$H-$theorems for the Brownian motion on the hyperbolic plane

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Abstract

We study $H-$theorems associated with the Brownian motion with constant drift on the hyperbolic plane. Since this random process verifies a linear Fokker-Planck equation, it is easy to show that, up to a proper scaling, its Shannon entropy is increasing over time. As a consequence, its distribution is converging to a maximum Shannon entropy distribution which is also shown to be related to the non-extensive statistics. In a second part, relying on a theorem by Shiino, we extend this result to the case of Tsallis entropies: we show that under a variance-like constraint, the Tsallis entropy of the Brownian motion on the hyperbolic plane is increasing provided that the non-extensivity parameter of this entropy is properly chosen in terms of the drift of the Brownian motion.

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I. INTRODUCTION

The existence of an $H$–theorem for a statistical system submitted to some constraints ensures that the Shannon entropy $h(f_t)$ of its probability density function (p.d.f.) $f_t$ at time $t$,

$$h(f_t) = -\int_{\mathbb{R}} f_t(x) \log f_t(x) \, dx,$$

is increasing with time. As a consequence, the asymptotic (stationary) p.d.f. $f_\infty(x)$ of the system is the maximum Shannon entropy p.d.f. that satisfies the given constraints.

The Shannon entropy is a particular member of a family of information measures called Tsallis entropies that were introduced [9] in 1988 by Tsallis in the context of statistical physics; they are defined as

$$h_q(f_t) = \frac{1}{1-q} \left( \int_{\mathbb{R}} f_t^q(x) \, dx - 1 \right);$$

where $q > 0$ is the non-extensivity parameter. It can be checked using l’Hospital rule that the Shannon entropy is the limit case $\lim_{q \to 1} h_q(f_t) = h(f_t)$.

A natural question then arises: under what conditions does an $H$–theorem extend to an $H_q$–theorem, where the Shannon entropy is replaced by a Tsallis entropy? Several studies have been devoted to this problem in the recent years: for example, Plastino et al [3] study the conditions of existence of an $H_q$–theorem for the following non-linear Fokker-Planck equation

$$\frac{\partial f_t}{\partial t} = -\frac{\partial}{\partial x} (K(x) f_t(x)) + \frac{1}{2} Q \frac{\partial^2}{\partial x^2} f_t^{2-q}(x)$$

for some parameter $q$, while Tsallis and Bukman [2] apply the same approach to the equation

$$\frac{\partial f_t^\mu}{\partial t} = -\frac{\partial}{\partial x} (F(x) f_t^\mu(x)) + D \frac{\partial^2}{\partial x^2} f_t^{\nu}(x)$$

for some positive parameters $\mu$ and $\nu$.

Our aim in this paper is to show the existence of both an $H$– and an $H_q$–theorem for the Brownian motion - more precisely its $x$–component - with constant drift on the hyperbolic plane. This study is simplified by the fact that this component of the Brownian motion satisfies a linear Fokker-Planck equation, for which the conditions of existence of $H$–theorems are well-known, as we will see below. However, even in this simple case, this study reveals an interesting link between the constant drift parameter of the Brownian motion, the constant negative curvature of the hyperbolic plane and the non-extensivity parameter $q$ that characterizes the entropy.
II. THE CLASSICAL ENTRepIC APPROACH TO THE LINEAR FOKkER-PLANCK EQUATION

A. General approach

In the case of a system described by a univariate p.d.f. $f_t(x)$ that verifies the linear Fokker-Planck equation

$$\frac{\partial f_t(x)}{\partial t} = -\frac{\partial}{\partial x} (K(x) f_t(x)) + \frac{\partial^2}{\partial x^2} (Q(x) f_t(x)), \quad (II.1)$$

where $K(x)$ is the drift function and $Q(x)$ the diffusion function, it can be shown (see for example [8]) that the relative entropy (or Kullback-Leibler divergence)

$$h(f_t \| g_t) = \int_{\mathbb{R}} f_t(x) \log \frac{f_t(x)}{g_t(x)} \, dx$$

between two any solutions of (II.1) decreases to 0 with time. More precisely, it holds

$$\frac{\partial}{\partial t} h(f_t \| g_t) = - \int_{\mathbb{R}} Q(x) f_t(x) \left( \frac{\partial}{\partial x} \log \frac{f_t(x)}{g_t(x)} \right)^2 \, dx \leq 0 \quad (II.2)$$

since the diffusion function is assumed positive. Thus entropy is decreasing and bounded, so that the limit distributions $f_\infty$ and $g_\infty$ verify

$$\int_{\mathbb{R}} Q(x) f_t(x) \left( \frac{\partial}{\partial x} \log \frac{f_t(x)}{g_t(x)} \right)^2 \, dx = 0,$$

which implies that $f_\infty$ and $g_\infty$ coincide. This proves the unicity of a stationary solution of the Fokker-Planck equation (II.1).

We note that the quantity

$$I(f \| g) = \int_{\mathbb{R}} f \left( \frac{\partial}{\partial x} \log \frac{f(x)}{g(x)} \right)^2 \, dx \quad (II.3)$$

is nothing but the relative Fisher information [10, eq. (174)] between the p.d.f.s $f$ and $g$; thus the integral that appears on the right-hand side of (II.2) is a weighted version of this relative Fisher information, with the diffusion function $Q(x)$ as the weighting function.

In order to deduce an $H-$theorem from this result, we denote as $g_\infty$ the stationary solution to (II.1), assuming that it exists. We then remark that the relative entropy between any solution $f_t$ of (II.1) and the stationary solution $g_\infty$ is related to the Shannon entropy of $f_t$ as

$$h(f_t \| g_\infty) = -h(f_t) - \int_{\mathbb{R}} f_t(x) \log g_\infty(x) \, dx.$$
Thus, provided that the solution $f_t$ verifies *at any time* the constraint

$$\int_{\mathbb{R}} f_t(x) \log g_\infty(x) \, dx = \eta_1$$

(II.4)

where $\eta_1$ is a constant, we deduce that as the relative entropy decreases to 0, the Shannon entropy of the solution $f_t$ increases with time to its maximum value.

We remark that, in a statistical physics framework, the relative entropy $h(f_t \| g_\infty)$ coincides with the free energy. Moreover, taking the limit $t \to +\infty$ in (II.4) shows that the constraint $\eta_1$ is also equal to the negentropy of the stationary solution $g_\infty$, namely

$$\int_{\mathbb{R}} g_\infty(x) \log g_\infty(x) \, dx = -h(g_\infty).$$

(II.5)

**B. The $x$–component of the Brownian motion in the Poincaré half-upper plane**

In [1], Comtet et al. derived the differential equation verified by the $x$–component $X_t$ of the Brownian motion with constant drift $\mu$ and constant diffusion constant $D$ in the Poincaré half-upper plane representation of the hyperbolic plane,

$$\frac{\partial}{\partial t} f_t(x) = D \frac{\partial}{\partial x} \left[ (1 + x^2) \frac{\partial}{\partial x} f(x) + (2\mu + 1) x f_t(x) \right].$$

This is a linear Fokker-Planck equation; in the notations of (II.1), the diffusion function is quadratic and positive, $Q(x) = D (1 + x^2)$ and the drift function is linear, $K(x) = D (1 - 2\mu) x$. Moreover, for a positive drift $\mu$, the asymptotic solution reads

$$g_\infty(x) = A_\mu (1 + x^2)^{-\mu - \frac{1}{2}}$$

(II.6)

with a normalization constant $A_\mu = \frac{\Gamma(\mu + \frac{1}{2})}{\Gamma(\mu) \Gamma(\frac{1}{2})}$. The constraint to be verified by the p.d.f. $f_t$ is thus deduced from the above results as

$$\int_{\mathbb{R}} f_t(x) \log (1 + x^2) \, dx = \eta_1 \forall t > 0.$$  

(II.7)

(note that the normalization constant $\log A_\mu$ and the constant $-\mu - \frac{1}{2}$ that appear in $\log g_\infty$ need not be taken into account).

The value of the constant $\eta_1$ can be easily computed as

$$\eta_1 = \psi \left( \mu + \frac{1}{2} \right) - \psi (\mu)$$

where $\psi$ is the digamma function.
We note that this constraint can be imposed by a simple scaling of the random process $X_t$ since the function
\[ a \mapsto \int f_t(x) \log \left(1 + ax^2\right) dx \]
is a bijection from $\mathbb{R}^+$ to $\mathbb{R}^+$. We also remark that this scaling does not require the existence of a variance for $X_t$; for example, the Cauchy p.d.f.
\[ f_C(x) = \frac{1}{\pi \left(1 + x^2\right)}, \quad x \in \mathbb{R} \]
- which is the asymptotic distribution of the Brownian motion without drift ($\mu = 0$) on the hyperbolic plane - has an infinite variance but finite nonlinear moment
\[ \int f_C(x) \log \left(1 + x^2\right) dx = 2 \log 2. \]

We deduce the following theorem.

**Theorem.** The Shannon entropy of the $x-$component, normalized according to (II.7), of the Brownian motion on the Poincaré half-upper plane representation of the hyperbolic plane increases over time.

### III. A GENERALIZATION TO THE TSALLIS ENTROPIES

#### A. General approach

The monotone behavior of the relative Shannon entropy between any two solutions $f_t$ and $g_t$ of the linear Fokker-Planck equation (II.1) has been extended by Shiino [7] to the case of the relative Tsallis entropy, defined as
\[
h_q(f_t\|g_t) = \frac{1}{q-1} \left( \int f^q(x)g^{1-q}(x) \, dx - 1 \right).
\]
More precisely, the derivative with respect to time of this relative entropy verifies [11]
\[
\frac{\partial}{\partial t} h_q(f_t\|g_t) = -q \int D(x) f_t(x) \left( \frac{f_t(x)}{g_t(x)} \right)^q \left( \frac{\partial}{\partial x} \log \frac{f_t(x)}{g_t(x)} \right)^2 \, dx \leq 0, \quad \forall q > 0. \tag{III.1}
\]
We note that this inequality holds for any positive value of $q$ and simplifies to (II.2) as $q \to 1$. Moreover, the right-hand side integral in (III.1) can be considered as a $q-$version of the relative Fisher information that appears in (II.2).
In order to deduce from this monotonicity an $H_q$-theorem, we need the additional assumption that the stationary solution $g_\infty$ can be written under the form

$$g_\infty(x) = C_q (1 + U(x))^{\frac{1}{1-q}}$$

for some specific value $q = q_*$ of the non-extensivity parameter. This assumption means that the p.d.f. $g_\infty$ is itself a maximum $q_*$-entropy pdf with constraint

$$\int g_\infty^{\prime q_*}(x) U(x) \, dx = \eta_{q_*} \quad \text{(III.2)}$$

for some constant value $\eta_{q_*}$. We refer the reader to [6] for the conditions on the diffusion and drift functions of the Fokker-Planck equation that ensure the validity of this assumption. In this case, we have

$$h_{q_*}(f_t \| g_\infty) = -C_{q_*}^{1-q_*} h_{q_*}(f_t) - \frac{C_{q_*}^{1-q_*}}{1-q_*} \int f_t^{q_*}(x) U(x) \, dx - \beta_{q_*}$$

where $\beta_{q_*} = -\frac{A_{q_*}^{1-q_*}}{1-q_*}$ is a constant. Thus, provided that the constraint

$$\int f_t^{q_*}(x) U(x) \, dx = \eta_{q_*}$$

is met at all times, we deduce that the $q_*$-entropy is increasing with time and reaches asymptotically the entropy of the stationary pdf $g_\infty$.

**B. The $x$-component of the Brownian motion in the Poincaré half-upper plane**

In the special case of the $x$-component $X_t$ of the Brownian motion in the Poincaré half-upper plane, the stationary solution (II.6) verifies the condition (III.2) with the value $q_*$ such that $-\mu - \frac{1}{2} = \frac{1}{1-q_*}$ and the function $U(x) = x^2$, namely

$$\frac{1}{1-q} \int x^2 f_t^{q_*}(x) \, dx = \eta_{q_*} \quad \text{(III.3)}$$

with

$$q_* = \frac{2\mu + 3}{2\mu + 1}, \quad 1 < q_* < 3. \quad \text{(III.4)}$$

As a consequence, since the $q$-relative entropy is decreasing, the $q$-entropy is increasing under the conditions that $q$ is chosen equal to $q_*$ as in (III.4) and that the constraint (III.3) is verified.
Moreover, by taking the limit as \( t \to +\infty \) in (III.3), the constraint \( \eta_{q_*} \) in (III.3) is computed as

\[
\eta_{q_*} = \frac{1}{1-q} \int_{\mathbb{R}} x^2 g_{q_*}^\infty (x) \, dx.
\]

The value of this constraint is

\[
\eta_{q_*} = \pi^{\frac{q_*}{2}} (q_* - 1) \left( \frac{\Gamma \left( \frac{q_* - 3}{2q_* - 2} \right)}{\Gamma \left( \frac{1}{q_* - 1} \right)} \right)^{q_*}.
\]

We note that this constraint can always be imposed by scaling: denoting \( \sigma^2_q(X_t) = (1-q)^{-1} \int x^2 f_t^q (x) \, dx \) the value of this constraint and \( \sigma^2_q(Y) \) the one corresponding to \( Y = aX \) with \( a > 0 \), then

\[
\sigma^2_q(Y) = a^{3-q} \sigma^2_q(X).
\]

We deduce the following result.

**Theorem.** With \( q = q_* \) as in (III.4), the \( q \)–entropy of the Brownian motion, normalized according to (III.3), is increasing with time.

**IV. NUMERICAL ILLUSTRATION**

The following figures depict ten realizations of a discretized version of this Brownian motion, with parameters \( D = 0.01, m = 0 \) and \( \mu = 1 \) on Figure 1 while \( D = 0.01, m = 3 \) and \( \mu = 7 \) on Figure. In both cases, the process starts from the point \( x = 0, y = 1 \) and the superimposed thick curve is the asymptotic probability density of its \( x \)–component. Without external drift (Figure 1), the random process wanders for a long time far away from the real axis before "falling” on it. The distribution of the ”landing points” on the real axis is thus very wide, in fact a Lorentz distribution with infinite variance. With an external drift (Figure 2), the process is forced to walk in the direction of the real axis, so that the landing points are more concentrated around 0, what is reflected by their narrow distribution, a \( q \)–Gaussian distribution with variance \( \sigma^2 = 0.2 \).
Figure 1: ten realizations of a discretized version of the Brownian motion on the hyperbolic plane, with parameters $D = 0.01$, $m = 0$ and $\mu = 1$

Figure 2: ten realizations of a discretized version of the Brownian motion on the hyperbolic plane, with parameters $D = 0.01$, $m = 3$ and $\mu = 7$

V. THE BROWNIAN MOTION IN THE UNIT DISK

Another representation of the hyperbolic space is the unit disk $\mathbb{D} = \{ w = re^{i\theta}, r \leq 1, 0 \leq \theta < 2\pi \}$ with the metric in polar coordinates

$$ds^2 = \frac{4}{(1 - r^2)^2} (dr^2 + r^2 d\theta^2).$$

There is a conformal mapping between the Poincaré upper half-plane $\mathbb{H}$ =
\[ z = x + iy, \ y > 0 \] and the unit disk \( \mathbb{D} \) defined as

\[ w = \frac{iz + 1}{z + i}. \]

Comtet et al show that the density of the radial component of the Brownian motion in the unit disk representation of the hyperbolic space converges to \( \delta (r - 1) \) as \( t \to +\infty \). Having no explicit Fokker-Planck for the radial part \( \theta_t \) of this process, we were unable to prove a corresponding \( H^- \) theorem. However, we show here that we can use the maximum entropy approach to derive the asymptotic distribution of the angular part \( \theta_t \), using the following result:

**Theorem.** If the random variable \( X \) has maximum entropy under the log-constraint \( E \log (1 + X^2) = \gamma \), then the random variable

\[ \tilde{X} = \frac{X}{\sqrt{1 + X^2}} \]

has maximum entropy under the constraint \( E \log \left( 1 - \tilde{X}^2 \right) = -\gamma \).

More precisely, if the p.d.f. of \( X \) reads

\[ f_X(x) = \frac{\Gamma \left( \mu + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \right) \Gamma (\mu)} \left( 1 + x^2 \right)^{-\mu - \frac{1}{2}}, \ x \in \mathbb{R} \]

then \( E \log (1 + X^2) = \psi \left( \mu + \frac{1}{2} \right) - \psi (\mu) \) and the p.d.f. of \( \tilde{X} \) reads

\[ f_{\tilde{X}}(\tilde{x}) = \frac{\Gamma \left( \mu + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \right) \Gamma (\mu)} \left( 1 - \tilde{x}^2 \right)^{\mu - 1}, \ -1 \leq \tilde{x} \leq 1 \]

with \( E \log \left( 1 - \tilde{X}^2 \right) = -E \log (1 + X^2) = \psi (\mu) - \psi \left( \mu + \frac{1}{2} \right) \).

Since in the asymptotic regime, \( f_r(r) = \delta (r - 1) \), the conformal mapping becomes

\[ \cos \theta = \frac{2X}{1 + X^2}, \ \sin \theta = \frac{X^2 - 1}{X^2 + 1}, \]

it can be easily verified that

\[ \cos \left( \frac{\theta}{2} - \frac{\pi}{4} \right) = \frac{X}{\sqrt{1 + X^2}} = \tilde{X} \]

so that a simple change of variable yields

\[ \theta \sim (1 - \sin \theta)^{\mu - 1} \]

as obtained in [1].

The asymptotic distribution of the angular part has thus maximum Tsallis entropy with parameter \( \tilde{q} \) such that

\[ \tilde{q} = \frac{\mu - 2}{\mu - 1} < 1. \]
VI. CONCLUSION

We have shown how to use the monotonicity of the Shannon or Tsallis relative entropies to deduce an $H$–theorem for the Shannon and Tsallis entropy, first in the general case and then in the case of the Brownian motion on the hyperbolic plane. Three remarkable results have been observed in this study: first, the natural drift induced by the negative constant curvature of the hyperbolic plane transforms the asymptotically Gaussian of the usual Brownian motion on the plane to the Cauchy distribution, which belongs to the extended family of Tsallis distributions. Secondly, the addition of a constant positive external drift transforms this Cauchy behavior to a non-extensive behavior with non-extensivity parameter directly related to the value of this drift. This appearance of non-extensive distributions in the context of an underlying curved space remains to be linked to physically relevant experiments and data. At last, the Tsallis distributions with $q > 1$ on the Poincaré half upper-plane realization of the hyperbolic plane transform, via the conformal mapping, into Tsallis distributions with $q < 1$ on the unit disk realization of the hyperbolic plane.

VII. ANNEX: PROOF OF SHIINO’S RESULT

Following Shiino’s notations, we consider

$$D_q(f_t \parallel g_t) = \int_{\mathbb{R}} f_t(x)^q g_t(x)^{1-q} dx$$

and omit the time and space variables for readability. The time derivative reads

$$\frac{\partial}{\partial t} D_q(f \parallel g) = q \int f^{q-1} \left( \frac{f}{g} \right)^{1-q} \left( -\frac{\partial}{\partial x} (Kf) + \frac{\partial^2}{\partial x^2} (Qf) \right) dx + (1 - q) \int f^{q-1} \left( \frac{f}{g} \right)^{1-q} \left( -\frac{\partial}{\partial x} (Kg) + \frac{\partial^2}{\partial x^2} (Qg) \right) dx.$$ 

Since $f$ and $g$ are both solutions of the Fokker-Planck equation (III.1), we deduce

$$\frac{\partial}{\partial t} D_q (f \parallel g) = q \int \left( \frac{f}{g} \right)^{q-1} \left( -\frac{\partial}{\partial x} (Kf) + \frac{\partial^2}{\partial x^2} (Qf) \right) dx + (1 - q) \int \left( \frac{f}{g} \right)^{q-1} \left( -\frac{\partial}{\partial x} (Kg) + \frac{\partial^2}{\partial x^2} (Qg) \right) dx.$$ 

The integrals with the drift function $K(x)$ can be integrated by parts, yielding respectively

$$-q \int \left( \frac{f}{g} \right)^{q-1} \frac{\partial}{\partial x} (Kf) = +q (q - 1) \int \left( \frac{f}{g} \right)^{q-2} \frac{\partial}{\partial x} \left( \frac{f}{g} \right) Kf.$$ 

\[10\]
and
\[-(1-q) \int \left( \frac{f}{g} \right)^q \frac{\partial}{\partial x} (Kg) = q(1-q) \int \left( \frac{f}{g} \right)^{q-1} \frac{\partial}{\partial x} \left( \frac{f}{g} \right) Kg \]
\[= q(1-q) \int \left( \frac{f}{g} \right)^{q-2} \frac{\partial}{\partial x} \left( \frac{f}{g} \right) Kf \]
so that their sum vanishes.

The integrals with the diffusion function \( Q(x) \) are also integrated by parts according respectively to
\[q \int \left( \frac{f}{g} \right)^{q-1} \frac{\partial^2}{\partial x^2} (Qf) = -q \int \frac{\partial}{\partial x} \left( \frac{f}{g} \right)^{q-1} \frac{\partial}{\partial x} (Qf) \]
\[= -q(q-1) \int \left( \frac{f}{g} \right)^{q-2} \frac{\partial}{\partial x} \left( \frac{f}{g} \right) \frac{\partial}{\partial x} (Qf) \]
\[= -q(q-1) \int \left( \frac{f}{g} \right)^{q-1} \frac{\partial}{\partial x} \left( \log \frac{f}{g} \right) \frac{\partial}{\partial x} (Qf) \]
\[= -q(q-1) \int Q \left( \frac{f}{g} \right)^{q-1} \frac{\partial}{\partial x} \left( \log \frac{f}{g} \right) \frac{\partial f}{\partial x} \]
and
\[(1-q) \int \left( \frac{f}{g} \right)^q \frac{\partial^2}{\partial x^2} (Qg) = -(1-q) \int \frac{\partial}{\partial x} \left( \frac{f}{g} \right)^q \frac{\partial}{\partial x} (Qg) \]
\[= -(1-q) q \int \left( \frac{f}{g} \right)^{q-1} \frac{\partial}{\partial x} \left( \frac{f}{g} \right) \frac{\partial}{\partial x} (Qg) \]
\[= -(1-q) q \int g \left( \frac{f}{g} \right)^{q-1} \frac{\partial}{\partial x} \left( \frac{f}{g} \right) \frac{\partial Q}{\partial x} \]
\[-(1-q) q \int Q \left( \frac{f}{g} \right)^{q-1} \frac{\partial}{\partial x} \left( \frac{f}{g} \right) \frac{\partial g}{\partial x} \]
Their sum consists in an integral with the diffusion function \( Q \)
\[(q-1) q \int Q \left( \frac{f}{g} \right)^{q-1} \left( \frac{\partial}{\partial x} \left( \frac{f}{g} \right) \frac{\partial g}{\partial x} - \frac{\partial}{\partial x} \left( \log \frac{f}{g} \right) \frac{\partial f}{\partial x} \right) \]
and an integral with its derivative
\[q(1-q) \int \left( \frac{f}{g} \right)^{q-1} \left( f \frac{\partial}{\partial x} \left( \log \frac{f}{g} \right) - g \frac{\partial}{\partial x} \left( \frac{f}{g} \right) \right) \frac{\partial Q}{\partial x} \]
The second integral is easily seen to vanish, while the first one can be simplified to
\[(q-1) q \int Q \left( \frac{f}{g} \right)^{q-1} \left( \frac{\partial}{\partial x} \left( \frac{f}{g} \right) \frac{\partial g}{\partial x} - \frac{\partial}{\partial x} \left( \log \frac{f}{g} \right) \frac{\partial f}{\partial x} \right) = -q(q-1) \int Q f \left( \frac{\partial}{\partial x} \log \frac{f}{g} \right)^2 . \]
Thus the Tsallis divergence $h_q(f\|g) = \frac{1}{q-1}D_q(f\|g)$ verifies the stated equality.

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[11] The proof of this result is omitted in [7]; we provide it in the annex for the interested reader.