Structured Condition Numbers of Structured Tikhonov Regularization Problem and their Estimations

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Abstract. Both structured componentwise and structured normwise perturbation analysis of the Tikhonov regularization are presented. The structured matrices under consideration include: Toeplitz, Hankel, Vandermonde, and Cauchy matrices. Structured normwise, mixed and componentwise condition numbers for the Tikhonov regularization are introduced and their explicit expressions are derived. For the general linear structure, we prove the structured condition numbers are smaller than their corresponding unstructured counterparts based on the derived expressions. By means of the power method and small sample condition estimation, the fast condition estimation algorithms are proposed. Our estimation methods can be integrated into Tikhonov regularization algorithms that use the generalized singular value decomposition (GSVD). The structured condition numbers and perturbation bounds are tested on some numerical examples and compared with their unstructured counterparts. Our numerical examples demonstrate that the structured mixed condition numbers give sharper perturbation bounds than existing ones, and the proposed condition estimation algorithms are reliable.

Keywords: Tikhonov regularization, structured matrix, condition number, componentwise, structured perturbation, small sample condition estimation.

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1 Introduction

For discrete ill-posed problems, the Tikhonov regularization (cf. [46]) reads

$$\min_x \left\{ \|Ax - b\|_2^2 + \lambda^2\|Lx\|_2^2 \right\}, \quad A \in \mathbb{R}^{m \times n} \quad \text{and} \quad L \in \mathbb{R}^{p \times n} \quad (1.1)$$
where $\lambda$ is the regularization parameter, which controls the weight between $\|Lx\|_2$ and the residual $\|Ax - b\|_2$. The matrix $L$ is typically the identity matrix $I_n$ or a discrete approximation to some derivation operator. Tikhonov regularization is also known as \textit{ridge regression} in statistics [7].

For the regularization problem (1.1), to ensure the uniqueness of the solution for any $\lambda > 0$, we always assume that $\text{rank}(L) = p \leq n \leq m$ and $\text{rank}
\begin{bmatrix} A \\ L \end{bmatrix} = n$ (cf. [7, §5]). The regularization problem (1.1) can be rewritten in the matrix form

$$
\min_x \left\| \begin{bmatrix} A \\ \lambda L \end{bmatrix} x - \begin{bmatrix} b \\ 0 \end{bmatrix} \right\|_2, 
$$

where $0$ is the zero vector. Since the normal equations corresponding to (1.2) are

$$
\left( A^\top A + \lambda^2 L^\top L \right) x = A^\top b, 
$$

we can obtain the following explicit expression for the Tikhonov regularized solution:

$$
x_\lambda = \left( A^\top A + \lambda^2 L^\top L \right)^{-1} A^\top b.
$$

Alternatively, the problem (1.1) can also be solved by the generalized singular value decomposition (GSVD) [22, 26, 47]. For rectangular matrices $A \in \mathbb{R}^{m \times n}$ and $L \in \mathbb{R}^{p \times n}$ with $\text{rank}(L) = p$ and $\text{rank}
\begin{bmatrix} A \\ L \end{bmatrix} = n$, the GSVD of $(A, L)$ is given by the pair of factorizations

$$
A = U \begin{bmatrix} \Sigma & 0 \\ 0 & I_{n-p} \end{bmatrix} RQ^\top 
\quad \text{and} \quad
L = V \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} RQ^\top,
$$

where $U \in \mathbb{R}^{m \times n}$ has orthonormal columns, $V \in \mathbb{R}^{p \times p}$, $Q \in \mathbb{R}^{n \times n}$ are orthogonal, $R$ is $n$-by-$n$, upper triangular and nonsingular, and $\Sigma$ and $S$ are $p$-by-$p$ diagonal matrices: $\Sigma = \text{Diag}(\sigma_1, \sigma_2, \ldots, \sigma_p)$ and $S = \text{Diag}(\mu_1, \mu_2, \ldots, \mu_p)$ with

$$
0 \leq \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_p < 1 \quad \text{and} \quad 1 \geq \mu_1 \geq \mu_2 \geq \ldots \geq \mu_p > 0,
$$

satisfying $\Sigma^2 + S^2 = I_p$. Then the \textit{generalized singular values} $\gamma_i$ of $(A, L)$ are defined by the ratios $\gamma_i = \sigma_i / \mu_i$ ($i = 1, 2, \ldots, p$). Once the GSVD is computed, the Tikhonov regularized solution can be obtained by [26, Chapter 4]

$$
x_\lambda = QR^{-1} \begin{bmatrix} F & 0 \\ 0 & I_{n-p} \end{bmatrix} \begin{bmatrix} \Sigma^\top & 0 \\ 0 & I_{n-p} \end{bmatrix} U^\top b, \quad F = \text{Diag}(f_1, f_2, \ldots, f_p),
$$

where $f_i = \gamma_i^2 / (\gamma_i^2 + \lambda^2)$ for $i = 1, 2, \ldots, p$, are called the \textit{filter factors} for the Tikhonov regularization [26, 27] and $\Sigma^\top$ is the Moore-Penrose inverse of $\Sigma$ [7].

In sensitivity analysis, condition numbers are of great importance because they measure the worst-case effect of small changes in the data on the solution. For the perturbation analysis of the linear least squares (LS) problem, the reader is referred to [2, 3, 4, 13, 14]. Arioli et al. [2] introduced a \textit{partial condition number} of the LS problem, which can be viewed as a condition number of a \textit{linear functional} of the LS problem. Baboulin et al. [3] have shown that the partial
condition numbers of the LS problem represent some quantities in statistics. For the perturbation analysis for the Tikhonov regularization, we refer to [20, 23] and references therein. Malyshev [38] adopted a unified theory to study the normwise condition numbers for the Tikhonov regularization. Chu et al. [12] investigated the componentwise perturbation analysis of the Tikhonov regularization problems and derived condition number expressions involving the Kronecker products, which can be of huge dimension even for small problems, preventing us from estimating the condition numbers while solving the Tikhonov regularization problem. In this paper, we consider the structured condition numbers for a linear functional of the Tikhonov regularization. Fast condition number estimation, which is important in practice, is discussed.

Structured matrix computation is a hot research topic; see [10, 39] and the references therein. The structured Tikhonov regularization problem was recently studied in [6, 11, 24]. Eldén gave a stable efficient algorithm for the Tikhonov regularization with triangular Toeplitz structure. Park and Eldén [42] devised fast algorithms for solving LS with Toeplitz structure, based on the generalization of the classical Schur algorithm, and discussed their stability properties. Also, Park and Eldén studied the stability analysis and fast algorithms for triangularization of rectangular Toeplitz matrices [41]. Hence, it is natural to investigate structured perturbations on the structured coefficient matrix, which lead to the structured condition numbers for the structured Tikhonov regularization problem. Structured condition numbers for several categories of structured matrices have been presented in [5, 8, 9, 13, 17, 28, 43, 44, 49, 50]. In this paper we derive explicit formulas for the condition numbers of the Tikhonov regularization problem, when perturbations of \((A, b)\) are measured by normwise or componentwise or a mixture of normwise and componentwise. To make our discussion general, we consider the condition number of \(Mx\), i.e., a linear function of the Tikhonov regularized solution, where \(M \in \mathbb{R}^{l \times n}\) and \(x \in \mathbb{R}^n\), \(l \leq n\). The common situations are the special cases, when \(M\) is the identity matrix (condition number of the Tikhonov regularized solution) or a canonical vector (condition number of one component of the solution). We obtain the expressions of the structured condition numbers in the absence of the Kronecker product, so that they can be estimated by the power method due to Hager [21] and Higham [29, 30], see [31, Chapter 15] for the detail, while solving the Tikhonov regularization problem.

Moreover, in this paper, we adopt the statistical condition estimation (SCE) method [32] for numerically estimating the condition of Tikhonov regularization problem. The SCE can be used to estimate the componentwise local sensitivity of any differentiable function at a given input data, which is flexible and accommodates a wide range of perturbation types such as structured perturbations. Thus SCE often provides less conservative estimates than the methods that do not exploit structures. The SCE method has been shown to be both reliable and efficient for many problems including linear systems [34], structured linear systems [35], linear least squares problems [33], eigenvalue problems [19, 37], matrix functions [32], the roots of polynomials [36], etc.

We follow the convention of representing a point \(x \in \mathbb{R}^n\) as a column vector. If \(x \in \mathbb{R}^n\) and \(y \in \mathbb{R}^m\), then \([x; y]\) is an \(m + n\) column vector by stacking \(x\) on top of \(y\). If \(A \in \mathbb{R}^{m \times n}\) and \(B \in \mathbb{R}^{m \times q}\), then \([A, B]\) denotes the matrix obtained by putting \(A\) and \(B\) side by side. The symbol \(',^\top\'\) denotes matrix transpose, \(\| \cdot \|_2\) is the spectral norm, \(\| \cdot \|_F\) is the Frobenius norm and \(\| \cdot \|_\infty\) is the infinity norm. The matrix \(\text{Diag}(d) \in \mathbb{R}^{q \times q}\) denotes a diagonal matrix with the vector \(d\)'s entries being its corresponding diagonal components. For any points \(a, b \in \mathbb{R}^n\), the vector \(c = \frac{a}{b}\) is obtained by componentwise division. In particular, \(b_i = 0\) assumes \(a_i = 0\), and in this case \(c_i = 0\). For a matrix \(A \in \mathbb{R}^{m \times n}\), we define \(\vec{v}(A) \in \mathbb{R}^{mn}\) by \(\vec{v}(A) = [a_1^\top, a_2^\top, \ldots, a_n^\top]^\top\), where \(A = [a_1, a_2, \ldots, a_n]\) with \(a_i \in \mathbb{R}^m\), \(i = 1, 2, \ldots, n\). The unvec operation is defined as \(A = \text{unvec}(v)\)
which sets the entries of $A$ to $a_{ij} = v_{i+(j-1)n}$ for $v = [v_1, v_2, \ldots, v_{mn}] \in \mathbb{R}^{1 \times mn}$. We define a permutation matrix $\Pi$ of order $mn$ so that $\Pi(\text{vec}(A)) = \text{vec}(A^\top)$. Let ‘$\otimes$’ denote the Kronecker product \cite{18}, i.e., $A \otimes B = [a_{ij}B] \in \mathbb{R}^{mn \times nq}$ for $A = (a_{ij}) \in \mathbb{R}^{mn \times n}$ and $B \in \mathbb{R}^{pq \times q}$. The notation $|A| \leq |B|$ means that $|a_{ij}| \leq |b_{ij}|$. For the Kronecker product, we recall the following properties, which can be found in \cite{18},

\[(A \otimes B)\top = A\top \otimes B\top, \quad |A \otimes B| = |A| \otimes |B|, \quad \text{vec}(AXB) = \left( B\top \otimes A \right) \text{vec}(X), \tag{1.5}\]

where $|A| = [|a_{ij}|]$ and $a_{ij}$ is the $(i,j)$-th entry of $A$.

This paper is organized as follows. We provide some preliminaries in Section 2, investigate matrices with linear structures in Section 3 and move to matrices with nonlinear structures in Section 4. The SCE-based condition estimation algorithms are proposed in Section 5. In Section 6, we demonstrate test results showing the sharpness of our structured condition numbers and effectiveness of the condition estimation algorithms. Finally, conclusions are drawn in the last section.

\section{Preliminaries}

In this section, we first recall the general (unstructured) condition number definitions \cite{17}. Then we consider the structured Tikhonov regularization problems, introduce structured perturbations, and define their structured condition numbers. Finally, we briefly describe the basic ideas of SCE.

\subsection{Structured condition numbers for the Tikhonov regularization}

For $x, a \in \mathbb{R}^p$ and $\varepsilon > 0$ we denote $S(a, \varepsilon) = \{ x \in \mathbb{R}^p \mid |x - a| \leq \varepsilon |a| \}$ and $T(a, \varepsilon) = \{ x \in \mathbb{R}^p \mid \|x - a\|_2 \leq \varepsilon \}$. For a function $F : \mathbb{R}^p \to \mathbb{R}^q$, we denote $\text{Dom}(F)$ as its domain. The following lemma defines general (unstructured) condition numbers.

\begin{lemma} \cite{17} \end{lemma}

Let $F : \mathbb{R}^p \to \mathbb{R}^q$ be a continuous mapping defined on an open set $\text{Dom}(F) \subset \mathbb{R}^p$. Let $a \in \text{Dom}(F)$ such that $a \neq 0$ and $F(a) \neq 0$.

\begin{enumerate}[(i)]
\item The mixed condition number of $F$ at $a$ is defined by

\[ m(F,a) = \lim_{\varepsilon \to 0} \sup_{x \in S(a,\varepsilon) \setminus \{a\}} \frac{\|F(x) - F(a)\|_\infty}{\|F(a)\|_\infty} \frac{1}{d(x,a)} = \frac{\|DF(a)\|_\infty |a|_\infty}{\|F(a)\|_\infty}, \]

where $DF(a)$ is the Fréchet derivative of $F$ at $a$ and $|a| = (|a_i|)$ with $a = [a_1, a_2, \ldots, a_p]^\top$.

\item Suppose $F(a) = (f_1(a), f_2(a), \ldots, f_q(a))$ is such that $f_j(a) \neq 0$ for $j = 1, 2, \ldots, q$. Then the componentwise condition number of $F$ at $a$ is

\[ c(F,a) = \lim_{\varepsilon \to 0} \sup_{x \in S(a,\varepsilon) \setminus \{a\}} \frac{d(F(x), F(a))}{d(x,a)} = \left\| \frac{DF(a)\|\| |a|}{\|F(a)\|_\infty} \right\|_\infty. \]

\item The normwise condition number of $F$ at $a$ is defined by

\[ \kappa(F,a) = \lim_{\varepsilon \to 0} \sup_{x \in T(a,\varepsilon) \setminus \{a\}} \frac{\|F(x) - F(a)\|_2}{\|x - a\|_2} \frac{\|a\|_2}{\|F(a)\|_2} = \frac{\|DF(a)\|_2 \|a\|_2}{\|F(a)\|_2}. \]
\end{enumerate}
In the following we assume that $\Delta A$ and $\Delta b$ are perturbations to $A$ and $b$ respectively, which satisfy $\text{rank} \left( \begin{bmatrix} A + \Delta A \\ L \end{bmatrix} \right) = n$. The perturbed counterpart of the problem (1.1) and its normal equations (1.3) are, respectively,

$$
\min_{x + \Delta x} \left\{ \| (A + \Delta A)(x + \Delta x) - (b + \Delta b) \|_2^2 + \lambda^2 \| L(x + \Delta x) \|_2^2 \right\},
$$

and

$$
\left( (A + \Delta A)^T (A + \Delta A) + \lambda^2 L^T L \right) (x_\lambda + \Delta x) = (A + \Delta A)^T (b + \Delta b).
$$

Then the perturbed Tikhonov regularized solution is given by

$$
x_\lambda + \Delta x = \left( (A + \Delta A)^T (A + \Delta A) + \lambda^2 L^T L \right)^{-1} (A + \Delta A)^T (b + \Delta b).
$$

Denoting

$$
P(A, \lambda) = \left( A^T A + \lambda^2 L^T L \right)^{-1},
$$

Chu et al. [12] define the non-structured mixed, componentwise, and normwise condition numbers for the Tikhonov regularization and obtain respectively

$$
m_{\text{Reg}} = \lim_{\epsilon \to 0} \sup_{|\Delta A| \leq \epsilon |A|} \frac{\| \Delta x \|_\infty}{\| x_\lambda \|_\infty} = \frac{\| [H(A, b) | \text{vec}(|A|) + |P(A, \lambda) A^T| |b|] \|_\infty}{\| x_\lambda \|_\infty},
$$

$$
c_{\text{Reg}} = \lim_{\epsilon \to 0} \sup_{|\Delta A| \leq \epsilon |A|} \frac{1}{\epsilon} \frac{\| \Delta x \|_\infty}{\| x_\lambda \|_\infty} = \frac{\left\| [H(A, b) | \text{vec}(|A|) + |P(A, \lambda) A^T| |b|] \right\|_\infty}{\left\| x_\lambda \right\|_\infty},
$$

$$
\text{cond}^F_{\text{Reg}} = \lim_{\epsilon \to 0} \sup_{\| \Delta A, \Delta b \| \leq \epsilon \| A, b \| F} \frac{\| \Delta x \|_2}{\epsilon \| x_\lambda \|_2} = \frac{\left\| [H(A, b), P(A, \lambda) A^T] \right\|_2}{\| [A, b] \|_F} \frac{\| A, b \|_F}{\| x_\lambda \|_2},
$$

where $H(A, b) = -x_\lambda^T \otimes \left[ P(A, \lambda) A^T \right] + \left[ P(A, \lambda) \otimes r_\lambda^T \right]$ and $r_\lambda = b - A x_\lambda$.

If we define a mapping

$$
\psi : [A, b] \in \mathbb{R}^{m \times n} \times \mathbb{R}^m \mapsto \left( A^T A + \lambda^2 L^T L \right)^{-1} A^T b \in \mathbb{R}^n
$$

then it is easy to see that the definitions in Lemma 1 are equivalent to (2.3)-(2.5), that is,

$$
m_{\text{Reg}} = m(\psi, [A, b]), \quad c_{\text{Reg}} = c(\psi, [A, b]), \quad \text{cond}^F_{\text{Reg}} := \kappa(\psi, [A, b]).
$$

When the coefficient matrix $A$ in (1.1) has some structures, such as Toeplitz, it is reasonable to assume that the perturbation $\Delta A$ in (2.1) has the same structure of $A$. Then $\Delta A$ is called structured perturbation [43, 44] on $A$. Usually a structured matrix $A \in \mathbb{R}^{m \times n}$ can be represented by fewer than $mn$ parameters. For example, an $m \times n$ Toeplitz matrix can be represented by its first column and last row, $m + n - 1$ parameters. Here we use a mapping to characterize this relationship. Let $\mathcal{S}$ be the set of structured matrices under consideration and $a$ the vector representing a structured matrix $A$, then we define a mapping

$$
g : a \in \mathbb{R}^k \mapsto A \in \mathcal{S}.
$$
In order to apply Lemma 1 to define the structured condition numbers for the Tikhonov regularization, we construct a mapping
\[
\phi : \ [a; b] \in \mathbb{R}^{k+m} \rightarrow M \left( A^\top A + \lambda^2 L^\top L \right)^{-1} A^\top b \in \mathbb{R}^l,
\]
where \( M \in \mathbb{R}^{l \times n} \), \( l \leq n \), is general. In particular, when \( M = e_i^\top \), the \( i \)-th column of the identity matrix, then we are interested in some particular component of \( x_\lambda \).

Let \( \Delta a \) be the perturbation on \( a \), then the structured perturbation matrix \( \Delta A \) on \( A \) in (2.1) is \( g(a + \Delta a) - g(a) \). Now we are ready to define the structured mixed, componentwise and normwise condition numbers for a linear functional of the structured Tikhonov regularization,

\[
m_S^{\text{Reg}}(A, b) := m(\phi, [a; b]) = \lim_{\epsilon \to 0} \sup_{\|\Delta a\| \leq \epsilon \|a\|} \| M \Delta x \|_\infty / \epsilon \| M x_\lambda \|_\infty,
\]

\[
c_S^{\text{Reg}}(A, b) := c(\phi, [a; b]) = \lim_{\epsilon \to 0} \sup_{\|\Delta a\| \leq \epsilon \|a\|} \frac{1}{\epsilon} \left( \frac{M \Delta x}{M x_\lambda} \right)_\infty,
\]

\[
\kappa_S^{\text{Reg}}(A, b) := \kappa(\phi, [a; b]) = \lim_{\epsilon \to 0} \sup_{\|\Delta a\| \leq \epsilon \|a\|} \frac{\| M \Delta x \|_2}{\epsilon \| M x_\lambda \|_2},
\]

where \( \Delta x \) is defined in (2.2).

Remark 1 Note that here \( g \) is a general mapping, in that it can represent any structure. When the structure in \( A \) is linear, such as symmetric, or Toeplitz, or Hankel, we can choose \( g \) a linear mapping, which will be discussed in Section 3. When \( A \) has a nonlinear structure such as Vandermonde or Cauchy, we can choose a nonlinear mapping \( g \) to define the structured condition numbers. Especially we can define the unstructured linear functional condition number for \( x_\lambda \) when we restrict \( S \) to be \( \mathbb{R}^{m \times n} \), which are generalizations of (2.3), (2.4) and (2.5), as follows

\[
m^{\text{Reg}}(A, b) = \lim_{\epsilon \to 0} \sup_{\|\Delta A\| \leq \epsilon \|A\|} \frac{1}{\epsilon} \left( \frac{M \Delta x}{M x_\lambda} \right)_\infty, \quad c^{\text{Reg}}(A, b) = \lim_{\epsilon \to 0} \sup_{\|\Delta a\| \leq \epsilon \|a\|} \frac{1}{\epsilon} \left( \frac{M \Delta x}{M x_\lambda} \right)_\infty,
\]

\[
\kappa^{\text{Reg}}(A, b) = \lim_{\epsilon \to 0} \sup_{\|\Delta A, \Delta b\| \leq \epsilon \|A, b\|} \frac{\| M \Delta x \|_2}{\epsilon \| M x_\lambda \|_2}.
\]

When \( M = I_n \), the above definitions reduce to (2.3), (2.4) and (2.5).

Finally, we give the well-known definitions reduce to (2.3), (2.4) and (2.5).

Lemma 2 Let \( E \in \mathbb{R}^{n \times n} \) and \( \| \cdot \| \) be any norm on \( \mathbb{R}^{n \times n} \), if \( \| E \| < 1 \), then \( I_n + E \) is nonsingular and its inverse can be expressed by

\[
(I_n + E)^{-1} = I_n - E + \mathcal{O}(\| E \|^2).
\]
2.2 Statistical condition estimation

In SCE, a small random perturbation is introduced to the input, and the change in the output, by an appropriate scaling, is measured as a condition estimate. Explicit bounds on the probability of the accuracy of the estimate exist [32]. The idea of SCE can be illustrated by a general real-valued function: \( f : \mathbb{R}^p \rightarrow \mathbb{R} \), and we are interested in the sensitivity at some input vector \( x \). By the Taylor theorem we have

\[
 f(x + \delta d) - f(x) = \delta (Df(x))^\top d + O(\delta^2),
\]

where \( \delta \) is a small scalar, \( \|d\|_2 = 1 \) and \( Df(x) \) is the Fréchet derivative of \( f \) at \( x \). Note that the quantity \( (Df(x))^\top d \) (denoted by \( Df(x; d) \)) is just the directional derivative of \( f \) with respect to \( x \) at the direction \( d \). It is easy to see that up to the first order in \( \delta \),

\[
 |f(x + \delta d) - f(x)| \approx \delta Df(x; d),
\]

then the local sensitivity can be measured by \( \|Df(x; d)\|_2 \). The condition numbers of \( f \) at \( x \) are mainly determined by the norm of the gradient \( Df(x) \) ([32]). According to [32], if we select \( d \) uniformly and randomly from the unit \( p \)-sphere \( S_{p-1} \) (denoted \( d \in \mathcal{U}(S_{p-1}) \)), then the expectation \( E(|Df(x; d)|/\omega_p) \) is \( \|Df(x)\|_2 \), where \( \omega_p \) is the Wallis factor. In practice, the Wallis factor can be approximated accurately [32] by

\[
 \omega_p \approx \sqrt{\frac{2}{\pi(p - \frac{1}{2})}}.
\]

Therefore, we can use

\[
 \nu = \frac{|Df(x; d)|}{\omega_p}
\]

as a condition estimator, which can estimate \( \|Df(x)\|_2 \) with high probability for the function \( f \) at \( x \) (see [32] for details), for example,

\[
 \text{Prob}\left( \frac{\|Df(x)\|_2}{\gamma} \leq \nu \leq \gamma \|Df(x)\|_2 \right) \geq 1 - \frac{2}{\pi\gamma} + O\left( \frac{1}{\gamma^2} \right),
\]

for \( \gamma > 1 \). We can use multiple samples of \( d \), denoted \( d_j \), to increase the accuracy [32]. The \( t \)-sample condition estimation is given by

\[
 \nu(k) = \frac{\omega_k}{\omega_p} \sqrt{|Df(x; d_1)|^2 + |Df(x; d_2)|^2 + \cdots + |Df(x; d_t)|^2},
\]

where \([d_1, d_2, \ldots, d_t]\) is orthonormalized after \( d_1, d_2, \ldots, d_t \) are selected uniformly and randomly from \( \mathcal{U}(S_{p-1}) \). The accuracy of \( \nu(2) \) is given by

\[
 \text{Prob}\left( \frac{\|\nabla f(x)\|_2}{\gamma} \leq \nu(2) \leq \gamma \|\nabla f(x)\|_2 \right) \approx 1 - \frac{\pi}{4\gamma^2}, \quad \gamma > 1.
\]

Usually, a few samples are sufficient for good accuracy. These results can be conveniently generalized to vector- or matrix-valued functions by viewing \( f \) as a map from \( \mathbb{R}^p \) to \( \mathbb{R}^q \). The operations \texttt{vec} and \texttt{unvec} can be used to convert between matrices and vectors, where each of the \( q \) entries of \( f \) is a scalar-valued function. Evaluating the matrix function at a slightly perturbed argument yields a local condition estimate for one component of the computed solution.
Lemma 3 The Fréchet derivative $D\phi([a; b])$ of function $\phi$ defined in (2.7) is given by

$$D\phi([a; b]) = MP(A, \lambda) \left[ v_1, v_2, \ldots, v_k, A^\top \right],$$

where $v_i = -A^\top S_i x_\lambda + S_i^\top r_\lambda$ for $i = 1, 2, \ldots, k$.

Proof. Let $\Delta A = g(\Delta a)$ and $\Delta b$ be perturbations on $A = g(a)$ and $b$ respectively. Firstly, denoting $A = (A + \Delta A)(A + \Delta A) + \lambda^2 L^\top L$ and recalling that $P(A, \lambda) = (A^\top A + \lambda^2 L^\top L)^{-1}$, we have

$$A = \left( A^\top A + \lambda^2 L^\top L \right) + \left( A^\top (\Delta A) + (\Delta A)^\top A \right) + (\Delta A)^\top (\Delta A)$$

$$= \left( A^\top A + \lambda^2 L^\top L \right) \left[ I_n + P(A, \lambda) \left( A^\top (\Delta A) + (\Delta A)^\top A \right) + P(A, \lambda) \left( (\Delta A)^\top (\Delta A) \right) \right].$$

If $\|\Delta A\|$ is sufficiently small, then $\|P(A, \lambda) (A^\top (\Delta A) + (\Delta A)^\top A + (\Delta A)^\top (\Delta A))\| < 1$, from Lemma 2, $A$ is nonsingular and its inverse

$$A^{-1} = \left[ I_n + P(A, \lambda) \left( A^\top (\Delta A) + (\Delta A)^\top A \right) + P(A, \lambda) \left( (\Delta A)^\top (\Delta A) \right) \right]^{-1} P(A, \lambda)$$

$$= P(A, \lambda) - P(A, \lambda) \left( A^\top (\Delta A) + (\Delta A)^\top A \right) P(A, \lambda) + O(\|\Delta A\|^2),$$

since $\|A^\top (\Delta A) + (\Delta A)^\top A\| = O(\|\Delta A\|)$ and $\|(\Delta A)^\top (\Delta A)\| = O(\|\Delta A\|^2)$. From (2.2), (3.3) and $x_\lambda = P(A, \lambda) A^\top b$, after some algebraic manipulation, we have

$$\Delta x = P(A, \lambda) \left( A^\top (\Delta b) + (\Delta A)^\top (b - A x_\lambda) - A^\top (\Delta A) x_\lambda \right) + O(\|\Delta A\|^2) + O(\|\Delta A\|\|\Delta b\|).$$

Omitting the second and higher order terms and applying the third equation in (1.5), we have

$$\Delta x \approx P(A, \lambda) \left( A^\top (\Delta b) + (\Delta A)^\top (b - A x_\lambda) - A^\top (\Delta A) x_\lambda \right)$$

$$= P(A, \lambda) \left[ \left( -(x_\lambda^\top \otimes A^\top) + (r_\lambda^\top \otimes I_n) \Pi \right) \vec(\Delta A) + A^\top (\Delta b) \right]$$

$$= P(A, \lambda) \left[ \left( -(x_\lambda^\top \otimes A^\top) + (I_n \otimes r_\lambda^\top) \right) \vec(\Delta A) + A^\top (\Delta b) \right],$$

(3.4)
recalling that \( r_\lambda = b - Ax_\lambda \).

Since \( \Delta A \) is a structured perturbation on \( A \), then \( \Delta A = g(\Delta a) \), i.e., there exist parameters \( \Delta a_1, \Delta a_2, \ldots, \Delta a_k \) such that \( \Delta A = \sum_{i=1}^{k} \Delta a_i S_i \). Denote \( \Delta a = [\Delta a_1, \Delta a_2, \ldots, \Delta a_k]^T \). From (3.4), we have

\[
\phi([a + \Delta a; b + \Delta b]) - \phi([a; b]) 
\approx MP(A, \lambda) \left\{ \left( - (x_\lambda^T \otimes A^T) + (I_n \otimes r_\lambda^T) \right) [\text{vec}(S_1), \ldots, \text{vec}(S_k)] \Delta a + A^T (\Delta b) \right\}
\]

\[
= MP(A, \lambda) \left[ - A^T S_1 x_\lambda + S_1^T r_\lambda, \ldots, - A^T S_k x_\lambda + S_k^T r_\lambda, A^T \right] \Delta v,
\]

where \( \Delta v = [\Delta a; \Delta b] \). By the definition of the Fréchet derivative, the lemma then can be proved. \( \square \)

**Theorem 1** Let \( A \in \mathcal{L}, b \in \mathbb{R}^m \) and \( x_\lambda = (A^T A + \lambda^2 L^T L)^{-1} A^T b = P(A, \lambda) A^T b \) be the Tikhonov regularized solution of (1.1). Then we obtain the structured normwise, componentwise, and mixed condition numbers:

\[
m_{\mathcal{L}}^{\text{Reg}}(A, b) = \left\| \sum_{i=1}^{k} a_i |MP(A, \lambda) (A^T S_i x_\lambda - S_i^T r_\lambda)| + |MP(A, \lambda) A^T| \right\| \infty \left/ \|M x_\lambda\|_\infty \right.,
\]

\[
c_{\mathcal{L}}^{\text{Reg}}(A, b) = \left\| \sum_{i=1}^{k} a_i |MP(A, \lambda) (A^T S_i x_\lambda - S_i^T r_\lambda)| + |MP(A, \lambda) A^T| \right\| \infty \left/ M x_\lambda \right.,
\]

\[
\kappa_{\mathcal{L}}^{\text{Reg}}(A, b) = \left\| MP(A, \lambda) \left[ S_1^T r_\lambda - A^T S_1 x_\lambda, \ldots, S_k^T r_\lambda - A^T S_k x_\lambda, A^T \right] \right\|_2 \left/ \|M x_\lambda\|_2 \right. .
\]

**Proof.** From Lemmas 1 and 3, we have

\[
m_{\mathcal{L}}^{\text{Reg}}(A, b) = \frac{\left\| \text{D} \phi ([a; b]) \left[ \begin{array}{c} a \\ b \end{array} \right] \right\|_\infty}{\| x_\lambda \|_\infty} = \frac{\| ||MP(A, \lambda) \nu_1; \nu_2; \ldots; \nu_k|| a | + |MP(A, \lambda) A^T| \| b \| \|_\infty}{\| M x_\lambda \|_\infty}
\]

\[
= \frac{\left\| \sum_{i=1}^{k} a_i |MP(A, \lambda) (A^T S_i x_\lambda - S_i^T r_\lambda)| + |MP(A, \lambda) A^T| \right\| \infty}{\| M x_\lambda \|_\infty}.
\]

and

Similarly, we can obtain explicit expressions of the structured componentwise and normwise condition numbers. \( \square \)

When \( \{S_k\} \) is the canonical basis for \( \mathbb{R}^{m \times n} \) in Theorem 1, we have the following compact forms of the unstructured condition numbers in Remark 1 for \( m^{\text{Reg}}(A, b), c^{\text{Reg}}(A, b) \) and \( \kappa^{\text{Reg}}(A, b) \).
Theorem 2 As stated before, we have the following expressions

\[
m_{\text{Reg}}(A, b) = \frac{\|MP(A, \lambda) \left[(I_n \otimes r_{\lambda}^T) - (x^T \otimes A^T)\right] \text{vec}(|A|) + |MP(A, \lambda)A^T| b\|_\infty}{\|Mx\|_\infty},
\]
\[
c_{\text{Reg}}(A, b) = \frac{\|MP(A, \lambda) \left[(I_n \otimes r_{\lambda}^T) - (x^T \otimes A^T)\right] \text{vec}(|A|) + |MP(A, \lambda)A^T| b\|_\infty}{\|Mx\|_2},
\]
\[
\kappa_{\text{Reg}}(A, b) = \frac{\|MP(A, \lambda) \left[(I_n \otimes r_{\lambda}^T) - (x^T \otimes A^T), A^T\right] \|_2 \sqrt{\|A\|_F^2 + \|b\|_2^2}}{\|Mx\|_2}.
\]

Proof. For the expression of \(\kappa_{\text{Reg}}(A, b)\) given in Theorem 1, let \(\{S_{ij} = e_i^{(m)}e_j^{(n)}\}^T\) be the canonical basis for \(\mathbb{R}^{m \times n}\), where \(e_j^{(n)}\) is the \(j\)-th column of the identity matrix \(I_n\), \(i = 1, 2, \ldots, m\) and \(j = 1, 2, \ldots, n\). Then we have the following simplified expression:

\[-S_{ij}^T r_{\lambda, i} A^T S_{ij} x_{\lambda} = -e_j^{(n)} r_{\lambda, (i)} + A^T e_i^{(m)} x_{\lambda, (j)},\]

where \(r_{\lambda, (i)}\) and \(x_{\lambda, (j)}\) are respectively the \(i\)-th and \(j\)-th components of \(r_{\lambda}\) and \(x_{\lambda}\). Now, fixing \(j\), we get

\[-e_j^{(n)} r_{\lambda, (i)} + A^T e_1^{(m)} x_{\lambda, (j)}, \ldots, -e_j^{(n)} r_{\lambda, (i)} + A^T e_n^{(m)} x_{\lambda, (j)} = -e_j^{(n)} r_{\lambda} + x_{\lambda, (j)} A^T,\]

which implies that

\[
\left[-S_{11}^T r_{\lambda} + A^T S_{11} x_{\lambda}, \ldots, -S_{1n}^T r_{\lambda} + A^T S_{1n} x_{\lambda}, -S_{12}^T r_{\lambda} + A^T S_{12} x_{\lambda}, \ldots, -S_{mn}^T r_{\lambda} + A^T S_{mn} x_{\lambda}\right]
\]

\[
= \left[-(e_1^{(n)} \otimes r_{\lambda}^T) + x_{\lambda, (1)} A^T, -(e_2^{(n)} \otimes r_{\lambda}^T) + x_{\lambda, (2)} A^T, \ldots, -(e_n^{(n)} \otimes r_{\lambda}^T) + x_{\lambda, (n)} A^T\right]
\]

\[
= \left[-(I_n \otimes r_{\lambda}^T) + (x^T \otimes A^T)\right].
\]

Applying the above equation to the expression of \(\kappa_{\text{Reg}}(A, b)\) in Theorem 1, we prove the third statement. The expressions of \(m_{\text{Reg}}(A, b)\) and \(c_{\text{Reg}}(A, b)\) can be obtained similarly. \(\square\)

Remark 2 If we choose \(M = I_n\), then \(m_{\text{Reg}}(A, b)\), \(c_{\text{Reg}}(A, b)\) and \(\kappa_{\text{Reg}}(A, b)\) respectively reduce to the expressions of \(m_{\text{Reg}}, c_{\text{Reg}}\) and \(\text{cond}_{\text{Reg}}\) in (2.3), (2.4) and (2.5).

How are the structured condition numbers compared to their unstructured counterparts? The cases of nonsingular matrix inversion and linear systems have been investigated in [8, 9, 43, 44, 45] and the references therein. In the following proposition, we will show that \(m_{\text{Reg}}(A, b)\) is smaller than \(m_{\text{Reg}}(A, b)\). The same is true for the componentwise and normwise condition numbers. Before that we need the following lemma for rectangular structured matrices. Its proof is omitted since it is similar to that of [43, Lemma 6.3].

Lemma 4 when \(A\) is a Toeplitz or Hankel matrix, and \(A = \sum_{i=1}^{k} a_i S_i\), then

\[\|a\|_2 \leq \sqrt{2}\|A\|_F.\]
Proposition 1  Suppose that the basis \( \{S_1, S_2, \ldots, S_k\} \) for \( \mathcal{L} \) satisfies \( |A| = \sum_{i=1}^{k} |a_i||S_i| \) for any \( A \in \mathcal{L} \) in (3.1), then
\[
m^{\text{Reg}}_\mathcal{L}(A, b) \leq m^{\text{Reg}}(A, b) \quad \text{and} \quad c^\mathcal{L}_{\text{Reg}}(A, b) \leq c^\text{Reg}(A, b).
\]
For the structured normwise condition number, when \( A \) is a Toeplitz or Hankel matrix, we have
\[
\kappa^\mathcal{L}_{\text{Reg}}(A, b) \leq \sqrt{2} \max \left\{ \max_{i=1,2,\ldots,k} \|S_i\|_F, 1 \right\} \kappa^\text{Reg}(A, b).
\]

Proof. From Theorem 1, using the monotonicity of the infinity norm, we have
\[
\begin{align*}
\left\| \sum_{i=1}^{k} |a_i| \left( MP(A, \lambda) \left( A^T S_i x_\lambda - S_i x_\lambda \right) + MP(A, \lambda) A^T \right) \right\|_\infty \\
= \left\| MP(A, \lambda) \left( A^T S_i x_\lambda - S_i x_\lambda, \ldots, A^T S_k x_\lambda - S_k x_\lambda \right) \right\|_\infty |a_i| + \left\| MP(A, \lambda) A^T \right\|_\infty |b| \\
= \left\| MP(A, \lambda) \left( (x^T \otimes A^T) - (I_n \otimes r^T_\lambda) \right) \right\|_\infty \left\| \left( x^T \otimes A^T, \ldots, x^T \otimes A^T, A^T \right) \right\|_\infty |a_i| + \left\| MP(A, \lambda) A^T \right\|_\infty |b| \\
\leq \left\| MP(A, \lambda) \left( (x^T \otimes A^T) - (I_n \otimes r^T_\lambda) \right) \right\|_\infty \sum_{i=1}^{k} |a_i| \| vec(S_i) \| + \left\| MP(A, \lambda) A^T \right\|_\infty |b| \\
= \left\| MP(A, \lambda) \left( I_n \otimes r^T_\lambda - (x^T \otimes A^T) \right) \right\|_\infty \| vec(\lambda) \| + \left\| MP(A, \lambda) A^T \right\|_\infty |b|,
\end{align*}
\]
for the last equality we use the assumption \( |A| = \sum_{i=1}^{k} |a_i||S_i| \). With the above inequality, and the expressions of \( m^{\text{Reg}}_\mathcal{L}(A, b), m^{\text{Reg}}(A, b), c^\mathcal{L}_{\text{Reg}}(A, b), c^\text{Reg}(A, b) \), it is easy to prove the first two inequalities in this proposition.

When \( A \) is a Toeplitz or Hankel matrix, the standard basis for the Toeplitz matrix subspace or the Hankel matrix subspace is orthogonal under the inner product \( \langle B_1, B_2 \rangle = \text{trace} \left( B_1^T B_2 \right) = [\text{vec}(B_1)]^T \text{vec}(B_2) \) for \( B_1, B_2 \in \mathbb{R}^{m \times n} \). It is easy to deduce that
\[
\begin{align*}
\left\| MP(A, \lambda) \left[ S_i^T r_\lambda - A^T S_i x_\lambda, \ldots, S_k^T r_\lambda - A^T S_k x_\lambda, A^T \right] \right\|_2 \\
= \left\| MP(A, \lambda) \left( (x^T \otimes A^T) - (I_n \otimes r^T_\lambda) \right) \right\|_2 |vec(S_1), \ldots, vec(S_k), A^T| \\
= \left\| MP(A, \lambda) \left( (x^T \otimes A^T) - (I_n \otimes r^T_\lambda) \right) \right\|_2 \left[ \begin{array}{c}
vec(S_1), \ldots, vec(S_k)
\end{array} \right] \left[ \begin{array}{l}
0
\end{array} \right] I_{m_1} \left[ \begin{array}{c}
0
\end{array} \right] I_{m_2} \\
\leq \left\| MP(A, \lambda) \left( (x^T \otimes A^T) - (I_n \otimes r^T_\lambda) \right) \right\|_2 \left[ \begin{array}{c}
vec(S_1), \ldots, vec(S_k)
\end{array} \right] \left[ \begin{array}{l}
0
\end{array} \right] I_{m_1} \left[ \begin{array}{c}
0
\end{array} \right] I_{m_2} \\
= \left\| MP(A, \lambda) \left( (x^T \otimes A^T) - (I_n \otimes r^T_\lambda) \right) \right\|_2 \left[ \begin{array}{c}
\max_{i=1,2,\ldots,k} \| S_i \|_F, 1
\end{array} \right],
\end{align*}
\]
where for the last equation we used the orthogonality of the basis \( \{S_i\} \). So from Lemma 4,

\[
\kappa_L^{\text{Reg}}(A, b) \leq \max \left\{ \max_{i=1, 2, \ldots, k} \|S_i\|_{F, 1} \right\} \left( \frac{\|MP(A, \lambda) [(x^\top A^\top) - (I_n \otimes r_\lambda^\top)] \cdot A^\top \|_2 \|a\|}{\|Mx_\lambda\|_2} \right)
\]

\[
\leq \max \left\{ \max_{i=1, 2, \ldots, k} \|S_i\|_{F, 1} \right\} \left( \frac{\|MP(A, \lambda) [(x^\top A^\top) - (I_n \otimes r_\lambda^\top)] \cdot A^\top \|_2 \sqrt{2 \|A\|_F^2 + \|b\|_2^2}}{\|Mx_\lambda\|_2} \right)
\]

which completes the proof of this proposition. \( \square \)

**Remark 3** Clearly, the assumption \(|A| = \sum_{i=1}^{k} |a_i||S_i|\) in Proposition 1 is satisfied for Toeplitz and Hankel matrices.

### 3.2 Condition number estimators

Efficiently estimating condition numbers is crucial in practice. The condition number \( \kappa_L^{\text{Reg}}(A, b) \), for example, involves the spectral norm of the \( l \times (m + k) \) matrix \( D \phi([a; b]) \), which can be expensive to compute when \( m \) or \( k \) is large. The power method can be used for fast condition number estimation [31, page 289]. Its major computation is the matrix-vector multiplications \( D \phi([a; b]) h_1 \) and \( D \phi([a; b])^\top h_2 \) with \( h_1 \in \mathbb{R}^{m+k} \) and \( h_2 \in \mathbb{R}^l \). To consider \( D \phi([a; b])^\top h \), for \( h \in \mathbb{R}^l \) and \( v_i \) defined in Lemma 3, denoting \( D = MP(A, \lambda)A^\top \) and using \( P(A, \lambda)^\top = P(A, \lambda) \), \( (1.5) \) and \( \Pi^\top = \Pi^{-1} \), we have

\[
v_i^\top P(A, \lambda)M^\top h = r_\lambda^\top S_i P(A, \lambda)M^\top h - x_\lambda^\top S_i A P(A, \lambda)M^\top h
\]

\[
= \text{vec} \left[ \left( h^\top MP(A, \lambda) \right)^\top \otimes r_\lambda^\top \right] \text{vec}(S_i) - \left[ \left( h^\top D \right) \otimes x_\lambda^\top \right] \text{vec}(S_i^\top)
\]

\[
= \left[ \left( h^\top MP(A, \lambda) \right)^\top \otimes r_\lambda^\top \right] \text{vec}(S_i) - \left[ \left( h^\top D \right) \otimes x_\lambda^\top \right] \Pi \text{vec}(S_i)
\]

\[
= \text{vec}(S_i^\top) \Pi^{-1} \left[ \Pi \text{vec}(r_\lambda h^\top MP(A, \lambda)) - \text{vec}(x_\lambda h^\top D) \right]
\]

\[
= \left[ \Pi \text{vec}(S_i) \right]^\top \text{vec} \left[ P(A, \lambda)M^\top h r_\lambda^\top - x_\lambda h^\top D \right]
\]

\[
= \left[ \text{vec}(S_i^\top) \right]^\top \text{vec} \left[ P(A, \lambda)M^\top h r_\lambda^\top - x_\lambda h^\top D \right]
\]

\[
= \text{trace} \left[ S_i \left( P(A, \lambda)M^\top h r_\lambda^\top - x_\lambda h^\top D \right) \right], \tag{3.5}
\]

where we applied \( [\text{vec}(A_1)]^\top \text{vec}(A_2) = \text{trace} (A_1^\top A_2) \) for the same dimensional matrices \( A_1 \) and \( A_2 \) in the last equality. It follows from (3.5) that

\[
D \phi([a; b])^\top h = \left( MP(A, \lambda) \left[ v_1, \ldots, v_k, A^\top \right] \right)^\top h = \begin{bmatrix} \vdots \vphantom{v_k^\top P(A, \lambda)M^\top h} \\ v_k^\top P(A, \lambda)M^\top h \end{bmatrix}
\]

\[
= \begin{bmatrix} a(h) \\ D^\top h \end{bmatrix}, \tag{3.6}
\]
where \( a(h) = \begin{bmatrix} \text{trace} \left( S_1(P(A, \lambda)M^T h r_{\lambda}^T - x_\lambda h^T D) \right), \ldots, \text{trace} \left( S_k(P(A, \lambda)M^T h r_{\lambda}^T - x_\lambda h^T D) \right) \end{bmatrix}^T \). It leads to the following proposition.

**Proposition 2** The adjoint operator of \( D\phi([a; b]) \), with the scalar products \( a_1^T a_2 + b_1^T b_2 \) and \( h^T h \) in \( \mathbb{R}^{k+m} \) and \( \mathbb{R}^l \) respectively, is
\[
D\phi([a; b])^* : h \in \mathbb{R}^l \mapsto \begin{bmatrix} a(h), \ D^T h \end{bmatrix} \in \mathbb{R}^k \times \mathbb{R}^m.
\]
Furthermore, when \( l = 1 \),
\[
\kappa_{\text{Reg}}^*(A, b) = \sqrt{\frac{\sum_{i=1}^{k} s_i^2 + \|D\|^2_2}{\|Mx_\lambda\|_2^2}}, \tag{3.7}
\]
where \( s_i = \text{trace} \left( S_i \left( P(A, \lambda)M^T r_{\lambda}^T - x_\lambda D \right) \right), i = 1, 2, \ldots, k \).

**Proof.** For any \( (\Delta a, \Delta b) \in \mathbb{R}^{k} \times \mathbb{R}^{m} \) and \( h \in \mathbb{R}^l \), from Lemma 3 and (3.6), we have
\[
\langle h, D\phi([a; b]) \cdot (\Delta a, \Delta b) \rangle = h^T \left( D\phi([a; b]) \cdot (\Delta a, \Delta b) \right) = h^T D\phi([a; b]) \begin{bmatrix} \Delta a \\ \Delta b \end{bmatrix} = (D\phi([a; b])^T h) \begin{bmatrix} \Delta a \\ \Delta b \end{bmatrix} = a(h)^T (\Delta a) + (D^T h)^T (\Delta b) = (D\phi([a; b])^* \cdot h, (\Delta a, \Delta b)),
\]
which proves the first part. For the second part, noticing that
\[
\|D\phi([a; b])\|_2 = \|D\phi([a; b])^T\|_2 = \max_{h \neq 0} \frac{\left\| \begin{bmatrix} a(h)^T, (D^T h)^T \end{bmatrix} \right\|_2}{\|h\|_2},
\]
and using (3.6), where \( h \in \mathbb{R} \) since \( l = 1 \), we can show that
\[
\|D\phi([a; b])\|_2 = \sqrt{\sum_{i=1}^{k} s_i^2 + \|D\|^2_2},
\]
which completes the proof.

**Remark 4** When \( l = 1 \), we compute the conditioning of the \( i \)-th component of the solution. In that case \( M \) is the \( i \)-th canonical vector of \( \mathbb{R}^{1 \times n} \) and, in (3.8), \( P(A, \lambda)M^T \) is the \( i \)-th column of \( P(A, \lambda) \) and \( D \) is the \( i \)-th row of \( P(A, \lambda)A^T \).

Using (3.2) and (3.6), we can now apply the iteration of the power method [31, page 289] in Algorithm 1 to compute the normwise condition number \( \kappa_{\text{Reg}}^*(A, b) \). In this algorithm, we assume that \( x_\lambda, r_{\lambda} \) and \( \lambda \) are available. When the GSVD (1.4) of \( (A, L) \) is available, a compact form of \( P(A, \lambda) \) is given by
\[
P(A, \lambda) = QR^{-1} \begin{bmatrix} (\Sigma^2 + \lambda^2 S^2)^{-1} & 0 \\ 0 & I_{n-p} \end{bmatrix} R^{-T} Q^T, \tag{3.9}
\]
which can be used to reduce the computational cost of the estimators of the normwise, mixed and componentwise condition numbers.
The power method for estimating $\kappa_{\text{Reg}}^A(A, b)$

Select initial vector $h \in \mathbb{R}^l$.

\begin{algorithm}
\begin{algorithmic}
\For{$p = 1, 2, \ldots$}
\State Using (3.9), calculate $P(A, \lambda)M^T h_{r_p}^\top - x_h h_{r_p}^\top D$. From (3.6), denote $a_p = a(h)$ and $b_p = D^\top h$.
\State Calculate $\nu = \|[a_p; b_p]\|_2$, let $a_p = a_p/\nu$ and $b_p = b_p/\nu$.
\State Let $A_p = \sum_{i=1}^k a_{p,(i)} S_i$, where $a_{p,(i)}$ is the $i$-th component of $a_p$.
\State Using (3.2) and (3.9), compute $h = MP(A, \lambda) \left( A_{\top} b_p + A_{\top} r_{\lambda} - A_{\top} A_{p} x_{\lambda} \right)$.
\EndFor
\end{algorithmic}
\end{algorithm}

The quantity $\nu$ computed by Algorithm 1 is an approximation of the largest eigenvalue of $D\phi([a; b]) D\phi([a; b])^\top$. When there is an estimate of the corresponding dominant eigenvector of $D\phi([a; b]) D\phi([a; b])^\top$, the initial $h$ can be set to this estimate, but in many implementations $h$ is initialized as a random vector. The algorithm is terminated by a sufficient number of iterations or by evaluating the difference between two consecutive values of $\nu$ and comparing it to a tolerance given by the user.

For the mixed and componentwise condition numbers, we note that

$$m_{\text{Reg}}^A(A, b) = \frac{\|MP(A, \lambda) [v_1, \ldots, v_k, A_{\top}] \text{Diag}([a; b])\|_{\infty}}{\|Mx_{\lambda}\|_{\infty}} = \frac{\|D\phi([a; b]) \text{Diag}([a; b])\|_{\infty}}{\|Mx_{\lambda}\|_{\infty}},$$

$$c_{\text{Reg}}^A(A, b) = \frac{\|MP(A, \lambda) [v_1, \ldots, v_k, A_{\top}] \text{Diag}([a; b])\|_{\infty}}{Mx_{\lambda}} = \frac{\|D\phi([a; b]) \text{Diag}([a; b])\|_{\infty}}{\|Mx_{\lambda}\|_{\infty}}.$$ 

The above equations show that we only need to estimate the infinity norm of $D\phi([a; b]) \text{Diag}([a; b])$. Since we have the adjoint operator of $D\phi([a; b])$ in (3.7), the power method for estimating one norm [31, page 292] can be used to estimate $m_{\text{Reg}}^A(A, b)$ as shown in Algorithm 2. Also, note that, from (3.6), for $h \in \mathbb{R}^l$,

$$D\phi([a; b]) \text{Diag}([a; b]) = \text{Diag}([a; b]) \begin{bmatrix} a(h) \\ b \odot (D^\top h) \end{bmatrix},$$

where `$\odot$' denotes the Hadamard (componentwise) product. In Algorithm 2, $\text{sign}(a)$ denotes the vector obtained by applying the sign function to each component of the vector $a$. We can estimate $c_{\text{Reg}}^A(A, b)$ similarly.

The main computational cost of Algorithm 1 or Algorithm 2 is the computation of solving several nonsingular triangular systems with the coefficient matrices $R$ and $R^\top$. If we have the GSVD of $(A, L)$ available, the computational cost is insignificant compared with the cost of solving the Tikhonov regularized problem. Thus, the estimators can be integrated into a GSVD based Tikhonov solver without compromising the overall computational complexity. Our methods can be readily modified for fast unstructured condition number estimation, which is not considered in [12].
Algorithm 2 The power method for estimating \( m_{\text{Reg}}(A, b) \)

Select initial vector \( h = l^{-1}e \in \mathbb{R}^l \).

\[
\text{for } p = 1, 2, \ldots \text{ do } \\
\quad \text{Using (3.9), calculate } P(A, \lambda)M^\top hr_{\lambda}^\top - x_{\lambda}h^\top D. \text{ From (3.7), compute } a(h) \text{ and } D^\top h. \\
\quad \text{Using (3.10), denote } \alpha_p = a \odot a(h) \text{ and } \beta_p = b \odot (D^\top h). \\
\quad \text{Let } \bar{\alpha}_p = \text{sign}(\alpha_p) \text{ and } \bar{\beta}_p = \text{sign}(\beta_p). \\
\quad \text{Compute } a_p = a \odot \bar{\alpha}_p, \ b_p = b \odot \bar{\beta}_p. \\
\text{Form } A_p = \sum_{i=1}^{k} a_{p,(i)} S_i, \text{ where } a_p = [a_{p,(1)}, a_{p,(2)}, \ldots, a_{p,(k)}]^\top. \\
\text{Using (3.1) and (3.9), compute } z = MP(A, \lambda)(A^\top b_p + A^\top p r_{\lambda} - A^\top A_p x_{\lambda}). \\
\quad \text{if } \|z\|_\infty \leq h^\top z \text{ then } \\
\quad \quad \gamma = \left\| \begin{bmatrix} \alpha_p \\ \beta_p \end{bmatrix} \right\|_1 \\
\quad \quad \text{quit} \\
\text{end if} \\
\quad h = e_j^{(l)}, \text{ where } |z_j| = \|z\|_\infty \text{ (smallest such } j). \\
\text{end for} \\
\] \( m_{\text{Reg}}(A, b) = \gamma. \)

4 Nonlinear Structures

In this section, we present the structured condition numbers of matrices with nonlinear structures, namely the Vandermonde matrices and the Cauchy matrices.

4.1 Vandermonde matrices

Let \( \text{VdM} \) be the class of \( m \times n \) Vandermonde matrices. If \( V = [v_{ij}] \in \text{VdM} \), then there exists \( a = [a_0, a_1, \ldots, a_{m-1}]^\top \in \mathbb{R}^n \) such that, for all \( i = 0, 1, \ldots, m-1 \) and \( j = 0, 1, \ldots, n-1 \), \( v_{ij} = a_j^i \).

We write \( V = g(a) \). Let \( \Delta a = (\Delta a_0, \Delta a_1, \ldots, \Delta a_{n-1})^\top \in \mathbb{R}^n \) be the perturbation on \( a \). Then we define the first order term \( \Delta V \) of \( g(a + \Delta a) - g(a) \). From

Lemma 5 ([13, Lemma 6]) An explicit expression of \( \Delta V \) is

\[
\Delta V = V_1 \text{Diag}(\Delta a), \quad \text{where } V_1 = \text{Diag}(c) \begin{bmatrix} 0 \\ V(1 : m - 1, :) \end{bmatrix}, \ c = [0, 1, \ldots, m - 1]^\top.
\]

Here \( V(1 : m - 1, :) \) is the \( (m - 1) \times n \) submatrix of \( V \) consisting of the first \( m - 1 \) rows of \( V \).

Lemma 6 The Fréchet derivative \( D\phi([a; b]) \) of function \( \phi \) defined in (2.7) is

\[
D\phi([a; b]) = MP(V, \lambda) \left[ -V^\top V_1 \text{Diag}(x_{\lambda}) + \text{Diag}(y), V^\top \right],
\]

where \( y = V_1^\top r_{\lambda} \) and \( r_{\lambda} = b - V x_{\lambda} \).
Proof. It follows from (3.4) and $\text{Diag}(a) z = \text{Diag}(z) a$ for vectors $a$ and $z$ of the same dimension,

$$\phi([a + \Delta a; b + \Delta b]) - \phi([a; b])$$

$$\approx MP(V, \lambda) \left[ (-x_\lambda^T \otimes V^T + I_n \otimes r_\lambda^T) \ \text{vec}(\Delta V) + V^T(\Delta b) \right]$$

$$= MP(V, \lambda) \left[ (-x_\lambda^T \otimes V^T + (r_\lambda^T \otimes I_n)I) \ \text{vec}(\Delta V) + V^T(\Delta b) \right]$$

$$= MP(V, \lambda) \left[ (-V^T(\Delta V)x_\lambda + (\Delta V)^T r_\lambda) + V^T(\Delta b) \right]$$

$$= MP(V, \lambda) \left[ -V^T V_1 \text{Diag}(\Delta a)x_\lambda + \text{Diag}(\Delta a) V_1^T r_\lambda \right] + V^T(\Delta b)$$

which completes the proof of this lemma. \qed

From Lemmas 1 and 6, we have the following theorem of structured condition numbers of the Vandermonde matrix.

Theorem 3 Let $V \in \text{VdM}$, $b \in \mathbb{R}^m$ and $x_\lambda = (V^T V + \lambda L^T L)^{-1} V^T b = P(V, \lambda)V^T b$ be the Tikhonov regularized solution of (1.1). Recall that $y = V_1^T r_\lambda$, then the structured condition numbers of the Vandermonde matrix are:

$$n_{\text{VdM}}(V, b) = \frac{\|MP(V, \lambda) [V^T V_1 \text{Diag}(x_\lambda) - \text{Diag}(y)]\| |a| + \|P(V, \lambda)V^T| b\|_\infty}{\|Mx_\lambda\|_\infty},$$

$$c_{\text{VdM}}(V, b) = \frac{\|MP(V, \lambda) [V^T V_1 \text{Diag}(x_\lambda) - \text{Diag}(y)]\| |a| + \|P(V, \lambda)V^T| b\|_\infty}{Mx_\lambda},$$

$$\kappa_{\text{VdM}}(V, b) = \frac{\|MP(V, \lambda) [\text{Diag}(y) - V^T V_1 \text{Diag}(x_\lambda), V^T]\|_2 \left\| \begin{bmatrix} a \end{bmatrix} \right\|_2}{\|Mx_\lambda\|_2^2}.$$  

In particular, when $l = 1$,

$$\kappa_{\text{VdM}}(V, b) = \frac{\sqrt{\|y \circ (P(V, \lambda) M^T) - x_\lambda \circ (V_1^T D_V)\|_2^2 + \|D_V\|_2^2} \left\| \begin{bmatrix} a \end{bmatrix} \right\|_2}{\|Mx_\lambda\|_2},$$

where $D_V = MP(V, \lambda)V^T$. The above expressions can be used to estimate $n_{\text{VdM}}(V, b)$, $c_{\text{VdM}}(V, b)$ and $\kappa_{\text{VdM}}(V, b)$ with lower dimensional input. We can devise algorithms similar to Algorithms 1 and 2 for estimating the condition numbers.
4.2 Cauchy matrices

Let Cauchy be the class of $m \times n$ Cauchy matrices. If $C = [c_{ij}] \in \text{Cauchy}$, then there exist $u = [u_1, u_2, \ldots, u_m]^T \in \mathbb{R}^m$ and $v = [v_1, v_2, \ldots, v_n]^T \in \mathbb{R}^n$, with $u_i \neq v_j$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ such that, for all $i \leq m$ and $j \leq n$,

$$c_{ij} = \frac{1}{u_i - v_j}.$$  

If $w = [u; v] \in \mathbb{R}^{m+n}$, then $C = g(w)$. Let $\Delta w = [\Delta u; \Delta v] = [\Delta u_1, \Delta u_2, \ldots, \Delta u_m, \Delta v_1, \ldots, \Delta v_n]^T \in \mathbb{R}^{m+n}$ be the perturbation on $w$. The first order term $\Delta C$ in $g(w + \Delta w) - g(w)$ is given by [13, Lemma 9]

$$\Delta C \approx \begin{bmatrix} \frac{\Delta u_i - \Delta v_j}{(u_i - v_j)^2} \end{bmatrix} = \text{Diag}(\Delta u)C_1 - C_1 \text{Diag}(\Delta v) \in \mathbb{R}^{m \times n},$$

where $C_1 = [1/(u_i - v_j)^2] \in \mathbb{R}^{m \times n}$.

Lemma 7 The Fréchet derivative $D\phi([w; b])$ of function $\phi$ defined in (2.7) is given by

$$D\phi([w; b]) = MP(C, \lambda) \begin{bmatrix} C_u, C_v, C^T \end{bmatrix},$$

where $C_u = C_1^T \text{Diag}(r_\lambda) - C^T \text{Diag}(z_1)$, $C_v = C^T C_1 \text{Diag}(x_\lambda) - \text{Diag}(z_2)$, $z_1 = C_1 x_\lambda$, $z_2 = C_1^T r_\lambda$ and $r_\lambda = b - C x_\lambda$.

Proof. Following the proof of Lemma 6, we can show that

$$\phi([w + \Delta w; b + \Delta b]) - \phi([w; b]) = MP(C, \lambda) \begin{bmatrix} C_1^T \text{Diag}(r_\lambda) - C^T \text{Diag}(z_1), C^T C_1 \text{Diag}(x_\lambda) - \text{Diag}(z_2), C^T \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta b \end{bmatrix}.$$  

Then the Fréchet derivative of $\phi$ at $[w; b]$ is

$$D\phi[w; b] = MP(C, \lambda) \begin{bmatrix} C_1^T \text{Diag}(r_\lambda) - C^T \text{Diag}(z_1), C^T C_1 \text{Diag}(x_\lambda) - \text{Diag}(z_2), C^T \end{bmatrix}.$$  

Theorem 4 Let $C \in \text{Cauchy}$, $b \in \mathbb{R}^m$ and $x_\lambda = (C^T C + \lambda L^T L)^{-1} C^T b = P(C, \lambda) C^T b$ be the solution of the Tikhonov regularization problem (1.1), then the structured condition numbers are:

$$m_{\text{Reg}}^\text{Cauchy}(C, b) = \frac{\|MP(C, \lambda) C_u\|_\infty + \|MP(C, \lambda) C_v\|_\infty + \|MP(C, \lambda) C^T\|_\infty \|b\|_\infty}{\|M x_\lambda\|_\infty},$$

$$c_{\text{Reg}}^\text{Cauchy}(C, b) = \frac{\left\|MP(C, \lambda) C_u\|_\infty + \|MP(C, \lambda) C_v\|_\infty + \|MP(C, \lambda) C^T\|_\infty \right\|_\infty}{\|M x_\lambda\|_\infty},$$

$$\kappa_{\text{Reg}}^\text{Cauchy}(C, b) = \frac{\|MP(C, \lambda) [C_u, C_v, C^T]\|_2 \left\|\begin{bmatrix} w \\ b \end{bmatrix}\right\|_2}{\|M x_\lambda\|_2}. $$
In particular, when \( l = 1 \),

\[
\kappa_{\text{Reg}}^{\text{Cauchy}}(C, b) = \frac{\sqrt{t^2 + s^2 + \|D_C\|^2}}{\|Mx_{\lambda}\|_2} \left\| \begin{bmatrix} u \\ v \\ b \end{bmatrix} \right\|_2,
\]

where \( t = \|r_{\lambda} \odot [C_1 P(C, \lambda) M^T] - z_1 \odot (D_C^T)\|_2 \) and \( s = \|x_{\lambda} \odot (C_1^T D_C^T) - z_2 \odot [P(C, \lambda) M^T]\|_2 \).

Similar to the case of the Vandermonde matrix, for the Cauchy matrix, the adjoint operator of \( D\phi([w; b]) \), using the scalar products \( u_1^T u_2 + v_1^T v_2 + b_1^T b_2 \) and \( h^T h \) on \( \mathbb{R}^{2m+n} \) and \( \mathbb{R}^l \) respectively, is

\[
D\phi([w; b])^* : h \in \mathbb{R}^l \mapsto \begin{bmatrix} u(h) \\ v(h) \\ D_C^T h \end{bmatrix} \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m,
\]

where \( u(h) = r_{\lambda} \odot [C_1 P(C, \lambda) M^T h] - z_1 \odot (D_C^T h), \ v(h) = x_{\lambda} \odot (C_1^T D_C^T h) - z_2 \odot [P(C, \lambda) M^T h] \) and \( D_C = MP(C, \lambda) C^T \).

In particular, when \( l = 1 \), we have

\[
\kappa_{\text{Reg}}^{\text{Cauchy}}(V, b) = \frac{\sqrt{t^2 + s^2 + \|D_C\|^2}}{\|Mx_{\lambda}\|_2} \left\| \begin{bmatrix} u \\ v \\ b \end{bmatrix} \right\|_2,
\]

where \( t = \|r_{\lambda} \odot [C_1 P(C, \lambda) M^T] - z_1 \odot (D_C^T)\|_2 \) and \( s = \|x_{\lambda} \odot (C_1^T D_C^T) - z_2 \odot [P(C, \lambda) M^T]\|_2 \).

Using the above expressions, the algorithms similar to Algorithms 1 and 2 for estimating \( m_{\text{Reg}}^{\text{Cauchy}}(C, b), c_{\text{Cauchy}}(C, b) \) and \( \kappa_{\text{Cauchy}}(C, b) \) can be obtained.

5 SCE for the Tikhonov Regularization Problem

In this section we use SCE to devise algorithms for the condition estimations of the structured and unstructured Tikhonov regularization problem, both the normwise and componentwise cases are considered.

5.1 SCE for normwise perturbations

For the unstructured Tikhonov regularization problem, we are interested in the condition estimation for the function \( \psi([A, b]) \) at the point \([A, b] \) defined in (2.6). Let \([A, b] \) be perturbed to \([A + \delta E \ b + \delta f] \) in the normal equations (1.3), where \( \delta \in \mathbb{R}, \ E \in \mathbb{R}^{m \times n} \) and \( f \in \mathbb{R}^m \) and \([E, f] \) has the Frobenius norm equal to one. According to Subsection 2.2, we first need to evaluate the directional derivative \( D\psi([A, b]; [E, f]) \) of \( \psi([A, b]) \) with respect to \([A, b] \) in the direction \([E, f] \). From the proof of Lemma 3, we have

\[
D\psi([A, b]; [E, f]) = P(A, \lambda)\left(A^T f + E^T r_\lambda - A^T Ex_\lambda\right).
\]

When we have the GSVD (1.4) of \((A, L)\), it is easy to deduce that

\[
D\psi([A, b]; [E, f]) = QR^{-1} \begin{bmatrix} (\Sigma^2 + \lambda^2 S^2)^{-1} & 0 \\ 0 & I_{n-p} \end{bmatrix} R^{-T} Q^T \left(A^T f + E^T r_\lambda - A^T Ex_\lambda\right). \tag{5.1}
\]

With the above result, we now use the results of Subsection 2.2 to obtain the SCE-based methods for estimating the condition of the Tikhonov regularization problems. Both the normwise and
componentwise perturbations are considered. Algorithm 3 computes an estimation of the normwise condition number. Inputs to the method are the matrices $A \in \mathbb{R}^{m \times n}$, $L \in \mathbb{R}^{p \times n}$, the vector $b \in \mathbb{R}^m$, the computed solution $x_\lambda$ and the parameter $\lambda$. The output is an estimation $\kappa_{\text{SCE}}^{(k)}$ of the normwise condition number $\text{cond}_{\text{Reg}}^F$. The method requires the GSVD (1.4) of $(A, L)$, which is generally computed when solving the Tikhonov regularization problem. The integer $k \geq 1$ refers to the number of perturbations of input data. Note that when $k = 1$, there is no need to orthonormalize the set of vectors in Step 1 of the method. In the following the standard normal distribution are denote by $\mathcal{N}(0,1)$, and for $B = (b_{ij}) \in \mathbb{R}^{p \times q}$, $|B|^2 = (|b_{ij}|^2) \in \mathbb{R}^{p \times q}$ and $\sqrt{|B|} = (\sqrt{|b_{ij}|}) \in \mathbb{R}^{p \times q}$.

**Algorithm 3** SCE for the Tikhonov regularization problem under normwise perturbations

1. Generate matrices $[E_1, f_1], [E_2, f_2], \ldots, [E_k, f_k]$ whose entries are random numbers in $\mathcal{N}(0,1)$, where $E_i \in \mathbb{R}^{m \times n}$, $f_i \in \mathbb{R}^m$. Use a QR factorization for the matrix
   \[
   \begin{bmatrix}
   \text{vec}(E_1) & \text{vec}(E_2) & \cdots & \text{vec}(E_k) \\
   f_1 & f_2 & \cdots & f_k
   \end{bmatrix}
   \]
   and form an orthonormal matrix $[\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_k]$. Each $q_i$ can be converted into the desired matrices $[\tilde{E}_i, \tilde{f}_i]$ with the unvec operation.

2. Calculate $D\psi([A, b]; [\tilde{E}_i, \tilde{f}_i])$ by (5.1), $i = 1, 2, \ldots, k$.

3. Compute the absolute condition vector
   \[
   \kappa_{\text{abs}}^{(k)} := \frac{\omega_k}{\omega_p} \sqrt{\left|D\psi([A, b]; [\tilde{E}_1, \tilde{f}_1])\right|^2 + \cdots + \left|D\psi([A, b]; [\tilde{E}_k, \tilde{f}_k])\right|^2}.
   \]

4. Compute the normwise condition estimation:
   \[
   \kappa_{\text{SCE}}^{(k)} := \frac{\|\kappa_{\text{abs}}^{(k)}\|_2 \|[A, b]\|_F}{\|x_\lambda\|_2}.
   \]

### 5.2 SCE for componentwise perturbations

Componentwise perturbations are relative to the magnitudes of the corresponding entries in the input arguments (e.g., the perturbation $\Delta A$ satisfies $|\Delta A| \leq \epsilon |A|$, see (2.3)). These perturbations may arise from input error or from rounding error, and hence are the most common perturbations encountered in practice. In fact, most of error bounds in LAPACK are componentwise since the perturbations of input data are componentwise in real world computing, see [1, section 4.3.2] for details. We often want to find the condition of a function with respect to componentwise perturbations on inputs. For the function
   \[
   \psi([A, b]) = \left( A^\top A + \lambda^2 L^\top L \right)^{-1} A^\top b,
   \]
SCE is flexible enough to accurately gauge the sensitivity of matrix functions subject to componentwise perturbations. Define the linear function

\[ h([B, d]) = [B, d] \circ [A, b], \quad B \in \mathbb{R}^{m \times n}, \quad d \in \mathbb{R}^m. \]

Let \( E \in \mathbb{R}^{m \times (n+1)} \) be the matrix of all ones, then \( h(E) = [A b] \) and

\[ h(E + [E, f]) = [A, b] + h([E, f]). \]

We know that \( h([E, f]) \) is a componentwise perturbation on \([A, b]\), and \( h \) converts a general perturbation \( E \) into componentwise perturbations on \([A, b]\). Therefore, to obtain the sensitivity of the solution with respect to relative perturbations, we simply evaluate the Fréchet derivative of \( \psi([A, b]) = \psi(h(E)) \)

with respect to \( E \) in the direction \([E, f]\), which is

\[
D(\psi \circ h) (E; [E, f])) = D\psi(h(E)) Dh(E; [E, f]) = D\psi([A, b])h ([E, f]) = D\psi ([A, b]; h ([E, f])),
\]

since \( h \) is linear. Thus, to estimate the condition of the Tikhonov regularization solution \( x_\lambda \) when perturbations are componentwise, we first generate the perturbations \( E \) and \( f \) and multiply them componentwise by the entries of \( A \) and \( b \), respectively. The remaining steps are the same as the corresponding steps in Algorithm 3, as shown in Algorithm 4.

### 5.3 SCE for structured perturbations

The SCE also is flexible for the condition estimation for structured Tikhonov regularization problem. We are interested in the condition estimation for the function \( \phi \) defined in (2.7), which defines the general function for structured Tikhonov regularization problem. Because SCE can estimate the condition of the each component of \( x_\lambda \), we only need to choose \( M = I_n \) in (2.7).

The key step in the SCE is the computation of the directional derivative \( D\phi([a; b]; [e; f]) \) of \( \phi([a; b]) \) with respect to \([a; b]\) in the direction \([e; f]\), where \( e \in \mathbb{R}^k \) and \( f \in \mathbb{R}^m \). We have derived the explicit expressions of the Fréchet derivative \( D\phi([a; b]) \) in Lemmas 3, 6 and 7 for a general linear structure, Vandermonde or Cauchy matrix. Based on Lemmas 3, 6 and 7, the three directional derivatives \( D\phi([a; b]; [e; f]) \) are:

\[
D\phi([a; b]; [e; f]) = P(A, \lambda) \left( A^\top f + E^\top r_\lambda - A^\top E x_\lambda \right), \quad e = (e_i) \in \mathbb{R}^k, \quad E = \sum_{i=1}^{k} e_i S_i,
\]

for linear structures,

\[
D\phi([a; b]; [e; f]) = MP(V, \lambda) \left( \text{Diag}(y)e - V^\top V_1 \text{Diag}(x_\lambda)e + V^\top f \right), \quad e \in \mathbb{R}^n,
\]

for Vandermonde matrices, and

\[
D\phi([a; b]; [e; f]) = MP(C, \lambda) \left( C_u e_1 + C_v e_2 + C^\top f \right), \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \in \mathbb{R}^{m+n},
\]

for Cauchy matrices, where \( V_1 \) is defined in Lemma 5, \( y = V_1^\top r_\lambda \), \( C_u \) and \( C_v \) are defined in Lemma 7. Based on those expressions, we can derive algorithms for structured normwise and componentwise condition estimation. The algorithms are similar to those of Algorithms 3 and 4, thus are omitted here.
Algorithm 4 SCE for the Tikhonov regularization problem under componentwise perturbations

1. Generate matrices $[E_1, f_1], [E_2, f_2], \ldots, [E_k, f_k]$ whose entries are random numbers in $N(0, 1)$, where $E_i \in \mathbb{R}^{m \times n}$, $f_i \in \mathbb{R}^m$. Use a QR factorization for the matrix
\[
\begin{bmatrix}
\text{vec}(E_1) & \text{vec}(E_2) & \cdots & \text{vec}(E_k)
\end{bmatrix}
\begin{bmatrix}
f_1 \\ f_2 \\ \vdots \\ f_k
\end{bmatrix}
\]
to form an orthonormal matrix $[q_1, q_2, \ldots, q_k]$. Each $q_i$ can be converted into the desired matrices $[E_i, f_i]$ with the unvec operation.

2. For $i = 1, \ldots, k$, set $[\tilde{E}_i, \tilde{f}_i]$ to the componentwise product of $[A, b]$ and $[E_i, f_i]$.

3. Calculate $D\psi([A, b]; [\tilde{E}_1, \tilde{f}_1])$ by (5.1), $i = 1, 2, \ldots, k$.

4. Compute the absolute condition vector
\[
c^{(k)}_{\text{abs}} := \frac{\omega_k}{\omega_p} \sqrt{D\psi([A, b]; [\tilde{E}_1, \tilde{f}_1])^2 + \cdots + D\psi([A, b]; [\tilde{E}_k, \tilde{f}_k])^2}.
\]

5. The mixed condition estimation $m^{(k)}_{\text{SCE}}$ and componentwise condition estimation $c^{(k)}_{\text{SCE}}$ are:
\[
m^{(k)}_{\text{SCE}} := \frac{\|c^{(k)}_{\text{abs}}\|_{\infty}}{\|x_\lambda\|_{\infty}} \quad \text{and} \quad c^{(k)}_{\text{SCE}} := \frac{\|c^{(k)}_{\text{abs}}\|_{\infty}}{\|x_\lambda\|_{\infty}}.
\]

6 Numerical Examples

In this section, we demonstrate our test results of some numerical examples to illustrate structured condition numbers and condition estimations presented in the previous sections. All the computations are carried out using MATLAB 8.1 with the REGULARIZATION TOOLS package [25] with the machine precision $2 \times 10^{-16}$.

For a structured matrix $A$, which is determined by the vector $a \in \mathbb{R}^k$, we generated the perturbed matrix $\hat{A}$ as follows. For $a \in \mathbb{R}^k$ and $b \in \mathbb{R}^m$, let $[s; f]$ be a random vector whose entries are uniformly distributed in the open interval $(-1, 1)$, where $s \in \mathbb{R}^k$ and $f \in \mathbb{R}^m$, the perturbations on $a$ and $b$ are respectively
\[
\Delta a_i = \varepsilon s_i a_i, \quad \Delta b_j = \varepsilon f_j b_j,
\]
then $\hat{A} = g(a + \Delta a)$ and $\hat{b} = b + \Delta b$. In our experiments, we set $\varepsilon = 10^{-8}$.

REGULARIZATION TOOLS package [25] includes four methods for determining the Tikhonov regularization parameter. For the Tikhonov regularization with continuous regularization parameter, the $L$-curve is a continuous curve as a parametric plot of the discrete smoothing (semi) norm $\|Lx_\lambda\|_2$ versus the corresponding residual norm $\|Ax_\lambda - b\|_2$, with the parameter $\lambda$ as the parameter. The corner of the $L$-curve appears for regularization parameters close to the optimal parameter that balances the regularization errors and perturbation errors in $x_\lambda$, which is the basis for the $L$-curve criterion for choosing the regularization parameter. Besides the $L$-curve criterion for
parameter-choice, a variety of parameter-choice strategies have been proposed, such as the discrepancy principle (Discrep. pr.) [40], generalized cross-validation (GCV) [48] and the quasi-optimality criterion (Quasi-opt) [40].

The Tikhonov regularization solution $x_\lambda$ was computed by the Matlab function tikhonov corresponding to $A$, $b$ in REGULARIZATION TOOLS package with different regularization parameters chosen by four classical criteria or by the predefined value. The perturbed solution $y_\lambda$ is obtained in the similar way to $x_\lambda$, but $y_\lambda$ corresponds to $\hat{A}$ and $\hat{b}$. Denote the error $\Delta x_\lambda = y_\lambda - x_\lambda$.

We compare the structured condition numbers with unstructured ones for various Tikhonov regularization parameters in the following examples.

**Example 1** ([44]) Let $A = g(a)$ be a $5 \times 5$ symmetric Toeplitz matrix which is defined

$$
A = g(a) = \begin{bmatrix}
0 & 0 & 1 + h & -1 & 1 \\
0 & 0 & 0 & 1 + h & -1 \\
1 + h & 0 & 0 & 0 & 1 + h \\
-1 & 1 + h & 0 & 0 & 0 \\
1 & -1 & 1 + h & 0 & 0 \\
\end{bmatrix}, \quad a = \begin{bmatrix}
0 \\
0 \\
1 + h \\
-1 \\
1 \\
\end{bmatrix}, \quad b = \begin{bmatrix}
0 \\
h \\
2(1 + h) \\
-1 \\
h \\
0 \\
\end{bmatrix}
$$

for $h = 10^{-3}$.

The above matrix $A$ is a square symmetric Toeplitz matrix and $g(a)$ is a square symmetric Toeplitz matrix whose first column is $a$. We can choose the basis $Z_i = g(e_i), i = 1, 2, \ldots, 5$, so that $T = \sum_{i=1}^5 a_iZ_i$. From Theorem 1, with the $Z_i$, we can get the expressions for $m_{\text{SymToep}}(A, b)$, $c_{\text{SymToep}}(A, b)$ and $\kappa_{\text{SymToep}}(A, b)$. The relative errors and condition numbers are shown in Table 1, where $M = I_5$.

Table 1 shows that the structured mixed condition numbers are much smaller than the corresponding unstructured ones, which give tight linear perturbation bounds. Both $c_{\text{SymToep}}(A, b)$ and $\kappa_{\text{SymToep}}(A, b)$ are smaller than the corresponding unstructured ones.

In Table 2, we choose $M = e_3^T$, where $e_3$ is the third column of $I_5$. In this case,

$$
\frac{\|M\Delta x_\lambda\|_\infty}{\|Mx_\lambda\|_\infty} = \frac{\|M\Delta x_\lambda\|}{\|Mx_\lambda\|_\infty}, \quad m_{\text{SymToep}}(A, b) = c_{\text{SymToep}}(A, b).
$$

| $\lambda$ | Discrep. pr. | $L$-curve | GCV | Quasi-opt |
|---|---|---|---|---|
| $\|\Delta x_\lambda\|_2$ | 6.3937 · $10^{-1}$ | 4.9988 · $10^{-4}$ | 4.9988 · $10^{-4}$ | 4.9988 · $10^{-4}$ |
| $\|\Delta x_\lambda\|_2$ | 9.1464 · $10^{-1}$ | 1.4820 | 1.1081 | 3.1918 |
| $\|\Delta x_\lambda\|_\infty$ | 9.3703 · $10^{-1}$ | 1.6158 | 1.3755 | 3.7359 |
| $e^{-1}\|\Delta x_\lambda\|_{\infty}$ | 2.29 · $10^6$ | 6.26 · $10^6$ | 1.55 · $10^6$ | 4.74 · $10^6$ |
| $\text{cond}_{\text{Reg}}^2$ in (2.5) | 3.3961 · $10^3$ | 4.4761 · $10^4$ | 4.4761 · $10^4$ | 4.4761 · $10^4$ |
| $m_{\text{Reg}}$ in (2.3) | 1.5204 · $10^3$ | 2.0035 · $10^4$ | 2.0035 · $10^4$ | 2.0035 · $10^4$ |
| $c_{\text{Reg}}$ in (2.4) | 9.8192 · $10^6$ | 1.6064 · $10^7$ | 1.6064 · $10^7$ | 1.6064 · $10^7$ |
| $\kappa_{\text{SymToep}}(A, b)$ | 1.0047 · $10^3$ | 1.3242 · $10^4$ | 1.3242 · $10^4$ | 1.3242 · $10^4$ |
| $m_{\text{SymToep}}(A, b)$ | 4.3765 | 4.4971 | 4.4971 | 4.4971 |
| $c_{\text{SymToep}}(A, b)$ | 9.8143 · $10^6$ | 1.6056 · $10^7$ | 1.6056 · $10^7$ | 1.6056 · $10^7$ |
bounds given by \( \kappa \) numbers in Algorithms 1 and 2 are denoted by a method. We set the maximal number of iterations to 10 in Algorithm 1. The estimated condition component of \( x \) display the values of \( \varepsilon_m \), \( \varepsilon_\kappa \), and \( \varepsilon_{\kappa_{\text{SymToep}}} \) and \( \varepsilon_{m_{\text{SymToep}}} \), where \( \| \cdot \|_{\infty} \) has better conditioning than the third one. From Table 2, we can see the quantities \( \varepsilon_{\kappa_{\text{SymToep}}} (A, b) \) and \( \varepsilon_{m_{\text{SymToep}}} (A, b) \), where \( \kappa_{\text{SymToep}} (A, b) \) give tighter perturbation bounds than \( \varepsilon_{\kappa_{\text{SymToep}}} (A, b) \), since they have the same order as that of the true relative perturbation bounds. Table 3 shows the results from different choices of \( M \), i.e., \( M = e_1 \) and \( M = e_3 \). For example, if we choose \( M = e_1 \), then we are interested in the conditioning of the first component of \( x_\lambda \). We display the values of \( m_{\text{SymToep}} (A, b) \) and \( \kappa_{\text{SymToep}} (A, b) \). From Table 3, we can say that the first component of \( x_\lambda \) has better conditioning than the third one.

At the end of this example, we use Algorithms 1 and 2 to illustrate the effectiveness of the power method. We set the maximal number of iterations to 10 in Algorithm 1. The estimated condition numbers in Algorithms 1 and 2 are denoted by \( \kappa_{\text{SymToep}}^{\text{Est}} (A, b) \) and \( m_{\text{SymToep}}^{\text{Est}} (A, b) \) respectively.

From Table 4, we can say that \( \kappa_{\text{SymToep}}^{\text{Est}} (A, b) \) and \( m_{\text{SymToep}}^{\text{Est}} (A, b) \) give good estimations for this specific \( A, L \) and \( b \), especially \( m_{\text{SymToep}}^{\text{Est}} (A, b) \) gives better estimation.

**Example 2** ([44]) Let \( A = g([c; r]) \) be the \( 6 \times 6 \) Hankel matrix defined by

\[
A = g([c; r]) = \begin{bmatrix}
h & 1 & 1 & -1 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & -1 & 1 & 1 \\
0 & 0 & -1 & 1 & 1 & 0 \\
\end{bmatrix}, \quad c = \begin{bmatrix}
h \\
1 \\
1 \\
1 \\
1 \\
0 \\
0 \\
\end{bmatrix}, \quad r = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}, \quad b = \begin{bmatrix}
h \\
2 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

for \( h = 10^{-3} \), where \( c \) is the first column of \( A \) and \( r \) is the last row of \( A \).
Table 4: \( L = I_5, M = I_5 \) and \( M = e_1^\top \).

| \( M = I_5 \) | Discrep. pr. | \( L \)-curve | GCV | Quasi-opt |
|----------------|--------------|--------------|-----|----------|
| \( \lambda \)  | 6.39 \cdot 10^{-4} | 5 \cdot 10^{-4} | 5.08 \cdot 10^{-4} | 5.08 \cdot 10^{-4} |
| \( \kappa_{\text{SymToep}}(A, b) \) | 1.5878783796 \cdot 10^{-3} | 2.0931466345 \cdot 10^{-3} | 2.0584876249 \cdot 10^{-3} | 2.0584876249 \cdot 10^{-3} |
| \( \kappa_{\text{Est}}^{\text{SymToep}}(A, b) \) | 7.5891802517 \cdot 10^{-2} | 1.0002500616 \cdot 10^{-3} | 9.8369583257 \cdot 10^{-2} | 9.8369583257 \cdot 10^{-2} |
| \( m_{\text{Est}}(A, b) \) | 7.6117517197 \cdot 10^{-2} | 1.0027483753 \cdot 10^{-3} | 9.8617760333 \cdot 10^{-2} | 9.8617760333 \cdot 10^{-2} |
| \( m_{\text{Est}}(A, b) \) | 7.6055529470 \cdot 10^{-2} | 1.0022493745 \cdot 10^{-3} | 9.8567031098 \cdot 10^{-2} | 9.8567031098 \cdot 10^{-2} |

Table 5: \( L = I_6, M = I_6 \).

| \( M = I_6 \) | Discrep. pr. | \( L \)-curve | GCV | Quasi-opt |
|----------------|--------------|--------------|-----|----------|
| \( \lambda \)  | 6.39 \cdot 10^{-4} | 5 \cdot 10^{-4} | 5.08 \cdot 10^{-4} | 5.08 \cdot 10^{-4} |
| \( \kappa_{\text{SymToep}}(A, b) \) | 1.5886924101 \cdot 10^{-3} | 2.0940875560 \cdot 10^{-3} | 2.0594198505 \cdot 10^{-3} | 2.0594198505 \cdot 10^{-3} |
| \( \kappa_{\text{Est}}^{\text{SymToep}}(A, b) \) | 3.7980426062 \cdot 10^{-2} | 5.0050048168 \cdot 10^{-3} | 4.9222129601 \cdot 10^{-2} | 4.9222129601 \cdot 10^{-2} |
| \( m_{\text{Est}}(A, b) \) | 7.605560483 \cdot 10^{-2} | 1.0022496243 \cdot 10^{-3} | 9.8567056493 \cdot 10^{-2} | 9.8567056493 \cdot 10^{-2} |
| \( m_{\text{Est}}(A, b) \) | 7.6055529470 \cdot 10^{-2} | 1.0022493745 \cdot 10^{-3} | 9.8567031098 \cdot 10^{-2} | 9.8567031098 \cdot 10^{-2} |

We can choose the basis \( Y_1 = g([e_1; 0]), \ldots, Y_5 = g([e_5; 0]), Y_6 = g([e_6; e_1]), Y_7 = g([0; e_2]), \ldots, Y_{11} = g([0; e_6]), \) so that \( A = \sum_{k=1}^{11} a_k Y_k. \) Again, from Theorem 1 with the \( Y_i, \) the expressions for \( m_{\text{Hankel}}(A, b), c_{\text{Hankel}}(A, b) \) and \( \kappa_{\text{Hankel}}(A, b) \) can be obtained.

From Table 5, we conclude that the structured mixed condition numbers can be much smaller than the corresponding unstructured condition numbers. Structured mixed condition numbers also give sharp perturbation bounds. The forward errors obtained by multiplying the structured mixed condition numbers with \( 10^{-8} \) are of the same order as that of the exact errors.

**Example 3 ([16])** Let \( V = g(a) \) be a \( 25 \times 10 \) Vandermonde matrix whose \((i, j)\)-entry is

\[
V_{ij} = \begin{pmatrix} i - 1 \\ 10 \end{pmatrix}^\top, \quad a = \begin{bmatrix} 1 \\ 10 \\ 2 \\ 10 \\ \ldots, 9 \\ 10 \\ 1 \end{bmatrix}^\top, \quad b \in \mathbb{R}^{25} \quad \text{with} \quad b_{2k-1} = -1, \ b_{2k} = 1.
\]

In Table 6, when \( \lambda \) is small, the problem is ill-conditioned under unstructured perturbations. The structured condition numbers are much smaller than the unstructured ones. The perturba-


Table 6: \( L = I_{10}, M = I_{10} \).

| \( \lambda \) | Discrep. pr. | \( L \)-curve | GCV | Quasi-opt |
|----------------|-------------|--------------|-----|-----------|
| \( \| \Delta x_{\lambda} \|_2 \) | 1.36 \cdot 10^{-5} | 6.31 \cdot 10^{-5} | 5.69 | 5.69 |
| \( \| \Delta x_{\lambda} \|_\infty \) | 2.7484 | 3.007 | 1.4131 | 1.4131 |
| \( c \) | 3.4057 | 2.6762 | 2.7502 | 2.7502 |
| \( \epsilon^{-1} \) | 2.2796 \cdot 10 | 2.1046 \cdot 10 | 5.3054 | 5.3054 |
| \( \text{cond}_{\text{Reg}} \) in (2.5) | 4.8637 \cdot 10^6 | 1.7445 \cdot 10^6 | 2.6028 \cdot 10 | 2.6028 \cdot 10 |
| \( m_{\text{Reg}} \) in (2.3) | 5.5816 \cdot 10^4 | 2.3085 \cdot 10^4 | 3.9279 \cdot 10 | 3.9279 \cdot 10 |
| \( c_{\text{Reg}} \) in (2.4) | 5.0645 \cdot 10^9 | 6.3061 \cdot 10^4 | 8.5328 \cdot 10 | 8.5328 \cdot 10 |
| \( \kappa_{\text{Vdm}}(A, b) \) | 4.7123 \cdot 10 | 5.2721 \cdot 10 | 1.2499 \cdot 10 | 1.2499 \cdot 10 |
| \( m_{\text{Vdm}}(A, b) \) | 1.4219 \cdot 10 | 1.4076 \cdot 10 | 2.0828 \cdot 10 | 2.0828 \cdot 10 |
| \( e_{\text{Vdm}}(A, b) \) | 1.8428 \cdot 10^4 | 6.1557 \cdot 10 | 4.0178 \cdot 10 | 4.0178 \cdot 10 |

The first-order unstructured asymptotic perturbation bounds severely overestimate the true relative errors in both normwise and componentwise cases for the numerical perturbation bounds. The first-order unstructured asymptotic perturbation bounds are much smaller than the unstructured one and they give sharp perturbation bounds. The first-order unstructured asymptotic perturbation bounds severely overestimate the true relative errors in both normwise and componentwise cases for the numerical examples of the discrepancy principle and \( L \)-curve methods. As for the last two columns, since the regularization parameter \( \lambda \) is large, the problems are well-conditioned. The structured condition numbers are of the same order as the unstructured ones. Both of them give sharp perturbation bounds.

**Example 4** ([13]) Let \( A = g(a) \) be a \( 10 \times 8 \) Cauchy matrix whose \((i, j)\)-entry is

\[
a_{ij} = \frac{1}{i + j - 1}, \quad a = [u_1, \ldots, u_{10}, v_1, v_2, \ldots, v_8]^\top, \quad \text{with} \; u_i = i, \; v_j = 1 - j,
\]

\[
b = [1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1]_\top.
\]

Then \( A \) is a rectangular Hilbert matrix.

From the second and third columns (Discrep. pr. and \( L \)-curve) of Table 7, we can see that the structured condition numbers are much smaller than the unstructured one and they give sharp perturbation bounds. The first-order unstructured asymptotic perturbation bounds severely overestimate the true relative errors in both normwise and componentwise cases for the numerical examples of the discrepancy principle and \( L \)-curve methods. As for the last two columns, since the regularization parameter \( \lambda \) is large, the problems are well-conditioned. The structured condition numbers are of the same order as that of the unstructured ones.

In the rest of this section, we will show our test results on the proposed SCE algorithms for the conditioning estimation of the Tikhonov regularization solution. Both the unstructured and structured Tikhonov regularization cases are considered. For the unstructured Tikhonov regularization, the test problems originally come from discretization of Fredholm integral equations of the first kind, and they lead to discrete ill-posed problems. We use the test problems included in the REGULARIZATION TOOLS package. In these numerical experiments, a discrete ill-posed problem \( Ax = b \) using one of the many built-in test problems is firstly generated; then the white noise is added to the right-hand side with a perturbation \( e \) whose elements are normally distributed with zero mean and standard deviation chosen such that the noise-to-signal ratio \( \| e \|_2 / \| b \|_2 = 10^{-4} \), thus producing a more ‘realistic’ problem.
We generated the perturbations $\Delta A = \varepsilon \times (E \odot A)$ and $\Delta b = \varepsilon \times (f \odot b)$, where $\varepsilon = 10^{-8}$, $E$ and $f$ are random matrices whose entries are uniformly distributed in the open interval $(-1, 1)$.

To measure the effectiveness of the estimators, we define the over-estimation ratios

$$
\begin{align*}
    r_{\kappa} &:= \frac{(k)}{\kappa_{SCE} \cdot \varepsilon} \cdot \frac{\|\Delta x_\lambda\|_2}{\|x_\lambda\|_2}, \\
    r_{m} &:= \frac{(k)}{m_{SCE} \cdot \varepsilon} \cdot \frac{\|\Delta x_\lambda\|_\infty}{\|x_\lambda\|_\infty}, \\
    r_{c} &:= \frac{(k)}{c_{SCE} \cdot \varepsilon} \cdot \frac{\|\Delta x_\lambda\|/\|x_\lambda\|_\infty}{\|\Delta x_\lambda\|/\|x_\lambda\|_\infty},
\end{align*}
$$

where $k$ is the subspace dimension in Algorithms 3 and 4, $\kappa_{SCE}^{(k)}$, $m_{SCE}^{(k)}$ and $c_{SCE}^{(k)}$ are the outputs from Algorithms 3 and 4. Typically the ratios in (0.1, 10) are acceptable [31, Chapter 19].

For unstructured Tikhonov regularization problems, we test the SCE for several classical ill-posed problems included in the REGULARIZATION TOOLS package: deriv2, shaw and wing. Those three examples give square coefficient matrices $A$ and right-hand side vectors $b$. For the matrix $L$ in (1.1), we chose the identity matrix and

$$
L_1 = \begin{bmatrix}
    1 & -1 & & \\
    & \ddots & \ddots & \\
    & & \ddots & -1 \\
    & & & 1
\end{bmatrix} \in \mathbb{R}^{(n-1) \times n}
$$

which approximates the first derivative operator. We adopted the following four values of the regularization parameter $\lambda$:

$$
0.1, \quad 6 \cdot 10^{-2}, \quad 1.7 \cdot 10^{-3}, \quad 1.7 \cdot 10^{-4}.
$$

In Table 8, we report the numerical results on the ratios $r_{\kappa}$, $r_{m}$ and $r_{c}$ for examples with various dimensions and choices of $L$. The table shows that the mixed condition estimation $m_{SCE}^{(k)}$ reflects the true error bound accurately, while the componentwise condition estimation $c_{SCE}^{(k)}$ gives accurate error bounds for most cases and the normwise condition estimation fails to reflect the true error bound accurately. Specifically, $r_{m}$ are between 0.94 and 16.33, implying that the condition estimation $m_{SCE}^{(k)}$ can be considered reliable [31]. The values of the componentwise condition estimation $c_{SCE}^{(k)}$ are within (16.54, 88.19) except for the case shaw, where $n = 512$, $L = I_n$, $k = 3$ with all choices.
of \( \lambda \), the case wing, where \( n = 256, L = I_n, k = 3, \lambda = 0.1, 6 \cdot 10^{-2} \) and the case shaw, where \( n = 256, L = L_1, k = 5, \lambda = 1.7 \cdot 10^{-3} \), indicating that the componentwise condition estimation \( c_{\text{SCE}}^{(k)} \) is effective for most cases. For the normwise condition estimation \( \kappa_{\text{SCE}}^{(k)} \), most of the values of \( r_\kappa \) are of order \( O(10^2) \), and even some of them are of order \( O(10^3) \), showing that The normwise condition estimation overly estimates for most of the cases.

For structured Tikhonov regularization cases, we tested the following Toeplitz matrix:

\[
A = (a_{i-j}) \in \mathbb{R}^{m \times n}, \quad a_{i-j} = \rho |i-j|.
\]

We used the right-hand side \( b = e \in \mathbb{R}^m \) and \( \rho = 0.99999 \). This Toeplitz matrix is also a symmetric matrix. The Tikhonov regularization parameter is determined by the four classical criteria. As discussed in Subsection 5.3, similar to Algorithms 3 and 4, we can use the SCE to obtain the structured normwise, mixed and componentwise condition estimations denoted by \( \kappa_{\text{SymToep,SCE}}^{(k)} \), \( m_{\text{SymToep,SCE}}^{(k)} \) and \( c_{\text{SymToep,SCE}}^{(k)} \) respectively. The perturbations \( \Delta a \) on \( a \) and \( \Delta b \) on \( b \) were generated as in (6.1). As in the previous example, let the overestimate ratios be defined by

\[
\begin{align*}
    r_{\kappa}^{\text{SymToep}} & := \frac{\kappa_{\text{SymToep,SCE}}^{(k)} \cdot \varepsilon}{\|\Delta x_\lambda\|_2 / \|x_\lambda\|_2}, \\
    r_{m}^{\text{SymToep}} & := \frac{m_{\text{SymToep,SCE}}^{(k)} \cdot \varepsilon}{\|\Delta x_\lambda\|_\infty / \|x_\lambda\|_\infty}, \\
    r_{c}^{\text{SymToep}} & := \frac{c_{\text{SymToep,SCE}}^{(k)} \cdot \varepsilon}{\|\Delta x_\lambda / x_\lambda\|_\infty},
\end{align*}
\]

which measure the reliability of the condition estimators. In this example, we always set \( L = I_n \).

In Table 9, except for two cases, the values of all the three ratios are of order \( O(10) \), implying that the SCE structured condition estimations are reliable.

### 7 Concluding Remarks

In this paper, we introduce the structured condition numbers for the structured Tikhonov regularization problem and derive their exact expressions without the Kronecker product. The structures considered include linear structures, such as Toeplitz and Hankel, and nonlinear structures, such as Vandermonde and Cauchy. We show that our structured condition numbers are smaller than unstructured condition numbers for Toeplitz and Hankel structures. Applying the power method, we devise fast algorithms for estimating the unstructured and structured condition number under normwise and componentwise perturbations, that can be integrated into a GSVD based Tikhonov regularization solver. We also investigate the SCE for estimating structured condition numbers. The numerical examples show that our structured mixed condition numbers give tight error bounds and the proposed condition estimations are reliable and efficient. A possible future research topic is to study the ratio between the structured and unstructured condition numbers for the structured Tikhonov regularization problem.

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Table 8: SCE for the Tikhonov regularization problem.

|       | $r_n$       | $r_m$       | $r_c$       |
|-------|-------------|-------------|-------------|
| deriv2, $n = 64$, $L = I_n$, $k = 5$ |             |             |             |
| $\lambda = 0.1$ | $7.7533 \times 10^2$ | $2.0847$ | $5.1984 \times 10^1$ |
| $\lambda = 6 \cdot 10^{-2}$ | $7.6234 \times 10^2$ | $2.1284$ | $4.7450 \times 10^1$ |
| $\lambda = 1.7 \cdot 10^{-3}$ | $6.7427 \times 10^2$ | $1.2970$ | $2.4927 \times 10^1$ |
| $\lambda = 1.7 \cdot 10^{-4}$ | $9.2294 \times 10^2$ | $1.1437$ | $3.9081 \times 10^1$ |
| deriv2, $n = 64$, $L = L_1$, $k = 5$ |             |             |             |
| $\lambda = 0.1$ | $9.2479 \times 10^2$ | $2.2288$ | $3.8756 \times 10^1$ |
| $\lambda = 6 \cdot 10^{-2}$ | $9.2057 \times 10^2$ | $2.5900$ | $3.6441 \times 10^1$ |
| $\lambda = 1.7 \cdot 10^{-3}$ | $6.2048 \times 10^2$ | $1.0458$ | $2.9864 \times 10^1$ |
| $\lambda = 1.7 \cdot 10^{-4}$ | $9.2686 \times 10^2$ | $1.5548$ | $3.4564 \times 10^1$ |
| wing, $n = 128$, $L = L_1$, $k = 5$ |             |             |             |
| $\lambda = 0.1$ | $2.3753 \times 10^4$ | $1.6337 \times 10^1$ | $1.6544 \times 10^1$ |
| $\lambda = 6 \cdot 10^{-2}$ | $1.4154 \times 10^4$ | $7.5189$ | $1.5910 \times 10^1$ |
| $\lambda = 1.7 \cdot 10^{-3}$ | $1.3068 \times 10^4$ | $1.6301$ | $2.2538 \times 10^1$ |
| $\lambda = 1.7 \cdot 10^{-4}$ | $9.6524 \times 10^2$ | $1.3917$ | $8.1316 \times 10^1$ |
| wing, $n = 256$, $L = I_n$, $k = 3$ |             |             |             |
| $\lambda = 0.1$ | $6.1606 \times 10^4$ | $1.0112$ | $2.6130 \times 10^2$ |
| $\lambda = 6 \cdot 10^{-2}$ | $5.4330 \times 10^4$ | $1.3470$ | $1.8994 \times 10^2$ |
| $\lambda = 1.7 \cdot 10^{-3}$ | $1.0126 \times 10^4$ | $1.8223$ | $6.2581 \times 10^1$ |
| $\lambda = 1.7 \cdot 10^{-4}$ | $1.0923 \times 10^4$ | $2.1715$ | $7.2018 \times 10^1$ |
| shaw, $n = 512$, $L = I_n$, $k = 3$ |             |             |             |
| $\lambda = 0.1$ | $1.6632 \times 10^4$ | $2.4174$ | $1.7467 \times 10^2$ |
| $\lambda = 6 \cdot 10^{-2}$ | $1.8554 \times 10^4$ | $2.8948$ | $3.7301 \times 10^2$ |
| $\lambda = 1.7 \cdot 10^{-3}$ | $1.0707 \times 10^4$ | $3.1782$ | $3.7127 \times 10^2$ |
| $\lambda = 1.7 \cdot 10^{-4}$ | $2.9693 \times 10^2$ | $1.6429$ | $1.8405 \times 10^2$ |
| shaw, $n = 256$, $L = L_1$, $k = 5$ |             |             |             |
| $\lambda = 0.1$ | $2.8617 \times 10^2$ | $2.9029$ | $8.8186 \times 10^1$ |
| $\lambda = 6 \cdot 10^{-2}$ | $3.1563 \times 10^2$ | $3.0425$ | $8.3908 \times 10^1$ |
| $\lambda = 1.7 \cdot 10^{-3}$ | $3.0292 \times 10^2$ | $1.0652$ | $2.0672 \times 10^2$ |
| $\lambda = 1.7 \cdot 10^{-4}$ | $1.1406 \times 10^2$ | $9.4368 \times 10^{-1}$ | $3.0579 \times 10^1$ |
Table 9: SCE for the structured Tikhonov regularization problem.

|                | $r_{\text{Sc}}$ | $r_{\text{Sm}}$ | $r_{\text{Sc}}$ |
|----------------|-----------------|-----------------|-----------------|
| $m = 100, n = 50, k = 3$ |                 |                 |                 |
| Discrep. pr. $\lambda = 2.21$ | 1.3427 · 10^{-2} | 1.2354          | 4.0474 · 10^{-2} |
| $L$-curve $\lambda = 6.19 · 10^{-2}$ | 1.8396 · 10^{-2} | 1.7149          | 2.9893 · 10^{-2} |
| GCV $\lambda = 1.35 · 10^{-4}$ | 1.8113 · 10^{-4} | 1.5819          | 2.0690 · 10^{-4} |
| Quasi-opt $\lambda = 7.48 · 10^{-1}$ | 1.1606 · 10^{-1} | 8.6429 · 10^{-1} | 5.7865 · 10^{-1} |
| $m = 300, n = 200, k = 3$ |                 |                 |                 |
| Discrep. pr. $\lambda = 4.71$ | 7.6537 · 10^{-1} | 4.1397          | 4.8346 · 10^{-1} |
| $L$-curve $\lambda = 1.10 · 10^{-1}$ | 5.6455 · 10^{-1} | 2.4765          | 7.8375 · 10^{-1} |
| GCV $\lambda = 3.22 · 10^{-4}$ | 3.2584 · 10^{-4} | 1.4440          | 8.8256 · 10^{-4} |
| Quasi-opt $\lambda = 4.49$ | 7.1689 · 10^{-1} | 3.6997          | 5.4007 · 10^{-1} |
| $m = 500, n = 300, k = 3$ |                 |                 |                 |
| Discrep. pr. $\lambda = 1.49 · 10^{-2}$ | 5.8732 · 10^{-2} | 1.9115          | 1.0526 · 10^{-2} |
| $L$-curve $\lambda = 1.03$ | 7.8779 · 10^{-1} | 3.2648          | 9.2462 · 10^{-1} |
| GCV $\lambda = 5.66 · 10^{-4}$ | 4.3191 · 10^{-1} | 1.6125          | 1.2357 · 10^{-1} |
| Quasi-opt $\lambda = 9.25$ | 1.0777 · 10^{-2} | 4.3658          | 6.8813 · 10^{-1} |

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