A SECOND-ORDER STOCHASTIC MAXIMUM PRINCIPLE FOR GENERALIZED MEAN-FIELD SINGULAR CONTROL PROBLEM

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Abstract. In this paper, we study the generalized mean-field stochastic control problem when the usual stochastic maximum principle (SMP) is not applicable due to the singularity of the Hamiltonian function. In this case, we derive a second order SMP. We introduce the adjoint process by the generalized mean-field backward stochastic differential equation. The keys in the proofs are the expansion of the cost functional in terms of a perturbation parameter, and the use of the range theorem for vector-valued measures.

1. Introduction. We consider the following optimal stochastic control problem of mean-field type with the state equation

\[ \begin{cases} dX_t = b(t, X_t, P_{X_t}, v_t)dt + \sigma(t, X_t, P_{X_t})dB_t, \\ X_0 = x, \end{cases} \]

and the cost functional

\[ J(v) = \mathbb{E} \left\{ \int_0^T h(t, X_t, P_{X_t}, v_t)dt + \Phi(X_T, P_{X_T}) \right\}, \]

where \( P_\xi \) denotes the law of the random variable \( \xi \).

The agent wishes to minimize his cost functional, namely, an admissible control \( u \in \mathcal{U} \) is said to be optimal if

\[ J(u) = \min_{v \in \mathcal{U}} J(v). \]

where \( \mathcal{U} \) is the set of all admissible controls to be defined later in Section 3.

About stochastic maximum principle (SMP), some pioneering works have been done by Pontryagin et al. [22]. They obtained Pontryagin’s maximum principle by using “spike variation”. Kushner ([14], [15]) studied the SMP in the framework of differential games. The authors of this paper are interested in finding a sufficient condition for optimality when the usual SMP is not applicable due to the singularity of the Hamiltonian function. In this case, we derive a second order SMP. We introduce the adjoint process by the generalized mean-field backward stochastic differential equation. The keys in the proofs are the expansion of the cost functional in terms of a perturbation parameter, and the use of the range theorem for vector-valued measures.

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when the diffusion coefficient does not depend on the control variable, and the cost functional consists of terminal cost only. Haussmann [10] gave a version of SMP when the diffusion of the state does not depend on the control variable. Arkin and Saksonov [1], Bensoussan [3] and Bismut [4], proved different versions of SMP under various setups. An SMP was obtained by Peng [21] in 1990. In that paper, first and second order variational inequalities are introduced, when the control domain need not to be convex, and the diffusion coefficient contains the control variable.

Pardoux and Peng [20] introduced non-linear backward stochastic differential equations (BSDE) in 1990. They showed that under appropriate assumptions, BSDE admits an unique adapted solution, and the associated comparison theorem holds. Buckdahn et al [6] obtained mean-field BSDE in a natural way as the limit of some high dimensional system of forward and backward stochastic differential equations. Li [16] studied SMP for mean-filed controls when the domain of the control is assumed to be convex. Under some additional assumptions, both necessary and sufficient conditions for the optimality of a control were proved. Buckdahn et al [7] studied generalized mean-field stochastic differential equations and the associated partial differential equations (PDEs). “Generalized” means that the coefficients depend on both the state process and its law. They proved that under appropriate regularity conditions on the coefficients, the SDE has a unique classical solution. Buckdahn et al. [5] obtained SMP for generalized mean-field system in 2016.

Sometimes, the Hamiltonian function becomes constant in the control variable, as we will see in the next example, which makes the aforementioned SMP not applicable.

**Example 1.1.** Consider the control problem with state equation:

\[
\begin{cases}
    dX_t^v = v_t dt + \{(X_t^v - 1) + \mathbb{E}[(X_t^v - 1)]\} dB_t, & v \in U := \{-1, 0, 1\}, \\
    X_0^v = 1,
\end{cases}
\] (3)

and cost functional:

\[
J(v) = \frac{1}{2} \mathbb{E}\{(X_T^v - 1) + \mathbb{E}[(X_T^v - 1)]\}^2.
\]

For the control \( u_t \equiv 0 \), \( X_t^u \equiv 1 \) is the unique solution of (3). It is clear that \( J(u) = 0 \), and hence, \( u \) is an optimal control. On the other hand, the first order adjoint processes satisfy the following equation:

\[
\begin{cases}
    dp_t = \{q_t + \mathbb{E}[q_t]\} dt - q_t dB_t \\
    p_T = 0.
\end{cases}
\] (4)

Clearly \((p_t, q_t) \equiv (0, 0)\) is the solution. Therefore

\[
H(t, X_t^u, P_t, p_t, q_t, v) \equiv 0, \quad v \in U,
\]

which makes the SMP useless in characterizing the optimal control \( u_t = 0 \). \(\square\)

Now, we discuss singular optimal stochastic controls defined as follows.

**Definition 1.2.** An admissible control \( \tilde{u}(\cdot) \) is singular on region \( V \) if \( V \subset U \) is of positive measure and for a.e. \( t \in [0, T] \) and \( v \in V \), we have for any \( v \in V \),

\[
H(t, X_t^\tilde{u}, P_t, p_t, q_t, \tilde{u}_t, v) = H(t, X_t^\tilde{u}, P_t, p_t, q_t, \tilde{u}_t, v), \quad a.s.
\] (5)

As we have seen in last example, the SMP is not very useful under singular control. Our goal is to derive further necessary condition for optimality. We shall call the original SMP as the first order SMP while the one we will derive as the second order one.
For second-order SMP of singular control problems, Bell [2], Gabasov [9], Kazemi-
Dehkordi [12], Krener [13], Mizukami and Wu [19] devoted themselves to the deter-
ministic case. Lu [17], interested in second order necessary conditions for stochastic
evolution system. Tang [23] studied the singular optimal control problem for sto-
chastic system with state equation
\[
\begin{aligned}
    dX_t &= b(t, X_t, v_t) dt + \sigma(t, X_t) dB_t, \\
    X_0 &= x.
\end{aligned}
\] 
(6)
and the cost functional
\[
    J(v) = \mathbb{E}\{ \int_0^T h(t, X_t, v_t) dt + \Phi(X_T) \},
\] 
(7)
By applying spike variation and vector-value measure theory, a second-order maxi-
mum principle is presented which involves the second-order adjoint process.

In this paper, we study the case when the state equation and the cost functional
are in generalized mean-field form. The rest of this paper is organized as follows: In
Section 2, we introduce the preliminaries about the generalized mean-field BSDEs.
In Section 3, we set up the formulation of the singular optimal stochastic control
problem and state the main result of the paper. Section 4 is devoted to the study
of the impact of the control actions on the state and the cost functional by using
Taylor’s expansion. In that section, we also present some estimations about the
state. In Section 5, the method in Section 4 is reused for the expansion of the cost
functional with respect to the control variable. Sections 6 is devoted to the proof
of the second order stochastic maximum principle.

2. Preliminaries. In this section, for the convenience of the reader, we state some
results of Buckdahn et al. [7] without proofs.

Let \(\mathcal{P}_2(\mathbb{R}^n)\) be the collection of all square integrable probability measures over
\((\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\), endowed with the 2-Wasserstein metric \(W_2\), which is defined as
\[
    W_2(P_\mu, P_\nu) = \inf \left\{ \left( \mathbb{E}[|\mu' - \nu'|^2] \right)^{\frac{1}{2}} \right\},
\]
for all \(\mu', \nu' \in L^2(\mathcal{F}_0; \mathbb{R}^d)\) with \(P_{\mu'} = P_\mu, P_{\nu'} = P_\nu\). Denote by \(L^2(\mathcal{F}; \mathbb{R}^n)\) the collec-
tion of all \(\mathbb{R}^n\)-valued square integrable random variables. The following definition is taken from Cardaliaguet [8].

**Definition 2.1.** A function \(f : \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R}\) is said to be differentiable in \(\mu \in \mathcal{P}_2(\mathbb{R}^n)\) if, the function \(\tilde{f} : L^2(\mathcal{F}; \mathbb{R}^n) \rightarrow \mathbb{R}\) given by \(\tilde{f}(\nu) = f(P_\mu)\) is differentiable (in Fréchet sense) at \(\nu_0\), defined by \(P_{\nu_0} = \mu\), i.e. there exists a linear continuous mapping \(D\tilde{f}(\nu_0) : L^2(\mathcal{F}; \mathbb{R}^n) \rightarrow \mathbb{R}\), such that
\[
    \tilde{f}(\nu_0 + \eta) - \tilde{f}(\nu_0) = D\tilde{f}(\nu_0)(\eta) + o(|\eta|_{L^2}),
\]
with \(|\eta|_{L^2} \rightarrow 0\) for \(\eta \in L^2(\mathcal{F}; \mathbb{R}^n)\).

According to the Riesz representation theorem, there exists a unique random
variable \(\theta_0 \in L^2(\mathcal{F}; \mathbb{R}^n)\) such that \(D\tilde{f}(\nu_0)(\eta) = (\theta_0, \eta)_{L^2} = \mathbb{E}[\theta_0\eta]\), for all \(\eta \in L^2(\mathcal{F}; \mathbb{R}^n)\). In [8] it has been proved that there is a Borel function \(h_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n\) such that \(\theta_0 = h_0(\nu_0)\) a.s. Then,
\[
    f(P_\mu) - f(P_{\nu_0}) = \mathbb{E}[h_0(\nu_0)(\nu - \nu_0)] + o(|\nu - \nu_0|_{L^2}), \quad \nu \in L^2(\mathcal{F}; \mathbb{R}^n).
\]
We call \(\partial_\nu f(\nu_0, y) := h_0(y)\), \(y \in \mathbb{R}^n\), the derivative of \(f : \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R}^n\) at \(P_{\nu_0}\). Note that \(\partial_\nu f(\nu_0, y)\) is \(P_{\nu_0}(dy)\)-a.s. uniquely determined.
For mean-field type SDE and BSDE, we introduce the following notations. Let \((\Omega', \mathcal{F}', \mathbb{P}')\) be a copy of the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). For each random variable \(\xi\) over \((\Omega, \mathcal{F}, \mathbb{P})\) we denote by \(\xi'\) a copy of \(\xi\) defined over \((\Omega', \mathcal{F}', \mathbb{P}')\). \(E[\cdot] = \int_{\Omega'} (\cdot) d\mathbb{P}'\) acts only over the variables \(\omega'\).

**Definition 2.2.** We say that \(f \in C_b^{1,1}(\mathcal{P}_2(\mathbb{R}^d))\) (continuously differentiable over \(\mathcal{P}_2(\mathbb{R}^d)\) with Lipschitz-continuous bounded derivative), if for all \(v \in L^2(\mathcal{F}, \mathbb{R}^d)\), there exists a \(\mathbb{P}_v\)-modification of \(\partial \mu f(P_v, \cdot)\), again denote by \(\partial \mu f(P_v, \cdot)\), such that \(\partial \mu f : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \longrightarrow \mathbb{R}^d\) is bounded and Lipschitz continuous, i.e., there is a real constant \(C\) such that

\[
\begin{align*}
  &i) \ |\partial \mu f(\mu, x)| \leq C, \quad \forall \mu \in \mathcal{P}_2(\mathbb{R}^d), x \in \mathbb{R}^d, \\
  &ii) \ |\partial \mu f(\mu, x) - \partial \mu f(\mu', x')| \leq C (W_2(\mu, \mu') + |x - x'|),
\end{align*}
\]

((8)) we denote by \(\partial \mu f(\cdot, x) : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \longrightarrow \mathbb{R}^d\) is a Lipschitz-continuous function. Then

**Definition 2.3.** We say that \(f \in C_b^{2,1}(\mathcal{P}_2(\mathbb{R}^d))\), \(g \in C_b^{1,1}(\mathcal{P}_2(\mathbb{R}^d))\) and

\[
\begin{align*}
  &i) \ (\partial \mu f)_{j} (\cdot, y) \in C_b^{1,1}(\mathcal{P}_2(\mathbb{R}^d))\), for all \(y \in \mathbb{R}^d, 1 \leq j \leq d\), and \(\partial \mu f : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \longrightarrow \mathbb{R}^d\) is bounded and Lipschitz-continuous; \\
  &ii) \ (\partial \mu g)(\mu, \cdot) : \mathbb{R}^d \longrightarrow \mathbb{R}^d\) is differentiable for every \(\mu \in \mathcal{P}_2(\mathbb{R}^d)\), and its derivative \(\partial \mu \partial \mu g : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \longrightarrow \mathbb{R}^d\) is bounded and Lipschitz-continuous.
\]

**Example 2.4.** For twice continuously differentiable functions \(h : \mathbb{R}^d \longrightarrow \mathbb{R}\) and \(g : \mathbb{R} \longrightarrow \mathbb{R}\) with bounded derivatives. Consider \(f (P_v) := g(\mathbb{E}[h(v)])\), \(v \in L^2(\mathcal{F}; \mathbb{R}^d)\). Then, given any \(v_0 \in L^2(\mathcal{F}; \mathbb{R}^d)\), \(\hat{f}(v) := f (P_v) = g(\mathbb{E}[h(v)])\) is Fréchet differentiable in \(v_0\), and

\[
\hat{f}(v_0 + \eta) - \hat{f}(v_0) = \int_0^1 g'(\mathbb{E}[h(v_0 + s\eta)])\mathbb{E}[h'(v_0 + s\eta)]ds = g'(\mathbb{E}[h(v_0)])\mathbb{E}[h'(v_0)] \eta + o(|\eta|^{1/2}) = g'(\mathbb{E}[h(v_0)])h'(v_0) \eta + o(|\eta|^{1/2}).
\]

So, \(D\hat{f}(v_0)(\eta) = \mathbb{E}[g'(\mathbb{E}[h(v_0)])h'(v_0)]\eta, \ \eta \in L^2(\mathcal{F}; \mathbb{R}^d)\), i.e.,

\[
\partial \mu f (P_v, y) = g'(\mathbb{E}[h(v_0)])((\partial \mu h)(y), y \in \mathbb{R}^d.
\]

Similarly, we see that

\[
\partial \mu f (P_v, x, y) = g''(\mathbb{E}[h(v_0)])((\partial \mu h)(x) \times (\partial \mu h)(y),
\]

and

\[
\partial \mu \partial \mu f (P_v, x, y) = g'''(\mathbb{E}[h(v_0)])(\partial \mu^2 h)(y).
\]
Let us now consider a complete probability space $(\Omega, \mathcal{F}, P)$ on which we define a 
$d$-dimensional Brownian motion $B = (B^1, \cdots, B^d) = (B_t)_{t \in [0,T]}$, where $T \geq 0$ 
represents an arbitrarily fixed time horizon. We make the following assumptions: There 
is a sub-$\sigma$-field $\mathcal{F}_0 \subset \mathcal{F}$ such that

i) the Brownian motion $B$ is independent of $\mathcal{F}_0$, and

ii) $\mathcal{F}_0$ is “rich enough”, i.e., $\mathcal{P}_2(\mathbb{R}^d) = \{P_0, \omega \in L^2(\mathcal{F}_0; \mathbb{R}^d)\}$.

By $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ we denote the filtration generated by $B$, completed and aug-
mented by $\mathcal{F}_0$.

Given deterministic Lipschitz functions $\sigma : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d \times d}$ and $b : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$, we consider for the initial state $(t, x) \in [0, T] \times \mathbb{R}^d$ and 
$\xi \in L^2(\mathcal{F}_t; \mathbb{R}^d)$ the stochastic differential equations (SDEs)

$$X_s^{t,\xi} = \xi + \int_t^s \sigma(X_r^{t,\xi}, P_{X_r^{t,\xi}})dB_r + \int_t^s \sigma(X_r^{t,\xi}, P_{X_r^{t,\xi}})dr, \quad s \in [t, T],$$

(9)

and

$$X_s^{t,x,\xi} = x + \int_t^s \sigma(X_r^{t,x,\xi}, P_{X_r^{t,x,\xi}})dB_r + \int_t^s \sigma(X_r^{t,x,\xi}, P_{X_r^{t,x,\xi}})dr, \quad s \in [t, T].$$

(10)

It is well-known that under the assumptions above both SDEs have unique solutions in $\mathcal{S}^2([t, T]; \mathbb{R}^d)$, which is the space of $\mathbb{F}$-adapted continuous processes $Y = (Y_s)_{s \in [t, T]}$ with $\mathbb{E}[\sup_{s \in [t, T]} |Y_s|^2] \leq \infty$.

**Hypothesis 2.1.** The couple of coefficients $(\sigma, b)$ belongs to $C_b^{1,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d \times d} \times \mathbb{R}^d)$, i.e., the components $\sigma_{i,j}, b_j$, $1 \leq i, j \leq d$, satisfy the following conditions:

i) $\sigma_{i,j}(x, \cdot), b_j(x, \cdot)$ belong to $C_b^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$, for all $x \in \mathbb{R}^d$

ii) $\sigma_{i,j}(\cdot, \mu), b_j(\cdot, \mu)$ belong to $C_b^{1,1}(\mathbb{R}^d)$, for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$

iii) The derivatives $\partial_j \sigma_{i,j}, \partial_j b_j : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d, \partial_\mu \sigma_{i,j}, \partial_\mu b_j : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$, are bounded and Lipschitz continuous.

**Hypothesis 2.2.** The couple of coefficient $(\sigma, b)$ belongs to $C_b^{2,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d \times d} \times \mathbb{R}^d)$, i.e., $(\sigma, b) \in C_b^{1,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d \times d} \times \mathbb{R}^d)$ and the components $\sigma_{i,j}, b_j$, $1 \leq i, j \leq d$, satisfies the following conditions:

i) $\partial_{x_k} \sigma_{i,j}(\cdot, \cdot), \partial_{x_k} b_j(\cdot, \cdot)$ belong to $C_b^{1,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$, for all $1 \leq k \leq d$;

ii) $\partial_{x_k} \sigma_{i,j}(\cdot, \cdot), \partial_{x_k} b_j(\cdot, \cdot)$ belong to $C_b^{1,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d)$, for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$

iii) All the derivatives of $\sigma_{i,j}, b_j$, up to order 2 are bounded and Lipschitz continuous.

The following theorem is taken from [7]. It gives the Itô formula related to a probability measure.

**Theorem 2.5.** Let $\Phi \in C_b^{2,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$. Then, under Hypothesis 2.2, for all 
$0 \leq t \leq s \leq T, x \in \mathbb{R}^d, \xi \in L^2(\mathcal{F}_t; \mathbb{R}^d)$ the Itô formula is satisfied as follow:

$$\Phi(X_s^{t,x,\xi}, P_{X_s^{t,x,\xi}}) - \Phi(x, P_x)$$

$$= \int_t^s \left( \sum_{i=1}^d \partial_{x_i} \Phi(X_r^{t,x,\xi}, P_{X_r^{t,x,\xi}}) b_i(X_r^{t,x,\xi}, P_{X_r^{t,x,\xi}}) ight) dr$$

$$+ \frac{1}{2} \sum_{i,j,k=1}^d \partial^2_{x_i x_j} \Phi(X_r^{t,x,\xi}, P_{X_r^{t,x,\xi}}) (\sigma_{i,k} \sigma_{j,k})(X_r^{t,x,\xi}, P_{X_r^{t,x,\xi}})$$

$$+ \mathbb{E} \left[ \sum_{i=1}^d \partial_{x_i} \Phi(X_r^{t,x,\xi}, P_{X_r^{t,x,\xi}}) (X_r^{t,x,\xi})' b_i((X_r^{t,x,\xi})', P_{X_r^{t,x,\xi}}) \right]$$

$$+ \mathbb{E} \left[ \Phi(X_s^{t,x,\xi}, P_{X_s^{t,x,\xi}}) - \Phi(x, P_x) \right]$$
The second-order derivatives with respect to \( \frac{\partial}{\partial \mu} \Phi (X_r^{t,x,P_t}, P_{X_r^{t,x}}, (X_r^{t,x})', (X_r^{t,x})'') \sigma_{i,k} \sigma_{j,k} (X_r^{t,x})', P_{X_r^{t,x}}) \) \\
+ \int_t^s \frac{1}{2} \sum_{i,j,k=1}^d \partial_{y_i} \partial_{y_j} \Phi (X_r^{t,x,P_t}, P_{X_r^{t,x}}) \sigma_{i,k} \sigma_{j,k} (X_r^{t,x})', P_{X_r^{t,x}}) dB_t^j, \ s \in [t,T]. \quad (11)

For simplicity, we will make use of the following notations concerning matrices. \( \mathbb{R}^{n \times d} \) the space of real matrices of \( n \times d \)-type, and \( \mathbb{R}_d^{n \times n} \) the linear space of the vectors of matrices \( M = (M_1, \cdots , M_d) \), with \( M_i \in \mathbb{R}_d^{n \times n} \), \( 1 \leq i \leq d \). Given any \( \alpha, \beta \in \mathbb{R}^n \), \( L, S \in \mathbb{R}_d^{n \times d} \), \( \gamma \in \mathbb{R}^d \) and \( M, N \in \mathbb{R}_d^{n \times n} \), we introduce the following notation: \( \alpha \beta = \sum_{i=1}^d \alpha_i \beta_i \in \mathbb{R} \), \( \alpha \times \beta = (\alpha_i \beta_j)_{1 \leq i,j \leq n} \); \( LS = \sum_{i=1}^d L_i S_i \in \mathbb{R} \), where \( L = (L_1, \cdots , L_d), S = (S_1, \cdots , S_d) \); \( ML = \sum_{i=1}^d M_i L_i \in \mathbb{R}_d^{n \times n} \); \( M \alpha \gamma = \sum_{i=1}^d (M_i \alpha) \gamma_i \in \mathbb{R}^n \); \( MN = \sum_{i=1}^d M_i N_i \in \mathbb{R}_d^{n \times n} \).

For mean-field type SDE and BSDE, we still have to introduce some notations. Let \((\Omega, \mathcal{F}, P), (\Omega, \mathcal{F}, \tilde{P})\) be two copies of the probability space \((\Omega, \mathcal{F}, P)\). For any random variable \( \xi \) over \((\Omega, \mathcal{F}, P)\), we denote by \( \xi \) and \( \xi \) its copies on \( \tilde{\Omega} \) and \( \bar{\Omega} \), respectively, which means that they have the same law as \( \xi \), but defined over \((\tilde{\Omega}, \mathcal{F}, \tilde{P})\) and \((\bar{\Omega}, \mathcal{F}, \bar{P})\), \( \tilde{E}[] = \int_{\Omega} (\cdot) d\tilde{P} \) and \( \bar{E}[] = \int_{\Omega} (\cdot) d\bar{P} \) act only over the variables from \( \tilde{\omega} \) and \( \bar{\omega} \), respectively.

3. Formulation of the singular optimal stochastic control problem and the main result. In this section, we formulate our generalized mean-field optimal control problem and state the main result of this article. Let \((\Omega, \mathcal{F}, P)\) be a probability space with filtration \( \mathcal{F}_t \). Suppose that \( B_t \) is a Brownian motion on \((\Omega, \mathcal{F}, P)\), where \( \mathcal{F} \) is the filtration generated by \( B_t \), augmented by all \( P \)-null sets. Let \( \mathcal{U} \) denote the admissible control set consisting of \( \mathcal{F}_t \)-adapted process \( u_t \), take values in \( U \), such that \( \sup_{0 \leq t \leq T} E[|u_t|^8] < \infty \), where \( U \) is a subset of \( \mathbb{R}^k \).

Let \( b : [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times U \rightarrow \mathbb{R}^n \), \( \sigma : [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R}^d \), \( h : [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times U \rightarrow \mathbb{R} \), and \( \Phi : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R} \).

The state equation and the cost functional are defined by (1) and (2). Throughout this paper, we make the following assumptions on the coefficients:

**Hypothesis 3.1.** (1) The functions \( b, \sigma, h, \Phi \) are differentiable with respect to \( (x, \mu, v) \). \( b, \sigma \) satisfy Lipschitz condition with respect to \( (x, \mu, v) \).

(2) The first-order derivatives with respect to \( (x, \mu) \) of \( b, \sigma \) are Lipschitz continuous and bounded.

(3) The first-order derivatives with respect to \( (x, \mu) \) of \( h, \Phi \) are Lipschitz continuous and bounded by \( C(1 + |x| + |v|) \).

(4) The second-order derivatives with respect to \( (x, \mu) \) of \( b, \sigma, h, \Phi \) are continuous and bounded. All the second-order derivatives are Borel measurable with respect to \( (t, x, \mu, v) \).

Suppose that \( u \) is an optimal control and \( X_u \) is the associated trajectory. We are to find the necessary conditions satisfied by \( u \). Firstly, we introduce the following abbreviations:

\[
\begin{align*}
b(t) :&= b(t, X_t^u, P_{X_t^u}, u_t), \quad b_x(t) := b_x(t, X_t^u, P_{X_t^u}, u_t), \\
\bar{b}(t) :&= b(t, X_t^u, P_{X_t^u}, \bar{u}_t), \quad \bar{b}_x(t) := b_x(t, X_t^u, P_{X_t^u}, \bar{u}_t), \\
b_{xx}(t) :&= b_{xx}(t, X_t^u, P_{X_t^u}, u_t), \quad b_{\mu}(t) := b_{\mu}(t, X_t^u, P_{X_t^u}, \bar{X}_t^u, u_t).
\end{align*}
\]
\[ \tilde{b}_\mu(t) := b_\mu(t, \tilde{X}^u_t, P_{\tilde{X}^u_t}, X^u_t, \tilde{u}_t), \]
\[ \tilde{b}_{\mu\mu}(t) := b_{\mu\mu}(t, \tilde{X}^u_t, P_{\tilde{X}^u_t}, X^u_t, \tilde{u}_t), \]
\[ b_{\mu}(t) := b_{\mu}(t, X^u_t, P_{X^u_t}, \tilde{X}^u_t, u_t), \]
\[ b_{\mu\mu}(t) := b_{\mu\mu}(t, X^u_t, P_{X^u_t}, \tilde{X}^u_t, u_t), \]
\[ b_{\mu}(t) := b_{\mu}(t, X^u_t, P_{X^u_t}, \tilde{X}^u_t, u_t), \]
\[ \triangle b(t; v) := b(t, X^u_t, P_{X^u_t}, v) - b(t), \]
\[ \triangle b_{\mu}(t; v) := b_{\mu}(t, X^u_t, P_{X^u_t}, v) - b_{\mu}(t). \]

Similar shorthand notations for the second-order derivatives and those about \( \sigma, h \) can also be introduced.

Consider the first order adjoint process
\[
\begin{align*}
-dp_t &= \left\{ -b_x(t)p_t + \sigma_x(t)q_t + h_x(t) \\
&\quad + \mathbb{E}\left[ \tilde{b}_\mu(t)\tilde{p}_t + \tilde{\sigma}_\mu(t)\tilde{q}_t + \tilde{h}_\mu(t) \right] \right\} dt - q_t dB_t, \\
 PT &= \Phi_x(X^u_T, P_{X^u_T}) + \mathbb{E}\left[ \Phi_\mu(X^u_T, P_{X^u_T}, X^u_T) \right].
\end{align*}
\]

According to Theorem 3.1 [6], this BSDE admit a unique adapted solution. We also denote the solution as \((p_t^u, q_t^u)\). Define the Hamiltonian as follows:
\[ H(t, x, \mu, p, q, v) = pb(t, x, \mu, v) + q\sigma(t, x, \mu) + h(t, x, \mu, v) \]
The following first-order SMP is obtained as a special case of [5].

**Theorem 3.1 (The First Order SMP).** Let Hypothesis 3.1 hold. Suppose that \( X^u_t \) is the associated trajectory of the optimal control \( u_t \) and \((p, q)\) is the solution to the mean-field backward stochastic differential equation (MFBSDE) (12). Then, there is a subset \( I_0 \subset [0, T] \) which is of full measure such that \( \forall t \in I_0 \),
\[
H(t, X^u_t, P_{X^u_t}, p_t, q_t, u_t) = \inf_{v \in \mathcal{U}} H(t, X^u_t, P_{X^u_t}, p_t, q_t, v), \quad a.s.. \tag{13}
\]

As we pointed out in the introduction, the aim of this article is to derive another SMP when the Hamiltonian function above becomes singular, and hence, the SMP above is not suitable for characterizing of the optimal control \( u_t \). To this end, we define the second-order adjoint process as follows:
\[
\begin{align*}
dP_t &= -\left\{ b_{xx}^u(t)p_t + b_{x\mu}^u(t)q_t + h_{xx}^u(t) \right\} dt + Q_t dB_t, \\
PT &= \Phi_{xx}(X^u_T, P_{X^u_T}) + 2\mathbb{E}\left[ \Phi_{x\mu}(X^u_T, P_{X^u_T}, X^u_T) \right] \\
&\quad + \mathbb{E}\left[ \Phi_{\mu\mu}(X^u_T, P_{X^u_T}, X^u_T) \right] + \mathbb{E}\left[ \Phi_{\mu\mu}(X^u_T, P_{X^u_T}, X^u_T) \right]. \tag{15}
\end{align*}
\]

Remark 1. By changing the terminal condition \( p_T \), we can always eliminate the terminal cost when deducing the variational inequality. In fact, the terminal condition \( p_T = 0 \) is due to the assumption that \( \Phi \equiv 0 \). Without this assumption, we only need to set
\[
P_T = \Phi_{xx}(X^u_T, P_{X^u_T}) + 2\mathbb{E}\left[ \Phi_{x\mu}(X^u_T, P_{X^u_T}, X^u_T) \right] \\
&\quad + \mathbb{E}\left[ \Phi_{\mu\mu}(X^u_T, P_{X^u_T}, X^u_T) \right] + \mathbb{E}\left[ \Phi_{x\mu}(X^u_T, P_{X^u_T}, X^u_T) \right].
\]
Finally, we present our main result in this article.

**Theorem 3.2.** Assume that Hypothesis 3.1 hold. Let \((X^u, u)\) be an optimal pair and let \(u\) be singular on the control region \(V\). Suppose that \((P, Q)\) is the unique adapted solution of equation (14). Then, there is a full measure subset \(I_0 \subset [0, T]\) such that at each \(t \in I_0\), \((X^u, u)\) satisfies, not only the first-order stochastic maximum principle, but also the following inequality

\[
\triangle H_x(t; v) \triangle b(t; v) + E \left[ \triangle H_\mu(t; v) \triangle \dot{b}(t; v) \right] \\
+ \triangle \dot{b}^*(t; v) P_t \triangle b(t; v) \geq 0, \quad \forall v \in U, \text{a.s..}
\]  

(16)

4. Quantitative analysis of the impact of control actions on the state.

In this section, we expand the state process according to different orders of the perturbation parameter \(d(u, v)\), a distance between the optimal control \(u\) and its perturbation \(v\).

**Lemma 4.1.** Under Hypothesis 3.1 on the coefficients, we have,

\[
E \sup_{0 \leq t \leq T} |X^v_t|^8 \leq K \left( 1 + E \left| \int_0^T |v_s| ds \right|^8 \right),
\]

Proof. By the state equation (1), for \(\tau \in [0, T]\) we have,

\[
E \sup_{0 \leq t \leq \tau} |X^v_t|^8 \leq K \left( |x|^8 \right. \\
\left. + \sup_{0 \leq t \leq \tau} \left| \int_0^t b(s, X^v_s, P_{X^v_s}, v_s) ds \right|^8 \\
+ \left( \int_0^\tau |\sigma(s, X^v_s, P_{X^v_s})|^2 ds \right)^4 \right) \\
\leq K \left( |x|^8 + E \int_0^\tau \sup_{0 \leq s \leq \tau} |X^v_s|^8 ds + E \left| \int_0^T |v_s| ds \right|^8 \right)
\]  

(17)

From Gronwall’s inequality, we then have the desired result. \(\Box\)

For \(v_i \in \mathcal{U}, i = 1, 2\), we define

\[
I(v_1, v_2) = \left\{ t \in [0, T] | P(\{ \omega : v_1(t) \neq v_2(t) \}) > 0 \right\}
\]

and \(d(v_1, v_2) = |I(v_1, v_2)|\) is the Lebesgue measure of \(I(v_1, v_2)\). Then, \((\mathcal{U}, d)\) is a metric space.

Given the optimal pair \((X^u, u)\), we now proceed to the perturbation \(X^v\) of \(X^u\). Let

\[
X^{v,1}_t = \int_0^t \left\{ b_x(s)X^{v,1}_s + E \left[ b_\mu(s)\dot{X}^{v,1}_s \right] + \triangle b(s, v) \right\} ds \\
+ \int_0^t \left\{ \sigma_x(s)X^{v,1}_s + E \left[ \sigma_\mu(s)\dot{X}^{v,1}_s \right] \right\} dB_s \\
:= \int_0^t b^1(s, v) ds + \int_0^t \sigma^1(s, v) dB_s,
\]

and

\[
X^{v,2}_t = \int_0^t \left\{ b_x(s)X^{v,2}_s + E \left[ b_\mu(s)\dot{X}^{v,2}_s \right] + \triangle b_x(s, v)X^{v,1}_s + E \left[ \triangle b_\mu(s, v)\dot{X}^{v,1}_s \right] \right. \\
+ \frac{1}{2} b_{xx}(s)X^{v,1}_s \times X^{v,1}_s + E \left[ b_{\mu\mu}(s)X^{v,1}_s \times \dot{X}^{v,1}_s \right] \\
\]
Then, there exists a

\[ + \frac{1}{2} \mathbb{E} \left[ b_{\mu}(s) X_{s}^{v,1} \times \dot{X}_{s}^{v,1} \right] + \frac{1}{2} \mathbb{E} \mathbb{E} \left[ b_{\mu}(s) \dot{X}_{s}^{v,1} \times \dot{X}_{s}^{v,1} \right] \] 

ds

\[ + \int_{0}^{t} \left\{ \sigma_{x}(s) X_{s}^{v,2} + \mathbb{E} \left[ \sigma_{\mu}(s) \dot{X}_{s}^{v,2} \right] \right\} ds \]

By Hypothesis 3.1 and the Burkholder-Davis-Gundy inequality, we have

\[ \mathbb{E} \sup_{0 \leq t \leq T} |X_{t}^{v,1}|^{2} \leq K d^{2}(v, u), \] 

\[ \mathbb{E} \sup_{0 \leq t \leq T} |X_{t}^{v,2}|^{2} \leq K d^{4}(v, u). \] 

**Lemma 4.2.** Assume that Hypothesis 3.1 holds. Then, there exists a \( K > 0 \), such that for any \( v(\cdot), u(\cdot) \in \mathcal{U} \), we have

\[ g_{1}(\tau) = \mathbb{E} \sup_{0 \leq t \leq \tau} |X_{t}^{v,1}|^{2}, \quad g_{2}(\tau) = \mathbb{E} \sup_{0 \leq t \leq \tau} |X_{t}^{v,2}|^{2}. \]

By Hypothesis 3.1 and the Burkholder-Davis-Gundy inequality, we have

\[ g_{1}(\tau) \leq K \left( \int_{0}^{\tau} g_{1}(s) ds + \mathbb{E} \int_{0}^{\tau} |\triangle b(s; v)| ds \right)^{2}, \] 

and

\[ g_{2}(\tau) \leq K \left( \int_{0}^{\tau} g_{2}(s) ds + |g_{1}(T)|^{2} \right) \]

\[ + \mathbb{E} \int_{0}^{T} |\triangle b(s; v)| ds \int_{0}^{T} |\triangle b(s; v)| ds \int_{0}^{T} |\triangle b(s; v)| ds \int_{0}^{T} |\triangle b(s; v)| ds \] 

The application of Grownwall’s inequality allows to obtain that

\[ \mathbb{E} \sup_{0 \leq t \leq T} |X_{t}^{v,1}|^{2} \leq K \left( \mathbb{E} \left[ \int_{0}^{T} |\triangle b(s; v)| ds \right]^{2} \right), \] 

\[ \mathbb{E} \sup_{0 \leq t \leq T} |X_{t}^{v,2}|^{2} \leq K \left( \mathbb{E} \left[ \int_{0}^{T} |\triangle b(s; v)| ds \right]^{4} + \mathbb{E} \int_{0}^{T} |\triangle b(s; v)| ds \int_{0}^{T} |\triangle b(s; v)| ds \int_{0}^{T} |\triangle b(s; v)| ds \int_{0}^{T} |\triangle b(s; v)| ds \right) \]
Notice that the first-order derivative \( b_x \) is bounded. Then, (26) implies the following estimate

\[
\mathbb{E} \sup_{0 \leq t \leq T} |X_t^{v,2}|^2 \leq K \left( \mathbb{E} \left| \int_0^T |\nabla b(s,v)| ds \right|^4 + d^4(u,v) \right).
\]

(27)

According to assumption about \( v \) and \( u \), then,

\[
\mathbb{E} \sup_{0 \leq t \leq T} |X_t^{v,1}|^2 \leq K d^2(v,u), \quad \text{and} \quad \mathbb{E} \sup_{0 \leq t \leq T} |X_t^{v,2}|^2 \leq K d^4(v,u).
\]

(28)

In fact, by Minkowski’s inequality, we have

\[
\mathbb{E} \left| \int_0^T \nabla b(s,v) ds \right|^4 \leq \int_0^T \left( \mathbb{E} \left| \nabla b(s,v) \right|^4 \right) ds \leq K d^4(u,v).
\]

(29)

The following lemma gives the order of \( X_t^{v*} \).

**Lemma 4.3.** Assume Hypothesis 3.1 holds. For \( v(\cdot) \in \mathcal{U} \) and Borel subset \( I_\rho \subset [0,T] \) with Lebesgue measure \( |I_\rho| \), define

\[
\dot{v}_t = v_t I_{\rho}(t) + u_t I_{[0,T}\setminus I_\rho}(t), \quad X_t^{v*} := X^*(t, \dot{v}).
\]

Then we have

\[
\mathbb{E} \sup_{0 \leq t \leq T} |X_t^{v*}|^2 = o(|I_\rho|^4)
\]

(30)

when \( |I_\rho| \to 0 \).

**Proof.** We introduce the following notations first

\[
\nabla b_{xx}(s; \lambda_\eta; v) := b_{xx}(s, X_s^v, \lambda_\eta X_s^{v,12}, P_{X_s^v} + \lambda_\eta X_s^{v,12}, v_s) - b_{xx}(s),
\]

\[
\nabla b_{x\mu}(s; \lambda_\eta; v) := b_{x\mu}(s, X_s^v, P_{X_s^v} + \lambda_\eta X_s^{v,12}, \dot{X}_s^v, \lambda_\eta \dot{X}_s^{v,12}, v_s) - b_{x\mu}(s),
\]

\[
\nabla b_{\mu x}(s; \lambda; v) := b_{\mu x}(s, X_s^v, P_{X_s^v} + \lambda X_s^{v,12}, \dot{X}_s^v, \mu \dot{X}_s^{v,12}, v_s) - b_{\mu x}(s),
\]

where \( X_s^{v,12} := X_s^{v,1} + X_s^{v,2} \). Similarly notations can be introduced with \( b \) replaced by \( \sigma \).

We now proceed to estimating \( X_t^{v*}(\dot{v}) \) defined by (20). By (1), (18) and (19), we have

\[
dX_t^{v*} = \alpha(t) dt + \beta(t) dB_t,
\]

(32)

where

\[
\alpha(t) = b(t, X_t^v, P_{X_t^v}, \dot{v}_t) - \left[ b(t) + b^1(t, \dot{v}) + b^2(t, \dot{v}) \right],
\]

\[
\beta(t) = \sigma(t, X_t^v, P_{X_t^v}) - \left[ \sigma(t) + \sigma^1(t, \dot{v}) + \sigma^2(t, \dot{v}) \right].
\]

We can represent \( \alpha(t) \) as follows.

\[
\alpha(t) = b(t, X_t^v, P_{X_t^v}, \dot{v}_t) - \left[ b(t) + \left\{ b_x(t) X_t^{v,1} + \mathbb{E} \left[ b_{x\mu}(t) \dot{X}_t^{v,1} \right] + \nabla b(t, \dot{v}) \right\} \right.
\]

\[
+ \left. \left\{ b_x(t) X_t^{v,2} + \mathbb{E} \left[ b_{\mu x}(t) \dot{X}_t^{v,2} \right] + \nabla b_x(t, \dot{v}) X_t^{v,1} + \mathbb{E} \left[ \nabla b_x(t, \dot{v}) \dot{X}_t^{v,1} \right] \right\} \right]
\]

(33)
\[+ \frac{1}{2} b_{xx}(t) X_t^{\hat{v},1} \times X_t^{\hat{v},1} + \hat{E} \left[ b_{x\mu}(t) X_t^{\hat{v},1} \times X_t^{\hat{v},1} \right] \]

\[+ \frac{1}{2} \hat{E} \left[ b_{\mu\mu}(t) \hat{X}_t^{\hat{v},1} \times \hat{X}_t^{\hat{v},1} \right] + \frac{1}{2} \hat{E} \left[ b_{\mu\mu}(t) \hat{X}_t^{\hat{v},1} \times \hat{X}_t^{\hat{v},1} \right] \right].

(33)

Denote

\[A(t; \hat{v}) = \frac{1}{2} b_{xx}(t)(X_t^{\hat{v},2} \times X_t^{\hat{v},2} + 2X_t^{\hat{v},1} \times X_t^{\hat{v},2}) + \frac{1}{2} \hat{E} \left[ \hat{b}_{\mu\mu}(t) \hat{X}_t^{\hat{v},1} \times X_t^{\hat{v},2} \right] + \frac{1}{2} \hat{E} \left[ \hat{b}_{\mu\mu}(t) \hat{X}_t^{\hat{v},2} \times X_t^{\hat{v},1} \right] + \frac{1}{2} \hat{E} \left[ \hat{b}_{\mu\mu}(t) \hat{X}_t^{\hat{v},2} \times \hat{X}_t^{\hat{v},2} \right] + \triangle b_x(t; \hat{v}) \hat{X}_t^{\hat{v},2} + \hat{E} \left[ \triangle b_x(t; \hat{v}) \hat{X}_t^{\hat{v},2} \right]

\[+ \hat{E} \left[ b_{xx}(t)(X_t^{\hat{v},2} \times X_t^{\hat{v},2} + X_t^{\hat{v},1} \times X_t^{\hat{v},2} + \hat{X}_t^{\hat{v},1} \times X_t^{\hat{v},2}) \right]

\[+ \int_0^1 \int_0^1 \lambda \left[ \triangle b_{xx}(t; \lambda\eta; \hat{v})d\lambda d\eta X_t^{\hat{v},12} \times X_t^{\hat{v},12} \right]

\[+ \int_0^1 \int_0^1 \hat{E} \left[ \hat{\lambda} \left( \triangle b_{\mu\mu}(t; \lambda\eta; \hat{v})d\lambda d\eta X_t^{\hat{v},12} \times X_t^{\hat{v},12} \right) \right]

\[+ \int_0^1 \int_0^1 \hat{E} \left[ \hat{\lambda} \left( \triangle b_{\mu\mu}(t; \lambda\eta; \hat{v})d\lambda d\eta \hat{X}_t^{\hat{v},12} \times \hat{X}_t^{\hat{v},12} \right) \right]

\[+ \hat{E} \left[ \int_0^1 \triangle b_{\mu\mu}(t; \lambda; \hat{v})d\lambda X_t^{\hat{v},12} \times \hat{X}_t^{\hat{v},12} \right].

(34)

It is easy to show that

\[\alpha(t) = b(t, X_t^{\hat{v}}, P_{X_t^{\hat{v}}}, \hat{v}_t) - \left[ b(t, X_t^{\hat{v}}, P_{X_t^{\hat{v}}} + X_t^{\hat{v},12}, \hat{v}_t) \right.

\[+ b_x(t, X_t^{\hat{v}}, P_{X_t^{\hat{v}}} + X_t^{\hat{v},12}, \hat{v}_t) X_t^{\hat{v},12}

\[+ \int_0^1 \int_0^1 \lambda b_{xx}(t, X_t^{\hat{v}} + \lambda\eta X_t^{\hat{v},12}, P_{X_t^{\hat{v}}} + X_t^{\hat{v},12}, \hat{v}_t)d\lambda d\eta X_t^{\hat{v},12} \times X_t^{\hat{v},12} \right]

\[+ A(t; \hat{v})

\[= b(t, X_t^{\hat{v}}, P_{X_t^{\hat{v}}}, \hat{v}_t) - b(t, X_t^{\hat{v}} + X_t^{\hat{v},12}, P_{X_t^{\hat{v}}} + X_t^{\hat{v},12}, \hat{v}_t) + A(t; \hat{v}).

Similarly, by setting

\[B(t; \hat{v}) := \frac{1}{2} \sigma_{xx}(t)(X_t^{\hat{v},2} \times X_t^{\hat{v},2} + 2X_t^{\hat{v},1} \times X_t^{\hat{v},2})

\[+ \frac{1}{2} \lambda \left[ \triangle \sigma_{xx}(t; \lambda\eta; \hat{v})d\lambda d\eta X_t^{\hat{v},12} \times X_t^{\hat{v},12} \right]

\[+ \frac{1}{2} \hat{E} \left[ \sigma_{\mu\mu}(t)(X_t^{\hat{v},2} \times X_t^{\hat{v},2} + X_t^{\hat{v},1} \times X_t^{\hat{v},2} + \hat{X}_t^{\hat{v},1} \times \hat{X}_t^{\hat{v},2}) \right]

\[+ \hat{E} \left[ \sigma_{\mu\mu}(t)(X_t^{\hat{v},2} \times \hat{X}_t^{\hat{v},2} + X_t^{\hat{v},1} \times \hat{X}_t^{\hat{v},2} + \hat{X}_t^{\hat{v},1} \times \hat{X}_t^{\hat{v},2}) \right]

\[+ \int_0^1 \int_0^1 \lambda \left[ \triangle \sigma_{xx}(t; \lambda\eta; \hat{v})d\lambda d\eta X_t^{\hat{v},12} \times X_t^{\hat{v},12} \right]

\[+ \int_0^1 \int_0^1 \hat{E} \left[ \hat{\lambda} \left( \triangle \sigma_{\mu\mu}(t; \lambda\eta; \hat{v})d\lambda d\eta X_t^{\hat{v},12} \times X_t^{\hat{v},12} \right) \right].

(35)
According to Gronwall’s inequality, we have
\begin{align}
\beta(t) &= \sigma(t, X_t^\hat{v}, P_{X_t^\hat{v}}) - \left[\sigma(t, X_t^u, P_{X_t^u} + X_t^{\hat{v}, 12}) + \sigma_x(t, X_t^u, P_{X_t^u} + X_t^{\hat{v}, 12}) X_t^{\hat{v}, 12}\right]
+ \int_0^t \int_0^1 \dot{\sigma}_x(t; \lambda; \hat{v}) d\lambda d\eta X_t^\hat{v} 
+ \tilde{E} \left[\int_0^1 (\Delta \sigma_x(t; \lambda; \hat{v}) d\lambda X_t^{\hat{v}, 12} \times \hat{X}_t^{\hat{v}, 12}) d\lambda X_t^{\hat{v}, 12} \times \hat{X}_t^{\hat{v}, 12}\right],
\end{align}
(35)
we have
\begin{align}
\beta(t) &= \sigma(t, X_t^\hat{v}, P_{X_t^\hat{v}}) - \sigma(t, X_t^u + X_t^{\hat{v}, 12}, P_{X_t^u} + X_t^{\hat{v}, 12}) + B(t; \hat{v}).
\end{align}
According to Hypothesis 3.1, we have
\begin{align}
|b(t, X_t^\hat{v}, P_{X_t^\hat{v}}; \hat{v}_t) - b(t, X_t^u + X_t^{\hat{v}, 12}, P_{X_t^u} + X_t^{\hat{v}, 12}; \hat{v}_t)|
\leq K \left(|X_t^{\hat{v}, s}| + W_2(P_{X_t^\hat{v}}, P_{X_t^u} + X_t^{\hat{v}, 12})\right).
\end{align}
Note that
\begin{align}
W_2(P_{X_t^\hat{v}}, P_{X_t^u} + X_t^{\hat{v}, 12})^2 \leq \mathbb{E} \left|X_t^\hat{v} - X_t^u - X_t^{\hat{v}, 12}\right|^2 = \mathbb{E} \left|X_t^{\hat{v}, s}\right|^2.
\end{align}
By Burkholder-Davis-Gundy inequality, for \(\tau \in [0, T]\), we obtain the following estimation
\begin{align}
\mathbb{E} \sup_{0 \leq t \leq \tau} |X_t^{\hat{v}, s}|^2 \leq \int_0^T \mathbb{E} \sup_{0 \leq \tau \leq s} |X_r^{\hat{v}, s}|^2 ds + \mathbb{E} \int_0^T |A(s; \hat{v})|^2 ds 
+ \mathbb{E} \int_0^T |B(s; \hat{v})|^2 ds.
\end{align}
(36)
According to Gronwall’s inequality, we have
\begin{align}
\mathbb{E} \sup_{0 \leq t \leq T} |X_t^{\hat{v}, s}|^2 \leq K \left(\mathbb{E} \int_0^T |A(s; \hat{v})|^2 ds + \mathbb{E} \int_0^T |B(s; \hat{v})|^2 ds\right).
\end{align}
(37)
About \(A(s; \hat{v})\) we have
\begin{align}
\mathbb{E} \int_0^T |A(s; \hat{v})|^2 ds 
\leq K \mathbb{E} \left(\sup_{0 \leq s \leq t} |X_s^{\hat{v}, 2}|^4 + \sup_{0 \leq s \leq t} |X_s^{\hat{v}, 1}|^2 \sup_{0 \leq s \leq t} |X_s^{\hat{v}, 2}|^2 
+ \sup_{0 \leq s \leq t} |X_s^{\hat{v}, 2}|^4 \int_0^T \left[|\Delta b_x(s; \hat{v})|^2 + |\Delta b_{\mu}(s; \hat{v})|^2\right] ds
+ \sup_{0 \leq s \leq t} |X_s^{\hat{v}, 1}|^4 \int_0^T \int_0^1 \int_0^1 |\lambda \Delta b_{xx}(s; \lambda \eta; \hat{v})|^2 d\lambda d\eta ds
+ \sup_{0 \leq s \leq t} |X_s^{\hat{v}, 1}|^4 \int_0^T \int_0^1 \int_0^1 \mathbb{E} \mathbb{E} \left|\lambda \Delta b_{\mu\mu}(s; \lambda \eta; \hat{v})\right|^2 d\lambda d\eta ds
+ \sup_{0 \leq s \leq t} |X_s^{\hat{v}, 1}|^4 \int_0^T \int_0^1 \int_0^1 \mathbb{E} \left|\lambda \Delta b_{\mu}(s; \lambda \eta; \hat{v})\right|^2 d\lambda d\eta ds\right).
\end{align}
Similarly, cost functional according to different order of the perturbation.

Expansion of the cost functional with respect to control variable.

5. Expansion of the cost functional with respect to control variable. In this section, we use the method of Lemma 4.3 again to study the expansion of the cost functional according to different order of the perturbation.

\[
+ \sup_{0 \leq s \leq t} |X^{\hat{\nu},1}_s|^4 \bar{E} \int_0^T \int_0^1 |\triangle b_{x\mu}(s; \lambda; \hat{v})|^2 d\lambda ds \].
\]

Note that

\[
E \left| \int_0^T |\triangle b(s; \hat{v})| ds \right|^8 \leq K |I_\rho|^8,
\]
similar estimates hold with \( b \) replaced by \( b_x \) and \( b_{\mu} \). Since \( X^{\hat{\nu},1}_t \to 0 \) as \( |I_\rho| \to 0 \), so we also have

\[
\triangle b_{xx}(s; \lambda\eta; \hat{v}) \to 0,
\]
replace \( b_{xx} \) by \( b_{\mu\mu}, b_{\mu\gamma} \) and \( b_{x\mu} \), we can get the similar result when \( |I_\rho| \to 0 \).

According to estimation of \( X^{\hat{\nu},1}_x, X^{\hat{\nu},2}_x \) in Lemma 4.2, we obtain

\[
E \int_0^T |A(s; \hat{v})|^2 ds
\]

\[
\leq K \left( E \left| \int_0^T |\triangle b(s; \hat{v})| ds \right|^8 + \sqrt{E \left| \int_0^T |\triangle b_x(s; \hat{v})| ds \right|^8}
\right.
\]

\[
+ \sqrt{E \left| \int_0^T |\triangle b_{x\mu}(s; \hat{v})| ds \right|^8} \times \left( \sqrt{E \left| \int_0^T |\triangle b(s; \hat{v})| ds \right|^8}\right)
\]

\[
+ \sqrt{E \int_0^T |\triangle b(s; \hat{v})| ds} + \sqrt{E \int_0^T |\triangle b_x(s; \hat{v})| ds} + \sqrt{E \int_0^T |\triangle b_{x\mu}(s; \hat{v})| ds} \right.
\]

\[
+ \sqrt{E \int_0^T |\lambda \triangle b_{xx}(s; \lambda \eta; \hat{v})| d\lambda d\eta ds} + \sqrt{E \int_0^T |\lambda \triangle b_{x\mu}(s; \lambda \eta; \hat{v})| d\lambda d\eta ds}
\]

\[
+ \sqrt{E \int_0^T |\lambda \triangle b_{x\mu}(s; \lambda \eta; \hat{v})| d\lambda d\eta ds} + \sqrt{E \int_0^T |\lambda \triangle b_{x\gamma}(s; \lambda \eta; \hat{v})| d\lambda d\eta ds}
\]

\[
+ \sqrt{E \int_0^T |\lambda \triangle b_{\mu\gamma}(s; \lambda \eta; \hat{v})| d\lambda d\eta ds} + \sqrt{E \int_0^T |\lambda \triangle b_{\mu\mu}(s; \lambda \eta; \hat{v})| d\lambda d\eta ds}
\]

\[
+ \sqrt{E \int_0^T |\lambda \triangle b_{\mu\nu}(s; \lambda \eta; \hat{v})| d\lambda d\eta ds} + \sqrt{E \int_0^T |\lambda \triangle b_{\mu\nu}(s; \lambda \eta; \hat{v})| d\lambda d\eta ds}
\]

\[
\left. + \sqrt{E \int_0^T |\lambda \triangle b_{\mu\nu}(s; \lambda \eta; \hat{v})| d\lambda d\eta ds} \right) = o \left( |I_\rho|^4 \right), \text{ as } |I_\rho| \to 0. \tag{39}
\]

Similarly

\[
E \int_0^T |B(s; \hat{v})|^2 ds = o \left( |I_\rho|^4 \right), \text{ as } |I_\rho| \to 0. \tag{40}
\]

Finally, by (37) we have the desire result. \( \square \)
Lemma 5.1. Assume that Hypothesis 3.1 holds. Define
\[
J^*(v.) = J(v.) - J(u.) - \mathbb{E} \int_t^t \left\{ h_x(s)X_s^{v.1} + \mathbb{E} \left[ \mu(s)X_s^{v.1} \right] + \Delta h(s, v)
+ h_x(s)X_s^{v.2} + \mathbb{E} \left[ \mu(s)X_s^{v.2} \right] + \Delta h_x(s, v)X_s^{v.1} + \mathbb{E} \left[ \Delta h(s, v)X_s^{v.1} \right]
+ \frac{1}{2} h_{xx}(s)X_s^{v.1} \times X_s^{v.1} + \mathbb{E} \left[ h_{x\mu}(s)X_s^{v.1} \times X_s^{v.1} \right]
+ \frac{1}{2} \mathbb{E} \left[ h_{\mu\mu}(s)X_s^{v.1} \times X_s^{v.1} \right] \right\} ds.
\]
(41)
Recall that \( \hat{v} \) is defined by (30) by \( I_\rho \). Then, when \( |I_\rho| \to 0 \), we have
\[
J^*(\hat{v}.): = o \left( |I_\rho|^2 \right).
\]
(42)
Proof. Denote
\[
Y_t^v = \int_0^t h(s, X_s^v, P_{X_s^v}, v_s) ds.
\]
(43)
By (20) and Lemma 4.3, we have
\[
Y_t^v = Y_t^u + Y_t^{v.1} + Y_t^{v.2} + Y_t^{v.s},
\]
(44)
where
\[
Y_t^v = \int_0^t h(s, X_s^v, P_{X_s^v}, v_s) ds,
\]
(45)
\[
Y_t^{v.1} = \int_0^t \left\{ h_x(s)X_s^{v.1} + \mathbb{E} \left[ \mu(s)X_s^{v.1} \right] + \Delta h(s, v) \right\} ds,
\]
(46)
\[
Y_t^{v.2} = \int_0^t \left\{ h_x(s)X_s^{v.2} + \mathbb{E} \left[ \mu(s)X_s^{v.2} \right] + \Delta h_x(s, v)X_s^{v.1} + \mathbb{E} \left[ \Delta h(s, v)X_s^{v.1} \right]
+ \frac{1}{2} h_{xx}(s)X_s^{v.1} \times X_s^{v.1} + \mathbb{E} \left[ h_{x\mu}(s)X_s^{v.1} \times X_s^{v.1} \right]
+ \frac{1}{2} \mathbb{E} \left[ h_{\mu\mu}(s)X_s^{v.1} \times X_s^{v.1} \right] \right\} ds.
\]
(47)
Then,
\[
J(v.) - J(u.) = \mathbb{E} Y_t^v - \mathbb{E} Y_t^u,
\]
(48)
and hence,
\[
J^*(\hat{v}.): = \mathbb{E} Y_t^v - \mathbb{E} Y_t^u - \mathbb{E} Y_t^{v.1} - \mathbb{E} Y_t^{v.2}.
\]
(49)
Using the same method in Lemma 4.3, we complete the proof. □

Now, we proceed to deriving the expansion of the perturbed cost function.
\[
X_t^{v.12} = \int_0^t \left\{ b_x(s)X_s^{v.1} + \mathbb{E} \left[ \mu(s)X_s^{v.1} \right] + \Delta b(s, v)
+ b_x(s)X_s^{v.2} + \mathbb{E} \left[ \mu(s)X_s^{v.2} \right] + \Delta b_x(s, v)X_s^{v.1} + \mathbb{E} \left[ \Delta b(s, v)X_s^{v.1} \right]
+ \frac{1}{2} h_{xx}(s)X_s^{v.1} \times X_s^{v.1} + \mathbb{E} \left[ h_{x\mu}(s)X_s^{v.1} \times X_s^{v.1} \right]
+ \frac{1}{2} \mathbb{E} \left[ h_{\mu\mu}(s)X_s^{v.1} \times X_s^{v.1} \right] \right\} ds.
\]
we can choose a suitable
Recalling that
Now we apply the range theorem for vector-valued measures due to
According to Lemma 4.1 and applying Itô’s formula to \( p_t X_t^{v,12} \), we obtain
Hence,
Now we apply the range theorem for vector-valued measures due to [18], to deduce the variational inequality.
Recall that \( \hat{v}_t \) is defined by (30). According to Lemma 4.1 [18], for any \( 0 < \rho < 1 \), we can choose a suitable \( I_\rho \subset [0, T] \) such that \(|I_\rho| = \rho T\).
\[
\rho \int_0^T \Delta b(s; v) ds = \int_{I_\rho} \Delta b(s; v) ds + \eta^*, \ E|\eta^*|^2 = o(\rho^4),
\]
\[
\rho \int_0^T E[\Delta H(s; v)] ds = \int_{I_\rho} E[\Delta H(s; v)] ds + o(\rho^2),
\]
and
\[
\rho \int_0^T \mathbb{E} \left\{ \Delta H_x(s;v)X_s^{v,1} + \tilde{E} \left[ \Delta H_\mu(s;v)\tilde{X}_s^{v,1} \right] + \Delta b^*(s;v)P_s X_s^{v,1} \right\} ds \\
= \int _{I_\rho} \mathbb{E} \left\{ \Delta H_x(s;v)X_s^{v,1} + \tilde{E} \left[ \Delta H_\mu(s;v)\tilde{X}_s^{v,1} \right] + \Delta b^*(s;v)P_s X_s^{v,1} \right\} ds \\
+ o(\rho^2). \tag{55}
\]

**Lemma 5.2.** For the $I_\rho$ above, $t \in [0, T]$, we also have
\[
\rho \int_0^t \Delta b(s;v) ds = \int _{I_\rho \cap [0,t]} \Delta b(s;v) ds + \eta^*_t, \quad \sup_{0 \leq t \leq T} \mathbb{E} |\eta^*_t|^2 = o(\rho^4). \tag{56}
\]

The proof of the above lemma is essentially the same as Lemma 4.1 [18]. For the convenience of readers, we present the proof here.

**Proof.** Let $\phi_i(\cdot) \in L^2 (\Omega;L^2(0,T;\mathbb{R}^k))$, $l_i \geq 1$, $i = 1, \cdots, k$. Suppose
\[
\sup_{0 \leq t \leq T} \mathbb{E} |\phi_1(t)|^2 < \infty
\]
. Given $0 < \rho < 1$ and set $\delta = \rho^2$, then there exists a $n > 0$, we can find a process $\phi^*_n(\cdot)$ in the form of
\[
\phi^*_n(t) = \sum_{i=0}^n \xi_i^ j I_{[t_j,t_{j+1})}(t), \quad 1 \leq i \leq k,
\]
with $0 = t_0 < t_1 < \cdots < t_{n+1} = T$, $\max |t_{i+1} - t_i| < \delta$, $\xi_i^ j$ being $\mathcal{F}_{t_j}$-measurable, s.t.
\[
\sup_{0 \leq t \leq T} \mathbb{E} |\phi_1(t) - \phi^*_n(t)|^2 < \delta. \tag{57}
\]
\[
\sum_{i=2}^k \mathbb{E} \left( \int_0^T |\phi_i(t) - \phi^*_n(t)|^2 dt \right) < \delta. \tag{58}
\]

Note that we can always choose the partition $\{t_j\}_{0 \leq j \leq n+1}$ independent of $i = 1, \cdots, k$. Now letting
\[
G = \bigcup_{j=0}^n [t_j, t_j + \rho(t_{j+1} - t_j)) \}
\]
It’s easy to see $|G| = \rho T$. Thus (53), (54), and (55) are proved by taking $\phi_i$ suitably. For any $s \in [0, T]$, we can always find a $m \geq 0$, s.t. $s \in [t_m, t_{m+1})$. Then we have
\[
\sup_{0 \leq s \leq T} \mathbb{E} \left| \int_0^T I_{(t \leq s)} \left( 1 - \frac{I_G(t)}{\rho} \right) \phi^*_n(t) dt \right|^2 \\
\leq \sup_{0 \leq s \leq T} \mathbb{E} \left| \sum_{j=0}^{m-1} \xi^ j \left( t_{j+1} - t_j \right) - \frac{\rho(t_{j+1} - t_j)}{\rho} \right| + \xi^m \left( s - t_m \right) - \frac{(s - t_m)}{\rho} \right|^2 \\
+ \xi^m \left( s - t_m \right) - \frac{(s - t_m)}{\rho} \right|^2 \\
+ \xi^m \left( s - t_m \right) - \frac{(s - t_m)}{\rho} \right|^2
\]
Thus, we can get
\begin{equation}
\sup_{0 \leq s \leq T} \mathbb{E} \left[ \int_0^T I_{\{t \leq s\}} \left( 1 - \frac{I_G(t)}{\rho} \right) \phi_1(t) dt \right]^2 
\leq K \delta^2 \rho^{-2} + K \delta^2,
\end{equation}
where $I_G(t)$ is the indicator function of $G$. Thus, for each $1 \leq i \leq k$,
\begin{align*}
\sup_{0 \leq s \leq T} \mathbb{E} \left[ \int_0^T I_{\{t \leq s\}} \left( 1 - \frac{I_G(t)}{\rho} \right) \phi_1(t) - I_{\{t \leq s\}} \left( 1 - \frac{I_G(t)}{\rho} \right) \phi'_i(t) dt \right]^2 
&\leq \sup_{0 \leq s \leq T} \mathbb{E} \left[ \int_0^T I_{\{t \leq s\}} \left( 1 - \frac{I_G(t)}{\rho} \right) \phi_1(t) - I_{\{t \leq s\}} \left( 1 - \frac{I_G(t)}{\rho} \right) \phi'_i(t) dt \right]^2 
&+ K \delta^2 \rho^{-2} 
\leq \mathbb{E} \left[ \int_0^T \left( 1 - \frac{I_G(t)}{\rho} \right) \left| \phi_1(t) - \phi'_i(t) \right| dt \right]^2 + K \delta^2 \rho^{-2}.
\end{align*}
Further more, inspired by (59) with $I_G$ given here.

**Lemma 5.3.** For any $\rho \in [0, 1]$, there exists a subset $I_\rho$ of $[0, T]$, such that
\begin{equation}
\lim_{\rho \to 0^+} \sup_{0 \leq t \leq T} \mathbb{E} \left[ \frac{X_{t}^{\rho, 1} - \rho X_{t}^{\nu, 1}}{\rho} \right]^2 = 0.
\end{equation}
where $\nu$ is defined by (30) with $I_\rho$ given here.

**Proof.** By (18) and lemma 5.2, we have
\begin{align*}
X_{t}^{\rho, 1} - \rho X_{t}^{\nu, 1} 
&= \int_0^t b_s(s) \left( X_{s}^{\rho, 1} - \rho X_{s}^{\nu, 1} \right) ds + \int_0^t \mathbb{E} \left[ b_\mu(s) \left( X_{s}^{\rho, 1} - \rho X_{s}^{\nu, 1} \right) \right] ds 
&\quad + \int_0^t \left( \Delta b(s; \hat{v}) - \rho \Delta b(s; \nu) \right) ds 
&\quad + \int_0^t \sigma_s(s) \left( X_{s}^{\rho, 1} - \rho X_{s}^{\nu, 1} \right) ds + \int_0^t \mathbb{E} \left[ \sigma_\mu(s) \left( X_{s}^{\rho, 1} - \rho X_{s}^{\nu, 1} \right) \right] dB_s.
\end{align*}
Thus, we can get
\begin{align*}
\mathbb{E} \left| X_{t}^{\rho, 1} - \rho X_{t}^{\nu, 1} \right|^2 
&\leq K \int_0^t \mathbb{E} \left| X_{s}^{\rho, 1} - \rho X_{s}^{\nu, 1} \right|^2 ds + \sup_{0 \leq t \leq T} \mathbb{E} \left[ \int_0^t \left( \Delta b(s; \hat{v}) - \rho \Delta b(s; \nu) \right) ds \right]^2.
\end{align*}
Notice that
\begin{equation}
K \sup_{0 \leq t \leq T} \mathbb{E} \left[ \frac{1}{\rho} \left( \int_0^t \left( \Delta b(s; \hat{v}) - \rho \Delta b(s; \nu) \right) ds \right)^2 \right] 
= K \sup_{0 \leq t \leq T} \mathbb{E} \left[ \frac{1}{\rho^2} \left( \int_0^t \left( \Delta b(s; \hat{v}) - \rho \Delta b(s; \nu) \right) ds \right)^2 \right] = o(1).
\end{equation}
By Gronwall's inequality, we get (61).
Lemma 5.4. For any \( \rho \in [0, 1] \), there exists a set \( I_\rho \in [0, T] \) and a matrix value process \( \Phi(t) \), s.t. \( X^{\Phi,1}_t \) is represented by the following

\[
X^{\Phi,1}_t = \Phi(t) \int_0^t \Phi^{-1}(s) \triangle b(s; \hat{v}) dt + A^*_t,
\]

where

\[
\lim_{\rho \to 0^+} \sup_{0 \leq t \leq T} \mathbb{E} \left[ \frac{A^*_t}{\rho} \right] = 0.
\]

Proof. Let \( \Phi(t) \) be the unique solution of the following matrix value SDE:

\[
\Phi(t) = I + \int_0^t \left\{ b_x(s) + \bar{E} [b_y(s)] \right\} \Phi(s) ds + \int_0^t \left\{ \sigma_x(s) + \bar{E} [\sigma_y(s)] \right\} \Phi(s) dB_s,
\]

and set

\[
\Psi(t) = I + \int_0^t \left\{ -b_x(s) + \bar{E} [b_y(s)] \right\} \Phi(s) ds + \int_0^t \left\{ \sigma_x(s) + \bar{E} [\sigma_y(s)] \right\} \Phi(s) dB_s.
\]

Applying Itô's formula to \( \Psi_t \Phi_{t,1} \), we can easily get \( d[\Psi_t \Phi_{t,1}] = 0 \), which means \( \Psi_t \Phi_{t,1} \equiv I \), i.e. \( \Psi_t = \Phi_{t,1}^{-1} \). Applying Itô's formula to \( d[\Psi_t X^{\Phi,1}_t] \), by Lemma 5.3, we then get our desire result. \( \square \)

Now we continue to derive the expansion. Applying Itô's formula to \( Y_t := X^{\Phi,1}_t X^{\Phi,1}_{*t} \), we have

\[
dY_t = \left\{ Y_t b^*_x(t) + X^{\Phi,1}_t \bar{E} \left[ X^{\Phi,1}_{*t,1} b^*_y(t) \right] + X^{\Phi,1}_t \triangle b^*(t; \hat{v}) + b_x(t) Y_t + \bar{E} \left[ b_y(t) X^{\Phi,1}_t \right] X^{\Phi,1}_t + \triangle b(t; v) X^{\Phi,1}_t + \left( \sigma_x(t) X^{\Phi,1}_t + \bar{E} \left[ \sigma_y(t) X^{\Phi,1}_t \right] \right) \left( X^{\Phi,1}_{*t,1} \sigma^*_x(t) + \bar{E} \left[ X^{\Phi,1}_{*t,1} \sigma^*_y(t) \right] \right) \right\} dt + \left\{ Y_t \sigma^*_x(t) + X^{\Phi,1}_t \bar{E} \left[ X^{\Phi,1}_{*t,1} \sigma^*_y(t) \right] + \sigma_x(t) Y_t + \bar{E} \left[ \sigma_y(t) X^{\Phi,1}_t \right] X^{\Phi,1}_{*t,1} \right\} dB_t.
\]

Applying Itô's formula to \( P_t Y_t \), according to (61), we obtain

\[
\mathbb{E} \int_0^T \text{Trace} \{ H_{xy}(s) X^{\Phi,1}_t X^{\Phi,1}_{*t} \} ds + 2 \mathbb{E} \bar{E} \int_0^T \text{Trace} \{ H_{xy}(s) X^{\Phi,1}_t \hat{X}^{\Phi,1}_{*t} \} ds + 2 \mathbb{E} \bar{E} \int_0^T \text{Trace} \{ H_{y}(s) \hat{X}^{\Phi,1}_t \} ds + \mathbb{E} \int_0^T \text{Trace} \{ P_s X^{\Phi,1}_t \triangle b^*(t; \hat{v}) + P_s \triangle b(s; \hat{v}) X^{\Phi,1}_{*t} \} ds + o(\rho^2) = 2 \mathbb{E} \int_0^T \left[ \triangle b^*(s; \hat{v}) P_s X^{\Phi,1}_t \right] ds + o(\rho^2).
\]
Finally, according to \((70)\), we conclude
\[
J(\hat{v}) - J(u) = \mathbb{E} \int_0^T \Delta H(s; v) ds + \mathbb{E} \int_0^T \Delta H_x(s; \hat{v}) X_{s}^{v,1} ds \\
+ \mathbb{E} \mathbb{E} \int_0^T \Delta H_\mu(s; \hat{v}) X_{s}^{v,1} ds \\
\mathbb{E} \int_0^T [\triangle b^*(s; \hat{v}) P_s X_{s}^{v,1}] ds + o(\rho^2). \tag{71}
\]
From Lemma 5.1, \((61)\) and \((71)\), we obtain
\[
J(\hat{v}) - J(u) = \mathbb{E} \int_0^T \Delta H(s; v) ds + \rho \mathbb{E} \int_0^T \Delta H_x(s; v) X_{s}^{v,1} ds \\
+ \mathbb{E} \mathbb{E} \int_0^T \Delta H_\mu(s; v) X_{s}^{v,1} ds \\
+ \rho \mathbb{E} \mathbb{E} \int_0^T \Delta H_\mu(s; v) X_{s}^{v,1} ds + \rho \mathbb{E} \int_0^T [\triangle b^*(s; v) P_s X_{s}^{v,1}] ds \\
+ o(\rho^2). \tag{72}
\]
Finally, according to \((55)\), we have
\[
J(\hat{v}) - J(u) = \rho \mathbb{E} \int_0^T \Delta H(s; v) ds + \rho^2 \mathbb{E} \int_0^T \Delta H_x(s; v) X_{s}^{v,1} ds \\
+ \rho^2 \mathbb{E} \int_0^T \Delta H_\mu(s; v) X_{s}^{v,1} ds + \rho^2 \mathbb{E} \int_0^T [\triangle b^*(s; v) P_s X_{s}^{v,1}] ds \\
+ o(\rho^2). \tag{73}
\]

6. The proofs of the stochastic maximum principle. Although the first-order SMP has been obtained by \([5]\), we give a proof here for completeness. In fact, after the preparation of the previous sections which will also be needed in the proof of the second-order SMP, this proof does not take too much extra effort.

**Proof of first-order SMP:** Since \((X^u, u)\) is an optimal pair of our system, it follows from \((73)\) that
\[
J(\hat{v}) - J(u) = \rho \mathbb{E} \int_0^T \Delta H(s; v) ds + o(\rho) \geq 0. \tag{74}
\]
for any \(\rho \in [0, T], \forall v(\cdot) \in \mathcal{U}\). Setting \(\rho \rightarrow 0^+\), we obtain
\[
\mathbb{E} \int_0^T \Delta H(s; v) ds \geq 0, \forall v(\cdot) \in \mathcal{U}. \tag{75}
\]
Then we can deduce that, for any fixed \(v \in \mathcal{U}\), there exists a null subset \(S^v \subset [0, T] \times \Omega\), such that for each \((t, \omega) \in (S^v)^c\),
\[
\Delta H(s; v) \geq 0. \tag{76}
\]
Otherwise, suppose that
\[ A = \{(s, \omega) : \triangle H(s; v^*) < 0\} \]
has positive measure in \([0, T] \times \Omega\), for a \(v^* \in \mathcal{U}\). Let
\[ \hat{v}^* = v^*1_A + u1_{A^c}. \]
Then,
\[ \mathbb{E} \int_0^T \triangle H(s; \hat{v}^*) ds = \mathbb{E} \int_0^T \triangle H(s; v^*) 1_A ds < 0. \quad (77) \]
This contradicts from (75).

Select a countable dense subset \(\{v^{(i)}\}_{i=1}^\infty \subset \mathcal{U}\), set
\[ S_0 = \bigcup_{i=1}^\infty S^{v^{(i)}}. \]
Then, \(S_0\) is a null subset of \([0, r] \times \Omega\), and for \((t, \omega) \in S := (S_0)^c\), we get
\[ \triangle H(s; v^{(i)}) \geq 0. \quad (78) \]
By Fubini’s theorem, it is easy to see that there exists a null subset \(T_0\) of \([0, T]\), such that \(\forall t \in T_0\), (78) holds \(a.s..\).

Finally, from the continuity of the function and the denseness of \(\{v^{(i)}\}_{i=1}^\infty\), we have for \(t \in (T_0)^c\),
\[ \triangle H(s; v) \geq 0, \ \forall v \in \mathcal{U}, a.s. \quad (79) \]

Now, we proceed to presenting the proof of the second-order stochastic maximum principle for singular generalized mean-field control problem.

**Proof of Theorem 3.2.** The optimality and the singularity imply that
\[ \triangle H(t; v) \equiv 0, \ \forall v \in \mathcal{V}. \]
According to (73), we have
\[
\begin{align*}
\mathbb{E} \int_{t_1}^{t_2} & \left\{ \triangle H_s(s; v)X_s^{v,1} + \hat{\mathbb{E}} \left[ \triangle H_{\mu}(s; v)X_s^{v,1} \right] + \triangle b^*(s; v)P_sX_s^{v,1} \right\} ds \\
\geq & \ 0, \ \forall v \in \mathcal{V}(t_1, t_2), a.s.,
\end{align*}
\]
where
\[
\mathcal{V}(t_1, t_2) := \left\{ v(\cdot) \in \mathcal{U}| v_1 \in \mathcal{V}, a.s., a.e., t \in [t_1, t_2]; v(t) = u(t), \right\}
\]
\[ t \in [0, T] \setminus [t_1, t_2]. \]

As in [11][23], denote by \(\{t_i\}_{i=1}^\infty\) the collection of all rational numbers in \([0, T]\), and \(\{v_k\}_{k=1}^\infty\) a dense subset of \(\mathcal{V}\). Because of the fact that \(\mathcal{F}_t\) is countability generated for \(t \in [0, T]\), we can assume \(\{A_{ij}\}_{i=1}^\infty\) generates \(\mathcal{F}_t, \ i = 1, 2, \cdots\). For any \(\tau \in [t, T]\) and \(\theta \in (0, T - \tau)\), write \(E_{\theta}^t = [\tau, \tau + \theta]\), and define
\[
v_{i,j}^k(t, \omega) = \left\{ v_k(t, \omega), \ (t, \omega) \in E_0^t \times A_{i,j}, \right\}
\]
\[ u(t, \omega), \ (t, \omega) \in (E_0^t \times A_{i,j})^c. \quad (81) \]

Let \(X_{ij}^{1k}\) be the solution to the equation (18) with respect to \(v_{i,j}^k(\cdot)\). Notice that we can always choose suitable \(I_\theta\), such that \(I_\theta \cap E_0^t = E_0^t\). So Lemma 5.4 holds for \(X_{ij}^{1k}\).
By Lemma 4.1 [25], lemma 4.2 and Lebesgue differential theorem, there is a null
subset $T_{ij}^k \subset [0, T]$ such that for $t \in (T_{ij}^k)^c$, we have

$$0 \leq \lim_{\theta \to 0^+} \frac{1}{\theta^2} \int_{\tau}^{\tau+\theta} \mathbb{E} \left\{ \triangle H_x(s; v_{ij}^k)X_{ij}^{1k}(s) + \tilde{\mathbb{E}} \left[ \triangle H_{\mu}(s; v_{ij}^k)\tilde{X}_{ij}^{1k}(s) \right] \right. $$

$$+ \triangle b^*(s; v_{ij}^k)P_s X_{ij}^{1k}(s) \left. \right\} ds $$

$$= \lim_{\theta \to 0^+} \frac{1}{\theta^2} \int_{\tau}^{\tau+\theta} \mathbb{E} \left\{ \triangle H_x(s; v_{ij}^k)\Phi(s) \int_{\tau}^{s} \Phi^{-1}(r) \triangle b(r; v_k)1_{A_{ij}} dr 
\right. $$

$$+ \tilde{\mathbb{E}} \left[ \triangle H_{\mu}(s; v_{ij}^k)\Phi(s) \int_{\tau}^{s} \Phi^{-1}(r) \triangle \tilde{b}(r; v_k)1_{A_{ij}} dr \right] $$

$$+ \triangle b^*(s; v_{ij}^k)P_s \Phi(s) \int_{\tau}^{s} \Phi^{-1}(r) \triangle b(r; v_k)1_{A_{ij}} dr \right\} ds $$

$$= \mathbb{E} \left\{ \triangle H_x(\tau; v_k) \triangle b(\tau; v_k)1_{A_{ij}} + \tilde{\mathbb{E}} \left[ \triangle H_{\mu}(\tau; v_k) \triangle \tilde{b}(\tau; v_k)1_{A_{ij}} \right] \right. $$

$$+ \triangle b^*(\tau; v_k)P_\tau \triangle b(\tau; v_k)1_{A_{ij}} \left. \right\}. \quad (82)$$

Set

$$T_0 = \bigcup_{1 \leq i,j,k \leq \infty} T_{ij}^k.$$  

Then, $T_0$ is a null subset of $[0, T]$. For $s \in [0, T] \setminus T_0$ and $i$, we deduce that

$$\mathbb{E} \left\{ \triangle H_x(s; v_k) \triangle b(s; v_k)1_{A_{ij}} + \tilde{\mathbb{E}} \left[ \triangle H_{\mu}(s; v_k) \triangle \tilde{b}(s; v_k)1_{A_{ij}} \right] \right. $$

$$+ \triangle b^*(s; v_k)P_s \triangle b(s; v_k)1_{A_{ij}} \left. \right\} \geq 0, \quad \forall j, k = 1, 2, \cdots, \quad (83)$$

which means

$$\mathbb{E} \left\{ \triangle H_x(s; v) \triangle b(s; v)1_A + \tilde{\mathbb{E}} \left[ \triangle H_{\mu}(s; v) \triangle \tilde{b}(s; v)1_A \right] \right. $$

$$+ \triangle b^*(s; v)P(s) \triangle b(s; v)1_A \left. \right\} \geq 0, \quad \forall v \in V, \ A \in \mathcal{F}_t. \quad (84)$$

By virtue of the continuity of the function and the denseness of $\{v_k\}_{k=1}^\infty$, we finish the proof. \hfill \Box

**Remark 2.** We now come back to Example 1.1. It is not hard to check that the second order adjoint process $(P_t, Q_t) \equiv (1, 0)$. Then

$$\triangle H_x(t; v) \triangle b(t; v) + \tilde{\mathbb{E}} \left[ \triangle H_{\mu}(t; v) \triangle \tilde{b}(t; v) \right] $$

$$+ \triangle b^*(t; v)P_t \triangle b(t; v) = v^2 \geq 0, \quad \forall v \in U, \ a.s..$$

So we can say $u_t \equiv 0$ is the only candidate for optimal controls.

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