Variational Worldline Approximation for the Relativistic Two-Body Bound State in a Scalar Model

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Abstract. We use the worldline representation of field theory together with a variational approximation to determine the lowest bound state in the scalar Wick-Cutkosky model where two equal-mass constituents interact via the exchange of mesons. Self-energy and vertex corrections are included approximately in a consistent way as well as crossed diagrams. Only vacuum-polarization effects of the heavy particles are neglected. In a path integral description of an appropriate current-current correlator an effective, retarded action is obtained by integrating out the meson field. As in the polaron problem we employ a quadratic trial action with variational functions to describe retardation and binding effects through multiple meson exchange. The variational equations for these functions are derived, discussed qualitatively and solved numerically. We compare our results with the ones from traditional approaches based on the Bethe-Salpeter equation and find an enhanced binding contrary to some claims in the literature. For weak coupling this is worked out analytically and compared with results from effective field theories. However, the well-known instability of the model, which usually is ignored, now appears at smaller coupling constants than in the one-body case and even when self-energy and vertex corrections are turned off. This induced instability is investigated analytically and the width of the bound state above the critical coupling is estimated.

1 Introduction

The bound state problem in Quantum Field Theory has a long history starting with the classic paper by Salpeter and Bethe more than 50 years ago. While the Bethe-Salpeter equation (BSe) is formally exact, its building blocks are only defined as infinite series of increasingly higher n-point Green functions. This hierarchy of coupled equations (called Dyson-Schwinger equations) has to
be truncated for practical purposes. In this way approximations are introduced which are hard to control and sometimes violate symmetries (e.g. gauge invariance). The most common approximation is the ladder approximation where bare propagators and vertices are used. There is a vast literature on the ladder BSe and its many deficiencies (e.g. existence of abnormal states, failure in the large mass limit etc.) [2]. Practical methods to include an extended set of diagrams (dressed propagators [3] or cross-ladder [4]) have been developed only recently.

Many other variants have been investigated over the years: 3-dimensional reductions (also called the “quasipotential” approach) [5, 6, 7], relativistic Hamiltonian models in light-front [8] and point form [9] quantization, Feynman-Schwinger representations [10] to name only a few. It is close to impossible (and not intended here) to give a comprehensive overview over all attempts and theoretical approaches available in the literature.

Apart from the theoretical interest, there are (at least) three areas of physics in which the relativistic bound state problem needs to be addressed: the first one is bound-state Quantum Electrodynamics (QED) where ultra-precise experimental data are available. However, from the theoretical point of view this is a special field since one can employ perturbation theory in powers of the fine-structure constant and its logarithm after binding between the constituents has been established. Impressive accuracy matching the experimental precision has been achieved in recent years [11].

There is another area for relativistic bound-state calculations where strong-coupling is required: the electromagnetic structure of nuclear few-body systems at intermediate and high energies. For example, relativistic effects in the electromagnetic form factor of the deuteron seem to be important already at surprisingly low momentum transfers [12, 13]. Unfortunately, these effects cannot be clearly separated from those of the phenomenological nuclear interaction and of the assumed form of the current operator.

Finally there is the area of hadronic physics where one has to understand how (light) relativistic quarks and gluons bind to form the low-energy mesons and baryons (possibly also hybrids and exotics). Here, Quantum Chromodynamics (QCD) at least provides a clear underlying theoretical framework. However, the problems of confinement, chiral symmetry breaking and strong coupling are formidable and have not been solved (analytically) at low energies. Lattice Gauge Theory (see, e.g. ref. [14]) is considered as the prime method to obtain gauge-invariant results from first principles, albeit with enormous numerical effort and problems of its own. The continous progress of lattice calculations not withstanding, considerable progress has also been made in the last years in solving the Dyson-Schwinger equations for Landau-gauge QCD under some simplifications [15] and in describing the low-lying hadrons as bound-state of quarks and gluons [16]. While phenomenologically quite successful and often going beyond the ladder BSe, these calculations still have limitations due to truncations, gauge dependence and the use of model propagators. In a similar way as in QED, remarkable theoretical progress has been achieved recently for special systems like heavy quarkonia by using the techniques of (nonrelativistic) effective field theories [17].

Given the twisted history of the relativistic bound-state problem, it might be
useful to consider it from a different angle, namely from the one provided by the *worldline variational* approach. Variational methods are widely used in molecular, atomic and nuclear physics as they offer accurate, non-perturbative results for bound and scattering states in a quantum-mechanical framework. In field theory their use is more restricted \[18\] since one cannot handle non-gaussian wave functionals as trial states. Nevertheless, there have been a number of variational calculations for relativistic bound-state systems, most notably by Darewych and collaborators \[19, 20\]. In contrast to these works we advocate here the worldline description of field theory which entails a huge reduction of the dynamical degrees of freedom. This is essential for a variational calculation in which only quadratic trial actions can be employed for analytical evaluations. The classic example is Feynman’s treatment of the polaron \[21\] – a non-relativistic field theory for an electron interacting with the phonons in a crystal – which still stands out among all approaches describing this system non-perturbatively for all coupling constants. The key to this success was that, in a path integral treatment of the problem, he could integrate out the phonons exactly leading to a retarded two-time action for the electron. For the latter a variational approximation then gave results unmatched by all other (analytical) methods. Shortly after this seminal paper, Mano, a student of Feynman’s, applied these results to a scalar relativistic theory \[22\]. However, his pioneering work was largely forgotten as was the description of relativistic systems by trajectories (worldlines). Recent times have seen a revival of worldline methods \[23\] under different names (e.g. Feynman-Schwinger representation) and for different purposes, e.g. for an efficient calculation of diagrams or effective actions. In a series of papers \([24, 25, 26, 27, 28, 29]\) we have extended the “polaron” variational approach to field theory as pioneered by Mano. This was first done in the context of the scalar Wick-Cutkosky (WC) model \[30\] and later extended to a more realistic fermionic theory, viz. QED \[31, 32\]. As an attempt to catch non-perturbative features of these field theories, the results have been encouraging: for example, the well-known instability of the WC model \[33\] was detected by the worldline variational approach, and in QED a fully gauge-invariant approximate result for the anomalous mass dimension of the electron was obtained \[34\].

Therefore we believe that a fresh look at the relativistic bound-state problem from the worldline variational perspective may be useful even in an unrealistic model like the WC model, which incidentally was developed just for the purpose of studying bound states \[1\]. In previous applications \[24 - 29\] we used a version of that scalar model which describes neutral “nucleon” fields $\Phi$ interacting with “meson” fields $\chi$. It is given by the Lagrangian (our metric signature is $+--$ and in general we put $\hbar = c = 1$).

$$
\mathcal{L}_1 = \frac{1}{2} (\partial_\mu \Phi)^2 - \frac{1}{2} M_0^2 \Phi^2 + \frac{1}{2} (\partial_\mu \chi)^2 - \frac{1}{2} m^2 \chi^2 + g \Phi^2 \chi.
$$

(1.1)

In order to compare with standard literature (e.g. Chap. 10 in Itzykson & Zuber \[35\]) it is advisable to switch to charged particles and to allow for different

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1 Frequently only the exact solution of the ladder BSe for massless exchange particles is called the “Wick-Cutkosky model”. Here we use this designation in a more general sense for a class of scalar models described by Lagrangians of the form \[1, 1, 3\].
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masses of the constituents. Then one needs two different types of nucleon fields
and therefore the model which we are going to use is described by the following
Lagrangian

\[ \mathcal{L}_2 = \sum_{i=1}^{2} \left( |\partial_\mu \Phi_i|^2 - M_i^2 |\Phi_i|^2 \right) + \frac{1}{2} (\partial_\mu \chi)^2 - \frac{1}{2} m^2 \chi^2 + \sum_{i=1}^{2} g_i |\Phi_i|^2 \chi. \] (1.2)

The coupling constants \( g_i \) are now twice as large as in the neutral case as can
be seen by collecting all pieces quadratic in the nucleon fields. Previously we
have considered \cite{36} a bound nucleon-antinucleon system in the neutral model
described by the Lagrangian \cite{14} while now we will investigate a bound two-
particle state described by the Lagrangian \cite{12}. Both models are, of course, a
far cry from reality and in the present treatment basically equivalent. Of course,
there are differences from the fact that in nucleon loops two types of nucleons
run around, i.e. that the determinant factor appears twice. However, since we
will work in the “quenched” approximation in the following this doesn’t matter.
Note that also the meson properties are unchanged in this approximation. For
simplicity the present work will be restricted mostly to the equal-mass case

\[ M_1 = M_2 \equiv M_0 \] (1.3)

where \( M_0 \) denotes the bare mass of the nucleon. First results from this new
approach have been presented in ref. \cite{36}. Here we will give a full account of
this formalism, test it in the nonrelativistic case, present improved numerical
calculations and explore special cases, in particular when self-energy and vertex
effects are switched off but crossed-ladder diagrams are retained.

Finally it is worthwhile to recall the historical and conceptual roots of the
worldline variational approach: for a single scalar particle it may be considered
as a relativistic version of the polaron problem, and hence the present work is
close in spirit (but not in the details) to investigating a bipolaron, the possible
bound state of two electrons in an ionic crystal. This occurs when the attractive
electron-phonon coupling overcomes the repulsive Coulomb interaction and has
been the topic of many theoretical papers \cite{37,38} as a proposed mechanism for
high-temperature superconductivity.

This paper is organized as follows: in the next Sect. we briefly overview
the well-known nonrelativistic results for the present model. We then study a
particular current-current correlator (or polarization propagator) whose poles
will give us the relativistic bound-state energies. Sect. 4 describes the variational
calculation of that object in the worldline formalism, including the retarded trial
action which we use. In the next two Sects. the variational equations for the
retardation functions which enter our trial action are derived and numerically
solved. The last Sect. contains our conclusions and the outlook for further work.
Most of the technical details such as the calculation of the various averages with
the trial action or numerical details are collected in several appendices.

2 Nonrelativistic Results

A useful guide for the results one may expect for the two-body relativistic bound-
state problem is provided by the nonrelativistic limit. Although this is quite
standard we briefly recall the essential facts and point to the differences in a relativistic treatment. It is well-known (and re-derived by path integral methods in appendix A.1) that in the nonrelativistic limit one obtains a Schrödinger equation for the two particles interacting via an attractive Yukawa potential

\[ V(x_1 - x_2) = -\frac{\alpha}{|x_1 - x_2|} e^{-m|x_1 - x_2|} \]  

(2.1)

where

\[ \alpha \equiv \frac{g_1 g_2}{16 \pi M_1 M_2} \]  

(2.2)

is the usual (dimensionless) coupling constant \(^2\). One of the simplifications of the nonrelativistic treatment is that center-of-mass and relative motions decouple. For massless mesons \((m = 0)\) the Yukawa potential in Eq. (2.1) becomes an attractive Coulombic potential with internal energy levels

\[ \epsilon_{n\text{nonrelat.}} \bigg|_{m=0} = \frac{\alpha^2 2(n+1)^2}{2(n+1)^2} m_{\text{red}}, \quad n = 0, 1, 2 \ldots \]  

(2.3)

where

\[ m_{\text{red}} = \frac{M_1 M_2}{M_1 + M_2} \]  

(2.4)

is the reduced mass of the system. If \(m \neq 0\) then first-order perturbation theory gives

\[ \epsilon_0 = -\frac{1}{2} \alpha^2 M_{\text{red}} \left[ 1 - 4\delta + 6\delta^2 - 8\delta^3 + O(\delta^4) \right] \]  

(2.5)

where

\[ \delta = \frac{m}{2\alpha M_{\text{red}}} \]  

(2.6)

is proportional to the ratio of Bohr radius to meson Compton wavelength. This shows that the finite range of the Yukawa potential, as expected, decreases the binding energy. Indeed, numerical calculations \[39\] demonstrate that an attractive Yukawa potential only develops bound states if

\[ \delta^{-1} = \frac{2 M_{\text{red}}}{m} > 1.67981. \]  

(2.7)

Thus for large enough coupling constants and/or small meson mass the WC model will most probably - inasmuch as the nonrelativistic approximation is an useful guideline - have bound states. Note that nonrelativistically the masses are always constants which are not changed by the interaction whereas relativistically the binding energy subtracts from the “weight” of the system. If the system has total momentum \(q\) then its total energy is

\[ M_1 + M_2 + \frac{q^2}{2(M_1 + M_2)} + \epsilon_0 \simeq \sqrt{(M_1 + M_2 + \epsilon_0)^2 + q^2} \]  

(2.8)

\(^8\text{Sometimes, e.g. in ref. }[35], \text{ denoted by } \lambda. \text{ Taking into account the different coupling terms for the case of neutral particles the present definition is the same as in refs. }[24] - [29].\)
and with $\epsilon_0 < 0$ the rest mass of a bound state is less than the sum of its constituent masses. Of course, the separation of center-of-mass and relative motion is no longer valid in the relativistic case, nor is the concept of a reduced mass. However, our approach of calculating the poles of suitable Green functions does not require these, and indeed automatically gives the correct nonrelativistic center-of-mass energy $q^2/(2(M_1 + M_2))$ as shown in appendix A.2.

3 The Current-Current Correlator

"In the relativistic approach, bound states and resonances are identified by the occurrence of poles in Green functions." (p. 481 in ref. [35]). Therefore in general one has to investigate a 4-point function of the form

$$G_{ij}^{(4)}(x_1, x_2, x_3, x_4) = \left\langle 0 \left| \mathcal{T}\left(\hat{\Phi}_j(x_1)\hat{\Phi}_i(x_2)\hat{\Phi}_j^+(x_3)\hat{\Phi}_i^+(x_4)\right)\right| 0 \right\rangle$$

with no external mesons and $i, j = 1, 2$. However, for the purpose of getting bound-state energies this object contains too much information and therefore it is advisable to consider the Fourier transform of a special 4-point function where arguments are pairwise identical. In the context of the model described by the Lagrangian (1.2) this could be

$$\Pi_{ij}(q) := -i \int d^4x e^{iq\cdot x} \left\langle 0 \left| \mathcal{T}\left(\hat{\Phi}_j(x)\hat{\Phi}_i(x)\hat{\Phi}_i^+(0)\hat{\Phi}_j^+(0)\right)\right| 0 \right\rangle_{\text{connected}}.$$  

Here we already have used translational invariance to set the second argument to zero. Equation (3.2) is a version of the polarization propagator in the language of many-body physics [40] or a 2-point correlator for the “current” operator

$$\hat{C}_{ij}^+(x) = \hat{\Phi}_j^+(x)\hat{\Phi}_i^+(x).$$

It is also (up to a sign and different currents) essentially the quantity considered by Shifman et al. in their work on QCD sum rules [41] to obtain hadronic masses from quarks and gluons. Note that we have taken a particular scalar current and that we are free to choose different ones: for example, $\hat{\Phi}_i^+(x)\partial_\mu\hat{\Phi}_j^+(x)$ may be used to study bound states with angular momentum 1. Due to covariance the present polarization propagator can only depend on invariants, i.e. $\Pi_{ij}(q) = \Pi_{ij}(q^2)$. We will concentrate on the case $i = 1, j = 2$ in which the operator (3.3) creates a nucleon of type 1 and one of type 2 out of the vacuum.

To see what the structure of the polarization propagator is we now look at its spectral representation. This is done in the standard fashion by inserting a complete set of states with four momenta $P_n$:

$$1 = \sum_n \langle P_n | P_n \rangle ; \quad P_n^2 = M_n^2$$

into Eq. (3.2). Using translational invariance one can perform the $x$-integration and obtains

$$\Pi_{ij}(q^2) = (2\pi)^3 \sum_n \frac{1}{q^2 - M_n^2 + i0} \left\langle P_n \left| \hat{C}_{ij}^+(0) \right| 0 \right\rangle^2.$$
showing that the polarization propagator indeed develops poles at \( q^2 = M_n^2 \). Here \( M_n \) is the mass of the intermediate state which can be reached from the ground state (the vacuum) by application of either the current \( \hat{C}_{ij}(0) \) or by \( \hat{C}_{ij}(0) \). Using the standard expansion of the field operators in terms of creation and annihilation operators for the free-field case, one easily sees that for it creates a two-particle state (consisting either of two nucleons or two antinucleons of type \( i \) and \( j \)) as intermediate state when acting on the vacuum.

### 3.1 Reduction to a Path integral Over Meson Fields

With no meson source the generating functional for our model is denoted by \( Z' \) and reads

\[
Z'[J^*, J] := \int D\Phi_i D\Phi_i^* D\chi \exp \left\{ i \int d^4 x \left[ \mathcal{L}_2 + \sum_i (J_i^* \Phi_i + \Phi_i^* J_i) \right] \right\}
\]

One may integrate out the nucleon fields to obtain

\[
Z'[J^*, J] = \text{const.} \int D\chi e^{i S_0[\chi]} \prod_{i=1}^2 \left( \frac{1}{\det \mathcal{O}_i(\chi)} \right) 
\times \exp \left[ -i \int d^4 x d^4 y J_i^*(x) \langle x | \mathcal{O}_i^{-1}(\chi) | y \rangle J_i(y) \right] \tag{3.7}
\]

where

\[
S_0[\chi] = \frac{1}{2} \int d^4 x \left[ (\partial \chi)^2 - m^2 \chi^2 \right] \tag{3.8}
\]

is the free meson action and

\[
\mathcal{O}_i(\chi) = -\partial^2 - M_i^2 + g_i \chi \tag{3.9}
\]

is the operator for the quadratic part of the nucleon action. Note that charged nucleon fields lead to a power of \(-1\) of the determinant instead of the usual \(-1/2\) for neutral fields. However, in the quenched approximation where this determinant is neglected this makes no difference.

The functional differentiation for

\[
\Pi_{ij}(q) = -i \int d^4 x e^{i q \cdot x} \frac{\delta^4 \ln Z'}{\delta J_i^*(x) \delta J_j^*(x) \delta J_i(0) \delta J_j(0)} \bigg|_{J=J^*=0} \tag{3.10}
\]

is most easily done by employing a cumulant expansion in terms of the sources in Eq. (3.7). This gives

\[
\ln Z' = \mathcal{O} \left( J^0, J^2 \right) + \frac{(-i)^2}{2} \sum_{k,l=1}^2 \prod_{i=1}^4 \left( \int d^4 x_i \right) J_k^*(x_1) J_k(x_2) J_l^*(x_3) J_l(x_4) 
\times R_{kl}(x_1, x_2; x_3, x_4) + \mathcal{O} \left( J^6 \right) \tag{3.11}
\]
where

\[ R_{kl}(x, y; x', y') = \langle \langle x | O^{-1}_k(\chi) | y \rangle \langle x' | O^{-1}_l(\chi) | y' \rangle \rangle - \langle \langle x | O^{-1}_k(\chi) | y \rangle \rangle \langle \langle x' | O^{-1}_l(\chi) | y' \rangle \rangle. \] (3.12)

Here we have defined the following average over the meson field

\[ \langle \langle A \rangle \rangle := \left\{ \int D\chi \exp \left[i S_0[\chi] - \sum_i \text{Tr} \ln O_i(\chi) \right] \right\}^{-1} \times \int D\chi A(\chi) \exp \left[i S_0[\chi] - \sum_i \text{Tr} \ln O_i(\chi) \right]. \] (3.13)

The functional differentiations can now be performed easily with the result

\[ \Pi_{ij}(q) = i \int d^4 x e^{iq \cdot x} \sum_{k=1}^2 \langle \langle x | O_{i}^{-1}(\chi) | x=0 \rangle \langle x | O_{j}^{-1}(\chi) | x=0 \rangle \rangle - \langle \langle x | O_{i}^{-1}(\chi) | x=0 \rangle \rangle \langle \langle x | O_{j}^{-1}(\chi) | x=0 \rangle \rangle \]. \] (3.14)

This describes the propagation of a nucleon (or an antinucleon) of type \( i \) and one of type \( j \) from the space-time point \( 0 \) to \( x \). In between they emit and absorb all types of mesons which is represented by the functional integral over the meson field \( \chi \).

The last term Eq. (3.14) subtracts the unconnected pieces and consists just of the product of usual propagators. Since this term does not contain any poles besides the usual one-particle poles we can drop it when we search for additional bound-state poles in the polarization propagator. Taking \( i, j = 1, 2 \) we therefore will investigate the simpler form

\[ \Pi(q) := i \int d^4 x e^{iq \cdot x} \langle \langle x | O_1^{-1}(\chi) | x=0 \rangle \ (x | O_2^{-1}(\chi) | x=0 \rangle \). \] (3.15)

### 3.2 Worldline Description of the Polarization Propagator

We now derive the worldline formulation for \( \Pi(q) \) given in Eq. (3.15) still for unequal masses. As in the case of the 2-point function this is only possible in the quenched approximation, i.e. by neglecting the determinant in the average

\[ \langle \langle A \rangle \rangle \longrightarrow \langle \langle A \rangle \rangle_{\text{quenched}} := \left\{ \int D\chi \exp \left(i S_0[\chi] \right) \right\}^{-1} \times \int D\chi A(\chi) \exp \left(i S_0[\chi] \right). \] (3.16)

Only then can one perform the path integral over the meson field \( \chi \) after using the Schwinger representation

\[ \frac{1}{O_i(\chi)} = \frac{1}{2i\kappa_0} \int_0^\infty dT \exp \left[ \frac{iT}{2\kappa_0} (-\partial^2 - M_i^2 + g_i \chi + i0) \right]. \] (3.17)
and the quantum-mechanical path-integral\(^3\)

\[
\langle y \mid \exp \left[ -iT \left( -\frac{\hat{p}^2}{2\kappa_0} - \frac{g_i}{2\kappa_0} \chi(\hat{x}) \right) \right] \mid x \rangle = \int_{x(0)=x}^{x(T)=y} D x \exp \left\{ i \int_0^T dt \left[ -\frac{\kappa_0}{2} x^2 + \frac{g_i}{2\kappa_0} \chi(x) \right] \right\}
\]

(3.18)

\(\times \), i.e. for both propagators. In this way one obtains

\[
\Pi(q) = i \int d^4x e^{ig \cdot x} \int_0^\infty \frac{dT_1 dT_2}{(2i\kappa_0)^2} \exp \left[ -\frac{i}{2\kappa_0} (M_1^2 T_1 + M_2^2 T_2) \right]
\]

\[
\times \prod_{i=1}^2 \left( \int_{x_i(0)=0}^{x_i(T_i)=x_i} D x_i \right) \exp \left\{ i \sum_{i=1}^2 \int_0^{T_i} dt_i \left( -\frac{\kappa_0}{2} x_i^2 \right) \right\}
\]

\[
\times \left\{ i \sum_{i=1}^2 \frac{g_i}{2\kappa_0} \int_0^{T_i} dt_i \chi(x_i(t_i)) \right\}. \quad (3.19)
\]

Writing the argument of the last exponential as

\[
b(z) = \frac{1}{2\kappa_0} \sum_{i=1}^2 g_i \int_0^{T_i} dt \delta(z - x_i(t))
\]

(3.20)

the required functional average over the meson field can now be performed since it is just a gaussian integral

\[
\int D\chi e^{iS_0[\chi]+i(b,\chi)} = \int D\chi \exp \left[ i \left( \frac{b}{\kappa_0} \right) \right] \exp \left[ -\frac{i}{2} \left( b, \frac{1}{\sqrt{-\partial^2 - m^2}} b \right) \right].
\]

(3.21)

Hence

\[
\langle \exp [i(b,\chi)] \rangle = : \exp \left( iS_{\text{int}}[x_1, x_2] \right)
\]

(3.22)

with

\[
S_{\text{int}}[x_1, x_2] = -\frac{1}{8\kappa_0^2} \sum_{i,j=1}^2 g_i g_j \int_0^{T_i} dt \int_0^{T_j} dt' \left< x_i(t) \right| \left( \frac{1}{-\partial^2 - m^2} \right) x_j(t') \right>
\]

\[
= -\sum_{i,j=1}^2 \frac{g_i g_j}{8\kappa_0^2} \int_0^{T_i} dt \int_0^{T_j} dt' \int \frac{d^4p}{(2\pi)^4} \exp \left[ -ip \cdot (x_i(t) - x_j(t')) \right]
\]

\[
\times \frac{1}{p^2 - m^2 + i\epsilon}. \quad (3.23)
\]

\(^3\kappa_0 > 0\) reparametrizes the proper time and can be considered as “mass” of the equivalent quantum-mechanical particle. In numerical applications we will set this parameter (more precisely: its euclidean counterpart) to 1 in which case all proper times have mass dimension \(-2\) as can be seen from Eq. (3.17). Another convenient choice is \(\kappa_0 = M\) so that the proper time has the same dimension as the ordinary time to which it reduces in the nonrelativistic limit.
The polarization propagator is therefore given by the double worldline path integral •10•

\[ \Pi(q) = i \int d^4x e^{iq \cdot x} \int_0^\infty \frac{dT_1 dT_2}{(2i\kappa_0)^2} \exp \left[ -\frac{i}{2\kappa_0} \left( M_1^2 T_1 + M_2^2 T_2 \right) \right] \times \prod_{i=1}^2 \left( \int_x \delta(x_i(T_i) - x) \right) \exp \left\{ i \sum_{i=1}^2 S_0[x_i] + i S_{\text{int}}[x_1, x_2] \right\} (3.24) \]

where

\[ S_0[x_i] = \int_0^{T_i} dt \left( -\frac{\kappa_0}{2} \dot{x}_i^2(t) \right), \quad i = 1, 2 \quad (3.25) \]

is the standard free action for each particle.

The perturbative expansion of the polarization propagator is shown in Fig. 1. Note that the worldlines of the particles are parametrized by their proper times over which one has to integrate; thus crossed and ladder-type diagrams are included on an equal footing as are self-energy and vertex corrections.

If Eq. (3.23) is split into terms with \( i = j \) and \( i \neq j \) one sees that the former generate the self-energies of each particle, while the latter describe the interaction between the nucleons by exchange of (any number of) mesons. The vertex corrections come automatically due to different values of the proper times; for example, if one self-energy meson is already “in the air” when another one is emitted to the other particle. A typical diagram which is contained in the compact expression (3.24) is depicted in Fig. 2a. A diagram which is omitted due to the quenched approximation (neglect of pair production) is shown in Fig. 2b. We normalize the path integrals by dividing and multiplying with

\[ i \int d^4x e^{iq \cdot x} \int_{x_1(0)=0}^{x_1(T_1)=x} \mathcal{D}x_1 \int_{x_2(0)=0}^{x_2(T_2)=x} \mathcal{D}x_2 \exp \left\{ i \sum_{i=1}^2 S_0[x_i] \right\} \]

\[ = \left( \frac{\kappa_0}{2\pi(T_1 + T_2)} \right)^2 \exp \left( i \frac{q^2}{2\kappa_0} \frac{T_1 T_2}{T_1 + T_2} \right) \quad (3.26) \]
which is easily obtained from
\[
\int_{x(0)=0}^{x(T)=x} Dx \ e^{iS_0[x]} = \langle x \bigg| \exp \left(i T \frac{p^2}{2\kappa_0}\right) \bigg| x = 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \exp \left(-ip \cdot x + i\frac{p^2T}{2\kappa_0} \right). \tag{3.27}
\]

Thus
\[
\Pi(q) = -\frac{1}{16\pi^2} \int_0^\infty \frac{dT_1 \ dT_2}{(T_1 + T_2)^2} \exp \left[ i \frac{q^2}{2\kappa_0} \frac{T_1 T_2}{T_1 + T_2} - i \frac{q^2}{2\kappa_0} (M_1^2 T_1 + M_2^2 T_2) \right] \\
\times \int \frac{\mathcal{D}(x_1, x_2) \ \exp(i\tilde{S})}{\mathcal{D}(x_1, x_2) \ \exp(i\tilde{S}_0)} \tag{3.28}
\]

where
\[
\int \mathcal{D}(x_1, x_2) \equiv \int d^4x \ e^{i\tilde{S} - i\tilde{S}_0} \int_{x(0)=0}^{x_1(T_1)=x} \mathcal{D}x_1 \int_{x(0)=0}^{x_2(T_2)=x} \mathcal{D}x_2 \tag{3.29}
\]
\[
\tilde{S} \equiv q \cdot x + S, \quad \tilde{S}_0 \equiv q \cdot x + \sum_{i=1}^2 S_0[x_i]. \tag{3.30}
\]

With no interaction the ratio of path integrals in Eq. (3.28) is unity and one obtains the free polarization propagator. Without performing the proper-time integrals in Eq. (3.28) one sees in this case that the exponential vanishes, i.e. there is no damping anymore if the ratio \( r = T_2/T_1 \) of proper times fulfills the quadratic equation
\[
r^2M_2^2 + r \left(M_2^2 + M_1^2 - q^2\right) M_1^2 = 0 . \tag{3.31}
\]
This only has positive, real solutions if

\[ q^2 \geq q_{\text{th}}^2 = (M_1 + M_2)^2. \]  

(3.32)

Hence there are (branch-point) singularities in the free polarization propagator starting from values of the external momentum \( q \) as given in Eq. (3.32): this is just the familiar cut in the polarization propagator which describes the two particles in the continuum. It is a cut because we integrate over two proper times, or equivalently over all the poles generated by one proper-time integration (say \( T_1 \)) depending on \( r \). In the interacting polarization propagator this cut also exists although it is modified in strength but not in position by the interactions between the unbound nucleon 1 and 2. However, for the bound state problem we are looking for an additional pole below \( q_{\text{th}}^2 \). As in the case of the two-point function we expect that such a pole may be generated by the undamped integration of an exponential over one proper time (combination).

4 Variational Calculation for the Worldline Polarization Propagator

We now apply the Feynman-Jensen variational principle to the last factor in Eq. (3.28), i.e. approximate

\[
\frac{\int \mathcal{D} \exp(i\tilde{S})}{\int \mathcal{D} \exp(i\tilde{S}_0)} = \frac{\int \mathcal{D} \exp(i\tilde{S}_1)}{\int \mathcal{D} \exp(i\tilde{S}_0)} \cdot \frac{\int \mathcal{D} \exp(i\tilde{S}_t) \exp(i(\tilde{S} - \tilde{S}_t))}{\int \mathcal{D} \exp(i\tilde{S}_t)}
\]

\[
\simeq \frac{\int \mathcal{D} \exp(i\tilde{S}_1)}{\int \mathcal{D} \exp(i\tilde{S}_0)} \cdot \exp \left[ i \left\langle \tilde{S} - \tilde{S}_t \right\rangle_t \right].
\]

(4.1)

In the last line the average is defined as

\[
\left\langle \tilde{S} - \tilde{S}_t \right\rangle_t = \frac{\int \mathcal{D}(x_1, x_2) \left( \tilde{S} - \tilde{S}_t \right) \exp(i\tilde{S}_t)}{\int \mathcal{D}(x_1, x_2) \exp(i\tilde{S}_t)}.
\]

(4.2)

It is essential for our approach that the RHS of Eq. (4.1) not only yields an approximation but is stationary w.r.t. arbitrary variations of the trial action \( S_t \). Because the path integrals have to be solvable we are restricted to quadratic, but retarded (i.e. non-local) trial actions.

4.1 The Trial Action

In the following we will use the Fourier path integral method as employed originally in refs. [24, 25]. Although there exist more general and elegant treatments [29], the expansion in Fourier modes has the advantage of being simple and straightforward.

It is convenient to rescale the times in the free and the interacting part of the action

\[
t = T_i \tau, \quad t' = T_j \tau'.
\]

(4.3)

\(^4\text{In the following these references will be denoted by (I) and (II), respectively.}\)
with $\tau, \tau' \in [0,1]$. The paths fulfilling the required boundary conditions can then be written as

$$x_i(\tau) = x \tau + \sum_{k=1}^{\infty} \frac{\sqrt{2T_i}}{k\pi} \alpha_k^{(i)} \sin (k\pi \tau)$$

(4.4)

where, of course, the Fourier coefficients $\alpha_k^{(i)}$ are 4-dimensional vectors. The free action becomes

$$S_0 = -\frac{\kappa_0}{2} \sum_{k=1}^{2} \left[ \frac{x^2}{T_i} + \sum_{i=1}^{\infty} \alpha_k^{(i)} \right], \quad \tilde{S}_0 = q \cdot x + S_0.$$  

(4.5)

This serves as a guide for a quadratic trial action which we choose as

$$\tilde{S}_t = \tilde{\lambda} q \cdot x - \frac{\kappa_0}{2} \sum_{i=1}^{2} \left[ A_0 x^2 \frac{1}{T_i} + \sum_{k=1}^{\infty} A_k^{(i)} \alpha_k^{(i)} \right] + \kappa_0 \sum_{k=1}^{\infty} B_k \alpha_k^{(1)} \cdot \alpha_k^{(2)}$$

(4.6)

where the last term accounts for the direct coupling of the two worldlines. Being diagonal in Fourier space this is not the most general quadratic action but in all known cases (polaron, WC model, QED) possible non-diagonal pieces do not contribute in the limit of infinite proper time(s). However, as shown in ref. [29], additional Lorentz structures built from external momenta (here the momentum transfer $q$) can lead to considerable improvements. Here we will not include such refinements and stick to the simple trial action (4.6). As shown in appendix A.2 for the nonrelativistic case, such a trial action leads to the correct center-of-mass energy and to an internal ground-state energy of the system exactly as if evaluated with a variational trial wave function of gaussian type. It may also be noted that in the relativistic case the ansatz (4.6) generalizes the one for a single nucleon, which showed the instability of the WC model in the one-particle sector.

4.2 The Pole of the Variational Polarization Propagator and Mano’s Equation

Since the trial action is at most quadratic in the dynamical variables, all path integrals needed for Feynman’s variational principle can be performed. The calculation of these averages is outlined in appendix B and gives the result (B.16)

$$\Pi^{\text{var}}(q) = -\frac{1}{16\pi^2} \int_0^{\infty} \frac{dT_1dT_2}{(T_1+T_2)^2} \exp \left\{ \frac{i}{2\kappa_0} \left[ -(M_1^2T_1 + M_2^2T_2) + \frac{T_1T_2}{T_1+T_2} q^2 (2\lambda - \lambda^2) - (T_1 + T_2) (\Omega_{12} + V) (T_1, T_2) \right] \right\}$$

(4.7)

where $\Omega_{12}$ and $V$ are defined in Eq. (B.17) and (B.18), respectively. The original variational parameter $\lambda$ has been replaced by

$$\lambda = \frac{\tilde{\lambda}}{A_0}.$$  

(4.8)
As in the case of the 2-point function (or the nonrelativistic polarization propagator) the variational expression simplifies considerably on the pole. However, now we have two times $T_1, T_2$ and it is not clear a priori which one, or which combination should go to infinity in order to produce the (single) pole. For the equal-mass case \[ \text{(4.3)} \] the solution simply is to introduce

$$T := \frac{T_1 + T_2}{2}, \quad s := T_1 - T_2. \tag{4.9}$$

Since $T_1 T_2 / (T_1 + T_2) = T/2 - s^2/(8T)$ we have from Eq. \[ \text{(4.7)} \]

$$\Pi_{\text{var}}(q) = -\frac{1}{64\pi^2} \int_0^\infty dT \exp \left\{ \frac{i}{4\kappa_0} T \left[ -4M_0^2 + q^2 \left( 2\lambda - \lambda^2 \right) + \frac{4\kappa_0}{i T} \ln g(q^2, T) \right] \right\} \tag{4.10}$$

where

$$g(q^2, T) = \frac{1}{T^2} \int_{-2T}^{+2T} ds \exp \left\{ \frac{i}{2\kappa_0} \left[ -\frac{s^2}{8T} q^2 (2\lambda - \lambda^2) - 2T \left( \Omega_{12} + V \right) \left( T + \frac{s}{2}, T - \frac{s}{2} \right) \right] \right\}. \tag{4.11}$$

It is obvious that an additional pole develops if $\ln g(q^2, T)$ has terms linear in $T$ for large $T$. This happens if both $\Omega_{12}$ and $V$ are independent of $T$ and $s$ in the limit $T \to \infty$. Then

$$\frac{4\kappa_0}{i T} \ln g(q^2, T) \xrightarrow{T \to \infty} -4 \left( \Omega_{12} + V \right) - \frac{4\kappa_0}{i} \frac{T^2}{T} + O\left( \frac{1}{T} \right) \tag{4.12}$$

because the remaining $s$-integral is convergent.

The proof that $\Omega_{12}, V$ become constants in the large $T$-limit follows along the same lines as in the 2-point case: for $T \to \infty$ we may assume that the coefficients $A_k = A(k\pi/T)$ and that the sums over Fourier coefficients become integrals over $E = k\pi/T$

$$\sum_k f(A_k) \longrightarrow \int_0^\infty dk f \left( A \left( \frac{k\pi}{T} \right) \right) = \frac{T}{\pi} \int_0^\infty dE f \left( A(E) \right). \tag{4.13}$$

In this way we obtain from Eq. \[ \text{(B.17)} \]

$$\Omega_{12} = \frac{d\kappa_0}{2i\pi} \int_0^\infty dE \left[ \ln \left( A^{(1)}(E)A^{(2)}(E) - B^2(E) \right) \right. \right.$$  

$$\left. + \frac{A^{(1)}(E) + A^{(2)}(E)}{A^{(1)}(E)A^{(2)}(E) - B^2(E)} - 2 \right]. \tag{4.14}$$

When the coupling function $B(E)$ vanishes we find that $\Omega_{12}|_{B=0} = \Omega_1 + \Omega_2$, i.e. the kinetic term reduces to the sum of usual kinetic terms for the self-energy of the individual nucleons (see ref. (I)).
To evaluate $V$ from Eq. (B.18) we go back to the original (unscaled) times $t_1 = T_1 \tau, t_2 = T_2 \tau'$ and note that for

$$\sigma := t_1 - t_2, \quad \Sigma := \frac{t_1 + t_2}{2} \implies \int_0^T dt_1 \int_0^T dt_2 \ldots = \int_{-\infty}^{+\infty} d\sigma \int_{\left|\sigma\right|/2}^{T-\left|\sigma\right|/2} d\Sigma \ldots$$

The integral over the relative time $\sigma$ remains whereas the integral over the total time $\Sigma$ produces the needed factor $T$. Therefore we obtain

$$V = \sum_{i,j=1}^{2} \frac{g_i g_j}{8\kappa_0} \int_{-\infty}^{+\infty} d\sigma \int d^4 p \frac{1}{(2\pi)^4} \frac{1}{p^2 - m^2 + i0} \exp \left\{ - \frac{i}{2\kappa_0} \left[ p^2 \mu_{ij}^2(\sigma) - \lambda p \cdot q \sigma \right] \right\}. \quad (4.16)$$

This only works if the quantity $\mu_{ij}^2$ defined in Eq. (B.15) becomes independent of the total time $\Sigma$ in the limit $T \to \infty$. Inspection of Eq. (B.15) together with the addition theorems of trigonometric functions shows that this is indeed the case:

$$\mu_{ij}^2(t_1, t_2; T_1, T_2) = \mu_{ij}^2(\sigma, \Sigma; s, T) \xrightarrow{T \to \infty} \mu_{ij}^2(\sigma) \quad (4.17)$$

with

$$\mu_{ij}^2(\sigma) = \frac{2}{\pi} \int_0^{\infty} dE \frac{1}{E^2} \frac{A^{(1)}(E) A^{(2)}(E) - B^2(E)}{A^{(1)}(E) A^{(2)}(E) - B^2(E) - B(E) \cos (E\sigma)} \left\{ 2 \delta_{ij} A^{(3-i)}(E) \sin^2 \left( \frac{E\sigma}{2} \right) + (1 - \delta_{ij}) \right\}. \quad (4.18)$$

This is the generalization of the pseudotime $\mu^2(\sigma)$ for the self-energy of a single nucleon encountered in previous work. Even the diagonal ($i = j$)-part now depends on both profile functions as well as on the coupling function $B(E)$. If the coupling vanishes then $\mu_{ii}^2$ reduces to the usual pseudotime for the $i$th particle. Similarly, the off-diagonal ($i \neq j$)-part also depends on both profile functions $A^{(1)}, A^{(2)}$. The mutual dependence reflects the change of properties of the one particle in the presence of the other (“medium effect”) and the vertex corrections mentioned before. Note also the missing factor 2 in front of the last term in the exponential of Eq. (4.16) compared to the single particle (2-point function) case. This means that on the average each particle in the momentum loop of the polarization propagator has momentum $q/2$ and – since $(q/2)^2 < M^2$ in the bound state – is slightly off-shell. Finally, it is evident that $\mu_{ij}^2 = \mu_{ji}^2$.

Collecting all terms in the exponent of Eq. (4.16) which are linear in $T$ we obtain (what we usually call) Mano’s equation

$$0 = -4M_0^2 + q^2 (2\lambda - \lambda^2) - 4 \left( \Omega_{12} + V \right). \quad (4.19)$$

determining a possible additional pole in the polarization propagator. In view of the previous remark it may also be written as

$$\left( \frac{q}{2} \right)^2 (2\lambda - \lambda^2) = M_0^2 + \Omega_{12} + V \quad (4.20)$$
which very much resembles Mano’s equation for the 2-point function. The missing factor 2 on the RHS in front of the kinetic and potential terms is contained in the definitions (B.17,B.18). One may split up the interaction as

\[ V = : \sum_{i=1}^{2} V_{ii} + 2 V_{12} \]  

(4.21)

with

\[ V_{ij} = \frac{g_i g_j}{4 \kappa_0} \int_0^\infty d\sigma \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i0} \exp \left[ \frac{i}{2\kappa_0} \left( p^2 \mu_{ij}(\sigma) - \lambda p \cdot q \sigma \right) \right]. \]  

(4.22)

Here we have used the fact that the pseudotime is even to restrict the \( \sigma \)-integration to the interval \([0, \infty]\). One may call \( V_{ii} \) the self-energy part and \( V_{12} \) the binding part of the interaction, although these effects are coupled as mentioned before.

Let us discuss Mano’s Eq. (4.20) qualitatively: without the kinetic and potential terms one would obtain \( \lambda = 1 \) as a stationary solution (as in the non-relativistic case) and \( q^2_* = q^2_{th} = 4M^2 \) would determine the beginning of the continuum branch-cut. The effects of the interaction can best be estimated in the euclidean formulation (\( \kappa_0 = i\kappa_E \)). Then for small coupling

\[ \Omega_{12} \approx \frac{d\kappa_E}{2\pi} \int_0^\infty dE \left\{ \sum_{i=1}^{2} \left[ \ln A^{(i)} + \frac{1}{A^{(i)}} - 1 \right] + \frac{B^2}{A^{(1)} A^{(2)}} \left( \frac{1}{A^{(1)}} + \frac{1}{A^{(2)}} - 1 \right) \right. \]

\[ + \mathcal{O}(B^4) \right\} \]  

(4.23)

which is positive since \( \ln A + 1/A - 1 \geq 0 \) for \( A > 0 \) and the second term adds for \( A^{(i)} < 2 \). Thus the kinetic term tends to push \( q^2_* \) above the threshold value. On the other hand, from Eq. (4.16) the potential term can be seen to be attractive: \( V < 0 \) since \( dp_0 = idp_4 \) and \( p^2 - m^2 = -(p^2_E + m^2) \). This is the usual competition between kinetic and potential energy and one may therefore expect a solution of Mano’s equation with \( q^2_* < 4M^2_{\text{phys}} \), i.e. a truly relativistic bound state for sufficient strong coupling. Since we know that the instability of the WC model prevents large coupling, the nonrelativistic criterion (2.7) does not apply anymore.

4.3 Regularization and Renormalization

As usual in quantum field theory divergences show up and have to be absorbed in physical, i.e. measured quantities (renormalization). Fortunately, the Wick-Cutkosky model is a super-renormalizable theory (Eq. (2.2) shows that the coupling constants \( g_i \) have the dimension of a mass) and in the quenched approximation only a mass renormalization is necessary. This has been done already for the 2-point function in refs. (I,II) and, indeed, the source of the divergence is again the self-energy of the particles. To be more specific we expect that only the diagonal pseudotime vanishes at small relative times:

\[ \mu_{ii}(\sigma) \xrightarrow{\sigma \to 0} \sigma \]  

(4.24)
In contrast, the non-diagonal part of the pseudotime stays constant (see appendix C):

\[ \mu_{12}^2(\sigma) \xrightarrow{\sigma \to 0} \text{const.} \]  

(4.25)

This means that for \( \sigma \to 0, p \to \infty \) the \( p \)-integrand for \( V_{ii} \) (but not for \( V_{12} \) !) loses its exponential suppression which leads to a (large \( p \) or small \( \sigma \)) UV-divergence. The solution is straightforward and simple (compared to a real, i.e. renormalizable theory with a dimensionless coupling constant): regularize the divergent expression, isolate the divergence and combine it with the (squared) bare mass of the particle. In ref. (I) we simply subtracted a particular term with the meson mass as a scale.

Another possibility for regularization is a cut-off at small proper time (proper-time regularization), i.e.

\[ \int_0^\infty d\sigma \ldots \to \int_1^{1/\Lambda^2} d\sigma \ldots \]  

(4.26)

It is understood that at the end of the calculation we let the cut-off \( \Lambda \) (with dimension mass squared) tend to infinity. We then write the regularized expression for \( V_{ii} \) as

\[
V_{ii} = \frac{g_i^2}{4\kappa_0} \int_{1/\Lambda^2}^\infty d\sigma \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i0} \exp \left[ \frac{i}{2\kappa_0} p^2 \sigma \right] 
\]

\[
+ \frac{g_i^2}{4\kappa_0} \int_{1/\Lambda^2}^\infty d\sigma \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i0} \left\{ \exp \left[ \frac{i}{2\kappa_0} (p^2 \mu_{ii}^2(\sigma) - \lambda p \cdot q \sigma) \right] 
- \exp \left[ \frac{i}{2\kappa_0} p^2 \sigma \right] \right\} =: V_{ii}^{\text{sing}} + V_{ii}^{\text{reg}}. \]  

(4.27)

The singular part can be evaluated easily:

\[
V_{ii}^{\text{sing}} = \frac{g_i^2}{4\kappa_0} \int_{1/\Lambda^2}^\infty d\sigma \int_0^{\infty} du' \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\kappa_0 i} \exp \left\{ \frac{i}{2\kappa_0} \left[ (p^2 - m^2)u' + p^2 \sigma \right] \right\} 
\]

\[
= - \frac{g_i^2}{32\pi^2} \int_0^{\infty} du' \frac{1}{u' + 1/\Lambda^2} \exp \left[ - \frac{i}{2\kappa_0} m^2 u' \right] = - \frac{g_i^2}{32\pi^2} \varepsilon E_1(z) \]  

(4.28)

where \( z = i m^2/(2\kappa_0 \Lambda^2) \) and \( E_1(z) \) is the exponential integral (ref. [42], Chap. 5). Using its expansion for small arguments (i.e. large cut-off \( \Lambda \)) one obtains

\[
V_{ii}^{\text{sing}} \bigg|_{\text{prop. time reg.}} \xrightarrow{\Lambda \to \infty} - \frac{g_i^2}{32\pi^2} \ln \left( \frac{\Lambda^2}{m^2} \right) + C \]  

(4.29)

where

\[
C \bigg|_{\text{prop. time reg.}} = \frac{g_i^2}{32\pi^2} \left[ \gamma_E + \ln \left( \frac{i}{2\kappa_0} \right) \right] \]  

(4.30)

is a constant specific for this regularization scheme. The divergent part plus the constant (or part of it) can be absorbed in the bare mass of each particle by defining the finite (intermediate) mass

\[
\tilde{M}_i^2 = M_0^2 - \frac{g_i^2}{16\pi^2} \ln \left( \frac{\Lambda^2}{m^2} \right) + 2C. \]  

(4.31)
Variational Worldline Approximation for the Relativistic Bound State

For $g_1 = g_2 = 2g, C = 0$ this is exactly the quantity $^5$ which has been calculated in refs. (I,II) by requiring a pole of the 2-point function at the physical mass $M$. The remaining, regularized interaction just provided a finite mass shift to the observed mass of the particle. Note that in $V_1^{\text{reg}}$ we now can send the cut-off to infinity since the integrals are convergent by construction. In this way all traces of the cut-off have disappeared, or better are hidden in the masses $\bar{M}_i$. As can be seen in Eq. (4.30) proper-time and other regularization methods with a cut-off should only be used in euclidean time ($\kappa_0 = i\kappa_E$) to make everything real. Other regularization schemes do not need that restriction: for example, after performing the $\sigma$-integration, one has in $d = 4 - 2\varepsilon$ dimensions

$$V_{ii}^{\text{sing}}|_{\text{dim}} = -\frac{g_i^2}{32\pi^2} \left( \frac{4\pi\nu^2}{m^2} \right)^\varepsilon \frac{\Gamma(\varepsilon)}{1 - \varepsilon} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 - m^2 + i0} \frac{2i\kappa_0}{p^2}$$

(4.32)

where $\nu$ is a mass parameter introduced for keeping the mass dimension of the coupling constant fixed in arbitrary dimensions. The $p$-integral is a standard one (see, e.g. appendix B in ref. [43]) and one obtains

$$V_{ii}^{\text{sing}}|_{\text{dim}} = -\frac{g_i^2}{32\pi^2} \left[ \frac{1}{\varepsilon} + 1 + \gamma_E + \ln(4\pi) + \ln \left( \frac{\nu^2}{m^2} \right) + \mathcal{O}(\varepsilon) \right]$$

(4.33)

which is independent of $\kappa_0$ and real. As usual one may combine the bare mass with the $1/\varepsilon$-divergence leading to the minimal subtraction (MS) scheme or include some of the constants (MS). In the Pauli-Villars regularization one stays in $d = 4$ dimensions but modifies the meson propagator

$$\frac{1}{p^2 - m^2 + i0} \rightarrow \frac{1}{p^2 - m^2 + i0} - \frac{1}{p^2 - \Lambda^2 + i0}.$$ 

(4.34)

This just amounts subtracting from Eq. (4.33) the same expression with $m^2 \rightarrow \Lambda^2$. Hence one obtains the same result as in Eq. (4.29) with no additional constant.

Whatever regularization one chooses, the divergent part of $V_{ii}$ can be combined with the (squared) bare masses of the nucleons. Of course, there is the ambiguity of adding possible constants to the singular part of the interaction and subtracting them from the regular part, leading to different renormalization schemes. The scheme used previously did not have a constant term in Eq. (4.29) and we will also choose this convention. However, it should be stressed that this only affects the relation between the unobservable bare mass and the intermediate mass and all physical results are independent of this choice.

### 4.4 Special Cases

Let us now consider the case where the two particles not only have the same mass but also the same coupling constant

\[ g_1 = g_2 \equiv g'. \] 

(4.35)

\(^5\)Note that in these references the intermediate mass has been called $M_1$. To avoid confusion with the bare mass of particle 1 we use $\bar{M}_1$, etc. in the present work.
Their dynamics being the same we can assume

\[ A^{(1)}(E) = A^{(2)}(E) \equiv A(E) \]  

(4.36)

and Mano’s Eq. reads

\[ \bar{M}^2 = \left( \frac{g}{2} \right)^2 (2\lambda - \lambda^2) - \Omega_{12} - 2 (V_{11}^{reg} + V_{12}) . \]  

(4.37)

Here

\[ \Omega_{12} = \frac{d\kappa_0}{2\pi i} \int_0^\infty dE \left\{ \ln \left[ A^2(E) - B^2(E) \right] + \frac{2A(E)}{A^2(E) - B^2(E)} - 2 \right\} \]  

(4.38)

and

\[ V_{1j} = \frac{g^2}{4\kappa_0} \int_0^\infty d\sigma \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i0} \left\{ \exp \left[ \frac{i}{2\kappa_0} \left( p^2 \mu_{1j}^2(\sigma) - \lambda p \cdot q\sigma \right) \right] - \delta_{1j} \exp \left[ \frac{i}{2\kappa_0} p^2 \sigma \right] \right\} \]  

(4.39)

for \( j = 1, 2 \). The pseudotimes are related to the profile functions by

\[ \mu_{11}^2(\sigma) = \frac{4}{\pi} \int_0^\infty dE \frac{A(E)}{E^2 A^2(E) - B^2(E)} \sin^2 \left( \frac{E\sigma}{2} \right) \]  

(4.40)

\[ \mu_{12}^2(\sigma) = \frac{2}{\pi} \int_0^\infty dE \frac{1}{E^2 A^2(E) - B^2(E)} \left[ A(E) - B(E) \cos(E\sigma) \right] . \]  

(4.41)

It is very useful to introduce the combinations

\[ A_{\pm}(E) : = A(E) \pm B(E) \]  

(4.42)

because then the kinetic term separates in two distinct pieces

\[ \Omega_{12} = \Omega[A_-] + \Omega[A_+] . \]  

(4.43)

Here

\[ \Omega[A] = \frac{d\kappa_0}{2i\pi} \int_0^\infty dE \left[ \ln A(E) + \frac{1}{A(E)} - 1 \right] \]  

(4.44)

is just the usual kinetic term encountered in the self-energy of a single nucleon (see (I)). The pseudotimes now become

\[ \mu_{11}^2(\sigma) = \frac{2}{\pi} \int_0^\infty dE \frac{1}{E^2} \left[ \frac{1}{A_-(E)} + \frac{1}{A_+(E)} \right] \sin^2 \left( \frac{E\sigma}{2} \right) \]  

(4.45)

\[ \mu_{12}^2(\sigma) = \frac{2}{\pi} \int_0^\infty dE \frac{1}{E^2} \left[ \frac{\sin^2(E\sigma/2)}{A_-(E)} + \frac{\cos^2(E\sigma/2)}{A_+(E)} \right] . \]  

(4.46)

Note the appearance of the cosine function in the ‘interaction pseudotime’ \( \mu_{12}^2 \) which leads to a finite value at \( \sigma = 0 \), in contrast to the ‘self-energy pseudotime’ \( \mu_{11}^2 \) which vanishes at that point. As explained before this requires a subtraction...
in $V_{11}$ but not in the interaction part $V_{12}$. We may perform the $p$-integration in the usual way by exponentiating the meson propagator (cf. Eq. (4.28)). Then we obtain explicitly in euclidean time ($\kappa_0 = i\kappa E$)

$$V_{11}^{\text{reg}} = -\frac{\alpha}{2\pi} M^2 \int_0^\infty d\sigma \int_0^1 du \left[ \frac{1}{\mu_{11}^2(\sigma)} e \left( m \mu_{11}(\sigma), \frac{\sigma \lambda q/\mu_{11}(\sigma)}{\mu_{11}(\sigma)}, u \right) - \frac{1}{\sigma} e \left( m \sqrt{\sigma}, 0, u \right) \right]$$

(4.47)

$$V_{12} = -\frac{Z\alpha}{2\pi} M^2 \int_0^\infty d\sigma \frac{1}{\mu_{12}^2(\sigma)} \int_0^1 du e \left( m \mu_{12}(\sigma), \frac{\sigma \lambda q/\mu_{12}(\sigma)}{\mu_{12}(\sigma)}, u \right)$$

(4.48)

where

$$e \left( m \mu_{1j}(\sigma), \frac{\sigma \lambda q/\mu_{1j}(\sigma)}{\mu_{1j}(\sigma)}, u \right) = \exp \left\{ -\frac{1}{2\kappa E} \left[ m^2 \mu_{1j}^2(\sigma) \frac{1-u}{u} + \frac{(\lambda q/2)^2 \sigma}{\mu_{1j}^2(\sigma)} u \right] \right\} =: e_j(u, \sigma).$$

(4.49)

is the function used in refs. (I,II). Here $q/2$ is a shorthand notation for $\sqrt{q/2}$ and the dimensionless coupling constant as defined in Eq. (2.2) is given by

$$\alpha = \frac{g^2}{16\pi M^2}. \quad (4.50)$$

In Eq. (4.47) we now explicitly use the renormalization scheme of ref. (I), i.e. subtract a simple exponential in the proper-time integral which has the meson mass $m$ as a scale. Subtraction at a different scale $\nu$ (which, e.g. is needed for $m = 0$) would add the term

$$\frac{\alpha}{2\pi} M^2 \int_0^\infty d\sigma \frac{1}{\sigma} \int_0^1 du \left[ \exp \left( -\frac{\nu^2}{2\kappa E} \frac{1-u}{u} \sigma \right) - \exp \left( -\frac{m^2}{2\kappa E} \frac{1-u}{u} \sigma \right) \right] = \frac{\alpha}{2\pi} M^2 \int_0^1 du \ln \frac{m^2}{\nu^2} = \frac{\alpha}{2\pi} M^2 \ln \frac{m^2}{\nu^2} \equiv C_\nu$$

(4.51)

on the RHS of Eq. (4.47). This is exactly the same term as obtained in Eq. (4.33) directly from dimensional regularization and amounts to choosing another constant $C$ in Eq. (4.31). Therefore one obtains a new intermediate mass

$$\bar{M}^2(\nu) = M_0^2 - \frac{\alpha}{\pi} M^2 \ln \left( \frac{\Lambda^2}{\nu^2} \right)$$

(4.52)

which is independent of $m$ but $\nu$-dependent. Obviously, the scale $\nu$ is arbitrary and the physical observables do not depend on it. Note also that $\Omega_{12}$ and $V_{1j}$ are independent of the reparametrization parameter $\kappa E$. This is because profile functions and pseudotimes scale as

$$A_{\pm}(\kappa E, E) \equiv A_{\pm}(\kappa E E) \implies \mu_{1j}^2(\kappa E, \sigma) \equiv \kappa_E^2 \mu_{1j}^2(\sigma/\kappa_E)$$

(4.53)

where the last relation follows from Eqs. (4.35), (4.40). Using the explicit forms of kinetic terms and potentials it is easily seen that these quantities do not depend
on the parameter $\kappa_E$. Mano’s equation then implies that also the variational parameter $\lambda$ is reparametrization-invariant.

In addition, in Eq. (4.48) we have introduced an artificial strength factor $Z$ which in the end has to be set to one but allows one to distinguish between the binding part and the radiative corrections. This is standard practice in QED bound-state calculations for atomic systems including hydrogen. In the present case, however, the factor $Z$ has no physical meaning since it does not reflect the different coupling constant of one of the particles; otherwise we would have to include it in the corresponding self-energy of this particle. With this artificial parameter to switch on and off one now can study several special cases:

(i) $Z = 0$ : Since there is no dependence on $\mu_{12}^2$ and $\mu_{11}^2$ is symmetric in $A_-, A_+$ (see Eq. (4.45)) both profile functions give the same contribution and Mano’s Eq. (13.7) for the 2-body case becomes identical to the one for the self-energy of a single nucleon.

(ii) $\alpha = 0, Z\alpha \neq 0$ : This corresponds to the case studied by practically all relativistic bound-state approaches (with the exception of atomic systems where one does perturbative QED calculations): neglect self-energy and vertex corrections. Naively one then expects no sign of an instability in the WC model but we will see that this is not the case. Since now $\bar{M} = M$ Mano’s Eq. simplifies to

$$M^2 = (q^2)^2 (2\lambda - \lambda^2) - \Omega[A_-] - \Omega[A_+] - 2V_{12}.$$ (4.54)

Still, when varying this equation one includes all crossed exchange diagrams (in variational approximation with our quadratic trial action ...), not just iteration of the ladder diagrams. It is unclear how one can simulate the ladder Bethe-Salpeter approximation in the present worldline approach as it naturally contains all orderings of internal lines. Similar remarks apply to approximations where a 3-dimensional reduction has been performed (6) or one particle has been put on the mass-shell (7).

(iii) $c \to \infty$ : For the nonrelativistic limit one could re-introduce the velocity of light $c$ and let it tend to infinity as in appendix A.1. A simpler method is to perform the limit $M \to \infty$ in the unsubtracted expression keeping the coupling constant $\alpha$ fixed. Setting $q = (2M + \epsilon, q)$ one sees that the term $p \cdot q$ in the exponential forces $p_0 \to 0$ in all other terms. The $p_0$-integration then produces a factor

$$2\pi \delta \left( \frac{\lambda q_0 \sigma}{2\kappa_0} \right) = 2\pi \delta(\sigma),$$ (4.55)

i.e. an instantaneous pion exchange. Therefore

$$V_{ij} \xrightarrow{c \to \infty} -4\pi\lambda_0 \alpha M^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 + m^2} \exp \left[ -\frac{i}{2\kappa_0} \mu_{1j}^2 \mu_{1j}^2(0) \right].$$ (4.56)

Because $\mu_{11}^2(0) = 0$ the unregularized self-energy interaction $V_{11}$ now is just a (divergent) constant which changes $M_0^2 \to M^2$ but has no effect on
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the dynamics of the particles (we anticipate \( \lambda \rightarrow 1 \) for \( M \rightarrow \infty \)). Writing

\[
\mu_1^2(0) = : 1/\omega \quad \text{Mano’s Eq. (4.37)} \]

then reads in the nonrelativistic limit

\[
M^2 = (M^2 + \epsilon_q M - q^2/4)^2 (2\lambda - \lambda^2) - \Omega_- - \Omega_+ - 2V_{12}(\omega) \quad (4.57)
\]

from which we immediately deduce by variation that \( A_-(E) = 1 \) as \( V_{12} \) does not depend on it anymore. Hence \( \Omega_- = 0 \). Assuming \( V_{12}(\omega) \ll M^2 \) one finds \( \lambda \rightarrow 1 \) as expected and therefore for the total nonrelativistic energy

\[
\epsilon_q = \frac{q^2}{4M} + \frac{1}{M} \left[ \Omega_+ + 2V_{12}(\omega) \right] . \quad (4.58)
\]

Variation w.r.t. \( A_+(E) \) shows that this profile function has the familiar (euclidean) form \( A_+(E) = 1 + \omega^2/E^2 \), in agreement with the nonrelativistic variational calculation in appendix A.2 (see Eq. A.50 for \( M_1 = M_2 \)). Consequently, Mano’s equation for the internal energy \( \epsilon_0 \) (with \( \kappa_0 = i\kappa_E \))

\[
\epsilon_0 = \frac{d}{4M} \kappa_E \omega - 4\pi Z\alpha \int \frac{d^3p}{(2\pi)^3} \frac{1}{\mathbf{p}^2 + m^2} \exp \left[ -\frac{\mathbf{p}^2}{2\kappa_E \omega} \right] \quad (4.59)
\]

is the correct nonrelativistic variational equation (A.55) for the Yukawa potential if we put \( d = 3 \). For the Coulomb case \((m = 0)\) where all calculations can be done analytically this gives the results

\[
\kappa_E \omega = \frac{8}{d^2\pi} (Z\alpha)^2 M^2 \quad (4.60)
\]

\[
\epsilon_0 = -\frac{4}{d\pi} (Z\alpha)^2 M^2/2 . \quad (4.61)
\]

However, there is no compelling reason to reduce the dimensionality to \( d = 3 \) by fiat and therefore we conclude that the nonrelativistic limit of the variational calculation is different from the variational calculation in the nonrelativistic limit which starts from the beginning with \( d = 3 \). This phenomenon can be traced back to the ansatz (4.6) for the trial action which is covariant but too rigid to allow a different treatment of time and space-like dynamical variables needed for the correct nonrelativistic limit. An obvious remedy is to give the trial action more freedom, for example by allowing more general Lorentz structures for the profile functions \( A(E) \rightarrow A^{\mu\nu}(E) = A_L(E) q^\mu q^\nu/q^2 + A_T(E) (g^{\mu\nu} - q^\mu q^\nu/q^2) \) and similarly for \( B(E) \). Indeed, the kinetic term then splits into \( \Omega_L/d + (d-1)\Omega_T/d \) (see Eq. (19) in ref. [29]) which would provide the necessary dimensional reduction for the nonrelativistic limit without destroying the covariance of the whole approach. Actually, as shown in appendix C of ref. [29], the variational principle demands such a general form if the profile functions are left completely free.

(iv) \( \alpha \rightarrow 0 \): Although our main interest is in the strong-coupling case a detailed investigation of the weak-coupling limit is worthwhile, for example to see whether the logarithmic terms of the ladder Bethe-Salpeter approximation (see, e.g. Eq. (10-78) in ref. [35]) also appear in the present approach.
Since the nonrelativistic limit coincides with the weak-coupling limit we can expect that Eq. (4.61) with \( d = 4 \) is the leading term for \( m = 0 \). Indeed, as shown in appendix 6.3, the binding energy in the worldline variational approach has the expansion

\[
\frac{\epsilon_0}{M/2} = \frac{(Z\alpha)^2}{\pi} \left[ 1 + \frac{7}{2} \frac{\alpha}{\pi} + \ldots \right] - \frac{(Z\alpha)^4}{\pi^2} \left[ 1 + \ldots \right] - \ldots \tag{4.62}
\]

This shows an increased binding caused both by radiative effects (indicated by the \( \alpha/\pi \)-terms) and by relativistic corrections (indicated by the \( (Z\alpha)^4 \)-term). Further discussion is postponed until Sect. 6.2.

5 Variational Equations

Here we derive the variational equations for the \( \lambda \)-parameter and profile functions \( A_{\pm}(E) \) from Mano’s equation (4.37). As this is straightforward we just quote the results: assuming that \( \kappa_E \) does not depend on \( \lambda \) variation w.r.t. \( \lambda \) gives

\[
\lambda = 1 - \frac{4}{q^2} \frac{\partial}{\partial \lambda} (V_{11} + V_{12}) \tag{5.1}
\]

with

\[
\frac{\partial V_{ij}}{\partial \lambda} = Z^{j-1} \frac{\alpha}{2\pi} M^2 \lambda \frac{q^2}{4\kappa_E} \int_0^\infty d\sigma \frac{\sigma^2}{\mu_{1j}^4(\sigma)} \int_0^1 du u e_j(u, \sigma). \tag{5.2}
\]

The explicit \( \lambda \)-factor in the derivative can be combined with the LHS of Eq. (5.1) to obtain the variational Eq. for \( \lambda \) in the form (cf. Eq. (139) in ref. [I]).

\[
\lambda = \left[ 1 + \frac{\alpha}{2\pi \kappa E} M^2 \int_0^\infty d\sigma \frac{\sigma^2}{\mu_{1j}^4(\sigma)} \int_0^1 du u \sum_{j=1}^2 Z^{j-1} \frac{\sigma^2}{\mu_{1j}^4(\sigma)} e_j(u, \sigma) \right]^{-1} \tag{5.3}
\]

This is still a nonlinear equation but is expected to better converge under iteration as part of the interaction is already included. In addition, Eq. (5.3) explicitly shows that \( 0 < \lambda \leq 1 \) since the integrand is positive.

Variation w.r.t. the profile functions \( A_{\pm}(E) \) gives

\[
A_-(E) = 1 + \frac{2}{\kappa_E E^2} \int_0^\infty d\sigma \frac{\sum_{j=1}^2 \delta V_{1j}}{\delta \mu_{1j}^2(\sigma)} \sin^2 \left( \frac{E\sigma}{2} \right) \tag{5.4}
\]

\[
A_+(E) = 1 + \frac{2}{\kappa_E E^2} \int_0^\infty d\sigma \left[ \frac{\delta V_{11}}{\delta \mu_{1j}^2(\sigma)} \sin^2 \left( \frac{E\sigma}{2} \right) + \frac{\delta V_{12}}{\delta \mu_{1j}^2(\sigma)} \cos^2 \left( \frac{E\sigma}{2} \right) \right] \tag{5.5}
\]

The functional derivative of \( V_{ij} \) w.r.t. the pseudotimes is easily found to be

\[
\frac{\delta V_{1j}}{\delta \mu_{2j}^2(\sigma)} = \frac{Z^{j-1} \alpha}{2\pi} \frac{M^2}{\mu_{1j}^4(\sigma)} \int_0^1 du \left[ 1 + \frac{m^2}{2\kappa_E \mu_{1j}^2(\sigma)} \frac{1-u}{u} - \frac{(\lambda q/2)^2 \sigma^2}{2\kappa_E \mu_{1j}^2(\sigma)} u \right] e_j(u, \sigma) \tag{5.6}
\]

\(^{6}\)One can choose a \( \lambda \)-dependent reparametrization parameter to simplify the potential energy term \( V \) at the price of making the kinetic term \( \Omega_{12} \) \( \lambda \)-dependent. Due to the virial theorem (see below) nothing new is obtained by such a procedure.
which should be compared with Eq. (140) in ref. [I]. With a suitable integration by parts the result can also be written as

$$\frac{\delta V_{1j}}{\delta \mu^{2}_{1j} (\sigma)} = Z^{j-1} \frac{\alpha}{2\pi M} \frac{1}{\mu^{1}_{1j} (\sigma)} \int_{0}^{1} du \, u \left[ 2 - \frac{\lambda q / 2 \sigma^{2}}{2\kappa E \mu^{1}_{1j} (\sigma)} u \right] e_{j} (u, \sigma)$$

(5.7)

and is particularly simple for massless pions:

$$\frac{\delta V_{1j}}{\delta \mu^{2}_{1j} (\sigma)} \bigg|_{m=0} = Z^{j-1} \frac{\alpha}{2\pi M} \frac{1}{\mu^{1}_{1j} (\sigma)} \exp \left[ -\frac{(\lambda q / 2 \sigma^{2})^{2}}{2\kappa E \mu^{1}_{1j} (\sigma)} \right].$$

(5.8)

Note that we have to deal with one-dimensional but highly non-linear integral equations. However, even without solving them they can be used to deduce the behaviour of $A_{\pm} (E), \mu^{2}_{1j} (\sigma)$ for small and large values of its arguments. This is derived in appendix C. Here we just summarize the results: at large $E$ both profile functions approach unity in the same way:

$$A_{\pm} (E) \xrightarrow{E \to \infty} 1 + \frac{\alpha}{4} \frac{M^{2}}{\kappa E} + \ldots. \quad (5.9)$$

The same holds for the pseudotimes in the limit $\sigma \to \infty$

$$\mu^{2}_{1j} (\sigma) \xrightarrow{\sigma \to \infty} \frac{\sigma}{2A_{-}(0)} + \frac{1}{\pi} \int_{0}^{\infty} dE \, \frac{1}{E^{2}} \left[ \frac{1}{A_{-}(E)} - \frac{1}{A_{-}(0)} + \frac{1}{A_{+}(E)} \right] + \ldots. \quad (5.10)$$

For $E, \sigma \to 0$ there is an essential difference between the self-energy and the interaction part: whereas $A_{-} (E)$ approaches a constant value $A_{-} (0)$, $A_{+} (E)$ diverges for small $E$:

$$A_{+} (E) \xrightarrow{E \to 0} \omega^{2}_{\text{var}} \frac{E}{E^{2}} + \text{const.} + \ldots, \quad \omega^{2}_{\text{var}} := \frac{2}{\kappa E} \int_{0}^{\infty} d\sigma \frac{\delta V_{12}}{\delta \mu^{2}_{12} (\sigma)}. \quad (5.11)$$

Therefore this term cannot be considered as “small” but has to be kept consistently. This is, of course, what we expect for a bound state which cannot be reached by perturbation theory from the free solution. Similarly, the pseudotimes behave very differently for small $\sigma$: $\mu^{2}_{11} (\sigma) \to \sigma$ as usual, but

$$\mu^{2}_{12} (\sigma) \xrightarrow{\sigma \to 0} \frac{2}{\pi} \int_{0}^{\infty} dE \, \frac{1}{E^{2} A_{+} (E)} \equiv \mu^{2}_{12} (0). \quad (5.12)$$

tends to a constant. This different behaviour in the UV-region was crucial for the mass renormalization in the WC model.

Given the solution of the variational equation it is possible to express the kinetic term $\Omega_{12} = \Omega_{-} + \Omega_{+}$ not only from its definition in terms of $A_{\pm} (E)$ but also from a “virial theorem” in which case it is given in terms of the pseudotimes. Indeed, following the derivation given in appendix E of ref. [29] one can derive straightforwardly that

$$\Omega^{\text{vir}}_{12} = 2 \int_{0}^{\infty} d\sigma \sum_{j=1}^{2} \frac{\delta V_{1j}}{\delta \mu^{2}_{1j} (\sigma)} \left[ \mu^{2}_{1j} (\sigma) - \sigma \frac{\partial \mu^{2}_{1j} (\sigma)}{\partial \sigma} \right].$$

(5.13)
Note that this relation only holds after variation and for the sum of the two kinetic terms. It is also possible to derive expressions for the individual terms but they are more involved and will not be considered here. The factor of “2” in front of Eq. (5.13) is due to our definition of \( \mu^2 \) in Eqs. (4.45,4.46), whose normalization is reduced by one half (compared to the usual definition of the pseudotime) but where one has to sum over two profile functions. Indeed, if either \( V_{11} = 0 \) or \( V_{12} = 0 \) one obtains the standard virial theorem in the polaron approach (see ref. (II)).

Another useful relation is obtained by differentiating Mano’s equation with respect to some parameter, say the artificial coupling strength \( Z \), and using the variational equations to greatly simplify the result. For this purpose we first note that the interaction potentials depend on the combination

\[
x := \lambda \frac{q}{2} \equiv \lambda \frac{\sqrt{q^2}}{2}
\]

and, of course, functionally on the pseudotimes, i.e. on the profile functions. Introducing this combination as a variational parameter instead of \( \lambda \), Mano’s equation therefore reads

\[
\tilde{M}^2 = qx - x^2 - (\Omega_+ + \Omega_-) - 2(V_{11}^{\text{reg}} + V_{12}) .
\]  

(5.15)

Note that \( \Omega_\pm \) is a functional of the profile functions which, after solving the variational equations, are complicated functions of \( Z \) as are the parameter \( x \) and the momentum \( q \). Thus by differentiation the RHS of Eq. (5.15) w.r.t. \( Z \) we obtain

\[
0 = \frac{\partial q}{\partial Z} x - 2 \frac{V_{12}}{Z} + \left\{ q - 2x - 2 \frac{\partial}{\partial x} (V_{11}^{\text{reg}} + V_{12}) \right\} \frac{\partial x}{\partial Z} - \sum_{j=\pm} \int_{0}^{\infty} dE \left\{ \frac{\delta}{\delta A_j(E)} \left[ \Omega_+ + \Omega_- + 2(V_{11}^{\text{reg}} + V_{12}) \right] \right\} \frac{\partial A_j(E)}{\partial Z} .
\]  

(5.16)

At first sight this looks rather complicated, but due to the variational equations the terms in the curly brackets vanish identically and after reintroducing the parameter \( \lambda \) we obtain the simple result

\[
\frac{\partial}{\partial Z} \left( \frac{q}{2} \right)^2 = \frac{2}{\lambda} \frac{V_{12}}{Z} .
\]  

(5.17)

This is our version of the “Feynman-Hellmann theorem” [44] for the worldline approach. Note that the RHS originates from the explicit \( Z \)-dependence of \( V_{12} \) and that the dependence of the variational solutions on that strength only enters implicitly. We use the Feynman-Hellmann theorem in appendix E.1 to derive the first terms in the weak-coupling expansion of the binding energy.
6 Results and Discussion

6.1 Numerical Solutions and Binding Energies

The system of coupled integral equations (5.3) - (5.5), together with Eqs. (5.7), (4.45) and (4.46) can be solved by iteration using similar methods as in ref. (II). The basic idea is to use a grid of gaussian integration points $E_i, \sigma_j$ on which profile functions and pseudotimes are evaluated and used in the corresponding $E, \sigma$-integrals. This is similar to the mesh methods employed in ref. [45] for quantum-mechanical eigenvalue problems. The necessary modifications for the worldline bound-state case are discussed in appendix D. To obtain stable and reliable results it was crucial to understand and match the analytical solutions at small and large values of the variables (see appendix C) to the numerical outcome.

After considerable effort we have developed a program which solves the variational bound-state equations numerically with sufficient accuracy. Setting $V_{12} = 0$ reproduced the one-body results – a necessary but unfortunately rather weak check of its correctness. More stringent is the “virial check”, i.e. comparing the total kinetic term (4.43) evaluated from the profile functions with the one in Eq. (5.13) obtained from the pseudotimes. To find a bound-state solution one has to search for values $q^2 < 4M^2$ where the LHS minus the RHS of Mano’s equation (4.37) changes sign. This was done by applying the *regula falsi* once the possible solution had been bracketed. The values of the intermediate renormalized mass $\bar{M}$ have been recalculated at the beginning of the program by setting $Z\alpha = 0$ and $\sqrt{q^2} = 2M$.

Figure 3a shows the behaviour of the profile functions $A_\pm(E)$ as functions of the variable $E$ after a bound-state solution had been found in this way at $\sqrt{q^2} = 1.99218M$ for $\alpha = 0.5$. As expected from our analytic results the two profile functions have a very different behaviour for small $E$ but it is also clearly seen that $A_-(E)$ differs appreciably from the one-body profile function obtained at the same coupling constant. This illustrates how binding affects the nucleon self-energies or – in other language – the in-medium effects. Similar changes can also be observed in Fig. 3b for the pseudotimes.

Table 1 gives the result for the binding energy

$$\epsilon = \sqrt{q^2} - 2M$$  \hspace{1cm} (6.1)$$

obtained for the mass ratio $^7$

$$\frac{m}{M} = 0.15$$  \hspace{2cm} (6.2)$$

and a range of coupling constants $\alpha$. The lowest value comes from the nonrelativistic criterion (A.61) which for the chosen parameters says that $\alpha > 0.4067$ in order to get a bound state in the Yukawa potential (alternatively no solution of Mano’s equation with $q^2 < 4M^2$ can be found for too small $\alpha$). An upper value

---

$^7$Our previous results [35] were obtained with masses $M = 0.939$ GeV and $m = 0.14$ GeV, i.e. $m/M = 0.149095$. In order to facilitate comparison with results in the literature we have repeated the calculations for this standard mass ratio.
Figure 3. (a) The profile functions $A_-(E)$ and $A_+(E)$, (b) the pseudotimes $\mu_{12}^2(\sigma)$ and $\mu_{11}^2(\sigma)$ for the bound-state solution at $\alpha = 0.5$ with reparametrization parameter $\kappa_E = 1$. For comparison the one-body results (subscript $o$) are also shown.

is determined by the critical value

$$\alpha_{\text{crit}}^{(1)} = 0.817$$

above which no real solutions of Mano’s equation in the one-body case are found; this is the variational sign of the instability of the WC model when

\[8^8\text{This value is slightly larger than the one obtained in ref. (II) because of the slightly higher mass ratio. Similar values have been found with truncated Dyson-Schwinger equations [46, 4].}\]


Table 1. Results for the binding energy $\epsilon = \sqrt{q^2} - 2M$ from the solution of the variational bound-state equations for $m/M = 0.15$ and different coupling constants. The value of the intermediate mass $\bar{M}$ of the single nucleon (renormalized at $\nu = m$) is also given. Calculations were performed with $3 \times 72$ gaussian points and the iteration was stopped when the relative deviation $\Delta$ (defined in Eq. (D.7)) was less than $10^{-3}$ or when it increased twice consecutively. In this case $\Delta_{\text{min}}$ is given. The column “vir. check” gives the relative deviation $(\Omega_{12}^{\text{up}} - \Omega_{12})/\Omega_{12}$. The width of the bound state above the critical coupling $\alpha^{(2)}_{\text{crit}} = 0.542$ is estimated from Eq. (6.25).

| $\alpha$ | $\bar{M}/M$ | $\Delta_{\text{min}}$ | vir. check | $\epsilon/M$ | $\Gamma/M$ |
|----------|--------------|----------------------|-------------|--------------|------------|
| 0.44     | 0.75757      | $< 10^{-3}$          | - 0.0019    | - 0.00063    | 0          |
| 0.45     | 0.75156      | $< 10^{-3}$          | - 0.0001    | - 0.00138    | 0          |
| 0.46     | 0.74553      | $< 10^{-3}$          | - 0.0015    | - 0.00232    | 0          |
| 0.47     | 0.73947      | $< 10^{-3}$          | - 0.0001    | - 0.00338    | 0          |
| 0.48     | 0.73339      | $< 10^{-3}$          | 0.0001      | - 0.00462    | 0          |
| 0.49     | 0.72728      | $< 10^{-3}$          | 0.0001      | - 0.00610    | 0          |
| 0.50     | 0.72114      | $< 10^{-3}$          | 0.0002      | - 0.00782    | 0          |
| 0.51     | 0.71498      | $< 10^{-3}$          | 0.0002      | - 0.00985    | 0          |
| 0.52     | 0.70879      | $< 10^{-3}$          | 0.0003      | - 0.01226    | 0          |
| 0.53     | 0.70258      | $< 10^{-3}$          | 0.0005      | - 0.01516    | 0          |
| 0.54     | 0.69634      | $< 10^{-3}$          | 0.0004      | - 0.01900    | 0          |
| 0.541    | 0.69572      | $< 10^{-3}$          | 0.0008      | - 0.01945    | 0          |
| 0.542    | 0.69509      | $< 10^{-3}$          | 0.0008      | - 0.01998    | 0          |
| 0.543    | 0.69446      | $1.8 \cdot 10^{-3}$  | 0.0033      | - 0.02048    | 0.00012    |
| 0.545    | 0.69321      | $4.4 \cdot 10^{-3}$  | 0.0083      | $\sim$ -0.0212 | 0.00060 |
| 0.55     | 0.69007      | $1.1 \cdot 10^{-2}$  | 0.021       | $\sim$ -0.0233 | 0.0026 |
| 0.56     | 0.68378      | $1.7 \cdot 10^{-2}$  | 0.048       | $\sim$ -0.0248 | 0.0088 |
| 0.57     | 0.67747      | $2.3 \cdot 10^{-2}$  | 0.069       | $\sim$ -0.0288 | 0.0170 |
| 0.58     | 0.67112      | $3.1 \cdot 10^{-2}$  | 0.091       | $\sim$ -0.0331 | 0.0269 |
| 0.59     | 0.66475      | $4.0 \cdot 10^{-2}$  | 0.118       | $\sim$ -0.0391 | 0.0382 |
| 0.60     | 0.65836      | $4.6 \cdot 10^{-2}$  | 0.143       | $\sim$ -0.0443 | 0.0508 |

the quantum corrections are taken into account.

However, already for $\alpha > 0.542$ one observes that the iteration only converges for a certain number of iterations and then the global measure of deviation defined in Eq. (D.7) starts to rise again. In the one-body case this was a signal for the instability of the WC model and here also we can interpret this phenomenon as the impossibility of finding real solutions of the variational equations for coupling constants larger than

$$\alpha^{(2)}_{\text{crit}} = 0.542 \quad .$$

Consequently the virial check deteriorates rapidly. We also observe that this (infamous) instability now occurs at lower values of the coupling constant. This
is not unreasonable as, in general (e.g. in a laser), transitions may be induced by the presence of other particles [47]: the higher the number of particles in a system the faster it decays. In ref. (II) we showed that above the critical coupling only complex solutions of Mano’s equation were possible where the imaginary parts determined the width of the particle. Similarly, in Sect. 6.3 we will give a rough, analytical estimate of $\alpha_{\text{crit}}^{(2)}$ and determine the width of the bound state above this critical coupling.

### 6.2 Comparison with Other Work

Of course, not only the possible width of the bound state due to the instability of the WC model is of interest but also the magnitude of the binding. Here our approach should be compared with the nonrelativistic variational method (using a non-retarded quadratic trial action, i.e. gaussian wavefunctions) and with the exact nonrelativistic results from solving the s-wave Schrödinger equation for a Yukawa potential in the reduced system. The results are listed in Table 2.

In addition, we also give binding energies from the ladder Bethe-Salpeter equation (BSe). To be more precise we display the coupling constant belonging to a given $\sqrt{q^2/2M}$ calculated from Efimov’s variational approximation [48] to the ladder BSe. Comparison with some exact numerical solutions available in the literature (e.g. ref. [49]) shows that this approximation with 2 variational parameters is surprisingly good. For example, for a binding energy of 0.01$M$ ref. [50] reports a necessary coupling constant $\alpha = 0.5716$ whereas Efimov’s approximation requires $\alpha = 0.5730$ $^9$. The 2-dimensional integrals needed in Eqs. (56), (58) of Efimov’s paper were evaluated by standard numerical Gauss-Legendre integration and the minimum was found easily with the help of the CERN routine MINUIT.

As can be seen in Fig. 4 the binding energies of the full variational approach including the self-energy and vertex corrections are much larger in magnitude than the results from the other approaches provided the coupling constant $\alpha$ is appreciably larger than the threshold value needed to support a bound state. Of course, in the present variational approximation this threshold value is too large due to the insufficient approximation of a Yukawa potential by a harmonic oscillator one. Also the comparison of nonrelativistic variational results with the exact values from the solution of the Schrödinger equation shows that quantitatively the variational approximation is not a very good one (for reasons that are discussed in appendix A.2 after eq. A.57). Nevertheless the field theoretical effects are clearly visible and produce much more binding. That this must be due to self-energy/vertex corrections can be inferred by comparison with Nieuwenhuis & Tjon’s Monte-Carlo calculation [52] which gives the two-body binding energy beyond the ladder approximation but still without self-energy and vertex corrections $^{10}$.

$^9$Actually, upon repeating the calculations listed in Table 3 of Efimov’s paper for $m/M = 0.1$, better values were found in a few cases because the author had missed the true minimum [51].

$^{10}$Unfortunately, the results there are only given in the form of Fig. 1 from which values have to be read off in an approximate way: $\sqrt{q^2/M} \approx 1.990 \pm 0.020, 1.981 \pm 0.013, 1.961 \pm 0.014, 1.921 \pm 0.018, 1.851 \pm 0.020, 1.770 \pm 0.016$ for $\alpha = 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$ (the abscissa in Fig. 1 is $4\alpha$). The BSe curve is in good agreement with the values obtained from Efimov’s approximation.
Table 2. The binding energies $\epsilon/M$ from the worldline variational approach with $m/M = 0.15$ compared with those from the 3-dimensional nonrelativistic (NR) variational method, the exact solution of the nonrelativistic Schrödinger equation and (Efimov’s variational approximation to) the ladder Bethe-Salpeter equation (lBSe). The last column gives the Monte-Carlo results (estimated errors of the last digits in parenthesis) of ref. [52] in which ladder and crossed-ladder graphs have been included (“generalized lBSe”) but self-energy and vertex corrections were still neglected.
Figure 4. The binding energy $\epsilon/M$ of the two-body bound state for the equal-mass case with $m/M = 0.15$ as a function of the dimensionless coupling constant defined in Eq. (4.50). The results from the worldline variational approach including self-energy and vertex corrections are compared with those from Efimov’s variational approximation to the ladder Bethe-Salpeter (IBS) equation [48]. Also shown are the Monte-Carlo results (with errors) from ref. [52] in the “generalized ladder approximation” to the Bethe-Salpeter equation (glBS). The thick bars for the worldline results represent the width of the bound state above the critical coupling estimated from Eq. (6.25).

Having introduced the artificial strength parameter $Z$ for the direct interaction it is, of course, possible to check the self-energy effects on the binding energy by considering the limit

$$\alpha \rightarrow 0, \quad Z\alpha \rightarrow \alpha'$$

(6.5)

in which the self-interaction $V_{11}$ vanishes. For the single nucleon we then have $\bar{M} = M$ and the variational equations (5.4, 5.5) simplify slightly because

$$A_-(E) + A_+(E) \bigg|_{\alpha=0} = 2 + \frac{\omega^2_{\var}}{E^2}.$$  

(6.6)

Also the asymptotic behaviour of the profile functions changes to $A_\pm(E) \rightarrow 1 + \omega^2_{\var}/(2E^2)$.

Fig. 5 shows a comparison of variational binding energies with and without self-energy effects as functions of the coupling constant $\alpha'$. It is seen that the binding energies without self-energy effects are much smaller especially near the threshold where the interaction is strong enough for a bound state. In this case
the particles move slowly, and without self-energy and vertex corrections the minimal coupling is therefore the same as in the nonrelativistic case. However, we have to correct for the fact that our trial action is too restricted to reduce to the correct 3-dimensional kinetic term (as discussed above) and therefore we have to use Eq. (A.61) with $d = 4$. This gives

$$\alpha'_{\text{min}}|_{\alpha=0} = 0.54207 \ .$$ (6.7)

Our numerical results obtained by solving the variational equations and collected in Table 3 are consistent with this minimal value, which is an additional, nice check on the correctness and stability of our numerical procedures. Note that the minimal coupling (6.7) for a bound state without self-energy effects numerically coincides with the value (6.4) found for the instability of the bound system with self-energy and vertex corrections. This points to an important role of the direct binding as a new “doorway” for the decay of the system.

As noted before, the variational calculation “without” still contains (approximately) all crossed diagrams and therefore leads to more binding than the ladder Bethe-Salpeter equation for sufficiently large $\alpha'$. Note that up to $\alpha' = 1$ no sign of an instability is found. However, for slightly higher coupling constants the variational calculation does not converge anymore and we find again a critical coupling

$$\alpha'_{\text{crit}}|_{\alpha=0} = 1.041 \ .$$ (6.8)
\[ \alpha' \quad \Delta_{\text{min}} \quad \text{vir. check} \quad \epsilon/M \quad \Gamma/M \]

| 0.55 | \(< 10^{-3}\) | -0.0013 | -0.00031 | 0 |
| 0.60 | \(< 10^{-3}\) | -0.0013 | -0.00252 | 0 |
| 0.65 | \(< 10^{-3}\) | -0.0012 | -0.00564 | 0 |
| 0.70 | \(< 10^{-3}\) | -0.0012 | -0.00973 | 0 |
| 0.75 | \(< 10^{-3}\) | -0.0007 | -0.01485 | 0 |
| 0.80 | \(< 10^{-3}\) | -0.0011 | -0.02116 | 0 |
| 0.85 | \(< 10^{-3}\) | -0.0008 | -0.02879 | 0 |
| 0.90 | \(< 10^{-3}\) | -0.0011 | -0.03807 | 0 |
| 0.95 | \(< 10^{-3}\) | 0.0003 | -0.04930 | 0 |
| 1.00 | \(< 10^{-3}\) | 0.0005 | -0.06353 | 0 |
| 1.04 | \(< 10^{-3}\) | 0.0013 | -0.07888 | 0 |
| 1.041 | \(< 10^{-3}\) | 0.0012 | -0.07942 | 0 |
| 1.042 | 1.7 \cdot 10^{-3} | 0.0028 | \sim -0.0798 | 0.00004 |
| 1.043 | 2.7 \cdot 10^{-3} | 0.0043 | \sim -0.0802 | 0.00012 |
| 1.045 | 4.4 \cdot 10^{-3} | 0.0072 | \sim -0.0810 | 0.00034 |
| 1.05 | 8.9 \cdot 10^{-3} | 0.015 | \sim -0.0832 | 0.00114 |
| 1.08 | 3.4 \cdot 10^{-2} | 0.055 | \sim -0.095 | 0.0103 |
| 1.10 | 5.1 \cdot 10^{-2} | 0.087 | \sim -0.104 | 0.0191 |

Table 3. Same as in Table 1 but for \(\alpha = 0, Z\alpha = \alpha'\), i.e. without self-energy and vertex corrections which implies \(M = \bar{M}\). The width of the unstable state above the critical coupling \(\alpha'_{\text{crit}} = 1.041\) is estimated from Eq. (6.29).

At first sight this is surprising as the instability of the WC model has been attributed to the self-energy effects which are absent in this limit and one would suspect a numerical instability. However, Table 3 shows that the virial test (checking how well the variational equations are fulfilled numerically) is excellent up to the critical value. In addition, the very same analysis which explains the induced instability in the full model (with self-energy effects) leads to the prediction \(\alpha' \sim 0.814\) for zero meson mass (see next Sect.). We are thus led to the conclusion that the WC model is “unstable in nearly any case” except for rather special approximations like the ladder approximation. The Monte-Carlo calculations by Nieuwenhuis & Tjon [52] may easily have missed this instability given their large error bars, systematic errors and the fact that they only consider coupling constants not larger than \(\alpha' = 0.9\).

The increased binding due to the self-energy/vertex corrections is also evident in the weak-binding case where (for \(m = 0\)) we have found in appendix F.1

\[ \epsilon_0 = -\frac{M}{2} \left[ \frac{(Z\alpha)^2}{\pi} K_{\text{var}} + \frac{(Z\alpha)^4}{\pi^2} + \ldots \right]. \quad (6.9) \]

with an enhancement factor

\[ K_{\text{var}} = 1 + \frac{7}{2} \frac{\alpha}{\pi} + \ldots \quad (6.10) \]
for the leading nonrelativistic term. That this is not an artefact of the variational approximation but also occurs in the full theory can be seen as follows: the one-loop vertex correction and the wavefunction renormalization for a free nucleon depicted in fig. 6 give rise to an effective coupling constant

\[ g' \rightarrow g'_\text{eff} = g' \left( 1 + \frac{\alpha}{\pi} + \ldots \right) \]  

(6.11)
as obtained in Eq. (58) of ref. [27]. As derived in appendix F.2, this is also the exact one-loop result since the first-order variational calculation reduces to that in the weak-coupling limit. Thus we have \( \alpha \rightarrow \alpha(1 + \alpha/\pi + \ldots)^2 \) and we therefore expect that the exact bound-state energy of the equal-mass WC model has the weak-binding expansion

\[ \epsilon_n \big|_{\text{exact}} = -\frac{M}{2} \frac{(Z\alpha)^2}{2(n + 1)^2} K_{\text{exact}} + \ldots \]  

(6.12)
with an enhancement factor

\[ K_{\text{exact}} = \left( 1 + \frac{\alpha}{\pi} + \ldots \right)^4 = 1 + 4 \frac{\alpha}{\pi} + \ldots . \]  

(6.13)
The result (6.10) from the variational worldline approach is in good agreement with this except that the numerical factor is 7/2 and not 4. Since the variational calculation underestimates the leading term—the nonrelativistic binding in units of the reduced mass is \((Z\alpha)^2/\pi\) instead of the correct Coulomb value \((Z\alpha)^2/2\)—such a small quantitative discrepancy is of no concern. Note that the procedure used to determine the exact enhancement factor (6.13) is equivalent to the matching procedure used in the application of effective field theories to bound-state problems [53]. Indeed, Eq. (2.2) in ref. [54] is exactly the nonrelativistic effective field theory for the massless WC model and in that language Eq. (6.11) determines the Wilson coefficient \(c_1\) to one-loop order.

However, the enhancement observed above is at variance with Ji’s claim [55] that inclusion of self-energy terms leads to less binding. This result was obtained in the light-cone formalism whose boosted states contain many equal-time Fock components. Therefore a direct comparison with equal-time methods is difficult although the physical results should be the same ...
The origin of the disagreement is not clear to us: it could be due to a missing nucleon mass renormalization \(^{11}\) or a shortcoming of Ji’s restricted Tamm-Dancoff approximation, which only retains (light-cone) states with at most one meson. In view of the fact that both variational and effective field theory predict more binding, serious doubts remain whether the light-cone calculation is correct and/or complete. This assessment is also supported by the results from the dressed ladder approximation which give more binding compared to the calculations with bare propagators (see Fig. 8 in ref. [3]). The observed enhancement of the effective coupling in the WC model is similar to the enhancement of the electron-photon coupling in QED; however, in the latter case only the magnetic coupling (the famous anomalous magnetic moment of the electron) is affected and not the electric charge which is conserved in a gauge theory.

The \(\mathcal{O}((Z\alpha)^4)\)-terms in Eq. (6.9) may be compared with the corresponding result from the Todorov equation, one of the relativistic quasi-potential equations which describe spinless binaries bound by a Coulomb potential. However, what is found in the textbooks (e.g. ref. [57], Eqs. (4.197) and (4.206)) corresponds to scalar QED, not to our coupling of mesons (“scalar photons”) to scalar particles. The latter case was evaluated by Brezin, Itzykson and Zinn-Justin who anticipated Todorov’s equation \(^{58}\) in a study of the relativistic eikonal approximation. This approach includes relativistic recoil and “an approximate summation of the crossed-ladder Feynman diagrams” but “does not include radiative corrections of the self-energy type” \(^{59}\). Using the (unnumbered) equation following Eq. (20) in that paper or Eq. (3.8) in ref. \([58]\) one obtains in the equal-mass case

\[
\epsilon_n \bigg|_{\text{Todorov}} = \left\{ 2M^2 \left[ 1 + \sqrt{1 - (Z\alpha/(n+1))^2} \right] \right\}^{1/2} - 2M \]

\[
= -\frac{M}{2} \left[ \frac{1}{2} \left( \frac{Z\alpha}{n+1} \right)^2 + \frac{5}{32} \left( \frac{Z\alpha}{n+1} \right)^4 + \ldots \right]. \quad (6.14)
\]

For \(n = 0\) this relativistic Balmer formula gives a coefficient \(5/32 = 0.156\) for the \((Z\alpha)^4\)-term whereas the variational result \([6.9]\) leads to \(1/\pi^2 = 0.101\). Thus both approaches predict an enhanced binding of the lowest-lying state also from relativistic effects. Although the variational calculation gives a smaller coefficient – as it does for the leading nonrelativistic \(\mathcal{O}((Z\alpha)^2)\)-term – it has the clear advantage of also containing self-energy and vertex corrections.

Comparison with the weak-binding expansions shows that in the variational approach all calculated expansion coefficients have the same sign but are smaller in magnitude than in the exact case. Therefore it is tempting to deduce that the true binding energy is always below the variational bound-state energy as in the standard Rayleigh-Ritz variational principle of quantum mechanics. However, we have not succeeded in proving this conjecture rigorously. The main obstacle is that renormalization requires the intermediate mass \(\bar{M}\) of a single nucleon as input in Mano’s equation \([4.37]\) and this quantity is the result of a variational calculation itself. Actually, as shown in Eq. (85) in ref. \([1]\), \(M^2\) is bounded from

\(^{11}\) For a recent discussion of renormalization in light-front dynamics see ref. \([56]\).
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below 12 and one may expect that a similar inequality also exists for the 2-body case. But this does not lead to a bound for the binding energy because this quantity is determined relative to the single-nucleon result. Note that this argument does not depend on whether the renormalization is finite or infinite: the minimum property of the variational calculation is also lost in the finite bipolaron problem [38].

6.3 Induced Instability

In order to understand the numerical results it is also useful to have some rough analytical insight into the solutions of the variational equations. In particular, we would like to understand the mechanism of the induced instability. Such an insight is provided by the ansatz

\[ A_-(E) = 1 \] (6.15)

with \( \lambda \) as the only free variational parameter. This was used in ref. (11) to determine the critical coupling beyond which no real solution of the variational equations exist anymore. However, in the two-body case the binding certainly has to be taken into account and the simplest choice is a harmonic oscillator-like form \( (\kappa E = 1) \)

\[ A_+(E) = 1 + \frac{\omega^2}{E^2}. \] (6.16)

This is also the nonrelativistic solution and has the correct form for \( E \to 0 \) (see Eq. (5.11)). Hence there are two variational parameters to be determined from varying Mano’s Eq. To allow an analytical treatment we furthermore assume \( m = 0 \) and weak binding

\[ \frac{\omega}{(q/2)^2} \simeq \frac{\omega}{M^2} \ll 1 \] (6.17)

together with the property that the relevant proper times are \( \mathcal{O}(1/M^2) \), i.e. \( \omega \sigma \ll 1 \). As shown in appendix E the critical coupling can then be worked out analytically and is given by

\[ \alpha_{\text{crit}} \simeq \frac{\pi}{8} \frac{(1 + \sqrt{1 + 3z})^3}{(1 + z + \sqrt{1 + 3z})^2}, \quad z := 2\pi Z^2. \] (6.18)

For \( Z = 0 \), i.e. for the one-body case, this reduces to the value \( \alpha_{\text{crit}} = \pi/4 = 0.7854 \) obtained in ref. (11). As a function of \( Z \) the critical coupling has the behaviour

\[ \alpha_{\text{crit}} \xrightarrow{Z \to 0} \frac{\pi}{4} \left[ 1 - \frac{\pi}{2} Z^2 + \frac{3\pi^2}{4} Z^4 + \ldots \right] \] (6.19)

\[ \alpha_{\text{crit}} \xrightarrow{Z \to \infty} \sqrt{\frac{27\pi}{128}} \frac{1}{Z} + \ldots \] (6.20)

which shows that the critical coupling for the bound-state system is reduced compared to the one-body case.

12 In the 1-body case the value of \( \bar{M} \) served as a measure of the goodness of different parametrizations for the profile function.
Figure 7. (a) The critical coupling $\alpha_{\text{crit}}$ and (b) the correction factor $f_{\text{corr}}$ as function of the artificial strength $Z$ of the binding interaction in the simplified treatment of the induced instability.

Indeed fig. 7 demonstrates that the reduction is monotonous with increasing $Z$. For $Z = 1$ one obtains

$$\left. \alpha_{\text{crit}} \right|_{Z=1} \simeq 0.4627$$

which is in reasonable agreement with the value $\alpha_{\text{crit}} = 0.542$ obtained from the numerical solution of the variational equations. It is also interesting that the low-$Z$ expansion (6.19) does not converge for the value $Z = 1$, a phenomenon which has also been observed in bound-state QED calculations [60].

It is also possible to determine approximately the width of the bound-state for $\alpha > \alpha_{\text{crit}}$ following the treatment in Sect. IV. A of ref. (II). To be specific we write

$$q^2 = \left( M^* - i \frac{\Gamma}{2} \right)^2, \quad M^* \simeq 2M$$

and introduce

$$\zeta := \lambda \frac{q}{2} \simeq \lambda \left( M - i \frac{\Gamma}{4} \right) =: \zeta_0 e^{-i\chi}.$$ (6.23)

A nonvanishing phase $\chi$ signals a complex (approximate) solution of the variational equations for coupling constants above the critical one. Near the critical coupling $\alpha_{\text{crit}}$ given in Eq. (6.18) the phase $\chi(\alpha)$ is determined by the implicit equation (see appendix E)

$$\alpha = \alpha_{\text{crit}} \left[ 1 + \alpha_2 \chi^2 + \mathcal{O}(\chi^4) \right]$$ (6.24)

In relativistic calculations it is more customary to denote the square of the mass of an unstable particle as $M^{*2} - iM^*\Gamma$ but this does not make any significant difference for the rough analytical estimate which we will derive.
where $\alpha_2$ is given in Eq. (6.20). This can be used to evaluate the width near the critical coupling for which one obtains

$$\Gamma \rightarrow \alpha \rightarrow \alpha_{\text{crit}} \frac{2}{3} 2M \left(\frac{\alpha - \alpha_{\text{crit}}}{\alpha_{\text{crit}}}\right)^{3/2} f_{\text{corr}}(Z) \quad (6.25)$$

with a correction factor

$$f_{\text{corr}}(Z) = \frac{\sqrt{1 + 3z(1 + \sqrt{1 + 3z})(1 + z + \sqrt{1 + 3z})^{1/2}}}{(1 + 2z + \sqrt{1 + 3z})^{3/2}}, \quad z \equiv 2\pi Z^2. \quad (6.26)$$

The limiting cases are

$$f_{\text{corr}}(Z) \rightarrow Z \rightarrow 0 \quad 1 + \frac{\pi}{2} \frac{19\pi^2}{8} Z^2 + \ldots, \quad f_{\text{corr}}(Z) \rightarrow Z \rightarrow \infty \quad \frac{3}{4} \sqrt{2} + \frac{1}{16} \sqrt{\frac{3}{\pi Z}} + \ldots \quad (6.27)$$

and Fig. 7b shows that this function is rising rapidly from $f_{\text{corr}}(0) = 1$ to its asymptotic value $f_{\text{corr}}(\infty) = 1.0607$. At $Z = 1$ its value is $f_{\text{corr}}(1) = 1.08845$. However, this enhancement of the width pales (and may be not even significant in the light of the drastic approximations which have been employed) in comparison with the factor 2 which distinguishes Eq. (6.25) from the one-body result. This factor 2 simply comes from the fact that the bound state has mass $M^* \simeq 2M$. Thus, the induced instability by the presence of the second particle not only shows up in the lower critical coupling constant but also in the width which is roughly doubled if one exceeds this coupling by the same relative amount as in the single-particle case.

What happens in the limit (6.5), i.e. for the case where the self-energy and vertex corrections are neglected? This can be easily analyzed by writing $Z = \alpha'/\alpha$ which goes to infinity when $\alpha$ tends to zero and $\alpha'$ is kept fixed. With the help of Eq. (6.20) we immediately obtain

$$\alpha_{\text{crit}} \rightarrow 9 \sqrt{\frac{\pi}{2}} \frac{\alpha_{\text{crit}}}{\alpha'_{\text{crit}}} \Rightarrow \alpha'_{\text{crit}} \simeq \sqrt{\frac{27\pi}{128}} = 0.8141 \quad (6.28)$$

which is close to the numerically observed value. Similarly above the critical coupling the estimate (6.25) for the width becomes

$$\Gamma \bigg|_{\alpha = 0} \simeq \sqrt{2M} \left(\frac{\alpha' - \alpha'_{\text{crit}}}{\alpha'_{\text{crit}}}\right)^{3/2}. \quad (6.29)$$

Therefore the induced instability also shows up if self-energy and vertex corrections are switched off.

7 Summary and Outlook

We have investigated the relativistic binding problem of two particles with equal mass in the scalar Wick-Cutkosky model employing worldline variational methods. Compared to the standard field-theoretical description the worldline (“particle”) representation of the system entails a huge reduction in the degrees of freedom, leading to a quantum-mechanical path integral over the trajectories of the
heavy particles ("nucleons") like in the famous polaron problem. While previous work in this approach concentrated on single-particle properties and processes this is the first application of the method to the inherently non-perturbative binding of two relativistic particles and can therefore be considered as a relativistic analogue of the bipolaron problem: although the delicate balance between repulsive two-body interaction and attractive self-energy is missing, there is the additional problem of non-perturbative renormalization. Fortunately, the WC model is a super-renormalizable theory so that – in the quenched approximation – only an identical mass renormalization both in the 1-nucleon and the 2-nucleon sector is necessary.

Whereas most bipolaron studies employed parametrized ansätze for the variational functions we used a quadratic trial action with free retardation functions which are determined by the variational principle itself. This variational principle takes the form of a specific equation ("Mano’s equation" for the two-body case) which is stationary under unrestricted variation of the variational functions and determines the lowest bound state in a fully covariant manner. No definition of center-of-mass coordinates, factorization of wave functions or choice of special frames is necessary. This was checked in the nonrelativistic case where the correct center-of-mass and internal energies were obtained from the variational calculation. In the relativistic case we have derived the (non-linear) variational equations, solved them numerically and studied the asymptotic behaviour of the variational functions for small and large values of their arguments.

In this approach the binding energy is not determined by an eigenvalue equation but by the specific value of the external variable \( q^2 \ll 4M^2 \) where both sides of Mano’s two-body equation are equal. We have obtained this value analytically for the special case of small coupling constant and zero mass of the exchanged particle ("pion") and numerically without any restriction below a critical coupling. In both cases we found more binding due to the effects of self-energy and vertex corrections which are included approximately in our calculation. This is in contrast to results from light-cone calculations [55] which claim that self-energy effects are repulsive. However, a variety of results obtained by different techniques – dressed-ladder Bethe-Salpeter equation \( \mathcal{R} \), effective field theories, our variational approach – all point to the opposite conclusion. We have identified the source of this increased binding as mainly due to the pion-nucleon coupling enhanced by vertex corrections, in a similar way as the anomalous magnetic moment of the electron or the muon is enhanced by radiative corrections. Additional binding also comes from crossed-ladder contributions as shown in ref. [52]. This was confirmed by artificially switching off the radiative effects in our program and retaining only ladder and crossed-ladder contributions to the binding.

However, whereas Quantum Electrodynamics has a stable ground state for not too strong fields the scalar Wick-Cutkosky model has not – a deficiency which too often is forgotten or neglected in bound-state calculations employing this model. Indeed, whereas previous worldline calculations in the one-particle sector already yielded a critical coupling above which no real solutions of the variational equations exist anymore, the two-particle bound state ceased to exist for even smaller coupling constants indicating an induced instability. Surprisingly
this instability also shows up when the self-energy and vertex corrections are neglected and only ladder and crossed-ladder diagrams are included. Although the variational equations in our approach are highly non-linear integral equations we were able to explain this phenomenon analytically by simple ansätze which lead to solvable, algebraic equations. Actually, the ability to combine numerical results with simple analytical insights seems to be a feature which the worldline variational approach shares with ordinary Quantum Mechanics: the virial theorem which was used to control the accuracy of the numerical calculations and the Feynman-Hellmann theorem employed for the weak-coupling expansion of the binding energy are other examples.

One of the main weaknesses of the present approach certainly is the use of a quadratic, oscillator-like trial action also for the direct interaction of the two particles which in the nonrelativistic limit reduces to a Yukawa potential. This limits the accuracy, in particular near the threshold where the coupling constant is strong enough to support a bound state. With the present (isotropic) trial action this deficiency is even worse since in the nonrelativistic (weak coupling) limit the kinetic term reduces to a 4-dimensional and not to a 3-dimensional theory. This could (and will) be easily improved by allowing anisotropic terms as was done in the one-particle sector in ref. [29] thus giving the trial action more flexibility. The necessary modifications of Mano’s equation are straightforward with a modest increase in numerical complexity due to the appearance of longitudinal and transverse components w.r.t. to the external momentum \( q \). Second-order corrections to the variational result can also be calculated rather straightforwardly [61] and typically reduce deviations from known exact results by a factor of 3 – 4. Still, one would like to obtain the correct quantum-mechanical binding energy for weak coupling also in this approach which could be possible following the work of Luttinger and Lu in the polaron case [62]. At the expense of applying Jensen’s inequality twice these authors have introduced a (variational) potential into the trial action and were able to reproduce the polaron energy for strong coupling where Feynman’s quadratic trial action leads to the largest deviations.

Further extensions may include the unequal-mass case, binding of more than two particles similar as in ref. [63] and inclusion of vacuum polarization effects, i.e. going beyond the quenched approximation (see ref. [64] for attempts in this direction). Calculating bound states in QED would be another step towards approaching a realistic theory although the present-day perturbative techniques are sufficient for the small electromagnetic coupling constant. Spin degrees of freedom and gauge invariance are no obstacles for a worldline description [31]. As we were able to extract the anomalous mass dimension of the electron, i.e. to determine the singularity structure of renormalization constants in the worldline variational approach [34], the necessary non-perturbative renormalization for a renormalizable theory like QED requires more work but should be feasible. It is clear, however, that the ultimate goal and challenge for a non-perturbative approach lies in the dynamics of quarks and gluons at low energies. Whereas color degrees of freedom can be included in a worldline formalism [65] it is unclear, at present, how to treat the non-abelian – and therefore nonquadratic – gluon action in a gauge-invariant way. Auxiliary-field methods in which three- and
four-gluon self-interactions are eliminated \[66\] may be a possible way to proceed as can be shown for the simpler case of a $\Phi^4$-theory \[67\]. We think that the results obtained so far in the worldline variational approach, in particular the novel way of treating the relativistic bound-state problem presented in this work, are encouraging enough to make an effort in this direction.

**Acknowledgement.** All diagrams were drawn by using JaxoDraw \[68\].

### Appendix A: Nonrelativistic Binding

#### A.1 The Nonrelativistic Limit

It has long been known that the exchange of a scalar particle (a meson with mass $m$) is approximately equivalent to an attractive Yukawa potential between the heavy nucleons with mass $M$. The usual derivation starts from the one-meson-exchange scattering amplitude in the static limit and equates it with the Born approximation of potential scattering (see e.g. ref. \[69\], appendix 10). A more consistent way is to integrate out the meson field in the generating functional

$$Z[J^*, J; j] = \int \prod_{i=1}^2 \mathcal{D}\Phi \exp \left\{ \int d^4x \left[ \mathcal{L}_2 + \sum_i (J_i^* \Phi_i + \Phi_i J_i + j \chi) \right] \right\}$$

(A.1)

for vanishing sources, keeping the velocity of light $c$ (and for completeness also Planck’s constant $\hbar$) and then letting $c \to \infty$ in order to obtain the nonrelativistic limit \[70\]. The velocity of light (re)appears in the following places

$$x_0 = ct \implies \partial^2 = \frac{1}{c^2} \frac{\partial^2}{\partial^2 t} - \Delta \quad \text{(d'Alembert operator)}$$

and it should be remembered that the action is the time integral over the Lagrange function

$$S = \int dt \int d^3x \mathcal{L} = \frac{1}{c} \int d^4x \mathcal{L}$$

(A.3)

entering as $\exp(iS/\hbar)$ in the path integral. The standard shifted gaussian integral then leads to an effective action for the nucleons

$$S_{\text{eff}}[\Phi_i^*, \Phi_i] = \frac{1}{c^2} \sum_i \int d^4x \Phi_i^*(x) \left( -\Box - \frac{M_i^2 c^2}{\hbar^2} \right) \Phi_i(x) - \frac{1}{c^2} \sum_{i,j} \frac{g_i g_j}{2}$$

$$\times \int d^4x d^4y \Phi_i^*(x) \Phi_i(y) \left( x \left( -\Box - \frac{m_i^2 c^2}{\hbar^2} \right)^{-1} - y \right) \Phi_j^*(y) \Phi_j(y)$$

(A.4)

which is a non-local $\Phi^4$-theory. Making the ansatz \[71\]

$$\Phi_i = \frac{\hbar \phi_i}{\sqrt{2M_i}} e^{-iM_i c^2 t/\hbar} \implies \frac{1}{c^2} \partial^2 \Phi_i = \left[ -\frac{M_i^2 c^2}{\hbar^2} \phi_i - \frac{2i M_i}{\hbar} \partial_t \phi_i + \frac{1}{c^2} \partial_t^2 \phi_i \right] \frac{\hbar}{\sqrt{2M_i}} e^{-iM_i c^2 t/\hbar}.$$  

(A.5)

and sending $c \to \infty$ \[14\] gives

$$S_{\text{eff}}[\phi_i^*, \phi_i] \to \frac{1}{c^2} \sum_i \int d^4x \phi_i^*(x) \left( i\hbar \partial_t + \frac{\hbar^2}{2M_i} \Delta \right) \phi(x) - \frac{1}{c^2} \sum_{i,j} \frac{\hbar^4 g_i g_j}{8M_i M_j}$$

$$\times \int d^2x d^4y \phi_i^*(x) \phi_i(x) \langle x \left( -\Box - \frac{m_i^2 c^2}{\hbar^2} \right)^{-1} | y \rangle \phi_j^*(y) \phi_j(y).$$

(A.6)

\[14\] Since the meson mass $m$ may be small or zero we do not perform the limit in the meson propagator.
The first term is the kinetic term of a system of nonrelativistic particles (of two types) whereas the last one describes their interactions. In the limit $c \to \infty$ the retardation of meson exchange (i.e. the dependence on $p_0$) is suppressed and one obtains an *instantaneous* interaction

\[
\left< x \left| \frac{1}{-\Box - m^2c^2/h^2 + i0} \right| y \right> = \int \frac{d^4p}{(2\pi \hbar)^4} \frac{\exp[-ip \cdot (x-y)/\hbar]}{p^2/\hbar^2 - m^2c^2/\hbar^2 + i0} \rightarrow \infty \hbar^2 \delta(x_0 - y_0) \int \frac{d^4q}{(2\pi \hbar)^4} \frac{\exp[ip \cdot (x-y)/\hbar]}{p^2 + m^2c^2} = -\delta(x_0 - y_0) \frac{1}{4\pi|x-y|} \exp\left(-\frac{m\hbar}{c}|x-y|\right).
\] (A.7)

ence, if the effective, nonrelativistic action is written in the standard many-body form for two-body potentials

\[
S_{\text{eff}}[\phi^*_i, \phi_i] = \int dt \left\{ \int d^3x \sum_i \phi^*_i(x, t) \left(i\hbar\partial_t + \frac{\hbar^2 \Delta}{2M_i}\right) \phi_i(x, t) - \frac{1}{2} \int d^3x d^3y \sum_{i,j} \phi^*_i(x, t) \phi^*_j(y, t) V_{ij}(x - y) \phi_j(y, t) \phi_i(x, t) \right\} \quad \text{(A.8)}
\]

one finds that an attractive Yukawa potential

\[
V_{ij}(x - y) = -\frac{\hbar^4 g_i g_j}{16\pi M_i M_j} \frac{\exp(-mc|x-y|/\hbar)}{|x-y|} \quad \text{(A.9)}
\]

with range $\hbar/(mc)$, the Compton wavelength of the exchanged meson, acts between the different types of nucleons.

### A.2 Variational Approximation for the Nonrelativistic Polarization Propagator

Since we make a variational approximation of the ground state energy of the relativistic system it is appropriate to assess the accuracy of our calculation in the nonrelativistic limit to which it should reduce for $c \to \infty$. The standard approach in Quantum Mechanics is, of course, the Rayleigh-Ritz variational principle which involves a trial wave function. Although it is obvious that a quadratic trial action corresponds to a gaussian trial wave function we will derive the corresponding approximation in close analogy to the field-theoretic (worldline) case. In particular, we do not use the simplification that the nonrelativistic Hamiltonian can be split up into a center-of-mass part and a relative part. Instead we will deal with the translationally invariant full system whose Hamiltonian reads

\[
\hat{H} = \frac{\mathbf{p}_1^2}{2M_1} + \frac{\mathbf{p}_2^2}{2M_2} + V(x_1 - x_2) \quad \text{(A.10)}
\]

and investigate a quantity where one projects on a fixed (total) momentum $\mathbf{q}$. This is just the nonrelativistic polarization propagator in $d = 3$ dimensions

\[
\Pi(\mathbf{q}, E) = \int d^3x e^{-i\mathbf{q} \cdot \mathbf{x}} \left< x_1 = x_2 = x \left| \frac{1}{E - \hat{H} + i0} \right| x_1 = x_2 = 0 \right> = -i \int d^3x e^{-i\mathbf{q} \cdot \mathbf{x}} \int_0^\infty dT \left< x_1 = x_2 = x \left| \exp \left[ iT \left( E - \hat{H} \right) \right] \right| x_1 = x_2 = 0 \right> \quad \text{(A.11)}
\]

which has poles at the total energy of the system

\[
\Pi(\mathbf{q}, E) = \sum_n |\psi_n(0)|^2 \left< E - \left[ \frac{\mathbf{q}^2}{2(M_1 + M_2)} + \epsilon_n \right] \right>^{-1} \quad \text{(A.12)}
\]

The time-evolution operator has the path integral representation

\[
\left< x_1 = x_2 = x \left| \exp \left[ -iT \left( \hat{H} - i0 \right) \right] \right| x_1 = x_2 = 0 \right> = \int_{x_1(0) = x_2(0) = x}^{x_1(T) = x} D^3x_1 D^3x_2 \exp \left\{ iS[x_1, x_2] \right\} \quad \text{(A.13)}
\]
where the action is

\[ S[x_1, x_2] = \int_0^T dt \left[ \frac{M_1}{2} \dot{x}_1^2 + \frac{M_2}{2} \dot{x}_2^2 - V(x_1 - x_2) \right]. \]  

(A.14)

Note that both particles have one common time - in contrast to the relativistic case. This facilitates the limit \( T \to \infty \) even for the case of unequal masses \( M_1 \neq M_2 \). For vanishing interaction the free polarization propagator can be calculated easily

\[ \Pi^{(0)}(q, E) = \frac{1}{(2\pi)^3} \left( \frac{\pi M_{\text{red}}}{i} \right)^{3/2} \int_0^\infty \frac{dT}{T^{3/2}} \exp \left[ iT \frac{q^2}{2(M_1 + M_2)} \right], \]

(A.15)

where \( M_{\text{red}} \) is the reduced mass of the system defined in Eq. (2.4). This is used for the normalization of the path integral in the general case. Thus we have

\[ \Pi(q, E) = \left( \frac{M_{\text{red}}}{4\pi t} \right)^{3/2} \int_0^\infty \frac{dT}{T^{3/2}} \exp \left[ iT \frac{q^2}{2(M_1 + M_2)} \right] \cdot \int \mathcal{D}(x_1, x_2) \exp(i\mathcal{S}) \]

(A.16)

where

\[ \int \mathcal{D}(x_1, x_2) = \int d^3x e^{-i\mathbf{q} \cdot \mathbf{x}} \int x_{1(T)=x} \mathcal{D}^3x_1 \int x_{2(T)=x} \mathcal{D}^3x_2. \]

(A.17)

\[ \mathcal{S} \equiv -\mathbf{q} \cdot \mathbf{x} + S[x_1, x_2], \quad \mathcal{S}_0 \equiv -\mathbf{q} \cdot \mathbf{x} + \sum_{i=1}^2 S_0[x_i] \]

(A.18)

\[ S_0[x_i] = \int_0^T dt \frac{M_i}{2} \dot{x}_i^2. \]

(A.19)

We now apply the Feynman-Jensen variational principle with a suitable (and manageable) trial action. For a proper choice we first look at the free action: with the Fourier expansion of the paths

\[ x_i(t) = x \frac{t}{T} + \sum_{k=1}^\infty \frac{2\sqrt{T}}{k\pi} a_k^{(i)} \sin \left( \frac{k\pi t}{T} \right) \]

(A.20)

the free action becomes

\[ \mathcal{S}_0 = -\mathbf{q} \cdot \mathbf{x} + \sum_{i=1}^2 \left[ x^2 \frac{M_i}{2T} + \sum_{k=1}^\infty a_k^{(i)} \right] - \sum_{k=1}^\infty B_k a_k^{(1)} \cdot a_k^{(2)}. \]

(A.21)

In the one-body case a rule of thumb is to “decorate” the free action with variational parameters to obtain a quadratic trial action which reduces to the free one if the interaction is switched off. In the two-body case we need an additional coupling term which should account for the interaction of the particles. Therefore we take as trial action

\[ \tilde{\mathcal{S}}_t = -\tilde{\lambda} \mathbf{q} \cdot \mathbf{x} + \sum_{i=1}^2 \left[ A_0 x^2 \frac{M_i}{2T} + \sum_{k=1}^\infty A_k^{(i)} a_k^{(i)} \right] - \sum_{k=1}^\infty B_k a_k^{(1)} \cdot a_k^{(2)}. \]

(A.22)

The Fourier coefficients \( A_0, A_k^{(i)}, B_k \) as well as the parameter \( \tilde{\lambda} \) are variational parameters. The relative minus sign between the \( A \) and \( B \) terms has been chosen for consistency with the relativistic ansatz. Since the trial action is at most quadratic the averages can be calculated analytically:

\[
\left< \mathcal{S}_0 - \tilde{\mathcal{S}}_t \right> = \frac{3}{2} \left[ \frac{1 - A_0}{A_0} + \sum_{k=1}^\infty \frac{2B_k^2}{A_k^{(1)} A_k^{(2)} - B_k^2} + \sum_{k=1}^\infty \frac{A_k^{(1)} + A_k^{(2)} - 2A_k^{(1)} A_k^{(2)} + A_k^{(1)} A_k^{(2)} - B_k^2}{A_k^{(1)} A_k^{(2)} - B_k^2} \right].
\]

(A.23)
Collecting the terms linear in $E$

However, this constant is still a functional of the variational functions

As usual the pole of the polarization propagator is determined by the limit

Here

is the Fourier transform of the potential between the two particles and the abbreviation

has been used. Thus we find

where the kinetic term is defined as

As usual the pole of the polarization propagator is determined by the limit $T \to \infty$ in which case the sums over Fourier coefficients become integrals over $E = k \pi t / T$ as in Eq. (4.13). In addition, we may use the fact that only those terms in the exponent which develop a linear $T$-dependence contribute to the pole position. This means that the $A_0$-terms can be dropped and that we can replace

in the argument of the function $\nu(T, t)$ making it a constant:

However, this constant is still a functional of the variational functions $A^{(i)}(E), B(E)$. Thus we obtain

Collecting the terms linear in $T$ in Eq. (A.28) and setting

\[ \lambda \equiv \frac{\hat{\lambda} \leftarrow}{A_0} \]
Mano’s equation reads
\[ E_0 = \frac{\mathbf{q}^2}{2(M_1 + M_2)} \left( 2\lambda - \lambda^2 \right) + (\Omega + V) =: E_{\text{CM}} + \epsilon_0. \] (A.35)

Due to our metric \((q^2 = q_0^2 - \mathbf{q}^2)\) there is now a different sign between the center-of-mass (CM) and internal parts compared with the relativistic case. Variation with respect to \(\lambda\) yields
\[ \lambda = 1 \] (A.36)

since \(\Omega\) and \(V\) are independent of \(\lambda\). This is crucial both for obtaining the correct center-of-mass energy
\[ E_{\text{CM}} = \frac{\mathbf{q}^2}{2(M_1 + M_2)} \] (A.37)

and for guaranteeing that the internal energy \(\epsilon_0\) is independent of the total momentum of the system.

To determine the optimal variational functions we now vary Mano’s equation (A.35) with respect to \(A^{(1)}(E), A^{(2)}(E)\) and \(B(E)\)
\[ \frac{\delta}{\delta A^{(\nu)}(E)} (\Omega + V) = 0, \quad \frac{\delta}{\delta B(E)} (\Omega + V) = 0. \] (A.38)

The variation of the individual terms gives
\[
\frac{\delta \Omega}{\delta A^{(i)}} = c_1 \left( \frac{A^{(3-i)}(A^{(1)}(A^{(2)} - B)^2 + A^{(3-i)^2})}{A^{(1)}(A^{(2)} - B)^2} \right), \quad i = 1, 2 \quad \text{(A.39)}
\]
\[
\frac{\delta \Omega}{\delta B} = c_1 \left( \frac{-2B}{A^{(1)}(A^{(2)} - B)^2} + \frac{2(A^{(1)} + A^{(2)} B)}{A^{(1)}(A^{(2)} - B)^2} \right) \quad \text{(A.40)}
\]
\[
\frac{\delta V}{\delta A^{(i)}} = c_2 \frac{E^2}{2 \sqrt{M_{2(i)}}} \left( A^{(1)}(A^{(2)} - B)^2 - \left( \sqrt{\frac{M_1}{M_2} A^{(1)} + \sqrt{\frac{M_2}{M_1} A^{(2)} + 2B} A^{(3-i)} \right) \right) \quad \text{(A.41)}
\]
\[
\frac{\delta V}{\delta B} = c_2 \frac{E^2}{2 \sqrt{M_{2(i)}}} \left( -2 \left( A^{(1)}(A^{(2)} - B)^2 - \left( \sqrt{\frac{M_1}{M_2} A^{(1)} + \sqrt{\frac{M_2}{M_1} A^{(2)} - 2B} B \right) \right) \right) \quad \text{(A.42)}
\]

where we have omitted the argument \(E\) in \(A^{(i)}(E)\) and \(B(E)\); \(c_1 = 3/(2\pi)\) is a numerical constant whereas the constant
\[
c_2 = \frac{1}{2\pi} \frac{1}{\sqrt{M_1 M_2}} \int \frac{d^3 p}{(2\pi)^3} B^2 \tilde{V}(p) e^{-\nu^2 p^2} \quad \text{(A.43)}
\]
depends on \(\nu\), i.e. on the variational functions. We observe the relations
\[
2B \frac{\delta \Omega}{\delta A^{(i)}} + A^{(3-i)} \frac{\delta \Omega}{\delta B} = c_1 \frac{2B}{A^{(1)}(A^{(2)} - B)^2} \quad \text{(A.44)}
\]
\[
2B \frac{\delta V}{\delta A^{(i)}} + A^{(3-i)} \frac{\delta V}{\delta B} = \frac{2}{E^2} \left( \frac{\sqrt{M_{3(i)}} + \frac{c_1}{c_2} E^2}{M_i} \right) \quad \text{(A.45)}
\]

which (by adding suitable combinations of the variational equations) gives us
\[
A^{(i)}(E) = B(E) \left( \sqrt{\frac{M_{3(i)}}{M_i} + \frac{c_1}{c_2} E^2} \right). \quad \text{(A.46)}
\]

If this is inserted into one of the variational equations one finds after some algebra the simple solution
\[
B(E) = \frac{c_2}{c_1} \frac{1}{E^2} \quad \text{(A.47)}
\]

and hence
\[
A^{(i)}(E) = 1 + \sqrt{\frac{M_{3(i)}}{M_i} c_2} \frac{1}{c_1 E^2}. \quad \text{(A.48)}
\]
Defining
\[ \omega^2 := -\frac{M_1 + M_2}{\sqrt{M_1 M_2} c_2^2} \]  \hspace{1cm} (A.49)
we finally have as solutions of the variational equations in real time
\[ A^{(1)}(E) = 1 - \frac{1}{1 + \frac{M_1}{M_2}} \frac{\omega^2}{E^2}, \quad B(E) = -\frac{\sqrt{M_1 M_2}}{M_1 + M_2} \frac{\omega^2}{E^2}. \]  \hspace{1cm} (A.50)

Note that for equal masses
\[ A^{(1)}(E)\bigg|_{M_1 = M_2} = A^{(2)}(E) = A(E) = 1 - \frac{1}{2} \frac{\omega^2}{E^2}, \quad B(E)\bigg|_{M_1 = M_2} = -\frac{1}{2} \frac{\omega^2}{E^2}. \]  \hspace{1cm} (A.51)
so that \( A(E) - B(E) = 1 \) and \( A(E) + B(E) = 1 - \omega^2/E^2 \). With Eq. (A.43) the definition (A.49) for the quantity \( \omega \) reads
\[ \omega^2 = \frac{-1}{3} \frac{M_1 + M_2}{M_1 M_2} \int \frac{d^3p}{(2\pi)^3} p^2 \tilde{V}(p) e^{-ip^2 \nu}. \]  \hspace{1cm} (A.52)
This is an implicit equation for \( \omega \) since \( \nu \) depends on \( \omega \). Indeed substituting the solutions (A.50) into Eq. (A.51) one obtains
\[ \nu = \frac{1}{2\pi} \frac{1}{\sqrt{M_1 M_2}} \int_0^\infty dE \frac{1}{E^2} \sqrt{\frac{M_1/M_2 + \sqrt{M_2/M_1}}{1 - \omega^2/E^2 + i0}} = -\frac{i}{4\omega} \left( \frac{1}{M_1} + \frac{1}{M_2} \right). \]  \hspace{1cm} (A.53)
Here we had to specify how the singularity at \( E = \omega \) has to be treated. This can be inferred from the original real-time path integral by noting that the profile functions need an infinitesimal positive imaginary part for convergence of the integral. This is equivalent to \( \omega^2 \rightarrow \omega^2 - i0 \), Feynman’s prescription for the causal propagator. Note that the RHS of Eq. (A.52) is positive for an attractive potential as anticipated. The implicit equation becomes more familiar if we consider Mano’s equation before variation w.r.t. \( \omega \). The kinetic term (A.32) is
\[ \Omega = \frac{3}{2\pi^2} \int_0^\infty dE \left[ \ln \left( \frac{1}{E^2} \right) + \frac{2 - \omega^2/E^2}{1 - \omega^2/E^2 + i0} - 2 \right] = \frac{3}{4} \omega. \]  \hspace{1cm} (A.54)
and therefore the internal energy in Mano’s equation (A.33) is determined from
\[ \epsilon_0 = \frac{3}{4} \omega + \int \frac{d^3p}{(2\pi)^3} \tilde{V}(p) \exp \left( -\frac{p^2}{4M_{\text{red}} \omega} \right). \]  \hspace{1cm} (A.55)
which is \textit{exactly} the Rayleigh-Ritz variational principle with the normalized gaussian trial wave function
\[ \psi_1(x) = \left( \frac{M_{\text{red}} \omega}{\pi} \right)^{3/4} \exp \left( -\frac{M_{\text{red}} \omega x^2}{2} \right). \]  \hspace{1cm} (A.56)
Indeed variation of Eq. (A.56) w.r.t. \( \omega \) leads to Eq. (A.52) if Eq. (A.55) is taken into account.

Thus – as expected – in the nonrelativistic limit the Feynman-Jensen variational principle with a quadratic trial action is equivalent to the Rayleigh-Ritz variational principle with a gaussian trial wavefunction \(^{16}\). Apart from the mass-dependent factors the profile functions (A.50) are recognized as the standard ones for a harmonic oscillator with frequency parameter \( \omega \). It is this variational parameter which has to be optimized for the specific potential \( V(x) \) in the quantum mechanical problem.

For the Yukawa potential \( V(x) = -\alpha \exp(-mx)/x, x = |x| \), a simple calculation gives
\[ \epsilon_0 \leq \epsilon_{\text{var}}(y) = M_{\text{red}} \alpha^2 \left[ \frac{d}{4} \frac{1}{y^2} - \frac{2}{\sqrt{\pi}} \frac{1}{y} + 2\alpha e^{\alpha^2 y^2} \text{erfc}(\alpha y) \right]. \]  \hspace{1cm} (A.57)
where \( y = \alpha \sqrt{M_{\text{red}}/\omega} \) is the appropriate dimensionless variational parameter and \( \delta = m/(2\alpha M_{\text{red}}) \) was already defined in Eq. (2.6). \( \text{erfc}(x) = 1 - \text{erf}(x) \) denotes the complimentary error function.

\(^{16}\) As we have worked in real time where the Feynman-Jensen variational principle only ensures stationarity, we have missed the minimum property of the method. However, this is easily established in euclidean time where we can use the Feynman-Jensen inequality.
error function and we have explicitly introduced the dimension $d (=3)$ in the kinetic-energy term. $d = 4$ then also covers the weak-coupling limit of the relativistic variational calculation where -- as discussed -- the rigid, four-dimensional trial action leads to an enhanced kinetic term and therefore to less binding in the nonrelativistic limit. For small $\delta$ one obtains

$$
\epsilon_{\text{var}} = M_{\text{red}} \alpha^2 \left[ -\frac{4}{d\pi} + 2\delta - d\delta^2 + \frac{d^2\pi}{8} \delta^3 + \ldots \right]
$$

(A.58)

which means that the exact Coulomb result in Eq. (2.5) is missed by 15% for $d = 3$ and by 36% for $d = 4$. However, the first-order (for $d = 4$) and (for $d = 3$) also the second-order correction due to the finite meson mass come out correctly. At first sight this looks not very impressive for the variational calculation predicts bound states for the Yukawa potential if

The variational energy becomes zero or positive when

$$
\delta \geq \delta_{\text{crit}} = \frac{4}{d\sqrt{\pi}} \frac{s_0}{1 + s_0^d}
$$

(A.59)

where $s_0 \equiv \delta y$ fulfills the transcendental equation

$$
\sqrt{\pi} s_0 \left( 1 + s_0^d \right) \exp \left( s_0^d \right) \text{erfc} \left( s_0 \right) - s_0^d = \frac{1}{2}.
$$

(A.60)

Numerically one finds $s_0 = 0.821324$ and therefore $\delta_{\text{crit}} = 1.10687/d$. This means that the variational calculation predicts bound states for the Yukawa potential if

$$
\alpha \frac{2M_{\text{red}}}{m} \geq \left\{ \begin{array}{ll}
2.71035 & : d = 3 \\
3.61380 & : d = 4
\end{array} \right.
$$

(A.61)

Due to the minimum property of the variational calculation this is always larger than the exact number $\frac{2\pi}{\sqrt{\pi}}$.

### Appendix B: Calculation of Averages

To perform path integrals involving the trial action (A.6) we need the “master” integral in $d$ dimensions

$$
I_M = \int d^d x \int D^d a^{(1)} D^d a^{(2)} e^{i\tilde{S}_M} = : \int \tilde{D} e^{i\tilde{S}_M}
$$

(B.1)

where

$$
\tilde{S}_M = p \cdot x - \frac{\kappa_0}{2} A_x x^2 - \frac{\kappa_0}{2} \sum_{i=1}^2 \sum_{k=1}^{\infty} A_k^{(i)} a_k^{(i)2} + \kappa_0 \sum_{k=1}^{\infty} B_k a_k^{(1)} \cdot a_k^{(2)} + \sum_{i=1}^2 \sum_{k=1}^{\infty} f_k^{(i)} \cdot a_k^{(i)}.
$$

(B.2)

This is practically the trial action with sources $f_k^{(i)}$ coupled to the modes which will be needed to evaluate the average of the interaction. With the (ubiquitous) gaussian integral

$$
\int d^d a \exp \left[ -i \frac{\kappa_0}{2} A a^2 + i b \cdot a \right] = \frac{\text{const.}}{A^{d/2}} \exp \left[ i \frac{b^2}{2\kappa_0} A \right]
$$

(B.3)

we get by completing the square

$$
I_M = \text{const.} A_x^{-d/2} \prod_{k=1}^{\infty} \left( A_k^{(1)} A_k^{(2)} - B_k^2 \right)^{-d/2} \exp \left\{ \frac{i}{2\kappa_0} \frac{p^2}{A_x} + \frac{i}{2\kappa_0} \sum_{k=1}^{\infty} A_k^{(1)} A_k^{(2)} - B_k^2 \right\}
$$

$$
\cdot \left[ A_k^{(2)} f_k^{(1)} + A_k^{(1)} f_k^{(2)} + 2 B_k f_k^{(1)} \cdot f_k^{(2)} \right] = : \text{const.} \cdot \exp \left( \frac{i}{2\kappa_0} \mathcal{F}_M \right),
$$

(B.4)

where $\mathcal{F}_M$ is a function of $p, A_x, A_k^{(i)}, B_k$ and $f_k^{(i)}$. We are now in a position to calculate the averages required in the Feynman-Jensen variational principle (2.1). First

$$
\frac{\int \tilde{D} \exp(i\tilde{S}_t)}{\int \tilde{D} \exp(i\tilde{S}_0)} = \exp \left[ \frac{i}{2\kappa_0} (\mathcal{F}_t - \mathcal{F}_0) \right].
$$

(B.5)
Similarly we may obtain

\[
\lim_{\kappa \to 0} \left\{ \frac{i}{2\kappa} \left[ q^2 T_1 T_2 \left( \frac{\dot{x}^2}{A_0} - 1 \right) + i\kappa \right] \left( \ln A_0 + \sum_{k=1}^{\infty} \ln \left( A_k^{(1)} A_k^{(2)} - B_k^2 \right) \right) \right\}.
\]

(B.6)

For the denominator (i.e. \( F_\alpha \)) one sets \( p = q, A_x = 1/T_1 + 1/T_2 \) and in both cases the sources are zero: \( f_k^{(i)} = f_k^{(j)} = 0 \). This gives

\[
\frac{\int \hat{D} \exp(i\hat{S}_0)}{\int \hat{D} \exp(iS_0)} = \exp \left\{ \frac{i}{2\kappa} \left[ q^2 T_1 T_2 \left( \frac{\dot{x}^2}{A_0} - 1 \right) + i\kappa \right] \left( \ln A_0 + \sum_{k=1}^{\infty} \ln \left( A_k^{(1)} A_k^{(2)} - B_k^2 \right) \right) \right\}.
\]

Similarly we may obtain

\[
\left\langle S_0 \right\rangle_t = \frac{1}{i} \left[ \frac{\partial}{\partial \lambda} + \frac{\partial}{\partial A_0} + \sum_{i=1}^{\infty} \frac{\partial}{\partial A_k^{(i)}} \right] \ln \int \hat{D} e^{i\hat{S}_t} = \left[ \frac{\partial}{\partial \lambda} + \frac{\partial}{\partial A_0} + \sum_{i=1}^{\infty} \frac{\partial}{\partial A_k^{(i)}} \right] \left\langle S_0 \right\rangle_t = \frac{1}{2\kappa_0} \left[ q^2 T_1 T_2 \left( \frac{\dot{x}^2}{A_0} - 1 \right) + i\kappa_0 d \left( 1 + \sum_{k=1}^{\infty} \frac{A_k^{(1)} + A_k^{(2)}}{A_k^{(1)} A_k^{(2)} - B_k^2} \right) \right].
\]

(B.7)

Finally we calculate the average of the interaction part

\[
\left\langle S_{int} \right\rangle_t = -\sum_{\alpha, \beta=1}^{2} \frac{g_\alpha g_\beta^2 T_1 T_2}{8\kappa_0^2} \int_0^1 d\tau d\tau' \int \frac{d^4p}{(2\pi)^4} \frac{1}{\sqrt{p^2 - m^2 + i0}} \langle \exp[-ip \cdot (x_\alpha(\tau) - x_\beta(\tau'))] \rangle.
\]

(B.9)

Since

\[
p \cdot (x_\alpha(\tau) - x_\beta(\tau')) = -p \cdot x (\tau - \tau') - \sum_{k=1}^{\infty} \left( \frac{\sqrt{2T_1}}{k\pi} \sin(k\pi\tau) A_k^{(1)} - \frac{\sqrt{2T_2}}{k\pi} \sin(k\pi\tau) A_k^{(2)} \right),
\]

the average of the last exponential in Eq. (B.9) is given by

\[
\exp \left[ \frac{i}{2\kappa_0} \left( F_{int}(\alpha, \beta) - F_t \right) \right]\)

(B.10)

where

\[
F_{int}(\alpha, \beta) = F_M \left[ p \to \tilde{\lambda}q - p(\tau - \tau'), A_x \to A_0(1/T_1 + 1/T_2), A_k^{(i)}, B_k, \right]
\]

\[
f_k^{(i)} = -p \frac{\sqrt{2T_1}}{k\pi} \left( \delta_{\alpha i} \sin(k\pi\tau) - \delta_{\beta i} \sin(k\pi\tau') \right).
\]

(B.12)

With the help of the master integral \( B.4 \) we obtain

\[
F_{int}(\alpha, \beta) - F_t = \frac{\sum_{1}^{\infty} \frac{1}{(k\pi)^2} A_k^{(2)} - B_k^2}{\sum_{1}^{\infty} 2T_1 A_k^{(2)} - B_k^2} \left[ \left( \delta_{\alpha i} \sin(k\pi\tau) - \delta_{\beta i} \sin(k\pi\tau') \right) \right].
\]

(B.13)
Introducing the modified variational parameter $\lambda$ from Eq. \eqref{eq:lambda}, we therefore have

$$
\langle S_{\text{tot}} \rangle_t = -\frac{1}{8\kappa_0} \sum_{\alpha,\beta=1}^2 g_\alpha g_\beta T_\alpha T_\beta \int_0^1 d\tau d\tau' \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i0} \cdot \exp \left\{ \frac{i}{2\kappa_0} \left[ \frac{1}{2} \mu_{\alpha\beta}(\tau, \tau'; T_1, T_2) - 2\lambda_p \cdot q \cdot \frac{T_1 T_2}{T_1 + T_2} (\tau - \tau') \right] \right\}
$$

where

$$
\mu_{\alpha\beta}(\tau, \tau'; T_1, T_2) = \frac{T_1 T_2}{A_0(T_1 + T_2)} (\tau - \tau')^2 + \sum_{k=1}^{\infty} \frac{2}{(k\pi)^2} \left( A_k^{(2)} A_k^{(2)} - B_k^2 \right) \cdot \left\{ \delta_{\alpha\beta} T_\alpha (A_k^{(2)} \delta_{\alpha\alpha} + A_k^{(1)} \delta_{\alpha\beta}) \left[ \sin(k\pi \tau) - \sin(k\pi \tau') \right]^2 \right.

\left. + (1 - \delta_{\alpha\beta}) \left[ T_\alpha A_k^{(3)} \sin^2(k\pi \tau) + T_\beta A_k^{(3)} \sin^2(k\pi \tau') \right] - 2B_k \sqrt{T_1 T_2} \sin(k\pi \tau) \sin(k\pi \tau') \right\}.
$$

Putting everything together we now have

$$
\Pi^{\text{var}}(q) = -\frac{1}{16\pi^2} \int_0^\infty \frac{dT_1dT_2}{(T_1 + T_2)^2} \exp \left\{ \frac{i}{2\kappa_0} \left[ -(M_1^2 T_1 + M_2^2 T_2) + \frac{T_1 T_2}{T_1 + T_2} q^2 (2\lambda - \lambda^2) \right] - (T_1 + T_2) \left( \Omega_{12} + V \right) (T_1, T_2) \right\}
$$

where we have defined

$$
\Omega_{12}(T_1, T_2) := \frac{-i\kappa_0}{T_1 + T_2} \left\{ \ln A_0 + \frac{1}{A_0} - 1 + \sum_{k=1}^{\infty} \left[ \ln \left( A_k^{(1)} A_k^{(2)} - B_k^2 \right) + \frac{A_k^{(1)} + A_k^{(2)}}{A_k^{(3)} - B_k^2} - 2 \right] \right\}
$$

and

$$
V(T_1, T_2) = -\frac{2\kappa_0}{T_1 + T_2} \langle S_{\text{tot}} \rangle_t.
$$

Equation \eqref{eq:var} still holds for all $q$, i.e. also away from the poles of the polarization propagator.

**Appendix C: Asymptotic Behaviour of Profile functions and Pseudotimes**

Here we derive the limiting behaviour of the solutions of the variational equations both for $E, \sigma \to 0$ and for $E, \sigma \to \infty$. As the pseudotimes are Fourier cosine transforms of the (inverse) profile functions we expect, of course, that the large $E$-behaviour of the one function is related to the small $\sigma$-behaviour of the other and vice versa.

We first consider the limit $E \to \infty$ in Eqs. \eqref{eq:var}, namely one would just replace the squared trigonometric functions by their average value 1/2 and obtain $A_{\pm} \to 1 + (\text{const} + \text{const})/E^2$ where the constants are determined by $\int_0^\infty d\sigma \delta V_{11} / \delta \mu_{11}^2(\sigma)$. However, for $j = 1$ this integral does not exist as $\mu_{11}^2 \to \sigma$ for $\sigma \to 0$ and $\delta V_{11} / \delta \mu_{11}^2$ diverges as $1/\sigma^2$ in that limit (see Eq. \eqref{eq:delta}). This is, of course the standard behaviour for the self-energy part and the way to obtain the proper asymptotic behaviour is well-known: introduce a factor $\sigma^2/\sigma^2$ into the $\sigma$-integral and change to the variable $x = E\sigma$.

We then obtain

$$
A_{\pm}(E) = 1 + \frac{2}{\kappa E^2} \int_0^\infty d(x/E) \left( \frac{E}{x} \right)^2 \left( \sigma^2 \frac{\delta V_{11}}{\delta \mu_{11}^2(\sigma)} \right)_{\sigma = x/E} \sin^2 \left( \frac{x}{2E} \right)
$$

$$
\text{as} \quad E \to \infty \quad \frac{\sigma \to 0}{\text{lim}} \quad \frac{\sigma^2}{\delta \mu_{11}^2(\sigma)} \int_0^\infty dx \frac{\sin^2(x/2E)}{x^2} = 1 + \frac{\alpha M^2}{4\kappa E} E + \mathcal{O} \left( \frac{\ln^2 E}{E^2} \right)
$$

\[ \text{C.1} \]
where in the last step the \( (\sigma \to 0) \)-limit of Eq. \( (5.7) \) has been used. It should be emphasized that this is exactly the same asymptotic behaviour as in the one-body (self-energy) case (see Eq. (143) in ref. [1]); the only new information here is that both profile functions become identical at asymptotic values of \( E \). However, they have quite a different behaviour at small \( E \): whereas \( A_-(E) \) approaches a constant value

\[
A_-(0) = 1 + \frac{1}{2\kappa E} \int_0^\infty d\sigma \frac{1}{\sigma^2} \sum_{j=1}^2 \frac{\delta V_{ij}}{\delta \mu_{1j}(\sigma)}, \quad (C.2)
\]

\( A_+(E) \) diverges for small \( E \):

\[
A_+(E) \xrightarrow{E \to 0} \frac{\omega^2_{\text{var}}}{E^2} + \text{const.}, \quad \omega^2_{\text{var}} := \frac{2}{\kappa E} \int_0^\infty d\sigma \frac{\delta V_{12}}{\delta \mu_{12}(\sigma)}, \quad (C.3)
\]
as expected from the nonrelativistic limit. Consequently the pseudotimes also have quite different behaviour at small \( \sigma \). First, from Eq. (4.45) it is seen that \( \mu_{11} \) has the standard self-energy behaviour

\[
\mu_{11}(\sigma) = \frac{2}{\pi} \int_0^\infty dx \frac{\sin^2(x/2)}{x^2} \left[ \frac{1}{A_-(x/\sigma)} + \frac{1}{A_+(x/\sigma)} \right] \xrightarrow{\sigma \to 0} \frac{4}{\pi} \sigma \int_0^\infty dx \frac{\sin^2(x/2)}{x^2} = \sigma. \quad (C.4)
\]

For the low-\( \sigma \) behaviour of \( \mu_{12} \) we may just set \( \sigma = 0 \) in the definition to obtain

\[
\mu_{12}(\sigma) \xrightarrow{\sigma \to 0} = \frac{2}{\pi} \int_0^\infty dE \frac{1}{E^2 A_+(E)} = \mu_{12}(0), \quad (C.5)
\]

where the integral exists due to the behaviour of \( A_+(E) \) at small \( E \). Thus for \( \sigma \to 0 \) the ‘interaction’ pseudotime stays constant and nonzero as assumed. Note that this constant is not the inverse of the \( \omega_{\text{var}} \) defined in Eq. (C.3) since \( A_+(E) \) deviates from \( 1 + \omega^2_{\text{var}}/E^2 \) at \( E > 0 \).

It is also possible to work out higher-order terms by considering

\[
\tilde{\mu}_{12}(\sigma) = \frac{1}{\pi} \int_0^\infty dE \left[ \frac{1}{A_-(E)} - \frac{1}{A_+(E)} \right] \cos (E\sigma). \quad (C.6)
\]

Because both profile functions have the same asymptotic behaviour for large \( E \) the integral also exists for \( \sigma = 0 \). Therefore we obtain after integration with the boundary condition \( \tilde{\mu}_{12}(0) = 0 \)

\[
\mu_{12}(\sigma) \xrightarrow{\sigma \to 0} \mu_{12}(0) + \frac{1}{2\pi} \int_0^\infty dE \left[ \frac{1}{A_-(E)} - \frac{1}{A_+(E)} \right] \cdot \sigma^2 + \ldots. \quad (C.7)
\]

For the self-energy part the situation is different:

\[
\tilde{\mu}_{11}(\sigma) = \frac{1}{\pi} \int_0^\infty dE \left[ \frac{1}{A_-(E)} + \frac{1}{A_+(E)} - 2 \right] \cos (E\sigma) \quad (C.8)
\]
does not exist for \( \sigma = 0 \) as the integrand behaves like \( E^{-1} \) for large \( E \) as can be seen from Eq. (C.2). This means that an expansion only in powers of \( \sigma \) is not possible and that logarithmic terms appear in higher order. The leading term is obtained by realizing that the cosine provides an upper limit const./\( \sigma \) for the \( E \)-integration. Therefore

\[
\tilde{\mu}_{11}(\sigma) \xrightarrow{\sigma \to 0} \frac{1}{\pi} \int_{\text{const.}/\sigma}^\infty dE \left[ -\frac{\alpha}{2\kappa E} \right] = \frac{\alpha}{2\pi} \frac{M^2}{\kappa E} \ln \frac{\sigma}{\sigma_0}, \quad (C.9)
\]

where \( \sigma_0 \) is some undetermined constant. Integration with the appropriate boundary conditions gives

\[
\tilde{\mu}_{11}(\sigma) \xrightarrow{\sigma \to 0} \sigma + \frac{\alpha}{4\pi} \frac{M^2}{\kappa E} \sigma^2 \ln \frac{\sigma}{\sigma_1} + \ldots \quad (C.10)
\]

where \( \sigma_1 \) is another constant to be determined at \( O(\sigma^2) \). In the framework of generalized functions one obtains the same result from the asymptotic behaviour of \( \tilde{\mu}_{11}(\sigma) \) in the definition (C.3) of the profile functions in the definition (C.10) of \( \mu_{11}(\sigma) \) (see Table 1 from ref. [12] !)

\[\text{Note the different definition of the Digamma function in this work compared to ref. [12].}\]
of the pseudotime \( \mu_{11}^2(\sigma) \) induces the \( \ln^2(E)/E^2 \)-remainder in the asymptotic expansion of the profile functions which is essential for the convergence of several integrals encountered above.

Finally we consider the pseudotimes for large \( \sigma \) by adding and subtracting the term \( 1/A_-(0) \) from the inverse pseudotimes in the integrand. This gives

\[
\mu_{11}^2(\sigma) = \frac{2}{\pi} \int_0^\infty dE \frac{\sin^2(E\sigma/2)}{E^2 A_-(0)} + \frac{2}{\pi} \int_0^\infty dE \frac{\sin^2(E\sigma/2)}{E^2} \left[ \frac{1}{A_-(E)} - \frac{1}{A_-(0)} + \frac{1}{A_+(E)} \right]
\]

\[\sigma \to \infty \]

\[
\frac{\sigma}{2A_-(0)} + \frac{1}{\pi} \int_0^\infty dE \frac{1}{E^2} \left[ \frac{1}{A_-(E)} - \frac{1}{A_-(0)} + \frac{1}{A_+(E)} \right]
\]

where the last integral exists due to the proper behaviour of the integrand at \( E = 0 \) and \( E = \infty \). Since the last line in Eq. (C.11) was obtained by replacing \( \sin^2(E\sigma/2) \) asymptotically by \( 1/2 \) it also holds for \( \mu_{12}^2 \) where \( 1/A_+(E) \) is weighted with \( \cos^2(E\sigma/2) \).

**Appendix D: Numerical Details**

Here we describe a few modifications and improvements of the numerical methods used in ref. (II) in order to achieve stable and reliable results for the bound-state problem. As before we have used Gauss-Legendre integration rules with \( n g \) gaussian points and \( n e \) extra divisions to evaluate

\[
\int_a^b dy f(y) \approx \frac{h}{2} \sum_{i=1}^{n_e} \sum_{j=1}^{n_g} w_j f(y_{ij}) , \quad y_{ij} = a + \left( i - \frac{1}{2} \right) h + \frac{h}{2} x_j , \quad h = \frac{b-a}{n_e}
\]

where \( x_j, w_j \) are the usual abscissae and weights for Gauss-Legendre numerical integration in the interval \([-1, 1]\) (see Eq. 25.4.30 in ref. [42]). A major problem is the calculation of the pseudotimes from the profile functions at large \( E, \sigma \) since the integrand \( \sin^2(E\sigma/2) \) leads to huge oscillations and loss of accuracy when the integrals are evaluated by mapping to a finite interval (e.g. by the transformation \( E = E_0 \tan \psi, \sigma = \eta_1^{-1} \tan \phi \), \( E_0 \) a suitable scale) and subsequent gaussian integration. This procedure was adopted in ref. (II) and found to be sufficient for the one-body case. Recently we have shown that one can eliminate the profile functions altogether and solve a nonlinear, delay-type equation for the pseudotime which bears a striking similarity to the classical Abraham-Lorentz equation [32]. This works perfectly well for the one-body case but would need a different strategy for the bound-state problem. Therefore for the present purposes we have decided to stay in the proven, conventional scheme but to take finite upper limits \( E_{\text{max}}, \sigma_{\text{max}} \) for the integrations. The upper limit for \( E \) is chosen such that the asymptotic contribution \( (D.2) \) at \( E = E_{\text{max}} \) is less than \( 5 \times \Delta \) where \( \Delta \) is the measure of deviation which should be reached in the iteration (see below). Thus for \( \kappa_E = 1 \)

\[
E_{\text{max}} = \frac{\alpha M^2}{4 5\Delta} .
\]

In all \( E \)-integrals the numerical integration is done up to \( E = E_{\text{max}} \) and the asymptotic contribution from \( E_{\text{max}} \) up to \( \infty \) is evaluated analytically and added to the numerical result. Similarly, an upper limit for the \( \sigma \)-integration is determined by requiring that the \( u \)-integral over the ubiquitous exponential \( e(u, \sigma) \) in Eq. (D.10)

\[
K(x, y) = \int_0^1 du \exp \left( -x \frac{1-u}{u} - y u \right) , \quad x = \frac{1}{2} m^2 \mu^2(\sigma) , \quad y = \frac{1}{2} \frac{(\lambda q/2)^2 \sigma^2}{\mu^2(\sigma)}
\]

should be smaller than \( e^{-15} = 3.1 \cdot 10^{-7} \). Using the large-\( \sigma \) limit of this expression from Eqs. (82) - (84) in ref. [29] we therefore have to take

\[
\sigma_{\text{max}} \approx \frac{15}{\ln M} .
\]

Also, in all \( \sigma \)-integrals the numerical integration is done up to \( \sigma = \sigma_{\text{max}} \) and the asymptotic contribution from \( \sigma_{\text{max}} \) till \( \infty \) is evaluated analytically and added to the numerical result. This strategy works very well as we know the large \( (E, \sigma) \)-behaviour of profile functions and pseudotimes (see appendix C).
The number of gaussian points \( n_g \) (typically 72) and subdivisions \( n_e \) (typically 3 – 4) should, of course, be adapted to the values of \( E_{\text{max}}, \sigma_{\text{max}} \): ideally at least one point should be on each oscillation of the trigonometric function \( \sin^2(E\sigma/2) = (1 - \cos(E\sigma))/2 \) for all values of \( E, \sigma \) but experience showed that \( E_{\text{max}} \sigma_{\text{max}}/(\pi n_g n_e) \approx 3 - 4 \) was sufficient.

In principle, the \( \mu \)-integral for \( K(x,y) \) in Eq. (5.12b) can be expressed in terms of a (non-standard) special function, i.e. either by a special Shkarofsky plasma dispersion function (Eq. (26) in ref. [3]) or by an incomplete MacDonald function of first order in Bessel form (as defined in Eq. (III.1.8) of ref. [4]). However, by doing so nothing is gained for practical computations. Therefore, for simplicity and ease of implementation, we have used a direct gaussian integration of the defining integral (D.2) with the same number of gaussian points and subdivisions as for the solution of the variational equations. Of course, for \( m = 0 \) this is not necessary as the \( \mu \)-integration can be done analytically.

The problem of numerical computation of integrals with oscillating integrands is also present when one tries to evaluate the virial theorem in the form of Eq. (5.13). Here the derivative term in the virial theorem (5.13) may be eliminated completely. However, Table 1 and 3 show that below the critical coupling such a form of the virial theorem is not needed as the virial check is satisfied within a factor of two of the relative deviation which measures how well the variational equations are fulfilled.

If one solves the coupled variational equations by iteration there is a crucial difference now compared to the one-body case in that one cannot start with \( x \) from Eq. (C.3) each time during the iteration.

The iteration was stopped when the global measure \( \Delta_i(x) \) was less than a prescribed value \( \Delta_{\text{var}} \); this is done by updating the coefficient \( \lambda \) and \( \omega \) care; this is done by updating the coefficient \( \lambda \) and \( \omega \) after each iteration was performed with a prescription given by Hauck et al. [7] which suppresses oscillating values from one iteration to the next by using an exponential weighing of old and new values:

\[
1 - e^{-\Delta_i(x)} x^{i+1} + e^{-\Delta_i(x)} x^{(i)} \rightarrow x^{(i+1)}.
\]

This is most effective in the early stages of the iteration when the relative deviation \( \Delta_i(x) \) is still large.

The iteration was stopped when the global measure \( \Delta_i \) was less than a prescribed value for which typically \( 10^{-3} \) was taken. Although this is far away from the \( 10^{-5} \) - accuracy which was easily obtained in the one-body case, it should be sufficient for the present purposes. Given the much more demanding numerical problems of the two-body case we believe that any improvement would need much more computing power (i.e. more integration points) and/or more efficient algorithms [10, 11] to solve the nonlinear integral equations.
Appendix E: Critical Ouling Constant and Width: Analytic Approximation

Here we show how one obtains the critical coupling from the simple ansatz $A_-(E) = 1, A_+(E) = 1 + \omega^2/E^2$, keeping $\lambda$ and $\omega$ as variational parameters. For this choice and $\kappa = 1$ one easily finds the kinetic terms and pseudotimes as

$$\Omega_- = 0 \quad , \quad \Omega_+ = \omega \quad , \quad \mu_{1j}^2(\sigma) = \frac{\sigma}{2} + \frac{1}{2\omega} \left[ 1 + (-)^j e^{-\omega \sigma} \right] \quad (j = 1, 2). \quad (E.1)$$

Furthermore we assume weak binding

$$\omega \left( \frac{2q}{M} \right)^2 \simeq \omega M^2 \ll 1 \quad (E.2)$$

and that $\omega \sigma \ll 1$ in all integrals over the proper time. Then we may take approximately

$$\mu_{11}^2(\sigma) \simeq \sigma \quad , \quad \mu_{12}^2(\sigma) \simeq \frac{1}{\omega} \quad (E.3)$$

and obtain after performing the $\sigma$-integration

$$V_{11} \simeq \frac{\alpha}{2\pi} M^2 \int_0^1 du \ln \left[ 1 + \frac{\left( \frac{\lambda q}{2} \right)^2 u^2}{m^2} \right] \quad (E.4)$$

$$V_{12} \simeq -Z\frac{\alpha}{\sqrt{2\pi}} M^2 \sqrt{\frac{\omega}{(\lambda q/2)^2}} \int_0^1 du \frac{1}{2\sqrt{u}} \exp \left[ -\frac{m^2}{2\omega} u \right]. \quad (E.5)$$

Equation (E.4) is identical with Eq. (46) in ref. [II] if we use the weak binding approximation $q/2 = M$. Mano’s equation then approximately reads

$$\tilde{M}^2 = (2\lambda - \lambda^3) \left( \frac{q}{2} \right)^2 - \omega - 2 (V_{11} + V_{12}) . \quad (E.6)$$

We now vary w.r.t. $\lambda$ and $\omega$ and obtain (for the case $m = 0$ where the $u$-integrals can be performed analytically)

$$\frac{1}{\lambda} = 1 + \frac{\alpha}{\pi} \frac{M^2}{(\lambda q/2)^2} \left[ 1 + Z \sqrt{\frac{\pi \omega}{2(\lambda q/2)^2}} \right] \quad (E.7)$$

$$\omega = \frac{(Z\alpha)^2}{2\pi} \frac{M^4}{(\lambda q/2)^2}. \quad (E.8)$$

Note that Eq. (E.8) reduces to the $d = 4$-result in Eq. (4.60) if we replace $\lambda q/2$ by $M$. Here, however we keep the $\lambda$-dependence which in the one-body case was essential for probing the instability of the WC model. Inserting Eq. (E.8) into the $\lambda$-Eq. (E.7) and taking $q/2 \approx M$ now gives a quartic equation for the variational parameter

$$\lambda^4 - \lambda^3 + \frac{\alpha}{\pi} \lambda^2 + \frac{(Z\alpha)^2}{2\pi} = 0 \quad (E.9)$$

which for $Z = 0$ reduces to the approximate one-body equation (149) in ref. [I]. To find the critical value of the coupling constant where this equation ceases to have real solutions we do not have to use the cumbersome explicit solutions; rather we observe that branching into the complex plane occurs when $\partial \lambda / \partial \alpha = \infty$. Therefore we may differentiate Eq. (E.8) w.r.t. to $\lambda$ and setting $\partial \alpha / \partial \lambda = 0$ we obtain

$$4\lambda^3_{\text{crit}} - 3\lambda_{\text{crit}} + \frac{(Z\alpha)^2}{\pi} = 0 \quad (E.10)$$

If this is put back into Eq. (E.9) we can solve for

$$\lambda_{\text{crit}} = \frac{1}{4} \left( 1 - z \right) \left[ 1 - 3z + \sqrt{1 + 3z} \right] \quad (E.11)$$

where

$$z = 2\pi Z^2 \quad (E.12)$$
and obtain the critical coupling as

$$\alpha_{\text{crit}} = \frac{\pi}{8} \frac{1 - 9z + (1 + 3z)^{3/2}}{(1 - z)^2}. \quad (E.13)$$

By simple algebraic manipulations this can be brought into the form \(E.13\).

To obtain an analytic estimate for the width of the ground state above the critical coupling one allows the product \(\zeta = \lambda q/2 = : \chi_0 \exp(-i\chi)\) to become complex. The quartic equation then reads

$$\frac{1}{\chi} = 1 + \frac{\alpha}{\pi} \frac{M^2}{\zeta^2} + \frac{(Z\alpha)^2}{2\pi} \frac{M^4}{\zeta^4}. \quad (E.14)$$

If this and the variational solution \(E.8\) are inserted into the approximation \(E.6\) we obtain

$$\zeta^2 = M^2 - 2\frac{\alpha}{\pi} M^2 - \frac{3(Z\alpha)^2}{2\pi} \frac{M^4}{\zeta^2} + \frac{\alpha}{\pi} M^2 \int_0^1 \frac{du}{\ln} \left[ 1 + \frac{\zeta^2}{m^2} \frac{u^2}{1 - u} \right]$$

which is the generalization of Eq. (48) in ref. [II]. For the imaginary part of this equation it is possible to set immediately the meson mass to zero \(\chi^2\) with the result

$$\zeta_0^2 \sin 2\chi \left[ 1 - \frac{3(Z\alpha)^2}{2\pi} \frac{M^4}{\zeta_0^2} \right] = \frac{2\alpha}{\pi} M^2 \chi \quad (E.16)$$

(compare with Eq. (49) in ref. [II]). Equation \(E.10\) allows the modulus \(\zeta_0\) to be determined in terms of the phase \(\chi\):

$$\zeta_0^2 = \frac{\alpha}{\pi} M^2 \frac{1}{2S(\chi)} \left[ 1 + \sqrt{1 + 6\pi Z^2 S^2(\chi)} \right], \quad S(\chi) := \frac{\sin 2\chi}{2\chi} \left( \frac{\chi}{\pi} \right)^{9/2} \left( 1 - \frac{2}{3} \chi^2 + \ldots \right) \quad (E.17)$$

For \(Z = 0\) this reduces to Eq. (49) in ref. [II]. The width is obtained from Eqs. \(E.23\) and \(E.11\)

$$M - i \frac{\Gamma}{4} = \zeta + \frac{(Z\alpha)^2}{2\pi} \frac{M^4}{\zeta^2} + \frac{\alpha}{\pi} \frac{M^2}{\zeta} \quad (E.18)$$

by taking real and imaginary parts:

$$M = \zeta_0 \cos \chi \left[ 1 + \frac{\alpha}{\pi} \frac{M^2}{\zeta_0^2} \right] + \frac{(Z\alpha)^2}{2\pi} \frac{M^4}{\zeta_0^2} \cos 3\chi \quad (E.19)$$

$$\Gamma = 4\zeta_0 \sin \chi \left[ 1 - \frac{\alpha}{\pi} \frac{M^2}{\zeta_0^2} \right] - 2\frac{(Z\alpha)^2}{\pi} \frac{M^4}{\zeta_0^2} \sin 3\chi. \quad (E.20)$$

With the help of Eq. \(E.16\) the last equation can be brought into the form

$$\Gamma = 4\zeta_0 \sin \chi \left[ 1 - S(\chi) \right] + \frac{2(Z\alpha)^2}{\pi} \frac{M^4}{\zeta_0^2} \left[ 3 \sin \chi S(\chi) - \sin 3\chi \right] \quad (E.21)$$

which shows that close to the critical coupling \((\chi \to 0)\) the width behaves as

$$\Gamma \sim \zeta_{\text{crit}}^2 \frac{8}{3} \left[ 1 + \frac{3(Z\alpha_{\text{crit}})^2}{2\pi} \frac{M^4}{\zeta_{\text{crit}}^4} \right] \chi^3. \quad (E.22)$$

Here

$$\zeta_{\text{crit}}^2 = \frac{\alpha_{\text{crit}}}{\pi} M^2 \frac{1}{2} \left[ 1 + \sqrt{1 + 6z} \right] \quad (E.23)$$

is the value of the modulus at the critical coupling \(\chi = 0\). The parameter \(z\) has been defined in Eq. \(E.12\).

It remains to determine \(\chi(\alpha)\). As in ref. (II) one obtains the inverse relation by combining Eq. \(E.11\) with Eq. \(E.12\). After some algebra this gives

$$\alpha = 2\pi S(\chi) [w(z, \chi)]^3 \cdot \left\{ \cos \chi [1 + S(\chi)] [w(z, \chi)]^2 + z S^2(\chi) [\cos 3\chi - 3 S(\chi) \cos \chi] \right\}^{-2} \quad (E.24)$$

\(^{18}\)This can be also done for the real part if we eliminate \(\bar{M}\) from Mano’s one-body equation, e.g. by setting \(Z = 0\) in Eq. \(E.13\).
where $w(z, \chi) = 1 + \sqrt{1 + 3zS(\chi)}$. This result is the generalization of Eq. (53) in ref. [II] to which it exactly reduces for $Z = z = 0$. Expanding $\alpha$ around the critical value in powers of $\chi$ yields

$$\alpha = \alpha_{\text{crit}} \left[ 1 + \alpha_2 \chi^2 + \ldots \right]$$

where

$$\alpha_2 = \frac{1 + 2z + \sqrt{1 + 3z}}{1 + z + \sqrt{1 + 3z}}.$$  

Finally instead of Eq. (57) in ref. [II] one gets from Eq. (E.22) the following width for the ground state

$$\Gamma \simeq \frac{2}{3} 2M \left( \frac{\alpha - \alpha_{\text{crit}}}{\alpha_{\text{crit}}} \right)^{3/2} f_{\text{crit}}(Z) + \ldots$$

where the explicit form of the correction factor is given in Eq. (E.20).

Appendix F: Weak-Binding Limit

Here we derive the weak-coupling limit for the binding energy in the case of massless pions (which is the original Wick-Cutkosky model). Setting $m = 0$ allows us to perform the calculations analytically. First this is done for the variational approximation and then in the potential version of effective field theory appropriate for the present model [78].

F.1 Variational Calculation

For the variational approach a convenient starting point is the Feynman-Hellmann theorem [53,77] for which one only has to evaluate

$$V_{12} = \frac{\alpha}{\pi} \frac{M^2}{(\lambda q/2)^2} \int_0^\infty d\sigma \frac{1}{\sigma^2} \left\{ 1 - \exp \left[ -\frac{(\lambda q/2)^2\sigma^2}{2\mu_{12}^2(\sigma)} \right] \right\}.$$  

(A.1)

Again, we have chosen to work in the reparametrization “gauge” $\kappa_E = 1$. From Eq. (E.1) we infer that approximately $\mu_{12}^2(\sigma) \simeq 1/\omega + \omega\sigma^2/4 + \ldots$ with $\omega = O((Z\alpha)^2)$ (see Eq. (E.8)). Therefore we insert the expansion

$$\mu_{12}^2(\sigma) = u_0 + u_2\sigma^2 + \ldots,$$  

(u.1)

into Eq. (A.1). A simple calculation then gives

$$V_{12} = \frac{Z\alpha}{\sqrt{2\pi}} \frac{M^2}{\lambda q/2} \frac{1}{\sqrt{u_0}} \left[ 1 - \frac{1}{2} \frac{u_2}{(\lambda q/2)^2} + \ldots \right].$$

(A.3)

We have to determine the coefficients $u_0, u_2$ from the variational equations. This we can do perturbatively except that the low-$E$ behaviour $A_i(E) \rightarrow \omega_{\text{var}}^2/E^2$ needs to be included to all orders to generate a bound state. Thus from Eq. (A.12) and Eq. (A.5) we have

$$u_0 = \frac{1}{\omega_{\text{var}}} \left[ \frac{1}{4} \int_0^\infty d\sigma \left( \delta V_{12} \right)^2 \right].$$

(A.4)

We have used $\omega_{\text{var}} \ll 1$ and the abbreviation $\delta V_{i,j} := \delta V_{ij}/\delta \mu_{12}^2(\sigma)$. If the result (A.4) is inserted into the equation defining $\omega_{\text{var}}^2$

$$\omega_{\text{var}}^2 \equiv \int_0^\infty d\sigma \delta V_{12} = \frac{Z\alpha}{\sqrt{2\pi}} \frac{M^2}{\lambda q/2} \frac{1}{u_0} \left[ 1 - \frac{1}{2} \frac{u_2}{(\lambda q/2)^2} + \ldots \right].$$

(A.5)

we obtain

$$\omega_{\text{var}}^2 \equiv \frac{Z\alpha}{\sqrt{2\pi}} \frac{M^2}{\lambda q/2} \left[ 1 - \frac{1}{4} \int_0^\infty d\sigma \left( \delta V_{11} - \delta V_{12} \right)^2 \right].$$

(A.6)
The correction terms may be safely expanded and the approximations \( \lambda \simeq 1, \; q/2 \simeq M, \; \mu_{1/2} \simeq \sigma, \; \mu_{11} \simeq 1/\omega, \; u_2 \simeq \omega/4, \; \omega \simeq (Z\alpha)^2/(2\pi) \) used without impunity. Thus we obtain

\[
\omega_{\text{var}} = \frac{(Z\alpha)^2}{2\pi} \frac{M^4}{(\lambda q/2)^2} \left[ 1 + \frac{3\alpha}{4\pi} - \frac{5}{16} \frac{(Z\alpha)^2}{\pi} + \ldots \right].
\] (F.7)

Note that the RHS of Eq. (F.6) is just \(-V_{12}/u_0\). Hence

\[
V_{12} \simeq -u_0 \omega_{\text{var}} = -\frac{(Z\alpha)^2}{2\pi} \frac{M^4}{(\lambda q/2)^2} \left[ 1 + \frac{\alpha}{2\pi} - \frac{5}{4} \frac{(Z\alpha)^2}{\pi^2} + \ldots \right].
\] (F.8)

It remains to determine the variational parameter \( \lambda \) up to order \( \alpha, (Z\alpha)^2 \).

This can be done by using the variational Eq. (6.3) with the zeroth-order approximations for the pseudotimes or simply by expanding Eq. (127) which is correct to that order. In both cases one obtains

\[
\lambda = 1 - \left[ \frac{\alpha}{\pi} + \frac{(Z\alpha)^2}{2\pi} \right] + \ldots,
\] (F.9)

which now allows application of the Feynman-Hellmann theorem

\[
\left( \frac{q}{2} \right)^2 \frac{\partial}{\partial Z} \left( \frac{q}{2} \right)^2 = -\frac{Z\alpha^2}{\pi} M^4 \left[ 1 + \frac{7}{2} \frac{\alpha}{\pi} + \frac{5}{8} \frac{\alpha^2}{\pi^2} + \ldots \right].
\] (F.10)

Integrating on both sides and using \( q/2 = M + \epsilon_0 \) gives

\[
(M + \epsilon_0)^4 = \text{const.} - \frac{(Z\alpha)^2}{\pi} M^4 \left[ 1 + \frac{7}{2} \frac{\alpha}{\pi} + \frac{5}{8} \frac{(Z\alpha)^4}{\pi^4} + \ldots \right].
\] (F.11)

The integration constant must equal \( M^4 \) because the two particles are unbound for \( Z = 0 \). Therefore the ground-state binding energy has the weak-coupling expansion

\[
\epsilon_0 = -\frac{M}{2} \left[ \frac{(Z\alpha)^2}{\pi} \left( 1 + \frac{7}{2} \frac{\alpha}{\pi} + \frac{(Z\alpha)^4}{\pi^4} + \ldots \right) \right].
\] (F.12)

### F.2 Effective Field-Theory Calculation

For simplicity, we consider here not the Lagrangian (127) but a system of identical nucleons described by the Lagrangian

\[
\mathcal{L} = \Phi^\dagger \left( -\partial^2 - M^2 + g' \chi \right) \Phi + \mathcal{L}_0(\chi)
\] (F.13)

where the last term is the free Lagrangian for the mesons. In the nonrelativistic limit \( (M \to \infty) \) we make the ansatz

\[
\Phi(x) = \frac{1}{\sqrt{2M}} e^{-imx_0} \phi(x, x_0 = t)
\] (F.14)

to describe particles (anti-particles would have a different sign in the phase factor and are omitted as explicit degrees of freedom in the following). This leads to

\[
\mathcal{L} = \phi^\dagger \left( i\partial_t + \frac{\Delta}{2M} + \frac{g'}{2M} \chi - \frac{1}{2M} \partial_t^2 \right) \phi + \mathcal{L}_0(\chi).
\] (F.15)

The action is the 4-dimensional integral over the Lagrangian (density) so that an integration by parts brings the last term in the brackets into the form \( \phi^\dagger \phi/(2M) \). Using the equation of motion for the nonrelativistic field \( \phi \)

\[
\left( -2iM\partial_t - \Delta + \partial_t^2 \right) \phi(x, t) = g' \chi \phi(x, t)
\] (F.16)

one obtains

\[
\dot{\phi}(x, t) \simeq \frac{i}{2M} (\Delta + g' \chi) \phi(x, t).
\] (F.17)

Therefore

\[
\mathcal{L}_3 = -\frac{1}{2M} \phi^\dagger \partial_t^2 \phi \longrightarrow \frac{1}{(2M)^2} \phi^\dagger (\Delta + g' \chi)^2 \phi
\] (F.18)
is the $1/M^3$-correction to the leading nonrelativistic Lagrangian. Its form is in agreement with Eq. (3.6) or Eq. (4.12) in ref. [79] and can also be easily obtained from the Foldy-Wouthuysen-Tani Hamiltonian $\phi^1 \sqrt{M^2 - \Delta - g' X} \phi$ by an expansion in inverse powers of the heavy mass $M$. Note that a “seagull” term $\phi^1 X' \phi$ appears in the nonrelativistic field theory.

Since one has changed the high-energy behaviour of the theory the correct effective Lagrangian (for particles) is

$$\mathcal{L}_{\text{eff}} = \phi^1 \left( i \partial_t + \frac{\Delta}{2M} + \frac{\Delta^2}{8M^2} + \frac{g'}{2M} c_1 \chi \right) \phi + \frac{1}{(2M)^2} \phi^1 \left[ g' c_2 \left( \Delta X + \chi \Delta \right) + g' c_3 \chi^2 + g' d_1 \left( \Delta \chi \right) \right] \phi + \ldots + \mathcal{L}_0(\chi). \quad (F.19)$$

This should be accurate up to order $M v^4$ in the (velocity) counting rules \(^{19}\) in which

$$\begin{align*}
\partial_t &= \mathcal{O}(M v^2) \quad \text{(nonrelativistic energy)} \\
\Delta &= \mathcal{O}(M^2 v^2) \Rightarrow \frac{\Delta}{2M} = \mathcal{O}(M v^2) \quad \text{(nonrelativistic kinetic energy)} \\
g' \frac{g}{2M} &= \sqrt{4\pi \alpha} = \mathcal{O}(v^{1/2}) \quad \text{(velocity in Bohr orbit = } \alpha) \\
\chi &= \mathcal{O}(M v^{3/2}) \Rightarrow \frac{g'}{2M} \chi = \mathcal{O}(M v^2) \quad \text{(potential energy).} \quad (F.20)
\end{align*}$$

In Eq. (F.19) the “Wilson-coefficients” $c_i = 1 + \mathcal{O}(g^2)$ encode the missing high-energy (short-distance) information. They are obtained by “matching”, i.e. by comparing physical amplitudes in the full and the effective theory. The coefficient $d_1$ is expected to be $\mathcal{O}(g^2)$ since it is not present in the tree-level calculations and this is confirmed by the explicit calculation below. Note that the kinetic terms $\Delta/(2M), \Delta^2/(8M^2)$ are not renormalized since they reflect the exact Lorentz symmetry of the underlying relativistic theory.

We determine the Wilson coefficients $c_1, c_2, d_1$ by evaluating the meson-nucleon scattering amplitude in both theories including the lowest-order radiative corrections. To do that we also need the residue $Z_r$ of the nucleon 2-point function

$$\frac{1}{p^2 - M_0^2 + \Sigma(p^2)} \rightarrow \frac{Z_r}{p^2 - M^2} \quad (F.21)$$

where $M^2 = M_0^2 + \Sigma(M^2)$ and

$$Z_r^{-1} = 1 + \left. \frac{\partial \Sigma(p^2)}{\partial p^2} \right|_{p^2 = M^2}. \quad (F.22)$$

Of course, in lowest order we have $Z_r^{(0)} = 1$. Using Muta’s conventions \(^{30}\) we obtain for the 1-loop self-energy (depicted in fig. 14)

$$\Sigma^{(1)}(p^2) = g'^2 \nu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{M_0^2 - (p - k)^2} \frac{1}{m^2 - k^2}$$

$$= g'^2 \Gamma(2 - d/2) \left(\frac{4\pi}{d - 2}\right)^{d/2} \int_0^1 dx \frac{1}{M_0^2 x + m^2(1 - x) - p^2 x(1 - x)} x^{3-d/2}. \quad (F.23)$$

Here we have introduced standard Feynman parameters and performed the diverging momentum integral in $d$ dimensions. In this order we may replace the bare mass $M_0$ by the physical mass $M$, differentiate w.r.t. $p^2$, expand and find

$$Z_r^{(1)} = -g'^2 \nu^{4-d} \frac{\Gamma(3 - d/2)}{(4\pi)^{d/2}} \int_0^1 dx \frac{x(1 - x)}{[M^2 x^2 + m^2(1 - x)]^{3-d/2}}$$

$$= g'^2 \frac{16\pi^2}{16\pi^2} \int_0^1 dx \frac{x(1 - x)}{M^2 x^2 + m^2(1 - x)}. \quad (F.24)$$

\(^{19}\)Since $v \equiv c\|/c$ the counting is equivalent to the expansion in appendix A.1 but the procedure is a systematic one which also allows to include loop effects.
which is UV-finite but IR-divergent for $m = 0$.

The tree-level truncated nucleon-meson $(2,1)$-point function (multiplied by $-i$) is just the coupling constant $g'$. The one-loop correction to this vertex function is shown in fig. 6b, and leads to

\[
\Gamma^{(1)}(p, q) = g'^3 \int \frac{d^4k}{(2\pi)^4} \frac{1}{M^2 - (p - k)^2} \frac{1}{M^2 - (p - k + q)^2} \frac{1}{m^2 - k^2}.
\]

(F.25)

For $q \to 0$ we obtain by standard techniques

\[
\Gamma^{(1)}(p, q \to 0) = \frac{g'^3}{16\pi^2} \int_0^1 dx \frac{x}{M^2 x + m^2 (1 - x) - p^2 x (1 - x)} + \mathcal{O}(g'^3 q^2).
\]

(F.26)

The physical amplitude for meson-nucleon scattering has two external nucleon legs and therefore requires twice a wavefunction renormalization constant $\sqrt{Z_r}$ applied to the truncated $(2,1)$-point function. Hence

\[
T(q \to 0) = Z_r \Gamma(p, q \to 0)|_{\rho^3 = M^2} = g'^3 \left[ 1 + \frac{g'^2}{6} \langle r^2 \rangle + \mathcal{O}(q^4) \right]
\]

(F.27)

\[
g'_{\text{eff}} = g' \left[ 1 + \frac{g'^2}{16\pi^2} \int_0^1 dx \frac{x^2}{M^2 x^2 + m^2 (1 - x)} + \ldots \right]
\]

(F.28)

Here $\langle r^2 \rangle$ is the root-mean square radius of the nucleon due to radiative corrections.

The meson-nucleon scattering amplitude in the nonrelativistic theory described by the Lagrangian \[13\] reads

\[
T_{\text{nonrel}}(p, q) = \left\{ c_1 \frac{g'}{2M} + \frac{g'}{(2M)^3} \left[ c_2 \left( -p^2 - (p + q)^2 \right) - d_1 q^2 \right] \right\} \sqrt{2E_p E_{p+q}}
\]

(F.29)

where the square-root factor is due to the different normalization of relativistic and nonrelativistic single-particle states \[31]. Expanding $E_p = \sqrt{M^2 + p^2}$ and $E_{p+q}$ for low three-momenta we obtain

\[
T_{\text{nonrel}}(p \to 0, q \to 0) = c_1 g' + (c_1 - c_2) g' \frac{p^2}{4M^2} + (p + q)^2 \frac{q^2}{4M^2} - d_2 g' \frac{q^2}{4M^2} + \ldots.
\]

(F.30)

Comparing with Eq. (F.24) we find for massless mesons

\[
c_1 = c_2 = \frac{g'_{\text{eff}}}{g'} = 1 + \frac{\alpha}{\pi} + \ldots,
\]

(F.31)

and $(q^2 \to -q^2$ when all momenta are small)

\[
d_1 = \frac{2M^2}{3} \frac{g'_{\text{eff}}}{g'} \langle r^2 \rangle = \mathcal{O}(g'^2).
\]

(F.32)

The coefficients $c_1, c_2$ account for an enhancement of the effective meson-nucleon coupling constant through loop effects similar as in Schwinger’s famous determination of the anomalous magnetic moment of the electron in QED. The coefficient $d_1$ describes the finite extension of nucleons due to the meson cloud and therefore is infrared divergent. This can also be seen from its explicit lowest-order perturbative expression as given in Eq. (60) of ref. \[26\] but we do not need it in the following.

In the next step one has to integrate out the mesons in order to reduce the relativistic bound-state problem to a quantum-mechanical one. This we do by solving the equation of motion for the meson field (operator) obtained from varying Eq. \[10\]

\[
(\partial + m^2) \chi_0 = \frac{Z g'}{2M} c_1 \phi \phi + \ldots
\]

(F.33)

Since in Eq. (F.31) we have determined $c_1$ only to one-loop order, i.e. to order $v$ in the counting rules, we have retained only the leading term and for consistency have neglected the $\mathcal{O}(v^2)$ corrections in the effective Lagrangian. Furthermore, we replace $g' \to Z g'$ to indicate that the mesons are from the other particle (self-energy effects vanish in the effective nonrelativistic
theory if dimensional regularisation is used \[82\]). For the case that no external mesons are present the solution of Eq. (F.33) is

\[ \chi_0(x, t) = Z g' \int \frac{d^4 y}{2 M_{c_1}} \left\{ \phi^\dagger(y) \phi(y) \right\} \]

\[ \simeq \frac{Z g'}{2 M_{c_1}} \int \frac{d^3 y}{|\Delta + m^2|} \left\{ \phi^\dagger(y, t) \phi(y, t) \right\} \quad \text{(F.34)} \]

According to the counting rules, the time derivative in the d’Alembertian is also suppressed by a factor \(\mathcal{O}(v^2)\) compared to the Laplacian. Equation (F.34) is therefore correct up to order \(Mv^5/2\) and after substitution into Eq. (F.19) gives the following effective (potential) Lagrangian

\[ \mathcal{L}_{\text{eff}} = \phi^\dagger(x, t) \left\{ i \partial_t + \frac{\Delta}{2M} - \frac{1}{2} \int d^3 y \phi^\dagger(y, t) V_{\text{eff}}(x - y) \phi(y, t) + \mathcal{O}(Mv^4) \right\} \phi(x, t) \quad \text{(F.35)} \]

where

\[ V_{\text{eff}}(x - y) = -c_1^2 \left( \frac{Z g'}{2 M} \right)^2 \left\{ \frac{1}{|\Delta + m^2|} \right\} = -c_1^2 \frac{Z \alpha}{|x - y|} e^{-m|x - y|} \quad \text{(F.36)} \]

is the usual Yukawa potential enhanced by a factor \(c_1^2\) from radiative corrections. In the Coulombic case \((m = 0)\) this means that the leading term for the binding energy is

\[ \epsilon_n = -\frac{M}{2} \frac{(Z \alpha)^2}{2(n + 1)^2} c_1^2 + \ldots = -\frac{M}{2} \frac{(Z \alpha)^2}{2(n + 1)^2} \left[ 1 + \frac{4 \alpha^2}{\pi} \right] + \mathcal{O}((Z \alpha)^4) \], \(n = 0, 1, 2, \ldots \) \quad \text{(F.37)} \]

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