Fractional combinatorial Calabi flow on surfaces

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July 30, 2021

Abstract

Using the fractional discrete Laplace operator for triangle meshes, we introduce a fractional combinatorial Calabi flow for discrete conformal structures on surfaces, which unifies and generalizes Chow-Luo’s combinatorial Ricci flow for Thurston’s circle packings, Luo’s combinatorial Yamabe flow for vertex scaling and the combinatorial Calabi flow for discrete conformal structures on surfaces. For Thurston’s Euclidean and hyperbolic circle packings on triangulated surfaces, we prove the longtime existence and global convergence of the fractional combinatorial Calabi flow. For vertex scalings on polyhedral surfaces, we do surgery on the fractional combinatorial Calabi flow by edge flipping under the Delaunay condition to handle the potential singularities along the flow. Using the discrete conformal theory established in [21-22], we prove the longtime existence and global convergence of the fractional combinatorial Calabi flow with surgery.

MSC (2020): 52C26

Keywords: Combinatorial Ricci flow; Combinatorial Calabi flow; Discrete conformal structure; Circle packing; Vertex scaling

1 Introduction

Since Chow-Luo’s introduction of the combinatorial Ricci flow for Thurston’s circle packings on surfaces [5], combinatorial curvature flows have been important approaches for finding geometric structures on low dimensional manifolds, which have lots of applications in geometric topology and practical applications [6,24,50]. The combinatorial curvature flows that have been extensively studied on surfaces include Chow-Luo’s combinatorial Ricci flow for Thurston’s circle packings [5], Luo’s combinatorial Yamabe flow for vertex scaling [29], the combinatorial Calabi flow for discrete conformal structures [8,10,14,47,54] and others, which were invented and studied independently in the history. Recently, Chow-Luo’s combinatorial Ricci flow for Thurston’s circle packings and Luo’s combinatorial Yamabe flow for vertex scalings on surfaces have been unified as the combinatorial
Ricci flow in the framework of discrete conformal structures on polyhedral surfaces [51]. In this paper, we introduce a fractional combinatorial Calabi flow for discrete conformal structures on polyhedral surfaces, which unifies and generalizes the combinatorial Ricci flow and combinatorial Calabi flow for discrete conformal structures on surfaces.

Suppose \((M, T)\) is a triangulated connected closed surface with the triangulation \(T = \{V, E, F\}\), where \(V, E, F\) represent the sets of vertices, edges and faces respectively. For simplicity, we set \(|V| = N\) and use \(i, \{ij\}, \triangle ijk\) to denote the elements in \(V, E, F\) respectively. \(\varepsilon : V \rightarrow \{0, 1\}\) and \(\eta : E \rightarrow \mathbb{R}\) are two weights defined on the sets of vertices and edges respectively. \((M, V)\) ((\(M, V, \varepsilon\)) respectively) is called as a marked surface (weighted marked surface respectively). The following unified notion of discrete conformality was proposed by Glickenstein et al.

**Definition 1** ([18, 20, 39, 51]). A discrete conformal structure on a weighted triangulated surface \((M, T, \varepsilon, \eta)\) is a map \(f : V \rightarrow \mathbb{R}\) determining a discrete polyhedral metric \(l : E \rightarrow (0, +\infty)\) with

\[
l_{ij} = \sqrt{\varepsilon_i e^{2f_i} + \varepsilon_j e^{2f_j} + 2\eta_{ij} e^{f_i + f_j}} \quad (1.1)
\]

in the Euclidean background geometry and

\[
l_{ij} = \cosh^{-1} \left( \sqrt{(1 + \varepsilon_i e^{2f_i})(1 + \varepsilon_j e^{2f_j}) + \eta_{ij} e^{f_i + f_j}} \right) \quad (1.2)
\]

in the hyperbolic background geometry.

To determine a discrete polyhedral metric on \((M, T)\), the map \(l : E \rightarrow (0, +\infty)\) should satisfy the triangle inequalities for every face \(\triangle ijk \in F\). The discrete conformal structure in Definition 1 unifies and generalizes the existing special types of discrete conformal structures on surfaces, including the tangential circle packings (\(\varepsilon \equiv 1, \eta \equiv 1\)), Thurston’s circle packings (\(\varepsilon \equiv 1, \eta \in [0, 1]\)), inversive distance circle packings (\(\varepsilon \equiv 1, \eta \in (-1, +\infty)\)), the vertex scaling (\(\varepsilon \equiv 0, \eta \in (0, +\infty)\)) and others. The discrete conformal structures in Definition 1 could be defined for more general settings, including \(\varepsilon_i = -1\) for some vertices \(i \in V\) and the spherical background geometry. Please refer to [20, 47, 51] for more information on this. In this paper, we focus on the case that \(\varepsilon : V \rightarrow \{0, 1\}\) and the Euclidean and hyperbolic background geometry.

Set

\[
u_i = f_i \quad (1.3)
\]

for any vertex \(i \in V\) in the Euclidean background geometry and

\[
u_i = \begin{cases} 
  f_i, & \text{if } \varepsilon_i = 0, \\
  \frac{1}{2} \log \left| \frac{\sqrt{1 + e^{2f_i} - 1}}{\sqrt{1 + e^{2f_i} + 1}} \right|, & \text{if } \varepsilon_i = 1,
\end{cases} \quad (1.4)
\]
in the hyperbolic background geometry. For simplicity, we also call $u$ defined by (1.3) and (1.4) as a Euclidean and hyperbolic discrete conformal structure respectively. The combinatorial Ricci flow for discrete conformal structure on surfaces \cite{5,29,51} is defined to be
\[\frac{du_i}{dt} = -(K - \overline{K})_i,\] (1.5)
where $K : V \to (-\infty,2\pi)$ is the combinatorial curvature with $K_i$ defined as $2\pi$ less the cone angle at $i \in V$, $\overline{K} : V \to (-\infty,2\pi)$ is a fixed function representing a target combinatorial curvature with $\sum_{i \in V} \overline{K}_i = 2\pi\chi(M)$ in the Euclidean background geometry and $\sum_{i \in V} \overline{K}_i > 2\pi\chi(M)$ in the hyperbolic background geometry. The combinatorial Ricci flow (1.5) introduced in \cite{51} unifies and generalizes Chow-Luo’s combinatorial Ricci flow for Thurston’s circle packings \cite{5} and Luo’s combinatorial Yamabe flow for vertex scaling \cite{29} on surfaces. The combinatorial Calabi flow for discrete conformal structures on surfaces \cite{8–10,14,47,54} is defined to be
\[\frac{du_i}{dt} = \Delta(K - \overline{K})_i,\] (1.6)
where $\Delta = -L = -\left(\frac{\partial K}{\partial u}\right)$ is the discrete Laplace operator for the discrete conformal structures in Definition 1. The discrete Laplace operator $\Delta$ is proved \cite{47} to be negative definite (negative semi-definite with rank $N - 1$ in the Euclidean background geometry) under the structure condition
\[\varepsilon_s \varepsilon_t + \eta_{st} > 0, \quad \forall\{st\} \in E,\]
\[\varepsilon_q \varepsilon_{st} + \eta_{qs} \eta_{qt} \geq 0, \quad \forall\{qst\} \in F.\] (1.7)
See also \cite{2,29,48} for the special case of vertex scaling and \cite{25,30,44,46,52} for the special case of inversive distance circle packings. In the case of vertex scaling, Gu-Luo-Wu \cite{23}, Luo-Sun-Wu \cite{31} and Wu-Zhu \cite{43} recently introduced the following type combinatorial curvature flow
\[\frac{du_i}{dt} = \Delta^{-1}(K_0 - \overline{K})_i,\] (1.8)
to study the convergence of discrete uniformization conformal factor to the smooth uniformization conformal factor, where $K_0$ is the initial combinatorial curvature. Note that the definitions of the three different combinatorial curvature flows (1.5), (1.6) and (1.8) involve three linear operators $-Id$, $\Delta$ and $\Delta^{-1}$ acting on $K - \overline{K}$ or $K_0 - \overline{K}$, which can be written in a unified form $\Delta^n = -L^n = -\left(\frac{\partial K}{\partial u}\right)^n$ with $n = 0,1,-1$ respectively. This motivates us to consider the following fractional combinatorial Laplace operator $\Delta^s$ for $s \in \mathbb{R}$.

As the matrix $L = \left(\frac{\partial K}{\partial u}\right)$ is symmetric and positive definite \cite{47} under the structure condition (1.7), by the Gram-Schmidt orthonormalization, there exists an orthonormal
matrix $P \in O(N)$ such that

$$L = \left( \frac{\partial K}{\partial u} \right) = P^T \cdot \text{diag}\{\lambda_1, \cdots, \lambda_n\} \cdot P,$$

where $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ are nonnegative eigenvalues of $L = (\frac{\partial K}{\partial u})$. For $s \in \mathbb{R}$, the fractional discrete Laplace operator $\Delta^s$ of order $2s$ \cite{1} is defined to be the matrix

$$\Delta^s = -L^s = -P^T \cdot \text{diag}\{\lambda_1^s, \cdots, \lambda_n^s\} \cdot P,$$  \hspace{1cm} (1.9)

where $0^s$ is set to be $0$ for any $s \in \mathbb{R}$.

The fractional Laplace operators have been one of the most studied research topics in the present century since the work of Caffarelli-Silvestre \cite{4} and Silvestre \cite{37}. It has lots of applications in harmonic analysis, fractional calculus, functional analysis and probability. Especially, it gives a good description of the approximation of discrete jump models to continuous jump models in random walk \cite{32}. Using the geometric fractional Laplace operator, fractional curvature flows have been studied in Riemannian geometry. A typical example is Jin-Xiong’s fractional Yamabe flow \cite{26} used to study the fractional Yamabe problem. The fractional discrete Laplace operator in (1.9) is a discrete analogue of the classical fractional Laplace operator, which has been extensively studied in complex networks. Please refer to \cite{35} and the references therein. Motivated by the three combinatorial curvature flows (1.5), (1.6) and (1.8), we introduce the following fractional combinatorial Calabi flow for discrete conformal structures on surfaces.

**Definition 2.** Suppose $(M, \mathcal{T}, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure condition (1.7). $s \in \mathbb{R}$ is a constant. The fractional combinatorial Calabi flow of order $s$ for the discrete conformal structures on $(M, \mathcal{T}, \varepsilon, \eta)$ is defined to be

$$\frac{du_i}{dt} = \Delta^s(K - K)_i,$$  \hspace{1cm} (1.10)

where $\Delta^s$ is the fractional discrete Laplace operator defined by (1.9).

**Remark 1.** The fractional combinatorial Calabi flow (1.10) unifies and generalizes Chow-Luo’s combinatorial Ricci flow for Thurston’s circle packings, Luo’s combinatorial Yamabe flow for vertex scaling and the combinatorial Calabi flow for discrete conformal structures on surfaces. Specially, in the cases of $s = 0$ and $1$, the fractional combinatorial Calabi flow (1.10) is reduced to the combinatorial Ricci flow (1.5) and combinatorial Calabi flow (1.6) respectively, which are gradient flows. Note that for generic $s \in \mathbb{R}$, the fractional combinatorial Calabi flow (1.10) is not a gradient flow. In the case of $s = -1$, the fractional combinatorial Calabi flow (1.10) is slightly different from the combinatorial curvature flow.
where $K$ depends on $t$ in (1.10) and $K_0$ is fixed in (1.8). Note that the fractional discrete Laplace operator $\Delta^s$ is a nonlocal operator in general (exceptional cases include $s \in \mathbb{Z}_{\geq 0}$), because the eigenvalues $\lambda_1, \cdots, \lambda_n$ globally depend on the elements of the matrix $L = \left(\frac{\partial K}{\partial u}\right)$. This implies that the fractional combinatorial Calabi flow (1.10) is a nonlocal combinatorial curvature flow in general. The fractional combinatorial Calabi flow (1.10) is different from the combinatorial $p$-th Calabi flow defined by discrete $p$-Laplace operator in [7, 28]. Motivated by Definition 2, we further introduce a fractional combinatorial Calabi flow for decorated and hyper-ideal hyperbolic polyhedral metrics on 3-dimensional manifolds in [42], where the basic properties of the flow are also established.

Using a fractional discrete Laplace operator for tangential sphere packing metrics on 3-dimensional manifolds, a similar fractional combinatorial curvature flow for $s \geq 0$ was previously introduced in [13].

As the eigenvalues $\lambda_1, \cdots, \lambda_n$ of $L = \left(\frac{\partial K}{\partial u}\right)$ are Lipschitz functions of the discrete conformal structures, the short time existence for the solution of fractional combinatorial Calabi flow (1.10) follows by the standard theory in ordinary differential equations. For the fractional combinatorial Calabi flow (1.10), we further have the following result.

**Theorem 1.1.** Suppose $(M, \mathcal{T}, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure condition (1.7). If there exists a discrete conformal structure $\overline{u}$ with combinatorial curvature $\overline{K}$, then for any $s \in \mathbb{R}$, there exists a positive constant $\delta > 0$ such that if $||u(0) - \overline{u}|| < \delta$ ($\sum_{i \in V} u_i(0) = \sum_{i \in V} \overline{u}_i$ additionally in the case of Euclidean background geometry), the solution $u(t)$ of the fractional combinatorial Calabi flow (1.10) exists for all time and converges exponentially fast to $\overline{u}$.

For generic initial discrete conformal structures in Definition 1, the solution of fractional combinatorial Calabi flow (1.10) on $(M, \mathcal{T}, \varepsilon, \eta)$ may develop singularities, which correspond to the triangles degenerate along the flow (1.10) or the conformal factors tend to infinity. In the case of $s = 0$, the second author [47] proved the longtime existence and global convergence for the solution of fractional combinatorial Calabi flow (1.10) by extending the flow through singularities. For $s \neq 0$, we do not have any unified approach to handle the singularities along the fractional combinatorial Calabi flow (1.10) for the discrete conformal structures in Definition 1. However, for Thurston’s circle packings on surfaces, we show that the singularities never develop along (1.10) and prove the following longtime existence and global convergence for the solution of fractional combinatorial Calabi flow (1.10).

**Theorem 1.2.** Suppose $(M, \mathcal{T}, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon \equiv 1$ and $\eta : E \to (-1, 1]$ satisfying the structure condition (1.7).
In the Euclidean background geometry, if there exists a discrete conformal structure \( \pi \in \mathbb{R}^N \) with combinatorial curvature \( K \), then for any \( s \in \mathbb{R} \) and any initial value \( u(0) \in \mathbb{R}^N \) with \( \sum_{i \in V} u_i(0) = \sum_{i \in V} u_i \), the solution of fractional combinatorial Calabi flow (1.10) exists for all time and converges exponentially fast to \( \pi \).

In the hyperbolic background geometry, if there exists a discrete conformal structure \( \pi \in \mathbb{R}_{<0}^N \) with combinatorial curvature \( K \), then for any \( s \in \mathbb{R} \) and initial value \( u(0) \in \mathbb{R}_{<0}^N \), the solution of fractional combinatorial Calabi flow (1.10) exists for all time and converges exponentially fast to \( \pi \).

Remark 2. If \( s = 0 \) and \( \eta \in [0, 1] \), the result in Theorem 1.2 is obtained by Chow-Luo [5] for combinatorial Ricci flow of Thurston’s circle packings. If \( s = 1 \) and \( \eta \in [0, 1] \), the result in Theorem 1.2 is reduced to the results obtained in [8–10, 14] for combinatorial Calabi flow of Thurston’s circle packings. Following Thurston’s arguments in [40], word by word, one can replace the condition on the existence of \( u \) with combinatorial curvature \( K \) in Theorem 1.2 by some linear equalities and inequalities on \( K \), which characterize the image of the curvature map \( K \) for Thurston’s circle packings with \( \eta : E \to (-1, 1) \) satisfying the structure condition (1.7). One can also refer to [5, 11, 45] for this.

For vertex scaling, the fractional combinatorial Calabi flow (1.10) on \((M, T, \varepsilon, \eta)\) may develop singularities. In this case, we do surgery on the fractional combinatorial Calabi flow (1.10) by edge flipping under the Delaunay condition to handle the potential singularities along (1.10), which was first introduced in [21, 22] for combinatorial Yamabe flow. Here we give a brief description of the surgery in the Euclidean background geometry. For a piecewise linear metric (PL metric for short in the following) defined on \((M, T)\), it is said to satisfy the Delaunay condition if for every edge \( \{ij\} \in E \) we have \( \theta^ij_k + \theta^ij_l \leq \pi \), where \( \theta^ij_k, \theta^ij_l \) are inner angles facing the edge \( \{ij\} \in E \) in the triangles \( \triangle ijk \) and \( \triangle ijl \) respectively. Along the fractional combinatorial Calabi flow (1.10) on \((M, T, \varepsilon, \eta)\), suppose the Delaunay condition is satisfied for \( t \in [0, T] \) and there exists an edge \( \{ij\} \in E \) and a constant \( \varepsilon > 0 \) such that for any \( t \in (T, T + \varepsilon) \), we have \( \theta^ij_k + \theta^ij_l > \pi \). In this case, we replace the edge \( \{ij\} \in E \) by a new edge \( \{kl\} \) to get a new triangulation \( T' \) at the time \( t = T \) and then evolve the fractional combinatorial Calabi flow (1.10) with the PL metric on \((M, T')\) at \( t = T \) as the initial metric on \((M, T')\). This process is called surgery by edge flipping under the Delaunay condition. The surgery by edge flipping could also be defined for piecewise hyperbolic metrics (PH metrics for short in the following) on surfaces under the hyperbolic Delaunay condition, which is defined to be \( \theta^ij_k + \theta^ij_l \leq \theta^ik_j + \theta^il_j + \theta^ik \) for adjacent triangles \( \triangle ijk, \triangle ijl \in F \). With the help of discrete conformal theory established in [21, 22], we prove the following result for the fractional combinatorial Calabi flow with surgery in the case of vertex scaling.
Theorem 1.3. Suppose \((M, V)\) is a connected closed marked surface.

(a) In the Euclidean background geometry, if \(K : V \to (-\infty, 2\pi)\) satisfies \(\sum_{i \in V} K_i = 2\pi \chi(M)\), then for any \(s \in \mathbb{R}\) and any initial PL metrics on \((M, V)\), the solution of fractional combinatorial Calabi flow with surgery on \((M, V)\) for vertex scaling exists for all time and converges exponentially fast.

(b) In the hyperbolic background geometry, if \(K : V \to (-\infty, 2\pi)\) satisfies \(\sum_{i \in V} K_i > 2\pi \chi(M)\), then for any \(s \in \mathbb{R}\) and any initial PH metrics on \((M, V)\), the solution of fractional combinatorial Calabi flow with surgery on \((M, V)\) for vertex scaling exists for all time and converges exponentially fast.

Remark 3. If \(s = 0\), the result in Theorem 1.3 was proved by Gu-Luo-Sun-Wu \[22\] in the Euclidean background geometry and by Gu-Guo-Luo-Sun-Wu \[21\] in the hyperbolic background geometry respectively for combinatorial Yamabe flow with surgery. If \(s = 1\), the result proved by the first author in \[41\] can not be applied directly to prove that the number of surgeries along the fractional combinatorial Calabi flow \((1.10)\) is finite. It is conceived that for any \(s \in \mathbb{R}\), this is true for fractional combinatorial Calabi flow with surgery.

For discrete conformal structures in Definition 1, one can introduce the notion of weighted Delaunay condition. For a PL or PH metric on \((M, T, \varepsilon, \eta)\) generated by discrete conformal structures in Definition 1, it is said to satisfy the \textit{weighted Delaunay condition} if \(\frac{\partial K}{\partial u} \leq 0\) for every edge \(\{ij\} \in E\). In the case of \(\varepsilon \equiv 0\), which corresponds to vertex scaling, the weighted Delaunay condition is equivalent to the standard Delaunay condition. Please refer to \[2, 3, 17–20, 27, 43, 47–49\] for more information on Delaunay condition and weighted Delaunay condition. For the fractional combinatorial Calabi flow \((1.10)\) of discrete conformal structures in Definition 1, one can also do surgery along the flow \((1.10)\) by edge flipping under the weighted Delaunay condition. We have the following conjecture on the global convergence of the solution of fractional combinatorial Calabi flow with surgery for discrete conformal structures in Definition 1.

Conjecture 1. Suppose \((M, V, \varepsilon)\) is a marked weighted connected closed surface with \(\varepsilon : V \to \{0, 1\}\) and there exists a PL or PH metric generated by a discrete conformal structure in Definition 1 with combinatorial curvature \(K\). For any \(s \in \mathbb{R}\) and any initial PL or PH metric on \((M, V, \varepsilon)\) generated by discrete conformal structures in Definition 1, the solution of fractional combinatorial Calabi flow with surgery exists for all time and converges exponentially fast after finite number of surgeries.
In Conjecture 1, the PL or PH metric generated by some discrete conformal structure in Definition 1 with combinatorial curvature $\overline{K}$ does not depend on the triangulations of $(M, V, \varepsilon)$. Conjecture 1 is a generalization of Conjecture 3 in [47] proposed by the second author on the convergence of combinatorial Ricci flow with surgery and combinatorial Calabi flow with surgery for discrete conformal structures in Definition 1. Note that for Thurston’s circle packings on a triangulated surface $(M, T)$, the weighted Delaunay condition is automatically satisfied, so we do not need to do surgery on the fractional combinatorial Calabi flow (1.10). For vertex scaling on a triangulated surface $(M, T)$, the weighted Delaunay condition is equivalent to the standard Delaunay condition. This implies that the surgery by edge flipping under the weighted Delaunay condition for vertex scaling is the same as the surgery by edge flipping under the standard Delaunay condition. Therefore, Theorem 1.2 and Theorem 1.3 provides strong supports for Conjecture 1.

The paper is organized as follows. In Section 2, we study the basic properties of the fractional combinatorial Calabi flow for discrete conformal structures on surfaces in Definition 1 and prove Theorem 1.1. In Section 3, we study the fractional combinatorial Calabi flow for Thurston’s circle packings on surfaces and prove Theorem 1.2. In Section 4, we study the fractional combinatorial Calabi flow for vertex scaling on surfaces and prove Theorem 1.3.

Acknowledgements

The authors thank Professor Feng Luo, Dr. Yanwen Luo, Dr. Wai Yeung Lam, Xiaoping Zhu, Professor Linlin Sun and Professor Changsong Deng and for communications. The research of the second author is supported by the Fundamental Research Funds for the Central Universities under grant no. 2042020kf0199.

2 Basic properties of fractional combinatorial Calabi flow for discrete conformal structures on surfaces

Recall the following result on the matrix $L = (\frac{\partial K}{\partial u})$ for the discrete conformal structures in Definition 1 which generalizes the results obtained in [2,25,29,30,41,46,48,52].

Theorem 2.1 ( [47]). Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated surface with the weights $\varepsilon : V \rightarrow \{0, 1\}$ and $\eta : E \rightarrow \mathbb{R}$ satisfying the structure condition (1.7).

(a) In the Euclidean background geometry, the matrix $L = (\frac{\partial K}{\partial u})$ is symmetric and positive semi-definite with rank $N - 1$ and kernel $\{t^T \mathbf{1} = t(1, \cdots, 1)^T \in \mathbb{R}^N | t \in \mathbb{R} \}$ for all nondegenerate Euclidean discrete conformal structures on $(M, T, \varepsilon, \eta)$. 
(b) In the hyperbolic background geometry, the matrix $L = \frac{\partial K}{\partial u}$ is symmetric and strictly positive definite for all nondegenerate hyperbolic discrete conformal structures on $(M, \mathcal{T}, \varepsilon, \eta)$.

As a direct consequence, we have the following results.

**Lemma 2.2.** Suppose $(M, \mathcal{T}, \varepsilon, \eta)$ is a weighted triangulated surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure condition (1.7).

(a) In the Euclidean background geometry, for any $s \in \mathbb{R}$, the matrix $L^s$ is symmetric and positive semi-definite with rank $N - 1$ and kernel \( \left\{ t1 = t(1, \cdots, 1)^T \in \mathbb{R}^N \mid t \in \mathbb{R} \right\} \) for all nondegenerate Euclidean discrete conformal structures on $(M, \mathcal{T}, \varepsilon, \eta)$.

(b) In the hyperbolic background geometry, for any $s \in \mathbb{R}$, the matrix $L^s$ is symmetric and strictly positive definite for all nondegenerate hyperbolic discrete conformal structures on $(M, \mathcal{T}, \varepsilon, \eta)$.

**Remark 4.** Note that for a nondegenerate Euclidean discrete conformal structure on $(M, \mathcal{T}, \varepsilon, \eta)$, we have $\sum_{i \in V} (K - \overline{K})_i = 2\pi \chi(M) - 2\pi \chi(M) = 0$, which implies $K - \overline{K} \in \text{Ker}(L^s)^\perp = \text{Im}(L^s) = 1^\perp$. By Lemma 2.2 restricted to $\text{Ker}(L^s)^\perp = \text{Im}(L^s) = 1^\perp$, $\Delta^s$ is a nonsingular linear operator. Specially, $\Delta^0|_{1^\perp} = -Id|_{1^\perp}$ and $\Delta^1|_{1^\perp} = -L|_{1^\perp}$. Combining this with $K - \overline{K} \in 1^\perp$ implies that the 0-order Euclidean fractional combinatorial Calabi flow (1.10) is the Euclidean combinatorial Ricci flow (1.5) and the 1-order Euclidean fractional combinatorial Calabi flow (1.10) is the Euclidean combinatorial Calabi flow (1.6). Paralleling results hold for the hyperbolic background geometry. Therefore, the fractional combinatorial Calabi flow (1.10) unifies and generalizes the combinatorial Ricci flow (1.5) and the combinatorial Calabi flow (1.6).

**Remark 5.** In the Euclidean background geometry, if $\overline{K_i} = K_{av} = \frac{2\pi \chi(M)}{N}$ for every $i \in V$, the fractional combinatorial Calabi flow (1.10) is equivalent to

$$\frac{du_i}{dt} = \Delta^s K_i$$

by Lemma 2.2 (a), which is the standard form of fractional combinatorial Calabi flow. As the results in this paper could be proved for prescribed combinatorial curvature $\overline{K}$, we introduce the fractional combinatorial Calabi flow in the form (1.10) to simplify the statements of the results.

By Lemma 2.2 (a), we further have the following property for the solution of the Euclidean fractional combinatorial Calabi flow (1.10).
Lemma 2.3. Suppose \((M, T, \varepsilon, \eta)\) is a weighted triangulated surface with the weights \(\varepsilon : V \rightarrow \{0, 1\}\) and \(\eta : E \rightarrow \mathbb{R}\) satisfying the structure condition (1.7). \(u(t)\) is a solution of the Euclidean fractional combinatorial Calabi flow (1.10). Then \(\sum_{i=1}^{N} u_i(t)\) is invariant along the Euclidean fractional combinatorial Calabi flow (1.10).

Proof. By direct calculations, we have

\[
\frac{d}{dt} \left( \sum_{i=1}^{N} u_i(t) \right) = - \sum_{i=1}^{N} L^s(K - \overline{K})_i = -1^T L^s(K - \overline{K}) = 0
\]

by Lemma 2.2 (a).

Lemma 2.3 implies that the solution of Euclidean fractional combinatorial Calabi flow (1.10) stays in the hyperplane \(\{u \in \mathbb{R}^N \mid \sum_{i=1}^{N} u_i = \sum_{i=1}^{N} u_i(0)\}\). Without loss of generality, we will assume \(u(0)\) is in the hyperplane \(\Sigma_0 := \{u \in \mathbb{R}^N \mid \sum_{i=1}^{N} u_i = 0\}\) for the Euclidean background geometry in the following.

Proof of Theorem 1.1. Set \(\Gamma_i(u) = \Delta^s(K - \overline{K})_i\). By assumption, \(\pi\) is an equilibrium point of the system \(\frac{du}{dt} = \Gamma(u)\). Furthermore, \(D\Gamma|_{u=\pi} = -L^{s+1}\). In the case of hyperbolic background geometry, \(D\Gamma\) is negative definite at \(\pi\) by Lemma 2.2 (b). Then the result in Theorem 1.1 for the hyperbolic background geometry is a direct application of the Lyapunov stability theorem ( [34] Chapter 5). In the case of Euclidean background geometry, \(D\Gamma|_{u=\pi} = -L^{s+1}\) is negative semi-definite with kernel \(\{t1 \in \mathbb{R}^N \mid t \in \mathbb{R}\}\) by Lemma 2.2 (a), which is perpendicular to the hyperplane \(\Sigma_0\). Restricted to \(\Sigma_0\), then the following of the proof for Theorem 1.1 in the Euclidean background geometry is also a direct application of the Lyapunov stability theorem.

3 Fractional combinatorial Calabi flow for Thurston’s circle packings on surfaces

Thurston’s circle packing [40] is a special type of discrete conformal structures on polyhedral surfaces with \(\varepsilon \equiv 1\) and \(\eta \in [0, 1]\) in Definition 1. Motivated by [41, 52, 53], we consider a generalization of Thurston’s original definition, which corresponds to \(\varepsilon \equiv 1\) and \(\eta \in (-1, 1]\) in Definition 1.

3.1 The case of Thurston’s Euclidean circle packings

In the Euclidean background geometry, the edge length defined by Thurston’s circle packing is given by

\[
l_{ij} = \sqrt{e^{2u_i} + e^{2u_j} + 2\eta_{ij} e^{u_i + u_j}}. \quad (3.1)
\]
We have the following result on the triangle inequality for the edge length defined by \( (3.1) \).

**Lemma 3.1** (Corollary 2.2). Suppose \( \eta : E \to (-1, 1] \) satisfies the structure condition \((1.7)\). Then for any \( u \in \mathbb{R}^N \), the triangle inequalities are satisfied for every triangle \( \triangle ijk \in F \).

Combining Theorem 2.1 and Lemma 3.1, the function \( F(u) = \int_0^u \sum_{i=1}^N (K_i - \overline{K}_i) du_i \) is a well-defined smooth convex function defined on \( \mathbb{R}^N \) with \( \nabla_u F = K - \overline{K} \) and \( \text{Hess}_u F = L \). If there exists \( \overline{u} \in \Sigma_0 \) with \( K(\overline{u}) = \overline{K} \), then \( \nabla_u F(\overline{u}) = 0 \), which implies \( F(\overline{u}) = \min_{u \in \Sigma_0} F(u) \) and \( \lim_{u \to \infty, u \in \Sigma_0} F(u) = +\infty \) by the convexity of \( F(u) \).

**Proof of Theorem 1.2 (a):** Along the fractional combinatorial Calabi flow \((1.10)\), we have
\[
\frac{d}{dt} F(u(t)) = \sum_{i=1}^N \nabla_{u_i} F \cdot \frac{du_i}{dt} = -(K - \overline{K})^T \cdot L^s \cdot (K - \overline{K}) \leq 0
\]
by Lemma 2.2 (a), which implies \( F(u(t)) \leq F(u(0)) \) along the fractional combinatorial Calabi flow \((1.10)\). Combining Lemma 2.3 and \( \lim_{u \to \infty, u \in \Sigma_0} F(u) = +\infty \), the solution \( u(t) \) of Euclidean fractional combinatorial Calabi flow \((1.10)\) stays in a compact subset \( \Omega \) of \( \Sigma_0 \), which further implies the solution \( u(t) \) of \((1.10)\) exists for all time.

As the solution \( u(t) \) of the Euclidean fractional combinatorial Calabi flow \((1.10)\) stays in a compact subset \( \Omega \subset \subset \Sigma_0 \) and \( L^{s+1} \) is strictly positive definite on \( \Sigma_0 \) by Lemma 2.2, the first nonzero eigenvalue of \( L^{s+1} \), which is a continuous function of the discrete conformal structures, has a positive lower bound \( \lambda \) along the Euclidean fractional combinatorial Calabi flow \((1.10)\). Therefore, for the combinatorial Calabi energy \( \overline{C}(t) := \sum_{i=1}^N (K_i - \overline{K}_i)^2 \), we have
\[
\frac{d}{dt} \overline{C}(t) = 2 \sum_{i=1}^N (K_i - \overline{K}_i) \frac{dK_i}{dt} = -2(K - \overline{K})^T \cdot L^s \cdot (K - \overline{K}) \leq -2\lambda \overline{C}(t),
\]
which implies \( \overline{C}(t) \leq e^{-2\lambda t} \overline{C}(0) \). As \( K|_{\Sigma_0} \) is a diffeomorphism from \( \Sigma_0 \) to \( K(\Sigma_0) \) by Theorem 2.1, this further implies that the solution \( u(t) \) of Euclidean fractional combinatorial Calabi flow \((1.10)\) converges exponentially fast to \( \overline{u} \).

**3.2 The case of Thurston’s hyperbolic circle packings**

In the hyperbolic background geometry, set \( e^{r_i} = \sinh r_i \) in \((1.2)\). Then the edge length \( l_{ij} \) defined by Thurston’s hyperbolic circle packing is given by
\[
\cosh l_{ij} = \cosh r_i \cosh r_j + \eta_{ij} \sinh r_i \sinh r_j.
\]
The map \( r : V \rightarrow (0, +\infty) \) is called Thurston’s hyperbolic circle packing metric. For simplicity of notations, set
\[
C_i = \cosh r_i, S_i = \sinh r_i
\]
in the following of this subsection.

**Lemma 3.2** ([47] Lemma 4.6). Suppose \( \eta_{ij} \in (-1, 1] \). For the edge length \( l_{ij} \) defined by (3.2), there exist \( C = C(\eta_{ij}) > 0 \) and \( C' = C'(\eta_{ij}) > 0 \) such that
\[
C(C_i C_j + S_i S_j) \leq \cosh l_{ij} \leq C'(C_i C_j + S_i S_j).
\]

Parallelling to Lemma 3.1 for Thurston’s Euclidean circle packing metrics, we have the following result on triangle inequalities for Thurston’s hyperbolic circle packing metrics.

**Lemma 3.3** ([52] Lemma 2.4, [44] Corollary 3.2). Suppose \( \eta : E \rightarrow (-1, 1] \) satisfies the structure condition (1.7). For any \( r \in \mathbb{R}^N_+ \), the triangle inequalities are satisfied for any triangle \( \triangle ijk \in F \) with edge length defined by (3.2).

By the definition of \( u_i \) in (1.4), we have
\[
u_i = \frac{1}{2} \log \frac{\cosh r_i - 1}{\cosh r_i + 1} = \log \tanh \frac{r_i}{2}
\]
for Thurston’s hyperbolic circle packing metrics, which implies \( u \in \mathbb{R}^N_- \) by Lemma 3.3. By Theorem 2.1 (b) and Lemma 3.3, the function
\[
F(u) = \int_{u_0}^u \sum_{i=1}^N (K_i - \overline{K}_i) du_i
\]
is a strictly convex function defined on \( \mathbb{R}^N_- \) with \( \nabla u F = K - \overline{K} \). If there exists \( \overline{u} \in \mathbb{R}^N_- \) with \( K(\overline{u}) = \overline{K} \), then \( \nabla u F(\overline{u}) = 0 \), which further implies that \( \lim_{u \to \infty, u \in \mathbb{R}^N_-} F(u) = +\infty \) by the strict convexity of \( F(u) \).

**Lemma 3.4.** Suppose \((M, \mathcal{T}, \varepsilon, \eta)\) is a weighted triangulated connected closed surface with the weights \( \varepsilon \equiv 1 \) and \( \eta : E \rightarrow (-1, 1] \) satisfying the structure condition (1.7). Suppose that there exists \( \overline{u} \in \mathbb{R}^N_- \) with \( K(\overline{u}) = \overline{K} \) in the hyperbolic background geometry and \( u(t) \) is a solution of the hyperbolic fractional combinatorial Calabi flow (1.10). Then \( u_i(t) \) is uniformly bounded from below along the hyperbolic fractional combinatorial Calabi flow (1.10) for every \( i \in V \).

**Proof.** Along the hyperbolic fractional combinatorial Calabi flow (1.10), we have
\[
\frac{d}{dt} F(u(t)) = \sum_{i=1}^N \nabla u_i F \cdot \frac{du_i}{dt} = -(K - \overline{K})^T \cdot L^s \cdot (K - \overline{K}) \leq 0
\]
by Lemma 2.2 (b), which implies $F(u(t)) \leq F(u(0))$ along the hyperbolic fractional combinatorial Calabi flow (1.10). Combining with the fact that $\lim_{u \to \infty, u \in \mathbb{R}^N_{\geq 0}} F(u) = +\infty$ under the existence of $\pi \in \mathbb{R}^N_{\geq 0}$ with $K(\pi) = \overline{K}$, we have $u(t)$ is uniformly bounded, which implies that $u_i(t)$ is uniformly bounded from below along the hyperbolic fractional combinatorial Calabi flow (1.10) for every $i \in V$. □

Remark 6. By (3.4), $r_i \in (0, +\infty)$ is a strictly nondecreasing function of $u_i \in (-\infty, 0)$ with $\lim_{u_i \to -\infty} r_i = 0$ and $\lim_{u_i \to 0} r_i = +\infty$. Under the existence of Thurston’s hyperbolic circle packing with combinatorial curvature $\overline{K}$, Lemma 3.4 implies that there exists a constant $R > 0$ such that $r_i(t) > R$ for all $i \in V$ along the fractional combinatorial Calabi flow (1.10). We will always assume $r_i \geq R > 0$ in the following of this subsection. By the proof of Lemma 3.4 one can also get a positive upper bound for $u_i(t)$. However, $u_i < 0$ by (3.4). Therefore, the positive upper bound of $u_i(t)$ is useless.

Lemma 3.5. Suppose $\eta_{ij} \in (-1, 1]$ for every edge $\{ij\} \in E$ and there exists a constant $R > 0$ such that $r_i > R$ for every $i \in V$. Then $\cosh l_{ij} - 1$ has a uniform lower bound $C = C(\eta, R) > 0$ for every edge $\{ij\} \in E$.

Proof. By Cauchy inequality, we have

$$
\cosh l_{ij} = \sqrt{(1 + \sinh^2 r_i)(1 + \sinh^2 r_j) + \eta_{ij} \sinh r_i \sinh r_j} \\
\geq 1 + \sinh r_i \sinh r_j + \eta_{ij} \sinh r_i \sinh r_j \\
= 1 + (1 + \eta_{ij}) \sinh r_i \sinh r_j,
$$

which implies

$$
\cosh l_{ij} - 1 \geq (1 + \eta_{ij}) \sinh r_i \sinh r_j \geq \min_{\{ij\} \in E} (1 + \eta_{ij}) \sinh^2 R > 0
$$

by $\eta_{ij} \in (-1, 1]$ for every edge $\{ij\} \in E$ and $r_i \geq R$ for every $i \in V$. We can take $C = \min_{\{ij\} \in E} (1 + \eta_{ij}) \sinh^2 R$. □

Recall that for Thurston’s hyperbolic circle packing metrics, the matrix $L = \left( \frac{\partial K}{\partial u} \right)$ could be decomposed as (14 Theorem 3.1)

$$
L = A + B,
$$

where $A$ is a diagonal matrix with

$$
A_{ii} = \frac{\partial}{\partial u_i} \left( \sum_{\Delta ijk \in F} \text{Area}(\Delta ijk) \right)
$$
and $B$ is a symmetric matrix with

$$B_{ij} = \begin{cases} -\left(\frac{\partial \theta_{jk}^i}{\partial u_j} + \frac{\partial \theta_{jl}^i}{\partial u_j}\right), & \text{if } j \sim i; \\ -\sum_{k \sim i} B_{ik}, & \text{if } j = i; \\ 0, & \text{otherwise}. \end{cases} \quad (3.5)$$

Further recall the following formula (52, Lemma 4.1, 44, Lemma 3.6) for the derivative of inner angle $\theta_{jk}^i$ in the triangle $\triangle ijk$

$$\frac{\partial \theta_{jk}^i}{\partial u_j} = \frac{\partial \theta_{jl}^i}{\partial u_j} = \frac{1}{A_{ijk} \sinh^2 l_{ij}} [C_k S_j^2 S_i^2 (1 - \eta_{ij}^2) + C_i S_j^2 S_k \gamma_{ijk} + C_j S_i^2 S_j S_k \gamma_{ijk}], \quad (3.6)$$

where $A_{ijk} = \sinh l_{ij} \sinh l_{ik} \sin \theta_{jk}^i$ and $\gamma_{ijk} = \eta_{jk} + \eta_{ij} \eta_{ik}$. By (3.6), $\eta \in (-1, 1]$ and the structure condition (1.7), we have

$$\frac{\partial \theta_{jk}^i}{\partial u_j} \geq 0, \quad \frac{\partial \theta_{jl}^i}{\partial u_j} \geq 0, \quad -B_{ij} = \frac{\partial \theta_{jk}^i}{\partial u_j} + \frac{\partial \theta_{jl}^i}{\partial u_j} \geq 0 \quad (3.7)$$

for $j \sim i$. Note that $\frac{\partial \theta_{jk}^i}{\partial u_j} = 0$ if and only if $\eta_{ij} = 1$ and $\eta_{jk} + \eta_{ik} = 0$, which is attainable.

By the following formula obtained by Glickenstein and Thoma (20, Proposition 9)

$$\frac{\partial}{\partial u_i} \text{Area}(\triangle ijk) = \frac{\partial \theta_{jk}^i}{\partial u_i} (\cosh l_{ij} - 1) + \frac{\partial \theta_{jl}^i}{\partial u_i} (\cosh l_{ik} - 1), \quad (3.8)$$

we have

$$A_{ii} = \sum_{j \sim i} \left(\frac{\partial \theta_{jk}^i}{\partial u_j} + \frac{\partial \theta_{jl}^i}{\partial u_j}\right) (\cosh l_{ij} - 1) = \sum_{j \sim i} (-B_{ij}) (\cosh l_{ij} - 1) \geq 0 \quad (3.9)$$

by (3.7), where $\triangle ijk$ and $\triangle ijl$ are adjacent triangles sharing the common edge $\{ij\} \in E$.

In the case that $r_i > R > 0$ for every $i \in V$, we further have the following stronger result on $A_{ii}$.

**Lemma 3.6.** Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon \equiv 1$ and $\eta : E \to (-1, 1]$ satisfying the structure condition (1.7). $r : V \to (0, +\infty)$ is a Thurston’s hyperbolic circle packing metric with $r_i \geq R > 0$ for any $i \in V$. Then there exist positive constants $a_1 = a_1(\eta, R)$ and $a_2 = a_2(\eta, R)$ such that $a_1 \leq A_{ii} \leq a_2$ for any $i \in V$.

**Proof.** We use $q \sim p$ to denote that there exist constants $C = C(\eta, R) > 0$ and $C' = C'(\eta, R) > 0$ such that $Cp \leq q \leq C'p$. Then what we need to prove is equivalent to $A_{ii} \sim 1$. 14
Note that under the condition \( r_i \geq R > 0 \) for any \( i \in V \), we have

\[
C_i \sim e^{r_i}, S_i \sim e^{r_i}, \cosh l_{ij} + 1 \sim \cosh l_{ij} \sim e^{r_i + r_j},
\]

(3.10)

where \( \cosh l_{ij} \sim e^{r_i + r_j} \) follows from Lemma \[52\] Combining \( \eta \in (-1, 1] \), the structure condition (1.7), (3.6) and (3.10), we have

\[
\frac{\partial \theta_{ij}^k}{\partial u_j} (\cosh l_{ij} - 1) \sim e^{r_i + r_j + r_k} A_{ijk} [(1 - \eta_{ij}^2) + \gamma_{ijk} + \gamma_{ijkl}].
\]

(3.11)

By the hyperbolic cosine law, we have

\[
A_{ijk}^2 = \sinh^2 l_{ij} \sinh^2 l_{ik} (1 - \cos^2 \theta_{ij}^k)
= \sinh^2 l_{ij} \sinh^2 l_{ik} - (\cosh l_{ij} \cosh l_{ik} - \cosh l_{j})^2
= (\cosh l_{ij} - 1)(\cosh l_{ik} - 1) - (\cosh l_{ij} \cosh l_{ik} - \cosh l_{j})^2
= 1 + 2 \cosh l_{ij} \cosh l_{ik} \cosh l_{jk} - \cosh^2 l_{ij} - \cosh^2 l_{ik} - \cosh^2 l_{jk}.
\]

(3.12)

Submitting (3.12) into (3.11) and by lengthy but direct calculations (refer to the proof of Lemma 2.4 in [52] or Lemma 3.1 in [44]), we have

\[
A_{ijk}^2 = 2 S_i^2 S_j^2 S_k^2 (1 + \eta_{ij} \eta_{ik} \eta_{jk}) + S_i^2 S_j^2 (1 - \eta_{ij}^2) + S_i^2 S_k^2 (1 - \eta_{ik}^2) + S_j^2 S_k^2 (1 - \eta_{jk}^2)
+ 2 C_j C_k S_i S_j S_k \gamma_{ijk} + 2 C_i C_k S_i S_j S_k \gamma_{ijk} + 2 C_i C_j S_i S_j S_k \gamma_{ijkl},
\]

which implies

\[
e^{-2(r_i + r_j + r_k)} A_{ijk}^2 \sim \left[ 1 + \eta_{ij} \eta_{ik} \eta_{jk} + \gamma_{ijk} + \gamma_{ijkl} + \gamma_{ijkl} + \eta_{ij}^2 \right] \]

(3.13)

by (3.10), \( \eta \in (-1, 1] \) and the structure condition (1.7). Note that

\[
1 + \eta_{ij} \eta_{ik} \eta_{jk} + \gamma_{ijk} + \gamma_{ijkl} = (1 + \eta_{ij})(1 + \eta_{ik})(1 + \eta_{jk}) > 0
\]

(3.14)

by \( \eta \in (-1, 1] \). Combining (3.13), (3.14) and \( r_i \geq R > 0 \), we have \( e^{-2(r_i + r_j + r_k)} A_{ijk}^2 \sim 1 \), which implies

\[
\frac{\partial \theta_{ij}^k}{\partial u_j} (\cosh l_{ij} - 1) \sim (1 - \eta_{ij}^2) + \gamma_{ijk} + \gamma_{ijkl}
\]

by (3.11). Similarly, \( \frac{\partial \theta_{ij}^k}{\partial u_j} (\cosh l_{ij} - 1) \sim (1 - \eta_{ij}^2) + \gamma_{ijkl} + \gamma_{ijl} \). Therefore,

\[
A_{ii} \sim \sum_{j \sim i} [2(1 - \eta_{ij}^2) + \gamma_{ijk} + \gamma_{ijkl} + \gamma_{ijl}]
\]

(3.15)
by (3.9). By \( \eta \in (-1, 1] \) and the structure condition (1.7), we have
\[
\sum_{j \sim i} (2(1 - \eta_{ij}^2) + \gamma_{ijk} + \gamma_{ijl} + \gamma_{jil}) \geq 0,
\]
where the equality is attained if and only if \( 1 - \eta_{ij}^2 = 0 \) and \( \gamma_{ijk} = \gamma_{ijl} = \gamma_{jil} = 0 \) for every vertex \( j \) adjacent to \( i \). By \( 1 - \eta_{ij}^2 = 0 \) and \( \eta \in (-1, 1] \), we have \( \eta_{ij} = 1 \) for every \( j \in V \) adjacent to \( i \), which further implies \( \gamma_{ijk} = \eta_{jk} + \eta_{ij} \eta_{jk} = 1 + \eta_{jk} > 0 \). This contradicts to \( \gamma_{ijk} = 0 \). Therefore, we have \( \sum_{j \sim i} (2(1 - \eta_{ij}^2) + \gamma_{ijk} + \gamma_{ijl} + \gamma_{jil}) > 0 \), which implies \( A_{ii} \sim 1 \) by (3.15). This completes the proof.

As a corollary of Lemma 3.6, we have the following estimate on the eigenvalues of
\[ L = \left( \frac{\partial K}{\partial u} \right). \]

**Corollary 3.7.** Suppose \((M, T, \varepsilon, \eta)\) is a weighted triangulated connected closed surface with the weights \( \varepsilon \equiv 1 \) and \( \eta : E \to (-1, 1] \) satisfying the structure condition (1.7). \( r : V \to (0, +\infty) \) is a Thurston’s hyperbolic circle packing metric with \( r_i \geq R > 0 \) for any \( i \in V \). Then there exist positive constants \( a_3 = a_3(\eta, R) \) and \( a_4 = a_4(\eta, R) \) such that the eigenvalues \( \lambda_1, \ldots, \lambda_N \) of \( L = \left( \frac{\partial K}{\partial u} \right) \) stay in a closed interval \([a_3, a_4] \subset (0, +\infty)\).

**Proof.** Combining Lemma 3.5 (3.7) and Glickenstein-Thomas’s formula (3.8), we have
\[
\frac{\partial}{\partial u_i} \text{Area}(\triangle ijk) \geq C_0(\frac{\partial \theta_{ij}^k}{\partial u_i} + \frac{\partial \theta_{ij}^k}{\partial u_j} + \frac{\partial \theta_{ij}^k}{\partial u_k})
\]
for some positive constant \( C_0 = C_0(\eta, R) \), which implies that
\[ A_{ii} = \sum_{\triangle ijk \in \mathcal{K}} \frac{\partial}{\partial u_i} \text{Area}(\triangle ijk) \geq C_0 \sum_{j \sim i} (-B_{ij}). \tag{3.16} \]

By (3.16) and the definition of \( B \) in (3.5), the matrix \( 3A - C_0B \) is diagonal dominant and thus positive definite, which implies \( B < \frac{3}{C_0} A \). Note that \( B \) is positive semi-definite by \( B_{ij} \leq 0 \) for \( j \sim i \) in (3.7), we have
\[ A \leq L = A + B < (1 + \frac{3}{C_0})A. \]

Then the result in the corollary follows by Lemma 3.6.

Using Corollary 3.7, we prove the following key lemma on the matrix \( L^s \).

**Lemma 3.8.** Suppose \((M, T, \varepsilon, \eta)\) is a weighted triangulated connected closed surface with the weights \( \varepsilon \equiv 1 \) and \( \eta : E \to (-1, 1] \) satisfying the structure condition (1.7). \( r : V \to (0, +\infty) \) is a Thurston’s hyperbolic circle packing metric with \( r_i \geq R > 0 \) for any \( i \in V \). Then for any \( s \in \mathbb{R} \), there exists a constant \( C = C(s, \eta, R) > 0 \) such that
\[
\sum_{j \in V, j \neq i} ((L^s)_{ij})^2 \leq C \sum_{j \in V, j \neq i} (L_{ij})^2 \frac{((L^s)_{ii})^2}{(L_{ii})^2}.
\]
for all \( i \in V \).

**Proof.** By the definition of \( L^s = (\frac{\partial K}{\partial u})^s \) and Lemma 2.2 (b), there exists an orthonormal matrix \( P \in O(N) \) such that

\[
L^s = P^T \cdot \text{diag}\{\lambda_1^s, \cdots, \lambda_N^s\} \cdot P = P^T \cdot \Lambda^s \cdot P,
\]

where \( \lambda_i > 0, i = 1, \cdots, N, \) are eigenvalues of \( L = (\frac{\partial K}{\partial u}) \) and \( \Lambda = \text{diag}\{\lambda_1, \cdots, \lambda_N\} \).

Assume \( P = (P_1, \cdots, P_N) \) where \( P_1, \cdots, P_N \) are orthonormal column vectors. Then

\[
(L^s)_{ij} = P_i^T \Lambda^s P_j,
\]

which implies

\[
\sum_{j=1}^{N} ((L^s)_{ij})^2 = (L^s L^s)_{ii} = (P^T \Lambda^2 s P)_{ii} = P_i^T \Lambda^2 s P_i.
\]

As \( P_i \) is a column unit vector, assume \( P_i = (x_1, \cdots, x_N)^T \in \mathbb{R}^N \) with \( \sum_{j=1}^{N} x_j^2 = 1 \). Then

\[
\sum_{j=1}^{N} ((L^s)_{ij})^2 = \sum_{j=1}^{N} \lambda_j^2 s x_j^2, \quad (L^s)_{ii} = \sum_{j=1}^{N} \lambda_j^2 s x_j^2.
\]

Therefore, we just need to prove that there exists \( C = C(s, \eta, R) > 0 \) such that

\[
\frac{\sum_{j=1}^{N} \lambda_j^2 s x_j^2}{(\sum_{j=1}^{N} \lambda_j^2 s x_j^2)^2} - 1 \leq C \left( \frac{\sum_{j=1}^{N} \lambda_j^2 s x_j^2}{(\sum_{j=1}^{N} \lambda_j^2 s x_j^2)^2} - 1 \right) \quad (3.17)
\]

for a unit vector \((x_1, \cdots, x_N)^T \in \mathbb{R}^N\). Note that

\[
\frac{\sum_{j=1}^{N} \lambda_j^2 s x_j^2}{(\sum_{j=1}^{N} \lambda_j^2 s x_j^2)^2} - 1 = \frac{(\sum_{j=1}^{N} \lambda_j^2 s x_j^2)(\sum_{j=1}^{N} x_j^2) - (\sum_{j=1}^{N} \lambda_j^2 s x_j^2)^2}{(\sum_{j=1}^{N} \lambda_j^2 s x_j^2)^2} = \frac{\sum_{j \neq k}(\lambda_j^2 s x_j^2 x_k^2 + \lambda_k^2 s x_j^2 x_k^2 - 2\lambda_j^2 s x_j^2 x_k^2)}{2(\sum_{j=1}^{N} \lambda_j^2 s x_j^2)^2} \leq \frac{1}{2\alpha^2 s} \sum_{j \neq k}(\lambda_j^2 - \lambda_k^2 s)^2 x_j^2 x_k^2,
\]

where Corollary 3.7 is used in the last line. Similarly, we have

\[
\frac{\sum_{j=1}^{N} \lambda_j^2 s x_j^2}{(\sum_{j=1}^{N} \lambda_j^2 s x_j^2)^2} - 1 = \frac{\sum_{j \neq k}(\lambda_j - \lambda_k)^2 x_j^2 x_k^2}{2(\sum_{j=1}^{N} \lambda_j^2 s x_j^2)^2} \geq \frac{1}{2\alpha^2 s} \sum_{j \neq k}(\lambda_j - \lambda_k)^2 x_j^2 x_k^2. \quad (3.19)
\]
By the mean value theorem, there exists $\xi$ between $\lambda_j$ and $\lambda_k$ such that $\lambda_j^s - \lambda_k^s = s\xi^{s-1}(\lambda_j - \lambda_k)$, which implies that there exists a constant $C' = C'(s, \eta, R) > 0$ such that
\[
|\lambda_j^s - \lambda_k^s| \leq C'|\lambda_j - \lambda_k|
\] (3.20)
by Corollary 3.7. Then (3.17) is a direct consequence of (3.18), (3.19) and (3.20). □

As an application of Lemma 3.8, we prove the following comparison between $(L^s)_{ii}$ and $\sum_{j \in V, j \neq i}|(L^s)_{ij}|$.

**Lemma 3.9.** Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon \equiv 1$ and $\eta : E \to (-1, 1]$ satisfying the structure condition (1.7). $r : V \to (0, +\infty)$ is a Thurston’s hyperbolic circle packing metric with $r_i \geq R > 0$ for any $i \in V$. For any $s \in \mathbb{R}$ and $\bar{C} > 0$, there exists a constant $R_1 = R_1(s, \eta, R, \bar{C}) > 0$ such that if $r_i > R_1$, then
\[
(L^s)_{ii} \geq \bar{C} \sum_{j \in V, j \neq i}|(L^s)_{ij}|.
\]

**Proof.** By Lemma 3.2, there exists $C_1 = C_1(\eta) > 0$ such that $\cosh l_{ij} \geq C_1(C_iC_j + S_iS_j) = C_1\cosh(r_i + r_j) > C_1\cosh r_i$. Therefore, for any $C_2 > 0$ (to be determined), there exists $R_1 = R_1(\eta, C_1, C_2) > 0$ such that if $r_i > R_1$, then $\cosh l_{ij} - 1 \geq C_2$ for every edge $\{ij\} \in E$ adjacent to $i$, which implies
\[
A_{ii} \geq C_2 \sum_{j \sim i} \left( \frac{\partial \theta^j}{\partial u_i} + \frac{\partial \theta^j}{\partial u_i} \right) = C_2 \sum_{j \sim i} (-B_{ij})
\] (3.21)
by (3.22). Therefore,
\[
L_{ii} = A_{ii} + \sum_{j \sim i} (-B_{ij}) \geq (C_2 + 1) \sum_{j \sim i} (-B_{ij}) = (C_2 + 1) \sum_{j \in V, j \neq i} |L_{ij}|
\]
by (3.21), which further implies
\[
L_{ii}^2 \geq (C_2 + 1)^2 \sum_{j \in V, j \neq i} |L_{ij}|^2.
\] (3.22)

Combining (3.22) and Lemma 3.8 we have
\[
\sum_{j \in V, j \neq i} |(L^s)_{ij}|^2 \leq C_3 \sum_{j \in V, j \neq i} |(L^s)_{ij}|^2 \leq \frac{C_3}{(C_2 + 1)^2},
\] (3.23)
where $C_3 = C_3(s, \eta, R) > 0$ is given by Lemma 3.8. The inequality (3.23) implies
\[
(L^s)_{ii} \geq \frac{C_2 + 1}{\sqrt{C_3}}|(L^s)_{ij}|
\]
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for $j \neq i$ and then

$$(L^s)_{ii} \geq \frac{C_2 + 1}{N \sqrt{C_3}} \sum_{j \in V, j \neq i} |(L^s)_{ij}|.$$ 

Set $C_2 = N \tilde{C} \sqrt{C_3} - 1 > 0$. Then if $r_i \geq R_1 = R_1(\eta, C_1, C_2) = R_1(s, \eta, R, \tilde{C})$, we have

$$(L^s)_{ii} \geq \tilde{C} \sum_{j \in V, j \neq i} |(L^s)_{ij}|.$$ 

$\square$

**Remark 7.** In the case of $s = 1$ and $\eta \in [0, 1]$, the result in Lemma 3.9 was proved by Ge-Hua [10].

We shall prove that the solution $u(t)$ of fractional combinatorial Calabi flow (1.10) for Thurston’s hyperbolic circle packings stays in a compact subset of $\mathbb{R}^{N_0}$. To prove this, we further need the following result on Thurston’s hyperbolic circle packing metrics.

**Lemma 3.10.** Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon \equiv 1$ and $\eta : E \to (-1, 1]$ satisfying the structure condition (1.7). $\triangle ijk$ is a triangle in $F$. For any $\epsilon > 0$, there exists a positive constant $R_2$ such that if $r_i > R_2$, then $\theta^{jk}_i < \epsilon$.

Lemma 3.10 has been proved for different cases in almost the same manner. We will not give another proof for Lemma 3.10 here. Readers interested in the proof could refer to [11, 12, 15, 16, 38, 47, 52].

**Proposition 3.11.** Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon \equiv 1$ and $\eta : E \to (-1, 1]$ satisfying the structure condition (1.7). Suppose that there exists $u \in \mathbb{R}^{N_0}$ such that $K(u) = K$ in the hyperbolic background geometry and $u(t), t \in [0, T)$, is a solution of the hyperbolic fractional combinatorial Calabi flow (1.10) with the maximal existing time $T \leq +\infty$. Then there exists a constant $c < 0$ such that $u_i(t) < c$ for every $i \in V$ along the hyperbolic fractional combinatorial Calabi flow (1.10).

**Proof.** By the relationship of $u_i$ and $r_i$ in (3.4), we just need to prove that $r_i(t)$ is uniformly bounded from above in $(0, +\infty)$ for every $i \in V$ along the hyperbolic fractional combinatorial Calabi flow (1.10), which is equivalent to $r(t)$ is bounded along (1.10).

By Lemma 3.3 and Remark 6 there exists a constant $R > 0$ such that $r_i(t) \geq R > 0$ along the fractional combinatorial Calabi flow (1.10) for every $i \in V$ under the existence of $\overline{u}$ with $K(\overline{u}) = K$. By Lemma 3.10 for $\epsilon = \frac{1}{2\pi}(2\pi - K_i) > 0$, there exists a constant $R_2 > 0$ with $R_2 \geq R$ such that if $r_i > R_2$, then $\theta_i^k < \epsilon$ for every triangle $\triangle ijk \in F$ at $i$, which implies

$$K_i - \overline{K}_i = 2\pi - \sum_{\Delta ijk \in F} \theta_i^k - \overline{K}_i > \frac{1}{2}(2\pi - \overline{K}_i).$$

(3.24)
By Lemma \ref{lem:3.9} for $\tilde{C} = \frac{2 \max_{v \in V}[(N+2)\pi + |K_v|]}{(2\pi - K_i)} > 0$, there exists $R_1 > 0$ with $R_1 \geq R$ such that if $r_i \geq R_1$, then
\[
(L^s)_{ii} \geq \tilde{C} \sum_{j \in V, j \neq i} |(L^s)_{ij}| = \frac{2 \max_{v \in V}[(N+2)\pi + |K_j|]}{(2\pi - K_i)} \sum_{j \in V, j \neq i} |(L^s)_{ij}|. \tag{3.25}
\]

Set $R_3 = \max(R_1, R_2, r_i(0) + 1)$. If $r_i \geq R_3$, then
\[
L^s(K - \overline{K})_i = (L^s)_{ii}(K_i - \overline{K})_i + \sum_{j \in V, j \neq i} (L^s)_{ij}(K_j - \overline{K})_j \\
\geq \frac{1}{2}(2\pi - \overline{K}_i)(L^s)_{ii} + \sum_{j \in V, j \neq i} (L^s)_{ij}(K_j - \overline{K})_j \\
\geq \max_{j \in V}[(N+2)\pi + |\overline{K}_j|] \sum_{j \in V, j \neq i} |(L^s)_{ij}| + \sum_{j \in V, j \neq i} (L^s)_{ij}(K_j - \overline{K})_j \tag{3.26} \\
= \sum_{j \in V, j \neq i} |(L^s)_{ij}| \left( \max_{k \in V}[(N+2)\pi + |\overline{K}_k|] - |K_j - \overline{K}_j| \right) \\
\geq 0,
\]
where \ref{eq:3.24} is used in the second line and \ref{eq:3.25} is used in the third line.

Suppose that along the fractional combinatorial Calabi flow \ref{eq:1.10}, $r(t), t \in [0, T)$, is not bounded. Then there exists at least one vertex $i_0 \in V$ such that $\lim sup_{t \to T^-} r_{i_0} = +\infty$. Without loss of generality, we can take $i_0 = i$. Therefore, there exists $t_0 \in (0, T)$ such that $r_i(t_0) > R_3$ by $\lim sup_{t \to T^-} r_i = +\infty$. Set
\[
a = \inf\{t < t_0 | r_i(s) > R_3, \forall s \in [t, t_0]\}.
\]

Then $a \in (0, t_0)$ by $R_3 \geq r_i(0) + 1$, $r_i(a) = R_3$ and $r_i(t) > R_3$ for all $t \in (a, t_0)$. Combining this with \ref{eq:3.26}, we have
\[
\frac{du_i}{dt} = -L^s(K - \overline{K})_i < 0
\]
for $t \in (a, t_0]$ along the hyperbolic fractional combinatorial Calabi flow \ref{eq:1.10}, which implies $\frac{dr_i}{dt} = \frac{du_i}{dt} < 0$ and then $r_i(t) < r_i(a) = R_3$ for $t \in (a, t_0]$. This contradicts to the fact that $r_i(t_0) > R_3$. Therefore, $\lim sup_{t \to T^-} r_i < +\infty$ for every $i \in V$, which implies $r_i(t)$ is bounded from above for every $i \in V$. \hfill \Box

As a direct corollary of Lemma \ref{lem:3.4} and Proposition \ref{prop:3.11}, we have the following result, which proves the longtime existence part of Theorem \ref{thm:1.2} (b).

**Corollary 3.12.** Suppose $(M, \mathcal{T}, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon \equiv 1$ and $\eta : E \to (-1, 1]$ satisfying the structure condition \ref{eq:1.7}. Suppose there exists $\pi \in \mathbb{R}^N_{\leq 0}$ such that $K(\pi) = \overline{K}$ in the hyperbolic background geometry.
Then the solution \( u(t) \) of the hyperbolic fractional combinatorial Calabi flow (1.10) stays in a compact subset of \( \mathbb{R}^N_{<0} \) and exists for all time.

The following of the proof for Theorem 1.2 (b), i.e. the convergence of the solution \( u(t) \) of hyperbolic fractional combinatorial Calabi flow (1.10) to \( \bar{u} \) in the case of Thurston’s hyperbolic circle packing metrics, is almost the same as that for Theorem 1.2 (a). We omit the details here.

4 Fractional combinatorial Calabi flow for vertex scaling on surfaces

Vertex scaling is a special type of discrete conformal structures in Definition 1 with \( \varepsilon \equiv 0 \). In the Euclidean background geometry, \( \varepsilon \equiv 0 \) implies \( l_{ij}^2 = \eta_{ij} e^{f_i + f_j} \) by Definition 1, which implies that

\[
\bar{l}_{ij} = l_{ij} e^{u_i + u_j}
\]

(4.1)

for two different PL metrics \( l \) and \( \bar{l} \) on \((M, \mathcal{T}, \varepsilon, \eta)\) with \( u = \bar{f} - f \). (4.1) is the definition of Euclidean vertex scaling on surfaces introduced independently by Luo [29] and Röcek-Williams [36]. In the hyperbolic background geometry, \( \varepsilon \equiv 0 \) implies \( \sinh^2 \frac{l_{ij}}{2} = \eta_{ij} e^{f_i + f_j} \) by Definition 1, which further implies that

\[
\sinh \frac{\bar{l}_{ij}}{2} = \frac{1}{2} e^{u_i + u_j}
\]

(4.2)

for two different PH metrics \( l \) and \( \bar{l} \) on \((M, \mathcal{T}, \varepsilon, \eta)\) with \( u = \bar{f} - f \). (4.2) is the definition of hyperbolic vertex scaling on surfaces introduced by Bobenko-Pinkall-Springborn [2]. By comparing with a fixed piecewise Euclidean or hyperbolic metric \( l_0 \) generated by vertex scaling on a triangulated surface, it is easy to check that (4.1) and (4.2) are equivalent to the definition of discrete conformal structures in Definition 1 with \( \varepsilon \equiv 0 \) in the Euclidean and hyperbolic background geometry respectively. In the following of this section, we will use (4.1) and (4.2) as the definition of Euclidean and hyperbolic vertex scaling respectively. The function \( u : V \to \mathbb{R} \) in (4.1) and (4.2) is a shift of the original discrete conformal factor \( f : V \to \mathbb{R} \) in Definition 1 by a constant function defined on \( V \) and also called as a discrete conformal factor.

4.1 The case of Euclidean vertex scaling

For the Euclidean vertex scaling in (4.1), the triangle inequalities are not always satisfied. As noted in Section 2, this causes that the fractional combinatorial Calabi flow (1.10) for Euclidean vertex scaling may develop singularities. To handle the potential singularities
along the fractional combinatorial Calabi flow (1.10) for Euclidean vertex scaling, we need
to do surgery on the flow (1.10) by edge flipping under the Delaunay condition. To prove
rigorously the longtime existence and convergence of the solution of Euclidean fractional
combinatorial Calabi flow with surgery, we need the discrete conformal theory developed
by Gu-Luo-Sun-Wu [22] for Euclidean vertex scaling. We briefly recall some main results
in [22] that we need in this paper. For more details, please refer to Gu-Luo-Sun-Wu’s
original work [22].

Gu-Luo-Sun-Wu [22] did not take the triangulations of \((M, V)\) as intrinsic structures
attached to the marked surface \((M, V)\), analogous to the coordinate charts on smooth
manifolds. The intrinsic structure on \((M, V)\) is the polyhedral metric, which is independent
of the triangulations. The combinatorial curvature \(K\) is also an intrinsic quantity for a
polyhedral metric on \((M, V)\). Based on this viewpoint, Gu-Luo-Sun-Wu [22] introduced
the following new definition of discrete conformality for PL metrics on \((M, V)\), which
allows the triangulations of \((M, V)\) to be changed by edge flipping under the Delaunay
condition.

**Definition 3** ([22] Definition 1.1). Two PL metrics \(d, d'\) on \((M, V)\) are discrete conformal
if there exist sequences of PL metrics \(d_1 = d, \ldots, d_m = d'\) on \((M, V)\) and triangulations
\(T_1, \ldots, T_m\) of \((M, V)\) satisfying

(a) (Delaunay condition) each \(T_i\) is Delaunay in \(d_i\),

(b) (Vertex scaling condition) if \(T_i = T_{i+1}\), there exists a function \(u : V \to \mathbb{R}\) so that if \(e\)
is an edge in \(T_i\) with end points \(v\) and \(v'\), then the lengths \(l_{d_{i+1}}(e)\) and \(l_{d_i}(e)\) of \(e\) in
\(d_i\) and \(d_{i+1}\) are related by \(l_{d_{i+1}}(e) = l_{d_i}(e)e^{u(v)+u(v')/2}\),

(c) if \(T_i \neq T_{i+1}\), then \((M, d_i)\) is isometric to \((M, d_{i+1})\) by an isometry homotopic to
identity in \((M, V)\).

The space of PL metrics discrete conformal to \(d\) on \((M, V)\) is called as the discrete con-
formal class of \(d\) and denoted by \(D(d)\). Gu-Luo-Sun-Wu [22] proved the following discrete
uniformization theorem for PL metrics with discrete conformality given by Definition 3.

**Theorem 4.1** ([22] Theorem 1.2). Suppose \((M, V)\) is a closed connected marked surface
and \(d\) is a PL metric on \((M, V)\). Then for any \(K : V \to (-\infty, 2\pi)\) with \(\sum_{v \in V} K(v) = 2\pi \chi(M)\),
there exists a PL metric \(\overline{d}\), unique up to scaling and isometry homotopic to the
identity on \((M, V)\), such that \(\overline{d}\) is discrete conformal to \(d\) and the discrete curvature of \(\overline{d}\)
is \(K\).

To prove Theorem 4.1 Gu-Luo-Sun-Wu [22] used the decorated Teichmüller space
theory established by Penner [33]. Denote the decorated Teichmüller space of all equiva-
lence class of decorated hyperbolic metrics on \(M - V\) by \(T_D(M - V)\) and the Teichmüller
space of all PL metrics on \((M, V)\) by \(T_{PL}(M, V)\). Gu-Luo-Sun-Wu \[22\] established the following correspondence between \(T_D(M - V)\) and \(T_{PL}(M, V)\).

**Theorem 4.2** (\[22\] Theorem 4.5, Corollary 4.7). There exists a \(C^1\)-diffeomorphism \(A : T_{PL}(M, V) \rightarrow T_D(M - V)\) such that \(A|_{D(d)} : D(d) \rightarrow \{p\} \times \mathbb{R}^V > 0\) is a \(C^1\)-diffeomorphism.

The construction of the map \(A\) is rather technical and we will not give the details of the construction here. What we need to use is the following property of the map \(A|_{D(d)} : D(d) \rightarrow \{p\} \times \mathbb{R}^V > 0\).

**Theorem 4.3** (\[22\] Proposition 5.2, \[29\] Theorem 1.2, Theorem 2.1, Corollary 2.3). Set \(u_i = \ln w_i\) for \(w = (w_1, w_2, \cdots, w_n) \in \mathbb{R}^n > 0\) and define

\[
F : \mathbb{R}^n \rightarrow (-\infty, 2\pi)^n
\]

\[u \mapsto K_{A^{-1}(p, w(-u))}\].

(4.3)

Then

1. for any \(k \in \mathbb{R}\), \(F(v + k(1, 1, \cdots, 1)) = F(v)\).

2. there exists a \(C^2\)-smooth convex function \(W = \int \sum_{i=1}^n F_i(u)du_i : \mathbb{R}^n \rightarrow \mathbb{R}\) so that its gradient \(\nabla W\) is \(F\) and the restriction \(W : \{u \in \mathbb{R}^n | \sum_{i=1}^n u_i = 0\} \rightarrow \mathbb{R}\) is strictly convex.

Suppose \(T\) is a triangulation of \((M, V)\), we use \(\Omega^T_D(d')\) to denote the admissible spaces of discrete conformal factors \(u\) such that \(T\) is Delaunay for \(d' \in D(d)\). As pointed out in \[22\], \(\mathbb{R}^n = \cup_T \Omega^T_D(d')\) is an finite analytic cell decomposition of \(\mathbb{R}^n\). This further implies that \(F\) in (4.3) is a \(C^1\) extension of \(K|_{\Omega^T_D(d')}\) for \(d' \in D(d)\). Furthermore, by Theorem 4.3, \((\frac{\partial F}{\partial u})\) is a positive semi-definite matrix with kernel \(\{t|t \in \mathbb{R}\}\). Therefore, we can define the following fractional discrete Laplace operator \(\tilde{\Delta}^s\) for discrete conformal factor \(u \in \mathbb{R}^N\).

**Definition 4.** Suppose \((M, V)\) is a marked surface with a PL metric \(d\) and \(s \in \mathbb{R}\) is a constant. The 2\(s\)-order Euclidean fractional discrete Laplace operator \(\tilde{\Delta}^s\) for \(u \in \mathbb{R}^n = \cup_T \Omega^T_D(d')\) is defined to be

\[
\tilde{\Delta}^s = -\left(\frac{\partial F}{\partial u}\right)^s.
\]

(4.4)

**Remark 8.** If \(s = 1\), the Euclidean fractional discrete Laplace operator \(\tilde{\Delta}^s\) in Definition 4 is the discrete Laplace operator introduced for discrete conformal factors \(u \in \mathbb{R}^N\) in \[22\][54]. By Definition 4, the fractional discrete Laplace operator \(\tilde{\Delta}^s = -\left(\frac{\partial F}{\partial u}\right)^s\) defined on \(\mathbb{R}^n = \cup_T \Omega^T_D(d')\) is a continuous extension of the fractional discrete Laplace operator \(\Delta^s = -\left(\frac{\partial K}{\partial u}\right)^s\) defined on \(\Omega^T_D(d')\). Similar to the case for Euclidean discrete Laplace
operator in\[3, 22, 54\], for a PL metric $d$ on $(M, V)$, the Euclidean fractional discrete Laplace operator $\tilde{\Delta}^s$ is an intrinsic operator in the sense that it is independent of the Delaunay triangulations of $(M, V)$ for $d$.

Using the fractional discrete Laplace operator $\tilde{\Delta}^s$ in Definition 4, the Euclidean fractional combinatorial Calabi flow (1.10) with surgery could be written as

$$\frac{du_i}{dt} = \tilde{\Delta}^s(F - \overline{K}).$$ (4.5)

As the right hand side of (4.5) is a $C^1$ function of $u \in \mathbb{R}^N$, the local existence for the solution of Euclidean fractional combinatorial Calabi flow with surgery (4.5) follows from the standard theory in ordinary differential equations.

Proof of Theorem 1.3 (a): By Theorem 4.1, Theorem 4.2 and Theorem 4.3, for $\overline{K} : V \to (-\infty, 2\pi)$ with $\sum_{i=1}^N \overline{K}_i = 2\pi \chi(M)$, there exists $\overline{\pi} \in \mathbb{R}^N$ up to a shift of a vector $t \mathbf{1}$, $t \in \mathbb{R}$, such that $F(\overline{\pi}) = \overline{K}$. Without loss of generality, we assume that $\sum_{i=1}^N \overline{\pi}_i = \sum_{i=1}^N u_i(0)$, where $u(0)$ is the initial value of (4.5).

By Theorem 4.3, the kernel of $\tilde{\Delta}$ is $\{t \mathbf{1} | t \in \mathbb{R}\}$, which implies the kernel of $\tilde{\Delta}^s$ is $\{t \mathbf{1} | t \in \mathbb{R}\}$. Then we have

$$\frac{d}{dt} \left( \sum_{i=1}^N u_i(t) \right) = \sum_{i=1}^N \tilde{\Delta}^s(F - \overline{K})_i = 1^T \tilde{\Delta}^s(F - \overline{K}) = 0$$

along (4.5) by $Ker(\tilde{\Delta}^s) = \{t \mathbf{1} | t \in \mathbb{R}\}$, which implies that $\sum_{i=1}^N u_i(t)$ is invariant along (4.5). Without loss of generality, assume that $u(0) \in \Sigma_0 = \{u \in \mathbb{R}^N | \sum_{i=1}^N u_i = 0\}$. By Theorem 4.2, we can define the following energy function

$$\overline{W}(u) = \int_{\overline{\pi}}^{u} \sum_{i=1}^N (F_i - \overline{K}_i) du_i,$$

which is a $C^2$ smooth convex function defined on $\mathbb{R}^N$ with $\nabla \overline{W} = F - \overline{K}$. Furthermore, by $F(\overline{\pi}) = \overline{K}$ and Theorem 4.3, we have $\nabla_u \overline{W}(\overline{u}) = 0$ and $\lim_{u \in \Sigma_0, u \to \infty} \overline{W}(u) = +\infty$.

Along the fractional combinatorial Calabi flow with surgery (4.5),

$$\frac{d\overline{W}(u(t))}{dt} = \sum_{i=1}^N \nabla_{u_i} \overline{W} \cdot \frac{du_i}{dt} = -(F - \overline{K})^T \left( \frac{\partial F}{\partial u} \right)^s (F - \overline{K}) \leq 0$$ (4.6)

by the property that $(\frac{\partial F}{\partial u})^s$ is positive semi-definite with kernel $\{t \mathbf{1} | t \in \mathbb{R}\}$, which implies that $\overline{W}(u(t)) \leq \overline{W}(u(0))$. Combining the fact that $u(t) \in \Sigma_0$ along (4.5) and $\lim_{u \in \Sigma_0, u \to \infty} \overline{W}(u) = +\infty$, this further implies that the solution $u(t)$ of the Euclidean fractional combinatorial Calabi flow with surgery (4.5) stays in a compact subset of $\Sigma_0$. Therefore, the solution $u(t)$ of (4.5) exists for all time. The following of the proof is the same as that for Theorem 1.2 (a). We omit the details here. □
4.2 The case of hyperbolic vertex scaling

For hyperbolic vertex scaling, we also need to do surgery by edge flipping under the hyperbolic Delaunay condition to handle the potential singularities along the fractional combinatorial Calabi flow \(1.10\). To prove the longtime existence and global convergence for the solution of fractional combinatorial Calabi flow with surgery in the case of hyperbolic vertex scaling, we need the discrete conformal theory established by Gu-Guo-Luo-Sun-Wu in [21]. Paralleling to the case of Euclidean vertex scaling, we only sketch the main results obtained by Gu-Guo-Luo-Sun-Wu in [21] for PH metrics that we need to use in this paper. For more details, please refer to Gu-Guo-Luo-Sun-Wu’s original work [21].

**Definition 5** ([21], Definition 1). Two PH metrics \(d, d'\) on a closed marked surface \((M, V)\) are discrete conformal if there exists sequences of PH metrics \(d_1 = d, d_2, \ldots, d_m = d'\) on \((M, V)\) and triangulations \(T_1, \ldots, T_m\) of \((M, V)\) satisfying

(a) (Delaunay condition) each \(T_i\) is Delaunay in \(d_i\),

(b) (Vertex scaling condition) if \(T_i = T_{i+1}\), there exists a function \(u : V \rightarrow \mathbb{R}\), called a conformal factor, so that if \(e\) is an edge in \(T_i\) with end points \(v\) and \(v'\), then the lengths \(x_{d_i}(e)\) and \(x_{d_{i+1}}(e)\) of \(e\) in metrics \(d_i\) and \(d_{i+1}\) are related by \(\sinh \frac{x_{d_{i+1}}(e)}{2} = e^{\frac{u(v)+u(v')}{2}} \sinh \frac{x_{d_i}(e)}{2}\),

(c) if \(T_i \neq T_{i+1}\), then \((M, d_i)\) is isometric to \((M, d_{i+1})\) by an isometry homotopic to the identity in \((M, V)\).

The space of PH metrics on \((M, V)\) discrete conformal to \(d\) is called as the conformal class of \(d\) and denoted by \(D(d)\). Gu-Guo-Luo-Sun-Wu [21] proved the following discrete uniformization theorem for PH metrics with discrete conformality given by Definition 5.

**Theorem 4.4** ([21], Theorem 3). Suppose \((M, V)\) is a closed connected surface with marked points and \(d\) is a PH metric on \((M, V)\). Then for any \(K : V \rightarrow (-\infty, 2\pi)\) with \(\sum_{v \in V} K(v) > 2\pi \chi(M)\), there exists a unique PH metric \(d'\) on \((M, V)\) so that \(d'\) is discrete conformal to \(d\) and the discrete curvature of \(d'\) is \(K\).

Following the Euclidean case in [22], Gu-Guo-Luo-Sun-Wu [21] denote the decorated Teichmüller space of all equivalence class of decorated hyperbolic metrics on \(M - V\) by \(T_D(M - V)\) and the Teichmüller space of all PH metrics on \((M, V)\) by \(T_{hp}(M, V)\). Based on Penner’s decorated Teichmüller space theory [33], Gu-Guo-Luo-Sun-Wu [21] established the following correspondence between \(T_D(M - V)\) and \(T_{hp}(M, V)\).

**Theorem 4.5** ([21] Theorem 22, Corollary 24). There exists a \(C^1\)-diffeomorphism \(A : T_{hp}(M, V) \rightarrow T_D(M - V)\) such that \(A|_{D(d)} : D(d) \rightarrow \{p\} \times \mathbb{R}^N_{>0}\) is a \(C^1\)-diffeomorphism.
Paralleling to the Euclidean case, Gu-Guo-Luo-Sun-Wu [21] further prove the following property of the map $A|D(d) : D(d) \to \{p\} \times \mathbb{R}_>^N$ in the hyperbolic background geometry.

**Theorem 4.6** ([21] Section 4.1, [2] Proposition 6.1.5). Set $u_i = \ln w_i$ for $w = (w_1, w_2, \cdots, w_N) \in \mathbb{R}_>^N$ and define

$$F : \mathbb{R}^n \to (-\infty, 2\pi)^N,$$

$$u \mapsto K_{A^{-1}(p, w(-u))}.$$  \hfill (4.7)

Then there exists a $C^2$-smooth strictly convex function $W = \int \sum_{i=1}^{N} F_i(u) du : \mathbb{R}^n \to \mathbb{R}$ with gradient $\nabla W = F$.

Similar to the Euclidean case, suppose $T$ is a triangulation of $(M, V)$, we use $\Omega_T^D(d')$ to denote the admissible spaces of hyperbolic discrete conformal factors $u$ such that $T$ is Delaunay for $d' \in D(d)$. Then $\mathbb{R}^N = \cup_T \Omega_T^D(d')$ is an finite analytic cell decomposition of $\mathbb{R}^N$ [21]. By Theorem 4.5 and Theorem 4.6 $F$ is a $C^1$ smooth function defined on $\mathbb{R}^N = \cup_T \Omega_T^D(d')$ and $(\partial F/\partial u)$ is a continuous strictly positive definite matrix defined for $u \in \mathbb{R}^N$. Specially, this implies that $(\partial F/\partial u)$ is independent of the Delaunay triangulation of $(M, V)$ for a PH metric $d' \in D(d)$. Following the Euclidean case, we can define the following hyperbolic fractional discrete Laplace operator.

**Definition 6.** Suppose $(M, V)$ is a marked surface with a PH metric $d$ and $s \in \mathbb{R}$ is a constant. The $2s$-order hyperbolic fractional discrete Laplace operator $\tilde{\Delta}^s$ for $u \in \mathbb{R}^n = \cup_T \Omega_T^D(d')$ is defined to be

$$\tilde{\Delta}^s = - \left( \frac{\partial F}{\partial u} \right)^s.$$  \hfill (4.8)

**Remark 9.** If $s = 1$, the hyperbolic fractional discrete Laplace operator $\tilde{\Delta}$ in Definition 6 is the hyperbolic discrete Laplace operator introduced for discrete conformal factors $u \in \mathbb{R}^N$ in [21,54]. By Definition 6, the fractional discrete Laplace operator $\tilde{\Delta}^s = -(\partial F/\partial u)^s$ defined on $\mathbb{R}^n = \cup_T \Omega_T^D(d')$ is a continuous extension of the fractional discrete Laplace operator $\Delta^s = -(\partial F/\partial u)^s$ defined on $\Omega_T^D(d')$. Similar to the hyperbolic discrete Laplace operator in [21,54], for a PH metric $d$ on $(M, V)$, the hyperbolic fractional discrete Laplace operator $\tilde{\Delta}^s$ is an intrinsic operator for $d$ in the sense that it is independent of the Delaunay triangulations of $(M, V)$ for $d$.

Using the hyperbolic fractional discrete Laplace operator $\tilde{\Delta}^s$ in Definition 6, the fractional combinatorial Calabi flow (1.10) with surgery for hyperbolic vertex scaling could be written as

$$\frac{du_i}{dt} = \tilde{\Delta}^s(F - K).$$  \hfill (4.9)
As the right hand side of (4.9) is a continuous function of $u \in \mathbb{R}^N$, the local existence for the solution of hyperbolic fractional combinatorial Calabi flow with surgery (4.9) follows from the standard theory of ordinary differential equations.

Proof of Theorem 1.3 (b): By Theorem 4.4, Theorem 4.5 and Theorem 4.6, for $K: V \to (-\infty, 2\pi)$ with $\sum_{i=1}^N K_i > 2\pi \chi(M)$, there exists a unique $\overline{u} \in \mathbb{R}^N$ such that $F(\overline{u}) = K$.

By Theorem 4.6, we can define the following energy function

$$W(u) = \int_{\pi}^{u} \sum_{i=1}^N (F_i - K_i) du_i,$$

which is a $C^2$ smooth strictly convex function defined on $\mathbb{R}^N$ with $\nabla W = F - \overline{K}$. Furthermore, by $F(\overline{u}) = \overline{K}$ and Theorem 4.6 we have $\nabla_u W(\overline{u}) = 0$ and $\lim_{u \to \infty} W(u) = +\infty$.

Along the hyperbolic fractional combinatorial Calabi flow with surgery (4.9), we have

$$\frac{dW(u(t))}{dt} = \sum_{i=1}^N \nabla_{u_i} W \cdot \frac{du_i}{dt} = -(F - \overline{K})^T \left( \frac{\partial F}{\partial u} \right)^s (F - \overline{K}) \leq 0 \quad (4.10)$$

by the property that $(\frac{\partial F}{\partial u})^s$ is a strictly positive definite matrix, which implies that $W(u(t)) \leq W(u(0))$ along (4.9). Combining with $\lim_{u \to \infty} W(u) = +\infty$, this further implies that the solution $u(t)$ of hyperbolic fractional combinatorial Calabi flow with surgery (4.9) stays in a compact subset of $\mathbb{R}^N$. Therefore, the solution $u(t)$ of (4.9) exists for all time. The following of the proof is the same as that for Theorem 1.2 (a). We omit the details here. □

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