Predicting Lifetime of Dynamical Networks Experiencing Persistent Random Attacks

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Empirical estimation of critical points at which complex systems abruptly flip from one state to another is among the remaining challenges in network science. However, due to the stochastic nature of critical transitions it is widely believed that critical points are difficult to estimate, and it is even more difficult, if not impossible, to predict the time such transitions occur [1–4]. We analyze a class of decaying dynamical networks experiencing persistent attacks in which the magnitude of the attack is quantified by the probability of an internal failure, and there is some chance that an internal failure will be permanent. When the fraction of active neighbors declines to a critical threshold, cascading failures trigger a network breakdown. For this class of network we find both numerically and analytically that the time to the network breakdown, equivalent to the network lifetime, is inversely dependent upon the magnitude of the attack and logarithmically dependent on the threshold. We analyze how permanent attacks affect dynamical network robustness and use the network lifetime as a measure of dynamical network robustness offering new methodological insight into system dynamics.

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Despite being largely robust to outside attacks and spontaneous fluctuations, during which a fraction of nodes and links can become either temporarily or permanently dysfunctional [5–7], most real-world complex systems ultimately collapse and thus have a finite lifetime. The collapse commonly occurs when, at a critical point, the output of the system reaches a critical threshold and the system abruptly shifts from one phase to another [8–10]. Examples of real-world complex systems that experience sudden collapse are numerous, e.g., the spread of disease in living organisms [11], the spread of political dissent in a human society, or the spread of a product sales pattern in economics.

In recent years, network science has been utilized to describe complex systems [12–21], and many dynamical and temporal networks have been proposed [22–28] to explain the complex dynamics that occur in real-world networks [11]. However, the large majority of studies on network science analysed separately either (a) network robustness [5] or (b) empirical indicators of critical transitions [11–13]. The latter group of studies focuses on a phenomenon known as critical slowing down, where as the system approaches a critical point, correlations and variance of the fluctuations increase. Here for (a) and (b) we propose a common theoretical framework based on early warning signals to predict a network catastrophe (collapse). We focus on an important class of networks characterized by nodes which flip between two intrinsic states where nodes have the ability to control the state of their neighbours. Examples for these networks range from cascading processes in interdependent networks [12, 13], epidemic spreading in scale-free networks [11], dynamical networks with spontaneous recovery [28] to the glassy dynamics of kinetically constrained models [29].

We find that introducing permanent failure produces networks that are continuously decaying, but the decline is continuous only up to a specific, critical point—the network lifetime ($t_c$)—when the network abruptly collapses. We find that this critical point is logarithmically dependent upon the threshold and inversely dependent on the size of the outside attack.

Results. So far, no single network model has been able to explain the process of finite-time decline with the possibility of spontaneous collapse, how ageing or continuous ongoing time-dependent attacks affects dynamical network robustness, and how long a network can continue to function before it collapses or how to predict the time of network collapse ($t_c$) prior to its occurrence. To address these three issues associated with dynamical networks experiencing ongoing time-dependent attacks, we propose a decaying mechanism in the evolution of a network. We base this mechanism on three assumptions.

(i) The function of a node $n_i$ is dependent upon its nearest neighbors. Generally speaking, a network is robust if its nodes are able to function even with a large
fraction of failed nearest neighbors, here denoted by $T_h$. If at time $t$ the fraction of the active neighbors of node $n_i$ is smaller than or equal to $T_h$, at time $t + 1$ node $n_i$ will become externally inactive with a probability $r$. We use a fractional threshold \[30\,31\], which is more appropriate than the absolute threshold \[28\,32\] when networks have heterogeneous degrees.

(ii) Each node can internally fail independent of other nodes, with a probability $p$ quantifying the magnitude of the external attack. This dynamical case is equivalent to the static network case when robustness is studied under simultaneous random or targeted attack \[5\,25\,33\,34\].

(iii) Although we assume that a node can recover from an internal failure after a finite period of time ($\tau$) has passed, because internal failure always carries the potential risk of being permanent, by $1 - q$ we define such an event. Thus, due to ongoing time-dependent attacks, when a network is attacked not every node will be able to recover. As in Ref. \[28\], a node can be considered active only if it is both internally and externally active. For a more complicated network, that includes adding new nodes and introduction of link failures, see Methods and Model extensions.

Note that the dynamical approach used to quantify the robustness of a network under time-dependent attacks can also be used to address the ageing process. In this case the probability $p$ serves as an internal network property \[28\]. Over time not every internally-failed node can recover, e.g., at the cell level in biology there may be a defective apoptotic process \[33\]. In seminal studies of the stability of large complex systems, Gardner and Ashby \[36\] (numerically) and May \[37\] (analytically) examined how they maintain stability up to some critical level of connectedness and then suddenly, as the level of connectedness increases, become unstable. In contrast, our focus is on the reverse process in which a continuous decrease in the number of functioning nodes and links between them, over time decreases network complexity. We describe the level of network functionality in terms of the fraction of the total number of nodes that are continuing to function \[28\] and the fraction of active links, $f_l$ \[30\].

To determine how $T_h$ and $q$ affect network functionality, we first generate three different random Erdős Renyi (ER) networks and scale-free networks (Barabási-Albert BA model), each with initial $N(t = 0) = 10,000$ nodes and an average degree $\langle k \rangle = 10$. Then each network is placed under persistent attack, quantified by $p$, where we allow nodes in the inactive phase to permanently fail with a probability $1 - q$. For varying values of $q$ for each decaying network, Fig. 1 shows the fraction of active nodes, the fraction of active links, and the average degree, each as a function of time. Figure 1 shows that when the threshold is fixed but the values of $q$ varied, the network decays continuously until at some specific time $t_c$, it abruptly and spontaneously collapses, a collapse brought on by a sudden large increase in the number of inactive links and nodes. Note that this sudden collapse does not require any parameter other than $q$. As expected, the more rapid the exponential decay, the sooner the network will collapse. When we compare the fractions of active nodes and links we see that, when the network collapse occurs, the fraction of active links decreases substantially more

![Figure 1](image-url)
than the fraction of active nodes, where the first fraction approximately equals $1 - r$.

Note that our analysis addresses both how long a dynamical network will function before collapse and how robust it will be under long-term continuous attack, i.e., the larger the network lifetime $t_c$, the more robust the network. The study of robustness of dynamic networks under continuous attack is highly relevant to the concerns of both researchers and practitioners. For instance, important goal of military science is determining how a military network can remain robust under persistent enemy attack; or a goal in finance is determining how a financial system can remain robust when a fraction of its banks fails over time. References [5–6] reported that scale-free networks are more robust than ER networks to multiple random and simultaneous attacks. Next we compare the robustness of decaying BA and ER networks.

Here the network robustness is defined in a dynamical way, where the more robust a network is, the longer it lasts. Figure 2 compares the dynamical robustness of decaying BA and ER networks under continuous long-term attack, measured by $p$, as a function of time. We use the same parameters for both types of networks. Using network lifetime to quantify dynamical network robustness, Fig. 2 shows that, even when nodes and links fail, BA networks are generally more robust than ER networks. A dynamical BA network is typically able to survive with higher values of $t_c$ than an ER network before it exhibits an abrupt drop in the fraction of its active nodes and links and its average degree. Figure 2(c) shows that for the $q$ values displayed, the average degree of a dynamical BA network lasts longer (i.e., $t_c$ is larger) than the average degree of a dynamical ER network.

To determine whether this difference in dynamical robustness is general or sample-dependent, we compare decaying BA and ER networks for different $q$ values. For fixed $T_h$ and $r$, and varying values of $q$, Fig. 2(a) shows the relationship between $t_c$, whose precise definition will be explained in Fig. 3 and $p$. The BA decaying networks generally exhibit a higher level of dynamical robustness than the ER decaying networks. We find similar dependence between $t_c$ and $p$ for varying network degrees.

The dynamical approach also allows the non-trivial possibility that, following an attack, nodes can remain permanently damaged, corresponding to the $q = 0$ case in which no recovery is allowed. This case is important because it allows us to compare our results with studies in which robustness is analysed in static networks under either simultaneous random or targeted attack [5–7, 25–34]. Note that when $q = 0$, the period $\tau$ becomes irrelevant. For a fixed $T_h = 0.5$ and $\langle k \rangle = 6$, Fig. 2(b) shows how the time $t_c$ is affected by the level of outside attack, quantified by $p$.

Again as in Figure 2(a), the larger the level of outside attack, the smaller the network crash time $t_c$ versus $p$ for (a) $q = 0.995$, $0.95 - 0.995$. The BA decaying networks exhibit higher level of robustness than the ER decaying networks. We use $r = 0.8$ and $\langle k \rangle = 6$. (b) For BA and ER decaying dynamical networks when a node cannot recover ($q = 0$), for $\langle k \rangle = 6$ the larger time $t_c$ (the larger dynamical robustness) we obtain for BA decaying network than for ER decaying network. Here, the larger time $t_c$, the longer the network lasts, the more robust the network.

**FIG. 2:** Dynamical robustness of BA and ER decaying networks for different values of $q$, where $t_c$ serves as a measure of dynamical robustness. We show $t_c$ versus $p$ for (a) $q = 0.995$, $0.95 - 0.995$. The BA decaying networks exhibit higher level of robustness than the ER decaying networks. We use $r = 0.8$ and $\langle k \rangle = 6$. (b) For BA and ER decaying dynamical networks when a node cannot recover ($q = 0$), for $\langle k \rangle = 6$ the larger time $t_c$ (the larger dynamical robustness) we obtain for BA decaying network than for ER decaying network. Here, the larger time $t_c$, the longer the network lasts, the more robust the network.

**FIG. 3:** Time of network crash $t_c$ versus the fraction of active links for different values of the network threshold, $T_h$ calculated for BA dynamical network. $t_c$ dramatically depends on $T_h$. 
work lifetime. Because no recovery is allowed for this case, the lifetime values in Fig. 2(b) are much smaller than in Fig. 2(a). When $q = 0$, decaying BA networks exhibit a significantly higher level of dynamical robustness than decaying ER networks, and the times $t_c$ calculated for BA networks are consistently larger than the times $t_c$ calculated for ER networks—the dependence between $t_c$ and $p$ follows a hyperbolic function.

In the decaying network approach, as previously stated, we quantify the network threshold in terms of the fraction of failed nearest neighbors a node can sustain and still work properly. If a node $n_i$ is initially linked to 10 nodes and needs at least 6 active neighbors to function, then as the network decays node $n_i$ will die when the number of its active neighbors drops to 5. Figure 3 shows a decaying BA network with fixed $q$ and $p$. The network lifetime $t_c$ is strongly dependent upon $T_h$, e.g., when $T_h$ is decreased from 0.5 to 0.2, $t_c$ changes by a factor of 10.

In a complex dynamical system, thresholds and critical points control the transition process between different states [38]. A phase transition occurs when a system reaches a system threshold. Using this perspective, the network decline and collapse numerically described above can also be treated analytically. Note that at time $t - \tau$, due to attacks, the network has $N(t - \tau)$ living nodes, both active and inactive, among which $pN(t - \tau)$ nodes switch to the internally inactive phase. The change in the number of living nodes is proportional to the probability that a node which internally failed at $t - \tau$ will not recover, $1 - q$, thus

$$dN(t - \tau) = -(1 - q) p N(t - \tau) dt.$$ (1)

The number of living nodes remaining is determined by an exponential decay $N(t) = N(0) \exp(-(1-q)p t)$ with a decay rate of $\lambda = (1-q)p$, where $N(t)/N(0)$ is the average fraction of living nodes equivalent to the average fraction of living neighbors of each node. For simplicity, let us assume that all nodes have the same initial degree $k$, remembering that node $n_i$ will be active if the number of its functioning neighbors is larger than $m = T_h k$. At time $t$, the fraction of $n_i$’s failed neighboring nodes $N_f(t)/N(t)$ among $N(t)$ living nodes previously estimated as $k(t) = k \exp(-(1-q)p t)$, can be approximated using the probabilities of internal and external failures, $p$ and $E[m, k(t)]$,

$$a = a(p, r, m, k(t)) = p + r (1 - p) E[m, k(t)],$$ (2)

where $E[m, k(t)] = \sum_{j=0}^{m} \binom{k(t)}{j} a^{k(t) - j} (1 - a)^j$. In the above figures we analysed the ratio between the living active and the initial number of nodes $N_a(t)/N(0)$, which is equivalent to

$$f_n = \frac{N_a(t)}{N(t)} = (1 - a) \exp[-(1 - q)p t].$$ (3)

Here $\exp[-(1 - q)p t]$ is due to the continuous decline of the network, and $1 - a$ indicates the sudden network

FIG. 4: Indicators for network crash. Shown are 3 indicators together with the fraction of active links obtained for (a) ER and (b) BA decaying network when $q = 0.99$. We show indicator I (mstd), moving standard deviation of fraction of active nodes (forward method) with window size $\tau = 50$ (the same $\tau$ as in recovery process), and indicator II (crash indicator), the average fraction of active neighbors in excess of threshold $T_h$. Approximately, when the average fraction of active neighbors exceeds the threshold, cascading failures trigger a network breakdown. The cascade lasts several decades. We also show the analytical indicator III ($t_c$). We use $p = 0.003$, $T_h = 0.5$, and $r = 0.8$.

FIG. 5: For the varying $q$ values, for both decaying ER and decaying BA network we show the size of the decrease of $f_n$ during the network crash, equal to $T_h - r$. 
crash occurring when the fraction of living nodes $k(t)/k$ approaches the threshold $T_h = m/k$. At this limit, from Eqs. (2) and (3) we see that $E[m, k(t)] \rightarrow 1$ and the probability of external failure dominates $p$.

Although predictive power is important in any scientific endeavour, it is widely assumed that predicting critical transitions is extremely difficult because any change in the dynamic state of a system immediately prior to reaching the critical point will be slight. Unlike early-warning indicators that utilize recent system outputs to detect impending system collapse, we use past data to estimate network parameters, from which we generate numerical simulations that describe the network state at any future time, including the moment of network failure $t_c$. Thus the dynamical approach allows us to precisely predict the network crash.

Figure 4 shows the predictive power of this dynamical approach when applied to both (a) ER and (b) BA networks in the process of decaying. It shows the fraction of active nodes for time scales that include the network crash and also two important values: (I) the standard deviation of the fraction of active nodes calculated using moving boxes similar to the approach proposed in Ref. [9] and (II) the average fraction of active neighbors—active both internally and externally—above the threshold $T_h$. Equations (2) and (3) provide a theoretical explanation for II, predicting that when the fraction of currently living nodes approaches the threshold that controls the external failures, the probability of critical transition increases and finally cascading failures lasting several decades trigger a network breakdown. Note that just prior to network crash, indicator I increases substantially and thus is an indicator of impending failure and a predictor of the time $t_c$, i.e., it predicts when network failure will occur. Reference [9] suggests that as a dynamical system approaches a critical threshold its state at any given moment increasingly resembles its previous state, implying an increase in variance and autocorrelation as reported for dynamical networks in Ref. [30]. We find that this predictive power holds for both BA and ER decaying networks for a wide variety of parameters. In Fig. 4 the values of II shift from positive to negative and the time when this occurs can also be used as a predictor of the time of network collapse $t_c$. In addition to the two numerical indicators, we also suggest a third analytical indicator (III), obtained by equating $\exp[-(1-q)pt]$ in Eq. (4) with $T_h$,

\[ t_c = -\ln(T_h)/p(1-q). \] (4)

In the decaying network approach, the proportional threshold $T_h$ controls both node-level crashes and “macro” network-level crashes, which makes $T_h$ a breakdown threshold. For a general set of network parameters, the time of network failure is the ratio between the closeness of the fraction of active nodes to the threshold, $-\ln(T_h)$ and the rapidity of the approach of the fraction to the threshold $p(1-q)$. In Figs. 4(a) and 4(b) we see that $t_c$ typically corresponds to the beginning of the network failure. Additionally, for varying values of $q$ in Fig. 5 for both decaying BA and ER networks, we find

FIG. 6: Statistics of indicator III for (a) decaying ER and (b) decaying BA network. We show how the first four moments of indicator III change over time just before, during and just after the crash. Just before the network crash, skewness and kurtosis dramatically and abruptly increase.

FIG. 7: Distribution of the fraction of active neighbors of each node above the threshold $T_h$ substantially changes during the network failure.
how much the fraction of active nodes decreases during the network crash. It decreases for $T_h - r$.

Figures (1) through (4) show that Eq. (4) agrees well with the numerical results that describe how $t_c$ is affected by $T_h$, $p$, and $q$. Figures (4)(a) and (4)(b) show that the values of indicators $II$ and $III$ are nearly identical, suggesting that during a network crash the time when the fraction of living nodes approaches the threshold is approximately identical to the time when the fraction of active neighboring nodes approaches the threshold, implying that in Eq. (3) the second term contributes significantly more than the first. Further we find that the time series of the fraction of active neighbors of each node above the threshold $T_h$ substantially changes over time. For this time series, calculated for each network in Figs. (4(a) and (4(b)) in addition to the average representing our indicator III, Fig. (7(a)-(b)) shows the higher moments, variance, skewness, and kurtosis. For different times prior, during, and after the network failure, Fig. (7) shows the distribution of the fraction of active neighbors of each node above the threshold $T_h$. Just prior to network failure, skewness and kurtosis and the distribution dramatically change.

In estimating the time of a future network crash, we calculate network parameters from its creation at time $t = 0$ to some time $t$ during which the fraction of active nodes $f_a$ decreases from 1 to some value $f'$. When we numerically generate the network we thus record the time $t$ at which $f_a$ reaches $f'$. Then the true prediction of future network collapse is not $t_c$ but $t_c - t$, i.e., $t_c$ represents the entire lifetime of the network, and our interest is in estimating the lifetime remaining after $t$.

**Conclusion.** In both the natural and social sciences a wide range of real-world complex networks have a finite lifetime characterized by abrupt shifts between phases. One of the differences between the behaviour of natural systems and social systems near a critical point is that the sudden random change commonly occurs at a fixed point in natural systems [39]. In contrast, in social sciences even a critical point is a random variable and, although there is an expectation that some radical change is imminently probable, it is generally assumed that it is not possible to predict the exact time of its occurrence. We have explored the predictive power for a particular class of dynamic non-equilibrium decaying networks.

In our network model, the same parameter, a proportional threshold $T_h$ responsible for node-level crashes, also controls “macro” network-level crashes. Our dynamical network study includes cases in which nodes can recover ($q \neq 0$) or remain in a failed state ($q = 0$) after a failure caused by the ageing process or by severe hostile time-dependent attacks. Model extensions are included in the Methods section. How successful our network might be when applied in practice depends first, on how capable we are to estimate the model parameters, especially the critical threshold $H$, and second, how good our network is as a proxy for a real-world complex system. In practice the second issue is very frequent when real-world stochastic systems composed of large number of units, such as financial, are modelled with theoretical stochastic processes, such as multivariate autoregressive for instance. Besides, real-world networks rather interact with other networks [12] [13] than act as a single network [11]. Due to interdependencies between different networks, a shock on a particular network may change not only its own network parameters, but also the parameters of all other networks. Clearly, since the time to the network breakdown depends on network parameters, any change in network parameters also affects our estimation on the timing of network collapse.

**Methods and Model Extensions**

**Phase-flipping with decay** Ref. [28] reported that introduction of both dynamical recovery and stochastic contiguous spreading leads to spontaneous collective network phase-flipping phenomena. For $q = 1$ case, we choose the network parameter set to enable phase-flipping between two stable states. When decay mechanism is included ($q \neq 1$), in Fig. (8) with decreasing parameter $q$, the fraction of living nodes gradually disappears over time and so the phase-flipping phenomena and collective network mode.

**Introduction of link failures.** For decaying networks, it is reasonably to assume that links as well as nodes may also fail at any time $t$. Thus, we additionally consider that every link can independently fail with probability $p_l$ (internal failures in links). We define that a link is “healthy” (active) if it is active both internally, controlled by $p_l$, and externally, controlled by $p$ in (ii). When a dynamical network is defined one should find out which part of the parameter space is characteristic for a stable network regime and which part of the space determines an unstable network regime. For our dynamical network defined by three probability parameters, $p$, $r$, and $p_l$, next, for a case when nodes always recover after being inactive for a while ($q = 1$), we provide analytical results for the 3D hysteresis in parameter space here enclosed by manyfolds comprising spinodals which separate regions of stability and instability. Thus, for an active node $i$ with $k_i$ neighbors, the link between $n_i$ and another node $n_j$ fails either due to an internal link failure ($X$) with probability $p_l$ or due to external failure of node $j$ ($Y$) equal to $a$—in the latter case according to definition, when $n_j$ is failed, all its links have failed. Thus, assuming $X$ and $Y$ are independent, the link is active with probability $(1 - a)(1 - p_l)$ and accordingly inactive with probability $1 - (1 - a)(1 - p_l) = a + p_l - ap_l \equiv \alpha_l$. Similar to (i), now when even the links can fail internally, we define that node $n_i$ is active only if there are more than $m$ active links (not nodes as in (i)). Thus by

$$E(k, m, \alpha_l) \equiv \sum_{j=0}^{m} a_k^{k-j} (1 - \alpha_l)^j \left( \frac{k}{k - j} \right), \quad (5)$$

we define the probability that node $n_i$’s neighborhood
is critically damaged with more than $m$ broken links. Finally, we derive the probability that a randomly chosen node $n_i$ with degree $k$ is inactive, equal to the fraction of inactive nodes,

$$a_k = a = p + r(1 - p)E(k, m, a_{\ell}),$$

where we apply probability theory $P(X + Y) = P(X) + P(Y) - P(X)P(Y)$ for two independent events. Clearly, this equation is derived under the assumption of independence of the external and internal failures is only approximately true since internal failures affect external failures. For an ER decaying network. Fig. 3 shows the predictive power for a case when internal link failures are considered. 

For simplicity for the moment we analyze random regular networks since Eq. (6) assumes that all nodes have the same degree. Fig. 4 shows the mean-field (MFT) prediction for the position of spinodals in 3D parameter space $(p_1, p_2, p_3)$ obtained from Eq. (6), where $p_1 = 1 - \exp(-pr)$ [28]. In 3D parameter space, the 3D hysteresis region is enclosed by planes comprising spinodals. The 3D space is obtained by fixing $p_\ell$ and calculating the 2D hysteresis region bounded with two spinodals merging at a critical point.

Addition of new nodes. In the growth of a network, the model can be additionally extended in a way that nodes can be also created not only destroyed. We can assume that at each time $t$, among $N(t)$ living nodes, each living node can create a new node with probability $p'$. Then Eq. (1) is modified to

$$dN(t - \tau) = (p' - (1 - q)p)N(t - \tau)dt,$$

and the system can not only exponentially decay, but also increase and even stay stable if $p' = (1 - q)p$.

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FIG. 8: Phase-flipping with decay mechanism for the decaying BA network with parameters $T_h = 0.50, \tau = 50$, the average degree $\langle k \rangle = 3$, 1000 nodes. With decreasing $q$, from 0.999995, 0.9999, and 0.999 (from top to bottom) we show that the average $f_n$ decays with time where the smaller $q$ the faster the decay.
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FIG. 9: Indicators for network crash, similar as in Fig.4 but now we include internal link failures. Shown are 3 indicators obtained for an ER decaying network when \( q = 0.99, \ T_h = 0.99, r = 0.8, p = 0.003, \) and \( p_{\ell} = 0.003. \)

FIG. 10: For \( q = 1 \) case, we show the 3D phase diagram in model parameter space \( (p_1, p_2, p_3) \) representing hysteresis region.