Temperature-doping phase diagram of layered superconductors

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Abstract

The superconducting properties of a layered system are analyzed for the cases of zero- and non-zero angular momentum of the pairs. The effective thermodynamic potential for the quasi-2D XY-model for the gradients of the phase of the order parameter is derived from the microscopic superconducting Hamiltonian. The dependence of the superconducting critical temperature $T_c$ on doping, or carrier density, is studied at different values of coupling and inter-layer hopping. It is shown that the critical temperature $T_c$ of the layered system can be lower than the critical temperature of the two-dimensional Berezinskii-Kosterlitz-Thouless transition $T_{BKT}$ at some values of the model parameters, contrary to the case when the parameters of the XY-model do not depend on the microscopic Hamiltonian parameters.

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I. INTRODUCTION

Theoretical description of the doping dependence of the superconducting properties of the high-temperature superconductors (HTSCs) is one of the most difficult problem of the modern condensed matter physics. Generally speaking, the complicated crystal structure of these materials, low-dimensional (quasi-2D) transport properties, the superconducting order parameter anisotropy, strong correlations and other properties result in the fact that many years after the discovery the microscopic mechanism of HTSC is not understood yet.

During the last years many models which take into account some of the cuprate properties have been proposed. The doping dependence of the superconducting properties at $T = 0$ in the $s$-wave pairing channel was studied for the 3D case in [1, 2, 3] and, particularly for the quasi-2D case [4]. For the 2D case this problem was studied at $T = 0$ (when a long-range superconducting order is still possible in a 2D system [6]) for the case of local attraction in [1, 2, 5], and for the phonon-mediated model [7] (for over-review see [8], for example). The $d$-wave pairing along with the $s$-wave one at $T = 0$ for the case of the extended Hubbard model with the next nearest neighbor attraction was studied in [9, 10] and for a 2D continuum model with short-ranger attraction and electron correlations - in the paper [11]. The properties of a model with doping dependent correlation length were studied recently in [12].

The 2D $s$-wave pairing at finite temperatures, when the Berezinskii-Kosterlitz-Thouless (BKT) transition can take place in superconducting system, was considered in [13, 14] for the case of the model with local attraction and in [15, 16] for the case of the electron-phonon pairing. The problem of the $s$-wave superconductivity with fluctuating order parameter phase in the 3D case was analyzed in [14, 17]. The effective action for slowly fluctuating $d$-wave superconducting order parameter for the 2D case was also analyzed in [18, 19, 20, 21].

However, it is known that the long-range order is impossible in the 1D and 2D systems with an order parameter which has a continuous symmetry [6]. Therefore, to get real phase transition with a long-range order and homogeneous order parameter one needs to take into account the inter-layer coupling $t_z$. The layered superconductivity is much more complicated since the possibility of the inter-layer fluxon and intra-layer vortex phase transitions with corresponding critical temperatures $T_f$ and $T_v$ must be analyzed. It was already shown [22, 23, 24, 25], that there is only one phase transition in such a system with the critical
temperature $T_c$ and $T_v < T_c < T_f \simeq 8T_v$. The critical temperature $T_c$ is equal to $T_v$, or, what is equivalent, to the temperature $T_{BKT}$ of the 2D BKT phase transition at $t_z = 0$. Then, this temperature value is increasing to the value $T_f$ with the inter-layer hopping $t_z$ growth. In the papers [22, 23, 24, 25] the phase order parameter effective Hamiltonian was studied in the presence of an external magnetic field and this model was mapped on the quasi-2D XY-model. The XY-model parameters $J_\parallel$ and $J_\perp$ were considered as phenomenological constants. It was shown that $T_{BKT} = \frac{\pi}{2}J_\parallel$ and $T_f \simeq 8T_{BKT}$.

In this paper we derive the effective XY-Hamiltonian from the initial Hamiltonian for the layered system of attracting fermions. In this case the parameters $J_\parallel$ and $J_\perp$ depend on the bare parameters - charge carrier density, coupling, pair angular momentum, temperature and the inter-layer hopping. As it will be shown below, this leads to the non-trivial dependence of the superconducting critical temperature $T_c$ on the model parameters. In particular, in general this temperature is different from the critical temperature of the 2D BKT transition and $T_c < T_{BKT}$ at some values of the model parameters, contrary to the results for the case when parameters $J_\parallel$ and $J_\perp$ don’t depend on the parameters of the microscopic Hamiltonian, and when the relation $T_c < T_{BKT}$ always holds at $J_\perp > 0$ [22, 23, 24, 25].

II. THE MODEL AND THE THERMODYNAMIC POTENTIAL

The model Hamiltonian for a layered superconducting system can be written as

$$H(\tau) = \sum_{\sigma, j} \int d^2r \psi_j^{\dagger}(\tau, r) \left[ -\frac{\nabla^2}{2m} + 2t_z - \mu \right] \psi_{j\sigma}(\tau, r) - \sum_{j_1, j_2} t_{mn} \int d^2r \psi_{j_1\sigma}^{\dagger}(\tau, r) \psi_{j_2\sigma}(\tau, r) \left[ -\frac{\nabla^2}{2m} + 2t_z - \mu \right] \psi_{j_2^{\dagger}\sigma}(\tau, r)$$

$$- \frac{1}{2} \sum_{\sigma, j} \int d^2r_1 d^2r_2 \psi_{j_1\sigma}(\tau, r_1) \psi_{j_2^{\dagger}\sigma}(\tau, r_2) V(r_1, r_2) \psi_{j\sigma}(\tau, r_1) \psi_{j\sigma}(\tau, r_2)$$

where $\psi_{j\sigma}(\tau, r)$ is a fermi-field in the with mass $m$ and spin $\sigma = \uparrow, \downarrow$, $\tau$ is an imaginary time and $j, r$ are layer number and intra-layer coordinate, correspondingly; $t_{j_1j_2} = t_z (\delta_{j_2,j_1+1} + \delta_{j_2,j_1-1})$ corresponds to the nearest inter-plane hopping. The free fermion dispersion relation in the momentum space has the following form

$$\xi(\mathbf{k}, k_z) = \frac{k^2}{2m} + 2t_z - 2t_z \cos(ak_z) - \mu,$$

where $\mathbf{k}$ is a 2D wave vector with a bandwidth $W$, and $k_z$ is the momentum in the inter-layer ($z$) direction, it changes in the interval $[0, 2\pi/a]$, where $a$ is the inter-layer spacing; $\mu$ is the
chemical potential of the system. In Eq. (1) interaction \( V(\mathbf{r}_1, \mathbf{r}_2) \) describes a non-retarded in-plane fermion attraction.

The partition function of the system is

\[
Z = \int D\psi^\dagger D\psi e^{-S}
\]

with the action

\[
S = \int_0^\beta d\tau \left[ \sum_{\sigma,j} \int d^2r \psi_j^\dagger(\tau, r) \partial_\tau \psi_j(\tau, r) + H(\tau) \right].
\]

To study the superconducting properties of the system with an arbitrary pairing symmetry the Hubbard-Stratonovich transformation with bilocal fields \( \phi_j(\tau, \mathbf{r}_1, \mathbf{r}_2) \) and \( \phi^\dagger_j(\tau, \mathbf{r}_1, \mathbf{r}_2) \) can be applied [26]:

\[
\exp \left[ \psi_j^\dagger(\tau, \mathbf{r}_2) \psi_j(\tau, \mathbf{r}_1) V(\mathbf{r}_1, \mathbf{r}_2) \psi_{\bar{j}}^\dagger(\tau, \mathbf{r}_1) \psi_{\bar{j}}(\tau, \mathbf{r}_2) \right] = \int D\phi^\dagger D\phi \exp \left[ - \int_0^\beta d\tau \sum_j \int d^2r_1 d^2r_2 \right.
\]

\[
\times \left( \frac{[\phi_j(\tau, \mathbf{r}_1, \mathbf{r}_2)]^2}{V(\mathbf{r}_1, \mathbf{r}_2)} - \phi_j^\dagger(\tau, \mathbf{r}_1, \mathbf{r}_2) \psi_j^\dagger(\tau, \mathbf{r}_2) - \psi_j^\dagger(\tau, \mathbf{r}_1) \phi_j^\dagger(\tau, \mathbf{r}_2) \right] \]

(5)

Let us introduce the Nambu spinor

\[
\Psi_j(\tau, \mathbf{r}) = \begin{pmatrix} \psi_j^\dagger(\tau, \mathbf{r}) \\ \psi_{\bar{j}}(\tau, \mathbf{r}) \end{pmatrix}, \Psi_j^\dagger(\tau, \mathbf{r}) = \begin{pmatrix} \psi_j^\dagger(\tau, \mathbf{r}) & \psi_{\bar{j}}^\dagger(\tau, \mathbf{r}) \end{pmatrix}.
\]

In this case the partition function can be written as

\[
Z = \int D\psi^\dagger D\psi D\phi^\dagger D\phi e^{-S(\psi^\dagger, \psi, \phi^\dagger, \phi)},
\]

where

\[
S(\psi^\dagger, \psi, \phi^\dagger, \phi) = \int_0^\beta d\tau \sum_{j_1,j_2} \int d^2r_1 \int d^2r_2 \{ \delta_{j_1,j_2} \frac{|\phi_j(\tau, \mathbf{r}_1, \mathbf{r}_2)|^2}{V(\mathbf{r}_1, \mathbf{r}_2)} \}
\]

\[
- \delta_{j_1,j_2} \Psi_j^\dagger(\tau, \mathbf{r}_1) \left\{ -\partial_\tau - \tau_z (\frac{\nabla^2}{2m} + 2t_z - \mu) \right\} \Psi_j(\tau, \mathbf{r}_2) - \Psi_{\bar{j}}^\dagger(\tau, \mathbf{r}_1) \Psi_{\bar{j}}(\tau, \mathbf{r}_2) + t_{j_1,j_2} \Psi_j^\dagger(\tau, \mathbf{r}_1) \tau_x \Psi_j(\tau, \mathbf{r}_2) - \delta_{j_1,j_2} \Psi_{\bar{j}}^\dagger(\tau, \mathbf{r}_1) \tau_x \Psi_{\bar{j}}(\tau, \mathbf{r}_2)
\]

(7)

where \( \tau_\pm = \frac{1}{2}(\tau_x \pm \tau_y) \) are the Pauli matrices.

In order to study the fluctuations of the order parameter phase and to map the corresponding superconducting effective action on the quasi-2D XY model, it is convenient to
make decomposition of $\psi_{\sigma,j}(\tau, r)$ $\psi_{\sigma,j}^\dagger(\tau, r)$ on their modulus $\chi_{\sigma,j}(\tau, r)$ and phase $\theta_j(\tau, r)$, which as it will be shown below is proportional to the order parameter phase:

$$
\psi_{\sigma,j}(\tau, r) = \chi_{\sigma,j}(\tau, r)e^{i\theta_j(\tau, r)/2},
$$

$$
\psi_{\sigma,j}^\dagger(\tau, r) = \chi_{\sigma,j}^\dagger(\tau, r)e^{-i\theta_j(\tau, r)/2}.
$$

In this case the Nambu operators are

$$
\Psi_j(\tau, r) = e^{i\tau z \theta_j(\tau, r)/2} \Upsilon_j(\tau, r),
$$

$$
\Psi_j^\dagger(\tau, r) = \Upsilon_j^\dagger(\tau, r)e^{-i\tau z \theta_j(\tau, r)/2},
$$

(8)

where $\Upsilon_j(\tau, r)$ and $\Upsilon_j^\dagger(\tau, r)$ are “neutral” Nambu spinor operators:

$$
\Upsilon_j(\tau, r) = \begin{pmatrix} \chi_{j\uparrow}(\tau, r) \\ \chi_{j\downarrow}(\tau, r) \end{pmatrix},
\Upsilon_j^\dagger(\tau, r) = \begin{pmatrix} \chi_{j\uparrow}(\tau, r) \\ \chi_{j\downarrow}(\tau, r) \end{pmatrix}.
$$

The order parameter can be expressed as

$$
\phi_j(\tau, r_1, r_2) = \Delta(\tau, r_1, r_2)e^{i\theta_j(\tau, r_1, r_2)}
$$

$$
\phi_j^\dagger(\tau, r_1, r_2) = \Delta(\tau, r_1, r_2)e^{-i\theta_j(\tau, r_1, r_2)},
$$

where we assume that the modulus of the order parameter $\Delta(\tau, r_1, r_2)$ does not depend on the layer index. It is also natural to assume that

$$
\phi_j(\tau, r_1, r_2) \simeq \Delta(\tau, r)e^{i\theta_j(\tau, R)},
$$

(9)

where $r = r_1 - r_2$ and $R = (r_1 + r_2)/2$ are the relative and the center of mass coordinates, correspondingly [21, 27]. The relation (9) means that the dynamics of the Cooper pairs is described by the order parameter modulus the symmetry of which depends, generally speaking, on the relative pair coordinate and the motion of the superconducting condensate is described by the order parameter phase, which changes slowly with the distance and can be described by center of mass coordinate. In this case it is easy to obtain

$$
\phi_j^\dagger(\tau, r_1, r_2)\Psi_j^\dagger(\tau, r_1)\tau_-\Psi_j(\tau, r_2) + \Psi_j^\dagger(\tau, r_1)\tau_+\Psi_j(\tau, r_2)\phi_j(\tau, r_1, r_2)
$$

$$
\simeq \Delta(\tau, r)Y_j^\dagger(\tau, r_1)\tau_x Y_j(\tau, r_2)
$$

(10)
Substituting (8), (9) and (10) into the expression for the partition function (6) it is easy to get

\[ Z = \int \Delta D \Delta D \theta e^{-\beta \Omega(\Delta, \theta)}, \]

where the thermodynamic potential is

\[ \beta \Omega(\Delta, \theta) = \int_0^\beta d\tau \int d^2r \frac{N \Delta(\tau, r)^2}{V(r)} - Tr ln G^{-1}, \]

\( N \) is number of the layers. The Nambu spinor Green function \( G \) can be expressed as

\[ G^{-1} = G^{-1} - \Sigma, \]

where \( G^{-1} \) is a part of the inverse Green’s function which does not depend on the order parameter phase:

\[ G^{-1}_{j_1,j_2}(\tau_1, \tau_2, r_1, r_2) = \langle \tau_1, r_1, j_1 | G^{-1} | \tau_2, r_2, j_2 \rangle \]

\[ = \delta_{j_1,j_2} \delta(r_1 - r_2) \delta(\tau_1 - \tau_2) \left[ -\partial_{\tau_1} - \tau_2 \left( -\frac{\nabla^2 r_1}{2m} + 2t - \mu \right) \right] \]

\[ -\delta_{j_2,j_1 \pm 1} \delta(r_1 - r_2) \delta(\tau_1 - \tau_2) \tau_z t_z + \delta_{j_1,j_2} \tau_x \Delta(\tau_1 - \tau_2, r_1 - r_2). \]

The self-energy \( \Sigma \) is the sum of the parts which come from the in-plane and inter-plane order parameter phase phase interaction \( \Sigma^\parallel \) and \( \Sigma^\perp \), respectively:

\[ \Sigma = \Sigma^\parallel + \Sigma^\perp, \]

where

\[ \Sigma^\parallel_{j_1,j_2}(\tau_1, \tau_2, r_1, r_2) = \langle \tau_1, r_1, j_1 | \Sigma^\parallel | \tau_2, r_2, j_2 \rangle \]

\[ = \delta_{j_1,j_2} \delta(r_1 - r_2) \delta(\tau_1 - \tau_2) \left[ i \tau_z \frac{\partial_{\tau_1} \theta_{j_1}(\tau_1, r_1)}{2} - i m \nabla_{r_1} \theta_{j_1}(\tau_1, r_1) \right] \]

\[ + \frac{\tau_z}{8m} \left( \nabla_{r_1} \theta_{j_1}(\tau_1, r_1) \right)^2 - i m \nabla_{r_1} \theta_{j_1}(\tau_1, r_1) \nabla_{r_1} \]

and

\[ \Sigma^\perp_{j_1,j_2}(\tau_1, \tau_2, r_1, r_2) = \langle \tau_1, r_1, j_1 | \Sigma^\perp | \tau_2, r_2, j_2 \rangle \]

\[ = -\delta_{j_2,j_1 \pm 1} \delta(r_1 - r_2) \delta(\tau_1 - \tau_2) \tau_z t_z (1 - \exp[-i \tau_z (\theta_{j_1}(\tau_1, r_1) - \theta_{j_2}(\tau_2, r_2))]). \]

The potential term of the thermodynamic potential is

\[ \beta \Omega_{pot}(\Delta) = \int_0^\beta d\tau \int d^2r \frac{N \Delta(\tau, r)^2}{V(r)} - Tr ln G^{-1}. \]

and the kinetic term can be expanded in powers of the self-energy \( \Sigma \):

\[ \beta \Omega_{kin}(\Delta, \theta) = Tr \sum_{n=1}^\infty \frac{1}{n} (G \Sigma)^n. \]
III. THE BKT TRANSITION IN THE 2D CASE

Let us begin with the case when there is no inter-plane coupling: $t_z = 0$. In this case the behavior in each plane is independent and the system undergoes the BKT transition. Let us assume that the order parameter phase fluctuations are small. In this case to get the thermodynamic potential up to the second order in $\nabla \theta$ we neglect all the terms in (20), except $n = 1, 2$. Also we neglect the time dependence of $\theta$ and the second derivative $\nabla^2 \theta$.

The effective potential in this case has the following structure (see, for example [8]):

$$\Omega(\Delta, \theta) = \Omega_{pot}(\Delta) + \frac{J_\parallel}{2} \int d^2 r (\nabla \theta)^2,$$

where

$$J_\parallel = \int \frac{d^2 k d k_z}{(2\pi)^3} \left( n_f(k) \frac{\xi(k)}{4m} - \frac{1}{16m^2 T} n_f(k) \frac{k^2}{\cosh^2(\sqrt{\xi(k)^2 + \Delta(k)^2}/2T)} \right),$$

(13)

and the momentum distribution function $n_f(k)$ is

$$n_f(k) = 1 - \tanh \left( \frac{\sqrt{\xi(k)^2 + \Delta(k)^2}}{2T} \right) \frac{\xi(k)}{\sqrt{\xi(k)^2 + \Delta(k)^2}}.$$

(14)

Free fermion spectrum $\xi(k)$ in (13) and (14) is defined by (2) at $t_z = 0$ in this case.

The minimization of the effective potential (12) at $\nabla \theta = 0$ with respect to the superconducting order parameter $\Delta(k)$ leads to the standard gap equation:

$$\Delta(p) = \int \frac{d^2 k d k_z}{(2\pi)^3} \frac{\Delta(k)}{2 \sqrt{\xi(k)^2 + \Delta(k)^2}} \tanh \left( \frac{\sqrt{\xi(k)^2 + \Delta(k)^2}}{2T} \right) V(p, k).$$

(15)

The minimization of the effective potential at $\nabla \theta = 0$ with respect to the chemical potential $\delta\Omega_{pot}/\delta \mu = -vn_f$ ($v$ is the volume of the system) gives the equation which connects $\mu$ and the particle density $n_f$ in the system, or the 2D Fermi energy $e_F = \pi n_f/m$:

$$n_f = \int \frac{d^2 k d k_z}{(2\pi)^3} n_f(k),$$

(16)

where the momentum distribution function $n_f(k)$ is defined in (14).

To search the solutions with different angular momenta $l$ of the pairs, we assume that the interaction potential has the following form:

$$V(p, k) = V \cos(l\varphi_p) \cos(l\varphi_k).$$

(17)
Below we shall use dimensionless coupling parameter $G = mV/(2\pi)$ for the numerical calculations.

In the case of the interaction $|17|$ the gap depends only on the momentum direction:

$$\Delta(p) = \Delta_l \cos(l\varphi_p),$$

where $\Delta_l$ is the amplitude of the superconducting gap in the case of the pair angular momentum equal to $l$. The solution of the gap equation together with the number equation at $\Delta_l = 0$ give the critical temperature of the mean-field superconducting transition $T_\Delta \equiv T_c^{MF}$ on the charge carrier density $n_f$. The solution of the equation

$$T = \frac{\pi}{2} J_\parallel(\Delta_l, \mu, T)$$

(18)

together with the gap equation and the number equation give the dependence of the critical temperature of the BKT-transition on the charge carrier density $n_f$. Equation $|18|$ is obtained by mapping $|12|$ on the corresponding thermodynamic potential of the 2D spin XY-model.

As it follows from the system $|15|$, $|16|$ and $|18|$, the solution for the $T_\Delta$ and $T_{BKT}$ do not depend on $l$ when $l \neq 0$ for the case of the simple interaction potential $|17|$. This follows from the fact that the $l$-dependence of the integral is only as $\cos^2(l\varphi)$ and from the identity

$$\int_0^{2\pi} \frac{d\varphi}{(2\pi)} F[\cos^2(l\varphi)] = \int_0^{2\pi} \frac{d\varphi}{(2\pi)} F[\cos^2(\varphi)] ,$$

where $F[\cos^2(l\varphi)]$ is an arbitrary function without singularities, and $l$ is an arbitrary non-zero integer number. Therefore it is necessary to analyze the solutions with $l = 0$ and $l = 1$.

The phase diagram of the system in the 2D case is presented in the Fig.1. The temperature $T_\Delta$ is much higher in the $s$-channel. However, $T_{BKT} \simeq e_F/8$ in both channels at small carrier density. This result can be easily obtained analytically from $|13|$ and $|18|$.

The doping dependence of the $T_{BKT}$ in the cases of $l = 0$ and $l \neq 0$ is presented in Figs. 2 and 3, correspondingly. The relation $T_{BKT} \simeq e_F/8$ holds up to higher values of the carrier density in the $s$-channel at fixed value of coupling. It means that the local pairs are bounded tighter in this case.
FIG. 1: Phase diagram of the 2D system in different pairing channels for the coupling parameter $G = 1$. The solid lines are $T_{\Delta}$ (the upper curve) and $T_{BKT}$ for the $s$-wave pairing channel. The dashed lines are the corresponding curves for the case $l \neq 0$. Here and below all quantities are normalized on the 2D free electron bandwidth $W$.

IV. TRANSITION IN THE CASE OF COUPLED LAYERS

Let us consider a system of coupled layers. The self-energy, proportional to the inter-layer coupling can be written as

$$\Sigma^\perp = t_z \tau_z \Sigma^\perp_1 + t_z \Sigma^\perp_2,$$

where

$$\Sigma^\perp_{1, j_1 j_2} (\tau_1, \tau_2, r_1, r_2) = \langle \tau_1, r_1, j_1 | \Sigma^\perp_1 | \tau_2, r_2, j_2 \rangle$$

$$= -\delta_{j_2, j_1 \pm 1} \delta (r_1 - r_2) \delta (\tau_1 - \tau_2) \cos (\theta_{j_1} (\tau_1, r_1) - \theta_{j_2} (\tau_2, r_2)),$$

$$\Sigma^\perp_{2, j_1 j_2} (\tau_1, \tau_2, r_1, r_2) = \langle \tau_1, r_1, j_1 | \Sigma^\perp_2 | \tau_2, r_2, j_2 \rangle$$

$$= \delta_{j_2, j_1 \pm 1} \delta (r_1 - r_2) \delta (\tau_1 - \tau_2) \sin (\theta_{j_1} (\tau_1, r_1) - \theta_{j_2} (\tau_2, r_2)).$$

Similarly to the 2D case, we assume that the phase of the order parameter changes slowly in the inter-layer direction. Therefore, the thermodynamic potential can be calculated up to the second order in $(\theta_j - \theta_{j \pm 1})$:

$$\Omega^\perp_{\text{kin}} = t_z TTr (G \tau_z \Sigma^\perp_1) + \frac{t_z^2}{2} TTr (G \Sigma^\perp_2 G \Sigma^\perp_2). \quad (19)$$
FIG. 2: The doping dependence of $T_{BKT}$ at $\ell \neq 0$ and different coupling parameters: $G = 0.5$ (dash-dotted line), $G = 1.0$ (dotted line) and $G = 2.0$ (dashed line). The solid line is the function $T_{BKT} = e_F/8$.

The terms proportional to $\Sigma^\parallel \Sigma^\perp$ and $\Sigma^\perp_1 \Sigma^\perp_2$ are zero due to reflection symmetry in $z$-direction.

To map the system on the quasi-2D XY-model with the nearest neighbor interaction we need to obtain

$$\Omega_{kin} = \frac{J^\parallel}{2} \sum_j \int d^2r (\nabla \theta_j)^2 + J_z \sum_j (1 - \cos(\theta_j - \theta_{j-1})).$$

This dependence comes from the first term in (19). The second term in (19) is proportional to $\sin(\theta_j - \theta_{j+1})\sin(\theta_j - \theta_{j+1})$, what is equivalent to the XY-model with the next nearest neighbor and next next nearest neighbor interactions. Therefore we neglect this term since it is of a higher order ($\sim t_z^2$) on the inter-layer hopping with respect to the first term (which is $\sim t_z$). However, if the coupling $t_z$ is not small this term can lead for important physical consequences (see for example an analysis for the 2D case [28]). Thus, the parameter $J_z$ is

$$J_z = t_z \int \frac{d^2k d k_z}{(2\pi)^3} n_f(k) \cos(ak_z).$$

Now we have obtained the kinetic part of the thermodynamic potential $\Omega_{kin}$ in the case of slowly fluctuating phase of the order parameter. This function is given by (20), where the parameters $J^\parallel$ and $J_z$ are given by (13) and (21). Similarly to (21), an additional integration over $k_z$ must be performed in (13).
The effective action (19) was studied in [22, 23, 24, 25] in the case when the parameters $J_{||}$ and $J_z$ are considered independent on the fermion Hamiltonian parameters. It was shown [23], that there is only one phase transition in such a system at $T_c$ which is bigger than the temperature of the BKT transition in the case of non-coupled layers $T_{BKT} = \left(\pi/2\right)J_{||}$. In the case of small coupling $T_c \simeq T_{BKT}$ and when $t_z$ is increasing to the inter-plane hopping value, $T_c$ is approaching to the value $T_{BKT} = 4\pi J_{||} \simeq 8T_{BKT} = T_f$ of the fluxon transition, when the inter-layer order starts to take place.

More precisely, the following expression for the effective free energy was considered

$$F = \frac{1}{8\pi} \int d^2r dz \{(\nabla \times A)^2 + \frac{1}{\lambda_e} \sum_j \left[\frac{\phi_0}{2\pi} \nabla \theta_j(r) - A(r, z)\right]^2 \delta(z - jd)\}$$

$$- \frac{J_z}{\xi_0^2} \int d^2r \cos[\theta_j(r) - \theta_{j-1}(r)] - \frac{2\pi}{\phi_0} \int_{(j-1)d}^{jd} A_z(r, z') dz' - E_c \sum_{j, r} s_j^2(r), \quad (22)$$

where $A(r, z)$ is the vector potential, $\phi_0 = hc/2e$ is the flux quantum, $E_c$ is the loss of the condensation energy in a volume $\xi_0^2d$, $\xi_0$ is the in-plane correlation length, $d_0$ is the thickness of each layer, and $d(> d_0)$ is the inter-layer distance. The field $s_j(r)$ describes vorticity of the lattice, $s_j(r) = 1$ if the vortex is present at the point, and $s_j(r) = 0$, otherwise. The length scale $\lambda_e$ is connected with the London in-plane penetration length $\lambda_L$ as $\lambda_e = \lambda_L^2/d_0$. It was shown by a renormalization group study [24, 25] that in a physical case $\lambda_e \gg d_0$ the
self-consistent equation which describes the dependence of the critical temperature $T_c$ on the free energy parameters $\xi_0$ has the form

$$
T_c \simeq \frac{\tau [E_c + (\tau/8)\ln(T_c/J_z)]}{E_c + \tau \ln(T_c/J_z)},
$$

(23)

where $\tau = \phi_0^2/4\pi e^2$ is connected with the BKT transition temperature as $\tau = 8T_{BKT}$.

The comparison of the expressions (22) and (20), gives the next self-consistent equation for the critical temperature $T_c$, which follows from (23):

$$
T_c \simeq 4\pi J\| \frac{E_c + (\pi J\|/2)\ln(T_c/J_z)}{E_c + (4\pi J\|)\ln(T_c/J_z)},
$$

(24)

where the in-plane correlation length $\xi_0$ is absorbed in the parameter $J_z$ (i.e. $t_z(a/\xi_0)^2 \rightarrow t_z$).

The parameter $E_c$ actually should be renormalized by including the influence of the inter-layer coupling on the vortex system $\xi_0$. It is considered here as a model parameter, which should be found experimentally, in particular its doping dependence should be taken into account. For calculation we use the value $E_c = 0.01W$ (for estimation of $E_c$ based on an amplitude dependent Ginzburg-Landau theory, see for example [29]).

It is interesting to note, that in the limit of very small carrier densities, when $J\| \simeq e_F \rightarrow 0$, the analytical solution for $T_c$ can be obtained $T_c \simeq 4\pi J\| \simeq e_F$. This is different from the one layer case when $T_c = T_{BKT} \simeq e_F/8$, independently on the pair angular momentum $l$. However, the region of extremely low carrier densities is not interesting from physical point of view.

To find the critical temperature $T_c$ one needs to solve the system of equations (15), (16) and (24) with functions $J_m(\mu,T,\Delta(T))$ and $J_m(\mu,T,\Delta(T))$ defined in (13) and (21). The numerical solutions show that $T_c < T_{BKT}$ at small carrier densities in the case of large values of the inter-layer hopping $t_z$ and not very strong coupling $G$ (Fig.4). It means that the dependence of the parameters $J\|$ and $J_\perp$ on coupling, carrier density and temperature leads to the non-trivial relation between $T_c$ and the 2D critical temperature $T_{BKT}$ at some values of model parameters, different from $T_c > T_{BKT}$, as it was predicted for the case of fixed $J\|$ and $J_\perp$.[22, 23, 24, 25].

In general, $T_c$ is growing with the inter-layer coupling $t_z$ (Figs.4,5). However, in the case of small carrier density the critical temperature is decreasing with $t_z$ growth when $l \neq 0$ (Fig.4, in the $l = 0$ case this effect takes place at smaller coupling $G$). It can be explained as a consequence of the fact, that the density of states on the Fermi level $\rho(e_F)$ at small
FIG. 4: The doping dependence of $T_c$ of the layered system in the case $l \neq 0$ at different values of the inter-layer hopping and $G = 1.0, E_c = 0.01$. The solid line is the corresponding 2D temperature $T_{BKT}$. The insert is the inter-layer hopping dependence of $T_c$ at $G = 1$ and $e_F = 0.05$.

FIG. 5: The same as in Fig.4 for the case $l = 0$.

carrier densities is decreasing when system tends to become three dimensional with $t_z$ growth ($\rho(e_F) \simeq \sqrt{e_F}$ in the 3D case and $\rho(e_F) = const$ in the 2D case). On the other hand, the role of the term $\sim t_z^2$ must be studied in addition at rather large values of $t_z$, when inter-layer hopping becomes of order of the intra-layer hopping, i.e. $t_z \simeq 0.1W$ (see again [28]).
V. CONCLUSIONS

To summarize, the doping dependence of the superconducting critical temperature of layered superconductors on the charge carrier density has been studied in cases of different angular momentum of the pairs $l$, coupling and inter-layer hopping. It has been shown that the critical temperature $T_c$ is smaller then the 2D critical temperature $T_{BKT}$ at some values of the model parameters, contrary to the XY-model with the parameters $J_\parallel$ and $J_\perp$ which do not depend on carrier density $n_f$, inter-particle coupling $V$ and the temperature of the system $T$. In particular, at small carrier densities $T_c \neq e_F/8$, contrary to the dependence of $T_{BKT}$ in the 2D case. The critical temperature $T_c$ is growing with $t_z$, except the case of non-zero angular momentum of the pairs at small carrier densities.

At the same time some questions are remained unresolved. In particular, the behavior of the system when the inter-layer coupling $t_z$ is not very small has to be studied and the doping dependence of the vortex condensation energy should be taken into account. This problems are planned to be studied in the future.

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