Einstein’s concept of a clock and clock paradox

Wang Guowen
College of Physics, Peking University, Beijing, China
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Abstract

A geometric illustration of the Lorentz transformations is given. According to similarity between space and time and correspondence between a ruler and a clock, like the division number in a moving ruler, the tick number of a moving clock is independent of its relative speed and hence invariant under the Lorentz transformations. So the hand of the moving clock never runs slow but the time interval between its two consecutive ticks contracts. Thus it is Einstein’s concept of slowing of the hands of moving clocks to create the clock paradox or twin paradox. Regrettably, the concept of the clock that Einstein retained is equivalent to Newton’s concept of absolute time that he rejected. This is a blemish in Einstein’s otherwise perfect special relativity.

Key words: special relativity, time contraction, time dilation, clock paradox, twin paradox, light clock

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1 Introduction

As well known, the concept of slowing of a moving clock as physical interpretation of relative time was first stated in Einstein’s 1905 paper [1] and the lifetime dilation of moving organisms mentioned in his 1911 paper [2]. This kind of concept has created the so-called clock paradox or twin paradox. The twin paradox concerns a pair of twins, one of which aboard a rocket flies into space for years at a speed fractionally less light speed and the other remains home. According Einstein’s concept, the travelling twin will be much younger than the other on returning. This seems to be a paradoxical consequence since motion is relative. As well known, Einstein “solved the paradox by taking into account the influence on clocks of the gravitational field, which is relative to (the accelerated system) $K’$.” [3] He meant that the acceleration and deceleration effects and the gravity effect are the causes creating an asymmetry of aging of the twins. The paradox has even led a minute minority of physicists, for example, Herbert Dingle [4], to argue that Einstein’s relativity must be false. One of the long debated questions is whether general relativity is required to solve the paradox. If special relativity is tenable as the overwhelming majority of physicists believes, what mistake might be ever made by Einstein?
In search of an answer to the question, the present author thinks of Einstein’s discussion of time measurement. He stated: “one determines the rate of the moving clock by comparing the position of the hands of this clock with the positions of the hands of other clocks at rest and if it moves with a velocity approximating the velocity of light, the hands of the clock would move forward infinitely slowly.” [2] He asserted also: “By definition, to measure the time interval during which an event takes place means to count the number of time periods indicated by the clock from the beginning till the end of the event in question.” [5] Clearly, his way to measure time is identical with Newton’s. It seems that the concept of the clock that he retained is equivalent to Newton’s concept of absolute time that he rejected. The author discovers that a geometric illustration of the Lorentz transformations and a graphic display of similarity between divisions of a ruler and ticks of a clock can make the matter clear.

2 Geometric illustration of the Lorentz transformations

Equivalently to Minkowski’s formulation of special relativity, we consider two four-dimensional systems \((x, y, x, w)\) and \((x', y', x', w')\) in relative translational motion and assume that the origin of the latter moves at the light speed \(c\) in accordance with the following equation:

\[
c^2 t^2 = x_0^2 + y_0^2 + z_0^2 + w_0^2, \quad w_0 = ct_0
\]

where \(\tau_0\) is the proper time. This equation expresses mathematically that a free object moves at the maximum speed \(c\) at which it can travel in the four dimensions \((x, y, x, w)\) where the variable \(w\) is much like the spatial ones, \(x, y\) and \(z\) [6]. From Eq.(1) we can draw a figure shown as Fig.1 to illustrate the Lorentz transformations. In this figure only two-dimensional systems \((x, w)\) and \((x', w')\) are drawn for the purpose of this article and \(x_0 = vt\) is given. From the projective relations of the coordinates in the figure, we can write intuitively with ease the Lorentz transformations for the points on the \(x'\) axis and an identity:

\[
x' = \frac{x - vt}{\gamma}, \quad \gamma = \cos \phi = \sqrt{1 - v^2/c^2}
\]

\[
t = \frac{\tau + x'v/c^2}{\gamma}
\]

\[
x = \frac{x' + vt}{\gamma}
\]

\[
\tau = \frac{t - xv/c^2}{\gamma}
\]

\[
c^2 t^2 + x'^2 = c^2 \tau^2 + x^2
\]

The last equation can be changed into

\[
c^2 t^2 - x^2 = c^2 \tau^2 - x'^2
\]
Thus, in general, including $y$ and $z$, we have
\[ c^2 t^2 - x^2 - y^2 - z^2 = c^2 \tau^2 - x'^2 - y'^2 - z'^2 \] (8)
For a photon or a light pulse, we have
\[ c^2 t^2 - x^2 - y^2 - z^2 = c^2 \tau^2 - x'^2 - y'^2 - z'^2 = 0 \] (9)
which describes the principle of constancy of light speed. The $\tau$ is often written as $t'$ in the literature of relativity.

Let a ruler be located on the $x'$ axis. From Fig.1 we can see directly the length contraction of the moving ruler:
\[ \rho = L\gamma = L \cos \phi \] (10)
Obviously the length contraction of the ruler is a reciprocal projection effect between the two equivalent inertial systems in relative motion and the division number of the ruler is independent of the relative speed $v$ and hence invariant under the Lorentz transformations. The length interval between two neighboring divisions of the moving ruler at the speed $v$ in the $x$ direction contracts by the factor $\gamma$. Similarly, let a clock be located at the origin of the moving system and synchronized with one clock at rest in the stationary system when the origins of the two systems are coincide at $t = 0$. For the moving clock, in his 1905 paper, Einstein derived from Eq.(3) the time contraction equation [1]
\[ \tau_0 = t\gamma = t \cos \phi \] (11)
which is similar to Eq.(10) in form. This contraction can also be seen directly in Fig.1. Its inverse process is referred to as dilation of the time $\tau_0$. According to similarity between space and time and correspondence between a ruler and a clock, it is evident that the ticks of the moving clock are nothing but the correspondences of the divisions of
the ruler. Obviously the length contraction of the ruler is merely attributed to contraction of the length interval between its two neighboring divisions, so, similarly, the time contraction of the moving clock is merely attributed to contraction of the time interval between its two consecutive ticks and the number of the ticks is also independent of the speed $v$ and hence invariant under the Lorentz transformations. But we are unable to employ an existing clock to record the time $\tau$ which depends on both the speed $v$ and the location $x'$. Therefore, in order to record it, we need devise an alternative clock.

3 Time $\tau$ and $\tau$-clock

The clocks that we can devise to indicate the time $\tau$ are shown in Fig.2. In the figure, on the largest circles on the clock faces, the arc length swept out by the hand represents time $t$ in the stationary system. The $\gamma$ values (0, 0.25, 0.5 and 0.75) on the clock’s hand represent the different relative speeds. The readings in seconds on the circle to which the sliding arrow points represent the duration of an event in the moving system judged by an observer in the stationary system. For example, for the case of relative speed 0.866$c$, namely, $\gamma = 0.5$, Fig.2(a) shows that the clock at the origin of the moving system ticks 0.5 seconds and Fig.2(b) and Fig.2(c) show that the clocks a light-second away from the origin on the negative and the positive $x'$ half-axes.

![Figure 2: The $\tau$-clocks moving in the $x$ direction, for example, running at $\gamma = 0.5$: (a) at the origin of the moving system, (b) and (c) at the positions a light-second away from the origin on the negative and the positive $x'$ half-axes.](image)

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The counterclockwise and clockwise angular displacements of the readings $\tau$ depend on the term $x'/c^2$ in Eq.(3). Thus this kind of clock has dual functions for recording both $t$ and $\tau$ simultaneously and hence meets the relativistic requirement of the equivalence of reciprocal “time measurements” by two different inertial observers. This kind of clock may be called as “$\tau$-clock” to distinguish it from traditional clocks which record only the time in the stationary system or systems at very low relative speed. With $\tau$-clocks both of the twins can think the other’s one ticks a less time period. No paradox! The $\tau$-clock is useless in practice but essentially important for properly understanding special relativity.
4 Behavior of a light clock

The discussion of the behavior of a moving light clock is also significant for understanding special relativity. Imagine a light clock which is composed of a pair of parallel mirrors reflecting light pulses separated by the distance \( d = cT_p/2 \) that light travels in half a second, here \( T_p = 1 \) second. There are a source of light pulses and a light detector on one mirror for producing every tick signal. When the vertical light clock moves at uniform speed \( v \) in the horizontal \( x \) direction, according to Eq.(9), we have

\[
 c^2\left(\frac{T_p}{2}\right)^2 - v^2\left(\frac{T_p}{2}\right)^2 - y^2\left(\frac{T_p}{2}\right) = 0 \tag{12}
\]

\[
 c^2\left(\frac{\tau_p}{2}\right)^2 - y^2\left(\frac{\tau_p}{2}\right) = 0 \tag{13}
\]

where \( y' = y \). Thus, different from the real length \( d \), the “relative path” of the pulse between the top and bottom mirrors in the \( y \) direction is

\[
 d' = y\left(\frac{T_p}{2}\right) = c\gamma T_p/2 = c\tau_p/2 = \gamma d = d \cos \phi \tag{14}
\]

that satisfies the equation of the right triangle

\[
 d'^2 + v^2 = c^2 \tag{15}
\]

Here \( d' \) is the contracted light path in the \( y \) direction judged by an observer in the stationary system. It is necessary to emphasize again that the time contraction of the moving light clock is merely attributed to the contraction of the period between its two consecutive ticks. This means that the light pulse hits the top mirror when the relative light path equals \( d' \) and the detector produces a tick signal when it gets to bottom mirror. Fig.3 illustrates the behavior of the light clock by relative travel paths of the light pulse for different \( \gamma \) values. Thus the explanation of time dilation by the light clock in textbooks, such as Feynman Lectures on Physics volume 1 [7], is wrong because of ignoring the relativistic contraction of the travel path of the light pulse.

![Figure 3: Illustration of the behavior of the light clock by relative travel paths of the light pulse for different \( \gamma \) values.](image)


5
5 Conclusion

We have seen that, like the division number in a moving ruler, the tick number of a moving clock is independent of its relative speed and hence invariant under the Lorentz transformations. So the hand of the moving clock never runs slow but the time interval between its two consecutive ticks contracts. Thus it is Einstein’s concept of slowing of the hands of moving clocks to create the clock paradox or twin paradox. Regrettably, his way to measure time $\tau$ is identical with Newton’s. In other words, the concept of the clock that he retained is equivalent to Newton’s concept of absolute time that he rejected.

Similar to the clock’s ticking, the heart beating of the travelling twin never runs really slow but according to special relativity the beating period between two consecutive beats contracts. Since the human lifetime may be considered as being basically determined by the total number of the heart beats, no one of the twins will become younger than the other when they reunite. Thus, the general relativity is not required to resolve the twin paradox. It is Einstein’s assertion of slowing of the hands of moving clocks to create the paradox and hence make many physicists believe slowing of all physical processes with the increased speed, including chemistry reactions, nuclear reactions, life process and others. This is a blemish in Einstein’s otherwise perfect special relativity.

Finally, the author would say that if the present conclusion is correct, it will prohibits us from believing the assertion that some related experiments [8,9] have confirmed the Einstein’s prediction of slowing of moving clocks that creates the clock paradox.

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