On Parallel Sections of a Vector Bundle

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Abstract

We consider when a smooth vector bundle endowed with a connection possesses non-trivial, local parallel sections. This is accomplished by means of a derived flag of subsets of the bundle. The procedure is algebraic and rests upon the Frobenius Theorem.
1 Introduction

A connection on a vector bundle is a type of differentiation that acts on vector fields. Its importance lies in the fact that given a piecewise continuous curve connecting two points on the underlying manifold, the connection defines a linear isomorphism between the respective fibres over these points. A renowned theorem of differential geometry states that when the Riemann curvature tensor of the connection vanishes, there exist local frames comprised of parallel sections. This paper presents a refinement of this result. That is, given a connection on a vector bundle we determine when there exist local parallel sections and we find the subbundle they generate. This is accomplished by means of an algebraic construction of a derived flag of subsets of the original vector bundle.

This question has been considered in the real analytic case by Trencevski (cf. [3]). Our solution, by contrast, follows from the Frobenius Theorem, which applies to smooth ($C^\infty$) data. Furthermore, while Trencevski’s method relies upon power series expansions we take a more geometric approach to the problem.

By applying our methods to the vector bundle of symmetric two-tensors over a manifold we obtain a solution to the problem of determining when a connection is locally a metric connection. For the case of surfaces, this has also been dealt with in [1].
2 The Existence of Parallel Sections

Let \( \pi : W \to M \) be a smooth vector bundle and \( W' \) a subset of \( W \) with the following two properties:

P1: For each \( x \in M \), \( W_x \cap W' \) is a linear subspace of \( W_x := \pi^{-1}(x) \).

P2: For each \( w \in W' \) there exists an open neighbourhood \( U \) of \( \pi(w) \) in \( M \) and a smooth local section \( X : U \subseteq M \to W' \subseteq W \) such that \( w = X(\pi(w)) \).

Let

\[ \nabla : \mathcal{A}^0(W) \to \mathcal{A}^1(W) \]

be a connection on \( W \), where \( \mathcal{A}^n(W) \) denotes the space of local sections \( U \subseteq M \to W \otimes \Lambda^n M \). Define a map

\[ \tilde{\alpha} : \mathcal{A}^0(W) \to \mathcal{A}^1(W/W') \]

by

\[ \tilde{\alpha} := \phi \circ \nabla, \]

where \( W/W' \) is the quotient of \( W \) and \( W' \) taken fibrewise and \( \phi : W \otimes T^*M \to (W/W') \otimes T^*M \) denotes the natural projection. For any local section \( X : U \subseteq M \to W' \subseteq W \) and differentiable function \( f : U \to \mathbb{R} \) we have \( \tilde{\alpha}(fX) = f\tilde{\alpha}(X) \). Thus, there corresponds to \( \tilde{\alpha} \) a map

\[ \alpha_{W'} : W' \to (W/W') \otimes T^*M \]

acting linearly on each fibre of \( W' \). \( \alpha_{W'} \) is the second fundamental 1-form of \( W' \).

Let \( V \) be any subset of \( W \) satisfying P1. Define \( S(V) \) to be the subset of \( V \) consisting of all elements \( v \) for which there exists a smooth
local section $X : U \subseteq M \to V \subseteq W$ such that $v = X(\pi(v))$. Then $\mathcal{S}(V)$ satisfies both P1 and P2.

We seek to construct the maximal flat subset $\tilde{W}$, of $W$. $\tilde{W}$ may be obtained as follows. Set

$$
V^{(0)} := \{ w \in W \mid R(\cdot)(w) = 0 \}
$$

$$
W^{(i)} := \mathcal{S}(V^{(i)})
$$

$$
V^{(i+1)} := \ker \alpha_{W^{(i)}}
$$

where $R : TM \otimes TM \otimes W \to W$ denotes the curvature tensor of $\nabla$. This gives a sequence

$$
W \supseteq W^{(0)} \supseteq W^{(1)} \supseteq \cdots \supseteq W^{(k)} \supseteq \cdots
$$

of subsets of $W$. Note that $W^{(i)}$ is not necessarily a vector bundle over $M$ since the dimension of the fibres may vary from point to point. For some $k \in \mathbb{N}$, $W^{(l)} = W^{(k)}$ for all $l \geq k$. Define $\tilde{W} = W^{(k)}$, with projection $\tilde{\pi} : \tilde{W} \to M$.

In order to extract information from $\tilde{W}$ we need some concept of regularity. Accordingly, we say that the connection $\nabla$ is regular at $x \in M$ if there exists a neighbourhood $U$ of $x$ such that $\tilde{\pi}^{-1}(U) \subseteq \tilde{W}$ is a vector bundle over $U$. $\nabla$ is regular if $\tilde{W}$ is a vector bundle over $M$. The dimension of the fibres of $\tilde{W}$, for regular $\nabla$, shall be denoted $\text{rank}\, \tilde{W}$.

**Theorem 1** Let $\nabla$ be a connection on the smooth vector bundle $\pi : W \to M$.

(i) If $X : U \subseteq M \to W$ is a local parallel section then the image of $X$ lies in $\tilde{W}$.
(ii) Suppose that $\nabla$ is regular at $x \in M$. Then for every $w \in \tilde{W}_x$ there exists a local parallel section $X : U \subseteq M \rightarrow \tilde{W}$ with $X(x) = w$.

**Proof:**

(i) follows directly from the definition of $\tilde{W}$.

(ii) Suppose that $\nabla$ is regular at $x \in M$ and let $w \in \tilde{W}_x$. By regularity, there exists a neighbourhood $U_1$ of $x$ and a frame $(X_1, ..., X_n)$ of $\tilde{\pi}^{-1}(U_1) \subseteq \tilde{W}$. By choosing a possibly smaller neighbourhood $U_2 \subseteq U_1$ of $x$ we can extend $(X_1, ..., X_n)$ to a frame $\mathcal{X} := (X_1, ..., X_n, ..., X_N)$ of $\pi^{-1}(U_2) \subseteq W$. Let $\omega = \omega^i_j$ denote the connection form of $\nabla$ with respect to $\mathcal{X}$: $\nabla_XX_j = \sum_{i=1}^N X_i\omega^i_j(X)$. Since $\tilde{W}$ has zero second fundamental 1-form,

$$\omega = \begin{pmatrix} \phi & \ast \\ 0 & \ast \end{pmatrix}$$

where $\phi$ is an $n \times n$ matrix of 1-forms. The curvature form $\Omega = \Omega^i_j$ of $\nabla$ with respect to $\mathcal{X}$ is

$$\Omega = d\omega + \omega \wedge \omega = \begin{pmatrix} d\phi + \phi \wedge \phi & \ast \\ \ast & \ast \end{pmatrix}$$

Since the curvature tensor $R$ is identically zero, when restricted to $TM \otimes TM \otimes \tilde{W}$, it follows that

$$d\phi + \phi \wedge \phi = 0$$

Therefore, by the Frobenius Theorem, there exists an $n \times n$ matrix of functions $A = A^i_j$ defined in a neighbourhood $U \subseteq U_2$ of $x$ such that $dA = -\phi \wedge A$ and $A(x) = I_{n \times n}$, the $n \times n$ identity matrix (cf. [4],
chp. 7, 2. Proposition 1., pg. 290). Let $c^j$, $1 \leq j \leq n$, be real scalars satisfyine $w = \sum_{j=1}^{n} X_j(x)c^j$. Define functions $f^i$ on $U$ by

$$f^i = \begin{cases} 
\sum_{j=1}^{n} A^j_i c^j & 1 \leq i \leq n \\
0 & n+1 \leq i \leq N.
\end{cases}$$

Let $X : U \to \tilde{W}$ be the local section of $\tilde{W}$ defined by $X := \sum_{i=1}^{N} X_i f^i$. Since $df + \omega \cdot f = 0$, $X$ is parallel. Moreover, $X(x) = w$.

q.e.d.

**Corollary 2** Let $\nabla$ be a regular connection on the smooth vector bundle $\pi : W \to M$. Then $(\tilde{W}, \nabla)$ is a flat vector bundle over $M$.

**Corollary 3** Let $\nabla$ be a connection on the smooth vector bundle $\pi : W \to M$, regular at $x \in M$. Then there are $\dim \tilde{W}_x$ independent local parallel sections in a neighbourhood of $x \in M$.

**Example** Consider the symmetric connection $\nabla$ on the 2-sphere, $M = S^2$, defined as follows: $\Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta$, $\Gamma_{\theta\phi}^\phi = \Gamma_{\phi\theta}^\phi = \cot \theta$ and all other Christoffel symbols are zero. Here $\theta$ and $\phi$ are the polar and azimuthal angles on $S^2$, respectively. Let

$$X_1 = d\theta \otimes d\theta$$
$$X_2 = d\phi \otimes d\phi$$
$$X_3 = d\theta \otimes d\phi + d\phi \otimes d\theta$$

be a basis of $W$, the symmetric elements of $T^*M \otimes T^*M$. The curvature terms $R_{\theta\phi} = \nabla_{\partial_\phi} \nabla_{\partial_\phi} - \nabla_{\partial_\theta} \nabla_{\partial_\theta}$ are

$$R_{\theta\phi}(X_1) = -(\sin^2 \theta)X_3$$
$$R_{\theta\phi}(X_2) = X_3$$
$$R_{\theta\phi}(X_3) = 2X_1 - 2(\sin^2 \theta)X_2$$
This gives $W^{(0)} = \text{span}(X_1 + (\sin^2 \theta)X_2)$. Non-zero local sections of $W^{(0)}$ are of the form $X = f(X_1 + (\sin^2 \theta)X_2)$ where $f$ is a smooth non-vanishing function defined on an open subset of $S^2$. The covariant derivative of $X$ is $\nabla X = X \otimes d\log |f|$ and so $W^{(1)} = W^{(0)}$. Thus $\tilde{W} = W^{(0)}$. Since $\tilde{W}$ is a rank one vector bundle over $S^2$ it follows that $\nabla$ is a locally metric connection; in fact, it is the Levi-Civita connection of the induced metric of the standard embedding of the two-sphere in three-dimensional Euclidean space.
References

[1] R. Atkins and Z. Ge, *An Inverse Problem in the Calculus of Variations and the Characteristic Curves of Connections on SO(3)-Bundles* Can. Math. Bull. (1995)

[2] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry I* (John Wiley & Sons, 1963)

[3] K. Trencevski, *On the Parallel Vector Fields in Vector Bundles* Tensor N.S. 60 (1998)

[4] M. Spivak, *Differential Geometry II* (Publish or Perish, 1970, 1979)