CHIRAL ANOMALIES AND TOPOLOGY*

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ABSTRACT

When a quantum field theory has a symmetry, global or local like in gauge theories, in the tree or classical approximation formal manipulations lead to believe that the symmetry can also be implemented in the full quantum theory, provided one uses the proper quantization rules. While this is often true, it is not a general property and therefore requires a proof because simple formal manipulations ignore the unavoidable divergences of perturbation theory.

The existence of invariant regularizations allows solving the problem in most cases but the combination of gauge symmetry and chiral fermions leads to subtle issues. Depending on the specific group and field content, anomalies are found: obstructions to the quantization of chiral gauge symmetries. Because anomalies take the form of local polynomials in the fields, are linked to local group transformations, but vanish for global (rigid) transformations one discovers that they have a topological character.

In these notes we review various perturbative and non-perturbative regularization techniques, and show that they leave room for possible anomalies when both gauge fields and chiral fermions are present. We determine the form of anomalies in simple examples. We relate anomalies to the index of the Dirac operator in a gauge background. We exhibit gauge instantons that contribute to the anomaly

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in the example of the $CP(N - 1)$ models and $SU(2)$ gauge theories. We briefly mentioned a few physical consequences.

For many years the problem of anomalies had been discussed only within the framework of perturbation theory. New non-perturbative solutions based on lattice regularization have recently been proposed. We describe the so-called overlap and domain wall fermion formulations.

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1 Symmetries, regularization, anomalies

*Divergences.* Symmetries of the classical lagrangian or tree approximation do not always translate into symmetries of the corresponding complete quantum theory. Indeed quantum field theories are affected by UV divergences that invalidate simple algebraic proofs.

The origin of UV divergences in field theory is double. First a field contains an infinite number of degrees of freedom. The corresponding divergences are directly related to renormalization group and reflect the property that even in renormalizable quantum field theories degrees of freedom remain coupled on all scales.

However another type of divergences can appear, which is related to the order between quantum operators and the transition between classical and quantum hamiltonians. Such divergences are already present in perturbation theory in ordinary quantum mechanics, for instance in the quantization of the geodesic motion on a manifold (like a sphere). Even in the case of forces linear in the velocities (like a coupling to a magnetic field) finite ambiguities are found. In local quantum field theory the problem is even more severe because for a scalar field for example the commutation between field operator $\hat{\phi}$ and conjugate momentum $\hat{\pi}$ takes the form

$$[\hat{\phi}(x), \hat{\pi}(y)] = i\hbar \delta^{d-1}(x-y).$$

In a local theory all operators are taken at the same point and thus the commutator is divergent except in quantum mechanics ($d = 1$ with our conventions).

Divergences of this nature thus are present as soon as derivative couplings are involved, or when fermions are present. They reflect the property that the knowledge of the classical theory is not sufficient, in general, to completely determine the quantized theory.

*Regularization.* Regularization is a useful intermediate step in the renormalization program that consists in modifying the initial theory at short distance, large momentum or otherwise to render perturbation theory finite. Note that from the point of view of Particle Physics all these modifications alter in some essential way the physical properties of the theory, and thus can only be considered as intermediate steps in the removal of divergences.

When a regularization can be found which preserves the symmetry of the initial classical action, a symmetric quantum field theory can be constructed.

Momentum cut-off regularization schemes, based on modifying propagators at large momenta, are specifically designed to cut the infinite number of degrees of freedom. With some care these methods will preserve formal symmetries.
of the un-renormalized theory that correspond to global (space-independent) linear group transformations. Problems may, however, arise when the symmetries correspond to non-linear or local transformations, like in the examples of non-linear $\sigma$ models or gauge theories, due to the unavoidable presence of derivative couplings. It is easy to verify that in this case regularizations that only cut momenta do not in general provide a complete regularization.

The addition of regulator fields has in general the same effect as modifying propagators, but offers a few new possibilities, in particular when regulator fields have the wrong spin–statistics connection. Fermion loops in a gauge background can be regularized by such a method.

Other methods have to be explored. In many examples dimensional regularization solves the problem because then the commutator between field and conjugated momentum taken at the same point vanishes. However in the case of chiral fermions dimensional regularization fails because chiral properties are specific to even dimensions.

Of particular interest is the method of lattice regularization, because it can be used, beyond perturbation theory, either to discuss the existence of a quantum field theory, or to determine physical properties of field theories by non-perturbative numerical techniques. One verifies that such a regularization indeed specifies an order between quantum operators. It therefore solves the ordering problem in non-linear $\sigma$-models or non-abelian gauge theories. However, again it fails in presence of chiral fermions: the manifestation of this difficulty takes the form of a doubling of the fermion degrees of freedom. Until recently this had prevented a straightforward numerical study of chiral theories.

Anomalies. That no conventional regularization scheme can be found in the case of gauge theories with chiral fermions is not surprising since we know examples of theories with anomalies, i.e. theories in which a local symmetry present at the in the tree or classical approximation cannot be implemented in the full quantum theory. Note that this creates obstructions to the construction of chiral gauge theories because exact gauge symmetry, and thus the absence of anomaly, is essential for the physical consistency of a gauge theory.

Note that we study in these lectures only local anomalies, which can be determined by perturbative calculations; peculiar global non-perturbative anomalies have also been exhibited.

Anomalies are local quantities because they are consequences of short distance singularities. They are responses to local (space-dependent) group transformations but vanish for a class of space-independent transformations. This gives them a topological character that is further confirmed by their relations with the index of the Dirac operator in a gauge background.

The recently discovered solutions of the Ginsparg–Wilson relation and the method of overlap fermions seem to provide an unconventional solution to the regularization problem in gauge theories with chiral fermions on the lattice. They evade the doubling problem of fermion because chiral transformations are no
longer strictly local on the lattice, and relate the problem of anomalies with
the invariance of the fermion measure. The absence of anomalies can then be
verified directly on the lattice, and this seems to confirm that the theories that
had been discovered anomaly-free in perturbation theory are also anomaly-free
in the non-perturbative lattice construction. Therefore the specific problem of
lattice fermions was in essence technical rather than reflecting an inconsistency
of chiral gauge theories beyond perturbation theory, as one may have feared.

Finally since these new regularization schemes have a natural implementation
in five dimensions under the form of domain wall fermions, it again opens the
doors to speculations about higher space dimensions.

We first discuss the advantages and shortcomings, from the point of view of
symmetries, of three regularization schemes, momentum cut-off, dimensional and
lattice regularization. We show that they leave room for possible anomalies when
both gauge fields and chiral fermions are present.

We then recall the origin and the form of anomalies, beginning with the sim-
plest example of the so-called abelian anomaly, i.e. the anomaly in the conserva-
tion of the abelian axial current in gauge theories. We relate anomalies to the
index of a covariant Dirac operator in the background of a gauge field.

In the two-dimensional $CP(N-1)$ models and in four-dimensional non-abelian
gauge theories we exhibit gauge instantons. We show that they can be classified
in terms of a topological charge, space integral of the chiral anomaly. The
existence of gauge field configurations that contribute to the anomaly have direct
physical implications, like possible strong CP violation and the solution to the
$U(1)$ problem.

We examine the form of the anomaly for a general axial current, and infer
conditions for gauge theories that couple differently to fermion chiral components
to be anomaly-free. A few physical applications are also briefly mentioned.

Finally the formalism of overlap fermions on the lattice and the role of the
Ginsparg–Wilson relation are explained. The alternative construction of domain
wall fermions is explained, starting from the basic mechanism of zero-modes in
supersymmetric quantum mechanics.

Conventions. Throughout these notes we work in euclidean space (with imag-
inary or euclidean time), and this also implies a formalism of euclidean fermions.

2 Momentum cut-off regularization

We first discuss methods that work in the continuum (compared to lattice meth-
ods) and at fixed dimension (unlike dimensional regularization). The idea then
is to modify field propagators beyond a large momentum cut-off to render all
Feynman diagrams convergent. However the regularization has to satisfy one
important condition: the inverse of the regularized propagator must remain a
smooth function of the momentum $p$. Indeed singularities in momentum vari-
ables generate, after Fourier transformation, contributions to the large distance
behaviour of the propagator, and regularization should modify the theory only at short distance.

2.1 Matter fields: propagator modification

Scalar fields. A simple modification of the propagator improves the convergence of Feynman diagrams at large momentum. For example in the case of the action of the self-coupled scalar field,

\[ S(\phi) = \int d^d x \left[ \frac{1}{2} \phi(x)(-\nabla_x^2 + m^2)\phi(x) + V_1(\phi(x)) \right], \quad (2.1) \]

the propagator in Fourier space \( \frac{1}{m^2 + p^2} \) can be replaced by

\[ \Delta_B(p) = \left( \frac{1}{p^2 + m^2} \right)_{\text{reg}}, \]

with

\[ \Delta_B^{-1}(p) = (p^2 + m^2) \prod_{i=1}^n (1 + p^2/M_i^2). \quad (2.2) \]

The masses \( M_i \) are proportional to the momentum cut-off \( \Lambda \),

\[ M_i = \alpha_i \Lambda, \quad \alpha_i > 0. \]

If the degree \( n \) is chosen large enough all diagrams become convergent. In the formal large cut-off limit, at parameters \( \alpha \) fixed, the initial propagator is recovered. This is the spirit of momentum or Pauli–Villars’s regularization.

Note that such a propagator cannot be derived from a hermitian hamiltonian. Hermiticity of the hamiltonian implies that if the propagator is, as above, a rational function, it must be a sum of poles with positive residues and thus cannot decrease faster than \( 1/p^2 \).

While this modification can be implemented also in Minkowski space because the regularized propagators decreases in all complex \( p^2 \) directions (except real negative), in euclidean time more general modifications are possible. Schwinger’s proper time representation suggests:

\[ \Delta_B(p) = \int_0^\infty dt \rho(t \Lambda^2) e^{-t(p^2 + m^2)}, \quad (2.3) \]

in which the function \( \rho(t) \) is positive (to ensure that \( \Delta_B(p) \) does not vanish and thus is invertible) and satisfies the condition

\[ |1 - \rho(t)| < C e^{-\sigma t} \quad (\sigma > 0) \quad \text{for} \ t \to +\infty. \]
By choosing a function $\rho(t)$ that decreases fast enough for $t \to 0$ the behaviour of the propagator can be arbitrarily improved. If $\rho(t) = O(t^n)$ the behaviour (2.2) is recovered. Another example is:

$$\rho(t) = \theta(t-1), \quad \text{(2.4)}$$

$\theta(t)$ being the step function, which leads to exponential decrease:

$$\Delta_B(p) = \frac{e^{-(p^2 + m^2)/\Lambda^2}}{p^2 + m^2}. \quad \text{(2.5)}$$

As the example shows, it is thus possible to find in this more general class propagators without unphysical singularities, but they do not follow from a hamiltonian formalism because continuation to real time becomes impossible.

**Spin 1/2 fermions.** For spin 1/2 fermions similar methods are applicable. To the free Dirac action

$$S_{F0} = \int d^d x \bar{\psi}(x)(\partial + m)\psi(x), \quad \text{(2.6)}$$

corresponds in Fourier representation the propagator $1/(m + i\not{p})$. We can replace it by the regularized propagator $\Delta_F(p)$,

$$\Delta_F^{-1}(p) = (m + i\not{p}) \prod_{i=1}^{n}(1 + p^2/M_i^2). \quad \text{(2.7)}$$

Note that we use the standard notation $\not{p} \equiv p_\mu \gamma_\mu$, with euclidean fermion conventions, analytic continuation to imaginary or euclidean time of the usual Minkowski fermions, and hermitian matrices $\gamma_\mu$.

**Remarks.** Momentum cut-off regularizations have several advantages: one can work at fixed dimension and in the continuum. However, two potential weaknesses have to be stressed:

(i) The generating functional of correlation function $Z(J)$ obtained by adding to the action (2.1) a source term for fields

$$S(\phi) \mapsto S(\phi) - \int d^d x J(x)\phi(x),$$

can be written,

$$Z(J) = \det^{1/2}(\Delta_B) \exp \left[ -\mathcal{V}_1 (\delta/\delta J) \right] \exp \left( \frac{1}{2} \int d^d x d^d y J(x)\Delta_B(x-y)J(y) \right), \quad \text{(2.8)}$$
where the determinant is generated by the gaussian integration, and

\[ V_I(\phi) \equiv \int d^d x V_I(\phi(x)). \]

None of the momentum cut-off regularizations described so far can deal with the determinant. As long as the determinant is a divergent constant that cancels in normalized correlation functions this is not a problem, but in the case of a determinant in the background of an external field (which generates a set of one-loop diagrams) this may become a serious issue.

(ii) This problem is related to another one: Some models have even in simple quantum mechanics divergences or ambiguities due to problem of order between quantum operators in products of position and momentum variables. A class of Feynman diagrams then cannot be regularized by this method. Quantum field theories where this problem occurs include models with non-linear or gauge symmetries.

**Global linear symmetries.** To implement symmetries of the classical action in the quantum theory we need a regularization scheme that preserves the symmetry. This requires some care but can always be achieved for linear global symmetries, i.e. symmetries that correspond to transformations of the fields of the form

\[ \phi_R(x) = R \phi(x), \]

where \( R \) is a constant matrix. The main reason is that in the quantum hamiltonian field operators and conjugate momenta are not mixed by the transformation, and therefore the order of operators is to some extent irrelevant. To take an example directly relevant here, a theory with massless fermions may, in four dimensions, have a chiral symmetry

\[ \psi_\theta(x) = e^{i \theta \gamma_5} \psi(x), \quad \bar{\psi}_\theta(x) = \bar{\psi}(x) e^{i \theta \gamma_5}. \]

The substitution \( (2.7) \) (for \( m = 0 \)) preserves chiral symmetry. Note the importance here of being able to work in fixed dimension four because chiral symmetry is defined only in even dimensions. In particular the invariance of the integration measure \( [d \bar{\psi}(x)d\psi(x)] \) relies on the property that \( \text{tr} \gamma_5 = 0 \).

### 2.2 Regulator fields

Regularization in the form \( (2.2) \) or \( (2.7) \) has another equivalent formulation based on the introduction of regulator fields.

**The scalar case.** In the case of scalar fields to regularize the action \( (2.1) \) for the scalar field \( \phi \) one introduces additional dynamical fields \( \phi_r, r = 1, \ldots, r_{\text{max}}, \) and considers the modified action \( S_{\text{reg.}}(\phi, \phi_{k}) \):

\[
S_{\text{reg.}}(\phi, \phi_{r}) = \int d^d x \left[ \frac{1}{2} \phi \left( -\nabla^2 + m^2 \right) \phi + \sum_{r} \frac{1}{2z_r} \phi_r \left( -\nabla^2 + M_r^2 \right) \phi_r + V_I(\phi + \sum_r \phi_r) \right].
\]  

\( (2.9) \)
With the action (2.9) any internal \(\phi\) propagator is replaced by the sum of the \(\phi\) propagator and all the \(\phi_r\) propagators \(z_r/(p^2 + M^2_r)\). For an appropriate choice of the constants \(z_r\), after integration over the regulator fields, the form (2.2) is recovered. Note that the condition of cancellation of the \(1/p^2\) contribution at large momentum implies

\[
1 + \sum_r z_r = 0.
\]

Therefore not all \(z_r\) can be positive and some of the fields \(\phi_r\) necessarily are unphysical.

**Fermions.** The fermion inverse propagator (2.7) can be written

\[
\Delta^{-1}_F(p) = (m + i\not{p}) \prod_{r=1}^{r_{\text{max}}} (1 + i\not{p}/M_r)(1 - i\not{p}/M_r).
\]

This indicates that again the same form can be obtained by a set of regulator fields \(\{\bar{\psi}_r, \psi_r\}\). One replaces the kinetic part of the action by

\[
\int d^d x \bar{\psi}(x)(\emptyset + m)\psi(x) \mapsto \int d^d x \bar{\psi}(x)(\emptyset + m)\psi(x) + \sum_{r, \epsilon = \pm} \frac{1}{z_r} \int d^d x \bar{\psi}_{r\epsilon}(x)(\emptyset + \epsilon M_r)\psi_{r\epsilon}(x). (2.10)
\]

In the same way in the interaction the fields \(\psi\) and \(\bar{\psi}\) are replaced by the sums

\[
\psi \mapsto \psi + \sum_{r, \epsilon} \psi_{r\epsilon}, \quad \bar{\psi} \mapsto \bar{\psi} + \sum_{r, \epsilon} \bar{\psi}_{r\epsilon}.
\]

Again for a proper choice of the constants \(z_r\), after integration over the regulator fields the form (2.7) is recovered. Note in particular that for \(m = 0\) one finds \(z_{r+} = z_{r-}\). This indicates how chiral symmetry is preserved by the regularization, although the regulators are massive: by fermion doubling. The fermions \(\psi_+\) and \(\psi_-\) are chiral partners. For a pair \(\psi \equiv (\psi_+, \psi_-)\), \(\bar{\psi} \equiv (\bar{\psi}_+, \bar{\psi}_-)\) the action can be written

\[
\int d^d x \bar{\psi}(x)(\emptyset \otimes 1 + M 1 \otimes \sigma_3)\psi(x),
\]

where the first matrix \(1\) and the Pauli matrix \(\sigma_3\) act in \(\pm\) space. The spinors then transform like

\[
\psi_\emptyset(x) = e^{i\theta \gamma_5 \otimes \sigma_1} \psi(x), \quad \bar{\psi}_\emptyset(x) = \bar{\psi}(x) e^{i\theta \gamma_5 \otimes \sigma_1},
\]

because \(\sigma_1\) anticommutes with \(\sigma_3\).
2.3 Abelian gauge theory

The problem of matter in presence of a gauge field can be decomposed into two steps, first matter in an external gauge field, and then the integration over the gauge field. For gauge fields we choose a covariant gauge, in such a way that power counting is the same as for scalar fields.

Charged fermions in a gauge background. The new problem that arises in presence of a gauge field is that only covariant derivatives are allowed, because gauge invariance is essential for the physical consistency of the theory. The regularized action in a gauge background now reads

\[
S(\bar{\psi}, \psi, A) = \int d^d x \bar{\psi}(x) (m + \slashed{D}) \prod_r \left(1 - \frac{\slashed{D}^2}{M_r^2}\right) \psi(x),
\]

where \( D_\mu \) is the covariant derivative

\[
D_\mu = \partial_\mu + i e A_\mu.
\]

Note that up to this point the regularization, unlike dimensional or lattice regularizations, preserves a possible chiral symmetry for \( m = 0 \).

The higher order covariant derivatives however generate new, more singular, gauge interactions and it is no longer clear whether the theory can be rendered finite.

Fermion correlation functions in the gauge background are generated by:

\[
Z(\bar{\eta}, \eta; A) = \int \left[d\psi(x)d\bar{\psi}(x)\right] \exp \left[-S(\bar{\psi}, \psi, A) + \int d^d x \bar{\eta}(x) \psi(x) + \bar{\psi}(x) \eta(x)\right],
\]

where \( \bar{\eta}, \eta \) are Grassmann sources. Integrating over fermions explicitly we obtain

\[
Z(\bar{\eta}, \eta; A) = Z_0(A) \exp \left[-\int d^d x d^d y \bar{\eta}(y) \Delta_F(A; y, x) \eta(x)\right],
\]

where \( Z_0(A) = \mathcal{N} \det \left[(m + \slashed{D}) \prod_r \left(1 - \frac{\slashed{D}^2}{M_r^2}\right)\right], \)

where \( \mathcal{N} \) is a gauge field-independent normalization and \( \Delta_F(A; y, x) \) the fermion propagator in an external gauge field.

Diagrams constructed from \( \Delta_F(A; y, x) \) belong to loops with gauge field propagators, and therefore can be rendered finite if the gauge field propagator can be improved, a condition that we check below. The other problem involves the determinant that generates closed fermion loops in a gauge background. Using \( \ln \det = \text{tr} \ln \) we find

\[
\ln Z_0(A) = \text{tr} \ln (m + \slashed{D}) + \sum_r \text{tr} \ln \left(1 - \frac{\slashed{D}^2}{M_r^2}\right) - (A = 0),
\]
or using the anticommutation of $\gamma_5$ with $\not{D}$

$$\det(\not{D} + m) = \det \gamma_5 (\not{D} + m) \gamma_5 = \det (m - \not{D}),$$

$$\ln Z_0(A) = \frac{1}{2} \text{tr} \ln (m^2 - \not{D}^2) + \sum_r \text{tr} \ln \left(1 - \not{D}^2/M_r^2\right) - (A = 0),$$

We see that the regularization has no effect from the point of view of power counting on the determinant, and therefore on one-loop diagrams of the form of fermion closed loops with external gauge fields, a problem that requires an additional regularization.

The fermion determinant. The fermion determinant can finally be regularized by adding to the action a boson regulator field with fermion spin, and therefore a propagator similar to $\Delta_F$ but with different masses

$$S_B(\bar{\phi}, \phi; A) = \int d^d x \bar{\phi}(x) (M_0^B + \not{D}) \prod_{r=1} \left(1 - \not{D}^2/(M_r^B)^2\right) \phi(x). \quad (2.14)$$

The integration over the boson ghost fields $\bar{\phi}, \phi$ adds to $\ln Z_0$ the quantity

$$\delta \ln Z_0(A) = -\frac{1}{2} \text{tr} \ln \left((M_0^B)^2 - \not{D}^2\right) - \sum_r \text{tr} \ln \left(1 - \not{D}^2/(M_r^B)^2\right) - (A = 0).$$

Expanding in inverse powers of $\not{D}$ one adjusts the masses to cancel as many powers as possible. However, the unpaired initial fermion mass $m$ is the source of a problem. The corresponding determinant can only be regularized with an unpaired boson $M_0^B$. In the chiral limit $m = 0$ we have two options: either we give a chiral charge to the boson field and the mass $M_0^B$ breaks chiral symmetry, or we leave it invariant in a chiral transformation. Then we find the determinant of the transformed operator

$$e^{i\theta \gamma_5(x)} \not{D} e^{i\theta \gamma_5(x)} (\not{D} + M_0^B)^{-1}.$$

For $\theta(x)$ constant $e^{i\theta \gamma_5}$ anticommutes with $\not{D}$ and cancels. Otherwise a non-trivial contribution remains. The method thus suggests possible difficulties with space-dependent chiral transformations.

Since actually the problem reduces to the study of a determinant in an external background one can study it directly, as we will starting with section 4. One examines whether it is possible to define some regularized form in a way consistent with chiral symmetry. When this is possible one then inserts the one-loop renormalized diagrams in the general diagrams regularized by the preceding cut-off methods.

Boson determinant in a gauge background. The boson determinant can be regularized by introducing a massive scalar charged fermion. It can also be
expressed in terms of the statistical operator using Schwinger’s representation
\( \langle \text{tr} \ln = \ln \det \rangle \)

\[
\ln \det H - \ln \det H_0 = \text{tr} \int_0^\infty \frac{dt}{t} \left[ e^{-tH_0} - e^{-tH} \right],
\]

where the operator \( H \) is analogous to a non-relativistic hamiltonian in a magnetic field,

\[ H = -D_\mu D_\mu + m^2, \quad H_0 = -\nabla^2 + m^2. \]

The integral over time is regularized by cutting it for \( t \) small, integrating from \( t = 1/\Lambda^2 \).

*The gauge propagator.* For the free gauge action in a covariant gauge usual
derivatives can be used because in an abelian theory the gauge field is neutral.
The tensor \( F_{\mu\nu} \) is gauge invariant and the action for the scalar \( \partial_\mu A_\mu \) is arbitrary.
Therefore the large momentum behaviour of the gauge field propagator can be
arbitrarily improved.

### 2.4 Non-abelian gauge theories

Compared with the abelian case, the new features of the non-abelian gauge
action are the presence of gauge field self-interactions and ghost terms. For
future purpose we define our notation. We introduce the covariant derivative, as
acting on matter field,

\[
D_\mu = \partial_\mu + A_\mu(x),
\]

where \( A_\mu \) is an anti-hermitian matrix, and the curvature tensor \( F_{\mu\nu} \)

\[
F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].
\]

The pure gauge action then is

\[
S(A_\mu) = -\frac{1}{4g^2} \text{tr} \int d^dx \text{tr} F_{\mu\nu}(x)F_{\mu\nu}(x),
\]

In the covariant gauge

\[
S_{\text{gauge}}(A_\mu) = -\frac{1}{2\xi} \int d^d x \text{tr}(\partial_\mu A_\mu)^2,
\]

the ghost field action takes the form:

\[
S_{\text{ghost}}(A_\mu, \bar{C}, C) = -\int d^d x \text{tr} \bar{C} \partial_\mu (\partial_\mu C + [A_\mu, C]).
\]

The ghost fields thus have a simple \( \delta_{ab}/p^2 \) propagator and canonical dimension
one in four dimensions.
The problem of regularization in non-abelian gauge theories has several features in common with the abelian case, as well as with the non-linear $\sigma$-model. The regularized gauge action takes the form:

$$S_{\text{reg.}}(A_\mu) = - \int d^d x \, \text{tr} \, F_{\mu\nu} P \left( \frac{D^2}{\Lambda^2} \right) F_{\mu\nu},$$  

(2.18)

in which $P$ is a polynomial of arbitrary degree. In the same way the gauge function $\partial_\mu A_\mu$ is changed into:

$$\partial_\mu A_\mu \mapsto Q \left( \frac{\partial^2}{\Lambda^2} \right) \partial_\mu A_\mu,$$

(2.19)

in which $Q$ is a polynomial of same degree as $P$. As a consequence both the gauge field propagator and the ghost propagator can be arbitrarily improved. However, as in the abelian case, the covariant derivatives generate new interactions that are more singular. It is easy to verify that the power counting of one-loop diagrams is unchanged while higher order diagrams can be made convergent by taking the degrees of $P$ and $Q$ large enough: Regularization by higher derivatives takes care of all diagrams except, as in all geometric models, some one-loop diagrams (and thus subdiagrams).

As with charged matter the one-loop diagrams have to be examined separately. For fermion matter it is however still possible as, in the abelian case, to add a set of regulator fields, massive fermions and bosons with spin. In the chiral situation the problem of the compatibility between the gauge symmetry and the quantization is reduced to an explicit verification of the WT identities for the one-loop diagrams. Note that the preservation of gauge symmetry is necessary for the cancellation of unphysical states in physical amplitudes, and thus essential to the physical relevance of the quantum field theory.

### 3 Other regularization schemes

The other regularization schemes we now discuss have the common property that they modify in some essential way the structure of space–time, dimension regularization because it relies on defining Feynman diagrams for non-integer dimensions, lattice regularization because continuum space is replaced by a discrete lattice.

#### 3.1 Dimensional regularization

Dimensional regularization involves continuation of Feynman diagrams in the parameter $d$ ($d$ is the space dimension) to arbitrary complex values, and therefore seems to have no meaning outside perturbation theory. However this regularization very often leads to the simplest perturbative calculations.
In addition it solves with the problem of commutation of quantum operators in local field theories. Indeed commutators for example in the case of a scalar field take the form

\[ [\hat{\phi}(x), \hat{\pi}(y)] = i\hbar \delta^{d-1}(x - y) = i\hbar(2\pi)^{1-d} \int d^{d-1}p \, e^{ip(x-y)}, \]

where \(\hat{\pi}(x)\) is the momentum conjugated to the field \(\hat{\phi}(x)\). As we have already stressed, in a local theory all fields are taken at same point, and therefore a commutation in the product \(\hat{\phi}(x)\hat{\pi}(x)\) generates a divergent contribution (for \(d > 1\)) proportional to

\[ \delta^{d-1}(0) = (2\pi)^{1-d} \int d^{d-1}p. \]

The rule of dimensional regularization imply the consistency of the change of variables \(p \rightarrow \lambda p\) and thus \(\int d^{d}p = 0\), in contrast to momentum regularization where it is proportional to a power of the cut-off. Therefore the order between operators becomes irrelevant because the commutator vanishes. Dimensional regularization thus is applicable to geometric models where these problems of quantization occur, like non-linear \(\sigma\) models or gauge theories.

Its use, however, requires some care in massless theories. For instance in a massless theory in two dimensions integrals of the form \(\int d^{d}k/k^{2}\) are met. They again vanish in dimensional regularization for the same reason. However here they correspond to an unwanted cancellation between UV and IR logarithmic divergences.

More important here, it is not applicable when some essential property of the field theory is specific to the initial dimension. An example is provided by theories containing fermions in which parity symmetry is violated.

**Fermions.** For fermions belonging to the fundamental representation of the spin group \(\text{Spin}(d)\) the strategy is the same. The spin problem can be reduced to the calculation of traces of \(\gamma\) matrices. Therefore only one additional prescription for the trace of the unit matrix is needed. There is no natural continuation since odd and even dimensions behave differently. Since no algebraic manipulation depends on the explicit value of the trace, any smooth continuation in the neighbourhood of the relevant dimension is satisfactory. A convenient choice is to take the trace constant. In even dimension as long as only \(\gamma_{\mu}\) matrices are involved no other problem arises. However no dimensional continuation that preserves all properties of \(\gamma_{d+1}\) can be found. This leads to serious difficulties if \(\gamma_{d+1}\) in the calculation of Feynman diagrams has to be expressed in terms of the product of all other \(\gamma\) matrices. For example in four dimensions \(\gamma_{5}\) is related to the other \(\gamma\) matrices by

\[ 4! \gamma_{5} = -\epsilon_{\mu_{1}\ldots\mu_{4}} \gamma_{\mu_{1}} \cdots \gamma_{\mu_{4}}, \quad (3.1) \]
where $\epsilon_{\mu_1\ldots\mu_4}$ is the complete antisymmetric tensor with $\epsilon_{1234} = 1$. Problems therefore arise in the case of gauge theories with chiral fermions, because the special properties of $\gamma_5$ are involved as we recall below. This difficulty is the source of chiral anomalies.

Since perturbation theory involves the calculation of traces, one possibility is to define $\gamma_5$ near four dimensions by

$$\gamma_5 = E_{\mu_1\ldots\mu_4} \gamma_{\mu_1} \ldots \gamma_{\mu_4}, \quad (3.2)$$

where $E_{\mu\nu\rho\sigma}$ is a completely antisymmetric tensor, which reduces to $-\epsilon_{\mu\nu\rho\sigma}/4!$ in four dimensions. It is easy to then verify that, with this definition, $\gamma_5$ anticommutes with the other $\gamma_\mu$ matrices only in four dimensions. If for example we evaluate the product $\gamma_\nu \gamma_5 \gamma_\nu$ in $d$ dimensions, we find:

$$\gamma_\nu \gamma_5 \gamma_\nu = (d - 8)\gamma_5.$$

Anticommuting properties of the $\gamma_5$ would have led to a factor $-d$ instead. Multiplied by a pole in $1/(d-4)$ consequence of UV divergences in one-loop diagrams, the additional contribution proportional to $d - 4$ may lead to a finite difference with the formal result.

The other option would be to keep the anticommuting property of $\gamma_5$, but the preceding example shows that this is contradictory with a form (3.2). Actually one verifies that the only consistent prescription for generic dimensions then is that the traces of $\gamma_5$ with any product of $\gamma_\mu$ matrices vanishes, and therefore is useless.

Finally an alternative possibility consists in breaking $O(d)$ symmetry and keeping the four $\gamma$ matrices of $d = 4$.

### 3.2 Lattice regularization

We have explained that Pauli–Villars’s regularization does not work for field theories in which the geometric properties generate interactions like models on homogeneous spaces (for example the non-linear $\sigma$-model) or gauge theories. In these theories some divergences are related to the problem of quantization and order of operators, which already appears in simple quantum mechanics. Other regularization methods then are needed. In many cases lattice regularization may be used.

**Lattice field theory.** To each site $x$ of the lattice are attached field variables corresponding to fields in the continuum. To the action $S$ in the continuum corresponds a lattice action, the energy of lattice field configurations in the language of classical statistical physics. The functional integral becomes a sum over configurations and the regularized partition function is the partition function of a lattice model.

All expressions in these notes will refer implicitly to a hypercubic lattice and we denote by $a$ the lattice spacing.
The advantages of lattice regularization are:

(i) Lattice regularization indeed corresponds to a specific choice of quantization.

(ii) It is the only established regularization that for gauge theories and other geometric models has a meaning outside perturbation theory. For instance the regularized functional integral can be calculated by numerical methods, like stochastic methods (Monte-Carlo type simulations) or strong coupling expansions.

(iii) It preserves most global or local symmetries with the exception of the space $O(d)$ symmetry that is replaced by a hypercubic symmetry (but this turns out not to be a major difficulty) and fermion chirality, which turns out to be a more serious problem, as we will show.

The main disadvantage is that it leads to rather complicated perturbative calculations.

3.3 Boson field theories

Scalar fields. To the action (2.1) for a scalar field $\phi$ in the continuum corresponds a lattice action, which is obtained in the following way: The euclidean lagrangian density becomes a function of lattice variables $\phi(x)$, where $x$ now is a lattice site. Locality can be implemented by considering lattice lagrangians that depend only on a site and its neighbours (though this is a too strong requirement; lattice interactions decreasing exponentially with distance are also local). Derivatives $\partial_\mu \phi$ of the continuum are replaced by finite differences, for example:

$$\partial_\mu \phi \mapsto \nabla^\text{lat}_\mu \phi = [\phi(x + an_\mu) - \phi(x)]/a,$$

where $a$ is the lattice spacing and $n_\mu$ the unit vector in the $\mu$ direction. The lattice action then is the sum over lattice sites.

With the choice (3.3) the propagator $\Delta_a(p)$ for the Fourier components of a massive scalar field is given by

$$\Delta_a^{-1}(p) = m^2 + \frac{2}{a^2} \sum_{\mu=1}^d \left(1 - \cos(ap_\mu)\right).$$

It is a periodic function of the components $p_\mu$ of the momentum vector with period $2\pi/a$. In the small lattice spacing limit the continuum propagator is recovered:

$$\Delta_a^{-1}(p) = m^2 + p^2 - \frac{1}{12} \sum_\mu a^2 p^4_\mu + O(p^6).$$

In particular hypercubic symmetry implies $O(d)$ symmetry at order $p^2$.

Gauge theories. Lattice regularization defines unambiguously a quantum theory. Therefore, once one has realized that gauge fields should be replaced by link
variables corresponding to parallel transport along links of the lattice, one can regularize a gauge theory.

The link variables $U_{xy}$ are group elements associated with the links joining the sites $x$ and $y$ on the lattice. The regularized form of $\int dxF_{\mu\nu}^2$ is the product of link variables along a closed curve on the lattice, the simplest being a square on a hypercubic lattice, leading to the well-known plaquette action, each square forming a plaquette. The typical gauge invariant lattice action corresponding to the continuum action of a gauge field coupled to scalar bosons then has the form:

$$S(U, \phi^*, \phi) = \beta \sum_{\text{plaquettes}} \text{tr} \, U_{xy} U_{yz} U_{zt} U_{tx} + \kappa \sum_{\text{links}} \phi^*_x U_{xy} \phi_y + \sum_{\text{sites}} V(\phi^*_x \phi_x),$$

(3.6)

where $x, y, \ldots$ denotes lattice sites, and $\beta$ and $\kappa$ are coupling constants. The action (3.6) is invariant under independent group transformations on each lattice site, lattice equivalents of the gauge transformations of the continuum theory. The measure of integration over the gauge variables is the group invariant measure on each site. Note that on the lattice and in a finite volume the gauge invariant action leads to a well-defined partition function because the gauge group is compact. However in the continuum or infinite volume limits the compact character of the group is lost. Even on the lattice regularized perturbation theory is defined only after gauge fixing.

We finally note that on the lattice the difficulties with the regularization do not come from the gauge field directly, but involve the gauge field only through the integration over chiral fermions.

### 3.4 Fermion and the doubling problem

We now review a few problems specific to relativistic fermions on the lattice. We consider the free action for a Dirac fermion:

$$S(\bar{\psi}, \psi) = \int d^d x \bar{\psi}(x) (\not\partial + m) \psi(x).$$

To regularize this action by a lattice and preserve chiral properties in the massless limit one can replace $\partial_\mu \psi(x)$ by

$$\nabla^\text{lat}_\mu \psi(x) = [\psi(x + an_\mu) - \psi(x - an_\mu)]/2a.$$

Then the inverse of the fermion propagator $\Delta$ for the Fourier components $\tilde{\psi}(p)$ of the field is:

$$\Delta^{-1}(p) = m + i \sum_\mu \gamma_\mu \frac{\sin ap_\mu}{a} \quad (3.7)$$

a periodic function of the components $p_\mu$ of the momentum vector. A problem then arises: the equations relevant to the small lattice spacing limit,

$$\sin(ap_\mu) = 0$$
have each two solutions \( p_\mu = 0 \) and \( p_\mu = \pi/a \) within one period, i.e. within the Brillouin zone \( 2\pi/a \). Therefore the propagator \( (3.7) \) propagates \( 2^d \) fermions. To remove this degeneracy it is possible to add to the regularized action an additional scalar term \( \delta S \) involving second derivatives:

\[
\delta S(\bar{\psi}, \psi) = \frac{1}{2} M \sum_{x, \mu} \left[ 2\bar{\psi}(x)\psi(x) - \bar{\psi}(x + an_\mu)\psi(x) - \bar{\psi}(x)\psi(x + an_\mu) \right]. \tag{3.8}
\]

After Fourier transformation the modified Dirac operator \( D_W \) reads

\[
D_W(p) = m + M \sum_\mu (1 - \cos ap_\mu) + \frac{i}{a} \sum_\mu \gamma_\mu \sin ap_\mu. \tag{3.9}
\]

The fermion propagator becomes:

\[
\Delta(p) = D_W^\dagger(p) \left( D_W(p) D_W^\dagger(p) \right)^{-1},
\]

with:

\[
D_W(p)D_W^\dagger(p) = \left[ m + M \sum_\mu (1 - \cos ap_\mu) \right]^2 + \frac{1}{a^2} \sum_\mu \sin^2 ap_\mu.
\]

Therefore the degeneracy between the different states has been lifted. For each component \( p_\mu \) that takes the value \( \pi/a \) the mass is increased by \( M \). If \( M \) is of order \( 1/a \) the spurious states are eliminated in the continuum limit. This is the recipe of Wilson’s fermions.

However a problem arises if one wants to construct a theory with massless fermions and chiral symmetry. Chiral symmetry implies for the Dirac operator \( D(p) \) anticommutation with \( \gamma_5 \)

\[
\{ D(p), \gamma_5 \} = 0, \tag{3.10}
\]

and therefore both the mass term and the term \( (3.8) \) are excluded. It remains possible to add various counter-terms and try to adjust them to recover chiral symmetry in the continuum limit. But then there is no \textit{a priori} guarantee that this is indeed possible and moreover calculations are plagued by fine tuning problems and cancellations of unnecessary UV divergences.

One could also think about modifying the fermion propagator by adding terms connecting fermions separated by more than one lattice spacing. But it has been proven that this does not solve the doubling problem. In fact this doubling of the number of fermion degrees of freedom is directly related to the problem of anomalies.

Since the most naive form of the propagator yields \( 2^d \) fermion states, one tries in practical calculations to reduce this number to a smaller multiple of two, using for instance the idea of staggered fermions introduced by Kogut and Susskind.

However the general picture has recently changed with the discovery of the properties of overlap fermions and solutions of the Ginsparg–Wilson relation, a topic we postpone, and on which we will come back in section 7.
4 The abelian anomaly

We have pointed out that none of the standard regularization methods can deal in a straightforward way with one-loop diagrams in the case of gauge fields coupled to chiral fermions. We now show that indeed chiral symmetric gauge theories, involving gauge fields coupled to massless fermions, can be found where the axial current is not conserved. The divergence of the axial current in a chiral theory, when it does not vanish, is called an anomaly. Anomalies in particular lead to obstructions to the construction of gauge theories when the gauge field couples differently to the two fermion chiral components.

Several examples are physically important like the theory of weak electromagnetic interactions, the electromagnetic decay of the π₀ meson, or the U(1) problem. We first discuss the abelian axial current, in four dimensions (the generalization to all even dimensions then is straightforward), and then the general non-abelian situation.

4.1 Abelian axial current and abelian vector gauge field

The only possible source of anomalies are one-loop fermion diagrams in gauge theories when chiral properties are involved. This reduces the problem to the discussion of fermions in background gauge fields, or equivalently to the properties of the determinant of the gauge covariant Dirac operator.

We thus consider the QED-like fermion action \( S(\bar{\psi}, \psi; A) \) for massless Dirac fermions \( \psi, \bar{\psi} \) in the background of an abelian gauge field \( A_\mu \)

\[
S(\bar{\psi}, \psi; A) = - \int d^4x \bar{\psi}(x) \slashed{D} \psi(x), \quad \slashed{D} \equiv \partial + ie A(x),
\]

and the corresponding functional integral

\[
\mathcal{Z}(A_\mu) = \int [d\psi d\bar{\psi}] \exp \left[ -S(\psi, \bar{\psi}; A) \right] = \det \slashed{D}.
\]

We can find regularizations that preserve gauge invariance,

\[
\psi(x) = e^{i\Lambda(x)} \psi'(x), \quad \bar{\psi}(x) = e^{-i\Lambda(x)} \bar{\psi}'(x), \quad A_\mu(x) = -\frac{1}{e} \partial_\mu \Lambda(x) + A'_\mu(x),
\]

and since the fermions are massless, chiral symmetry. We would therefore naively expect the corresponding axial current to be conserved. However the proof of current conservation involves space-dependent chiral transformations, and therefore steps that cannot be regularized without breaking the local symmetry.

Under a space-dependent chiral transformation

\[
\psi_\theta(x) = e^{i\theta(x)\gamma_5} \psi(x), \quad \bar{\psi}_\theta(x) = \bar{\psi}(x) e^{i\theta(x)\gamma_5},
\]

\[
(4.3)
\]

\[
(4.4)
\]
the action becomes

$$S_\theta(\bar{\psi}, \psi; A) = -\int d^4x \bar{\psi}_\theta(x)\mathcal{D}\psi_\theta(x) = S(\bar{\psi}, \psi; A) + \int d^4x \partial_\mu \theta(x)J_\mu^5(x),$$

where $J_\mu^5(x)$, the coefficient of $\partial_\mu \theta$, is the axial current,

$$J_\mu^5(x) = i\bar{\psi}(x)\gamma_5\gamma_\mu\psi(x). \quad (4.5)$$

After the transformation (4.4) $Z(A_\mu)$ becomes:

$$Z(A_\mu, \theta) = \det [e^{i\gamma_5\theta(x)}\mathcal{D} e^{i\gamma_5\theta(x)}]. \quad (4.6)$$

Note that $\ln[Z(A_\mu, \theta)]$ is the generating functional of connected $\partial_\mu J_\mu^5$ correlation functions in an external field $A_\mu$.

Since $e^{i\gamma_5 \theta}$ has a determinant that is unity, one would naively conclude that $Z(A_\mu, \theta) = Z(A_\mu)$ and therefore that the current $J_\mu^5(x)$ is conserved. This is a conclusion we now check by an explicit calculation of the expectation value of $\partial_\mu J_\mu^5(x)$ in the case of the action (4.1).

**Remarks.**

(i) For any regularization that is consistent with the hermiticity of $\gamma_5$

$$|Z(A_\mu, \theta)|^2 = \det \begin{bmatrix} e^{i\gamma_5\theta(x)}\mathcal{D} & e^{i\gamma_5\theta(x)} \end{bmatrix} \det \begin{bmatrix} e^{-i\gamma_5\theta(x)}\mathcal{D}^\dagger & e^{-i\gamma_5\theta(x)} \end{bmatrix} = \det (\mathcal{D}\mathcal{D}^\dagger),$$

and thus $|Z(A_\mu, \theta)|$ is independent of $\theta$. Therefore an anomaly can appear only in the imaginary part of $\ln Z$.

(ii) We have shown that one can find a regularization with regulator fields such that gauge invariance is maintained, and the determinant is independent of $\theta$ for $\theta(x)$ constant.

(iii) If the regularization is gauge invariant $Z(A_\mu, \theta)$ is also gauge invariant. Therefore a possible anomaly will also be gauge invariant.

(iv) $\ln Z(A_\mu, \theta)$ is connected and 1PI. Short distance singularities thus take the form of local polynomials in the fields and sources. Since a possible anomaly is a short distance effect it must also take the form of a local polynomial of $A_\mu$ and $\partial_\mu \theta$ constrained by parity and power counting, $A_\mu$ and $\partial_\mu \theta$ having dimension one and no mass being available,

$$\ln Z(A_\mu, \theta) - \ln Z(A_\mu, 0) = i \int d^4x \mathcal{L}(A, \partial \theta; x),$$

where $\mathcal{L}$ is the sum of monomials of dimension four. At order $\theta$ only one is available:

$$\mathcal{L}(A, \partial \theta; x) \propto e^2 \epsilon_{\mu\nu\rho\sigma} \partial_\mu \theta(x) A_\nu(x) \partial_\rho A_\sigma(x),$$
where \( \epsilon_{\mu \nu \rho \sigma} \) is the complete antisymmetric tensor with \( \epsilon_{1234} = 1 \). A simple integration by parts and anti-symmetrization shows that

\[
\int d^4x \mathcal{L}(A, \partial \theta; x) \propto e^2 \epsilon_{\mu \nu \rho \sigma} \int d^4x F_{\mu \nu}(x) F_{\rho \sigma}(x),
\]

an expression that is gauge invariant.

The coefficient of \( \theta(x) \) is the expectation value in an external gauge field of \( \partial_\mu J_5^\mu(x) \), the divergence of the axial current. It is thus determined up to a multiplicative constant,

\[
\langle \partial_\lambda J_5^\lambda(x) \rangle \propto e^2 \epsilon_{\mu \nu \rho \sigma} \partial_\mu A_\nu(x) \partial_\rho A_\sigma(x) \propto e^2 \epsilon_{\mu \nu \rho \sigma} F_{\mu \nu}(x) F_{\rho \sigma}(x),
\]

where \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic tensor, and we denote by \( \langle \bullet \rangle \) expectation values with respect to the measure \( e^{-\mathcal{S}(\bar{\psi}, \psi; A)} \).

Since the possible anomaly is independent up to a multiplicative factor of the regularization, it must indeed be a gauge invariant local function of \( A_\mu \).

To find the multiplicative factor, which is the only regularization dependent feature, it is sufficient to calculate the coefficient of the term quadratic in \( A \) in the expansion of \( \langle \partial_\lambda J_5^\lambda(x) \rangle \) in powers of \( A \). We define the three-point function in momentum representation:

\[
\Gamma^{(3)}_{\lambda \mu \nu}(k; p_1, p_2) = \frac{\delta}{\delta A_\mu(p_1)} \frac{\delta}{\delta A_\nu(p_2)} \langle J_5^\lambda(k) \rangle \bigg|_{A=0},
\]

(4.7)

\[
= \frac{\delta}{\delta A_\mu(p_1)} \frac{\delta}{\delta A_\nu(p_2)} i \text{tr} \left[ \gamma_5 \gamma_\lambda D^{-1}(k) \right] \bigg|_{A=0}.
\]

\( \Gamma^{(3)} \) is the sum of the two Feynman diagrams of figure 1.

The contribution of diagram (a) is:

\[
(a) \leftrightarrow \frac{e^2}{(2\pi)^4} \text{tr} \left[ \int d^4q \gamma_5 \gamma_\lambda \gamma_\mu (q + k')^{-1} \gamma_\nu (q' - p_2)^{-1} \gamma_\nu (q' - q)^{-1} \right],
\]

(4.8)

and the contribution of diagram (b) is obtained by exchanging \( p_1, \gamma_\nu \leftrightarrow p_2, \gamma_\nu \).
Power counting tells us that the function $\Gamma^{(3)}$ may have a linear divergence that, due to the presence of the $\gamma_5$ factor, must be proportional to $\epsilon_{\lambda\mu\nu\rho}$, symmetric in the exchange $p_1, \gamma_\nu \leftrightarrow p_2, \gamma_\nu$, and thus proportional to

$$\epsilon_{\lambda\mu\nu\rho} (p_1 - p_2)_\rho.$$  \hfill (4.9)

On the other hand by commuting $\gamma_5$ we notice that $\Gamma^{(3)}$ is formally a symmetric function of the three sets of external arguments. A divergence breaks the symmetry between external arguments. Therefore a symmetric regularization of the kind we will adopt in the first calculation leads to a finite result. The result is not ambiguous because a possible ambiguity again is proportional to (4.9).

Similarly if the regularization is consistent with gauge invariance the vector current is conserved

$$p_1 \mu \Gamma^{(3)}_{\lambda\mu\nu} (k; p_1, p_2) = 0.$$  \hfill (4.10)

Applied to the divergent part the equation implies

$$-p_1 \mu p_2 \rho \epsilon_{\lambda\mu\nu\rho} = 0,$$

which cannot be satisfied. Therefore the sum of the two diagrams is finite. Finite ambiguities must also have the form (4.9) and thus are also forbidden by gauge invariance. All regularizations consistent with gauge invariance must give the same answer.

Therefore there are two possibilities:

(i) The divergence $k_\lambda \Gamma^{(3)}_{\lambda\mu\nu} (k; p_1, p_2)$ in a regularization respecting the symmetry between the three arguments vanishes. Then both $\Gamma^{(3)}$ is gauge invariant and the axial current is conserved.

(ii) The divergence of the symmetric regularization does not vanish. Then it is possible to add to $\Gamma^{(3)}$ a term proportional to (4.9) to restore gauge invariance but this term breaks the symmetry between external momenta: the axial current is not conserved, an anomaly is present.

4.2 Explicit calculation

*Momentum regularization.* The calculation can be done using one of the various gauge invariant regularizations, for example momentum cut-off regularization or dimensional regularization with $\gamma_5$ being defined as in dimension four and thus no longer anticommuting with other $\gamma$ matrices. Instead we choose a regularization that preserves the symmetry between the three external arguments and global chiral symmetry, but breaks gauge invariance, modifying the fermion propagator:

$$(q)^{-1} \mapsto (q)^{-1} \rho(\varepsilon q^2),$$

where $\varepsilon$ is the regularization parameter ($\varepsilon \to 0_+$), $\rho(z)$ is a positive differentiable function such that $\rho(0) = 1$, and decreasing fast enough for $z \to +\infty$, at least like $1/z$. 
Then current conservation and gauge invariance are compatible only if the divergence $k_\lambda R^{(3)}_{\lambda\mu\nu}(k; p_1, p_2)$ vanishes.

It is convenient to consider directly the contribution $C^{(2)}(k)$ of order $A^2$ to $\langle k_\lambda J^5_\lambda(k) \rangle$, which sums the two diagrams:

$$C^{(2)}(k) = e^2 \int d^4 p_1 d^4 p_2 A_\mu(p_1) A_\nu(p_2) \int \frac{d^4 q}{(2\pi)^4} \rho(\varepsilon(q + k)^2) \rho(\varepsilon(q - p_2)^2) \rho(\varepsilon q^2)$$

$$\times \text{tr} \left[ \gamma_5 k'(q + \bar{k})^{-1} \gamma_\mu (q - \bar{p}_2)^{-1} \gamma_\nu \bar{q}^{-1} \right],$$

(4.11)

because the calculation then suggests how the method generalizes to arbitrary even dimensions.

We first transform the expression, using the identity:

$$k'(q + \bar{k})^{-1} = 1 - \bar{q}(q + \bar{k})^{-1}. \tag{4.12}$$

Then

$$C^{(2)}(k) = C_1^{(2)}(k) + C_2^{(2)}(k),$$

with

$$C_1^{(2)}(k) = e^2 \int d^4 p_1 d^4 p_2 A_\mu(p_1) A_\nu(p_2) \int \frac{d^4 q}{(2\pi)^4} \rho(\varepsilon(q + k)^2) \rho(\varepsilon(q - p_2)^2) \rho(\varepsilon q^2)$$

$$\times \text{tr} \left[ \gamma_5 \gamma_\mu (q - \bar{p}_2)^{-1} \gamma_\nu \bar{q}^{-1} \right], \tag{4.13}$$

and

$$C_2^{(2)}(k) = -e^2 \int d^4 p_1 d^4 p_2 A_\mu(p_1) A_\nu(p_2) \int \frac{d^4 q}{(2\pi)^4} \rho(\varepsilon(q + k)^2) \rho(\varepsilon(q - p_2)^2) \rho(\varepsilon q^2)$$

$$\times \text{tr} \left[ \gamma_5 \gamma_\nu (q + \bar{k})^{-1} \gamma_\mu (q - \bar{p}_2)^{-1} \gamma_\nu \bar{q}^{-1} \right]. \tag{4.14}$$

In $C_2^{(2)}(k)$ we use the cyclic property of the trace and the commutation of $\gamma_\nu \bar{q}^{-1}$ and $\gamma_5$ to cancel the propagator $\bar{q}^{-1}$ and obtain

$$C_2^{(2)}(k) = -e^2 \int d^4 p_1 d^4 p_2 A_\mu(p_1) A_\nu(p_2) \int \frac{d^4 q}{(2\pi)^4} \rho(\varepsilon(q + k)^2) \rho(\varepsilon(q - p_2)^2) \rho(\varepsilon q^2)$$

$$\times \text{tr} \left[ \gamma_5 \gamma_\nu (q + \bar{k})^{-1} \gamma_\mu (q - \bar{p}_2)^{-1} \right]. \tag{4.15}$$

We then shift $q \mapsto q + p_2$ and interchange $(p_1, \mu)$ and $(p_2, \nu)$,

$$C_2^{(2)}(k) = -e^2 \int d^4 p_1 d^4 p_2 A_\mu(p_1) A_\nu(p_2) \int \frac{d^4 q}{(2\pi)^4} \rho(\varepsilon(q - p_2)^2) \rho(\varepsilon q^2) \rho(\varepsilon(q + p_1)^2)$$

$$\times \text{tr} \left[ \gamma_5 \gamma_\mu (q - \bar{p}_2)^{-1} \gamma_\nu \bar{q}^{-1} \right]. \tag{4.16}$$
We see that the two terms $C_1^{(2)}$ and $C_2^{(2)}$ would cancel in the absence of regulators. This corresponds to the formal proof of current conservation. However without regularization the integrals diverge and these manipulations are not legitimate.

Instead here we find a non-vanishing sum due to the difference in regulating factors:

$$C^{(2)}(k) = e^2 \int d^4 p_1 d^4 p_2 A_\mu(p_1) A_\nu(p_2) \int \frac{d^4 q}{(2\pi)^4} \rho(\varepsilon(q-p_2)^2) \rho(\varepsilon q^2)$$

$$\times \operatorname{tr} \left[ \gamma_5 \gamma_\mu(q - p_2) -\gamma_\nu \gamma^{-1} \right] \left[ \rho(\varepsilon(q+k)^2) - \rho(\varepsilon(q+p_1)^2) \right]. \quad (4.17)$$

After evaluation of the trace $C^{(2)}$ becomes (using (B.1)):

$$C^{(2)}(k) = -4e^2 \int d^4 p_1 d^4 p_2 A_\mu(p_1) A_\nu(p_2) \int \frac{d^4 q}{(2\pi)^4} \rho(\varepsilon(q-p_2)^2) \rho(\varepsilon q^2)$$

$$\times \epsilon_{\mu\nu\rho\sigma} \frac{p_2 \rho q_\sigma}{q^2(q-p_2)^2} \left[ \rho(\varepsilon(q+k)^2) - \rho(\varepsilon(q+p_1)^2) \right]. \quad (4.18)$$

Contributions coming from finite values of $q$ cancel in the $\varepsilon \to 0$ limit. Due the cut-off the relevant values of $q$ are of order $\varepsilon^{-1/2}$. We therefore rescale $q$ accordingly $q \varepsilon^{1/2} \to q$ and find

$$C^{(2)}(k) = -4e^2 \epsilon_{\mu\nu\rho\sigma} \int d^4 p_1 d^4 p_2 A_\mu(p_1) A_\nu(p_2) I_{\rho\sigma}(p_1, p_2),$$

with

$$I_{\rho\sigma}(p_1, p_2) \sim \int \frac{d^4 q}{(2\pi)^4} p_2 \rho q_\sigma \rho^2(\varepsilon q^2) \rho'(\varepsilon q^2) \left[ 2\varepsilon q_\lambda (k-p_1) \right]. \quad (4.19)$$

The identity:

$$\int d^4 q q_\alpha q_\beta f(q^2) = \frac{1}{4} \delta_{\alpha\beta} \int d^4 q q^2 f(q^2),$$

transforms the integral into:

$$I_{\rho\sigma}(p_1, p_2) \sim -\frac{1}{2} p_2 \rho(p_1 + p_2) \int \frac{\varepsilon d^4 q}{(2\pi)^4 q^2} \rho^2(\varepsilon q^2) \rho'(\varepsilon q^2). \quad (4.20)$$
The remaining integral can be calculated explicitly (we recall $\rho(0) = 1$)
\[
\int \frac{\varepsilon d^4q}{(2\pi)^4q^2} \rho^2(\varepsilon q^2) \rho'(\varepsilon q^2) = \frac{1}{8\pi^2} \int_0^\infty \varepsilon q dq \rho^2(\varepsilon q^2) \rho'(\varepsilon q^2) = -\frac{1}{48\pi^2},
\]
and yields a result independent of the function $\rho$. We finally obtain:
\[
\langle k_\lambda J^5_\lambda(k) \rangle = -\frac{e^2}{12\pi^2} \varepsilon_{\mu\nu\rho\sigma} \int d^4p_1 d^4p_2 p_1\mu A_\nu(p_1)p_2\rho A_\sigma(p_2), \tag{4.21}
\]
and therefore from the definition \(\ref{eq:4.7}\)
\[
k_\lambda \Gamma^{(3)}_{\lambda\mu\nu}(k; p_1, p_2) = \frac{e^2}{6\pi^2} \varepsilon_{\mu\lambda\rho\sigma} p_1\rho p_2\sigma . \tag{4.22}
\]
This non-vanishing result implies that any definition of the determinant $\det D$ breaks at least either current conservation or gauge invariance. Since gauge invariance is essential to the consistency of a gauge theory we choose to break current conservation. Exchanging arguments, we obtain the value of $p_1\mu \Gamma^{(3)}_{\lambda\mu\nu}(k; p_1, p_2)$:
\[
p_1\mu \Gamma^{(3)}_{\lambda\mu\nu}(k; p_1, p_2) = \frac{e^2}{6\pi^2} \varepsilon_{\lambda\nu\rho\sigma} k_\rho p_2\sigma . \tag{4.23}
\]
If instead we had used a gauge invariant regularization, the result for $\Gamma^{(3)}$ would have differed by a term $\delta \Gamma^{(3)}$ proportional to \(\ref{eq:4.9}\):
\[
\delta \Gamma^{(3)}_{\lambda\mu\nu}(k; p_1, p_2) = K \varepsilon_{\lambda\mu\nu}(p_1 - p_2)_\rho . \tag{4.24}
\]
The constant $K$ then is determined by the condition of gauge invariance
\[
p_1\mu \left[ \Gamma^{(3)}_{\lambda\mu\nu}(k; p_1, p_2) + \delta \Gamma^{(3)}_{\lambda\mu\nu}(k; p_1, p_2) \right] = 0 ,
\]
which yields
\[
p_1\mu \delta \Gamma^{(3)}_{\lambda\mu\nu}(k; p_1, p_2) = -\frac{e^2}{6\pi^2} \varepsilon_{\lambda\nu\rho\sigma} k_\rho p_2\sigma \Rightarrow K = \frac{e^2}{(6\pi^2)} . \tag{4.25}
\]
This gives an additional contribution to the divergence of the current
\[
k_\lambda \delta \Gamma^{(3)}_{\lambda\mu\nu}(k; p_1, p_2) = \frac{e^2}{3\pi^2} \varepsilon_{\mu\lambda\rho\sigma} p_1\rho p_2\sigma . \tag{4.26}
\]
Therefore in a QED-like gauge invariant field theory with massless fermions the axial current is not conserved: this is called the chiral anomaly. For any gauge invariant regularization one finds
\[
k_\lambda \Gamma^{(3)}_{\lambda\mu\nu}(k; p_1, p_2) = \left( \frac{e^2}{2\pi^2} = \frac{2\alpha}{\pi} \right) \varepsilon_{\mu\nu\rho\sigma} p_1\rho p_2\sigma . \tag{4.27}
\]
The equation (4.27) can be rewritten after Fourier transformation as an axial current non-conservation equation:

$$\partial_\lambda J^5_\lambda(x) = -i \frac{\alpha}{4\pi} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x). \quad (4.28)$$

Since global chiral symmetry is not broken, the integral over the whole space of the anomalous term must vanish. This condition is indeed verified since the anomaly can immediately be written as a total derivative:

$$\epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = 4 \partial_\mu (\epsilon_{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma). \quad (4.29)$$

The space integral of the anomalous term depends only on the behaviour of the gauge field at boundaries, and this property already indicates a connection between topology and anomalies.

The equation (4.28) also implies:

$$\ln \det \left[ e^{i\gamma_5 \theta(x)} \mathcal{P} e^{i\gamma_5 \theta(x)} \right] = \ln \det \mathcal{P} - i \frac{\alpha}{4\pi} \int d^4x \theta(x) \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x). \quad (4.30)$$

Remark. One might be surprised that in the calculation the divergence of the axial current does not vanish, though the regularization of the fermion propagator seems to be consistent with chiral symmetry. The reason is simple: if we add for example higher derivative terms to the action, the form of the axial current is modified and the additional contributions cancel the term we have found.

In the form we have presented it the calculation generalizes without difficulty to general even dimensions $2n$. Note simply that the permutation $(\eta_1, \mu) \leftrightarrow (\eta_2, \nu)$ in equation (4.16) is replaced by a cyclic permutation. If gauge invariance is maintained the anomaly in the divergence of the axial current $J^S_\lambda(x)$ in general is

$$\partial_\lambda J^S_\lambda(x) = -2i \frac{e^n}{(4\pi)^n n!} \epsilon_{\mu_1 \nu_1 \ldots \mu_n \nu_n} F_{\mu_1 \nu_1} \ldots F_{\mu_n \nu_n}, \quad (4.31)$$

where $\epsilon_{\mu_1 \nu_1 \ldots \mu_n \nu_n}$ is the completely antisymmetric tensor, and $J^S_\lambda \equiv J^{(2n+1)}_\lambda$ is the axial current.

Boson regulator. We have seen that we could also regularize by adding massive fermions and bosons with spin, the unpaired boson affecting transformation properties under space-dependent chiral transformations. Denoting by $\phi$ the boson field and by $M$ its mass, we perform in the regularized functional integral a change of variables of the form of a space-dependent chiral transformation acting in the same way on the fermion and boson field. The variation $\delta S$ of the action at first order in $\theta$ is

$$\delta S = \int d^4x \left[ \partial_\mu \theta(x) J^5_\mu(x) + 2i M \theta(x) \bar{\phi}(x) \gamma_5 \phi(x) \right],$$
with
\[ J^5_\mu(x) = i\bar{\psi}(x)\gamma_5\gamma_\mu\psi(x) + i\bar{\phi}(x)\gamma_5\gamma_\mu\phi(x). \]

Expanding in \( \theta \) and identifying the coefficient of \( \theta(x) \) we therefore obtain the equation
\[
\langle \partial_\mu J^5_\mu(x) \rangle = 2iM \langle \bar{\phi}(x)\gamma_5\phi(x) \rangle = -2iM \text{tr} \gamma_5 \langle x | \mathcal{D}^{-1} | x \rangle .
\] (4.32)

The divergence of the axial current comes here from the boson contribution. We know that in the large \( M \) limit it becomes quadratic in \( A \). Expanding the r.h.s. in powers of \( A \), keeping the quadratic term we find after Fourier transform:
\[
C^{(2)}(k) = -2iMe^2 \int d^4p_1 d^4p_2 A_\mu(p_1)A_\nu(p_2) \int \frac{d^4q}{(2\pi)^4} \times \text{tr} \left[ \gamma_5(\not{q} + \not{k} - iM)^{-1}\gamma_\mu(\not{q} - \not{p}_2 - iM)^{-1}\gamma_\nu(\not{q} - iM)^{-1} \right].
\] (4.33)

The apparent divergence of this contribution is regularized by formally vanishing diagrams that we do not write, but which justify the following formal manipulations.

In the trace the formal divergences cancel and one obtains
\[
C^{(2)}(k) \sim 8M^2e^2\epsilon_{\mu\nu\rho\sigma} \int d^4p_1 d^4p_2 p_1\mu p_2\sigma A_\mu(p_1)A_\nu(p_2) \times \frac{1}{(2\pi)^4} \int \frac{d^4q}{(q^2 + M^2)^3}.
\]
The limit \( M \to \infty \) corresponds to remove the regulator. The limit is finite because after rescaling of \( q \) the mass can be eliminated. One finds:
\[
C^{(2)}(k) \sim \frac{e^2}{4\pi^2}\epsilon_{\mu\nu\rho\sigma} \int d^4p_1 d^4p_2 (p_2)p_1\mu p_2\sigma A_\mu(p_1)A_\nu(p_2),
\]
in agreement with equation (4.27).

**Point-splitting regularization.** Another calculation, based on regularization by point splitting, gives further insight into the mechanism that generates the anomaly. We thus consider the non-local operator
\[
J^5_\mu(x,a) = i\bar{\psi}(x - a/2)\gamma_5\gamma_\mu\psi(x + a/2) \exp \left[ ie \int_{x-a/2}^{x+a/2} A_\lambda(s)ds_\lambda \right],
\] (4.34)
in the limit \( |a| \to 0 \). To avoid a breaking of rotation symmetry by the regularization, before taking the limit \( |a| \to 0 \) we will average over all orientations of the vector \( a \). The multiplicative gauge factor (parallel transporter) ensures gauge
invariance of the regularized operator (transformations (4.3)). The divergence of the operator for $|a| \to 0$ then becomes

$$
\partial^\mu J^5_\mu(x,a) \sim -ea_\lambda \bar{\psi}(x-a/2)\gamma_5\gamma_\mu F_\mu\lambda(x)\psi(x+a/2)
$$

$$
\times \exp \left[ ie \int_{x-a/2}^{x+a/2} A_\lambda(s)ds \right],
$$

(4.35)

where the $\psi, \bar{\psi}$ field equations have been used. We now expand the expectation value of the equation in powers of $A$. The first term vanishes. The second term is quadratic in $A$ and yields

$$
\langle \partial^\mu J^5_\mu(x,a) \rangle \sim ie^2a_\lambda F_\mu\lambda(x) \int d^4y A_\nu(y+x) \text{tr} \gamma_5 \Delta_F(y-a/2)\gamma_\nu \Delta_F(-y-a/2)\gamma_\mu
$$

where $\Delta_F(y)$ is the fermion propagator:

$$
\Delta_F(y) = -\frac{i}{(2\pi)^4} \int d^4k \frac{k^\rho}{k^2} = \frac{1}{2\pi^2} \frac{y^\rho y^\sigma}{y^4}.
$$

(4.36)

We now take the trace. The propagator is singular for $|y| = O(|a|)$ and therefore we can expand $A_\nu(x+y)$ in powers of $y$. The first term vanishes for symmetry reasons ($y \mapsto -y$), and we obtain

$$
\langle \partial^\mu J^5_\mu(x,a) \rangle \sim \frac{ie^2}{\pi^4} \epsilon_{\mu\nu\tau\sigma}a_\lambda F_\mu\lambda(x)\partial_\rho A_\nu(x) \int d^4y \frac{y_\rho y_\sigma a_\tau}{|y+a/2|^4|y-a/2|^4}.
$$

The integral over $y$ gives a linear combination of $\delta_{\rho\sigma}$ and $a_\rho a_\sigma$ but the second term gives a vanishing contribution due to $\epsilon$ symbol. It follows

$$
\langle \partial^\mu J^5_\mu(x,a) \rangle \sim \frac{ie^2}{3\pi^4} \epsilon_{\mu\nu\tau\rho}a_\lambda a_\sigma F_\mu\lambda(x)\partial_\rho A_\nu(x) \int d^4y \frac{y^2-\frac{(y \cdot a)^2}{a^2}}{|y+a/2|^4|y-a/2|^4}.
$$

After integration we then find

$$
\langle \partial^\mu J^5_\mu(x,a) \rangle \sim \frac{ie^2}{4\pi^2} \epsilon_{\mu\nu\tau\rho} \frac{a_\lambda a_\sigma}{a^2} F_\mu\lambda(x) F_\rho\nu(x).
$$

Averaging over the $a$ directions we see that the divergence is finite for $|a| \to 0$, and thus finally

$$
\lim_{|a| \to 0} \langle \partial^\mu J^5_\mu(x,a) \rangle = \frac{ie^2}{16\pi^2} \epsilon_{\mu\nu\lambda\rho} F_\mu\lambda(x) F_\rho\nu(x),
$$

and we recover the result (4.28).
On the lattice an averaging over $a_{\mu}$ is produced by summing over all lattice directions. Because the only expression quadratic in $a_{\mu}$ that has the symmetry of the lattice is $a^2$ the same result is found: the anomaly is lattice-independent.

A direct physical application. In a phenomenological model of Strong Interaction physics, where the a $SU(2) \times SU(2)$ chiral symmetry is softly broken by the pion mass, in the absence of anomalies the divergence of the neutral axial current is proportional to the $\pi_0$ field (corresponding to the neutral pion). A short calculation then shows the decay of $\pi_0$ into two photons would vanish a zero momentum. The axial anomaly\(^\text{[4.28]}\) gives instead a non-vanishing contribution to the decay, in good agreement with experimental data.

Chiral gauge theory. A gauge theory is consistent only if the gauge field is coupled to a conserved current. An anomaly that affects the current destroys gauge invariance at the quantum level. Therefore the theory with axial gauge symmetry, where the action in the fermion sector reads

$$S(\bar{\psi}, \psi; B) = -\int d^4x \bar{\psi}(x)(\partial + ig\gamma_5 B)\psi(x), \quad (4.37)$$

is inconsistent. Indeed current conservation applies to the $BBB$ vertex at one-loop order. Because now the three point vertex is symmetric the divergence is given by the expression\(^\text{[4.21]}\), and thus does not vanish.

More generally the anomaly prevents the construction of a theory that would have both an abelian gauge vector and axial symmetry, where the action in the fermion sector would read

$$S(\bar{\psi}, \psi; A, B) = -\int d^4x \bar{\psi}(x)(\partial + ieA + i\gamma_5 g B)\psi(x). \quad (4.38)$$

A way to solve both problems is to cancel the anomaly by introducing another fermion of opposite chiral coupling. With more fermions other combinations of couplings are possible. Note, however that in the purely axial gauge theory it is easy to verify that a theory with two fermions of opposite chiral charges can be rewritten as a vector theory by combining differently the chiral components of both fermions.

4.3 Two dimensions

As an exercise we verify by explicit calculation the general expression\(^\text{[4.31]}\) in the special example of dimension two

$$\partial_\mu J^3_\mu = -i\frac{e}{2\pi} \epsilon_{\mu\nu} F_{\mu\nu}. \quad (4.39)$$

The general form of the l.h.s. is again dictated by locality and power counting: the anomaly must have canonical dimension two. The explicit calculation requires
some care since massless fields may lead to IR divergences in two dimensions. One thus gives a mass \( m \) to fermions, which breaks chiral symmetry explicitly, and takes the massless limit at the end of the calculation. The calculation involves only one diagram

\[
\Gamma^{(2)}_{\mu\nu}(k, -k) = \frac{\delta}{\delta A_\nu(-k)} \langle J_3^\mu(k) \rangle \bigg|_{A=0} = \frac{\delta}{\delta A_\nu(-k)} i \text{tr} \left[ \gamma_3 \gamma_\mu D^{-1}(k) \right] \bigg|_{A=0} = \frac{e}{(2\pi)^2} \text{tr} \gamma_3 \gamma_\mu \int \frac{d^2 q}{i q + m} \frac{1}{\gamma_\nu} \frac{1}{i q + i k + m}.
\]

Here the \( \gamma \)-matrices are simply the ordinary Pauli matrices. Then

\[
k_\mu \Gamma^{(2)}_{\mu\nu}(k, -k) = \frac{e}{(2\pi)^2} \text{tr} \gamma_3 i k \int \frac{d^2 q}{i q + m} \frac{1}{\gamma_\nu} \frac{1}{i q + i k + m}.
\]

We use the method of the boson regulator, which leads to the two-dimensional analogue of equation (4.32). It here leads to the calculation of the difference between two diagrams (analogues of equation (4.33)) due to the explicit chiral symmetry breaking

\[
C_\mu(k) = 2m \frac{e}{(2\pi)^2} \text{tr} \gamma_3 \int \frac{d^2 q}{i q + m} \frac{1}{\gamma_\nu} \frac{1}{i q + i k + m} - (m \mapsto M)
\]

\[
= 2m \frac{e}{(2\pi)^2} \text{tr} \gamma_3 \int \frac{d^2 q}{(q^2 + m^2)} \frac{1}{(k + q)^2} - (m \mapsto M).
\]

In the trace again the divergent terms cancel

\[
C_\mu(k) = 4em^2 \epsilon_{\mu\nu} k_\nu \frac{1}{(2\pi)^2} \int \frac{d^2 q}{(q^2 + m^2)} \frac{1}{(k + q)^2} - (m \mapsto M).
\]

The two contributions are now separately convergent. When \( m \to 0 \) the \( m^2 \) factor dominates the logarithmic IR divergence and the contribution vanishes. In the second term in the limit \( M \to \infty \) one obtains

\[
C_\mu(k) \bigg|_{m \to 0, M \to \infty} \sim -4eM^2 \epsilon_{\mu\nu} k_\nu \frac{1}{(2\pi)^2} \int \frac{d^2 q}{(q^2 + M^2)^2} = \frac{e}{\pi} \epsilon_{\mu\nu} k_\nu,
\]

in agreement with equation (4.39).

4.4 Non-abelian vector gauge fields and abelian axial current

We still consider an abelian axial current but now in the framework of a non-abelian gauge theory. The fermion fields transform non-trivially under a gauge group \( G \) and \( A_\mu \) is the corresponding gauge field. The action is:

\[
S(\bar{\psi}, \psi; A) = - \int d^4x \bar{\psi}(x) D\psi(x), \quad (4.40)
\]
with the convention (2.13) and:

$$D = \partial + A.$$  \hfill (4.41)

In a gauge transformation of unitary matrix \(g(x)\) the gauge field \(A_\mu\) and the Dirac operator become

$$A_\mu(x) \mapsto g(x) \partial_\mu g^{-1}(x) + g(x) A_\mu(x) g^{-1}(x) \Rightarrow D \mapsto g^{-1}(x) D g(x).$$ \hfill (4.42)

The axial current \(J_5^\mu\)

$$J_5^\mu(x) = i \bar{\psi}(x) \gamma_5 \gamma_\mu \psi(x),$$

is still gauge invariant. Therefore no new calculation is needed; the result is completely determined by dimensional analysis, gauge invariance and the preceding abelian calculation that yields the term of order \(A^2\),

$$\partial_\lambda J_5^\lambda(x) = -\frac{i}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr} F_{\mu\nu} F_{\rho\sigma},$$ \hfill (4.43)

in which \(F_{\mu\nu}\) now is the corresponding curvature \((2.16)\). Again this expression must be a total derivative. One indeed verifies:

$$\epsilon_{\mu\nu\rho\sigma} \text{tr} F_{\mu\nu} F_{\rho\sigma} = 4 \epsilon_{\mu\nu\rho\sigma} \partial_\mu (A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma).$$ \hfill (4.44)

4.5 Anomaly and eigenvalues of the Dirac operator

We assume that the spectrum of \(D\), the Dirac operator in a non-abelian gauge field (equation (4.41)), is discrete (putting temporarily the fermions in a box if necessary) and call \(d_n\) and \(\varphi_n(x)\) the corresponding eigenvalues and eigenvectors:

$$D \varphi_n = d_n \varphi_n.$$ \hfill (4.45)

For a unitary or orthogonal group the massless Dirac operator is anti-hermitian; therefore the eigenvalues are imaginary and the eigenvectors orthogonal. In addition we choose them with unit norm.

The eigenvalues are gauge invariant, because in a gauge transformation of unitary matrix \(g(x)\) the Dirac operator transforms like in (4.42), and thus simply

$$\varphi_n(x) \mapsto g(x) \varphi_n(x).$$

The anticommutation \(\gamma_5 D + D \gamma_5 = 0\) implies

$$D \gamma_5 \varphi_n = -d_n \gamma_5 \varphi_n.$$ \hfill (4.46)

Therefore either \(d_n\) is different from zero, and \(\gamma_5 \varphi_n\) is an eigenvector of \(D\) with eigenvalue \(-d_n\), or \(d_n\) vanishes. The eigenspace corresponding to the eigenvalue
0 then is invariant under $\gamma_5$, which can be diagonalized: the eigenvectors of $\mathbf{D}$ can be chosen eigenvectors of definite chirality, i.e. eigenvectors of $\gamma_5$ with eigenvalue $\pm 1$,

$$\mathbf{D} \varphi_n = 0, \quad \gamma_5 \varphi_n = \pm \varphi_n.$$ 

We call $n_+$ and $n_-$ the dimensions of the eigenspace of positive and negative chirality respectively.

We now consider the determinant of the operator $\mathbf{D} + m$ regularized by mode truncation (mode regularization):

$$\det_N (\mathbf{D} + m) = \prod_{n \leq N} (d_n + m), \quad (4.47)$$

keeping the $N$ lowest eigenvalues of $\mathbf{D}$ (in modulus), with $N - n_+ - n_-$ even, in such a way that the corresponding subspace is $\gamma_5$ invariant.

The regularization is gauge invariant because the eigenvalues of $\mathbf{D}$ are gauge invariant.

Note that in the truncated space

$$\text{tr} \gamma_5 = n_+ - n_-.$$ 

(4.48)

The trace of $\gamma_5$ equals $n_+ - n_-$, the index of the Dirac operator $\mathbf{D}$. A non-vanishing index thus endangers axial current conservation.

In a chiral transformation (4.4) with constant $\theta$ the determinant of $(\mathbf{D} + m)$ becomes

$$\det_N (\mathbf{D} + m) \mapsto \det_N (e^{i\theta \gamma_5 (\mathbf{D} + m)} e^{i\theta \gamma_5}).$$

We now consider the various eigenspaces.

If $d_n \neq 0$ the matrix $\gamma_5$ is represented by the Pauli matrix $\sigma_1$ in the sum of eigenspaces corresponding to the two eigenvalues $\pm d_n$ and $\mathbf{D} + m$ by $d_n \sigma_3 + m$. The determinant in the subspace then is

$$\det (e^{i\theta} \sigma_1 (d_n \sigma_3 + m) e^{i\theta} \sigma_1) = \det e^{2i\theta \sigma_1} \det (d_n \sigma_3 + m) = m^2 - d_n^2,$$

because $\sigma_1$ is traceless.

In the eigenspace of dimension $n_+$ of vanishing eigenvalues $d_n$ with eigenvectors with positive chirality, $\gamma_5$ is diagonal with eigenvalue 1 and thus

$$m^{n_+} \mapsto m^{n_+} e^{2i\theta n_+}.$$ 

Similarly in the eigenspace of chirality $-1$ and dimension $n_-$

$$m^{n_-} \mapsto m^{n_-} e^{-2i\theta n_-}.$$ 

We conclude

$$\det_N (e^{i\theta \gamma_5 (\mathbf{D} + m)} e^{i\theta \gamma_5}) = e^{2i\theta (n_+ - n_-)} \det_N (\mathbf{D} + m),$$
The ratio of both determinants is independent of $N$. Taking the limit $N \to \infty$ we find
\[
\det \left[ (e^{i\gamma_5 \theta} (\mathcal{D} + m) e^{i\gamma_5 \theta}) (\mathcal{D} + m)^{-1} \right] = e^{2i\theta(n_+ - n_-)}.
\] (4.49)

Note that the l.h.s. of equation (4.49) is obviously 1 when $\theta = n\pi$, which implies that the coefficient of $2\theta$ in the r.h.s. must indeed be an integer.

The variation of $\ln \det (\mathcal{D} + m)$
\[
\ln \det \left[ (e^{i\gamma_5 \theta} (\mathcal{D} + m) e^{i\gamma_5 \theta}) (\mathcal{D} + m)^{-1} \right] = 2i\theta (n_+ - n_-),
\]
at first order in $\theta$ is related to the variation of the action (4.1) and thus to the expectation value of the integral of the divergence of the axial current, $\int d^4x \langle \partial_{\mu} J_{\mu}^5(x) \rangle$ in four dimensions. In the limit $m = 0$ it is thus related to the space integral of the chiral anomaly (4.43).

We have thus found a local expression giving the index of the Dirac operator:
\[
-\frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \int d^4x \, \text{tr} \, F_{\mu\nu} F_{\rho\sigma} = n_+ - n_-.
\] (4.50)

Concerning this result several comments can be made:

(i) At first order in $\theta$ in the absence of regularization we have calculated ($\ln \det = \text{tr} \ln$)
\[
\ln \det \left[ 1 + i\theta \left( \gamma_5 + (\mathcal{D} + m) \gamma_5 (\mathcal{D} + m)^{-1} \right) \right] \sim 2i\theta \text{tr} \, \gamma_5,
\]
where we have used the cyclic property of the trace. Since the trace of the matrix $\gamma_5$ vanishes we could expect naively a vanishing result. But trace here means trace in $\gamma$ space and in coordinate space, and $\gamma_5$ really stands here for $\gamma_5 \delta(x - y)$. The mode regularization yields a well-defined finite result for the ill defined product $0 \times \delta^d(0)$.

(ii) The property that the integral (4.50) is quantized shows that the form of the anomaly is related to topological properties of the gauge field since the integral does not change when the gauge field is deformed continuously. The integral of the anomaly over the whole space thus depends only on the behaviour at large distances of the curvature tensor $F_{\mu\nu}$ and the anomaly must a total derivative as equation (4.44) confirms.

(iii) One might be surprised that $\det \mathcal{D}$ is not invariant under global chiral transformations. However we have just established that when the integral of the anomaly does not vanish, $\det \mathcal{D}$ vanishes. This explains that to give a meaning to the r.h.s. of equation (4.49) we have been forced to introduce a mass to find a non-trivial result. The determinant of $\mathcal{D}$ in the subspace orthogonal to eigenvectors with vanishing eigenvalue, even in presence of a mass, is chiral invariant by parity doubling, but for $n_+ \neq n_-$ not the determinant in the eigenspace of eigenvalue
zero because the trace of $\gamma_5$ does not vanish in the eigenspace (equation (4.48)). In the limit $m \to 0$ the complete determinant vanishes but not the ratio of determinants for different values of $\theta$ because the powers of $m$ cancel.

(iv) The discussion of the index of the Dirac operator is valid in any even dimension. Therefore the topological character and the quantization of the space integral of the anomaly are general.

5 Instantons, anomalies and $\theta$-vacua

We now discuss the role of instantons in several examples where the classical potential has a periodic structure with an infinite set of degenerate minima. We exhibit their topological character, and in the presence of gauge fields relate them to anomalies and the index of the Dirac operator. Instantons imply that the eigenstates of the hamiltonian depend on an angle $\theta$. In the quantum field theory the notion of $\theta$-vacuum emerges.

5.1 The periodic cosine potential

As a first example of the role of instantons when topology is involved we consider a simple hamiltonian with a periodic potential:

$$H = -\frac{g^2}{2} \left( \frac{d}{dx} \right)^2 + \frac{1}{2g} \sin^2 x. \quad (5.1)$$

The potential has an infinite number of degenerate minima $x = n\pi$, $n \in \mathbb{Z}$. Each minimum is an equivalent starting point for a perturbative calculation of the eigenvalues of $H$. Periodicity implies that the perturbative expansions are identical to all orders in $g$, a property that seems to imply that the quantum hamiltonian has an infinite number of degenerate eigenstates. In reality we know that the exact spectrum of the hamiltonian $H$ is not degenerate, as a result of barrier penetration. Instead it is continuous and has, at least for $g$ small enough, a band structure.

The structure of the ground state. To characterize more precisely the structure of the spectrum of the hamiltonian (5.1) we introduce the operator $T$ that generates an elementary translation of one period $\pi$

$$T\psi(x) = \psi(x + \pi).$$

Since $T$ commutes with the hamiltonian,

$$[T, H] = 0, \quad (5.2)$$

both operators can be diagonalized simultaneously. Because the eigenfunctions of $H$ must be bounded at infinity, the eigenvalues of $T$ are pure phases. Each eigenfunction of $H$ thus is characterized by an angle $\theta$ (pseudo-momentum) eigenvalue of $T$:

$$T|\theta\rangle = e^{i\theta}|\theta\rangle. \quad (5.3)$$
The corresponding eigenvalues $E_n(\theta)$ are periodic functions of $\theta$ and for $g \to 0$ are close to the eigenvalues of the harmonic oscillator

$$E_n(\theta) = n + 1/2 + O(g).$$

To all orders in powers of $g$ $E_n(\theta)$ is independent of $\theta$ and the spectrum of $H$ is infinitely degenerate. Exponentially small contributions due to instantons lift the degeneracy and introduce a $\theta$ dependence. To each value of $n$ then corresponds a band when $\theta$ varies in $[0, 2\pi]$.

Path integral representation. The spectrum of $H$ can be extracted from the calculation of the quantity $Z_\ell$

$$Z_\ell(\beta) = \text{tr} T^\ell e^{-\beta H} = \frac{1}{2\pi} \sum_{n=0}^{\infty} \int d\theta \ e^{-i\ell \theta} e^{-\beta E_n(\theta)}. $$

Indeed

$$Z(\theta, \beta) \equiv \sum_\ell e^{i\ell \theta} Z_\ell(\beta) = \sum_n e^{-\beta E_n(\theta)}, $$

(5.4)

where $Z(\theta, \beta)$ is the partition function restricted to states with a fixed $\theta$ angle.

The path integral representation of $Z_\ell(\beta)$ differs from the representation of the partition function $Z_0(\beta)$ only by the boundary conditions. The operator $T$ has the effect of translating the argument $x$ in the matrix element $\langle x' | \text{tr} e^{-\beta H} | x \rangle$ before taking the trace:

$$Z_\ell(\beta) = \int_{x(\beta/2) = x(-\beta/2) + \ell \pi} [dx(t)] \exp \left[-S(x)\right],$$

(5.5)

$$S(x) = \frac{1}{2g} \int_{-\beta/2}^{\beta/2} \left[ \dot{x}^2(t) + \sin^2(x(t)) \right] dt.$$  (5.6)

A careful study of the trace operation in the case of periodic potentials shows that $x(-\beta/2)$ varies over only one period (see Appendix A1).

Therefore from (5.4) we derive the path integral representation of $Z(\theta, \beta)$:

$$Z(\theta, \beta) = \sum_\ell \int_{x(\beta/2) = x(-\beta/2) + \ell \pi} [dx(t)] \exp \left[-S(x) + i\ell \theta\right]$$

$$= \int_{x(\beta/2) = x(-\beta/2) (\text{mod } \pi)} [dx(t)] \exp \left[-S(x) + \frac{\theta}{\pi} \int_{-\beta/2}^{\beta/2} dt \dot{x}(t)\right].$$  (5.7)

Note that $\ell$ is a topological number since two trajectories with different values of $\ell$ cannot be related continuously. In the same way

$$Q = \frac{1}{\pi} \int_{-\beta/2}^{\beta/2} dt \dot{x}(t),$$

(5.8)
is a topological charge; it depends on the trajectory only through the boundary conditions.

For $\beta$ large and $g \to 0$ the path integral is dominated by the constant solutions 
$x_c(t) = 0 \mod \pi$ corresponding to the $\ell = 0$ sector. A non-trivial $\theta$ dependence

\[ \int dt \left[ \dot{x}(t) \pm \sin(x(t)) \right]^2 \geq 0 \Rightarrow \mathcal{S} \geq |\cos(x(+\infty)) - \cos(x(-\infty))| / g. \quad (5.9) \]

The action is finite only if $x(\pm\infty) \in \{0, \pi\}$. The non-vanishing value of the l.h.s.

\[ \dot{x}_c = \pm \sin x_c \Rightarrow x_c(t) = 2 \tan^{-1} e^{\pm(t-t_0)}, \quad (5.10) \]

and the corresponding classical action then is:

\[ \mathcal{S}(x_c) = 2/g. \quad (5.11) \]

The instanton solutions belong to the $\ell = \pm 1$ sector and connect two consecutive minima of the potential. They yield the leading contribution to barrier penetration for $g \to 0$. An explicit calculation yields

\[ E_0(g) = E_{\text{pert.}}(g) - \frac{4}{\sqrt{\pi} g} e^{-2/g} \cos \theta [1 + O(g)]. \]

5.2 Instantons and anomaly: $CP(N - 1)$ models

We now consider field theories, the two-dimensional $CP(N - 1)$ models, where again instantons and topology play a role and the semi-classical vacuum has a similar periodic structure. The new feature is the relation between the topological charge and the two-dimensional chiral anomaly.

We here mainly describe the nature of the instanton solutions and refer the reader to the literature for a more detailed analysis. Note that the explicit calculation of instanton contributions in the small coupling limit in the $CP(N - 1)$ models, as well as in the non-abelian gauge theories we discuss in section 5.3, remains to large extent an unsolved problem. Due to the scale invariance of the classical theory, instantons depend on a scale (or size) parameter. Instanton contributions then involves the running coupling constant at the instanton size. Both families of theories are UV asymptotically free. Therefore the running coupling is small for small instantons and the semi-classical approximation is justified. However, in the absence of any IR cut-off, the running coupling becomes large for large instantons, and it is unclear whether a semi-classical approximation remains valid.
The \( CP(N-1) \) manifolds. We consider a \( N \)-component complex vector \( \varphi \) of unit length,
\[
\bar{\varphi} \cdot \varphi = 1.
\] (5.12)
This condition characterizes a space isomorphic to the quotient space \( U(N)/U(N-1) \). In addition two vectors \( \varphi \) and \( \varphi' \) are considered equivalent if
\[
\varphi' \equiv \varphi \iff \varphi'_\alpha = e^{iA} \varphi_\alpha.
\] (5.13)
This condition characterizes the symmetric space and complex Grassmannian manifold \( U(N)/U(1)/U(N-1) \). It is isomorphic to the manifold \( CP(N-1) \) (for \( N-1 \)-dimensional Complex Projective), which is obtained from \( \mathbb{C}^N \) by the equivalence relation
\[
z_\alpha \equiv z'_\alpha \quad \text{if} \quad z'_\alpha = \lambda z_\alpha
\]
where \( \lambda \) belongs to the Riemann sphere (compactified complex plane).

The \( CP(N-1) \) models. A symmetric space admits a unique metric, up to a multiplicative factor, and this leads to a unique action with two derivatives. One form of the unique symmetric classical action is:
\[
S(\varphi, A_\mu) = \frac{1}{g} \int d^2x \bar{\varphi} \cdot D_\mu \varphi,
\] (5.14)
in which \( g \) is a coupling constant and \( D_\mu \) the covariant derivative:
\[
D_\mu = \partial_\mu + iA_\mu.
\] (5.15)
The field \( A_\mu \) is a gauge field for the \( U(1) \) transformations
\[
\varphi'(x) = e^{iA(x)} \varphi(x), \quad A'_\mu(x) = A_\mu(x) - \partial_\mu \Lambda(x).
\] (5.16)
The action is obviously \( U(N) \) symmetric and the gauge symmetry ensures the equivalence (5.13).

Since the action contains no kinetic term for \( A_\mu \) the gauge field is not a dynamical field but only an auxiliary field that can be integrated out. The action is quadratic in \( A \) and the gaussian integration results in replacing in the action \( A_\mu \) by the solution of the \( A \)-field equation:
\[
A_\mu = i\bar{\varphi} \cdot \partial_\mu \varphi,
\] (5.17)
where the equation (5.12) has been used. After this substitution the field \( \bar{\varphi} \cdot \partial_\mu \varphi \) acts as a composite gauge field.

For what follows however we find more convenient to keep \( A_\mu \) as an independent field.
Instantons. To prove the existence of locally stable non-trivial minima of the action the following Bogomolnyi inequality can be used (note the analogy with (5.9)):

$$\int d^2x |D_\mu \varphi \mp i\epsilon_{\mu\nu} D_\nu \varphi|^2 \geq 0,$$

(5.18)

($\epsilon_{\mu\nu}$ being the antisymmetric tensor, $\epsilon_{12} = 1$). After expansion the inequality can be cast in the form

$$S(\varphi) \geq 2\pi |Q(\varphi)|/g,$$

(5.19)

with

$$Q(\varphi) = -\frac{i}{2\pi} \epsilon_{\mu\nu} \int d^2x D_\mu \varphi \cdot \overline{D_\nu \varphi} = \frac{i}{2\pi} \int d^2x \epsilon_{\mu\nu} D_\nu D_\mu \varphi \cdot \bar{\varphi}.$$  

(5.20)

Then

$$i\epsilon_{\mu\nu} D_\nu D_\mu = \frac{1}{2} i\epsilon_{\mu\nu} [D_\nu, D_\mu] = \frac{1}{2} F_{\mu\nu},$$

(5.21)

where $F_{\mu\nu}$ is the curvature

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$  

Therefore, using (5.12),

$$Q(\varphi) = \frac{1}{4\pi} \int d^2x \epsilon_{\mu\nu} F_{\mu\nu}.$$  

(5.22)

The integrand is proportional to the two-dimensional abelian chiral anomaly (4.39), and thus is a total divergence

$$\frac{1}{2} \epsilon_{\mu\nu} F_{\mu\nu} = \partial_\mu \epsilon_{\mu\nu} A_\nu.$$  

Substituting this form into equation (5.22) and integrating in a large disk of radius $R$ one obtains:

$$Q(\varphi) = \frac{1}{2\pi} \lim_{R \to \infty} \oint_{|x|=R} d\mu A_\mu(x).$$

(5.23)

$Q(\varphi)$ thus depends only on the behaviour of the classical solution for $|x|$ large and is a topological charge. Finiteness of the action demands that at large distances $D_\mu \varphi$ vanishes and therefore

$$D_\mu \varphi = 0 \Rightarrow [D_\mu, D_\nu] \varphi = F_{\mu\nu} \varphi = 0,$$

Since $\varphi \neq 0$ this equation implies that $F_{\mu\nu}$ vanishes and thus $A_\mu$ is a pure gauge (and $\varphi$ a gauge transform of a constant vector)

$$A_\mu = \partial_\mu \Lambda(x) \Rightarrow Q(\varphi) = \frac{1}{2\pi} \lim_{R \to \infty} \oint_{|x|=R} d\mu \partial_\mu \Lambda(x).$$

(5.24)
The topological charge measures the variation of the angle $\Lambda(x)$ on a large circle, which is a multiple of $2\pi$ because $\varphi$ is regular. One is thus led to the consideration of the homotopy classes of mappings from $U(1)$, i.e. $S_1$ to $S_1$, which are characterized by an integer $n$, the winding number. This is equivalent to the statement that the homotopy group $\pi_1(S_1)$ is isomorphic to the additive group of integers $\mathbb{Z}$.

Then

$$Q(\varphi) = n \implies S(\varphi) \geq 2\pi|n|/g.$$ (5.25)

The equality $S(\varphi) = 2\pi|n|/g$ corresponds to a local minimum and implies that the classical solutions satisfy first order partial differential (self-duality) equations:

$$D_\mu \varphi = \pm i\epsilon_{\mu\nu} D_\nu \varphi.$$ (5.26)

For each sign there is really only one equation for instance $\mu = 1, \nu = 2$. It is simple to verify that both equations imply the $\varphi$-field equations, and combined with the constraint (5.12) the $A$-field equation (5.17). In the complex coordinates $z = x_1 + i x_2$, $\bar{z} = x_1 - i x_2$ they can be written

$$\partial_z \varphi_\alpha(z, \bar{z}) = -i A_z(z, \bar{z}) \varphi_\alpha(z, \bar{z}),$$

$$\partial_{\bar{z}} \varphi_\alpha(z, \bar{z}) = -i A_{\bar{z}}(z, \bar{z}) \varphi_\alpha(z, \bar{z}).$$

Exchanging the two equations just amounts to exchange $\varphi$ and $\bar{\varphi}$. We therefore solve only the second equation

$$\varphi_\alpha(z, \bar{z}) = \kappa(z, \bar{z}) P_\alpha(z),$$

where $\kappa(z, \bar{z})$ is a particular solution of

$$\partial_{\bar{z}} \kappa(z, \bar{z}) = -i A_{\bar{z}}(z, \bar{z}) \kappa(z, \bar{z}).$$

Vectors solutions of the equations (5.26) are proportional to holomorphic or anti-holomorphic (depending on the sign) vectors (this reflects the conformal invariance of the classical field theory). The function $\kappa(z, \bar{z})$, which gauge invariance allows to choose real (this corresponds to the $\partial_\mu A_\mu = 0$ gauge), then is constrained by the condition (5.12)

$$\kappa^2(z, \bar{z}) P \cdot \bar{P} = 1.$$ (5.12)

The asymptotic conditions constrain the functions $P_\alpha(z)$ to be polynomials. Common roots to all $P_\alpha$ would correspond to non-integrable singularities for $\varphi_\alpha$, and therefore are excluded by the condition of finiteness of the action. Finally if the polynomials have maximal degree $n$, asymptotically

$$P_\alpha(z) \sim c_\alpha z^n \Rightarrow \varphi_\alpha \sim \frac{c_\alpha}{\sqrt{c \cdot \bar{c}}} (z/\bar{z})^{n/2}.$$
When the phase of $z$ varies by $2\pi$, the phase of $\varphi_\alpha$ varies by $2n\pi$, showing that the corresponding winding number is $n$.

The structure of the semi-classical vacuum. In contrast to our analysis of periodic potentials in quantum mechanics, we have here discussed the existence of instantons without reference to the structure of the classical vacuum. To find an interpretation of instantons in gauge theories, it is useful to express the results in the temporal gauge. Then classical minima of the potential correspond to fields $\varphi(x_1)$, where $x_1$ is only the space variable, gauge transforms of a constant vector:

$$\varphi(x_1) = e^{i\Lambda(x_1)} \mathbf{v}, \quad \mathbf{v} \cdot \mathbf{v} = 1.$$ 

Moreover if the vacuum state is invariant under space reflection $\varphi(+\infty) = \varphi(-\infty)$ and thus

$$\Lambda(+\infty) - \Lambda(-\infty) = 2\nu\pi \quad \nu \in \mathbb{Z}.$$ 

Again $\nu$ is a topological number that classifies degenerate classical minima, and the semi-classical vacuum thus has a periodic structure. This analysis is consistent with Gauss’s law that only implies that states are invariant under infinitesimal gauge transformations, and therefore under gauge transformations of the class $\nu = 0$ that are continuously connected to the identity.

We now consider a large rectangle with extension $R$ in the space direction and $T$ in the euclidean time direction and by a smooth gauge transformation continue the instanton solution to the temporal gauge. Then the variation of the pure gauge comes entirely from the sides at fixed time. One finds for $R \to \infty$,

$$\Lambda(+\infty, 0) - \Lambda(-\infty, 0) - [\Lambda(+\infty, T) - \Lambda(-\infty, T)] = 2n\pi.$$ 

Therefore instantons interpolate between different classical minima. Like in the case of the cosine potential, one projects onto a proper quantum eigenstate, the “$\theta$-vacuum” corresponding to an angle $\theta$ by adding, in analogy with the expression (5.7), a topological term to the classical action

$$S(\varphi) \mapsto S(\varphi) + i\frac{\theta}{4\pi} \int d^2 x \epsilon_{\mu\nu} F^\mu_{\nu}.$$ 

Remark. Replacing in the topological charge $Q$ the gauge field by the explicit expression (5.17) we find

$$Q(\varphi) = \frac{i}{2\pi} \int d^2 x \epsilon_{\mu\nu} \partial_\mu \bar{\varphi} \cdot \partial_\nu \varphi = \frac{i}{2\pi} \int d^2 x \, d\bar{\varphi}_\alpha \wedge d\varphi_\alpha,$$

where the notation of exterior differential calculus has been used. We recognize the integral of a two-form, a symplectic form and $4\pi Q$ is the area of a 2-surface.
embedded in $CP(N-1)$. A symplectic form is always closed, here it is also exact, so that $Q$ is the integral of a one-form (equation (5.23))

$$Q(\varphi) = \frac{i}{2\pi} \int \bar{\varphi}_\alpha d\varphi_\alpha = \frac{i}{4\pi} \int (\bar{\varphi}_\alpha d\varphi_\alpha - \varphi_\alpha d\bar{\varphi}_\alpha).$$

The $O(3)$ non-linear $\sigma$-model. The $CP(1)$ model is locally isomorphic to the $O(3)$ non-linear $\sigma$-model, with the identification

$$\phi^i(x) = \bar{\varphi}_\alpha(x) \sigma^i_{\alpha\beta} \varphi_\beta(x).$$

(5.27)

where $\sigma^i$ are the three Pauli matrices.

Using for example an explicit representation of Pauli matrices, one verifies indeed

$$\phi^i(\varphi^j) = 1, \quad \partial_\mu \phi^i(\varphi^j) = 4 \bf{D}_\mu \varphi \cdot \bf{D}_\mu \varphi.$$

Therefore the field theory can be expressed in terms of the field $\phi^i$, and takes the form of the non-linear $\sigma$-model. The fields $\phi$ are gauge invariant, and the whole physical picture is a picture of confinement of the charged scalar “quarks” $\varphi_\alpha(x)$ and the propagation of the $\phi^i$ neutral bound states.

Instantons in the $\phi$ description take the form of $\phi$ configurations with uniform limit for $|x| \to \infty$. They thus define a mapping from the compactified plane topologically equivalent to $S_2$ to the sphere $S_2$ (the $\phi^i$ configurations). Since $\pi_2(S_2) = \mathbb{Z}$ the $\varphi$ and $\phi$ pictures are consistent.

In the example of $CP(1)$ a solution of winding number 1 is

$$\varphi_1 = \frac{1}{\sqrt{1 + \bar{z}z}} , \quad \varphi_2 = \frac{z}{\sqrt{1 + \bar{z}z}} .$$

Translating the $CP(1)$ minimal solution into the $O(3)$ $\sigma$-model language one finds

$$\phi_1 = \frac{z + \bar{z}}{1 + \bar{z}z} , \quad \phi_2 = \frac{1}{i} \frac{z - \bar{z}}{1 + \bar{z}z} , \quad \phi_3 = \frac{1 - \bar{z}z}{1 + \bar{z}z} .$$

This defines a stereographic mapping of the plane onto the sphere $S_2$, as one verifies by setting $z = \tan(\eta/2) e^{i\theta}, \eta \in [0, \pi]$.

In the $O(3)$ representation the topological charge $4\pi Q$ has the interpretation of an area of surface in $S_2$, itself embedded in $\mathbb{R}^3$:

$$Q = \frac{i}{2\pi} \int d\bar{\varphi}_\alpha \wedge d\varphi_\alpha = \frac{1}{8\pi} \epsilon_{ijk} \int \phi_i d\phi_j \wedge \phi_k \equiv \frac{1}{8\pi} \epsilon_{\mu\nu} \epsilon_{ijkl} \int d^2x \phi_i \partial_\mu \phi_j \partial_\nu \phi_k .$$

The result is a multiple of the area of the sphere $S_2$, which in this interpretation explains the quantization.
5.3 Instantons and anomaly: non-abelian gauge theories

We now consider non-abelian gauge theories in four dimensions. Again gauge field configurations can be found that contribute to the chiral anomaly and for which therefore the r.h.s. of equation (4.50) does not vanish. A specially interesting example is provided by instantons, i.e. finite action solutions of euclidean field equations.

To discuss this problem it is sufficient to consider pure gauge theories and the gauge group $SU(2)$ since a general theorem states that for a Lie group containing $SU(2)$ as a subgroup the instantons are those of the $SU(2)$ subgroup.

In the absence of matter fields it is convenient to use a $SO(3)$ notation. The gauge field $A_\mu$ is a vector that is related to the element $A_\mu$ of the Lie algebra used previously as gauge field by

$$A_\mu = -\frac{1}{2}iA_\mu \cdot \sigma,$$

(5.28)

where $\sigma_i$ are the three Pauli matrices. The gauge action then reads:

$$S(A_\mu) = \frac{1}{4g^2} \int [F_{\mu\nu}(x)]^2 d^4x,$$

(5.29)

($g$ is the gauge coupling constant) where the curvature $F_{\mu\nu}$ is also a vector:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \times A_\nu.$$

(5.30)

The corresponding classical field equations are

$$D_\nu F_{\nu\mu} = \partial_\nu F_{\nu\mu} + A_\nu \times F_{\nu\mu} = 0.$$

(5.31)

The existence and some properties of instantons in this theory follow from considerations analogous to those presented for the $CP(N-1)$ model.

We define the dual of the tensor $F_{\mu\nu}$ by

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}.$$

(5.32)

Then the Bogomolnyi inequality

$$\int d^4x \left[ F_{\mu\nu}(x) \pm \tilde{F}_{\mu\nu}(x) \right]^2 \geq 0,$$

(5.33)

implies:

$$S(A_\mu) \geq 8\pi^2 |Q(A_\mu)|/g^2,$$

(5.34)

with

$$Q(A_\mu) = \frac{1}{32\pi^2} \int d^4x F_{\mu\nu} \cdot \tilde{F}_{\mu\nu}.$$

(5.35)
The expression $Q(A_\mu)$ is proportional to the integral of the chiral anomaly, here written in $SO(3)$ notation. We have shown that the quantity $F_{\mu\nu} \cdot \tilde{F}_{\mu\nu}$ is a pure divergence (equation (4.44)): 

$$F_{\mu\nu} \cdot \tilde{F}_{\mu\nu} = \partial_\mu V_\mu,$$

with

$$V_\mu = -4 \epsilon_{\mu\nu\rho\sigma} \text{tr} \left( A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma \right)$$

and

$$V_\mu = 2 \epsilon_{\mu\nu\rho\sigma} \left[ A_\nu \cdot \partial_\rho A_\sigma + \frac{1}{3} A_\nu \cdot (A_\rho \times A_\sigma) \right].$$

The integral thus depends only on the behaviour of the gauge field at large distances and its values are quantized (equation (4.50)). Here again, as in the $CP(N-1)$ model, the bound involves a topological charge, $Q(A_\mu)$.

We now use equation (4.44) and Stokes theorem

$$\int_D d^4x \partial_\mu V_\mu = \int_{\partial D} d\Omega \, \hat{n}_\mu V_\mu,$$

where $d\Omega$ is the measure on the boundary $\partial D$ of the four-volume $D$ and $\hat{n}_\mu$ the unit vector normal to $\partial D$. We take for $D$ a sphere of large radius $R$ and finds for the topological charge $Q$,

$$Q(A_\mu) = \frac{1}{32\pi^2} \int d^4x \text{tr} F_{\mu\nu} \cdot \tilde{F}_{\mu\nu} = \frac{1}{32\pi^2} R^3 \int_{r=R} d\Omega \, \hat{n}_\mu V_\mu,$$

The finiteness of the action implies that the classical solution must asymptotically become a pure gauge, i.e. with our conventions,

$$A_\mu = -\frac{1}{2} i A_\mu \cdot \sigma = g(x) \partial_\mu g^{-1}(x) + O(|x|^{-2}) \quad |x| \to \infty.$$

The element $g$ of the $SU(2)$ group can be parametrized in terms of Pauli matrices

$$g = u_4 + i u \cdot \sigma,$$

where $(u_4, u)$ is a four-component real vector belonging the unit sphere $S_3$:

$$u_4^2 + u^2 = 1.$$

Since $SU(2)$ is topologically equivalent to the sphere $S_3$, the pure gauge configurations on a sphere of large radius $|x| = R$ define a mapping from $S_3$ to $S_3$. Such mappings belong to different homotopy classes that are characterized by an integer called the winding number. Here we identify the homotopy group $\pi_3(S_3)$, which again is isomorphic to the additive group of integers $\mathbb{Z}$. 

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The simplest one to one mapping corresponds to an element $g(x)$ of the form

$$g(x) = \frac{x_4 + i\mathbf{x} \cdot \mathbf{\sigma}}{r}, \quad r = (x_4^2 + \mathbf{x}^2)^{1/2},$$  \hspace{1cm} (5.40)

and thus

$$A^i_m \sim 2(x_4 \delta_{im} + \epsilon_{imk} x_k) r^{-2}, \quad A_4^i = -2x_i r^{-2}.$$

Note that the transformation

$$g(x) \mapsto U_1 g(x) U_2^\dagger = g(Rx),$$

where $U_1$ and $U_2$ are two constant $SU(2)$ matrices, induces a $SO(4)$ rotation of matrix $R$ of the vector $x_\mu$. Then

$$U_2 \partial_\mu g^\dagger(x) U_1^\dagger = R_{\mu\nu} \partial_\nu g^\dagger(Rx), \quad U_1 g(x) \partial_\mu g^\dagger(x) U_1^\dagger = g(Rx) R_{\mu\nu} \partial_\nu g^\dagger(Rx),$$

and therefore

$$U_1 A_\mu(x) U_1^\dagger = R_{\mu\nu} A_\nu(Rx).$$

Introducing this relation into the definition (5.36a) of $V_\mu$ we verify that the dependence in the matrix $U_1$ cancels in the trace and thus $V_\mu$ transforms like a four-vector. Since only one vector is available, and taking into account dimensional analysis we conclude

$$V_\mu \propto x_\mu / r^4.$$  \hspace{1cm} (5.41)

For $R \to \infty$ $A_\mu$ approaches a pure gauge (equation (5.38)), and therefore $V_\mu$ can be transformed into

$$V_\mu \sim -\frac{1}{3} \epsilon_{\mu\nu\rho\sigma} A_\nu \cdot (A_\rho \times A_\sigma).$$

It is sufficient to calculate $V_1$. We can choose $\rho = 3, \sigma = 4$ and multiply by a factor six to take into account all other choices. Then

$$V_1 = 16 \epsilon_{ijk} (x_4 \delta_{2i} + \epsilon_{i2i} x_i)(x_4 \delta_{3j} + \epsilon_{j3m} x_m) x_k / r^6 = 16x_1 / r^4,$$

and thus

$$V_\mu \sim 16x_\mu / r^4 = 16\hat{n}_\mu / R^3.$$

The powers of $R$ in equation (5.37) cancel and since $\int d\Omega = 2\pi^2$ the value of the topological charge is simply

$$Q(A_\mu) = 1.$$  \hspace{1cm} (5.42)

If we compare this result with equation (1.50) we see that we have indeed found the minimal action solution.
Without explicit calculation we know already from the analysis of the index of the Dirac operator that the topological charge is an integer:

\[ Q(\mathbf{A}_\mu) = \frac{1}{32\pi^2} \int d^4 x \mathbf{F}_{\mu\nu} \cdot \tilde{\mathbf{F}}_{\mu\nu} = n. \quad (5.43) \]

As in the case of the \( CP(N-1) \) model this result has a geometric interpretation. In general in the parametrization (5.39),

\[ V_\mu \sim \frac{8}{3} \epsilon_{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} u_\alpha \partial_\nu u_\beta \partial_\rho u_\gamma \partial_\sigma u_\delta. \]

A few algebraic manipulations starting from

\[ \int_{S_3} R^3 d\Omega \, \hat{\n}_\mu V_\mu = \frac{1}{6} \epsilon_{\mu\nu\rho\sigma} \int V_\mu du_\nu \wedge du_\rho \wedge du_\sigma, \]

then yield

\[ Q = \frac{1}{12\pi^2} \epsilon_{\mu\nu\rho\sigma} \int u_\mu du_\nu \wedge du_\rho \wedge du_\sigma, \quad (5.44) \]

where the notation of exterior differential calculus again has been used. The area \( \Sigma_p \) of the sphere \( S_{p-1} \) in the same notation can be written

\[ \Sigma_p = \frac{2\pi^{p/2}}{\Gamma(p/2)} \frac{1}{(p-1)!} \epsilon_{\mu_1...\mu_p} \int u_{\mu_1} du_{\mu_2} \wedge ... \wedge du_{\mu_p}, \]

when the vector \( u_\mu \) describes the sphere \( S_{p-1} \) only once. In the l.h.s. of equation (5.44) one thus recognizes an expression proportional to the area of the sphere \( S_3 \). Because in general \( u_\mu \) describes \( S_3 \) \( n \) times when \( x_\mu \) describes \( S_3 \) only once a factor \( n \) is generated.

The inequality (5.35) then implies

\[ S(\mathbf{A}_\mu) \geq 8\pi^2 |n|/g^2. \quad (5.45) \]

The equality, which corresponds to a local minimum of the action, is obtained for fields satisfying the self-duality equations

\[ \mathbf{F}_{\mu\nu} = \pm \tilde{\mathbf{F}}_{\mu\nu}. \quad (5.46) \]

These equations, unlike the general classical field equations (5.31), are first order partial differential equations and therefore easier to solve. The one-instanton solution, which depends on an arbitrary scale parameter \( \lambda \), is

\[ A^i_m = \frac{2}{r^2 + \lambda^2} (x_4 \delta_{im} + \epsilon_{imk} x_k), \quad m = 1, 2, 3, \quad A^4_4 = -\frac{2x_4}{r^2 + \lambda^2}. \quad (5.47) \]
The semi-classical vacuum. We now proceed in analogy with the analysis of the \( CP(N-1) \) model. In the temporal gauge \( A_4 = 0 \) the classical minima of the potential correspond to gauge field components \( A_i, \ i = 1, 2, 3, \) which are pure gauge functions of the three space variables \( x_i: \)

\[
A_m = -\frac{1}{2} i A_m \cdot \sigma = g(x_i) \partial_m g^{-1}(x_i) .
\]

(5.48)

The structure of the classical minima is related to the homotopy classes of mappings of the group elements \( g \) into compactified \( \mathbb{R}^3 \) (because \( g(x) \) goes to a constant for \( |x| \to \infty \)), i.e. again of \( S_3 \) into \( S_3 \) and thus the semi-classical vacuum has a periodic structure. One verifies that the instanton solution (5.47), transported into the temporal gauge by a gauge transformation, connects minima with different winding numbers. Therefore, as in the case of the \( CP(N-1) \) model, to project onto a \( \theta \)-vacuum, one adds a term to the classical action of gauge theories:

\[
S_\theta(A_\mu) = S(A_\mu) + \frac{i \theta}{32\pi^2} \int d^4 x \ F_{\mu \nu} \cdot \tilde{F}_{\mu \nu} ,
\]

(5.49)

and then integrates over all fields \( A_\mu \) without restriction. At least in the semi-classical approximation, the gauge theory depends on an additional parameter, the angle \( \theta \). For non-vanishing values of \( \theta \) the additional term violates CP conservation, and is at the origin of the strong CP violation problem, because if \( \theta \) does not vanish experimental bounds are consistent only with unnaturally small values.

5.4 Fermions in an instanton background

We now apply this analysis to QCD, the theory of Strong Interactions, where \( N_F \) Dirac fermions \( Q, \bar{Q} \), the quark fields, are coupled to non-abelian gauge fields \( A_\mu \) corresponding to the \( SU(3) \) colour group. The action can be written (returning here to standard \( SU(3) \) notation)

\[
S(A_\mu, Q, \bar{Q}) = -\int d^4 x \left[ \frac{1}{4g^2} \text{tr} F_{\mu \nu}^2 + \sum_{f=1}^{N_f} Q_f (\mathcal{D} + m_f) Q_f \right] .
\]

The existence of abelian anomalies and instantons has several physical consequences. We mention here two of them.

The strong CP problem. According to the analysis of Section 4.5 only configurations with a non-vanishing index of the Dirac operator contribute to the \( \theta \)-term. Then the Dirac operator has at least one vanishing eigenvalue. If one fermion field is massless, the determinant resulting from the fermion integration thus vanishes, the instantons do not contribute to the functional integral and the strong CP violation problem is solved. However such an hypothesis seems to
be inconsistent with experimental data on quark masses. Another scheme was based on a scalar field, the axion, which unfortunately has remained experimentally invisible.

The solution of the $U(1)$ problem. Experimentally it is observed that masses of pseudo-scalar mesons are smaller or even much smaller (in the case of pions) than the masses of the corresponding scalar mesons. This strongly suggests an approximate chiral symmetry corresponding to small quark $u$ and $d$ masses, and a more badly violated chiral symmetry corresponding to the strange quark, realized in a phase of spontaneous symmetry breaking. This picture is confirmed by its many other physical consequences.

In a theory in which the quarks would be massless the action would have a chiral $U(N_F) \times U(N_F)$ symmetry, in which $N_F$ is the number of flavours. The spontaneous breaking of chiral symmetry to its diagonal subgroup $U(N_F)$ leads to expect $N_F^2$ Goldstone bosons associated with all axial currents (corresponding to the generators of $U(N) \times U(N)$ that do not belong to the remaining $U(N)$ symmetry group). If the masses of quarks are non-vanishing but small one expects this picture to survive approximately, with instead of Goldstone bosons light pseudo-scalar mesons. From the preceding analysis we know that the axial current corresponding to the $U(1)$ abelian subgroup has an anomaly. The WT identities which imply the existence of Goldstone bosons correspond to constant group transformations and therefore involve only the space integral of the divergence of the current. Since the anomaly is a total derivative one might have expected the integral to vanish. However non-abelian gauge theories have configurations that give non-vanishing values of the form $\int \text{d}^4x \gamma_5 \bar{Q} \gamma_5 Q$, which are the variations of the mass terms in an infinitesimal chiral transformation. If the contributions of $\mathcal{M}_f$ vanish when $m_f \to 0$, as one would normally expect, then a situation of approximate chiral symmetry is realized (in a symmetric or spontaneously broken phase). However if we integrate over fermions

$$\mathcal{M}_f = m_f \int \text{d}^4x \bar{Q}_f(x) \gamma_5 Q_f(x),$$

which are the variations of the mass terms in an infinitesimal chiral transformation. If the contributions of $\mathcal{M}_f$ vanish when $m_f \to 0$, as one would normally expect, then a situation of approximate chiral symmetry is realized (in a symmetric or spontaneously broken phase). However if we integrate over fermions
first, at fixed gauge fields we find (disconnected) contributions proportional to

\[ \langle \mathcal{M}_f \rangle = m_f \text{tr} \gamma_5 (\mathcal{D} + m_f)^{-1}. \]

We have shown in section 4.3 that for topologically non-trivial gauge field configurations \( \mathcal{D} \) has zero eigenmodes, which for \( m_f \to 0 \) give the leading contributions

\[ \langle \mathcal{M}_f \rangle = m_f \sum_n \int d^4x \varphi_n^*(x) \gamma_5 \varphi_n(x) \frac{1}{m_f} + O(m_f) \]

\[ = (n_+ - n_-) + O(m_f). \]

These contributions do not vanish for \( m_f \to 0 \) and are responsible, after integration over gauge fields, for a violation of chiral symmetry.

6 Non-abelian anomaly

We first consider the problem of conservation of a general axial current in a non-abelian vector gauge theory, and then the issue of obstruction to gauge invariance in chiral gauge theories.

6.1 General axial current

We now discuss the problem of the conservation of a general axial current in the example of an action with \( N \) massless Dirac fermions in the background of non-abelian vector gauge fields

\[ S(\psi, \bar{\psi}; A) = - \int d^4x \bar{\psi}(x) \mathcal{D}_i \psi_i(x). \]  

(6.1)

In the absence of gauge fields the action \( S(\psi, \bar{\psi}; 0) \) has a \( U(N) \times U(N) \) symmetry corresponding to the transformations:

\[ \psi' = \left[ \frac{1}{2} (1 + \gamma_5) U_+ + \frac{1}{2} (1 - \gamma_5) U_- \right] \psi, \]

(6.2)

\[ \bar{\psi}' = \bar{\psi} \left[ \frac{1}{2} (1 + \gamma_5) U_+^\dagger + \frac{1}{2} (1 - \gamma_5) U_-^\dagger \right], \]

(6.3)

where \( U_\pm \) are \( N \times N \) unitary matrices. We denote by \( t^\alpha \) the anti-hermitian generators of \( U(N) \),

\[ U = 1 + \theta_\alpha t^\alpha + O(\theta^2). \]

We consider gauge fields coupled to all vector currents, corresponding to diagonal subgroup of \( U(N) \times U(N) \) corresponding to \( U_+ = U_- \),

\[ A_\mu = t^\alpha A_\mu^\alpha. \]
We define axial currents in terms of infinitesimal space-dependent chiral transformation:

\[ U_\pm = 1 \pm \theta_\alpha(x)t^\alpha + O(\theta^2) \Rightarrow \delta \psi = \theta_\alpha(x)\gamma_5 t^\alpha \psi, \quad \delta \bar{\psi} = \theta_\alpha(x)\bar{\psi}\gamma_5 t^\alpha. \]

The variation of the action then reads

\[ \delta S = \int d^4x \left\{ J_5^\alpha(x)\partial_\mu \theta_\alpha(x) + \theta_\alpha(x)\bar{\psi}(x)\gamma_5 \gamma_\mu [A_\mu, t^\alpha]\psi(x) \right\}, \quad (6.4) \]

where \( J_5^\alpha(x) \) is the axial current:

\[ J_5^\alpha(x) = \bar{\psi}\gamma_5 \gamma_\mu t^\alpha \psi. \quad (6.5) \]

Since the gauge group has a non-trivial intersection with the chiral group, the commutator \([A_\mu, t^\alpha]\) no longer vanishes

\[ [A_\mu, t^\alpha] = A_\beta^\mu f_{\beta\alpha\gamma} t^\gamma, \]

where the \( f_{\beta\alpha\gamma} \) are the totally antisymmetric structure constants of the Lie algebra of \( U(N) \). Thus

\[ \delta S = \int d^4x \theta_\alpha(x) \left\{ -\partial_\mu J_5^\alpha(x) + f_{\beta\alpha\gamma} A_\beta^\mu(x)J_5^\gamma(x) \right\}. \quad (6.6) \]

The classical current conservation equation is replaced by a gauge covariant conservation equation:

\[ D_\mu J_5^\alpha = 0, \quad (6.7) \]

where we have defined the covariant divergence of the current by

\[ (D_\mu J_5^\alpha)_\alpha = \partial_\mu J_5^\alpha + f_{\alpha\beta\gamma} A_\mu^\beta J_5^\gamma. \]

In the contribution to the anomaly the terms quadratic in the gauge fields are modified, compared to the expression (4.44), only by the appearance of a new geometric factor. Then the complete form of the anomaly is dictated by gauge covariance. One finds:

\[ D_\lambda J_5^\alpha(x) = -\frac{1}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr} t^\alpha F_{\mu\nu} F_{\rho\sigma}. \quad (6.8) \]

This result for the most general chiral and gauge transformations. If we restrict both groups in such a way that the gauge group has an empty intersection with the chiral group the anomaly becomes proportional to \( \text{tr} t^\alpha \), where \( t^\alpha \) are the generators of the chiral group \( G \times G \), and is therefore different from zero only for the abelian factors of \( G \).
6.2 Obstruction to gauge invariance

We now consider a non-abelian gauge field coupled to left or right-handed fermions for instance:

\[ S(\bar{\psi}, \psi; A) = - \int d^4x \bar{\psi}(x) \frac{1}{2} (1 + \gamma_5) D\psi(x), \]  

(6.9)

(the discussion with \( \frac{1}{2}(1-\gamma_5) \) is similar).

The gauge theory can be consistent only if the partition function

\[ Z(A_\mu) = \int [d\psi d\bar{\psi}] \exp \left[ -S(\psi, \bar{\psi}; A) \right] \]  

(6.10)

is gauge invariant.

We introduce the generators \( t^\alpha \) of the gauge group in the fermion representation and define the corresponding current \( J_\mu \) as:

\[ J_\mu^\alpha(x) = \bar{\psi} \frac{1}{2} (1 + \gamma_5) \gamma_\mu t^\alpha \psi. \]  

(6.11)

The invariance of \( Z(A_\mu) \) under an infinitesimal gauge transformation again leads to a covariant conservation equation for the current:

\[ \langle D_\mu J_\mu \rangle = 0, \]

with

\[ D_\mu = \partial_\mu + [A_\mu, \cdot]. \]

The calculation of the quadratic contribution to the anomaly is simple: the first regularization adopted for the calculation in section 4.2 is also suited to the present situation since the current-gauge field three-point function is symmetric in the external arguments. The group structure is reflected by a simple geometric factor. The global factor can be taken from the abelian calculation. It differs from result (4.21) by a factor \( 1/2 \) that comes from the projector \( \frac{1}{2}(1 + \gamma_5) \). The general form of the term of degree three in the gauge field can also easily be found, but the calculation of the global factor is somewhat tedious. We argue in the section 6.3 that it can be obtained from consistency conditions. The complete expression reads:

\[ (D_\mu J_\mu(x))^\alpha = - \frac{1}{24\pi^2} \partial_\mu \epsilon_{\mu\nu\rho\sigma} tr \left[ t^\alpha \left( A_\nu \partial_\rho A_\sigma + \frac{1}{2} A_\nu A_\rho A_\sigma \right) \right]. \]  

(6.12)

If the projector \( \frac{1}{2}(1 + \gamma_5) \) is replaced by \( \frac{1}{2}(1 - \gamma_5) \) the sign of the anomaly changes.

Unless this term vanishes identically there is an obstruction to the construction of the gauge theory. The first term is proportional to \( d_{\alpha\beta\gamma} \):

\[ d_{\alpha\beta\gamma} = \frac{1}{2} tr \left[ t^\alpha \left( t^\beta t^\gamma + t^\gamma t^\beta \right) \right]. \]  

(6.13)
The second term involves the product of four generators, but taking into account the antisymmetry of the \( \epsilon \) tensor, one product of two consecutive can be replaced by a commutator. Therefore the term is also proportional to \( d_{\alpha\beta\gamma} \).

For a unitary representation the generators \( t^\alpha \) are, with our conventions, antihermitian. Therefore the coefficients \( d_{\alpha\beta\gamma} \) are purely imaginary:

\[
d_{\alpha\beta\gamma}^* = \frac{1}{2} \operatorname{tr} \left[ t^\alpha \left( t^\beta t^\gamma + t^\gamma t^\beta \right) \right]^\dagger = -d_{\alpha\beta\gamma}.
\] (6.14)

For all real (the \( t^\alpha \) antisymmetric) or “pseudo-real” (\( t^\alpha = -S t^\alpha S^{-1} \)) representations these coefficients vanish. It follows that the only non-abelian groups that can lead to anomalies in four dimensions are: \( SU(N) \) for \( N \geq 3 \), \( SO(6) \) and \( E_6 \).

### 6.3 Wess–Zumino consistency conditions

In section 6.2 we have calculated the part of the anomaly that is quadratic in the gauge field and asserted that the remaining part could be obtained from geometric arguments. The method is based on BRS transformations. The anomaly is the variation of a functional under an infinitesimal gauge transformation. This implies compatibility conditions, which here are constraints on the form of the anomaly.

One convenient way to express these constraints is to express the nilpotency of BRS transformations. In a BRS transformation the variation of the gauge field \( A_\mu \) takes the form:

\[
\delta_{\text{BRS}} A_\mu(x) = D_\mu C(x) \bar{\epsilon},
\] (6.15)

where \( C \) is a (fermion) ghost field and \( \bar{\epsilon} \) an anticommuting constant. The corresponding variation of \( \ln Z(A_\mu) \) is:

\[
\delta_{\text{BRS}} \ln Z(A_\mu) = -\int d^4 x \, \langle J_\mu(x) \rangle \, D_\mu C(x) \bar{\epsilon}.
\] (6.16)

The anomaly equation has the general form:

\[
\langle D_\mu J_\mu(x) \rangle = A(A_\mu; x).
\] (6.17)

In terms of \( A \) the equation (6.16), after an integration by parts, can be rewritten:

\[
\delta_{\text{BRS}} \ln Z(A_\mu) = \int d^4 x \, A(A_\mu; x) \, C(x) \bar{\epsilon}.
\] (6.18)

We then calculate the BRS variation of \( AC \). We therefore need also the BRS transformation of the fermion ghost \( C(x) \):

\[
\delta_{\text{BRS}} C(x) = \bar{\epsilon} \, C^2(x).
\] (6.19)
We express that the square of the BRS operator $\delta_{\text{BRS}}$ vanishes (like a cohomology operator), and thus that $A\mathcal{C}$ is BRS invariant

$$\delta_{\text{BRS}}^2 = 0 \Rightarrow \delta_{\text{BRS}} \int d^4x \mathcal{A}(A_{\mu};x)\mathcal{C}(x) = 0.$$ 

This condition yields a constraint on the possible form of anomalies that determines the term cubic in $A$ in the r.h.s. of equation (5.12) completely. One can verify

$$\delta_{\text{BRS}} \epsilon_{\mu\nu\rho\sigma} \int d^4x \text{tr} \left[ \mathcal{C}(x) \partial_{\mu} \left( A_{\nu} \partial_{\rho} A_{\sigma} + \frac{1}{2} A_{\nu} A_{\rho} A_{\sigma} \right) \right] = 0.$$ 

Explicitly, after integration by parts, the equation takes the form

$$\epsilon_{\mu\nu\rho\sigma} \int d^4x \left\{ \partial_{\mu} \mathcal{C}^2(x) A_{\nu} \partial_{\rho} A_{\sigma} + \partial_{\mu} \mathcal{C} D_{\nu} C \partial_{\rho} A_{\sigma} + \partial_{\mu} C A_{\nu} \partial_{\rho} C \partial_{\sigma} C \right\} = 0.$$ 

The terms linear in $A$ cancels automatically

$$\epsilon_{\mu\nu\rho\sigma} \int d^4x \left( \partial_{\mu} \mathcal{C} \partial_{\nu} C \partial_{\rho} A_{\sigma} + \partial_{\mu} C A_{\nu} \partial_{\rho} \partial_{\sigma} C \right) = 0,$$

after integrating by parts the first term and using the antisymmetry of the $\epsilon$ symbol.

In the same way the cubic terms cancel (one has to remember the anticommuting properties of $C$)

$$\epsilon_{\mu\nu\rho\sigma} \int d^4x \left\{ \left( \partial_{\mu} C C + C \partial_{\mu} C \right) A_{\nu} A_{\rho} A_{\sigma} + \partial_{\mu} C \left( [A_{\nu}, C] C A_{\rho} A_{\sigma} + A_{\nu} D_{\rho} C A_{\sigma} + A_{\nu} A_{\rho} D_{\sigma} C \right) \right\} = 0.$$ 

It is only the quadratic terms that gives a relation between the quadratic and cubic terms in the anomaly, both contributions being proportional to

$$\epsilon_{\mu\nu\rho\sigma} \int d^4x \partial_{\mu} \mathcal{C} \partial_{\nu} C A_{\rho} A_{\sigma}.$$
7 Lattice fermions: Ginsparg–Wilson relation

**Notation.** We now return to the problem of lattice fermions discussed in section 3.4. For convenience we set the lattice spacing $a = 1$ and use for the fields the notation $\psi(x) \equiv \psi_x$.

**Ginsparg–Wilson relation.** It had been noted, many years ago, that a potential way to avoid the doubling problem while still retaining chiral properties in the continuum limit was to look for lattice Dirac operators $D$ that, instead of anticommuting with $\gamma_5$, would satisfy the relation

$$D^{-1}\gamma_5 + \gamma_5 D^{-1} = \gamma_5 1 \quad (7.1)$$

where $1$ stands for the identity both for lattice sites and in the algebra of $\gamma$-matrices. More explicitly

$$(D^{-1})_{xy}\gamma_5 + \gamma_5 (D^{-1})_{xy} = \gamma_5 \delta_{xy}.$$  

More generally the r.h.s. can be replaced by any local positive operator on the lattice. Locality on the lattice is defined by a decrease that is at least exponential when the point $x, y$ are separated. The anti-commutator being local, it is expected that it does not affect correlation functions at large distance and that chiral properties are recovered in the continuum limit. Note that when $D$ is the Dirac operator in a gauge background the condition $(7.1)$ is gauge invariant.

However, only recently have lattice Dirac operators solutions to the Ginsparg–Wilson relation $(7.1)$ been discovered, because the demands that both $D$ and the anticommutator $\{D^{-1}, \gamma_5\}$ should be local, seemed difficult to satisfy, specially in the most interesting case of gauge theories.

Note that while the relation $(7.1)$ implies some generalized form of chirality on the lattice, as we now show, it does not guarantee the absence of doublers, as examples illustrate. But the important point is that in this class solutions can be found without doublers.

### 7.1 Chiral symmetry and index

We first discuss the main properties of a Dirac operator satisfying relation $(7.1)$, and exhibit a generalized form of chiral transformations on the lattice.

Using the relation, quite generally true for an euclidean Dirac operator (consequence of hermiticity and reflection symmetry),

$$D^\dagger = \gamma_5 D \gamma_5, \quad (7.2)$$

one can rewrite the relation $(7.1)$, after multiplication by $\gamma_5$,

$$D^{-1} + (D^{-1})^\dagger = 1,$$
and therefore
\[ D + D^\dagger = DD^\dagger = D^\dagger D. \] (7.3)

This implies that the lattice operator \( D \) has an index, and in addition
\[ S = 1 - D, \] (7.4)
is unitary
\[ SS^\dagger = 1. \] (7.5)
The eigenvalues of \( S \) lie on the unit circle. The eigenvalue one corresponds to the pole of the Dirac propagator.

Note also the relations
\[ \gamma_5 S = S^\dagger \gamma_5, \quad (\gamma_5 S)^2 = 1. \] (7.6)
The matrix \( \gamma_5 S \) is hermitian and \( \frac{1}{2}(1 \pm \gamma_5 S) \) are two orthogonal projectors. If \( D \) is a Dirac operator in a gauge background these projectors depend on the gauge field.

It is then possible to construct lattice actions that have a chiral symmetry that corresponds to local but non point-like transformations. In the abelian example,
\[ \psi'_x = \sum_y \left( e^{i\theta \gamma_5 S} \right)_{xy} \psi_y, \] (7.7)
\[ \bar{\psi'}_x = \bar{\psi}_x e^{i\theta \gamma_5}. \] (7.8)

Indeed,
\[ \bar{\psi'}_x D \psi'_x = \bar{\psi}_x D \psi_x \iff e^{i\theta \gamma_5} D e^{i\theta \gamma_5 S} = D \iff D e^{i\theta \gamma_5 S} = e^{-i\theta \gamma_5} D. \]

Using the second relation (7.3) we expand the exponentials and reduce the equation to
\[ D \gamma_5 S = -\gamma_5 D, \] (7.9)
which is another form of relation (7.1).

These transformations, however, no longer leave the integration measure over the fermion fields,
\[ \prod_x d\psi_x d\bar{\psi}_x, \]
automatically invariant. The jacobian \( J \) of the change of variables \( \psi \mapsto \psi' \) is
\[ J = \text{det} e^{i\theta \gamma_5} e^{i\theta \gamma_5 S} = 1 + 2i\theta \text{tr} \gamma_5 (1 - D/2) + O(\theta^2), \] (7.10)
where trace means trace in the space of \( \gamma \) matrices and in the lattice indices. This leaves open the possibility of generating the expected anomalies, when the
Dirac operator of the free theory is replaced by the covariant operator in the background of a gauge field, as we now show.

**Eigenvalues of Dirac operator in a gauge background.** We briefly discuss the index of a lattice Dirac operator \( D \) satisfying the relation (7.1), in a gauge background. We assume that its spectrum is discrete (this is certainly true on a finite lattice where \( D \) is a matrix). The operator \( D \) is related by (7.4) to a unitary operator \( S \) whose eigenvalues have modulus one. Therefore if we denote by \( |n\rangle \) its \( n \)th eigenvector,

\[
D |n\rangle = (1 - S) |n\rangle = (1 - e^{i\theta_n}) |n\rangle \Rightarrow D^\dagger |n\rangle = (1 - e^{-i\theta_n}) |n\rangle .
\]

Then using equation (7.2), we infer

\[
D \gamma_5 |n\rangle = (1 - e^{-i\theta_n}) \gamma_5 |n\rangle .
\]

The discussion that follows then is analogous to the discussion of Section 4.5 to which we refer for details. We note that when the eigenvalues are not real, \( \theta_n \neq 0 \) (mod \( \pi \)), \( \gamma_5 |n\rangle \) is an eigenvector different from \( |n\rangle \) because the eigenvalues are different. Instead in the two subspaces corresponding to the eigenvalues 0 and 2, we can choose eigenvectors with definite chirality

\[
\gamma_5 |n\rangle = \pm |n\rangle .
\]

We call below \( n_\pm \) the number of eigenvalues 0, and \( \nu_\pm \) the number of eigenvalues 2 with chirality \( \pm 1 \).

Note that on a finite lattice \( \delta_{xy} \) is a finite matrix, and thus

\[
\text{tr} \, \gamma_5 \delta_{xy} = 0 .
\]

Therefore

\[
\text{tr} \, \gamma_5 (2 - D) = - \text{tr} \, \gamma_5 D ,
\]

which implies

\[
\sum_n \langle n | \gamma_5 (2 - D) |n\rangle = - \sum_n \langle n | \gamma_5 D |n\rangle . \quad (7.11)
\]

In the equation all complex eigenvalues cancel because the vector \( |n\rangle \) and \( \gamma_5 |n\rangle \) are orthogonal. The sum reduces to the subspace of real eigenvalues, where the eigenvectors have definite chirality. In the l.h.s. only the eigenvalue 0 contributes, and in the r.h.s. only the eigenvalue 2. We find

\[
n_+ - n_- = - (\nu_+ - \nu_-) .
\]

The equation tells us that the difference between the number of states of different chirality in the zero eigenvalue sector is cancelled by the difference in the massive sector of eigenvalue two.
**Remark.** It is interesting to note the relation between the spectrum of $D$ and $\gamma_5D$, which from relation (7.2) is a hermitian matrix,

$$\gamma_5D = D^\dagger \gamma_5 = (\gamma_5D)^\dagger,$$

and thus is diagonalizable with real eigenvalues. It is easy to verify the two equations, the second being obtained by changing $\theta$ in $\theta + 2\pi$,

$$\gamma_5D(1 - i e^{i\theta_n/2} \gamma_5) \langle n | = 2 \sin(\theta_n/2) (1 - i e^{i\theta_n/2} \gamma_5) \langle n |,$$

$$\gamma_5D(1 + i e^{i\theta_n/2} \gamma_5) \langle n | = -2 \sin(\theta_n/2) (1 + i e^{i\theta_n/2} \gamma_5) \langle n |,$$

which implies that the eigenvalues are paired $\pm \sin(\theta_n/2)$ except for $\theta_n = 0 \mod \pi$ where $\langle n |$ and $\gamma_5 \langle n |$ are proportional. For $\theta_n = 0$ $\gamma_5D$ has also eigenvalue 0. For $\theta_n = \pi$ $\gamma_5D$ has eigenvalue $\pm 2$ depending on the chirality of $|n \rangle$.

In the same way

$$\gamma_5(2 - D)(1 + i e^{i\theta_n/2} \gamma_5) \langle n | = 2 \cos(\theta_n/2) (1 + e^{i\theta_n/2} \gamma_5) \langle n |,$$

$$\gamma_5(2 - D)(1 - i e^{i\theta_n/2} \gamma_5) \langle n | = -2 \cos(\theta_n/2) (1 - e^{i\theta_n/2} \gamma_5) \langle n |.$$

**Jacobian and lattice anomaly.** The variation of the jacobian (7.10) can now be evaluated. Opposite eigenvalues of $\gamma_5(2 - D)$ cancel. The eigenvalues $\theta_n = \pi$ give factors one. Only $\theta_n = 0$ gives a non-trivial contribution

$$J = \det e^{i\theta \gamma_5(2-D)} = e^{2i\theta(n_+ - n_-)}.$$

The quantity $\text{tr} \gamma_5(2 - D)$, coefficient of the term of order $\theta$, is a sum of terms that are local, gauge invariant, pseudoscalar and topological as the continuum anomaly (4.43) since

$$\text{tr} \gamma_5(2 - D) = \sum_n \langle n | \gamma_5(2 - D) | n \rangle = 2(n_+ - n_-).$$

**Non-abelian generalization.** We now consider the non-abelian chiral transformations

$$\psi_U = \left[ \frac{1}{2} (1 + \gamma_5S)U_+ + \frac{1}{2} (1 - \gamma_5S)U_- \right] \psi,$$

$$\bar{\psi}_U = \bar{\psi} \left[ \frac{1}{2} (1 + \gamma_5S)U_+^\dagger + \frac{1}{2} (1 - \gamma_5S)U_-^\dagger \right],$$

where $U_\pm$ are matrices belonging to some unitary group $G$

$$U = 1 + \Theta + O(\Theta^2).$$
We note that this amounts to define differently chiral components of $\bar{\psi}$ and $\psi$, for $\psi$ the definition being even gauge field dependent.

We assume that $G$ is a vector symmetry of the fermion action, and thus the Dirac operator commutes with the elements $\Theta$ of the Lie algebra

$$[D, \Theta] = 0.$$  

Then again the relation (7.1) in the form (7.9) implies the invariance of the fermion action

$$\bar{\psi} U_D \psi U = \bar{\psi} D \psi.$$  

The jacobian of an infinitesimal chiral transformation $\Theta = \Theta_+ = -\Theta_-$ is

$$J = 1 + \text{tr} \gamma_5 \Theta (2 - D) + O(\Theta^2).$$  

**Wess–Zumino consistency conditions.** To determine anomalies in the case of gauge fields coupling differently to fermion chiral components one can on the lattice also play with the nilpotency of BRS transformations, which then take the form

$$\delta U_{xy} = \bar{\varepsilon} (C_x U_{xy} - U_{xy} C_y),$$

$$\delta C_x = \bar{\varepsilon} C_x^2,$$

instead of (6.15, 6.19). Moreover the matrix elements $D_{xy}$ of the gauge covariant Dirac operator transform like $U_{xy}$.

### 7.2 Explicit construction: Overlap fermions

An explicit solution to the Ginsparg–Wilson relation without doublers can be derived from operators $D_W$ that share the properties of the Wilson–Dirac operator of equation (3.9), i.e. which avoid doublers at the price of breaking chiral symmetry explicitly. Setting

$$A = 1 - D_W/M,$$

where $M > 0$ is a mass parameter, one takes (the idea of overlap fermions)

$$S = A (A^\dagger A)^{-1/2} \Rightarrow D = 1 - A (A^\dagger A)^{-1/2}.$$  

With this ansatz $D$ has a zero eigenmode when $A (A^\dagger A)^{-1/2}$ has the eigenvalue one. This can happen when $A$ and $A^\dagger$ have the same eigenvector with a positive eigenvalue.

**Free fermions.** We now verify the absence of doublers in the absence of gauge fields. The Fourier representation has the general form

$$D_W(p) = \alpha(p) + i \gamma_\mu \beta_\mu(p),$$  

(7.15)
where $\alpha(p)$ and $\beta_\mu(p)$ are real, periodic, smooth functions such that

$$\beta_\mu(p) \sim p_\mu \quad \text{as} \quad |p| \to 0,$$

and $\alpha(p) > 0$ for all values of $p_\mu$ that correspond to doublers, i.e. such that $\beta_\mu(p) = 0$ for $|p| \neq 0$.

In the case of the operator (7.1.15) a short calculation shows that the determinant of $D$ vanishes when

$$\sqrt{\left(M - \alpha(p)\right)^2 + \beta_\mu^2(p) - M + \alpha(p)} + \beta_\mu^2(p) = 0.$$

This implies $\beta_\mu(p) = 0$, an equation that necessarily admits doubler solutions, and

$$|M - \alpha(p)| = M - \alpha(p).$$

The solutions to this equation depend on the value of $\alpha(p)$ with respect to $M$ for the doubler modes, i.e. for the values of $p$ such that $\beta_\mu(p) = 0$. If $\alpha(p) \leq M$ the equation is automatically satisfied and the corresponding doubler survives. As mentioned in the introduction, the relation (7.1.1) alone does not guarantee the absence of doublers. If instead $\alpha(p) > M$ the equation implies $\alpha(p) = M$, which is impossible. Therefore by rescaling $\alpha(p)$, if necessary, we can keep the wanted $p_\mu = 0$ mode, while eliminating all doublers, which then correspond to the eigenvalue two for $D$, and the doubling problem is solved, at least in a free theory.

In presence of a gauge field the construction can be generalized and works provided the plaquette action is sufficiently close to one.

Remark. Let us stress that, if it seems that the doubling problem has been solved from the formal point of view, from the numerical point of view the calculation of the operator $(A^\dagger A)^{-1/2}$ in a gauge background represents a major challenge.

8 Supersymmetric quantum mechanics and domain wall fermions

Because the construction of lattice fermions without doublers we have just described is somewhat artificial, one may wonder whether there is a context in which they would appear more naturally. Therefore we now briefly outline how a similar lattice Dirac operator can be generated by embedding first four-dimensional space in a larger five dimensional space. This is the method of domain wall fermions.

Because the general idea behind domain wall fermions has emerged first in another context, as a preparation, we first recall a few properties of the spectrum of the hamiltonian in supersymmetric quantum mechanics, a topic also related to the index of the Dirac operator (section 4.5), and stochastic dynamics or Fokker–Planck equation.
8.1 Supersymmetric quantum mechanics

We consider a first order differential operator $\mathfrak{D}$ that is a $2 \times 2$ matrix ($\sigma_i$ still are the Pauli matrices):

$$\mathfrak{D} \equiv \sigma_1 d_x - i\sigma_2 A(x). \quad (8.1)$$

The function $A(x)$ is real, and thus the operator $\mathfrak{D}$ is anti-hermitian.

The operator $\mathfrak{D}$ shares several properties of the Dirac operator of section 4.5. In particular it satisfies

$$\sigma_3 \mathfrak{D} + \mathfrak{D} \sigma_3 = 0,$$

and thus has an index. We introduce the operator $D$

$$D = d_x + A(x) \Rightarrow D^\dagger = -d_x + A(x), \quad (8.2)$$

and

$$Q = D \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow Q^\dagger = D^\dagger \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$ 

Then

$$\mathfrak{D} = Q - Q^\dagger, \quad Q^2 = (Q^\dagger)^2 = 0.$$ 

Thus $\{Q, Q^\dagger\}$ are generators of the simplest form of supersymmetry, and the supersymmetric hamiltonian $H$ is given by

$$H = QQ^\dagger + Q^\dagger Q = -\mathfrak{D}^2 = \begin{pmatrix} D^\dagger D & 0 \\ 0 & DD^\dagger \end{pmatrix}. \quad (8.3)$$

We note that $H$ is a positive operator. The eigenvectors of $H$ have the form $\psi_+(x)(1,0)$ and $\psi_-(x)(0,1)$ and satisfy

$$D^\dagger D |\psi_+\rangle = \varepsilon_+ |\psi_+\rangle, \quad \text{or} \quad DD^\dagger |\psi_-\rangle = \varepsilon_- |\psi_-\rangle, \quad \varepsilon_\pm \geq 0, \quad (8.4)$$

where

$$D^\dagger D = -d_x^2 + A^2(x) - A'(x), \quad DD^\dagger = -d_x^2 + A^2(x) + A'(x).$$

Moreover if $x$ belongs to a bounded interval or $A(x) \to \infty$ for $|x| \to \infty$ then the spectrum of $H$ is discrete.

Multiplying the first equation (8.4) by $D$, we conclude that if $D |\psi_+\rangle \neq 0$, and thus $\varepsilon_+$ does not vanish, it is an eigenvector of $DD^\dagger$ with eigenvalue $\varepsilon_+$, and conversely. Therefore except for a possible ground state with vanishing eigenvalue, the spectrum of $H$ is doubly degenerate.

This observation is consistent with the analysis of section 4.3 applied to the operator $\mathfrak{D}$. We know from that either eigenvectors are paired $|\psi\rangle, \sigma_3 |\psi\rangle$ with
opposite eigenvalues $\pm i \sqrt{\varepsilon}$, or they correspond to the eigenvalue zero and can be chosen with definite chirality

$$\mathcal{D} |\psi\rangle = 0, \quad \sigma_3 |\psi\rangle = \pm |\psi\rangle.$$ 

It is convenient to now introduce the function $S(x)$:

$$S'(x) = A(x), \quad (8.5)$$

and for simplicity discuss only the situation of operators on the entire real line.

We assume that

$$S(x)/|x| \geq \ell > 0.$$ 

Then the function $S(x)$ is such that $e^{-S(x)}$ is a normalizable wave function: $\int dx \ e^{-2S(x)} < \infty$. In the stochastic interpretation $e^{-2S(x)}$ is the equilibrium distribution.

When $e^{-S(x)}$ is normalizable we know one eigenvector with vanishing eigenvalue and chirality $+1$, which corresponds to the isolated ground state of $\mathcal{D}^\dagger \mathcal{D}$

$$\mathcal{D} |\psi_+, 0\rangle = 0 \Leftrightarrow \mathcal{D} |\psi_+\rangle = 0, \quad \sigma_3 |\psi_+, 0\rangle = |\psi_+, 0\rangle,$$

with

$$\psi_+(x) = e^{-S(x)}.$$ 

On the other hand the formal solution of $\mathcal{D}^\dagger |\psi_-\rangle = 0$,

$$\psi_-(x) = e^{S(x)}$$ 

is not normalizable, and therefore no eigenvector with negative chirality is found.

We conclude that the operator $\mathcal{D}$ has only one eigenvector with zero eigenvalue corresponding to positive chirality: the index of $\mathcal{D}$ is one. Note that expressions for the index of the Dirac operator in a general background have been derived. In the present example they yield

$$\text{Index} = \frac{1}{2} \left[ \text{sgn} \ A(+\infty) - \text{sgn} \ A(-\infty) \right],$$

in agreement with the explicit calculation.

The resolvent. For later purpose it is useful to exhibit some properties of the resolvent $\mathcal{G}$:

$$\mathcal{G} = (\mathcal{D} - k)^{-1},$$

for real values of the parameter $k$. Parametrizing $\mathcal{G}$ as a $2 \times 2$ matrix

$$\mathcal{G} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix},$$
one obtains
\[
G_{11} = -k (D^\dagger D + k^2)^{-1} \quad G_{21} = -D (D^\dagger D + k^2)^{-1}
\]
\[
G_{22} = -k (D D^\dagger + k^2)^{-1} \quad G_{12} = D^\dagger (D D^\dagger + k^2)^{-1}.
\]
For \( k^2 \) real one verifies \( G_{21} = -G_{12}^\dagger \).

A number of properties then follow directly from the analysis of \( A^2 \).

When \( k \to 0 \) only \( G_{11} \) has a pole, \( G_{11} = O(1/k) \), \( G_{22} \) vanishes as \( k \) and \( G_{12}(x, y) = -G_{21}(y, x) \) have finite limits:
\[
\mathcal{G}(x, y) \sim \begin{pmatrix} -\frac{1}{k} \psi_+(x) \psi_+(y)/\|\psi_+\|^2 & -G_{21}(y, x) \\ G_{12}(x, y) & 0 \end{pmatrix} \sim -\frac{1}{2k} \frac{\psi_+(x) \psi_+(y)}{\|\psi_+\|^2} (1 + \sigma_3).
\]

Another limit of interest is the limit \( y \to x \). The non-diagonal elements are discontinuous but the limit of interest for domain wall fermions is the average of the two limits
\[
\mathcal{G}(x, x) = \frac{1}{2} (1 + \sigma_3) G_{11}(x, x) + \frac{1}{2} (1 - \sigma_3) G_{22}(x, x) + i \sigma_2 G_{12}(x, x).
\]

When the function \( A(x) \) is odd, \( A(-x) = -A(x) \), in the limit \( x = 0 \) the matrix \( \mathcal{G}(x, x) \) reduces to
\[
\mathcal{G}(0, 0) = \frac{1}{2} (1 + \sigma_3) G_{11}(0, 0) + \frac{1}{2} (1 - \sigma_3) G_{22}(0, 0).
\]

**Examples.**

(i) In the example of the function \( S(x) = \frac{1}{2} x^2 \), the two components of the hamiltonian \( H \) become
\[
D D^\dagger = -d_x^2 + x^2 + 1, \quad D^\dagger D = -d_x^2 + x^2 - 1.
\]

We recognize two shifted harmonic oscillators and the spectrum of \( D \) contains one eigenvalue zero, and a spectrum of opposite eigenvalues \( \pm i \sqrt{2n}, n \geq 1 \).

(ii) Another example useful for later purpose is \( S(x) = |x| \). Then \( A(x) = \epsilon(x) \) (\( \epsilon(x) \) is the sign function), and \( A'(x) = 2 \delta(x) \). The two components of the hamiltonian \( H \) become
\[
D D^\dagger = -d_x^2 + 1 + 2 \delta(x), \quad D^\dagger D = -d_x^2 + 1 - 2 \delta(x).
\]

Here one finds one isolated eigenvalue zero, and a continuous spectrum \( \epsilon \geq 1 \).

(iii) A less singular but similar example that can be solved analytically corresponds to \( A(x) = \mu \tanh(x) \), where \( \mu \) is for instance a positive constant. It leads to the potentials
\[
V(x) = A^2(x) \pm A'(x) = \mu^2 - \frac{\mu(\mu \mp 1)}{\cosh^2(x)}.
\]

The two operators have a continuous spectrum starting at \( \mu^2 \) and a discrete spectrum
\[
\mu^2 - (\mu - n)^2, \quad n \in \mathbb{N} \leq \mu, \quad \mu^2 - (\mu - n - 1)^2, \quad n \in \mathbb{N} \leq \mu - 1.
\]
8.2 Field theory in two dimensions

A natural realization in quantum field theory of such a situation corresponds to a two-dimension model of a Dirac fermion in the background of a static soliton (finite energy solution of field equations).

We consider the action $S(\bar{\psi}, \psi, \varphi)$, $\psi, \bar{\psi}$ being Dirac fermions, and $\varphi$ a scalar boson:

$$S(\bar{\psi}, \psi, \varphi) = \int dxdt \left[ -\bar{\psi} (\partial + m + M\varphi) \psi + \frac{1}{2} (\partial_\mu \varphi)^2 + V(\varphi) \right].$$

We assume that $V(\varphi)$ has degenerate minima, like $(\varphi^2 - 1)^2$ or $\cos \varphi$ and field equations therefore admit soliton solutions $\varphi(x)$, static solitons being the instantons of the one-dimension quantum $\varphi$ model.

Let us now study the spectrum of the corresponding Dirac operator

$$D = \sigma_1 \partial_x + \sigma_2 \partial_t + m + M\varphi(x).$$

We assume for definiteness that $\varphi(x)$ go from $-1$ for $x = -\infty$ to $+1$ for $x = +\infty$, a typical example being

$$\varphi(x) = \tanh(x).$$

Since time translation symmetry remains, we can introduce the (euclidean) time Fourier components and study

$$D = \sigma_1 dx + i\omega \sigma_2 + m + M\varphi(x).$$

The zero eigenmodes of $D$ are also the solutions of the eigenvalue equation

$$D |\psi\rangle = \omega |\psi\rangle, \quad D = \omega + i\sigma_2 D = \sigma_3 dx + i\sigma_2 (m + M\varphi(x)),$$

which differs from equation (8.1) by an exchange between the matrices $\sigma_3$ and $\sigma_1$. The possible zero eigenmodes of $D (\omega = 0)$ thus satisfy

$$\sigma_1 |\psi\rangle = \epsilon |\psi\rangle, \quad \epsilon = \pm 1,$$

and therefore are proportional to $\psi_\epsilon(x)$ solution of

$$\epsilon \psi_\epsilon' + (m + M\varphi(x))\psi_\epsilon = 0.$$

This equation has a normalizable solution only if $|m| < |M|$ and $\epsilon = +1$. Then we find one fermion zero-mode.

A soliton solution breaks space translation symmetry, but leads to a zero-mode which would generate IR divergences. The zero-mode is removed by taking the position of the soliton as a collective coordinate and translation symmetry then is restored. The implications of the fermion zero-mode then require further analysis. It is found that it is associated with a double degeneracy of the soliton state, which carries 1/2 fermion number.
8.3 Domain wall fermions

Continuum formulation. One now considers four-dimensional space (but the strategy applies to all even dimensional spaces) as a surface embedded in five-dimensional space. We denote by $x_\mu$ the usual four coordinates, and $t$ the coordinate in the fifth dimension. Physical space corresponds to $t = 0$. We then study the five-dimensional Dirac operator $\mathcal{D}$ in the background of a classical scalar field $\varphi(t)$ that depends only on $t$. The fermion action reads

$$S(\bar{\psi}, \psi) = -\int dt \, d^4x \, \bar{\psi}(t,x) \mathcal{D}\psi(t,x),$$

with

$$\mathcal{D} = \partial + \gamma_5 d_t + M\varphi(Mt).$$

Since translation symmetry in four-space is not broken, we can work in Fourier representation

$$\mathcal{D} = ip_\mu \gamma_\mu + \gamma_5 d_t + M\varphi(Mt),$$

where the parameter $M$ is a mass large with respect to the mass of all physical particles.

To find the mass spectrum corresponding to $\mathcal{D}$, it is convenient to write it

$$\mathcal{D} = \gamma_p \left[ i |p| + \gamma_p \gamma_5 d_t + \gamma_p M\varphi(Mt) \right],$$

where $\gamma_p = p_\mu \gamma_\mu / |p|$ and thus $\gamma_p^2 = 1$. The eigenvectors with vanishing eigenvalue of $\mathcal{D}$ are also those of the operator $\mathcal{D}^*$

$$\mathcal{D}^* = i\gamma_p \mathcal{D} + |p| = i\gamma_p \gamma_5 d_t + i\gamma_p M\varphi(Mt),$$

with eigenvalue $|p|$.

We then note that $i\gamma_p \gamma_5$, $\gamma_p$ and $-\gamma_5$ are hermitian matrices that form a representation of the algebra of Pauli matrices. The operator $\mathcal{D}^*$ can then be compared with the operator (8.1), and $M\varphi(Mt)$ corresponds to $A(x)$. Under the same conditions $\mathcal{D}$ has an eigenvector with an isolated vanishing eigenvalue corresponding to an eigenvector with positive chirality. All other eigenvalues, for dimensional reasons are proportional to $M$ and thus correspond to fermions of large masses. Moreover the eigenfunction with eigenvalue zero decays on a scale $t = O(1/M)$. Therefore for $M$ large one is left with a fermion that has a single chiral component, confined on the $t = 0$ surface.

One possible interpretation of the function $\varphi(t)$ is that $\varphi(t)$ is a solution of field equations and connects two minima $\varphi = \pm 1$ of the $\varphi$ potential. In the limit of very sharp transition one is led to the hamiltonian (8.6). Note that this interpretation is possible only for an even dimension $d \geq 4$; in dimension two like in the two-dimensional field theory, zero-modes forbid a static wall.
More precise results follow from the study of section 8.1. We have noticed that \( G(t_1, t_2; p) \), the inverse of the Dirac operator in Fourier representation, has a short distance singularity for \( t_2 \to t_1 \) in the form of a discontinuity. This is here an artifact of treating the fifth dimension differently from the four others. In real space for the function \( G(t_1, t_2; x_1 - x_2) \) with separate points on the surface, \( x_1 \neq x_2 \), the limit \( t_1 = t_2 \) corresponds to points in five dimensions that do not coincide and this singularity is absent. A short analysis shows that this amounts in Fourier representation to take the average of the limiting values (a property that can easily be verified for the free propagator). Then if \( \varphi(t) \) is an odd function one finds for \( t_1 = t_2 = 0 \)

\[
D^{-1}(p) = \frac{i}{2p} \left[ d_1(p^2)(1 + \gamma_5) + (1 - \gamma_5)p^2d_2(p^2) \right],
\]

where \( d_1, d_2 \) are regular functions of \( p^2 \). Therefore \( D^{-1} \) anticommutes with \( \gamma_5 \) and chiral symmetry is realized in the usual way. If however \( \varphi(t) \) is of more general type one finds

\[
D^{-1} = \frac{i}{2p} \left[ d_1(p^2)(1 + \gamma_5) + (1 - \gamma_5)p^2d_2(p^2) \right] + d_3(p^2),
\]

where \( d_3 \) is regular. As a consequence

\[
\gamma_5D^{-1} + D^{-1}\gamma_5 = 2d_3(p^2)\gamma_5,
\]

which is a form of Wilson–Ginsparg’s relation because the r.h.s. is local.

**Domain wall fermions: lattice.** We now replace four-dimensional continuum space by a lattice but keep the fifth dimension continuous. We replace the Dirac operator by the Wilson–Dirac operator (7.13) to avoid doublers. In Fourier representation we find

\[
D = \alpha(p) + i\beta_\mu(p)\gamma_\mu + \gamma_5d_t + M\varphi(Mt).
\]

This has the effect of replacing \( p_\mu \) by \( \beta_\mu(p) \) and shifting \( M\varphi(Mt) \leftrightarrow M\varphi(Mt) + \alpha(p) \). To ensure the absence of doublers we require that for the values for which \( \beta_\mu(p) = 0 \) and \( p \neq 0 \) none of the solutions to the zero eigenvalue equation is normalizable. This is realizes if for \( |t| \to \infty \) \( \varphi(t) \) is bounded, for instance

\[
|\varphi(t)| \leq 1
\]

and \( M < |\alpha(p)| \).

The inverse Dirac operator on the surface \( t = 0 \) takes the general form

\[
D^{-1} = i\beta \left[ \delta_1(p^2)(1 + \gamma_5) + (1 - \gamma_5)\delta_2(p^2) \right] + \delta_3(p^2),
\]
where $\delta_1$ is the only function that has a pole for $p = 0$, and where $\delta_2, \delta_3$ are regular. The function $d_3$ does not vanish even if $\varphi(t)$ is odd because the addition of $\alpha(p^2)$ breaks the symmetry. We then always find Wilson–Ginsparg’s relation

$$\gamma_5 \mathcal{D}^{-1} + \mathcal{D}^{-1} \gamma_5 = 2\delta_3(p^2) \gamma_5$$

More explicit expressions can be obtained in the limit $\varphi(t) = \text{sgn}(t)$ (a situation analogous to (8.3)), using the analysis of section A2.

Of course to simulate domain walls on the computer one has also to discretize the fifth dimension.

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APPENDICES

A1 Trace formula for periodic potentials

We consider a hamiltonian $H$ corresponding to a real periodic potential $V(x)$ with period $X$:

$$V(x + X) = V(x).$$  \hfill (A1.1)

Eigenfunctions $\psi_\theta(x)$ are then also eigenfunctions of the translation operator $T$:

$$T \psi_\theta(x) \equiv \psi_\theta(x + X) = e^{i\theta} \psi_\theta(x).$$  \hfill (A1.2)

We first restrict space to a box of size $NX$ with periodic boundary conditions. This implies a quantization of the angle $\theta$

$$e^{iN\theta} = 1 \Rightarrow \theta = \theta_p \equiv \frac{2\pi p}{N}, \quad 0 \leq p < N.$$  \hfill (A1.3)

We call $\psi_{p,n}$ the normalized eigenfunctions of $H$ corresponding to the band $n$ and the pseudo-momentum $\theta_p$

$$\int_0^{NX} dx \, \psi_{p,m}^*(x) \psi_{q,n}(x) = \delta_{mn}\delta_{pq},$$

and $E_n(\theta_p)$ the corresponding eigenvalues. Reality implies:

$$E_n(\theta) = E_n(-\theta).$$  \hfill (A1.4)

This leads to a decomposition of the identity operator in $[0, NX]$

$$\delta(x - y) = \sum_{p,n} \psi_{p,n}(x) \psi_{p,n}^*(y).$$

We now consider an operator $O$ that commutes with $T$

$$[T, O] = 0 \Rightarrow \langle x | O | y \rangle = \langle x + X | O | y + X \rangle.$$

Then

$$\langle q, n | O | p, m \rangle = \int_0^{NX} dx \, dy \, \psi_{q,n}^*(x) \langle x | O | y \rangle \psi_{p,m}(y) = \delta_{pq}O_{mn}(\theta_p).$$

Its trace can be written

$$\text{tr} O = \int_0^{NX} dx \, \langle x | O | x \rangle = N \int_0^X dx \, \langle x | O | x \rangle = \sum_{p,n} O_{nn}(\theta_p).$$
We then take the infinite box limit $N \to \infty$. Then

$$\frac{1}{N} \sum_p \to \frac{1}{2\pi} \int_0^{2\pi} d\theta,$$

and thus we find

$$\int_0^X dx \langle x | O | x \rangle = \sum_n \frac{1}{2\pi} \int_0^{2\pi} O_{nn}(\theta) d\theta. \quad (A1.5)$$

We now apply this general result to the operator

$$O = T^\ell e^{-\beta H}.$$

Then

$$\int_0^X \langle x | T^\ell e^{-\beta H} | x \rangle dx = \frac{1}{2\pi} \sum_n \int_0^{2\pi} e^{i\ell\theta - \beta E_n(\theta)} d\theta, \quad (A1.6)$$

which using the definition of $T$ can be rewritten

$$\int_0^X \langle x + \ell X | e^{-\beta H} | x \rangle dx = \frac{1}{2\pi} \sum_n \int_0^{2\pi} e^{i\ell\theta - \beta E_n(\theta)} d\theta. \quad (A1.7)$$

In the path integral formulation this leads to a representation of the form

$$\int_{x(\beta/2) = x(-\beta/2) + \ell X} \exp \left[-S(x)\right] \frac{dx(t)}{t} = \frac{1}{2\pi} \sum_n \int_0^{2\pi} e^{i\ell\theta - \beta E_n(\theta)} d\theta, \quad (A1.8)$$

where $x(-\beta/2)$ varies only in $[0, X]$, justifying the representation $(5.3)$.

**A2 Resolvent of the hamiltonian in supersymmetric QM**

The resolvent $G(z) = (H + z)^{-1}$ of the hermitian operator $H$, 

$$dH = -d_x^2 + V(x),$$

where $-z$ is below the spectrum of $H$, satisfies the differential equation:

$$(-d_x^2 + V(x) + z)G(z; x, y) = \delta(x - y). \quad (A2.1)$$

We recall how $G(z; x, y)$ can be expressed in terms of two independent solutions of the homogeneous equation

$$(-d_x^2 + V(x) + z)\varphi_{1,2}(x) = 0. \quad (A2.2)$$
If we partially normalize by the wronskian $W$,

$$W(\varphi_1, \varphi_2) \equiv \varphi'_1(x) \varphi_2(x) - \varphi_1(x) \varphi'_2(x) = 1,$$

and moreover impose the boundary conditions

$$\varphi_1(x) \to 0 \text{ for } x \to -\infty, \quad \varphi_2(x) \to 0 \text{ for } x \to +\infty,$$

then it is easily verified that $G(z; x, y)$ is given by

$$G(z; x, y) = \varphi_1(y) \varphi_2(x) \theta(x - y) + \varphi_1(x) \varphi_2(y) \theta(y - x). \quad (A2.3)$$

If the potential is an even function $V(-x) = V(x)$,

$$\varphi_2(x) \propto \varphi_1(-x).$$

We now apply this result to the operator

$$H = DD^\dagger \quad \text{with} \quad z = -k^2.$$

The functions $\varphi_i$ then satisfy

$$(DD^\dagger + k^2) \varphi_i(x) = \equiv \ [-d^2 + A^2(x) + A'(x) + k^2] \varphi_i(x) = 0,$$

and the equation (A2.3) yields the resolvent $G_-(k^2; x, y)$.

The corresponding solutions for the operator $D^\dagger D + k^2$ follow since

$$D^\dagger (DD^\dagger + k^2) \varphi_i = 0 = (D^\dagger D + k^2) D^\dagger \varphi_i = 0.$$

Setting

$$\chi_i(x) = D^\dagger \varphi_i(x),$$

we calculate the wronskian for normalization purpose

$$W(\chi_1, \chi_2) \equiv \chi'_1(x) \chi_2(x) - \chi_1(x) \chi'_2(x) = -k^2.$$

Thus the corresponding resolvent $G_+$ reads

$$G_+(k^2; x, y) = -\frac{1}{k^2} \left[ \chi_1(y) \chi_2(x) \theta(x - y) + \chi_1(x) \chi_2(y) \theta(y - x) \right]. \quad (A2.4)$$

The limits $x = y$ are

$$G_-(k^2; x, x) = \varphi_1(x) \varphi_2(x), \quad G_+(k^2; x, x) = -\frac{1}{k^2} \chi_1(x) \chi_2(x).$$
Note that these functions satisfy third order linear differential equations. If the potential is even, here this implies that \( A(x) \) is odd, \( G_{\pm}(k^2; x, x) \) are even functions.

Finally we also need \( D^\dagger G_{\pm}(k^2; x, y) \):

\[
D^\dagger G_{\pm}(k^2; x, y) = \phi_1(y)D^\dagger \phi_2(x)\theta(x - y) + \phi_2(y)D^\dagger \phi_1(x)\theta(y - x).
\]

We note that \( D^\dagger G_{\pm}(k^2; x, y) \) is not continuous at \( x = y \):

\[
\lim_{y \to x^+} D^\dagger G_{\pm}(k^2; x, y) = \phi_2(y)D^\dagger \phi_1(x), \quad \lim_{y \to x^-} D^\dagger G_{\pm}(k^2; x, y) = \phi_1(x)D^\dagger \phi_2(x),
\]

and therefore from the wronskian,

\[
\lim_{y \to x^-} D^\dagger G_{\pm}(k^2; x, y) - \lim_{y \to x^+} D^\dagger G_{\pm}(k^2; x, y) = 1.
\]

The half sum is given by

\[
\overline{D^\dagger G_{\pm}(k^2; x, x)} = \frac{1}{2} \lim_{y \to x^-} D^\dagger G_{\pm}(k^2; x, y) + \frac{1}{2} \lim_{y \to x^+} D^\dagger G_{\pm}(k^2; x, y)
\]

\[
= \frac{1}{2}D^\dagger \phi_1(x)\phi_2(x) + \frac{1}{2}D^\dagger \phi_2(x)\phi_1(x)
\]

\[
= \frac{1}{2}(\phi_1\phi_2)'(x) + A(x)\phi_1(x)\phi_2(x).
\]

This function is odd when \( A(x) \) is odd.

Finally in the limit \( k \to 0 \) one finds

\[
\phi_1(x) = N e^{S(x)} \int_{-\infty}^{x} du \ e^{-2S(u)}, \quad \phi_2(x) = N e^{S(x)} \int_{x}^{\infty} du \ e^{-2S(u)},
\]

with

\[
N^2 \int_{-\infty}^{+\infty} du \ e^{-2S(u)} = 1.
\]

Moreover

\[
D^\dagger \phi_1(x) = -N e^{-S(x)}, \quad D^\dagger \phi_2(x) = N e^{-S(x)}.
\]

Therefore, as expected

\[
G_{\pm}(k^2; x, y) \sim \frac{1}{k^2} N^2 e^{-S(x) - S(y)}.
\]

Finally

\[
D^\dagger G_{\pm}(0; x, y) = N^2 \left[ \theta(x - y) e^{-S(x) + S(y)} \int_{-\infty}^{y} du \ e^{-2S(u)} + (x \leftrightarrow y) \right],
\]

and therefore

\[
\overline{D^\dagger G_{\pm}(0; x, x)} = \frac{1}{2} N^2 \int_{-\infty}^{\infty} dt \ \text{sgn}(x - t) e^{-2S(t)}.
\]