COMMUTATION RELATIONS AND
HYPERCYCLIC OPERATORS

VITALY E. KIM

Abstract

In this paper we establish hypercyclicity of continuous linear operators on $H(\mathbb{C})$ that satisfy certain commutation relations.

1 Introduction and statement of the main results

Let $X$ be a topological vector space. A continuous linear operator $T : X \to X$ is said to be hypercyclic if there exists an element $x \in X$ such that its orbit $\text{Orb}(T, x) = \{T^n x, \ n = 0, 1, 2, \ldots\}$ is dense in $X$. Let us denote by $H(\mathbb{C})$ the space of all entire functions with the uniform convergence topology. For $\lambda \in \mathbb{C}$ we will denote by $S_\lambda$ the translation operator on $H(\mathbb{C})$: $S_\lambda f(z) \equiv f(z + \lambda)$. The first examples of hypercyclic operators are due to G. D. Birkhoff and G. MacLane. In 1929, Birkhoff [4] showed that the translation operator $S_\lambda$ is hypercyclic on $H(\mathbb{C})$ if $\lambda \neq 0$. In 1952, MacLane [14] proved that the differentiation operator is hypercyclic on $H(\mathbb{C})$. A significant generalization of these results was proved by G. Godefroy and J. H. Shapiro [8] in 1991. They showed that every convolution operator on $H(\mathbb{C}^N)$ that is not a scalar multiple of the identity is hypercyclic. Recall that a convolution operator on $H(\mathbb{C}^N)$ is a continuous linear operator that commutes with all translations (or equivalently, commutes with each partial differentiation operator).

Some classes of non-convolution hypercyclic operators on $H(\mathbb{C})$ (or $H(\mathbb{C}^N)$) can be found in [1], [6], [15], [3], [12]. More information on hypercyclic operators can be found in [9], [2], [10].

In the present paper we prove the following result.

**Theorem 1.1.** Let $T : H(\mathbb{C}) \to H(\mathbb{C})$ be a continuous linear operator that satisfies the following conditions:

1. $\ker T \neq \{0\}$;
2. $T$ satisfies the commutation relation:

$$[T, \frac{d}{dz}] = aI,$$  \hspace{1cm} (1)

where $a \in \mathbb{C} \setminus \{0\}$ is some constant and $I$ is identity operator.

Then $T$ is hypercyclic.

Remark 1.2. In Theorem 1.1 we consider a constant $a \neq 0$. Note that if $a = 0$ in (1) then $T$ is a convolution operator on $H(\mathbb{C})$ that is hypercyclic by a Theorem of Godefroy and Shapiro [8].

We also prove the following more general result.

**Theorem 1.3.** Let $T : H(\mathbb{C}) \to H(\mathbb{C})$ satisfies the conditions of Theorem 1.1 and let $L$ be an entire function such that $L \not\equiv \text{const}$ and $L(T)$ is a continuous operator from $H(\mathbb{C})$ to itself. Then $L(T)$ is hypercyclic.

The paper is organised as follows. In Section 2 we formulate and prove a theorem that establishes the completeness of translates of certain entire functions. In Section 3 we prove Theorems 1.1 and 1.3 by application of the results of Section 2 and the Godefroy-Shapiro criterion. In Section 4 we give a description of operators that are hypercyclic by Theorem 1.1 or 1.3 and provide some examples.

## 2 Complete systems of translates of entire functions

Recall that the system of entire functions $\{f_\lambda(z), \lambda \in \Lambda \subset \mathbb{C}\}$ is called complete in $H(\mathbb{C})$ if $\text{span}\{f_\lambda, \lambda \in \Lambda\} = H(\mathbb{C})$. The connection between complete systems of entire functions and hypercyclic operators on $H(\mathbb{C})$ is established by the Godefroy-Shapiro criterion which was exhibited by G. Godefroy and J. H. Shapiro in [8] (see also e.g. [2, Ch. 1]):

**Theorem 2.1** (Godefroy-Shapiro criterion). Let $T$ be a continuous linear operator on $H(\mathbb{C})$. Suppose that there is a system of entire functions $\{f_\lambda\}_{\lambda \in \mathbb{C}}$ such that:

1. $T[f_\lambda] = \lambda f_\lambda$, $\forall \lambda \in \mathbb{C}$;
Then $T$ is hypercyclic

In this section we prove the following result.

**Theorem 2.2.** Let $T : H(\mathbb{C}) \to H(\mathbb{C})$ satisfies the conditions of Theorem 1.1 for some $a \neq 0$. Then the system $\{S_\lambda f\}_{\lambda \in \Lambda}$ is complete in $H(\mathbb{C})$ for any function $f \in \ker T \setminus \{0\}$ and any set $\Lambda \subset \mathbb{C}$ that has an accumulation point in $\mathbb{C}$.

**Proof.** Let $T$ satisfy the conditions of Theorem 1.1. Then there is an entire function $f$ such that $f \in \ker T$ and $f \not\equiv 0$. From (1) it follows that

$$[[T, \frac{d^n}{dz^n}] = an \frac{d^{n-1}}{dz^{n-1}}, n = 0, 1, \ldots$$

(see e.g. [5, §16]). Hence,

$$Tf^{(n)} = an f^{(n-1)}, n = 0, 1, \ldots \tag{2}$$

Recall that every convolution operator $M_L$ on $H(\mathbb{C})$ can be represented as a linear differential operator of, generally speaking, infinite order with constant coefficients:

$$M_L[f](z) = \sum_{k=0}^{\infty} d_k f^{(k)}(z), \quad (3)$$

where $L(\lambda) = \sum_{k=0}^{\infty} d_k \lambda^k$ is an entire function of exponential type. The function $L$ is called a characteristic function of the convolution operator (3). A convolution operator (3) can be also represented in the form

$$M_L[f](\lambda) = (F_L, S_\lambda f), \quad (4)$$

where $F_L$ is a continuous linear functional on $H(\mathbb{C})$ such that $(F_L, \exp(\lambda z)) = L(\lambda)$ (see e.g. [13]).

Let $\{\lambda_j\}_{j=1}^{\infty}$ be the set of zeros of $L$. Denote by $m_j$ the multiplicity of zero $\lambda_j$. Then $\ker M_L$ contains exponential monomials $\exp(\lambda_j z)$, $b = 0, 1, \ldots, m_j - 1, j \in \mathbb{N}$ and does not contain any other exponential monomials. In particular, $\ker M_L$ does not contain any
exponential monomial of the kind \( z^p \exp(\mu z) \) if \( L(\mu) \neq 0 \). By the result of L. Schwartz \cite{16}, every differentiation-invariant closed linear subspace \( V \) of \( H(\mathbb{C}) \) coincides with the closure of the linear span of exponential monomials that are contained in \( V \). Hence,

\[
\ker M = \text{span}\{ z^b \exp(\lambda_j z), j \in \mathbb{N}, b = 0, 1, \ldots, m_j - 1 \}.
\]

Denote by \( \text{Conv}[H(\mathbb{C})] \) the class of all convolution operators (4) on \( H(\mathbb{C}) \) except the operator of multiplication by zero.

Let \( \Lambda \) be an arbitrary subset of \( \mathbb{C} \) that has an accumulation point in \( \mathbb{C} \). In order to prove that the system \( \{ S_\lambda f \}_{\lambda \in \Lambda} \) is complete in \( H(\mathbb{C}) \), it is enough to prove that \( f \not\in \ker M \) for any \( M \in \text{Conv}[H(\mathbb{C})] \). Indeed, if the system \( \{ S_\lambda f \}_{\lambda \in \Lambda} \) is not complete in \( H(\mathbb{C}) \), then according to the Approximation Principle (see e.g. \cite{7, Ch. 2}) there is a non-zero continuous linear functional \( F \) on \( H(\mathbb{C}) \) such that \( (F, S_\lambda f) = 0, \forall \lambda \in \Lambda \) and hence, \( f \in \ker M \) for some \( M \in \text{Conv}[H(\mathbb{C})] \).

Assume that \( f \in \ker M_{\Phi} \) for some \( M_{\Phi} \in \text{Conv}[H(\mathbb{C})] \) with the characteristic function \( \Phi(\lambda) = \sum_{k=0}^{\infty} b_k \lambda^k \), i.e.

\[
\sum_{k=0}^{\infty} b_k f^{(k)}(z) \equiv 0.
\]

Then, obviously, \( T^n M_{\Phi} f = 0, \forall n \in \mathbb{N} \). Using (2) we obtain

\[
T^n M_{\Phi} [f](z) \equiv a^n \sum_{k=n}^{\infty} b_k \frac{k!}{(k-n)!} f^{(k-n)}(z).
\]

Hence,

\[
\sum_{k=n}^{\infty} b_k \frac{k!}{(k-n)!} f^{(k-n)}(z) \equiv 0, \ n = 0, 1, \ldots \tag{5}
\]

We see that \( f \) satisfies the infinite system of convolution equations (5) with the characteristic functions \( \Phi^{(n)} \), \( n = 0, 1, \ldots \). Let us note that \( \Phi \not\equiv 0 \), since by assumption \( M_{\Phi} \in \text{Conv}[H(\mathbb{C})] \).

Denote \( W = \bigcap_{n=0}^{\infty} \ker M_{\Phi^{(n)}} \). It is easy to see that \( W \) is differentiation-invariant closed linear subspace of \( H(\mathbb{C}) \). Assume that \( W \) contains some exponential monomial of the kind \( z^p \exp(\mu z) \). Then \( \Phi^{(n)}(\mu) = 0, n = 0, 1, \ldots \). But this is impossible, since by assumption \( \Phi \not\equiv 0 \). Hence, \( W \) does not contain any functions of the kind \( z^p \exp(\mu z) \). Then, by the
result of L. Schwartz [16], W is trivial, i.e. $W = \{0\}$. We have a contradiction to the fact that $f \in W$ and $f \neq 0$. Hence, $f \not\in \ker M_\Phi$. 

\begin{remark}
In the proof of Theorem 2.2 we use the result of L. Schwartz [16] on spectral synthesis in $H(\mathbb{C})$. But we cannot prove the analogue of Theorem 2.2 for $H(\mathbb{C}^N)$ ($N > 1$) using the same method because there is a counter-example on spectral synthesis in $H(\mathbb{C}^N)$ by D. Gurevich [17]. In particular Gurevich showed that there is a non-trivial differentiation-invariant subspace $V$ in $H(\mathbb{C}^N)$ that does not contain any exponential monomial. We are going to discuss the analogues of Theorems 2.2, 1.1 and 1.3 for $H(\mathbb{C}^N)$ in one of the next papers.
\end{remark}

3 Proofs of Theorems 1.1 and 1.3

It is enough to prove Theorem 1.3, since Theorem 1.1 is a special case of Theorem 1.3 when $L(z) \equiv z$. Let $T : H(\mathbb{C}) \to H(\mathbb{C})$ and $L \in H(\mathbb{C})$ satisfy the conditions of Theorem 1.3. Let $f \in \ker T \setminus \{0\}$. Using the Taylor expansion we can write:

$$S_\lambda f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!} \lambda^n.$$ 

Then from (2) it follows that

$$T S_\lambda f = \sum_{n=0}^{\infty} \frac{T[f^{(n)}(z)]}{n!} \lambda^n = \sum_{n=1}^{\infty} \frac{a^n f^{(n-1)}(z)}{n!} \lambda^n = a \lambda \sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!} \lambda^n.$$ 

Hence, $T S_\lambda f = a \lambda S_\lambda f$ and $L(T)f = L_a(\lambda) S_\lambda f$, $\forall \lambda \in \mathbb{C}$, where $L_a(\lambda) \equiv L(a\lambda)$. Since $L_a \not\equiv \text{const}$, then both sets $\{\lambda \in \mathbb{C} : |L_a(\lambda)| < 1\}$ and $\{\lambda \in \mathbb{C} : |L_a(\lambda)| > 1\}$ are non-empty open subsets of $\mathbb{C}$. Hence, $L(T)$ is hypercyclic by Theorems 2.2 and 2.1.

4 Operators that satisfy Theorem 1.1 or 1.3

In this section we give the examples of linear continuous operators on $H(\mathbb{C})$ which are hypercyclic by Theorem 1.1 or 1.3. First, we shall describe the class of linear continuous operators on $H(\mathbb{C})$ that satisfy the condition 2 of Theorem 1.1.
Theorem 4.1. A continuous linear operator $T : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ satisfies the commutation relations (1) if and only if $T = M - azI$, where $M$ is a convolution operator (3) on $H(\mathbb{C})$.

Proof. 1) Let $T = M - azI$, where $M$ is an arbitrary convolution operator (3) on $H(\mathbb{C})$. Then we have $[T, \frac{d}{dz}] = [M, \frac{d}{dz}] - [azI, \frac{d}{dz}] = aI$.

2) Let $T$ satisfies the commutation relation (1), i.e. $[T, \frac{d}{dz}] = aI$. Let us define the operator $T_0 = -azI$. Then $[T_0, \frac{d}{dz}] = aI$. Denote $M = T - T_0$. Then $[M, \frac{d}{dz}] = [T - T_0, \frac{d}{dz}] = 0$. Hence, $M$ is a convolution operator on $H(\mathbb{C})$, and $T = T_0 + M = M - azI$. \qed

Let $T = M - azI$, where $M$ is a convolution operator. Then $T$ is hypercyclic by Theorems 1.1 and 4.1 if there exists an entire function $f$ such that $f \not\equiv 0$ and $f \in \ker T$. Here we provide some examples of such functions and corresponding hypercyclic operators.

Example a) Let $T = \frac{d}{dz} - zI$. If $f(z) = C e^{z^2/2}$, where $C$ is some constant, then $f \in \ker T$. Hence, $T$ is hypercyclic by Theorem 1.1. Consider the operator

$$L(T) = \frac{d^2}{dz^2} + (2z - 1)\frac{d}{dz} + (z^2 - z - 1)I,$$

where $L(\lambda) = \lambda^2 + \lambda$. Then the operator (6) is hypercyclic by Theorem 1.3.

b) Let $T = \frac{d^2}{dz^2} - zI$. If $f(z) = C_1 Ai(z) + C_2 Bi(z)$, where $C_1$, $C_2$ are some constants, and $Ai$, $Bi$ are Airy functions (see e.g. [17]), then $f \in \ker T$. Hence, $T$ is hypercyclic by Theorem 1.1.

Now we will describe the class of hypercyclic differential operators of infinite order with polynomial coefficients. This class contains as a partial case all convolution operators on $H(\mathbb{C})$ that are not scalar multiples of the identity.

Theorem 4.2. Let $T = \frac{d}{dz} - azI$, where $a \in \mathbb{C}$ is some constant. Let $L(z) = \sum_{k=0}^{\infty} d_k \lambda^k$ be an arbitrary entire function of exponential type such that $L \not\equiv$ const. Then the operator

$$L(T) = \sum_{k=0}^{\infty} d_k T^k$$

is hypercyclic operator on $H(\mathbb{C})$. \qed
Proof. Let’s show that $L(T)f \in H(\mathbb{C})$, $\forall f \in H(\mathbb{C})$. We can represent $f$ in the form $f(z) = e^{az^2/2}g(z)$. Then $T^kf(z) = e^{az^2/2}g^{(k)}(z)$, $k = 0, 1, \cdots$. Hence, $L(T)f(z) = e^{az^2/2}M_Lg(z) \in H(\mathbb{C})$, where $M_L$ is a convolution operator on $H(\mathbb{C})$ generated by $L$. Since $T$ satisfies the conditions of Theorem 1.1, then $L(T)$ is hypercyclic by Theorem 1.3. 

References

[1] R. Aron, D. Markose, On universal functions. J. Korean Math. Soc. 41:1 (2004), 65–76.

[2] F. Bayart, E. Matheron, Dynamics of linear operators. Cambridge University Press, 2009.

[3] J. J. Betancor, M. Sifi, K. Trimeche, Hypercyclic and chaotic convolution operators associated with the Dunkl operators on $C$. Acta Math. Hungar. 106 (2005), 101–116.

[4] G. D. Birkhoff, Demonstration d’un theoreme elementaire sur les fonctions entieres. C. R. Acad. Sci. Paris 189 (1929), 473–475.

[5] L. D. Faddeev, O. A. Yakubovskii, Lectures on Quantum Mechanics for Mathematics Students. AMS, 2009.

[6] G. Fernandez, A. A. Hallack, Remarks on a result about hypercyclic non-convolution operators. J. Math. Anal. Appl. 309:1 (2005), 52–55.

[7] R. E. Edwards, Functional analysis: theory and applications. Dover Publications, 1995.

[8] G. Godefroy, J. H. Shapiro, Operators with dense, invariant, cyclic vector manifolds. J. Funct. Anal. 98:2 (1991), 229–269.

[9] K.-G. Grosse-Erdmann, Universal families and hypercyclic operators. Bull. Amer. Math. Soc. 36:3 (1999), 345–381.
[10] K.-G. Grosse-Erdmann, A. Peris Manguillot, *Linear chaos*. Springer, 2011.

[11] D. I. Gurevich, *Counterexamples to a problem of L. Schwartz* (in Russian). Funkts. Anal. Prilozh. 9:2 (1975), 29–35 [English translation: Funct. Anal. Appl. 9:2 (1975), 116–120].

[12] V. E. Kim, *Hypercyclicity and chaotic character of generalized convolution operators generated by Gelfond-Leontev operators* (in Russian). Mat. Zametki 85:6 (2009), 849–856 [English translation: Math. Notes 85:6 (2009), 807–813].

[13] A. S. Krivosheev, V. V. Napalkov, *Complex analysis and convolution operators* (in Russian). Usp. Mat. Nauk 47:6 (1992), 3–58 [English translation: Russ. Math. Surv. 47:6 (1992), 1–56].

[14] G. MacLane, *Sequences of derivatives and normal families*. J. Anal. Math. 2 (1952), 72–87.

[15] H. Petersson, *Supercyclic and hypercyclic non-convolution operators*. J. Operator Theory 55:1 (2006), 133–151.

[16] L. Schwartz, *Theorie generale des fonctions moyenne-periodique*. Ann. Math. 48:4 (1947), 867–929.

[17] O. Vallee, M. Soares, *Airy functions and applications to physics*. Imperial College Press, 2004.

Vitaly E. Kim  
Institute of Mathematics with Computing Centre,  
Russian Academy of Sciences,  
112 Chernyshevsky str.,  
450008 Ufa, Russia  
e-mail: kim@matem.anrb.ru