COTANGENT BUNDLES WITH GENERAL NATURAL KÄHLER STRUCTURES

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We give the conditions under which an almost Hermitian structure \((G, J)\) of general natural lift type on the cotangent bundle \(T^*M\) of a Riemannian manifold \((M, g)\), is Kählerian. First, we obtain the algebraic conditions under which the manifold \((T^*M, G, J)\) is almost Hermitian. Next we get the integrability conditions for the almost complex structure \(J\), then the conditions under which the associated 2-form is closed. The manifold \((T^*M, G, J)\) is Kählerian if it is almost Kählerian and the almost complex structure \(J\) is integrable. It follows that the family of Kählerian structures of the above type on \(T^*M\) depends on three essential parameters (one is a certain proportionality factor, the other two are parameters involved in the definition of \(J\)).

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1. INTRODUCTION

The fundamental differences between the geometry of the cotangent bundle \(T^*M\) and that of the tangent bundle \(T M\) of a Riemannian manifold \((M, g)\), are due to the different construction of lifts to \(T^*M\), which cannot be defined just as in the case of \(T M\) (see [14]).

The possibility to consider vertical, complete and horizontal lifts on \(T^*M\) leads to interesting geometric structures that were studied in the last years (see [5], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18]). Due to the duality tangent bundle-cotangent bundle, some of these results are very much similar to previous results for the tangent bundles.

In the present paper we study the conditions under which a structure \((G, J)\) of general natural lift type on the cotangent bundle \(T^*M\) of a Riemannian manifold \(M\), is a Kählerian structure. The similar problem for the tangent bundle was treated by Oproiu [6] (see also [7] for the diagonal case). Part of the results from the papers [12], [15], [16], can be obtained as particular cases from those in the present paper.

First, we consider a tensor field \(J\) of type \((1, 1)\) on \(T^*M\) which is a general natural lift of the Riemannian metric \(g\). The condition on \(J\) to define an almost
complex structure on $T^*M$ leads to certain algebraic relations between the parameters involved in its definition. Four out of the eight parameters involved in the definition of $J$ may be obtained as (rational) functions of the other four parameters. The integrability condition for the almost complex structure $J$ implies that the base manifold must have constant sectional curvature. Then some other relations fulfilled by the parameters and their derivatives are obtained, so that the number of the essential parameters involved in the definition of the integrable almost complex structure is 2.

We next study the conditions under which a Riemannian metric $G$ which is a general natural lift of $g$, is Hermitian with respect to $J$. We get that the structure $(G, J)$ on $T^*M$ is almost Hermitian if and only if the coefficients $c_1, c_2, c_3$ involved in the definition of the metric $G$ are proportional to the coefficients $a_1, a_2, a_3$ involved in the definition of the almost complex structure $J$, with proportionality factor $\lambda$, and the combinations $c_1 + 2t d_1$, $c_2 + 2t d_2$, $c_3 + 2t d_3$ are proportional to $a_1 + 2t b_1$, $a_2 + 2t b_2$, $a_3 + 2t b_3$, with proportionality factor of the form $\lambda + 2t \mu$. The main result is that the structure $(G, J)$ is Kählerian if and only if $\mu = \lambda'$ and the coefficients $b_1, b_2, b_3$ of the almost complex structure $J$ can be expressed as certain rational functions of $a_1, a_2, a_3$ and their derivatives. The condition for $(G, J)$ to be Kählerian is obtained from the property for $(G, J)$ to be almost Kählerian and the integrability property for $J$.

Some quite long computations have been done by using the MATHEMATICA package RICCI for tensor calculations.

The manifolds, tensor fields and other geometric objects we consider in this paper are assumed to be differentiable of class $C^\infty$ (i.e., smooth). We use the computations in local coordinates in a fixed local chart but many results may be expressed in an invariant form by using the vertical and horizontal lifts. The well known summation convention is used throughout this paper, the range of the superscripts and subscripts $h, i, j, k, l, m, r$ being always $\{1, \ldots, n\}$.

2. PRELIMINARY RESULTS

Let $(M, g)$ be a smooth $n$-dimensional Riemannian manifold and denote its cotangent bundle by $\pi : T^*M \to M$. Recall that there is a structure of a $2n$-dimensional smooth manifold on $T^*M$, induced from the structure of smooth $n$-dimensional manifold of $M$. From every local chart $(U, \varphi) = (U, x^1, \ldots, x^n)$ on $M$, a local chart $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \ldots, q^n, p_1, \ldots, p_n)$ is induced on $T^*M$ as follows. For a cotangent vector $p \in \pi^{-1}(U) \subset T^*M$, the first $n$ local coordinates $q^1, \ldots, q^n$ are the local coordinates of its base point $x = \pi(p)$ in the local chart $(U, \varphi)$ (in fact, we have $q^i = \pi^*x^i = x^i \circ \pi$, $i = 1, \ldots, n$).
The last $n$ local coordinates $p_1, \ldots, p_n$ of $p \in \pi^{-1}(U)$ are the vector space coordinates of $p$ with respect to the natural basis $(dx_{\pi(p)}^1, \ldots, dx_{\pi(p)}^n)$, defined by the local chart $(U, \varphi)$, i.e. $p = p_idx_{\pi(p)}^i$.

An $M$-tensor field of type $(r, s)$ on $T^*M$ is defined by sets of $n^{r+s}$ components (functions depending on $q^i$ and $p_i$), with $r$ superscripts and $s$ subscripts assigned to induced local charts $(\pi^{-1}(U), \Phi)$ on $T^*M$, such that the local coordinate change rule is that of the local coordinate components of a tensor field of type $(r, s)$ on the base manifold $M$ (see [4] for details in the case of the tangent bundle). A usual tensor field of type $(r, s)$ on $M$ may be thought as an $M$-tensor field of type $(r, s)$ on $T^*M$. If the considered tensor field on $M$ is covariant only, the corresponding $M$-tensor field on $T^*M$ may be identified with the induced (pulledback by $\pi$) tensor field on $T^*M$.

Some useful $M$-tensor fields on $T^*M$ can be obtained as follows. Let $v, w : [0, \infty) \rightarrow \mathbb{R}$ be smooth functions and let $\|p\|^2 = g_{\pi(p)}^{-1}(p, p)$ be the square of the norm of the cotangent vector $p \in \pi^{-1}(U)$ ($g^{-1}$ is the tensor field of type $(2,0)$ with components $(g^{ij}(x))$, which are the entries of the inverse of the matrix $(g_{ij}(x))$ defined by the components of $g$ in the local chart $(U, \varphi)$). The components $v g_{ij}(\pi(p))$, $w(\|p\|^2)p_i p_j$ define $M$-tensor fields of types $(0,2)$, $(1,1)$, $(0, 2)$, respectively, on $T^*M$. Similarly, the components $v g^{kl}(\pi(p))$, $g^{0i} = p_i g^{hi}$, $w(\|p\|^2)g^{0i}g^{hi}$ define $M$-tensor fields of type $(2,0)$, $(1,0)$, $(2,0)$, respectively, on $T^*M$. Of course, all the above components are considered in the induced local chart $(\pi^{-1}(U), \Phi)$.

The Levi Civita connection $\tilde{\nabla}$ of $g$ defines a direct sum decomposition

$$(2.1) \quad TT^*M = VT^*M \oplus HT^*M.$$ 

of the tangent bundle to $T^*M$ into vertical distributions $VT^*M = \text{Ker } \pi_*$ and the horizontal distribution $HT^*M$.

If $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \ldots, q^n, p_1, \ldots, p_n)$ is a local chart on $T^*M$ induced from the local chart $(U, \varphi) = (U, x^1, \ldots, x^n)$, then the local vector fields $\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}$ on $\pi^{-1}(U)$ define a local frame for $VT^*M$ over $\pi^{-1}(U)$ and the local vector fields $\frac{\delta}{\delta q^1}, \ldots, \frac{\delta}{\delta q^n}$ define a local frame for $HT^*M$ over $\pi^{-1}(U)$, where

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} + \Gamma^0_{ih} \frac{\partial}{\partial p_h}, \quad \Gamma^0_{ih} = p_k \Gamma^k_{ih},$$

and $\Gamma^k_{ih}(\pi(p))$ are the Christoffel symbols of $g$.

The set of vector fields $\{\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}, \frac{\delta}{\delta q^1}, \ldots, \frac{\delta}{\delta q^n}\}$ defines a local frame on $T^*M$, adapted to the direct sum decomposition (2.1).
We consider the energy density defined by $g$ in the cotangent vector $p$, namely,

$$
(2.2) \quad t = \frac{1}{2} \|p\|^2 = \frac{1}{2}g_{\pi(p)}^{-1}(p, p) = \frac{1}{2}g^{ik}(x)p_ip_k, \quad p \in \pi^{-1}(U).
$$

We have $t \in [0, \infty)$ for all $p \in T^*M$.

Roughly speaking, the natural lifts have as coefficients functions of the density energy only ([1], [2], [3]).

From now on we shall work in a fixed local chart $(U, \varphi)$ on $M$ and in the induced local chart $(\pi^{-1}(U), \Phi)$ on $T^*M$.

We can easily prove

**Lemma 2.1.** If $n > 1$ and $u, v$ are smooth functions on $T^*M$ such that either $u\gamma_{ij} + v\pi_p\partial_j = 0$, $u\gamma^{ij} + v\partial_i^0g^{ij} = 0$, or $u\delta_j + v\partial^0p_j = 0$ on the domain of any induced local chart on $T^*M$, then $u = 0$, $v = 0$.

### 3. THE INTEGRABILITY OF THE ALMOST COMPLEX STRUCTURES

If $X$ is a vector field $X \in \mathcal{X}(M)$, then $g_X$ is the 1-form on $M$ defined as $g_X(Y) = g(X, Y), \forall Y \in \mathcal{X}(M)$. Using the local chart $(U, x^1, \ldots, x^n)$, on $M$ we can write $X = X^i \frac{\partial}{\partial x^i}$. Then $g_X = g_{ij}X^jdx^i$, $g_{\frac{\partial}{\partial x^i}} = g_{ij}dx^j$.

Let us consider a 1-form $\theta \in \Lambda^1(M)$. Then $\theta^* = g_{\theta}^{-1}$ is a vector field on $M$ defined by the musical isomorphism $g(\theta^*, Y) = \theta(Y), \forall Y \in \mathcal{X}(M)$. If $\theta = \theta_i dx^i$ then $\theta^* = g^{ij}\theta_j \frac{\partial}{\partial x^i}$ on the local chart $(U, x^1, \ldots, x^n)$ or $M$.

For $p \in T^*M$ we consider the vector $p^*\theta$ tangent to $M$ at $\pi(p)$.

The Liouville vector field $p^V$ on $T^*M$ is, given by $p^V_p = p^i \frac{\partial}{\partial p_i}$ at every point $p$ of the induced local chart $(\pi^{-1}(U), \Phi)$ on $T^*M$.

The similar horizontal vector field $(p^*)^H$ on $T^*M$ is given by $(p^*)^H = g^{0i}\frac{\partial}{\partial q^i}$ at every point $p$ of the induced local chart $(\pi^{-1}(U), \Phi)$ on $T^*M$ (recall that $g^{0i} = p_0g^{hi}$).

Consider real valued smooth functions $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ defined on $[0, \infty) \subset \mathbb{R}$. We define a general natural tensor $J$ of type $(1, 1)$ on $T^*M$ by its action on the horizontal and vertical lifts on $T^*M$ by

$$
\begin{align*}
JX^H_p &= a_1(t)(g_X)^V_p + b_1(t)p(X)p^V_p + a_4(t)x^H_p + b_4(t)p(X)(p^*)^H_p, \\
J\theta^V_p &= a_3(t)\theta^V_p + b_3(t)g_{\pi(p)}^{-1}(p, \theta)p^V_p - a_2(t)(\theta^*)^H_p - b_2(t)g_{\pi(p)}^{-1}(p, \theta)(p^*)^H_p,
\end{align*}
$$

at every point $p$ of the induced local chart $(\pi^{-1}(U), \Phi)$ on $T^*M$, $\forall X \in \mathcal{X}(M), \forall \theta \in \Lambda^1(M)$.  

With respect to the local adapted frame \( \{ \frac{\delta}{\delta q^i}, \frac{\partial}{\partial p^i} \}_{i=1, \ldots, n} \), the expressions of \( J \) are

\[
\begin{align*}
J \frac{\delta}{\delta q^i} &= a_1(t) g_{ij} \frac{\partial}{\partial p_j} + b_1(t) p_i C + a_4(t) \frac{\delta}{\delta q^j} + b_4(t) p_i \tilde{C}, \\
J \frac{\partial}{\partial p_i} &= a_3(t) \frac{\partial}{\partial p_j} + b_3(t) g^{ij} C - a_2(t) g^{ij} \frac{\delta}{\delta q^j} - b_2(t) g^{ij} \tilde{C},
\end{align*}
\]

(3.1)

where we have denoted by \( C = p^V \) the Liouville vector-field on \( T^* M \) and by \( \tilde{C} = (p^*)^H \), the corresponding horizontal vector field on \( T^* M \).

We may also write

\[
\begin{align*}
J \frac{\delta}{\delta q^i} &= J_{ij}^{(1)} \frac{\partial}{\partial p_j} + J_{ij}^{(2)} \frac{\delta}{\delta q^j}, \\
J \frac{\partial}{\partial p_i} &= J_{ij}^{(3)} \frac{\partial}{\partial p_j} - J_{ij}^{(2)} \frac{\delta}{\delta q^j},
\end{align*}
\]

(3.2)

where

\[
\begin{align*}
J_{ij}^{(1)} &= a_1(t) g_{ij} + b_1(t) p_i p_j, & J_{ij}^{(2)} &= a_4(t) \delta_i^j + b_4(t) g^{ij} p_i, \\
J_{ij}^{(3)} &= a_3(t) \delta_i^j + b_3(t) g^{ij} p_j, & J_{ij}^{(2)} &= a_2(t) g^{ij} + b_2(t) g^{ij} g^{ij}.
\end{align*}
\]

**Theorem 3.1.** A natural tensor field \( J \) of type \((1, 1)\) on \( T^* M \) given by (3.1) or (3.2) defines an almost complex structure on \( T^* M \) if and only if \( a_4 = -a_3, b_4 = -b_3 \) and the coefficients \( a_1, a_2, a_3, b_1, b_2 \) and \( b_3 \) are related by

\[
a_1 a_2 = 1 + a_3^2, \quad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1 + (a_3 + 2tb_3)^2.
\]

**Remark.** By conditions (3.3), the coefficients \( a_1, a_2, a_1 + 2tb_1, a_2 + 2tb_2 \) have the same sign and cannot vanish. We assume that \( a_1 > 0, a_2 > 0, a_1 + 2tb_1 > 0, a_2 + 2tb_2 > 0 \) for all \( t \geq 0 \).

**Remark.** Relations (3.3) allow to express two of the coefficients \( a_1, a_2, a_3, b_1, b_2, b_3 \), as functions of the other four; e.g., we have

\[
a_2 = \frac{1 + a_3^2}{a_1}, \quad b_2 = \frac{2a_3 b_3 - a_2 b_1 + 2t b_2}{a_1 + 2tb_1}.
\]

The integrability condition for the above almost complex structure \( J \) on a manifold \( M \) is characterized by the vanishing of its Nijenhuis tensor field \( N_J \) defined by

\[
N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]
\]

for all vector fields \( X \) and \( Y \) on \( M \).

**Theorem 3.2.** Let \((M, g)\) be an \( n \) (> 2)-dimensional connected Riemannian manifold. The almost complex structure \( J \) defined by (3.1) on \( T^* M \) is
integrable if and only if $(M, g)$ has a constant sectional curvature $c$ and the coefficients $b_1$, $b_2$, $b_3$ are given by

$$
\begin{align*}
(b_1 &= \frac{2c^2 a_2' + 2cta_1 a_2' + a_1 a_4' - c + 3ca_3^2}{a_1 - 2ta_1' - 2cta_2 - 4ct^2 a_2'}, \\
b_2 &= \frac{2ta_2^2 - 2ta_1' a_2' + ca_1^2 + 2cta_2 a_3' + a_1 a_2'}{a_1 - 2ta_1' - 2cta_2 - 4ct^2 a_2'}, \\
b_3 &= \frac{a_1 a_3' + 2ca_2 a_3 + 4cta_3 a_3 - 2cta_2 a_3}{a_1 - 2ta_1' - 2cta_2 - 4ct^2 a_2'}.
\end{align*}
$$

(3.5)

**Proof.** From the condition $N_J(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}) = 0$ we obtain that the horizontal component of this Nijenhuis bracket vanishes if and only if

$$a_2' = \frac{a_2 a_3' + 2a_3 b_2 - a_2 b_3}{2(a_3 + tb_3)},$$

while the vertical component vanishes if and only if

$$
\begin{align*}
(a_1 a_2' - a_1 b_2 + 2a_3 b_3 t)(\delta^h_j p_i - \delta^h_i p_j) - a_2^2 g^{jk} R^h_{kij} - a_2 b_2 g^{ij} (p_j R^h_{kij} - p_i R^h_{kji}) &= 0.
\end{align*}
$$

(3.6)

Taking into account that the curvature of the base manifold does not depend on $p$, we differentiate with respect to $p_k$ in (3.7). Considering the value of this derivative at $p = 0$, we get

$$R^h_{kij} = c(\delta^h_j g_{ki} - \delta^h_i g_{kj}),$$

(3.8)

where

$$c = \frac{a_1(0)}{a_2^2(0)}(b_2(0) - a_2'(0)),$$

which is a function depending on $q_1, \ldots, q^n$ only. According to Schur’s theorem, $c$ must be a constant when $n > 2$ and $M$ is connected.

Using the condition of constant sectional curvature for the base manifold, from $N_J(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}) = 0$ we obtain

$$
\begin{align*}
a_1' &= \frac{a_1 b_1 + c(1 - 3a_3^2 - 4ta_3 b_3)}{a_1 + 2tb_1}, \\
a_3' &= \frac{a_1 b_3 - 2ca_2 (a_3 + tb_3)}{a_1 + 2tb_1}.
\end{align*}
$$

(3.9)

If we replace in (3.6) the expression of $a_3'$ and relations (3.4), we get

$$a_2' = \frac{2a_3 b_3 - a_2 b_1 - ca_2^2}{a_1 + 2tb_1}.$$

(3.10)
The values of \(a'_1, a'_3\) in (3.9) and \(a'_2\) in (3.10) fulfil the vanishing condition of the vertical component of the Nijenhuis bracket \(N_J\left(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\right)\). The same expressions also fulfil the relation

\[
a_1a'_2 + a'_2a_2 = 2a_3a'_3,
\]

obtained by differentiating with respect to \(t\) the first relation (3.3).

We can solve the system given by (3.6) and (3.9), with respect to \(b_1, b_2, b_3\). Taking (3.11) into account, we get relations (3.5), which fulfil identically the expression of \(b_2\) from (3.4).

Remark. In the diagonal case, where \(a_3 = 0\), we have \(b_3 = 0\), too, and

\[
a_2 = \frac{1}{a_1}, \quad b_1 = \frac{a_1a'_1 - c}{a_1 - 2ta'_1}, \quad b_2 = \frac{c - a_1a'_1}{a_1(a'_1 - 2ct)}.
\]

We thus retrieve some results from [13] and [16], papers which have treated the diagonal case only.

4. KÄHLER STRUCTURES OF GENERAL NATURAL LIFT TYPE ON THE COTANGENT BUNDLE

In this section we introduce a Riemannian metric \(G\) of general natural lift type on the cotangent bundle \(T^*M\), defined by

\[
G_p(X^H, Y^H) = c_1(t)g_{\pi(p)}(X, Y) + d_1(t)p(X)p(Y),
\]

\[
G_p(\theta^V, \omega^V) = c_2(t)g^{-1}_{\pi(p)}(\theta, \omega) + d_2(t)g^{-1}_{\pi(p)}(p, \theta)g^{-1}_{\pi(p)}(p, \omega),
\]

\[
G_p(X^H, \theta^V) = G_p(\theta^V, X^H) = c_3(t)\theta^V + d_3(t)p(X)g_{\pi(p)}(p, \theta),
\]

\[
\forall X, Y \in \mathcal{X}(M), \forall \theta, \omega \in \Lambda^1(M), \forall p \in T^*M.
\]

In local coordinates expressions (4.1) become

\[
G \left( \frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j} \right) = c_1(t)g_{ij} + d_1(t)p_ip_j = G_{ij}^{(1)},
\]

\[
G \left( \frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j} \right) = c_2(t)g^{ij} + d_2(t)g^{0i}g^{0j} = G_{ij}^{(2)},
\]

\[
G \left( \frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j} \right) = G \left( \frac{\delta}{\delta q^i}, \frac{\partial}{\partial p_j} \right) = c_3(t)\delta^j_i + d_3(t)p_i g^{0j} = G^j_i,
\]

where \(c_1, c_2, c_3, d_1, d_2, d_3\) are six smooth functions of the density energy on \(T^*M\). The conditions for \(G\) to be positive definite are met if

\[
(c_1 + 2td_1) > 0, \quad c_2 + 2td_2 > 0, \quad (c_1 + 2td_1)(c_2 + 2td_2) - (c_3 + 2td_3)^2 > 0.
\]
The metric $G$ is almost Hermitian with respect to the general almost complex structure $J$, if $G(JX, JY) = G(X, Y)$ for all vector fields $X, Y$ on $T^*M$.

We now prove the result below.

**THEOREM 4.1.** The family of natural Riemannian metrics $G$ on $T^*M$ such that $(T^*M, G, J)$ is an almost Hermitian manifold is given by (4.2) provided that the coefficients $c_1, c_2, c_3, d_1, d_2, d_3$ are related to the coefficients $a_1, a_2, a_3, b_1, b_2, b_3$ by the proportionality relations

$$\frac{c_1}{a_1} = \frac{c_2}{a_2} = \frac{c_3}{a_3} = \lambda$$

(4.4)

$$\frac{c_1 + 2td_1}{a_1 + 2tb_1} = \frac{c_2 + 2td_2}{a_2 + 2tb_2} = \frac{c_3 + 2td_3}{a_3 + 2tb_3} = \lambda + 2t\mu,$$

(4.5)

where the proportionality coefficients $\lambda > 0$ and $\lambda + 2t\mu > 0$ are functions of $t$.

**Proof.** We use the local adapted frame $\{\frac{\delta}{\delta q^i}, \frac{\partial}{\partial p_j}\}_{i,j=1,\ldots,n}$. The metric $G$ is almost hermitian if and only if

$$\begin{cases}
G \left( J \frac{\delta}{\delta q^i}, J \frac{\delta}{\delta q^j} \right) = G \left( \frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j} \right), \\
G \left( J \frac{\partial}{\partial p_i}, J \frac{\partial}{\partial p_j} \right) = G \left( \frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j} \right), \\
G \left( J \frac{\partial}{\partial p_i}, J \frac{\delta}{\delta q^j} \right) = G \left( \frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j} \right).
\end{cases}$$

(4.6)

By Lemma 2.1, the coefficients of $g_{ij}, g^{ij}, \delta^i_j$ in conditions (4.6) must vanish. It follows that the parameters $c_1, c_2, c_3, d_1, d_2, d_3$ from the definition of the metric $G$ satisfy a homogeneous linear system of the form

$$\begin{cases}
(a_2^2 - 1)c_1 + a_1^2 c_2 - 2a_1 a_3 c_3 = 0, \\
a_2^2 c_1 + (a_3^2 - 1)c_2 - 2a_2 a_3 c_3 = 0, \\
a_2 a_3 c_1 + a_1 a_3 c_2 - 2a_1 a_2 c_3 = 0.
\end{cases}$$

(4.7)

The nontrivial solutions of (4.7) are given by (4.4).

From the vanishing condition of the coefficients of $p_i p_j$, $g^{ij} g^{pj}$, $g^{pi} p_j$ in (4.6), we obtain a much more complicated system fulfilled by $d_1, d_2, d_3$. In order to get a certain similitude with system (4.7) fulfilled by $c_1, c_2, c_3$, we multiply the new equations by $2t$ and substract the corresponding equations of system (4.7). The new system can be written in a form in which the new
unknowns are \( c_1 + 2td_1, c_2 + 2td_2, c_3 + 2td_3 \), namely,
\[
\begin{align*}
[(a_3 + 2tb_3)^2 - 1](c_1 + 2td_1) + (a_1 + 2tb_1)^2(c_2 + 2td_2) \\
-2(a_1 + 2tb_1)(a_3 + 2tb_3)(c_3 + 2td_3) = 0,
\end{align*}
\]
\[
\begin{align*}
(a_2 + 2tb_2)^2(c_1 + 2td_1) + [(a_3 + 2tb_3)^2 - 1](c_2 + 2td_2) \\
-2(a_2 + 2tb_2)(a_3 + 2tb_3)(c_3 + 2td_3) = 0,
\end{align*}
\]
\[
\begin{align*}
(a_2 + 2tb_2)(a_3 + 2tb_3)(c_1 + 2td_1) + (a_1 + 2tb_1)(a_3 + 2tb_3)(c_2 + 2td_2) \\
-2(a_1 + 2tb_1)(a_2 + 2tb_2)(c_3 + 2td_3) = 0.
\end{align*}
\]
Then the nonzero solutions are given by (4.5).

Conditions (4.3) are fullfield, due to properties (3.4) of the coefficients of the almost complex structure \( J \).

The differential of \( \Omega \) is given by (4.5) are
\[
\begin{align*}
d_1 &= \lambda b_1 + \mu(a_1 + 2tb_1), \\
d_2 &= \lambda b_2 + \mu(a_2 + 2tb_2), \\
d_3 &= \lambda b_3 + \mu(a_3 + 2tb_3).
\end{align*}
\]

**Remark.** In the case where \( a_3 = 0 \), we have \( c_3 = d_3 = 0 \) and we obtain the almost Hermitian structure considered in [13] and [16]. Moreover, if \( \lambda = 1 \) and \( \mu = 0 \), we obtain the almost K"{a}hlerian structure considered in the same papers.

Considering the two-form \( \Omega \) defined by the almost Hermitian structure \( (G, J) \) on \( T^*M \) as \( \Omega(X, Y) = G(X, JY) \) for all vector fields \( X, Y \) on \( T^*M \), we obtain

**Proposition 4.2.** The expression of the 2-form \( \Omega \) in the local adapted frame \( \{ \delta^{ij}, \frac{\partial}{\partial p^j} \} \) on \( T^*M \), is given by
\[
\Omega \left( \frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j} \right) = 0, \quad \Omega \left( \frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j} \right) = 0, \quad \Omega \left( \frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j} \right) = \lambda \delta^{ij} + \mu g^{0i} p_j
\]
or, equivalently,
\[
\Omega = (\lambda \delta^{ij} + \mu g^{0i} p_j) \text{D}p_i \wedge dq^j,
\]
where \( \text{D}p_i = dp_i - \Gamma^h_{ia} dq^h \) is the absolute differential of \( p_i \).

Next, by calculating the exterior differential of \( \Omega \), we obtain

**Theorem 4.3.** The almost Hermitian structure \( (T^*M, G, J) \) is almost K"{a}hlerian if and only if \( \mu = \lambda' \).

**Proof.** The differential of \( \Omega \) is
\[
d\Omega = (d\lambda \delta^{ij} + d\mu g^{0i} p_j + \mu d g^{0i} p_j + \mu g^{0i} dp_j) \wedge Dp_i \wedge dq^j + (\lambda \delta^{ij} + \mu g^{0i} p_j) dDp_i \wedge dq^j.
\]
We first obtain the expressions of \( d\lambda, d\mu, dg^{0i}, \) and \( dDp_i \) as
\[
d\lambda = \lambda' g^{0k} Dp_h, \quad d\mu = \mu' g^{0k} Dp_h, \quad dg^{0i} = g^{hi} Dp_h - \Gamma^i_{j0} dq^j,
\]
\[
dDp_i = \frac{1}{2} R^0_{ijhk} dq^h \wedge dq^j - \Gamma^i_{lj} Dp_h \wedge dq^j.
\]
Substituting these relations in \( d\Omega \), using the properties of the external product, the symmetry of \( g^{ij}, \Gamma^h_{ij} \) and the Bianchi identities, we get
\[
d\Omega = \frac{1}{2} (\lambda' - \mu) p_k (g^{kh} \delta_j^i - g^{ki} \delta_j^h) Dp_h \wedge Dp_i \wedge dq^j.
\]
Hence \( d\Omega = 0 \) if and only if \( \mu = \lambda' \). □

Remark. The family of general natural almost Kählerian structures on \( T^*M \) depends thus on five essential coefficients \( a_1, a_3, b_1, b_3, \lambda \), which must satisfy the supplementary conditions \( a_1 > 0, a_1 + 2tb_1 > 0, \lambda > 0, \lambda + 2t\mu > 0 \).

Combining the results from Theorems 4.1, 3.2 and 4.3 we can state

**Theorem 4.4.** An almost Hermitian structure \( (G, J) \) of general natural lift type on \( T^*M \) is Kählerian if and only if the almost complex structure \( J \) is integrable (see Theorem 3.2) and \( \mu = \lambda' \).

Remark. The family of general natural Kählerian structures on \( T^*M \) depends on three essential coefficients \( a_1, a_3, \lambda \), which must satisfy the supplementary conditions \( a_1 > 0, a_1 + 2tb_1 > 0, \lambda > 0, \lambda + 2t\lambda' > 0 \), where \( b_1 \) is given by (3.5). Examples of such structures can be found in [12], [13], [16].

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