Approximate recovery with locality and symmetry constraints

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Numerous quantum many-body systems are characterized by either fundamental or emergent constraints—such as gauge symmetries or parity superselection for fermions—which effectively limit the accessible observables and realizable operations. Moreover, these constraints combine non-trivially with the potential requirement that operations be performed locally. The combination of symmetry and locality constraints influence our ability to perform quantum error correction in two counterposing ways. On the one hand, they constrain the effect of noise, limiting its possible action over the quantum system. On the other hand, these constraints also limit our ability to perform quantum error correction, or generally to reverse the effect of a noisy quantum channel. We show that, as in the unconstrained setting, the optimal decoding fidelity can be expressed in terms of a dual fidelity characterizing the information available in the environment.

I. INTRODUCTION

A usual assumptions in quantum information theory literature, is that all self-adjoint operators on a Hilbert space can be in principle measured. However, in many systems of interest, such as fermions or gauge fields, accessible observables are limited to certain subalgebras. Fermions provide the simplest example for this; only observables commuting with fermion parity are considered physical. More generally, gauge theories, which describe the dynamics of elementary particles with remarkable accuracy and elegance, postulate that physical observables are limited to a certain subalgebra. The resulting effect is equivalent as long as the energy of the system is high in the presence of corresponding constraints.

Despite the ubiquitous nature of gauge symmetries, superselection rules and locality constraints, a general quantum information framework for studying the interplay of these constraints is in its infancy. The pivotal role that quantum information and particularly quantum error correction (QEC) is playing in recent developments of both condensed matter and high energy physics urgently demands the development of a solid framework. Significant progress has been made in interpreting entanglement in the presence of superselection rules and gauge constraints [3, 13]. Here we consider the information-disturbance tradeoff and its application to QEC, where it can be used to characterize which communication channels (representing noise) can be reversed on a given code. In particular, we will mention two areas where this will immediately prove useful.

In the condensed-matter side, a plethora of work has recently been dedicated to the classification of symmetry protected topological phases (SPTs) [9, 11] and symmetry enriched topological phase (SETs). These generalize the notion of topological order to a settings where a symmetry or gauge constraints are imposed. An SPT phase is a phase which would be trivial if the imposed symmetry were allowed to be broken yet become “disconnected” from the trivial (product) phase if the symmetry is imposed. In contrast, SETs are phases disconnected from the trivial phase even if the symmetry constraint is lifted. As the stability for topological order essentially requires QEC conditions [12, 13], it is only natural that this connection extend to the symmetry protected setting. Indeed, the seminal examples for an SPT phase is Kitaev’s Majorana chain [14], which is a gapped phase with topological degeneracy protected by fermionic parity conservation and geometric locality and is currently being pursued experimentally as a candidate qubit [15]. This first examples, as well as more recent constructions [16] are being explored as candidate systems for quantum information storage.

Within high energy physics, recent progress in holography provides the second natural arena for our results. In particular, the realization that the bulk/boundary mapping in holography presented properties of a QEC [17] has lead to vigorous debate with respect to the role of symmetries and gauge constraints. Whereas qualitative features have been reproduced in the context of traditional QEC theory [18, 21], it has been argued that gauge constraints play an essential role [22, 23]. In fact, the boundary theory in holography is a gauge quantum field theory and thus the validity QEC assumptions must be re-examined in the presence of corresponding constraints.

In a general approach to constrained systems, the allowed observables form a $\ast$-algebra $\mathcal{A}$, namely the set is closed under multiplication and the operation of taking the adjoint. Moreover, the observables local to a certain region of space form a $\ast$-subalgebra of $\mathcal{A}$ which is
not necessarily associated with a tensor factor of the full Hilbert-space.

Technically, if they are infinite-dimensional, these algebras require some additional mathematical structure, such as that of a C$^*$-algebra or von Neumann algebra, but here, for simplicity, we consider only algebras of finite-dimensional matrices, closed under the conjugate-transpose ($\dagger$-algebras) where those concepts are all equivalent. This is appropriate for systems of fermions on finite lattices, but will require some generalization to be applicable to lattice gauge theory to account for the fact that the Hilbert space of the gauge field on each edge is not finite-dimensional for Lie groups.

A natural starting point would be the operator-algebra quantum error correction (OAQEC) [24] (a synthesis of the theory of noiseless subsystems and subsystem codes [25][26]) because it provides sufficient conditions for a quantum channel to be reversible on a given code when one only cares about a given $\dagger$-algebra of observable. We instead consider a broad generalization of this approach to the approximate setting introduced in Ref. [27] (Theorem 1), based on techniques borrowed from Ref. [28] (information-disturbance tradeoff). This approach is a particular formalization of the general fact that quantum information can be recovered after the action of a channel if and only if it is not available in the environment (as characterized by the complementary channel).

In section II, we show that these results can be adapted to the case where the recovery map is required to be “physical”, in that it does not reveal information outside of the allowed observables (Corollary 4). Indeed, constraints on physical observables also affect channels as these should not enable the indirect measurement of unphysical observables. In addition, if we require that our channels be acting locally to some region of space, then they must leave unchanged all the observables acting outside that region.

In section IV we further extend them to a situation where the recovery map is required to be local, in that it fixes observables associated with a complementary region of space (Theorem 1 and Corollary 2). This requires a concept of local complementary map, defined in Section IV B.

II. PRELIMINARIES

In this section we review material required to present and illustrate our results. In particular section II A reviews the algebra for fermions which allows providing the simplest examples beyond tensor product Hilbert spaces with a genuine physical motivation. Section II B we propose a notion of physicality of a quantum channel as derived form the physicality from the algebras of observables. Finally, we review the main result of Ref. [27][29] and exemplify how they comprise traditional QEC conditions. The current work can be seen as a natural generalization of these result to a setting where symmetry constraints and locality are imposed on the channels involved.

A. Fermions

As an example of system with constraint, let us consider a system of spinless fermions on a lattice with set of vertices $\Omega = \{1, \ldots, N\}$. One associates to each site an annihilation operator $a_i$. These operators generate a minimal $\dagger$-algebra such that $a_ia_j + a_ja_i^\dagger = \delta_{ij}1$ and $a_ia_j + a_ja_i = 0$ for all $i, j$. Before considering locality, this algebra is isomorphic to that of all operators acting on a Hilbert space $H$ of dimension $2^N$ (which can be regarded as the Fock space corresponding to $N$ modes with a basis $(a_{i_{N}}^\dagger)^{i_{N}} \cdots (a_{i_1}^\dagger)^{i_1}|0\rangle$, where $n_1, \ldots, n_N \in \{0, 1\}$ count the “number of fermions” at each sites, and $|0\rangle$ is the Fock vacuum).

For any region of space corresponding to the subset of vertices $\omega \subseteq \Omega$, we want to interpret the $\dagger$-subalgebra generated by the operators $a_i$ for $i \in \omega$ as characterizing the observables local to $\omega$. These algebras, however, do not commute for disjoint subsets. In order to make sure that observables in disjoint regions are jointly measurable, and hence commute, we declare that only observables which are even order polynomials in the annihilation operators are physical.

This is equivalent to saying that the physical operators are those that commute with the parity observable $C$ which has eigenvalues 1 or $-1$ for states with an even or respectively an odd number of fermions, and hence is referred to as the parity superselection rule [4][50].

Hence, for every region of space $\omega \subseteq \Omega$, we assign a physical subalgebra $A_\omega$ of operators which are functions of the operators $a_i$, $i \in \omega$, and commute with $C$. Specifically, these algebras have the form $A_\omega \simeq M_1 \oplus M_{-1}$, where $M_{\pm 1}$ are full matrix algebras of dimensions $k$ by $k$ with $k = 2^{|\omega|-1}$. They correspond to all operators acting on the Hilbert space sectors with an even, respectively odd, number of fermions on region $\omega$. This is an instance of a local quantum theory as defined in algebraic quantum field theory (but simpler since space is discrete).

B. Physical channels

Let us consider a channel, also known as completely-positive and trace-preserving (CPTP) maps, $\mathcal{N} : B(H) \rightarrow B(K)$, where $H$ and $K$ are finite-dimensional Hilbert spaces, and $B(H)$ denotes the set of all operators on $H$. Also let $A$ and $B$ be $\dagger$-algebras of operators acting on $H$ and $K$ respectively, which represent the physical observables. Below, we always assume that these algebras contain the identity on their respective Hilbert spaces (in general, an algebra’s identity element could be a projector on the Hilbert space).

For $\mathcal{N}$ to be physical, it should be such that the recipient cannot gain information about unphysical ob-
servables. This is most easily expressed in the Heisen-
weg weaker condition \[34\].

In the literature, variations of these questions have been
whether the channel can be reversed on a single state
whether the effect of this channel can be reversed, i.e.,
all density matrices \(\rho\)
jector
channel sufficiency
In the presence of constraints, what we require is the
maxima are taken over all CPTP maps, and
where the maxima are taken over all CPTP maps, and
\(\tilde{\mathcal{N}}\) and \(\tilde{\mathcal{M}}\) are any channels complementary to \(\mathcal{N}\) and \(\mathcal{M}\)
respectively. This holds taking \(F\) to be either (a) the
entanglement fidelity \(F_{\rho}\) or (b) the worst-case entanglement fidelity \(F_W\).

Specifically, the two fidelity measures considered are
defined as follows.

(a) The **entanglement fidelity** compares the effect of
two channels on a single state, while accounting for
the possible loss of entanglement with a reference system:
\[ F_{\rho}(\mathcal{N}, \mathcal{M}) := f((\mathcal{N} \otimes \text{id})(\psi), (\mathcal{M} \otimes \text{id})(\psi)) \]  

(b) **Worst-case entanglement fidelity**: Alternatively,
channels can be compared on a code, that is, a
subspace \(\mathcal{H}_0\) of \(\mathcal{H}\) defined by a canonical isometry
\(W : \mathcal{H}_0 \to \mathcal{H}\), and can be characterized using
the worst-case entanglement fidelity
\[ F_W(\mathcal{N}, \mathcal{M}) := \min_{\rho} f((\mathcal{N}W \otimes \text{id})(\psi_{\rho}), (\mathcal{M}W \otimes \text{id})(\psi_{\rho})), \]  

where \(W(\rho) = W\rho W^\dagger\), and \(\psi_{\rho}\) is any purification
of \(\rho\).

Both channel fidelities can be used to construct
distances satisfying the triangle inequality, such as the
Bures distance. Note that if \(\rho \in \mathcal{B}(\mathcal{H}_0)\), then by definition
\(F_W \leq F_{\rho}\) as suggested by the names. Below, all we need
is the fact that both fidelities are monotonic under the
left action of any channel, i.e.,
\[ F(\mathcal{R}\mathcal{N}, \mathcal{R}\mathcal{M}) \geq F(\mathcal{N}, \mathcal{M}) \]  

for any channels \(\mathcal{R}, \mathcal{N}, \mathcal{M}\).

A complementary channel \(\tilde{\mathcal{N}}\) of \(\mathcal{N}\) can be built as follows.
The Stinespring dilation theorem states that there is an isometry
\(V : \mathcal{H} \to \mathcal{K} \otimes \mathcal{L}\) such that \(\mathcal{N}(\rho) = \text{Tr}_\mathcal{L} V\rho V^\dagger\),
where \(\text{Tr}_\mathcal{L}\) is the partial trace over \(\mathcal{L}\). Let \(|i\rangle\)
denote elements of a basis of \(\mathcal{L}\), then we obtain the Kraus
operators \(E_i = (1 \otimes |i\rangle\langle i|) V\). Reciprocally, \(V = \sum_i E_i \otimes |i\rangle\).
Any such dilation gives us a complementary channel
\(\tilde{\mathcal{N}}(\rho) = \text{Tr}_\mathcal{K} V\rho V^\dagger\). We also call a channel \(\tilde{\mathcal{N}}\)
complementary to \(\mathcal{N}\) if there exists channels \(\mathcal{R}\) and \(\mathcal{S}\)
such that \(\mathcal{N} = \mathcal{R}\tilde{\mathcal{N}}\) and \(\tilde{\mathcal{N}} = \mathcal{S}\tilde{\mathcal{N}}\),
where \(\tilde{\mathcal{N}}\) has the above form. (This is the equivalence relation defined in Ref. [29].)
In order to illustrate the use of theorem 1 for channel
reversal, we first present the setting of perfect recovery
in traditional subspace QEC.

**Example 2** (Subspace QEC). Consider the case when both fidelities
in Eq. (1) are maximal using \(F = F_W\), and
for \(\mathcal{M} = \text{id}\) (i.e., we wish \(\mathcal{R}\) to recover all information
initially available in the code defined by \(V\)). In this case,
we can use \(\tilde{\mathcal{M}} = \text{Tr}\). Hence the channel \(\mathcal{S}\) to be
optimized on the right hand side of Eq. (1) is just a state \(\sigma\),
since it is applied to the one-dimensional density matrix
\(1 : \mathcal{S}(1) = \sigma\). Eq. (1) means that \(\mathcal{N}\) is exactly correctable
on the code defined by \(W\) if and only if there exists a state
\(\sigma\) such that \(\tilde{\mathcal{N}}(W\rho W^\dagger) = \sigma \text{Tr}(\rho)\).

In terms of an explicit expression for \(\mathcal{N}\),
\[ \mathcal{N}(\rho) = \sum_i E_i \rho E_i^\dagger \]  
and
\[ \tilde{\mathcal{N}}(\rho) = \sum_{ij} \text{Tr}(\rho E_i^\dagger E_j)|i\rangle\langle j|, \]  
where \(\psi \equiv |\psi\rangle\langle \psi|\) denotes any purification of \(\rho\),
and \(f\) is the fidelity \(f(\rho, \sigma) = \text{Tr}(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})\). This quantity can also be used to bound the average
fidelity with respect to an ensemble averaging to \(\rho\)\[35\].
this means that for all \(i, j\), \(W^\dagger E_i E_j W = (i|j)W^\dagger\), which are the Knill-Laflamme conditions for quantum error correction [32].

**Example 3 (OAQEC).** When \(\mathcal{M} = \mathcal{P}_A\) is the projector on a \(\dagger\)-algebra \(A\): the condition from maximum fidelity yields that \(A\) is correctable on the code defined by the isometric encoding \(W(\rho) = WPW^\dagger\) if and only if

\[
\hat{N}W = SP_AW
\]

for some channel \(S\), where we used the fact that a channel complementary to \(\mathcal{P}_A\) is \(\hat{P}_A = \mathcal{P}_{\hat{A}}\): the projector on the commutant of \(\hat{A}\).

To recover the original formulation of operator algebra QEC (OAQEC) [24, 36], we use \(W = 1\), but replace \(N\) by \(N\hat{W}\) where now \(\hat{W}\) is the encoding isometry. We obtain that there is a channel \(R\) such that \(RN\hat{W} = \mathcal{P}_A\) if and only if there is a channel \(S\) such that \(N\hat{W} = SP_{\hat{A}}\). It is easy to see that we can then use \(S = N\hat{W}\). The resulting condition is that the range of \((N\hat{W})^\dagger\) be inside \(A'\). Expressed in terms of Kraus operators, this is the result of Ref. [24]. This also characterizes subsystem codes [26] when the algebra is a factor.

### III. REVERSAL ON CONSTRAINED SYSTEMS

In the present section, we address the question of channel reversal for constrained systems, and provide some instructive examples.

#### A. Reversal and constrained systems

In the presence of constraints, the problem with using the duality relation given by Eq. (1) is that the optimization on the left hand side is over channels \(R\) which may not be physical.

Recall that we defined \(\mathcal{P}\) and \(\mathcal{Q}\) as the channels projecting respectively on the source and target’s physical algebra. By substituting \(\mathcal{Q}\mathcal{N}\) for \(\mathcal{N}\) in Eq. (1), we obtain

\[
\max_{\mathcal{R}} F(\mathcal{RQ}\mathcal{N}, \mathcal{M}) = \max_S F(\mathcal{QN}, S\mathcal{M}).
\]

The recovery channel \(\mathcal{R}' = \mathcal{R}\mathcal{Q}\) is properly physical since \(\mathcal{P}\mathcal{R}'\mathcal{Q} = \mathcal{P}\mathcal{R}\mathcal{Q} = \mathcal{P}\mathcal{R}'\).

But does this correspond to the optimization over all physical recovery maps? Let us specialize this to the case where \(\mathcal{M} = \mathcal{P}\mathcal{M}\). For instance, this is the case if \(\mathcal{M}\) is the projector on any subalgebra of \(A\).

Suppose \(\mathcal{R}\) is any physical recovery channel. Then because of the contractivity of the Bures distance and the fact that \(\mathcal{P}^2 = \mathcal{P}\),

\[
F(\mathcal{PRQN}, \mathcal{PM}) = F(\mathcal{PRQN}, \mathcal{PM}) \geq F(\mathcal{RQN}, \mathcal{PM}).
\]

Therefore, if \(\mathcal{R}\) is any physical optimal recovery channel then so is \(\mathcal{R}' = \mathcal{PR}\mathcal{Q}\). We conclude that:

**Corollary 4.** For any physical channel \(\mathcal{N}\) from a system with physical algebra projector \(\mathcal{P}\) to one with projector \(\mathcal{Q}\), and any channel \(\mathcal{M}\).

\[
\max_{\mathcal{R}\text{ physical}} F(\mathcal{RN}, \mathcal{PM}) = \max_S F(\mathcal{QN}, S\mathcal{PM}).
\]

where the optimization on the left hand side is over channels \(R\) which are physical, i.e., such that \(\mathcal{PR}\mathcal{Q} = \mathcal{P}\) and the right hand side optimization over channels \(S\) is unconstrained.

If \(\mathcal{P}\) denotes the projector on \(\dagger\)-algebra \(A\), then \(\hat{\mathcal{P}}\) can be taken as the projector on the commutant \(A'\) [29, 37]. Moreover, \(\hat{\mathcal{P}}\mathcal{M}(\rho) = (\hat{\mathcal{P}} \otimes \text{id}_E)(V\rho V^\dagger)\) where \(V\) is the isometry from the Stinespring dilation of \(\mathcal{M}\), and \(E\) the environment, or ancilla for this distillation. The same can be done to obtain \(\hat{\mathcal{Q}}\).

Corollary 4 holds whether we replace \(F\) by \(F_W\) or \(F_P\). For instance, with \(F = F_W\) and \(\mathcal{M} = \text{id}\), the left-hand side of Eq. (10) is the worst-case fidelity of recovery for states within the code-space defined by \(W\).

In contrast, the entanglement fidelity \(F_e\) provides a lower bound [35] on how well recovery fares on average with respect to an ensemble represented by \(\rho\). This bound was invoked in Ref. [23], to evaluate the QEC properties of a thermal CFT ensemble as, in this setting, it would be much better behaved than \(F_W\) going to the setting of an infinite dimensional Hilbert space.

#### B. Example: exact reversal for commuting constraints

Let us consider a case where the physical algebra takes the form \(A = \mathcal{B}(\mathbb{C}^{n_1}) \oplus \cdots \oplus \mathcal{B}(\mathbb{C}^{n_d})\), with each superselection sector characterized by a projector \(P_i\) of rank \(n_i\). There is a corresponding “charge” observable \(C = \sum_i c_i P_i\), \(c_i \neq c_j\). For instance, for a system of fermions, \(C\) would be the parity observable. Alternatively, this algebra may arise from requiring that observables commute with self-adjoint operators \(L_i\) which all commute with each other, such as in an Abelian gauge theory.

The projector \(\mathcal{P}\) on \(A\) represents a “blind measurement” of \(C\):

\[
\mathcal{P}\rho = \sum_i P_i \rho P_i.
\]

A Stinespring dilation of \(\mathcal{P}\) is given by the isometry \(\sum_i P_i \otimes |i\rangle\) where the extra system records the measurement outcome. It follows that

\[
\hat{\mathcal{P}}\rho = \sum_i \text{Tr}(P_i \rho)|i\rangle\langle i|.
\]

This is the quantum-to-classical channel characterizing the measurement of \(C\). Hence, the map \(S\) in Eq. (10) prepares a quantum state depending on the classical outcome \(i\) of the global charge measurement.
Let us take \( Q = \mathcal{P}, \mathcal{N} \) as in \((5)\) and \( M = \mathcal{P} \) in Eq. \((10)\), meaning that we wish to recover all the physical information. Using the dilation isometry \( V = \sum_i V_i \otimes |i\rangle \), we have

\[
\widehat{\mathcal{N}}(\rho) = (\widehat{\mathcal{P}} \otimes \text{id})(V \rho V^\dagger) = \sum_{jnm} \text{Tr}(P_j E_m \rho E_n^\dagger P_j) |n\rangle \langle m| \otimes |j\rangle \langle j|.
\]

(13)

In this example the map \( S \) is of the form

\[
S(|i\rangle \langle i|) = \sum_j \sigma_{ji} \otimes |j\rangle \langle j|
\]

(14)

where \( \sigma_{ji} \geq 0 \) and \( \sum_j \text{Tr} \sigma_{ji} = 1 \).

Let us consider the implication of Eq. \((10)\) for exact reversal of \(\mathcal{N}\) on a subspace defined by the isometry \( W \). For exact reversal, \( F_W \) equals to 1 exactly when \( F_\rho \) equals 1, provided \( \rho \) has full rank on the code space defined by \( W \).

Here both \( \mathcal{R} \) and \( \mathcal{N} \) act on \( \mathcal{H} \) and the dilation \( \psi \) of \( \rho \) is defined on an extended Hilbert space \( \mathcal{H} \otimes \mathcal{J} \). The channel is exactly correctable in this case when

\[
(\widehat{\mathcal{P}} \mathcal{N} \otimes \text{id})(\psi) = (S \mathcal{P} \otimes \text{id})(\psi).
\]

(15)

Or in other words, for all \( j, n, m \),

\[
W^\dagger E_n^\dagger P_j E_m W = \sum_i (n|\sigma_{ji}|m) W^\dagger P_i W.
\]

(16)

This condition can be reformulated as follows:

**Corollary 5.** A necessary condition for the channel \( \mathcal{N} \) to be correctable on the code with isometry \( V \), for a system with superselection charge \( C = \sum_i c_i P_i \), is that there exist complex numbers \( c_{ijnm} \) such that

\[
W^\dagger E_n^\dagger P_j E_m W = \sum_i c_{ijnm} W^\dagger P_i W.
\]

(17)

This necessary condition also becomes sufficient if \( [C, WW^\dagger] = 0 \), i.e., if the change observable \( C \) commutes with the code projector \( WW^\dagger \). Indeed, this is equivalent to \( P_i W W^\dagger P_j = 0 \) for \( i \neq j \). In other words, it is sufficient if all states in the code respect the superselection criterion, which is natural in this context.

**IV. LOCAL REVERSIBILITY**

In Refs. \([38, 39]\), the unconstrained dual optimization relation Eq. \((1)\) for \( \mathcal{M} = \text{id} \) was adapted to characterize local recoverability, i.e., with the constraint that the recovery channel be local to a subsystem. This was later applied \([39]\), to study local recoverability in the setting of an isometry \( W \) defined by a MERA circuit \([40]\). In this section, we generalize the notion of local recovery to a setting where the physical local algebra can be arbitrary \( \dagger \)-algebras extending the information-disturbance approach of Refs. \([27, 29]\).

**A. Local channels**

Let us consider a local net of algebra, i.e., a map assigning each region of space \( \omega \) to a \( \dagger \)-subalgebra \( \mathcal{A}_\omega \) such that \( \mathcal{A}_\omega \subseteq \mathcal{A}_\omega' \) whenever \( \omega \subseteq \omega' \), and \( \mathcal{A}_\omega \) and \( \mathcal{A}_\omega' \) commute whenever \( \omega \cap \omega' = \emptyset \). Also we assume that \( 1 \in \mathcal{A}_\omega \) for all \( \omega \).

A channel \( \mathcal{N} \) local to \( \omega \) should be such that \( \mathcal{N}^\dagger(\mathcal{A}_\omega) \subseteq \mathcal{A}_\omega \), so that an observer with access to \( \omega \) cannot learn about observables which are not local to \( \omega \) due to the action of the channel. In addition, \( \mathcal{N}^\dagger \) should fix the observables \( \mathcal{A}_\omega \) local to the complement set \( \omega^c \), so that an observer having access to \( \omega^c \) cannot learn that anything happened.

One may also require that additional observables be fixed, up to the full commutant of \( \mathcal{A}_\omega \) (which is not in general equal to \( \mathcal{A}_\omega^c \), as can be seen for fermions). We want to be agnostic towards such choices. Hence we define locality generally as follows:

**Definition 2.** We say that channel \( \mathcal{N} \) (from \( \mathcal{H} \) to itself) is local to an algebra \( \mathcal{A} \), with effective complement \( B \subseteq \mathcal{A}^c \) if \( \mathcal{N}^\dagger(\mathcal{A}) \subseteq \mathcal{A} \) and \( \mathcal{N}^\dagger(B) = B \) for all \( B \in \mathcal{B} \). If \( \mathcal{B} = \mathcal{A}^c \) (commutant of \( \mathcal{A} \)), we say that \( \mathcal{N} \) is strongly local.

If \( \mathcal{A} = \mathcal{A}_\omega \), the effective complement \( B \) is to be distinguished from the algebra \( \mathcal{A}_\omega^c \) on the complementary region. We may set \( \mathcal{B} = \mathcal{A}_\omega^c \) in the above definition, in which case we would say that the channel is weakly local.

In order to better understand the implications of this definition, we need the following known fact:

**Proposition 6.** If a channel \( \mathcal{N}(\rho) = \sum_i E_i \rho E_i^\dagger \) fixes a \( \dagger \)-algebra \( \mathcal{B} \) (i.e., \( \mathcal{N}^\dagger(B) = B \) for all \( B \in \mathcal{B} \)) then \([E_i, B] = 0\) for all \( B \in \mathcal{B} \) and all \( i \).

**Proof.** Given that any finite-dimensional \( \dagger \)-algebra is spanned by its projectors, we only need to show this for projectors \( P \) in \( \mathcal{B} \). We have \( \sum_i E_i P E_i^\dagger = P \). Multiplying this equation on both side by \( P^\perp = 1 - P \) we get \( \sum_i P^\perp E_i P E_i^\dagger = 0 \). Taking the expectation value with respect to any vector, we can deduce that for all \( i \), \( P E_i P = 0 \). Similarly, since \( P^\perp \) is in \( \mathcal{B} \), we have \( P^\perp E_i P = 0 \). Together this implies \( PE_i = E_i P \).

This implies that, requiring a channel to be local to \( \mathcal{A} \) with maximal complement \( \mathcal{A}' \) is equivalent to asking for its Kraus operators to lie in \( \mathcal{A} = \mathcal{A}'^c \) (via the double-commutant theorem).

Furthermore, we also have that for \( \mathcal{N} \) local to \( \mathcal{A} \) with complement \( \mathcal{B} \) then

\[
\mathcal{N}^\dagger(AB) = \mathcal{N}^\dagger(A)B
\]

(18)

for all \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \).

This also allows us to characterize the local channels in a way that will be useful below. Let \( \mathcal{P}_{W'} \) be the projector on the commutant \( \mathcal{B}' \). We can implement it as an integral
over that Haar measure in the group of unitary operators within $\mathcal{B}$:

$$\mathcal{P}_{\mathcal{B}'}(X) = \int_{U \in \mathcal{B}} UXU^\dagger dU, \quad (19)$$

or using a unitary 1-design $U_i \in \mathcal{B}$, $i = 1, \ldots, n$:

$$\mathcal{P}_{\mathcal{B}'}(X) = \frac{1}{n} \sum_i U_i XU_i^\dagger. \quad (20)$$

Hence, we have the Stinespring dilation

$$\mathcal{P}_{\mathcal{B}'}(X) = V_B^\dagger (1 \otimes X)V_B, \quad (21)$$

with

$$V_B = \frac{1}{\sqrt{n}} \sum_i |i\rangle \otimes U_i. \quad (22)$$

Suppose $\mathcal{N}(\rho) = \sum_i E_i \rho E_i^\dagger$, and hence has a dilation

$$\mathcal{V}_N = \sum_i E_i \otimes |i\rangle; \quad \mathcal{N}^\dagger(X) = V_N^\dagger (X \otimes 1)V_N.$$ If $\mathcal{N}^\dagger$ fixes $\mathcal{B}$, then since $[E_i, U_j] = 0$, we obtain

$$(V_B \otimes 1)V_N = (1 \otimes V_N)V_B, \quad (23)$$

because, when extended, both sides are proportional to

$$\sum_{ij} |i\rangle \otimes U_i E_j \otimes |j\rangle = \sum_{ij} |i\rangle \otimes E_j U_i \otimes |j\rangle. \quad (24)$$

Tracing-out the third tensor factor on both sides of the equation yields

$$\mathcal{V}_B \circ \mathcal{N} = (\text{id} \otimes \mathcal{N}) \circ \mathcal{V}_B, \quad (25)$$

where we used $\mathcal{V}_B(\rho) = V_B \rho V_B^\dagger$. Similarly, tracing-out the first tensor factor yields

$$(\mathcal{P}_{\mathcal{B}'} \otimes \text{id}) \circ \mathcal{V}_N = (\text{id} \otimes \mathcal{N}) \circ \mathcal{V}_B. \quad (26)$$

B. Local complementary channels

In order to generalize Theorem 1, we need a notion of local complementary channel.

Definition 3. Let $\mathcal{N}$ be a channel local to $\mathcal{A}$ with complement $\mathcal{B}$. Given a Stinespring dilation $\mathcal{N}^\dagger(X) = V^\dagger(X \otimes 1_E)V$, where $V$ is an isometry: $V^\dagger V = 1$, We define a corresponding local complementary channel $\mathcal{N}^\dagger_{\mathcal{B}}$ by

$$\mathcal{N}^\dagger_{\mathcal{B}}(B \otimes E) = V^\dagger (\mathcal{P}_B(B) \otimes E)V. \quad (27)$$

This definition will be justified by the fact that such local complementary channel appears naturally in Theorem 7, where it plays the same role as the normal complementary channel in Theorem 1.

We can, nevertheless motivate it intuitively as follows. Consider a channel from Alice to Bob. The complementary channel represents all the information that can possibly be recovered by a third party—the “environment”—simultaneously to Bob receiving his information. If, however, Bob cannot access the subsystem defined by the algebra $\mathcal{B}$ in the output of the channel (because he is a local observer), then that system should also be counted as part of the environment.

A local complementary channel can be expressed in terms of a standard complementary channel through

$$\mathcal{\hat{N}}^\dagger_{\mathcal{B}} = \mathcal{P}_{\mathcal{B}'} \circ \mathcal{N}, \quad (28)$$

where $\mathcal{P}_{\mathcal{B}'}$ denotes the projector on the commutant $\mathcal{B}'$ of the algebra $\mathcal{B}$. This follows from the fact that $\mathcal{P}_{\mathcal{B}'} \mathcal{N}^\dagger(X \otimes E) = V^\dagger (\mathcal{P}_{\mathcal{B}'}(X) \otimes E)V$, where $V$ is an isometry such that $\mathcal{N}^\dagger(Y) = V^\dagger (Y \otimes 1)V$.

C. Condition for local reversibility

We now have the tools needed to generalize Theorem 1 to local channels. Let us first consider only the constraint that a local channel must fix some algebra. This is also equivalent to considering only strong locality, since if $\mathcal{N}^\dagger$ fixes $\mathcal{B}$, then it must also map $\mathcal{B}'$ into $\mathcal{B}'$ (since its Kraus operators must then all belong to $\mathcal{B}'$).

Theorem 7. Let $\mathcal{N}$ and $\mathcal{M}$ be two channels such that both $\mathcal{N}^\dagger$ and $\mathcal{M}^\dagger$ fix the algebra $\mathcal{B}$. Then

$$\max_{\mathcal{R}^\dagger \text{ fixes } \mathcal{B}} F(\mathcal{R}\mathcal{N}, \mathcal{M}) = \max_{\mathcal{S}^\dagger \text{ fixes } \mathcal{B}} F(\mathcal{N}^\dagger_{\mathcal{B}}, \mathcal{S}\mathcal{M}^\dagger_{\mathcal{B}}). \quad (29)$$

Proof. Let $\mathcal{V}_B$ be an isometry dilating the projector on $\mathcal{B}'$ as in Section IV A. Using the fact that the fidelity is invariant under post-processing by an isometry, and then Eq. (25),

$$F(\mathcal{R}\mathcal{N}, \mathcal{M}) = F(\mathcal{V}_B \mathcal{R}\mathcal{N}, \mathcal{V}_B \mathcal{M}) = F(\text{id} \otimes \mathcal{R}\mathcal{N}) \mathcal{V}_B, (\text{id} \otimes \mathcal{M}) \mathcal{V}_B). \quad (30)$$

If we use the entanglement fidelity $F_\rho$, then this last term is just

$$F_{\mathcal{P}_{\mathcal{B}'}(\rho)}(\mathcal{R}\mathcal{N}, \mathcal{M}). \quad (31)$$

We can then apply Theorem 1 to this quantity, to obtain

$$\max_\mathcal{R} F_{\mathcal{P}_{\mathcal{B}'}(\rho)}(\mathcal{R}\mathcal{N}, \mathcal{M}) = \max_\mathcal{S} F_{\mathcal{P}_{\mathcal{B}'}(\rho)}(\mathcal{\hat{N}}, \mathcal{S}\mathcal{M}). \quad (32)$$

This also works for the worst-case fidelity $F_W$, but we need a cosmetically stronger version of Theorem 1 that would hold when minimizing over states of the form $\mathcal{P}_{\mathcal{B}'}(\rho)$ rather than all states. But since this set is also convex, the proof in Ref. 27 (or Ref. 29 with more details) works unchanged.
Using Eq. [26], we obtain, both for $F$ replaced by $F_p$ or $F_W$, 
\[
F((\text{id} \otimes \mathcal{R} \mathcal{N}) \mathcal{V}_B, (\text{id} \otimes \mathcal{M}) \mathcal{V}_B) = F((\text{id} \otimes \mathcal{N}) \mathcal{V}_B, (\text{id} \otimes \mathcal{S} \mathcal{M}) \mathcal{V}_B) = F((\mathcal{P}_B \otimes \text{id}) \mathcal{V}_N, (\mathcal{P}_B' \otimes \mathcal{S}) \mathcal{V}_M) 
\]
(33)
\[
= F((\mathcal{P}_B \otimes \text{id}) \mathcal{V}_N, (\mathcal{P}_B \otimes \mathcal{S}) \mathcal{V}_M) = F(\mathcal{N}^{\mathcal{B}}, (\text{id} \otimes \mathcal{S}) \mathcal{N}^{\mathcal{B}}).
\]
In the second-to-last step, we used the fact that $\mathcal{P}_B$ is complementary to $\mathcal{P}_B'$, and hence is equivalent to $\mathcal{P}_C$ to a reversible post-processing, which cannot change the value of the fidelity. 

We can now combine this with the approach of Section III to obtain a dual condition for the local correctability of a channel. Where locality is defined with respect to an algebra $A$ with commuting complement $B$.

Corollary 8. Let $\mathcal{P}$ be the projector on $A \cup B$ (the algebra generated by $A \cup B$) and $\mathcal{N}$ a channel local to $A$ with complement $B$, then 
\[
\max_{\mathcal{R} \text{ local}} F(\mathcal{R} \mathcal{N}, \mathcal{P}) = \max_{\mathcal{S}^{|i| \text{ fixes } B}} F(\mathcal{P} \mathcal{R} \mathcal{N}^{\mathcal{B}}, \mathcal{S} \mathcal{P}^{\mathcal{B}}),
\]
(34)
where the maximization on the left-hand side is over channels $\mathcal{R}$ which are local in the same sense as for $\mathcal{N}$.

Proof. From Theorem 7, the right hand side of Eq. (34) is equal to 
\[
\max_{\mathcal{R} \text{ local}} F(\mathcal{R} \mathcal{P} \mathcal{R} \mathcal{N}, \mathcal{P}) = F(\mathcal{P} \mathcal{R} \mathcal{P} \mathcal{P} \mathcal{N}, \mathcal{P}) \geq F(\mathcal{R} \mathcal{N}, \mathcal{P}).
\]
But the set of channels $\mathcal{R} \mathcal{P}$ where $\mathcal{R}$ reaches the maximum in this expression must include some which are local, since by monotonicity of the fidelity, the local map $\mathcal{P} \mathcal{R} \mathcal{P}$ can only perform better than $\mathcal{R} \mathcal{P}$. Also, if $\mathcal{R}$ is a local channel which maximizes $F(\mathcal{R} \mathcal{N}, \mathcal{P})$, then does $\mathcal{P} \mathcal{R} \mathcal{P}$ since then 
\[
F(\mathcal{P} \mathcal{R} \mathcal{P} \mathcal{P} \mathcal{N}, \mathcal{P}) = F(\mathcal{P} \mathcal{R} \mathcal{N}, \mathcal{P}) \geq F(\mathcal{R} \mathcal{N}, \mathcal{P}).
\]
We observe that $\mathcal{P} \mathcal{N}^{\mathcal{B}} = \mathcal{P}_C \mathcal{N}$ and we can use $\mathcal{P} \mathcal{N}^{\mathcal{B}} = \mathcal{P} \mathcal{B}$, where $B_c = B' \cap (A \cup B)$ is the commutant of $B$ relative to $A \cup B$.

We can also be more explicit about the structure of these algebras. Since $A$ and $B$ commute with each other, the intersection 
\[
\mathcal{I} := A \cap B
\]
(37)
is a commutative $\dagger$-algebra. Hence it is spanned by a complete family of projectors $P_i$, $i = 1, \ldots, n$, such that $P_i P_j = \delta_{ij} P_i$ and $\sum_i P_i = 1$.

The algebras $A$ and $B$ must be block-diagonal in terms of the sectors $H_i = P_i H$, $i = 1, \ldots, N$, and, since they commute, they must take the form 
\[
\mathcal{A} = \bigoplus_{i=1}^N A_i \otimes 1_{m_i} \quad \text{and} \quad \mathcal{B} = \bigoplus_{i=1}^N 1_{n_i} \otimes B_i
\]
(38)
where the $i$th term of the direct sum is supported on the sector $H_i$, and $A_i$ and $B_i$ are themselves $\dagger$-algebras.

Then the relative commutant of $B$ in $A \cup B$ has the form 
\[
B_c = \bigoplus_i A_i \otimes (Z(B_i) - 1),
\]
(39)
where $Z(B_i) = B_i \cap B_i'$ is a commutative algebra: the center of $B_i$.

D. Example: standard tensor product

To see how this works, let us first consider the meaning of Eq. (34) being equal to one (maximal) for channels which are local in the usual sense of a tensor product of Hilbert space, say system $H_A \otimes H_B$. The algebras are $A = B(H_A) \otimes 1$ and $B = 1 \otimes B(H_B)$. The channel $N$ being local to system $A$ implies $N = N_A \otimes \text{id}$.

The problem considered here is whether the specific noise channel $N$ can be corrected by a channel acting within $A$. This is not to be confused with the more standard problem of recovering arbitrary noise channels on $A$ without restricting the locality of the recovery operation.

Let us define $D_{\sigma}$ to be a fully depolarizing channel, with constant output state $\sigma$: $D_{\sigma}(\rho) = \sigma \text{ Tr } \rho$ for all $\rho$.

We observe that $B^c = A = B(H_A) \otimes 1$ and $(B^c)^c = B = 1 \otimes B(H_A)$. Hence 
\[
\mathcal{P}_{B^c} = \text{id} \otimes D_{1/d_B} \quad \text{and} \quad \mathcal{P}_{(B^c)^c} = \mathcal{P}_{(B^c)^c} = \mathcal{P}_{1/d_A} \otimes \text{id},
\]
(40)
where $d_A$ and $d_B$ are the dimensions of $H_A$ and $H_B$ respectively. It follows that 
\[
\mathcal{P} \mathcal{N}^{\mathcal{B}} = \mathcal{P} \mathcal{B} \mathcal{N} = N_A \otimes D_{1/d_B} = \mathcal{N}_A \otimes \text{id}.
\]
(41)
Moreover, for any channel $S$, $S \circ D_{1/d_A} = D_{\sigma}$, where $\sigma = S(1/d_A)$. Hence, Corollary 8 tells us that 
\[
\max_{\mathcal{R}} F(\mathcal{R} N_A \otimes \text{id}_B, \text{id}_{AB}) = \max_{\mathcal{R}} F(\mathcal{N}_A \otimes \text{id}_B, D_{\sigma} \otimes \text{id}_B).
\]
(42)
This is almost exactly like the QEC conditions without the locality constraint: the channel is reversible if and only if the local environment gets no information. The difference is that the code is defined as a subspace of the joint system $AB$ instead of just system $A$.

To understand this in more detail, let us expand the conditions resulting for the fidelity $F = F_W$ being maximal on both sides. If we write this condition in terms of the Kraus operators $E_i$ of $N_A$, we obtain that the condition is that there exists a state $\sigma$ such that for all operators $E \otimes B$, 
\[
\sum_{ij} W_i^\dagger (E_i^\dagger E_j \otimes B) W_j \langle i | E_j | j \rangle = W_i^\dagger (1 \otimes B) W \text{ Tr } (\sigma E).
\]
(43)
Similarly, \( \tilde{\omega} \) parity \( C \) is the global parity observable.

The commutant is just

\[
W_m^\dagger E_j W_n = \lambda_{ij} W_m^\dagger W_n,
\]

where \( W_n := (1 \otimes \langle n|)W \).

E. Example: fermions and strong locality

Let us unpack Corollary 8 for fermions, and with the strong locality requirement \( \overline{\mathcal{B}} = \mathcal{A}' \).

Firstly, if \( \mathcal{B} = \mathcal{A}' \), then \( \mathcal{B}^c = \mathcal{A} \), and

\[
\overline{\mathcal{P}N}^B = \overline{\mathcal{P}\mathcal{N}A}.
\]

Similarly, \( \overline{\mathcal{P}B} = \overline{\mathcal{P}\mathcal{A}'} \).

Let \( \mathcal{A} = \mathcal{A}_\omega \) be a local algebra for fermions for a region \( \omega \). A charge generating the center \( \mathcal{I} = \mathcal{A} \cap \mathcal{A}' \) is the parity \( C_\omega \) of the number of fermions in the region \( \omega \). The commutant is just \( \mathcal{A}' = \mathcal{I} \vee \mathcal{A}_\omega^c = \text{span}\{C, \mathcal{A}_\omega^c\} \), where \( C \) is the global parity observable.

Corollary 8 tells us that the optimal fidelity for local reversal of \( \mathcal{N} \) is equal to

\[
\max S F(\overline{\mathcal{P}\mathcal{A}'}, \mathcal{S}\mathcal{P}\mathcal{A}') \text{ where } \mathcal{S}' \text{ must fix } \mathcal{A}'.
\]

But the channel \( \mathcal{S} \) needs only be defined on \( \mathcal{A}' \) since it acts after a projection on it. Since its adjoint must fix \( \mathcal{A}' \), it can be assumed to be of the form

\[
\mathcal{S}'(B \otimes E) = \sum_i \text{Tr} (\rho_i E) P_i BP_i,
\]

for any \( B \in \mathcal{A}' \) and \( E \) acting on the environment from the dilation of \( \mathcal{N} \). The only freedom are the two arbitrary fixed states \( \rho_i, i \in \{-1, 1\} \). Here, \( P_1 \) and \( P_{-1} \) are the projectors on even and odd parity for the region \( \omega \).

If \( E_i \) are Kraus operators for \( \mathcal{N} \), the code \( W \) is then exactly locally correctable if and only if there exist two states \( \rho_{\pm 1} \) such that for all \( B \in \mathcal{A}' \), and all \( E \),

\[
\sum_{ij} \langle i|E|j \rangle W_m^\dagger E_j B E_j W = W_m^\dagger \mathcal{S}'(B \otimes E) W
\]

\[
= \sum_k \text{Tr}(\rho_k E) W_m^\dagger P_k B P_k W,
\]

or, equivalently, for all \( i, j \), any parity \( k \), and for all \( B \in \mathcal{A}' \), there exist \( \lambda_{ijk} \in \mathbb{C} \) such that

\[
W_m^\dagger E_j B_k W = \lambda_{ijk} W_m^\dagger B_k W,
\]

where \( B_k = P_k BP_k \) commute with the operators \( E_i \).

The only difference with Eq. (44) is that \( B \) is restricted to the algebra \( \mathcal{A}' \) which is a direct sum of two factors, corresponding to the two values for the parity of the region \( \omega \).

For instance, if the code defined by \( W \) is restricted to states of a fixed parity, then the conditions are of the same form as in the Section IV D.

We see that these conditions do not reduce to those of Section III B when \( \omega \) is the whole system. This is because the strong locality condition is non-trivial in this limit, as the channel is still required to fix the global parity operator, which commutes with \( \mathcal{A}_0 \). This is what makes the above conditions simpler.

V. CONCLUSIONS

We have extended the result of Refs. [27,29] on approximate channel recovery to settings where the channels are restricted by symmetry and/or locality constraints. The results obtained preserve the elegant duality structure of the information disturbance trade-off. Namely, they take the form of a dual optimization representing the information which is not accessible to the environment. Although we do not have a general solution for the dual optimization, it is often simpler, and has a solution in important specific situations. As the duality itself has already proven useful conceptually [17,19,28,38,41], we expect that our results will be widely applicable to the QEC aspects of symmetry protected topological phases as well as in some more realistic realizations of holography.

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