Convergence of sequences in $\ell_2(P)$ with respect to a partial metric

Annisa Rahmita Soemarsono and Mahmud Yunus
Department of Mathematics, Institut Teknologi Sepuluh Nopember, Surabaya, Indonesia
E-mail: annisarahmitas@gmail.com, yunusm@matematika.its.ac.id

Abstract. Partial metric spaces are generalization of metric space. The distance from a point to itself need not be zero in partial metric space. By the properties of metric and partial metric space, we have the analogue of the two spaces. Using the analogue, we construct sequences in $\ell_2(P)$ with respect to a partial metric. We then investigate the convergence of sequences in $\ell_2(P)$. In this work, we obtain that the convergence of sequences in $\ell_2(\mathbb{N})$ can be established in $\ell_2(P)$ with respect to a partial metric.

1. Introduction
Partial metric spaces are generalization of metric spaces. The concept of partial metric space is introduced by Matthews [2]. He defines that the distance from a point to itself need not be zero. From the properties of partial metric, Matthews has an analogue of metric and partial metric space by defining a function. The function is induced by a partial metric such that it is a metric. The results of Matthews’s research are extended by Heckmann [3] to construct sequences in partial metric space. He need partially ordered set ($P$) for constructing sequences in a partial metric space. Then, Waszkiewicz [4] continue the study of partially ordered set in partial metric space.

From the last researches presented in [2], [3] and [4], Kadak, et al [1] construct sequences in partial metric space with respect to partial ordering. In their research, we know that convergence of sequences in metric space is useful for investigating the convergence of sequences in partial metric space. If sequences converge to a point in a metric space with usual metric $d_P$ induced by a partial metric $p$, then we can identify the convergence of sequences in a partial metric space.

Space $\ell_2(\mathbb{N})$ is one of well-known metric spaces containing convergent sequences [6]. In this paper, using the analogue of metric and partial metric space, we ensure that sequences in the metric space $\ell_2(\mathbb{N})$ also converge in $\ell_2(P)$ with respect to a partial metric.

2. Preliminaries
In this section, we give the concepts of partial metric and partially ordered set required to identify the convergence of sequences in $\ell_2(P)$ with respect to a partial metric space.

2.1. Partial metric space
Matthews [2] introduces a concept which the distance from a point to itself need not be zero. The concept is a generalization of metric space, and is called as partial metric space. There is a
slight difference between metric and partial metric space. One of metric axioms state that the 
distance \( d(x, y) \) of two points \( x \) and \( y \) must be zero if \( x = y \). This condition is not always true in 
partial metric space. It makes an axiom of partial metric for which if there are two same points 
\( x = y \), then \( p(x, y) \) need not be zero. By extending the metric axioms, Matthews [2] defines 
partial metric space.

**Definition 2.1** A function \( p : X \times X \rightarrow \mathbb{R}^+ \cup \{0\} \) is a partial metric on a nonempty set \( X \) 
if for \( x, y, z \in X \) satisfies the following axioms:

(P1) \( x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y); \)
(P2) \( p(x, x) \leq p(x, y); \)
(P3) \( p(x, y) = p(y, x); \)
(P4) \( p(x, z) \leq p(x, y) + p(y, z) - p(y, y). \)

A pair \((X, p)\) is called a partial metric space such that \( X \) is a nonempty set and \( p \) is a partial 
metric on \( X \).

One of the examples is defined by a function \( p : \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+ \cup \{0\} \) with 
\( p(x, y) = \max \{x, y\}, \forall x, y \in \mathbb{R}^+ \cup \{0\}. \) The partial metric is required to show that a partial 
metric can induce a metric.

2.2. Analogue of metric and partial metric

Partial metric space is a generalization of metric space, therefore some properties of metric 
space remain exist in partial metric space. This condition makes an analogue of metric and 
partial metric. The analogue is showed by a function \( d^p \) induced by a partial metric \( p \) with 
\( p(x, y) = \max \{x, y\}. \) As consequence, the function \( d^p \) is a metric on \( X \) [2].

**Theorem 2.2** If \( p \) is a partial metric on a nonempty set \( X \), then the function \( d^p : X \times X \rightarrow \mathbb{R} \) 
with 
\[
d^p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \forall x, y \in X
\]
is a metric on \( X \).

The function represents the distance of two points in the usual case as \( d^p(x, y) = |x - y| \). By 
the analogue of the two spaces, we can ensure the convergence of sequences in a partial metric 
space.

2.3. Partially ordered set

Matthews [2] defines a partially ordered set in a partial metric space.

**Definition 2.3** A partially ordered set (poset) is a pair \((X, \sqsubseteq)\) such that \( \sqsubseteq \) is a partial 
ordering on a nonempty set \( X \). For each partial metric \( p : X \times X \rightarrow \mathbb{R}^+ \cup \{0\} \) where \( \sqsubseteq_p \) 
is a binary relation over \( X \) such that \( x \sqsubseteq_p y \) if and only if \( p(x, x) = p(x, y) \) for all \( x, y \in X \).

By Definition 2.3, it can be proved that \( \sqsubseteq_p \) is a partial ordering for each partial metric \( p \). 
Accordingly, the binary relation \( \sqsubseteq_p \) satisfies the following properties:

(i) \( x \sqsubseteq_p y \) (reflexivity);
(ii) If \( x \sqsubseteq_p y \) and \( y \sqsubseteq_p x \), then \( x = y \) (antisymmetry);
(iii) If \( x \sqsubseteq_p y \) and \( y \sqsubseteq_p z \), then \( x \sqsubseteq_p z \) (transitivity).

Matthews [2] also define the partial metric \( \max \{a, b\} \) (or \( \min \{a, b\} \)) over the nonnegative 
reals with the partial ordering \( \sqsubseteq_{\max} \) (or \( \sqsubseteq_{\min} \)). For intervals \([a, b] \sqsubseteq_p [c, d] \) if and only if \([c, d] \) is 
a subset of \([a, b] \).
2.4. Space $\ell_2(\mathbb{N})$

In the space $\ell_2(\mathbb{N})$, the distance function $d : \ell_2(\mathbb{N}) \times \ell_2(\mathbb{N}) \rightarrow \mathbb{R}$ defined as follow [6]:

$$d(x, y) = \left( \sum_{i=1}^{\infty} |x_i - y_i|^2 \right)^{1/2}, \forall x, y \in \ell_2(\mathbb{N}).$$  

(2)

The space $\ell_2(\mathbb{N})$ is one of metric spaces containing convergent sequences.

2.5. Space $\ell_2(P)$

Kadak, et al [1] define a distance function $p_q$ induced by the metric $d^p$ defined in Equation 1.

**Proposition 2.4** A distance function $p_q : l_q(P) \times l_q(P) \rightarrow \mathbb{R}^+ \cup \{0\}$ for $1 \leq q < \infty$ defined by

$$p_q(x, y) = \left( \sum_{i=0}^{\infty} d^p(x_i, y_i)^q \right)^{1/q}$$

(3)

where $x, y \in l_q(P)$, is a partial metric. A pair $(l_q(P), p_q)$ is a partial metric space.

As consequence of Proposition 2.4, we get a pair $(\ell_2(P), p_2)$ is a partial metric space with the partial metric $p_2 : \ell_2(P) \times \ell_2(P) \rightarrow \mathbb{R}^+ \cup \{0\}$ defined by

$$p_2(x, y) = \left( \sum_{i=0}^{\infty} d^p(x_i, y_i)^2 \right)^{1/2}, \forall x, y \in \ell_2(P).$$

(4)

3. Results and Discussions

In this section, we present how to investigate the convergence of sequences in $\ell_2(P)$ with respect to a partial metric space. By the analogue of metric and partial metric, we can identify the convergence of sequences in a partial metric space with respect to metric such that we also can ensure the convergence of sequences in $\ell_2(P)$ with respect to a partial metric.

3.1. Convergence of sequences in a partial metric space with respect to a metric

A sequence $\{x_n\}$ in a metric space $(X, d^p)$ converges to a point $x \in X$ if for every $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $d^p(x_n, x) < \epsilon$ for all $n > n_0$. By the analogue of metric and partial metric, we have $\{x_n\}$ converges to a point $x \in X$ in a partial metric space $(X, p)$ with respect to $d^p$.

**Lemma 3.1** If a sequence $\{x_n\}$ converges to a point $x \in X$ in a metric space $(X, d^p)$, then $\{x_n\}$ converges to a point $x \in X$ in a partial metric space $(X, p)$.

**Proof.** On the previous section, we know that $d^p$ is a metric induced by a partial metric $p$. If $\{x_n\}$ is a sequence in a metric space $(X, d^p)$ and converges to a point $x \in X$, then for every $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$d^p(x_n, x) = 2p(x_n, x) - p(x_n, x_n) - p(x, x) < \epsilon.$$  

(5)

We know that $p$ is a partial metric, therefore it satisfies the axioms (P2) and (P3). Consequently, $0 \leq p(x_n, x_n) \leq p(x_n, x)$ and $0 \leq p(x, x) \leq p(x, x_n) = p(x_n, x)$. Furthermore, Equation 5 is only satisfied if $p(x_n, x) < \epsilon$. Hence, for every $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$p(x_n, x) < \epsilon.$$  

(6)
From Equation 6, we get \( \{ x_n \} \) also converges in a partial metric space \((X,p)\) if \( \{ x_n \} \) converges in a metric space \((X,d)\). In the other words, \( \{ x_n \} \) converges in a partial metric space \((X,p)\) with respect to \(d^p\).

We identify that Equation 1 can be generalized. From Equation 1, we obtain that the coefficients of the right-hand side are fixed i.e the coefficients of \(p(x,y), p(x,x)\) and \(p(y,y)\) are 2, 1 and 1. Then, we generalize Equation 1 by choosing the constants \(a,b,c\) \(\in\mathbb{R}\) such that \(a \geq b + c\), where \(a\) is the coefficient of \(p(x,y)\), \(b\) is the coefficient of \(p(x,x)\) and \(c\) is the coefficient of \(p(y,y)\). Moreover, we can induce the metric \(d^p\) using the constants \(a,b,c\).

**Theorem 3.2** If \(p\) is a partial metric on a nonempty set \(X\), the function \(d^p : X \times X \to \mathbb{R}\) defined by
\[
d^p(x,y) = ap(x,y) - bp(x,x) - cp(y,y), \forall x,y \in X
\]
is a metric on \(X\) with \(a,b,c \in \mathbb{R}\) such that \(a \geq b + c\).

**Proof.** We use the contradiction to prove the theorem. Choose \(a,b,c \in \mathbb{R}\) such that \(a < b + c\). On Theorem 3.2, we know that \(p\) is a partial metric on \(X\), therefore it satisfies (P2) and (P3). Moreover, \(0 \leq p(x,x) \leq p(x,y)\) and \(0 \leq p(y,y) \leq p(x,y)\). If \(p(x,x) = p(x,y) = p(y,y)\), then
\[
d^p(x,y) = ap(x,y) - bp(x,x) - cp(y,y) = (a - (b + c))p(x,y).
\]

In Equation 8, we obtain that \(d^p(x,y) < 0\). This condition contradicts with the fact that \(p\) is a partial metric. It proves that the metric \(d^p\) can be induced by a partial metric \(p\) if \(a\) is the coefficient of \(p(x,y)\), \(b\) is the coefficient of \(p(x,x)\) and \(c\) is the coefficient of \(p(y,y)\) such that \(a \geq b + c\) for \(a,b,c \in \mathbb{R}\). In the other words, we show that the coefficients of the right-hand side of Equation 1 are not fixed.

From Theorem 3.2, we obtain that a sequence \(\{ x_n \}\) converges to a point \(x \in X\) in a partial metric space \((X,p)\) with respect to \(d^p\) defined in Theorem 3.2.

### 3.2. Convergence of sequences in \(\ell_2(P)\) with respect to a partial metric

In Theorem 2.2, we know that the function \(d^p\) represents the distance of two points in the usual case where \(d^p(x,y) = |x - y|\). As consequence, we denote the usual metric \(d\) defined in Equation 2 as the metric \(d^p\). Moreover, by Equation 2, we obtain that \((\ell_2(\mathbb{R}),d^p)\) is a metric space containing convergent sequences.

Let \(\{ x_n \}\) is a sequence in a metric space \((\ell_2(\mathbb{R}),d^p)\). We say that \(\{ x_n \}\) converges to a point \(a \in \ell_2(\mathbb{R})\) if for every \(\epsilon > 0\), there exists an \(n_0 \in \mathbb{N}\) such that \(d^p(x_n,a) < \epsilon\) for all \(n > n_0\). From Lemma 3.1, we can ensure that \(\{ x_n \}\) also converges in a partial metric space \((\ell_2(P),p_2)\) with respect to a partial metric \(p_2\) defined in Equation 4.

The principle of space \(\ell_2(P)\) is almost same with \(\ell_2(\mathbb{R})\). If the sequences in \(\ell_2(\mathbb{R})\) are formed by elements of \(\mathbb{N}\), then the sequences in \(\ell_2(P)\) are formed by elements of partially ordered set \(P\). We need to choose a partially ordered set in a partial metric space.

**Proposition 3.3** A closed interval \([a,b]\) with the partial ordering \(\sqsubseteq_p\) denoted as a pair \(([a,b],\sqsubseteq_p)\), is a partially ordered set for a partial metric \(p : [a,b] \times [a,b] \to \mathbb{R}^+ \cup \{0\}\) defined by
\[
p([s,t],[u,v]) = \max \{t,v\} - \min \{s,u\},
\]
for all \([s,t],[u,v] \in ([a,b] : a \leq b)\).

**Proof.** By Definition 2.3, we show that \(P\) defined in Proposition 3.3 is a partially ordered set. First, we show that the partial ordering \(\sqsubseteq_p\) satisfies reflexivity. Let \(x = [s,t]\), therefore \(p(x,x) = t - s = p(x,x)\). It represents that \(x \sqsubseteq_p x\) for all \(x \in ([a,b] : a \leq b)\).
Second, we present that the partial ordering \( \sqsubseteq_p \) satisfies antisymmetry. For all \( x, y \in \{ [a, b] : a \leq b \} \) where \( x = [s, t] \) and \( y = [u, v] \), if \( x \sqsubseteq_p y \), then \( p(x, x) = t - s = p(x, y) \). It is satisfied only if \( s \leq u \) and \( t \geq v \). If \( y \sqsubseteq_p x \), then \( p(y, y) = v - u = p(y, x) \). We know that \( p \) defined in Equation 9 is a partial metric such that it satisfies (P3). As consequence, we get \( p(y, y) = v - u = p(x, y) \). It is only satisfied if \( s \geq u \) and \( t \leq v \). Furthermore, we obtain that \( s = u \) and \( t = v \). It represents that \( x = [s, t] = [u, v] = y \).

Third, we indicate that the partial ordering \( \sqsubseteq_p \) satisfies transitivity. For all \( x, y, z \in \{ [a, b] : a \leq b \} \) where \( x = [s, t] \), \( y = [u, v] \) and \( z = [m, n] \), if \( x \sqsubseteq_p y \) and \( y \sqsubseteq_p z \), then \( p(x, x) = t - s = p(x, y) \). It is satisfied only if \( u \leq m \) and \( v \geq n \). Therefore, we get \( s \leq m \) and \( t \geq n \) such that \( p(x, z) = t - s = p(x, x) \). Consequently, we get \( x \sqsubseteq_p z \).

Let \( \{x_n\} \) is a sequence in \( \ell_2(P) \) with respect to a partial metric defined in Equation 9. First, we investigate that a metric can be induced by a partial metric. By the analogue of metric and partial metric in Equation 1, we obtain that the function \( d^p \) is a metric induced by partial metric \( p \) defined in Equation 9.

**Theorem 3.4** A function \( d^p : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) for all \( x, y \in \mathbb{R} \) defined by

\[
d^p(x, y) = |x - y|, \tag{10}
\]

is a metric induced by a partial metric \( p : [a, b] \times [a, b] \to \mathbb{R}^+ \cup \{0\} \) for all \( [s, t], [u, v] \in \{ [a, b] : a \leq b \} \), where

\[
p([s, t], [u, v]) = \max \{ t, v \} - \min \{ s, u \}. \tag{11}
\]

**Proof.** It is obvious that the function \( d^p \) defined in Equation 10 is a metric on \( \mathbb{R} \). Now, we show that the metric \( d^p \) can be induced by a partial metric \( p \) defined in Equation 11. From Equation 1, we obtain that

\[
d^p(x, y) = 2 \left( \max \{ t, v \} - \min \{ s, u \} \right) - (t - s) - (v - u) \tag{12}
\]

There is four cases of Equation 12.

- **Case 1:**
  For \( [s, t] \cap [u, v] = \emptyset \), we have \( d^p(x, y) = (u - s) - (t - v) \) where \( s < u \) and \( t < v \) and \( d^p(x, y) = (s - u) - (v - t) \) where \( s > u \) and \( t > v \).

- **Case 2:**
  For \( [s, t] \cap [u, v] \neq \emptyset \), we have \( d^p(x, y) = (u - s) - (t - v) \) where \( s < u \) and \( t < v \) and \( d^p(x, y) = (s - u) - (v - t) \) where \( s > u \) and \( t > v \).

- **Case 3:**
  For \( [s, t] \subseteq [u, v] \), we have \( d^p(x, y) = (s - u) - (t - v) \) where \( s > u \) and \( t < v \).

- **Case 4:**
  For \( [u, v] \subseteq [s, t] \), we have \( d^p(x, y) = (u - s) - (v - t) \) where \( s < u \) and \( t > v \).

From the four cases above, we have \( d^p(x, y) = |s - u| - |t - v| \) for all \( s, t, u, v \in \mathbb{R} \). We consider that \( |s - u| - |t - v| = |t - v| - |s - u| \), therefore \( d^p(x, y) = |s - u| - |t - v| = |t - v| - |s - u| \).

Let \( x = |s - u| \in \mathbb{R} \) and \( y = |t - v| \in \mathbb{R} \), then we obtain that \( d^p(x, y) = x - y = y - x = |x - y| \). It represents that the metric \( d^p \) defined in Equation 10 can be induced by a partial metric \( p \) defined in Equation 11. \( \Box \)

Let \( \{x_n\} \) is a sequence in \( \ell_2(P) \) where \( P \) is partially ordered set defined in Proposition 3.3. We know that \( \{x_n\} \) converges to a point \( a \in \ell_2(\mathbb{N}) \) such that for every \( \epsilon > 0 \), there exists an \( n_0 \in \mathbb{N} \) satisfying \( d^p(x_n, a) < \epsilon \) for all \( n > n_0 \). Consequently, for every \( \epsilon > 0 \), there exists an \( n_0 \in P \) such that \( p_2(x_n, a) < \epsilon \) for all \( n > n_0 \). It concludes that \( \{x_n\} \) also converges to a point \( a \in \ell_2(P) \) with respect to a partial metric \( p_2 \) induced by partial metric \( p \) defined in Equation 11.
Corollary 3.5 A sequence \( \{x_n\} \) is called to converge in the partial metric space \((\ell_2(P), p_2)\) if for every \( \epsilon > 0 \), there exists an \( n_0 \in P \) such that \( p_2(x_n, x) < \epsilon \) for all \( n > n_0 \).

4. Concluding Remarks
We have presented that the convergence of sequences in a metric space can be constructed to a partial metric space. We notice the analogue of metric and partial metric to ensure the convergence of sequences in a partial metric space using a partial metric with respect to partial ordering. For a particular case, we show that the convergence of sequences in the metric space \((\ell_2(\mathbb{N}), d_2)\) can be established in the partial metric space \((\ell_2(P), p_2)\).

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