IRREDUCIBLE CHARACTERS FOR
ALGEBRAIC GROUPS IN
CHARACTERISTIC THREE. II

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ABSTRACT

In this note, we determine the irreducible characters for the simple algebraic groups of type $A_4$ and $D_4$ over an algebraically closed field $K$ of characteristic 3, by using a theorem of Xi Nanhua [1] and the MATLAB software.

The determination of all irreducible characters is a big theme in the modular representations of algebraic groups and related finite groups of Lie type. But so far only a little is known concerning it in the case when the characteristic of the base field is less than the Coxeter number. Carter and Lusztig described certain constructions, the raising and lowering operators, for $\mathfrak{sl}(n, \mathbb{C})$ in [2] and further developed in [3] in order to determine nonzero homomorphisms between certain Weyl modules, and to obtain bases for

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certain Verma modules and all finite-dimensional irreducible $\mathfrak{sl}(n, \mathbb{C})$-modules. Gilkey-Seitz gave an algorithm to compute part of characters of $L(\lambda)$’s with $\lambda \in X_1(T)$ for $G$ being of type $G_2$, $F_4$, $E_6$, $E_7$ and $E_8$ in characteristic 2 and even in larger primes in [4]. Dowd and Sin gave all characters of $L(\lambda)$’s with $\lambda \in X_1(T)$ for all groups of rank less than or equal to 4 in characteristic 2 in [5]. They got their results by using the “standard” Gilkey-Seitz algorithm and computer. An element $e_{(p^n - 1)p - \lambda} \in u^-_n$ for each irreducible module $L(\lambda)$ with $\lambda \in X_n(T)$ was defined in [6, §39.1, p. 304] and [1, p. 239]. This element could be used in constructing a certain basis for $L(\lambda)$, computing dim $L(\lambda)$, and determining ch $(L(\lambda))$. In this way, we determined all irreducible characters for the special linear groups $SL(5, K)$, $SL(6, K)$ and $SL(7, K)$, the special orthogonal group $SO(7, K)$ and the symplectic group $Sp(6, K)$ over an algebraically closed field $K$ of characteristic 2 in [7], [8] and [9] and for the special orthogonal group $SO(7, K)$ and the symplectic group $Sp(6, K)$ over an algebraically closed field $K$ of characteristic 3 in [10]. In the present note, we shall work out all irreducible characters for the simple algebraic groups of type $A_4$ and $D_4$ over an algebraically closed field $K$ of characteristic 3. We shall freely use the notations in [9] without further comments.

1. PRELIMINARIES

Let $G$ be the simple algebraic group of type $A_4$ or $D_4$ over an algebraically closed field $K$ of characteristic 3. Take a Borel subgroup $B$ and a maximal torus $T$ of $G$ with $T \subset B$. Let $X(T)$ be the character group of $T$, which is also called the weight lattice of $G$ with respect to $T$. Let $R \subset X(T)$ be the root system associated to $(G, T)$, and choose a positive root system $R_+$ in such a way that $-R_+$ corresponds to $B$. Let $S = \{ x_1, x_2, x_3, x_4 \}$ be the set of simple roots of $G$ such that

$$R_+ = \{ x_1, x_2, x_3, x_4, x_{12} = x_1 + x_2, x_{23} = x_2 + x_3, x_{34} = x_3 + x_4, x_{123} = x_1 + x_2 + x_3, x_{234} = x_2 + x_3 + x_4, x_{1234} = x_1 + x_2 + x_3 + x_4 \}$$

for $G$ being the simple algebraic group of type $A_4$, and

$$R_+ = \{ x_1, x_2, x_3, x_4, x_{12} = x_1 + x_2, x_{23} = x_2 + x_3, x_{24} = x_2 + x_4, x_{123} = x_1 + x_2 + x_3, x_{124} = x_1 + x_2 + x_4, x_{134} = x_1 + x_3 + x_4, x_{234} = x_2 + x_3 + x_4, x_{1234} = x_1 + x_2 + x_3 + x_4 \}$$
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for $G$ being the simple algebraic group of type $D_4$. Let $\omega_i (1 \leq i \leq 4)$ be the fundamental weights of $G$ such that $\langle \omega_i, \omega_j^\vee \rangle = \delta_{ij}$, the Kronecker delta, and denote by $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ the weight $\lambda = \lambda_1 \omega_1 + \lambda_2 \omega_2 + \lambda_3 \omega_3 + \lambda_4 \omega_4$ with $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{Z}_+$, the integer ring. Then the dominant weight set is as follows:

$$X(T)_+ = \{ (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in X(T) \mid \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \}.$$ 

Let $W = N_G(T)/T$ be the Weyl group and let $W_3$ be the affine Weyl group of $G$. It is well-known that for $\lambda \in X(T)_+$, $H^0(\lambda)$ is the induced $G$-module from the 1-dimensional $B$-module $K_i$ which contains a unique irreducible $G$-submodule $L(\lambda)$ of the highest weight $\lambda$. In this way, $X(T)_+$ parameterizes the finite-dimensional irreducible $G$-modules. We set $\text{ch}(\lambda) = \text{ch}(H^0(\lambda))$ and $\text{ch}_1(\lambda) = \text{ch}(L(\lambda))$ for all $\lambda \in X(T)_+$. Moreover, $\text{ch}(\lambda)$ is given by the Weyl character formula, and for $\lambda \in X(T)_+$, we have

$$\text{ch}(\lambda) = \frac{\sum_{w \in W} \text{det}(w)e(\lambda + \rho))}{\sum_{w \in W} \text{det}(w)e(\rho)}.$$ 

For $\lambda = (a, b, c, d) \in X_1(T)$, when $G$ is the simple algebraic group of type $A_4$ we have

$$\dim H^0(a, b, c, d) = \frac{1}{252} (a + 1)(b + 1)(c + 1)(d + 1)(a + b + 2) \times (b + c + 2)(c + d + 2)(a + b + c + 3) \times (b + c + d + 3)(a + b + c + d + 4),$$

and when $G$ is the simple algebraic group of type $D_4$ we have

$$\dim H^0(a, b, c, d) = \frac{1}{252} (a + 1)(b + 1)(c + 1)(d + 1)(a + b + 2) \times (b + c + 2)(b + d + 2)(a + b + c + 3) \times (b + c + d + 3)(a + b + d + 3) \times (a + b + c + d + 4)(a + 2b + c + d + 5).$$

Let $F^n$ be the $n$-th Frobenius morphism of $G$ with $G_n \subset G$ the scheme-theoretic kernel of $F^n$. Let $V^{[n]}$ be the Frobenius twist for any $G$-module $V$. It is well-known that $V^{[n]}$ is trivial regarding as a $G_n$-module. Moreover, any $G$-module $M$ has such a form if the action of $G_n$ on $M$ is trivial. Let

$$X_n(T) = \{ (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in X(T)_+ \mid \lambda_1, \lambda_2, \lambda_3, \lambda_4 < 3^n \}.$$
Then the irreducible $G$-modules $L(\lambda)$'s with $\lambda \in X_\mu(T)$ remain irreducible regarded as the $G_\mu$-modules. On the other hand, any irreducible $G_\mu$-module is isomorphic to exactly one of them.

For $\lambda \in X(T)_+$, we have the unique decomposition

$$\lambda = \lambda^0 + 3^n \lambda^1 \quad \text{with} \quad \lambda^0 \in X_\mu(T), \lambda^1 \in X(T)_+.$$  

Then the Steinberg tensor product theorem tells us that

$$L(\lambda) \cong L(\lambda^0) \otimes L(\lambda^1)^{|\mu|}.$$  

Therefore we can determine all the characters $\text{ch}_3(\lambda)$ with $\lambda \in X(T)_+$ by using the Steinberg tensor product theorem, provided that all the characters $\text{ch}_3(\lambda)$ with $\lambda \in X_1(T)$ are known.

Recall the strong linkage principle in [11]. We define a strong linkage relation $\mu \uparrow \lambda$ in $X_\mu(T)$ if $L(\mu)$ occurs as a composition factor in $H^0(\lambda)$. Then $H^0(\lambda)$ is irreducible when $\lambda$ is a minimal weight in $X(T)_+$ with respect to the partial ordering determined by the strong linkage relations.

Let $\mathfrak{g}$ be the simple Lie algebra over $\mathbb{C}$ which has the same type as $G$, and $\mathfrak{h}$ the universal enveloping algebra of $\mathfrak{g}$. Let $e_\alpha, f_\alpha, h_\alpha(\alpha \in R_+, i = 1, 2, 3, 4)$ be a Chevalley basis of $\mathfrak{g}$. We also denote $e_{s_\alpha}, f_{s_\alpha}$ by $e_i, f_i$, respectively, where $I \in A = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1324\}$ for $G$ being the simple algebraic group of type $A_4$, and $I \in D = \{1, 2, 3, 4, 12, 23, 24, 123, 124, 234, 1324\}$ for $G$ being the simple algebraic group of type $D_4$, respectively. The Kostant $\mathbb{Z}$-form $\mathfrak{h}_\mathbb{Z}$ of $\mathfrak{h}$ is the $\mathbb{Z}$-subalgebra of $\mathfrak{h}$ generated by the elements $e_2^{(k)} := e_2^k / k!$, $f_2^{(k)} := f_2^k / k!$ for $\alpha \in R_+$ and $k \in \mathbb{Z}_+$. Set

$$\binom{h_\alpha + c}{k} := \frac{(h_\alpha + c)(h_\alpha + c - 1) \cdots (h_\alpha + c - k + 1)}{k!}.$$  

Then $\binom{h_\alpha + c}{k}$ is a $\mathbb{Z}_+$ for $i = 1, 2, 3, 4$, $c \in \mathbb{Z}$, $k \in \mathbb{Z}_+$. Define $\mathfrak{u}_k := \mathfrak{h}_\mathbb{Z} \otimes K$ and call $\mathfrak{u}_k$ the hyperalgebra over $K$ associated to $\mathfrak{g}$. Let $\mathfrak{u}^+, \mathfrak{u}^0, \mathfrak{u}^-$ be the positive part, negative part, zero part of $\mathfrak{u}_k$, respectively. They are generated by $e_2^{(k)}, f_2^{(k)}$, and $\binom{h_\alpha}{k}$, respectively. By abuse of notations, the images in $\mathfrak{u}_k$ of $e_2^{(k)}, f_2^{(k)}, \binom{h_\alpha}{k}$, etc. will be denoted by the same notations, respectively. The algebra $\mathfrak{u}_k$ is a Hopf algebra, and $\mathfrak{u}_k$ has a triangular decomposition $\mathfrak{u}_k = \mathfrak{u}_k^+ \mathfrak{u}_k^0 \mathfrak{u}_k^-$. Given a positive integer $n$, let $\mathfrak{u}_n$ be the subalgebra of $\mathfrak{u}_k$ generated by the elements $e_2^{(k)}, f_2^{(k)}, \binom{h_\alpha}{k}$ for $\alpha \in R_+, i = 1, 2, 3, 4$ and $0 \leq k < 3^n$. In particular, $\mathfrak{u}_1 = \mathfrak{u}_1^+$ is precisely the restricted enveloping algebra of $\mathfrak{g}$. Denote by $\mathfrak{u}_n^+, \mathfrak{u}_n^-, \mathfrak{u}_n^0$ the positive part, negative part, zero part
of $u_n$, respectively. Then we have also a triangular decomposition $u_n = u_{n-1} u_n^{1} u_n^{2}$. Given an ordering in $R_+$, it is known that the PBW-type bases for $\Pi_K$ resp. for $u_n$ have the form of

$$\prod_{\alpha \in R_+} f^{(\alpha)}_1 \prod_{i=1}^{s} \left( \frac{h_i}{b_i} \right) \prod_{\alpha \in R_+} e^{(\alpha)}_2$$

with $a_1, h_i, b_i \in \mathbb{Z}_+$ resp. with $0 \leq a_1, b_i, c_2 < 3^g$.

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in X_n(T)$. We set $\lambda_I = \sum_{I \in I} \lambda_i$ for $I \in A$ or $D$, here each element $I$ is also viewed as a certain set of simple roots. Following [6] and [1], we define an element $\xi_2$ in $u_n$ by

$$\xi_2 = f_1^{(\lambda_1)} f_2^{(\lambda_2)} f_3^{(\lambda_3)} f_4^{(\lambda_4)} f_2^{(\lambda_2)} f_3^{(\lambda_3)} f_4^{(\lambda_4)} f_2^{(\lambda_2)} f_3^{(\lambda_3)} f_4^{(\lambda_4)}$$

for $G$ being the simple algebraic group of type $A_4$, and

$$\xi_2 = f_1^{(\lambda_1)} f_2^{(\lambda_2)} f_3^{(\lambda_3)} f_4^{(\lambda_4)} f_2^{(\lambda_2)} f_3^{(\lambda_3)} f_4^{(\lambda_4)} f_2^{(\lambda_2)} f_3^{(\lambda_3)} f_4^{(\lambda_4)}$$

for $G$ being the simple algebraic group of type $D_4$. As a special case of [1, Theorems 6.5 and 6.7], we have

**Theorem 1.** Assume that $g$ is a simple Lie algebra of the simple algebraic group of type either $A_4$ or $D_4$ over an algebraically closed field $K$ of characteristic $3$. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in X_n(T)$.

(i) The element $\xi_2$ lies in $u_n$.

(ii) Let $\mathcal{H}$ be the left ideal of $\Pi_K$ generated by the elements $e_i^{(k)}$ and $(b_i)_{k} = \binom{(\lambda_i, \alpha)}{k} f_i^{(k)}$ ($i = 1, 2, 3, 4$, $k \geq 1$, $k_i \geq 3^g$) and the elements $f \in u_n$ with $f_{\mathcal{H}}(3^{g-1}) = 0$. Then $\Pi_K/\mathcal{H} \cong L(\lambda)$ (Note that $L(\lambda)$ has a $\Pi_K$-module structure, which is irreducible).

(iii) As a $u_n$-module, $L(\lambda)$ is isomorphic to $u_n \xi_2^{(3^{g-1})}$.

By abuse of notations, the images in $\Pi_K/\mathcal{H} \cong L(\lambda)$ of $f_i^{(k_i)}$ and $f_i^{(k_i)}$ will be denoted by the same notations. We shall use this theorem to compute the multiplicities of the weight spaces for all the dominant weight of $L(\lambda)$, to compute $\dim L(\lambda)$, and to determine $\text{ch} (L(\lambda)) = \text{ch}_3 (\lambda)$ ($\lambda \in X_1(T)$) in this note, when $G$ is the simple algebraic group of type either $A_4$ or $D_4$. 

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2. CHARACTERS OF THE IRREDUCIBLE MODULES OF G

From now on we shall assume that \( n = 1 \). Denote by \( V^* \) the dual module of \( V \), then we have by the duality that \( \text{ch} \mathcal{H}^\lambda = \text{ch}(\omega_0 \lambda) \) and \( \text{ch} L(\lambda) \) is also known that the graph automorphism group for \( G \) being the simple algebraic group of type \( D_4 \) is isomorphic to \( S_5 \), the symmetric group on 3 letters. For each \( \sigma \in S_5 \), it defines an automorphism \( \sigma \) of the Dynkin graph of type \( D_4 \), which permutes elements of the set \( \{ x_1, x_3, x_4 \} \). In the same way, \( \sigma \) defines a permutation of the set \( \{ \omega_1, \omega_3, \omega_4 \} \), which induces an automorphism \( \sigma \) of \( X(T) \). Then we get the bijections \( \sigma \) of the sets \( \{ \text{ch}(\lambda) | \lambda \in X(T)^+ \} \), \( \{ \text{ch}_3(\lambda) | \lambda \in X(T)^+ \} \) and \( \{ e(\lambda) | \lambda \in X(T)^+ \} \), respectively. It says that

\[
\text{ch}(\sigma(\lambda)) = \sigma(\text{ch}(\lambda)) \quad \text{and} \quad \text{ch}_3(\sigma(\lambda)) = \sigma(\text{ch}_3(\lambda)) \quad \text{and} \quad e(\sigma(\lambda)) = \sigma(e(\lambda)).
\]

So we list all main results only for part of the highest weights in this note.

First of all, an easy calculation shows the following lemmas:

**Lemma 2.** There are 33 strong linkage classes in \( X(T)^+ \) for the simple algebraic group of type \( A_4 \). Their representatives are \((0,0,0,0),(1,0,0,1),(0,1,0,2),(2,0,1,0),(1,0,2,2),(1,0,0,0),(0,0,0,1),(0,0,1,2),(2,1,0,0),(2,2,0,0),(0,0,2,2),(1,2,2,2),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,1,2,0),(2,0,0,0),(0,1,2,0),(0,1,0,0),(1,2,0,0),(1,2,0,0),(0,2,0,0),(0,1,0,0),(0,0,1,0),(0,0,2,0),(2,1,0,0),(2,0,1,0),(2,2,0,0),(1,2,0,0),(2,0,0,0),(1,2,0,0),(1,2,0,0),(2,2,0,0),(0,2,2,2),(2,2,2,2). Moreover, one has

\[
\text{ch}(\lambda) = \sigma(\text{ch}(\lambda)) \quad \text{and} \quad \text{ch}_3(\lambda) = \sigma(\text{ch}_3(\lambda)) \quad \text{and} \quad e(\lambda) = \sigma(e(\lambda)).
\]

**Lemma 3.** There are 23 strong linkage classes in \( X(T)^+ \) for the simple algebraic group of type \( D_4 \). Their representatives are \((0,0,0,0),(2,0,0,0),(0,0,2,0),(0,0,0,2),(1,0,0,0),(0,0,1,0),(0,0,0,1),(0,0,1,1),(1,0,1,0),(2,1,0,0),(0,1,2,0),(0,1,0,2),(2,0,1,1),(1,0,2,1),(0,1,0,0),(0,0,0,2),(0,0,1,3),(0,0,0,3),(2,0,0,2),(1,2,1,1),(2,0,2,0),(1,0,2,0),(0,1,2,1),(1,0,1,3),(0,0,3,2),(1,1,2,2),(0,2,1,0),(4,0,1,0),(2,2,1,2),(1,0,1,2),(2,0,2,1).
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Moreover, one has

\begin{align*}
(0, 0, 0, 0) & \mapsto (0, 1, 0, 0) \mapsto (1, 0, 0, 1) \mapsto (2, 0, 2, 0) \mapsto (1, 0, 1, 1) \mapsto (1, 0, 1, 3) \mapsto (2, 1, 2, 2) \mapsto (2, 0, 0, 0) \mapsto (0, 2, 0, 0) \mapsto (2, 0, 0, 0) \mapsto (1, 0, 2, 2, 2) \mapsto (1, 0, 0, 0) \mapsto (1, 0, 0, 2) \mapsto (0, 1, 1, 1) \mapsto (1, 1, 2, 0) \mapsto (3, 0, 2, 0) \mapsto (2, 1, 1, 1) \mapsto (1, 3, 0, 0) \mapsto (3, 2, 0, 0) \mapsto (2, 2, 1, 1) \mapsto (2, 0, 1, 0) \mapsto (3, 0, 0, 0) \mapsto (1, 2, 0, 0) \mapsto (0, 2, 1, 1) \mapsto (0, 1, 1, 3) \mapsto (0, 0, 3, 3) \mapsto (1, 1, 2, 2).
\end{align*}

Furthermore, the elements \( f_I (I \in A \text{ or } I \in D) \) satisfy the following commutator relations:

\begin{align*}
f_1 f_2 &= f_2 f_1 + f_{12}, \quad f_2 f_3 = f_3 f_2 + f_{23}, \\
f_3 f_4 &= f_4 f_3 + f_{34}, \quad f_{12} f_3 = f_3 f_{12} + f_{123}, \\
f_{23} f_4 &= f_4 f_{23} + f_{234}, \quad f_{13} f_3 = f_3 f_{13} + f_{132}, \\
f_{23} f_4 &= f_4 f_{23} + f_{234}, \quad f_1 f_2 = f_2 f_1 + f_{1234}, \\
f_{12} f_3 &= f_3 f_{12} + f_{1234}, \quad f_{12} f_4 = f_4 f_{12} + f_{1234}, \\
f_{13} f_4 &= f_4 f_{13} + f_{1324}, \quad f_{12} f_4 = f_4 f_{12} + f_{1234}, \\
f_{12} f_3 &= f_3 f_{12} + f_{1234}, \quad f_{12} f_4 = f_4 f_{12} + f_{1234}, \\
f_{13} f_4 &= f_4 f_{13} + f_{1324}, \quad f_{12} f_4 = f_4 f_{12} + f_{1234}, \\
f_{13} f_4 &= f_4 f_{13} + f_{1324}, \quad f_{12} f_4 = f_4 f_{12} + f_{1234}, \\
f_{13} f_4 &= f_4 f_{13} + f_{1324}, \quad f_{12} f_4 = f_4 f_{12} + f_{1234}, \\
f_{13} f_4 &= f_4 f_{13} + f_{1324}, \quad f_{12} f_4 = f_4 f_{12} + f_{1234}, \\
f_{13} f_4 &= f_4 f_{13} + f_{1324}, \quad f_{12} f_4 = f_4 f_{12} + f_{1234}, \\
f_{13} f_4 &= f_4 f_{13} + f_{1324}, \quad f_{12} f_4 = f_4 f_{12} + f_{1234}, \\
f_{13} f_4 &= f_4 f_{13} + f_{1324}, \quad f_{12} f_4 = f_4 f_{12} + f_{1234}, \\
f_{13} f_4 &= f_4 f_{13} + f_{1324}, \quad f_{12} f_4 = f_4 f_{12} + f_{1234}, \\
& \quad \text{for all the other } I, I' \in A,
\end{align*}

for \( G \) being the simple algebraic group of type \( A_4 \), and

\begin{align*}
f_2 f_1 &= f_1 f_2 + f_{12}, \quad f_3 f_2 = f_2 f_3 + f_{23}, \\
f_3 f_4 &= f_4 f_3 + f_{34}, \quad f_{12} f_3 = f_3 f_{12} + f_{123}, \\
f_2 f_3 &= f_3 f_2 + f_{234}, \quad f_1 f_3 = f_3 f_1 + f_{132}, \\
f_2 f_3 &= f_3 f_2 + f_{234}, \quad f_1 f_3 = f_3 f_1 + f_{132}, \\
f_2 f_3 &= f_3 f_2 + f_{234}, \quad f_1 f_3 = f_3 f_1 + f_{132}, \\
f_2 f_3 &= f_3 f_2 + f_{234}, \quad f_1 f_3 = f_3 f_1 + f_{132}, \\
f_2 f_3 &= f_3 f_2 + f_{234}, \quad f_1 f_3 = f_3 f_1 + f_{132}, \\
f_2 f_3 &= f_3 f_2 + f_{234}, \quad f_1 f_3 = f_3 f_1 + f_{132}, \\
f_2 f_3 &= f_3 f_2 + f_{234}, \quad f_1 f_3 = f_3 f_1 + f_{132}, \\
f_2 f_3 &= f_3 f_2 + f_{234}, \quad f_1 f_3 = f_3 f_1 + f_{132}, \\
f_2 f_3 &= f_3 f_2 + f_{234}, \quad f_1 f_3 = f_3 f_1 + f_{132}, \\
& \quad \text{for all the other } I, I' \in D,
\end{align*}

for \( G \) being the simple algebraic group of type \( D_4 \).

Now we can prove our main theorems. Let \( e(v) = \sum_{w \in W} w(v) \) be the sum of weights of the \( W \)-orbit of \( v \) for all \( v \in X(T)_+ \). It is well-known that \( \{ \text{ch}(v) \mid v \in X(T)_+ \} \), \( \{ \text{ch}_3(v) \mid v \in X(T)_+ \} \) and \( \{ e(v) \mid v \in X(T)_+ \} \) form bases of \( \mathbb{Z}[X(T)]^W \), the \( W \)-invariant subring of \( \mathbb{Z}[X(T)] \), respectively. According to the Weyl character formula and the Freudenthal multiplicity formula, we
get a change of basis matrix $A = (a_{i \ell})_{\ell, v \in X(T)_+}$ from \( \{ e(v) | v \in X(T)_+ \} \) to \( \{ ch(v) | v \in X(T)_+ \} \), which is a triangular matrix with 1 on its diagonal, i.e.,

$$ch(\lambda) = \sum_{v < \lambda, v \in X(T)_+} a_{i \ell} e(v)$$

with \( a_{i \ell} = 1 \) (cf. [12]). Based on our computation, we get another change of basis matrix $B = (b_{i \ell})_{\ell, v \in X(T)_+}$ from \( \{ e(v) | v \in X(T)_+ \} \) to \( \{ ch_3(v) | v \in X(T)_+ \} \), which is also a triangular matrix with 1 on its diagonal.

Let us mention our computation of $B$ more detailed. First of all, we compute \( \xi_{2p-\lambda} \) for any \( \lambda \in X_1(T) \). It is well known that for each dominant weight \( v \) of \( H^0(\lambda) \), \( \beta = \lambda - v \) can be expressed in terms of sum of positive roots, and there exist many ways to do so. Each way corresponds to an element \( f_{\beta} \xi_{2p-\lambda} \) in \( u_n \). Then we compute various \( f_{\beta} \xi_{2p-\lambda} \)'s generate the weight space \( L(\lambda)_v \) of the irreducible submodule \( L(\lambda) \) of \( H^0(\lambda) \). Therefore, we can easily determine the dimension of \( L(\lambda)_v \), provided that we compute the rank of the set of all these non-zero \( f_{\beta} \xi_{2p-\lambda} \)'s. It can be reduced to compute the rank of a corresponding matrix. Finally, we obtain the formal character of \( L(\lambda) \), which can be written as a linear combination of \( e(v)'s \) with non-negative integer coefficients. That is

$$ch_3(\lambda) = \sum_{v < \lambda, v \in X(T)_+} b_{i \ell} e(v)$$

with \( b_{i \ell} = 1 \). In this way, we get the second matrix $B$.

For example, we assume that $G$ is the simple algebraic group of type $A_4$ and \( \lambda = (0, 1, 1, 0) \). Then we have by the table in [12, p. 76]

$$ch(0, 1, 1, 0) = e(0, 1, 1, 0) + 2e(1, 0, 0, 1) + 5e(0, 0, 0, 0).$$

It is easy to see that

$$\xi = \xi_{2p-\lambda} = \xi_{2(2112)} = f_1^{(2)} f_2^{(3)} f_3^{(4)} f_4^{(6)} f_5^{(2)} f_6^{(2)} f_7^{(3)} f_8^{(2)}.$$ 

For \( v = (1, 0, 0, 1) \), we have \( \lambda - v = (-1, 1, 1, -1) = \xi_2 + \xi_3 \). First we compute each of the set \( S_\sigma = \{ f_2 f_3 \xi, f_2 f_3 \xi \} \). Then we compute the rank of the set \( S_\sigma \), which is equal to 1. So we have \( dim L(0, 1, 1, 0, 0, 0, 0, 1) = 1 \). For \( \mu = (0, 0, 0, 0, 0) \), we have \( \lambda - \mu = (0, 1, 1, 0) = 2\xi_2 + 2\xi_3 + 2\xi_4 \). We compute each of the set \( S_\mu = \{ f_1 f_2 f_3 f_4 \xi, f_1 f_2 f_3 f_4 \xi, f_1 f_2 f_3 f_4 \xi, f_1 f_2 f_3 f_4 \xi, f_1 f_2 f_3 f_4 \xi, f_1 f_2 f_3 f_4 \xi \} \).
IRREDUCIBLE CHARACTERS. II

$f_{1234}f_{2}f_{3}f_{4}$, and then we compute the rank of the set $S_{\mu}$, which is equal to 1. So we have $\dim L(0, 1, 1, 0)_{(0, 0, 0, 0)} = 1$. Finally, we obtain

\[ \text{ch}_3(0, 1, 1, 0) = e(0, 1, 1, 0) + e(1, 0, 0, 1) + e(0, 0, 0, 0), \]

and

\[ \text{ch}(0, 1, 1, 0) = \text{ch}_3(0, 1, 1, 0) + \text{ch}_3(1, 0, 0, 1). \]

When $\lambda$ lies in $X(T)_+$ but not in $X_1(T)$, we can also compute the formal character $\text{ch}_3(\lambda)$ by using the Steinberg tensor product theorem. For $\lambda \in X(T)_+$, we have the unique decomposition

\[ \lambda = \lambda^0 + 3\lambda^1 \quad \text{with } \lambda^0 \in X_1(T), \lambda^1 \in X(T)_+. \]

Then the Steinberg tensor product theorem tells us that

\[ \text{ch}_3(\lambda) = \text{ch}_3(\lambda^0) \cdot \text{ch}_3(3\lambda^1). \]

Therefore, we can determine all characters $\text{ch}_3(\lambda)$ with $\lambda \in X(T)_+$, provided that all characters $\text{ch}_3(\lambda)$ with $\lambda \in X_1(T)$ are known. For example, when $G$ being the simple algebraic group of type $D_4$ and $\lambda = (1, 0, 3, 1)$, we have

\[ \text{ch}_3(1, 0, 3, 1) = \text{ch}_3(1, 0, 0, 1) \cdot \text{ch}_3(0, 0, 3, 0) \\
= e(1, 0, 0, 1) + 3e(0, 0, 1, 0) \cdot e(0, 0, 3, 0) \\
= e(1, 0, 3, 1) + e(1, 0, 0, 1) + e(2, 0, 2, 0) + e(0, 0, 2, 2) + 3e(0, 0, 4, 0) + 3e(0, 1, 2, 0) + 3e(0, 0, 2, 0). \]

We list the two matrices $A$ and $B$ in the attached tables. When $G$ is the simple algebraic group of type $A_4$, these matrices can be found in Table 1, 2 and 3. When $G$ is the simple algebraic group of type $D_4$, these matrices can be found in Table 4 and 5. One should read these tables in such a way that

\[ \text{TABLE 1} = \begin{pmatrix} a_1 & 0 \\ a_2 & a_3 \end{pmatrix}, \quad \text{TABLE 2} = \begin{pmatrix} b_1 & 0 \\ b_2 & b_3 \end{pmatrix}, \]

\[ \text{TABLE 3} = \begin{pmatrix} c_1 & 0 \\ c_2 & c_3 \end{pmatrix}; \]
and

\[
\begin{align*}
\text{TABLE 4} &= \begin{pmatrix} d_1 & 0 & 0 \\ d_2 & d_3 & 0 \\ d_4 & d_5 & d_6 \end{pmatrix}, \\
\text{TABLE 5} &= \begin{pmatrix} e_1 & 0 & 0 \\ e_2 & e_3 & 0 \\ e_4 & e_5 & e_6 \end{pmatrix}.
\end{align*}
\]

Table 1.

| \(a_1\) | 0000 | 1 | 1 |
|--------|------|---|---|
|        | 1001 | 4 | 1 | 1 |
|        | 0110 | 1 | 1 | 1 | 1 |
|        | 0102 | 4 | 1 | 1 | 1 |
|        | 1200 | 4 | 1 | 1 | 1 |
|        | 0021 | 4 | 1 | 1 | 1 |
|        | 2002 | 4 | 1 | 1 | 1 |
|        | 3100 | 4 | 1 | 1 | 1 |
|        | 1003 | 4 | 1 | 1 | 1 |
|        | 1111 | 4 | 1 | 1 | 1 |
|        | 1030 | 4 | 1 | 1 | 1 |
|        | 0301 | 4 | 1 | 1 | 1 |
|        | 3011 | 4 | 1 | 1 | 1 |
|        | 1103 | 4 | 1 | 1 | 1 |
|        | 0220 | 4 | 1 | 1 | 1 |
|        | 2201 | 4 | 1 | 1 | 1 |
|        | 1022 | 4 | 1 | 1 | 1 |

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Table 1. Continued

| 3003 | 4 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
|      | 20 | 10 | 4 | 4 | 4 | 4 | 1 | 1 | 1 |
|      | 2120 | 24 | 18 | 13 | 10 | 8 | 6 | 6 | 6 |
|      | 30 | 21 | 14 | 10 | 9 | 6 | 6 | 6 | 3 |
|      | 0212 | 24 | 18 | 13 | 8 | 10 | 6 | 6 | 6 |
|      | 30 | 21 | 14 | 9 | 10 | 6 | 6 | 6 | 3 |

$a_2$

| 1310 | 1 | 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
|      | 24 | 18 | 14 | 10 | 8 | 9 | 6 | 6 | 3 |
|      | 0131 | 1 | 1 | 3 | 1 | 1 | 1 | 1 | 1 |
|      | 2112 | 29 | 24 | 18 | 12 | 12 | 9 | 9 | 4 |
|      | 65 | 44 | 29 | 20 | 20 | 12 | 12 | 14 | 6 |
|      | 2031 | 1 | 1 | 2 | 3 | 1 | 1 | 1 | 1 |
|      | 1302 | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 1 |
|      | 1221 | 64 | 49 | 37 | 28 | 28 | 21 | 21 | 9 |
|      | 80 | 59 | 44 | 32 | 32 | 24 | 24 | 24 | 9 |

$a_3$

| 1310 | 1 |
|      | 1 |
|      | 0131 | 1 |
|      | 2112 | 1 |
|      | 2031 | 1 |
|      | 1302 | 1 |
|      | 1221 | 1 |

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Table 2.

\[
\begin{array}{ccccccccccc}
\hline
b_1 \\
1000 & 1 \\
0101 & 3 & 1 \\
0020 & 1 & 1 & 1 \\
2001 & 3 & 1 & . & 1 \\
0012 & 3 & 2 & 1 & . & 1 \\
1110 & 3 & 2 & 1 & 1 & . & 1 \\
0004 & . & . & . & . & . & 1 \\
0300 & . & . & . & . & . & . & 1 \\
3010 & . & . & 1 & . & . & . & . & 1 \\
1102 & 9 & 4 & 1 & 3 & 1 & 1 & . & 1 \\
2200 & 6 & 4 & 3 & 3 & . & 2 & . & 1 & 1 & . & 1 \\
1021 & 12 & 8 & 6 & 3 & 3 & 2 & . & . & . & 1 & . & 1 \\
3002 & 1 & . & . & . & . & . & . & 1 & . & . & . & 1 \\
4100 & . & . & 1 & 1 & . & . & . & . & 1 & . & . & . & 1 \\
0211 & 9 & 6 & 4 & 4 & 3 & 3 & . & 1 & . & . & . & 1 \\
1013 & 16 & 10 & 6 & 6 & 3 & 4 & . & 2 & . & . & . & 1 & . & 1 \\
0130 & 12 & 9 & 6 & 3 & 6 & 2 & 3 & . & 2 & . & . & . & 1 & . & 1 \\
2111 & 10 & 7 & 6 & 4 & 3 & 3 & . & 1 & . & 2 & . & . & 1 & . & 1 \\
0203 & 14 & 9 & 5 & 6 & 5 & 3 & 2 & 1 & . & . & . & . & . & . & 1 \\
2030 & 16 & 12 & 9 & 7 & 6 & 5 & . & 1 & 2 & 3 & 1 & 3 & 1 & . & 1 & 1 & 1 & 1 \\
1301 & 19 & 14 & 10 & 10 & 4 & 8 & . & 6 & 3 & 3 & 3 & 2 & 1 & . & . & 1 & . & 1 \\
\hline
\end{array}
\]

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Table 2. Continued

| $h_2$  | 3  | 2  | 4  | 1  | 4  | 1  | 3  | 1  | 3  | 1  | 3  | 1  | 1  | 1  | 1  | 1  |
|--------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 4011   | 20 | 12 | 6  | 12 | 3  | 6  | 2  | 6  | 3  | 2  | 1  | 3  | 2  | 1  | 3  | 2  |
| 0122   | 21 | 16 | 12 | 9  | 9  | 7  | 3  | 3  | 5  | 4  | 2  | 2  | 1  | 1  | 1  | 1  |
| 2103   | 4  | 2  | 1  | 3  | 3  | 2  | 1  | 2  | 1  | 2  | 1  | 2  | 1  | 1  | 1  | 1  |
| 1220   | 30 | 21 | 15 | 15 | 9  | 11 | 6  | 4  | 7  | 3  | 5  | 3  | 4  | 2  | 2  | 1  |
| 3201   | 1  | 1  | 3  | 4  | 3  | 1  | 3  | 1  | 3  | 1  | 3  | 1  | 1  | 1  | 1  | 1  |
| 2022   | 24 | 18 | 14 | 15 | 5  | 11 | 6  | 8  | 4  | 6  | 3  | 3  | 3  | 2  | 2  | 1  |
| 0041   | .  | 1  | .  | 1  | .  | 1  | .  | 1  | .  | 1  | .  | 1  | .  | 1  | .  | 1  |
| 2417   | 9  | 7  | 6  | 5  | 4  | 4  | 1  | 2  | 1  | 2  | 1  | 2  | 1  | 1  | 1  | 1  |
| 0410   | 20 | 10 | 4  | 10 | 4  | 4  | 1  | 1  | 4  | 1  | 1  | 4  | 1  | 1  | 1  | 1  |
| 3120   | 14 | 11 | 8  | 8  | 5  | 7  | 6  | 3  | 4  | 3  | 3  | 2  | 3  | 1  | 2  | 1  |
| 1212   | 36 | 26 | 18 | 21 | 12 | 14 | 6  | 10 | 9  | 6  | 6  | 6  | 3  | 4  | 2  | 4  |
| 2310   | 63 | 45 | 31 | 33 | 22 | 23 | 7  | 12 | 9  | 16 | 6  | 10 | 6  | 8  | 5  | 2  |
| 1131   | 23 | 01 | .  | 3  | 1  | .  | 2  | 1  | 2  | 1  | 1  | 2  | 1  | 2  | 1  | 2  |
| 0402   | 56 | 44 | 36 | 30 | 26 | 24 | 8  | 12 | 8  | 18 | 6  | 16 | 6  | 10 | 6  | 8  |
| 3112   | 8  | 8  | 6  | 6  | 3  | 4  | 6  | 3  | 4  | 2  | 4  | 4  | 1  | 1  | 1  | 1  |
| 0321   | 80 | 56 | 38 | 44 | 26 | 29 | 8  | 12 | 20 | 20 | 12 | 12 | 14 | 6  | 8  | 6  |
| 3031   | 50 | 38 | 29 | 29 | 29 | 21 | 23 | 6  | 15 | 11 | 17 | 9  | 13 | 9  | 11 | 5  |
| 2302   | 58 | 46 | 36 | 37 | 21 | 30 | 6  | 21 | 18 | 17 | 15 | 13 | 10 | 6  | 11 | 5  |
| 2221   | 117| 90 | 69 | 72 | 51 | 55 | 15 | 33 | 32 | 41 | 24 | 31 | 24 | 9  | 25 | 12 |
|         | 117| 90 | 69 | 72 | 51 | 55 | 15 | 33 | 32 | 41 | 24 | 31 | 24 | 9  | 25 | 12 |

(continued)
Table 2. Continued

| 1 | 0 | 0 | 2 | 0 | 1 | 0 | 0 | 3 | 1 | 2 | 1 | 3 | 4 | 0 | 1 | 0 | 2 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 1 | 0 | 3 | 0 | 1 | 2 | 0 | 0 | 1 | 2 | 0 | 1 | 1 | 2 |
| 0 | 0 | 2 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 2 | 0 | 0 | 1 | 1 | 3 | 1 | 0 |
| 0 | 1 | 0 | 1 | 2 | 0 | 4 | 0 | 0 | 2 | 0 | 1 | 2 | 0 | 1 | 3 | 0 | 1 | 3 |

\[ b_3 \]

| 4011 | . | . | 1 |
| . | . | . | 1 |
| 0122 | . | . | . | 1 |
| . | . | . | 1 |
| 2103 | . | . | . | 1 |
| . | . | . | 1 |
| 1220 | 1 | 1 | . | . | 1 |
| 1 | 1 | . | . | 1 |
| 3201 | . | 1 | . | . | 1 |
| . | 1 | . | . | 1 |
| 0041 | . | . | 1 | . | . | 1 |
| . | . | . | 1 | . | . | 1 |
| 2022 | 1 | . | 1 | . | . | 1 |
| 1 | . | 1 | . | . | 1 |
| 4003 | . | . | . | 1 | . | . | . | 1 |
| . | . | 1 | . | . | . | 1 |
| 0410 | . | 1 | . | . | 1 | . | . | . | 1 |
| 1 | 2 | . | . | . | 1 | . | . | . | 1 |
| 3120 | 1 | . | 2 | . | . | 1 | . | . | . | 1 |
| 2 | 1 | 2 | . | . | 1 | 1 | . | . | . | 1 |
| 1212 | 1 | 2 | 2 | 2 | 1 | . | 1 | . | . | 1 |
| 1 | 2 | 2 | 2 | 1 | . | 1 | . | . | 1 |
| 2310 | . | 3 | . | . | 1 | 1 | . | . | . | 1 |
| 2 | 4 | 2 | . | 2 | 2 | . | . | 1 | 1 |
| 1131 | 3 | . | 1 | . | 1 | . | 1 | . | . | . | 1 |
| 4 | 2 | . | 4 | 2 | 2 | . | 2 | 2 | . | 1 | 1 |
| 0402 | . | 3 | . | . | 1 | . | . | . | 1 | . | . | . | 1 |
| 1 | 3 | . | 1 | 1 | 1 | . | 1 | . | 1 | . | . | . | 1 |
| 3112 | 1 | . | 3 | . | 1 | 2 | . | 1 | 1 | . | 1 | . | . | . | 1 |
| 2 | 2 | 4 | 2 | 4 | 1 | 2 | 2 | . | 2 | 2 | . | 1 | 1 | . | 1 |
| 0321 | . | 3 | . | 1 | 2 | . | . | . | 2 | . | 1 | . | 1 | . | 1 |
| 4 | 5 | . | 3 | 3 | 4 | . | 1 | 2 | . | 2 | 2 | . | 1 | 1 | . | 1 |
| 3031 | 1 | . | . | . | . | . | . | . | . | . | . | . | 1 | . | . | . | . | 1 |
| 6 | 2 | 3 | 3 | 3 | 2 | 2 | 1 | 3 | 1 | . | 2 | 1 | . | 1 | . | 1 |
| 2302 | . | 4 | . | 1 | . | 1 | 1 | . | . | 1 | . | 1 | 1 | . | . | . | . | 1 |
| 2 | 6 | 3 | 3 | 3 | 3 | 2 | 3 | . | 2 | 1 | 1 | 1 | 2 | 1 | . | 1 | . | 1 |
| 2221 | 10 | 10 | 7 | 7 | 7 | 8 | 5 | 3 | 5 | 3 | 2 | 4 | 4 | 2 | 2 | 1 | 2 | 1 | 1 | 1 |
| 10 | 10 | 7 | 7 | 7 | 8 | 5 | 3 | 5 | 3 | 2 | 4 | 4 | 2 | 2 | 1 | 2 | 1 | 1 | 1 |

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Table 2. Continued

| 2 | 1 | 4 | 0 | 2 | 1 | 3 | 0 | 2 | 4 | 0 | 3 | 2 | 0 | 2 | 4 | 0 | 3 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 3 | 0 | 1 | 1 | 2 | 2 | 0 | 0 | 0 | 4 | 1 | 2 | 3 | 1 | 4 | 1 | 3 | 0 | 3 | 2 |
| 3 | 0 | 1 | 2 | 0 | 2 | 0 | 4 | 2 | 0 | 1 | 2 | 1 | 3 | 0 | 1 | 2 | 3 | 0 | 2 |
| 0 | 1 | 1 | 2 | 3 | 0 | 1 | 1 | 2 | 3 | 0 | 0 | 2 | 0 | 1 | 2 | 2 | 1 | 1 | 2 | 1 |

Table 3.

| C | 1 |
|---|---|
| 0100 | 1 |
| 2000 | 1 | 1 |
| 1011 | 1 | 1 |
| 0011 | 2 | 1 |
| 0003 | 1 |
| 1101 | 4 | 3 | 1 | 1 |
| 1020 | 4 | 1 | 3 | 1 | 1 |
| 3001 | 1 | 1 |
| 0210 | 6 | 3 | 3 | 2 | 1 | 1 |
| 1012 | 9 | 3 | 6 | 3 | 2 | 1 |
| 2110 | 6 | 4 | 4 | 3 | 2 | 1 | 1 |
| 0202 | 4 | 3 | 3 | 2 | 1 | 1 |
| 1004 | 1 | 1 |
| 1300 | 1 |
| 4010 | 1 | 1 |
| 0121 | 9 | 6 | 6 | 3 | 4 | 3 |
| 2102 | 18 | 12 | 9 | 3 | 6 | 2 |
| 3200 | 1 |
| 0040 | 4 | 3 | 3 | 1 | 2 | 2 |
| 2021 | 12 | 9 | 9 | 3 | 6 | 5 |

(continued)
Table 3. Continued

|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
| 4002 | 2 | 3 . . | 1 . 3 . . | 1 . . . | 1 . | . . . | 1 . |
| 10 | 10 | 4 1 | 4 1 1 1 1 | 1 1 . | 1 . | . . . | 1 . |
| 0113 | . | 1 1 1 . . . | 1 . 1 1 . . . | . . . | 1 . | . . . | 1 . |
| 14 | 8 11 | 8 6 4 | . 2 4 | . 2 . 2 | . 1 . | . . . | 1 . |
|   |   |   |   |   |   |   |   |
| 0 | 2 0 | 0 1 | 1 3 0 | 1 2 0 1 | 1 4 0 2 3 0 2 4 0 |
| 1 | 0 0 0 | 0 1 0 0 2 0 | 1 2 0 | 3 0 1 1 2 0 0 0 1 |
| 0 | 0 1 | 0 0 2 0 1 | 1 1 0 0 0 1 2 0 0 4 2 0 1 |
| 0 | 0 1 3 | 1 0 1 | 0 2 0 2 4 0 0 1 2 0 0 1 2 3 |
| 1211 | 19 | 13 | 13 | 6 10 | 7 4 | 5 4 | 3 3 . 1 . 2 2 . . 1 . |
| 34 | 24 22 | 7 16 | 10 6 8 5 4 4 | . 2 . 2 . 2 | . 1 . |
| 0032 | . . | . . 1 | . 1 | . . . . . . | 1 . 1 . |
| 0993 | 6 7 | 4 5 | 4 . 3 3 . 2 1 | . 2 1 . |
| 2013 | 3 3 6 | 1 . . . | 3 1 3 . | 1 . . |
| 24 | 12 18 | 12 9 6 3 | 2 6 2 2 3 | . 1 2 . |
| 1130 | . . | 1 . . 1 | . . . | 1 . 1 . |
| 24 | 16 18 | 8 12 | 10 4 6 6 3 4 | 1 . 4 2 . |
| 0401 | . . . | . . 1 . . 3 | 1 1 . 3 | . . . . |
| 12 | 9 9 | 3 7 5 3 5 2 | 3 2 . 3 | 1 1 . 1 . |
| 3111 | 2 3 3 . | 2 1 3 | 1 2 . . 2 | 1 1 . 1 1 |
| 40 | 32 26 | 8 20 | 12 14 8 6 8 4 | . 2 4 2 4 2 | . 2 2 |
| 1203 | 3 4 3 | 4 2 | 1 . 3 | 2 3 . . 1 |
| 32 | 24 | 21 | 12 15 8 6 6 | 8 3 6 3 1 | 2 3 . |
| 0320 | 1 . . | 1 . . 1 | 1 1 . 1 . |
| 20 | 14 14 | 6 11 | 8 5 7 5 4 4 | . 3 . 3 3 | 1 2 |
| 3030 | . . 3 | . . . . . . . . . . . . . . . . 1 . . . 1 |
| 23 | 16 | 18 | 10 12 9 7 5 6 5 3 | 1 2 3 3 1 1 3 1 |
| 2301 | . . | . . 1 2 . 3 | 1 1 . 3 | 1 1 1 . |
| 28 | 22 21 | 6 17 13 | 10 11 5 8 4 | . 6 3 3 3 3 2 1 |
| 1122 | 40 | 28 31 | 16 22 17 9 11 | 12 7 8 4 3 | . 6 5 . 1 4 |
| 54 | 36 42 | 24 28 22 9 | 14 16 7 10 6 3 | . 8 5 . 2 4 |
| 3103 | 4 . | . . . . . . . . . . . . . . . . 1 . |
| 40 | 30 24 | 12 18 9 12 | 6 9 6 6 3 1 3 | 2 6 1 | . 2 3 |
| 2220 | 45 36 | 33 15 26 19 | 15 15 12 11 10 | . 6 4 7 7 3 3 3 5 3 |
| 45 | 36 33 | 15 26 19 | 15 15 12 11 10 | . 6 4 7 7 3 3 3 5 3 |
| 0312 | 1 4 . | 2 . 3 3 | 2 2 . 3 | . 1 . . |
| 42 | 33 31 | 16 24 17 | 12 14 12 9 10 4 6 | . 6 6 . 1 4 |
| 3022 | 9 6 7 | 4 4 3 6 | . 2 4 . 1 . 4 | . 3 3 2 3 |
| 54 | 36 42 | 24 27 21 15 | 11 15 11 8 6 3 4 6 8 3 1 6 3 |
| 2212 | 93 72 | 69 36 54 | 39 30 31 28 22 | 22 29 12 8 12 14 16 6 3 10 6 |
| 99 | 78 72 | 36 57 | 40 33 | 32 28 23 22 9 12 9 14 16 6 3 10 6 |
Table 3. Continued

|   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 2 | 0 | 0 | 1 | 1 | 3 | 0 | 1 | 2 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 2 | 0 | 1 | 2 | 0 |
| 0 | 0 | 1 | 0 | 0 | 2 | 0 | 1 | 1 | 1 | 0 | 0 |
| 0 | 0 | 1 | 3 | 1 | 0 | 1 | 0 | 2 | 0 | 2 | 4 |
| 0 | 0 | 1 | 3 | 1 | 0 | 1 | 0 | 2 | 0 | 2 | 4 |

\[c^3\]

|   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1211 | . | 1 |
| . | 1 |
| 0032 | . | 1 |
| 1 . | 1 |
| 2013 | 1 . | 1 |
| 1 . | 1 |
| 1130 | . . | 1 |
| . 1 . | 1 |
| 0401 | . 1 . | 1 |
| . 1 . | 1 |
| 3111 | . . . | 1 |
| . 1 . | 1 |
| 1203 | 2 . 1 . | 1 |
| 2 1 . 1 | 1 |
| 0320 | . 1 . 1 1 | 1 |
| . 2 . 1 1 | 1 |
| 3030 | . . . . . . 1 |
| . 1 . 1 . 1 | 1 |
| 2301 | . 1 . 1 1 | 1 |
| . 2 . 1 1 | 1 |
| 1122 | 3 2 1 2 1 . 1 | 1 |
| 4 2 2 2 1 . 1 | 1 |
| 3103 | . . 1 . . . . . . 1 |
| 2 1 . 2 . 1 1 | 1 |
| 2220 | . 4 . 2 1 2 . 1 1 1 | 1 |
| . 4 . 2 1 2 . 1 1 1 | 1 |
| 0312 | . 2 . 2 . 1 1 | 1 |
| 3 4 1 2 1 2 . 2 1 . 1 | 1 |
| 3022 | . . 1 . 2 . 1 . 1 . 1 | 1 |
| 3 2 1 3 1 . 2 1 . 1 1 1 | 1 |
| 2212 | 7 8 3 5 2 2 4 4 1 1 2 2 2 1 1 1 1 |
| 7 8 3 5 2 2 4 4 1 1 2 2 2 1 1 1 1 |

\[c^3\]

|   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 2 | 1 | 0 | 3 | 1 | 0 | 3 | 2 | 1 |
| 1 | 2 | 0 | 0 | 1 | 4 | 1 | 2 | 3 | 0 | 3 | 1 |
| 1 | 1 | 3 | 1 | 3 | 0 | 1 | 0 | 2 | 3 | 0 | 2 |
| 3 | 1 | 2 | 3 | 0 | 1 | 1 | 3 | 0 | 1 | 2 | 3 |

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Table 4.

| \( d_i \) | 0000 | 1 | 1 |
|-----------|------|---|---|
| 0100      | 4    | 1 |
| 2000      | 3    | 1 |
| 0020      | 3    | 1 |
| 0002      | 3    | 1 |
| 1011      | 10   | 6 |
| 0200      | 3    | 2 |
| 2100      | 15   | 8 |
| 0120      | 15   | 8 |
| 0102      | 15   | 8 |
| 4000      | 6    |
| 2020      | 6    |
| 0040      | 6    |
| 1111      | 17   |
| 0300      | 4    |
| 3011      | 29   |
| 1031      | 43   |
| 1013      | 43   |

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\begin{table}
\begin{tabular}{cccccccccccc}
\toprule
\multicolumn{14}{c}{Table 4. Continued} \\
\midrule
\multicolumn{14}{c}{} \\
\addlinespace[1.5ex]\end{tabular}
\end{table}

\begin{table}
\begin{tabular}{cccccccccccc}
\toprule
$d_2$ & 36 & 26 & 21 & 16 & 16 & 12 & 8 & 6 & 4 & 4 & 4 & 3 & 3 & 2 \\
\addlinespace[1.5ex]\midrule
\multicolumn{14}{c}{IRREDUCIBLE CHARACTERS. II} \\
\addlinespace[1.5ex]\end{tabular}
\end{table}

\begin{table}
\begin{tabular}{cccccccccccc}
\toprule
\multicolumn{14}{c}{} \\
\addlinespace[1.5ex]\end{tabular}
\end{table}
Table 4. Continued

| 1131 | . | 4 | 4 | 4 | 4 | 5 | 4 | 2 | 4 | 2 | . | 3 | 4 | 3 | 3 | . | 2 |
|------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 192  | 144| 104| 112| 104| 78 | 56| 36| 48 | 36| 8  | 30| 24| 24 | 30| 8 | 19|
| 1113 | . | 4 | 4 | 4 | 4 | 5 | 4 | 2 | 2 | 4 | . | 3 | 3 | 4 | 2|
| 192  | 144| 104| 104| 112| 78 | 56| 36| 36 | 48 | 8  | 24| 8  | 30| 30| 24| 19|
| 2300 | 12 | 4 | 4 | . | . | 1 | 1 | . | . | . | . | . | . | . | . | . | 1|
| 120  | 87 | 70 | 60 | 60 | 47 | 36| 29 | 23 | 23 | 12 | 18 | 6  | 18| 18 | 6 | 14|

\[
\begin{array}{cccccccccccccccc}
\begin{array}{cccccccccccccccc}
0 & 0 & 2 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 4 & 2 & 0 & 2 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 2 & 0 & 2 & 4 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 2 & 4 & 1 & 1
\end{array}
\end{array}
\]

\[d_h\]

| 2200 | 1 | 1 | . | . | 1 |
|------|---|---|---|---|---|
| 1    | 1 | . | . | 1 |
| 0220 | 1 | . | 1 | . | . | 1 |
| 1    | 1 | . | . | 1 |
| 0202 | 1 | . | . | 1 | . | 1 |
| 1    | 1 | . | 1 | . | 1 |
| 4100 | 1 | . | 1 | . | . | 1 |
| 2    | 2 | . | 1 | . | . | 1 |
| 2120 | 1 | 2 | 2 | . | 1 | 1 | . | . | 1 |
| 1    | 2 | 2 | . | 1 | 1 | . | . | 1 |
| 0140 | . | . | 1 | . | . | 1 | 1 | . | . | 1 |
| 1    | . | 2 | . | 1 | . | . | 1 |
| 2102 | 1 | 2 | . | 2 | 1 | 1 | . | . | 1 |
| 1    | 2 | . | 2 | 1 | 1 | . | . | 1 |
| 0122 | 1 | . | 2 | 2 | . | 1 | 1 | . | . | 1 |
| 1    | . | 2 | 2 | . | 1 | 1 | . | . | 1 |
| 0104 | . | . | 1 | . | . | 1 | . | . | . | . | 1 |
| 1    | . | 2 | . | 1 | . | . | 1 |
| 4020 | . | 3 | . | 1 | . | 1 | 1 | . | . | . | 1 |
| 1    | 3 | 1 | . | 1 | 1 | . | . | 1 | . | . | . | 1 |
| 2040 | . | 3 | . | 1 | . | 1 | 1 | . | . | . | . | 1 |
| 1    | 1 | 3 | . | 1 | 1 | . | . | 1 | . | . | . | 1 |
| 4002 | . | 3 | . | 1 | . | 1 | . | . | 1 | 1 | . | . | . | . | . | 1 |
| 1    | 3 | . | 1 | 1 | . | 1 | 1 | . | . | 1 | . | . | . | . | . | 1 |
| 2022 | 1 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | . | . | . | . | 1 |
| 1    | 3 | 3 | 3 | 1 | 1 | 1 | 1 | 1 | . | . | . | . | 1 |
| 0042 | . | 3 | . | 1 | . | 1 | 1 | 1 | . | . | . | . | 1 |
| 1    | . | 3 | 1 | 1 | 1 | . | . | 1 | . | . | . | . | 1 |
| 2004 | . | . | 3 | . | 1 | . | 1 | 1 | 1 | . | . | . | . | . | . | . | 1 |
| 1    | 1 | 3 | . | 1 | 1 | . | . | 1 | 1 | . | . | . | . | . | . | . | 1 |
| 0024 | . | . | 3 | . | 1 | . | 1 | 1 | 1 | . | . | . | . | . | . | . | . | 1 |
| 1    | . | 1 | 3 | . | 1 | 1 | . | . | 1 | 1 | . | . | . | . | . | . | . | 1 |
**Table 4. Continued**

| 1211 5 4 4 4 3 3 3 . 2 . 2 2 . . . . 1 . . . 1 |
| 8 5 5 5 4 4 4 . 2 . 2 2 . . . . 1 . . . 1 |
| 0400 4 . . . . . . . . 1 1 1 . . . . . . . . 1 |
| 5 2 2 2 2 2 2 . 1 . 1 1 . . . . . . . 1 . 1 |
| 3111 . 3 1 1 2 . . 2 1 . 1 1 . . . . . . . . 1 |
| 8 14 6 6 8 4 4 4 4 . 4 2 . 2 . 2 . 2 . . . 1 |
| 1131 . 1 3 1 . 2 . . 1 2 . 1 . 1 . 1 1 . . . 1 |
| 1113 . 1 1 3 . . 2 . . 1 1 2 . . 1 1 1 . . . 1 |
| 8 6 6 4 4 8 4 . 4 4 2 4 . . 2 . 2 2 . . . 1 |
| 2300 3 . . . 1 1 1 . . . . 1 1 1 . . . . . . . 1 1 |
| 9 8 5 5 7 4 4 2 3 . 3 3 . 1 . 1 2 . . 2 1 |
| 0 3 1 1 2 0 0 4 2 0 2 0 0 4 2 4 2 0 2 0 1 0 |
| 3 0 0 0 2 2 2 2 1 1 1 1 1 1 0 0 0 0 0 0 2 4 |
| 0 1 3 1 0 2 0 0 2 4 0 2 0 2 4 0 2 4 0 2 1 0 |
| 0 1 1 3 0 0 2 0 0 0 2 2 4 0 0 2 2 4 4 1 0 |

**d4**

| 0320 12 4 . 4 . 1 . 1 . . . . . . . . 1 |
| 120 87 60 70 60 47 36 23 29 23 6 18 12 18 18 6 14 |
| 0302 12 4 . . 4 . 1 . . 1 . . . . . . 1 |
| 120 87 60 60 70 47 36 23 23 29 6 18 6 18 18 12 14 |
| 3031 . . . 3 . . . 3 . . . 3 . . . 3 . . . 3 |
| 156 122 94 94 94 71 52 40 40 37 18 30 18 27 27 10 19 |
| 3013 . . . 3 . . . 3 . . . 3 . . . 3 . . . 3 |
| 156 122 94 94 94 71 52 40 37 40 18 27 10 30 27 18 19 |
| 1033 . . . 3 . . . 3 . . . 3 . . . 3 . . . 3 |
| 156 122 94 94 94 71 52 37 40 40 10 27 18 27 30 18 19 |
| 0222 174 136 108 105 105 82 63 47 49 49 15 36 21 36 39 21 28 |
| 237 184 141 141 141 106 78 55 61 61 15 42 24 42 48 24 32 |
| 2202 174 136 105 105 105 82 63 49 47 49 21 36 15 39 36 21 28 |
| 237 184 141 141 141 106 78 61 55 61 24 42 15 48 42 24 32 |
| 2220 174 136 105 105 108 82 63 49 49 47 21 39 21 36 36 15 28 |
| 237 184 141 141 141 108 78 61 61 55 24 48 24 42 42 15 32 |
| 2122 347 280 225 225 225 178 137 107 107 107 45 82 82 82 45 61 |
| 472 370 288 288 288 223 172 129 129 129 54 96 54 96 96 54 71 |
| 0 0 2 0 0 1 0 2 0 0 4 2 0 2 0 0 1 |
| 0 1 0 0 0 0 2 1 1 1 0 0 0 0 0 0 1 |
| 0 0 0 2 0 1 0 0 2 0 0 2 4 0 2 0 1 |
| 0 0 0 0 2 1 0 0 0 2 0 0 0 2 2 4 1 |

(continued)
Table 4. Continued

\[
\begin{array}{cccccccccccccccc}
\text{0320} & 3 & . & . & 1 & 1 & 1 & . & . & . & 1 & . & . & . & . & . & 1 & 1 \\
& 9 & 5 & 8 & 5 & 4 & 7 & 4 & . & 3 & 2 & 3 & 3 & . & 1 & . & 2 & 1 & . & 2 & 1 \\
\text{0302} & 3 & . & . & 1 & 1 & 1 & . & 1 & . & . & . & . & . & . & . & . & 1 & 1 \\
& 9 & 5 & 5 & 8 & 4 & 4 & 7 & . & 3 & 3 & 3 & 2 & . & 2 & 1 & 1 & 2 & 1 \\
\text{3031} & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & . & 1 & . \\
& 7 & 12 & 12 & 6 & 7 & 7 & 3 & 3 & 7 & 3 & 3 & 3 & . & 3 & 3 & 1 & 3 & 1 & . & 1 \\
\text{3013} & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & . & 1 & . \\
& 7 & 12 & 6 & 12 & 7 & 3 & 7 & 3 & 3 & 7 & 3 & 3 & 3 & . & 3 & 3 & 3 & 1 & 1 & . \\
\text{1033} & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & . & 1 & . \\
& 7 & 6 & 12 & 12 & 3 & 7 & 7 & . & 3 & 3 & 7 & 3 & . & 1 & . & 3 & 3 & 1 & 3 & 1 & . \\
\text{0222} & 16 & 12 & 16 & 16 & 10 & 12 & 12 & . & 7 & 4 & 7 & 9 & 4 & . & 3 & . & 5 & 3 & 3 & 4 & 1 \\
& 17 & 12 & 18 & 18 & 10 & 13 & 13 & . & 7 & 4 & 7 & 10 & 4 & . & 3 & . & 5 & 3 & 3 & 4 & 1 \\
\text{2202} & 16 & 16 & 12 & 16 & 12 & 10 & 12 & 4 & 7 & . & 9 & 7 & 4 & 3 & . & 3 & 5 & 3 & 3 & 4 & 1 \\
& 17 & 18 & 12 & 18 & 13 & 10 & 13 & 4 & 7 & . & 10 & 7 & 4 & 3 & . & 3 & 5 & 3 & 3 & 4 & 1 \\
\text{2220} & 16 & 16 & 16 & 12 & 12 & 12 & 10 & . & 4 & 9 & 4 & 7 & 7 & . & 3 & 3 & 3 & 3 & . & . & . & . & 4 & 1 \\
& 17 & 18 & 18 & 12 & 13 & 13 & 10 & 4 & 10 & 4 & 7 & 7 & . & 3 & 3 & 3 & 3 & . & . & . & . & 4 & 1 \\
\text{2122} & 33 & 34 & 34 & 34 & 24 & 24 & 24 & 24 & 8 & 18 & 8 & 18 & 18 & 8 & 8 & 6 & 6 & 6 & 13 & 6 & 6 & 8 & 1 \\
& 36 & 38 & 38 & 38 & 26 & 26 & 26 & 26 & 9 & 19 & 9 & 19 & 19 & 9 & 9 & 6 & 6 & 6 & 14 & 6 & 6 & 8 & 1 \\
\hline
\end{array}
\]

\[
\begin{array}{cccccccccccccccc}
\text{d}_5 & 0 & 3 & 1 & 1 & 2 & 0 & 0 & 4 & 2 & 0 & 2 & 0 & 0 & 4 & 2 & 4 & 2 & 0 & 2 & 0 & 1 & 0 \\
& 3 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 \\
& 0 & 1 & 3 & 1 & 0 & 2 & 0 & 0 & 2 & 4 & 0 & 2 & 0 & 2 & 4 & 0 & 2 & 4 & 0 & 2 & 1 & 0 \\
& 0 & 1 & 1 & 3 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 2 & 4 & 0 & 0 & 2 & 2 & 4 & 4 & 1 & 0 \\
\hline
\text{d}_6 & 3111 & 1 & 1 \\
& 1131 & . & 1 \\
& 1113 & . & . & 1 \\
& 2300 & . & . & 1 \\
& 1 & . & 1 \\
\hline
\text{0320} & . & . & . & 1 \\
& . & 1 & . & 1 \\
\hline
\text{0302} & . & . & . & 1 \\
& . & 1 & . & 1 \\
\text{3031} & . & . & . & . & 1 \\
& 1 & 1 & . & . & 1 \\
\text{3013} & . & . & . & . & 1 \\
& 1 & 1 & . & . & 1 \\
\text{1033} & . & . & . & . & 1 \\
& 1 & 1 & . & . & 1 \\
\text{0222} & 2 & 2 & . & 1 & 1 & . & . & 1 & 1 \\
\end{array}
\]
### Table 4. Continued

| . 2 2 . 1 1 . . 1 1 |
|---------------------|
| 2202 2 . 2 1 . 1 . 1 . . 1 |
| 2 . 2 1 . 1 . 1 . . 1 |
| 2220 2 2 . 1 1 . 1 . . . . 1 |
| 2 2 . 1 1 . 1 . . . . 1 |
| 2122 4 4 4 1 1 1 2 2 2 1 1 1 1 |
| 4 4 4 1 1 1 2 2 2 1 1 1 1 |
| 3 1 1 2 0 0 3 3 1 0 2 2 2 |
| 1 1 1 3 3 3 0 0 0 2 2 2 1 |
| 1 3 1 0 2 0 3 1 3 2 0 2 2 |
| 1 1 3 0 0 2 1 3 3 2 2 0 2 |

#### Table 5.

| $e_1$ |
|--------|
| 1000   |
| 1      |
| 1      |
| 0011   |
| . 3 1  |
| . 3 1  |
| . 3 1 1 |
| 1100   |
| . 3 1 1 |
| . 1 1 1 1 1 |
| 3000   |
| . . 1 |
| . 3 1 1 1 |
| . 6 2 1 |
| 1020   |
| . 6 3 1 1 |
| . 6 3 1 1 |
| . 6 3 1 1 |
| 1002   |
| . 6 3 1 1 |
| . 6 3 1 1 |
| . 6 3 1 1 |
| 0111   |
| . 4 3 2 1 1 1 |
| 17 9 4 2 2 1 |
| 2011   |
| . 2 1 2 3 3 3 1 1 |
| 21 12 7 3 3 3 1 1 |
| 21 12 7 3 3 3 1 1 |
| 0031   |
| . . 1 1 1 1 |
| 10 6 3 3 1 1 1 |
| 0013   |
| . . 1 1 1 1 |
| . 10 6 3 1 3 1 1 |
| 1200   |
| . 12 8 6 1 3 3 2 1 1 1 |
| 21 11 7 2 3 3 2 1 1 |
| 3100   |
| . 1 4 1 1 1 1 |
| 18 10 6 6 3 3 2 2 2 1 1 |
| 1120   |
| . 16 12 8 3 6 4 3 3 2 1 1 |
| 33 21 12 3 9 6 4 2 2 1 1 |
| 1102   |
| . 16 12 8 3 4 6 4 3 3 2 1 1 |
| 33 21 12 3 6 9 4 2 2 1 1 |

(continued)
Table 5. Continued

| 5000 | 1 . 1 3 . . . 1 . . . 1 . 1 . 1 |
|------|---------------------------------|
| 6    | 3 3 3 3 1 1 1 1 . . 1 1 . 1 1 |
| 3020 | . . 1 3 . 1 . 1 . . . 1 . . . 1 |
| 22   | 15 10 6 6 6 3 3 3 . 1 . 1 1 . 1 |
| 1040 | . . 1 . . 3 . 1 . 3 . . . 1 . . . 1 |
| 3002 | . . 1 3 1 . . 1 . . . 1 . . . 1 |
| 1022 | 24 18 12 3 9 9 6 2 3 3 3 1 . 1 . 1 |
| 1004 | . . 1 . . 3 1 . . 3 . . . 1 . . . 1 |
| 0211 | 36 26 16 6 12 12 8 4 3 3 3 . 2 2 . . 1 . 1 |
|      | 1 . 0 1 3 1 1 0 2 0 0 1 . . 1 1 5 1 3 1 1 0 |
|      | 0 0 1 0 0 0 0 0 0 0 2 . 1 1 . 1 0 0 0 0 0 2 |
|      | 0 1 0 0 2 0 1 1 3 1 0 0 0 2 0 0 2 4 0 2 0 1 |
|      | 0 1 0 0 3 2 1 1 1 0 0 0 2 0 0 0 2 4 1 1 |

\(e_2\)

| 2111 | 50 36 26 12 16 16 12 8 4 4 6 2 3 3 . 1 . 1 2 |
|------|---------------------------------|
| 101  | 69 48 24 30 30 19 14 6 6 8 4 4 4 . 2 2 2 |
| 0131 | . 1 2 . 3 . 1 1 3 . . 1 . . . 1 . 1 |
| 54   | 38 25 8 21 15 12 6 9 3 4 . 4 2 . 2 2 |
| 0113 | . 1 2 . . 3 1 1 . 3 . . . 1 . . . 1 |
| 54   | 38 25 8 15 21 12 6 3 9 4 . 4 2 . 2 2 |
| 1300 | 4 . . . . . . . . . . . . . . . . 1 . . . |
| 56   | 36 26 12 16 16 12 8 4 4 7 2 3 3 . 1 . 1 2 |
| 4011 | 3 3 6 10 3 3 2 6 . . 3 6 1 1 3 3 . 3 |
| 54   | 38 29 21 18 18 12 12 3 3 7 7 3 3 3 3 . 3 1 |
| 2031 | 3 2 . 6 3 3 3 1 6 1 . . 3 . . 1 3 . 1 |
| 78   | 57 40 18 30 27 19 12 6 7 3 7 3 . 3 3 1 3 |
| 0051 | 4 3 2 . 6 . 3 1 6 . 2 . 3 . . 3 3 1 3 |
| 2115 | 10 3 10 6 6 3 6 1 3 . 3 1 . . 3 1 |
| 2013 | 3 2 . 3 6 3 3 1 1 6 . . 3 . . 1 1 1 |
| 78   | 57 40 18 27 30 19 12 6 7 3 3 3 . 1 . 3 3 |
| 0033 | 3 . 3 . . . . . . . . . . . . . . . . |
| 39   | 29 21 10 15 15 10 6 6 6 3 . 3 3 . . 1 . 3 |
| 0015 | 4 3 2 . . 6 3 1 . 6 2 . 3 . . . 3 |
| 2115 | 10 3 6 10 6 3 1 6 3 . 1 3 . . 1 3 |
| 3200 | 1 3 3 3 1 1 2 2 . . 1 2 1 1 1 1 . 1 |
| 62   | 42 34 24 20 20 15 13 5 5 9 7 4 4 2 3 . 3 3 |
| 1220 | 108 78 54 24 42 36 27 18 15 9 13 4 10 7 . 3 3 3 5 |
| 108  | 78 54 24 42 36 27 18 15 9 13 4 10 7 . 3 3 3 5 |
Table 5. Continued

|    | 1202 | 108  | 78   | 54   | 24   | 36   | 42   | 27   | 18   | 9    | 15   | 13   | 4    | 7    | 10   | .    | .    |
|----|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
|    |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 1140| 4    | 6    | 4    |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 3102| 3    | 6    | 9    | 15   | 9    | 3    | 4    | 8    | 3    |      |      |      |      |      |      |      |
| 1712| 99   | 66   | 31   | 45   | 45   | 34   | 25   | 18   | 18   | 8    | 13   |      |      |      |      |
| 204 | 152  | 111  | 54   | 81   | 81   | 45   | 31   | 26   | 12   | 15   | 16   | 12   |      |      |      |
| 100 | 4    | 6    | 4    |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 0311| 12   | 4    |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 147 | 107  | 77   | 37   | 57   | 57   | 43   | 29   | 21   | 21   | 24   | 8    | 16   | 16   |      |      |
| 3022| 3    | 6    | 5    | 6    | 6    | 5    | 4    | 3    | 3    | 4    | 2    | 2    | 3    | 1    | 3    |
| 174 | 133  | 102  | 60   | 75   | 75   | 55   | 42   | 27   | 27   | 29   | 19   | 19   | 19   | 19   |

\[e_3\]

|    | 2111 |      | 1    | 1    |
|----|------|------|------|------|
| 0131|      |      |      | 1    |
| 0113| 1    |      |      | 1    |
| 1300| 1    |      |      | 1    |
| 4011|      |      |      | 1    |
| 2031|      |      |      | 1    |
| 0051| 1    | 1    |      | 1    |
| 2013| 3    |      |      | 1    |
| 0033|      |      |      | 1    |
| 0015| 3    | 1    |      | 1    |

(continued)
Table 5. Continued

3200 . . 1 . . . 1 . . . . . 1
.2 1 . .1 1 . . . . . 1
1220 . 4 2 2 . 1 . 1 . . . . . 1
.4 2 2 . 1 . 1 . . 1
1202 3 4 2 . 2 1 . . . 1 . . . . 1
.4 2 2 . 1 . 1 . . 1
3120 . . 2 . . . 2 1 . . . . . 1
.4 4 2 . 1 2 2 . . 1 1 . 1
1140 . 1 1 2 . . . 1 1 . . . . 1
.4 2 4 . 1 . 2 2 . . 1 1 . 1
3102 . . 2 . . . 2 . . 1 . . 1 . . . . 1
.4 4 2 . 1 2 2 . . 1 1 . 1
1122 4 6 4 3 3 1 . 2 . 2 1 . . 1 1 . . 1
.6 8 4 4 4 4 1 . 2 . 2 2 . . 1 1 . 1
1104 3 1 1 . 2 . . . 1 . 1 . . 1 . . 1
.9 4 2 . 4 1 . . 2 2 . . 1 1 . 1
3040 . . . . . . . . . . 1 . . . . . . 1
.3 3 3 . 1 1 3 1 . . 1 1 . 1 1 . 1
0311 . 3 1 . . 3 . . . . . . . . . . 1
.4 4 5 3 3 4 . 2 . 2 1 . . 2 2 . . 1 1 . 1
3022 1 . 2 . . . 2 1 . . 1 . . 1 . . 1 . . 1
.6 7 7 3 3 1 1 3 . 3 1 . 1 1 1 1 . 1 .

\(e_d\)

1042 12 15 16 12 21 9 13 9 21 3 6 2 12 2 3
129 93 70 36 57 51 40 27 30 18 19 6 19 12 . 6
3004 . . . . . . . . . . . . . . . . . .

1024 12 15 16 12 9 21 13 9 3 21 6 2 2 12 2 .
129 93 70 36 51 57 40 27 18 30 19 6 12 19 . 3
2211 217 168 130 78 100 100 76 59 42 42 45 25 32 32 6 18
306 231 177 110 129 129 96 75 48 48 55 31 37 37 7
213 12 14 14 4 9 15 10 8 3 12 6 2 4 8 . 3
174 134 100 50 72 84 59 40 27 42 31 11 21 27 . 9
2031 12 14 4 14 4 15 9 10 8 12 3 6 2 8 4 . 3
174 134 100 50 84 72 59 40 42 27 31 11 27 21 . 9
0133 . . . . . . . . . . . . . . . . . .
189 151 119 70 93 93 71 52 45 45 37 16 31 31 . 12
2131 16 21 20 12 19 19 17 12 16 9 8 4 12 7 . 6
342 270 209 120 167 159 124 92 81 63 68 36 58 46 8 30
IRREDUCIBLE CHARACTERS. II

Table 5. Continued

| 2113 | 16 | 21 | 20 | 12 | 19 | 19 | 17 | 12 | 9 | 16 | 8 | 4 | 7 | 12 | . | 3 |
|------|----|----|----|----|----|----|----|----|---|----|---|---|---|----|---|---|
| 342  | 270| 209| 120| 159| 167| 124| 92 | 63 | 81 | 68 | 36 | 46 | 58 | 8 | 24 |
| 1400 | 12 | 4  | 5  | .  | .  | 1  | 2  | .  | 3  | 1  | 1  | .  | .  | .  | .  | .  |
| 126  | 91 | 71 | 41 | 51 | 51 | 41 | 31 | 21 | 21 | 17 | 17 | 3  | 9  | .  | .  |
| 2033 | 3  | 3  | 9  | .  | .  | 3  | .  | 3  | .  | .  | .  | .  | .  | .  | .  | .  |
| 310  | 249| 198| 120| 157| 157| 123| 93 | 75 | 75 | 71 | 37 | 55 | 55 | 10 | 27 |
| 1302 | 24 | 12 | 5  | .  | .  | 7  | 3  | 2  | .  | 3  | 6  | 1  | 3  | 3  | .  |
| 273  | 213| 163| 93 | 123| 133| 99 | 75 | 51 | 63 | 60 | 30 | 42 | 48 | 6 | 24 |
| 1320 | 24 | 12 | 5  | .  | 7  | 3  | 2  | 3  | .  | 6  | 1  | 3  | 3  | .  | .  |
| 273  | 213| 163| 93 | 133| 123| 99 | 75 | 63 | 51 | 60 | 30 | 48 | 42 | 6  | 24 |
| 1222 | 570| 456| 363| 222| 288| 288| 175| 141| 141| 138| 73 | 108| 108| 15 | 57 |
| 594  | 474| 375| 225| 297| 297| 234| 177| 144| 144| 139| 73 | 109| 109| 15 | 57 |

(continued)

\( e_3 \)

| 1042 | 12 | .  | 7  | .  | 3  | 1  | 7  | 1  | .  | .  | 3  | 3  | .  | .  | 1  | 1  |
|------|----|----|----|----|----|----|----|----|---|----|---|---|---|----|---|---|
| 1204 | .  | 1  | .  | .  | .  | .  | .  | .  | .  | .  | 1  | .  | .  | .  | .  | .  |
| 1024 | .  | 3  | 7  | 12 | 3  | 1  | 1  | 7  | .  | .  | 3  | 3  | 3  | .  | 1  |
| 3004 | .  | 6  | 6  | 6  | 3  | 3  | 3  | 1  | 1  | .  | 3  | .  | 1  | 1  | 1  |
| 1024 | .  | 3  | 7  | 12 | 3  | 1  | 1  | 7  | .  | .  | 3  | 3  | 3  | .  | 1  |
| 2211 | 9  | 18 | 23 | 9  | 18 | 13 | 7  | 7  | 7  | 4  | 5  | .  | 3  | 3  | 4  | 2  |
| 9  | 21 | 26 | 9  | 20 | 15 | 7  | 7  | 8  | 5  | 5  | 5  | .  | 4  | 4  | 2  |
| 1302 | 24 | 12 | 5  | .  | 7  | 3  | 2  | 3  | 6  | 1  | 3  | 3  | 3  | .  | 1  |
| 1222 | 570| 456| 363| 222| 288| 288| 175| 141| 141| 138| 73 | 108| 108| 15 | 57 |
| 1024 | .  | 3  | 7  | 12 | 3  | 1  | 1  | 7  | .  | .  | 3  | 3  | 3  | .  | 1  |
| 1222 | 570| 456| 363| 222| 288| 288| 175| 141| 141| 138| 73 | 108| 108| 15 | 57 |

(continued)
Table 5. Continued

|   | 48 | 57 | 85 | 48 | 65 | 45 | 37 | 37 | 27 | 12 | 26 | 9 | 26 | 21 | 9 | 10 | 20 | 20 | 7 |
|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 22 |  1 |  3 |  1 |  1 |  0 |  2 |  0 |  0 |  1 |  4 |  2 |  0 |  2 |  0 |  0 |  3 |  1 |  1 |  3 |
| 23 |  0 |  0 |  0 |  2 |  1 |  1 |  1 |  3 |  0 |  0 |  0 |  0 |  0 |  0 |  2 |  2 |  2 |  1 |
| 24 |  4 |  0 |  2 |  0 |  1 |  1 |  3 |  1 |  0 |  1 |  3 |  5 |  1 |  3 |  1 |  0 |  2 |  0 |  2 |
| 25 |  0 |  2 |  2 |  4 |  1 |  1 |  3 |  0 |  1 |  1 |  3 |  3 |  5 |  0 |  0 |  2 |  0 |  0 |  2 |

References

Ye and Zhou
In all these tables, the left column indicates $\chi(\lambda)$’s or $\chi_3(\lambda)$’s, and the bottom row indicates $e(\nu)$’s. For each weight $\lambda \in X(T)_+$, we can find two rows in tables. The upper row corresponds to all integers $b_{i\nu}$ and the lower row corresponds to all integers $a_{i\nu}$. Therefore, we can easily get the third change of basis matrix $D = AB^{-1}$ from $\{\chi_3(\nu) \mid \nu \in X(T)_+\}$ to $\{\chi(\nu) \mid \nu \in X(T)_+\}$, which is still a triangular matrix with 1 on its diagonal. The matrix $D$ gives the decomposition patterns of various $H^0(\lambda)$ with $\lambda \in X(T)_+$. In this way, we obtain the following theorems.

**Theorem 4.** When $G$ is the simple algebraic group of type $A_4$, let $\Lambda = \{(2,0,0,2), (1,2,0,0), (0,0,2,1), (1,1,1,1), (2,1,1,2), (2,1,2,0), (2,2,1,0), (0,1,1,0), (0,2,2,0), (1,2,2,1), (2,0,2,0), (2,0,0,1), (2,0,0,0), (1,0,0,2), (1,1,1,1), (0,0,1,1), (2,1,2,0), (1,2,1,2), (1,1,2,1), (1,0,2,0), (2,0,1,1), (2,1,1,2), (2,2,2,0), (0,0,1,1), (1,1,1,0), (1,1,0,1), (1,0,1,1), (2,1,1,0), (0,1,1,2), (2,0,2,0), (2,0,1,0), (2,1,1,1), (2,0,2,1), (1,1,2,0), (1,0,0,0), (2,2,1,2), (2,2,1,1), (2,2,2,2), (2,1,2,2) \in X_1(T)\). Then $H^0(\lambda)$ is an irreducible $G$-module for all $\lambda \in X_1(T) \setminus \Lambda$, and

\[
\begin{align*}
\chi(2,0,0,2) &= \chi_3(2,0,0,2) + \chi_3(0,0,0,0), \\
\chi(1,2,0,0) &= \chi_3(1,2,0,0) + \chi_3(0,0,0,0), \\
\chi(1,1,1,1) &= \chi_3(1,1,1,1) + \chi_3(2,0,0,2) + \chi_3(1,2,0,0) + \chi_3(0,0,2,1) + \chi_3(0,0,0,0), \\
\chi(2,1,1,2) &= \chi_3(2,1,1,2) + \chi_3(3,0,0,3) + \chi_3(3,0,1,1) + \chi_3(1,1,0,3) + 2\chi_3(1,1,1,1) + \chi_3(3,1,0,0) + \chi_3(0,0,1,3) + \chi_3(1,2,0,0) + \chi_3(0,0,2,1) + \chi_3(2,0,0,2) + 3\chi_3(0,0,0,0), \\
\chi(2,1,2,0) &= \chi_3(2,1,2,0) + \chi_3(0,1,0,2), \\
\chi(0,1,1,0) &= \chi_3(0,1,1,0) + \chi_3(1,0,0,1), \\
\chi(0,2,2,0) &= \chi_3(0,2,2,0) + \chi_3(0,1,1,0), \\
\chi(1,2,2,1) &= \chi_3(1,2,2,1) + \chi_3(0,2,2,0) + \chi_3(0,1,1,0), \\
\chi(0,0,2,0) &= \chi_3(0,0,2,0) + \chi_3(1,0,0,0), \\
\chi(2,0,0,1) &= \chi_3(2,0,0,1) + \chi_3(1,0,0,0), \\
\chi(1,1,1,0) &= \chi_3(1,1,1,0) + \chi_3(2,0,0,1) + \chi_3(0,0,2,0) + \chi_3(1,0,0,0), \\
\chi(0,2,1,1) &= \chi_3(0,2,1,1) + \chi_3(0,3,0,0) + \chi_3(1,1,1,0) + \chi_3(2,0,0,1) + \chi_3(0,0,2,0), \\
\chi(0,2,1,0) &= \chi_3(0,2,1,0) + \chi_3(0,3,0,0) + \chi_3(1,1,1,0) + \chi_3(2,0,0,1) + \chi_3(0,0,2,0), \\
\chi(2,2,1,0) &= \chi_3(2,2,1,0) + \chi_3(0,3,0,0) + \chi_3(1,1,1,0) + \chi_3(2,0,0,1) + \chi_3(0,0,2,0), \\
\chi(2,2,2,0) &= \chi_3(2,2,2,0) + \chi_3(0,3,0,0) + \chi_3(1,1,1,0) + \chi_3(2,0,0,1) + \chi_3(0,0,2,0). 
\end{align*}
\]
\[
\begin{align*}
\text{ch}(1, 2, 1, 2) &= \text{ch}_3(1, 2, 1, 2) + \text{ch}_3(0, 2, 0, 3) + \text{ch}_3(3, 0, 1, 0) \\
&\quad + \text{ch}_3(0, 2, 1, 1) + \text{ch}_3(0, 3, 0, 0) + \text{ch}_3(1, 1, 1, 0) \\
&\quad + \text{ch}_3(2, 0, 0, 1) + \text{ch}_3(0, 0, 2, 0) + \text{ch}_3(1, 0, 0, 0), \\
\text{ch}(1, 1, 0, 2) &= \text{ch}_3(1, 1, 0, 2) + \text{ch}_3(0, 0, 1, 2), \\
\text{ch}(2, 1, 1, 1) &= \text{ch}_3(2, 1, 1, 1) + \text{ch}_3(2, 2, 0, 0) + \text{ch}_3(3, 0, 0, 2) \\
&\quad + \text{ch}_3(1, 1, 0, 2) + \text{ch}_3(0, 0, 1, 2) \\
\text{ch}(2, 0, 2, 2) &= \text{ch}_3(2, 0, 2, 2) + \text{ch}_3(1, 0, 1, 3) + \text{ch}_3(1, 0, 2, 1), \\
\text{ch}(0, 0, 1, 1) &= \text{ch}_3(0, 0, 1, 1) + \text{ch}_3(0, 1, 0, 0), \\
\text{ch}(1, 1, 0, 1) &= \text{ch}_3(1, 1, 0, 1) + \text{ch}_3(0, 0, 1, 1) + \text{ch}_3(0, 1, 0, 0), \\
\text{ch}(2, 1, 1, 0) &= \text{ch}_3(2, 1, 1, 0) + \text{ch}_3(3, 0, 0, 1) + \text{ch}_3(1, 1, 0, 1) \\
&\quad + \text{ch}_3(0, 0, 1, 1), \\
\text{ch}(0, 2, 0, 2) &= \text{ch}_3(0, 2, 0, 2) + \text{ch}_3(1, 1, 0, 1) + \text{ch}_3(0, 0, 0, 3) \\
&\quad + \text{ch}_3(0, 0, 1, 1), \\
\text{ch}(1, 2, 1, 1) &= \text{ch}_3(1, 2, 1, 1) + \text{ch}_3(1, 3, 0, 0) + \text{ch}_3(0, 2, 0, 2) \\
&\quad + \text{ch}_3(2, 1, 1, 0) + \text{ch}_3(3, 0, 0, 1) \\
&\quad + \text{ch}_3(1, 1, 0, 1) + \text{ch}_3(0, 0, 1, 1), \\
\text{ch}(1, 0, 2, 0) &= \text{ch}_3(1, 0, 2, 0) + \text{ch}_3(2, 0, 0, 0), \\
\text{ch}(0, 1, 2, 1) &= \text{ch}_3(0, 1, 2, 1) + \text{ch}_3(1, 0, 2, 0), \\
\text{ch}(1, 1, 2, 2) &= \text{ch}_3(1, 1, 2, 2) + \text{ch}_3(0, 0, 3, 2) + \text{ch}_3(0, 1, 1, 3) \\
&\quad + \text{ch}_3(0, 1, 2, 1) + \text{ch}_3(1, 0, 0, 4) + \text{ch}_3(1, 0, 2, 0) \\
&\quad + \text{ch}_3(2, 0, 0, 0), \\
\text{ch}(2, 0, 2, 1) &= \text{ch}_3(2, 0, 2, 1) + \text{ch}_3(1, 0, 1, 2), \\
\text{ch}(2, 2, 1, 2) &= \text{ch}_3(2, 2, 1, 2) + \text{ch}_3(4, 0, 1, 0) + \text{ch}_3(2, 1, 1, 0).
\end{align*}
\]
IRREDUCIBLE CHARACTERS. II

\[
\begin{align*}
\dim L(0, 0, 1, 1) &= 30, & \dim L(2, 1, 0, 2) &= 1215, & \dim L(2, 2, 2, 0) &= 8505, \\
\dim L(2, 0, 0, 0) &= 15, & \dim L(1, 0, 2, 0) &= 195, & \dim L(2, 2, 1, 2) &= 19260, \\
\dim L(0, 1, 2, 1) &= 855, & \dim L(1, 1, 2, 2) &= 6810, & \dim L(2, 2, 2, 2) &= 59049.
\end{align*}
\]

**Theorem 5.** When \( G \) is the simple algebraic group of type \( D_4 \), let \( \Lambda = \{1, 0, 1, 1\}, (2, 0, 2, 0), (2, 0, 0, 2), (0, 0, 2, 2), (1, 1, 1, 1), (2, 0, 2, 2), (2, 1, 2, 2), (0, 2, 0, 0), (2, 2, 0, 0), (0, 2, 2, 2), (1, 2, 0, 2), (1, 1, 0, 0), (1, 1, 2, 0), (1, 1, 0, 2), (0, 1, 0, 2), (0, 1, 2, 1), (2, 1, 0, 1), (2, 1, 1, 0), (2, 1, 1, 1), (1, 1, 2, 1), (1, 1, 0, 2), (1, 2, 0, 0), (0, 2, 0, 2), (0, 2, 1, 1), (0, 2, 1, 0), (1, 2, 1, 0), (1, 2, 2, 0), (1, 2, 1, 2), (2, 1, 2, 0), (2, 1, 2, 0), (2, 0, 0, 2), (2, 0, 2, 0), (1, 0, 2, 2), (2, 0, 1, 0), (2, 0, 2, 0), (1, 2, 2, 2), (2, 2, 1, 2), (2, 2, 2, 0), \} \subset X_1(T). \) Then \( H^0(\lambda) \) is an irreducible \( G \)-module for all \( \lambda \in X_1(T) \setminus \Lambda \), and

\[
\begin{align*}
\text{ch}(1, 0, 1, 1) &= \text{ch}_3(1, 0, 1, 1) + \text{ch}_3(0, 1, 0, 0), \\
\text{ch}(2, 0, 2, 0) &= \text{ch}_3(2, 0, 2, 0) + \text{ch}_3(1, 0, 1, 1), \\
\text{ch}(1, 1, 1, 1) &= \text{ch}_3(1, 1, 1, 1) + \text{ch}_3(2, 0, 0, 2) + \text{ch}_3(2, 0, 2, 0) \\
&\quad + \text{ch}_3(0, 0, 2, 2) + 2\text{ch}_3(1, 0, 1, 1) + 2\text{ch}_3(0, 1, 0, 0) \\
&\quad + \text{ch}_3(0, 0, 0, 0), \\
\text{ch}(2, 0, 2, 2) &= \text{ch}_3(2, 0, 2, 2) + \text{ch}_3(3, 0, 1, 1) + \text{ch}_3(1, 0, 3, 1) \\
&\quad + \text{ch}_3(1, 0, 1, 3) + 2\text{ch}_3(1, 1, 1, 1) + \text{ch}_3(2, 0, 0, 2) \\
&\quad + \text{ch}_3(2, 0, 2, 0) + \text{ch}_3(0, 0, 2, 2) + \text{ch}_3(1, 0, 1, 1) \\
&\quad + 2\text{ch}_3(0, 1, 0, 0) + 2\text{ch}_3(0, 0, 0, 0), \\
\text{ch}(2, 1, 2, 2) &= \text{ch}_3(2, 1, 2, 2) + \text{ch}_3(2, 0, 2, 2) + \text{ch}_3(0, 1, 4, 0) \\
&\quad + \text{ch}_3(0, 1, 0, 4) + \text{ch}_3(4, 1, 0, 0) + \text{ch}_3(3, 0, 1, 1) \\
&\quad + \text{ch}_3(1, 0, 3, 1) + \text{ch}_3(1, 0, 1, 3) + 2\text{ch}_3(0, 3, 0, 0) \\
&\quad + \text{ch}_3(0, 0, 2, 2) + \text{ch}_3(1, 0, 1, 1) + 2\text{ch}_3(0, 1, 0, 0) \\
&\quad + 3\text{ch}_3(0, 0, 0, 0), \\
\text{ch}(0, 2, 0, 0) &= \text{ch}_3(0, 2, 0, 0) + \text{ch}_3(2, 0, 0, 0) + \text{ch}_3(0, 0, 2, 0) \\
&\quad + \text{ch}_3(0, 0, 0, 2), \\
\text{ch}(2, 2, 0, 0) &= \text{ch}_3(2, 2, 0, 0) + \text{ch}_3(4, 0, 0, 0) + \text{ch}_3(0, 2, 0, 0) \\
&\quad + \text{ch}_3(2, 0, 0, 0) + \text{ch}_3(0, 0, 2, 0) + \text{ch}_3(0, 0, 0, 2), \\
\text{ch}(1, 2, 1, 1) &= \text{ch}_3(1, 2, 1, 1) + \text{ch}_3(2, 2, 0, 0) + \text{ch}_3(0, 2, 2, 0) \\
&\quad + \text{ch}_3(0, 2, 0, 2) + \text{ch}_3(2, 0, 0, 0) + \text{ch}_3(0, 0, 2, 0) \\
&\quad + \text{ch}_3(0, 0, 0, 2),
\end{align*}
\]
\[ \text{ch}(0, 1, 2, 2) = \text{ch}_3(0, 1, 2, 2) + \text{ch}_3(2, 1, 0, 0), \]
\[ \text{ch}(0, 2, 2, 2) = \text{ch}_3(0, 2, 2, 2) + \text{ch}_3(0, 1, 2, 2), \]
\[ \text{ch}(1, 0, 2, 2) = \text{ch}_3(1, 0, 2, 2) + \text{ch}_3(2, 0, 1, 1), \]
\[ \text{ch}(1, 2, 2, 2) = \text{ch}_3(1, 2, 2, 2) + \text{ch}_3(1, 0, 2, 2), \]
\[ \text{ch}(1, 1, 0, 0) = \text{ch}_3(1, 1, 0, 0) + \text{ch}_3(0, 0, 1, 1), \]
\[ \text{ch}(1, 2, 0, 0) = \text{ch}_3(1, 2, 0, 0) + \text{ch}_3(3, 0, 0, 0) + \text{ch}_3(1, 1, 0, 0) + 2\text{ch}_3(0, 0, 1, 1), \]
\[ \text{ch}(0, 2, 1, 1) = \text{ch}_3(0, 2, 1, 1) + \text{ch}_3(1, 2, 0, 0) + \text{ch}_3(1, 1, 0, 0) + 2\text{ch}_3(0, 0, 1, 1), \]
\[ \text{ch}(1, 1, 2, 2) = \text{ch}_3(1, 1, 2, 2) + \text{ch}_3(0, 0, 3, 3) + \text{ch}_3(0, 1, 1, 3) + \text{ch}_3(0, 1, 3, 1) + 2\text{ch}_3(0, 2, 1, 1) + \text{ch}_3(1, 0, 0, 4) + \text{ch}_3(1, 0, 4, 0) + 2\text{ch}_3(1, 2, 0, 0) + \text{ch}_3(3, 1, 0, 0) + 2\text{ch}_3(3, 0, 0, 0) + \text{ch}_3(1, 1, 0, 0) + 2\text{ch}_3(0, 0, 1, 1), \]
\[ \text{ch}(0, 1, 1, 1) = \text{ch}_3(0, 1, 1, 1) + \text{ch}_3(1, 0, 0, 2) + \text{ch}_3(1, 0, 2, 0) + \text{ch}_3(1, 0, 0, 0), \]
\[ \text{ch}(1, 1, 2, 0) = \text{ch}_3(1, 1, 2, 0) + \text{ch}_3(0, 0, 3, 1) + \text{ch}_3(0, 1, 1, 1) + \text{ch}_3(1, 0, 0, 2) + \text{ch}_3(1, 0, 2, 0) + \text{ch}_3(1, 0, 0, 0), \]
\[ \text{ch}(2, 1, 1, 1) = \text{ch}_3(2, 1, 1, 1) + \text{ch}_3(3, 0, 0, 2) + \text{ch}_3(3, 0, 2, 0) + \text{ch}_3(0, 0, 3, 1) + \text{ch}_3(0, 0, 1, 3) + \text{ch}_3(1, 1, 2, 0) + \text{ch}_3(1, 1, 0, 2) + \text{ch}_3(0, 1, 1, 1) + \text{ch}_3(1, 0, 0, 2) + \text{ch}_3(1, 0, 2, 0) + 3\text{ch}_3(1, 0, 0, 0), \text{ch}(2, 2, 1, 1) + \text{ch}_3(2, 1, 1, 1) + \text{ch}_3(3, 0, 0, 2) + \text{ch}_3(3, 0, 2, 0) + \text{ch}_3(1, 1, 0, 2) + \text{ch}_3(1, 1, 0, 2) + \text{ch}_3(0, 0, 3, 1) + \text{ch}_3(0, 0, 1, 3) + 2\text{ch}_3(1, 0, 0, 0), \]

\[
\begin{align*}
\dim L(0, 0, 0, 0) &= 1, & \dim L(1, 0, 1, 1) &= 322, & \dim L(1, 1, 1, 1) &= 1841, \\
\dim L(0, 1, 0, 0) &= 28, & \dim L(2, 0, 2, 0) &= 518, & \dim L(2, 0, 2, 2) &= 8132, \\
\dim L(2, 0, 0, 0) &= 35, & \dim L(2, 2, 0, 0) &= 3948, & \dim L(2, 1, 2, 2) &= 96755, \\
\dim L(0, 2, 0, 0) &= 195, & \dim L(0, 1, 2, 2) &= 8343, & \dim L(1, 2, 1, 1) &= 13776, \\
\dim L(1, 0, 0, 0) &= 8, & \dim L(1, 1, 2, 0) &= 1896, & \dim L(1, 1, 2, 2) &= 24240, \\
\dim L(0, 0, 1, 1) &= 56, & \dim L(2, 0, 1, 1) &= 1296, & \dim L(0, 2, 2, 2) &= 43254, \\
\dim L(1, 1, 0, 0) &= 104, & \dim L(1, 2, 0, 0) &= 1176, & \dim L(2, 2, 1, 1) &= 58920.
\end{align*}
\]
IRREDUCIBLE CHARACTERS. II

\[ \dim L(1,0,2,0) = 224, \quad \dim L(0,2,1,1) = 4768, \quad \dim L(1,2,2,2) = 196344, \]
\[ \dim L(0,1,1,1) = 384, \quad \dim L(1,0,2,2) = 3240, \quad \dim L(1,2,2,0) = 18144, \]
\[ \dim L(2,1,0,0) = 567, \quad \dim L(2,1,1,1) = 7600, \quad \dim L(2,2,2,2) = 531441, \]

**Remark.** There are other computer programs available to compute the characters such as Scott in Virginia and Laurantzen in Aarhus.

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