Possible crossover from BCS superconductivity to Bose-Einstein condensate in quark matter

Hiroaki Abuki

Abstract. The possibility of the crossover from the BCS pairing to the Bose-Einstein condensate (BEC) of diquarks with going down in density is discussed in the framework of in the Nambu Jona-Lasinio (NJL) model. We find that the quark matter at moderate density may be close to the intermediate of the crossover, the precursory regime to the BEC phase.

Keywords: quark matter, superconducting, Bose-Einstein condensate

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INTRODUCTION

It is now established that the ground state of QCD at extremely high baryon density is in the color-flavor locked superfluid [1] where quarks with all the three flavors participate in the Cooper pairing. The appearance of the superfluid CFL is completely due to the BCS mechanism saying that any arbitrary weak attraction between quarks leads to the tachyonic Cooperon on the Fermi surface at sufficiently low temperature.

For the compact star phenomenology, however, one needs to consider other candidates with less symmetric pairing because the strange quark mass is not so much smaller compared to quark chemical potential which is at most of order 500MeV even at the center of stars. The finite value of strange quark mass causes a stress on the CFL phase. This kinematic effect is known to bring about a rich variety of phases at moderate densities; see [2, 3] for the NJL model studies of QCD phase diagram.

In addition to such kinematic effect, there is the other key ingredient which plays an important role at low densities; that is the strong coupling nature due to the asymptotic freedom of QCD. Going down in density, quark-gluon interaction becomes large, and this may lead a modification in superconducting phases. In fact, the ladder QCD calculation of the coherence length indicates that the Cooper pair size decreases significantly toward low density and it could be of order of inter-quark spacing at $\mu = 500$MeV [4, 5]. This strongly suggests the crossover from the BCS-type weak coupling superconductivity to the Bose-Einstein Condensate (BEC) of tightly bound quark pairs [6, 7, 8, 9, 10]. Another possible interesting phenomenon is the formation of pseudogap above the critical temperature [11]. These two scenarios indicate the existence of non-trivial (non-Wigner) phase above the critical temperature at strong coupling.

Major studies done in past mainly concentrate on the spectral analysis of collective modes. If the fluctuation is so large, however, there must be its feedback to the thermodynamics, the equation of state, for example. Such strong modification of thermodynamics due to fluctuations may bring about remarkable astrophysical consequences.
In this short article, after briefly summarizing the application of the Nozières Schmitt-Rink (NSR) theory [12] to the NJL model following our detailed analysis [6, 7], we discuss the relativistic BCS-BEC crossover paying a particular attention to the significance of the fluctuation feedback to the quark matter thermodynamics.

APPLICATION OF THE NOZIÈRES SCHMITT-RINK THEORY TO THE NAMBU JONA-LASINIO MODEL

We here consider a general relativistic four-fermion model with a point attraction [6],

\[ \mathcal{L}[\psi, \bar{\psi}] = \bar{\psi} (i\gamma^\mu - m + \gamma_0 \mu) \psi + G \psi^\dagger \gamma_5 C \psi \psi^T i \gamma_5 C \psi, \]

where \( \psi, m \) denote the Dirac spinor field and its mass, \( \mu \) is the chemical potential to adjust the asymmetry between particle and antiparticle, and \( G \) parameterizes the strength of attraction. The extension to the NJL model with isospin doublet, three colors is straightforward and the results will be given later. After introducing Hubbard-Stratonovich fields \( \Delta(\tau, x) \) for \( i \psi^T \gamma_5 C \psi \), the fermion can be integrated out:

\[ Z = Z_0 \int \mathcal{D} \Delta \mathcal{D}^* \exp \left( -S_{\text{eff}}[\Delta, \Delta^*] \right), \]

\( Z_0 = e^{-\beta \Omega_0(\mu, T)} \) is the free fermion part of the partition function, while \( S_{\text{eff}}[\Delta, \Delta^*] \) is the effective action for the collective bosonic fields. According to [12], we include the effect of fluctuation up to the second order in \( \Delta \). We have

\[ S_{\text{eff}}[\Delta, \Delta^*] = T \sum_n \int \frac{dp}{(2\pi)^3} \left( \frac{1}{G} - \chi(i\omega_n, p) \right) \left| \Delta(i\omega_n, p) \right|^2 \]

where \( \omega_n \) is fermionic Matsubara frequency. Then the field \( \Delta \) can be integrated out and the pressure of the system leads to

\[ p(\mu, T) = p_0(\mu, T) + p_{\text{fluc}}(\mu, T), \]

where \( p_0(\mu, T) = 2T \sum_{\sigma = \pm} \int \frac{dp}{(2\pi)^3} \ln \left( 1 + e^{-|p^2 + m^2 - \sigma \mu|/T} \right) \) is the free fermion contribution, and \( p_{\text{fluc}} \) corresponds to the fluctuation contribution defined by

\[ p_{\text{fluc}}(\mu, T) = T \sum_N \int \frac{dK}{(2\pi)^3} \ln \left( 1 - G \chi_{\mu, T}(i\Omega_N, K) \right). \]

Here, \( \chi_{\mu, T}(i\Omega_N, K) = \text{F.T.} \left[ \langle T_\tau \left[ \psi^T i \gamma_5 C \psi(\tau, x) \right] \left[ \psi^T i \gamma_5 C \psi(0, 0) \right]^T \right] \) is the Cooperon at one loop, and \( \Omega_N \) is bosonic frequency. (See [6, 7, 11] for the explicit expression.) Note that, since we are approaching \( T_c \) from above, the overall factor 1/2 is dropped because the phase and amplitude fluctuation contribute equally to the partition function.

When the temperature (chemical potential) is decreased (increased) from the normal phase, the Cooperon becomes tachyonic at some critical point. The thermodynamic stability requires \( 1 - G \chi_{\mu, T}(0, K) > 0 \). Because the function takes minimum at \( K = 0 \) for the system without density imbalance, the condition of criticality is simply

\[ 1 - G \chi_{\mu, T}(0, 0) = 0, \]
which is nothing but the Thouless criterion. This condition generates a one-dimensional line in the \((\mu, T)\)-plane which we call the Thouless line. To see how large the fluctuation effect on the thermodynamics is, it is better to move on to the canonical ensemble. This corresponds to determine the value of \(\mu_c\) by means of

\[
\frac{\partial P_0}{\partial \mu} (\mu_c, T_c) + \frac{\partial P_{\text{fluc}}}{\partial \mu} (\mu_c, T_c) = \frac{k_F^3}{3\pi^2},
\]

where \(k_F\) parameterizes the total density of the system. The first term is the free fermion contribution while the second term is the fluctuation contribution. We will see later that, the second term gives a significant contribution even in the BCS side.

From above two basic equations, we numerically obtain \(T_c\) and \(\mu_c\) as a function of density and coupling, i.e., \(T_c(G, k_F, \Lambda)\) and \(\mu_c(G, k_F, \Lambda)\) where \(\Lambda\) is an appropriate momentum cutoff. Before going into numerical computations, let us briefly summarize the effect of including \(N_c = 3\) colors and \(N_f = 2\) flavors. If we assume the attraction in the isoscalar and color anti-triplet channel, the Thouless criterion Eq. (5) is not affected. On the other hand, the number condition Eq. (6) is modified as follows.

\[
N_f N_c \frac{\partial P_0}{\partial \mu} (\mu_c, T_c) + \frac{N_c(N_c - 1)}{2} \frac{\partial P_{\text{fluc}}}{\partial \mu} (\mu_c, T_c) = N_f N_c \frac{k_F^3}{3\pi^2},
\]

where \(k_F\) is redefined so that each fermion species has a density \(\frac{k_F^3}{3\pi^2}\) when the interaction is turned off. We see that the fluctuation contribution is multiplied by a kinematic factor \(d_B \equiv \frac{N_c(N_c - 1)}{2}\), representing the fact that the system has \(d_B\) collective modes belonging to the antisymmetric representation of \(SU_c(3)\). From the parametric dependence, we see that the fluctuation dominates the thermodynamics in the \(N_c \to \infty\) limit.

In numerical calculations, we set \(\frac{mc}{\hbar a} = 0.2\). In addition, we mainly study the crossover detail for a relativity parameter, \(\frac{\hbar k_F}{mc} = 0.2\), leaving to see its density/mass dependence later. Further, we use the modified coupling \(g = \frac{G_Rc}{G_R}\) instead of bare coupling \(G\).

In Fig. 1(a), we show \(T_c, \mu_c\) as a function of \(g\). When \(g \lesssim -0.2\), \(T_c\) is well approximated by the mean field result indicated by \(T_{\text{MF}}\). In the mean field approximation (without \(P_{\text{fluc}}\)), \(\mu_{\text{MF}} \approx E_F\) by neglecting a tiny correction of order \(T^2/E_F\). As a consequence, \(T_{\text{MF}}\) is determined almost by the Thouless criterion. As \(g\) is increased and the unitary point \(g = 0\) is approached, the mean field result starts to deviate from real \(T_c\). This means that fluctuation contribution in the number equation Eq. (7) grows gradually and it cannot be ignored anymore. In fact, the quark density coming from fluctuation grows as seen in fig. (b). When the coupling exceeds \(g = 0.07\), \(\mu_c\) gets lower than fermion mass \(m\).

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1 Note that even in the known strong coupling system, the nuclear matter consisting of neutron and proton, this parameter is of order \(\frac{\hbar k_F}{mc} = \mathcal{O}(10^{-3})\) at the Mott transition point [13].

2 The regularized coupling \(G_R\) is introduced by \(-\frac{1}{G_R} = \frac{1}{a} - \chi_{0,0}(2m, 0)\), and \(G_{Rc}\) stands for the critical regularized coupling for zero mass boson at vacuum, i.e., \(-\frac{1}{G_{Rc}} = \chi_{0,0}(0, 0) - \chi_{0,0}(2m, 0)\). \(G_R\) is related to the scattering length \(a_s\) by \(G_R = \frac{4\pi\hbar^2}{m a_s}\). Then we see that the zero-binding bound state forms at \(g = 0\) (the unitary limit) and it becomes massless at \(g = 1\) at vacuum. See [7] for the detail.
and the in-medium bound state appears accordingly. (See [6, 7] for the detailed spectral analysis.) Then the system goes into the BEC phase where \( T_c \) is nearly saturated to a constant and \( T_{MF} \) completely fails. This saturating behaviour suggests that \( T_c \) is determined by the number equation Eq. (7) because it does not explicitly depend on \( G \). In fact, the in-medium boson mass is shown to be twice of \( \mu_c (M_B = 2\mu_c) \) at \( T = T_c \), and provided \( M_B \gg T_c \) (nonrelativistic), Eq. (7) can be approximated by

\[
N_c N_f \frac{k_F^3}{3\pi^2} \sim \frac{N_c(N_c-1)}{2} \int \frac{dK}{(2\pi)^3} \frac{2}{e^{k^2/2M_B T_c} - 1} \equiv 2d_B \zeta \left( \frac{3}{2} \right) \left( \frac{M_B T_c}{2\pi} \right)^{\frac{3}{2}}.
\]

The factor 2 in the integrand comes from the fact one diquark consists of two quarks.

This gives an approximation \( T_c \sim T_{BEC}^{\text{NR}} = \frac{2}{\pi^{1/3}} \left( \frac{N_c-1}{N_f} \right) \frac{\zeta}{\sqrt{\pi \pi}} \frac{k_F^2}{M_B} \). In fig. (a), this formula with \( M_B = 2\mu_c \) is tried by the thin line, which agrees very well with real \( T_c \). The rightarrow indicates the nonrelativistic strong coupling limit of \( T_c \), evaluated by \( T_{BEC}^{\text{NR}} \) with \( M_B = 2m \). In the current framework, \( T_c \) does not saturate to this value and slightly increases due to the binding effect. This is a residual relativistic effect that the binding energy of diquarks can become as large as the order of the constituent fermion mass.

As \( g \) is increased and \( g \gg 1 \) is approached, \( T_{BEC}^{\text{NR}} \) starts to fail and the system eventually goes into the new regime, the relativistic BEC (RBEC) [14]. This is because the nonrelativistic approximation \( 2\mu_c \gg T_c \) is no longer valid there due to the large binding effect. In fact, the diquark mass is smaller than temperature, \( 2\mu_c \ll T_c \), for \( g \gtrsim 1 \), and therefore anti-diquarks contribute to the thermodynamics. By only taking the stable boson and antiboson contributions in Eq. (7), we get an approximation \( T_c \sim T_{BEC}^{\text{RL}} = \frac{1}{\pi} \sqrt{\frac{\zeta}{M_B} N_f N_c - 1} \) in the same way as [14]. This formula with \( M_B = 2\mu_c \) is tried by thin line in FIG. 1(a), which fairly agrees with the real \( T_c \). However, in contrast to the RBEC of the elementary boson [14], our composite boson system has fermionic degrees of freedom due to the competition between the internal energy and entropy [7]. For this reason, the agreement is not so good as that in the BEC region. FIG. 1(b) confirms this picture; quark density
from the fermion sector again comes to play a major role in the deep RBEC region.

Remarkably, in the (R)BEC region, there is a non-trivial phase above $T_c$, the preformed boson phase, up to $T_{\text{diss}}$. $T_{\text{diss}}$, shown by the dashed line in FIG. 1(a), characterizes the ionization of diquarks. Needless to say, the system in the preformed boson phase differs much from a pure Fermi gas although the symmetry is restored.

Interestingly enough, the definite crossover appears also in the dynamic equation for the pair excitation [7]. By expanding $S_{\text{eff}}$ up to the quartic order in $\Delta$ and performing the low energy/long wavelength expansion, we obtain the dynamic equation near $T_c$

$$\frac{d}{dt}\Delta(t, x) = \frac{\delta F_{\text{eff}}[\Delta, \Delta^*]}{\delta \Delta^*(t, x)} = \left[ a_0 - \frac{T_c - T}{T_c} + \frac{c}{4m} \nabla_x^2 - b_0 |\Delta(t, x)|^2 \right] \Delta(t, x), \quad (9)$$

where $d$ (complex) and $\{a_0, c, b_0\}$ (real) are the low energy coefficients. If we define the complex effective mass by $M_{\text{eff}} = \frac{2m d}{c}$, its real part coming from the particle-hole asymmetry represents the propagating piece of the fluctuating pair field [16], while the imaginary part expresses its diffusive nature. The real and imaginary parts of $M_{\text{eff}}$ as a function of $g$ are depicted in FIG. 2(a). In the BCS regime, the pair mode is diffusive, but the magnitude of damping decreases significantly as the unitary point $g = 0$ is approached. When the system goes into the BEC phase, the imaginary part vanishes and fluctuation becomes a pure propagating mode due to a bound state gap. However $M_{\text{eff}}$ does not saturate to $2m$ in contrast to the nonrelativistic calculation [15]; it gets smaller towards the RBEC phase, which is also due to the relativistic binding effect.

Fig. 2(b) shows how the crossover characteristics of $T_c$ is affected by the increase of relativity parameter $\frac{\hbar k_F}{mc}$. This corresponds to decreasing $m$ or, increasing density $k_F$.

From bottom to top, $\frac{\hbar k_F}{mc} = 0.2 \times n$ with $n = 1, 2, \cdots, 12$. Several notes are in order.

(i) The BCS/BEC crossover point $m = \mu_c$ indicated by the large point shifts to higher value of $g$. This is due to the Pauli-blocking effect which prevents the formation of in-medium bound state at high (small) density (mass). At the same time, the crossover characteristics of $T_c$ gets smeared. (ii) The universal thermodynamics of the unitarity point is absent in the relativistic system as noticed in [9]. $T_c$ explicitly depends on the
additional parameter $\frac{\hbar k_F}{mc}$. In fact, the pressure at zero temperature is deduced to take a form

$$p = \frac{2k_F^3}{15\pi^2} \frac{k_F^2}{2m} f\left(\frac{1}{k_Fa_s}, \frac{\hbar k_F}{mc}, \frac{\hbar}{mc}\right)$$

with $f$ denoting a dimensionless function; we observe only in the $m \to \infty$ limit the nonrelativistic universal behaviour at $\frac{1}{k_Fa_s} = 0$ is recovered.

Let us finally discuss in which regime the actual quark matter does exist. The modified coupling $g$ is a function of $\Lambda$ and $m$. It is easy to see $g \to 1$ with $m \to \infty$. Also $g \to -\infty$ when $m \to 0$ with a natural assumption that a bare coupling $G\Lambda^2$ is less than $\pi^2$, the critical value for the dynamical Majorana mass generation at vacuum. As there should be the point $g = 0$ in between, we conclude that for any fixed value for $G < \pi^2$, the system is BCS-like for $m \to 0$ and it is BEC-like at sufficiently large $m$. If we fix $G$ to the usually adopted value, $3/4$ of an appropriate scaler coupling [2, 3], $g = 0$ corresponds to $m/\Lambda \approx 0.53$. This is much larger mass compared to the current quark masses in agreement with the recently appeared paper [16]. We conclude that somewhat exotic conditions must be satisfied to have the diquark BEC in QCD phase diagram; (i) the diquark coupling is stronger than expected, and/or (ii) in-medium quark mass is larger than its perturbative estimate. Interestingly, two lattice studies, one about $qq$ interaction [17] and the other for in-medium quasiquark mass [18] are encouraging. Also it is worth noting that even if the BEC phase cannot be reached, the fluctuation feedback to quark matter thermodynamics will be significant. In fact, it can be neglected only in the weak coupling limit as noted in [10]. If so it may bring about a remarkable modification of the structure of possible quark, or hybrid stars. Exploring possible BCS/BEC crossover in quark matter with more realistic situations taken into account, as well as looking for its astrophysical consequences clearly deserves further investigations.

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REFERENCES

1. K. Rajagopal and F. Wilczek, arXiv:hep-ph/0011333; M. G. Alford, Ann. Rev. Nucl. Part. Sci. 51, 131 (2001); T. Schäfer, arXiv:hep-ph/0304281; D. H. Rischke, Prog. Part. Nucl. Phys. 52, 197 (2004).
2. H. Abuki, M. Kitazawa and T. Kunihiro, Phys. Lett. B 615, 102 (2005) [arXiv:hep-ph/0412382]; H. Abuki and T. Kunihiro, Nucl. Phys. A 768, 118 (2006) [arXiv:hep-ph/0509172].
3. S. B. Ruester et al., Phys. Rev. D 72, 034004 (2005); D. Blaschke et al., Phys. Rev. D 72, 065020 (2005).
4. M. Matsuzaki, Phys. Rev. D 62, 017501 (2000) [arXiv:hep-ph/9910541].
5. H. Abuki, T. Hatsuda and K. Itakura, Phys. Rev. D 65, 074014 (2002) [arXiv:hep-ph/0109013]; K. Itakura, Nucl. Phys. A 715, 859 (2003).
6. Y. Nishida and H. Abuki, Phys. Rev. D 72, 096004 (2005) [arXiv:hep-ph/0504083].
7. H. Abuki, Nucl. Phys. A 791, 117 (2007) [arXiv:hep-ph/0605081].
8. K. Nawa, E. Nakano and H. Yabu, Phys. Rev. D 74, 034017 (2006); J. Deng, A. Schmitt and Q. Wang, arXiv:nucl-th/0611097; A. H. Rezaeian and H. J. Pirner, Nucl. Phys. A 779, 197 (2006).
9. L. He and P. Zhuang, Phys. Rev. D 75, 096003 (2007) [arXiv:hep-ph/0703042].
10. L. He and P. Zhuang, arXiv:0705.1634 [hep-ph].
11. M. Kitazawa et al., Phys. Rev. D 65, 091504 (2002); Phys. Rev. D 70, 056003 (2004); Prog. Theor. Phys. 114, 117 (2005).
12. P. Nozières and S. Schmitt-Rink, J. Low. Temp. Phys. 59 195 (1985).
13. U. Lombardo et al., Phys. Rev. C 64, 064314 (2001).
14. H.E. Haber and H.A. Weldon, Phys. Rev. Lett. 46, 1497 (1981); J.I. Kapusta, Phys. Rev. D 24, 426 (1981).
15. R. Haussmann, Phys. Rev. B 49, 12975 (1994).
16. M. Kitazawa, D. H. Rischke and I. A. Shovkovy, arXiv:0707.3966 [nucl-th]; arXiv:0709.2235 [hep-ph].
17. A. Nakamura and T. Saito, Prog. Theor. Phys. 112, 183 (2004) [arXiv:hep-lat/0406038].
18. P. Petreczky et al., Nucl. Phys. Proc. Suppl. 106, 513 (2002).