Comments on behaviour of solutions of elliptic quasi-linear problems in a neighbourhood of boundary singularities

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Abstract: We have investigated the behaviour of solutions of elliptic quasi-linear problems in a neighbourhood of boundary singularities in bounded and unbounded domains. We found exponents of the solution’s decreasing rate near the boundary singularities.

Keywords: Elliptic equations, Quasi-linear problems, Boundary singularities

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1 Introduction

This is a brief description of results of [1–3] with comments and improvements, which were presented at the International conference on differential equations dedicated to the 110th anniversary of Ya. B. Lopatynsky. In these articles we have investigated the behaviour of solutions of quasi-linear elliptic problems in a neighborhood of boundary singularities in bounded and unbounded domains. We have found exponents of the solution’s decreasing rate of the type $j_{u_0}/|x|^\alpha$, near the boundary singularities.

Let $G \subset \mathbb{R}^n$ be an unbounded domain (see Fig. 1) with boundary $\partial G$ that is a smooth surface everywhere except at the origin $O$ and near $O$ it is a conical surface, $n \geq 2$. We assume that $G = G^R \cup G_R$, where $G^R = G^d_0 \cup G^R_0$ and $G^d_0$ is a rotational cone with the vertex at $O$ and the aperture $\omega_0 \in (0, \pi)$, $d \ll 1$, $G_R = \{x = (r, \omega) \in \mathbb{R}^n \mid r \in (R, \infty), \omega \in \Omega \subset S^{n-1}, n \geq 2\}, R \gg 1, S^{n-1}$ is the unit sphere.

We introduce the following notations for a domain $G$ which has a boundary conical point:

- $\Omega := G \cap S^{n-1}$, where $S^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$;
- $\partial \Omega$: the boundary of $\Omega$;
- $G^d_0 := G \cap \{(r, \omega) : 0 \leq r < b, \omega \in \Omega\}$: a layer in $\mathbb{R}^n$;
- $\Gamma^d_0 := \partial G \cap \{(r, \omega) : 0 \leq r < b, \omega \in \partial \Omega\}$: the lateral surface of $G^d_0$;
- $G^d := G \setminus G^d_0$, $\Gamma^d := \partial G \setminus \Gamma^d_0$.

We use standard function spaces: $C^k(\overline{G})$ with the norm $|u|_{k, \partial G}$; $L_p(G)$ with the norm $\|u\|_{p, G}$, $p \geq 1$; the Sobolev space $W^{k, p}(G)$ with the norm $\|u\|_{W^{k, p}(G)}$ for integer $k \geq 0$; the weighted Sobolev-Kondratiev space $V^{k, \alpha}_p(G)$ for
integer $k \geq 0$, $1 < p < \infty$ and $\alpha \in \mathbb{R}$ with the finite norm $\|u\|_{V^k_p\alpha(G)} = \left( \int_G \sum_{|\beta| \leq k} r^{\alpha + p(|\beta| - k)} |D^\beta u|^p \, dx \right)^{\frac{1}{p}}$

and the space $V^{k-\frac{1}{p}}_{p,\alpha}(\Gamma)$ of functions $\varphi$, given on $\partial G$, with the norm $\|\varphi\|_{V^{k-\frac{1}{p}}_{p,\alpha}(\partial G)} = \inf \|\varphi\|_{V^k_{p,\alpha}(G)}$, where the infimum is taken over all functions $\varphi$ such that $\varphi|_{\partial G} = \varphi$ in the sense of traces.

### 2 Oblique derivative problem

In [1] we have investigated the behaviour of strong solutions to the oblique derivative problem for the general second-order quasi-linear elliptic equation in a neighbourhood of a conical boundary point of an $n$-dimensional bounded domain, $n \geq 2$. In the case of the linear equation we refer to [4, 5].

Local maximum principle for strong solutions to the elliptic quasi-linear oblique derivative problem in convex rotational cones has been obtained by Lieberman in [6]. He [7] and Trudinger [8] have obtained local gradient bound estimate and local Hölder gradient estimate of strong solutions in any sub-domain with a $C^2$ boundary portion of the domain.

The results obtained in [1] are a generalization and improvement of results of [9] on the case of the oblique boundary condition. We consider the oblique derivative problem for the elliptic second-order linear equation:

\[
\begin{aligned}
& a^{ij}(x, u, u_x)u_{x_i}u_{x_j} + a(x, u, u_x) = 0, \quad a^{ij} = a^{ji}, \quad x \in G^R_0, \\
& \frac{\partial u}{\partial \mathbf{n}} + \chi(\omega) \frac{\partial u}{\partial r} + \frac{1}{|\mathbf{x}|} \gamma(\omega) u = g(x), \quad x \in \partial G^R_0 \setminus \mathcal{O},
\end{aligned}
\]  

\begin{equation}
(QL)
\end{equation}

where $\mathbf{n}$ denotes the unit exterior normal vector to $\partial G^R_0 \setminus \mathcal{O}$, $(r, \omega)$ are spherical coordinates in $\mathbb{R}^n$ with pole $\mathcal{O}$; repeated indices are understood as summation from 1 to $n$.

**Definition 2.1.** A function $u$ is called a strong solution of problem $(QL)$ if $u \in W^{2,q}_{loc}(G^R_0 \setminus \mathcal{O}) \cap W^{1}(G^R_0) \cap C^0\left(G^R_0 \setminus \mathcal{O}\right)$, $q > n$ and satisfies the equation for almost all $x \in G^R_0$ for all $\varepsilon > 0$ as well as the boundary condition in the sense of traces on $\partial G^R_0 \setminus \mathcal{O}$. We assume that $M_0 = \max_{x \in G^R_0} |u(x)|$ is known (see e.g. Theorem 13.1 [7]).

### 2.1 Assumptions

Let $\mathcal{M} = \{(x, u, z) : x \in G^R_0, u \in \mathbb{R}, z \in \mathbb{R}^n\}$. With regard to problem $(QL)$ we assume that the following conditions are satisfied on $\mathcal{M}$:

**A** $a^{ij}(x, u, z) \in W^{1,q}(\mathcal{M})$, $q > n$; $a(x, u, z)$, $\frac{\partial a(x, u, z)}{\partial u}$ are Caratheodory functions;
(B) the uniform ellipticity

\[ \forall \xi \in \mathbb{R}^n, \quad |a(x, u, z)\xi_i \xi_j| \leq \mu |\xi|^2, \]

with the ellipticity constants \( \mu \geq \nu > 0; \quad a^{ij}(0, 0, 0) = \delta^i_j, \quad i, j = 1, \ldots, n \) - the Kronecker symbol;

(C) \( \frac{\partial u}{\partial n} \leq 0; \)

(D) \( \gamma(\omega), \chi(\omega) \in C^1(\Omega); \) there exist numbers \( s > 1, \chi_0 \geq 0, \gamma_0 > \frac{\tan \alpha_0}{\pi} \geq 0 \) and \( \gamma_1 \geq \gamma(\omega) \geq \gamma_0 > 0, \quad 0 \leq \chi(\omega) \leq \chi_0 \) as well as nonnegative constants \( \mu_1, k_1, g_0, g_1 \) and functions \( b(x), f(x) \in L^q_{\text{loc}}(G), q \geq n \) such that the inequalities

\[ |a(x, u, z) + \frac{\partial a(x, u, z)}{\partial u}| \leq \mu_1 |z|^2 + b(x)|z| + f(x), \]

\[ 0 \leq b(x), f(x) \leq k_1 |x|^s - 2, \quad |g(x)| \leq g_0 |x|^{s-1}, \quad |\nabla g| \leq g_1 |x|^{s-2}, \]

hold;

(E) coefficients of problem \((QL)\) satisfy such conditions that guarantee \( u \in C^{1+\overline{\nu}}(G') \) and the existence of the local a priori estimate

\[ |u|_{1+\overline{\nu}, G'} \leq M_1, \quad \overline{\nu} \in (0, 1). \]

for any smooth \( G' \subset \subset G_0^R \setminus \partial \) (see Theorems 13.13 and 13.14 [7]).

2.2 The main result

Theorem 2.2 [11]. Let \( u \) be a strong solution of problem \((QL)\) and \( \lambda > 1 \) be the smallest positive eigenvalue of the following eigenvalue problem for the Laplace - Beltrami operator \( \Delta_{\omega} \) on the unit sphere

\[ \begin{cases} \Delta_{\omega} \psi + \lambda(\lambda + n - 2)\psi(\omega) = 0, & \omega \in \Omega, \\ \frac{\partial \psi}{\partial \nu} + (\lambda \chi(\omega) + \gamma(\omega)) \psi(\omega) = 0, & \omega \in \partial \Omega. \end{cases} \quad (EVP) \]

where \( \nu \) is the unit exterior normal to \( \partial G_0^1 \) at the points of \( \partial \Omega \). Suppose that assumptions (A) - (E) are satisfied. Then there exist numbers \( d > 0, c_0, c_1 \) independent of \( u \) such that

(1)

\[ |u(x)| \leq c_0 \left\{ \begin{array}{ll} |x|^{s} & \text{if } s > \lambda, \\ |x|^{s} \ln^{3} \frac{1}{|x|} & \text{if } s = \lambda, \\ |x|^{s} & \text{if } s < \lambda, \end{array} \right. \quad x \in G_0^d, \]

(2)

\[ |\nabla u(x)| \leq c_1 \left\{ \begin{array}{ll} |x|^{s-1} & \text{if } s > \lambda, \\ |x|^{s-1} \ln^{3} \frac{1}{|x|} & \text{if } s = \lambda, \\ |x|^{s-1} & \text{if } s < \lambda, \end{array} \right. \quad x \in G_0^d. \]

- If \( 4 + \lambda(\lambda - 2) > 0 \) then \( u(x) \in V_{q,4-n}^2(G_0^d) \) and there exist numbers \( d > 0, c_2 \) independent of \( u \) such that

(3)

\[ \|u(x)\|_{V_{q,4-n}^2(G_0^d)} \leq c_2 \left\{ \begin{array}{ll} e^{s-2} \frac{1}{d} & \text{if } s > \lambda, \\ e^{\lambda-2} \frac{1}{d} \ln^{3} \frac{1}{|x|} & \text{if } s = \lambda, \\ e^{s-2} \frac{1}{d} & \text{if } s < \lambda, \end{array} \right. \]

and

- if \( 1 < \lambda < 2, q > \frac{n}{\lambda - 1} \) then

(4)

\[ u(x) \in C^2(G_0^d) \] if \( s \geq \lambda, \)

\[ u(x) \in C^2(G_0^d) \] if \( s < \lambda. \]
Theorem 2.3 ([4, 5]). There exists the smallest positive eigenvalue $\lambda$ of problem (EVP), which satisfies the following inequalities

$$0 < \lambda < \sqrt{\left(\frac{\pi}{\omega_0}\right)^2 + \left(\frac{n-2}{2}\right)^2} - \frac{n-2}{2}$$

for $n \geq 3$.

### 3 Quasi-linear nonlocal Robin problem

In [3] we have investigated the behaviour of weak solutions for the nonlocal Robin problem with quasi-linear elliptic divergence second-order equations in a plain domain in a neighbourhood of the boundary corner point $O$. In the case of the linear equation we refer to [10, 11].

Here, we consider a different eigenvalue problem (see (EVP2)) and derive a new Friedrichs-Wirtinger type inequality adapted to the quasi-linear elliptic problem considered in [3]. It allows us to improve the main result of [3] (see Theorem 3.7): the exponent of weak solution behavior (see inequality (20)) in a neighborhood of an angular point $O$ in the case $B < 0$ is better than that one obtained in [3].

We consider the type of nonlocal problems, where the support of nonlocal terms intersects the boundary. Namely, the situation in which a part $\Gamma$ of domain boundary $\partial G$ is mapped by transformation $\gamma$ on $\gamma(\Gamma)$ and $\gamma(\Gamma) \cap \partial G \neq \emptyset$.

Let us consider the domain $G^R_0 \subset \mathbb{R}^2$. Moreover, let $\Gamma^+_+ \text{ and } \Gamma^-_-$ be the part of boundary $\partial G^R_0$ for which $x_2 > 0$ and $x_2 < 0$ respectively. We assume, that $\partial G^R_0 = \Gamma^+_+ \cup \Gamma^-_-$ is a smooth curve everywhere except at the origin $O \in \partial G^R_0$ and near the point $O$ curves $\Gamma^\pm$ are lateral sides of an angle with the measure $\omega_0 \in [0, \pi)$ and the vertex at $O$. Let $\Sigma_0 = G^R_0 \cap \{x_2 = 0\}$, where $O \in \Sigma_0$. Furthermore, let $\gamma$ be a diffeomorphism mapping of $\Gamma^+_+$ onto $\Sigma_0$. Additionally, we suppose that there exists $d > 0$ such that in the neighbourhood of $\Gamma^d_0$, the mapping $\gamma$ is the rotation about the origin $O$ and $-\omega_0/2$ is the angle of rotation. It means $\gamma(\Gamma^d_0) = \Sigma_0 = G^d_0 \cap \Sigma_0$.

We have considered a quasi-linear elliptic equation with the nonlocal boundary condition connecting the values of the unknown function $u$ on the boundary part $\Gamma^+_+$ with its values on $\Sigma_0$:

$$
\begin{align*}
-\frac{d}{dx_1}(u^q |\nabla u|^{m-2} u_{x_1}) + a_0 r^{-m} u |q^+ u|^{q+m-2} - \mu u |u^{q-2}| \nabla u^m &= f(x), & x \in G^R_0; \\
|u|^q |\nabla u|^{m-2} &\frac{\partial u}{\partial n} + \beta_+ r^{m-1} u |u^{q+m} = \gamma(\gamma(x))|u(\gamma(x))|^{q+m-2} = g(x, u), & x \in \Gamma^+_+; \\
|u|^q |\nabla u|^{m-2} &\frac{\partial u}{\partial n} + \beta_- r^{-m} u |u^{q-m} = h(x, u), & x \in \Gamma^-_-. 
\end{align*}
$$

(QL2)

Here $q \geq 0$, $m > 1$, $\mu \geq 0$, $a_0 \geq 0$, $\beta_+ > 0$, $b \geq 0$ are given numbers and $\tilde{n}$ denotes the unit outward with respect to $G^R_0$ normal to $\partial G^R_0 \setminus \{O\}$.

Recall that we are dealing with the nonlocal problem and the boundary $\partial G^R_0$ is non smooth, so the formulation of problem (QL2) does not make sense in general. To make our formulation precise, we give the definition of the weak solution of (QL2).

**Definition 3.1.** A function $u$ is called a weak solution of problem (QL2) provided that $u \in C^0(\overline{G^R_0}) \cap V^1_{m,0}(G^R_0)$ and for all functions $\eta \in C^0(\overline{G^R_0}) \cap V^1_{m,0}(G^R_0)$ the following integral identity

$$
\int_G |u|^q |\nabla u|^{m-2} u_{x_i} \eta_{x_i} dx + \int_G \left[ a_0 \frac{1}{r^m} u |u^{q+m-2} - \mu u |u^{q-1}| \nabla u^m sgn u \right] \eta(x) dx
+ \beta_+ \int_{\Gamma^+_+} \frac{1}{r^{m-1}} u |u^{q+m-2} \eta(x) ds + b \int_{\Gamma^-_-} \frac{1}{r^{m-1}} u(\gamma(x)) |u(\gamma(x))|^{q+m-2} \eta(x) ds
$$

for all $\eta \in C^0(\overline{G^R_0}) \cap V^1_{m,0}(G^R_0)$.
+ \beta_- \int_{\Gamma_-} \frac{1}{r^{m-1}} |u|^{\eta + m-2} \eta(x) ds = \int_G f(x) \eta(x) dx + \int_{\Gamma_+} g(x, u) \eta(x) ds + \int_{\Gamma_-} h(x, u) \eta(x) ds

is satisfied.

### 3.1 The Friedrichs-Wirtinger type inequality

We consider the following eigenvalue problem:

\begin{equation}
\begin{aligned}
&\left( |\psi'(\omega)|^{m-2} \psi'(\omega) \right)' + \partial |\psi(\omega)|^{m-2} \psi(\omega) = 0, \quad \omega \in \Omega \\
&\left| \psi'\left( \frac{\alpha \omega}{2} \right) \right|^{m-2} \psi'\left( \frac{\alpha \omega}{2} \right) + \beta_+ \left| \psi\left( \frac{\alpha \omega}{2} \right) \right|^{m-2} \psi\left( \frac{\alpha \omega}{2} \right) + b |\psi(0)|^{m-2} \psi(0) = 0 \quad \text{(EVP2)}
\end{aligned}
\end{equation}

with $m \geq 2$, $\beta_+ > 0$, $b > 0$, which consists in determining all values $\partial$ (eigenvalues) for which (EVP2) has nonzero weak solutions (eigenfunctions) $\psi(\omega)$.

**Definition 3.2.** A function $\psi$ is called a weak solution of problem (EVP2) provided that $\psi \in W^{1,m}(\Omega) \cap C^0(\overline{\Omega})$ and satisfies the integral identity

\begin{equation}
\int_\Omega \left( |\psi'(\omega)|^{m-2} \psi'(\omega) \eta'(\omega) - \partial |\psi(\omega)|^{m-2} \psi(\omega) \eta(\omega) \right) d\omega + \beta_+ \left| \psi\left( \frac{\alpha \omega}{2} \right) \right|^{m-2} \psi\left( \frac{\alpha \omega}{2} \right) \eta\left( \frac{\alpha \omega}{2} \right)
\end{equation}

\begin{equation}
+ \beta_- \left| \psi\left( \frac{-\alpha \omega}{2} \right) \right|^{m-2} \psi\left( \frac{-\alpha \omega}{2} \right) \eta\left( \frac{-\alpha \omega}{2} \right) + b |\psi(0)|^{m-2} \psi(0) \eta\left( \frac{\alpha \omega}{2} \right) = 0
\end{equation}

for all $\eta(\omega) \in W^{1,m}(\Omega) \cap C^0(\overline{\Omega})$.

**Theorem 3.3** (Friedrichs-Wirtinger's type inequality). Let $\Omega \subset S$ be an arc, $m \geq 2$, $\partial$ be the eigenvalue of problem (EVP2) and $\psi \in W^{1,m}(\Omega) \cap C^0(\overline{\Omega})$ be the corresponding eigenfunction. Then for any $u \in W^{1,m}(\Omega) \cap C^0(\overline{\Omega})$, $u \not\equiv \text{const} \neq 0$ the inequality

\begin{equation}
\partial \int_\Omega |u(\omega)|^m d\omega \leq \int_\Omega |u'(\omega)|^m d\omega + B u\left( \frac{\alpha \omega}{2} \right)^m + \beta_- u\left( \frac{-\alpha \omega}{2} \right)^m \quad \text{(F - W)}
\end{equation}

holds, where

\begin{equation}
B = b \psi(0) |\psi(0)|^{m-2} \psi\left( \frac{\alpha \omega}{2} \right) \psi\left( \frac{\alpha \omega}{2} \right)^{-m} + \beta_+.
\end{equation}

**Proof.** Let us first derive the estimate (F - W) for functions $u \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ and $\psi \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. Setting $u(\omega) = \psi(\omega) v(\omega)$ we obtain

\begin{equation}
|u'(\omega)|^m = |(\psi(\omega) v(\omega))'|^m = \left[ (v'(\omega) \psi(\omega) + v(\omega) \psi'(\omega))^2 \right]^\frac{m}{2}
\end{equation}

\begin{equation}
= \left[ \psi^2(\omega) v^2(\omega) + 2 \psi'(\omega) \psi(\omega) v'(\omega) v(\omega) + \psi^2(\omega) v^2(\omega) \right]^\frac{m}{2}
\end{equation}

\begin{equation}
|v(\omega)|^m |\psi'(\omega)|^m \left[ 1 + \frac{\psi'(\omega) v'(\omega)}{\psi(\omega) v(\omega)} + \frac{\psi^2(\omega) v^2(\omega)}{\psi(\omega) v(\omega) v^2(\omega)} \right]^\frac{m}{2}.
\end{equation}

By Bernoulli inequality $(1 + x)^\alpha \geq 1 + \alpha x$, with $\alpha \geq 1$ and $x > -1$ from (6) we have

\begin{equation}
|u'(\omega)|^m \geq |v(\omega)|^m |\psi'(\omega)|^m \left( 1 + \frac{m}{2} \left[ \frac{v'(\omega) \psi(\omega)}{\psi(\omega) v(\omega)} + \frac{\psi^2(\omega) v^2(\omega)}{\psi(\omega) v(\omega) v^2(\omega)} \right] \right)
\end{equation}

\begin{equation}
\geq |v(\omega)|^m |\psi'(\omega)|^m + m |v(\omega)|^{m-2} v(\omega) v'(\omega) \psi(\omega) v(\omega) \psi'(\omega) |\psi'(\omega)|^{m-2}.
\end{equation}
Next, in virtue of \((|v(\omega)|^m)' = m|v(\omega)|^{m-1} v(\omega)v'(\omega)\) and from \((EVP2)\) it follows that:

\[
\left( |v(\omega)|^m \psi'(\omega) |\psi'(\omega)|^{m-2} \right)' = (|v(\omega)|^m)' \psi(\omega) |\psi'(\omega)|^{m-2} \\
+ |v(\omega)|^m |\psi'(\omega)|^m + |v(\omega)|^m \psi(\omega) \left( |\psi'(\omega)|^{m-2} \psi'(\omega) \right)' \\
m|v(\omega)|^{m-2} v(\omega) |\psi'(\omega)|^m |\psi'(\omega)|^{m-2} + |v(\omega)|^m |\psi'(\omega)|^m \\
+ |v(\omega)|^m \psi(\omega) \left( |\psi'(\omega)|^{m-2} \psi'(\omega) \right)' = |v(\omega)|^m |\psi'(\omega)|^m \\
m|v(\omega)|^{m-2} v(\omega) |\psi'(\omega)|^m |\psi'(\omega)|^{m-2} - \vartheta |v(\omega)|^m |\psi(\omega)|^m.
\]

(8)

Now, from (7), (8) and definition of function \(u(\omega)\) we get the inequality

\[
\vartheta |u(\omega)|^m \leq |u'(\omega)|^m - \left( |v(\omega)|^m \psi(\omega) |\psi'(\omega)|^{m-2} \right)'.
\]

(9)

Integrating (9) over \(\Omega\) and since \(\psi(\omega)\) is a solution of \((EVP2)\) we obtain

\[
\vartheta \int_{\Omega} |u(\omega)|^m d\omega \leq \int_{\Omega} |u'(\omega)|^m d\omega - |v(\omega)|^m \psi(\omega) |\psi'(\omega)|^{m-2} \left| u(\omega) = \frac{\alpha_0}{\omega - \frac{\alpha_0}{2}} \right|_\omega = \frac{\alpha_0}{\omega - \frac{\alpha_0}{2}} - \frac{\alpha_0}{\omega - \frac{\alpha_0}{2}} \\
= \int_{\Omega} |u'(\omega)|^m d\omega + \beta_+ \left| v \left( \frac{\alpha_0}{2} \right) \right|^m \left| \psi \left( \frac{\alpha_0}{2} \right) \right|^m + \beta_- v \left( - \frac{\alpha_0}{2} \right) \left| \psi \left( - \frac{\alpha_0}{2} \right) \right|^m \\
+ b v \left( \frac{\alpha_0}{2} \right) \left| \psi \left( \frac{\alpha_0}{2} \right) \right|^2 \left| \psi(\omega) |\psi'(\omega)|^{m-2} \right| \\
+ \left( \beta_+ + b \psi(0) |\psi(0)|^{m-2} \left| u \left( \frac{\alpha_0}{2} \right) \right|^m \right) u \left( \frac{\alpha_0}{2} \right) |\psi(\omega)|^m + \beta_- u \left( - \frac{\alpha_0}{2} \right) |\psi(\omega)|^m.
\]

Thus, we get \((F - W)\) for smooths functions. The extension to arbitrary functions in \(W^{1,m}(\Omega) \cap C^0(\overline{\Omega})\) follows by straight-forward approximation argument. \(\square \)

**Remark 3.4.** In virtue of inequality \((F - W), Corollary 2.2[3] for any \(v \in V_{m,0}^1(G_0^d) \cap C^0(G_0^d) \) and \(q \in (0, d)\), we have

\[
\int_{G_0^d} |v|^m d\omega \leq \frac{C}{\vartheta} \left\{ \int_{G_0^d} |\nabla v|^m d\omega + B \int_{\Gamma_0^+} \frac{|v|^m}{r^{m-1}} ds + \beta_- \int_{\Gamma_0^-} \frac{|v|^m}{r^{m-1}} ds \right\}
\]

\((H - W)\)

where \(\vartheta\) is the least positive eigenvalue of \((EVP2)\) problem.

### 3.2 Assumptions

With regard to problem \((QL2)\) we assume that the following conditions are satisfied:

(a) let \(p > m > 1, 0 \leq \mu < \frac{q+m-1}{m-1}\) be given numbers; \(g(x, u)\) be the Caratheodory and continuously differentiable with respect to variable \(u\) function \(\Gamma_+ \times \mathbb{R} \to \mathbb{R}\), \(h(x, u)\) be the Caratheodory and continuously differentiable with respect to variable \(u\) function \(\Gamma_- \times \mathbb{R} \to \mathbb{R}\);

(b) \(\frac{\partial g(x, u)}{\partial u} \leq 0, \frac{\partial g(x, u)}{\partial u} \leq 0; \)

(c) \(f(x) \in L_\mu^q(G_0^R); g(x, 0), h(x, 0) \in W^{1,p}_{m-1}(G_0^R)\) and there exist positive constants \(g_1, h_1\) such that

\[
\left| \frac{\partial g(x, u)}{\partial u} \right| \leq g_1 |u|^{m-1}, \quad x \in \Gamma_+; \quad \left| \frac{\partial h(x, u)}{\partial u} \right| \leq h_1 |u|^{m-1}, \quad x \in \Gamma_-;
\]

(d) \(0 \leq b < \frac{m(m-1)^{m-2} - (m-1)(q+1)}{q \left( \frac{1}{m+1} - \frac{q+1}{2} \right) }; \frac{m\beta_q}{1 + (m-1)2^{m-1}} \)

For the following we will use the numbers:
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\( K \leq \min \left\{ \frac{m-1}{q+m-1} \right\} \) with \( \varsigma = \frac{m-1}{q+m-1} \).

(10)

\( K := \min \left\{ \frac{m-1}{q+m-1} \right\} \) with \( \varsigma = \frac{m-1}{q+m-1} \).

(11)

\[ \mathcal{Z}(m) = m \sqrt{\frac{m}{2}} \left( \frac{2}{m+2} \right)^{\frac{m+2}{2m}}. \]

(12)

\[ k_\sigma := \sup_{\varphi > 0} \varphi^{-ma} \int_{G_0^d} |f(x)|^{\frac{m}{m-1}} \sigma \left\{ \int_{\Omega} |v(x,0)|^{\frac{m}{m-1}} ds + \int_{\Omega} |h(x,0)|^{\frac{m}{m-1}} ds \right\}, \quad \sigma > 1; \]

(13) and

\[ \mathcal{S}(\varphi) := \begin{cases} K_+ \frac{m}{\mathcal{Z}(m)}, & \text{if } B > 0 \\ K_+ \frac{m}{\mathcal{Z}(m)}, & \text{if } B \leq 0, \end{cases} \]

where \( \mathcal{S} \) is defined by (12).

\[ k_\sigma := \sup_{\varphi > 0} \varphi^{-ma} \int_{G_0^d} |f(x)|^{\frac{m}{m-1}} \sigma \left\{ \int_{\Omega} |v(x,0)|^{\frac{m}{m-1}} ds + \int_{\Omega} |h(x,0)|^{\frac{m}{m-1}} ds \right\}, \quad \sigma > 1; \]

(13) and

\[ \mathcal{S}(\varphi) := \begin{cases} K_+ \frac{m}{\mathcal{Z}(m)}, & \text{if } B > 0 \\ K_+ \frac{m}{\mathcal{Z}(m)}, & \text{if } B \leq 0, \end{cases} \]

where \( \mathcal{S} \) is defined by (12).

3.3 Integral estimates

Lemma 3.5. Let \( B \) be defined by (5) and \( G_0^d \) be a sector. Let \( v \in C^0(G_0^d) \cap V_{m,0}^1(G_0^d) \) for almost all \( \varphi \in (0, d) \) and

\[ V_\pm(\varphi) = \int_{G_0^d} |\nabla v|^{\frac{m}{m-1}} dx + \alpha_\pm \int_{r_0^d} |v(x)|^{\frac{m}{m-1}} ds + \beta_\pm \int_{r_0^d} |v(x)|^{\frac{m}{m-1}} ds, \]

(14)

where

\[ \alpha_\pm = \begin{cases} 1, & \text{if } B > 0 \\ 0, & \text{if } B \leq 0. \end{cases} \]

Let \( \mathcal{Z} \) be the smallest positive eigenvalue of problem (EVP2). Then for almost all \( \varphi \in (0, d) \)

\[ \int_{\Omega} \varphi \frac{\partial v}{\partial r} |\nabla v|^{m-2} \bigg|_{r=\varphi} ds \leq \mathcal{Z}(m) \cdot \frac{\varphi}{m \mathcal{S}(m)} V_\pm(\varphi), \quad m \geq 2, \]

(15)

where \( \mathcal{Z} \) is defined by (12).

Proof. We repeat the proof of Lemma 2.3[3] applying inequality \((F - W)\) instead of \((W)_m\) and taking \( \beta_\pm = \alpha_\pm B \).

\[ \mathcal{S}(\varphi) := \begin{cases} K_+ \frac{m}{\mathcal{Z}(m)}, & \text{if } B > 0 \\ K_+ \frac{m}{\mathcal{Z}(m)}, & \text{if } B \leq 0, \end{cases} \]

where \( \mathcal{S} \) is defined by (12).

Theorem 3.6. Let \( B \) be defined by (5) and \( u \) be a weak solution of problem (QL). \( \mathcal{S} \) be the smallest positive eigenvalue of problem (EVP2). Let us assume that assumptions (a)-(d) are satisfied. In addition, assume that there exists real number \( k_\sigma \geq 0 \) defined by (13). Then there exist \( d \in (0, 1) \) and a constant \( c > 0 \) independent of \( u \) such that for any \( \varphi \in (0, d) \)

\[ \int_{G_0^d} |u|^{\frac{m}{m-1}} |\nabla u|^{m-1} dx + \int_{r_0^d} \frac{1}{r^{m-1}} |u|^{\frac{m}{m-1}} ds + \int_{r_0^d} \frac{1}{r^{m-1}} |u|^{\frac{m}{m-1}} ds \leq c \Theta^m(\varphi), \]

(15)

where

\[ \Theta(\varphi) = \begin{cases} \frac{\varphi}{m \mathcal{S}(m)}, & \sigma > \mathcal{S} \\ \varphi \ln \left( \frac{\varphi}{m \mathcal{S}(m)} \right), & \sigma = \mathcal{S} \\ \varphi \sigma, & \sigma < \mathcal{S} \end{cases}, \quad \sigma > \mathcal{S}, \sigma = \mathcal{S}, \sigma < \mathcal{S}. \]
Proof. The proof we starting from estimates the integrals on the right hand side of the inequality (5.8) of Theorem 4.2[3]. By the Young inequality with parameters  \( m, \frac{m}{m-1} \), we get
\[
\int_{r_0^0} |v||g(x,0)|ds \leq \frac{1}{m} \int_{r_0^0} |v|^m ds + \frac{m-1}{m} \int_{r_0^0} |g(x,0)|^{\frac{m}{m-1}} ds, \quad \forall \delta > 0.
\]
Similarly, with \( \delta = 1 \), we have
\[
\int_{r_0^0} |v||h(x,0)|ds \leq \frac{1}{m} \int_{r_0^0} |v|^m ds + \frac{m-1}{m} \int_{r_0^0} |h(x,0)|^{\frac{m}{m-1}} ds
\]
and
\[
\int_{G_0^0} vf(x)dx \leq \frac{1}{m} \int_{G_0^0} |v|^m dx + \frac{m-1}{m} \int_{G_0^0} |f(x)|^{\frac{m}{m-1}} dx \leq \frac{\theta^m}{m\partial} V_+(\varrho) + \frac{m-1}{m} \int_{G_0^0} |f(x)|^{\frac{m}{m-1}} dx,
\]
in virtue of inequality \( (H - W) \). Thus, by Lemma 3.5 and with regard to definition (14), from inequality (5.8) of Theorem 4.2[3], we have
\[
\left[ s^{m-1}(1 - \mu\xi) - \frac{m-1}{m} \omega_0^{m-1} \int_{G_0^0} |\nabla v|^m dx + \left[ \beta_+ - \frac{1 + (m-1)2^{m-1}}{m} \right] \int_{r_0^0} |v|^m ds \right.
\]
\[+ \beta_- \int_{r_0^0} \frac{1}{r^{m-1}} |v|^m ds \leq s^{m-1} \frac{\nu(m)}{m\partial} \varrho V_+^{\nu}(\varrho) + \frac{m-1}{m} \int_{G_0^0} |f(x)|^{\frac{m}{m-1}} dx + \frac{m-1}{m} \int_{G_0^0} |g(x,0)|^{\frac{m}{m-1}} ds
\]
\[+ \int_{r_0^0} |h(x,0)|^{\frac{m}{m-1}} ds + \frac{1}{m} \left( \frac{\theta^m}{m\partial} V_+(\varrho) + \delta^m \int_{r_0^0} |v|^m ds + \int_{r_0^0} |v|^m ds \right), \quad \forall \delta > 0. \quad (16)
\]
Now, there are two cases to consider: \( B > 0 \) and \( B \leq 0 \). Firstly, let \( B > 0 \). Therefore
\[
\int_{r_0^0} |v|^m ds + \int_{r_0^0} |v|^m ds \leq \kappa \varrho^{m-1} V_+(\varrho),
\]
where \( \kappa = \max \left\{ \beta_+, \beta_- \right\} \). Hence, setting \( \delta = 1 \), by assumption (iv), Lemma 3.5 with regard to definition (14) and (13), from (16) it follows that
\[
(\kappa_+ - \delta(\varrho)) V_+(\varrho) \leq \frac{\nu(m)}{m\partial} \varrho^{m-1} V_+^{\nu}(\varrho) + \frac{m-1}{m} \kappa \varrho^{m\sigma}, \quad (17)
\]
where \( \delta(\varrho) = \text{const} \ (m, \varrho, \partial, \beta_-) \cdot \varrho^{m-1} \) and \( \kappa_+ \) is defined by (11).

Next, let \( B \leq 0 \). Setting in (16)
\[
\delta = \left[ \frac{m}{2d^{m-1}} \left( \beta_+ - \frac{1 + (m-1)2^{m-1}}{m} \right) \right],
\]
by assumption (d) and \( \frac{1}{d^{m-1}} \leq \frac{1}{r^{m-1}} \), we have
\[
\left[ s^{m-1}(1 - \mu\xi) - \frac{m-1}{m} \omega_0^{m-1} \int_{G_0^0} |\nabla v|^m dx + \frac{1}{2d^{m-1}} \left[ \beta_+ - \frac{1 + (m-1)2^{m-1}}{m} \right] \int_{r_0^0} |v|^m ds \right.
\]
\[+ \beta_- \int_{r_0^0} \frac{1}{r^{m-1}} |v|^m ds \leq s^{m-1} \frac{\nu(m)}{m\partial} \varrho V_+^{\nu}(\varrho) + \frac{m-1}{m} \left( \int_{G_0^0} |f(x)|^{\frac{m}{m-1}} dx + c_1(m, \beta_+) \int_{r_0^0} |g(x,0)|^{\frac{m}{m-1}} ds
\]
Finally, by \( \int_{\Gamma_0^\infty} |v|^m ds \leq \varrho^{m-1} V_{-}(\varrho) \), Lemma 3.5 and with regard to definition (14) and (13), from (18) it follows that

\[
(K_{-} - \delta(\varrho)) V_{-}(\varrho) \leq \frac{\Xi(m)}{m \varrho^p} s^{m-1} \varrho V_{-}(\varrho) + \frac{m-1}{m} k \varrho^2 \varrho^{m\sigma},
\]

where \( \delta(\varrho) = \text{const} \ (m, q, \varrho, \beta, B) \cdot \varrho^{m-1} \), and \( K_{-} \) is defined by (10).

Thus, (17) and (19) we can write as follows

\[
(K_{\pm} - \delta(\varrho)) V_{\pm}(\varrho) \leq \frac{\Xi(m)}{m \varrho^p} s^{m-1} \varrho V_{\pm}(\varrho) + \frac{m-1}{m} k \varrho^2 \varrho^{m\sigma}.
\]

The rest of the proof follows verbatim the proof of Theorem 4.2 [3], starting from the paragraph following inequality (5.14).

\[\square\]

### 3.4 The main result

Main result in this part of work is the following statement:

**Theorem 3.7.** Let \( u \) be a weak solution of problem \((QL_2)\) and assumptions \((a) - (d)\) are fulfilled. Let us assume that \( M_0 = \max_{x \in \overline{\Omega}} |u(x)| \) is known (see e.g. Theorem 3.1 [3]). In addition, suppose that there exist a real number \( k_{\sigma} > 0 \) defined by (13) and \( K \geq 0 \) such that

\[K := \sup_{\varrho > 0 \Theta(\varrho)} \frac{\varrho^{m-1}}{\Theta(\varrho)} \left( \frac{m(p-2)}{m(p-1)} \| f(x) \|_{\varrho^{-1} \varrho'} \right) + \frac{m-1}{m} \varrho^{p-2} \left( || g(x, 0) ||_{\varrho^{-1} \varrho'} G_0^{\varrho} + || h(x, 0) ||_{\varrho^{-1} \varrho'} G_0^{\varrho} \right) + \frac{m^2-1}{m^2} \varrho^{p-2} \left( || \nabla g(x, 0) ||_{\varrho^{-1} \varrho'} G_0^{\varrho} + || \nabla h(x, 0) ||_{\varrho^{-1} \varrho'} G_0^{\varrho} \right),
\]

where \( \Theta(\varrho) \) is defined by (15). Then there exist \( d \in (0, 1) \) and a constant \( C > 0 \) independent of \( u \) such that

\[|u(x)| \leq C \left( |x|^{1-\frac{2}{m+2}} \Theta(|x|) \right)^{\frac{m-1}{m+1}}, \quad \forall x \in G_0^d.
\]

**Proof.** The estimate (20) we derive analogously to (6.1.7)[12] in virtue of Theorem 3.2[3] and above proved Theorem 3.6 and the inequality \((H - W)\).

\[\square\]

### 4 Boundary value problems near the infinity

In [2] we consider the following boundary value problems for quasi-linear elliptic divergence equations:

\[
\begin{align*}
- \frac{d}{dx_1} a_1(x, u, \nabla u) + b(x, u, \nabla u) &= 0, \quad x \in G_d, \\
\alpha(x) \frac{\partial u}{\partial n} + \frac{1}{|x|^m} &\gamma \left( \frac{1}{|x|^m} \right) u^{q+m-2} = g(x, u), \quad x \in \partial G_d;
\end{align*}
\]

\((QL_3)\)

where: \( d > 0, a_1 : G_d \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, b : G_d \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, m > 1, q \geq 0, \alpha(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R}^d; \\
1, & \text{if } x \notin \mathbb{R}^d \end{cases} \) is the part of the boundary \( \partial G_d \), where the Dirichlet boundary condition is posed, \( \gamma : (0, +\infty) \to (0, +\infty) \) and \( \frac{\partial u}{\partial n} = a_1(x, u, \nabla u) \cos(\bar{n}, x_1), \bar{n} \) denotes the unit outward with respect to \( G_d \) normal to \( \partial G_d \).

Our aim was to find an exponent of \((QL_3)\) weak solutions’ decreasing rate at the infinity (in the case of the linear equation we refer to [13, 14]).
Definition 4.1. A function $u$ is called a weak solution of problem $(QL_5)$ provided that $u \in \mathcal{C}^0(\overline{G}_d) \cap V^1_{m,0}(G_d)$, \( \lim_{|x| \to \infty} u(x) = 0 \), and satisfies the integral identity

$$
\int_{G_d} \left\{ a_i(x, u, u_x) \eta(x) + b(x, u, u_x) \eta(x) \right\} dx + \int_{\partial G_d} \alpha(x) \frac{\gamma(\omega)}{m-1} u|u|^{q+m-2} \eta(x) ds = \int_{\partial G_d} \alpha(x) g(x, u) \eta(x) ds
$$

for all functions $\eta \in \mathcal{C}^0(\overline{G}_d) \cap V^1_{m,0}(G_d)$ such that \( \lim_{|x| \to \infty} \eta(x) = 0 \).

4.1 Assumptions

Let $m < n < p$, $0 \leq \mu < \frac{q+m-1}{m-1}$ be given numbers; $a_0(x), a_1(x), b_0(x)$ be non-negative measurable functions. We assume that:

1. \( \int \sum_{i=1}^{n} a_i^2(x, u, \xi) + \sum_{i=1}^{n} \left| \frac{\partial a_i(x, u, \xi)}{\partial u} \right|^2 \leq |u|^q |\xi|^{m-1} + a_1(x); \quad a_1(x) \in L_{\frac{p}{m-1}}(G_d) \cap L_{\frac{m}{m-1}}(G_d); \)

2. \( b(x, u, \xi) \geq |u|^q |\xi|^{m-1} - a_0(x); \quad a_0(x) \in L_{\frac{p}{m}}(G_d); \)

3. \( |b(x, u, \xi)| \leq |u|^q |\xi|^{m-1} + b_0(x); \quad b_0(x) \in L_{\frac{m}{m-1}}(G_d) \cap L_1(G_d); \)

4. \( \frac{\partial b(x, u, \xi)}{\partial u} \leq 0; \)

5. \( \gamma(\omega) \geq \gamma_0 > 0 \) on $\partial G_d$.

In addition, suppose that the functions $a_i(x, u, \xi)$ are continuously differentiable with respect to $u, \xi$ variables in $\mathfrak{M}_{q,0} = \mathfrak{M} \times [-M_0, M_0] \times \mathbb{R}^n$, $\varrho > R$ and satisfy in $\mathfrak{M}_{q,0}$ the following conditions

6. \( (m-1) u q a_{ij}(x, u, \xi) \frac{\partial a_{ij}(x, u, \xi)}{\partial u} = q a_{ij}(x, u, \xi) \xi_j; \quad i = 1, \ldots, n; \)

7. \( \sum_{i=1}^{n} \left| a_i(x, u, u_x) - |u|^q |\nabla u|^{m-2} u_{x_i} \right|^2 \leq A \left( \frac{1}{|x|} \right) |u|^q |\nabla u|^{m-1}, \)

where $A(t)$ is a monotonically increasing and Dini-continuous at zero function.

4.2 The main result

Suppose that there are finite numbers $k_s$ and $K$ such that

$$
k_s = \sup_{\varrho > R} q^{m} s \left\{ \int_{G_\varrho} \frac{q(m+1)}{m(m+1)(m+m-1)} (a_0(x)) \frac{m+1}{m+m-1} d x + \int_{\Gamma_\varrho} \frac{m}{m-1} (b_0(x)) \frac{m}{m-1} d s \right. \left. + \int_{\Gamma_\varrho} \alpha(x)(g(x, 0)) \frac{m}{m-1} d s \right\}, \quad s > 0;
$$

$$
K = \sup_{\varrho > R} q^{m} s \left\{ q^{m-1} \left( \frac{q+1}{m+1} \right) \left( \frac{q+1}{m+1} \right) \|a_0(x)\|_{\frac{m}{m-1}} \frac{q+1}{m+1} G_\varrho \\
+ q^{1-\frac{m}{m-1}} \|b_0(x)\|_{\frac{m}{m-1}} G_\varrho + q^{1-\frac{m}{m-1}} \|a_0(x)\|_{\frac{m}{m-1}} G_\varrho + \|g(x, 0)\|_{\frac{m}{m-1}} G_\varrho \right\}, \quad s > 0,
$$

where

$$
\Theta(\varrho) = \begin{cases} 
q^{1-\frac{m}{m-1}} \frac{q+1}{m+1} \frac{q+1}{m+1} \ln \varrho, & s > \frac{1}{\theta(m)}, \frac{m+1}{q+1} \frac{q+1}{m-1}; \\
q^{1-\frac{m}{m-1}} \ln \varrho, & 0 < s < \frac{1}{\theta(m)}, \frac{m+1}{q+1} \frac{q+1}{m-1}; \\
q^{-s}, & \theta(m), \frac{q+1}{m+1} \frac{q+1}{m-1}.
\end{cases}
$$
\[ \Xi(m) = \begin{cases} 
\sqrt{\frac{m}{2}} \cdot \left( \frac{2}{m+1} \right)^{\frac{m+2}{2m}}, & m \geq 2; \\
(m - 1) \frac{m-1}{m^2}, & 1 < m \leq 2 \end{cases} \]

and \( \vartheta \) is the smallest positive eigenvalue of the eigenvalue problem for the \( m \)-Laplace-Beltrami operator on the unit sphere:

\[
\begin{aligned}
\text{div}_\omega (|\nabla_\omega \psi|^{m-2} \nabla_\omega \psi) + \vartheta (m)|\psi|^{m-2} \psi &= 0, & \omega \in \Omega; \\
\alpha(\omega)|\nabla_\omega \psi|^{m-2} \frac{\partial \psi}{\partial n} + \psi |\psi|^{m-2} \psi(\omega) &= 0, & \omega \in \partial \Omega.
\end{aligned}
\]

**Theorem 4.2** ([2]). Let \( u \) be a weak solution of problem \((QL_3)\) and assumptions \((1)-(7)\) are satisfied. Suppose, in addition, that \( g(x,0) \in L^{j-1} \left( \partial G_d \right) \), \( 1 < j < \frac{n-1}{m-1} \), \( d > 0 \). Then there exist \( \bar{R} > R \gg 1 \) and a constant \( C_0 > 0 \) such that for all \( x \in G_{\bar{R}} \)

\[ |u(x)| \leq C_0 \left( |x|^{1-\# \Theta(|x|)} \right)^{\frac{m-1}{q+m-1}}. \]

### 5 The ideas of proofs

The ideas of proofs of Theorem 2.2, Theorem 3.7 and Theorem 4.2 are based on the deduction of new inequalities of Friedrichs-Wirtinger type with exact constants as well as some integral-differential inequalities adapted to our problems. The precise exponents of the solution’s decrease rate depend on these exact constants. For details we refer to [1–3].

The existence of the smallest positive eigenvalue of problem \((EVP)\) for \( n = 3 \) was proved in [4]. The ideas of proof of this theorem are based on the Legendre spherical harmonics (see [4]) and the Gegenbauer functions.

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