GIBBS’ STATES FOR MOSER-CALOGERO POTENTIALS

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We present two independent approaches for computing the thermodynamics for classical particles interacting via the Moser–Calogero potential. Combining the results we propose the form of equation of state or, what is equivalent, the asymptotics of the Jacobian between volume elements corresponding two symplectic structures on the phase space.

1. Introduction

Non-equilibrium statistical mechanics studies properties of dynamical systems describing big or infinite systems of interacting particles. The Hamiltonian structure of the equations allows one to define on the phase space so-called Gibbsian measures, invariant under the dynamics. These measures are usually constructed from the basic Hamiltonian (=energy) and other classical conserved quantities such as the number of particles, integrals of the momentum and the angular momentum. They constitute a finite parameter family, so-called grand–canonical ensemble. At present there are four known 1D hierarchies with similar structure of the ensemble in the generic case and the same number (2) of integrable exceptions, as it is explained below.

Statistical mechanics of 1D particles is well studied, [1]. In the generic case of Lenard-Jones potential the Gibbs' states form 3 parameter family. In 1975-76, Moser and Calogero discovered two repulsive potentials $U(y) = x^{-2}, \sinh^{-2} x$ which are integrable by the inverse spectral transform, [2, 3]. Additional Gibbs' states from additional integrals of the motion different from the classical ones for such potentials were constructed by Chulaevskii [4]. In the paper of Gurevich [5], the exceptional character of Moser–Calogero potentials was proved.

Statistical mechanics of the 1D semi-linear wave equation with restoring force was studied by McKean–Vaninsky [6, 7]. The phase space of the dynamical system is a product of two function spaces. In this phase space, a Gibbsian measure was constructed and the existence of flow was proved, as was the invariance of the measure under the flow.

Also, two integrable equations were considered: linear Klein-Gordon and Sinh-Gordon. These equations have integrals of local densities as an additional conserved quantities of motion. These integrals produce infinitely many additional Gibbs’
states. Klein and Sinh-Gordon can be considered as analogs of Moser-Calogero potentials.

In such integrable equations the conserved quantities prevent any Gibbs’ state from being ergodic in the periodic case. On the entire line, these are destroyed on the support of the measure. As it was proved by McKean, for the sinh-Gordon equation, the Gibbs’ state constructed from the basic Hamiltonian is ergodic. It is believed that all other mutually singular Gibbs’ states are ergodic too.

Another field model where it is possible to carry out this program is the de-focusing nonlinear Schrödinger equation with the power nonlinearity. Again, there are two integrable equations, the linear and the cubic cases, with additional Gibbs’ states.

Recently, another hierarchy of so-called modified KdV equations with similar properties was discovered. Now all these one-dimensional hierarchies fit nicely into the table.

| $H = \sum \frac{\dot{q}^2}{2} + \sum U(\Delta q)$ | $H = \int \frac{\dot{p}^2 + q^2}{2} + F(q)$ | $H = \int |\dot{\psi}|^2 + F(|\psi|^2)$ | $H = \int \frac{q^2}{2} + F(q)$ |
|-------------------------------------------------|-------------------------------------------------|---------------------------------|---------------------------------|
| $H=$energy                                      | $H=$energy                                      | $H=$energy                      | $H=$energy                      |
| $P = \sum p=$momentum                           | $P = \int p\dot{q}=$momentum                    | $P = \int \dot{\psi} \bar{\psi}=$momentum | $P = \int q^2=$momentum         |
| $N=\sharp$ of particles                         | $N = \int |\psi|^2=\sharp$ of particles           |                                 |                                 |
| $e^{-\beta H - \alpha N - \gamma P}$          | $e^{-\beta H - \gamma P}$                      | $e^{-\beta H - \alpha N - \gamma P}$ | $e^{-\beta H - \gamma P}$      |
| $\frac{1}{q^2}$                                 | $q^2 \cosh q$                                   | $|\psi|^2$                      | $|\psi|^4$                      |

The first line of the table displays the Hamiltonians of these four hierarchies. For particles, the potential $U(\cdot)$ is assumed to be Lenard-Jones type. For PDE hierarchies $F(\cdot)$ is the potential of a restoring force; it is an even function increasing at infinity. The next three lines are generic classical conserved quantities. The NLS-hierarchy and m-KdV hierarchy do not have any analog of $N=\sharp$ of particles. The next line is the symbol for the Gibbs’ measure; it is a three/two parameter family. The last line presents integrable potentials where additional conserved quantities produce an additional Gibbs’ states.

All three integrable nonlinear equations are so-called AKNS systems, related to some selfadjoint problem for the Dirac operator. These are all the nonlinear equations related to such spectral problem.

Additional structure of integrability allows one to obtain detailed information about the Gibbs’ states. In section 2 we describe the decomposition of the canonical measure for the cubic Schrödinger equation previously obtained in [11]. In section
3, taking such decomposition for granted for Moser–Calogero potentials, we compute the thermodynamic limit of the specific free energy, the average energy per particle and the equation of state. In the last section 4 we present an alternative approach in the case of the rational potential $2x^{-2}$ and formulate the conjecture on the explicit form of the equation of state and the asymptotics of the Jacobian.

2. Gibbs’ States in Action-Angle Variables.

For the cubic Schrödinger equation, the global action-angle variables $0 \leq I_k, 0 \leq \phi_k < 2\pi$, $k = ..., -1, 0, 1, ...$, associated with the basic symplectic structure $\omega$ were constructed by McKean–Vaninsky. The next step is to express the measure $e^{-H}d\text{vol}$, where $H = \int_0^1 |\psi'|^2 + 2|\psi|^4$ in such coordinates. The main ingredient is the trace formula $H = \sum I'_k$, where the $I'$s are periods of some Abelian differential on the Riemann surface. Presumably they are the actions, but relative to some higher symplectic structure $\omega'$. On a formal level,

$$e^{-H}d\text{vol} = e^{-\sum I'} \prod dI d\phi = e^{-\sum I'} \text{Jac} \prod dI' d\phi,$$

where Jac is the Jacobian between the variables $I$ and $I'$. Therefore,

$$\text{Jac}^{-1} e^{-H}d\text{vol} = e^{-\sum I'} \prod dI' d\phi,$$

i.e. $I'$s and $\psi'$s are independent, the $I'$s are exponential and the $\phi'$s are uniform. Such a decomposition was proved by McKean–Vaninsky. It can be useful in computing different thermodynamical quantities, as explained below.

3. Thermodynamics of Moser-Calogero Potentials.

Consider particles on the line interacting with the potential $U = 2x^{-2}$ (the rational case) or $2\sinh^{-2}x$ (the trigonometric case). To construct the Gibbs’ state for an infinite system of such particles, we periodize the potential as in $U_{\omega_1}(\bullet) = \sum_k U(\bullet + k\omega_1)$. In our case, $U_\omega$ is simply the Weierstrass $2\wp$ with possibly infinite second imaginary period $\omega_2$ (the rational case). Now $N$ particles on the circle of the perimeter $\omega_1$ are governed by the Hamiltonian

$$H(q, p) = \sum_{k=1}^N \frac{p_k^2}{2} + \sum_{k \neq j; k, j = 1}^N 2\wp(q_k - q_j).$$

The partition function is defined as

$$Z(N, \omega_1, \beta) = \frac{1}{N!} \int_M e^{-\beta H(q, p)} d^N q d^N p,$$
where the integral is taken over the phase space $M = ([0, \omega_1] \times R^1)^N$ and $\beta = T^{-1}$ is the reciprocal temperature. In the thermodynamic limit $N$ and $\omega_1$ tend to $\infty$, but their ratio $\omega_1/N$ tends to some constant: the specific volume $v > 0$. It is well known that, in the limit, the specific free energy per particle

$$\psi(v, \beta) = \lim_{N \to \infty} \frac{1}{N} \log Z(N, \omega_1, \beta)$$

exists in the limiting ensemble. The average energy per particle and the pressure can be computed by simple differentiation of $\psi$

$$< E > = -\frac{\partial \psi(v, \beta)}{\partial \beta}, \quad p = T \frac{\partial \psi(v, \beta)}{\partial v}.$$

Assuming the possibility of decomposition for Gibbs’ states described in section 2, we compute $\psi(v, \beta)$ by some version of the method of stationary phase.

Obvious rotational symmetry allows one to reduce the dimension of the phase space. Indeed, according to the classical prescription, Arnold

$$I_1 = \frac{1}{2\pi} \int p \ dq = \frac{1}{2\pi} \int \sum p_k dq_k = \frac{\omega_1}{2\pi} \sum p_k = \frac{\omega_1}{2\pi} P,$$

where $P$ is the total momentum. The corresponding angle $\varphi_1$ is just the usual on the circle.

The change of variables $q_k \to q'_k$ and $p_k \to p'_k + \frac{P_0}{N}$ makes the total momentum $P'$ vanish. The construction of the action-angle variables $I_2, \varphi_2, \cdots, I_N, \varphi_N$ on the reduced phase space is based on the KP–1 equation, see [14]. We do not need an explicit representation for them now.

To compute the partition function, the domain of integration is similarly reduced:

$$Z(N, \omega_1, \beta) = \frac{1}{N!} \int_M e^{-\beta H(q, p)} \int_{R^1} \delta(P(q, p) - P_0) dP_0 \ d^N q \ d^N p$$

$$= \int_{R^1} dP_0 \frac{1}{N!} \int_M e^{-\beta H(q', p')} \delta(P(q, p) - P_0) d^N q \ d^N p.$$ 

The canonical transformation $q_k \to q'_k$ and $p_k \to p'_k + \frac{P_0}{N}$ produces

$$H(q, p) = H(q', p') + \frac{P_0^2}{2N}, \quad P = P' + P_0$$

and

$$Z = \int_{R^1} dP_0 \frac{1}{N!} \int_M e^{-\beta H(q', p') - \beta P_0^2/2N} \delta(P') d^N q' \ d^N p'$$

$$= \sqrt{2\pi NT} \frac{1}{N!} \int_M e^{-\beta H(q, p)} \delta(P) d^N q \ d^N p.$$. 
Now make a canonical transformation to the action-angle variables $I_1, \varphi_1, \cdots, I_N, \varphi_N$. Then

$$Z = \sqrt{2\pi NT} \frac{1}{N!} \int_{\text{range of } I_1, \cdots, I_N \times [0,2\pi)^N} e^{-\beta H(I_2, \cdots, I_N)} \delta(I_1) \frac{2\pi}{\omega_1} d\varphi_1 \prod_{k=2}^N dI_k d\varphi_k.$$  

Integrating out $I_1$ and all the angles we obtain

$$Z = \sqrt{2\pi NT} \frac{\omega_1}{2\pi} (2\pi)^N \frac{1}{N!} \int_{\text{range of } I_2, \cdots, I_N} e^{-\beta H(I_2, \cdots, I_N)} \prod_{k=2}^N dI_k.$$  

To proceed farther we have to make few assumptions. The arguments in favor of them are supplied at the end of this section.

**Assumption 1.** The energy $H$ can be expressed as $H = \sum_{k=2}^N I'_k$ ("trace formula"), where $I'_2, \cdots, I'_N$ are the actions relative to some higher symplectic structure $\omega'$.  

**Assumption 2.** The range of the variables $I'_2, \cdots, I'_N$ is the rectangular domain $\{(I'_2, \cdots, I'_N) : I'_k \min \leq I'_k, \quad k = 2, \cdots, N\}$, as shown in the picture

Then $H_{\min} = \sum I'_\min$ and in the thermodynamic limit, $H_{\min}/N \to h(v)$, where $h(v) = \sum_{k \neq 0} U(kv)$ is the specific energy of one particle in the ground state with the specific volume $v$ of the limiting infinite system.

**Assumption 3.** The Jacobian between the variables $I_2, \cdots, I_N$ and $I'_2, \cdots, I'_N$ in the thermodynamic limit has the asymptotics

$$N^{-1} \log \text{Jac}(I'_2, \cdots, I'_N, \omega_1, N) = \log N - s(v, \beta) + o(1)$$

in probability relative to the Gibbs’ ensemble. The nontrivial part of the asymptotics $s(v, \beta)$, we call entropy, by analogy with statistical mechanics, Lanford [13].

Now we employ the trace formula:

$$Z(N, \omega_1, \beta) = \sqrt{2\pi NT} \frac{\omega_1}{2\pi} (2\pi)^N \frac{1}{N!} \int_{\text{range of } I'_2, \cdots, I'_N} e^{-\beta \sum I'_k} \text{Jac}(I'_2, \cdots, I'_N) \prod_{k=2}^N dI'_k.$$  

For the specific free energy in the thermodynamic limit, we obtain finally

$$\psi(v, \beta) = \lim N^{-1} \log Z(N, \omega_1, \beta)$$
\[ \log 2 \pi + N^{-1} \log N + \int_{\text{range of } I_2', \ldots, I_N'} e^{-\beta N \log N - N s(v, \beta)} \prod_{k=2}^{N} dI_k' + o(1) \]

\[ = \log 2 \pi + N^{-1} \log N + \int_{\text{range of } I_2', \ldots, I_N'} e^{-\beta \sum I_k' \log N} \prod_{k=2}^{N} dI_k' \]

\[ + \log N - s(v, \beta) + o(1) \]

\[ = \log 2 \pi + 1 + N^{-1} \log e^{-\beta H_{\text{min}}} - s(v, \beta) + o(1) \]

\[ = \log 2 \pi + 1 - \beta h(v) - \log \beta - s(v, \beta). \]

The average energy is

\[ \langle E \rangle = -\frac{\partial \psi(v, \beta)}{\partial \beta} = T + h(v) + s'_\beta(v, \beta). \]

The equation of state has the form

\[ p = T \frac{\partial \psi(v, \beta)}{\partial v} = -T s'_v(v, \beta) - h'_v(v). \]

These formulas provide a bridge between two unsolved problems: the asymptotics of the Jacobian determines the thermodynamic functions and vice versa. We will try to guess an explicit form of \( s(v, \beta) \) in the next section.

Now we explain why we expect our assumptions to be true.

**Assumption 1.** The trace formula appears: in the linear problem, \( \mathtt{7} \), as the Parsevall identity and, in the cubic Schrödinger case \( \mathtt{11} \), as a result of contour integration on the Riemann surface. We expect such formula here too, with \( I' \)’s being a periods of some Abelian differential.

**Assumption 2.** A general fact due to Atiyah-Guillemin-Sternberg states that, for a compact symplectic phase space with a torus action, the image of the momentum map is a convex polytope. The shape of the momentum map here is suggested by the previous work on PDE and by the fact that the energy in the ground state is strictly positive.

**Assumption 3.** To explain why the \( \text{Jac}(I_2', \ldots, I_N', \omega_1, N) \) becomes a constant in thermodynamic limit, we have to define all objects under consideration on one probability space. Consider a new phase space \( M \) for the infinite-particle system on the entire line. \( M \) contains all possible positions and velocities of particles, with the sole restriction that any finite region in the configuration space contains a finite number of particles (so-called locally finite).

Let \( M_{\omega_1} \subset M \) comprise all spatially \( \omega_1 \)-periodic configurations. On \( M_{\omega_1} \), let \( \mu_{\omega_1}(\bullet) \) be a Gibbs’ state with periodic boundary conditions. Namely, consider projection of any configuration from \( M_{\omega_1} \) into a finite volume, from \( 0 \) to \( \omega_1 \). Any configuration is determined by the projection. Define on the projections a Gibbs’ state with periodic boundary condition. It induces a measure \( \mu_{\omega_1}(\bullet) \) on \( M_{\omega_1} \) and
on the big space $M$, too. A limiting Gibbs’ state $\mu_\infty(\bullet)$ on $M$ is obtained from $\mu_{\omega_1}(\bullet)$ by passing to the limit $\omega_1 \to \infty$.

On the big space $M$ acts the one-parameter group of spatial translations $\tau_b, b \in \mathbb{R}^1$. Obviously, $\mu_{\omega_1}(\bullet)$ and $M_{\omega_1}$ itself are invariant under the action of $\tau$. The limiting measure $\mu_\infty(\bullet)$ is also invariant under such translations and even ergodic.

For any such $M_{\omega_1}$ we can define a Jacobian $\text{Jac}(I'_2, \ldots, I'_N, \omega_1, N)$; it can be considered on the whole $M$ as a function defined almost everywhere with respect to $\mu_{\omega_1}(\bullet)$. The Jacobians are $\tau$–invariant functions. If they have a limit defined almost everywhere with respect $\mu_\infty(\bullet)$, then it is $\tau$-invariant and must be a constant due to the ergodicity of this measure. The exact form of the asymptotics is suggested by [16].

4. Alternative Approach. The Rational Case.

Gallavotti-Marchioro [17] and also Francoise [18], considered the system of $N$ particles on the line with the Hamiltonian

$$H(q, p) = \sum_{k=1}^{N} \frac{p_k^2}{2} + \frac{\lambda^2 q_k^2}{2} + \sum_{k \neq j: k, j = 1}^{N} 2(q_k - q_j)^{-2}.$$ 

They obtained a closed expression for the partition function:

$$Z(N, \lambda, \beta) = \frac{1}{N!} \int_{\mathbb{R}^N \times \mathbb{R}^N} e^{-\beta H} d^N q d^N p = \frac{1}{N!} \left( \frac{2\pi}{\beta \lambda} \right)^N \exp(-\beta \lambda N(N-1)).$$

Now, pass to the limit with $N \to \infty$ and assume $\lambda \sim N^{-1} c(v, \beta)$, where $c(v, \beta)$ is some unknown function. With this scaling $H$ grow like $N$ in the statistical ensemble. In the limit, we obtain an infinite–particle system with specific volume $v$ and temperature $\beta$. For the specific free energy we have

$$\psi(v, \beta) = -N^{-1} \log \frac{1}{N!} \left( \frac{2\pi N}{\beta c} \right)^N e^{-\beta c(N-1)} + o(1)$$

$$= -N^{-1} \log N! + \log 2\pi + \log N - \log \beta - \log c - \beta c + o(1)$$

$$= \log 2\pi + 1 - \beta c - \log \beta - \log c.$$ 

The expression for the specific free energy contains the unknown function $c(v, \beta)$, but the role of the two unknown functions $c(v, \beta)$ and $s(v, \beta)$ is different. Comparing these equations we conjecture $c(v, \beta) = h(v)$ and $s(v, \beta) = \log h(v)$. In the rational case $h(v)$ can be computed explicitly

$$h(v) = \sum_{k \neq 0} U(kv) = \sum_{k \neq 0} \frac{2}{k^2 v^2} = \frac{2\pi^2}{3v^2}.$$ 

Therefore

$$p = -T \frac{\partial}{\partial v} \log h(v) - h'_v = \frac{2T}{v} + \frac{4\pi^2}{3v^3}, \quad \langle E \rangle = T + h(v) = T + \frac{2\pi^2}{3v^2}.$$
These equations are similar to the ideal gas for which \( p = T/v \) and \( \langle E \rangle = T/2 \).
In the trigonometric case \( (U(x) = 2 \sinh^{-2} x) \) we expect similar results.

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References

1. D. Ruelle. *Statistical Mechanics: Rigorous results.* (W.A. Benjamin, Inc., New York-Amsterdam, 1969).
2. J. Moser. Three Integrable Hamiltonian Systems Connected with Isospectral Deformations. *Adv. in Math.* 1975, vol. 116, 2, pp. 197-220.
3. F. Calogero. Exactly Solvable One Dimensional Many-Body Problems. *Lettere al Nuovo Cimento*, 1975, vol. 13, 11, pp. 411-416.
4. V.A. Chulavevskii. Method of Inverse Scattering in Statistical Mechanics. *Funkts. Anal. Prilozh.*, 1983, vol. 17, 1, pp. 53-62.
5. B.M. Gurevich. Asymptotically additive integrals of the motion of particles with pairwise interaction. *Trans. Moscow Math. Society.* 1990, vol. 52, pp. 177-232.
6. H.P. McKean and K.L. Vaninsky. Statistical mechanics of nonlinear wave equations. *Trends and perspectives and applied mathematics.* (Springer, New York, 1994), Appl. Math. Sci., 100. pp. 239-264.
7. K.L. Vaninsky. Invariant Gibbs’ measures of the Klein-Gordon equation. in *Proc. of Symp. in Pure Mathematics*, eds. M. Pinsky and M. Cranston, (AMS 1995), vol 57, pp. 495–509.
8. H.P. McKeane. Statistical mechanics of nonlinear wave equations. 3. Metric transitivity for hyperbolic sine-Gordon. *J. of Stat Phys.* 79, 1995, no. 3-4, pp. 731-737.
9. H.P. McKeane. Statistical mechanics of nonlinear wave equations. 4. Cubic Schrödinger. *Comm. Math. Phys.*, vol. 168, 1995, no. 3, pp. 479-491.
10. J. Bourgain. Periodic Nonlinear Schrödinger Equation and Invariant Measures. *Comm. Math. Phys.*, 166, 1994, pp. 1-26.
11. H.P. McKeane and K.L. Vaninsky. Action-angle variables for the cubic Schrödinger equation. to appear in *CPAM*.
12. H.P. McKeane and K.L. Vaninsky. Cubic Schrödinger: the petit canonical ensemble in action-angle variables. to appear in *CPAM*.
13. V.I. Arnold. *Mathematical Methods of Classical Mechanics* (Springer–Verlag, Berlin–Heidelberg–New York, 1978).
14. I.M. Krichever. Elliptic solutions of the Kadomtsev–Petviashvili equation and integrable systems of particles. *Funct. Anal. Appl.* 14, (1980), pp 282-290.
15. O.E. Lanford. Entropy and Equilibrium States in Classical Statistical Mechanics. *Lect. Notes in Phys.* 20, 1973, pp. 1-113.
16. K.L. Vaninsky. Symplectic structures and volume elements in the function space for the nonlinear Schrödinger equation. (*paper in preparation*).
17. G. Gallavotti and C. Marchioro, On the calculation of the integral, *J. of Math. Analysis and Applications*, vol 44, 1973, pp. 661-675.
18. J.-P. Francoise. Canonical partition function of Hamiltonian systems and stationary phase formula. *Comm. Math. Phys.*, 117, 1988, pp. 37-47.