THE DUAL QUANTUM GROUP FOR THE QUANTUM GROUP ANALOGUE OF THE NORMALIZER OF $SU(1, 1)$ IN $SL(2, \mathbb{C})$

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Abstract. The quantum group analogue of the normalizer of $SU(1, 1)$ in $SL(2, \mathbb{C})$ is an important and non-trivial example of a non-compact quantum group. The general theory of locally compact quantum groups in the operator algebra setting implies the existence of the dual quantum group. The first main goal of the paper is to give an explicit description of the dual quantum group for this example involving the quantized enveloping algebra $U_q(\mathfrak{su}(1, 1))$. It turns out that $U_q(\mathfrak{su}(1, 1))$ does not suffice to generate the dual quantum group. The dual quantum group is graded with respect to commutation and anticommutation with a suitable analogue of the Casimir operator characterized by an affiliation relation to a von Neumann algebra. This is used to obtain an explicit set of generators. Having the dual quantum group the left regular corepresentation of the quantum group analogue of the normalizer of $SU(1, 1)$ in $SL(2, \mathbb{C})$ is decomposed into irreducible corepresentations. Upon restricting the irreducible corepresentations to $U_q(\mathfrak{su}(1, 1))$-representation one finds combinations of the positive and negative discrete series representations with the strange series representations as well as combinations of the principal unitary series representations. The detailed analysis of this example involves analysis of special functions of basic hypergeometric type and, in particular, some results on these special functions are obtained, which are stated separately.

The paper is split into two parts; the first part gives almost all of the statements and the results, and the statements in the first part are independent of the second part. The second part contains the proofs of all the statements.

Contents

Preamble 2
1. Introduction 2
2. Von Neumann algebraic quantum groups 8
3. The quantum group analogue of the normalizer of $SU(1, 1)$ in $SL(2, \mathbb{C})$ 11
4. The von Neumann algebra for the dual quantum group 16
5. The decomposition of the left regular corepresentation 20
6. Results for special functions of basic hypergeometric type 27
Proofs
7. Extensions of the generators of $U_q(\mathfrak{su}(1, 1))$ 27
8. The Casimir operator 41
9. Generators of the dual von Neumann algebra $\hat{M}$ 57
10. Unitary corepresentations 67
11. Identities for special functions 79
Appendix A. Operators and von Neumann algebras 87
Appendix B. Special functions 88
Preamble

The proofs of the statements in the paper are technical. To enhance the readability of the paper, the paper is essentially split into two parts. The first part contains all the statements and can be read independently from the second part containing the proofs. Moreover, Section 6 is independent of the remainder of the paper, and in Section 8 we state explicit results for special functions of basic hypergeometric type. This section is meant for people interested in special functions. For the convenience of the reader we have added an index, which includes references to notations frequently used.

1. Introduction

On the one hand, the general theory of quantum groups has its roots in approaches in the axiomatizations of generalizations of groups such that the Pontryagin-van Kampen duality for locally compact abelian groups extends to this wider class. On the other hand, a large class of explicit and interesting quantum groups arose from various cases, e.g. \(R\)-matrices as solutions of the Yang-Baxter equation and the RTF-formalism. For the quantum groups related to compact groups arising in this way the duality is formulated on the level of Hopf algebra duality between the quantized function algebra and the quantized enveloping algebra. For the historic development of the general theory for locally compact quantum groups we refer to the papers –especially the introductions– [39], [40], [46], and books [15], [52]. For the development of quantum groups involving the Yang-Baxter equation and the RTF-formalism we refer to the books [9], [16], [25]. It has turned out that many of the examples arising in this way fit into the general theory of quantum groups, especially for the quantum group analogues of compact groups. These quantum groups can usually be analyzed in an algebraic way. For quantum group analogues of non-compact groups the situation is not so clear.

As it turns out the Hopf algebra arising from the standard \(R\)-matrix for \(SL(2, \mathbb{C})\) has three different \(*\)-structures [15], and we consider a \(*\)-structure making the Hopf algebra into a Hopf \(*\)-algebra as the choice of an appropriate real form. The compact case, corresponding to the quantum group analogue of \(SU(2)\), has been studied extensively, see [1], [10], [25] and references given there. This is also the basic example of a quantum group having an intimate link with special functions of basic hypergeometric type [17], see [3], [16], [23], [19], as well as [28]. Then there is the non-compact case associated to the non-compact group \(SU(1, 1)\) and a non-compact case associated to the group \(SL(2, \mathbb{R})\). Although \(SU(1, 1) \cong SL(2, \mathbb{R})\) as Lie groups the corresponding Hopf \(*\)-algebras for the deformed case are different. The subject of this paper is the Hopf \(*\)-algebra associated to the group \(SU(1, 1)\), in which the deformation parameter \(q\) is real. For the case of the Hopf \(*\)-algebra associated to \(SL(2, \mathbb{R})\) the deformation parameter is on the unit circle, and the situation changes dramatically, see [4] and for recent progress on the level of associated special functions see van de Bult [1].
In this paper we focus on the Hopf $\ast$-algebra associated to $SU(1, 1)$, which is recalled in Section 3. We also recall that Woronowicz [56] showed that there was no way to extend the comultiplication of this Hopf $\ast$-algebra in analytic way, i.e. to the level of operators on Hilbert spaces. Based on work of Korogodsky [37] and Woronowicz [57] it is possible to show that there exists a quantum group analogue of the normalizer of $SU(1, 1)$ in $SL(2, \mathbb{C})$ in the context of the definition of Kustermans and Vaes [10], [11] (see also [39] and [52] for an introduction) on the level of a von Neumann algebraic quantum group. This has been shown in [30], where special functions of basic hypergeometric type proved to be essential in the construction. The purpose of this paper is to give an explicit description of the dual quantum group and to decompose the left regular corepresentation into irreducible corepresentations for this explicit quantum group. In the decomposition of the left regular corepresentation we see the analogy with the group case, since in the left regular representation of the group $SU(1, 1) \simeq SL(2, \mathbb{R})$ only discrete series representations and principal unitary series occur. However, in the quantum group case the discrete series are no longer split up into a positive discrete series and a negative discrete series.

Some of these results have been announced in [31], and in this paper we give full proofs of these statements. This paper can be read independently from [31]. After browsing the paper it should be clear to the casual reader that making the general quantum group machinery work for this specific case is a very technical business. However, we believe that this is worthwhile since $SU(1, 1) \simeq SL(2, \mathbb{R})$ is one of the most important non-compact Lie groups [13], [21], [26], [34], [42], [50] and any reasonable quantum group theory has to have the example of a quantum group analogue of $SU(1, 1)$. Moreover, we hope that understanding this example may also lead to other non-trivial examples of non-compact quantum groups and related quantum homogeneous spaces, such as quantum group analogues of $SU(n, 1)$, and related homogeneous spaces $SU(n, 1)/S(U(n) \times U(1))$. Moreover, in the operator algebra context $K$-theory is available, and the first step in this direction is taken [12]. In particular, one can ask for a $K$-theoretic approach to discrete series representations in this setting. We expect that the link with special functions can lead to new and deep results in the theory of special functions, and we have included some highly non-trivial examples in Section 6, but we expect that the relation is deeper and not yet fully exploited. E.g. the link with twisted primitive elements, suitable Cartan type decompositions and (associated) spherical functions and corresponding transform as indicated in [33] can be studied from an operator algebraic point of view, see also [4], Ch. 3] for a more general study of the Plancherel measure in this context. Having the decomposition of the left regular representation available it is now also natural to consider other questions, e.g. can we decompose tensor products, describe the intertwiners in terms of special functions, etc? We are confident that the interpretation of discrete series representations in this context gives a solution to indeterminacy problems related to certain tensor product decompositions of infinite dimensional representations of $U_q(\mathfrak{su}(1, 1))$, see cf. [19], [18] for cases where the indeterminacy is absent.

We now describe the contents of the paper. In Sections 2-3 we recall the necessary background on general locally compact quantum groups in the von Neumann algebraic setting and the specific example that we study. In Section 2 we recall the Kustermans-Vaes approach to locally compact quantum groups on the von Neumann algebraic level, which is the framework
for this paper, and Section 2 is mainly based on [11]. Next in Section 3 we give a concise description of the Hopf ∗-algebra and the quantum group analogue of the normalizer of $SU(1,1)$ in $SL(2,\mathbb{C})$ in the context of Section 4. Section 3 is based on [30]. In Section 4 we give an explicit description of the dual quantum group in the case of the quantum group analogue of the normalizer of $SU(1,1)$ in $SL(2,\mathbb{C})$. In particular we show that the generators of the quantized universal enveloping algebra $U_q(\mathfrak{su}(1,1))$ can be realized as unbounded operators affiliated to the von Neumann algebra of the dual quantum group. We discuss how the (suitable extension of the) Casimir operator can be used to find sufficiently many generators. It turns out that the self-adjoint extension of the algebraically defined symmetric, but not essentially self-adjoint, operator is characterized by affiliation to the von Neumann algebra for the dual quantum group. We also show that comultiplication defined on the von Neumann algebraic group coincides with the comultiplication of the Hopf ∗-algebra $U_q(\mathfrak{su}(1,1))$. In Section 5 we collect some interesting new (as far as we are aware) results for special functions of basic hypergeometric type which are byproducts of the approach taken. In particular, the results discussed in Section 3 can be read independently by someone only interested in special functions, but the proofs are dependent on the rest of the paper. Sections 4–6 describe the results of this paper in detail and form the core of the paper. All the main results and its background can be obtained from Sections 2–6. The gist of the main results are obtained when reading only this part of the paper, which can also be viewed as a very extended introduction. The proofs of all statements in these sections are given in the remainder of the paper consisting of Sections 7–11. In Appendix A we recall some notation and terminology of von Neumann algebras, whereas we recall the necessary details of the special functions involved in Appendix B. In Appendix C we discuss a specific example of a Jacobi operator, whereas Appendix D contains nitty-gritty proofs of some intermediate lemmas.

2. Von Neumann algebraic quantum groups

In this section we recall the definition of the von Neumann algebraic quantum groups and related results. So we work with a theory on the quantum group analogue of locally compact groups in the realm of operator algebras. We summarize the main features, and we discuss the group case for a unimodular Lie group $G$. The proofs of all statements can be found in the papers [40], [41] by Kustermans and Vaes. Introductory texts on this subject are [39], [54], see also [52]. In Section 3 we describe the example we study, namely the von Neumann algebraic quantum group associated to the normalizer of $SU(1,1)$ in $SL(2,\mathbb{C})$, which is essentially recalling the results of [30].

Definition 2.1. Consider a von Neumann algebra $M$ together with a unital normal ∗-homomorphism $\Delta: M \to M \otimes M$ (the comultiplication) such that $(\Delta \otimes \text{Id})\Delta = (\text{Id} \otimes \Delta)\Delta$ (coassociativity). Moreover, if there exist two normal semi-finite faithful weights $\varphi$, $\psi$ on $M$ such that

$$\varphi((\omega \otimes \text{Id})\Delta(x)) = \varphi(x)\omega(1), \quad \forall \omega \in M^+_\varphi, \forall x \in \mathcal{M}^+_\varphi \quad (\text{left invariance}),$$

$$\psi((\text{Id} \otimes \omega)\Delta(x)) = \psi(x)\omega(1), \quad \forall \omega \in M^+_\psi, \forall x \in \mathcal{M}^+_\psi \quad (\text{right invariance}),$$
then \((M, \Delta)\) is a von Neumann algebraic quantum group.

Note that we suppress \(\varphi\) and \(\psi\) from the notation \((M, \Delta)\) for a von Neumann algebraic quantum group.

The notation in Definition 2.1 follows the standard notation for weights, tensor products and preduals, see e.g. [22], [31], which are briefly recalled in Appendix A. We recall here the basic constructions for weights, since the related modular objects play an important role, see [51]. In particular, a weight \(\varphi\) is a map \(\varphi: \mathcal{M}_+ \to [0, \infty]\), \(\mathcal{M}_+\) being the cone of positive elements in \(\mathcal{M}\), such that \(\varphi(x + y) = \varphi(x) + \varphi(y)\) and \(\varphi(\lambda x) = \lambda \varphi(x)\) for \(\lambda \geq 0\). Then 
\[
\mathcal{M}^+ = \{x \in \mathcal{M}_+ \mid \varphi(x) < \infty\}, \quad \mathcal{N} = \{x \in \mathcal{M} \mid \varphi(x^* x) < \infty\}
\]

is a left ideal and \(\mathcal{M}\) is the linear span of \(\mathcal{M}^+\). Then \(\mathcal{M} = \mathcal{N}^* \mathcal{N}\), and \(\varphi\) extends uniquely to \(\mathcal{M}\). The weight \(\varphi\) is faithful if \(\varphi(x) \neq 0\) for all non-zero \(x \in \mathcal{M}_+\). The weight \(\varphi\) is semifinite if \(\mathcal{M}\) is \(\sigma\)-strong-\(*\) dense in \(\mathcal{M}_+\) or \((\mathcal{M})' = \mathcal{M}\). The weight \(\varphi\) is normal if \(\varphi(\sup_x x_\lambda) = \sup_\lambda \varphi(x_\lambda)\) for any bounded increasing net \(\{x_\lambda\}_{\lambda \in \Lambda}\) in \(\mathcal{M}_+\), and this can be reformulated in various different ways. Normal semifinite faithful weight is abbreviated to nsf weight.

A GNS-construction for a weight is similar to a GNS-construction for a state. A GNS-construction for a weight \(\varphi\) is a triple \((\mathcal{H}, \pi, \Lambda)\) consisting of a Hilbert space \(\mathcal{H}\), a \(\ast\)-homomorphism \(\pi: \mathcal{M} \to \mathcal{B}(\mathcal{H})\) and a linear map \(\Lambda: \mathcal{N} \to \mathcal{H}\) such that

1. \(\Lambda(\mathcal{N})\) is dense in \(\mathcal{H}\);
2. \(\langle \Lambda(a), \Lambda(b) \rangle = \varphi(b^* a)\) for all \(a, b \in \mathcal{N}\);
3. \(\pi(x) \Lambda(a) = \Lambda(x a)\) for all \(x \in \mathcal{M}, a \in \mathcal{N}\).

In case \(\varphi\) is a nsf weight, the representation \(\pi\) is injective, normal and nondegenerate, and \(\Lambda\) is closed for the \(\sigma\)-strong-\(*\) topology on \(\mathcal{M}\) and the norm topology of \(\mathcal{H}\). In case we want to stress the dependence on the weight \(\varphi\) we use the notation \(\mathcal{M}_\varphi^+, \mathcal{M}_\varphi, \mathcal{N}_\varphi, \mathcal{H}_\varphi, \pi_\varphi, \Lambda_\varphi\) as in Definition 2.1.

The weight \(\varphi\), respectively \(\psi\), in Definition 2.1 is the left, respectively right, Haar weight for the von Neumann algebraic quantum group \((M, \Delta)\). It can be shown that the left and right Haar weights are unique up to a constant.

In this paper, we mainly deal with the von Neumann algebra \(M\) and the corresponding von Neumann algebra \(\hat{M}\) for the dual von Neumann algebraic quantum group, see Theorem 2.3, and the weights do not play a big role, but the associated modular operator, modular conjugation and modular automorphism group plays an important role. In order to obtain the properties of these operators, consider the GNS-representation for \(\varphi\) and the antilinear map from \(\Lambda(\mathcal{N} \cap \mathcal{N}^*) \subset \mathcal{H}\) to itself defined by \(\Lambda(x) \mapsto \Lambda(x^*)\). This map has polar decomposition \(J \nabla^{-1/2}\), where \(J: \mathcal{H} \to \mathcal{H}\) is an antilinear isometry and \(J^2 = 1\). \(J\) is the modular conjugation and the (generally unbounded) self-adjoint operator \(\nabla\) is the modular operator associated with the weight \(\varphi\). Then

\[
J \pi(M) J = \pi(M)', \quad \nabla^{it} \pi(M) \nabla^{-it} = \pi(M), \quad t \in \mathbb{R}. \tag{2.1}
\]

Here \(\pi(M)' = \{x \in \mathcal{B}(\mathcal{H}) \mid xy = yx \ \forall y \in \pi(M)\}\) is the commutant of \(\pi(M)\). For a nsf weight \(\pi\) is faithful, and then we identify \(\pi(M)\) with \(M\), so that (2.1) gives \(JMJ = M', \nabla^{it} M \nabla^{-it} = M, \ t \in \mathbb{R}\). Then \(\sigma_t(x) = \nabla^{it} x \nabla^{-it}, x \in M\), defines a strongly continuous one-parameter group \(\sigma\) of \(\ast\)-automorphisms on \(M\) for the nsf weight \(\varphi\). It is the modular automorphism group \(\sigma = \sigma_\varphi\) for the nsf weight \(\varphi\).
Having the GNS-construction for the left invariant nsf weight $\varphi$ we define

$$W^*\left(\Lambda(a) \otimes \Lambda(b)\right) = \left(\Lambda \otimes \Lambda\right)(\Delta(b)(a \otimes 1)),$$

then $W$ is a unitary operator on $\mathcal{H} \otimes \mathcal{H}$, which is known as the multiplicative unitary and is instrumental in the development of locally compact quantum groups, as pointed out initially in [3]. Identifying $M$ with $\pi(M)$, we obtain $\Delta(x) = W^*(1 \otimes x)W$ for all $x \in M$, so that the multiplicative unitary implements the comultiplication.

**Remark 2.2.** To see how groups are included in this definition take a group $G$, which for convenience we assume to be a unimodular Lie group. Then the von Neumann algebra $M \cong L^\infty(G)$ is acting by multiplication operators on the Hilbert space $L^2(G)$, defined with respect to the left Haar measure $d_l g$. So we consider $M$ as a subalgebra of $B(L^2(G))$. Then $\varphi(f) = \int_G f(g) d_l g$ for $f \in L^\infty(G) \cap L^2(G) = \mathcal{M}$, and the corresponding GNS-construction of $\varphi$ is $(L^2(G), \text{Id}, \Lambda)$ where $\Lambda : \mathcal{N}_\varphi = L^2(G) \cap L^\infty(G) \to L^2(G)$, $x \mapsto x$. In this case the predual is $M_\star = L^1(G) \subset M^\star$ by considering $L^1 \ni f \mapsto (L^\infty(G) \ni x \mapsto \int_G f(g)x(g) d_l g)$ and $M \cong (M_\star)^\star$ and the $\sigma$-weak topology is the $\sigma(M, M_\star)$-topology. For $f, h \in L^2(G)$ a normal functional $\omega_{f,h}$ is defined as the matrix element $\omega_{f,h}(x) = \langle xf, h \rangle = \int_G x(g)f(g)h(g)d_l g$. In this case the multiplicative unitary $W$ is

$$W : L^2(G) \otimes L^2(G) \cong L^2(G \times G) \to L^2(G) \otimes L^2(G) \cong L^2(G \times G)$$

$$(W^*f)(g,h) = f(g, gh), \quad (Wf)(g,h) = f(g, g^{-1}h).$$

Particular to the unimodular Lie group case is that the antipode $S : M \to M$, $(Sx)(g) = x(g^{-1})$ is bounded, but in the general case it is not. To indicate how the antipode can be obtained from the invariant weight in the general case note that

$$\int_G x(h^{-1}g) y(g) d_l g = \int_G x(g) y(hg) d_l g$$

so that

$$S : (\text{Id} \otimes \varphi)(\Delta(x)(1 \otimes y)) \mapsto (\text{Id} \otimes \varphi)((1 \otimes x)\Delta(y)).$$

This in particular gives the key to defining the antipode $S$ on a von Neumann algebraic quantum group as an unbounded operator. A basic result is a polar decomposition of the antipode. To be precise, there exists a unique $\ast$-anti-automorphism $R : M \to M$ and a unique strongly continuous one-parameter group of $\ast$-automorphisms $\tau : \mathbb{R} \to \text{Aut}(M)$ satisfying

$$S = R\tau_{-i/2}, \quad R^2 = \text{Id}, \quad \tau_t R = R\tau_t \quad \forall t \in \mathbb{R}. \quad (2.2)$$

$R$ is known as the unitary antipode, and $\tau$ the scaling group. One can show that $\varphi R$ is a right invariant nsf weight, and one can make the choice $\psi = \varphi R$ for the right Haar weight, which we assume from now on.

An interesting result in the theory of locally compact quantum group is duality, see Theorems 2.3 and 2.4. For the dual locally compact quantum group we have

$$\hat{M} = \{\omega \otimes \text{Id}(W) \mid \omega \in B(\mathcal{H})_\star\} \subset B(\mathcal{H}), \quad (2.3)$$

where the closure is with respect to the $\sigma$-strong-$\ast$ topology and $\mathcal{H}$ is the GNS-space for the left invariant weight $\varphi$. 
The analogue of the left regular representation of a Lie group preserves algebra \( \hat{\omega} \) for the GNS-space \( H \). Because of this, we have for the multiplicative unitary the relations

\[
\hat{R}(x) = \hat{J} x^* \hat{J}, \quad \forall x \in M, \quad \hat{\hat{R}}(x) = J x^* J, \quad \forall x \in \hat{M}. \tag{2.4}
\]

It follows from Theorem 2.3 that the multiplicative unitary \( \hat{w} = \Sigma W^* \Sigma \) for the dual von Neumann algebraic quantum group is \( \hat{w} \) of a von Neumann algebra acting on \( H \) is a unitary corepresentation. A unitary corepresentation \( U \) of a von Neumann algebraic quantum group on a Hilbert space \( H \) is a unitary element \( U \in M \otimes B(H) \) such that \( (\Delta \otimes \Id)(U) = \int G \hat{U}_{13} U_{23} \in M \otimes M \otimes B(H) \), where the standard leg-numbering is used in the right hand side. In particular, it follows from the relations \( W_{12} W_{13} W_{23} = W_{23} W_{12} \) and \( \Delta(x) = W^*(1 \otimes x)W \) that the multiplicative unitary \( W \) defines a unitary corepresentation of \( M \) on the GNS-space. This corepresentation is the analogue of the left regular representation of a Lie group \( G \) on the Hilbert space \( L^2(G) \). A closed subspace \( L \subseteq H \) for the unitary corepresentation is an invariant subspace if \( (\omega \otimes \Id)(U) \) preserves \( L \) for all \( \omega \in M_* \). In particular, it follows from Definition 2.3 that an invariant subspace is precisely the closed subspace invariant for the action of the dual von Neumann algebra \( \hat{M} \), since it is generated by \( (\omega \otimes \Id)(W) \), \( \omega \in M_* \). A unitary corepresentation \( U \) in the Hilbert space \( H \) is irreducible if there are only trivial (i.e. equal to \( \{0\} \) or the whole Hilbert space \( H \)) invariant subspaces. In particular, \( \{ (\omega \otimes \Id)(U) \mid \omega \in M_* \}'' = B(H) \) implies that \( U \) is an irreducible unitary corepresentation.

The nice feature of the von Neumann algebraic quantum groups is the following theorem, due to Kustermans and Vaes [10], [11], which is a far-reaching generalization of the Pontryagin-van Kampen duality.

**Theorem 2.4.** \( (\hat{M}, \hat{\Delta}) = (M, \Delta) \).

**Remark 2.5.** We finish by discussing some of the above in the case of a unimodular Lie group \( G \) continuing Remark 2.2. Identify \( \omega \in M_* \) with a function \( k \in L^1(G) \), then \( (\omega \otimes \Id)(W) \in B(L^2(G)) \) is the convolution operator \( f \mapsto k * f \), \( (k * f)(g) = \int_G k(s)f(s^{-1}g) \, ds \). Then the product in \( \hat{M} \) corresponds to the convolution product, and the dual left invariant weight on such a convolution operator is evaluation of the kernel at the identity of the group \( G \). To see that the corepresentation associated to the multiplicative unitary corresponds to the left
regular representation, say $\lambda$, we check

$$\left(\left(\left(\omega_{f_1, f_2} \otimes \text{Id}\right)(W)\right) f_3\right)(h) = \langle W(f_1 \otimes f_3), f_2 \rangle_1(h) = \int_G (W(f_1 \otimes f_3))(g, h) \bar{f}_2(g) \, dg$$

$$= \int_G f_1(g) \, \bar{f}_2(g) \, f_3(g^{-1} h) \, dg = \left(\lambda(f_1 \bar{f}_2) f_3\right)(h).$$

Since the normal functional $\omega_{f_1, f_2}$ corresponds to $f_1 \bar{f}_2 \in L^1(G)$, the required result follows.

3. THE QUANTUM GROUP ANALOGUE OF THE NORMALIZER OF $SU(1, 1)$ IN $SL(2, \mathbb{C})$

In this section we recall the von Neumann algebraic quantum group for which we calculate the dual von Neumann algebraic quantum group, and for which we decompose the left regular corepresentation. Except for the last paragraph, all the results described are taken from [30].

The Lie group $SU(1, 1) \cong SL(2, \mathbb{R})$ is one of the most important non-compact Lie groups. On the level of Hopf algebras, a classification of real forms of the quantized universal enveloping algebra $U_q(\mathfrak{sl}(2, \mathbb{C}))$ results in three different real forms, i.e. Hopf $*$-algebras; the compact case $U_q(\mathfrak{su}(2))$ for $0 < q < 1$, which is extensively studied [1], [16], [25], [43]; the non-compact case $U_q(\mathfrak{sl}(2, \mathbb{R}))$ with $q$ on the unit circle, see e.g. the previously mentioned books and [7]; and the non-compact case $U_q(\mathfrak{su}(1, 1))$ for $0 < q < 1$. In these cases there is a related dual Hopf $*$-algebra which is a deformation of the algebra of polynomials on the related group. We refer to the books [9], [16], [25], [43], as well as to [8], [45], [33] for more information and references. However, as Woronowicz [56] proved, there is no C$^*$-algebra interpretation for the related Hopf $*$-algebra with a well-defined comultiplication. Later, Korogodsky [37] indicates how the ill-defined comultiplication could be avoided. With the introduction of the theory of von Neumann algebraic quantum [40], [41] it is natural to ask whether or not this important example can be incorporated in the theory of von Neumann algebraic quantum groups. As it turns out the answer is yes, and the key to the solution is using special functions.

All statements of this section are proved in [30], except (3.6) for which a direct proof is given.

Throughout the paper, we fix a number $0 < q < 1$. Define $A_q$ to be the unital $*$-algebra generated by elements $\alpha$, $\gamma$ and $e$ and relations

$$\alpha^\dagger \alpha - \gamma^\dagger \gamma = e \quad \alpha \alpha^\dagger - q^2 \gamma^\dagger \gamma = e \quad \gamma^\dagger \gamma = \gamma \gamma^\dagger$$

$$\alpha \gamma = q \gamma \alpha \quad \alpha^\dagger \gamma = q \gamma^\dagger \alpha$$

$$e^\dagger = e \quad e^2 = 1 \quad \alpha e = e \alpha \quad \gamma e = e \gamma$$

(3.1)

where $\dagger$ denotes the $*$-operation on $A_q$ (in order to distinguish this kind of adjoint with the adjoints of possibly unbounded operators in Hilbert spaces). In case we take $e = 1$ in (3.1) we obtain the $*$-algebra which is usually associated with the algebra of polynomials on the quantum analogue of $SU(1, 1)$, see [15], [38], [33]. The additional generator $e$ has been introduced by Korogodsky [37].

For completeness we give the Hopf $*$-algebra structure on $A_q$. By $A_q \otimes A_q$ we denote the algebraic tensor product. There exists a unique unital $*$-homomorphism $\Delta : A_q \to A_q \otimes A_q$
such that
\[ \Delta(\alpha) = \alpha \otimes \alpha + q (e \gamma^\dagger) \otimes \gamma \quad \Delta(\gamma) = \gamma \otimes \alpha + (e \alpha^\dagger) \otimes \gamma \quad \Delta(e) = e \otimes e \] (3.2)

The counit \( \varepsilon : \mathcal{A}_q \to \mathbb{R} \) and antipode \( S : \mathcal{A}_q \to \mathcal{A}_q \) are given by
\[ S(\alpha) = e \alpha^\dagger \quad S(\alpha^\dagger) = e \alpha \quad S(\gamma) = -q \gamma \quad S(\gamma^\dagger) = -\frac{1}{q} \gamma^\dagger \quad S(e) = e \] (3.3)

This makes \( \mathcal{A}_q \) into a Hopf \(*\)-algebra.

To see that for \( q = 1 \) we obtain the Hopf \(*\)-algebra of polynomials on the group \( SU(1,1) \) (when restricting to the sub-Hopf \(*\)-algebra \( \mathcal{A}_1 \) given by \( e = 1 \)) and on the normalizer \( N_{SL(2,\mathbb{C})}(SU(1,1)) \) of \( SU(1,1) \) in \( SL(2,\mathbb{C}) \) we recall
\[ SU(1,1) = \left\{ g \in SL(2,\mathbb{C}) \mid g^* J g = J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a & c \\ \bar{c} & \bar{a} \end{pmatrix} \mid a, c \in \mathbb{C}, \ |a|^2 - |c|^2 = 1 \right\} \]
and we let \( \alpha(g) = a, \gamma(g) = c \). Similarly,
\[ N_{SL(2,\mathbb{C})}(SU(1,1)) = \left\{ g \in SL(2,\mathbb{C}) \mid g^* J g = \pm J \right\} = \left\{ \begin{pmatrix} a & c \\ \bar{c} & \bar{a} \end{pmatrix} \mid a, c \in \mathbb{C}, \ \varepsilon \in \{\pm 1\}, |a|^2 - |c|^2 = \varepsilon \right\} = SU(1,1) \cup SU(1,1) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]
and we put \( \alpha(g) = a, \gamma(g) = c, e(g) = \varepsilon \).

The following result by Woronowicz \[56\] states that one cannot expect a suitable quantum group on an operator algebra level arising from Hopf \(*\)-algebra \( \mathcal{A}_q \) (i.e. with \( e = 1 \) in (3.1)). In Theorem \[3.1\] a representation of \( \mathcal{A}_q \) consists of two closed operators \( \alpha \) and \( \gamma \) acting in a Hilbert space \( H \) such that the domains of \( \alpha, \gamma, \alpha^*, \gamma^* \) are equal, say \( D \), and such that the relations in (3.1) are represented in a weak sense, e.g. \( \alpha \gamma = q \gamma \alpha \) is translated by \( \langle \gamma v, \alpha^* w \rangle = q \langle \alpha v, \gamma^* w \rangle \) for all \( v, w \in D \), etc.

**Theorem 3.1** (Woronowicz \[56\]). For \((\alpha^1, \gamma^1)\), resp. \((\alpha^2, \gamma^2)\), closed operators on an infinite dimensional Hilbert space \( H^1 \), resp. \( H^2 \), representing the relations, there exist no closed operators \( \alpha, \gamma \) acting on \( H^1 \otimes H^2 \) representing the relations and extending \( \alpha^1 \otimes \alpha^2 + q (\gamma^1)^* \otimes \gamma^2 \), \( \gamma^1 \otimes \alpha^2 + (\alpha^1)^* \otimes \gamma^2 \), such that \( \alpha^*, \gamma^* \) extend \( (\alpha^1)^* \otimes (\alpha^2)^* + q \gamma^1 \otimes (\gamma^2)^* \), \((\gamma^1)^* \otimes (\alpha^2)^* + \alpha^1 \otimes (\gamma^2)^* \).

Theorem 3.1 is a negative result, but Korogodsky \[37\] pointed out how to proceed by adding the additional generator \( e \).

It is not hard to represent the commutation relations (3.1) by unbounded operators acting on the Hilbert space \( H = L^2(\mathbb{T}) \oplus L^2(I_q) \), where \( I_q = -q^N \cup q^Z \) and equipped with the counting measure. Here \( \mathbb{T} = \{ z \in \mathbb{C} \mid |z| = 1 \} \) denotes the unit circle, \( \mathbb{N} = \{ 1, 2, \ldots \} \) and \( \mathbb{N}_0 = \{ 0, 1, 2, \ldots \} \). If \( p \in I_q \), we define \( \delta_p(x) = \delta_{x,p} \) for all \( x \in I_q \), so the family \( \{ \delta_p \mid p \in I_q \} \) is the natural orthonormal basis of \( L^2(I_q) \). For \( L^2(\mathbb{T}) \) we have the natural orthonormal basis \( \{ \zeta^m \mid m \in \mathbb{Z} \} \), with \( \zeta \) the identity function on \( \mathbb{T} \). Then \( \{ \zeta^m \otimes \delta_p \mid m \in \mathbb{Z}, p \in I_q \} \) is an orthonormal basis for \( H \). Define linear operators \( \alpha_0, \gamma_0, e_0 \) on the space \( E \) of finite linear
combinations of $\zeta^m \otimes \delta_p$ by
\[
\begin{align*}
\alpha_0(\zeta^m \otimes \delta_p) &= \sqrt{\text{sgn}(p)} + p^{-2} \zeta^m \otimes \delta_{p^*}, \\
\gamma_0(\zeta^m \otimes \delta_p) &= p^{-1} \zeta^{m+1} \otimes \delta_p, \\
e_0(\zeta^m \otimes \delta_p) &= \text{sgn}(p) \zeta^m \otimes \delta_p.
\end{align*}
\]
for all $p \in I_q, m \in \mathbb{Z}$. The actions of $\alpha_0^\dagger$ and $\gamma_0^\dagger$ on $E$ can be given in a similar fashion by taking formal adjoints, and these satisfy the relations (3.1), and give a faithful representation of the algebra $A_q$. Then [30, §2] the operators $\alpha_0, \gamma_0$ are closable with densely defined closed unbounded operators $\alpha, \gamma$ as their closure. Moreover, the adjoints $\alpha^*$ and $\gamma^*$ are the closures of $\alpha_0^\dagger, \gamma_0^\dagger$. Let $e$ be the closure of $e_0$, then $e$ is a bounded linear self-adjoint operator on $H$. As discussed by Woronowicz [57] and in [30], it is not sufficient to consider the von Neumann algebra generated by $\alpha, \gamma$ and $e$ in order to obtain a well-defined comultiplication. Consider the linear map $T: \zeta^m \otimes \delta_p \mapsto \zeta^m \otimes \delta_{-p}, T \in B(H)$, where we take $\delta_p = 0$ in case $p \notin I_q$, and let $u$ be its partial isometry.

**Definition 3.2.** $M$ is the von Neumann algebra in $B(H)$ generated by $\alpha, \gamma, e$ and $u$.

By definition, see Appendix A.4, $\alpha$ and $\gamma$ are affiliated to $M$.

It can be shown [30, Lemma 2.4 (3)] that $M = L^\infty(\mathbb{T}) \otimes B(L^2(I_q))$. We define the operators
\[
\Phi(m, p, t): \zeta^r \otimes \delta_x \mapsto \delta_{xt} \zeta^{m+r} \otimes \delta_p,
\]
for $m, r \in \mathbb{Z}, p, t, x \in I_q$.

A straightforward calculation gives
\[
\Phi(m_1, p_1, t_1) \Phi(m_2, p_2, t_2) = \delta_{p_2,t_1} \Phi(m_1 + m_2, p_1, t_2), \quad \Phi(m, p, t)^* = \Phi(-m, p, t)
\]
In particular the finite linear span of the operators $\Phi(m, p, t)$ form a $\sigma$-weakly dense $\ast$-subalgebra in $M$.

In order to show that $M$ is the von Neumann algebra of a von Neumann algebraic quantum group we need to define the comultiplication $\Delta$ and the left and right invariant nsf weights $\varphi$ and $\psi$ such that the requirements of Definition 2.1 are met. We start with the construction of the left invariant nsf weight by writing down its GNS-construction. Define $\text{Tr} = \text{Tr}_{L^\infty(\mathbb{T})} \otimes \text{Tr}_{B(L^2(I_q))}$ on $M$, where $\text{Tr}_{L^\infty(\mathbb{T})}$ and $\text{Tr}_{B(L^2(I_q))}$ are the canonical traces on $L^\infty(\mathbb{T})$, i.e. $\text{Tr}_{L^\infty(\mathbb{T})}(f) = \int_\mathbb{T} f(\zeta) d\zeta$ with normalization $\text{Tr}_{L^\infty(\mathbb{T})}(1) = 1$, and on $B(L^2(I_q))$, normalized by $\text{Tr}_{B(L^2(I_q))}(P) = 1$ for any rank one orthogonal projection. Note that $\text{Tr}$ is a tracial weight on $M$ so in particular its modular group is trivial. For $\text{Tr}$ we have the following GNS-construction:

- a Hilbert space $\mathcal{K} = H \otimes L^2(I_q) = L^2(\mathbb{T}) \otimes L^2(I_q) \otimes L^2(I_q)$ equipped with the orthonormal basis $\{f_m,p,t | m \in \mathbb{Z}, p, t \in I_q\}$;
- a unital $\ast$-homomorphism $\pi: M \to B(\mathcal{K})$, $\pi(a) = a \otimes \text{Id}_{L^2(I_q)}$ for $a \in M$;
- $\Lambda_{\text{Tr}}: \mathcal{N}_{\text{Tr}} \to \mathcal{K}$, $a \mapsto \sum_{p \in I_q} (a \otimes \text{Id}_{L^2(I_q)}) f_{0,p,p}$.

We define the left invariant nsf weight $\varphi$ formally as $\varphi(x) = \text{Tr}(|\gamma| x |\gamma|)$ with the operator $|\gamma|$ affiliated to $M$. We proceed by defining the set $D$ as the set of elements of $x \in M$ such that $x|\gamma|$ extends to a bounded operator on $H$, denoted by $x|\gamma|$, and such that $x|\gamma| \in \mathcal{N}_{\text{Tr}}$, and for $x \in D$ we put $\Lambda(x) = \Lambda_{\text{Tr}}(x|\gamma|)$. The set $D$ is then a core for the operator $\Lambda$ which is closable for the $\sigma$-strong-$\ast$-norm topology.

**Definition 3.3.** The nsf weight $\varphi$ on $M$ is defined by its GNS-construction $(\mathcal{K}, \pi, \Lambda)$.
Remark 3.4. From the general theory of nsf weights as recalled in Section 2 we know that \( \varphi \) comes with a modular automorphism group \( \sigma \), a modular conjugation \( J \) and modular operator \( \nabla \). In particular, as established in [30, §4], we have:

- \( \sigma_t(x) = |\gamma|^{2it}x|\gamma|^{-2it} \) for all \( x \in M, t \in \mathbb{R} \);
- \( \Phi(m, p, t) \in \mathcal{N}_\varphi \) and \( \Lambda(\Phi(m, p, t)) = |t|^{-1}f_{mpt} \);
- \( \Phi(m, p, t) \) is analytic for \( \sigma \) and \( \sigma_z(\Phi(m, p, t)) = |p|^{-1}t|^{2iz} \Phi(m, p, t) \) for all \( z \in \mathbb{C} \);
- \( Jf_{mpt} = f_{-m, t, p} \);
- \( f_{mpt} \) in the domain of \( \nabla \) and \( \nabla f_{mpt} = |p|^{-1}t^2 f_{mpt} \).

Remark 3.5. Note that in particular we can use \( \pi \) to identify \( M \subset B(H) \) with its image \( \pi(M) \subset B(K) \). From now on we use this identification, and we work with \( M \) realized as von Neumann algebra in \( B(K) \).

In [30, §4] it is observed that the right invariant weight \( \psi = \varphi \), so it remains to construct the comultiplication which we give using the multiplicative unitary \( W \in B(K \otimes K) \). We give an explicit expression for \( W^* \in B(K \otimes K) \) in terms of basic hypergeometric series \( a_p \) in (7.10). The functions \( a_p(\cdot, \cdot) \) are recalled in Definition 6.2, and the unitarity of the multiplicative unitary \( W \) is closely related to orthogonality properties of these functions \( a_p \). Then the comultiplication is given by, recall Remark 3.5 that we view \( M \subset B(K) \),

\[
\Delta(x) = W^*(1 \otimes x)W, \quad x \in M. \tag{3.5}
\]

In fact, this formula has led to the definition of the multiplicative unitary in (7.10), since the functions \( a_p \) are interpreted as Clebsch-Gordan coefficients for the tensor product decomposition of the representations considered in (3.4). We refer to [30, §3] for a more elaborate discussion of this motivation.

Theorem 3.6. The pair \( (M, \Delta) \) is a von Neumann algebraic quantum group.

Theorem 3.6 is [30, Thm. 4.9], and the really hard part is to prove the coassociativity \( \text{Id} \otimes \Delta \circ \Delta = \Delta \otimes \text{Id} \circ \Delta \). For this part the choice of sign \( s(\cdot, \cdot) \) in Definition 7.2 of the function \( a_p \) is essential. It should be noted that the results are obtained in different order in [30] than presented here.

All of the above is included in [30], but we additionally need the action of the dual modular conjugation \( \hat{J} \) in the GNS-space \( K \). Explicitly, we have

\[
\hat{J}f_{m, p, t} = \text{sgn}(p)^{\chi(p)} \text{sgn}(t)^{\chi(t)} (-1)^m f_{-m, p, t}, \quad p, t \in I_q, \; m \in \mathbb{Z}. \tag{3.6}
\]

This can be proved from the results in [30] as follows. Since the right invariant weight equals the left invariant weight, we have \( \hat{J}\Lambda(x) = \Lambda(R(x)^*) \) for \( x \in \mathcal{N} \), see [41, Prop. 2.11]. Using [30, Prop. 4.14] for the explicit expression of the unitary antipode \( R \) we see that applying this expression with \( x = \Phi(m, p, t) \) gives (3.6).

4. The von Neumann algebra for the dual quantum group

The general theory as described in Section 2 shows that there is a dual von Neumann algebraic quantum group associated to the von Neumann algebraic quantum group \( (M, \Delta) \) associated to the normalizer of \( SU(1, 1) \) in \( SL(2, \mathbb{C}) \), see Theorem 2.3. Since we have the
von Neumann algebra $M$ explicitly given by Definition 3.2 and Theorem 3.6 it is natural to ask for an explicit description in terms of generators for the von Neumann algebra $\hat{M}$ of the dual von Neumann algebraic quantum group. On the level of Hopf algebras, there is a duality between $A_q^t$ and the quantized universal enveloping algebra $U_q(\mathfrak{su}(1,1))$, see [13] and [8]. So it is natural to expect that the quantized enveloping algebra $U_q(\mathfrak{su}(1,1))$ plays a role in an explicit description of $\hat{M}$, but also that $U_q(\mathfrak{su}(1,1))$ will not suffice to describe $\hat{M}$. This is made explicit in Theorem 4.13.

Let us first recall the quantized universal enveloping algebra $U_q(\mathfrak{su}(1,1))$ in order to fix the notation. The study of $U_q(\mathfrak{su}(1,1))$ goes back to Vaksman and Korogodskii [53], and Masuda et al. [17], see also Burban and Klimyk [8]. Its representation theory is also needed in this paper, and we recall the irreducible admissible representations in Section 8, where we decompose the GNS-space with respect to the $U_q(\mathfrak{su}(1,1))$-action. For general information on quantized universal enveloping algebras one can consult e.g. [9], [16], [25], [43], [49].

Recall that $U_q(\mathfrak{su}(1,1))$ is the complex unital $*$-algebra generated by $K, K^{-1}, E$ and $F$ subject to

$$KK^{-1} = 1 = K^{-1}K, \quad KE = qEK, \quadKF = q^{-1}FK, \quad FE - EF = \frac{K^2 - K^{-2}}{q - q^{-1}} \quad (4.1)$$

and where the $*$-structure is defined by $K^* = K, E^* = F$. Since we assume $0 < q < 1$, the $*$-structure is easily seen to be compatible with (4.1). (We identify $(A, B, C, D)$ of [33] by $(K, E, -F, K^{-1})$ and compared to the notation of [13] we have $e = E, f = -F$ and $k = K$.) The algebra $U_q(\mathfrak{su}(1,1))$ has more structure, since it can be made into a Hopf $*$-algebra. For completeness we recall the action of the antipode $S$ and the comultiplication $\Delta$ on the generators;

$$S(K) = K^{-1}, \quad S(E) = -q^{-1}E, \quad S(F) = -qF, \quad S(K^{-1}) = K. \quad (4.2)$$

and

$$\Delta(K) = K \otimes K, \quad \Delta(E) = K \otimes E + E \otimes K^{-1} \quad \Delta(F) = K \otimes F + F \otimes K^{-1}, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}. \quad (4.3)$$

The Casimir element

$$\Omega = \frac{1}{2} \left((q^{-1} - q)^2FE - qK^2 - q^{-1}K^{-2}\right) = \frac{1}{2} \left((q^{-1} - q)^2EF - qK^{-2} - q^{-1}K^2\right) \quad (4.4)$$

is a central self-adjoint element in $U_q(\mathfrak{su}(1,1))$. In fact, we use a slightly renormalized version of the operator used in [43]. If $C$ denotes the element introduced in [43], Part II, (1.9)], one has $\Omega = -\frac{1}{2} \left((q^{-1} - q)^2\right)C - 1$. The Casimir element $\Omega$ generates the center of $U_q(\mathfrak{su}(1,1))$. In order to represent the algebra $U_q(\mathfrak{su}(1,1))$ on the Hilbert space $\mathcal{K}$ of the GNS-representation some care has to be taken, since the operators are in general unbounded. We define the dense subspace $\mathcal{K}_0$ of $\mathcal{K}$ as the linear subspace consisting of finite linear combinations of the orthonormal basis elements $f_{mpt}$, see the definition in Section 3. Equivalently $\mathcal{K}_0$ can also be viewed as the linear span of elements of the form $\zeta^m \otimes f$ with $m \in \mathbb{Z}$, $f \in \mathcal{K}(I_q \times I_q)$, where $\mathcal{K}(I_q \times I_q)$ is the space of compactly supported function on $I_q \times I_q$. Note that $\mathcal{K}_0$ is dense in $\mathcal{K}$ and that $\mathcal{K}_0$ inherits the inner product of $\mathcal{K}$, so we can look at the space of adjointable
operators \( L^+(\mathcal{K}_0) \) for \( \mathcal{K}_0 \), see [13, Prop. 2.1.8]. Recall that

\[ L^+(\mathcal{K}_0) = \{ T : \mathcal{K}_0 \to \mathcal{K}_0 \text{ linear} \mid \exists S : \mathcal{K}_0 \to \mathcal{K}_0 \text{ linear so that } \langle Tx, y \rangle = \langle x, Sy \rangle \ \forall x, y \in \mathcal{K}_0 \}. \]

The \(*\)-operation in \( L^+(\mathcal{K}_0) \) will be denoted by \( \dagger \).

**Definition 4.1.** We define operators \( U \) realization of \([9]\), implies that the \(*\)-representation of \( U \) and the elements from \( \{ q^{1/2} \} \) for all \( q \in \mathbb{C} \) and \( t \in I_q \).

Using (2.4) we see that (4.7) is in correspondence with (4.2).

**Proposition 4.2.** We have

\[ (q - q^{-1}) E_0 f_{\text{mpt}} = \text{sgn}(t) \left( q^{-m+1/2} \frac{|p|}{|t|} \sqrt{1 + \kappa(q^{-1})} f_{m+1,p,q^{-1}t} \right) - \text{sgn}(p) \left( q^{m+1} \frac{|t|}{|p|} \sqrt{1 + \kappa(p)} f_{m+1,q,p,t} \right) \quad (4.5) \]

and \( K_0 f_{\text{mpt}} = q^{-m} |p/t|^{3/2} f_{\text{mpt}} \) for all \( m \in \mathbb{Z} \), \( p, t \in I_q \).

Here \( \text{sgn} \) denotes the sign, and \( \kappa(x) = \text{sgn}(x)x^2 \), see Definition 6.1.

**Definition 4.2.** We define operators \( E_0, K_0 \) in \( L^+(\mathcal{K}_0) \) by

\[ (q - q^{-1}) E_0 f_{\text{mpt}} = \text{sgn}(t) \left( q^{-m+1/2} \frac{|p|}{|t|} \sqrt{1 + \kappa(t)} f_{m+1,p,q^{-1}t} \right) - \text{sgn}(p) \left( q^{m+1/2} \frac{|t|}{|p|} \sqrt{1 + \kappa(p)} f_{m+1,q,p,t} \right) \quad (4.6) \]

and for all \( m \in \mathbb{Z} \), \( p, t \in I_q \).

At this point we observe that modular conjugation \( J \) preserves \( \mathcal{K}_0 \), since \( J f_{\text{mpt}} = f_{-m,t,p} \)

**Proposition 4.2.** We have

\[ K_0 E_0 = q E_0 K_0 \quad \text{and} \quad E_0^\dagger E_0 - E_0 E_0^\dagger = \frac{K_0^2 - K_0^{-2}}{q - q^{-1}}. \]

and the elements from \( \{ K_0^m E_0^l E_0^\dagger \}_{m \in \mathbb{Z}, k, l \in N_0} \) are linearly independent.

**Proposition 4.2.** We have

\[ (q - q^{-1}) K_0 f_{\text{mpt}} = \text{sgn}(t) \left( q^{-m+1/2} \frac{|p|}{|t|} \sqrt{1 + \kappa(q^{-1})} f_{m+1,p,q^{-1}t} \right) - \text{sgn}(p) \left( q^{m+1/2} \frac{|t|}{|p|} \sqrt{1 + \kappa(p)} f_{m+1,q,p,t} \right) \quad (4.5) \]

and for all \( m \in \mathbb{Z} \), \( p, t \in I_q \).

At this point we observe that modular conjugation \( J \) preserves \( \mathcal{K}_0 \), since \( J f_{\text{mpt}} = f_{-m,t,p} \)

**Proposition 4.2.** We have

\[ K_0 E_0 = q E_0 K_0 \quad \text{and} \quad E_0^\dagger E_0 - E_0 E_0^\dagger = \frac{K_0^2 - K_0^{-2}}{q - q^{-1}}. \]

and the elements from \( \{ K_0^m E_0^l E_0^\dagger \}_{m \in \mathbb{Z}, k, l \in N_0} \) are linearly independent.

Proposition 4.2 implies that there exists a unique unital \(*\)-representation \( \rho : U_q(\mathfrak{su}(1, 1)) \to L^+(\mathcal{K}_0) \) so that \( \mathcal{E} \to E_0 \) and \( \mathcal{K} \to K_0 \), hence \( \mathcal{K}_0 \) is turned into a \( U_q(\mathfrak{su}(1, 1))\)-module. Define \( \mathcal{U} \) to be the unital \(*\)-subalgebra of \( L^+(\mathcal{K}_0) \) generated by \( K_0, K_0^{-1} \) and \( E_0 \). This is a \(*\)-representation of \( U_q(\mathfrak{su}(1, 1)) \) by unbounded operators in the sense of [13, Ch. 8], so that in particular each element of \( \mathcal{U} \) is closable. The Poincaré-Birkhoff-Witt theorem, see e.g. [15], implies that the \(*\)-representation \( U_q(\mathfrak{su}(1, 1)) \to L^+(\mathcal{K}_0) \) is faithful and \( \mathcal{U} \) is a concrete realization of \( U_q(\mathfrak{su}(1, 1)) \).

An essential role in the representation theory of \( U_q(\mathfrak{su}(1, 1)) \) is played by the Casimir operator \( \Omega_0 = \rho(\Omega) \), i.e.

\[ \Omega_0 = \frac{1}{2} \left( (q - q^{-1})^2 E_0^\dagger E_0 - q K_0^2 - q^{-1} K_0^{-2} \right) = \frac{1}{2} \left( (q - q^{-1})^2 E_0 E_0^\dagger - q^{-1} K_0^2 - q K_0^{-2} \right). \]
By Definition 4.1 and (4.6) we have the explicit expression
\[ 2 \Omega_0 f_{mpt} = -\text{sgn}(pt) \sqrt{(1 + \kappa(p))(1 + \kappa(t))} f_{m,qp,qt} \]
\[ + (q^{-1}p|t| + q^{-1}t|p|) f_{mpt} - \text{sgn}(pt) \sqrt{(1 + \kappa(q^{-1}p))(1 + \kappa(q^{-1}t))} f_{m,q^{-1}p,q^{-1}t} \]
(4.8)
for all \( m \in \mathbb{Z} \) and \( p, t \in I_q \).

Recall that we are using a renormalized version (and terminology) of the operator used in [45]. The renormalization is chosen in such a way that the continuous spectrum of the relevant self-adjoint extension of \( \Omega_0 \) is given by \([-1, 1]\) and the point spectrum of this extension has a maximal degree of symmetry with respect to the origin.

Not \( K_0, E_0 \) and \( \Omega_0 \) are the operators relevant to the dual locally compact quantum group \((\hat{\mathcal{M}}, \hat{\Delta})\) introduced in Section 2, but rather the right closed extensions of these operators. Now \( K_0 \) is essentially self-adjoint, so it is clear what extension of \( K_0 \) to use. At this moment, it is not clear what kind of extension of \( E_0 \) we need, but Proposition 4.4 shows that the closure of \( E_0 \) is the natural extension in this setting. Next the Casimir operator is discussed.

**Definition 4.3.** We define the densely defined, closed, linear operators \( E \) and \( K \) in \( \mathcal{K} \) as the closures of \( E_0 \) and \( K_0 \) respectively.

One expects at least that \( K \) and \( E \) are affiliated to the dual von Neumann algebra \( \hat{\mathcal{M}} \). This is indeed the case.

**Proposition 4.4.** \( K \) is an injective positive self-adjoint operator in \( \mathcal{K} \). The operators \( K \) and \( E \) are affiliated to the von Neumann algebra \( \hat{\mathcal{M}} \).

Note that the spectrum \( \sigma(K) \) consists of \( q^{\mathbb{Z}} \cup \{0\} \). Moreover, \( E^* \) is the closure of \( E_0^\dagger \), and there exists a characterization of \( E \) given in Proposition 8.4.

Next we want to define the Casimir operator on \( \mathcal{K} \) as the right extension of \( \Omega_0 \). Since \( \Omega_0^\dagger = \Omega_0 \), it is natural to look for a self-adjoint extension of \( \Omega_0 \) to be this right extension. But \( \Omega_0 \) is not essentially self-adjoint, which is discussed in Section 8, thus, unlike the cases \( E \) and \( K \), we can not merely use the closure of \( \Omega_0 \).

**Definition 4.5.** We define the Casimir operator \( \Omega \) as the closure of the operator
\[ \frac{1}{2} \left( (q - q^{-1})^2 E^* E - q K^2 - q^{-1} K^{-2} \right). \]

At this point it is not clear that Definition 4.5 makes sense.

**Theorem 4.6.** The Casimir operator \( \Omega \) is a well-defined self-adjoint operator. The Casimir operator commutes strongly with the unbounded operators \( E \) and \( K \). Moreover, the Casimir operator \( \Omega \) is the unique self-adjoint extension of \( \Omega_0 \) that is affiliated to the von Neumann algebra \( \hat{\mathcal{M}} \).

The proofs of these statements are given Section 8. At the same time it will emerge that \( \Omega \) is not the closure of \( \Omega_0 \).

The Casimir element \( \Omega_0 \) belongs to the center of \( \mathcal{U} \), and hence commutes with \( E_0 \) and \( K_0 \) in \( \mathcal{L}^+(\mathcal{K}_0) \). On the Hilbert space level, this result has an analogue to the extent that the Casimir operator \( \Omega \) strongly commutes with \( E \) and \( K \), see Theorem 4.10. However, since \((\hat{\mathcal{M}}, \hat{\Delta})\) is...
a quantization of the normalizer of $SU(1, 1)$ in $SL(2, \mathbb{C})$, and not of $SU(1, 1)$, it is to be expected that the Casimir operator does not commute with all elements of $\hat{M}$. Indeed, the Casimir operator satisfies a graded commutation relation with the elements of $\hat{M}$, i.e. there exists a decomposition $\hat{M} = \hat{M}_+ \oplus \hat{M}_-$ such that the Casimir operator commutes with the elements of $\hat{M}_+$ and anti-commutes with elements of $\hat{M}_-$, see Proposition 4.8.

In order to formulate the graded commutation relation involving the Casimir operator we provide $K$ and $\hat{M}$ with a natural $\mathbb{Z}_2$-grading.

**Definition 4.7.** We define the closed subspaces $\mathcal{K}_+, \mathcal{K}_- \subseteq K$ as

$$\mathcal{K}_\pm = \text{Span}\{ f_{m,p,t} \mid m \in \mathbb{Z}, p, t \in I_q \text{ so that } \text{sgn}(pt) = \pm \},$$

So $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$. We define the $\sigma$-weakly closed subspaces $\hat{\mathcal{M}}_+, \hat{\mathcal{M}}_- \subseteq \hat{M}$ as

$$\hat{\mathcal{M}}_+ = \{ x \in \hat{M} \mid x \mathcal{K}_\pm \subseteq \mathcal{K}_\pm \} \quad \text{and} \quad \hat{\mathcal{M}}_- = \{ x \in \hat{M} \mid x \mathcal{K}_\pm \subseteq \mathcal{K}_\mp \}.$$

Then $\hat{\mathcal{M}}_+$ is a von Neumann algebra, and $\hat{\mathcal{M}}_-$ is a self-adjoint subspace so that $\hat{\mathcal{M}}_- \hat{\mathcal{M}}_+ \subseteq \hat{\mathcal{M}}_- \cap \hat{\mathcal{M}}_+$. In order to get a real $\mathbb{Z}_2$-grading on $\hat{M}$, we need the following result.

**Proposition 4.8.** $\hat{M} = \hat{\mathcal{M}}_+ \oplus \hat{\mathcal{M}}_-$. Let $x \in \hat{\mathcal{M}}_+$ and $y \in \hat{\mathcal{M}}_-$, then $x \Omega \subseteq \Omega x$ and $y \Omega \subseteq -\Omega y$.

Proposition 4.8 implies that $E$ and $K$ do not suffice to generate $\hat{M}$ because of Theorem 4.9. In order to determine $\hat{M}$, Proposition 4.8 also provides the key ingredient once we have determined the spectral decomposition of $\Omega$ explicitly. Indeed, Proposition 4.8 implies that elements of $\hat{M}$ can be described by mapping (generalized) eigenvectors for the eigenvalue $\lambda$ of the Casimir operator to (generalized) eigenvectors for the eigenvalue $\pm \lambda$ of the Casimir operator. For this we have to study the Casimir operator restricted to suitable invariant subspaces on which the spectrum of $\Omega$ has simple spectrum, which is done in Section 8.

We define bounded operators on the Hilbert space $\mathcal{K}$ of the GNS-representation using the multiplicative unitary $W \in B(\mathcal{K} \otimes \mathcal{K})$. Using the normal functionals $\omega_{f,g} \in B(\mathcal{K})^*$ defined by $\omega_{f,g}(x) = \langle x f, g \rangle$, $f, g \in \mathcal{K}$, we define

$$Q(p_1, p_2, n) = (\omega_{f,g} \otimes \text{Id})(W^*): \mathcal{K} \to \mathcal{K}, \quad f = f_{0,p_1,1}, \ g = f_{n,p_2,1}. \quad (4.9)$$

**Proposition 4.9.** The operators $Q(p_1, p_2, n) \in B(\mathcal{K})$, $p_1, p_2 \in I_q$, $n \in \mathbb{Z}$, are in $\hat{M}$ and the linear span is strong-* dense in $\hat{M}$. Moreover, $Q(p_1, p_2, n) \in \hat{M}_{\text{sgn}(p_1 p_2)}$.

Proposition 4.9 is the key to the proof of Proposition 4.8, and describes sufficiently many elements of $\hat{M}$.

Since the operators $Q(p_1, p_2, n)$ span $\hat{M}$ linearly, we calculate the structure constants.

**Proposition 4.10.** For $p_1, p_2, r_1, r_2 \in I_q$, $n, m \in \mathbb{Z}$, we have $Q(p_1, p_2, n) Q(r_1, r_2, m) = 0$ in case $|\frac{p_2}{p_1}| \neq q^n$ or $|\frac{r_2}{r_1}| \neq q^n$. In case $|\frac{p_2}{p_1}| = q^m$ and $|\frac{r_2}{r_1}| = q^n$ we have

$$Q(p_1, p_2, n) Q(r_1, r_2, m) = \sum_{x_1, x_2 \in I_q} a_{x_1}(r_1, p_1) a_{x_2}(r_2, p_2) Q(x_1, x_2, n + m)$$

where the coefficients $a_{x_i}(r_i, p_i)$, $i = 1, 2$, are defined in Definition 4.8.

Since $\hat{M}' = \hat{J} \hat{M} \hat{J}$, see (2.1) for the dual von Neumann algebra, Proposition 4.8 leads to the following.
Corollary 4.11. The operators $\hat{J}Q(p_1,p_2,n)\hat{J} \in B(\mathcal{K})$, $p_1,p_2 \in I_q$, $n \in \mathbb{Z}$, are in $\hat{M}'$ and the linear span is strongly dense in $\hat{M}'$.

The main problem in proving Proposition 4.9 is that the operators $Q(p_1,p_2,n)$ do not preserve the dense subspace $\mathcal{K}_0$. We have the following polar-type decomposition of these operators.

Lemma 4.12. For fixed $p_1,p_2 \in I_q$, $n \in \mathbb{Z}$, there exists an orthogonal projection $P = P(p_1,p_2,n) \in B(\mathcal{K})$, a continuous function $H(\cdot) = H(\cdot;p_1,p_2,n)$ and a partial isometry $U = U_n^{sgn(p_1),sgn(p_2)}$ so that

$$Q(p_1,p_2,n) = UH(\Omega)P.$$ 

Since the elements $H(\Omega)$ and $P$, as elements of the spectral decomposition of $K$, are in the von Neumann algebra generated by $E$ and $K$, we only need to incorporate the partial isometries. Now we can state the main theorem of this section, which gives an explicit description of the von Neumann algebra for the dual locally compact quantum group.

Theorem 4.13. The von Neumann algebra $\hat{M}$ is generated by $K$, $E$, $U^+_0$, $U^-_0$.

It is interesting to connect the comultiplication of the dual quantum group as in Theorem 2.3 with the comultiplication (4.3) of the quantized universal enveloping algebra.

Proposition 4.14. We have $\hat{\Delta}(K) = K \otimes K$, and

$$K_0 \otimes E_0 + E_0 \otimes K_0^{-1} \subset \hat{\Delta}(E) \quad \text{and} \quad K_0 \otimes E_0^\dagger + E_0^\dagger \otimes K_0^{-1} \subset \hat{\Delta}(E^*).$$

In Proposition 4.14 the left hand side denotes the algebraic tensor product of the unbounded operators which are defined on the domain $\mathcal{K}_0 \otimes \mathcal{K}_0 \subset \mathcal{K} \otimes \mathcal{K}$. So we see that the comultiplication of the dual quantum group corresponds to the comultiplication of the quantized universal enveloping algebra, see (3.2). Note that for an element $x$ affiliated to $\hat{M}$ we can calculate $\hat{\Delta}(x)$ as an affiliated element of $\hat{M} \otimes \hat{M}$.

We can also calculate the comultiplication on the elements $Q(p_1,p_2,n)$ spanning $\hat{M}$, see Proposition 4.9, using the pentagonal equation.

Proposition 4.15. For $p_1,p_2 \in I_q$, $n \in \mathbb{Z}$, we have

$$\hat{\Delta}(Q(p_1,p_2,n)) = \sum_{m \in \mathbb{Z},p \in I_q} Q(p,p_2,n-m) \otimes Q(p_1,p,m),$$

where the sum converges in the $\sigma$-weak-topology of $\hat{M} \otimes \hat{M}$.

The action of the unitary antipode $\hat{R}$ and of the $*$-operator on the generators $Q(p_1,p_2,n)$ of $\hat{M}$ is given in Corollary 7.4.

5. The decomposition of the left regular corepresentation

As remarked in Section 2 the multiplicative unitary acting in the GNS-representation of the left invariant weight is the analogue of the left regular representation. For the Lie group $SU(1,1) \cong SL(2,\mathbb{R})$, the decomposition into irreducible representations involves the principal unitary series and the discrete series, see e.g. [13], [21], [31], [12], [50]. The decomposition is
obtained by considering the action of the Casimir operator, since its eigenspaces give invariant subspaces as the Casimir operator is a central element. Our next goal is to decompose the left regular corepresentation given by the multiplicative unitary $W$ acting in the GNS-representation $K$ into irreducible corepresentations. We want to proceed in a similar fashion, but as follows from Proposition 4.8 we need to combine two eigenspaces of the Casimir operator. We first consider the discrete part, and next the continuous part.

In Section 5 we decompose the GNS-space $K$ into irreducible representations for $U_q(\mathfrak{su}(1,1))$ by decomposing the action of the Casimir operator, and since its generators are related to affiliated operators to $\hat{M}$ we expect that this is a building block in the decomposition. In this section we describe the decomposition explicitly, and for each corepresentation in the decomposition of the left regular corepresentation we indicate its decomposition as $U_q(\mathfrak{su}(1,1))$-representation using its representations as described in Section 8.4.

In order to find the decomposition of the left regular corepresentation we have to look for invariant subspaces of $(\omega \otimes \text{Id})(W)$, $\omega \in M_\ast$, which are the generators of $\hat{M}$. By Proposition 4.8 we can restrict to eigenspaces for the Casimir operator for the eigenvalues $\lambda$ and $-\lambda$. By considering combinations of such eigenspaces in suitable invariant subspaces for the Casimir operator we can determine invariant subspaces, hence irreducible corepresentations occurring in the decomposition of the left regular corepresentation. In this approach we have to distinguish between eigenvalues $\lambda$ of the Casimir operator $\Omega$ satisfying $|\lambda| > 1$, leading to the analogue of discrete series representations of $SU(1,1)$, and those satisfying $|\lambda| \leq 1$, leading to the analogue of principal unitary series representations of $SU(1,1)$. The case $\lambda = 0$ has to be considered separately.

In Section 5.1 we discuss the analogue of the discrete series representations, and in Section 5.2 we discuss the analogue of the principal unitary series representations. For the precise description of the results we need to use some notation that is used in the proofs.

5.1. Unitary corepresentations: discrete series. In order to be able to describe the results we need to consider the discrete spectrum of the Casimir operator. The complete spectrum of the Casimir operator $\Omega$ is described in Section 8, where for suitable $\Omega$-invariant subspaces $K(p,m,\varepsilon,\eta) \subset K$ the spectral decomposition of $\Omega|_{K(p,m,\varepsilon,\eta)}$ is discussed in detail. The spectrum is simple and consists of a continuous part $[-1,1]$ and a discrete part depending on $K(p,m,\varepsilon,\eta)$ for $p \in \mathbb{Q}$, $m \in \mathbb{Z}$, $\varepsilon,\eta \in \{\pm 1\}$. We refer to (7.1) for the definition of these subspaces. Throughout this subsection we fix $p \in \mathbb{Q}$, $\lambda \in -\mathbb{N} \cup \mathbb{N}$ and set $\mu(\lambda) = \frac{1}{2}(\lambda + \lambda^{-1})$. Thus, $x$ is an isolated point of the spectrum of the Casimir operator $\Omega$ if $x \in \sigma_d(\Omega)$, see Section 8.3. We denote $e_{m,\eta}(p,x) \in D(\Omega) \cap K(p,m,\varepsilon,\eta)$ to be the eigenvector of the Casimir operator $\Omega$ of the eigenvalue $\varepsilon\eta x$ in the subspace $K(p,m,\varepsilon,\eta)$ of the GNS-space. We note that $e_{m,\eta}(p,x) \neq 0$ if and only if $\Omega$ has an eigenvector with eigenvalue $\varepsilon\eta x$ inside $K(p,m,\varepsilon,\eta)$. By the results proved in Section 8.3 the eigenspace of $\Omega$ restricted to $K(p,m,\varepsilon,\eta)$ is at most one-dimensional, so that $e_{m,\eta}(p,x)$ is defined up to phase-factor after putting $\|e_{m,\eta}(p,x)\| = 1$. The precise choice is given in Section 10.1.

Recall we have to find closed invariant subspaces for the action of $\hat{M}$, and we can define closed invariant subspaces in terms of the eigenvectors of the Casimir operator $\Omega$. This is straightforward once we have described the actions of the generators of $\hat{M}$ on the eigenvectors of $\Omega$ in Lemma 10.1.
Lemma 5.1. We define the closed subspace $\mathcal{L}_{p,x}$ of $\mathcal{K}$ as

$$\mathcal{L}_{p,x} = \text{Span}\{ e_{m}^{\varepsilon,\eta}(p, x) \mid m \in \mathbb{Z}, \varepsilon, \eta \in \{ - , + \} \}.$$  

The space $\mathcal{L}_{p,x}$ is an invariant subspace of the corepresentation $W$ of $(M, \Delta)$. If $\mathcal{L}_{p,x} \neq \{ 0 \}$ we say that that $(p, x)$ determines a discrete series corepresentation of $(M, \Delta)$. The element $W_{p,x} = W|_{\mathcal{K} \otimes \mathcal{L}_{p,x}}$ is a unitary corepresentation of $(M, \Delta)$ on $\mathcal{L}_{p,x}$.

Using the explicit actions of the generators of $\hat{M}$ as described in Theorem 4.13 on the eigenvectors of the Casimir operator we can classify the values of $(p, x)$ such that $\mathcal{L}_{p,x}$ is a discrete series corepresentation of $(M, \Delta)$. The result is the following.

Proposition 5.2. Consider $p \in \mathbb{Q}^{\ast}$ and $x = \mu(\lambda)$ where $\lambda \in -q^{2l+1}p \cup q^{2l+1}p$ and $|\lambda| > 1$. Let $j, l \in \mathbb{Z}$ be such that $|\lambda| = q^{l-2}p^{-1} = q^{l+2}p$, so $l < j$. Then $(p, x)$ determines a discrete series corepresentation of $(M, \Delta)$ in the following 3 cases, and these are the only cases:

(i) If $x > 0$, in which case

$$\{ e_{m}^{+ +}(p, x) \mid m \in \mathbb{Z} \} \cup \{ e_{m}^{+ -}(p, x) \mid m \in \mathbb{Z}, m \leq l \} \cup \{ e_{m}^{- +}(p, x) \mid m \in \mathbb{Z}, m \geq j \}$$

is an orthonormal basis for $\mathcal{L}_{p,x}$.

(ii) If $x < 0$, $l \geq 0$ and $j > 0$, in which case

$$\{ e_{m}^{+ -}(p, x) \mid m \in \mathbb{Z} \} \cup \{ e_{m}^{+ +}(p, x) \mid m \in \mathbb{Z}, m \leq l \} \cup \{ e_{m}^{- -}(p, x) \mid m \in \mathbb{Z}, m \geq j \}$$

is an orthonormal basis for $\mathcal{L}_{p,x}$.

(iii) If $x < 0$, $l < 0$ and $j \leq 0$, in which case

$$\{ e_{m}^{+ -}(p, x) \mid m \in \mathbb{Z} \} \cup \{ e_{m}^{+ +}(p, x) \mid m \in \mathbb{Z}, m \leq l \} \cup \{ e_{m}^{- +}(p, x) \mid m \in \mathbb{Z}, m \geq j \}$$

is an orthonormal basis for $\mathcal{L}_{p,x}$.

Proposition 5.2 gives a complete list of discrete corepresentations occurring in the left regular corepresentation. In each of the cases listed in Proposition 5.2 we can consider the representation of $\hat{M}$ as a representation of $U_{q}(\mathfrak{su}(1, 1))$ (by unbounded operators in the sense of [18]), and then, by comparing the action of $E$ and $K$ as given in Lemma 10.1, with the listing in Section 8.4, we see that $\mathcal{L}_{p,x}$ in case (i), (ii) and (iii) of Proposition 5.2 corresponds to

$$\pi_{1}^{S} = D_{\frac{1}{2}(\chi(p)-1)+\epsilon(p)}^{+} \oplus D_{-\frac{1}{2}(\chi(p)-\epsilon(p)-l)}^{-} \oplus D_{\frac{1}{2}(\chi(p)+j)}^{+}$$

(5.1)

as $U_{q}(\mathfrak{su}(1, 1))$-module, where the decomposition corresponds to the order of the orthonormal basis. The notation for the $U_{q}(\mathfrak{su}(1, 1))$-modules is as in Section 8.4. Here $\chi(p) \in \mathbb{Z}$ is defined in Definition 5.1 and $\epsilon(p) = \frac{1}{2} \chi(p) \mod 1$, so $\epsilon(p) = 0$ for $p \in q^{2l}$ and $\epsilon(p) = \frac{1}{2}$ for $p \in q^{2l+1}$, see (8.25). So we see that a discrete series corepresentation in the left regular corepresentation decomposes in the same way as sum of three $U_{q}(\mathfrak{su}(1, 1))$-representations involving a strange series representation in combination with a positive and negative discrete series representation.

Proposition 5.3. Assume that $(p, x)$ determines a discrete series corepresentation of $(M, \Delta)$. Then $W_{p,x}$ is an irreducible corepresentation of $(M, \Delta)$.
5.2. **Unitary corepresentations: principal series.** Next we discuss the irreducible corepresentations of \((M, \Delta)\) in the left regular corepresentation \(W\) corresponding to the continuous spectrum of the Casimir operator \(\Omega\). We cannot obtain these representations by restriction to closed subspaces, so we have to use another approach.

Motivated by Lemma \([10.1]\) and the admissible irreducible representations of \(U_q(\mathfrak{su}(1,1))\) as discussed in Section \([5.1]\) we define for \(x = \cos \theta \in [-1, 1] \) and \(p \in \mathbb{Z}\) a Hilbert space \(\mathcal{L}_{p,x}\) by

\[
\mathcal{L}_{p,x} = \bigoplus_{\varepsilon, \eta \in \{-, +\}} \ell^2_{\varepsilon, \eta}(p, x),
\]

where each space \(\ell^2_{\varepsilon, \eta}(p, x)\) denotes a copy of \(\ell^2(\mathbb{Z})\) with standard orthonormal basis \(\{e^\varepsilon_m(p, x) \mid m \in \mathbb{Z}\}\). We define operators \(K, E, U_0^+, U_0^-\) on \(\mathcal{L}_{p,x}\) by

\[
K e^\varepsilon_m(p, x) = p^{\frac{1}{2}} q^m e^\varepsilon_m(p, x),
\]

\[
(q^{-1} - q)E e^\varepsilon_m(p, x) = q^{-m-\frac{1}{2}} p^{-\frac{1}{2}} \left(1 + \varepsilon \eta p q^{2m+1}\right) e^\varepsilon_m(p, x),
\]

\[
U_0^+ e^\varepsilon_m(p, x) = \eta (-1)^{v(p)} e^{\varepsilon_2, -\varepsilon_1}(p, x),
\]

\[
U_0^- e^\varepsilon_m(p, x) = \varepsilon \eta (p) (-1)^m e^{\varepsilon_1, -\varepsilon_2}(p, x).
\]

Here \(\chi(p) = \log_q(p)\) is defined in Definition \([11.1]\) and \(v(p)\) is defined in \([10.1]\). Explicitly, for \(p = q^{2k}\) or \(p = q^{2k-1}\) with \(k \in \mathbb{Z}\) we have \(v(p) = k\). The operators \(E\) and \(K\) are unbounded closable operators with dense core the finite linear combinations of the orthonormal basis vectors \(e^\varepsilon_m(p, x), m \in \mathbb{Z}, \varepsilon, \eta \in \{-, +\}\). The operators \(U_0^+\) and \(U_0^-\) are bounded; they are isometries.

**Proposition 5.4.** The operators \(E, K, U_0^+, U_0^-\) defined by \((5.2)\) generate a von Neumann algebra \(\hat{M}_{p,x}\) that is isomorphic to \(\hat{M}\). Consequently, \((5.2)\) determines a unitary corepresentation \(W_{p,x}\) of \((M, \Delta)\). The corepresentation \(W_{p,x}\) is reducible, and its decomposition into irreducible corepresentations is given by

\[
W_{p,x} = W_{p,x}^1 \oplus W_{p,x}^2, \quad \text{in case} \ x \neq 0, \ \text{or} \ p \in q^{2\mathbb{Z}+1},
\]

\[
W_{p,0} = W_{p,0}^{1,1} \oplus W_{p,0}^{1,2} \oplus W_{p,0}^{2,1} \oplus W_{p,0}^{2,2}, \quad \text{in case} \ p \in q^{2\mathbb{Z}}.
\]

**Remark 5.5.** Denoting the corresponding invariant subspaces by \(\mathcal{L}_{p,x}^j\) and \(\mathcal{L}_{p,0}^{j,k}\) of Proposition \(5.4\), which are described explicitly in Section \([10.2]\), we can consider these irreducible constituents of Proposition \(5.4\) as representations of \(U_q(\mathfrak{su}(1,1))\). If we consider the irreducible representations \(W_{p,x}^j, j = 1, 2, \) of \(\hat{M}\) as representations of \(U_q(\mathfrak{su}(1,1))\), they decompose into irreducible principal series \(U_q(\mathfrak{su}(1,1))\)-representations as \(\pi_{b(-x), \varepsilon(p)} \oplus \pi_{b(x), \varepsilon(p)}\), where \(b(x)\) is determined by \(\mu(q^{2ib(x)}) = x \) and \(\varepsilon(p) = \frac{1}{2} \chi(p) \mod 1\), as in the decomposition of the discrete series subcorepresentation of \(W\) into \(U_q(\mathfrak{su}(1,1))\)-modules in Section \(5.1\) and the representations of \(U_q(\mathfrak{su}(1,1))\) are described in Section \(8.4\). Similarly, it follows that for \(x = 0\) and \(\varepsilon(p) = 0\), the irreducible representations \(W_{p,0}^{j,k}, j; k = 1, 2\), of \(\hat{M}\) can be considered as irreducible principal series representations \(\pi_{b(0), 0}\) of \(U_q(\mathfrak{su}(1,1))\), where \(b(0) = -\frac{x}{4 \ln q}\). This follows directly from the explicit description of the spaces \(\mathcal{L}_{p,x}^j\) and \(\mathcal{L}_{p,0}^{j,k}\) in Section \([10.2]\) and \((5.2)\) compared to the listing of irreducible representations of \(U_q(\mathfrak{su}(1,1))\) in Section \(8.4\).
In Section 8 we discuss for suitable $\Omega$-invariant subspaces $K(p, m, \varepsilon, \eta) \subset K$ the spectral decomposition of $\Omega|_{K(p, m, \varepsilon, \eta)}$, and we denote by $K_c(p, m, \varepsilon, \eta) \subset K(p, m, \varepsilon, \eta) \subset K$ the subspace corresponding to the continuous spectrum $[-1, 1]$ of $\Omega|_{K(p, m, \varepsilon, \eta)}$.

**Proposition 5.6.** For $p \in q\mathbb{Z}$ let $K_c(p) \subset K$ be the subspace defined by

$$K_c(p) = \bigoplus_{\varepsilon, \eta \in \{-, +\}} K_c(p, m, \varepsilon, \eta),$$

then

$$W|_{K\otimes K_c(p)} \cong \int_{-1}^{1} W_{p,x} \, dx.$$ 

For direct integrals of (co)representations we refer to [48, Ch.8].

5.3. **Decomposition of the left regular corepresentation.** Since $K = K_c \oplus K_d$ with $K_d$, respectively $K_c$, the subspace corresponding to the discrete, respectively continuous, spectrum of the Casimir operator, we find by combining Propositions 5.3 and 5.6 the following decomposition of the left regular corepresentation $W$ of $(M, \Delta)$.

**Theorem 5.7.**

$$W \cong \bigoplus_{p \in q\mathbb{Z}} \left( \int_{-1}^{1} W_{p,x} \, dx \oplus \bigoplus_{x \in \sigma_d(\Omega_p)} W_{p,x} \right),$$

where $\Omega_p = \bigoplus_{\varepsilon, \eta \in \{-, +\}} \Omega|_{K(p, m, \varepsilon, \eta)}$ and $\sigma_d$ denotes the discrete spectrum.

It is well-known that in the decomposition of the left regular representation of $SU(1,1) \cong SL(2,\mathbb{R})$ the discrete series representation and the principal unitary series occur, and in this sense Theorem 5.7 is the appropriate analogue of this result. In case of the group $SU(1,1) \cong SL(2,\mathbb{R})$ we also have complementary series representations, which do not occur in the decomposition of the left regular representation, but which can be obtained by continuation from the principal unitary series representation. For the quantum group analogue of the normalizer of $SU(1,1)$ in $SL(2,\mathbb{C})$ we have a similar result. So we can obtain unitary complementary series corepresentations of $(M, \Delta)$, and the approach is sketched in Section 10.3.

6. **Results for special functions of basic hypergeometric type**

This section is separately readable from the remainder of the paper. This section is meant to give a couple of examples of rather complicated identities for special functions of basic hypergeometric type $1\varphi_1$ and type $2\varphi_1$, see [17]. We assume that the reader of this section is familiar with the notation for basic hypergeometric series [17], but the definition is recalled in Appendix B. In the first subsection we introduce the notation for special functions, and we recall some elementary properties. The first subsection introduces notation and special functions that are used throughout the paper, whereas the following subsections give explicit highly non-trivial results for these special functions. These identities follow from the quantum group theoretic interpretation.
6.1. Definition of some special functions. The set of natural numbers (without 0) will be denoted by \( \mathbb{N} \) and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). We write, as in Section 3, \( I_q = -q^\mathbb{N} \cup q^\mathbb{N} \). We use the following functions frequently.

Definition 6.1. (i) \( \chi : -q^\mathbb{Z} \cup q^\mathbb{Z} \rightarrow \mathbb{Z} \) such that \( \chi(x) = \log_q(|x|) \) for all \( x \in -q^\mathbb{Z} \cup q^\mathbb{Z} \); (ii) \( \kappa : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \kappa(x) = \text{sgn}(x) x^2 \) for all \( x \in \mathbb{R} \); (iii) \( \nu : -q^\mathbb{Z} \cup q^\mathbb{Z} \rightarrow \mathbb{R}^+ \) such that \( \nu(t) = q^{\frac{1}{2}((\chi(t)-1)(\chi(t)-2)} \) for all \( t \in -q^\mathbb{Z} \cup q^\mathbb{Z} \); (iv) \( s : \mathbb{R}_0 \times \mathbb{R}_0 \rightarrow \{-1, 1\} \) is defined such that

\[
s(x, y) = \begin{cases} 
-1 & \text{if } x > 0 \text{ and } y < 0 \\
1 & \text{if } x < 0 \text{ or } y > 0
\end{cases}
\]

for all \( x, y \in \mathbb{R}_0 = \mathbb{R} \setminus \{0\} \). (v) \( \mu : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\} \) such that \( \mu(y) = \frac{1}{2}(y + y^{-1}) \) for all \( y \in \mathbb{C} \setminus \{0\} \).

For \( a, b, z \in \mathbb{C} \), we define

\[
\Psi \left( \begin{array}{c} a \\ b \\ \end{array} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q^n; q)_\infty}{(q; q)_n} (-1)^n q^{\frac{1}{2}n(n-1)} z^n = (b; q)_\infty \varphi_1 \left( \begin{array}{c} a \\ b, q, z \end{array} \right). \tag{6.1}
\]

This is an entire function in \( a, b \) and \( z \). Here we have used the standard notation for basic hypergeometric series [17], or see Appendix B.3.

We use the normalization constant \( c_q = (\sqrt{2} q^{q^2 - q^{-2}})_\infty^{-1} \). Then the following definition is [44, Def. 3.1], and the notations as in Definition 6.1 are used.

Definition 6.2. If \( p \in I_q \), we define the function \( a_p : I_q \times I_q \rightarrow \mathbb{R} \) such that \( a_p \) is supported on the set \( \{ (x, y) \in I_q \times I_q \mid \text{sgn}(xy) = \text{sgn}(p) \} \) and is given by

\[
a_p(x, y) = c_q s(x, y) (-1)^{x(p)} (-\text{sgn}(y))^{x(y)} |y| \nu(py/x) \sqrt{\frac{(-\kappa(p), -\kappa(y); q^2)_\infty}{(-\kappa(x); q^2)_\infty}} \times \Psi \left( \begin{array}{c} -q^2 \kappa(y) \\ q^2 \kappa(x/y) \end{array} ; q^2, q^2 \kappa(x/p) \right)
\]

for all \( (x, y) \in I_q \times I_q \) satisfying \( \text{sgn}(xy) = \text{sgn}(p) \).

The functions \( a_p(x, y) \) for \( p, x, y \in I_q \) have been introduced in [40, §3], motivated by their occurrence as Clebsch-Gordan coefficients. Depending on the choices of the sign, these functions can be identified with well-known special functions of basic-hypergeometric type. In particular, for \( \text{sgn}(x) = \text{sgn}(y) \) the functions \( a_p(x, y) \) can be identified with the \( q \)-Laguerre polynomials in case \( \text{sgn}(x) = \text{sgn}(y) = -1 \) and with the associated big \( q \)-Bessel functions in case \( \text{sgn}(x) = \text{sgn}(y) = +1 \), see [40]. The \( q \)-Laguerre polynomials correspond to an indeterminate moment problem, and the big \( q \)-Bessel functions form a complementary orthogonal basis to the orthogonal polynomials for an explicit solution to the moment problem corresponding to Ramanujan’s \( _1 \psi_1 \)-summation formula, see [10] for details. For \( \text{sgn}(x) = -\text{sgn}(y) \), the functions \( a_p(x, y) \) can be matched with Al-Salam–Carlitz polynomials and \( q \)-Charlier polynomials, see [27] for their definition.

For completeness we recall the orthogonality properties of these functions, see [40, Prop. 3.2, 3.3]. For \( \theta \in -q^\mathbb{Z} \cup q^\mathbb{Z} \) we define \( \ell_\theta = \{ (x, y) \in I_q \times I_q \mid y = \theta x \} \).
Proposition 6.3. Consider $\theta \in -q^Z \cup q^Z$. Then the family $\{ a_p | \ell_\theta \mid p \in I_\theta \}$ such that $\text{sgn}(p) = \text{sgn}(\theta)$ is an orthonormal basis for $l^2(I_\theta)$. In particular,
\[
\sum_{x \in I_\theta \text{ so that } \theta x \in I_\theta} a_p(x, \theta x) a_r(x, \theta x) = \delta_{p,r}, \quad p, r \in I_\theta.
\]

Proposition 6.4. Consider $\theta \in -q^Z \cup q^Z$ and define $J = q^Z \subset I_\theta$ if $\theta > 0$ and $J = -q^N \subset I_\theta$ if $\theta < 0$. For every $(x, y) \in \ell_\theta$ we define the function $e_{(x,y)} : J \to \mathbb{R}$ such that $e_{(x,y)}(p) = a_p(x, y)$ for all $p \in J$. Then the family $\{ e_{(x,y)} \mid (x, y) \in \ell_\theta \}$ forms an orthonormal basis for $l^2(J)$. In particular,
\[
\sum_{p \in J} a_p(x, \theta x) a_p(y, \theta y) = \delta_{x,y}, \quad x, y \in I_\theta.
\]

For convenience we state the following symmetry relations for the functions $a_p(x, y)$, see [30, Prop. 3.5]:
\[
a_p(x, y) = (-1)^{\chi(x,y)\text{sgn}(x)\chi(x)} \left| \frac{y}{p} \right| a_y(x, p);
\]
\[
a_p(x, y) = \text{sgn}(p)^{\chi(x)\text{sgn}(y)\chi(y)} a_p(y, x);
\]
\[
a_p(x, y) = (-1)^{\chi(x,y)\text{sgn}(y)} x \left| \frac{y}{p} \right| a_x(p, y).
\]

6.2. Summation and transformation formulas for $a_p(x, y)$. The functions $a_p(x, y)$, which as noted above are closely related to some well-known orthogonal polynomials of basic hypergeometric type, are used in the definition of the so-called multiplicative unitary $W$, see (7.10). In the general theory of locally compact groups, the multiplicative unitary $W$ plays an important role. In particular, it satisfies the pentagonal equation, a relation that is essential in proving Propositions 4.10 and 1.13. The result in these propositions lead to operator identities in suitable Hilbert spaces, and taking matrix coefficients then essentially lead to Theorems 6.3 and 6.8 in this section. The details of the proofs are given in Section 11.2.

6.2.1. Representing the structure of $\hat{M}$. By taking the non-trivial structure constants of Proposition 4.10 and considering matrix coefficients at both sides we obtain the following theorem.

Theorem 6.5. For $p_1, p_2, r_1, r_2 \in I_q$, $l, n, m \in \mathbb{Z}$, $\varepsilon, \eta \in \{ \pm \}$ and with $z \in I_q$ so that $\text{sgn}(z) = \varepsilon$ and $\varepsilon \eta pq^l z \in I_q$ and with $w \in I_q$ so that $\text{sgn}(w) = \varepsilon \text{sgn}(r_1p_1)$ and $\varepsilon \eta \text{sgn}(r_1p_1r_2p_2) pq^{l+m+n} w \in I_q$ we have
\[
\sum_{x \in I_q \text{ so that } \text{sgn}(x) = \text{sgn}(r_1p_1) \text{ and } |x|\text{sgn}(r_2p_2) pq^{2l+m+n} \in I_q} a_z(x, w) a_x(r_1, p_1) a_{|x|\text{sgn}(r_2p_2) pq^{2l+m+n}}(r_2, p_2) \times a_{\varepsilon \eta pq^l z}(|x|\text{sgn}(r_2p_2) pq^{2l+m+n}, \text{sgn}(r_1p_1r_2p_2) \varepsilon \eta pq^{l+m+n} w) = \delta_{l, r_1}^{|p_1| \varepsilon p_2, -2l-m, \delta_{m, p_2}^{|p_1| \varepsilon p_2, -2l-2m-n} \times \sum_{u \in I_q \text{ so that } \text{sgn}(u) = \text{sgn}(r_1) \varepsilon \text{ and } \varepsilon \eta \text{sgn}(r_1r_2) pq^{l+m+n} w \in I_q}} a_z(r_1, u) a_u(p_1, w) a_{\varepsilon \eta pq^l z}(r_2, \varepsilon \eta \text{sgn}(r_1r_2) pq^{l+m+n} u) \times a_{\varepsilon \eta \text{sgn}(r_1r_2) pq^{l+m+n} u}(p_2, \text{sgn}(r_1p_1r_2p_2) \varepsilon \eta pq^{l+m+n} w),
\]
where the series on both sides converge absolutely.

Remark 6.6. (i) The formula of Theorem 6.5 contains many special cases involving $q$-Laguerre polynomials, big $q$-Bessel functions, Al-Salam–Carlitz polynomials and $q$-Charlier polynomials as special cases by suitable specializing the signs in the formula. Note moreover that in all cases the sums are essentially sums over $q^Z$ or $q^N$. For each particular choice of the signs the square roots occurring in Definition 5.2 in Theorem 5.5 will cancel or can be taken together. It would be of interest to find a direct analytic proof.

(ii) As stated before, the functions $a_p(x, y)$ can be interpreted as Clebsch-Gordan coefficients related to representations of the quantized function algebra, which has no classical counterpart. For the case of the quantum $SU(2)$ group the corresponding Clebsch-Gordan coefficients are Wall polynomials, which are special cases of little $q$-Jacobi polynomials and also can be interpreted as $q$-analogues of Laguerre polynomials, see [35]. The classical Clebsch-Gordan coefficients also satisfy summation formulas involving the product of four Clebsch-Gordan coefficients, see e.g. [55, Ch. 8.7], but the structure of the summations is quite different. Relations as in Theorems 6.5 and 6.8, if proved directly, might give a hint of proving directly that the corresponding $q$-analogues of the Racah coefficients are zero at the appropriate places, leading to a direct proof of the coassociativity for $M$, see the discussion [30, p. 289].

Theorem 6.5 can be used to obtain positivity results for sums where the summands have four of the functions $a_p(x, y)$. The result is contained in Corollary 6.7. We give the case corresponding to the $q$-Laguerre polynomials explicitly, and we refer to Askey [2, Lecture 5] for more information on the related positivity results for the Laguerre polynomials. The $q$-Laguerre polynomials are defined by

$$L_n^{(\alpha)}(x; q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \varphi_1 \left( \frac{q^{-n}}{q^{\alpha+1}}, q, -q^{1+\alpha}x \right), \quad (6.3)$$

in this application we only consider the case $\alpha = 0$.

Corollary 6.7. For $r_1, r_2 \in I_q$, $l, m \in \mathbb{Z}$ and with $z \in I_q$ so that $\text{sgn}(z) = \varepsilon$ and $\varepsilon \eta |z_1^2| |q^{-m-l}z| \in I_q$ and we have

$$(-\eta)^{I+m}(\eta \text{sgn}(r_1))(\eta \text{sgn}(r_2))(\varepsilon \eta)^{\chi(z)} \sum_{x \in q^Z} x^2 a_x(r_1, r_1) a_x(z, z) a_{xq^{-m-l}r_1^2}(r_2, r_2) a_{xq^{-m-l}r_1^2}(\varepsilon \eta) \frac{r_2}{r_1} |q^{-m-l}z, \varepsilon \eta| \frac{r_2}{r_1} |q^{-m-l}z| > 0$$

and for $a \in \mathbb{Z}$ and $n_1, n_2, n_3, n_4 \in \mathbb{N}_0$ we have

$$\sum_{k \in \mathbb{Z}} \frac{q^k}{(-q^k, -q^{k+a}; q)_\infty} L_{n_1}^{(0)}(q^k; q) L_{n_2}^{(0)}(q^k; q) L_{n_3}^{(0)}(q^{k+a}; q) L_{n_4}^{(0)}(q^{k+a}; q) > 0.$$

Note that the sum is closely related to one of the orthogonality measures for the $q$-Laguerre polynomials, which correspond to an indeterminate moment problem. A similar positivity result can be obtained for the $q$-Bessel functions involved.
6.2.2. Representing the comultiplication in $\hat{M}$. The explicit expression for $\hat{\Delta}$ in the dual quantum group $\hat{M}$ as given in Proposition 4.15 or better the expression (7.23) in the proof of Proposition 4.15 leads to a formula for its matrix elements. The result is the following theorem.

Theorem 6.8. For fixed $r \in q^\mathbb{Z}$, $m_1, m_2, M, n \in \mathbb{Z}$, $p_1, p_2 \in I_q$, $\varepsilon_1, \varepsilon_2, \eta_1, \eta_2, \sigma \in \{\pm\}$ and for $z_1, z_2, w_1, w_2 \in I_q$ satisfying

$$\text{sgn}(z_i) = \varepsilon_i, \ (i = 1, 2), \ \varepsilon_1\eta_1 q^{m_1} r z_1 \in I_q, \ \varepsilon_2\eta_2 q^{r - 2m_1 - m_2 - n} \frac{z_2 p_2}{r |p|} \in I_q,$$

$$\text{sgn}(w_1) = \text{sgn}(p_1)\varepsilon_1, \ \text{sgn}(w_2) = \sigma\varepsilon_2, \ \sigma\text{sgn}(p_1)\varepsilon_1\eta_1 q^{m_1 + M} r_1 w_1 \in I_q,$$

$$\sigma\text{sgn}(p_2)\varepsilon_2\eta_2 q^{r - 2m_1 - m_2 - M} \frac{w_2 p_2}{r |p|} \in I_q$$

and such that $a_{z_1}(p_1, w_1) \neq 0$ we have

$$\frac{1}{w_2^2} a_{ep_1\eta_1 q^{m_1} r z_1} (\sigma |p_1| q^{r - 2m_1 + M}, \varepsilon_1\eta_1 \sigma \text{sgn}(p_1) w_1 r q^{m_1 + M}) a_{z_2} (\sigma |p_1| q^{2m_1 + M}, w_2)$$

$$\times a_{\varepsilon_2\eta_2 \frac{p_2}{|p_1|} q^{-2m_1 - m_2 - n} (p_2, \varepsilon_2\eta_2 \frac{p_2}{|p_1|} q^{r - 2m_1 - m_2 - M})}$$

$$\sum_{y, x \in I_q \text{ so that } \text{sgn}(y) = \varepsilon_2\eta_1 \text{ and } \text{sgn}(p_1 p_2) q^{n x w_1} / z_1 \in I_q, \ \varepsilon_1\varepsilon_2\eta_1\eta_2 q^{-m_1 - m_2} y / r z_1 \in I_q}$$

$$\times a_{\varepsilon_2\eta_2 \frac{2 p_2}{|p_1|} q^{-2m_1 - m_2 - n} (x, \varepsilon_1\varepsilon_2\eta_1\eta_2 q^{-m_1 - m_2} y / r z_1) a_{z_2} (p_2, \text{sgn}(p_1 p_2) q^{n x w_1} / z_1)}$$

$$\times a_{\sigma\text{sgn}(p_2)\varepsilon_2\eta_2 \frac{w_2 p_2}{|p_1|} q^{-2m_1 - m_2 - M} (\text{sgn}(p_1 p_2) q^{n x w_1} / z_1, \varepsilon_1\varepsilon_2\eta_1\eta_2 q^{-m_1 - m_2} y / r z_1)}$$

where the left-hand-side is considered to be zero in case $\sigma |p_1| q^{2m_1 + M} \notin I_q$. The series converges absolutely.

Remark 6.9. (i) First note that the largest part of Remark 6.6(i) is also applicable to Theorem 6.8, except for the fact that the summation is more involved. Viewing the summation as a sum over an area in $I_q \times I_q \subset \mathbb{R}^2$ (with $x$ on the horizontal axis and $y$ on the vertical axis), we see that the summation area is a subset of $I_q \times I_q$ bounded by a vertical line and a hyperbola. Depending on the sign choices there are eight possibilities for the location of the vertical line and the hyperbola.

(ii) Theorem 6.8 follows from the operator identity in Proposition 4.15, but the single term in the left hand side of Theorem 6.8 corresponds to summation on the left hand side of Proposition 4.15, whereas the double sum on the right hand side of Theorem 6.8 corresponds to the single term on the right hand side of Proposition 4.15.

(iii) Since the results in Theorems 6.3 and 6.8 both reflect the pentagonal equation for the multiplicative unitary, one might expect the resulting identities to be equivalent by using the orthogonality relations of Propositions 6.3 and 6.4. However, this is not the case as follows by considering the dependence of both results on the free parameters.
6.3. Formulas involving $2\varphi_1$-series. In Section 3 we show that with respect to the spectral decomposition of the Casimir operator $\Omega$ the operators $Q(p_1, p_2, n)$ generating $\hat{M}$, see Proposition 4.9, act by multiplication by a $2\varphi_1$-series up to a sign change in the argument. Since we also have another explicit expression for the action of $Q(p_1, p_2, n)$ by Lemma 7.1, we have two different explicit expressions for the action of $Q(p_1, p_2, n)$. This leads to the following theorem, where the functions $\Psi$ are essentially $1\varphi_1$-functions as defined in (6.1). Actually, we have written out two of several options depending on several sign choices.

**Theorem 6.10.** Let $m, n \in \mathbb{Z}$, $p_1, p_2 \in q^2$ and $\lambda \in \mathbb{T}$.

(i) For $k \in \mathbb{Z}$,

$$\sum_{l=-\infty}^{\infty} (-1)^{l+k+n} \left( \frac{p_2^2 q^{2n-2k-3}}{p_2^2 q^{2n-2k-3}} \rho \right) \left( q^{1-n} p_1 \lambda / p_2, q^{1-n} p_1 / p_2 ; q^2, -q^2-2l \right)$$

$$\times \left( q^{2-2m-2n} ; q^2 \right)_n 2\varphi_1 \left( q^{1-n} p_1 \lambda / p_2, q^{1-n} p_1 / p_2 ; q^2, -q^2-2l \right)$$

$$\times \left( -q^{2-2l} q^{2+2k} / p_2^2 ; q^2, q^{2+2k} / p_2^2 \right) \Psi \left( -q^{2-2l+2m} p_1^2 / p_2^2 ; q^2, q^{2+2k-2m-2n} / p_2^2 \right)$$

$$= p_2^2 q^{2n-3k} q^{-k^2} \left( q^2, -q^2 / p_2^2, q^2 \right)_n \left( p q^{1-n} \lambda / p_2, p q^{1-n} \lambda / p_2 ; q^2 \right)_n$$

$$\times \left( q^{2-2n} ; q^2 \right)_n 2\varphi_1 \left( p q^{1-n} \lambda / p_2, p q^{1-n} \lambda / p_2 ; q^2, -q^2 / p_2^2 \right)$$

where the sum converges absolutely.

(ii) Assume $q^{-m} p_2 / p_1 \leq 1$ and $q^{-m-n} p_2 / p_1 \leq 1$, then for $k \in \mathbb{N}_0$,

$$\sum_{l=0}^{\infty} \frac{p_2^2 q^{2-k-l} q^{(l-k)(l-k-1)}}{(q^2, q^2)_l} 3\varphi_2 \left( q^{-2l}, q^{1+n} \lambda / p_1, q^{1+n} \lambda ; q^2, q^2 \right)$$

$$\times \Psi \left( q^{-4l} q^{2+2k} / p_2^2 ; q^2, -q^4+2k-2m-2n / p_2^2 \right)$$

$$= q^{2n(k-n+1)} q^{-n(n-1)} p_2^2 q^{2m+2n} \left( q^{2m} p_1^2 / p_2^2, q^{2n} \left( q^{2m} p_1^2 / p_2^2, q^{2n} \right) ; q^2, q^2 \right)_n$$

$$\times \left( q^{2+2n} ; q^2 \right)_n 2\varphi_1 \left( q^{1-n} p_2 \lambda / p_1, q^{1-n} p_2 \lambda ; q^2, -q^2 / p_2^2 \right)$$

$$\times \left( q^{2+2k} ; q^2 \right)_n 3\varphi_2 \left( q^{-2k}, q^{1-n} p_2 \lambda / p_1, q^{1-n} p_2 \lambda ; q^{2-2m-2n} p_2^2 / p_2^2, q^2 \right)_n$$

where the sum converges absolutely.

**Remark 6.11.** (i) The $2\varphi_1$-function inside the sum in Theorem 6.10(i) is essentially the little $q$-Jacobi function $f(q; q^{-2m-2n}, q^{-n} p_1 / p_2; -q^2 ; q^2)$, see (3.28), and the summations formula remains valid if $\mu(\lambda)$ is a discrete mass point of the corresponding orthogonality
measure \( \nu \), see Appendix B.3. In Theorem 6.10(ii) the \( 3\varphi_2 \)-series is essentially an Al-Salam–Chihara polynomial, and the same remark applies using the orthogonality measure described in Appendix B.3. Note that the \( 3\varphi_2 \)-series can be transformed to a \( 2\varphi_1 \)-series by (B.3).

(ii) If we multiply the formula (i) by \( f_\nu(\mu(\lambda); q^{2-2m-2n}, q^{1-n}p_1/p_2; -q^2|q^2) \) and we use the orthogonality relations, see Appendix B.3, it follows that the above identity is equivalent to an integral identity of the form \( \int 2\varphi_1 2\varphi_1 \varphi_1 d\nu = \Psi \Psi \). The integral can be written as an integral over \([-1, 1]\) plus an infinite sum. The same remark applies for (ii) but this time using the orthogonality relations, see Appendix B.4, for the Al-Salam–Chihara polynomials.

(iii) Note that we can view the \( \Psi \)-functions as \( q \)-analogues of the Bessel function, cf. the discussion in Section 6.1 and since we can do the same for the \( 2\varphi_1 \)-series involved in (i) we may also consider Theorem 6.10(i) as an identity for \( q \)-Bessel functions.

The following result follows from the structure constants formula of Proposition 4.10. Note that Theorem 6.3 also follows from Proposition 4.10, but now we use again the fact that we can realize \( Q(p_1, p_2, n) \) as multiplication operators by a \( 2\varphi_1 \)-series up to a sign-change in the argument.

**Theorem 6.12.** Let \( \lambda \in \mathbb{T}, p_1, p_2, r_1, r_2 \in I_q, n, m \in \mathbb{Z}, \) and assume that \( |p_1| = q^{m} \) and \( |p_2| = q^{n} \). Then

\[
\sgn(r_1)^2 \frac{1 - \sgn(p_1)}{\sgn(r_2)^2} \frac{1 - \sgn(p_2)}{r_1 r_2 |r_1 r_2| \nu(r_1) \nu(r_2) \nu(p_1) \nu(p_2)} \times (q^2, -\sgn(r_1) r_1^2, -\sgn(r_2) r_2^2, -\sgn(p_2) q^2 / r_2^2, -\sgn(p_2) q^2 / p_2^2, q^2) \infty
\]

\[
= \frac{(-\sgn(r_1 p_1) q^{m-n-1} / \lambda, -\sgn(r_1 p_1) q^{3+m+n+1} / \lambda, -\sgn(r_1 r_2) \lambda q^{3-n} / p_1 p_2; q^2) \infty}{(-\sgn(r_1 r_2 p_1 p_2) q^{m+n+1} / \lambda, -\sgn(r_1 r_2 p_1 p_2) q^{1-m-n} / \lambda, -\sgn(r_1 r_2) p_1 |p_2| q^{n-1} / \lambda; q^2) \infty}
\]

\[
\times (\sgn(p_1 p_2) q^{2+2n}; q^2) \infty 2\varphi_1 \left( \frac{sgn(r_1 r_2 p_1 p_2) q^{1+m+n} / \lambda, sgn(r_1 r_2) p_2 q^{1+n} / \lambda}{sgn(p_1 p_2) q^{2+2n}} ; q^2, -\sgn(p_2) q^2 / p_2^2 \right)
\]

\[
\times (\sgn(r_1 r_2) q^{2+2m}; q^2) \infty 2\varphi_1 \left( \frac{r_2 q^{1+m} / r_1 \lambda, r_2 q^{1-m} / \lambda}{\sgn(r_1 r_2) q^{2+2m}} ; q^2, -\sgn(r_2) q^2 / r_2^2 \right)
\]

\[
= \sum_{(x_1, x_2) \in A} x_2^{m+n} |x_1|^2 \nu(x_1) \nu(x_1 p_1 / r_1) \nu(x_2 p_2 / r_2) |(sgn(r_1 r_2 p_2) q^{-2m-2n} x_1)|
\]

\[
\times (-\sgn(r_1 p_1) x_1^2, -\sgn(r_2 p_2) x_2^2, -\sgn(r_1 p_2) q^2 / x_1^2, -\sgn(r_1 r_2 p_1 p_2) q^{2+2m+2n}; q^2) \infty
\]

\[
\times 2\varphi_1 \left( \frac{sgn(r_1 r_2 p_2 p_2) q^{1+m+n+1} / \lambda, sgn(r_1 r_2 p_1 p_2) q^{1+m+n+1}}{sgn(r_1 r_2 p_1 p_2) q^{2+2m+2n}} ; q^2, -\sgn(r_2 p_2) q^2 / x_2^2 \right)
\]

\[
\times \Psi \left( \frac{-\sgn(p_1) q^2 / p_1^2}{sgn(r_1 p_1) q^{2+2m} / p_1^2}; q^2, sgn(p_1) \frac{q^2 r_1^2}{x_1^2} \right) \Psi \left( \frac{-\sgn(p_2) q^2 / p_2^2}{sgn(r_2 p_2) q^{2+2m} / p_2^2}; q^2, sgn(p_2) \frac{q^2 r_1^2}{x_1^2} \right)
\]

where the sum converges absolutely. Here \( A \subset I_q \times I_q \) is given by

\[
A = \{ (x_1, x_2) \in I_q \times I_q \mid \sgn(x_1) = \sgn(p; r_1), \ sg(x_2) = \sgn(p_2; r_2), |x_1| = |x_2| \}.
\]

From Theorem 6.12 we obtain another positivity result.
Corollary 6.13. Let \( p_1, p_2 \in I_q \) and \( \lambda \in \mathbb{T} \), then
\[
0 < \sum_{x \in q^2} \nu(x)^2 (-x^2, q^2_\infty) 2\varphi_1 \left( \frac{q/\lambda, q\lambda}{q^2}, q^2, -\frac{q^2}{x^2} \right) \times \Psi \left( -\text{sgn}(p_1)q^2/p^2_1 \right) \times q^2, \text{sgn}(p_1)q^2/p^2_1 \right) \Psi \left( -\text{sgn}(p_2)q^2/p^2_2 \right) \times q^2, \text{sgn}(p_2)q^2/p^2_2 \right) .
\]

6.4. Biorthogonality relations for 2\( \varphi_1 \)-functions. We have explicit expressions for the matrix elements of the principal series corepresentations \( W \) of the quantum group. Let
\[
\text{Corollary 6.13.}
\]
\[\text{Theorem 6.14.}\]
\[\text{Remark 6.15.}\]

Let \( m \in \mathbb{Z} \) and \( \lambda \in \mathbb{C} \setminus \{0\} \), and define \( s(\cdot, \cdot; \lambda, m) : I_q \times I_q \rightarrow \mathbb{C} \) by
\[
s(p_1, p_2; \lambda, m) = p_2^{(m+1)/2} \nu(p_1p_2q^m) |p_1p_2\nu(p_1)\nu(p_2)e_q^2 \sqrt{(-\kappa(p_1), -\kappa(p_2); q^2_\infty)} \times (q^2, q^2_\infty) \times \text{sgn}(p_2)q^2 q^m/\lambda, -\text{sgn}(p_1)q^{-m-1}/\lambda, -\text{sgn}(p_2)q^{m+3}; q^2_\infty) \times (\kappa(p_1)p_2^2 q^{m+2}; q^2_\infty) 2\varphi_1 \left( \frac{\text{sgn}(p_1)p_2^2 q^{m+2}; q^2_\infty} {\kappa(p_1)p_2^2 q^{m+2}; q^2_\infty} \right),
\]

for \( p_1, p_2 \in I_q \). From this expression it is not clear that the function is defined for all values of \( p_2 \in I_q \), but an application of Jackson’s transformation formula \[17\] (III.4) shows how to extend to all values of \( p_2 \in I_q \).

Theorem 6.14. The following biorthogonality relations hold:
\[
\sum_{p_1 \in I_q} s(p_1, p_2; \lambda, m) s(p_1, p'_2; \lambda^{-1}, m) = \delta_{p_2, p'_2},
\]
\[
\sum_{p_2 \in I_q} s(p_1, p_2; \lambda, m) s(p_1, p'_2; \lambda^{-1}, m) = \delta_{p_1, p'_2}.
\]

Remark 6.15. The two biorthogonality relations Theorem 6.14 are actually equivalent. Also, for \( \lambda \in \mathbb{T} \) the biorthogonality relations are orthogonality relations.

Proofs

7. Extensions of the generators of \( U_q(\mathfrak{su}(1, 1)) \)

7.1. Decomposition of the GNS-space. The operators \( K_0 \) and \( E_0 \), and therefore also \( \Omega_0 \), are defined on the dense subspace \( \mathcal{K}_0 \) of the Hilbert space \( \mathcal{K} \) of the GNS-construction for the left-invariant weight \( \varphi \). In order to obtain the right closures of the operators \( K_0, E_0, \Omega_0 \) we first give a convenient decomposition of \( K_0 \).

Let \( p \in q^\mathbb{Z} \), \( m \in \mathbb{Z} \) and \( \varepsilon, \eta \in \{-, +\} \), and define
\[
J(p, m, \varepsilon, \eta) = \{ z \in I_q \mid \varepsilon\eta q^m p \in I_q \text{ and } \text{sgn}(z) = \varepsilon \},
\]
\[
\mathcal{K}_0(p, m, \varepsilon, \eta) = \text{span}\{ f_m, \varepsilon \eta q^m p_z, z \in J(p, m, \varepsilon, \eta) \}.
\]
We denote by $\mathcal{K}(p, m, \varepsilon, \eta)$ the closure of $\mathcal{K}_0(p, m, \varepsilon, \eta)$ inside $\mathcal{K}$. Then $\mathcal{K}(p, m, \varepsilon, \eta) \cong \ell^2(J(p, m, \varepsilon, \eta))$, and we consider $v \in \mathcal{K}(p, m, \varepsilon, \eta)$ as a function $v: J(p, m, \varepsilon, \eta) \to \mathbb{C}$ by setting
\[ v(z) = \langle v, f_{-m, \varepsilon q^m p z, z} \rangle, \quad z \in J(p, m, \varepsilon, \eta). \] (7.2)

By convention, for $z \in \pm q^Z \setminus J(p, m, \varepsilon, \eta)$ we set $v(z) = 0$. Note that $J(p, m, \varepsilon, \eta) = I_q^+ = q^Z$ if $\varepsilon = \eta = +$. If $\varepsilon = -$ or $\eta = -$, then $J(p, m, \varepsilon, \eta)$ is a bounded $q$-halfline with 0 as only accumulation point. In this case $J(p, m, \varepsilon, \eta)$ is of the form $\varepsilon C(p, m)q^N$ for some constant $C(p, m) \in q^N$ depending on $p$ and $m$. Explicitly,
\[ C(p, m) = \begin{cases} 1, & \text{if } (\varepsilon = -, \eta = +) \text{ or } (\varepsilon = -, \eta = - \text{ and } q^mp \leq 1), \\ q^{-m}p^{-1} & \text{if } (\varepsilon = +, \eta = -) \text{ or } (\varepsilon = -, \eta = - \text{ and } q^mp \geq 1). \end{cases} \]

In particular, the sign of the bounded $q$-halfline is determined by $\varepsilon$. Note that for the modular conjugation $J$ we have
\[ J: \mathcal{K}(p, m, \varepsilon, \eta) \to \mathcal{K}(\frac{1}{p}, -m, \eta, \varepsilon), \quad J f_{-m, \varepsilon q^m p z, z} = f_{m, \eta q^{-m} p^{-1}(\varepsilon q^m p z), \varepsilon q^m p z}. \] (7.3)

We have an algebraic direct sum decomposition
\[ \mathcal{K}_0 = \bigoplus_{\varepsilon, \eta \in \{-, +\}} \mathcal{K}_0(p, m, \varepsilon, \eta). \]

By Definition 4.1 and (4.9) the actions of $K_0$, $E_0$ and $E_0^\dagger$ on the basis elements of $\mathcal{K}_0(p, m, \varepsilon, \eta)$ are given explicitly by
\[ K_0 f_{-m, \varepsilon, \eta q^m p z, z} = q^m \sqrt{p} f_{-m, \varepsilon, \eta q^m p z, z}, \]
\[ (q - q^{-1}) E_0 f_{-m, \varepsilon, \eta q^m p z, z} = \varepsilon q^m (pq) \frac{1}{2} \sqrt{1 + \varepsilon z^2 q^{-2}} f_{m, \eta q^{-m} p^{-1}(\varepsilon q^m p z), z/q} \]
\[ - \eta q^{-m} (pq) \frac{1}{2} \sqrt{1 + \eta q^{2m} p^2 z^2} f_{m-1, \eta q^{m+1} p z, z}, \] (7.4)
\[ (q - q^{-1}) E_0^\dagger f_{-m, \varepsilon, \eta q^m p z, z} = \varepsilon q^m (p/q) \frac{1}{2} \sqrt{1 + \varepsilon z^2} f_{m+1, \eta q^{-m-1} p(z/q), zq} \]
\[ - \eta q^{-m} (p/q) \frac{1}{2} \sqrt{1 + \eta q^{2m-2} p^2 z^2} f_{m+1, \eta q^{m-1} p z, z}, \]
so that
\[ K_0 = q^m \sqrt{p} \operatorname{Id}: \mathcal{K}_0(p, m, \varepsilon, \eta) \to \mathcal{K}_0(p, m, \varepsilon, \eta), \]
\[ E_0: \mathcal{K}_0(p, m, \varepsilon, \eta) \to \mathcal{K}_0(p, m+1, \varepsilon, \eta), \]
\[ E_0^\dagger: \mathcal{K}_0(p, m, \varepsilon, \eta) \to \mathcal{K}_0(p, m-1, \varepsilon, \eta), \]
\[ \Omega_0: \mathcal{K}_0(p, m, \varepsilon, \eta) \to \mathcal{K}_0(p, m, \varepsilon, \eta). \] (7.6)

For the action of $\Omega_0$ on the basis elements, see (4.8).

Proof of Proposition 4.2. We need to show that the relations
\[ K_0 E_0 = q E_0 K_0, \quad E_0^\dagger E_0 - E_0 E_0^\dagger = \frac{K_0^2 - K_0^{-2}}{q - q^{-1}}. \]
are valid when acting on the basis elements $f_{mpt}$ of $K_0$. The first relation follows immediately from (7.4). Using (4.5) and (4.6), we obtain
\[
(q - q^{-1})^2 \left( E_0^\dagger E_0 - E_0 E_0^\dagger \right) f_{mpt} = \left[ q^m \left| \frac{t}{p} \right| \left( q^{-1}(1 + \kappa(p)) - q(1 + \kappa(q^{-1}p)) \right) + q^{-m} \left| \frac{p}{t} \right| \left( q(1 + \kappa(q^{-1}t)) - q^{-1}(1 + \kappa(t)) \right) \right] f_{mpt}
\]
and then Definition 4.3 proves the second relation.

In order to prove the linear independence of the operators $K_0^n E_0^k (E_0^\dagger)^l$, $n \in \mathbb{Z}$, $k, l \in \mathbb{N}_0$, we assume that the sum
\[
\sum_{n \in \mathbb{Z}} c_{nkl} K_0^n E_0^k (E_0^\dagger)^l
\]
with only finitely many non-zero coefficients $c_{nkl}$, equals zero as operator on $K_0$. By (7.4) we have
\[
K_0^n E_0^k (E_0^\dagger)^l = (q^{m-l+k} p^\frac{1}{2})^n E_0^k (E_0^\dagger)^l : K_0(p, m, \varepsilon, \eta) \rightarrow K_0(p, m - l + k, \varepsilon, \eta).
\]
So for fixed $r \in \mathbb{Z}$, the sum $\sum_{k = l - r} c_{nkl} (q^{m-r+k} p^\frac{1}{2})^n E_0^k (E_0^\dagger)^l : K_0(p, m, \varepsilon, \eta) \rightarrow K_0(p, m + r, \varepsilon, \eta)$ equals zero. We fix such an $r$ and take $\varepsilon = + = \eta$. From (7.4) and (7.5) we see that
\[
E_0^k (E_0^\dagger)^k-r f_{-m,q^m p z,z} = \sum_{r = -k} a^n_{mpz} f_{-m-r,q^m+rp z,q^k-zq}^* \text{ for certain coefficients } a^n_{mpz}. \text{ Let } k_0 \text{ be the maximum of the } k \text{'s such that } c_{n,k,k-r} \neq 0, \text{ then it follows that}
\]
\[
0 = \left\langle \sum_{n \in \mathbb{Z}} c_{n,k,l} (q^{m+r} p^\frac{1}{2})^n E_0^k (E_0^\dagger)^l f_{-m,q^m p z,z}, f_{-m-r,q^m+rp z,q-zq} \right\rangle
\]
\[
= \sum_{m \in \mathbb{Z}} c_{n,k_0-k-r} (q^{m+r} p^\frac{1}{2})^n \left\langle E_0^{k_0} (E_0^\dagger)^{k_0-r}, f_{-m,q^m p z,z}, f_{-m-r,q^m+rp z,q-zq} \right\rangle
\]
The coefficient $a^n_{mpz, k_0} = \langle E_0^{k_0} (E_0^\dagger)^{k_0-r} f_{-m,q^m p z,z}, f_{-m-r,q^m+rp z,q-zq} \rangle$ can be explicitly calculated from (7.4) and (7.5) as the product of $\langle (E_0^\dagger)^{k_0-r} f_{-m,q^m p z,z}, f_{-m+k_0-r,q^m+rp z,q-zq} \rangle$ and $\langle E_0^{k_0} f_{-m+k_0-r,q^m-k_0+rp z,q-zq}, f_{-m-r,q^m+rp z,q-zq} \rangle$. These matrix coefficients are non-zero for all $p, m, z$ and can be calculated explicitly in terms of $q$-shifted factorials. This leaves us with the identity $\sum_{m \in \mathbb{Z}} c_{n,k_0-k-r} (q^{m+r} p^\frac{1}{2})^n = 0$ for all $m$ and $p$, from which we conclude that the coefficients $c_{n,k_0,k-r}$ are zero. \hfill \Box

After these considerations we can start considering the closures of $E_0$ and $K_0$. From the results in Appendix A.2 it follows that the closure of $K_0$ is given by the direct sum of $q^n \sqrt{p} \text{ Id}'_{K_0(p, m, \varepsilon, \eta)}$, see also (7.14).

Let us now consider the closure of $E_0$. Since
\[
\langle E_0 v, w \rangle = \langle v, E_0^\dagger w \rangle, \quad \forall v, w \in K_0,
\]
(7.7)
we see that $\mathcal{K}_0 \in D(E_0^*)$, so that $E_0^*$ is densely defined. This means that $E_0$ is closable, and its closure is $E = (E_0^*)^*$, and similarly for $E_0^\dagger$. From (7.7) one obtains

$$E_0 \subset (E_0^\dagger)^* \implies E \subset (E_0^*)^*, \quad \text{and } E_0^\dagger \subset E_0^*.$$  \hspace{1cm} (7.8)

Moreover, putting $E_{p,m}^{\varepsilon,\eta} = E|_{\mathcal{K}(p,m,\varepsilon,\eta)}$ we see from the explicit action (7.4) of $E_0$ on the basis elements of $\mathcal{K}_0$ that the closure of $E_0|_{\mathcal{K}_0(p,m,\varepsilon,\eta)}$ gives $E_{p,m}^{\varepsilon,\eta} : \mathcal{K}(p,m,\varepsilon,\eta) \to \mathcal{K}(p,m+1,\varepsilon,\eta)$. It follows that

$$E = \bigoplus_{\varepsilon,\eta \in \{-,+\}} E_{p,m}^{\varepsilon,\eta},$$  \hspace{1cm} (7.9)

and so $v \in D(E)$ if and only if $P_{p,m}^{\varepsilon,\eta} v \in D(E_{p,m}^{\varepsilon,\eta})$ for all $\varepsilon, \eta \in \{\pm\}$, $p \in \mathbb{Z}^\varepsilon$, $m \in \mathbb{Z}$, where $P_{p,m}^{\varepsilon,\eta} \in B(\mathcal{K})$ is the orthogonal projection onto $\mathcal{K}(p,m,\varepsilon,\eta)$, see Appendix A.2. The operator $E^*$, and the closures of $E_0^\dagger$ and $(E_0^*)^*$ have similar decompositions.

Examining coefficients in (7.4), we see that $E_0|_{\mathcal{K}_0(p,m,\varepsilon,\eta)}$ extends to a bounded operator $E_{p,m}^{\varepsilon,\eta} : \mathcal{K}(p,m,\varepsilon,\eta) \to \mathcal{K}(p,m+1,\varepsilon,\eta)$ unless $\varepsilon = + = \eta$. Similarly, from (7.5) it follows that $E_0^\dagger|_{\mathcal{K}_0(p,m,\varepsilon,\eta)}$ extends to a bounded operator $\mathcal{K}(p,m,\varepsilon,\eta) \to \mathcal{K}(p,m-1,\varepsilon,\eta)$ unless $\varepsilon = + = \eta$, and this bounded operator is indeed equal to the adjoint $(E_{p,m}^{\varepsilon,\eta})^* : \mathcal{K}(p,m,\varepsilon,\eta) \to \mathcal{K}(p,m-1,\varepsilon,\eta)$. The case $\varepsilon = + = \eta$ is more delicate, and we study this case later on in Section 7.4.

### 7.2. The multiplicative unitary and related operators.

Next we study the operators $Q(p_1, p_2, n) \in B(\mathcal{K})$, defined by (4.9), restricted to the subspaces $\mathcal{K}(p,m,\varepsilon,\eta)$. The definition of the operators $Q(p_1, p_2, n)$ involves the multiplicative unitary $W \in B(\mathcal{K} \otimes \mathcal{K})$. For our purposes useful description of $W$ in terms of the functions $a_\mu(\cdot, \cdot)$ can be found in [3] Prop. 4.5, 4.10:

$$W^*(f_{m_1,p_1,t_1} \otimes f_{m_2,p_2,t_2}) = \sum_{y,z \in I_q} \left| \frac{t_2}{y} \right| a_{t_2}(p_1, y)a_{t_2}(z, \text{sgn}(p_2 t_2)yzq^{m_2}/p_1)$$

$$\times f_{m_1+m_2-\chi(p_1 t_2)/z,t_1} \otimes f_{\chi(p_1 t_2)/z}$$  \hspace{1cm} (7.10)

The functions $a_\mu(\cdot, \cdot)$ are defined in Definition 6.2. For convenience we state the corresponding result for $W$ as well, which follows directly from (7.10):

$$W(f_{m_1,p_1,t_1} \otimes f_{m_2,p_2,t_2}) = \sum_{r,s \in I_q} \left| \frac{s}{t_2} \right| a_s(\text{sgn}(rp_2 t_2)sp_1 q^{m_2}, t_2) a_r(p_1, p_2)$$

$$\times f_{m_1-\chi(sp_2 t_2),sp_1 q^{m_2},t_1} \otimes f_{m_2+\chi(sp_2 t_2),r,s}.$$  \hspace{1cm} (7.11)

**Lemma 7.1.** Let $p \in \mathbb{Z}^\varepsilon$, $p_1, p_2 \in I_q$, $n, m \in \mathbb{Z}$, and $\varepsilon, \eta \in \{-,+\}$. If $q^{2m}p \neq q^{-n}\|p_2/p_1\|$, then

$$Q(p_1, p_2, n)(\mathcal{K}(p,m,\varepsilon,\eta)) = \{0\}.$$  \hspace{1cm} (7.12)

If $q^{2m}p = q^{-n}\|p_2/p_1\|$, then

$$Q(p_1, p_2, n) : \mathcal{K}(p,m,\varepsilon,\eta) \to \mathcal{K}(p,m+n,\text{sgn}(p_1)\varepsilon, \text{sgn}(p_2)\eta),$$  \hspace{1cm} (7.13)
and $Q(p_1, p_2, n)$ is given explicitly by

$$Q(p_1, p_2, n)f = (-1)^{m'}(\eta')^{\chi(p_1p_2)+m}|p_1p_2| q^{n_p}$$

$$\times \sum_{w \in J(p, m', \varepsilon', \eta')} \left(\frac{\varepsilon'\eta'}{\varepsilon}\chi(w)\right) \sum_{z \in J(p, m, \varepsilon, \eta)} \frac{f(z)}{|z|} a_{p_1}(z, w) a_{p_2}(\varepsilon \eta q^{m} p z, \varepsilon'\eta' q^{m'} p w) f_{-m', \varepsilon'\eta' q^{m'} p w, w},$$

where $f \in K(p, m, \varepsilon, \eta)$, $\varepsilon' = \text{sgn}(p_1)\varepsilon$, $\eta' = \text{sgn}(p_2)\eta$, $m' = m + n$.

Recall that $f(z) = \langle f, f_{-m'\varepsilon'\eta' q^{m'+p}z, z} \rangle$ for $f \in K(p, m, \varepsilon, \eta)$ using the convention (7.2).

By the definition of $K_{\varepsilon}$, see Definition 4.7, we have $K_{\varepsilon} = \bigoplus_{p \in q^2, \varepsilon, \eta = \pm} K(p, m, \varepsilon, \eta)$, and then Lemma 7.1 implies that

$$Q(p_1, p_2, n): K_{\varepsilon} \to K_{\text{sgn}(p_1)p_2\varepsilon}, \quad \varepsilon \in \{-, +\}.$$ This proves the last statement of Proposition 4.9 assuming we know that $Q(p_1, p_2, n) \in \hat{M}$.

Recall the action (3.6) of the dual modular conjugation $\hat{J}$, so that

$$\hat{J} f_{-m'\varepsilon'\eta' q^{m'+p}z, z} = \eta^{+\chi(p)\varepsilon}(z)(-1)^m f_{-(m), \varepsilon'\eta' q^{m'} z, z}$$

and thus $\hat{J}: K(p, m, \varepsilon, \eta) \to K(q^{m'} p, -m, \varepsilon, \eta)$. Now Lemma 7.1 implies the following.

**Corollary 7.2.** Let $p \in q^2, \, p_1, p_2 \in P_q, \, m, n \in \mathbb{Z}$ and $\varepsilon, \eta \in \{-, +\}$. If $p \neq q^{-n}|p_2/p_1|$, then

$$\hat{J}Q(p_1, p_2, n)\hat{J}(K(p, m, \varepsilon, \eta)) = \{0\}.$$ If $p = q^{-n}|p_2/p_1|$, then

$$\hat{J}Q(p_1, p_2, n)\hat{J}: K(p, m, \varepsilon, \eta) \to K(q^{n} p, m - n, \text{sgn}(p_1)\varepsilon, \text{sgn}(p_2)\eta)$$

and

$$\hat{J} Q(p_1, p_2, n) \hat{J} f = (-1)^{m\varepsilon} \eta^{\chi(p)} |p_1p_2| q^{n_p} \sum_{w \in J(q^{2n} p, -m, \varepsilon, \eta')} \frac{1}{|w|}$$

$$\times \left(\sum_{z \in J(p, m, \varepsilon, \eta)} \frac{f(z)}{|z|} (\varepsilon\eta \chi(z)) a_{p_1}(z, w) a_{p_2}(\varepsilon\eta q^{m} p z, \varepsilon'\eta' q^{m+n} p w) \right) f_{-m, \varepsilon'\eta' q^{m+n} p w, w},$$

where $f \in K(p, m, \varepsilon, \eta)$, $\varepsilon' = \text{sgn}(p_1)\varepsilon$ and $\eta' = \text{sgn}(p_2)\eta$.

Again we postpone the proof until the end of this subsection.

Let us state the matrix elements of $Q(p_1, p_2, n)$ and $\hat{J} Q(p_1, p_2, n)$ $\hat{J}$ explicitly:

$$\langle Q(p_1, p_2, n) f_{uvw}, f_{irs} \rangle = \delta_{u-l, n} \delta_{l, \chi(p_1p_2w)} \delta_{r, \text{sgn}(uvw)s_{p_2q^n/p_1}} |w| \frac{a_w(p_1, s) a_v(p_2, r)}{s} \right),$$

$$\langle \hat{J} Q(p_1, p_2, n) \hat{J} f_{uvw}, f_{irs} \rangle = \delta_{-l, u} \delta_{l, \chi(p_2w/p_1v)} \delta_{r, \text{sgn}(uvw)s_{p_2q^n/p_1}} \text{sgn}(r) \chi(r) \text{sgn}(s) \chi(s)$$

$$\times \text{sgn}(v) \chi(v) \text{sgn}(w) \chi(w)(-1)^{l+u} |w| \frac{a_w(p_1, s) a_v(p_2, r)}{s} \right),$$

(7.13)

to which one may apply the symmetry relations (5.2).

The remainder of this subsection is devoted to the proofs of Lemma 7.1 and Corollary 7.2.
Proof of Lemma 7.1. We start by considering matrix elements of the more generally defined operator
\[
(\omega f_{m_1,p_1,t_1}, f_{m_2,p_2,t_2} \otimes \text{Id})(W^*) \in B(K),
\]
with \(m_1, m_2 \in \mathbb{Z}\) and \(p_1, p_2, t_1, t_2 \in I_q\). For \(n_1, n_2 \in \mathbb{Z}\) and \(r_1, r_2, s_1, s_2 \in I_q\) we have
\[
\begin{align*}
\langle (\omega f_{m_1,p_1,t_1}, f_{m_2,p_2,t_2} \otimes \text{Id})(W^*) f_{n_1,r_1,s_1} f_{n_2,r_2,s_2} \rangle \\
= \langle W^* f_{m_1,p_1,t_1} \otimes f_{n_1,r_1,s_1}, f_{m_2,p_2,t_2} \otimes f_{n_2,r_2,s_2} \rangle \\
= \delta_{t_1,t_2} \delta_{n_1-n_2,m_1-m_2} \delta_{n_2,\chi(p_1 r_1/s_1 p_2)} \delta_{r_2,s_2} \text{sgn}(r_1 s_1) s_2 p_2 q^{n_1}/p_1 \\
\times |s_1/s_2| a_{s_1}(p_1,s_2) a_{r_1}(p_2, \text{sgn}(r_1 s_1) s_2 p_2 q^{n_1}/p_1),
\end{align*}
\]
where we used expression (7.10) for \(W^*\). The dependence on \(t_1, t_2 \in I_q\) and \(m_1, m_2 \in \mathbb{Z}\) of the right hand side occurs only in the first two Kronecker deltas, so by (7.9) we have
\[
(\omega f_{m_1,p_1,t_1}, f_{m_2,p_2,t_2} \otimes \text{Id})(W^*) = \delta_{t_1,t_2} Q(p_1,p_2,m_2 - m_1).
\]
(7.14)
We see that it suffices to restrict to the case \(t_1 = t_2 = 1, m_1 = 0, m_2 = n\), and we switch to the basis elements of \(K(p,m,\varepsilon,\eta,\kappa)\), see Section 7.4, i.e., we replace \((n_1, r_1, s_1)\) by \((-m, \varepsilon \eta q^m p z, z)\) and \((n_2, r_2, s_2)\) by \((-m', \varepsilon' \eta' q^{m'} p' z', z')\), where \(p, p' \in q\mathbb{Z}\) and \(\varepsilon, \eta, \varepsilon', \eta' \in \{+, -\}\). Then we find
\[
\begin{align*}
\langle Q(p_1,p_2,n) f_{-m,\varepsilon \eta q^m p z, z}, f_{-m',\varepsilon' \eta' q^{m'} p' z', z'} \rangle \\
= \delta_{n,m'} \delta_{-m'-m+\chi(p_1 p_2), p} \delta_{\varepsilon' \eta' q^{m'} p', \varepsilon \eta q^m p} \delta_{m,2} \bigg| z \bigg| a_{z}(p_1, z') a_{\varepsilon \eta q^m p z, \varepsilon' \eta' q^{m'} p' z'}.
\end{align*}
\]
The first two Kronecker deltas always give zero unless \(m' = n + m = -m - \chi(p_1 p_2)\), or equivalently \(q^{2m} p = q^{-n}|p_2/p_1|\), which is the first statement of Lemma 7.4. Assuming that this condition is valid we see the third Kronecker delta becomes \(\delta_{\varepsilon' \eta' q^{m'} p', \varepsilon \eta q^m p} \delta_{m,2} \). Since \(p, p' \in q\mathbb{Z}\), we find that we need \(p = p'\) and \(\varepsilon' \eta' = \text{sgn}(p_1 p_2) \varepsilon \eta\). Assuming these conditions and using the last symmetry of (7.2) we find that
\[
\begin{align*}
\langle Q(p_1,p_2,n) f_{-m,\varepsilon \eta q^m p z, z}, f_{-m',\varepsilon' \eta' q^{m'} p' z', z'} \rangle \\
= (-1)^{m+n}(\varepsilon' \eta') \chi^{(z')}(\eta') \chi(p)+m+n \frac{p_1 p_2}{q^m p} \bigg| z \bigg| a_{p_1}(z, z') a_{p_2}(\varepsilon \eta q^m p z, \varepsilon' \eta' q^{m'} p' z').
\end{align*}
\]
Now the product of the functions \(a_p\) is zero unless \(\varepsilon' = \text{sgn}(p_1) \varepsilon\) and \(\eta' = \text{sgn}(p_2) \eta\), see Definition 7.2. So in case \(q^{2m} p = q^{-n}|p_2/p_1|\) we find \(Q(p_1,p_2,n): K(p,m,\varepsilon,\eta) \rightarrow K(p,m+n,\varepsilon \text{sgn}(p_1), \eta \text{sgn}(p_2))\) and with \(m' = m + n\), \(\varepsilon' = \varepsilon \text{sgn}(p_1), \eta' = \eta \text{sgn}(p_2)\) we find
\[
Q(p_1,p_2,n) f_{-m,\varepsilon \eta q^m p z, z} = (-1)^{m'} (\eta')^{(z')} \chi(p)+m' \frac{|p_1 p_2|}{q^m p} \bigg| z \bigg| \sum_{z' \in J(p,m',\varepsilon',\eta')} \bigg(\frac{\varepsilon' \eta'}{|z'|} a_{p_1}(z, z') a_{p_2}(\varepsilon \eta q^m p z, \varepsilon' \eta' q^{m'} p' z') \bigg) f_{-m',\varepsilon' \eta' q^{m'} p' z', z'}.
\]
This gives the required expression leading to the last statement of Lemma 7.1 after taking into account \(q^{2m} p = q^{-n}|p_2/p_1|\). \qed
Proof of Corollary 7.2. The first statements are immediate from Lemma 7.1 and 7.12, and assuming the condition \( p = q^{-m}[p_2/p_1] \) we get, with \( \varepsilon' = \text{sgn}(p_1)\varepsilon \) and \( \eta' = \text{sgn}(p_2)\eta \),
\[
\hat{J}Q(p_1, p_2, n)\hat{J} \sum_{m,m' \in Z} (\varepsilon\eta)\chi(z)\eta'^{m' + \chi(p)}(1)^m[p_1, p_2] \frac{1}{q^{m'p}} Q_{m'p p z, z} \times \sum_{z' \in \hat{J}(q^{m'p}, m, \varepsilon', \eta')} \frac{1}{|z'|} a_{p_1}(z, z') a_{p_2}(\varepsilon\eta q^{m'p} z, \varepsilon'\eta' q^{m' + n} p z') f_{-(m-n), \varepsilon'\eta' q^{m' + n} p z', z'}
\]
using (7.12), (7.13) and \( Jq^{2m'p}, m - n, \varepsilon', \eta' \) = \( J(q^{2m'p}, m - n, \varepsilon', \eta') \). This implies the last statement of Corollary 7.2.

7.3. A basis for the dual von Neumann algebra. In this subsection we give a proof of Proposition 4.9 and Corollary 4.11. For this we use the description of \( \hat{M} \) as in (2.3).

Lemma 7.3. The operators \( Q(p_1, p_2, n), p_1, p_2 \in I_q, n \in Z, \) are in \( \hat{M} \), and the linear span of the operators \( Q(p_1, p_2, n), p_1, p_2 \in I_q, n \in Z, \) is strongly dense in \( \hat{M} \). Moreover, for \( x \in \hat{M} \) there exists a net \( \{ x_i \}_{i \in I} \) in this linear span such that \( x_i \to x \) in the strong \(*\)-topology with \( \| x_i \| \leq \| x \| \).

Lemma 7.3 proves Proposition 4.9 except for the last statement, which was proved in Section 7.2 after Lemma 7.1. By the general Tomita-Takesaki theory, see [41, Vol. II, cf. (2.1)], we have that the commutant satisfies \( \hat{M}' = J\hat{M}J \), and so Corollary 4.11 follows.

Proof. By (2.3) and Theorem 2.3 we have to consider
\[
(\omega_{f_{m_1, p_1, t_1}, f_{m_2, p_2, t_2}} \otimes \text{Id})(W) = (\omega_{f_{m_1, p_1, t_1}, f_{m_2, p_2, t_2}} \otimes \text{Id}) \left( (\hat{J} \otimes J) W^* (\hat{J} \otimes J) \right)
\]
\[
= J (\omega_{j f_{m_1, p_1, t_1}, j f_{m_2, p_2, t_2}} \otimes \text{Id}) (W^*) J
\]
\[
= \text{sgn}(p_1)\chi(p_1)\text{sgn}(p_2)\chi(p_2)\text{sgn}(t_1)\chi(t_1)\text{sgn}(t_2)\chi(t_2) (1)^{m_1 + m_2} J (\omega_{f_{m_1, p_1, t_1}, f_{m_2, p_2, t_2}} \otimes \text{Id}) (W^*) J
\]
using (2.3), \( \hat{J}^2 = \text{Id} \), \( \langle \hat{J} f, \hat{J} g \rangle = \langle g, f \rangle \), \( J \) being antilinear, and (3.6). It follows from the proof of Lemma 7.1, in particular from (7.14), that we can restrict to the case \( t_1 = t_2 = 1, m_1 = 0 \), \( m_2 = -n \). By (2.3) and \( J^2 = \text{Id} \) we see that, up to a sign, \( Q(p_1, p_2, n) \) equals \( J (\omega \otimes \text{Id})(W) J \) for \( \omega \in B(K) \). Recall from (2.4) that the unitary antipode \( \hat{R} \) for the dual quantum group is given by \( \hat{R}(x) = Jx^*J \), so that for \( x \in \hat{M} \) we have \( JxJ = \hat{R}(x^*) \in \hat{M} \). Now we see that \( Q(p_1, p_2, n) \in \hat{M} \).

In order to prove the density statement, we recall that there exists a dense \(*\)-subalgebra \( M^2 \) of the predual \( M_* \) such such that \{ \( (\omega \otimes \text{Id})(W) \mid \omega \in M^2 \) \} is \( \sigma\)-strong-\(*\) dense \(*\)-subalgebra of \( \hat{M} \), see [41, p. 79]. The subspace \( M^2 \) consists of those normal functionals \( \omega \) such that \( \omega \circ S \) is again a normal functional, where \( \omega(x) = \omega(x^*) \), and the \(*\)-operator for \( \omega \in M^2 \) defined as \( \omega^* = \omega \circ S \). Now we apply the Kaplansky density theorem, see e.g. [51, Vol. I, Ch. II, Thm. 4.8], to obtain a net \( \{ \omega_i \}_{i \in I} \) in \( M^2 \) with the properties \( \| (\omega_i \otimes \text{Id})(W) \| < \| x^* \| = \| x \| \) for all \( i \in I \) and such that \( (\omega_i \otimes \text{Id})(W) \to x^* \) in the strong-\(*\) topology, so that also \( (\omega_i \otimes \text{Id})(W^*) \to x \) in the strong-\(*\) topology.

Let \( L \) be the linear span of the normal functionals \( \omega_{f_{m_1, p_1, t_1}, f_{m_2, p_2, t_2}} \) for \( p_1, p_2, t_1, t_2 \in I_q \) and \( m_1, m_2 \in Z \), then \( L \) is norm dense in \( M_* \) and \( \omega_{f,g} = \omega_{g,f} \) so \( L \) is closed under \( \omega \mapsto \omega \). Now
define the index set $I_0 = I \times \mathbb{N}$, and make this a directed (or upward filtering) set by the product order, i.e. $(i_1, k_1) \leq (i_2, k_2)$ whenever $i_1 \leq i_2$ in $I$ and $k_1 \leq k_2$. For $j = (i, k) \in I_0$ we can pick $\eta_j \in L$ such that $\|\langle \eta_j \otimes \text{Id} \rangle(W^*) - (\hat{\omega}_j \otimes \text{Id})(W^*)\| \leq 1/k$ and $\|\langle \eta_j \otimes \text{Id} \rangle(W^*)\| < \|x\|$. For such $j \in I_0$ set $x_j = \langle \eta_j \otimes \text{Id} \rangle(W^*)$ in the linear span of the operators $Q(p_1, p_2, n)$, $p_1, p_2 \in I_q$, $n \in \mathbb{Z}$, and the net $\{x_j\}_{j \in I_0}$ satisfies all required properties.

\begin{proof}

Note that the statements are equivalent because $Q(p_1, p_2, n) \in \hat{M}$ by Lemma 7.3 and $\hat{R}x = Jx^*J$ for $x \in \hat{M}$, see (2.4).

For $f, g \in K$ we set $T = (\omega_{f, g} \otimes \text{Id})(W^*)$. Then $T^* = (\omega_{g, f} \otimes \text{Id})(W)$, so that $(\hat{J} \otimes J)(W^*)(\hat{J} \otimes J) = W$ gives

$$T^* = J(\omega_{f, g} \otimes \text{Id})(W^*)J$$

as in the first part of the proof of Lemma 7.3. Specializing $f = f_{0, p_1, 1}$, $g = f_{n, p_2, 1}$ gives $T = Q(p_1, p_2, n)$, and using the action (3.6) of $\hat{J}$ on $f_m$ we obtain

$$Q(p_1, p_2, n)^* = (-1)^n \text{sgn}(p_1)^{\chi(p_1)} \text{sgn}(p_2)^{\chi(p_2)} J(\omega_{f_{n, p_2, 1}, f_{0, p_1, 1}} \otimes \text{Id})(W^*) J$$

$$= (-1)^n \text{sgn}(p_1)^{\chi(p_1)} \text{sgn}(p_2)^{\chi(p_2)} J Q(p_2, p_1, n) J,$$

where the last equality follows from (7.14).

We finish the subsection by establishing the structure constants for the operators $Q(p_1, p_2, n)$ as a linear basis for $\hat{M}$. First observe that as elements of $B(K)$

$$((\omega_{f, g} \otimes \text{Id})(W^*))((\omega_{\xi, \eta} \otimes \text{Id})(W^*)) = (\omega_{\xi, \eta} \otimes \omega_{f, g} \otimes \text{Id})(W_{13}^* W_{13}^*)$$

for arbitrary vectors $f, g, \xi, \eta \in K$. Using the pentagonal equation $W_{12} W_{13} W_{23} = W_{23} W_{12}$ this can be rewritten in the compact form

$$((\omega_{f, g} \otimes \text{Id})(W^*))((\omega_{\xi, \eta} \otimes \text{Id})(W^*)) = (\omega_{W_{12}(\xi \otimes f), W_{13}(\eta \otimes g)} \otimes \text{Id})(\text{Id} \otimes W^*). \quad (7.16)$$

\begin{proof}[Proof of Proposition 4.11]

We start with the choice $f = f_{0, p_1, 1}$, $g = f_{n, p_2, 1}$, $\xi = f_{0, r_1, 1}$, $\eta = f_{m, r_2, 1}$, so that the left hand side of (7.16) equals $Q(p_1, p_2, n) Q(r_1, r_2, m)$. In order to evaluate the right hand side of (7.16) we use (7.11), which leads to

$$\sum_{x_1, y_1 \in I_q} \sum_{x_2, y_2 \in I_q} |y_1 y_2| a_{x_1}(r_1, p_1) a_{x_2}(r_2, p_2)$$

so that $\text{sgn}(x_1, y_1, r_1, 1) \text{sgn}(x_2, y_2, r_2, 1) \text{sgn}(x_1, y_1, r_1, 1) a_{y_2} \text{sgn}(x_2, y_2, r_2, 1)$

$$\times (f_{-\chi(y_1 p_1), \text{sgn}(x_1, y_1, r_1, 1)} f_{m-\chi(y_2 p_2), \text{sgn}(x_2, y_2, r_2, 1)}) (W^*). \quad (7.17)$$

The proof is complete.

\end{proof}
The inner product in the summand of (7.17) leads to $\delta_{-\chi(y_1p_1,m-\chi(y_2p_2)\delta_{\text{sgn}(x_1p_1,y_1r_1,\text{sgn}(x_2p_2)y_2r_2q^n)}$, whereas, by (7.14), the last term in the summand is $\delta_{y_1,y_2}Q(x_1,x_2,n+\chi(p_2)-\chi(p_1))$. Combining this we see that the last two terms in the summand of (7.17) equal

$$
\delta_{\chi(p_2),m+\chi(p_1)}\delta_{\text{sgn}(x_1p_1),r_1,\text{sgn}(x_2p_2,y_2r_2q^n)\delta_{y_1,y_2}Q(x_1,x_2,n+m),
$$

which is zero in case $|p_1|^2 \neq q^m$ independent of $x_1, y_1, x_2, y_2$.

Assuming $|p_1|^2 = q^m$ and inserting this into (7.17) leads to

$$
\sum_{x_1,x_2 \in I_q} a_{x_1}(r_1,p_1)a_{x_2}(r_2,p_2)Q(x_1,x_2,n+m)\times \left( \sum_{y_1 \in I_q \text{ so that } \text{sgn}(x_1p_1)y_1r_1=\text{sgn}(x_2p_2)y_2r_2q^n \in I_q} \right)
\left( y_1^2 a_{y_1}(\text{sgn}(x_1p_1)y_1r_1,1) a_{y_1}(\text{sgn}(x_2p_2)y_2r_2q^n,1) \right),
$$

where empty sums are zero. For the expression in (7.18) to be non-zero result we require $\text{sgn}(x_1) = \text{sgn}(r_1p_1)$ and $\text{sgn}(x_2) = \text{sgn}(r_2p_2)$, see Definition 6.2. Then we see that $\text{sgn}(x_1p_1)y_1r_1 = y_1|r_1|$ and $\text{sgn}(x_2p_2)y_2r_2q^n = y_1|r_2|q^n$, and so the inner sum is zero unless $|y_1|^2 = q^n$. In this case the inner sum equals

$$
\sum_{y_1 \in I_q \text{ so that } y_1|r_1| \in I_q} y_1^2 (a_{y_1}(y_1|r_1|,1))^2 = \sum_{y_1 \in I_q \text{ so that } y_1|r_1| \in I_q} (a_{1}(y_1, y_1|r_1|))^2 = 1,
$$

where the first equality follows from the symmetry relations (6.3), and the second equality is a special case of Proposition 6.3 (with $p = 1$ and $\theta = |r_1|$).

Collecting the results finishes the proof of Proposition 4.10. \hfill \Box

7.4. Affiliation of $K$ and $E$ to $\hat{M}$. The purpose of this subsection is to prove Proposition 4.4. First we focus on the operator $K$.

By Definition 4.3 $K$ is the closure of $(K_0, K_0)$, with $K_0$ given by Definition 4.1. Since $K_0$ acts diagonally on basis elements $f_{mpt}, m \in \mathbb{Z}, p,t \in I_q$, we find from Definition 4.1

$$
D(K) = \left\{ \sum_{m \in \mathbb{Z}, p,t \in I_q} c_{mpt} f_{mpt} \mid \sum_{m \in \mathbb{Z}, p,t \in I_q} |c_{mpt}|^2 q^{-m} \left| \frac{p}{t} \right|^p \right< \infty \right\},
$$

$$
K \left( \sum_{m \in \mathbb{Z}, p,t \in I_q} c_{mpt} f_{mpt} \right) = \sum_{m \in \mathbb{Z}, p,t \in I_q} c_{mpt} q^{-\frac{m}{2}} |\frac{p}{t}|^{\frac{1}{2}} f_{mpt}.
$$

It is now straightforward from (7.19) to check that $K$ is an injective positive self-adjoint operator, establishing the first statement of Proposition 4.4. We now prove the second statement for the operator $K$.

**Proposition 7.5.** $K$ is affiliated to $\hat{M}$.

*Proof.* Note that $K$ restricted to $K_0(p,m,\varepsilon,\eta)$ acts as $q^m \sqrt{p} \text{Id}$ by Definition 4.1 and (7.1). It follows that $K(p,m,\varepsilon,\eta) \subset D(K)$. So $\hat{J}Q(p_1,p_2,n)\hat{f}_{mpt} \in D(K)$ by Corollary 7.2 and

$$
K(\hat{J}Q(p_1,p_2,n)\hat{f}_{mpt}) = \hat{J}Q(p_1,p_2,n)\hat{K}f_{mpt}
$$
since the action of $K$ on $\mathcal{K}(p, m, \varepsilon, \eta)$ is the same as on $\mathcal{K}(q^{2n}p, m - n, \text{sgn}(p_1)\varepsilon, \text{sgn}(p_2)\eta)$ in case $p = q^{-n}[p_2/p_1]$. In case this is not true, both sides equal zero.

Since $K_0$ is a core for $K$, we can take for $f \in D(K)$ a sequence $K_0 \ni f_i \to f$ and $Kf_i \to g = Kf$. Then $\hat{J} Q(p_1, p_2, n) \hat{J} f_i \to \hat{J} Q(p_1, p_2, n) \hat{J} f$ by continuity, and $K \hat{J} Q(p_1, p_2, n) \hat{J} f_i = \hat{J} Q(p_1, p_2, n) \hat{J} Kf_i \to \hat{J} Q(p_1, p_2, n) \hat{J} g$. Since $K$ is closed, we conclude $\hat{J} Q(p_1, p_2, n) \hat{J} f \in D(K)$ and $K \hat{J} Q(p_1, p_2, n) \hat{J} f = \hat{J} Q(p_1, p_2, n) \hat{J} Kf$. This means

$$\hat{J} Q(p_1, p_2, n) \hat{J} K \subset K \hat{J} Q(p_1, p_2, n) \hat{J},$$

so $K$ commutes with the generators of $\hat{M}'$, see Appendix A.3.

To see that $K$ commutes with an arbitrary element $T \in \hat{M}'$, pick $T_i$ from the linear span of $\hat{J} Q(p_1, p_2, n) \hat{J}$ such that $T_i \to T$ strongly, see Corollary 4.11. Take any $f \in D(K)$, so that $T_i f \to Tf$ and since $T_i f \in D(K)$ (by $T_i K \subset KT_i$) we have $KT_i f = T_i Kf \to TKf$ by the strong convergence. Again by the closedness of $K$ we conclude that $Tf \in D(K)$ and $KTf = TKf$, or $TK \subset KT$. Since $T \in \hat{M}'$ is arbitrary, $K$ is affiliated to $\hat{M}$, see Appendix A.4.

In order to show that $E$ is affiliated to $\hat{M}$ we need to work more carefully. We start with a useful property of the operator $E_0$.

**Lemma 7.6.** Let $p_1, p_2 \in I_q$ and $n \in \mathbb{Z}$. Then

$$\langle \hat{J} Q(p_1, p_2, n) \hat{J} v, E_0^\dagger w \rangle = \langle \hat{J} Q(p_1, p_2, n) \hat{J} E_0 v, w \rangle, \quad \forall v, w \in K_0.$$

We relegate the proof of Lemma 7.6 to Appendix D.1, since it is a tedious check. By Lemma 7.6 we have for the closure $E$ of $E_0$ the equality

$$\langle \hat{J} Q(p_1, p_2, n) \hat{J} v, E_0^\dagger w \rangle = \langle \hat{J} Q(p_1, p_2, n) \hat{J} E v, w \rangle$$

for $v, w \in K_0$. Now fix $v = f_m \in K_0(p, m, \varepsilon, \eta)$ and put $u = \hat{J} Q(p_1, p_2, n) \hat{J} v$. It follows that $u \in D((E_0^\dagger)^*)$ and

$$(E_0^\dagger)^* u = \hat{J} Q(p_1, p_2, n) \hat{J} E f_m \in K_0(p, m, \varepsilon, \eta).$$

(7.20)

This equality can be extended in the following way.

**Lemma 7.7.** Let $u = \hat{J} Q(p_1, p_2, n) \hat{J} f_m \in K_0(p, m, \varepsilon, \eta)$. Then $\langle (E_0^\dagger)^* u, w \rangle = \langle u, E^\dagger w \rangle$ for all $w \in D(E^\dagger)$.

Before proving Lemma 7.7 we show how it implies that $E$ is affiliated to $\hat{M}$, which finishes the proof of Proposition 4.4.4.

**Proposition 7.8.** $E$ is affiliated to $\hat{M}$.

**Proof.** Since $E$ is the closure of $E_0$, it follows from Lemma 7.6 that $u \in D(E^{**}) = D(E)$, and

$$E \hat{J} Q(p_1, p_2, n) \hat{J} f_m \in K_0(p, m, \varepsilon, \eta) \Rightarrow E \hat{J} Q(p_1, p_2, n) \hat{J} E \hat{J} Q(p_1, p_2, n) \hat{J} f_m \in K_0(p, m, \varepsilon, \eta).$$

(7.20)

This shows that $E \hat{J} Q(p_1, p_2, n) \hat{J} = \hat{J} Q(p_1, p_2, n) \hat{J} E$ on $K_0$. Now the proof is finished as in the last stage of Proposition 4.4.4 using the closedness of $E$, $K_0$ being a core for $E$, and the strong-* denseness of the operators $\hat{J} Q(p_1, p_2, n) \hat{J}$ in $\hat{M}'$ by Corollary 4.11. □
Before we turn to the proof of Lemma 7.7, recall the decomposition (7.3) of \( E \) into operators \( E_{p,m}^{\varepsilon,\eta} : \mathcal{K}(p,m,\varepsilon,\eta) \to \mathcal{K}(p,m+1,\varepsilon,\eta) \). The operators \( E_{p,m}^{\varepsilon,\eta} \) are bounded, unless \( \varepsilon = + = \eta \).

We study the case \( \varepsilon = + = \eta \) by considering truncated inner products. Define for \( x \in q^Z \) a truncated inner product by

\[
\langle v, w \rangle_x = \sum_{z \in \mathcal{L}(p,m,+,+)} v(z) \overline{w(z)}, \quad v, w \in \mathcal{K}(p,m,+,+).
\]

(7.21)

For \( x \to \infty \) this gives back the inner product on \( \mathcal{K}(p,n,+,+). \) Let us remark that all coefficients in (7.4) and (7.5) remain bounded for \( z \to 0, z \in q^Z, \) so we do not need to consider a truncated inner product of the form (7.21) with the terms \( z \leq y \) cut off, for some \( y \in q^Z, y < x. \)

**Lemma 7.9.** Let \( w \in D(E^*) \cap \mathcal{K}(p,m,+,+), u \in D((E_0^1)^*) \cap \mathcal{K}(p,m-1,+,+), \) then, with \( x \in q^Z, \)

\[
\langle (E_0^1)^* u, w \rangle_x - \langle u, E^* w \rangle_x = \frac{q^{m-1}(pq)^{1/2}}{q - q^{-1}} \sqrt{1 + x^{-2}q^2} \frac{1}{q} u(x/q) w(x)
\]

using the convention (7.2).

**Proof.** By (7.4) and (7.3) for the case \( \varepsilon = + = \eta, \) using the boundedness of the coefficients as \( z \to 0, z \in q^Z, \) we obtain

\[
(q - q^{-1}) \left( \langle (E_0^1)^* u, w \rangle_x - \langle u, E^* w \rangle_x \right) = \sum_{z \in q^Z, z \leq x} \left( q^{m-1}(pq)^{1/2} \sqrt{1 + z^2q^{-2}} \frac{u(z)}{q} w(z) - q^{1-m}(pq)^{-1/2} \sqrt{1 + q^{2m-2}p^2z^2} u(z) w(z) \right)
\]

\[
- \sum_{z \in q^Z, z \leq x} \left( q^m(p/q)^{1/2} \sqrt{1 + z^2} q^{-1} u(z) w(qz) - q^{-m}(p/q)^{-1/2} \sqrt{1 + q^{2m-2}p^2z^2} u(z) w(z) \right)
\]

\[
= q^{m-1}(pq)^{1/2} \sqrt{1 + x^{-2}q^{-2}} \frac{u(x/q)}{q} w(x)
\]

giving the required expression. \( \square \)

The following result will be useful when we want to take the limit \( x \to \infty, x \in q^Z, \) in the previous lemma. Recall the convention (7.2).

**Lemma 7.10.** (i) Let \( v = Q(p_1,p_2,n)\hat{J}_{m,n,\varepsilon,\eta} \) and assume \( q^{2m}p = q^{-n}|p_2/p_1|, \) so that \( v \in \mathcal{K}(p,n,+,\varepsilon,\eta) \) is non-zero. If \( \varepsilon = + = \eta \) then there exists a continuous function \( h: \mathbb{R}_{\geq 0} \to \mathbb{R} \) such that \( x v(x) = h(x^{-2}) \) for \( x \in I_q^+ \). In case \( m + n = 0, h: \mathbb{R}_{\geq 0} \to \mathbb{R} \) is differentiable, in particular at 0.

(ii) Let \( u = \hat{J} Q(p_1,p_2,n)\hat{J}_{m,n,\varepsilon,\eta} \) and assume \( p = q^{-n}|p_2/p_1|, \) so that \( u \in \mathcal{K}(q^{2n}p,m-n,\varepsilon,\eta) \) is non-zero. If \( \varepsilon = + = \eta \) then there exists a continuous function \( h: \mathbb{R}_{\geq 0} \to \mathbb{R} \) such that \( x u(x) = h(x^{-2}) \) for \( x \in I_q^+ \). In case \( m-n = 0, h: \mathbb{R}_{\geq 0} \to \mathbb{R} \) is differentiable, in particular at 0.
Proof. We prove the second statement; the first statement is proved in the same way. It follows from Corollary 7.2 or (7.13) that $u \in \mathcal{K}(q^{2n}p, m - n, \varepsilon, +, +)$ and for $x \in I^+_q$

$$x u(x) = (-1)^m \eta^{m+\chi(p)} (\varepsilon \eta) \chi(z) \left[ \frac{p_1 p_2}{q^{m}p|z|} \right] a_{p_1}(z, x) a_{p_2}(\theta z, \theta' x)$$

where $\theta = \varepsilon q^{m}p, \theta' = q^{m+n}p$. Lemma 3.1 gives $a_{p}(z, x) = x^{\chi(p/\varepsilon)} f_1(x^{-2})$ as well as $a_{p}(z, x) = x^{\chi(p/\varepsilon)} f_2(x^{-2})$ for certain differentiable functions $f_1, f_2: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ using the last equation of the symmetry relations (3.2) and then Lemma 3.1. Now we find, with $C$ a generic non-zero constant not depending on $x$,

$$x u(x) = C x^{\chi(z/p_1)} f_1(x^{-2}) (\theta' x)^{\chi(p_2/\theta z)} f_2((\theta')^{-2} x^{-2}) = C x^{\chi(p_2/p_1)} f_1(x^{-2}) f_2((\theta')^{-2} x^{-2})$$

using $|p_2/\theta p_1| = q^{m-n}$ as follows from the assumption $q^m p = |p_2|$. This proves the statement in case $m - n \geq 0$, since we can take $h(t) = C t^{\frac{(m-n)}{2}} f_1(t) f_2((\theta')^{-2} t)$. In case $n = m$ the statement on the differentiability of $h$ follows immediately.

Similarly, we find, for other functions $f_1, f_2$,

$$x u(x) = C x^{\chi(p_1/\varepsilon)} f_1(x^{-2}) (\theta x)^{\chi(p_2/\theta z)} f_2((\theta')^{-2} x^{-2}) = C x^{\chi(p_2/\theta)} f_1(x^{-2}) f_2((\theta')^{-2} x^{-2})$$

using $|p_2/\theta p_1| = q^{m-n}$ again. This proves the statement in case $m - n \leq 0$, since we can take $h(t) = C t^{\frac{(m-n)}{2}} f_1(t) f_2((\theta')^{-2} t)$.

We are now ready to prove Lemma 7.4.

Proof of Lemma 7.4. We set $p' = q^{2n}p$, $m' = m - n$, $\varepsilon' = \text{sgn}(p_1)\varepsilon$, $\eta' = \text{sgn}(p_2)\eta$, then $u = J Q(p_1, p_2, \eta) J f_{-m, \varepsilon q^{m}p, z} \in \mathcal{K}(p', m', \varepsilon', \eta')$ by Lemma 7.2. Using the decomposition of $(E_0)^t$, cf. (7.3),

$$(E_0^t)^* = \bigoplus_{\alpha, \beta \in \{-+, +\}} (E_0^t|_{\mathcal{K}_0(\alpha, t, \alpha, \beta)})^*,$$

we find $u \in D((E_0^t|_{\mathcal{K}_0(p', m'+1, \varepsilon', \eta')}^*)^*).$ Using the similar decomposition for $E^*$ we find that $w' = P_{p', m'+1} w \in D((E_{p'}^t, \eta')^*)$, where $P_{p, m, \eta} \in B(\mathcal{K})$ is the orthogonal projection onto $\mathcal{K}(p, m, \varepsilon, \eta)$ as in Section 7.1. This gives

$$\langle (E_0^t)^* u, w \rangle - \langle u, E^* w \rangle = \langle (E_0^t|_{\mathcal{K}_0(p', m'+1, \varepsilon', \eta')}^*) u, w' \rangle - \langle u, (E_{p'}^t, \eta')^* w' \rangle.$$

In case $\varepsilon' = -$ or $\eta' = -$, $E_{p'}^t, \eta'$ is bounded. Therefore $(E_{p'}^t, \eta')^*$ is the unique continuous extension of $E_{0|\mathcal{K}_0(p', m'+1, \varepsilon', \eta')}^*$, so $(E_{0|\mathcal{K}_0(p', m'+1, \varepsilon', \eta')}^*)^* = E_{p', \eta'}^t$ and hence the right hand side is zero, as required.

It remains to consider the case $\varepsilon' = + = \eta'$. In this case we consider the truncated inner product. Using Lemma 7.3 we find for $x \in q^2 = I^+_q$,

$$\langle (E_0^t|_{\mathcal{K}_0(p', m'+1, +, +)}^*) u, w' \rangle_x - \langle u, (E_{p'}^t, \eta')^* w' \rangle_x = \frac{q^{m'-1}(p'q)^{\frac{1}{2}}}{q - q^{-1}} \frac{1 + x^{-2}q^2}{\sqrt{q - q^{-1}}} \frac{x}{q} u(x/q) w'(x).$$
and we need to show that the right hand side tends to zero as $x \to \infty$ through $I_q^+$. Since $w' \in K(p', m' + 1, +, +) \cong \ell^2(I_q^+)$ it follows that $w'(x) \to 0$ as $x \to \infty$, so the required result follows from Lemma 7.10 which implies $\frac{x}{q} u(x/q)$ is bounded as $x \to \infty$ in $I_q^+$. □

7.5. **The comultiplication on $\hat{M}$.** In order to calculate the action of the comultiplication of the dual quantum group on the elements $Q(p_1, p_2, n)$, we note that this can be done in greater generality. First observe

$$W ((\omega_{f,g} \otimes \text{Id})(W^*) \otimes \text{Id}) W^* = (\omega_{f,g} \otimes \text{Id} \otimes \text{Id})(W_{13}^* W_{12}^*) = (\omega_{f,g} \otimes \text{Id} \otimes \text{Id})(W_{13}^* W_{12}^*).$$

(7.22)

The first equality is straightforward, and the second follows from the pentagonal equation for the multiplicative unitary, see Section 2. Using an orthonormal basis $\{e_k\}$ for the Hilbert space $K$, so that we have $\langle x, y \rangle = \sum_k \langle x, e_k \rangle \langle e_k, y \rangle$ we get

$$\sum \hat{\Delta}((\omega_{f,g} \otimes \text{Id})(W^*)) \Sigma = (\omega_{f,g} \otimes \text{Id} \otimes \text{Id})(W_{13}^* W_{12}^*) = \sum_k (\omega_{f,e_k} \otimes \text{Id})(W^*) \otimes (\omega_{e_k,g} \otimes \text{Id})(W^*).$$

(7.23)

using the definition of $\hat{\Delta}$ and notation as in Theorem 2.3.

**Proof of Proposition 4.14.** We use the general formula (7.23) with $f = f_{0,p_1,1}$, $g = f_{n,p_2,1}$, the orthonormal basis $f_{m,p,t}$ ($m, p, t \in I_q$) and next (7.14) to rewrite the right hand side in terms of the operators $Q(p_1, p_2, n)$. The series converges in the von Neumann algebra $\hat{M} \otimes \hat{M}$, so that we find convergence in the $\sigma$-weak topology. □

Next we prove the link between the comultiplication $\hat{\Delta}$ of the dual quantum group $\hat{M}$ and the comultiplication (3.2) $\Delta$ of the Hopf $*$-algebra $U_q(\mathfrak{su}(1,1))$ as given in Proposition 4.14.

**Proof of Proposition 4.14.** The comultiplication for the dual locally compact quantum group is given by $\hat{\Delta}(x) = \sum W(x \otimes 1) W^* \Sigma, x \in \hat{M}$, see Theorem 2.3. We use the same formula for the elements $K$ and $E$ affiliated to $\hat{M}$, see Proposition 4.14. In order to prove that $\hat{\Delta}(K) = K \otimes K$ we need to show $\sum W(K \otimes 1) W^* \Sigma = K \otimes K$, or $W(K \otimes 1) W^* = \sum K \otimes K \Sigma = K \otimes K$. So it suffices to check that $(K \otimes 1) W^* = W^* (K \otimes K)$.

Now by (7.10) and (7.19) we check this formula first by evaluating it on the orthonormal basis of $K \otimes K$. So for arbitrary $m_1, m'_1, m_2, m'_2 \in \mathbb{Z}, p_1, p'_1, p_2, p'_2 \in I_q, m_1, m'_1, m_2, m'_2 \in I_q$

$$\langle W^*(f_{m_1,p_1,t_1} \otimes f_{m'_1,p'_1,t'_1}), (K \otimes 1)(f_{m_2,p_2,t_2} \otimes f_{m'_2,p'_2,t'_2}) \rangle = \sqrt{\frac{P_2}{t_2}} q^{-\frac{1}{2}m_2}$$

$$\times \delta_{t_1,t_2} \delta_{|p_1|/p_1^2|p_2|/p_2^2} q^{m_2} \delta_{m_1+m_2+m'_2} \delta_{\text{sgn}(p'_1 t'_1) q^{m_1} p_2/p_1, p'_2/t'_2} \left| \frac{t'_1}{t'_2} \right| \alpha_{t_1}(p_1, t_2) a_{p_1}(p_2, p'_2)$$
and similarly

\[
\langle W^*(K \otimes K)(f_{m,p,t_1} \otimes f_{m',p',t_1'}), f_{m_2,p_2,t_2} \otimes f_{m_2',p_2',t_2'} \rangle = \sqrt{\frac{p_1p_1'}{t_1t_1'}} q^{-\frac{1}{2}(m_1+m_1')} \delta_{t_1,t_2} \delta_{p_1,p_1'|p_2,p_2'|} q^{m_2} \delta_{m_1+m_1'-m_2-m_2'} \delta_{\text{sgn}(p_1)p_1,q^{m_1}p_{21}p_1,p_2,p_2'} \left| \frac{t_1'}{t_2} \right| a_{t_1'}(p_1,t_2') a_{p_1'}(p_2,p_2').
\]

These expressions are equal by inspection using the Kronecker deltas. The linear span of elements \(f_{m,p,t}\) forms a core for the operator \(K\). So it follows that

\[
\langle W^*(f_{m,p_1,t_1} \otimes f_{m',p_1',t_1'}), (K \otimes 1)w \rangle = \langle W^*(K \otimes K)(f_{m,p_1,t_1} \otimes f_{m',p_1',t_1'}), w \rangle
\]

for all \(w \in D(K \otimes 1)\). Hence, \(W^*(K_0 \otimes K_0) \subset (K \otimes 1)W^*\), thus \(K \otimes K \subset W(K \otimes 1)W^*\) using that \(K\) is self-adjoint and that the closure of \(K_0 \otimes K_0\) equals \(K \otimes K\). Since both operators are self-adjoint, this inclusion is an equality. This proves the statement for \(\Delta(K)\).

Let us now prove the more complicated second statement. Choose \(p_1, p_1', t_1, t_1' \in I_q\) and \(m_1, m_1' \in \mathbb{Z}\). Take also \(p_2, p_2', t_2, t_2' \in I_q\) and \(m_2, m_2' \in \mathbb{Z}\). Since \(E_0 \subseteq E\) and \(E_0^* \subseteq E^*\), (4.10) and (7.10) imply that

\[
\langle q - q^1 \rangle \langle W^*(E_0 \otimes K_0 + K_0^{-1} \otimes E_0)(f_{m_1,p_1,t_1} \otimes f_{m_1',p_1',t_1'}), f_{m_2,p_2,t_2} \otimes f_{m_2',p_2',t_2'} \rangle
\]

\[
= \delta_{t_1,t_2} \delta_{p_1,p_1'|p_2,p_2'|} q^{m_1+m_1'-m_2-m_2'} \delta_{\text{sgn}(p_1)p_1',q^{m_1}p_{21}p_1,p_2,p_2'} \left| \frac{t_1'}{t_2} \right| a_{t_1'}(p_1,t_2') a_{p_1'}(p_2,p_2')
\]

\[
- \delta_{t_1,t_2} \delta_{p_1,p_1'|p_2,p_2'|} q^{m_1+m_1'-m_2-m_2'} \delta_{\text{sgn}(p_1)p_1',q^{m_1}p_{21}p_1,p_2,p_2'} \left| \frac{t_1'}{t_2} \right| a_{t_1'}(p_1,t_2') a_{p_1'}(p_2,p_2')
\]

\[
\times \text{sgn}(p_1) q^{1/2(m_1-m_1'-1)} \left| \frac{t_1p_1'}{p_1t_1'} \right| \sqrt{1 + \kappa(q^{-1}t_1')} \left| \frac{t_1'}{t_2'} \right| a_{t_1'}(p_1,t_2') a_{p_1'}(p_2,p_2') \quad (7.24)
\]

\[
+ \delta_{t_1,t_2} \delta_{p_1,p_1'|p_2,p_2'|} q^{m_1+m_1'-m_2-m_2'} \delta_{\text{sgn}(p_1)p_1',q^{m_1}p_{21}p_1,p_2,p_2'} \left| \frac{t_1'}{t_2} \right| a_{t_1'}(p_1,t_2') a_{p_1'}(p_2,p_2')
\]

\[
\times \text{sgn}(p_1) q^{1/2(m_1-m_1'-1)} \left| \frac{t_1p_1'}{p_1t_1'} \right| \sqrt{1 + \kappa(q^{-1}t_1')} \left| \frac{t_1'}{t_2'} \right| a_{t_1'}(p_1,t_2') a_{p_1'}(p_2,p_2')
\]

\[
- \delta_{t_1,t_2} \delta_{p_1,p_1'|p_2,p_2'|} q^{m_1+m_1'-m_2-m_2'} \delta_{\text{sgn}(p_1)p_1',q^{m_1}p_{21}p_1,p_2,p_2'} \left| \frac{t_1'}{t_2} \right| a_{t_1'}(p_1,t_2') a_{p_1'}(p_2,p_2')
\]

\[
\times \text{sgn}(p_1) q^{1/2(m_1-m_1'-1)} \left| \frac{t_1p_1'}{p_1t_1'} \right| \sqrt{1 + \kappa(p_1')} \left| \frac{t_1'}{t_2'} \right| a_{t_1'}(p_1,t_2') a_{p_1'}(p_2,p_2').
\]
Lemma 8.1. Study the commutation relation between that the above equality holds. Thus, we see that (7.26) holds.

\[ |t\rangle |\rho\rangle \text{ on the right hand side of (7.24) agree. Thus in order to prove that the left hand sides of (7.25) and } (7.24) \text{ agree, we need to show that, under the conditions } t_1 = t_2, m_1 + m'_1 = m_2 + m'_2 + 1, |p_1p_1'/p_2t'_1| = q^{m'_1-1} p_2/p_1 = p'_2/t'_2, \]

\[ 0 = \text{sgn}(p_2) q^{m'_2+1} |t_2/p_2|^{1/2} \sqrt{1 + \kappa(q^{-1}p_2)} a_{t'_1} (p_1, t'_2) a_{p'_1} (q^{-1}p_2, p'_2) \]

\[ - \text{sgn}(p_1) q^{m_1-m'_1-1} |p_1'/p_2t'_1|^{1/2} \sqrt{1 + \kappa(p_2)} a_{t'_1} (qp_2, t'_2) a_{p'_1} (p_2, p'_2) \]

\[ + \text{sgn}(t'_1) q^{m_1-m'_1-1} |t_1p_1'/p_2t'_1|^{1/2} \sqrt{1 + \kappa(q^{-1}t'_1)} a_{q^{-1}t'_1} (p_1, t'_2) a_{p'_1} (p_2, p'_2) \]

\[ - \text{sgn}(p'_1) q^{m_1-m'_1-1} |t_1t'_1/p_1p_1'|^{1/2} \sqrt{1 + \kappa(p'_1)} a_{t'_1} (p_1, t'_2) a_{q^{-1}p'_1} (p_2, p'_2) \]

For this purpose we can use the q-contiguous relations

\[ \text{sgn}(p) \sqrt{1 + \kappa(p)} a_{q^{-1}p}(x, y) = \text{sgn}(x) \sqrt{1 + \kappa(q^{-1}x)} a_{p}(q^{-1}x, y) - \frac{x}{q} a_{p}(x, y) \]

and

\[ \text{sgn}(p) \sqrt{1 + \kappa(q^{-1}p)} a_{q^{-1}p}(x, y) = \text{sgn}(x) \sqrt{1 + \kappa(x)} a_{p}(qx, y) - \frac{x}{q} a_{p}(x, y) \]

for all \( x, y, p \in I_q \) which follow from Lemma 3.2 and the symmetry relations (3.2). If one uses the first equality to replace \( a_{q^{-1}p} (p_2, p'_2) \) and the second one to replace \( a_{q^{-1}t'_1} (p_1, t'_2) \) one checks that the above equality holds. Thus, we see that (7.26) holds.

The linear span of elements \( f_{m,p,t} \) forms a core for \( E^* \). So it follows that

\[ \langle W^* (f_{m_1,p_1,t_1} \otimes f_{m'_1,p'_1,t'_1}), (E^* \otimes 1) v \rangle = \langle W^* (E_0 \otimes K_0 + K_0^{-1} \otimes E_0) (f_{m_1,p_1,t_1} \otimes f_{m'_1,p'_1,t'_1}), v \rangle \]

for all \( v \in D(E^* \otimes 1) \). Hence, \( W^* (E_0 \otimes K_0 + K_0^{-1} \otimes E_0) \subset (E \otimes 1) W^* \), or \( E_0 \circ K_0 + K_0^{-1} \circ E_0 \subset W(E \otimes 1) W^* \Rightarrow K_0 \circ E_0 + E_0 \circ K_0^{-1} \subset \Sigma W(E \otimes 1) W^* \Sigma = \Delta(E) \).

The last statement of Proposition 4.14 is proved in the same way. \( \square \)

8. The Casimir operator

8.1. Definition of the Casimir operator. In this section we prove Theorem 4.13. In order to show that the Casimir operator \( \Omega \) as defined in Definition 4.13 is well-defined, we need to study the commutation relation between \( K \) and \( E \).

Lemma 8.1. If \( s \in \mathbb{R} \), then \( K^{is} E = q^{is} E K^{is} \). Consequently, \( K \) and \( E^* E \) strongly commute.
Proposition 8.2. The Casimir operator \( \Omega \) we see that the closure \( \Omega \) of the adjoint operator. Moreover, \( \Omega \) is affiliated to \( \hat{E} \) and commutes strongly with \( K \). By Definition 4.1, since \( E \) is the closure of \( E_0 \), with domain \( D(E_0) \) the finite linear span of the \( f_{mpt} \), and \( K \) is bounded, this implies \( K^{\ast} E \subseteq q^{i s} E K^{i s} \). Using Proposition 4.4 we multiply this result with the bounded operator \( E \) of the \( f_{mpt} \) and \( E \) by Definition 4.1. Since \( E \) are self-adjoint extensions of \( S \), the operators \( E \) commute with all spectral projections of the self-adjoint operator \( E \). In particular, \( K \) and \( E \) are resolvent commuting, hence they strongly commute, see Appendix A.3. \( \square \)

Lemma 8.1 leads to a proof of a part of Theorem 4.6.

Proposition 8.2. The Casimir operator \( \Omega \) as defined in Definition 4.3 is a well-defined self-adjoint operator. Moreover, \( \Omega \) is affiliated to \( \hat{M} \) and commutes strongly with \( K \) and \( E \).

Proof. Since \( K \) and \( E \) are strongly commuting self-adjoint operators, see Appendix A.3, we see that the closure \( \Omega \) of

\[
\frac{1}{2} \left( (q - q^{-1})^2 E E - q K^2 - q^{-1} K^{-2} \right).
\]

is a well-defined self-adjoint operator. Moreover, by Appendix A.4 and Proposition 4.4, the operators \( K^2 \), \( K^{-2} \) and \( E \) are affiliated to \( \hat{M} \). It follows that \( \Omega \) is affiliated to \( \hat{M} \) and that \( \Omega \) commutes strongly with \( K \) and \( E \).

In order to show that \( \Omega \) also strongly commutes with \( E \) we first need some preliminary results. Along the way we also prove the last statement of Theorem 4.6.

Recall the decomposition of the Hilbert space \( \mathcal{K} \) into components \( \mathcal{K}(p, m, \varepsilon, \eta) \) with \( p \in q^{\mathbb{Z}} \), \( m \in \mathbb{Z} \) and \( \varepsilon, \eta \in \{-, +\} \), and the corresponding decomposition (7.9) of the operator \( E \) into operators \( E_{p,m}^{\varepsilon,\eta} \).

Lemma 8.3. Let \( p \in q^{\mathbb{Z}} \), \( m \in \mathbb{Z} \) and \( \varepsilon, \eta \in \{-, +\} \). Then

\[
(E_{p,m}^{\varepsilon,\eta})^* E_{p,m}^{\varepsilon,\eta} = E_{p,m-1}^{\varepsilon,\eta} (E_{p,m-1}^{\varepsilon,\eta})^* + \frac{q^{2m} p - q^{-2m} p^{-1}}{q - q^{-1}} \text{Id}.
\]

Proof. Proposition 4.2 implies that

\[
E_{0}^\dagger|_{\mathcal{K}_{0}(p,m+1,\varepsilon,\eta)} E_{0}|_{\mathcal{K}_{0}(p,m,\varepsilon,\eta)} = E_{0}^\dagger|_{\mathcal{K}_{0}(p,m,\varepsilon,\eta)} E_{0}|_{\mathcal{K}_{0}(p,m,\varepsilon,\eta)} + \frac{q^{2m} p - q^{-2m} p^{-1}}{q - q^{-1}} \text{Id}. \tag{8.1}
\]

If \( \varepsilon = - \) or \( \eta = - \), the lemma follows from this equality by the continuity of the operators involved.

It remains to deal with the case \( \varepsilon = \eta = + \). From (8.1) we see that the operators

\[
S_1 = (E_{p,m}^{++,\varepsilon})^* E_{p,m}^{++,\varepsilon} \quad \text{and} \quad S_2 = E_{p,m-1}^{++,\varepsilon} (E_{p,m-1}^{++,\varepsilon})^* + \frac{q^{2m} p - q^{-2m} p^{-1}}{q - q^{-1}} \text{Id}
\]

are both self-adjoint extensions of \( S := E_{0}^\dagger|_{\mathcal{K}_{0}(p,m++,\varepsilon,\eta)} E_{0}|_{\mathcal{K}_{0}(p,m++,\varepsilon,\eta)} \). We will prove that they are the same by linking \( S \) to a Jacobi operator, which is studied in Appendix C.
Set \( \theta = q^m p \). By (4.5) and (4.6), we get for \( v \in \mathcal{K}_0(p, m, +, +) \) and \( x \in I_q^+ \),

\[
(q - q^{-1})^2 (Su)(x) = [q^{m+1} \theta (1 + q^{-2}x^2) + q^{-(m+1)} \theta^{-1} (1 + \theta^2 x^2)] v(x) - \sqrt{(1 + q^{-2}x^2)} (1 + \theta^2 q^{-2}x^2) v(qx) - \sqrt{(1 + x^2)} (1 + \theta^2 x^2) v(qx)
\]

Let \( \{e_k\}_{k \in \mathbb{Z}} \) be the standard orthonormal basis of \( \ell^2(\mathbb{Z}) \), and let \( \mathcal{K}(\mathbb{Z}) \) be the dense subspace consisting of finite linear combinations of the \( e_k \)'s. For \( k \in \mathbb{Z} \) we denote \( f_k = \langle f, e_k \rangle_{\ell^2(\mathbb{Z})} \) for any \( f \in \ell^2(\mathbb{Z}) \). We define the unitary transformation \( U : \ell^2(\mathbb{Z}) \to \mathcal{K}(p, m, +, +) \) so that \( (Uf)(q^k) = f_k \) for all \( f \in \mathcal{K}(\mathbb{Z}) \) and \( k \in \mathbb{Z} \). So \( U^*SU \in \text{End}(\mathcal{K}(\mathbb{Z})) \) is given by

\[
(q - q^{-1})^2 (U^*SUf)_k = [q^{m+1} \theta (1 + q^{2(k-1)}) + q^{-(m+1)} \theta^{-1} (1 + \theta^2 q^{2k})] f_k - \sqrt{(1 + q^{2(k-1)}) (1 + \theta^2 q^{2(k-1)})} f_{k-1} - \sqrt{(1 + q^{2k}) (1 + \theta^2 q^{2k})} f_{k+1}
\]

for all \( f \in \mathcal{K}(\mathbb{Z}) \), \( k \in \mathbb{Z} \). After a close inspection, one sees that

\[
U^*SU = (q - q^{-1})^{-2} ((q^{m+1} \theta + q^{-m-1} \theta^{-1}) I_d - 2L)
\]

where \( L = L(q^{2+2|m|}, \theta^{-1}, -q^2 | q^2 \rangle \langle q^2 | q^2) \) is the Jacobi operator of Appendix B.3, see (B.33).

If \( m \neq 0 \), then \( c = q^{2+2|m|} \leq q^4 \) which by Theorem B.15 implies that \( L \) and thus \( S \) is essentially self-adjoint. Therefore \( S_1 = S_2 \) in this case.

Now assume that \( m = 0 \), so \( c = q^2 \). In this case \( L \) is not essentially self-adjoint, but we can use Theorem C.1 to prove that \( S_1 \) and \( S_2 \) are equal. From Proposition 7.8 or Proposition 4.4 we know that \( E \) is affiliated to \( \hat{M} \), implying that \( E^*E \) and \( EEE^* \) are also affiliated to \( \hat{M} \). This guarantees that

\[
\hat{J} Q(1, p, 0) \hat{J} X \subseteq X \hat{J} Q(1, p, 0) \hat{J}
\]

for \( X = E^*E \) and \( X = EEE^* \), see Appendix A.4. Since \( f_{0,p,1} \) belongs to \( D(E^*E) \) and \( D(EEE^*) \), it follows that the vector \( w := \hat{J} Q(1, p, 0) \hat{J} f_{0,p,1} \) belongs to \( D(E^*E) \) and \( D(EEE^*) \). As a consequence, \( w \) belongs to \( D((E^+_{p,0})^*E^+_{p,0}) = D(S_1) \) and to \( D(E^+_{p,-1}) (E^+_{p,-1})^* = D(S_2) \).

As in the proof of Lemma C.10(ii) we see that Corollary C.2 implies

\[
w = \hat{J} Q(p_1, p_2, n) \hat{J} = \sum_{x \in J(p, 0, +, +)} \frac{1}{x} a_1(1, x) a_p(p, x) f_{0, px, x}
\]

so that \( x w(x) = a_1(1, x) a_p(p, x) = h(x^{-2}) \). By Lemma B.1 the function \( h : \mathbb{R}_{\geq 0} \to \mathbb{R} \) is differentiable and \( h(0) \neq 0 \).

So \( U^*w \) belongs to \( D(U^*S_1 U) \) and \( D(U^*S_2 U) \) and \( (U^*w)_-k = (q^{2k})^{\frac{1}{2}} h(q^{2k}) \) for all \( k \in \mathbb{Z} \). Since \( U^*S_1 U \) and \( U^*S_2 U \) are both self-adjoint extensions of the operator

\[
(q - q^{-1})^{-2} ((q^{m+1} \theta + q^{-m-1} \theta^{-1}) I_d - 2L)
\]

Theorem C.1 now guarantees that \( U^*S_1 U = U^*S_2 U \) and we are done.

Using Lemma C.3 we can describe the relation between \( E^* \) and \( E_0^\dagger \), and we can give a characterization of the operator \( E \).

**Proposition 8.4.**

(i) \( E^* \) is the closure of \( E_0^\dagger \).

(ii) \( E \) is the unique closed, linear operator in \( \mathcal{K} \) so that \( E_0 \subseteq E \) and \( E_0^\dagger \subseteq E^* \).
Lemma 8.5. Let \( p \in q^Z, m \in \mathbb{Z}, \varepsilon, \eta \in \{-, +\} \). By Lemma 8.3 there exists a constant \( c \in \mathbb{R} \) such that
\[
(E^\varepsilon_\eta)^* E^\varepsilon_\eta = E^\varepsilon_\eta + c \text{Id}.
\]
Thus, \( D((E^\varepsilon_\eta)^* E^\varepsilon_\eta) = D((E^\varepsilon_\eta)^* E^\varepsilon_\eta_{m-1})^* \) and since these sets form a core for \( E^\varepsilon_\eta \) and \( (E^\varepsilon_{m-1})^* \) respectively, (8.2) implies that \( D(E^\varepsilon_{m-1}) = D((E^\varepsilon_{m-1})^*) \) and \( \|E^\varepsilon_{m-1} v\|^2 = \|(E^\varepsilon_{m-1})^* v\|^2 + c\|v\|^2 \) for all \( v \in D(E^\varepsilon_{m-1}) \). Because \( K_0(p, m, \varepsilon, \eta) \) is a core for \( E^\varepsilon_{m-1} \), this in turn guarantees that \( K_0(p, m, \varepsilon, \eta) \) is a core for \( (E^\varepsilon_{m-1})^* \). In other words, \( (E^\varepsilon_{m-1})^* \) is the closure of \( E_0^\dagger|\mathcal{K}_0(p, m, \varepsilon, \eta) \). Thus,
\[
E^* = \bigoplus_{p \in q^Z, m \in \mathbb{Z}} \bigoplus_{\varepsilon, \eta \in \{-, +\}} (E^\varepsilon_\eta)^* = \bigoplus_{p \in q^Z, m \in \mathbb{Z}} \bigoplus_{\varepsilon, \eta \in \{-, +\}} E_0^\dagger|\mathcal{K}_0(p, m, \varepsilon, \eta)
\]
\[
= \left( \bigoplus_{p \in q^Z, m \in \mathbb{Z}} \bigoplus_{\varepsilon, \eta \in \{-, +\}} E_0^\dagger|\mathcal{K}_0(p, m, \varepsilon, \eta) \right) = E_0^\dagger.
\]
For the second statement, we take a closed linear operator \( F \) in \( \mathcal{K} \) such that \( E_0 \subseteq F \) and \( E_0^\dagger \subseteq F^* \). Since, by definition, \( E \) is the closure of \( E_0 \) and \( F \) is a closed extension of \( E_0 \), we must have that \( E \subseteq F \). By part (i) we know that \( E^* \) is the closure of \( E_0^\dagger \). Since \( F^* \) is a closed extension of \( E_0^\dagger \), this implies that \( E^* \subseteq F^* \) and by taking the adjoint of this inclusion, we see that \( F \subseteq E \). Thus, \( F = E \). 

We define, for \( p \in q^Z, m \in \mathbb{Z}, \varepsilon, \eta \in \{-, +\} \), self-adjoint operators in \( \mathcal{K}(p, m, \varepsilon, \eta) \) by
\[
\Omega_{\varepsilon, \eta}^\varepsilon_\eta = \frac{1}{2} \left( (q - q^{-1})^2 (E^\varepsilon_\eta)^* E^\varepsilon_\eta - (q^{2m+1} + q^{-2m-1} p^{-1}) \text{Id} \right).
\]
Now we have the following decomposition of the Casimir operator;
\[
\Omega = \bigoplus_{\varepsilon, \eta \in \{-, +\}} \Omega_{\varepsilon, \eta}^\varepsilon_\eta, \tag{8.4}
\]
see Appendix A.2.

Lemma 8.5. Let \( p \in q^Z, m \in \mathbb{Z}, \varepsilon, \eta \in \{-, +\} \).

(i) If \( (\varepsilon = - \text{ or } \eta = -) \) or \( (\varepsilon = \eta = + \text{ and } m \neq 0) \), then \( \Omega_{\varepsilon, \eta}^{\varepsilon_\eta} \) is the closure of the essentially self-adjoint operator \( \Omega_0|\mathcal{K}_0(p, m, \varepsilon, \eta) \).

(ii) \( \Omega_{p,0,0}^+ \) is a self-adjoint extension of \( \Omega_0|\mathcal{K}_0(p, 0, +, +) \).

Let us remark that \( \Omega_{p,0,0}^+ \) is not the closure of \( \Omega_0|\mathcal{K}_0(p, 0, +, +) \).

Proof. Take \( p \in q^Z, m \in \mathbb{Z} \) and \( \varepsilon, \eta \in \{-, +\} \). If \( \varepsilon = - \text{ or } \eta = - \), then (L4) and (T1) imply that \( \Omega_0|\mathcal{K}_0(p, m, \varepsilon, \eta) \) is bounded, hence essentially self-adjoint, and \( \Omega_{\varepsilon, \eta}^{\varepsilon_\eta} \) must be the closure of \( \Omega_0|\mathcal{K}_0(p, m, \varepsilon, \eta) \).

Now assume that \( \varepsilon = \eta = + \) and set \( \theta = q^mp \). As in the second half of the proof of Lemma 8.3, one sees that \( \Omega_0|\mathcal{K}_0(p, m, \varepsilon, \eta) \) is unitarily equivalent to \(-L\), where \( L = L(c, d, z \mid q) \) is the Jacobi operator of Appendix C in base \( q^2 \) and with parameters \( c = q^{2m+1|z|}, d = \theta^{-1} q^{|m|+1} \) and
If $m \neq 0$, this implies, see [29, Prop. 4.5.3] and Appendix C, that $\Omega_0|_{\mathcal{K}_0(p,m, \varepsilon, \eta)}$ is essentially self-adjoint and $\Omega_{p,\eta}^{\varepsilon,m}$ must be the closure of $\Omega_0|_{\mathcal{K}_0(p,m, \varepsilon, \eta)}$.

If $m = 0$ the reasoning of the last part of the proof of Proposition 8.3 shows, since $\Omega$ is affiliated to $\hat{M}$, that $\Omega_0^{+,+}$ must be unitarily equivalent to the self-adjoint extension of $-L_q$ described in Theorem C.1.

The proof of Lemma 8.5 and the last statement of Theorem C.1 lead to the following result, which will be useful later on and for this reason it is stated separately. Again we use the convention (7.2).

**Lemma 8.6.** Consider $p \in q\mathbb{Z}$, $m \in \mathbb{Z}$, $\varepsilon, \eta \in \{-, +\}$ and $v \in D(\Omega_0^* \cap \mathcal{K}(p, m, \varepsilon, \eta))$. Assume moreover that if $m = 0$ and $\varepsilon = \eta = +$, there exists a function $h : \mathbb{R}_{\geq 0} \to \mathbb{R}$ that is differentiable at 0 and satisfies $v(x) = x^{-1} h(x^{-2})$ for all $x \in I_+^q$. Then $v$ belongs to $D(\Omega)$ and $\Omega v = \Omega_0^* v$.

We are now ready to prove the last statement of Theorem 4.6.

**Proposition 8.7.** The Casimir operator $\Omega$ is the unique self-adjoint extension of $\Omega_0$ that is affiliated to $\hat{M}$.

**Proof.** Choose a self-adjoint operator $C$ in $\mathcal{K}$ so that $C$ is affiliated to $\hat{M}$ and $\Omega_0 \subseteq C$. We have to show that $C = \Omega$.

We divide $\Omega_0$ into two parts. For this purpose define

$$L = \left\{ (p, m, \varepsilon, \eta) \mid p \in q\mathbb{Z}, m \in \mathbb{Z}, \varepsilon, \eta \in \{-, +\} \right. \left. \text{s.t. } (\varepsilon = - \text{ or } \eta = -) \text{ or } (\varepsilon = \eta = + \text{ and } m \neq 0) \right\}.$$

Now set

$$\Omega_0^{(1)} = \sum_{(p, m, \varepsilon, \eta) \in L} \Omega_0|_{\mathcal{K}_0(p,m,\varepsilon,\eta)} \quad \text{and} \quad \Omega_0^{(2)} = \sum_{p \in q\mathbb{Z}} \Omega_0|_{\mathcal{K}_0(p,0,+,+)}$$

and define respective self-adjoint extensions

$$\Omega^{(1)} = \bigoplus_{(p, m, \varepsilon, \eta) \in L} \Omega_{\varepsilon, \eta}^{m,p} \quad \text{and} \quad \Omega^{(2)} = \bigoplus_{p \in q\mathbb{Z}} \Omega_{p,0}^{+,+}.$$

By Lemma 8.3 we know that $\Omega_0^{(1)}$ is essentially self-adjoint with closure $\Omega^{(1)}$. Since

$$\text{Ker}(\Omega_0^* \pm i \text{Id}) = \text{Ker}(\Omega^{(1)} \oplus (\Omega_0^{(2)})^* \pm i \text{Id})$$

$$= \text{Ker}(\Omega^{(1)} \pm i \text{Id}) \oplus (\Omega_0^{(2)})^* \pm i \text{Id})$$

$$= \{0\} \oplus \text{Ker}(\Omega_0^{(2)} \pm i \text{Id})$$,

the theory of self-adjoint extensions via the deficiency spaces, see [14, §XII.4], implies the existence of a self-adjoint extension $D$ of $\Omega_0^{(2)}$ so that $C = \Omega^{(1)} \oplus D$.

We have seen in Proposition 7.4 that $K$ is affiliated to $\hat{M}$ implying that $\hat{J}K\hat{J}$ is affiliated to $\hat{M'}$. By Definitions 1.1, 4.3 and (1.0), we know that $\mathcal{K}_0$ is a core for $\hat{J}K\hat{J}$ and $\hat{J}K\hat{J}f_{m,\varepsilon,\eta}^{\pm\eta} = \sqrt{D} f_{m,\varepsilon,\eta}^{\pm\eta}$ for all $p \in q\mathbb{Z}$, $m \in \mathbb{Z}$ and $\varepsilon, \eta \in \{-, +\}$. Thus, for each
\( p \in q^\mathbb{Z} \), the orthogonal projection \( P_p \) of \( \mathcal{K} \) onto \( \bigoplus_{m \in \mathbb{Z}, \varepsilon, \eta \in \{-, +\}} \mathcal{K}(p, m, \varepsilon, \eta) \) belongs to \( \hat{M}' \), since it is the spectral projection of \( \hat{J}K\hat{J} \) with respect to the eigenvalue \( \sqrt{p} \). Because \( C \) is affiliated to \( \hat{M} \), the operator \( C \) commutes with each projection \( P_p \). As a consequence, there exists for every \( p \in q^\mathbb{Z} \) a self-adjoint extension \( D_p \) of \( \Omega|_{\mathcal{K}(p, 0, 0)} \) so that \( D = \bigoplus_{p \in q^\mathbb{Z}} D_p \). As in the proof of Lemma 8.5, the fact that \( C \) is affiliated to \( \hat{M} \) implies for every \( p \in q^\mathbb{Z} \) that \( D_p \) is unitarily equivalent to the self-adjoint extension described in Theorem C.1 and hence, \( D_p = \Omega_{p,0}^{++} \). Thus, we conclude that \( \Omega = C \).

To finish the proof of Theorem 4.6 we need to prove the following result.

**Proposition 8.8.** The operators \( E \) and \( \Omega \) strongly commute.

Before embarking on the proof of Proposition 8.8 we first collect all the elements for the proof of Theorem 4.6.

**Proof of Theorem 4.6.** By Proposition 8.3 the Casimir operator is a well-defined self-adjoint operator affiliated to \( \hat{M} \), and by Proposition 8.7 the Casimir operator is the unique self-adjoint extension of \( \Omega_0 \) affiliated to \( \hat{M} \). By Proposition 8.2 the Casimir operator commutes strongly with \( K \), and by Proposition 8.8 it also commutes strongly with \( E \).

**Proof of Proposition 8.8.** By Proposition 8.2 the Casimir operator \( \Omega \) is self-adjoint, and we have to prove that

\[
E_\Omega(B) E \subset E E_\Omega(B)
\]

for all Borel sets \( B \subset \mathbb{R} \), where \( E_\Omega \) is the spectral decomposition of \( \Omega \), see Appendix A.3. Using the decompositions (7.3), (8.3), (8.4) and Lemma 8.3 it suffices to show

\[
E_{\Omega,\varepsilon,\eta}^{p,m+1}(B) E_{\Omega,\varepsilon,\eta}^{p,m} \subset E_{\Omega,\varepsilon,\eta}^{p,m} E_{\Omega,\varepsilon,\eta}^{p,m+1}(B).
\]

for \( p \in q^\mathbb{Z} \), \( m \in \mathbb{Z} \), \( \varepsilon, \eta \in \{-, +\} \). Then, by (8.3) and Lemma 8.3, we get —being careful regarding the domains involved—

\[
2 \Omega_{p,m+1}^{\varepsilon,\eta} E_{p,m}^{\varepsilon,\eta} = \left[ (q - q^{-1})^2 (E_{p,m+1}^{\varepsilon,\eta})^* E_{p,m+1}^{\varepsilon,\eta} - (q^{2m+3} p + q^{-2m-3} p^{-1}) \text{Id} \right] E_{p,m}^{\varepsilon,\eta}
= \left[ (q - q^{-1})^2 E_{p,m}^{\varepsilon,\eta} (E_{p,m}^{\varepsilon,\eta})^* + (q - q^{-1}) (q^{2m+2} p - q^{-2m-2} p^{-1}) \text{Id} - (q^{2m+3} p + q^{-2m-3} p^{-1}) \text{Id} \right] E_{p,m}^{\varepsilon,\eta}
= \left[ (q - q^{-1})^2 E_{p,m}^{\varepsilon,\eta} (E_{p,m}^{\varepsilon,\eta})^* - (q^{2m+1} p + q^{-2m-1} p^{-1}) \text{Id} \right] E_{p,m}^{\varepsilon,\eta}
= E_{p,m}^{\varepsilon,\eta} \left[ (q - q^{-1})^2 (E_{p,m}^{\varepsilon,\eta})^* E_{p,m}^{\varepsilon,\eta} - (q^{2m+1} p + q^{-2m-1} p^{-1}) \text{Id} \right]
= 2 E_{p,m}^{\varepsilon,\eta} \Omega_{p,m}^{\varepsilon,\eta}.
\]

Take the polar decomposition \( E_{p,m}^{\varepsilon,\eta} = \hat{U}_{p,m}^{\varepsilon,\eta} |E_{p,m}^{\varepsilon,\eta}| \). Since

\[
|E_{p,m}^{\varepsilon,\eta}| = \left( (E_{p,m}^{\varepsilon,\eta})^* E_{p,m}^{\varepsilon,\eta} \right)^{1/2} : \mathcal{K}(p, m, \varepsilon, \eta) \to \mathcal{K}(p, m, \varepsilon, \eta),
\]

\[
\hat{U}_{p,m}^{\varepsilon,\eta} : \mathcal{K}(p, m, \varepsilon, \eta) \to \mathcal{K}(p, m + 1, \varepsilon, \eta),
\]

(8.3) implies that \( |E_{p,m}^{\varepsilon,\eta}| \) and \( \Omega_{p,m}^{\varepsilon,\eta} \) strongly commute. Choose \( v \in D(|E_{p,m}^{\varepsilon,\eta}|^3) \). So \( v \in D(\hat{U}_{p,m}^{\varepsilon,\eta} |E_{p,m}^{\varepsilon,\eta}|) \cap D(\hat{U}_{p,m}^{\varepsilon,\eta} |\Omega_{p,m}^{\varepsilon,\eta}|) \) by (8.3), implying that \( \Omega_{p,m}^{\varepsilon,\eta} (|E_{p,m}^{\varepsilon,\eta}| v) = |E_{p,m}^{\varepsilon,\eta}| (\Omega_{p,m}^{\varepsilon,\eta} v) \). Since
v ∈ D(E^{ε,η}_{p,m} Ω^{ε,η}_{p,m}), (8.3) implies that \( \tilde{U}^{ε,η}_{p,m} (|E^{ε,η}_{p,m}| v) \in D(Ω^{ε,η}_{p,m+1}) \) and
\[
Ω^{ε,η}_{p,m+1} \tilde{U}^{ε,η}_{p,m} (|E^{ε,η}_{p,m}| v) = \tilde{U}^{ε,η}_{p,m} |E^{ε,η}_{p,m}| Ω^{ε,η}_{p,m} v = \tilde{U}^{ε,η}_{p,m} Ω^{ε,η}_{p,m} (|E^{ε,η}_{p,m}| v).
\]
If \( w \in \text{Ker}[E^{ε,η}_{p,m}] \), then by (8.3) \( Ω^{ε,η}_{p,m} w = -(q^{2m+1}p + q^{-2m-1}p^{-1}) w \), thus \( \tilde{U}^{ε,η}_{p,m} Ω^{ε,η}_{p,m} w = 0 = Ω^{ε,η}_{p,m+1} \tilde{U}^{ε,η}_{p,m} w \).

Now \( \text{Ker}[E^{ε,η}_{p,m}] + [\text{Im}|E^{ε,η}_{p,m}| \cap D((E^{ε,η}_{p,m})^*E^{ε,η}_{p,m})] \) is a core for \((E^{ε,η}_{p,m})^*E^{ε,η}_{p,m}\) and thus for \( Ω^{ε,η}_{p,m} \), as follows by using the spectral decomposition of \( |E^{ε,η}_{p,m}| \). Consequently the above results and the closedness of \( Ω^{ε,η}_{p,m+1} \) imply that \( \tilde{U}^{ε,η}_{p,m} Ω^{ε,η}_{p,m} \subseteq Ω^{ε,η}_{p,m+1} \tilde{U}^{ε,η}_{p,m} \). Now \( |E| = \bigoplus |E^{ε,η}_{p,m}| \), and \( \tilde{U} = \bigoplus \tilde{U}^{ε,η}_{p,m} \) give the polar decomposition \( E = \tilde{U} |E| \), see Appendix A.3, and we get \( \tilde{U} Ω \subseteq Ω \tilde{U} \), hence \( \tilde{U} E_{Ω}(B) = E_{Ω}(B) \tilde{U} \) for any Borel set \( B \subseteq \mathbb{R} \) by the spectral theorem. It follows that \( E \) and \( Ω \) strongly commute. \( \square \)

8.2. **Graded commutation relations for the Casimir operator.** This subsection is devoted to the proof of Proposition 4.8. The first statement of this proposition is an immediate consequence of Proposition 4.9, which we already proved in Section 7.3. Recall the subspaces \( M_+, M_- \subset \hat{M} \) defined in Definition 4.4. Note that Proposition 4.4 implies that \( M_\pm \) is the strong-* closure of
\[
\text{Span}\{Q(p_1, p_2, n) \mid p_1, p_2 \in I_q, n \in \mathbb{Z} \text{ so that } \text{sgn}(p_1 p_2) = \pm\}
\] (8.6)
Next we investigate the graded commutation relations of the Casimir operator \( Ω \) with the elements \( Q(p_1, p_2, n) \) generating the von Neumann algebra \( \hat{M} \), see Lemma 7.3, as stated in Proposition 4.8. The hard computations are contained in the following lemma, whose proof is postponed to Appendix D.2.

**Lemma 8.9.** For \( u, v \in K_0 \), \( p_1, p_2 \in I_q \) and \( n \in \mathbb{Z} \), we have
\[
\langle Q(p_1, p_2, n) u, Ω_0 v \rangle = \text{sgn}(p_1 p_2) \langle Q(p_1, p_2, n) Ω_0 u, v \rangle.
\]

**Lemma 8.10.** Let \( x \in \hat{M}_+ \) and \( y \in \hat{M}_- \), then \( x Ω_0 \subset Ω x \) and \( y Ω_0 \subset −Ω y \).

**Proof.** Consider \( p_1, p_2, p, t \in I_q, n, m \in \mathbb{Z} \). From Lemma 8.9 it follows that the vector \( v = Q(p_1, p_2, n) f_{m,p,t} \) belongs to \( D(Ω^n_0) \) and
\[
Ω^n_0 v = \text{sgn}(p_1 p_2) Q(p_1, p_2, n) Ω_0 f_{m,p,t}.
\]
By Lemma 7.3 the vector \( v \in K(p, m+n, ε\text{sgn}(p_1), η\text{sgn}(p_2)) \), and if \( m+n = 0, ε\text{sgn}(p_1) = +, η\text{sgn}(p_2) = + \), there exists by Lemma 7.1 a function \( h : \mathbb{R}_{≥0} → \mathbb{C} \) that is differentiable in 0 and satisfies \( v(x) = x^{-1} h(x^{-2}) \) for all \( x \in I_q^+ \). From Lemma 8.6 we now conclude that \( v \in D(Ω) \) and that \( Ω v = Ω^n_0 v \), hence
\[
\text{sgn}(p_1 p_2) Q(p_1, p_2, n) Ω_0 \subset Ω Q(p_1, p_2, n).
\]
Now for \( x \in \hat{M}_+ \) and \( y \in \hat{M}_- \) the lemma follows from the closedness of \( Ω \) and (8.6) \( \square \)

We need to improve the commutation relations from Lemma 8.10 to come to the second statement of Proposition 4.8. To do this we need the following lemma.
Lemma 8.11. Consider a Hilbert space $H$, a self-adjoint operator $A$ in $H$ and a partial isometry $U$ on $H$ for which the final projection $UU^*$ commutes with $A$. Then $U^*AU$ is self-adjoint.

Proof. First we show that $U^*AU$ is densely defined. Set $P = U^*U$ and $Q = UU^*$. Since $QA \subseteq AQ$, we have that $U(U^*D(A)) = QD(A) \subseteq D(A)$ implying that $U^*D(A) \subseteq D(U^*AU)$. Clearly, $(1 - P)H \subseteq D(U^*AU)$ thus $U^*D(A) + (1 - P)H \subseteq D(U^*AU)$ from which it follows that $U^*AU$ is densely defined.

Next we need to verify the self-adjointness. Let $v, w \in D(U^*AU)$, then, since $A$ is self-adjoint,

$$\langle U^*AUv, w \rangle = \langle AUv, w \rangle = \langle Uv, AW \rangle = \langle v, U^*AUw \rangle.$$ 

Thus, $U^*AU$ is symmetric. To prove that $U^*AU$ is self-adjoint, choose $v \in D((U^*AU)^*)$. If $w \in D(A)$, then $Qw \in D(A)$ and $A(Qw) = Q(Aw)$. Thus,

$$\langle Uv, Aw \rangle = \langle v, U^*Aw \rangle = \langle v, U^*QAw \rangle = \langle v, U^*AQw \rangle = \langle v, (U^*AU)U^*w \rangle = \langle (U^*AU)^*v, U^*w \rangle = \langle U(U^*AU)^*v, w \rangle.$$ 

This implies $Uv \in D(A^*) = D(A)$, so that $v \in D(U^*AU)$. From this we conclude that $(U^*AU)^* = U^*AU$. 

We are now in a position to prove the graded commutation relations of the Casimir.

Proposition 8.12. Let $x \in \hat{M}_+$ and $y \in \hat{M}_-$, then $x \Omega \subset \Omega x$ and $y \Omega \subset -\Omega y$.

Proof of Proposition 8.12. By Proposition 4.8, already established in Section 7.3, we obtain the decomposition $M = \hat{M}_+ \oplus \hat{M}_-$. The final statement of Proposition 4.8 is Proposition 8.12.

Proof of Proposition 8.12. First we deal with $\hat{M}_+$. Choose a unitary $u \in \hat{M}_+$. From Lemma 8.10 we know that $u \Omega_0 \subseteq \Omega u$, thus $\Omega_y \subseteq u^* \Omega u$. Since $u^* \Omega u$ is a self-adjoint extension of $\Omega_y$ that is affiliated with $\hat{M}$, Proposition 8.7 guarantees that $\Omega = u^* \Omega u$, or in other words, $u \Omega = \Omega u$. Since each element in $\hat{M}_+$ is a linear combination of such unitary elements, we get that $x \Omega \subseteq \Omega x$ for all $x \in \hat{M}_+$, proving the first statement.

Next choose $y \in \hat{M}_-$ and consider the polar decomposition $y = v |y|$ of $y$. We are going to show that $v \in \hat{M}_-$. Since $y^* \in \hat{M}_-$, the operator $y^*y$ is in the von Neumann algebra $\hat{M}_+$, hence $|y| = (y^*y)^{\frac{1}{2}} \in \hat{M}_+$. Take $e \in \mathcal{K}_+$. Since $|y| \mathcal{K}_+ \subseteq \mathcal{K}_+$, there exists $e_1 \in \overline{|y| \mathcal{K}_+}$ and $e_2 \in \mathcal{K}_+$ with $e_2 \perp |y| \mathcal{K}_+$ so that $e = e_1 + e_2$. Since also $|y| \mathcal{K}_- \subseteq \mathcal{K}_-$, we see that $e_2 \perp |y| \mathcal{K}_-$, implying that $ve = ve_1 + ve_2 = ve_1$, since $v$ acts as zero on $(\text{Im}|y|)$. Because $y = v |y|$ and $y \mathcal{K}_+ \subseteq \mathcal{K}_+$, it follows that $ve \in \mathcal{K}_-$, similarly, $v \mathcal{K}_- \subseteq \mathcal{K}_+$. Hence, $v \in \hat{M}_-$. It follows that the initial projection $p = v^*v$ and final projection $q = vv^*$ belong to $\hat{M}_+$. This implies that $p \Omega \subseteq \Omega p$ and $q \Omega \subseteq \Omega q$ by the first part of this proposition.

Because $v \in \hat{M}_-$, we have that $v \Omega_0 \subseteq -\Omega v$ by Lemma 8.10, implying that $p \Omega_0 \subseteq -v^* \Omega v$. We also have that $(1 - p) \Omega_0 \subseteq \Omega (1 - p)$. Thus, $\Omega_0 \subseteq -v^* \Omega v + \Omega (1 - p)$.

Since the final projection of $v$ commutes with the self-adjoint operator $\Omega$, the operator $v^* \Omega v$ is also self-adjoint by Lemma 8.11. Because of the same reason, $\Omega (1 - p)$ is self-adjoint.
Therefore, as the orthogonal sum of self-adjoint operators, the operator $-v^* \Omega v + \Omega (1 - p)$ is a self-adjoint extension of $\Omega$.

Since $\Omega$ is affiliated to $\hat{M}$ and $v, p \in \hat{M}$, one sees that $-v^* \Omega v + \Omega (1 - p)$ is affiliated to $\hat{M}$. Hence, $\Omega = -v^* \Omega v + \Omega (1 - p)$ by Proposition 8.7. If $e \in D(\Omega)$, this equality implies that $ve \in D(\Omega)$, $(1 - p)e \in D(\Omega)$ and $\Omega e = -v^* \Omega ve + \Omega (1 - p)e$. Thus, using $vp = v, qv = v$,

$$v \Omega e = -q \Omega ve + v p \Omega (1 - p)e = -\Omega q ve + v \Omega p (1 - p)e = -\Omega q ve = -\Omega ve.$$

Thus, we have proved that $v \Omega \subseteq -\Omega v$. Since $y = v |y|$ we conclude that $y \Omega \subseteq -\Omega y$. □

8.3. Spectral decomposition of the Casimir operator. From (the proof of) Lemma 8.10 it follows that $Q(p_1, p_2, n)$ maps eigenvectors for eigenvalue $x$ of $\Omega$ in $\mathcal{K}(p, m, \varepsilon, \eta)$ to multiples of eigenvectors of $\Omega$ in $\mathcal{K}(p, m + n, \text{sgn}(p_1)\varepsilon, \text{sgn}(p_2)\eta)$ for the eigenvalue $\text{sgn}(p_1 p_2)x$ or to zero.

So, it will be convenient to have an alternative description of the GNS-space $\mathcal{K}$ corresponding to the spectral decomposition of $\Omega$. This alternative description has the advantage that the action of the operators $E$ and, of course, $\Omega$, is far more transparent. Moreover, it leads to the direct integral decomposition of the left regular corepresentation of $(\mathcal{M}, \Delta)$ into irreducible unitary representations, see Section 7.1.

The description of the spectral decomposition of $\Omega$ relies on certain special functions which can be written in terms of basic hypergeometric series: the Al-Salam–Chihara polynomials and the little $q$-Jacobi functions. The main properties of these special functions needed in this subsection are given in Appendices B.4 and B.5. The spectral decomposition of the Casimir operator $\Omega$ immediately leads to the decomposition of the GNS-space $\mathcal{K}$ as a $U_q(\mathfrak{su}(1, 1))$-module. This is done in Section 8.4.

The Casimir operator $\Omega$ is a self-adjoint extension of $\Omega_0 \in \mathcal{L}^+(\mathcal{K}_0)$. Let $p \in q^\mathbb{Z}$, $m \in \mathbb{Z}$, $\varepsilon, \eta \in \{-, +\}$. It follows from (4.8) that $\Omega_0|_{\mathcal{K}_0(p, m, \varepsilon, \eta)}$ is basically a Jacobi operator, i.e., a tridiagonal operator on $\ell^2(\mathbb{N}_0)$ or $\ell^2(\mathbb{Z})$. The spectral decomposition of these specific Jacobi operators can be described in terms of Al-Salam–Chihara polynomials in case of $\ell^2(\mathbb{N}_0)$, and in terms of little $q$-Jacobi functions in case of $\ell^2(\mathbb{Z})$. Whether $\mathcal{K}_0(p, m, \varepsilon, \eta)$ can be identified with $\ell^2(\mathbb{N}_0)$ or $\ell^2(\mathbb{Z})$ depends on the sign of the parameters $\varepsilon$ and $\eta$, see the beginning of Section 8.1. We need to distinguish between four different cases.

Let us recall from (4.7) that the modular conjugation $J : \mathcal{K} \to \mathcal{K}$, defined by $J : f_{m, p, t} \mapsto f_{-m, t, p}$, satisfies that $E_0^2J = -E_0J$ and $JK_0 = K_0^{-1}J$, and consequently $J\Omega_0 = \Omega_0J$.

Note that $J : \mathcal{K}(p, m, \varepsilon, \eta) \to \mathcal{K}(p^{-1}, -m, \varepsilon, \eta)$, since $J f_{-m, \eta q^{-m}p^{-1}y, y} = f_{m, \eta q^{-m}p^{-1}y, y}$ with $y = \varepsilon q^m p z$ and $\text{sgn}(y) = \eta$. We will use this to reduce the number of cases that we need to consider.

8.3.1. The case $\varepsilon = +$ and $\eta = -$. Recall that $\mathcal{K}(p, m, \varepsilon, \eta) \cong \ell^2(J(p, m, \varepsilon, \eta))$. In the case under consideration,

$$J(p, m, +, -) = \{ z \in I_q \mid -q^m p z \in I_q, \text{sgn}(z) = + \}$$

can be labeled by $\mathbb{N}_0$ using $n = m + \chi(p) + \chi(z) - 1$. Now put $e_n = f_{-m, \eta q^m p z, z}$ using this identification, then (4.8) leads to

$$2\Omega_0 e_n = \sqrt{(1 - q^{2n+2})(1 + p^{-2}q^{2n+2-2m})} e_{n+1} + p^{-1}q^{2n+1-2m}(q^{2m} - 1) e_n + \sqrt{(1 - q^{2n})(1 + p^{-2}q^{2n-2m})} e_{n-1}. \quad (8.7)$$
Comparing this with the Jacobi operator $J(a, b \mid q)$ for the Al-Salam–Chihara polynomials, 
(B.13), see also (B.11), we see that $2\Omega_0 = J(q/p, -q^{1-2m}/p \mid q^2)$. By Theorem [B.13] and 
(B.13) $2\Omega_0$ extends uniquely to a bounded self-adjoint operator on $K(p, m, +, -)$, and it has 
continuous spectrum $[-1, 1]$ and discrete spectrum $\sigma_d(p, m, +, -) = \mu(D(p, m, +, -))$ where 
$D(p, m, +, -) = D(q/p, -q^{1-2m}/p|q^2)$, using the notation of (B.17). The multiplicity of the 
(generalized) eigenspaces is one.

Let $I(p, m, +, -) = I(q/p, -q^{1-2m}/p|q^2)$, see (B.17). We define the operator 
\[
\Upsilon_{p,m}^{+,-}: K(p, m, +, -) \to L^2(I(p, m, +, -)),
\]
\[
f_{-m,-pq^{m+p},z} \mapsto g_{z}(\cdot\mid p, m, +, -) = (-1)^m h_{m + \chi(p) + \chi(z) - 1}(\cdot\mid p, -q^{1-2m}/p|q^2)
\]
in terms of Al-Salam–Chihara polynomials using the notation as in (B.18). Then $\Upsilon_{p,m}^{+,-}$ gives 
the spectral decomposition of the action of the Casimir operator on $K(p, m, +, -)$, so $\Upsilon_{p,m}^{+,-}$ 
is a unitary intertwiner of the Casimir operator with the multiplication operator $M(x)$ on 
$L^2(I(p, m, +, -))$. Here, and elsewhere, $M(g)$ denotes the operator of multiplication by the 
factor $-1)^m$ in (8.8) is not of importance for the spectral decomposition of the Casimir operator, 
but is inserted in order to avoid signs later on when we decompose $K$ as a $U_q(\mathfrak{su}(1,1))$-module.

8.3.2. The case $\varepsilon = -$ and $\eta = +$. Using the modular conjugation $J$, the case $\varepsilon = -$ 
and $\eta = +$ can be reduced to the case $\varepsilon = +$ and $\eta = -$. Define $I(p, m, -,+) = I(p^{-1}, -m, +, -) = 
I(pq, -pq^{1+2m}|q^2)$ using the notation (B.17), then 
\[
\Upsilon_{p,m}^{+,-} = (-1)^m \Upsilon_{p,1-m}^{+,-} \circ J: K(p, m, -,+) \to L^2(I(p, m, -,+)),
\]
\[
f_{-m,-pq^{m+p},z} \mapsto g_{z}(\cdot\mid p, m, -,+) = h_{\chi(z) - 1}(\cdot\mid pq, -pq^{1+2m}|q^2)
\]
gives the intertwiner of the action of the Casimir operator $\Omega_0: K(p, m, -,+) \to K(p, m, -,+)$ 
with $M(x)$. As before $\Omega_0$ has a unique extension to a bounded self-adjoint operator with 
multiplicity one for the (generalized) eigenspaces.

Combining Sections 8.3.1 and 8.3.2 we see that for $\varepsilon, \eta \in \{-, +\}$, $\varepsilon \neq \eta$, we have the 
following description of the discrete spectrum $\mu(D(p, m, \varepsilon, \eta))$:
\[
D(p, m, \varepsilon, \eta) = \{ q^{1+2r}p^{-\varepsilon} \mid r \in \mathbb{N}_0, q^{1+2r}p^{-\varepsilon} > 1 \}
\]
\[
\cup \{ -q^{1+2r}p^{-\varepsilon} \mid r \in \mathbb{N}, r \geq -\varepsilon m, q^{1+2r}p^{-\varepsilon} > 1 \}.
\]

8.3.3. The case $\varepsilon = -$ and $\eta = -$. In this case the $q$-interval $J(p, m, -,-) = \{ z \in I_q \mid 
q^{m+p}z \in I_q, \text{sgn}(z) = - \}$ can be labeled by $\mathbb{N}_0$. If we put $z = -q^{n+1}$, $n \in \mathbb{N}_0$, then we need 
m + \chi(p) + n \in \mathbb{N}_0 in order to have $q^{m+p}z \in I_q$. So we have to consider two cases: 
m + \chi(p) \geq 0 and $m + \chi(p) \leq 0$. Since the modular conjugation $J$ changes the sign of $m + \chi(p)$ we can 
restrict to $m + \chi(p) \geq 0$, and obtain the other case using $J$.

We assume $m + \chi(p) \geq 0$ and put $z = -q^{n+1}$, $n \in \mathbb{N}_0$, so that $n$ labels $J(p, m, -,-)$. Put 
e_n = f_{-m,-pq^{m+p},-n+1}, then the expression (4.9) for $\Omega_0f_{mpt}$ gives 
\[
2(-\Omega_0)e_n = \sqrt{(1 - q^{2n+2})(1 - p^2q^{2n+2} + 2)}e_{n+1} + pq^{2n+1}(1 + q^{2m})e_n
\]
\[
+ \sqrt{(1 - q^{2n})(1 - p^2q^{2m+2})}e_{n-1}.
\]
which we recognize using (B.13) and (B.11) as the Jacobi operator \( J(pq, pq^{1+2m} \mid q^2) \) for the Al-Salam–Chihara polynomials. So \( \Omega_0 \) uniquely extends to a bounded self-adjoint operator. Put \( I(p, m, -, -) = -I(pq, pq^{1+2m} \mid q^2) \), see (B.14), then

\[
\forall_{p, m, -} : \mathcal{K}(p, m, -) \rightarrow L^2(I(p, m, -)),
\]

\[
f_{-m, q^m p^z, z} \mapsto g_z(\cdot ; p, m, -) = (-1)^m h_{\chi(z)}(- \cdot ; pq, pq^{1+2m} \mid q^2).
\]

Intertwines the action of the Casimir operator with the multiplication operator \( M(x) \) on \( L^2(I(p, m, -)) \) for \( m + \chi(p) \geq 0 \). Note that we take the normalized Al-Salam–Chihara polynomials with a minus sign in front of the argument because of the minus sign in front of \( \Omega_0 \) in (8.11).

In case \( m + \chi(p) \leq 0 \) we define \( I(p, m, -) = I(p^{-1}, -m, -) \) and

\[
\forall_{p, m} = (-1)^{m+\chi(p)} \forall_{p^{-1}, -m} \circ J : \mathcal{K}(p, m, -) \rightarrow L^2(I(p, m, -)),
\]

\[
f_{-m, pq^m z, z} \mapsto g_z(\cdot ; p, m, -) = (-1)^{\chi(p)} h_{m+\chi(p)+\chi(z)-1}(- \cdot ; q/p, q^{1-2m}/p \mid q^2).
\]

This gives two definitions in case \( m + \chi(p) = 0 \) or \( q^n p = 1 \), and it is straightforward to check that they coincide. Now we have the intertwiner for the action of the Casimir operator with the multiplication operator \( M(x) \) on \( L^2(I(p, m, -)) \) for all \( m \in \mathbb{Z} \) and \( p \in q^\mathbb{Z} \). Let us remark that the discrete spectrum \( \mu(D(p, m, -)) \) is given explicitly by

\[
D(p, m, -) = \begin{cases} 
\{ -q^{1+2r} p \mid r \in \mathbb{N}_0, q^{1+2r} p > 1 \} \\
\cup \{ -q^{1+2(r+m)} p \mid r \in \mathbb{N}_0, q^{1+2(r+m)} p > 1 \}, & pq^m \leq 1, \\
\{ -q^{1+2r} p^{-1} \mid r \in \mathbb{N}_0, q^{1+2r} p^{-1} > 1 \} \\
\cup \{ -q^{1+2(r-m)} p^{-1} \mid r \in \mathbb{N}_0, q^{1+2(r-m)} p^{-1} > 1 \}, & pq^m \geq 1.
\end{cases}
\]

In both cases at most one of the two sets is non-empty.

8.3.4. The case \( \varepsilon = + \) and \( \eta = + \). In this case \( J(p, m, +, +) \) can be labeled by \( \mathbb{Z} \). We put \( z = q^n, n \in \mathbb{Z} \), and \( e_n = f_{-m, pq^n + m, q^n} \), then (8.8) gives

\[
2(-\Omega_0) e_n = \sqrt{(1 + q^{2n})(1 + p^2 q^{2m+2n})} e_{n+1} - pq^{2n-1}(1 + q^{2m}) e_n \\
+ \sqrt{(1 + q^{2n-2})(1 - p^2 q^{2m+2n-2})} e_{n-1}.
\]

Comparing this with (B.33) we recognize \( -2\Omega_0 \) as the (doubly infinity) Jacobi operator \( \Lambda(q^{2-2m}, q^{-2m}/p, -q^2 \mid q^2) \) for the little \( q \)-Jacobi functions. Let us remark that there are other choices for the parameters which, of course, all lead to the same result; we can identify \( -2\Omega_0 \) also with \( \Lambda(q^{2m+2}, q/p, -q^2 \mid q^2) \), \( \Lambda(q^{2m+2}, pq^{m+1}, -q^{2-2m}/p^2 \mid q^2) \) or \( \Lambda(q^{2-2m}, pq, -q^{2-2m}/p^2 \mid q^2) \). Because of these symmetries we can obtain the spectral decomposition of a self-adjoint extension of \( \Omega_0 \) in the case \( m > 0 \) from the case \( m < 0 \).

Let us first assume that \( m \leq 0 \). By Theorem B.15 the unbounded operator \( \Omega_0 \) is essentially self-adjoint for \( m < 0 \), so in this case \( \Omega_0 \) has a unique self-adjoint extension \( C \). The spectral decomposition of \( C \) is described in Theorem B.15. For \( m = 0 \) we choose the self-adjoint extension \( C \) of \( \Omega \mid_{\mathcal{K}(p, 0, +, +)} \) with spectral decomposition as described in Theorem C.1. The multiplicity of the (generalized) eigenspaces is one. Put \( I(p, m, +, +) = -I(q^{2-2m}, p^{-1} q^{1-2m}, -q^2 \mid q^2) \) using

\[
\forall_{p, m} : \mathcal{K}(p, m, +, +) \rightarrow L^2(I(p, m, +, +)),
\]

\[
f_{-m, pq^m z, z} \mapsto g_z(\cdot ; p, m, +, +) = (-1)^m h_{\chi(z)}(- \cdot ; pq, pq^{1+2m} \mid q^2)
\]
the notation as in (3.36), then for \( m \leq 0 \)
\[
\Upsilon_{p,m}^{+,+} : \mathcal{K}(p,m,+,+) \to L^2(I(p,m,+,+)),
\]
\[
f_{-m,q^m z\epsilon,p m} \mapsto g_z(\cdot;p,m,+,+) = (-1)^m j_{\chi(z)}(-\cdot;q^{2-2m},p^{-1}q^{1-2m};-q^2 | q^2)
\]
intertwines the action of the Casimir operator with the multiplication operator \( M(x) \) on \( L^2(I(p,m,+,+)) \), using the notation (3.37). To this end we need to argue that \( C \) agrees with \( \Omega_{p,m}^{+,+} \), which is clear in the case \( m \neq 0 \). For \( m = 0 \) we recall from the proof of Lemma 8.3 that \( \Omega_{p,0}^{+,+} \) is the self-adjoint extension of \( \Omega_0|_{\mathcal{K}(p_0,0,+,+)} \) described in Theorem C.1, as is \( C \).

Note that we take in (8.16) the normalized little \( q \)-Jacobi functions with a minus sign in front of the argument because of the minus sign in front of \( \Omega_0 \) in (8.14).

For \( m \geq 0 \) define \( I(p,m,+,+) = I(p^{-1},-m,+,+) \) and
\[
\Upsilon_{p,m}^{+,+} = (-1)^m \Upsilon_{p^{-1},-m}^{+,+} \circ U : \mathcal{K}(p,m,+,+) \to L^2(I(p,m,+,+)),
\]
\[
f_{m,pq^m z\epsilon,p m} \mapsto g_z(\cdot;p,m,+,+) = j_{m+\chi(p)+\chi(z)}(-\cdot;q^{2+2m},pq^{1+2m};-q^2 | q^2).
\]

Note that this corresponds to the symmetry of the corresponding Jacobi operator,
\[
L(q^{2-2m},q^{-2m}/p,-q^2 | q^2) = L(q^{2+2m},pq^{1+2m},-q^2 | q^2),
\]
as we observed earlier. As before, for \( m \geq 0 \) the operators \( \Upsilon_{p,m}^{+,+} \) intertwine the action of the Casimir operator with the multiplication operator \( M(x) \) on \( L^2(I(p,m,+,+)) \).

It may seem that we now have two definitions for \( \Upsilon_{p,0}^{+,+} \), but it follows from (3.48) that they coincide.

Finally, let us give an explicit description of the discrete spectrum \( \mu(D(p,m,+,+)) \):
\[
D(p,m,+,+) = \{ q^{1+2k} p | k \in \mathbb{Z}, q^{1+2k} p > 1 \} \cup \{ -q^{1+2r} p | r \in \mathbb{Z}, r \geq \max\{0,m\}, q^{1+2r} p > 1 \} \cup \{ -q^{1+2r} p^{-1} | r \in \mathbb{Z}, r \geq \max\{0,-m\}, q^{1+2r} p^{-1} > 1 \}.
\]
The last two sets are finite and at most one of them is non-empty, while the first set is infinite.

8.3.5. The spectral decomposition of \( \Omega \). Gathering the results from the four different cases \( \varepsilon = \pm \) and \( \eta = \pm \), we obtain the spectral decomposition of the Casimir operator \( \Omega \).

**Theorem 8.13.** There exists a unique unitary operator
\[
\Upsilon : \mathcal{K} \to \bigoplus_{p \in \mathbb{Z}, m \in \mathbb{Z}, \varepsilon, \eta \in \{\pm\}} L^2(I(p,m,\varepsilon,\eta)),
\]
\[
\Upsilon(f_{m,\varepsilon,\eta q^m z\epsilon,p m}) = g_z(\cdot;p,m,\varepsilon,\eta), \quad \forall z \in J(p,m,\varepsilon,\eta),
\]
so that for \( p \in q^Z, m \in \mathbb{Z} \) and \( \varepsilon, \eta \in \{\pm\} \), we have \( \Upsilon(\mathcal{K}(p,m,\varepsilon,\eta)) = L^2(I(p,m,\varepsilon,\eta)) \).

Let \( \Upsilon_{p,m}^{\varepsilon,\eta} : \mathcal{K}(p,m,\varepsilon,\eta) \to L^2(I(p,m,\varepsilon,\eta)) \) be the restriction of \( \Upsilon \) to \( \mathcal{K}(p,m,\varepsilon,\eta) \). Then, for \( p \in q^\mathbb{Z}, m \in \mathbb{Z} \) and \( \varepsilon, \eta \in \{\pm\}, \)
\[
\Upsilon_{p,m}^{\varepsilon,\eta} \cdot \Omega_{p,m}^{\varepsilon,\eta} (\Upsilon_{p,m}^{\varepsilon,\eta})^* = M(x) \quad \text{on } L^2(I(p,m,\varepsilon,\eta)).
\]

Here \( I(p,m,\varepsilon,\eta) = [-1,1] \cup \sigma_d(p,m,\varepsilon,\eta) \) with \( \sigma_d(p,m,\varepsilon,\eta) = \mu(D(p,m,\varepsilon,\eta)) \) and with \( D(p,m,\varepsilon,\eta) \) given in Section 8.3 for the various choices of \( p, m, \varepsilon \) and \( \eta \).
8.4. The decomposition of the GNS-space $\mathcal{K}$ as a $U_q(\mathfrak{su}(1, 1))$-module. The (generalized) eigenspaces of the Casimir operator correspond to invariant subspaces under the action of $U_q(\mathfrak{su}(1, 1))$. In this way, the spectral decomposition of $\Omega$ from Section 5.3 leads to the decomposition of the GNS-space $\mathcal{K}$ into irreducible $*$-representations of $U_q(\mathfrak{su}(1, 1))$. Let us first recall these representations of $U_q(\mathfrak{su}(1, 1))$.

The $*$-representations of $U_q(\mathfrak{su}(1, 1))$ require unbounded operators, and for this we use the theory as developed in [48, Ch. 8]. In particular this means that for such a representation $\pi$ in a Hilbert space $V$ there exists a common dense domain $\mathcal{D} \subset V$, which is invariant for $\pi(X)$ for all $X \in U_q(\mathfrak{su}(1, 1))$, such that the relations of (4.1) remain valid when acting on $v \in \mathcal{D}$. Moreover, we require $\langle \pi(X)v, w \rangle_V = \langle v, \pi(X^*)w \rangle_V$ for all $v, w \in \mathcal{D}$. It follows that each $\pi(X)$, $X \in U_q(\mathfrak{su}(1, 1))$, is closable.

Admissible representations of $U_q(\mathfrak{su}(1, 1))$ are $*$-representations in a Hilbert space $V$ acting by unbounded operators, such that $V$ decomposes into finite-dimensional eigenspaces for the action of $\mathfrak{K}$, and such that the eigenvalues of $\mathfrak{K}$ are of the form $q^k$, $k \in \frac{1}{2}\mathbb{Z}$. Then the following irreducible admissible representations exhaust the list, see e.g. [49], [13]. In each of these cases the common invariant dense domain is the subspace of finite linear combinations of the basis vectors $e_n$.

Note that each of these admissible irreducible representations is completely determined by the eigenvalue of the Casimir operator $\Omega$ on $V$ and the spectrum of $\mathfrak{K}$.

Positive discrete series. The representation space is $l^2(\mathbb{N}_0)$ with orthonormal basis $\{e_n\}_{n \in \mathbb{N}_0}$. Let $k \in \frac{1}{2}\mathbb{N}$, define the action of the generators by

\[
\mathbf{K} \cdot e_n = q^{k+n} e_n, \quad \mathbf{K}^{-1} \cdot e_n = q^{-k-n} e_n, \\
(q^{-1} - q) \mathbf{E} \cdot e_n = q^{\frac{1}{2}k-n} \sqrt{(1-q^{2n+2})(1-q^{4k+2n})} e_{n+1}, \\
(q^{-1} - q) \mathbf{F} \cdot e_n = q^{\frac{1}{2}k-n} \sqrt{(1-q^{2n})(1-q^{4k+2n-2})} e_{n-1},
\]

with the convention $e_{-1} = 0$. This representation is denoted by $D_k^+$ and $D_k^+(\Omega) = -\mu(q^{1-2k})$.

Negative discrete series. The representation space is $l^2(\mathbb{N}_0)$ with orthonormal basis $\{e_n\}_{n \in \mathbb{N}_0}$. Let $k \in \frac{1}{2}\mathbb{N}$, and define the action of the generators by

\[
\mathbf{K} \cdot e_n = q^{-k-n} e_n, \quad \mathbf{K}^{-1} \cdot e_n = q^{k+n} e_n, \\
(q^{-1} - q) \mathbf{E} \cdot e_n = q^{\frac{1}{2}k-n} \sqrt{(1-q^{2n+2})(1-q^{4k+2n-2})} e_{n-1}, \\
(q^{-1} - q) \mathbf{F} \cdot e_n = q^{\frac{1}{2}k-n} \sqrt{(1-q^{2n+2})(1-q^{4k+2n})} e_{n+1},
\]

with the convention $e_{-1} = 0$. This representation is denoted by $D_k^-$ and $D_k^-(\Omega) = -\mu(q^{1-2k})$.

Principal series. The representation space is $l^2(\mathbb{Z})$ with orthonormal basis $\{e_n\}_{n \in \mathbb{Z}}$. Let $0 \leq b \leq \frac{\pi}{2\ln q}$ and $\varepsilon \in \{0, \frac{1}{2}\}$ and assume $(b, \varepsilon) \neq (0, \frac{1}{2})$. The action of the generators is defined by

\[
\mathbf{K} \cdot e_n = q^{n+\varepsilon} e_n, \quad \mathbf{K}^{-1} \cdot e_n = q^{-n-\varepsilon} e_n, \\
(q^{-1} - q) \mathbf{E} \cdot e_n = q^{\frac{1}{2}n-\varepsilon} \sqrt{(1-q^{2n+1+2\varepsilon+2ib})(1-q^{2n+1+2\varepsilon-2ib})} e_{n+1}, \\
(q^{-1} - q) \mathbf{F} \cdot e_n = q^{\frac{1}{2}n-\varepsilon} \sqrt{(1-q^{2n-1+2\varepsilon+2ib})(1-q^{2n-1+2\varepsilon-2ib})} e_{n-1}.
\]
We denote the representation by $\pi_{b,\varepsilon}$. In case $(b, \varepsilon) = (0, \frac{1}{2})$ this still defines an admissible unitary representation. It splits as the direct sum $\pi_{\pi_{b,\varepsilon}} \cong D_k^+ \oplus D_k^-$ of a positive and negative discrete series representation by restricting to the invariant subspaces span$\{e_n \mid n \geq 0\}$ and to span$\{e_n \mid n < 0\}$. We keep this convention for $\pi_{\pi_{b,\varepsilon}}$. Note that $\pi_{b,\varepsilon}(\Omega) = \mu(q^{2n}) = \cos(-2b \ln q)$.

**Strange series.** The representation space is $\ell^2(\mathbb{Z})$ with orthonormal basis $\{e_n\}_{n \in \mathbb{Z}}$. Let $\varepsilon \in \{0, \frac{1}{2}\}$, and $a > 0$. The action of the generators is defined by

$$
\begin{align*}
\mathbf{K} \cdot e_n &= q^{n+\varepsilon} e_n, \\
\mathbf{K}^{-1} \cdot e_n &= q^{-n-\varepsilon} e_n, \\
(q^{-1} - q) \mathbf{E} \cdot e_n &= q^{-n-\varepsilon-\frac{1}{2}} \sqrt{(1 + q^{2n+2\varepsilon+1+2n})(1 + q^{2n+2\varepsilon-2n+1})} e_{n+1}, \quad (8.23) \\
(q^{-1} - q) \mathbf{F} \cdot e_n &= q^{-n-\varepsilon+\frac{1}{2}} \sqrt{(1 + q^{2n+2\varepsilon-1+2n})(1 + q^{2n+2\varepsilon+2n-1})} e_{n-1}.
\end{align*}
$$

We denote this representation by $\pi^{\text{S}}_{a,\varepsilon}$. Note that $\pi^{\text{S}}_{a,\varepsilon}(\Omega) = \mu(q^{2n})$.

**Complementary series.** This series of representations acts in $\ell^2(\mathbb{Z})$. The actions of the generators follow from the action $(8.22)$ by putting $\varepsilon = 0$ and formally replacing $-\frac{1}{2} + ib$ by $\lambda$ and taking $-\frac{1}{2} < \lambda < 0$. This series of representations does not play a role in this paper.

We define $\tau(\mathbf{K}) = \mathbf{K}^{-1}$, $\tau(\mathbf{K}^{-1}) = \mathbf{K}$, $\tau(\mathbf{E}) = -\mathbf{F}$, $\tau(\mathbf{F}) = -\mathbf{E}$. From (4.4) we check that $\tau$ extends to an involutive algebra homomorphism $\tau: U_q(\mathfrak{su}(1, 1)) \to U_q(\mathfrak{su}(1, 1))$. From (4.4) it is clear that $\tau(\Omega) = \Omega$. Composing an irreducible admissible representation with the involutive algebra automorphism $\tau: U_q(\mathfrak{su}(1, 1)) \to U_q(\mathfrak{su}(1, 1))$ gives an admissible irreducible representation of $U_q(\mathfrak{su}(1, 1))$. This easily gives

$$
D_k^+ \circ \tau \cong D_k^-, \quad \pi_{b,\varepsilon} \circ \tau \cong \pi_{b,\varepsilon}, \quad \pi^{\text{S}}_{a,\varepsilon} \circ \tau \cong \pi^{\text{S}}_{a,\varepsilon}.
$$

Denoting the orthonormal bases in the representations on the left hand side of $(8.24)$ by $\{e_n\}$ we can describe the unitary intertwiners as $e_n^* \mapsto (-1)^n e_n$ in the first case and as $e_n^* \mapsto (-1)^n e_{-n-2\varepsilon}$ for the last two cases.

Recall that the modular conjugation $J: \mathcal{K} \to \mathcal{K}$ satisfies that $E_0^1 J = -E_0 J$ and $JK_0 = K_0^{-1} J$, and consequently $J\Omega_0 = \Omega_0 J$. This implies that $J$ implements the involutive algebra automorphism $\tau: U_q(\mathfrak{su}(1, 1)) \to U_q(\mathfrak{su}(1, 1))$.

The spectral decomposition of the Casimir operator $\Omega$ from Section 8.3 gives a decomposition of $\mathcal{K}$ into invariant subspaces for the action of $\Omega$. Let $p \in \mathbb{Z}$ and $\varepsilon, \eta \in \{-, +\}$. It follows from (7.3) that the space $\mathcal{K}_0(p, \varepsilon, \eta) = \bigoplus_{m \in \mathbb{Z}} \mathcal{K}_0(p, m, \varepsilon, \eta)$ is invariant for the action of $U_q(\mathfrak{su}(1, 1))$. We denote by $\pi_\mathcal{K}(p, \varepsilon, \eta)$ the representation of $U_q(\mathfrak{su}(1, 1))$ on $\mathcal{K}(p, \varepsilon, \eta)$. In the following we decompose $\pi_\mathcal{K}(p, \varepsilon, \eta)$ in terms of irreducible admissible $\varepsilon$-representations of $U_q(\mathfrak{su}(1, 1))$ using the spectral decomposition of $\Omega$: $\mathcal{K}(p, m, \varepsilon, \eta) \to \mathcal{K}(p, m, \varepsilon, \eta)$ from Section 8.3. As before, we have to distinguish four cases depending on the signs of $\varepsilon$ and $\eta$. It turns out that the representation label $\varepsilon$ for the principal and strange series representations occurring in the decomposition of $\pi_\mathcal{K}(p, \varepsilon, \eta)$, depends on the parameter $p$. For this reason we define $\varepsilon: q^\mathbb{Z} \to \{0, \frac{1}{2}\}$ by

$$
\varepsilon(p) = \frac{1}{2} \chi(p) \mod 1.
$$

(8.25)
8.4.1. The case $\varepsilon = +$ and $\eta = -$. In this case the spectral decomposition of the Casimir operator acting on $\mathcal{K}(p, m, +, -)$ is determined by (8.8). From the explicit action of $E_{0}^{\dagger}$ (4.4), (8.8) and Lemma [3.14] we obtain

\[
(q^{-1} - q) \ Y_{p,m-1}^{+,+} (E_{p,m-1}^{+})^{*} (Y_{p,m}^{+,+})^{*} = q^{\frac{1}{2}-m} p^{-\frac{1}{2}} M(\sqrt{1 + 2 x q^{2m-1} p + q^{4m-2} p^2}) : L^{2}(I(p, m, +, -)) \rightarrow L^{2}(I(p, m - 1, +, -)).
\]  

(8.26)

Note that $L^{2}(I(p, m, +, -)) = L^{2}(I(p, m - 1, +, -))$, unless $q^{1-2m}/p > 1$. In this case $I(p, m - 1, +, -) = I(p, m, +, -) \setminus \{\mu(-q^{1-2m}/p)\}$, and the multiplication operator is zero for the point $\mu(-q^{1-2m}/p)$. So the multiplication operator in (8.26) is well-defined.

From (7.6) we have $\Upsilon_{+}^{+,+} K (\Upsilon_{p,m}^{+,+})^{*} = q^{m} p^{\frac{1}{2}} I$ so (8.26) and (1.4) give

\[
(q^{-1} - q) \ Y_{p,m+1}^{+,+} E_{p,m}^{+} (Y_{p,m}^{+,+})^{*} = q^{\frac{1}{2}-m} p^{-\frac{1}{2}} M(\sqrt{1 + 2 x q^{2m+1} p + q^{4m+2} p^2}) : L^{2}(I(p, m, +, -)) \rightarrow L^{2}(I(p, m + 1, +, -)).
\]  

(8.27)

This can also be derived directly from a similar identity for the Al-Salam–Chihara polynomials.

$\mathcal{K}(p, +, -)$ is not an admissible representation of $U_{q}(\mathfrak{su}(1, 1))$, since the $K$-eigenspaces are not finite dimensional. However, since the actions of $E$ and $E^{*}$ in (8.26), (8.27) match the actions given in the list of irreducible $*$-representations for $U_{q}(\mathfrak{su}(1, 1))$, we can still determine the decomposition explicitly. The possible eigenvalues of the Casimir and the eigenvalues of $K$ then determine the decomposition. In Theorem [8.14] we deal with the positive discrete series representations, since $I(p, m - 1, +, -) \subset I(p, m, +, -)$ for $m$ large enough implying that $K$ acts as the creation operator. The direct integral and direct sums of representations of $U_{q}(\mathfrak{su}(1, 1))$ by unbounded operators in Theorem [8.14] uses the construction of [18, Ch. 8].

**Theorem 8.14.** The decomposition of $\pi_{\mathcal{K}}(p, +, -)$ into irreducible admissible $*$-representations is given by

\[
\pi_{\mathcal{K}}(p, +, -) \cong \int_{0}^{-\pi/2 \ln q} \pi_{b, \varepsilon(p)} db \oplus \bigoplus_{l \in \mathbb{Z}, \chi(p) > 1} D_{l + \frac{1}{2} \chi(p)} \oplus \bigoplus_{l \in \mathbb{N}_{0}, \chi(p) < -1} \pi_{q^{1+2l} \varepsilon(p)}.
\]

8.4.2. The case $\varepsilon = -$ and $\eta = +$. In Section [8.3.2] we obtained the spectral decomposition of $\Omega$ in this case from the case $\varepsilon = +$ and $\eta = -$ using the modular conjugation $J$. For the actions of $E$ and $E^{*}$ we can do the same. Using $J E_{0} = -E_{0}^{\dagger} J$ we obtain from (8.26) and (8.27)

\[
(q^{-1} - q) \ Y_{p,m+1}^{-,+} E_{p,m}^{+} (Y_{p,m}^{- Ebayke}^{+,+})^{*} = q^{\frac{1}{2}-m} p^{-\frac{1}{2}} M(\sqrt{1 + 2 x q^{2m+1} p + q^{4m+2} p^2}) : L^{2}(I(p, m, -, +)) \rightarrow L^{2}(I(p, m + 1, -, +)),
\]  

(8.28)

This can also be proved in the same way as (8.26) and (8.27).

From $\epsilon(p^{-1}) = \epsilon(p)$ and (8.27) we obtain from Theorem [8.14] the following decomposition of $\mathcal{K}(p, -, +)$ as $U_{q}(\mathfrak{su}(1, 1))$-module.
Theorem 8.15. Let $\epsilon(p) = \frac{1}{2} \chi(p) \mod 1$. The decomposition of $\pi_K(p, -, +)$ into irreducible admissible $*$-representations is given by

$$\pi_K(p, -, +) \cong \int_{0}^{-\pi/2 \ln q} \pi_{b, \epsilon(p)}(u) dB \oplus \bigoplus_{l \in \mathbb{Z}} D_{l-\frac{1}{2} \chi(p)}^{-} \oplus \bigoplus_{l \in \mathbb{N}_0} \mathcal{S}^{p+q}_{\pi_{q+2l, \epsilon(p)}}.$$ 

8.4.3. The case $\epsilon = -$ and $\eta = -$. Similar as in the case $\epsilon = +$ and $\eta = -$ we find

$$(q^{-1} - q) \mathcal{Y}^{--}_{p,m+1} E^--_{p,m} (\mathcal{Y}^{--}_{p,m})^* = q^{-\frac{1}{2} - m} p^{-\frac{1}{2}} M(\sqrt{1 + 2q^{2m+1}p + q^{4m+2}p^2})$$

: $L^2(I(p, m, -,-)) \to L^2(I(p, m + 1, -,-))$, 

$$(q^{-1} - q) \mathcal{Y}^{--}_{p,m-1} (E^--_{p,m-1})^* (\mathcal{Y}^{--}_{p,m})^* = q^{\frac{1}{2} - m} p^{-\frac{1}{2}} M(\sqrt{1 + 2q^{2m-1}p + q^{4m-2}p^2})$$

: $L^2(I(p, m, -,-)) \to L^2(I(p, m - 1, -,-)).$ 

(8.29)

In the first equation we assume $m + \chi(p) \geq 0$, and in the second we require $m + \chi(p) > 0$.

It is now a matter of bookkeeping to keep track of the discrete spectrum of $\Omega$ in $\mathcal{K}(p, -, -)$ in order to find the discrete summands in the decomposition of $\mathcal{K}(p, -, -)$ as $U_q(\mathfrak{su}(1,1))$-module. Note that for $pq > 1$ there is always discrete spectrum for $m$ large, so that $E$ acts as the creation operator and hence we have positive discrete series representations in the decomposition. Similarly, $q > p$ leads to the occurrence of negative discrete series representations in the decomposition.

Theorem 8.16. The decomposition of $\pi_K(p, -, -)$ into irreducible admissible representations is given by

$$\pi_K(p, -, -) \cong \int_{0}^{-\pi/2 \ln q} \pi_{b, \epsilon(p)}(u) dB \oplus \bigoplus_{l \in \mathbb{N}_0} D^{+}_{l-\frac{1}{2} \chi(p)-l} \oplus \bigoplus_{l \in \mathbb{N}_0} D^{-}_{l+\frac{1}{2} \chi(p)-l'}.$$ 

Note that at least one of the direct sums in the decomposition is empty.

8.4.4. The case $\epsilon = +$ and $\eta = +$. In this case the spectral decomposition of the Casimir operator restricted to $\mathcal{K}(p,m,+,+)$ is described in Section 8.3.4. From Lemma 8.16 we obtain

$$(q^{-1} - q) \mathcal{Y}^{++,+}_{p,m+1} E^{++,+}_{p,m} (\mathcal{Y}^{++,+}_{p,m})^* = q^{\frac{1}{2} - m} p^{-\frac{1}{2}} M(\sqrt{1 + 2q^{2m+1}p + q^{4m+2}p^2})$$

: $L^2(I(p, m, +, +)) \to L^2(I(p, m + 1, +, +))$, 

$$(q^{-1} - q) \mathcal{Y}^{++,+}_{p,m-1} (E^{++,+}_{p,m-1})^* (\mathcal{Y}^{++,+}_{p,m})^* = q^{\frac{1}{2} - m} p^{-\frac{1}{2}} M(\sqrt{1 + 2q^{2m-1}p + q^{4m-2}p^2})$$

: $L^2(I(p, m, +, +)) \to L^2(I(p, m - 1, +, +)).$ 

(8.30)

We have to be careful in establishing the equality in (8.30) because of the unboundedness of the operators involved. From the way we defined $\mathcal{Y}^{++,+}_{p,m}$ in Section 8.3.4, we conclude that the operators on the left hand side of (8.30) are restrictions of the ones on the right hand side. Let us denote the operator on the left hand side of the first equality in (8.30) by $S$ and
the operator on the right hand side of this equality by $T$. So $S \subseteq T$. Then, by \cite{8.3} and the result from Section \ref{8.3.4},

\[ S^* S = 2 \Upsilon_{p,m} \big( \Upsilon_{p,m} \big)^* + (q^{2m+1}p + q^{-2m-1}p^{-1})\text{Id} \]

\[ = 2 M(x) + (q^{2m+1}p + q^{-2m-1}p^{-1})\text{Id} = T^* T, \]

implying that $|S| = |T|$, and as a consequence, $D(S) = D(T)$.

It is now a matter of bookkeeping to keep track of the discrete spectrum of $\Omega$ in $\mathcal{K}(p, +, +)$ in order to find the discrete summands in the decomposition of $\mathcal{K}(p, +, +)$ as $U_p(\mathfrak{su}(1,1))$-module. Note that for $pq > 1$ there is always a discrete spectrum for $m$ large, so that $E_0$ acts as the creation operator and hence we have positive discrete series representations in the decomposition. Similarly, $q > p$ leads to the occurrence of negative discrete series representations in the decomposition. These two cases correspond to the (possibly empty) finite sequence of discrete mass points in the spectral measure of the Casimir operator \cite{8.16}, \cite{8.17}. The infinite sequence of discrete mass points that is always present in the spectral decomposition of the Casimir operator on $\mathcal{K}(p, m, +, +)$ for all $m \in \mathbb{Z}$ corresponds to strange series representations.

**Theorem 8.17.** The decomposition of $\pi_\mathcal{K}(p, +, +)$ into irreducible admissible representations is given by

\[ \pi_\mathcal{K}(p, +, +) \cong \bigoplus_{\epsilon} \int_0^{-\pi/2\ln q} \pi_{b,\epsilon(p)} \, db \bigoplus \bigoplus_{l \in \mathbb{Z}} \pi_{pq^{1+2l},\epsilon(p)}^{S\epsilon} \bigoplus_{l \in \mathbb{N}_0} D^-_{-\chi(p)/2 - l} \bigoplus_{l \in \mathbb{N}_0} D^+_{-\chi(p) - l} \]

Note that at least one of the finite direct sums in the decomposition is empty.

### 9. Generators of the dual von Neumann algebra $\hat{M}$

By Theorem \ref{4.6}, $E$ and $K$ strongly commute with the Casimir $\Omega$. Since there are elements in $\hat{M}$ that anti-commute with $\Omega$, see Proposition \ref{13.8}, $\hat{M}$ cannot be generated by $E$ and $K$ alone. So we need to find extra operators that, together with $E$ and $K$, generate the dual von Neumann algebra $\hat{M}$. It is the purpose of this section to describe a generating set for $\hat{M}$, i.e., to prove Theorem \ref{14.3}. We do so by establishing a generator $Q(p_1, p_2, n)$ of $\hat{M}$, see \cite{13.9} and Proposition \ref{13.9} as the composition of a partial isometry and an operator expressed in terms of the Casimir operator. The partial isometries occurring in this way give us the required additional generators for the dual von Neumann algebra $\hat{M}$.

Throughout this section we fix $p_1, p_2 \in I_q, p \in \mathbb{N}$, $m, n \in \mathbb{Z}, \epsilon, \eta \in \{-, +\}$. Furthermore, we set $m' = m + n$, $\epsilon' = \epsilon \text{sgn}(p_1)$ and $\eta' = \eta \text{sgn}(p_2)$, and assume $q^{2m}p = q^{-n}|p_2/p_1|$, unless explicitly stated otherwise. In this case the operator $Q(p_1, p_2, n) : \mathcal{K}(p, m, \epsilon, \eta) \to \mathcal{K}(p, m', \epsilon', \eta')$ is non-zero by Lemma \ref{7.1}.
9.1. A polar-type decomposition for $Q(p_1, p_2, n)$. In this subsection we establish a polar-type decomposition for the element $Q(p_1, p_2, n)$. Since operators of this form span $\hat{M}$ by Proposition 4.3, we can obtain the generators of $\hat{M}$. By Proposition 4.8 the operator $Q(p_1, p_2, n) \in \hat{M}$ commutes or anti-commutes with the Casimir operator $\Omega$, hence $Q(p_1, p_2, n)$ sends a (generalized) eigenvectors of $\Omega$ to another (generalized) eigenvector. In order to avoid working with generalized eigenvectors, we consider an operator $T$, acting on $L^2$-functions on the spectrum of $\Omega$, that is unitarily equivalent to $Q(p_1, p_2, n)$. We determine the explicit action of $T$ by investigating how $T$ affects the asymptotic behaviour of certain functions. Having explicitly the action of $T$, we can compute explicitly how $T^*T$ acts, and this leads to the polar decomposition of $T$. This in turn leads to a polar-type decomposition for $Q(p_1, p_2, n)$.

In order to find explicitly the action of the operator $T$ as described above (and defined later on by (9.1)), we need a result on the asymptotic behaviour of certain functions. In order to formulate the result we define the following function:

$$S(t; p_1, p_2, n) = (\text{sgn}(p_2))^{n} |p_1 p_2| c_q^2 q^n \sqrt{(-\kappa(p_1), -\kappa(p_2); q^2)_{\infty}} \times \sum_{z \in \text{sgn}(p_1) q^2} (\text{sgn}(p_1 p_2) t)^{\chi(z)} \frac{1}{|z|} \nu(p_1, \frac{p_2 q^n}{z}) \nu(p_2 q^n) 1 \varphi_1 \left( \begin{array}{c} -q^2 / \kappa(p_1) \\ 0 \end{array} ; q^2, q^2 \kappa(z) \right) \times 1 \varphi_1 \left( \begin{array}{c} -q^2 / \kappa(p_2) \\ 0 \end{array} ; q^2, q^2 \kappa(\text{sgn}(p_1 p_2) q^{-n} z) \right),$$

where the sum is absolutely convergent. Clearly, $S(\cdot ; p_1, p_2, n)$ is analytic on $\mathbb{C} \setminus \{0\}$. This function is studied in Appendix B.3 in some more detail. Two properties of $S$ that we need here are given in the following lemma.

**Lemma 9.1.** The analytic function $S(\cdot ; p_1, p_2, n) : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ satisfies the following properties:

(i) $S(t; p_1, p_2, n) = (-q)^n \text{sgn}(p_1)^{\chi(p_1)} \text{sgn}(p_2)^{\chi(p_2)} + n \text{sgn}(p_1 p_2) S(\text{sgn}(p_1 p_2) t^{-1}; p_1, p_2, -n)$

(ii) $S(t; p_1, p_2, n)$ is a multiple of a $2 \varphi_1$-function:

$$S(t; p_1, p_2, n) = p_2^n q^{\frac{1}{2} n(n-1)} |p_1 p_2| \nu(p_1) \nu(p_2) c_q^2 \sqrt{(-\kappa(p_1), -\kappa(p_2); q^2)_{\infty}} \times \left( \begin{array}{c} q^2, -q^2 / \kappa(p_2), -t q^{3-n} / p_1 p_2, -p_1 p_2 q^{n-1} / t, p_1 q^{1-n} / p_2 t \end{array} ; q^2 \right)_{\infty} \times \left( \begin{array}{c} -p_1 |p_2| q^{-n-1} / t, -t q^{n+3} / p_1 p_2, |p_1| q^{1+n} / |p_2| t \end{array} ; q^2 \right)_{\infty} \times \left( \begin{array}{c} q^{2+2n} / p_2 q^{1+n} / p_1 t, p_2 q^{1+n} / p_1 \end{array} ; q^2 \right)_{\infty} 2 \varphi_1 \left( \begin{array}{c} p_2 q^{1+n} / p_1 t, p_2 q^{1+n} / p_1 \\ \text{sgn}(p_1 p_2) q^{2+2n} \end{array} ; q^2, -q^2 / \kappa(p_2) \right).$$

See Appendix B.3, and in particular Proposition B.10 and Lemma B.11, for a proof of Lemma 9.1.

It turns out to be useful to split the function $S$ in a part that is symmetric in $t$ and $t^{-1}$, and a part that is not.
Lemma 9.2. For \( x = \mu(t) \), define

\[
B(t; p_1, p_2, n) = \begin{cases}
  t^n (-|p_1| q^{1-n}/|p_2| t; q^2)_\infty, & \chi(p_1p_2) + n \text{ even}, \\
  t^n (-|p_1| q^{1-n}/p_2/t; q^2)_\infty 1 - \text{sgn}(p_2)t^{-1} 1 - t^{-1}, & \chi(p_1p_2) + n \text{ odd},
\end{cases}
\]

\[
h(x) = \begin{cases}
  \text{sgn}(p_2)^{\frac{1}{2}}(\chi(p_1p_2)-n+2) (qt, q/t; q^2)_\infty, & \chi(p_1p_2) + n \text{ even}, \\
  \text{sgn}(p_2)^{\frac{1}{2}}(\chi(p_1p_2)-n+3) (t, t^{-1}; q^2)_\infty, & \chi(p_1p_2) + n \text{ odd},
\end{cases}
\]

and

\[
N(x; p_1, p_2, n) = h(x) q^{2n} p_2 q^{\frac{1}{2}n(n-1)} |p_1p_2|^{1-n} \nu(p_1) \nu(p_2) q^2
\times (q^2, -q^2/\kappa(p_2), \text{sgn}(p_1p_2)q^{2+2n}; q^2)_\infty \sqrt{(-\kappa(p_1), -\kappa(p_2); q^2)_\infty}
\times 2^{\varphi_1} \left( -|p_2| q^{1+n}/|p_1| t, -|p_2| t q^{1+n}/|p_1|; q^2, -q^2/\kappa(p_2) \right),
\]

then \( S(-\text{sgn}(p_1p_2)t; p_1, p_2, n) = B(t; p_1, p_2, n) N(x; p_1, p_2, n) \).

Proof. According to \( \chi(p_1p_2) + n \) being even or odd, we set

\[
\chi(p_1p_2) + n = \begin{cases}
  2 - 2k, & \\
  3 - 2l
\end{cases}
\]

for \( k, l \in \mathbb{Z} \). Using the \( \theta \)-product identity (7.1) we find

\[
\frac{(t q^{2-n}/|p_1p_2|, |p_1p_2| q^{-n}/t; q^2)_\infty}{(t q^{2-n}/|p_1| p_2, p_1 p_2 q^{-n}/|p_1| q^{-n}/t; q^2)_\infty} = \begin{cases}
  \text{sgn}(p_2)^{k+n} t^n q^{n(n-1)} q^{2nk} (qt, q/t; q^2)_\infty, & \\
  \text{sgn}(p_2)^{l+n} t^n q^{n(n-1)} q^{2nt} (t, t^{-1}; q^2)_\infty
\end{cases}
\]

then the result follows from Lemma 9.1(ii). The expression for \( N \) is manifestly symmetric in \( t \) and \( t^{-1} \), so \( N \) is indeed a function of \( x = \mu(t) \).

In the following lemmas, and in the rest of this section, we use the notation \( f(z) \sim g(z) \) as \( z \to 0 \), for \( \lim_{z \to 0} (f(z) - g(z)) = 0 \). We are now ready to formulate the asymptotic behaviour we need later on.

Lemma 9.3. Let \( f: J(p, m, \varepsilon, \eta) \to \mathbb{C} \) be bounded, and consider the function

\[
g(w) = (-1)^m (\eta')^{\chi(p_1p_2)+m} q^{m} p_2^{2} \frac{\varepsilon' \eta' \chi(w)}{|w|}
\times \sum_{z \in J(p, m, \varepsilon, \eta)} \frac{f(z)}{|z|} a_{p_1}(z, w) a_{p_2}(\varepsilon \eta q^m p z, \varepsilon' \eta' q^m p w),
\]
for \( w \in J(p, m', \varepsilon', \eta') \).

(1) If \( f(z) \sim A t^{-x(z)} \) as \( z \to 0 \) for some \( A \in \mathbb{C} \) and \( t \in \mathbb{C} \), \( |t| > 1 \), then
\[
g(w) \sim A t^{-x(w)} \eta^n s(\varepsilon, \varepsilon') s(\eta, \eta') S(\varepsilon \eta/t; p_1, p_2, n), \quad \text{as } w \to 0.
\]

(2) If \( f(z) \sim \Re(A e^{-i\psi(x(z))}) \) as \( z \to 0 \) for some \( A \in \mathbb{C} \) and \( |\psi| \in (0, \pi) \), then
\[
g(w) \sim \eta^n s(\varepsilon, \varepsilon') s(\eta, \eta') \Re(A e^{-i\psi(x(w))} S(\varepsilon \eta e^{-i\psi}; p_1, p_2, n)), \quad \text{as } w \to 0.
\]

The proof of Lemma \( 7.3 \) is given in Appendix \( 7.3 \).

Using the unitary operators \( \Upsilon_{p,m}^{\varepsilon,\eta} : \mathcal{K}(p, m, \varepsilon, \eta) \to L^2(I(p, m, \varepsilon, \eta)) \) from Section \( 8.3 \), we define an action of the generators \( Q(p_1, p_2, n) \) of \( M \) on the space \( L^2(I(p, m, \varepsilon, \eta)) \) by
\[
T(p_1, p_2, n) = \Upsilon_{p,m}^{\varepsilon,\eta} Q(p_1, p_2, n)(\Upsilon_{p,m}^{\varepsilon,\eta})^* : L^2(I(p, m, \varepsilon, \eta)) \to L^2(I(p, m', \varepsilon', \eta')),
\]
where \( m' = m + n, \varepsilon' = sgn(p_1) \varepsilon \) and \( \eta' = sgn(p_2) \eta \). Recall that we assume \( q^{2n} p = q^{-n} [p_2/p_1] \).

If this condition is not satisfied, we see from Lemma \( 7.1 \) that the operator \( T(p_1, p_2, n) \) is trivially zero. Since \( Q(p_1, p_2, n) \Omega \subseteq sgn(p_1 p_2) \Omega Q(p_1, p_2, n) \), we have
\[
T(p_1, p_2, n) M(x) \subseteq sgn(p_1 p_2) M(x) T(p_1, p_2, n)
\]
for any \( x \) in the spectrum of \( \Omega \). For \( g \in L^2(I(p, m, \varepsilon, \eta)) \) this implies
\[
(T(p_1, p_2, n) g)(x) = C(x) g(sgn(p_1 p_2)x),
\]
for a certain bounded measurable function \( C : I(p, m', \varepsilon', \eta') \to \mathbb{C} \). It follows immediately that \( C(x) = 0 \) if \( sgn(p_1 p_2)x \notin I(p, m, \varepsilon, \eta) \), which can only happen in case \( x \in \sigma_d(p, m', \varepsilon', \eta') \).

The set
\[
\{g_z(\cdot; p, m, \varepsilon, \eta) | z \in J(p, m, \varepsilon, \eta)\}
\]
is an orthonormal basis for \( L^2(I(p, m, \varepsilon, \eta)) \). Recall that the functions \( g_z(x; p, m, \varepsilon, \eta) \) are defined in terms of Al-Salam–Chihara polynomials or little \( q \)-Jacobi functions, see Section \( 8.3 \). From the asymptotic behaviour of these special functions, see (B.22), (B.23), (B.40) and (B.41), it follows that the functions \( g_z(x; p, m, \varepsilon, \eta) \) satisfy
\[
g_z(\mu(\lambda); p, m, \varepsilon, \eta) \sim \begin{cases} A(\lambda)(-\varepsilon \eta \lambda)^{-x(z)}, & \lambda \in D(p, m, \varepsilon, \eta), \\ \Re(A(\lambda)(-\varepsilon \eta \lambda)^{-x(z)}), & \lambda \in \mathbb{T}, \end{cases}
\]
as \( z \to 0 \), for a certain \( A(\lambda) = A(\lambda; p, m, \varepsilon, \eta) \in \mathbb{C} \). In general the functions \( A \) are only defined on \( \mathbb{T}_0 = \mathbb{T} \setminus \{-1, 1\} \). The function \( A(\lambda) \) has an explicit expression in terms of the \( c \)-functions for the corresponding special functions, for instance
\[
A(e^{i\psi}; p, m, +, -) = (-1)^m e^{-i\psi(m+\chi(p)-1)} \sqrt{\frac{2}{\pi \sin \psi}} \frac{c(e^{-i\psi}; q/p, -q^{1-2m}/p \mid q^2)}{|c(e^{-i\psi}; q/p, -q^{1-2m}/p \mid q^2)|}.
\]
for \( 0 < |\psi| < \pi \), which follows from (8.8) and (B.22), and we have similar expressions in the other cases. For convenience we have written down the explicit formulas for \( A(p, m, \varepsilon, \eta) \) in Appendix \( 3.4 \). With the help of the explicit action of \( Q(p_1, p_2, n) \) on the basis elements of \( \mathcal{K}(p, m, \varepsilon, \eta) \) given in Lemma \( 7.4 \), and with Lemma \( 7.3 \), we can now compute explicitly the function \( C \). The following notation will be useful: for \( n, m \in \mathbb{Z}, p \in q^n, \varepsilon, \eta, \sigma, \tau \in \{-, +\} \), we set
\[
X_{n,\sigma,\tau}^m(p, m, \varepsilon, \eta) = \mathbb{T} \cup \left( D(p, m + n, \sigma \varepsilon, \tau \eta) \cap \sigma \tau D(p, m, \varepsilon, \eta) \right).
\]
(9.3)
Lemma 9.4. Let \( g \in L^2(I(p, m, \varepsilon, \eta)) \), \( X = X_{n}^{\text{sgn}(p_1), \text{sgn}(p_2)}(p, m, \varepsilon, \eta) \), then for almost all \( x = \mu(\lambda) \in I(p, m', \varepsilon', \eta') \)

\[
(T(p_1, p_2, n)g)(x) = C(x)g(\text{sgn}(p_1 p_2) x),
\]

where \( C = C(\cdot; m, \varepsilon, \eta; p_1, p_2, n) \) is given by

\[
C(\mu(\lambda)) = \begin{cases} 
\frac{\frac{1}{2}(1-\text{sgn}(p_1)) \eta \left(1-\text{sgn}(p_2)\right) + n}{2} \cdot S(-\text{sgn}(p_1 p_2) \lambda; p_1, p_2, n) \\
\times A(\lambda; p, m', \varepsilon', \eta') \\
\frac{A(\text{sgn}(p_1 p_2) \lambda; p, m, \varepsilon, \eta)}{A(\text{sgn}(p_1 p_2) \lambda; p, m, \varepsilon, \eta)} \\
\end{cases}
\]

\( \lambda \in X, \) \\
otherwise.

Note that the expression on the right hand side is not obviously symmetric with respect to interchanging \( \lambda \) and \( \lambda^{-1} \), but it is since the function \( C \) only depends on \( x = \mu(\lambda) \).

Proof. We assume \( \lambda \in X \). We know that (9.4) is valid for some function \( C \) and for all \( g \in L^2(I(p, m, \varepsilon, \eta)) \). We choose \( g = g_z = g_z(\cdot; p, m, \varepsilon, \eta) \). Since the function \( C \) is independent of \( z \), we can determine the function \( C \) by letting \( z \to 0 \).

Using \( \Upsilon_{p,m}^\varepsilon f_{-m,\varepsilon'\eta'q^m pz,z} = g_z(\cdot; p, m, \varepsilon, \eta) \), it follows immediately from Lemma 9.3 and (9.4) that

\[
C(\cdot)g_z(\text{sgn}(p_1 p_2) \cdot) = T(p_1, p_2, n)g_z(\text{sgn}(p_1 p_2) \cdot; p, m, \varepsilon, \eta) \\
= (-1)^{m'}(\varepsilon')^x(p_1 p_2) + m q^{n+m} / |z| \\
\times \sum_{w \in \mathcal{J}(p, m', \varepsilon', \eta')} \frac{(\varepsilon' q^m) \chi(w)}{|w|} a_{p_1} (z, w) a_{p_2} (\varepsilon' \eta' q^m p z, \varepsilon' \eta' q^m p w) g_w(\cdot; p, m', \varepsilon', \eta'),
\]
as an identity in \( L^2(I(p, p', \varepsilon', \eta')) \). Since \( g_z \) is a real-valued function, we see that the function \( C \) is real-valued almost everywhere. From (9.2) it follows that \( w \mapsto g_w(x; p, m', \varepsilon', \eta') \) is bounded for all \( x \in I(p, p', \varepsilon', \eta') \setminus \{ \pm 1 \} \). This implies that the sum

\[
\sum_{w \in \mathcal{J}(p, p', \varepsilon', \eta')} \frac{(\varepsilon' q^m) \chi(w)}{|w|} a_{p_1} (z, w) a_{p_2} (\varepsilon' \eta' q^m p z, \varepsilon' \eta' q^m p w) g_w(x; p, m', \varepsilon', \eta')
\]
converges for all \( x \in I(p, m', \varepsilon', \eta') \setminus \{ \pm 1 \} \). Using symmetry relations (6.2) for the functions \( a_{p_1} \), we have

\[
C(x)g_z(\text{sgn}(p_1 p_2) x) = (-1)^{m'} \text{sgn}(p_1) \chi(p_1) \text{sgn}(p_2) \chi(p_2) \eta^{m'} + \chi(p_1 p_2) (\varepsilon')^x q^m p |z| \\
\times \sum_{w \in \mathcal{J}(p, m', \varepsilon', \eta')} \frac{1}{|w|} a_{p_1} (w, z) a_{p_2} (\varepsilon' q^m p w, \varepsilon' \eta' q^m p z) g_w(x; p, m', \varepsilon', \eta').
\]

Let \( z \to 0 \) in this expression using Lemma 9.3 and the asymptotic behaviour (9.2) of \( g_z \), then for \( \lambda \in \mathbb{T}_0 \),

\[
C(\mu(\lambda)) \Re \left( A(\text{sgn}(p_1 p_2) \lambda) \left( - \text{sgn}(p_1 p_2) \varepsilon \lambda \right)^{-\chi(z)} \right) \sim \frac{(-1)^n q^n (\eta')^n \text{sgn}(p_1) \chi(p_1) \text{sgn}(p_2) \chi(p_2) s(\varepsilon, \varepsilon') s(\eta, \eta')}{\Re \left( (A'(\lambda)(-\varepsilon' \eta' \lambda)^{-\chi(z)} S(-\lambda^{-1}; p_1, p_2, -n) \right)},
\]
and for $\lambda \in X_d(p, m', \varepsilon', \eta'; n)$,
\[
C(\mu(\lambda)) A(\text{sgn}(p_1p_2)\lambda) \left( -\text{sgn}(p_1p_2)\varepsilon\lambda \right)^{-\chi(\varepsilon)} \sim \\
(-1)^n q^n (\eta')^n \text{sgn}(p_1) \chi(p_1) \text{sgn}(p_2) \chi(p_2) s(\varepsilon, \varepsilon') s(\eta, \eta') \\
\times A'(\lambda)(-\varepsilon' \eta' \lambda)^{-\chi(\varepsilon')} S(-\lambda^{-1}; p_1, p_2, -n),
\]
where we use the shorthand notation $A'(\lambda) = A(\lambda; p, m', \varepsilon', \eta')$. Applying the first symmetry for the function $S(\cdot; p_1, p_2, n)$ from Lemma 9.4, using $s(\varepsilon, \varepsilon') = \varepsilon^{\frac{1}{2}(1 - \text{sgn}(p_1))}$ and similarly for $s(\eta, \eta')$, and using the fact that $C$ is real-valued, the result follows. \hfill \Box

**Remark 9.5.** Lemma 9.4 immediately gives nontrivial summation formulas for special functions. We work this out in Section 11.1.

Next we consider the polar decomposition for $Q(p_1, p_2, n)$. We need the following lemma.

**Lemma 9.6.** For $p_1, p_2 \in I_q$, $n \in \mathbb{Z}$, we have
\[
Q(p_1, p_2, n)^* = (-q)^n \text{sgn}(p_1) \chi(p_1) \text{sgn}(p_2) \chi(p_2) Q(p_1, p_2, -n).
\]

**Proof.** Using the matrix elements (7.13) and their symmetries following from (6.2), it is straightforward to check that the matrix elements
\[
\langle f_{mpt}, Q(p_1, p_2, n)f_{lrs} \rangle \quad \text{and} \quad \langle Q(p_1, p_2, -n) f_{mpt}, f_{lrs} \rangle
\]
agree up to the factor $(-q)^n \text{sgn}(p_1) \chi(p_1) \text{sgn}(p_2) \chi(p_2)$ for all $m, p, t, l, r, s$. \hfill \Box

Alternatively, one can also use Corollary 7.4 and $J f_{mpt} = f_{-m, t, p}$, see Section 3, to prove Lemma 9.6 using (2.4).

From Lemma 9.6 it follows that
\[
T(p_1, p_2, n)^* = (-q)^n \text{sgn}(p_1) \chi(p_1) \text{sgn}(p_2) \chi(p_2) T(p_1, p_2, -n). \quad (9.5)
\]
Combining this with Lemma 9.4 we find for $g \in L^2(I(p, m, \varepsilon, \eta))$,
\[
(T(p_1, p_2, n)^* T(p_1, p_2, n) g)(\mu(\lambda)) \\
= (-q)^n \text{sgn}(p_1) \chi(p_1) \text{sgn}(p_2) \chi(p_2) + n + 1 S(-\lambda; p_1, p_2, n) S(-\text{sgn}(p_1p_2)\lambda; p_1, p_2, -n) g(\mu(\lambda)), \\
= S(-\lambda; p_1, p_2, n) S(-\lambda^{-1}; p_1, p_2, n) g(\mu(\lambda)),
\]
where $\lambda \in X = X_n^{\text{sgn}(p_1), \text{sgn}(p_2)}(p, m, \varepsilon, \eta)$. The last equality follows from a symmetry relation from Lemma 9.4. Note that this implies $S(-\lambda; p_1, p_2, n) S(-\lambda^{-1}; p_1, p_2, n) \geq 0$. Furthermore, for $\lambda \notin X$ we have
\[
(T(p_1, p_2, n)^* T(p_1, p_2, n) g)(\mu(\lambda)) = 0.
\]
Now we define for $x = \mu(\lambda) \in I(p, m, \varepsilon, \eta)$,
\[
L(x; p_1, p_2, n) = \begin{cases} \\
\sqrt{S(-\lambda; p_1, p_2, n) S(-\lambda^{-1}; p_1, p_2, n)}, & \lambda \in X, \\
0, & \text{otherwise},
\end{cases}
\]
and we define a partial isometry \( V(p_1, p_2, n) : L^2(I(p, m, \varepsilon, \eta)) \rightarrow L^2(I(p, m', \varepsilon', \eta')) \) by

\[
(V(p_1, p_2, n) g)(x) = \begin{cases} 
  \frac{C(x)g(\text{sgn}(p_1 p_2)x)}{L(\text{sgn}(p_1 p_2)x)}, & x \in \mu(X), \\
  0, & \text{otherwise},
\end{cases}
\]

where \( C \) is given in Lemma 9.4. We remark that for \( \lambda \in X \) it follows from Lemma 9.1 that \( L \) is a multiple of the absolute value of a \( 2\phi_1 \)-function. Now from Lemma 9.4 we find the polar decomposition of \( T(p_1, p_2, n) : L^2(I(p, m, \varepsilon, \eta)) \rightarrow L^2(I(p, m', \varepsilon', \eta')) \):

\[
T(p_1, p_2, n) = V(p_1, p_2, n)|T(p_1, p_2, n)|,
\]

where

\[
(|T(p_1, p_2, n)| \ g)(x) = \begin{cases} 
  L(x)g(x), & x \in \mu(X), \\
  0, & \text{otherwise},
\end{cases}
\]

for \( g \in L^2(I(p, m, \varepsilon, \eta)), \ x \in I(p, m, \varepsilon, \eta) \) and the set \( X \) is given by (9.3). Note that \( |T(p_1, p_2, n)| = 0 \) on \( L^2(I(p, m, \varepsilon, \eta)) \) if \( p \neq q^{-n-2m}|p_2/p_1| \). We can now describe explicitly the polar decomposition for \( Q(p_1, p_2, n) \).

**Proposition 9.7.** The operators \( U(p_1, p_2, n) \) and \( |Q(p_1, p_2, n)| \) in the polar decomposition \( Q(p_1, p_2, n) = U(p_1, p_2, n)|Q(p_1, p_2, n)| \) are given by

\[
|Q(p_1, p_2, n)| = L(\Omega), \quad \text{and} \quad U(p_1, p_2, n) = \Upsilon^* V(p_1, p_2, n) \Upsilon.
\]

We are going to define a partial isometry closely related to \( V(p_1, p_2, n) \) which is more convenient for us. Let us first have a closer look at the function \( \frac{L}{L(\text{sgn}(p_1 p_2)x)} \) appearing in the definition of \( V(p_1, p_2, n) \). Using Lemma 9.2 we find (omitting dependence on certain parameters in the notation)

\[
L(x) = \sqrt{B(\sigma \tau \lambda)B(\sigma \tau \lambda^{-1})}|N(\sigma \tau x)|, \quad x = \mu(\lambda),
\]

where \( \sigma = \text{sgn}(p_1) \) and \( \tau = \text{sgn}(p_2) \). Here we use that \( N(x) \) is symmetric in \( \lambda \) and \( \lambda^{-1} \), hence real-valued, and consequently \( B(\lambda)B(\lambda^{-1}) \) is positive. This shows that \( \frac{C(x)}{L(\text{sgn}(p_1 p_2)x)} \) can be written as

\[
\varepsilon^{\frac{1}{2}(1-\sigma)}\eta^{\frac{1}{2}(1-\tau)+n} \frac{A'(\lambda)}{A(\sigma \tau \lambda)} \frac{B(\lambda)}{\sqrt{B(\lambda)B(\lambda^{-1})}} \text{sgn}(N(x)).
\]

This expression, in particular the factor \( \text{sgn}(N(x)) \), is not very convenient for us, therefore we are going to consider the partial isometry \( \text{sgn}(N(\cdot))V(p_1, p_2, n) \). Let us introduce the following functions:

\[
E(\lambda; p, m) = \frac{(-q^{1-2m}/p\lambda; q^2)_\infty}{\sqrt{(-q^{1-2m}/p\lambda, -q^{1-2m}\lambda/p; q^2)_\infty}}, \quad G(\lambda; p, m, \varepsilon, \eta) = A(\lambda; p, m, \varepsilon, \eta)E(\lambda; p, m),
\]

\[
\nu^\tau_n(\lambda; p) = \frac{(\tau \lambda)^{\epsilon(p)}}{\lambda^{\epsilon(p)-n}}, \quad (9.6)
\]
where $\epsilon(p)$ is defined by (8.25). In particular, for $\epsilon(p) = \frac{1}{2}$ we have $\nu^0(\lambda; p) = \mp i$ for $\lambda \in \mathbb{T}^\pm$. With these functions, we define for $n \in \mathbb{Z}$, $\sigma, \tau \in \{-, +\}$, a partial isometry $V_{n}^{\sigma, \tau}: L^2(I(p, m, \varepsilon, \eta)) \to L^2(I(p, m + n, \sigma\eta, \tau\eta))$ closely related to $V(p_1, p_2, n)$ by

$$V_{n}^{\sigma, \tau} g(x) = \begin{cases} \mathbb{I}_{\mathbb{Z}}^{2(1-\sigma) + n} V_n(\lambda; p) \frac{G(\lambda; p, m + n, \sigma\varepsilon, \tau\eta)}{G(\sigma\tau\lambda; p, m, \varepsilon, \eta)} g(\sigma\tau x), & \lambda \in X, \\ 0, & \text{otherwise.} \end{cases}$$

(9.7)

where $g \in L^2(I(p, m, \varepsilon, \eta))$, and $X = X_{n}^{\sigma, \tau}(p, m, \varepsilon, \eta)$. Let us remark that

$$\nu_n^{\text{sgn}(p_2)}(\lambda; p) = \frac{E(\lambda; p, m + n)}{E(\text{sgn}(p_1p_2)\lambda; p, m)} \frac{B(\lambda; p_1p_2, n)}{\sqrt{B(\lambda; p_1p_2, n)B(\lambda^{-1}; p_1p_2, n)}},$$

for $p = q^{-n-2m}|p_2/p_1|$, so that $V_n^{\text{sgn}(p_1), \text{sgn}(p_2)} = \text{sgn}(N(\cdot; p_1, p_2, n))V(p_1, p_2, n)$. We also denote

$$V_{n}^{\text{sgn}(p_1), \text{sgn}(p_2)} : \bigoplus_{\varepsilon, \eta \in \{\pm\}} L^2(I(p, m, \varepsilon, \eta)) \to \bigoplus_{\varepsilon, \eta \in \{\pm\}} L^2(I(p, m, \varepsilon, \eta))$$

by summing $V_{n}^{\sigma, \tau}: L^2(I(p, m, \varepsilon, \eta)) \to L^2(I(p, m + n, \sigma\eta, \tau\eta))$.

We now arrive at the following polar-type decomposition for the operators $Q(p_1, p_2, n)$.

**Proposition 9.8.** Let $m, n \in \mathbb{Z}$, $p_1, p_2 \in I_q$, $\varepsilon, \eta \in \{-, +\}$, and assume $p = q^{-n-2m}|p_2/p_1|$. For $\sigma, \tau \in \{-, +\}$ we define a partial isometry $U_n^{\sigma, \tau} = \Upsilon V_n^{\sigma, \tau} \Upsilon$, so that

$$U_n^{\sigma, \tau}|_{K(p, m, \varepsilon, \eta)} = (\Upsilon_{p, m+n}^{\sigma\varepsilon, \tau\eta} V_n^{\sigma, \tau}(\Upsilon_{p, m}^{\eta, \varepsilon})) : K(p, m, \varepsilon, \eta) \to K(p, m + n, \sigma\varepsilon, \tau\eta)$$

(9.8)

Furthermore, we define a continuous function $H = H(\cdot; p_1, p_2, n)$ by

$$H(x; p_1, p_2, n) = \frac{1}{\nu_n^{\text{sgn}(p_2)}(\text{sgn}(p_1p_2)\lambda; p)} \frac{E(\lambda; p, m)}{E(\text{sgn}(p_1p_2)\lambda; p, m + n)} S(-\lambda; p_1p_2, n), \quad x = \mu(\lambda),$$

and we denote by $P = P(p_1, p_2, n) \in B(K)$ the spectral projection of $K$ corresponding to the eigenvalue $\sqrt{q^{-n}|p_2/p_1|}$. Then

$$Q(p_1, p_2, n) = U_n^{\text{sgn}(p_1), \text{sgn}(p_2)} H(\Omega) P.$$

Again the right hand side defining $H$ is not obviously symmetric with respect to interchanging $\lambda$ and $\lambda^{-1}$, but it is as can be observed either from the proof of Proposition 9.8 or by observing that the $\lambda$-dependent part in (9.9) is indeed symmetric with respect to $\lambda \leftrightarrow \lambda^{-1}$. Observe that $E(\lambda; p, m) = E(\lambda; pq^{2m}, 0)$ and $pq^{2m} = q^{-n}|p_2/p_1|$. Also, $\nu_n^{\text{sgn}(p_2)}(\text{sgn}(p_1p_2)\lambda; p)$ does not depend on $p$ and $m$ since $\epsilon(p) = \epsilon(q^{-n-2m}|p_2/p_1|) = \epsilon(q^{-n}|p_2/p_1|)$ by (8.25), so $H$, as a function of $x$, only depends on the parameters $p_1, p_2$ and $n$.

Note that Proposition 9.8 proves Lemma 4.12.
Proof. From (9.13), Lemma 9.4 and Theorem 8.13 we find
\[
\left(\Upsilon_{p,m'}^{\varepsilon,\eta} Q(p_1, p_2, n) \left(\Upsilon_{p,m}^{\varepsilon,\eta}\right)^* g\right)(x)
= \varepsilon \frac{1}{2} (1-\sigma) \eta \frac{1}{2} (1-\tau) + n \nu'(\lambda; p) \frac{G(\lambda; p, m', \varepsilon', \eta')}{G(\sigma \tau \lambda; p, m, \varepsilon, \eta)} H(\sigma \tau x) g(\sigma \tau x)
= \left( (\Upsilon_{p,m'}^{\varepsilon,\eta})^* U_{\tau}^{\sigma} H(\Omega) \Upsilon_{p,m}^{\varepsilon,\eta} g\right)(x),
\]
hence \(Q(p_1, p_2, n) = U_{\tau}^{\sigma} H(\Omega)\) on \(K(p, m, \varepsilon, \eta)\). Now observe that \(P\) is the orthogonal projection onto
\[
\bigoplus_{\varepsilon, \eta \in \{-, +\}} K(p, m, \varepsilon, \eta), \quad p = q^{-2m} q^{-n} |p_2/p_1|,
\]
then the result follows. \(\square\)

From this proposition it follows that the function \(C(x)\) from Lemma 9.4 can be written as
\[
C(x) = \varepsilon \frac{1}{2} (1-\sigma) \eta \frac{1}{2} (1-\tau) + n \nu'(\lambda; p) \frac{G(\lambda; p, m', \varepsilon', \eta')}{G(\sigma \tau \lambda; p, m, \varepsilon, \eta)} H(\sigma \tau x; p_1, p_2, n), \quad (9.9)
\]
for \(\sigma = \text{sgn}(p_1)\) and \(\tau = \text{sgn}(p_2)\). Let us give two identities for the function \(C\) that will be useful later on. The first identity follows from the structure formula in Proposition 1.10 for the linear basis \(\{Q(p_1, p_2, n) \mid p_1, p_2 \in I_q, n \in \mathbb{Z}\}\) for the von Neumann algebra \(\hat{M}\). This formula implies a product formula for the function \(C\) that is useful later on. The second identity is a consequence of Lemma 9.6.

**Lemma 9.9.** Let \(p_1, p_2, r_1, r_2 \in I_q, k, m, n \in \mathbb{Z}, \varepsilon, \eta \in \{-, +\}\) and \(y \in [-1, 1]\).

(i) Assume \(|\frac{p_1}{p_2}| = q^n\) and \(|\frac{r_1}{r_2}| = q^n\), then the following product formula holds:
\[
C(y; k + m, \text{sgn}(r_1)\varepsilon, \text{sgn}(r_2)\eta; p_1, p_2, n) C(\text{sgn}(p_1 p_2) y; k, \varepsilon, \eta; r_1, r_2, m) = \sum_{x_1, x_2 \in I_q, \text{sgn}(x_1) = \text{sgn}(p_1 r_1), \text{sgn}(x_2) = \text{sgn}(p_2 r_2) \overline{|x_1| = |x_2|}} a_{x_1}(r_1, p_1) a_{x_2}(r_2, p_2) C(y; k, \varepsilon, \eta; x_1, x_2, m + n).
\]

(ii) The following symmetry relation holds:
\[
C(y; m, \varepsilon, \eta; p_1, p_2, n) = (-q)^n \text{sgn}(p_1) \chi(p_1) \text{sgn}(p_2) \chi(p_2) C(\text{sgn}(p_1 p_2) y; m + n, \text{sgn}(p_1)\varepsilon, \text{sgn}(p_2)\eta; p_1, p_2, -n).
\]

**Proof.** (i) From (9.14) it follows that the operators \(T(p_1, p_2, n)\) satisfy the same structure formula as the operators \(Q(p_1, p_2, n)\), see Proposition 1.10. Let \(p = q^{-2k-m-n}\), then \(p = q^{-2(k+m)-n}\) and \(p = q^{-2(k+m)-n}\). Applying the structure formula to a function \(g \in L^2(\hat{M}^{(p, k, \varepsilon, \eta)})\) and using the action of \(T(p_1, p_2, n)\) as multiplication by the function \(C\) from Lemma 9.4, we obtain
\[
C(y; k + m, \text{sgn}(r_1)\varepsilon, \text{sgn}(r_2)\eta; p_1, p_2, n) C(\text{sgn}(p_1 p_2) y; k, \varepsilon, \eta; r_1, r_2, m) g(\text{sgn}(p_1 p_2 r_1 r_2) y)
= \sum_{x_1, x_2 \in I_q} a_{x_1}(r_1, p_1) a_{x_2}(r_2, p_2) C(y; k, \varepsilon, \eta; x_1, x_2, m + n) g(\text{sgn}(x_1 x_2) y)
\]
Observe that $T(x_1, x_2, n + m) = 0$ on $L^2(I(p, k, \varepsilon, \eta))$ unless $p = q^{-2k-n-m}r_2$, which implies that the sum is only over $x_1, x_2 \in I_q$ satisfying $|x_1| = |x_2|$. Furthermore, by Definition 1.2 we have $a_x(p, r) = 0$ if $\text{sgn}(x) \neq \text{sgn}(pr)$, so we may write $\text{sgn}(x_1x_2) = \text{sgn}(p_1p_2r_1r_2)$ in the above sum, since the terms where this is not true do not contribute to the sum. Finally, since $g$ was chosen arbitrarily, the result follows.

(ii) Write out $\langle T(p_1, p_2, n)f, g \rangle = \langle f, T(p_1, p_2, n)^*g \rangle$ for suitable functions $f$ and $g$, using Lemma 9.1(i) and (ii). Using the fact that $f$ and $g$ are chosen arbitrarily and continuity in $y$ of the function $C(y)$, the identity follows. Alternatively, the second identity can also be derived from Lemma 9.1(i). \hfill \Box

9.2. Generators of $\hat{M}$. The main step towards finding a generating set for $\hat{M}$ is the polar-type decomposition for $Q(p_1, p_2, n)$ from Proposition 9.8. The partial isometries $U_{n}^{\sigma, \tau}$, $\sigma, \tau \in \{-, +\}$, $n \in \mathbb{Z}$, from Proposition 9.8 give us the required extra generators for the dual von Neumann algebra $\hat{M}$. First we show that the operators $U_{n}^{\sigma, \tau}$ belong to the von Neumann algebra $\hat{M}$.

Proposition 9.10. For $l \in \mathbb{Z}$ and $\sigma, \tau \in \{-, +\}$, the operator $U_{l}^{\sigma, \tau}$ belongs to $\hat{M}$.

Proof. Since $Q(p_1, p_2, n) \in \hat{M}$ by Proposition 4.9, the polar decomposition $Q(p_1, p_2, n) = U(p_1, p_2, n) |Q(p_1, p_2, n)|$ of Proposition 4.7 gives that $U(p_1, p_2, n) \in \hat{M}$, $|Q(p_1, p_2, n)| \in \hat{M}$. Recall that $U(p_1, p_2, n) = \Upsilon V(p_1, p_2, n) \Upsilon$, and that

$$V_{n}^{\text{sgn}(p_1), \text{sgn}(p_2)} = \text{sgn}(N(\cdot; p_1, p_2, n)) V(p_1, p_2, n).$$

Define the Borel sets $A = \{x \in \mathbb{R} \mid N(x; p_1, p_2, n) > 0\}$, $B = \{x \in \mathbb{R} \mid N(x; p_1, p_2, n) < 0\}$, so that

$$V_{n}^{\text{sgn}(p_1), \text{sgn}(p_2)} = M(\chi_A(\cdot)) V(p_1, p_2, n) - M(\chi_B(\cdot)) V(p_1, p_2, n)$$

and

$$U_{n}^{\text{sgn}(p_1), \text{sgn}(p_2)} = \Upsilon^* V_{n}^{\text{sgn}(p_1), \text{sgn}(p_2)} \Upsilon = \Upsilon^* M(\chi_A) \Upsilon \Upsilon^* V(p_1, p_2, n) \Upsilon - \Upsilon^* M(\chi_B) \Upsilon \Upsilon^* V(p_1, p_2, n) \Upsilon = E_\Omega(A) U(p_1, p_2, n) - E_\Omega(B) U(p_1, p_2, n)$$

where $\chi_A$ is the indicator function of the set $A$ and $\Omega$ is the spectral decomposition of the Casimir operator using Theorem 8.13. Since the Casimir operator $\Omega$ is affiliated to $\hat{M}$ by Theorem 4.4, it follows that the spectral projections $E_\Omega(A), E_\Omega(B) \in \hat{M}$. Since we already noted that $U(p_1, p_2, n) \in \hat{M}$, we see that $U_{n}^{\text{sgn}(p_1), \text{sgn}(p_2)} \in \hat{M}$. \hfill \Box

We can now show that the partial isometries $U_{n}^{\sigma, \tau}$ provide the extra generators for $\hat{M}$ that we need. The following properties are useful.

Lemma 9.11. Let $m, n, n' \in \mathbb{Z}$, $p \in q^{\mathbb{Z}}$ and $\varepsilon, \eta, \sigma, \tau \in \{-, +\}$, then the partial isometries $U_{n}^{\sigma, \tau} : \mathcal{K}(p, m, \varepsilon, \eta) \to \mathcal{K}(p, m + n, \sigma \varepsilon, \tau \eta)$ satisfy the following properties:

(i) $U_{n+n'}^{++} = U_{n}^{++} U_{n'}^{++}$;
(ii) $U_{n}^{-+} = U_{n}^{-+} U_{0}^{-+}$;
(iii) $U_{n}^{+-} = U_{0}^{+-} U_{n}^{+-}$;
(iv) $U_{n}^{--} = U_{n}^{+-} U_{0}^{-+}$.
(v) \((U^\sigma)\ast_n^* = \sigma^{n+1}T(p)\sigma^{n+1}U_n^\sigma\ast_n^*\).

**Proof.** This follows directly from the definition of \(U_n^\sigma\ast_n^*\), see (9.7) and (9.8). For the computation of \((U^\sigma\ast_n^*)\ast_n^*\) it is useful to observe that \(\nu_{\sigma,\nu}(\lambda; p)\nu_{\sigma,\nu}(\sigma T\lambda; p)\) is equal to \(-1\) for \(\sigma = T = 1\) and \(\epsilon(p) = \frac{1}{2}\), and it is equal to 1 in all other cases. \qed

Now we can finally show that the von Neumann algebra \(\hat{M}\) is generated by \(K, E, U_0^+\) and \(U_0^-\).

**Proof of Theorem 4.13.** From Propositions 9.8, 9.10 and Lemma 9.11 it follows that \(\hat{M}\) is generated by \(K, \Omega, U_1^+, U_0^+\) and \(U_0^-\). Using (9.8) and writing \(A(\lambda)\) explicitly, using the appropriate \(c\)-functions, we find for \(x = \mu(\lambda) \in I(p, m, \varepsilon, \eta) \cap I(p, m + 1, \varepsilon, \eta)\)

\[G(\lambda; p, m + 1, \varepsilon, \eta) = \eta G(\lambda; p, m, \varepsilon, \eta),\]

hence \((V_1^+ g)(x) = g(x),\) so we see from (8.27), (8.28), (8.29) and (8.30) that \(U_1^+ = \Upsilon V_1^+ \Upsilon\) is the partial isometry in the polar decomposition of \(E\). Then, using Definition 4.5 for \(\Omega\), it follows that \(\hat{M}\) is generated by \(K, E, U_0^+\) and \(U_0^-\). \qed

10. **Unitary corepresentations**

In this section we need the function \(v : q^Z -\to \mathbb{Z}\) defined by

\[v(t) = \frac{1}{2} \chi(t) + \epsilon(t), \quad t \in q^Z.\]  

(10.1)

So if \(t = q^{2k}\) or \(t = q^{2k-1}\) for some \(k \in \mathbb{Z}\), then \(v(t) = k\).

Recall from Section 3 that we assume \(p \in q^Z, m \in \mathbb{Z}\) and \(\varepsilon, \eta \in \{-, +\}\). Let \(K_d(p, m, \varepsilon, \eta)\) denote the closed subspaces of \(K(p, m, \varepsilon, \eta)\) spanned by all the eigenvectors of \(\Omega\) in \(K(p, m, \varepsilon, \eta)\), and denote its orthogonal complement by \(K_c(p, m, \varepsilon, \eta)\), so that we have a decomposition \(K = K_c \oplus K_d\) corresponding to the continuous and discrete spectrum of \(\Omega\). The unitary operator \(\Upsilon_{p, m}^{\varepsilon, \eta}\) restricted to \(K_d(p, m, \varepsilon, \eta)\) or \(K_c(p, m, \varepsilon, \eta)\) is again a unitary operator mapping into \(\ell^2(\sigma_d(p, m, \varepsilon, \eta))\), respectively \(L^2([-1, 1])\).

10.1. **Discrete series.** In this subsection we assume that \(x \in \sigma_d(p, m, \varepsilon, \eta)\). For \(\varepsilon, \eta \in \{-, +\}\), \(p \in q^Z\) and \(m \in \mathbb{Z}\), we define an element \(e_{m, p}^{\varepsilon, \eta}(p, x) \in K_d(p, m, \varepsilon, \eta)\) by

\[e_{m, p}^{\varepsilon, \eta}(p, x) = \sum_{z \in J(p, m, \varepsilon, \eta)} g_z(\varepsilon \eta x; p, m, \varepsilon, \eta) f_{-m, \varepsilon \eta p, z} z.\]

Since \(\{\delta_x \mid x \in \sigma_d(p, m, \varepsilon, \eta)\}\) is an orthonormal basis for \(\ell^2(\sigma_d(p, m, \varepsilon, \eta))\), it follows from unitarity of \(\Upsilon_{p, m}^{\varepsilon, \eta}\) that the set \(\{e_{m, p}^{\varepsilon, \eta}(p, x) \mid \varepsilon \eta x \in \sigma_d(p, m, \varepsilon, \eta)\}\) is an orthonormal basis for \(K_d(p, m, \varepsilon, \eta)\). This can also be seen directly from (3.19) and (3.38). Moreover, from Theorem 8.13 we see that \(e_{m, p}^{\varepsilon, \eta}(p, x)\) is an eigenvector of \(\Omega\) for eigenvalue \(\varepsilon \eta x\).
Lemma 10.1. The actions of the generators of $\hat{M}$ on $e^{ε,η}_{m,n}(p, x)$ are given by

$$K e^{ε,η}_{m,n}(p, x) = p^2 q^m e^{ε,η}_{m,n}(p, x),$$

$$(q^{-1} - q) E e^{ε,η}_{m,n}(p, x) = q^{-m+\frac{1}{2}} p^{-\frac{1}{2}} \sqrt{1 + 2 \varepsilon \eta} x q^{2m+1} p + q^{4m+2p^2} e^{ε,η}_{m+1}(p, x),$$

$$U_0^{+} e^{ε,η}_{m,n}(p, x) = \eta (-1)^{v(p)} e^{ε,η}_{m,n}(p, x),$$

$$U_0^{-} e^{ε,η}_{m,n}(p, x) = \varepsilon \eta^{\chi(p)} (-1)^{m} e^{ε,η}_{m,n}(p, x).$$

Proof. The action of $K$ follows from (7.19); the action of $E$ follows from (8.27), (8.28), (8.29) and (8.30). To determine the action of $U_0^{+}$ we observe that

$$V_0^{+} \delta_{ε\nu x}(μ(γ)) = \eta \nu^0(γ;p) \frac{G(γ;p, m, ε, -η)}{G(γ;p, m, ε, η)} \delta_{ε\nu x}(−μ(γ)) = \eta (-1)^{v(p)} \delta_{ε\nu x}(μ(γ)),$$

by Lemma 10.2, see below. Applying $\Upsilon^*$ gives us

$$U_0^{+} e^{ε,η}_{m,n}(p, x) = \eta (-1)^{v(p)} e^{ε,η}_{m,n}(p, x).$$

The action of $U_0^{-}$ is calculated in the same way. □

Lemma 10.2. We have

$$\nu^0(λ;p) \frac{G(λ;p, m, ε, -η)}{G(−λ;p, m, ε, η)} = (-1)^{v(p)} \quad \nu^0(λ;p) \frac{G(λ;p, m, ε, -η)}{G(−λ;p, m, ε, η)} = (-1)^{m} η^{χ(p)}.$$

Proof. We treat here the formula for $λ \in \mathbb{T}$, $ε = η = +$, and $m \leq 0$ in detail. The formulas corresponding to the other cases are obtained from similar computations. Note that, by construction, all formulas are equal to $±1$.

Assume $λ \in \mathbb{T}$, $ε = η = +$, and $m \leq 0$. From writing $A(λ;p, m, +, ±)$ and $E(λ;p, m)$ in terms of $q$-shifted factorials and canceling common factors, we obtain (see (9.4))

$$\frac{G(λ;p, m, +, −)}{G(−λ;p, m, +, +)} =$$

$$\frac{λ^{1−m−χ(p)}(qλ/p; q^2)_∞}{(pqλ, −q^{2m−1}pλ, −q^{2m−1}pλ; q^2)_∞} \sqrt{(qpλ^{±1}, −q^{3−2m}λ^{±1}/p, −q^{2m−1}pλ^{±1}/q^2; q^2)_∞}. $$

Recall here that $(aλ^{±1}; q)_∞ = (aλ, a/λ; q)_∞$, which is strictly positive for $0 \neq a \in \mathbb{R}$ and $λ \in \mathbb{T}$. Now assume $ε(p) = \frac{1}{2}$. Using the $θ$-product identity (B.4) we may write

$$(qλ/p; q^2)_∞ = (-1)^{v(p)} λ^{v(p)} q^{−v(p)\chi(p)(1/λ, q^2λ; q^2)_∞},$$

$$(-q^{3−2m}/pλ, −q^{2m−1}pλ; q^2)_∞ = λ^{1−m−v(p)} q^{−m+v(p)(1/m+v(p)(1/λ, −q^2/λ; q^2)_∞},$$

from which it follows that

$$\frac{G(λ;p, m, +, −)}{G(−λ;p, m, +, +)} = (-1)^{v(p)} \frac{λ^{1/λ, q^2λ; q^2}_∞}{(−λ, −q^2/λ; q^2)_∞} \sqrt{(−λ^{±1}, −q^{2λ^{±1}; q^2}_∞). \sqrt{(λ^{±1}, q^2λ^{±1}; q^2)_∞}}.$$
Using the identity \((aq; q)_\infty = (a; q)_\infty/(1 - a)\), and using \((a^{\pm 1}; q) > 0\) for \(a \in \mathbb{T}\), the above expression reduces to
\[
(-1)^{\nu(p)} \frac{1 + \lambda}{1 - \lambda} \sqrt{\frac{(1 - \lambda)(1 - \lambda^{-1})}{(1 + \lambda)(1 + \lambda^{-1})}}.
\]
From this expression we finally obtain
\[
G(\lambda; p, m, +, -) = \begin{cases} 
    i (-1)^{\nu(p)} & \lambda \in \mathbb{T}^+, \\
    i (-1)^{\nu(p)+1} & \lambda \in \mathbb{T}^-.
\end{cases}
\]
Using \(\nu_0^-(\lambda; p) = \mp i\) for \(\lambda \in \mathbb{T}^\pm\), the result follows for the case \(\epsilon(p) = \frac{1}{2}\).

Next we assume \(\epsilon(p) = 0\). The \(\theta\)-product identity (B.3) gives in this case
\[
(q\lambda/p; q^2)_\infty = (-1)^{\nu(p)} (q/\lambda)^{\nu(p)} q^{-\nu(p)(\nu(p)-1)} (q\lambda^{\pm 1}; q^2)_\infty /
(pq/\lambda; q^2)_\infty.
\]
Using \(q^{-1-2m}/p\lambda, -q^{-2m-1}p\lambda; q^2)_\infty = (q\lambda)^{1-m-n}(q^{-m-n}(m+n-2)-q^{-n})(q\lambda^{\pm 1}; q^2)^2).

Now all q-shifted factorials become symmetric in \(\lambda\) and \(\lambda^{-1}\), hence positive, and this leads to
\[
G(\lambda; p, m, +, -) = (-1)^{\nu(p)} \frac{(q\lambda^{\pm 1}; q^2)_\infty}{(pq\lambda^{\pm 1}, -q\lambda^{\pm 1}; q^2)_\infty} \sqrt{(pq\lambda^{\pm 1}, -q\lambda^{\pm 1}; q^2)^2} = (-1)^{\nu(p)}.
\]
This proves the result in case \(\epsilon(p) = 0\).

Notice that the invariance of \(\mathcal{L}_{p,x}\) as defined in Lemma 5.1 follows from the fact that for \(p_1, p_2 \in I_q\) and \(n \in \mathbb{Z}\), the operator
\[
Q(p_1, p_2, n): \mathcal{K}(p, m, \varepsilon, \eta) \to \mathcal{K}(p, m + n, \text{sgn}(p_1) \varepsilon, \text{sgn}(p_2) \eta)
\]
and \(\text{sgn}(p_1 p_2) Q(p_1, p_2, n) \Omega \subseteq \Omega Q(p_1, p_2, n)\) for all \(p_1, p_2 \in I_q\) and \(n \in \mathbb{Z}\), see Lemma 7.1 and Proposition 11.9. This proves that \(\mathcal{L}_{p,x}\) is an invariant subspace for \(\hat{M}\), hence it gives rise to a corepresentation of \((\hat{M}, \Delta)\). Since \(W\) is the multiplicative unitary, its restriction \(W_{p,x}\) is also unitary. This proves Lemma 5.1.

In order to prove Proposition 5.2, we have to do some bookkeeping, based on the discrete spectrum of the Casimir operator \(\hat{\Omega}\) acting on \(\mathcal{K}(p, m, \varepsilon, \eta)\) given as \(\sigma_d(p, m, \varepsilon, \eta) = \{\mu(\lambda) \mid \lambda \in D(p, m, \varepsilon, \eta)\}\), where the set \(D(p, m, \varepsilon, \eta)\) is given explicitly in (8.10), (8.14), (8.18). So we have to keep track which of the eigenvectors \(e_m^{\varepsilon, \eta}(p, x)\) in \(\mathcal{L}_{p,x}\) correspond to eigenvalues in the spectrum of \(\hat{\Omega}\) in \(\mathcal{K}(p, m, \varepsilon, \eta)\).

**Proof of Proposition 5.2.** Note that \(p = q^{-l-j}\) and \(|\lambda| = q^{1+l+j}\). Since \(|\lambda| > 1\), it follows that \(l < j\). In order to see that \(|\lambda| \in q^{2\mathbb{Z}+1}p\), consider \(x = \mu(\lambda)\) where \(\lambda \in -q^{-m} \cup q^{-m}\). It follows from (8.10), (8.14) and (8.18) that if there exist \(m \in \mathbb{Z}\) and \(\varepsilon, \eta \in \{-, +\}\) such that \(\mathcal{K}(p, m, \varepsilon, \eta) \ni e_m^{\varepsilon, \eta}(p, x) \neq 0\), then \(|\lambda| \in q^{2\mathbb{Z}+1}p\).

Assume first that \(x > 0\). It follows from (8.13) that an eigenvector \(e_m^{\varepsilon}(p, x) \in \mathcal{K}(p, m, +, +)\) is non-zero if and only if \(x = \mu(q^{1+2k}p)\) such that \(q^{1+2k}p > 1\). So such an eigenvector exists for all \(m \in \mathbb{Z}\). It follows from (8.14) that such an eigenvector \(e_m^{\varepsilon}(p, x) \in \mathcal{K}(p, m, +, +)\) does not exist, since the discrete spectrum of \(\hat{\Omega}\) on \(\mathcal{K}(p, m, -\varepsilon, -\eta)\) is always negative. A check shows that eigenvectors \(e_m^{\varepsilon}(p, x) \in \mathcal{K}(p, m, +, +)\) satisfying \(\hat{\Omega} e_m^{\varepsilon}(p, x) = -\mu(q^{1+2k}p) e_m^{\varepsilon}(p, x)\) exist precisely when \(k \geq -m\). Similarly, eigenvectors \(e_m^{\varepsilon}(p, x) \in \mathcal{K}(p, m, +, +)\) satisfying
and we consider them separately.

We see that \( K \) is affiliated to \( \hat{M} \) by our choice of \( m \).

By considering the partial isometry \( e_m(p,x) \) in Proposition 5.2, we find that such eigenvectors occur for the eigenvalue \( x = \mu(q^{1+2l}p) \) in \( K(p,m,-,+) \) precisely when \( l \geq 0 \) and such eigenvectors occur for the eigenvalue \( x = \mu(q^{1-2l}p^{-1}) \) in \( K(p,m,+,--) \) precisely when \( j \leq 0 \). So these cases cannot occur simultaneously, and we consider them separately.

From (8.18) we find eigenvectors \( e_m(p,x) \) in case \( l \geq \max(0,m) \) or \( j \leq 0 \). Using (8.14) we see that \( e_m(p,x) \) is an eigenvector for the eigenvalue \( x \) precisely when \( l \geq 0 \) or \( l \geq m \) for the case \( pq^m \leq 1 \), i.e. \( l + j \leq m \). In case \( pq^m \geq 1 \), or \( l + j \geq m \), we see that \( e_m(p,x) \) is an eigenvector for the eigenvalue \( x \) precisely when \( j \leq 0 \) or \( j \leq m \).

Assume \( l \geq 0 \), and hence \( j \geq 0 \). Then we find no eigenvectors of type \( e_m(p,x) \) and \( e_m(p,x) \) for all \( m \in \mathbb{Z} \) by (8.11). We find \( e_m(p,x) \) for all \( m \in \mathbb{Z} \) with \( m \leq l \) by considering the case \( m \leq 0 \) and \( m \geq 0 \) separately in (8.18). Consider now (8.14). In case \( m \leq l + j \) (or \( q^m p \geq 1 \)) we find eigenvectors \( e_m(p,x) \) for \( j \leq 0 \), which is excluded in this case, or \( j \leq m \).

So in total we get \( e_m(p,x) \) for \( j \leq m \leq l + j \). In case \( m \geq l + j \) (or \( q^m p \leq 1 \)) we find eigenvectors \( e_m(p,x) \) for \( l \geq m \), which is excluded since it implies \( j \leq 0 \), and for \( l \geq 0 \). So we find \( e_m(p,x) \) for \( m \geq l + j \). Combining we find \( e_m(p,x) \) for all \( m \in \mathbb{Z} \) with \( m \geq j \). This gives case (ii) of Proposition 5.2. Case (iii) is obtained similarly by analyzing \( j \leq 0 \) and hence \( l < 0 \).

Next we show that the corresponding unitary corepresentations of \((M,\Delta)\) are irreducible.

**Proof of Proposition 5.3.** We have already observed that \( L_{p,x} \) is invariant. Consider \( L_{p,x} \) with the convention \( p = q^{-l-j} \) with \( l < j \) as in Proposition 5.2, and assume for the moment that \( l + 1 \neq j \), or \( l + 1 < j \). We claim that is possible to choose \( \varepsilon,\eta \in \{\pm\}, m \in \mathbb{Z} \) so that

1. \( 0 \neq e_{m,\varepsilon,\eta}(p,x) \in L_{p,x} \)
2. \( e_{m,\varepsilon,\eta}(p,x) \notin L_{p,x} \) for \((s,t) = (-,0), (+,0), (-,0)\).

Take \( m \in \mathbb{Z} \) such that \( l < m < j \), which is possible by the assumption \( l + 1 < j \). By Proposition 5.2 we find by inspection that \( e_m(p,x) \) in case (i), \( e_m(p,x) \) in case (ii) and \( e_m(p,x) \) in case (iii) gives the required choice.

Recall that \( K \) is affiliated to \( \hat{M} \), see Proposition 1.1. Thus, if \( P \) denotes the spectral projection of \( K \) with respect to the eigenvalue \( p^{1/2} q^m \), we find \( P \in \hat{M} \). So \( P|_{L_{p,x}} \) is the orthogonal projection onto the closed subspace spanned by \( \{e_{s,t}(p,x) \mid s,t \in \{-,0\} \} \). But by our choice of \( m,\varepsilon \) and \( \eta \), this implies that \( P|_{L_{p,x}} \) is the orthogonal projection onto \( e_{m,\varepsilon,\eta}(p,x) \).

So we can look at the invariant subspace of \( L_{p,x} \) generated by the vector \( e_{m,\varepsilon,\eta}(p,x) \).

Consider the closure \( L \) of \( \{T|_{L_{p,x}} e_{m,\varepsilon,\eta}(p,x) \mid T \in \hat{M} \} \), so that \( L \) is an invariant subspace of \( L_{p,x} \) and \( L \neq \{0\} \). Using Lemma 10.1, the partial isometry \( V \) in the polar decomposition of \( E \) maps \( e_{m,\varepsilon,\eta}(p,x) \) to \( e_{m+1,\varepsilon,\eta}(p,x) \) if this corresponds to an eigenvector of \( \Omega \) in \( K(p,m+1,\varepsilon,\eta) \) and to zero if this is not the case. Using Lemma 10.1 and the fact that the partial isometry \( V \in \hat{M} \) by Proposition 1.3, we see that all other vectors in the three lists for \( L_{p,x} \) in Proposition 5.2 can be reached by repeated application of \( V, V^*, U_0^{-} \) and \( U_0^{+} \). Hence \( L = L_{p,x} \), and irreducibility follows.
In case $l + 1 = j$ we cannot establish that $P|_{\mathcal{L}_{p,x}}$ is the orthogonal projection on a single vector in the lists as in Proposition 5.2, but we can view it, by taking $m = l$, as an orthogonal projection on the subspace $\mathbb{C}e_{l+1}^{-}(p,x) \oplus \mathbb{C}e_{l+1}^{+}(p,x)$ in case (i) of Proposition 5.2, on $\mathbb{C}e_{l+1}^{-}(p,x) \oplus \mathbb{C}e_{l+1}^{+}(p,x)$ in case (ii) and on $\mathbb{C}e_{l+1}^{-}(p,x) \oplus \mathbb{C}e_{l+1}^{+}(p,x)$ in case (iii). Now use the fact that the partial isometry $V \in \hat{M}$ of $E$ kills the second vector in each of these spaces to see that the range of the composition $V|_{\mathcal{L}_{p,x}}P|_{\mathcal{L}_{p,x}}$ has dimension 1 spanned by $e_{l+1}^{+}(p,x)$ in case (i), by $e_{l+1}^{-}(p,x)$ in case (ii) and by $e_{l+1}^{+}(p,x)$ in case (iii). Now we can argue as in the case $l + 1 > j$ above to find that $\mathcal{L}_{p,x}$ is irreducible. \hfill \Box

10.2. Principal series. We start by recalling the definition of Section 6. Let $x = \cos \theta \in [-1, 1]$ and $p \in qZ$. We define a Hilbert space $\mathcal{L}_{p,x}$ by

$$\mathcal{L}_{p,x} = \bigoplus_{\varepsilon, \eta \in \{-, +\}} \ell^2_{\varepsilon, \eta}(p,x),$$

where each space $\ell^2_{\varepsilon, \eta}(p,x)$ denotes a copy of $\ell^2(\mathbb{Z})$ with standard orthonormal basis $\{e_{m,\eta}(p,x) \mid m \in \mathbb{Z}\}$. For convenience we recall the definition of the operators $K, E, U_0^{+}, U_0^{-}$ on $\mathcal{L}_{p,x}$ as given in (5.2):

$$K e_{m,\eta}(p,x) = p^{\frac{1}{2}}q^{m} e_{m,\eta}(p,x),$$

$$(q^{-1} - q)E e_{m,\eta}(p,x) = q^{-m+\frac{1}{2}}p^{-\frac{1}{2}}[1 + \varepsilon\eta q^{2m+1}e^{i\theta}]e_{m+1,\eta}(p,x),$$

$$U_0^{+} e_{m,\eta}(p,x) = \eta (-1)^{\varepsilon(p)} e_{m,-\eta}(p,x),$$

$$U_0^{-} e_{m,\eta}(p,x) = \varepsilon\eta^{\chi(p)}(-1)^{m} e_{m,-\eta}(p,x).$$

The operators $E$ and $K$ are unbounded closable operators with dense domain the finite linear combinations of the orthonormal basis vectors $e_{m,\eta}(p,x)$, $m \in \mathbb{Z}$, $\varepsilon, \eta \in \{-, +\}$. The operators $U_0^{+}$ and $U_0^{-}$ are bounded; they are isometries.

Remark 10.3. It is useful to observe that each subspace $\ell^2_{\varepsilon, \eta}(p,x)$ of $\mathcal{L}_{p,x}$ is a principal series $U_q(\mathfrak{su}(1,1))$-module as defined by (8.22). The above defined actions of $K$ and $E$ on $e_{m,\eta}(p,x)$ coinide with the actions of $K$ and $E$ in the principal series representation $\pi_{b(x)}$ on the standard basis vector $e_{m+k}$, where $\mu(q^{2b}) = -\varepsilon\eta x$ and $p = q^{2k+2\varepsilon(p)}$. Using $\Omega = \frac{1}{2}((q^{-1} - q)^2E^*E - qK^2 - q^{-1}K^2)$ it can be verified that $\Omega e_{m,\eta}(p,x) = \varepsilon\eta x e_{m,\eta}(p,x)$. Furthermore, the discrete series corepresentations from Lemma 10.1 can (formally) be obtained from (10.2) by taking $\varepsilon\eta x$ in the discrete spectrum of $\Omega$.

The operators (10.2) generate a von Neumann algebra $\hat{M}_{p,x}$. We can construct the elements $Q_{p,x}(p_1, p_2, n)$, $p_1, p_2 \in I_p$, $n \in \mathbb{Z}$ for $\hat{M}_{p,x}$, basically by reversing the arguments that led to the proof of Theorem 4.13. Let us first define operators $U_{n,\sigma,\tau}^{\sigma,\tau} : \mathcal{L}_{p,x} \rightarrow \mathcal{L}_{p,x}$ for $n \in \mathbb{Z}$ and $\sigma, \tau \in \{-, +\}$ as follows. We set $U_{0,\sigma}^{\sigma} = \text{Id}$, and we define $U_{1,\sigma}^{\sigma}$ as the partial isometry in the polar decomposition of $E$, i.e., $U_{1,\sigma}^{\sigma}e_{m,\eta}(p,x) = e_{m+1,\eta}(p,x)$. Now we define $U_{n,\sigma,\tau}^{\sigma,\tau}$, $n \in \mathbb{Z}$,
\[ \sigma, \tau \in \{-, +\}, \text{ recursively by} \]
\[
\begin{align*}
U_{n}^{++} &= U_{n-1}^{++}, & n \in \mathbb{N}, \\
U_{n}^{+-} &= U_{n}^{0+}U_{n}^{++}, & n \in \mathbb{N}, \\
U_{n}^{-+} &= U_{n}^{+}U_{n}^{0-}, & n \in \mathbb{N}, \\
U_{n}^{--} &= U_{n}^{+}U_{n}^{-}, & n \in \mathbb{N}_0, \\
U_{n}^{\sigma\tau} &= \sigma^{n+1}\tau^{(p-1)(1-\sigma)+1}(U_{n}^{\sigma\tau})^*, & n \in \mathbb{N}.
\end{align*}
\]

From \((10.2)\), Lemma \(10.2\) and the identity \(G(\lambda; p, m + 1, \varepsilon, \eta) = \eta G(\lambda; p, m, \varepsilon, \eta)\), see the proof of Theorem \(4.13\) at the end of Section \(9.2\), we find
\[
U_{n}^{\sigma\tau}e^{\varepsilon\eta}_{m}(p, x) = \varepsilon^{\frac{1}{2}(1-\sigma)}\eta^{\frac{1}{2}(1-\tau)+n}\nu_{n}(\lambda; p) \frac{G(\lambda; p, m + n, \sigma\varepsilon, \tau\eta)}{\lambda G(\lambda; p, m, \varepsilon, \eta)} e^{\sigma\varepsilon, \tau\eta}_{m+n}(p, x), \tag{10.3}
\]
where \(\mu(\lambda) = x\). Now for \(p_1, p_2 \in I_q\) we set \(\sigma = \text{sgn}(p_1), \tau = \text{sgn}(p_2)\), and we define
\[
Q_{p,x}(p_1, p_2, n) = U_{n}^{\sigma\tau}H(\Omega; p_1, p_2, n)P(p_1, p_2, n),
\]
where \(P(p_1, p_2, n)\) is the spectral projection of \(K\) defined in \((10.2)\) corresponding to the eigenvalue \(\sqrt{q^{-n}|p_2/p_1|}\), and \(H\) is the function defined in Proposition \(9.8\).

**Lemma 10.4.** The operators \(Q_{p,x}(p_1, p_2, n)\) have the following properties:

(i) \(Q_{p,x}(p_1, p_2, n)\) acts on the standard basisvectors of \(L_{p,x}\) by
\[
Q_{p,x}(p_1, p_2, n)e^{\varepsilon\eta}_{m}(p, x) = \begin{cases} 
0, & \text{if } |p_2| \neq q^n \text{ or } |p_1| \neq q^n, \\
C(\sigma\varepsilon\eta x; m, \varepsilon, \eta; p_1, p_2, n) e^{\sigma\varepsilon, \tau\eta}_{m+n}(p, x), & q^{2m} \neq q^{-n}\frac{|p_2|}{|p_1|},
\end{cases}
\]
where \(C\) is the function given by \((9.8)\).

(ii) In \(M_{p,x}\) we have
\[
Q_{p,x}(p_1, p_2, n)Q_{p,x}(r_1, r_2, m) = 0,
\]
if \(|p_2| \neq q^n\) or \(|p_1| \neq q^n\), and
\[
Q_{p,x}(p_1, p_2, n)Q_{p,x}(r_1, r_2, m) = \sum_{x_1, x_2 \in I_q} a_{x_1}(r_1, p_1) a_{x_2}(r_2, p_2) Q_{p,x}(x_1, x_2, n + m).
\]

(iii) The adjoint of \(Q_{p,x}(p_1, p_2, n)\) in \(M_{p,x}\) is given by
\[
Q_{p,x}(p_1, p_2, n)^* = (-q)^n \text{sgn}(p_1)^{\chi(p_1)} \text{sgn}(p_2)^{\chi(p_2)} Q_{p,x}(p_1, p_2, -n).
\]

**Proof.** (i) First note that \(P(p_1, p_2, n)\) is the orthogonal projection onto
\[
\text{Span}\left\{e^{\varepsilon\eta}_{m}(p, x) \mid q^{2m} = q^{-n}\frac{|p_2|}{|p_1|}, \varepsilon, \eta \in \{-, +\}\right\}.
\]
The explicit action of \(Q_{p,x}(p_1, p_2, n)\) on an orthonormal basisvector \(e^{\varepsilon\eta}_{m}(p, x)\) now follows from \((10.3)\) and \((9.3)\).
(ii) By (i) the product of two $Q_{p,x}$ operators is given by

$$Q_{p,x}(p_1, p_2, n)Q_{p,x}(r_1, r_2, m)e^{\varepsilon \eta}(p, x) =$$

$$C(y; k + m, \text{sgn}(r_1)\varepsilon, \text{sgn}(r_2)\eta; p_1, p_2, n)C(\text{sgn}(p_1 p_2) y, k, \varepsilon, \eta; r_1, r_2, m) e^{\varepsilon' \eta'}(p, x)$$

if $q^{2k} = q^{-m}\frac{p_1}{p_2}$ and $q^{2k+2m} = q^{-n}\frac{p_2}{p_1}$, and it is zero otherwise. Here $y = \text{sgn}(p_1 p_2) r_1 r_2 \varepsilon \eta x$, $\varepsilon' = \text{sgn}(r_1 p_1) \varepsilon$ and $\eta' = \text{sgn}(r_2 p_2) \eta$. Now we use the product formula for the function $C$ from Lemma 9.9, then it follows that

$$Q_{p,x}(p_1, p_2, n)Q_{p,x}(r_1, r_2, m)e^{\varepsilon \eta}(p, x) =$$

$$\sum_{x_1, x_2 \in I_q} a_{x_1}(r_1, p_1)a_{x_2}(r_2, p_2)C(y; k, \varepsilon, \eta; x_1, x_2, m + n) e^{\varepsilon \eta}(p, x).$$

if $|\frac{p_1}{p_2}| = q^m$ and $|\frac{p_2}{p_1}| = q^n$, and the product is zero otherwise. Observe that inside the sum on the right hand side the condition $q^{2k} = q^{-n-m}\frac{p_2}{p_1}$ is satisfied because $|x_1| = |x_2|$, and since $Q_{p,x}(x_1, x_2, n + m) = 0$ otherwise, the product formula for two $Q_{p,x}$ operator follows.

(iii) The adjoint of $Q_{p,x}(p_1, p_2, n)$ follows from (i) and the symmetry property for $C$ from Lemma 9.9.

**Proposition 10.5.** Let $p_1, p_2 \in I_q$, $n \in \mathbb{Z}$, and let $\omega_{f,g} \in B(K)_*$ be the normal functional given by $\omega_{f,g}(y) = \langle yf, g \rangle$ for $y \in B(K)$, where $f = f_{0,p_1,1}$ and $g = f_{n,p_2,1}$. Then there exists a unique unitary corepresentation $W_{p,x} \in M \otimes B(L_{p,x})$ such that

$$(\omega_{f,g} \otimes \text{Id})(W_{p,x}^*) = Q_{p,x}(p_1, p_2, n).$$

The proof of this proposition follows from Lemmas 10.6 - 10.9.

**Lemma 10.6.** Assume $p_1, t_1 \in I_q$, $m, m \in \mathbb{Z}$ and $\varepsilon, \eta \in \{+, -\}$. There exists a unique co-isometry $W_{p,x} \in M \otimes B(L_{p,x})$ such that

$$W_{p,x}(f_{m_1, p_1 t_1} \otimes e^{\varepsilon \eta}(p, x)) =$$

$$\sum_{p \in I_q} C(\text{sgn}(p_1 p_2) \varepsilon \eta x; m, \varepsilon, \eta; p_1, p_2, \chi(p_2/p_1 p) - 2m)$$

$$\times f_{\chi(p_2/p_1 p) - m - 2m, p_2 t_1} e^{\varepsilon \eta\chi(p_2/p_1 p) - m}(p, x).$$

**Proof.** We set $W_{\mathcal{T}} = (\text{Id} \otimes \Upsilon)W(\text{Id} \otimes \Upsilon^*)$. For $i = 1, 2$, assume $m_i, n_i \in \mathbb{Z}$, $p_i, t_i \in I_q$, $\varepsilon_i, \eta_i \in \{+, -\}$, $p \in q_i \mathbb{Z}$, $g_i \in L^2(I(p, m_i, \varepsilon_i, \eta_i))$. Recall from (7.4) that $(\omega_{f_{n_1 t_1}, f_{n_2 t_2}} \otimes \text{Id})(W_{\mathcal{T}}^*) = \delta_{t_1 t_2} Q(p_1, p_2, n_2 - n_1)$, then by Lemma 7.4 and (7.1) we have

$$\langle W_{\mathcal{T}}^*(f_{n_1 p_1 t_1} \otimes g_1), f_{n_2 p_2 t_2} \otimes g_2 \rangle = \langle (\omega_{f_{n_1 t_1}, f_{n_2 t_2}} \otimes \text{Id})(W_{\mathcal{T}}^*) g_1, g_2 \rangle$$

$$= \delta_{t_1 t_2} \delta_{\text{sgn}(p_1) \varepsilon_1 \varepsilon_2} \delta_{\text{sgn}(p_2) \eta_1 \eta_2} \delta_{m_1 + n_2 - n_1, m_2} \langle C(\cdot; m_1, \varepsilon_1, \eta_1; p_1, p_2, n_2 - n_1) g_1(\text{sgn}(p_1 p_2 \cdot)), g_2 \rangle,$$

if $p = q^{n_1 - n_2 - 2m_2}\frac{p_2}{p_1}$, and the expression is equal to zero otherwise. Now it follows that

$$W_{\mathcal{T}}^*(f_{n_1 p_1 t_1} \otimes g_1) = \sum_{p \in I_q} f_{k_1, p_2 t_1} \otimes C(\cdot; m_1, \varepsilon_1, \eta_1; p_1, p_2, k_1 - n_1) g_1(\text{sgn}(p_1 p_2 \cdot)),$$
where \( k_i = k_i(p_2) \in \mathbb{Z}, i = 1, 2 \), is determined by \( p = q_i^{n_i-k_i,2m_i} |\frac{p_2}{p_1}| \).

Let \( (\delta_n)_{n \in \mathbb{N}} \) be a sequence of nonnegative real-valued continuous functions on \([-1, 1]\) that approximate the Dirac \( \delta \)-distribution \( \delta(\cdot-x) \). In particular, the functions \( \delta_n \) have the following property:

\[
\lim_{n \to \infty} \int_{-1}^{1} \delta_n(u) f(u) du = f(x),
\]

for a continuous function \( f \) on \([-1, 1]\). We write \( \delta_n(\cdot; p, m_i, \varepsilon_i, \eta_i) \) for the function \( \delta_n(\varepsilon_i \eta_i \cdot) \) considered as a function in \( L^1(I(p, m_i, \varepsilon_i, \eta_i)) \). In particular, \( \delta_n(x; p, m_i, \varepsilon_i, \eta_i) = 0 \) for \( x \notin [-1, 1] \). We set \( g_i = \sqrt{\delta(\cdot; p, m_i, \varepsilon_i, \eta_i)} \), then by unitarity of \( W_t \),

\[
\delta_{n_1n_2} = \sum_{p_3 \in I_q} \int_{-1}^{1} g_1(\text{sgn}(p_1p_3)x) g_2(\text{sgn}(p_2p_3)x) C(x; m_1, \varepsilon_1, p_1, p_2, k_1 - n_1) C(x; m_2, \varepsilon_2, p_2, p_3, k_2 - n_2) dx
\]

Here \( k_i = k_i(p_3) \). Note that \( C \) is real-valued on \([-1, 1]\). Since the function \( C \) is continuous on \([-1, 1]\), we find from letting \( n \to \infty \),

\[
\delta_{n_1n_2} = \sum_{p_3 \in I_q} \int_{-1}^{1} C(\text{sgn}(p_3)y; m_1, \varepsilon_1, p_1, p_2, m_1 + \chi(p_2/p_1)) \times C(\text{sgn}(p_3)y; m_1, \varepsilon_1, p_1, p_2, m_1 + \chi(p_2/p_1) - 2m_1 + \chi(p_2/p_1)) \delta_{n_1n_2}
\]

\( \delta_{n_1n_2} \) converges in \( M(\zeta^{n} \otimes \text{Id}_{L^2(I_q)} \otimes \text{Id}) \). Absolute convergence of this sum is obtained from Lemma B.12, see also the proof of Lemma 10.7. Now it follows that \( W_{p,x}^{*} \) defined by (10.4) is an isometry.

Furthermore, from the explicitly formula for \( W_{p,x}^{*} \) we see that \( W_{p,x}^{*} \) commutes with \( M(\zeta^{n} \otimes \text{Id}_{L^2(I_q)} \otimes \text{Id}) \). Therefore

\[
W_{p,x}^{*} \in \left( L^\infty(\mathbb{T}) \otimes \mathcal{C} \text{Id}_{L^2(I_q)}(\mathbb{T}) \otimes \text{Id}_{L^2(I_q)} \right)' = L^\infty(\mathbb{T}) \otimes B(L^2(I_q)) \otimes \mathcal{C} \text{Id}_{L^2(I_q)} \otimes B(L_{p,x}),
\]

so we have indeed \( W_{p,x} \in M \otimes B(L_{p,x}) \), by Remark 3.3 and the observation recalled after Definition 3.2. 

**Lemma 10.7.** Let \( m_1, m_2 \in \mathbb{Z}, p_1, p_2, t_1, t_2 \in I_q \), then

\[
(\omega_{f_{m_1t_1t_1},f_{m_2t_2t_2}} \otimes \text{Id})(W_{p,x}^{*}) = \delta_{t_1t_2} Q_{p,x}(p_1, p_2, m_2 - m_1).
\]

**Proof.** Let \( m \in \mathbb{Z} \) and \( \varepsilon, \eta \in \{-, +\} \). From Lemma 10.6 we find

\[
(\omega_{f_{m_1t_1t_1},f_{m_2t_2t_2}} \otimes \text{Id})(W_{p,x}^{*}) e_{m}^{\varepsilon}(p, x) =
\]

\[
\delta_{t_1t_2} \delta_{\varepsilon}(p_2/p_1)^{m_1-2m_2} C(\text{sgn}(p_1p_2)\varepsilon; m, \varepsilon, \eta; p_1, p_2, m_2 - m_1) e_{m+m_2-m_1}^{\text{sgn}(p_1)\varepsilon, \text{sgn}(p_2)\eta}(p, x).
\]

Compare this with Lemma 10.4(i), then the result follows. 

\( \square \)
Lemma 10.8. \( W_{p,x} \) is a corepresentation of \((M, \Delta)\), i.e.,
\[
(\Delta \otimes \text{Id})(W_{p,x}) = (W_{p,x})_{12}(W_{p,x})_{23}.
\]

**Proof.** We use the structure formula for the \( Q_{p,x} \) operators from Lemma 10.4. For \( i = 1, 2 \) let \( m_i, n_i \in \mathbb{Z} \) and \( p_i, r_i, s_i, t_i \in I_q \). Define for \( i = 1, 2 \) the elements \( f_i, g_i \in \mathcal{K} \) by \( f_i = f_{m_i, p_i, t_i} \) and \( g_i = f_{m_i, r_i, s_i} \). By Lemmas 10.4 and 10.7 we have
\[
\left((\omega_{f_1, f_2} \otimes \text{Id})(W_{p,x}^*)\right)\left((\omega_{g_1, g_2} \otimes \text{Id})(W_{p,x}^*)\right) = \delta_{t_1 t_2} \delta_{s_1 s_2} \sum_{x_1, x_2 \in I_q} a_{x_1}(r_1, p_1) a_{x_2}(r_2, p_2) Q_{p,x}(x_1, x_2, n + m),
\]
where \( n = n_2 - n_1 \) and \( m = m_2 - m_1 \). Similar as in the proof of Proposition 10.10, see §7.3, it now follows that
\[
\left((\omega_{f_1, f_2} \otimes \text{Id})(W_{p,x}^*)\right)\left((\omega_{g_1, g_2} \otimes \text{Id})(W_{p,x}^*)\right) = (\omega_{W(g_1 \otimes f_1), W(g_2 \otimes f_2)} \otimes \text{Id})(1 \otimes W_{p,x}^*),
\]
where \( W \in B(\mathcal{K} \otimes \mathcal{K}) \) denotes the multiplicative unitary. We rewrite the right hand side as
\[
(\omega_{g_1, g_2} \otimes \omega_{f_1, f_2} \otimes \text{Id})((W^* \otimes 1)(1 \otimes W_{p,x}^*)(W \otimes 1)),
\]
then we conclude that \( W_{12}(W_{p,x}^*)_{23} W_{12} = (W_{p,x}^*)_{23}(W_{p,x}^*)_{13} \). Using \( \Delta(y) = W^*(1 \otimes y)W \) for \( y \in M \), it follows that \( (\Delta \otimes \text{Id})(W_{p,x}) = (W_{p,x})_{13}(W_{p,x})_{23} \). \( \square \)

**Lemma 10.9.** \( W_{p,x} \) is unitary.

**Proof.** For \( i = 1, 2 \) let \( m_i, n_i \in \mathbb{Z} \), \( p_i, t_i \in I_q \), \( \varepsilon_i, \eta_i \in \{-, +\} \). Using Lemma 10.6 we find
\[
W_{p,x}(f_{m_2, p_2, t_2} \otimes e_{n_2}^{\varepsilon_2, \eta_2}(p, x)) = \\
\sum_{p_1 \in I_q} C(\varepsilon_2 \eta_2; \chi(p_2/p_1 p) - n_2, \text{sgn}(p_1) \varepsilon_2, \text{sgn}(p_2) \eta_2; p_1, p_2, 2n_2 - \chi(p_2/p_1 p))
\times f_{\chi(p_2/p_1 p) + m_2 - 2n_2, p_1, t_2} \otimes e_{\chi(p_2/p_1 p) - n_2}^{\text{sgn}(p_1) \varepsilon_2, \text{sgn}(p_2) \eta_2}(p, x).
\]
Let \( W_T \) be defined as in the proof of Lemma 10.6, and for \( i = 1, 2 \) let \( g_i \in L^2(I(p, n_i, \varepsilon_i, \eta_i)) \). Using (9.3) it follows that
\[
\left\langle f_{m_1, p_1, t_1} \otimes g_1, W_T(f_{m_2, p_2, t_2} \otimes g_2) \right\rangle = (-q)^{m_2 - m_1} \text{sgn}(p_1) \chi(p_1) \text{sgn}(p_2) \chi(p_2) \left\langle f_{m_2, p_2, t_2} \otimes g_1, W_T^*(f_{m_1, p_1, t_1} \otimes g_2) \right\rangle.
\]
In the same way we find from Lemmas 10.7 and 10.4(iii)
\[
\left\langle f_{m_1, p_1, t_1} \otimes e_{n_1}^{\varepsilon_1, \eta_1}(p, x), W_{p,x}(f_{m_2, p_2, t_2} \otimes e_{n_2}^{\varepsilon_2, \eta_2}(p, x)) \right\rangle = (-q)^{m_2 - m_1} \text{sgn}(p_1) \chi(p_1) \text{sgn}(p_2) \chi(p_2) \left\langle f_{m_2, p_2, t_2} \otimes e_{n_1}^{\varepsilon_1, \eta_1}(p, x), W_{p,x}^*(f_{m_1, p_1, t_1} \otimes e_{n_2}^{\varepsilon_2, \eta_2}(p, x)) \right\rangle.
\]
It is now straightforward to check that \( W_{p,x} \) is obtained from \( W_T \) in the same as \( W_{p,x}^* \) from \( W_T^* \) in the proof of Lemma 10.6. Then it follows that \( W_{p,x} \) is an isometry, as is \( W_{p,x}^* \), hence \( W_{p,x} \) is unitary. \( \square \)
It is a direct consequence of the proof of Lemma 10.6 that the corepresentations \( W_p, x \) occur as principal series in the left regular corepresentation \( W \) of \((M, \Delta)\) as in Proposition 5.6. Let us give the intertwiner explicitly.

First observe that we have

\[
K_c(p) = \bigoplus_{\varepsilon, \eta \in \{-, +\}} K(p, \varepsilon, \eta) \cap K_c,
\]

where

\[
K(p, \varepsilon, \eta) = \bigoplus_{m \in \mathbb{Z}} K(p, m, \varepsilon, \eta).
\]

We define

\[
I_{p, \varepsilon, \eta} : K_c(p, \varepsilon, \eta) \to \int_{-1}^{1} \ell_{\varepsilon, \eta}^2(p, x) \, dx
\]

\[
f_m \mapsto \int_{-1}^{1} \left( \Upsilon_{p, m}^{\varepsilon, \eta} f_m \right)(\varepsilon \eta x) e_{\varepsilon, \eta}^m(p, x) \, dx,
\]

where \( f_m = f_m(p, \varepsilon, \eta) \in K_c(p, m, \varepsilon, \eta) \). The intertwiner \( I_p : K(p) \to \int_{-1}^{1} L_{p, x} \, dx \) which implements the equivalence in Proposition 5.6 is given by

\[
I_p = \bigoplus_{\varepsilon, \eta \in \{-, +\}} I_{p, \varepsilon, \eta}.
\]

Remark 10.10. In Section 8.4 the \( U_q(\mathfrak{su}(1, 1)) \)-representations \( \pi_{K_c}(p, \varepsilon, \eta) \) on \( K_c(p, \varepsilon, \eta) \) are decomposed into irreducible \( \ast \)-representations. Let \( \pi_{K_c}(p, \varepsilon, \eta) \) be the \( U_q(\mathfrak{su}(1, 1)) \)-representation \( \pi_{K}(p, \varepsilon, \eta) \) restricted to \( K_c(p, \varepsilon, \eta) \). Using the decompositions from Section 8.4, see Theorems 8.14, 8.15, 8.16 and 8.17, we see that

\[
\pi_{K_c}(p, \varepsilon, \eta) \cong \int_{-1}^{1} \pi_{b(-\varepsilon \eta x), \varepsilon(p)} \, dx,
\]

where \( b(y), y \in [-1, 1] \), is the unique number in \([0, -\pi^2 \ln q]\) determined by \( \mu(q^{2ib(y)}) = y \). Here we regard \( \ell_{\varepsilon, \eta}^2(p, x) \) as a \( U_q(\mathfrak{su}(1, 1)) \)-module as explained in Remark 10.3. The operators \( I_{p, \varepsilon, \eta} \) are the precisely the intertwiners for the above equivalence. Here we regard \( \ell_{\varepsilon, \eta}^2(p, x) \) as \( U_q(\mathfrak{su}(1, 1)) \)-modules corresponding to the principal series \( \pi_{b(-\varepsilon \eta x), \varepsilon(p)} \) as explained in Remark 10.3.

Next we decompose the principal series corepresentations into irreducible corepresentations. We need the following closed subspaces of \( \mathcal{L}_{p, x} \):

\[
\mathcal{L}_{p, x}^1 = \overline{\text{Span}}\left\{ e_m^+(p, x) + i\chi(p)e_m^-(p, x), e_m^+(p, x) - i\chi(p)e_m^-(p, x) \mid m \in \mathbb{Z}\right\},
\]

\[
\mathcal{L}_{p, x}^2 = \overline{\text{Span}}\left\{ e_m^+(p, x) - i\chi(p)e_m^-(p, x), e_m^+(p, x) + i\chi(p)e_m^-(p, x) \mid m \in \mathbb{Z}\right\}.
\]

Lemma 10.11. The spaces \( \mathcal{L}_{p, x}^j, j = 1, 2 \), are orthogonal \( W_{p, x} \)-invariant subspaces of \( \mathcal{L}_{p, x} \).

Proof. The orthogonality of \( \mathcal{L}_{p, x}^1 \) and \( \mathcal{L}_{p, x}^2 \) is immediate from their definitions. Using the actions (10.2) of the generators of \( \hat{M} \) it is a straightforward exercise to check that \( \mathcal{L}_{p, x}^j, j = 1, 2 \), are \( W_{p, x} \)-invariant. \( \Box \)
For \( j = 1, 2 \), we denote by \( W^j_{p,x} \) the restriction of \( W^1_{p,x} \) to the subspace \( L^j_{p,x} \).

**Proposition 10.12.** For \( x \neq 0 \) the corepresentations \( W^j_{p,x}, j = 1, 2 \), are irreducible.

**Proof.** We prove the irreducibility of \( W^1_{p,x} \) in case \( x \neq 0 \), for \( W^2_{p,x} \) the proof is similar. Let \( L \) be a nonzero closed \( W^1_{p,x} \)-invariant subspace of \( L^1_{p,x} \). We choose a nonzero vector \( v \in L \). For \( k \in \mathbb{Z} \), let \( P_k \) denote spectral projection of \( K \) onto the eigenspace corresponding to the eigenvalue \( q^k, k \), i.e., \( P_k \) is the orthogonal projection onto \( \text{Span}\{e^{\sigma \eta}(p, x) \mid \sigma, \eta \in \{-, +\}\} \), see (10.12). We have \( v = \sum_{k \in \mathbb{Z}} P_k v \), and since \( v \neq 0 \), there exists an \( m \in \mathbb{Z} \) such that \( P_m v \neq 0 \). Since \( K \) is affiliated to \( M \), the projection \( P_m \) belongs to \( M \), implying \( P_m v \in L \). Now let \( C_{\pm x} \) denote the spectral projections of the Casimir \( \Omega \) onto the eigenspaces corresponding to the eigenvalues \( \pm x \), then \( P_m v = C_x P_m v + C_{-x} P_m v \), so one of the vectors \( C_{\pm x} P_m v \) is nonzero. Let us assume \( C_x P_m v \) is nonzero, then it is a nonzero multiple of \( e^{\pm}(x, p) + i \chi(p) e_{-}(x, p) \), and it belongs to \( L \) since \( C_x \in \hat{M}_{p,x} \). Applying \( U^\pm \) shows that \( e^{\pm}(x, p) - i \chi(p) e_{+}(x, p) \in L \). Finally, applying the isometries in the polar decompositions of \( E \) and \( E^* \) repeatedly, we find that the vectors \( e^{\pm}(x, p) + i \chi(p) e_{-}(x, p) \) and \( e^{\pm}(x, p) - i \chi(p) e_{+}(x, p) \) belong to \( L \) for any \( k \in \mathbb{Z} \), hence \( L = L^1_{p,x} \). If \( C_x P_m v = 0 \), then \( C_{-x} P_m v \neq 0 \), and similar arguments show again that \( L = L^1_{p,x} \).

In the proof of Proposition 10.12, we used the Casimir operator \( \Omega \) to distinguish between the spaces \( \text{Span}\{e^{\pm}(p, x) \mid m \in \mathbb{Z}\} \) and \( \text{Span}\{e^{\pm}(p, x) \mid m \in \mathbb{Z}\} \). For \( x = 0 \) we can no longer do this, because now the restriction of \( \Omega \) is the zero operator, so it is possible that there are nontrivial irreducible subspaces inside \( L^1_{p,0} \) and \( L^2_{p,0} \). We define the following closed subspaces of \( L_{p,0}^j \):

\[
L_{1,0,0}^1 = \overline{\text{Span}}\left\{ e^+(p, 0), i \chi(p) e^{-}(p, 0) + i \chi(p+1) e^-+ e^-(p, 0) + i(-1)^{\chi(p)} e^-+ e^+(p, 0) \mid m \in \mathbb{Z}\right\}, \\
L_{1,0,0}^2 = \overline{\text{Span}}\left\{ e^+(p, 0), i \chi(p) e^{-}(p, 0) - i \chi(p+1) e^+e^- e^-(p, 0) + i(-1)^{\chi(p)} e^+e^- e^+(p, 0) \mid m \in \mathbb{Z}\right\}, \\
L_{2,0,0}^1 = \overline{\text{Span}}\left\{ e^+(p, 0), - i \chi(p) e^{-}(p, 0) + i \chi(p+1) e^+e^- e^-(p, 0) + i(-1)^{\chi(p)} e^+e^- e^+(p, 0) \mid m \in \mathbb{Z}\right\}, \\
L_{2,0,0}^2 = \overline{\text{Span}}\left\{ e^+(p, 0), - i \chi(p) e^{-}(p, 0) - i \chi(p+1) e^+e^- e^-(p, 0) + i(-1)^{\chi(p)} e^+e^- e^+(p, 0) \mid m \in \mathbb{Z}\right\}.
\]

Observe that \( L_{j,0,0}^i = L_{j,0,0}^i \oplus L_{j,0,0}^2 \), for \( j = 1, 2 \).

**Proposition 10.13.** Assume \( x = 0 \). (i) For \( \chi(p) \) odd, i.e., \( \epsilon(p) = \frac{1}{2} \), the corepresentations \( W^j_{p,0} \), with \( j = 1, 2 \), are irreducible. (ii) For \( \chi(p) \) even, i.e., \( \epsilon(p) = 0 \), the corepresentations \( W^{j,k}_{p,0} = W_{p,0}|_{L_{j,k,0}} \), with \( j, k \in \{1, 2\} \), are irreducible.

Before proving Proposition 10.13, we note that Propositions 10.5, 10.12, 10.13 prove Proposition 5.4.

**Proof.** We prove the proposition for \( j = 1 \). The case \( j = 2 \) is proved in the same way. Let \( L \) be a nonzero closed \( W^1_{p,0} \)-invariant subspace of \( L^1_{p,0} \). In the same way as in the proof of Proposition 10.12 it follows that the vectors

\[
f_m(c_m) = e^+e^- + i \chi(p) e^{-}e^- + c_m e^+e^- - c_m i \chi(p) e^+e^- , \quad m \in \mathbb{Z},
\]
are in $L$ for some (yet to be determined) constant $c_m$, and every vector in $L$ can be expanded in terms of the vectors $f_m(c_m)$. Here we use the shorthand notation $e_m^{ε,η} = e_m^{ε,η}(p,0)$, $ε, η \in \{-, +\}$. Applying $U_0^{+}$ we find

$$U_0^{+} f_m(c_m) = c_m i^p (-1)^m \left[ e_m^{++} + i \chi(p) e_m^{--} + (-1) \chi(p) e_m^{+} - (-i) \chi(p) e_m^{-} \right].$$

Since $U_0^{+} f_m(c_m)$ must be in $L$, we see that $c_m$ satisfies $c_m = (-1) \chi(p) + 1 c_m^{-1}$, so that

$$c_m = ± i \chi(p) + 1.$$

Let us write $f_m^1 = f_m(i \chi(p) + 1)$ and $f_m^2 = f_m(-i \chi(p) + 1)$, then $L_{p,0}^{1,j} = \text{Span} \{ f_m^j \mid m \in \mathbb{Z} \}$ for $j = 1, 2$. Observe that $Ef_m^j = d_m f_m^{j+1}$ for some constant $d_m$. Let us assume that $f_m^1 \in L$. Applying $U_0^{+}$ to $f_m^1$ gives us

$$U_0^{+} f_m^1 = (-1)^{\chi(p)} \left[ e_m^{++} - i \chi(p) e_m^{+} - i \chi(p) e_m^{--} + (-1) \chi(p) e_m^{-} \right]$$

$$= \begin{cases} 
(-1)^{\chi(p)} i^{\chi(p)+1} f_m^1, & \chi(p) \text{ even}, \\
(-1)^{\chi(p)} i^{\chi(p)+1} f_m^2, & \chi(p) \text{ odd},
\end{cases}$$

so for $\chi(p)$ even we have $L = L_{p,0}^{1,1}$, and for $\chi(p)$ odd we have $L = L_{p,0}^{1,1} \oplus L_{p,0}^{1,2} = L_{p,0}^{1,1}$. If we assume $f_m^2 \in L$, we find in the same way that $L = L_{p,0}^{1,2}$ for $\chi(p)$ even. □

10.3. Complementary series. Let $p \in q^{2Z}$, i.e., $ε(p) = 0$, and let $x = ± \mu(λ)$ with $λ \in (q, 1)$. Note that $x$ is not in the spectrum of the Casimir operator in the left regular corepresentation, which is described in Section 8.1. Let $L_{p,x} = \bigoplus_{ε, η \in \{-, +\}} L_{p,x}^{ε,η}(p, x)$ with orthonormal basis \{ $e_{n,ε}^{ε,η}(p, x)$ \mid $n \in \mathbb{Z}$, $ε, η \in \{-, +\}$ \}, similar as for the principal series corepresentations. We define a unitary corepresentation $W_{p,x} \in M \otimes B(L_{p,x})$ by

$$W_{p,x}(f_{m_2,p_2,t_2} \otimes e_{n_2}^{ε_2,η_2}(p_2, x)) =$$

$$\sum_{p_1 \in I_q} C(ε_1, η_1; χ(p_2/p_1p) - n_2, \text{sgn}(p_1) ε_2, \text{sgn}(p_2) η_2; p_1, p_2, 2n_2 - χ(p_2/p_1p))$$

$$\times f_χ(p_2/p_1p) + m_2 - 2n_2, p_2, t_2 \otimes e^{\text{sgn}(p_1) ε_2, \text{sgn}(p_2) η_2}_{χ(p_2/p_1p) - n_2}(p, x).$$

with the function $C$ from Proposition 9.4. Initially, the function $C$, as a function of $x$, is only defined on the spectrum of $Ω$, but using the explicit expressions for $A$ and $S$ (see the definition of $C$ in Proposition 9.4), we can also define $C$ for $x = ± \mu(λ)$ with $λ \in (q, 1)$. Observe that the denominator of $A(λ; p, m, ε, η)$ contains factors with the square root of $(-q^{1-2n} λ/p, -q^{1-2n}/p λ; q^2)_{∞}$ for a certain $n \in \mathbb{Z}$. Assume $-λ \in (q, 1)$, then this infinite product is positive for $p \in q^{2Z}$, but for $p \in q^{2Z+1}$ it is not. For this reason we require that $p \in q^{2Z}$ or $ε(p) = 0$, see (8.23). This corresponds nicely with the situation for the principal unitary and complementary series representations of $SU(1, 1)$ and $U_q(sl(1, 1))$, see Section 8.4. Formally the above defined corepresentation corresponds to the definition of the principal series corepresentation $W_{p,x}$ from Lemma 10.6. In particular, the actions of the
generators of \( \hat{M} \) on the basisvectors \( \epsilon_{m}^{p,x}(p, x) \) are given by

\[
K \epsilon_{m}^{p,x}(p, x) = p^{\frac{1}{2}} q^{m} \epsilon_{m}^{p,x}(p, x),
\]

\[
(q^{-1} - q) E \epsilon_{m}^{p,x}(p, x) = q^{-m+\frac{1}{2}} p^{-\frac{1}{2}} \sqrt{1 + 2\varepsilon \eta x p q^{2m+1} + q^{4m+2} p^{2}} \epsilon_{m+1}^{p,x}(p, x),
\]

\[
U_{0}^{+} \epsilon_{m}^{p,x}(p, x) = \eta (-1)^{\nu(p)} \epsilon_{m}^{p,x}(-\eta)(p, x),
\]

\[
U_{0}^{+} \epsilon_{m}^{p,x}(p, x) = \varepsilon \eta^{\lambda(p)} (-1)^{m} \epsilon_{m}^{p,x}(p, x).
\]

We call \( W_{p,x} \) the complementary series corepresentation of \( (M, \Delta) \). In order to show that this is indeed a unitary operator, we need to find orthogonality relations and dual orthogonality relations for the functions \( C \) in case \( x = \pm \mu(\lambda) \) with \( \lambda \in (q, 1) \). These relations are obtained in Corollary 11.4 from the orthogonality relations for the function \( C \) by analytic continuation. The fact that \( W_{p,x} \) is indeed a corepresentation is proved along the same lines as for the principal series corepresentations. Here we need to show that the product identity from Lemma 9.9 remains valid for \( x = \pm \mu(\lambda) \) with \( \lambda \in (q, 1) \). This is done in Lemma 11.1.

Finally, in the same way as Proposition 10.12 it can be proved that the subcorepresentations \( W_{p,x}^j = W_{p,x} \big| \mathcal{L}_{p,x}^j \), with the subspaces \( \mathcal{L}_{p,x}^j \), \( j = 1, 2 \), defined as in (10.3), are irreducible.

11. Identities for special functions

11.1. Summation formulas from the action of \( Q(p_{1}, p_{2}, n) \). We start by proving the summation formulas in Section 6.3, which essentially follow by the action of \( Q(p_{1}, p_{2}, n) \) with respect to the spectral decomposition of the Casimir operator.

Proof of Theorem 6.10. In Lemma 9.3 we computed how the operator \( T(p_{1}, p_{2}, n) \) defined by (9.1) acts on functions in \( L^{2}(I(p, m, \varepsilon, \eta)) \). In this computation we actually proved a summation formula involving the functions \( a_{p}(z, w) \) and the orthonormal functions \( g_{z}(x; p, m, \varepsilon, \eta) \), which are essentially Al-Salam–Chihara polynomials and little \( q^{2} \)-Jacobi functions. Here we write out explicitly, i.e., in terms of basic hypergeometric functions, the summation formula corresponding to the case \( \varepsilon = \eta = + \); in this case both \( g_{z} \)-functions appearing in the formula are little \( q \)-Jacobi functions, i.e., non-terminating \( \varphi_{1} \)-functions. This is a rather tedious, but straightforward computation. The second case follows similarly using \( \varepsilon = \eta = - \). □

The product formula from Lemma 9.9 leads to the summation formula in Theorem 6.12 with the same structure as the formula from Theorem 6.10.

Proof of Theorem 6.12. We first write the formula from Lemma 9.9 in terms of the \( S \)-functions:

\[
\text{sgn}(r_{1})^{\frac{1}{2}(1-\text{sgn}(p_{1}))} \text{sgn}(r_{2})^{\frac{1}{2}(1-\text{sgn}(p_{2}))} + \frac{1}{n} S(\text{sgn}(r_{1} r_{2}) \lambda; p_{1}, p_{2}, n) S(\lambda; r_{1}, r_{2}, m) = \sum_{(x_{1}, x_{2}) \in \mathcal{A}} a_{x_{1}}(r_{1}, p_{1}) a_{x_{2}}(r_{2}, p_{2}) S(\lambda; x_{1}, x_{2}, m + n), \quad (11.1)
\]

where we denoted \( y = -\text{sgn}(p_{1} p_{2} r_{1} r_{2}) \mu(\lambda) \) and canceled common factors. Now the result follows from expressing the \( S \)-functions as \( \varphi_{1} \)-functions and the \( a_{x} \)-functions as \( \Psi \)-functions, see Proposition 9.10 and Definition 6.2. □
Observe that the sum in Theorem 6.12 is actually a single sum. If we denote \( x_1 = \text{sgn}(p_1 r_1) q^k \), then \( x_2 = \text{sgn}(p_2 r_2) q^k \), and we can write the above sum as a sum over \( k \) where \( k \in \mathbb{N} \) if \( \text{sgn}(p_1 r_1) = - \) or \( \text{sgn}(p_2 r_2) = - \), and \( k \in \mathbb{Z} \) if \( \text{sgn}(p_1 r_1) = \text{sgn}(p_2 r_2) = + \). Furthermore, if we set \( r_1 = p_1 \), \( r_2 = p_2 \) (this implies \( m = -n \)), and we use the second symmetry relation from Lemma 3.11 for the function \( S \), the left hand side in (11.1) contains the product \( S(\lambda^{-1}; p_1, p_2, n) S(\lambda; p_1, p_2, n) \), which is positive (this corresponds to the operator \( Q(p_1, p_2, n)^* Q(p_1, p_2, n) \)). So we find

\[
(-1)^{\lambda(p_2/p_1)} \text{sgn}(p_1) \text{sgn}(p_2) \sum_{(x_1, x_2) \in A} a_{x_1}(p_1, p_1) a_{x_2}(p_2, p_2) S(\lambda; x_1, x_2, 0) > 0.
\]

This leads to Corollary 6.13.

For the definition of the complementary series corepresentations of \((M, \Delta)\) in Section 10.3, the following lemma is crucial for showing that it is indeed a corepresentation.

**Lemma 11.1.** The identity from Theorem 6.12 is also valid for \( \lambda \in (q, 1) \). Consequently, if \( n + m \in 2\mathbb{Z} \), the product formula for the function \( C \) in Lemma 9.9(i) also holds for \( y = \mu(\lambda) \) with \( \lambda \in (q, 1) \).

**Proof.** First we prove that the identity from Theorem 6.12, or equivalently (11.1), is also valid for \( \lambda \in (q, 1) \). Actually, we prove a stronger result: the identity (11.1) holds for all \( \lambda \in \mathbb{C} \setminus \{0\} \). Recall that, for \( p_1, p_2, q, n \in \mathbb{Z} \), the function \( S(\cdot, p_1, p_2, n) \) is analytic on \( \mathbb{C} \setminus \{0\} \). So clearly the left hand side of (11.1) is analytic in \( \lambda \) on \( \mathbb{C} \setminus \{0\} \). We show that the right hand side of (11.1) also defines an analytic function, then the result follows from analytic continuation.

Let \( r_1, r_2, p_1, p_2 \in I_q \) and \( n, m \in \mathbb{Z} \). Assume \( \lambda \in K \subset \mathbb{C} \setminus \{0\} \) where \( K \) is a compact set, then there exists a constant \( B > 0 \) such that \( |\lambda| < B \) and \( |\lambda^{-1}| < B \). We define \( g : q^\mathbb{Z} \to \mathbb{R} \) by

\[
g(x) = \begin{cases} q^{3\chi(x)} B \chi(x), & x \leq q, \\ q^{-(n+m)\chi(x)}, & x \geq 1. \end{cases}
\]

By Lemma 3.12 there exists a constant \( C > 0 \) such that

\[
|S(\lambda; x_1, x_2, n + m)| < C g(|x_1|),
\]

for \( (x_1, x_2) \in A \). Furthermore, by Lemma 3.4 there exists a constant \( D' > 0 \) such that

\[
|a_x(r, p)| \leq D' q^{\chi(x)\chi(r/p) - \frac{3}{2}} q^{\frac{1}{2} \chi(x)^2}, \quad x, r, p \in I_q,
\]

so that

\[
|a_{x_1}(r_1, p_1) a_{\text{sgn}(r_2 p_2)}(x_1, r_2, p_2)| \leq D q^{\chi(x_1)\chi(r_1 r_2/p_1 p_2) - \frac{3}{2}} q^{\chi(x_1)^2},
\]

for some constant \( D > 0 \). Because of the factor \( q^{\chi(x)^2} \) the sum

\[
\sum_{x} q^{\chi(x)\chi(r_1 r_2/p_1 p_2) - 3} q^{\chi(x)^2} g(x)
\]

converges absolutely. Here the sum is over \( x \in q^\mathbb{N} \) in case \( \text{sgn}(r_1 p_1) = - \) or \( \text{sgn}(r_2 p_2) = - \), and over \( x \in q^\mathbb{Z} \) in case \( \text{sgn}(r_1 p_1) = \text{sgn}(r_2 p_2) = + \). It follows that the right hand side of (11.1) converges uniformly on \( K \), hence it is analytic on \( K \).
Finally, let $\pm \lambda \in (q, 1)$. We multiply (11.1) by
\[
\varepsilon^\frac{1}{2}(1-\text{sgn}(p_1r_1)), \eta^\frac{1}{2}(1-\text{sgn}(p_2r_2)) A(-\text{sgn}(p_1r_1p_2r_2)\lambda; p, k + m + n, \text{sgn}(p_1r_1)\varepsilon, \text{sgn}(p_2r_2)\eta)
\]
then we obtain the desired product formula for the function $C$ as in Lemma 11.1(i), with $y = -\text{sgn}(p_1r_1p_2r_2)\mu(\lambda)$.

Let us remark that with the same arguments as in the proof of Lemma 11.1 it follows that the product formula for the function $C$ holds for all $\lambda \in (0, 1) \setminus q^{\mathbb{N}_0}$ if $n + m \in 2\mathbb{Z}$. Note that the points $\lambda \in q^{\mathbb{N}_0}$ correspond to discrete series corepresentations, and at these points the product formula is of course also valid. In this case the functions $A$ are essentially square roots of residues of $c$-functions.

11.2. Biorthogonality relations. In the proof of Lemma 10.6 we obtained orthogonality relations for the function $C$ for the case $x = \mu(\lambda)$ with $\lambda \in \mathbb{T}$. These relations lead to biorthogonality relations for the $S$ functions, which by analytic continuation hold for all $\lambda \in \mathbb{C} \setminus \{0\}$. We need these biorthogonality relations for $\pm \lambda \in (q, 1)$ in order to show that the complementary series corepresentations are unitary.

Lemma 11.2. Let $\lambda \in \mathbb{C} \setminus \{0\}$ and $m \in \mathbb{Z}$. The set
\[
\{ p_2 \mapsto S(\text{sgn}(p_1)\lambda; p_1, p_2, \chi(p_1p_2) + m) \mid p_1 \in I_q \}
\]
is a basis for $\ell^2(I_q)$ with dual basis
\[
\{ p_2 \mapsto S(\text{sgn}(p_1)\lambda^{-1}; p_1, p_2, \chi(p_1p_2) + m) \mid p_1 \in I_q \}.
\]
Similarly, the set
\[
\{ p_1 \mapsto S(\text{sgn}(p_1)\lambda; p_1, p_2, \chi(p_1p_2) + m) \mid p_2 \in I_q \}
\]
is a basis for $\ell^2(I_q)$ with dual basis
\[
\{ p_1 \mapsto S(\text{sgn}(p_1)\lambda^{-1}; p_1, p_2, \chi(p_1p_2) + m) \mid p_2 \in I_q \}.
\]

Proof. First assume $x = \mu(\lambda)$ with $\lambda \in \mathbb{T}_0$. From unitarity of $W_{p,x}$ and the explicit expressions for $W_{p,x}^*$ and $W_{p,x}$, we obtain orthogonality and dual orthogonality relations for the matrix elements $C$. Indeed, from writing out $W_{p,x} W_{p,x}^* [f_{m_1p_1t_1} \otimes e^{\text{sgn}(p_1)\varepsilon, \eta}(p, x)] = f_{m_1p_1t_1} \otimes e^{\text{sgn}(p_1)\varepsilon, \eta}(p, x)$ we find, for $p_1' \in I_q$ and $y = \varepsilon \eta x$,
\[
\delta_{p_1p_1'} = \sum_{p_2 \in I_q} C(\text{sgn}(p_2)y; m - \chi(p_1), \text{sgn}(p_1)\varepsilon, \eta; p_1, p_2, \chi(p_2p_1/p) - 2m)
\]
\[
\times C(\text{sgn}(p_2)y; m - \chi(p_1'), \text{sgn}(p_1')\varepsilon, \eta; p_1', p_2, \chi(p_2p_1'/p) - 2m),
\]
and from writing out $W_{p,x} W_{p,x}^* [f_{m_2p_2t_2} \otimes e^{\text{sgn}(p_2)\eta}(p, x)] = f_{m_2p_2t_2} \otimes e^{\text{sgn}(p_2)\eta}(p, x)$ we find, for $p_2' \in I_q$ and $y = \varepsilon \eta x$,
\[
\delta_{p_2p_2'} = \sum_{p_1 \in I_q} C(\text{sgn}(p_2)y; m - \chi(p_1), \text{sgn}(p_1)\varepsilon, \eta; p_1, p_2, \chi(p_1p_2/p) - 2m)
\]
\[
\times C(\text{sgn}(p_2'y); m - \chi(p_1), \text{sgn}(p_1)\varepsilon, \eta; p_1, p_2', \chi(p_1p_2'/p) - 2m).
\]
Expressing the functions $C$ in terms of the functions $S$, see Lemma 11.4, the first orthogonality relation gives, for $\lambda \in \mathbb{T}$,
\[
\delta_{p_1\bar{p}_1'} = \sum_{p_2 \in I_\lambda} \frac{A(\text{sgn}(p_2)\lambda; p, \chi(p_2/p) - m, \varepsilon, \text{sgn}(p_2)\eta)A(\text{sgn}(p_2)\lambda^{-1}; p, \chi(p_2/p) - m, \varepsilon, \text{sgn}(p_2)\eta)}{A(\text{sgn}(p_1)\lambda; p, m - \chi(p_1), \text{sgn}(p_1)\varepsilon, \eta)A(\text{sgn}(p_1')\lambda^{-1}; p, m - \chi(p_1'), \text{sgn}(p_1')\varepsilon, \eta)} \times S(-\text{sgn}(p_1)\lambda; p_1, p_2, \chi(p_1p_2/p) - 2m)S(-\text{sgn}(p_1')\lambda^{-1}; p_1', p_2, \chi(p_1p_2/p) - 2m).
\]

We use $A(\lambda)A(\lambda^{-1}) = |A(\lambda)|^2 = 1$, then we obtain
\[
\delta_{p_1\bar{p}_1'} = \sum_{p_2 \in I_\lambda} S(-\text{sgn}(p_1)\lambda; p_1, p_2, \chi(p_1p_2/p) - 2m)S(-\text{sgn}(p_1')\lambda^{-1}; p_1', p_2, \chi(p_1p_2/p) - 2m).
\]

From Lemma 11.12(iii) and (iv) it follows this sum converges uniformly in $\lambda$ on any compact set of $\mathbb{C} \setminus \{0\}$. Since the function $S$ is analytic for $\lambda \in \mathbb{C} \setminus \{0\}$, by analytic continuation the orthogonality relations are valid for all $\lambda \in \mathbb{C} \setminus \{0\}$. In the same way we find from the second orthogonality relations for the functions $C$, for $\lambda \in \mathbb{C} \setminus \{0\}$,
\[
\delta_{p_2\bar{p}_2'} = \sum_{p_1 \in I_\lambda} S(-\text{sgn}(p_1)\lambda; p_1, p_2, \chi(p_1p_2/p) - 2m)S(-\text{sgn}(p_1')\lambda^{-1}; p_1', p_2, \chi(p_1p_2/p) - 2m)
\]

In order to show uniform convergence here, we also need the third symmetry relation for $S$ from Lemma 11.11. Now replace $-\lambda$ by $\lambda$, and $-\chi(p) - 2m$ by $m$, then we have biorthogonality relations in $\ell^2(I_\lambda)$ for the functions $S(\text{sgn}(p_1)\lambda; p_1, p_2, \chi(p_1p_2) + m)$ with respect to $p_1$ and $p_2$, which implies that they form a basis for $\ell^2(I_\lambda)$.

The biorthogonality relations in Theorem 11.14 follow from Lemma 11.2 using
\[
s(p_1, p_2; \lambda, m) = S(\text{sgn}(p_1)\lambda; p_1, p_2, \chi(p_1p_2) + m).
\]

**Remark 11.3.** By the third symmetry relation for $S$ from Lemma 11.11 the two biorthogonality relations for $S$ from Lemma 11.2 are actually equivalent. It is also useful to observe that for $\lambda \in \mathbb{T}$ the biorthogonality relations are orthogonality relations.

To prove unitarity for the complementary series corepresentations we need to write the biorthogonality relations from Lemma 11.2 in case $\pm \lambda \in (q, 1)$, as orthogonality relations for the functions $C$.

**Corollary 11.4.** For $m \in \mathbb{Z}$, $p \in q^{2\mathbb{Z}}$, $\varepsilon, \eta \in \{-, +\}$, and $y = \pm \mu(\lambda)$ with $\lambda \in (q, 1)$, the following orthogonality relations hold:
\[
\delta_{p_1\bar{p}_1'} = \sum_{p_2 \in I_\lambda} C(\text{sgn}(p_2)\lambda; m - \chi(p_1), \text{sgn}(p_1)\varepsilon, \eta, p_1, p_2, \chi(p_2p_1/p) - 2m)
\]
\[
\times C(\text{sgn}(p_2)\lambda; m - \chi(p_1'), \text{sgn}(p_1')\varepsilon, \eta, p_1', p_2, \chi(p_2p_1'/p) - 2m),
\]
\[
\delta_{p_2\bar{p}_2'} = \sum_{p_1 \in I_\lambda} C(\text{sgn}(p_2)\lambda; m - \chi(p_1), \text{sgn}(p_1)\varepsilon, \eta, p_1, p_2, \chi(p_1p_2/p) - 2m)
\]
\[
\times C(\text{sgn}(p_2')\lambda; m - \chi(p_1), \text{sgn}(p_1)\varepsilon, \eta, p_1, p_2, \chi(p_1p_2/p) - 2m).
\]
Proof. This follows from Lemma 11.2 and the observations that
\[ C(\text{sgn}(p_2)y; m - \chi(p_1), \text{sgn}(p_1)\varepsilon, \eta; p_1, p_2, \chi(p_2) p_1/p - 2m) = g(\lambda)S(-\lambda; p_1, p_2, \chi(p_1 p_2) + m), \]
where \( g(\lambda) \) is given by
\[ g(\lambda) = \text{sgn}(p_1)\varepsilon^{1 - \text{sgn}(p_1)\eta^2} A(\text{sgn}(p_2)\lambda; p, \chi(p_2) p/p - \varepsilon, \text{sgn}(p_2)\eta) \]
and from \( A(\lambda)A(\lambda^{-1}) = 1 \), which follows from the definitions of \( A \), see §4.6.

11.3. Proof of the summation and transformation theorems. In this subsection we prove Theorems 6.5 and 6.8. The theorems are reflections of the structure constants for the multiplicative unitary \( C \) of the quantum group acting on \( D(q(p_1, p_2, n)) \), see Proposition 4.10. Inspection of the proofs, see Section 7.3, shows that both results follow from the pentagonal equation \( W_{12}W_{13}W_{23} = W_{23}W_{12} \) for the multiplicative unitary \( W \). However, as remarked in Remark 6.9, the results in Theorems 6.5 and 6.8 cannot be obtained from each other.

Proof of Theorem 6.5. We start with the result of Proposition 4.10 and we next let the corresponding operator identity act on \( f_{-l, \eta p q_d^{l+m}, \eta} \in D(p, l, \varepsilon, \eta) \). Lemma 4.4 shows that
\[ Q(p_1, p_2, n)Q(r_1, r_2, m): D(p, l, \varepsilon, \eta) \rightarrow D(p, l + m + n, \text{sgn}(r_1 p_1)\varepsilon, \text{sgn}(r_2 p_2)\eta) \]
is non-zero precisely if \( q^{2l} p = q^{-m} |\frac{r_1}{p_1}| \) and \( q^{2l+2m} p = q^{-n} |\frac{r_2}{p_2}| \). In particular, in case \( q^{-n} |\frac{r_2}{p_2}| \neq q^{-m} |\frac{r_1}{p_1}| \) we find \( Q(p_1, p_2, n)Q(r_1, r_2, m) = 0 \).

In order to calculate the appropriate matrix coefficient we proceed for \( f_{-l-m-n, \eta p q_d^{l+m+n}, \eta} \in D(p, l + m + n, \text{sgn}(r_1 p_1)\varepsilon, \text{sgn}(r_2 p_2)\eta) \) as
\[ \langle Q(p_1, p_2, n)Q(r_1, r_2, m), f_{-l, \eta p q_d^{l+m+n}, \eta} \rangle = \sum_{u \in J(p, l + m, \varepsilon \text{sgn}(r_1), \eta \text{sgn}(r_2))} \langle Q(r_1, r_2, m), f_{-l-m-n, \eta p q_d^{l+m+n}, \eta} \rangle \times \langle Q(p_1, p_2, n), f_{-l-m-n, \eta p q_d^{l+m+n}, \eta} \rangle \]
using the orthogonal basis for the intermediate space \( D(p, l + m, \varepsilon \text{sgn}(r_1), \eta \text{sgn}(r_2)) \). In this sum we can use (7.13) twice, and using (7.1) we find that this equals
\[ \delta_{\eta p q_d^{l+m+n}} \sum_{u \in I_q \text{ so that } \text{sgn}(u) = \text{sgn}(r_1)\eta \text{ and } \text{sgn}(r_2 p_2)q_d^{l+m+n}u \in I_q} \frac{z}{w} a_z(r_1, u) a_u(p_1, w) \]
(11.2)
\times a_{\eta p q_d^{l+m+n}}(r_2, \eta \text{sgn}(r_1 r_2) p q_d^{l+m+n} u) a_{\eta p q_d^{l+m+n}}(r_1 r_2 p q_d^{l+m+n} u, \text{sgn}(r_1 p_1 r_2 p_2)\varepsilon \eta p q_d^{l+m+n} u)

Next observe
\[ Q(x_1, x_2, m + n): D(p, l, \varepsilon, \eta) \rightarrow D(p, l + m + n, \text{sgn}(x_1)\varepsilon, \text{sgn}(x_2)\eta) \]
is non-zero only if \( q^{2l} p = q^{-m-n} |\frac{r_2}{p_2}| \), so that the double sum in Proposition 4.10 reduces to a single sum. Moreover, by Definition 6.3 shows that in the sum the functions \( a_{x_i}(r_i, p_i) \) for
Proof of Corollary 6.7. □

0, so we assume $q$ by Lemma 7.1, see in particular (7.15). We are interested in the case

$$
\sum_{x_1, x_2 \in I_q} a_{x_1}(r_1, p_1) a_{x_2}(r_2, p_2) \langle Q(x_1, x_2, m + n) f_{-l, \varepsilon \eta \rho q}\rangle_{z, z}, f_{-l-m-n, \varepsilon \eta \rho q(r_1 p_1 r_2 p_2)\varepsilon \eta \rho q^{l+m+n} w, w}\rangle
$$

and this reduces to a single sum and the summand is evaluated by (7.13). By eliminating $x_2$ and renaming $x_1$ by $x$ we see that this equals

$$
\sum_{x \in I_q \text{ so that } \varepsilon \eta \rho q(r_1 p_1) \text{ and } x | \varepsilon \eta \rho q(r_2 p_2)\varepsilon \eta \rho q^{l+m+n} \in I_q} \frac{z}{w} a_x(r_1, p_1) a_x(x, w) a(x|\varepsilon \eta \rho q(r_2 p_2)\varepsilon \eta \rho q^{l+m+n} (r_2, p_2) \rangle
$$

(11.3)

Finally, equating (11.2) and (11.3) gives the result, where the conditions on the parameters in Theorem 6.7 follows from the fact that the matrix elements are taken with respect to vectors in the GNS Hilbert space.

Proof of Corollary 6.7. Observe that by Lemma 6.6

$$
Q(r_1, r_2, -m) Q(r_1, r_2, m) = (-q)^m \varepsilon \eta \rho q(r_1) \varepsilon \eta \rho q(r_2) Q(r_1, r_2, m)^* Q(r_1, r_2, m)
$$

so that

$$
(-q)^m \varepsilon \eta \rho q(r_1) \varepsilon \eta \rho q(r_2) \langle Q(r_1, r_2, -m) Q(r_1, r_2, m) f_{-l, \varepsilon \eta \rho q} z, z, f_{-l, \varepsilon \eta \rho q} z, z\rangle
$$

$$
= \|Q(r_1, r_2, m) f_{-l, \varepsilon \eta \rho q} z, z\|^2 = \frac{(r_1 r_2)^2}{q^{2l} z^2 p^2}
$$

(11.5)

(11.4)

by Lemma 7.1, see in particular (7.13). We are interested in the case $Q(r_1, r_2, m) f_{-l, \varepsilon \eta \rho q} z, z \neq 0$, so we assume $q^{2l} p = q^{-m} |r_1|^2$. The case that this sum can equal zero, is already covered by Theorem 6.7. Since the right hand side is obviously positive, and the left hand side is (up to the factor in front) equal to (11.3) with $p_1, p_2, n, w$ replaced by $r_1, r_2, -m, z$. Since we assume $q^{2l} p = q^{-m} |r_1|^2$ we replace $p$ by $q^{-m-2l} |r_1|^2$, and moreover, we use the third symmetry of (5.2) twice, to find

$$
(-q)^m \varepsilon \eta \rho q(r_1) \varepsilon \eta \rho q(r_2) (r_1 r_2)^2 z \varepsilon \eta (x z / r_1) q^l
$$

$$
\sum_{x \in \mathcal{Q}^2 \text{ so that } xq^{-m} |r_1|^2 \in \mathcal{Q}^2} x^2 a_x(r_1, r_1) a_x(z, z) a_{xq^{-m} |r_1|^2} (r_2, r_2) a_{xq^{-m} |r_1|^2} (\varepsilon \eta) \langle r_2 | q^{-m-l} z, z, \varepsilon \eta \rangle r_1
$$

(11.5)

$$
= \|Q(r_1, r_2, m) f_{-l, \varepsilon \eta \rho q} z, z\|^2 > 0
$$

where the right hand side can be evaluated explicitly as a sum of squares by (11.4) with $p$ replaced by $q^{-m-2l} |r_1|^2$. This proves the general statement of Corollary 6.7 since the condition on the summation parameter $x$ is always satisfied.
For the final statement on $q$-Laguerre polynomials we observe that for $\text{sgn}(p) = +$ and $\text{sgn}(y) = -$, or $y = -q^{1+k}$, $k \in \mathbb{N}_0$, we have from Definition 4.1 and (B.3)
\[ a_p(-q^{1+k}, -q^{1+k}) = c_q(-1)^{\chi(p)}q^{1+k}\nu(p)\sqrt{(-p^2;q^2)_\infty}L_k^{(0)}(q^2p^{-2};q^2). \]
So we choose $r_1 = -q^{1+a}$, $r_2 = -q^{1+b}$, $z = -q^{1+c}$, $\varepsilon \eta |_{r_1}^{r_2} |q^{-m-l}z = -q^{1+d}$ with $a, b, c, d \in \mathbb{N}_0$, so we replace $l$ by $c + b - a - m - d$ and take $\varepsilon = -$, $\eta = -$. We replace $m$ by $b - a - e$ with $e \in \mathbb{Z}$, discard the positive $x$-independent terms and find
\[
\sum_{x \in q^2} x^2 \nu(x)^2 \nu(xq^e)^2 (-x^2, -x^2q^{2e}; q^2) \infty \sum_{x \in q^2} \frac{q^{2k}}{(-q^{2k}; q^2)_\infty} L_a^{(0)}(q^{2k}; q^2) L_c^{(0)}(q^{2k}; q^2) L_b^{(0)}(q^{2k}; q^2) L_d^{(0)}(q^{2k}; q^2) > 0.
\]
Now putting $x = q^{1-k}$, $k \in \mathbb{Z}$, and using the theta-product identity (B.1) twice and not taking into account the $k$-independent positive terms we find
\[
\sum_{k \in \mathbb{Z}} \frac{q^{2k}}{(-q^{2k}; q^2)_\infty} L_a^{(0)}(q^{2k}; q^2) L_c^{(0)}(q^{2k}; q^2) L_b^{(0)}(q^{2k}; q^2) L_d^{(0)}(q^{2k}; q^2) > 0.
\]
Relabeling and switching to base $q$ proves the required statement. 

Proof of Theorem 6.8. For the proof it is easier to start by conjugating the result of Proposition 4.13 with the flip operator to obtain
\[
\sum_{p \in I_q, m \in \mathbb{Z}} \tilde{Q}(p_1, p, m) \otimes \tilde{Q}(p, p_2, n - m) = W(Q(p_1, p_2, n) \otimes \text{Id}) W^*, \quad (11.6)
\]
which is a consequence of the proof of Proposition 4.13. We let both sides act on
\[
f_{-m_1, \varepsilon_1 \eta_1, q^{m_1} r_1, \varepsilon_1 \eta_1, z_1} \otimes f_{-m_2, \varepsilon_2 \eta_2, q^{m_2} r_2, \varepsilon_2 \eta_2} \in \mathcal{K}(r_1, m_1, \varepsilon_1 \eta_1) \otimes \mathcal{K}(r_2, m_2, \varepsilon_2 \eta_2)
\]
and we take inner products with
\[
f_{-m_1 - M, \sigma \text{sgn}(p_1) \varepsilon_1 \eta_1 q^{m_1} r_1, \varepsilon_1 \eta_1, w_1} \otimes f_{-m_2 - n + M, \sigma \text{sgn}(p_2) \varepsilon_2 \eta_2 q^{m_2} r_2, \varepsilon_2 \eta_2}
\in \mathcal{K}(r_1, m_1 + M, \text{sgn}(p_1) \varepsilon_1 \eta_1) \otimes \mathcal{K}(r_2, m_2 + n - M, \varepsilon_2 \eta_2, \text{sgn}(p_2) \eta_2).
\]
Then the sum over $I_q$ and $\mathbb{Z}$ reduces to a single term by a double application of (11.13). Indeed, we find that we need $m = M$ and $\text{sgn}(p) = \sigma$ for a non-zero contribution, but also both the conditions $q^{2m_1 + M} = |\frac{p}{p_1}|$ and $q^{2m_2 + n - M} = |\frac{p_2}{p_2}|$ need to be satisfied. So for the matrix element of the left hand side of (11.6) to have a single non-zero term we require $r_1 r_2 q^{2m_1 + m_2} = q^{-n}|\frac{p_1}{p_2}|$, and in this case the left hand side equals
\[
\left| \frac{z_1 z_2}{w_1 w_2} \right| a_{z_1}(p_1, w_1) a_{z_2}(p_2, w_2) a_{\varepsilon_1 \eta_1 q^{m_1} r_1, \varepsilon_1 \eta_1, w_1} a_{\varepsilon_2 \eta_2 q^{m_2} r_2, \varepsilon_2 \eta_2, w_2}
\times a_{z_1}(\sigma |p_1| r_1 q^{2m_1 + M}, w_1) a_{\varepsilon_1 \eta_1 q^{m_1} r_1, \varepsilon_1 \eta_1, w_1} a_{\varepsilon_2 \eta_2 q^{m_2} r_2, \varepsilon_2 \eta_2, w_2}
\times a_{z_1}(\sigma |p_1| r_1 q^{2m_1 + M}, w_1) a_{\varepsilon_1 \eta_1 q^{m_1} r_1, \varepsilon_1 \eta_1, w_1} a_{\varepsilon_2 \eta_2 q^{m_2} r_2, \varepsilon_2 \eta_2, w_2} (11.7)
\]
where we have chosen to eliminate $r_2$. Here all arguments of the function $a_p(x, y)$ are indeed elements of $I_q$, except possible $\sigma |p_1| r_1 q^{2m_1 + M}$ and in case $\sigma |p_1| r_1 q^{2m_1 + M} \notin I_q$ the expression has to be read as zero.
In order to calculate the same matrix element for the right hand side of (11.4) we rewrite this matrix element as

$$\left\langle \left( Q(p_1, p_2; n) \otimes \text{Id} \right) W^* (f_{-m_1, \varepsilon_1 \eta_1 q^{m_1} r_1 z_1} \otimes f_{-m_2, \varepsilon_2 \eta_2 q^{m_2} r_2 z_2}) \right\rangle \right.$$ \hspace{1cm} (11.8)

$$W^* (f_{-m_1 - M, \varepsilon \text{sgn}(p_1)} \varepsilon_1 \eta_1 q^{m_1 + M} r_1 w_1, w_1 \otimes f_{-m_2 - n + M, \varepsilon \text{sgn}(p_2)} \varepsilon_2 \eta_2 q^{m_2 + n - M} r_2 w_2, w_2).$$

In this expression we use (7.10) twice, with parameters $y_1, x_1$ (instead of $y, z$ as in (7.10)) for the action of $W^*$ in the left leg of the inner product and with parameters $y_2, x_2$ for the action of $W^*$ in the left leg of the inner product. The resulting four-fold sum has the advantage that the inner product factorizes, and we obtain

$$\sum_{y_1 y_2} \left| \begin{array}{c} z_2 u_2 \\ y_1 y_2 \end{array} \right| a_{z_2} (\varepsilon_1 \eta_1 q^{m_1} r_1 z_1, y_1) a_{\varepsilon_2 \eta_2 q^{m_2} r_2 z_2} (x_1, \varepsilon_1 \varepsilon_2 \eta_1 \eta_2 y_1 y_2 q^{-m_1 - m_2} / r_1 z_1)$$

$$\times a_{w_2} (\varepsilon \text{sgn}(p_1) \varepsilon_1 \eta_1 q^{m_1 + M} r_1 w_1, y_2)$$

$$\times a_{\varepsilon \text{sgn}(p_2) \varepsilon_2 \eta_2 q^{m_2 + n - M} r_2 w_2} (x_2, \varepsilon_1 \varepsilon_2 \eta_1 \eta_2 \varepsilon \text{sgn}(p_1 p_2) y_2 x_2 q^{-m_1 - m_2 - n} / r_1 w_1)$$

$$\times \langle Q(p_1, p_2; n) f_{-2m_1 - 2m_2 - \chi(r_1 r_2 z_1 / x_1), x_1, z_1}, f_{-2m_1 - 2m_2 - n - \chi(r_1 r_2 w_1 / x_2), x_2, w_1} \rangle$$

$$\times \langle f_{m_1 + m_2 + \chi(r_1 r_2 z_1 / x_1), \varepsilon_1 \varepsilon_2 \eta_1 \eta_2 q^{-m_1 - m_2 - n} y_1 y_2}, f_{m_1 + m_2 + n, \varepsilon \text{sgn}(p_1 p_2) q^{-m_1 - m_2 - n} y_2 x_2 / r_1 w_1, y_2} \rangle$$

where the sum is four-fold: $y_1, x_1, y_2, x_2 \in I_q$ so that $\varepsilon_1 \varepsilon_2 \eta_1 \eta_2 q^{-m_1 - m_2 + \chi(r_1 r_2 z_1 / x_1) / r_1 z_1} \in I_q$ and $\varepsilon_1 \varepsilon_2 \eta_1 \eta_2 \varepsilon \text{sgn}(p_1 p_2) y_2 x_2 q^{-m_1 - m_2 - n} / r_1 w_1 \in I_q$.

The final term in the summand (11.3) gives three Kronecker delta's, which lead to the reduction of the four-fold sum to a double(!) sum since $y_2 = y_1$ and $x_2 = \varepsilon \text{sgn}(p_1 p_2) q^n x_1 w_1 / z_1$ are required. Substituting this in the matrix element of $Q(p_1, p_2, n)$ in the summand in (11.3) gives

$$\langle Q(p_1, p_2; n) f_{-2m_1 - 2m_2 - \chi(r_1 r_2 z_1 / x_1), x_1, z_1}, f_{-2m_1 - 2m_2 - n - \chi(r_1 r_2 z_1 / x_1), \varepsilon \text{sgn}(p_1 p_2) q^n x_1 w_1 / z_1, w_1} \rangle$$

and by (7.13) this equals zero unless $r_1 r_2 = q^{-2m_1 - 2m_2 - n}$. In case this condition holds we see that the matrix coefficient of $Q(p_1, p_2; n)$ equals

$$\left| \begin{array}{c} z_1 \\ w_1 \end{array} \right| a_{z_2} (p_1, w_1) a_{w_2} (p_2, \varepsilon \text{sgn}(p_1 p_2) q^n x_1 w_1 / z_1).$$
Eliminating again $r_2$ and using this we find that (11.9) equals

$$\sum_{y_1, x_1 \in I_q} \left| \frac{z_2 w_2}{y_1^2} \right| a_{z_2} \left( \epsilon_1 \eta_1 q^{m_1} r_1 z_1, y_1 \right) \times a_{\epsilon_2 q^{2m_2} - m_2 - r_1 \eta_1 q^{-m_1 - m_2} y_1 x_1} \left( x_1, \epsilon_1 \eta_1 q^{m_1} r_1 z_1 \right) a_{w_2} \left( \sigma \operatorname{sgn}(p_1) \epsilon_1 \eta_1 q^{m_1 + m_2} r_1 w_1, y_1 \right) \times a_{\eta_2 q^{-2m_1 - m_2} y_1 x_1} \left( \sigma \operatorname{sgn}(p_2) \epsilon_2 q^{-2m_2} w_2 x_1 \right) \left( \eta_1 q^{-m_1 - m_2} y_1 x_1 \right) \times \left| \frac{z_1}{w_1} \right| a_{z_1} \left( p_1, w_1 \right) a_{x_1} \left( p_2, \sigma \operatorname{sgn}(p_1 p_2) q^r x_1 w_1 \right) \right).$$

(11.10)

Equating (11.7) and (11.10) and canceling common factors and relabeling $r_1, x_1, y_1$ by $r, x, y$ then proves Theorem 6.8 except for the sign constraint on $y$ in the sum. This follows from Definition 6.2.

\[\square\]

APPENDIX A. OPERATORS AND VON NEUMANN ALGEBRAS

A.1. von Neumann algebras. Let $H$ be a Hilbert space, and $B(H)$ the space of bounded linear operators equipped with the operator norm $\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$. Apart from the topology induced by the operator norm, there are various other topologies on $B(H)$. A net $\{T_i\}_{i \in I}$ converges strongly to $T$ if $\{T_i x\}_{i \in I}$ converges to $Tx$ for all $x \in H$. A net $\{T_i\}_{i \in I}$ converges weakly to $T$ if $\{(T_i x, y)\}_{i \in I}$ converges to $(Tx, y)$ for all $x, y \in H$. A net $\{T_i\}_{i \in I}$ converges strongly-* to $T$ if $\{T_i\}_{i \in I}$ converges strongly to $T$ and $\{T_i^*\}_{i \in I}$ converges strongly to $T^*$.

A von Neumann algebra is a unital *-subalgebra $M$ of $B(H)$ which is closed for the weak topology. A fundamental property is that $M$ equals its bicommutant $M''$. The elements of the form $T^*T$ form the cone of positive elements, denoted by $M_+$. A *-homomorphism is unital when it maps unit to unit.

A linear functional $\omega: M \to \mathbb{C}$ is normal if $\omega: M_1 \to \mathbb{C}$ is continuous with respect to the weak topology, where $M_1$ is the closed unit ball with respect to the operator norm. The space of normal functionals form the predual $M_*$ which is a norm-closed subspace of the dual $M^*$. The cone of positive normal functionals is denoted $M^+_*$. Then $M = (M_*)^*$ and the $\sigma$-weak topology on $M$ is the $\sigma(M, M_*)$-topology. The $\sigma$-strong-* topology is the locally convex vector topology induced by the seminorms $p_\omega(T) = \sqrt{\omega(T^*T)}$, $p_\omega^*(T) = \sqrt{\omega(TT^*)}$ for all $\omega \in M_+^*$. A unital *-homomorphism $\pi: M \to N$, $M$ and $N$ von Neumann algebras is normal if $\omega \pi \in M_*$ for all $\omega \in N_*$.

The tensor product of the von Neumann algebras $M \subset B(H)$ and $N \subset B(K)$ is the weak closure $M \otimes N$ of the algebraic tensor product $M \circ N \subset B(H \otimes K)$. For $\omega \in M_*$, $\eta \in N_*$ we have $\omega \otimes \eta \in (M \otimes N)_*$ as the unique element extending the algebraic tensor product $\omega \otimes \eta$.

A.2. Summation of operators. If we use the symbol $\oplus$ without further mention we mean the completed version. Let $(H_i)_{i \in I}$ be a family of Hilbert spaces and define the Hilbert space $H = \oplus_{i \in I} H_i$. Suppose that a permutation $\sigma: I \to I$ and for every $i \in I$ a closed, densely
defined, linear operator $T_i$ from $H_i$ into $H_{v(i)}$ is given. Then $\oplus_{i \in I} T_i$ denotes the closed, densely defined, linear operator in $H$ with domain

$$\{ v \in H \mid v_i \in D(T_i) \text{ for each } i \in I \text{ and } \sum_{i \in I} \| T_i(v_i) \|^2 < \infty \}$$

and so that $(\oplus_{i \in I} T_i)(v) = \sum_{i \in I} T_i(v_i)$ for all $v \in D(\oplus_{i \in I} T_i)$. Also recall that $T^* = \oplus_{i \in I} T_i^*$.

It is also worthwhile to remember that $T^*T = \oplus_{i \in I} T_iT_i$ and $|T| = \oplus_{i \in I} |T_i|$.

A.3. Commutation. Let $H$ be a Hilbert space. Consider two linear operators $S$, $T$ acting in a Hilbert space $H$. We say that $S \subseteq T$ if $D(S) \subseteq D(T)$ and $Sv = Tv$ for all $v \in D(S)$.

Let $T$ a densely defined, closed, linear (possibly unbounded) operator in $H$. If $S \in B(H)$, we say that $S$ and $T$ commute if $ST \subseteq TS$. If $N$ is a (possibly unbounded) self-adjoint operator in $H$, we say that $T$ and $N$ strongly commute if $T$ commutes with every spectral projection of $N$. If $T$ and $N$ are both (possibly unbounded) self-adjoint operators, then $T$ and $N$ commute strongly if and only if their spectral projections commute. This is also known as resolvent commuting self-adjoint operators. In this case $T + N$ is a closable operator and its closure $\overline{T + N}$ is self-adjoint.

A.4. Affiliation and unbounded generators. If $M$ is a von Neumann algebra on $H$, then a densely defined closed linear operator $T$ is affiliated to $M$ (in the von Neumann algebraic sense) if and only if $TU = UT$ for each unitary $U$ in the commutant $M'$. Then $T$ is affiliated with $M$ if and only if $T$ commutes with every element of $M'$. Moreover, if $T$ is affiliated with $M$, then so are $T^*$ and $T^*T$. If $T$ is a positive invertible operator affiliated to $M$, then so is $T^{-1}$. Also, if $T$ and $N$ are self-adjoint operators that are affiliated with $M$ and $T$ and $N$ commute strongly, then $\overline{T + N}$ is affiliated with $M$.

For $T_1, \ldots, T_n$ closed, densely defined (possibly unbounded) linear operators acting on a Hilbert space $H$ we define the von Neumann algebra

$$N = \{ x \in B(H) \mid xT_i \subseteq T_ix, \text{ and } xT_i^* \subseteq T_i^*x \ \forall \ i \}'.$$

Then $N$ is the smallest von Neumann algebra so that $T_1, \ldots, T_n$ are affiliated to $N$, and we call $N$ the von Neumann algebra generated by $T_1, \ldots, T_n$.

**Appendix B. Special functions**

B.1. Basic hypergeometric functions. Here we recall standard notations from the theory of basic hypergeometric functions, see for instance [17].

We fix a parameter $q \in (0, 1)$. The $q$-shifted factorials are defined by

$$(x; q)_\infty = \prod_{k=0}^{\infty} (1 - xq^k), \quad (x; q)_n = \frac{(x; q)_\infty}{(xq^n; q)_\infty}, \quad x \in \mathbb{C}, \ n \in \mathbb{Z}.$$ 

In particular, for $n \in \mathbb{N}$ we have $(x; q)_n = (1 - x)(1 - qx) \cdots (1 - qx^{n-1})$. Considered as a function of $x$, the $q$-shifted factorial $(x; q)_\infty$ is an entire function. Moreover, $(x; q)_\infty = 0$ if and only if $x \in q^{-\mathbb{N}_0}$. For products of $q$-shifted factorials we use the shorthand notation

$$(x_1, x_2, \ldots, x_k; q)_n = (x_1; q)_n(x_2; q)_n \cdots (x_k; q)_n, \quad n \in \mathbb{Z} \cup \{\infty\}.$$
A formula that we frequently use is the $\theta$-product identity:

$$(q^k x, q^{1-k}/x; q)_\infty = (-x)^{-k} q^{-k(k-1)/2} (x, q/x; q)_\infty, \quad x \in \mathbb{C} \setminus \{0\}, \ k \in \mathbb{Z}. \quad (B.1)$$

For $r, s \in \mathbb{N}_0$ the basic hypergeometric series is defined by

$$r \varphi_s \left( x_1, x_2, \ldots, x_r ; y_1, y_2, \ldots, y_s ; q, z \right) = \sum_{k=0}^{\infty} \left( x_1, x_2, \ldots, x_r ; q \right)_k (q, y_1, y_2, \ldots, y_s ; q)_k \left( -1 \right)^k q^{k(k-1)/2} z^k.$$  

Here we assume $x_i \in \mathbb{C}$ for $i = 1, \ldots, r$, $y_i \in \mathbb{C} \setminus q^{-\mathbb{N}_0}$ for $i = 1, 2, \ldots, s$, and $z \in \mathbb{C}$. If $r \leq s$, the series converges absolutely for all $z \in \mathbb{C}$. If $r = s + 1$, the series converges absolutely for $|z| < 1$. In case $r > s + 1$, the definition of the basic hypergeometric series only makes sense if $x_i \in q^{-\mathbb{N}_0}$ for some $i \in \{1, 2, \ldots, r\}$, i.e., if the series terminates.

### B.2. The functions $a_p$.

The functions $a_p(x, y)$ for $x, y, p \in I_q^+$ have been introduced in Definition 6.2, and these functions play a crucial role in the whole construction. We need some more properties of these functions which are described in this subsection.

We need to study the case $a_p(x, y)$ for $y \in I_q^+ = q^{2}$. This is contained in the following lemma.

**Lemma B.1.** For $y \in I_q^+$ there exists a differentiable function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $a_p(x, y) = y^{\chi(p/x)} f(y^{-2})$. Moreover, $f(0) = 0$ unless $0 < x/p \leq 1$, and in that case $f(0) \neq 0$.

**Proof.** Assume $y \in I_q^+$, so that $\text{sgn}(y) = +$. So in particular, $a_p(x, y) = 0$ for $\text{sgn}(x) \neq \text{sgn}(p)$ by Definition 6.2 and in this case we can take $f$ identically equal to zero.

In case $\text{sgn}(x) = \text{sgn}(p)$ we rewrite the $y$-dependent part in Definition 6.2 before the $\Psi$-function,

$$y \nu(py/x) \sqrt{(-y^2; q^2)_\infty} = \nu(p/x) \sqrt{(-1, -q^2; q^2)_\infty} \frac{y^{\chi(p/x)}}{\sqrt{(-q^2/y^2; q^2)_\infty}}$$

using the theta-product identity (B.1). Now using $s(x, y) = 1$ we find

$$a_p(x, y) = y^{\chi(p/x)} f(y^{-2}),$$

$$f(z) = C(p, x) \frac{1}{\sqrt{(-q^2 z; q^2)_\infty}} \Psi \left( -q^2 z \ ; q^2 \kappa(x) z ; q^2, q^2 x^2/p^2 \right),$$

$$C(p, x) = c_q (-1)^{\chi(p) + \chi(x)} \nu(p/x) \sqrt{(-1, -q^2, -\kappa(p); q^2\kappa(x))}.$$  

This gives the required differentiable function $f$, which is well-defined on $(-q^{-2}, \infty)$ and even real-analytic. The value of $f(0)$ is

$$f(0) = C(p, x) \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2; q^2)_n} \left( -\frac{q^2 x^2}{p^2} \right)^n = C(p, x) \left( q^2 x^2/p^2; q^2 \right)_\infty$$

by (17), (II.2)], and this is zero if $x/p > 1$ since $x/p \in q^{2}$ and non-zero otherwise.  

The following contiguous relations are useful.
Prop. 3.9], see also [30, (6.3)]. If we apply the second contiguency relation with Lemma B.3.

A proof of the second equality can be found in the second half of the proof of [30, 90 WOLTER GROENEVELT, ERIK KOELINK AND JOHAN KUSTERMANS]

Lemma B.2. Consider x, y, p ∈ I_q. Then

\[ \sqrt{1 + \kappa(q^{-1}x)} a_p(q^{-1}x, y) = (xy/p) \ a_p(x, y) - \text{sgn}(y) q^{-1} \sqrt{1 + \kappa(y)} a_p(x, qy) \]

and

\[ \sqrt{1 + \kappa(x)} a_p(qx, y) = (xy/p) \ a_p(x, y) - \text{sgn}(y) q \sqrt{1 + \kappa(q^{-1}y)} a_p(x, q^{-1}y). \]

Proof. A proof of the second equality can be found in the second half of the proof of [30, Prop. 3.9], see also [30, (6.3)]. If we apply the second contiguency relation with x and y interchanged, we get

\[ \sqrt{1 + \kappa(y)} a_p(qy, x) = (xy/p) \ a_p(x, y) - \text{sgn}(y) q \sqrt{1 + \kappa(q^{-1}x)} a_p(y, q^{-1}x) \]

and the first contiguency relation follows from the second equality in (B.2). \( \square \)

The following identity is essentially the second-order q-difference equation for \( \varphi_1 \)-functions.

Lemma B.3. Consider x, y, p ∈ I_q. Then

\[ (\kappa(p) - \kappa(y) + \frac{y^2p^2}{x^2}) a_p(x, y) + \frac{yp}{x} \sqrt{1 + \kappa(q^{-1}p)} a_{q^{-1}p}(x, y) + q \frac{yp}{x} \sqrt{1 + \kappa(p)} a_{qp}(x, y) = 0. \]

Proof. This equation holds trivially if \( py/x < 0 \). From now on we assume that \( py/x > 0 \). We know that the Ψ-functions satisfy the following q-difference equation for a, b, c, z ∈ \( \mathbb{C} \) (see the proof of Lemma 2.1 of [10], or take a limit in [13, Ex.1.13])

\[ (c - az) \ \Psi(a; c; q^2, 2z) + (z - (c + q^2)) \ \Psi(a; c; q^2, z) + q^2 \ \Psi(a; c; q^2, z/q^2) = 0. \]

Hence,

\[ (q^2 \kappa(x/y) + q^4 x^2/y^2 p^2)) \ \Psi(-q^2/\kappa(y); q^2 \kappa(x/y); q^2, q^2 \kappa(x/q^{-1}p)) \]

\[ + (q^2 \kappa(x/y) - q^2 + q^2 \kappa(x/p)) \ \Psi(-q^2/\kappa(y); q^2 \kappa(x/y); q^2, q^2 \kappa(x/p)) \]

\[ + q^2 \ \Psi(-q^2/\kappa(y); q^2 \kappa(x/y); q^2, q^2 \kappa(x/qp)) = 0. \]

Multiplying this equation with \( y^2p^2/q^2x^2 (-1)^{\chi(p)+1} \nu(py/x) \) and using the fact that \( \nu(py/x) = q^{-2} (py/x) \nu(q^{-1}py/x) = q^{2} (x/py) \nu(qpy/x) \), we get that

\[ (\kappa(p) - \kappa(y) + p^2y^2/x^2) (-1)^{\chi(p)} \nu(py/x) \ \Psi(-q^2/\kappa(y); q^2 \kappa(x/y); q^2, q^2 \kappa(x/p)) \]

\[ + (py/x) (1 + \kappa(q^{-1}p)) (-1)^{\chi(q^{-1}p)} \nu(q^{-1}py/x) \ \Psi(-q^2/\kappa(y); q^2 \kappa(x/y); q^2, q^2 \kappa(x/q^{-1}p)) \]

\[ + q (py/x) (-1)^{\chi(qp)} \nu(qpy/x) \ \Psi(-q^2/\kappa(y); q^2 \kappa(x/y); q^2, q^2 \kappa(x/qp)) = 0. \]

Multiplying this with \( \sqrt{(\kappa(p); q^2)_{\infty}} \), it follows that

\[ 0 = (\kappa(p) - \kappa(y) + p^2y^2/x^2) (-1)^{\chi(p)} \nu(py/x) \]

\[ \times \sqrt{(\kappa(p); q^2)_{\infty}} \ \Psi(-q^2/\kappa(y); q^2 \kappa(x/y); q^2, q^2 \kappa(x/p)) \]

\[ + \frac{py}{x} \sqrt{1 + \kappa(q^{-1}p)} (-1)^{\chi(q^{-1}p)} \nu(q^{-1}py/x) \]

\[ \times \sqrt{(\kappa(q^{-1}p); q^2)_{\infty}} \ \Psi(-q^2/\kappa(y); q^2 \kappa(x/y); q^2, q^2 \kappa(x/q^{-1}p)) \]

\[ + \frac{qpy}{x} \sqrt{1 + \kappa(p)} (-1)^{\chi(qp)} \nu(qpy/x) \]

\[ \times \sqrt{(\kappa(qp); q^2)_{\infty}} \ \Psi(-q^2/\kappa(y); q^2 \kappa(x/y); q^2, q^2 \kappa(x/qp)). \]
Now the lemma follows from Definition B.2.

We also need a few estimates involving the functions $a_p(x, y)$.

**Lemma B.4.** Consider $p \in I_q$ and $r, s \in q^2$. Then, there exists a constant $D > 0$ so that

$$|a_p(x, y)| \leq D \nu(p/y) |x|^{\chi(p/y)}$$

for all $x, y \in I_q$ satisfying $|x| \geq r$ and $|y| \leq s$.

**Proof.** If $\text{sgn}(xy) = \text{sgn}(p)$ (otherwise $a_p(x, y) = 0$), then the symmetry relation (B.2) and Definition B.2 imply that

$$|a_p(x, y)| = |a_p(y, x)| = c_q \psi \left( \frac{q^2}{x} \frac{\kappa(p)}{y} ; q^2 \kappa(y/p) \right) \left| \frac{x}{y} \nu(p/x/y) \sqrt{(-\kappa(x); q^2)_{\infty}} \right|.$$ 

and $|x| \nu(p/x/y) = \nu(qx) \nu(p/y) |x|^{\chi(p/y)}$ by Definition B.1. Now observe that for $x > 0$,

$$\sqrt{(-\kappa(x); q^2)_\infty} \nu(qx) = \frac{\sqrt{2} (q^2; q^2)_\infty}{\sqrt{(-q^2/x^2; q^2)_\infty}},$$

by the $\theta$-product identity (B.1). Furthermore, for $x < 0$, the set $\{ x \in I_q \mid |x| \geq r \}$ is finite. Hence, it is clear that there exists a constant $D > 0$ so that $|a_p(x, y)| \leq D \nu(p/y) |x|^{\chi(p/y)}$ for all $x, y \in I_q$ satisfying $|x| \geq r$ and $|y| \leq s$.

**Lemma B.5.** Consider $p, y \in I_q$, $\alpha > 0$ and $r \in [1, \infty)$. Then, the family $(|x|^{-\alpha} a_p(x, y))_{x \in I_q}$ belongs to $\ell^r(I_q)$.

**Proof.** Since $|a_p(x, y)| = |a_p(y, x)|$ by (B.2), Lemma B.4 implies the existence of a constant $D > 0$ so that $|x^{-\alpha} a_p(x, y)| \leq D \nu(p/x) |y|^{\chi(p/x)-\alpha}$ for all $x \in I_q$ satisfying $|x| \leq q$.

Next we need an estimate for $|x| \geq 1$. If $p/y \geq 1$, Lemma B.4 assures the existence of $E > 0$ so that $|a_p(x, y)| \leq E$ for all $x \in I_q$ satisfying $x \geq 1$. If on the other hand, $p/y < 1$, Lemma B.4 and the fact that $|a_p(x, y)| = |y/p| |a_p(x, p)|$ by (B.2), guarantee also in this case the existence of $E > 0$ so that $|a_p(x, y)| \leq E$ for all $x \in I_q$ satisfying $x \geq 1$. Hence, the lemma follows.

**B.3. The function $S(t; p_1, p_2, n)$.** The following function is defined as an infinite sum of certain limits of the functions $a_p$. Let $p_1, p_2 \in I_q$, $n \in \mathbb{Z}$. The function $S(\cdot; p_1, p_2, n) : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ is defined by

$$S(t; p_1, p_2, n) =$$

$$C \sum_{x \in \text{sgn}(p_1)q^2} (\text{sgn}(p_1)p_2)^{\chi(x)} \frac{1}{|z|} \nu\left( p_1 \right) \nu\left( \frac{p_2 q^n}{z} \right) 1 \varphi_1 \left( \begin{array}{ccc} q^2 & -q^2 / \kappa(p_1) & 0 \\ 0 & q^2 & \kappa(sgn(p_1)p_2)q^{-n} \end{array} \right) \times 1 \varphi_1 \left( \begin{array}{ccc} -q^2 / \kappa(p_2) & 0 \\ 0 & q^2 & \kappa(sgn(p_1)p_2)q^{-n} \end{array} \right),$$

where

$$C = C(p_1, p_2, n) = (\text{sgn}(p_2))^{n} |p_1 p_2| c_q^2 q^n \sqrt{(-\kappa(p_1), -\kappa(p_2); q^2)_{\infty}}.$$
The sum is absolutely convergent, so $S(\cdot; p_1, p_2, n)$ is an analytic function on $\mathbb{C} \setminus \{0\}$. The function $S(t; p_1, p_2, n)$ can be written as a $2\varphi_1$-function. To see this we need a few lemmas.

In the following lemma the special case $b = q$ is obtained by Koornwinder and Swarttouw as a $q$-analogue of Graf’s addition formula for Bessel functions \cite[4.10]{Graf}. The proof of Lemma \ref{lem:4} runs along the same lines as the proof used in \cite{Graf}.

**Lemma B.6.** For $c \in q\mathbb{Z}$, $|u| < 1$, and $|bu/w| < 1$,

$$\sum_{n=-\infty}^{\infty} u^n q^{\frac{1}{2}n(n-1)} 1\varphi_1\left(\frac{u}{0}; q, cq^n \right) 1\varphi_1\left(\frac{v}{0}; q, bq^n \right) = \frac{(q, u, -w, -q/w, -cu/w, bq/c; q)_{\infty}}{(-bu/w, -c/w, -wq/c; q)_{\infty}} 2\varphi_1\left(\frac{-bu/w, -wq/cu}{bq/c}; q, u \right).$$

Other expressions for the sum in the above lemma, for values of $u, w, b$ not satisfying the above conditions, can be obtained using transformation formulas for $2\varphi_1$-series.

**Proof.** Assume $|y| < 1, |sb/x| < |t| < |y^{-1}|$ and $|y| < |t|$. We write the product of the following $1\psi_1$-function and $1\varphi_0$-function as a double series;

$$1\psi_1\left(\frac{x}{sy} b; q, y t \right) 1\varphi_0\left(\frac{x/s}{y} b; q, -y/t \right) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{(x/sy; q)_n (x/s; y; q)_k}{(b; q)_n (q; q)_k} (-1)^k y^{n+k} t^{n-k}.$$ Renaming $n = m + k$, the sum over $k$ can be written as a $2\varphi_1$-series. Using Ramanujan’s $1\psi_1$-summation formula \cite[(II.29)]{Ramanujan} and the $q$-binomial formula \cite[(II.3)]{Ramanujan}, we obtain

$$\frac{(q, bsy/x, xt/s, qs/xt; q)_{\infty}}{(b, qsy/x, y t, bs/xt; q)_{\infty}} (-y/t; q)_{\infty} = \sum_{m=-\infty}^{\infty} \frac{(x/sy; q)_m (yt)_m}{(b; q)_m (q; q)_m} 2\varphi_1\left(\frac{x/sy, xq^m/yt}{bq^m}; q, -y^2 \right).$$  (B.3)

We consider this formula as the Laurent expansion of the left hand side considered as a function of $t$.

Let us consider two special cases of (B.3). Letting $y \to 0$, we obtain

$$\frac{(q, xt/s, qs/xt, -xs/t; q)_{\infty}}{(b, bs/xt; q)_{\infty}} = \sum_{m=-\infty}^{\infty} \left( -\frac{xt}{s} \right)^m q^{\frac{1}{2}m(m-1)} \frac{1}{(b; q)_m} 1\varphi_1\left(\frac{-}{bq^m}; q, -q^m x^2 \right) = \frac{1}{(b; q)_{\infty}} \sum_{m=-\infty}^{\infty} \left( -\frac{xt}{s} \right)^m q^{\frac{1}{2}m(m-1)} 1\varphi_1\left(\frac{-}{b}; q, bq^m \right).$$

In the last line we used the transformations

$$1\varphi_1\left(\frac{z}{0}; q, c \right) = (c, z; q)_{\infty} 2\varphi_1\left(0, 0; q, z \right) = (c; q)_{\infty} 1\varphi_1\left(\frac{-}{c}; q, cq \right),$$  (B.4)

which follow from Heine’s $2\varphi_1$-transformations \cite[(III.1), (III.3)]{Heine} by letting $a, b \to 0$.

For the second special case we observe that in the above calculations the assumption $|sb/x| < |t|$ was needed for absolute convergence of the bilateral $1\psi_1$-series. In case $b = q$ this series
can be written as a unilateral series, a \(1\varphi_0\)-series, and then the assumption \(|sb/x| < |t|\) is no longer needed. Now setting \(b = q\) and \(x = 0\), we find
\[
\frac{1}{(yt, -y/t; q)_\infty} = \sum_{m=-\infty}^\infty \frac{(yt)^m}{(q; q)_m} 2\varphi_1 \left( 0, 0 \mid q^{1+m} ; q, -y^2 \right)
\]
\[
= \frac{1}{(q, -y^2; q)_\infty} \sum_{m=-\infty}^\infty (yt)^m 1\varphi_1 \left( -y^2 0 \mid q, q^{1+m} \right) \tag{B.5}
\]
where we used \((B.4)\), and for the last equality we used the \(t \leftrightarrow -t^{-1}\) invariance and reversed the sum.

Multiplying our two special cases of \((B.3)\), we obtain a second expression for the Laurent expansion of the left hand side of \((B.3)\) considered as a functions of \(t\);
\[
\frac{(q, bsy/x, xt/s, qs/xt, -xs/t; q)_\infty}{(b, qsy/x, yt, bs/xt, -y/t; q)_\infty}
\]
\[
= \frac{(bsy/x; q)_\infty}{(q, -y^2, b, sqy/x; q)_\infty} \sum_{k=-\infty}^\infty \left( -\frac{t}{y} \right)^k 1\varphi_1 \left( -y^2 0 \mid q, q^{1-k} \right)
\]
\[
\times \sum_{n=-\infty}^\infty \left( -\frac{xt}{s} \right)^n q^{2n(n-1)} 1\varphi_1 \left( -x^2/b 0 \mid q, bq^n \right).
\]
Here we used \(n + k = m\). Comparing coefficients of \(t\) in \((B.3)\) and the above formula, and, to get rid of the squares, replacing \((-y^2, -x^2/b, xy/s)\) by \((u, v, w)\), we obtain
\[
\sum_{n=-\infty}^\infty w^n q^{2n(n-1)} 1\varphi_1 \left( u 0 \mid q, q^{1+n-m} \right) 1\varphi_1 \left( v 0 \mid q, bq^n \right) =
\]
\[
u^m(q, u, -qu/w, -w/u, bq^m; q)_\infty 2\varphi_1 \left( -bv/w, -qw^m/u \mid bq^{m} ; q, u \right).
\]
Observe that by the \(\theta\)-product identity \((B.1)\),
\[
u^m(q, u, -qu/w, -w/u; q)_\infty = w^m q^{2m(m-1)} (-uq^{1-m}/w; q)_\infty = \frac{(w, -q/w, -uq^{1-m}/w; q)_\infty}{(-q^{1-m}/w, -qw^m; q)_\infty},
\]
then the result follows from writing \(q^{1-m} = c\). \(\square\)
Remark B.7. We can prove a slightly more general result along the same lines as the proof of Lemma B.6, starting with the product

\[ 1\psi_1 \left( \frac{x}{sy} b ; q, yt \right) \bar{1}\psi_1 \left( \frac{x}{s} d ; q, \frac{y}{t} \right). \]

This leads to the identity

\[ \sum_{k=-\infty}^{\infty} \left( \frac{xy}{s} \right)^k q^{\frac{1}{2}k(k-1)} 1\varphi_1 \left( -\frac{x}{b} ; q, bq^k \right) 2\varphi_2 \left( -y^2, dy/xs ; q, q^{1-m+k} \right) = \]

\[ (-y^2)^m \left( \frac{x}{sy} q \right)_m (d, -y^2, syq/x ; q)_\infty \]

\[ \left( \frac{b}{q} \right)_m (bsy/x ; q)_\infty \]

\[ 2\psi_2 \left( xq^m/sy, xs/y ; bq^m, d ; q, -y^2 \right). \]

For \( d = q \) this is equivalent to the result from Lemma B.6.

The following lemma shows that the result of Lemma B.6 remains valid for \( c \not\in q^{-Z} \), if we assume \( u \in q^{-\mathbb{N}_0} \). The \( 2\varphi_1 \)-series in Lemma B.6 does not converge in this case, but it can be obtained from the \( 2\varphi_1 \)-series in the following Lemma by an application of Heine’s transformation \([17, (III.2)]\).

Lemma B.8. For \( u = q^{-\mathbb{N}_0} \) and \( |bu/w| < 1 \),

\[ \sum_{n=-\infty}^{\infty} w^n q^{\frac{1}{2}n(n-1)} 1\varphi_1 \left( \frac{u}{0} ; q, cq^n \right) 1\varphi_1 \left( \frac{v}{0} ; q, bq^n \right) = \]

\[ (q, -w, -q/w, -cu/w ; q)_\infty \]

\[ \left( -c/w ; q \right)_\infty \]

\[ 2\varphi_1 \left( -wq/cu, v ; q, -bu/w \right). \]

Proof. Let us denote the infinite sum on the left hand side by \( S \). We write \( u = q^{-k} \) with \( k \in \mathbb{N}_0 \), then by definition of the \( 1\varphi_1 \)-series, we have

\[ S = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{k} \sum_{l=0}^{\infty} \frac{(q^{-k}, q)_m (v ; q)_l (q ; q)_l}{(q ; q)_m (q ; q)_l} q^{\frac{1}{2}m(m-1)} (-c)^m q^{\frac{1}{2}l(l-1)} (-b)^l q^{\frac{1}{2}m(n-1)} (wq^{m+l})^n. \]

This double sum converges absolutely, so we may first sum over \( n \). Using Jacobi’s triple product identity \([17, (II.28)]\) we find

\[ \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n(n-1)} (wq^{m+l})^n = (q, -wq^{m+l}, -q^{1-m-l}/w ; q)_\infty \]

\[ = w^{-(m+l)} q^{-\frac{1}{2}m(m-1)} q^{-\frac{1}{2}l(l-1)} q^{-lm} (q, -w, -q/w ; q)_\infty. \]

Here the second equality follows from the \( \theta \)-product identity \([B.1]\). Now \( S \) reduces to

\[ S = (q, -w, -q/w ; q)_\infty \sum_{l=0}^{k} \frac{(v ; q)_l}{(q ; q)_l} (-b/w)^l \frac{(q^{-k} ; q)_m}{(q ; q)_m} (-c q^{-l}/w)^m. \]
The sum over $m$ can be evaluated with the $q$-binomial formula \cite[(II.3)]{L}
\[
\sum_{m=0}^{k} \frac{(q^{-k}; q)_m}{(q; q)_m} (-cq^{-l}/w)^m = \frac{(-cq^{-l-k}/w; q)_\infty}{(-cq^{-l}/w; q)_\infty} = \frac{(-cq^{-l-k}/w; q)_l(-cq^{-k}/w; q)_\infty}{(-cq^{-l}/w; q)_l(-c/w; q)_\infty}
\]
which becomes a multiple of a single sum,
\[
S = \frac{(q, -w, -q/w, -cq^{-k}/w; q)_\infty}{(-c/w; q)_\infty} \sum_{l=0}^{\infty} \frac{(-wq^{1+k}/c, v; q)_l}{(q, -wq/c; q)_l} \left( -\frac{bq^{-k}}{w} \right)^l.
\]

The sum is the $2\varphi_1$-series in the lemma. \hfill \Box

Remark B.9. In Lemmas \ref{lemma_3} and \ref{lemma_5} the sum $\Sigma$ on the left hand side has an obvious symmetry $(u, c) \leftrightarrow (v, b)$. On the right hand side this symmetry is not at all obvious, so there must be a $2\varphi_1$-transformation behind this symmetry. Let us see how the symmetry follows from known transformation formulas.

Applying the three-term transformation formula \cite[(III.31)]{L} we find
\[
(bq/c; q)_\infty 2\varphi_1 \left( \frac{-bw/v, -wq/cu}{bq/c}; q, u \right) = \frac{(v, bq/c, c/b; q)_\infty}{(-cv/wq, -wq/bv; q)_\infty} 2\varphi_1 \left( \frac{-wq/cv/q/v}{-q^2 w/cv}; q, -wq/bu \right) + \frac{c (v, cq/b, -wq/cu, -qw/cv, -bw/wq, -q^2 w/bw; q)_\infty}{b (u, -wq/bu, -qw/bv, -uwc/wq, -q^2 w/uwc; q)_\infty} 2\varphi_1 \left( \frac{-wq/bv, -cu/w}{cq/b}; q, v \right),
\]

where we also applied Heine’s transformation \cite[(III.3)]{L} for the second $2\varphi_1$ on the right hand side. Observe that the second $2\varphi_1$-function on the right hand side is the same as the $2\varphi_1$-function on the left hand side after the substitutions $(u, v, c, b) \leftrightarrow (v, u, b, c)$, which is exactly the symmetry we are looking for. This shows that the first $2\varphi_1$-function on the right hand side must vanish, which implies the condition $v \in q^{-\mathbb{N}_0}$ or $c/b \in q\mathbb{Z}$. Assuming one of these conditions, the symmetry $(u, c) \leftrightarrow (v, b)$ for $\Sigma$ is still not clear at this point, because of all the $q$-shifted factorials in front of the $2\varphi_1$-function. To take care of these factors we need to apply the $\theta$-product identity \cite[(B.3)]{B} several times. Let us assume that $v = q^{-k}$, $k \in \mathbb{N}_0$, then
\[
\frac{(-cu/w, -wq/cu; q)_\infty}{(-q^2 w/cuv, -uvc/wq; q)_\infty} = \left( \frac{-wq}{cu} \right)^{k+1} q^{\frac{k(k+1)}{2}},
\]
\[
\frac{(-bw/wq, -q^2 w/bw; q)_\infty}{(-bu/w, -wq/bu; q)_\infty} = \left( \frac{-bu}{wq} \right)^{k+1} q^{-\frac{k(k+1)}{2}},
\]
\[
\frac{(-wq/cv; q)_\infty}{(-c/w, -wq/c; q)_\infty} = \left( \frac{-c/wq}{cq/w; q}_\infty \right)^{k} q^{\frac{k(k-1)}{2}},
\]
\[
\frac{1}{(-wq/bv; q)_\infty} = \left( \frac{-wq}{b} \right)^{k} q^{\frac{k(k-1)}{2}} \frac{(-bw/w; q)_\infty}{(-b/w, -qw/b; q)_\infty}.
\]
which leads to
\[
\Sigma = \frac{(q,-w,-q/w,cq/b,-bv/w,v;q)_{\infty}}{(-uc/w,-b/w,-wq/b;q)_{\infty}} 2\varphi_1 \left( -wq/bv,-cu/w \middle| cq/b ; q,v \right).
\]

Comparing this with the right hand side in Lemma B.6, the symmetry \((u,c) \leftrightarrow (v,b)\) is now clear. In case \(b/c \in q^\mathbb{Z}\) similar computations must be used.

Observe that the conditions \(b/c \in q^\mathbb{Z}\) and \(v \in q^{-\mathbb{N}_0}\) correspond to Lemmas B.6 and B.8, respectively.

We are now ready to obtain a \(2\varphi_1\)-expression for the function \(S(t;p_1,p_2,n)\).

**Proposition B.10.** The function \(S(t;p_1,p_2,n)\) defined by (B.2) can be written as a multiple of a \(2\varphi_1\)-function:
\[
S(t;p_1,p_2,n) = p_2^n q^{\frac{n(n-1)}{2}} |p_1| p_2 |\nu(p_1)\nu(p_2)| q^2 \sqrt{(-\kappa(p_1),-\kappa(p_2);q^2)_{\infty}}
\times \frac{(q^2,-q^2/\kappa(p_2),-tq^{3-n}/p_1 p_2,-p_1 p_2 q^{n-1}/t,p_1 q^{-n}/p_2 t;q^2)_{\infty}}{(p_1 q^{1+n}/p_2 t,-p_1 p_2 q^{-n-1}/t,-tq^{n+3}/|p_1| p_2;q^2)_{\infty}}
\times (\text{sgn}(p_1 p_2) q^{2+2n};q^2)_{\infty} 2\varphi_1 \left( \frac{p_2 q^{1+n}/p_1 t,p_2 t q^{1+n}/p_1}{\text{sgn}(p_1 p_2) q^{2+2n}} ; q^2, -q^2/\kappa(p_2) \right).
\]

**Proof.** We substitute \(z = \text{sgn}(p_1) q^k, k \in \mathbb{Z}\), in (B.2), then
\[
S(t;p_1,p_2,n) = K \sum_{k=-\infty}^{\infty} \left( \frac{t q^{3-n}}{p_1 p_2} \right)^k q^{k(k-1)} \psi_1 \left( -\text{sgn}(p_1) q^2/p_1^2 ; q^2, \text{sgn}(p_1) q^{2+2k} \right)
\times \psi_1 \left( -\text{sgn}(p_2) q^2/p_2^2 ; q^2, \text{sgn}(p_2) q^{2+2k-2n} \right),
\]
with \(K = q^{\frac{n(n-1)}{2}} p_1^n p_2^n |\nu(p_1)\nu(p_2)| c_q^2 \sqrt{(-\kappa(p_1),-\kappa(p_2);q^2)_{\infty}}\).

Now we apply Lemmas B.6 and B.8 with \(q\) replaced by \(q^2\), and
\[
w = \frac{t q^{3-n}}{p_1 p_2}, \quad u = -\text{sgn}(p_2) q^2/p_2^2, \quad v = -\text{sgn}(p_1) q^2/p_1^2, \quad b = \text{sgn}(p_1) q^2, \quad c = \text{sgn}(p_2) q^{2-2n},
\]
to obtain the desired expression. \(\square\)

The function \(S(t;p_1,p_2,n)\) can be written in terms of several other \(2\varphi_1\)-functions using the following result.

**Lemma B.11.** The function \(S(t;p_1,p_2,n)\) satisfies the following symmetry relations:
\[
S(t;p_1,p_2,n) = (qt)^n S(t;p_2,p_1,-n)
= (-q)^n \text{sgn}(p_1)^{\chi(p_1)} \text{sgn}(p_2)^{\chi(p_2)+n} \text{sgn}(p_1 p_2) S(\text{sgn}(p_1 p_2) t^{-1}; p_1, p_2, -n),
= (-t)^n \text{sgn}(p_1)^{\chi(p_1)+n} \text{sgn}(p_2)^{\chi(p_2)} \text{sgn}(p_1 p_2) S(\text{sgn}(p_1 p_2) t^{-1}; p_2, p_1, n).
\]

**Proof.** The first symmetry relation follows from replacing the summation variable \(z\) by \(z q^n\) in definition (B.2).
Comparing coefficients of \( t \) in (B.5) gives the transformation formula
\[
1\varphi_1 \left( \frac{a}{0} ; q, q^{1+n} \right) = a^{-n} \ 1\varphi_1 \left( \frac{a}{0} ; q, q^{1-n} \right), \quad n \in \mathbb{Z}.
\]
Furthermore, as a special case of [30, Prop. 6.6] we have
\[
1\varphi_1 \left( \frac{q^{-n}}{0} ; q, qy \right) = y^n 1\varphi_1 \left( \frac{q^{-n}}{0} ; q, q/y \right), \quad n \in \mathbb{N}_0, \ y \in \mathbb{C} \setminus \{0\}.
\]
To both \( 1\varphi_1 \)-functions in (B.2) we apply one of the above transformations; the second one in case the \( 1\varphi_1 \) is a terminating series, the first transformation otherwise. Now we change the summation variable from \( z \) to \( z^{-1} \) to obtain the second symmetry relation.

The third relation follows from combining the first two relations.

Proposition B.10 and the symmetry relations from Lemma B.11 imply transformation formulas between the \( 2\varphi_1 \)-series involved. For instance, the first symmetry relation in Lemma B.11 together with an application of the \( \theta \)-product identity (B.1), corresponds to the transformation described in Remark B.9.

We also need the following asymptotic results for the function \( S \).

**Lemma B.12.** Assume \( t \in \mathbb{C} \setminus \{0\} \) and \( k, n \in \mathbb{Z} \).

(i) For \( k \to -\infty \),
\[
S(t; q^k, q^k, n) = \mathcal{O}(q^{-nk}).
\]

(ii) Let \( \sigma, \tau \in \{-, +\} \), then there exist constants \( C_1, C_2 \) independent of \( k \), such that
\[
S(t; \sigma q^k, \tau q^k, n) = (\sigma \tau q^3)^k \left( C_1 t^{-k} + C_2 t^k \right) \left( 1 + \mathcal{O}(q^{2k}) \right),
\]
for \( k \to \infty \).

(iii) Let \( p_1 \in I_q \) and \( \tau \in \{-, +\} \), then for \( k \to \infty \),
\[
S(t; p_1, \tau q^k, k + n) = \mathcal{O}(q^k).
\]

(iv) Let \( p_1 \in I_q \), then for \( k \to -\infty \),
\[
S(t; p_1, q^k, k + n) = \mathcal{O} \left( q^{\frac{1}{2} k^2} (p_1 t q^{-n-1})^k \right).
\]

**Proof.** (i) We use Proposition B.10 to write \( S(t; q^k, q^k, n) \) as a multiple of a \( 2\varphi_1 \)-series. Using the \( \theta \)-product identity (B.1) we find
\[
\left( -t q^{3-n-2k}, -q^{n-1+2k}/t; q^2 \right)_\infty = q^{-2nk} t^n
\]
and
\[
q^{2k} \nu(q^k)^2 (-\kappa(q^k), -q^2/\kappa(q^k); q^2)_\infty = q^2 (-1, -q^2; q^2)_\infty,
\]
so that
\[
S(t; q^k, q^k, n) = c_q^2 2^{-nk} q^{1/2(n-1)} \left( -1, -q^2, q^2, q^{1-n}/t; q^2 \right)_\infty \left( 1+n/t; q^2 \right)_\infty
\]
\[
\times \left( q^{2+2n}; q^2 \right)_\infty 2\varphi_1 \left( q^{1+n}/t, t q^{1+n}/q^{2+2n}; q^2, q^2, q^{2-2k} \right).
\]
From this expression it is clear that $S(t; q^k, q^n, n) = \mathcal{O}(q^{-nk})$ for $k \to -\infty$.

(ii) Write $S(t; \sigma q^k, \tau q^k, n)$ as a multiple of a $2\varphi_1$-function using Proposition $[\text{B.10}]$. Using the three-term transformation formula $[\text{[17]}, \text{III.32}]$ and the $\theta$-product identity $([\text{B.1}])$ we find

\[
S(t; \sigma q^k, \tau q^k, n) = c^2_n (\sigma q^3)^k \sqrt{(-\sigma q^{2k}, -\tau q^{2k}; q^2)_{\infty}} \frac{(q^2, \sigma \tau q^{1-n}/t, -\sigma \tau q^{1-n}/t, -\sigma \tau q^{3-n}/q^2)_{\infty}}{(q^{n+1}/t, -\sigma q^{n-1}/t, -\sigma \tau q^{3-n}/q^2)_{\infty}} \times 
\left\{ t^{-k} \frac{\sigma \tau q^{1+n}/t}{(q^2, -\tau q^{2k}; q^2)_{\infty}} 2\varphi_1 \left( \sigma \tau q^{1+n}/t, q^{1-n}/t, q^2, -\sigma q^2 \right) + t^k \frac{\sigma \tau q^{1+n}/t}{(t^2, -\tau q^{2k}; q^2)_{\infty}} 2\varphi_1 \left( \sigma \tau q^{1+n}/t, q^{1-n}/t, q^2, -\sigma q^2 \right) \right\}.
\]

From this expression the result follows.

(iii) By Proposition $[\text{B.10}]$ and $[\text{[17]}, \text{III.4}]$ there exists a constant $C_1$, which is independent of $k$, such that

\[
S(t; p_1, \tau q^k, k + n) = C_1 (\tau p_1 q^k)^{k(n+k)} q^{\frac{1}{2}(k^2+1)} q^{\frac{1}{2}(k+1)(k+2)} \sqrt{(-\tau q^{2k}; q^2)_{\infty}} \times (\tau p_1 q^{1-2k-n}/t, -\tau p_1 q^{2k+n-1}/t, -\tau q^{3-n-2k}/p_1; q^2)_{\infty} \times (\tau \text{sgn}(p_1) q^{2+2k+2n}; q^2)_{\infty} 2\varphi_2 \left( \tau q^{1+2k+n}/p_1 t, |p_1| q^{1+n}/t, \tau \text{sgn}(p_1) q^{2+2k+2n}, -q^{3+n}/p_1 t; q^2, -t q^{3+n}/p_1 \right).
\]

Using the $\theta$-product identity $([\text{B.1}])$ twice, we find

\[
(\tau p_1 q^{1-2k-n}/t, -\tau p_1 q^{2k+n-1}/t, -\tau q^{3-n-2k}/p_1; q^2)_{\infty} = C_2 \frac{(1)^{k} q^{-2nk} q^{-2k(k-1)} (\tau q^{1+2k+n}/p_1; q^2)_{\infty}}{(-\tau q^{1+2k+n}/p_1; q^2)_{\infty}}.
\]

Now we see that for large $k$ there exists a constant $C_3$, independent of $k$, such that

\[
|S(t; p_1, \tau q^k, k + n)| \leq C_3 q^k.
\]

(iv) Assume $k < 0$. By Proposition $[\text{B.10}]$ we have

\[
S(t; p_1, q^k, -n - k) = C_1 q^{-k(n+k)} q^{\frac{1}{2}(n+k)(n+k+1)} q^{\frac{1}{2}(k^2+1)} q^{k(n+1)} \sqrt{(-\tau q^{2k}; q^2)_{\infty}} \times \frac{(-q^{2-k}, \text{sgn}(p_1) q^{2-2n-2k}; q^2)_{\infty}}{(|p_1| q^{1-n-2k}/t, -p_1 q^{2k+n-1}/t, -q^{3-n-2k}/p_1; q^2)_{\infty}} 2\varphi_1 \left( q^{1-n}/p_1 t, t q^{1-n}/p_1, \text{sgn}(p_1) q^{2-2n-2k}; q^2, -q^{2-k} \right).
\]

for a certain constant $C_1$ independent of $k$. Using the $\theta$-product identity $([\text{B.1}])$ we have

\[
(-p_1 q^{2k+n-1}/t, -t q^{3-2k-n}/p_1; q^2)_{\infty} = C_2 \left( \frac{t}{p_1 q^{n-1}} \right)^k q^{-k(k-1)},
\]

\[
(-q^{2k}; q^2)_{\infty} = C_3 \frac{q^{-k(k-1)}}{(-q^{2-k}; q^2)_{\infty}},
\]

so that, for large $|k|$, there is a constant $C_4$ such that

\[
|S(t; p_1, q^k, -n - k)| \leq C_4 q^{\frac{1}{2}k^2} q^{(n-\frac{3}{2})k} |p_1/t|^k.
\]

Now the result follows from the second symmetry relation in Lemma $[\text{B.11}]$. □
B.4. **Al-Salam–Chihara polynomials.** The spectral analysis of Jacobi operators on \( \ell^2(\mathbb{N}_0) \) and \( \ell^2(\mathbb{Z}) \) plays an essential role in this paper. We refer to Berezanskii [4, Ch.7], Pruitt [17], Masson and Repka [14], Kakehi [23], see also [33, App. A], for general information on Jacobi operators on \( \ell^2(\mathbb{Z}) \). We use [29] for general reference. The spectral decomposition of the Jacobi operators we encounter are described with the help of certain special functions, namely the Al-Salam–Chihara polynomials and the little \( q \)-Jacobi functions. In this subsection we collect some results and notations for the Al-Salam–Chihara polynomials. Results for little \( q \)-Jacobi functions are given in the next subsection.

The Al-Salam–Chihara polynomials were introduced by Al-Salam and Chihara in [1] to classify all orthogonal polynomials satisfying a convolution type property. These polynomials also have been studied by Askey and Ismail [3, §3]. The Al-Salam–Chihara polynomials form a subfamily of the Askey-Wilson polynomials Askey and Wilson [1], Gasper and Rahman [17, §§7.5-7].

Consider \( a, b \in \mathbb{R} \setminus \{0\} \). For \( n \in \mathbb{N}_0 \), the Al-Salam–Chihara polynomials \( P_n(\cdot; a, b \mid q) : \mathbb{C} \to \mathbb{C} \) are defined by

\[
P_n(\mu(y); a, b \mid q) = a^{-n}(ab; q)_n \phi_2 \left( \frac{q^{-n}, ay, a/y}{ab, 0}; q, q \right) = (a/y; q)_n y^n \phi_1 \left( \frac{q^{-n}, by}{q^{1-n}y/a}; q, q/y \right),
\]

for \( y \in \mathbb{C} \setminus \{0\} \). The equality in (B.6) follows from [17, (III.7)] and holds if \( q^{1-n}y/a \notin q^{-\mathbb{N}_0} \). We see that for \( x = \mu(y) \in \mathbb{R} \) the polynomials \( P_n(x) = P_n(x; a, b \mid q) \) are real-valued. The Al-Salam–Chihara polynomials satisfy the three-term recurrence relation

\[
2x P_n(x) = P_{n+1}(x) + q^n(a+b) P_{n}(x) + (1-q^n)(1-abq^{n-1}) P_{n-1}(x)
\]

with initial condition \( P_{-1}(x) = 0 \), \( P_{0}(x) = 1 \). From this relation we see that the Al-Salam–Chihara polynomials are symmetric in \( a \) and \( b \). Favard’s Theorem gives that these polynomials are orthogonal with respect to a positive measure on the real line for \( ab < 1 \), which from now on we assume to hold. The measure can be determined from the asymptotic behaviour of the Al-Salam–Chihara polynomials as the degree tends to infinity. This behaviour is determined by

\[
\frac{(abq^n; q)_{\infty}}{(q, ab; q)_{\infty}} P_n(\mu(y); a, b \mid q) =
\]

\[
c(y; a, b \mid q) y^n \phi_1 \left( \frac{ay, by}{qy^2}; q, q^{n+1} \right) + c(y^{-1}; a, b \mid q) y^{-n} \phi_1 \left( \frac{a/y, b/y}{qy^{-2}}; q, q^{n+1} \right),
\]

valid if \( y^2 \notin q^\mathbb{Z} \), where

\[
c(y; a, b \mid q) = \frac{(a/y, b/y; q)_{\infty}}{(y^{-2}; q)_{\infty}(q, ab; q)_{\infty}}.
\]

We extend the \( c \)-function \( c(\cdot; a, b \mid q) \) by continuity to all points of \( \mathbb{C} \) where possible.

The asymptotic behaviour can be obtained as a limiting case \( (b, c \to 0) \) of the asymptotic behaviour of the Askey-Wilson polynomials [17, (7.5.9)], or by using [17, (3.3.5)] with \( (a, b, c, z) \mapsto (ay, by, qy^2, q^{n+1}) \) and next [17, (1.4.6)]; [B.7] and the \( \theta \)-product identity [B.4].
See also [3, §3.1] for the asymptotic behaviour using Darboux’s method including the cases \( x = \pm 1 \).

The corresponding orthonormal Al-Salam–Chihara polynomials \( p_n(\cdot ; a, b \mid q) : \mathbb{C} \to \mathbb{C} \) are defined by

\[
p_n(x; a, b \mid q) = \frac{1}{\sqrt{(q, ab;q)_n}} P_n(x; a, b \mid q)
\]

for all \( x \in \mathbb{C} \). The orthonormal Al-Salam–Chihara polynomials satisfy the recurrence relation

\[
2xp_n(x) = c_n p_{n+1}(x) + d_n p_n(x) + c_{n-1} p_{n-1}(x),
\]

\[
c_n = \sqrt{(1 - q^{n+1})(1 - abq^n)}, \quad d_n = q^n(a + b),
\]

and initial conditions \( p_{-1}(x) = 0, p_0(x) = 1 \). Note that the coefficients \( c_n \) and \( d_n \) are bounded, since we assume \( 0 < q < 1 \). Under our assumption \( ab < 1 \) the Al-Salam–Chihara polynomials are orthogonal with respect to a positive measure on \( \mathbb{R} \);

\[
\int_{\mathbb{R}} p_n(x; a, b \mid q)p_m(x; a, b \mid q) \, dm(x; a, b \mid q) = \delta_{n,m},
\]

where the measure \( dm(\cdot ; a, b \mid q) \) is defined by

\[
\int_{\mathbb{R}} f(x) \, dm(x; a, b \mid q) = (q; ab;q)_\infty \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty}{(ae^{i\psi}, ae^{-i\psi}, be^{i\psi}, be^{-i\psi}; q)_\infty} \int_0^\pi f(\cos \psi) \, d\psi
\]

\[
+ \sum_{r \in \mathbb{N}_0 \atop |aq'| > 1} f(\mu(aq')) w_r(a; b \mid q) + \sum_{r \in \mathbb{N}_0 \atop |bq'| > 1} f(\mu(bq')) w_r(b; a \mid q),
\]

with

\[
w_r(a; b \mid q) = \frac{(a; q)_\infty (a^2; ab;q)_r (1 - a^2 q^{2r})}{(b/a; q)_\infty (qaq/b; q)_r (1 - a^2)} q^{-r^2 a^{-3r} b^{-r}}.
\]

Note that the weight function in (B.12) is very explicit. It can be rewritten in terms of the \( c \)-function (B.9) as

\[
\int_{\mathbb{R}} f(x) \, dm(x; a, b \mid q) = \frac{1}{2\pi} \int_0^\pi f(\cos \psi) \, d\psi
\]

\[
+ \sum_{s \in D} f(\mu(s)) \text{Res}_{w=s} \frac{1}{wc(w; a, b \mid q)c(w^{-1}; a, b \mid q)},
\]

where the set \( D \) is given by

\[
D = D(a, b \mid q) = \{ s \in \mathbb{C} \mid |s| > 1, c(s; a, b \mid q) = 0 \},
\]

and we assume that the zeroes of the \( c \)-function in \( D \) are simple. The two sets of discrete mass points in the measure in (B.12) are finite. If \( ab > 0 \), at most one of the sets of discrete mass points can occur, since we also assume \( ab < 1 \). If \( ab < 0 \), then both series of discrete mass points can occur.

Consider the corresponding Jacobi operator on \( \ell^2(\mathbb{N}_0) \) equipped with the standard orthonormal basis \( \{e_n\}_{n=0}^{\infty} \),

\[
2Je_n = c_n e_{n+1} + d_n e_n + c_{n-1} e_{n-1},
\]

(B.15)
with $c_n$ and $d_n$ as in (B.14), initially defined on the dense domain of finite linear combinations of the basis vectors. Since the coefficients are bounded, $J$ extends uniquely to a bounded self-adjoint operator on $\ell^2(\mathbb{N}_0)$. If we need to stress the dependence on the parameters, we write $J = J(a, b \mid q)$. The resolution of the identity for the self-adjoint extension of $J$ can be described with the orthonormal Al-Salam–Chihara polynomials and the corresponding orthogonality measure.

**Theorem B.13.** The Jacobi operator $J$ extends uniquely to a bounded self-adjoint operator on $\ell^2(\mathbb{N}_0)$. Let $E_J$ be the resolution of the identity for the self-adjoint extension of $J$, then for any Borel set $\mathcal{B} \subset \mathbb{R}$ and $u = \sum_{n=0}^{\infty} u_n e_n, v = \sum_{n=0}^{\infty} v_n e_n \in \ell^2(\mathbb{N}_0)$ we have

$$
\langle E_J(\mathcal{B})u, v \rangle_{\ell^2(\mathbb{N}_0)} = \int_{\mathcal{B}} F_J u(x) \overline{F_J v(x)} \, dm(x; a, b \mid q), \\
F_J u(x) = \sum_{n=0}^{\infty} u_n p_n(x; a, b \mid q).
$$

(B.16)

For the purposes in this paper we want to rewrite the orthogonality relations (B.13) for the Al-Salam–Chihara polynomials as orthogonality relations on $L^2(I(a, b \mid q))$, where $I(a, b \mid q)$ is the support of $dm(\cdot; a, b \mid q)$, so

$$
I(a, b \mid q) = [-1, 1] \cup \mu(D(a, b \mid q)), \\
D(a, b \mid q) = \{aq^r \mid r \in \mathbb{N}_0, |aq^r| > 1\} \cup \{bq^r \mid r \in \mathbb{N}_0, |bq^r| > 1\},
$$

(B.17)
in accordance with (B.14). On $[-1, 1]$ we take the Lebesgue measure, and on the discrete part we take the counting measure. Now define for $0 < |\psi| < \pi$

$$
h_n(\cos \psi; a, b \mid q) = \sqrt{\frac{1}{2\pi |\sin \psi|}} p_n(\cos \psi; a, b \mid q),
$$

$$
h_n(\mu(eq^r); a, b \mid q) = \sqrt{w_r(e; f \mid q)} p_n(\mu(eq^r); a, b \mid q),
$$

where $e$ is either $a$ or $b$, and $f$ is the other parameter, and $|eq^r| > 1$ with $r \in \mathbb{N}_0$. So $\{h_n(\cdot; a, b \mid q)\}_{n=0}^{\infty}$ is an orthonormal basis for $L^2(I(a, b \mid q))$. It follows in particular that

$$
\sum_{n=0}^{\infty} h_n(\mu(x); a, b \mid q) h_n(\mu(y); a, b \mid q) = \delta_{x,y}, \quad x, y \in D(a, b \mid q),
$$

(B.19)

so that the functions $h_n(\mu(x); a, b \mid q), n \in \mathbb{N}_0$, have $\ell^2$-norm 1 for $x \in D(a, b \mid q)$. The orthogonality relations (B.19) can also be proved directly using the $q$-binomial theorem and the $q$-Saalschütz formula [17, (II.3),(II.12)], and it is related to a discrete measure on $q^{-\mathbb{N}_0}$ for which only a finite number of moments exist.

The polynomials $p_n(\cdot; a, b \mid q)$ and the $c$-function are symmetric in $a, b$, which implies the symmetry relation

$$
h_n(\cdot; a, b \mid q) = h_n(\cdot; b, a \mid q).
$$

(B.20)

Another symmetry that we need is

$$
h_n(\cdot; a, b \mid q) = (-1)^n h_n(-\cdot; -a, -b; q),
$$

(B.21)

which follows from writing out explicitly $h_n$ as a multiple of a $2\varphi_1$-function.
The asymptotic behaviour of the orthonormal basis of $L^2(I(a, b \mid q))$ as the degree $n$ tends to $\infty$ can be obtained from (B.8). For $0 < |\psi| < \pi$ we find

$$h_n(\cos \psi; a, b \mid q) = \frac{2}{\pi |\sin \psi|} \Re \left( e^{i\psi} c(e^{i\psi}; a, b \mid q) \right) (1 + O(q^n)), \quad n \to \infty, \quad \text{(B.22)}$$

and see [3, §3.1] for the case $x = \pm 1$. Observe that the expression is symmetric with respect to $\psi \leftrightarrow -\psi$. On the discrete side the zeroes of the $c$-function make the first term on the right hand side of (B.8) vanish, so that the behaviour of $h_n$ is given by

$$h_n(\mu(aq^r); a, b \mid q) = (aq^r)^{-n} \sqrt{w_r(a; b \mid q) c(1/qa^r; a, b \mid q) \left(1 + O(q^n)\right)}, \quad n \to \infty. \quad \text{(B.23)}$$

This implies $h(x; a, b \mid q) \in l^2(\mathbb{N}_0)$ for $x$ in the discrete spectrum. The expression for $h_n(\mu(bq^k); a, b \mid q)$ follows from (B.23) by interchanging $a$ and $b$ in the right hand side. We can also reformulate (B.23) as

$$h_n(\mu(s); a, b \mid q) = s^{-n} \sqrt{\operatorname{Res}_{w=s} \frac{c(w^{-1}; a, b \mid q)}{w c(w; a, b \mid q)}} (1 + O(q^n)), \quad n \to \infty, \quad \text{(B.24)}$$

for $s \in D(a, b \mid q)$, assuming such zeroes of the $c$-function are simple.

In this paper we need a certain contiguous relations for the Al-Salam–Chihara polynomials. The contiguous relation can be looked upon as an operator that can be used for a Darboux factorization of the Jacobi operator $J$.

**Lemma B.14.** The orthonormal basis functions $h_n(x; a, b \mid q)$ satisfy

$$\sqrt{1 - 2bx + b^2} h_n(x; a, b \mid q) = \sqrt{1 - abq^n} h_n(x; a, bq \mid q) - b \sqrt{1 - q^n} h_{n-1}(x; a, bq \mid q),$$

for $x \in I(a, b \mid q)$.

**Proof.** From the connection coefficient formula [4, §6], [17, §7.6] it follows that

$$P_n(x; a, b \mid q) = P_n(x; a, bq \mid q) - b(1 - q^n) P_{n-1}(x; a, bq \mid q). \quad \text{(B.25)}$$

This can also be obtained directly from the second explicit expression of $P_n$ in (B.20) by writing out the $2\varphi_1$-function as a sum, and using the identity $(by; q)_k = (bqy; q)_k - by(1 - q^k)(bqy; q)_k$. Rewriting (B.23) for the orthonormal basis $h_n(x; a, b \mid q)_k$, $x \in I(a, b \mid q)$, gives the desired relation. For $x = \cos \psi$ this follows directly from (B.10), (B.18), and for $x$ in the discrete spectrum this is a consequence of

$$\frac{w_r(a; bq \mid q)}{w_r(a; b \mid q)} = \frac{(1 - abq^r)(1 - q^{-r}b/a)}{1 - ab}, \quad \frac{w_{r-1}(bq; a \mid q)}{w_r(b; a \mid q)} = \frac{(1 - b^2q^r)(1 - q^{-r})}{1 - ab}.$$

Here we use the convention that $h_n(x; a, b \mid q) = 0$ for $x \notin I(a, b \mid q)$. \hfill \Box

**B.5. Little $q$-Jacobi functions.** In this subsection we collect the results and notations for the little $q$-Jacobi functions needed in this paper. The little $q$-Jacobi functions are the kernel of an explicit transform pair that is related to the spectral analysis of the hypergeometric $q$-difference equation, and they arise as matrix elements for the quantum $SU(1, 1)$ group, see [15]. References for this subsection are Kakehi [23], Kakehi et al. [24], and also [32, App. A], [29], [32].
The hypergeometric \( q \)-difference equation, see [17, Exerc. 1.13], can be rewritten as
\[
(c - abz) u(qz) + ((a + b)z - c - q) u(z) + (q - z) u(z/q) = 0
\] (B.26)
for a function \( u(z) \) and one explicit solution of (B.26) is \( u(z) = 2 \varphi_1(a, b; c, q, z) \).

Using the hypergeometric \( q \)-difference we find solutions to
\[
2x f_k(x) = (1 - q^{1+k}/z) f_{k+1}(x) + q^k c + q/dz f_k(x) + (1 - cq^k/d^2z) f_{k-1}(x),
\] (B.27)
where we assume from now on that \( z < 0, c > 0, \) and \( d \in \mathbb{R} \setminus \{0\} \). For more general sets of parameters, see [33, App. A]. Indeed, we find the solution,
\[
f_k(\mu(y)) = (c, z, q/z; q)\infty d^{-k} 2\varphi_1 \left( \frac{dy, d/y}{c}, q, zq^{-k} \right),
\] (B.28)
where we from now on assume \( 0 < c < 1 \) in order to avoid complications for \( c \in q^{-\mathbb{N}_0} \).

We use the notation \( f_k(x) = f_k(x; c, d; z \mid q) \) if we want to stress the dependence on the parameters. Note that replacing \( c \) and \( d \) by \( q^2/c \) and \( qd/c \) leaves (B.27) invariant, hence \( f_k(x; q^2/c, qd/c; z \mid q) \) is also a solution to (B.27), as can also be checked directly from (B.26). These solutions are linearly independent for \( c \neq q \).

The equation (B.27) can also be viewed for \( k \geq 0 \) as the recurrence relation for the (suitably renormalized) associated Al-Salam–Chihara polynomials, and the description of the solution space matches Gupta, Ismail and Masson [20].

Next we define
\[
F_k(y) = y^k 2\varphi_1 \left( \frac{dy, qdy/c}{qy^2}, c, q^{1+k}c/d^2z, \right), \quad y^2 \notin q^{-\mathbb{N}},
\] (B.29)
then, for \( y \neq \pm 1, F_k(y) \) and \( F_k(y^{-1}) \) define two linearly independent solutions to (B.28) as follows easily from (B.26). We use the notation \( F_k(y^{\pm 1}) = F_k(y^{\mp 1}; c, d; z \mid q) \) if we want to stress the dependence on the parameters. Note that \( F_k(y^{\pm 1}) \) are invariant under replacing \( c \) and \( d \) by \( q^2/c \) and \( qd/c \). Since the solution space to (B.27) is two-dimensional there are relations between the solutions; in particular,
\[
f_k(\mu(y)) = c(y) F_k(y) + c(y^{-1}) F_k(y^{-1}), \quad c(y) = \frac{(c/dy, d/y, dz, y, q/dyz; q)\infty}{(y^{-2}; q)\infty}
\] (B.30)
which follows from [17, (4.3.2)] for \( y^2 \notin q^{-\mathbb{N}} \). As in the previous subsection we extend this \( c \)-function by continuity to all points of \( \mathbb{C} \) where possible. We use the notation \( c(y; c, d; z \mid q) \) if we want to stress the dependence on the parameters. Note that this \( c \)-function is different from the one for the Al-Salam–Chihara polynomials in Section B.4. In this subsection \( c(y) \) is defined by (B.30).

The corresponding orthonormal recurrence relation, i.e., the normalization which makes the corresponding Jacobi operator symmetric, is
\[
2x u_k(x) = a_k u_{k+1}(x) + b_k u_k(x) + a_{k-1} u_{k-1}(x),
\]
\[
a_k = \sqrt{\left( 1 - \frac{q^{k+1}}{z} \right) \left( 1 - \frac{cq^{k+1}}{d^2z} \right)}, \quad b_k = \frac{q^k(c + q)}{dz},
\] (B.31)
Note that we assume \( z < 0, 0 < c < 1, d \in \mathbb{R} \setminus \{0\} \), so that the square root is well-defined. We put

\[
\rho_k^2 = \frac{(cq^{1+k}/d^2z; q)_\infty}{(q^{1+k}/z; q)_\infty} = \frac{\left(\frac{c}{d^2}\right)^{1-k}\left(zq^{-k}, d^2z/c, cq/d^2z; q\right)_\infty}{(d^2q^{-k}/c, z, q/z; q)_\infty},
\]

where the second expression follows from the \( \theta \)-product identity (B.31), then \( u_k(z) = \rho_k f_k(z) \) satisfies (B.31) if and only if \( f_k(z) \) satisfies (B.27). We use the notation \( \rho_k(c, d; z \mid q) \) if we want to stress the dependence on the parameters. Now the following orthogonality relations hold;

\[
\int_{\mathbb{R}} \rho_k f_k(x) \rho_l f_l(x) \, dv(x; c, d; z \mid q) = \delta_{k,l}, \tag{B.33}
\]

where the measure \( dv \) is defined by

\[
\int_{\mathbb{R}} g(x) \, dv(x; c, d; z \mid q) = \frac{1}{2\pi} \int_{0}^{\pi} g(\cos \psi) \frac{d\psi}{|c(e^{i\psi})|^2} + \sum_{r \in \mathbb{Z}, |q^{1-r}/dz| > 1} g(\mu(q^{1-r}/dz))v_r + \sum_{r \in \mathbb{Z}, |q^{1-r}/dz| > 1} g(\mu(q^{1-r}/dz))w_r + \sum_{r \in \mathbb{Z}, |q^{1-r}/dz| > 1} g(\mu(q^{1-r}/dz))w'_r,
\]

with

\[
c(y) = c(y; c, d; z \mid q), \quad v_r = \frac{-(1 - q^{2-2r}/d^2z^2) (d^2z^{2(1-r)} q^{-2(r-2)(r-1)} - (q, c, d^2z/c, z, q^{1-r}/dz, d^2q^{r-1}; q)_\infty)}{(q, q, cq^{1-r}/d^2z, q^{1-r}/z, qzq^{-1}, q^{1-r}/dz, d^2q^{r-1}; q)_\infty}, \]

\[
w_r = \frac{(d^2/c^2; q)_\infty (1 - c^2q^{2r}/d^2) (c^2/d^2, c; q)_r}{(q, c, d^2/c, q/c, d^2z/c, qz/c; q)_\infty (1 - c^2/d^2) (q, cq/d^2; q)_r c^{-r}},
\]

\[
w'_r = \frac{(d-2; q)_\infty (1 - d^2q^{2r}/d^2) (d^2, c; q)_r}{(q, c, d^2/c, d^2z/c, qz/c; q)_\infty (1 - d^2) (q, qd^2/c; q)_r c^{-r}}.
\]

If we want to stress the dependence on the parameters we use the notation \( w_r(c, d; z \mid q) \), \( w'_r(c, d; z \mid q) \) and \( v_r(c, d; z \mid q) \) for the weights in (B.33). Note that at most one of the last two sets of discrete mass points can occur, since we assume \( 0 < c < 1 \). The first set of discrete mass points always occurs. The orthogonality measure (B.33) can be rewritten in terms of the c-function;

\[
\int_{\mathbb{R}} g(x) \, dv(x; c, d; z \mid q) = \frac{1}{2\pi} \int_{0}^{\pi} g(\cos \psi) \frac{d\psi}{|c(e^{i\psi})|^2} + \sum_{s \in D} g(\mu(s)) \text{Res}_{w=s} \frac{1}{w c(w)c(w^{-1})}, \tag{B.34}
\]

where we assume that the zeroes of the c-function are simple, and where the set \( D \) is defined by

\[
D = D(c, d; z \mid q) = \{s \in \mathbb{C} \mid |s| > 1, c(s) = 0\}.
\]

See Kakehi [23], and [33 App. A] for a bit more general situation, [29] for an introduction, and [32] for a general scheme of function transforms with basic hypergeometric kernel of which (B.33) is part.
Denote by $L$ the corresponding (doubly infinite) Jacobi operator on $\ell^2(\mathbb{Z})$ with orthonormal basis $\{e_k\}_{k \in \mathbb{Z}}$, i.e.,

$$2L e_k = a_k e_{k+1} + b_k e_k + a_{k-1} e_{k-1},$$

with $a_k$ and $b_k$ defined as in (B.31), and $L$ initially defined on the dense domain of finite linear combinations of the basis vectors. We write $L = L(c, d, z \mid q)$ if we need to stress the dependence on the parameters. The operator $L$ is unbounded, because the coefficients tend to $\pm\infty$ as $k \to -\infty$. Its adjoint is given by the same formula (B.35) with its maximal domain, i.e. $D^* = \{v = \sum_k v_k e_k \in \ell^2(\mathbb{Z}) \mid \sum_k (a_k v_{k+1} + b_k v_k + a_{k-1} v_{k-1}) e_k \in \ell^2(\mathbb{Z})\}$. From Section 4.5 of [29] we have the following result. Note that we need to switch from the basis $e_k$ to $e_{-k}$ of $\ell^2(\mathbb{Z})$ for the correspondence with [29].

**Theorem B.15.** The operator $L$ is essentially self-adjoint for $0 < c \leq q^2$. In this case the resolution of the identity $E_L$ for the unique self-adjoint extension of $L$ is given by

$$\langle E_L(B)u, v \rangle_{\ell^2(\mathbb{Z})} = \int_B \mathcal{F}_L u(x) \overline{\mathcal{F}_L v(x)} \, d\nu(x; c, d; z \mid q), \quad \mathcal{F}_L u(x) = \sum_{k=-\infty}^{\infty} u_k \rho_k f_k(x),$$

for any Borel set $B \subset \mathbb{R}$ and any $u = \sum_k u_k e_k, v = \sum_k v_k e_k \in \ell^2(\mathbb{Z})$.

In [29], Prop.4.5.3, it is also proved that $L$ has deficiency indices $(1, 1)$ in case $q^2 < c < 1$, $c \neq q$, hence $L$ has self-adjoint extensions. In the proof linear independence of certain functions $w f(z)$ and $w g(z)$ (see [29]) is used, which is no longer valid in case $c = q$. The special case $c = q$ is also needed in this paper, and we treat this case in Appendix C.

In this paper it is convenient to rewrite the orthogonality relations (B.33) as orthogonality relations on $L^2(I(c, d; z \mid q))$, where $I(c, d; z \mid q)$ is the support of $d\nu(\cdot; c, d; z \mid q)$. So

$$I(c, d; z \mid q) = [-1, 1] \cup \mu(D(c, d; z \mid q)),$$

$$D(c, d; z \mid q) = \left\{ dq^r \mid r \in \mathbb{N}_0, |aq^r| > 1 \right\} \cup \left\{ c \frac{dq^r}{d} \mid r \in \mathbb{N}_0, \left| \frac{c}{d} q^r \right| > 1 \right\}$$

(B.36)

in accordance with (B.34). On $[-1, 1]$ we take the Lebesgue measure, and on the discrete part we take the counting measure. We now define the function $j_k(x; c, d; z \mid q) \in L^2(I(c, d; z \mid q))$ by

$$j_k(\cos \psi; c, d; z \mid q) = \frac{\rho_k(c, d; z \mid q) f_k(\cos \psi; c, d; z \mid q)}{\sqrt{2\pi}|\sin \psi||c(e^{i\psi}; c, d; z \mid q)|}, \quad 0 < |\psi| < \pi,$$

$$j_k(\mu(q^{1-r}/dz); c, d; z \mid q) = \sqrt{v_k(c, d; z \mid q) \rho_k(c, d; z \mid q)} f_k(\mu(q^{1-r}/dz); c, d; z \mid q),$$

$$j_k(\mu(cq^r/d); c, d; z \mid q) = \sqrt{w_k(c, d; z \mid q) \rho_k(c, d; z \mid q)} f_k(\mu(cq^r/d); c, d; z \mid q),$$

$$j_k(\mu(dq^r); c, d; z \mid q) = \sqrt{w_k(c, d; z \mid q) \rho_k(c, d; z \mid q)} f_k(\mu(dq^r); c, d; z \mid q),$$

so that $\{j_k(\cdot; c, d; z \mid q)\}_{k \in \mathbb{Z}}$ yields an orthonormal basis for $L^2(I(c, d; z \mid q))$. We use the convention that $j_k(x; c, d; z \mid q) = 0$ for $x \notin I(c, d; z \mid q)$. In particular this implies that

$$\sum_{k \in \mathbb{Z}} j_k(\mu(x); c, d; z \mid q) j_k(\mu(y); c, d; z \mid q) = \delta_{x,y}, \quad x, y \in D(c, d; z \mid q),$$

(B.38)
so that \( \{j_k(\mu(x); c, d; z \mid q)\}_{k \in \mathbb{Z}} \) has \( \ell^2 \)-norm 1 for \( x \in D(c, d; z \mid q) \).

The asymptotic behaviour of \( j_k(x; c, d; z \mid q) \) as \( k \to -\infty \) follows from

\[
\rho_k f_k(x) = (\text{sgn}(d)\sqrt{c})^k (c, d^2 z/c, cq/d^2 z; q)_{\infty}(1 + \mathcal{O}(q^{-k})), \quad x \in \mathbb{C},
\]

which is an immediate consequence of \((\text{B.28})\) and \((\text{B.32})\). For the asymptotic behaviour as \( k \to \infty \) we use \((\text{B.30})\), \((\text{B.29})\), \((\text{B.32})\), and proceed analogously as in the derivation of \((\text{B.22})\). This gives

\[
j_k(\cos \psi; c, d; z \mid q) = \sqrt{\frac{2}{\pi |\sin \psi|}} \frac{\Re(e^{i\psi}; c, d; z \mid q)e^{ik\psi}}{|c(e^{i\psi}; c, d; z \mid q)|}(1 + \mathcal{O}(q^k)), \quad k \to \infty,
\]

for \( 0 < |\psi| < \pi \). Note that the expression is symmetric with respect to \( \psi \mapsto -\psi \). The asymptotic behaviour in the discrete mass points as \( k \to \infty \) follows similarly as \((\text{B.23})\). The behaviour is \( \ell^2 \), and for \( k \to \infty \) we have

\[
\begin{align*}
    j_k(\mu(q^{1-r}/dz); c, d; z \mid q) &= (q^{1-r}/dz)^{-k} \sqrt{u_r(c, d; z \mid q)} c(q^{r-1}dz; c, d; z \mid q) (1 + \mathcal{O}(q^k)), \\
    j_k(\mu(cq^r/dz); c, d; z \mid q) &= (cq^r/dz)^{-k} \sqrt{w_r(c, d; z \mid q)} c(q^{-r}/c; d, z \mid q) (1 + \mathcal{O}(q^k)), \\
    j_k(\mu(dq^r); c, d; z \mid q) &= (dq^r)^{-k} \sqrt{w_r'(c, d; z \mid q)} c(q^{-r}/d; c, d; z \mid q) (1 + \mathcal{O}(q^k)).
\end{align*}
\]

We can rewrite \((\text{B.41})\), cf. \((\text{B.24})\),

\[
\begin{align*}
    j_k(\mu(s); c, d; z \mid q) &= s^{-k} \sqrt{\frac{\Re_{w=s}}{\text{Res}} \frac{c(w^{-1}; c, d; z \mid q)}{w c(w; c, d; z \mid q)}} (1 + \mathcal{O}(q^k)), \quad k \to \infty,
\end{align*}
\]

for \( s \in D(c, d; z \mid q) \) assuming the zeroes of the \( c \)-function are simple.

We will need a contiguous relation for the normalized little \( q \)-Jacobi functions, which can be obtained from the \( q \)-derivative of the \( 2\varphi_1 \)-series.

**Lemma B.16.** The orthonormal basis functions \( j_k(x; c, d; z \mid q) \) satisfy

\[
\sqrt{1 - 2x/d + d^2} j_k(x; qc, qd; z \mid q) = \frac{1}{d} \sqrt{1 - \frac{q^k}{z} j_{k-1}(x; c, d; z \mid q)} - \sqrt{1 - \frac{cq^k}{d^2} z j_k(x; c, d; z \mid q)},
\]

for \( x \in I(c, d; z \mid q) \).

**Proof.** A direct calculation, or see \([17]\), Exerc. 1.12, shows that

\[
f_k(x; c, d; z \mid q) - \frac{1}{d} f_{k+1}(x; c, d; z \mid q) = z(1 - 2dx + d^2) f_k(x; qc, qd; z \mid q).
\]

Rewriting \((\text{B.43})\) for the orthonormal basis \( j_k(x; c, d; z \mid q) \) then gives the desired contiguous relation. For \( x = \cos \psi \) this is immediate from \((\text{B.37})\), \((\text{B.32})\) and \((\text{B.33})\). For \( x \) in the discrete
Combining gives the following special case

\[ \frac{w_r(qc, qd; z \mid q)}{w_r(c, d; z \mid q)} = d^2 z^2 (1 - c q^r)(1 - d^2 q^{-r} / c), \]

\[ \frac{w'_{r-1}(qc, qd; z \mid q)}{w'_r(c, d; z \mid q)} = d^2 z^2 (1 - d^2 q^r)(1 - q^{-r}), \]

\[ \frac{v_{r-1}(qc, qd; z \mid q)}{v_r(c, d; z \mid q)} = d^2 z^2 (1 - d^2 z q^{-r-1})(1 - q^{-1+r} / z). \]

\[ \square \]

Yet another result for the little \( q \)-Jacobi functions needed in this paper is related to a symmetry property that follows from Heine's transformation [17, (1.4.6)] and analytic continuation:

\[ z \varphi_1 \left( \frac{dy, d/y}{c} ; q, z q^{-k} \right) = \left( zd^2 q^{-k} / c ; q \right)_\infty \varphi_1 \left( cy/d, c/dy ; q, q^{-k} zd^2 \frac{c}{c} \right). \]  \hspace{1cm} (B.44)

Together with (B.28) and (B.32) this implies the symmetry

\[ \rho_k(c, d; z \mid q) f_k(x; c, d; z \mid q) = \rho_k(c, \frac{c}{d}; \frac{zd^2}{c} \mid q) f_k(x; c, d; zd^2 / c \mid q). \]  \hspace{1cm} (B.45)

The action on the parameters is an involution, and \( I(x; c, d; z \mid q) = I(c, c/d; zd^2 / c \mid q). \) Moreover, we have

\[ c(y; c, d; z \mid q) = c(y; c, \frac{c}{d}; \frac{zd^2}{c} \mid q), \]

\[ v_k(c, d; z \mid q) = v_k(c, \frac{c}{d}; \frac{zd^2}{c} \mid q), \]

\[ w_k(c, d; z \mid q) = w'_k(c, \frac{c}{d}; \frac{zd^2}{c} \mid q), \]

which implies

\[ j_k(x; c, d; z \mid q) = j_k(x; c, c/d; zd^2 / c \mid q). \]  \hspace{1cm} (B.46)

This shows that in the special case \( d^2/c \in \mathbb{Z}^2 \), we can transfer the multiplication by a power of \( q \) in \( z \) to a shift in the index \( k \). Using (B.41) we obtain for \( p \in \mathbb{Z} \)

\[ \rho_k(c, d; z q^{-p} \mid q) f_k(x; c, d; z q^{-p} \mid q) = (-dz)^p q^{-\frac{1}{2} p(p+1)} \rho_{k+p}(c, d; z \mid q) f_{k+p}(x; c, d; z \mid q), \]

\[ c(y; c, d; z q^{-p} \mid q) = (-dzy)^p q^{-\frac{1}{2} p(p+1)} c(y; c, d; z \mid q), \]

\[ v_r(c, d; z q^{-p} \mid q) = (dz)^{-2p} q^{p(p+1)} v_{r-p}(c, d; z \mid q), \]

\[ w_r(c, d; z q^{-p} \mid q) = (dz)^{-2p} q^{p(p+1)} w_r(c, d; z \mid q), \]

\[ w'_r(c, d; z q^{-p} \mid q) = (dz)^{-2p} q^{p(p+1)} w'_r(c, d; z \mid q). \]

Moreover, \( I(c, d; z q^{-p} \mid q) = I(c, d; z \mid q) \) and so

\[ j_k(x; c, d; z q^{-p} \mid q) = (\text{sgn}(d))^p j_{k+p}(x; c, d; z \mid q). \]  \hspace{1cm} (B.47)

Combining gives the following special case

\[ j_k(x; q, q^{\frac{k}{2}(1-p)}; z \mid q) = j_{k+p}(x; q, q^{\frac{k}{2}(1+p)}; z \mid q), \]  \hspace{1cm} (B.48)
B.6. Explicit formulas for the function $A$. Here we write out explicitly the functions $A = A(\cdot; p, m, \epsilon, \eta)$, $p \in q\mathbb{Z}$, $m \in \mathbb{Z}$ and $\epsilon, \eta \in \{-, +\}$. These functions are used in §9.2 for the description of the polar decomposition of the elements $Q(p_1, p_2, n) \in \hat{M}$, and they are used later on in §10.1 and §10.2 to describe explicitly the actions of the generators of $\hat{M}$ on $L_{p,x}$ in the discrete series and principal series corepresentations. The functions $A$ are essentially special cases of the $c$-functions for Al-Salam–Chihara polynomials and little $q$-Jacobi functions, divided by their absolute value. We only give the formulas for $A(\lambda)$ with $\lambda = e^{i\psi} \in \mathbb{T}_0$.

For $\epsilon = +$, $\eta = -$,

$$A(\lambda; p, m, +, -) = (-1)^m \lambda^{1-m-\chi(p)} \sqrt{\frac{2}{\pi |\sin \psi|}} \frac{(q\lambda/p, -q^{-1-2m}\lambda/p; q^2)_\infty}{(\lambda^2; q^2)_\infty} \left(\frac{(\lambda^{\pm 2}; q^2)_\infty}{(q\lambda^{\pm 1}/p, -q^{-1-2m}\lambda^{\pm 1}/p; q^2)_\infty}\right)^{\frac{1}{2}},$$

and for $\epsilon = -$, $\eta = +$,

$$A(\lambda; p, m, -, +) = \lambda \sqrt{\frac{2}{\pi |\sin \psi|}} \frac{(pq\lambda, -pq^{1+2m}\lambda; q^2)_\infty}{(\lambda^2; q^2)_\infty} \left(\frac{(\lambda^{\pm 2}; q^2)_\infty}{(pq\lambda^{\pm 1}, -pq^{1+2m}\lambda^{\pm 1}; q^2)_\infty}\right)^{\frac{1}{2}}.$$

For $\epsilon = \eta = -$,

$$A(\lambda; p, m, -, -) = (-1)^{m+1} \lambda^{1-m-\chi(p)} \sqrt{\frac{2}{\pi |\sin \psi|}} \frac{(-pq\lambda, -pq^{1+2m}\lambda; q^2)_\infty}{(\lambda^2; q^2)_\infty} \left(\frac{(\lambda^{\pm 2}; q^2)_\infty}{(-pq\lambda^{\pm 1}, -pq^{1+2m}\lambda^{\pm 1}; q^2)_\infty}\right)^{\frac{1}{2}},$$

for $\chi(p) + m \geq 0$, and for $\chi(p) + m < 0$,

$$A(\lambda; p, m, -, -) = (-1)^{m+1} \lambda^{1-m-\chi(p)} \sqrt{\frac{2}{\pi |\sin \psi|}} \frac{(-q\lambda/p, -q^{-1-2m}\lambda/p; q^2)_\infty}{(\lambda^2; q^2)_\infty} \left(\frac{(\lambda^{\pm 2}; q^2)_\infty}{(-q\lambda^{\pm 1}/p, -q^{-1-2m}\lambda^{\pm 1}/p; q^2)_\infty}\right)^{\frac{1}{2}}.$$

For $\epsilon = \eta = +$,

$$A(\lambda; p, m, +, +) = (-1)^{m+\chi(p)} \lambda^{m-\chi(p)} \frac{(-q\lambda/p, -pq^{1+2m}\lambda/pq^{2+2m}/\lambda, q^{-1-2m}\lambda/p; q^2)_\infty}{(\lambda^2; q^2)_\infty} \times \sqrt{\frac{2}{\pi |\sin \psi|}} \left(\frac{(\lambda^{\pm 2}; q^2)_\infty}{(-q\lambda^{\pm 1}/p, -pq^{1+2m}\lambda^{\pm 1}, pq^{2+2m}\lambda^{\pm 1}, q^{-1-2m}\lambda^{\pm 1}/p; q^2)_\infty}\right)^{\frac{1}{2}},$$
for \( m \geq 0 \), and for \( m < 0 \),

\[
A(\lambda; p, m, +, +) = (-1)^m \sqrt{\frac{2}{\pi |\sin \psi|}} \frac{(-pq\lambda, -q^{1-2m}\lambda/p, q^{3-2m}/\lambda p, pq^{-1+2m}\lambda; q^2)_{\infty}}{\left(\sqrt{\frac{\lambda^2}{q^2}}\right)_{\infty}} \times \left(\frac{\lambda^{\pm 2}; q^2)_{\infty}}{(-pq\lambda^{\pm 1}, -q^{1-2m}\lambda^{\pm 1}/p, q^{3-2m}\lambda^{\pm 1}/p, pq^{-1+2m}\lambda^{\pm 1}; q^2)_{\infty}}\right)^{\frac{1}{2}}.
\]

**Appendix C. Special case of a Jacobi operator**

In this section we study the special case \( c = q \) of the Jacobi operator \( L = L_c = L(c, d, z \mid q) \) defined by (B.35). For special choices of \( c, d \) and \( z \), the operator \( L \) is a certain restriction of \( E_0^\dagger E_0 \) or the Casimir operator (see Section 8.3). The operator \( L(q, d, z \mid q) \) that we consider in this subsection corresponds to the case \( \varepsilon = \eta = +, m = 0 \).

Let \( \mathcal{F}(\mathbb{Z}) \) be the space of complex-valued functions on \( \mathbb{Z} \). We study the linear operator \( \tilde{L}_c: \mathcal{F}(\mathbb{Z}) \to \mathcal{F}(\mathbb{Z}) \), given by

\[
2 (\tilde{L}_c u)_k = a_{k-1}(c) u_{k-1} + b_k(c) u_k + a_k(c) u_{k+1}
\]

for all \( u \in \mathcal{F}(\mathbb{Z}) \) and \( k \in \mathbb{Z} \). The coefficients \( a_k(c) \) and \( b_k(c) \) are given (B.31), and we write \( a_k(c), b_k(c) \) instead of \( a_k, b_k \) to stress the dependence on the parameter \( c \). Recall from Section B.3 that \( d \in \mathbb{R} \setminus \{0\} \) and \( z \in (-\infty, 0) \), so that both terms in the square root are positive, and \( a_k, b_k \in \mathbb{R} \). We define the linear operator \( L: K(\mathbb{Z}) \to K(\mathbb{Z}) \) as the restriction of \( \tilde{L}_c \) to \( K(\mathbb{Z}) \), the linear subspace of finite linear combinations of basis vectors, i.e., the subspace of compactly supported functions in \( \mathcal{F}(\mathbb{Z}) \). Then \( (L_c, K(\mathbb{Z})) \) is an unbounded symmetric operator on the Hilbert space \( \ell^2(\mathbb{Z}) \). Moreover, the unboundedness occurs as \( k \to -\infty \), since in this case the coefficients \( a_k(c) \) and \( b_k(c) \) grow exponentially. Note that for \( k \to \infty \) the coefficients \( a_k(c) \) and \( b_k(c) \) remain bounded.

In this subsection we need the Wronskian associated to the Jacobi operator \( L \);

\[
[u, v]_k = a_k(u_{k+1}v_k - u_kv_{k+1}), \tag{C.1}
\]

see (29, (4.2.3)). Two eigenfunctions \( u, v \) of \( L \) are linearly independent if and only if \( [u, v] \neq 0 \).

The remainder of this subsection furnishes the proof the following result.

**Theorem C.1.** Consider \( u \in \ell^2(\mathbb{Z}) \) so that \( \tilde{L}_q(u) \in \ell^2(\mathbb{Z}) \) and so that there exists a function \( f: \mathbb{R}_{\geq 0} \to \mathbb{C} \) that is differentiable in \( 0 \) and satisfies \( f(0) \neq 0 \) and \( u_{-k} = q^{\frac{k}{2}} f(q^k) \) for all \( k \in \mathbb{N} \). Then there exists a unique self-adjoint extension \( T \) of \( L_q \) so that \( u \in D(T) \). Moreover, if \( v \in \ell^2(\mathbb{Z}) \), \( \tilde{L}_q(v) \in \ell^2(\mathbb{Z}) \) and if there exists a function \( g: \mathbb{R}_{\geq 0} \to \mathbb{C} \) that is differentiable in \( 0 \) and satisfies \( v_{-k} = q^{\frac{k}{2}} g(q^k) \) for all \( k \in \mathbb{N} \), then \( v \in D(T) \) as well.

The resolution of the identity \( E_T \) for the self-adjoint extension \( T \) of \( L_q \) is given by

\[
\langle E_T(B)u, v \rangle_{\ell^2(\mathbb{Z})} = \int_B \mathcal{F}_Tu(x)\mathcal{F}_Tv(x)\,dv(x; q, d; z \mid q), \quad \mathcal{F}_Tu(x) = \sum_{k=-\infty}^{\infty} u_k\rho_kf_k(x),
\]

for any Borel set \( B \subset \mathbb{R} \) and any \( u = \sum_k u_ke_k, v = \sum_k v_ke_k \in \ell^2(\mathbb{Z}) \).
Observe that the resolution of the identity is the same as in Theorem \[3.13\] with \(c = q\).

For the proof we need the eigenfunctions of the operator \(L_c\). For \(c \in (0, 1)\) and \(y \in \mathbb{C} \setminus \{0\}\), let us denote

\[
\tilde{f}(c, y)_k = \rho_k(c, d; z \mid q)f_k(\mu(y); c, d; z \mid q),
\]

\[
\tilde{g}(c, y)_k = \rho_k(c, d; z \mid q)f_k(\mu(y); q^2/c, qd/c; z \mid q),
\]

where \(f_k\) and \(\rho_k\) are defined by (B.28) and (B.32), respectively. From Section B.3 we know that \(\tilde{f}\) and \(\tilde{g}\) are both solutions of the eigenvalue equation \(L_cu = \mu(y)u\). Another solution is the function

\[
\tilde{F}(c, y)_k = \rho_k(c, d; z \mid q)F_k(y; c, d; z \mid q),
\]

see (B.29) for the definition of \(F_k\).

In [29, Section 4.5] it is shown that the operator \(L_c\) has deficiency indices \((1, 1)\) in case \(q^2 < c < 1, q \neq c\). The proof of this fact relies on the fact that the functions \(\tilde{f}(c, y)\) and \(\tilde{g}(c, y)\) are both in the space \(\{u \mid L^*u = zu, \sum_{k=1}^\infty |u_k|^2 < \infty\}\) for \(z = \mu(y) \in \mathbb{C} \setminus \mathbb{R}\). In case \(c = q\), we have \(\tilde{f}(q, y) = \tilde{g}(q, y)\), so we must provide another eigenvector for \(\tilde{L}_q\).

**Definition C.2.** Let \(y \in \mathbb{C} \setminus \{0\}\). We define \(\tilde{h}(y) \in \mathcal{F}(\mathbb{Z})\)

\[
\tilde{h}(y)_k = \lim_{c \to q} \frac{\tilde{f}(c, y)_k - \tilde{g}(c, y)_k}{c - q}, \quad \text{for all } k \in \mathbb{Z}.
\]

For \(c \in (q^2, 1)\), we have

\[
\tilde{L}_c \left( \frac{\tilde{f}(c, y) - \tilde{g}(c, y)}{c - q} \right) = \mu(y) \frac{\tilde{f}(c, y) - \tilde{g}(c, y)}{c - q}.
\]

Since the coefficients \(a_k(c), b_k(c)\) of \(\tilde{L}_c\) depend continuously on \(c\), and \(\mu(y)\) is independent of \(c\), the above equality together with Definition C.2 imply that \(\tilde{L}_q \tilde{h}(y) = \mu(y) \tilde{h}(y)\).

Let us establish the asymptotics of \(\tilde{h}(y)_k\) as \(k \to -\infty\).

**Lemma C.3.** Consider \(y \in \mathbb{C} \setminus \{0\}\). Then there exists a convergent sequence \((r_k)_{k=1}^\infty\) in \(\mathbb{C}\) and a differentiable function \(f : \mathbb{R}^+ \to \mathbb{C}\) such that \(f(0) \neq 0\) and \(\tilde{h}(y)_{-k} = q^k (r_k + k f(q^k))\) for all \(k \in \mathbb{N}\).

**Proof.** Define the \(C^\infty\)-functions \(B, C : (q^2, 1) \times [0, \infty) \to \mathbb{C}\) such that

\[
B(c, x) = \varphi_1 \left( \frac{dy}{c}, \frac{d/y}{c} ; q, xz \right) \quad \text{and} \quad C(c, x) = \varphi_1 \left( \frac{qdy/c, qd/y_c}{q^2/c} ; q, xz \right)
\]

for all \(c \in (q^2, 1), x \in \mathbb{R}^+\). We have for \(c \in (q^2, 1), k \in \mathbb{Z}\), that

\[
\tilde{f}(c, y)_{-k} - \tilde{g}(c, y)_{-k} = w_{-k}(c) (B(c, q^k) - (q/c)^k C(c, q^k)),
\]

where

\[
w_{-k}(c) = (c, z, q/z; q) \varphi_{-k} c, d, z \mid q).
\]

Therefore,

\[
\tilde{h}(y)_{-k} = w_{-k}(q) \left( (\partial_1 B)(q, q^k) - (\partial_1 C)(q, q^k) + (k/q) C(q, q^k) \right).
\]
Now define the $C^\infty$-function $D : (q^2, 1) \times [0, \infty) \to \mathbb{C}$ such that

$$D(c, x) = (c; q)^{\infty} \sqrt{(zx; q)^{\infty}(d^2 z/c, qc/d^2 z, z, q/z; q)^{\infty}} (d^2 zx/c; q)^{\infty}$$

for all $c \in (q^2, 1), x \in \mathbb{R}^+$. Now (B.32) shows that $w_{-k}(c) = c^{\frac{k}{2}} D(c, q^k)$ for all $c \in (q^2, 1)$. Thus,

$$\tilde{h}(y)_{-k} = q^{\frac{k}{2}} D(q, q^k) ((\partial_1 B)(q, q^k) - (\partial_1 C)(q, q^k)) + q^{-1} k q^{\frac{k}{2}} D(q, q^k) C(q, q^k).$$

Note that $q^{-1} D(q, 0) C(q, 0) = q^{-1} D(q, 0) = q^{-1} (c; q)^{\infty} \sqrt{(d^2 z/q, q^2/d^2 z, z, q/z; q)^{\infty}} > 0$. So the lemma follows.

**Lemma C.4.** Let $y \in \mathbb{C} \setminus \mathbb{R}, |y| < 1$. Then $\tilde{F}(q, y)$ belongs to $\ell^2(\mathbb{Z})$ and there exists a convergent sequence $(r_k)_{k=1}^{\infty}$ in $\mathbb{C}$ and a differentiable function $h : \mathbb{R}^+ \to \mathbb{C}$ so that $h(0) \neq 0$ and $\tilde{F}(q, y)_{-k} = q^{\frac{k}{2}} (r_k + k h(q^k))$ for all $k \in \mathbb{N}$.

**Proof.** Definition (B.28) and (B.32) imply that $\tilde{f}(q, y)_{-k}/q^k$ converges as $k \to \infty$. Since $\tilde{f}(q, y)_{-k}/kq^{\frac{k}{2}}$ converges to 0 as $k \to 0$ and, by Lemma C.3, $\tilde{h}(y)_{-k}/kq^{\frac{k}{2}}$ converges to a non-zero number as $k \to 0$, we conclude that $\tilde{f}(q, y)$ and $\tilde{h}(y)$ are linearly independent.

Because $\tilde{f}(q, y), \tilde{h}(y)$ and $\tilde{F}(q, y)$ belong to the eigenspace of $\tilde{L}_q$ for the eigenvalue $\mu(y)$, and since such an eigenspace is always two-dimensional, there exist complex numbers $\lambda$ and $\nu$ so that $\tilde{F}(q, y) = \lambda \tilde{f}(q, y) + \nu \tilde{h}(y)$. Clearly, this gives $[\tilde{f}(q, y), \tilde{F}(q, y)] = \nu [\tilde{f}(q, y), \tilde{h}(y)]$, see (C.1). By [29, last Eq. of (4.5.4)] we know that $[\tilde{f}(q, y), \tilde{F}(q, y)] \neq 0$, implying that $\nu \neq 0$. Hence, Lemma C.3 and the remarks in the beginning of this proof guarantee the existence of a convergent sequence $(r_k)_{k=1}^{\infty}$ in $\mathbb{C}$ and a differentiable function $h : \mathbb{R}^+ \to \mathbb{C}$ so that $h(0) \neq 0$ and $\tilde{F}(q, y)_{-k} = q^{\frac{k}{2}} (r_k + k h(q^k))$ for all $k \in \mathbb{N}$. So we immediately get that $\tilde{F}(q, y)_k$ is $\ell^2$ as $k \to -\infty$. Definition (B.29) and (B.32) imply that $\tilde{F}(q, y)_k$ is $\ell^2$ as $k \to \infty$, since $|y| < 1$. So we conclude that $\tilde{F}(q, y) \in \ell^2(\mathbb{Z})$.

Note that Lemma C.4 applies to $y = (1 - \sqrt{2}) i$, so $\mu(y) = i$. Since $\tilde{F}(q, y)$ belongs to $\ell^2(\mathbb{Z})$, the vector $\tilde{F}(q, y)$ belongs to $D(L_q^*$ and $L_q^*(\tilde{F}(q, y)) = i \tilde{F}(q, y)$. This implies that $L_q$ is not essentially self-adjoint.

**Lemma C.5.** Let $f, g : \mathbb{R} \to \mathbb{C}$ be functions that are differentiable in $0$, $(r_k)_{k=1}^{\infty}$ a sequence in $\mathbb{R}$ such that $(r_k q^k)_{k=1}^{\infty}$ converges to 0. Then $(r_k (f(q^{-k}) g(q^k) - f(q^k) g(q^{-k}))_{k=1}^{\infty}$ converges to 0.

**Proof.** For $k \in \mathbb{N}$, write

$$r_k (f(q^{-k}) g(q^k) - f(q^k) g(q^{-k})) = (r_k q^k) \left( \frac{f(q^{-k}) - f(q^k)}{q^k} g(q^k) + f(q^k) \frac{g(q^k) - g(q^{-k})}{q^k} \right)$$

and observe that

$$\left( \frac{f(q^{-k}) - f(q^k)}{q^k} \right)_{k=1}^{\infty} \quad \text{and} \quad \left( \frac{g(q^k) - g(q^{-k})}{q^k} \right)_{k=1}^{\infty}$$

are bounded because $f$ and $g$ are differentiable in 0. □
We are now ready to prove Theorem C.1.

Proof of Theorem C.1. We set \( y = (1 - \sqrt{2}) i \), then \( \mu(y) = i \). Consider \( \lambda \in \mathbb{T} \). We define a linear operator \( T_{\lambda} \) in \( l^2(\mathbb{Z}) \) such that

\[
D(T_{\lambda}) = \left\{ w \in l^2(\mathbb{Z}) \mid \tilde{L}_q(w) \in l^2(\mathbb{Z}) \text{ and } \lim_{k \to \infty} [w, \lambda \tilde{F}(q, y) + \tilde{\lambda} \tilde{F}(q, \bar{y})]_{-k} = 0 \right\}
\]

and \( T_{\lambda} \) is the restriction of \( \tilde{L}_q \) to \( D(T_{\lambda}) \). Here we use the Wronskian \([.,.]\) defined by (C.1). We know by [29, Lemma (4.2.3)] that \( T_{\lambda} \) is a self-adjoint extension of \( L_q \) and that every self-adjoint extension arises in this way.

By Lemma C.3 there exists a convergent sequence \( (r_k)_{k=1}^{\infty} \) in \( \mathbb{C} \), a differentiable function \( h: \mathbb{R}^+ \to \mathbb{C} \) such that \( h(0) \neq 0 \) and \( \tilde{F}(q, y)_{-k} = q^k (r_k + k h(q^k)) \) for all \( k \in \mathbb{N} \). Take \( v \in l^2(\mathbb{Z}) \) such that \( \tilde{L}_q(v) \in l^2(\mathbb{Z}) \) and such that there exists a function \( g: \mathbb{R}^+ \to \mathbb{C} \) that is differentiable in 0 and satisfies \( v_{-k} = q^{\frac{k}{2}} g(q^k) \) for all \( k \in \mathbb{N} \). Let us calculate \( \lim_{k \to \infty} [v, \tilde{F}(q, y)]_{-k} \).

For \( k \in \mathbb{N} \),

\[
q^{\frac{k}{2}+\frac{1}{2}} (v_{-k+1} \tilde{F}(q, y)_{-k} - v_{-k} \tilde{F}(q, y)_{-k+1}) = q^{k-1} g(q - k h(q^k)) - g(q) (r_{k-1} + (k - 1) h(q^{k-1}))
\]

\[
= (g(q^{k-1}) r_k - g(q^k) r_{k-1}) + k (g(q^{k-1}) h(q^k) - g(q^k) h(q^{k-1})) + g(q) h(q^{k-1})
\]

The first term converges to 0, since \( g \) is continuous and \( \{r_k\}_k \) is convergent. Since \( \{q^{k} r_k\}_{k=1}^{\infty} \) converges to 0, Lemma C.5 implies that the second term of the above sum converges to 0 as \( k \to \infty \). Therefore the above expression converges to \( g(h(0)) \) as \( k \to \infty \). Since \( a_{-k}(c) = q^{k} \frac{\sqrt{2}}{d|z|} (1 + O(q^k)) \), this implies that \( \lim_{k \to \infty} [v, \tilde{F}(q, y)]_{-k} = \frac{q}{d|z|} g(h(0)) \).

Since \( \tilde{F}(q, y)_k = \tilde{F}(q, \bar{y})_k \) for all \( k \in \mathbb{Z} \) by the assumptions \( z < 0 \) and \( d \in \mathbb{R} \setminus \{0\} \), we see that

\[
\lim_{k \to \infty} [v, \lambda \tilde{F}(q, y) + \tilde{\lambda} \tilde{F}(q, \bar{y})]_{-k} = \frac{q}{d|z|} g(h(0)) \lambda h(0) + \tilde{\lambda} h(0))
\]

\[
= 2 \frac{q}{d|z|} g(h(0)) \Re(\lambda h(0))
\]

If we use this equality for \( v = u \) and \( g = f \), we see that \( u \) belongs to the domain of \( T_{\lambda} \) if and only if \( \Re(\lambda h(0)) = 0 \). Notice that such a \( \lambda \) clearly exists and is determined up to a sign, but that \( T_{\lambda} = T_{-\lambda} \). So we have proved the existence and uniqueness of the self-adjoint extension \( T \). Equation (C.3) also guarantees that an element \( v \) satisfying the properties described in the lemma belongs to \( D(T) \).

The spectral decomposition of a self-adjoint extension \( T \) of the Jacobi operator \( L_{c^2} \), for \( 0 < c \leq q^2 \), is determined in [29, §4.5] from eigenfunctions \( \Phi_y \) and \( \phi_y \) for eigenvalue \( \mu(y) \), \( 0 < |y| < 1 \), such that \( \Phi(y) \in l^2(\mathbb{N}) \) and \( \phi(y) \in l^2(\mathbb{N}) \), see [29, §4.3.2]. Here \( \phi(y) \), extended to \( l^2(\mathbb{Z}) \) by setting \( \phi(y)_k = 0 \) for \( k \geq 0 \), must be an element of the domain of \( T \). In case \( 0 < c \leq q^2 \) we have \( \Phi(y) = \tilde{F}(c, y) \) and \( \phi(y) = \tilde{f}(c, y) \), and these functions determine the spectral decomposition of \( L_{c^2} \) from Theorem C.1. In order to find the spectral decomposition of \( T_{\lambda} \) we need to find the right choices of \( \Phi(y) \) and \( \phi(y) \) in this case. Note that there is only one eigenfunction of \( L_q \) for eigenvalue \( \mu(y) \) in \( l^2(\mathbb{N}) \), namely \( \tilde{F}(q, y) \), so \( \Phi(y) = \tilde{F}(q, y) \). There are two eigenfunctions in \( l^2(\mathbb{N}) \), namely \( \tilde{f}(q, y) \) and \( h(q, y) \), so \( \phi(y) \) is a linear combination.

Proof of Lemma 7.6.

D.1. Proof. Assume first that \( T \) it suffices to show that there exists a function \( g \) : \( \mathbb{R}_{\geq 0} \to \mathbb{C} \), differentiable in 0, such that \( f(q, y)_{-k} = q^{k/2} g(q^k) \) for all \( k \in \mathbb{N} \). But this follows directly from the definition of \( f(q, y) \), see (C.2), (B.28) and (B.32).

Appendix D. Proofs of some lemmas

D.1. Proof of Lemma 7.6. We prove the following result. Let \( p_1, p_2 \in I_q \) and \( n \in \mathbb{Z} \). Then

\[
\langle \hat{J} Q(p_1, p_2, n) \hat{J} v, E_0^t w \rangle = \langle \hat{J} Q(p_1, p_2, n) \hat{J} E_0 v, w \rangle, \quad \forall v, w \in \mathcal{K}_0.
\]

Proof. Assume first that \( v = f_{mpt} \) and \( w = f_{ir} \) for \( m, l \in \mathbb{Z} \) and \( p, t, r, s \in I_q \). Then (4.4), (7.13) and the last symmetry of (6.2) imply

\[
(q - q^{-1}) \langle \hat{J} Q(p_1, p_2, n) \hat{J} v, E_0^t w \rangle = \delta_{\chi(p_1 p_2 t), -m-n} \delta_{\chi(p_1 p_2 t) p_q} \langle p_1 p_2 / p | (-1)^m \text{sgn}(p) \chi(p) \text{sgn}(t) \rangle \chi(t)
\]

\[
\times \left[ \text{sgn}(s) q^{-\frac{m+1}{2}} |r/s|^\frac{1}{2} \sqrt{1 + \kappa(s)} |q s|^{-1} a_{p_1}(t, q s) a_{p_2}(p, r) \right.
\]

\[
\left. - \text{sgn}(r) q^{\frac{m+1}{2}} |s/r|^\frac{1}{2} \sqrt{1 + \kappa(q^{-1}r)} |s|^{-1} a_{p_1}(t, s) a_{p_2}(q^{-1}r) \right].
\]

Because of the presence of the three Kronecker deltas, we can replace \( |r/s| q^{-l-1} \) by \( |p/t| q^{-m} \).

This gives

\[
(q - q^{-1}) \langle \hat{J} Q(p_1, p_2, n) \hat{J} v, E_0^t w \rangle = \delta_{\chi(p_1 p_2 t), -m-n} \delta_{\chi(p_1 p_2 t) p_q} \langle p_1 p_2 / p | (-1)^m \text{sgn}(p) \chi(p) \text{sgn}(t) \rangle \chi(t)
\]

\[
\times \left[ \text{sgn}(s) q^{-\frac{m+1}{2}} |p/t|^\frac{1}{2} \sqrt{1 + \kappa(s)} |q s|^{-1} a_{p_1}(t, q s) a_{p_2}(p, r) \right.
\]

\[
\left. - \text{sgn}(r) q^{\frac{m+1}{2}} |t/p|^\frac{1}{2} \sqrt{1 + \kappa(q^{-1}r)} |s|^{-1} a_{p_1}(t, s) a_{p_2}(q^{-1}r) \right].
\]

For the other side of the required equation we similarly derive from (1.3), (7.13) and the last symmetry of (6.2) that

\[
(q - q^{-1}) \langle \hat{J} Q(p_1, p_2, n) \hat{J} E_0 v, w \rangle = \delta_{\chi(p_1 p_2 t), -m-n} \delta_{\chi(p_1 p_2 t) p_q} \langle p_1 p_2 / p | (-1)^m \text{sgn}(p) \chi(p) \text{sgn}(t) \rangle \chi(t)
\]

\[
\times \left[ -q^{-\frac{m+1}{2}} |p/t|^\frac{1}{2} \sqrt{1 + \kappa(q^{-1}t)} |s|^{-1} a_{p_1}(q^{-1}t, s) a_{p_2}(p, r) \right.
\]

\[
\left. + q^{\frac{m+1}{2}} |t/p|^\frac{1}{2} \sqrt{1 + \kappa(p)} |q s|^{-1} a_{p_1}(t, s) a_{p_2}(q p, r) \right].
\]

Comparing this expression with (D.1) we see that we need the \( q \)-contiguous relations of Lemma B.2. Using the first equality of Lemma B.2 for \( a_{p_1}(q^{-1}t, s) \) and the second equality of Lemma

of these two functions. We show that \( \phi(y) = \tilde{f}(q, y) \) is the right choice for \( \phi(y) \) here. This implies that the spectral decomposition of \( T_\lambda \) is the same as the spectral decomposition of \( L \) from Theorem B.13 (with \( c = q \), of course). We only need to show that \( \phi(y) \in D(T_\lambda) \), so it suffices to show that there exists a function \( g : \mathbb{R}_{\geq 0} \to \mathbb{C} \), differentiable in 0, such that \( \tilde{f}(q, y)_{-k} = q^{k/2} g(q^k) \) for all \( k \in \mathbb{N} \). But this follows directly from the definition of \( \tilde{f}(q, y) \), see (C.2), (B.28) and (B.32). □
for \( a_{pq}(qp, r) \) gives
\[
(q - q^{-1}) \left( \hat{J} Q(p_1, p_2, n) \hat{J} E_0 v, w \right) = \delta \chi(p_1/p_2t, -m-n \delta m+n-1, l \delta \gamma(pq, (p_2/p_1)q^{-m+1}s, r) \left| p_1p_2/p \right| (-1)^m \gamma(p) \gamma(p) \gamma(q) \gamma(t) \times \left[ \gamma(s) q^{-n-l} |p/t|^{\frac{1}{2}} \sqrt{1 + \gamma(s) |q|^{-1} a_{pq}(t, s) a_{pq}(p, r)} \right. \\
- \left( st/qp_1 \right) q^{-n-l} |p/t|^{\frac{1}{2}} |s|^{-1} a_{pq}(t, s) a_{pq}(p, r) \\
- \gamma(s) q^{-n-l} |t/p|^{\frac{1}{2}} \sqrt{1 + \gamma(q^{-1}r) |s|^{-1} a_{pq}(t, s) a_{pq}(p, q^{-1}r)} \\
+ \left( pr/qp_2 \right) q^{-n-l} |t/p|^{\frac{1}{2}} |s|^{-1} a_{pq}(t, s) a_{pq}(p, r) \right].
\]

Comparing this expression with (D.1) we see that
\[
(q - q^{-1}) \left( \hat{J} Q(p_1, p_2, n) \hat{J} E_0 v, w \right) = (q - q^{-1}) \left( \hat{J} Q(p_1, p_2, n) \hat{J} v, E_0^t w \right) \\
+ \delta \chi(p_1/p_2t, -m-n \delta m+n-1, l \delta \gamma(pq, (p_2/p_1)q^{-m+1}s, r) \left| p_1p_2/p \right| (-1)^m \gamma(p) \gamma(p) \gamma(q) \gamma(t) |pt|^{\frac{1}{2}} \\
\times |sgn(t) (s/p_1) q^{-m-l} + \gamma(p) (r/p_2) q^{m-l} |
\]

If the Kronecker \( \delta \)-function \( \delta \gamma(pq, (p_2/p_1)q^{-m+1}s, r) \) is non-zero, then the term in square brackets equals 0, thus \( \langle \hat{J} Q(p_1, p_2, n) \hat{J} E_0 v, w \rangle = \langle \hat{J} Q(p_1, p_2, n) \hat{J} v, E_0^t w \rangle \) for \( v = f_{mp} \) and \( w = f_{vs} \).

By linearity the lemma holds for all \( v, w \in K_0 \).

**D.2. Proof of Lemma 8.9.** Here we prove the following result: For \( u, v \in K_0, \; p_1, p_2 \in I_q \) and \( n \in \mathbb{Z} \), we have
\[
\langle Q(p_1, p_2, n) u, \Omega_0 v \rangle = \gamma(p_1p_2) \langle Q(p_1, p_2, n) \Omega_0 u, v \rangle. \tag{D.2}
\]

The proof depends on properties of the functions \( a_{pq}(\cdot, \cdot) \). One of the properties is the second-order \( q \)-difference equation from Lemma 3.3. The other properties we need are essentially the contiguous relations from Lemma B.2. We state these relations in the following lemma.

**Lemma 3.1.** Consider \( x, y, p \in I_q \), then
\[
\sqrt{1 + \gamma(y/q)} a_p(x, y/q) = \frac{py}{qx} a_p(x, y) + \sqrt{1 + \gamma(p)} a_{pq}(x, y),
\]
and
\[
\sqrt{1 + \gamma(y)} a_p(x, qy) = \frac{py}{x} a_p(x, y) + \sqrt{1 + \gamma(p/q)} a_{pq}(x, y).
\]

**Proof.** One uses the last equation of (D.2) to write \( a_{pq}(x, q^{-1}y) \) in terms of \( a_p(x, p^{-1}y) \). Then apply the second relation of Lemma B.2 and use (D.2) again to obtain the first equality. The second equality is proved in the same way using the first relation of Lemma B.2.

**Proof of (D.2).** Let \( l, m, n \in \mathbb{Z} \) and \( p_1, p_2, p, r, \sigma, \tau \in I_q \). We will establish
\[
\langle Q(p_1, p_2, n) f_{mp}, \Omega_0 f_{lr} \rangle = \gamma(p_1p_2) \langle Q(p_1, p_2, n) \Omega_0 f_{mp}, f_{lr} \rangle \tag{D.3}
\]
by writing out both sides of this identity in terms of matrix coefficients (7.13) of \( Q(p_1, p_2, n) \).
Let us first consider the left hand side, which we call $S_L$ for convenience, of (D.3). From the explicit action (4.8) of $\Omega_0$ on $f_{mpt}$ we find

$$2S_L = (q^{l-1} |s| + q^{-l-1} |s|) \langle Q(p_1, p_2, n) f_{m,p,t}, \tilde{f}_{l,r,s} \rangle$$

$$- \operatorname{sgn}(rs) \sqrt{(1 + \kappa(r))(1 + \kappa(s))} \langle Q(p_1, p_2, n) f_{m,p,t}, \tilde{f}_{l,q,r,s} \rangle$$

$$- \operatorname{sgn}(rs) \sqrt{(1 + \kappa(q^{-1}r))(1 + \kappa(q^{-1}s))} \langle Q(p_1, p_2, n) f_{m,p,t}, \tilde{f}_{l,q^{-1}r,q^{-1}s} \rangle.$$

In terms of the matrix coefficients (7.13) of $Q(p_1, p_2, n)$, we have

$$2S_L = \delta_{[p_1/p_2l],q^{m-n}} \delta_{m-n,t} \delta_{\operatorname{sgn}(pt)(p_2/p_1)q^m} s, r$$

$$\times \left[ (q^{l-1} |s| + q^{-l-1} |s|) \left| \frac{t}{s} \right| a_t(p_1, s) a_p(p_2, r) \right.$$

$$- \operatorname{sgn}(rs) \sqrt{(1 + \kappa(r))(1 + \kappa(s))} \left| \frac{t}{q s} \right| a_t(p_1, q s) a_p(p_2, qr)$$

$$- \operatorname{sgn}(rs) \sqrt{(1 + \kappa(r/q))(1 + \kappa(s/q))} \left| \frac{t q}{s} \right| a_t(p_1, s/q) a_p(p_2, r/q) \right].$$

From the $q$-contiguous relations of Lemma [D.7] it follows that

$$\sqrt{(1 + \kappa(r))(1 + \kappa(s))} a_t(p_1, q s) a_p(p_2, qr) =$$

$$\left( \frac{t s}{p_1} a_t(p_1, s) + \sqrt{1 + \kappa(t/q)} a_t/q(p_1, s) \right) \left( \frac{p r}{p_2} a_p(p_2, r) + \sqrt{1 + \kappa(p/q)} a_p/q(p_2, r) \right)$$

and

$$\sqrt{(1 + \kappa(r/q))(1 + \kappa(s/q))} a_t(p_1, s/q) a_p(p_2, r/q) =$$

$$\left( \frac{t s}{q p_1} a_t(p_1, s) + \sqrt{1 + \kappa(t) a_t/q(p_1, s)} \right) \left( \frac{p r}{q p_2} a_p(p_2, r) + \sqrt{1 + \kappa(p/q)} a_p(q q)(p_2, r) \right),$$

which implies

$$2S_L = \delta_{[p_1/p_2l],q^{m-n}} \delta_{m-n,t} \delta_{\operatorname{sgn}(pt)(p_2/p_1)q^m} s, r$$

$$\times \left[ (q^{l-1} |s| + q^{-l-1} |s|) \left| \frac{t}{s} \right| a_t(p_1, s) a_p(p_2, r) \right.$$

$$- \operatorname{sgn}(rs) \left| \frac{t}{s} \right| q^{-1} \sqrt{(1 + \kappa(t/q))(1 + \kappa(p/q))} a_t/q(p_1, s) a_p/q(p_2, r)$$

$$- \operatorname{sgn}(rs) \left| \frac{t}{s} \right| q \sqrt{(1 + \kappa(t))(1 + \kappa(p))} a_q(p_1, s) a_q(p_2, r)$$

$$- \operatorname{sgn}(rs) \left| \frac{t}{s} \right| \left( \frac{p r}{q p_2} A_t(p_1, s) a_p(p_2, r) + \frac{t s}{q p_1} A_p(p_2, r) a_t(p_1, s) \right) \right],$$

where

$$A_x(y, z) = \frac{xz}{y} a_x(y, z) + \sqrt{1 + \kappa(x/q)} a_{x/q}(y, z) + q \sqrt{1 + \kappa(x)} a_{xq}(y, z).$$
The expression of $A_z(y, z)$ simplifies by Lemma 3.2 to

$$\frac{xz}{y} A_z(y, z) = (\kappa(z) - \kappa(x)) a_x(y, z).$$

Since $\delta_{|p_1p/p_2|, q^{m-n}} \delta_{m-n, l} \delta_{\text{sgn}(p_1/p_2 | q^{m,s,r})} = 0$ unless $pr = \text{sgn}(p_1 p_2) q^{m+l} st p_2^2 / p_1^2$, we now get

$$2S_L = \delta_{|p_1p/p_2|, q^{m-n}} \delta_{m-n, l} \delta_{\text{sgn}(p_1/p_2 | q^{m,s,r})} \begin{bmatrix} (q^{l-1} r | s ) + q^{-l-1} s | r | \end{bmatrix} \begin{bmatrix} t \cr s \end{bmatrix} a_t(p_1, s) a_p(p_2, r)$$

$$- \text{sgn}(rs) \begin{bmatrix} t \cr s \end{bmatrix} q^{-1} (1 + \kappa(t/q))(1 + \kappa(p/q)) a_{t/q}(p_1, s) a_{p/q}(p_2, r)$$

$$- \text{sgn}(rs) \begin{bmatrix} t \cr s \end{bmatrix} q (1 + \kappa(t))(1 + \kappa(p)) a_{qp}(p_1, s) a_{qp}(p_2, r)$$

$$- \text{sgn}(rs) q^{m+l-1} \begin{bmatrix} \frac{tp_2}{sp_1} \end{bmatrix} (\kappa(s) - \kappa(t)) a_t(p_1, s) a_p(p_2, r)$$

$$- \text{sgn}(rs) q^{m-l-1} \begin{bmatrix} \frac{tp_1}{sp_2} \end{bmatrix} (\kappa(r) - \kappa(p)) a_p(p_2, r) a_t(p_1, s) \right].$$

Unless $\text{sgn}(p_1 p_2) = \text{sgn}(rs) \text{sgn}(pt)$, $|p_2/p_1| q^m = |r/s|$ and $q^l |p_2/p_1| = |p/t|$, the above expression is zero. Thus,

$$2S_L = \delta_{|p_1p/p_2|, q^{m-n}} \delta_{m-n, l} \delta_{\text{sgn}(p_1/p_2 | q^{m,s,r})} \text{sgn}(p_1 p_2) \begin{bmatrix} t \cr s \end{bmatrix}$$

$$\begin{bmatrix} (q^{m-1} p | t ) + q^{-m-1} t | p | \end{bmatrix} a_t(p_1, s) a_p(p_2, r)$$

$$- \text{sgn}(pt) q^{-1} (1 + \kappa(t/q))(1 + \kappa(p/q)) a_{t/q}(p_1, s) a_{p/q}(p_2, r)$$

$$- \text{sgn}(pt) q (1 + \kappa(t))(1 + \kappa(p)) a_{qp}(p_1, s) a_{qp}(p_2, r)$$

$$- \text{sgn}(pt) q^{m+l-1} \begin{bmatrix} \frac{tp_2}{sp_1} \end{bmatrix} (\kappa(s) - \kappa(t)) a_t(p_1, s) a_p(p_2, r)$$

$$- \text{sgn}(pt) q^{m-l-1} \begin{bmatrix} \frac{tp_1}{sp_2} \end{bmatrix} (\kappa(r) - \kappa(p)) a_p(p_2, r) a_t(p_1, s) \right].$$

Next we write out the right hand side $S_R$ of (D.3). Using the action (1.8) of $\Omega_0$ again, we see that

$$2S_R = (q^{m-1} p | t ) + q^{-m-1} t | p | \langle Q(p_1, p_2, n) f_{m,p,t}, f_{l,r,s} \rangle$$

$$- \text{sgn}(pt) \sqrt{(1 + \kappa(p))(1 + \kappa(t))} \langle Q(p_1, p_2, n) f_{m,q,p,t}, f_{l,r,s} \rangle$$

$$- \text{sgn}(pt) \sqrt{(1 + \kappa(q^{-1} p))(1 + \kappa(q^{-1} t))} \langle Q(p_1, p_2, n) f_{m,q^{-1},p^{-1},q^{-1} t}, f_{l,r,s} \rangle.$$

Writing this out in terms of the matrix coefficients of $Q(p_1, p_2, n)$, see (7.13), we obtain

$$2S_R = \delta_{|p_1p/p_2|, q^{m-n}} \delta_{m-n, l} \delta_{\text{sgn}(p_1/p_2 | q^{m,s,r})} \begin{bmatrix} t \cr s \end{bmatrix}$$

$$\begin{bmatrix} (q^{m-1} p | t ) + q^{-m-1} p | t | \end{bmatrix} a_t(p_1, s) a_p(p_2, r)$$

$$- \text{sgn}(pt) q^{-1} (1 + \kappa(q^{-1} t))(1 + \kappa(q^{-1} p)) a_{q^{-1} t}(p_1, s) a_{q^{-1} p}(p_2, r)$$

$$- \text{sgn}(pt) q (1 + \kappa(t))(1 + \kappa(p)) a_{q t}(p_1, s) a_{q p}(p_2, r) \right].$$
Comparing this with (D.4) we see that $S_L = S_R$, hence (D.3) holds.

D.3. **Proof of Lemma 9.3.** We prove the following result: Let $f : J(p, m, \varepsilon, \eta) \to \mathbb{C}$ be bounded, and consider the function

$$g(w) = (-1)^m (\eta')^{\chi(p_1 p_2) + m} q^{n + m} p_1^2 \frac{(\varepsilon' \eta')^{\chi(w)}}{|w|} \sum_{z \in J(p, m, \varepsilon, \eta)} \frac{f(z)}{|z|} a_{p_1}(z, w) a_{p_2}((\varepsilon \eta) q^m p z, \varepsilon' \eta' q^{m'} p w),$$

for $w \in J(p, m', \varepsilon', \eta')$.

1. If $f(z) \sim A t^{-\chi(z)}$ as $z \to 0$ for some $A \in \mathbb{C}$ and $t \in \mathbb{C}$, $|t| > 1$, then

$$g(w) \sim A t^{-\chi(w)} \eta^n s(\varepsilon, \varepsilon') s(\eta, \eta') S(\varepsilon \eta / t; p_1, p_2, n),$$

as $w \to 0$.

2. If $f(z) \sim \Re(A e^{-i \psi \chi(z)})$ as $z \to 0$ for some $A \in \mathbb{C}$ and $\psi \in \mathbb{R}$, then

$$g(w) \sim \eta^n s(\varepsilon, \varepsilon') s(\eta, \eta') \Re(A e^{-i \psi \chi(w)} S(\varepsilon \eta e^{-i \psi}; p_1, p_2, n)),$$

as $w \to 0$.

Here we use the notation $f(z) \sim g(z)$ as $z \to 0$, for $\lim_{z \to 0} (f(z) - g(z)) = 0$. The function $S(\cdot; p_1, p_2, n)$ is defined by (B.2).

**Proof.** The proof is based on splitting the sum in $g(w)$, and taking limits in both parts of the sum using Tannery’s theorem, i.e., the dominated convergence theorem for infinite sums.

First of all, the boundedness of $f$ together with Lemma [B.3] implies that the sum by which $g(w)$ is defined is absolutely convergent. Let us denote $\theta = \varepsilon \eta q^m p$, $\theta' = \varepsilon' \eta' q^{m'} p$ and $r = \min\{q, q/|\theta|\}$. Now we split the sum for $g$ into a part with $|z| > r$ and a part with $|z| \leq r$. First we consider the part with $|z| > r$. We define, for $y \in J(p, m', \varepsilon', \eta')$,

$$B(y) = \frac{(\varepsilon' \eta')^{-\chi(y)}}{|y|} \sum_{z \in J(p, m, \varepsilon, \eta), |z| > r} \frac{1}{|z|} a_{p_1}(z, y) a_{p_2}(\theta z, \theta' y) f(z).$$

By Lemma [B.4] there exists a constant $D > 0$ so that

$$|a_{p_1}(z, y) a_{p_2}(\theta z, \theta' y)| \leq D \nu(p_1/y) \nu(p_2/\theta'y) |z|^\chi(p_1/y) |\theta z|^\chi(p_2/\theta'y),$$

for all $z \in J(p, m, \varepsilon, \eta)$ and $y \in J(p, m', \varepsilon', \eta')$ satisfying $|z| > r$ and $|y| < r$. Since, by assumption, $f$ is bounded, inequality (D.4) and Tannery’s theorem imply that $B(y) \to 0$ as $y \to 0$.

Next consider the remaining sum over $z \in J(p, m, \varepsilon, \eta)$, $|z| \leq r$, for all $y \in J(p, m', \varepsilon', \eta')$. We go over to a new summation parameter $x = z/y$, so that it follows from $\sgn(z) = \varepsilon$ that
We now consider the asymptotic behaviour of 

\[ C(y) = \text{sgn}(p_1). \]

This gives

\[
\frac{(\varepsilon' \eta') \chi(y)}{|y|} \sum_{x \in J(p, m, \varepsilon, \eta) \mid |x| \leq r} \frac{1}{|z|} a_{p_1}(z, y) a_{p_2}(\theta z, \theta' y) f(z)
\]

\[ = \frac{(\varepsilon' \eta') \chi(y)}{|y|} \sum_{x \in \text{sgn}(p_1)q^x \mid |x| \leq \frac{r}{|y|}} \frac{1}{|yx|} a_{p_1}(yx, y) a_{p_2}(\theta yx, \theta' y) f(yx). \]  

\tag{D.6}

Let \( F : J(p, m, \varepsilon, \eta) \to \mathbb{C} \) be a bounded function such that

\[
f(y) = \begin{cases} 
    t^{-\chi(y)} F(y), & \text{if } f(w) \sim A t^{-\chi(w)}, \text{ as } w \to 0, \\
    \Re(e^{-i\psi(x)} F(y)), & \text{if } f(w) \sim \Re(A e^{-i\psi(x)}), \text{ as } w \to 0.
\end{cases}
\]

Observe that this implies \( \lim_{y \to 0} F(y) = A \). Now for \( y \in J(p, m', \varepsilon', \eta') \) and \( |t| \geq 1 \), we define

\[
C(y; t) = \frac{(\varepsilon' \eta') \chi(y)}{|y|} \sum_{x \in \text{sgn}(p_1)q^x \mid |x| \leq \frac{r}{|y|}} \frac{1}{|yx|} a_{p_1}(yx, y) a_{p_2}(\theta yx, \theta' y) t^{-\chi(x)} F(yx). \]  

\tag{D.7}

We now consider the asymptotic behaviour of \( C(y; t) \) as \( y \to 0 \).

Let us first see that we can take termwise limits in \( \text{(D.7)} \). For \( x \in \text{sgn}(p_1)q^x \) satisfying \( |x| \leq \frac{r}{|y|} \), we have by Definition \( \text{[7.2]} \),

\[
\frac{(\varepsilon' \eta') \chi(y)}{|y^2x|} a_{p_1}(yx, y) a_{p_2}(\theta yx, \theta' y) t^{-\chi(x)} F(yx)
\]

\[ = \frac{(\varepsilon' \eta') \chi(y)}{|y^2x|} c^2_q s(\varepsilon, \varepsilon') s(\eta, \eta') (-1)^{\chi(p_1p_2)} (-\varepsilon')^{\chi(yx)} (-\eta')^{\chi(\theta yx)} |y|^2 |\theta'| t^{-\chi(x)} F(yx)
\]

\[
\times \nu(p_1/x) \nu(p_2 q^n/x) \sqrt{(-\kappa(p_1), -\kappa(p_2); q^2)_{\infty}} \sqrt{(-\kappa(y), -\kappa(\theta y); q^2)_{\infty}}
\]

\[
\times \Psi\left( -q^2/\kappa(p_1) \right) \Psi\left( -q^2/\kappa(p_2) \right) q^2 \kappa(\theta yx/p_1) q^2 \kappa(\theta yx/p_2) q^2 \kappa(\text{sgn}(p_1p_2)q^{-n}x)
\]

\[
= (-1)^{\chi(p_1p_2)+m} s(\varepsilon, \varepsilon') s(\eta, \eta') (\eta')^{\chi(p)+m} q^m p \cdot c^2_q q^n \sqrt{(-\kappa(p_1), -\kappa(p_2); q^2)_{\infty}}
\]

\[
\times (\varepsilon' \eta')/t \chi(x) F(yx) \nu(p_1/x) \nu(p_2 q^n/x) \sqrt{(-\kappa(y), -\kappa(\theta y); q^2)_{\infty}}
\]

\[
\times |x|^{-1} (q^2 \kappa(yx/p_1), q^2 \kappa(\theta yx/p_2); q^2)_{\infty}
\]

\[
\times 1 \varphi_1\left( -q^2/\kappa(p_1) \right) \varphi_1\left( -q^2/\kappa(p_2) \right) q^2 \kappa(yx/p_1) q^2 \kappa(yx/p_2) q^2 \kappa(\text{sgn}(p_1p_2)q^{-n}x),
\]  

\tag{D.8}
Assuming for the moment that we can apply Tannery’s Theorem, we see from the last expression that \( C(y; t) \) converges to
\[ A \left( -1 \right)^{p(p_2 + p)} \frac{q^m P}{p_1 p_2} \left( \eta' \right)^{p(p) + m} \text{sgn}(p_2) \, s(\varepsilon', \varepsilon) \, s(\eta', \eta) \, S(\varepsilon \eta / t; p_1, p_2, n) \]
as \( y \to 0 \), using (B.2). This proves the lemma.

In order to be able to apply Tannery’s Theorem, we need to estimate the summand by a term independent of \( y \). For small \( x \) such an estimate follows from (D.8), since \( F(y) \to A \) as \( y \to 0 \) and the functions \( \nu \) are small. It remains to give an estimate for large \( x \) uniformly for \( |y| \leq q^l \) for some \( l \in \mathbb{Z} \). By (6.2) we have
\[
\frac{1}{|y^2 x|} |a_{p_1}(y, x)||a_{p_2}(\theta y, x)| |t^{-\chi(x)} F(yx)|
= \frac{1}{|y^2 x|} |a_{p_1}(y, x)||a_{p_2}(\theta y, x)| |t^{-\chi(x)} F(yx)|
= c_q^2 |\theta x t^{-\chi(x)} F(yx)| \sqrt{(\kappa(y), -\kappa(y'; \theta); q^2)_{\infty}} \nu(p_1 x) \nu(p_2 q^{-n} x)
\times \sqrt{(\kappa(y), -\kappa(\theta y'; \theta); q^2)_{\infty}} \Psi \left( -q^2 / \kappa(p_1) \; q^2 \kappa(y/p_1) ; q^2, q^2 / \kappa(x) \right)
\times \Psi \left( -q^2 / \kappa(p_2) \; q^2 \kappa(\theta y/p_2) ; q^2, q^2 \kappa(\text{sgn}(p_1 p_2) q^n / x) \right).
\]
The \( \Psi \)-functions are bounded for \( |x| \) large and \( |y| \leq q^l \). Put \( |x| = q^{-k} \), then using the boundedness of \( F \) and the \( \theta \)-product identity (B.1), we find
\[
|t^{-\chi(x)} F(yx)| \nu(p_1 x) \nu(p_2 q^{-n} x) \sqrt{(\kappa(y), -\kappa(\theta y'; \theta); q^2)_{\infty}}
\leq D_1 |t x^{-\chi(x)} \nu(p_1 x) \nu(p_2 q^{-n} x) \sqrt{(q^{2l-2k} - |\theta| q^{2l-2k}); q^2)_{\infty}}
= D_2 |t q^{n+1+l} / p_1 p_2 | \theta | q^{2l-k} \sqrt{(-q^{2l-1} / |\theta| ; q^2)_{k}}
\leq D_3 |t q^{n+1+l} / p_1 p_2 | \theta | q^{2l-k},
\]
where the constants \( D_i \) are independent of \( x \). We see that for \( l \) large enough this gives us the desired estimate. \( \square \)
Index

A(λ; p, m, ε, η), 60
A_q, 8
a_p(x, y), 21
affiliation, 88

basic hypergeometric series, 89
C(x; m, ε, η; p_1, p_2, n), 51
χ(x), 21
Casimir operator, 14
commutant, 7
commutation of operators, 88
comultiplication, 4
corepresentation, 7
complementary series, 78
decomposition of the left regular, 20
discrete series, 17, 67
irreducible, 7
principal series, 19, 71
unitary, 7

D(p, m, ε, η), 52
D^+_k, 53
D^-_k, 53
dual locally compact quantum group, 7
dual modular conjugation, 11

e(p), 18, 54
f_{mp}, 10
rφ_s-series, 88
G(λ; p, m, ε, η), 93
GNS-construction, 6

H(x; p_1, p_2, n), 58, 64
Haar weight, 7
Hopf ∗-algebra A_q, 8

I_q, 8, 21
invariant subspace, 8

J(p, m, ε, η), 28
j, 11
K, 14
K, 10
K(p, ε, η), 54
K(p, m, ε, η), 28

κ(p, m, ε, η), 28
κ(x), 21
L(c, d, z | q), 107
L^2(I(p, m, ε, η), 52
ℓ_θ, 21
M(x), 50
M^+, 7
M, 7
M^+, 15
M_-, 15
μ, 21
modular automorphism group, 7
modular conjugation, 7
modular operator, 7
multiplicative unitary, 7

N(x; p_1, p_2, n), 59
N_{SL(2, C)}(SU(1, 1)), 9
N, 7
N_0, 7
ν(x), 21
ν^∗_n(λ; p), 53
normal functional, 87
normal homomorphism, 87

Ω, 14
Ω_{p,m}, 43

Ψ, 21
π_κ(p, ε, η), 53
π^∗_{a, ε}, 54
π^∗_{b, ε}, 54
pentagonal identity, 7
Pontryagin dual, 7
predual M_+, 87

Q(p_1, p_2, n), 15
q-shifted factorials, 88
quantized universal enveloping algebra
U_q(su(1, 1)), 12
quantum group, 7

S(t; p_1, p_2, n), 28, 91
SU(1, 1), 7
Σ, 7
σ_d(p, m, ε, η), 52
s(x, y), 21
scaling group, 6
σ-strong-∗ topology, 87
σ-weak topology, 87
strong ∗-topology, 87
strong commutation of operators, 88
strong topology, 87
structure constants, 15

\[ T(p_1, p_2, n), \]
\[ \mathbb{T}, \]
\[ \mathbb{T}_0, \]
tensor product, 87

\[ U_q(\mathfrak{su}(1, 1)), \]
\[ U_q(\mathfrak{su}(1, 1))-\text{representations} \]
complementary series, 54
negative discrete series, 53
\[ \pi_K(p, \varepsilon, \eta), \]
positive discrete series, 53
principal series, 53
strange series, 53

\[ v(t), \]
unitary antipode, 6, 7

\[ V(p_1, p_2, n), \]
\[ V_{\sigma, \tau}^n, \]
von Neumann algebra, 87
von Neumann algebra generated by operators, 88
von Neumann algebraic quantum group, 5

weak topology, 87
weight, 3
faithful, 3
GNS-construction, 3
normal, 3
nsf, 3
semifinite, 3

\[ X_{n}^{\sigma, \tau}(p, m, \varepsilon, \eta), \]
\[ \Upsilon_{\varepsilon, \eta}^{p, m}, \]
\[ \Upsilon, \]
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