GLOBAL WELL-POSEDNESS AND TIME-DECAY ESTIMATES OF THE COMPRESSIBLE NAVIER-STOKES-KORTEWEG SYSTEM IN CRITICAL BESOV SPACES

NOBORU CHIKAMI AND TAKAYUKI KOBAYASHI

Abstract. We consider the compressible Navier-Stokes-Korteweg system describing the dynamics of a liquid-vapor mixture with diffuse interphase. The global solutions are established under linear stability conditions in critical Besov spaces. In particular, the sound speed may be greater than or equal to zero. By fully exploiting the parabolic property of the linearized system for all frequencies, we see that there is no loss of derivative usually induced by the pressure for the standard isentropic compressible Navier-Stokes system. This enables us to apply Banach's fixed point theorem to show the existence of global solution. Furthermore, we obtain the optimal decay rates of the global solutions in the $L^2(\mathbb{R}^d)$-framework.

1. Compressible Navier-Stokes-Korteweg system

We consider the following barotropic compressible Navier-Stokes-Korteweg system endowed with an internal local capillarity:

$$\begin{cases}
\partial_t \rho + \text{div} (\rho u) = 0, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) + \nabla (P(\rho)) = \text{div} (\tau(\rho, \nabla u) + K(\rho)), \\
(\rho, u)|_{t=0} = (\rho_0, u_0),
\end{cases}$$

(1.1)

where $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $d \geq 2$, $\rho = \rho(t, x) \in \mathbb{R}_+$ and $u = u(t, x) \in \mathbb{R}^d$ are unknowns representing the fluid density and the velocity vector field, respectively. The viscous strain tensor $\tau$, in general, is given by

$$\tau = \tau(\rho, Du) = 2\mu(\rho)D(u) + \lambda(\rho)\text{div } u \text{ Id},$$

with Id denoting an identity matrix and $D(u)$ designating the deformation tensor defined by

$$D(u) := \frac{1}{2}(\nabla u + {}^t(\nabla u)) \quad \text{with} \quad (\nabla u)_{ij} := \partial_i u_j,$$

where $^t(\nabla u)$ denotes the transposed matrix of $\nabla u$. For simplicity we assume that the viscosity coefficients $\lambda$ and $\mu$ are given constants satisfying $\mu > 0$ and $\lambda + 2\mu > 0$. We write $\mathcal{L}u = \text{div} (2\mu Du) + \nabla (\lambda \text{div } u)$. When $\mu$ and $\lambda$ are constants, then we see that $\text{div} (\tau) = \mathcal{L}u = \mu \Delta u + (\mu + \lambda)\nabla \text{div } u$. The pressure is assumed to be barotropic, i.e., $P$ is a sufficiently smooth function of $\rho$.

In the general Korteweg system, the capillary term is given by $\text{div} (K(\rho))$, where $K(\rho)$ is the Korteweg tensor defined by

$$K(\rho) = \frac{\kappa(\rho)}{2}(\Delta \rho^2 - |\nabla \rho|^2)\text{Id} - \kappa(\rho)\nabla \rho \otimes \nabla \rho,$$

where $\kappa(\rho)$ is a sufficiently smooth function of $\rho$.

Date: December 31, 2018.
where $\nabla \rho \otimes \nabla \rho$ stands for the tensor product $(\partial_j \rho \partial_k \rho)_{jk}$. The coefficient $\kappa$ may depend on $\rho$ in general. Here, we assume the case of positive constant coefficient $\kappa(\rho) \equiv \kappa$. We assume that there is no vacuum, and $\rho$ tends to a positive constant $\rho_*$ at spatial infinity.

The system describes the dynamics of a liquid-vapor mixture in the setting of the Diffuse Interface approach: between the two phases lies a thin region of continuous transition and the phase changes are described through the variations of the density (with for example a Van der Waals pressure). Historically, the system is first proposed by J. D. Van der Waals (1893), and later Korteweg tensor is introduced by D. J. Korteweg (1901). The system considered here is a version reformulated by J. E. Dunn and J. Serrin [13] in 1985.

An important features of System (1.1) is that the pressure is non-monotone in general. Simply put, $P'(\rho_*)$ is no longer a positive number, which inevitably leads to linear instability for some value of $\rho_*$. As such, most mathematical results (in the whole space) until now deal with the condition that $P'(\rho_*)$ is strictly positive, except for local-in-time results.

There are numerous works dedicated to the study of the system with the capillarity term. For known well-posedness results for System (1.1), global smooth solutions are established by [14, 15]. Decay estimates along with the analysis of diffusive wave phenomenon are considered by T. Kobayashi and K. Tsuda [16]. The latter also obtains the existence of time-periodic solutions along with their stability in [20]. See also [17,19] for the existence and decay results. In the critical framework, R. Danchin and B. Desjardins [11] first obtain the well-posedness in critical Besov spaces. Especially, they consider a general non-monotone pressure to show local existence of the solution. For results on non-local capillary terms and convergence to various models, we refer to the works by F. Charve and B. Haspot [4–6]. In [18], T. Tang and H. Gao obtain the global solution for System (1.1) in $\mathbb{T}^d$ under a non-monotone pressure law.

Two difficulties usually arise in the analysis of the compressible viscous fluids. One is the loss of derivative induced by the pressure term and the other is the appearance of the second order nonlinear term. Both of them preclude the application of standard semi-group theory and necessitate the laborious energy estimates. In contrast to the compressible viscous fluids without the capillary term, however, System (1.1) is known to have one mathematical upside, which is that there is no longer a loss of derivative due to the parabolic smoothing effect for both the velocity and the density.

We may check that a scaling invariance holds for System (1.1): namely a family of rescaled functions $(\rho, u) \rightarrow (\rho_\nu, u_\nu)$ with

$$
\rho_\nu(t, x) = \rho(\nu^2 t, \nu x) \quad \text{and} \quad u_\nu(t, x) = \nu u(\nu^2 t, \nu x)
$$

leaves (1.1) invariant provided that $P$ is changed into $\nu^2 P$. The pressure term, to some extent, can be regarded as a lower order term in our setting and therefore it is possible to solve the system in the scale-critical function spaces, i.e., the spaces in which the norm is kept invariant under the transforms (1.2). Moreover, we clarify in this paper that it suffices to assume that the pressure is merely non-negative, i.e., $P'(\rho_*) \geq 0$, in order to obtain global solutions in critical Besov spaces. We establish a priori estimates for the linearized system that is uniform with respect to the parameter $P'(\rho_*) \geq 0$ (See Lemma 2.5). We also show the decay estimates of such global small solutions under additional $L^1(\mathbb{R}^d)$-type regularity assumption.

We remark that although we focus our analysis in the $L^2(\mathbb{R}^d)$-setting, it is not difficult to extend our decay results to the $L^p(\mathbb{R}^d)$-framework in the spirit of [12]. In fact, F. Charve, R. Danchin and J. Xu have recently shown the existence of solution for System
\textbf{(CNSK)} in $L^p(\mathbb{R}^d)$-framework along with the Gevrey analyticity of the solution \cite{7}. As a corollary of the analyticity, they obtain decay estimates of higher order norms with respect to a critical data. Note that contrary to \cite{7} our decay estimates deal with the decay rates of critical norms with respect to a $L^1(\mathbb{R}^d)$ data, which naturally requires more labour and cannot be deduced from their results. Furthermore, our results for the critical case $P'(\rho_*) = 0$ seems completely new.

1.1. \textbf{Notations and function spaces.} Hereafter, we denote by $L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, the standard Lebesgue spaces on the $d$-dimensional Euclidean space $\mathbb{R}^d$, and by $L^p(\mathbb{Z})$ the set of sequences with summable $p$-th powers. Unless otherwise stated, we assume $d \geq 2$ throughout the paper. We abbreviate $\| \cdot \|_{L^p(\mathbb{R}^d)} = \| \cdot \|_L^p$ whenever the dimension $d$ is not relevant, and denote the time-space Lebesgue norm by $\| \cdot \|_{L^1(0,T;L^p)} = \| \cdot \|_{L^1(0,T;L^p)}$. An infinitely differentiable, complex-valued function $f$ on $\mathbb{R}^d$ is called a Schwartz function if for every pair of multi-indices $\alpha$ and $\beta$ there exists a positive constant $C_{\alpha,\beta}$ such that $\sup_{x \in \mathbb{R}^d} |x^{\alpha} \partial^\beta f| = C_{\alpha,\beta} < \infty$. We denote the set of all Schwartz functions on $\mathbb{R}^d$ by $S(\mathbb{R}^d)$ and its topological dual by $S'(\mathbb{R}^d)$. Elements in $S'(\mathbb{R}^d)$ are called tempered distributions.

We define the fractional derivative operator by

$$\Lambda^s := (-\Delta)^{s/2} = \mathcal{F}^{-1} \cdot |\cdot|^s \mathcal{F}$$

for $s \in \mathbb{R}$, where $\mathcal{F} u = \hat{u}$ denotes the Fourier transform of $u \in S'(\mathbb{R}^d)$.

We define a subspace of $S'(\mathbb{R}^d)$ which lays a basic foundation of our analysis. For the details, we refer to \cite{2}.

**Definition 1.1** (\cite{2} P22). We denote by $S'_0(\mathbb{R}^d)$ the space of tempered distributions $u$ such that $\lim_{\lambda \to \infty} \| \mathcal{F}^{-1} [\theta(\lambda \cdot) \hat{u}] \|_{L^\infty} = 0$ for any $\theta$ in $S(\mathbb{R}^d)$.

In what follows, we denote by $C$ a generic constant that may change from line to line. We write $A \cong B$ whenever there exists a positive constant $c$ such that $c^{-1} A \leq B \leq cA$. We also denote $A \leq B$ whenever there exists arbitrary harmless constant $C$ such that $A \leq CB$. Constants depending on parameters $a, b, \ldots, c$ are denote by $C(a, b, \ldots, c)$. We agree that $p'$ always denotes the Hölder conjugate of $p$, i.e., $\frac{1}{p'} + \frac{1}{q'} = 1$, with the convention $\frac{1}{\infty} = 0$. Vectors in this article are understood as column vectors in general; however, we omit the sign of transpose $t$ whenever it is clear from the context or simply irrelevant. We express a function $u$ of variables $(t,x)$ as $u$, $u(t)$ or $u(t,x)$ depending on circumstances.

1.1.1. **Besov spaces.** Let $\{\phi_j\}_{j \in \mathbb{Z}}$ be a homogeneous Littlewood-Paley dyadic decomposition. Namely, let $\hat{\phi} \in S(\mathbb{R}^d)$ be a non-negative, radially symmetric function that satisfies

$$\text{supp} \ \hat{\phi} \subset \{ \xi \in \mathbb{R}^d; 2^{-1} |\xi| \leq 2 \} \quad \text{and} \quad \sum_{j \in \mathbb{Z}} \hat{\phi}(2^{-j} \xi) = 1 \quad \text{for all} \quad \xi \neq 0.$$

Setting $\hat{\phi}_j(\xi) := \hat{\phi}(2^{-j} \xi)$ and $\hat{\Phi}_{j-1}(\xi) := 1 - \sum_{k \geq j} \hat{\phi}_k(\xi)$ for $j \in \mathbb{Z}$, we have

$$\text{supp} \ \hat{\Phi}_{j-1} \subset \{ \xi \in \mathbb{R}^d; |\xi| \leq 2^j \} \quad \text{and} \quad \text{supp} \ \hat{\phi}_j \subset \{ \xi \in \mathbb{R}^d; 2^{j-1} \leq |\xi| \leq 2^{j+1} \}$$

for all $j \in \mathbb{Z}$. Given an element $u$ in $S'_0(\mathbb{R}^d)$ and a (homogeneous) Littlewood-Paley decomposition $\{\phi_j\}_{j \in \mathbb{Z}}$, we define the (homogeneous) Littlewood-Paley localization operator by

$$\hat{\Delta}_ju := \mathcal{F}^{-1}[\hat{\phi}_j \hat{u}] \quad \text{(1.3)}$$
and the frequency cut-off operator by \( \hat{S}_j u := \sum_{j' \leq j} \Delta_{j'} u = F^{-1}[\hat{\Phi}_j \hat{u}] \) for \( j \in \mathbb{Z} \).

**Definition 1.2** (Homogeneous Besov space). For \( s \in \mathbb{R} \) and \( 1 \leq p, \sigma \leq \infty \) we define the homogeneous Besov spaces \( \dot{B}^s_{p,\sigma}(\mathbb{R}^d) \) as follows:

\[
\dot{B}^s_{p,\sigma}(\mathbb{R}^d) := \{ u \in S'_0(\mathbb{R}^d) : \| u \|_{B^s_{p,\sigma}(\mathbb{R}^d)} < \infty \}, \quad \| u \|_{B^s_{p,\sigma}(\mathbb{R}^d)} := \left\{ \left\| 2^j \| \hat{\Delta}_j u \|_{L^p} \right\|_{j \in \mathbb{Z}} \right\|_{\ell^\sigma}.
\]

For notational convenience, we abbreviate \( \| \cdot \|_{\dot{B}^s_{p,\sigma}(\mathbb{R}^d)} = \| \cdot \|_{\dot{B}^s_{p,\sigma}} \).

More properties of Besov spaces may be found in e.g. [2].

For some fixed \( j_0 \in \mathbb{Z} \), we denote by \( u_h := \hat{S}_{j_0} u \) the low frequencies of \( u \), and by \( u_h := u - u_h \) the high frequencies of \( u \). We also need the notation

\[
\| u \|_{B^j_{2,1}} := \sum_{j \leq j_0} 2^{js} \| \hat{\Delta}_j u \|_{L^2} \quad \text{and} \quad \| u \|_{B^j_{2,1}} := \sum_{j \geq j_0 - 1} 2^{js} \| \hat{\Delta}_j u \|_{L^2}.
\]

The small overlap between low and high frequencies are to ensure that

\[
\| u \|_{B^j_{2,1}} \leq C \| u \|_{B^j_{2,1}} \quad \text{and} \quad \| u_h \|_{B^j_{2,1}} \leq C \| u \|_{B^j_{2,1}}.
\]

Lastly, we introduce the so-called Chemin-Lerner spaces.

**Definition 1.3** (Chemin-Lerner spaces. [2][9]). For \( t_1, t_2 > 0, s \in \mathbb{R}, 1 \leq \rho, \sigma \leq \infty \), we set

\[
\| u \|_{\tilde{L}^\rho(t_1, t_2; \dot{B}^s_{p,\sigma})} := \left\{ \left\| 2^j \| \hat{\Delta}_j u \|_{L^\rho(t_1, t_2; L^p)} \right\|_{j \in \mathbb{Z}} \right\|_{\ell^\sigma}.
\]

We then define the space \( \tilde{L}^\rho(t_1, t_2; \dot{B}^s_{p,\sigma}(\mathbb{R}^d)) \) as the set of tempered distributions \( u \) over \( (t_1, t_2) \times \mathbb{R}^d \) such that \( \lim_{j \to -\infty} \hat{S}_j u = 0 \) in \( S'( (t_1, t_2) \times \mathbb{R}^d) \) and \( \| u \|_{\tilde{L}^\rho(t_1, t_2; \dot{B}^s_{p,\sigma})} < \infty \).

When \( t_1 = 0 \) and \( t_2 = T \), we denote \( \tilde{L}^\rho_T(\dot{B}^s_{p,\sigma}(\mathbb{R}^d)) = \tilde{L}^\rho(0, T; \dot{B}^s_{p,\sigma}(\mathbb{R}^d)) \) with the shortened expression of the norm \( \| \cdot \|_{\tilde{L}^\rho_T(\dot{B}^s_{p,\sigma})} \).

In what follows, we agree that

\[
\tilde{C}([0, T]; \dot{B}^s_{2,1}) := \{ v \in C([0, T]; \dot{B}^s_{2,1}) : \| v \|_{\tilde{L}^\infty_T(\dot{B}^s_{2,1})} < \infty \}.
\]

**1.2. Momentum formulation and main results.** Setting \( m := \rho u \), we recast System (1.1) in the momentum formulation, which reads as follows:

\[
\begin{aligned}
\partial_t \rho + \text{div} \, m &= 0, \\
\partial_t m + \text{div} \, (\rho^{-1} m \otimes m) + \nabla P &= \mathcal{L}(\rho^{-1} m) + \text{div} \, (K(\rho)), \\
(m, \rho)|_{t=0} &= (\rho_0, m_0).
\end{aligned}
\]

(CNSK)

The clear difference from the velocity formulation is that now the mass conservation law is linear while the capillary term is nonlinear. Also notable is that there is \( \rho^{-1} \) inside the second-order term, thereby requiring more regularity to the density \textit{a priori}. This poses no obstacle in the actual analysis since, as is well-known, there is a parabolic smoothing effect for the density in System (CNSK). In this paper, we perform our analysis in the momentum formulation. In fact, the momentum formulation turns out to be essential for our analysis for the case \( \gamma = 0 \); Particularly, the fact the the nonlinear terms may be written in divergence form greatly helps our analysis in \( d = 3, \) as we see in the proof of Lemma 3.4.
1.2.1. Global existence. Throughout this paper, we set \( \gamma := P'(\rho_*) \). We first state a result of well-posedness for \( d \geq 2 \) and \( \gamma \geq 0 \), which is contained in \([11]\). Under smallness of the initial data, the solution is globally extended when \( \gamma > 0 \).

**Theorem 1.4** \([11]\). Let \( d \geq 2 \) and let \( P \) be a smooth function such that \( \gamma \geq 0 \). For sufficiently small initial data such that

\[
((\gamma + \Lambda)(\rho_0 - \rho_*), m_0) \in \dot{B}^\frac{4}{d-1}_{2,1}(\mathbb{R}^d),
\]

there exists a time \( T > 0 \) and a unique solution \((\rho, m)\) to (CNSK) satisfying

\[
((\gamma + \Lambda)(\rho - \rho_*), m) \in \mathcal{C}([0, T]; \dot{B}^\frac{4}{d-1}_{2,1}(\mathbb{R}^d)) \cap L^1(0, T; \dot{B}^\frac{4}{d-1}_{2,1}(\mathbb{R}^d)).
\]

Moreover, if \( \gamma > 0 \) then \( T = \infty \) and there exists a constant \( C \) depending on \( d, \mu, \lambda, \rho_* \) and \( P \) such that

\[
\|((\gamma + \Lambda)(\rho - \rho_*), m)\|_{L^\infty(0, \infty; \dot{B}^\frac{4}{d-1}_{2,1})} + \|\Lambda^2((\gamma + \Lambda)(\rho - \rho_*), m)\|_{L^1(0, \infty; \dot{B}^\frac{4}{d-1}_{2,1})} \leq C\|((\gamma + \Lambda)(\rho_0 - \rho_*), m_0)\|_{\dot{B}^\frac{4}{d-1}_{2,1}}.
\]

**Remark 1.5.** As observed in \([11]\), as far as a local solution is concerned, the smallness of the initial data and the condition \( \gamma \geq 0 \) may be removed.

The question remains whether we may obtain a global solution when the sound speed is zero, i.e., when \( \gamma = 0 \). It turns out that restricting the dimension to \( d \geq 3 \), we have the global well-posedness for the case under an additional low-frequency assumption.

**Theorem 1.6.** Let \( d \geq 3 \) and let \( P \) be a smooth function such that \( \gamma = 0 \). For sufficiently small initial data such that

\[
(\Lambda(\rho_0 - \rho_*), m_0) \in \dot{B}^{\frac{4}{d-3}}_{2,1}(\mathbb{R}^d) \cap \dot{B}^{\frac{4}{d-1}}_{2,1}(\mathbb{R}^d),
\]

there exists a unique solution \((\rho, m)\) to (CNSK) satisfying

\[
(\Lambda(\rho - \rho_*), m) \in \mathcal{C}([0, \infty); \dot{B}^{\frac{4}{d-3}}_{2,1}(\mathbb{R}^d) \cap \dot{B}^{\frac{4}{d-1}}_{2,1}(\mathbb{R}^d)) \cap L^1(0, \infty; \dot{B}^{\frac{4}{d-1}}_{2,1}(\mathbb{R}^d) \cap \dot{B}^{\frac{4}{d+1}}_{2,1}(\mathbb{R}^d)).
\]

Moreover, there exists a positive constant \( C \) depending on \( d, \mu, \lambda, \rho_* \) and \( P \) such that

\[
\|\langle \Lambda(\rho - \rho_*), m\rangle\|_{L^\infty(0, \infty; \dot{B}^{\frac{4}{d-1}}_{2,1})} + \|\Lambda^2\langle (\Lambda(\rho - \rho_*), m)\rangle\|_{L^1(0, \infty; \dot{B}^{\frac{4}{d-1}}_{2,1})} \leq C\|\langle \Lambda(\rho_0 - \rho_*), m_0\rangle\|_{\dot{B}^{\frac{4}{d-1}}_{2,1}}.
\]

**Remark 1.7.** In both Theorem 1.4 and Theorem 1.6, the space \( \mathcal{C}(0, T; \dot{B}^s_{2,1}(\mathbb{R}^d)) \) is defined by Definition 1.5.

Let us remark that although we focus our analysis for the barotropic case, our technique can be also applied to non-isentropic cases. The results can also be naturally extended in the settings of torus and variable capillarity and viscosity coefficients.

1.2.2. Decay estimates. Next, we clarify the decay rates of the solution constructed by Theorem 1.4 and Theorem 1.6.
We first state a result for the case $\gamma > 0$. Setting $\langle t \rangle := (1 + t)$, we introduce an auxiliary norm
\[
D(t) := \sup_{s \in (-\frac{d}{2}, \frac{d}{2} + 1]} \| (\langle t \rangle)^{\frac{1}{2}(s + \frac{d}{2})}((\gamma + \Lambda)(\rho - \rho_*), m) \|^t_{L^\infty_t(B^d_{2, 1})} \]
\[
+ \| \tau^\alpha \Lambda^2((\gamma + \Lambda)(\rho - \rho_*), m) \|^h_{L^\infty_t(B^d_{2, 1})}. \tag{1.6}
\]
Here and in what follows, we agree that the notation $\langle \tau \rangle^\sigma f$, $\sigma \in \mathbb{R}$, designates the function $(\tau, x) \mapsto (\tau)^\sigma f(\tau, x)$. The notations $\| \cdot \|^t_{L^\infty_t(B^d_{2, 1})}$ and $\| \cdot \|^h_{L^\infty_t(B^d_{2, 1})}$ are defined analogously to (1.4).

Under additional assumptions on the low frequencies of the initial data, one may obtain time-decay estimates that are reminiscent of those of the standard compressible Navier-Stokes equations (see [12]).

**Theorem 1.8.** Let $\gamma > 0$. Let the data $(a_0, m_0)$ satisfy the assumptions of Theorem 1.4. There exist a frequency-threshold $j_0$ only depending on $d$, $\mu$, $\lambda$, $\kappa$ and $\gamma$ and a positive constant $c$ so that if in addition
\[
D_0 := \| (\rho_0 - \rho_*) \|^t_{B^d_{2, \infty}} + \| m_0 \|^t_{B^d_{2, \infty}} \leq c
\]
then the global solution $(a, m)$ given by Theorem 1.4 satisfies for all $t \geq 0$,
\[
D(t) \leq C(D_0 + \| (\Lambda(\rho_0 - \rho_*), m_0) \|^h_{B^{d, -1}_{2, 1}}) \tag{1.7}
\]
where $D(t)$ is defined by (1.6) and $\alpha := \frac{d}{2} + \frac{1}{2} - \varepsilon$ for sufficiently small $\varepsilon > 0$.

We next state the result for the case $\gamma = 0$. We set
\[
\widetilde{D}(t) := \sup_{s \in (-\frac{d}{2}, \frac{d}{2} + 1]} \| (\langle t \rangle)^{\frac{1}{2}(s + \frac{d}{2})}(\Lambda(\rho - \rho_*), m) \|^t_{L^\infty_t(B^d_{2, 1})} + \| \tau^\alpha \Lambda^2(\Lambda(\rho - \rho_*), m) \|^h_{L^\infty_t(B^d_{2, 1})}. \tag{1.8}
\]
For the case $\gamma = 0$, we have the following.

**Theorem 1.9.** Let $\gamma = 0$. Let the data $(\rho_0, m_0)$ satisfy
\[
(\Lambda(\rho_0 - \rho_*), m_0) \in \dot{B}^{\frac{d}{2}}_{2, \infty}(\mathbb{R}^d) \cap \dot{B}^{\frac{d}{2} - 1}_{2, 1}(\mathbb{R}^d). \tag{1.9}
\]
There exist a frequency-threshold $j_0$ only depending on $d$, $\mu$, $\lambda$ and $\kappa$ and a positive constant $c$ so that if in addition
\[
\widetilde{D}_0 := \| (\rho_0 - \rho_*) \|^t_{B^{\frac{d}{2}, \infty}} + \| m_0 \|^t_{B^{\frac{d}{2}, \infty}} \leq c
\]
then the global solution $((\rho - \rho_*), m)$ given by Theorem 1.4 satisfies for all $t \geq 0$,
\[
\widetilde{D}(t) \leq C(\widetilde{D}_0 + \| (\Lambda(\rho_0 - \rho_*), m_0) \|^h_{B^{\frac{d}{2}, -1}_{2, 1}}) \tag{1.10}
\]
where $\widetilde{D}(t)$ is defined by (1.8) and $\alpha := \frac{d}{2} + \frac{1}{2} - \varepsilon$ for sufficiently small $\varepsilon > 0$.

**Remark 1.10.** Note that since $d \geq 3$, we have $-\frac{d}{2} \leq \frac{d}{2} - 3$, which implies $\| u \|^t_{B^{\frac{d}{2} - 3}_{2, 1}} \leq \| u \|^t_{B^{\frac{d}{2}}_{2, \infty}}$ for $d \geq 3$. Thus, the regularity assumption (1.9) in Theorem 1.9 is stronger than...
that of Theorem 1.6 and the corresponding small global solution is bound to naturally arise.

The rest of the paper is organized as follows: in the second section, we study the linearized problem of (CNSK). The third section is devoted to the proof of Theorem 1.4 and Theorem 1.6. In the fourth section, we prove Theorem 1.8 and Theorem 1.9. In the Appendix, we collect some properties of Besov spaces.

2. Linearized Problem

Linearization around a constant state \((\rho_s, 0)\) gives

\[
\begin{aligned}
\partial_t a + \text{div } m &= 0, \\
\partial_t m - \frac{1}{\rho_s} \mathcal{L} m - \kappa \rho_s \nabla a + \gamma \nabla a &= \mathcal{L} (Q(a)m) + \text{div } (K(a)), \\
- \frac{1}{\rho_s} \text{div } (m \otimes m) - \text{div } (Q(a)m \otimes m) + (P'(a + \rho_s) - \gamma) \nabla a, \\
\end{aligned}
\]

(2.1)

where we put \(\gamma := P'(\rho_s)\), \(a := \rho - \rho_s\) and \(Q(a) := \frac{a}{\rho(a + \rho_s)}\). We split \(m\) into compressible and incompressible components, i.e., \(m = w + \mathcal{P} \perp u\), with \(w := \mathcal{P} m\) where \(\mathcal{P}\) and \(\mathcal{P} \perp\) are the projectors onto divergence-free and potential vector-fields, respectively \((w := (\text{Id} + \nabla \text{div} (-\Delta)^{-1}) m)\). Setting \(v := (\nabla^2 \Delta)^{-1} \text{div } m = (\nabla^2 \Delta)^{-1} \text{div } \mathcal{P} \perp m\), we have

\[
\begin{aligned}
\partial_t a + \Lambda_2 v &= 0, \\
\partial_t v - \nu \Delta v - \gamma \Lambda a - \kappa \rho_s \Lambda^3 a &= 0, \\
\partial_t w - \mu \Delta w &= 0, \\
\end{aligned}
\]

(2.2)

where

\[
\nu := \frac{\nu}{\rho_s}, \quad \gamma := 2\mu + \lambda, \quad \mu := \frac{\mu}{\rho_s}. 
\]

(2.3)

We investigate the properties of the above linearized system.

2.1. Maximal regularity estimates in Besov spaces. Let us consider the following standard heat equation with a constant diffusive coefficient \(\nu > 0\):

\[
\begin{aligned}
\partial_t u - \nu \Delta u &= f, \quad t > 0, \quad x \in \mathbb{R}^d, \\
\quad u|_{t=0} &= u_0, \quad x \in \mathbb{R}^d, \\
\end{aligned}
\]

(2.4)

where the outer force \(f: \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}\) and the initial data \(u_0: \mathbb{R}^d \rightarrow \mathbb{R}\) are given functions or tempered distributions. Recall the following estimate for a linear heat equation, which can be found in literatures such as [2] P157 and [10] P7.

Lemma 2.1. Let \(T > 0\), \(s \in \mathbb{R}\) and \(1 \leq p, r, \sigma \leq \infty\). Assume that \(u_0 \in \dot{B}^s_{p,\sigma}(\mathbb{R}^d)\) and \(f \in \dot{L}^{-\frac{s}{2}}_r(\dot{B}^{s-\frac{2}{p}+\frac{2}{r}}_{p,\sigma}(\mathbb{R}^d))\). Then (2.4) has a unique solution \(u\) in \(\dot{L}^{-\frac{s}{2}}_r(\dot{B}^{s-\frac{2}{p}+\frac{2}{r}}_{p,\sigma}(\mathbb{R}^d)) \cap \dot{L}^\infty_T(\dot{B}^{s+\frac{2}{r}}_{p,\sigma}(\mathbb{R}^d))\) and there exists a constant \(C\) depending only on \(q\) and \(r\) such that for all \(q \in [r, \infty]\), we have

\[
\nu^{\frac{1}{q}} ||u||_L^\infty_T(\dot{B}^{s+\frac{2}{r}}_{p,\sigma}) \leq C \left( ||u_0||_\dot{B}^s_{p,\sigma} + \nu^{\frac{1}{q}-\frac{1}{r}} ||f||_L^\infty_T(\dot{B}^{s-\frac{2}{p}+\frac{2}{r}}_{p,\sigma}) \right).
\]

If in addition \(\sigma\) is finite, then \(u\) belongs to \(C([0, T); \dot{B}^s_{p,\sigma}(\mathbb{R}^d))\).

Our aim here is to prove a similar estimate for the linearized system of (CNSK).
Two difficulties usually arise in the analysis of the compressible viscous fluids. One is the loss of derivative induced by the pressure term and the other is the appearance of the second order nonlinear term. Both of them preclude the application of standard semigroup theory and necessitate laborious energy estimates. In contrast to the compressible viscous fluids without the capillary term, System (CNSK) is known to have a pleasant mathematical upside, which is that there is no longer a loss of derivative due to the parabolic smoothing effect for the density.

From hereon, we consider a linear hyperbolic-parabolic equation

\[
\begin{cases}
\partial_t a + \text{div} \ m = f, \\
\partial_t m - \frac{1}{\rho_*} \nabla m + \gamma \nabla a - \kappa \rho_* \nabla \Delta a = g
\end{cases}
\]  

(2.5)

with initial data \((a_0, m_0)\) and derive an a priori estimate in Besov spaces using the method of Lyapunov in the frequency space.

**Lemma 2.2.** Let \(T > 0, \ s \in \mathbb{R}, \ 1 \leq r, \sigma \leq \infty, \ \gamma \geq 0, \)

\[((\gamma + \Lambda) a_0, m_0) \in \hat{B}^{s}_{2,\sigma}(\mathbb{R}^d) \quad \text{and} \quad ((\gamma + \Lambda) f, g) \in \hat{L}^r(0,T; \hat{B}^{s+\frac{\gamma}{2}}_{2,\sigma}(\mathbb{R}^d)).\]

Suppose that \((a, m)\) is a solution for (2.5) such that

\[((\gamma + \Lambda) a, m) \in \hat{L}^q(0,T; \hat{B}^{s+\frac{\gamma}{2}}_{2,\sigma}(\mathbb{R}^d)) \quad \text{for} \quad q \in [r, \infty] \]

with initial data \((a_0, m_0)\). Then there exists some positive constant \(C\) depending only on \(\rho_*, \ \lambda, \ \mu, \ \kappa, \ d, \ q, \ r \) and \(\sigma\) such that for all \(q \in [r, \infty]\), the following estimate hold:

\[
\|((\gamma + \Lambda) a, m)\|_{\hat{L}^q_r(\hat{B}^{s+\frac{\gamma}{2}}_{2,\sigma})} \leq C \left( \|((\gamma + \Lambda) a_0, m_0)\|_{\hat{B}^{s}_{2,\sigma}} + \|((\gamma + \Lambda) f, g)\|_{\hat{L}^r(\hat{B}^{s+\frac{\gamma}{2}}_{2,\sigma})} \right),
\]

for all \(0 < t \leq T\).

**Remark 2.3.** Note that the above constant does not depend the parameter \(\gamma\) and holds for both cases \(\gamma > 0\) and \(\gamma = 0\). It is clear from the above estimates that one needs to impose a stronger control on the low frequencies at a linear level when \(\gamma > 0\), which is in accordance with the known result for the isentropic compressible fluid.

**Proof of Lemma 2.2.** The argument below is inspired by [10]. However, the resulting smoothing effects, as expected, are strikingly better than those of the compressible Navier-Stokes system. After projecting \(m\) onto the compressible part by \(v = \Lambda^{-1} \text{div} \ m\) as before, we first treat the homogeneous system of the first two equations of System (2.2):

\[
\begin{cases}
\partial_t a + \Lambda v = 0, \\
\partial_t v - \nu \Delta v - \gamma \Lambda a - \kappa \rho_* \Lambda^3 a = 0
\end{cases}
\]

where \(\gamma \geq 0\). Taking the Fourier transform, System (2.6) is transformed into

\[
\frac{d}{dt} \begin{bmatrix} \hat{a} \\ \hat{v} \end{bmatrix} = \hat{A}(\xi) \begin{bmatrix} \hat{a} \\ \hat{v} \end{bmatrix} \quad \text{with} \quad \hat{A}(\xi) := \begin{bmatrix} 0 & -1 \\ (\gamma + \kappa \rho_*)|\xi|^2 & \nu |\xi|^2 \end{bmatrix}. \]

The characteristic equation for the matrix \(A(\xi)\) is

\[
X^2 + \nu |\xi|^2 X + (\gamma + \kappa \rho_*)|\xi|^2 = 0,
\]
and $\hat{A}(\xi)$ has two distinct eigenvalues given by:

$$
\lambda_{\pm} = \lambda_{\pm}(\xi) := -\frac{\nu}{2}|\xi|^2 \left(1 \pm \sqrt{1 - \frac{4(\gamma|\xi| + \kappa\rho_*|\xi|^3)|\xi|}{\nu^2|\xi|^4}}\right)
$$

$$
= -\frac{\nu}{2}|\xi|^2 \left(1 \pm \sqrt{1 - \frac{4\kappa\rho_*}{\nu^2} - \frac{4\gamma}{\nu^2}|\xi|^2}\right);
$$

Recall that $\gamma \geq 0$. From the above formula, we readily gather the following:

1. If $\nu^2 < 4\kappa\rho_*^3$, then both $\lambda_{\pm}(\xi)$ are two complex conjugated eigenvalues

$$
\lambda_{\pm}(\xi) = -\frac{\nu}{2}|\xi|^2 \left(1 \pm iS(|\xi|)\right) \quad \text{with} \quad S(|\xi|) = \sqrt{\frac{4\kappa\rho_*}{\nu^2} - 1 + \frac{4\gamma}{\nu^2}|\xi|^2},
$$

regardless of the choice of $\gamma \geq 0$, for all frequencies. If $\gamma > 0$, we have

$$
\lambda_{\pm}(\xi) = -\frac{\nu}{2}|\xi|^2 \left(1 \pm iS(|\xi|)\right) = -\frac{\nu}{2}|\xi|^2 \mp i\sqrt{\gamma}|\xi| + O(|\xi|^3)
$$

as $|\xi| \to 0$ and

$$
\lambda_{\pm}(\xi) = -\frac{\nu}{2}|\xi|^2 \left(1 \pm iS(|\xi|)\right) = -\frac{\nu}{2}|\xi|^2 \mp i\sqrt{\gamma}|\xi| + iO(1)
$$

as $|\xi| \to \infty$. If $\gamma = 0$, we have

$$
\lambda_{\pm}(\xi) = -\frac{\nu}{2} \left(1 \pm i \sqrt{\frac{4\kappa\rho_*}{\nu^2} - 1}\right)|\xi|^2.
$$

Thus, we may expect parabolic smoothing for both components throughout all frequencies.

2. If $\nu^2 = 4\kappa\rho_*^3$ and $\gamma > 0$, then both $\lambda_{\pm}(\xi)$ are two complex conjugated eigenvalues

$$
\lambda_{\pm}(\xi) = -\frac{\nu}{2}|\xi|^2 \left(1 \pm i \frac{\sqrt{4\gamma}}{\sqrt{\nu^2|\xi|^2}}\right) = -\frac{\nu}{2}|\xi|^2 \left(1 \pm i \frac{2\sqrt{\gamma}}{\nu|\xi|}\right) = -\frac{\nu}{2}|\xi|^2 \pm i\sqrt{\gamma}|\xi|
$$

for all frequencies. If $\nu^2 = 4\kappa\rho_*^3$ and $\gamma = 0$, then we have a (real) double root

$$
\lambda_{\pm}(\xi) = -\frac{\nu}{2}|\xi|^2.
$$

3. If $\nu^2 > 4\kappa\rho_*^3$ and $\gamma > 0$ then we have two regions of distinct behaviors.

i. In the low frequency ($|\xi| < \frac{4\gamma\rho_*^2}{\nu^2 - 4\kappa\rho_*^3}$), $\lambda_{\pm}(\xi)$ are two complex conjugated eigenvalues

$$
\lambda_{\pm}(\xi) = -\frac{\nu}{2}|\xi|^2 \left(1 \pm i \left(\frac{2\rho_*\sqrt{\gamma}}{\nu|\xi|} - \frac{\nu^2 - 4\kappa\rho_*^3}{4\kappa\rho_*\nu\sqrt{\gamma}}|\xi| + O(|\xi|^3)\right)\right)
$$

whose real parts are negative and

$$
\lambda_{\pm}(\xi) = -\frac{\nu}{2}|\xi|^2 \mp i\sqrt{\gamma}|\xi| + iO(|\xi|^3) \quad \text{as} \quad |\xi| \to 0.
$$
On the other hand, in the high frequency ($|\xi| > \frac{4\gamma \rho^2}{\nu^2 - 4\kappa \rho^2}$), the two real eigenvalues can be expressed as

$$\lambda_{\pm}(\xi) = -\frac{\nu}{2}|\xi|^2 \left( 1 \pm R(|\xi|) \right) \quad \text{with} \quad R(|\xi|) = \sqrt{1 - \frac{4\kappa \rho^2}{\nu^2} - \frac{4\gamma}{|\xi|^2}}$$

whose real parts are negative and

$$\lambda_{\pm}(\xi) = -\frac{\nu}{2}|\xi|^2 \left( 1 \pm \left( 1 - \frac{4\kappa \rho^2}{\nu^2} \right)^{\frac{1}{2}} \right) + O(1) \quad \text{as} \quad |\xi| \to \infty$$

If $\nu^2 > 4\kappa \rho^2$ and $\gamma = 0$, then both $\lambda_{\pm}(\xi)$ are real and purely parabolic:

$$\lambda_{\pm}(\xi) = -\frac{\nu}{2} \left( 1 \pm \sqrt{1 - \frac{4\kappa \rho^2}{\nu^2}} \right) |\xi|^2$$

We observe that there are two striking differences from the barotropic compressible fluid: One is that the system is linearly stable even when the sound speed is zero, i.e., $\gamma = 0$. Another is that the solutions in all frequency modes likely exhibit parabolic smoothing effect. These two features directly stem from the presence of the Korteweg tensor.

According to the above ansatz, we seek to derive the natural regularities for $(\hat{a}, \hat{v})$ by the construction of the Lyapunov function in the Fourier space. From (2.7), we immediately obtain the following sets of equalities:

$$\frac{1}{2} \frac{d}{dt} |\hat{a}|^2 + |\xi| \Re(\hat{v} \hat{a}) = 0,$$

and

$$\frac{1}{2} \frac{d}{dt} |\hat{v}|^2 + |\xi|^2 |\hat{v}|^2 - (\gamma + \kappa \rho \xi^2) |\xi| |\hat{a}|^2 + \nu |\xi|^2 |\Re(\hat{v} \hat{a})| = 0,$$

where $\Re z$ denotes the real part of a complex number $z$ and $(\cdot, \cdot)$ is the inner product in $\mathbb{C}$ (that is $(z_1, z_2) := z_1 \overline{z_2}$ with $\overline{z}$ denoting the complex conjugate of $z$).

The proof below holds for $\gamma \geq 0$. We fix some real parameter $\eta > 0$. By adding $(\gamma + \kappa \rho |\xi|^2) \times (2.7)$, $\frac{d}{dt}$ and $-\eta |\xi| \times (2.9)$, we discover that

$$\frac{1}{2} \frac{d}{dt} \left( (\gamma + \kappa \rho |\xi|^2) |\hat{a}|^2 + |\hat{v}|^2 - 2\eta |\xi| \Re(\hat{a} \hat{v}) \right)$$

$$\quad + (\nu - \eta) |\xi|^2 |\hat{v}|^2 + \eta (\gamma + \kappa \rho |\xi|^2) |\xi|^2 |\hat{a}|^2 - \eta \nu |\xi|^3 |\Re(\hat{a} \hat{v})| = 0.$$  

By Young’s inequality, for all $\varepsilon > 0$ we obtain the following lower bound for the last three terms of the above:

$$(\nu - \eta) |\xi|^2 |\hat{v}|^2 + \eta (\gamma + \kappa \rho |\xi|^2) |\xi|^2 |\hat{a}|^2 - \eta \nu |\xi|^3 |\Re(\hat{a} \hat{v})| \geq \left( \nu - \eta \left( 1 + \frac{\varepsilon}{\rho} \right) \right) |\xi|^2 |\hat{v}|^2 + \eta (\gamma + (\kappa \rho - \varepsilon \nu) |\xi|^2) |\xi|^2 |\hat{a}|^2.$$  

for all frequencies. Taking $\varepsilon = \frac{\kappa \rho^3}{2\nu}$, we have $\kappa \rho - \varepsilon \nu = \frac{\rho_s^3}{\nu^2}$. Hence, taking $\eta$ such that

$$\eta < \left( \frac{\rho_s^3}{\nu^2} + \frac{2\nu}{\kappa \rho^2} \right)^{-1},$$
we obtain
\[(\nu - \eta)|\xi|^2 |\hat{\nu}|^2 + \eta(\gamma + \kappa \rho_* |\xi|^2) |\hat{a}|^2 - \eta \nu |\xi|^2 \Re(\hat{a} \hat{\nu}) \geq C_0(\nu, \kappa, \rho_*) |\xi|^2 ((\gamma + |\xi|^2) |\hat{a}|^2 + |\hat{\nu}|^2)\]
for some constant $C_0(\nu, \kappa, \rho_*)$ depending only on $\nu$, $\kappa$ and $\rho_*$. In the same way, there exists a positive constant $C_1(\nu, \kappa, \rho_*)$ such that
\[
C_1(\nu, \kappa, \rho_*)^{-1} ((\gamma + |\xi|^2) |\hat{a}|^2 + |\hat{\nu}|^2) \leq (\gamma + \kappa \rho_* |\xi|^2) |\hat{a}|^2 + |\hat{\nu}|^2 - 2\eta |\xi| \Re(\hat{a} \hat{\nu}) \leq C_1(\nu, \kappa, \rho_*) ((\gamma + |\xi|^2) |\hat{a}|^2 + |\hat{\nu}|^2).
\]
Indeed, the upper bound is trivial; In order to prove the lower bound, we use Young’s inequality to show
\[
(\gamma + \kappa \rho_* |\xi|^2) |\hat{a}|^2 + |\hat{\nu}|^2 - 2\eta |\xi| \Re(\hat{a} \hat{\nu}) \geq \gamma |\hat{a}|^2 + (\kappa \rho_* - 2\eta \varepsilon') |\xi|^2 |\hat{a}|^2 + \left(1 - \frac{2\eta}{\varepsilon'}\right) |\hat{\nu}|^2
\]
for all $\varepsilon' > 0$. Therefore, taking $\varepsilon' = \frac{\kappa \rho_*}{4\eta}$ and $\eta \leq \frac{2\sqrt{2}}{\kappa \rho_*}$ for example, we obtain the lower bound, and thus, (2.11). The above computations combined with (2.10) imply
\[
\frac{1}{2} \frac{d}{dt} \mathcal{L}^2 + |\xi|^2 (\gamma \hat{a}, |\xi| \hat{a}, \hat{\nu})^2 \leq 0 \quad \text{with} \quad \mathcal{L}^2 := (\gamma + \kappa \rho_* |\xi|^2) |\hat{a}|^2 + |\hat{\nu}|^2 - 2\eta |\xi| \Re(\hat{a} \hat{\nu}).
\]
By (2.11), there exists a constant $\bar{c} > 0$ independent of $|\xi|$ such that
\[
\frac{1}{2} \frac{d}{dt} \mathcal{L}^2 + \bar{c} |\xi|^2 \mathcal{L}^2 \leq 0,
\]
which implies the exponential stability
\[
\mathcal{L}^2(t) \leq e^{-\bar{c}|\xi|^2 t} \mathcal{L}^2(0),
\]
for all $t \geq 0$. This combined with (2.11) again gives
\[
|\langle \hat{\gamma} \hat{a}, |\xi| \hat{a}, \hat{\nu} \rangle| \leq e^{-\bar{c}|\xi|^2 t} |\langle \hat{\gamma} \hat{a}, |\xi| \hat{a}, \hat{\nu} \rangle|(0).
\]

Granted with the above estimate in the frequency space, it is now straightforward to complete the proof of the lemma: it suffices to multiply the Littlewood-Paley decomposition $\hat{\phi}_j$, perform exactly the same computations as above on
\[
(\gamma \hat{\phi}_j \hat{a}, 2^j \hat{\phi}_j \hat{a}, \hat{\phi}_j \hat{\nu})
\]
to obtain a bound similar to (2.12):
\[
\mathcal{L}^2(t) \leq e^{-\bar{c}|\xi|^2 t} \mathcal{L}^2(0),
\]
with
\[
\mathcal{L}^2 := (\gamma + \kappa \rho_* 2^j |\hat{\phi}_j \hat{a}|^2 + |\hat{\phi}_j \hat{\nu}|^2 - 2\eta 2^j \Re(\hat{\phi}_j \hat{a} \hat{\phi}_j \hat{\nu}).
\]
Similarly to (2.11), we have
\[
\mathcal{L}^2 \cong |\langle \hat{\gamma} \hat{\phi}_j \hat{a}, 2^j \hat{\phi}_j \hat{a}, \hat{\phi}_j \hat{\nu} \rangle|,
\]
so we have obtained
\[
|\langle \hat{\gamma} \hat{\phi}_j \hat{a}, 2^j \hat{\phi}_j \hat{a}, \hat{\phi}_j \hat{\nu} \rangle| \leq e^{-\bar{c}|\xi|^2 t} |\langle \hat{\gamma} \hat{\phi}_j \hat{a}, 2^j \hat{\phi}_j \hat{a}, \hat{\phi}_j \hat{\nu} \rangle|(0).
\]
Then taking $L^2(\mathbb{R}^d)$ norms of both sides of the resulting inequality, we obtain
\[
\| \langle \hat{\gamma} \hat{\phi}_j \hat{a}, 2^j \hat{\phi}_j \hat{a}, \hat{\phi}_j \hat{\nu} \rangle \|_{L^2} \leq e^{-\bar{c}|\xi|^2 t} \| \langle \hat{\gamma} \hat{\phi}_0 \hat{a}_0, 2^j \hat{\phi}_0 \hat{a}_0, \hat{\phi}_j \hat{\nu}_0 \rangle \|_{L^2}.
\]
For $\gamma > 0$, this implies that for any integer $m$ there exist constants depending only on $m$ such that the action of the semigroup $e^{tA}$ is given by

$$
\|e^{t\hat{A}}(\hat{\gamma}\hat{j}\hat{a}_0, \hat{\phi}_j\hat{v}_0)\|_{L^2} \leq Ce^{-\gamma^2jt} \|\gamma\hat{j}\hat{a}_0, \hat{\phi}_j\hat{v}_0\|_{L^2} \quad \text{for } j \leq m \tag{2.13}
$$

and

$$
\|e^{t\hat{A}}(2^j\hat{\phi}_j\hat{a}_0, \hat{\phi}_j\hat{v}_0)\|_{L^2} \leq Ce^{-\gamma^2jt} \|2^j\hat{\phi}_j\hat{a}_0, \hat{\phi}_j\hat{v}_0\|_{L^2} \quad \text{for } m < j, \tag{2.14}
$$

where $(\hat{a}_0, \hat{v}_0) = (\hat{a}, \hat{v})(0)$. Thanks to (2.13) and (2.14), we deduce

$$
\|e^{t\hat{A}}(\gamma\hat{\phi}_j\hat{a}_0, 2^j\hat{\phi}_j\hat{a}_0, \hat{\phi}_j\hat{v}_0)\|_{L^2} \leq Ce^{-\gamma^2jt} \|\gamma\hat{\phi}_j\hat{a}_0, 2^j\hat{\phi}_j\hat{a}_0, \hat{\phi}_j\hat{v}_0\|_{L^2} \quad \text{for all } j \in \mathbb{Z}. \tag{2.15}
$$

To handle the source term, splitting the frequency regions into low and high and resorting to Duhamel’s formula with the aid of (2.13) and (2.14), we obtain

$$
\|\gamma\hat{\phi}_j\hat{a}_0, 2^j\hat{\phi}_j\hat{a}_0, \hat{\phi}_j\hat{v}_0\|_{L^2} \lesssim e^{-\gamma^2jt} \|\gamma\hat{\phi}_j\hat{a}_0, 2^j\hat{\phi}_j\hat{a}_0, \hat{\phi}_j\hat{v}_0\|_{L^2} + \int_0^t e^{-\gamma^2j(t-\tau)} \|\gamma\hat{\phi}_j\hat{f}, 2^j\hat{\phi}_j\hat{f}, \hat{\phi}_j\hat{g}\|_{L^2}d\tau.
$$

Then, time-integration along with the convolution inequality leads to a frequency-localized time-space estimate. Summing up the resulting inequality over $j \in \mathbb{Z}$ with a weight $2^j$, we obtain the estimate for the compressible part. Since the incompressible component of the velocity $w := (\text{Id} - \Lambda^{-1}\text{div})u$ satisfies a mere heat equation

$$
\partial_tw - \mu \Delta w = (\text{Id} - \Lambda^{-1}\text{div})g,
$$

the standard parabolic estimate (Lemma 2.1) for $w$ yields the desired estimates for $w$ as well. \hfill \square

As a corollary of the proof of Lemma 2.2, we may deduce the following estimate which is used to establish decay estimates.

**Lemma 2.4.** Let $d \geq 2$, $s_1, s_2 \in \mathbb{R}$ and $s_1 > s_2$. Let $e^{tK}$ be the semigroup associated to System (2.5) with $f \equiv g \equiv 0$ and initial data $(a_0, m_0) \in B^s_{2,\infty}(\mathbb{R}^d)$. Then there exists a constant $C$ depending on $d$, $s_1$ and $s_2$ such that the following estimate holds:

$$
\|e^{tK}(\gamma + \Lambda)a_0, m_0\|_{\dot{B}^s_{2,1}} \leq Ct^{-s_1-s_2} \|((\gamma + \Lambda)a_0, m_0)\|_{\dot{B}^s_{2,\infty}} \quad \text{for all } t > 0.
$$

**Proof.** From (2.15) and the corresponding estimate for $w$, we readily deduce

$$
\|e^{tK}(\Delta_j(\gamma + \Lambda)a_0, \Delta_jm_0)\|_{L^2} \leq Ce^{-\gamma^2jt} \|\Delta_j(\gamma + \Lambda)a_0, \Delta_jm_0\|_{L^2}
$$

for all $j \in \mathbb{Z}$. By Lemma 5.2 with $r \equiv 1$, we readily have

$$
\left\{2^{js_1}e^{tK}(\Delta_j(\gamma + \Lambda)a_0, \Delta_jm_0)\right\}_{j \in \mathbb{Z}} \leq C t^{-s_1-s_2} \left\{2^{js_1}e^{-\gamma^2jt} \|((\gamma + \Lambda)a_0, m_0)\|_{\dot{B}^s_{2,\infty}} \right\}_{j \in \mathbb{Z}}.
$$

\hfill \square

3. **Nonlinear Problem**

In this section, we show the existence and uniqueness of global solution to the nonlinear problem (2.1).
3.1. **Banach’s fixed point theorem.** We first prove the following auxiliary lemma, which is a generalization of Lemma 2.1 in [8] to bi- and trilinear operators.

**Lemma 3.1.** Let \((X, \| \cdot \|_X)\) be a Banach space. Let \(B : X \times X \to X\) be a bilinear continuous operator with norm \(K_2\) and \(T : X \times X \times X \to X\) be a trilinear continuous operators with norm \(K_3\). Let further \(L : X \to X\) be a continuous linear operator with norm \(N < 1\). Then for all \(y \in X\) such that
\[
\|y\|_X < \min \left\{ \frac{1 - N}{2}, \frac{(1 - N)^2}{2(2K_2 + 3K_3)} \right\},
\]
Equation \(x = y + L(x) + B(x, x) + T(x, x, x)\) has a unique solution \(x\) in the ball \(B^X_R(0)\) of center 0 and radius \(R := \min \left\{ 1, \frac{1 - N}{2K_2 + 3K_3} \right\}\). In addition, \(x\) satisfies
\[
\|x\|_X \leq \frac{2}{1 - N} \|y\|_X. 
\]

**Proof.** Although this is standard, we prove the lemma for completeness. Letting \(\Phi : x \mapsto y + L(x) + B(x, x) + T(x, x, x)\), we prove that \(\Phi\) is a contraction map in some ball. Let \(I := \|y\|_X\) and let \(x\) be in some closed ball \(B^X_R(0)\) of \(X\). Assuming \(R \leq 1\), we have
\[
\|\Phi(x)\|_X \leq \|y\|_X + \|L(x)\|_X + \|B(x, x)\|_X + \|T(x, x, x)\|_X
\leq I + NR + (K_2 + K_3)R^2.
\]

Let \(R\) be defined by
\[
R := \frac{2}{1 - N} I. 
\]
Then we have from (3.2)
\[
\|\Phi(x)\|_X \leq \left( \frac{1 - N}{2} + N + (K_2 + K_3)R \right) R.
\]

Thus, suffices to take \(R\) such that
\[
R \leq \min \left\{ 1, \frac{1 - N}{2(K_2 + K_3)} \right\}, \tag{3.4}
\]
for \(\Phi\) to be a self-map on the closed ball \(B^X_R(0)\).

On the other hand, we have for any \(x_1, x_2 \in B^X_R(0)\) and for \(R \leq 1\),
\[
\|\Phi(x_1) - \Phi(x_2)\|_X \leq \|L(x_1 - x_2)\|_X + \|B(x_1 - x_2, x_1)\|_X + \|B(x_1, x_1 - x_2)\|_X \\
+ \|T(x_1 - x_2, x_1, x_1)\|_X + \|T(x_1, x_1 - x_2, x_1)\|_X + \|T(x_2, x_2, x_1 - x_2)\|_X
\leq (N + 2K_2R + 3K_3R^2)\|x_1 - x_2\|_X \leq (N + (2K_2 + 3K_3)R)\|x_1 - x_2\|_X.
\]

Hence, in order for \(\Phi\) to be a contraction, \(N + (2K_2 + 3K_3)R\) has to hold; this amounts to taking \(R\) such that
\[
R < \min \left\{ 1, \frac{1 - N}{2K_2 + 3K_3} \right\}. \tag{3.5}
\]
Thus, from (3.3), (3.4) and (3.5), Banach’s fixed point theorem is applicable for all \(y \in X\) such that
\[
\frac{2}{1 - N} \|y\|_X < \min \left\{ 1, \frac{1 - N}{2K_2 + 3K_3} \right\}, \text{ i.e., } \|y\|_X < \min \left\{ \frac{1 - N}{2}, \frac{(1 - N)^2}{2(2K_2 + 3K_3)} \right\},
\]

The uniqueness of the fixed point in the ball naturally follows. Routine computations as in the stability estimate lead to \((3.1)\). \[\]  

3.2. **Global results in** \(L^2(\mathbb{R}^d)\)-based spaces when \(\gamma > 0\). Lemma \textsuperscript{2.2} yields the following.

**Corollary 3.2.** Let \(d \geq 2\), \(T > 0\), \(s \in \mathbb{R}\), \(\gamma \geq 0\) and
\[(\gamma + \Lambda)f, g) \in \tilde{L}^{1}(0, T; \dot{B}_{2,\sigma}^{s}(\mathbb{R}^{d})).\]
Assume that \((a, m)\) is a solution for \((2.5)\) such that
\[(\gamma + \Lambda)a, m) \in \mathcal{L}^{\infty}(0, T; \dot{B}_{2,\sigma}^{s}(\mathbb{R}^{d})) \cap \tilde{L}^{1}(0, T; \dot{B}_{2,\sigma}^{s+2}(\mathbb{R}^{d})).\]
Then there exists some positive constant \(C\) depending only on \(\rho_{*}, P, \lambda, \mu, \kappa\) and \(d\) such that the following estimate hold: if \(\gamma \geq 0\), then
\[
\begin{align*}
\|((\gamma + \Lambda)a, m)\|_{\tilde{L}^{\infty}(\dot{B}_{2,\sigma}^{s})} + \|\Lambda^2((\gamma + \Lambda)a, m)\|_{\tilde{L}^{1}(\dot{B}_{2,\sigma}^{s})} \\
\leq C((\|((\gamma + \Lambda)a, m, m)\|_{\dot{B}_{2,\sigma}^{s}} + \|((\gamma + \Lambda)f, g)\|_{\tilde{L}^{1}(\dot{B}_{2,\sigma}^{s})})
\end{align*}
\]
for all \(0 < t \leq T\).

Given initial data \((a_0, m_0)\) such that \(((\gamma + \Lambda)a_0, m_0) \in \dot{B}_{2,1}^{\frac{d}{2}-1}(\mathbb{R}^{d})\), in order to solve \((2.1)\), it suffices to show that the map
\[
\Phi : (b, n) \mapsto (a, m)
\]
with \((a, m)\) the solution to
\[
\begin{align*}
\partial_{t}a + \text{div} m &= 0, \\
\partial_{t}m - \frac{1}{\rho_{*}}\mathcal{L}m - \kappa \rho_{*} \nabla \Delta a + \gamma \nabla a &= N(b, n), \\
(a, m)_{t=0} &= (a_0, m_0),
\end{align*}
\]
where
\[
N(b, n) := \mathcal{L}(Q(b)n) + \kappa b \nabla \Delta b - \frac{1}{\rho_{*}}\text{div} (n \otimes n)
- \text{div} (Q(b)n \otimes n) + (P'(b + \rho_{*}) - \gamma) \nabla b,
\]
has a fixed point in a ball of a suitable metric space.

We define a Banach space \(CL_T\) by
\[
CL_T := \tilde{L}^{\infty}(0, T; \dot{B}_{2,1}^{\frac{d}{2}-1}(\mathbb{R}^{d})) \cap \tilde{L}^{1}(0, T; \dot{B}_{2,1}^{\frac{d}{2}+1}(\mathbb{R}^{d}))
\]
equipped with a norm
\[
\|(a, m)\|_{CL_T} := \|((\gamma + \Lambda)a, m)\|_{\tilde{L}^{\infty}(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|\Lambda^2((\gamma + \Lambda)a, m)\|_{\tilde{L}^{1}(\dot{B}_{2,1}^{\frac{d}{2}-1})}.
\]
By interpolation, we may readily see that \((a, m) \in CL_T\) further satisfies
\[
((\gamma + \Lambda)a, m) \in L^{q}(0, T; \dot{B}_{2,1}^{\frac{d}{2}-1+\frac{2}{q}}(\mathbb{R}^{d})) \quad \text{for} \quad q \in [1, \infty].
\]
Recall that \(N\) defined in \((3.8)\) is a combination of bi-and-trilinear terms. We have the following nonlinear estimates.
Lemma 3.3. Let $d \geq 2$ and $T > 0$. Let $N$ be as defined in $(3.8)$ and let $P$ be a smooth function such that $\gamma \geq 0$. Then there exists a positive constant $C$ depending only on $\rho_\ast$, $P$, $\lambda$, $\mu$, $\kappa$ and $d$ such that

$$
\|N(b,n)\|_{L^1_t(B^\frac{d}{2}+1)} \leq C(\|b\|_{CLT}^2 + \|n\|_{CLT}^3)
$$

(3.10)

for all $(b,n) \in CLT$, $0 \leq t \leq T$, provided further that $\gamma > 0$.

Furthermore, there exists a positive constant $C$ depending only on $\rho_\ast$, $P$, $\lambda$, $\mu$, $\kappa$ and $d$ such that

$$
\|N(b,n)\|_{L^1_t(B^\frac{d}{2}+1)} \leq C((1 + t)\|b\|_{CLT}^2 + \|n\|_{CLT}^3)
$$

(3.11)

for all $(b,n) \in CLT$, $0 \leq t \leq T$, provided further that $\gamma = 0$.

Proof. Taking $q = 1, 2, \infty$ in (3.9), we in particular have

$$
(\gamma b, Ab, n) \in \tilde{L}^\infty(0, T; \dot{B}^{\frac{d}{2}+1}_2(\mathbb{R}^d)) \cap \tilde{L}^2(0, T; \dot{B}^{\frac{d}{2}}_2(\mathbb{R}^d)) \cap \tilde{L}^1(0, T; \dot{B}^{\frac{d}{2}+1}_2(\mathbb{R}^d))
$$

(3.12)

for all $(b,n) \in CLT$. Note that $Q$ satisfies $Q'(b) = \frac{1}{(b + \rho_\ast)^2}$ and $Q'(0) = \frac{1}{\rho_\ast^2} > 0$. Thus, thanks to Lemma 5.7 and Lemma 5.5 we have

$$
\|Q(b)n\|_{L^1_t(B^\frac{d}{2}+1)} \leq C\left(\|\nabla(Q(b))n\|_{L^1_t(B^\frac{d}{2}+1)} + \|Q'(0)(\nabla b)n\|_{L^1_t(B^\frac{d}{2}+1)} + \|Q(b)n\|_{L^1_t(B^\frac{d}{2}+1)}
$$

\[\leq C \left(\|b\|_{L^\infty_t(B^\frac{d}{2}+1)} \|\nabla b\|_{L^1_t(B^\frac{d}{2}+1)} \|n\|_{L^1_t(B^\frac{d}{2}+1)} + \|\nabla b\|_{L^1_t(B^\frac{d}{2}+1)} \|n\|_{L^1_t(B^\frac{d}{2}+1)} + \|Q(b)n\|_{L^1_t(B^\frac{d}{2}+1)}\right).\]

(3.13)

Now, a standard interpolation argument ensures that the right hand-side of (3.13) is bounded by

$$
\|b\|_{CLT}^2 + \|n\|_{CLT}^3.
$$

Similarly, using (3.12), we have

$$
\|\kappa b\nabla \Delta b\|_{L^1_t(B^\frac{d}{2}+1)} \leq C\|b\|_{L^\infty_t(B^\frac{d}{2}+1)} \|\nabla \Delta b\|_{L^1_t(B^\frac{d}{2}+1)} \leq C\|b\|_{L^\infty_t(B^\frac{d}{2}+1)} \|\Delta b\|_{L^1_t(B^\frac{d}{2}+1)}; \quad (3.14)
$$

$$
\left\|\frac{1}{\rho_\ast} \text{div} (n \otimes n)\right\|_{L^1_t(B^\frac{d}{2}+1)} \leq C\|n \otimes n\|_{L^1_t(B^\frac{d}{2}+1)} \leq C\|n\|_{L^2_t(B^\frac{d}{2}+1)}^2; \quad (3.15)
$$

$$
\|\text{div} (Q(b)n \otimes n)\|_{L^1_t(B^\frac{d}{2}+1)} \leq C\|Q(b)n \otimes n\|_{L^1_t(B^\frac{d}{2}+1)} \leq C\|b\|_{L^\infty_t(B^\frac{d}{2}+1)} \|n\|_{L^2_t(B^\frac{d}{2}+1)}^2. \quad (3.16)
$$

For the pressure term when $\gamma > 0$,

$$
\|P'(b + \rho_\ast) - P'(\rho_\ast)\|_{L^1_t(B^\frac{d}{2}+1)} \leq C\|P'(b + \rho_\ast) - P'(\rho_\ast)\|_{L^\infty_t(B^\frac{d}{2}+1)} \|\nabla b\|_{L^1_t(B^\frac{d}{2}+1)} \leq C\|b\|_{L^\infty_t(B^\frac{d}{2}+1)} \|\nabla b\|_{L^1_t(B^\frac{d}{2}+1)};
$$

(3.17)
where we have used the regularity for $\gamma b$ in (3.12). Thus, combining (3.13), (3.14), (3.15), (3.16), and (3.17), we have
\[
\|N(b, n)\|_{L^1_t(B_{x,1}^{d-1})} \lesssim (\|b\|_{L^\infty_t(B_{x,1}^d)} + 1) \|\nabla b\|_{L^1_t(B_{x,1}^d)} \|n\|_{L^1_t(B_{x,1}^d)} \\
+ \|b\|_{L^\infty_t(B_{x,1}^d)} \left( \|\text{div} n\|_{L^1_t(B_{x,1}^d)} + \|\Delta b\|_{L^1_t(B_{x,1}^d)} \right) \\
+ \|n\|^2_{L^1_t(B_{x,1}^d)} \left( 1 + \|b\|_{L^\infty_t(B_{x,1}^d)} \right) + t \|b\|^2_{L^1_t(B_{x,1}^d)},
\]
provided that $\gamma > 0$. Thus, we have (3.10) if $\gamma > 0$.

If, on the other hand, $\gamma = 0$, we only have
\[
\Lambda b \in \tilde{L}^\infty(0, T; B_{x,1}^d(\mathbb{R}^d)) \cap \tilde{L}^2(0, T; \tilde{B}_{x,1}^d(\mathbb{R}^d)) \cap \tilde{L}^1(0, T; \tilde{B}_{x,1}^{d+1}(\mathbb{R}^d)).
\]
We may write the term $P'(b + \rho_*)\nabla b = \nabla (P(b + \rho_*))$ and
\[
\|\nabla (P(b + \rho_*))\|_{L^1_t(B_{x,1}^{d-1})} \leq C\|P(b + \rho_*)\|_{L^1_t(B_{x,1}^d)} = C\|P(b + \rho_*) - P(\rho_*)\|_{L^1_t(B_{x,1}^d)} \leq C\|b\|_{L^1_t(B_{x,1}^d)},
\]
thanks to Lemma 5.7. However, we have no control on $\|b\|_{L^1_t(B_{x,1}^d)}$, so the above only implies
\[
\|\nabla (P(b + \rho_*))\|_{L^1_t(B_{x,1}^{d-1})} \leq Ct\|b\|_{L^\infty_t(B_{x,1}^d)}.
\]
Alternatively, we have
\[
\|P'(b + \rho_*)\nabla b\|_{L^1_t(B_{x,1}^{d-1})} \leq C\|P'(b + \rho_*)\|_{L^1_t(B_{x,1}^d)} \|\nabla b\|_{L^\infty_t(B_{x,1}^{d-1})} \\
\leq C\|b\|_{L^1_t(B_{x,1}^d)} \|\nabla b\|_{L^\infty_t(B_{x,1}^{d-1})} \leq Ct\|\nabla b\|^2_{L^\infty_t(B_{x,1}^{d-1})}.
\]
Either way, we may not close the estimates globally-in-time if $\gamma = 0$ with our given regularity in $CL_T$; Combining (3.13), (3.14), (3.15), (3.16), and (3.18), we have
\[
\|N(b, n)\|_{L^1_t(B_{x,1}^{d-1})} \lesssim (\|b\|_{L^\infty_t(B_{x,1}^d)} + 1) \|\nabla b\|_{L^1_t(B_{x,1}^d)} \|n\|_{L^1_t(B_{x,1}^d)} \\
+ \|b\|_{L^\infty_t(B_{x,1}^d)} \left( \|\text{div} n\|_{L^1_t(B_{x,1}^d)} + \|\Delta b\|_{L^1_t(B_{x,1}^d)} \right) \\
+ \|n\|^2_{L^1_t(B_{x,1}^d)} \left( 1 + \|b\|_{L^\infty_t(B_{x,1}^d)} \right) + t \|b\|^2_{L^1_t(B_{x,1}^d)}
\]
in this case. Thus, we have (3.11) if $\gamma = 0$. This completes the proof of Lemma 3.3. □

**Proof of Theorem 1.4.** The time-continuity follows from the standard density argument as in the corresponding proof for the linear heat equation. The proof of Theorem 1.4 readily
follows from Corollary \[3.2\] and Lemma \[3.3\] and Lemma \[3.1\]. As the proof is standard, we only sketch the steps. We let \((a_L, m_L)\) be the solution to the linear equation:
\[
\begin{aligned}
\partial_t a_L + \text{div} m_L &= 0, \\
\partial_t m_L - \frac{1}{\rho_*} L m_L - \kappa \rho_* \nabla \Delta a_L + \gamma \nabla a_L &= 0, \\
(a, m)|_{t=0} &= (a_0, m_0)
\end{aligned}
\]
and let \((\tilde{a}, \tilde{m}) := (a - a_L, m - m_L)\). Then Corollary \[3.2\] with \(s \equiv \frac{d}{2} - 1\) and \(\sigma \equiv 1\) gives
\[
\|(a, m)\|_{CLT} \leq \|(a_L, m_L)\|_{CLT} + \|\tilde{a}, \tilde{m}\|_{CLT} \\
\leq C(\|((\gamma + \Lambda) a_0, m_0)\|_{B_{\infty}^{\frac{d}{2}-1}} + \|N(b, n)\|_{\tilde{L}_1^d(B_{2,1}^\gamma)}),
\]
for all \((b, n) \in CLT\). Lemma \[3.3\] ensures that \(N(b, n)\) can be regarded as a combination of bi-and-trilinear continuous operators in \(CLT\). We define a ball in \(CLT\) centered at the origin by
\[
B_{R}^{CLT}(0) := \{(a, m) \in CLT; \|(a, m)\|_{CLT} \leq R\},
\]
where \(R > 0\). Lemma \[3.1\] now ensures the existence of a unique solution in the ball \(B_{R}^{CLT}(0)\) for a sufficiently small data. Note that \(T = \infty\) is allowed for in the case \(\gamma > 0\), thanks to the global estimate in Lemma \[3.3\]. By \[3.1\], \(\|(a, m)\|_{CLT} \leq C(\|((\gamma + \Lambda) a_0, m_0)\|_{B_{\infty}^{\frac{d}{2}-1}})\) is also ensured. \[\square\]

3.3. Global solution when \(\gamma = 0\). From hereon, we focus on the case \(\gamma = 0\). When \(\gamma > 0\), the term \(\gamma \nabla \rho\) induces a damping effect, which provides the control of \(\|\nabla b\|_{\tilde{L}_1^d(B_{2,1}^\gamma)}\).

The lack of this time-integrability in the case \(\gamma = 0\) is troublesome, especially when we control the pressure term. As shown by \[3.18\], we need the control of \(\|\nabla b\|_{\tilde{L}_1^d(B_{2,1}^\gamma)}\), whereas we only have \(\nabla b \in \tilde{L}_1^d(B_{2,1}^\gamma)\) in \(CLT\). In order to compensate this lack of global control, we impose stronger regularity on the low-frequency, which is of lower order in terms of the scaling, to gain the necessary global control of the density.

To handle this case, we have to change the expressions of the capillary and pressure terms. We linearize the capillary term while keeping it in the divergence form:
\[
\text{div} K(\rho) = \kappa \rho_* \nabla \Delta a + \text{div} (K(\rho)),
\]
where as before we denoted \(a := \rho - \rho_*\). Similarly, we write the pressure as follows: By Taylor expansion, we have
\[
P(a + \rho_*) = P(\rho_*) + P'(\rho_*) a + a \bar{P}(a),
\]
where \(\bar{P} : \mathbb{R}_+ \to \mathbb{R}\) is smooth and vanishes at 0. Using the assumption that \(P'(\rho_*) = 0\), we have
\[
\nabla P(a + \rho_*) = \nabla (a \bar{P}(a)).
\]
With this expression, we obtain the same linear structure with nonlinear terms all written in divergence form:
\[
N(b, n) = \mathcal{L} (Q(b)n) + \text{div} (K(b)) - \frac{1}{\rho_*} \text{div} (n \otimes n) - \text{div} (Q(b)n \otimes n) + \nabla (b \bar{P}(b)).
\]
(3.20)

This expression enables us to close necessary product estimates in the Besov space of regularity index \(s = \frac{d}{2} - 3\) by a sole restriction \(d \geq 3\).
Given initial data \((a_0, m_0)\) such that \((\Lambda a_0, m_0) \in B^d_{2,1}(-1)(\mathbb{R}^d)\), we show that the map 
\(\Phi : (b, n) \mapsto (a, m)\) defined in (3.6) with \((a, m)\) the solution to (3.7) with \(\gamma = 0\), i.e.,
\[
\begin{align*}
\partial_t a + \text{div} m &= 0, \\
\partial_t m - \frac{1}{\rho_*} \mathcal{L} m - \kappa \rho_* \nabla \Delta a &= N(b, n), \\
(a, m)_{t=0} &= (a_0, m_0),
\end{align*}
\]
where \(N = N(b, n)\) is as defined in (3.20), has a fixed point in a ball of a suitable metric space. To define such a space, for \(d \geq 3\), we let \(\overline{CL_T}\) be given by
\[
\overline{CL_T} := L^\infty(0, T; \dot{B}^d_{2,1}(-3) \cap \dot{B}^d_{2,1}(-1)(\mathbb{R}^d)) \cap \overline{L^1}(0, T; \dot{B}^d_{2,1}(-1)(\mathbb{R}^d) \cap \dot{B}^d_{2,1}(-1)(\mathbb{R}^d))
\]
equipped with a norm
\[
\| (a, m) \|_{\overline{CL_T}} := \| (\Lambda a, m) \|_{\dot{L}^\infty(B^d_{2,1}(-3) \cap B^d_{2,1}(-1))} + \| \Lambda^2 (\Lambda a, m) \|_{\dot{L}^1(B^d_{2,1}(-3) \cap B^d_{2,1}(-1))}.
\]
By interpolation, we may readily see that \((a, m) \in \overline{CL_T}\) satisfies (3.9) and, additionally,
\[
(\Lambda a, m) \in \overline{L}^q(0, T; \dot{B}^{d-3+\frac{2}{q}}_{2,1}(\mathbb{R}^d)) \quad \text{for} \quad q \in [1, \infty].
\]
Then we have the following global nonlinear estimates for the case \(\gamma = 0\).

**Lemma 3.4.** Let \(d \geq 3\) and \(T > 0\). Let \(N\) be as defined in (3.8) and let \(P\) be a smooth function such that \(\gamma = 0\). Then there exists a positive constant \(C\) depending only on \(\rho_*\), \(P\), \(\lambda\), \(\mu\) and \(d\) such that
\[
\| N(b, n) \|_{\dot{L}^1(B^d_{2,1}(-3) \cap B^d_{2,1}(-1))} \leq C(\| (b, n) \|_{\overline{CL_T}}^2 + \| (b, n) \|_{\overline{CL_T}}^3)
\]
for all \((b, n) \in \overline{CL_T}\) and \(0 \leq t \leq T\).

**Remark 3.5.** The restriction \(d \geq 3\) for (3.22) is used to close product estimates in \(B^d_{2,1}(-1)(\mathbb{R}^d)\).

**Proof.** Let \(d \geq 3\). Note that by (3.12), we still have
\[
(\Lambda b, n) \in \overline{L}^\infty(0, T; \dot{B}^{d-1}_{2,1}(\mathbb{R}^d)) \cap \overline{L}^2(0, T; \dot{B}^d_{2,1}(\mathbb{R}^d)) \cap \overline{L}^1(0, T; \dot{B}^{d+1}_{2,1}(\mathbb{R}^d)).
\]
From this, it is clear that we have (3.13), (3.14), (3.15) and (3.16) as before. Furthermore, \((b, n) \in \overline{CL_T}\) implies (3.21):
\[
(\Lambda b, n) \in \overline{L}^\infty(0, T; \dot{B}^{d-3}_{2,1}(\mathbb{R}^d)) \cap \overline{L}^2(0, T; \dot{B}^d_{2,1}(\mathbb{R}^d)) \cap \overline{L}^1(0, T; \dot{B}^{d+1}_{2,1}(\mathbb{R}^d)).
\]
To handle the last term \(\| \nabla(P(b + \rho_*)) \|_{\dot{L}^1(B^d_{2,1}(-1))}\), we use the above additional low-frequency assumption on \(b\). We have
\[
\| P'(b + \rho_*) \nabla b \|_{\dot{L}^1(B^d_{2,1}(-1))} \leq C \| P'(b + \rho_*) \|_{\dot{L}^1(B^d_{2,1}(-1))} \| \nabla b \|_{\dot{L}^\infty(B^d_{2,1}(-1))} \| b \|_{\dot{L}^1(B^d_{2,1})}
\]
\[
\leq C \| b \|_{\dot{L}^\infty(B^d_{2,1})} \| b \|_{\dot{L}^1(B^d_{2,1})},
\]
thanks to (3.12) and (3.23). Notice that, here, we do not need to use the expression (3.20). Combining all information, we have

\[ \| N(b, n) \|_{L_t^1(B_{2,1}^{d-3})} \lesssim (\| b \|_{L_t^\infty(B_{2,1}^{d})} + 1) \| \nabla b \|_{L_t^1(B_{2,1}^{d})} \| n \|_{L_t^\infty(B_{2,1}^{d})} \]

\[ + \| b \|_{L_t^\infty(B_{2,1}^{d})} \left( \| \text{div} n \|_{L_t^1(B_{2,1}^{d})} + \| b \|_{L_t^1(B_{2,1}^{d})} + \| \Delta b \|_{L_t^1(B_{2,1}^{d})} \right) \]

\[ + \| n \|_{L_t^2(B_{2,1}^{d})} \left( 1 + \| b \|_{L_t^\infty(B_{2,1}^{d})} \right). \]

In order to complete the proof of (3.22), we still have to bound \( \| N(b, n) \|_{L_t^1(B_{2,1}^{d-3})} \) under the assumption that \( d \geq 3 \). Thanks to (3.12), (3.23), Lemma 5.7, Lemma 5.5 we have

\[ \| \mathcal{L}(Q(b)n) \|_{L_t^1(B_{2,1}^{d-3})} \leq C \| \nabla(Q(b))n + Q(b)\text{div}n\text{Id} \|_{L_t^1(B_{2,1}^{d-2})} \]

\[ \leq C \|(Q'(b) - Q'(0))\nabla b)n \|_{L_t^1(B_{2,1}^{d-2})} + \| Q'(0)\nabla b)n \|_{L_t^1(B_{2,1}^{d-2})} + \| \nabla b \|_{L_t^1(B_{2,1}^{d-2})} \]

\[ \leq C \left( \| b \|_{L_t^\infty(B_{2,1}^{d})} \| \nabla b \|_{L_t^1(B_{2,1}^{d})} \| n \|_{L_t^2(B_{2,1}^{d-2})} + \| \nabla b \|_{L_t^1(B_{2,1}^{d-2})} \| n \|_{L_t^2(B_{2,1}^{d-2})} \right) \]

\[ + \| b \|_{L_t^\infty(B_{2,1}^{d})} \| n \|_{L_t^1(B_{2,1}^{d-1})}; \quad (3.24) \]

\[ \| \text{div}(K(b)) \|_{L_t^1(B_{2,1}^{d-3})} \leq C \| \Delta b^2 - |\nabla b|^2\text{Id} - \kappa \nabla a \otimes \nabla a \|_{L_t^1(B_{2,1}^{d-2})} \]

\[ \lesssim \| b^2 \|_{L_t^1(B_{2,1}^{d})} + \| \nabla b \|_{L_t^1(B_{2,1}^{d})} + \| \text{div} b \|_{L_t^1(B_{2,1}^{d})} \]

\[ \lesssim \| b \|_{L_t^2(B_{2,1}^{d})} + \| \nabla b \|_{L_t^1(B_{2,1}^{d-2})} \| b \|_{L_t^1(B_{2,1}^{d})} \]

\[ \lesssim \| \Delta b \|_{L_t^1(B_{2,1}^{d-1})} + \| \text{div} b \|_{L_t^1(B_{2,1}^{d-2})} \| \text{div} b \|_{L_t^1(B_{2,1}^{d})}; \quad (3.25) \]

\[ \left\| \frac{1}{\rho} \text{div}(n \otimes n) \right\|_{L_t^1(B_{2,1}^{d-3})} \leq C \| n \otimes n \|_{L_t^1(B_{2,1}^{d-2})} \leq C \| n \|_{L_t^\infty(B_{2,1}^{d})} \| n \|_{L_t^1(B_{2,1}^{d})}; \quad (3.26) \]

\[ \| \text{div}(Q(b)n \otimes n) \|_{L_t^1(B_{2,1}^{d-3})} \leq C \| Q(b)n \otimes n \|_{L_t^1(B_{2,1}^{d-2})} \]

\[ \leq C \| Q(b)n \|_{L_t^\infty(B_{2,1}^{d})} \| n \|_{L_t^1(B_{2,1}^{d})} \| n \|_{L_t^1(B_{2,1}^{d})}; \quad (3.27) \]

\[ \| \nabla (b\tilde{P}(b)) \|_{L_t^1(B_{2,1}^{d-3})} \leq |b\tilde{P}(b)|_{L_t^1(B_{2,1}^{d-2})} \leq C \| b \|_{L_t^1(B_{2,1}^{d})} \| \tilde{P}(b) \|_{L_t^\infty(B_{2,1}^{d})} \]

\[ \leq C \| b \|_{L_t^\infty(B_{2,1}^{d})} \| \text{div} b \|_{L_t^1(B_{2,1}^{d-1})}; \quad (3.28) \]

Note here that if we use the expression (3.3), then instead of (3.25) and (3.28), we end up with using Lemma 5.5 with \( s = \frac{d}{2} - 3 \), which requires \( d \geq 4 \).
Summing up (3.24), (3.25), (3.26), (3.27) and (3.28), we have

\[ \|N(b,n)\|_{L^1_t(B^{\frac{d}{2} - 1}_{2,1})} \lesssim \|b\|_{L^\infty_t(B^{\frac{d}{2}}_{2,1})} (\|\nabla b\|_{L^2_t(B^{\frac{d}{2}}_{2,1})} \|n\|_{L^2_t(B^{\frac{d}{2} - 2}_{2,1})} + \|n\|_{L^2_t(B^{\frac{d}{2} - 2}_{2,1})} \|n\|_{L^2_t(B^{\frac{d}{2}}_{2,1})}) \]

+ \|b\|_{L^\infty_t(B^{\frac{d}{2}}_{2,1})} (\|n\|_{L^1_t(B^{\frac{d}{2} - 1}_{2,1})} + \|b\|_{L^1_t(B^{\frac{d}{2}}_{2,1})})

+ \|n\|_{L^2_t(B^{\frac{d}{2} - 2}_{2,1})} (\|\nabla b\|_{L^1_t(B^{\frac{d}{2}}_{2,1})} + \|n\|_{L^2_t(B^{\frac{d}{2} - 2}_{2,1})}) + \|\nabla b\|_{L^\infty_t(B^{\frac{d}{2} - 3}_{2,1})} \|b\|_{L^1_t(B^{\frac{d}{2}}_{2,1})}, \]

which completes the proof of (3.22).

Proof of Theorem 1.6. Similarly to Theorem 1.4, the proof of Theorem 1.6 readily follows from Corollary 3.2, Lemma 3.4 and Lemma 3.1. The details are left to the readers.

4. Proof of Decay Estimates

In this section, we prove the decay estimates for the global small solutions that have been obtained in the previous section.

Proof of Theorem 1.8. The proof below is inspired by [12]. For notational simplicity, in the following, we denote

\[ X(t) := \|(a, m)\|_{CL^T} = \|((\gamma + \Lambda)a, m)\|_{L^\infty_t(B^{\frac{d}{2} - 1}_{2,1})} + \|\Lambda^2((\gamma + \Lambda)a, m)\|_{L^\infty_t(B^{\frac{d}{2} - 1}_{2,1})}. \]

and \( X_0 := \|(\gamma + \Lambda)a_0, m_0)\|_{B^{\frac{d}{2} - 1}_{2,1}}. \)

Firstly, note that for a solution to (CNSK) constructed by Theorem 1.4 with \( D(t) \leq \infty \), we also have

\[ \|(\tau)^{\alpha}((\gamma + \Lambda)a, m)\|_{L^\infty_t(B^{\frac{d}{2} - 1}_{p,1})} \lesssim \|(\gamma + \Lambda)a, m)\|_{L^\infty_t(B^{\frac{d}{2} - 1}_{p,1})} + \|\tau^{\alpha}((\gamma + \Lambda)a, m)\|_{L^\infty_t(B^{\frac{d}{2} + 1}_{p,1})} \lesssim X(t) + D(t) \leq \infty. \]

(4.1)

Step 1: Bounds for the low frequencies. We denote the semigroup associated to the left hand-side of System (2.5) by \( e^{tK} \) and let

\[ U = \begin{bmatrix} (\gamma + \Lambda)a \\ m \end{bmatrix}, \quad U_0 = \begin{bmatrix} (\gamma + \Lambda)a_0 \\ m_0 \end{bmatrix} \quad \text{and} \quad F = F(a, m) = \begin{bmatrix} 0 \\ N(a, m) \end{bmatrix}, \]

where \( N = N(a, m) \) is as defined in (3.20). By Duhamel’s formula, we have the expression

\[ U(t) = e^{tK}U_0 + \int_0^t e^{(t-\tau)K}F(a, m)(\tau)d\tau. \]

Since \( ((\gamma + \Lambda)a_0, m_0) \in B^{-\frac{d}{2}}_{2,\infty}(\mathbb{R}^d) \), the action of the semigroup on \( B^s_{2,1}(\mathbb{R}^d) \) given by Lemma 2.3 implies

\[ \|e^{tK}U_0\|_{B^s_{2,1}} \leq Ct^{-\frac{1}{2}(s + \frac{d}{2})} \|(\gamma + \Lambda)a_0, m_0)\|_{B^{-\frac{d}{2}}_{2,\infty}}. \]

In addition, we have

\[ \|e^{tK}U_0\|_{B^s_{2,1}} \leq \sum_{j \leq j_0} 2^j \Delta_j e^{tK}U_0\|_{L^2} \leq \sum_{j \leq j_0} 2^{j(s + \frac{d}{2})}2^{-j\frac{d}{2}}\|\Delta_j U_0\|_{L^2} \leq C\|U_0\|_{B^{-\frac{d}{2}}_{2,\infty}} \]

provided that \( s + \frac{d}{2} > 0 \), where \( C \) depends only on \( s \), \( d \) and \( j_0 \). Thus, we have

\[ \|e^{tK}U_0\|_{B^s_{2,1}} \leq C(t)^{-\frac{1}{2}(s + \frac{d}{2})}\|U_0\|_{B^{-\frac{d}{2}}_{2,\infty}} \quad \text{for} \quad \|U_0\|_{B^{-\frac{d}{2}}_{2,\infty}}. \]
By (4.2) and the integral representation, we have

\[
\|U(t)\|_{B^s_{2,1}}^t = \|(\gamma + \Lambda)a, m)(t)\|_{B^s_{2,1}}^t \leq e^{tK}U_0\|_{B^s_{2,1}}^t + \int_0^t e^{(t-\tau)K}N(a, m)(\tau)\|_{B^s_{2,1}}^t d\tau
\]

\[
\leq C \left( \langle t \rangle^{-\frac{1}{2}(s+\frac{d}{2})} U_0\|_{B^s_{2,1}}^t + \int_0^t \langle t-\tau \rangle^{-\frac{1}{2}(s+\frac{d}{2})} N(a, m)(\tau)\|_{B^s_{2,1}}^t d\tau \right),
\]

(4.3)

where the constant \(C\) depends on \(j_0\) (but stays bounded as long as \(-\infty < j_0\)). It now comes down to bounding the nonlinear terms with our prescribed regularity by \(X(t)\) and \(D(t)\) defined in (1.6). We suppose now that \(\alpha\) in \(D(t)\) is some real number to be decided later.

In what follows, we are going to be frequently using the fact that

\[
\sup_{0 \leq \tau \leq t} \langle \tau \rangle^\frac{d}{2} \|(\gamma + \Lambda)a, m)(\tau)\|_{B^s_{2,1}}^t \lesssim X(t) + D(t).
\]

(4.4)

Indeed, we have, thanks to (4.1),

\[
\|u(t)\|_{B^s_{2,1}}^0 \leq \|u(t)\|_{B^s_{2,1}}^0 + \|u(t)\|_{B^s_{2,1}}^0 \leq \|u(t)\|_{B^s_{2,1}}^0 + \|u(t)\|_{B^s_{2,1}}^0 \leq \langle \tau \rangle^{-\frac{d}{2}} \|u(t)\|_{B^s_{2,1}}^0 + \langle \tau \rangle^{-\alpha} \|u(t)\|_{B^s_{2,1}}^0
\]

\[
\lesssim \langle \tau \rangle^{-\frac{d}{2}} D(t) + \langle \tau \rangle^{-\alpha} (X(t) + D(t)),
\]

for \(u \in \{(\gamma + \Lambda)a, m\}\) if \(\frac{d}{2} - 1 \geq 0\), i.e., \(d \geq 2\). Inequality (4.4) follows if

\[
\alpha \geq \frac{d}{4}.
\]

\(\text{Estimate of } \mathcal{L}(Q(a)m): \) We have

\[
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s+\frac{d}{2})} \|\mathcal{L}(Q(a)m)(\tau)\|_{B^s_{2,1}}^t d\tau \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s+\frac{d}{2})} \|Q(a)(\tau)\|_{B^s_{2,1}}^t d\tau
\]

\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s+\frac{d}{2})} \|a(\tau)\|_{B^s_{2,1}}^t \|m(\tau)\|_{B^s_{2,1}}^t d\tau
\]

thanks to Lemma 5.6. Splitting the frequencies of \(m\), we have, for the integral pertaining \(\|m_l(\tau)\|_{B^s_{2,1}}^t\),

\[
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s+\frac{d}{2})} \|a(\tau)\|_{B^s_{2,1}}^t \|m_l(\tau)\|_{B^s_{2,1}}^t d\tau \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s+\frac{d}{2})} \langle \tau \rangle^{-\alpha} \langle \tau \rangle^{-\frac{1}{2}(s+\frac{d}{2})} d\tau
\]

\[
\times \sup_{0 \leq \tau \leq t} \left( \langle \tau \rangle^\frac{d}{2} \|a(\tau)\|_{B^s_{2,1}}^t \right) \sup_{0 \leq \tau \leq t} \left( \langle \tau \rangle^\frac{1}{2} \|m(\tau)\|_{B^s_{2,1}}^t \right)
\]

\[
\lesssim \langle t \rangle^{-\frac{1}{2}(s+\frac{d}{2})} (X(t) + D(t)) D(t),
\]

thanks to (4.4), Lemma 5.1 and \(\frac{1}{2}(s + \frac{d}{2}) < 1 + \frac{d}{2}\), i.e., \(s < 2 + \frac{d}{2}\), which is automatically satisfied since \(s \leq \frac{d}{2} + 1\).
As for the high frequencies, we first let $0 \leq t \leq 2$. Then we have $\langle t - \tau \rangle \simeq \langle \tau \rangle \simeq \langle t \rangle \simeq 1$ for all $0 \leq \tau \leq t \leq 2$, which gives

$$
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \|a(\tau)\|_{B^{s,1}_{2,1}} \|m(\tau)\|_{B^{s,1}_{2,1}} d\tau \lesssim \int_0^t \langle \tau \rangle^{-\frac{d}{4}} \|m(\tau)\|_{B^{4,1}_{2,1}} d\tau \times D(t)
$$

$$
\lesssim \int_0^t \|m(\tau)\|_{B^{4+1,1}_{2,1}} d\tau \times D(t)
$$

$$
\lesssim \langle t \rangle^{-\frac{1}{2}(s + \frac{d}{2})} X(t) D(t).
$$

We next treat the case $t \geq 2$. We split the integrals into $I_1$ and $I_2$ as follows:

$$
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \|a(\tau)\|_{B^{s,1}_{2,1}} \|m(\tau)\|_{B^{s,1}_{2,1}} d\tau
$$

$$
= \left( \int_0^1 + \int_1^t \right) \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \|a(\tau)\|_{B^{s,1}_{2,1}} \|m(\tau)\|_{B^{s,1}_{2,1}} d\tau =: I_1 + I_2.
$$

Note that if $0 \leq \tau \leq 1$ and $t \geq 2$, we have and $\langle t \rangle \leq 2t \leq 2(t - \tau)$. As such, we have

$$
I_1 \lesssim \int_0^1 \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \|a(\tau)\|_{B^{s,1}_{2,1}} \|m(\tau)\|_{B^{s,1}_{2,1}}^h d\tau \times D(t)
$$

$$
\lesssim \langle t \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \int_0^1 \|m(\tau)\|_{B^{4+1,1}_{2,1}} d\tau \times D(t)
$$

$$
\lesssim \langle t \rangle^{-\frac{1}{2}(s + \frac{d}{2})} X(t) D(t).
$$

On the other hand, since $\langle \tau \rangle^\alpha \leq 2^\alpha \tau^\alpha$ if $1 \leq \tau \leq t$, we have

$$
I_2 \leq \int_1^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \|a(\tau)\|_{B^{s,1}_{2,1}} \|m(\tau)\|_{B^{s,1}_{2,1}}^h d\tau
$$

$$
\lesssim \int_1^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \|a(\tau)\|_{B^{s,1}_{2,1}} \|m(\tau)\|_{B^{s,1}_{2,1}}^h \langle \tau \rangle^{-\frac{d}{4} - \alpha} d\tau \times D(t)
$$

$$
\lesssim \int_1^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \langle \tau \rangle^{-\frac{d}{4} - \alpha} d\tau \times D(t)^2 \lesssim \langle t \rangle^{-\frac{1}{2}(s + \frac{d}{2})} D(t)^2
$$

by Lemma 5.1 and $\frac{1}{2}(s + \frac{d}{2}) \leq \frac{d}{4} + \alpha$, i.e.,

$$
s \leq 2\alpha.
$$

Thus, we obtain

$$
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \|L(Q(a)m)(\tau)\|_{B^{s,1}_{2,\infty}} d\tau \lesssim \langle t \rangle^{-\frac{1}{2}(s + \frac{d}{2})} (X(t) + D(t)) D(t),
$$

provided that $\alpha$ satisfies (4.5) and (4.8).

Estimate of div $(K(a))$: Since div $(K(a)) = a \nabla \Delta a$, we have

$$
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \|\text{div } K(a)(\tau)\|_{B^{s,1}_{2,\infty}} d\tau \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \|a \nabla \Delta a\|_{B^{s,1}_{2,\infty}} d\tau
$$

$$
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \|a(\tau)\|_{B^{s,1}_{2,1}} \|\nabla \Delta a(\tau)\|_{B^{s,1}_{2,1}} d\tau
$$
thanks to the limiting case of the product rule (Lemma 5.6). We split the frequencies of
the higher order term as before: \( \| a(\tau) \|_{B^2_{2,1}} \leq \| a(t)(\tau) \|_{B^2_{2,1}} + \| a_h(\tau) \|_{B^2_{2,1}} \). Note that we have
\( \frac{1}{2} (s + \frac{d}{2}) \leq \frac{d}{2} + \frac{1}{2} (2 + \frac{d}{2}) \), i.e., \( s \leq 2 + \frac{d}{2} \), which is satisfied by assumption, so Lemma 5.1 implies
\[
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \| a(\tau) \|_{B^2_{2,1}} \| a(t)(\tau) \|_{B^2_{2,1}} \, d\tau
\leq \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \| a(\tau) \|_{B^2_{2,1}}^h \, d\tau \times (X(t) + D(t)) D(t)
\leq \langle t \rangle^{-\frac{1}{2}(s + \frac{d}{2})} (X(t) + D(t)) D(t),
\]
thanks to (4.4) and the definition of \( D(t) \).

The bounds pertaining to the term \( \| a_h(\tau) \|_{B^2_{2,1}} \) can be carried out similarly to the bounds for \( \mathcal{L}(Q(a)m) \). Let \( 0 \leq t \leq 2 \) first. Since \( \langle t - \tau \rangle \simeq \langle \tau \rangle \simeq \langle t \rangle \simeq 1 \) for all \( 0 \leq \tau \leq t \leq 2 \), we have
\[
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \| a(\tau) \|_{B^2_{2,1}} \| a_h(\tau) \|_{B^2_{2,1}} \, d\tau \leq \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \| a(\tau) \|_{B^2_{2,1}}^h \, d\tau \times D(t)
\leq \int_0^t \| a(\tau) \|_{B^2_{2,1}}^h \, d\tau \times D(t)
\leq \langle t \rangle^{-\frac{1}{2}(s + \frac{d}{2})} X(t) D(t)
\]
by the definition of \( X(t) \) and \( D(t) \). We next treat the case \( t \geq 2 \). Splitting the time-integral, we have
\[
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \| a(\tau) \|_{B^2_{2,1}} \| a_h(\tau) \|_{B^2_{2,1}} \, d\tau
= \left( \int_0^1 + \int_1^t \right) \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \| a(\tau) \|_{B^2_{2,1}} \| a_h(\tau) \|_{B^2_{2,1}} \, d\tau =: I_3 + I_4.
\]
We have
\( I_3 \lesssim t^{-\frac{1}{2}(s + \frac{d}{2})} \int_0^1 \| a(\tau) \|_{B^2_{2,1}} \, d\tau \times D(t) \lesssim \langle t \rangle^{-\frac{1}{2}(s + \frac{d}{2})} X(t) D(t) \)
by a similar calculation to (4.6). On the other hand, for \( I_4 \), we have
\( I_4 \lesssim \langle t \rangle^{-\frac{1}{2}(s + \frac{d}{2})} D(t)^2 \)
similarly to (4.7), thanks to Lemma 5.1 and \( \frac{1}{2} (s + \frac{d}{2}) \leq \frac{d}{4} + \alpha \), i.e., (4.8). Thus, we obtain
\[
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \| \text{div} K(a(\tau)) \|_{B^{\frac{d}{2}}_{2,\infty}} \, d\tau \lesssim \langle t \rangle^{-\frac{1}{2}(s + \frac{d}{2})} (X(t) + D(t)) D(t),
\]
provided that \( \alpha \) satisfies (4.5) and (4.8).

**Estimate of \( \frac{1}{t^\alpha} \text{div} (m \otimes m) \):** We have
\[
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \| \text{div} (m \otimes m)(\tau) \|_{B^{\frac{d}{2}}_{2,\infty}} \, d\tau \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \| (m \otimes m)(\tau) \|_{B^{\frac{d}{2}}_{2,\infty}} \, d\tau
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \| m(\tau) \|_{B^1_{2,1}} \| m(\tau) \|_{B^1_{2,1}} \, d\tau
\]
by Lemma \[5.6\]. Splitting the frequencies of the higher order term by \( \|m(t)\|_{B^1_{2,1}} \leq \|m_l(t)\|_{B^1_{2,1}} + \|m_h(t)\|_{B^1_{2,1}} \), we may carry out the estimates similarly to previous computations. Lemma \[5.1\] gives

\[
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \|m(t)\|_{B^0_{2,1}} \|m_l(t)\|_{B^1_{2,1}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \|m_l(t)\|_{B^1_{2,1}} \, d\tau \times D(t) \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \|m_l(t)\|_{B^1_{2,1}} \, d\tau \times (X(t) + D(t)) \times D(t) \\
\lesssim (t)^{-\frac{1}{2}(s + \frac{d}{2})} (X(t) + D(t)) \times D(t),
\]
	hanks{thanks to \[4.4\] and the definition of \( D(t) \).}

For the high frequencies \( \|m_h(t)\|_{B^1_{2,1}} \), let \( 0 \leq t \leq 2 \), so that we have

\[
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \|m(t)\|_{B^0_{2,1}} \|m_h(t)\|_{B^1_{2,1}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{d}{2}} \|m(t)\|_{B^1_{2,1}} \, d\tau \times D(t) \\
\lesssim \int_0^t \|m(t)\|_{B^{d+1}_{2,1}} \, d\tau \times D(t) \\
\lesssim (t)^{-\frac{1}{2}(s + \frac{d}{2})} X(t) \times D(t)
\]

by the definition of \( X(t) \) and \( D(t) \). We next treat the case \( t \geq 2 \). Splitting the time-integral, we have

\[
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \|m(t)\|_{B^0_{2,1}} \|m_h(t)\|_{B^1_{2,1}} \, d\tau \\
= \left( \int_0^1 + \int_1^t \right) \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \|m(t)\|_{B^0_{2,1}} \|m_h(t)\|_{B^1_{2,1}} \, d\tau =: I_5 + I_6.
\]

We have

\[
I_5 \lesssim t^{-\frac{1}{2}(s + \frac{d}{2})} \int_0^1 \|m(t)\|_{B^{d+1}_{2,1}} \, d\tau \times D(t) \lesssim (t)^{-\frac{1}{2}(s + \frac{d}{2})} X(t) \times D(t)
\]

by a similar calculation to \[4.6\]. For \( I_6 \), we have

\[
I_6 \lesssim (t)^{-\frac{1}{2}(s + \frac{d}{2})} D(t)^2
\]

similarly to \[4.7\], thanks to Lemma \[5.1\] and \( \frac{1}{2}(s + \frac{d}{2}) \leq \frac{d}{4} + \alpha \), i.e., \[4.8\]. Thus, we have in the end

\[
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \|\text{div} (m \otimes m)(\tau)\|_{B^{d}_{2,\infty}} \, d\tau \lesssim (t)^{-\frac{1}{2}(s + \frac{d}{2})} (X(t) + D(t)) \times D(t), \quad (4.11)
\]

provided that \( \alpha \) satisfies \[4.5\] and \[4.8\].
**Estimate of \(\text{div} (Q(a)m \otimes m)\):** Lemma 5.6 and the composition estimate (Lemma 5.7) give

\[
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \| \text{div} (Q(a)m \otimes m)(\tau) \|_{B_{2, \infty}^{-\frac{d}{2}}}^t d\tau
\leq \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \| (Q(a)m \otimes m)(\tau) \|_{B_{2, \infty}^{-\frac{d}{2}}}^t d\tau
\leq \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \| a(\tau) \|_{B_{2,1}^{\frac{d}{2}}} \| (m \otimes m)(\tau) \|_{B_{2,1}^{\frac{d}{2}}}^t d\tau.
\]

At this point, we notice that \(\|a(\tau)\|_{B_{2,1}^{\frac{d}{2}}} \leq X(t)\) and

\[
\|m \otimes m\|_{B_{2,1}^{\frac{d}{2}}} = \sum_{ij} \|m_i m_j\|_{B_{2,1}^{\frac{d}{2}}} \leq \sum_{ij} \|\nabla (m_i m_j)\|_{B_{2,1}^{\frac{d}{2}}}
\]

by Bernstein inequality, so the situation coincides with that of \(\text{div} (m \otimes m)\). By the same computation as that for (4.11), we readily obtain

\[
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \| \text{div} (Q(a)m \otimes m)(\tau) \|_{B_{2, \infty}^{-\frac{d}{2}}}^t d\tau \leq \langle t \rangle^{-\frac{1}{2}(s + \frac{d}{2})} X(t)(X(t) + D(t))D(t),
\]

provided that \(\alpha\) satisfies (4.5) and (4.8).

**Estimate of \((P'(a) - P'(0))\nabla a\):** Lemma 5.6 and Lemma 5.7 give

\[
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \| (P'(a)(\tau)) - P'(0)\|_{B_{2, \infty}^{-\frac{d}{2}}}^t d\tau
\leq \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \| P'(a)(\tau)) - P'(0)\|_{B_{2,1}^{\frac{d}{2}}}^t d\tau
\leq \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \| a(\tau) \|_{B_{2,1}^{\frac{d}{2}}}^t d\tau.
\]

By the same computation as that for (4.9), we obtain

\[
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s + \frac{d}{2})} \| (P'(a)(\tau)) - P'(0)\|_{B_{2, \infty}^{-\frac{d}{2}}}^t d\tau \leq \langle t \rangle^{-\frac{1}{2}(s + \frac{d}{2})}(X(t) + D(t))D(t),
\]

provided that \(\alpha\) satisfies (4.5) and (4.8).

In the end, inserting (4.9), (4.10), (4.11), (4.12) and (4.13) into (4.3), we deduce that

\[
\sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{1}{2}(s + \frac{d}{2})} \| U(\tau) \|_{B_{2,1}^{\frac{d}{2}}}^t \leq C \left( \| U_0 \|_{B_{2, \infty}^{-\frac{d}{2}}}^t + X(t)(X(t) + D(t))D(t) \right),
\]

given that (4.5) and (4.8) hold, where the constant \(C\) depends on \(d\), \(\mu\), \(\lambda\), \(\kappa\) and \(\gamma\) and the frequency-threshold \(j_0\) can be arbitrary.
Step 2: Bounds for the high frequencies. Rewriting System (CNSK) in terms of the weighted unknowns \((t^a, t^m)\), we have
\[
\begin{aligned}
\partial_t(t^a) + \text{div}(t^m) &= \alpha t^{a-1}, \\
\partial_t(t^m) - \frac{1}{\rho_*} \mathcal{L}(t^m) - \kappa \rho_* \nabla(t^a) + \gamma \nabla(t^a) &= \alpha t^{a-1} + t^a F(a, m), \quad (4.15)
\end{aligned}
\]
Using Lemma 2.2 with \(a \equiv t^a, m \equiv t^m, f \equiv \alpha t^{a-1} a, g \equiv \alpha t^{a-1} v + t^a F(a, m), s \equiv \frac{d}{2} + 1\) and \(q \equiv r \equiv \infty\), we have
\[
\|\tau^a((\gamma + \Lambda)a, m)\|^h_{L^\infty_t(B^{d+1}_{2,1})} \lesssim \|(\gamma + \Lambda)(\alpha t^{a-1} a) + \alpha t^{a-1} m + t^a F(a, m)\|^h_{L^\infty_t(B^{d+1}_{2,1})},
\]
where the constant appearing on the right hand-side does not depend on \(j_0\).

We first handle the lower order linear terms on the right hand-side of the above:
\[
\|\tau^a((\gamma + \Lambda)a, m)\|^h_{L^\infty_t(B^{d+1}_{2,1})} \lesssim \|(\gamma + \Lambda)(\alpha t^{a-1} a) + \alpha t^{a-1} m\|^h_{L^\infty_t(B^{d+1}_{2,1})}.
\]
For \(u \in \{(\gamma + \Lambda) a, m\}\), we have
\[
\|\alpha t^{a-1} u\|^h_{L^\infty_t(B^{d+1}_{2,1})} \leq \|u\|^h_{L^\infty_t(B^{d+1}_{2,1})} \leq X(t)
\]
when \(0 \leq \tau \leq t \leq 2\) and
\[
\|\alpha t^{a-1} u\|^h_{L^\infty_t(0,1;B^{d+1}_{2,1})} \leq \|u\|^h_{L^\infty_t(0,1;B^{d+1}_{2,1})} \leq X(t)
\]
when \(t \geq 2\) and \(0 \leq \tau \leq 1\). Moreover, when \(t \geq 2\) and \(1 \leq \tau \leq t\), by the frequency cut-off of the norm, we have
\[
\|\alpha t^{a-1} u\|^h_{L^\infty_t(1,t;B^{d+1}_{2,1})} = \alpha \sum_{j \geq j_0} 2^j(\frac{d}{2}+1)2^{-2j}\|\alpha t^{a-1} u\|_{L^\infty_t(1,t;L^2)}
\]
\[
\leq \alpha 2^{-2j_0} \sum_{j \geq j_0} 2^j(\frac{d}{2}+1)\|\alpha t^{a-1} u\|_{L^\infty_t(0,t;L^2)}
\]
\[
\leq \alpha 2^{-2j_0}\|\alpha t^{a-1} u\|^h_{L^\infty_t(B^{d+1}_{2,1})}.
\]
Taking \(j_0 \in \mathbb{Z}\), such that \(\alpha 2^{-2j_0} < \frac{1}{4}\), for instance, we may absorb the contributions of these terms into the left hand-side. As \(\alpha\) only depends on the dimension and the parameter \(\varepsilon\), so does \(j_0\). Thus, we arrive at
\[
\|\tau^a((\gamma + \Lambda)a, m)\|^h_{L^\infty_t(B^{d+1}_{2,1})} \lesssim X(t) + \|\alpha t^{a-1} F(a, m)\|^h_{L^\infty_t(B^{d+1}_{2,1})},
\]
where the frequency-threshold \(j_0\) depends only on the dimension \(d\) and the given constants such as \(\mu, \lambda, \kappa\) and \(\gamma\). It now comes down to estimating the nonlinear terms.

\underline{Estimate of \(L(Q(a)m)\)}:
\[
\|\tau^a L(Q(a)m)\|^h_{L^\infty_t(B^{d+1}_{2,1})} \lesssim \sum_{ij} \|\tau^a \partial_i \partial_j(Q(a)m)\|^h_{L^\infty_t(B^{d+1}_{2,1})}
\]
\[
\lesssim \sum_{ij} \|\tau^a(\partial_i \partial_j(Q(a)m) + 2\partial_i(Q(a)) \partial_j m + Q(a) \partial_i \partial_j m)\|^h_{L^\infty_t(B^{d+1}_{2,1})}.
\]
We have
\[ \| \tau^\alpha \partial_t \partial_j (Q(a)) m \|_{L^\infty_t(B^{d-1}_{2,1})} \lesssim \| \tau^\alpha \partial_t \partial_j (Q(a)) \|_{L^\infty_t(B^{d}_{2,1})} \| m \|_{L^\infty_t(B^{d-1}_{2,1})} \lesssim \| \tau^\alpha a \|_{L^\infty_t(B^{d}_{2,1})}^{\frac{d}{2}} \| m \|_{L^\infty_t(B^{d-1}_{2,1})} \lesssim \| X(t) \| \tau^\alpha a \|_{L^\infty_t(B^{d+2}_{2,1})}; \]
\[ \| \tau^\alpha \partial_t (Q(a)) \partial_j m \|_{L^\infty_t(B^{d-1}_{2,1})} \lesssim \| \partial_t (Q(a)) \|_{L^\infty_t(B^{d-1}_{2,1})} \| \tau^\alpha \partial_j m \|_{L^\infty_t(B^{d}_{2,1})} \lesssim \| a \|_{L^\infty_t(B^{d}_{2,1})} \| \tau^\alpha m \|_{L^\infty_t(B^{d-1}_{2,1})}; \]
\[ \| \tau^\alpha Q(a) \partial_\alpha \partial_j m \|_{L^\infty_t(B^{d-1}_{2,1})} \lesssim \| Q(a) \|_{L^\infty_t(B^{d}_{2,1})} \| \tau^\alpha \partial_\alpha \partial_j m \|_{L^\infty_t(B^{d-1}_{2,1})} \lesssim \| a \|_{L^\infty_t(B^{d}_{2,1})} \| \tau^\alpha m \|_{L^\infty_t(B^{d+1}_{2,1})}; \]
Note that we have
\[ \| \tau^\alpha a \|_{L^\infty_t(B^{d+2}_{2,1})} \lesssim \| \langle \tau \rangle^\alpha a \|_{L^\infty_t(B^{d+2-\varepsilon}_{2,1})} + \| \tau^\alpha a \|_{L^\infty_t(B^{d+2}_{2,1})} \lesssim D(t), \]
provided that \( \alpha \leq \frac{1}{2}(d + 2 - \varepsilon + \frac{d}{2}) \), i.e.,
\[ \alpha \leq \frac{1}{2}(d + 2 - \varepsilon) = \frac{d}{2} + 1 - \frac{\varepsilon}{2}. \]
We also have
\[ \| \tau^\alpha m \|_{L^\infty_t(B^{d+1}_{2,1})} \lesssim \| \langle \tau \rangle^\alpha m \|_{L^\infty_t(B^{d+1-\varepsilon}_{2,1})} + \| \tau^\alpha m \|_{L^\infty_t(B^{d+1}_{2,1})} \lesssim D(t), \]
provided that \( \alpha \leq \frac{1}{2}(d + 1 - \varepsilon + \frac{d}{2}) \), i.e.,
\[ \alpha \leq \frac{1}{2}(d + 1 - \varepsilon). \] (4.17)
Thus, we have
\[ \| \tau^\alpha L(Q(a)) m \|_{L^\infty_t(B^{d-1}_{2,1})} \lesssim X(t) D(t), \] (4.18)
provided that \( \alpha \) satisfies (4.17).

**Estimate of \( \text{div} (K(a)) \):** Since \( \text{div} (K(a)) = a \nabla \Delta a \), we have
\[ \| \tau^\alpha \text{div} (K(a)) \|_{L^\infty_t(B^{d-1}_{2,1})} \lesssim \| \tau^\alpha a \nabla \Delta a \|_{L^\infty_t(B^{d-1}_{2,1})} \lesssim \| a \|_{L^\infty_t(B^{d}_{2,1})} \| \tau^\alpha \Delta a \|_{L^\infty_t(B^{d}_{2,1})} \lesssim X(t) \| \tau^\alpha \Delta a \|_{L^\infty_t(B^{d}_{2,1})}. \]
We have
\[ \| \tau^\alpha \Delta a \|_{L^\infty_t(B^{d}_{2,1})} \lesssim \| \tau^\alpha a \|_{L^\infty_t(B^{d+2-\varepsilon}_{2,1})} + \| \tau^\alpha \Delta a \|_{L^\infty_t(B^{d}_{2,1})}, \]
provided that \( \alpha \leq \frac{1}{2}(d + 2 - \varepsilon + \frac{d}{2}) \), i.e., \( \alpha \leq \frac{1}{2}(d + 2 - \varepsilon) \), which is satisfied if (4.17) holds. Thus
\[ \| \tau^\alpha \text{div} (K(a)) \|_{L^\infty_t(B^{d-1}_{2,1})} \lesssim X(t) D(t) \] (4.19)
holds, provided that \( \alpha \) satisfies (4.17).
\textbf{Estimate of } \frac{1}{\rho_*} \text{div} (m \otimes m): \text{ We have }

\[ \| \tau^\alpha \text{div} (m \otimes m) \|^h_{L_t^\infty (B_{2,1}^{d+1})} = \| \tau^\alpha (m \cdot \nabla m + m \text{div} m) \|^h_{L_t^\infty (B_{2,1}^{d+1})} \اعداد كاذبة
\leq \| m \|^H_{L_t^\infty (B_{2,1}^{d+1})} \| \tau^\alpha \nabla m \|^H_{L_t^\infty (B_{2,1}^{d})} \اعداد كاذبة
\leq X(t) \| \tau^\alpha \nabla m \|^H_{L_t^\infty (B_{2,1}^{d})}.
\]

As for the decay functional, we deduce

\[ \| \tau^\alpha \nabla m \|^H_{L_t^\infty (B_{2,1}^{d})} \leq \| \tau^\alpha m \|^H_{L_t^\infty (B_{2,1}^{d+1-\varepsilon})} + \| \tau^\alpha m \|^H_{L_t^\infty (B_{2,1}^{d+1})}, \]
\[ \alpha \leq \frac{1}{2} (\frac{d}{2} + 1 - \varepsilon + \frac{d}{2}), \text{ i.e., } (4.17). \text{ Thus, we have }
\]

\[ \| \tau^\alpha \text{div} (m \otimes m) \|^H_{L_t^\infty (B_{2,1}^{d+1})} \leq X(t)D(t), \quad (4.20) \]

provided that (4.17) holds.

\textbf{Estimate of } \text{div} (Q(a)m \otimes m): \text{ We have }

\[ \| \tau^\alpha \text{div} (Q(a)m \otimes m) \|^H_{L_t^\infty (B_{2,1}^{d+1})} \leq \| \tau^\alpha Q(a)m \otimes m \|^H_{L_t^\infty (B_{2,1}^{d+1})} \اعداد كاذبة
\leq \| a \|^H_{L_t^\infty (B_{2,1}^{d+1})} \| \tau^\alpha m \otimes m \|^H_{L_t^\infty (B_{2,1}^{d+1})} \اعداد كاذبة
\leq X(t) \sum_j \| \tau^\alpha \partial_j (m \otimes m) \|^H_{L_t^\infty (B_{2,1}^{d+1})}. \]

By (4.20), we obtain

\[ \| \tau^\alpha \text{div} (Q(a)m \otimes m) \|^H_{L_t^\infty (B_{2,1}^{d+1})} \leq X(t)D(t), \quad (4.21) \]

provided that (4.17) holds.

\textbf{Estimate of } (P'(a) - P'(0))\nabla a: \text{ We have }

\[ \| \tau^\alpha (P'(a) - P'(0))\nabla a \|^H_{L_t^\infty (B_{2,1}^{d+1})} \leq \| a \|^H_{L_t^\infty (B_{2,1}^{d+1})} \| \tau^\alpha \nabla a \|^H_{L_t^\infty (B_{2,1}^{d+1})} \اعداد كاذبة
\leq X(t) \| \tau^\alpha \nabla a \|^H_{L_t^\infty (B_{2,1}^{d+1})}, \]

in which we deduce for the decay functional

\[ \| \tau^\alpha \nabla a \|^H_{L_t^\infty (B_{2,1}^{d+1})} \leq \| \tau^\alpha a \|^H_{L_t^\infty (B_{2,1}^{d+1-\varepsilon})} + \| \tau^\alpha a \|^H_{L_t^\infty (B_{2,1}^{d+1})}, \]

if (4.17) holds. Thus, this implies

\[ \| \tau^\alpha \text{div} (Q(a)m \otimes m) \|^H_{L_t^\infty (B_{2,1}^{d+1})} \leq X(t)D(t), \quad (4.22) \]

provided that (4.17) holds.
By (4.5), (4.8) and (4.17), we now know that the following inequality holds:

$$
\|\tau^\alpha ((\gamma+\Lambda)a, m)\|_{L^\infty(B_{2,1}^{\frac{d}{2}+1})}^h \leq C(X(t) + (X(t) + D(t))D(t)).
$$

(4.23)

**Conclusion.** By (4.5), (4.8) and (4.17), we now know that $\alpha$ must satisfy

$$
\max \left\{ \frac{d}{4}, \frac{s}{2} \right\} \leq \alpha \leq \frac{1}{2}(d+1-\varepsilon)
$$

for some arbitrary small $\varepsilon > 0$. Thus, the choice of $\alpha = \frac{1}{2}(d+1-\varepsilon)$ is justified under our assumption on $s$.

Putting together (4.14) and (4.23), we finally arrive at

$$
D(t) \lesssim D_0 + X(t) + X(t)^2 + D(t)^2.
$$

As Theorem 1.4 ensures that $X(t) \lesssim X_0$ with $X_0$ being small, we may now conclude that (1.7) is fulfilled for all time provided that $D_0$ and $X_0$ are small enough. This concludes Theorem 1.8.

**Proof of Theorem 7.9.** As much of the argument is similar to the proof of Theorem 1.8, we only sketch the proof here. As before, for notational simplicity, we denote

$$
\bar{X}(t) := \|\langle a, m \rangle\|_{C^1L_T} = \|\langle \Lambda a, m \rangle\|_{L^\infty(B_{2,1}^{\frac{d}{2}-1})} + \|\Lambda^2\langle a, m \rangle\|_{L^1(B_{2,1}^{\frac{d}{2}-1})},
$$

and $\bar{X}_0 := \|\langle \Lambda a_0, m_0 \rangle\|_{B_{2,1}^{\frac{d}{2}-1}}$.

Note that, for any solution to (CNSK) with $\gamma = 0$ constructed by Theorem 1.6 if in addition $\bar{D}(t) < \infty$, we have

$$
\|\langle \tau \rangle^\alpha \langle \Lambda a, m \rangle\|_{L^\infty(B_{2,1}^{\frac{d}{2}-3} \cap B_{2,1}^{\frac{d}{2}-1})}^h \lesssim \|\langle \Lambda a, m \rangle\|_{L^\infty(B_{2,1}^{\frac{d}{2}-3} \cap B_{2,1}^{\frac{d}{2}-1})}^h + \|\tau^\alpha \langle \Lambda a, m \rangle\|_{L^\infty(B_{2,1}^{\frac{d}{2}+1})}^h \lesssim \bar{X}(t) + \bar{D}(t) < \infty.
$$

(4.24)

In addition, similarly to (4.4), we have

$$
\sup_{0 \leq \tau \leq t} \|\langle \tau \rangle^{\frac{d}{2}} \langle \Lambda a, m \rangle(\tau)\|_{B_{2,1}^0} \lesssim \bar{X}(t) + \bar{D}(t).
$$

Starting from (4.3), our task is to bound

$$
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s+\frac{d}{2})} \|N(a, m)(\tau)\|_{B_{2,\infty}^{\frac{d}{4}}} d\tau,
$$

provided that (4.5) holds.

Identical calculations leading to (4.9), (4.10), (4.11), (4.12) and (4.13) yield

$$
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s+\frac{d}{2})} \|L(Q(a)m)(\tau)\|_{B_{2,\infty}^{\frac{d}{4}}} d\tau \lesssim \langle t \rangle^{-\frac{1}{2}(s+\frac{d}{2})} (\bar{X}(t) + \bar{D}(t)) \bar{D}(t),
$$

(4.25)

$$
\int_0^t \langle t - \tau \rangle^{-\frac{1}{2}(s+\frac{d}{2})} \|\text{div } K(a)(\tau)\|_{B_{2,\infty}^{\frac{d}{4}}} d\tau \lesssim \langle t \rangle^{-\frac{1}{2}(s+\frac{d}{2})} (\bar{X}(t) + \bar{D}(t)) \bar{D}(t),
$$

(4.26)
\[
\int_0^t (t - \tau)^{-\frac{1}{2} (s + \frac{d}{2})} \| \text{div} (m \otimes m)(\tau) \|_{B^\frac{d}{2} \infty}^f d\tau \lesssim \langle t \rangle^{-\frac{1}{2} (s + \frac{d}{2})} (\tilde{X}(t) + \tilde{D}(t)) \tilde{D}(t), \tag{4.27}
\]

\[
\int_0^t (t - \tau)^{-\frac{1}{2} (s + \frac{d}{2})} \| \text{div} (Q(a) m \otimes m)(\tau) \|_{B^\frac{d}{2} \infty}^f d\tau \lesssim \langle t \rangle^{-\frac{1}{2} (s + \frac{d}{2})} \tilde{X}(t)(\tilde{X}(t) + \tilde{D}(t)) \tilde{D}(t)
\]

and

\[
\int_0^t (t - \tau)^{-\frac{1}{2} (s + \frac{d}{2})} \| \nabla (a \tilde{P}(a)) (\tau) \|_{B^\frac{d}{2} \infty}^f d\tau \lesssim \langle t \rangle^{-\frac{1}{2} (s + \frac{d}{2})} (\tilde{X}(t) + \tilde{D}(t))^2, \tag{4.29}
\]

respectively, provided that \( \alpha \) satisfies (4.5) and (4.8). In the end, by (4.25), (4.26), (4.27), (4.28) and (4.29), we obtain

\[
\sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{1}{2} (s + \frac{d}{2})} \| U(\tau) \|_{B^\frac{d}{2} 1}^f \leq C \left( \| U_0 \|_{B^\frac{d}{2} \infty}^f + \tilde{X}(t)(\tilde{X}(t) + \tilde{D}(t)) \tilde{D}(t) \right), \tag{4.30}
\]

where the constant \( C \) depends on \( d, \mu, \lambda \) and \( \kappa \) and the frequency-threshold \( j_0 \) can be arbitrary.

The bounds for the high frequencies are also carried out in the same way as the proof for Theorem \( \text{Theorem 1.8} \). Applying Lemma \( \text{Lemma 2.2} \) to (4.15) with \( \gamma \equiv 0 \) and employing the same frequency cut-off argument leading to (4.16), we obtain

\[
\| \tau^\alpha (\Lambda a, m) \|^h_{L^\infty_t (B^\frac{d}{2} 1)} \lesssim \tilde{X}(t) + \| \tau^\alpha F(a, m) \|^h_{L^\infty_t (B^\frac{d}{2} 1)},
\]

where the frequency-threshold \( j_0 \) depends only on the dimension \( d \) and the given constants such as \( \mu, \lambda, \kappa \).

We begin estimating the nonlinear terms in the high-frequency regime. This too is carried out in the same way as before. By the same computations leading to (4.18), (4.19), (4.20), (4.21) we may easily see

\[
\| \tau^\alpha \mathcal{L}(Q(a)m) \|^h_{L^\infty_t (B^\frac{d}{2} 1)} \lesssim \tilde{X}(t) \tilde{D}(t), \tag{4.31}
\]

\[
\| \tau^\alpha \text{div} (K(a)) \|^h_{L^\infty_t (B^\frac{d}{2} 1)} \lesssim \tilde{X}(t) \tilde{D}(t), \tag{4.32}
\]

\[
\| \tau^\alpha \text{div} (m \otimes m) \|^h_{L^\infty_t (B^\frac{d}{2} 1)} \lesssim \tilde{X}(t) \tilde{D}(t), \tag{4.33}
\]

\[
\| \tau^\alpha \text{div} (Q(a)m \otimes m) \|^h_{L^\infty_t (B^\frac{d}{2} 1)} \lesssim \tilde{X}(t) \tilde{D}(t), \tag{4.34}
\]

respectively, provided that \( \alpha \) satisfies (4.17).

Lastly, to estimate the pressure term, we have to slightly modify the calculation of (4.22). We have

\[
\| \tau^\alpha (P'(a) - P'(0)) \nabla a \|^h_{L^\infty_t (B^\frac{d}{2} 1)} \lesssim \| a \|^h_{L^\infty_t (B^\frac{d}{2} 1)} \| \tau^\alpha \nabla a \|^h_{L^\infty_t (B^\frac{d}{2} 1)} \]

\[
\lesssim \tilde{X}(t) \| \tau^\alpha \nabla a \|^h_{L^\infty_t (B^\frac{d}{2} 1)} \]

\[
\lesssim \tilde{X}(t)(\tilde{X}(t) + \tilde{D}(t)), \tag{4.35}
\]
where we have used (4.24) in the last inequality. Thus, combining (4.31), (4.32), (4.33), (4.34) and (4.35), we obtain the bound for the high-frequency regime.

\[ \|\tau^a(\Lambda u, m)\|_{L_t^\infty(B_{r,1}^{d+1})} \leq C(\tilde{X}(t) + (\tilde{X}(t) + \tilde{D}(t))\tilde{D}(t)). \]

(4.36)

Now it is clear that (1.10) follows from (4.30) and (4.36). This concludes Theorem 1.9.

5. Appendix

5.1. Calculus facts. The following inequalities are useful in the proof of decay estimates.

**Lemma 5.1** ([12]). Let \( t := \sqrt{1 + t^2} \). For any positive real numbers \( a, b \), there exists a positive constant \( C \) such that the following inequalities hold: If \( \max(a, b) > 1 \), then

\[ \int_0^t \langle \tau \rangle^{-a} \langle t - \tau \rangle^{-b} d\tau \leq C(t)^{-\min(a, b)} \text{ for all } t \geq 0. \]

Recall that we have \( \langle t \rangle \approx (1+t) \approx \max(1,t) \). In practical situation, we choose either of the above equivalent quantities that fit our convenience.

The following is well-known.

**Lemma 5.2** ([2] P73). Let \( c_0 \) be a positive real number. For any positive real number \( r \), there exists a positive constant \( C(r) \) depending on \( r \) such that

\[ \sup_{t \geq 0} \sum_{k \in \mathbb{Z}} (2^k \frac{1}{r^2}) e^{-c_0 2^k t} \leq C(r). \]

5.2. Praproduction estimates. We introduce the notation of Bony’s paradifferential calculus. Recall Definition 1.3 and the operators

\[ \tilde{\Delta}_j u = F^{-1} [\hat{\phi}_j \hat{u}], \quad \tilde{\Delta}_j u = \sum_{|j-k| \leq 2} \hat{\Delta}_k u \quad \text{and} \quad \hat{\Delta}_j u = \sum_{j' \leq j} \tilde{\Delta}_j u \quad \text{for } j \in \mathbb{Z}. \]

Given \( u, v \in S' (\mathbb{R}^d) \) we have a formal decomposition of the pointwise product \( uv \):

\[ uv = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j v \hat{\Delta}_j u + \sum_{j \in \mathbb{Z}} \hat{\Delta}_j (uv) = \tilde{T}_u v + \tilde{T}_v u + \tilde{R}(u, v). \]

One may easily verify that, with our choice of \( \phi \), the following relations hold:

\[ \Delta_j \tilde{\Delta}_k u = 0 \quad \text{if} \quad |k - j| \geq 3 \quad \text{and} \quad \Delta_k \tilde{\Delta}_j (u) = 0 \quad \text{if} \quad |k - j| \geq 4. \]

(5.1)

Property (5.1) readily implies

\[ \Delta_k (\tilde{T}_u v) = \sum_{|j - k| \leq 3} \Delta_k (\hat{\Delta}_j u \hat{\Delta}_j v) \quad \text{and} \quad \Delta_k (\tilde{R}(u, v)) = \sum_{j \geq k-4} \Delta_k (\tilde{\Delta}_j u \tilde{\Delta}_j v). \]

For the details of the paradifferential calculus, we refer to Chapter 2 of [2]. The up-to-date homogeneous paraprod estimates are the following.

**Lemma 5.3** ([1]). Let \( s_1, s_2 \in \mathbb{R}, 1 \leq p, p_1, p_2 \leq \infty, 1 \leq \sigma, \sigma_1, \sigma_2 \leq \infty \) with \( \frac{1}{\sigma} = \frac{1}{\sigma_1} + \frac{1}{\sigma_2} \) and further assume that \( \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \leq 1, 1 \leq \lambda \leq \infty \) with \( p_1 \leq \lambda \). Then there exists a positive constant \( C \) depending on \( d, p, p_1, p_2, s_1, s_2, \sigma_1, \sigma_2 \), such that we have

\[ \|\tilde{T}_u v\|_{B_{p,\sigma}^{s_1 + s_2 + \frac{4 - d}{p_1} - \frac{d}{p_2}}} \leq C \left\{ \begin{array}{ll} \|u\|_{B_{p_1}^{s_1 + \sigma_1}} \|v\|_{B_{p_2}^{s_2 + \sigma_2}} & \text{if } s_1 + \frac{d}{\lambda} \leq \frac{d}{p_1}, \\
\|u\|_{B_{p_1}^{s_1}} \|v\|_{B_{p_2}^{s_2}} & \text{if } s_1 + \frac{d}{\lambda} = \frac{d}{p_1}. \end{array} \right. \]
Lemma 5.4 ([1]). Let \( s_1, s_2 \in \mathbb{R}, 1 \leq p_1, p_2 \leq \infty, 1 \leq \sigma, \sigma_1, \sigma_2 \leq \infty, \frac{1}{\sigma} = \frac{1}{\sigma_1} + \frac{1}{\sigma_2}, \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \) and
\[
s_1 + s_2 + d \inf \left( 0, 1 - \frac{1}{p_1} - \frac{1}{p_2} \right) > 0.
\]
Then there exists a constant \( C > 0 \) depending on \( d, p, p_1, p_2, \sigma, \sigma_1 \) and \( \sigma_2 \) such that
\[
\| \tilde{R}(u, v) \|_{\dot{B}_{\sigma, \infty}^{s_1 + s_2, \sigma_1} \dot{B}_{\sigma, \infty}^{s_2, \sigma_2}} \leq C \| u \|_{\dot{B}_{p_1, 1}^{s_1}} \| v \|_{\dot{B}_{p_2, 1}^{s_2}}.
\]

Let furthermore \( \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \) and \( s_1 + s_2 = 0 \). Then there exists a constant \( C > 0 \) depending on \( d, p, p_1, p_2, s_1 \) and \( s_2 \) such that
\[
\| \tilde{R}(u, v) \|_{\dot{B}_{p_1, 1}^{s_1} \dot{B}_{p_2, 1}^{s_2}} \leq C \| u \|_{\dot{B}_{p_1, 1}^{s_1}} \| v \|_{\dot{B}_{p_2, 1}^{s_2}}.
\]

Lemma 5.3 and Lemma 5.4 immediately lead to the following product estimates, which are used throughout the paper.

Lemma 5.5. Let \( d \geq 1, s_1, s_2 \leq \frac{d}{2} \) and \( s_1 + s_2 \geq 0 \). Then it holds that
\[
\| uv \|_{\dot{B}_{2, 1}^{s_1 + s_2 - \frac{d}{2}}} \leq C \| u \|_{\dot{B}_{2, 1}^{s_1}} \| v \|_{\dot{B}_{2, 1}^{s_2}}.
\]

For the limiting case \( s_1 + s_2 = 0 \), we have the following.

Lemma 5.6. Let \( s_1, s_2 \in \mathbb{R}, s_1 \leq \frac{d}{2}, s_2 \leq \frac{d}{2} \) and \( s_1 + s_2 \geq 0 \). Then there exists a constant \( C > 0 \) depending on \( d, s_1 \) and \( s_2 \) such that
\[
\| uv \|_{\dot{B}_{2, \infty}^{s_1 + s_2 - \frac{d}{2}}} \leq C \| u \|_{\dot{B}_{2, 1}^{s_1}} \| v \|_{\dot{B}_{2, 1}^{s_2}}.
\]

We recall a standard composition estimate in Besov spaces.

Lemma 5.7 ([12]). Let \( F : I \to \mathbb{R} \) be a smooth function (with \( I \) an open interval of \( \mathbb{R} \) containing 0) vanishing at 0. Then for any \( s > 0, 1 \leq p \leq \infty \) and interval \( J \) compactly supported in \( I \) there exists a constant \( C \) such that
\[
\| F(a) \|_{\dot{B}_{p, 1}^{s}} \leq C \| a \|_{\dot{B}_{p, 1}^{s}}
\]
for any \( a \in \dot{B}_{p, 1}^{d}(\mathbb{R}^d) \) with values in \( J \).

In the case \( s > - \min \left( \frac{d}{p}, \frac{d}{p'} \right) \) if in addition to the above hypotheses we have \( a \in \dot{B}_{p, 1}^{d}(\mathbb{R}^d) \) then \( F(a) \in \dot{B}_{p, 1}^{d}(\mathbb{R}^d) \cap \dot{B}_{p, 1}^{s}(\mathbb{R}^d) \) and
\[
\| F(a) \|_{\dot{B}_{p, 1}^{s}} \leq C \| a \|_{\dot{B}_{p, 1}^{s}} \left( \| F'(0) \|_{\dot{B}_{p, 1}^{s}} + C \| a \|_{\dot{B}_{p, 1}^{s}} \right).
\]

Acknowledgement

The authors are indebted to Professor H. Abels for his valuable comments. The first author is supported by JSPS Grant-in-Aid for Young Scientists (B) 17K14216 and Grant-in-Aid for JSPS Research Fellow 18J00557. The second author is supported by JSPS Grant-in-Aid for Scientific Research (C) 18K03368 and (B) 16H03945.
References

[1] Abidi, H. and Paicu, M., Existence globale pour un fluide inhomogène, Ann. Inst. Fourier (Grenoble) 57 (2007), 883–917.

[2] Bahouri, H., Chemin, J.-Y., and Danchin, R., Fourier analysis and nonlinear partial differential equations, Grundlehren der Mathematischen Wissenschaften, vol. 343, Springer, Heidelberg, 2011.

[3] Charve, F. and Danchin, R., A global existence result for the compressible Navier-Stokes equations in the critical $L^p$ framework, Arch. Ration. Mech. Anal. 198 (2010), no. 1, 233–271.

[4] Charve, F. and Haspot, B., Convergence of capillary fluid models: from the non-local to the local Korteweg model, Indiana Univ. Math. J. 60 (2011), no. 6, 2021–2059.

[5] , On a Lagrangian method for the convergence from a non-local to a local Korteweg capillary fluid model, J. Funct. Anal. 265 (2013), no. 7, 1264–1323.

[6] Charve, F., Local in time results for local and non-local capillary Navier-Stokes systems with large data, J. Differential Equations 256 (2014), no. 7, 2152–2193.

[7] Charve, F., Danchin, R., and Xu, J., Gevrey analyticity and decay for the compressible Navier-Stokes system with capillarity, arXiv:1805.01764v1.

[8] Chemin, J.-Y. and Gallagher, I., Wellposedness and stability results for the Navier-Stokes equations in $\mathbb{R}^3$, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), no. 2, 599–624.

[9] Chemin, J.-Y. and Lerner, N., Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes, J. Differential Equations 121 (1995), 314–328.

[10] Danchin, R., Fourier analysis methods for the compressible Navier-Stokes equations, arXiv:1507.02637.

[11] Danchin, R. and Desjardins, B., Existence of solutions for compressible fluid models of Korteweg type, Ann. Inst. H. Poincaré Anal. Non Linéaire 18 (2001), no. 1, 97–133 (English, with English and French summaries).

[12] Danchin, R. and Xu, J., Optimal time-decay estimates for the compressible Navier-Stokes equations in the critical $L^p$ framework, Arch. Ration. Mech. Anal. 224 (2017), no. 1, 53–90.

[13] Dunn, J. E. and Serrin, J., On the thermomechanics of interstitial working, Arch. Rational Mech. Anal. 88 (1985), no. 2, 95–133.

[14] Hattori, H. and Li, D., Global solutions of a high-dimensional system for Korteweg materials, J. Math. Anal. Appl. 198 (1996), no. 1, 84–97.

[15] , Solutions for two-dimensional system for materials of Korteweg type, SIAM J. Math. Anal. 25 (1994), no. 1, 85–98.

[16] Kobayashi, T. and Tsuda, K., Global existence and time decay estimate of solutions to the compressible two phase flow system under critical condition, preprint.

[17] Kotschote, M., Existence and time-asymptotics of global strong solutions to dynamic Korteweg models, Indiana Univ. Math. J. 63 (2014), no. 1, 21–51.

[18] Tang, T. and Gao, H., On the compressible Navier-Stokes-Korteweg equations, Discrete Contin. Dyn. Syst. Ser. B 21 (2016), no. 8, 2745–2766.

[19] Tan, Z. and Zhang, R., Optimal decay rates of the compressible fluid models of Korteweg type, Z. Angew. Math. Phys. 65 (2014), no. 2, 279–300.

[20] Tsuda, K., Existence and stability of time periodic solution to the compressible Navier-Stokes-Korteweg system on $\mathbb{R}^3$, J. Math. Fluid Mech. 18 (2016), no. 1, 157–185.
