Determination of the right-hand side term in the degenerate parabolic equation with two variables

V L Kamynin\(^1\) and A B Kostin\(^1\)

\(^1\)National Research Nuclear University MEPhI (Moscow Engineering Physics Institute), 31 Kashirskoe shosse, 115409 Moscow, Russia  
E-mail: vlkamynin2008@yandex.ru, abkostin@yandex.ru

Abstract. We find two type conditions sufficient for unique solvability of inverse problem of source determination in degenerate parabolic equation with two independent variables.

1. Introduction
In the present work we study the unique solvability of the inverse problem of determination of a pair of functions \(\{u(t, x); p(x)\}\) satisfying in the rectangle \(Q \equiv [0, T] \times [0, l]\) the parabolic equation

\[
\rho(t, x)u_t - u_{xx} - b(t, x)u_x - c(t, x)u = g(t, x)p(x) + r(t, x),
\]

the initial and boundary conditions

\[
u(0, x) = u_0(x), \quad x \in [0, l], \quad u(t, 0) = u(t, l) = 0, \quad t \in [0, T],
\]

and the additional condition of integral observation

\[
l(u) \equiv \int_0^T u(t, x)\chi(t) \, dt = \varphi(x), \quad x \in [0, l].
\]

In what follows we use Lebesgue, Sobolev and Hölder spaces with corresponding norms in usual sense. For convenience we denote the space \(L_2(0, l)\) by \(E\) and the norm in the space \(L_q(0, l)\) by \(\| \cdot \|_q, \, q \in [1, \infty]\). We will also use the notation \(G(x) = \int_0^T g(t, x)\chi(t) \, dt \equiv l(g)(x)\) and the well–known Poincaré–Steklov inequality, which, for \(n = 1\) is in the form

\[
\|z\|_2 \leq \frac{1}{\pi} \|z_x\|_2 \quad \text{for all} \quad z \in W^1_2(0, l).
\]

In what follows we assume that the functions appearing in the input data of the problem (1)–(3) are measurable and satisfy the following conditions:

\[
0 \leq \rho(t, x) \leq \rho_1, \quad (t, x) \in Q; \quad \frac{1}{\rho} \in L_q(Q), \quad q > 1, \quad \|\frac{1}{\rho}\|_{L_q(Q)} \leq \rho_2; \quad (A)
\]

\[
\begin{aligned}
\{ b \in L_2(0, T; L_\infty(0, l)); \quad & b^2/\rho, \, c^2/\rho \in L_\infty(Q); \\
\|b\|_{L_2(0, T; L_\infty(0, l))} & \leq K_b; \quad \|b^2/\rho\|_{L_\infty(Q)} \leq K_{b, \rho}; \quad \|c^2/\rho\|_{L_\infty(Q)} \leq K_{c, \rho}; \quad (B)
\end{aligned}
\]
\( u_0 \in W^1_2(0, l), \quad \varphi \in W^2_2(0, l), \quad \varphi(0) = \varphi(l) = 0, \quad \|\varphi''\|_2 \leq K^*_\varphi; \quad (C) \)

\[
\begin{align*}
&g, \quad g^2/\rho \in L_1(0, T; L_\infty(0, l)); \\
r \in L_1(0, T; E), \quad r^2/\rho \in L_1(Q); \\
\chi \in L_\infty(0, T), \quad \|\chi\|_{L_\infty(0, T)} \leq K_\chi, \\
&\int_0^T g(t, x) \chi(t) dt \geq g_0 > 0, \quad |\rho(T, x)\chi(T)| \leq \rho_3, \quad |\rho(t, x)\chi(0)| \leq \rho_4,
\end{align*}
\]

\[
\begin{align*}
&\|c\chi\|_{L_2(0, T; L_\infty(0, l))} \leq K_{e, \chi}; \\
&\|\rho(t, x)\|_{L_2(0, T; L_\infty(0, l))} \leq K_{p, \chi}; \\
&\text{either } \frac{\rho_1}{\rho} \in L_\infty(Q), \quad \text{or } \rho_t \in L_1(Q) \quad \text{and } \rho_t(t, x) \leq 0, \quad (t, x) \in Q. \quad (E)
\end{align*}
\]

**Definition 1.** By the generalized solution of the problem (1)–(3) we mean the pair of functions \( \{u(t, x); p(x)\} \) with

\[
u \in C^{0, \alpha}(Q) \cap L_\infty(0, T; W^1_2(0, l)) \cap W^{1, 2}_s(Q), \quad s > 1, \quad \alpha \in (0, 1), \quad p \in L_2(0, l) \equiv E,
\]

such that this pair satisfies the equation (1) almost everywhere in \( Q \) and the function \( u(t, x) \) satisfies the conditions (2), (3) in the classical sense.

By virtue of condition (A) the equation (1) is a degenerate parabolic equation. Inverse problems for degenerate parabolic equations are important in applications in engineering and financial mathematics (see, for example, \[1, 2, 3\]). Earlier the inverse problem (1)–(3) was investigated in \[4\] but with more stringent assumptions on the input data functions and in the different class than in the present one. Let us also note the papers \[5, 6\] where inverse problem for equation of the type (1) but with unknown function \( p(t) \) and with another type of overdetermination condition was considered.

Let \( p \in E \) be a known function and let us consider the direct problem (1), (2) with a given function \( p(x) \) in the right-hand side of equation (1). For convenience, we introduce the notation

\[
f(t, x) = g(t, x)p(x) + r(t, x), \quad \text{so that } \quad f^2/\rho \in L_1(Q),
\]

then the equation (1) takes the form

\[
\rho(t, x)u_t - u_{xx} - b(t, x)u_x - c(t, x)u = f(t, x), \quad (t, x) \in Q. \quad (5)
\]

In our investigation we need the following result from \[4\] concerning the direct problem (5), (2).

**Theorem 1.** Let the conditions (A)–(E) hold and \( p \in E \). Set

\[
q^* = \frac{2q}{q + 1}, \quad \lambda^* = 3(K_{b, \rho} + \frac{f^2}{\pi^2}K_{c, \rho}). \quad (6)
\]

Then there exists a generalized solution \( u(t, x) \) of the problem (5), (2) with \( u \in W^{1, 2}_{q^*}(Q) \). Moreover this solution is unique and it satisfies the estimates:

\[
\sup_{0 \leq t \leq T} \|u_x(t, \cdot)\|_2^2 + \|\rho u_t^2\|_{L_1(Q)} \leq e^{\lambda^*T}(\|u_0\|_2^2 + 3\|f^2/\rho\|_{L_1(Q)}), \quad (7)
\]
\[ \|u\|_{W^{1,2}_x(Q)}^2 \leq C_1, \]  
\[ |u(t_1, x_1) - u(t_2, x_2)| \leq C_2|x_1 - x_2|^{1/2} + C_3|t_1 - t_2|^{2q_1-1}, \quad \text{for all } (t_1, x_1), (t_2, x_2) \in Q, \]  
where \( C_1, C_2, C_3 = \text{const} > 0 \) depend only on \( l, T, \rho_1, \rho_2, \|u_0\|_2, K_{b,\rho}, K_{c,\rho}, K_{g,\rho}, K_{r,\rho} \) and on \( \|p\|_2. \]

2. Unique solvability of inverse problem. The first variant of sufficient conditions

In this section we assume that the input data of the inverse problem (1)–(3) satisfy conditions (A) – (E), constants \( q^* > 1 \) and \( \lambda^* \) are defined in (6). Let us derive the operator equation for the unknown function \( p(x) \).

Let pair of functions \( \{u(t, x); p(x)\} \) be any generalized solution of the inverse problem (1)–(3). Let us multiply equation (1) by \( \chi(t) \) and integrate over \([0, T]\). Taking into account conditions (2), (3) after integrating by parts we obtain the relation

\[ p(x) = \frac{1}{G(x)} \left[ \rho(T, x)\chi(T)u(T, x) - \int_0^T (c(t, x)\chi(t) + (\rho(t, x)\chi(t)))u(t, x)\, dt - \int_0^T b(t, x)\chi(t)u_x(t, x)\, dt \right] - \frac{1}{G(x)} \left[ \varphi''(x) + 1(r)(x) + (\rho(0, x)\chi(0))u_0(x) \right]. \]

In view of this relation let us introduce the operator \( A : E \rightarrow E \) by the formula

\[ Ap = \frac{1}{G(x)} \left[ \rho(T, x)\chi(T)u(T, x) - \int_0^T (c(t, x)\chi(t) + (\rho(t, x)\chi(t)))u(t, x)\, dt - \int_0^T b(t, x)\chi(t)u_x(t, x)\, dt \right] - \frac{1}{G(x)} \left[ \varphi''(x) + 1(r)(x) + (\rho(0, x)\chi(0))u_0(x) \right], \]

where \( p(x) \) is an arbitrary function in \( E \), and \( u = u(t, x; p) \) is a solution of direct problem (1), (2) with given \( p(x) \) in the right-hand side of equation (1). In view of theorem 1 solution \( u(t, x; p) \) exists and is unique and the operator \( A \) is well–defined. By virtue (11) the relation (10) can be written as

\[ p = Ap. \]

**Lemma 1.** Let conditions (A)–(E) hold. Then the operator equation (12) is equivalent to the inverse problem (1)–(3) in the following sense. If pair \( \{u(t, x); p(x)\} \) is a generalized solution of the inverse problem, then \( p(x) \) satisfies (12). Conversely, if \( p \in E \) is a solution of operator equation (12), and \( u = u(t, x; p) \) is a solution of direct problem (1), (2) with this \( p \), then the pair \( \{u(t, x); p(x)\} \) is a generalized solution of inverse problem (1)–(3).

The proof of this lemma is standard — see, for example, [4].

**Theorem 2.** Let the conditions (A)–(E) hold, the numbers \( q^* \) and \( \lambda^* \) are defined in (6). Suppose that

\[ \beta = \frac{12}{90} \exp(\lambda^* T)K_{g,\rho} \left[ \frac{t^2}{\pi^2}(\rho_3^2 + TK_{c,\rho}^2 + TK_{\rho,\lambda}^2) + TK_{\lambda}^2K_b^2 \right] < 1. \]

Then there exists a generalized solution of inverse problem (1)–(3) and \( u \in W_{q^*}^{1,2}(Q) \). Moreover, such a solution is unique and the following estimates

\[ \|p\|_2 \leq \frac{1}{(1 - \sqrt{\beta})90} \left\{ \exp(\lambda^* T/2) \left[ \frac{t^2}{\pi^2}(\rho_3^2 + TK_{c,\rho}^2 + TK_{\rho,\lambda}^2) + TK_{\lambda}^2K_b^2 \right]^{1/2}(\|u_0\|_2 + 3K_{r,\rho})^{1/2} + K_{r,\rho}^* + K_{\lambda}K_{r,\lambda}T^{1/2} + \rho_4 \|u_0\|_2 \right\}, \]
In the present section we assume that in the equation (1) \( b \). Solvability of inverse problem. The second variant of sufficient conditions for example, [7, p. 43]). Estimate (15), (8) and (9) were proved in theorem 1.

From (18), (19) taking into account assumption (13) we derive the contractibility of operator \( A \) and thus the operator equation (12) is uniquely solvable. Therefore in view of lemma 1 there exists a generalized solution of the inverse problem (1)–(3) and it is unique. The estimate (14) and thus the operator equation (12) is uniquely solvable. Therefore in view of lemma 1 there exists a generalized solution of the inverse problem (1)–(3) and it is unique. The estimate (14)

\[
\sup_{0 \leq t \leq T} \|u_x(t, \cdot)\|_2^2 \leq \exp(\lambda^*T) \left[ \|u'_0\|_2^2 + 6K_{g,p}\|p\|_2^2 + 6K_{r,p} \right],
\]

(15)
as well as (8) and (9) hold.

Proof. Suppose that \( p^{(1)}, p^{(2)} \in E \) and \( u^{(i)}(t, x) = u(t, x; p^{(i)}) \) for \( i = 1, 2 \) are solutions of corresponding direct problems (1), (2). Set \( v(t, x) = u^{(1)}(t, x) - u^{(2)}(t, x) \) and \( y(x) = p^{(1)}(x) - p^{(2)}(x) \). Then the pair \( \{v(t, x); y(x)\} \) satisfies the relations

\[
\rho(t, x)v_t - v_{xx} - b(t, x)v_x - c(t, x)v = g(t, x)y(x), \quad (t, x) \in Q,
\]

(16)

\[
v(0, x) = 0, \quad x \in (0, l) \quad v(t, 0) = v(t, l) = 0, \quad t \in [0, T].
\]

(17)

By virtue of (11), the conditions (A)–(E) and inequality (4) we have

\[
\|A^{(1)} - A^{(2)}\|_2 \leq \frac{4}{g_0^2} \left[ \rho_2^2 \|v(T, \cdot)\|_2^2 + K_{\chi}^2 K_{\rho}^2 \|v_x\|_2^2 + (K_{\chi}^2 + K_{\rho}^2) \|v\|_2^2 \right] \leq
\]

\[
\leq \frac{4}{g_0^2} \left[ \frac{\lambda^2}{\pi^2} (p_{\chi}^2 + T K_{\chi}^2 + T K_{\rho}^2) + T K_{\chi}^2 K_{\rho}^2 \right] \sup_{0 \leq t \leq T} \|v_x(t, \cdot)\|_2^2.
\]

(18)

On the other hand due to the estimate(7) applied to the function \( v(t, x) \) via relations (16), (17) we obtain that

\[
\sup_{0 \leq t \leq T} \|v_x(t, \cdot)\|_2^2 \leq 3 \exp(\lambda^*T) \int_0^T \sup_{0 \leq x \leq l} g(t, x) \ dt \cdot \int_0^l g^2(x) \ dx \leq
\]

\[
\leq 3K_{g,p} \exp(\lambda^*T) \cdot \|p^{(1)} - p^{(2)}\|_2^2.
\]

(19)

From (18), (19) taking into account assumption (13) we derive the contractibility of operator \( A \) and thus the operator equation (12) is uniquely solvable. Therefore in view of lemma 1 there exists a generalized solution of the inverse problem (1)–(3) and it is unique. The estimate (14) follows from the well-known error estimate of the first approximation in iterative method (see, for example, [7, p. 43]). Estimate (15), (8) and (9) were proved in theorem 1.

3. Solvability of inverse problem. The second variant of sufficient conditions

In the present section we assume that in the equation (1) \( b(t, x) \equiv b(x) \) and the conditions (A)–(E) hold.

Performing the transformation of the right-hand side of equation (1)

\[
g(t, x)p(x) \equiv \frac{g(t, x)}{G(x)} G(x)p(x) \equiv \tilde{g}(t, x)\tilde{p}(x),
\]

we reduce our problem (1)–(3) to the equivalent problem of the same form, but for which the equality \( I(\tilde{g})(x) \equiv 1, \quad x \in [0, l] \), holds. Thus, without loss of generality we assume that in the initial problem (1)–(3) \( G(x) \equiv 1 \), while we retain the previous notation for the input data and unknown functions.

Denote by \( z^0(t, x) \) the solution of direct problem (1), (2) for \( p(x) \equiv 0 \) (so that \( z^0(t, x) = u(t, x; 0) \)). Let us introduce the new unknown function \( u(t, x) - z^0(t, x) \) for which we retain the notation \( u(t, x) \) and we obtain the inverse problem of finding the pair \( \{u(t, x); p(x)\} \) from the conditions

\[
\rho(t, x)u_t - u_{xx} - b(x)u_x - c(t, x)u = g(t, x)p(x), \quad (t, x) \in [0, T] \times [0, l],
\]

(20)
\[ u(0, x) = 0, \quad x \in [0, l]; \quad u(t, 0) = u(t, l) = 0, \quad t \in [0, T], \quad (21) \]

\[ \int_0^T u(t, x) \chi(t) \, dt = \varphi(x) - \mathbf{1}(z^0)(x) \equiv \tilde{\varphi}(x), \quad x \in [0, l]. \quad (22) \]

Obviously \( \tilde{\varphi} \in W^2_2(0, l) \cap W^1_4(0, l) \). Further in present section we investigate just the problem \((20)-(22)\). The results obtained for it, obviously, will also hold for the original inverse problem \((1)-(3)\).

Let us consider in \( E \equiv L_2(0, l) \) the cone of nonnegative functions

\[ E_+ = \{ p \in L_2(0, l) \mid p(x) \geq 0, \quad x \in [0, l] \}. \]

This cone is closed and reproducing (see, for example, [8]). We define the linear operator \( B : E \rightarrow E \) by the formula

\[ Bp = \rho(T, x) \chi(T) u(T, x; p) - \int_0^T [(\rho(t, x) \chi(t))_t + c(t, x) \chi(t)] u(t, x; p) \, dt, \quad (23) \]

where \( u = u(t, x; p) \) is a solution of direct problem \((20), (21)\) with chosen function \( p \in E \) in the right-hand side of \((20)\). According to the theorem 1 the operator \( B \) is defined on the whole space \( E \) and is bounded, i.e. \( B \in \mathcal{L}(E) \) and following estimates of solutions of direct problem \((20), (21)\)

\[ u \in C(Q), \quad \|u(t, \cdot)\|_2 \leq \text{const} \cdot \|p\|_2 \quad \text{for all} \quad t \in [0, T], \quad (24) \]

\[ \|u_x(t, \cdot)\|_2 \leq \text{const} \cdot \|p\|_2 \quad \text{for all} \quad t \in [0, T], \quad (25) \]

hold. Let us consider the operator equation of the second kind in the space \( E \):

\[ (I - B)p = \psi, \quad (26) \]

where \( \psi(x) = -\tilde{\varphi}''(x) - b(x)\tilde{\varphi}'(x), \psi \in E \) by virtue to conditions \((B)-(D)\). As in [4] we prove the following lemma.

**Lemma 2.** Let conditions \((A)-(E)\) hold. Then the inverse problem \((20)-(22)\) is equivalent to the operator equation \((26)\).

In addition to \((A)-(E)\) we assume that the following conditions are satisfied:

\[ \chi(t) \geq 0, \quad t \in [0, T]; \quad g(t, x) \geq 0, \quad (\rho(t, x) \chi(t))_t + c(t, x) \chi(t) \leq 0, \quad (t, x) \in Q. \quad (27) \]

In the lemmas below we establish a number of properties for operator \( B \) from which the solvability of the equation \((26)\) will follow.

**Lemma 3.** Let conditions \((A)-(E)\) and \((27)\) hold. Then \( B \) is a positive operator.

The proof is done in [4].

**Lemma 4.** Let conditions \((A)-(E)\) hold. Then \( B \in \mathcal{L}(E) \) is a compact operator.

**Proof.** We represent \( B \) in the form

\[ Bp = B_1p + B_2p, \quad \text{where} \quad B_1p = \rho(T, x) \chi(T) u(T, x; p), \]

\[ B_2p = - \int_0^T [(\rho(t, x) \chi(t))_t + c(t, x) \chi(t)] u(t, x; p) \, dt \equiv \int_0^T R(t, x) u(t, x; p) \, dt. \]
By virtue of the estimate (25) the operator $\tilde{B}_1 p = u(t, x; p)$ transforms each bounded set of space $E$ to a set bounded in $W^2_2(0, l)$ which is compact in $E$. Then the operator $B_1$ is also compact, since it is a product of the operator bounded in $E$ on the compact operator.

Let us consider the operator $B_2$. Taking into account conditions $(B)$, $(D)$, it follows that $R(t, x) \equiv -\sqrt{\rho(c/\sqrt{p})} x \in L_1(0, T; L_\infty(0, l))$. We extend $R(t, x)$ by zero for $t > T$ and introduce for it the mean Steklov functions

$$R^h(t, x) = \frac{1}{h} \int_t^{t+h} R(\tau, x) d\tau.$$ 

Using continuous in mean for functions from $L_1(0, T; L_\infty(0, l))$ (see [9, chapter 4, §1]) it is easy to check that

$$\int_0^T \|R(t, \cdot) - R^h(t, \cdot)\|_\infty dt \to 0, \quad h \to 0 + .$$ (28)

We define a family of operators $B_2^h : E \to E$ by the formula

$$B_2^h p = \int_0^T R^h(t, x) u(t, x; p) dt.$$ 

Since $R^h \in C(0, T; L_\infty(0, l))$, $u \in C(Q)$ and estimate (24) hold it is obvious that $B_2^h \in \mathcal{L}(E)$. By virtue of the estimates (24), (25), the operators $R^h p = R^h(t, x) u(t, x; p)$ are compact in $E$ for any $t \in [0, T]$, since they are the products of the bounded operators on the compact operator. Therefore the operators $B_2^h$ are also compact as they are the limits of Riemann integral sums in the norm of $\mathcal{L}(E)$.

The simplest estimate shows that according to (28)

$$\|B_{2} p - B_{2}^h p\|_E \leq \text{const} \cdot \|p\|_E \cdot \int_0^T \|R(t, \cdot) - R^h(t, \cdot)\|_\infty dt \to 0, \quad h \to 0 + .$$

Therefore $B_{2}^h \to B_2$ in $\mathcal{L}(E)$, and hence $B$ is compact operator in $E$ (see [7, p. 191]). Lemma is proved.

From lemmas 2, 4 we obtain the following statement well–known in the theory of inverse problems for uniformly parabolic equations (see [10, 11], etc.).

**Theorem 3.** Let conditions $(A)$–$(E)$ hold. Then the inverse problem (20)–(22) is equivalent to the operator equation of second kind with compact operator in $E$, i.e. this inverse problem has a Fredholm property.

The proof of following lemma is done in [4].

**Lemma 5.** Let conditions $(A)$–$(E)$ and (27) hold. Suppose that $\chi(t) > 0$, $t \in [0, T]$. Then the spectral radius $r(B)$ of the operator $B$ is less then one, i.e. $r(B) < 1$.

Now we obtain a theorem on unique solvability of the inverse problem (20)–(22).

**Theorem 4.** Let conditions $(A)$–$(E)$, (27) hold. Suppose that $\chi(t) > 0$, $t \in [0, T]$, and function $\tilde{\varphi} \equiv \varphi - I(z^0) \in W^2_2(0, l) \cap \hat{W}^1_2(0, l)$. Then there exists a generalized solution of inverse problem (1)–(3). Moreover, such a solution is unique and the estimate

$$\|p\|_E \leq C \cdot \|\psi\|_E ,$$ (29)

where $\psi(x) = -\tilde{\varphi}''(x) - b(x)\tilde{\varphi}'(x)$, as well as estimates (8), (9) hold.

**Proof.** By virtue of lemma 5 $r(B) < 1$, therefore the operator equation of second kind (26) has a unique solution $p \in E$ and the estimate of stability (29) holds. Then by virtue of lemma 2 the inverse problem (20)–(22) has a generalized solution which is unique. The estimates for $u$ follow from (29) and the estimates for the solution of direct problem from theorem 1.
4. Examples

Let us present some examples of inverse problems for which the above-proved theorems hold.

Example 1. Consider the inverse problem

\[(T - t)^{2/3} u_t - u_{xx} = p(x)T, \quad (t, x) \in Q,\]

\[u(0, x) = u_0(x), \quad x \in [0, l]; \quad u(0, t) = u(l, t) = 0, \quad t \in [0, T],\]

\[\frac{1}{T} \int_0^T u(t, x) \, dt = \varphi(x), \quad x \in [0, l],\]

with arbitrary functions \(u_0 \in W^{1,2}(0, l)\) and \(\varphi \in W^{2,2}(0, l), \varphi(0) = \varphi(l) = 0.\)

Obviously, for this problem all the conditions of the theorem 4 are satisfied. Therefore the inverse problem (30)–(32) is uniquely solvable.

Example 2. Consider the inverse problem for the equation

\[(T - t)^{2/3} (x + 1) u_t - u_{xx} + (T - t)^{1/3} u_x + (T - t)^{1/3} u = p(x), \quad (t, x) \in Q,\]

with initial and boundary conditions (31) and additional condition (32). It is easy to verify that for this inverse problem the conditions (A)–(E) are fulfilled. The condition (13) can be written in the form

\[36l^2 \exp\left(\frac{3T(\pi^2 + l^2)}{\pi^2}\right) \left[\frac{4l^2(l^2 + 3l + 3)}{9\pi^2 T^{1/3}} + \frac{3T(\pi^2 + l^2)}{5\pi^2}\right] < 1.\]

Thus the condition (13) of theorem 2 is certainly valid if \(T\) is fixed and \(l\) is sufficiently small. So in this case the inverse problem (33), (31), (32) has a generalized solution which is unique.

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