AN $\alpha$-NUMBER CHARACTERIZATION OF $L^p$ SPACES ON UNIFORMLY RECTIFIABLE SETS

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ABSTRACT. We give a characterization of $L^p(\sigma)$ for uniformly rectifiable measures $\sigma$ using Tolsa’s $\alpha$-numbers, by showing, for $1 < p < \infty$ and $f \in L^p(\sigma)$, that 
\[ \|f\|_{L^p(\sigma)} \sim \left( \int_0^\infty \left( \alpha_{f_\sigma}(x,r) + |f|_{x,r,\sigma}(x,r) \right)^2 \frac{dr}{r} \right)^{\frac{1}{2}}. \]

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1. INTRODUCTION

We say a measure $\mu$ in $\mathbb{R}^n$ is $d$-rectifiable if $\mu$ is absolutely continuous with respect to $d$-dimensional Hausdorff measure and we may exhaust $\mu$-almost all of $\mathbb{R}^n$ by countably many Lipschitz graphs. It is a classical result that, for $\mu$-a.e. $x \in \mathbb{R}^n$, the densities $\mu(B(x,r))/r^{d}$ stabilize as $r \to 0$ in the sense that they converge to a nonzero constant and so on small scales the measure $\mu$ scales like $d$-dimensional Lebesgue measure. What’s more is that the shape of the measure $\mu$ also stabilizes: as we zoom in on $x$, if we set $\mu_{x,r}(A) = \mu(rA + x)$, then $\mu_{x,r}r^{-d}$ converges weakly to $d$-dimensional Lebesgue measure on some $d$-dimensional plane.
In [Tol09], Tolsa quantified how much a uniformly rectifiable measure can deviate from resembling planar Lebesgue measure. Recall that a measure $\mu$ is uniformly rectifiable (UR) if firstly, it is Ahlfors $d$-regular, meaning there is $A > 0$ so that
\[
A^{-1}r^d \leq \sigma(B(x,r)) \leq Ar^d \quad \text{for all } x \in \text{supp } \sigma, \ 0 < r < \text{diam}(\text{supp } \sigma),
\]
and $\sigma$ has big pieces of Lipschitz images (BPLI), meaning there are constants $L, c > 0$ so that for each $x \in \text{supp } \sigma$ and $0 < r < \text{diam}(\text{supp } \sigma)$ there is an $L$-Lipschitz mapping $f : B_d(0,r) \to B(x,r)$ so that $\mu(f(B_d(0,r)))$. We say a set $E \subseteq \mathbb{R}^n$ is UR if $\mathcal{H}^d\!|_E$ is UR.

Before stating Tolsa’s result, we will describe how he measures the planarity of a measure. First, they are used to study higher co-dimensional analogues of harmonic measure. See also [DS91] where they are used to characterize rectifiability of sets of finite measure in terms of existence of principal values for the Riesz transform, and [DEM18, Fen20, DM20] where they are used to study higher co-dimensional analogues of harmonic measure.

To define a distance between measures. For two measures $\mu$ and $\nu$ and a ball $B$ we define
\[
F_B(\sigma, \nu) := \sup \left\{ \left| \int \phi \, d\sigma - \int \phi \, d\nu \right| : \phi \in \text{Lip}_1(B) \right\},
\]
where $\text{Lip}_1(B)$ is the set of 1-Lipschitz functions supported in $B$. This is a variant of the Wasserstein 1-distance from mass transport theory. See [Mat95, Chapter 14] for a discussion about this distance.

For a (possibly real-valued) measure $\mu$ and $d \in \mathbb{N}$, if $B = B(x,r)$ we define
\[
\alpha^d_\mu(x,r) = \alpha^d_\mu(B) := \frac{1}{r^{d+1}} \inf_{c \in \mathbb{R}, L} F_B(\mu, c\mathcal{H}^d\!|_L).
\]
We will often omit the superscript $d$, as it will be fixed throughout.

**Theorem 1.1** ([Tol09, Theorem 1.2]). An Ahlfors $d$-regular measure $\sigma$ is UR if and only if the measure $\alpha^d_\mu(x,r)^2\sigma(x) \frac{dr}{r}$ is a Carleson measure, meaning that for all balls $B$ centered on $\text{supp } \sigma$ with $0 < r_B < \text{diam}(\text{supp } \sigma)$,
\[
\int_0^{r_B} \int_B \alpha^d_\mu(x,r)^2 d\sigma(x) \frac{dr}{r} \leq C\sigma(B)
\]
for some fixed $C > 0$.

Estimates on $\alpha$-type numbers are particularly useful in studying rectifiability. From a geometric viewpoint, they give quite a lot of information about the shape of a measure. David and Semmes [DS91] gave an earlier characterization of UR sets in terms of a Carleson measure condition on $\beta$-numbers, which are quantities like $\alpha$-numbers except they only measure the average distance of a measure to a plane, so while a measure could be very close to lying on a plane, its mass could be very unevenly distributed resulting in a large $\alpha$-number. The additional information provided by the $\alpha$-numbers was crucial for the main result of [Tol09], where Tolsa improved on the work in [DS91] by expanding the class of Calderon-Zygmund operators on UR sets that were known to be bounded. See also [Tol08] where $\alpha$-numbers are used to characterize rectifiability of sets of finite measure in terms of existence of principal values for the Riesz transform, and [DEM18, Fen20, DM20] where they are used to study higher co-dimensional analogues of harmonic measure.

The purpose of this note is to extend Tolsa’s result to measures that are not Ahlfors regular, but are given by $L^p$ functions defined on UR sets.

Given a Radon measure $\sigma$, $f \in L_{loc}^1(\sigma)$, and a ball $B = B(x,r)$ with $\sigma(B(x,r)) > 0$ set
\[
f_B = f_{x,r} = \frac{\int_B f \, d\sigma}{\sigma(B)}.
\]

**Main Theorem.** Let $\sigma$ be a UR measure and $f \in L^p(\sigma)$ where $1 < p < \infty$. Then
\[
\|f\|_{L^p(\sigma)} \sim \left( \int_0^\infty \left( \alpha f_{\sigma}(x,r) + |f|_{x,r} \alpha_{\sigma}(x,r) \right)^2 \frac{dr}{r} \right)^{\frac{1}{2}},
\]
with the implicit constant depending on $p$ and $\sigma$. 


Sharpness of the result. An interesting aspect of our result is the presence of two terms that comprise our square function. We don’t know whether the result holds for general UR sets without the second term. Neither of the terms bounds the other in the pointwise sense: one could be zero while the other is nonzero. On the other hand, we don’t know whether the norm of the square function involving only \( \alpha_f \) dominates the one involving only \( |f|_{x,r} \alpha_\sigma \). The reverse inequality is certainly not true, as the latter square function vanishes if \( \sigma \) is the Lebesgue measure on \( \mathbb{R}^d \).

**Question.** Let \( \sigma \) be a UR measure and \( f \in L^p(\sigma) \) where \( 1 < p < \infty \). Do we have

\[
\|f\|_{L^p(\sigma)} \lesssim \left\| \left( \int_0^\infty \alpha_{f\sigma}(x,r)^2 \frac{dr}{r} \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)} \quad \text{?} 
\]

(1.3)

Equivalently, is it true that

\[
\left\| \left( \int_0^\infty (|f|_{x,r} \alpha_\sigma(x,r))^2 \frac{dr}{r} \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)} \lesssim \left\| \left( \int_0^\infty \alpha_{f\sigma}(x,r)^2 \frac{dr}{r} \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)} \quad \text{?}
\]

The answer to the question above is obviously affirmative in the flat case, i.e. \( \sigma = H^d \llcorner L \) for \( L \) a \( d \)-dimensional plane. It is also positive if \( \sigma \) is an Ahlfors \( d \)-regular measure on a \( d \)-dimensional plane \( L \), i.e. \( \sigma = g \mathcal{H}^d \llcorner L \) for some \( g \) satisfying \( A^{-1} \leq g \leq A \). Indeed, let \( \tilde{\sigma} = \mathcal{H}^d \llcorner L \), so that \( f\sigma = fg\tilde{\sigma} \). In that case, by the Main Theorem

\[
\|f\|_{L^p(\sigma)} \sim_A \|fg\|_{L^p(\tilde{\sigma})} \sim_p \left( \int_0^\infty \alpha_{fg\sigma}(x,r)^2 \frac{dr}{r} \right)^{\frac{1}{2}} L^p(\tilde{\sigma}) \sim_A \left( \int_0^\infty \alpha_{f\sigma}(x,r)^2 \frac{dr}{r} \right)^{\frac{1}{2}} L^p(\sigma) .
\]

Finally, one could show that (1.3) is true for “sufficiently flat” UR measures \( \sigma \). What we mean by this is that if the constant \( C \) from Theorem 1.1 is sufficiently small, then some variant of Carleson’s embedding theorem can be used\(^1\) to show that

\[
\left\| \left( \int_0^\infty (|f|_{x,r} \alpha_\sigma(x,r))^2 \frac{dr}{r} \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)} \lesssim \mu C \|f\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)} .
\]

This means that the second term from the square function in (1.2) can essentially be absorbed by the left hand side. To make this more rigorous, one should perhaps track the dependence of the implicit constants in (1.2) on the UR constants of \( \sigma \) with more diligence than we did. However, the implicit constants can only get better as \( \sigma \) becomes flatter, and they certainly cannot blow-up as the Carleson constant \( C \) goes to 0: if \( \sigma \) satisfies the Carleson condition of Theorem 1.1 with some \( C \), then it also satisfies it with constant \( C' \) for every \( C' \geq C \).

**Related work.** While our focus has been in the Ahlfors regular setting, \( \alpha \)-numbers have also been used to study measures in more general settings. In [ATT18], it was shown that pointwise doubling measures \( \mu \) were \( d \)-rectifiable on the set where the square function \( \int_0^\infty \alpha_\mu(x,r)^2 \frac{dr}{r} \) was finite, resolving a question left open in [ADT16]. This paper also exposed some limitations with working with \( \alpha \)-numbers, as a counterexample showed that the same result is not true for general measures. However, the second author of this paper obtained such a generalization in [Dab19] using a different \( \alpha \)-number, which measures distance between a measure and planar measure using the Wasserstein 2-metric, which Tolsa had earlier used to give a characterization of UR measures in

\(^1\)For \( p = 2 \) use e.g. [Tol14, Theorem 5.8], for \( p \neq 2 \) one can show a corresponding statement by proving an appropriate good-lambda inequality, in the spirit of what we do in Section 4 (but simpler).
So while using the Wasserstein 2-distance allows one to get a more complete picture, the \( \alpha \)-number in Theorem 1.1 has a more transparent definition and thus is easier to work with.

In [Orp18] Orponen uses a similar square function to characterize when two measures on the real line (one being doubling) are absolutely continuous, however among a few of the differences between the \( \alpha \)-numbers he uses and ours, while we compare distance between a measure and a plane, his numbers compare the distance between the two measures, which is another interesting direction.

**Organization of the article.** In Section 2 we introduce the necessary tools and make some initial reductions. We define also \( Jf \), a dyadic variant of the square function from the Main Theorem, see (2.4). We show that \( \|Jf\|_2 \lesssim \|f\|_2 \) in Section 3. The proof uses martingale difference operators, and it is inspired by how Theorem 1.1 was originally proved, see [Tol09, Section 4]. In Section 4 we use the estimate \( \|Jf\|_2 \lesssim \|f\|_2 \) and an appropriate good-lambda inequality to conclude that \( \|Jf\|_p \lesssim \|f\|_p \) for general \( 1 < p < \infty \).

Finally, in Section 5 we prove \( \|f\|_p \lesssim \|Jf\|_p \). To do that we use the Littlewood-Paley theory of David, Journé and Semmes [DJS85].

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2. **Preliminaries**

2.1. **Notation.** In our estimates we will write \( f \lesssim g \) to denote \( f \leq Cg \) for some constant \( C \) (the so-called “implicit constant”). If the implicit constant depends on a parameter \( t \), i.e. \( C = C(t) \), we will write \( f \lesssim_t g \). The notation \( f \sim g \) and \( f \sim_t g \) stands for \( g \lesssim f \lesssim g \) and \( g \lesssim_t f \lesssim_t g \), respectively.

To make the notation lighter, we will usually not track the dependence of \( C \) on dimensions \( n, d \), on the Ahlfors regularity constant of \( \sigma \), or the parameter \( 1 < p < \infty \).

Given \( x \in \mathbb{R}^n \) and \( r > 0 \) we denote by \( B(x, r) \) the open ball centered at \( x \) with radius \( r \). Conversely, given a ball \( B \) (either open or closed, and either \( n \) or \( d \)-dimensional), \( r_B \) and \( z_B \) denote the radius and the center of \( B \), respectively.

For simplicity, we will sometimes write \( \|f\|_p := \|f\|_{L^p(\sigma)} \).

In the introduction we introduced the notation \( f_B \) to signify the average of \( f \) over a ball \( B \) with respect to \( \sigma \). For general Borel sets \( E \subset \mathbb{R}^d \) with \( \sigma(E) > 0 \) and \( f \in L^1_{loc}(\sigma) \) we will write

\[
\langle f \rangle_E = \frac{\int_E f \, d\sigma}{\sigma(E)}
\]

For a finite set \( I \) we will write \( \#I \) to denote the cardinality of \( I \).

If \( v, w \in \mathbb{R}^n \), then \( v \cdot w \) denotes their scalar product.

Given \( E, F \subset \mathbb{R}^d \), \( \text{dist}_H(E, F) \) stands for the Hausdorff distance between \( E \) and \( F \).
2.2. Adjacent systems of cubes. As usual, we will work with a family of subsets of \( \text{supp} \sigma =: \Sigma \) that in many ways resemble the family of dyadic cubes on \( \mathbb{R}^d \). For this reason we will call these sets “cubes”. Many different systems of cubes have been constructed throughout the years, beginning with the work of David [Dav88] and Christ [Chr90]. In our proof it will be convenient to use adjacent systems of cubes constructed by Hytönen and Tapiola [HT14]. One should think of them as a generalization of the translated dyadic grids in \( \mathbb{R}^d \), widely used to perform the “1/3 trick”.

First, we will say that a family \( \mathcal{D} \) of Borel subsets of \( \Sigma \) satisfies the usual properties of David-Christ cubes if \( \mathcal{D} = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k \), and for each \( k \in \mathbb{Z} \):

- (a) for \( P, Q \in \mathcal{D}_k \), \( P \neq Q \), we have \( \sigma(P \cap Q) = \emptyset \),
- (b) the sets in \( \mathcal{D}_k \) cover \( \Sigma \):
  \[
  \Sigma = \bigcup_{Q \in \mathcal{D}_k} Q,
  \]
- (c) for each \( Q \in \mathcal{D}_k \) and each \( l \geq k \)
  \[
  Q = \bigcup_{P \in \mathcal{D}_l : P \subset Q} P,
  \]
- (d) there exists \( 0 < \delta < 1 \) (independent of \( k \)) such that each \( Q \in \mathcal{D}_k \) has a center \( z_Q \in Q \) satisfying
  \[
  B\left(z_Q, \frac{\delta^k}{\delta}\right) \cap \Sigma \subset Q \subset B(z_Q, 3\delta^k) \cap \Sigma.
  \tag{2.1}
  \]

Consequently, as long as \( \delta^k \leq \text{diam}(\Sigma) \), we have \( \sigma(Q) \sim \delta^{kd} \). Set \( \ell(Q) := \delta^k \).
- (e) the cubes \( Q \in \mathcal{D}_k \) have thin boundaries, that is, there exists \( \gamma \in (0, 1) \) such that for \( \eta \in (0, 0.1) \) we have
  \[
  \sigma(\{x \in \Sigma : \text{dist}(x, Q) < \eta \ell(Q)\}) \leq \eta^\gamma \sigma(Q). \tag{2.2}
  \]

Remark 2.1. Note that in the above we assume \( \mathcal{D}_k \) to be defined for all \( k \in \mathbb{Z} \). In the case of unbounded \( \Sigma \), this translates to having arbitrarily large cubes as \( k \to -\infty \). In the case of compact \( \Sigma \), there exists some \( k_0 \) such that for all \( k \leq k_0 \) we have \( \mathcal{D}_k = \{\Sigma\} \). However, in our proof we will assume that \( \Sigma \) is unbounded, see Lemma 2.5.

In our setting, the results [HT14, Theorem 2.9, Theorem 5.9] can be summarized as follows.

Lemma 2.2. Let \( \sigma \) be a \( d \)-Ahlfors regular measure on \( \mathbb{R}^n \). Then, there exist \( 1 \leq N < \infty \) and a small constant \( 0 < \delta < 0.01 \), depending only on the Ahlfors regularity constants of \( \sigma \), such that the following holds. Let \( \Omega = \{1, \ldots, N\} \). For each \( \omega \in \Omega \) we have a system of cubes \( \mathcal{D}(\omega) \) satisfying the usual properties of David-Christ cubes, and additionally, for all \( x \in \text{supp} \sigma \) and \( r > 0 \) there are \( \omega \in \Omega \), \( k \in \mathbb{Z} \) and \( Q \in \mathcal{D}_k(\omega) \) with
  \[
  B(x, r) \cap \text{supp} \sigma \subset Q.
  \]

and
  \[
  \ell(Q) = \delta^k \sim_\delta r.
  \]

Remark 2.3. The construction in [HT14] is valid for general (geometrically) doubling metric spaces, possibly with no underlying measure space structure. The constants \( N \) and \( \delta \) from Lemma 2.2 depend on the doubling constant of the metric space. Hytönen and Tapiola construct two different kinds of cubes, which they call open and closed cubes, see [HT14, Theorem 2.9]. In the above we consider closed cubes, so that properties (b), (c) and (d) follow immediately from [HT14, Theorem 2.9]. To get the property (a) one uses the fact that interiors of \( P \) and \( Q \) are disjoint by [HT14, (2.11)], and then \( \sigma(\partial P) = \sigma(\partial Q) = 0 \) follows from (e). To prove the thin boundaries property (e) one may adapt the proof of Christ [Chr90, pp. 610–612] together with Ahlfors regularity of \( \sigma \). We omit the details.
From now on, let us fix a uniformly rectifiable measure $\sigma$, with $\Sigma = \text{supp } \sigma$. Let $\Omega$, $\delta$ and $\mathcal{D}(\omega)$ be as in Lemma 2.2. For simplicity, in our estimates we will not track the dependence of implicit constants on $\delta$.

For all $\omega \in \Omega$ and $Q \in \mathcal{D}_k(\omega)$ we will write
\[
\mathcal{D}(Q) := \{ P \in \mathcal{D}(\omega) : P \subset Q \},
\]
\[
\text{Ch}(Q) := \mathcal{D}(Q) \cap \mathcal{D}_{k+1}(\omega).
\]

The elements of Ch(Q) will be called children of Q, and Q will be called their parent.

Set
\[
B_Q := B(z_Q, 4\ell(Q)),
\]
so that $Q \subset B_Q \cap \Sigma$, and whenever $P \in \mathcal{D}(Q)$ we also have $B_P \subset B_Q$.

Fix some $\omega_0 \in \Omega$, and set
\[
\mathcal{D} := \mathcal{D}(\omega_0).
\]

This will be our system of reference. Given $Q \in \mathcal{D}$ we define $\omega(Q) \in \Omega$ to be the index such that there exists $R(Q) \in \mathcal{D}(\omega(Q))$ satisfying $B_Q \cap \Sigma \subset R(Q)$ and $\ell(R(Q)) \sim \ell(Q)$. If there is more than one such $\omega$, we simply choose one. We define also $\mathcal{G}(\omega) \subset \mathcal{D}$ as the family of cubes $Q \in \mathcal{D}$ such that $\omega(Q) = \omega$. Clearly,
\[
\bigcup_{\omega \in \Omega} \mathcal{G}(\omega) = \mathcal{D}.
\]

2.3. $\alpha$-numbers. In proving the main theorem, it will be more convenient to work with dyadic versions of the $\alpha$-numbers. Below we will introduce the notation needed for this framework. Given a Radon measure $\mu$ we denote by $L^\mu_{x,r}$ a minimizing $d$-plane for $\alpha\mu(x,r)$, and by $c^{\mu}_{x,r}$ the corresponding constant. They may be non-unique, in which case we just choose one of the minimizers. Set $P^\mu_{x,r} = H^d \mathcal{L} L^\mu_{x,r}$ and $\mathcal{L}^\mu_{x,r} = c^{\mu}_{x,r} P^\mu_{x,r}$. If $B = B(x,r)$ we will also write $L^\mu_B$, $c^\mu_B$ etc.

For $Q \in \mathcal{D}$ and a Radon measure $\mu$ we set
\[
\alpha_\mu(Q) := \alpha_\mu(B_Q).
\]

We will write $L^\mu_Q := L^\mu_{B_Q}$, $c^\mu_Q := c^\mu_{B_Q}$ etc.

Observe that whenever $B_1 \subset B_2$ are balls, we have $\text{Lip}_1(B_1) \subset \text{Lip}_1(B_2)$, and so if $r_{B_1} \geq C r_{B_1}$, then
\[
\alpha_\mu(B_1) = \frac{1}{r_{B_1}^{d+1}} \inf_{c \in \mathbb{R}^d} F_{B_1}(\mu, c\mathcal{H}^d \mathcal{L} L) \leq \frac{1}{r_{B_1}^{d+1}} F_{B_1}(\mu, \mathcal{L}^\mu_{B_2}) \leq \frac{1}{r_{B_1}^{d+1}} F_{B_2}(\mu, \mathcal{L}^\mu_{B_2}) \sim C \alpha_\mu(B_2). \tag{2.3}
\]

Consider the following square function:
\[
J(x) = \left( \sum_{x \in Q \in \mathcal{D}} \alpha_\sigma(Q)^2 + |f|_{B_Q}^2 \alpha_\sigma(Q)^2 \right)^{1/2}.
\tag{2.4}
\]

The Main Theorem will follow from the following dyadic version:

**Theorem 2.4.** Let $\sigma$ be a uniformly rectifiable measure with unbounded support, and let $f \in L^p(\sigma)$ for some $1 < p < \infty$. Then
\[
\|Jf\|_{L^p(\sigma)} \sim \|f\|_{L^p(\sigma)}.
\]

First, let us show why we may assume that $\text{supp } \sigma$ is unbounded.

**Lemma 2.5.** It suffices to only prove Main Theorem in the case that $\text{supp } \sigma$ is unbounded.
Indeed, since \( \text{supp} \mu = \sigma + \mathcal{P}^d \mathcal{L}(\mathbb{R}^d \setminus \mathbb{B}) \). It is not hard to show that \( \mu \) is also UR. If Main Theorem holds for UR measures of unbounded support, then it holds for \( \mu \). Let \( f \in L^p(\sigma) \subseteq L^p(\mu) \) and let

\[ \theta^f_\sigma(x, r) := \alpha_{f \sigma}(x, r) + |f|_{x,r} \alpha_{\sigma}(x, r), \]

so that, by the Main Theorem,

\[ \left\| \left( \int_0^\infty \theta^f_\sigma(x, r)^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(\mu)} \sim \|f\|_{L^p(\mu)} = \|f\|_{L^p(\sigma)}. \tag{2.5} \]

Observe that

\[ \theta^f_\sigma(x, r) = \theta^f_\mu(x, r) \text{ for } x \in \text{supp } \sigma \text{ and } 0 < r < 2. \]

Thus,

\[ \left\| \left( \int_0^2 \theta^f_\sigma(x, r)^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(\sigma)} = \left\| \left( \int_0^2 \theta^f_\mu(x, r)^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(\mu)} \leq \left\| \left( \int_0^\infty \theta^f_\mu(x, r)^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(\mu)} \lesssim \|f\|_{L^p(\sigma)}. \tag{2.5} \]

Furthermore, we claim that for any \( x \in \text{supp } \sigma \) and \( r > 2 \) we have

\[ \theta^f_\sigma(x, r) \lesssim r^{-d} |f|_\mathcal{B}. \tag{2.6} \]

Indeed, since \( \text{supp } f \subset \text{supp } \sigma \subset \mathbb{B} \),

\[ \alpha_{f \sigma}(x, r) \leq \frac{1}{r^{d+1}} F_B(x, r)(f \sigma, 0) \leq \frac{1}{r^d} \int_{\mathbb{B}} |f| \, d\sigma \sim \frac{1}{r^d} |f|_\mathcal{B}, \tag{2.7} \]

and also

\[ |f|_{x,r} \alpha_{\sigma}(x, r) \lesssim \frac{1}{r^d} \int_{\mathbb{B}} |f| \, d\sigma \sim \frac{1}{r^d} |f|_\mathcal{B}. \]

It follows from (2.6) that

\[ \int_2^\infty \theta^f_\sigma(x, r)^2 \frac{dr}{r} \leq \int_2^\infty \left( \frac{|f|_\mathcal{B}^2}{r^{d+1}} \right) \frac{dr}{r^{d+1}} \lesssim |f|_\mathcal{B}^2, \]

and so

\[ \left\| \left( \int_0^\infty \theta^f_\sigma(x, r)^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)} + \int_{\mathbb{B}} |f| \, d\sigma \lesssim \|f\|_{L^p(\sigma)}. \]

To finish the proof we now need to show the reverse inequality. Notice that since \( f \) is supported on \( \text{supp } \sigma \), \( \alpha_{f \mu}(x, r) = \alpha_{f \sigma}(x, r) \) for all \( x \in \text{supp } \sigma \) and \( r > 0 \). We can argue just as in (2.7) to get that for \( x \in \text{supp } \mu \) and \( r \geq 2 \),

\[ \alpha_{f \mu}(x, r) = \alpha_{f \sigma}(x, r) \lesssim \frac{1}{r^{d+1}} F_B(x, r)(f \sigma, 0) \lesssim \frac{|f|_\mathcal{B}}{r^d} \lesssim \frac{|f|_\mathcal{B}^2}{r^{2d}} \alpha_\sigma(2\mathbb{B}), \]

where we also used \( \alpha_\sigma(2\mathbb{B}) \sim 1 \).
Hence,
\[
\int_{B} \left( \int_{0}^{x} \alpha f_{\mu}(x, r)^{2} \frac{dr}{r} \right)^{\frac{\theta}{2}} d\mu(x)
\]
\[
\lesssim \int_{B} \left( \int_{0}^{2} \alpha f_{\mu}(x, r)^{2} \frac{dr}{r} \right)^{\frac{\theta}{2}} d\mu(x) + \int_{B} \left( \int_{2}^{x} \alpha f_{\mu}(x, r)^{2} \frac{dr}{r} \right)^{\frac{\theta}{2}} d\mu(x)
\]
\[
\lesssim \int_{B} \left( \int_{0}^{2} \alpha f_{\sigma}(x, r)^{2} \frac{dr}{r} \right)^{\frac{\theta}{2}} d\mu(x) + \int_{B} \left( \int_{2}^{x} (|f|_{2\alpha\sigma}(2\mathcal{B}))^{2} \frac{dr}{r} \right)^{\frac{\theta}{2}} d\mu(x)
\]
\[
\lesssim \int_{B} \left( \int_{0}^{2} \alpha f_{\sigma}(x, r)^{2} \frac{dr}{r} \right)^{\frac{\theta}{2}} d\mu(x) + (|f|_{2\alpha\sigma}(2\mathcal{B}))^{p} \lesssim \int_{B} \left( \int_{0}^{x} \theta^{\frac{\theta}{2}}(x, r)^{2} \frac{dr}{r} \right)^{\frac{\theta}{2}} d\mu(x)
\]
where we used (2.3) in the final inequality.

Furthermore, for \( x \in \mathbb{R}^{d} \setminus 4\mathcal{B} \), if \( \alpha f_{\mu}(x, r) \neq 0 \), then \( r \geq |x|/2 \) and so
\[
\int_{\mathbb{R}^{d} \setminus 4\mathcal{B}} \left( \int_{0}^{\infty} \alpha f_{\mu}(x, r)^{2} \frac{dr}{r} \right)^{\frac{\theta}{2}} d\mu(x) = \sum_{j=2}^{\infty} \int_{\mathbb{R}^{d} \setminus (2^{j+1} \mathcal{B} \setminus 2^{j} \mathcal{B})} \left( \int_{|x|/2}^{\infty} (|f|_{2\alpha\sigma}(2\mathcal{B}))^{2} \frac{dr}{r} \right)^{\frac{\theta}{2}} d\mu(x)
\]
\[
\lesssim (|f|_{2\alpha\sigma}(2\mathcal{B}))^{p} \sum_{j=2}^{\infty} \int_{\mathbb{R}^{d} \setminus (2^{j+1} \mathcal{B} \setminus 2^{j} \mathcal{B})} |x|^{-pd} d\mu(x)
\]
\[
\lesssim (|f|_{2\alpha\sigma}(2\mathcal{B}))^{p} \lesssim \int_{B} \left( \int_{0}^{x} |f|^{2}(x, r)^{2} \frac{dr}{r} \right)^{\frac{\theta}{2}} d\mu(x)
\]
again using (2.3). These two estimates imply
\[
\left\| \left( \int_{0}^{\infty} \alpha f_{\mu}(x, r)^{2} \frac{dr}{r} \right)^{1/2} \right\|_{L^{p}(\mu)} \lesssim \left\| \left( \int_{0}^{x} \theta^{\frac{\theta}{2}}(x, r)^{2} \frac{dr}{r} \right)^{1/2} \right\|_{L^{p}(\sigma)}.
\]  
(2.8)

Note that for \( x \in \text{supp} \sigma \) and \( r < 2 \), we have \( \mathcal{P}_{x,r}^{\sigma} = \mathcal{P}_{x,r}^{\mu} \). For \( r \geq 2 \), notice that \( \alpha_{\sigma}(2\mathcal{B}) \sim 1 \), and so
\[
\alpha_{\mu}(x, r) \leq \frac{1}{r^{d+1}} B_{x}(\mathcal{B})(\mu, \mathcal{P}_{x,r}^{\sigma}) = \frac{1}{r^{d+1}} B_{x}(\mathcal{B})(\sigma, \mathcal{P}_{x,r}^{\sigma} \mathcal{L} \mathcal{B}) \lesssim r^{-d} \lesssim \frac{\alpha_{\sigma}(2\mathcal{B})}{r^{d}},
\]
hence
\[
|f|_{2\alpha_{\mu}, \sigma}^{\mu}(x, r) \leq |f|_{2\alpha_{\sigma}(2\mathcal{B})} \frac{\alpha_{\sigma}(2\mathcal{B})}{r^{2d}},
\]
where \( |f|_{2\alpha_{\mu}, \sigma}^{\mu} = \int_{B(x,r)} f d\mu/\mu(B(x,r)) \). Thus, just as how we proved (2.8), we can show
\[
\left\| \int_{0}^{\infty} (|f|^{2}_{x,r}) \alpha_{\mu}(x, r)^{2} \frac{dr}{r} \right\|_{L^{p}(\mu)} \lesssim \left\| \int_{0}^{\infty} (|f|^{2}_{x,r}) \alpha_{\sigma}(x, r)^{2} \frac{dr}{r} \right\|_{L^{p}(\sigma)}.
\]
This, (2.8) and (2.5) imply the desired estimate:
\[
\|f\|_{L^{p}(\sigma)} \lesssim \left\| \left( \int_{0}^{x} \theta^{\frac{\theta}{2}}(x, r)^{2} \frac{dr}{r} \right)^{1/2} \right\|_{L^{p}(\sigma)}.
\]

**Proof of the Main Theorem using Theorem 2.4.** By Lemma 2.5, we may assume that \( \text{supp} \sigma = \Sigma \) is unbounded, so that Theorem 2.4 holds.
Let \( x \in \Sigma, \ r > 0 \). Let \( k \in \mathbb{Z} \) be such that \( \delta^{k+1} < r \leq \delta^k \), and let \( Q \) be a cube in \( D_k \) containing \( x \). Recall that \( Q \subset B(z_Q, 3\ell(Q)) \). Since \( r \leq \ell(Q) \), we have
\[
B(x, r) \subset B(z_Q, 3\ell(Q) + r) \subset B(z_Q, 4\ell(Q)) = B_Q.
\]
Hence, by (2.3),
\[
\alpha_{f_\sigma}(x, r) \lesssim \alpha_{f_\sigma}(Q).
\]
We also have \( |f|_{x,r} \lesssim |f|_{B_Q} \), and so
\[
|f|_{x,r,\alpha_\sigma(x, r)} \lesssim |f|_{B_Q, \alpha_\sigma(Q)}.
\]
Consequently,
\[
\int_{\delta^k}^{\delta^{k+1}} \alpha_{f_\sigma}(x, r) + |f|_{x,r,\alpha_\sigma(x, r)}^2 \frac{dr}{r} \lesssim \alpha_{f_\sigma}(Q)^2 + |f|_{B_Q, \alpha_\sigma(Q)}^2.
\]
Summing over \( k \in \mathbb{Z} \) yields
\[
\int_0^\infty \alpha_{f_\sigma}(x, r) + |f|_{x,r,\alpha_\sigma(x, r)}^2 \frac{dr}{r} \lesssim \sum_{x \in Q \in D} \alpha_{f_\sigma}(Q)^2 + |f|_{B_Q, \alpha_\sigma(Q)}^2.
\]
Similarly, for \( x \in \Sigma, \ r > 0, \delta^{k+1} < r \leq \delta^k \), we may consider a cube \( Q \in D_{k+2} \) such that \( x \in Q \subset B_Q \subset B(x, r) \). Mimicking the estimates above, one gets
\[
\sum_{x \in Q \in D} \alpha_{f_\sigma}(Q)^2 + |f|_{B_Q, \alpha_\sigma(Q)}^2 \lesssim \int_0^\infty \alpha_{f_\sigma}(x, r) + |f|_{x,r,\alpha_\sigma(x, r)}^2 \frac{dr}{r}.
\]
Putting the two estimates together, we get the comparability of the dyadic and continuous variants of the square function:
\[
\lf f(x) \r^2 = \sum_{x \in Q \in D} \alpha_{f_\sigma}(Q)^2 + |f|_{B_Q, \alpha_\sigma(Q)}^2 \sim \int_0^\infty \alpha_{f_\sigma}(x, r) + |f|_{x,r,\alpha_\sigma(x, r)}^2 \frac{dr}{r}.
\]
\[\square\]

Theorem 2.4 will follow from the results from the next three sections. From now on we assume that \( \sigma \) is a uniformly rectifiable measure with unbounded support.

3. \( \| f \|_2 \lesssim \| f \|_2 \)

First, we prove the estimate \( \| J f \|_p \lesssim \| f \|_p \) in the case \( p = 2 \).

**Proposition 3.1.** Let \( f \in L^2(\sigma) \). Then
\[
\sum_{Q \in D} (\alpha_{f_\sigma}(Q)^2 + |f|_{B_Q, \alpha_\sigma(Q)}^2) \ell(Q)^d \lesssim \| f \|^2_{L^2(\sigma)}.
\]

Our main tool in the proof of Proposition 3.1 are the martingale difference operators associated to systems of cubes \( D(\omega) \).

Given \( \omega \in \Omega, \ Q \in D(\omega) \), and \( f \in L^1_{\text{loc}}(\sigma) \) we set
\[
\Delta_Q f = \sum_{P \in \text{Ch}(Q)} \langle f \rangle_P 1_P - \langle f \rangle_Q 1_Q.
\]

Observe that all \( \Delta_Q f \) have zero mean, i.e. \( \int \Delta_Q f \ d\sigma = 0 \).

It is well known (see e.g. [Gra14, Chapter 6.4]) that given \( f \in L^2(\sigma) \) and some system of cubes \( D(\omega) \) we have
\[
f = \sum_{Q \in D(\omega)} \Delta_Q f.
\]
with the convergence understood in the $L^2$ sense. It is crucial that $\sigma(\Sigma) = \infty$, so that $f + C \in L^2(\sigma)$ if and only if $C = 0$ (in the case $\sigma(\Sigma) < \infty$ one would have to subtract from the left hand side above the average of $f$).

Note that $\Delta_Q f$ are mutually orthogonal in $L^2(\sigma)$, so that

$$\|f\|_{L^2(\sigma)}^2 = \sum_{Q \in \mathcal{D}(\omega)} \|\Delta_Q f\|_{L^2(\sigma)}^2$$ \hspace{1cm} (3.1)

Moreover, if $Q \in \mathcal{D}(\omega)$, then for $\sigma$-a.e. $x \in Q$

$$f(x) = \langle f \rangle_Q + \sum_{P \in \mathcal{D}(Q)} \Delta_P f(x).$$ \hspace{1cm} (3.2)

**Lemma 3.2.** Suppose $Q \in \mathcal{D}$, and let $R = R(Q) \in \mathcal{D}(\omega(Q))$ be as in Section 2.1. Then, for $f \in L^2(\sigma)$ we have

$$\alpha_{f,\sigma}(Q) \lesssim |\langle f \rangle_R| \alpha_{\sigma}(R) + \sum_{P \in \mathcal{D}(R)} \ell(P)^{1+\delta/2} \|\Delta_P f\|_2.$$

**Proof.** Let $\varphi \in \text{Lip}_1(B_Q)$ and consider a candidate for $L^\sigma_Q$ of the form $\langle f \rangle_R L^\sigma_Q$. For all $x \in B_Q \cap \text{supp} \sigma$ we have $x \in R$, so that using (3.2)

$$\left| \int \varphi(x) f(x) \, d\sigma(x) - \langle f \rangle_R \int \varphi(x) \, d\sigma_Q(x) \right| = \left| \int \varphi(x) \langle f \rangle_R + \sum_{P \in \mathcal{D}(R)} \varphi(x) \Delta_P f(x) \, d\sigma(x) - \int \varphi(x) \langle f \rangle_R \, d\sigma_Q(x) \right|$$

$$\leq |\langle f \rangle_R| \left| \int \varphi(x) \, d\sigma(x) - \int \varphi(x) \, d\sigma_Q(x) \right| + \sum_{P \in \mathcal{D}(R)} \left| \int \varphi(x) \Delta_P f(x) \, d\sigma(x) \right| =: I_1 + I_2.$$

It is clear that

$$I_1 \leq |\langle f \rangle_R| \alpha_{\sigma}(Q) \ell(Q)^{d+1} \lesssim |\langle f \rangle_R| \alpha_{\sigma}(R) \ell(Q)^{d+1},$$

which gives rise to the first term on the right hand side of (3.3).

Concerning $I_2$, we use the zero mean property of martingale difference operators, and the fact that $\varphi \in \text{Lip}_1(B_Q)$, to get

$$I_2 = \sum_{P \in \mathcal{D}(R)} \left| \int \left( \varphi(x) - \varphi(z_P) \right) \Delta_P f(x) \, d\sigma(x) \right| \leq \sum_{P \in \mathcal{D}(R)} \int \left| \varphi(x) - \varphi(z_P) \right| \|\Delta_P f\|_1 \, d\sigma(x)$$

$$\lesssim \sum_{P \in \mathcal{D}(R)} \ell(P) \|\Delta_P f\|_1 \lesssim \sum_{P \in \mathcal{D}(R)} \ell(P)^{1+\delta/2} \|\Delta_P f\|_2.$$

Dividing by $\ell(Q)^{d+1}$ and taking supremum over $\varphi \in \text{Lip}_1(B_Q)$ yields (3.3). \hfill $\Box$

**Proof of Proposition 3.1.** We begin by noting that, since $\sigma$ is uniformly rectifiable, $\alpha_{\sigma}(Q)^2 \ell(Q)^d$ is a Carleson measure by the results from [Tol09], see Theorem 1.1. Therefore, the estimate

$$\sum_{Q \in \mathcal{D}} |f|^2_{B_Q} \alpha_{\sigma}(Q)^2 \ell(Q)^d \lesssim \|f\|_{L^2(\sigma)}^2$$

follows immediately from Carleson’s embedding theorem, see e.g. [Tol14, Theorem 5.8], and we only need to estimate the sum involving $\alpha_{f,\sigma}(Q)$.

Observe that for each $\omega \in \Omega$ and $R \in \mathcal{D}(\omega)$ there is at most a bounded number of cubes $Q \in \mathcal{D}$ such that $R(Q) = R$. 

Fix some $\omega \in \Omega$. Recall that $\mathcal{G}(\omega)$ is the family of cubes $Q \in \mathcal{D}$ such that $\omega(Q) = \omega$. We apply (3.3) and the observation above to get

$$\sum_{Q \in \mathcal{G}(\omega)} \alpha_{f,\sigma}(Q)^2\ell(Q)^d \lesssim \sum_{R \in \mathcal{D}(\omega)} |(f)_R|^2 \alpha_\sigma(R)^2\ell(R)^d + \sum_{R \in \mathcal{D}(\omega)} \left( \sum_{P \in \mathcal{D}(R)} \frac{\ell(P)^{1+d/2}}{\ell(R)^{1+d/2}} \|\Delta_P f\|_2 \right)^2 =: S_1 + S_2.$$ 

Concerning $S_1$, we may use Carleson's embedding theorem again to estimate $S_1 \lesssim \|f\|_2^2$.

Moving on to $S_2$, we apply Cauchy-Schwarz inequality to get

$$S_2 \leq \sum_{R \in \mathcal{D}(\omega)} \left( \sum_{P \in \mathcal{D}(R)} \frac{\ell(P)}{\ell(R)} \|\Delta_P f\|_2^2 \right) \left( \sum_{P \in \mathcal{D}(R)} \frac{\ell(P)^{d+1}}{\ell(R)^{d+1}} \right).$$

It is easy to see that, due to Ahlfors regularity of $\sigma$, $\sum_{P \in \mathcal{D}(R)} \frac{\ell(P)^{d+1}}{\ell(R)^{d+1}} \lesssim 1$. Thus,

$$S_2 \leq \sum_{R \in \mathcal{D}(\omega)} \sum_{P \in \mathcal{D}(R)} \frac{\ell(P)}{\ell(R)} \|\Delta_P f\|_2^2 = \sum_{P \in \mathcal{D}(\omega)} \|\Delta_P f\|_2^2 \sum_{R \supset P} \frac{\ell(P)}{\ell(R)} \lesssim \sum_{P \in \mathcal{D}(\omega)} \|\Delta_P f\|_2^2 \lesssim \|f\|_2^2.$$

Putting the estimates above together we arrive at

$$\sum_{Q \in \mathcal{G}(\omega)} \alpha_{f,\sigma}(Q)^2\ell(Q)^d \lesssim \|f\|_2^2.$$ 

Summing over all $\omega \in \Omega$ (recall that $\# \Omega$ is bounded) we get the desired estimate. \qed

4. $\|Jf\|_p \lesssim \|f\|_p$ for $1 < p < \infty$

In this section we use the estimate $\|Jf\|_2 \lesssim \|f\|_2$ to prove $\|Jf\|_p \lesssim \|f\|_p$ for general $1 < p < \infty$. More precisely, we will show a localized version of the estimate, which implies the global estimate via a limiting argument.

Fix an arbitrary $Q_0 \in \mathcal{D}$ and set

$$J_0 f(x) := \left( \sum_{x \in Q \in \mathcal{D}(Q_0)} \alpha_{f,\sigma}(Q)^2 + \|f\|_{B_{Q_0}^2}^2 \alpha_\sigma(Q)^2 \right)^{1/2}.$$ 

**Proposition 4.1.** Let $1 < p < \infty$ and $f \in L^p(\sigma)$. Then,

$$\|J_0 f\|_{L^p(Q_0)} \lesssim \|f\|_{L^p(B_{Q_0})}.$$ 

The proposition follows easily from a good-lambda inequality stated below. Let $M$ denote the non-centered maximal Hardy-Littlewood operator with respect to $\sigma$, i.e.

$$M f(x) = \sup \{|f|_B : x \in B, \ B \text{ is a ball}\}.$$ 

Since $\sigma$ is Ahlfors regular, the operator $M$ is bounded on $L^p(\sigma)$ for $p > 1$, see e.g. [To14, Theorem 2.6, Remark 2.7].

**Lemma 4.2.** Let $f \in L^1_{loc}(\sigma)$. For any $\alpha > 1$ there exists $\varepsilon = \varepsilon(\alpha) > 0$ such that for all $\lambda > 0$

$$\sigma(\{x \in Q_0 : \ J_0 f(x) > \alpha \lambda \}, M f(x) \leq \varepsilon \lambda) \leq \frac{9}{10} \sigma(\{x \in Q_0 : \ J_0 f(x) > \lambda\}). \quad (4.1)$$

Let us show how to use the above to prove Proposition 4.1.
Proof of Proposition 4.1. Note that $J_0 f = J_0(f \mathbb{1}_{B_{Q_0}})$, so without loss of generality we may assume that $\text{supp } f \subset B_{Q_0}$. Let $\alpha = \alpha(p) > 1$ be so close to 1 that $0.9\alpha^p < 0.95$, and let $\varepsilon = \varepsilon(\alpha)$ be as in Lemma 4.2. We use the layer cake representation to get

$$
\int_{Q_0} J_0 f(x)^p \, d\sigma(x) = p \int_0^\infty \lambda^{p-1} \sigma(\{x \in Q_0 : J_0 f(x) > \lambda\}) \, d\lambda
$$

$$
= p \alpha^p \int_0^\infty \lambda^{p-1} \sigma(\{x \in Q_0 : J_0 f(x) > \alpha \lambda\}) \, d\lambda
$$

$$
\leq p \alpha^p \int_0^\infty \lambda^{p-1} \sigma(\{x \in Q_0 : J_0 f(x) > \alpha \varepsilon \lambda\}) \, d\lambda
$$

$$
+ p \alpha^p \int_0^\infty \lambda^{p-1} \sigma(\{x \in Q_0 : Mf(x) > \varepsilon \lambda\}) \, d\lambda
$$

$$
\leq \frac{19}{20} p \alpha^p \int_0^\infty \lambda^{p-1} \sigma(\{x \in Q_0 : J_0 f(x) > \lambda\}) \, d\lambda
$$

Absorbing the first term from the right hand side into the left hand side, we arrive at

$$
\int_{Q_0} J_0 f(x)^p \, d\sigma(x) \leq 20 \alpha^p \varepsilon^{-p} \int Mf(x)^p \, d\sigma(x).
$$

We use the $L^p$ boundedness of $M$ and the assumption $\text{supp } f \subset B_{Q_0}$ to conclude

$$
\int_{Q_0} J_0 f(x)^p \, d\sigma(x) \lesssim_{\alpha, \varepsilon} \int_{B_{Q_0}} f(x)^p \, d\sigma(x).
$$

The remainder of this section is dedicated to proving Lemma 4.2.

4.1. Preliminaries. Fix $\alpha > 1$ and $\lambda > 0$. First, we set

$$
E_\lambda = \{x \in Q_0 : J_0 f(x) > \lambda\}.
$$

Consider the covering of $E_\lambda$ with a family of cubes $\mathcal{C}_\lambda \subset \mathcal{D}(Q_0)$ such that for every $S \in \mathcal{C}_\lambda$ we have

$$
\sigma(S \cap E_\lambda) \geq 0.99 \sigma(S)
$$

and $S$ is the maximal cube with this property. Since the cubes from $\mathcal{C}_\lambda$ are pairwise disjoint, to get (4.1) it is enough to find $\varepsilon = \varepsilon(\alpha)$ such that for each $S \in \mathcal{C}_\lambda$ we have

$$
\sigma(\{x \in S : J_0 f(x) > \alpha \lambda, Mf(x) \leq \varepsilon \lambda\}) \leq \frac{8}{10} \sigma(S). \tag{4.2}
$$

Fix $S \in \mathcal{C}_\lambda$. Without loss of generality assume that

$$
\sigma(\{x \in S : Mf(x) \leq \varepsilon \lambda\}) > \frac{8}{10} \sigma(S), \tag{4.3}
$$

otherwise there is nothing to prove.

Given $x \in S$, we split the sum from the definition of $J_0 f(x)$ into two parts:

$$
J_0 f(x)^2 = \sum_{x \in Q \in \mathcal{D}(S)} \left( \alpha f_\sigma(Q)^2 + |f|_{B_{Q_0}}^2 \alpha_\sigma(Q)^2 \right) + \sum_{S \subset Q \in \mathcal{D}(Q_0)} \left( \alpha f_\sigma(Q)^2 + |f|_{B_{Q_0}}^2 \alpha_\sigma(Q)^2 \right)
$$

$$
= J_1 f(x)^2 + J_2 f(x)^2. \tag{4.4}
$$
Clearly, \( J_2 f(x) \equiv J_2 f \) is just a constant. By the definition of \( G \), there exists \( y \in \hat{S} \) (where \( \hat{S} \) is the parent of \( S \)) such that \( y \not\in E_\lambda \). By the definition of \( E_\lambda \), we get that

\[
J_2 f \leq J_0 f(y) \leq \lambda.
\]

We will show the following.

**Lemma 4.3.** There exists a set \( S_1 \subset S \) such that \( \sigma(S_1) \geq 0.5\sigma(S) \) and

\[
\int_{S_1} J_1 f(x)^2 \, d\sigma(x) \lesssim \varepsilon^2 \lambda^2 \sigma(S_1)
\]

The estimate (4.2) follows from the above easily. Indeed, using Chebyshev, we can find \( S_2 \subset S_1 \) such that for all \( x \in S_2 \) we have \( J_1 f(x) \lesssim \varepsilon \lambda \) and \( \sigma(S_2) \geq 0.5\sigma(S_1) \geq 0.2\sigma(S) \). Then, choosing \( \varepsilon = \varepsilon(\alpha) \) small enough, (4.4) gives \( J_0 f(x)^2 \leq \lambda^2 + C\varepsilon^2 \lambda^2 \leq \alpha^2 \lambda^2 \) on \( S_2 \), so that

\[
\sigma(\{x \in S : J_0 f(x) > \alpha \lambda, \ Mf(x) \leq \varepsilon \lambda\}) \leq \sigma(S \setminus S_2) \leq \frac{8}{10} \sigma(S).
\]

So our goal is to prove Lemma 4.3.

### 4.2. Calderón-Zygmund decomposition.

Let \( R = R(S) \) be as in Section 2.2, so that \( B_S \cap \text{supp} \sigma \subset R \). We consider a variant of the Calderón-Zygmund decomposition of \( f \mathbb{1}_R \) with respect to \( D(R) \) at the level of \( 2\varepsilon \lambda \).

First, let \( \{Q_j\}_j \subset D(R) \) be maximal cubes satisfying \( |f|_{B_{Q_j}} \geq 2\varepsilon \lambda \). Note that for all \( x \in Q_j \) (and recalling that \( M \) is the non-centered maximal function) we have

\[
Mf(x) \geq |f|_{B_{Q_j}} \geq 2\varepsilon \lambda.
\]

Hence, \( \bigcup_j Q_j \subset \{x \in S : Mf(x) \geq 2\varepsilon \lambda\} \), and so

\[
\sigma(R \setminus \bigcup_j Q_j) \geq \sigma(S \setminus \bigcup_j Q_j) \geq \sigma(\{x \in S : Mf(x) \leq \varepsilon \lambda\}) \geq \frac{8}{10} \sigma(S) \sim \ell(S)^d \sim \ell(R)^d.
\]

In particular, \( Q_j \neq R \) for all \( j \). Thus, by the maximality of \( Q_j \) we get easily

\[
|f|_{B_{Q_j}} \sim \varepsilon \lambda.
\]

We define \( g \in L^\infty(\sigma) \) by

\[
g(x) = f(x) \mathbb{1}_R \cup \bigcup_j Q_j(x) + \sum_j (f)_{Q_j} \mathbb{1}_{Q_j}(x).
\]

From the definition of \( Q_j \) and (4.6) it follows that \( \|g\|_{\infty} \lesssim \varepsilon \lambda \). We define also \( b \in L^1(\sigma) \) as

\[
b(x) = \sum_j (f(x) - (f)_{Q_j}) \mathbb{1}_{Q_j}(x) =: \sum_j b_j(x).
\]

Note that \( f = g + b \) and for all \( j \) we have \( \int_{Q_j} b_j \, d\sigma = 0 \).

### 4.3. Definition of \( S_1 \).

We set \( S_1 = S \setminus N_\eta \), where \( N_\eta \) is some small neighbourhood of \( \bigcup_j Q_j \). To make this more precise, given a small \( \eta > 0 \) we define \( N_\eta = \bigcup_j N_{\eta,j} \), where

\[
N_{\eta,j} = \{x \in \text{supp} \sigma : \text{dist}(x, Q_j) < \eta \ell(Q_j)\}.
\]

The thin boundaries property of \( D(2.2) \) gives

\[
\sigma(N_{\eta,j} \setminus Q_j) \lesssim \eta^2 \sigma(Q_j)
\]
for some $\gamma \in (0, 1)$. From (4.5) and the fact that $\sigma(S) \sim \sigma(R)$ we get

$$
\sigma(S \setminus N_\eta) \geq \sigma(S \setminus \bigcup_j Q_j) - \sum_j \sigma(N_{\eta,j} \setminus Q_j) \geq (4.5) \frac{8}{10} \sigma(S) - \sum_j \eta^\gamma \sigma(Q_j) \\
\geq \frac{8}{10} \sigma(S) - \eta^\gamma \sigma(R) \geq \frac{8}{10} \sigma(S) - C \eta^\gamma \sigma(S) = \left(\frac{8}{10} - C \eta^\gamma\right) \sigma(S).
$$

Here $C$ depends only on the implicit constant in $\sigma(S) \sim \sigma(R)$, which in turn depends on the Ahlfors regularity constant of $\sigma$ and on the parameters from the definition of the system $D$.

Choosing $\eta$ so small that $C \eta^\gamma < 0.1$, we get that $S_1 = S \setminus N_\eta$ satisfies

$$
\sigma(S_1) \geq \frac{7}{10} \sigma(S).
$$

4.4. Estimating $J_1f$. Now, we will show that

$$
\int_{S_1} J_1f(x)^2 \, d\sigma(x) \lesssim \varepsilon^2 \lambda^2 \sigma(S_1)
$$

(4.7)

Recall that

$$
J_1f(x)^2 = \sum_{x \in Q \in D(S)} \alpha_{f_\sigma}(Q)^2 + \sum_{x \in Q \in D(S)} |f|^2_{B_\eta} \alpha_{\sigma}(Q)^2 =: J'_1f(x)^2 + J''_1f(x)^2.
$$

First we deal with $J''_1f$. Observe that for all $Q \in D(S)$ intersecting $S_1$ we have

$$
|f|_{B_\eta} \lesssim \varepsilon \lambda.
$$

(4.8)

Indeed, let $y \in Q \cap S_1$, and let $P \in D(R)$ be such that $y \in P$, $\ell(Q) \sim \ell(P)$, and $B_Q \subset B_P$. By the maximality of $Q_j$ and the fact that $P \setminus \bigcup_j Q_j \neq \emptyset$ we get $|f|_{B_\eta} \leq 2 \varepsilon \lambda$. Estimate (4.8) follows from the inclusion $B_Q \subset B_P$.

Using (4.8) as well as Theorem 1.1 we get

$$
\int_{S_1} \sum_{x \in Q \in D(S)} |f|^2_{B_\eta} \alpha_{\sigma}(Q)^2 \, d\sigma(x) \lesssim \varepsilon^2 \lambda^2 \sum_{Q \in D(S)} \alpha_{\sigma}(Q)^2 \sigma(Q \cap S_1) \\
\lesssim \varepsilon^2 \lambda^2 \sum_{Q \in D(S)} \alpha_{\sigma}(Q)^2 \sigma(Q) \lesssim \varepsilon^2 \lambda^2 \sigma(S) \sim \varepsilon^2 \lambda^2 \sigma(S_1).
$$

Thus, we are only left with showing

$$
\int_{S_1} J'_1f(x)^2 \, d\sigma(x) = \int_{S_1} \sum_{x \in Q \in D(S)} \alpha_{f_\sigma}(Q)^2 \, d\sigma(x) \lesssim \varepsilon^2 \lambda^2 \sigma(S_1).
$$

(4.9)

Lemma 4.4. For $Q \in D(S)$ we have

$$
\alpha_{f_\sigma}(Q) \lesssim \alpha_{g_\sigma}(Q) + \varepsilon \lambda \sum_{j : Q_j \cap B_Q \neq \emptyset} \frac{\ell(Q_j)^{d+1}}{\ell(Q)^{d+1}}.
$$

Proof. Let $\varphi \in \text{Lip}_1(B_Q)$. Then, using the decomposition $f(y) = g(y) + b(y)$ valid for all $y \in R \supset B_S \supset \supp \sigma \supset B_Q \supset \supp \sigma$,

$$
\left| \int \varphi(y) f(y) \, d\sigma(y) - \int \varphi(y) \, d\mathcal{L}^{\sigma}_{Q}(y) \right| \\
\leq \left| \int \varphi(y) g(y) \, d\sigma(y) - \int \varphi(y) \, d\mathcal{L}^{\sigma}_{Q}(y) \right| + \left| \int \varphi(y) b(y) \, d\sigma(y) \right| \\
\lesssim \ell(Q)^{d+1} \alpha_{g_\sigma}(Q) + \sum_j \int \varphi(y) b_j(y) \, d\sigma(y).
$$
Concerning the second term on the right hand side, recall that \( \int b_j \, d\sigma = 0 \) and that supp \( b_j \subset Q_j \), keeping that in mind, denoting by \( x_j \) the center of \( Q_j \), we estimate in the following way:

\[
\sum_j \left| \int \varphi(y)b_j(y) \, d\sigma(y) \right| = \sum_j \left| \int (\varphi(y) - \varphi(x_j))b_j(y) \, d\sigma(y) \right|
\]

\[
\leq \sum_j \int |(\varphi(y) - \varphi(x_j))b_j(y)| \, d\sigma(y) \lesssim \sum_{j:Q_j \cap B_Q \neq \emptyset} \ell(Q_j) \int |b_j(y)| \, d\sigma(y)
\]

\[
= \sum_{j:Q_j \cap B_Q \neq \emptyset} \ell(Q_j) \int_{Q_j} |f(y) - \langle f \rangle_{Q_j}| \, d\sigma(y) \lesssim \sum_{j:Q_j \cap B_Q \neq \emptyset} \ell(Q_j)^{d+1} |\langle f \rangle_{Q_j}|
\]

\[
\lesssim \varepsilon \lambda \sum_{j:Q_j \cap B_Q \neq \emptyset} \ell(Q_j)^{d+1}.
\]

Together with the previous string of estimates, taking supremum over all \( \varphi \in \text{Lip}_1(B_Q) \), we get

\[
\alpha_{f\sigma}(Q) \lesssim \alpha_{g\sigma}(Q) + \varepsilon \lambda \sum_{j:Q_j \cap B_Q \neq \emptyset} \frac{\ell(Q_j)^{d+1}}{\ell(Q_j)^{d+1}}.
\]

An immediate consequence of Lemma 4.4 is the estimate

\[
\int_{S_1} J_1 f(x)^2 \, d\sigma(x)
\]

\[
\lesssim \int_{S_1} J_1 g(x)^2 \, d\sigma(x) + \varepsilon^2 \lambda^2 \int_{S_1} \sum_{x \in \mathcal{D}(S)} \left( \sum_{j:Q_j \cap B_Q \neq \emptyset} \frac{\ell(Q_j)^{d+1}}{\ell(Q_j)^{d+1}} \right)^2 \, d\sigma(x). \tag{4.10}
\]

Using Proposition 3.1 and the fact that \( \|g\|_\infty \lesssim \varepsilon \lambda, \) supp \( g \subset R \), we get

\[
\int_{S_1} J_1 g(x)^2 \, d\sigma(x) \leq \|Jg\|_2^2 \lesssim \|g\|_p^2 \lesssim \|g\|_\infty^2 \sigma(R) \lesssim \varepsilon^2 \lambda^2 \sigma(R) \sim \varepsilon^2 \lambda^2 \sigma(S_1). \tag{4.11}
\]

Moving on to the second term from the right hand side of (4.10), denote by Tree \( \subset \mathcal{D}(S) \) the family of cubes contained in \( S \) that intersect \( S_1 \). We have

\[
\int_{S_1} \sum_{x \in \mathcal{D}(S)} \left( \sum_{j:Q_j \cap B_Q \neq \emptyset} \frac{\ell(Q_j)^{d+1}}{\ell(Q_j)^{d+1}} \right)^2 \, d\sigma(x) \leq \sum_{Q \in \text{Tree}} \sigma(Q) \left( \sum_{j:Q_j \cap B_Q \neq \emptyset} \frac{\ell(Q_j)^{d+1}}{\ell(Q_j)^{d+1}} \right)^2 \tag{4.12}
\]

Cauchy-Schwarz

\[
\lesssim \sum_{Q \in \text{Tree}} \ell(Q)^{-d-2} \left( \sum_{j:Q_j \cap B_Q \neq \emptyset} \ell(Q_j)^{d+2} \right) \left( \sum_{j:Q_j \cap B_Q \neq \emptyset} \ell(Q_j)^{d} \right)
\]

Note that since \( Q \in \text{Tree} \), we have \( Q \cap S_1 \neq \emptyset \). By the definition of \( S_1 \), this implies that for all \( j \) such that \( Q_j \cap B_Q \neq \emptyset \) we have \( \ell(Q) \gtrsim \eta \ell(Q_j) \). Indeed, if \( \ell(Q) \ll \eta \ell(Q_j) \), then \( B_Q \cap Q_j \neq \emptyset \) implies \( Q \subset N_{\eta j} \), which would contradict \( Q \cap S_1 \neq \emptyset \).

By the observation above, we have some \( C = C(\eta) \) such that if \( B_Q \cap Q_j \neq \emptyset \), then \( Q_j \subset CB_Q \). Consequently,

\[
\sum_{j:Q_j \cap B_Q \neq \emptyset} \ell(Q_j)^{d} \lesssim \sum_{j:Q_j \subset CB_Q} \sigma(Q_j) \lesssim \sigma(CB_Q) \sim \eta \ell(Q)^{d}.
\]
Thus, the right hand side of (4.12) can be estimated by
\[
\sum_{Q \in \text{Tree}} \ell(Q)^{-2} \sum_{j : Q_j \cap B_Q \neq \emptyset} \ell(Q_j)^{d+2} = \sum_{j} \ell(Q_j)^{d+2} \sum_{Q \in \text{Tree} : Q_j \cap B_Q \neq \emptyset} \ell(Q)^{-2}.
\] (4.13)
As noted above, \(Q_j \cap B_Q \neq \emptyset\) implies \(\ell(Q) \gtrsim \ell(Q_j)\). Hence,
\[
\sum_{Q \in \text{Tree} : Q_j \cap B_Q \neq \emptyset} \ell(Q)^{-2} \lesssim \ell(Q_j)^{-2},
\]
where we used the fact that the sum above is essentially a geometric series. Putting this together with (4.13) and (4.12), we get
\[
\int_{S_1} \sum_{x \in D(S)} \left( \sum_{j : Q_j \cap B_Q \neq \emptyset} \frac{\ell(Q_j)^{d+1}}{\ell(Q)^{d+1}} \right)^2 d\sigma(x) \lesssim \sum_j \ell(Q_j)^d \lesssim \ell(R)^d \sim \sigma(S_1).
\]
Together with (4.10) and (4.11) this gives the desired estimate (4.9):
\[
\int_{S_1} J_1^j f(x)^2 d\sigma(x) \lesssim \varepsilon^2 \lambda^2 \sigma(S_1).
\]
This finishes the proof of Lemma 4.3.

5. \(\|f\|_p \lesssim \|Jf\|_p\) FOR \(1 < p < \infty\)

In this section we show the second inequality of Theorem 2.4.

**Proposition 5.1.** Let \(f \in L^p(\sigma)\) for some \(1 < p < \infty\). Then
\[
\|f\|_{L^p(\sigma)} \lesssim \|Jf\|_{L^p(\sigma)}.
\] (5.1)

### 5.1. Littlewood-Paley theory

Our main tool will be the Littlewood-Paley theory for spaces of homogeneous type developed by David, Journé and Semmes in [DJS85]. We follow the way it was paraphrased (in English) in [Tol17, Section 15].

For \(r > 0\), \(x \in \Sigma\), and \(g \in L^1_{\text{loc}}(\sigma)\), let
\[
D_r g(x) = \frac{\phi_r * (g \sigma)(x)}{\phi_r * \sigma(x)} - \frac{\phi_{2r} * (g \sigma)(x)}{\phi_{2r} * \sigma(x)}
\]
where \(\phi_r(y) = r^{-d} \phi(y/r)\) and \(\phi\) is a radially symmetric smooth nonnegative function supported in \(B(0, 1)\) with \(\int_{\mathbb{R}^n} \phi = 1\).

For a function \(g \in L^1_{\text{loc}}(\sigma)\) and \(r > 0\), we denote
\[
S_r g(x) = \frac{\phi_r * (g \sigma)(x)}{\phi_r * \sigma(x)},
\]
so that
\[
D_r g = S_r g - S_{2r} g.
\]
Let \(W_r\) be the operator of multiplication by \(1/S_r^* 1\). We consider the operators
\[
\hat{S}_r = S_r W_r S_r^* \quad \text{and} \quad \hat{D}_r = \hat{S}_r - \hat{S}_{2r}.
\]
Note that \(\hat{S}_r\), and thus \(\hat{D}_r\), are self-adjoint and \(\hat{S}_r 1 \equiv 1\), so that
\[
\hat{D}_r 1 = \hat{D}_r^* 1 = 0.
\] (5.2)
Let \(s_r(x, y)\) the kernel of \(S_r\) with respect to \(\sigma\), that is, so we can write
\[
S_r g(x) = \int s_r(x, y) g(y) \, d\sigma(y).
\]
Observe that
\[ s_r(x, y) = \frac{1}{\phi_r \ast \sigma(x)} \phi_r(x - y) \]
and the kernel of \( \mathcal{S}_r \) is
\[ \mathcal{S}_r(x, y) = \int s_r(x, z) \frac{1}{S_r^*1(z)} s_r(y, z) \, d\sigma(z). \]

We claim that the kernel \( \tilde{d}_r(x, \cdot) \) for the operator \( \tilde{D}_r \) is supported in \( B(x, 4r) \) and satisfies the Lipschitz bounds
\[ |\tilde{d}_r(x, y) - \tilde{d}_r(x, z)| \leq |y - z| r^{-d-1}. \]

Indeed, let \( x, x' \in \text{supp} \sigma \). Since \( \phi_r \) is \( Cr^{-d-1} \)-Lipschitz and \( \sigma \) is Ahlfors regular,
\[ |\phi_r \ast \sigma(x) - \phi_r \ast \sigma(x')| = \left| \int (\phi_r(x - y) - \phi_r(x' - y)) \, d\sigma(y) \right| \lesssim \frac{|x - x'|}{r}. \]

Thus, for \( y \in \text{supp} \sigma \),
\[ |s_r(x, y) - s_r(x', y)| \leq \frac{|\phi_r(x - y) - \phi_r(x' - y)|}{\phi_r \ast \sigma(x)} + \frac{1}{\phi_r \ast \sigma(x)} - \phi_r \ast \sigma(x') \]
\[ \lesssim \frac{|x - x'|}{r^{d+1}} + r^{-d} \frac{|\phi_r \ast \sigma(x) - \phi_r \ast \sigma(x')|}{\phi_r \ast \sigma(x)^2} \sim \frac{|x - x'|}{r^{d+1}}. \]

Hence,
\[ \mathcal{S}_r(x, y) - \mathcal{S}_r(x', y) = \left| \int (s_r(x, z) - s_r(x', z)) \frac{1}{S_r^*1(z)} s_r(y, z) \, d\sigma(z) \right| \]
\[ \lesssim \frac{|x - x'|}{r^{d+1}} \int \frac{1}{S_r^*1(z)} s_r(y, z) \, d\sigma(z) \lesssim \frac{|x - x'|}{r^{d+1}} \]
where in the last line we used the fact that \( \int s_r(y, z) \, d\sigma(z) = 1 \) and
\[ S_r^*1(z) = \int \frac{\phi_r(x - z)}{\phi_r \ast \sigma(x)} \, d\sigma(x) \geq \int_{B(z, r/2)} \frac{r^{-d}}{\phi_r \ast \sigma(x)} \, d\sigma(x) \sim 1. \]

Since \( \tilde{d}_r = \mathcal{S}_r - \tilde{S}_r \) and is symmetric, this proves (5.3). Moreover, notice that if \( x \in \text{supp} \sigma \), \( \text{supp} s_r(x, \cdot) \subseteq B(x, r) \), and so the integrand of \( \mathcal{S}_r \) is nonzero only when \( z \in B(x, r) \cap B(y, r) \), meaning \( |x - y| \leq 2r \), and so \( \text{supp} \tilde{S}_r \subseteq B(x, 2r) \), hence \( \text{supp} \tilde{d}_r \subseteq B(x, 4r) \), which proves our claim.

**Theorem 5.2.** [DJS85] Let \( r_k = 2^{-k} \), and \( g \in L^p(\sigma) \), \( 1 < p < \infty \), we have
\[ \|g\|_{L^p(\sigma)} \sim \left\| \left( \sum_{k \in \mathbb{Z}} |\tilde{D}_r g|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)}. \]

The original result is stated for \( p = 2 \), but this case implies the other cases (see for example the proof of [Tol01, Corollary 6.11]).

Let \( D_k := \tilde{D}_r g \), \( d_k := d_r g \). By (5.4), it is clear that to prove (5.1), it suffices to show that
\[ \left\| \left( \sum_{k \in \mathbb{Z}} |\tilde{D}_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)} \lesssim \|J f\|_{L^p(\sigma)}. \]
Similarly, without loss of generality we may assume that $L$ had and so (if we had Preliminaries. Fix $x \in \Sigma $, $k \in \mathbb{Z} $, and let $Q \in \mathcal{D} $ be the smallest cube containing $x$ and such that supp $\tilde{d}_k(x, \cdot) \subset 0.5B_Q$. Then, $\ell(Q) \sim r_k$ and

$$ |\tilde{D}_k f(x)| \lesssim \alpha_{f \sigma}(Q) + |f|_{B_Q} \alpha_{\sigma}(Q). \tag{5.5} $$

The remainder of this section is devoted to the proof of this lemma.

5.2. Preliminaries. Fix $x \in \Sigma $, $k \in \mathbb{Z} $, and let $Q$ be as above. As noted just above (5.3), we have $\tilde{d}_k(x, \cdot) \subset B(x, 4r_k)$, and so $\ell(Q) \sim r_k$ follows immediately.

We make a few simple reductions.

Remark 5.4. Without loss of generality we may assume that $\alpha_{\sigma}(Q) \leq \varepsilon$ for some small $\varepsilon$. Indeed, if we had $\alpha_{\sigma}(Q) \geq \varepsilon$, then using (5.3) and the fact that supp $\tilde{d}_k(x, \cdot) \subset B_Q$

$$ |\tilde{D}_k f(x)| = \left| \int \tilde{d}_k(x, y) f(y) \, d\sigma(y) \right| \leq \left\| \tilde{d}_k(x, \cdot) \right\|_{\infty} \int_{B_Q} |f(y)| \, d\sigma(y) \lesssim \ell(Q)^{-d} \int_{B_Q} |f(y)| \, d\sigma(y) \sim |f|_{B_Q} \lesssim |f|_{B_Q} \alpha_{\sigma}(Q), $$

and so in this case (5.5) holds. From now on we assume $\alpha_{\sigma}(Q) \leq \varepsilon$.

Remark 5.5. Similarly, without loss of generality we may assume that $L^f_{\sigma} \cap 0.5B_Q \neq \emptyset$. If we had $L^f_{\sigma} \cap 0.5B_Q = \emptyset$, then $L^f_{\sigma} \cap \text{supp } \tilde{d}_\nu(x, \cdot) = \emptyset$ so that

$$ \int \tilde{d}_k(x, y) \, dL^f_{\sigma}(y) = 0. $$

This implies

$$ |\tilde{D}_k f(x)| = \left| \int \tilde{d}_k(x, y) f(y) \, d\sigma(y) \right| \lesssim \alpha_{f \sigma}(B), $$

and so (5.5) is true also in this case.

Recall that $c^f_{Q}$, $c^\sigma_{Q}$ are the constants minimizing $\alpha_{f \sigma}(Q)$, $\alpha_{\sigma}(Q)$, respectively. Since $\sigma$ is Ahlfors regular and $\alpha_{\sigma}(Q) \leq \varepsilon$, we get by [ATT18, Lemma 3.3]

$$ c^\sigma_{Q} \sim 1. \tag{5.6} $$

To show (5.5) we begin by using (5.2) and the triangle inequality:

$$ |\tilde{D}_k f(x)| = \left| \int_{\Sigma} \tilde{d}_k(x, y) f(y) \, d\sigma(y) \right| \overset{(5.2)}{=} \left| \int_{\Sigma} \tilde{d}_k(x, y) f(y) \, d\sigma(y) - \frac{c^f_{\sigma}}{c^\sigma_{Q}} \int_{\Sigma} \tilde{d}_k(x, y) \, d\sigma(y) \right| $$

$$ \leq \left| \int_{\Sigma} \tilde{d}_k(x, y) f(y) \, d\sigma(y) - \int_{L^f_{\sigma}} \tilde{d}_k(x, y) \, dL^f_{\sigma}(y) \right| $$

$$ + \left| \int_{L^f_{\sigma}} \tilde{d}_k(x, y) \, dL^f_{\sigma}(y) - \frac{c^f_{\sigma}}{c^\sigma_{Q}} \int_{L^f_{\sigma}} \tilde{d}_k(x, y) \, dL^f_{\sigma}(y) \right| $$

$$ + \frac{c^f_{\sigma}}{c^\sigma_{Q}} \left| \int_{L^f_{\sigma}} \tilde{d}_k(x, y) \, dL^\sigma_{\sigma}(y) - \int_{\Sigma} \tilde{d}_k(x, y) \, d\sigma(y) \right| =: (I) + (II) + (III). \tag{5.7} $$

Using the Lipschitz property of $\tilde{d}_k$ (5.3) we immediately get that $(I) \lesssim \alpha_{f \sigma}(Q)$, and that

$$ (III) \lesssim \frac{c^f_{\sigma}}{c^\sigma_{Q}} \alpha_{\sigma}(Q) \overset{(5.6)}{\sim} \frac{c^f_{\sigma}}{c^\sigma_{Q}} \alpha_{\sigma}(Q). \tag{5.8} $$
Lemma 5.6. We have $|c_{Q}^{f_{\sigma}}| \lesssim |f|_{B_{Q}}$.

Proof. Indeed, if we had $|c_{Q}^{f_{\sigma}}| \geq \Lambda |f|_{B_{Q}}$ for some big $\Lambda > 10$, then $\hat{c}_{Q}^{f_{\sigma}} = 0$ would be a better competitor for a constant minimizing $\alpha_{f_{\sigma}}(Q)$. To see that, note that for any $\varphi \in \text{Lip}_{1}(B_{Q})$

$$\left| \int \varphi \, d\sigma - 0 \right| \leq C\ell(Q)^{d+1}|f|_{B_{Q}}.$$  

That is, $F_{B_{Q}}(f_{\sigma}, 0) \leq C\ell(Q)^{d+1}|f|_{B_{Q}}$. On the other hand, taking a positive $\psi \in \text{Lip}_{1}(B_{Q})$ such that $\psi(x) = \ell(Q)$ for $x \in 0.7B_{Q}$, and using the assumption $L_{Q}^{f_{\sigma}} \cap 0.5B_{Q} \neq \emptyset$ we get

$$\alpha_{f_{\sigma}}(Q)\ell(Q)^{d+1} \geq \int \psi \, d\sigma - c_{Q}^{f_{\sigma}} \int_{L_{Q}^{f_{\sigma}}} \psi \, d\mathcal{H}^{d} \geq |c_{Q}^{f_{\sigma}}| \ell(Q)\mathcal{H}^{d}(0.7B_{Q} \cap L_{Q}^{f_{\sigma}}) - \left| \int \psi \, d\sigma \right|$$

$$\geq \tilde{C} \Lambda |f|_{B_{Q}} \ell(Q)^{d+1} - C\ell(Q)^{d+1}|f|_{B_{Q}} \geq \frac{\tilde{C}\Lambda}{2} |f|_{B_{Q}} \ell(Q)^{d+1} > F_{B_{Q}}(f_{\sigma}, 0),$$

assuming $\Lambda$ big enough. This contradicts the optimality of $c_{Q}^{f_{\sigma}}$. \hfill \Box

Using the lemma above and (5.8) we get

$$(III) \lesssim |f|_{B_{Q}} \alpha_{\sigma}(Q).$$

Hence, by (5.7), to finish the proof of (5.5) it remains to show that

$$(II) = \left| c_{Q}^{f_{\sigma}} \right| \left| \int_{L_{Q}^{\sigma}} \tilde{d}_{k}(x, y) \, d\mathcal{H}^{d}(y) - \int_{L_{Q}^{f_{\sigma}}} \tilde{d}_{k}(x, y) \, d\mathcal{H}^{d}(y) \right| \lesssim \alpha_{f_{\sigma}}(Q) + |f|_{B_{Q}} \alpha_{\sigma}(Q).$$

This can be seen as an estimate of how far from each other the planes $L_{Q}^{f_{\sigma}}$ and $L_{Q}^{\sigma}$ are.

The inequality above follows immediately from Proposition 5.7 proven in the next subsection, together with the already established estimate $\left| c_{Q}^{f_{\sigma}} \right| \lesssim |f|_{B_{Q}}$.

5.3. Angles between planes approximating $f_{\sigma}$ and $\sigma$. In the following proposition we do not use uniform rectifiability in any way, and so we state it for a general Ahlfors regular measure $\mu$. Recall that given a ball $B$ we defined $\mathcal{P}_{B}^{\mu} = \mathcal{H}^{d} \llcorner L_{B}^{\mu}$.

Proposition 5.7. Let $\mu$ be an Ahlfors d-regular measure on $\mathbb{R}^{n}$, and let $f \in L_{1, \text{loc}}^{1}(\mu)$. Let $x \in \text{supp} \mu$, $r > 0$, $B = B(x, r)$, and suppose that $L_{B}^{f_{\mu}} \cap 0.5B \neq \emptyset$. Then,

$$|c_{B}^{f_{\mu}}| \frac{1}{r^{d+1}} F_{B}(\mathcal{P}_{B}^{f_{\mu}}, \mathcal{P}_{B}^{\mu}) \lesssim \alpha_{f_{\mu}}(B) + |c_{B}^{f_{\mu}}| \alpha_{\mu}(B). \quad (5.9)$$

In the proof of Proposition 5.7 we will use the following lemma.

Lemma 5.8. Let $B = B(x, r)$ and let $L_{1}$, $L_{2}$ be two $d$-planes intersecting $0.5B$. Set $\mathcal{P}_{1} = \mathcal{H}^{d} \llcorner L_{1}$, $\mathcal{P}_{2} = \mathcal{H}^{d} \llcorner L_{2}$. Then,

$$\frac{1}{r^{d}} F_{B}(\mathcal{P}_{1}, \mathcal{P}_{2}) \lesssim \text{dist}_{H}(L_{1} \cap B, L_{2} \cap B). \quad (5.10)$$

Proof. First, set

$$D = \frac{\text{dist}_{H}(L_{1} \cap B, L_{2} \cap B)}{r}.$$  

Note that we always have $F_{B}(\mathcal{P}_{1}, \mathcal{P}_{2}) \lesssim r^{d+1}$ so that if $D \gtrsim 1$, then (5.10) follows trivially. Hence, without loss of generality we may assume that $D \leq \varepsilon$ for some $\varepsilon > 0$ to be fixed later.
We claim that if $\varepsilon$ is chosen small enough (depending only on $n$, $d$), then there exists an isometry $A : L_1 \to L_2$ such that for $y \in B \cap L_1$ we have $|y - A(y)| \lesssim Dr$. To see that, let $y_1 \in L_1 \cap B$ be arbitrary. Set $y_2 = \pi_{L_2}(y_1)$. Clearly,

$$|y_1 - y_2| \leq Dr \leq \varepsilon r.$$ 

Let $v_1, \ldots, v_d$ be an orthonormal basis of the linear plane $L'_1 := L_1 - y_1$. For $i = 1, \ldots, d$ define

$$w_i := \pi_{L_2}(y_1 + v_i) - y_2 \in L_2 - y_2 := L'_2.$$

In fact, since $y_2 = \pi_{L_2}(y_1)$, we have $w_i = \pi_{L_2}(v_i)$. It is easy to see that for all $v \in L'_1$ we have

$$|\pi_{L'_2}(v) - v| \lesssim D|v|.$$ 

Hence, $|w_i - v_i| \lesssim D \leq \varepsilon$ and for $i \neq j$

$$|w_i \cdot w_j| = |(w_i - v_i) \cdot (w_j - v_j) + (w_i - v_i) \cdot v_j + v_i \cdot (w_j - v_j)| \lesssim D \leq \varepsilon.$$

Choosing $\varepsilon$ small enough (depending only on dimensions), we get easily that $\{w_i\}$ is a basis of $L'_2$. Moreover, if $\{\tilde{w}_i\}$ is the orthonormal basis of $L'_2$ constructed from $\{w_i\}$ using the Gram-Schmidt process, then it follows from the estimates above that for all $i = 1, \ldots, d$

$$|\tilde{w}_i - v_i| \lesssim D.$$

We define the map $A : L_1 \to L_2$ as the unique isometry such that $A(y_1) = y_2$ and $A(y_1 + v_i) = y_2 + \tilde{w}_i$. It follows immediately from basic linear algebra that for $y \in L_1 \cap B$ we have $|y - A(y)| \lesssim Dr$.

Now, let $\varphi \in \text{Lip}_1(B)$. We have

$$\left| \int_{L_1} \varphi(y) \, dH^d(y) - \int_{L_2} \varphi(y) \, dH^d(y) \right| \leq \int_{L_1} |\varphi(y) - \varphi(A(y))| \, dH^d(y) \lesssim \int_{L_1 \cap B} Dr \, dH^d(y) \lesssim Dr^{d+1}. $$

Taking supremum over $\varphi \in \text{Lip}_1(B)$ finishes the proof. \hfill \square

**Proof of Proposition 5.7.** For simplicity of notation we will usually omit the subscript $B$, i.e. we will write $L^\mu := L^\mu_B$, $c^\mu := c^\mu_B$, and so on.

Without loss of generality we can assume that $c^\mu \geq 0$. Indeed, if that was not the case we could consider $g = -f$. Then the plane and constant $L^g = L^\mu$, $c^g = -c^\mu \geq 0$ are minimizing for $\alpha_{g\mu}(B)$, and we have $\alpha_{g\mu}(B) = \alpha_{f\mu}(B)$. Thus, proving (5.9) for $g$ is equivalent to proving it for $f$, and $c^g \geq 0$.

Note that we always have $F_B(\mathcal{P}^\mu, \mathcal{P}^f) \leq r^{d+1}$ so that if $\alpha_{\mu}(B) \gtrsim 1$, then (5.9) is trivial. Assume that $\alpha_{\mu}(B) \lesssim \varepsilon$ for some small $\varepsilon > 0$ (depending on dimensions and Ahlfors regularity constants), to be fixed later.

Note that if $\varepsilon$ is small enough, then one can use Ahlfors regularity of $\mu$ to conclude that $L^\mu \cap 0.5B \neq \emptyset$ (see for example [To09, Lemma 3.1]). We use this observation, the assumption $L^f \cap 0.5B \neq \emptyset$ and (5.10) to estimate

$$c^f \frac{1}{r^{d+1}} F_B(\mathcal{P}^\mu, \mathcal{P}^f) \lesssim c^\mu \frac{\text{dist}_H(L^\mu \cap B, L^f \cap B)}{r} =: c^\mu D.$$

Our aim is to show that

$$c^f \mu D \lesssim c^\mu \alpha_{\mu}(B) + \alpha_{f\mu}(B). \tag{5.11}$$

Let $0 < \eta < 0.01$ be some dimensional constant. Note that, since $L^f \cap 0.5B \neq \emptyset$, the set $L^f \cap 0.9B$ is a $d$-dimensional ball with $H^d(L^f \cap 0.9B) \sim r^d$. We claim that we can find a $d$-dimensional ball $B_0$ contained in $L^f \cap 0.9B$, of radius $\eta r$ (in particular $r_{B_0} \sim \eta r_B$), and such that

$$\text{dist}(z, L^\mu) \geq 10\eta Dr \quad \text{for all } z \in B_0. \tag{5.12}$$
Indeed, if there was no such ball, i.e. if for all $d$-dimensional balls $B_0 \subset L^{\mu} \cap 0.9 B$ of radius $\eta r$ there was some $z \in B_0$ with $\text{dist}(z, L^{\mu}) \leq 10 \eta Dr$, then it would follow easily from the definition of Hausdorff distance, and from the fact that $L^{\mu}$ and $L^{\mu}$ are $d$-planes intersecting $0.5 B$, that

$$\text{dist}_H(L^{\mu} \cap B, L^{\mu} \cap B) \lesssim \eta Dr = \eta \text{dist}_H(L^{\mu} \cap B, L^{\mu} \cap B).$$

For $\eta$ small enough, this is a contradiction. We omit the details, which can be readily filled in e.g. using [AT15, Lemma 6.4].

Consider an open neighbourhood of $B_0$ given by

$$U := \{ y \in \mathbb{R}^n : \text{dist}(y, B_0) < \eta Dr \},$$

and also for $\lambda > 0$ set

$$\lambda U := \{ y \in \mathbb{R}^n : \text{dist}(y, B_0) < \lambda \eta Dr \}.$$ 

Since $D \leq 1$, one should think of $U$ as an $n$-dimensional pancake around $B_0$ of thickness $\eta Dr$, so that the smaller $D$, the flatter the pancake. Note that by (5.12) for all $0 < \lambda < 10$ we have $\lambda U \cap L^\mu = \emptyset$, and also $\lambda U \subset B$ because $B_0 \subset 0.9 B$.

Let $\varphi : \mathbb{R}^d \to [0, \eta Dr]$ be a function satisfying $\varphi \equiv \eta Dr$ in $U$, $\text{supp} \varphi \subset 2U$, and $\text{Lip}(\varphi) \leq 1$. Clearly, $\varphi \in \text{Lip}_1(B)$, and so

$$\left| \int \varphi f \, d\mu - \int \varphi \, dL^\mu \right| \leq \alpha_{f\mu}(B)r^{d+1}. \tag{5.13}$$

Furthermore, note that $\varphi \equiv \eta Dr$ on $B_0$, so that

$$\int \varphi \, dL^\mu = c^{f\mu} \int L_\mu \varphi \, dH^d = c^{f\mu} \eta Dr \, H^d(B_0) = C(d)c^{f\mu} Dr^{d+1}.$$ 

Together with (5.13) this implies

$$\int \varphi f \, d\mu \geq C(\eta, d)c^{f\mu} Dr^{d+1} - \alpha_{f\mu}(B)r^{d+1}. \tag{5.14}$$

Recall that we are trying to prove $c^{f\mu} D \lesssim c^{f\mu} \alpha(\mu)(B) + \alpha_{f\mu}(B)$. If we had $c^{f\mu} D \leq \Lambda \alpha_{f\mu}(B)$ for some $\Lambda = \Lambda(\eta, d) > 100$, then there is nothing to prove. So without loss of generality assume that $c^{f\mu} D \geq \Lambda \alpha_{f\mu}(B)$. In that case (5.14) gives

$$\int \varphi f \, d\mu \gtrsim \eta c^{f\mu} Dr^{d+1}. \tag{5.15}$$

Now we define a modified version of $\varphi$. Recall that $\text{supp} \varphi \subset 2U$. For all $y \in \text{supp} \mu \cap 2U$ let $B_y = B(\mu, \eta Dr / 5)$. We use the $5r$ covering theorem to extract from $\{ B_y \}_{y \in \text{supp} \mu \cap 2U}$ a subfamily of pairwise disjoint balls $\{ B_i \}_{i \in I}$ such that $\text{supp} \mu \cap 2U \subset \bigcup_i 5B_i$. Note that $\bigcup_i 10B_i \subset 4U$, and in particular, $\bigcup_i 10B_i \cap L^\mu = \emptyset$. Moreover, the balls $10B_i$ have bounded intersection. Thus, we may consider a partition of unity

$$\Psi = \sum_{i \in I} \psi_i,$$

such that $\text{supp} \psi_i \subset 10B_i$ for each $i \in I$, $\Psi \equiv 1$ on $\bigcup_i 5B_i$, and $\text{Lip} \Psi \lesssim (\eta Dr)^{-1}$.

Consider $\Phi = \varphi \Psi$. We have

$$\| \nabla \Phi \|_\infty \lesssim \| \nabla \varphi \|_\infty \| \Psi \|_\infty + \| \varphi \|_\infty \| \nabla \Psi \|_\infty \lesssim 1 + \eta Dr(\eta Dr)^{-1} = 1.$$

Hence, $C\Phi \in \text{Lip}_1(B)$ for some $C \sim 1$, so that

$$\left| \int \Phi f \, d\mu - \int \Phi \, dL^\mu \right| \leq C^{-1} \alpha_{f\mu}(B)r^{d+1}. \tag{5.16}$$

On the other hand, observe that $\Psi \equiv 1$ on $\text{supp} \varphi \cap \text{supp} \mu$. By (5.15)

$$\int \Phi f \, d\mu = \int \varphi f \, d\mu \gtrsim \eta c^{f\mu} Dr^{d+1}.$$
Together with (5.16) this gives
\[
\int \Phi \, d\mathcal{L}^f \mu \geq C(\eta)c^{d+1} \mathcal{H}^d r^{d+1} - C^{-1} \alpha f_{\perp}(B)r^{d+1} \lesssim \eta \, c^{d+1} \mathcal{H}^d r^{d+1},
\]
where we used once again the additional assumption \(c^{d+1} \geq \Lambda \alpha f_{\perp}(B)\) we made along the way (and choosing \(\Lambda\) large).

Now we will show that
\[
\int_{L^f} \Phi \, d\mathcal{H}^d \lesssim \eta \, \alpha(\mu) r^{d+1}.
\]
(5.18)

Since \(\mathcal{L}^f = c^{d+1} \mathcal{H}^d L^f\), together with (5.17) this will give \(c^{d+1} \mathcal{H}^d \lesssim \eta \, c^{d+1} \alpha(\mu)\), and so the proof of (5.11) will be finished.

Recall that \(\Phi \subset \text{supp } \Psi \subset \bigcup_{i \in I} 10B_i\), and that \(\|\Phi\|_{\infty} \leq \|\Psi\|_{\infty} = \eta Dr\). Hence,
\[
\int_{L^f} \Phi \, d\mathcal{H}^d \lesssim \eta \, Dr \sum_{i \in I} \mathcal{H}^d (L^f \cap 10B_i) \lesssim \eta \, \#(I(Dr)^{d+1}).
\]

To estimate \#\(I\) we will use Ahlfors regularity of \(\mu\). Recall that \(\{B_i\}_{i \in I}\) are pairwise disjoint, they are centered at points from \(\text{supp } \mu \cap 2U\), and \(r(B_i) = \eta D/5\). Thus,
\[
\#(I(rD)^{d+1} \sim \eta \sum_{i \in I} \mu(B_i) = \mu \left( \bigcup_{i \in I} B_i \right).
\]

On the other hand, since the balls \(\{B_i\}\) are centered at points from \(2U\), we have \(\bigcup_{i \in I} B_i \subset 3U\) and
\[
\mu \left( \bigcup_{i \in I} B_i \right) \leq \mu (3U).
\]

To bound \(\mu (3U)\) consider \(\varphi \in \text{Lip}_1(B)\) such that \(\varphi \geq 0, \varphi \equiv \eta D\) on \(3U\) and \(\text{supp } \varphi \subset 4U\). Recalling that \(4U \cap L^\mu = \emptyset\), we arrive at
\[
rD \mu (3U) \lesssim \eta \int \varphi \, d\mu = \left| \int \varphi \, d\mu - \int \varphi \, d\mathcal{L}^f \right| \leq \alpha(\mu) r^{d+1}.
\]

Putting all the estimates above together we get (5.18):
\[
\int_{L^f} \Phi \, d\mathcal{H}^d \lesssim \eta \, \#(I(Dr)^{d+1} \lesssim \eta \, rD \mu (3U) \lesssim \eta \, \alpha(\mu) r^{d+1}.
\]

\[\square\]

REFERENCES

[ADT16] J. Azzam, G. David, and T. Toro. Wasserstein distance and the rectifiability of doubling measures: part I. Math. Ann., 364(1-2):151–224, 2016, arXiv:1408.6645. doi:10.1007/s00208-015-1206-z, 3

[AT15] J. Azzam and X. Tolsa. Characterization of \(\alpha\)-rectifiability in terms of Jones’ square function: Part II. Geom. Funct. Anal., 25(5):1371–1412, 2015, arXiv:1501.01572. doi:10.1007/s00039-015-0334-7, 21

[ATT18] J. Azzam, X. Tolsa, and T. Toro. Characterization of rectifiable measures in terms of \(\alpha\)-numbers. Preprint, to appear in Trans. Amer. Math. Soc., 2018, arXiv:1808.07661. doi:10.1090/tran/7170, 3, 18

[Chr90] M. Christ. A \(T(1)\) theorem with remarks on analytic capacity and the Cauchy integral. Colloq. Math., 60:601–628, 1990, doi:10.4064/cm-60-2-601-628, 5

[Dab19] D. Dąbrowski. Necessary condition for rectifiability involving Wasserstein distance \(W_2\). Preprint, to appear in Int. Math. Res. Not. IMRN, 2019, arXiv:1904.11000, doi:10.1093/imrn/rnaa012, 3

[Dav88] G. David. Moreaux de graphes lipschitziens et intégrales singulières sur une surface. Rev. Mat. Iberoam., 4(1):73–114, 1988, doi:10.4171/RMI/64, 5

[DEM18] G. David, M. Engelstein, and S. Mayboroda. Square functions, non-tangential limits and harmonic measure in co-dimensions larger than one. Preprint, to appear in Duke Math. J., 2018, arXiv:1808.08882, 2

[DJS85] G. David, J.-L. Journé, and S. Semmes. Opérateurs de Calderón-Zygmund, fonctions para-accretives et interpolation. Rev. Mat. Iberoam., 1(4):1–56, 1985, doi:10.4171/RMI/17, 4, 16, 17

[DM20] G. David and S. Mayboroda. Harmonic measure is absolutely continuous with respect to the Hausdorff measure on all low-dimensional uniformly rectifiable sets. arXiv preprint, 2020, arXiv:2006.14661, 2
AN α-NUMBER CHARACTERIZATION OF LP SPACES

[DS91] G. David and S. Semmes. Singular integrals and rectifiable sets in \( \mathbb{R}^n \): Au-delà des graphes lipschitziens. *Astérisque*, 193, 1991. doi:10.24033/ast.68.

[Fen20] J. Feneuil. Absolute continuity of the harmonic measure on low dimensional rectifiable sets. *arXiv preprint*, 2020, arXiv:2006.03118.

[Gra14] L. Grafakos. *Classical Fourier Analysis*, volume 249 of *Grad. Texts in Math*. Springer-Verlag New York, 3 edition, 2014. doi:10.1007/978-1-4939-1194-3.

[HT14] T. Hytönen and O. Tapiola. Almost Lipschitz-continuous wavelets in metric spaces via a new randomization of dyadic cubes. *J. Approx. Theory*, 185:12–30, 2014, arXiv:1310.2047. doi:10.1016/j.jat.2014.05.017.

[Mat95] P. Mattila. *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*, volume 44 of *Cambridge Stud. Adv. Math*. Cambridge Univ. Press, 1995. doi:10.1017/CBO9780511623813.

[Orp18] T. Orponen. Absolute continuity and α-numbers on the real line. *Anal. PDE*, 12(4):969–996, 2018, arXiv:1703.02935. doi:10.2140/apde.2019.12.969.

[Tol01] X. Tolsa. Littlewood–Paley theory and the \( T(1) \) theorem with non-doubling measures. *Adv. Math.*, 164(1):57–116, 2001, arXiv:math/0006039. doi:10.1006/aima.2001.2011.

[Tol08] X. Tolsa. Principal values for Riesz transforms and rectifiability. *J. Funct. Anal.*, 254(7):1811–1863, 2008, arXiv:0708.0199. doi:j.jfa.2007.07.020.

[Tol09] X. Tolsa. Uniform rectifiability, Calderón-Zygmund operators with odd kernel, and quasiorthogonality. *Proc. Lond. Math. Soc. (3)*, 98(2):393–426, 2009, arXiv:0805.1053. doi:10.1112/plms/pdn035.

[Tol12] X. Tolsa. Mass transport and uniform rectifiability. *Geom. Funct. Anal.*, 22(2):478–527, 2012, arXiv:1103.1543. doi:10.1007/s00039-012-0160-0.

[Tol14] X. Tolsa. *Analytic capacity, the Cauchy transform, and non-homogeneous Calderón-Zygmund theory*, volume 307 of *Progress in Mathematics*. Birkhäuser, 2014. doi:10.1007/978-3-319-00596-6.

[Tol17] X. Tolsa. Rectifiable measures, square functions involving densities, and the Cauchy transform. *Mem. Amer. Math. Soc.*, 245(1158), 2017, arXiv:1408.6979. doi:10.1090/memo/1158.

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