant $C_2(s, \alpha, d)$ such that for any $f \in H^s(\mathbb{R}^d)$ and $N \geq 2s + 2 - \alpha \log_m d$,
\[
\|(I - S^N_{\phi, \varepsilon})f\|_{L^2} \leq \|(I - S^N_{\phi})f\|_{L^2} + C_2(s, \alpha, d)\|f\|_{H^s(\mathbb{R}^d)} \|\varepsilon\|_{\max} m^{-N\frac{(\alpha + 2)d}{2\alpha + 2} + d/2},
\] (3.10)
where
\[
\|\varepsilon\|_{\max} := \max \{ \|\varepsilon\|_{\ell^s(\mathbb{Z}^d)}, \|\varepsilon\|_{\ell^s(\mathbb{Z}^d)}^{\alpha/2} \}.
\] (3.11)
Proof. We only need to prove that
\[
\|(S_N^{\alpha} - S_0^\alpha)f\|_{L^2} \leq C_2(s, \alpha, d)\|f\|_{H^s(\mathbb{R}^d)}\|\eta\|_{max}m^{-N(\frac{4s+(\alpha-2)d}{2s-\alpha}+d)/2}.
\] (3.12)
The above inequality will be proved in Appendix B. \qed

Lemma 3.4. Suppose that both \(\phi \in H^s(\mathbb{R}^d)\) and \(\tilde{\phi} \in H^{-s}(\mathbb{R}^d)\) are M-refinable, where \(\varsigma > s > d/2\). Moreover, a sequence \(\eta \in C_0^\infty(\mathbb{R}^d)\) with \(0 < \alpha < \min\{2s - d, 2\}\). Then there exists a constant \(C_2(s, \varsigma, \alpha, d) > 0\) such that for any \(N \geq \frac{s+2-\alpha}{2s-\alpha+2}\log_m d, \lambda \in \mathbb{R}^d\) and every \(f \in H^s(\mathbb{R}^d)\),
\[
\|(I - S_{\phi,2}^N)(f - f(\cdot + M^{-N}\lambda))\|_{L^2} \leq \hat{C}_2(s, \varsigma, \alpha, d)\|f\|_{H^s(\mathbb{R}^d)}m^{-N\theta(s, \varsigma, \alpha, \varsigma)}\|\lambda\|_2^2,
\] (3.13)
where \(\theta(s, \varsigma, \alpha, \varsigma) := \varsigma + \min\{(\varsigma - s)/2, (4s+(\alpha-2)d)/(2s-\alpha+2)\} + d/2\) with \(\varsigma\) defined in (3.8).

Proof. We first estimate \(\|(I - S_{\phi,2}^N)(f - f(\cdot + M^{-N}\lambda))\|_{L^2}\) as follows:
\[
\|(I - S_{\phi,2}^N)(f - f(\cdot + M^{-N}\lambda))\|_{L^2} \\
\leq \|(I - S_\phi^N)(f - f(\cdot + M^{-N}\lambda))\|_{L^2} + \|(S_{\phi,2}^N - S_\phi^N)(f - f(\cdot + M^{-N}\lambda))\|_{L^2} \\
\leq \|(I - S_\phi^N)(f - f(\cdot + M^{-N}\lambda))\|_{L^2} + C_2(s, \alpha, d)\|\eta\|_{max}m^{-N(\frac{4s+(\alpha-2)d}{2s-\alpha}+d)/2}\|f - f(\cdot + M^{-N}\lambda))\|_{H^s(\mathbb{R}^d)},
\] (3.14)
where the first and second inequalities are derived from the triangle inequality, Lemma 3.3 and (3.12), respectively.

Invoking (2.8), we get
\[
\|(I - S_{\phi,2}^N)(f - f(\cdot + M^{-N}\lambda))\|_{L^2} \leq C(s, \varsigma)m^{-N(s-s)/2}\|f - f(\cdot + M^{-N}\lambda))\|_{H^s(\mathbb{R}^d)}.
\] (3.15)
On the other hand,
\[
\|f - f(\cdot + M^{-N}\lambda))\|_{H^s(\mathbb{R}^d)} \\
= \left[\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\mathcal{F}(\xi)(1 - e^{i(MT)^{-N}\lambda}\xi)|^2(1 + \|\xi\|_2^2)^{s}d\xi\right]^{1/2} \\
\leq \left[\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} 4|\sin((MT)^{-N}\lambda \cdot \xi/2)|^{2}\mathcal{F}(\xi)^2(1 + \|\xi\|_2^2)^{s}d\xi\right]^{1/2} \\
\leq \left[\frac{2^{2-2s}}{(2\pi)^d} \|\mathcal{F}(\xi)\|_{L^2}^{2s} \int_{\mathbb{R}^d} \|\xi\|_2^2 \mathcal{F}(\xi)^2(1 + \|\xi\|_2^2)^{s}d\xi\right]^{1/2} \\
\leq 2^{1-s}m^{-N\varsigma}\|\lambda\|_2\|f\|_{H^s(\mathbb{R}^d)},
\] (3.16)
where the first and third inequalities are derived from \(\varsigma \leq 1\) and \(\varsigma \leq \varsigma - s\), respectively. Now by (3.14)–(3.16) we have
\[
\|(I - S_{\phi,2}^N)(f - f(\cdot + M^{-N}\lambda))\|_{L^2} \\
\leq (C(s, \varsigma)m^{-N(s-s)/2} + C_2(s, \alpha, d)\|\eta\|_{max}m^{-N(\frac{4s+(\alpha-2)d}{2s-\alpha}+d)/2}2^{1-s}m^{-N\varsigma}\|\lambda\|_2\|f\|_{H^s(\mathbb{R}^d)} \\
\leq 2^{1-s}m\max\{C(s, \varsigma), C_2(s, \alpha, d)\}\|f\|_{H^s(\mathbb{R}^d)}m^{-N\theta(s, \varsigma, \alpha, \varsigma)}\|\lambda\|_2^2.
\] (3.17)
Define \(\hat{C}_2(s, \varsigma, \alpha, d) := 2^{1-s}\max\{C(s, \varsigma), C_2(s, \alpha, d)\}\) to conclude the proof of (3.13). \qed

3.2 Proof of Theorem 3.1

By Parseval identity, we have
\[
(f, m^{Nd/2}\tilde{\phi}(M^{N} \cdot -k - \varepsilon_k + \lambda)) = m^{Nd/2}\tilde{\phi}(M^{N} \cdot -k - \varepsilon_k))
\]
Moreover, the perturbation sequence \( \phi \) is defined as follows:

\[
\phi(s, \lambda, d) = \lambda \epsilon^{-1} \sum_{k \in \mathbb{Z}^d} \phi(M^s \cdot (k - \epsilon_k + \lambda)).
\]

By using (3.18) and the triangle inequality, the error \( \| (I - S_{\phi_{\#}}^N) f \|_{L^2} \) is estimated as follows:

\[
\begin{align*}
\| (I - S_{\phi_{\#}}^N) f \|_{L^2} & \leq \| (I - S_{\phi_{\#}}^N) f \|_{L^2} + \| S_{\phi_{\#}}^N \|_{L^2} (f - f(\cdot + M^{-N}\lambda)) \|_{L^2} \\
& \leq \| (I - S_{\phi_{\#}}^N) f \|_{L^2} + \| (I - S_{\phi_{\#}}^N)(f - f(\cdot + M^{-N}\lambda)) \|_{L^2} + \| f - f(\cdot + M^{-N}\lambda) \|_{L^2} \\
& =: I_1 + I_2,
\end{align*}
\]

where

\[
I_1 = \| (I - S_{\phi_{\#}}^N) f \|_{L^2}, \quad I_2 = \| (I - S_{\phi_{\#}}^N)(f - f(\cdot + M^{-N}\lambda)) \|_{L^2} + \| f - f(\cdot + M^{-N}\lambda) \|_{L^2}.
\]

Since \( \xi - \lambda \in L^\alpha(\mathbb{Z}^d) \), it follows from Lemma 3.4, (3.13) and (3.16) that

\[
I_2 \leq \tilde{C}_2(s, \xi, \alpha, d) \| f \|_{H^s(\mathbb{R}^d)} m^{-N\theta(s, \xi, \alpha, \zeta)} \| \lambda \|_{\mathbb{Z}^d}^\zeta + 2^1 \zeta \| \lambda \|_{\mathbb{Z}^d}^\zeta m^{-N\xi} \| f \|_{H^s(\mathbb{R}^d)}.
\]

By Lemma 3.3, (3.10) and \( \zeta \leq \left( \frac{4s + (\alpha - 2)d}{2s - \alpha + 2} \right), \) we have

\[
I_1 \leq \| (I - S_{\phi_{\#}}^N) f \|_{L^2} + C_2(s, \alpha, d) \| \xi - \lambda \|_{\mathbb{Z}^d} m^{-N\xi} \| f \|_{H^s(\mathbb{R}^d)}.
\]

Now it follows from (3.19)–(3.21) that

\[
\begin{align*}
\| (I - S_{\phi_{\#}}^N) f \|_{L^2} & \leq I_1 + I_2 \\
& \leq \| (I - S_{\phi_{\#}}^N) f \|_{L^2} + \| (\tilde{C}_2(s, \xi, \alpha, d) + 2^1 \zeta) \| \lambda \|_{\mathbb{Z}^d}^\zeta + C_2(s, \alpha, d) \| \xi - \lambda \|_{\mathbb{Z}^d} m^{-N\xi} \| f \|_{H^s(\mathbb{R}^d)}.
\end{align*}
\]

Define

\[
C_3(s, \xi, \alpha, d) := \max \{ \tilde{C}_2(s, \xi, \alpha, d) + 2^1 \zeta, C_2(s, \alpha, d) \}
\]

to conclude the proof.

4 Approximations to functions in Sobolev spaces by nonuniform sampling

This section starts with the definition of the sum rule of a refinable function. Let the \( M \)-refinable function \( \phi \in H^s(\mathbb{R}^d) \) be defined via the \( M \)-refinement equation: \( \tilde{\phi}(MT) = \tilde{a}(\cdot)\tilde{\phi}(\cdot) \). We say that \( \phi \) has \( \kappa + 1 \) sum rules if \( \tilde{a}(\xi + j) = O(\| \xi \|_2) \) as \( \xi \to 0 \), where any \( j \in \{ (MT)^{-1}\mathbb{Z}^d/\mathbb{Z}^d \} \) with \( j \neq 0 \) is as in (3.1). For the relationship between the sum rule of \( \phi \) and the approximation order of the shift-invariant space generated from \( \phi \), readers can refer to [25].

With the help of Theorem 3.1 we establish the approximation in the following theorem, which states that any function in \( H^s(\mathbb{R}^d) \) (where \( s > d/2 \)) can be stably reconstructed by its nonuniform sampling.

**Theorem 4.1.** Suppose that \( \phi \in H^s(\mathbb{R}^d) \) is \( M \)-refinable and has \( \kappa + 1 \) sum rules where \( d/2 < s < \kappa + 1 \). Moreover, the perturbation sequence \( \xi \) is as in (3.4) for some \( \lambda \in \mathbb{R}^d \) and \( 0 < \alpha < \min\{ 2s - d, 2 \} \), where \( d/2 < s < \zeta \), and

\[
N \geq \frac{2s + 2 - \alpha}{2 - \alpha} \log_m d
\]

is arbitrary. Then there exists a positive constant \( C_0(s, \xi, \alpha, d) \) such that for any \( f \in H^s(\mathbb{R}^d) \),

\[
\left\| f - \sum_{k \in \mathbb{Z}^d} f(M^{-N}(k + \epsilon_k))\phi(M^{-N}\cdot k) \right\|_{L^2}
\]
\[ \leq C_0(s, \varsigma, \alpha, d) \| f \|_{H^s(\mathbb{R}^d)} [m^{-\varsigma(s-N)} + m^{-N\varsigma} (\| \lambda \|_2^\varsigma + \| \| - \lambda \|_{\max}^\varsigma)], \tag{4.1} \]

where \( \varsigma \) is defined in (3.8), and as in Theorem 3.1,
\[ \| \| - \lambda \|_{\max} = \max \{ \{ \| \delta - \lambda \| \}_{k \in \mathbb{Z}^d} \}, \| \{ \| \delta - \lambda \| \}_{k \in \mathbb{Z}^d} \} \{ \| \{ \| \delta - \lambda \| \}_{k \in \mathbb{Z}^d} \} \}. \]

**Proof.** Define a distribution \( \Delta \) on \( \mathbb{R}^d \) by
\[ \Delta(x_1, x_2, \ldots, x_d) := \delta(x_1) \otimes \delta(x_2) \otimes \cdots \otimes \delta(x_d), \tag{4.2} \]
where \( \delta \) is the delta distribution on \( \mathbb{R} \), and \( \otimes \) is the tensor product. It follows from \( \delta \equiv 1 \) that \( \Delta \in H^{-\mu}(\mathbb{R}^d) \) is \( M \)-refinable for any \( \mu > d/2 \). Since \( \phi \) has \( \kappa + 1 \) sum rules, by MEP [37, Algorithm 4.1] we can design a pair of dual \( M \)-framelet systems \( X^*(\phi; \psi^1, \psi^2, \ldots, \psi^m) \) and \( X^{-s}(\Delta; \psi^1, \psi^2, \ldots, \psi^m) \) such that \( \psi^1, \psi^2, \ldots, \psi^m \) have \( \kappa + 1 \) vanishing moments. By the sampling property of \( \delta \), we have
\[ \langle f, \Delta \rangle = f(0). \tag{4.3} \]

By combining (3.2) and (4.3), the operators \( S^N_\phi \) and \( S^N_{\psi_{\mathcal{L}}} \) defined in (1.10) and (3.6) can be expressed by
\[ S^N_\phi f = \sum_{k \in \mathbb{Z}^d} \{ f(M^{-N} k) \} \phi(M^N \cdot k), \quad S^N_{\psi_{\mathcal{L}}} f = \sum_{k \in \mathbb{Z}^d} \{ f(M^{-N} (k + \varepsilon_k)) \} \phi(M^N \cdot k). \tag{4.4} \]

It follows from Theorem 2.3, (2.8), Theorem 3.1 and (3.7) that
\[ \left\| f - \sum_{k \in \mathbb{Z}^d} \{ f(M^{-N} (k + \varepsilon_k)) \} \phi(M^N \cdot k) \right\|_{L^2} \leq \| f \|_{H^s(\mathbb{R}^d)} C(s, \varsigma, m^{-\varsigma(s-N)} + C_3(s, \varsigma, \alpha, d) \| f \|_{H^s(\mathbb{R}^d)} m^{-N\varsigma} (\| \lambda \|_2^\varsigma + \| \| - \lambda \|_{\max}^\varsigma)], \]
\[ \leq C_0(s, \varsigma, \alpha, d) \| f \|_{H^s(\mathbb{R}^d)} [m^{-\varsigma(s-N)} + m^{-N\varsigma} (\| \lambda \|_2^\varsigma + \| \| - \lambda \|_{\max}^\varsigma)], \]
where \( C_0(s, \varsigma, \alpha, d) := \max \{ C(s, \varsigma), C_3(s, \varsigma, \alpha, d) \}. \]

**Remark 4.2.** Suppose that the perturbation sequence \( \varepsilon^{(N)} = \{ \varepsilon_k^{(N)} \}_{k \in \mathbb{Z}^d} \) at the scale \( N \) satisfies
\[ \left( \sum_{k \in \mathbb{Z}^d} \| \varepsilon_k^{(N)} - \lambda^{(N)} \|_2^\varsigma \right)^{1/\alpha} < \infty \]
for \( \lambda^{(N)} \in \mathbb{R}^d \). If
\[ \| \lambda^{(N)} \|_2^\varsigma + \| \varepsilon^{(N)} - \lambda^{(N)} \|_{\max} = o(m^N\varsigma), \tag{4.5} \]
then it follows from Theorem 4.1 and (4.1) that
\[ \lim_{N \to \infty} \sum_{k \in \mathbb{Z}^d} \{ f(M^{-N} (k + \varepsilon_k)) \} \phi(M^N \cdot k) = f. \]

### 5 Comparison with prior work: The scope of application, sampling flexibility and estimation techniques

The sampling-based approximation in Theorem 4.1 and (4.1) enjoys the two required properties: (i) the scope of application: the entire space \( H^s(\mathbb{R}^d) \) with \( \varsigma > d/2 \); (ii) flexibility of sampling: the approximation is conducted by the nonuniform sampling \( \{ f(M^{-N} (k + \varepsilon_k)) \} \), where the sequence \( \varepsilon = \{ \varepsilon_k \} \) just needs to satisfy (3.4) and is not necessary in \( \ell^\alpha(\mathbb{Z}^d) \).

There are some papers addressing the approximation in \( H^s(\mathbb{R}^d) \) with \( s > d/2 \) (see, for example, [1, 5, 6, 13, 14, 17, 19, 22], [29, 30, 35, 36, 44, 53] and the references therein). In this section, we will make comparisons between the results in the present paper and the ones in the literature on the aspects of the scope of application, flexibility of sampling and estimation techniques.
5.1 Comparison on the applicable scope

There are many approximations to smooth functions in $H^s(\mathbb{R}^d)$ ($s > d/2$) in the literature such as [5, 17, 19, 22, 29, 30, 35]. These approximations are derived from shift-invariant spaces, and they only hold for smooth functions, but not for the entire space $H^s(\mathbb{R}^d)$. Clearly, there are many functions, including

$$\phi(x_1, x_2) := B_2(x_1)B_2(x_2)$$

in $H^s(\mathbb{R}^d)$, that are not smooth, where $B_2 := \chi_{(0,1)} \ast \chi_{(0,1)}$ is the cardinal B-spline of order 2 with $\ast$ being the convolution operation. More precisely, it follows from Han [27] that $\phi(x_1, x_2)$ is in $H^\mu(\mathbb{R}^2)$ with $1 < \mu < 3/2$. In [36], we established the approximation to the function $f$ that satisfies

$$|\hat{f}(\xi)| \leq C(1 + \|\xi\|_2)^{\frac{d-a}{2}}$$

for every $\xi \in \mathbb{R}^d$. \hspace{1cm} (5.1)

By Comparison 2.5, however, there are many functions in $H^s(\mathbb{R}^d)$ that do not satisfy (5.1), i.e., the above approximations do not hold for the entire space $H^s(\mathbb{R}^d)$. Instead the approximation in Theorem 4.1 holds for the entire space $H^s(\mathbb{R}^d)$.

5.2 Comparison on the flexibility of sampling

There exist many approximations for Sobolev spaces available in the literature (see [6, 36, 44]). But the sampling points used for these approximations are uniform. Bouchot and Hamm [5, 19] recently constructed the approximation to the univariate functions in $H^s(\mathbb{R}) \cap C^n(\mathbb{R})$ by nonuniform sampling. More precisely, the nonuniform samples are the values $\{f(hx_k)\}_{k \in \mathbb{Z}}$ such that the approximation error is $O(h^n)$, where the sequence $\{x_k\}_{k \in \mathbb{Z}}$ is strictly increasing such that $\{e^{i k x}\}_{k \in \mathbb{Z}}$ constitutes a Riesz base for $L^2[-\pi, \pi]$. By [4, 19, 40, 50], a necessary condition for a sequence $\{x\}_{k \in \mathbb{Z}}$ to be a Riesz-type sequence is that there exist constants $0 < q \leq Q < \infty$ such that

$$q \leq x_{k+1} - x_k \leq Q. \hspace{1cm} (5.2)$$

A classical sufficient condition for (5.2) is Kadec’s 1/4-theorem (see [33]), which states that if $|x_k - x_{k-1}| \leq 1/4$ then $\{x_k\}_{k \in \mathbb{Z}}$ is a Riesz-type sequence. Instead our approximation in (4.1) is conducted by the nonuniform samples $\{f(m^{-N}(k + \varepsilon_k))\}_{k \in \mathbb{Z}}$ (for the case of dimension $d = 1$, the dilation matrix $M$ degenerates to $m$). Note that the sequence

$$\{x_k\}_{k \in \mathbb{Z}} := \{k + \varepsilon_k\}_{k \in \mathbb{Z}} \subseteq \mathbb{R}$$

just needs to satisfy (3.4). Clearly, many sequences satisfying (3.4) are Riesz-type ones such as

$$\{x_k\}_{k \in \mathbb{Z}} = \left\{k + a_0 + \frac{1}{25 + k^2}\right\}_{k \in \mathbb{Z}}$$

with $0 < a_0 < 1/5$. However, there are also many sequences which satisfy (3.4) but are not Riesz-type ones such as $\{x_k\}_{k \in \mathbb{Z}} = \{k + \lambda + \theta_k\}_{k \in \mathbb{Z}}$ with

$$\|\{\theta_k\}_k\|_\infty < \frac{(1 + \lambda - Q)}{2} \quad \text{and} \quad \sum_{k \in \mathbb{Z}} \theta_k^2 < \infty,$$

and $\{x_k\}_{k \in \mathbb{Z}} = \{k + \lambda + \theta_k\}_{k \in \mathbb{Z}}$ with

$$\|\{\theta_k\}_k\|_\infty < \min \left\{\frac{1 + \lambda}{2}, \frac{q - 1 - \lambda}{2}\right\} \quad \text{and} \quad \sum_{k \in \mathbb{Z}} \theta_k^2 < \infty,$$

where $Q$ and $q$ are as in (5.2), i.e., the choice of sampling sequences in the paper is quite different from those in [5, 19].
5.3 Comparison on the estimation techniques

As mentioned in Subsections 5.1 and 5.2, the approximation in the present paper is different from that in [36] on the aspects of the applicable scope and sampling flexibility. Besides the two aspects we next compare the estimation techniques of the present paper with that used in [36].

(i) The estimate of $\| (I - S_N^\phi ) f \|$. The error $\| (I - S_N^\phi ) f \|_{L^2(\mathbb{R}^d)}$ given in [36, Theorem 3.2] depends on the pointwise decay of $\hat{f}(\xi)$ as mentioned in (5.1). However, it is very difficult to exactly find out the pointwise decay of $\hat{f}(\xi)$ from the samples of $f$. Instead, our estimation in Lemma 2.1 just relies on the global convergence

$$ \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 (1 + \| \xi \|_2^2)^s d\xi < \infty, $$

and the approximation established in Theorem 2.3 holds for all the functions in $H^s(\mathbb{R}^d)$.

(ii) The error estimate of $\| (I - S_N^\phi ) f \|_{L^2}$. The approximations in the two lemmas were not established in [36].

6 Numerical simulation

In this section, numerical simulations are conducted to check the efficiency of the approximation formula in Theorem 4.1 and (4.1). The perturbation sequence $\xi$ in (4.1) is denoted by

$$ \xi = \{ \varepsilon_k := \theta_k + \lambda \}_{k \in \mathbb{Z}^d}, \quad (6.1) $$

where $\theta_k$ is random.

6.1 One-dimensional case

Let $\phi = B_3$, the cardinal B-spline of order 3 defined by $B_3 := \chi_{(0,1]} \ast \chi_{(0,1]} \ast \chi_{(0,1]}$. Moreover, let the target function

$$ f(x) := \frac{\sin x}{x} + \frac{1}{3} B_2(x - 2) - \frac{1}{6} \cos((x - 3)^2) B_2(x - 3) $$

$$ + \frac{1}{2} \cos((x - 4)^2) B_2(x - 4) - \frac{1}{2} \cos((x - 5)^2) B_2(x - 5). \quad (6.2) $$

By $\hat{B}_2(\xi) = (1 - e^{-i\xi})^2$, it is easy to check that $f \in H^s(\mathbb{R})$ where $1/2 < s < 3/2$. Our aim of this subsection is to use Theorem 4.1 and (4.1) to approximate $f$ on $[-100, 100]$. Let $\lambda$ in (6.1) be 1 and the i.i.d. random variables $\{ \theta_k \}$ obey the standard Gaussian distribution $\mathcal{N}(0, 1)$. Then

$$ f \approx \sum_{k = -100}^{100} f(2^{-N}(k + 1 + \theta_k)) \phi(2^N \cdot -k). \quad (6.3) $$

The approximation error of (6.3) is defined as

$$ \text{error}_N = \left[ \sum_{i \in \Lambda} \left( f(x_i) - \sum_{k = -100}^{100} f(2^{-N}(k + 1 + \theta_k)) \phi(2^N x_i - k) \right)^2 / \sum_{j \in \Lambda} |f(x_j)|^2 \right]^{1/2}, \quad (6.4) $$

where

$$ \{ x_i \}_{i \in \Lambda} = \{-100 + 0.01i : i = 0, 1, \ldots, 20000 \}. $$

For each scale $N \in \{ 1, 2, \ldots, 10 \}$, the approximation scheme in (6.3) is conducted for 500 trials, and the maximum of the 500 errors is recorded in Figure 1. It is witnessed in Figure 1 that the series in (6.3) converges to $f$ on $[-100, 100]$ as $N$ tends to $\infty$. 
6.2 Two-dimensional case

Let \( \phi(x_1, x_2) := B_3(x_1)B_3(x_2) \) and the target function \( f(x_1, x_2) := \frac{1}{20 + x_2} + B_2(x_1)B_2(x_2) \). Clearly, \( \phi \) is 2I3-refinable. By [27], it is easy to check that \( \phi \in H^\mu(\mathbb{R}^2) \) and \( f \in H^\nu(\mathbb{R}^2) \), where \( 1 < \mu < 5/2 \) and \( 1 < \nu < 3/2 \). Let \( \lambda \) in (6.1) be 0.5 and the i.i.d. random variables \( \{\theta_k\} \) obey the standard Gaussian distribution. Then by Theorem 4.1 and (4.1), we can approximate \( f \) on \([-2, 2]^2\) via

\[
  f \approx \sum_{k_1 = -2^{N+1}}^{2^{N+1}} \sum_{k_2 = -2^{N+1}}^{2^{N+1}} f(2^{-N}(k + 0.5 + \theta_k))(2^N \cdot -k),
\]

where \( k = (k_1, k_2) \). The relative reconstruction error is defined as

\[
  \text{error}_N = \left( \sum_{i \in \Lambda} \left| f(x_i) - \sum_{k_1 = -2^{N+1}}^{2^{N+1}} \sum_{k_2 = -2^{N+1}}^{2^{N+1}} f(2^{-N}(k + 0.5 + \theta_k))(2^N x_i - k) \right|^2 / \sum_{j \in \Lambda} |f(x_j)|^2 \right)^{1/2},
\]

**Figure 1** Reconstruction efficiency of the one-dimensional target

**Figure 2** Reconstruction efficiency of the two-dimensional target
where
\[ \{x_i\}_{i \in \Lambda} = \left\{ \frac{2}{250} \ell : \ell = -250, \ldots, 250 \right\} \times \left\{ \frac{2}{250} \ell' : \ell' = -250, \ldots, 250 \right\} \]
with \( \times \) being the Cartesian product. For each scale \( N \in \{1, 2, \ldots, 10\} \), the approximation scheme in (6.5) is conducted for 500 trials, and the maximum error of the 500 errors is recorded in Figure 2. It is witnessed in Figure 2 that the series in (6.6) converges to \( f \) as \( N \) tends to \( \infty \).

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Appendix A  Proof of Lemma 3.2

We first establish the upper bound of $\sum_{\|j\|_1 \geq m^J} \|j\|_1^{-2s}$, and then prove (3.9) by the norm equivalence in $\mathbb{R}^d$. For $\|j\|_1 \geq m^J$, it is clear that there exists at least a component of $j$ such that it is not smaller than $m^J/d$. Then

$$\{ j \in \mathbb{Z}^d : \|j\|_1 \geq m^J \} \subseteq \bigcup_{k=1}^{d} \{ j = (j_1, j_2, \ldots, j_d) : |j_k| \geq m^J/d, j \in \mathbb{Z}, \ell \neq k \}. \quad (A.1)$$
By (A.1), we have
\[
\sum_{\|j\| \geq m^d} \|j\|_1^{2s} \leq d \left[ \sum_{\|j\| > m^d} \sum_{j_2 \in \mathbb{Z}} \cdots \sum_{j_d \in \mathbb{Z}} \frac{1}{(|j_1| + |j_2| + \cdots + |j_d|)^{2s}} \right],
\]
(A.2)

where \( [x] \) denotes the smallest integer that is larger than \( x \).

Note that the sum on the right-hand side of (A.2) has nothing to do with the signs of the components of \( j \). Then
\[
\sum_{\|j\| > m^d} \sum_{j_2 \geq 0} \cdots \sum_{j_d \geq 0} \frac{1}{(|j_1| + |j_2| + \cdots + |j_d|)^{2s}}
\]

\[
\leq 2^{d-1} \sum_{\|j\| > m^d} \sum_{j_2 \geq 0} \cdots \sum_{j_d \geq 0} \frac{1}{(|j_1| + |j_2| + \cdots + |j_d|)^{2s}}
\]

\[
\leq 2^{d-1} \left[ \sum_{n=1}^{d-1} \left( \frac{d-1}{n} \right) I_{1,n} + I_2 \right],
\]
(A.3)

where \( \binom{0}{n} = 0 \) with \( n > 0 \), and
\[
I_{1,n} = \sum_{\|j\| > m^d} \sum_{j_2 \geq 1} \cdots \sum_{j_{2n-1} \geq 1} \frac{1}{(|j_1| + |j_2| + \cdots + |j_{2n-1}|)^{2s}}, \quad I_2 = \sum_{j_1 = [m^d / d]}^{\infty} \frac{1}{j_1^2}.
\]

For any \( a > 0, N \geq 1 \) and \( t > 1 \), it is easy to check that
\[
\sum_{n=N}^{\infty} \frac{1}{(a+n)^t} \leq \int_{N-1}^{\infty} \frac{1}{(a+x)^t} dx = \frac{1}{t-1} \frac{1}{(a+N-1)^{t-1}}.
\]
(A.4)

Applying (A.4) with \( N = 1 \), we obtain
\[
I_{1,n} \leq \prod_{l=1}^{n} \frac{1}{2s-l} \sum_{j_1 = [m^d / d]}^{\infty} \frac{1}{j_1^{2s-n}} \leq \sum_{j_1 = [m^d / d]}^{\infty} \frac{1}{j_1^{2s-d+1}} \prod_{l=1}^{n} \frac{1}{2s-l}.
\]
(A.5)

Using (A.4) again, we have
\[
\sum_{j_1 = [m^d / d]}^{\infty} \frac{1}{j_1^{2s-d+1}} = \sum_{j_1 = [m^d / d]+1}^{\infty} \frac{1}{j_1^{2s-d+1}} + \frac{1}{[m^d / d]^{2s-d+1}}
\]

\[
\leq \frac{1}{[m^d / d]^{2s-d}} \left( \frac{1}{2s-d} + \frac{1}{[m^d / d]} \right).
\]
(A.6)

Similarly,
\[
\sum_{j_1 = [m^d / d]}^{\infty} \frac{1}{j_1^{2s-d}} \leq \frac{1}{[m^d / d]^{2s-1}} \frac{1}{2s-1} + \frac{1}{[m^d / d]^{2s}}.
\]
(A.7)

Combining (A.2), (A.3) and (A.5)-(A.7), we have
\[
\sum_{\|j\| \geq m^d} \|j\|_1^{2s} \leq 2^{d-1} \left[ \sum_{n=1}^{d-1} \left( \frac{d-1}{n} \right) \prod_{l=1}^{n} \frac{1}{2s-l} \frac{1}{[m^d / d]^{2s-d}} \left( \frac{1}{2s-d} + \frac{1}{[m^d / d]} \right) \right]
\]

\[
+ \frac{1}{[m^d / d]^{2s-1}} \frac{1}{2s-1} + \frac{1}{[m^d / d]^{2s}}
\]

\[
\leq 2^{d-1} 2^{2s-d} \left[ \sum_{n=1}^{d-1} \left( \frac{d-1}{n} \right) \prod_{l=1}^{n} \frac{1}{2s-l} \left( \frac{1}{2s-d} + 1 \right) + \frac{1}{2s-1} + 1 \right] m^{-J(2s-d)}.
\]
(A.8)
It follows from $\|j\|_2 \leq \|j\|_1 \leq \sqrt{d} \|j\|_2$ that

$$\{ j \in \mathbb{Z}^d : \|j\|_2 \geq m^J \} \subseteq \{ j \in \mathbb{Z}^d : \|j\|_1 \geq m^J \}. \quad \text{(A.9)}$$

Then

$$\sum_{\|j\|_2 \geq m^J} \|j\|_2^{-2s} \leq \sum_{\|j\|_1 \geq m^J} \|j\|_2^{-2s} \leq d^{-s} \sum_{\|j\|_1 \geq m^J} \|j\|_1^{-2s}, \quad \text{(A.10)}$$

where the first and second inequalities are derived from (A.9) and $\|j\|_1 \leq \sqrt{d} \|j\|_2$, respectively. Now by (A.8) and (A.10), the proof of (3.9) can be concluded.

### Appendix B  Proof of (3.12)

By the direct computation, we get

$$\left\| \langle f, m^{Nd/2} \widehat{\phi}(M^N \cdot -k) - m^{Nd/2} \widehat{\phi}(M^N \cdot -k - \varepsilon_k) \rangle \right\|^2 \leq \frac{m^{-Nd}}{(2\pi)^d} \left| \int_{\mathbb{R}^d} \widehat{\phi}(\varepsilon_k) e^{i(M^N \cdot -k \cdot \xi)} (1 - e^{i(M^N \cdot -k \cdot \xi)}) d\xi \right|^2 \leq \frac{m^{-Nd}}{(2\pi)^d} \int_{\mathbb{R}^d} \left( 1 + \|\xi\|^2_2 \right)^{-s} \left| 1 - e^{i(M^N \cdot -k \cdot \xi)} \right|^2 d\xi,$$

where the inequality is derived from the Cauchy-Schwarz inequality,

$$I_1(J) = \sum_{\|j\|_2 \geq m^J} \int_{\mathbb{T}^d} (1 + \|\xi + 2j\pi\|^2_2)^{-s} \left| 1 - e^{i(M^N \cdot -k \cdot (\xi + 2j\pi))} \right|^2 d\xi,$$

and

$$I_2(J) = \sum_{\|j\|_2 \leq m^J} \int_{\mathbb{T}^d} (1 + \|\xi + 2j\pi\|^2_2)^{-s} \left| 1 - e^{i(M^N \cdot -k \cdot (\xi + 2j\pi))} \right|^2 d\xi$$

with $J > 0$ to be optimally selected. The two quantities $I_1(J)$ and $I_2(J)$ are estimated as follows:

$$I_1(J) = 4 \sum_{\|j\|_2 \geq m^J} \int_{\mathbb{T}^d} (1 + \|\xi + 2j\pi\|^2_2)^{-s} \left| 1 - e^{i(M^N \cdot -k \cdot (\xi + 2j\pi))} \right|^2 d\xi \leq 4 \sum_{\|j\|_2 \geq m^J} \int_{\mathbb{T}^d} (1 + \|\xi + 2j\pi\|^2_2)^{-s} \|\xi + 2j\pi\|^2_2 d\xi \leq 4 \left( M^N \varepsilon_k \right)^{\alpha} \sum_{\|j\|_2 \geq m^J} \int_{\mathbb{T}^d} (1 + \|\xi + 2j\pi\|^2_2)^{-s} d\xi \leq 4 \left( M^N \varepsilon_k \right)^{\alpha} \sum_{\|j\|_2 \geq m^J} \left( \sqrt{d} + \|j\|_2 \right)^{\alpha} \int_{\mathbb{T}^d} (1 + \|\xi + 2j\pi\|^2_2)^{-s} d\xi \leq 4 \left( M^N \varepsilon_k \right)^{\alpha} (2\pi)^d \sum_{\|j\|_2 \geq m^J} \left( \sqrt{d} + \|j\|_2 \right)^{\alpha} \sum_{\|j\|_2 \geq m^J} \|j\|_2^{-2(s-\alpha/2)} \leq 4 \left( M^N \varepsilon_k \right)^{\alpha} (2\pi)^d \sum_{\|j\|_2 \geq m^J} \|j\|_2^{-2(s-\alpha/2)}$$
where \( \hat{\mathcal{C}}(s, d, \alpha) = 4\pi^{d-2s+\alpha}2^{d+\alpha} \hat{\mathcal{C}}(s, d) \), the second and last inequalities are derived from \( \alpha \leq 2 \), Lemma 3.2 and (3.9), respectively. The quantity \( I_2 \) is estimated as follows:

\[
I_2(J) \leq \sum_{\|J\|_2 < m J} \int_{T^d} (1 + \|\xi + 2\pi j\|_2^2)^{-s} \left| 1 - e^{i(M^T)^{-N} \varepsilon_k \cdot (\xi + 2j\pi)^2} \right|^2 d\xi \\
\leq (2\pi)^d \sum_{\|J\|_2 < m J} \max_{\xi \in [0, 2\pi]^d} \left| 1 - e^{i(M^T)^{-N} \varepsilon_k \cdot (\xi + 2j\pi)^2} \right|^2 \\
\leq 4\|\varepsilon_k\|_2^2 (2\pi)^d m^{-2N+(2+d)J},
\]

i.e.,

\[
I_1(J) = O(m^{-[J(2s-\alpha-d)+N\alpha]}) \quad \text{and} \quad I_2(J) = O(m^{-2N+(2+d)J}).
\]

Therefore,

\[
I_1(J) + I_2(J) = O(m^{-\min\{J(2s-\alpha-d)+N\alpha, 2N-(2+d)J\}}).
\]

It is easy to check that if choosing \( J = \frac{2-\alpha}{2s+2-\alpha} N \), then the convergence rate in (B.4) is optimal. Incidentally, Lemma 3.2 requires that \( m^J \geq d \). Therefore, by \( N \geq \frac{2s+2-\alpha}{2} \log_d m \) the choice for \( J = \frac{2-\alpha}{2s+2-\alpha} N \) is feasible. Now for this choice, we have

\[
I_1(J) + I_2(J) = O(m^{-N \frac{4s+(\alpha-2)d}{2s+\alpha+2}}) \quad (B.5)
\]

Combining (B.1)–(B.3) and (B.5), we have

\[
|\langle f, m^{Nd/2} \hat{\phi} (M^N \cdot -k) - m^{Nd/2} \hat{\phi} (M^N \cdot -k - \varepsilon_k) \rangle|^2 \\
\leq \langle f, m^{-Nd} \hat{\phi} (M^N \cdot -k) \rangle^2 \\
\leq \langle f, m^{-Nd} \hat{\phi} (M^N \cdot -k) \rangle^2 \\
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\leq \langle f, m^{-Nd} \hat{\phi} (M^N \cdot -k) \rangle^2 \\
\leq \langle f, m^{-Nd} \hat{\phi} (M^N \cdot -k) \rangle^2 \\
\[
\| \tilde{\phi} \|_{L^\infty(\mathbb{T}^d)} \sum_{k \in \mathbb{Z}^d} |(f, m^{N + d/2} \tilde{\phi}(M^N \cdot -k) - m^{N + d/2} \tilde{\phi}(M^N \cdot -k - \varepsilon_k))|^2
\]
\[
\leq \frac{\| f \|_{H^s(\mathbb{R}^d)}^2}{(2\pi)^d} \| \tilde{\phi} \|_{L^\infty(\mathbb{T}^d)}^2 \| [\tilde{\phi}, \tilde{\phi}]_0 \|_{L^\infty(\mathbb{T}^d)} C_3(s, \alpha, d) \| \varepsilon \|_{\max}^2 \frac{m^{-N(\frac{d+1}{2} - \frac{3\alpha}{4})^d}}{2\pi^{d-2}} C_3(s, \alpha, d),
\]
where
\[
\phi_{N,k} = m^{N + d/2} \phi(M^N \cdot -k),
\]
and \( \| \varepsilon \|_{\max} \) is defined in (3.11). Now we choose
\[
C_2(s, \alpha, d) := \| \tilde{\phi} \|_{L^\infty(\mathbb{R}^d)} \sqrt{\frac{\| [\tilde{\phi}, \tilde{\phi}]_0 \|_{L^\infty(\mathbb{T}^d)}}{(2\pi)^{d-2}}} C_3(s, \alpha, d)
\]
to conclude the proof of (3.12).