EXISTENCE OF A PERIOD TWO SOLUTION OF A DELAY
DIFFERENTIAL EQUATION

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ABSTRACT. We consider the existence of a symmetric periodic solution for the following distributed delay differential equation

\[ x'(t) = -f \left( \int_0^1 x(t-s)ds \right), \]

where \( f(x) = r \sin x \) with \( r > 0 \). It is shown that the well studied second order ordinary differential equation, known as the nonlinear pendulum equation, derives a symmetric periodic solution of period 2, expressed in terms of the Jacobi elliptic functions, for the delay differential equation. We here apply the approach in Kaplan and Yorke (1974) for a differential equation with discrete delay to the distributed delay differential equation.

1. Introduction. Infinite dimensional dynamical systems described by delay differential equations, where the derivative of an unknown function depends on the past values of the function, have been studied, see the standard monographs [3, 6]. Many qualitative results for periodic solutions of delay differential equations have been obtained since 1970 (see e.g. [13, 15] and Chapter XV in [3]). In the paper [7], Kaplan and Yorke study the existence of a periodic solution of period 4 for the following delay differential equation

\[ x'(t) = -f(x(t-1)), \]

(1.1)

where \( f : \mathbb{R} \to \mathbb{R} \) is a continuous and odd function. The authors show the existence of a periodic solution \( x \) with symmetry \( x(t) = -x(t+2) \). Thus the period of the solution is 4 and satisfies the following Hamiltonian system of ordinary differential equations

\[ x'(t) = -f(y(t)), \quad y'(t) = f(x(t)), \]

(1.2)

where \( y(t) := x(t-1) \). See also [4, 5, 9, 16] and references therein for generalizations of the result.

In the paper [12] the author studies the existence of a period 2 solution for a Hutchinson-Wright type differential equation with distributed delay. Motivated by the study [7], the author derives a system of ordinary differential equations that the period 2 solution satisfies. It is a surprising finding that an associated ordinary differential equation finally becomes the Duffing equation. As a result, the delay differential equation has a period 2 solution, expressed in terms of the Jacobi elliptic functions ([2, 10]).
In this paper we consider the existence of a periodic solution of the following delay differential equation

\[ x'(t) = -f \left( \int_0^1 x(t-s)ds \right). \] (1.3)

By a suitable change of variables, the delay differential equation studied in [12] becomes the equation (1.3) with \( f(x) = r(e^x - 1) \), \( r > 0 \). In the paper [8], for a general distributed delay differential equation including the equation (1.3), the author studies the existence of a symmetric periodic solution using the fixed point theorem. Theorem 1.1 in [8] implies the existence of a symmetric period 2 solution for the equation (1.3) with an odd function \( f \). In this paper, we study the distributed delay differential equation (1.3) with \( f(x) = r \sin x \), \( x \in \mathbb{R} \) with \( r > 0 \). This particular choice of the function allows us to obtain a concrete form of the periodic solution, observing a relation between the equation (1.3) and a system of ordinary differential equations, which is equivalent to the nonlinear pendulum equation ([11, 14]). We show that the existence of a closed symmetric trajectory of the system of ordinary differential equations implies the existence of a symmetric periodic solution for the delay differential equation (1.3) and vice versa.

The paper is organized as follows. In Section 2, we characterize a symmetric period 2 solution for the equation (1.3) in Definition 2.1 using the notion of special symmetric periodic solution (SSPS), analogously to the symmetric periodic solution, which is found in [7] and is further studied in [4, 5] in general, for the discrete delay differential equation (1.1). Every SSPS of the equation (1.3) is related to a closed symmetric trajectory of a system of ordinary differential equations, which is equivalent to the nonlinear pendulum equation, see Proposition 2.1. In particular, we see that every closed symmetric trajectory of minimal period 2 for the system of ordinary differential equations yields an SSPS for the distributed delay differential equation (1.3). Explicit form of the periodic solution is given in terms of the Jacobi elliptic function. In Section 3, we study a condition for the existence of the period 2 solution expressed in terms of the Jacobi elliptic functions. Thus we obtain an SSPS expressed in terms of the Jacobi elliptic functions for the equation (1.3). In Section 4, we discuss our results. In Appendix, we briefly introduce the Jacobi elliptic functions.

2. The pendulum equation and the distributed delay differential equation.

In [4, 5] the authors characterize a special symmetric periodic solution for a discrete delay differential equation including the equation (1.1) as a special case. Following [4, 5], let us analogously define a special symmetric periodic solution (SSPS) for the distributed delay differential equation (1.3).

**Definition 2.1.** For the equation (1.3) with \( f(x) = r \sin x \) where \( r > 0 \), \( x : \mathbb{R} \to \mathbb{R} \) is called a special symmetric periodic solution (SSPS) if

- \( x \) solves the equation (1.3) for some \( r > 0 \),
- \( x(t) \neq 0 \) for \( 0 \leq t < \frac{1}{2} \) and \( x \left( \frac{1}{2} \right) = 0 \),
- \( x(t-1) = -x(t) \) for all \( t \in \mathbb{R} \).

It is easy to see that every SSPS is periodic with minimal period 2.

We show that every SSPS of the equation (1.3) is related to a closed trajectory of the following system of ordinary differential equations...
The solution of (2.1) with the initial condition (2.2) defines a closed symmetric trajectory around the origin (2.1) in the \((x, y)\) plane, if (2.4) holds.

Solution profile of an SSPS (2.11) expressed in terms of the Jacobi elliptic function for (1.3).

**Figure 2.1.** Symmetric periodic solutions of (2.1) and (1.3)

\[
\begin{align*}
  x'(t) &= -f(y(t)), \\
  y'(t) &= 2x(t),
\end{align*}
\] (2.1a)
\[y''(t) = -2f(y(t)).\]

which is equivalent to the nonlinear pendulum equation.

\[
x(t)^2 + 2r \sin^2 \frac{y(t)}{2} = a^2
\]

for any \(t \in \mathbb{R}\). For the sake of brevity, we here restrict our attention to the case that

\[0 < a < \sqrt{2r}\]

holds. If (2.4) holds, then (2.3) denotes a closed symmetric trajectory around the origin in the \((x, y)\) plane, see Figure 2.1 (A) (see also [11, 14]).

**Proposition 2.1.** Every SSPS for the equation (1.3) satisfies the system (2.1) and the first component of every closed symmetric trajectory around the origin of minimal period 2 for the system (2.1) yields an SSPS for the equation (1.3). Thus the equation (1.3) has an SSPS if and only if the system (2.1) has a closed symmetric trajectory around the origin of minimal period 2.

**Proof.** Let \(x(t)\) be an SSPS of the equation (1.3). For \(y(t) := \int_0^1 x(t-s)ds\), we have

\[y'(t) = x(t) - x(t-1) = 2x(t).\]
Therefore, the SSPS of the equation (1.3) solves the system (2.1). Noting that $x(t)$ oscillates about 0, one can see that $(x(t), y(t))$ is a closed symmetric trajectory around the origin of period 2.

Next let $(x(t), y(t))$ be a closed symmetric trajectory around the origin of minimal period 2 for the system (2.1). One sees that $(-x(t), -y(t))$ is also the same trajectory, hence there exists $c$ such that $x(t + c) = -x(t)$, $0 \leq c < 2$. Since $x(t + 2c) = -x(t + c) = x(t)$, we find $c = 1$. Therefore, we have $x(t + 1) = x(t - 1) = -x(t)$, hence

$$y'(t) = 2x(t) = x(t) - x(t - 1).$$

From direct substitution $(x, y) = (x(t - t), -y(t - t))$ and $(x, y) = (-x(t), y(t))$ also solve the system (2.1). Let us denote by $c^* \in [0, 2)$ such that $x(c^*) = 0$ for the solution $(x(t), y(t))$ of (2.1) with the initial condition (2.2). Consider the solution $(x(c^* + t), y(c^* + t))$. Since $(-x(c^* - t), y(c^* - t))$ is also solutions of the system (2.2), uniqueness of the solution of (2.1) with the initial condition (2.1) implies $(x(c^* + t), y(c^* + t)) = (-x(c^* - t), y(c^* - t))$. Then one has $x(2c^*) = -x(0) = x(1)$, thus $c^* = \frac{1}{2}$. Therefore, $x(t + \frac{1}{2}) = -x(t - \frac{1}{2})$ with $x(0) = a$ i.e., $x$ is an odd function about $t = \frac{1}{2}$, thus $\int_0^1 x(s)ds = 0$ follows. It then implies that

$$y(t) = \int_{t-1}^t x(s)ds.$$

Consequently, the first equation (2.1a) implies that $x(t)$ is an SSPS for the delay differential equation (1.3).

We now seek a period 2 solution of (2.1). It is well known that the solution of the pendulum equation is expressed in terms of the Jacobi elliptic functions. Following [10], let us derive the solution of the system (2.1). From (2.1b), using (2.1b), one obtains

$$\frac{(y'(t))^2}{4} = a^2 - 2r \sin \frac{y(t)}{2}. \quad (2.5)$$

We then define

$$z(t) = \sin \frac{y(t)}{2}. \quad (2.6)$$

We differentiate both sides of (2.6). Using the equation (2.5), we obtain

$$(z'(t))^2 = (1 - z^2) \left( a^2 - 2rz^2 \right), \quad (2.7)$$

with the initial condition

$$z(0) = 0, \quad z'(0) = a. \quad (2.8)$$

It is known that, under the condition (2.4), the equation (2.7) with the initial condition (2.8) can be solved as

$$z(t) = \frac{a}{\sqrt{2r}} \text{sn} \left( \sqrt{2rt}, \frac{a}{\sqrt{2r}} \right), \quad (2.9)$$

where $\text{sn}$ denotes the Jacobi elliptic function (see Appendix A). We denote by $K(k)$ the complete elliptic integrals of the first kind [2, 10]:

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta.$$
for $0 \leq k < 1$. Since the period of $\text{sn} \left( t, k \right)$ is $4K(k)$, the period of the solution (2.9) is given as

$$\tau_a := \frac{4}{\sqrt{2r}} K \left( \frac{a}{\sqrt{2r}} \right).$$

(2.10)

Therefore, for the system (2.1) with the initial condition (2.2), one has

$$x(t) = \text{acn} \left( \sqrt{2rt}, \frac{a}{\sqrt{2r}} \right),$$

(2.11)

$$y(t) = 2 \arcsin z(t) = 2 \arcsin \left( \frac{a}{\sqrt{2r}} \text{sn} \left( \sqrt{2rt}, \frac{a}{\sqrt{2r}} \right) \right).$$

(2.12)

where $\text{cn}$ denotes the Jacobi elliptic function (see Appendix A). Note that the formula (2.12) is obtained from the definition (2.6). Then (2.11) is obtained by (2.12), (2.1b) and the properties of the Jacobi elliptic functions.

3. Existence of an SSPS. Let us consider the initial condition (2.2) such that

$$\tau_a = 2,$$

(3.1)

where $\tau_a$ is given as in (2.10).

**Lemma 3.1.** There exists a unique $a \in (0, \sqrt{2r})$ such that (3.1) holds if and only if $r > \frac{\pi^2}{2}$ holds.

**Proof.** Since the complete elliptic integrals of the first kind $K$ is a strictly increasing function such that

$$K(0) = \frac{\pi}{2} < \lim_{k \to 1^{-}} K(k) = \infty,$$

one can see that

$$\lim_{a \to 0^+} \tau_a = \pi \sqrt{\frac{2}{r}} < \lim_{a \to \sqrt{2r}^-} \tau_a = \infty.$$

If $r > \frac{\pi^2}{2}$, then $\lim_{a \to 0^+} \tau_a < 2$ holds. From the monotonicity of $K$, we obtain the conclusion. \(\square\)

Now it is straightforward to obtain the following result from Proposition 2.1 and Lemma 3.1.

**Theorem 3.1.** Let us assume that $r > \frac{\pi^2}{2}$ holds. Then the equation (1.3) with $f(x) = r \sin x$ has a unique SSPS, up to time-translation, expressed as in (2.11), where $a \in (0, \sqrt{2r})$ is a unique root of (3.1).

Figure 2.1b shows the profile of an SSPS for the equation (1.3) with $r = 8$. We also note that the equation (1.3) has countably many symmetric periodic solutions due to the periodicity of the function $f(x) = r \sin x$. Let $x(t)$ be a solution of (1.3). Since $f$ is a periodic function such that

$$f(x) = f(x + 2n\pi), \ n \in \mathbb{Z},$$

$x(t) + 2n\pi, \ n \in \mathbb{Z}$ is also a solution.
4. Discussion. In this paper, we study the existence of an SSPS of the distributed delay differential equation (1.3) with \( f(x) = r \sin x \). Such a symmetric periodic solution has been studied for a discrete delay differential equation (1.1), assuming that \( f \) is an odd function since the study by Kaplan and York [7]. The symmetric periodic solution for a discrete delay differential equation is analyzed in detail ([4, 5, 9, 16]). Compared to discrete delay differential equations, few results for distributed delay differential equations concerning the periodic solution are available. See [1] for an application of the Hamiltonian system (1.2) to obtain a symmetric periodic solution of Volterra type integral equation (renewal equation). In the paper [8], the author studies a distributed delay differential equation including the equation (1.3) as a special case. The author constructs a Poincare map of which the fixed point is a symmetric periodic solution of the distributed delay differential equation. Essentially the main result, Theorem 1.1, in the paper [8] implies the existence of an SSPS for the equation (1.3), see also Remark 3.6 in [8]. Note that, in the paper [8], the positive feedback condition, namely \( xf(x) > 0 \) for \( x \neq 0 \), is assumed. If one studies a periodic solution around the origin locally, the assumption may be weakened. In this paper, paying attention to the equation (1.3) with a special nonlinear function \( f(x) = r \sin x \), we prove that the existence of an SSPS expressed in terms of the Jacobi elliptic function for the equation (1.3).

We expect that the results obtained in this paper hold for distributed delay differential equation (1.3) with a class of odd functions \( f \). In the forthcoming paper we generalize these results to a class of distributed delay differential equations (1.3). By a suitable change of variables, the delay differential equation studied in [12] becomes (1.3) with \( f(x) = r(e^x - 1) \), which is not an odd function. This example indicates that the symmetry of the nonlinear function \( f \) may not be necessary for the existence of period 2 solution. Our future work also involves the analysis of Hopf bifurcation for the equation (1.3) that derives the stability of the periodic solution, which emerges from the Hopf bifurcation. Stability of the SSPS obtained in Theorem 3.1 is an open problem. For some \( r > \frac{\pi^2}{2} \), numerical simulations suggest that the SSPS attracts two different solutions, see Figure 4.1.

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Appendix A. Jacobi elliptic functions. We briefly introduce the Jacobi elliptic functions. See [2] for detail. See also [10], where the Jacobi elliptic functions are defined as the solutions of a system of ordinary differential equations.

The incomplete elliptic integrals of the first kind and second kind are respectively given as

$$F(\varphi, k) = \int_0^{\varphi} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta,$$
$$E(\varphi, k) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \theta} d\theta$$

for $\varphi \in \mathbb{R}$ and $0 \leq k < 1$. Here $k$ is a parameter called the modulus. The complete elliptic integrals of the first kind and second kind are given as

$$K(k) := F\left(\frac{\pi}{2}, k\right), \quad E(k) := E\left(\frac{\pi}{2}, k\right).$$
The amplitude function \( \text{am} \) is defined as the inverse function of the elliptic integral of the first kind, fixing the modulus \( k \), i.e.,

\[
\text{am} \left( F(\varphi, k), k \right) = \varphi.
\]

Then the Jacobi elliptic functions \( \text{sn}, \text{cn} : \mathbb{R} \rightarrow [-1, 1] \) are respectively defined as

\[
\text{sn} \left( t, k \right) = \sin \left( \text{am} \left( t, k \right) \right),
\]

\[
\text{cn} \left( t, k \right) = \cos \left( \text{am} \left( t, k \right) \right).
\]

One then sees that the period of \( \text{sn} \) and \( \text{cn} \) is given as \( 4K(k) \). Then the Jacobi elliptic function \( \text{dn} \) is defined by

\[
\text{dn} \left( t, k \right) = \sqrt{1 - k^2 \text{sn} \left( t, k \right)}.
\]

The period of \( \text{dn} \) is \( 2K(k) \).

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