A FOURIER CONTINUATION FRAMEWORK
FOR HIGH-ORDER APPROXIMATIONS

AKASH ANAND

Abstract. It is well known that approximation of functions on $[0,1]$ whose periodic extension is not continuous fail to converge uniformly due to rapid Gibbs oscillations near the boundary. Among several approaches that have been proposed toward the resolution of Gibbs phenomenon in recent years, a Fourier continuation (FC) based approximation scheme has been suggested by Bruno and collaborators in the context of certain PDE solvers where approximation grids used are equispaced. While the practical efficacy of FC based schemes in obtaining a high-order numerical solution of PDEs is well known, theoretical convergence analyses largely remain unavailable. The primary objective of this paper is to take a step in this direction where we analyze the convergence rates of a Fourier continuation framework for approximations based on discrete functional data coming from equispaced grids. In this context, we explore a certain two-point Hermite interpolation strategy for constructing Fourier continuations that, not only simplifies the implementation of such approximations but also makes possible a rigorous analysis of its numerical properties. In particular, we show that the approximations converge with order $r+1$ for functions coming from a subspace of $C^{r+1}([0,1])$, the space of $r$-times continuously differentiable function whose $r$th derivative is Lipschitz continuous. We also demonstrate that theoretical rates are indeed achieved in practice, through a variety of numerical experiments.

1. Introduction

Given a function $f \in C^{r-1}([0,1])$, the space of $r$-times continuously differentiable real valued functions whose $r$th derivative is Lipschitz continuous, we seek to approximate it using a trigonometric polynomial of the form

\begin{equation}
\sum_{k=-n}^{n} c_k(f)e^{2\pi ikx/b}
\end{equation}

with $c_{-k} = \overline{c_k}$ for some $b \geq 1$. In particular, we are interested in approximations obtained as a truncated Fourier series, that we denote by $T_{n,b}(f)$. It is well known that if $f$ satisfies $f(0) = f(1)$, then the trigonometric polynomial obtained by truncating its Fourier series (with $b = 1$) converges uniformly. In fact, if all its derivatives up to order $r$ satisfy

\begin{equation}
f^{(\ell)}(0) = f^{(\ell)}(1) \quad 0 \leq \ell \leq r,
\end{equation}

then the approximation errors converge according to:

$$\|f - T_{n}f\|_{\infty,[0,1]} = \max_{x \in [0,1]}|f(x) - T_{n}f(x)| = O\left(\frac{\log n}{n^{r+1}}\right).$$

If $f \in C^{r}([0,1]) \cap C^{r+2}_{pw}([0,1])$, a subspace of $C^{r-1}([0,1])$, and additionally satisfies eq. (2), then the rate of convergence improves further to $O(n^{-(r+1)})$. Note that $f \in C^{r}_{pw}([a,b])$ if and only if $f^{(\ell)} \in C^{r}_{pw}([a,b])$, the space of piecewise continuous functions on $[a,b]$. Recall that a function $g \in C^{r}_{pw}([a,b])$ if there are finitely many, say $n_d$, open disjoint intervals $(a_j, a_{j+1})$ with $a_0 = a$ and $a_{n_d} = b$, such that $g|_{(a_j, a_{j+1})}$ extends as a continuous function to $[a_j, a_{j+1}]$. For simplicity, we denote the space $C^{r}([a,b]) \cap C^{r+2}_{pw}([a,b])$ by $D^{r,1}([a,b])$ and its subspace where functions additionally satisfy

$f^{(\ell)}(a) = f^{(\ell)}(b), \quad 0 \leq \ell \leq r,$

by $D^{r,1}_{0}([a,b])$. For example, $f_0(x) = |x - 1/2|$ is in $D^{0,1}_{0}([0,1])$, $f_1(x) = (x - 1/2)|x - 1/2|$ is in $D^{1,1}_{0}([0,1])$ and $f_2(x) = (x - 1/2)^2|x - 1/2|$ is in $D^{2,1}_{0}([0,1])$.

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The Fourier series approximations, on the other hand, fail to converge uniformly when \( f(0) \neq f(1) \) due to rapid oscillations near boundary, known as the Gibb’s phenomenon \([12,13,20,25]\) (see fig. 1 for an example) — development of effective strategies for its alleviation has remained a subject matter of much ongoing research.

Several approximation approaches have been proposed to overcome the difficulty of Gibb’s oscillations. These include schemes that utilize Fourier or physical space filters \([15]\) as well as those that project the partial Fourier sums onto suitable functional spaces. For example, the Gegenbauer projection technique \([14,19]\) utilizes a space spanned by Gegenbauer polynomials. In Fourier-Padé approximations, partial Fourier sums are approximated by rational trigonometric functions \([7,10,11]\). Techniques based on extrapolation algorithms \([5]\) have also been used. Several Fourier continuation (or extension) approaches have also been proposed that seek to find a trigonometric polynomial of the form \([1]\) with \( b > 1 \). Such schemes rely on smoothly continuing \( f \) on \([0,1]\) to \( f_c \) on \([0,b]\) or \([1-b,1]\) for a suitable choice of \( b \) in such a way that \( f_c \equiv f \) on \([0,1]\) and \( f^{(\ell)}(0) = f^{(\ell)}(b) \) for all integers \( 0 \leq \ell \leq r \) for some \( r > 0 \). Once such an \( f_c \) has been produced, the restriction of its truncated Fourier series to \([0,1]\) serves as an approximation to \( f \) (see Figure 3). For some examples where Fourier extension ideas have been used and discussed in various contexts, see \([3,4,8,9,21,23]\).

More recently, an algorithmic construction for Fourier continuation has been suggested by Bruno and Lyon in \([6]\) in the context of certain PDE solvers where approximation grids used are equispaced. While the efficacy of this approach has been established through its application in various partial differential equation solvers (for example, see \([1,2,22]\)), owing to the algorithmic nature of this scheme, numerical analysis of these methods, to a large extent, is intractable and consequently, theoretical convergence rates remain unavailable.

The primary objective of this paper is to analyze a Fourier continuation framework for eliminating Gibbs oscillations from approximations based on discrete functional data coming from equispaced grids. Indeed, a Fourier continuation based high-order approximation becomes most relevant when the underlying grid is uniform, allowing for efficient calculations using FFT. In this context, we explore the construction based on the Hermite interpolation, that not only simplifies implementation of the approximation but also makes possible a rigorous analysis of its numerical properties. Toward this, in section 2 we discuss the general framework for constructing such continuations where we also present theoretical convergence rates for approximations of functions in \( D^{r,1}([0,1]) \) by corresponding truncated Fourier series arising out of the continuation framework. In section 3 we review the Hermite polynomial based Fourier continuation approach and show that this construction indeed falls within the general framework of section 2. We thus conclude that the theoretical convergence rates obtained remain valid in the context of Hermite polynomial based scheme. We then numerically verify this, in the discrete setting, in section 4 through a variety of computational experiments.

![Figure 1](image.png)

**Figure 1.** Approximations \( T_{n,1}(f) \) of \( f(x) = x \) obtained as truncated Fourier series exhibit Gibb’s phenomenon.
Figure 2. Continuation $f_c(x)$ of $f(x) = x$ on $[0, 1]$ to $[-1, 1]$ using a polynomial of degree $2r+1$ so that all derivatives up to order $r$ are continuous and $f_c^{(\ell)}(-1) = f_c^{(\ell)}(1)$ for $0 \leq \ell \leq r$.

2. A framework for Fourier continuation analysis

As described above, the Fourier continuation framework for approximation of a function $f \in D^{r,1}([0,1])$ can be viewed as a two step procedure, namely,

(1) **continuation**: for a given $b > 1$, construct a function $f_c : [1 - b, 1] \to \mathbb{R}$ such that the following conditions hold:

\begin{align*}
(3) & \quad f_c(x) = f(x), \quad \text{for all } x \in [0, 1], \\
(4) & \quad f_c(x) \in D_b^{r,1}([1 - b, 1]).
\end{align*}

We illustrate this step in fig. 2 for $b = 2$ where we continue $f(x) = x$ on $[0, 1]$ to the interval $[-1, 1]$ with varying degree of smoothness as controlled by $r$. The explicit construction used in these examples are discussed in section 3.

(2) **Fourier approximation**: for an $n \in \mathbb{N}$ and $x \in [0, 1]$,

\[ f(x) \approx T_{n,b}(f)(x) = \sum_{k=-n}^{n} c_k(f_c)e^{2\pi ikx/b}, \]

where

\[ c_k(f_c) = \frac{1}{b} \int_{1-b}^{1} f_c(x)e^{-2\pi ikx/b} dx. \]
In fig. 3 we show three Fourier approximations that correspond to \( n = 2, 4 \) and 8 to a continuation of \( f(x) = x \) shown in fig. 2 where the continued function is in \( C^{15,1}([-1, 1]) \) (in fact, in \( D^{15,1}_0([-1, 1]) \)). Note the absence of Gibbs oscillations in these approximations in contrast to those shown in fig. 1.

We analyze the approximation properties of the above strategy under the assumption that \( f_c \) is of the form

\[
\begin{align*}
f_c(x) &= \begin{cases} f(x), & x \in [0, 1], \\ \mathcal{L}_r(F)(x), & x \in [1 - b, 0], \end{cases}
\end{align*}
\]

with

\[
F = \begin{bmatrix} f(0) & f^{(1)}(0) & \cdots & f^{(\ell)}(0) & \cdots & f^{(r)}(0) \\ f(1) & f^{(1)}(1) & \cdots & f^{(\ell)}(1) & \cdots & f^{(r)}(1) \end{bmatrix}
\]

and a bounded linear operator \( \mathcal{L}_r : M_{2,r+1}(\mathbb{R}) \to \mathbb{D}^{r,1}(1-b,0) \), where \( M_{m,n}(\mathbb{R}) \) denotes the normed linear space of all \( m \times n \) real matrices with the norm given by

\[
\|A\|_{\text{max}} = \max_{jk} |a_{jk}|.
\]

Moreover, the linear operator \( \mathcal{L}_r \) is required to satisfy the derivative conditions that read

\[
\begin{align*}
(\mathcal{L}_r(F))^{(\ell)}(0) &= f^{(\ell)}(0) \\
(\mathcal{L}_r(F))^{(\ell)}(1-b) &= f^{(\ell)}(1)
\end{align*}
\]

for \( 0 \leq \ell \leq r \). Note that \( \mathcal{L}_r(F)(x) \) assumes the form

\[
\mathcal{L}_r(F)(x) = \sum_{m=0}^r L_m^0(x)f^{(m)}(0) + \sum_{m=0}^r L_m^1(x)f^{(m)}(1)
\]
for some functions $L_m^0, L_m^1 \in C^{r,1}([1-b,0])$, where the derivative conditions eq. (8)-eq. (9) require that they satisfy

\begin{equation}
\begin{aligned}
(L_m^0)^{(\ell)}(0) &= \delta_{\ell m}, \\
(L_m^1)^{(\ell)}(0) &= 0, \\
(L_m^1)^{(\ell)}(1-b) &= \delta_{\ell m},
\end{aligned}
\tag{11}
\end{equation}

\begin{equation}
\begin{aligned}
(L_m^0)^{(\ell)}(1-b) &= 0, \\
(L_m^1)^{(\ell)}(1-b) &= \delta_{\ell m},
\end{aligned}
\tag{12}
\end{equation}

for $0 \leq \ell \leq r$.

Remark 2.1. While in our discussions, we extend $f$ to the left of the interval $[0,1]$, that is, to the interval $[1-b,1]$, a similar right continuation, that is, to the interval $[0,b]$ also works analogously.

Lemma 2.2. If $g \in D_0^{r,1}([1-b,1])$, then, for all $n > 0$, we have

$$\|T_{n,b}(g) - g\|_{\infty,[0,1]} \leq \frac{C}{n^{r+1}},$$

for a positive constant $C$ independent of $n$.

Proof. Note that the Fourier coefficients $c_k(g)$, for $k \neq 0$, upon $(r+1)$ integration by parts, are given by

$$c_k(g) = \frac{1}{b} \left( \frac{b}{2\pi ik} \right)^{r+1} \int_{1-b}^{1} g^{(r+1)}(x) e^{-2\pi ikx/b} dx.$$  

As $g^{(r+1)}$ is piecewise differentiable with finitely many jump discontinuities, say at $a_j$, $j = 0, 1, \ldots, n_d$, with $a_j < a_{j+1}$, $a_0 = 1-b$ and $a_{n_d} = 1$, an application of integration by parts to each of these subintervals $[a_j, a_{j+1}]$ yields

\begin{equation}
\begin{aligned}
\frac{1}{b} \left( \frac{b}{2\pi ik} \right)^{r+2} \sum_{j=0}^{n_d} \left( g^{(r+1)}(a_{j+1}) - g^{(r+1)}(a_j) \right) e^{-2\pi ik(a_j)/b} \\
&\quad + \frac{1}{b} \left( \frac{b}{2\pi ik} \right)^{r+2} \int_{1-b}^{1} g^{(r+2)}(x) e^{-2\pi ikx/b} dx
\end{aligned}
\tag{13}
\end{equation}

where $f(a \pm)$ denotes $\lim_{h \to 0^+} f(a \pm h)$. The left hand limit at $x = 1 - b$ and right hand limit at $x = 1$, respectively, are obtained as $g^{(r+1)}(a_0) = g^{(r+1)}(a_{n_d})$ and $g^{(r+1)}(a_{n_d}) = g^{(r+1)}(a_0)$. The result now follows from eq. (13) and the following inequality:

$$\|T_{n,b}(g) - g\|_{\infty,[0,1]} \leq \|T_{n,b}(g) - g\|_{\infty,[1-b,1]} \leq \sum_{|k| \geq n+1} |c_k(g)|.$$

Clearly, as constructed in eq. (5), the continuation $f_c \in D_0^{r,1}([1-b,1])$, and therefore, its truncated Fourier series converges according to the rate obtained in lemma 2.2. This construction, of course, assumes that the boundary data matrix $F$ is available exactly, as might be the case in many applications. However, in many other cases, especially when approximations are being constructed from a discrete functional data, complete boundary information may not available explicitly and are obtained indirectly using numerical approximations. Consequently, the matrix $F$ used in the continuation process may be inexact, which in turn, introduces additional inaccuracies in the Fourier continuation approximations. To study the effect of inexact data matrix on errors, we begin by denoting the approximate continuation

\begin{equation}
\hat{f}_c(x) = \begin{cases}
    f(x), & x \in [0,1], \\
    L_r(F)(x), & x \in [1-b,0],
\end{cases}
\tag{14}
\end{equation}

corresponding to $\hat{F} \in M_{2,r+1}(\mathbb{R})$, $\hat{F} \neq F$. We note that $\hat{f}_c$ as defined in eq. (14), typically, is not in $C^{r,1}([1-b,1])$ and, in fact, can by discontinuous if the first column of $F$ differs from that of $F$. Obviously, the exact knowledge of one or more columns of $F$ has favorable impact on the regularity of $\hat{f}_c$. For example, in a typical discrete setting, while the inexact derivative calculations result in $\hat{F}$ to carry numerical error, availability of boundary data $f(0)$ and $f(1)$ makes it possible to choose $\hat{F}$ so that its first column matches exactly with that of $F$ thus making $\hat{f}_c$ continuous at $x = 0$ and satisfy $\hat{f}_c(1-b) = \hat{f}_c(1)$. To formalize this
Let \( \mathbf{F} = (f_{jk})_{0 \leq j \leq 1, 0 \leq k \leq r} \) be \( r \)-exact (or simply exact) with respect to \( f \) if \( f_{jk} = f^{(k)}(j), j \in \{0, 1\}, 0 \leq k \leq r \); for \( 0 \leq s < r \), a matrix \( \mathbf{F} \) is \( s \)-exact with respect to \( f \) if the first \((s+1)\) columns of \( \mathbf{F} \) agree exactly with those of \( f \) but they differ in \((s+2)\)th column, that is, \( f_{jk} = f^{(k)}(j), j \in \{0, 1\}, 0 \leq k \leq s \) and \( f_{j(s+1)} \neq f^{(s+1)}(j) \) for \( j = 0 \) or \( j = 1 \).

It is straightforward to see that if \( \mathbf{F} \) is \( s \)-exact with respect to \( f \) then for \( \ell = 0, \ldots, s \), \( f_{c}^{(\ell)} \) is continuous at \( x = 0 \) and \( f_{c}^{(\ell)}(1-b) = f_{c}^{(\ell)}(1) \).

**Lemma 2.4.** Let \( f \in C^{r+1}([0,1]) \) and \( f_c \) be its continuation as given in eq. \( (5) \). If the approximate continuation \( \tilde{f}_c \) given in eq. \( (14) \) corresponds to \( \tilde{f} \in M_{2,r+1}(\mathbb{R}) \) that is \( s \)-exact with respect to \( f \), \( 0 \leq s \leq r \), then, \( \tilde{f}_c \in C^0([1-b,1]). \)

**Lemma 2.5.** Let \( f \in C^{r+1}([0,1]), f_c \) and \( \tilde{f}_c \) be as given in eq. \( (5) \) and eq. \( (14) \) respectively where \( \tilde{f} \in M_{2,r+1}(\mathbb{R}) \) is \( s \)-exact with respect to \( f \), \( 0 \leq s \leq r \). Then, there is a constant \( C > 0 \) independent of \( n \), such that

\[
\|T_{n,b}(-\tilde{f}_c) - T_{n,b}(f_c)\|_{\infty,[0,1]} \leq Cn^{-s+1}\|\tilde{F} - F\|_{\max}.
\]

**Proof.** Clearly,

\[
\|T_{n,b}(-\tilde{f}_c) - T_{n,b}(f_c)\|_{\infty,[0,1]} = \|T_{n,b}(\tilde{f}_c - f_c)\|_{\infty,[0,1]} \leq \sum_{|k| \geq n+1} c_k(\tilde{f}_c - f_c).
\]

Now, from the derivative conditions eq. \( (8) \)-eq. \( (9) \), and \((s + 1)\) times integrate by parts, we get

\[
c_k(\tilde{f}_c - f_c) = \frac{1}{b} \left( \frac{b}{2\pi i k} \right)^{s+1} \sum_{m=s+1}^{r} \left( \tilde{f}_{0m} - f^{(m)}(0) \right) \int_{1-b}^{0} \left( L_m^0 \right)^{(s+1)}(x)e^{-2\pi i k x/b} dx
\]

\[
+ \frac{1}{b} \left( \frac{b}{2\pi i k} \right)^{s+1} \sum_{m=s+1}^{r} \left( \tilde{f}_{1m} - f^{(m)}(1) \right) \int_{1-b}^{0} \left( L_m^1 \right)^{(s+1)}(x)e^{-2\pi i k x/b} dx.
\]

Now, using eq. \( (15) \) and the fact that \( (L_m^0)^{(s+1)} \) and \( (L_m^1)^{(s+1)} \) are Lipschitz continuous, we conclude the result.

The next result shows that the Fourier continuation approximations converge rapidly and the rate of convergence is tied only to the smoothness of \( f \) and the order of accuracy in the derivative approximations.

**Theorem 2.6.** Let \( f \in D^{r+1}([0,1]), f_c \) and \( \tilde{f}_c \) be as given in eq. \( (5) \) and eq. \( (14) \) respectively where \( \tilde{f} \in M_{2,r+1}(\mathbb{R}) \) is \( s \)-exact with respect to \( f \), \( 0 \leq s < r \). Then there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
\|T_{n,b}(\tilde{f}_c) - f\|_{\infty,[0,1]} \leq C_1 \|\tilde{F} - F\|_{\max} + C_2 n^{r+1}
\]

for all \( n \geq 1 \). In particular, if \( \|\tilde{F} - F\|_{\max} = O(1/n^p) \), \( p + s > r \), then

\[
\|T_{n,b}(\tilde{f}_c) - f\|_{\infty,[0,1]} = O\left( \frac{1}{n^{r+1}} \right)
\]

whereas, if \( p + s \leq r \), then

\[
\|T_{n,b}(\tilde{f}_c) - f\|_{\infty,[0,1]} = O\left( \frac{1}{n^{p+s+1}} \right)
\]

**Proof.** The result follows from lemma \( 2.2 \), lemma \( 2.5 \) and the triangle inequality

\[
\|T_{n,b}(\tilde{f}_c) - f\|_{\infty,[0,1]} \leq \|T_{n,b}(\tilde{f}_c) - T_{n,b}(f_c)\|_{\infty,[0,1]} + \|T_{n,b}(f_c) - f\|_{\infty,[0,1]}.
\]

In the next section, we review a well known explicit Fourier continuation strategy based on two point Hermite interpolation that falls within the framework we discussed here.
3. THE CONSTRUCTION BASED TWO POINT HERMITE INTERPOLATION

In this section, we investigate the Fourier extension strategy using two point Hermite interpolation that has been used before in various contexts (for examples, see [8, 23, 24]). While one could work with any $b > 1$ for the construction, we restrict our presentation to the choice $b = 2$. For a matrix $\hat{F} = \{f_{jk}\}_{0 \leq j \leq 1, 0 \leq k \leq r} \in M_{2, r+1}(\mathbb{R})$, we introduce the polynomial $P_r(\hat{F})(x)$ of degree $2r + 1$ given by

\[ P_r(\hat{F})(x) = (1 + x)^{r+1} \sum_{m=0}^{r} \frac{\hat{f}_{0m}}{m!} \sum_{n=0}^{r-m} (-1)^n \binom{r + n}{n} x^{m+n} \]

\[ + (-x)^{r+1} \sum_{m=0}^{r} \frac{\hat{f}_{1m}}{m!} \sum_{n=0}^{r-m} \binom{r + n}{n} (1 + x)^{m+n}. \]

We note that $P_r(\hat{F})(x)$ can be expressed as

\[ P_r(\hat{F})(x) = \sum_{m=0}^{r} \hat{f}_{0m} P^0_m(x) + \sum_{m=0}^{r} \hat{f}_{1m} P^1_m(x) \]

with

\[ P^0_m(x) = \frac{1}{m!} x^m (1 + x)^{r+1} \sum_{n=0}^{r-m} (-1)^n \binom{r + n}{n} \]

and

\[ P^1_m(x) = \frac{1}{m!} (1 + x)^{m} (-x)^{r+1} \sum_{n=0}^{r-m} (1 + x)^n \binom{r + n}{n} \]

Before we show that the $P_r$ defined above indeed is a bounded linear operator that satisfies derivative conditions eq. [8]-eq. [9], we observe the following useful fact.

**Lemma 3.1.** Let $n$ be a positive integer and $r$ be an integer with $n \leq r + 1$. Then, we have

\[ \sum_{k=0}^{n} (-1)^k \binom{r+1}{k} \binom{r+n-k}{r} = 0. \]

**Proof.** The identity eq. [17] follows from the observation that, for $m \geq 0$, we have

\[ \binom{r+m}{r} = \frac{(-1)^m}{m!} \frac{d^m}{dx^m} \frac{1}{(1 + x)^{r+1}} \bigg|_{x=0} \]

and that, for $0 \leq k \leq r + 1$,

\[ \binom{r+1}{k} = \frac{1}{k!} \frac{d^k}{dx^k} (1 + x)^{r+1} \bigg|_{x=0}. \]

Indeed,

\[ 0 = \frac{d^n}{dx^n} \left[ (1 + x)^{r+1} \frac{1}{(1 + x)^{r+1}} \right] = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k} (1 + x)^{r+1} \left[ \frac{d^{n-k}}{dx^{n-k}} \frac{1}{(1 + x)^{r+1}} \right], \]

and the result follows. \( \square \)

**Theorem 3.2.** The operator $P_r$ defines a bounded linear operator from $M_{2, r+1}$ to $C^{r+1}([-1, 0])$ that also satisfies derivative conditions eq. [8]-eq. [9].

**Proof.** The linearity of $P_r$ is obvious from the definition. Now, for $-1 \leq x \leq 0$,

\[ |P_r(\hat{F})(x)| \leq \sum_{m=0}^{r} \left| \frac{\hat{f}_{0m}}{m!} \right| \sum_{n=0}^{r-m} \binom{r + n}{n} + \sum_{m=0}^{r} \left| \frac{\hat{f}_{1m}}{m!} \right| \sum_{n=0}^{r-m} \binom{r + n}{n} \]

\[ \leq \left\| \hat{F} \right\|_{\text{max}} \left( \sum_{m=0}^{r} \frac{1}{m!} \left( 2r + 1 - m \right) \binom{r+1-m}{r-m} + \sum_{m=0}^{r} \frac{1}{m!} \left( 2r + 1 - m \right) \binom{2r+1-m}{r-m} \right) \]

\[ \leq 2 \left( 2r + 2 \right) \left\| \hat{F} \right\|_{\text{max}}. \]
The boundedness of $P_r$ thus follows with $\|P_r\| \leq 2(2r+2)$ where $\|\cdot\|$ is the induced operator norm. Now,

$$
(P^0_m)^{(\ell)}(x) = \frac{\ell!}{m!} \sum_{k=0}^{\ell} \binom{r+1}{k} (1+x)^{r+1-k} \sum_{n=(\ell-k+m)}^{r-m} (-1)^n \binom{r+n}{n} \binom{m+n}{\ell-k} x^{n-(\ell-k-m)}
$$

(18)

and

$$
(P^1_m)^{(\ell)}(x) = (-1)^{r+1} \frac{\ell!}{m!} \sum_{k=0}^{\ell} \binom{r+1}{k} x^{r+1-k} \sum_{n=(\ell-k+m)}^{r-m} (-1)^n \binom{r+n}{n} \binom{m+n}{\ell-k} (1+x)^{n-(\ell-k-m)},
$$

(19)

where $(x)_+ = \max(0, x)$. It is clear from eq. (18) that $(P^0_m)^{(\ell)}(-1) = 0$ for all $0 \leq \ell \leq r$. Also, if $\ell < m$, then $(P^1_m)^{(\ell)}(0) = 0$. For $\ell \geq m$, we have

$$
(P^0_m)^{(\ell)}(0) = \frac{\ell!}{m!} \sum_{k=0}^{\ell-m} (-1)^{\ell-k-m} \binom{r+1}{k} \binom{r+\ell-k-m}{\ell-k-m}.
$$

Clearly, $(P^0_m)^{(m)}(0) = 1$. For $\ell > m$, on the other hand, it follows from lemma 3.1 that $(P^0_m)^{(\ell)}(0) = 0$. Thus, $(P^0_m)^{(\ell)}$ satisfy eq. (11). Using eq. (19), one similarly sees that $(P^1_m)^{(\ell)}(0) = 0$ and $(P^1_m)^{(\ell)}(-1) = \delta_{\ell m}$, that is, $(P^1_m)^{(\ell)}$ satisfy eq. (12).

In the light of Theorem theorem 3.2 we utilize $P_r(F)$ for a continuation of $f$ given by

$$
f_c(x) = \begin{cases} f(x), & x \in [0, 1] \\ P_r(F)(x), & x \in [-1, 0]. \end{cases}
$$

(20)

The truncated Fourier series that, in this setting, reads

$$
T_{n,2}(f)(x) = \sum_{k=-n}^{n} c_k(f_c)e^{\pi ikx}
$$

(21)

with

$$
c_k(f_c) = \frac{1}{2} \int_{-1}^{1} f_c(x)e^{-\pi ikx} dx.
$$

(22)

is then used as an approximation to $f$ on $[0, 1]$.

4. A DISCRETE APPROXIMATION PROBLEM AND NUMERICAL EXAMPLES

We now consider the problem of constructing a Fourier continuation approximation for the case where discrete functional data is available on an equispaced grid on the interval $[0, 1]$ that has relevance in many applications including certain PDE solvers. Such a grid of size $n+1$ has its $j$th grid point at $x_j = j/n$ where the corresponding function value $f(x_j)$ is assumed to be known and are denoted by $f_j$ for $j = 0, \ldots, n$. The functional data, in this case, is continued to the equispaced grid on $[-1, 1]$ according to

$$
(f^*_c)_j = \begin{cases} f_j, & j = 0, \ldots, n \\ P_r(F_{n,p})(j/n), & j = -n, \ldots, -1, \end{cases}
$$

(23)

where the 0-exact boundary data matrix $F_{n,p}$ is obtained as

$$
f_{0,m} = D_{n,p}^m(f_j(x_0)) \quad \text{and} \quad f_{1,m} = D_{n,p}^m(f_j(x_n)).
$$

for $1 \leq m \leq r$, using forward and backward finite difference derivative operators $D_{n,p}^m(f)$ and $D_{n,p}^m(f)$ respectively of order of accuracy $p$ for approximations of $f^m$ whose generic form reads

$$
D_{n,p}^m(f)(x_\ell) = (\pm m)^m \sum_{k=0}^{m-p-1} (a^m_p)_k f_{\ell \pm k}
$$

(24)

where

$$
D_{n,p}^m(f)(x_\ell) = (\pm m)^m \sum_{k=0}^{m-p-1} (a^m_p)_k f_{\ell \pm k}
$$

(24)
for appropriately chosen constants \( (a_p^m)_k \). Now, using \( \hat{f}_c^d \), as obtained in eq. (23), we compute

\[
(24) \quad c_k^d(\hat{f}_c^d) = \frac{1}{2n} \sum_{j=-n}^{n-1} (\hat{f}_c^d)_j e^{-\pi ij/n},
\]

for \( k = -n, \ldots, n - 1 \), to arrive at the interpolating Fourier continuation approximation for the discrete problem given by

\[
(25) \quad \mathcal{T}_{n,2}^d(\hat{f}_c^d)(x) = \sum_{k=-n}^{n-1} c_k^d(\hat{f}_c^d)e^{\pi ikx}.
\]

Note that the coefficients \( c_k^d(\hat{f}_c^d) \) can be computed in \( O(n \log n) \) computational time using the fast Fourier transform (FFT).

To guage the accuracy of such approximations, we begin by obtaining an estimate for \( \|\mathcal{T}_{n,2}^d(\hat{f}_c^d) - \mathcal{T}_{n,2}(f_c)\|_{\infty,[0,1]} \). Toward this, the following straightforward calculation

\[
\mathcal{T}_{n,2}^d(\hat{f}_c^d)(x) - \mathcal{T}_{n,2}(f_c)(x) = \sum_{k=-n}^{n-1} \left( c_k^d(\hat{f}_c^d) - c_k(f_c) \right) e^{\pi ikx} - c_n(f_c)e^{\pi nx}
\]

\[
= \sum_{k=-n}^{n-1} \left( c_k^d(\hat{f}_c^d) - c_k^d(f_c^d) \right) e^{\pi ikx} + \sum_{k=-n}^{n-1} \left( c_k^d(f_c^d) - c_k(f_c) \right) e^{\pi ikx} - c_n(f_c)e^{\pi nx}
\]

\[
= \sum_{k=-n}^{n-1} e^{\pi ikx} \frac{1}{2n} \sum_{j=-n}^{n-1} \left( (\hat{f}_c^d)_j - (f_c^d)_j \right) e^{-\pi ij/k/n}
\]

\[
+ \sum_{k=-n}^{n-1} \left( \frac{1}{2n} \sum_{j=-n}^{n-1} (f_c^d)_j e^{-\pi ij/k/n} - c_k(f_c) \right) e^{\pi ikx} - c_n(f_c)e^{\pi nx}
\]

\[
= \sum_{k=-n}^{n-1} e^{\pi ikx} \frac{1}{2n} \sum_{j=-n}^{n-1} \sum_{\ell=-\infty}^{\infty} c_\ell(f_c - f_c)e^{\pi ij(\ell-k)/n}
\]

\[
+ \sum_{k=-n}^{n-1} \left( \frac{1}{2n} \sum_{j=-n}^{n-1} \sum_{\ell=-\infty}^{\infty} c_\ell(f_c)e^{\pi ij(\ell-k)/n} - c_k(f_c) \right) e^{\pi ikx} - c_n(f_c)e^{\pi nx}
\]

\[
= \sum_{k=-n}^{n-1} \sum_{\ell=-\infty}^{\infty} c_{k+2\ell n}(\hat{f}_c - f_c)e^{\pi i(k+2\ell n)x} + \sum_{k=-n}^{n-1} \sum_{\ell=-\infty}^{\infty} c_k(\hat{f}_c - f_c)e^{\pi ikx} - c_n(f_c)e^{\pi nx},
\]

and the fact that \( \hat{f}_c(x) = f_c(x) \) for \( x \in [0,1] \) yields

\[
\|\mathcal{T}_{n,2}^d(\hat{f}_c^d) - \mathcal{T}_{n,2}(f_c)\|_{\infty,[0,1]} \leq 2 \sum_{|\ell| \geq n} |c_\ell(\hat{f}_c - f_c)| + \sum_{|\ell| \geq n} |c_\ell(f_c)|.
\]

Thus, we have

\[
\|\mathcal{T}_{n,2}^d(\hat{f}_c^d) - \mathcal{T}_{n,2}(f_c)\|_{\infty,[0,1]} \leq O\left( \frac{1}{n} \right) \|F_{n,p} - F\|_{\max} + O\left( \frac{1}{n^{1+r}} \right),
\]

9
Table 1. Convergence study for approximations of \( f(x) = \sin(20x) \) using the derivative approximations of order \( p = 3 \) and the continuation polynomial of degree \( 2r + 1 \) for \( r = 1, 2 \) and 3.

| \( n \) | \( r = 1 \) | \( r = 2 \) | \( r = 3 \) |
|---|---|---|---|
| 2^6 | \( 1.17 \times 10^{-3} \) | \( 2.39 \times 10^{-4} \) | \( 1.93 \times 10^{-4} \) |
| 2^7 | \( 3.20 \times 10^{-4} \) | \( 1.90 \times 10^{-5} \) | \( 1.24 \times 10^{-5} \) |
| 2^8 | \( 8.24 \times 10^{-5} \) | \( 1.68 \times 10^{-6} \) | \( 7.85 \times 10^{-7} \) |
| 2^9 | \( 2.07 \times 10^{-5} \) | \( 1.73 \times 10^{-7} \) | \( 9.73 \times 10^{-8} \) |
| 2^{10} | \( 5.20 \times 10^{-6} \) | \( 2.15 \times 10^{-8} \) | \( 3.09 \times 10^{-9} \) |
| 2^{11} | \( 1.22 \times 10^{-6} \) | \( 2.69 \times 10^{-9} \) | \( 7.99 \times 10^{-10} \) |
| 2^{12} | \( 2.92 \times 10^{-7} \) | \( 4.27 \times 10^{-10} \) | \( 8.10 \times 10^{-11} \) |

where the second term in the last inequality follows from the fact that Fourier coefficients \( c_k(g) \) decay as \( |k|^{-3} \) for \( g \in D_0^{r,1}([-1, 1]) \) while the first term results from the fact that, for \( k \neq 0 \), we have

\[
\frac{1}{2\pi^2k^2} \left( \sum_{m=0}^{r} (f_{0.m} - f_m^0)(P_m^0)'(0) + \sum_{m=0}^{r} (f_{1.m} - f_m^1)(P_m^1)'(0) \right) - \frac{(-1)^k}{2\pi^2k^2} \left( \sum_{m=0}^{r} (f_{0.m} - f_m^0)(P_m^0)'(-1) + \sum_{m=0}^{r} (f_{1.m} - f_m^1)(P_m^1)'(-1) \right) - \frac{1}{2\pi^2k^2} \int_{-1}^{0} \left( \sum_{m=0}^{r} (f_{0.m} - f_m^0)(P_m^0)''(x) + \sum_{m=0}^{r} (f_{1.m} - f_m^1)(P_m^1)''(x) \right) e^{-\pi kx} dx.
\]

Finally, using the inequality

\[
\|\mathcal{T}_{n,2}(\hat{f}_c) - f\|_{\infty,[0,1]} \leq \|\mathcal{T}_{n,2}(\hat{f}_c) - \mathcal{T}_n(\hat{f}_c)\|_{\infty,[0,1]} + \|\mathcal{T}_n(\hat{f}_c) - f\|_{\infty,[0,1]}
\]

in conjunction of theorem 2.6, we see that

\[
\|\mathcal{T}_{n,2}(\hat{f}_c) - f\|_{\infty,[0,1]} \leq O\left( \frac{1}{n^{p+1}} \right) + O\left( \frac{1}{n^{r+1}} \right) = O\left( \frac{1}{n^{\min\{p,r\}+1}} \right)
\]

4.1. Numerical examples. We now discuss some numerical experiments to demonstrate that theoretical convergence rates obtained above are indeed achieved in practice. Toward this, we consider the problem of approximating a function \( f(x) \) on \([0, 1]\) using the functional data on a uniform grid of size \( n \). We record the relative approximation error \( e_n \) that is obtained as

\[
e_n = \max_{\frac{n}{N} \leq j \leq N} |\mathcal{T}_{n,2}(\hat{f}_c)(z_j) - f(z_j)|/\max_{\frac{n}{N} \leq j \leq N} |f(z_j)|
\]

where \( N = 2^{13} \) and \( z_j = j/N \) are the evaluations points on a large uniform grid where approximate and exact values are compared.

In the first set of experiments, we study the effect of \( p \) and \( r \) on the rate of convergence as \( n \) increases. The results in table 1 and table 2 for a smooth function \( f(x) = \sin(20x) \) clearly show that the numerical rate of convergence indeed matches the theoretical rate \( \min\{p, r\} + 1 \). Moreover, as expected, the quality of approximations remains satisfactory even for highly oscillatory functions, as seen in table 3.

Next, we take the function \( f(x) = |x - 1/3|(x - 1/3)^2 \in D^{2,1}([0, 1]) \), where the convergence rate increases as the order of derivative approximations improves, but only up to cubic convergence, as seen in table 4.
The results in table 3 confirm that, unlike the previous smooth cases, increasing the value of \( r \) in the Fourier continuation approximation beyond 2 does not bring additional gains in terms of convergence speed.

Finally, we conclude this section by looking at approximation quality of the proposed approach for \( f(x) = \sin(20x) \) using the derivative approximations of order \( p = 4 \) and the continuation polynomial of degree \( 2r + 1 \) for \( r = 2, 3 \) and 4.

5. Concluding remarks

In this paper, we analyzed a Fourier approximation strategy for non-periodic functions that, to avoid Gibbs oscillations, utilizes a construction for their smooth continuation to a larger interval so that the continued function is periodic. We were able to show that such approximations indeed converge with high-order. In particular, we investigated the two-point Hermite polynomial based continuation strategy and found that they are not only simple to implement but also high-order accurate. Further, in the discrete setting where functional data is available only on an equispaced grid, this construction was utilized to obtain interpolatory trigonometric approximations that converge with high-order and has \( O(n \log n) \) computational complexity.

| \( n \) | \( r = 2 \) | \( r = 3 \) | \( r = 4 \) |
|-------|-------|-------|-------|
|       | \( e_n \) | \( e_{n-1} \) | \( e_n \) | \( e_{n-1} \) | \( e_n \) | \( e_{n-1} \) |
| \( 2^6 \) | 1.42 \times 10^{-4} | — | 6.94 \times 10^{-5} | — | 4.03 \times 10^{-5} | — |
| \( 2^7 \) | 1.28 \times 10^{-5} | 11.10 | 2.53 \times 10^{-6} | 27.39 | 1.42 \times 10^{-6} | 28.32 |
| \( 2^8 \) | 1.44 \times 10^{-6} | 8.85 | 1.02 \times 10^{-7} | 24.81 | 4.59 \times 10^{-8} | 31.04 |
| \( 2^9 \) | 1.75 \times 10^{-7} | 8.23 | 4.64 \times 10^{-9} | 22.02 | 1.44 \times 10^{-9} | 31.84 |
| \( 2^{10} \) | 2.16 \times 10^{-8} | 8.13 | 2.32 \times 10^{-10} | 19.96 | 4.51 \times 10^{-11} | 31.94 |
| \( 2^{11} \) | 2.60 \times 10^{-9} | 8.01 | 1.27 \times 10^{-11} | 18.28 | 1.32 \times 10^{-12} | 34.16 |
| \( 2^{12} \) | 3.37 \times 10^{-10} | 8.00 | 7.46 \times 10^{-13} | 17.04 | 7.67 \times 10^{-14} | 17.21 |

**Table 2.** Convergence study for approximations of \( f(x) = \sin(20x) \) using the derivative approximations of order \( p = 4 \) and the continuation polynomial of degree \( 2r + 1 \) for \( r = 2, 3 \) and 4.

| \( n \) | \( k = 50 \) | \( k = 100 \) | \( k = 200 \) |
|-------|-------|-------|-------|
|       | \( e_n \) | \( e_{n-1} \) | \( e_n \) | \( e_{n-1} \) | \( e_n \) | \( e_{n-1} \) |
| \( 2^6 \) | 6.78 \times 10^{-2} | — | 4.55 \times 10^{-1} | — | 1.15 \times 10^{-9} | — |
| \( 2^7 \) | 1.02 \times 10^{-3} | 66.28 | 6.61 \times 10^{-2} | 6.89 | 4.73 \times 10^{-1} | 2.43 |
| \( 2^8 \) | 3.03 \times 10^{-6} | 337 | 1.32 \times 10^{-3} | 50.07 | 1.07 \times 10^{-1} | 4.44 |
| \( 2^9 \) | 7.21 \times 10^{-8} | 42.08 | 6.87 \times 10^{-6} | 192 | 3.94 \times 10^{-3} | 27.07 |
| \( 2^{10} \) | 1.95 \times 10^{-9} | 36.95 | 1.42 \times 10^{-7} | 48.44 | 8.10 \times 10^{-9} | 486 |
| \( 2^{11} \) | 5.45 \times 10^{-11} | 35.79 | 3.86 \times 10^{-9} | 36.74 | 1.12 \times 10^{-7} | 72.48 |
| \( 2^{12} \) | 1.63 \times 10^{-12} | 33.41 | 1.14 \times 10^{-10} | 33.86 | 3.50 \times 10^{-9} | 31.93 |

**Table 3.** Convergence study for approximations of \( \exp(-2 \cos{kx}) \) with \( k = 50, 100 \) and 200 using the derivative approximations of order \( p = 4 \) and the continuation polynomial of degree 9.
Our numerical experiments validate the performance of this scheme in terms of approximation quality and that the theoretical convergence rates are attained in practice.

While this work focussed mainly on investigating the approximation properties of Fourier continuation strategy, a future step in this direction of significant interest would be to analyze its use in PDE solvers and study the corresponding convergence rates.

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| n   | p = 1   | p = 2   | p = 3   |
|-----|---------|---------|---------|
|     | e_n     | e_n/e_{n-1} | e_n     | e_n/e_{n-1} | e_n     | e_n/e_{n-1} |
| 2^6 | 1.54 × 10^{-4} | —       | 3.20 × 10^{-6} | —       | 3.29 × 10^{-6} | —       |
| 2^7 | 3.88 × 10^{-5} | 3.97    | 4.02 × 10^{-7} | 7.94    | 4.18 × 10^{-7} | 7.88    |
| 2^8 | 9.74 × 10^{-6} | 3.98    | 5.05 × 10^{-8} | 7.97    | 5.26 × 10^{-8} | 7.94    |
| 2^9 | 2.43 × 10^{-6} | 4.01    | 6.32 × 10^{-9} | 7.98    | 6.59 × 10^{-9} | 7.98    |
| 2^{10} | 6.08 × 10^{-7} | 4.00    | 7.81 × 10^{-10} | 8.10    | 8.17 × 10^{-10} | 8.06    |
| 2^{11} | 1.46 × 10^{-7} | 4.16    | 9.77 × 10^{-11} | 8.00    | 1.02 × 10^{-10} | 7.99    |
| 2^{12} | 3.65 × 10^{-8} | 4.00    | 1.22 × 10^{-11} | 8.00    | 1.28 × 10^{-11} | 8.00    |

Table 4. Convergence study for approximations of \( f(x) = |x - 1/3|(x - 1/3)^2 \) using the continuation polynomial of degree 5 and the derivative approximations of order \( p = 1, 2 \) and 3.

| n   | p = 1   | p = 2   | p = 3   |
|-----|---------|---------|---------|
|     | e_n     | e_n/e_{n-1} | e_n     | e_n/e_{n-1} | e_n     | e_n/e_{n-1} |
| 2^6 | 1.56 × 10^{-4} | —       | 2.60 × 10^{-6} | —       | 8.13 × 10^{-7} | —       |
| 2^7 | 3.91 × 10^{-5} | 4.00    | 3.15 × 10^{-7} | 8.25    | 1.01 × 10^{-7} | 8.03    |
| 2^8 | 9.79 × 10^{-6} | 4.00    | 3.88 × 10^{-8} | 8.12    | 1.27 × 10^{-8} | 8.01    |
| 2^9 | 2.44 × 10^{-6} | 4.02    | 4.79 × 10^{-9} | 8.09    | 1.58 × 10^{-9} | 8.00    |
| 2^{10} | 6.09 × 10^{-7} | 4.00    | 5.97 × 10^{-10} | 8.03    | 1.98 × 10^{-10} | 8.00    |
| 2^{11} | 1.46 × 10^{-7} | 4.17    | 7.15 × 10^{-11} | 8.35    | 2.27 × 10^{-11} | 8.72    |
| 2^{12} | 3.66 × 10^{-8} | 4.00    | 8.93 × 10^{-12} | 8.01    | 2.84 × 10^{-12} | 7.99    |

Table 5. Convergence study for approximations of \( f(x) = |x - 1/3|(x - 1/3)^2 \) using the continuation polynomial of degree 7 and the derivative approximations of order \( p = 1, 2 \) and 3.
| $n$ | $\epsilon = 1$ | $\epsilon = 0.1$ | $\epsilon = 0.01$ |
|-----|----------------|-----------------|------------------|
|     | $e_n$          | $e_n/e_{n-1}$   | $e_n$            | $e_n/e_{n-1}$ | $e_n$ | $e_n/e_{n-1}$ |
| $2^6$ | $1.43 \times 10^{-9}$ | — | $1.39 \times 10^{-7}$ | — | $2.06 \times 10^{-1}$ | — |
| $2^7$ | $4.24 \times 10^{-11}$ | $33.67$ | $4.07 \times 10^{-9}$ | $34.12$ | $3.02 \times 10^{-2}$ | $6.81$ |
| $2^8$ | $1.29 \times 10^{-12}$ | $32.81$ | $1.21 \times 10^{-10}$ | $33.53$ | $5.98 \times 10^{-4}$ | $50.52$ |
| $2^9$ | $3.99 \times 10^{-14}$ | $32.43$ | $3.68 \times 10^{-12}$ | $32.99$ | $1.92 \times 10^{-7}$ | $3123$ |
| $2^{10}$ | $9.55 \times 10^{-15}$ | $4.17$ | $1.11 \times 10^{-13}$ | $33.07$ | $2.97 \times 10^{-14}$ | $6.46 \times 10^6$ |

Table 6. Convergence study for approximations of $f_1(x) = ((x - 1/3)^2 + \epsilon^2)^{-1}$ with $\epsilon = 1, 0.1$ and 0.01 using the derivative approximations of order $p = 4$ and the continuation polynomial of degree 9.

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Akash Anand, Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur, UP 208016
E-mail address: akasha@iitk.ac.in