On Variational Principles for Gravitating Perfect Fluids

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Abstract

The connection is established between two different action principles for perfect fluids in the context of general relativity. For one of these actions, $S$, the fluid four–velocity is expressed as a sum of products of scalar fields and their gradients (the velocity–potential representation). For the other action, $\bar{S}$, the fluid four–velocity is proportional to the totally antisymmetric product of gradients of the fluid Lagrangian coordinates. The relationship between $S$ and $\bar{S}$ is established by expressing $S$ in Hamiltonian form and identifying certain canonical coordinates as ignorable. Elimination of these coordinates and their conjugates yields the action $\bar{S}$. The key step in the analysis is a point canonical transformation in which all tensor fields on space are expressed in terms of the Lagrangian coordinate system supplied by the fluid. The canonical transformation is of interest in its own right. It can be applied to any physical system that includes a material medium described by Lagrangian coordinates. The result is a Hamiltonian description of the system in which the momentum constraint is trivial.
I. INTRODUCTION

Many different variational principles have been devised that describe perfect fluids in the context of general relativity. Some make extensive use of the Lagrangian coordinate system, the system of spacetime coordinates defined by the fluid flow lines and the proper time along those flow lines [1]. The use of special coordinates violates the spirit of general relativity. More importantly, the Lagrangian time coordinate does not, in general, label spacelike hypersurfaces and is not suitable for the development of a Hamiltonian version of the action. Some perfect fluid actions rely on constrained variations in which the field variables are not freely varied [2]. In practice it is unclear whether or not such an action can be expressed in Hamiltonian form. Moreover, the usual relationship between the variational principle and its associated boundary value problem [3] is absent because typically the boundary conditions that follow from the action must be imposed on the parameters of the constrained field variations rather than on the field variables themselves.

Some perfect fluid actions employ a set of spacetime scalar fields as the basic field variables. The fluid four–velocity is then constructed from combinations of these fields and their gradients. This is the velocity–potential or Clebsch potential [4] formalism—for general relativistic perfect fluids this approach was first discussed by Tam and O’Hanlon [5] and by Schutz [6]. (See also Ref. [7].) In these papers the theory of Pfaff forms [8] was used to minimize the number of velocity potentials. Thus, the fluid four–velocity $U_\alpha$ was expressed in terms of four scalar fields, $\varphi, \vartheta, W, \text{and } Z$, as
\[
\mu U_\alpha = -\varphi,_\alpha - s\vartheta,_\alpha + WZ,_{\alpha},
\]
where $\mu$ is the fluid chemical potential and $s$ is the entropy per particle. The velocity potential formalism was further developed in Ref. [9] where it was argued that locally the fluid velocity can be expressed as in Eq. (1.1), but globally the representation (1.1) must allow for potentials that are not single–valued functions on spacelike hypersurfaces. On the other hand, if the velocity potentials $\varphi, \vartheta, W, \text{and } Z$ are required to be single–valued then the representation (1.1) places a restriction on the velocity $U_\alpha$. In particular, it can be shown that the helicity\(^1\) for an isentropic fluid ($s = \text{const}$) necessarily vanishes when the velocity is given by expression (1.1) with single–valued potentials.

The shortcomings of the velocity–potential representation (1.1) are avoided by enlarging the number of potentials and writing [9]
\[
\mu U_\alpha = -\varphi,_\alpha - s\vartheta,_\alpha + W_k Z^k,_{\alpha},
\]
where the index $k$ takes the values 1, 2, 3. Thus there are eight scalar potentials $\varphi, \vartheta, W_k, \text{and } Z^k$. In this case the fields $Z^k$ can be interpreted as Lagrangian coordinates for the fluid. That is, the values of the three fields $Z^k(y)$ serve as a set of labels that specify which fluid flow line passes through the spacetime point $y$. Perfect fluid actions that are based on

\(^1\)Helicity is defined as the integral over a spacelike hypersurface of $-V \wedge dV$, where $V := \mu U$ is the Taub current [10] expressed as a one–form.
the velocity–potential representation (1.2), with $Z^k$ interpreted as Lagrangian coordinates, will be denoted by $S$. There are several such actions corresponding to different functional expressions for the fluid equation of state [9]. These actions have several features in common with yet another class of perfect fluid actions first discussed by Carter [11] and by Kijowski et al. [12]. (See also Ref. [13].) These actions will be denoted by $\bar{S}$. They are distinguished by the fact that they are functionals only of the Lagrangian coordinates $Z^k$. In this case, the four–velocity is constructed from the antisymmetric product of gradients of the Lagrangian coordinates,

$$U^\alpha \sim \epsilon^{\alpha\beta\sigma\rho} Z^1_{,\beta} Z^2_{,\sigma} Z^3_{,\rho},$$

where $\epsilon^{\alpha\beta\sigma\rho}$ is obtained by raising indices on the spacetime volume form $\epsilon^{\alpha\beta\sigma\rho}$.

The results obtained in Ref. [9] suggest that the actions $S$ and $\bar{S}$ are closely related. The first objective of this paper is to establish this relationship.

In brief, the connection between the actions $S$ and $\bar{S}$ is described as follows. Both of the actions $S$ and $\bar{S}$ can be put into standard Hamiltonian form in which the Hamiltonian is a linear combination of contributions to the Hamiltonian and momentum constraints. For $S$, the variables $s$ and $W_k$ are proportional to the momenta conjugate to $\vartheta$ and $Z^k$, respectively, so the phase space variables are $\varphi$, $\vartheta$, $Z^k$, and their canonical conjugates. By a point canonical transformation, the Hamiltonian action $S$ can be expressed in terms of a new set of canonical variables. The new canonical coordinates $\varphi$ and $\vartheta$, that in an appropriate sense replace $\varphi$ and $\vartheta$, are ignorable (cyclic) and can be eliminated along with their canonical conjugates by Routh’s procedure [3]. The remaining variables, after application of the inverse canonical transformation, are the Lagrangian coordinates $Z^k$ and their conjugates. The resulting action is the Hamiltonian form of $\bar{S}$.

The key step in the analysis is the point canonical transformation. This transformation is described in terms of a mapping $Z : \Sigma \rightarrow \mathcal{S}$ from the space manifold $\Sigma$ to the abstract “fluid space” manifold $\mathcal{S}$ whose points $\zeta \in \mathcal{S}$ are the individual fluid flow lines [12]. The canonical variables $Z^k(x)$ collectively express the mapping $Z$ in terms of local coordinates $x^a$ on $\Sigma$ and $\zeta^k$ on $\mathcal{S}$:

$$Z : \Sigma \rightarrow \mathcal{S} \quad \text{by} \quad x^a \mapsto \zeta^k = Z^k(x).$$

Under the canonical transformation, the canonical coordinate $Z^k(x)$ is replaced by its inverse $X^a(\zeta)$, which is the local coordinate expression of the inverse mapping $X := Z^{-1}$. Whereas $\zeta^k = Z^k(x)$ specifies which flow line passes through the space point $x^a$, $x^a = X^a(\zeta)$ specifies the spatial location of the flow line $\zeta^k$.

Under the canonical transformation all other tensor fields are mapped from space $\Sigma$ to the fluid space $\mathcal{S}$ by $Z$ and its inverse $X$. Accordingly, the original set of canonical variables are referred to as the $\Sigma$–variables, while the new set of canonical variables are referred to as the $\mathcal{S}$–variables. In particular, the $\Sigma$–variables $\varphi(x)$ and $\vartheta(x)$ are mapped into the $\mathcal{S}$–variables $\varphi(\zeta) := \varphi(X(\zeta))$ and $\vartheta(\zeta) := \vartheta(X(\zeta))$. Their canonical conjugates $\Pi_\varphi(\zeta)$ and $\Pi_\vartheta(\zeta)$ are the fluid particle number per unit coordinate cell $d^3\zeta$ and the fluid entropy per unit coordinate cell $d^3\zeta$. Since the particle number and entropy do not change within a flow tube (defined by a neighborhood of a point $\zeta$ in the fluid space), the momenta $\Pi_\varphi(\zeta)$
and $\Pi_\varphi(\zeta)$ are conserved. In the variational principle the conservation of $\Pi_\varphi$ and $\Pi_\theta$ is a consequence of the ignorable property of $\varphi$ and $\theta$.

The second objective of this paper is to present the details of the canonical transformation just described. This transformation can be applied to any system that includes a material medium described by Lagrangian coordinates $Z^k$. The result is a canonical formulation of the system in which the variable $X^a(\zeta)$ and its conjugate $P_a(\zeta)$ are the only ones that depend on the space manifold $\Sigma$ for their definitions. Consequently the momentum constraint, which is the canonical generator of spatial diffeomorphisms, depends only on the variables $X^a$ and $P_a$. Thus the momentum constraint expressed in the canonical $\mathcal{S}$–variables is relatively trivial. This result, expressed in a slightly different form, was used in Ref. [14] to simplify the formal Dirac quantization of gravity coupled to a pressureless perfect fluid (dust). In particular, the operator condition on the quantum state functional that follows from the classical momentum constraint is trivially solvable. The canonical transformation between $\Sigma$–variables and $\mathcal{S}$–variables is also utilized in Ref. [15], where the action for an elastic medium of clocks [16] is analyzed.

Two perfect fluid actions of the velocity–potential type [9] are presented in Section 2a. For one of these actions, the equation of state is specified by giving $\rho(n, s)$, the fluid energy density $\rho$ as a function of particle number density $n$ and entropy per particle $s$. For the other action, the equation of state is specified by giving $p(\mu, s)$, the fluid pressure $p$ as a function of chemical potential $\mu$ and entropy per particle $s$. Symmetries of these actions that preserve the form of the velocity–potential expression (1.2) are presented. The Hamiltonian form of the action with $\rho(n, s)$ is given in Ref. [9], while the Hamiltonian form of the action with $p(\mu, s)$ is derived in Appendix A. These results are merged in Section 2b. The result is an action $\bar{S}$ whose associated Hamiltonian is implicitly defined through the solution to an auxiliary equation that relates the Eulerian and Lagrangian particle number densities. The distinction between the choice of $\rho(n, s)$ or $p(\mu, s)$ as the equation of state arises through the choice of $n$ or $\mu$ as the independent variable in the auxiliary equation. Conserved Noether charges associated with the symmetries of the action are displayed. The discussion in Section 2c addresses the results obtained from an equation of state in which $n$ and temperature $T$ (or $\mu$ and $nT$) are the independent thermodynamical variables.

The canonical transformation from the $\Sigma$–variables to the $\mathcal{S}$–variables is presented in Section 3a, and applied to the perfect fluid and gravitational actions in Section 3b. A key mathematical derivation is presented in Appendix B. The momentum constraint and its role as the generator of space $\Sigma$ diffeomorphisms is discussed in Section 3c. The conserved charge that generates fluid space $\mathcal{S}$ diffeomorphisms is discussed in Section 3d.

In Section 4a the canonical coordinates $\varphi$ and $\vartheta$ are identified as ignorable variables and the pairs $\varphi, \Pi_\varphi$ and $\vartheta, \Pi_\vartheta$ are eliminated by a straightforward application of Routh’s procedure. The resulting action is denoted $\bar{S}$. The symmetries and conserved charges of $\bar{S}$ are discussed. In Section 4b the canonical transformation is reversed for the remaining variables yielding an expression for the action in terms of $\Sigma$–variables. The Lagrangian form of $\bar{S}$ is obtained in Section 4c for the equation of state $\rho(n, s)$. The fluid four–velocity is expressed as the combination (1.3) of gradients of the Lagrangian coordinates. In Section 4d a hybrid action is constructed by eliminating the ignorable pair $\varphi, \Pi_\varphi$, and retaining the pair $\vartheta, \Pi_\vartheta$ as dynamical variables. This yields the action discussed in Ref. [12].
II. PERFECT FLUID ACTION WITH VELOCITY–POTENTIALS

A. Lagrangian Actions

The local expression of the first law of thermodynamics can be written as

\[ d\rho = \mu \, dn + nT \, ds \ , \]  

(2.1)

where \( \rho \) is the energy density, \( n \) is the number density, \( s \) is the entropy per particle, \( \mu \) is the chemical potential, and \( T \) is the temperature. Pressure is defined by \( p := n\mu - \rho \). These quantities characterize the thermodynamical state of a fluid in the fluid rest frame.

Equation (2.1) shows that the fluid equation of state can be specified by the function \( \rho(n, s) \). An action principle of the velocity–potential type that describes a relativistic perfect fluid with equation of state \( \rho(n, s) \) is [9]

\[ S_F[J^\alpha, \varphi, \vartheta, s, W_k, Z^k; \gamma_{\alpha\beta}] = \int d^4y \left\{ -\sqrt{-\gamma} \rho(\sqrt{-\gamma} J^\alpha + s \vartheta_{,\alpha} - W_k Z^k_{,\alpha}) \right\} \ . \]  

(2.2)

Here, \( \gamma_{\alpha\beta} \) is the spacetime metric and the magnitude of \( J^\alpha \) is denoted by \( |J| := \sqrt{-J^\alpha \gamma_{\alpha\beta} J^\beta} \).

The variable \( J^\alpha \) is interpreted as the densitized particle number flux vector; that is, the fluid four–velocity is defined by

\[ U^\alpha := \frac{J^\alpha}{n \sqrt{-\gamma}} \]  

(2.3)

where \( n := |J| / \sqrt{-\gamma} \) is the number density. The equation of motion obtained by varying the action with respect to \( J^\alpha \) is

\[ \mu U_\alpha = -\varphi_{,\alpha} - s \vartheta_{,\alpha} + W_k Z^k_{,\alpha} \ . \]  

(2.4)

This is the velocity–potential expression (1.2). The potential \( \vartheta \) that appears here is the “thermasy” [17]. The equation of motion that follows from varying \( s \), along with the first law (2.1), show that the gradient of \( \vartheta \) along the fluid flow lines is the temperature:

\[ \vartheta_{,\alpha} U^\alpha = T \ . \]  

(2.5)

The equations of motion obtained by varying \( W_k \) imply

\[ Z^k_{,\alpha} U^\alpha = 0 \ , \]  

(2.6)

so that the Lagrangian coordinates \( Z^k \) are indeed constant along the fluid flow lines. From these results it follows that the gradient of the variable \( \varphi \) along the flow lines is given by

\[ \varphi_{,\alpha} U^\alpha = \mu - Ts \ , \]  

(2.7)

which is the chemical free energy of the fluid.
The stress–energy–momentum tensor obtained from the action (2.2) has the form appropriate for a relativistic perfect fluid. Moreover, it is shown in Ref. [9] that the equations of motion that follow from this action correctly encode the familiar perfect fluid equations, namely, the conservation of particle number and entropy per particle along the fluid flow lines, and the Euler equation relating the fluid acceleration to the gradient of pressure.

The first law of thermodynamics also can be written in the form

\[
\frac{dp}{n d\mu - nT ds} = \frac{n d\mu}{n d\mu - nT ds} = \frac{n T ds}{n d\mu - nT ds},
\]

with the energy density defined by \(\rho := n\mu - p\). This expression shows that the fluid equation of state can be specified by the function \(p(\mu, s)\). An action principle of the velocity–potential type that describes a relativistic perfect fluid with equation of state \(p(\mu, s)\) is [9]

\[
S_F[V^\alpha, \varphi, \vartheta, s, W_k, Z^k; \gamma_{\alpha\beta}] = \int d^4y \sqrt{-\gamma}\left\{p(|V|, s) - \frac{\partial p(|V|, s)}{\partial |V|} \left[|V| - \frac{V^\alpha}{|V|}(\varphi, s + s\vartheta, s - W_k Z^k_s, s)\right]\right\}.
\]

The variable \(V^\alpha\) is interpreted as the Taub current [10], so the fluid four–velocity is defined by

\[
U^\alpha := \frac{V^\alpha}{\mu}
\]

where \(\mu := |V|\) is the chemical potential. The equation of motion obtained by varying \(V^\alpha\) yields the velocity–potential expression

\[
V_\alpha = -\varphi, s - s\vartheta, s + W_k Z^k_s, s,
\]

which coincides with Eq. (2.4). The velocity potentials that appear in the action (2.9) have the same physical interpretation as those that appear in the action (2.2).

As shown in Ref. [9], the action (2.9) yields the correct stress–energy–momentum tensor and the correct perfect fluid equations of motion. A simpler version of this action is obtained by replacing \(V^\alpha\) with the solution (2.11) of the \(V^\alpha\) equation of motion. This yields \(S^F = \int d^4y \sqrt{-\gamma}p(|V|, s)\) where \(V^\alpha\) is shorthand notation for the right–hand side of Eq. (2.11). Both versions lead to the same action in Hamiltonian form; the derivation in Appendix A uses the action in Eq. (2.9) as a starting point.

The actions (2.2) and (2.9) are invariant under certain symmetry transformations that preserve the form of the velocity potential expression on the right–hand sides of Eqs. (2.4), (2.11). These symmetries include [9,14] invariance with respect to a change of Lagrangian coordinate labels,

\[
Z^k = \Xi^k(Z'), \quad W'_k = \frac{\partial \Xi^\ell(Z')}{\partial Z^k} W_\ell,
\]

invariance with respect to a deformation or “tilt” of the constant–\(\varphi\) surfaces,

\[
\varphi' = \varphi + \Phi(Z), \quad W'_k = W_k + \frac{\partial \Phi(Z)}{\partial Z^k},
\]

which are described by the variables \(\Phi(Z)\) and \(\Xi^k(Z')\).
and invariance with respect to a deformation of the constant–\(\vartheta\) surfaces,

\[
\vartheta' = \vartheta + \Theta(Z), \quad W_k' = W_k + s \frac{\partial \Theta(Z)}{\partial Z^k}.
\] (2.12c)

The invariances (2.12a–c) give rise to conserved Noether currents and charges. For the invariance (2.12a), the Noether current is related to the fluid momentum in the directions orthogonal to \(\varphi_a + s\vartheta_a\) [14]. The Noether currents associated with the invariances (2.12b) and (2.12c) are expressions of the conserved particle number current and the conserved entropy current [9].

**B. Hamiltonian Action**

The Hamiltonian form of the action (2.2) is derived in Ref. [9], and the Hamiltonian form of the action (2.9) is derived in Appendix A. These results are summarized as follows. The Hamiltonian action is

\[
S^F[\varphi, \Pi, \vartheta, \Pi_\vartheta, Z^k, P_k; g_{ab}, N^\perp, N^a] = \int_{\mathcal{I}} \int_{\Sigma} dt d^3x \left( \Pi_\varphi \dot{\varphi} + \Pi_{\vartheta} \dot{\vartheta} + P_k \dot{Z}^k - N^\perp H_F^\perp - N^a H_F^a \right),
\] (2.13)

where \(g_{ab}\) is the metric on space \(\Sigma\) and \(N^\perp\) and \(N^a\) are the lapse function and shift vector, respectively. The fluid contributions to the momentum and Hamiltonian constraints are

\[
H_F^a = \Pi_\varphi \varphi_a + \Pi_{\vartheta} \vartheta_a + P_k Z^k_a,
\] (2.14a)

\[
H_F^\perp = \sqrt{\mu^2 \Pi_\varphi^2 + H_F^a g^{ab} H_F^b} - \sqrt{g} \rho.
\] (2.14b)

In the expression for \(H_F^\perp\), \(\mu\) and \(p\) are determined as functions of the canonical variables as follows. First, choose an equation of state, either \(\rho(n, s)\) or \(p(\mu, s)\). For the case \(\rho(n, s)\), define

\[
\mu(n, s) := \frac{\partial \rho(n, s)}{\partial n}, \quad p(n, s) := n \frac{\partial \rho(n, s)}{\partial n} - \rho(n, s);
\] (2.15a)

for the case \(p(\mu, s)\), define

\[
n(\mu, s) := \frac{\partial p(\mu, s)}{\partial \mu}.
\] (2.15b)

Next solve the equation

\[
\frac{\Pi_\varphi}{\sqrt{g}} = n \sqrt{1 + \frac{H_F^a g^{ab} H_F^b}{\mu^2 \Pi_\varphi^2}},
\] (2.16)

for either \(n\) or \(\mu\) (depending on the equation of state used) as a function of \(\Pi_\varphi, H_F^a, g_{ab}\), and \(s\). Now make the identification
\[
\begin{align*}
\text{s} := \frac{\Pi_\varphi}{\Pi_\vartheta} \tag{2.17}
\end{align*}
\]

for the entropy per particle \(s\). In this way either \(n\) or \(\mu\) (depending on the equation of state used) is determined as a function of \(\Pi_\varphi, \Pi_\vartheta, H_a^F\), and \(g_{ab}\). This result along with the equation of state and the definitions (2.15a) or (2.15b) determine \(\mu\) and \(p\), and subsequently \(H_a^F\), as functions of \(\Pi_\varphi, \Pi_\vartheta, H_a^F\), and \(g_{ab}\).

The momentum \(\Pi_\varphi\) is the particle number on \(\Sigma\) per unit coordinate cell \(d^3x\). The auxiliary equation (2.16) relates the Eulerian number density \(\Pi_\varphi/\sqrt{g}\) (the number density per unit proper volume) to the Lagrangian number density \(n\). The square root factor in Eq. (2.16) is just the relativistic gamma factor \(\sqrt{1 + U_a g_{ab} U_b}\), where \(U_a = -H_a^F/(\mu \Pi_\varphi)\) are the spatial components of the fluid four-velocity.

The Noether charges associated with the symmetries (2.12a) are
\[
Q[\xi] = -\int_\Sigma d^3x P_k(x) \xi^k(Z(x)) , \quad \xi^k(Z) := \frac{\partial \Xi^k(Z, \sigma)}{\partial \sigma} \bigg|_{\sigma=0} , \tag{2.18}
\]
where \(\Xi(Z, \sigma)\) denotes a one-parameter subgroup of transformations (2.12a). Likewise, the Noether charges associated with the symmetries (2.12b) are
\[
Q[\phi] = \int_\Sigma d^3x \Pi_\varphi(x) \phi(Z(x)) , \quad \phi(Z) := \frac{\partial \Phi(Z, \sigma)}{\partial \sigma} \bigg|_{\sigma=0} , \tag{2.19}
\]
and the charges associated with the symmetries (2.12c) are
\[
Q[\theta] = \int_\Sigma d^3x \Pi_\vartheta(x) \theta(Z(x)) , \quad \theta(Z) := \frac{\partial \Theta(Z, \sigma)}{\partial \sigma} \bigg|_{\sigma=0} . \tag{2.20}
\]
The charges (2.18–20) generate the infinitesimal versions of the symmetries (2.12) through the Poisson brackets. (The derivation of the Hamiltonian action shows that \(W_k = -P_k/\Pi_\varphi\).)

In particular, the charge (2.18) is the canonical generator of \(\text{Diff} S\), fluid space diffeomorphisms, where the fluid space vector field \(\xi^k(\zeta)\) is an element of the Lie algebra of \(\text{Diff} S\).

\section*{C. Other Equations of State}

For the sake of completeness, it is appropriate to finish this section with a comment on other possible expressions for the equation of state. Observe that the first law as expressed in Eq. (2.8) is obtained from the first law (2.1) by a Legendre transformation in \(n\) and \(\mu\). A Legendre transformation can be performed in \(T\) and \(s\) instead, which leads to the first law in the form
\[
d(na) = (\mu - Ts)dn - ns dT . \tag{2.21}
\]
Here \(a = \rho/n - Ts\) is the physical free energy. This result suggests the use of an equation of state of the form \(a(n, T)\). The perfect fluid action with this equation of state is obtained.
by replacing \( \rho(n, s) \) in Eq. (2.2) with \( na(n, T) + nTs \). The temperature \( T \) is treated as a new variable; the equation of motion obtained by varying \( T \) reproduces the definition of \( s \) from the first law, namely,

\[
s = -\frac{\partial a(n, T)}{\partial T}.
\] (2.22)

The Hamiltonian form of the resulting action coincides with the Hamiltonian action (2.13) with \( na(n, T) + nTs \) replacing \( \rho(n, s) \). However, this action is not truly in Hamiltonian form, since the extra variable \( T \) appears. \( T \) can be eliminated by inverting Eq. (2.22) for \( T(n, s) \) and substituting the result back into the Hamiltonian. Through this process the equation of state \( a(n, T) \) is replaced by the equation of state \( \rho(n, s) \), and the Hamiltonian action is converted back into the action (2.13) with equation of state \( \rho(n, s) \). In a similar manner, a Legendre transformation in \( nT \) and \( s \) can be applied to the first law (2.8), leading to an equation of state in which \( p + nTs \) is given as a function of \( \mu \) and \( nT \). The corresponding action can be obtained from Eq. (2.9). In passing to the Hamiltonian form of this action the variable \( nT \) is eliminated, resulting in the action (2.13) with equation of state \( p(\mu, s) \).

III. CANONICAL S–VARIABLES

A. The Canonical Transformation

The main points of this section are best illustrated with the perfect fluid coupled to other fields. Thus, consider a perfect fluid coupled to the gravitational field. The action for the combined system is obtained by adding the gravitational action to the fluid action \( S^F \). The result, in Hamiltonian form, is

\[
S[\varphi, \Pi_\varphi, \theta, \Pi_\theta, Z^k, P_k, g_{ab}, p^{ab}, N^\perp, N^a] = \int_{\mathbb{R}} dt \int \Sigma d^3x \left( \Pi_\varphi \dot{\varphi} + \Pi_\theta \dot{\theta} + P_k \dot{Z}^k + p^{ab} \dot{g}_{ab} - N^\perp H^\perp - N^a H_a \right),
\] (3.1)

where \( p^{ab} \) is the momentum conjugate to the metric \( g_{ab} \). The Hamiltonian and momentum constraints are

\[
H^\perp = H^G + H^F, \\
H_a = H^G_a + H^F_a,
\] (3.2a)

where \( H^F \) and \( H^G \) are the fluid contributions (2.14) and \( H^G \) and \( H^G_a \) are the familiar gravitational field contributions.

\(^2\)One can attempt to retain \( T \) as a canonical variable by defining its conjugate momentum \( P_T \). However, this leads to a primary constraint \( P_T = 0 \) and a secondary constraint which is Eq. (2.22) written in terms of the canonical variables. This pair of constraints is second class, and its elimination is equivalent to the result obtained by solving Eq. (2.22) for \( T \) and substituting the solution back into the Hamiltonian.
The canonical variables that appear in the action (3.1) are the pairs $Z_k(x)$ and $P_k(x)$, $\varphi(x)$ and $\Pi_\varphi(x)$, $\vartheta(x)$ and $\Pi_\vartheta(x)$, and $g_{ab}(x)$ and $p_{ab}(x)$. These are the canonical $\Sigma$–variables. Note that $\varphi(x)$, $\Pi_\varphi(x)$, $\vartheta(x)$, $\Pi_\vartheta(x)$, $g_{ab}(x)$, and $p_{ab}(x)$ are (time $t$–dependent) tensor fields on $\Sigma$. The variable $Z_k(x)$ is the local coordinate chart expression of a ($t$–dependent) mapping $Z : \Sigma \rightarrow \mathcal{S}$ from the space manifold $\Sigma$ to the fluid space $\mathcal{S}$. Its conjugate $P_k(x)$ is the coordinate expression of a ($t$–dependent) mapping from $\Sigma$ to $T^* \mathcal{S}$, with density weight 1 on $\Sigma$. In this section an alternative set of canonical variables, the $\mathcal{S}$–variables, is described. The $\mathcal{S}$–variables utilize the fact that on $\Sigma$ the fluid particles naturally define a preferred set of spatial coordinates.

First, introduce the $t$–dependent mapping $X$, which is the inverse of $Z$:

$$X : \mathcal{S} \rightarrow \Sigma \ , \quad X := Z^{-1} .$$

The mapping $X$ specifies the spatial coordinates of the fluid flow lines; that is, $x^a = X^a(\zeta)$ is the spatial location of the fluid flow line whose Lagrangian coordinate label is $\zeta^k$. The mappings $Z$ and $X$ induce mappings of tensor fields from $\Sigma$ to $\mathcal{S}$. In particular, for the scalars $\varphi$, $\vartheta$ and the covariant metric tensor $g_{ab}$, the corresponding tensors on $\mathcal{S}$ are the pullbacks of those fields by $X$:

$$\varphi := X^* \varphi = \varphi \circ X ,$$

$$\vartheta := X^* \vartheta = \vartheta \circ X ,$$

$$g := X^* g .$$

In local coordinates, these definitions become

$$\varphi(\zeta) = \varphi(X(\zeta)) ,$$

$$\vartheta(\zeta) = \vartheta(X(\zeta)) ,$$

$$g_{k\ell}(\zeta) = X^a, k(\zeta) X^b, \ell(\zeta) g_{ab}(X(\zeta)) .$$

The variables $g_{k\ell}(\zeta)$ are the components of the metric tensor

$$ds^2 = g_{k\ell}(\zeta) d\zeta^k d\zeta^\ell$$

on the fluid space $\mathcal{S}$. That is, $g_{k\ell}(\zeta)$ measures the proper distance $ds$ in space $\Sigma$ between neighboring fluid particles with Lagrangian coordinate labels $\zeta^k$ and $\zeta^k + d\zeta^k$.

The canonical transformation from the $\Sigma$–variables to the $\mathcal{S}$–variables is a point transformation of the canonical coordinates defined by

$$\left( Z^k(x), \varphi(x), \vartheta(x), g_{ab}(x) \right) \mapsto \left( X^a(\zeta), \varphi(\zeta), \vartheta(\zeta), g_{k\ell}(\zeta) \right) .$$

The momenta conjugate to the new coordinates $X^a(\zeta)$, $\varphi(\zeta)$, $\vartheta(\zeta)$ and $g_{k\ell}(\zeta)$ are identified by imposing the invariance of the canonical one–form, or equivalently, the invariance of

$$\int_\Sigma d^3 x \left\{ P_k(x) \dot{Z}^k(x) + \Pi_\varphi(x) \dot{\varphi}(x) + \Pi_\vartheta(x) \dot{\vartheta}(x) + p_{ab}(x) \dot{g}_{ab}(x) \right\} .$$

Note that $x^a$ and $\zeta^k$ play the role of continuous indices that label the different canonical coordinates and momenta.
As mappings between the space manifold $\Sigma$ and the fluid space manifold $S$, $Z^k(x)$ and $X^a(\zeta)$ play a special role in the canonical transformation. Consequently a separate analysis will be presented for the terms $P_k \dot{Z}^k$ in Eq. (3.8). The remaining terms in Eq. (3.8) are constructed from tensor fields on $\Sigma$, and their analysis is presented in Appendix B.

To begin, use the fact that $Z \circ X$ is the identity on $S$ to express $\dot{Z}^k(x)$ in terms of $\dot{X}^a(\zeta)$. In local coordinates, we have

$$Z^k(X(\zeta)) = \zeta^k, \quad \text{(3.9)}$$

and differentiation with respect to $t$ yields

$$\dot{Z}^k(x) \bigg|_{x=X(\zeta)} = -\left(Z_{k,a}(x) \bigg|_{x=X(\zeta)} \right) \dot{X}^a(\zeta). \quad \text{(3.10)}$$

This equation can be written more compactly as $\dot{Z}^k(X(\zeta)) = -Z_{k,a}(X(\zeta)) \dot{X}^a(\zeta)$. Now use this result along with a change of integration variables $x^a = X^a(\zeta)$ in the first term of expression (3.8) to obtain

$$\int_{\Sigma} d^3x P_k(x) \dot{Z}^k(x) = \int_{S} d^3\zeta \left| \frac{\partial X(\zeta)}{\partial \zeta} \right| P_k(X(\zeta)) \dot{Z}^k(X(\zeta)) \quad \text{(3.11a)}$$

$$= -\int_{S} d^3\zeta \left| \frac{\partial X(\zeta)}{\partial \zeta} \right| P_k(X(\zeta)) Z_{k,a}(X(\zeta)) \dot{X}^a(\zeta). \quad \text{(3.11b)}$$

Here, $|\partial X(\zeta)/\partial \zeta|$ is the Jacobian of the change of variables.

The remaining terms in Eq. (3.8) are rewritten by making use of the following result, which is derived in Appendix B. Consider an arbitrary $t$–dependent tensor field $\chi$ on $\Sigma$, and its conjugate tensor density $\pi$. (Indices on $\chi$ and $\pi$ are suppressed, and summation of indices is implied in expressions such as $\pi \chi_a$.) Also let $H_a(x; \pi, \chi)$ denote the contribution from $\chi$ and $\pi$ to the momentum constraint. The key result from Appendix B is

$$\int_{\Sigma} d^3x \pi(x) \dot{\chi}(x) = \int_{S} d^3\zeta \left( \pi(\zeta) \dot{\chi}(\zeta) - H_k(\zeta; \pi, \chi) Z_{k,a}(X(\zeta)) \dot{X}^a(\zeta) \right), \quad \text{(3.12)}$$

where $\chi(\zeta), \pi(\zeta)$, and $H_k(\zeta; \pi, \chi)$ are the tensors (tensor densities) on $S$ that correspond to $\chi(x), \pi(x)$, and $H_a(x; \pi, \chi)$, respectively.

The results (3.11) and (3.12) can be applied to expression (3.8) with the result

---

3 The change of integration variables $x^a = X^a(\zeta)$, which involves the new canonical coordinate $X^a(\zeta)$, is described more precisely as follows. The integrand on the left–hand side of Eq. (3.11a) is a three–form on $\Sigma$, call it $\omega$. Its counterpart on $S$ is the three–form $\omega := X^* \omega$ obtained by the pullback mapping $X^*$. In terms of local coordinates, $\omega_{klm}(\zeta) = X^a_{k,l}(\zeta) X^b_{,m}(\zeta) \omega_{abc}(X(\zeta))$. Equation (3.11a) just expresses the fundamental identity $\int_{\Sigma} \omega = \int_{S} \omega$ in terms of local coordinates. Thus, the change of variables $x^a = X^a(\zeta)$ in Eq. (3.11) arises because the integrand (the three–form) is being mapped from $\Sigma$ to $S$. This is the way in which the continuous index set $x^a$ is changed to the continuous index set $\zeta^k$. 

11
\[
\int \Sigma d^3x \left\{ P_k(x) \dot{Z}^k(x) + \Pi_\varphi(x) \dot{\varphi}(x) + \Pi_\theta(x) \dot{\theta}(x) + p^{ab}(x) \dot{g}_{ab}(x) \right\}
= \int_S d^3\zeta \left\{ -H_k(\zeta)Z^k_{\cdot a}(X(\zeta)) \dot{X}^a(\zeta) + \Pi_\varphi(\zeta) \dot{\varphi}(\zeta) + \Pi_\theta(\zeta) \dot{\theta}(\zeta) + p^{k\ell}(\zeta) \dot{g}_{k\ell}(\zeta) \right\}.
\]  
(3.13)

Here, \( \Pi_\varphi(\zeta), \Pi_\theta(\zeta), \) and \( p^{k\ell}(\zeta) \) are the fluid space tensor densities that correspond to the fields \( \varphi(x), \theta(x), \) and \( p^{ab}(x) \):

\[
\Pi_\varphi(\zeta) := \left| \frac{\partial X(\zeta)}{\partial \zeta} \right| \Pi_\varphi(X(\zeta)) ,
\]  
(3.14a)

\[
\Pi_\theta(\zeta) := \left| \frac{\partial X(\zeta)}{\partial \zeta} \right| \Pi_\theta(X(\zeta)) ,
\]  
(3.14b)

\[
p^{k\ell}(\zeta) := \left| \frac{\partial X(\zeta)}{\partial \zeta} \right| Z^{\cdot k}_{a}(X(\zeta))Z^{\cdot \ell}_{b}(X(\zeta)) p^{ab}(X(\zeta)) .
\]  
(3.14c)

Also, \( H_k(\zeta) \) is the fluid space covariant vector density obtained by mapping the full momentum constraint \( H_a(x) \) from \( \Sigma \) to \( S \):

\[
H_k(\zeta) := \left| \frac{\partial X(\zeta)}{\partial \zeta} \right| X^{a,k}(\zeta) H_a(X(\zeta)) .
\]  
(3.15)

Equation (3.13) shows that the momentum conjugate to \( \varphi(\zeta) \) is \( \Pi_\varphi(\zeta) \), which is the fluid particle number on \( S \) per unit coordinate cell \( d^3\zeta \). Likewise the momentum conjugate to \( \theta(\zeta) \) is \( \Pi_\theta(\zeta) \), the fluid entropy on \( S \) per unit coordinate cell \( d^3\zeta \). The momentum conjugate to \( g_{k\ell}(\zeta) \) is \( p^{k\ell}(\zeta) \), the expression of the gravitational momentum in the fluid coordinate system \( \zeta^k \). Finally, the momentum conjugate to \( X^a(\zeta) \) is defined by

\[
P_a(\zeta) := -H_k(\zeta) Z^{\cdot a}_{\cdot k}(X(\zeta)) = - \left| \frac{\partial X(\zeta)}{\partial \zeta} \right| H_a(X(\zeta)) ,
\]  
(3.16)

and is proportional to the momentum constraint (3.2b) for the system.

**B. The Action as a Functional of \( S \)-Variables**

With the results (3.13)–(3.16), the action (3.1) becomes

\[
S[X^a, P_a, \varphi, \Pi_\varphi, \theta, \Pi_\theta, g_{k\ell}, p^{k\ell}, N^\perp, N^a]
= \int_{\mathbb{R}} dt \int_S d^3\zeta \left\{ P_a(\zeta) \dot{X}^a(\zeta) + \Pi_\varphi(\zeta) \dot{\varphi}(\zeta) + \Pi_\theta(\zeta) \dot{\theta}(\zeta) + p^{k\ell}(\zeta) \dot{g}_{k\ell}(\zeta)
- N^\perp(X(\zeta)) H^\perp(\zeta) + N^a(X(\zeta)) P_a(\zeta) \right\} ,
\]  
(3.17)

where the Hamiltonian constraint (3.2a) is expressed in the fluid space \( S \) as

\[
H^\perp(\zeta) := \left| \frac{\partial X(\zeta)}{\partial \zeta} \right| H^\perp(X(\zeta)) ,
\]

\[
= H^\perp + \mu^2 \Pi_\varphi^2 + H^\perp k g^{k\ell} H^\ell - \sqrt{g} p .
\]  
(3.18)
The gravitational part $H^G(\zeta)$ of the Hamiltonian constraint is obtained from $H^G(x)$ by replacing the gravitational $\Sigma$–variables $g_{ab}(x)$, $\rho^b(x)$ with the corresponding $S$–variables $g_{k\ell}(\zeta)$, $p^{k\ell}(\zeta)$. For the perfect fluid part of the Hamiltonian constraint (3.18), $\mu$ and $p$ are determined as before: First choose an equation of state $\rho(n, s)$ or $\rho(\mu, s)$ and define the relevant thermodynamical variables as in Eqs. (2.15). Next, solve the equation

$$\frac{\Pi_\varphi}{\sqrt{g}} = n \sqrt{1 + \frac{H_{\mu}' g^{k\ell} H_{p}' }{\mu^2 \Pi_\varphi^2}}$$

(3.19)

for $n$ or $\mu$ as a function of $\Pi_\varphi(\zeta)$, $H^F_k(\zeta)$, $g_{k\ell}(\zeta)$, and $s$. With the identification $s := \Pi_\delta/\Pi_\varphi$, then $\mu$ and $p$ are determined as functions of $\Pi_\varphi(\zeta)$, $\Pi_\delta(\zeta)$, $H^F_k(\zeta)$, and $g_{k\ell}(\zeta)$. The fluid space density $H^F_k(\zeta)$ that appears here and explicitly in the Hamiltonian constraint (3.18) is the fluid contribution to the momentum constraint, mapped to $S$. It is expressed as a function of the canonical $S$–variables through the solution of Eq. (3.16) with the identification $H_k := H_k^G + H_k^F$. This yields

$$H^F_k(\zeta) = -P_a(\zeta) X^a,k(\zeta) - H^G_k(\zeta),$$

(3.20)

where $H^G_k(\zeta)$ is the gravitational contribution to the momentum constraint, mapped to $S$. Thus, $H^F_k(\zeta)$ depends only on the gravitational canonical variables $g_{k\ell}(\zeta)$ and $\pi^{k\ell}(\zeta)$.

We can make a further change of variables in the action by defining fluid space tensors that correspond to the lapse function and shift vector:

$$N^\perp(\zeta) := N^\perp(X(\zeta)),$$

(3.21a)

$$N^k(\zeta) := N^a(X(\zeta)) Z^{k,a}(X(\zeta)).$$

(3.21b)

The action then becomes

$$S[X^a, P_a, \varphi, \Pi_\varphi, \delta, \Pi_\delta, g_{k\ell}, p^{k\ell}, N^\perp, N^k]$$

$$= \int_{\mathbb{R}} dt \int_S d^3\zeta \left\{ P_a(\zeta) \dot{X}^a(\zeta) + \Pi_\varphi(\zeta) \dot{\varphi}(\zeta) + \Pi_\delta(\zeta) \dot{\delta}(\zeta) + p^{k\ell}(\zeta) \dot{g}_{k\ell}(\zeta) - N^\perp(\zeta) H^\perp(\zeta) - N^k(\zeta) H_k(\zeta) \right\},$$

(3.22)

where the momentum constraint

$$H_k(\zeta) := -P_a(\zeta) X^a,k(\zeta) = 0$$

(3.23)

from Eq. (3.16) depends only on the fluid coordinates $X^a$ and their conjugates $P_a$.

The Hamiltonian constraint $H_\perp(x)$ that appears in the action (3.1) is smeared with a spatial scalar $N^\perp(x)$. This defines a canonical generator,

$$H[N^\perp] := \int_\Sigma d^3x N^\perp(x) H_\perp(x) = \int_S d^3\zeta N^\perp(X(\zeta)) H_\perp(\zeta),$$

(3.24)

which, as shown here, can be written either in terms of the canonical $\Sigma$–variables or the canonical $S$–variables. On the other hand, the Hamiltonian constraint that appears in the
action (3.22) is smeared with a fluid space scalar \( N^\perp(\zeta) \). The corresponding canonical generator is

\[
H[N^\perp] := \int_{\Sigma} d^3x \, N^\perp(Z) H_\perp(x) = \int_{\Sigma} d^3x \, N^\perp(Z(x)) H_\perp(x) .
\]  

(3.25)

Now, the Poisson bracket of any canonical variable \( F \) with the smeared constraint \( H[N^\perp] \) equals

\[
\{ F, H[N^\perp] \} = \int_{\Sigma} d^3x \, N^\perp(X) \{ F, H_\perp(\zeta) \} + \{ F, N^\perp(X(\zeta)) \} H_\perp(\zeta) 
\]

\[
= \{ F, H[N^\perp] \} + \int_{\Sigma} d^3\zeta \, \{ F, N^\perp(X(\zeta)) \} H_\perp(\zeta) ,
\]  

(3.26)

where \( N^\perp \) and \( N^\perp \) are related as in Eq. (3.21a). This result shows that the brackets \( \{ F, H[N^\perp] \} \) and \( \{ F, H[N^\perp] \} \) are equal modulo the constraints \( H_\perp(x) = 0 = H_\perp(\zeta) \). Thus, on the constraint surface, the Poisson bracket of \( F \) with \( H[N^\perp] \) gives the change in \( F \) under a displacement of the hypersurface \( \Sigma \) by a proper time \( N^\perp(Z(x)) \) in the direction orthogonal to the hypersurface.

**C. Momentum Constraint and \( \text{Diff}_\Sigma \)**

The momentum constraint \( H_a(x) \) smeared with an externally prescribed vector field \( N^a(x) \) on \( \Sigma \) can be written either in terms of the canonical \( \Sigma \)-variables or the canonical \( \mathcal{S} \)-variables:

\[
H[\vec{N}] := \int_{\Sigma} d^3x \, N^a(x) H_a(x) = - \int_{\Sigma} d^3\zeta \, N^a(X(\zeta)) P_a(\zeta) .
\]

(3.27)

\( H[\vec{N}] \) is the generator of \( \text{Diff}_\Sigma \) through the Poisson brackets. That is, for any canonical variable \( F \), the Poisson bracket

\[
\{ F, H[\vec{N}] \}
\]

(3.28)

gives the change in \( F \) due to an infinitesimal diffeomorphism of \( \Sigma \) generated by the vector field \( \vec{N} \). If \( F \) is expressed solely in terms of \( \Sigma \)-variables, then the Poisson bracket is most conveniently computed using the \( \Sigma \)-variables and the first integral in Eq. (3.27). If \( F \) is expressed solely in terms of \( \mathcal{S} \)-variables, then the Poisson bracket is most conveniently computed using the \( \mathcal{S} \)-variables and the second integral in Eq. (3.27). If \( F \) is a tensor field on \( \Sigma \), then the Poisson bracket (3.28) yields the Lie derivative \( \mathcal{L}_{\vec{N}} F \).

Observe that the \( \mathcal{S} \)-variables \( \varphi, \Pi_\varphi, \vartheta, \Pi_\vartheta, g, \) and \( p \) have vanishing Poisson brackets with \( H[\vec{N}] \). This simply reflects the fact that they are tensor fields on the fluid space \( \mathcal{S} \) and are invariant under \( \text{Diff}_\Sigma \). Among the canonical \( \mathcal{S} \)-variables, only \( X^a \) and \( P_a \) have nontrivial transformation properties under \( \text{Diff}_\Sigma \):

\[
\{ X^a(\zeta), H[\vec{N}] \} = - N^a(X(\zeta)) ,
\]

(3.29a)

\[
\{ P_a(\zeta), H[\vec{N}] \} = P_b(\zeta) N^b_{\ a}(X(\zeta)) .
\]

(3.29b)
The transformation (3.29a) can be verified as follows. Let \( \mathcal{N}_\sigma : \Sigma \to \Sigma \) denote the one-parameter family of diffeomorphisms of \( \Sigma \) with generator

\[
\vec{N} := \left. \frac{d\mathcal{N}_\sigma}{d\sigma} \right|_{\sigma = 0}.
\] (3.30)

Also let \( \mathcal{N}_\sigma^* \) generically denote the action of \( \mathcal{N}_\sigma \) on the canonical field variables. For the fields \( Z^k(x) \), viewed as scalars on \( \Sigma \), the action of \( \mathcal{N}_\sigma \) is given by the pullback \( \mathcal{N}_\sigma^* Z^k = Z^k \circ \mathcal{N}_\sigma \). The associated infinitesimal diffeomorphism is then

\[
\left. \frac{d}{d\sigma} \left( \mathcal{N}_\sigma^* Z^k(x) \right) \right|_{\sigma = 0} = Z^k, a(x)N^a(x) = \{Z^k(x), H[\vec{N}]\}.
\] (3.31)

Correspondingly, the action of \( \text{Diff}_\Sigma \) on the mapping \( Z \) is defined by \( \mathcal{N}_\sigma^* Z = Z \circ \mathcal{N}_\sigma \). Since \( X \) is the inverse of \( Z \), the action of \( \text{Diff}_\Sigma \) on \( X \) is given by \( \mathcal{N}_\sigma^* X = N^{-1}_\sigma \circ X \). For the associated infinitesimal diffeomorphism, this yields

\[
\left. \frac{d}{d\sigma} \left( \mathcal{N}_\sigma^* X^a(\zeta) \right) \right|_{\sigma = 0} = -N^a(X(\zeta)) ,
\] (3.32)

which agrees with Eq. (3.29a).

The transformation (3.29b) can be checked by recalling that \( P_a(\zeta) \) is a mapping from \( S \) to \( T^* \Sigma \) with density weight 1 on \( S \). Thus, \( P_a(\zeta)X^a,k(\zeta) \) is a covariant vector density on \( S \) and is invariant under \( \text{Diff}_\Sigma \). Given the transformation (3.29a) for \( X^a(\zeta) \), Eq. (3.29b) is the required result for the vanishing of the Poisson bracket \( \{P_a(\zeta)X^a,k(\zeta), H[\vec{N}]\} \).

The smeared momentum constraint (3.27) appears as a term in the action functionals (3.1) and (3.17). The action (3.22) employs the Lagrange multiplier \( N^k(\zeta) \) and defines a smeared constraint

\[
H[\vec{N}] := \int_S d^3\zeta \quad N^k(\zeta)H_k(\zeta) = \int_\Sigma d^3x \quad N^k(Z(x))X^a,k(Z(x))H_a(x) .
\] (3.33)

For any canonical variable \( F \), the Poisson brackets \( \{F, H[\vec{N}]\} \) and \( \{F, H[\vec{N}]\} \) are equal modulo the constraints \( H_a(x) = 0 = H_k(\zeta) \). Thus, the smeared constraint (3.33) is the canonical generator of \( \text{Diff}_\Sigma \) on the constraint surface \( H_a(x) = 0 = H_k(\zeta) \).

D. Noether Charges

The Noether charges \( Q[\phi] \) and \( Q[\theta] \) from Section 2b can be expressed in terms of the \( \Sigma \)–variables or the \( S \)–variables:

\[
Q[\phi] = \int_\Sigma d^3x \quad \phi(Z(x))\Pi_\phi(x) = \int_S d^3\zeta \quad \phi(\zeta)\Pi_\phi(\zeta) ,
\] (3.34a)

\[
Q[\theta] = \int_\Sigma d^3x \quad \theta(Z(x))\Pi_\theta(x) = \int_S d^3\zeta \quad \theta(\zeta)\Pi_\theta(\zeta) .
\] (3.34b)

\( Q[\phi] \) and \( Q[\theta] \) generate canonical transformations that yield changes in the dynamical variables due to deformations of the hypersurfaces of constant \( \varphi \) and \( \vartheta \), respectively. Among the canonical \( S \)–variables, only \( \varphi \) is affected by \( Q[\phi] \) and only \( \vartheta \) is affected by \( Q[\theta] \).
The Noether charge

\[ Q[\xi] = - \int_{\Sigma} d^3x \xi^k(Z(x))P_k(x) \]  

(3.35)

from Eq. (2.18) can be written in terms of the \( S \)-variables as follows. First, write the momentum constraint as

\[ H_a(x) = P_k(x)Z^k,\alpha(x) + H_a^+(x) , \]  

(3.36)

where \( H_a^+(x) \) contains the contributions from the \( \Sigma \) tensor fields \( \varphi, \Pi, \psi, \Pi_\varphi, g_{ab}, \) and \( p^{ab} \). This expression can be solved for \( P_k(x) \),

\[ P_k(X(\zeta)) = H_a(X(\zeta))X^a,k(\zeta) - H^a_+(X(\zeta))X^a,k(\zeta) , \]  

(3.37)

and the result inserted into Eq. (3.35) along with a change of integration variables \( x^a = X^a(\zeta) \). The first term can be rewritten using Eqs. (3.15) and (3.23), while the second term involves a straightforward mapping of \( \Sigma \)-tensor fields to the fluid space. The result,

\[ Q[\xi] = \int_{\Sigma} d^3\zeta \xi^k(\zeta)\left(P_a(\zeta)X^{a,k}(\zeta) + H^+_k(\zeta)\right) , \]  

(3.38)

can be recognized as the canonical generator of fluid space diffeomorphisms, \( \text{Diff}\mathcal{S} \). That is, for any canonical variable \( F \), the Poisson bracket \( \{F, Q[\xi]\} \) gives the change in \( F \) due to an infinitesimal diffeomorphism of \( \mathcal{S} \) generated by the vector field \( \xi(\zeta) \). If \( F \) is one of the fluid space tensor fields then the Poisson bracket \( \{F, Q[\xi]\} \) equals the Lie derivative \( \mathcal{L}_\xi F \). For the variables \( X^a(\zeta) \) and \( P_a(\zeta) \), the transformation generated by \( Q[\xi] \) yields

\[ \{X^a(\zeta), Q[\xi]\} = \xi^k(\zeta)X^{a,k}(\zeta) , \]  

(3.39a)

\[ \{P_a(\zeta), Q[\xi]\} = \left(\xi^k(\zeta)P_a(\zeta)\right)_k . \]  

(3.39b)

Thus, \( X^a(\zeta) \) transforms as a set of scalar fields on \( \mathcal{S} \) and \( P_a(\zeta) \) transforms as a set of scalar densities on \( \mathcal{S} \).

IV. PERFECT FLUID ACTION \( S \)

A. Routh’s Procedure

Given an equation of state \( \rho(n, s) \) or \( p(\mu, s) \) and the corresponding definitions (2.15), the Hamiltonian constraint \( H_\perp(\zeta) \) of Eq. (3.18) depends explicitly on the canonical \( \mathcal{S} \)-variables \( g_{k\ell}(\zeta), \pi^{k\ell}(\zeta), \) and \( \Pi, \) and also on \( s, H_k^F, \) and \( n \) or \( \mu \) (depending on the equation of state used). The variable \( n \) or \( \mu \) is determined as a function of \( H_k^F, s, \) and the canonical \( \mathcal{S} \)-variables \( g_{k\ell}(\zeta) \) and \( \Pi, \) through the auxiliary equation (3.19). The entropy per particle \( s \) is defined by \( s := \Pi(\zeta)/\Pi(\zeta), \) and \( H_k^F \) is determined as a function of the canonical \( \mathcal{S} \)-variables \( X^a(\zeta), P_a(\zeta), g_{k\ell}(\zeta), \) and \( \pi^{k\ell}(\zeta) \) according to Eq. (3.20). Thus, the Hamiltonian constraint \( H_\perp(\zeta) \) depends on the canonical \( \mathcal{S} \)-variables \( g_{k\ell}(\zeta), \pi^{k\ell}(\zeta), X^a(\zeta), P_a(\zeta), \Pi, \) and \( \Pi(\zeta) \). It does not depend on \( \varphi(\zeta) \) or \( \psi(\zeta) \). Likewise, the momentum constraint
\( H_k(\zeta) \) of Eq. (3.23) does not depend on \( \varphi(\zeta) \) or \( \vartheta(\zeta) \). It follows that the Hamiltonian
\[
\int_S d^3 \zeta \left( N^\perp(\zeta) H^\perp(\zeta) + N^k(\zeta) H_k(\zeta) \right)
\]
for the combined perfect fluid/gravity system does not depend on \( \varphi(\zeta) \) or \( \vartheta(\zeta) \); these variables are therefore ignorable (cyclic).

Since \( \varphi(\zeta) \) and \( \vartheta(\zeta) \) are ignorable, they and their canonical conjugates can be eliminated as dynamical variables through the standard procedure due to Routh [3]. Accordingly, the equations of motion for \( \varphi(\zeta) \) and \( \vartheta(\zeta) \) imply that \( \Pi_\varphi(\zeta) \) and \( \Pi_\vartheta(\zeta) \) are constant in time. Then \( \Pi_\varphi(\zeta) \) and \( \Pi_\vartheta(\zeta) \) can be interpreted as fixed, prescribed fluid space densities, and the total time derivative terms \( \Pi_\varphi \dot{\varphi} \) and \( \Pi_\vartheta \dot{\vartheta} \) can be discarded from the action (3.22). The resulting action is
\[
\bar{S}[X^a, P_a, g_{k\ell}, N^\perp, N^k] = \int_{\mathcal{S}} dt \int_S d^3 \zeta \left\{ P_a(\zeta) \dot{X}^a(\zeta) + p^{k\ell}(\zeta) \dot{g}_{k\ell}(\zeta) - N^\perp(\zeta) H^\perp(\zeta) - N^k(\zeta) H_k(\zeta) \right\} . \tag{4.1}
\]
The Hamiltonian and momentum constraints that appear here are defined by the same expressions (3.18), (3.23) as before, but now \( \Pi_\varphi(\zeta) \) and \( \Pi_\vartheta(\zeta) \) are viewed as fixed densities on the fluid space \( \mathcal{S} \).

The symmetries (2.12b) and (2.12c) of the \( S \) action are absent from the new action \( \bar{S} \), and the corresponding charges (3.34) play no role. On the other hand, the action \( \bar{S} \), like \( S \), is invariant under arbitrary diffeomorphisms of the fluid space \( \mathcal{S} \) as long as \( \Pi_\varphi(\zeta) \) and \( \Pi_\vartheta(\zeta) \) are transformed as densities. In this sense the description of perfect fluids provided by \( \bar{S} \) is independent of the choice of coordinates \( \zeta^k \) on \( \mathcal{S} \). However, this invariance involves a transformation of the fixed quantities \( \Pi_\varphi(\zeta) \) and \( \Pi_\vartheta(\zeta) \) in addition to a transformation of the dynamical variables. Therefore it does not, in general, correspond to a conserved Noether charge. Only the subset of fluid space diffeomorphisms \( \xi \) that satisfy
\[
\mathcal{L}_\xi \Pi_\varphi = 0, \quad \mathcal{L}_\xi \Pi_\vartheta = 0 \tag{4.2}
\]
give rise to a conserved charge [3]. This charge is given by expression (3.38) with the terms proportional to \( \Pi_\varphi(\zeta) \) and \( \Pi_\vartheta(\zeta) \) dropped:
\[
Q[\xi] = \int_{\mathcal{S}} d^3 \zeta \xi^k(\zeta) \left( P_a(\zeta) X^a_{,k}(\zeta) + H^c_k(\zeta) \right) . \tag{4.3}
\]
This charge generates a symmetry of \( \bar{S} \) through the Poisson brackets for any fluid space vector \( \xi \) that leaves the particle number density \( \Pi_\varphi(\zeta) \) and entropy density \( \Pi_\vartheta(\zeta) \) invariant under Lie transport, Eq. (4.2).

**B. \( \Sigma \)–Variables for \( \bar{S} \)**

Now apply the point canonical transformation
\[
(X^a(\zeta), g_{k\ell}(\zeta)) \mapsto (Z^k(x), g_{ab}(x)) \tag{4.4}
\]
to the action (4.1). For the coordinates \( X^a(\zeta), g_{k\ell}(\zeta) \), this is the inverse of the point transformation (3.7) from the canonical \( \Sigma \)–variables to the canonical \( S \)–variables. The term
\[ P_a(\zeta) \dot{X}^a(\zeta) \] in the action can be rewritten using Eq. (3.10). Application of the relationship (3.12) (derived in Appendix B) to the gravitational kinetic term leads to

\[
\int_S d^3 \zeta \mathbf{p}^{k\ell}(\zeta) \dot{g}_{k\ell}(\zeta) = \int \sum d^3 x \left( p^{ab}(x) \dot{g}_{ab}(x) - H_a^c(x) X^a_{,b} (Z(x)) \dot{Z}^b(x) \right). \tag{4.5}
\]

Together these results yield

\[
\int_S d^3 \zeta \left( P_a(\zeta) \dot{X}^a(\zeta) + \mathbf{p}^{k\ell}(\zeta) \dot{g}_{k\ell}(\zeta) \right) = \int \sum d^3 x \left( P_k(x) \dot{Z}^k(x) + p^{ab}(x) \dot{g}_{ab}(x) \right), \tag{4.6}
\]

where the momentum conjugate to \( Z^k(x) \) is

\[
P_k(x) := -\left| \frac{\partial Z(x)}{\partial x} \right| \left( P_a(Z(x)) X^a_{,b} (Z(x)) + H_a^c(Z(x)) \right). \tag{4.7}
\]

Note that the momentum \( P_k(x) \) defined here differs from the momentum (which was given the same notation) \( P_k(x) \) of Section 2b. This is clear from the simple observation that Eq. (3.16), which defines the relationship between \( P_a(\zeta) \) and the “old” \( P_k(x) \), contains the variables \( \varphi, \vartheta \), and their conjugates. These variables do not appear in the relationship (4.7) between \( P_a(\zeta) \) and the “new” \( P_k(x) \). Therefore the new canonical variables \( Z^k(x), P_k(x), g_{ab}(x), \) and \( p^{ab}(x) \) are not simply a subset of the canonical \( \Sigma \)-variables that appear in the Hamiltonian version (3.1) of the velocity–potential action \( S \). Nevertheless, the set \( Z^k(x), P_k(x), g_{ab}(x), \) and \( p^{ab}(x) \) defined here will be referred to as \( \Sigma \)-variables in order to distinguish them from the \( S \)-variables \( X^a(\zeta), P_a(\zeta), g_{k\ell}(\zeta), \) and \( \mathbf{p}^{k\ell}(\zeta) \).

The canonical transformation (4.4) can be accompanied by a change of variables (3.21) in which the Lagrange multipliers \( \mathbf{N}^\perp(\zeta) \) and \( \mathbf{N}^k(\zeta) \) are replaced by \( N^\perp(x) \) and \( N^a(x) \). Then the action (4.1) becomes

\[
\tilde{S}[Z^k, P_k, g_{ab}, p^{ab}, N^\perp, N^k] = \int \sum dt \int d^3 x \left\{ P_k(x) \dot{Z}^k(x) + p^{ab}(x) \dot{g}_{ab}(x) - N^\perp(x) H_\perp(x) - N^a(x) H_a(x) \right\}, \tag{4.8}
\]

where the momentum constraint is

\[
H_a(x) := \left| \frac{\partial Z(x)}{\partial x} \right| H_k(Z(x)) Z^a_{,b} (Z(x)) = P_k(x) Z^k_{,a} (x) + H_a^c(x). \tag{4.9}
\]

The perfect fluid contribution

\[
H_\perp^\perp(x) := \left| \frac{\partial Z(x)}{\partial x} \right| H_\perp(Z(x)) Z^k_{,a} (x) = P_k(x) Z^k_{,a} (x), \tag{4.10}
\]

differs from the previous expression (2.14a) in that the terms involving the coordinates \( \varphi, \vartheta \), and their conjugates are absent.

The Hamiltonian constraint in the action (4.8) can be written in terms of the \( \Sigma \)-variables as

\[
H_\perp(x) := \left| \frac{\partial Z(x)}{\partial x} \right| H_\perp(Z(x)) = H_\perp^c(x) + \sqrt{\mu^2 \Pi_\varphi} + H_\perp^\varphi g^{ab} H_\perp^a - \sqrt{y_p}. \tag{4.11}
\]
Here, $\Pi_\varphi(x)$ is a function of $Z(x)$ defined by
\[
\Pi_\varphi(x) := \left| \frac{\partial Z(x)}{\partial x} \right| \Pi_\varphi(Z(x)),
\]
with $\Pi_\varphi(\zeta)$ a fixed density on $S$. In $H_\perp(x)$, $\mu$ and $p$ are determined as functions of $s$ and either $n$ or $\mu$ by the equation of state, either $\rho(n, s)$ or $p(\mu, s)$, and the associated definitions (2.15). The entropy per particle $s$ is a function of $Z(x)$, $s = s(Z(x))$, as determined by the fixed scalar function
\[
s(\zeta) := \frac{\Pi_\varphi(\zeta)}{\Pi_\varphi(\zeta)}.
\]
on the fluid space $S$. The auxiliary equation
\[
\frac{\Pi_\varphi}{\sqrt{g}} = n \sqrt{1 + \frac{H_a^F g^{ab} H_b^F}{\mu^2 \Pi_\varphi^2}}
\]
determines either $n$ or $\mu$, depending on the equation of state. The auxiliary equation here is formally identical to Eq. (2.16), but now $H_a^F(x)$ and $\Pi_\varphi(x)$ are defined by Eqs. (4.10) and (4.12), respectively. Thus, $n$ or $\mu$ is determined as a function of $g_{ab}(x)$, $P_k(x)$ (which appears in $H_a^F(x)$), and $Z^k(x)$ (which appears in $\Pi_\varphi(x)$, $s(Z(x))$, and $H_a^F(x)$).

With the help of Eq. (4.7) the conserved charge (4.3) can be written in terms of the $\Sigma$-variables as
\[
Q[\vec{\xi}] = -\int_\Sigma d^3x \xi^k(Z(x))P_k(x).
\]
Again, $\vec{\xi}$ must satisfy the conditions (4.2). The symmetry generated by $Q[\vec{\xi}]$ with such a vector $\vec{\xi}$ corresponds to the transformations $\delta Z^k = -\xi^k(Z)$ and $\delta P_k = P_k \xi^{\ell} \delta \xi_k^\ell(Z)$.

C. Lagrangian $\bar{S}$ Action

The Lagrangian form of the action (4.8) is obtained by eliminating the momenta $P_k(x)$ and $p^{ab}(x)$. For the gravitational variables, elimination of $p^{ab}(x)$ yields the usual Einstein–Hilbert action. For the perfect fluid variables, variation of the action with respect to $P_k(x)$ gives the equation of motion
\[
0 = \dot{Z}^k - N^a Z^k_{,a} - \frac{\mu}{\sqrt{g}} \left( \frac{n}{\Pi_\varphi} \frac{\partial \mu}{\partial P_k} - \frac{\partial p}{\partial P_k} + \frac{nH_a^F g^{ab} Z^b_{,b}}{\mu \Pi_\varphi^2} \right),
\]
where the auxiliary equation (4.14) has been used to eliminate the expression $H_a^F g^{ab} H_b^F$. By applying the relationships (2.15) and using the fact that $s = s(Z(x))$ is independent of $P_k(x)$, one finds that the two terms involving $\partial \mu/\partial P_k$ and $\partial p/\partial P_k$ cancel one another. Thus, the $P_k(x)$ equation of motion becomes
\[
\dot{Z}^k - N^a Z^k_{,a} = \frac{\mu}{\sqrt{g}} \frac{H_a^F g^{ab} Z^b_{,b}}{\mu \Pi_\varphi^2}.
\]
With this result, the perfect fluid contribution to the action (4.8) reduces to

\[ \bar{S}^F[Z^k; g_{ab}, N^\perp, N^a] = \int_{\mathcal{B}^1} dt \int_{\Sigma} d^3x \ N^\perp \sqrt{g} (p - n\mu) \ . \] (4.18)

Here, \( n, \mu, \) and \( p \) are determined by the equation of state, the relationships (2.15), and the solution of the auxiliary equation (4.14). In the auxiliary equation, \( P_k(x) \) is replaced by the solution of the equation of motion (4.17).

The action (4.18) can be put in covariant form by interpreting \( t \) and \( x^a \) as spacetime coordinates \( y^0 = t, y^a = x^a \). The spacetime metric \( \gamma_{\alpha\beta} \) is constructed from the spatial metric, lapse, and shift according to the usual ADM decomposition,

\[ \sqrt{-\gamma} = \sqrt{\gamma} \ . \] (4.19)

This implies \( N^\perp \sqrt{g} = \sqrt{-\gamma} \). If the equation of state is specified by the function \( \rho(n, s) \), the relationships (2.15) show that the covariant action \( \bar{S}^F \) becomes

\[ \bar{S}^F[Z^k; \gamma_{\alpha\beta}] = -\int d^4y \sqrt{-\gamma} \rho(n, s) \ . \] (4.20)

In order to provide for a covariant specification of \( n \), first rewrite Eq. (4.12) as

\[ \Pi_\varphi = \sqrt{g} \epsilon^{abc} Z^1_{,a} Z^2_{,b} Z^3_{,c} \Pi_\varphi(Z) = \frac{1}{3!} \sqrt{g} \epsilon^{abc} \eta_{k\ell m}(Z) Z^k_{,a} Z^\ell_{,b} Z^m_{,c} \ , \] (4.21)

where \( \epsilon^{abc} \) is obtained by raising indices on the volume form on \( \Sigma \). Also, \( \eta_{k\ell m} \) are the components of a three–form

\[ \eta = \frac{1}{3!} \eta_{k\ell m} dZ^k \wedge dZ^\ell \wedge dZ^m = \Pi_\varphi d^3Z \] (4.22)

on the fluid space \( S \), so that \( \eta_{123}(Z) = \Pi_\varphi(Z) \) is the fixed density on \( S \). Now consider the spacetime vector density

\[ J^\alpha := -\frac{1}{3!} \sqrt{-\gamma} \epsilon^{\alpha\beta\sigma\rho} \eta_{k\ell m} Z^k_{,\beta} Z^\ell_{,\sigma} Z^m_{,\rho} \ , \] (4.23)

where \( \epsilon^{\alpha\beta\sigma\rho} \) is obtained by raising indices on the spacetime volume form. Note that \( J^\alpha \) is tangent to the fluid flow lines, since it is orthogonal to the gradients of the Lagrangian coordinates. The \( t \) component of \( J^\alpha \) is \( J^t = \Pi_\varphi \), while the spatial components are

\[ J^a = -\frac{1}{2} \sqrt{g} \epsilon^{abc} \eta_{k\ell m} \dot{Z}^k_{,b} Z^\ell_{,c} Z^m_{,c} \ . \] (4.24)

Now, the equation of motion (4.17) for \( P_k(x) \) can be used to obtain the result

\[ J^a + N^a J^t = -\left( \frac{N^\perp \sqrt{g} n}{\mu} \right) g^{ab} H^\varphi_b \ . \] (4.25)
Using the ADM split (4.19), the squared magnitude of $J^\alpha$ becomes

\[ |J|^2 := -J^\alpha \gamma_{\alpha\beta} J^\beta = (N^\perp J^\Gamma)^2 - g_{ab}(J^a + N^a J^t)(J^b + N^b J^t) \]

\[ = (n\sqrt{-\gamma})^2 \] (4.26)

where, again, the auxiliary equation (4.14) has been used. Thus, $J^\alpha$ is the densitized particle number flux vector, $J^\alpha = n\sqrt{-\gamma} U^\alpha$. In the action (4.20), $n$ is the function of $Z^k$ defined by $n := |J|/\sqrt{-\gamma}$, with $J^\alpha$ defined by Eq. (4.23). The action (4.20) is discussed in Ref. [9], where it is confirmed that the variation $\delta \overline{S}_F$ leads to the correct perfect fluid stress–energy–momentum tensor and equations of motion.

The Lagrangian action $\overline{S}$, like its canonical counterpart, is invariant under fluid space diffeomorphisms that leave the particle number density and entropy density invariant. That is, the action $\overline{S}$ is invariant under transformations $\delta Z^k = -\xi^k(Z)$ that satisfy

\[ \mathcal{L}_\xi \eta = 0 \, , \quad \mathcal{L}_\xi s = 0 \] (4.27)

Indeed, with the definitions of $n$ and $s$ as functions of $Z^k$, it follows that

\[ \delta n = -\frac{1}{3!} U^\alpha \epsilon^{\alpha\beta\rho} (\mathcal{L}_\xi \eta_{k\ell m}) Z^k_{\,\beta} Z^\ell_{\,\sigma} Z^m_{\,\rho} \, , \] (4.28a)

\[ \delta s = -\mathcal{L}_\xi s \, . \] (4.28b)

Thus, $\delta n$ and $\delta s$, and in turn $\delta \overline{S}$, vanish for $\xi$ satisfying Eq. (4.27).

If the equation of state is specified by the function $p(\mu, s)$, then the integrand in the action (4.18) can be expressed as the product of $\sqrt{-\gamma}$ and $p(\mu, s) - \mu n(\mu, s)$, where $n(\mu, s) := \partial p(\mu, s)/\partial \mu$. The chemical potential $\mu$ can be defined by the solution of the equation $n(\mu, s) = |J|/\sqrt{-\gamma}$. This gives $\mu(n, s)$, where $n$ is now viewed as shorthand notation for $|J|/\sqrt{-\gamma}$. By substituting $\mu(n, s)$ for $\mu$, one finds that the integrand reduces to the product of $\sqrt{-\gamma}$ and $p(\mu(n, s), s) - \mu(n, s) n(\mu(n, s), s) = -\rho(n, s)$, and the resulting action simply coincides with the action (4.20) described above. Note that the process of solving for $\mu(n, s)$ and substituting the result back into the integrand is equivalent to performing a Legendre transformation in which the equation of state is changed from $p(\mu, s)$ to $\rho(n, s)$. Thus, we reach the obvious conclusion that a Lagrangian $\overline{S}$–type action for a perfect fluid with a given equation of state $p(\mu, s)$ can be obtained by rewriting the equation of state as a function $\rho(n, s)$ and then using the action (4.20).

**D. Hybrid Action $\overline{S}_F[Z^k; \varphi; \gamma_{\alpha\beta}]$**

It is possible, of course, to eliminate just one of the ignorable coordinates, either $\varphi$ or $\vartheta$, from the action $S$ while retaining the other as a dynamical variable. This leads to a hybrid action that is, loosely speaking, half way between the velocity–potential type action $S$ and the action $\overline{S}$ discussed above. In this subsection I will treat the case in which $\varphi$ and its conjugate are eliminated, and $\vartheta$ and its conjugate are retained. This leads to an action of the $\overline{S}$ type in the sense that the fluid four–velocity is expressed in terms of the totally antisymmetric product of gradients of $Z^k$, as in Eq. (1.3).
The Hamiltonian constraint associated with the Lagrange multiplier \( N \) respectively. Associated with the Lagrange multiplier \( \vartheta \) generates a deformation of the constant \( \vartheta \) Eqs. (4.11), (4.12), and (4.14). In this case the entropy per particle, the action becomes
\begin{equation}
\delta Z = \int d^3x \Pi(\vartheta(Z(x))) \Theta(Z(x)) + H^a(Z(x)) ,
\end{equation}
resulting action can be expressed in terms of \( \Sigma \)-variables by applying the point canonical transformation
\begin{equation}
\left( X^a(\vartheta(\varrho)), \vartheta(\varrho), g_{ab}(\varrho) \right) \mapsto \left( Z^k(x), \vartheta(x), g_{ab}(x) \right) .
\end{equation}
The momenta conjugate to \( \vartheta \), \( g_{ab} \), and \( Z^k(x) \) are \( \Pi_\varrho(x) \), \( p^{ab}(x) \), and
\begin{equation}
P_k(x) := -\frac{1}{\varrho} \frac{\partial Z(x)}{\partial x} \left( P(\varrho(Z(x))) X^a_k(Z(x)) + \Pi_\varrho(Z(x)) \vartheta_k(Z(x)) + H^C_k(Z(x)) \right) ,
\end{equation}
respectively. Associated with the Lagrange multiplier \( N^a(x) \) is the momentum constraint
\begin{equation}
H^a(x) = H^a_\varrho(x) + H^a_C(x) \text{ with the perfect fluid contribution}
\end{equation}
The Hamiltonian constraint associated with the Lagrange multiplier \( N^a(x) \) is given by Eqs. (4.11), (4.12), and (4.14). In this case the entropy per particle,
\begin{equation}
s := \frac{\Pi_\varrho(x)}{\Pi_\varphi(x)} ,
\end{equation}
is a function of \( \Pi_\varrho(x) \) and \( Z^k(x) \) (through \( \Pi_\varphi(x) \)).
The symmetries of this action are generated through the Poisson brackets by the conserved charges
\begin{equation}
Q[\Theta] = \int d^3x \Pi_\varrho(x) \Theta(Z(x)) ,
\end{equation}
\begin{equation}
Q[\xi] = \int d^3x P_k(x) \xi^k(Z(x)) ,
\end{equation}
where \( \xi \) leaves the fluid space number density invariant, \( \mathcal{L}_\xi \Pi_\varphi = 0 \). The charge (4.33a) generates a deformation of the constant \( \vartheta \) surfaces, \( \delta \vartheta = \delta \Theta(Z) \), while the charge (4.33b) generates a fluid space diffeomorphism, \( \delta Z^k = -\xi^k(Z) \).

The action can be written in Lagrangian form by eliminating the momenta \( P_k(x) \), \( \Pi_\varrho(x) \), and \( p^{ab}(x) \). Elimination of \( p^{ab}(x) \) leads to the usual Einstein–Hilbert action for the gravitational field. Varying the action with respect to \( P_k(x) \) produces the equation of motion (4.16), which reduces to Eq. (4.17). With this result, the perfect fluid contribution to the action becomes
\begin{equation}
\tilde{S}^F[Z^k, \vartheta, \Pi_\varrho; g_{ab}, N^\perp, N^a] = \int dt \int d^3x \left( \dot{\vartheta} - N^a \vartheta_a - N^\perp \sqrt{g_n} \frac{nH^F_k g^{ab} \vartheta_b}{\mu \Pi^2_\varphi} \right) .
\end{equation}

At this stage it is convenient to make a change of variables in which \( s = \Pi_\varphi/\Pi_\varphi \) replaces \( \Pi_\varrho \) as an independent variable. (Thus, for the moment, the momentum \( \Pi_\varrho \) will not be
eliminated from the action, but will be hidden in the new variable $s$.) Assuming now an
equation of state of the form $\rho(n, s)$, the action is

$$\tilde{S}^F[Z^k, \vartheta, s; g_{ab}, N^\perp, N^\alpha] = \int d^4y \left\{ -N^\perp \sqrt{g} \rho(n, s) + s\Pi_\varphi \left( \dot{\vartheta} - N^\alpha \vartheta, _\alpha - N^\perp \sqrt{g} n H^F_{\alpha} g^{ab} \vartheta, _b \right) \right\} . \quad (4.35)$$

Again, define the vector density $J^\alpha$ of Eq. (4.23). With the result (4.25) it is straightforward
to verify that

$$J^\alpha \vartheta, _\alpha = \Pi_\varphi \left( \dot{\vartheta} - N^\alpha \vartheta, _\alpha - N^\perp \sqrt{g} n H^F_{\alpha} g^{ab} \vartheta, _b \right) , \quad (4.36)$$

so that the action simplifies to

$$\tilde{S}^F[Z^k, \vartheta, s; \gamma_{\alpha\beta}] = \int d^4y \left\{ -\sqrt{-\gamma} \rho(n, s) + sJ^\alpha \vartheta, _\alpha \right\} . \quad (4.37)$$

Here, $n = |J|/\sqrt{-\gamma}$ depends on $Z^k$ through the definition (4.23) for $J^\alpha$.

The entropy per particle $s$ (and hence the momentum $\Pi_\varphi$) can be eliminated from the
action (4.37) by use of the solution of the $s$ equation of motion, namely,

$$0 = -\sqrt{-\gamma} \frac{\partial \rho(n, s)}{\partial s} + J^\alpha \vartheta, _\alpha . \quad (4.38)$$

From the first law Eq. (2.1), we have $\partial \rho/\partial s = nT$ so that $\vartheta$ is related to temperature by
$T = \vartheta, _\alpha U^\alpha$ (Eq. (2.5)). It follows that the solution of the $s$ equation of motion (4.38) has the
form $s(n, T)$, the entropy per particle as a function of number density $n$ and temperature $T$. If this result is substituted into the action (4.37), the integrand becomes the product of
$\sqrt{-\gamma}$ and a function of $n$ and $T$,

$$n a(n, T) := nT s(n, T) - \rho(n, s(n, T)) . \quad (4.39)$$

The function $a(n, T)$ is the physical free energy of the fluid. The action (4.37) therefore
reduces to

$$\tilde{S}^F[Z^k, \vartheta; \gamma_{\alpha\beta}] = -\int d^4y \sqrt{-\gamma} n a(n, T) , \quad (4.40)$$

where $n = |J|/\sqrt{-\gamma}$ and $T = \vartheta, _\alpha J^\alpha /|J|$ with $J^\alpha$ defined as a function of $Z^k$ by Eq. (4.23).

This action is invariant under the symmetry transformations $\delta \vartheta = \theta(Z)$ and $\delta Z^k = -\xi^k(Z)$,
where $\xi$ leaves the fluid space three–form $\eta$ invariant under Lie transport, $\mathcal{L}_\xi \eta = \theta$.

The action (4.40) is the one discussed by Kijowski et al. [12]. The equation of state
is specified by giving the physical free energy $a$ as a function of number density $n$ and
temperature $T$. 

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APPENDIX A: HAMILTONIAN ACTION WITH EQUATION OF STATE \( P(\mu, s) \)

Let \( t \) denote a scalar function on spacetime whose gradient is nonzero and (future pointing) timelike. The \( t = \) constant surfaces foliate spacetime into spacelike hypersurfaces with unit normal \( n_{\alpha} = -N^\perp_{\alpha} \), where \( N^\perp := (-t_{\alpha} \gamma^{\alpha\beta} t_{\beta})^{-1/2} \) defines the lapse function. Also introduce the time vector field \( t^\alpha \) such that \( t_{\alpha} t^\alpha = 1 \) and the projection tensor \( g_{\alpha\beta}^T := \delta_{\alpha\beta} + n_{\alpha} n_{\beta} \) onto the leaves of the foliation. With the shift vector defined by \( N^\alpha := g_{\alpha\beta}^T t^\beta \), the time vector field can be written as \( t^\alpha = N^\perp n^\alpha + N^\alpha \).

Using the above relationships, the vector \( V^\alpha \) can be decomposed as

\[
V^\alpha = (g_{\beta}^\alpha - n_{\alpha} n_{\beta})V^\beta = (gV)^\alpha + (t^\alpha - N^\alpha)V^t ,
\]

where the definitions \( (gV)^\alpha := g_{\alpha\beta}^T V^\beta \) and \( V^t := t_{\alpha} V^\alpha \) are used. It follows that the chemical potential \( |V| \) can be written as

\[
|V| := \sqrt{-V^\alpha \gamma_{\alpha\beta} V^\beta} = \sqrt{(N^\perp V^t)^2 - (gV)^\alpha (gV)^\alpha} .
\]

The action (2.9) then becomes

\[
S^F = \int d^4y \sqrt{-\gamma} \left\{ -\rho(|V|, s) + \frac{n(|V|, s)}{|V|} (V^t t^\alpha + (gV)^\alpha - V^t N^\alpha)(\varphi_{,\alpha} + s \vartheta_{,\alpha} - W_k Z^k_{,\alpha}) \right\} ,
\]

where the definitions \( n(\mu, s) := \partial p(\mu, s) / \partial \mu \) and \( \rho(\mu, s) := \mu n(\mu, s) - p(\mu, s) \) have been used.

Now tie the spacetime coordinates to the foliation by choosing \( t \) as the time coordinate and choosing spatial coordinates such that \( \partial / \partial t \) is the time vector field \( t^\alpha \). The \( t \)-components of \( (gV)^\alpha \) and \( N^\alpha \) vanish, and the action becomes

\[
S^F[(gV)^\alpha, V^t, \varphi, \vartheta, s, W_k, Z^k; N^\perp, N^\alpha, g_{ab}] = \int \mathbb{R}^t \int d^3x \sqrt{-g} \left\{ -\rho(|V|, s) + \frac{n(|V|, s)}{|V|} V^t (\varphi + s \vartheta - W_k \dot{Z}^k) + \frac{n(|V|, s)}{|V|} ((gV)^a - V^t N^a)(\varphi_{,a} + s \vartheta_{,a} - W_k Z^k_{,a}) \right\} .
\]

The form of this expression suggests a change of variables in which \( V^t, s, \) and \( W_k \) are replaced by

\[
\Pi_{\varphi} := N^\perp \sqrt{g} \frac{n(|V|, s)}{|V|} V^t , \quad (A.5a)
\]

\[
\Pi_{\vartheta} := N^\perp \sqrt{g} \frac{n(|V|, s)}{|V|} V^t s , \quad (A.5b)
\]

\[
P_k := -N^\perp \sqrt{g} \frac{n(|V|, s)}{|V|} V^t W_k . \quad (A.5c)
\]

These variables will become the momenta conjugate to \( \varphi, \vartheta, \) and \( Z^k \), respectively.

Now introduce the variables
\[ \xi^a := \frac{(gV)^a}{V^t}, \]  

\text{(A.6)}

which can be viewed as replacements for \((gV)^a\). In terms of the new variables (A.5), (A.6), the action becomes

\[ S^F[\xi^a, \varphi, \Pi_\varphi, \vartheta, \Pi_\vartheta, Z^k, P_k; N^\bot, N^a, g_{ab}] \]
\[ = \int_{\mathbb{R}} dt \int_{\Sigma} d^3x \left\{ \Pi_\varphi \dot{\varphi} + \Pi_{\vartheta} \dot{\vartheta} + P_k \dot{Z}^k - N^\bot \sqrt{g} \rho(|V|, \Pi_\varphi/\Pi_\vartheta) + (\xi^a - N^a) H^F_a \right\}, \]  

\text{(A.7)}

where \(s = \Pi_\vartheta/\Pi_\varphi\) (as follows from Eqs. (A.5a) and (A.5b)) and \(H^F_a\) is the fluid contribution to the momentum constraint,

\[ H^F_a := \Pi_\varphi \varphi_a + \Pi_{\vartheta} \vartheta_a + P_k Z^k_a. \]  

\text{(A.8)}

In the action (A.7) \(|V|\) is determined as follows. Definition (A.6) can be combined with Eq. (A.2) to yield

\[ V^t = \frac{|V|}{\sqrt{(N^\bot)^2 - \xi^a \xi_a}}. \]  

\text{(A.9)}

Then the momentum (A.5a) becomes

\[ \Pi_\varphi = N^\bot \sqrt{g} \frac{n(|V|, \Pi_\vartheta/\Pi_\varphi)}{\sqrt{(N^\bot)^2 - \xi^a \xi_a}}. \]  

\text{(A.10)}

This can be viewed as an equation that determines \(|V|\) as a function of \(\Pi_\varphi, \Pi_\vartheta, \xi^a, N^\bot,\) and \(g_{ab}\).

The next step is to eliminate \(\xi^a\) from the action (A.7) by substituting the solution of the \(\xi^a\) equations of motion. The \(\xi^a\) equations of motion are found by varying \(S\) with respect to \(\xi^a\), keeping the other variables fixed. For this calculation, the variation of \(|V|\) is needed. From Eq. (A.10) this is determined to be

\[ \frac{\partial n(|V|, \Pi_\vartheta/\Pi_\varphi)}{\partial |V|} \delta |V| = - \left( \frac{n(|V|, \Pi_\vartheta/\Pi_\varphi)}{(N^\bot)^2 - \xi^b \xi_b} \right) \xi_a \delta \xi^a. \]  

\text{(A.11)}

Using this result and the relationship \(\partial \rho(\mu, s)/\partial \mu = \mu \partial n(\mu, s)/\partial \mu\) in the variation of the action (A.7), the \(\xi^a\) equations of motion are obtained:

\[ \frac{|V|}{\sqrt{(N^\bot)^2 - \xi^b \xi_b}} \xi_a = - \frac{H^F_a}{\Pi_\varphi}. \]  

\text{(A.12)}

Here, \(|V|\) is considered to be a function of \(\Pi_\varphi, \Pi_\vartheta, \xi^a, N^\bot,\) and \(g_{ab}\) through the solution of Eq. (A.10). Equation (A.12) can be squared and, after a bit of algebra, written as

\[ \frac{N^\bot}{\sqrt{(N^\bot)^2 - \xi^b \xi_b}} = \sqrt{1 + \frac{H^F_a g_{ab} H^F_b}{|V|^2 \Pi_\varphi^2}}. \]  

\text{(A.13)}
Then Eq. (A.10) becomes

\[ \Pi_{\phi} \sqrt{g} = n(|V|, \Pi_{\theta}/\Pi_{\phi}) \sqrt{1 + H_a^F g^{ab} H_b^F / |V|^2 \Pi_{\phi}^2}. \]  

(A.14)

This equation determines \(|V|\) as a function of \(\Pi_{\phi}, \Pi_{\theta}, H^F_a,\) and \(g_{ab}\).

The implicit solution (A.12) of the \(\xi^a\) equations of motion can be inserted into the action (A.7) to eliminate the variable \(\xi^a\). Note that \(\xi^a\) only appears in the fluid contribution to the Hamiltonian constraint; this is

\[ H^F_\perp = \sqrt{g} |V| n(|V|, \Pi_{\theta}/\Pi_{\phi}) - \frac{\xi^a H^a_\perp}{N_\perp} - \sqrt{g} p(|V|, \Pi_{\theta}/\Pi_{\phi}), \]  

(A.15)

where \(\rho(\mu, s) = \mu n(\mu, s) - p(\mu, s)\) has been used. The term involving the number density \(n\) can be rewritten using Eq. (A.10), and the term involving \(\xi^a H^F_a\) can be rewritten by contracting Eq. (A.12) with \(\xi^a\). This leads to the expression

\[ H^F_\perp = \Pi_{\phi} |V| \frac{N_\perp}{\sqrt{(N_\perp)^2 - \xi^b \xi_b}} - \sqrt{g} p(|V|, \Pi_{\theta}/\Pi_{\phi}), \]  

(A.16)

which, when combined with Eq. (A.13), becomes

\[ H^F_\perp = \sqrt{|V|^2 \Pi_{\phi}^2 + H_a^F g^{ab} H_b^F} - \sqrt{g} p(|V|, \Pi_{\theta}/\Pi_{\phi}). \]  

(A.17)

Here, \(|V|\) is determined as a function of \(\Pi_{\phi}, \Pi_{\theta}, H^F_a,\) and \(g_{ab}\) through Eq. (A.14).

With the result (A.17) the action (A.7) is brought to standard Hamiltonian form with contributions (A.17) and (A.8) to the Hamiltonian and momentum constraints. The chemical potential \(|V|\) that appears in \(H^F_\perp\) is determined by the auxiliary equation (A.14). This agrees with the action described in Eqs. (2.13)–(2.17) (for the equation of state \(p(\mu, s)\)) of the main text.

**APPENDIX B: DERIVATION OF EQUATION (3.12)**

In this appendix the parameter dependence of tensors and mappings will be denoted by subscripts. Thus, for example, the one−parameter family of mappings from \(\mathcal{S}\) to \(\Sigma\) is denoted \(X_t\), and the generic one−parameter family of tensors on \(\Sigma\) is denoted \(\chi_t\).

Let \(V_t\) denote the vector field on \(\Sigma\) defined in terms of local coordinates by

\[ V_t^a(x) := \left( \frac{d}{dt} X_t^a(\zeta) \right) |_{\zeta = Z_t(x)}. \]  

(B.1)

For each value of \(t\), the integral curves of \(V_t\) define a one−parameter \(\sigma\) family of diffeomorphisms \(N_{t,\sigma}: \Sigma \to \Sigma\). By definition \(N_{t,\sigma}\) satisfies the differential equation

\[ \frac{dN_{t,\sigma}}{d\sigma} = V_t(N_{t,\sigma}), \]  

(B.2)
where $N_{t,0} = I_\Sigma$ is the identity mapping on $\Sigma$. By combining the previous two equations, we obtain

$$\frac{d}{d\sigma} N_{t,\sigma}^\alpha(x) \bigg|_{\sigma=0} = \left( \frac{d}{dt} X_t^\alpha(\zeta) \right) \bigg|_{\zeta=Z_t(x)} = \frac{d}{d\sigma} X_t^\alpha(Z_t(x)) \bigg|_{\sigma=0}. \quad \text{(B.3)}$$

It readily follows that up to order $\sigma^2$, $N_{t,\sigma}$ is given by

$$N_{t,\sigma} = X_{t+\sigma} \circ Z_t + O(\sigma^2). \quad \text{(B.4)}$$

The $O(\sigma^2)$ corrections to $N_{t,\sigma}$ are not needed in the analysis that follows.

Consider first the case in which $\chi_t$ is a one–parameter family of scalar fields on $\Sigma$. The Lie derivative of $\chi_t$ with respect to $V_t$ is given by

$$\left( \mathcal{L}_{V_t}\chi_t \right) \bigg|_x := \frac{d}{d\sigma} \chi_t(N_{t,\sigma}(x)) \bigg|_{\sigma=0} = \frac{d}{d\sigma} \chi_t(X_{t+\sigma}(\zeta)) \bigg|_{\sigma=0, \ z=Z_t(x)}. \quad \text{(B.5)}$$

In order to simplify this expression, denote the pullback of $\chi_t(x)$ to the fluid space $\mathcal{S}$ by $\chi_t(\zeta) := \chi_t(X_t(\zeta))$ and compute

$$\frac{d}{dt} \chi_t(\zeta) = \frac{d}{d\sigma} X_{t+\sigma}(\zeta) \bigg|_{\sigma=0} \quad = \frac{d}{d\sigma} \chi_t(X_{t+\sigma}(\zeta)) \bigg|_{\sigma=0} \quad = \left( \frac{d}{dt} \chi_t(x) \right) \bigg|_{x=X_t(\zeta)} + \frac{d}{d\sigma} \chi_t(X_{t+\sigma}(\zeta)) \bigg|_{\sigma=0}. \quad \text{(B.6)}$$

Thus, the Lie derivative (B.5) equals

$$\left( \mathcal{L}_{V_t}\chi_t \right) \bigg|_x = -\frac{d}{dt} \chi_t(x) + \left( \frac{d}{dt} \chi_t(\zeta) \right) \bigg|_{\zeta=Z_t(x)}. \quad \text{(B.7)}$$

This result expresses the Lie derivative of $\chi_t$ along $V_t$ in terms of the derivatives of $\chi_t$ and $X_t$ with respect to the parameter $t$.

The result (B.7) can be extended to arbitrary one–parameter families of tensor fields $\chi_t$. For example, consider $\chi_t(x)$ to be a covariant vector and compute

$$\left( \mathcal{L}_{V_t}\chi_t \right) \bigg|_x := \frac{d}{d\sigma} \left( N_{t,\sigma}^\alpha \chi_t(N_{t,\sigma}(x)) \right) \bigg|_{\sigma=0} \quad = Z_t^\alpha \frac{d}{d\sigma} \left( X_{t+\sigma}^\alpha \chi_t(X_{t+\sigma}(\zeta)) \right) \bigg|_{\sigma=0, \ z=Z_t(x)}. \quad \text{(B.8)}$$

Here, Eq. (B.4) has been used to express

$$N_{t,\sigma}^\alpha = Z_t^\alpha \circ X_{t+\sigma} + O(\sigma^2). \quad \text{(B.9)}$$

A calculation similar to that in Eq. (B.6) yields

$$\frac{d}{dt} \chi_t(\zeta) = X_t^\alpha \left( \frac{d}{dt} \chi_t(x) \right) \bigg|_{x=X_t(\zeta)} + \frac{d}{d\sigma} \left( X_{t+\sigma}^\alpha \chi_t(X_{t+\sigma}(\zeta)) \right) \bigg|_{\sigma=0}. \quad \text{(B.10)}$$
for the fluid space covector $\chi_t(\zeta) := X_t^*\chi_t(X_t(\zeta))$. Therefore the Lie derivative of $\chi_t$ is
\[
\left(\mathcal{L}_{V_t}\chi_t\right)\bigg|_x = -\frac{d}{dt}\chi_t(x) + Z_t^*(\frac{d}{dt}\chi_t(\zeta))\bigg|_{\zeta=Z_t(x)} .
\] (B.11)
Likewise, the Lie derivative of a contravariant vector is given by
\[
\left(\mathcal{L}_{V_t}\chi_t\right)\bigg|_x = -\frac{d}{dt}\chi_t(x) + X_t'(\frac{d}{dt}\chi_t(\zeta))\bigg|_{\zeta=Z_t(x)} ,
\] (B.12)
where $X_t'$ is the derivative mapping of contravariant vectors.

Armed with the results (B.7), (B.11), (B.12) and their generalizations to arbitrary tensors, Eq. (3.12) is obtained as follows. For definiteness, consider the case in which $\chi_t(x)$ is a one–parameter family of covariant vectors and $\pi_t(x)$ denotes the canonically conjugate one–parameter family of contravariant vector densities on $\Sigma$. Contract Eq. (B.11) with $\pi_t(x)$ and integrate over $\Sigma$ to obtain
\[
\int_{\Sigma} d^3x \pi_t(x)\frac{d}{dt}\chi_t(x) = \int_{\Sigma} d^3x \left(\pi_t(x)Z_t^*(\frac{d}{dt}\chi_t(\zeta))\bigg|_{\zeta=Z_t(x)} - \pi_t(x)\left(\mathcal{L}_{V_t}\chi_t\right)\bigg|_x\right)
\]
\[
- \int_{\Sigma} d^3x \left(\pi_t(x)Z_t^*(\frac{d}{dt}\chi_t(\zeta))\bigg|_{\zeta=Z_t(x)} - V_t^a(x)H_a(\zeta; \pi_t, \chi_t)\right) .
\] (B.13)
Here, $H_a(x; \pi_t, \chi_t)$ is the contribution from $\chi_t$ and $\pi_t$ to the momentum constraint. (Integration by parts is used to obtain the second line of Eq. (B.13) from the first. It is assumed that the space manifold $\Sigma$ is compact without boundary, so that no boundary terms appear.) Now rewrite the right–hand side of Eq. (B.13) as an integral over the fluid space $S$ by changing integration variables, $x^a = X_t^a(\zeta)$. This gives
\[
\int_{\Sigma} d^3x \pi_t(x)\frac{d}{dt}\chi_t(x) = \int_{S} d^3\zeta \left(\pi_t(\zeta)\frac{d}{dt}\chi_t(\zeta) - V_t^k(\zeta)H_k(\zeta; \pi_t, \chi_t)\right) .
\] (B.14)
The vector field $V_t(\zeta)$ is defined in terms of local coordinates by
\[
V_t^k(\zeta) := \left(V_t^a(x)Z_t^k_{,a}(x)\right)\bigg|_{x=X_t(\zeta)} = \left(\frac{d}{dt}X_t^a(\zeta)\right)\bigg|_{x=X_t(\zeta)} ,
\] (B.15)
where the definition (B.1) has been used. Inserting this expression for $V_t(\zeta)$ into Eq. (B.14) yields Eq. (3.12) from the main text. The derivation for tensors of arbitrary rank follows along similar lines.
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