BREAKING OF 1RSB IN RANDOM MAX-NAE-SAT

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ABSTRACT. For several models of random constraint satisfaction problems, it was conjectured by physicists and later proved that a sharp satisfiability transition occurs. For random $k$-SAT and related models it happens at clause density $\alpha = \alpha_{\text{sat}} \times 2^k$. Just below the threshold, further results suggest that the solution space has a “1RSB” structure of a large bounded number of near-orthogonal clusters inside $\{0,1\}^N$. In particular it is known that $\alpha_{\text{sat}}$, the exact satisfiability threshold, increases between zero and $\alpha_{\text{sat}}$, with high probability the solution space is a nonempty (random) subset of constraints. This is a combinatorial optimization problem on the random energy landscape defined by the problem instance. For a simplified variant, the strong refutation problem, there is strong evidence that an algorithmic transition occurs around $\alpha = N^{1/2-\epsilon}$. For $\alpha$ bounded in $N$, a very precise estimate of the max-sat value was obtained by Achlioptas, Naor, and Peres (2007), but it is not sharp enough to indicate the nature of the energy landscape. Later work (Sen, 2016; Panchenko, 2016) shows that for $\alpha$ very large (roughly, $\Omega(4^k)$) the max-sat value approaches the mean-field (complete graph) limit: this is conjectured to have an “1RSB” structure where near-optimal configurations form clusters within clusters, in an ultrametric hierarchy of infinite depth inside $\{0,1\}^N$. A stronger form of 1RSB was shown in several recent works to have algorithmic implications (again, in complete graphs). Consequently we find it of interest to understand how the model transitions from 1RSB near $\alpha_{\text{sat}}$ to (conjecturally) 1RSB for large $\alpha$. In this paper we show that in the random regular $k$-NAE-SAT model, the 1RSB description breaks down already above $\alpha \times 4^k/k^2$. This is proved by an explicit perturbation in the 2RSB parameter space. The choice of perturbation is inspired by the “bug proliferation” mechanism proposed by physicists (Montanari and Ricci-Tersenghi, 2003; Krzakala, Pagnani, and Weigt, 2004), corresponding roughly to a percolation-like threshold for a subgraph of dependent variables.

1. Introduction

A random constraint satisfaction problem (random csp), broadly construed, is any problem specified by $N$ variables subject to $M$ random constraints. We shall consider a prototypical example, random regular $k$-NAE-SAT, where an instance $\mathcal{G}_N$ involves $N$ binary variables $x_i \in \{0,1\}$, subject to $M = N\alpha$ random constraints such that each constraint involves a subset of $k$ variables (the formal definition is below). In the satisfiable regime $0 \leq \alpha \leq \alpha_{\text{sat}}$, with high probability the solution space is a nonempty (random) subset $S(\mathcal{G}_N) \subseteq \{0,1\}^N$. It is predicted by physicists [KMR’07] to undergo a precise series of sharp structural transitions as $\alpha$ increases between zero and $\alpha_{\text{sat}}$. Several of these predictions have now been supported by rigorous results: for example, we point to works on solution geometry [AR06, AC08, MRT11], the exact satisfiability threshold $\alpha_{\text{sat}}$ [AM02, CP12, DSS16], the number of solutions [SSZ16], and associated inference problems [CKPZ18]. In particular it is known that $\alpha_{\text{sat}} \approx 2^{k-1} \ln 2 – O(1)$.

In this paper we consider the unsatisfiable regime $\alpha > \alpha_{\text{sat}}$, where with high probability the solution space $S(\mathcal{G}_N)$ is empty. It then becomes natural to study the max-satisfiable value (or ground state energy)

$$e_{\text{min}}(\mathcal{G}_N) \equiv \frac{1}{N} \min \left\{ \# \{ \text{constraints violated by} \ x \} : x \in \{0,1\}^N \right\}.$$ 

The computer science literature on the max-satisfiability problem has primarily focused on the regime where $\alpha = \alpha_N$ diverges in $N$. In this case, an easy bound gives $e_{\text{min}}(\mathcal{G}_N) \approx (1 – \sigma_N(1))\alpha_N/2^{k-1}$, which allows for a simple phrasing of the so-called strong refutation problem [Fei02]: is there an efficiently computable bound $e_{\text{alg}}(\mathcal{G}_N) \leq e_{\text{min}}(\mathcal{G}_N)$ (for any $\mathcal{G}_N$) such that $e_{\text{alg}}(\mathcal{G}_N) \approx (1 + \sigma_N(1))\alpha_N/2^{k-1}$ with high probability for random $\mathcal{G}_N$? An efficient (spectral) strong refutation algorithm exists above $\alpha_N \approx N^{k/2-1}$ ([CGL07]), and extended by [AOW15]). On the other hand, within a large family of convex programming algorithms (as defined by the sum-of-squares hierarchy) it has been shown that many problems of this kind are solvable in subexponential but not polynomial time for $1 \ll \alpha_N \ll N^{k/2-1}$ [Gri01, Sch08, RRS17, KMOV17].

In the regime where $\alpha$ does not diverge with $N$, very good bounds on $e_{\text{min}}(\mathcal{G}_N)$ are given by [ANP07], as we will review below. However, the bounds are not quite precise enough to give information about the nature of the energy landscape. More recent results in the spin glass literature [Sen18, Pan18] show that for $\alpha$ very large (roughly,
\( \Omega(64^k) \) the max-sat value approaches the \textit{mean-field} (complete graph) limit, which is given by a Parisi-type variational formula [AC17] (in the physics literature see [LP01, CLP02]). The solution of the mean-field variational formula is conjectured to be “full replica symmetry breaking” (FRSB), e.g., by analogy with the zero-temperature Sherrington–Kirkpatrick model [ACZ17]. A stronger version of FRSB has been shown in several recent works (in mean-field settings) to have algorithmic implications [AM18, Sub18, Mon18]. By contrast, results near the satisfiability threshold [DSS16, SSZ16] are consistent only with “one-step replica symmetry breaking” (1RSB). This is to say that as \( \alpha \) increases from \( \alpha_{\text{sat}} \) to \( \infty \), the model must transition from 1RSB to FRSB; and one may even speculate further on whether the \( N^{k/2-1} \) threshold in the algorithmic literature relates to a transition in the type of FRSB.

In this paper we study a phenomenon which is proposed in the physics literature as the first transition beyond \( \alpha_{\text{sat}} \) in the type of rsn. It is predicted to occur at an explicit value \( \alpha_{\text{Ga}} \) [MR03, KPW04] (termed the \textit{Gardner transition}, after [Gar85]) — by a mechanism of \textit{bug proliferation}, which we describe below. A simple consequence of this prediction is that the ground state energy would coincide with the 1RSB value \( e_{1\text{RSB}} \) up to \( \alpha_{\text{Ga}} \), but not thereafter. Our main result is a rigorous upper bound on this transition:

\textbf{Theorem 1.1.} For all \( k \geq k_0 \) (where \( k_0 \) is an absolute constant), if \( \mathcal{G}_N \) is an instance of random regular \( k \)-\textsc{nae-sat} on \( N \) variables subject to \( N \alpha \) constraints (Definition 1.2), and \( \mathbb{E} \) is expectation over \( \mathcal{G}_N \), then the quantity

\[
\liminf_{N \to \infty} \left\{ \mathbb{E}[e_{\text{min}}(\mathcal{G}_N)] - e_{1\text{RSB}} \right\}
\]

is well-defined and nonnegative for all \( \alpha_{\text{sat}} \leq \alpha \leq 4^k/k \). It is strictly positive for all \( \alpha_{\text{Ga}} \leq \alpha \leq 4^k/k \) where \( \alpha_{\text{Ga}} = 4^k/k \).

\begin{equation}
\alpha_{\text{Ga}} = \frac{\alpha(1 - p_{\text{max}})}{2^k - 1}
\end{equation}

for \( 0 \leq p_{\text{max}} \leq 1 \). The first assertion of the theorem, the nonnegativity of (1), improves on the best previous upper bound on \( p_{\text{max}} \) by a factor \( 1 - \Omega(x) \) where the correction \( \Omega(x) \) reflects the typical sizes of \textit{clusters} of near-max-satisfiable configurations. We give the basic intuition for this correction in §1.2, and show in (12) that in the regime \( 2^k k^2 \ll \alpha \ll 4^k/k \) we expect a correction \( x \geq \Omega(1/d^{1/2}) \). In §2.3 (Corollary 2.10) we state a more precise bound for all \( \alpha_{\text{sat}} \leq \alpha \leq 4^k/k \).

The result relies on an abstract “interpolation bound” proved in [SSZ16], which was adapted from a combination of prior works [GT02, Gue03, PT04, BGT13, Gam14]. Its main consequence, for our purpose, is stated in Proposition 1.10 below; it involves an optimization over parameters \( 0 \leq y_1 \leq y_2 \) and over a large space of probability measures \( Q \). We prove Theorem 1.1 by direct analysis of the bound in a specific region of \((y_1, y_2, Q)\). This seems to bear some resemblance to the approach of [ACZ17], although only at a high level. Our explicit choice of perturbation is based on the “bug proliferation” mechanism proposed by physicists [MR03, KPW04], which we detail in the introductory section below. We leave as an open question to prove the matching lower bound, i.e., to show that

\[
\lim_{N} \mathbb{E}[e_{\text{min}}(\mathcal{G}_N)] = e_{1\text{RSB}}
\]

for all \( \alpha_{\text{sat}} \leq \alpha \leq \alpha_{\text{Ga}} \).

In the remainder of this introductory section we present some guiding heuristics for this model, leading to the formal definitions of \( e_{1\text{RSB}} \) and \( \alpha_{\text{Ga}} \). Our discussion is based primarily on [ANP07], together with the two papers from the physics literature that describe the bug proliferation mechanism: of the latter, one studies a similar model as here for \( k = 3, 4 \) [MR03], while the other studies the \( q \)-coloring model [KRW04]. We will focus on the combinatorial intuition for \( k \)-\textsc{nae-sat} which simplifies when \( k \) is large. At the end of this section we outline the proof of Theorem 1.1. Before proceeding further, we formally define the model:

\textbf{Definition 1.2} (random regular \textsc{nae-sat}). Let \( d, k \) be positive integers, and assume \( N \) is a positive integer such that \( M = Nd/k \) is also integer. A \textit{random} \( d \)-\textit{regular} \( k \)-\textsc{nae-sat} instance on \( N \) variables is encoded by a random bipartite graph \( \mathcal{G}_N \). The vertex set of \( \mathcal{G}_N \) is partitioned into \( V = \{v_1, \ldots, v_N\} \) (\textit{variables}) and \( F = \{a_1, \ldots, a_M\} \) (constraints or \textit{clauses}). The two sets \( V, F \) are joined by a set \( E \) of random edges, generated according to the “configuration model”: give \( d \) half-edges to each \( v \in V \), give \( k \) half-edges to each \( a \in E \), then take a uniformly random matching between the \( V \)-incident and \( F \)-incident half-edges to form a total of \( Nd = Mk \) edges. Note that the sampling procedure can result in multi-edges, so \( \mathcal{G}_N \) is more precisely a multi-graph. Finally, assign to each \( e \in E \) an independent label \( L_e \) sampled uniformly from \{0, 1\}. We denote the instance as \( \mathcal{G}_N = (V, F, E, L) \). For \( e \in E \) we
write $v(e)$ for the incident variable, and $a(e)$ for the incident clause. We write
\[ \nabla v \equiv \delta a(e) \setminus e = \{ \text{edges incident to } e \text{ through a clause} \}, \]
\[ \nabla e \equiv \delta v(e) \setminus e = \{ \text{edges incident to } e \text{ through a variable} \}. \]

For any variable $v \in V$ we write $\delta v$ for the ordered $d$-tuple of edges incident to $v$, and $\delta a$ for the ordered $k$-tuple of clauses $(a(e))_{e \in \delta a}$. For any clause $a \in F$ we write $\delta a$ for the ordered $k$-tuple of edges incident to $a$, and $\delta a$ for the ordered $k$-tuple of variables $(v(e))_{e \in \delta a}$. If $a \in F$ and $v \in V$ are neighbors joined by a single edge $e$ (as will most often be the case) then we write $e \equiv (av)$. Given a variable assignment $x \in \{0, 1\}^N$, a clause $a \in F$ is violated if and only if the $k$-tuple $(L_e \oplus x(v(e)))_{e \in \delta a}$ is all equal (all 0 entries or all 1 entries). A solution of $G_N$ is a variable $x \in \{0, 1\}^N$ that violates no clauses.

**Definition 1.3** (energy landscape and max-satisfiable value). Given an instance $G_N$ generated as in Definition 1.2, its **energy landscape** or **Hamiltonian** is simply the total count of violated clauses: for $x \in \{0, 1\}^N$,
\[ \mathcal{H}_N(x) = \sum_{a \in F} \mathcal{H}_a(x) = \sum_{a \in F} \sum_{e \in \delta a} 1\left\{ e \in \delta a : L_e \oplus x(v(e)) = 1 \right\} \in \{0, k\} \right\}. \tag{2} \]

Note that $\mathcal{H}_N$ is a random function on $\{0, 1\}^N$ determined by the instance $G_N$. The solutions of $G_N$ are precisely the zeroes of $\mathcal{H}_N$. The **max-satisfiable value** (ground state energy) of $G_N$ is
\[ e_{\text{min}}(G_N) \equiv \frac{E_{\text{min}}(G_N)}{N} \equiv \frac{1}{N} \min \left\{ \mathcal{H}_N(x) : x \in \{0, 1\}^N \right\}. \]

Note that $0 \leq e_{\text{min}}(G_N) \leq \alpha \equiv d/k$, and $e_{\text{min}}(G_N)$ is positive if and only if $G_N$ has no proper solutions. We will hereafter use the shorthand $e_{\text{min}}(\alpha) \geq e$ to mean that $\liminf_N E[e_{\text{min}}(G_N)] \geq e$, and $e_{\text{min}}(\alpha) \leq e$ to mean that $\limsup_N E[e_{\text{min}}(G_N)] \geq e$. We will then write $\alpha_{\text{max}}(p) \equiv \alpha''$ to mean that $p_{\text{max}}(\alpha') < p$ for all $\alpha' > \alpha$, and similarly $\alpha_{\text{max}}(p) \equiv \alpha''$ to mean that $p_{\text{max}}(\alpha') > p$ for all $\alpha' < \alpha$.

**Remark 1.4.** Physicists predict that a broad family of random CSPs (including $(\text{NAE})$-SAT, proper coloring, and independent set) exhibit qualitatively similar phase diagrams ([KMR+07] and refs. therein). The existing rigorous literature has proved different aspects of these predictions in different models, including for at least six closely related variants of the model specified in Definition 1.2: namely, random regular $k$-$\text{NAE}$-SAT, random regular $k$-$\text{SAT}$, random regular $k$-$\text{SAT}$, random $k$-$\text{SAT}$, random regular $k$-hypergraph bicoloring, random $k$-hypergraph bicoloring. Throughout this introduction, to simplify the discussion we will (nonrigorously) transfer all existing results to the setting of random regular $k$-$\text{NAE}$-SAT. It is not unreasonable to expect that a result proved in any of the other models can be reproved in random regular $k$-$\text{NAE}$-SAT, which is mathematically the simplest of all the six. Certainly, however, none of our formal results relies on this assumption.

To explain the basic intuitions underlying this paper, in §1.1 we review the first moment bound of [ANP07] in the setting of random regular $k$-$\text{NAE}$-SAT. We then explain in §1.2 why the first moment bound is loose, and a rough heuristic approximation. In §1.3 we explain that when $\alpha$ is not too large the heuristic correction is a reasonable approximation, but it should fail beyond some threshold $\alpha \approx 4^k/k^3$. In §1.4 and §1.5 we explain the more refined heuristic provided by the 1RSB combinatorial framework. This leads to the formal definitions of $\alpha_{\text{1rs}}$ and $\alpha_{\text{sat}}$, in §1.6 and §1.7 respectively. Finally, in §1.8 we state the interpolation bound and describe the proof approach.

### 1.1 First moment bound

Throughout this paper we write $f_{n,k} \equiv g_{n,k}$ to indicate that $C^{-1} \leq f_{n,k}/g_{n,k} \leq C$ for a constant $C$ not depending on $n, k$. We write $f \ll g$ to indicate that $\lim_{k \to \infty} f/g = 0$. We parametrize
\[ c \equiv \frac{\alpha}{2^k \ln 2}, \quad e \equiv \frac{\alpha(1 - p)}{2^k \alpha}. \tag{3} \]

To explain the above parametrization of $e$, consider an instance $G_N$ of $d$-regular random $k$-$\text{NAE}$-SAT, and let $\mathcal{H}_N$ be its Hamiltonian defined by (2) above. For any fixed $x \in \{0, 1\}^N$, the number of constraints that it violates is distributed as $\mathcal{H}_N(x) \sim \text{Bin}(M, 1/2^{k-1})$, so $E[\mathcal{H}_N(x)/N] = \alpha/2^{k-1}$. Therefore it is certainly the case that $E[e_{\text{min}}(G_N)] \leq \alpha/2^{k-1}$, so it is natural to parametrize energies as in (3). Now, following [ANP07], for any given energy level $0 \leq e \leq \alpha$, and any $0 < \eta \leq 1$, we can consider
\[ X_{e, \eta} = \sum_{x \in \{0, 1\}^N} 1\left\{ \frac{\mathcal{H}_N(x)}{N} \leq e \right\} \eta^{\mathcal{H}_N(x)} \geq \# \left\{ x \in \{0, 1\}^N : \frac{\mathcal{H}_N(x)}{N} \leq e \right\} \equiv Y_e. \tag{4} \]
If $E$ is expectation over the random instance $\mathcal{G}_N$, then
\[
E[X_{e,\eta}] = \frac{2^N}{n^{\eta N e}} \left(1 - \frac{2\eta}{2^k} + \frac{2\eta}{2^k}\right)^{Na} \exp\left\{N f_{\eta}(\alpha, e)\right\}.
\] (5)

If $\alpha, e$ are fixed, a stationary point of $f$ as a function of $\eta$ is given by
\[
\eta = \eta(\alpha, e) = \frac{e(2^{k-1} - 1)}{\alpha - e} = \left(\frac{1-p}{2^{k-1} - (1-p)}\right) \equiv \eta(p).
\] (6)

Setting $f_{\eta}(p)(\alpha, e) = 0$ gives the relation $\alpha = \alpha_{\text{ubd}}(p) \equiv c(p) \cdot 2^{k-1} \ln 2$ where
\[
c(p) = \frac{1}{(2^{k-1} - (1-p)) \ln \frac{2^{k-1} - (1-p)}{2^{k-1} - 1} + (1-p) \ln(1-p)} \leq \frac{1}{p + (1-p) \ln(1-p)}.
\] (7)

One can check (by differentiating) that $c(p)$ is strictly decreasing with respect to $p$, with $c(1) = 1$ and $c(p) \uparrow \infty$ as $p \downarrow 0$. Therefore the inverse function is well-defined for all $\alpha \geq 2^{k-1} \ln 2$, and we denote it as
\[
e_{\text{ubd}}(\alpha) \equiv \frac{\alpha(1-p_{\text{ubd}}(\alpha))}{2^{k-1}} \equiv (\alpha_{\text{ubd}})^{-1}(\alpha).
\] (8)

Since $f_{\eta}$ is decreasing in $\alpha$, we conclude that $f_{\eta}(p)(\alpha, e) < 0$ for all $\alpha > \alpha_{\text{ubd}}(p)$. For any such $\alpha$, Markov’s inequality gives that $P(Y_e \leq \lambda \leq \alpha) \leq EY_e \leq EX_{e,\eta}$ is exponentially small with respect to $N$. We can summarize the above as

**Lemma 1.5.** If $\mathcal{G}_N$ is random regular $k$-NAE-SAT on $N$ variables subject to $N\alpha$ constraints, then
\[
\lim_{N \to \infty} \inf_{N \to \infty} E[e_{\min}(\mathcal{G}_N)] = \lim_{N \to \infty} \inf_{N \to \infty} E\left[\min \left\{e \geq 0 : Y_e > 0\right\}\right] \geq e_{\text{ubd}}(\alpha)
\] (9)

as defined by (8). (In the shorthand of Definition 1, we have $e_{\min} \geq e_{\text{ubd}}(\alpha)$.)

Lemma 1.5 is the first moment bound from [ANP07], transferred to the setting of regular nae-SAT. Recalling (3), if $c$ is large then the expression (8) for $p_{\text{ubd}}$ can be approximated by
\[
1 - c = g(p_{\text{ubd}}) = p_{\text{ubd}} + (1 - p_{\text{ubd}}) \ln(1 - p_{\text{ubd}}) \approx \frac{(p_{\text{ubd}})^2}{2} \left(1 + O(p_{\text{ubd}})\right),
\]
so that $p_{\text{ubd}} = (2/c)^{1/2} + O(1/c)$. We point out that [ANP07] studies the more difficult model of random $k$-SAT, and their main result is a much more challenging lower bound, which is done by the second moment method. Translating their full result to our model would give
\[
\alpha_{\text{ubd}}(p) \left(1 + O\left(\frac{k}{2^{k/2}}\right)\right) \leq \alpha_{\max}(p) \leq \alpha_{\text{ubd}}(p).
\] (10)

We will not seek to rigorously prove the lower bound in (10), since we expect it to be easier than the lower bound already achieved by [ANP07]. The more interesting open problem is to establish that $e_{\text{ubd}}$ is tight for $\alpha \leq \alpha_{\text{Ga}}$.

1.2. **Clustering of near-max-satisfiable configurations.** We next describe the intuition for why the first moment bound (9) cannot be exactly sharp. Suppose for the sake of argument that it is. Let $e = e_{\text{ubd}}$ and $\eta = \eta(p_{\text{ubd}})$ as above. Any $x \in \{0, 1\}^N$ that contributes to $X_{e,\eta}$ will be max-satisfiable, so it certainly must satisfy the weaker condition of being **locally max-satisfiable**, in the sense that flipping any single variable $x_v$ cannot decrease the number of violated constraints. Explicitly, let $F_0$ be the number of clauses incident to $v$ which are satisfied only if $x_v = 0$:
\[
F_0 = \#\left\{e \in dv : \#x \in v(e) : Lx \oplus v(x) = 1 \right\} = k - 1,
\]
and similarly $F_1$. The spin $x_v = 0$ is locally max-satisfiable if and only if $F_0 \geq F_1$. Let $X_{e,\eta}(x, \ell_0, \ell_1)$ denote the contribution to $X_{e,\eta}$ from configurations $x$ with $(x_0, F_0, F_1) = (x, \ell_0, \ell_1)$. By taking expectation only over the edge labels $L_e$ around the clauses neighboring $v$, we find
\[
E\left\{X_{e,\eta}(0, \ell_0, \ell_1)\right\} = C_N \mathbf{1}(\ell_0 \geq \ell_1) \left(\frac{d}{\ell_0, \ell_1}\right)^{d - \ell_0 - \ell_1} \left(\frac{2\eta}{2^k}\right)^{\ell_1},
\] (11)
where $C_N$ is a factor not depending on $\ell_0, \ell_1$, and for any $a_1 + \ldots + a_t \leq b$ we abbreviate
\[
\binom{b}{a_1, \ldots, a_t} = \frac{b!}{a_1! \cdots b_1! (b - a_1 - \ldots - a_t)!}.
\]
Summing (11) over \( \ell_0 \geq \ell_1 \), we find that the total expected contribution to \( X_{e,n} \) from configurations with \( x_v = 0 \) is

\[
\mathbb{E}\left\{ X_{e,n}(x_v = 0) \right\} = C_N \sum_{d \leq \ell_0} \left( \frac{d}{\ell_0} \right) \left( 1 - \frac{4}{2^k} \right)^{d-\ell} \left( \frac{2(1 + \eta)}{2^k} \right)^{\ell} P_{n,\ell},
\]

\[
P_{n,\ell} = \mathbb{P}\left( \text{Bin} \left( \ell, \frac{\eta}{1 + \eta} \right) \leq \frac{\ell}{2} \right).
\]

Simply using the crude bound \( 1/2 \leq P_{n,\ell} \leq 1 \) gives

\[
\mathbb{E}\left\{ X_{e,n}(x_v = 0) \right\} \approx C_N \left( 1 - \frac{4}{2^k} + \frac{2(1 + \eta)}{2^k} \right)^d.
\]

Now note that if \( F_0 = F_1 \) then variable \( v \) is free, meaning that flipping \( x_v \) alone does not change the total number of violated constraints. Summing (11) over \( \ell_0 = \ell_1 = \ell/2 \) gives

\[
\mathbb{E}\left\{ X_{e,n}(x_v = 0, v \text{ is free}) \right\} = C_N \sum_{\ell \text{ even}} \left( \frac{d}{\ell} \right) \left( 1 - \frac{4}{2^k} \right)^{d-\ell} \left( \frac{4\eta^{1/2}}{2^k} \right)^{\ell} P_{r,\ell}
\]

where \( P_{r,\ell} = \mathbb{P}(\text{Bin}(\ell, 1/2) = \ell/2) \propto 1/\ell^{1/2} \). Now assume that \( c \) is large, so \( p \) is small and we see from (6) that \( \eta \approx 1 \). Without the factor \( P_{r,\ell} \), the above sum is dominated by \( \ell \propto d\eta^{1/2}/2^k \propto d^{2k}/2^k \). Accounting for \( P_{r,\ell} \) results in

\[
\mathbb{E}\left\{ X_{e,n}(x_v = 0, v \text{ is free}) \right\} \approx \frac{C_N}{(d/2^k)^{1/2}} \left( 1 - \frac{4}{2^k} + \frac{4\eta^{1/2}}{2^k} \right)^d.
\]

(A more careful version of this calculation appears in Section 2.) This would suggest the typical fraction of variables that are free is something like

\[
\pi_\ell \propto \frac{\mathbb{E}\left\{ X_{e,n}(x_v = 0, v \text{ is free}) \right\}}{\mathbb{E}\left\{ X_{e,n}(x_v = 0) \right\}} \approx \frac{1}{(d/2^k)^{1/2}} \left( 1 - \frac{4}{2^k} + \frac{4\eta^{1/2}}{2^k} \right)^d \left( 1 - \frac{4}{2^k} + \frac{2(1 + \eta)}{2^k} \right)^d.
\]

If we assume that \( k^2 \ll_k c \ll_k 2^{k/2} \), then the above simplifies to

\[
\pi_\ell \propto \frac{1}{(d/2^k)^{1/2}} \left( 1 - \frac{2(1 - \eta^{1/2})^2/2^k}{1 - 4/2^k + 2(1 + \eta)/2^k} \right)^d.
\]

Suppose the configuration \( x \) has order \( N/d^{1/2} \) free variables. Suppose for simplicity that they do not interact, meaning that flipping any subset of free variables does not change the total number of violated constraints. We will examine the validity of this supposition in §1.3, but we simply grant it for now. This would mean that for a typical max-satisfiable configuration \( x \) we can find at least \( 2^N \eta \) nearby configurations \( x' \) with \( \mathcal{H}_N(x) = \mathcal{H}_N(x') \). But this would mean \( \mathbb{E}X_{e,n} \geq 2^N \eta \), in contradiction with our choice of \( e = e_{\text{bd}} \) and \( \eta = \eta(e_{\text{bd}}) \) which ensures that \( \mathbb{E}X_{e,n} \) is exponentially small in \( N \). This suggests that \( e_{\text{bd}} \) (or equivalently its inverse \( \alpha_{\text{bd}} \)) cannot be tight bounds; our main theorem verifies this by establishing the lower bound \( e_{\text{min}} > e_{\text{bd}} \). The above calculation suggests that \( \exp\{Nf_p(\alpha, e)\} \) overestimates the typical value of \( X_{e,n} \) by at least a factor \( 2^{N\eta} \), where \( \pi_\ell \approx 1/d^{1/2} \), which suggests, in the regime \( k^2 \ll_k c \ll_k 2^{k/2} \), that \( \alpha_{\text{max}}(p) \) is exponentially small in \( 1/d^{1/2} \).

\[
\alpha_{\text{max}}(p) \leq \left( 1 - \frac{1}{d^{1/2}} \right) \alpha_{\text{bd}}(p).
\]

In §2.3 (Corollary 2.10) we prove a rigorous bound which covers the full regime \( \alpha_{\text{sat}} \leq \alpha \leq 4^k/k \), and agrees with (12) for \( k^2 \ll_k c \ll_k 2^{k/2} \). In fact in this regime we conjecture the estimate \( \Omega(1/d^{1/2}) \) to be tight.
1.3. **Percolation of dependent free variables.** We now revisit the above assumption that the free variables do not interact. Take a clause $a$ with no incident multi-edges (as will be the case for most clauses), and suppose it neighbors two free variables $v \neq w$. If the values of $L_{uv} \oplus x_v$ for $u \in \Delta a \setminus \{v, w\}$ are all 0 or all 1, then $x_v$ and $x_w$ are linked, meaning they cannot both be arbitrarily flipped without increasing the number of violated constraints. For a free variable $v$, the number of linked free variables $w$ sharing a clause with $v$ is (on average, heuristically)

$$r = \left(\frac{(d-1)(k-1)}{d} \times \frac{1}{2^{k-2}} \leq \frac{d^{1/2}k}{2^k}, \right)$$

where the factor $(d-1)(k-1)$ accounts for the branching factor of the underlying graph $\mathcal{G}_N$. We view the process of linked free variables as a dependent percolation on $\mathcal{G}_N$ spreading at rate $r$ given by (13). As long as the rate is small, corresponding to $d \ll k^4$, we would expect the percolation to be subcritical, in the sense that the free subgraph — the subgraph of $\mathcal{G}_N$ induced by free variables and linking clauses — is mostly a forest of $O(1)$-sized trees. Moreover, roughly a $(1-r)$-fraction of free variables should be isolated (not linked to any other frees), so for small $r$ it is a reasonable approximation to assume that none of the free variables interact.

As we detail in §1.4 below, in the context of the current problem, the 1RSB framework is simply a convenient combinatorial model for the free subgraph, which captures the effect of free variables on $e_{\text{min}}$ in a well-organized manner. It yields the prediction that the limiting ground state energy is exactly $e_{\text{min}} = e_{1\text{RSB}}(\alpha)$, where $e_{1\text{RSB}}(\alpha)$ is an explicit function defined below in Proposition 1.8. The threshold $\alpha_{\text{Ga}}$, given formally by Proposition 1.9, is an explicit prediction of the exact percolation threshold for the 1RSB combinatorial model. The derivation of $e_{1\text{RSB}}$ relies crucially on the assumption that the free subgraph is essentially a forest, which should not be the case beyond $\alpha_{\text{Ga}}$.

This is the basic intuition for our main result which verifies that $e_{1\text{RSB}}$ is indeed incorrect beyond $\alpha_{\text{Ga}}$.

We remark that it is a much more challenging problem to obtain a sharper estimate of $e_{\text{min}}$ in the regime $\alpha > \alpha_{\text{Ga}}$. The main result that we know of was obtained for the random $k$-sat model [Pan18] (see also [Sen18]) by comparison with mean-field limits [LP01, CLP02]; from the discussion in [Pan18] the estimate requires roughly $\alpha \geq \Omega(4^k)$. A related result was obtained for the max-cut problem by [DMS17], for random graphs of large degree. It remains an difficult challenge to understand the regime between the mean-field (i.e., complete graph) limits and $\alpha_{\text{Ga}}$.

Having laid out the basic intuitions for the model, we next proceed to define the 1RSB combinatorial framework. We emphasize that the 1RSB model itself is a heuristic, which plays no formal role in the proof of our main result. We introduce it because it is the quickest way to motivate the exact definitions of $e_{1\text{RSB}}$ and $\alpha_{\text{Ga}}$. We point to [MM09] for an introductory account and further references on the 1RSB framework.

1.4. **Combinatorial model of near-max-satisfiable clusters.** Following our earlier discussion, we now restrict attention to the subspace $Q(\mathcal{N}) \subseteq \{0, 1\}^N$ of configurations that are locally max-satisfiable. Define a graph on vertex set $Q$ by putting an edge between $x$ and $x'$ if and only if they differ in a single coordinate and $\mathcal{N}(x) = \mathcal{N}(x')$.

A (locally max-satisfiable) cluster is any subset $\omega \subseteq Q$ that constitutes a maximal connected component in that graph. The 1RSB heuristic models a cluster as follows:

**Definition 1.6** (warning configurations). Suppose $\mathcal{N} = (V, F, E, \mathcal{L})$ is any $d$-regular $k$-nae-sat problem instance. A warning configuration on $\mathcal{N}$ is an element $\omega \in \{0, 1, \emptyset\}^{2E}$ which assigns a pair $\omega_e \equiv (\hat{\omega}_e, \check{\omega}_e)$ to each edge $e \in E$, satisfying conditions that we now specify. We take the convention throughout that $x \oplus \emptyset \equiv \emptyset$. Define

$$\ell(\hat{\omega}_1, \ldots, \hat{\omega}_{d-1}) \equiv \# \left\{ 1 \leq i \leq d-1 : \hat{\omega}_i = x \right\}.$$

Then $\omega$ is a valid warning configuration if and only if it satisfies variable relations

$$\hat{\omega}_e = \hat{\omega}(\hat{\omega}_g : g \in \bar{V}(e)) = \begin{cases} 0 & \ell_0(\hat{\omega}_g : g \in \check{V}(e)) > \ell_1(\hat{\omega}_g : g \in \check{V}(e)) , \\ 1 & \ell_0(\hat{\omega}_g : g \in \check{V}(e)) < \ell_1(\hat{\omega}_g : g \in \check{V}(e)) , \\ \emptyset & \ell_0(\hat{\omega}_g : g \in \check{V}(e)) = \ell_1(\hat{\omega}_g : g \in \check{V}(e)) , \end{cases}$$

(14)
as well as clause relations

\[ \dot{\omega}_e = \hat{\omega}_e \left( \dot{\omega}_\bar{g} : g \in \bar{V}(e) \right) = \begin{cases} 0 & L_e \otimes \dot{\omega}_\bar{g} = L_e \otimes 1 \text{ for all } g \in \bar{V}(e), \\ 1 & L_e \otimes \dot{\omega}_\bar{g} = L_e \otimes 0 \text{ for all } g \in \bar{V}(e), \\ \text{otherwise} & \end{cases} \]

(15)

for all \( e \in E \). We may write \( \hat{\omega}_e \equiv \hat{\omega}_e \left( \dot{\omega}_\bar{g} : g \in \bar{V}(e) \right) \) to emphasize the number of arguments. Given \( \varphi \) let

\[ \eta \equiv \eta(\varphi) \equiv \left( \hat{\omega}_e : e \in \delta V \right) \in \{0, 1, \mathbf{f}\}^N. \]

If \( \#\{v \in V : \eta_v = \mathbf{f}\} \leq N/k^2 \) then we say that \( \varphi \) is near-frozen.

Under the 1RSB heuristic, there is essentially a bijective correspondence

\[ \{ \text{locally max-satisfiable clusters } \omega \subseteq Q \subseteq \{0, 1\}^N \} \leftrightarrow \{ \text{near-frozen warning configurations } \varphi \in \{0, 1, \mathbf{f}\}^{2E} \} \]

(16)

between clusters \( \omega \) and near-frozen warning configurations \( \varphi \). A loose characterization of the correspondence is that \( \eta \equiv \eta(\varphi) \) encodes the smallest subcube of \( \{0, 1\}^N \) containing \( \omega ; \eta_v \in \{0, 1\} \) if and only if \( x_v = x_v \) for all \( x \in \omega \), and \( \eta_v = \mathbf{f} \) if and only if \( x_v \) takes both values \( \{0, 1\} \). A more precise interpretation is that

\[ \dot{\omega}_e = \text{variable-to-clause warning} \text{ along } e \]

= locally optimal choice within \( \omega \) of \( x_{v(e)} \) in absence of edge \( e \),

\[ \dot{\omega}_e = \text{clause-to-variable warning} \text{ along } e \]

= locally optimal choice within \( \omega \) of \( x_{v(e)} \) “in absence of” edges \( \hat{V}(e) \),

where \( \mathbf{f} \) means that both spins \( \{0, 1\} \) are locally optimal. Under this interpretation, the \( \dot{\varphi}, \dot{\omega} \) must then satisfy local consistency relations, which are the so-called warning propagation (WP) equations (15) and (14). The near-frozen restriction rules out configurations such as \( \varphi = \mathbf{f}^2 \) (all messages \( \mathbf{f} \) which we do not expect to correspond to any actual cluster.

1.5. Tree formula for the max-satisfiable value. To give an explicit calculation, let \( \tau = (V', F', E') \) be a finite bipartite tree (representing an \( O(1) \)-sized subgraph of \( G_N \)) with variables at its leaves. Say \( \tau \) has a frozen boundary, in the sense that \( \dot{\omega}_e \in \{0, 1\} \) is fixed at every leaf edge \( e \). By applying the maps \( \hat{\omega}_e, \hat{\omega}_e \) recursively inwards from the leaves, we see that there is exactly one valid warning configuration \( \varphi \) on \( \tau \) that is consistent with the boundary condition. Let \( E_{\min}(\tau) \) be the minimum number of clauses violated by any configuration \( \varphi \in \{0, 1\}^{V'} \) with \( x_{v(e)} = \dot{\omega}_e \) at the leaves. We next explain that \( E_{\min}(\tau) \) can be computed by a simple dynamic-programming-type method.

Let \( E'' \) be the set of non-leaf edges of \( \tau \). For any \( e \in E'' \) we let \( \hat{\tau}_e \) be the component containing \( a(e) \) in \( \tau \setminus \hat{V}(e) \), and let \( \hat{\tau}_e \) be the component containing \( v(e) \) in \( \tau \setminus e \). Let \( \hat{E}_e = E_{\min}(\hat{\tau}_e) \) and \( \hat{E}_e = E_{\min}(\hat{\tau}_e) \). If \( V'' \) denotes the non-leaf variables of \( \tau \), around any \( v \in V'' \) we have

\[ E_{\min}(\tau) = \phi(\dot{\omega}_{\delta_v}) + \sum_{e \in \delta_v} \hat{E}_e, \quad \phi(\dot{\omega}_{\delta_v}) \equiv \min \left( \ell_0(\dot{\omega}_{\delta_v}), \ell_1(\dot{\omega}_{\delta_v}) \right). \]

(17)

Similarly, around any clause \( a \in F' \), we have

\[ E_{\min}(\tau) = \phi(\dot{\omega}_{\delta_a}) + \sum_{e \in \delta_a} \hat{E}_e, \quad \phi(\dot{\omega}_{\delta_a}) \equiv \min \left( \hat{\omega}_e \left( \dot{\omega}_g : g \in \bar{V}(e) \right) \right). \]

(18)

We sometimes write \( \phi \equiv \phi_d \) and \( \phi \equiv \phi_k \) to emphasize the number of arguments. Finally, for any \( e \in E'' \) we have

\[ E_{\min}(\tau) = \phi(\dot{\omega}_e) + \hat{E}_e, \quad \phi(\dot{\omega}_e) \equiv \min \left( \dot{\omega}_e, \dot{\omega}_e = 1 \right). \]

(19)

By summing over the internal vertices and subtracting over the internal edges, we arrive at

\[ \sum_{v \in V''} \phi(\dot{\omega}_{\delta_v}) + \sum_{a \in F'} \phi(\dot{\omega}_{\delta_a}) - \sum_{e \in E''} \phi(\dot{\omega}_e) = \left( \#V'' + \#F' - \#E'' \right) E_{\min}(\tau) = E_{\min}(\tau), \]

(20)

where the last equality uses that \( \tau \) is a tree. Thus the max-satisfiable value of a tree with frozen boundary is a sum of local functionals \( \phi, \dot{\phi}, \dot{\phi} \) of the warning configuration.
The 1RSB heuristic further assumes that for near-frozen warning configurations, the entire graph $\mathcal{G}_N = (V, F, E, L)$ can essentially be carved into trees with frozen boundaries. (In reality, even in the regime where free variables do not percolate, a typical warning configuration may contain a bounded number of small cycles of free warnings which do not admit a tree decomposition. However these few cycles should only affect the number of violated clauses by $O(1)$, so can be ignored in the heuristic analysis.) Then, by summing (20) over the components of the tree decomposition, we conclude that $y$ corresponds to a cluster $\omega \subseteq \{0, 1\}^N$ at energy level

$$E_{\min}(\omega; \mathcal{G}_N) \equiv \min \left\{ \mathcal{H}_N(\overline{x}) : \overline{x} \in \omega \right\} = \phi(\overline{w}) \equiv \sum_{\nu \in V} \phi(\hat{\overline{w}}_{\nu\nu}) + \sum_{a \in F} \phi(L \oplus \hat{\overline{w}}_{a}) - \sum_{e \in E} \phi(\overline{w}_e).$$

(21)

This is the main advantage of the $y$ encoding; it allows us to read off $E_{\min}(\omega; \mathcal{G}_N)$ as a sum of local terms.

1.6. Explicit 1RSB prediction. Going back to the bijection (16), we can take a parameter $y \geq 0$ and consider

$$3(y) = \sum_\omega \exp \left\{ -y E_{\min}(\omega; \mathcal{G}_N) \right\} = \sum_\omega \exp \left\{ -y \phi(\overline{w}) \right\}$$

(22)

where the first sum goes over clusters, while the second sum goes over near-frozen warning configurations. The corresponding probability measure on warning configurations is given by

$$\mu_y(\overline{w}) = \frac{\exp \left\{ -y \phi(\overline{w}) \right\}}{3(y)}.$$  

(23)

This is sometimes called the survey propagation or $SP_y$ model, and can be viewed as a refinement of the reweighting $y^{\mathcal{G}_N(\omega)}$ discussed in §1.1. The “lifting” from $y^{\mathcal{G}_N(\omega)}$ to $e^{-y\phi(\overline{w})}$ represents one level of replica symmetry breaking. The 1RSB solution to the original model is given by the replica symmetric solution to the “lifted” model (23). This sometimes goes by the name of survey propagation (sp). In particular, the 1RSB (sp) equations are simply the replica symmetric or belief propagation (bp) equations for the lifted model. They can be defined as a pair of mappings on the space

$$\mathcal{M} \equiv \left\{ \text{probability measures } q \text{ on } \{0, 1, \emptyset\} \text{ satisfying } q(0) = q(1) \right\}.$$  

(24)

The clause survey propagation takes $\hat{q} \in \mathcal{M}$ and outputs

$$[\hat{SP}_y(\hat{q})](\hat{w}) = \sum_{\hat{\omega}} 1 \left\{ \hat{w} = \hat{w}_p \oplus \hat{w}_{\omega} \right\} \prod_{i=1}^{k-1} \hat{q}(\hat{u}_i),$$

(25)

where the sum goes over $\hat{\omega} \in \{0, 1, \emptyset\}^{k-1}$, and $\hat{SP}_y(\hat{q})$ is a probability measure on $\{0, 1, \emptyset\}$, and in fact $\hat{SP}_y(\hat{q}) \in \mathcal{M}$. The variable survey propagation takes $\hat{q} \in \mathcal{M}$ and outputs

$$[\hat{SP}_y(\hat{q})](\hat{w}) = \frac{1}{\hat{z}} \sum_{\hat{\omega}} 1 \left\{ \hat{w} = \hat{w}_p \oplus \hat{w}_{\omega} \right\} \exp \left\{ -y \phi_{d-1}(\hat{w}) \right\} \prod_{i=1}^{d-1} \hat{q}(\hat{u}_i),$$

(26)

where the sum is over $\hat{\omega} \in \{0, 1, \emptyset\}^{d-1}$, and $\hat{z}$ is the normalization such that $\hat{SP}_y(\hat{q}) \in \mathcal{M}$. Let $SP_y = \hat{SP}_y \circ \hat{SP}_y$. Now, recalling (3), we hereafter restrict consideration to parameters $y \geq 0$ satisfying

$$\gamma \equiv 2e \left( 1 - \frac{1}{e^{y/2}} \right)^2 = 1.$$  

(27)

Note $\gamma = c \min\{1, y^2\}$. If $c$ is large then (27) forces $y = 1/c^{1/2}$. If $c \propto 1$ then it only forces that $y \geq \Omega(1)$. Define

$$\mathcal{M}^\star \equiv \left\{ q \in \mathcal{M} : q(\emptyset) \leq \frac{1}{k^2} \right\},$$  

(28)

$$\mathcal{M}^\nu \equiv \left\{ q \in \mathcal{M}^\star : q(\emptyset) = \frac{2^{-ky/2}}{(\max\{ce^{y/2}, 1\}^{1/2})} \right\}.$$  

(29)

We prove the following result on fixed points of the $SP_y$ recursion:

**Proposition 1.7** (proved in Section 2). Suppose $\alpha = c2^{k-1} \ln 2$ with $\alpha_{\text{sat}} \leq \alpha \leq 4^k/k$, and suppose $y \geq 0$ satisfies (27). Then in the set $\mathcal{M}^\star$ there is a unique $\hat{q}_y$ satisfying the fixed-point equation $\hat{q}_y = SP_y(\hat{q}_y)$. It must further lie in the smaller domain $\mathcal{M}^\nu$. 
Let $\hat{q} = \hat{q}_y$ be as given by Proposition 1.7, and denote $\hat{\eta} \equiv \hat{\eta}_y \equiv \hat{\eta}_y(\hat{q}_y)$. Recall the local functionals $\hat{\phi}, \hat{\phi}, \hat{\varphi}$ from (17), (18), (19). We can define three probability measures $-\hat{\nu}_y$ over $\hat{\omega} \in \{0, 1, f\}^d, \hat{\nu}_y$ over $\hat{\omega} \in \{0, 1, f\}^k$, and lastly $\hat{\nu}_y$ over $w = (\hat{\omega}, \hat{\omega}) \in \{0, 1\}^2$ — as follows:

\[
\hat{\nu}_y(\hat{\omega}) = \frac{1}{\hat{\beta}_y(\hat{q})} \exp \left\{ - y \hat{\phi}_{d}(\hat{\omega}) \right\} \prod_{i=1}^{d} \hat{q}(\hat{\omega}_i),
\]

(30)

\[
\hat{\nu}_y(\hat{\omega}) = \frac{1}{\hat{\beta}_y(\hat{q})} \exp \left\{ - y \hat{\phi}_{d}(\hat{\omega}) \right\} \prod_{i=1}^{d} \hat{q}(\hat{\omega}_i),
\]

(31)

\[
\hat{\nu}_y(\hat{\omega}) = \frac{1}{\hat{\beta}_y(\hat{q}, \hat{q})} \exp \left\{ - \hat{\varphi}(\hat{\omega}, \hat{\omega}) \right\} \hat{q}(\hat{\omega}) \hat{q}(\hat{\omega}).
\]

(32)

Under the sp heuristic, the local marginals of the measure (23) are approximately given by the $v$: for instance,

\[
\hat{\mu}_y \left( w \in \{0, 1, f\}^d : \hat{\nu}_y = \hat{\omega} \right) \approx \hat{\nu}_y(\hat{\omega}).
\]

The corresponding energy level can be obtained by averaging (21) with respect to the $v$: this gives

\[
e(y) = \sum \hat{\phi}_{d}(\hat{\omega}) \hat{\nu}_y(\hat{\omega}) + \alpha \sum \hat{\phi}_{k}(\hat{\omega}) \hat{\nu}_y(\hat{\omega}) - d \sum \hat{\varphi}(w) \hat{\nu}_y(\hat{\omega}).
\]

(33)

The sp heuristic further predicts that $N^{-1} \ln \hat{\beta}(y)$ converges (for a suitable range of $y$, to be discussed in Remark 2.8) to the replica symmetric formula,

\[
\hat{\beta}(y) = \ln \hat{\beta}_y(\hat{q}) + \alpha \left\{ \ln \hat{\beta}_y(\hat{q}) - k \ln \hat{\beta}_y(\hat{q}, \hat{q}) \right\},
\]

(34)

where $\hat{\beta}_y, \hat{\beta}_y$, and $\hat{\beta}_y$ are the normalizing constants from (30), (31), and (32). Now, returning to (22), suppose that we had an “energetic complexity function” function $\hat{\Gamma}(y)$ such that

\[
\hat{\beta}_y \equiv \left\{ w : \phi(w) \approx N \right\} \approx E \hat{\beta}_y \approx \exp \left\{ N \hat{\beta}(y) \right\}
\]

where the interpretation for $\hat{\beta}(y) < 0$ is that $E \hat{\beta}_y$ is exponentially small with respect to $N$ so $\hat{\beta}_y = 0$ whp. Then we would expect

\[
\exp \left\{ N \hat{\beta}(y) \right\} \approx \hat{\beta}(y) \approx \exp \left\{ N \max \left\{ \hat{\beta}(y) \right\} \right\},
\]

that is to say, given $\hat{\beta}$ we can obtain $\hat{\beta}$ by taking the Legendre dual. Of course, we are in the opposite situation: we already obtained explicit expressions (33) and (34) for $e(y)$ and $\hat{\beta}(y)$, but we do not know $\hat{\beta}$. We therefore formally define the energetic complexity function as $\hat{\beta}(y) = \hat{\beta}(y) + ye(y)$. (While the informal complexity $\hat{\beta}$ is a function of $e$, the formal complexity $\hat{\beta}$ is a function of $y$.) Recall (27) and let

\[
\hat{\beta}(y) \equiv e \left( 1 - \frac{1 + y}{e} \right).
\]

(35)

It is straightforward to verify that $\hat{\beta}(y)/2 \leq \Gamma(y) \leq \hat{\beta}(y)$ for all $y \geq 0$; see Figure 1a. For small $y$ (corresponding, via (27), to large $e$) we have

\[
\hat{\beta}(y) = \hat{\beta}(y) \left\{ 1 - \hat{\beta}(y) \right\}.
\]

For $y \geq \Omega(1)$ (corresponding, via (27), to $e \approx 1$) we have instead

\[
2\hat{\beta}(y) = \hat{\beta}(y) \left\{ 1 + \hat{\beta}(y) \right\}.
\]

The following proposition formally defines the 1RSB formula.

**Proposition 1.8** (proved in Section 2). Suppose $k \geq k_0$ and $\kappa_{\text{sat}} \leq \alpha \leq 4^k/k$; and denote $e = \alpha/(2^{k-1} \ln 2)$. Then the function $\hat{\beta}(y) = \hat{\beta}(y) + ye(y)$ is continuous on the range of $y$ satisfying (27), with at least one root $y_\ast$. Any such root $y_\ast$ satisfies the estimate

\[
\Gamma(y_\ast) = 1 + O \left( \frac{1}{e^{\alpha(\frac{1}{2})}} \right).
\]

(36)
The 1RSB ground state energy $e_{\text{1RSB}}(\alpha)$ is defined as the infimum of $e(y_*)$ over all roots $y$ in the range (27).

![Image](image.png)

(a) $\gamma(y)/2 \leq \Gamma(y) \leq \gamma(y)$ for all $y \geq 0$.\[\Gamma(y_*) \doteq 1\] for both rs and 1rsb solutions (36).

(b) Upper curve: $\gamma(y_*)$ as a function of $c \geq 1$.
Lower curve: $\exp(-y_*)$ as a function of $c \geq 1$.

Figure 1. Approximate parameters of the 1RSB solution. At clause density $\alpha = ek^{k-1} \ln 2$, the max-satisfiable value is $c = \alpha(1 - p)/2^{k-1}$, where $1 - p = \eta = e^{-y}$ is given approximately by the lower curve in panel (b). At this precision it is consistent with the replica symmetric (rs) solution (cf. the estimate of [ANP07]). A more precise comparison between rs and 1RSB is given by Corollary 2.10.

We expect $y_*$ to be unique (depending only on $k$ and $\alpha$), but we do not have a proof for this. In §2.2 (Remark 2.7) we discuss how uniqueness of $y_*$ relates to other physical properties of the system.

We note that the estimate (36) is a rather lossy approximation of $e_{\text{1RSB}}$. In fact, on its own it does not carry more information than the first moment [ANP07] bound: observe from (5) that $\eta = \eta(p) = (1 - p)[1 + O(p/2^k)]$.

Substituting into (7) gives

$$\frac{1}{e(p)} = \left(1 - \eta + \eta \ln \eta\right) \left(1 + O\left(\frac{1}{e^{2ck\eta}}\right)\right) = \left(1 - \frac{1 + y}{e^y}\right) \left(1 + O\left(\frac{1}{e^{2ck\eta}}\right)\right),$$

simply by taking $y = -\ln \eta$. Thus more care is needed to obtain a comparison such as (12) with the first moment. Towards this end, let us comment briefly on what Proposition 1.8 implies for $\hat{q}(f)$. Recall from Proposition 1.7 that $\hat{q} = \hat{q}_y \in M^Y$, meaning (see (29)) that

$$\hat{q}(f) \geq \frac{2^{-k\gamma/2}}{(ek^{-y/2}, 1)^{1/2}} \geq \frac{1}{2^{k\gamma/2}} \left(\min \left\{\frac{ek^{-y/2}}{2}, 1\right\}\right)^{1/2} \leq O\left(\frac{1}{2^{k\gamma/2}}\right)$$

for $\gamma = \gamma(y)$. It follows from (27) that $ek^{-y/2} = O(1)$ if and only if $y \geq 2 \ln k - O(1)$. For such $y_* \geq 2 \ln k - O(1)$, the estimate (36) implies

$$c = \Gamma(y_*) \left(1 - \frac{1 + y_*}{e^{y_*}}\right)^{-1} = 1 - \frac{O(\ln k)}{k^2},$$

meaning $c$ is only slightly above the satisfiability threshold. In this regime

$$\frac{k\gamma(y_*)}{2} = \frac{k\gamma(y_*)}{2\Gamma(y_*)} \Gamma(y_*) = k \left(1 + O\left(\frac{1}{e^{y_*/2}}\right)\right) \left(1 + O\left(\frac{1}{e^{2ck\eta}}\right)\right) = k + O(1),$$

so $\hat{q}(f) \geq 2^{-k}$. This is consistent with estimates slightly below the satisfiability threshold obtained by [DSS16].

We now discuss $y \leq 2 \ln k + O(1)$. In general, for any fixed $c$ the value $\Gamma(y)$ is strictly increasing in $y$, therefore $y_*$ must be roughly decreasing with $c$ (modulo the error in the estimate (36)). The ratio $\gamma(y)/\Gamma(y)$ is a function of $y$ alone, and is increasing in $y$. Therefore, as $c$ increases, $\gamma(y_*) = \gamma(y_*)/\Gamma(y_*)$ decreases smoothly, from $\gamma \doteq 2$ to $\gamma \doteq 1$ (Figure 1b). For $\Omega(1/k^2) \leq y \leq 2 \ln k + O(1)$ we have $\hat{q}(f) = kO(1)/2^{k\gamma/2}$, which is roughly increasing as $c$ decreases if we ignore the $kO(1)$ factor. Finally, if $y = O(1/k^2)$ then

$$\frac{k\gamma(y_*)}{2} = \frac{k\gamma(y_*)}{2\Gamma(y_*)} \Gamma(y_*) = \frac{k}{2} \left(1 + O(1)\right) \left(1 + O\left(\frac{1}{e^{2ck\eta}}\right)\right) = \frac{k}{2} + O(1),$$

so in this regime we have

$$\hat{q}(f) \geq \frac{y}{(2^k k )^{1/2}} \geq \frac{1}{(2^k ck)^{1/2}} \leq \frac{1}{d^{1/2}}.$$
which matches with (12).

1.7. Explicit Gardner threshold. We now describe the exact predicted threshold \( \alpha_{Ga} \) for the stability of the 1RSB solution. Recall the loose calculation (13) of the branching rate of linked freees. One can refine this by considering the rate of “bug proliferation” [MR03, KPW04] in the warning model: if a warning incoming to a vertex is changed, it may change an outgoing warning, and one can calculate the branching rate of this process. Explicitly, let

\[
\hat{\omega}_{ai} : 2 \leq a \leq d, 2 \leq i \leq k \equiv (\hat{\omega}_j)_{1 \leq j \leq b} \equiv \hat{\omega}_{1:b} \in \{0, 1, \cdot \}^b
\]

where we have abused notation and made the identification \( \hat{\omega}_{ai} \equiv \hat{\omega}_{(a-2)(k-1)+i-1} \). Recall the mappings \( \hat{W} \) and \( \hat{W}^\phi \) defined in (14) and (15). Define \( \hat{\omega}_a \equiv \hat{W}(\hat{\omega}_{a,2}, \ldots, \hat{\omega}_{a,k}) \) for each \( 2 \leq a \leq d \), and then let

\[
\hat{W}(\hat{\omega}_{1:b}) \equiv \hat{W}(\hat{\omega}_{2}, \ldots, \hat{\omega}_{d}) \\
\hat{W}^\phi(\hat{\omega}_{1:b}) \equiv \hat{W}^\phi(\hat{\omega}_{2}, \ldots, \hat{\omega}_{d})
\]

Let \( \hat{q}_y \) be as given by Proposition 1.7. Then, for \( \hat{\nu}, \hat{\psi}, \hat{\omega}, \hat{\psi} \in \{0, 1, \cdot \} \), let

\[
B_{\hat{\nu}, \hat{\omega}, \hat{\psi}} \equiv \sum_{\hat{\omega}_{1:b}} 1 \left\{ \hat{\nu} = \hat{W}(\hat{\omega}, \hat{\psi}_{\hat{\omega}_{1:b}}) \right\} \exp \left\{ -y \hat{\psi}(\hat{\omega}, \hat{\omega}_{1:b}) \right\} \hat{q}_y(\hat{\omega}) \prod_{i=2}^b \hat{q}_y(\hat{\omega}_i)
\]

This defines a \( 9 \times 9 \) matrix \( B \), which is the stability matrix for our model. We let \( B_x \) be the \( 6 \times 6 \) submatrix with row and column indices in \( \{ (\hat{\nu}, \hat{\omega}) : \hat{\omega} \neq \hat{\omega} \} \), and let \( \lambda \equiv \lambda_y(\alpha) \) be the largest eigenvalue of \( B_x \). The physics literature [MR03, KPW04] proposes that the 1RSB solution is correct as long as \( b \lambda_y(\alpha) \) (a refinement of (13)) is less than one at \( y = y_*(\alpha) \). We extract its large-\( k \) behavior in the following:

**Proposition 1.9** (proved in Section 3). The Gardner threshold \( \alpha_{Ga} \) can be formally defined as

\[
\alpha_{Ga} \equiv \sup \left\{ \alpha \leq 4^k/k : b \lambda_y(\alpha) > 1 \text{ for some } y_* \text{ satisfying (27) with } \Xi(y_*) = 0 \right\}.
\]

The large-\( k \) behavior is given by \( \alpha_{Ga} \asymp 4^k/k^3 \).

1.8. Interpolation bound. As mentioned before, our proof of Theorem 1.1 is based on a general interpolation upper bound, in the spirit of [GT02, Gue03, PT04, BGT13, Gam14]. The precise bound that we use, as we now describe, is a generalization of a similar result in [SSZ16]. Let \( \Omega \) be the space of probability measures on \( \{0, 1, \cdot \} \). We write \( \rho \) for elements of \( \Omega \), and \( Q \) for probability measures over \( \Omega \). Similarly as above, we will abuse notation and write

\[
\left( \hat{\omega}_{ai} : 1 \leq a \leq d, 2 \leq i \leq k \right) \equiv (\hat{\omega}_j)_{1 \leq j \leq D} \equiv \hat{\omega}_{1:D}
\]

where \( D \equiv d(k-1) \). Let \( \hat{\omega}_a \equiv \hat{W}(\hat{\omega}_{a,2}, \ldots, \hat{\omega}_{a,k}) \) and \( \hat{W}(\hat{\omega}_{1:D}) \equiv \hat{W}(\hat{\omega}_{1}, \ldots, \hat{\omega}_{d}) \). Define

\[
\mathcal{G}(y_1, y_2, Q) = \int \left\{ \sum_{\hat{\omega}_{1:k}} \exp(-y_2 \hat{W}(\hat{\omega}_{1:k})) \prod_{j=1}^k \rho_j(\hat{\omega}_j) \right\} \prod_{j=1}^k dQ(\rho_j),
\]

\[
\mathcal{W}(y_1, y_2, Q) = \int \left\{ \sum_{\hat{\omega}_{1:D}} \exp(-y_2 \hat{W}(\hat{\omega}_{1:D})) \prod_{j=1}^D \rho_j(\hat{\omega}_j) \right\} \prod_{j=1}^D dQ(\rho_j).
\]

For \( 0 \leq y_1 \leq y_2 \), the zero-temperature 2RSB functional is defined by

\[
\Phi^{2RSB}(y_1, y_2, Q) \equiv \frac{1}{y_1} \ln \mathcal{W}(y_1, y_2, Q) - \frac{\alpha(k-1)}{y_1} \ln \mathcal{G}(y_1, y_2, Q).
\]

A heuristic derivation of \( \Phi^{2RSB} \) is presented in Section 4, but we briefly describe it here. For simplicity assume \( \alpha \equiv d/k \) is an integer, and let \( \tilde{\mathcal{G}}_{N/2} \) be an instance of random \( d \)-regular \( k \)-NAE-SAT on \( N \) variables. Remove \( \alpha(k-1) \) clauses and their incident edges at random, and call the resulting graph \( \tilde{\mathcal{G}}_{N+1/2} \); it is still a \( k \)-NAE-SAT instance on \( N \) variables, but is no longer \( d \)-regular since some variables have open “slots” (missing edges). Then introduce a new variable \( v \equiv u_{N+1} \), together with \( d \) new edges. For each new clause, add one new edge connecting the clause to
\(v\), and \(k - 1\) new edges connecting the clause to the open “slots” in \(\mathcal{H}_{N+1/2}\). Then the resulting graph \(\mathcal{H}_{N+1}\) is an instance of random \(d\)-regular \(k\)-NAE-SAT on \(N + 1\) variables. For \(\beta \geq 0\) we can consider
\[
\mu_\beta(x) \equiv \frac{\exp\{-\beta \mathcal{H}_{N+1/2}(x)\}}{Z_{N+1/2}(\beta)}
\]
where \(Z_{N+1/2}(\beta)\) is the normalizing constant that makes \(\mu_\beta\) a probability measure over \(x \in \{0, 1\}^N\). The structure of \(\mu_\beta\) is not known. However, by analogy with other models [Der81, Der85, Rue87, Par79, Pan13a], a natural simplifying assumption is that it has a **hierarchical (ultrametric) structure** with Poisson–Dirichlet weights on each level of the hierarchy. This means that the \(\ell\)-point marginals of \(\mu_\beta\), for bounded \(\ell\), converge in the large-\(N\) limit to an explicit form: for a two-level hierarchy,
\[
\mu_\beta(x_1, \ldots, x_\ell) \approx \sum_{s,l_1} \nu_{sl_1} \prod_{i=1}^{\ell} w_{sl_1}(x_i),
\]
where the \(w_{sl_1}\) are sampled recursively as follows. Let \(\mathcal{P}_0 \equiv \mathcal{P}\) be the space of probability measures over \(\{0, 1\}\), and for \(r \geq 1\) let \(\mathcal{P}_r\) be the space of probability measures over \(\mathcal{P}_{r-1}\). Let \(Q_\beta \in \mathcal{P}_2\). Let \((r_{s,i})_{s,i} \) be i.i.d. samples from law \(Q_\beta\). For each \(i\) and each \(s\), let \((w_{sl_1})_{s,l_1} \geq 1\) be a sequence of i.i.d. samples from \(r_{s,i}\). Note \(r_{s,i} \in \mathcal{P}_1\) so \(w_{sl_1} \in \mathcal{P}\). Independently, \((w_{sl_1})_{s,l_1} \geq 1\) are random weights sampled from the law of a **Ruelle probability cascade** (RPC) with parameters \(0 < m_1 < m_2 < 1\) — a two-level version of the standard Poisson–Dirichlet process (see [Pan13b, Ch. 2] and Section 4). Under assumption (44), and taking \(\beta \to \infty\) with \(m_1 \beta \to y_1\), one has
\[
\begin{align*}
\lim_{\beta \to \infty} \frac{1}{\beta} \ln \frac{Z_N(\beta)}{Z_{N+1/2}(\beta)} &= \frac{a(k-1)}{y_1} \ln G(y_1, y_2, Q), \\
\lim_{\beta \to \infty} \frac{1}{\beta} \ln \frac{Z_{N+1}(\beta)}{Z_{N+1/2}(\beta)} &= \frac{1}{y_1} \ln \mathcal{W}(y_1, y_2, Q),
\end{align*}
\]
where \(Q\) is a probability measure over \(\Omega\) obtained as a projection of \(Q_\beta\). The basic idea is as follows:

a. Project \(w \in \mathcal{P}\) to \(w \in \{0, 1, f\}\) where \(\{w \text{ near } 1_0\}\) maps to \(w = 0\), \(\{w \text{ near } 1_1\}\) maps to \(w = 1\), and the remaining \(w \in \mathcal{P}\) map to \(w = f\). Denote this mapping \(\pi : \mathcal{P} \to \{0, 1, f\}\).

b. Project \(r \in \mathcal{P}_1\) to \(\rho \in \Omega\) via the pushforward, \(\rho(w) = (\pi_\rho)(w) = r(\pi^{-1}(w))\).

c. Project \(Q_\beta \in \mathcal{P}_2\) to a probability measure \(Q\) over \(\Omega\) via another pushforward, \(Q = (\pi_\xi)^1 Q_\beta = Q_\beta \circ (\pi_\xi)^{-1}\).

The details are given in Section 4. Combining the above relations gives the heuristic approximation
\[
-e_{\min} = \lim_{\beta \to \infty} \frac{1}{N\beta} \ln Z_N(\beta) = \lim_{\beta \to \infty} \frac{1}{\beta} \ln \frac{Z_{N+1}(\beta)}{Z_N(\beta)} = \Phi^{\text{RSB}}(y_1, y_2, Q).
\]

The following proposition shows that one side of the approximation can be made rigorous:

**Proposition 1.10** (proved in Section 4). For any parameters \(0 \leq y_1 \leq y_2\) and any probability measure \(Q\) over \(\Omega\), we have a corresponding zero-temperature 2RSB bound \(\inf_{y \geq 0} \Phi^{\text{RSB}}(y, y_2, Q) \leq \Phi^{\text{RSB}}(y_1, y_2, Q)\).

The detailed heuristic derivation of \(\Phi^{\text{RSB}}\), as well as the proof of Proposition 1.10, are given in Section 4. There are two simple ways in which \(\Phi^{\text{RSB}}\) can degenerate:

I. The probability measure \(Q\) is fully supported on a single element \(\rho \in \Omega\). In this case \(\Phi^{\text{RSB}}(y_1, y_2, Q)\) depends only on \(y_2\) and \(\rho\), so we can define \(\Phi^{\text{RSB}}(y_1, y_2, Q) \equiv \Phi^{\text{RSB}}(y_2, \rho)\).

II. The probability measure \(Q\) decomposes as \(Q = \rho_0 Q_0 + \rho_1 Q_1 + \rho_2 Q_2\) where each \(Q_k\) is fully supported on the single element \(1_k \in \Omega\). In this case we have \(\Phi^{\text{RSB}}(y_1, y_2, Q) = \Phi^{\text{RSB}}(y_1, \rho)\).

One can verify by straightforward algebraic manipulations that
\[
\Phi^{\text{RSB}}(y, \hat{\gamma}_y) = \frac{\hat{\gamma}(\gamma)}{y}.
\]

Thus an immediate consequence of Proposition 1.10 is that
\[
\inf_{y \geq 0} \Phi^{\text{RSB}}(y, y_2, Q) \leq \inf_{y \geq 0} \Phi^{\text{RSB}}(y, \hat{\gamma}_y) = \inf_{y \geq 0} \frac{\hat{\gamma}(\gamma)}{y},
\]
(45)
where $Q_\Pi \equiv \hat{q}_y(0)Q_0 + \hat{q}_y(1)Q_1 + \hat{q}_y(\bar{f})Q_{\bar{f}}$. It has been observed in the physics literature [MR03, KPW04] that linearizing the stationarity equations (equivalently, the 2RSB cavity equations)

$$\partial \Phi^{2\text{RSB}}(y_1, y_2, Q) \overline{\partial Q} = 0$$

around $Q = Q_\Pi$ gives rise to the stability matrix $B_\Pi$ introduced in §1.1. To prove Theorem 1.1, we show that an explicit perturbation of $(y, y, Q_\Pi)$ decreases the value of $\Phi^{2\text{RSB}}$ as soon as the top eigenvalue of $B_\Pi$ exceeds $1/b$. While the physics literature certainly hints that this would be the case, to our knowledge this rigorous connection between the Gardner eigenvalue and the stability of the 2RSB functional has not been previously established.

### Organization of paper.
In Section 2 we prove Propositions 1.7 and 1.8, as well as the general version of (12). In Section 3 we prove Proposition 1.9 and Theorem 1.1. Finally, in Section 4 we review the 2RSB heuristic and prove Proposition 1.10.

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#### 2. Tree recursions and 1RSB value

In this section we analyze the survey propagation (sp) recursions introduced in §1.6. The proof of Proposition 1.7 appears in §2.1. We give some discussion on the sp formula as a variational problem in §2.2. The comparison with the first moment bound (extending (12)) appears in §2.3, and the proof of Proposition 1.8 appears in §2.4. Some technical lemmas are deferred to §2.5–2.6. Recalling (3), we assume throughout the paper that $\alpha = d/k = c^{2k-1} \ln 2$ with $\alpha_{\text{sat}} \leq \alpha \leq 4^k/k$. We always restrict our consideration to parameters $y \geq 0$ satisfying (27). From now on we will often suppress $y$ from the notation, e.g. we write $\text{sp} \equiv \text{sp} \circ \text{sp}$ rather than $\text{sp}_y \equiv \text{sp}_y \circ \text{sp}_y$.

#### 2.1. Contraction of survey propagation.
Recall (24) that $M$ denotes the space of all probability measures $q$ on $\{0, 1, \bar{f}\}$ with the symmetry $q(0) = q(1)$; and recall (28) and (29) the definitions of $M' \subseteq M^* \subseteq M$. Let

$$M^* \equiv \left\{ q \in M^* : q(\bar{f}) = e^{-\Theta(k)} c^{1/2} \right\}$$

Recall (29) and note that $M' \subseteq M^*$: if $c$ is large then (27) forces $y \propto 1/c^{1/2}$, so

$$\frac{e^{-\Theta(k)}}{c^{1/2}} \leq \frac{2^{-ky/2}}{(ck)^{1/2}} \leq \frac{2^{-ky/2}}{(\max\{ck e^{-y/2}, 1\})^{1/2}} \leq \frac{2^{-ky/2}}{e^{1/2}}.$$ 

If $c \gg 1$ then it forces $y \geq \Omega(1)$, so

$$\frac{e^{-\Theta(k)}}{c^{1/2}} \leq \frac{2^{-ky/2}}{(ck)^{1/2}} \leq \frac{2^{-ky/2}}{(\max\{ck e^{-y/2}, 1\})^{1/2}} \leq \frac{2^{-ky/2}}{e^{1/2}}.$$

The next two propositions, which will be proved in this subsection, summarize the contractive behavior of the sp recursion in our regime of interest.

#### Proposition 2.1.
Suppose $\alpha = c^{2k-1} \ln 2$ with $\alpha_{\text{sat}} \leq \alpha \leq 4^k/k$, and suppose $y \geq 0$ satisfies (27). Then for any $\hat{q} \in M^*$ we have $\text{sp}_y(\hat{q}) \in M' \subseteq M^*$.

#### Proposition 2.2.
Suppose $\alpha = c^{2k-1} \ln 2$ with $\alpha_{\text{sat}} \leq \alpha \leq 4^k/k$, and suppose $y \geq 0$ satisfies (27). Then the derivative of the survey propagation map satisfies the estimate

$$\Omega \left( \frac{e^{-\Theta(k)}}{c^{1/2}} \right) \leq \left[ \frac{d[sp_y(\hat{q})](\bar{f})}{d\hat{q}(\bar{f})} \right] \leq O \left( \frac{1}{c^{k/2}} \right)$$

uniformly over all $\hat{q} \in M^*$.

#### Proof of Proposition 1.7.
It follows from Proposition 2.1 that if $\hat{q}(\bar{f}) = 1/k^2$ then $[\text{sp}(\hat{q})](\bar{f}) < \hat{q}(\bar{f})$. On the other hand, an easy calculation gives that if $\hat{q}(\bar{f}) = 0$ then $[\text{sp}(\hat{q})](\bar{f}) > \hat{q}(\bar{f})$. It follows by the intermediate value theorem that a fixed point $[\text{sp}(\hat{q})](\bar{f}) = \hat{q}(\bar{f})$ must exist with $\hat{q}(\bar{f}) \leq 1/k^2$. By Proposition 2.1, any such fixed point must in fact lie in $M' \subseteq M^*$. It then follows from Proposition 2.2 that the fixed point is unique. \qed
To prove Propositions 2.1 and 2.2, we first take advantage of the symmetry condition \( q(0) = q(1) \) that defines \( \mathcal{M} \): it means that the mapping from \( \hat{q} \in \mathcal{M} \) to \( \hat{q} = \hat{\Phi}(\hat{q}) \) to \( \hat{q} = \hat{\Phi}(\hat{q}) \) can be naturally expressed as univariate mappings from \( x \) to \( w \) to \( \hat{x} \) where \( x \equiv \hat{q}(f) \), \( w \equiv 1 - \hat{q}(f) \), and \( \hat{x} \equiv \hat{q}(f) \). First, the clause recursion (25) can be rewritten as
\[
 w = w(x) = \frac{2(1-x)^{k-1}}{2^{k-1}} \leq \frac{4}{2^k}.
\]
(46)
To simplify the variable recursion (26), first write
\[
 AM \equiv \frac{1 + e^{-y}}{2}, \quad GM \equiv \frac{1}{\exp(y/2)},
\]
and use these to define (for \( 0 \leq \ell \leq d - 1 \)) the binomial weights
\[
 A_\ell \equiv A_{d-1,\ell}(w) \equiv \binom{d-1}{\ell} (w \cdot AM)^\ell (1 - w)^{d-1-\ell},
\]
\[
 G_\ell \equiv G_{d-1,\ell}(w) \equiv \binom{d-1}{\ell} (w \cdot GM)^\ell (1 - w)^{d-1-\ell}.
\]
Note that \( 1/2 \leq AM \leq 1 \) always, but \( GM \) can be arbitrarily small. For each \( \ell \) we also define
\[
 P_\ell \equiv P(\text{Bin}(\ell, \frac{1}{1+e^y}) < \frac{\ell}{2}), \quad (47)
\]
\[
 Q_\ell \equiv P(\text{Bin}(\ell, \frac{1}{1+e^y}) = \frac{\ell}{2}), \quad (48)
\]
\[
 S_\ell \equiv P(\text{Bin}(\ell, \frac{1}{2}) = \frac{\ell}{2}), \quad (49)
\]
Note that \( P_\ell = 0 \) for \( \ell = 0 \), \( S_\ell = 0 \) for all \( \ell \) odd, and \( A_\ell Q_\ell = G_\ell S_\ell \). Let
\[
 z_0 \equiv z_0(w) \equiv \sum_{\ell=0}^{d-1} A_\ell P_\ell \equiv \sum_{\ell=0}^{d-1} A_{d-1,\ell}(w) P_\ell,
\]
\[
 z_\ell \equiv z_\ell(w) = \sum_{\ell=0}^{d-1} A_\ell Q_\ell = \sum_{\ell=0}^{d-1} G_\ell S_\ell.
\]
(50)
(51)
Let \( \hat{z}(w) \equiv 2z_0(w) + z_\ell(w) \). The variable recursion (26) can be rewritten as
\[
 \hat{x}(w) = \frac{\hat{z}_\ell(w)}{2z_0(w) + \hat{z}_\ell(w)} = \frac{\hat{z}_\ell(w)}{z(w)}.
\]
(52)
To analyze the recursion, we let \( L_{AM} \) and \( L_{GM} \) be random variables with distribution
\[
 P(L_{AM} = \ell) = \frac{A_\ell P_\ell}{z_0} = \frac{1}{z_0} \binom{d-1}{\ell} (w \cdot AM)^\ell (1 - w)^{d-1-\ell} P(\text{Bin}(\ell, \frac{1}{1+e^y}) < \frac{\ell}{2}),
\]
(53)
\[
 P(L_{GM} = \ell) = \frac{G_\ell S_\ell}{z_\ell} = \frac{1}{z_\ell} \binom{d-1}{\ell} (w \cdot GM)^\ell (1 - w)^{d-1-\ell} P(\text{Bin}(\ell, \frac{1}{2}) = \frac{\ell}{2}),
\]
(54)
for \( 0 \leq \ell \leq d - 1 \). For comparison, let \( \hat{L}_{AM} \) and \( \hat{L}_{GM} \) be random variables with distributions
\[
 P(\hat{L}_{AM} = \ell) = \frac{A_\ell}{(1 - w + w \cdot AM)^{d-1}} \quad \text{for } 0 \leq \ell \leq d - 1,
\]
(55)
\[
 P(\hat{L}_{GM} = \ell) = \frac{G_\ell}{(1 - w + w \cdot GM)^{d-1}} \quad \text{for } 0 \leq \ell \leq d - 1.
\]
(56)
Let \( \ell_{AM} \equiv E\hat{L}_{AM} \) and \( \ell_{GM} \equiv E\hat{L}_{GM} \).
Lemma 2.3 (proved in §2.6). If $\alpha = c 2^{k-1} \ln 2$ with $\alpha_{\text{sat}} \leq \alpha \leq 4^k / k$, and $w = 1/2^k$, then we have

\[ 1 - O\left( \frac{1 - \tilde{z}_t(w)}{\exp\{\Omega(1/k)\}} \right) \leq \frac{1}{\tilde{z}_q(w)} \leq 1, \tag{57} \]

\[ \frac{1}{(1 - w + w \cdot AM)^{d-1}} \leq \frac{1}{(1 - w + w \cdot GM)^{d-1}} \leq \frac{1}{\max\{d w \cdot GM, 1\}^{1/2}}. \tag{58} \]

uniformly over $y \geq 0$ satisfying (27).

Proof of Proposition 2.1. We start from $\tilde{q} \in M$, which means $x = \tilde{q}(f) \leq 1/k^2$. This maps to $\tilde{q} = \tilde{S}(\tilde{q})$, where we can see readily from (46) that

\[ 1 - \tilde{q}(f) = w(x) = \frac{4}{2^k} \left( 1 + O\left( \frac{1}{k} \right) \right). \]

This then maps to $\tilde{q} = \tilde{S}(\tilde{q})$. Substituting the bounds of Lemma 2.3 into (52) gives

\[ \tilde{q}(f) \equiv \tilde{x}(w) \leq \frac{1}{\exp(-d w (AM - GM))} \leq \frac{2^{-k/2}}{(\max\{d w \cdot GM, 1\})^{1/2}}. \]

which implies $\tilde{q} \in M^* \subseteq M^*$. □

We next provide a simple (and rather crude) estimate on $\mathbb{E}L_{AM}$ and $\mathbb{E}L_{GM}$.

Lemma 2.4 (proved in §2.6). If $\alpha = c 2^{k-1} \ln 2$ with $\alpha_{\text{sat}} \leq \alpha \leq 4^k / k$, and $w = 1/2^k$, then we have

\[ |\mathbb{E}L_{AM} - \lambda_{AM}| \leq O((d w)^{1/2}) = O((ck)^{1/2}), \]

\[ |\mathbb{E}L_{GM} - \lambda_{GM}| \leq \min\{d w \cdot GM, (d w \cdot GM)^{1/2}\} \leq O((ck)^{1/2}). \]

uniformly over $y \geq 0$ satisfying (27).

Lemma 2.4 immediately gives $\mathbb{E}(L_{AM} - L_{GM}) \leq O((ck)^{1/2})$. We do not expect this bound to be tight, but it will suffice for our purposes. We also obtain the following lower bound:

Lemma 2.5 (proved in §2.6). In the setting of Lemma 2.4 we also have $\mathbb{E}(L_{AM} - L_{GM}) \geq \Omega(k)$. Proof of Proposition 2.2. From the clause sp recursion (46) we calculate

\[ -w'(x) = \frac{2(k - 1)(1 - x)^{k-2}}{2^k - 2} = kw\left( 1 + O\left( \frac{1}{k} \right) \right). \]

From the variable sp recursion (52) we calculate

\[ -\tilde{x}'(w) = \frac{2\tilde{z}_0\tilde{z}_t}{(2\tilde{z}_0 + \tilde{z}_t)^2} \left( \frac{1}{\tilde{z}_0 \cdot d w} - \frac{1}{\tilde{z}_t \cdot d w} \right) = \frac{1 - \tilde{x} \cdot \mathbb{E}(L_{AM} - L_{GM})}{w(1 - w)}. \]

Combining these gives

\[ \frac{d[Sp(\tilde{q}(f))]}{d\tilde{q}(\tilde{f})} = \tilde{x}'(w(x))w'(x) = \left( 1 + O\left( \frac{1}{k} \right) \right) k\tilde{x}(1 - \tilde{x}) \cdot \mathbb{E}(L_{AM} - L_{GM}) \left( \frac{1}{1 - w} \right). \]

It then follows from Lemmas 2.4 and 2.5 that

\[ \Omega(k^2 \tilde{x}) \leq \frac{d[Sp(\tilde{q}(f))]}{d\tilde{q}(\tilde{f})} \leq O\left( \frac{k\tilde{x}(ck)^{1/2}}{1 - w} \right). \]

Since $\tilde{q} \in M^* \subseteq M^*$, it follows from Proposition 2.1 that $\tilde{q} \in M^*$ as well. The claimed bound then follows from the definition of $M^*$. □
2.2. Replica symmetric formulas for survey propagation model. We first evaluate the formula (34) for \( \bar{\gamma}(y) \). Recall that \( \tilde{q}_y \in \mathcal{M}^* \) is the solution given by Proposition 1.7, and \( \tilde{q}_y \equiv \tilde{S}_y(q) \). We will now express all formulas in terms of \( x_y \equiv \tilde{q}(f) \) and \( w_y \equiv 1 - \tilde{q}(f) \). The normalizing constants of (31) and (32) are equal:

\[ \tilde{\gamma}(\tilde{q}_y) = 1 - \frac{2(1-x_y)^k(1-e^{-y})}{2^k} = 1 - \frac{(1-x_y)w_y(1-e^{-y})}{2} = \tilde{\gamma}(\tilde{q}_y, \tilde{q}_y). \]  

(61)

Meanwhile the normalizing constant of (30) is

\[ \tilde{\gamma}(\tilde{q}) = 2 \left( \tilde{\gamma}_0(w) + \frac{\tilde{\gamma}(\tilde{q})}{2} \right) \equiv \tilde{\gamma}(w) \]

(62)

where \( \tilde{\gamma}_0(w) \) and \( \tilde{\gamma}(\tilde{q}) \) are defined similarly to (53) and (48), but with \( d-1 \) in place of \( d \):

\[ \tilde{\gamma}_0(w) = \sum_{\ell=0}^d A_{d,\ell}(w)P_{\ell}, \quad \tilde{\gamma}(\tilde{q}) = \sum_{\ell=0}^d G_{d,\ell}(w)P_{\ell}. \]

Then we have \( \bar{\gamma}(y) = F(x_y, w_y, y) \) for

\[ F(x, w, y) \equiv \ln \tilde{\gamma}(w) + \alpha \ln \left( 1 - \frac{2(1-x)^k(1-e^{-y})}{2^k} \right) - \alpha k \ln \left( 1 - \frac{(1-x)w(1-e^{-y})}{2} \right). \]

(63)

Since \( \tilde{\gamma}(\tilde{q}_y) \) and \( \tilde{\gamma}(\tilde{q}_y, \tilde{q}_y) \) are equal as noted above, there is more than one way to define \( F(x, w, y) \) such that \( \bar{\gamma}(y) = F(x_y, w_y, y) \). We have chosen the representation (63) because it satisfies the following:

Lemma 2.6. For any given \( y > 0 \), if the pair \((x, w)\) satisfies the equations \( \psi = \psi(x) \) and \( x = \bar{x}(w) \) as in (46) and (52) (where the second relation (52) depends also on \( y \)), then \((x, w)\) is a stationary point of \((x, w) \leftrightarrow F(x, w, y)\).

The proof of Lemma 2.6 is deferred to §2.5, but we now point out its main consequences: first, it gives

\[ \bar{\gamma}'(y) = \frac{\partial F}{\partial x}(x_y, w_y, y) \frac{dx_y}{dy} + \frac{\partial F}{\partial w}(x_y, w_y, y) \frac{dw_y}{dy} + \frac{\partial F}{\partial y}(x_y, w_y, y) = \frac{\partial F}{\partial y}(x_y, w_y, y). \]

The right-hand side above is equal to \(-e(y)\), as defined by (33). Recalling (45), it follows that

\[ \frac{d}{dy} \left( \frac{\bar{\gamma}(y)}{y} \right) = \frac{\bar{\gamma}(y) - y\bar{\gamma}'(y)}{y^2} = \frac{\bar{\gamma}(y) + ye(y)}{y^2} = \frac{\bar{\gamma}(y) + ye(y)}{y^2}, \]

which is to say that a stationary point of \( \bar{\gamma}(y)/y \) corresponds precisely to a root of \( \bar{\gamma}(y)/y^2 \). Moreover

\[ \Sigma'(y) = \frac{d}{dy} \left( \bar{\gamma}(y) + ye(y) \right) = \bar{\gamma}'(y) + e(y) + ye'(y) = ye'(y). \]

Recall the statement of Proposition 1.8; we see from the above that if \( e'(y) < 0 \) then \( y_* \) is unique (depending only on \( k \) and \( \alpha \)). Since we do not prove that \( e'(y) < 0 \), Proposition 1.8 contains a weaker statement that \( \Sigma \) crosses from positive to negative in a certain range of \( y \). However, we now comment in some more detail on basic physical interpretations which lead us to expect \( e'(y) < 0 \):

Remark 2.7. As we already commented before, a salient open question is to prove the matching lower bound of Theorem 1.1, i.e., to prove that \( \lim_N \mathbb{E}[e_{\text{min}}(\phi_N)] = e_{\text{rsb}} \) for all \( \alpha_{\text{sat}} \leq \alpha \leq \alpha_{\text{Ga}} \). A natural approach is to validate the sp heuristic by proving that \( N^{-1} \ln \bar{\gamma}(y) \) indeed converges (for suitable \( y \), as we discuss in Remark 2.8 below) to the predicted value \( \bar{\gamma}(y) \) defined by (34). This is believed to hold, although we do not have a proof in the present paper. However we now review how this prediction relates to other basic properties of \( \bar{\gamma}(y) \). First, it is reasonable to also expect (by interchanging limits) that

\[ \bar{\gamma}'(y) = \lim_{N \to \infty} \frac{d}{d} \ln \bar{\gamma}(y) \bigg|_N = - \lim_{N \to \infty} \frac{1}{\bar{\gamma}(y)} \sum_2 \varphi(y) \exp \left( -y \varphi \right) - \lim_{N \to \infty} \frac{\langle \varphi \rangle_N}{N} \]

where \( \langle \varphi \rangle_N \) denotes the average of \( \varphi(y) \) with respect to the measure \( \mu_y \) defined by (23). This is consistent with the above observation that \( \bar{\gamma}'(y) = -e(y) \), since \( e(y) \) is precisely the replica symmetric prediction for \( \lim_N \langle \varphi \rangle_N / N \). We furthermore expect that

\[ -e'(y) = \bar{\gamma}''(y) = \lim_{N \to \infty} \frac{d^2}{dy^2} \ln \bar{\gamma}(y) \bigg|_N = \lim_{N \to \infty} \frac{\langle \varphi^2 \rangle_N - \langle \varphi \rangle_N^2}{N}. \]
is a strictly positive constant, because it is reasonable to guess that the variance of \( q(y) \), normalized by \( N \), converges to a positive constant. This explains why we expect the root \( y^* \) to be unique (depending only on \( k, \alpha \)).

**Remark 2.8.** Continuing from the previous remark, we also expect that

\[
\lim_{y \to 0} \mathcal{F}(y) = \lim_{N \to \infty} \frac{\ln \mathcal{Z}(0)}{N} \in (0, \infty)
\]

— this is because \( \mathcal{Z}(0) \) simply counts the number of valid warning configurations, which should grow exponentially in \( N \) for any fixed \( k, \alpha \). This means \( \mathcal{F}(y) / y \to \infty \) as \( y \downarrow 0 \). For the sake of discussion, suppose that \( e(y) \) is strictly decreasing, so that \( \Sigma(e(y)) = \Xi(y) \) is well-defined. For all \( y \geq 0 \) we have \( \mathcal{F}(y) = \Sigma(e(y)) - ye(y) \) where \( e(y) \) decreases in \( y \), and reaches \( e_{1SB} \) at \( y = y^* \), where we have

\[
\frac{\mathcal{F}(y^*)}{y^*} = \frac{\Sigma(e(y^*)) - y^* e_{1SB}}{y^*} = -e_{1SB}.
\]

We now make precise the range of \( y \) where the sp heuristic should hold. We expect that for all \( y \geq 0 \) we have

\[
\lim_{N \to \infty} \frac{\ln \mathcal{Z}(y)}{N} \leq \lim_{N \to \infty} \frac{\ln \mathcal{E}\mathcal{Z}(y)}{N} = \mathcal{F}(y),
\]

but that the two limits are equal only for \( y \leq y^* \). For \( y > y^* \) we expect that \( e(y) < e_{1SB}, \Sigma(e(y)) < 0 \), and

\[
\lim_{N \to \infty} \frac{\ln \mathcal{Z}(y)}{N} = \max_\epsilon \left\{ \Sigma(e) - ye : \Sigma(e) \geq 0 \right\} < \max_\epsilon \left\{ \Sigma(e) - ye \right\} = \lim_{N \to \infty} \frac{\ln \mathcal{E}\mathcal{Z}(y)}{N} = \mathcal{F}(y).
\]

In particular, if \( e_{\infty} \) is the smallest \( \epsilon \) for which \( \Sigma(e) > -\infty \), we expect that

\[
\lim_{y \to \infty} \frac{\mathcal{F}(y)}{y} = -e_{\infty} > -e_{1SB}.
\]

This would imply that for \( y \geq 0 \) the function \( \mathcal{F}(y) / y \) is not minimized near the extremes \( y = 0 \) or \( y = \infty \), and so must be minimized at some finite \( y \). We saw above that a stationary point of \( \mathcal{F}(y) / y \) corresponds exactly to a root of \( \Sigma(e(y)) \). In this case, the characterization of \( e_{1SB} \) in Proposition 1.8 could be equivalently expressed as

\[
-e_{1SB}(\alpha) = \inf \left\{ \frac{\mathcal{F}(y)}{y} : y \geq 0 \right\},
\]

which matches the upper bound (45). While the definition of \( e_{1SB} \) in Proposition 1.8 may be more natural from a combinatorial point of view, the characterization (64) is closer to standard spin glass conventions.

We emphasize that Remarks 2.7 and 2.8 are speculative: it remains open to prove the various limits stated there, or to formally establish that (64) is equivalent to the definition from Proposition 1.8. We have included the remarks only for the purpose of elaborating on the physical content of basic properties of the functions \( \mathcal{F}(y), e(y), \) and \( \Xi(y) \). We now proceed to formally prove our claims on the 1SB ground state energy formula.

### 2.3. Comparison with known upper bound

We first establish the comparison (12) between the 1SB bound and the first moment upper bound. As above, let \( \hat{q}_y \) be as given by Proposition 1.7, \( \hat{q}_y \equiv \hat{q}(y) \). Denote \( x \equiv x_y \equiv \hat{q}_y(\xi) \) and \( w \equiv w_y \equiv 1 - \hat{q}_y(\xi) \). Let

\[
\mathcal{F}\mathcal{S}(y) = \ln 2 + a \ln \left( 1 - \frac{2(1 - e^{-y})}{2^k} \right),
\]

and note from (5) that we can express

\[
f_\eta(\alpha, e) = \mathcal{F}\mathcal{S} \left( \ln \frac{1}{\eta} \right) + e \ln \frac{1}{\eta}. \]

Towards the proof of (12), we first establish the following comparison between \( \mathcal{F}(y) \) and \( \mathcal{F}\mathcal{S}(y) \):

**Lemma 2.9.** Under the conditions of Proposition 1.7, we have \( \mathcal{F}(y) \leq \mathcal{F}\mathcal{S}(y) - \Omega(x_y) \) where \( x_y \equiv \hat{q}_y(\xi) \).

**Proof.** With some simple algebraic manipulations we can express

\[
\mathcal{F}(y) - \mathcal{F}\mathcal{S}(y) = -\ln \left( 1 + \frac{\hat{3}_\xi(w)}{2\hat{3}_\xi(w) + \hat{3}_\xi(w)} \right) + \ln \frac{\hat{3}_\xi(w) + \hat{3}_\xi(w)}{(1 - w(1 - \Lambda))w} \left( 1 - \frac{w(1 - \Lambda)}{1 - (4/2^k)(1 - \Lambda)} - k \right) \ln \frac{1 - w(1 - \Lambda)}{1 - (1 - x)w(1 - \Lambda)} \right) \right) .
\]

(65)
We will argue that the dominant contribution comes from the first term. To this end, recall from Proposition 1.7 that
\[ X \asymp \frac{2^{-k\gamma/2}}{(\max\{ke^{-\gamma/2}, 1\})^{1/2}}. \]

From the equation \( w = w(x) \) (see (46)) we have \( w = 4[1 - (k - 1)x + O(k^2x^2)]/2^k \). It follows that
\[
\ln \frac{1 - w(1 - AM)}{1 - (4/2^k)(1 - AM)} = \frac{4}{2^k}(1 - AM)(k - 1)x \left( 1 + O(kx) \right),
\]
\[
(k - 1) \ln \frac{1 - w(1 - AM)}{1 - (1 - x)w(1 - AM)} = \frac{4}{2^k}(1 - AM)(k - 1)x \left( 1 + O(kx) \right).
\]

By combining these we see that the second line of (65) is bounded by
\[
O\left( \frac{4}{2^k}(1 - AM)(k - 1)x \cdot kx \right) = O\left( e(1 - AM)k^2x^2 \right) \leq \frac{O(x)}{\exp(\Omega(k))},
\]
where the last bound above uses the bound on \( x \) together with the observation that \( 1 - AM \approx \min\{1, y\} \approx 1/e^{1/2} \) by (27). Next we estimate \( \hat{\delta}_0(w) + \frac{1}{2}\delta(w) \). Explicitly,
\[
\hat{\delta}_0(w) = \sum_{\ell=0}^{d} \binom{d}{\ell} (w \cdot AM)\ell (1 - w)^{d-\ell} P_{\ell}, \tag{66}
\]
\[
\hat{\delta}(w) = \sum_{\ell=0}^{d} \binom{d}{\ell} (w \cdot AM)\ell (1 - w)^{d-\ell} Q_{\ell}, \tag{67}
\]
with \( P_{\ell} \) as in (47) and \( Q_{\ell} \) as in (48). We have
\[
R_{\ell} \equiv P_{\ell} + Q_{\ell} = \mathbb{P}\left( \text{Bin}\left( \ell, \frac{1}{e^{\gamma} + 1} \right) \leq \frac{\ell}{2} \right) \leq 1,
\]
so clearly \( \hat{\delta}_0(w) + \frac{1}{2}\delta(w) < (1 - w(1 - AM))^{d/2} \), which means that the second term in the first line of (65) is negative.

Finally, the estimates of Lemma 2.3 and Proposition 2.1 remain valid with \( d \) in place of \( d - 1 \), so the first term in the first line of (65) is
\[
\leq -\Omega \left( \frac{\hat{\delta}(w)}{2\hat{\delta}_0(w) + \frac{1}{2}\delta(w)} \right) \leq -\Omega(x).
\]
The claim follows. \( \square \)

Recalling (3), (6), (7), (27), and (29), we define
\[
x(p) \equiv \left( \frac{\exp\{-k(\ln 2) \cdot 2c(p)(1 - \eta(p))^{1/2}\}}{\max\{c(p)k\eta(p)^{1/2}, 1\}} \right)^{1/2}.
\]

We extend (12) to the following bound:

**Corollary 2.10.** For \( \Omega(k/2^k) \leq p^2 \leq 1 \) (corresponding to \( 1 \leq c(p) \leq 2^k/k \)) we have
\[
\alpha_{\max}(p) \leq \left( 1 - \Omega(x(p)) \right) \alpha_{\text{abd}}(p)
\]

for \( p \) defined by (3) and \( x(p) \) defined by (68).

**Proof.** Parametrize \( e = \alpha(1 - p)/2^{k-1} \) as in (3), and let \( y = y_\eta = -\ln \eta \) where \( \eta = \eta(p) \) as in (6). Recall that for this particular choice \( y = y_\eta \) we have \( f_\eta(\alpha, e) = \tilde{\gamma}(y) + ye \). Then Lemma 2.9 gives
\[
\tilde{\gamma}(y_\eta) + y_\eta e \leq f_\eta(\alpha, e) - \Omega(x(p)) = \ln 2 - \Omega(x(p)) - \alpha \left( \frac{(1 - p)\ln \eta}{2^{k-1}} - \ln \left( 1 - \frac{1 - \eta}{2^{k-1}} \right) \right).
\]

By essentially the same calculation as (7), the last expression above will be negative for all \( \alpha \) larger than
\[
\alpha(\eta) \equiv \frac{2^{k-1}(\ln 2 - \Omega(x(p)))}{(2^{k-1} - (1 - p))\ln \frac{2^{k-1} - (1 - p)}{2^{k-1} - (1 - p)} + (1 - p)\ln p} \leq \left( 1 - \Omega(x(p)) \right) \alpha_{\text{abd}}(p).
\]

We extend (12) to the following bound:

**Corollary 2.10.** For \( \Omega(k/2^k) \leq p^2 \leq 1 \) (corresponding to \( 1 \leq c(p) \leq 2^k/k \)) we have
\[
\alpha_{\max}(p) \leq \left( 1 - \Omega(x(p)) \right) \alpha_{\text{abd}}(p)
\]

for \( p \) defined by (3) and \( x(p) \) defined by (68).
From our interpolation bound (45), if $\overline{\mathcal{Y}}(y') + y'$ is negative for any $y' \geq 0$, then

\[-e_{\min} \leq \inf_{y \geq 0} \frac{\overline{\mathcal{Y}}(y)}{y} \leq \frac{\overline{\mathcal{Y}}(y')}{y'} < -e.\]

Since this applies for all $\alpha > \overline{\alpha}(p)$, we conclude $\overline{\alpha}(p) \leq \overline{\alpha}(p)$, proving the claim. \[\square\]

For $k^2 \ll k \ll k^4/k$ it is straightforward to verify that $x(p) = 1/d^{1/2}$, so Corollary 2.10 subsumes (12). One can also verify that for all $p$ in the stated range we have

\[1 \leq 2c(p)(1 - \eta(p)^{1/2})^2 \leq \frac{2(1 - (1 - \eta(p)^{1/2})^2}{p + (1 - p) \ln(1 - p)} \leq 2,\]

and substituting into (68) gives $x(p) \leq 1/2^{k/2}$. This confirms that the improved upper bound of Corollary 2.10 does not contradict the lower bound (10) (which, as we remarked before, is the analogue of the [ANP07] lower bound for this model).

2.4. Ground state energy. In §2.3 we effectively considered $\alpha$ in terms of $e_{\text{ISB}}$, and set the parameter $y$ exactly to match $\ln \eta$ from the first moment calculation; this give a relatively easy way to obtain the comparison (12). We now proceed to the proof of Proposition 1.8 where the main difficulty is to solve for $y$ for which $\overline{\mathcal{Z}}(y) = 0$.

**Lemma 2.11.** In the setting of Proposition 1.7,

\[\overline{\mathcal{Y}}(y) = \ln 2 \left(1 - c(1 - e^{-y})\right) + O\left(\frac{1}{e^{\Omega(k)}}\right).\]

**Lemma 2.12.** In the setting of Proposition 1.7,

\[y e(y) = \frac{c y \ln 2}{e^y} + O\left(\frac{1}{e^{\Omega(k)}}\right).\]

The two lemmas immediately imply Proposition 1.8:

**Proof of Proposition 1.8.** It follows from Lemma 2.11 and 2.12 that the energetic complexity function satisfies

\[\frac{\overline{\mathcal{Z}}(y)}{\ln 2} = \frac{\overline{\mathcal{Y}}(y) + ye(y)}{\ln 2} = 1 - c \left(1 - \frac{1 + y}{e^y}\right) + O\left(\frac{1}{e^{\Omega(k)}}\right) = 1 - \Gamma(y) + O\left(\frac{1}{e^{\Omega(k)}}\right),\]

and the claim follows. \[\square\]

In the remainder of this subsection we prove the preceding two lemmas. As before, we let $\hat{q} \equiv \hat{q}_y$ be the solution of Proposition 1.7, and $\hat{q} \equiv \hat{q}_y \equiv \hat{q}_y(f)$. Let $x \equiv x_y \equiv \hat{q}_y(f)$ and $w \equiv w_y \equiv 1 - \hat{q}_y(f)$.

**Proof of Lemma 2.11.** Recalling (61) and (63), we have (with some rearranging)

\[\overline{\mathcal{Y}}(y) = \ln 2 + \ln \left(\frac{3_0(w)}{2} + \frac{3_2(w)}{2}\right) - \alpha(k - 1) \ln \left(1 - (1 - x)w(1 - AM)\right)\]

\[= \ln 2 + \ln \left(\frac{3_0(w)}{1 - (1 - x)w(1 - AM)}\right) + O(x_d(w)) + \alpha \ln \left(1 - (1 - x)w(1 - AM)\right),\]

where we defined $x_d(w)$ similarly as (52) but with $d$ in place of $d - 1$:

\[x_d(w) \equiv \frac{3_0(w)}{2z_0(w) + z_2(w)}.\]

The estimates of Lemma 2.3 apply equally well with $d$ in place of $d - 1$, so $x_d(w) = x$, and

\[\ln \left(\frac{3_0(w)}{1 - (1 - x)w(1 - AM)}\right) = -d \ln \left(\frac{1 - (1 - x)w(1 - AM)}{1 - w(1 - AM)}\right) + O\left(\frac{1}{e^{\Omega(k)}}\right)\]

\[= O\left(d w(1 - AM)x + \frac{1}{e^{\Omega(k)}}\right) = O\left(e^{1/2}x + \frac{1}{e^{\Omega(k)}}\right) = O\left(\frac{1}{e^{\Omega(k)}}\right),\]

where we used (27) to estimate $1 - AM = \min\{1, y\} = 1/e^{1/2}$, and then obtained the final bound on $x$ using the result from Proposition 1.7 that $\hat{q}_y \in \mathcal{M}^\ast$. Substituting into the expression for $\overline{\mathcal{Y}}(y)$ and simplifying further gives

\[\overline{\mathcal{Y}}(y) = \ln 2 + \alpha \ln \left(1 - (1 - x)w(1 - AM)\right) + O\left(\frac{1}{e^{\Omega(k)}}\right) = \ln 2 - \alpha w(1 - AM) + O\left(\frac{1}{e^{\Omega(k)}}\right).\]
Recalling (46), we have \( w = w(x) = [1 - O(kx)]^{4/2^k} \), while \( \alpha = c2^{k-1} \ln 2 \). The result follows. \( \Box \)

**Proof of Lemma 2.12.** Recall the definition (33) of \( e(y) \): it consists of a variable term, minus a clause term, minus an edge term. We estimate these separately:

**Clause and edge terms.** From the \( sp \) equations, the clause and edge terms of (33) agree, and can be simplified as

\[
\sum_{z} \tilde{\phi}_z(z) V_y(z) = \sum_{x} \tilde{\phi}_x(x) V_y(x) = \frac{2 \hat{q}_y(0) \hat{q}_y(1) e^{-y}}{1 - 2 \hat{q}_y(0) \hat{q}_y(1) (1 - e^{-y})} = \frac{w(\alpha - \frac{1}{2})}{(1 - x)^{-1} - w(1 - \alpha M)} \equiv \tilde{e}(x, w).
\]

Recall from Proposition 1.7 that \( \hat{q}_y \in M^i \subseteq M^* \). Expanding with respect to \( x \) and \( w \) gives

\[
a(k-1) \tilde{e}(x, w) = \frac{\alpha(k-1) w(\alpha - \frac{1}{2})}{1 - w(1 - \alpha M)} \bigg( 1 + O(x) \bigg) = \frac{\alpha(k-1) w / (2e^y)}{1 - w(1 - \alpha M)} + O\left( \frac{dwx}{e^y} \right)
\]

\[
= \frac{\alpha(k-1) w}{2e^y} + O\left( \frac{dwx}{e^y} \right) \bigg( w(1 - \alpha M) + x \bigg) = \frac{\alpha(k-1) w}{2e^y} + O\left( \frac{dwx}{e^y} \right) \bigg( e^{1/2}k y \bigg) e^{1/2} e^{O(k)}
\]

where (similarly as in the proof of Lemma 2.11) we used (27) to estimate \( \alpha M - 1 = \min\{1, y\} \approx 1/e^{1/2} \), and used \( \hat{q}_y \in M^* \) to bound \( x \). Multiplying by \( y \) and simplifying gives

\[
y \left( \frac{\alpha(k-1) w}{2e^y} - \frac{\alpha(k-1) w}{2e^y} \right) = O\left( \frac{dwx}{e^y} \right) \bigg( e^{1/2}k y \bigg) e^{1/2} e^{O(k)} = O\left( \frac{1}{e^{1/2} k} \right)
\]

(70), where we again made use of (27).

**Variable term.** Define \( L_{d,AM}, L_{d,GM}, \tilde{L}_{d,AM}, \tilde{L}_{d,GM} \) similarly to (53), (54), (55), (56), but with \( d \) in place of \( d-1 \):

\[
\mathbf{P}(L_{d,AM} = \ell) = \frac{A_{d,\ell}(w)}{\tilde{h}_0(w)} \bigg( \frac{w}{1-w} \bigg) \mathbf{P}(L_{d,GM} = \ell) = \frac{G_{d,\ell}(w)}{\tilde{h}_1(w)} \bigg( \frac{w}{1-w} \bigg) \frac{dwx}{e^y},
\]

(71)

for all \( 0 \leq \ell \leq d \), and with \( \tilde{h}_0(w) \) and \( \tilde{h}_1(w) \) as defined by (66) and (67). Let \( x_d(w) \) be as defined by (69). Conditional on \( L_{d,AM} = \ell \), let \( X \sim \text{Bin}(\ell, p = 1/1 + e^y) \). Then the variable term in (33) can be simplified as

\[
\sum_{y} \tilde{\phi}_y \tilde{V}_y = \frac{1}{\tilde{h}_0(\tilde{q})} \sum_{\ell \in \ell_1} \min\{\ell_0, \ell_1\} \bigg( \frac{d}{\ell_0, \ell_1} \bigg) \frac{\hat{q}_y(0) \hat{q}_y(1) e^{-y}}{\exp\{y \min\{\ell_0, \ell_1\}\}}
\]

\[
\bigg( 1 - x_d(w) \bigg) \mathbf{E}\left( \mathbf{P}(X < \frac{L_{d,AM} - 1}{2}) \right) = x_d(w) \frac{E_{L_{d,GM}} - 1 - O\left( \frac{1}{e^{1/2} k} \right)}{2e^y}.
\]

Let \( \ell_{d,AM} \equiv \tilde{E}_{d,AM} \) and \( \ell_{d,GM} \equiv \tilde{E}_{d,GM} \). We will show in §2.6 (Lemma 2.14) that

\[
\mathbf{E}\left( \mathbf{E}(X \mid X < \frac{L_{d,AM} - 1}{2}) \right) = \frac{dwx}{e^y} \left( 1 - O\left( \frac{1}{e^{1/2} k} \right) \right).
\]

(72)

Next, note that changing \( d - 1 \) to \( d \) does not affect the analysis of Lemma 2.4, so we have \( E_{L_{d,GM}} = O(\ell_{d,GM}) \) in general, and \( E_{L_{d,GM}} = \ell_{d,GM} + O((\ell_{d,GM})^{-1/2}) \) if \( \ell_{d,GM} \) is large. We also note that

\[
x_d(w) \bigg( \frac{\ell_{d,GM}}{2} - \ell_{d,AM} \bigg) = x_d(w) dw \min\{1, y\} = \frac{dwx}{e^{1/2} e^y} = \frac{\ell_{d,GM} x}{e^{1/2}},
\]

where the estimate of \( \min\{1, y\} \) comes from (27). Consequently, if \( \ell_{d,GM} \) is large, we have

\[
x_d(w) \bigg( \frac{E_{L_{d,GM}}}{2} - \mathbf{E}\left( \mathbf{E}(X \mid X < \frac{L_{d,AM} - 1}{2}) \right) \bigg) = O\left( \frac{\ell_{d,GM} x}{e^{1/2}} \right),
\]

(73)

On the other hand, since \( \ell_{GM} \approx c k / e^{1/2} \), if \( \ell_{GM} = O(1) \) then we must have \( e^y \geq \Omega(c k) \geq \Omega(k) \), in which case (27) forces \( c \approx 1 \). In this case we will simply bound

\[
x_d(w) \bigg( \frac{E_{L_{d,GM}}}{2} - \mathbf{E}\left( \mathbf{E}(X \mid X < \frac{L_{d,AM} - 1}{2}) \right) \bigg) = O\left( \frac{dwx}{e^y} + \ell_{d,GM} x \right) = O\left( \frac{\ell_{d,GM} x}{e^{1/2}} \right).
\]

(74)
By substituting (72), (73), and (74) into the above explicit formula for \( \hat{\epsilon}(w) \), we obtain
\[
y \left( \hat{\epsilon}(w) - \frac{d \hat{w}}{2e^y} \right) = O \left( \frac{dw y}{e^{1/2} e^y e^{\Omega(k)}} + \frac{\ell_{d,GM} x y}{e^{1/2}} + \frac{x y 1 \{ \ell_{d,GM} \geq \Omega(1) \}}{(\ell_{d,GM})^{1/2}} \right) = O \left( \frac{1}{e^{\Omega(k)}} \right),
\] where the last estimate can be justified as follows. Since \( \ell_{d,GM} \leq k/\epsilon y^{1/2} \), if \( \ell_{d,GM} \geq \Omega(1) \) then \( y = O(k) \). Thus
\[
xy 1 \{ \ell_{d,GM} \geq \Omega(1) \} = O(kx) = O \left( \frac{1}{e^{\Omega(k)}} \right).
\]

If \( y \geq \Omega(1) \) then (27) forces \( c \approx 1 \), so
\[
\frac{dy w y}{e^{1/2} e^y e^{\Omega(k)}} + \frac{\ell_{d,GM} x y}{e^{1/2}} = O \left( \frac{k y e^{-\Omega(k)}}{e y e^{\Omega(k)}} + \frac{ky}{e y/2} \right) = O \left( \frac{1}{e^{\Omega(k)}} \right).
\]
If \( y \) is small then (27) forces \( c \approx 1/y^2 \), so
\[
\frac{dy w y}{e^{1/2} e^y e^{\Omega(k)}} + \frac{\ell_{d,GM} x y}{e^{1/2}} = O \left( \frac{k}{e^{\Omega(k)}} + kx \right) = O \left( \frac{1}{e^{\Omega(k)}} \right).
\]
This justifies the last step of (75).

Combined. It follows from (70) and (75) together that
\[
ye(y) = y \left( \hat{\epsilon}(w) - \alpha(k-1)\hat{\epsilon}(x, w) \right) = \frac{y \alpha w}{2e^y} + O \left( \frac{1}{e^{\Omega(k)}} \right) = \frac{cy \ln 2}{e^y} + O \left( \frac{1}{e^{\Omega(k)}} \right),
\]
where the last step uses that \( w = [1 - O(kx)]4/2^k \) and \( c = 2^{k-1} \ln 2 \).

2.5. Stationarity equations. We now derive stationarity equations for \( F(x, w, y) \).

Proof of Lemma 2.6. It is straightforward to check with \( w(x) \) as defined by (46) we have
\[
\frac{\partial F}{\partial x}(x, w, y) = \frac{\alpha k(1 - e^{-y})}{2} \left( \frac{w(x)}{1 - (1 - x)w(x)(1 - AM)} - \frac{w}{1 - (1 - x)w(1 - AM)} \right)
\]
which is zero if \( w = w(x) \). The partial derivative with respect to \( w \) is slightly more involved. We first consider the normalizing constant from (30), which is written above as (62), and with more explicit expressions given in (66) and (67). Denote
\[
\hat{q}_y(\ell_0, \ell_1) \equiv \frac{1}{2(w_y)} \left( d - 1 \right) \frac{\hat{q}_y(0)^{\ell_0} \hat{q}_y(1)^{\ell_1}}{\exp \left( y \min \{ \ell_0, \ell_1 \} \right)}.
\]
We will write, for instance, \( \hat{q}_y(\ell_0, \ell_1 + 2) \) for the sum of \( \hat{q}_y(\ell_0, \ell_1) \) over all pairs \( (\ell_0, \ell_1) \) satisfying \( \ell_0 \geq \ell_1 + 2 \).

With this notation, \( \hat{q}_y(0) = \hat{q}_y(1) = \hat{q}_y(\ell_0 = \ell_1) + 1 \) while \( \hat{q}_y(\ell) = \hat{q}_y(\ell_0 = \ell_1) \). By decomposing (30) according to the first warning \( \hat{u}_1 \), we can compare it with the normalizing constant \( z(w) \) from the sr recursion (52):
\[
\hat{q}_y(\ell_0, \ell_1) \equiv \frac{1}{2(w_y)} \left( d - 1 \right) \frac{\hat{q}_y(0)^{\ell_0} \hat{q}_y(1)^{\ell_1}}{\exp \left( y \min \{ \ell_0, \ell_1 \} \right)}.
\]
We will write, for instance, \( \hat{q}_y(\ell_0, \ell_1 + 2) \) for the sum of \( \hat{q}_y(\ell_0, \ell_1) \) over all pairs \( (\ell_0, \ell_1) \) satisfying \( \ell_0 \geq \ell_1 + 2 \).

With this notation, \( \hat{q}_y(0) = \hat{q}_y(1) = \hat{q}_y(\ell_0 = \ell_1) + 1 \) while \( \hat{q}_y(\ell) = \hat{q}_y(\ell_0 = \ell_1) \). By decomposing (30) according to the first warning \( \hat{u}_1 \), we can compare it with the normalizing constant \( z(w) \) from the sr recursion (52):
\[
\hat{q}_y(\ell_0, \ell_1) \equiv \frac{1}{2(w_y)} \left( d - 1 \right) \frac{\hat{q}_y(0)^{\ell_0} \hat{q}_y(1)^{\ell_1}}{\exp \left( y \min \{ \ell_0, \ell_1 \} \right)}.
\]
Combining with the symmetries \( \hat{q}_y(0) = \hat{q}_y(1) \) and \( \hat{q}_y(0) = \hat{q}_y(1) \) gives (after some algebra)
\[
\hat{q}_y(\ell_0) = 2 \hat{q}_y(0) + \hat{q}_y(1) = 1 - w_y(1 - x_y)(1 - AM) .
\]
In fact, by essentially the same derivation it holds for all \( w \) that
\[
\hat{q}_y(\ell_0) = 1 - w(1 - x(w))(1 - AM) .
\]
with \( \hat{x}(w) \) as in (52). Next, differentiating the above expressions for \( \hat{z}_0(w) \) and \( \hat{z}_4(w) \) gives

\[
(30)'(w) = \sum_{\ell=0}^{d} \left( d \right) \ell(w \cdot AM)^{\ell-1}(1-w)^{d-\ell} \mathbf{P}_\ell \frac{\ell - dw}{w(1-w)} \\
= -\frac{d30(w)}{1-w} + dw \cdot AM \sum_{\ell=1}^{d} \left( d \right) \ell-1(w \cdot AM)^{\ell-1}(1-w)^{d-\ell} \mathbf{P}_\ell \\
= -\frac{d30(w)}{1-w} + dw \cdot AM \sum_{\ell=1}^{d} \left( d \right) \ell-1(w \cdot AM)^{\ell-1}(1-w)^{d-\ell} \mathbf{P}_\ell \\
\]

(77)

\[
(34)'(w) = -\frac{d34(w)}{1-w} + dw \cdot AM \sum_{\ell=1}^{d} \left( d \right) \ell-1(w \cdot AM)^{\ell-1}(1-w)^{d-\ell} \mathbf{Q}_\ell. \\
\]

(78)

For integers \( i \geq 1 \) let \( I_i \) be i.i.d. Bernoulli random variables with \( \mathbb{E}I_i = p \equiv 1/(e^y + 1) \). For any finite subset \( S \) of positive integers let \( Y(S) \) be the sum of \( I_i \) over \( i \in S \). Abbreviating \( \ell \equiv \{1, \ldots, \ell\} \), we have \( \mathbf{P}_\ell \equiv \mathbb{P}(E_\ell) \) where \( E_\ell \equiv \{ Y(\ell) \} < \ell/2 \}, \) and \( \mathbf{Q}_\ell \equiv \mathbb{P}(F_\ell) \) where \( F_\ell \equiv \{ Y(\ell) \} = \ell/2 \}. \) We then consider two cases:

a. If \( \ell \) is odd, then \( E_{\ell-1} = \{ Y(\ell-1) \} < \ell-1/2 \} = \{ Y(\ell-1) \} \leq \ell/2 - 1 \}, \) and it implies

\[
Y(\ell) \leq Y(\ell-1) + 1 \leq \frac{\ell-1}{2} < \frac{\ell}{2}, \\
\]

so \( E_{\ell-1} \subseteq E_\ell \). On the event \( E_\ell \setminus E_{\ell-1} \) we must have

\[
\frac{\ell-1}{2} \leq Y(\ell-1) \leq Y(\ell) < \frac{\ell}{2}, \\
\]

which means \( Y(\ell-1) = Y(\ell) = \ell/2 \). Therefore \( E_\ell \setminus E_{\ell-1} = F_{\ell-1} \cap \{ I_\ell = 0 \} \), and so

\[
\mathbf{P}_\ell + \frac{\mathbf{Q}_\ell}{2} = \mathbf{P}_{\ell-1} + (1-p)\mathbf{Q}_{\ell-1}, \\
\]

where the first equality holds simply because \( \mathbf{Q}_\ell = 0 \) for \( \ell \) odd.

b. If \( \ell \) is even, then \( E_{\ell-1} = \{ Y(\ell-1) \} < \ell-1/2 \} = \{ Y(\ell-1) \} \leq \ell/2 - 1 \}, \) and it implies

\[
Y(\ell) \leq Y(\ell-1) + 1 \leq \frac{\ell}{2}, \\
\]

so \( E_{\ell-1} \subseteq E_\ell \cup F_\ell \). On the event \( (E_\ell \cup F_\ell) \setminus E_{\ell-1} \) we must have

\[
\frac{\ell-1}{2} \leq Y(\ell-1) \leq Y(\ell) \leq \frac{\ell}{2}, \\
\]

which means \( Y(\ell-1) = Y(\ell) = \ell/2 \). Therefore \( (E_\ell \cup F_\ell) \setminus E_{\ell-1} = \{ Y(\ell-1) = \ell/2 \} \cap \{ I_\ell = 0 \} \), and so

\[
\mathbf{P}_\ell + \frac{\mathbf{Q}_\ell}{2} = \mathbf{P}_{\ell-1} + (1-p)\mathbf{P}\left(\text{Bin}(\ell-1, p) = \frac{\ell}{2}\right). \\
\]

Recalling the definition of \( \mathbf{Q}_\ell \) then gives

\[
\mathbf{P}_\ell + \frac{\mathbf{Q}_\ell}{2} = \mathbf{P}_{\ell-1} + (1-p)\mathbf{P}\left(\text{Bin}(\ell-1, p) = \frac{\ell}{2}\right) - \frac{1}{2}\mathbf{P}\left(\text{Bin}(\ell, p) = \frac{\ell}{2}\right) = \mathbf{P}_{\ell-1}, \\
\]

where the last step is by a simple algebraic manipulation of the binomial coefficients.

Combining (77) and (78) gives

\[
\hat{x}'(w) = -\frac{d34(w)}{1-w} + \frac{2d \cdot AM}{1-w} \sum_{\ell=1}^{d} \left( d \right) \ell-1(w \cdot AM)^{\ell-1}(1-w)^{d-\ell} \left( \mathbf{P}_\ell + \frac{\mathbf{Q}_\ell}{2} \right). \\
\]

Substituting the above expressions for \( \mathbf{P}_\ell + \mathbf{Q}_\ell/2 \), then re-indexing \( \ell - 1 \) as \( \ell \), gives

\[
\hat{x}'(w) = -\frac{d34(w)}{1-w} + \frac{2d \cdot AM}{1-w} \sum_{\ell=0}^{d-1} \left( d \right) \ell(w \cdot AM)^{\ell}(1-w)^{d-\ell} \left( \mathbf{P}_\ell + (1-p)\mathbf{Q}_\ell \right) \\
= -\frac{d34(w)}{1-w} + \frac{2d \cdot AM}{1-w} \left( z_0(w) + (1-p)\hat{z}_4(w) \right) \\
= -\frac{d34(w)}{1-w} + \frac{d \cdot AM \cdot \hat{x}(w)}{1-w} \left( 1 + \frac{(1-AM)\hat{x}(w)}{AM} \right). \\
\]
Finally, combining with (76) gives
\[ \frac{\hat{j}'(w)}{\hat{j}(w)} = -d[1 - w(1 - \hat{x}(w))(1 - AM)] + d[AM + (1 - AM)\hat{x}(w)] = \frac{-d(1 - AM)(1 - \hat{x}(w))}{1 - w(1 - \hat{x}(w))(1 - AM)}, \]
where again the last equality is by some simple algebra. Altogether we obtain
\[ \frac{\partial F}{\partial w}(x, w, y) = d(1 - AM)\left\{ \frac{1 - x}{1 - (1 - x)w(1 - AM)} - \frac{1 - \hat{x}(w)}{1 - (1 - \hat{x}(w))w(1 - AM)} \right\}, \]
which is zero if $\hat{x}(w) = x$. \hfill \Box

2.6. Binomial estimates. We now prove the technical estimates used earlier in this section. Recall the classical binomial Chernoff bounds: if $X \sim \text{Bin}(r, \hat{p})$ then it holds for any $t \geq 0$ that
\[ \Pr\{X \geq rp(1 + t)\} \leq \exp\left\{ -tR - \frac{rpt^2}{2 + t} \right\} \leq \exp\left\{ -\frac{rpt^2}{3} \right\}. \tag{79} \]
In the lower tail a simpler bound holds: for all $0 \leq t \leq 1$, we have
\[ \Pr\{X \leq rp(1 - t)\} \leq \exp\left\{ -tR - \frac{rpt^2}{2} \right\} \leq \exp\left\{ -\frac{rpt^2}{8} \right\}. \tag{80} \]
Throughout, $R(x|p) \equiv x \ln(x/p) + (1 - x)\ln(1 - x)/(1 - p)$, the binary relative entropy function. We will make frequent use of (79) and (80) in the remainder of this section.

Lemma 2.13. Let $P_\ell, S_\ell$ be as defined by (47) and (49). For all $0 \leq \ell \leq d - 1$ we have
\[ S_\ell = 1\{\ell \text{ even}\} \left( \frac{2}{\pi \ell} \right)^{1/2} \left[ 1 + O\left( \frac{1}{1 + \ell} \right) \right]. \]
It holds uniformly over all $y \geq 0$ that
\[ 1 - P_\ell \leq \exp\left\{ -tR - \frac{\ell y^2}{8} \right\} \leq \frac{\ell y^2}{8} \tag{81} \]
for $y \geq 0$ small enough.

Proof. The estimate on $S_\ell$ follows immediately from Stirling’s approximation. The binomial Chernoff bound gives
\[ 1 - P_\ell \leq \exp\left\{ -tR - \frac{\ell y^2}{8} \right\} \leq \exp\left\{ -\ell \min\{y, y^2\} \right\} \]
uniformly over all $y \geq 0$. The estimate for small $y$ follows by Taylor expanding the relative entropy function. \hfill \Box

We now turn to the proofs of Lemmas 2.3 and 2.4 which were introduced in §2.1.

Proof of Lemma 2.3. As before we abbreviate $\hat{z}_0 \equiv \hat{z}_0(w)$ and $\hat{z}_\ell \equiv \hat{z}_\ell(w)$.

Bounds on $\hat{z}_0$. From the definition (53), together with the trivial bound $P_\ell \leq 1$, we immediately conclude
\[ \frac{\hat{z}_0}{(1 - w + w \cdot AM)^{d-1}} \leq 1. \]
By the lower bound on $P_\ell$ from Lemma 2.13, we also have
\[ 1 - \frac{\hat{z}_0}{(1 - w + w \cdot AM)^{d-1}} \leq \frac{(1 - w + \exp\left\{ -\ell \min\{y, y^2\} \right\}) w \cdot AM}{1 - w + w \cdot AM} \]
\[ = \left( 1 - \frac{w \cdot AM\left[ 1 - \exp\left\{ -\ell \min\{y, y^2\} \right\} \right]}{1 - w + w \cdot AM} \right)^{d-1} \leq \frac{1}{\exp\{\ell \min\{y^2, 1\}\}} \leq \frac{1}{\exp\{\Omega(k)\}}, \]
since $dw \min\{y^2, 1\} = ck \min\{y^2, 1\} \leq k$ by (27). This proves (57).

Bounds on $\hat{z}_\ell$. From the definition (54), we have trivially
\[ \frac{1}{\exp\{\ell \text{ even} \}} \leq \left( \frac{1 - w}{1 - w + w \cdot GM} \right)^{d-1} \leq \frac{\hat{z}_\ell}{(1 - w + w \cdot GM)^{d-1}} \leq 1, \]
where the left-hand side is $\Omega(1)$ as long as $d w \cdot GM = O(1)$. It remains to consider what happens when $d w \cdot GM$ is large. From the estimate of Lemma 2.13 we have

$$\frac{z}{(1 - w + w \cdot GM)^{d-1}} = \mathbb{E} \left[ 1\{L_{GM} \text{ even}\} \left( \frac{2}{\pi L_{GM}} \right)^{1/2} \left\{ 1 + O \left( \frac{1}{1 + L_{GM}} \right) \right\} \right].$$

Since $L_{GM} \gg d w \cdot GM$, it follows from the Chernoff bound (79) that for a large enough absolute constant $C$,

$$\mathbb{P} \left( |L_{GM} - L_{AM}| \geq C(L_{GM} \ln L_{GM})^{1/2} \right) \leq \frac{1}{(L_{GM})^{20}},$$

(where the power 10 is somewhat arbitrary, but large enough for our purposes). This implies

$$\frac{z}{(1 - w + w \cdot GM)^{d-1}} = \left( \frac{2}{2\pi L_{GM}} \right)^{1/2} \left( 1 + O \left( \frac{(L_{GM} \ln L_{GM})^{1/2}}{L_{GM}} \right) \right) \mathbb{P}(L_{GM} \text{ even}) + O \left( \frac{1}{(L_{GM})^{1/2}} \right).$$

We then note that

$$\mathbb{P}(L_{GM} \text{ even}) = \frac{1 + \mathbb{E}[(-1)^{L_{GM}}]}{2} = \frac{1}{2} \left( 1 + \left( \frac{1 - w - w \cdot GM}{1 - w + w \cdot GM} \right)^{d-1} \right) = \frac{1}{2} + O \left( \frac{1}{\exp \{ \Omega(L_{GM}) \}} \right).$$

Combining with the previous estimate gives

$$\frac{z}{(1 - w + w \cdot GM)^{d-1}} = \left( \frac{2}{2\pi L_{GM}} \right)^{1/2} \left( 1 + O \left( \frac{(L_{GM} \ln L_{GM})^{1/2}}{(L_{GM})^{1/2}} \right) \right) \leq \frac{1}{(L_{GM})^{1/2}},$$

from which (58) follows.\[\square\]

**Proof of Lemma 2.4.** We abbreviate $p = 1/(e^y + 1)$ as well as

$$p_{AM} \equiv \frac{w \cdot AM}{1 - w + w \cdot AM}, \quad p_{GM} \equiv \frac{w \cdot GM}{1 - w + w \cdot GM}.$$

Recall the definitions (53) and (54) of $L_{AM}$ and $L_{GM}$.

**Bounds on $\mathbb{E}L_{AM}$.** By Jensen’s inequality we have

$$\left( \mathbb{E}L_{AM} - \ell_{AM} \right)^2 \leq \mathbb{E} \left( L_{AM} - \ell_{AM} \right)^2 = \sum_{\ell = 0}^{d-1} \frac{A_{\ell} L_{AM} \ell - \ell_{AM})^2}{2z} = \sum_{\ell = 0}^{d-1} \frac{A_{\ell} L_{AM} \ell - \ell_{AM})^2}{(1 - w + w \cdot AM)^{d-1}}\] (82)

where the last estimate is by (57) from Lemma 2.3. Since $P_{\ell} \leq 1$, the last expression above is upper bounded by

$$\sum_{\ell = 0}^{d-1} \frac{A_{\ell} (\ell - \ell_{AM})^2}{(1 - w + w \cdot AM)^{d-1}} = \text{Var} L_{AM} = (d - 1)p_{AM}(1 - p_{AM}) = (d - 1)p_{AM} = dw,$$

which proves the first claim (59).

**Bounds on $\mathbb{E}L_{GM}$ when $L_{GM}$ is large.** Abbreviate $J$ for the integers between $L_{GM}/2$ and $2L_{GM}$. Recalling the estimate (58) from Lemma 2.3, we have

$$\left| \mathbb{E}L_{GM} - L_{GM} \right| \leq \left| \mathbb{E}L_{GM} - L_{AM} \right| + \sum_{\ell \in J} \frac{\ell - \ell_{GM} |G_{\ell} S_{\ell}}{2z} + \sum_{\ell \in J} \frac{(L_{GM} \ell - \ell_{GM} |G_{\ell}}{1 - w + w \cdot GM)^{d-1}.$$ (83)

The lower tail Chernoff bound (80) gives

$$\sum_{\ell \leq \ell_{GM}/2} (L_{GM} \ell - \ell_{GM} |G_{\ell}(1 - w + w \cdot GM)^{d-1} \leq (L_{GM})^{3/2} \mathbb{P}\left( \text{Bin}(d - 1, p_{GM}) \leq \frac{L_{GM}}{2} \right) \leq \frac{(L_{GM})^{3/2}}{\exp \{ \Omega(L_{GM}) \}}.$$ (84)

The upper tail Chernoff bound (79) gives

$$\sum_{\ell \geq 2L_{GM}} (L_{GM} \ell - \ell_{GM} |G_{\ell}(1 - w + w \cdot GM)^{d-1} \leq \sum_{j \geq 0} (L_{GM})^{1/2}(L_{GM} + j) \mathbb{P}\left( \text{Bin}(d - 1, p_{GM}) \geq 2L_{GM} + j \right)
\leq \frac{(L_{GM})^{3/2}}{\exp \{ \Omega(L_{GM}) \}} + \sum_{j \geq 0} \frac{(L_{GM})^{1/2}j}{\exp \{ \Omega(L_{GM} + j) \}} \leq \frac{(L_{GM})^{3/2}}{\exp \{ \Omega(L_{GM}) \}}.$$
By Lemma 2.13 we have $S_\ell \approx 1/(\ell_{\text{GM}})^{1/2}$ uniformly over $\ell \in J$. We then have (similarly to (82), and using (58) again)

$$\left( \frac{1}{2\ell} \sum_{\ell \in J} |\ell - \ell_{\text{GM}}|G_\ell S_\ell \right)^2 \leq \frac{1}{2\ell} \sum_{\ell \in J} (\ell - \ell_{\text{GM}})^2 G_\ell S_\ell \approx \sum_{\ell \in J} \frac{(\ell - \ell_{\text{GM}})^2 G_\ell}{(1 - w + w \cdot \text{GM})^{d-1}},$$

$$\leq \sum_{\ell=0}^{d-1} \frac{(\ell - \ell_{\text{GM}})^2 G_\ell}{(1 - w + w \cdot \text{GM})^{d-1}} = \text{Var} \bar{L}_{\text{GM}} = (d - 1)p_{\text{GM}}(1 - p_{\text{GM}}) \times dw,$$

Substituting these estimates back into (83) gives the claim (60) in the case that $\ell_{\text{GM}}$ is large.

**Bounds on $\mathbb{E}L_{\text{GM}}$ when $\ell_{\text{GM}} = O(1)$.** Write $\bar{P}$ for the law of $\bar{L}_{\text{GM}}$ conditioned to be even. It is straightforward to check that $S(\ell) = S_\ell$ is nonincreasing with respect to $\ell$ even. It follows that $L_{\text{GM}}$ and $S(L_{\text{GM}})$ have nonpositive covariance under $\bar{P}$; to see this, let $L, L'$ be independent samples from the law $\bar{P}$, and note

$$\text{Cov}_{\bar{P}} \left(S(L_{\text{GM}})L_{\text{GM}}, L_{\text{GM}}\right) = \frac{1}{2} \mathbb{E}\left(\left(S(L) - S(L')\right)\left(L - L'\right)\right) \leq 0,$$  \hfill (85)

where the last inequality holds since the random variable inside the expectation is nonpositive almost surely. Thus

$$\mathbb{E}L_{\text{GM}} = \mathbb{E}\left[\frac{\mathbb{E}[S(L_{\text{GM}})L_{\text{GM}}]}{\mathbb{E}[S(L_{\text{GM}})]}\right] \leq \mathbb{E}\left(\frac{L_{\text{GM}}}{\mathbb{P}(L_{\text{GM}} \text{ even})}\right) \leq \mathbb{E}(L_{\text{GM}}) \leq \mathbb{P}(L_{\text{GM}} \text{ even}) = O(\ell_{\text{GM}}),$$ \hfill (86)

by the assumption that $\ell_{\text{GM}} \approx dw \cdot \text{GM} = O(1)$. This proves (60) in the case $\ell_{\text{GM}} = O(1)$.

**Proof of Lemma 2.5.** We first use simple correlation inequalities (such as (86)) to obtain one-sided improvements on the bounds of Lemma 2.4.

**Upper bound on $\mathbb{E}L_{\text{GM}}$.** Recall from (86) that $\mathbb{E}L_{\text{GM}} \leq \mathbb{E}\bar{L}_{\text{GM}}$, which is $O(\ell_{\text{GM}})$ in the case $\ell_{\text{GM}} = O(1)$. If $\ell_{\text{GM}}$ is large, then we can use the binomial moment-generating function to estimate

$$\mathbb{E}\left((-1)^{\bar{L}_{\text{GM}}}ight) = (1 - 2p_{\text{GM}})^{d-1} = \left(1 - \frac{w - w \cdot \text{GM}}{1 - w + w \cdot \text{GM}}\right)^{d-1} = O\left(\frac{1}{\exp\{\ell_{\text{GM}}\}}\right),$$

$$\mathbb{E}\left(L_{\text{GM}}(-1)^{\bar{L}_{\text{GM}}}ight) = \frac{d}{d\theta} \left(1 - p_{\text{GM}} + p_{\text{GM}}e^{\theta}\right)^{d-1}\bigg|_{\theta = i\pi} = O\left(\frac{\ell_{\text{GM}}}{\exp\{\ell_{\text{GM}}\}}\right) = O\left(\frac{1}{\exp\{\ell_{\text{GM}}\}}\right).$$

Rearranging these bounds gives

$$\mathbb{P}(L_{\text{GM}} \text{ even}) = \frac{1 + \mathbb{E}[(-1)^{\bar{L}_{\text{GM}}}] - \mathbb{E}[L_{\text{GM}}(-1)^{\bar{L}_{\text{GM}}}]}{2} = \frac{1}{2} + O\left(\frac{1}{\exp\{\ell_{\text{GM}}\}}\right),$$

$$\mathbb{E}(L_{\text{GM}}) = \frac{\mathbb{E}\bar{L}_{\text{GM}} + \mathbb{E}L_{\text{GM}}(-1)^{\bar{L}_{\text{GM}}}}{2\mathbb{P}(L_{\text{GM}} \text{ even})} = \ell_{\text{GM}} + O\left(\frac{1}{\exp\{\ell_{\text{GM}}\}}\right).$$ \hfill (87)

We therefore conclude that if $\ell_{\text{GM}}$ is large then

$$\mathbb{E}L_{\text{GM}} \leq \ell_{\text{GM}} + O\left(\frac{1}{\exp\{\ell_{\text{GM}}\}}\right),$$ \hfill (88)

which is an improvement on the upper bound on $\mathbb{E}L_{\text{GM}}$ obtained in Lemma 2.4.

**Lower bound on $\mathbb{E}L_{\text{GM}}$.** As in the proof of Lemma 2.6, for integers $i \geq 1$ let $l_i$ be i.i.d Bernoulli random variables with $\mathbb{E}l_i = p \equiv 1/(e^y + 1)$. For any finite subset $S$ of positive integers let $Y(S)$ be the sum of $l_i$ over $i \in S$. Abbreviating $[\ell] \equiv \{1, \ldots, \ell\}$, we have $P_\ell \equiv P(E_\ell)$ where $E_\ell \equiv \{Y([\ell]) < \ell/2\}$. We then note that

$$E_{\ell+2} \setminus E_{\ell} = \begin{cases}
\ell/2 \leq Y([\ell]) \leq Y([\ell + 2]) < \ell/2 \\
Y([\ell]) = \ell/2, Y([\ell + 1, \ell_2]) = 0
\end{cases},$$

$$E_{\ell} \setminus E_{\ell+2} = \begin{cases}
\ell/2 - 1 \leq Y([\ell + 2]) - 2 \leq Y([\ell]) < \ell/2 \\
Y([\ell]) = \ell/2 - 1, Y([\ell + 1, \ell_2]) = 2
\end{cases}.$$


Writing \( p_\ell(k) \equiv \mathbb{P}(\text{Bin}(\ell, p) = k) \), the above implies

\[
P_{\ell+2} - P_\ell = p_\ell \left( \left\lfloor \frac{\ell}{2} \right\rfloor (1 - p)^2 - p_\ell \left( \left\lfloor \frac{\ell}{2} \right\rfloor - 1 \right) p^2 \right) = p_\ell \left( \left\lfloor \frac{\ell}{2} \right\rfloor (1 - p)^2 \left[ 1 - \frac{e^{-y \left\lceil \ell/2 \right\rceil}}{\ell + 1 - \left\lceil \ell/2 \right\rceil} \right] \right) \geq 0,
\]

since \( 2 \left\lfloor \ell/2 \right\rfloor \leq \ell + 1 \) and \( y \geq 0 \). Now let \( P \) and \( \bar{P} \) denote the laws of the binomial random variable \( \bar{L}_{AM} \) conditioned to be even and odd, respectively. Write \( P(\ell) \equiv P_\ell \). The bound (89) implies (by the same argument as in (85)) that \( \bar{L}_{AM} \) and \( \bar{P}(L_{AM}) \) have nonnegative covariance under \( \bar{P} \) and under \( P \). Rearranging gives

\[
\mathbb{E}(L_{AM} \mid L_{AM} \text{ even}) \geq \mathbb{E} \bar{L}_{AM},
\]

\[
\mathbb{E}(L_{AM} \mid L_{AM} \text{ odd}) \geq \mathbb{E} \bar{L}_{AM}.
\]

(90)

and similarly \( \mathbb{E}(L_{AM} \mid L_{AM} \text{ odd}) \geq \mathbb{E} \bar{L}_{AM} \). By the argument of (87) we have

\[
P(\bar{L}_{AM} \text{ even}) = \frac{1}{2} + O \left( \frac{1}{\exp\{\Omega(ck)\}} \right),
\]

\[
\bar{E}(L_{AM}) = \ell_{AM} + O \left( \frac{1}{\exp\{\Omega(ck)\}} \right) = \bar{E}(L_{AM}),
\]

having used that \( \ell_{AM} \approx dw \approx ck \). This gives

\[
\mathbb{E} L_{AM} \geq \ell_{AM} - O \left( \frac{1}{\exp\{\Omega(ck)\}} \right)
\]

(91)

which is an improvement on the lower bound on \( \mathbb{E} L_{AM} \) obtained in Lemma 2.4.

**Conclusion.** First of all we note that (27) implies

\[
\ell_{AM} - \ell_{GM} = \frac{(d - 1)w(1 - w)(AM - GM)}{(1 - w + w \cdot AM)(1 - w + w \cdot GM)} \approx dw(AM - GM) \geq \Omega(k).
\]

If \( \ell_{GM} \) is large, then combining with (88) and (91) gives

\[
\mathbb{E}(L_{AM} - L_{GM}) \geq \ell_{AM} - \ell_{GM} - O \left( \frac{1}{\exp\{\Omega(\ell_{GM})\}} \right) \geq \Omega(k).
\]

If \( \ell_{GM} = O(1) \) then \( \mathbb{E}(L_{AM} - L_{GM}) \geq \ell_{AM} - O(\ell_{GM}) \geq \Omega(ck) - O(1) \geq \Omega(k) \). The result follows. \( \square \)

The next lemma was used in the proof of Proposition 1.8 (in §2.4).

**Lemma 2.14.** In the setting of Proposition 1.8, let \( L_{d,AM} \) and \( \bar{L}_{d,AM} \) be the random variables with laws given by (71). Conditional on \( L_{d,AM} = \ell \) let \( X \sim \text{Bin}(\ell, (1/2) + 1) \). Then

\[
\mathbb{E} \left( \mathbb{E} \left( X \middle| X < \frac{L_{d,AM}}{2} \right) \right) = \frac{dw}{2e^y} \left( 1 + O \left( \frac{1}{e^{1/2} e^{\Omega(ck)}} \right) \right),
\]

(On the left-hand side above, the inner expectation over \( X \) conditional on \( L_{d,AM} \), and the outer expectation is over \( L_{d,AM} \).)

**Proof.** Let \( p = 1/(1 + e^y) \), and \( X \sim \text{Bin}(\ell, p) \). We also define \( P_{AM} \) as before, so \( \ell_{d,AM} \equiv \mathbb{E} \bar{L}_{d,AM} = dp_{AM} \), and

\[
(\mathbb{E} \bar{L}_{d,AM}) P = \frac{dp_{AM} e^{y/2}}{1 - w(1 - AM)} = \frac{dw}{2e^y} \left( 1 + O(w(1 - AM)) \right) = \frac{dw}{2e^y} \left( 1 + O \left( \frac{1}{2e^{y/2}} \right) \right),
\]

where the last step used (27) to estimate \( 1 - AM \approx \min\{1, y\} \approx 1/e^{y/2} \). We shall now consider the expectation of \( X \) conditioned on \( L_{d,AM} \), then average over the law of \( L_{d,AM} \).

Conditional on \( L_{d,AM} = \ell \). Abbreviate \( E(\ell) \equiv \mathbb{E}(X \mid X < \ell/2) \). Then

\[
E(\ell) - \ell P = \frac{E(X - EX; X < \ell/2)}{P(X < \ell/2)} = -E(X - EX; X \geq \ell/2) \geq \ell(1/2 - p) - \ell P(X < \ell/2) \geq 0.
\]

(93)

Now suppose for the moment that \( \ell \approx \ell_{d,AM} \). By Lemma 2.13 and (27), for such \( \ell \) we have

\[
P \left( X \geq \frac{\ell}{2} \right) = 1 - P_{\ell} \leq \exp \left( -\Omega \left( \ell \min\{y, y^2\} \right) \right) \leq \exp \left( -\Omega \left( \ell \min\{1, y^2\} \right) \right) \leq \frac{1}{e^{\Omega(k)}}.
\]

If \( y \) is small then \( \ell(1/2 - p) \approx \ell y \), so for a large enough constant \( C \) we can bound

\[
\mathbb{E} \left( X - EX; X \geq \ell \frac{\ell}{2} \right) \leq 2C\ell y P \left( X \geq \ell \frac{\ell}{2} \right) + \ell P \left( X \geq \ell \frac{\ell}{2} + C\ell y \right) \leq O \left( \frac{\ell y}{\exp\{\Omega(\ell y^2)\}} \right) \leq O \left( \frac{\ell P}{e^{1/2} e^{\Omega(k)}} \right),
\]
where the last step uses that for small $y$ we have $p \approx 1$, and $\ell y^2 \approx cky^2 \approx k$ by (27). On the other hand, if $y \geq \Omega(1)$ then $\ell(1/2 - p) \sim \ell$, and we can instead bound

$$E\left(X - EX; X \geq \frac{\ell}{2}\right) \leq \ell P\left(X \geq \frac{\ell}{2}\right) \leq O\left(\frac{\ell}{c^{1/2}e^{\Omega(ky)}}\right),$$

where the last step uses that for large $y$ we have $p \approx e^{-y}$, and $c = 1$ by (27). Substituting into (93) gives

$$E(\ell) = E\left(X; X < \frac{\ell}{2}\right) \geq \ell P\left(1 - O\left(\frac{1}{c^{1/2}e^{\Omega(k\max(1, y))}}\right)\right)$$

for $\ell \leq \ell_d, \text{AM}$, and for all $y \geq 0$ satisfying (27).

Averaging over $L_d, \text{AM}$. Combining (93) with the lower tail Chernoff bound (80) gives

$$E\left(E\left(L_{d, \text{AM}}\right) \leq \frac{\ell_{d, \text{AM}}}{2}\right) \leq E\left(L_{d, \text{AM}}; L_{d, \text{AM}} \leq \frac{\ell_{d, \text{AM}}}{2}\right) \leq O\left(\frac{\ell_{d, \text{AM}}}{\exp\{\Omega(\ell_{d, \text{AM}})\}}\right).$$

Combining (93) with the upper tail Chernoff bound (80) gives

$$E\left(E\left(L_{d, \text{AM}} \geq 2\ell_{d, \text{AM}}\right) \leq E\left(L_{d, \text{AM}}; L_{d, \text{AM}} \geq 2\ell_{d, \text{AM}}\right) \leq \sum_{j \geq 0} O\left(\frac{2\ell_{d, \text{AM}} + j}{\exp\{\Omega(\ell_{d, \text{AM}} + j)\}}\right).$$

Now abbreviate $f$ for the integers between $\ell_{d, \text{AM}}/2$ and $2\ell_{d, \text{AM}}$. Combining (93) with the last two bounds gives

$$0 \leq E\left(L_{d, \text{AM}}p - E(L_{d, \text{AM}})\right) \leq E\left(L_{d, \text{AM}}p - E(L_{d, \text{AM}}); L_{d, \text{AM}} \in [f]\right) + O\left(\frac{\ell_{d, \text{AM}}p}{\exp\{\Omega(\ell_{d, \text{AM}})\}}\right).$$

Recall that $\ell_{d, \text{AM}} \approx ck$. Combining with (94) gives that the right-hand side above is

$$\leq O\left(\frac{\ell_{d, \text{AM}}}{c^{1/2}\Omega(k \max(1, y))} + \frac{\ell_{d, \text{AM}}p}{\Omega(ck)}\right) \leq O\left(\frac{\ell_{d, \text{AM}}p}{c^{1/2}\Omega(k)}\right).$$

Rearranging this bound gives

$$(E\hat{L}_{d, \text{AM}})p\left\{-O\left(\frac{1}{c^{1/2}\Omega(k)}\right)\right\} \leq E[E(\hat{L}_{d, \text{AM}})] \leq (E\hat{L}_{d, \text{AM}})p,$$

and the claim follows by recalling (92).

3. Gardner threshold and 2RSB perturbation

In this section we evaluate the Gardner threshold and prove our main result. The explicit evaluation of the stability matrices is given in §3.1, and the asymptotics of the Gardner eigenvalue are extracted in §3.2 to prove Proposition 1.9. The proof of the main result Theorem 1.1 is completed in §3.3–3.6. Let $\hat{q}_y$ be as given by Proposition 1.7, and $\hat{q}_y \equiv \mathbb{P}(\hat{q}_y)$. For the most part we will suppress $y$ from the notation and write simply $\hat{q}_y \equiv \hat{q}_y(\hat{w})$ and $\psi_q \equiv \hat{q}_y(\hat{w})$. For any integers $a \leq b$ we write $x_{a:b} \equiv (x_a, \ldots, x_b).

3.1. Evaluation of the stability matrix

We decompose the stability matrix (39) as a product of two matrices, as follows. Define the clause stability matrix to be the $9 \times 9$ matrix $\hat{B}$ with entries

$$\hat{B}_{\hat{w}, \hat{w}} = \frac{\hat{q}_y \sum_{\hat{w}_{j:k}} \mathbf{1}\left\{\hat{w} = \mathbb{P}(\hat{w}_{j:k})\right\}}{\hat{q}_y \sum_{\hat{w}_{j:k}} \mathbf{1}\left\{\hat{w} = \mathbb{P}(\hat{w}_{j:k})\right\}} \prod_{j=3}^k \rho_{q_{j}}.$$

$$\hat{q}_y \sum_{\hat{w}_{j:k}} \mathbf{1}\left\{\hat{w} = \mathbb{P}(\hat{w}_{j:k})\right\} \prod_{j=3}^k \rho_{q_{j}} = \frac{\hat{q}_y \sum_{\hat{w}_{j:k}} \mathbf{1}\left\{\hat{w} = \mathbb{P}(\hat{w}_{j:k})\right\}}{\hat{q}_y \sum_{\hat{w}_{j:k}} \mathbf{1}\left\{\hat{w} = \mathbb{P}(\hat{w}_{j:k})\right\}} \prod_{j=3}^k \rho_{q_{j}} = \hat{q}_y \sum_{\hat{w}_{j:k}} \mathbf{1}\left\{\hat{w} = \mathbb{P}(\hat{w}_{j:k})\right\} \prod_{j=3}^k \rho_{q_{j}} = \hat{q}_y \sum_{\hat{w}_{j:k}} \mathbf{1}\left\{\hat{w} = \mathbb{P}(\hat{w}_{j:k})\right\} \prod_{j=3}^k \rho_{q_{j}}.$$
where the last equality is the definition of a $9 \times 9$ matrix $\hat{N}$. Similarly, define the variable stability matrix to be the $9 \times 9$ matrix $B$ with entries

$$
\hat{B}_{\hat{s},\hat{a}} = \sum_{s,d} \left\{ \hat{w} = \mathcal{W}(\hat{\omega}_{s,d}) \right\} \frac{1}{\mathcal{D}(s,d)} \left( \prod_{i=3}^{d} \psi_{s,i} \right) \left( \prod_{i=2}^{d} \psi_{s,i} \right) = \frac{1}{z\rho_{a}} \sum_{s,d} \left\{ \hat{w} = \mathcal{W}(\hat{\omega}_{s,d}) \right\} \frac{1}{\mathcal{D}(s,d)} \left( \prod_{i=3}^{d} \psi_{s,i} \right) \left( \prod_{i=2}^{d} \psi_{s,i} \right) \hat{N}_{\hat{s},\hat{a}} ,
$$

where the last equality is the definition of a $9 \times 9$ matrix $\hat{N}$. The full stability matrix is $B = \hat{B}\hat{B}$, and we define

$$
B_{\hat{r},\hat{a}} = \frac{\rho_{a} \hat{N}_{\hat{r},\hat{a}}}{z\rho_{a}}.
$$

The pattern of non-zero entries in $\hat{B}$ and $B$ is shown in Figure 2. For instance, the last two rows of $\hat{B}$ are identically zero because there is no choice of $\hat{w}$, $\hat{s}$, $\hat{w}_{3,k}$ such that $0 = \mathcal{W}(\hat{\omega}_{3,k})$ while $1 = \mathcal{W}(\hat{\omega}_{3,k})$.

![Figure 2: The stability matrices $\hat{B}$ and $B$. Only the top left $7 \times 7$ submatrices will be used.](image)

We will not use the last two columns of either matrix, so we will only evaluate the entries in the top left $7 \times 7$ submatrices, which we denote $\hat{B}_{7\times7}$ and $B_{7\times7}$. Clearly, it is sufficient to evaluate $N_{7\times3}$ and $N_{7\times7}$. We have

$$
\hat{N}_{\hat{r},\hat{a}} = 1 , \\
\hat{N}_{\hat{r},00} = \hat{N}_{\hat{r},0t} = \hat{N}_{\hat{r},ot} = 1 - (\rho_{a})^{k-2} , \\
\hat{N}_{\hat{0},11} = \hat{N}_{\hat{t},01} = \hat{N}_{\hat{t},1t} = (\rho_{a})^{k-2}.
$$

All other entries of $\hat{N}_{7\times7}$ (hence $B_{7\times7}$) are defined by the symmetry between 0 and 1. To calculate $N_{7\times7}$, define

$$
S_{0} \equiv \sum_{\ell_{1},\ell_{t}} \{ \ell_{1} = \ell_{t} \} \left( \frac{d-2}{\ell_{0},\ell_{t}} \frac{\psi_{0}^{d-2-(\ell_{0}+\ell_{t})} \psi_{1}^{d-2-\ell_{t}}}{\exp(y\ell_{0})} \right).
$$

Let $S_{1}$ and $S_{2}$ be defined in the same way as $S_{0}$, but with $\{ \ell_{1} = \ell_{t} + 1 \}$ and $\{ \ell_{1} \geq \ell_{t} + 2 \}$ respectively in place of $\{ \ell_{1} = \ell_{t} \}$. We will express the entries of $N$ in terms of the $S_{i}$. Note that

$$
\dot{z} = S_{0} + 2 \left( 1 - (1 - e^{-y})\psi_{0} \right) (S_{1} + S_{2})
$$

In the first three rows of $N_{7\times7}$ we have

$$
N_{0,0} = N_{0,01} = N_{0,0t} = S_{1} + S_{2} , \\
N_{0,0} = N_{0,0t} = N_{0,0t} = (1 - (\rho_{a})^{k-2})S_{0} , \\
N_{0,0,1} = (\rho_{a})^{k-2}S_{0} + S_{1} + S_{2} , \\
N_{0,0,0} = (\rho_{a})^{k-2}S_{1} + S_{2} + (\rho_{a})^{k-2}S_{2}e^{-y} , \\
N_{0,0,0} = (1 - (\rho_{a})^{k-2})(S_{1} + S_{2}) + (\rho_{a})^{k-2}S_{2}e^{-y} ,
$$

All other entries in the first three rows of $N_{7\times7}$ are determined by the symmetry between 0 and 1.
3.2. **Gardner eigenvalue and auxiliary matrices.** Now let $B_{4 \times 4}$ be the $4 \times 4$ submatrix of $B$ given by row and column indices in $\{f_0, f_1, 0f, 1f\}$ (in the center of Figure 2b): the corresponding entries of $N$ are given by

$$N_{4 \times 4} = \begin{pmatrix} f_0 & f_1 & 0f & 1f \\ f_1 & 0f & f_0 & 1f \\ 0f & f_0 & 1f & f_1 \\ 1f & 0f & f_1 & 0f \end{pmatrix} \cdot (\rho_0)^{k-2}.$$ 

From this it is easy to calculate that the largest eigenvalue of $B_{4 \times 4}$ (hence of the $6 \times 6$ matrix $B$) is

$$\lambda = \frac{(\rho_0)^{k-2}(S_0 + S_1 e^{-y/2})}{z}. \tag{97}$$

This is precisely the same $\lambda$ that appears in the statement of Proposition 1.9. Moreover, this $\lambda$ corresponds to a (right) eigenvector $\xi \in \mathbb{R}^9$ of the full $9 \times 9$ matrix $B$, given explicitly by

$$\xi^t = \left( \begin{array}{cccccccc} 1 & -2\rho_0 e^{-y/2} & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right). \tag{98}$$

**Proof of Proposition 1.9.** Recalling (95), we can rewrite $S_0$ in terms of $x, w$ as

$$S_0 = \sum_{\ell=0}^{d-2} \binom{d-2}{\ell} (w \cdot gM)^\ell (1-w)^{d-2-\ell} \binom{\ell}{1/2}.$$ 

Similarly we can write $S_1, S_2$ in terms of $x, w$ as

$$S_1 = \sum_{\ell=0}^{d-2} \binom{d-2}{\ell} (w \cdot gM)^\ell (1-w)^{d-2-\ell} \binom{\ell}{1/2},$$

$$S_2 = \sum_{\ell=0}^{d-2} \binom{d-2}{\ell} (w \cdot AM)^\ell (1-w)^{d-2-\ell} \binom{\ell}{1/2}.$$ 

The estimates of Lemma 2.3 apply equally well with $d-2$ in place of $d-1$, so

$$S_0 + \frac{S_1}{e^{y/2}} \approx S_0 \approx \frac{(1-w+w \cdot gM)^{d-2}}{(\max\{1,dw \cdot gM\})^{1/2}},$$

and from (96) we conclude

$$\hat{z} = S_0 + 2\frac{1-(1-e^{-y})w}{2} (S_1 + S_2) \approx S_0 + S_1 + S_2 \approx (1-w+w \cdot AM)^{d-2}.$$ 

Substituting into (97) gives

$$\lambda \approx \frac{1}{2^{d}(\max\{1,dw \cdot gM\})^{1/2}} \frac{(1-w+w \cdot gM)^{d-2}}{(1-w+w \cdot AM)^{d-2}} \approx \frac{x}{2^k},$$

therefore $b \lambda \approx dk \lambda \approx ck^2 x$. From Proposition 1.7 and 1.8 we have that $x$ is exponentially small with respect to $k$ for all $\alpha_{\text{sat}} \leq \alpha \leq 4^k$, so $b \lambda \geq 1$ cannot occur before $c \geq e^\Omega(k)$. In this regime the estimate (36) (and the discussion leading to (38)) implies $x \approx 1/d^{1/2}$, therefore

$$b \lambda \approx ck^2 x \approx \frac{ck^2}{d^{1/2}} \approx \left( \frac{ck^3}{2^k} \right)^{1/2}.$$ 

This crosses one at $c \approx 2^k/k^3$, corresponding to $\alpha_{\text{Ga}} \approx 4^k/k^3$ as claimed. \hfill \Box

We now define some auxiliary matrices which will have a role in what follows. Let $P$ be the $9 \times 9$ symmetric matrix with entries $P_{y \hat{z}, \hat{y} \hat{z}} = 1\{y = \hat{u}, \hat{z} = \hat{v}\}$. Let $\Pi, \Xi, \Gamma$ be the $9 \times 9$ matrices with entries

$$\Pi_{\hat{y} \hat{z}, \hat{y} \hat{z}} = \psi_{\hat{y} \hat{z}} \exp\{-y \bar{\phi}(\hat{u} \hat{z})\},$$

$$\Xi_{\hat{y} \hat{z}, \hat{y} \hat{z}} = \psi_{\hat{y} \hat{z}} \exp\{y \bar{\phi}(\hat{u} \hat{z}) - y \bar{\phi}(\hat{u} \hat{z}) - y \bar{\phi}(\hat{u} \hat{z})\},$$

$$\Gamma_{\hat{y} \hat{z}, \hat{y} \hat{z}} = \psi_{\hat{y} \hat{z}} \exp\{-y \bar{\phi}(\hat{u} \hat{z})\}\{\phi(\hat{u} \hat{z}) - \phi(\hat{u} \hat{z})\}. $$
We will write $\Theta$ be the $9 \times 9$ matrix with entries

$$\Theta_{\hat{\omega}, \hat{\omega}} \equiv \{ \hat{\nu} = \hat{\omega} \} = \sum_{\omega} \psi_{\omega} \rho_{\omega} \exp(y_{\hat{\omega}}(\hat{\omega})) \equiv \{ \hat{\nu} = \hat{\omega} \} = \sum_{\omega} \frac{\Pi_{\hat{\nu}, \hat{\omega}} \Pi_{\hat{\omega}, \hat{\nu}}}{\Pi_{\hat{\nu}, \hat{\nu}}}.$$  

We remark for later use that $\Pi \hat{B}, \Xi \hat{B},$ and $\Gamma \hat{B}$ are all symmetric matrices, as are $\Pi \hat{B}, \Xi \hat{B},$ and $\Gamma \hat{B}$. It also follows from the definitions that $P(\Pi - \Xi), (\Pi - \Xi) P,$ and $P \Gamma P$ are all identically zero. As a result, for any vector $\delta \in \mathbb{R}^9$ satisfying $\delta = \rho \delta$, we have $P \rho \delta = P \Gamma \delta = 0$, and $\Pi \delta = \Xi \delta$. Since $\delta = \rho \delta$ implies $\hat{B} \delta = \hat{B} \delta$, two further consequences are that $(\hat{B} \delta, \Gamma \delta) = (\hat{B} \delta) \Gamma \delta = 0$ and similarly $(\hat{B} \delta, \Gamma \delta) = (\hat{B} \delta)^{\dagger} P \Gamma P \delta = 0$. We record also that $\Pi \xi$ is identically zero, while

$$\begin{align*}
(\hat{B} \xi, \Gamma \xi) &= (1 - e^{-y/2}) e^{-y/2} (1 - x) x^2 \nu, \\
(\hat{B} \xi, \Xi \xi) &= (1 - e^{-y/2}) (1 - \nu e^{-y/2}) (1 - x) \nu.
\end{align*}$$

We will use all these observations in what follows to complete the proof of our main result Theorem 1.1.

### 3.3. Perturbation around type II degeneracy

Recall the discussion of §1.8. As suggested by the physics literature [MR03, KPW04], we evaluate the zero-temperature 2RSB functional (42) on a slight perturbation of case II, as follows. Let $y_1 = y$ and take $y_2 > y_1$ such that $\nu \equiv y_1 / y_2$ is close to 1, or equivalently that $\zeta \equiv 1 - \nu$ is small. Suppose

$$Q = \sum_{\omega} \rho_{\omega}(1 + \delta_{\omega})Q_{\omega}.$$  

where each $Q_{\omega}$ is a probability measure on $\Omega$ whose support is contained in a small neighborhood of $1_{\omega}$. This means that if $\rho$ is sampled from $Q_{\omega}$, then $f \equiv f_{\rho} \equiv \rho - 1_{\omega}$ is a signed measure all of whose weights are small. Let

$$\epsilon_{\omega} \equiv \int \left( \rho(\hat{\omega}) - 1 \{ \hat{\omega} = \hat{\omega} \} \right) dQ_{\omega}(\rho) = \int f_{\rho}(\hat{\omega}) dQ_{\omega}(\rho);$$

this quantity captures the "average mass sent from $\hat{\omega}$ to $\hat{\omega}$.” Finally, for any $\hat{\nu}, \hat{\xi}, \hat{\nu}$ define the scalar product

$$\gamma_{\hat{\nu}, \hat{\omega}} \equiv \int \left( \rho(\hat{\nu}) - 1 \{ \hat{\omega} = \hat{\omega} \} \right) \left( \rho(\hat{\xi}) - 1 \{ \hat{\xi} = \hat{\xi} \} \right) dQ_{\omega}(\rho) = \int f_{\rho}(\hat{\nu}) dQ_{\omega}(\rho).$$

In order for $Q, Q_0, Q_1, Q_2$ to be all valid probability measures on $\Omega$, we must have

$$\sum_{\omega} \rho_{\omega} \delta_{\omega} = 0, \quad \sum_{\omega} \epsilon_{\omega} = 0$$

for all $\hat{\omega}$. We will write $\epsilon$ for the vector in $\mathbb{R}^9$ with entries $\epsilon_{\omega}$. It will be convenient also to define vectors $\delta, \pi \in \mathbb{R}^9$ where

$$\delta_{\omega} = 1 \{ \hat{\omega} = \hat{\omega} \} \delta_{\omega}, \quad \pi_{\omega} = 1 \{ \hat{\omega} = \hat{\omega} \} \delta_{\omega} \epsilon_{\omega}.$$

Note that $\delta$ can have at most three nonzero entries, and the same holds for $\pi$. Let $Y$ be the $9 \times 9$ matrix with entries $Y_{\hat{\nu}, \hat{\omega}} = 1 \{ \hat{\nu} = \hat{\omega} \} Y_{\hat{\nu}, \hat{\omega}}$. For our purposes, the vectors $\delta, \epsilon, \pi$ encode the key summary statistics of $Q$. We will assume that all entries of $\delta$ and $\epsilon$ are $O(\zeta^2)$, while all entries of $\pi$ are $O(\zeta^4)$. Let $Q_{\Xi}$ be the degenerate measure described in case II in §1.8, corresponding to $\delta, \epsilon, \pi$ all zero.

**Proposition 3.1.** Suppose $y_1 = y < y_2$ such that $\nu \equiv y_1 / y_2$ is close to 1. Let $\zeta \equiv 1 - \nu$, and take $Q$ as in (101) such that for all $\omega$ we have $\| \rho - 1_{\omega} \|_\infty = O(\zeta^2)$ uniformly over all $\rho \in \text{supp } Q_\omega$. If $Q$ has summary statistics $\delta, \epsilon, \pi, \gamma$, then

$$\Phi_{2\text{RSB}}(y_1, y_2, Q) = \Phi_{2\text{RSB}}(y, Q_{\Xi}) + \frac{d(k-1)}{2} \left( \hat{B} \tau, \frac{(\Pi - y \zeta \Gamma \nu - \zeta \Xi)}{\nu} (bB - I) \right)$$

$$- \frac{d(k-1)(d-k-1)}{2} \left( 1_{\omega}, P(\Pi - y \nu^{-1} \zeta \Gamma) \right)^2 + O(\zeta^6),$$

where $\tau \equiv \delta + \nu (\epsilon + \pi) \in \mathbb{R}^9.
3.4. Perturbed clause functional. In this subsection we analyze the clause 2RSB functional $G(y_1, y_2, Q)$ for $Q$ near $Q_{II}$, and show how the clause stability matrix $B$ arises.

**Lemma 3.2.** In the setting of Proposition 3.1,

$$G(y_1, y_2, Q) = G(y, y, Q_{II}) + k \left\{ \frac{k \nu \zeta}{2} G_3(1_\nu, (Y \otimes \Theta) 1_\nu) \right\}$$

where $\otimes$ denotes the Hadamard (entrywise) matrix product.

**Proof.** Recall the definition (40). We abbreviate $G \equiv G(y_1, y_2, Q)$, $G_{II} \equiv G(y, y, Q_{II})$, and $\Delta G \equiv G - G_{II}$. We also write $\rho(\hat{w}_{1:k})$ as shorthand for the $k$-fold product $\rho_{w_1} \cdots \rho_{w_k}$. Expanding according to the definition (101) gives

$$G = \sum_{\hat{w}_{1:k}} \rho(\hat{w}_{1:k}) \prod_{j=1}^k (1 + \delta_{\hat{w}_j}) \left\{ \prod_{i=1}^k \hat{y}_{\hat{w}_{1:k}(i)} \right\} \prod_{j=1}^k \rho_j(\hat{w}_j) \right\} dQ_{\hat{w}_j}.$$

For $\rho_j$ sampled from $Q_{\hat{w}_j}$, let $f_j \equiv \rho_j - 1_{\nu}$. Notice that the inner sum above, over configurations $\hat{w}_{1:k}$, is dominated by the contribution from the case $\hat{w}_{1:k} = \hat{w}_{1:k}$. We can expand it to second order (with respect to the $f_j$) as a sum of three terms $I_0, I_1, I_2$: the contribution from $\hat{w}_{1:k} = \hat{w}_{1:k}$ is

$$I_0 = \frac{1}{\exp(y_2 \hat{w}_{1:k})} \prod_{j=1}^k (1 + f_j(\hat{w}_j))$$

The contribution from configurations $\hat{w}_{1:k}$ that differ from $\hat{w}_{1:k}$ in a single coordinate is

$$I_1 = \sum_{i=1}^k \sum_{\hat{w}_{1:k}} \frac{1}{\exp(y_2 \hat{w}_{1:k})} f_i(\hat{w}_i) \left\{ \prod_{j=1}^k \rho_j(\hat{w}_j) \right\} dQ_{\hat{w}_j} = O(\zeta^6),$$

where $\hat{w}_{1:k}^{[i]}$ refers to $\hat{w}_{1:k}$ with the $i$-th entry dropped. The contribution from configurations $\hat{w}_{1:k}$ that differ from $\hat{w}_{1:k}$ in two coordinates is

$$I_2 = \sum_{1 \leq i < j \leq k} \frac{1}{\exp(y_2 \hat{w}_{1:k})} f_i(\hat{w}_i) f_j(\hat{w}_j) = O(\zeta^6),$$

where $\hat{w}_{1:k}^{[i,j]}$ refers to $\hat{w}_{1:k}$ with the $i$-th and $j$-th entries dropped. It is convenient for us to rearrange the terms and express $I_0 + I_1 + I_2 = \exp(-y_2 \hat{w}(\hat{w}_{1:k}))((1 + J_3 + J_4)$ where

$$J_1 = \sum_{j=1}^k \sum_{\hat{w}_{1:k}} \frac{\exp(-y_2 \hat{w}(\hat{w}_{1:k}))}{\exp(-y_2 \hat{w}(\hat{w}_{1:k}))} f_j(\hat{w}_j) = O(\zeta^2),$$

$$J_2 = \sum_{1 \leq i < j \leq k} \frac{\exp(-y_2 \hat{w}(\hat{w}_{1:k}))}{\exp(-y_2 \hat{w}(\hat{w}_{1:k}))} f_i(\hat{w}_i) f_j(\hat{w}_j) = O(\zeta^4).$$

Since $(1 + t)^v = 1 + vt + (1 - v)t^2/2 + O(t^3)$ for small $t$, we can expand

$$(1 + J_1 + J_2)^v = 1 + vJ_1 + vJ_2 - \frac{v(1 - v)}{2} (J_1)^2 + O(\zeta^6).$$

Since $J_1$ involves a sum over $1 \leq i \leq k$, its square is a double sum, and we can further decompose $(J_3)^2 = J_3 + J_4$ where $J_3$ captures the diagonal terms while $J_4$ captures the off-diagonal terms. Substituting this expansion back into the definition of $G$ results in the decomposition

$$G = G_0 + kvG_1 + \left( \frac{k}{2} \right)^v G_2 - \frac{k(1 - v)}{2} G_3 - \left( \frac{k}{2} \right)^v (1 - v) G_4 + O(\zeta^6).$$

(105)
We now proceed to evaluate the $\mathcal{G}_i$, beginning with $\mathcal{G}_0$ which is the value when $e$ and $\Upsilon$ are zero:

$$
\mathcal{G}_0 = \sum_{\hat{u}_{1:k}} \frac{\rho(\hat{u}_{1:k})}{\exp(y\hat{\varphi}(\hat{u}_{1:k}))} \prod_{j=1}^k (1 + \delta_{\hat{u}_j}) = \sum_{\hat{u}_{1:k}} \frac{\rho(\hat{u}_{1:k})}{\exp(y\hat{\varphi}(\hat{u}_{1:k}))} \left\{ 1 + k\delta_{\hat{u}_1} + \left( \frac{k}{2} \right) \delta_{\hat{u}_1} \delta_{\hat{u}_2} \right\} + O(\zeta^6)
$$

$$
= \mathcal{G}_\Pi + k(1, P\Pi\delta) + \left( \frac{k}{2} \right) (\tilde{\mathbf{B}}\delta, \Pi\delta) + O(\zeta^6).
$$

For future use we denote the two scalar products appearing in the last expression above as $\mathcal{G}_{0,1}$ and $\mathcal{G}_{0,2}$, so that

$$
\mathcal{G}_0 = \mathcal{G}_\Pi + k\mathcal{G}_{0,1} + \left( \frac{k}{2} \right) \mathcal{G}_{0,2}.
$$

By symmetry among the coordinates $1 \leq j \leq k$, the average of $J_1$ is $k\mathcal{G}_1$ where

$$
\mathcal{G}_1 = \sum_{\hat{u}_{1:k}} \rho(\hat{u}_{1:k}) \prod_{j=1}^k (1 + \delta_{\hat{u}_j}) \sum_{\hat{u}_{1:k}} \epsilon_{\hat{u}_{1:k}} \exp\left\{ - (y_2 - y)(\hat{\varphi}(\hat{u}_{2:k}) - \hat{\varphi}(\hat{u}_{1:k})) \right\}
$$

$$
= \sum_{\hat{u}_{1:k}} \rho(\hat{u}_{1:k}) \prod_{j=1}^k (1 + \delta_{\hat{u}_j} + (k - 1)\delta_{\hat{u}_j}) \sum_{\hat{u}_{1:k}} \epsilon_{\hat{u}_{1:k}} \left\{ 1 - y_2(1 - \nu)(\hat{\varphi}(\hat{u}_{1:k}) - \hat{\varphi}(\hat{u}_{1:k})) \right\} + O(\zeta^6)
$$

$$
= (1, \nu (\Pi - y\nu^{-1}\zeta\Gamma)(e + \pi)) + (k - 1)(\tilde{\mathbf{B}}\delta, (\Pi - y\zeta\Gamma)e) + O(\zeta^6).
$$

For future use we denote the two scalar products appearing in the last expression above as $\mathcal{G}_{1,1}$ and $\mathcal{G}_{1,2}$, so that

$$
k\mathcal{G}_1 = k\mathcal{G}_{1,1} + 2\left( \frac{k}{2} \right) \mathcal{G}_{1,2} + O(\zeta^6).
$$

The average of $J_2$ is $\left( \frac{k}{2} \right) \mathcal{G}_2$ where

$$
\mathcal{G}_2 = \sum_{\hat{u}_{1:k}} \frac{\rho(\hat{u}_{1:k})}{\exp(y\hat{\varphi}(\hat{u}_{1:k}))} \sum_{\hat{u}_{1:k}} \epsilon_{\hat{u}_{1:k}} \epsilon_{\hat{u}_{2:k}} \exp\left\{ (y_2 - y)(\hat{\varphi}(\hat{u}_{2:k}) - \hat{\varphi}(\hat{u}_{1:k})) \right\} + O(\zeta^6)
$$

$$
= \left( \tilde{\mathbf{B}}e, (\Pi - y\zeta\Gamma)e \right) + O(\zeta^6).
$$

The average of $J_3$ is $k\mathcal{G}_3$ where

$$
\mathcal{G}_3 = \sum_{\hat{u}_{1:k}, \hat{u}_{2:k}} \rho(\hat{u}_{1:k}) \exp(y\hat{\varphi}(\hat{u}_{1:k})) \gamma_{\hat{u}_{1:k}} \epsilon_{\hat{u}_{1:k}} \epsilon_{\hat{u}_{2:k}} \exp\left\{ (y_2 - y)(\hat{\varphi}(\hat{u}_{2:k}) - \hat{\varphi}(\hat{u}_{1:k})) \right\} + O(\zeta^5)
$$

$$
= \sum_{\hat{u}_{1:k}, \hat{u}_{2:k}} \gamma_{\hat{u}_{1:k}} \epsilon_{\hat{u}_{1:k}} \epsilon_{\hat{u}_{2:k}} \exp\left\{ y\hat{\varphi}(\hat{u}_{1:k}) \right\} + O(\zeta^5) = (1, (\Upsilon \otimes \Theta)1) + O(\zeta^5)
$$

The average of $J_4$ is $\left( \frac{k}{2} \right) \mathcal{G}_4$ where

$$
\mathcal{G}_4 = \sum_{\hat{u}_{1:k}, \hat{u}_{2:k}} \frac{\rho(\hat{u}_{1:k})}{\exp(y\hat{\varphi}(\hat{u}_{1:k}))} \epsilon_{\hat{u}_{1:k}} \epsilon_{\hat{u}_{2:k}} \exp\left\{ y\hat{\varphi}(\hat{u}_{2:k}) \right\} + O(\zeta^5) = \left( \tilde{\mathbf{B}}e, \Xi e \right) + O(\zeta^5).
$$

Collecting terms gives

$$
\Delta \mathcal{G} = k \left\{ \mathcal{G}_{0,1} + v \mathcal{G}_{1,1} + \frac{v\zeta}{2} \mathcal{G}_3 \right\} + \left( \frac{k}{2} \right) \left\{ \mathcal{G}_{0,2} + 2v \mathcal{G}_{1,2} + v \mathcal{G}_3 + \frac{v\zeta}{2} \mathcal{G}_4 \right\} \equiv k\Delta_1 + \left( \frac{k}{2} \right) \Delta_2.
$$

Recall that $\Pi\tilde{\mathbf{B}}, \Xi\tilde{\mathbf{B}},$ and $\Gamma\tilde{\mathbf{B}}$ are all symmetric matrices. We have

$$
\Delta_1 = \left( 1, \nu (\Pi - y\nu^{-1}\zeta\Gamma)(\delta + v(e + \pi)) \right) - \frac{v\zeta}{2} \mathcal{G}_3 (1, (\Upsilon \otimes \Theta)1) + O(\zeta^5),
$$

having used that $P\Gamma\delta$ is identically zero. We also have

$$
\Delta_2 = (\tilde{\mathbf{B}}\delta, (\Pi - y\zeta\Gamma)e) + 2v (\tilde{\mathbf{B}}\delta, (\Pi - y\zeta\Gamma)e) + v (\tilde{\mathbf{G}}e, (\Pi - y\zeta\Gamma)e) - \zeta(\tilde{\mathbf{B}}e, \Xi e) + O(\zeta^5).
$$
Recall from the discussion at the end of §3.2 that \((\hat{B}\delta, \Gamma\delta) = 0\), so we can freely add any multiple of \((\hat{B}\delta, \Gamma\delta)\) to the above. Recall also that \(\Pi\delta = \Xi\delta\), so we can also freely interchange \((\Pi\delta, \hat{B}\epsilon)\) with \((\Xi\delta, \hat{B}\epsilon)\). Using these identities, and absorbing some errors into the \(O(\zeta^6)\) term, we can “complete the square” and obtain

\[
\Delta_2 = \frac{(\hat{B}(\delta + e), (\Pi - y\Gamma)(\delta + e))}{\nu} - \zeta(\hat{B}(\delta + e), \Xi(\delta + e)) + O(\zeta^6) .
\]  

(109)

The claimed result follows.  

3.5. **Perturbed variable functional.** In this subsection we analyze the variable 2RSB functional \(\mathcal{W}(y_1, y_2, Q)\) for \(Q \approx Q_\text{II}\), and show how the full stability matrix \(B\) arises.

**Lemma 3.3.** In the setting of Proposition 3.1,

\[
\mathcal{W}(y_1, y_2, Q) = \frac{\mathcal{W}(y_1, y_2, Q_\text{II})}{\hat{z}} + d(k - 1) \left( \sum_{\bar{y}_1, \bar{y}_2} \rho(\bar{y}_1, \bar{y}_2) \right) - \frac{d(k - 1)\nu\zeta}{2} \mathcal{G}_1(1_\Omega, (Y \otimes \Theta)1_\Omega)
\]

where \(\hat{z}\) is the normalizing constant in the \(\hat{S}_y\) recursion (26).

**Proof.** This is similar to the proof of Lemma 3.2, although slightly more involved because the input warnings are propagated two layers to the root. We recall the definition (41), and continue to write \(D \equiv d(k - 1)\). We abbreviate \(\mathcal{W} \equiv \mathcal{W}(y_1, y_2, Q)\), \(\mathcal{W}_\beta \equiv \mathcal{W}(y_1, y_2, Q_\text{II})\), and \(\Delta \mathcal{W} \equiv \mathcal{W} - \mathcal{W}_\beta\). Then

\[
\mathcal{W} \equiv \sum_{\bar{y}_1, \bar{y}_2} \rho(\bar{y}_1, \bar{y}_2) \int \left\{ \sum_{\hat{s}_1, \hat{s}_2} \exp(-y_2\varphi(\hat{s}_1, \hat{s}_2)) \prod_{j=1}^D \rho_j(\hat{s}_j) \right\} v D \mathcal{Q}_j(\rho_j).
\]

As in the proof of Lemma 3.2, the inner sum over \(\hat{s}_1, \hat{s}_2\) is dominated by the \(\hat{s}_1, \hat{s}_2 = \hat{s}_1, \hat{s}_2\) term. It can be expanded similarly as \(\exp(-y_2\varphi(\hat{s}_1, \hat{s}_2))(1 + J_1 + J_2 + J_2, y_1) + O(\zeta^6)\) where the first-order correction is

\[
J_1 = \sum_{j=1}^D \sum_{s_j} \frac{\exp(-y_2\varphi(\hat{s}_1, \hat{s}_2))}{\exp(-y_2\varphi(\hat{s}_1, \hat{s}_2))} f_j(\hat{s}_j) = O(\zeta^2),
\]

and we now split the second-order correction into two components:

\[
J_2 = \sum_{a=1}^d \sum_{1 \leq i < k} \frac{\exp(-y_2\varphi(\hat{s}_a, \hat{s}_b))}{\exp(-y_2\varphi(\hat{s}_1, \hat{s}_2))} f_a(\hat{s}_a) f_b(\hat{s}_b) = O(\zeta^4),
\]

\[
J_{2, xy} = \sum_{1 \leq a < b \leq d} \sum_{i=2}^k \frac{\exp(-y_2\varphi(\hat{s}_a, \hat{s}_b))}{\exp(-y_2\varphi(\hat{s}_1, \hat{s}_2))} f_a(\hat{s}_a) f_b(\hat{s}_b) = O(\zeta^4).
\]

We further decompose \((J_2)^2 = J_3 + J_4 + J_{4, xy}\) where \(J_3\) is the contribution to the double sum from diagonal terms \((ai, ai)\), \(J_4\) is the contribution from off-diagonal terms \((ai, aj)\) with \(i \neq j\), and \(J_{4, xy}\) is the contribution from off-diagonal terms \((ai, bj)\) with \(a \neq b\). Altogether it gives

\[
(1 + J_1 + J_2 + J_{2, xy}) = 1 + v J_1 + v J_2 + v J_{2, xy} - \frac{v \zeta}{2} J_3 - \frac{v \zeta}{2} J_4 - \frac{v \zeta}{2} J_{4, xy} + O(\zeta^6).
\]

Then, similarly to (105), we have a corresponding expansion

\[
\mathcal{W} = \mathcal{W}_0 + d(k - 1)\nu\mathcal{W}_1 + d \left( \frac{k - 1}{2} \right) v\mathcal{W}_2 + \left( \frac{d}{2} \right) (k - 1)^2 v\mathcal{W}_{2, xy}
\]

\[
- d(k - 1)\frac{v \zeta}{2} \mathcal{W}_3 - d \left( \frac{k - 1}{2} \right) v\mathcal{W}_4 - \left( \frac{d}{2} \right) (k - 1)^2 v\mathcal{W}_{4, xy} + O(\zeta^6).
\]
Recalling (106) and (107), we have similarly
\[ W_0 = W_0' + d(k - 1)W_{0,1} + d \binom{k - 1}{2}W_{0,2} + \binom{d}{2}(k - 1)^2W_{0,2,xx}, \]
\[ d(k - 1)W_1' = d(k - 1)W_{1,1}' + d \binom{k - 1}{2}W_{1,2}' + 2d \binom{d}{2}(k - 1)^2W_{1,2,xx}' \]
Recalling that \( \hat{z} \) is the normalizing constant in the \( \hat{s}_y \) recursion (26), we have
\[ \hat{z} = \frac{W_{0,1}}{G_{0,1}} = \frac{W_{0,2}}{G_{0,2}} = \frac{W_{1,1}}{G_{1,1}} = \frac{W_{1,2}}{G_{1,2}} = \frac{W_2}{G_2} = \frac{W_3}{G_3} = \frac{W_4}{G_4}, \]
so it remains to calculate \( W_{0,2,xx}, W_{1,2,xx}, W_{2,xx}, \) and \( W_{4,xx}. \) Recalling that \( B \equiv \hat{B}\hat{B} \), we have
\[ W_{0,2,xx}/\hat{z} = (\hat{B}\delta, \Pi B\delta) + O(\zeta^5), \]
\[ W_{1,2,xx}/\hat{z} = (\hat{B}\delta, (\Pi - y\zeta\Gamma)B\epsilon) + O(\zeta^6), \]
\[ W_{2,xx}/\hat{z} = (\hat{B}\epsilon, (\Pi - y\zeta\Gamma)B\epsilon) + O(\zeta^6), \]
\[ W_{4,xx}/\hat{z} = (\hat{B}\epsilon, \Xi B\epsilon) + O(\zeta^5). \]
Collecting terms gives
\[ \Delta W/\hat{z} = d(k - 1)\Delta_1 + \binom{k - 1}{2}\Delta_2 + \binom{d}{2}(k - 1)^2\left\{ W_{0,2,xx} + 2vW_{1,2,xx} + v^2W_{2,xx} - \nu_{xx} W_{4,xx} \right\} \]
for \( \Delta_1, \Delta_2 \) as defined by (108). We denote the last coefficient above (in braces) as \( \Delta_{2,xx}. \) Similarly to (109) we have
\[ \Delta_{2,xx} = \frac{(\hat{B}(\delta + \epsilon), (\Pi - y\zeta\Gamma)B(\delta + \epsilon))}{\nu} - \zeta(\hat{B}(\delta + \epsilon), \Xi B(\delta + \epsilon)) + O(\zeta^6). \]
The result follows. \( \square \)

3.6. Proof of main theorem. We now prove Proposition 3.1 and deduce our main result Theorem 1.1.

Proof of Proposition 3.1. We abbreviate \( \Delta \Phi \equiv \Phi_{x\text{SB}}(y_1, y_2, Q) - \Phi_{x\text{SB}}(y, y, Q_0) \). By substituting the estimates of Lemmas 3.2 and 3.3 into (42), and recalling that \( \ln(1 + x) = x - x^2/2 + O(x^3) \) for small \( x \), we find
\[ \Delta \Phi = \frac{d(k - 1)}{2}(b\Delta_{2,xx} - \Delta_2) - \frac{d(k - 1)(d^2 - d - k)}{2}(\Delta_1)^2 + O(\zeta^6). \]
The result follows. \( \square \)

Proof of Theorem 1.1. Recall from §3.2 that the 9 \times 9 stability matrix \( B \equiv \hat{B}\hat{B} \) has eigenvalue \( \lambda \) given explicitly by (97), which is the same as the \( \lambda \) that appears in Proposition 1.9. Associated to this \( \lambda \) is a right eigenvector \( \xi \in \mathbb{R}^9 \) of \( B \), given explicitly by (98). We now note that this vector can be split as \( \xi = \sigma + \alpha \) where
\[ \left( \begin{array}{c} \sigma \, \sigma^t \end{array} \right) = \left( \begin{array}{cccccccc} 2(1 - e^{-y/2})\rho_0 & -(1 - e^{-y/2})\rho_1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\rho_0 & -\rho_1 e^{y/2} & -\rho_1 e^{-y/2} & \rho_0 & \rho_0 & \rho_0 & \rho_1 e^{y/2} & \rho_1 e^{-y/2} & \rho_1 e^{y/2} \end{array} \right). \]
These vectors satisfy (cf. (104)) the constraints
\[ \sum_\alpha \rho_\alpha \alpha \sigma_\alpha = 0, \quad \sum_\alpha \alpha \sigma_\alpha = 0 \text{ for all } \hat{\alpha}. \]
We apply Proposition 3.1 with \( \delta = \zeta^2 \alpha \) and \( \epsilon = \zeta^2 \nu^{-1} \sigma \), so that \( \tau = \delta + \nu(\epsilon + \pi) = \zeta^2 \xi + O(\zeta^4) \); this gives
\[ \frac{\Delta \Phi}{d(k - 1)/2} = (b\lambda - 1)\zeta^4 \left( \hat{B}\tau, \left( \frac{\Pi - y\zeta\Gamma}{\nu} - \zeta \Xi \right) \xi \right) - (d^2 - d - k)\zeta^4 \left( \mathbf{1}_9, \mathbf{P}(\Pi - y\nu^{-1}\zeta\Gamma)\xi \right)^2 + O(\zeta^6) \]
where, as before, we abbreviate \( \Delta \Phi \equiv \Phi_{x\text{SB}}(y_1, y_2, Q) - \Phi_{x\text{SB}}(y, y, Q_0) \). It follows from our earlier calculation (100) that \( \Delta \Phi \) is negative whenever \( b\lambda > 1 \). The result follows by applying Proposition 1.10. \( \square \)
4. Interpolation bound

In this final section we give the proof of Proposition 1.10. In order to keep our presentation somewhat self-contained, in §4.1 we give the heuristic derivation of the (positive-temperature) 2RSB functional, in the setting of random regular \textsc{nae-sat}. In §4.2 we review a general interpolation bound proved in prior work, and use it to deduce Proposition 1.10.

4.1. Heuristic derivation of 2RSB functional. The heuristic derivation in this subsection expands on the outline presented in §1.8; and is a simple application of the well-known “cavity method.” There are too many instances of the method to be adequately cited here, but we point out a few influential works [MP01, YFW01, ASS03]. Our discussion is based on [Pan13b], and we follow similar notation. For \( m \in (0, 1) \) we shall write \( \Pi \sim \mathcal{P}(m) \) to mean that \( \Pi \) is a Poisson point process on \( (0, \infty) \) with intensity measure

\[
m \, dx \over x^{m+1}.
\]

The key property of \( \mathcal{P}(m) \) is the following scaling relation:

**Lemma 4.1** ([Pan13b, Thm. 2.6]). Let \((\Omega, \mathcal{F}, \mathcal{P})\) be a probability space, and \((X, Y) : \Omega \rightarrow (0, \infty) \times S\) a pair of random elements on \(\Omega\), where \((S, \mathcal{I})\) is a measurable space. Suppose \(\mathbb{E}(X^m) < \infty\) and let \(\nu_m\) be the measure on \(S\) defined by

\[
\nu_m(B) = \frac{\mathbb{E}(X^m \mathcal{1}\{Y \in B\})}{\mathbb{E}(X^m)}.
\]

Suppose \(\Pi \sim \mathcal{P}(m)\), and let \((u_n)_{n \geq 1}\) denote the points of \(\Pi \) arranged in decreasing order. Let \((X_n, Y_n)_{n \geq 1}\) be an i.i.d. sequence of copies of \((X, Y)\), independent from \(\Pi\). Then \((u_n X_n, Y_n)_{n \geq 1}\) is again a Poisson process, and has the same intensity measure as the process \(\mathbb{E}(X^m)^{1/m} u_n, Y_n)_{n \geq 1}\) where \((Y_n)_{n \geq 1}\) is a sequence of i.i.d. samples from \(\nu_m\) that are also independent from \(\Pi\).

The following discussion generalizes trivially to any finite number of levels of replica symmetry breaking, but for concreteness we consider only two levels. Fix 2RSB parameters \(0 < m_1 < m_2 < 1\). Let \(\Pi \sim \mathcal{P}(m_2)\), and let \((u_s)_{s \geq 1}\) denote the points of \(\Pi \) arranged in decreasing order. For all integers \(s \geq 1\) let \(\Pi_s\) be an independent sample from \(\mathcal{P}(m_2)\), and let \((u_{st})_{t \geq 1}\) denote the points of \(\Pi_s\) arranged in decreasing order. Let \(w_{st} \equiv u_s u_{st}\), and let

\[
v_{st} \equiv \sum_{s', t' \geq 1} \frac{w_{st}}{w_{s', t'}}.
\]

The doubly infinite array \(\nu \equiv (v_{st})_{s \geq 1, t \geq 1}\) gives the weights of a 2-level \textbf{Ruelle probability cascade} with parameters \(m_1, m_2\). We hereafter abbreviate this as \(\nu \sim \text{RPC}(m_1, m_2)\).

For simplicity we continue to assume that \(\alpha \equiv \nu / k\) is an integer. Let \(\mathcal{H}_N, \mathcal{H}_{N+1/2}, \mathcal{H}_{N+1}\) be as defined in §1.8. For \(\beta \geq 0\) we consider the Gibbs measure \(\mu_\beta\) defined by (43), using the Hamiltonian of \(\mathcal{H}_{N+1/2}\). We assume that the finite-dimensional marginals of \(\mu_\beta\) are given by (44), which we repeat here for convenience:

\[
\mu_\beta(x_1, \ldots, x_{\ell}) \approx \sum_{s \geq 1, t \geq 1} v_{st} \prod_{i=1}^\ell v_{st} \prod_{i=1}^\ell w_{st,i}(x_i).
\]

We sample the weights \(v_{st}\) from the RPC\((m_1, m_2)\) law, as defined by (110). We recall that the \(w_{st,i}\) are generated recursively, as follows. Let \(\mathcal{H}_0 \equiv \mathcal{H}\) be the space of probability measures over \(\{0, 1\}\), and for \(r \geq 1\) let \(\mathcal{H}_r\) be the space of probability measures over \(\mathcal{H}_{r-1}\). Let \(Q \in \mathcal{H}_2\). Let \((r_{s,i})_{s,i}\) be i.i.d. samples from law \(Q\). For each \(i\) and each \(s\), let \((w_{st,i})_{t \geq 1}\) be a sequence of i.i.d. samples from \(R_{s,i}\). Note \(R_{s,i} \in \mathcal{H}_s\) so \(w_{st,i} \in \mathcal{H}_s\). Recall that \(\mathcal{H}_{N+1/2}\) is formed by deleting from \(\mathcal{H}_N\) a set of \(d(1 - 1/k)\) random clauses, which we denote \(\mathcal{F}'\). Then

\[
\ln \frac{Z_N(\beta)}{Z_{N+1/2}(\beta)} \approx \ln \sum_{s \geq 1} u_s \sum_{t \geq 1} u_{st} \prod_{a \in \mathcal{F}'} \left\{ \prod_{i \in a} \exp(-\beta \mathcal{H}_s(x_{a,i})) \right\} \prod_{i \in a} w_{st,i}(x_i) = \ln \sum_{s \geq 1} u_s \sum_{t \geq 1} u_{st} \prod_{a \in \mathcal{F}'} \hat{X}_{a,\beta}(w_{st,a})
\]

where

\[
\hat{X}_{a,\beta}(x) \equiv \sum_{s \geq 1} u_s \sum_{t \geq 1} u_{st} \prod_{i \in a} \exp(-\beta \mathcal{H}_s(x_{a,i})) \prod_{i \in a} w_{st,i}(x_i).
\]
where $\hat{X}_{a,\beta}(w_{st,sa})$ is a random function (depending on the labels $L_w$ of the edges $e \in \delta a$) of the $k$-tuple of measures $w_{st,sa} \equiv (w_{st,i})_{i \in \partial a}$. Taking expectations and applying Lemma 4.1 for the $(u_s)_{s \geq 1}$ gives

$$E \ln \frac{Z_N(\beta)}{Z_{N+1/2}(\beta)} \approx E \ln \sum_{s \geq 1} u_s \prod_{a \in F'} E_a \left( \hat{X}_{a,\beta}(w_{s,\partial a})^{m_2} \right)^{1/m_2} \sum_{s \geq 1} u_s - E \ln \sum_{s \geq 1} u_s \sum_{t \geq 1} u_t$$

where $E_a$ denotes expectation conditional on the $r_{s,i}$ and on the $\{0,1\}$-labels of the edges incident to $F'$. Let $E'$ denote expectation conditional only on the $\{0,1\}$ edge labels. Applying Lemma 4.1 again for $(u_s)_{s \geq 1}$ gives

$$E \ln \frac{Z_N(\beta)}{Z_{N+1/2}(\beta)} \approx E \left[ \prod_{a \in F'} E_a \left( \hat{X}_{a,\beta}(w_{1,\partial a})^{m_2} \right)^{1/m_2} \sum_{s \geq 1} u_s \sum_{t \geq 1} u_t \right] - E \ln \sum_{s \geq 1} u_s \sum_{t \geq 1} u_t = \frac{\alpha(k-1)}{m_1} E \ln G_{\beta,m_1,m_2}(\hat{Q})_{m_1},$$

where the last equality defines $G_{\beta}$. We can write it more explicitly as

$$G_{\beta,m_1,m_2}(\hat{Q}) = \int \left( \int \hat{X}_{a,\beta}(w_{1,k})^{m_2} \prod_{i=1}^k dr_i(w_i) \right)^{m_1/m_2} \prod_{i=1}^k d\hat{Q}(r_i),$$

(112)

$$\hat{X}_{a,\beta}(w_{1,k}) = \sum_{x_{ik}} \exp(-\beta \mathcal{H}_a(x_{ik})) \prod_{i=1}^k w_i(x_i).$$

We emphasize that the comparison (111) holds under the heuristic (44). Under the same assumption we can likewise derive a comparison between $\hat{G}_{N+1/2}$ and $\hat{G}_{N+1}$ — since this is very similar to the preceding calculation, we omit the details and simply state the result. Write $D \equiv d(k-1)$ as before, and denote

$$(x_{aj} : 1 \leq a \leq d, 2 \leq j \leq k) \equiv x_{1:D}.$$  

Let $\mathcal{H}_1, \ldots, \mathcal{H}_d$ be the Hamiltonians for $d$ random clauses, and let

$$\mathcal{H}(x_{0:D}) \equiv \mathcal{H}(x_{0}, x_{1:D}) \equiv \sum_{a=1}^d \mathcal{H}_a(x_a, x_{a2}, \ldots, x_{ak}).$$

Then, analogously to (111), we have the comparison

$$E \ln \frac{Z_{N+1}(\beta)}{Z_{N+1/2}(\beta)} \approx E \ln \frac{W_{\beta,m_1,m_2}(\hat{Q})_{m_1}}{m_1},$$

(113)

where the explicit form of $W_{\beta}$ is given, analogously to (112), by

$$W_{\beta,m_1,m_2}(\hat{Q}) = \int \left( \int X_{\beta}(w_{1:D})^{m_1} \prod_{i=1}^D dr_i(w_i) \right)^{m_1/m_2} \prod_{i=1}^D d\hat{Q}(r_i),$$

$$X_{\beta}(w_{1:D}) = \sum_{x_{0:D}} \exp(-\beta \mathcal{H}(x_{0:D})) \prod_{i=1}^D w_i(x_i).$$

Combining (111) and (113) gives, under the heuristic assumption (44), the comparison

$$E \ln \frac{Z_{N+1}(\beta)}{Z_N(\beta)} \approx \frac{1}{m_1} E \left\{ \ln W_{\beta,m_1,m_2}(\hat{Q}) - \alpha(k-1) \ln G_{\beta,m_1,m_2}(\hat{Q}) \right\} \equiv \Phi_{\beta,m_1,m_2}(\hat{Q}),$$

(114)

where the last identity defines the (positive-temperature) 2RSB functional $\Phi_{\beta,m_1,m_2}$. As we review next, one side of (114) can be made rigorous via an interpolation bound (Proposition 4.2 below).
4.2. General interpolation bound. Let $\mathcal{G}_N$ be an instance of random $d$-regular $k$-NAE-SAT on $N$ variables, with Hamiltonian $\mathcal{H}_N$. As before, let

$$Z_N(\beta) \equiv \sum_\chi \exp \left\{-\beta \mathcal{H}_N(\chi)\right\}$$

(115)

where the sum goes over $\chi \in \{0,1\}^N$. The following is a direct consequence of prior results:

**Proposition 4.2 (proved in [SSZ16, §E.4]).** Let $\mathcal{G}_N$ be an instance of random $d$-regular $k$-NAE-SAT on $N$ variables, and let $Z_N(\beta)$ be as in (115). If $E$ denotes expectation over the law of $\mathcal{G}_N$, then

$$E \ln Z_N(\beta) \leq \frac{\Phi_{\beta,m_1,m_2}(Q)}{\beta} + O\left(\frac{1}{N^{1/3}}\right),$$

for all $0 < m_1 < m_2 < 1$ and all $Q \in \mathcal{P}_2$, uniformly over all $\beta \geq 0$.

To conclude, we take $\beta \to \infty$ to deduce the zero-temperature bound Proposition 1.10:

**Proof of Proposition 1.10.** Let $w^0 \equiv 1_0$, $w^1 \equiv 1_1$, and $w^f \equiv (1_0 + 1_1)/2$. Thus, for each $w \in \{0, 1, f\}$ we have defined an element $w^0 \in \mathcal{P}$. Recall that $\Omega$ is the space of probability measures over $\{0, 1, f\}$: for each $\rho \in \Omega$ we define $r^\rho \in \mathcal{P}$ which is supported only on the three points $w^0$, $w^0$, $w^f$:

$$r^\rho \equiv \sum_{\rho \in \{0,1,f\}} \rho(w) 1_{w^\rho}.$$ 

Finally, if $Q$ is a probability measure over $\rho \in \Omega$, we let $\bar{Q}$ be the induced law of $r^\rho$. (Formally $\bar{Q} = r_{\bar{Q}} Q$ if $\bar{r}$ denotes the mapping $\rho \mapsto r^\rho$.) Then, as $\beta \to \infty$ we have

$$\hat{X}_{\beta,\rho}(w_{1,k})^{y_{j,k}/\beta} \to \exp\{-y_2 \phi(w_{1,k})\},$$

$$X_{\beta}(w_{1,D})^{y_{j,k}/\beta} \to \exp\{-y_2 \phi(w_{1,D})\}.$$ 

It follows from this that as $\beta \to \infty$ we have

$$C_{\beta,y_1,y_2}Q(\hat{\mathcal{G}}(y_1,y_2,Q), W_{\beta,y_1,y_2}Q(\mathcal{W}(y_1,y_2,Q).$$

Therefore $\beta^{-1} \Phi_{\beta,y_1,y_2}(\bar{Q}) \to \Phi^{\text{IRSB}}(y_1,y_2,Q)$, and the result follows.

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