Holomorphic D7-Branes and Flavored $\mathcal{N} = 1$ Gauge Theories

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Abstract

We consider D7-branes in the gauge theory/string theory correspondence, using a probe approximation. The D7-branes have four directions embedded holomorphically in a non-compact Calabi-Yau 3-fold (which for specificity we take to be the conifold) and their remaining four directions are parallel to a stack of D3-branes transverse to the Calabi-Yau space. The dual gauge theory, which has $\mathcal{N} = 1$ supersymmetry, contains quarks which transform in the fundamental representation of the gauge group, and we identify the interactions of these quarks in terms of a superpotential. By activating three-form fluxes in the gravity background, we obtain a dual gauge theory with a cascade of Seiberg dualities. We find a supersymmetric supergravity solution for the leading backreaction effects of the D7-branes, valid for large radius. The cascading theory with flavors exhibits the interesting phenomenon that the rate of the cascade slows and can stop as the theory flows to the infrared.
1 Introduction

The gauge theory/string theory correspondence \[1, 2, 3\] furnishes, in principle, a powerful set of tools for understanding gauge theories at strong coupling by performing computations in a dual string theory at weak coupling. However, the correspondence is only well-understood in systems where the string background is highly symmetric and nearly flat, while we expect that the duals to many interesting gauge theories (such as large-\(N\) QCD or SQCD) will not have these properties. It is therefore an interesting challenge to study less symmetric string backgrounds, and in particular to study backgrounds with reduced supersymmetry.

There is considerable evidence that the string theory dual to the pure \(\mathcal{N} = 1\) \(SU(N)\) supersymmetric gauge theory in four dimensions is related, at least in the infrared, to a geometry similar to that of a warped deformed conifold with flux \[4, 5, 6\]. This gauge theory exhibits chiral symmetry breaking and confinement at low energies. One interesting generalization of the supersymmetric pure glue theory is a gauge theory with added flavors. In the string dual, the gluon degrees of freedom come from 3-3 strings living on a stack of D3-branes, while the flavors come from additional open strings stretching to branes of higher dimension. These additional branes are usually D7-branes, but in some setups, the additional branes may be D5-branes. In any event, the important feature seems to be that the added branes must be extended along the radial AdS direction; then, volume factors suppress the dynamics of the NN strings on these “flavor branes”, which then contribute states to the gauge theory with global symmetries rather than gauge symmetries. The supergravity dynamics of some related systems were studied in \[7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17\]. The papers \[11, 13\] studied D7-branes on the conifold, and \[10\] studied a similar system with added orientifolds; in this paper we will continue this study and hopefully provide some new insights.

One of the ultimate goals of studying D7-branes in AdS compactifications is to study chiral symmetry breaking in the dual field theory. If the added quarks are massless, then the gauge theory with \(K\) flavors possesses a global symmetry \(SU(K) \times SU(K)\). In QCD, this symmetry is spontaneously broken to its diagonal \(SU(K)\) subgroup in the infrared. We will not reach the goal of finding the relevant infrared supergravity solution, but we will be able to find a solution valid at asymptotically large radius (but not so large that the backreaction of the D7-branes at infinity
becomes important; a similar approximation was used in \cite{8,9}.) We hope to convince the reader that even this partial solution contains interesting physics.

The form of the paper is as follows. In Section 2, we will review the geometry of the conifold. We then proceed in Section 3 to add D3-branes to the conifold, warping it, and to add D7-branes as probes in the resulting warped geometry. We identify the corresponding field theory and study its renormalization group flows for a simple embedding, and briefly consider some more general D7-brane embeddings. In Section 4 we add three-form fluxes to the warped conifold, and for large radius we find an explicitly supersymmetric supergravity solution including the leading backreaction of the D7-branes. The dual field theory exhibits a cascade of Seiberg dualities, for which the rate of the cascade decreases as the theory runs to low energy; this behavior appears as a radial dependence of the number of units of 3-form flux in supergravity. We consider the T-dual of our model in Section 5 (mostly summarizing work of others) and conclude in Section 6 with several open questions.

2 Geometry of the Conifold

In this section we briefly review the geometry of the conifold in order to fix notation. Useful references are \cite{13,19,20,21,22,23}.

The conifold is a non-compact Calabi-Yau 3-fold, defined by the equation

\[ z_1 z_2 - z_3 z_4 = 0 \]  

in \( C^4 \). Because Eqn. (1) is invariant under an overall real rescaling of the coordinates, this space is a cone, whose base is the Einstein space \( T^{1,1} \) \cite{13,19}. The metric on the conifold may be cast in the form \cite{18}

\[ ds_6^2 = dr^2 + r^2 ds_{T^{1,1}}^2, \]

where

\[ ds_{T^{1,1}}^2 = \frac{1}{9} \left( d\psi + \sum_{i=1}^2 \cos \theta_i d\phi_i \right)^2 + \frac{1}{6} \sum_{i=1}^2 \left( d\theta_i^2 + \sin^2 \theta_i d\phi_i^2 \right) \]

is the metric on \( T^{1,1} \). Here \( \psi \) is an angular coordinate which ranges from 0 to \( 4\pi \), while \( (\theta_1, \phi_1) \) and \( (\theta_2, \phi_2) \) parametrize two \( S^2 \)s in a standard way. This form of the metric shows that \( T^{1,1} \) is a \( U(1) \) bundle over \( S^2 \times S^2 \).
These angular coordinates are related to the \( z_i \) variables by

\[
\begin{align*}
    z_1 &= r^{3/2} e^{i/2(\psi - \phi_1 - \phi_2)} \sin(\theta_1/2) \sin(\theta_2/2), \\
    z_2 &= r^{3/2} e^{i/2(\psi + \phi_1 + \phi_2)} \cos(\theta_1/2) \cos(\theta_2/2), \\
    z_3 &= r^{3/2} e^{i/2(\psi + \phi_1 - \phi_2)} \cos(\theta_1/2) \sin(\theta_2/2), \\
    z_4 &= r^{3/2} e^{i/2(\psi - \phi_1 + \phi_2)} \sin(\theta_1/2) \cos(\theta_2/2).
\end{align*}
\]

(4) (5) (6) (7)

It is also sometimes helpful to consider a set of “homogeneous” coordinates \( A_a, B_b \) where \( a, b = 1, 2 \), in terms of which the \( z_i \) are

\[
\begin{align*}
    z_1 &= A_1 B_1, & z_2 &= A_2 B_2, \\
    z_3 &= A_1 B_2, & z_4 &= A_2 B_1.
\end{align*}
\]

(8) (9)

With this parameterization the \( z_i \) obviously solve the defining equation of the conifold.

We may also parameterize the conifold in terms of an alternative set of complex variables \( w_i \), given by

\[
\begin{align*}
    z_1 &= w_1 + i w_2, & z_2 &= w_1 - i w_2, \\
    z_3 &= -w_3 + i w_4, & z_4 &= -w_3 - i w_4.
\end{align*}
\]

(10)

The conifold equation may now be written as

\[
\sum w_i^2 = 0
\]

(11)

and we identify the \( T^{1,1} \) base of the cone as the intersection of the conifold with the surface

\[
\sum |w_i|^2 = r^3.
\]

(12)

Notice that \( T^{1,1} \) described in this way is explicitly invariant under \( SO(4) \simeq SU(2) \times SU(2) \) rotations of the \( w_i \) coordinates and under an overall phase rotation. Thus the symmetry group of the \( T^{1,1} \) is \( SU(2) \times SU(2) \times U(1) \).

A useful basis of 1-forms consists of the following holomorphic forms and their complex conjugates:

\[
\begin{align*}
    \lambda &= 3 \frac{dr}{r} + i \zeta, \\
    \sigma_1 &= \cot(\theta_1/2) (d\theta_1 - i \sin(\theta_1) d\phi_1), \\
    \sigma_2 &= \cot(\theta_2/2) (d\theta_2 - i \sin(\theta_2) d\phi_2),
\end{align*}
\]

(13) (14) (15)
where \( \zeta = d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2 \) is the one-form associated with the \( U(1) \) fiber of \( T^{1,1} \). A convenient shorthand notation is

\[
\Omega_{ij} = d\theta_i \wedge \sin(\theta_j) d\phi_j. \tag{16}
\]

The unusual factors of \( \cot \frac{\theta_i}{2} \) appearing in Eq.\((15)\) are present so that

\[
d\sigma_k = i\Omega_{kk}. \tag{17}
\]

To check supersymmetry, we will also need the Kähler form

\[
J = -r^2 \left( \frac{1}{3} \frac{dr}{r} \wedge \zeta - \frac{1}{6}(\Omega_{11} + \Omega_{22}) \right). \tag{18}
\]

Finally, for reasons that will become clear, we will often use the relation

\[
\frac{dz_1}{z_1} = \frac{1}{2}(\lambda + \sigma_1 + \sigma_2). \tag{19}
\]

## 3 Embedding Flavor Branes in \( \text{AdS}_5 \times T^{1,1} \)

We begin our study of D7-branes by attempting to embed them in the model of Klebanov and Witten\(^{19}\). This model is a particularly simple \( N = 1 \) gauge/gravity dual, obtained by placing a stack of \( N \) D3-branes near a conifold singularity. The branes source the RR 5-form flux and warp the geometry:

\[
ds^{2}_{10} = h(r)^{-1/2}(dx_\mu dx^\mu + dr^2) + h(r)^{1/2}r^2 ds^2_{T^{1,1}}, \tag{20}
\]

\[
h(r) = \frac{L^4}{r^4} \tag{21}
\]

\[
g^*_s F_5 = d^4 x \wedge dh^{-1} + \star (d^4 x \wedge dh^{-1}) \tag{22}
\]

\[
L^4 = \frac{27}{4} \pi g_s N \alpha'^2. \tag{23}
\]

The dual field theory has gauge group \( SU(N) \times SU(N) \) and matter fields \( A_{1,2}, B_{1,2} \) which transform in the bifundamental color representations \((N, \bar{N})_c\) and \((\bar{N}, N)_c\). The theory also has a superpotential

\[
W = \lambda Tr(A_i B_j A_k B_l) e^{ik} e^{jl}. \tag{24}
\]

By solving the F-term equations for this superpotential, we can see that we obtain supersymmetric vacua for arbitrary diagonal \( A_{1,2} \) and \( B_{1,2} \), so that the moduli space of the field theory is precisely that of \( N \) D3-branes placed on a conifold.
Now we would like to modify this field theory by the inclusion of fundamental matter, which should correspond on the dual string theory side to the addition of D7-branes. These D7-branes fill the 4 dimensions tangent to the D3-branes and must also wrap 4 dimensions in the conifold. It is natural to suppose that we will obtain a supersymmetric solution if the equation specifying the embedding is holomorphic \[11\] – then the submanifold corresponding to the D7-brane worldvolume inherits a complex structure and a closed Kähler form from the original Calabi-Yau space (and should therefore inherit some fraction of the original supersymmetry. See, for example, \[24\] \[25\].) Let us start then with the simple holomorphic equation \(z_1 = 0\), where \(z_1\) is one of the complex variables in the defining equation of the conifold \((1)\).

Note that in terms of the homogeneous coordinates \(A, B\), there are two branches of our D7-brane, \(A_1 = 0\) and \(B_1 = 0\). There is an \(SU(K)\) flavor symmetry associated with each branch of a stack of \(K\) D7-branes, so we expect the existence of a global \(SU(K) \times SU(K)\) flavor symmetry. Moreover, cancellation of gauge anomalies requires that we add two flavors to each gauge group, with opposite chiralities. We denote the resulting four sets of flavors as \(q, \bar{q}, Q, \bar{Q}\), and indicate their color and flavor representations in Table 1. We propose that the corresponding gauge invariant and flavor invariant terms in the superpotential are

\[
W_{\text{flavors}} = h q A_1 Q + g \bar{q} B_1 \bar{Q}.
\]

If the number of flavors \(K\) is much smaller than the size of the gauge group \(N\), then the dimensions of the flavor superfields are determined (to leading order in \(1/N\)) by the following argument. Because the theory including D7-brane probes is invariant

\[1\] Our embedding equation is in some sense “one-half” of the embedding of \[11\], who embedded D7-branes by the equation \(z_1 z_2 = 0\), in our coordinates. It differs also from the embedding of \[13\], which is not holomorphic, even in the limit where the RR 3-form flux is turned off.

| Field | \(SU(N_c) \times SU(N_c)\) | \(SU(K) \times SU(K)\) |
|-------|-----------------|-----------------|
| \(q\) | \((N, 1)\)       | \((K, 1)\)       |
| \(\bar{q}\) | \((\bar{N}, 1)\) | \((1, K)\)       |
| \(Q\)  | \((1, N)\)      | \((\bar{K}, 1)\) |
| \(\bar{Q}\) | \((1, \bar{N})\)| \((1, \bar{K})\)|

Table 1: Representation structure of the added \(\mathcal{N} = 1\) flavors.
under the rescaling $z_i \rightarrow \beta z_i$, the field theory should be classically scale invariant. In the conformal theory without any flavors, the $A$ and $B$ fields have dimension $3/4$. Therefore, power counting in the superpotential requires that the $q, Q$ superfields each have dimension $9/8$ (plus quantum corrections.)

It is worth noting that the D7-branes embedded by a holomorphic equation, as we have described here, are topologically trivial. This topological triviality is essential for RR charge conservation; a topologically nontrivial wrapping would necessitate the presence of anti-D7-branes or orientifold planes.

It is straightforward to add masses to the flavors, at the cost of breaking the $SU(K) \times SU(K)$ flavor symmetry to its diagonal $SU(K)$ subgroup. The relevant terms in the superpotential are

$$W_{masses} = \mu_1 \tilde{q} \tilde{\tilde{q}} + \mu_2 \tilde{Q} \tilde{\tilde{Q}}.$$  \hspace{1cm} (26)

To translate these masses to the D7-brane probe picture, it is helpful to rewrite $W_{flavors} + W_{masses}$ in the following matrix form:

$$W_{flavors} + W_{masses} = \left( \begin{array}{cc} \tilde{q} & \tilde{Q} \\ \tilde{\tilde{q}} & \tilde{\tilde{Q}} \end{array} \right) \left( \begin{array}{cc} \mu_1 & hA_1 \\ gB_1 & \mu_2 \end{array} \right) \left( \begin{array}{c} q \\ Q \end{array} \right).$$ \hspace{1cm} (27)

A useful technique for relating the string theory and field theory is to probe the string background with D-branes; for simplicity, let us consider a single D3-brane probe, so that the fields $A_1$ and $B_1$ are just scalars. Giving $A, B$ vacuum expectation values, we can think of the square matrix in (27) as a mass matrix for the quarks. When the D3-brane and D7-brane intersect, some of the quarks, which arise as 3-7 strings, become massless. In other words, the determinant of the mass matrix $hgA_1B_1 - \mu_1\mu_2$ should vanish when the D3-brane probe is on the D7-brane locus, or with an appropriate redefinition,

$$z_1 - \mu^2 = 0.$$ \hspace{1cm} (28)

Thus Eq. (28) is also an appropriate embedding equation for a D7-brane which gives massive flavors (when the D3-branes are located at the tip of the cone.)

To obtain further evidence for the validity of our construction, let us study the axion and dilaton fields sourced by our D7-brane; in particular, we can obtain RG flows in the field theory from the running of the dilaton in supergravity. It is convenient to consider the axion $C_0$ and dilaton $\Phi$ in the complex combination $\tau = C_0 + i e^{-\Phi}$. 

7
Given the embedding condition $z_1 = 0$, a natural guess for the combined dilaton-axion system is that

$$\tau \sim \log(z_1).$$  \hspace{1cm} (29)

With the normalization condition

$$\int_{S^3} F^1 = N_{D7} = K$$ \hspace{1cm} (30)

we see that

$$C_0 = \frac{K}{4\pi} (\psi - \phi_1 - \phi_2).$$ \hspace{1cm} (31)

Thus we have

$$\tau = \frac{i}{g_s} + \frac{K}{2\pi i} \log z_1,$$ \hspace{1cm} (32)

which gives the correct $SL(2,\mathbb{Z})$ monodromy $\tau \to \tau + K$ upon circling a stack of $K$ D7-branes; the first term has been chosen to give the correct value of the dilaton when $K = 0$. When $K \neq 0$, Eq.(32) shows that the dilaton is given by

$$e^{-\Phi} = \frac{1}{g_s} - \frac{3K}{4\pi} \log r - \frac{K}{2\pi} \log \left( \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right).$$ \hspace{1cm} (33)

The holomorphic dilaton-axion described in Eq.(32) is manifestly a supersymmetric solution of the supergravity equations of motion with the three-form fluxes set to zero. In the probe approximation we are using, we ignore the backreaction on the geometry and RR five-form. The singularity of (32) is acceptable for our purposes, as it corresponds to the presence of a D7-brane. As is usually the case with D7-branes, for small enough $z_1$ the dilaton may become negative; it would be interesting to find a solution using F-theory [26] that avoids this problem.

Recall that the couplings for the two gauge groups are determined as follows\(^2\) [19, 20]:

$$\frac{4\pi^2}{g_1^2} + \frac{4\pi^2}{g_2^2} = \frac{\pi}{e^{\Phi}},$$ \hspace{1cm} (34)

\(^2\)Strictly speaking, these formulae have been derived for the case of branes at orbifold singularities and the corresponding gauge theories. For systems such as our conifold theory, there is no proof from first principles, but the RG flows have been checked in several cases.
\[ \left[ \frac{4\pi^2}{g_1^2} - \frac{4\pi^2}{g_2^2} \right] e^\Phi = \frac{1}{2\pi\alpha'} \left( \int_{S^2} B_2 \right) - \pi \pmod{2\pi}. \] (35)

In this theory with no three-form fluxes in supergravity, we may set \( g_1 = g_2 = g_{YM} \). Making the identification of the AdS radius \( r \) as the renormalization scale \( \Lambda \) \[1\], we find that
\[
\frac{\partial}{\partial \log \Lambda} \frac{8\pi^2}{g_{YM}^2} = -\frac{3K}{4}. \] (36)

Let us compare this RG equation with the Shifman-Vainshtein \( \beta \)-functions \[27, 28\]:
\[
\frac{\partial}{\partial \log \Lambda} \frac{8\pi^2}{g_{YM}^2} = 3N - 2N(1 - 2\gamma_{A,B}) - K(1 - 2\gamma_q). \] (37)

Each gauge group has \( 2N \) effective flavors coming from the \( A, B \) bifundamentals and \( K \) effective flavors from half of the \( q, Q \) fields. With the anomalous dimensions \( \gamma_{A,B} = -1/4 \), the terms proportional to \( N \) cancel, as necessary for conformal invariance of the background. However, we argued earlier that the dimension of the \( q \) fields is \( 9/8 \), corresponding to \( \gamma_q = 1/8 \). With this assignment we see that the RG flows from supergravity and field theory agree precisely at this order in the \( 1/N \) expansion, provided that the anomalous dimensions of the \( A, B \) bifundamental fields do not receive corrections of order \( K/N \). It would be nice to check explicitly that such corrections do not appear.

One indication that this is the case comes from supergravity – from Einstein’s equations one sees that the metric receives leading backreaction corrections of order \( K^2/N^2 \), so at first order in \( K/N \) the spacetime geometry is anti-de Sitter. Therefore we should expect the field theory to be conformal including terms of order \( K/N \), as we have assumed. The RG flows are related by supersymmetry to the \( \theta \)-angles of the gauge theory. This relationship was studied on the conifold in \[29\]; more detailed supergravity analysis appears in \[30, 31, 32\].

There is a second possibility for the superpotential that will produce the same D7-brane geometry (the holomorphic embedding \( z_1 = \mu^2 \)) from the probe D3-brane perspective – consider the addition of only two flavors, \( q \) and \( Q \), with the superpotential
\[
W_{\text{flavors}} = q(A_1 B_1 + \mu_1)\bar{q}. \] (38)

However, this superpotential actually arises as a special case of the cubic superpotential \[25\] considered earlier, by adding a mass term of the form \( \mu \). Integrating out the the \( Q, \bar{Q} \) flavors, one obtains the quartic superpotential \[38\].
Another way to motivate the gauge theory construction presented here is to consider a related $C^2/Z_2$ orbifold theory, where standard arguments [33] allow us to construct the field theory explicitly. Then we can obtain the conifold field theory by turning on a particular mass deformation [19]. The relevant orbifold background may be defined by starting in flat ten-dimensional space and then orbifolding by the discrete symmetry

$$X^{6,7,8,9} \rightarrow -X^{6,7,8,9}. \quad (39)$$

An equivalent definition is to start with a $C^3$ space parametrized by the coordinates $z_1, z_2, z_3$ and to then consider the submanifold defined by the equation

$$z_1 z_2 - z_3^2 = 0. \quad (40)$$

The resulting four-manifold, tensored with six-dimensional flat space, is the same as the orbifold defined by Eq. (39). We may “solve” Eq. (40) by introducing complex variables $A, B$ such that

$$z_1 = A^2, \quad z_2 = B^2, \quad z_3 = AB. \quad (41)$$

The invariance of the $z_i$ coordinates under $A, B \rightarrow -A, -B$ expresses the $Z_2$ orbifold action.

By placing $N$ D3-branes transverse to this orbifold, we will obtain an $SU(N) \times SU(N)$ SCFT with $N = 2$ supersymmetry, and bifundamental matter superfields $A_{1,2}$ and $B_{1,2}$ (there must be two of each because of the $N = 2$ supersymmetry.) The quiver diagram for this theory consists of two nodes, corresponding to a stack of D3-branes and its mirror image under the orbifold action. The vector multiplets arise as strings connecting each node to itself, while the matter fields arise as strings connecting different nodes. It was shown in [19] that if one deforms this theory by giving masses to the adjoint scalars in the $N = 2$ vector multiplets, and subsequently integrates out the adjoints, the theory becomes the $N = 1$ conifold theory described earlier.

To add flavors, we now consider the addition of D7-branes, with 4 directions tangent to the D3-branes. One simple way to embed these branes is to wrap the remaining 4 directions on the entire orbifold [8, 9]. An alternative is to consider a holomorphic embedding equation (such as $z_1 = 0$) so that the D7 fills two dimensions of the orbifold and the remaining six spacetime directions. Because the homogeneous
coordinates that parametrize the orbifold are double-valued, including a D7-brane in this way can actually be thought of as the inclusion of two fractional D7-branes – a D7 and its orbifold mirror image (another way to think about this is to deform the defining equation of the orbifold to \( z_1 z_2 = z_3^2 - \varepsilon^2 \). Then \( z_1 = 0 \) has two branches, \( z_3 = \pm \varepsilon \).) It is then clear from considering the quiver diagram that we should obtain superpotential interactions of the form of Eq. (25), and that these terms survive when we integrate out the adjoints to arrive at the \( N = 1 \) theory of interest.

The arguments in this section should be more or less unchanged for the case of D7-branes in the “generalized conifold” geometries discussed in \cite{34}. In particular, the dimensions of the bifundamental matter fields are still 3/4, so the flavors should still have dimension 9/8, and the RG flow equations should work out the same way. It might be interesting to study conifolds in other dimensions \cite{35,36}.

### 3.1 Other Embedding Equations

Though we have obtained sensible results for the simple embedding equation \( z_1 = \mu^2 \), it is clearly interesting to consider a more general polynomial \( P(z_i) \). In this section we present some simple considerations on how one might realize such D7-brane embeddings might be realized in the dual field theory.

The simplest generalization of the D7-branes considered thus far arises from the symmetries of the conifold geometry. The base of the conifold possesses a geometric \( SO(4) \) symmetry which is spontaneously broken by the probe D7-brane; therefore we may perform such rotations to obtain alternative brane embeddings with the same shape but different positions in the conifold (the remaining geometric \( U(1) \) symmetry corresponds to multiplication of the mass parameter \( \mu \) by a phase.) These rotations are simple to describe in terms of the \( w_i \) coordinates of \cite{10}, which transform simply under \( SO(4) \). The embedding equation \( z_1 = \mu^2 \) becomes \( w_1 + iw_2 = \mu^2 \), and under a general \( SO(4) \) transformation this becomes

\[
\sum_{j=1}^{4} a_j w_j + ib_j w_j = \mu^2
\]

with the constraint that

\[
\sum a_j b_j = 0, \quad \sum a_j^2 = \sum b_j^2,
\]
for real $a_j, b_j$. Because all these embeddings are defined holomorphically, with respect to the same complex structure, we expect that they preserve supersymmetry.

Turning to the field theory, we also find that these rotated D7-branes are easy to describe with $SO(4)$ realized as the $SU(2) \times SU(2)$ action on the matter fields $A_a, B_a$. The relevant group action now sends $hA_1$ and $gB_2$ in the superpotential (25) to $h_1A_1 + h_2A_2$ and $g_1B_1 + g_2B_2$, respectively.

However, there are many holomorphic embeddings which do not satisfy the constraints (43); it is natural to ask if these other embeddings are allowed as well. For specificity, let us consider the explicit example $w_1 = z_1 + z_2 = 0$ (this equation was studied in [10] with the addition of orientifold planes.) Though we cannot realize this embedding as the determinant of a $2 \times 2$ matrix of the form in (27) we can obtain it from a $4 \times 4$ matrix:

$$
\begin{pmatrix}
0 & 0 & A_1 & A_2 \\
0 & 1 & 0 & 0 \\
B_1 & 0 & 1 & 0 \\
B_2 & 0 & 0 & 1
\end{pmatrix}
$$

(44)

which (heuristically) corresponds to having 2 D7-branes, whose 7-7 strings receive expectation values to give the necessary coupling constants in the field theory.

We can also realize this embedding equation $w_1 = 0$ by considering the related $C^2/Z_2$ orbifold theory with D3-branes transverse to the orbifold. For this theory it has been shown that one can embed D7-branes by wrapping four worldvolume dimensions on the entire orbifold, and placing the other 4 dimensions parallel to the D3-branes. In the field theory, the quarks that arise are coupled to the adjoint scalar field $\Phi$, breaking the supersymmetry from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ (there are actually two such embeddings, corresponding to a choice of orientation in the string background, and corresponding to a choice of which of the two adjoint scalars $\Phi$ and $\tilde{\Phi}$ to which the quarks are coupled in the field theory.) The superpotential is

$$
W = g \text{Tr} ((A_1B_1 + A_2B_2)\Phi) + g \text{Tr} ((B_1A_1 + B_2A_2)\tilde{\Phi}) + gq\Phi Q.
$$

(45)

Now, by giving masses to the adjoints,

$$
W \rightarrow W + \frac{m}{2} \Phi^2 - \frac{m}{2} \tilde{\Phi}^2
$$

(46)

and integrating them out, we obtain a superpotential

$$
W = Tr(A_iB_jA_kB_l)\epsilon^{ik}\epsilon^{jl} - (q\bar{q})^2 - 2q(A_1B_1 + A_2B_2)\bar{q}.
$$

(47)
The locus on which there are massless flavors is clearly \( w_1 = 0 \). Also, because the embedding equation is still scale invariant, we should expect the theory to be classically conformal. Thus the flavors all have dimension 3/4. The Shifman-Vainshtein \( \beta \)-functions with this assignment agree with supergravity if, in addition to a nontrivial dilaton, there are some 2-form potentials turned on. Note that the alternative quartic superpotential (47) does not actually represent new physics, as it can be obtained from the cubic superpotential defined by (44) by integrating out a set of flavors, and adding the interaction \((q \tilde{q})^2\), which becomes marginal in the IR. It would be nice to find the explicit supergravity solution for this D7-brane embedding on the conifold, and also to determine whether the cubic interactions encoded in (44) can be realized directly from string theory.

It is also possible to consider higher-order polynomials in the \( z_i \). If the polynomial factors into lower-order polynomials, then the separate factors have a natural interpretation as disjoint D7-branes. On the other hand, if the polynomial does not factor, then the picture changes slightly. For example, let us consider the polynomial \( z_1 z_2 - \mu^4 = 0 \). Once again, we may realize this embedding equation as the determinant of a mass matrix:

\[
\begin{pmatrix}
0 & \mu & A_1 & 0 \\
\mu & 0 & 0 & A_2 \\
B_1 & 0 & \mu & 0 \\
0 & B_2 & 0 & \mu
\end{pmatrix}
\]

(48)

At high energies, or equivalently at large AdS radius, \( z_1 \) and \( z_2 \) are large compared to the mass scale \( \mu^2 \) and so the theory seems to have eight quark superfields. At low energies, the mass perturbations allow us to integrate out some of the flavors. The analogous string picture is that we start with two D7-branes embedded by \( z_1 = 0 \) and \( z_2 = 0 \), which intersect at the tip of the conifold. By turning on appropriate 7-7 strings, the D7-branes fuse near the intersection.

4 Fractional D3-branes and Flavored Cascades

We can obtain an interesting generalization of the conifold theory described in the previous section by introducing three-form fluxes on the conifold. These models were studied in a series of papers [37, 38, 4] and were shown to be dual to an \( N = 1 \) theory with gauge group \( SU(N + M) \times SU(N) \), where \( N \) is the number of units of 5-form
flux and \( M \) is the number of units of 3-form flux in the background. The 3-form flux is sourced by D5-branes which are wrapped on the \( S^2 \) in \( T^{1,1} \); in the literature these wrapped D5-branes are often called “fractional D3-branes.”

In the large radius limit, the supergravity solution including these 3-form fluxes (but with the axion and dilaton constant) is

\[
\begin{align*}
    ds_{10}^2 &= h(r)^{-1/2}(dx_\mu dx^\mu + dr^2) + h(r)^{1/2}r^2 ds_{T^{1,1}}^2, \\
    h(r) &= \frac{27}{4r^4}gs\alpha'^2 \left(N + \frac{3}{2\pi}gsM^2 \log(r/r_0)\right), \\
    g_s\tilde{F}_5 &= d^4x \wedge dh^{-1} + \ast(d^4x \wedge dh^{-1}), \\
    F_3 &= \frac{M\alpha'}{2}\omega_3, \\
    H_3 &= \frac{3g_sM\alpha'}{2}dr \wedge \omega_2.
\end{align*}
\]

This solution possesses a naked singularity at small \( r \); a nonsingular solution was found in \[4\] by deforming the conifold. Though it would certainly be interesting to study D7-branes on a deformed conifold, we will not attempt to do so here.

Another feature of the supergravity solution with fractional branes, which will be quite important for this paper, is the logarithm in the warp factor \( h(r) \), and the corresponding logarithmic running of the 5-form flux \( \tilde{F}_5 \). The number of units of 5-form flux is dual to the number of colors in the field theory, so the radial dependence of \( \tilde{F}_5 \) implies that the gauge groups effectively decrease in size as the theory undergoes renormalization group flow from the ultraviolet to the infrared:

\[
N_{\text{eff}} = N + \frac{3}{2\pi}gsM^2 \log(r/r_0).
\]

This phenomenon arises from a “cascade” of Seiberg dualities, which we review briefly here, without dwelling on technical details. Suppose that we begin with a gauge theory with gauge group \( SU(N + M) \times SU(N) \). The \( SU(N + M) \) factor is coupled to \( 2N \) effective flavors, while the \( SU(N) \) factor couples to \( 2(N + M) \) effective flavors. From the NSVZ \( \beta \)-functions, it is clear that the \( SU(N + M) \) factor of the gauge group flows toward strong coupling in the infrared, while the \( SU(N) \) factor flows to weak coupling. When the \( SU(N + M) \) factor becomes strongly coupled, we may perform a Seiberg duality transformation, under which a strongly coupled gauge theory with gauge group \( SU(N_c) \) and \( N_f \) flavors becomes a weakly coupled theory with gauge group \( SU(N_f - N_c) \) and \( N_f \) flavors (there are also mesonic states, which acquire mass
and are irrelevant to our discussion.) In the present case, the duality sends the first gauge group to $SU(2N - (N + M)) = SU(N - M)$. After the duality transformation, we are left with an $SU(N) \times SU(N - M)$ gauge theory with $A, B$ bifundamental flavors; in other words, the number of colors $N$ has effectively decreased by $M$. Because the process just described will now repeat until $N \sim M$, it is known as a "cascade."

We will again attempt to find a supergravity solution corresponding to a D7-brane with the embedding $z_1 = 0$, using the axion-dilaton ansatz found in the previous section. Once again, for our purposes the only requirements are that the axion-dilaton is holomorphic and that it has the correct monodromy. What is different in this case is that the background three-form fluxes will induce three-form charges in the D7-branes; we will compute the leading order corrections to the fluxes, but treat the branes themselves as probes.

The addition of fundamental flavors has a pronounced effect on the pattern of duality cascades, which we will attempt to reproduce in this section from supergravity. With $K$ D7-branes, the number of effective flavors coupled to each gauge group increases by $K$. If we start with a gauge group $SU(N + M) \times SU(N)$, then because the first gauge group factor has $2N + K$ effective flavors, it is Seiberg-dual in the infrared to an $SU(2N + K - (N + M)) = SU(N - M + K)$ gauge theory. The second gauge group factor remains $SU(N)$, so we see that in addition to a decrease of the overall number of colors, the difference in the size of the gauge groups has decreased from $M$ to $M - K$. As we continue to follow the renormalization group, we see that at each step of the duality cascade, the strength of the cascade, $M$, decreases by $K$. A very similar phenomenon was described in the paper [39], who performed a field theory analysis of del Pezzo theories with bifundamental matter; it would be interesting to understand the stringy description of their duality cascade.

The difference in the size of the gauge groups, $M$, will decrease by increments of $K$ until it is smaller than or equal to $K$. If at this point $N$ is still greater than zero, we should have the $SU(N) \times SU(N)$ theory with $K$ flavors considered in Section 3. An alternative possibility is that $N$ may decrease to zero, but with a finite $M$ left over. Then we would expect the field theory to be that of $\mathcal{N} = 1$ $SU(M)$ SYM with $K$ flavors.
4.1 Three-form Fluxes and RG Flow

We now turn to the three-form fields $F_3$ and $H_3$. It is convenient to consider these fields in the complex combination $G_3 = F_3 - \tau H_3$. The supersymmetric solutions that we will study satisfy the conditions that $G_3$ is imaginary self-dual, has index structure $(2,1)$, and is primitive (contractions with the Kähler form vanish), that the three-forms and five-form satisfy Bianchi identities, and that the metric is a “warped product.” [41, 42]

To take advantage of the simplifications made possible by supersymmetry, we introduce a basis of imaginary self-dual $(2,1)$ forms for the conifold:

\[
\begin{align*}
\eta_1 &= \lambda \wedge \omega_2 \\
\eta_2 &= \frac{1}{2} \lambda \wedge (\sigma_1 \wedge \bar{\sigma}_2 - \sigma_2 \wedge \bar{\sigma}_1) \\
&= \cot(\theta_1/2) \cot(\theta_2/2) \lambda \wedge (d\theta_1 \wedge d\theta_2 + \sin(\theta_1) d\phi_1 \wedge \sin(\theta_2) d\phi_2) \\
\eta_3 &= \left( \frac{dr}{r} \wedge \zeta + \frac{1}{2} \Omega_{22} \right) \wedge \sigma_1, \\
\eta_4 &= \left( \frac{dr}{r} \wedge \zeta + \frac{1}{2} \Omega_{11} \right) \wedge \sigma_2, \\
\eta_5 &= \bar{\lambda} \wedge \sigma_1 \wedge \sigma_2 \\
&= \bar{\lambda} \wedge (d\theta_1 \wedge d\theta_2 - \sin(\theta_1) d\phi_1 \wedge \sin(\theta_2) d\phi_2 - i(\Omega_{12} - \Omega_{21})).
\end{align*}
\]

It is a happy coincidence that all these $(2,1)$ forms are primitive as well. In this basis, the background $G_3$ is just

\[
G_3^{(0)} = F_3^{(0)} - \frac{i}{g_s} H_3^{(0)} = -i \frac{M\alpha'}{2} \eta_1.
\]

Under exterior differentiation, these 3-forms give

\[
\begin{align*}
d(\eta_1) &= 0 \\
d(\eta_2) &= \frac{i}{2} \lambda \wedge (\Omega_{22} \wedge (\bar{\sigma}_1 - \sigma_1) - \Omega_{11} \wedge (\bar{\sigma}_2 - \sigma_2)) \\
d(\eta_3) &= \frac{dr}{r} \wedge \Omega_{22} \wedge \sigma_1 + \frac{dr}{r} \wedge \zeta \wedge \Omega_{11} - \frac{i}{2} \Omega_{11} \wedge \Omega_{22} \\
d(\eta_4) &= \frac{dr}{r} \wedge \Omega_{11} \wedge \sigma_2 + \frac{dr}{r} \wedge \zeta \wedge \Omega_{22} - \frac{i}{2} \Omega_{11} \wedge \Omega_{22} \\
d(\eta_5) &= -2i \bar{\lambda} \wedge \omega_2 \wedge (\sigma_1 + \sigma_2)
\end{align*}
\]

The three-form $G_3$ satisfies a Bianchi identity because the fields $F_3$ and $H_3$ are
derived from potentials:

\[ dG_3 = -d\tau \wedge H_3 = -\frac{3g_sMK\alpha'}{4\pi i} \frac{dz_1}{z_1} \wedge \frac{dr}{r} \wedge \omega_2 + O((g_sK)^2). \]  

(66)

Notice that the background three-form and the terms we will need to add to satisfy the Bianchi identity are antisymmetric under the exchange of the two two-spheres. This suggests that the forms \( \eta_3 \) and \( \eta_4 \) should appear only in the combination \( \eta_3 - \eta_4 \).

A three-form satisfying (66) is

\[ P_3 = \frac{3g_sMK\alpha'}{16\pi i}(4\log(r)\eta_1 - \eta_3 + \eta_4). \]  

(67)

There is also a closed 3-form which satisfies the supersymmetry conditions and has an acceptable singularity structure (logarithmic singularities at \( r = 0 \) and \( z_1 = 0 \)):

\[ Q_3 = \eta_3 - \eta_4 - \frac{2}{3}\log(z_1))\eta_1 + \frac{i}{6}\eta_5. \]  

(68)

The expression (67) is a particular solution of the Bianchi identity, while (68) is a homogeneous solution. To identify the linear combination of \( P_3 \) and \( Q_3 \) that gives the physical solution, we should look at the expected singularity structure of \( G_3 \) in the presence of D7-branes. Under the shift \( \psi \rightarrow \psi + 4\pi \), the complex scalar transforms as \( \tau \rightarrow \tau + K \), corresponding to the usual \( SL(2,Z) \) monodromy around \( K \) D7-branes. However, under this monodromy the complex three-form \( G_3 \) does not transform (for a general \( SL(2,Z) \) transformation, \( \tau \rightarrow \frac{a\tau + b}{c\tau + d} \) and \( G_3 \rightarrow \frac{G_3}{(c\tau + d)^2} \)). Thus the leading correction to \( G_3 \) is given by Eq. (67), with no additional contribution from the homogeneous solution (68):

\[ G_3 = \frac{M\alpha'}{2i} \left[ \left( 1 + \frac{3g_sK}{2\pi} \log r \right) \eta_1 + \frac{3g_sK}{8\pi} (\eta_4 - \eta_3) \right]. \]  

(69)

Having determined the three-form fluxes from supergravity, let us now investigate their effect in the dual field theory. The RR 3-form flux \( \tilde{F}_3 = F_3 - C_0H_3 = \frac{G_3 + \bar{G}_3}{2} \) is

\[ \tilde{F}_3 = \frac{M\alpha'}{2} \left[ \left( 1 + \frac{3g_sK}{2\pi} \log r \right) \zeta \wedge \omega_2 \right. \\
+ \frac{3g_sK}{8\pi} \left( \frac{dr}{r} \wedge \zeta + \frac{1}{2}(\Omega_{11} + \Omega_{22}) \right) \wedge (1 + \cos \theta_1)\phi_1 - (1 + \cos \theta_2)\phi_2 \left. \right]. \]  

(70)

Integrating this flux over the topologically nontrivial 3-cycle of \( T^{1,1} \), \( \omega_3 = \zeta \wedge \omega_2 \), we find that the number of units of flux varies logarithmically as a function of the radius \( r \):

\[ M_{eff}(r) = M \left( 1 + \frac{3g_sK}{2\pi} \log r \right). \]  

(71)
This logarithmic running, which is a central result of this paper, is quite similar to the running of the five-form flux (31) in the background with no D7-branes. Now, in addition to the variation of the number of colors, we see that the nontrivial axion and dilaton cause the difference in the size of the gauge group factors to decrease as we follow the RG flow from the ultraviolet to the infrared. Moreover, the rate of the decrease agrees with field theory expectations: for \( r \to r e^{-\frac{2\pi}{3g_s M}} \), \( N_{\text{eff}} \) decreases by \( M \), while \( M_{\text{eff}} \) decreases by \( K \).

Recall that the equations (34) and (35) relate the dilaton and \( B \) field to the field-theoretic renormalization group flow. The NS-NS 3-form flux \( H_3 \) is given in terms of the complex 3-form \( G_3 \) by

\[
H_3 = \frac{\bar{G}_3 - G_3}{\tau - \bar{\tau}}. \tag{72}
\]

Working to leading order in \( g_s K \), one finds after some algebra that

\[
H_3 \simeq \frac{3g_s M \alpha'}{2} \left[ (1 + \frac{9g_s K}{4\pi} \log r + \frac{g_s K}{2\pi} \log(\frac{\theta_1}{2} \sin \frac{\theta_2}{2})) \frac{dr}{r} \wedge \omega_2 
+ \frac{g_s K}{8\pi} \left( \frac{dr}{r} \wedge \zeta - \frac{1}{2} d\zeta \right) \wedge (\cot \frac{\theta_2}{2} d\theta_2 - \cot \frac{\theta_1}{2} d\theta_1) \right]. \tag{73}
\]

The corresponding two-form potential \( B_2 \) is given by

\[
B_2 = \frac{3g_s M \alpha'}{2} \left[ (\log r + \frac{9g_s K}{8\pi} \log^2 r + \frac{g_s K}{4\pi} (1 + 2 \log r) \log(\frac{\theta_1}{2} \sin \frac{\theta_2}{2})) \omega_2 
+ \frac{g_s K \log r}{8\pi} \zeta \wedge (\cot \frac{\theta_2}{2} d\theta_2 - \cot \frac{\theta_1}{2} d\theta_1) \right], \tag{74}
\]

and the form relevant for the RG flow equation (35) is

\[
e^{-\Phi} B_2 = \frac{3M \alpha'}{2} \left( \log r + \frac{3g_s K}{8\pi} \log^2 r \right) \omega_2 + \ldots \tag{75}
\]

where the ellipsis denotes terms that do not affect the RG flow at linear order in \( g_s K \). Thus

\[
\frac{\partial}{\partial \log \Lambda} \left[ \frac{4\pi^2}{g_1^2} - \frac{4\pi^2}{g_2^2} \right] = 3M \left( 1 + \frac{3g_s K}{4\pi} \log r \right), \tag{76}
\]

From the dilaton, we see that the sum of the gauge couplings, given by (34), varies as

\[
\frac{\partial}{\partial \log \Lambda} \left[ \frac{4\pi^2}{g_1^2} + \frac{4\pi^2}{g_2^2} \right] = -3K \frac{K}{4}. \tag{77}
\]
The leading terms give precisely the one-loop SV $\beta$-functions. The subleading (in $1/N$) logarithmic term incorporates some of the effects of a decreasing number of colors, but I have not found a convincing argument to explain its normalization (which may be related to a choice of renormalization scheme.)

4.2 Five-form and Warp Factor

The usual ansatz for the RR five-form and the geometric warp factor is

$$g_s \tilde{F}_5 = d^4 x \wedge dh^{-1} + \star (d^4 x \wedge dh^{-1})$$

which is manifestly self-dual. $\tilde{F}_5$ satisfies a Bianchi identity,

$$d\tilde{F}_5 = H_3 \wedge F_3 = \frac{G_3 \wedge \tilde{G}_3}{r - \tau}$$

$$= \frac{3g_s(M\alpha')^2}{8} \left( 1 + \frac{15g_sK}{4\pi} \log r + \frac{g_sK}{2\pi} \log(\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}) \right)$$

$$\times \frac{dr}{r} \wedge \zeta \wedge \Omega_{11} \wedge \Omega_{22} + O((g_sK)^2).$$

(79)

The warp factor $h$ which satisfies the equations of motion is (in the standard near-horizon limit, and at linear order in $g_sK$)

$$h(r, \theta_1, \theta_2) = \frac{L^4}{r^4} \left( 1 + \frac{3g_sM^2}{2\pi N} \log r (1 + \frac{3g_sK}{2\pi} (\log r + \frac{1}{2}) + \frac{g_sK}{4\pi} \log(\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}) \right).$$

(80)

To study the cascade at leading order, we may discard the angular terms. Then we find that the effective number of units of five-form flux is

$$N_{eff}(r) = N + \frac{3g_sM^2}{2\pi} (\log r + \frac{3g_sK}{2\pi} \log^2 r).$$

(82)

Under the radial rescaling $r \to e^{-\frac{3g_sM_{eff}}{g_sK} r}$, $N_{eff}$ decreases by $M_{eff} - K$ units, in agreement with the argument from Seiberg duality (up to linear order in $g_sK$.)

5 T-dual Type IIA Brane Configurations

There is another class of stringy constructions of the gauge theories studied in this paper, to which we now turn. These models are based on brane configurations in type IIA superstring theory [43], and it was argued in [44] that the T-dual of the conifold
theory with fractional branes is just such a collection of branes. Some features of the
gauge theory are quite transparent from the perspective of these brane constructions,
so in this section we will review some relevant results of these models, mostly following
the papers [45, 46, 47]. Throughout this section, we work in ten flat spacetime
dimensions, and take the $x^6$ direction to be compactified on a circle.

Let us first describe the brane construction corresponding to the conifold gauge
theory without fundamental flavors. We place one NS5-brane along the 012345 direc-
tions (which for conciseness we will call an NS brane), and another NS5-brane along
the 012389 directions (which we will call an NS’ brane.) We will also place D4-branes
along the 01236 directions. At generic positions, the D4-branes must wrap the en-
tire $x^6$ circle. However, because it is possible for D4-branes to end on NS5- branes, we
may also have half-D4-branes extending from the NS brane to the NS’ brane. Clearly,
there are two types of such half-D4-branes; moreover, two half-D4-branes of different
types may fuse to become a regular D4-brane, which is then free to move to generic
values of the 45789 coordinates. On each stack of half-D4-branes, there is a four-
dimensional gauge theory at low energies with gauge group $SU(N_i), i = 1, 2$, where
$N_i$ is the number of branes in each stack. There are also bifundamental matter fields
arising from strings connecting the two stacks of half-D4-branes. As a result, the
matter content is precisely that of the conifold theory described in sections 3 and 4.

By performing a T-duality in the $x^6$ direction, the half-D4-branes become fractional
D3-branes, while the NS5-branes T-dualize into the geometry of the conifold.

To add fundamental flavors to this model, we should add branes of higher dimen-
sion. The most natural choice is to add D6-branes. For example, we may embed
a D6-brane along the 0123457 directions, and take it to be coincident with the NS
brane. This brane configuration preserves $\mathcal{N} = 1$ supersymmetry in the gauge the-
ory. Moreover, we find that there will be $g, \tilde{g}, Q, \tilde{Q}$ flavors from 4-6 strings which are
completely analogous to the flavors found in Section 3. Of course, now the T-duality
along the $x^6$ direction takes the D6-brane to a D7-brane. The paper [46] also derived
the superpotential (25); they found moreover that the couplings $g$ and $h$ in (25) are
related by $g = -h$. Note that the D6-branes split into two halves on the NS brane;
however, it is necessary for both halves to be present for RR charge conservation.
This is analogous to the necessity of flavors with both positive and negative chirality
for cancellation of gauge anomalies; in the IIB picture the analogous requirement is
that the holomorphic embedding equation for the D7-branes be expressible in terms
of the $z_i$ variables (rather than an equation of the form $A_1 = 0$, for example.)

We can now explain (heuristically) the duality cascade in this theory from the perspective of the IIA construction via brane creation, first without fundamentals. If the sizes of the stacks of half-D4-branes are unequal, then the D4-brane tension will cause the NS and NS’ branes to bend. At some point, the NS and NS’ branes may reach equal values of $x^6$; this corresponds to a divergence of the coupling of the $SU(N + M)$ gauge group. We can remove this divergence by moving the NS’ brane around the circle. As the NS and NS’ brane cross, the stack of $N + M$ half-D4-branes shrinks to zero size and then regrows, changing its orientation to a stack of anti-branes in the process. However, the stack of $N$ half-D4-branes now extends more than once around the circle; in the region with the $N + M$ anti-branes sees a stack of $2N$ D4-branes; the branes then annihilate to give $N - M$ half-D4-branes. Thus we have a transition taking the gauge group $SU(N + M) \times SU(N)$ to $SU(N) \times SU(N - M)$.

With this picture in mind, the analogous cascade with D6-branes is easy to describe. As the NS’ brane crosses the NS brane and D6-brane, there is an additional D4-brane created due to the D6. Thus instead of having $2N - (N + M) = N - M$ D4-branes, we will have $2N - (N + M) + K = N - M + K$ D4-branes. This picture reproduces the pattern of Seiberg dualities described in Section 4.

6 Prospects

In this paper we have described supergravity solutions containing D7-branes which are dual to $\mathcal{N} = 1$ supersymmetric gauge theories with fundamental flavors. There are many questions to be resolved, and we shall collect some of them here.

- We expect that the holomorphic D7-brane configurations described in this paper are supersymmetric, but it would be nice to explicitly verify this claim from the standpoint of $\kappa$-symmetry on the D7-brane worldvolume.

- Another loose end in this paper concerns higher order corrections to the anomalous dimensions of the $A$, $B$, and $q$ fields. These could perhaps be analyzed in supergravity along the lines of [49], or in field theory.

- It would be interesting to study D7-brane fluctuations using the DBI action, as was done in similar setups in [12, 13]. This calculation would give a check
on stability of the brane configuration, and would also give the spectrum of mesonic operators in the field theory. Running the AdS/CFT correspondence in the other direction, it might also be possible to generate the complete space of D7-brane configurations from field theory, as was done for giant gravitons by [48].

- The supergravity solutions given in section 4 are only valid at intermediate values of the radial coordinate $r$, so it would be nice to find a globally well-defined solution using F-theory. With a solution valid in the far infrared, it might become possible to study phenomena associated with chiral symmetry breaking. It would also be interesting to study nonperturbative objects such as baryons, which arise from D3-branes wrapped on the blown-up 3-cycle of the deformed conifold. Such an object would be a true baryon, made of fundamental quarks (as opposed to dibaryons or baryon vertices with nondynamical quarks, as have been previously considered in the literature.)

- An F-theory solution with a compact CY four-fold would also be interesting as an ultraviolet completion of the cascading theory with flavors. Warped compact versions of the Klebanov-Strassler solution [3] have been considered in the context of possible solutions to the hierarchy problem [50]. The cascade studied in this paper suggests that warped compactifications with flavor have naturally small gauge groups at low energies; even if the numbers of units of flux $N$ and $M$ are both large in the ultraviolet, they both decrease via duality cascades until they are of order $K$ in the infrared. We leave the elucidation of these models for future work.

**Acknowledgments**

I am grateful to Igor Klebanov and Chris Herzog for collaboration in the early stages of this project and for many useful discussions. It is also a pleasure to thank Chris Beasley, Sameer Murthy, and Edward Witten for helpful conversations. This material is based upon work supported by the National Science Foundation Grants No. PHY-0243680 and PHY-0140311. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.
References

[1] J. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231, hep-th/9711200.

[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B428 (1998) 105, hep-th/9802109.

[3] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253, hep-th/9802150.

[4] I. R. Klebanov and M. Strassler, JHEP 0008 (2000) 052, hep-th/0007191.

[5] J. Maldacena and C. Nunez, Phys. Rev. Lett. 86 (2001) 588, hep-th/0008001.

[6] C. Vafa, hep-th/0008142.

[7] O. Aharony, A. Fayyazuddin and J. M. Maldacena, JHEP 9807, (1998) 013 hep-th/9806159.

[8] M. Grana and J. Polchinski, Phys. Rev. D65, 126005 (2002) hep-th/0106014.

[9] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda and R. Marotta, Nucl. Phys. B621, 157 (2002) hep-th/0107057.

[10] S. G. Naculich, H. J. Schnitzer and N. Wyllard, Nucl. Phys. B 638, 41 (2002) arXiv:hep-th/0204023; S. G. Naculich, H. J. Schnitzer and N. Wyllard, Int. J. Mod. Phys. A 17, 2567 (2002) arXiv:hep-th/0106020.

[11] A. Karch and E. Katz, JHEP 0206, (2002) 043 hep-th/0205236.

[12] M. Kruczenski, D. Mateos, R. C. Myers and D. J. Winters, hep-th/0304032.

[13] T. Sakai and J. Sonnenschein, hep-th/0305049.

[14] H. Nastase, arXiv:hep-th/0305069.

[15] X. J. Wang and S. Hu, JHEP 0309, 017 (2003) arXiv:hep-th/0307218.

[16] C. Nunez, A. Paredes and A. V. Ramallo, JHEP 0312, 024 (2003) arXiv:hep-th/0311201.

[17] J. Babington, J. Erdmenger, N. J. Evans, Z. Guralnik and I. Kirsch, Phys. Rev. D 69, 066007 (2004) arXiv:hep-th/0306018.
[18] P. Candelas and X. de la Ossa, *Nucl. Phys.* B**342** (1990) 246.

[19] I. R. Klebanov and E. Witten, *Nucl. Phys.* B**536** (1998) 199, [hep-th/9807080](https://arxiv.org/abs/hep-th/9807080).

[20] D. Morrison and R. Plesser, *Adv. Theor. Math. Phys.* **3** (1999) 1, [hep-th/9810201](https://arxiv.org/abs/hep-th/9810201).

[21] R. Minasian and D. Tsimpis, *Nucl. Phys.* B**572** (2000) 499, [hep-th/9911042](https://arxiv.org/abs/hep-th/9911042).

[22] A. Ceresole, G. Dall’Agata, R. D’Auria, and S. Ferrara, *Phys. Rev. D* **61** (2000) 066001, [hep-th/9905226](https://arxiv.org/abs/hep-th/9905226).

[23] K. Ohta and T. Yokono, *JHEP* **0002** (2000) 023, [hep-th/9912266](https://arxiv.org/abs/hep-th/9912266).

[24] K. Becker, M. Becker and A. Strominger, Nucl. Phys. B **456**, 130 (1995) [arXiv:hep-th/9507158](https://arxiv.org/abs/hep-th/9507158).

[25] H. Ooguri, Y. Oz and Z. Yin, Nucl. Phys. B **477**, 407 (1996) [arXiv:hep-th/9606112](https://arxiv.org/abs/hep-th/9606112).

[26] B. R. Greene, A. D. Shapere, C. Vafa and S. T. Yau, Nucl. Phys. B **337**, 1 (1990).

[27] M. Shifman and A. Vainshtein, *Nucl. Phys.* B**277** (1986) 456.

[28] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, *Nucl. Phys.* B**229** (1983) 381.

[29] I. R. Klebanov, P. Ouyang and E. Witten, *Phys. Rev.* D**65**, (2002) 105007 [hep-th/0202056](https://arxiv.org/abs/hep-th/0202056).

[30] M. Krasnitz, JHEP **0212**, 048 (2002) [arXiv:hep-th/0209163](https://arxiv.org/abs/hep-th/0209163).

[31] M. Bianchi, O. DeWolfe, D. Z. Freedman and K. Pilch, *JHEP* **0101**, (2001) 021, [hep-th/0009156](https://arxiv.org/abs/hep-th/0009156).

[32] A. Brandhuber and K. Sfetsos, *JHEP* **0012**, (2000) 014, [hep-th/0010048](https://arxiv.org/abs/hep-th/0010048).

[33] M. R. Douglas and G. W. Moore, [hep-th/9603167](https://arxiv.org/abs/hep-th/9603167).

[34] S. Gubser, N. Nekrasov and S. Shatashvili, *JHEP* **9905**, 003 (1999) [hep-th/9811230](https://arxiv.org/abs/hep-th/9811230).
[35] C. P. Herzog and I. R. Klebanov, Phys. Rev. D 63, 126005 (2001) hep-th/0101020

[36] M. Cvetic, G. W. Gibbons, H. Lu and C. N. Pope, Commun. Math. Phys. 232, 457 (2003) hep-th/0012011. M. Cvetic, G. W. Gibbons, H. Lu and C. N. Pope, Nucl. Phys. B 606, 18 (2001) hep-th/0101096. M. Cvetic, G. W. Gibbons, H. Lu and C. N. Pope, Nucl. Phys. B 617, 151 (2001) hep-th/0102185.

[37] I. R. Klebanov and N. Nekrasov, Nucl. Phys. B 574 (2000) 263, hep-th/9911096

[38] I. R. Klebanov and A. Tseytlin, Nucl. Phys. B 578 (2000) 123, hep-th/0002159

[39] S. Franco, A. Hanany, Y. H. He and P. Kazakopoulos, arXiv:hep-th/0306092

[40] N. Seiberg, Nucl. Phys. B 435, 129 (1995) arXiv:hep-th/9411149.

[41] M. Grana and J. Polchinski, Phys. Rev. D 63 (2001) 026001, hep-th/0009211

[42] S. S. Gubser, hep-th/0010010

[43] A. Hanany and E. Witten, Nucl. Phys. B 492, 152 (1997) arXiv:hep-th/9611230.

[44] K. Dasgupta and S. Mukhi, Nucl. Phys. B 551, 204 (1999) arXiv:hep-th/9811139.

[45] J. H. Brodie and A. Hanany, Nucl. Phys. B 506, 157 (1997) arXiv:hep-th/9704043.

[46] I. Brunner, A. Hanany, A. Karch and D. Lust, Nucl. Phys. B 528, 197 (1998) arXiv:hep-th/9801017.

[47] J. Park, R. Rabadan and A. M. Uranga, Nucl. Phys. B 570, 3 (2000) hep-th/9907074.

[48] C. E. Beasley, JHEP 0211, 015 (2002) arXiv:hep-th/0207125.

[49] S. Frolov, I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. B 620, 84 (2002) arXiv:hep-th/0108106.

[50] S. B. Giddings, S. Kachru and J. Polchinski, hep-th/0105097