I. INTRODUCTION

With the advancement of technology and miniaturization of digital circuits, managing and controlling thermal emissions in the small scale has gained utmost importance. The miniature circuits are prone to damage due to even minute thermal and electrical fluctuations. Devices to control the flow of electricity have existed and been improved upon since the invention of the vacuum diode [1–3]. Over the past few decades, heat conduction in such circuits has been intensively studied [4–8] and devices for regulating heat flow have been envisioned [9–13].

An electrical transistor [1–3, 14–17] is a three-terminal device – one of the terminals modulates (controls) the flow of electricity through the other two. The thermal counterpart of the electrical transistor – thermal transistor [18–20] – regulates the heat flow in a circuit. These devices can work as heat switches and modulators. Thermal transistors have been designed by utilizing gas-liquid transitions [21], suspended graphene [22], and many more systems [23–26]. A recent experiment has successfully implemented the thermal transistor using a thin layer of MoS$_2$ that is effective even at room temperature [27].

The analysis of thermal devices at the quantum level, especially from the viewpoint of quantum information processing and quantum computers has resulted in the development of individual quantum systems [28–30] over the past few years. Such devices include photon rectifiers [31], photon transistors [32, 33], quantum rectifiers [34, 35], quantum refrigerators and heat engines [36–45] among others. The aim of practical realization of such devices has made quantum thermodynamics indispensable to the field [46–53]. The macroscopic laws of thermodynamics were taken over to the quantum scale in, e.g., Refs. [54–61]. Furthermore, the possibility of interaction of a quantum system with its environment has given rise to several studies of quantum systems that talk with heat baths [62–64].

In recent years, there have been several proposals for various types of quantum thermal machines. A quantum thermal transistor (QTT) constructed using two-level systems is demonstrated in [65, 66] and one with coupling between a two-level and a three-level system in [67]. A QTT using superconducting circuits is designed in [68], where this device is said to have properties similar to the conventional semiconductor transistor. A QTT with a giant heat amplification phenomenon in the presence of a strong system-bath coupling limit by applying the polaron transformed Redfield equation combined with full counting statistics is investigated in [69].

In [65], a quantum thermal transistor has been proposed using an arrangement of three two-level systems, each paired with a different bosonic heat bath. Dynamic control of heat flow in two terminals is achieved by controlling the temperature of the bath attached to the third terminal. A similar device is discussed in [67]. Another such thermal machine with qubit-qutrit coupling between the systems is shown to behave as a heat current stabilizer [70]. See [71–75] for more work in this direction. Realization of a quantum thermal transistor using defects in solid-state matter, an optomechanical system, or transmon qubits has been suggested in Ref. [74]. A quantum thermal transistor using superconducting circuits is discussed in Ref. [68]. Similar setups can potentially be used to implement the system discussed in this paper.

A quantum thermal device operating in the steady state can take a finite amount of time to reach close to this regime. Since qubits are susceptible to environmental fluctuations, there may exist situations where such a delay could result in decoherence and loss in control of the qubits in question, thereby limiting its operating conditions. To avoid these difficulties, the operation of such devices in the transient regime is studied; in certain cases, this is more beneficial than restricting to the steady state regime. See [41–43] for examples of transient quantum devices. Here, we have studied whether a quantum thermal transistor can function efficiently in the transient regime and what advantages it provides over those operating in the steady state. In particular, we identify parameter regimes where a quantum thermal transistor is necessarily transient.

The rest of the paper is arranged as follows. We have discussed a model of the quantum thermal transistor and some properties that provide a good transistor effect, by comparing the operations of quantum and classical transistors in Sec. II.
The working of a transient quantum transistor is discussed in Sec. III. Certain cases of transient transistors and their amplification properties are included in Secs. III A and III B. The interesting phenomenon of necessarily transient quantum thermal transistors is presented in Sec. IV. In these sections, we discuss the variation of the dynamic amplification factor with temperature of the bath connected to the “base” qubit. Section IV A is about the change of amplification factor with time. Finally, in Sec. V, we discuss the difference of magnitudes of the two amplification factors with respect to temperature of the base-qubit bath for fixed time and with times for fixed temperature of all the baths. A conclusion is presented in Sec. VII.

II. UNDERLYING TOOLS

A quantum thermal transistor can be structured by three two-level systems (TLS) interacting with each other and each system connected to a thermal bath. This system was considered in the steady state regime in [65], while we consider it at fixed temperature of all the baths. A conclusion is presented in Sec. VII.

The Hamiltonian of the three bosonic baths is given by

$$H_{\text{bath}} = \sum_k \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k$$

and the system-bath interaction Hamiltonian has the form -

$$H_{\text{sys-bath}} = \sigma_z^\dagger \sum_k \hbar g_k (\hat{a}_k^\dagger \sigma_z + \hat{a}_k \sigma_z^\dagger)$$

Considering the rotating wave approximation, we have only taken the energy conserving terms. Here, $\hat{a}_k^\dagger (\hat{a}_k)$ is the bosonic annihilation (creation) operator, and $X = A, B, C$. This would mean that the valid transitions are the ones where only one spin is flipped. Thus there are only 12 possible transitions, as discussed in [65]. The system-bath coupling parameters $\omega_{AB}$ and $\omega_{BC}$ are assumed to be equal (and = $\delta$), and the third one, $\omega_{CA}$, is taken to be zero. So, the eigenenergies of $H_{\text{sys}}$ reduce to $E_1 = \delta \hbar$, $E_2 = 0$, $E_3 = -\delta \hbar$, $E_4 = 0$, $E_5 = 0$, $E_6 = -\delta \hbar$, $E_7 = 0$, $E_8 = \delta \hbar$. The allowed transitions for bath $A$ are $1 \leftrightarrow 5$, $2 \leftrightarrow 6$, $3 \leftrightarrow 7$, $4 \leftrightarrow 8$, for bath $B$ are $1 \leftrightarrow 3$, $2 \leftrightarrow 4$, $5 \leftrightarrow 7$, $6 \leftrightarrow 8$, and for bath $C$ are $1 \leftrightarrow 2$, $3 \leftrightarrow 4$, $5 \leftrightarrow 6$, $7 \leftrightarrow 8$, and the corresponding transition eigenenergies are $\hbar \omega_{ij} = E_i - E_j$ and $\omega_{ij} = -\omega_{ji}$.

In the Born-Markov approximation, the master equation of the system obeys the dynamic evolution,

$$\frac{d\rho}{dt} = -\frac{i}{\hbar} [H_{\text{sys}}, \rho] + \sum_{X=A,B,C} \mathcal{L}_X [\rho],$$

with $t$ representing time. For all further discussions, we have rescaled time as $\delta t$, and used the symbol $t$ for it. $\rho$ is the density matrix of the composite system and the Lindblad operators $\mathcal{L}_X [\rho]$ can be written as in [64]:

$$\mathcal{L}_X [\rho] =$$

$$\sum_{\omega \geq 0} \mathcal{J}(\omega) \left[ 1 + n_X^\omega \right] A_X^\dagger (\omega) \rho A_X (\omega) - \frac{1}{2} \left[ \rho, A_X^\dagger (\omega) A_X (\omega) \right]$$

$$+ \mathcal{J}(\omega) n_X^\omega \left[ A_X^\dagger (\omega) \rho A_X (\omega) - \frac{1}{2} \left[ \rho, A_X (\omega) A_X^\dagger (\omega) \right] \right].$$

Here, $n_X^\omega = \frac{1}{2} \text{sech}^2 \left( \frac{\hbar \omega}{2 \omega_0} \right)$ is the Bose-Einstein distribution and $\mathcal{J}(\omega)$ is the Ohmic spectral function which is taken to be $\mathcal{J}(\omega) = \frac{\hbar \omega}{\omega_0}$, where $\omega_0$ is a constant having the unit of frequency. Here we have taken $g_\omega = \kappa \omega$, where $\kappa$ is a dimensionless constant. The Lindblad operators, which are the decomposed form of the system operators ($\sigma_z^\dagger$) in the nondegenerate eigenbasis of $H_{\text{sys}}$, are given by

$$A_X (\omega) = \sum_{i,j \geq 2} \frac{1}{\omega = \omega_{ij}} |i\rangle \langle i| \sigma_z^\dagger |j\rangle \langle j|.$$
where $|i\rangle$ and $|j\rangle$ are elements of the nondegenerate eigenbasis of $H_{sys}$ and $A_{X}(\omega)$ is an eigenoperator of $H_{sys}$ corresponding to the energy eigenvalue $-\hbar\omega$, i.e., $[H_{sys}, A_{X}(\omega)] = -\hbar\omega A_{X}(\omega)$.

The heat current is defined as the amount of entropy exchanged per unit time between the open system and the environment. The flow of heat current is caused by the change of the internal energy $E = Tr(H_{sys}p)$, which results from dissipative effects [76]. So, heat current can be expressed as

$$J_{X} = Tr(H_{sys}L_{X}[p]).$$

(7)

Let us define $X = A, C$. It provides a measure of the efficiency of the system in working as a transistor. For the quantum thermal transistor to work, the heat flow between one of the three TLSs and the bath that it is coupled with (in our case, $J_B$) should control the other two heat currents ($J_A$ and $J_C$). This would require that a small change in $J_B$ results in relatively large changes in $J_A$ and $J_C$, giving rise to the transistor effect. This is analogous to a semiconductor transistor where a change in the base current controls the flow of emitter and collector currents. In the case of the QTT, the dynamic amplification factor ($\alpha$) is defined as

$$\alpha_X = \frac{\partial J_X}{\partial J_B} = \frac{\partial J_X}{\partial J_B},$$

(8)

where $X = A, C$. It provides a measure of the efficiency of the system in working as a transistor. For the quantum thermal transistor to work, the heat flow between one of the three TLSs and the bath that it is coupled with (in our case, $J_B$) should control the other two heat currents ($J_A$ and $J_C$). This would require that a small change in $J_B$ results in relatively large changes in $J_A$ and $J_C$, so that we have $|\alpha| > 1$. A higher value of $|\alpha|$ would mean that significantly better control of $J_A$ and $J_C$ has been achieved. Similar measures for efficiency have also been used in Refs. [65–75]. This amplification factor is analogous to the base transport factor ($\beta = \frac{\Delta T_B}{\Delta T}$), which is the ratio of change in collector to change in base current in the classical transistor. For a good transistor, $\beta \approx 100$, whereas in the quantum case, the three TLS-bath setup behaves like a transistor if $|\alpha_X| > 1$ and a good one if $|\alpha_X|$ has a sufficiently large value. Note however that the $\beta$ for a classical transistor and $\alpha$ for its quantum counterpart are not directly comparable. The elements of $\alpha$ are the heat currents corresponding to the exchanges of heat between the TLSs and their respective baths, and is not a result of the movement of any particle (electrons in case of a classical transistor). For the steady state situation, $\frac{\partial J_B}{\partial T_B} = 0$, thereby yielding $\sum_{X=A,B,C} J_X = 0$. So, $\alpha_A + \alpha_C + 1 = 0$ and when $\alpha_A$ and $\alpha_C$ have large values, then $\alpha_A \approx -\alpha_C$. This feature is lacking in the case of general evolved states. Furthermore, for certain values of $T_B$, the dynamic amplification attains very high values [65]. For a steady state quantum thermal transistor, $\alpha$ can reach to infinity for a certain value of $T_B$. This is due to the fact that, at such points, the denominator, $\frac{\partial J_B}{\partial T_B}$, of the expression for $\alpha$ vanishes (with the numerator remaining finite). This is reminiscent of the Coulomb blockade where the electrical conductance vanishes at low bias voltages. See Fig. 1.

In this paper, we have obtained the partial derivatives of the numerator and denominator of Eq. (8) numerically, using the five-point (midpoint) numerical formula. The first-order derivative of a function $f(x)$ at point $x_0$ is then given as

$$f'(x_0) = \frac{f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)}{12h},$$

(9)

An error of order $h^4$ appears here, which we neglect for $h$ of the order 0.001. In our case, the numerator and the denominator of Eq. (8) are functions of $T_B$, and $T_A = 0.2$ and $T_C = 0.02$ as in [65].

III. A QUANTUM TRANSISTOR IN TRANSIENT REGIME

A steady state QTT functions efficiently in the region $0 < T_B < 0.15$, where a slight change in $J_B$ results in large changes in $J_A$ and $J_C$; this gives us high values of $\alpha$. $\frac{\partial J_B}{\partial T_B}$ reduces with respect to $T_B$, for which the amplification factors shoot to infinity at $T_B \approx 0.12$. Therefore, at $T_B \approx 0.12$, the magnitude of amplification factor shoots to infinity and then again reduces; it falls to nearly equal to zero after $T_B > 1.15$ and the system no longer works as a transistor. See [65] for a detailed discussion of the quantum thermal transistor operating in the steady state regime.

It is possible that the time taken by the system to reach the steady state is quite large and continuing the time evolution for such a long duration may result in some noise or fluctuations in the tuning parameters of the setup. This may result in straying outside the ideal working parameters and the system may no longer serve our purpose. This could be overcome by operating the QTT in the transient regime, where the time of evolution of the system is smaller than the time taken to evolve into the steady state and the operation of transistor is better or at least as good as the steady state regime. Such a
QT T could prove to be advantageous over the same operating
in the steady state regime. But it is not true that, from every
initial state, such a transient QT T could be obtained.

In this paper we discuss some cases where the transient
regime is beneficial over the steady state of a QT T. We have
taken different initial states and studied their amplification
properties in the transient regime. We find that there exist
several states that follow a nature similar to that of the steady
state QT T. For these initial states, transistor effect is visible
even for smaller times. There are also some states for which
the dynamic amplification dies out for much smaller values of
T B. These states are not very good for our purposes, but not
useless either, because, although the operating region of the
transistor is T B < 0.15, the amplification factors have much
larger values; these are better transistors for a very small re-

gion of T B. Nonetheless, there exist states for which the tran-
sistor effect is retained even after T B ≥ 0.2. These are of
particular interest as they promise more efficiency than that of
the steady state QT T without the long wait to reach the steady
state. In the next subsections, we discuss some such initial
states and the corresponding transistor effect they show.

A. Paradigmatic initial states for reaching the transient regime

We start with the well-known three-qubit Greenberger-
Horne-Zeilinger state (GHZ state [77]), which has the form

\[ |\text{GHZ} \rangle = \frac{|000 \rangle + |111 \rangle}{\sqrt{2}}. \]

The nature of the amplification factor for different times with
the GHZ state as the initial state in the time evolution is il-
lustrated in Fig. 2. As is evident from the plots, for times
satisfying 0.1 ≤ t ≤ 0.3, the amplification factor is more than
that of the steady region QT T, but only for T B < 0.8. We
remember that time is used in this paper in the unit of δ, so
that t/δ has the unit of time. If we further increase time, the
amplification factor steadily decreases to a very small value;
at t ≈ 0.8, the transistor is not working at all, after which the
amplification factor increases again; it decreases once more
and coincides with the steady state when t ≈ 6, after which
it remains the same. Thus we can have a good transient QT T
with better efficiency than the steady state QT T but for a
reduced range of T B.

Another paradigmatic genuine three-party entangled three-
qubit state is the W state [78, 79] (see also [80]),

\[ |\text{W} \rangle = \frac{|001 \rangle + |010 \rangle + |100 \rangle}{\sqrt{3}}. \]

Since the evolution of the W state follows a trend similar to
that of the GHZ state, the rest of the discussion about the W
state is included in Appendix A. A discussion about the state
|000 \rangle as the initial state is also given in the same appendix.

The other two initial product states we have studied are
|001 \rangle and |011 \rangle. For these, we get appreciably good QT T
which has a significant advantage over a steady state QT T be-
beyond T B ≥ 0.2. This advantage in the behavior of the QT T
when |001 \rangle is chosen as the initial state is shown in Fig. 3. For
t = 0.1, we have a very large amplification factor as well as
a larger operating region of T B. As time increases, the ampli-
fication factor reduces and ultimately reaches the steady state
for times given by t ≈ 10. (Please refer to Appendix C for the
plots corresponding to certain intermediate times, viz. t = 3
and t = 6.) During the time evolution, it remains better than
the steady state QT T, and it always gives a larger operating re-
gion of T B until it goes to the steady state regime. For |011 \rangle
as the initial state (see Fig. 4), the transient transistor gives better
efficiency and a larger operating region of T B than that of the
steady state transistor for small times (t = 0.1). For further
increase of time, the amplification factor reduces. At t = 3, it
provides a slightly larger operating region with a lesser ampli-
fication factor than the steady state transistor. For t = 6 (see
Appendix C), it is not at all a better transistor than the steady
state one and, for t ≈ 10, it reaches to the steady state case.
For both the cases, the transient transistors can be considered
as necessarily transient QT T for t = 0.1. Some more exam-
ple of necessarily transient transistors are discussed in Sec.
IV.

B. Random initial states

Until now, we have studied the QT T in the transient regime
by considering a few paradigmatic three-qubit states as initial
states of the time evolution. There are of course a large num-
ber of other states that could be explored. It is not clear now if
we should classify them for their usefulness in the QT T task.
With respect to their entanglement content, these states can
broadly be divided into product, biseparable, GHZ- and W-
class states [79]. We expect that exploring random elements of
these classes of states would provide an intuitive picture of ap-
proximately the entire space of states. We analyzed 350 Haar
uniformly generated states belonging to either one of these
classes.

The three-qubit genuinely multiparty entangled pure states
are all GHZ-class states except for a set of measure zero [79].
The general form of the GHZ-class state is therefore the same
as the general three-qubit pure state, up to a set of measure
zero, and the latter is given by the following, which we chris-
ten as |ψ⟩GHZ:

\[ |ψ⟩_{\text{GHZ}} = a |000 \rangle + b |001 \rangle + c |010 \rangle + d |100 \rangle
+ a_1 |011 \rangle + b_1 |110 \rangle + c_1 |101 \rangle + d_1 |111 \rangle, \]

where a, b, c, d, a_1, b_1, c_1, d_1 are the complex coefficients,
constrained by the normalization condition.

For generating W-class states Haar uniformly, we take a
different approach. The general form of a W-class state is
[79]

\[ |ψ⟩_W = a |001 \rangle + b |010 \rangle + c |100 \rangle + d |000 \rangle, \]

and states that are local unitarily connected to it, where
a, b, c, d are arbitrary real numbers up to the normalization
constant. As an easier alternative to the Haar-uniform genera-
tion of the W-class states, we proceed as follows: (a) generate
FIG. 2. Transient three-qubit quantum thermal transistor with the time evolution beginning in the Greenberger-Horne-Zeilinger (GHZ) state. The panels exhibit the nature of amplification factors with the change of $T_B$ for different dimensionless times. The initial state is taken as the GHZ state. The axes represent dimensionless parameters in all panels. We remember that time is used in this paper in the unit of $\delta$, so that the times mentioned in the panels when divided by $\delta$ has the unit of time.

FIG. 3. Nature of amplification factors with the change of $T_B$ for different dimensionless times. The initial state is taken as $|001\rangle$. Both axes represent dimensionless parameters in all panels. For the same cases at $Time = 3$ and $Time = 6$, see Appendix C.

The above state with complex $a$, $b$, $c$, $d$; (b) generate the above state with complex $a$, $b$, $c$, $d$ but with $|0\rangle$ and $|1\rangle$ replaced by $\sigma_x$ eigenvectors; (c) repeat the same using $\sigma_y$ eigenvectors. In this way, we have a fair chance of having a picture of the entire $W$ class.

In the biseparable class of states, two out of the three TLS are entangled in each state. There can therefore be three types of biseparable states: separable in the $A : BC$ cut, or the $B : AC$ cut, or the $AB : C$ cut. The general form of a biseparable state that is separable in the $AB : C$ cut is

$$|\psi\rangle_{AB:C} = (a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle) \otimes (a_1|0\rangle + b_1|1\rangle).$$

Similarly for the $B : AC$ and $A : BC$ cuts.

The next (and final) is the class of 3-qubit product states which have the general form

$$|\psi\rangle_{Product} = (a|0\rangle + b|1\rangle) \otimes (a_1|0\rangle + b_1|1\rangle) \otimes (a_2|0\rangle + b_2|1\rangle).$$
We studied 50 Haar-uniformly generated states each from the classes GHZ, the three types of W-class states mentioned above, biseparable states separable in the \( A : BC \) cut, biseparable states separable in the \( AB : C \) cut, and product states, making a total of 350 states. We set the time so that \( t = 0.1 \), because many of these states reach their steady states rather quickly. Moreover, having a transient QTT at a short time is advantageous.

For these 350 initial states, we get three types of transient QTTs. One of them has the same operating efficiency as discussed in Sec. III for GHZ, \( W \) and \( |000\rangle \) as initial states. They have a good amplification property in a smaller region of \( T_B \), but have a better efficiency than that of the steady state transistor. This type of transient QTT is depicted in panels (a) and (b) of Fig. 5. The second type of transient transistor is shown in the last two panels of Fig. 5. These have an operating region of \( T_B \) that is similar to that of the steady state transistor, but has a larger value of amplification factor, which makes us conclude that these types of transistors are more advantageous than a steady state transistor. The third type of transient transistor is one of the main interests of our paper, and is discussed in the succeeding section. Note that we can categorize the transient transistors depending on their operating region with respect to \( T_B \), but we cannot categorize them depending on the chosen class from which the initial state is used. The three types of transistors can be found in any one of the classes of states we have mentioned above. This indicates that the initial entanglement content of the system seems to have little to do with its efficiency for the quantum task chosen at hand.

**IV. NECESSARILY TRANSIENT QUANTUM TRANSISTOR**

We found that, for certain states, we get significant amplification even for \( T_B \geq 0.2 \), where the transistor no longer functions in the steady state regime. As a consequence, these types of transistors can be considered as *necessarily transient quantum transistors*, and potentially form a beneficial quantum device not available in the steady state regime. Such states can belong to either of the four classes of states discussed in the preceding subsection, viz. the GHZ and \( W \) classes, biseparable states and product states. The transistors using initial product states \( |001\rangle \) and \( |011\rangle \) can also be included in this category of necessarily transient ones, as already discussed in Sec. III. Now, we shall discuss four such initial states and the operating regions of their providing transient thermal transistors. It may be noted here that within the literature of quantum refrigerators, transient refrigerators were considered in Refs.
and necessarily transient refrigerators were reported in Ref. [43].

One such initial state is taken from the Haar-uniformly generated $GHZ$-class states, denoted as $|\psi_{GHZ}^{'}\rangle$, and which has the form in Eq. (12). The amplification factor for $0 < T_B < 0.8$ is shown in Fig. 6 (a). It is clearly visible that in this case, the transient QTT has much better amplification in the region $0 < T_B < 0.15$ than that of a steady-state transistor. Moreover, the $GHZ$-class transient transistor also has a good amplification beyond $T_B = 0.15$. The efficiency of the transistor in $0.25 \leq T_B \leq 0.75$ is magnified in Fig. 6 (a'). The amplification is not very good in $0.4 \leq T_B \leq 0.6$, but it is still about 10. This is not appreciably good, but if we are constrained to be in the region $0.3 \leq T_B \leq 0.6$, then a transient QTT is possible using this state, although a steady-state one cannot be used.

The next example of a necessarily transient quantum transistor is obtained from an initial biseparable state $|\psi_{AB:C}^{'}\rangle$, that is separable in the $AB : C$ cut, and has the form in Eq. (14). This case is depicted in Fig. 6 (b), and this shows a nature similar to the previously-discussed $GHZ$-class transient transistor. The major difference is that in the $GHZ$ case, after reducing near $T_B = 0.3$, the amplification factor again rises and reaches an appreciably large value for $T_B > 0.6$, but in this biseparable case, the amplification factor reduces near $T_B = 0.4$ and does not increase again. This is clear in Fig. 6 (b'), which is a magnified version of Fig. 6 (b). The operating region of this transient transistor is $0 < T_B < 0.4$, which is not bad at all.

Necessarily transient quantum thermal transistor can also be obtained from the $W$-class of states. One example of such a state is $|\psi_{W}^{'}\rangle$, having the form in Eq. (13), with the corresponding results being given in Fig. 6 (c). From Fig. 6 (c) and its magnified version (c'), we can see that the operating region of the transient transistor having the initial state $|\psi_{W}^{'}\rangle$ is $0 < T_B < 0.7$ which is the same as that of the necessarily transient transistor having the initial state $|\psi_{GHZ}^{'}\rangle$ that we have already discussed. The long-$T_B$ behavior is qualitatively the same in these two cases (compare Figs. 6 (a') and (c')) but the short-$T_B$ behavior is interestingly different (compare Figs. 6 (a) and (c)). Despite this qualitative difference at short $T_B$, the quantity of amplification provided by necessarily transient transistors having $|\psi_{GHZ}^{'}\rangle$ and $|\psi_{W}^{'}\rangle$ as their initial states are almost the same before they shoot to a very large value and go to zero respectively.

The last one we discuss here is a necessarily transient transistor having a product state as its initial state. The product state $|\psi_{product}^{'}\rangle$ has the structure as in Eq. (15). Figures 6(d) and 6(d') [magnified version of (d)] show that this necessarily transient transistor provides a similar nature of amplification factor to that of the $W$-class state for small $T_B$, but as in $|\psi_{AB:C}^{'}\rangle$ discussed above, the rising of the amplification factor twice is missing here. This transient transistor has the smallest operating region $0 < T_B < 0.18$ compared to the preceding three, but can be categorized in the necessarily transient transistor.

The exact forms of the four initial states which we have considered for the analyses in the figures, viz. $|\psi_{GHZ}^{'}\rangle$, $|\psi_{AB:C}^{'}\rangle$, $|\psi_{W}^{'}\rangle$, $|\psi_{product}^{'}\rangle$, are given in Appendix B. For these four initial states, we can structure a necessarily transient transistor.

A note about the preparation of the initial states is in order here. The product state, $|\psi_{product}^{'}\rangle$, can be created by locally magnetizing the three spins-1/2 system in the appropriate directions. This requires local preparation of the states of each two-level system separately, and can be done on almost all physical platforms. The other three states can be prepared by local filtering operations on the $GHZ$, $W$, or two-qubit maximally entangled states (“Bell” states). See [82, 83] in this regard. Preparation of $GHZ$, $W$, and Bell states has been attained in several physical systems, and requires non-trivial entangling operations, whose exact form depends on the physical system used. See, e.g., Refs. [84–92], and references therein.

There are many states with this advantageous ability, but we discussed only four of them. It is important to note that all the necessarily transient QTTs have the amplification properties that are similar to any of these four. In our examples, we found that the transient transistors having the initial states $|\psi_{GHZ}^{'}\rangle$ and $|\psi_{W}^{'}\rangle$ illustrated in Figs. 6(a) and 6(c) as the most advantageous compared to other examples we have discussed. This does not mean that all necessarily transient transistor having $GHZ$- and $W$-class states as their initial states always have the better advantage over the other two classes. This fact is clearly visible from Figs. 3 and 4. Both are necessarily transient transistors having two different initial product states, but the behavior of the amplification factors are completely different, and yet the operating region is almost the same. The difference in operating regions of necessarily transient transistors having initial product states can be understood by comparing Figs. 3, 4, and 6(d). We can conclude that the four panels in Fig. 6 are not the representatives of their corresponding classes, but they are the representatives of the four types of necessarily transient quantum thermal transistors.

A. Variation of amplification factor with time

Until now in this section, we discussed some cases of necessarily transient QTTs at $t = 0.1$. At this time, the system studied remains in transient regime and is far from its steady state sector. In all previous plots of amplification factors at different times, we observed that all the transistors are in their transient regime at $t = 0.1$, and we get better efficiency around that time compared to later times. But it is necessary to identify the range of time for which we can have a better transistor than the steady state one. Thus observing the time evolution of this system is also crucial before we can consider it as a transient QTT. It also helps us to have an idea about how fast the system reaches the steady state regime. We have already discussed for which value of time the transient transistors having initial states as $GHZ$, $W$, and $|000\rangle$ provide good efficiencies, and that the necessarily transient transistors having initial states $|001\rangle$ and $|011\rangle$ give better efficiencies as well as a sufficiently large operating region in $T_B$, in Sec. III. Here we analyze the time evolution of the amplification factor of the necessarily transient transistor having the initial states
FIG. 6. Necessarily transient quantum transistors. Operating regions of necessarily transient transistors are exhibited for the initial states chosen from four classes of states: (a) $|\psi_{GHZ}\rangle$, (b) $|\psi_{AB,C}\rangle$, (c) $|\psi_W\rangle$, and (d) $|\psi_{product}\rangle$. To visualize the operating region beyond the capability of a steady state transistor, magnified views are shown in (a'), (b'), (c'), (d'), respectively, in the relevant regions. All axes represent dimensionless parameters.

FIG. 7. Time evolution of amplification factor for different $T_B$ for initial states taken as (a) $|\psi_{GHZ}\rangle$ and (b) $|\psi_W\rangle$. All the axes represent dimensionless parameters. In Appendix C, we present plots also for $T_B = 0.13$ and $T_B = 0.26$.

$|\psi_{GHZ}\rangle$ in Fig. 7 (a), and having $|\psi_W\rangle$ as the initial state in Fig. 7 (b). We have studied the time evolution by fixing $T_B$ at four values, viz. 0.05, 0.13, 0.26 and 0.36. (Plots for $T_B = 0.13$ and $T_B = 0.26$ are in Appendix C.) The values of $\alpha_A$ and $\alpha_C$ for these values of $T_B$ for a steady state transistor are given below:

- $T_B = 0.05 : \alpha_A \approx -149, \quad \alpha_C \approx 148$;
- $T_B = 0.13 : \alpha_A \approx 206, \quad \alpha_C \approx -207$;
- $T_B = 0.26 : \alpha_A \approx -0.4, \quad \alpha_C \approx -0.6$;
- $T_B = 0.36 : \alpha_A \approx -0.5, \quad \alpha_C \approx -0.5$.
We are interested in those times for which we can get a better or at least the same efficiency as in the steady state case. At $T_B = 0.05$, the transistor having the initial state $|\psi_{GHZ}\rangle$ has a good efficiency (better or the same as the steady state efficiency) for $0 < t < 3$, very close to $t = 6$, and for the further times where it has already reached the steady state (for our purposes). So, it is good to stay in the region $0 < t < 1$ and very close to $t = 6$ for achieving a good amplification. For $|\psi_W\rangle$ as the initial it is better to stay in the region $0 < t < 1$. At $T_B = 0.13$, for both the initial states, $|\psi_{GHZ}\rangle$ and $|\psi_W\rangle$, it is better to stay in $0 < t < 1$. For $|\psi_W\rangle$, it is also good to stay at $t$ close to 6. For $t > 6$, for both the states, the transistor reaches the steady state operating regime. At $T_B = 0.26$ and $t = 0.36$, we can see that for both the cases, it is better to stay at times near zero, because for a very small time, the amplification factor reduces very fast and goes to a very small magnitude (0.5). Hence, for the two most useful necessarily transient QTTs, we see that we can get better efficiencies if we stay at times close to zero for all $T_B$. In all our previous studies (in this section), we were at $t = 0.1$, which is closer to zero, and so providing a good amplification factor.

V. RELATION BETWEEN $\alpha_A$ AND $\alpha_C$

We have already stated the relation between $\alpha_A$ and $\alpha_C$ for transistors operating in the steady state regime in Sec. II. The relation, $\alpha_A + \alpha_C + 1 = 0$, is however not valid in the transient regime. For transient sectors, the relation between $\alpha_A$ and $\alpha_C$, obtained from Eq. (4), is given by

$$\alpha_A + \alpha_C + 1 = \left(\frac{\partial J_B}{\partial T_B}\right)^{-1} \text{Tr}\left[H_{sys} \frac{\partial}{\partial T_B} \frac{dp}{dt}\right].$$  \hspace{1cm} (16)

For the steady state regime, $\frac{dp}{dt} = 0$, whereby the right-hand-side (RHS) of Eq. (16) reduces to zero. Till now, we were concentrating on both the amplification factors together and found that when amplification is low then both $\alpha_A$ and $\alpha_C$ are low, and when amplification is high then both are high, but we did not have any idea whether $\alpha_A$ and $\alpha_C$ have almost the same magnitudes in sectors where they are very high. For the steady state case, it is always true that $\alpha_A \approx \alpha_C$ when amplification is high, but what is the scenario for the transient regime? This is the question that we analyze in this section. The difference of the magnitudes of $\alpha_A$ and $\alpha_C$ for different values of $T_B$ is depicted in Fig. 8(a), and the same for different values of time is depicted in Fig. 8(b). These are the scattered plots of $|\alpha_A| - |\alpha_C|$ for the 350 initial states chosen randomly. From Fig. 8(a), we infer that, when $T_B$ is low, the differences of the magnitudes of $\alpha_A$ and $\alpha_C$ are almost the same for all states and it is not very high (being $\approx 10$). In all the cases studied, we have seen that, for low $T_B$ and at $t = 0.1$, amplification factors are sufficiently high and so the difference in magnitudes of the amplification factors was not visible in the previous plots. Now, for further increase of $T_B$ the probability of having a greater difference in magnitudes of the amplification factors increases and for $0.12 \leq T_B \leq 0.15$ there is a finite probability to have this difference very high. For $T_B = 0.15$, we have got the maximum $|\alpha_A| - |\alpha_C| \approx 6000$ with a very small probability. The probability of having a smaller difference in magnitudes is greater. We have set the range of $|\alpha_A| - |\alpha_C|$ axis from 0 to 100 to show that the probability of having a smaller difference is always greater but there is a finite probability of having a larger difference for $T_B$ near 0.12 – 0.15, which is not shown in this figure. This large difference is also not visible in the previous plots because for this region of $T_B$, amplification factors are very large and the range of the $\alpha_X$ axes were not so large. If we further increase $T_B$, the probability of having greater $|\alpha_A| - |\alpha_C|$ reduces and, for most states, it comes to below 20.

Next we discuss how the difference $|\alpha_A| - |\alpha_C|$ changes with time for a fixed $T_B$. We have fixed $T_B$ at 0.08, because from the preceding observations, we have found that at $T_B = 0.08$, all the transient thermal transistors have a good amplification. From Fig. 8 (b), we can see that the differences of the magnitudes of the two amplification factors for different times at a fixed $T_B$ follows a similar nature as they have with different $T_B$ at fixed time. For smaller times, all the 350 states have very small differences between $\alpha_A$ and $\alpha_C$ in magnitude. So, in the cases where we were at $t = 0.1$, we got $\alpha_A \approx \alpha_C$ for $T_B = 0.08$. Now, if we go ahead on the time axis, the probability of having greater $|\alpha_A| - |\alpha_C|$ increases and, for $2 < t < 6$, we can have a finite probability to have some initial states for which this difference is very high. The maximum $|\alpha_A| - |\alpha_C|$ we have obtained is $\approx 60000$ for $t = 4.8$. For $t > 6$, the differences reduce for all states and are very near to zero, implying – within Haar-uniform generation accuracy and within the set of physical quantities studied – that all states reach the steady state regime for $t > 6$.

VI. PHYSICAL REALISATION

In this short section, we aim to provide indications to potential physical systems where the phenomenon discussed can be implemented, and also discuss the corresponding numbers in real units. There are some previous works in which proposals of experimental setups for similar ends are given. See Refs. [68, 74]. We will briefly discuss the experimental range of values of the parameters of our theoretical setup in a potential superconducting qubit implementation.

The decoherence time of a superconducting qubit lies between $50 - 100 \mu s$ [81]. So, for our system to be useful for a superconducting qubit set-up, we must work in this timespan. We have taken our dimensionless time as $t = \delta t$. Let us take $\delta t_{max} = 50 \mu s$. For $t = 0.1$, $\delta_{min} = 2 MHz$. Now, the dimensionless quantity, $T_X$, is given by

$$T_X = \frac{K_B T_X}{\hbar \delta}.$$  \hspace{1cm} (17)

If we choose $T_B = 0.2$, we would have $T_B|_{min} = 3.04 nK$. For smaller values of $t$, carefully choosing $\delta$ can help in controlling the heat flow of systems with temperatures of $O(mK)$ to $O(K)$.

If we restrict $t$ to $O(ns)$, $\delta$ will be in the $GHz$ range, and temperature around $mK$. In case of a superconducting qubit,
gate lengths are of $O(10-100\,\text{ns})$, carrying out $10^4$ operations per coherence time [81]. Therefore, the three TLS-bath QTT set-up can be useful for controlling the heat flow of such a system. A QTT using a superconducting transmon qubit is proposed in [68], where $\delta = O(GHz)$.

VII. CONCLUSION

We were interested to find whether the three two-level systems setup, which behaves like a thermal transistor in the steady state regime, can also operate in the transient regime, and whether there is any advantage to do so. From our analysis of several paradigmatic families of three-qubit states, including the Greenberger-Horne-Zeilinger and W states, as initial states of time evolution of the device of three qubits and three baths, we find transient thermal transistors having an efficiency better than the steady state one but having a smaller or equal operating region, as well as necessarily transient transistors having better amplification capability as well as a larger span of operating region.

VIII. ACKNOWLEDGEMENT

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Appendix A: Paradigmatic initial states: $W$ and $|000\rangle$ states

The nature of variation of amplification factor for the $W$ state (Fig. 9) follows a trend similar to that of the $GHZ$ state (Sec. III A). For small time ($t = 0.1$), the setup behaves as a good transistor in the region $0 < T_B < 0.8$, and for longer times the values of the amplification factor reduce and the operating region of the transistor in $T_B$ approaches the steady state region and reaches the steady state value at $t \approx 6$. The main difference with the GHZ state is that the amplification factor is never less than the steady state and thus the transient transistor is always better than a steady state transistor for the same discrete values of time, but for a smaller region of $T_B$.

FIG. 8. Variation of $||\alpha_A| - |\alpha_C||$ with (a) $T_B$ for time fixed at $t = 0.1$, and (b) time for $T_B$ fixed at 0.08. All the axes represent dimensionless parameters.

FIG. 9. Transient three-qubit quantum thermal transistor with the time evolution beginning in the $W$ state. The panels exhibit the nature of amplification factors with the change of $T_B$ for different dimensionless times. The initial state is taken as the $W$ state. The axes represent dimensionless parameters in all panels.
This means that the amplification factor reduces and is almost zero for $t = 0.8$ and it again has a better amplification after $t = 0.9$ and, at $t \approx 6$, it reaches the steady state. As in the previous two cases, the operating region of the transient transistor is bounded for a small region of $T_B$. Therefore, for this state, too, the higher amplifications for smaller $T_B$ values is an interesting feature for going over to the transient regime.

The plots in Figs. 6 (b) and (b') are obtained from an initial biseparable state, $|\psi_{ABC}'\rangle$, that has the form in Eq. (14) with

$$a = (-0.2506, -1.2750),$$
$$b = (0.4573, 0.0094),$$
$$c = (1.1436, 0.5672),$$
$$d = (-0.9806, 1.2475),$$
$$a_1 = (-0.7718, 0.4604),$$
$$b_1 = (0.2562, -0.3517).$$  \hfill (B2)

The $W$-class state, $|\psi_{W}'\rangle$, which results in the dynamics corresponding to the plots in Figs. 6 (c) and (c'), has the form in Eq. (13) with

$$a = (-0.6549, -1.5778),$$
$$b = (0.1125, -0.4555),$$
$$c = (0.8575, -0.4032),$$
$$d = (-0.5980, -1.0251).$$  \hfill (B3)

Finally, Fig. 6 (d) and (d') correspond to the product initial state, $|\psi_{Product}'\rangle$, that has the form as in Eq. (15) with

$$a = (0.7938, -0.4108),$$
$$b = (1.6511, 0.8510),$$
$$a_1 = (-0.5692, 1.3391),$$
$$b_1 = (-0.5365, -0.3410),$$
$$a_2 = (-2.4324, -1.0312),$$
$$b_2 = (-1.1394, -0.7807).$$  \hfill (B4)

Appendix C: Behavior at intermediate times

The plots of amplification factors for different intermediate times for the initial states, $|001\rangle$ and $|011\rangle$, are given in Figs. 11 and 12. Please refer to Sec. III A for the details.
FIG. 12. Nature of amplification factors with the change of $T_B$ for different dimensionless times. The initial state is taken as $|011\rangle$. Both axes represent dimensionless parameters in all panels. For the same cases at $Time = 0.1$ and $Time = 10$, see Fig. 4.

FIG. 13. Time evolution of amplification factor for different $T_B$ for initial states taken as (a) $|\psi_{GHZ}\rangle$ and (b) $|\psi_W\rangle$. All the axes represent dimensionless parameters. Plots for $T_B = 0.05$ and $T_B = 0.36$ are in Fig. 7.

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