Metastability Driven by Soft Quantum Fluctuation Modes

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Abstract  The semiclassical Euclidean path integral method is applied to compute the low temperature quantum decay rate for a particle placed in the metastable minimum of a cubic potential in a finite time theory. The classical path, which makes a saddle for the action, is derived in terms of Jacobian elliptic functions whose periodicity establishes the one-to-one correspondence between energy of the classical motion and temperature (inverse imaginary time) of the system. The quantum fluctuation contribution has been computed through the theory of the functional determinants for periodic boundary conditions. The decay rate shows a peculiar temperature dependence mainly due to the softening of the low lying quantum fluctuation eigenvalues. The latter are determined by solving the Lamè equation which governs the fluctuation spectrum around the time dependent classical bounce.

Keywords  Low temperature decay rate · Path integral methods · Quantum fluctuations · Soft modes

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1 Introduction

Quantum tunneling from a metastable state through a potential barrier is a fundamental nonlinear phenomenon occurring in many branches of the physical sciences [1–9]. The standard semiclassical approach to metastability has been formulated since long by Langer [10] and Coleman [11, 12] in the fields of statistical and nuclear physics respectively. The main idea underlying such approach consists in selecting a classical
background which solves the Euler-Lagrange equation and makes a saddle point for
the action. Around the background, the quantum fluctuations are treated in quadratic
approximation and their spectrum is obtained by solving a Schrödinger like stability
equation whose potential is given by the second spatial derivative of the metastable
potential. The semiclassical method finds a concise and powerful description in the
Euclidean path integral formalism [13, 14] in which the time for the bounce to per-
form a full excursion (inside the classically allowed region) is a measure of the in-
verse temperature of the system. In the standard treatments of metastability [10–12],
it is assumed that such time is infinite and therefore the decay rate formula holds,
strictly speaking, only at \( T = 0 \). For applications to specific systems however, a
precise knowledge of the decay rate at finite \( T \) (but within the quantum tunneling
regime) may be of great interest. To this purpose one has to build the finite time
theory of metastability for specific nonlinear potentials, setting the crossover temper-
ature between (low \( T \)) quantum and (high \( T \)) activated regimes and find the shape of
the decay rate when the crossover is approached from below. Focusing on a widely
investigated model in nonlinear science, a particle in the one dimensional cubic po-
tential, I present in Sect. 2 the finite time solution of the Euler-Lagrange equation in
terms of the powerful Jacobian elliptic functions formalism [15]: this generalizes the
well known infinite time bounce which is recovered asymptotically. I emphasize that
the system is taken as non dissipative therefore the temperature should not be viewed
here as a property of the heat bath [16–18], but rather as a measure of the system
size along the time axis. Section 3 is devoted to the computation of the classical ac-
tion. Section 4 describes the method of the semiclassical path integral and presents
the calculation of the overall quantum fluctuation contribution through the theory of
the functional determinants. Section 5 solves the stability equation for the periodic
potential defined by the classical background. This permits to obtain analytically the
lowest quantum fluctuation eigenvalues as a function of the finite time/temperature.
It is shown in Sect. 6 that the softening of such eigenvalues close to the crossover
largely determines the peculiar shape of the decay rate and its deviation from the
prediction of the standard zero \( T \) theory. The conclusions are drawn in Sect. 7.

2 Cubic Potential Model

To begin, consider a particle of mass \( M \) in the one dimensional cubic anharmonic
potential:

\[
V(x) = \frac{M\omega^2}{2}x^2 - \frac{\gamma}{3}x^3, \tag{1}
\]

plotted in Fig. 1(a) for \( \hbar \omega = 20 \text{ meV} \) and \( M = 10^3m_e \), \( m_e \) being the electron mass.
Say \( a \) the position of the top of barrier whose height is \( V(a) = \gamma a^3/6 \) with \( \gamma =
M\omega^2/a \). Let’s take throughout the paper, \( a = 1 \text{ Å} \). At \( x = 0 \) the particle is in a local
minimum from which it cannot escape classically. Thus, in the real time formalism,
the classical equation of motion admits only the trivial solution \( x_{cl} = 0 \). Physically
however such local minimum is metastable as quantum fluctuations allow the particle
to explore the abyss at \( x \geq 3a/2 \). In fact, a non trivial classical solution can be found