Dependence Balance Based Outer Bounds for Gaussian Networks with Cooperation and Feedback

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Joint work with Ravi Tandon.
The Multiple Access Channel with Generalized Feedback

- Feedback increases capacity of MAC [Gaarder-Wolf 1975].
- MAC with generalized feedback (MAC-GF) [Carleial 1982].
- Achievable schemes: [Carleial 1982] [Willems-van der Meulen 1983].
- Converse: cut-set outer bound.
Cut-set Outer Bound for MAC-GF

- Cut-set outer bound:

\[ CS = \left\{ (R_1, R_2) : R_1 \leq I(X_1; Y, Y_{F_2}|X_2) \right. \]

\[ R_2 \leq I(X_2; Y, Y_{F_1}|X_1) \]

\[ R_1 + R_2 \leq I(X_1, X_2; Y) \right\} \]

for some \( p(x_1, x_2) \) such that

\[ p(x_1, x_2, y, y_{F_1}, y_{F_2}) = p(x_1, x_2)p(y, y_{F_1}, y_{F_2}|x_1, x_2). \]

- Cut-set outer bound permits **arbitrary** input distributions \( p(x_1, x_2) \).

- Only optimal for the Gaussian MAC-FB (\( Y_{F_1} = Y_{F_2} = Y \)) [Ozarow 1984].
Motivation for a New Outer Bound

- Cut-set bound may not be tight in general.
- Possible reason: $CS$ permits arbitrary correlation between channel inputs.
- Rates of some of these input distributions may not be achievable.
- Need for a converse which restricts the set of input distributions.
MAC with feedback related to single output two-way channel (TWC).

Dependence balance bounds for the TWC [Hekstra-Willems 1989].
  - Outer bounds for DMC MAC-FB ($Y_{F_1} = Y_{F_2} = Y$) [Tandon-Ulukus 2008].
  - Generalizing this approach: a new outer bound for MAC-GF.
Proof of the dependence balance constraint:

\[
0 \leq I(W_1; W_2| Y_{F_1}^n, Y_{F_2}^n) = I(W_1; W_2| Y_{F_1}^n, Y_{F_2}^n) - I(W_1; W_2) = -I_3(W_1; W_2; Y_{F_1}^n, Y_{F_2}^n)
\]

\[
= \sum_{i=1}^{n} \left[ -H(Y_{F_1i}, Y_{F_2i}| Y_{F_1}^{i-1}, Y_{F_2}^{i-1}) + H(Y_{F_1i}, Y_{F_2i}| W_1, Y_{F_1}^{i-1}, Y_{F_2}^{i-1}) + H(Y_{F_1i}, Y_{F_2i}| W_2, Y_{F_1}^{i-1}, Y_{F_2}^{i-1}) - H(Y_{F_1i}, Y_{F_2i}| W_1, W_2, Y_{F_1}^{i-1}, Y_{F_2}^{i-1}) \right]
\]

\[
= \sum_{i=1}^{n} I(X_{1i}; X_{2i}| Y_{F_1i}, Y_{F_2i}, Y_{F_1}^{i-1}, Y_{F_2}^{i-1}) - \sum_{i=1}^{n} I(X_{1i}; X_{2i}| Y_{F_1}^{i-1}, Y_{F_2}^{i-1}) = n(I(X_1; X_2| Y_{F_1}, Y_{F_2}, T_1, T_2) - I(X_1; X_2| T_1, T_2))
\]
Dependence Balance Constraint

- Main steps in the proof:
  - Seemingly trivial inequality \( 0 \leq I(W_1; W_2 | Y_{F_1}^n, Y_{F_2}^n) \).
  - Symmetry of \( I_3(A; B; C) = I(A; B) - I(A; B|C). \)
  - Encoding functions: \( X_{1i} = f_{1i}(W_1, Y_{F_1}^{i-1}), X_{2i} = f_{2i}(W_2, Y_{F_2}^{i-1}). \)
  - Conditioning reduces entropy.

- Defining:

\[
X_1 = X_{1Q}, \quad X_2 = X_{2Q}, \quad Y_{F_1} = Y_{F_1Q}, \quad Y_{F_2} = Y_{F_2Q}, \\
T_1 = (Q, Y_{F_1}^{Q-1}), \quad T_2 = (Q, Y_{F_2}^{Q-1}), \quad Q \sim \text{Unif}\{1, \ldots, n\}.
\]

- Resulting DB constraint:

\[
I(X_1; X_2 | T_1, T_2) \leq I(X_1; X_2 | Y_{F_1}, Y_{F_2}, T_1, T_2)
\]

- Interpretation:
  - Dependence consumed: \( I(X_1; X_2 | T_1, T_2). \)
  - Dependence produced: \( I(X_1; X_2 | Y_{F_1}, Y_{F_2}, T_1, T_2). \)
Parallel Channel Extension of Dependence Balance

- For any \( p^+(z|x_1, x_2, y, y_{F_1}, y_{F_2}) \)
- Starting from

\[
0 \leq I(W_1; W_2|Y^n_{F_1}, Y^n_{F_2}, Z^n) - I(W_1; W_2)
\]

- Rest of the proof similar to the proof of DB.
- Defining:

\[
X_1 = X_{1Q}, X_2 = X_{2Q}, Y_{F_1} = Y_{F_1Q}, Y_{F_2} = Y_{F_2Q}, Z = Z_Q
\]

\[
T_1 = (Q, Y^{Q-1}_{F_1}, Z^{Q-1}), T_2 = (Q, Y^{Q-1}_{F_2}, Z^{Q-1}), Q \sim \text{Unif}\{1, \ldots, n\}.
\]

- Choice of \( Z \) effects the definition of auxiliary random variables.
- Resulting DB constraint:

\[
I(X_1; X_2|T_1, T_2) \leq I(X_1; X_2|Z, Y_{F_1}, Y_{F_2}, T_1, T_2)
\]

- Selecting \( Z = \phi \) yields the regular DB constraint.
A Typical Rate Constraint

- Rate upper bounds for MAC-GF:

\[
\begin{align*}
nR_1 &= H(W_1) \\
&= I(W_1; Y^n, Y^n_{F2}, Z^n) + H(W_1| Y^n, Y^n_{F2}, Z^n) \\
&\leq I(W_1; Y^n, Y^n_{F2}, Z^n) + n\epsilon_1^{(n)} \\
&\quad \ldots \\
&= \sum_{i=1}^{n} I(X_{1i}; Y_i, Y^n_{F2i}, Z_i|X_{2i}, Y^{i-1}_{F2}, Z^{i-1}) + n\epsilon_1^{(n)} \\
&= nl(X_1; Y, Y^n_{F2}, Z|X_{2}, T_2) + n\epsilon_1^{(n)} \\
&= nl(X_1; Y, Y^n_{F2}|X_{2}, T_2) + n \underbrace{I(X_1; Z|X_{2}, Y, Y^n_{F2}, T_2)}_{L_1} + n\epsilon_1^{(n)}
\end{align*}
\]

- \( L_1 \) is the excess rate compared to the rate \( R_1 \) for the choice \( Z = \phi \).
- **Tradeoff** in restricting set of \( p(t_1, t_2, x_1, x_2) \) and information leaks \( L_1, L_2 \).
- \( Z = \phi \), yields regular DB constraint and \( L_1 = L_2 = 0 \).
- \( Z = X_1 \), smallest permissible set of distributions, but \( L_1 \geq 0 \).
Application of Dependence Balance to MAC with Output Feedback

- MAC with noiseless output feedback: $Y_{F_1} = Y_{F_2} = Y$.
- Selecting $Z = X_1$ yields the following outer bound:

$$\mathcal{DB} = \bigcup_{X_1 \rightarrow T \rightarrow X_2} \left\{ (R_1, R_2) : R_1 \leq I(X_1; Y|X_2, T) + H(X_1|Y, X_2, T) \right\}$$

$$R_2 \leq I(X_2; Y|X_1, T)$$
$$R_1 \leq I(X_1; Y|X_2)$$
$$R_1 + R_2 \leq I(X_1, X_2; Y)$$

- For the class of MAC such that $X_1 = g(X_2, Y)$, $L_1 = H(X_1|X_2, Y, T) = 0$.
- $\mathcal{DB}$ matches the Cover-Leung achievable rate region [1981].
- $\mathcal{DB}$ equals the feedback capacity region for this class [Willems 1982].
- For the class for which $X_1 \neq g(X_2, Y)$ [Tandon-Ulukus 2008].
  - $\mathcal{DB}$ strictly improves upon the cut-set bound.
  - Explicit evaluation of bounds using composite functions.
A New Outer Bound for MAC-GF using Dependence Balance

- New outer bound:

\[ \mathcal{DB}^{MAC} = \{(R_1, R_2) : R_1 \leq I(X_1; Y, Y_{F_2}|X_2, T_2) \]
\[ R_2 \leq I(X_2; Y, Y_{F_1}|X_1, T_1) \]
\[ R_1 + R_2 \leq I(X_1, X_2; Y, Y_{F_1}, Y_{F_2}|T_1, T_2) \]
\[ R_1 + R_2 \leq I(X_1, X_2; Y) \}\]

for some \( p(t_1, t_2, x_1, x_2) \) such that

\[ p(t_1, t_2, x_1, x_2, y, y_{F_1}, y_{F_2}) = p(t_1, t_2, x_1, x_2)p(y, y_{F_1}, y_{F_2}|x_1, x_2) \]

and such that

\[ I(X_1; X_2|T_1, T_2) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T_1, T_2) \]

- **Difficult** to evaluate, two auxiliary random variables \( (T_1, T_2) \).

- **Modify** \( \mathcal{DB}^{MAC} \) according to the channel model in consideration.
  - MAC with noisy feedback (MAC-NF).
  - MAC with user cooperation (MAC-UC).
The Gaussian MAC with Noisy Feedback (MAC-NF)

- Channel model:
  
  \[ Y = X_1 + X_2 + Z \]
  
  \[ Y_{F1} = Y + Z_1 \]
  
  \[ Y_{F2} = Y + Z_2 \]

- Feedback signals \( (Y_{F1}, Y_{F2}) \) are degraded versions of \( Y \).
- Cut-set bound **not sensitive** to \( \sigma^2_{Z_1}, \sigma^2_{Z_2} \) for MAC-NF.
- For all \( \sigma^2_{Z_1}, \sigma^2_{Z_2} \), \( CS = C_{Ozarow} \), i.e., capacity with noiseless feedback.
- Need a converse reflecting the **uselessness** of feedback as \( \sigma^2_{Z_1}, \sigma^2_{Z_2} \to \infty \).
**DB Outer Bound for Gaussian MAC-NF**

- Use the special channel structure.
- Modify $\mathcal{DB}^{MAC}$ to express it in terms of one auxiliary random variable, $T$.
- Modified outer bound:

$$\mathcal{DB}_{NF}^{MAC} = \{(R_1, R_2) : R_1 \leq I(X_1; Y|X_2, T)$$
$$R_2 \leq I(X_2; Y|X_1, T)$$
$$R_1 + R_2 \leq I(X_1, X_2; Y|T)\}$$

for some $p(t, x_1, x_2)$ for which

$$p(t, x_1, x_2, y, y_{F_1}, y_{F_2}) = p(t, x_1, x_2)p(y|x_1, x_2)p(y_{F_1}|y)p(y_{F_2}|y)$$

and

$$I(X_1; X_2|T) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T)$$

where the random variable $T$ can be restricted to $|T| \leq |X_1||X_2| + 3$. 
The Gaussian MAC with User Cooperation (MAC-UC)

Channel model:

\[ Y = \sqrt{h_{10}}X_1 + \sqrt{h_{20}}X_2 + Z \]
\[ Y_{F_1} = \sqrt{h_{21}}X_2 + Z_1 \]
\[ Y_{F_2} = \sqrt{h_{12}}X_1 + Z_2 \]

- Suitable model for an uplink wireless setting.
- User cooperation diversity [Sendonaris-Erkip-Aazhang 2003].
- Cut-set bound sensitive to \( \sigma_{Z_1}^2, \sigma_{Z_2}^2 \) but not sensitive enough.
- As \( \sigma_{Z_1}^2, \sigma_{Z_2}^2 \to \infty \), \( CS \to C_{Ozarow} \), i.e., capacity with noiseless feedback.
\textbf{DB} Outer Bound for Gaussian MAC-UC

- Use the special channel structure.
- Modify $DB^{MAC}$ to express in terms of one auxiliary random variable, $T$.
- Modified outer bound:

\[ DB^{MAC}_{UC} = \{(R_1, R_2) : R_1 \leq I(X_1; Y, Y_{F_2}|X_2, T) \]
\[ R_2 \leq I(X_2; Y, Y_{F_1}|X_1, T) \]
\[ R_1 + R_2 \leq I(X_1, X_2; Y, Y_{F_1}, Y_{F_2}|T) \]
\[ R_1 + R_2 \leq I(X_1, X_2; Y) \}\]

for some $p(t, x_1, x_2)$ for which

\[ p(t, x_1, x_2, y, y_{F_1}, y_{F_2}) = p(t, x_1, x_2)p(y|x_1, x_2)p(y_{F_1}|x_2)p(y_{F_2}|x_1) \]

and

\[ I(X_1; X_2|T) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T) \]

where the random variable $T$ can be restricted to $|T| \leq |X_1||X_2| + 3$. 
Difficulty in Evaluation of $DB_{NF}^{MAC}$ and $DB_{UC}^{MAC}$

- For Gaussian channels, we need to consider joint densities $p(t, x_1, x_2)$.
- The only densities permitted are those which satisfy $DB$ constraint

$$I(X_1; X_2|T) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T)$$

- Typical difficulty (shortcoming of maximum entropy theorem)
  - Consider a $NG$ triple $(T, X_1, X_2)$ with density $p(t, x_1, x_2)$ satisfying $DB$.
  - Covariance matrix of $p(t, x_1, x_2)$ is $Q$.
  - There does not exist a $JG$ triple with covariance matrix $Q$ satisfying $DB$.

- Problem: Maximum entropy theorem fails beyond this point.
- Solution: A new approach to evaluate $DB$ based bounds.
A Systematic Approach for Evaluation of $\mathcal{DB}_{NF}^{MAC}$ and $\mathcal{DB}_{UC}^{MAC}$

Partition the set of densities $\mathcal{P}$:

$\mathcal{P} = \{ p(t, x_1, x_2) : E[X_1^2] \leq P_1, E[X_2^2] \leq P_2 \}$

$\mathcal{P}_G = \{ p(t, x_1, x_2) \in \mathcal{P} : (T, X_1, X_2) \text{ are } JG \}$

$\mathcal{P}_{NG} = \{ p(t, x_1, x_2) \in \mathcal{P} : (T, X_1, X_2) \text{ are } NG \}$

Further partition $\mathcal{P}_G$ and $\mathcal{P}_{NG}$:

$\mathcal{P}^D_B_G = \{ p(t, x_1, x_2) \in \mathcal{P}_G : (T, X_1, X_2) \text{ satisfy } (DB) \}$

$\mathcal{P}^D_B_{NG} = \{ p(t, x_1, x_2) \in \mathcal{P}_{NG} : (T, X_1, X_2) \text{ satisfy } (DB) \}$
A Systematic Approach for Evaluation of $DB_{NF}^{MAC}$ and $DB_{UC}^{MAC}$

Further partition the set $\mathcal{P}_{NG}^{DB}$:

$\mathcal{P}_{NG}^{DB(a)} = \{ p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB} : \text{cov. matrix of } p(t, x_1, x_2) \text{ is } Q \text{ and there exists a } JG \left( T_G, X_{1G}, X_{2G} \right) \text{ with cov. matrix } Q \text{ satisfying } (DB) \}$

$\mathcal{P}_{NG}^{DB(b)} = \{ p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB} : \text{cov. matrix of } p(t, x_1, x_2) \text{ is } Q \text{ and there does not exist a } JG \left( T_G, X_{1G}, X_{2G} \right) \text{ with cov. matrix } Q \text{ satisfying } (DB) \}$
Evaluation of $\mathcal{DB}_{NF}^{MAC}$ and $\mathcal{DB}_{UC}^{MAC}$

- **Claim:** Jointly Gaussian $p(t, x_1, x_2)$ satisfying $DB$ are sufficient to evaluate outer bounds.

- **Proof:**
  - **Common step for noisy feedback and user cooperation:**
    For every $p(t, x_1, x_2)$ in $\mathcal{P}_{NG}^{DB(a)}$, there exists a jointly Gaussian triple $(T_G, X_{1G}, X_{2G})$ with covariance matrix $Q$ satisfying $DB$ and yielding larger rates.
    All rates of distributions in $\mathcal{P}_{NG}^{DB(a)}$ are covered by distributions in $\mathcal{P}_G^{DB}$.

  - **Main step:**
    For every $p(t, x_1, x_2)$ in $\mathcal{P}_{NG}^{DB(b)}$ with covariance matrix $Q$, we show that there exists a jointly Gaussian triple $(T_G, X_{1G}, X_{2G})$ with covariance matrix $S$ satisfying $DB$ and yielding larger rates.
    All rates of distributions in $\mathcal{P}_{NG}^{DB(b)}$ are covered by distributions in $\mathcal{P}_G^{DB}$.

- Therefore, it suffices to consider only $\mathcal{P}_G^{DB}$ in evaluating our $DB$ bounds.
Evaluation of $DB_{NF}^{MAC}$: Main Step

- Recall the definition of the set $P_{NG}^{DB(b)}$:

  $$P_{NG}^{DB(b)} = \{ p(t, x_1, x_2) \in P_{NG}^{DB} : \text{cov. matrix of } p(t, x_1, x_2) \text{ is } Q \text{ and there does not exist a } JG(T_G, X_{1G}, X_{2G}) \text{ with cov. matrix } Q \text{ satisfying } (DB) \}$$

- Facts at hand:

  $$I(X_1; X_2|T) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T)$$
  $$I^Q(X_{1G}; X_{2G}|T_G) > I^Q(X_{1G}; X_{2G}|Y_{F_1}, Y_{F_2}, T_G)$$

- Maximum entropy theorem tells us:

  $$R_1 \leq I(X_1; Y|X_2, T) \leq I^Q(X_{1G}; Y|X_{2G}, T_G)$$
  $$R_2 \leq I(X_2; Y|X_1, T) \leq I^Q(X_{2G}; Y|X_{1G}, T_G)$$
  $$R_1 + R_2 \leq I(X_1, X_2; Y|T) \leq I^Q(X_{1G}, X_{2G}; Y|T_G)$$
Evaluation of $DB_{NF}^{MAC}$: Main Step

- Recall the definition of the set $\mathcal{P}_{NG}^{DB(b)}$:

  \[\mathcal{P}_{NG}^{DB(b)} = \{ p(t,x_1,x_2) \in \mathcal{P}_{NG}^{DB} : \text{cov. matrix of } p(t,x_1,x_2) \text{ is } Q \text{ and there does not exist a } JG (T_G, X_{1G}, X_{2G}) \text{ with cov. matrix } Q \text{ satisfying } (DB)\}\]

- Facts at hand:

  \[
  I(X_1; X_2|T) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T) \\
  I^Q(X_{1G}; X_{2G}|T_G) > I^Q(X_{1G}; X_{2G}|Y_{F_1}, Y_{F_2}, T_G)
  \]

- $DB$ with multivariate Costa’s EPI [Payaro-Palomar, ISIT 08] tells us:

  \[
  R_1 \leq I(X_1; Y|X_2, T) \leq I^Q(X_{1G}; Y|X_{2G}, T_G) \\
  R_2 \leq I(X_2; Y|X_1, T) \leq I^Q(X_{2G}; Y|X_{1G}, T_G) \\
  R_1 + R_2 \leq I(X_1, X_2; Y|T) \leq f(Q) < I^Q(X_{1G}, X_{2G}; Y|T_G)
  \]
Evaluation of $DB_{NF}^{MAC}$: Main Step

- Recall the definition of the set $P_{NG}^{DB(b)}$:

$$P_{NG}^{DB(b)} = \{ p(t, x_1, x_2) \in P_{NG}^{DB} : \text{cov. matrix of } p(t, x_1, x_2) \text{ is } Q \text{ and there does not exist a } JG(T_G, X_{1G}, X_{2G}) \text{ with cov. matrix } Q \text{ satisfying } (DB) \}$$

- Facts at hand:

$$I(X_1; X_2|T) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T)$$
$$I^Q(X_{1G}; X_{2G}|T_G) > I^Q(X_{1G}; X_{2G}|Y_{F_1}, Y_{F_2}, T_G)$$

- Construct a cov. matrix $S$ such that:

$$R_1 \leq I(X_1; Y|X_2, T) \leq I^Q(X_{1G}; Y|X_{2G}, T_G)$$
$$R_2 \leq I(X_2; Y|X_1, T) \leq I^Q(X_{2G}; Y|X_{1G}, T_G)$$
$$R_1 + R_2 \leq I(X_1, X_2; Y|T) \leq I^S(X_{1G}, X_{2G}; Y|T_G) < I^Q(X_{1G}, X_{2G}; Y|T_G)$$
Evaluation of $\mathcal{DB}_{NF}^{MAC}$: Main Step

- Recall the definition of the set $\mathcal{P}_{NG}^{DB(b)}$:

  $$\mathcal{P}_{NG}^{DB(b)} = \{ p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB} : \text{cov. matrix of } p(t, x_1, x_2) \text{ is } Q \text{ and there does not exist a } JG (T_G, X_{1G}, X_{2G}) \text{ with cov. matrix } Q \text{ satisfying } (DB) \}$$

- Facts at hand:

  $$I(X_1; X_2|T) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T)$$

  $$I^Q(X_{1G}; X_{2G}|T_G) > I^Q(X_{1G}; X_{2G}|Y_{F_1}, Y_{F_2}, T_G)$$

- Construct a cov. matrix $S$ such that:

  $$R_1 \leq I(X_1; Y|X_2, T) \leq I^Q(X_{1G}; Y|X_{2G}, T_G) \leq I^S(X_{1G}; Y|X_{2G}, T_G)$$

  $$R_2 \leq I(X_2; Y|X_1, T) \leq I^Q(X_{2G}; Y|X_{1G}, T_G) \leq I^S(X_{2G}; Y|X_{1G}, T_G)$$

  $$R_1 + R_2 \leq I(X_1, X_2; Y|T) \leq I^S(X_{1G}, X_{2G}; Y|T_G) < I^Q(X_{1G}, X_{2G}; Y|T_G)$$
Evaluation of $DB_{NF}^{MAC}$: Main Step

- Recall the definition of the set $\mathcal{P}_{NG}^{DB(b)}$:

$$\mathcal{P}_{NG}^{DB(b)} = \{ p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB} : \text{cov. matrix of } p(t, x_1, x_2) \text{ is } Q \text{ and there does not exist a } \mathcal{JG} \ (T_G, X_{1G}, X_{2G}) \text{ with cov. matrix } Q \text{ satisfying } (DB) \}$$

- Facts at hand:

$$I(X_1; X_2 | T) \leq I(X_1; X_2 | Y_{F1}, Y_{F2}, T)$$

$$I^Q(X_{1G}; X_{2G} | T_G) > I^Q(X_{1G}; X_{2G} | Y_{F1}, Y_{F2}, T_G)$$

- Construct a cov. matrix $S$ such that:

$$R_1 \leq I(X_1; Y | X_2, T) \leq I^Q(X_{1G}; Y | X_{2G}, T_G) \leq I^S(X_{1G}; Y | X_{2G}, T_G)$$

$$R_2 \leq I(X_2; Y | X_1, T) \leq I^Q(X_{2G}; Y | X_{1G}, T_G) \leq I^S(X_{2G}; Y | X_{1G}, T_G)$$

$$R_1 + R_2 \leq I(X_1, X_2; Y | T) \leq I^S(X_{1G}, X_{2G}; Y | T_G)$$
Evaluation of $DB_{NF}^{MAC}$: Main Step

- Recall the definition of the set $\mathcal{P}_{NG}^{DB(b)}$:

$$\mathcal{P}_{NG}^{DB(b)} = \{ p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB} : \text{cov. matrix of } p(t, x_1, x_2) \text{ is } Q \text{ and there does not exist a } \mathcal{JG} (T_G, X_{1G}, X_{2G}) \text{ with cov. matrix } Q \text{ satisfying (DB)} \}$$

- Facts at hand:

$$I(X_1; X_2|T) \leq I(X_1; X_2|Y_{F1}, Y_{F2}, T)$$

$$I^Q(X_{1G}; X_{2G}|T_G) > I^Q(X_{1G}; X_{2G}|Y_{F1}, Y_{F2}, T_G)$$

- Construct a cov. matrix $S$ such that:

$$R_1 \leq I(X_1; Y|X_2, T) \leq I^Q(X_{1G}; Y|X_{2G}, T_G) \leq I^S(X_{1G}; Y|X_{2G}, T_G)$$

$$R_2 \leq I(X_2; Y|X_1, T) \leq I^Q(X_{2G}; Y|X_{1G}, T_G) \leq I^S(X_{2G}; Y|X_{1G}, T_G)$$

$$R_1 + R_2 \leq I(X_1, X_2; Y|T) \leq I^S(X_{1G}, X_{2G}; Y|T_G)$$

- Such that any $(T_G, X_{1G}, X_{2G})$ with cov. matrix $S$ satisfies,

$$I^S(X_{1G}; X_{2G}|T_G) \leq I^S(X_{1G}; X_{2G}|Y_{F1}, Y_{F2}, T_G)$$
A typical covariance matrix for $P_1 = P_2 = 1$

$$Q = \begin{bmatrix}
1 & \rho_1 T \rho_2 T + \alpha_0 & \rho_1 T \\
\rho_1 T \rho_2 T + \alpha_0 & 1 & \rho_2 T \\
\rho_1 T & \rho_2 T & 1
\end{bmatrix}$$

For $\alpha_0 \leq 0$, and $g(\alpha_0) < 0$, we construct $S$ as,

$$S = \begin{bmatrix}
1 & \rho_1 T \rho_2 T & \rho_1 T \\
\rho_1 T \rho_2 T & 1 & \rho_2 T \\
\rho_1 T & \rho_2 T & 1
\end{bmatrix}$$

The function $g(\alpha)$ is given as

$$g(\alpha) = I^{Q(\alpha)}(X_{1G}; X_{2G} | Y_{F1}, Y_{F2}, T_G) - I^{Q(\alpha)}(X_{1G}; X_{2G} | T_G)$$

Any $(T_G, X_{1G}, X_{2G})$ with cov. matrix $S$ satisfies $DB$. 
Evaluation of $\mathcal{DB}^{MAC}_{NF}$: Construction of $S$

- A typical covariance matrix for $P_1 = P_2 = 1$

$$Q = \begin{bmatrix} 1 & \rho_1 \rho_2 + \alpha_0 & \rho_1 \\ \rho_1 \rho_2 + \alpha_0 & 1 & \rho_2 \\ \rho_1 & \rho_2 & 1 \end{bmatrix}$$

- For $\alpha_0 > 0$, and $g(\alpha_0) < 0$, we construct $S$ as,

$$S = \begin{bmatrix} 1 & \rho_1 \rho_2 + \alpha^* & \rho_1 \\ \rho_1 \rho_2 + \alpha^* & 1 & \rho_2 \\ \rho_1 & \rho_2 & 1 \end{bmatrix}$$

- The function $g(\alpha)$ is given as

$$g(\alpha) = I^{Q(\alpha)}(X_{1G}; X_{2G} | Y_{F1}, Y_{F2}, T_G) - I^{Q(\alpha)}(X_{1G}; X_{2G} | T_G)$$

- Any $(T_G, X_{1G}, X_{2G})$ with cov. matrix $S$ satisfies $DB$ with equality.
 Explicit Evaluation of $\mathcal{DB}_{NF}^{MAC}$

- We only need to consider probability distributions in $\mathcal{P}^{DB}_{G}$.
- I.e., consider only Gaussian $(T, X_1, X_2)$ satisfying

$$I(X_1; X_2|T) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T)$$
Explicit Evaluation of $\mathcal{DB}_{NF}^{MAC}$

- We only need to consider probability distributions in $\mathcal{P}_G^{DB}$.
- I.e., consider only Gaussian $(T, X_1, X_2)$ satisfying

\[ I(X_1; X_2 | T) \leq I(X_1; X_2 | Y_{F_1}, Y_{F_2}, T) \]
DB_{NF}^{MAC} Outer Bound

- Final expression for the outer bound:

\[
DB_{NF}^{MAC} = \bigcup_{Q \in Q^{DB}} \left\{ (R_1, R_2) : R_1 \leq \frac{1}{2} \log \left( 1 + \frac{f_1(Q)}{\sigma_Z^2} \right) \right. \\
R_2 \leq \frac{1}{2} \log \left( 1 + \frac{f_2(Q)}{\sigma_Z^2} \right) \\
R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{f_3(Q)}{\sigma_Z^2} \right) \right\}
\]

where

- \( f_1(Q) = \text{Var}(X_{1G}|X_{2G}, T_G) \)
- \( f_2(Q) = \text{Var}(X_{2G}|X_{1G}, T_G) \)
- \( f_3(Q) = \text{Var}(X_{1G}|T_G) + \text{Var}(X_{2G}|T_G) + 2\text{Cov}(X_{1G}, X_{2G}|T_G) \)

and \( Q^{DB} \) is the set of such \( 3 \times 3 \) matrices satisfying

\[
f_3(Q) \leq f_1(Q) + f_2(Q) + \frac{f_1(Q)f_2(Q)}{\left( \sigma_Z^2 + \frac{\sigma_{Z1}^2 \sigma_{Z2}^2}{(\sigma_{Z1}^2 + \sigma_{Z2}^2)} \right)}
\]
Limiting Behavior of $\mathcal{DB}^{\text{MAC}}_{\text{NF}}$ and $\mathcal{CS}$ Bounds

- Cut-set bound is not sensitive to $\sigma^2_{Z_1}, \sigma^2_{Z_2}$.
  - For all $\sigma^2_{Z_1}, \sigma^2_{Z_2}$, $\mathcal{CS} = C_{\text{Ozarow}}$, i.e., capacity with noiseless feedback.

- Our bound depends on the noise variances.
  - As $\sigma^2_{Z_1}, \sigma^2_{Z_2} \to 0$:
    We recover capacity with noiseless feedback [Ozarow 1984].
    $$f_3(Q) \leq f_1(Q) + f_2(Q) + \frac{f_1(Q)f_2(Q)}{\sigma^2_Z}$$

  - As $\sigma^2_{Z_1}, \sigma^2_{Z_2} \to \infty$:
    We recover capacity without feedback.
    $$f_3(Q) \leq f_1(Q) + f_2(Q)$$
Illustration of Bounds for Gaussian MAC with Noisy Feedback

- Cut-set bound is **insensitive** to the feedback noise variances $\sigma_{Z_1}^2, \sigma_{Z_2}^2$.
- $DB_{MAC}^{NF}$ **depends** on the feedback noise variances $\sigma_{Z_1}^2, \sigma_{Z_2}^2$.
  - As $\sigma_{Z_1}^2, \sigma_{Z_2}^2 \to 0$, $DB_{MAC}^{NF} \to CS = C_{Ozarow}$.
  - As $\sigma_{Z_1}^2, \sigma_{Z_2}^2 \to \infty$, $DB_{MAC}^{NF} \to C_{No-Feeback}$.

\[ P_1 = P_2 = \sigma_Z^2 = 1 \text{ and } \sigma_{Z_1}^2 = \sigma_{Z_2}^2 = 2, 5, 10. \]
Evaluation of $\mathcal{DB}^{MAC}_{UC}$: Main Step

- For user cooperation, $DB$ is equivalent to

\[
I(Y_{F_1}; X_1 | T) = 0 \\
I(Y_{F_2}; X_2 | Y_{F_1}, T) = 0
\]

- A jointly Gaussian $(T, X_1, X_2)$ satisfies $DB$ iff $X_1 \rightarrow T \rightarrow X_2$.
  - Equivalently, covariance matrix $Q$ of $(T, X_1, X_2)$ is such that $\rho_{12} = \rho_{1T}\rho_{2T}$.

- Consider any $p(t, x_1, x_2) \in \mathcal{DB}_{NG}^{DB(b)}$ with a cov. matrix $Q$ satisfying

\[
I(Y_{F_1}; X_1 | T) = 0
\]

  - This implies $\rho_{12} \neq \rho_{1T}\rho_{2T}$. 

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Evaluation of $\mathcal{DB}^{MAC}_{UC}$: Construction of $S$

- Construct a triple $(T', X_1, X_2)$ with a covariance matrix $S$ by selecting
  \[ T' = E[X_1 | T] \]

- The selection related to [Bross-Wigger-Lapidoth ISIT 2008].

- For such selection,
  \[ E[X_1X_2] = \frac{E[X_1T']E[X_2T']}{{\text{Var}}(T')} \]

- We also have,
  \[ E[X_1X_2] = \rho_{12} \sqrt{P_1P_2} \]
  \[ E[X_1T'] = \rho_{1T'} \sqrt{P_1P_{T'}} \]
  \[ E[X_2T'] = \rho_{2T'} \sqrt{P_2P_{T'}} \]

- Implying that covariance matrix $S$ of $(T', X_1, X_2)$ satisfies
  \[ \rho_{12} = \rho_{1T'} \rho_{2T'} \]

- Any jointly Gaussian $(T'_G, X_{1G}, X_{2G})$ with covariance matrix $S$ satisfies $DB$. 
Consider any jointly Gaussian \((T'_G, X_1G, X_2G)\) with covariance matrix \(S\).

\[
I(X_1G; Y, Y_{F2} | X_2G, T'_G) = h(Y, Y_{F2} | X_2G, T'_G) - h(Y, Y_{F2} | X_1G, X_2G, T'_G)
\]
\[
= h(\sqrt{h_{10}}X_1G + Z, \sqrt{h_{12}}X_1G + Z_2 | X_2G, T'_G)
\]
\[
- h(Y, Y_{F2} | X_1G, X_2G, T'_G)
\]
\[
\geq h(\sqrt{h_{10}}X_1 + Z, \sqrt{h_{12}}X_1 + Z_2 | X_2, T'))
\]
\[
- h(Y, Y_{F2} | X_1G, X_2G, T'_G)
\]
\[
\geq h(\sqrt{h_{10}}X_1 + Z, \sqrt{h_{12}}X_1 + Z_2 | X_2, T', T))
\]
\[
- h(Y, Y_{F2} | X_1, X_2, T)
\]
\[
= I(X_1; Y, Y_{F2} | X_2, T)
\]

Similarly,

\[
I(X_2G; Y, Y_{F1} | X_1G, T'_G) \geq I(X_2; Y, Y_{F1} | X_1, T)
\]

\[
I(X_1G, X_2G; Y, Y_{F1}, Y_{F2} | T'_G) \geq I(X_1, X_2; Y, Y_{F1}, Y_{F2} | T)
\]

\[
I(X_1G, X_2G; Y) \geq I(X_1, X_2; Y)
\]
Evaluation of $DB^{MAC}_{UC}$: Main Step

- Recall the definition of the set $\mathcal{P}_{\text{NG}}^{DB(b)}$:

\[ \mathcal{P}_{\text{NG}}^{DB(b)} = \{ p(t, x_1, x_2) \in \mathcal{P}_{\text{NG}}^{DB} : \text{cov. matrix of } p(t, x_1, x_2) \text{ is } Q \text{ and there does not exist a } \mathcal{J}G \left( T_G, X_{1G}, X_{2G} \right) \text{ with cov. matrix } Q \text{ satisfying } (DB) \} \]

- Facts at hand:

\[
I(X_1; X_2|T) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T) \\
I^Q(X_{1G}; X_{2G}|T_G) > I^Q(X_{1G}; X_{2G}|Y_{F_1}, Y_{F_2}, T_G)
\]

- Construct a cov. matrix $S$ such that:

\[
R_1 \leq I^Q(X_{1G}; Y, Y_{F_2}|X_{2G}, T_G) \leq I^S(X_{1G}; Y, Y_{F_2}|X_{2G}, T_G) \\
R_2 \leq I^Q(X_{2G}; Y, Y_{F_1}|X_{1G}, T_G) \leq I^S(X_{2G}; Y, Y_{F_1}|X_{1G}, T_G) \\
R_1 + R_2 \leq I^Q(X_{1G}, X_{2G}; Y, Y_{F_1}, Y_{F_2}|T_G) \leq I^S(X_{1G}, X_{2G}; Y, Y_{F_1}, Y_{F_2}|T_G) \\
R_1 + R_2 \leq I^Q(X_{1G}, X_{2G}; Y) \leq I^S(X_{1G}, X_{2G}; Y)
\]

- Such that any $(T_G, X_{1G}, X_{2G})$ with cov. matrix $S$ satisfies,

\[
I^S(X_{1G}; X_{2G}|T_G) = I^S(X_{1G}; X_{2G}|Y_{F_1}, Y_{F_2}, T_G)
\]
We only need to consider probability distributions in $\mathcal{P}_G^{DB}$. 

I.e., consider only Gaussian $(T, X_1, X_2)$ satisfying $X_1 \rightarrow T \rightarrow X_2$. 
We only need to consider probability distributions in $\mathcal{P}_G^{DB}$.

I.e., consider only Gaussian $(T, X_1, X_2)$ satisfying $X_1 \rightarrow T \rightarrow X_2$. 

Explicit Evaluation of $\mathcal{DB}_{UC}^{MAC}$
Final expression for the outer bound:

\[
DB_{\text{MAC}}^{\text{UC}} = \bigcup_{(\rho_1 T, \rho_2 T) \in [0,1] \times [0,1]} \left \{ (R_1, R_2) : R_1 \leq \frac{1}{2} \log (1 + f_1(\rho_1 T)) \right. \\
\left. R_2 \leq \frac{1}{2} \log (1 + f_2(\rho_2 T)) \right. \\
\left. R_1 + R_2 \leq \frac{1}{2} \log (1 + f_3(\rho_1 T, \rho_2 T)) \right. \\
\left. R_1 + R_2 \leq \frac{1}{2} \log (1 + f_4(\rho_1 T, \rho_2 T)) \right. \}
\]

where

\[
f_1(\rho_1 T) = (1 - \rho_1^2 T) P_1 \left( \frac{h_{10}}{\sigma_Z^2} + \frac{h_{12}}{\sigma_{Z_2}^2} \right), \quad f_2(\rho_2 T) = (1 - \rho_2^2 T) P_2 \left( \frac{h_{20}}{\sigma_Z^2} + \frac{h_{21}}{\sigma_{Z_1}^2} \right)
\]

\[
f_3(\rho_1 T, \rho_2 T) = f_1(\rho_1 T) + f_2(\rho_2 T) + (1 - \rho_1^2 T)(1 - \rho_2^2 T) P_1 P_2 \beta
\]

\[
f_4(\rho_1 T, \rho_2 T) = \frac{(h_{10} P_1 + h_{20} P_2 + 2 \rho_1 T \rho_2 T \sqrt{h_{10} h_{20} P_1 P_2})}{\sigma_Z^2}
\]

and

\[
\beta = \frac{(h_{12} h_{21} \sigma_Z^2 + h_{20} h_{12} \sigma_{Z_1}^2 + h_{10} h_{21} \sigma_{Z_2}^2)}{\sigma_Z^2 \sigma_{Z_1}^2 \sigma_{Z_2}^2}
\]
Limiting Behavior of $DB^\text{MAC}_{UC}$ and $CS$ Bounds

- As $\sigma^2_{Z_1}, \sigma^2_{Z_2} \to 0$:
  - Both $DB^\text{MAC}_{UC}$ and $CS$ degenerate to the total cooperation line.

$$R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \frac{h_{10}P_1 + h_{20}P_2 + 2\sqrt{h_{10}h_{20}P_1P_2}}{\sigma^2_Z}\right)$$

- As $\sigma^2_{Z_1}, \sigma^2_{Z_2} \to \infty$:
  - $DB^\text{MAC}_{UC} \to$ capacity region without cooperation.

$$f_1(\rho_{1T}) = \frac{(1 - \rho^2_{1T})h_{10}P_1}{\sigma^2_Z} \quad (1)$$

$$f_2(\rho_{2T}) = \frac{(1 - \rho^2_{2T})h_{20}P_2}{\sigma^2_Z} \quad (2)$$

$$f_3(\rho_{1T}, \rho_{2T}) = f_1(\rho_{1T}) + f_2(\rho_{2T}) \quad (3)$$

$$< \frac{(h_{10}P_1 + h_{20}P_2 + 2\rho_{1T}\rho_{2T} \sqrt{h_{10}h_{20}P_1P_2})}{\sigma^2_Z} \quad (4)$$

- $CS \to$ capacity region with output feedback [Ozarow 1984].
Illustration of Bounds for Gaussian MAC with User Cooperation

\[ P_1 = P_2 = \sigma_Z^2 = \sigma_{Z_1}^2 = \sigma_{Z_2}^2 = 1 \text{ and } h_{10} = h_{20} = 1, h_{12} = 3, h_{21} = 2. \]

- Cut-set bound is sensitive to cooperation noise variances \( \sigma_{Z_1}^2, \sigma_{Z_2}^2 \).
- \( DB_{UC}^{MAC} \) is more sensitive to feedback noise variances \( \sigma_{Z_1}^2, \sigma_{Z_2}^2 \).
  - As \( \sigma_{Z_1}^2, \sigma_{Z_2}^2 \to 0, DB_{UC}^{MAC} \to CS \) (degenerates to the total cooperation line).
  - As \( \sigma_{Z_1}^2, \sigma_{Z_2}^2 \to \infty, DB_{UC}^{MAC} \to C_{No-Cooperation} \).
The Gaussian IC with User Cooperation (IC-UC)

Channel model:

\[ Y_1 = X_1 + \sqrt{b}X_2 + N_1 \]
\[ Y_1 = \sqrt{a}X_1 + X_2 + N_2 \]
\[ Y_{F_1} = \sqrt{h_{21}}X_2 + Z_1 \]
\[ Y_{F_2} = \sqrt{h_{12}}X_1 + Z_2 \]

Similar outer bound for IC-UC.

It suffices to consider \( \mathcal{P}_G^{DB} \) when evaluating \( DB_{UC}^{IC} \).

Sum-rate \( DB \) based bound for IC-NF [Gastpar-Kramer 2006].
Illustration of Bounds for Gaussian IC with User Cooperation

\[ P_1 = P_2 = \sigma^2_{N_1} = \sigma^2_{N_2} = \sigma^2_{Z_1} = \sigma^2_{Z_2} = 1 \quad \text{and} \quad a = b = 0.5, h_{12} = h_{21} = 0.1. \]
$P_1 = P_2 = \sigma_{N_1}^2 = \sigma_{N_2}^2 = \sigma_{Z_1}^2 = \sigma_{Z_2}^2 = 1$ and $a = b = 0.5$, $h = h_{12} = h_{21}$. 
Conclusions

- Obtained a new outer bound for MAC-GF.
- Application of the new outer bound for two channel models:
  - Gaussian MAC with noisy feedback.
  - Gaussian MAC with user cooperation.
- Similar results for IC-GF.
- A new approach for evaluating bounds involving auxiliary random variables.
- For all non-zero values of $\sigma_{Z_1}^2$, $\sigma_{Z_2}^2$, our $DB$ bounds strictly improve over the cut-set bound.