Efficient algorithms for optimization problems involving semi-algebraic range searching

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Abstract

We present a general technique, based on parametric search with some twist, for solving a variety of optimization problems on a set of semi-algebraic geometric objects of constant complexity. The common feature of these problems is that they involve a ‘growth parameter’ $r$ and a semi-algebraic predicate $\Pi(o, o'; r)$ of constant complexity on pairs of input objects, which depends on $r$ and is monotone in $r$, meaning that if $\Pi(o, o'; r_1)$ is true then $\Pi(o, o'; r_2)$ is true for any $r_2 > r_1$. One then defines a graph $G(r)$ whose edges are all the pairs $(o, o')$ for which $\Pi(o, o'; r)$ is true, and seeks the smallest value of $r$ for which some graph-monotone property holds for $G(r)$.

Problems that fit into this context include (i) the reverse shortest path problem in unit-disk graphs, recently studied by Wang and Zhao [17], (ii) the same problem for weighted unit-disk graphs, with a decision procedure recently provided by Wang and Xue [16], (iii) extensions of these problems to three and higher dimensions, (iv) the discrete Fréchet distance with one-sided shortcuts in higher dimensions, extending the study by Ben Avraham et al. [7], (v) perfect matchings in intersection graphs: given, e.g., a set of fat ellipses of roughly the same size, find the smallest value $r$ such that if we expand each of the ellipses by $r$ (either additively or multiplicatively), the resulting intersection graph contains a perfect matching, (vi) generalized distance selection problems: given, e.g., a set of disjoint segments, find the $k$th smallest distance among the pairwise distances determined by the segments, for a given (sufficiently small but superlinear) parameter $k$ and an appropriate definition of distance between segments, and (vii) the maximum-height independent towers problem, in which we want to erect vertical towers of maximum height over a 1.5-dimensional terrain so that no pair of tower tops are mutually visible.

We obtain significantly improved solutions for problems (i), (ii) and (vi), and new efficient solutions to the other problems, which do not appear to have been studied earlier.

In general, our technique, when applicable, produces solutions that are significantly more efficient than those obtained by parametric search (or one of the alternative techniques), and unlike parametric search it does not require any parallelism.

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1 Introduction

In this work we generalize, in a very broad and comprehensive manner, a technique that was originally developed in Ben Avraham et al. [7] for solving the discrete Fréchet distance problem with one-sided shortcuts. We show that the high-level idea behind the technique is sufficiently versatile, so that, with suitable and rather nontrivial enhancements, it can be applied to many other optimization problems that have the underlying structure described in the abstract. Specifically, the input to these problems is a set \( S \) of \( n \) semi-algebraic geometric objects of constant complexity, in the plane or in higher dimensions, and a semi-algebraic predicate \( \Pi(o,o';r) \) of constant complexity on pairs of input objects, that also depends on a ‘growth parameter’ \( r > 0 \). The predicate \( \Pi \) is assumed to be monotone in \( r \), meaning that if \( \Pi(o,o';r_1) \) is true for some pair \( o,o' \in S \) then \( \Pi(o,o';r_2) \) is true for any \( r_2 > r_1 \). Thus each pair \( o,o' \in S \) has an associated critical value \( r_{o,o'} \), which is \( \min\{r \mid \Pi(o,o';r) \text{ is true}\} \) (in all our applications the minimum is attained). Define a ‘proximity graph’ \( G(r) \) on \( S \), whose edges are all the pairs \( (o,o') \) for which \( \Pi(o,o';r) \) is true. The optimization problem is to find the smallest value \( r^* \) of \( r \) for which some graph-monotone property holds for \( G(r^*) \).

As an example that clarifies these concepts consider the reverse shortest path problem for unit-disk graphs, recently studied by Wang and Zhao [17]. In this problem we are given a set \( P \) of \( n \) points in the plane, two points \( s,t \in P \), and an integer parameter \( k \), and we want to find the smallest value of \( r \) for which there is a path of at most \( k \) edges between \( s \) and \( t \) in \( G(r) \), which, in this special case, is the graph over \( P \), whose edges are all the pairs \( (u,v) \) of points of \( P \) at distance at most \( r \). The predicate \( \Pi(u,v;r) \) is simply \( \|u-v\| \leq r \), where \( \|\cdot\| \) denotes the Euclidean distance; clearly \( \Pi \) is monotone in \( r \), and the existence of a path of length \( k \) in \( G(r) \) is a graph-monotone property.

Wang and Zhao present a solution to this reverse shortest path problem that runs in \( O^*(n^{5/4}) \) time (where the \( O^*(\cdot) \) notation hides subpolynomial factors), in which the decision procedure takes linear time. We improve their result and obtain an algorithm that runs in randomized expected \( O^*(n^{6/5}) \) time.

We will shortly list many additional problems that fall into our context and the improved or new performance bounds that our technique yields.

Roughly speaking, one can think of our technique as a variant of parametric search, which aims to find the optimum value \( r^* \) of \( r \), on which the associated decision procedure is based. We assume that this procedure, which determines whether its input parameter \( r \) is larger than, smaller than, or equal to \( r^* \), is efficient, but we do not have an efficient way to parallelize it, which is required by standard parametric search to make the optimization procedure efficient. That is, if we could parallelize the decision procedure, say with polylogarithmic depth, we could apply standard parametric search and solve the optimization problem in time that is roughly the same as the cost of the decision procedure, up to a polylogarithmic factor. There are variants of parametric search, e.g., an expander-based approach [12], or techniques based on random sampling [10], where parallelism is not required. Nevertheless, standard implementations of these techniques also do not seem to achieve the improved bounds that we obtain here. For example, in a sampling-based approach, we want to take a random sample of a few critical values of \( r \), and run binary search through these values to narrow the range that contains the optimum \( r^* \). This is difficult to do when we only have a subrange of values from which we want to sample.

Standard parametric search simulates the execution of the decision procedure at (the unknown) \( r^* \), so that it determines the outcome of each comparison made by the procedure at \( r^* \),
by comparing $r^{\ast}$ with the $O(1)$ critical values of the comparison, at which it changes its sign, applying the (unsimulated) decision procedure itself at each (or at some) of these critical values to resolve the comparison. Instead, our variant employs a different approach, and simulates the decision procedure without resolving each of its comparisons right away. This results in a bifurcation tree, in which all possible outcomes ($r > r^{\ast}$, $r < r^{\ast}$, $r = r^{\ast}$) are explored. Only at certain steps during the execution we stop the construction of the tree, resolve all the comparisons that the algorithm has encountered, and are represented in the tree, follow the unique path in the tree that results from these comparisons (which represents a portion of the execution of the decision procedure at $r = r^{\ast}$), and then repeat this process, until the entire simulation of the decision procedure is completed.

In general, this technique would be too expensive, but we ensure its efficiency by applying it within a range $I = (\alpha, \beta]$ of values of $r$ that contains $r^{\ast}$ and only a prespecified small number $L$ of additional critical values. This will ensure that in most of the comparisons that the bifurcation tree encounters there will be no bifurcation because the relevant value of $r$ will lie outside $I$, so its relation to $r^{\ast}$ is known. The ‘interval-shrinking’ stage, which precedes the bifurcation-based simulation of the decision procedure, is performed using a fairly extensive generalization of the technique in [7]. That technique was designed in [7] for the special case where the critical values are distances between points in the plane, but we show that it can be extended to any kind of criticalities, as long as the critical values can be specified by a constant-complexity semi-algebraic predicate, which is the general situation that we assume here. This expansion of the context in which this technique can be applied is one of the main contributions of our paper.

The running time of the original procedure in [7], for distances between points in the plane, is $O^{\ast}(n^{4/3}/L^{1/3})$. In the more general setup considered here, the performance depends on the number of degrees of freedom of the input objects and on the cost of the decision procedure. For example, for distances between points in $\mathbb{R}^3$ and a near-linear decision procedure, the performance is $O^{\ast}(n^{3/2}/L^{1/2})$. See below for full details.

Combining the interval-shrinking step with the bifurcation-tree procedure, a careful choice of $L$ results in an efficient algorithm, whose running time depends on the number of degrees of freedom of the input objects and on the complexity of the decision procedure. For example, for distances between points in the plane and for a near-linear-time decision procedure, the algorithm runs in $O^{\ast}(n^{6/5})$ time.

We demonstrate the applicability of the resulting technique to numerous optimization problems that we now proceed to list.

(i) **The reverse shortest path problem for unit-disk graphs.** This already mentioned problem was recently studied by Wang and Zhao [17]. Recall that in this problem we are given a set $P$ of $n$ points in the plane, two points $s, t \in P$, and an integer parameter $k$, and we want to find the smallest value of $r$ for which there is a path of at most $k$ edges between $s$ and $t$ in $G(r)$, which is the graph over $P$, whose edges are all the pairs $(u, v)$ of points of $P$ at distance at most $r$. Wang and Zhao present a solution that runs in $O^{\ast}(n^{5/4})$ time, in which the decision procedure takes linear time. We improve their result and obtain an algorithm that runs in $O^{\ast}(n^{6/5})$ randomized expected time.

(ii) **The reverse shortest path problem for weighted unit-disk graphs.** We then consider the weighted variant of this problem, in which we want to find the minimum value $r^{\ast}$ such that $G(r^{\ast})$, defined as above, contains a path from $s$ to $t$ of length at most $w$, where each edge of $G(r^{\ast})$ has a weight equal to its Euclidean length, and the length of the path is the sum of the weights of its edges. The decision problem for this task, in which we specify $r$ and seek the shortest
path in $G(r)$ from $s$ to $t$, has been considered by Wang and Xue [16], who gave an algorithm that takes $O(n \log^2 n)$ time. Applying our technique and using this decision procedure, we solve the weighted variant of the reverse shortest path problem for unit-disk graphs in $O^*(n^{6/5})$ randomized expected time. Previously, Wang and Zhao [17] observed that this variant can be solved in $O(n^{4/3} \log^2 n)$ time, and mentioned the question of whether the $O(n^{4/3})$-time barrier can be broken as an interesting open problem.

(iii) Extensions of the reverse shortest path problems to three and higher dimensions. We then consider the reverse shortest-path problems from [17] and [16], in three and higher dimensions. To the best of our knowledge, distance selection, on which the method is based for the case of distances between points, has not been explicitly studied in higher dimensions. This is a simple example of the way in which the technique of [7] can be extended. More involved examples will be listed shortly.

The extension of each of the two reverse shortest-path problems (unweighted and weighted) to higher dimensions is interesting also because the decision procedure turns out to be more expensive and takes superlinear time, which is nonetheless faster than the cost of distance selection, on which the interval-shrinking part is based. As we argue, this suffices for obtaining an improved solution, whose running time lies ‘in between’ the two costs.

We solve both problems in three dimensions in $O^*(n^{17/12})$ randomized expected time. We sketch an extension of this result to any $d$ dimensions for the unweighted case, which runs in $O^*(n^{(6d-4)(d+1)/2}n^{d^2/(d+1)+d})$ randomized expected time, which is indeed faster than the cost $O^*(n^{2d/(d+1)})$ of distance selection.

(iv) The discrete Fréchet distance with shortcuts in higher dimensions. We then consider the Fréchet problem from [7] in $\mathbb{R}^d$, for any $d \geq 3$. In contrast with the reverse shortest path problem, the Fréchet distance problem has a decision procedure which is linear in any dimension. The cost of the distance selection procedure keeps growing with the dimension. The cost of the optimization procedure keeps growing too, but is always significantly smaller than the cost of distance selection. Concretely, our algorithm runs in $O^*(n^{(4d-2)/(3d-1)})$ randomized expected time.

(v) Perfect matchings in intersection and proximity graphs. We next consider applications of a different kind, involving intersection graphs of geometric objects. In these problems the critical values, determined by pairs of objects, are more involved than inter-point distances, and the underlying monotone graph property is the existence of a perfect matching in the graph.

Let $U$ be a set of $n = 2m$ disks in the plane with radii in the interval $[1, \Psi]$ for some constant parameter $\Psi$. The problem is to find the smallest value $r^*$ so that if we blow up each disk about its center by $r^*$, either additively or multiplicatively, the intersection graph of the expanded disks has a perfect matching.

Bonnet et al. [9] present an algorithm for computing a matching in the intersection graph $G$ of $U$ that, with high probability, is a maximum matching. The algorithm runs in $O(\Psi^6 n \log^{11} n + \Psi^{12 \omega \log n^{\omega/2}}) = O(n^{\omega/2})$ expected time, where $\omega \approx 2.37286$ is the exponent of matrix multiplication.

Let $U_r$ denote the set of the disks of $U$, each expanded by $r$. That is, we either add $r$ to its radius or multiply it by $r$. We apply the procedure of [9] to the intersection graph $G(r)$ of $U_r$ as a decision procedure for the search of the optimum value $r^*$. We observe that the requirement

\footnote{More precisely, the technique is based on a time-improving modification of the procedure for distance selection, namely the interval-shrinking procedure mentioned earlier.}
that the radii are between 1 and Ψ does not cause a problem. Indeed, when the blow-up of the disk radii is multiplicative, the ratio between the largest and smallest radii does not change and remains Ψ. When the blow-up is additive, the ratio only goes down, assuming that \( r^* > 0 \) (that is, the intersection graph of the original disks does not contain a perfect matching).

We show that one can compute the smallest \( r^* \) such that \( G(r^*) \) contains a perfect matching, in \( O^*(n^{3/4}+\omega/4) \approx O(n^{1.3432}) \) randomized expected time. To appreciate this bound, one should compare it with the appropriate bound for critical value selection, which is \( O^*(n^{3/2}) \); see a remark following Theorem 2.4.

**(vi) Generalized distance selection problems.** The next kind of applications that we consider are generalizations of the distance selection problem in the plane. Let \( S \) be a set of \( n \) pairwise disjoint segments in the plane. We define the distance between a pair of segments as the minimum value \( r \) such that, if we expand the length of each of the segments about its center by \( r \), then they intersect. The expansion can be either additive (i.e., we add \( r \) to each of the segment lengths) or multiplicative (i.e., we multiply each of the segment lengths by \( r \)). We want to find the \( k \)'th smallest distance among the \( \binom{n}{2} \) pairwise distances determined by the segments in \( S \), for a given parameter \( k \). This is equivalent to finding the smallest \( r \) for which there are exactly \( k \) vertices (i.e., intersection points) in the arrangement of \( S_r \), where \( S_r \) denotes the set of the segments of \( S \), each expanded by \( r \).

On the one hand, this problem can be solved, using more standard techniques (that we discuss later in the paper), in \( O^*(n^{8/5}) \) time. On the other hand, the corresponding decision problem — given \( r \), is the number of vertices in the arrangement of \( S_r \) greater than, equal to, or smaller than \( k \) — can be solved, using line sweeping, in \( O((n+k)\log n) \) time. Our technique allows us to ‘combine’ the two results and obtain a solution to the selection problem, which is more efficient than the best known one, provided that \( k \) is not too large. More precisely, we can find the \( k \)'th smallest distance in \( S \) in \( O^*(n^{8/11}k^{6/11}) \) randomized expected time, which is more efficient than the standard solution when \( k = o(n^{8/5}) \).

As another example, consider a set \( D \) of \( n \) pairwise disjoint disks of arbitrary radii in the plane. We seek the smallest \( r \), such that, if we expand each of the disks in \( D \) by \( r \) (i.e., we either add \( r \) to each of the radii, or multiply each of the radii by \( r \)), we get \( k \) intersecting pairs of disks, for some pre-specified integer \( 1 \leq k \leq \binom{n}{2} \). In this case, the problem can be solved using more standard techniques (that we discuss later) in \( O^*(n^{3/2}) \) time, and the corresponding decision problem can be solved using (a careful implementation) of the line-sweeping technique in \( O((n+k)\log n) \) time. Again, our technique allows us to ‘combine’ these results and obtain an algorithm that finds the smallest \( r \) such that there are \( k \) intersecting pairs in \( D_r \), where \( D_r \) is the set of the disks in \( D \), each expanded by \( r \). The algorithm runs in \( O^*(n^{3/4}k^{1/2}) \) randomized expected time, which is faster than the standard solution when \( k = o(n^{3/2}) \).

**(vii) The maximum-height independent towers problem.** Finally, in Section 4 we consider the maximum-height independent towers problem, in which we are given an \( x \)-monotone polygonal line \( T \) with \( n \) vertices, which we think of as a 1.5-dimensional terrain, and a set \( Q \) of \( m \) points on \( T \), and the goal is to find the maximum height \( h^* \) such that if we erect a vertical tower of height \( h^* \) at each point of \( Q \), the tips of the towers are mutually invisible (i.e., no segment connecting any two of them lies fully above \( T \)). This problem has a decision procedure (given \( h \), determine whether all the tower tips are mutually invisible), due to Ben Moshe et al. \([S]\), that runs in near-linear time. Using this procedure, we show that the optimization problem can be solved in \( O^*(n^{5/3}) \) randomized expected time.

The application of our technique to the maximum-height independent towers problem is
2 The underlying machinery: Interval shrinking and bifurcation trees

The material in this section provides the infrastructure, in rather full generality, on which our algorithms are based.

2.1 Shrinking the interval of critical values

Distances in a planar point set. We first recall the earlier result of [7], which handles the case of distances for planar point sets. We have a set $P$ of $n$ points in the plane, and a parameter $L \ll (n^2)$. The goal is to find an interval $(\alpha, \beta) \subset \mathbb{R}$ that (a) contains the optimum value $r^*$, and (b) contains at most $L$ distances determined by $P$.

Theorem 2.1 (Ben Avraham et al. [7]). Let $P$ and $L$ be as above. We can construct an interval $I = (\alpha, \beta) \subset \mathbb{R}$, such that $I$ contains the optimum value $r^*$, and $I$ contains at most $L$ distances determined by the points of $P$. The expected running time is $O^*(n^{4/3}L^{1/3} + D(n))$, where $D(n)$ is the cost of the decision procedure.

Remark. The term $O^*(D(n))$ in the above bound replaces the term $O(n \log n)$ in [7]. It comes from applying the decision procedure a logarithmic number of times in the interval shrinking algorithm. The cost of these applications is $O(n \log n)$ in [7] but in general it is $O(D(n) \log n) = O^*(D(n))$.

The general case of interval shrinking. In the general case we have an input set $S$ of $n$ semi-algebraic objects of constant complexity in any dimension $\mathbb{R}^d$ (e.g., in the plane we may have segments, disks, ellipses, etc.), and a semi-algebraic predicate $\Pi(o, o'; r)$ of constant complexity, over pairs $o, o'$ of objects of $S$, which is monotone in an additional ‘growth parameter’ $r$. Each pair $o, o'$ defines a critical value $r_{o,o'}$, which is the minimum $r$ for which $\Pi(o, o'; r)$ is true. (As already mentioned, the minimum indeed exists in all our applications.)

The general interval-shrinking problem goes as follows. We are given a parameter $L \ll \binom{n}{2}$ and seek an interval $I = (\alpha, \beta) \subset \mathbb{R}$, such that $I$ contains the optimum value $r^*$, and $I$ contains at most $L$ critical values determined by the objects of $S$.

The interval-shrinking procedure of [7] is related to the classical distance selection algorithm of [3, 12], and our general procedure is similarly related to a corresponding critical-value selection problem, in which we want to compute the $k$-th smallest critical value determined by pairs of objects of $S$. This problem is of independent interest, and can be solved using a suitable simpler variant of the technique presented below, as will be noted later.

The general interval-shrinking problem can be solved by adapting and extending the high-level approach in [7]. The technical details, though, are different and more involved, as they
rely on recent techniques for semi-algebraic range searching. Specifically, let \( t \) be the number of degrees of freedom of the objects in \( S \), namely the number of real parameters needed to specify an object. In the generalization of the main step of the procedure of \cite{7}, we have an interval \( I = (r_1, r_2) \), known to contain \( r^* \), and we want to determine whether it contains at most \( L \) critical values. If this is the case we terminate the procedure and return \( I \). If not, we want to shrink \( I \) into a subinterval \( I' \) that contains at most some fixed fraction of the number of the critical values in \( I \) (while still containing \( r^* \)).

To perform this step we reduce the problem to the following problem of batched range searching with semi-algebraic ranges. We represent the objects of \( S \) as \( n \) points in \( \mathbb{R}^l \), denoting the resulting set of points as \( P \), and also map each object \( o \in S \) to the range

\[
\sigma_o = \{ u \in \mathbb{R}^l \mid r_1 < r_{u,o} \leq r_2 \},
\]

which, by our assumptions, is semi-algebraic of constant complexity\footnote{Here \( u \) is an arbitrary point in \( \mathbb{R}^l \), designating an arbitrary possible range, not necessarily in \( S \). This reduction still works even when we let \( \sigma_o \) include points \( u \) for which there is no actual range that \( u \) designates.}. Let \( \Sigma \) denote the resulting collection of ranges. We have \( |P| = |\Sigma| = n \), but it will be more convenient to consider a bipartite version of the problem, in which \( P \) and \( \Sigma \) come from different subsets of \( S \) and may have different sizes. Put \( m = |P| \) and \( n = |\Sigma| \). Note also that the problem has a symmetric dual version, in which we map the ranges of \( \Sigma \) to points in \( \mathbb{R}^l \), and the points of \( P \) to constant-complexity semi-algebraic ranges in \( \mathbb{R}^l \). Both versions will be used in the algorithm.

**The semi-algebraic range-searching mechanism.** We adapt the range searching machinery of \cite{14} (or of \cite{2}), but, in order to gain efficiency, we need to terminate the hierarchical process prematurely. For this we have to go into the inner workings of the technique of \cite{14}. Concretely, we recall the following main technical result of \cite{14} (where the notations have been changed to avoid overloading of notations introduced earlier and to conform to our setup):

**Theorem 2.2** (Matoušek and Patáková \cite{14} Theorem 1.1). For every integer \( t > 1 \) there is a constant \( K \) such that the following hold. Given an \( n \)-point set \( P \subset \mathbb{R}^l \) and a parameter \( s > 1 \), there are numbers \( s_1, s_2, \ldots, s_t \in [s, s^K] \), positive integers \( \xi_1, \xi_2, \ldots, \xi_t \), a partition

\[
P = P^* \cup \bigcup_{i=1}^{t} \bigcup_{j=1}^{\xi_i} P_{ij}
\]

of \( P \) into disjoint subsets, and for every \( i, j \), a connected set \( S_{ij} \subset \mathbb{R}^l \) containing \( P_{ij} \), such that \( |P_{ij}| \leq n/s_i \) for all \( i, j \), \( |P^*| \leq s^K \), and the following holds:

If \( h \in \mathbb{R}[x_1, \ldots, x_l] \) is a polynomial of degree bounded by a constant \( D_0 \), and \( X = Z(h) \) is its zero set, then, for every \( i = 1, 2, \ldots, t \), the number of the sets \( S_{ij} \) crossed by \( h \) (intersected by but not contained in) \( X \) is at most \( O \left( s_i^{1-1/t} \right) \), with the implicit constant also depending on \( D_0 \).

The interval-shrinking procedure is carried out as follows. We fix a constant parameter \( s \). If \( m \geq n \) we apply Theorem 2.2 to \( P \subset \mathbb{R}^l \), and if \( m \leq n \) we apply Theorem 2.2 to the set \( \Sigma^* \) of points dual to the ranges of \( \Sigma \). Assume without loss of generality that \( m \geq n \), and follow the notations in the theorem.

For each \( i = 1, 2, \ldots, t \), each of the boundary surfaces of the ranges in \( \Sigma \) crosses at most \( O \left( s_i^{1-1/t} \right) \) sets \( S_{ij} \). Denote by \( \Sigma_{ij} \) (resp., \( \Sigma^0_{ij} \)) the subset of ranges whose boundary surfaces cross \( S_{ij} \) (resp., that fully contain \( S_{ij} \), and thus \( P_{ij} \)). Put \( q_{ij} = |\Sigma_{ij}| \). Then we have, for each
$i, \sum_{j=1}^{\xi_i} q_{ij} = O\left(ns_i^{1-1/t}\right)$. For each $i = 1, \ldots, t$ and each $j = 1, \ldots, \xi_i$, we (implicitly) form the biclique $P_{ij} \times \Sigma_{ij}^0$, and output its vertex sets, and face a subproblem involving $P_{ij}$ and $\Sigma_{ij}$, of respective sizes at most $n/s_i$ and $q_{ij}$, which we solve recursively, possibly switching to the dual setup, depending on which of these two sizes is larger. We stop the recursion when both of the sizes become smaller than $L$. 

At the bottom of recursion, we also output (the vertex sets of) the corresponding bicliques $P_{ij} \times \Sigma_{ij}$. We note that the first kind of bicliques $P_{ij} \times \Sigma_{ij}^0$ has the property that every edge $(o, o')$ in any such biclique satisfies $r_1 < r_{o, o'} \leq r_2$, but this is not necessarily the case for the second kind of bicliques $P_{ij} \times \Sigma_{ij}$. This is the setup that the machinery in [7] processes.

The efficiency of the procedure is determined by two comparable quantities: the overall size $K(m, n)$ of the vertex sets of the bicliques that we output, and the time $T(m, n)$ used by the procedure. It follows from the definition of the procedure and from Theorem 2.2 that the two quantities are asymptotically the same, so we will analyze just one of them, say $K(m, n)$.

**Lemma 2.3.** For any $\varepsilon > 0$ there is a constant $A$ that depends on $\varepsilon$, such that, for any set of at most $m$ points in $\mathbb{R}^t$ and any set of at most $n$ ranges in $\mathbb{R}^t$, of the type considered above, we have

$$K(m, n) \leq A \left( \frac{m^{t/(t+1)+\varepsilon} n^{t/(t+1)+\varepsilon}}{L^{(t-1)/(t+1)}} + m^{1+\varepsilon} + n^{1+\varepsilon} \right).$$

**Proof.** The proof is by induction. The base case $m, n \leq L$ is trivial since then $K(m, n) = O(m + n)$ (this is the overall size of the vertex sets of the second kind of bicliques that the procedure outputs; the linear bound follows since $s$ is assumed to be constant—see below). Consider then the case where, say, $m \geq n$ and $m > L$, and assume that (1) holds for all smaller values $m' \leq m$, $n' < n$ and $m' < m$, $n' \leq n$. Apply Theorem 2.2 to the primal setup (we would apply it in the dual in the complementary case where $n \geq m$ and $n > L$), and use the notations in the theorem. Since $s$ is a constant, the nonrecursive cost of the procedure is at most $B(m + n)$, where $B$ is a constant that depends on $s$ and the various other constant parameters of the setup. This also takes care of processing $P^*$, which is of constant size. By induction hypothesis, for each $i$ and $j$, the cost of the recursive processing of $P_{ij}$ and $\Sigma_{ij}$ is at most

$$A \left( p_{ij}^{t/(t+1)+\varepsilon} q_{ij}^{t/(t+1)+\varepsilon} L^{(t-1)/(t+1)} + p_{ij}^{1+\varepsilon} + q_{ij}^{1+\varepsilon} \right),$$

where $p_{ij} = |P_{ij}|$. Observe that we have $p_{ij} \leq m/s_i$ for each $j$, and the quantities $m_i := \sum_{j=1}^{\xi_i} p_{ij}$ satisfy $\sum_{i=1}^{d} m_i \leq m$ (since the decomposition in Theorem 2.2 is into disjoint subsets). Recall also that $\sum_{j=1}^{\xi_i} q_{ij} \leq cns_i^{1-1/t}$ for every $i$ and for a suitable absolute constant $c$.

We now sum the bounds in (2) over $j$ for each fixed $i$. Using Hölder’s inequality, the sum is
upper bounded by
\[
A \left( \sum_{j=1}^{s_i} \frac{p_{ij}^{t/(t+1)+\varepsilon} q_{ij}}{L(t-1)/(t+1)} + \sum_{j=1}^{s_i} p_{ij}^{1+\varepsilon} + \sum_{j=1}^{s_i} q_{ij}^{1+\varepsilon} \right)
\leq A \left( \frac{(m/s_i)^{(1-1/t)}}{L(t-1)/(t+1)} + \sum_{j=1}^{s_i} p_{ij}^{1+\varepsilon} + \sum_{j=1}^{s_i} q_{ij}^{1+\varepsilon} \left( cn_{s_i}^{1-1/t} \right)^{t/(t+1)+\varepsilon} + m_i^{1+\varepsilon} \right)
\leq A \left( \frac{m^{t/(t+1)+\varepsilon} n^{t/(t+1)+\varepsilon}}{L(t-1)/(t+1)} + \sum_{j=1}^{s_i} p_{ij}^{1+\varepsilon} + \sum_{j=1}^{s_i} q_{ij}^{1+\varepsilon} \frac{\epsilon/(t+1)+\varepsilon}{s_i^{(1+\varepsilon)/t}} + m_i^{1+\varepsilon} \right).
\]
Assuming that \(s_i\) is chosen sufficiently large, we may assume that the factor \(\frac{\epsilon/(t+1)+\varepsilon}{s_i^{(1+\varepsilon)/t}}\) is smaller than \(1/(4d)\). The problematic factor is \(Q := (cs_i^{1-1/t})^{1+\varepsilon}\) in the last term, which in general will be larger than 1, making the induction step problematic. To address this issue, one can easily verify that, under the assumption that \(m \geq n\) and \(m > L\), we have
\[
n \leq \frac{m^{t/(t+1)}}{L(t-1)/(t+1)} n^{t/(t+1)} L(t-1)/(t+1), \quad \text{or} \quad n^{1+\varepsilon} \leq \frac{m^{t/(t+1)+\varepsilon} n^{t/(t+1)+\varepsilon}}{L^{t-1}(1+\varepsilon)/(t+1)}.
\]
This allows us to write
\[
AQn^{1+\varepsilon} \leq A \frac{m^{t/(t+1)+\varepsilon} n^{t/(t+1)+\varepsilon}}{L^{t-1}/(t+1)} \cdot \frac{Q}{m^{\epsilon/(t+1)+\varepsilon} n^{\epsilon/(t+1)} L^{t-1} e/(t+1)},
\]
and we can make the second factor smaller than \(1/(4d)\) by assuming that \(m, n, L\) are sufficiently large. (When \(m\) and \(n\) are small, \(1\) will hold by choosing \(A\) sufficiently large.)

We now sum up the modified bounds (in which the inequality in [3] is substituted into the bound) over \(i = 1, \ldots, d\), and obtain the overall bound
\[
A \sum_{i=1}^{d} \left( \frac{m^{t/(t+1)+\varepsilon} n^{t/(t+1)+\varepsilon}}{L^{t-1}/(t+1)} \cdot \left( \frac{1}{4d} + \frac{1}{4d} \right) + m_i^{1+\varepsilon} \right) \leq A \left( \frac{m^{t/(t+1)+\varepsilon} n^{t/(t+1)+\varepsilon}}{L^{t-1}/(t+1)} + Am^{1+\varepsilon} \right).
\]
This, and the similar analysis of the symmetric case \(n \geq m\) and \(n > L\), establish the induction step and thus complete the proof of the lemma. \(\square\)

The lemma implies the following main result of this section.

**Theorem 2.4.** Given a set \(S\) of \(n\) semi-algebraic geometric objects of constant complexity (in any dimension) with \(t\) degrees of freedom, a growth parameter \(r\) and an \(r\)-monotone predicate \(\Pi(o, o'; r)\) over pairs \(o, o' \in S\), and a parameter \(n \leq L \ll \binom{t}{2}\), we can find an interval \(I\) that contains the optimum value \(r^*\) and contains at most \(L\) critical values determined by pairs of objects in \(S\), in randomized expected time \(O^*(n^{2t/(t+1)} / L^{t-1}/(t+1) + D(n))\).

**Remark.** A simplified version of the above machinery, in which we run the range searching algorithm to completion, yields an algorithm for critical value selection, in which we want to compute the \(k\)th smallest critical value among those determined by a set of \(n\) objects, as defined above, that runs in randomized expected time \(O^*(n^{2t/(t+1)})\).

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Shrinking the interval of critical distances in three and higher dimensions. As an illustration of the general machinery described above, consider the case where the objects are points in $\mathbb{R}^d$, for any $d \geq 3$, and the critical values are the distances between the points. In this special case the problem involves range searching in $t = d$ dimensions; the ranges are actually spherical shells with inner radius $r_1$ and outer radius $r_2$. Theorem 2.4 then implies:

**Theorem 2.5.** Given a set $P$ of $n$ points in $\mathbb{R}^d$, for any $d \geq 2$, and a parameter $n \leq L \ll \binom{d}{2}$, we can find an interval $I$ that contains the optimum value $r^*$ and contains at most $L$ distances determined by $P$, in randomized expected time $O^*(n^{2d/(d+1)} / L^{(d-1)/(d+1)} + D(n))$.

In particular, for $d = 3$ the expected running time is $O^*(n^{3/2}/L^{1/2} + D(n))$.

**Remark.** We have focused here on examples where the critical values are inter-point distances, but the machinery in Theorem 2.4 is clearly much more widely applicable, and several of our applications, given in subsequent sections, will use the theorem in more general contexts.

### 2.2 Bifurcation-tree construction

We now consider, in fairly full generality, the second step of the algorithm, which simulates the decision procedure using bifurcation trees. This part is similar to the corresponding procedure in [7], but requires several modifications to make it more broadly applicable.

Consider an optimization problem $\text{OPT}$ that involves a set $S$ of $n$ objects as above, and seeks the minimum value $r^*$ of some real parameter $r$ at which some property (that is monotone in $r$) holds for the set of critical values determined by pairs of objects of $S$, with respect to an $r$-monotone predicate $\Pi(o,o';r)$ as defined above. We assume that the decision procedure accesses its parameter $r$ only through comparison tests against concrete values. In particular, this is how the simulation of the decision procedure at the unknown $r^*$ accesses $r^*$. Let $D(n)$ denote an upper bound on the number of such comparisons in any execution of the decision procedure. We also assume that the overall running time of the decision procedure is $O(D(n))$.

Assume that we have already applied the interval-shrinking procedure for some value $L$ that will be determined shortly, and denote by $I$ the resulting shrunken interval. We choose another parameter $s$, and simulate the execution of the decision procedure at (the unknown) $r^*$. At each comparison test that we encounter, with some concrete value $r_0$, we check whether $r_0$ lies inside or outside $I$. If $r_0$ lies outside $I$ we know the result of the comparison (because $r^* \in I$), and follow the unique way in which the simulation of the decision procedure at $r^*$ proceeds; we denote this step as an *outdegree-one* step. If $r_0$ lies inside $I$, we bifurcate, following an execution path at which $r^* < r_0$, and a path at which $r^* > r_0$, referring to this step as an *outdegree-two* step. There is also the possibility that $r^* = r_0$, but in this case (if it is the correct one) the decision procedure terminates, so this case does not cause any further expansion of the tree beyond this test.

We continue to expand the resulting bifurcation tree $T$ in this manner. We stop each branch of the tree (a path of the tree that starts at a bifurcation node and consists of only outdegree-one nodes) when it accumulates $s$ nodes beyond its bifurcation node. If we reach another bifurcation before encountering $s$ outdegree-one nodes, we bifurcate there and start the count afresh for each of the two outgoing paths. We stop the entire construction of the tree either when each of its terminal branches contains $s$ outdegree-one nodes, or when the tree comes to contain $D(n)$ nodes, whichever happens earlier. When we stop the construction, we end a phase of the algorithm. At that time we collect the outdegree-two nodes of $T$ that we have created, and run a binary
search through the sequence of the corresponding critical values of $r$, using the unsimulated
decision procedure to guide the search. This yields two consecutive critical $r$-values $r_1, r_2 \in I$
that enclose $r^*$. This allows us to continue the simulation of the decision procedure, in a unique
manner, from its state at the root of $T$ to its state at the suitable leaf. We also replace $I$ by
the smaller interval $(r_1, r_2)$, which still contains $r^*$. We then start a new phase, executed in the
same manner, from the final state of the previous phase, and with the new, further shrunken
interval $I$, and continue to do so until we complete the simulation of the decision procedure, or,
more precisely, when we reach a comparison that has $r^*$ as a critical value, an event that must
occur during the simulation.

Suppose that we stop because of the first condition, namely each branch of the tree that ends
at a leaf contains $s$ outdegree-one nodes; this is a ‘successful’ phase. In particular, this implies
that, for every value in $I$ encountered in this phase, the decision procedure has made at least
$s$ comparisons during this phase with critical values outside $I$. It follows that the simulated
procedure will run to completion after at most $D(n)/s$ successful phases. Since the cost of
each phase is $O(D(n) \log n)$ (to construct the tree and to run the binary search through its
bifurcations), the overall cost of the successful phases is $O \left( \frac{D^2(n) \log n}{s} \right)$.

Consider next the case where we stop because of the second condition, namely $T$ contains
$D(n)$ nodes; this is an ‘unsuccessful’ phase. If we denote by $x$ the number of outdegree-two
nodes in $T$, then the length of each branch of the tree between two consecutive outdegree-two
nodes is at most $s$, so $D(n) \leq xs$, or $x \geq D(n)/s$. In other words, in an unsuccessful phase we
encounter at least $D(n)/s$ critical values in $I$, and the binary search through them will shrink $I$
further, to a subinterval that contains none of these values (except for its two endpoints). Hence
the number of unsuccessful phases is at most $L/(D(n)/s) = Ls/D(n)$, and their overall cost is

$$O \left( \frac{LsD(n) \log n}{D(n)} \right) = O \left( Ls \log n \right).$$

We balance these two bounds by choosing $s = D(n)/L^{1/2}$, noting that this value of $s$ is at
least 1, and thereby obtain the following proposition.

**Proposition 2.6.** The overall cost of the bifurcation tree portion of the algorithm, following a
suitable interval-shrinking step that produces an interval containing $r^*$ and at most $L$ critical
values of $r$, is $O \left( L^{1/2} D(n) \log n \right)$.

Note that this part of the algorithm is fairly general and is independent of the specific problem
at hand. All it requires is that we have an interval $(\alpha, \beta)$ that is known to contain at most some
number $L$ of critical values. (It also assumes, or rather requires, that the decision procedure
accesses the parameter $r$ only via comparisons, which can also be sign tests of constant-degree
polynomials in $r$.)

### 2.3 The overall algorithm

We now balance the cost of the bifurcation-tree part with that of the interval-shrinking part.
The actual balancing depends on the number $t$ of degrees of freedom of the input objects. For
problems involving distances between points in the plane, the overall cost is $O^* \left( n^{4/3}/L^{1/3} +
L^{1/2}D(n) \right)$, so we choose $L^{5/6} = n^{4/3}/D(n)$, or $L = n^{8/5}/D(n)^{6/5}$, making the overall cost of the
algorithm $O^* \left( n^{4/5}D(n)^{2/5} \right)$.

When the decision procedure takes (near) linear time, the running time is $O^* \left( n^{6/5} \right)$. The
new bound lies strictly between $D(n)$ and $O^* \left( n^{4/3} \right)$ for $D(n) = o(n^{4/3})$ and $D(n) = \omega(n)$.
For problems involving distances in three dimensions, the cost of the interval-shrinking procedure is $O^*(n^{3/2}/L^{1/2})$ (see Theorem 2.5), so the balancing leads to choosing $L = n^{3/2}/D(n)$, making the overall cost of the algorithm $O^*(n^{3/4}D(n)^{1/2})$. For linear-time decision procedures, this becomes $O^*(n^{5/4})$. In full generality, we have:

**Theorem 2.7.** Let $\text{Opt}$ be an optimization problem on a set $S$ of $n$ semi-algebraic objects of constant complexity, in any dimension, which have $t$ degrees of freedom. Assume that $\text{Opt}$ depends on critical values determined by pairs of objects of $S$ and a growth parameter $r$, in terms of an $r$-monotone predicate $\Pi(o, o'; r)$ as defined above. Assume further that $\text{Opt}$ has a decision procedure that is based (solely) on comparisons involving critical values determined by pairs of objects of $S$, so that it uses at most $D(n)$ such comparisons, and runs in overall $O(D(n))$ time.

We can then solve $\text{Opt}$ by an algorithm that runs in $O^*(n^{2t/(3t-1)}D(n)^{(2t-2)/(3t-1)})$ randomized expected time.

## 3 Applications

### 3.1 The reverse shortest path problem for unit-disk graphs

![Figure 1: The smallest value $r^*$ for which $G(r^*)$ has a path between $s$ and $t$ of length at most 6.](image)

In this problem we are given a set $P$ of $n$ points in the plane, two points $s, t \in P$, and a parameter $k \leq n - 1$, and the goal is to compute the smallest value $r^*$ so that there exists a path between $s$ and $t$ of length at most $k$ in the graph $G(r^*)$, which is the graph over $P$ whose edges are all the pairs $p, q \in P$ such that $\text{dist}(p, q) \leq r^*$; see Figure 1.

This problem has recently been studied in Wang and Zhao [17], who presented a solution that runs in $O^*(n^{5/4})$ time. Wang and Zhao use a decision procedure for this problem, due to Chan and Skrepetos [11], where $r$ is specified and the goal is to determine whether $G(r)$ contains a path from $s$ to $t$ of length at most $k$. The procedure is based on a careful execution of BFS in $G(r)$, and runs in $D(n) = O(n)$ time (after an initial sorting step).

Chan and Skrepetos’ procedure constructs a uniform grid of cell size $r/\sqrt{2}$, where $r$ is the input parameter, and distributes the points of $P$ among the grid cells. Naïvely implemented (using the floor function, say), this step does not conform to our requirements, that the data be accessed only via comparisons. Nevertheless we can distribute the points of $P$ using simple binary search. To do so, let $a$ denote the distance $\text{dist}(s, t)$. Note that $r^* \leq a$, because $G(a)$
has a single edge connecting \( s \) to \( t \). We also have \( r^* \geq a/k \), because for smaller values of \( r^* \) there is no path of length \( k \) in \( G(r^*) \) that connects \( s \) and \( t \) (this is a consequence of the triangle inequality). We therefore have \( a/k \leq r^* \leq a \). Assume then that \( r \) satisfies these inequalities (other values of \( r \) can be rejected right away), and observe that every point on a \( k \)-path from \( s \) to \( t \) in \( G(r) \) must be at distance at most \( \Theta(ka) \) from \( s \). This implies that any such path must be fully contained in a subgrid of size \( O(k^2) \times O(k^2) \) around \( s \). Hence, using binary search through the \( x \)- and \( y \)-coordinates of this subgrid we can distribute the relevant points of \( P \) in the cells of that subgrid, using only comparisons.

Plugging this into our mechanism, we obtain

**Theorem 3.1.** The reverse shortest path problem for unit-disk graphs, as just formulated, can be solved in \( O^*(n^{6/5}) \) randomized expected time.

By setting \( k = n - 1 \) we get the following corollary, which is of independent interest.

**Corollary 3.2.** Given \( P \) and \( s, t \in P \), as above, one can find the smallest value \( r^* \) so that there exists a path between \( s \) and \( t \) in \( G(r^*) \), in \( O^*(n^{6/5}) \) randomized expected time.

### 3.2 The reverse weighted shortest path problem for unit-disk graphs

In the weighted variant of the reverse shortest path problem, we want to find the minimum value \( r^* \) such that \( G(r^*) \), defined as above, contains a path from \( s \) to \( t \) of length at most \( w \), where each edge of \( G(r^*) \) has a weight, equal to its Euclidean distance, and the length of the path is the sum of the weights of its edges.

The decision procedure for this problem has been studied by Wang and Xue [16], who showed that, for a given \( r \) and a source point \( s \), all shortest paths in \( G(r) \) from \( s \) to the other points of \( P \) can be computed in \( O(n \log^2 n) \) time.

As in the unweighted case, the decision procedure constructs a uniform grid of cell size \( r/2 \), and distributes the points of \( P \) among the grid cells. We modify the procedure as we did in the unweighted case, and then apply our machinery to the modified procedure. More precisely, we now have \( a \leq w \) (by the triangle inequality), and \( \frac{a}{n} \leq r^* \leq a \leq w \), where the left inequality is again a consequence of the triangle inequality. For any \( \frac{a}{n} \leq r \leq a \), any point on a path from \( s \) to \( t \) in \( G(r) \) must lie at distance at most \( an \) from \( s \), which implies that any such path is fully contained in an \( O(n^2) \times O(n^2) \) grid around \( s \), and the same argument as before, replacing \( k \) by \( n \), lets us distribute the points of \( P \) in this subgrid by binary search, using only comparisons.

Plugging this into our mechanism, we obtain

**Theorem 3.3.** The reverse weighted shortest path problem for unit-disk graphs, as just formulated, can be solved in \( O^*(n^{6/5}) \) randomized expected time.

### 3.3 Perfect matching in disk intersection graphs

Let \( \mathcal{U} \) be a set of \( n = 2m \) disks in the plane with radii in the interval \([1, \Psi]\) for some constant parameter \( \Psi \). The problem is to find the smallest value \( r^* \) so that if we blow up each disk about its center by \( r^* \), either additively or multiplicatively, the intersection graph of the expanded disks has a perfect matching, see Figure 2.
Figure 2: A set $\mathcal{U}$ of six disks (drawn in black). In this example, $r$ is a multiplicative parameter, and $r^*$ is determined by the pair $U_1, U_3$. That is, if we blow up each of the disks of $\mathcal{U}$ by $r^*$, then the intersection graph of the blown up disks (drawn in red) has a perfect matching (drawn in blue), and $r^*$ is the smallest such value.

Bonnet et al. [9] present an algorithm for computing a matching in the intersection graph $G$ of $\mathcal{U}$ that, with high probability, is a maximum matching. The algorithm runs in

$$O(\Psi^6 n \log^{11} n + \Psi^{12}\omega n^{\omega/2}) = O(n^{\omega/2})$$

expected time, where $\omega \approx 2.37286$ is the exponent of matrix multiplication. By running the algorithm $O(\log n)$ times, we can make its failure probability sufficiently small (polynomially small in $n$).

Let $\mathcal{U}_r$ denote the set of the disks of $\mathcal{U}$, each expanded by $r$. That is, we either add $r$ to its radius or multiply it by $r$. We apply the procedure of [9] to the intersection graph $G(r)$ of $\mathcal{U}_r$ as a decision procedure for the search of the optimum value $r^*$. We observe that the requirement that the radii are between 1 and $\Psi$ does not cause a problem. Indeed, when the blow-up of the disk radii is multiplicative, the ratio between the largest and smallest radii does not change and remains $\Psi$. When the blow-up is additive, the ratio only goes down, assuming that $r^* > 0$ (that is, that the intersection graph of the original disks does not contain a perfect matching).

The critical values here are values of $r$ at which two expanded disks touch one another, see Figure 2. Specifically, when the blow-up is multiplicative, the critical value induced by a pair of disks, with respective centers $c, c'$ and radii $\rho, \rho'$, is $|cc'|/(\rho + \rho')$. When the blow-up is additive, the critical value is $\frac{1}{2} (|cc'| - \rho - \rho')$. This is the first application where the critical values are not distances.

Plugging $t = 3$ (the number of degrees of freedom of a disk) and $D(n) = O^*(n^{\omega/2})$ in the general bound given in Theorem 2.4, we obtain:

**Theorem 3.4.** In the above setting, one can compute, with arbitrarily large probability, the smallest $r^*$ such that $G(r^*)$ contains a perfect matching, in $O^*(n^{3/4+\omega/4}) \approx O(n^{1.3432})$ randomized expected time.

Perfect matching in intersection graphs of fat planar objects of similar size. Actually, the $O^*(n^{\omega/2})$ bound mentioned above holds for any set of fat objects of similar sizes, provided
that one can perform some basic operations on these objects efficiently. For example, suppose that $\mathcal{U}$ is a set of $n = 2m$ ellipses in the plane, such that for each ellipse in $\mathcal{U}$, the ratio between its major and minor axes is at most $\Psi_1$, and the length of its major axis is in $[1, \Psi_2]$, for some constant parameters $\Psi_1, \Psi_2$. Then one can compute the smallest $r^*$ such that $G(r^*)$ contains a perfect matching, in $O^*(n^{(5+2\omega)/7}) \approx O^*(n^{1.3922})$ expected time (here the number of degrees of freedom for an ellipse is $t = 5$ and $D(n) = O^*(n^{\omega/2})$). If the ratio between the major and minor axes of the ellipses is some constant $c$, then $t$ reduces to 4, and the running time improves to $O^*(n^{(8+3\omega)/11}) \approx O^*(n^{1.3744})$ randomized expected time.

3.4 Generalized distance selection in the plane

Let $S$ be a set of $n$ pairwise disjoint segments in the plane. We define the distance between a pair of segments as the minimum value $r$ such that, if we expand the length of each of the segments about its center by $r$, then they intersect; see Figure 4. The expansion can be either additive (i.e., we add $r$ to each of the segment lengths) or multiplicative (i.e, we multiply each of the segment lengths by $r$). We want to find the $k$'th smallest distance among the $\binom{n}{2}$ pairwise distances determined by the segments in $S$, for a given parameter $1 \leq k \leq \binom{n}{2}$. This is equivalent to finding the smallest $r$ for which there are exactly $k$ vertices (i.e., intersection points) in the arrangement of $S_r$, where $S_r$ denotes the set of the segments of $S$, each expanded by $r$; see Figure 5.

On the one hand, this problem can be solved using the algorithm in the remark following Theorem 2.4. In this case we have $t = 4$ degrees of freedom to represent a segment, which makes
the algorithm run in $O^*(n^{8/5})$ randomized expected time. On the other hand, the corresponding decision problem — given $r$, is the number of vertices in the arrangement of $S_r$ greater than, equal to, or smaller than $k$ — can be solved, using line sweeping, in $O((n + k) \log n)$ time. (We either perform the sweep to completion, when there are at most $k$ intersections between the expanded segments, or stop it when the number of intersections exceeds $k$.) Our technique allows us to combine the generalized interval shrinking procedure with the efficient decision procedure, and obtain a solution to the selection problem, which is more efficient than the aforementioned naïve solution when $k$ is not too large. More precisely, if $k = n^{1+\delta}$, we can find the $k$'th smallest distance in $S$ in randomized expected time $O^*(n^{(14+6\delta)/11})$. Since $n^\delta = k/n$, this can be rewritten as $O^*(n^{8/11\delta^6/11})$, and this is faster than the naïve solution when $k = o(n^{8/5})$.

Note that the critical value determined by a pair of segments $e, e'$ is $\max\{r, r'\}$, where $r$ is the growth parameter at which $e(r)$ touches the line supporting $e'$, and $c(r)$ is the segment $e$ expanded by $r$, and $r'$ is defined in a symmetric manner, exchanging $e$ and $e'$. This is thus another instance where the critical values are not distances.

As another example, consider a set $D$ of $n$ disjoint disks of arbitrary radii in the plane. We seek the smallest $r$ such that, if we expand each of the disks in $D$ by $r$ (i.e., we either add $r$ to each of the radii, or multiply each of the radii by $r$), then the number of intersecting pairs of disks in the resulting scene is exactly $k$, for some integer parameter $1 \leq k \leq \binom{n}{2}$. In other words, as in Section 3.3 the distance between disks $D = D(c, r)$ and $D' = D'(c', r')$ in $D$ is $r = (|cc'| - (r + r'))/2$ in the additive variant, and $r = |cc'|/(r + r')$ in the multiplicative variant. We seek the $k$’th smallest distance determined by the disks in $D$.

In this case, the problem can be solved using the same algorithm as in the previous case. Here disks have $t = 3$ degrees of freedom, so the algorithm runs in $O^*(n^{3/2})$ randomized expected time. The corresponding decision problem can be solved using line sweeping, as above, in $O((n + k) \log n)$ time. Again, our technique allows us to combine the general interval-shrinking mechanism with this efficient decision procedure, to obtain a more efficient solution, provided that $k$ is not too large. More precisely, if $k = n^{1+\delta}$, we can find the $k$’th smallest distance in $D$ in $O^*(n^{(5+2\delta)/4}) = O^*(n^{3/4}k^{1/2})$. This is faster than the naïve solution when $k = o(n^{3/2})$.
We have considered only two of the many settings in which one can obtain improved bounds for selecting the $k$‘th smallest distance, for a sufficiently small parameter $k$, analogously defined for a set of pairwise-disjoint geometric objects in the plane. The results that we did obtain above are summarized in the following theorem.

**Theorem 3.5.** (i) For $k = o(n^{8/5})$, one can compute the $k$‘th smallest distance in a set $S$ of $n$ pairwise-disjoint segments in the plane, as defined above, in $O^*(n^{8/11}k^{6/11})$ randomized expected time.

(ii) For $k = o(n^{3/2})$, one can compute the $k$‘th smallest distance in a set $D$ of $n$ pairwise-disjoint disks in the plane, as defined above, in $O^*(n^{3/4}k^{1/2})$ randomized expected time.

### 3.5 The discrete one-sided Fréchet distance with shortcuts in higher dimensions

We can extend the analysis in [7] to higher-dimensional spaces $\mathbb{R}^d$. We have two sequences $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ of points in $\mathbb{R}^d$. (We assume for simplicity that $|A| = |B| = n$.) The goal is to find a sequence of pairs $(a_{i_1}, b_{j_1}), (a_{i_2}, b_{j_2}), \ldots, (a_{i_k}, b_{j_k})$, so that (i) $i_1 = 1, j_1 = 1, i_k = n$, and $j_k = n$, (ii) both sequences $(i_1, \ldots, i_k), (j_1, \ldots, j_k)$ are weakly monotone increasing, (iii) $i_{t+1} = i_t$ or $i_t + 1$ for each $t < k$, and $\max_k \{\text{dist}(a_{i_t}, b_{j_k})\}$ is minimized. The latter quantity is called the discrete one-sided Fréchet distance with shortcuts (where the shortcuts are allowed only for $B$), and is denoted as $\text{dfds}(A, B)$. See [7] for more details.

This is another instance of an optimization problem whose critical values are distances between points, except that here we are concerned only with ‘bichromatic’ distances between the points of $A$ and those of $B$. The decision procedure is, given a threshold $\varepsilon > 0$, to determine whether $\text{dfds}(A, B) \leq \varepsilon$. As shown in [7], this procedure can be performed in $O(n)$ time, by simply searching for the existence of a weakly monotone path of a certain kind, from entry $(1, 1)$ to entry $(n, n)$ in a zero-one $n \times n$ matrix, whose $(i, j)$-entry is 0 (resp., 1) if $\text{dist}(a_i, b_j) > \varepsilon$ (resp., $\text{dist}(a_i, b_j) \leq \varepsilon$). Checking for the existence of such a path can be done in linear time, exactly as in [7]. Clearly, this part takes linear time in any dimension.

Plugging $t = d$ and $D(n) = O(n)$ in the general bound given in Theorem 2.7 we obtain:

**Theorem 3.6.** The discrete one-sided Fréchet distance with shortcuts between two sets of $n$ points in $\mathbb{R}^d$ can be computed in $O^*(n^{(4d-2)/(3d-1)})$ randomized expected time.

### 3.6 The reverse shortest path problem for unit-ball graphs in three dimensions

The unweighted and weighted variants of this problem are obvious extensions to three dimensions of the corresponding reverse shortest path problems in the plane. That is, we are given a set $P$ of $n$ points in $\mathbb{R}^3$, two points $s, t \in P$, and a parameter $1 \leq k \leq n - 1$ (or, in the weighted variant, a length $w \geq \text{dist}(s, t)$), and the goal is to compute the smallest value $r^*$ so that there exists a path between $s$ and $t$ of at most $k$ edges (resp., of length at most $w$) in the graph $G(r^*)$ over $P$, whose edges are all the pairs $p, q \in P$ such that $\text{dist}(p, q) \leq r^*$.

Unlike the problem of the discrete one-sided Fréchet distance with shortcuts, whose decision procedure runs in $O(n)$ time in any fixed dimension $d$, the running time of the suitable extension to higher dimensions of both the decision procedure of [11], for the unweighted planar reverse shortest path problem, and that of [16], for its weighted variant, increases in $d$ as we move to
higher dimensions. We show below that both procedures can be adapted to three dimensions, so that the resulting running time is $O^*(n^{4/3})$ for both variants.

Plugging this bound into the three-dimensional case in Theorem 2.7, the overall running time of both algorithms is $O^*(n^{5/4}D(n)^{1/2}) = O^*(n^{17/12})$. That is, we have

**Theorem 3.7.** The unweighted reverse shortest path problem, as well as its weighted variant, for unit-ball graphs in three dimensions on an input set $P$ of $n$ points can be solved in $O^*(n^{17/12})$ randomized expected time.

To complete the presentation, we briefly describe how to adapt the planar decision procedures to three dimensions.

**The unweighted variant.** Most steps of the procedure of [11] carry over to three dimensions with straightforward modifications, without affecting their asymptotic cost, except for the following step in the efficient BFS implementation, which in three dimensions reads as follows. We have a set $S$ of $u$ congruent spheres (say for concreteness, of radius 1), whose centers lie on one side of some given axis-parallel plane, say below the $xy$-plane, and a set $Q$ of $v$ points that lie on the other side of the plane (above the $xy$-plane), and the goal is to report all points of $Q$ that lie below the upper envelope of $S$.

This step can be implemented to run in $O^*(u^{2/3}v^{2/3} + u + v)$ time, as follows. We pass to the dual space, where the points of $Q$ are mapped to unit spheres centered at these points, and the spheres of $S$ become points (the centers of these spheres). Denote the set of dual spheres as $Q^*$ and the set of dual points as $S^*$. Now the goal is to determine, for each sphere in $Q^*$, whether it contains a point of $S^*$; this holds if and only if the primal point of $Q$ lies below the upper envelope of $S$.

This dual problem can be solved by lifting the spheres and points to $\mathbb{R}^4$, so that the spheres become hyperplanes, and the problem becomes that of testing each lower halfspace bounded by such a hyperplane for emptiness (of the lifted points). As shown in [13], this can be done in $d = 4$ dimensions using $s$ storage and $O^*(s)$ preprocessing time, so that each emptiness query takes $O^*(u/s^{1/4d/2}) = O^*(u/s^{1/2})$ time, for a total cost of $O^*(s + uv/s^{1/2})$. With a suitable choice of $s$ this becomes $O^*(u^{2/3}v^{2/3} + u + v)$, as claimed.

By a suitable adaptation of the technique of [11], the overall cost of this step of the decision procedure is $O^*(n^{4/3})$, and this dominates its total running time $D(n)$.

**The weighted variant.** The procedure of [16] (for points in the plane) is a carefully implemented version of Dijkstra’s algorithm. It uses a grid of cell size $\Theta(r)$ and properties of unit disk graphs, in a clever manner, to compute a shortest-path tree from $s$ in near-linear time, even though the number of edges in $G(r)$ could be quadratic. Informally, and not very precisely, if $c$ is the next point to be processed, then instead of using it to update (the shortest-path information of) its neighbors, as in Dijkstra’s algorithm, the procedure uses all the points in $P \cap \sigma$ to update all their neighbors, where $\sigma$ is the grid cell containing $c$.

A close and more rigorous examination of this procedure, leads to the conclusion that it can be adapted to three dimensions, so that, similar to the planar version, its running time $D(n)$ is determined by the time needed to perform a given sequence of $n$ operations, where each operation is either an insertion of a point in $\mathbb{R}^3$ with some additive weight, or a nearest-neighbor query with respect to the current set of weighted points. Alternatively, again similar to the planar version and since we are ignoring subpolynomial factors, the running time of the procedure in three dimensions is determined by the best bound for the (static) bichromatic
additively-weighted nearest neighbors problem in 3-space, in which one is given a set \( R \) of \( u \) red points and a set \( B \) of \( v \) additively-weighted blue points, where \( u + v = n \), and the goal is to compute for each red point its additively-weighted nearest blue point. Agarwal et al. \cite{5} (see also \cite{3}) solve this problem (for non-weighted points) in \( O^*(u^{2/3}v^{2/3} + u + v) = O^*(n^{4/3}) \) time. However, this bound also holds in the additively-weighted variant: It reduces to vertical ray shooting in a convex polytope defined as the intersection of \( v \) halfspaces in \( \mathbb{R}^5 \), using a clever lifting transform from three to five dimensions described in Aurenhammer \cite{6}; see also \cite{4}. This latter task, in \( d = 5 \) dimensions, can be solved with \( s \) storage in time

\[
O^*(s + \frac{uw}{s^{1/[d/2]}}) = O^*(s + \frac{uw}{s^{1/2}}),
\]

which becomes \( O^*(n^{4/3}) \) with a suitable choice of \( s \).

These arguments complete the proof of Theorem 3.7 in both the unweighted and weighted scenarios.

**Higher dimensions.** One can extend the result for unweighted unit-ball graphs in any higher dimension. To achieve a nontrivial performance bound here, we note that the decision procedure can be implemented to run faster than the cost of distance selection. Indeed, in the unweighted version of the problem, the decision procedure reduces to halfspace emptiness queries in \( d + 1 \) dimensions, as in the three-dimensional case described above, whereas the distance selection reduces to range searching (with spherical shells) in \( d \) dimensions. With \( n \) objects and \( s \) storage, the former task has query time \( O^*(n/s^{1/[(d+1)/2]}) \), whereas the latter task has query time \( O^*(n/s^{1/d}) \). Since \( (d+1)/2 \) is always smaller than \( d \), the decision procedure should indeed be faster than distance selection. Concretely, choosing the right value of \( s \), the cost of the decision procedure is \( O^*(n^{2(d+1)/2d}) \). Plugging this into Theorem 2.7, the overall algorithm runs in

\[
O^*(n^{2d/(3d-1)}D((n)^{(2d-2)/(3d-1)}) = O^*(n^{(6d-4)/(3d+1) + 2d/(3d+1)})
\]

randomized expected time, which is indeed smaller than the cost \( O^*(n^{2d/(d+1)}) \) of the distance selection. It would be interesting to obtain a similar bound for the weighted case too.

### 4 Seeing the most without being seen

In this section we present another application of our technique, in which the critical values are determined by *triples* of input points, rather than by pairs. Specifically, we study the following maximum-height independent towers problem. Let \( T = (p_1, \ldots, p_n) \) be a 1.5-dimensional terrain, namely a bounded \( x \)-monotone polygonal line (polyline for short). For any two points \( a, b \) above \( T \), we say that \( a \) and \( b \) see each other if the line segment \( ab \) lies strictly above \( T \) (we do not care what happens below \( T \)). Let \( Q \) be a set of \( m \) points on \( T \). The problem is to compute the maximum height \( h^* \), such that if we place a tower of height \( h^* \) at each of the points \( q \in Q \) (which is a vertical segment of length \( h^* \) whose bottom endpoint is \( q \)), then the tips of these towers do not see each other. We assume that \( h^* > 0 \), which implies that there must be at least one vertex of \( T \) between any two consecutive points in \( Q \) (in their \( x \)-order), and thus \( m < n \). See Figure 4.

Given any set \( Q \) of \( m \) points above \( T \), Ben-Moshe et al. \cite{8} present an \( O((n+m) \log m) \)-time algorithm to determine whether there exist two points (from \( Q \)) that see each other. We will
The graph $h^*$ is positioned at each of the points $q \in Q$. Then there must exist a pair of points $q_i, q_j \in Q$, such that the segment $e$ between the tips of the corresponding towers passes through a vertex $p_k$ of $T$, and none of the vertices of $T$ between $q_i$ and $q_j$ lies above $e$; see Figure 6. In other words, the vertical distance between $p_k$ and the segment $q_iq_j$ is $h^*$, and it is the maximum vertical distance between any intermediate vertex and $q_iq_j$. This implies that $h^*$ belongs to the set $D = \{\text{vert}(q_i, q_j) \mid q_i, q_j \in Q\}$, where $\text{vert}(q_i, q_j)$ is the maximum vertical distance between the segment $q_iq_j$ and the intermediate vertices $p_k$ between $q_i$ and $q_j$ that lie above $q_iq_j$.

We use a variant of a technique due to Varadarajan [15] for the following path simplification problem. Given $T$ as above, and a parameter $1 \leq k \leq n - 1$, find an $x$-monotone polyline $T'$ with at most $k$ edges that best approximates $T$, where the vertices of $T'$ must form a subset of the vertices of $T$ and include $p_1$ and $p_n$. The deviation associated with such a polyline $T'$ is the maximum vertical distance between a vertex of $T$ and $T'$, and the goal is to minimize this deviation.

Varadarajan’s solution consists of appropriate decision and optimization procedures, both with running time $O^*(n^{4/3})$. We first sketch a non-trivial adaptation of Varadarajan’s optimization procedure, which, combined with the aforementioned decision procedure of Ben-Moshe et al. [8], yields an $O^*(n^{4/3})$ algorithm for our towers problem. We then apply our machinery, exploiting the relative efficiency of the decision problem, to obtain an improved $O^*(n^{6/5})$ algorithm.

**An $O^*(n^{4/3})$ algorithm.** Let $E_Q = \{q_iq_j \mid q_i, q_j \in Q\}$ be the set of edges of the complete graph over $Q$. For an edge $q_iq_j \in E_Q$ and a height $h \geq 0$, we denote the segment connecting the tips of the towers of height $h$ based at $q_i$ and $q_j$, by $q_iq_j(h)$, and set $E_Q(h) = \{q_iq_j(h) \mid q_iq_j \in E_Q\}$. We seek the largest value $h^*$, such that each of the edges in $E_Q(h^*)$ is a non-visibility edge, in the sense that it intersects $T$. The potential $h_{ij}$ of an edge $q_iq_j \in E_Q$ is the maximum height $h$ such that $q_iq_j(h)$ is a non-visibility edge. By definition, $h^* = \min_{i < j} h_{ij}$.

**Constructing the non-visibility edges.** We describe an algorithm for constructing the set of all edges in $E_Q$ whose potential lies in some prescribed range $I = (h_1, h_2]$. The algorithm partitions $T$ at its median vertex $p_\mu$ (so $\mu = \lfloor n/2 \rfloor$) into a left portion $T^L$ and a right portion $T^R$, each consisting of at most $\lceil n/2 \rceil$ edges. It solves the problem (of constructing non-visibility edges $q_iq_j$ with potential in $I$) recursively on $T^L$ and on $T^R$, and then computes, in compact...
form, the set of all edges $q_iq_j$, with $q_i \in T^L$ and $q_j \in T^R$, whose potential is in $I$. The partition splits the edges of $T$ evenly between the two subproblems, but not necessarily the points of $Q$. We denote by $Q^L$ (resp., $Q^R$) the subset of points of $Q$ contained in $T^L$ (resp., in $T^R$), and put $m_L = |Q^L|$ and $m_R = |Q^R|$, so $m_L + m_R = m$.

![Figure 7](image)

**Figure 7**: The conditions for the potential of $q_iq_j$ to be in $(h_1, h_2)$. Here $q_iq_j$ is below $\tau_i^+$ (albeit above $\tau_i^-$) and above both $\tau_i^-$ and $\tau_j^-$, so the potential of $q_iq_j$ is in $(h_1, h_2)$.

To find the non-visibility edges between $Q^L$ and $Q^R$ we construct, for each vertex $p_k$, a pair of points $u_k, v_k$ lying on the downward vertical ray emanating from $p_k$ at respective distances $h_1$ and $h_2$ from $p_k$. For each point $q_i \in Q^L$ (resp., $q_j \in Q^R$), we consider the set $T^L_i = \{p_k \mid k' \leq k \leq \mu\}$ (resp., $T^R_j = \{p_k \mid \mu \leq k \leq k'\}$), where $k'$ is the smallest index of a vertex of $T$ to the right of $q_i$ (resp., the largest index of a vertex of $T$ to the left of $q_j$). Let $H_i^-$ denote the upper convex hull of $\{v_k \mid p_k \in T^L_i\}$, and let $H_i^+$ denote the upper convex hull of $\{u_k \mid p_k \in T^L_i\}$. Symmetrically, let $H_j^-$ denote the upper convex hull of $\{v_k \mid p_k \in T^R_j\}$, and let $H_j^+$ denote the upper convex hull of $\{u_k \mid p_k \in T^R_j\}$. Let $\tau_i^-$ (resp., $\tau_i^+$) denote the upper rightward-directed tangent ray from $q_i$ to $H_i^-$ (resp., to $H_i^+$). Symmetrically, let $\tau_j^-$ (resp., $\tau_j^+$) denote the upper leftward-directed tangent ray from $q_j$ to $H_j^-$ (resp., to $H_j^+$). Note that $\tau_i^-$ always lies clockwise to $\tau_i^+$, and that $\tau_j^-$ always lies counterclockwise to $\tau_j^+$. See Figure 7 for an illustration.

It is easily seen that, by construction, the potential of $q_iq_j$ is in $I$ if and only if the following two conditions both hold.

(i) $q_iq_j$ passes either below $\tau_i^-$ or below $\tau_j^+$.

(ii) $q_iq_j$ passes above both $\tau_i^-$ and $\tau_j^-$; it may overlap one of these rays.

Passing to the dual plane, $q_iq_j$ becomes the intersection point $(q_iq_j)^*$ of the dual lines $q_i^*$ and $q_j^*$. The rays $\tau_i^-$ and $\tau_i^+$ are mapped to two respective points $(\tau_i^-)^*$ and $(\tau_i^+)^*$ on $q_i^*$, with $(\tau_i^-)^*$ lying to the left of $(\tau_i^+)^*$. Symmetrically, the rays $\tau_j^-$ and $\tau_j^+$ are mapped to two respective points $(\tau_j^-)^*$ and $(\tau_j^+)^*$ on $q_j^*$, with $(\tau_j^-)^*$ lying to the right of $(\tau_j^+)^*$. Condition (i) translates to the condition that $(q_iq_j)^*$ lies either to the left of $(\tau_i^+)^*$ along $q_i^*$, or to the right of $(\tau_j^+)^*$ along $q_j^*$. Condition (ii) translates to the condition that $(q_iq_j)^*$ lies to the right of $(\tau_i^-)^*$ along $q_i^*$, and to the left of $(\tau_j^-)^*$ along $q_j^*$. See Figure 8
Figure 8: The dual setup for Conditions (i) and (ii). Condition (i) holds for $q_i$ but not for $q_j$.

We thus face a variant of the classical red-blue segment intersection problem (see, e.g., [I]), in which we have a collection $R$ of red segments $r_i := (\tau_i^-)^*(\tau_i^+)^*$, for $q_i \in Q^L$, and a collection $B$ of blue segments $b_j := (\tau_j^-)^*(\tau_j^+)^*$, for $q_j \in Q^R$, and we want to collect, in compact form, all pairs $(r_i, b_j) \in R \times B$ that satisfy Conditions (i) and (ii). As in the segment intersection problem, this can be done using a four-level batched halfplane range searching structure, where each level enforces one of the four sub-conditions in (i) and (ii), each of which amounts to requiring an endpoint of one segment (say, red) to lie in a suitable side of the line supporting another segment (say, blue). Using constant-size primal-dual construction, where at each level and each recursive phase, primal or dual, one structure, each of which is a recursive batched halfplane range searching data structure. We use $O$ portions to the right of $q$ to all four kinds of tangent lines, we compute all the desired red and blue segments $r_i := (\tau_i^-)^*(\tau_i^+)^*$ and $b_j := (\tau_j^-)^*(\tau_j^+)^*$. As just mentioned, the red-blue segment interaction mechanism constructs a four-level data structure, each of which is a recursive batched halfplane range searching data structure. We use a primal-dual construction, where at each level and each recursive phase, primal or dual, one set of segments is represented by points and the other set by halfplanes. Using constant-size cuttings, each bipartite graph constructed at that phase corresponds to a cell $\tau$ of the cutting, and consists of all the points in $\tau$ and of all the halfplanes containing $\tau$.

Shrinking the critical interval. However, our real goal is to obtain a shrunken interval $I$ that contains $h^*$ and at most $L$ other critical distances, for some suitable value of $L$. We solve this...
problem by first solving the inverse problem: Given an interval \( I = (h_1, h_2] \), determine whether \( I \) contains at most \( L \) critical distances. Actually we only require that \( I \) contains at most \( L \) critical distances determined by pairs in \( Q^L \times Q^R \). We then handle recursively the distances determined for \( Q^L \times Q^L \) and \( Q^R \times Q^R \), with the goal of having at most \( L/2 \) distances in \( I \) for each of the two instances. Continuing recursively in this manner, either all the bounds on the number of critical distances in \( I \) are determined to hold, and then \( I \) contains at most \( O(L \log n) \) critical distances, or we reach a recursive instance for which \( I \) contains too many critical distances. In this case we can shrink \( I \), similar to the way it was done in the planar case of distances. At the end of the process we obtain a shrunken interval that contains \( h^* \) and at most \( O(L \log n) \) other critical distances.

This counting procedure is implemented using the batched range searching mechanism reviewed above. However, to gain efficiency, we run this recursion only until the size of each subproblem is roughly \( L \). The details of the algorithm are almost identical to those given in Section 2.1, with obvious modifications. One less trivial modification is that the potential associated with a segment \( q_i q_j \), namely the quantity \( \text{vert}(q_i, q_j) \) defined above, is not readily available, and has to be computed on the fly.

The bifurcation-tree procedure is essentially identical to the one in Section 2.2, with a few straightforward modifications. Combining the interval-shrinking and the bifurcation-tree parts, as we did earlier, yields the following theorem, which summarizes our result.

**Theorem 4.1.** The maximum-height independent towers problem can be solved in \( O^*(n^6/5) \) randomized expected time.

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