On partitions avoiding right crossings

Sherry H. F. Yan, Yuexiao Xu
Department of Mathematics, Zhejiang Normal University, Jinhua 321004, P.R. China
huifangyan@hotmail.com

Abstract. Recently, Chen et al. derived the generating function for partitions avoiding right nestings and posed the problem of finding the generating function for partitions avoiding right crossings. In this paper, we derive the generating function for partitions avoiding right crossings via an intermediate structure of partial matchings avoiding 2-right crossings and right nestings. We show that there is a bijection between partial matchings avoiding 2-right crossing and right nestings and partitions avoiding right crossings.

Key words: Partial matchings, partitions, 2-right crossings, right nestings, right crossings.

AMS Mathematical Subject Classifications: 05A05, 05C30.

1 Introduction

This paper is concerned with the enumeration of set partitions that avoid right crossings. Recall that a partition $\pi$ of $[n] = \{1, 2, \ldots, n\}$ can be represented as a diagram with vertices drawn on a horizontal line in increasing order. For a block $B$ of $\pi$, we write the elements of $B$ in increasing order. Suppose that $B = \{i_1, i_2, \ldots, i_k\}$. Then we draw an arc from $i_1$ to $i_2$, an arc from $i_2$ to $i_3$, and so on. Such a diagram is called the linear representation of $\pi$. If $(i, j)$ is an arc in the diagram of $\pi$ with $i < j$, we call vertex $i$ an opener, and call vertex $j$ a transient if there is an arc $(j, k)$ for some integer $k$ such that $i < j < k$; otherwise, we call vertex $j$ a closer. A partial matching is a partition for which each block contains at most two elements. A partial matching without singletons is called a perfect matching.

A nesting of a partition is a pair of arcs $(i_1, j_1)$ and $(i_2, j_2)$ with $i_1 < i_2 < j_2 < j_1$. We call such a nesting a left nesting if $i_2 = i_1 + 1$. Similarly, we call it a right nesting if $j_1 = j_2 + 1$. A crossing of a partition is a pair of arcs $(i_1, j_1)$ and $(i_2, j_2)$ with $i_1 < i_2 < j_1 < j_2$ and we can define left crossing and right crossing analogously to how it was defined for nesting arcs. A neighbor alignment of a partition is a pair of arcs $(i_1, j_1)$ and $(i_2, j_2)$ such that $i_2 = j_1 + 1$.

Zagier [7] derived the generating function for perfect matchings avoiding left nestings and right nestings, which was introduced by Stoimenow [6]. Recently, Bousquet-Mélou et al. [1] constructed bijections between perfect matchings avoiding left nestings and right nestings and three other classes of combinatorial objects, that is, unlabeled $(2 + 2)$-free posets, permutations avoiding a certain pattern and ascent sequences. Dukes and Parviainen [4] presented a bijection between ascent sequences and upper triangular matrices whose non-negative entries...
are such that all rows and columns contain at least one non-zero entry. Claesson and Linusson \[3\] established a bijection between perfect matchings avoiding left nestings and permutations and showed that perfect matchings avoiding left crossings are equinumerous with perfect matchings avoiding left nestings. They also introduced the notion of \textit{k-left nesting} and conjectured that the number of perfect matchings of \([2n]\) without 2-left nestings is equal to the number of perfect matchings of \([2n]\) without left nestings and right nestings. This conjecture was confirmed by Levande \[5\]. Recall that two arcs \((i_1, j_1)\) and \((i_2, j_2)\) form a \textit{k-left nesting} if \(i_1 < i_2 < j_2 < j_1\) and \(i_2 - i_1 \leq k\). Similarly, we call it a \textit{k-right nesting} if \(i_1 < i_2 < j_2 < j_1\) and \(j_1 - j_2 \leq k\). Note that 1-left nesting is exactly a left nesting and 1-right nesting is exactly a right nesting. Similarly, we can define \textit{k-left crossing} and \textit{k-right crossing} analogously to how it was defined for nesting arcs. The left nestings, left crossings, right crossings, right nestings, neighbor alignments are called neighbor patterns.

Recently, Chen et al. \[2\] derived the generating functions for partial matchings avoiding neighbor alignments and for partial matchings avoiding neighbor alignments and left nestings. They obtained the generating function for partitions avoiding right nestings by presenting a bijection between partial matchings avoiding three neighbor patterns (left nestings, right nestings and neighbor alignments) and partitions avoiding right nestings. In general, the number of partitions of \([n]\) avoiding right nestings is not equal to the number of partitions of \([n]\) avoiding right nestings. In this paper, we derive the generating functions for partitions avoiding right nestings via an intermediate structure of partial matchings avoiding 2-right crossings and right nestings.

Denote by \(\mathcal{M}(n, k)\) the set of partial matchings of \([n]\) with \(k\) arcs. The set of partial matchings in \(\mathcal{M}(n, k)\) with no 2-right crossings and right nestings is denoted by \(\mathcal{P}(n, k)\). Let \(\mathcal{T}(n, k)\) be the set of partitions of \([n]\) with \(k\) arcs. Denote by \(\mathcal{CT}(n, k)\) the set of partition in \(\mathcal{T}(n, k)\) with no right crossings.

Denote by \(P(n, k)\) and \(\mathcal{CT}(n, k)\) the cardinalities of the sets \(\mathcal{P}(n, k)\) and \(\mathcal{CT}(n, k)\), respectively.

We obtain the generating function formula for the numbers \(P(n + k - 1, k)\) by establishing a bijection between partial matchings avoiding 2-right crossings and right nestings and a certain class of integer sequence. Moreover, we show that there is a correspondence between \(\mathcal{P}(n + k - 1, k)\) and \(\mathcal{CT}(n, k)\).

\textbf{Theorem 1.1} We have

\[
\sum_{n \geq 1} \sum_{k = 0}^{n-1} P(n + k - 1, k)x^ny^k = \sum_{n \geq 1} \frac{x^n(1 + xy)^{\binom{n}{2}}}{\prod_{k=0}^{n-1}(1 - (1 + xy)^{k+1}xy)}.
\] \hspace{1cm} (1.1)

\textbf{Theorem 1.2} We have

\[
\sum_{n \geq 1} \sum_{k = 0}^{n-1} \mathcal{CT}(n, k)x^ny^k = \sum_{n \geq 1} \frac{x^n(1 + xy)^{\binom{n}{2}}}{\prod_{k=0}^{n-1}(1 - (1 + xy)^{k+1}xy)}.
\] \hspace{1cm} (1.2)
Partial matchings avoiding 2-right crossings and right nestings

In this section, we construct a bijection between partial matchings avoiding 2-right crossings and right nestings and a certain class of integer sequences. As a consequence, we obtain the bivariate generating function for the number of partial matchings of \( n + k - 1 \) with \( k \) arcs and containing no 2-right crossing and right nestings.

Let \( x = x_1 x_2 \ldots x_n \) be an integer sequence. Denote by \( \max(x) \) the maximum element of \( x \). For \( 1 < i \leq n \), an element \( x_i \) of \( x \) is said to be a left-to-right maximum if \( x_i > \max(x_1 x_2 \ldots x_{i-1}) \). For \( n \geq 1 \), let \( S(n, k) \) be the set of integer sequences \( x = x_0 x_1 \ldots x_{n-1} \) with \( n - k \) left-to-right maxima satisfying that

- \( x_0 = 0 \);
- for all \( 1 \leq i \leq n - 1 \), \( 0 \leq x_i \leq \max(x_0 x_1 \ldots x_{i-1}) + 1 \);
- for all \( 0 \leq i \leq n - 1 \), if \( 0 \leq x_i < \max(x_0 x_1 \ldots x_{i-1}) \), then \( x_i < x_{i-1} \).

Denote by \( S(n) \) the set of such integer sequences with \( n \) left-to-right maxima. Denote by \( S(n, k) \) the cardinality of the set \( S(n, k) \). Let \( f(x, y) \) be the generating function for the numbers \( S(n, k) \). We derive the following generating function formula of \( f(x, y) \).

**Theorem 2.1**

\[
f(x, y) = \sum_{n \geq 1} \sum_{k=0}^{n-1} S(n, k) x^n y^k = \sum_{n \geq 1} \frac{x^n(1 + xy)^{(2)}}{\prod_{k=0}^{n-1} (1 - (1 + xy)^{k+1})}. \tag{2.1}
\]

**Proof.** Let \( x \) be a sequence in \( S(n) \) with \( n \) left-to-right maxima. It can be uniquely decomposed as \( 0 w_0 w_1 w_2 \ldots (n - 1) w_{n-1} \). For all \( 0 \leq i \leq n - 1 \), each \( w_i \) is a (possibly empty) integer sequence whose elements are nonnegative integers less than or equal to \( i \). For all \( 1 \leq i \leq n - 1 \), suppose that there are \( i_i \) occurrences of \( i \)'s in \( w_i \), then \( w_i \) reads as \( m_0 i_1 m_1 i_2 \ldots i_{i_i} \), where for all \( 0 \leq j \leq i_i \), the sequence \( m_j \) is a (possibly empty) decreasing sequence whose elements are nonnegative integers less than \( i \). Since \( w_0 \) is a (possibly empty) sequence of 0's. Hence, \( w_0 \) contributes \( \frac{1}{1-xy} \) to the generating function \( f(x, y) \), while for all \( 1 \leq i \leq n - 1 \), \( w_i \) contributes \( \frac{(1+xy)^i}{1-(1+xy)^i} \) to the generating function \( f(x, y) \). Furthermore, each left-to-right maximum contributes \( x \) to the generating function \( f(x, y) \). Summing over all \( n \geq 1 \), we derive the generating function formula (2.1). This completes the proof. \( \square \)

Now we proceed to construct a bijection between the set \( P(n + k - 1, k) \) and the set \( S(n, k) \).

**Theorem 2.2** There exists a bijection between the set \( P(n + k - 1, k) \) and the set \( S(n, k) \).
Proof. Let $M$ be a partial matching in $\mathcal{P}(n+k-1,k)$ of $[n+k-1]$ with $k$ arcs and containing no 2-right crossings and right nestings. We wish to generate a sequence $\alpha(M) = x_0x_1x_2\ldots x_{n-1}$ from $M$ recursively. First, we remove the labels of the closers and relabel the vertices of $M$ in the natural order. Denote by $\overline{M}$ the obtained diagram. For all $1 \leq i \leq n-1$, let $\mathcal{O}(i)$ be the set of openers of arcs whose closers are left to vertex $i$ of $\overline{M}$. Denote by $\mathcal{O}(n)$ the set of openers of all arcs of $\overline{M}$. Set $x_0 = 0$ and assume that we have obtained $x_{i-1}$. Now we proceed to generate $x_i$ in the following manner.

Case 1. If there is no closer immediately after vertex $i$, then let $x_i = \max(x_0x_1\ldots x_{i-1}) + 1$.

Case 2. If there is an arc whose closer is immediately after vertex $i$ and opener is labelled with $j$ and $j$ is the $m$-th minimum element of the set $[i] \setminus \mathcal{O}(i)$, then let $x_i = m - 1$.

We claim that for all $0 \leq i \leq n-1$, $\max(x_0x_1x_2\ldots x_i) = i - |\mathcal{O}(i+1)|$. We prove this claim by induction on $i$. Obviously, the claim holds for $i = 0$. So let $i \geq 1$ and assume that the claim holds for $i-1$. In Case 1, we have $\max(x_0x_1x_2\ldots x_i) = \max(x_0x_1\ldots x_{i-1}) + 1 = i - |\mathcal{O}(i)| + 1$. Since there exist no closers immediately after vertex $i$ in $\overline{M}$, we are led to $i - |\mathcal{O}(i+1)| = i - 1 - |\mathcal{O}(i)| + 1 = \max(x_0x_1x_2\ldots x_i)$. In Case 2, we have $0 \leq x_i \leq i - |\mathcal{O}(i)| - 1 = \max(x_0x_1\ldots x_{i-1})$, which yields that $\max(x_0x_1\ldots x_i) = \max(x_0x_1\ldots x_{i-1}) = i - 1 - |\mathcal{O}(i)|$. Since there is exactly one closer immediately after vertex $i$, we have $i - |\mathcal{O}(i+1)| = i - 1 - |\mathcal{O}(i)| = \max(x_0x_1x_2\ldots x_i)$. Therefore, the claim holds for all $0 \leq i \leq n-1$. This yields that $0 \leq x_i \leq \max(x_0x_1x_2\ldots x_{i-1}) + 1$ for all $1 \leq i \leq n-1$ and $\max(x_0x_1\ldots x_{n-1}) = n - 1 - |\mathcal{O}(n)| = n - 1 - k$, which implies that there are $n - k$ left-to-right maxima in $\alpha(M)$.

In order to prove that the obtained sequence $\alpha(M) \in \mathcal{S}(n,k)$, it remains to show that in the sequence $\alpha(M)$ if $0 \leq x_i < \max(x_1x_2\ldots x_{i-1})$, then $x_i < x_{i-1}$. According to the construction of the map $\alpha$, there exists an arc whose closer is immediately after vertex $i$. Suppose that the opener of this arc is labelled with $p$ in $\overline{M}$. Since $x_i < \max(x_1x_2\ldots x_{i-1}) = i - 1 - |\mathcal{O}(i)|$, we are led to $p < i$. Obviously, when $x_{i-1} = \max(x_1x_2\ldots x_{i-1})$, we have $x_i < x_{i-1}$. Now suppose that $x_{i-1} < \max(x_1x_2\ldots x_{i-1})$. According to the construction of the map $\alpha$, there exists an arc whose closer is immediately after vertex $i-1$. Suppose that the opener of this arc is labelled with $q$ in $\overline{M}$. Since there are no 2-right crossings in $M$, we have $p < q$. It is easy to check that $\mathcal{O}(i) = \mathcal{O}(i-1) \cup \{q\}$. From the construction of the map $\alpha$, we have $x_i < x_{i-1}$. Hence, the obtained sequence $\alpha(M) \in \mathcal{S}(n,k)$.

Conversely, given a sequence $x = x_0x_1x_2\ldots x_{n-1} \in \mathcal{S}(n,k)$, we wish to construct a partial matching of $[n+k-1]$ with $k$ arcs. First, we arrange $n-1$ vertices on a horizontal line and label them in the natural order. For $i = 1, 2, \ldots, n-1$, at step $i$, we insert at most one closer after vertex $i$ described as follows:

(i) if $x_i = \max(x_0x_1x_2\ldots x_{i-1}) + 1$, then we do nothing for vertex $i$;
(ii) if \(0 \leq x_i \leq \max(x_0, x_1, x_2, \ldots, x_{i-1})\), then we insert an arc whose opener is the \((x_i + 1)\)-th vacant vertex (from left to right) of the set \([i]\) and closer is immediately after vertex \(i\), where a vacant vertex is a vertex which is not joined to any arc.

Finally, we relabel the vertices in the natural order to get a partial matching.

By induction on \(i\) (\(0 \leq i \leq n - 1\)), it is easy to see that after step \(i\), the number of vacant vertices of the set \([i]\) is equal to \(\max(x_0, x_1, x_2, \ldots, x_{i-1})\). So (ii) is valid and the construction of the inverse map is well defined. Since there are exactly \(n - k\) left-to-right maxima in \(x\), the obtained partial matching is a matching of \([n + k - 1]\) with \(k\) arcs. Note that we add at most one closer immediately after each vertex \(i\) for all \(1 \leq i \leq n - 1\). Thus there is no two consecutive closers in the resulting partial matchings, which implies that there are no two right crossings and right nestings. Since when \(0 \leq x_i < \max(x_1, x_2, \ldots, x_{i-1})\), we have \(x_i < x_{i-1}\), there are no 2-right crossings. Hence, the resulting partial matching is in the set \(\mathcal{P}(n + k - 1, k)\). This implies that the above map \(\alpha\) is a bijection.

For example, let \(M = \{\{1, 6\}, \{2, 3\}, \{4, 12\}, \{5, 10\}, \{7, 8\}, \{9\}, \{11\}\} \in \mathcal{P}(12, 5)\). We obtain a diagram \(\overline{M}\) from \(M\) by removing the labels of the closers and relabeling the vertex of \(M\) in the natural order, see Figure 1. We have \(\mathcal{O}(1) = \emptyset, \mathcal{O}(2) = \emptyset, \mathcal{O}(3) = \{2\}, \mathcal{O}(4) = \{2\}, \mathcal{O}(5) = \{1, 2\}, \mathcal{O}(6) = \{1, 2, 5\}, \mathcal{O}(7) = \{1, 2, 4, 5\}\) and \(\mathcal{O}(8) = \{1, 2, 3, 4, 5\}\). So the corresponding sequence is \(\alpha(M) = 01120210\).

Conversely, given the integer sequence \(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7 = 01120210 \in S(8, 5)\), the construction of the corresponding partial matching is illustrated in Figure 2.

Combining Theorems 2.1 and 2.2, we are led to the generating function formula (1.1).

### 3 Partitions with no right crossings

In this section, we present a bijection between the set \(\mathcal{P}(n + k - 1, k)\) and the set \(\mathcal{CT}(n, k)\). As a result, we derive the generating functions for partitions avoiding right crossings. Before we present the bijection, we should recall the notion of 2-paths defined by Chen et al. [2]. Recall that a pair of two arcs \((i, j)\) and \((j, k)\) with \(i < j < k\) in the diagram of a partition is said to be a 2-path.

**Theorem 3.1** There is a bijection between the set \(\mathcal{P}(n + k - 1, k)\) and the set \(\mathcal{CT}(n, k)\). Moreover, this bijection transforms the number of neighbor alignments of a partial matching to the number of transients of a partition.
Proof. Let $M$ be a partial matching in $P(n + k - 1, k)$. we may reduce it to a partition by the following procedure.

- Change neighbor alignments to 2-paths from left to right until there are no more neighbor alignments, see Figure 3 for an illustration. More precisely, suppose that there is a neighbor alignment consisting of two arcs $(i, j)$ and $(j + 1, k)$. We change the arc $(j + 1, k)$ to $(j, k)$ and delete the vertex $j + 1$.

- Delete the singleton immediately after each closer, except for the last closer.

- Relabel the vertices in the natural order.

Denote by $P$ the obtained partition. We claim that there are no right crossings in the resulting partition $P$. Suppose that there are two crossing arcs $(i, j)$ and $(k, j + 1)$ with $i < k < j$ in the resulting partition. Since we have either deleted
an opener or a singleton immediately after each closer of $M$, except for the last
closer, the two crossing arcs forms a 2-right crossing in the partial matching $M$,
a contradiction. Hence, the claim is proved.

Conversely, given a partition $P' \in \mathcal{CT}(n, k)$, we wish to construct a partial
matching $M' \in \mathcal{R}(n+k-1, k)$. First, we add a vertex after each closer, except for
the last closer. Then, we change each pair of arcs $(i, j)$ and $(j, k)$ with $i < j < k$
into a neighbor alignment. Finally, we relabel the vertices in the natural order to
get the partial matching $M'$.

From the construction of the inverse map, we know that we have either added
a singleton or an opener after each closer of the partition $P'$. Hence, there are
no right crossings and right nestings in the resulting partial matching. We claim
that there are no 2-right crossings in the resulting partial matching $M'$. Suppose
that there are two crossing arcs $(i, j)$ and $(k, j+2)$ with $i < k < j$ in the resulting
partial matching $M'$. Then $j+1$ is either a singleton or an opener. From the
construction of the reverse map, the vertex $j+1$ is an added vertex. This implies
that the two crossing arcs is a right crossing in the partition $P'$, a contradiction.
Hence, the claim is proved. This implies that there is a bijection between the set
$\mathcal{P}(n+k-1, k)$ and the set $\mathcal{CT}(n, k)$. Obviously, this bijection transforms the
number of neighbor alignments of a partial matching to the number of transients
of a partition. This completes the proof.

Combining Theorems 1.1 and 3.1 we derive the generating function Formula
(1.2).

Figure 4 gives an example of a partial matching $M \in \mathcal{P}(12, 5)$ and its corre-
sponding partition $P \in \mathcal{CT}(8, 5)$.

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References

[1] M. Bousquet-Mélou, A. Claesson, M. Dukes, S. Kitaev, (2 + 2)-free posets,
ascent sequences and pattern avoiding permutations, J. Combin. Theory
Ser. A 117 (2010), 884–909.
[2] W.Y.C. Chen, N.J.Y. Fan and A.F.Y. Zhao, Partitions and partial matchings avoiding neighbor patterns, *Europ. J. Combin.*, to appear.

[3] A. Claesson, S. Linusson, *n*! matchings, *n*! posets, *Proc. Amer. Math. Soc.* **139** 2011, 435–449.

[4] M. Dukes, R. Parviainen, Ascent sequences and upper triangular matrices containing non-negative integers, *Electron. J. combin.* **17** (2010), R53.

[5] P. Levande, Two new interpretations of the Fishburn numbers and their refined generating functions, arXiv: math.CO /1006.3013.

[6] A. Stoimenow, Enumeration of chord diagrams and an upper bound for Vassiliev invariants, *J. Knot Theory Ramifications* **7** (1998), 93–114.

[7] D. Zagier, Vassiliev invariants and a strange identity related to the Dedeking eta-function, *Topology* **40** (2001), 945–960.