Prior Swapping for Data-Independent Inference

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Abstract

While Bayesian methods are praised for their ability to incorporate useful prior knowledge, in practice, priors that allow for computationally convenient or tractable inference are more commonly used. In this paper, we investigate the following question: for a given model, is it possible to use any convenient prior to infer a false posterior, and afterwards, given some true prior of interest, quickly transform this result into the true posterior?

We present a procedure to carry out this task: given an inferred false posterior and true prior, our algorithm generates samples from the true posterior. This transformation procedure, which we call “prior swapping” works for arbitrary priors. Notably, its cost is independent of data size. It therefore allows us, in some cases, to apply significantly less-costly inference procedures to more-sophisticated models than previously possible. It also lets us quickly perform any additional inferences, such as with updated priors or for many different hyperparameter settings, without touching the data. We prove that our method can generate asymptotically exact samples, and demonstrate it empirically on a number of models and priors.

1. Introduction

There are many cases in Bayesian modeling where a certain choice of prior distribution allows for computationally simple or tractable inference. For example,

- Conjugate priors yield posteriors with a known parametric form and therefore allow for non-iterative, exact inference [9, 14].

- Certain priors yield models with tractable conditional or marginal distributions, which allows efficient approximate inference algorithms to be applied (e.g. Gibbs sampling [50, 10, 65], sampling in collapsed models [60, 52, 38], or mean-field variational methods [56, 52]).

- Simple parametric priors (e.g. the normal distribution) allow for computationally cheap density queries, maximization, and sampling [2], which can allow for easier use in iterative inference algorithms (e.g. Metropolis-Hastings [29, 8], gradient-based MCMC [34], or sequential Monte Carlo [12, 41]).

- Certain priors mitigate issues of identifiability, and allow for simpler posteriors without multiple modes [43].

However, more sophisticated priors that provide a better depiction of observed or expert knowledge do not, in general, allow for the above inference techniques. Instead, researchers must resort to more general and computationally expensive inference methods. This can encourage the use of convenient priors in practice, rather than priors that might yield a more realistic or accurate inference, which is a criticism of Bayesian methods [15].

In this paper, we investigate the following question: for a given model, is it possible to use any convenient prior to infer a false posterior, and afterwards, given some true prior of interest, quickly transform this result into the true posterior?
Intuitively, our strategy is the following: for a given model, we first choose any computationally convenient false prior, and perform inference, which returns the false posterior. We then use our inferred false posterior, a true prior of interest, and the false prior, to efficiently produce samples from the true posterior via a method we call prior swapping.

One key attribute of prior swapping is that it runs without touching any data. Existing general inference algorithms are iterative and data-dependent: parameter updates at each iteration involve data, and the computational complexity or quality of each update depends on the amount of data used. Thus, inference on larger datasets takes more time. Furthermore, in practice, we often want to perform inference for a number of different priors or hyperparameter settings; for each, some data-dependent inference algorithm must be run to compute a new result.

In contrast, prior swapping algorithms are data-independent—i.e. parameter updates do not involve data, and neither the complexity nor quality of each update depends on data size. We therefore advocate doing difficult inference in two steps: first, perform data-dependent inference using the most computationally convenient prior for a given model, and then, for all future priors of interest (e.g. complex priors or a range of prior hyperparameter settings), perform quick, data-independent prior swapping. We summarize the advantages of this approach:

1. There often exists some false posterior that can be computed more efficiently than the true posterior; in some cases, we can therefore apply significantly less-costly inference procedures to more-sophisticated models than previously possible.
2. The prior swapping procedure runs independently of the data; this can allow for considerable speed-ups when doing inference on large datasets or ranges of hyperparameter values.
3. We can often maintain theoretical guarantees of existing approximate inference algorithms; e.g. when inferring the false posterior with an asymptotically-exact method (such as sampling), prior swapping can return asymptotically-exact true posterior samples.
4. We can update our prior, or incorporate new prior information, in an online fashion, without redoing data-dependent inference.
5. This is a black box procedure, which can be run directly on the output of most existing inference methods to add richer prior information to inference results.

2. Prior Swapping

Suppose we have a dataset of $n$ real, finite-dimensional vectors $x^n = \{x_1, \ldots, x_n\} \subset \mathbb{R}^n$, and we are interested in a family of models specified by the likelihood function $f_{x|\theta}(x^n|\theta)$, parameterized by a real, $d$-dimensional vector $\theta \in \mathbb{R}^d$. Suppose we have a prior distribution over the space of model parameters $\theta$, with probability density function (PDF) $f_\theta(\theta)$. The likelihood and prior define a joint model given by the PDF $f_{\theta,x}(\theta,x^n) = f_\theta(\theta)f_{x|\theta}(x^n|\theta)$. In Bayesian inference, we are interested in computing the posterior distribution—i.e. a conditional distribution of this joint model—with PDF

$$f_{\theta|x}(\theta|x^n) = \frac{f_{\theta}(\theta)f_{x|\theta}(x^n|\theta)}{\int f_{\theta}(\theta)f_{x|\theta}(x^n|\theta) \, d\theta} = \frac{1}{Z} f_{\theta} f_{x|\theta}(x^n|\theta). \quad (1)$$

Suppose we’ve chosen a different prior distribution $f_\phi(\theta)$, which we refer to as a false prior (while we refer to $f_\theta(\theta)$ as the true prior). We can now can define a new joint model $f_{\phi,x}(\theta,x^n) = f_\phi(\theta)f_{x|\theta}(x^n|\theta)$, with posterior (conditional) PDF $f_{\phi|x}(\theta|x^n) = \frac{1}{Z} f_{\phi}(\theta)f_{x|\theta}(x^n|\theta)$. We refer to this second posterior distribution as a false posterior.

The goal of our method is to first infer a false posterior and then leverage it to infer a true posterior. To carry out this transformation, we use the prior swap function $f_{\tilde{\phi}}$, which we define as the false posterior density multiplied by the true prior density and divided by the false prior density, i.e.

$$f_{\tilde{\phi}}(\theta) = \frac{f_{\phi|x}(\theta|x^n)f_\theta(\theta)}{f_\phi(\theta)} \propto f_{\phi}(\theta)f_{x|\theta}(x^n|\theta)f_\theta(\theta) \propto f_{\phi|x}(\theta|x^n). \quad (2)$$

Note that $f_{\tilde{\phi}}$ is proportional to the true posterior density $f_{\theta|x}$. However, depending on how we
An analytic expression for the false posterior is computed first.

A pseudocode. Here, each iteration only requires evaluating a few simple analytic expressions, and the complexity of the expression does not grow as inference proceeds. For example, this is the case if we approximate inference strategies such as variational inference and expectation propagation methods.

**Fixed-complexity false posterior.** As a first case, suppose that we have computed an analytic expression \( f_{\phi|x}^* \) for the false posterior density with a fixed-complexity parametric form (i.e., the complexity of the expression does not grow as inference proceeds). For example, this is the case if we

- Compute \( f_{\phi|x} \) exactly (in a closed form) via a conjugate prior.
- Generate samples from \( f_{\phi|x} \), via MCMC or other sampling methods, then use these samples to compute a parametric density estimate \( \hat{f}_{\phi|x} \).
- Apply optimization-based approximate inference methods, such as variational inference or expectation propagation, that return analytic approximate posteriors \( f_{\phi|x}^{\text{VI}} \) or \( f_{\phi|x}^\text{EP} \).

In each case, while we use some data-dependent algorithm to infer the false posterior, our resulting expression \( f_{\phi|x}^* \) is not a function of the data. Given this expression, consider the functions

\[
f_{\theta|x}^*(\theta) = f_{\theta|x}^*(\theta) f_{\phi}(\theta) \quad \text{and} \quad \nabla_{\theta} \log f_{\theta|x}^*(\theta) = \nabla_{\theta} \left( \log f_{\phi|x}^*(\theta) + \log f_{\phi}(\theta) - \log f_{\phi}(\theta) \right).
\]

We can use these to draw posterior samples extremely efficiently with a variety of standard MCMC algorithms. At each iteration in MCMC, to draw a new parameter, we must evaluate a target function associated with the posterior density. For example, we evaluate a function proportional to \( f_{\theta|x}(\theta|x^n) \) in Metropolis-Hastings (MH) \([29, 8]\) and \( \nabla_{\theta} \log f_{\theta|x}(\theta|x^n) \) in gradient-based MCMC methods (such as Langevin dynamics (LD) \([46]\), Hamiltonian Monte Carlo (HMC) \([34, 19]\), and other approximate inference strategies such as variational inference and expectation propagation methods.

**Consistent false posterior.** Suppose our false posterior inference procedure yields a set of samples \( \{\hat{\theta}_t\}_{t=1}^T \sim f_{\phi|x} \). Above, we proposed the strategy of computing a fixed-complexity parametric density estimate \( f_{\phi|x}^* \), and plugging this into the prior swap function (which we will denote \( f_{\phi|x}^p \)). This \( f_{\phi|x}^p \) is, in general, an inconsistent estimate of \( f_{\phi|x} \), and using it in prior swapping yields asymptotically-biased samples from \( f_{\theta|x} \). Here, we aim to answer the question: given samples from \( f_{\phi|x} \), can we develop an efficient method that returns asymptotically-exact samples from \( f_{\theta|x} \)

![Figure 1: Illustration of prior swapping, showing the prior swap density \( f_{\text{ps}} \propto f_{\theta|x} \), its factors (\( f_{\phi} \), \( f_{\phi|x} \), and \( f_{\theta} \)), and samples from this density.](image)
Suppose we instead use a consistent false posterior density estimate, such as a nonparametric \[58, 21\] or semiparametric \[18\] estimate. We will prove (Sec. 2.1) that plugging a consistent estimate for \(f_{\theta|x}\) into the prior swap function yields asymptotically-exact samples. However, the cost of these consistent estimates grows with the number of samples \(T\); typically, evaluating their PDF or gradient has a complexity of \(O(T)\). Using these estimates for \(f_{\theta|x}\) in the above prior swapping procedure therefore has a complexity of \(O(T)\) per iteration, which is costly for large \(T\).

We instead propose the following prior swapping method, which still yields asymptotically-exact samples from \(f_{\theta|x}\), yet does not require a significant increase in computation: first generate approximate posterior samples using \(f_{ps}^P\) (as above), and then correct these samples via algorithms from the parallel MCMC literature (described below), designed to sample from the product of densities.

To motivate this method, we choose a consistent semiparametric false posterior estimate \(\hat{f}_{s|x}\) (see \[18\] for background and consistency guarantees), which can be viewed as the product of a parametric density estimate \(\hat{f}_{s|x}^P\) and a nonparametric correction function. This is written

\[
\hat{f}_{s|x}^P(\theta) = \frac{1}{T} \sum_{t=1}^{T} \left[ \frac{1}{\hat{h}^d} K \left( \frac{||\theta - \hat{\theta}_t||}{\hat{h}} \right) \right] \hat{f}_{s|x}^P(\theta), 
\]

where we use \(K\) to denote a probability density kernel, with bandwidth \(\hat{h}\), where \(\hat{h} \to 0\) as \(T \to \infty\) (see \[58\] for details on probability density kernels and bandwidth selection). A general parametric family for \(\hat{f}_{s|x}^P\) that we can use in nearly all cases is the family of false prior distributions. These are typically simple parametric distributions over the correct parameter space, which contain parameterizations in a broad vicinity of \(f_{\theta|x}\), and for which there exist efficient parameter estimation algorithms. Given \(\hat{f}_{s|x}^P\), we write the semiparametric prior swap function as

\[
\hat{f}_{ps}^s(\theta) = \frac{\hat{f}_{s|x}^P(\theta)f_\theta(\theta)}{f_{\theta|x}(\theta)} = \frac{1}{T} \sum_{t=1}^{T} \left[ \frac{1}{\hat{h}^d} K \left( \frac{||\theta - \hat{\theta}_t||}{\hat{h}} \right) \right] \hat{f}_{s|x}^P(\theta)f_\theta(\theta)
\]

\[
= \left[ \frac{\hat{f}_{s|x}^P(\theta)f_\theta(\theta)}{f_{\theta|x}(\theta)} \right] \left[ \frac{1}{T} \sum_{t=1}^{T} K \left( \frac{||\theta - \hat{\theta}_t||}{\hat{h}} \right) \right] = \left[ \frac{\hat{f}_{ps}^P(\theta)}{f_{\theta|x}(\theta)} \right] \left[ \frac{1}{T} \sum_{t=1}^{T} w_t K \left( \frac{||\theta - \hat{\theta}_t||}{\hat{h}} \right) \right] \right].
\]

Hence, the prior swap function is proportional to the product of two densities: the parametric prior swap density, and a correction density. We can easily generate samples from both of the densities that comprise \(\hat{f}_{ps}^s\)—the former with (fixed-complexity) parametric prior swapping and the latter by sampling from components in the correction density with frequency proportional to the components’ weights.

We therefore turn to sample combination methods for efficiently generating samples from the product of densities \[48, 37, 42\]. Given two sets of samples \(\{\theta_{t}^1\}_{t=1}^{T}\) \(\sim f_1(\theta)\) and \(\{\theta_{t}^2\}_{t=1}^{T}\) \(\sim f_2(\theta)\), these algorithms aim to return \(\text{Prod}(\{\theta_{t}^1\}_{t=1}^{T}, \{\theta_{t}^2\}_{t=1}^{T}) = \{\theta_{t}\}_{t=1}^{T}\) \(\sim \frac{1}{T} f_1 f_2(\theta)\). These methods are efficient because each density product sample can be generated without iterating through either set of \(T\) input samples; typically, only a single sample from each input density is required. Each sample can therefore be produced with constant \(O(1)\) complexity.

We summarize the full asymptotically-exact prior swapping procedure in Alg. 1. In the appendix, we give pseudocode for the density product sample combination algorithms, summarize the complexity of all methods, and also discuss how this semiparametric prior swapping method allows for easy incorporation of observed prior information.

**Algorithm 1: Asymptotically-exact prior swapping**

**Input:** False posterior samples \(\{\hat{\theta}_t\}_{t=1}^{T} \sim f_{\theta|x}(\theta|x^n)\).

**Output:** Samples \(\{\theta_{t}\}_{t=1}^{T} \sim \hat{f}_{ps}^s(\theta) \Rightarrow f_{\theta|x}(\theta|x^n)\).

1. Estimate \(\hat{f}_{ps}^P\) using \(\{\hat{\theta}_t\}_{t=1}^{T}\) \(\Rightarrow \text{parametric estimate.}

2. Sample \(\{\hat{\theta}_t\}_{t=1}^{T} \sim \hat{f}_{ps}^P(\theta) \Rightarrow \text{parametric prior swapping.}

3. Sample \(\{\theta_{t}^2\}_{t=1}^{T} \sim \frac{1}{T} \sum_{t=1}^{T} \hat{f}_{s|x}^P(\hat{\theta}_t) K \left( \frac{||\theta - \hat{\theta}_t||}{\hat{h}} \right) \Rightarrow \text{parametric prior swapping.}

4. Sample \(\{\theta_{t}\}_{t=1}^{T} \sim \hat{f}_{ps}^S(\theta)\) via \(\text{Prod}(\{\theta_{t}^1\}_{t=1}^{T}, \{\theta_{t}^2\}_{t=1}^{T})\).
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This setting also has some relation to importance sampling (IS) [2]. For example, one could use the false posterior as an IS proposal (i.e. simply reweight and then use samples from $f_{\phi|x}$). However, in practice, performance would only be adequate for false posteriors that are very similar to the true posterior (and not for arbitrary $f_{\phi|x}$), especially in high dimensions [35, 54]. We show this empirically by comparing against this strategy in Sec. 3.

2.1 Theoretical Guarantees
Here we give theoretical guarantees about the correctness of the samples generated via prior swapping. First, note that if we have an exact analytic expression $f^*_\phi|x$ for the false posterior, the prior swap function is proportional to the true posterior, i.e. $f^*_{ps} = f_{ps} \propto f_{\phi|x}$.

Using $f^*_{ps}$ in standard MCMC algorithms therefore carries out MCMC on an exact true posterior target and comes with existing guarantees, such as producing asymptotically-exact posterior samples [8, 45, 37].

In the second setting, to prove that we generate asymptotically-exact samples given a consistent false posterior estimator, we need to show, as $T \to \infty$, that $\hat{f}^s_{ps}$ is consistent for $f_{ps}$ and that Alg. 1 indeed draws samples from $\hat{f}^s_{ps}$.

**Theorem 2.1.** Given false posterior samples $\{\hat{\theta}_t\}_{t=1}^T \sim f_{\phi|x}$ and $h \propto T^{-1/(4+d)}$, the estimator $\hat{f}^s_{ps}$ (Eq. 6) is consistent for $f_{ps}$, i.e. its mean-squared error satisfies

$$\sup_{f_{ps} \in \mathcal{P}(2,L)} \mathbb{E} \left[ \int (\hat{f}^s_{ps}(\theta) - f_{ps}(\theta))^2 d\theta \right] \leq \frac{c}{T^{4/(4+d)}}$$

for some $c > 0$ and $0 < h \leq 1$.

**Theorem 2.2.** The procedure given in Alg. 1 generates samples from $\hat{f}^s_{ps}$ (Eq. 6) as $T \to \infty$.

Proofs for both theorems and definition of $\mathcal{P}(2,L)$ are given in the appendix.

3. Empirical Results

We show empirical results on Bayesian generalized linear models (including linear and hierarchical logistic regression) with sparsity and heavy tailed priors, on latent factor models (including mixture models and topic models) with relational priors over factors (e.g. diversity-encouraging, agglomerate-encouraging, etc.), and feedforward neural networks, where we show hyperparameter tuning of weight-decay L2 regularization via prior swapping with normal priors. We aim to demonstrate empirically that prior swapping allows us to apply less-costly inference algorithms to more-complex models than was previously possible, and that it efficiently yields correct samples. We compare the following inference procedures:

- **MCMC on the true posterior**: MCMC sampling algorithms run directly on $f_{\theta|x}$.
- **VI on the true posterior**: variational inference algorithms run directly on $f_{\theta|x}$.
- **False posterior**: using the inferred false posterior $f_{\phi|x}$.
- **Importance sampling with false posterior proposal**: using $f_{\phi|x}$ as a proposal density and running importance sampling.
- **Prior swapping (fixed-complexity)**: prior swapping using a fixed-complexity expression $f^*_{\phi|x}$, including exactly-computed $f^*_{\phi|x}$, parametric estimates $f^p_{\phi|x}$, and approximations $f^VI_{\phi|x}$.
- **Prior swapping (consistent)**: prior swapping via Alg. 1 using a semiparametric $\hat{f}^s_{\phi|x}$.
- **Prior swapping to MAP**: optimization, using the prior swap function, to a MAP point estimate.
- **False-true hybrid prior** prior swapping on $f_{\phi|x}(\theta)f_{\phi}(\theta)$ without dividing out the false prior.
- **Normal approximation**: prior swapping using normal approximations for $f_{\phi|x}$, $f_{\theta}$, and $f_{\phi}$. 

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which closed form inference is possible), and then use prior swapping to quickly convert the result to logistic 

\[ \text{hierarchical Bayesian logistic regression model} \]
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Student's t, and VerySparse (with PDF VerySparse can write as

can provide a cheap yet accurate solution: first use a more-tractable prior (such as a normal prior, in

e.g., the \( L \))

dimensionality, their ability to produce models with greater interpretability, predictive accuracy,

and parsimony. For example, the \( L_1 \) norm has been used to induce sparsity with great effect [53, 39], and has been shown
to be equivalent to a mean-zero independent Laplace prior [53, 49, 32, 6]. In a Bayesian setting, inference given a sparsity prior can be difficult, and researchers often resort to computationally

tensive methods (such as Hamiltonian Monte Carlo) or biased posterior approximations (e.g. expectation propagation [31]) that make factorization or parametric assumptions [49, 16]. We provide a cheap yet accurate solution: first use a more-tractable prior (such as a normal prior, in which closed form inference is possible), and then use prior swapping to quickly convert the result to the posterior given a sparsity prior.

Our first set of experiments are on sparse Bayesian linear regression models [49, 30], which we can write as

\[ y_i = X_i \theta + \epsilon_i, \epsilon_i \sim N(0, \sigma^2), \theta \sim f_\theta, i = 1, \ldots, n. \]

For \( f_\theta \), we compute results on on Laplace, Student's t, and VerySparse (with PDF VerySparse(\( \sigma \)) = \prod_{i=1}^{d} \frac{1}{\Delta_i} \exp\{-|\theta|^{\alpha}/\sigma\} [49] ) priors. Here, a normal \( f_\theta \) is conjugate and allows for exact inference. Our second set of experiments are on a hierarchical Bayesian logistic regression model [20, 13], which we write as

\[ y_i \sim \text{Bern}(p_i), p_i = \text{logistic}(X_i \theta), \theta \sim f_\theta, i = 1, \ldots, n. \]

We will consider a hierarchical \( f_\theta = N(0, \alpha^{-1} I), \alpha \sim \text{Gamma}(\gamma, 1). \)

To assess performance, we use two metrics: posterior error and test error. To compute posterior error, we run a single chain of MCMC on the true posterior for one million steps, and use these samples as groundtruth (after removing the first quarter as burn in). We then compare these groundtruth samples with those returned by our inference methods. Specifically, we compute the Euclidean error, which we report for classification models, is the

\[ \hat{f}(\omega) = \frac{1}{n} \sum_{i=1}^{n} \text{groundtruth}(\omega_i) - \text{predicted}(\omega_i) \]

as groundtruth (after removing the first quarter as burn in). We then compare these groundtruth
estimates. For time point, we collect samples drawn before this time point, remove the first quarter as burn in, and

compute the above metrics.

### 3.1 Sparsity Inducing and Heavy Tailed Priors in Generalized Linear Models

Sparsity-encouraging regularizers have gained a high level of popularity over the past decade due to

t heir ability to produce models with greater interpretability, predictive accuracy, and parsimony. For

example, the \( L_1 \) norm has been used to induce sparsity with great effect [53, 39], and has been shown

to be equivalent to a mean-zero independent Laplace prior [53, 49, 32, 6]. In a Bayesian setting, inference given a sparsity prior can be difficult, and researchers often resort to computationally

tensive methods (such as Hamiltonian Monte Carlo) or biased posterior approximations (e.g. expectation propagation [31]) that make factorization or parametric assumptions [49, 16]. We provide a cheap yet accurate solution: first use a more-tractable prior (such as a normal prior, in which closed form inference is possible), and then use prior swapping to quickly convert the result to the posterior given a sparsity prior.

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For \( f_\theta \), we compute results on on Laplace, Student's t, and VerySparse (with PDF VerySparse(\( \sigma \)) = \prod_{i=1}^{d} \frac{1}{\Delta_i} \exp\{-|\theta|^{\alpha}/\sigma\} [49] ) priors. Here, a normal \( f_\theta \) is conjugate and allows for exact inference. Our second set of experiments are on a hierarchical Bayesian logistic regression model [20, 13], which we write as

\[ y_i \sim \text{Bern}(p_i), p_i = \text{logistic}(X_i \theta), \theta \sim f_\theta, i = 1, \ldots, n. \]

We will consider a hierarchical \( f_\theta = N(0, \alpha^{-1} I), \alpha \sim \text{Gamma}(\gamma, 1). \)

![Figure 2: Bayesian sparse linear models: (a-b) Convergence plots showing that prior swapping (run on closed form inferred \( f_\phi | x \)) performs accurate inference much faster than comparison methods and is robust to the choice of \( f_\phi \). Other comparison methods yield inferences that are biased or slow to converge, and these can worsen for different \( f_\phi \). (c) Inferred 1-d density marginals when the true prior sparsity is increased. (d) Prior swapping results for a variety of different sparsity priors.](image)
Figure 3: Bayesian hierarchical logistic regression: (a-b) Wall time and test error comparisons for varying data size \( n \). As \( n \) is increased, wall time remains constant for prior swapping but grows for standard inference methods. (c-d) Wall time and test error comparisons for varying model dimensionality \( d \). (e-g) Wall time and test error comparisons for inferences on a set of prior hyperparameters \( \gamma \in [1, 1.05] \). Here, a single \( f_{\phi|x} (\gamma = 1.025) \) is used for prior swapping on all other hyperparameters.

Here, we also use a normal \( f_\phi \) (see [20, 1] for examples of convenient inference in Bayesian logistic regression under normal priors). In these experiments, we generate synthetic data from the models in order to show results under varying \( n \) and \( d \). For comparison methods, we use MH for MCMC, and a mean field approximation [13] for VI.

In Fig. 2 we show results for sparse linear models, with \( n=10,000 \) observations, and \( d=20 \) dimensions. In (a) and (b) we show convergence plots and see that prior swapping performs faster inference (by a few orders of magnitude) than MCMC. We also see that semiparametric prior swapping (Alg. 1) achieves nearly identical performance as prior swapping on the exactly computed \( f_{\phi|x} \). The other comparison methods yield posteriors that are incorrect or very slow to converge. In (b) we halve the variance of our false prior, which hurts performance of the comparison methods, but leaves prior swapping unchanged. In (c) we show 1-d density marginals as we increase the prior sparsity, and in (d) we show prior swapping results for different sparsity priors.

In Fig. 3, we show results for hierarchical logistic regression. In (a) and (b) we vary the number of observations \( n=10-120,000 \) and see that prior swapping has a constant wall time while the wall times of both MCMC and VI increase with \( n \). In (b) we see that the prior swapping methods achieve the same test error as the standard inference methods. In (c) and (d) we vary the number of dimensions \( d=1-40 \). In this case, all methods have increasing wall time, and again the test errors match. In (e), (f), and (g), we vary the prior hyperparameter \( \gamma \in [1,1.05] \). For prior swapping, we infer a single \( f_{\phi|x} \) (using \( \gamma = 1.025 \)) with both MCMC and VI, and compute all other hyperparameter results using this \( f_{\phi|x} \). This demonstrates that prior swapping can quickly infer correct results over a range of hyperparameters. Here, the asymptotically-exact semiparametric prior swapping method matches the test error of MCMC slightly better than the parametric method.

### 3.2 Relational Priors over Factors in Latent Variable Models

Many latent variable models in machine learning—such as mixture models, topic models, probabilistic matrix factorization, and various others—involve a set of latent factors (e.g. components or topics). Often, we’d like to use priors that encourage interesting behaviors among the factors. For example, we might want dissimilar factors via a diversity-promoting prior [63, 24, 62] or for the factors to show some sort of sparsity pattern [28, 23, 64]. Inference in such models is often computationally expensive or designed on a case-by-case basis [63, 23, 64].
Collapsed Gibbs

1. Origin(s): \( f_s = N(\theta_s, \sigma^2) \), \( \forall j \) 
   \( \triangleright \) Push factors to origin.
2. Agglom(s): \( f_s = N(\theta_s, \sigma^2) \), \( \forall i, j \) 
   \( \triangleright \) Clump factors together.
3. Chain(s): \( f_s = N(\theta_s, \sigma^2) \), \( \forall i \) 
   \( \triangleright \) Clump factors in sequence.
4. Diverse(s): \( f_s = \mathcal{N}(1/\theta_s, 0) \), \( \forall i, j \) 
   \( \triangleright \) Separate factors.
5. SparseOrigin(s): \( f_s = \text{Lap}(\theta_s, 0) \), \( \forall i \) 
   \( \triangleright \) Push factors to origin (along few dimensions)
6. SparseAgglom(s): \( f_s = \text{Lap}(\theta_s - \theta_j, 0) \), \( \forall i, j \) 
   \( \triangleright \) Clump factors together (along few dimensions).
7. SparseChain(s): \( f_s = \text{Lap}(\theta_s - \theta_s, 0, \sigma^2) \), \( \forall i \) 
   \( \triangleright \) Clump factors in sequence (along few dimensions).
8. Origin-Diverse(s): \( f_s = \text{Origin}(\theta_s) - \text{Diverse}(\theta_s) \) 
   \( \triangleright \) Push factors to origin, maintain separation.

Figure 4: Latent factor models: (a) Prior swapping results for relational priors (defined in (b)) over components in a mixture model. (c) Prior swapping with a diversity-promoting true prior on an LDA topic model (Simple English Wikipedia corpus) to separate redundant topic clusters; the top 6 words per topic are shown. In (a, c) we also show wall times for the initial inference and prior swapping.

However, when conjugate priors are placed over the factor parameters, collapsed Gibbs sampling can be applied. In this method, the factor parameters are integrated out, leaving only a subset of variables; on these, the conditional distributions can be computed analytically, which allows for Gibbs sampling over these variables. Afterwards, samples of the collapsed factor parameters can be computed.

Hence, we propose the following strategy: first, assign a prior for the factor parameters that allows for collapsed Gibbs sampling; afterwards, reconstruct the factor samples and apply prior swapping for more complex relational priors over the factors. We can thus perform convenient inference in the collapsed model, yet apply more sophisticated priors to variables in the uncollapsed model.

We first show results on a Gaussian mixture model (GMM) \([27, 51]\), written \( x_i \sim \mathcal{N}(\mu_{z_i}, \Sigma_{z_i}) \), \( z_i \sim \text{Dir}(\alpha) \), \( \mu_{m_i} \sim \mu_\theta \), \( i = 1, \ldots, n \). Using a normal \( f_\phi \) over \( \mu_{m_i} \) allows for collapsed Gibbs sampling. We also show results on a topic model (latent Dirichlet allocation (LDA) \([3]\)) for text data (for the form of this model, see \([3, 55]\)). Here, using a Dirichlet \( f_\phi \) over topics allows for collapsed Gibbs sampling. For mixture models, we generate synthetic data from the above model (\( n=10,000 \), \( d=2 \), \( M=9 \)), and for topic models, we use the Simple English Wikipedia\(^*\) corpus (\( n=27,443 \) documents, \( \text{vocab}=10,192 \) words) \([36]\), and set \( M=400 \) topics.

In Fig. 4 we show results for mixture and topic models. In (a) we show inferred posteriors over GMM components for a number of relational priors, which we define in (b). In (c), we apply the diversity-promoting prior to LDA, to separate redundant topics. Here, we show two topic-clusters (“geography” and “family”) in \( f_{\phi_{ls}} \), which are separated into thematically-similar yet distinct topics after prior swapping. In (a) and (c) we also show wall times of the inference methods.

\(^*\) https://simple.wikipedia.org/
3.3 Tuning L2 Regularization (Weight Decay) in Deep Neural Networks

Learning neural networks with weight decay (L2 regularization) can be viewed as finding the MAP point estimate of a Bayesian neural network model with a normal prior [47, 33, 17]. Since the prior swap function can be used as an objective for optimization to a MAP estimate, we aim to use prior swapping for quick hyperparameter tuning of weight decay in neural networks. We will compare this to finding the optimal weight decay via stochastic gradient descent [4] and stochastic gradient Langevin dynamics [59], two popular methods for learning and inference in neural networks. These stochastic gradient methods have only a weak dependence on data at each iteration; however, their updates may be noisy or suboptimal [22, 57, 44, 11], while prior swapping updates involve exact gradients (of the prior swap function) without any stochasticity. Furthermore, note that we can use stochastic gradient inference methods in conjunction with prior swapping (i.e. to compute $f_{\phi|x}$, which we will do here).

We run our experiment on an eight-layer fully connected deep neural network with 400 nodes per layer, yielding a model with 1,280,410 dimensions. For data, we use the MNIST† handwritten digits classification dataset ($n=60,000$, $d=784$) [25]. We consider a family of true priors $f_\theta = \mathcal{N}(0, \sigma^2 I)$ over neural network parameters (for weight decay), and aim to compute MAP point estimates over a range of hyperparameters $\sigma^2$. Due to the high dimensionality, we make a parametric approximation $f_{\phi|x}$ for the false posterior, which we assume to be normal with a diagonal covariance matrix. We use SGLD to infer $\hat{f}_{\phi|x}$ (choosing a single $\sigma^2$ to parameterize $f_\phi$). For prior swapping, we use this $\hat{f}_{\phi|x}$ to learn MAP estimates for all other $\sigma^2$ values in the range.

In Fig 5 we show wall time and test error for the comparison methods over the set of weight decay parameters ($\sigma^2=50$-$10,000$). For prior swapping, we perform gradient ascent optimization using the gradient log prior swap function (Eq. 4). We see in (c) that all methods yield the same optimal parameter ($\sigma^2=550$) with the lowest test error. However, in (a) and (b) we see that prior swapping takes less time than SGD and SGLD.

4. Conclusion

We have presented procedures to carry out the task of prior swapping: given any false posterior (computed using some convenient false prior) and a true prior of interest, our algorithms generate samples from the true posterior. Empirically, we have demonstrated prior swapping on a number of models and priors, and have shown that it can (1) quickly generate correct samples, (2) allow for less-costly data-dependent inference algorithms to be applied to more-complex models than previously possible, (3) allow for quick model selection or hyperparameter tuning, and (4) be used directly on top of existing inference algorithms to add richer prior information to pre-computed inference results, without having to revisit the data. Theoretically, we have shown that, given a stream of false posterior samples, this strategy can be used to generate asymptotically exact samples from the true posterior.

We furthermore hope that prior swapping can be successfully implemented as a black box method, and paired with existing automatic inference engines or probabilistic programming languages [5, 61, 40, 26]. Since prior swapping does not require tuning and can be applied directly to the output of inference yielded by these frameworks, it has potential to aid in general purpose automated inference with more-realistic and difficult priors.

†. http://yann.lecun.com/exdb/mnist/
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Appendix for "Prior Swapping for Data-Independent Inference"

Appendix A. Proofs of Theoretical Guarantees

Here, we prove the two theorems stated in Sec. 2.1.

Throughout this analysis, we assume that we have $T$ samples $\{\tilde{\theta}_t\}_{t=1}^T \subset \mathcal{X} \subset \mathbb{R}^d$ from the false-posterior $f_{\phi|x}$, and that $h \in \mathbb{R}_+$ denotes the bandwidth of our semiparametric false-posterior density estimator $f_{\phi|x}^\dagger$. Let Hölder class $\Sigma(2, L)$ on $\mathcal{X}$ be defined as the set of all $f$ times differentiable functions $f : \mathcal{X} \to \mathbb{R}$ whose derivative $f^{(l)}$ satisfies

$$|f^{(l)}(\theta) - f^{(l)}(\theta')| \leq L |\theta - \theta'|^{2-l} \quad \text{for all } \theta, \theta' \in \mathcal{X}.$$ 

We also define the class of densities $\mathcal{P}(2, L)$ to be

$$\mathcal{P}(2, L) = \left\{ f \in \Sigma(2, L) \biggm| f \geq 0, \int f(\theta) d\theta = 1 \right\}.$$ 

We also assume that the false-posterior density $f_{\phi|x}$ is bounded, i.e. that there exists some $b > 0$ such that $f_{\phi|x}(\theta) \leq b$ for all $\theta \in \mathbb{R}^d$.

**Theorem 2.1.** Given false posterior samples $\{\tilde{\theta}_t\}_{t=1}^T \sim f_{\phi|x}$ and $h \asymp T^{-1/(4+d)}$, the estimator $\hat{f}_{\hat{\theta}_T}^\dagger$ (Eq. 6) is consistent, i.e. its mean-squared error satisfies

$$\sup_{f_{\theta} \in \mathcal{P}(2, L)} \mathbb{E} \left[ \int \left( \hat{f}_{\hat{\theta}_T}^\dagger(\theta) - f_{\theta}(\theta) \right)^2 d\theta \right] \leq \frac{c}{T^{4/(4+d)}}$$

for some $c > 0$ and $0 < h \leq 1$.

**Proof.** To prove mean-square consistency of our semiparametric prior swap estimator $\hat{f}_{\hat{\theta}_T}^\dagger$, we give a bound on the mean-squared error (MSE), and show that it tends to zero as we increase the number of samples drawn from the false-posterior. To prove this, we bound the bias and variance of the estimator, and use this to bound the MSE.

We first bound the bias of our semiparametric prior swap estimator. For any $f_{\theta} \in \mathcal{P}(2, L)$, we can write the bias as

$$\mathbb{E} \left[ \hat{f}_{\hat{\theta}_T}^\dagger \right] - f_{\theta} = \mathbb{E} \left[ \hat{f}_{\hat{\theta}_T}^\dagger \frac{f_\theta}{f_\phi} - f_{\phi|x} \frac{f_\theta}{f_\phi} \right] = \frac{f_\phi}{f_\phi} \mathbb{E} \left[ \hat{f}_{\phi|x} \right] - f_{\phi|x} \leq c h^2$$

for some $c > 0$, where we have used the fact that $\mathbb{E} \left[ \hat{f}_{\phi|x} \right] - f_{\phi|x} \leq \hat{c} h^2$ for some $\hat{c} > 0$ (given in [18, 58]).

We next bound the variance of our semiparametric prior swap estimator. For any $f_{\theta} \in \mathcal{P}(2, L)$, we can write the variance of our estimator as

$$\text{Var} \left[ \hat{f}_{\hat{\theta}_T}^\dagger \right] = \text{Var} \left[ \hat{f}_{\phi|x} \frac{f_\theta}{f_\phi} \right] = \frac{f_\theta^2}{f_\phi^2} \text{Var} \left[ \hat{f}_{\phi|x} \right] \leq \frac{c}{T h^d}$$
for some \( c > 0 \), where we have used the facts that \( \text{Var} \left[ \hat{f}^s_{\phi|x} \right] \leq \frac{c}{Th^d} \) for some \( c > 0 \) and \( \mathbb{E} \left[ \hat{f}^s_{\phi|x} \right]^2 \leq \tilde{c} \) for some \( \tilde{c} > 0 \) (given in [18, 58]).

Next, we will use these two results to bound the mean-squared error of our semiparametric prior swap estimator, which shows that it is mean-square consistent.

We can write the mean-squared error as the sum of the variance and the bias-squared, and therefore,

\[
\mathbb{E} \left[ \int \left( \hat{f}^s_{ps}(\theta) - f_{ps}(\theta) \right)^2 d\theta \right] \leq c_1 h^2 + \frac{c_2}{Th^d} \leq \frac{c}{T^4/(4+d)}
\]

for some \( c > 0 \), using the fact that \( h \approx T^{-1/(4+d)} \).

\[\square\]

**Theorem 2.2.** The procedure given in Alg. 1 generates samples from \( \hat{f}^s_{ps} \) (Eq. 6) as \( T \to \infty \).

**Proof.** In Alg. 1, note that line 2 is equivalent to standard MCMC (such as Metropolis-Hastings or Hamiltonian Monte Carlo) run on a posterior target function proportional to \( \hat{f}^p_{ps} \), and thus yields samples from \( \hat{f}^p_{ps} \) as \( T \to \infty \) (see [8, 2, 34] for MCMC guarantees).

In line 3, we can sample from \( \frac{1}{T} \sum_{t=1}^{T} f^p_{\phi|x}(\bar{\theta}_t)^{-1}K \left( \frac{\left\| \theta - \bar{\theta}_t \right\|}{h} \right) \) exactly via the following procedure:

For each sample \( s \),

1. Draw an index \( i \in \{1, \ldots, T\} \): \( i \sim \text{Cat} \left( \left\{ f^p_{\phi|x}(\bar{\theta}_t)^{-1} \right\}_{t=1}^{T} \right) \).

2. Draw the sample \( \theta_i \): \( \theta_i \sim K \left( \frac{\left\| \theta - \bar{\theta}_i \right\|}{h} \right) \).

We thus have two streams of asymptotically-exact samples, which is exactly the setting of the product density sample combination procedures [48, 37, 42], which run on the output of MCMC algorithms to generate asymptotically-exact samples from the distribution with PDF proportional to the product of the PDFs of the Markov chains’ stationary distributions (see [37], Sec. 5). We give pseudocode for these sample combination procedures in the “Sample Combination Algorithm Pseudocode” section of the appendix. Therefore, as \( T \to \infty \), line 4 of Alg. 1 yields samples from \( \hat{f}^s_{ps} \propto \left[ f^p_{ps} \right] \left[ \frac{1}{T} \sum_{t=1}^{T} f^p_{\phi|x}(\bar{\theta}_t)^{-1}K \left( \frac{\left\| \theta - \bar{\theta}_t \right\|}{h} \right) \right] \). \[\square\]

**Appendix B. Computational Complexity Summary**

Here we summarize the complexity of our prior swapping algorithms. Assume that we have run an existing inference algorithm to compute either a fixed-complexity analytic expression for the false-posterior density, or to draw \( T \) false-posterior samples, given some dataset of \( n \) observations. Suppose that we wish to compute \( S \) samples from the true-posterior.

Given a fixed-complexity analytic expression for the false-posterior, each step (i.e. generating each sample) in an MCMC algorithm requires a constant \( O(1) \) number of operations, and drawing \( S \) samples via our procedure therefore requires \( O(S) \) operations.

Given \( T \) samples from the false-posterior, the asymptotically-exact (semiparametric) procedure in Alg. 1 must perform the fixed-complexity prior swapping procedure on a parametric false posterior \( f_{\phi|x} \) (\( O(S) \) operations for \( S \) samples), draw \( S \) samples from the correction density (\( O(T) \) operations to compute component weights and \( O(S) \) operations to draw \( S \) samples, for a total complexity of \( O(S + T) \)), then apply a density product sample combination algorithm [48, 37, 42] (\( O(S) \) operations for \( S \) samples). Therefore, the asymptotically-exact (semiparametric) procedure has a total complexity of \( O(S + T) \) to draw \( S \) samples.

In general, given the false-posterior inference result, none of the prior swapping algorithms depend on the number of data points \( n \).
Appendix C. Prior Swapping Pseudocode (for Fixed-Complexity $f_{\theta|\phi}$)

Here we give pseudocode for the prior swapping procedure, given a fixed-complexity (parametric) false posterior expression $f_{\theta|\phi}$, using the prior swap functions $f_{\theta|\phi}$ and $\nabla_{\theta} \log f_{\theta|\phi}$ (Eqs. 3 and 4).

In Alg. 2, we show prior swapping via the Metropolis-Hastings algorithm, which makes use of $f_{\theta|\phi}$. In Alg. 3 we show prior swapping via Hamiltonian Monte Carlo, which makes use of $\nabla_{\theta} \log f_{\theta|\phi}$. A special case of Alg. 3, which occurs when we set the number of simulation steps to $L = 1$ (in line 6), is prior swapping via Langevin dynamics.

Algorithm 2: Prior swapping via Metropolis-Hastings.

```
Input: Prior swap function $f_{\theta|\phi}$, and proposal $q$.
Output: Samples $\{\theta_t\}_{t=1}^T \sim f_{\theta|\phi} \propto f_{\theta|\phi}$ as $T \to \infty$.
1 Initialize $\theta_0$. ▶ Initialize Markov chain.
2 for $t = 1, \ldots, T$ do
3     Draw $\theta_t \sim q(\theta_t | \theta_{t-1})$. ▶ Propose new sample.
4     Draw $u \sim \text{Unif}([0, 1])$.
5     if $u < \min \left\{ \frac{f_{\theta|\phi}(\theta_t)q(\theta_t | \theta_{t-1})}{f_{\theta|\phi}(\theta_{t-1})q(\theta_{t-1} | \theta_{t-2})} \right\}$ then
6         Set $\theta_t \leftarrow \theta_{t-1}$. ▶ Accept proposed sample.
7     else
8         Set $\theta_t \leftarrow \theta_{t-1}$. ▶ Reject proposed sample.
```

Algorithm 3: Prior swapping via Hamiltonian Monte Carlo.

```
Input: Prior swap function $f_{\theta|\phi}$, its gradient $\nabla_{\theta} \log f_{\theta|\phi}$, and step-size $\epsilon$.
Output: Samples $\{\theta_t\}_{t=1}^T \sim f_{\theta|\phi} \propto f_{\theta|\phi}$ as $T \to \infty$.
1 Initialize $\theta_0$. ▶ Initialize Markov chain.
2 for $t = 1, \ldots, T$ do
3     Draw $r_t \sim \mathcal{N}(0, I)$.
4     Set $(\tilde{\theta}_0, \tilde{r}_0) \leftarrow (\theta_{t-1}, r_{t-1})$.
5     Set $\tilde{r}_0 \leftarrow \tilde{r}_0 + \frac{\epsilon}{2} \nabla_{\theta} \log f_{\theta|\phi}(\tilde{\theta}_0)$.
6     for $l = 1, \ldots, L$ do
7         Set $\tilde{\theta}_l \leftarrow \tilde{\theta}_{l-1} + \epsilon \tilde{r}_{l-1}$.
8         Set $\tilde{r}_l \leftarrow \tilde{r}_{l-1} + \epsilon \nabla_{\theta} \log f_{\theta|\phi}(\tilde{\theta}_l)$.
9     Draw $u \sim \text{Unif}([0, 1])$.
10    if $u < \min \left\{ \frac{f_{\theta|\phi}(\tilde{\theta}_L)q(\tilde{\theta}_L | \tilde{r}_L)}{f_{\theta|\phi}(\tilde{\theta}_{L-1})q(\tilde{\theta}_{L-1} | \tilde{r}_{L-1})} \right\}$ then
11       Set $\theta_t \leftarrow \tilde{\theta}_L$. ▶ Accept proposed sample.
12    else
13       Set $\theta_t \leftarrow \tilde{\theta}_{t-1}$. ▶ Reject proposed sample.
```
Appendix D. Sample Combination Algorithm Pseudocode

Here we give pseudocode for a product density sample combination algorithm \cite{48,37,42}. We will write this algorithm for our setting (i.e. for generating samples from the product of two densities) though these algorithms are typically more general. We mainly follow Alg. 1 from \cite{37}.

The intuitive idea behind these algorithms is the following: given two sets of samples \( \{ \theta_i^1 \}_{i=1}^T \sim f_1(\theta) \) and \( \{ \theta_i^2 \}_{i=1}^T \sim f_2(\theta) \), they aim to return \( \text{Prod}(\{ \theta_i^1 \}_{i=1}^T, \{ \theta_i^2 \}_{i=1}^T) = \{ \theta_i \}_{i=1}^T \sim \frac{1}{2} f_1 f_2(\theta) \). At each iteration in this algorithm, four steps are carried out:

1. Choose one of the two input sample sets (uniformly at random).
2. Re-draw a sample from the chosen sample set.
3. Accept this drawn sample, or reject it and keep the previously drawn sample.
4. Save a noisy average of the two current samples (one from each sample set).

We give the pseudocode for this procedure in Alg. 4.

\begin{algorithm}[h]
\caption{Density product sample combination (Prod) algorithm \cite{37}.}
\begin{algorithmic}
\State \textbf{Input:} \( \{ \theta_i^1 \}_{i=1}^T \sim f_1(\theta) \) and \( \{ \theta_i^2 \}_{i=1}^T \sim f_2(\theta) \).
\State \textbf{Output:} \( \{ \theta_i \}_{i=1}^T \sim \frac{1}{2} f_1 f_2(\theta) \) as \( T \to \infty \).
\State \begin{algorithmic}
\State \begin{align*}
\text{Draw} & (k_1, k_2) \sim \text{Unif}\{1, \ldots, T\}. & \triangleright \text{Initialize sample indices.} \\
\text{Set} & h \leftarrow T^{-1/(4+d)}. \\
\text{for} & s = 1, \ldots, T \text { do} \\
\text{Set} & (c_1, c_2) \leftarrow (k_1, k_2). \\
\text{Draw} & m \sim \text{Unif}\{1, 2\}. & \triangleright \text{Choose an input sample set.} \\
\text{Draw} & c_m \sim \text{Unif}\{1, \ldots, T\}. & \triangleright \text{Draw sample index from chosen set.} \\
\text{Draw} & u \sim \text{Unif}\{0, 1\}. \\
\text{if} & u < \frac{1}{\mathcal{N}(\theta_m^1, \theta_m^2, \bH)} \text{\then} \\
\text{Set} & (k_1, k_2) \leftarrow (c_1, c_2). & \triangleright \text{Accept chosen sample.} \\
\text{Draw} & \epsilon \sim \mathcal{N}\left(0, \frac{\bH}{2}\right). \\
\text{Set} & \theta_s \leftarrow \frac{1}{2} \left( \theta_{k_1}^1 + \theta_{k_2}^2 \right) + \epsilon. & \triangleright \text{Compute an output sample.}
\end{align*}
\end{algorithmic}
\end{algorithmic}
\end{algorithm}

Appendix E. Incorporating observed prior information

The asymptotically-exact method for prior swapping allows for a related way to more easily incorporate observed prior information. Often, instead of (or in addition to) an analytic expression for our prior beliefs, we have prior observations, such as previously observed outcomes of what we aim to infer. Incorporating information such as this can lead to better data-driven priors. Here, we will extend our asymptotically-exact prior swapping procedure to this setting.

Suppose that we only have access to samples \( \{ \tilde{\theta}_i^1 \}_{i=1}^T \) from the true-prior \( f_\theta \), in addition to false-posterior samples \( \{ \tilde{\theta}_i^2 \}_{i=1}^T \) from \( f_{\phi|x} \). In this case, we can apply a density product sample combination algorithm \cite{48,37,42} (described in the "Sample Combination Algorithm Pseudocode" section of the appendix) to generate samples \( \{ \tilde{\theta}_i \}_{i=1}^T \) from \( f_{\text{prod}} \propto f_{\phi|x} f_\theta \), and then construct a semiparametric estimate \cite{18} of this density (the same one used in defining the asymptotically-exact prior swapping method), written

\[ \hat{f}_{\text{prod}}^p(\theta) = \frac{1}{T} \sum_{t=1}^T \frac{1}{h} K \left( \frac{\| \theta - \tilde{\theta}_t^1 \|}{h} \right) \hat{f}_{\text{prod}}(\tilde{\theta}_t), \]  

(7)
We then define the prior swap function for observed prior samples, \( \hat{f}_o^{\theta} \), to be

\[
\hat{f}_o^{\theta}(\theta) = \frac{\hat{f}_s^{\theta}(\theta)}{f_\phi(\theta)} = \frac{1}{T} \sum_{t=1}^{T} \frac{1}{h} K \left( \frac{\|\theta - \tilde{\theta}_t\|}{h} \right) \frac{\hat{f}_p^{\theta}(\theta)}{f_p^{\theta}(\tilde{\theta}_t)} f_\phi(\theta)
\]

\[
= \left[ \frac{\hat{f}_p^{\theta}(\theta)}{f_\phi(\theta)} \right] \left[ \frac{1}{T} \sum_{t=1}^{T} w'_t K \left( \frac{\|\theta - \tilde{\theta}_t\|}{h} \right) \right]
\]

where we’ve defined \( w'_t = \left( \frac{\hat{f}_p^{\theta}(\tilde{\theta}_t)h^d}{h} \right)^{-1} \). As in the asymptotically-exact prior swapping procedure, \( \hat{f}_o^{\theta} \) is proportional to the product of two densities, both of which we can easily sample from (note that we can sample from \( \frac{\hat{f}_p^{\theta}(\tilde{\theta}_t)}{f_\phi(\theta)} \) using the fixed-complexity parametric prior swapping method). As before, we then apply the density product sample combination methods to generate samples from \( \hat{f}_o^{\theta} \).

We summarize this full procedure in Alg 5.

**Algorithm 5:** Asymptotically-exact prior swapping (observed prior samples)

**Input:** Samples \( \{\tilde{\theta}_1\}^{T}_{t=1} \sim f_\phi(\theta) \) and \( \{\tilde{\theta}_2\}^{T}_{t=1} \sim f_\phi(\theta|x|) \).

**Output:** Samples \( \{\theta\}^{T}_{t=1} \sim \hat{f}_o^{\theta}(\theta) \rightarrow f_\phi(\theta|x|) \) as \( T \rightarrow \infty \).

1. Sample \( \{\tilde{\theta}_1\}^{T}_{t=1} \sim f_{pnum}^{\theta}(\theta) \) via Prod \( \{\tilde{\theta}_1\}^{T}_{t=1}, \{\tilde{\theta}_2\}^{T}_{t=1} \).
2. Estimate \( \hat{f}_p^{\theta} \) using \( \{\tilde{\theta}_1\}^{T}_{t=1} \) ▷ parametric estimate.
3. Sample \( \{\theta_1\}^{T}_{t=1} \sim \hat{f}_o^{\theta}(\theta) \) ▷ parametric prior swapping.
4. Sample \( \{\theta_2\}^{T}_{t=1} \sim \frac{1}{T} \sum_{t=1}^{T} w'_t K \left( \frac{\|\theta - \tilde{\theta}_t\|}{h} \right) \).
5. Sample \( \{\theta\}^{T}_{t=1} \sim \hat{f}_o^{\theta}(\theta) \) via Prod \( \{\theta_1\}^{T}_{t=1}, \{\theta_2\}^{T}_{t=1} \).