On the Volterra integral equation for the remainder term in the asymptotic formula on the associated Euler totient function

by

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Abstract. J.Kaczorowski and K.Wiertelak considered the integral equation for remainder terms in the asymptotic formula for the Euler totient function and for the twisted Euler $\phi$-function. In 2013, J.Kaczorowski defined the associated Euler totient function which extends the above two functions and proved an asymptotic formula for it. In the present paper, first, we consider the Volterra integral equation for the remainder term in the asymptotic formula for the associated Euler totient function. Secondly, we solve the Volterra integral equation and we split the error term in the asymptotic formula for the associated Euler totient function into two summands called arithmetic and analytic part respectively.

1. Introduction

J.Kaczorowski and K.Wiertelak obtained a decomposition for the remainder term in the asymptotic formula for a generalization of the Euler totient function (see [3]): For a non-principal real Dirichlet character $\chi (\mod q), q > 2$, let $\varphi(n, \chi)$ denote the twisted Euler $\phi$-function

\begin{equation}
\varphi(n, \chi) = n \prod_{p | n} \left(1 - \frac{\chi(p)}{p}\right).
\end{equation}

Let

\begin{equation}
E(x, \chi) = \sum_{n \leq x} \varphi(n, \chi) - \frac{x^2}{2L(2, \chi)}
\end{equation}

and

\begin{equation}
E_1(x, \chi) = \begin{cases} 
E(x, \chi) & (x \notin \mathbb{N}), \\
\frac{1}{2}(E(x - 0, \chi) + E(x + 0, \chi)) & \text{(otherwise)}
\end{cases}
\end{equation}

be the corresponding error terms. Here, as usual, $L(s, \chi)$ denotes the Dirichlet $L$-function associated to $\chi$. It is easy to see that $E(x, \chi) = O(x \log x)$ for $x \geq 2$. Hence $x^2/(2L(2, \chi))$ is the main term in (1.2). Let $s(x)$ be the sawtooth function

\begin{equation}
s(x) = \begin{cases} 
0 & (x \in \mathbb{Z}), \\
\frac{1}{2} - \{x\} & \text{(otherwise)},
\end{cases}
\end{equation}

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where \([x] = x - [x]\) is the fractional part of a real number \(x\). We write for \(x \geq 0\)

\[
(1.5) \quad f(x, \chi) = \sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n} \left(\frac{x}{n}\right),
\]

\[
(1.6) \quad g(x, \chi) = \sum_{n=1}^{\infty} \mu(n)\chi(n) \left\{\frac{x}{n}\right\} \left(\frac{x}{n} - 1\right),
\]

where \(\mu(n)\) denotes the Möbius function. J.Kaczorowski and K.Wiertelak considered the Volterra integral equation of second type for (1.3) and solved it.

**Theorem 1.1** (Theorem 1.1 in [3]). The solution of the following Volterra integral equation of second type

\[
(1.7) \quad F(x, \chi) - \int_{0}^{\infty} K(x, t)F(t, \chi)dt = E_{1}(x, \chi) \quad (x \geq 0),
\]

where

\[
K(x, t) = \begin{cases} 
1/t & (0 < t \leq x), \\
0 & (0 \leq x < t),
\end{cases}
\]

is the function

\[
(1.8) \quad F(x, \chi) = (f(x, \chi) + A)x,
\]

where \(A\) is an arbitrary constant.

(Probably, the term \(Ax\) is missing to give the general solution as noted in [1].) Also, J.Kaczorowski and K.Wiertelak splitted (1.3) into two sum-mands as follows:

**Theorem 1.2** (Theorem 1.2 in [3]). For \(x \geq 0\) we have

\[
(1.9) \quad E_{1}(x, \chi) = xf(x, \chi) + \frac{1}{2} g(x, \chi).
\]

By (1.9), \(E_{1}(x, \chi)\) can be splitted as follows:

\[
(1.10) \quad E_{1}(x, \chi) = E^{AR}(x, \chi) + E^{AN}(x, \chi),
\]

where

\[
(1.11) \quad E^{AR}(x, \chi) = xf(x, \chi) \quad \text{and} \quad E^{AN}(x, \chi) = \frac{1}{2} g(x, \chi)
\]

with \(f(x, \chi)\) and \(g(x, \chi)\) given by (1.5) and (1.6) respectively. We call \(E^{AR}(x, \chi)\) and \(E^{AN}(x, \chi)\) the arithmetic part and analytic part of \(E_{1}(x, \chi)\) respectively.

J.Kaczorowski defined the associated Euler totient function for the generalized \(L\)-functions including the Riemann zeta function, Dirichlet \(L\)-functions and obtained the asymptotic formula (see [4]): By a polynomial Euler product we mean a function \(F(s)\) of a complex variable \(s = \sigma + it\) which for \(\sigma > 1\) is defined by a product of the form

\[
(1.12) \quad F(s) = \prod_{p} F_{p}(s) = \prod_{p} \prod_{j=1}^{d} \left(1 - \frac{\alpha_{j}(p)}{p^{s}}\right)^{-1},
\]
where \( p \) runs over primes and \( |\alpha_j(p)| \leq 1 \) for all \( p \) and \( 1 \leq j \leq d \). We assume that \( d \) is chosen as small as possible, i.e. that there exists at least one prime number \( p_0 \) such that
\[
\prod_{j=1}^{d} \alpha_j(p_0) \neq 0.
\]
Then \( d \) is called the Euler degree of \( F \). For \( F \) in (1.12) we define the associated Euler totient function as follows :
\[
\varphi(n, F) = n \prod_{p \mid n} F_p(1)^{-1} \quad (n \in \mathbb{N}).
\]
Let
\[
\gamma(p) = p \left(1 - \frac{1}{F_p(1)}\right),
\]
\[
C(F) = \frac{1}{2} \prod_p \left(1 - \frac{\gamma(p)}{p^2}\right).
\]
J.Kaczorowski obtained the asymptotic formula for the error term in the asymptotic formula for (1.13).

**Theorem 1.3** (Theorem 1.1 in [4]). For a polynomial Euler product \( F \) of degree \( d \) and \( x \geq 1 \) we have
\[
\sum_{n \leq x} \varphi(n, F) = C(F)x^2 + O(x(\log 2x)^d).
\]

Let us put
\[
E(x, F) = \sum_{n \leq x} \varphi(n, F) - C(F)x^2
\]
and
\[
\alpha(n) = \mu(n) \prod_{p \mid n} \gamma(p),
\]
where \( \gamma(p) \) is defined by (1.14). The main aim of the present paper is to consider the Volterra integral equation of second type associated with \( \varphi(n, F) \), and to prove the results similar to Theorem 1.1 and 1.2.

2. Main theorems

For a polynomial Euler product \( F \) of degree \( d \), let
\[
E_2(x, F) := \begin{cases} E(x, F) & (x \notin \mathbb{N}) \\ \frac{1}{2}(E(x - 0, F) + E(x + 0, F)) & (\text{otherwise}) \end{cases}
\]
be the corresponding error terms. As in [3], we consider the following Volterra integral equation of second type for (2.1) as follows :
\[
F_1(x, F) - \int_0^x F_1(t, F) \frac{dt}{t} = E_2(x, F) \quad (x \geq 0),
\]
where \( F_1(x, F) \) is the unknown function. For every \( x \geq 0 \), let
\[
f_1(x, F) = \sum_{n=1}^{\infty} \frac{\alpha(n)}{n} s\left(\frac{x}{n}\right),
\]
where \( s(x) \) is the same as in (1.4). When \( x \) is a positive integer, the following fact holds for \( f_1(x, F) \).

**Fact 2.1.** For positive integer \( N \),

\[
(2.4) \quad f_1(N, F) = \frac{1}{2} (f_1(N + 0, F) + f_1(N - 0, F)).
\]

**Proof of Fact 2.1.** Let \( N \) be a positive integer. By elementary calculations, we have

\[
(2.5) \quad f_1(N + 0, F) = \frac{1}{2} \sum_{n \leq N+1 \atop n\mid N} \frac{\alpha(n)}{n} + \sum_{n \leq N+1 \atop n\nmid N} \frac{\alpha(n)}{n} \left( \frac{1}{2} - \left\{ \frac{N+0}{n} \right\} \right) + \sum_{n > N+1} \frac{\alpha(n)}{n} \left( \frac{1}{2} - \frac{N}{n} \right),
\]

\[
(2.6) \quad f_1(N - 0, F) = -\frac{1}{2} \sum_{n \leq N+1 \atop n\nmid N} \frac{\alpha(n)}{n} + \sum_{n \leq N+1 \atop n\mid N} \frac{\alpha(n)}{n} \left( \frac{1}{2} - \left\{ \frac{N-0}{n} \right\} \right) + \sum_{n > N+1} \frac{\alpha(n)}{n} \left( \frac{1}{2} - \frac{N}{n} \right).
\]

Since

\[
\left\{ \frac{N+0}{n} \right\} + \left\{ \frac{N-0}{n} \right\} = 2 \left\{ \frac{N+0}{n} \right\}
\]

for \( n \) which does not divide a positive integer \( N \), adding (2.5) and (2.6) we have

\[
(2.7) \quad \frac{1}{2} (f_1(N+0, F) + f_1(N-0, F)) = \sum_{n \leq N+1 \atop n\mid N} \frac{\alpha(n)}{n} \left( \frac{1}{2} - \left\{ \frac{N+0}{n} \right\} \right) + \sum_{n > N+1} \frac{\alpha(n)}{n} \left( \frac{1}{2} - \frac{N}{n} \right).
\]

By (2.3), the right-hand side of (2.7) corresponds to \( f_1(N, F) \). \( \Box \)

Moreover, to assure the convergence of the series (2.3), we assume that the series

\[
(2.8) \quad \sum_{n=1}^{\infty} \frac{\alpha(n)}{n}
\]

converges, where \( \alpha(n) \) is the same in (1.18).

**Theorem 2.2.** For every complex number \( A \), the function

\[
(2.9) \quad F_1(x, F) = (f_1(x, F) + A)x \quad (x \geq 0),
\]

is a solution of the integral equation (2.2) and these exhaust all solutions of (2.2).

As usual, in case we say a function \( F_1(x, F) \) is a solution of (2.2), we assume that the integral in (2.2) exists in the sense that the limit

\[
(2.10) \quad \lim_{\epsilon \to 0^+} \int_{\epsilon}^{\infty} |F_1(t, F)| \frac{dt}{t}
\]

exists. We use the same convention throughout this paper. Also, the meaning of the integral in (2.11) should also be interpreted in this sense. For every \( x \geq 0 \), let

\[
(2.11) \quad g_1(x, F) = \sum_{n=1}^{\infty} \alpha(n) \left\{ \frac{x}{n} \right\} \left( \left\{ \frac{x}{n} \right\} - 1 \right).
\]
Theorem 2.3. For \( x \geq 1 \) we have
\[
E_2(x, F) = xf_1(x, F) + \frac{1}{2}g_1(x, F).
\]

We split \( E_2(x, F) \) into the arithmetic part and the analytic part as follows:
\[
E_2(x, F) = E^{AR}(x, F) + E^{AN}(x, F),
\]
where
\[
E^{AR}(x, F) = xf_1(x, F) \quad \text{and} \quad E^{AN}(x, F) = \frac{1}{2}g_1(x, F).
\]

3. Remarks and auxiliary lemmas

We prepare some remarks and auxiliary lemmas.

Remark 3.1 (P33 in [4]). For every positive \( \epsilon \), \( \alpha(n) \ll n^\epsilon \). Hence the series
\[
\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s}
\]
absolutely converges for \( \sigma > 1 \). Since \( \alpha(n) \) is multiplicative by (1.18), we have
\[
\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s} = 2C(F).
\]

Remark 3.2 (Lemma 2.2 in [4]). The series
\[
\sum_{n=1}^{\infty} \frac{\varphi(n, F)}{n^s}
\]
converges absolutely for \( \sigma > 2 \) and in this half-plane we have
\[
\sum_{n=1}^{\infty} \frac{\varphi(n, F)}{n^s} = \zeta(s - 1) \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s}.
\]
In particular,
\[
\varphi(n, F) = n \sum_{m|n} \frac{\alpha(m)}{m}.
\]

We define the auxiliary function for \( x \geq 0 \) by
\[
R(x, F) = E_2(x, F) - xf_1(x, F).
\]

Lemma 3.3. For all positive \( x \),
\[
R(x, F) = - \int_{0}^{x} f_1(t, F) dt.
\]

Proof. We can prove that \( R(x, F) \) is a continuous function in the same way as in Lemma 1 of [1]. For positive \( x \) which is not an integer, take derivatives of the both sides of (3.6). Since \( x \) is not a positive integer, we have \( E_2(x, F) = E'(x, F) = -2C(F)x \). Therefore we have
\[
R'(x, F) = -2C(F)x - f_1(x, F) - xf_1'(x, F).
\]
Since $x$ is positive and not an integer, we have $\{x/n\}' = 1/n$ (see [2], P2691). Considering the hypothesis on the series (2.8), Remark 3.1, and the fact that $x$ is positive and not an integer, differentiating term by term we obtain

$$\frac{d}{dx} \sum_{n=1}^{\infty} \frac{\alpha(n)}{n} \left(\frac{x}{n}\right) = \sum_{n=1}^{\infty} \frac{\alpha(n)}{n} \frac{d}{dx} \left(\frac{1}{2} - \left\{\frac{x}{n}\right\}\right)$$

$$= -2C(F).$$

Consequently, we have

$$R'(x, F) = -f_1(x, F)$$

for $x$ which is positive and not an integer. Since $R(0, F) = 0$ by (2.3) and $R(x, F)$ is continuous for all positive $x$, we have (3.7) for all positive $x$. □

**Lemma 3.4.** Let $G$ be a complex-valued function defined on $[0, \infty)$ satisfying

(3.8) \[ \int_{0}^{x} |G(t)| \frac{dt}{t} < +\infty \]

and the integral equation

(3.9) \[ G(x) - \int_{0}^{x} G(t) \frac{dt}{t} = 0 \]

for all $x \geq 0$. Then we have

(3.10) \[ G(x) = A x \]

for some complex number $A$.

**Proof.** This is Lemma 2 in [1]. □

4. **Proof of main theorems.**

First we prove Theorem 2.1 for $x$ which is positive and not an integer. Let a function $F_1(x, F)$ be a solution of the Volterra integral equation of second type (2.2) satisfying the condition (2.10). Using Lemma 3.3, from (3.6) we have

(4.1) \[ E_2(x, F) - xf_1(x, F) = -\int_{0}^{x} f_1(t, F) dt. \]

Since $x$ is positive and not an integer, $E_2(x, F) = E(x, F)$. By (2.2), we have

$$\int_{0}^{x} (F_1(t, F) - tf_1(t, F)) \frac{dt}{t} = F_1(x, F) - xf_1(x, F).$$

Using Lemma 3.4, we have the solution

$$F_1(x, F) = (f_1(x, F) + A)x.$$

Conversely, if we assume that $F_1(x, F)$ is a function of type (2.9). Then, by (3.6) and (3.7),

$$F_1(x, F) - \int_{0}^{x} F_1(t, F) \frac{dt}{t} = xf_1(x, F) - \int_{0}^{x} f_1(t, F) dt$$

$$= xf_1(x, F) + R(x, F)$$

$$= E(x, F).$$
Therefore, the function $F_1(x, F)$ of type (2.5) is a solution of (2.2) for positive and not an integer $x$. Also, the function $f_1(x, F)$ is a locally bounded. In fact, by the hypothesis (2.8) and (3.2)

$$f_1(x, F) = \sum_{n=1}^{\infty} \frac{\alpha(n)}{n} \left( \frac{x}{n} \right)$$

$$= \sum_{n=1}^{\infty} \frac{\alpha(n)}{n} \left( \frac{1}{2} - \left( \frac{x}{n} \right) \right)$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{\alpha(n)}{n} - \sum_{n=1}^{\infty} \frac{\alpha(n)}{n} \left( \frac{x}{n} \right)$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{\alpha(n)}{n} - \sum_{n=1}^{\infty} \frac{\alpha(n)}{n} \left( \frac{x}{n} - \left[ \frac{x}{n} \right] \right)$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{\alpha(n)}{n} - 2C(F)x + \sum_{n \leq x} \frac{\alpha(n)}{n} \left[ \frac{x}{n} \right].$$

It is clear that the function $F_1(x, F)$ satisfies the condition (2.10).

Next we prove Theorem 2.1 for $x$ which is a positive integer. Let a function $F_1(x, F)$ be the solution of the equation (2.2) satisfying the condition (2.10). Since $x$ is a positive integer, $E_2(x, F) = \frac{1}{2}(E(x + 0, F) + E(x - 0, F))$. By continuity of $R(x, F)$ for all positive $x$ and (2.4), we obtain

$$F_1(x, F) - \int_0^x F_1(t, F) \frac{dt}{t} = \frac{1}{2}(E(x + 0, F) + E(x - 0, F))$$

(4.2)

$$= R(x, F) + xf_1(x, F).$$

Using (3.7) and Lemma 3.4, we see that the function (2.9) is the solution of (2.2). Conversely, if we assume that $F_1(x, F)$ is a function of type (2.9). By substituting it into the left hand side of (2.2), we have

$$F_1(x, F) - \int_0^x F_1(t, F) \frac{dt}{t} = f_1(x, F) - \int_0^x f_1(t, F) dt.$$  

Using (2.2),

(4.3)  

$$E(x + 0, F) = F_1(x + 0, F) - \int_0^x F_1(t, F) \frac{dt}{t},$$

(4.4)  

$$E(x - 0, F) = F_1(x - 0, F) - \int_0^x F_1(t, F) \frac{dt}{t}.$$  

By (4.2), (4.3), (2.9) and (2.4), we have

$$\frac{1}{2}(E(x + 0, F) + E(x - 0, F)) = f_1(x, F) - \int_0^x f_1(t, F) dt.$$  

Since $x$ is a positive integer, the left hand side corresponds to $E_2(x, F)$. Therefore, Theorem 2.1 also holds for all positive integer $x$. $\square$

Let us prove Theorem 2.2. By lemma 3.3 it is enough to show that for $x \geq 1$ we have

(4.5)  

$$\int_0^x f_1(t, F) dt = -\frac{1}{2}g_1(x, F).$$
This can be done as follows: Recalling Lemma 3.3, we have

\[(4.6) \quad R(x, F) = -\frac{x}{2} \sum_{n=1}^{\infty} \frac{\alpha(n)}{n} \chi + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\alpha(n)}{n} x^2 - \frac{1}{2} \sum_{n \leq s} \frac{\alpha(n)}{n} \int_{t}^{\infty} \left( \frac{t}{n} - \left\{ \frac{t}{n} \right\} \right) dt.\]

For \(x > 0\),

\[
\int_{0}^{x} |t| dt = \frac{1}{2} x^2 + \frac{1}{2} x
\]

(see [2], P2692) and hence we have

\[
(4.7) \quad \int_{n}^{\infty} \left( \frac{t}{n} \right) dt = \frac{n}{2} \left( \left\{ \frac{x}{n} \right\}^2 + \left[ \frac{x}{n} \right] - 1 \right).
\]

By substituting it into (4.6), we have

\[
R(x, F) = -\frac{1}{2} \sum_{n=1}^{\infty} \alpha(n) \frac{x}{n} + \frac{1}{2} \sum_{n=1}^{\infty} \alpha(n) \frac{x^2}{n^2} - \frac{1}{2} \sum_{n \leq s} \alpha(n) \left( \frac{x}{n} - \left\{ \frac{x}{n} \right\} - 1 \right) \left\{ \frac{x}{n} \right\}
\]

\[
= -\frac{1}{2} \sum_{n=1}^{\infty} \alpha(n) \frac{x}{n} + \frac{1}{2} \sum_{n=1}^{\infty} \alpha(n) \frac{x^2}{n^2} - \frac{1}{2} \sum_{n=1}^{\infty} \alpha(n) \left( \frac{x}{n} - \left\{ \frac{x}{n} \right\} - 1 \right) \left\{ \frac{x}{n} \right\}
\]

\[
= -\frac{1}{2} \sum_{n=1}^{\infty} \alpha(n) \left( \frac{x}{n} \right) + \frac{1}{2} \sum_{n=1}^{\infty} \alpha(n) \left( \frac{x}{n} \right)^2
\]

\[(4.8) \quad = \frac{1}{2} g_1(x, F).\]

The proof is complete. \(\square\)

Theorems 2.1 and 2.2 are generalizations of Theorems 1.1 and 1.2. This can be seen as follows: If \(F\) is the Dirichlet \(L\)-function \(L(s, \chi)\) in (1.13) and then by (1.14) the associated Euler totient function \(\varphi(n, F)\) corresponds to the twisted Euler \(\varphi\)-function \(\varphi(n, \chi)\). Since the Euler degree of \(L(s, \chi)\) equals to 1, we have \(\gamma(p) = \chi(p)\) in (1.14). By (1.18), we have \(\alpha(n) = \mu(n) \chi(n)\). Therefore we have

\[
(4.9) \quad f_1(x, F) = \sum_{n=1}^{\infty} \frac{\mu(n) \chi(n)}{n} s \left( \frac{x}{n} \right) = f(x, \chi),
\]

\[
(4.10) \quad g_1(x, F) = \sum_{n=1}^{\infty} \mu(n) \chi(n) \left( \frac{x}{n} \right) \left( \left\{ \frac{x}{n} \right\} - 1 \right) = g(x, \chi).
\]

Hence, (4.9) and (4.10) correspond to (1.5) and (1.6) respectively.

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