The Generalized Smale Conjecture for 3-manifolds with genus 2 one-sided Heegaard splittings

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1 Introduction

The Smale Conjecture, proved by Hatcher [12], asserts that if $M$ is the 3-sphere with the standard constant curvature metric, the inclusion $\text{Isom}(M) \to \text{Diff}(M)$ from the isometry group to the diffeomorphism group is a homotopy equivalence. The Generalized Smale Conjecture asserts this whenever $M$ is a closed 3-manifold with a metric of constant positive curvature. All known examples of closed 3-manifolds with finite fundamental group admit such metrics, whose isometry groups are easily determined. For many manifolds of constant positive curvature, $\text{Isom}(M) \to \text{Diff}(M)$ is known to be bijective on the set of path components [3], [25], et. al.

The Generalized Smale Conjecture is an instance of the general principle, first realized by Thurston, that 3-manifold topology is profoundly affected by the existence and behavior of geometric structures. In the positive curvature case, the Generalized Smale Conjecture suggests that not only the topological structure but also the group of all smooth automorphisms is controlled by the geometry. For (compact) 3-manifolds whose interiors have constant negative curvature and finite volume, the analogous expectation holds true, at least when the manifolds are sufficiently large, since by Mostow Rigidity,
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Waldhausen’s Theorem, and work of Hatcher discussed below, the composition $\text{Isom}(M) \to \text{Out}(\pi_1(M)) \to \text{Diff}(M)$ is a homotopy equivalence. In contrast, when the manifold has interior of constant negative curvature and infinite volume, or has constant zero curvature, a diffeomorphism will not in general be isotopic to an isometry (said differently, the diffeomorphism group may have more components than the isometry group). Even in these cases, however, Waldhausen’s Theorem and Hatcher’s work show that $\text{Isom}(M) \to \text{Diff}(M)$ is always a homotopy equivalence when one restricts to the connected components of the identity diffeomorphism.

Our approach to the Generalized Smale Conjecture, for the cases we will consider here, is based on work of Hatcher as extended by Ivanov. For sufficiently large $\mathbb{P}^2$-irreducible 3-manifolds, Hatcher ([11], combined with [12]), extending earlier work of Laudenbach [18], proved that the components of $\text{Diff}(M \text{ rel } \partial M)$ are contractible. The main part of the argument is to show that the space of imbeddings of a 2-sided incompressible surface $F$ that are disjoint from a parallel copy of $F$ is a deformation retract of the space of all imbeddings of $F$ (isotopic to the inclusion relative to $\partial F$).

For manifolds that are not sufficiently large and therefore do not contain a 2-sided incompressible surface, one may try to use a 1-sided incompressible surface instead. If $M$ is orientable, irreducible, and not sufficiently large, and contains a 1-sided incompressible surface $K$, then by theorem 4 of [23], $M - K$ is an open handlebody. When the complement of a 1-sided surface $K$ in $M$ is an open handlebody, we say that $(M, K)$ is a 1-sided Heegaard splitting. The genus of the splitting is the (nonorientable) genus of $K$. Ivanov [14], [15] extended Hatcher’s results to a restricted class of 3-manifolds with genus 2 1-sided Heegaard splittings, enabling him to determine the homotopy type of their diffeomorphism groups. In particular, the Generalized Smale Conjecture is implied for these manifolds.

The manifolds considered by Ivanov are those for which no Seifert fibering is nonsingular on the complement of any vertical Klein bottle. The remaining cases are:

I. Those for which every Seifert fibering nonsingular on the complement of $K$ restricts to the “meridinal” (nonsingular) fibering of $K$. These have binary dihedral fundamental groups.

II. Those for which every Seifert fibering nonsingular on the complement of
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$K$ restricts to the “longitudinal” (two exceptional fibers) fibering of $K$. These are the lens spaces $L(4n, 2n - 1), n \geq 2$.

III. The lens space $L(4, 1)$ (which admits both meridional and longitudinal fiberings nonsingular on the complement of $K$).

In this paper, we prove the Conjecture for cases I and II. More precise statements and an outline of the argument will be given in section §4 after fixing notation.

Ivanov’s announced results were used in [6] to construct examples of homeomorphisms of reducible 3-manifolds that are homotopic but not isotopic. Our results show that their construction applies to a larger class of 3-manifolds. In [22], our work was applied to the classification problem for 3-manifolds which have metrics of positive Ricci curvature and universal cover $S^3$.

The Generalized Smale Conjecture has attracted the interest of physicists studying the theory of quantum gravity. Certain physical configuration spaces can be realized as the quotient space of a principal $Diff_1(M, x_0)$-bundle with contractible total space, where $Diff_1(M, x_0)$ denotes the subspace of $Diff(M, x_0)$ that induce the identity on the tangent space to $M$ at $x_0$. (This group is homotopy equivalent to $Diff(M \# D^3 \text{ rel } \partial D^3)$.) Consequently the loop space of the configuration space is weakly homotopy equivalent to $Diff_1(M, x_0)$. Physical significance of $\pi_0(Diff(M))$ for quantum gravity was first pointed out in [3]. See also [1], [3], [9], [13], [27], [29]. The significance of some higher homotopy groups of $Diff(M)$ is examined in [8].

An earlier version of this paper was circulated in the late 1980’s, and indeed has been cited a number of times in the scientific literature. In this new version, the essential mathematical content is unchanged, but a considerable amount of detail has been added. Also, various “folk” theorems about fibrations of spaces of diffeomorphisms and imbeddings, heavily used in our arguments, have been put on firm ground by the work in [17].

2 Notation and statement of results

Let $K$ be a Klein bottle and let $P$ be the orientable $I$-bundle over $K$ with boundary the torus $T$. Let $R$ be a solid torus containing a meridinal 2-disc whose boundary $C$ is a meridinal circle lying in $\partial R$. Fix a presentation $\pi_1(K) = \langle a, b \mid b^{-1}ab = a^{-1} \rangle$. The four homotopy classes of essential simple closed
curves on $K$ are $b$, $ab$, $a$, and $b^2$, with $b$ and $ab$ orientation-reversing and $a$ and $b^2$ orientation-preserving. The free abelian group $\pi_1(\partial P)$ is generated by (loops homotopic in $P$ to) $a$ and $b^2$. For a pair $(m, n)$ of relatively prime integers, the 3-manifold $M(m, n)$ is formed by identifying $\partial R$ and $\partial P$ in such a way that $C$ is attached along a simple closed curve representing the element $a^m b^{2n}$. Since $M(-m, n) = M(m, n)$ and $M(-m, -n) = M(m, n)$, we can and always will assume that both $m$ and $n$ are positive. The fundamental group of $M(m, n)$ has presentation $\langle a, b \mid b^{-1}ab = a^{-1}, a^m b^{2n} = 1 \rangle$.

From [24] we have the following facts. If $m = 1$, then $M(1, n)$ is the lens space $L(4n, 2n - 1)$. Suppose that $m \neq 1$. If $m$ is even, then $\pi_1(M(m, n)) \cong D_{4m} \times C_n$, the direct product of the binary dihedral group $D_{4m} = \langle x, y \mid x^2 = (xy)^2 = y^{2n} \rangle$ and a cyclic group. Finally, if $m > 1$ is odd, write $4n = 2^n n_1$ where $n_1$ is odd. Then $\pi_1(M(m, n)) \cong D_{2^k, m} \times C_{n_1}$, where $D_{2^k, m}$ is the generalized dihedral group $\langle x, y \mid x^{2^k} = y^m = 1, x^{-1}yx = y^{-1} \rangle$. Note that when $m$ is odd, there is an isomorphism from $D_{4m}^*$ to $D_{4m}$ given by sending $x$ to $x$ and $y$ to $yx^2$.

In [24], it was announced that for $n \neq 1$, the inclusion $\text{Isom}(M(m, n)) \to \text{Diff}(M(m, n))$ is a homotopy equivalence, and a detailed proof for the case $m \neq 1$ and $n \neq 1$ was given in [14]. In the remaining sections of this paper, we will prove this result for the cases where $m = 1$ or $n = 1$, except for that of $m = n = 1$, where $M(1, 1) = L(4, 1)$. In section 4, we compute the isometry groups of the $M(m, n)$. In section 5, we reduce the Conjecture in the case of the $M(m, n)$ to proving that the inclusion from the space of fiber-preserving imbeddings of $K$ into $M$ to the space of all imbeddings is a weak homotopy equivalence on the connected components of the inclusion. That is, a parameterized family $\{K_t\}_{t \in D^k}$ of imbedded Klein bottles isotopic to $K$ can be deformed to a fiber-preserving family. The proof of this assertion occupies the final two sections. In contrast to the 2-sided case, one cannot avoid parameters for which $K_t$ intersects $K$ non-transversely and these configurations must be analyzed, but it is enough to consider “generic position” non-transverse configurations, as described in [13]. For these we show in theorem 5.3 that for each parameter value $t$ one can find a concentric fibered torus $T_u$ in a neighborhood of $K$ which meets $K_t$ transversely in circles that are either inessential in $T_u$ or cover imbedded circles in $K_t$. In section 5, we complete the argument, using the methods of Hatcher to eliminate inessential intersections with the concentric solid tori in $M - K$, then deforming the $K_t$ to be fiber-preserving inductively.
over the skeleta of a triangulation of the parameter space.

This program fails in a fundamental way in the case of \( L(4,1) \) because no Seifert fibering of \( L(4,1) \) is preserved by all isometries. Below we will point out more precisely the steps where the arguments break down.

### 3 Calculation of isometry groups

The finite subgroups of SO(4) that act freely on \( S^3 \) were worked out by Hopf and Seifert-Threlfall, and reformulated using quaternions by Hattori. We review this as described in [30] (see especially pp. 226-227). There is a quotient map \( F: \text{Spin}(3) \times \text{Spin}(3) \to \text{SO}(4) \), where \( \text{Spin}(3) \cong \text{SU}(2) \) can be identified with the group of unit quaternions, and \( F(a,b) \) acts on \( \mathbb{R}^4 \) by

\[
F(a,b)(q) = aqb^{-1}.
\]

The kernel of \( F \) is \( W = \{(1,1), (-1,-1)\} \). The center of SO(4) has order 2, and is generated by \([(1,-1)]\) which acts as the antipodal map on \( S^3 \). Let \( G \) be a finite subgroup of SO(4) acting freely on \( S^3 \), and let \( M = S^3/G \). If \( G \) has even order, then it must contain the antipodal map. As explained in [19], the images of the two Spin(3) factors in SO(4) can be described as the groups of “right rotations” and “left rotations”, which commute and intersect only in the antipodal map. Let \( G^* = F^{-1}(G) \), and let \( G_1 \) and \( G_2 \) be the projections of \( G^* \) into the factors of Spin(3) \( \times \) Spin(3). For \( q \neq 0 \), \( F(a,b)(q) = q \) if and only if \( a = q bq^{-1} \), i.e. if and only if \( a \) is conjugate to \( b \) in Spin(3). Thus \( G \) acts freely on \( S^3 \) if and only if \( G^* \) has no pair \((a,b) \notin W\) such that \( a \) is conjugate to \( b \) in Spin(3). Upon detailed examination, this implies that at least one of \( G_i \), say \( G_2 \), is cyclic and hence is contained in a circle subgroup \( S \) of Spin(3). Thus \( F(S) \) is contained in the normalizer of \( G \). This implies that there is an action of \( S^1 \) by isometries on \( M \), which determines a Seifert fibering of \( M \). In [19], the explicit imbeddings of the various \( G \) into SO(4) are given, and we will refer to these when we work out the isometry groups of some of the quotient manifolds.

Since O(4) is the full group of isometries of \( S^3 \), the isometry group of \( M \) is the quotient \( \text{Norm}(G)/G \) where \( \text{Norm}(G) \) is the normalizer of \( G \) in O(4). We are especially interested in \( \text{isom}(M) \), the connected component of the identity in \( \text{Isom}(M) \). Let \( \text{norm}(G^*) \) denote the connected component of the identity in the normalizer of \( G^* \) in Spin(3) \( \times \) Spin(3). Clearly the connected component of the identity in \( \text{Isom}(M) \) is \( \text{norm}(G^*)/(G^* \cap \text{norm}(G^*)) \). For computing this, we observe that \( \text{norm}(G^*) = \text{norm}(G_1) \times \text{norm}(G_2) \), where \( \text{norm}(G_i) \) denotes the...
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connected component of the identity in the normalizer of $G_i$ in Spin(3). When $G_i$ is cyclic of order 2, $\text{norm}(G_i) = \text{Spin}(3)$. When $G_i$ is cyclic of order greater than 2, $\text{norm}(G_i)$ is the unique $S^1$-subgroup of Spin(3) that contains $G_i$. When $G_i$ is noncyclic, $\text{norm}(G_i)$ is just the identity element.

We now specialize to the manifolds $M(m, n)$ described in section 2. Let $G = \pi_1(M(m, n))$.

Case I. $G$ is cyclic of order 4.

For $(a, b) \in G^*$, not both $a$ and $b$ can be of order 4, since they would then be conjugate in Spin(3). Therefore one of the $G_i$ is $C_4$ and the other is $C_2$, so $\text{norm}(G^*)$ is $S^1 \times \text{Spin}(3)$ and $\text{isom}(L(4, 1))$ is $S^1 \times \text{SO}(3)$.

Case II. $G$ is cyclic of order $4n$, $n \geq 2$.

Since the quotient is $L(4n, 2n - 1)$, one of the $G_i$ is cyclic of order 4 and the other is cyclic of order at least $2n$. To see this, we can use the description of the action of SO(4) on $S^3$ described in [19]. Let $R(\theta)$ be the rotation in SO(2) through an angle $\theta$, and let $M(\theta_1, \theta_2)$ be the orthogonal sum of $R(\theta_1)$ and $R(\theta_2)$ in SO(4). Let $\theta_0 = 2\pi/(4n)$. Then an element which generates the $C_{4n}$-action whose quotient is $L(4n, 2n - 1)$ is $M(\theta_0, (2n - 1)\theta_0) = M(n\theta_0, n\theta_0)M(-(n - 1)\theta_0, (n - 1)\theta_0)$. The first is an element of order 4 in the “right rotations”. The second is an element of order $4n$ or $2n$ in the “left rotations.”, according as $n$ is even or odd. Therefore $\text{norm}(G^*)$ is $S^1 \times S^1$ and $\text{isom}(L(4n, 2n - 1)) = S^1 \times S^1$.

Case III. $G \cong D_{4m}^* \times C_n$.

According to [19] we may take $G_1 \cong D_{4m}^*$ and $G_2 \cong C_{2n}$. If $n = 1$, then $\text{norm}(G^*)$ is Spin(3), so $\text{isom}(M(m, n)) = \text{SO}(3)$. If $n \geq 3$, then $\text{norm}(G^*)$ and $\text{isom}(M(m, n))$ are $S^1$.

Case IV. $G \cong D_{(2^k, m)} \times C_{n_1}$, $k \geq 3$.

From the first paragraph on p. 111 of [19] (which applies only when $k \geq 3$, not for $k \geq 2$ as stated there) we have $G_1 \cong C_{2^kn_1}$ and $G_2 \cong D_{4m}^*$. Therefore $\text{norm}(G^*)$ and $\text{isom}(M(m, n))$ are $S^1$.

Calculation of $\pi_0(\text{Diff}(M(m, n)))$, implying that $\pi_0(\text{Isom}(M(m, n)))$ is isomorphic to $\pi_0(\text{Diff}(M(m, n)))$, was done in [3], [4], and [24]. We summarize the information we have collected so far in the following table, where as above $k$ and $n_1$ are defined by $4n = 2^kn_1$ with $n_1$ odd.
4 Homotopy type of the space of diffeomorphisms

In the space of smooth imbeddings of the Klein bottle in $M = M(m, n)$, denote by $\text{imb}(K, M)$ the connected component of the inclusion of the “standard” Klein bottle $K_0$, which will be defined below when we give a more precise description of the Seifert fiberings we will be using. We will assume that exactly one of $m$ or $n$ is equal to 1.

When $m=1$, so that $M$ is a lens space, there is a Seifert fibering with two exceptional orbits of type $(2,1)$ contained in $K_0$. The quotient 2-orbifold is the sphere with two cone points of order 2. When $n=1$, $M$ is a binary dihedral space and there is a Seifert fibering which is nonsingular with orbit space $\mathbb{R}P^2$. In both cases, $K_0$ is a union of fibers. In the subspace of $\text{imb}(K, M)$ consisting of those imbeddings which take fibers of $K_0$ to fibers of $M$, let $\text{imb}_f(K, M)$ denote the connected component of the inclusion. These are called the fiber-preserving imbeddings.

Our main result shows that parameterized families of imbeddings of $K$ in $M$ can be deformed to families of fiber-preserving imbeddings.

**Theorem 4.1** If either $m \neq 1$ or $n \neq 1$, then the inclusion $\text{imb}_f(K, M) \to \text{imb}(K, M)$ is a weak homotopy equivalence.

The proof will be given in sections 5 and 6. From theorem 4.1, we can deduce the Generalized Smale Conjecture for these classes of 3-manifolds.
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**Theorem 4.2** For the binary dihedral spaces $M(m, 1)$, $m \geq 2$, the inclusion from $\text{Isom}(M(m, 1))$ to $\text{Diff}(M(m, 1))$ is a homotopy equivalence, consequently $\text{Diff}(M(m, 1))$ is homotopy equivalent to $\text{SO}(3) \times S_3$ or $\text{SO}(3) \times C_2$ according as $m = 2$ or $m > 2$.

**Theorem 4.3** For the lens spaces $M(1, n) = L(4n, 2n - 1)$, $n \geq 2$, the inclusion from $\text{Isom}(M(1, n))$ to $\text{Diff}(M(1, n))$ is a homotopy equivalence, consequently $\text{Diff}(M(1, n))$ is homotopy equivalent to $S^1 \times S^1 \times C_2 \times C_2$.

Before beginning the proofs, we will need a more precise description of the Seifert fiberings that are invariant under the isometries. Assume first that $m = 1$ so that $M$ is a lens space. A generating element of $\pi_1(M)$ was given explicitly in Case I in section 2, and is a product $M(n\theta_0, n\theta_0)M(-(n-1)\theta_0, (n-1)\theta_0)$ where $\theta_0 = 2\pi/(4n)$. The action of the left rotations contains an $S^1$-subgroup $S_L$ which contains the cyclic subgroup $C_{2n}$ of $\pi_1(M)$ generated by the element $M(2\theta_0, -2\theta_0)$; explicitly, it is the group of elements of the form $M(\theta, -\theta)$. The orbits of the action of $S_L$ are the fibers of a Hopf fibering of $S^3$ by geodesic circles. Since the right rotations commute with the left rotations, the action of the right rotations is fiber-preserving. The quotient space of the Hopf fibering is $S^2$ on which the right rotations act via a quotient map $g: R \to \text{SO}(3)$ described explicitly on p. 105 of [19]. In particular, $g(M(n\theta_0, n\theta_0))$ is an element of order 2 ($M(\pi, \pi)$ is the kernel of $g$), which acts on $S^2$ with two fixed points, corresponding to the two orbits of the $S_L$-action left invariant by $M(n\theta_0, n\theta_0)$. The quotient of $S^3$ by $C_{2n}$ is $L(2n, 1)$, and the $C_2$-action induced by $M(\theta_0, (2n - 1)\theta_0)$ preserves exactly two orbits which become the two $(2, 1)$ exceptional orbits of $L(4n, 2n - 1)$. Now let $K_0$ be the preimage of a great circle of $S^2$ through the two fixed points of $g(M(n\theta_0, n\theta_0))$. This is a totally geodesic vertical Klein bottle in $M$. As explained in section 2, $\text{isom}(M) = S^1 \times S^1$ where one $S^1$-factor is the vertical action on $M$ induced by $S_L$ and the other comes from the $S^1$-action induced on $M$ by the $S^1$-subgroup $S_R$ that contains $M(n\theta_0, n\theta_0)$. Restricted to $K_0$, these isometries give all the isometric fiber-preserving imbeddings of $K_0$ in $M$: the vertical reimbeddings are given by the restriction of $S_L$, while the action of $S_R$ moves $K_0$ through all the Klein bottles that are preimages of great circles of $S^2$ that pass through the two fixed points of $g(M(n\theta_0, n\theta_0))$. Summarizing, if we denote the isometric fiber-preserving imbeddings of $K_0$ by $\text{isom}_f(K, M)$, we have shown that the restriction map $\text{isom}(M) \to \text{isom}_f(K, M)$ is a homeomorphism.
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Suppose now that \( n = 1 \). From section 2, \( \pi_1(M) \cong D^*_4 \) and from Case III of section 2 we may assume that \( D^*_4 \) is a subgroup of the right rotations. The Seifert fibering invariant under \( \text{isom}(M) \) is obtained as follows (see pp. 112-113 of [19]). There is an \( S^1 \)-subgroup \( S \) of the group of right rotations which contains the index 2 subgroup \( C^*_{2m} \) of \( D^*_4 \). There is an order 4 right rotation \( \delta \) which conjugates each element of \( S \) to its inverse, and \( D^*_4 \) is generated by \( C^*_{2m} \) and \( \delta \). The orbits of \( S \) are preserved by \( \delta \) and determine the fibering of \( M \). Now let \( p: S^3 \to S^2 \) be the Hopf map whose point preimages are the orbits of \( S \). On \( S^2 \), \( \delta \) induces the antipodal map. Fix one of the invariant circles of \( \delta \). Its image under \( p \) is a great circle \( C \), and we let \( K_0 \) be the image of \( p^{-1}(C) \) in \( M \). The induced fibering on \( K_0 \) is the nonsingular one by meridinal fibers, and for this metric on \( K_0 \), \( \text{isom}_f(K_0) \) is \( S^1 \) in which the order 2 element takes each fiber to itself by the monodromy of \( K_0 \) regarded as a \( S^1 \)-fibering over \( p(C) \). As seen in section 2, the isometries of \( M \) are induced by the left rotations. In particular, the circle subgroup of the group of left rotations that leaves \( p^{-1}(C) \) invariant restricts to \( \text{isom}_f(K_0) \) on \( K_0 \). As in the case \( m = 1 \), the restriction determines a homeomorphism \( \text{isom}(M) \to \text{isom}_f(K, M) \).

Proof of Theorems 4.2 and 4.3 assuming Theorem 4.1: Since \( \text{Diff}(M) \) has the homotopy type of a CW-complex [21], it is enough to prove that the inclusion is a weak homotopy equivalence. As mentioned in section 2, the inclusion is known to induce a bijection on path components, so we will restrict attention to the connected components of the identity homeomorphism.

From corollary 8.7 of [17], restriction of diffeomorphisms to imbeddings defines a fibration

\[ \text{Diff}_f(M \text{ rel } K_0) \cap \text{diff}_f(M) \to \text{diff}_f(M) \to \text{imb}_f(K, M). \]

Since any diffeomorphism in this fiber is orientation-preserving, it cannot interchange the sides of \( K_0 \). Therefore the fiber may be identified with a subspace consisting of path components of \( \text{Diff}_f(S^1 \times D^2 \text{ rel } S^1 \times \partial D^2) \). By theorem 5.2 of [17], there is a fibration

\[ \text{Diff}_v(S^1 \times D^2 \text{ rel } S^1 \times D^2) \to \text{Diff}_f(S^1 \times D^2 \text{ rel } S^1 \times D^2) \to \text{Diff}(D^2 \text{ rel } \partial D^2), \]

whose fiber is the group of vertical diffeomorphisms that take each fiber to itself. The base is contractible by [26]. The fiber is contractible, this is seen by lifting diffeomorphisms to the infinite cyclic cover \( \mathbb{R} \times D^2 \) and canonically
and equivariantly deforming the lifts to preserve \( \{0\} \times D^2 \), and then to be the identity. We conclude that \( \text{Diff}_f(S^1 \times D^2 \text{ rel } S^1 \times \partial D^2) \) and hence also \( \text{Diff}_f(M \text{ rel } K_0) \) are contractible. Therefore our fibration from above becomes

\[
\text{Diff}_f(M \text{ rel } K_0) \to \text{Diff}_f(M) \to \text{imb}_f(K, M)
\]

with contractible fiber. Similarly there is a fibration

\[
\text{Diff}_f(M \text{ rel } K_0) \to \text{Diff}_f(M) \to \text{imb}_f(K, M)
\]

The fact that it is a fibration comes from [20] and the contractibility of the fiber uses [10]. We can now fit these into a diagram

\[
\begin{array}{ccc}
\text{Diff}_f(M \text{ rel } K_0) & \to & \text{Diff}_f(M) \\
\downarrow & & \downarrow \\
\text{Diff}_f(M \text{ rel } K_0) & \to & \text{Diff}_f(M) \\
& & \\
& & \\
& & \text{imb}_f(K, M)
\end{array}
\]

The vertical maps are inclusions. By theorem 4.1, the right hand vertical arrow is a weak homotopy equivalence. Since the fibers are both contractible, it follows that \( \text{Diff}_f(M) \to \text{imb}_f(K, M) \), \( \text{Diff}(M) \to \text{imb}(K, M) \), and \( \text{Diff}_f(M) \to \text{Diff}(M) \) are weak homotopy equivalences.

Let \( \mathcal{O}_K \) be the image of \( K_0 \) in the quotient orbifold \( \mathcal{O} \) of the fibering on \( M \). When \( m = 1 \), \( \mathcal{O}_K \) is a silvered interval imbedded as half of a great circle connecting the two order 2 cone points of \( \mathcal{O} \). When \( n = 1 \), \( \mathcal{O}_K \) is an \( \mathbb{RP}^1 \) which is the image of a great circle of \( S^2 \) in \( \mathcal{O} = \mathbb{RP}^2 \). Each element of \( \text{isom}_f(K, M) \) projects to an isometric imbedding of orbifolds of \( \mathcal{O}_K \) in \( \mathcal{O} \).

Denote by \( \text{imb}(\mathcal{O}_K, \mathcal{O}) \) the connected component of the inclusion in the space of orbifold imbeddings, and let a subscript \( v \) as in \( \text{Diff}_v(K_0) \) indicate the vertical maps that take each fiber to itself. We have a fibration of groups

\[
\text{Isom}_v(K_0) \cap \text{isom}_f(K, M) \to \text{isom}_f(K, M) \to \text{isom}(\mathcal{O}_K, \mathcal{O})
\]

When \( m = 1 \), this sequence is readily seen to be \( S^1 \to S^1 \times S^1 \to S^1 \). When \( n = 1 \), \( \text{isom}(\mathcal{O}_K, \mathcal{O}) \) can be identified with the unit tangent space of \( \mathbb{RP}^2 \), and \( \text{Isom}_v(K_0) \cap \text{isom}_f(K, M) = C_2 \) generated by applying the monodromy (of the \( S^1 \)-bundle \( K_0 \to p(C) \) described above) in each fiber. In this case, the sequence is \( C_2 \to \text{SO}(3) \to T_1(\mathbb{RP}^2) \) (topologically this is \( C_2 \to \mathbb{RP}^3 \to L(4,1) \)). Using inclusions as the vertical maps, we have a diagram of fibrations

\[
\begin{array}{ccc}
\text{Isom}_v(K_0) \cap \text{isom}_f(K, M) & \to & \text{isom}_f(K, M) \\
\downarrow & & \downarrow \\
\text{Diff}_v(K_0) \cap \text{imb}_f(K, M) & \to & \text{imb}_f(K, M)
\end{array}
\]

with contractible fiber.
where the bottom sequence is a fibration by theorem 8.9 of \[17\].

Suppose first that \( n = 1 \). Using theorem 4 of \[7\], the right-hand vertical arrow of the diagram is a homotopy equivalence. The two components of \( \text{Diff}_v(K_0) \) are contractible, each containing a unique isometry, and thus the left-hand vertical arrow is a homotopy equivalence. It follows that the middle vertical arrow is a weak homotopy equivalence. When \( m = 1 \), \( \text{Isom}_v(K_0) \) and \( \text{Diff}_v(K_0) \) have two components, but elements of one component reverse the direction of the fiber so are not contained in \( \text{imb}_f(K, M) \). The identity component \( \text{isom}_v(K_0) \) is homeomorphic to \( S^1 \), and the inclusion \( \text{isom}_v(K_0) \rightarrow \text{diff}_v(K_0) \) is a homotopy equivalence, although the full details of this are lengthy. Again the middle arrow is a weak homotopy equivalence.

We have seen that \( \text{isom}(M) \rightarrow \text{isom}_f(K, M) \) is a homeomorphism. (This fails when \((m, n) = (1, 1)\), since then \( \text{isom}(M) \) does not preserve any Seifert fibering of \( M \).) We now have a diagram of inclusions

\[
\begin{array}{ccc}
\text{isom}(M) & \longrightarrow & \text{isom}_f(K, M) \\
\downarrow & & \downarrow \\
\text{diff}(M) & \longrightarrow & \text{imb}_f(K, M) \\
\downarrow & & \downarrow \\
\text{diff}(M) & \longrightarrow & \text{imb}(K, M)
\end{array}
\]

in which all arrows except the one from \( \text{isom}(M) \) to \( \text{diff}(M) \) have been shown to be weak homotopy equivalences; it follows that it and the composite \( \text{isom}(M) \rightarrow \text{diff}(M) \) are weak homotopy equivalences as well.

\( \text{Theorems 4.2 and 4.3 assuming Theorem 4.1} \)

5 Generic Position Configurations

A smoothly imbedded (connected) 2-manifold \( T \) in a closed 3-manifold \( M \) has either a product neighborhood \( T \times [-1, 1] \), or a 2-fold covering from \( \tilde{T} \times [-1, 1] \) to a tubular neighborhood of \( T \). In the former case, let \( T_u \) denote \( T \times \{u\} \), and in the latter let \( T_u \) denote the image of \( \tilde{T} \times \{u\} \). We call the \( T_u \) horizontal levels of the neighborhood of \( T \).

Let \( S \) and \( T \) be smoothly imbedded closed surfaces in a closed 3-manifold \( M \). A point \( x \) in \( S \cap T \) is called a regular point if \( S \) is transverse to \( T \) at \( x \), otherwise it is a singular point. Following section 5 of \[13\], we call \( x \) a singular
§5. Generic Position Configurations

point of finite multiplicity if $S \cap T$ meets a small neighborhood $U$ of $x$ in a finite even number of smooth arcs running from $x$ to $\partial U$, transversely except at $x$ (cf. Fig. 3, p. 1653 of [15]). Then, either $S \cap T \cap U = \{x\}$ or $x$ is a saddle tangency of $S$ and $T$.

We say that the surfaces are in generic position if all singular points of intersection are of finite multiplicity. A parameterized family in $\text{Imb}(S, M)$ is said to be in generic position relative to $T$ if each of the imbeddings in the family has this property.

**Proposition 5.1** Suppose that $F: D^k \to \text{Imb}(S, M)$ is a parameterized family of imbeddings. Assume either that $F(t)(S)$ is in generic position relative to $T$ for all $t$ in $\partial D^k$, or that $F(t)(S) = T$ for all $t$ in $\partial D^k$. Then $F$ is homotopic relative to $\partial D^k$ to a map $G: D^k \to \text{Imb}(S, M)$ so that $G = F$ on $\partial D^k$ and $G(t)(S)$ is in generic position with respect to $T$ for all $t \in \text{int}(D^k)$. Moreover, for each $t \in \text{int}(D^k)$ there exists $u_0 > 0$ so that $G(t)(S)$ is transverse to $T_u$ for all $0 < u \leq u_0$, where $T_u$ are horizontal levels in a tubular neighborhood of $T$. 

For a discussion of this proposition, we refer the reader to lemma (5.2) and remark (5.3) of [15]. The map $G$ may be chosen arbitrarily close to $F$, although we will not need to do so.

Suppose now that $L_0$ is a 1-sided surface in $M$, and as above let $L_u$ denote the horizontal levels of a tubular neighborhood of $L_0$. A piecewise-linearly imbedded surface $S$ in $M$ is said to be flattened (with respect to $L_0$ and the choice of the $L_u$) if it satisfies the following conditions.

1. There is a 4-valent graph $\Gamma$ (possibly with components which are circles) contained in $L_0$ such that $S \cap L_0$ consists of the closures of some of the connected components of $L_0 - \Gamma$.

2. Each point $p$ in the interior of an edge of $\Gamma$ has a neighborhood $U$ for which the quadruple $(U, U \cap L_0, U \cap S, p)$ is PL homeomorphic to the configuration $(\mathbb{R}^3, \{(x, y, z) \mid z=0\}, \{(x, y, z) \mid \text{either } z=0 \text{ and } y \geq 0, \text{ or } y=0 \text{ and } z \geq 0\}, \{0\})$ (see Figure 1(a)).

3. Each vertex $v$ of $\Gamma$ has a neighborhood $U$ for which the quadruple $(U, U \cap L_0, U \cap S, v)$ is PL homeomorphic to the configuration $(\mathbb{R}^3, \{(x, y, z) \mid z=0\}, \{(x, y, z) \mid \text{either } z=0 \text{ and } xy \leq 0, \text{ or } y=0 \text{ and } z \geq 0, \text{ or } x=0 \text{ and } z \leq 0\}, \{0\})$ (see Figure 1(b)).
Lemma 5.2 Let $N$ be a 3-manifold containing a smoothly imbedded surface $L_0$, and let $S_1$ be a smoothly imbedded surface in $N$ which meets $L_0$ in generic position and meets each $L_u$ transversely, for $0 < u \leq u_0$. Then given $\epsilon > 0$ there is a PL isotopy $S_t$ from $S_1$ to a PL imbedded surface $S_0$ such that

1. Each $S_t$ is within distance $\epsilon$ of the inclusion $S_1$.
2. Each $S_t$ is transverse to $L_u$ for $0 < u \leq u_0$.
3. $S_0$ is flattened.

Proof of 5.2: The isotopy will move points monotonically with respect to $u$ levels. We first describe it near a singular point $x$ of $S_1 \cap L_0$. In a neighborhood $U$ of $x$, $S_1 \cap L_0$ consists of $x$ together with a (possibly empty) collection of arcs $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$ emanating from $x$. With respect to some fixed Riemannian metric for $N$, there is a neighborhood of $x$ for which the angle of intersection of $S_1$ with $L_0$ is small; the isotopy moves points only within an $\epsilon$ neighborhood of the $\alpha_i$ and decreases these angles to 0 everywhere in a neighborhood of $x$ (or pushes a 2-disc neighborhood of $x$ in $S_1$ down to a 2-disc neighborhood of $x$ in $L_0$, if there are no arcs). At the end of the initial isotopy, say for $1 \geq t \geq 1/2$, there is a neighborhood $U$ of $x$ for which $S_{1/2} \cap L_0 \cap U$ is a regular neighborhood in $L_0$ of $\cup_{i=1}^{2n} \alpha_i$. These isotopies may be performed simultaneously near each singular point of intersection. The remainder of the isotopy will take place in an $\epsilon$ neighborhood of the original (open) edges of $S_1 \cap L_0$. At the end of this isotopy, the intersection will be locally a regular neighborhood of the original edges, except that on some of the edges it might be necessary to introduce a point where the configuration is as in Figure 1(b)—this is necessary only when the flattenings at the singular points at the ends of the edge are in opposite senses. Again, these remaining isotopies may be performed in disjoint neighborhoods of the original edges.

We call an isotopy as in lemma 5.2 a flattening isotopy. By property (2), the collection of intersection circles in $L_u$ for $0 < u \leq u_0$ is changed only by isotopy in $L_u$. After flattening, each of these circles projects along $S_0$ to an immersed circle lying in $\Gamma$, having a transverse self-intersection at each of its double points (which can occur only at vertices of $\Gamma$.)
§5. Generic Position Configurations

Now we specialize to the standard Klein bottle \( K_0 \subseteq M \) and a parameterized family \( F: D^k \to \text{imb}(K, M) \). We denote the imbedding \( F(t) \) by \( F_t \) and its image by \( K_t \). Assume for all \( t \in \partial D^k \) either that \( K_t = K_0 \), or that \( K_t \) is transverse to \( K_0 \). By proposition 5.1, we may deform \( F \) to be in generic position relative to \( K_0 \) over \( \text{int}(D^k) \).

**Theorem 5.3** Suppose that \( M = M(m, n) \) with one but not both of \( m \) or \( n \) equal to 1, and let \( K_t \) be a parameterized family in generic position for each \( t \in \text{int}(D^k) \). Then for each \( t \in \text{int}(D^k) \), there exists \( u > 0 \) so that \( K_t \) is transverse to \( T_u \) and each circle of \( K_t \cap T_u \) is either inessential in \( T_u \), or represents \( a \) or \( b^2 \) in \( \pi_1(T_u) \).

It follows immediately that no circle of \( K_t \cap T_u \) is homotopic in \( T_u \) to the meridian. Moreover, it follows that no intersection circle is homotopic in \( T_u \) to a longitude of \( R_u \) which is not homotopic to a fiber of the Seifert fibering. For when \( n = 1 \), the longitudes are \( a(a^m b^2)^k \), where \( k \) is an arbitrary integer, and when \( m = 1 \) they are \( b^2(a b^{2n})^k \). (In \( L(4,1) \), however, an \( a \) circle is a longitude of \( R_u \) which is not homotopic to a fiber of the fibering with two \((2,1)\) orbits, while a \( b^2 \) circle is a longitude not homotopic to a fiber of the nonsingular fibering.)

The proof will produce \( u \) so that \( K_t \) is transverse to \( T_u \) for \( 0 < s \leq u \), but we will not need this property.

**Proof of 5.3:** Suppose first that the intersection \( K_t \cap K_0 \) is transverse. Since \( K_t \) must meet every nearby level \( T_u \) transversely, it intersects \( P_u \) in Möbius bands and annuli. Consequently the projection of \( T_u \) onto \( K_0 \) maps circles of intersection of \( K_t \cap T_u \) onto circles of \( K_t \cap K_0 \) either homeomorphically or by two-fold coverings. Only inessential and \( a \) and \( b^2 \) circles can be preimages of imbedded circles in \( K_0 \).

Suppose now that \( K_t \) intersects \( K_0 \) in some singular points. By proposition 5.1, \( K_t \) is transverse to \( T_u \) for all \( u \leq u_0 \) for some \( u_0 \). By lemma 5.2, we can flatten \( K_t \) near \( K_0 \), without changing either the transversality of \( K_t \) and \( T_u \) or the homotopy classes of the loops in \( K_t \cap T_u \). Then, \( K_t \cap K_0 \) consists of a valence 4 graph \( \Gamma \), which is the image of the collection of disjoint simple closed curves \( K_t \cap T_u \) under a 2-fold covering projection, together with some of the complementary regions of \( \Gamma \) in \( K_0 \), which we will call the faces. Each edge of \( \Gamma \) lies in exactly one face. We may choose the \( I \)-fibering so that \( K_t \cap P_u \) lies in the union of \( K_t \cap K_0 \) and the \( I \)-fibers that meet \( \Gamma \).
§5. Generic Position Configurations

Suppose for contradiction that one of the circles in $K_t \cap T_u$ represents $a^{k\ell}b^2$ with $k\ell \neq 0$. Since $K_t$ is geometrically incompressible (if not, then $M$ would contain an imbedded projective plane), there is an isotopy of $K_t$ which eliminates the circles of $K_t \cap T_u$ that are inessential in $T_u$, without altering the remaining circles or destroying the flattened position of $K_t \cap P_u$. So we may assume that $K_t \cap T_u$ consists of disjoint circles each representing $a^{k\ell}b^2$. Since $K_t$ is isotopic to $K_0$, each loop in $T_u$ has even algebraic intersection number with $K_t \cap T_u$, so there is an even number of these circles; denote them by $A_1, A_2, \ldots, A_{2r}$.

The vertices of $\Gamma$ are the images of the intersections of $\bigcup A_i$ with $\bigcup \tau(A_i)$, where the involution $\tau$ is the covering transformation for the covering map from $T_u$ to $K$ that determines the $I$-fibering of $P_u$. Now $\bigcup A_i$ and $\bigcup \tau(A_i)$ meet transversely; the number of intersections is at least $|\bigcup (A_i) \cdot (\bigcup \tau(A_i))| = |(2r a^{k\ell}b^2) \cdot (2r a^{k\ell}b^{-2})| = 4r^2|2k\ell|$.

Since each vertex of $\Gamma$ is covered by two intersections, $\Gamma$ has at least $4r^2|k\ell|$ vertices.

Notice that each edge of $\Gamma$ runs between two distinct vertices, since the $A_i$ are disjoint and imbedded and do not cover imbedded loops in $K$. Moreover, each face contains an even number of edges, since it can be lifted to $T_u$ with successive edges lying alternately in $\bigcup A_i$ and $\bigcup \tau(A_i)$. In particular, no face is a 1-gon, or is a 2-gon with its vertices identified. Therefore each face that is a 2-gon can be eliminated by an isotopy, yielding a new $K_t$ in flattened position (see Fig. 2). So we may assume that each face contains at least 4 vertices. Finally, observe that the Euler characteristic of $K_t \cap P_u$ is at least $-2r$, since $\chi(K_t) = 0$ and $K_t \cap P_u$ has exactly $2r$ boundary components. Letting $V$, $E$, and $F$ denote the number of vertices, edges, and faces of $K_t \cap K_0$, we have $E = 2V$ and $F \leq V/2$ (since each edge lies in exactly one face and each face has at least 4 edges). Therefore $-2r \leq \chi(K_t \cap P_u) = \chi(K_t \cap K) \leq -V/2$. Since $V \geq 4r^2|k\ell|$, it follows that $r|k\ell| \leq 1$, forcing $r = |k\ell| = 1$, $\chi(K_t \cap K) = -2$, $V = 4$, and $F = 2$. That is, $K_t \cap K$ consists of two faces, each a 4-gon, meeting at their four vertices. Moreover, $\Gamma$ is the image of two imbedded circles $ab^2$ or $ab^{-2}$ circles each projecting to a loop with one self-intersection. This forces the two faces of $K_t \cap K_0$ to meet each other as shown in Fig. 3. It follows that $K_t \cap P_u$ is a twice-punctured Klein bottle (shading the complementary faces would yield a twice-punctured torus). But then $ab^2$ or $ab^{-2}$ bounds a disc
6 Parameterization

We now complete the proof of theorem 4.1. Since \( \text{imb}(K, M) \) and \( \text{imb}_f(K, M) \) are connected, we have \( \pi_0(\text{imb}(K, M), \text{imb}_f(K, M)) = 0 \). To prove that the higher relative homotopy groups vanish, we begin with a parameterized family, which we may take to be a smooth map \( D^k \to \text{imb}(K, M) \), where \( k \geq 1 \), which takes all points of \( \partial D^k \) to the standard inclusion. By abuse of notation, we confuse the imbedding corresponding to the point \( t \in D^k \) with its image, denoting both by \( K_t \). By proposition 5.1, we may assume that each \( K_t \) is in general position with \( K_0 \). By theorem 5.3 and the remark following its statement, there is for each \( t \) a value \( u > 0 \) so that

1. \( K_t \) is transverse to \( T_u \).
2. No intersection circle of \( K_t \) with \( T_u \) is a meridian, or a longitude not homotopic in \( T_u \) to a fiber of the Seifert fibering.

Each intersection circle that bounds a (necessarily unique) 2-disc in \( T_u \) also bounds a unique 2-disc in \( K_t \), since \( K_t \) is geometrically incompressible, and if either of the two discs is innermost among all such discs, then their union bounds a unique 3-ball in \( M \). Only very routine modifications are needed to the procedure of Hatcher described in [11] to deform the family, keeping it fixed on \( \partial D^k \), so that for each \( t \in D^k \), there is a value \( u > 0 \) so that in addition to (1) and (2) we have

2' No intersection circle of \( K_t \) with \( T_u \) is inessential in \( T_u \).

In fact, the argument is somewhat simpler, since there is only a unique 3-ball across which the 2-discs can be pushed to eliminate the inessential circles; moreover since these 3-balls cannot contain essential loops of \( T_u \) in their boundaries, the deformations can be chosen so as not to affect the intersections which are essential in the \( T_u \). As in [11], it is necessary to pass to new levels, but
by choosing these very close to previously chosen levels we can ensure that no
new kinds of intersection circles arise.

Since \( K_t \) is incompressible, it follows that \( K_t \cap R_u \) consists of annuli (M"obius
bands cannot occur because orientation-reversing loops in \( K_t \) are dual to \([K_0] \in
H_2(M; \mathbb{Z}/2)\)).

Consider the annuli \( K_t \cap R_u \) whose boundary circles are not longitudes.
Each such annulus is parallel in \( R_u \) to a uniquely determined annulus in \( T_u \).
We again use the procedure of [11] to pull these annuli out of the \( R_u \). That
is, deform the family, keeping it fixed on \( \partial D^k \), so that for each \( t \in D^k \), there
is a value \( u > 0 \) so that

\[
(1) \quad K_t \text{ is transverse to } T_u.
\]

\( (2'') \) Every intersection circle of \( K_t \) with \( T_u \) is homotopic in \( T_u \) to a fiber of
the Seifert fibering.

The adaptation of the argument is routine; the annuli play the role of the
regions called \( D_M(c_i) \) in Hatcher’s paper, and the cross-sectional picture of
the regions between the annuli and the \( T_u \) is exactly as in the figure on p. 429
of [11]. Again, it is necessary to pass to new levels, but no new kinds of
intersection circles need arise.

To complete the argument, we require two technical lemmas.

**Lemma 6.1** Let \( T \) be a torus with a fixed \( S^1 \)-fibering, and let \( C_n = \cup_{i=1}^n S_i \)
be a union of \( n \) distinct fibers. Then \( \text{imb}_f(C_n, T) \to \text{imb}(C_n, T) \) is a weak
homotopy equivalence.

**Proof of 6.1:** Fix a basepoint \( s_0 \) in \( S_n \). Consider the diagram

\[
\begin{array}{ccc}
\text{imb}_f(S_n, T \text{ rel } s_0) & \to & \text{imb}_f(S_n, T) \\
\downarrow & & \downarrow \\
\text{imb}(S_n, T \text{ rel } s_0) & \to & \text{imb}(S_n, T)
\end{array}
\]

The top row is a fibration by corollary 9.6 of [11], and the bottom row is a fibra-
tion by [20]. The fiber of the first row is homeomorphic to \( \text{Diff}_+(S_n \text{ rel } s_0) \), the
group of orientation-preserving diffeomorphisms, which is contractible. The
fiber of the second row is contractible using [11]. Therefore the middle vertical
arrow is a weak homotopy equivalence. For \( n = 1 \), this completes the proof.
§6. Parameterization

Inductively, let \( A \) be the annulus that results from cutting \( T \) along \( S_n \) and consider a similar diagram, where \( \text{imb}(S_{n-1}, A) \) denotes the imbeddings with image in the interior of \( A \), and so on.

\[
\begin{align*}
\text{imb}_f(S_{n-1}, A \text{ rel } s_0) & \rightarrow \text{imb}_f(S_{n-1}, A) \rightarrow \text{imb}(s_0, \text{int}(A)) \\
& \downarrow \downarrow \downarrow \\
\text{imb}(S_{n-1}, A \text{ rel } s_0) & \rightarrow \text{imb}(S_{n-1}, A) \rightarrow \text{imb}(s_0, \text{int}(A))
\end{align*}
\]

The fibers are contractible, so the middle vertical arrow is a weak homotopy equivalence. Now we examine another diagram.

\[
\begin{align*}
\text{imb}_f(C_{n-1}, A \text{ rel } S_{n-1}) & \rightarrow \text{imb}_f(C_{n-1}, A) \rightarrow \text{imb}_f(S_{n-1}, A) \\
& \downarrow \downarrow \downarrow \\
\text{imb}(C_{n-1}, A \text{ rel } S_{n-1}) & \rightarrow \text{imb}(C_{n-1}, A) \rightarrow \text{imb}(S_{n-1}, A)
\end{align*}
\]

The first row is a fibration by corollary 6.5 of [17] and the second is a fibration by [20]. The right vertical arrow was shown to be a weak homotopy equivalence by the previous diagram, and the left one is a weak homotopy equivalence by induction, so the middle one is also. The proof is now completed by the diagram

\[
\begin{align*}
\text{imb}_f(C_n, T \text{ rel } S_n) & \rightarrow \text{imb}_f(C_n, T) \rightarrow \text{imb}_f(S_n, T) \\
& \downarrow \downarrow \downarrow \\
\text{imb}(C_n, T \text{ rel } S_n) & \rightarrow \text{imb}(C_n, T) \rightarrow \text{imb}(S_n, T)
\end{align*}
\]

Lemma 6.2. Let \( \Sigma \) be a compact 3-manifold with nonempty boundary and having a fixed Seifert fibering. Let \( F \) be a compact 2-manifold properly imbedded in \( \Sigma \), such that \( F \) is a union of fibers. Let \( \text{imb}_{\partial f}(F, \Sigma) \) be the connected component of the inclusion in the space of (proper) imbeddings for which the image of \( \partial F \) is a union of fibers. Then \( \text{imb}_f(F, \Sigma) \rightarrow \text{imb}_{\partial f}(\tilde{F}, \Sigma) \) is a weak homotopy equivalence.

To prove lemma 6.2, we need a preliminary result.

Lemma 6.3. The following maps induced by restriction are fibrations.
6. Parameterization

(i) \( \text{imb}(F, \Sigma) \to \text{imb}(\partial F, \partial \Sigma) \)

(ii) \( \text{imb}_{\partial f}(F, \Sigma) \to \text{imb}_f(\partial F, \partial \Sigma) \)

(iii) \( \text{imb}_f(F, \Sigma) \to \text{imb}_f(\partial F, \partial \Sigma) \).

Proof of 6.3: Part (ii) follows from part (i) since \( \text{imb}_{\partial f}(F, \Sigma) \) is the preimage of \( \text{imb}_f(\partial F, \partial \Sigma) \) under the fibration of part (i). Parts (i) and (iii) are cases of corollaries 9.3 and 9.4 of [17].

Proof of 6.2: First we use the following fibration from theorem 8.3 of [17],

\[
\text{Diff}_v(\Sigma \text{ rel } \partial \Sigma) \cap \text{diff}_f(\Sigma \text{ rel } \partial \Sigma) \to \text{diff}_f(\Sigma \text{ rel } \partial \Sigma) \to \text{diff}(\mathcal{O} \text{ rel } \partial \mathcal{O})
\]

where \( \mathcal{O} \) is the quotient orbifold of \( \Sigma \) and as usual \( \text{Diff}_v \) indicates the diffeomorphisms that take each fiber to itself. The orbifold diffeomorphism group of \( \mathcal{O} \) is homotopy equivalent to a subspace consisting of path components of the diffeomorphism group of the 2-manifold \( B \) obtained by removing the cone points from \( \mathcal{O} \). Since \( \partial B \) is nonempty, \( \text{diff}(B \text{ rel } \partial B) \) and therefore \( \text{diff}(\mathcal{O} \text{ rel } \partial \mathcal{O}) \) are contractible. Moreover, since \( \pi_1(\text{diff}(\mathcal{O} \text{ rel } \partial \mathcal{O})) \) is trivial, the homotopy exact sequence of the fibration shows that \( \text{Diff}_v(\Sigma \text{ rel } \partial \Sigma) \cap \text{diff}_f(\Sigma \text{ rel } \partial \Sigma) \) is connected so equals \( \text{diff}_v(\Sigma \text{ rel } \partial \Sigma) \). It is not difficult to see that each component of \( \text{Diff}_v(\Sigma \text{ rel } \partial \Sigma) \) is contractible (see lemma 10.4 of [17] for a similar argument), so we conclude that \( \text{diff}_f(\Sigma \text{ rel } \partial \Sigma) \) is weakly contractible.

Next, consider the diagram

\[
\begin{array}{ccc}
\text{Diff}_f(\Sigma \text{ rel } F \cup \partial \Sigma) \cap \text{diff}_f(\Sigma \text{ rel } \partial \Sigma) & \to & \text{diff}_f(\Sigma \text{ rel } \partial \Sigma) \\
\downarrow & & \downarrow \\
\text{Diff}(\Sigma \text{ rel } F \cup \partial \Sigma) \cap \text{diff}(\Sigma \text{ rel } \partial \Sigma) & \to & \text{diff}(\Sigma \text{ rel } \partial \Sigma) \\
\end{array}
\]

where the rows are fibrations by corollaries 8.7 and 3.6 of [17]. From above, the components of \( \text{Diff}_f(\Sigma \text{ rel } \partial \Sigma) \) and (by cutting along \( F \)) the components of \( \text{Diff}_f(\Sigma \text{ rel } F \cup \partial \Sigma) \) are weakly contractible. By [17], the components of \( \text{Diff}(\Sigma \text{ rel } \partial \Sigma) \) and \( \text{Diff}(\Sigma \text{ rel } F \cup \partial \Sigma) \) are weakly contractible. Therefore to show that \( \text{imb}_f(F, \Sigma \text{ rel } \partial F) \to \text{imb}(F, \Sigma \text{ rel } \partial F) \) is a weak homotopy equivalence it is sufficient to show that \( \pi_0(\text{Diff}_f(\Sigma \text{ rel } F \cup \partial \Sigma) \cap \text{diff}_f(\Sigma \text{ rel } \partial \Sigma)) \to \pi_0(\text{Diff}(\Sigma \text{ rel } F \cup \partial \Sigma) \cap \text{diff}(\Sigma \text{ rel } \partial \Sigma)) \) is bijective. It is
surjective because every diffeomorphism of a Seifert-fibered 3-manifold which is fiber-preserving on the (non-empty) boundary is isotopic relative to the boundary to a fiber-preserving diffeomorphism (lemma VI.19 of [14]). It is injective because fiber-preserving diffeomorphisms that are isotopic are isotopic through fiber-preserving diffeomorphisms (see [28]).

The proof is completed by the following diagram in which the rows are fibrations by parts (iii) and (ii) of lemma 6.3, and we have verified that the left vertical arrow is a weak homotopy equivalence.

\[
\begin{array}{ccc}
\text{imb}_f(F, \Sigma \text{ rel } \partial F) & \to & \text{imb}_f(F, \Sigma) \\
\downarrow & & \downarrow \\
\text{imb}(F, \Sigma \text{ rel } \partial F) & \to & \text{imb}_{\partial f}(F, \Sigma)
\end{array}
\]

6.2

We can now complete the proof of theorem 4.1 by deforming the family to a fiber-preserving family. Since conditions (1) and (2″) must remain true in a neighborhood of \( t \), we can cover \( D^k \) by convex \( k \)-cells \( B_j \), having corresponding levels \( u_j \) for which (1) and (2″) hold throughout \( B_j \). It is convenient to rename the \( B_j \) so that \( u_1 < u_2 < \ldots < u_r \). Choose a PL triangulation \( \Delta \) of \( D^k \) sufficiently fine so that each \( i \)-cell lies in at least one of the \( B_j \). We will proceed by increasing induction to deform the family of Klein bottles to be vertical over the \( i \)-skeleta of \( \Delta \). It will never be necessary to change the imbeddings over the boundary of \( D^k \).

Suppose first that \( \tau \) is a 0-simplex of \( \Delta \). Let \( j_1 < j_2 < \ldots < j_s \) be the values of \( j \) for which \( \tau \subseteq B_j \). By (2″) each intersection circle of \( K_\tau \) with each \( T_{j_q} \) is isotopic in \( T_{j_q} \) to a fiber of the Seifert fibering. Also, each is an orientation-preserving simple closed curve in \( K_\tau \), so must be isotopic in \( M \) to the \( a \) loop or the \( b^2 \) loop in \( K_0 \). When \( m = 1 \), \( b^2 \) is the generic fiber of \( M \), and \( a \) is not isotopic in \( M \) to \( b^2 \) since \( a = b^{2n} \) and \( n \neq 1 \). When \( n = 1 \), \( a \) is the fiber of \( M \), and \( b^2 \) is not isotopic to \( a \) since \( a^m = b^2 \) and \( m \neq 1 \). In either case, the intersection circles are isotopic in \( K_\tau \) to a fiber on \( K_\tau \) (the images of the fibers of \( K_0 \) under the imbedding corresponding to \( \tau \)). Therefore we may deform the parameterized family near \( \tau \) so that each \( K_\tau \cap T_{j_q} \) consists of fibers on \( T_{j_q} \) that are images of fibers of \( K_0 \). Then, using lemma 6.2 successively on
the solid torus $R_u$, the product regions $\overline{R_{u_j} - R_{u_{j-1}}}$ for $j = j_1, j_2, \ldots, j_2$, and the twisted $I$-bundle $P_{u_1}$, deform $K_\tau$ to be fiber-preserving.

Inductively, suppose that $K_t$ is vertical for each $t$ lying in any $i$-simplex of $\Delta$. Let $\tau$ be an $(i + 1)$-simplex of $\Delta$, and let $j_1 < j_2 < \ldots < j_s$ be the values of $j$ for which $\tau$ lies in $B_j$. For each $t \in \partial \tau$, $K_t$ is vertical. By lemma 6.1 applied to each parameterized family $K_t \cap T_{j_q}$, we may assume that $K_t \cap T_{j_q}$ consists of fibers. Again using lemma 6.2 and proceeding from $R_{u_{js}}$ to $P_{u_1}$, deform the family on $\tau$, keeping it fixed over $\partial \tau$, to be vertical for all points in $\tau$. This completes the induction step and the proof of theorem 4.1.

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