A characterization of the standard Reeb flow

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Abstract. Among the topological conjugacy classes of the continuous flows \( \{ \phi^t \} \) whose orbit foliations are the planar Reeb foliation, there is one special class called the standard Reeb flow. We show that \( \{ \phi^t \} \) is conjugate to the standard Reeb flow if and only if \( \{ \phi^{\lambda t} \} \) for any \( \lambda > 0 \).

Key words: Reeb foliations, flows, topological conjugacy.

1. Introduction

Let

\[ P = \{ (\xi, \eta) | \xi \geq 0, \eta \geq 0 \} - \{ (0, 0) \}. \]

A nonsingular flow \( \{ \Phi^t \} \) on \( P \) defined by

\[ \Phi^t(\xi, \eta) = (e^t \xi, e^{-t} \eta) \]

is called the standard Reeb flow. In this note the oriented foliation \( \mathcal{R} \) whose leaves are the orbits of \( \{ \Phi^t \} \) with the orientation given by the time direction is called the Reeb foliation. A continuous flow on \( P \) with orbit foliation \( \mathcal{R} \) is called an \( \mathcal{R} \)-flow. The topological conjugacy classes of \( \mathcal{R} \)-flows \( \{ \phi^t \} \) are classified in [L] in the following way. Let \( \gamma_1 : [0, \infty) \to P \) (resp. \( \gamma_2 : [0, \infty) \to P \)) be a continuous path such that \( \gamma_1(0) \in \{ \xi = 0 \} \) (resp. \( \gamma_2(0) \in \{ \eta = 0 \} \)) which intersects every interior leaf of \( \mathcal{R} \) at exactly one point. Then one can define a continuous function

\[ f_{\{ \phi^t \}, \gamma_1, \gamma_2} : (0, \infty) \to \mathbb{R} \]

by setting that \( f_{\{ \phi^t \}, \gamma_1, \gamma_2}(x) \) is the time needed for the flow \( \{ \phi^t \} \) to move

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from the point $\gamma_1(x)$ until it reaches a point on the curve $\gamma_2$. Then $f_{\{\phi^t\},\gamma_1,\gamma_2}$ belongs to the following space

$$E = \left\{ f : (0, \infty) \to \mathbb{R} | f \text{ is continuous and } \lim_{x \to 0} f(x) = \infty \right\}.$$ 

Of course $f_{\{\phi^t\},\gamma_1,\gamma_2}$ depends upon the choices of $\gamma_1$ and $\gamma_2$. There are two umbiguitities, one coming from the parametrization of $\gamma_1$, and the other coming from the positions of $\gamma_1$ and $\gamma_2$. Let $H$ be the space of homomorphisms of $[0, \infty)$ and $C$ the space of continuous functions on $[0, \infty)$. Define an equivalence relation $\sim$ on $E$ by

$$f \sim f' \iff f' = f \circ h + k, \quad \exists h \in H, \quad \exists k \in C.$$ 

Then clearly the equivalence class of $f_{\{\phi^t\},\gamma_1,\gamma_2}$ does not depend on the choice of $\gamma_1$ and $\gamma_2$. Moreover it is an invariant of the topological conjugacy classes of $\mathcal{R}$-flows. Thus if we denote by $\mathcal{E}$ the set of the topological conjugacy classes of the $\mathcal{R}$-flows, then there is a well defined map

$$\iota : \mathcal{E} \to E/\sim.$$ 

The main result of [L] states that $\iota$ is a bijection. In particular any $f \in E$ is obtained as $f = f_{\{\phi^t\},\gamma_1,\gamma_2}$ for some $\mathcal{R}$-flow $\{\phi^t\}$ and paths $\gamma_i$.

Clearly any strictly monotone function of $E$ belongs to a single equivalence class, and this corresponds to the standard Reeb flow $\{\Phi^t\}$. The purpose of this note is to show the following characterization of the standard Reeb flow.

**Theorem 1** An $\mathcal{R}$-flow $\{\phi^t\}$ is topologically conjugate to the standard Reeb flow $\{\Phi^t\}$ if and only if $\{\phi^{\lambda t}\}$ is topologically conjugate to $\{\phi^t\}$ for any $\lambda > 0$.

Of course the only if part is immediate. We shall show the if part in the next section.

**Remark 1.1** A single $\lambda$ is not enough for Theorem 1. In fact there is an $\mathcal{R}$-flow $\{\phi^t\}$ not topologically conjugate to $\{\Phi^t\}$ such that $\{\phi^{2t}\}$ is topologically conjugate to $\{\phi^t\}$. This will be given in Example 2.4 below.

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2. Proof of the if part

The equivalence class of \( f \in E \) is determined by how \( f(x) \) oscillates while it tends to \( \infty \) as \( x \to 0 \). So to measure the degree of oscillation of \( f \in E \), define a nonnegative valued continuous function \( f^* \) defined on \((0,1]\) by

\[
f^*(x) = \max(f|_{[x,1]}) - f(x).
\]

Then we have the following lemma.

**Lemma 2.1**

1. If \( \lambda > 0 \), then \((\lambda f)^* = \lambda f^*\).
2. If \( c \) is a constant, then \((f + c)^* = f^*\).
3. If \( h \in H \), then there is \( 0 < a < 1 \) such that \((f \circ h)^* = f^* \circ h \) on \((0,a)\).
4. If \( k \in C \) and \( x \to 0 \), then \((f + k)^*(x) - f^*(x) \to 0\).
5. There is a sequence \( \{x_n\} \) tending to 0 such that \( f^*(x_n) = 0\).

**Proof.** Points (1) and (2) are immediate. To show (3) notice that

\[
(f \circ h)^*(x) = \max(f|_{[h(x),h(1)]}) - f(h(x)) \quad \text{and} \quad f^* \circ h(x) = \max(f|_{[h(x),1]}) - f(h(x)).
\]

Since \( f(x) \to \infty \) \((x \to 0)\), both maxima coincide for small \( x \).

Let us show (4). By (2) we only need to show (4) assuming that \( k(0) = 0 \). Now given \( \epsilon > 0 \), there is \( \delta > 0 \) such that if \( 0 < x < \delta \), then \( |k(x)| < \epsilon \). Choose \( \eta > 0 \) small enough so that if \( 0 < x < \eta \), then we have

\[
f(x) \geq \max(f|_{[\delta,1]}) \quad \text{and} \quad (f + k)(x) \geq \max((f + k)|_{[\delta,1]}).
\]

This implies that for \( x \in (0,\eta) \),

\[
|f^*(x) - (f + k)^*(x)| \\
\leq |f(x) - (f + k)(x)| + |\max((f + k)|_{[x,\delta]}) - \max(f|_{[x,\delta]})| < 2\epsilon.
\]

This shows (4). Finally (5) follows from the assumption \( f(x) \to \infty \) as \( x \to 0 \). \( \square \)
For $f \in E$ define an invariant $\sigma(f) = \limsup_{x \to 0} f^*(x)$ which takes value in $[0, \infty]$. In fact $\sigma(f)$ coincides with the invariant $A(f)$ defined in [L] and used to show that $E$ is uncountable.

**Lemma 2.2** Assume $f, f' \in E$ and $\lambda > 0$.

1. We have $\sigma(\lambda f) = \lambda \sigma(f)$.
2. If $f \sim f'$, then $\sigma(f) = \sigma(f')$. In particular $f$ corresponds to the standard Reeb flow if and only if $\sigma(f) = 0$.

**Proof.** Clearly (1) follows from Lemma 2.1 (1), while the first statement of (2) is an easy consequence of Lemma 2.1 (3) and (4). To show the last statement, assume $\sigma(f) = 0$. Extend the function $f^*$ defined on $(0, 1]$ to $[0, \infty)$ by letting $f^* = 0$ on $\{0\} \cup (1, \infty)$.

Since $\sigma(f) = 0$, $f^*$ is continuous, i.e. $f^* \in C$. Thus $f \sim f + f^*$, and the latter is (weakly) monotone near 0. Still adding a suitable function, one can show that $f$ is equivalent to a function $g$ which is strictly monotone on the whole $(0, \infty)$ such that $g(x) \to 0$ ($x \to \infty$). Clearly such functions are mutually equivalent by a pre-composition of some $h \in H$, and correspond to the standard Reeb flow $\{\Phi^t\}$. \[\square\]

Now since

$$f\{\phi^{\lambda t}, \gamma_1, \gamma_2\} = \lambda^{-1} f\{\phi^{t}, \gamma_1, \gamma_2\}, \tag{2.1}$$

for $\lambda > 0$, Theorem 1 reduces to the following proposition.

**Proposition 2.3** If $f \in E$ and $f \sim \lambda f$ for any $\lambda > 0$, then $\sigma(f) = 0$.

The rest of the paper is devoted to the proof of Proposition 2.3. But before starting, let us mention an example for Remark 1.1.

**Example 2.4** By (2.1) and the main result of [L], it suffices to construct a function $f \in E$ such that $f(x/2) = 2f(x)$ and that $\sigma(f) = \infty$. Set for example

$$f(x) = \frac{1}{x} 2^{\sin(2\pi \log_2 x)}.$$
The following lemma, roughly the same thing as the linearization in one dimensional local dynamics, plays a crucial role in what follows.

**Lemma 2.5** Assume $f \in E$ satisfies $\lambda f = f \circ h + k$ for some $h \in H$, $k \in C$ and $\lambda > 1$. Then 0 is an attracting fixed point of $h$ and there exists $f_\infty \in E$ such that $f_\infty - f \in C$, $\lambda f_\infty = f_\infty \circ h$ and $f_\infty(x) \to 0$ ($x \to \infty$).

**Proof.** Any equivalence class of $E$ has a representative $f$ such that $f|_{[1, \infty)}$ is bounded. (2.2)

So it is no loss of generality to assume that the function $f$ in the lemma satisfies (2.2). We can also assume that $k(0) = 0$, by adding a suitable constant to $f$ if necessary. Choose $a' \in (0, 1)$ so that if $a \in (0, a')$,

$$f(a) > \frac{2}{\lambda - 1} \max(|k||_{[0,1]}).$$

Then we have

$$f \circ h(a) > \frac{\lambda + 1}{2} f(a), \quad \forall a \in (0, a').$$

(2.3)

If $a$ is sufficiently near 0, we have

$$f(a) > \sup(f|_{[1, \infty)}).$$

If furthermore $f^*(a) = 0$, then

$$\{x|f(x) > f(a)\} \subset (0, a).$$

Thus (2.3) implies $h(a) < a$ for such $a$. But this allows us to use (2.3) repeatedly for $h^n(a)$ ($n = 1, 2, \ldots$) instead of $a$, showing that $f \circ h^n(a) \to \infty$ as $n \to \infty$. Clearly this implies that $[0, a]$ is contained in the attracting domain of an attractor 0 of the homeomorphism $h$, showing the first point of Lemma 2.5.

For the rest of the proof, let us divide the argument into two cases according to the dynamics of $h$. First assume that the whole line $[0, \infty)$ is the attracting domain of 0. Let
\[ f_n(x) = \lambda^{-n} f(h^n(x)). \]

Then we have
\[ f_{n+1}(x) - f_n(x) = -\lambda^{-n-1} k(h^n(x)), \]
showing that \( f_n \to f_\infty \) uniformly on compact subsets of \((0, \infty)\) for some continuous function \( f_\infty \). Now since
\[ \lambda f_{n+1}(x) = f_n(h(x)), \]
we have
\[ \lambda f_\infty = f_\infty \circ h. \]

We also have
\[ |f(x) - f_\infty(x)| \leq \sum_{n=0}^{\infty} \lambda^{-n-1}|k(h^n(x))|. \]

The continuity of \( k \), together with the assumption \( k(0) = 0 \), implies that
\[ \lim_{x \to 0} |f(x) - f_\infty(x)| = 0, \]
showing that \( f_\infty - f \in C \).

Finally since \( h^{-n}(x) \to \infty \) \((n \to \infty)\) and
\[ f_\infty \circ h^{-n}(x) = \lambda^{-n} f_\infty(x), \quad \forall x \in (0, \infty), \]
we have \( f_\infty(x) \to 0 \) \((x \to \infty)\).

Next assume there is a fixed point \( b \) of \( h \) such that \((0, b)\) is an attracting domain of \( 0 \). Thus we have \( h^{-n}(x) \to b \) \((n \to \infty)\) for any \( x \in (0, b) \).

The same argument as above shows the existence of a continuous function \( f_\infty \) on \((0, b)\). Since
\[ f_\infty \circ h^{-n}(x) = \lambda^{-n} f_\infty(x), \quad \forall x \in (0, b), \]
we have
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\[ \lim_{x \uparrow b} f_\infty(x) = 0. \]

Now extend \( f_\infty \) by setting \( f_\infty = 0 \) on \([b, \infty)\). \qed

Let us start the proof of Proposition 2.3. Assume \( f \in E \) satisfies \( f \sim 2^{1/N} f \) for any \( N \in \mathbb{N} \). Applying Lemma 2.5, \( f \) can be changed within the equivalence class to one which satisfies the condition of \( f_\infty \) for \( \lambda = 2 \). We also assume for contradiction that \( \sigma(f) > 0 \). Then by Lemma 2.2 (1) it follows that \( \sigma(f) = \infty \).

Thus the proof of Proposition 2.3 reduces to showing that there is no \( f \in E \) which satisfies the following assumption.

**Assumption 2.6** A function \( f \in E \) satisfies

\[
2f = f \circ h, \quad \exists h \in H, \quad f(x) \to 0 \ (x \to \infty), \tag{2.4}
\]

\[
2^{1/N} f - f \circ h_N \in C \quad \exists h_N \in H, \quad \forall N \geq 2 \quad \text{and} \quad \sigma(f) = \infty. \tag{2.5}
\]

Define

\[
E_0 = \{ f \in E | f(x) \to 0 \ (x \to \infty) \}.
\]

Henceforth all the functions dealt with will be in \( E_0 \), and the following definition is more convenient. For \( f \in E_0 \) define

\[
f^\sharp(x) = \max(f|_{[x, \infty)}) - f(x).
\]

Clearly \( f^\sharp \) and \( f^\ast \) are the same near 0 and Lemma 2.1 (1), (4) and (5) hold also for \( f^\sharp \), while (3) becomes stronger. In summary we have:

**Lemma 2.7** Assume \( f, f' \in E_0 \).

1. If \( \lambda > 0 \), then \( (\lambda f)^\sharp = \lambda f^\sharp \).
2. If \( h \in H \), then \( (f \circ h)^\sharp = f^\sharp \circ h \).
3. If \( f' - f \in C \) and \( x \to 0 \), then \( f^\sharp(x) - (f')^\sharp(x) \to 0 \).
4. There is a sequence \( \{x_n\} \) tending to 0 such that \( f^\sharp(x_n) = 0 \).

Hereafter \( f \) is always to be a function satisfying Assumption 2.6. Thus we have
2f^♯ = f^♯ \circ h. \quad (2.7)

Fix \( N \) for a while and let \( h_1 = h_N^N \). Notice that by Lemma 2.5 both \( h \) and \( h_1 \) have 0 as their attractors and that

\[
f \circ h - f \circ h_1 = 2f - f \circ h_1 = \sum_{\nu=0}^{N-1} 2^{(N-\nu-1)/N} \left( 2^{1/N} f \circ h^\nu_N - f \circ h^\nu_{N+1}^\nu \right) \in C.
\]

The following is an easy corollary of Lemma 2.7.

**Corollary 2.8**  We have

\[
\lim_{x \to 0} |f^♯ \circ h(x) - f^♯ \circ h_1(x)| = 0.
\]

Our overall strategy is to show that \( f^♯ \) is too much oscillating in a fundamental domain of \( h \), thanks to condition (2.5). For that purpose first of all we have to compare the dynamics of \( h \) and \( h_1 \) near the common attractor 0 and to show that they have more or less the same fundamental domains.

**Lemma 2.9**  Either there exists a sequence \( \{a_n\} \) such that \( a_n \to 0 \) and that \( h_2(a_n) \leq h_1(a_n) \leq h(a_n) \) or there exists a sequence \( \{a_n\} \) such that \( a_n \to 0 \) and that \( h_2^2(a_n) \leq h(a_n) \leq h_1(a_n) \).

**Proof.**  If there is a sequence \( \{a_n\} \) such that \( a_n \to 0 \) and that \( h(a_n) = h_1(a_n) \), there is nothing to prove. So there are two cases to consider. One is when \( h_1(x) < h(x) \) for any small \( x \), and the other \( h_1(x) > h(x) \).

For the moment assume the former. In way of contradiction assume the contrary of the assertion of the lemma. This is equivalent to saying that \( h_1(x) < h^2(x) \) for any small \( x \). For small \( x \), let \( y = y(x) \in [h_1(x), x] \) be any point which gives \( \max(f^♯|_{[h_1(x), x]}) \). Notice that \( f^♯(y) \) can be as large as we wish by choosing \( x \) even smaller. Then since \( f^♯(h^2(y)) = 4f^♯(y) > f^♯(y) \), the point \( h^2(y) \) is contained in

\[
[h^2 \circ h_1(x), h^2(x)] - (h_1(x), x) = [h^2 \circ h_1(x), h_1(x)] \subset [h^2_1(x), h_1(x)].
\]

The last inclusion follows from the assumption for a contradiction.

Put \( h^2(y) = h_1(z) \) for some \( z = z(x) \in [h_1(x), x] \). Then we have
\[ f^\sharp \circ h_1(z) = 4f^\sharp(y) \geq 4f^\sharp(z) \quad \text{and} \quad f^\sharp \circ h(z) = 2f^\sharp(z). \quad (2.8) \]

If we choose \( x \) near enough to 0, then the associated \( z = z(x) \) is also near, and thus

\[ |2f^\sharp(z) - f^\sharp \circ h_1(z)| = |f^\sharp \circ h(z) - f^\sharp \circ h_1(z)| \]

can be arbitrarily small by Corollary 2.8. Then we have

\[ f^\sharp(z) \approx \frac{1}{2} f^\sharp \circ h_1(z) = 2f^\sharp(y) \gg 1 \]

for any such \( z = z(x) \). On the other hand \( z(x) \) can be arbitrarily near to 0, and thus (2.8) contradicts Corollary 2.8.

The opposite case where \( h(x) < h_1(x) \) for any small \( x \) can be dealt with similarly by considering \( f' \in E_0 \), equivalent to \( f \), such that \( 2f' = f' \circ h_1 \), instead of \( f \).

Now fix a large number \( N \) and choose \( f_1 \in E_0 \) such that

\[ f_1 - f \in C, \quad 2^{1/N}f_1 = f_1 \circ h_N. \]

The existence of such \( f_1 \) is guaranteed by Lemma 2.5 applied to \( \lambda = 2^{1/N} \). We have then

\[ 2^{1/N}f_1^\sharp = f_1^\sharp \circ h_N. \quad (2.9) \]

Together with Lemma 2.9 which asserts that the fundamental domain of \( h_N^N \) is more or less comparable with that of \( h \), this implies that \( f_1^\sharp \) is oscillating in an extremely high frequency for \( N \) big. We are going to get a contradiction from this.

We still assume (2.4) for \( f \). According to Lemma 2.9, there are two cases to consider. One is when there is a sequence \( a_n \to 0 \) such that \( h^2(a_n) \leq h_N^N(a_n) \leq h(a_n) \), the other being \( h_N^N(a_n) \leq h(a_n) \leq h_N^N(a_n) \).

Assume for the moment that the former holds for infinitely many \( N \). Let \( x_1^1 \) be the largest point such that \( x_1^1 \leq a_n \) and \( f_1^\sharp(x_1^1) = 0 \). Notice that by Lemma 2.7 (5) and the equation (2.9), we have

\[ x_1^1 \in (h_N(a_n), a_n]. \quad (2.10) \]
Then again by (2.9) \( f_1^\# \) vanishes at the points \( x_n^\nu = h_N^{\nu-1}(x_n^1) \) for any \( 1 \leq \nu \leq N \). Let \( y_n^\nu \) be any point in \([x_n^2, x_n^1]\) at which \( f_1^\# \) takes the maximal value and let \( y_n^\nu = h_N^{\nu-1}(y_n^1) \) for \( 1 \leq \nu \leq N - 1 \). By (2.10) the order of these points are as follows.

\[
h^2(a_n) < h_N^N(a_n) \leq x_n^N < y_n^{N-1} < \cdots < y_n^\nu < x_n^\nu < \cdots < y_n^1 < x_n^1 \leq a_n.
\]

Notice that \( y_n^\nu \) is a point in \([x_n^{\nu+1}, x_n^\nu]\) at which \( f_1^\# \) takes the maximal value, and

\[
f_1^\#(y_n^\nu) = 2^{(\nu-1)/N} f_1^\#(y_n^1).
\]

We also have

\[
f_1^\#(y_n^\nu) \geq \frac{1}{2} \max (f_1^\#|_{h_N^N(a_n), a_n} ). \tag{2.11}
\]

In fact on one hand

\[
\max (f_1^\#|_{x_n^N, a_n}) = f_1^\#(y_n^{N-1}) = 2^{(N-2)/N} f_1^\#(y_n^1) \leq 2 f_1^\#(y_n^1).
\]

On the other hand

\[
\max (f_1^\#|_{h_N^N(a_n), x_n^N}) \leq 2^{(N-1)/N} \max (f_1^\#|_{x_n^2, x_n^1}) \leq 2 f_1^\#(y_n^1),
\]

because

\[
h_N^{-N+1}[h_N^N(a_n)), x_n^N] = [h_N(a_n), x_n^1] \subset [x_n^2, x_n^1].
\]

Henceforth we focus our attention to the other homeomorphism \( h \in H \).

There is a sequence \( \{m_n\} \) of integers such that the points \( h^{-m_n}(a_n) \) belong to a fixed fundamental domain in the basin of 0 for \( h \). Notice that \( m_n \to \infty \) since \( a_n \to 0 \). Passing to a subsequence if necessary, we may assume that

\[
h^{-m_n}(a_n) \to a, \quad h^{-m_n}(x_n^\nu) \to x^\nu \text{ and } h^{-m_n}(y_n^\nu) \to y^\nu,
\]

for some points \( a, x^\nu \) and \( y^\nu \). There is an ordering

\[
h^2(a) \leq x^N \leq y^{N-1} \leq \cdots \leq y^\nu \leq x^\nu \leq \cdots \leq y^1 \leq x^1 \leq a.
\]
We shall show that \( f^\#(x^\nu) = 0 \) and that \( f^\#(y^\nu) \) is bounded away from 0 with a bound independent of \( N \). Since these points can be taken in the same compact interval \([h^2(a), a]\), this will contradict the continuity of \( f^\# \).

By Lemma 2.7 (4), \( f^1_1(x^\nu_n) = 0 \) implies \( f^1(x^\nu_n) \leq 1 \) for any large \( n \). Therefore by (2.7)

\[
f^\#(h^{-m_n}(x^\nu_n)) \leq 2^{-m_n},
\]

showing that \( f^\#(x^\nu) = 0 \).

On the other hand since \( h_N^N(a_n) \leq h(a_n) \), we have by (2.11)

\[
f^\#_1(y^\nu_n) \geq \frac{1}{2} \max (f^\#_1|_{h_N^N(a_n), a_n}) \geq \frac{1}{2} \max (f^\#_1|_{h(a_n), a_n}),
\]

and therefore again by Lemma 2.7 (4), for any large \( n \),

\[
f^\#(y^\nu_n) \geq \frac{1}{2} \max (f^\#|_{h(a_n), a_n}) - 1.
\]

Let \( M = \max(f^\#|_{[h(a), a]}) \) and notice that \( M > 0 \) since \( \sigma(f) > 0 \) (2.6) and by (2.7).

For any large \( n \), the interval \( h^{-m_n}[h(a_n), a_n] \) is near \([h(a), a]\), and is composed of a subinterval of \([h(a), a]\) and the iterate by \( h^{\pm 1} \) of the complementary subinterval, and therefore

\[
\max (f^\#|_{h^{-m_n}[h(a_n), a_n]}) \geq M/2.
\]

This implies by (2.7)

\[
\max (f^\#|_{[h(a_n), a_n]}) \geq \frac{1}{2} M 2^{m_n},
\]

showing that for any large \( n \)

\[
f^\#(y^\nu_n) \geq \frac{1}{4} M 2^{m_n} - 1.
\]

This concludes that
as is desired.

The opposite case where \( h_N^2(a_n) \leq h(a_n) \leq h_N^N(a_n) \ (\exists a_n \to 0) \) holds for infinitely many \( N \) can be dealt with in a similar way, although the argument is not completely symmetric.

References

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