Abstract—In this work, cyclic-skew-cyclic codes and sum-rank BCH codes are introduced. Cyclic-skew-cyclic codes are characterized as left ideals of a suitable non-commutative finite ring, constructed using skew polynomials on top of polynomials (or vice versa). Single generators of such left ideals are found, and they are used to construct generator matrices of the corresponding codes. The notion of defining set is introduced, using pairs of roots of skew polynomials on top of polynomials. A lower bound (called sum-rank BCH bound) on the minimum sum-rank distance is given for cyclic-skew-cyclic codes whose defining set contains certain consecutive pairs. Sum-rank BCH codes, with prescribed minimum sum-rank distance, are then defined as the largest cyclic-skew-cyclic codes whose defining set contains such consecutive pairs. The defining set of a sum-rank BCH code is described, and a lower bound on its dimension is obtained. Thanks to it, tables are provided showing that sum-rank BCH codes beat previously known codes for the sum-rank metric for binary $2 \times 2$ matrices (i.e., codes whose codewords are lists of $2 \times 2$ binary matrices, for a wide range of list lengths that correspond to the code length). Finally, a decoder for sum-rank BCH codes up to half their prescribed distance is obtained.

Index Terms—BCH codes, cyclic codes, Gabidulin codes, linearized Reed-Solomon codes, rank metric, Reed-Solomon codes, skew-cyclic codes, sum-rank metric.

I. INTRODUCTION

Codes correcting errors and/or erasures with respect to the sum-rank metric have found applications in universal error correction and security in multishot network coding [1], [2], rate-diversity optimal space-time codes [3], [4], partial-MDS codes for repair in distributed storage [5], compartmented secret sharing [6], and private information retrieval on partial-MDS-coded databases [7]. They may also find applications in extending McEliece’s public-key cryptosystem [8], or in a multishot or multilayer version of crisscross error and erasure correction, extending [9]. The sum-rank metric is also of theoretical interest as it recovers the Hamming metric [10] and the rank metric [11], [12] as particular cases (see Subsection II-B).

Similarly to other metrics, the objective is to obtain codes with large size, large minimum sum-rank distance and small finite-field size, together with an efficient error-correcting algorithm (unless we are only concerned with erasures). As we show next, one of the main difficulties in applying the known code constructions in practice is that they are defined over rather large finite fields.

A few constructions of convolutional codes for the sum-rank metric are known [13]–[17], but they are tailored mainly for erasure correction under erasure patterns common in streaming scenarios. Other constructions include concatenations of Hamming-metric convolutional codes with rank-metric block codes [18], and multi-level constructions over rank-metric block codes [1]. However, for such constructions, the number $m$ of rows per matrix in the definition of the sum-rank metric (see Subsection II-B) needs to be large in order to use meaningful rank-metric component codes. Moreover, the parameter $m$ appears as an exponent in the finite-field sizes of such constructions (convolutional code constructions [13]–[17] suffer also from large finite-field size exponents). Therefore, finite fields of size $2^2 = 4$ ($m = 2$), for instance, are not attainable by such techniques for non-trivial code parameters.

Note that small values for $m$, such as $m = 2$ or $3$, are also of interest in the applications (they correspond to $m$ outgoing links in multishot linearly coded networks [1], [2], and to locality $m$ in locally repairable codes [5]).

The first known block codes whose minimum sum-rank distance attains the Singleton bound and have sub-exponential field sizes are linearized Reed-Solomon codes [19], [20]. Such codes recover as particular cases (generalized) Reed-Solomon codes [21] and Gabidulin codes [9], [12] whenever the sum-rank metric recovers the Hamming metric and the rank metric, respectively. Linearized Reed-Solomon codes may be defined over finite fields of size $\Theta(\ell^m)$, where $m$ is as above and $\ell$ is the number of terms in the sum defining the sum-rank metric (see Subsection II-B) or, equivalently, the list size if we see codewords as lists of matrices [5]. Furthermore, $m$ can be arbitrary small, e.g., $m = 1, 2$ or $3$, while the code length $n = \ell m$, dimension $k$ and minimum sum-rank distance $d = n - k + 1$ may be arbitrarily large. However, $\ell$ still grows linearly as either $n$, $k$ or $d$ grows, and small finite-field sizes such as $2^2 = 4$ are not attainable either by such codes.

To tackle this issue, subfield subcodes of linearized Reed-Solomon codes were considered in [5]. Sec. VII. Their minimum sum-rank distance, over the smaller finite-field extension, is at least the minimum sum-rank distance, over the larger finite-field extension, of the linearized Reed-Solomon codes (see Subsection II-B and Lemma 29). However, the only estimate on their dimension (i.e., code size) considered in [5] is the well-known Delsarte’s lower bound [22] (see [22]). Note that the sum-rank metric for subfield subcodes considered in [5] is not the corresponding extension of the rank metric defined over subfield subcodes considered in [23].

In this work, we describe two new families of linear codes: Cyclic-skew-cyclic codes (Definition 4) and one of its subfamilies, sum-rank BCH codes (Definition 30). The latter codes have a prescribed minimum sum-rank distance thanks to being subfield subcodes of certain linearized Reed-Solomon...
codes. They can be defined over arbitrarily small finite fields, including \(2^2 = 4\) for \(m = 2\) and for arbitrarily large \(\ell\), with code length \(n = \ell m\) (for such parameters, codewords can be seen as lists of \(2 \times 2\) binary matrices with list length \(\ell\), see [4] and Appendix A). In addition, we show that their dimension is in most cases higher than that obtained by Delsarte’s lower bound (see Appendix A) for a given prescribed minimum sum-rank distance. To the best of our knowledge, the code dimensions that we obtain in Appendix A are the highest known so far for \(m = 2\) and the finite-field size \(2^2 = 4\) (i.e., codewords are lists of \(2 \times 2\) binary matrices), for the given code lengths and minimum sum-rank distances.

A. Similar Codes in the Literature

Here, we discuss related codes beyond other known codes endowed with the sum-rank metric [1, 13–18], which were discussed above.

Cyclic-skew-cyclic codes form a family of codes that recover as particular cases classical cyclic codes [24, Sec. 7.2] [25, Ch. 4] and skew-cyclic codes [12, Page 6] [26, Def. 1], whenever the sum-rank metric recovers the Hamming metric and the rank metric, respectively (note that skew-cyclic codes are also good for the Hamming metric [26]). Mathematically, cyclic-skew-cyclic codes can be characterized simultaneously as classical cyclic codes over some non-commutative finite ring, and as skew-cyclic codes over some commutative finite ring (Theorem 1), using polynomials on top of skew polynomials [9], or vice versa [17].

The study of cyclic codes over general finite rings was initiated in [27]. Not many works exist treating cyclic codes over non-commutative finite rings [28–30], and they deal with general properties rather than particular code constructions.

A wide range of works deal with skew-cyclic codes over commutative finite rings that are not fields [31–34]. However, no work has yet treated skew-cyclic codes over the commutative finite ring \(\mathbb{F}[x]/(x^\ell - 1)\), to the best of our knowledge.

Cyclic convolutional codes [35, 36] are similar to cyclic-skew-cyclic codes in that they also make use of skew polynomials over the commutative finite ring \(\mathbb{F}[x]/(x^\ell - 1)\), for a finite field \(\mathbb{F}\). However, they yield (infinite) convolutional codes rather than (finite) block codes.

Sum-rank BCH codes are both cyclic-skew-cyclic codes and subfield subcodes of certain linearized Reed-Solomon codes. Furthermore, they recover classical BCH codes [37, 38] and rank-metric BCH codes [39], whenever the sum-rank metric recovers the Hamming metric and the rank metric, respectively.

B. Main Results

We introduce cyclic-skew-cyclic codes in Definition 4 and characterize them as left ideals over a suitable non-commutative finite ring in Theorem 1.

In Theorem 2 we show that such ideals are generated by a unique element satisfying certain properties, called minimal generator skew polynomial (Definition 7). We show how to use such a left-ideal generator to obtain a generator matrix of the corresponding code in Theorem 3.

We then introduce the notion of defining set (Definition 21) and show that they characterize the corresponding cyclic-skew-cyclic code (Theorem 4). In Theorem 5 we show how to obtain the dimension of a cyclic-skew-cyclic code from its defining set.

In Theorem 6 we obtain a family of cyclic-skew-cyclic linearized Reed-Solomon codes. Using such codes, we obtain in Theorem 7 a lower bound (called sum-rank BCH bound) on the minimum sum-rank distance of cyclic-skew-cyclic codes whose defining set contains consecutive pairs. Sum-rank BCH codes are then defined as the largest codes satisfying such a property (Definition 30).

In Theorem 8 we describe the defining set of a sum-rank BCH code based on the pairs in its definition. Using this result and Theorem 5 we obtain in Theorem 9 a lower bound on the dimension of a sum-rank BCH code that can be easily computed from the pairs in its definition.

In Subsection VII-D we show that sum-rank BCH codes can be decoded with respect to the sum-rank metric up to half their prescribed distance by decoding the larger linearized Reed-Solomon code.

In Appendix A we obtain tables with lower and upper bounds on the parameters of sum-rank BCH codes for \(m = 2\) and the finite-field size \(2^2 = 4\). These tables show that the introduced sum-rank BCH codes beat previously known code dimensions for a given prescribed distance, for a wide range of code lengths.

C. Organization

The remainder of the manuscript is organized as follows. In Section II we provide some preliminaries, mainly the definitions of the sum-rank metric and skew polynomial rings. In Section III we introduced cyclic-skew-cyclic codes and characterize them as left ideals. In Section IV we find a single generator for such left ideals and the corresponding generator matrix. In Section V we introduce defining sets of cyclic-skew-cyclic codes, after defining appropriate evaluation maps for skew polynomials. We conclude the section by computing the dimension of a cyclic-skew-cyclic code from its defining set. In Section VI we revisit linearized Reed-Solomon codes and find a subfamily of such codes formed by cyclic-skew-cyclic codes. In Section VII we provide the sum-rank BCH bound and the definition of sum-rank BCH codes. We then describe their defining sets and conclude by giving a lower bound on their dimensions. The section concludes by showing how to decode sum-rank BCH codes up to half their prescribed distance. Section VIII concludes the manuscript.

II. Preliminaries and General Setting

A. Main Finite Fields in This Work

In this work, \(q_0\) denotes the power of a prime number \(p\). We also fix positive integers \(m\), \(\ell\) and \(s\), and denote \(q = q_0^s\).

We will assume that the finite field with \(q\) elements \(\mathbb{F}_q = \mathbb{F}_{q_0^s}\) contains all \(\ell\)th roots of unity, i.e., all roots of the polynomial \(x^\ell - 1 \in \mathbb{F}_p[x]\). This holds if, and only if, \(q - 1\) is divisible
by $\ell$, see [25] p. 110]. Observe that, if $\ell = 1$, then we may assume that $s = 1$. We refer to [40] for generalities on finite fields.

Throughout the manuscript, we will consider sum-rank metrics (see Definition 1 below), for which we will consider the two finite-field extensions

$$F_{q_0} \subseteq F_{q_0^n} \quad \text{and} \quad F_q \subseteq F_{q^n}.$$  

To relate these four fields, the following directed graph might be helpful, where $K \rightarrow L$ means that the field $L$ is a field extension of the field $K$, i.e., $K$ is a subfield of $L$:

$$F = F_{q_0^n} \quad \longrightarrow \quad F_q \quad \longrightarrow \quad F_{q_0}.$$  

Since we will define our main codes over $F_{q_0^n}$, we will simply denote $F = F_{q_0^n}$.  

B. The Sum-Rank Metric

We fix another positive integer $N$ and define a code length partition

$$n = \ell N = N + N + \cdots + N.$$  

The integer $n$ will be the length of our main codes, and $\ell$ will be the length partition that we will use in order to define the sum-rank metric, which was defined in [1, Sec. III-D] under the name extended rank distance.

Definition 1 (Sum-Rank Metric [1]). Let $K \subseteq L$ be a field extension (in our case, $F_{q_0} \subseteq F_{q_0^n}$ or $F_q \subseteq F_{q^n}$). Consider vectors $c \in L^n$ to be subdivided according to the partition $n = \ell N$ as in (2):

$$c = (c^{(0)}, c^{(1)}, \ldots, c^{(\ell-1)}) \in L^n,$$

for $i = 0, 1, \ldots, \ell - 1$. We define the sum-rank weight for the extension $K \subseteq L$ (i.e., the pair $(K, L)$) and length partition $n = \ell N$ as in (2) (i.e., the pair $(\ell, N)$) as the map $\text{wt}_{SR} : L^n \rightarrow \mathbb{N}$ given by

$$\text{wt}_{SR}(c) = \sum_{i=0}^{\ell-1} \dim_K \left( \langle c^{(i)} \rangle_{0, \ldots, \ell-1} \right).$$

for all $c \in L^n$ subdivided as above. Here, $\dim_K(\cdot)$ denotes dimension over $K$ and $\langle \cdot \rangle_K$ denotes $K$-linear span. We define the sum-rank distance for the same field extension and length partition as the map $d_{SR} : L^n \times L^n \rightarrow \mathbb{N}$ given by

$$d_{SR}(c, d) = \text{wt}_{SR}(c - d),$$

for all $c, d \in L^n$. Finally, for a code $C \subseteq L^n$ (a code will simply be any subset of $L^n$, linear or not), we define its minimum sum-rank distance as

$$d_{SR}(C) = \min \{ d_{SR}(c, d) \mid c, d \in C, c \neq d \}.$$  

To explain the name sum-rank metric, observe that, if $m = \dim_K(L) < \infty$, then

$$\text{wt}_{SR}(c) = \sum_{i=0}^{\ell-1} \text{Rk}_K \left( \begin{pmatrix} c^{(i)}_{0,0} & \cdots & c^{(i)}_{0,n-1} \\ c^{(i)}_{1,0} & \cdots & c^{(i)}_{1,n-1} \\ \vdots & \ddots & \vdots \\ c^{(i)}_{m-1,0} & \cdots & c^{(i)}_{m-1,n-1} \end{pmatrix} \right),$$

where $\text{Rk}_K(\cdot)$ denotes rank over $K$, $c \in L^n$ is subdivided as in Definition 1 and

$$c^{(i)}_j = \sum_{h=0}^{m-1} c^{(i)}_{h,j} \beta_h,$$

where $\beta_h \in K$, for $h = 0, 1, \ldots, m - 1$, for $j = 0, 1, \ldots, N - 1$ and for $i = 0, 1, \ldots, \ell - 1$, and where $\{\beta_0, \beta_1, \ldots, \beta_{m-1}\}$ is an ordered basis of $L$ over $K$. With such a matrix representation, we may consider $c \in L^n$ as a list of $\ell$ matrices

$$(C_1, C_2, \ldots, C_\ell) \in (K^{m \times N})^\ell.$$  

Considering codes as subsets of $(K^{m \times N})^\ell$, endowed with the sum-rank metric as above, may be important in some applications, but becomes cumbersome in our study.

The sum-rank metric recovers the Hamming metric if $m = N = 1$ [19, Ex. 36] and the rank metric if $\ell = 1$ [19, Ex. 37]. See also [5, Sec. II-A]. We will show throughout the paper how our results particularize to these two important cases.

In most applications of the sum-rank metric, the objective is to obtain codes having simultaneously a large size and a large minimum sum-rank distance, for a given code alphabet (pair $(K, L)$) and length (pair $(\ell, N)$). As in the classical Hamming-metric case [25] Th. 2.4.1, there is a trade-off between the size and minimum sum-rank distance of any code (linear or not) given by the Singleton bound [5, Cor. 2]. We refer to [24], [25] for generalities on codes.

Proposition 2 ([5]). With notation as in Definition 1 for a finite-field extension $K \subseteq L$, and for a code $C \subseteq L^n$ (which may be linear or not), it holds that

$$|C| \leq |L|^{n - d_{SR}(C) + 1}.$$  

Codes achieving equality in (4) are called maximum sum-rank distance (MSRD) codes. Maximum rank distance (MRD) codes (e.g., Gabidulin codes [9], [12]) are also MSRD codes, but any MRD code requires $|L| \geq 2^N$, exponential in the parameters $n$, $\ell$ and $N$. The first MSRD codes for sub-exponential field sizes $|L|$ in $n$ and $\ell$ are linearized Reed-Solomon codes [19], [20]. See Subsection VI-A. They require $|L| = \Theta(\ell^N)$. By considering subfield subcodes, one can reduce the base $\ell$ at the expense of possibly reducing code size (relative to field size) and minimum sum-rank distance [5, Sec. VII]. The objective of this manuscript is to study the structure of one family of subfield subcodes of linearized Reed-Solomon codes (Section VII). In particular, we will obtain a better estimate on their dimensions than previously known [5, Cor. 9]. This will lead to codes with larger size and minimum sum-rank distance than previous codes with $|L| \ll \ell^N$. 
To consider subfield subcodes and the sum-rank metric, we will consider the two finite-field extensions \( \mathbb{F}_{q_0} \subseteq \mathbb{F}_{q_0^n} \) and \( \mathbb{F}_q \subseteq \mathbb{F}_q^n \), as in Subsection II-A. Since we will only consider the length partition \( n = \ell N \) as in (2), for \( \ell \) and \( N \) fixed, we do not need to write the dependency of the sum-rank metric on the field extension and length partition. We will simply denote by

\[
\begin{align*}
\text{wt}_{SR} &: \mathbb{F}_{q_0}^n \rightarrow \mathbb{N}, \\
\text{wt}^{(i)}_{SR} &: \mathbb{F}_{q_0}^n \rightarrow \mathbb{N},
\end{align*}
\]

the sum-rank weights for the extensions \( \mathbb{F}_q \subseteq \mathbb{F}_q^n \) and \( \mathbb{F}_{q_0} \subseteq \mathbb{F}_{q_0^n} \), respectively (analogously for the corresponding metrics).

### C. Skew Polynomial Rings

Skew polynomial rings over division rings (i.e., commutative or non-commutative fields) were originally defined by Ore \[41\]. However, in this work, we will need to consider skew polynomial rings over commutative rings which are not necessarily integral domains. We refer to \[42\] for basic Algebra and to \[43\] for non-commutative Algebra.

Let \( R \) be an arbitrary commutative ring (with identity) and let \( \sigma : R \rightarrow R \) be a ring automorphism (we always assume that \( \sigma(1) = 1 \)). The skew polynomial ring \( R[z; \sigma] \) is defined as the free module over \( R \) with basis \( \{ z^i \mid i \in \mathbb{N} \} \), being \( \mathbb{N} = \{0, 1, 2, \ldots\} \), and with product given by the rule \( z^i z^j = z^{i+j} \), for all \( i, j \in \mathbb{N} \), and the rule

\[
z a = \sigma(a)z,
\]

for all \( a \in R \). Then \( R[z; \sigma] \) is a ring with identity \( 1 = z^0 \). It is commutative if, and only if, \( \sigma = \text{Id} \) (the identity automorphism), and it is an integral domain if, and only if, so is \( R \). Furthermore, if \( R \) is a field, then \( R[z; \sigma] \) is both a left and right Euclidean domain.

To define skew polynomial rings, we will mostly consider the field automorphism

\[
\sigma : \mathbb{F}_{q_0}^n \rightarrow \mathbb{F}_{q_0}^n,
\]

for all \( \sigma \). We will also consider \( \sigma \) restricted to some subfields of \( \mathbb{F}_{q_0}^n \), and we will also extend \( \sigma \) to polynomial rings over such subfields. Regardless of its domain, we will always use the letter \( \sigma \), as its definition can be easily inferred from the context.

### III. CYCLIC-SKew-CYCLIC CODES

#### A. The Definition

Recall that \( \mathbb{F} = \mathbb{F}_{q_0}^n \), as defined in \[1\], and that we consider vectors in \( \mathbb{F}^n \) to be subdivided as follows:

\[
c = (c^{(0)}, c^{(1)}, \ldots, c^{(\ell-1)}) \in \mathbb{F}^n, \quad \text{where}
\]

\[
c^{(i)} = (c^{(i)}_0, c^{(i)}_1, \ldots, c^{(i)}_{N-1}) \in \mathbb{F}^N,
\]

for \( i = 0, 1, \ldots, \ell - 1 \). With this notation, we may define the following operators.

**Definition 3 (Shifting Operators).** The cyclic inter-block shifting operator \( \varphi : \mathbb{F}^n \rightarrow \mathbb{F}^n \) is defined as

\[
\varphi(c^{(0)}, c^{(1)}, \ldots, c^{(\ell-1)}) = (c^{(\ell-1)}, c^{(0)}, \ldots, c^{(\ell-2)}).
\]

The skew-cyclic intra-block shifting operator \( \phi : \mathbb{F}^n \rightarrow \mathbb{F}^n \) is defined as

\[
\phi(c^{(0)}, c^{(1)}, \ldots, c^{(\ell-1)}) = \left( \psi(c^{(0)}), \psi(c^{(1)}), \ldots, \psi(c^{(\ell-1)}) \right),
\]

where \( \psi : \mathbb{F}^N \rightarrow \mathbb{F}^N \) is the classical skew-cyclic shifting operator, given by

\[
\psi(c_0, c_1, \ldots, c_{N-1}) = (\sigma(c_{N-1}), \sigma(c_0), \ldots, \sigma(c_{N-2})).
\]

The operators in Definition 3 can be trivially extended to any field \( L \) and endomorphism \( \sigma : L \rightarrow L \). These operators depend on the length partition \( n = \ell N \), the field \( L \) and the field endomorphism \( \sigma \). However, we will not write this dependency for simplicity.

The classical cyclic shifting operator \[24\] Sec. 7.2 \[25\] Ch. 4 is recovered from \( \varphi \) by setting \( N = 1 \), whereas the classical skew-cyclic shifting operator \[12\] Page 6 \[26\] Def. 1 is recovered from \( \phi \) by setting \( \ell = 1 \). Moreover, \( \phi \) and \( \varphi \) become the identity maps if \( m = N = 1 \) and \( \ell = 1 \), respectively.

We may now define cyclic-skew-cyclic codes.

**Definition 4 (Cyclic-Skew-Cyclic Codes).** We say that a code \( C \subseteq \mathbb{F}^n \) is a cyclic-skew-cyclic code, or a CSC code for short, if it is a linear code (linear over \( \mathbb{F} \)), and

\[
\varphi(C) \subseteq C \quad \text{and} \quad \phi(C) \subseteq C.
\]

Analogously for any field \( L \), instead of \( \mathbb{F} \), and any field endomorphism \( \sigma : L \rightarrow L \).

By the observations above, classical cyclic codes \[24\] Sec. 7.2 \[25\] Ch. 4 and classical skew-cyclic codes \[12\] Page 6 \[26\] Def. 1 are recovered from CSC codes by setting \( m = N = 1 \) and \( \ell = 1 \), respectively.

#### B. Algebraic Characterizations

We now give an algebraic description of CSC codes, which in turn recovers those of classical cyclic codes and skew-cyclic codes. First, we define

\[
R' = \frac{S'[x]}{(x^\ell - 1)}, \quad \text{where} \quad S' = \frac{\mathbb{F}[z; \sigma]}{(z^N - 1)}.
\]

The use of the prime symbol is due to the fact that we first consider \( R' \), since it is simpler to describe and corresponds to the length partition in \[3\]. However, we will use an alternative ring throughout the manuscript. We will assume that \( S' \) is a ring, which holds if, e.g., \( m \) divides \( N \). We are considering the polynomial ring \( S'[x] \) as usual, where the variable \( x \) commutes with all elements. In particular, we have that

\[
zx = xz.
\]

Ideals in \( S'[x] \) are assumed to be left ideals. Hence \((x^\ell - 1)\) denotes the left ideal of \( S'[x] \) generated by \( x^\ell - 1 \in S'[x] \), even though this ideal in particular is two-sided (since \( xz = zx \) and \( xa = ax \) for all \( a \in \mathbb{F} \)).
We may identify \( \mathbb{F}^n \) with \( \mathcal{R}' \), as vector spaces over \( \mathbb{F} \), via the vector space isomorphism \( \mu : \mathbb{F}^n \rightarrow \mathcal{R}' \) given by

\[
\mu \left( c^{(0)}, c^{(1)}, \ldots, c^{(\ell-1)} \right) = \left( \sum_{i=0}^{\ell-1} \left( \sum_{j=0}^{N-1} c^{(i)}_j z^j \right) x^i \right),
\]

where \( c^{(i)} = (c^{(i)}_0, c^{(i)}_1, \ldots, c^{(i)}_{N-1}) \in \mathbb{F}^N \), for \( i = 0, 1, \ldots, \ell - 1 \). We will often denote

\[
c(x, z) = \mu(c).
\]

It can be shown that \( \mathcal{R}' \) is naturally isomorphic to

\[
\mathcal{R} = \mathcal{S}[z; \sigma] \quad \text{where} \quad \mathcal{S} = \frac{\mathbb{F}[x]}{(x^\ell - 1)},
\]

where \( \sigma : \mathbb{F} \rightarrow \mathbb{F} \) is extended uniquely to \( \sigma : \mathcal{S} \rightarrow \mathcal{S} \) by setting \( \sigma(x) = x \).

This is the first time that we need to extend \( \sigma \) to a larger ring. First note that \( \sigma \) can be uniquely extended to \( \sigma : \mathbb{F}[x] \rightarrow \mathbb{F}[x] \), satisfying that \( \sigma(x) = x \). Second, \( \sigma : \mathbb{F}[x] \rightarrow \mathbb{F}[x] \) can be uniquely extended to

\[
\sigma : \frac{\mathbb{F}[x]}{(m(x))} \rightarrow \frac{\mathbb{F}[x]}{(m(x))},
\]

satisfying that \( \sigma(x) = x \), where \( m(x) \in \mathbb{F}[x] \), if and only if,

\[
\sigma(m(x)) \in (m(x)).
\]

This is clearly the case for \( m(x) = x^\ell - 1 \) since \( \sigma(x^\ell - 1) = x^\ell - 1 \). We will see less trivial cases later on.

Observe that \( \mathcal{S} \) is a commutative ring, but it is not an integral domain in general, thus \( \mathcal{R} \) is not an integral domain in general. Note that, in this case, the fact that \( xz = xz \) comes from rule (6) in the definition of skew polynomial rings and \( \sigma(x) = x \).

We may also identify \( \mathbb{F}^n \) with \( \mathcal{R} \) as vector spaces over \( \mathbb{F} \). In this case, we may use \( \nu : \mathbb{F}^n \rightarrow \mathcal{R} \) given by

\[
\nu \left( c^{(0)}, c^{(1)}, \ldots, c^{(\ell-1)} \right) = \left( \sum_{i=0}^{\ell-1} \left( \sum_{j=0}^{N-1} c^{(i)}_j x^i \right) z^j \right),
\]

where \( c^{(i)} = (c^{(i)}_0, c^{(i)}_1, \ldots, c^{(i)}_{N-1}) \in \mathbb{F}^N \), for \( i = 0, 1, \ldots, \ell - 1 \).

We have the following algebraic characterization of CSC codes.

**Theorem 1.** Let \( \mathcal{C} \subseteq \mathbb{F}^n \) be a code, which we do not assume to be linear. The following are equivalent:

1. \( \mathcal{C} \subseteq \mathbb{F}^n \) is a CSC code.
2. \( \mu(\mathcal{C}) \subseteq \mathcal{R}' \) is a left ideal of \( \mathcal{R}' \).
3. \( \nu(\mathcal{C}) \subseteq \mathcal{R} \) is a left ideal of \( \mathcal{R} \).

**Proof.** For all \( c \in \mathbb{F}^n \), it holds that

\[
x \mu(c) = \mu(\varphi(c)) \quad \text{and} \quad z \mu(c) = \mu(\phi(c)).
\]

Hence \( \mathcal{C} \) is \( \mathbb{F} \)-linear, \( \varphi(\mathcal{C}) \subseteq \mathcal{C} \) and \( \phi(\mathcal{C}) \subseteq \mathcal{C} \) if, and only if, \( \mu(\mathcal{C}) \) is \( \mathbb{F} \)-linear, \( x \mu(\mathcal{C}) \subseteq \mu(\mathcal{C}) \) and \( z \mu(\mathcal{C}) \subseteq \mu(\mathcal{C}) \). The latter three conditions are equivalent to \( \mu(\mathcal{C}) \) being a left ideal of \( \mathcal{R}' \). This proves the equivalence of items 1 and 2. The equivalence with item 3 is analogous.

Observe that Theorem 1 recovers the classical algebraic characterizations of cyclic codes and skew-cyclic codes by setting \( m = N = 1 \) and \( \ell = 1 \), respectively. In general, Theorem 1 states that a CSC code is simultaneously a cyclic code over the non-commutative finite ring \( \mathcal{S}' \), and a skew-cyclic code over the commutative finite ring \( \mathcal{S} \).

Unless otherwise stated, we will identify \( C, \mu(C) \) and \( \nu(C) \), and we will denote by \( C \) both the CSC code in \( \mathbb{F}^n \) and the left ideal of \( \mathcal{R}' \) or \( \mathcal{R} \), indistinctly.

**IV. GENERATORS OF CYCLIC-SKEW-CYCLIC CODES**

**A. Finding a Single Generator as a Left Ideal**

Cyclic codes over an arbitrary (commutative or not) finite ring are not always principal ideals [28], i.e., they do not always have a single generator. In fact, being principal is equivalent to being splitting [28, Th. 3.2]. A similar result holds for cyclic convolutional codes [44, Th. 3.5] [58, Sec. 4], which make use of skew polynomials but are not CSC codes.

In this subsection, we will show that \( \mathcal{R} \) (thus \( \mathcal{R}' \)) is a principal left ideal ring (PLIR), assuming that \( p \) does not divide \( \ell \) (i.e., \( \ell^{-1} \) has simple roots in \( \mathbb{F}_q \)). To that end, we will find a minimal generator skew polynomial of a CSC code that will enable us to obtain a generator matrix (Subsection IV.B) and obtain the defining set (Subsection IV.B) of the CSC code.

From now on, let the polynomials \( m_1(x), m_2(x), \ldots, m_t(x) \in \mathbb{F}[x] \) form the unique irreducible decomposition

\[
x^\ell - 1 = m_1(x)m_2(x) \cdots m_t(x)
\]

of \( x^\ell - 1 \) in the polynomial ring \( \mathbb{F}[x] \).

Recall from Subsection I.A that we are assuming that \( \mathbb{F}_q = \mathbb{F}_q^x \) contains all roots of \( x^\ell - 1 \). Recall from Subsection II.C that \( \sigma : \mathbb{F} \rightarrow \mathbb{F} \) is given by \( \sigma(a) = a^q \), for \( a \in \mathbb{F} \). Combining these two facts, we deduce that \( \sigma(a) = a \), for every root \( a \) of \( x^\ell - 1 \). Since the roots of \( m_i(x) \) are a subset of the roots of \( x^\ell - 1 \), its coefficients lie in \( \mathbb{F}_q \) too, or in other words,

\[
\sigma(m_i(x)) = m_i(x),
\]

for all \( i = 1, 2, \ldots, t \). Hence, as in (12), we may extend \( \sigma \) to

\[
\sigma : \frac{\mathbb{F}[x]}{(m_i(x))} \rightarrow \frac{\mathbb{F}[x]}{(m_i(x))};
\]

satisfying \( \sigma(x) = x \), for \( i = 1, 2, \ldots, t \).

Assume from now on that \( x^\ell - 1 \) has simple roots (i.e., \( p \) does not divide \( \ell \)), then \( m_i(x) \neq m_j(x) \) if \( i \neq j \). By the Bézout identities in the polynomial ring \( (\mathbb{F} \cap \mathbb{F}_q^x)[x] \) over the finite field \( \mathbb{F} \cap \mathbb{F}_q^x \), there exist \( a_i(x), b_i(x) \in (\mathbb{F} \cap \mathbb{F}_q^x)[x] \) such that

\[
a_i(x) \left( \frac{x^\ell - 1}{m_i(x)} \right) + b_i(x)m_i(x) = 1,
\]

for \( i = 1, 2, \ldots, t \). Define the \( i \)-th primitive idempotent [25, Sec. 4.3] as

\[
e_i(x) = a_i(x)m_1(x) \cdots m_{i-1}(x)m_{i+1}(x) \cdots m_t(x) = a_i(x) \left( \frac{x^\ell - 1}{m_i(x)} \right) \in (\mathbb{F} \cap \mathbb{F}_q^x)[x],
\]
for $i = 1, 2, \ldots, t$. Note that $e_i(x)$ is not idempotent in $\mathbb{F}[x]$, but its image in $\mathbb{F}[x]/(x^t - 1)$ is (see [17] below). Note also that, since $e_i(x) \in [\mathbb{F} \cap \mathbb{F}_{q_0}][x]$, then 

$$\sigma(e_i(x)) = e_i(x),$$

for $i = 1, 2, \ldots, t$, and in fact, it lies in the center of $\mathcal{R}$ (or $\mathcal{R}'$) as it is constant in $z$.

We may now state and prove the following result.

**Lemma 5.** Define the rings

$$\mathcal{R}_i = \mathcal{S}_i[z; \sigma] = \mathbb{F}[x] / (m_i(x)),$$

for $i = 1, 2, \ldots, t$. The natural maps

$$\rho : \mathcal{R} \longrightarrow \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_t,$$

and analogously for $\tau$, are well-defined ring isomorphisms. In addition, it holds that

$$\rho(e_i(x)) = \tau(e_i(x)) = 0, \ldots, 1, \ldots, 0 \in \mathbb{F}^t,$$

where $e_i \in \mathbb{F}^t$ has all of its components equal to 0 except its $i$th component, which is equal to 1, for $i = 1, 2, \ldots, t$.

Finally, given $f(x, z) \in \mathcal{R}_i$, for $i = 1, 2, \ldots, t$, the unique $f(x, z) \in \mathcal{R}$, such that $\rho(f(x, z)) = (f_1(x, z), f_2(x, z), \ldots, f_t(x, z))$, is given by

$$f(x, z) = \sum_{i=1}^t e_i(x) \bar{f}_i(x, z) = \sum_{i=1}^t \bar{f}_i(x, z)e_i(x),$$

where $\bar{f}_i(x, z) \in \mathcal{R}$ is such that its projection onto $\mathcal{R}_i$ is $f_i(x, z)$. Analogously for $\tau$.

**Proof.** By the Chinese Remainder Theorem [24] Sec. 10.9, Th. 5], the natural map

$$\left( \begin{array}{c} \mathbb{F}[x] \\ (x^t - 1) \end{array} \right) \longrightarrow \left( \begin{array}{c} \mathbb{F}[x] \\ (m_1(x)) \end{array} \right) \times \cdots \times \left( \begin{array}{c} \mathbb{F}[x] \\ (m_t(x)) \end{array} \right)$$

is a ring isomorphism. The map $\tau$ is just extending such a ring isomorphism to the corresponding skew polynomial rings. This can be done since $\sigma(x) = x$, for $\sigma$ defined over all domains $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_t$.

Next, there is a natural surjective ring morphism

$$(\mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_t) \longrightarrow \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_t.$$ 

By composing it with $\tau$, we obtain the ring morphism

$$\rho_0 : \left( \frac{\mathbb{F}[x]}{(x^t - 1)} \right) [z; \sigma] \longrightarrow \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_t.$$

Finally, we can extend $\rho_0$ to $\rho$ by noting that $\rho_0(z^N - 1) = 0$. This shows that $\rho$ is well-defined and a ring morphism.

The fact that $\rho$ and $\tau$ are bijective, i.e., ring isomorphisms, can be shown coefficient-wise by using that $[18]$ is a ring isomorphism. Similarly, the other statements of the theorem can be proven coefficient-wise using the Chinese Remainder Theorem.

The following codes will be useful thanks to Lemma 5.

**Definition 6.** Let $\pi_i : \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_t \longrightarrow \mathcal{R}_i$ denote the projection map onto $\mathcal{R}_i$, for $i = 1, 2, \ldots, t$. Given a CSC code $\mathcal{C} \subseteq \mathcal{R}$, we define its $i$th skew-cyclic component as

$$\mathcal{C}(i) = \pi_i(\rho(\mathcal{C})) \subseteq \mathcal{R}_i = \mathcal{S}_i[z; \sigma] / (z^N - 1),$$

which is a skew-cyclic code of length $N$ over the field $\mathcal{S}_i$ (recall that $m_i(x)$ is irreducible) with field automorphism $\sigma : \mathcal{S}_i \longrightarrow \mathcal{S}_i$, for $i = 1, 2, \ldots, t$.

We may now state and prove the main result of this subsection.

**Theorem 2.** If $\mathcal{C} \subseteq \mathcal{R}$ is a CSC code, then it holds that

$$\rho(\mathcal{C}) = \mathcal{C}(1) \times \mathcal{C}(2) \times \cdots \times \mathcal{C}(t).$$

In addition, there exist unique $g(x, z), h(x, z) \in \mathcal{S}[z; \sigma]$ satisfying the following three properties:

1) Their projections onto $\mathcal{S}_i[z; \sigma]$ are monic in $z$, for $i = 1, 2, \ldots, t$.

2) It holds that

$$\mathcal{C} = (g(x, z))$$

in the ring $\mathcal{R} = \mathcal{S}[z; \sigma] / (z^N - 1)$.

3) It holds that

$$g(x, z)h(x, z) = h(x, z)g(x, z) = z^N - 1$$

in the ring $\mathcal{S}[z; \sigma]$.

In particular, $\mathcal{C}$ is a principal left ideal and $\mathcal{R}$ and $\mathcal{R}'$ are PIRs.

Furthermore, the images of $g(x, z)$ and $h(x, z)$ in $\mathcal{S}_i[z; \sigma]$, denoted by $g_i(x, z), h_i(x, z) \in \mathcal{S}_i[z; \sigma]$, are the minimal generator and check skew polynomials, respectively, of the $i$th skew-cyclic component $\mathcal{C}(i) \subseteq \mathcal{R}_i$, for $i = 1, 2, \ldots, t$. In particular, it holds that

$$g(x, z) = \sum_{i=1}^t e_i(x) \bar{g}_i(x, z) = \sum_{i=1}^t \bar{g}_i(x, z)e_i(x),$$

$$h(x, z) = \sum_{i=1}^t e_i(x) \bar{h}_i(x, z) = \sum_{i=1}^t \bar{h}_i(x, z)e_i(x),$$

in the ring $\mathcal{S}[z; \sigma]$, for any $\bar{g}_i(x, z), \bar{h}_i(x, z) \in \mathcal{S}[z; \sigma]$ such that their projections onto $\mathcal{S}_i[z; \sigma]$ are $g_i(x, z)$ and $h_i(x, z)$, respectively.

**Proof.** It follows directly from Lemma 5 (mainly [17]) and the corresponding results for skew-cyclic codes (for instance, [26], Lemma 1 or [45] Th. 1, 2)).

Theorem 2 motivates the following definition.
Definition 7. Let $\mathcal{C} \subseteq \mathcal{R}$ be a CSC code, and let $g(x, z), h(x, z) \in S[z; \sigma]$ be as in Theorem 2. We say that $g(x, z)$ is the minimal generator skew polynomial of $\mathcal{C}$, and $h(x, z)$ is the minimal check skew polynomial of $\mathcal{C}$.

As expected, Definition 7 above recovers the classical definition of minimal generator and check skew polynomials when $t = 1$ [45, Th. 1]. However, in the classical cyclic case $m = N = 1$, it holds that

$$
g(x, z) = \sum_{i \in I} e_i(x) + \sum_{j \in [t]\setminus I} e_j(x)(z - 1),$$

$$
h(x, z) = \sum_{i \in I} e_j(x) + \sum_{i \in I} e_i(x)(z - 1),$$

for a set $I \subseteq [t]$. Hence the image of $g(x, z)$ in $\mathcal{R}$ coincides with the unique idempotent generator of $\mathcal{C}$ [25, Th. 4.3.2], which is $\sum_{i \in I} e_i(x) \in F[x]/(x^\ell - 1)$ [25, Th. 4.3.8 (vi)].

B. Finding a Generator Matrix and the Dimension

We next obtain a basis of a CSC code as a vector space from its minimal generator skew polynomial. The assumptions and notation will be as in Subsection 4A.

Theorem 3. Let $\mathcal{C} \subseteq \mathcal{R}$ be a CSC code, and let $g(x, z) \in S[z; \sigma]$ be its minimal generator skew polynomial. Given $c \in F^n$ and $c(x, z) = \nu(c) \in \mathcal{R}$, it holds that $c(x, z) \in \mathcal{C}$ if, and only if, there exist coefficients

$$
\lambda_{u,v}^{(i)} \in F,
$$

for $u = 0, 1, \ldots, d_i - 1$, for $v = 0, 1, \ldots, k_i - 1$, for $i = 1, 2, \ldots, t$, such that

$$
c(x, z) = \sum_{i=1}^{t} \left( \sum_{v=0}^{d_i-1} \sum_{u=0}^{k_i-1} \lambda_{u,v}^{(i)} x^{u} z^{v} \right) e_i(x)g(x, z)
$$

inside the ring $\mathcal{R}$, where

$$
k_i = N - \deg_z(g_i(x, z)) \quad \text{and} \quad d_i = \deg_x(m_i(x)),
$$

for $i = 1, 2, \ldots, t$, denoting by $\deg_x(\cdot)$ the degree in $x$, and similarly for other variables. Furthermore, the coefficients in (21) are unique among coefficients in $F$ satisfying (22).

In particular, a basis of $\mathcal{C}$ over $F$ is formed by the skew polynomials

$$
x^u z^v e_i(x)g(x, z) \in \mathcal{R},
$$

for $u = 0, 1, \ldots, d_i - 1$, for $v = 0, 1, \ldots, k_i - 1$, for $i = 1, 2, \ldots, t$.

Proof. By Theorem 2 and the corresponding result for skew-cyclic codes (see, e.g., [45, Th. 1]), it holds that

$$
g_i(x, z), zg_i(x, z), \ldots, z^{k_i-1}g_i(x, z) \in \mathcal{R}_i
$$

form a basis of the skew-cyclic code $\mathcal{C}^{(i)} \subseteq \mathcal{R}_i = S_i[z; \sigma]/(z^N - 1)$ over the field $S_i$, for $i = 1, 2, \ldots, t$. Furthermore, we obtain a basis of $\mathcal{S}_i$ over $F$ formed by

$$
1, x, x^2, \ldots, x^{d_i-1} \in \mathcal{S}_i,
$$

for $i = 1, 2, \ldots, t$. Thus we deduce the result by using Theorem 2 and

$$
\rho(e_i(x)g(x, z)) = e_i g_i(x, z),
$$

for $i = 1, 2, \ldots, t$.

Remark 8. It may happen that $k_i = 0$ for some $i = 1, 2, \ldots, t$. This means that $g_i(x, z) = z^N - 1$ and $\mathcal{C}^{(i)} = \{0\}$. This also means that the $i$th term in the sum (22) does not exist.

The first important consequence of the previous theorem is obtaining a generator matrix of a CSC code over its defining field, $F$.

Corollary 9. Let $\mathcal{C} \subseteq \mathcal{R}$ be a CSC code, and let $g(x, z) \in S[z; \sigma]$ be its minimal generator skew polynomial. Taking images in $\mathcal{R}$, let

$$
e_i(x)g(x, z) = \left( \sum_{j=0}^{N-1} \sum_{h=0}^{\ell-1} g_i^{(h)}(x^{k_i}) z^j \right) \in \mathcal{R},
$$

and define

$$
g_i = (g_i^{(0)}, g_i^{(1)}, \ldots, g_i^{(\ell-1)}) \in F^n, \quad \text{where}
$$

$$
g_i^{(h)} = (g_i^{(h)}(0), g_i^{(h)}(1), \ldots, g_i^{(h)}(k_i)),
$$

for $h = 0, 1, \ldots, \ell - 1$ and $i = 1, 2, \ldots, t$. Then the vectors $\varphi^u(\varphi^v(g_i)) \in F^n$, for $u = 0, 1, \ldots, d_i - 1$, for $v = 0, 1, \ldots, k_i - 1$, for $i = 1, 2, \ldots, t$, form a basis of $\mathcal{C} \subseteq \mathcal{R}$ over $F$, and thus they form the rows of a generator matrix of $\mathcal{C}$ over $F$.

The second important consequence is finding the dimension of a CSC code from its minimal generator skew polynomial.

Corollary 10. Let $\mathcal{C} \subseteq \mathcal{R}$ be a CSC code. Let $g(x, z) \in S[z; \sigma]$ be its minimal generator skew polynomial, and let $g_i(x, z) \in S_i[z; \sigma]$ be its projection onto $S_i[z; \sigma]$, for $i = 1, 2, \ldots, t$. It holds that

$$
\dim_F(\mathcal{C}) = \sum_{i=1}^{t} \dim_F(\mathcal{C}^{(i)})
$$

$$
= \sum_{i=1}^{t} \deg_x(m_i(x)) (N - \deg_z(g_i(x, z)))
$$

$$
= n - \sum_{i=1}^{t} \deg_x(m_i(x)) \deg_z(g_i(x, z)).
$$

(Recall that $0 \leq \deg_z(g_i(x, z)) \leq N$ and $\deg_z(g_i(x, z)) = N$ if, and only if, $g_i(x, z) = z^N - 1$, for $i = 1, 2, \ldots, t$.)

Remark 11. Note that $\deg_z(g_i(x, z))$ is needed in Corollaries 9 and 10. It is left to the reader to verify that

$$\deg_z(e_i(x)g(x, z)) = \deg_z(g_i(x, z)),
$$

considering $e_i(x)g(x, z) \in S[z; \sigma]$ and $g_i(x, z) \in S_i[z; \sigma]$, for $i = 1, 2, \ldots, t$.

We conclude by showing briefly which generator matrix one obtains from (22) in the cases $m = N = 1$ (classical cyclic codes) and $t = 1$ (skew-cyclic codes).
First, if \( \ell = 1 \), then \( t = 1 \), \( d_1 = 1 \), \( e_1(x) = 1 \), \( S \cong F \) naturally, and we may consider \( g(x, z) = g(z) \in F[z; \sigma] \). In this case, the basis rows in (24) are

\[
g, \phi(g), \phi^2(g), \ldots, \phi^{k-1}(g) \in F^N,
\]
where \( g \in F^N \) is formed by the first \( N \) coefficients of \( g(z) \) in \( z \), and \( k = N - \deg_z(g(z)) \). Hence we recover the generator matrix of a skew-cyclic code as well as the well-known formula for its dimension (see, e.g., items 3 and 4 in [45, Th. 1]).

Finally, if \( m = N = 1 \), then as shown in (20), we have that

\[
g(x, z) = \sum_{i \in \ell} e_i(x) + \sum_{j \in [t] \backslash \ell} e_j(x)(z - 1),
\]
for a set \( I \subseteq [t] \). Then the rows of the generator matrix of \( C \), given in (24), are

\[
e_{i_1}, \varphi(e_{i_1}), \varphi^2(e_{i_1}), \ldots, \varphi^{d_i-1}(e_{i_1}) \in F^t, \quad \text{for } \ i \in \ell,
\]
where \( e_{i_1} \in F^t \) is formed by the coefficients of \( e_i(x) \in F[x]/(x^t - 1) \), for \( i = 1, 2, \ldots, t \). This generator matrix corresponds to obtaining a classical cyclic code as a direct sum of the minimal cyclic codes generated by the primitive idempotents \( e_i(x) \), for \( i \in \ell \), and then appending the generator matrices of these minimal cyclic codes obtained from their corresponding idempotent as in [25, Th. 4.3.6].

V. SETS OF ROOTS OF CYCLIC-SKew-CYCLIC CODES

In this section, we provide a basic theory of defining sets of CSC codes that we will need later. Throughout this section, we will assume that \( N = m \), since this is the case of interest for defining sum-Rank BCH codes (Section VII).

A. Defining Appropriate Evaluation Maps

In this subsection, we define the evaluation maps that we will consider in order to define roots of skew polynomials in \( R \cong R' \). They will enable us to consider defining sets.

We start by revisiting the main definition of arithmetic evaluation of skew polynomials from [46, 47], which reads as follows.

**Definition 12 ([46, 47])**. Given a skew polynomial \( f(z) \in F_q^{m}[z; \sigma] \) and a field element \( \alpha \in F_q^{m} \), we define the evaluation of \( f(z) \) at \( \alpha \) as the unique element \( f(\alpha) \in F_q^{m} \) such that there exists a skew polynomial \( g(z) \in F_q^{m}[z; \sigma] \) satisfying that

\[
f(z) = g(z)(z - \alpha) + f(\alpha).
\]

Definition 12 is consistent by the right Euclidean division property of the skew polynomial ring \( F_q^{m}[z; \sigma] \) [41].

We now turn to the important class of linearized polynomials [40, Sec. 3.4].

**Definition 13**. Let \( K \) be a finite field of characteristic \( p \). We say that a conventional polynomial \( G(y) \in K[y] \) is a linearized polynomial with coefficients in \( K \) and automorphism \( \sigma : K \rightarrow K \), given by \( \sigma(a) = a^q \), for \( a \in K \), if it has the form

\[
G(y) = G_0y + G_1y^q + G_2y^{q^2} + \cdots + G_dy^{q^d},
\]
where \( G_0, G_1, \ldots, G_d \in K \). We denote the set of such linearized polynomials by \( L_qK[y] \). Finally, given a skew polynomial of the form

\[
f(z) = f_0 + f_1z + f_2z^2 + \cdots + f_dz^d \in K[z; \sigma],
\]
where \( f_0, f_1, \ldots, f_d \in K \), we define its associated linearized polynomial as

\[
f^{\sigma}(y) = f_0y + f_1y^q + f_2y^{q^2} + \cdots + f_dy^{q^d} \in L_qK[y].
\]

The set \( L_qK[y] \) forms a ring with usual addition and with composition of maps as multiplication. As the reader may check, this ring is isomorphic as a ring to the skew polynomial ring \( K[z; \sigma] \) by mapping a skew polynomial to its associated linearized polynomial.

The following result connects the arithmetic evaluation of skew polynomials from Definition 12 with the conventional evaluation of linearized polynomials. This result can be easily deduced from [47, Lemma 2.4] and is also a particular case of [19, Lemma 24].

**Lemma 14**. Given a skew polynomial \( f(z) \in F_q^{m}[z; \sigma] \), it holds that

\[
f(1^\beta) = f^{\sigma}(\beta)^{\sigma^{-1}},
\]
for all \( \beta \in F_q^{m} \setminus \{0\} \), where \( f^{\sigma}(\beta) \) is the conventional evaluation of the associated linearized polynomial \( f^\sigma(y) \in L_qF_q^{m}[y] \) in \( \beta \), and we denote by

\[
1^\beta = \sigma(\beta)^{\sigma^{-1}}
\]
the conjugate of 1 with respect to \( \beta \) (see Subsection VII-A).

We may now define the evaluation maps on the roots of unity that we will use to provide defining sets of CSC codes.

**Definition 15**. Given \( a \in F_q \) and \( \beta \in F_q^{m} \setminus \{0\} \) such that \( a^\ell = 1 \), we define the evaluation maps

\[
\text{Ev}_a : F_q^{m}[x]/(x^\ell - 1) \rightarrow F_q^{m}[x]/(x - a) \cong F_q^{m} \quad \text{and}
\]

\[
\text{Ev}_a^\beta : F_q^{m}[z; \sigma]/(z^m - 1) \rightarrow F_q^{m}[z; \sigma]/(z - \sigma(\beta)^{\sigma^{-1}}) \cong F_q^{m}
\]
as the natural projection maps onto the corresponding quotient left modules.

Observe that the evaluation maps in Definition 15 are well-defined since

\[
(x^\ell - 1) \subseteq (x - a) \quad \text{and} \quad (z^m - 1) \subseteq (z - \sigma(\beta)^{\sigma^{-1}}),
\]
where the first inclusion follows from \( a^\ell = 1 \), and the second inclusion follows from \( \sigma^m(\beta) = \beta^{\sigma^m} = \beta \) and Lemma 14.

As expected, it holds that

\[
\text{Ev}_a(f(x)) = f(a) \quad \text{and} \quad \text{Ev}_a^\beta(g(z)) = g(1^\beta),
\]
for \( f(x) \in F_q^{m}[x]/(x^\ell - 1) \) and \( g(z) \in F_q^{m}[z; \sigma]/(z^m - 1) \).

We will also find it useful to consider the following partial evaluation map and evaluation skew-cyclic codes.
Definition 16. Given \( a \in \mathbb{F}_q \) such that \( a^\ell = 1 \), we define the partial evaluation map

\[
\text{Ev}_{a,z} : \frac{\mathbb{F}_q[x]}{(x^m - 1)} [z; \sigma] \rightarrow \frac{\mathbb{F}_q[x]}{(x^m - 1)} [z; \sigma] \cong \frac{\mathbb{F}_q[z; \sigma]}{(z^m - 1)}
\]

as the ring morphism given by

\[
\text{Ev}_{a,z} (f_0(x) + f_1(x)z + \cdots + f_{m-1}(x)z^{m-1}) = f_0(a) + f_1(a)z + \cdots + f_{m-1}(a)z^{m-1},
\]

where \( f_0(x), f_1(x), \ldots, f_{m-1}(x) \in \mathbb{F}_q[x]/(x^\ell - 1) \). Finally, we will denote

\[
f(a, z) = \text{Ev}_{a,z}(f(x, z)) = \frac{\mathbb{F}_q[z; \sigma]}{(z^m - 1)},
\]

for \( f(x, z) \in \frac{\mathbb{F}_q[x]}{(x^m - 1)} [z; \sigma] \).

We will use the same definition and notation for

\[
\text{Ev}_{a,z} : \frac{\mathbb{F}_q[x]}{(x^m - 1)} [z; \sigma] \rightarrow \frac{\mathbb{F}_q[x]}{(x^m - 1)} [z; \sigma] \cong \frac{\mathbb{F}_q[z; \sigma]}{(z^m - 1)}.
\]

It is left to the reader to prove that \( \text{Ev}_{a,z} \) is a ring morphism. To this end, the reader might find it useful to use that \( \sigma(a) = a^\ell = a \) (a \( \in \mathbb{F}_q \) by assumption) to show that

\[
\text{Ev}_{a,z}(f(x, z)g(x, z)) = f(a, z)g(a, z),
\]

for \( f(x, z), g(x, z) \in \frac{\mathbb{F}_q[z; \sigma]}{(z^m - 1)} \).

Assume that \( p \) does not divide \( \ell \) (i.e., the roots of \( x^\ell - 1 \in \mathbb{F}_p[x] \) are simple). Observe that, if \( a \in \mathbb{F}_q \) is such that \( a^\ell = 1 \), then

\[
f(a, z) = \frac{\mathbb{F}_q[z; \sigma]}{(z^m - 1)}, \quad \text{for } f(x, z) \in \frac{\mathbb{F}_q[x]}{(x^m - 1)} [z; \sigma].
\]

Recall that we are denoting \( d_i = \text{deg}_x(m_i(x)) \) (Theorem 13 and \( F = \mathbb{F}_q^m \)), where \( m_i(x) \in \mathbb{F}_q[x] \) is the irreducible component of \( x^\ell - 1 \) in \( \mathbb{F}_q[x] \) such that \( m_i(a) = 0 \) (see Subsection IV.A). We may now give the following useful definition.

Definition 17. Assume that \( p \) does not divide \( \ell \). Given a CSC code \( C \subseteq \mathbb{R} \) and an element \( a \in \mathbb{F}_q \) such that \( a^\ell = 1 \), we define its evaluation skew-cyclic code on \( a \) as

\[
C(a) = \{ c(a, z) \mid c(x, z) \in C \} \subseteq \frac{\mathbb{F}_q^{m_i}[z; \sigma]}{(z^m - 1)},
\]

which is a skew-cyclic code of length \( m \) over the field \( \mathbb{F}_q^{m_i} \) with field automorphism \( \sigma : \mathbb{F}_q^{m_i} \rightarrow \mathbb{F}_q^{m_i} \), where \( m_i(x) \in \mathbb{F}_q[x] \) is the irreducible component of \( x^\ell - 1 \) in \( \mathbb{F}_q[x] \) such that \( m_i(a) = 0 \), and where \( d_i = \text{deg}_x(m_i(x)) \).

We have the following identification between \( C(a) \) and \( C^{(i)} \) (Definition 5), whose connection is the fact that \( m_i(a) = 0 \). This also proves that \( C(a) \) is a skew-cyclic code.

Proposition 18. Assume that \( p \) does not divide \( \ell \). Let \( a \in \mathbb{F}_q \) be such that \( a^\ell = 1 \), and let \( m_i(x) \in \mathbb{F}_q[x] \) be the irreducible component of \( x^\ell - 1 \) in \( \mathbb{F}_q[x] \) such that \( m_i(a) = 0 \). Consider the partial evaluation maps defined with the following domains and codomains:

\[
\text{Ev}_{a,z} : \mathbb{S}_{i} [z; \sigma] = \frac{\mathbb{F}_q[x]}{(m_i(x))} [z; \sigma] \rightarrow \mathbb{F}_q^{m_i} [z; \sigma], \quad (25)
\]

\[
\text{Ev}_{a,z} : \mathbb{R}_{i} = \frac{\mathbb{F}_q[x]}{(m_i(x))} [z; \sigma] \rightarrow \mathbb{F}_q^{m_i} [z; \sigma], \quad (26)
\]

where \( d_i = \text{deg}_x(m_i(x)) \). Then the maps (25) and (26) are ring isomorphisms, which moreover satisfy that

\[
\text{Ev}_{a,z} (\sigma^{(i)}) = C(a),
\]

for any CSC code \( C \subseteq \mathbb{R} \) (Definitions 6 and 17). In particular, it holds that

\[
\text{dim}_F \sigma^{(i)} = \text{dim}_F (C(a)).
\]

Furthermore, if \( g(x, z) \in \mathbb{S}_{i} [z; \sigma] \) is the minimal generator skew polynomial of \( C \), and \( g_i(x, z) \) is its image in \( \mathbb{S}_{i} [z; \sigma] \) (the minimal generator skew polynomial of \( C^{(i)} \) by Theorem 2), then we have that

\[
g(a, z) = g_i(a, z) \in \mathbb{F}_q^{m_i} [z; \sigma],
\]

which moreover is the minimal generator skew polynomial of \( C(a) \).

Proof. The fact that the maps \( \text{Ev}_{a,z} \) are well-defined and ring morphisms can be proven in the same way as for the map in Definition 16. The fact that they are ring isomorphisms can be seen from the fact that the natural evaluation map

\[
\text{Ev}_a : \frac{\mathbb{F}_q[x]}{(m_i(x))} \rightarrow \mathbb{F}_q^{m_i}
\]

is a field isomorphism. The fact that \( g(a, z) = g_i(a, z) \) follows from (19) and

\[
e_j (a) = \delta_{i,j},
\]

where \( \delta_{i,j} \) is the Kronecker delta, for \( j = 1, 2, \ldots, t \), which in turn follows from (15) and (16). The rest of the statements follow directly from \( g(a, z) = g_i(a, z) \), the definitions and the fact that the maps (25) and (26) are ring isomorphisms.

What we will want from Proposition 18 is the following important consequence. It follows directly from Lemma 5 and Proposition 13.

Corollary 19. Assume that \( p \) does not divide \( \ell \). Let \( x^\ell - 1 = m_1(x)m_2(x) \cdots m_t(x) \) be the irreducible decomposition of \( x^\ell - 1 \) in \( \mathbb{F}_q[x] \), and choose \( a_1, a_2, \ldots, a_t \in \mathbb{F}_q \) such that \( m_i(a_k) = 0 \), for \( i = 1, 2, \ldots, t \). Denote \( a = (a_1, a_2, \ldots, a_t) \in \mathbb{F}_q^t \); then the map

\[
\mathbb{E}_{a,z} : \frac{\mathbb{F}_q[x]}{(x^m - 1)} [z; \sigma] \rightarrow \frac{\mathbb{F}_q[m_1[z; \sigma]}{(z^m - 1)} \times \cdots \times \frac{\mathbb{F}_q[m_t[z; \sigma]}{(z^m - 1)}
\]

|| \[ \mathbb{R} \].
given by \( \text{Ev}_{a,z}(f(x,z)) = (f(a_1, z), f(a_2, z), \ldots, f(a_t, z)) \), for \( f(x,z) \in \mathcal{R} \), is a ring isomorphism. In particular, for any CSC code \( \mathcal{C} \subseteq \mathcal{R} \), it holds that

\[
\text{Ev}_{a,z} \mathcal{C} = C(a_1) \times C(a_2) \times \cdots \times C(a_t),
\]

and, for a given \( c(x,z) \in \mathcal{R} \), the following are equivalent:

1. \( c(x, z) \in \mathcal{C} \).
2. \( c(a, z) \in C(a) \), for all \( a \in \mathbb{F}_q \) such that \( a^t = 1 \).
3. \( c(a_i, z) \in C(a_i) \), for all \( i = 1, 2, \ldots, t \).

We conclude this subsection by defining total evaluation maps.

**Definition 20.** Given \( a \in \mathbb{F}_q \) and \( \beta \in \mathbb{F}_q^m \setminus \{0\} \) such that \( a^t = 1 \), we define the total evaluation map

\[
\text{Ev}_{a,\beta} : \left( \frac{\mathbb{F}_q[z]}{(z - \sigma(\beta))^{a^t - 1}} \right) \rightarrow \mathbb{F}_q^m
\]

as the composition map

\[
\text{Ev}_{a,\beta} = \text{Ev}_\beta \circ \text{Ev}_{a,z}.
\]

By Lemma 14, the total evaluation map is also given by

\[
\text{Ev}_{a,\beta}(f(x_0) + f_1(x_1) z + \cdots + f_{m-1}(x) z^{m-1}) = f_0(a) + f_1(a) \sigma(\beta) z^{-1} + \cdots + f_{m-1}(a) \sigma^{m-1}(\beta) z^{-1} = \left(f_0(a) \beta + f_1(a) \sigma(\beta) + \cdots + f_{m-1}(a) \sigma^{m-1}(\beta)\right) z^{-1},
\]

where \( f_0(x_0), f_1(x_1), \ldots, f_{m-1}(x) \in \mathbb{F}_q^m[x]/(x^t - 1) \). We will sometimes use the notation

\[
f(a^t \beta) = \text{Ev}_{a,\beta}(f(x,z)), \quad \text{for } f(x,z) \in \left( \frac{\mathbb{F}_q[z]}{(z - \sigma(\beta))^{a^t - 1}} \right).
\]

**B. The Defining Set**

In this subsection, we show that the zeros of the minimal skew polynomial generator define a CSC code, as in the particular cases of classical cyclic codes [25 Th. 4.4.2] and skew-cyclic codes [45 Th. 2]. We will use such defining sets to obtain the sum-rank BCH bound (Theorem 7) and a lower bound on the dimensions of sum-rank BCH codes (Theorem 9).

We start by the following definition.

**Definition 21 (Defining Set).** Given a CSC code \( \mathcal{C} \subseteq \mathcal{R} \) with minimal generator skew polynomial \( g(x, z) \in S[z;\sigma] \), we define the **defining set** of \( \mathcal{C} \) as

\[
\mathcal{T}_\mathcal{C} = \{(a, \beta) \in \mathbb{F}_q \times (\mathbb{F}_q^m \setminus \{0\}) | a^t = 1, \text{Ev}_{a,\beta}(g(x,z)) = 0 \}.
\]

As the name suggests, the defining set of a CSC code actually defines the CSC code. This is gathered in Theorem 4 below. We will need the following lemma, which follows from the product rule [47 Th. 2.7], but can easily be proven from scratch.

**Lemma 22.** Given \( a \in \mathbb{F}_q \) and \( \beta \in \mathbb{F}_q^m \setminus \{0\} \) such that \( a^t = 1 \), it holds that

\[
\text{Ev}_{a,\beta}(f(x,z)g(x,z)) = 0 \quad \text{if } \text{Ev}_{a,\beta}(g(x,z)) = 0,
\]

for \( f(x,z), g(x,z) \in \left( \frac{\mathbb{F}_q^m[z]}{(z^m - 1)} \right)[z;\sigma] \).

**Theorem 4.** Assume that \( p \) does not divide \( \ell \). Given a CSC code \( \mathcal{C} \subseteq \mathcal{R} \), the following hold:

1. Given \( c(x, z) \in \mathcal{R} \), it holds that \( c(x, z) \in \mathcal{C} \) if, and only if, \( c(a, z^\ell) = 0 \), for all \( (a, \beta) \in \mathcal{T}_\mathcal{C} \).
2. Given another CSC code \( \tilde{\mathcal{C}} \subseteq \mathcal{R} \), it holds that \( \mathcal{C} = \tilde{\mathcal{C}} \) if, and only if, \( \mathcal{T}_\mathcal{C} = \mathcal{T}_{\tilde{\mathcal{C}}} \).

**Proof.** Item 2 follows immediately from item 1, hence we only prove item 1.

First, if \( c(x, z) \in \mathcal{C} \), then \( c(a, z^\ell) = 0 \) by Lemma 22 since \( g(x,z) \) divides \( c(x,z) \) on the right and \( g(a, z^\ell) = 0 \), for all \( (a, \beta) \in \mathcal{T}_\mathcal{C} \).

Conversely, assume that \( c(a, z^\ell) = 0 \), for all \( (a, \beta) \in \mathcal{T}_\mathcal{C} \). Fix an element \( a \in \mathbb{F}_q \) such that \( a^t = 1 \). By the assumption on \( c(x, z) \) and the definition of \( \mathcal{T}_\mathcal{C} \), it holds that \( c(a, z^\ell) = 0 \), for all \( a \in \mathbb{F}_q \setminus \{0\} \) such that \( g(a, z^\ell) = \text{Ev}_\beta^\ell(g(a, z)) = 0 \).

By the corresponding result for skew-cyclic codes [45 Th. 2], we have that \( c(a, z) \in \mathcal{C}(a) \), since \( g(a, z) \) is the minimal generator skew polynomial of \( \mathcal{C}(a) \) (Proposition 18). Thus, we have shown that \( c(a, z) \in \mathcal{C}(a) \), for all \( a \in \mathbb{F}_q \) such that \( a^t = 1 \). By Corollary 19 we conclude that \( c(x, z) \in \mathcal{C} \), and we are done.

We conclude by computing the dimension of a CSC code from its defining set.

**Theorem 5.** Assume that \( p \) does not divide \( \ell \). Let \( \mathcal{C} \subseteq \mathcal{R} \) be a CSC code. For \( a \in \mathbb{F}_q \) such that \( a^t = 1 \), define

\[
\mathcal{T}_\mathcal{C}(a) = \{(a, \beta) \in \mathbb{F}_q \times (\mathbb{F}_q^m \setminus \{0\}) | a^t = 1, \text{Ev}_\beta^\ell(g(a, z)) = 0 \}
\]

which satisfies that \( \mathcal{T}_\mathcal{C}(a) \cup \{0\} \subseteq \mathbb{F}_q^m \) is a vector space over \( \mathbb{F}_q \). Let \( x^t - 1 = m_1(x) m_2(x) \cdots m_t(x) \) be the irreducible decomposition of \( x^t - 1 \) in \( \mathbb{F}_q^m \), and choose \( a_1, a_2, \ldots, a_t \in \mathbb{F}_q \) such that \( m_i(a_i) = 0 \), for \( i = 1, 2, \ldots, t \). It holds that

\[
\text{dim}_\mathcal{C} = \sum_{i=1}^t \deg_x(m_i) \left( m_i - \text{deg}_x(\mathcal{T}_\mathcal{C}(a_i) \cup \{0\}) \right)
\]

\[
= n - \sum_{i=1}^t \deg_x(m_i) \text{dim}_\mathcal{C}(\mathcal{T}_\mathcal{C}(a_i) \cup \{0\}).
\]

**Proof.** The fact that \( \mathcal{T}_\mathcal{C}(a) \cup \{0\} \subseteq \mathbb{F}_q^m \) is a vector space over \( \mathbb{F}_q \), for \( a \in \mathbb{F}_q \) such that \( a^t = 1 \), follows directly from Lemma 4.

Fix \( a \in \mathbb{F}_q \) such that \( a^t = 1 \). Let \( g(x,z) \in S[z;\sigma] \) be the minimal generator skew polynomial of \( \mathcal{C} \). By Proposition 18 \( g(a,z) \in \mathbb{F}_q^m \) is the minimal generator skew polynomial of \( \mathcal{C}(a) \).

Consider the extended code over \( \mathbb{F}_q^m \)

\[
\mathcal{C}(a) \otimes \mathbb{F}_q^m = \{ \lambda c(a, z) | \lambda \in \mathbb{F}_q^m, c(a, z) \in \mathcal{C}(a) \} \subset \left( \frac{\mathbb{F}_q^m[z;\sigma]}{(z^m - 1)} \right).
\]
which is also a skew-cyclic code. Using item 5 in [45, Th. 2], we deduce that $g(a, z) \in \mathbb{F}_q[z; \sigma]$ is also the minimal generator skew polynomial of $\mathcal{C}(a) \otimes \mathbb{F}_q^n$, since it divides $z^m - 1$ on the right. Furthermore, using Lemma 13 we can see that $T_C(a) \cup \{0\}$ is the set of roots of the linearized polynomial associated to $g(a, z) \in \mathbb{F}_q^n[z; \sigma]$ (Definition 13). Thus, applying item 2 in [45, Th. 2] on the skew-cyclic code $\mathcal{C}(a) \otimes \mathbb{F}_q^n$, we deduce that its dimension over $\mathbb{F}_q^n$ is given by
\[
\dim_{\mathbb{F}_q^n}(\mathcal{C}(a) \otimes \mathbb{F}_q^n) = m - \dim_{\mathbb{F}_q}(T_C(a) \cup \{0\}). \tag{29}
\]

The reader can also verify the identities
\[
\dim_{\mathbb{F}_q^n}(\mathcal{C}(a) \otimes \mathbb{F}_q^n) = \dim_{\mathbb{F}_q}[\mathcal{C}(a)], \quad \dim_{\mathbb{F}_q}[\mathcal{C}(a)] = \deg(m_i(x)) \dim_{\mathbb{F}_q}[\mathcal{C}(a)]. \tag{30}
\]

Combining Corollary 19 with (29) and (30) for $a_1, a_2, \ldots, a_\ell$, we conclude that
\[
\dim_{\mathbb{F}_q^n}(\mathcal{C}) = \sum_{i=1}^\ell \dim_{\mathbb{F}_q^n}(\mathcal{C}(a_i))
= \sum_{i=1}^\ell \deg(m_i(x)) \left(m - \dim_{\mathbb{F}_q}(T_C(a_i) \cup \{0\})\right),
\]
and we are done. \qed

To conclude, we discuss the cases $\ell = 1$ and $m = 1$.

First, in the skew-cyclic case $\ell = 1$, Definition 21 coincides with [45, Def. 2] after removing the redundant unique root of unity $a = 1$. Note that [45, Def. 2] uses the associated linearized polynomial (Definition 13) of the minimal generator skew polynomial. Finally, Theorems 4 and 5 recover items 3 and 2 in [45, Th. 2], respectively.

We now turn to the classical cyclic case $m = 1$. With notation as in (20), the reader may verify (using that $e_j(a) = \delta_{i,j}$ if $m_i(a) = 0$) that
\[
T_C = \{ \{a, \beta\} \in \mathbb{F}_q \times (\mathbb{F}_q \setminus \{0\}) \mid m_i(a) = 0, \text{ for some } i \notin I\}.
\]
In other words, after removing the second component $\beta \in \mathbb{F}_q \setminus \{0\}$, $T_C$ is exactly the set of roots of unity that are roots of the idempotent generator of $\mathcal{C}$. Such set is precisely the set of roots of the minimal generator polynomial of $\mathcal{C}$. Thus Definition 21 coincides with the standard definition of defining set of a classical cyclic code [24, Sec. 7.5] [25, Sec. 4.4]. Furthermore, Theorems 4 and 5 recover items (iii) and (iv) in [25, Th. 4.4.2].

VI. CICLIC-SKEW-CYCLIC LINEARIZED RS CODES

In this subsection, we revisit linearized Reed-Solomon codes [19, 20] and provide one of their subfamilies formed by CSC codes. They will be crucial for proving the sum-rank BCH bound (Theorem 7) and defining sum-rank BCH codes (Definition 30).

Recall that $\mathbb{F} = \mathbb{F}_{q^n}$, by definition (1), and that $q = q_0^s$ for some positive integer $s$. In this section, we will only work with the finite-field extension $\mathbb{F}_q \subseteq \mathbb{F}_{q^n}$.

A. Revisiting Linearized RS Codes

As before, we also consider the length partition $n = \ell N$ (Subsection 4.1.B). However, we need to assume that $1 \leq \ell \leq q - 1$ and $1 \leq N \leq m$. Therefore $n \leq (q - 1)m$.

Recall that we consider $\sigma : \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$ given by $\sigma(a) = a^q$, for all $a \in \mathbb{F}_{q^n}$. We need to define linear operators as in [19, Def. 20].

Definition 23 (Linear operators [19]). Fix $a \in \mathbb{F}_{q^n}$, and define its $i$th norm as $N_i(a) = \sigma_{i-1}(a) \cdots \sigma(a)a$, for $i \in \mathbb{N}$. Define the $\mathbb{F}_q$-linear operator $D_a^i : \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$ by
\[
D_a^i(b) = \sigma^i(b)N_i(a),
\]
for $b \in \mathbb{F}_{q^n}$ and $i \in \mathbb{N}$.

We say that $a, b \in \mathbb{F}_{q^n}$ are conjugate (with respect to $\mathbb{F}_{q^n}[z; \sigma]$) if there exists $c \in \mathbb{F}_{q^n} \setminus \{0\}$ such that $b = \sigma(c)c^{-1}a$ (see [46, 47, Eq. (25)]). Let $A = \{a_0, a_1, \ldots, a_{\ell - 1}\} \subseteq \mathbb{F}_{q^n} \setminus \{0\}$ be a set of $\ell$ pair-wise non-conjugate elements. Let $B_i = \{\beta_{i,0}, \beta_{i,1}, \ldots, \beta_{i,N-1}\} \subseteq \mathbb{F}_{q^n}$ be a set of $N$ elements of $\mathbb{F}_{q^n}$ that are linearly independent over $\mathbb{F}_q$, for $i = 0, 1, \ldots, \ell - 1$. Denote $B = (B_0, B_1, \ldots, B_{\ell - 1})$ and define the matrix
\[
D(A, B) = (D_0|D_1| \ldots |D_{\ell - 1}) \in \mathbb{F}_{q}^{k \times n}, \tag{31}
\]
where
\[
D_i = \begin{pmatrix}
\beta_{i,0} & \beta_{i,1} & \cdots & \beta_{i,N-1} \\
D_{a_0}(\beta_{i,0}) & D_{a_1}(\beta_{i,1}) & \cdots & D_{a_{\ell - 1}}(\beta_{i,N-1}) \\
D_{a_0}^2(\beta_{i,0}) & D_{a_1}^2(\beta_{i,1}) & \cdots & D_{a_{\ell - 1}}^2(\beta_{i,N-1}) \\
\vdots & \vdots & \ddots & \vdots \\
D_{a_0}^{k-1}(\beta_{i,0}) & D_{a_1}^{k-1}(\beta_{i,1}) & \cdots & D_{a_{\ell - 1}}^{k-1}(\beta_{i,N-1})
\end{pmatrix},
\]
for $i = 0, 1, \ldots, \ell - 1$, for $k = 1, 2, \ldots, n$. Then the following definition is a particular case of [19, Def. 31].

Definition 24 (Linearized Reed-Solomon codes [19]). For $k = 1, 2, \ldots, n$, we define the linearized Reed-Solomon code of dimension $k$, set of pair-wise non-conjugate elements $A$ and linearly independent sets $B$, as the linear code $C^n_k(A, B) \subseteq \mathbb{F}_{q^n}$ with generator matrix $D(A, B) \in \mathbb{F}_{q}^{k \times n}$ as in (31).

The following result is [19, Th. 4] and states that linearized Reed-Solomon codes attain equality in (4), expressing size in terms of dimension.

Proposition 25 ([19]). For $k = 1, 2, \ldots, n$, the linearized Reed-Solomon code $C^n_k(A, B) \subseteq \mathbb{F}_{q^n}$ in Definition 24 is a $k$-dimensional $\mathbb{F}_{q^n}$-linear MSR code for the finite-field extension $\mathbb{F}_q \subseteq \mathbb{F}_{q^n}$ and length partition $n = \ell N$ as in (2). That is, it satisfies
\[
d_{SR}(C^n_k(A, B)) = n - k + 1.
\]

As observed in [19, Sec. 3] and [2] Subsec. IV-A1], linearized Reed-Solomon codes recover (generalized) Reed-Solomon codes (21) when $m = N = 1$, and they recover Gabidulin codes (9, 12) when $\ell = 1$. These are the cases when the sum-rank metric particularizes to the Hamming metric and the rank metric, respectively.
B. Non-Conjugate Roots of Unity

The next lemma is a general result on \( \ell \)th roots of unity and conjugacy with respect to \( \mathbb{F}_{q^m}[z; \sigma] \), but we have not been able to find it in the literature. It will enable us to obtain linearized Reed-Solomon codes that are also CSC codes (Subsection VI-C), which in turn will allow us to obtain Sum-Rank BCH codes (Subsection VII-A).

Lemma 26. Recall that we are throughout the manuscript that \( \mathbb{F}_q \) contains all \( \ell \)th roots of unity. Assume also that there are exactly \( \ell \) distinct \( \ell \)th roots of unity, that is, \( p \) does not divide \( \ell \). Then the following conditions are equivalent:

1) The roots of \( x^\ell - 1 \in \mathbb{F}_p[x] \) are pair-wise non-conjugate with respect to \( \mathbb{F}_{q^m}[z; \sigma] \) (see Subsection VI-A).

2) \( \ell \) and \( m \) are coprime.

Proof. By assumption, we have \( \ell \) distinct roots of \( x^\ell - 1 \), all inside \( \mathbb{F}_q \). Let \( a \in \mathbb{F}_q \) be a primitive \( \ell \)th root of unity, meaning that all the \( \ell \)th roots of unity are given by \( a^0, a^1, \ldots, a^{\ell-1} \). Such primitive roots always exist [25 Page 105] [40 Sec. 2.4] (see also the next subsection). By Hilbert’s Theorem 90 [40 Ex. 2.33], there exist two distinct \( \ell \)th roots of unity \( a^i \) and \( a^j \) that are conjugate if, and only if,

\[
a^{im} = (a^j)^{\frac{m-1}{q-1}} = (a^j)^{\frac{m-1}{q-1}} = a^{jm},
\]

where \( 0 \leq i < j \leq \ell - 1 \). Now, since \( a \) is a primitive \( \ell \)th root of unity, such a condition may happen if, and only if, \( \ell \) and \( m \) share a common factor.

Remark 27. Note that, as a particular case of Lemma 26, the \( q - 1 \) elements in \( \mathbb{F}_q \setminus \{0\} \) are pair-wise non-conjugate with respect to \( \mathbb{F}_{q^m}[z; \sigma] \) if, and only if, \( q - 1 \) and \( m \) are coprime (note that the assumptions in Lemma 26 above are trivially satisfied for \( \ell = q - 1 \)).

Remark 28. Applying the definition of conjugacy directly (see Subsection VI-A), without making use of Hilbert’s Theorem 90, the reader may arrive at the fact that item 1 in Lemma 26 is also equivalent to \( \ell \) and \( \frac{m-1}{q-1} \) being coprime. Indeed, this latter condition is equivalent to \( \ell \) and \( m \) being coprime, under the assumption that \( \ell \) divides \( q - 1 \) (i.e., \( \mathbb{F}_q \) contains all \( \ell \)th roots of unity), as in Lemma 26 above. To see this, note that

\[
\frac{q^m - 1}{q - 1} - m = \sum_{i=1}^{m-1} (q^i - 1)
\]

is divisible by \( \ell \) if \( q - 1 \) is divisible by \( \ell \). In such a case, a factor of \( \ell \) divides \( \frac{q^m - 1}{q - 1} \) if, and only if, it divides \( m \).

C. Finding CSC Linearized RS Codes

Combining Definition 24 and Lemma 26, we will describe a subfamily of linearized Reed-Solomon codes formed by CSC codes, which in addition recovers classical cyclic (generalized) Reed-Solomon codes and skew-cyclic Gabidulin codes when setting \( m = N = 1 \) and \( \ell = 1 \), respectively.

In this subsection, we assume that: (1) \( N = m \); (2) \( \ell \) and \( m \) are coprime; (3) \( \ell \) and \( q \) are coprime; and (4) \( x^\ell - 1 \) has all of its \( \ell \) distinct roots in \( \mathbb{F}_q \) (i.e., \( \ell \) divides \( q - 1 \)). In the classical cyclic case \( m = 1 \), condition 2 is trivially satisfied, whereas in the skew-cyclic case \( \ell = 1 \), conditions 2, 3 and 4 are all trivially satisfied. Note that condition 2 is satisfied whether \( \ell = 1 \) or \( m = 1 \).

Let \( a \in \mathbb{F}_q \) be a primitive \( \ell \)th root of unity, meaning that the set of roots of \( x^\ell - 1 \) is given by

\[
A = \{a^0, a^1, a^2, \ldots, a^{\ell-1}\} \subseteq \mathbb{F}_q \setminus \{0\}.
\]

Such a primitive root always exists [25 Page 105] [40 Sec. 2.4]. Since we are assuming that \( \ell \) is coprime with both \( q \) and \( m \), the set \( A \) is formed by \( \ell \) distinct and pair-wise non-conjugate elements by Lemma 26.

Next, let \( \beta \in \mathbb{F}_{q^m} \) be a normal element of the extension \( \mathbb{F}_q \subseteq \mathbb{F}_{q^m} \). In other words,

\[
\{\beta, \sigma(\beta), \sigma^2(\beta), \ldots, \sigma^{m-1}(\beta)\} \subseteq \mathbb{F}_{q^m}
\]

forms a basis of \( \mathbb{F}_{q^m} \) over \( \mathbb{F}_q \). Recall that normal elements exist for any extension of finite fields [40 Th. 3.73]. We fix an integer \( b \geq 0 \). For \( i = 0, 1, \ldots, \ell - 1 \), note that the element \( \beta a^i \in \mathbb{F}_{q^m} \) is also normal. Denote the corresponding basis by

\[
B_i = \{\beta a^i, \sigma(\beta)a^i, \ldots, \sigma^{m-1}(\beta)a^i\} \subseteq \mathbb{F}_{q^m}
\]

(recall that \( a^i \in \mathbb{F}_q \), thus \( \sigma(a^i) = a^{\sigma(i)} \)). We define \( B = (B_0, B_1, \ldots, B_{\ell-1}) \).

For \( k = 1, 2, \ldots, n \), the corresponding linearized Reed-Solomon code \( C_k^\ell(A, B) \subseteq \mathbb{F}_{q^m}^n \) (Definition 24) is given by the generator matrix \( D(A, B) = (D_0|D_1|\ldots|D_{\ell-1}) \in \mathbb{F}_{q^m}^{nk \times n} \), where \( n = \ell m \) and \( D_i \in \mathbb{F}_{q^m}^{nk \times m} \) is given by

\[
\begin{pmatrix}
\beta a^i \\
\sigma(\beta)a^{i+b_1} \\
\sigma^2(\beta)a^{i+b_2} \\
\vdots \\
\sigma^{k-1}(\beta)a^{i+b_{k-1}}
\end{pmatrix}
\begin{pmatrix}
\beta a^{i+b_1} \\
\sigma(\beta)a^{i+b_1} \\
\sigma^2(\beta)a^{i+b_1} \\
\vdots \\
\sigma^{k-1}(\beta)a^{i+b_1}
\end{pmatrix}
\begin{pmatrix}
\sigma(\beta)a^{i+b_2} \\
\sigma(\beta)a^{i+b_2} \\
\sigma^2(\beta)a^{i+b_2} \\
\vdots \\
\sigma^{k-1}(\beta)a^{i+b_2}
\end{pmatrix}
\begin{pmatrix}
\sigma^2(\beta)a^{i+b_2} \\
\sigma^2(\beta)a^{i+b_2} \\
\sigma(\beta)a^{i+b_2} \\
\vdots \\
\sigma^{k-1}(\beta)a^{i+b_2}
\end{pmatrix}
\vdots
\begin{pmatrix}
\sigma^{k-1}(\beta)a^{i+b_{k-1}} \\
\sigma^{k-1}(\beta)a^{i+b_{k-1}} \\
\sigma^{k-1}(\beta)a^{i+b_{k-1}} \\
\vdots \\
\sigma^{k-1}(\beta)a^{i+b_{k-1}}
\end{pmatrix}
\end{pmatrix}
\]

for all \( i = 0, 1, \ldots, \ell - 1 \).

We conclude with the main result of this section, whose proof is left to the reader.

Theorem 6. For \( k = 1, 2, \ldots, n \), the linearized Reed-Solomon code \( C_k^\ell(A, B) \subseteq \mathbb{F}_{q^m}^n \) as above is a CSC code in \( \mathbb{F}_{q^m}^n \) with field automorphism \( \sigma: \mathbb{F}_{q^m} \longrightarrow \mathbb{F}_{q^m} \) (Definition 7).

VII. Sum-Rank BCH Codes

Throughout this section, we will make the same assumptions as in the beginning of Subsection VI-C that is: (1) \( N = m \); (2) \( \ell \) and \( m \) are coprime; (3) \( \ell \) and \( q \) are coprime; and (4) \( \ell \) divides \( q - 1 \), i.e., \( x^\ell - 1 \) has all of its \( \ell \) distinct roots in \( \mathbb{F}_q \) (\( q = q_m \)).
A. The Sum-Rank BCH Bound Leading to SR-BCH Codes

In this subsection, we provide a lower bound on the minimum sum-rank distance of CSC codes based on their defining set (Definition 21), which will allow us to define Sum-Rank BCH codes (Definition 30).

In order to prove our bound, we need the following lemma, which is [5, Th. 7]. Recall from Subsection VI-B that we denote by \( d_{SR} \) and \( d_{SR}^0 \) the sum-rank metrics over the finite-field extensions \( \mathbb{F}_q \subseteq \mathbb{F}_{q^m} \) and \( \mathbb{F}_{q^m} \subseteq \mathbb{F}_{q^{m'}} \), respectively, for the length partition \( n = \ell m \) as in (2) for \( N = m \).

Lemma 29 ([5]). For a code \( C \subseteq \mathbb{F}_{q^m} \) (we do not assume that it is linear), it holds that
\[
d^0_{SR}(C \cap \mathbb{F}^n) \geq d_{SR}(C).
\] (33)

Theorem 7 (Sum-Rank BCH bound). Let \( a \in \mathbb{F}_q \) be a primitive \( \ell \)th root of unity and let \( \beta \in \mathbb{F}_{q^m} \) be a normal element of the extension \( \mathbb{F}_q \subseteq \mathbb{F}_{q^m} \) (see Subsection VI-C). Let \( C \subseteq \mathbb{F}_{q^m} \) be a CSC code. If the defining set \( T_C \) of \( C \) (Definition 21) contains the consecutive pairs in \( \mathbb{F}_q \times (\mathbb{F}_{q^m} \setminus \{0\}) \),
\[
(a^{b+\ell-1}, \sigma(\beta)), \ldots, (a^{b+\ell-2}, \sigma(\beta)),
\] (34)
for integers \( b \geq 0 \) and \( 2 \leq \ell \leq n \), then it holds that
\[
d^0_{SR}(C) \geq \delta.
\]

Proof. If \( C \) satisfies the hypothesis, then by Theorem 4 and equation (27), it holds that
\[
C \subseteq (C_{\delta-1}^\langle, (A, B)^\perp \rangle) \cap \mathbb{F}^n,
\] (35)
where \( C_{\delta-1}^\langle, (A, B) \rangle \) is the \((\delta - 1)\)-dimensional linearized Reed-Solomon code (Definition 24) with \( A = \{1, a, a^2, \ldots, a^{\ell-1}\} \) and \( B = (B_0, B_1, \ldots, B_{\ell-1}) \) given by (32).

By [48, Th. 5], the dual \( C_{\delta-1}^\langle, (A, B) \rangle^\perp \subseteq \mathbb{F}_{q^m} \) is also an MSR code, hence
\[
d_{SR}(C_{\delta-1}^\langle, (A, B) \rangle^\perp) = n - (n - \delta + 1) + 1 = \delta.
\] (36)

Combining (35), (36) and Lemma 29 we conclude that \( d^0_{SR}(C) \geq \delta \).

This bound recovers the well-known BCH bound [24, Sec. 7.6, Th. 8] [25, Th. 4.5.3] (originally [37], [38]) in the classical cyclic case \( m = 1 \) and its rank-metric version [39, Prop. 1] in the skew-cyclic case \( \ell = 1 \). Note that [39, Prop. 1] is given for lengths \( N \geq m \), whereas here we only consider \( N = m \).

We may now define Sum-Rank BCH codes with prescribed distance.

Definition 30 (Sum-Rank BCH codes). Let \( a \in \mathbb{F}_q \) be a primitive \( \ell \)th root of unity and let \( \beta \in \mathbb{F}_{q^m} \) be a normal element of the finite-field extension \( \mathbb{F}_q \subseteq \mathbb{F}_{q^m} \) (see Subsection VI-C). Fix integers \( b \geq 0 \) and \( 2 \leq \delta \leq n \). We define the corresponding Sum-Rank BCH code (or SR-BCH code for short) with prescribed distance \( \delta \) as
\[
C_\delta(a^b, \beta) = (C_{\delta-1}^\langle, (A, B)^\perp \rangle) \cap \mathbb{F}^n,
\]
where \( C_{\delta-1}^\langle, (A, B) \rangle \subseteq \mathbb{F}_{q^m} \) is as in Definition 24 for \( A = \{1, a, a^2, \ldots, a^{\ell-1}\} \) and \( B = (B_0, B_1, \ldots, B_{\ell-1}) \) given by \( B_i = \{\beta a^i, \sigma(\beta) a^i, \ldots, \sigma^{m-1}(\beta) a^i\} \), for \( i = 0, 1, \ldots, \ell - 1 \).

SR-BCH codes recover classical BCH codes [24, Sec. 7.6] [25, Sec. 5.1] when \( m = 1 \) (in that case, \( C_\delta(a^b, \beta) = C_\delta(a^b, 1) \) for any \( \beta \in \mathbb{F}_{q^m} \setminus \{0\} \)), and they recover rank-metric BCH codes [39, Page 272] [45, Def. 7] when \( \ell = 1 \) for the code length \( N = m \). The latter are full-length skew-cyclic Gabidulin codes [43, Sec. 5.2] if \( s = 1 \). Moreover, SR-BCH codes form a subfamily of sum-rank alternant codes [8, Def. 12], which in turn recover classical alternant codes [24, Ch. 12] when \( m = 1 \).

The motivation for defining SR-BCH codes as in Definition 30 is to obtain the largest CSC code (hence hopefully having maximum possible code size) for a prescribed distance in view of Theorem 7. This is shown in the following result.

Proposition 31. With assumptions and notation as in Definition 30 the SR-BCH code \( C_\delta(a^b, \beta) \subseteq \mathbb{F}^n \) is a CSC code. Moreover, it is the largest CSC code in \( \mathbb{F}^n \), with respect to set inclusion, whose defining set contains the pairs in (34).

Proof. First, the linearized Reed-Solomon code \( C_{\delta-1}^\langle, (A, B) \rangle \) is a CSC code by Theorem 6. The reader may check that its dual code and the restriction of such a dual code to \( \mathbb{F}^n \) are also CSC codes. Thus \( C_\delta(a^b, \beta) \) is a CSC code.

Finally, if \( C \subseteq \mathbb{F}^n \) is another CSC code whose defining set contains the pairs in (34), then it holds that \( C \subseteq C_\delta(a^b, \beta) \) by the proof of Theorem 7 and we are done.

B. The Defining Set of a SR-BCH Code

In this subsection, we will give a method for finding the defining set of a SR-BCH code directly from the pairs in (34), without explicitly computing its minimal generator skew polynomial (Theorem 8). Thanks to Theorem 8 we will give, in the following subsection, a lower bound on the dimension of SR-BCH codes that is easy to compute from the pairs (34).

We will need the notion of minimal linearized polynomial, which we consider in a slightly more general form than in [40, Sec. 3.4].

Definition 32. Given an extension \( K \subseteq L \) of finite fields of characteristic \( p \) and an arbitrary set \( B \subseteq L \), we say that \( G(y) \in \mathcal{L}_q[K[y]] \) (Definition 13) is the minimal linearized polynomial of \( B \) in \( \mathcal{L}_q[K[y]] \) if it is the linearized polynomial of minimum degree in \( \mathcal{L}_q[K[y]] \) such that it is monic and \( G(\beta) = 0 \), for all \( \beta \in B \).

The minimal linearized polynomial of a given set exists for any finite-field extension of characteristic \( p \) (to prove this, consider, e.g., \( y^e - y \in \mathcal{L}_q[K[y]] \), where \( e = \log_p(q) \log_p(|L|) \). Its uniqueness follows from the next lemma, whose proof follows the same lines as [40, Th. 3.68] and is left to the reader.

Lemma 33. With notation as in Definition 32 let \( F(y), G(y) \in \mathcal{L}_q[K[y]] \) be such that \( F(\beta) = 0 \), for all \( \beta \in B \), and \( G(y) \) is the minimal linearized polynomial of \( B \) in \( \mathcal{L}_q[K[y]] \). If \( F(y) \) and \( G(y) \) are the associated linearized polynomials of \( f(z), g(z) \in K[z; \sigma] \), respectively (see Definition 72), then \( g(z) \) divides \( f(z) \) on the right in \( K[z; \sigma] \).

We will also need the following three auxiliary lemmas.
Lemma 34. Let $x^t - 1 = m_1(x)m_2(x) \cdots m_t(x)$ be the irreducible decomposition of $x^t - 1$ in $\mathbb{F}[x]$, and choose $a_i \in \mathbb{F}_q$ such that $m_i(a_i) = 0$, for $i = 1, 2, \ldots , t$. For each $i = 1, 2, \ldots , t$, let $B_i = \{\beta_{i,1}, \beta_{i,2}, \ldots , \beta_{i,k_i}\} \subseteq \mathbb{F}_q^m \setminus \{0\}$ be a set that does not need to be linearly independent over $\mathbb{F}_q$.

Let $G_i(y)$ be the minimal linearized polynomial of $B_i$ in $\mathbb{L}_q[\mathbb{F}_q^m,y]$ (Definition 22), and assume that it is the associated linearized polynomial of $g_i(z) \in \mathbb{F}_q^m[z;\sigma]$ (Definition 13), for $i = 1, 2, \ldots , t$. Finally, let $\tilde{g}_i(x,z) \in \mathbb{S}[z;\sigma]$ be such that its projection onto $\mathbb{S}_i[z;\sigma] = g_i(x,z) = \text{Ev}_{\beta_{i,j}}(g_i(z))$, where $\text{Ev}_{\beta_{i,j}} : \mathbb{S}_i[z;\sigma] \rightarrow \mathbb{F}_q^{k_i}[z;\sigma]$ is the ring isomorphism from (35), for $i = 1, 2, \ldots , t$. Then the skew polynomial

$$g(x,z) = \sum_{i=1}^{t} e_i(x)\tilde{g}_i(x,z) \in \mathbb{S}[z;\sigma]$$

is the minimal generator skew polynomial of the largest CSC code $C \subseteq R$, with respect to set inclusion, whose defining set $T_C$ contains the pairs

$$(a_i, \beta_{i,j}) \in \mathbb{F}_q \times (\mathbb{F}_q^m \setminus \{0\}),$$

for $j = 1, 2, \ldots , k_i$, for $i = 1, 2, \ldots , t$.

Proof. Fix an index $i = 1, 2, \ldots , t$. By Lemma 33 and the paragraph prior to it, $g_i(z) \in \mathbb{F}_q^{k_i}[z;\sigma]$ exists, it is unique and furthermore, it divides $z^m - 1$ on the right in $\mathbb{F}_q^{k_i}[z;\sigma]$, since $\beta_{i,j}^{q^m} = \beta_{i,j}$, for $j = 1, 2, \ldots , k_i$. Since $g_i(z)$ divides $z^m - 1$ on the right in $\mathbb{F}_q^{k_i}[z;\sigma]$, it is the minimal generator skew polynomial of the skew-cyclic code that it generates, $C_i = (g_i(z)) \subseteq \mathbb{F}_q^{k_i}[z;\sigma]/(z^m - 1)$, by item 5 in (33) Th. 2. Therefore, $g_i(x,z) \in \mathbb{S}_i[z;\sigma]$ is the minimal generator skew polynomial of $C = \text{Ev}_{\beta_{i,j}}^{-1}(C_i) \subseteq R_i$,

since $\text{Ev}_{\beta_{i,j}}$, given as in (26), is a ring isomorphism preserving degrees in $z$.

By Theorem 2, $g(x,z) \in \mathbb{S}[z;\sigma]$, given as in (37), is the minimal generator skew polynomial of the CSC code $C = \beta^{-1}(C^{(1)} \times C^{(2)} \times \cdots \times C^{(t)}) \subseteq R$.

It is only left to prove that $C$ is the largest CSC code in $R$ whose defining set $T_C$ contains the pairs in (38). By Proposition 18 it holds that $C_i = C(a_i)$, for $i = 1, 2, \ldots , t$. Let $c(x,z) \in R$ be such that $c(a_i, \beta_{i,j}) = 0$, for $j = 1, 2, \ldots , k_i$, for $i = 1, 2, \ldots , t$. By Lemma 33, $g_i(z)$ divides $c(a_i,z)$ on the right in $R$, thus $c(a_i,z) \in C(a_i)$, for $i = 1, 2, \ldots , t$. By Corollary 13 we conclude that $c(x,z) \in C$, and we are done. \hfill $\square$

Lemma 35. Let $B = \{\beta_1, \beta_2, \ldots , \beta_k\} \subseteq \mathbb{F}_q^m$ be an arbitrary set, let $d \in \mathbb{Z}$ be a divisor of $s$, and define

$$U = \left\{ \beta_0^{q^m d} | u = 0, 1, \ldots , s/d - 1 \right\},$$

and

$$V = \langle U \rangle_{\mathbb{F}_q} \subseteq \mathbb{F}_q^m.$$
for \( h = 0, 1, \ldots, m - 1 \). Thus, equation (39) holds if, and only if, we have that

\[
0 = (g_0(a) \beta + g_1(a) \sigma(\beta) + \cdots + g_{m-1}(a) \sigma^{m-1}(\beta)) q_i^m
\]

\[
= g_0(a) \gamma_0^m \beta^{\gamma_0^m} + \cdots + g_{m-1}(a) \gamma_0^m \sigma^{m-1}(\beta) q_i^m
\]

\[
= g_0(a) \gamma_0^m \beta^{\gamma_0^m} + \cdots + g_{m-1}(a) \gamma_0^m \sigma^{m-1}(\beta) q_i^m,
\]

and the result follows.

We may finally find the defining set of a SR-BCH in terms of the pairs (34), without explicitly computing its minimal generator skew polynomial.

**Theorem 8.** Let \( a \in \mathbb{F}_q \) be a primitive \( \ell \)th root of unity and let \( \beta \in \mathbb{F}_{q^m} \) be a normal element of the extension \( \mathbb{F}_q \subseteq \mathbb{F}_{q^m} \) (see Subsection VI-C). Fix integers \( b \geq 0 \) and \( 2 \leq \delta \leq n \). Let \( x^\ell - 1 = m_1(x)m_2(x) \cdots m_k(x) \) be the irreducible decomposition of \( x^\ell - 1 \) in \( \mathbb{F}[x] \). Define

\[
\mathcal{J}_i = \{ j \in \mathbb{N} \mid 0 \leq j \leq \delta - 2, m_i(a^{b+j}) = 0 \}
\]

\[
= \{ j_1, j_2, \ldots, j_\ell \},
\]

where \( k_i = |\mathcal{J}_i| \), and choose an arbitrary \( \tilde{j}_i \in \mathcal{J}_i \), for \( i = 1, 2, \ldots, t \). There exist integers \( h_1, h_2, \ldots, h_k \in \mathbb{Z} \), satisfying that \( 0 \leq h_\lambda \leq d_i - 1 \) and

\[
b + \tilde{j}_i \equiv (b + j_\lambda) \gamma_0^m \quad (\mod \ell), \tag{40}
\]

for \( \lambda = 1, 2, \ldots, k_i \), for \( i = 1, 2, \ldots, t \). Define the \( \mathbb{F}_q \)-linear vector subspace of \( \mathbb{F}_{q^m} \):

\[
\mathcal{V}_i = \left\{ \beta^{\gamma_0^m} v \in \{ s_j \lambda + m(\mu_i + h_\lambda) (\mod \ell) \mid \right.
\]

\[
u = 0, 1, \ldots, s \}
\]

\[
\left. u_1, u_2, \ldots, u_{\ell} = 1, \lambda = 1, 2, \ldots, k_i \right\} \subseteq \mathbb{F}_{q^m},
\]

for \( i = 1, 2, \ldots, t \). The following properties hold for the corresponding SR-BCH code \( C_{\delta}(a^b, \beta) \subseteq \mathbb{F}^n \) (Definition 30):

1. **The defining set of** \( C_{\delta}(a^b, \beta) \) **satisfies that**

\[
T_{C_{\delta}(a^b, \beta)}(a^{b+j}) = \mathcal{V}_i \setminus \{0\},
\]

**with notation as in (28), for** \( i = 1, 2, \ldots, t \).

2. **The dimension of** \( C_{\delta}(a^b, \beta) \) **over** \( \mathbb{F} \) **is**

\[
\dim_{\mathbb{F}}(C_{\delta}(a^b, \beta)) = n - \sum_{i=1}^{k} \deg_x(m_i(x)) \dim_{\mathbb{F}_q}(\mathcal{V}_i).
\]

**Proof.** Denote \( C = C_{\delta}(a^b, \beta) \) for simplicity.

The existence of \( h_1, h_2, \ldots, h_k \in \mathbb{Z} \) satisfying (40) and \( 0 \leq h_\lambda \leq d_i - 1 \), for \( \lambda = 1, 2, \ldots, k_i \), for \( i = 1, 2, \ldots, t \), follows from item 1 in Lemma 36 and the fact that \( a \) is a primitive \( \ell \)th root of unity. By item 2 in Lemma 36 it holds that

\[
\beta^{\gamma_0^m} \in T_C(a^{b+j}), \beta^{\gamma_0^m + m \lambda} \in T_C(a^{b+j}) \gamma_0^m = T_C(a^{b+j}).
\]

In other words, the fact that \( T_C \) contains the pairs in (34) is equivalent to the fact that \( T_C \) contains the pairs \( (a^{b+j}, \beta^{\gamma_0^m + m \lambda}) \), for \( \lambda = 1, 2, \ldots, k_i \), for \( i = 1, 2, \ldots, t \), hence, by Lemma 33 since \( C \) is the largest CSC code containing such pairs (Proposition 31), we deduce that the linearized polynomial associated to the minimal generator skew polynomial of \( C(a^{b+j}) \) is the minimal linearized polynomial of

\[
E_i = \{ \beta^{\gamma_0^m} \mid v \in \{ s_j \lambda + m h_\lambda (\mod \text{sm}) \mid \lambda = 1, 2, \ldots, k_i \} \subseteq \mathbb{F}_{q^m}.
\]

in \( \mathbb{F}_{q^m}[\gamma_0^m] \). By Lemma 35 such a minimal linearized polynomial is

\[
G_i(y) = \prod_{\beta \in \mathcal{V}_i} (y - \beta),
\]

for \( i = 1, 2, \ldots, t \). By the definition of \( T_C(a^{b+j}) \) (see (28), Lemma 14 and Proposition 18) we conclude that item 1 holds, i.e., \( T_C(a^{b+j}) = \mathcal{V}_i \setminus \{0\} \), for \( i = 1, 2, \ldots, t \).

Finally, item 2 follows from item 1 and Theorem 5.

**Remark 37.** If, for some \( i = 1, 2, \ldots, t \), it holds that \( \mathcal{J}_i = \emptyset \), then \( \mathcal{B}_i = \emptyset \) and \( \mathcal{V}_i = \{0\} \). Hence such a term does not appear in the sum in item 2 in Theorem 8.

As usual, we conclude by discussing the cases \( m = 1 \) and \( \ell = 1 \). In the classical cyclic case \( m = 1 \), Theorem 8 simply says that the defining set of a BCH code is the union of the cyclotomic sets that have a non-empty intersection with the pairs in (34) [25, Eq. (5.1)]. In the skew-cyclic case \( \ell = 1 \), rank-metric BCH codes were also defined in terms of defining sets in (45, Def. 7). However, the description in Theorem 8 is new in this case to the best of our knowledge. Note that, setting \( s = 1 \) if \( \ell = 1 \), then \( t = 1 \) and \( \mathcal{V}_1 = \{ \beta, \sigma(\beta), \ldots, \sigma^{d-2}(\beta) \} \subseteq \mathbb{F}_q \), hence Theorem 8 is consistent with (45, Th. 6) for full-length skew-cyclic Gabidulin codes.

**C. A Lower Bound on the Dimension of a SR-BCH Code**

In this subsection, we will make use of Theorem 8 to obtain a simple lower bound on the dimension of SR-BCH codes. This bound only makes use of the first component of the pairs in (34), and can be easily computed by using the corresponding cyclotomic sets. We will show how to use it in Example 38 and we will provide tables for a wide range of parameters in Appendix A. In those tables, it can be seen how our lower bound provides codes with a higher dimension for a given minimum sum-rank distance than previously known.

**Theorem 9.** With assumptions and notation as in Theorem 8 it holds that

\[
\dim_{\mathbb{F}}(C_{\delta}(a^b, \beta)) \geq n - \sum_{i=1}^{k} d_i \min \left\{ m, \frac{s k_i}{d_i} \right\},
\]

where \( d_i = \deg_x(m_i(x)) \) as in Theorem 8 and

\[
k_i = \left\{ j \in \mathbb{N} \mid 0 \leq j \leq \delta - 2, m_i(a^{b+j}) = 0 \right\} \geq 0,
\]

for \( i = 1, 2, \ldots, t \).
We now briefly discuss the bound (41). First, in the classical cyclic case \( m = 1 \), the bound (41) is an equality and becomes the well-known formula
\[
\dim \mathcal{C}_d(a^b, 1) = n - \sum_{i=1}^{t} d_i \varepsilon_i,
\]
where \( \varepsilon_i = 1 \) if there exists an integer \( j \) such that \( 0 \leq j \leq \delta - 2 \) and \( m_i(a^{b+j}) = 0 \), and \( \varepsilon_i = 0 \) otherwise, for \( i = 1, 2, \ldots, t \). In the skew-cyclic case \( \ell = 1 \), setting \( s = 1 \) (which can always be done), we recover the dimension of full-length \( (N = m) \) skew-cyclic Gabidulin codes
\[
\dim \mathcal{C}_d(1, \beta) = n - k_1 = n - \delta + 1,
\]
since \( t = s = d_1 = 1 \) and \( k_1 = \delta - 1 < n = m \).

As in the classical cyclic case, SR-BCH codes are in general subfield subcodes of duals of linearized Reed-Solomon codes. Therefore, we may also apply Delsarte’s lower bound (22) (see also [5, Cor. 9]) on the dimension of SR-BCH codes, obtaining
\[
\dim \mathcal{C}_d(a^b, \beta) \geq n - s(\delta - 1). \tag{42}
\]
However, the bound (41) is always tighter, since \( \sum_{i=1}^{t} k_i = \delta - 1 \), thus
\[
n - \sum_{i=1}^{t} d_i \min \left\{ m, \frac{sk_i}{d_i} \right\} \geq n - \sum_{i=1}^{t} sk_i = n - s(\delta - 1). \tag{43}
\]
Observe that equality holds in (43) if, and only if, \( sk_i \leq md_i \), for \( i = 1, 2, \ldots, t \).

Finally, the bound (41) is always at least 0 and at most the Singleton bound (4) for the prescribed distance \( \delta \):
\[
0 \leq n - \sum_{i=1}^{t} d_i \min \left\{ m, \frac{sk_i}{d_i} \right\} \leq n - \delta + 1. \tag{44}
\]
The first inequality in (43) can be deduced from
\[
\sum_{i=1}^{t} d_i \min \left\{ m, \frac{sk_i}{d_i} \right\} \leq m \sum_{i=1}^{t} d_i = m\ell = n,
\]
where equality holds if, and only if, \( md_i \leq sk_i \), for \( i = 1, 2, \ldots, t \). The second inequality in (44) can be deduced from
\[
n - \sum_{i=1}^{t} d_i \min \left\{ m, \frac{sk_i}{d_i} \right\} \leq n - \sum_{i=1}^{t} k_i = n - \delta + 1,
\]
where \( k_i \leq d_i m \), for \( i = 1, 2, \ldots, t \). It is left to the reader to show that (41) may be equal to the Singleton bound (4) for the prescribed distance \( \delta \) if, and only if, \( s = 1 \), which is the case in which SR-BCH codes coincide with CSC linearized Reed-Solomon codes as in Theorem 6.

We conclude by giving an example of how to compute the bound (41). This method can easily be automated, and in Appendix A we provide some tables with values for this bound obtained by a simple implementation in C++.

**Example 38.** Let \( q_0 = 2 \), \( m = 2 \), thus \( q_0^m = 4 \), \( s = 4 \), \( \ell = 15 \), thus \( n = 30 \). We first compute the 4-cyclotomic sets modulo 15, i.e., the sets of powers of \( a \) in the roots of \( m_i(x) = \prod_{b=0}^{d_i-1} (x - a^b) \), for \( i = 1, 2, \ldots, t \). Note that we do not need to know \( t \) nor \( m_i(x) \). These cyclotomic sets are \{0\}, \{1, 4\}, \{2, 8\}, \{3, 12\}, \{5\}, \{6, 9\}, \{7, 13\}, \{10\} and \{11, 14\} (see [25] Sec. 4.1). Choose \( \delta = 5 \) and \( b = 1 \). Then the first components of the pairs in (44) are \( a^1, a^2, a^3 \) and \( a^4 \). Thus we may assume that \( k_1 = 2 \) and \( d_1 = 2 \) (corresponding to \( a^1 \) and \( a^4 \)), \( k_2 = k_3 = 1 \) and \( d_2 = d_3 = 2 \) (corresponding to \( a^2 \) and \( a^3 \), respectively), being all other \( k_i = 0 \). Thus the lower bound (41) on the dimension of \( \mathcal{C}_d(a^\beta, \beta) \) yields
\[
n - d_1 m - sk_2 - sk_3 = 18.
\]
On the other hand, the Singleton bound (4) and Delsarte’s bound (42) provide 26 and 14 as upper and lower bounds on the dimension of the corresponding SR-BCH (see Table IV). In particular, we beat the previous known lower bound (42).

**D. Decoding SR-BCH Codes with respect to the Sum-Rank Metric**

We conclude this manuscript by noting that we may decode SR-BCH codes up to half their prescribed distance by considering them as subfield subcodes of an appropriate linearized Reed-Solomon code.

We start with the following result, whose proof follows the same lines as [2, Th. 4], and can be of interest by itself.

**Proposition 39.** Let the assumptions and notation be as in Definition 20. In such a case, there exist an integer \( c \geq 0 \) and a normal element \( \gamma \in \mathbb{F}_{q^n} \) of the finite-field extension \( \mathbb{F}_q \subseteq \mathbb{F}_{q^n} \) such that
\[
\mathcal{C}_{n-1}^\gamma(A, B) = \mathcal{C}_{n-\delta+1}^\gamma(A, B'),
\]
where \( \mathcal{C}_{n-\delta+1}^\gamma(A, B') \subseteq \mathbb{F}_{q^n} \) is as in Definition 24 for \( A = \{1, a, a^2, \ldots, a^{\ell-1}\} \) and \( B' = (B'_0, B'_1, \ldots, B'_{\ell-1}) \) given by \( B'_i = \{\gamma a^i, \sigma(a^i), \sigma^2(a^i), \ldots, \sigma^{m-1}(\gamma)a^i\} \), for \( i = 0, 1, \ldots, \ell - 1 \).

**Proof.** As in the proof of [2, Th. 4], there exist \( \alpha_0(i), \alpha_1(i), \ldots, \alpha_{m-1}(i) \in \mathbb{F}_{q^n} \) that are linearly independent over \( \mathbb{F}_q \), for \( i = 0, 1, \ldots, \ell - 1 \), such that
\[
\mathcal{C}_{n-1}^\gamma(A, B) = \langle \alpha \rangle_{\mathbb{F}_{q^m}},
\]
where \( \alpha = (\alpha(0), \alpha(1), \ldots, \alpha(\ell-1)) \in \mathbb{F}_{q^n}^m \), and \( \alpha(i) = (\sigma(0), \sigma(1), \ldots, \sigma(\ell-1)) \in \mathbb{F}_{q^m}^m \), for \( i = 0, 1, \ldots, \ell - 1 \). It holds that \( \mathcal{C}_{n-1}^\gamma(A, B) \) is a CSC code, thus there exist elements \( \lambda, \mu \in \mathbb{F}_{q^m} \) such that
\[
\sigma(\alpha_j(i)) = \lambda \alpha_j(i) \quad \text{and} \quad \sigma(\alpha_{j+1}(i)) = \mu \alpha_j(i),
\]
for \( j = 0, 1, \ldots, m - 1 \) and \( i = 0, 1, \ldots, \ell - 1 \).

The reader may check that \( \mu = 1, \lambda^m = 1 \) and \( \sigma(\lambda) = \lambda \).

From \( \mu = 1 \), we deduce that there exists an integer \( c \geq 0 \) such that \( \mu = a^c \) (\( a \) is a primitive \( \ell \)th root of unity). From Hilbert’s Theorem 90 [40, Ex. 2.33], there exists \( \nu \in \mathbb{F}_{q^m} \{\{0\} \) such that \( \lambda = \sigma(\nu)/\nu \). Next, define \( \tilde{\gamma} = \nu^{-1}\alpha_0(0) \in \mathbb{F}_{q^m} \). It holds that \( \tilde{\gamma} \) is a normal element of the extension \( \mathbb{F}_q \subseteq \mathbb{F}_{q^n} \), since \( \alpha_0(0), \alpha_1(0), \ldots, \alpha_{m-1}(0) \in \mathbb{F}_{q^n} \) are linearly independent over \( \mathbb{F}_q \). We also deduce that
\[
\sum_{i=0}^{t-1} \sum_{j=0}^{m-1} \sigma^j(\tilde{\gamma}) a^{i} \lambda^j a^{i} \lambda^{i+1}(\beta) = 0,
\]
(45)
for \( l = 0, 1, \ldots, n - 2 \). Finally, let \( \gamma = \sigma^{-n+\delta}(\overline{\gamma}) \), which is also a normal element of the extension \( \mathbb{F}_q \subseteq \mathbb{F}_{q^m} \). Applying \( \sigma^{-u} \) on \([43]\), for \( u = 0, 1, \ldots, n - \delta \), we conclude, as in the proof of \([2] \) Th. 4], that \( C_{\delta-1}(A,B)^{\perp} = C_{n-\delta+1}(A,B)^{\perp} \), and we are done. \( \square \)

A general form of dual codes of linearized Reed-Solomon codes was recently given in \([49] \) Th. 3.2.10]. We have decided to include Proposition \([39]\] since it explicitly treats the particular case of CSC linearized Reed-Solomon codes. Furthermore, we have presented a direct proof that does not need the general theory in \([49] \).

We conclude that \( C_2(a,b,\beta) = C_{n-\delta+1}(A,B)^{\perp} \cap \mathbb{F}_q^n \), with assumptions and notation as in Definition \([30]\] and Proposition \([39]\]. Now, we may simply use a decoder for \( C_{n-\delta+1}(A,B)^{\perp} \subseteq \mathbb{F}_q^n \) with respect to the sum-rank metric for \( \mathbb{F}_q \subseteq \mathbb{F}_{q^m} \), to decode \( C_2(a,b,\beta) \subseteq \mathbb{F}^n \) with respect to the sum-rank metric for \( \mathbb{F}_q \subseteq \mathbb{F}_{q^m} \).

To see why this approach works, observe that, if \( y = c + e \in \mathbb{F}^n \), where \( c \in C_2(a,b,\beta) \), and \( e \in \mathbb{F}^n \) is such that \( wt_{SR}^0(e) \leq (\delta - 1)/2 \), then

\[
wt_{SR}(e) \leq wt_{SR}^0(e) \leq \frac{\delta - 1}{2}
\]

(see \([5]\) for the notation). Thus a decoder for \( C_{n-\delta+1}(A,B)^{\perp} \) that corrects up to \( \lfloor (\delta - 1)/2 \rfloor \) errors with respect to the sum-rank metric \( wt_{SR} \) yields \( c \in C_{n-\delta+1}(A,B)^{\perp} \cap \mathbb{F}_q^n \), and we are done.

This approach yields the same complexity as the corresponding decoder of \( C_{n}^\sigma(A,B) \), also over the larger field \( \mathbb{F}_{q^m} \). Examples of such decoders include \([50], \) \([22] \) Sec. V \( ] \) and \([51]\). They are all extensions of the classical Welch-Berlekamp decoder \([52]\], given in decreasing order of computational complexity.

**VIII. CONCLUSION AND OPEN PROBLEMS**

In this work, we have introduced the novel families of cyclic-skew-cyclic codes and sum-rank BCH codes. We have studied their structure, obtaining: (1) The minimal generator skew polynomial of a CSC code, with its corresponding generator matrix, (2) the defining set of a CSC code, after carefully considering different evaluation maps, (3) obtained a lower bound (sum-rank BCH bound) on the minimum sum-rank distance of certain CSC codes, and (4) the defining set of sum-rank BCH codes from the pairs in their definition. We have also seen that sum-rank BCH codes can be decoded up to half their prescribed distance by considering them as subfield subcodes of linearized Reed-Solomon codes.

Using their prescribed distance (Theorem \([7]\]) and a lower bound on their dimensions based on their defining sets (Theorem \([9]\)), we obtained in Appendix \( \boxed{A} \) tables of parameters of sum-rank BCH codes, for the finite field \( \mathbb{F}_4 \) and for \( m = 2 \), beating previously known codes.

We now list some open problems for future research:

1) We have made several assumptions on the parameters of CSC codes throughout different sections of the manuscript (see, e.g., the beginning of Section \[VII\]). It would be interesting to study CSC codes lifting one or more of these assumptions.

2) The sum-rank BCH bound (Theorem \([7]\)) may admit extensions such as the Hartmann-Tzeng bound \([25] \) Th. 4.5.6] or the van Lindt-Wilson technique \([25] \) Th. 4.5.10]. During the review of this manuscript, this problem was partially solved in \([53]\), where Hartmann-Tzeng bounds and Roos bounds were given for the sum-rank metric.

3) It would be of interest to find faster decoders of sum-rank BCH codes as in the classical cyclic case, such as the Peterson-Gorenstein-Zierler decoder or the Berlekamp-Massey decoder \([25] \) Sec. 5.4].

4) It would be of interest to find better estimates on the dimension of a sum-rank BCH code than in Theorem \([3]\) or better, a simpler formula to exactly compute such a dimension than in Theorem \([8]\).

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**APPENDIX A**

### TABLES FOR BOUND (41) ON SOME NARROW-SENSE PRIMITIVE SR-BCH CODES

In this appendix, we provide tables of values for the Singleton upper bound (4), our lower bound (41), and Delsarte’s lower bound (42). All of these are bounds on the dimension dim(C) of a SR-BCH code \(C \subseteq \mathbb{F}_q^n\). We only consider the parameters \(m = 2, q_0 = 2\), and different values of \(\ell, s\) and \(n\).

If we represent codewords as lists of matrices, as in (3), then we may see the SR-BCH codes considered in this appendix as subsets

\[
C \subseteq \left( \mathbb{F}_2^{2\times 2} \right)^\ell,
\]

but being linear over \(\mathbb{F}_2\), seen as subsets \(C \subseteq \mathbb{F}_q^n\), \(n = 2\ell\).

We consider \(\ell = q - 1 = 2^s - 1\), extending the concept of primitive BCH codes [25 Sec. 5.1], and \(b = 0, 1\), being the latter case \(b = 1\) an extension of narrow-sense BCH codes [25 Sec. 5.1]. Note also that, since \(\ell = q - 1\) is odd, then \(\ell\) and \(m = 2\ell\) are coprime, and all assumptions at the beginning of Section VII are satisfied.

Bold numbers on the column corresponding to Theorem 9 (41) mean that the bound (41) beats (42). It is important to notice that the considered Singleton bound (4) is with respect the prescribed distance \(\delta\), whereas the exact minimum sum-rank distance \(d\) of the corresponding sum-rank BCH code may be strictly higher. For this reason, it may happen that \(n - d + 1 < n - \delta + 1\) in some cases. Thus the column corresponding to (4) may not represent the tightest Singleton bound for the corresponding dimension.
TABLE I
$q_0 = 2, m = 2, q_0^m = 4, s = 1, \ell = 1, n = 2.$

| $\delta$ | $b$ | Singleton (4) | Theorem 9 (11) | Delsarte (42) |
|----------|-----|---------------|----------------|--------------|
| 2        | 0   | 1             | 1              | 1            |
| 2        | 1   | 1             | 1              | 1            |

TABLE II
$q_0 = 2, m = 2, q_0^m = 4, s = 2, \ell = 3, n = 6.$

| $\delta$ | $b$ | Singleton (4) | Theorem 9 (11) | Delsarte (42) |
|----------|-----|---------------|----------------|--------------|
| 2        | 0   | 5             | 4              | 4            |
| 2        | 1   | 5             | 4              | 4            |
| 3        | 0   | 4             | 2              | 2            |
| 3        | 1   | 4             | 2              | 2            |

TABLE III
$q_0 = 2, m = 2, q_0^m = 4, s = 3, \ell = 7, n = 14.$

| $\delta$ | $b$ | Singleton (4) | Theorem 9 (11) | Delsarte (42) |
|----------|-----|---------------|----------------|--------------|
| 2        | 0   | 13            | 12             | 11           |
| 2        | 1   | 13            | 11             | 11           |
| 3        | 0   | 12            | 9              | 8            |
| 3        | 1   | 12            | 8              | 8            |
| 4        | 0   | 11            | 6              | 5            |
| 4        | 1   | 11            | 5              | 5            |
| 5        | 0   | 10            | 3              | 2            |
| 5        | 1   | 10            | 5              | 2            |
| 6        | 0   | 9             | 3              | -1           |
| 6        | 1   | 9             | 2              | -1           |
| 7        | 0   | 8             | 0              | -4           |
| 7        | 1   | 8             | 2              | -4           |
$q_0 = 2, m = 2, q_0^{3} = 4, s = 4, \ell = 15, n = 30.$

| $\delta$ | $b$ | Singleton | Theorem 9 (41) | Delsarte (42) |
|----------|-----|-----------|----------------|---------------|
| 2        | 0   | 29        | 28             | 26            |
| 2        | 1   | 29        | 26             | 26            |
| 3        | 0   | 28        | 24             | 22            |
| 3        | 1   | 28        | 22             | 22            |
| 4        | 0   | 27        | 20             | 18            |
| 4        | 1   | 27        | 18             | 18            |
| 5        | 0   | 26        | 16             | 14            |
| 5        | 1   | 26        | 18             | 14            |
| 6        | 0   | 25        | 16             | 10            |
| 6        | 1   | 25        | 16             | 10            |
| 7        | 0   | 24        | 14             | 6             |
| 7        | 1   | 24        | 12             | 6             |
| 8        | 0   | 23        | 10             | 2             |
| 8        | 1   | 23        | 8              | 2             |
| 9        | 0   | 22        | 6              | -2            |
| 9        | 1   | 22        | 8              | -2            |
| 10       | 0   | 21        | 6              | -6            |
| 10       | 1   | 21        | 8              | -6            |
| 11       | 0   | 20        | 6              | -10           |
| 11       | 1   | 20        | 6              | -10           |
| 12       | 0   | 19        | 4              | -14           |
| 12       | 1   | 19        | 2              | -14           |
| 14       | 0   | 17        | 0              | -22           |
| 14       | 1   | 17        | 2              | -22           |
TABLE V

$q_0 = 2, m = 2, q_0^m = 4, s = 5, \ell = 31, n = 62.$

| $\delta$ | $b$ | Singleton \( (3) \) | Theorem \( (9) \) \( (11) \) | Delsarte \( (12) \) |
|---|---|---|---|---|
| 2 | 0 | 61 | 60 | 57 |
| 2 | 1 | 61 | 57 | 57 |
| 3 | 0 | 60 | 55 | 52 |
| 3 | 1 | 60 | 52 | 52 |
| 4 | 0 | 59 | 50 | 47 |
| 4 | 1 | 59 | 47 | 47 |
| 5 | 0 | 58 | 45 | 42 |
| 5 | 1 | 58 | 47 | 42 |
| 6 | 0 | 57 | 45 | 37 |
| 6 | 1 | 57 | 42 | 37 |
| 7 | 0 | 56 | 40 | 32 |
| 7 | 1 | 56 | 37 | 32 |
| 8 | 0 | 55 | 35 | 27 |
| 8 | 1 | 55 | 32 | 27 |
| 10 | 0 | 53 | 30 | 17 |
| 10 | 1 | 53 | 27 | 17 |
| 12 | 0 | 51 | 25 | 7 |
| 12 | 1 | 51 | 22 | 7 |
| 14 | 0 | 49 | 20 | -3 |
| 14 | 1 | 49 | 17 | -3 |
| 18 | 0 | 45 | 5 | -23 |
| 18 | 1 | 45 | 7 | -23 |
| 22 | 0 | 41 | 5 | -43 |
| 22 | 1 | 41 | 7 | -43 |
| 26 | 0 | 37 | 0 | -63 |
| 26 | 1 | 37 | 2 | -63 |
| 30 | 0 | 33 | 0 | -83 |
| 30 | 1 | 33 | 2 | -83 |
TABLE VI

$q_0 = 2, m = 2, q_0^m = 4, s = 6, \ell = 63, n = 126.$

| $\delta$ | $b$ | Singleton | Theorem | Delsarte |
|-------|-----|-----------|---------|----------|
| 2     | 0   | 125       | 124     | 120      |
| 2     | 1   | 125       | 120     | 120      |
| 3     | 0   | 124       | 118     | 114      |
| 3     | 1   | 124       | 114     | 114      |
| 4     | 0   | 123       | 112     | 108      |
| 4     | 1   | 123       | 108     | 108      |
| 5     | 0   | 122       | 106     | 102      |
| 5     | 1   | 122       | 108     | 102      |
| 6     | 0   | 121       | 106     | 96       |
| 6     | 1   | 121       | 102     | 96       |
| 7     | 0   | 120       | 100     | 90       |
| 7     | 1   | 120       | 96      | 90       |
| 10    | 0   | 117       | 88      | 72       |
| 10    | 1   | 117       | 84      | 72       |
| 14    | 0   | 113       | 70      | 48       |
| 14    | 1   | 113       | 66      | 48       |
| 22    | 0   | 105       | 52      | 0        |
| 22    | 1   | 105       | 52      | 0        |
| 30    | 0   | 97        | 26      | -48      |
| 30    | 1   | 97        | 28      | -48      |
| 38    | 0   | 89        | 14      | -96      |
| 38    | 1   | 89        | 16      | -96      |
| 46    | 0   | 81        | 6       | -144     |
| 46    | 1   | 81        | 8       | -144     |
| 54    | 0   | 73        | 0       | -192     |
| 54    | 1   | 73        | 2       | -192     |
| 62    | 0   | 65        | 0       | -240     |
| 62    | 1   | 65        | 2       | -240     |
$q_0 = 2$, $m = 2$, $q_0^m = 4$, $s = 7$, $\ell = 127$, $n = 254$.

| $\delta$ | $b$ | Singleton [3] | Theorem [9] [11] | Delarte [42] |
|-----------|-----|---------------|------------------|--------------|
| 2         | 0   | 253           | 252              | 247          |
| 2         | 1   | 253           | 247              | 247          |
| 3         | 0   | 252           | 245              | 240          |
| 3         | 1   | 252           | 240              | 240          |
| 4         | 0   | 251           | 238              | 233          |
| 4         | 1   | 251           | 233              | 233          |
| 5         | 0   | 250           | 231              | 226          |
| 5         | 1   | 250           | 233              | 226          |
| 6         | 0   | 249           | 231              | 219          |
| 6         | 1   | 249           | 226              | 219          |
| 7         | 0   | 248           | 224              | 212          |
| 7         | 1   | 248           | 219              | 212          |
| 10        | 0   | 245           | 210              | 191          |
| 10        | 1   | 245           | 205              | 191          |
| 14        | 0   | 241           | 189              | 163          |
| 14        | 1   | 241           | 184              | 163          |
| 22        | 0   | 233           | 154              | 107          |
| 22        | 1   | 233           | 149              | 107          |
| 30        | 0   | 225           | 112              | 51           |
| 30        | 1   | 225           | 107              | 51           |
| 38        | 0   | 217           | 91               | -5           |
| 38        | 1   | 217           | 86               | -5           |
| 46        | 0   | 209           | 70               | -61          |
| 46        | 1   | 209           | 65               | -61          |
| 54        | 0   | 201           | 42               | -117         |
| 54        | 1   | 201           | 44               | -117         |
| 62        | 0   | 193           | 28               | -173         |
| 62        | 1   | 193           | 23               | -173         |