A NOTE ON GENERIC PROJECTIONS

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Abstract. Let \( X \subseteq \mathbb{P}^N = \mathbb{P}_K^N \) be a subvariety of dimension \( n \) and \( P \in \mathbb{P}^N \) a generic point. If the tangent variety \( \text{Tan} X \) is equal to \( \mathbb{P}^N \) then for generic points \( x, y \) of \( X \) the projective tangent spaces \( t_x X \) and \( t_y X \) meet in one point \( P = P(x, y) \). The main result of this paper is that the rational map \( (x, y) \mapsto P(x, y) \) is dominant. In other words, a generic point \( P \) is uniquely determined by the ramification locus \( R(\pi_P) \) of the linear projection \( \pi_P : X \to \mathbb{P}^{N-1} \).

1. Introduction

Let \( X \subseteq \mathbb{P}^N = \mathbb{P}_K^N \) be a subvariety of dimension \( n \) and \( \Lambda \subseteq \mathbb{P}^N \) a linear subspace of codimension \( k + 1 \) with \( k \geq n \). In \([FM2]\) we studied the question how the ramification locus \( R(\pi_\Lambda) \) of the linear projection \( \pi_\Lambda : X \to \mathbb{P}^k \) with center \( \Lambda \) varies with \( \Lambda \). More precisely we considered the problem as to how \( \Lambda \) is determined by \( R(\pi_\Lambda) \).

The motivation for this question comes from the study of the Stückrad-Vogel cycle. If we denote by \( v = v(X, X) = \sum_i v_i \) the Stückrad-Vogel selfintersection cycle of \( X \) then by \([VGa, FM1]\) the cycle \( v_{2n-k-1} \) may be interpreted as a ramification cycle of the generic projection \( \pi_\Lambda : X \to \mathbb{P}^k \), where \( \Lambda_U \subseteq \mathbb{P}_L^N \) is the generic linear subspace of codimension \( k + 1 \) given by the equations \( \sum_j U_{ij} x_j = 0 \), \( 0 \leq i \leq k \), over the purely transcendental extension field \( L := K(U_{ij}) \).

In \([FM2]\) we proved that the cycle \( v_{2n-k-1} \) has maximal transcendence degree if and only if the rational map \( \Lambda \mapsto R(\pi_\Lambda) \) from the Grassmannian \( G(N - k - 1, N) \) of \((N - k - 1)\)-planes in \( \mathbb{P}^N \) to the Hilbert scheme \( \text{Hilb}_X \) of \( X \) is generically finite. In particular we treated in that paper the case of smooth surfaces in \( \mathbb{P}^4 \) and the projection from a point, i.e. \( n = 2, N = 4 \) and \( \Lambda = \{ P \} \). Under a suitable positivity condition on the normal bundle \( N_{X/\mathbb{P}^4}(-1) \) we showed that \( R(\pi_P) \) uniquely determines \( P \), and from this we deduced that \( v_0(X, X) \) has maximal transcendence degree.

In a letter to the second author C. Ciliberto improved this by showing that for any nondegenerate smooth surface \( X \) in \( \mathbb{P}^4 \) the same result remains true. His essential idea was to use the second fundamental form of the surface to conclude that the rational map

\[
\begin{align*}
X \times X & \longrightarrow \mathbb{P}^4 \\
(x, y) & \longmapsto t_x X \cap t_y X
\end{align*}
\]

\[ (*) \]

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HUBERT FLENNER AND MIRELLA MANARESI

is dominant, where \( t_x \) and \( t_y \) are the projective tangent spaces.

In this paper we show how one can further generalize Ciliberto’s result to any \( n \)-dimensional subvariety \( X \) of \( \mathbb{P}^{2n} \). More precisely we show that if \( \text{Tan} \ X = \mathbb{P}^{2n} \) then the map \((*)\) is again surjective, where \( \text{Tan} \ X \) denotes the tangent variety of \( X \), i.e. the closure of the union of all projective tangent spaces \( t_x \) at smooth points of \( X \). Note that if \( X \) is smooth then by a result of Fulton and Lazarsfeld \([FL]\) \( \text{Tan} \ X = \mathbb{P}^{2n} \) if and only if \( \text{Sec} \ X = \mathbb{P}^{2n} \), where \( \text{Sec} \ X \) is the secant variety of \( X \) (see also \([FOV\ 1.3\ and\ 4.3.12]\)). As a corollary we obtain that the St"uckrad-Vogel cycle \( v_o(X,X) \) has maximal transcendence degree under the assumptions above.

Throughout this paper we work over the field \( K = \mathbb{C} \) of complex numbers. We note that the results remains valid over any algebraically closed field of characteristic 0 by standard arguments. In the following we will use the notations of our previous paper \([FM2]\). As a general reference on the St"uckrad-Vogel cycle we refer the reader to \([FOV]\).

2. Transcendence degree of the St"uckrad-Vogel cycle

Let \( X \) be an \( n \)-dimensional subvariety of \( \mathbb{P}^N \) with \( N = 2n \). Let \( x \in X \) be a generic point and let \( U \cong \mathbb{C}^N \) be an affine neighbourhood of \( x \) in \( \mathbb{P}^N \) such that \( x \) corresponds to \( 0 \in \mathbb{C}^N \). After a linear change of coordinates we can write \( X \) in a complex neighbourhood \( V \) of \( \hat{x} = 0 \) as a graph

\[
W \ni u \mapsto (u, f(u)) \in V
\]

with \( f_u(0) = f(0) = 0 \), where \( W \) is a neighbourhood of \( 0 \) in \( \mathbb{C}^N \).

With these notation we will show the following explicit criterion as to when \( \text{Tan} \ X \) has maximal dimension.

Lemma 2.1. The following are equivalent.

(a) \( \text{Tan} \ X = \mathbb{P}^N \);

(b) \( f_u(0) \cdot u \) is a nondegenerate matrix for a generic point \( u \in \mathbb{C}^n \).

Proof. Assume (a) is satisfied. Consider the map

\[
W \times \mathbb{C}^n \cong T(X \cap V) \xrightarrow{\text{can}} \mathbb{C}^N
\]

\[
(u, \xi) \longmapsto (u + \xi, f(u) + f_u(u) \cdot \xi),
\]

where \( T(X \cap V) \) denotes the tangent bundle of \( X \cap V \). By (a) the differential of this map has maximal rank, i.e. the matrix

\[
\begin{pmatrix}
E_n \\
f_u(0) + f_{uu}(0) \cdot \xi & f_u(0)
\end{pmatrix}
\]

is maximal rank, where \( E_n \) is the \( n \)-dimensional identity matrix. Hence (b) follows, and by reversing the argument it also follows that (b) implies (a). \( \square \)

We can now formulate the main result of this paper.

Theorem 2.2. Let \( X \) be an \( n \)-dimensional subvariety of \( \mathbb{P}^N \) with \( N = 2n \). If the tangent variety \( \text{Tan} \ X \) is equal to \( \mathbb{P}^N \), then a generic point \( P \in \mathbb{P}^N \) is uniquely determined by the ramification locus \( R(\pi_P) \) of the linear projection from \( P \).

Proof. By assumption, \( \text{Tan} \ X = \mathbb{P}^N \) and so in particular \( \text{Sec} \ X = \mathbb{P}^N \). Thus, using Terracini’s lemma \([FOV\ Proposition\ 4.3.2]\), for general points \( x, y \in X \) the
intersection \( t_x X \cap t_y X \) consists of just one point \( P = P(x, y) \). We need to show that the rational map

\[
\begin{align*}
X \times X & \longrightarrow \mathbb{P}^N \\
(x, y) & \longmapsto P(x, y)
\end{align*}
\]

is dominant. Let \( x \) be a general point of \( X \). Clearly it suffices to prove that the rational map

\[
\begin{align*}
X & \longrightarrow T_x X \\
y & \longmapsto P(x, y)
\end{align*}
\]

is dominant. We may assume that \( x = [1 : 0 : \ldots : 0] \) so that the affine open set \( U = \{ x_0 = 1 \} \cong \mathbb{C}^N \) is an affine open neighbourhood of \( x \); note that \( x \) then corresponds to the origin in \( \mathbb{C}^N \). After a linear change of coordinates we can write \( X \) in a (complex) neighbourhood of \( x \hat{} = 0 \) as a graph

\[
u \mapsto (u, f(u)) \in \mathbb{C}^N
\]

with \( f(0) = 0, \ f_u(0) = 0 \). With \( y = (u, f(u)) \) the point

\[
P = P(0, y) = (p(u), 0) \in T_x X,
\]

is the intersection of the linear subspaces \( \mathbb{C}^N \times 0 \) and the image of \( \xi \mapsto (u + \xi, f(u) + f_u(u) \cdot \xi) \). This leads to \( u + \xi = p(u) \), i.e. \( \xi = p(u) - u \) and

\[
0 = f(u) + f_u(u) \cdot \xi = f(u) + f_u(u)(p(u) - u).
\]

By \ref{2.1} the matrix \( f_u(u) \cdot \xi \) is nondegenerate. As

\[
f_u(u) = f_u(u) \cdot u + \text{higher order terms}
\]

the matrix \( f_u(u) \) is invertible for a general and sufficiently small point \( u \). Thus we can write

\[
p(u) = u - f_u(u)^{-1}f(u).
\]

We need to show that \( u \mapsto p(u) \) has maximal rank. The differential of this map is

\[
\begin{align*}
\eta & \mapsto \eta - f_u(u)^{-1}f_u(u)\eta + f_u(u)^{-1}f_{uu}(u)f_u(u)^{-1}\eta \\
& = f_u(u)^{-1}f_{uu}(u)f_u(u)^{-1}\eta.
\end{align*}
\]

Thus this map has maximal rank, since \( \tilde{\eta} = f_u(u)^{-1}\eta \) is a vector in general position, so by the lemma \( f_{uu}(0) \cdot \tilde{\eta} \) has maximal rank. As

\[
f_{uu}(u)\tilde{\eta} = f_{uu}(0)\tilde{\eta} + \text{higher order terms}
\]

it follows that also \( f_{uu}(u)\tilde{\eta} \) has maximal rank for \( u \) sufficiently small. \( \square \)

Using \cite[Lemma 3.3]{FM2} we have the following corollary.

**Corollary 2.3.** Let \( X \) be an \( n \)-dimensional subvariety of \( \mathbb{P}^N \) with \( N = 2n \). If the tangent variety \( \text{Tan} X \) is equal to \( \mathbb{P}^N \), then the Stückrad Vogel cycle \( v_0 = v_0(X, X) \) has maximal transcendence degree.

In the case of surfaces in \( \mathbb{P}^4 \) we recover the result of Ciliberto mentioned in the introduction.

**Corollary 2.4.** (C.Ciliberto) If \( X \subseteq \mathbb{P}^4 \) is a nondegenerate smooth surface, then a general point \( P \in \mathbb{P}^N \) is uniquely determined by the ramification locus \( R(\pi_P) \) of the linear projection from \( P \).
Proof. If $\text{Tan } X \neq \mathbb{P}^1$ then $\dim \text{Tan } X = \dim \text{Sec } X = \dim X + 1$ and so by $\text{FOV}, 4.6.6(1)$, $\text{Sec } X$ is a linear subspace of dimension 3, which contradicts the fact that $X$ is nondegenerate. Thus $\underline{2.2}$ implies the result. □

Remark 2.5. We note that Ciliberto’s argument is different from our proof. He uses the fact that for a nondegenerate surface which is not a cone the second fundamental form is nondegenerate.

Remark 2.6. The example 4.8 given in $\text{FM}_2$ is not a counterexample to the maximality of the transcendence degree of the Stückrad-Vogel cycle as was claimed there. The mistake comes from the fact that the dimension of the family of lines $\mathbb{P}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^m$ of bidegree $(1,1)$ was not calculated correctly. It is $2m + 1$ (and not $2m - 2$), since such a map is given by $(\alpha, j)$, where $\alpha$ is an automorphism of $\mathbb{P}^1$ and $j : \mathbb{P}^1 \hookrightarrow \mathbb{P}^m$ is the embedding of a line.

References

[CI] Ciliberto, C.: Letter dated from April 2001
[FM1] Flenner, H.; Manaresi, M.: Intersections of projective varieties and generic projections. Manuscripta Math. 92, 273-286 (1997)
[FM2] Flenner, H.; Manaresi, M.: Variation of ramification loci of generic projections. Math. Nachr. 194, 79-92 (1998)
[FL] Fulton, W.; Lazarsfeld, R.: Connectivity and its applications in algebraic geometry. In: Algebraic Geometry, University of Illinois at Chicago Circle, 1980. Lecture Notes in Mathematics, Volume 862. Berlin Heidelberg New York: Springer 1980, 26-92
[FOV] Flenner, H.; O’Carroll, L.; Vogel, W.: Joins and intersections. Monographs in Mathematics, Springer Verlag, Berlin–Heidelberg–New York, 1999.
[vGa] Gastel, L.J. van: Excess intersections and a correspondence principle. Invent. Math. 103, 197-221 (1991)