ON CANTOR SETS AND DOUBLING MEASURES

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Abstract. For a large class of Cantor sets on the real-line, we find sufficient and necessary conditions implying that a set has positive (resp. null) measure for all doubling measures of the real-line. We also discuss same type of questions for atomic doubling measures defined on certain midpoint Cantor sets.

1. Introduction and notation

Our main goal in this paper is to study the size of Cantor sets on the real-line $\mathbb{R}$ from the point of view of doubling measures. Recall that a measure $\mu$ on a metric space $X$ is called doubling if there is a constant $c < \infty$ such that

$$0 < \mu(B(x, 2r)) \leq c \mu(B(x, r)) < \infty$$

for all $x \in X$ and $r > 0$. Here $B(x, r)$ is the open ball with centre $x \in X$ and radius $r > 0$. We note that the collection of doubling measures on $\mathbb{R}$, and more generally, on any complete doubling metric space where isolated points are not dense, is rather rich. For instance, given $\varepsilon > 0$, there are doubling measures on $\mathbb{R}$ having full measure on a set of Hausdorff and packing dimension at most $\varepsilon$. See [Hei01], [Tuk89], [Wu98], [KRS12].

Let $D(\mathbb{R})$ be the collection of all doubling measures on $\mathbb{R}$ and denote

$$\mathcal{T} = \{ C \subset \mathbb{R} : \mu(C) = 0 \text{ for all } \mu \in D(\mathbb{R}) \},$$
$$\mathcal{F} = \{ C \subset \mathbb{R} : \mu(C) > 0 \text{ for all } \mu \in D(\mathbb{R}) \}.$$

In the literature, the sets in $\mathcal{F}$ have been called quasisymmetrically thick [SW98], [Hei01], thick for doubling measures [HWW09], and very fat [BHM01] and those in $\mathcal{T}$ have been termed quasisymmetrically null [SW98], [Hei01], null for doubling measures [HWW09], and thin [BHM01]. We call $C \subset \mathbb{R}$ thin if $C \in \mathcal{T}$ and fat if $C \in \mathcal{F}$.

In this paper, we address the problems of finding sufficient and/or necessary conditions for a Cantor set $C \subset \mathbb{R}$ to be fat (resp. thin). These problems arise naturally from the study of compression and expansion properties of quasisymmetric maps $f : \mathbb{R} \to \mathbb{R}$, see [Hei01] 13.20. A related problem is to characterise those subsets $U \subset \mathbb{R}$ which carry nontrivial doubling measures ([Hei03] Open problem 1.18); If $C \subset \mathbb{R}$ is a fat Cantor set, then it is easy to see that $U = \mathbb{R} \setminus C$ does not carry nontrivial doubling measures. For if it did, then one could extend any doubling measure $\mu$ on $U$ to $\mathbb{R}$ by letting $\mu(C) = 0$, and this would contradict $C$ being fat.

The first author was supported by OTKA grant no. 72655. The second author was supported by the Academy of Finland (project #126976).
We begin by discussing thinness and fatness for the middle interval Cantor sets $C(\alpha_n)$ determined via sequences $(\alpha_n)_{n=1}^\infty$, $0 < \alpha_n < 1$, as follows: We first remove an open interval of length $\alpha_1$ from the middle of $I_{1,1} = [0,1]$ and denote the remaining two intervals by $I_{2,1}$ and $I_{2,2}$. At the $k$:th step, $k \geq 2$, we have $2^{k-1}$ intervals $I_{k,1}, \ldots, I_{k,2^{k-1}}$ of length $\ell_k = 2^{-k+1} \prod_{n=1}^{k-1} (1 - \alpha_n)$ and we remove an interval of length $\alpha_k \ell_k$ from the middle of each $I_{k,i}$. Finally, the middle interval Cantor set $C(\alpha_n)$ is defined by

$$C = \bigcap_{k \in \mathbb{N}} \bigcup_{i=1}^{2^k} I_{k,i}.$$

The theorem below follows by combining results of Wu [Wu93, Theorem 1], Staples and Ward [SW98, Theorem 1.4], and Buckley, Hanson, and MacManus [BHM01, Theorem 0.3]. For $0 < p < \infty$, we denote by $\ell^p$ the set of all sequences $(\alpha_n)_{n=1}^\infty$, $0 < \alpha_n < 1$, for which $\sum_{n=1}^\infty \alpha_n^p < \infty$.

**Theorem 1.1.** Let $C = C(\alpha_n)$. Then

1. $C$ is thin if and only if $(\alpha_n) \notin \bigcup_{0 < p < \infty} \ell^p$.
2. $C$ is fat if and only if $(\alpha_n) \in \bigcap_{0 < p < \infty} \ell^p$.

In a recent paper, Han, Wang, and Wen [HWW09] generalised Theorem 1.1 for a broader collection of (still very symmetric) Cantor sets. Related results on thin and fat sets may be found in [Wu93, KW95, Wu98, SW98, BHM01, WWW08], and [ORST2].

The known proofs for Theorem 1.1 and its generalisation in [HWW09] rely heavily on the symmetries of the sets $C(\alpha_n)$. In this paper, we wish to consider analogues of Theorem 1.1 for Cantor sets with much less symmetry. To be more precise, we introduce the following notation. Suppose that for each $n \in \mathbb{N}$, we have a collection of closed intervals $\mathcal{I}_n = \{I_{n,i}\}_i$ with mutually disjoint interiors and open intervals $\mathcal{J}_n = \{J_{n,i} \subset I_{n,i}\}$ such that each $I_{n+1,i}$ is a subset of some $I_{n,j}$, $\bigcup_{n=1}^\infty \mathcal{I}_n = \bigcup_{n=1}^\infty \mathcal{J}_n$ and that $\sup_{n\in\mathbb{N}} |I_{n,j}| \to 0$ as $n \to \infty$. We also assume that $\bigcup_{i=1}^\infty I_i$ is bounded. We refer to $\{\mathcal{I}_n, \mathcal{J}_n\}_n$ as a Cantor construction. The resulting Cantor set is given by

$$C = C(\mathcal{I}_n, \mathcal{J}_n) = \bigcap_{n=1}^\infty \bigcup_{i=1}^{2^n} I_{n,i}.$$

Given the collections $\mathcal{I}_n$ and $\mathcal{J}_n$ as above, we also denote $\mathcal{I} = \bigcup_{n=1}^\infty \mathcal{I}_n$ and $\mathcal{J} = \bigcup_{n=1}^\infty \mathcal{J}_n$. If there exists $0 < c < 1$ so that $cI_{n,i} \cap J_{n,i} \neq \emptyset$ for all $I_{n,i}$, we say that our Cantor construction (and set) is nice.

Here $cI_{n,i}$ denotes the interval concentric with $I_{n,i}$ and with length $c|I_{n,i}|$. Furthermore, given a sequence $0 < \alpha_n < 1$, we say that the Cantor set $C = C(\mathcal{I}_n, \mathcal{J}_n)$ is $(\alpha_n)$-porous if $|J_{n,i}| \geq \alpha_n |I_{n,i}|$ for all $I_{n,i} \in \mathcal{I}_n$ and $(\alpha_n)$-thick, if $|J_{n,i}| \leq \alpha_n |I_{n,i}|$ for all $I_{n,i}$. Finally, $C$ is called $(\alpha_n)$-regular if $\lambda \alpha_n |I_{n,i}| \leq |J_{n,i}| \leq \Lambda \alpha_n |I_{n,i}|$ for all $I_{n,i}$ (here $0 < \lambda \leq \Lambda < \infty$ are constants that do not depend on $n$ nor $i$). We underline that these definitions do not refer only to the set $C$ but also to the construction of $C$ via $\{\mathcal{I}_n, \mathcal{J}_n\}_n$.

**Remarks 1.2.** a) Using our notation, it is possible that a Cantor set $C$ contains isolated points as some of the intervals $I_{n,i}$ could be degenerated. We allow this for technical reasons although in most interesting cases, e.g if $C$ is nice, the set $C$ is a

\[1\text{Geometrically, this only means that if the removed holes } J_{n,i} \text{ are small, then they cannot lie too close to the boundary of } I_{n,i}.\]
true Cantor set in the sense that it has no isolated points.

b) Observe that in our definitions, we do not impose any conditions on the number
or relative size of the intervals $I_{n+1,i} \subset I_{n,i}$. Note also that $I_{n+1,i} \in \mathcal{T}_{n+1}$ does not
have to be a component of any $I_{n,j} \setminus J_{n,j}$.

c) We formulate our results for Cantor sets, but it is reasonable to speak about
$(\alpha_n)$-porosity and $(\alpha_n)$-thickness for general subsets of $\mathbb{R}$ and not only for the ones
obtained from Cantor constructions. Roughly speaking, $A \subset \mathbb{R}$ is $(\alpha_n)$-porous if it
is contained in an $(\alpha_n)$-porous Cantor set and $(\alpha_n)$-thick, if it contains an $(\alpha_n)$-
thick Cantor sets. See [Wu98] and [SW98] for more details. In Section 4 we provide
a notion of $(\alpha_n)$-porosity which is useful in any metric space.

Our main result concerning doubling measures and Cantor sets is the following

**Theorem 1.3.** Suppose that $C = C(I_n, J_n)$ is a nice Cantor set. Then, for each
$0 < p < \infty$, there is $\mu \in D(\mathbb{R})$ and $0 < \lambda \leq \Lambda < \infty$ so that

$$
\lambda \left( \frac{|I_{n,i}|}{|I_{n,i}|} \right)^p \leq \frac{\mu(I_{n,i})}{\mu(I_{n,i})} \leq \Lambda \left( \frac{|I_{n,i}|}{|I_{n,i}|} \right)^p,
$$

for each $I_{n,i}$.

**Remark 1.4.** This result is interesting already for the middle interval Cantor sets
$C(\alpha_n)$. After the submission of this paper, we were informed that for uniform
Cantor sets, the result has been proved independently by Peng and Wen. See [PW11]
for the precise formulation of their result.

Let us now discuss what can be said about the validity of Theorem 1.3 for the
general Cantor sets $C(I_n, J_n)$. Observe that Theorem 1.3 includes the following four
statements:

I. If $(\alpha_n) \notin \bigcup_{0 < p < \infty} p \mathbb{R}$, then $\mu(C) = 0$ for all $\mu \in D(\mathbb{R})$.

II. If $(\alpha_n) \in \bigcup_{0 < p < \infty} p \mathbb{R}$, then there is $\mu \in D(\mathbb{R})$ with $\mu(C) > 0$.

III. If $(\alpha_n) \in \bigcap_{0 < p < \infty} p \mathbb{R}$, then $\mu(C) > 0$ for all $\mu \in D(\mathbb{R})$.

IV. If $(\alpha_n) \notin \bigcap_{0 < p < \infty} p \mathbb{R}$, then there is $\mu \in D(\mathbb{R})$ so that $\mu(C) = 0$.

The Claim I holds for general $(\alpha_n)$-porous sets $A \subset \mathbb{R}$ as shown by Wu [Wu98, Theorem 1]. In fact, her result remains true in all metric spaces. We provide a
simple proof in Lemma 1.1. The Claim III is a special case of a more general
result of Staples and Ward [SW98, Theorem 1.4]. They proved that if $C \subset \mathbb{R}$ is
$(\alpha_n)$-thick for some $(\alpha_n) \in \bigcap_{0 < p < \infty} p \mathbb{R}$, then $C$ is fat.

Our new results in Section 2 deal with the Claims IV and II. These are
the claims whose earlier proofs rely on the symmetries of $C(\alpha_n)$. We show that
if $\sum_{n=1}^{\infty} \alpha_n = \infty$ for some $p > 0$, and $C$ is a nice $(\alpha_n)$-porous Cantor set, then $C \notin \mathcal{F}$. On the other hand, if there is $p < \infty$ with $\sum_{n=1}^{\infty} \alpha_n < \infty$, and if $C$ is a nice
$(\alpha_n)$-thick Cantor set, then $C \notin \mathcal{F}$. Putting all these results together, we arrive at
a complete analogue of Theorem 1.3 for nice $(\alpha_n)$-regular Cantor sets $C \subset \mathbb{R}$. The
proofs of our results in Section 2 are all based on Theorem 1.3.

In the last part of the paper in Section 5 we discuss purely atomic doubling
measures. Recall that a measure $\mu$ on a metric space $X$ is called purely atomic,
if there is a countable set $F \subset X$ so that $\mu(X \setminus F) = 0$. Purely atomic doubling
measures have reached some attention recently, see e.g. [KW95], [Wu98], [LWW07],
and [WWW08].
Denote by $F_X$ the set of isolated points of a metric space $X$ and let $E_X = X \setminus F_X$. If $E_X$ is nowhere dense, it is reasonable to ask if there are purely atomic doubling measures on $X$ and on the other hand, what conditions guarantee that all doubling measures on $X$ are purely atomic. We will treat these questions for a class of metric spaces obtained by adding the midpoints of the intervals $J \in \mathcal{J}$ to the Cantor sets $C = C(\mathbb{I}_{\alpha_n}, \mathbb{I}_{\mathcal{J}_n})$. If $C$ is $(\alpha_n)$-regular, we will classify in terms of the sequence $(\alpha_n)$, which of the corresponding midpoint sets carry purely atomic doubling measures. We find a characterisation of the same nature for all doubling measures being purely atomic. The result, Theorem 2.1, is analogous to Theorem 1.1. We will also answer two questions on atomic doubling measures posed by Kaufman and Wu [KWW07], and Lou, Wen, and Wu [LWW07].

We finish this section with some notation. By a measure on (a metric space) $X$, we always mean a Borel regular outer measure, defined on all subsets of $X$. If $A \subset X$, we denote by $\mu|_A$ the restriction of $\mu$ to $A$ given by $\mu|_A(B) = \mu(A \cap B)$ for $B \subset X$. For an interval $I \subset \mathbb{R}$, we denote by $\partial I$ the set of its endpoints. We adopt the convention that $0 < c < \infty$ always denotes a constant that only depends on parameters which should be clear from the context. Sometimes we write $c = c(a, \ldots, b)$ to emphasize that $c$ depends only on the values of $a, \ldots, b$. For notational convenience, the exact value of $c$ may vary even inside a given chain of inequalities. Given a family of numbers $0 < A_\alpha, B_\alpha < \infty$, parametrised by $\alpha$, we denote $A_\alpha \lesssim B_\alpha$ if there is a constant $c$ so that $A_\alpha \leq c B_\alpha$ for all $\alpha$. By $A_\alpha \approx B_\alpha$ we mean that $A_\alpha \lesssim B_\alpha$ and $B_\alpha \lesssim A_\alpha$.

2. Results for $(\alpha_n)$-porous and $(\alpha_n)$-thick sets

Our new results concerning $(\alpha_n)$-porous and $(\alpha_n)$-thick Cantor sets are based on Theorem 1.3.

**Theorem 2.1.** Suppose that $C = C(\mathbb{I}_{\alpha_n}, \mathbb{I}_{\mathcal{J}_n})$ is nice and $(\alpha_n)$-porous for some $\alpha_n \notin \bigcap_{0 < p < \infty} \ell^p$. Then there is $\mu \in \mathcal{D}(\mathbb{R})$ with $\mu(C) = 0$.

**Proof.** We may assume that $C \subset [0, 1]$. Choose $p > 0$ such that $\sum_{n=1}^{\infty} \alpha_n^p = \infty$. Let $\mu$ be a doubling measure given by Theorem 1.3. Then $\mu(J_{n,i}) = \left(|J_{n,i}|/|I_{n,i}|\right)^p \mu(I_{n,i})$. As $|J_{n,i}| \geq \alpha_n |I_{n,i}|$, we get

\begin{equation}
\mu(J_{n,i}) \gtrsim \alpha_n^p \mu(I_{n,i}).
\end{equation}

This gives, for some $c > 0$,

\[\mu\left([0, 1] \setminus \bigcup_{j=1}^{n} \bigcup_{i} J_i\right) \leq (1 - c \alpha_n^p) \mu\left([0, 1] \setminus \bigcup_{j=1}^{n} \bigcup_{i} J_i\right)\]

for all $n \in \mathbb{N}$ and consequently,

\[\mu(C) = \mu\left([0, 1] \setminus \bigcup_{n=1}^{\infty} J_n\right) \leq \mu([0, 1]) \prod_{n=1}^{\infty} \left(1 - c \alpha_n^p\right) = 0,
\]

as $\sum_{n=1}^{\infty} \alpha_n^p = \infty$. \hfill \Box

**Theorem 2.2.** Suppose that $C = C(\mathbb{I}_{\alpha_n}, \mathbb{I}_{\mathcal{J}_n}) \subset \mathbb{R}$ is nice $(\alpha_n)$-thick for some $\alpha_n \in \bigcup_{0 < p < \infty} \ell^p$. Then there is $\mu \in \mathcal{D}(\mathbb{R})$ with $\mu(C) > 0$.

**Proof.** The proof is very similar to the proof of Theorem 2.1 and thus we skip the details. The estimate (2.1) gets replaced by $\mu(J_{n,i}) \lesssim \alpha_n^p \mu(I_{n,i})$ and this leads to $\mu(C) > 0$ when $\sum_{n=1}^{\infty} \alpha_n^p < \infty$. \hfill \Box
Putting together Theorems 2.1 and 2.2 and the results of Wu [Wu98, Theorem 1], and Staples and Ward [SW98, Theorem 1.4] mentioned earlier, we get the following classification for the thinness and fatness of nice \((\alpha_n)-\)regular Cantor sets.

**Corollary 2.3.** If \(C \subset \mathbb{R}\) is a nice \((\alpha_n)-\)regular Cantor set, then
\begin{enumerate}[(1)]  
  \item \(C\) is thin if and only if \((\alpha_n) \notin \bigcup_{0 < p < \infty} \ell^p\).
  \item \(C\) is fat if and only if \((\alpha_n) \in \bigcap_{0 < p < \infty} \ell^p\).
\end{enumerate}

3. **Proof of Theorem 1.3**

We begin with the following simple lemma.

**Lemma 3.1.** For all \(0 < p < \infty\) there is \(c = c(p) < \infty\) and a doubling measure \(\mu\) on \([0, 1]\) such that
\[
    c^{-1} t^p \leq \mu[0, t] = \mu[1 - t, 1] \leq ct^p
\]
for all \(0 < t < 1\).

**Proof.** We obey the following construction. Let \(m\) be an integer so large that \(2^{-mp + 1} < 1\). Define
\[
    \mu[0, 2^{-m}] = \mu[1 - 2^{-m}, 1] = 2^{-mp}
\]
and let \(\mu\) be uniformly distributed on \([2^{-m}, 1 - 2^{-m}]\) with total measure \(1 - 2^{-mp + 1}\). For each integer \(k \geq 2\), put
\[
    \mu[0, 2^{-km}] = \mu[1 - 2^{-km}] = 2^{-kmp}
\]
and let \(\mu\) be uniformly distributed on the interval \([2^{-km}, 2^{-(k-1)m}]\) (resp. \([1 - 2^{-(k-1)m}, 1 - 2^{-km}]\)) with total measure \(2^{-((k-1)m)mp - 2^{-kmp}}\). It is now easy to see that \(\mu\) has the required properties. \(\square\)

We now start to prove Theorem 1.3. We assume without loss of generality that \(\inf C = 0\) and \(\sup C = 1\). Fix \(0 < p < \infty\) and let \(c < 1\) be a constant so that
\[
    c I_{n,i} \cap J_{n,i} \neq \emptyset
\]
for all \(I_{n,i} \in \mathcal{I}\) (such a constant exists since the Cantor construction is nice). From now on, in this proof, all constants of comparability will only depend on \(p\) and \(\bar{c}\).

Let \(\eta > 0\) be a small constant so that \(\eta^p < \frac{1}{4}\). We start by dividing the interval \([0, 1]\) into construction intervals\(^2\) of level 1 and gaps of level 1 as follows. For all integers \(k \geq 2\), we choose gaps \(J, J' \in \mathcal{J}\) so that \(J \cap [2^{-k}, 2^{-k+1}] \neq \emptyset\) and \(J' \cap [1 - 2^{-k+1}, 1 - 2^{-k}] \neq \emptyset\). Denote the union of all these gaps by \(G^b_{1}\). Let also \(G^{\bar{c}}_{1} = \{J_{n,i} : I_{n,i} \cap \{0, 1\} \neq \emptyset\}\) and \(G_{1} = G^b_{1} \cup G^{\bar{c}}_{1}\). Call the elements of \(G_{1}\) gaps of level one and their complementary intervals the construction intervals of level one. Denote the collection of all construction intervals of level one by \(\mathcal{C}_{1}\).

Next we describe how the total measure \(\mu[0, 1] = 1\) is distributed among the construction intervals and gaps of level 1. Denote by \(G^{\bar{c}}_{1}\) the rightmost gap for which \(\text{dist}(G^{\bar{c}}_{1}, 0) < \eta\) and by \(G_{1}\) the leftmost gap so that \(\text{dist}(G_{1}, 1) < \eta\). Let \(G_{n}\) be the gaps between \(G^{\bar{c}}_{1}\) and \(G_{1}\) and \(K_{1}, \ldots, K_{n+1}\) the complementary intervals in between \(G^{\bar{c}}_{1}, G_{1}, \ldots, G_{n}, G_{1}\). It is possible that \(G^{\bar{c}}_{1} = G_{1}\) (if there is a huge gap in the middle) and in this case, the collection \(\{G_{1}, \ldots, G_{n}, K_{1}, \ldots, K_{n+1}\}\) is considered to be empty.

**Claim 1.** \(n \leq c\).

\(^2\)This refers to the construction of the measure rather than construction of the set \(C\).
us further denote the sub-construction intervals of $I$ by letting $K_r$ be the rightmost gap inside $I$. We continue the construction inductively inside $K_r$ by letting $G_r = \{0, \text{dist}(0, G_r(1)) < \eta |K_r|^{1/p}\}$. Define $K_r = \{0, \text{dist}(0, G_r(1))\}$ and $\mu(K_r) = (|K_r|/|K_r|)^p \mu(K_r) = |K_r|^{1/p}$. If $U$ is one of the gaps of level 1 between $G_r$ and $G_r$ (resp. $G_r$ and $G_r$) or one of the complementary intervals of level 1 in between these gaps, we put $\mu(U) = \gamma |U|^{1/p}$, where $\gamma$ is a constant defined so that the total measure of $K_r$ remains unchanged. A similar argument as in the proof of Claim implies again that $\gamma \approx 1$. We continue the construction inductively inside $K_r$ by letting $G_r$ be the rightmost gap inside $K_r$ for which $\text{dist}(0, G_r(1)) < \eta |K_r|^{1/p}$, $K_r = \{0, \text{dist}(0, G_r(1))\}$, $\mu(K_r) = (|K_r|/|K_r|)^p \mu(K_r) = |K_r|^{1/p}$ and so on. Continuing in this manner, we eventually get to define the measure of each gap and construction interval of level one. See Figure 1.

We proceed with the mass distribution process inside the construction intervals of level one. For such an interval $I$, we consider gaps $G_r = \{J_{n,i} \subset I : I_{n,i} \cap \partial I \neq \emptyset\}$ and also let $G_r$ consist of a dyadic sequence of gaps defined similarly as $G_r = G_r$ was defined for $I = [0, 1]$. More precisely, if $I = [a, b]$, for each $k \geq 2$ we choose gaps $J, J' \in \mathcal{J}$ so that $J \cap [a + 2^{-k}(b - a), a + 2^{-k+1}(b - a)] \neq \emptyset$ and $J' \cap [b - 2^{-k+1}(b - a), b - 2^{-k}(b - a)] \neq \emptyset$. Put $G_r = G_r \cup G_r$. We call the elements of $G_r$ the gaps of $I$. Their complementary intervals inside $I$ are called the sub-construction intervals of $I$. The mass $\mu(I)$ is distributed for the gaps and construction intervals of level two inside $I$ by the same procedure as the unit mass was distributed for the gaps and construction intervals of level one. The only difference is, that we replace $1 = \mu([0, 1])$ by $\mu(I)$. We repeat this process inductively for all construction intervals of all levels. We denote by $\mathcal{G}_n$ the set of all gaps of level $n$ and by $\mathcal{C}_n$ the collection of construction intervals of level $n$. Observe that the construction intervals do not have to be covering intervals (i.e. members of $\mathcal{I}$). So most likely, $\mathcal{C}_n \neq \mathcal{I}_n$ and also $\mathcal{G}_n \neq \mathcal{J}_n$ even though $\bigcup_n \mathcal{G}_n = \bigcup_n \mathcal{J}_n = \mathcal{J}$. Let us further denote $\mathcal{C} = \bigcup_{n=1}^\infty \mathcal{C}_n$.

We have now defined the measure of all the gaps and construction intervals and we may use a standard mass distribution principle, see e.g. [En90] Proposition 1.7, to define the measure $\mu|_{\mathcal{C}}$. Inside the gaps the measure will be distributed in the following way:
following manner: Let $G = [a, b] \in \mathcal{J}$. Then we let $\mu|_G$ be a doubling measure on $G$ given by a scaled version of Lemma 3.1 so that
\begin{equation}
(3.2) \quad \mu[a, a + t] = \mu[b - t, b] \approx \left(\frac{t}{|G|}\right)^p \mu(G)
\end{equation}
for all $0 < t < b - a$. By the proof of Lemma 3.1, this may be done in such a way that the doubling constant of $\mu|_G$ is independent of $G \in \mathcal{J}$. This completes the construction of $\mu$. To complete the proof of Theorem 1.3, we have to show that $\mu$ is doubling and satisfies (1.3).

Our next claim follows directly from the way $\mu$ is defined.

Claim 2. Let $K \in \mathcal{C}_n$ and $I \subset K$, $I \in \mathcal{C}_{n+1} \cup \mathcal{G}_{n+1}$. Then
\[
\mu(I) \approx \left(\frac{|I|}{|K|}\right)^p \mu(K).
\]

If $K = [a, b]$, then for all $0 < t < 1$, \[
\mu[a, a + t(b - a)] \approx t^p \mu(K) \approx \mu[b - t(b - a), b].
\]

Denote $N = \{0, 1\} \cup \bigcup_{G \in \mathcal{J}} \partial G$.

Claim 3. Suppose that $I \subset [0, 1]$ is an interval with $I \cap N \neq \emptyset$ and let $K \in \mathcal{C}$ be the shortest construction interval containing $I$. Then
\[
\mu(I) \approx \left(\frac{|I|}{|K|}\right)^p \mu(K).
\]

Proof of claim 3. Denote $K = [a, b]$ and let $c > 1$ be a constant from the Claim 2 so that
\begin{equation}
(3.3) \quad \frac{t^p \mu(K)}{|c| |K|^p} \leq \mu[a, a + t], \mu[b - t, b] \leq \frac{ct^p \mu(K)}{|K|^p}
\end{equation}
for all $0 < t < |K|$. Fix $\varepsilon = \varepsilon(p) > 0$ so that $\varepsilon^p \leq 1/(2c^2)$.

Assume first, that $\text{dist}(I, \{a, b\}) \geq \varepsilon |I|$. Let $K_1, \ldots, K_n$ be the sub-construction intervals of $K$ intersecting $I$ and $G_1, \ldots, G_m$ ($m \in \{n - 1, n, n + 1\}$) the gaps of $K$ intersecting $I$. It may happen that $U \setminus I \neq \emptyset$ for some (but at most two) $U \in \{K_1, \ldots, K_n, G_1, \ldots, G_m\}$. In this case, we replace $U$ by $U \cap I$ in the calculation below. As $\text{dist}(I, \{a, b\}) \geq \varepsilon |I|$, it follows as in the proof of Claim 4 that $n, m \leq c$. Using Claim 2 and (3.2), it now follows that
\[
\mu(I) = \sum_{i=1}^n \mu(K_i) + \sum_{j=1}^m \mu(G_j) \approx \sum_{i=1}^n \left(\frac{|K_i|}{|K|}\right)^p \mu(K) + \sum_{j=1}^m \left(\frac{|G_j|}{|K|}\right)^p \mu(K) \approx \left(\frac{|I|}{|K|}\right)^p \mu(K).
\]

Suppose then that $\delta = \text{dist}(I, \{a, b\}) < \varepsilon |I|$. We may assume by symmetry, that $\text{dist}(a, I) < \varepsilon |I|$. The claimed upper bound now follows from Claim 2 since
\[
\mu(I) \leq \mu[a, a + 2|I|] \approx \left(\frac{2|I|}{|K|}\right)^p \mu(K).
\]
For the lower bound, we use (3.3) to obtain
\[ \mu(I) = \frac{1}{|K|} \left( \frac{1}{c} |I|^p - c \delta^p \right) \]
\[ \geq (|I|/|K|)^p \mu(K) \left( \frac{1}{c} - c \varepsilon \right) \approx (|I|/|K|)^p \mu(K). \]

where the last estimate follows from the choice of \( \varepsilon \).

Now we are ready to verify (1.1). Fix \( J = J_n, i \in J \), and let \( K \) be the smallest construction interval containing \( I = I_n, i \). By Claim 3 we have
\[ \mu(I) \approx \left( \frac{|I|}{|K|} \right)^p \mu(K). \]

If \( J \) is a gap of \( K \), it follows from Claim 2 that \( \mu(J) \approx (|J|/|K|)^p \mu(K) \). Combining this with (3.4), we get \( \mu(J)/\mu(I) \approx (|J|/|I|)^p \). If \( J \) is not a gap of \( K \), we argue as follows: Since \( K \) is the smallest construction interval containing \( I \), there is a gap of \( K \) intersecting \( I \). Thus, if \( K' \) is the sub-construction interval of \( K \) containing \( J \), we have \( I \cap \partial K' \neq \emptyset \) and consequently \( J \in \mathcal{G}_K \). Now, using Claim 2 we obtain
\[ \mu(I) \approx \left( \frac{|I|}{|K|} \right)^p \mu(K) = \left( \frac{|J|}{|K'|} \right)^p \mu(K) \]
and it follows as above that \( \mu(J)/\mu(I) \approx (|J|/|I|)^p \). Whence, (1.1) follows.

It remains to show that \( \mu \) is doubling on \([0, 1]\). For this, it is clearly enough to show that
\[ \mu(I_1) \approx \mu(I_2) \]
if \( I_1 \) and \( I_2 \) are closed sub-intervals of \([0, 1]\) with equal length and \( I_1 \cap I_2 \neq \emptyset \). Let \( I_1 \) and \( I_2 \) be such intervals aligned from left to right. If \( I_1 \cup I_2 \subset G \) for some \( G \in \mathcal{G} \) and (3.5) follows from the way \( \mu|_G \) was defined.

Suppose next that \( I_1 \cap N \neq \emptyset \neq I_2 \cap N \). Let \( K_1 \) (resp. \( K_2 \)) be the smallest construction interval containing \( I_1 \) (resp. \( I_2 \)). Then \( K_1 \subset K_2 \) or \( K_2 \subset K_1 \) since \( I_1 \cap I_2 \neq \emptyset \) and any two construction intervals are either disjoint or within each other. We may assume without loss of generality, that \( K_1 \subset K_2 \).

**Claim 4.** If \( K_1 \in C_n \) and \( K_2 \in C_m \), then \( n \leq m + 3 \).

**Proof of Claim 4.** If \( K_1 = K_2 \), we are done, so assume \( K_2 \setminus K_1 \neq \emptyset \). Let \( G \) be the leftmost gap of \( K_2 \) that intersects \( I_2 \) and let \( K \in C_{m+1} \) be the construction interval next to \( G \) on the lefthand side. As \( I_1 \subset K_2 \) and \( G \cap I_1 = \emptyset \) (otherwise \( K_1 = K_2 \)), the intervals \( K \) and \( G \) are well defined. Moreover, we have \( I_1 \subset K \). Consider the collection \( \mathcal{G}_K \). If \( I_1 \cup \mathcal{G}_K \neq \emptyset \), it follows that \( K_1 = K \) (i.e. \( n = m + 1 \)) and we are done. Otherwise, there are two consecutive gaps \( G_1, G_2 \in \mathcal{G}_K \) and a sub-construction interval of \( K \) denoted by \( K' \in C_{m+2} \) in between \( G_1 \) and \( G_2 \) so that

\[ \mu(I_1) \approx \mu(K') \approx \mu(I_2). \]
\( I_1 \subset K' \). Let us denote by \( a \) the right endpoint of \( G_1 \) (= left endpoint of \( K' \)), by \( b \) the left endpoint of \( G_2 \) (= right endpoint of \( K' \)), and by \( c \) the right endpoint of \( K \) (= left endpoint of \( G \)), see Figure 2. From the way \( G'_K \) is constructed, it follows that \(|a-c| < 4|b-c| \) and so

\[
|K'| = |a-b| < 3|b-c| \leq 3|I_2| = 3|I_1|.
\]

We also know, by considering \( G'_K \), that all sub-construction intervals of \( K' \) have length at most \(|K'|/4 \) and similarly their sub-construction intervals are shorter than \(|K'|/4 < \frac{3}{4}|I_1| \). Thus, \( |I_1| \) cannot be contained in a construction interval of level \( m + 4 \) and the claim follows.

The Claims 2 and 4 now imply that \( \mu(K_1) \approx (|K_1|/|K_2|)^p \mu(K_2) \). On the other hand, by Claim 3, we have \( \mu(I_1) \approx (|I_1|/|K_1|)^p \mu(K_1) \) as well as \( \mu(I_2) \approx (|I_2|/|K_2|)^p \mu(K_2) \). Putting these estimates together implies

\[
\mu(I_1) \approx \left( \frac{|I_1|}{|K_1|} \right)^p \mu(K_1) \approx \left( \frac{|I_1|}{|K_1|} \right)^p \left( \frac{|K_1|}{|K_2|} \right)^p \mu(K_2) = \left( \frac{|I_2|}{|K_2|} \right)^p \mu(K_2) \approx \mu(I_2).
\]

Suppose finally, that only one of the intervals \( I_1 \) or \( I_2 \), say \( I_2 \), hits \( N \). Then \( I_1 \) is a subset of a gap \( G = [a, b] \) and \( \delta = \text{dist}(I_1, b) \leq |I_1| \). Letting \( I_3 = I_1 + \delta \), we have \( \mu(I_3) \approx \mu(I_3) \) as \( \mu_G \) is doubling. On the other hand, since \( I_3 \cap N \neq \emptyset \neq I_2 \cap N \), and \( I_2 \cap I_3 \neq \emptyset \), we already know that \( \mu(I_3) \approx \mu(I_2) \). Combining these estimates, we get \( \mu(I_1) \approx \mu(I_2) \). This completes the proof of Theorem 1.3.

It is natural to ask if we could drop the word “nice” from the assumptions in Theorems 1.3, 2.1 and 2.2 and in Corollary 2.3. The Proposition below shows (by choosing a fat Cantor set as \( E \)) that, at least, in Theorems 1.3 and 2.1 this is not possible. We do not know if one could remove this assumption from Theorem 2.2.

**Proposition 3.2.** If \( E \subset \mathbb{R} \) is nowhere dense and \( 0 < p < 1 \), then there is a Cantor set \( C \supset E \) which is \((\alpha_n)\)-porous for some \((\alpha_n) \notin \ell^p\).

**Proof.** We construct inductively the required intervals \( I_{n,i} \) and \( J_{n,i} \) that satisfy \( E \subset [0, 1] \setminus \bigcup_{n,i} J_{n,i} \).

**Step 1:** Pick any subinterval \( G \subset [0, 1] \setminus E \) of length \( \leq \frac{1}{2} \) so that \( G \cap \left[ \frac{1}{2}, \frac{3}{4} \right] \neq \emptyset \) and denote \( r = |G| \). Choose a number \( M_1 \in \mathbb{N} \) so that

\[
M_1^{1-p}(r/2)^p \geq 1.
\]

Let \( J_1, \ldots, J_{2M_1} \) be disjoint open sub-intervals of \( G \) with length \( \delta = r/(2M_1) \), enumerated from left to right. Define \( \alpha_1 = \alpha_2 = \ldots = \alpha_{M_1} = \delta \). From (3.6), we get \( \sum_{n=1}^{M_1} \alpha_n^p \geq 1 \). If \( a \) is the centre point of \( G \), define \( I_1 = \{[0, a], [a, 1]\} \),

\[
J_1 = \{J_{M_1}, J_{M_1+1}\}, J_2 = \{[0, a-\delta], [a+\delta, 1]\}, J_2 = \{J_{M_1-1}, J_{M_1+2}\}, \ldots, J_{M_1} = \{[0, a-r/2+\delta], [a+r/2-\delta, 1]\}, J_{M_1} = \{J_1, J_{2M_1}\}.
\]

**Step 2:** Suppose that \( M_1, \ldots, M_{m-1} \in \mathbb{N} \) as well as the collections \( I_j, J_j \) for \( 1 \leq j \leq \sum_{k=1}^{m-1} M_k \) have been defined. We now perform the step 1 construction inside each of the elements of \( \sum_{k=1}^{m-1} M_k \). The number \( M_m \), as well as \( \alpha_n \) for \( \sum_{k=1}^{m-1} M_k < n \leq \sum_{k=1}^{m} M_k \) will be determined according to the smallest relative
gap chosen inside the intervals $I \in \mathcal{I}_{\sum_{k=1}^{m-1} M_k}$, and we choose the number $M_m$ so large, that

$$\sum_{k=1}^{m-1} M_k \geq \alpha_n^p.$$ 

It is now evident from the construction, that $(\alpha_n) \notin \ell^p$ and that the set $C = \bigcap_{j=1}^{\infty} \bigcup J_j$ is $(\alpha_n)$-porous.

\[ \square \]

**Remark 3.3.** To formally fulfill the requirement $\bigcup \mathcal{I}_{n+1} = \bigcup \mathcal{I}_n \setminus \bigcup \mathcal{J}_n$ we should add to each $\mathcal{I}_{n+1}$ the boundary points of the deleted intervals $J \in \mathcal{J}_n$ and also emptysets as their “holes” to $\mathcal{J}_{n+1}$. For those readers who consider this cheating, we suggest to modify the construction so that $\bigcup \mathcal{I}_{n+1} = \bigcup \mathcal{I}_n \setminus \bigcup \mathcal{J}_n$ holds and the resulting Cantor set $C = C(\mathcal{I}_n, \mathcal{J}_n)_n$ contains no isolated points. It is also possible to modify the construction so that $(\alpha_n) \notin \bigcup_{0 \leq q < 1} \ell^q$.

4. A LEMMA ON $(\alpha_n)$-POROUS SETS IN METRIC SPACES

For the purpose of proving results for midpoint Cantor sets in Section 5 we present here a metric space version of Wu’s result on $(\alpha_n)$-porous sets being null for all doubling measures if $(\alpha_n) \notin \bigcup_{0 < p < \infty} \ell^p$. Her argument to prove the result in $\mathbb{R}$ readily works in much general situations once we find a reasonable definition of $(\alpha_n)$-porosity to use. There are basically two options: If one wants that the covering collection consists of distinct elements, then one has to use more general covering objects than just balls or intervals. The second option, which is more useful for us, is to relax the disjointness condition a bit and still keep using coverings with balls. For an analogous result using the first mentioned option, see [Leh10 Theorem 4.9].

We say that a subset $E \subset X$ of a metric space $X$ is $(\alpha_n)$-porous for a sequence $(\alpha_n)_{n=1}^{\infty}$, $0 < \alpha_n < 1$, if there is a constant $N \in \mathbb{N}$ and a sequence of (finite or countably infinite) coverings $\mathcal{B}_n = \{B_{n,j}\}_i$ of $E$ by balls $B_{n,j} = B(x_{n,j}, r_{n,j})$ with the following properties:

(P1) Each $B_{n,j}$ contains a sub ball $B'_{n,j} = B(y_{n,j}, \alpha_n r_{n,j}) \subset B_{n,j} \setminus E$.

(P2) Each point $x \in X$ belongs to at most $N$ different balls $B'_{n,j}$.

It is clear that if $C = C(\mathcal{I}_n, \mathcal{J}_n) \subset \mathbb{R}$ is $(\alpha_n)$-porous in the sense defined in the introduction, then it is also $(\alpha_n)$-porous in the sense of the above definition.

**Lemma 4.1.** Let $X$ be a metric space. If $(\alpha_n) \notin \bigcup_{0 < p < \infty} \ell^p$, and $E \subset X$ is $(\alpha_n)$-porous, then $\mu(E) = 0$ for all doubling measures $\mu$ on $X$.

**Proof.** Let $\mathcal{B}_n$ be the coverings that fulfill the $(\alpha_n)$-porosity conditions (P1) and (P2) and let $\mu$ be a doubling measure on $X$ with doubling constant $1 < c < \infty$. Without loss of generality, we may assume that $E$ is bounded and that $B_{n,j} \subset B$ for some fixed ball $B \subset X$. For each $n$, let $k_n$ be the smallest integer so that $k_n \geq -\log(\alpha_n) + 1$. Then $B_{n,i} \subset B(y_{n,j}, 2^{k_n} \alpha_n r_{n,j})$ for all $B_{n,i} \in \mathcal{B}_n$ and thus the doubling condition gives

$$\mu(E) \leq \sum_i \mu(B_{n,i}) \leq c^{-\log(\alpha_n) + 1} \sum_i \mu(B'_{n,i}) = c \alpha_n^{-p} \sum_i \mu(B'_{n,i}),$$
where \( p = \log c > 0 \). Let \( \varepsilon > 0 \). To complete the proof it suffices to find \( n \in \mathbb{N} \) so that \( \sum_i \mu(B'_{n,i}) \leq \varepsilon \alpha_n^p \). But if this is not the case, then (12) yields

\[
\infty > \mu(B) \geq \frac{1}{N} \sum_n \sum_i \mu(B'_{n,i}) \geq \varepsilon \sum_{n=1}^{\infty} \alpha_n^p = \infty
\]
giving a contradiction. \( \square \)

5. Purely atomic doubling measures

5.1. On midpoint Cantor sets. In this subsection, we show how the theorems of Section 2 can be turned to theorems on atomic doubling measures for certain class of midpoint Cantor sets.

For each Cantor set \( C = C(I_n, J_n) \), we define a midpoint Cantor set \( M = M(I_n, J_n) \) by letting \( M = C \cup J \in J \{ x \} \), where \( x \) is the centre point of \( J \in J \). If \( C \) is a middle interval Cantor set \( C = C(\alpha_n) \), we denote the corresponding midpoint Cantor set by \( M(\alpha_n) \). We consider each such \( M \) as a metric space, with the inherited Euclidean metric.

For these midpoint Cantor sets, we verify the following results analogous to the results obtained for doubling measures on the real-line.

**Theorem 5.1.** Suppose that \( C = C(I_n, J_n) \) is a Cantor set and let \( M = M(I_n, J_n) \). Then:

1. If \( C \) is \((\alpha_n)\)-porous for some \((\alpha_n) \notin \bigcup_{0<p<\infty} \ell^p \), then all doubling measures on \( M \) are purely atomic.

Suppose further that \( C \) is nice and let \( c \) be a constant so that \( |J_{n,i}| \cap cI_{n,i} \neq \emptyset \) for all \( I_{n,i} \). If

\[
|J_{n,i}| < \frac{1-c}{3} |I_{n,i}| \quad \text{for each } I_{n,i},
\]

then also the following holds:

2. If \( C \) is \((\alpha_n)\)-thick for some \((\alpha_n) \notin \bigcup_{0<p<\infty} \ell^p \), then there are doubling measures \( \mu \) on \( M \) with \( \mu(C) > 0 \).
3. If \( C \) is \((\alpha_n)\)-thick for some \((\alpha_n) \notin \bigcap_{0<p<\infty} \ell^p \), then there are no purely atomic doubling measures on \( M \).
4. If \( C \) is \((\alpha_n)\)-porous and \((\alpha_n) \notin \bigcap_{0<p<\infty} \ell^p \), then there are purely atomic doubling measures on \( M \).
5. Finally, suppose that \( C \) is nice and \((\alpha_n)\)-regular. Then all doubling measures on \( M \) are purely atomic if and only if \((\alpha_n) \notin \bigcup_{0<p<\infty} \ell^p \). There are no purely atomic doubling measures on \( M \) if and only if \((\alpha_n) \in \bigcap_{0<p<\infty} \ell^p \).

Our main tool to prove Theorem 5.1 is the following lemma. We denote by \( \delta_x \) the Dirac unit mass located at \( x \in \mathbb{R} \).

**Lemma 5.2.** Suppose that \( C = C(I_n, J_n) \) is a nice Cantor set and assume that (5.1) holds. Let \( M = M(I_n, J_n) \). If \( \mu \) is a doubling measure on \([ \inf C, \sup C ]\), we may define a doubling measure \( \nu \) on \( M \) by setting \( \nu = \mu|_{\inf C} + \sum_{J \in J} \mu(J) \delta_{x,J} \). On the other hand, if \( \nu \) is a doubling measure on \( M \), there is a doubling measure \( \mu \) on \([ \inf C, \sup C ]\) so that \( \nu|_{\inf C} = \mu|_{\inf C} \) and \( \mu(J) = \nu(x,J) \) for all \( J \in J \).

Before starting to prove Lemma 5.2, we state a couple of auxiliary results. The first one is a direct consequence of the doubling property.
Lemma 5.3. Let \( \mu \) be a doubling measure on a metric space \( X \) and let \( 1 < \Lambda < \infty \). Suppose that \( x, y \in X, d(x, y) \leq \Lambda r, \) and \( 1/\Lambda \leq r/s \leq \Lambda \). Then \( \mu(B(x, r)) \approx \mu(B(y, s)) \) where the constants of comparability only depend on \( \Lambda \) and the doubling constant of \( \mu \).

Lemma 5.4. Under the assumptions of Lemma 5.2, there is \( c > 0 \) so that the following holds: If \( J, J' \in \mathcal{J} \) and \( K \) is the interval between \( J \) and \( J' \), then \( |K| \geq c \min\{|J|, |J'|\} \).

Proof. Let \( I \) (resp. \( I' \)) be the smallest interval from \( I \) containing \( J \) (resp. \( J' \)). Then \( I \subset I', I' \subset I \) or \( I \cap I' = \emptyset \). In any case, \( J \cap I' = \emptyset \) or \( J' \cap I = \emptyset \). We may assume that \( J' \cap I = \emptyset \). Using Lemma 5.3, we get \( |K| = \operatorname{dist}(J, J') \geq \operatorname{dist}(J, \partial I) \geq (1 - c)|I|/2 - |J| \geq |I|(1 - c)/6 > |J|(1 - c)/6 \).

Proof of Lemma 5.2. We assume without loss of generality that \( \inf C = 0, \sup C = 1 \). By \( B(x, r) \) we denote the Euclidean interval \( B(x, r) = [x - r, x + r] \) whereas \( B_M(x, r) = B(x, r) \cap M \), for \( x \in M \).

To prove the first assertion, suppose that \( \mu \) is a doubling measure on \([0, 1]\) and let \( \nu \) be defined as in the lemma. We have to verify that \( \nu \) is a doubling measure on \( M \). Fix \( x \in M \) and \( r > 0 \). If \( B(x, 2r) \cap C = \emptyset \), we have \( B_M(x, r) = B_M(x, 2r) = \{x\} \) and nothing to prove. Proving that \( \nu \) is doubling thus reduces to showing the following. If \( x \in M \) and \( B(x, 2r) \cap C \neq \emptyset \), then

\[
\begin{align*}
\nu(B_M(x, 2r)) &\lesssim \mu(B(x, r)) \quad \text{and} \\
\nu(B_M(x, r)) &\gtrsim \mu(B(x, r)).
\end{align*}
\]

We may write \( \nu(B_M(x, 2r)) = \mu[a, b] + \nu(E) \), where \( a = \inf(B(x, 2r) \cap C) \), \( b = \sup(B(x, 2r) \cap C) \) and \( E \) is either empty or contains one or two isolated points of \( M \). By the construction of \( \nu \), we have \( \nu(E) \leq \mu(B(x, 4r)) \) and thus \( \nu(B_M(x, 2r)) \leq \mu[a, b] + \mu(B(x, 4r)) \leq \mu(B(x, 2r)) + \mu(B(x, 4r)) \)

\[
\lesssim \mu(B(x, r))
\]

since \( \mu \) is doubling. Thus 5.2 follows.

To show 5.3, assume first that \( B(x, r/2) \cap C \neq \emptyset \). If we let \( a = \inf(B(x, r) \cap C) \), \( b = \sup(B(x, r) \cap C) \), then Lemma 5.4 implies \( |b - a| \gtrsim r \) and thus \( \nu(B_M(x, r)) \gtrsim \nu[a, b] \gtrsim \mu(B(x, r)) \)

by Lemma 5.3. If \( B(x, r/2) \cap C = \emptyset \), then \( B(x, r/2) \subset J \) for some \( J \in \mathcal{J} \) with \( |J| \geq r \), and we have

\[
\nu(B_M(x, r)) \geq \nu\{x\} = \nu(J) \gtrsim \mu(B(x, r)).
\]

Thus we have 5.3 and it follows that \( \nu \) is a doubling measure on \( M \).

To give the details for the latter claim of the lemma requires a bit more work. Consider a doubling measure \( \nu \) on \( M \). We define \( \mu \) by the following procedure: Let \( c > 0 \) be the constant of Lemma 5.4 and choose \( 1/(1 + c) < t < 1 \). For \( J = [x - r, x + r] \in \mathcal{J} \), consider its division to Whitney type sub-intervals

\[
\begin{align*}
J_k^+ &= [x + r - (1 - t)^k r, x + r - (1 - t)^{k+1} r], \\
J_k^- &= [x - r + (1 - t)^{k+1} r, x - r + (1 - t)^k r].
\end{align*}
\]
that

\[ T \] formalize some of the claims for

\[ T \] are valid for

\[ T \]

\[ u \]

\[ K \]

\[ 5.4 \]

\[ \text{Since} \] \[ t > \frac{1}{2} \text{total measure} \]

\[ \text{dist}(\cdot) \]

\[ K' \]

\[ 1 - t \]

\[ u_2 \]

\[ \approx \]

\[ y \]

\[ (1 - 2g)K' \]

\[ \text{Figure 3. Illustration for the proof of (i).} \]

for \( k \in \{0, 1, 2, \ldots \} \). Next define

\[ m_{J_k^+} = \nu([x + r + (1 - t)^{k+1}r, x + r + (1 - t)^k r] \cap M), \]

\[ m_{J_k^-} = \nu([x - r - (1 - t)^k r, x - r - (1 - t)^{k+1} r] \cap M). \]

If \( K \) is one of the intervals \( J_k^+ \) or \( J_k^- \), let \( \mu|K \) be uniformly distributed on \( K \) with total measure

\[ \mu(K) = \frac{m_K \nu\{x_J\}}{\nu((2J \cap M) \setminus \{x_J\})}. \]

Observe that the scaling factor \( \nu\{x_J\}/\nu((2J \cap M) \setminus \{x_J\}) \) is bounded away from 0 and \( \infty \) as \( \nu \) is doubling. Thus we have

\[ \mu(K) \approx m_K \]

(5.4)

for all \( k \in \{J_k^+, J_k^-\}_{k=0}^{\infty} \). To complete the definition of \( \mu \), we set \( \mu|C = \nu|C \).

It is now evident that \( \mu(J) = \nu\{x_J\} \) for each \( J \in \mathcal{J} \) and it remains to show that \( \mu \) is doubling. For this purpose, we prove the following chain of claims. We formulate some of the claims for \( u_2 \) and \( J_k^+ \) but due to symmetry, similar claims are valid for \( u_1 \) and \( J_k^- \) as well.

Let \( J = [u_1, u_2] \in \mathcal{J} \). Then

(i) For each \( J_k^+ \), there is \( y \in M \) with \( y - u_2 \approx |J_k^+| \) such that \( \nu(B_M(y, |J_k^+|)) \approx \mu(J_k^+) \).

(ii) If \( K_0 \) and \( K_1 \) are two consecutive intervals among \( J_k^+ \), \( J_k^- \), then \( m_{K_0} \approx m_{K_1} \).

(iii) \( \mu(J_k^+) \approx \mu(\cup_{n > k} J_k^+) \)

(iv) If \( 0 < s < |J|/2 \), then \( \nu([u_2, u_2 + s] \cap M) \approx \mu[u_2 - s, u_2] \)

(v) If \( I \) is an interval with \( I \cap C \neq \emptyset \) and \( \kappa > 1 \), then \( \mu(I) \leq c(\kappa, t)\nu(\kappa I \cap M) \).

(vi) If \( 0 < s < |J| \), then \( \mu[u_2, u_2 + s] \approx \nu([u_2, u_2 + s] \cap M) \).

We now start to prove the claims (i)-(vi). Let \( c > 0 \) be the constant of Lemma 5.4. Since \( t > 1/(1 + c) \), we may choose \( g = g(t) > 0 \) such that \( 1 - t + gt < c(1 - 2g)t \).

Let \( K = J_k^+ \). By scaling, we may assume that \( (1 - t)^{k+1}r = 1 \) so that \( |K| = t \) and \( \text{dist}(K, u_2) = 1 - t \). Denote

\[ K' = [u_2 + (1 - t), u_2 + 1], \]

\[ (1 - 2g)K' = [u_2 + (1 - t) + gt, u_2 + 1 - gt]. \]

It follows from Lemma 5.4 and the choice of \( t \) and \( g \) that \( (1 - 2g)K' \cap M \neq \emptyset \). Thus, we may choose \( y \in M \) so that \( B_M(y, gt) \subset K' \). See Figure 3. Using the doubling property of \( \nu \), we get

\[ \mu(K) \approx \nu(K' \cap M) \geq \nu(B_M(y, gt)) \geq \nu(B_M(y, t)) \geq \nu(K' \cap M) \approx \mu(K) \]

As \( (1 - t) \leq |y - u_2| \leq 1 \), we have \( |y - u_2| \approx t = |J_k^+| \) and (i) follows.

Let \( K_0 \) and \( K_1 \) be two consecutive intervals among \( \{J_k^+, J_k^-\} \), and \( y_0, y_1 \in M \) be points given by (i). Then \( |y_0 - y_1| \lesssim |K_0| \approx |K_1| \) and combined with (5.4)
and Lemma [5.3] we get $m_{K_0} \approx \mu(K_0) \approx \nu(B_M(y_0, |K_0|)) \approx \nu(B_M(y_1, |K_1|)) \approx \mu(K_1) \approx m_K$, implying \[(i).

For $k = 0, 1, 2, \ldots$, let $y_k \in M$ be a point satisfying \[(i). Since $\nu$ is doubling, we get (using (5.4) and Lemma 5.3)

$$
\mu \left( \bigcup_{n=k+1}^{\infty} J_n^+ \right) = \mu[u_2 - \frac{1}{k+1} J_k^+, u_2] \approx \sum_{n>k} m_{J_n^+}
$$

Combining this with \[(i) yields \[(iv).\n
For each $\kappa > 0$, by (i)--(iii), we see that

$$
\mu \left( \bigcup_{n=k+1}^{\infty} J_n^+ \right) = \nu \left( [u_2, u_2 + \frac{1}{k+1} |J_k^+|] \cap M \right) \leq \nu(B_M(y_k, |J_k^+|)) \approx \mu(J_k^+).
$$

On the other hand, using (ii) we see that

$$
\mu(J_k^+) \approx m_{J_k^+} \approx m_{J_{k+1}^+} \leq \mu(\bigcup_{n=k+1}^{\infty} J_n^+) \text{ and \[(i)] follows.}
$$

By construction, we have

$$
\mu[u_2 - \frac{1}{k+1} J_k^+, u_2] \approx \sum_{n>k} m_{J_n^+} = \nu[u_2, u_2 + \frac{1}{k+1} |J_k^+|].
$$

Combining this with \[(i) yields \[(v).\n
Let $s < |J|$ and let $K$ be the largest interval among $\{J_k^+\}$ contained in $[u_2-s, u_2]$. With the help of \[(i)--(iii), we see that

$$
\mu[u_2-s, u_2] \approx \mu(K) \approx m_K \leq \nu([u_2, u_2 + s] \cap M)
$$

(And similarly $\mu[u_1, u_1 + s] \lesssim \nu([u_1-s, u_1] \cap M)$. To prove \[(vi), we apply this observation for the components of $I \cap C$ to obtain $\mu(I \cap C) \lesssim \nu(3I \cap M)$. Choosing

$$
y \in I \cap C, we have
$$

$$
\mu(I) = \nu(I \cap C) + \mu(I \cap C) \lesssim 2\nu(3I \cap M) \lesssim c(\kappa, t) \nu \left( B_M(y, \frac{\kappa}{2} I) \right)
$$

$$
\leq c(\kappa, t) \nu(\kappa I \cap M)
$$

for each $\kappa > 1$.

To prove \[(vii), let $v = \sup C \cap [u_2, u_2 + s]$. Using Lemma [5.3] we may find $y \in M$ and $r \gtrsim s$ so that $B_M(y, r) \subset [u_2, v]$. Now

$$
\nu([u_2, u_2 + s] \cap M) \lesssim \nu(B_M(y, r)) \leq \nu([u_2, v] \cap M) \approx \mu[u_2, v] \lesssim \mu[u_2, u_2 + s].
$$

On the other hand, we have $\mu[u_2, v] = \nu([u_2, v] \cap M)$ and

$$
\mu[v, u_2 + s] \lesssim \nu([v, u_2 + s] \cap M) \lesssim \nu(B_M(y, r)) \leq \nu([u_2, u_2 + s] \cap M)
$$

using \[(vi). Thus \[(vi) follows and we have verified all the claims \[(i)--(vi).

Let $I_1, I_2 \subset [0, 1]$ be two adjacent closed intervals of the same length. To finish the proof we have to show that

$$
(5.5) \quad \mu(I_1) \approx \mu(I_2).
$$

To achieve this goal, we consider several different cases and subcases.

**Case a:** Both intervals $I_1$ and $I_2$ are contained in a gap $J = [u_1, u_2] \subset J$. Let $K = \{J_k^+, J_k^- \colon k = 0, 1, 2, \ldots\}.$

**Subcase a1:** If both intervals $I_1$ and $I_2$ intersect at most 2 intervals of $K$, the estimate \[(5.5) follows directly from \[(i).

**Subcase a2:** If both intervals $I_i$ intersect at least 3 elements of $K$, let $K_i$ be the largest element $K \in K$ contained in $I_i$. Then, it follows from \[(i) and \[(ii) that $\mu(I_i) \approx \mu(K_i)$. On the other hand, there is at most one interval $K \in K$ in between $K_1$ and $K_2$ and thus, using \[(ii) once again, we get $\mu(K_1) \approx \mu(K_2)$.

**Subcase a3:** Suppose that $I_1$ intersects at least three sub-intervals $K \in K$ whereas $I_2$ intersects at most two of them. Again, letting $K_1$ be the largest element of $K$
contained in $I_1$, we have $\mu(I_1) \approx \mu(K_1)$. Now, if $K_2 \in \mathcal{K}$ and $K_2 \cap I_2 \neq \emptyset$, there are at most two intervals of $\mathcal{K}$ in between $K_1$ and $K_2$. Thus, from (11) we get $m_{K_2} \leq m_{K_1}$, giving $\mu(I_1) \approx \mu(I_2)$.

Case b: $I_1$ is contained in a gap but $I_2 \cap C \neq \emptyset$. We may assume by symmetry that $I_1 = [a, b]$, $I_2 = [b, c]$ (where $c - b = b - a$). Let $d = \inf(I_2 \cap C)$.

Subcase b1: If $d - b \geq c - d$, the claims (11) and (12) imply $\mu[b, c] \approx \mu[b, d]$ and from the case a and Lemma 5.3 we obtain $\mu[b, d] \approx \mu[a, b]$.

Subcase b2: If $d - b \leq c - d$, we first use the case a to get $\mu[a, b] \approx \mu[d - (c - d), d]$ and then (11) and (12) to conclude $\mu[d - (c - d), d] \approx \mu[d, c] \approx \mu[b, c]$.

Case c: $I_1 \cap C \neq \emptyset \neq I_2 \cap C$. By symmetry, we assume again that $I_1 = [a, b]$, $I_2 = [b, c]$, and denote $r = b - a = c - b$. Let $v_1 = \inf(I_1 \cap C)$, $v_2 = \sup(I_1 \cap C)$, $v_3 = \inf(I_2 \cap C)$, and $v_4 = \sup(I_2 \cap C)$.

Subcase c1: If $v_2 - v_1 \geq r/2$ and $v_4 - v_3 \geq r/2$, we can find $y_1 \in M$ so that $B_M(y_1, r/8) \subset [v_1, v_2] \cap M$ and

$$\mu(I_1) \geq \nu(v_1, v_2) = \nu([v_1, v_2] \cap M) \approx \nu(B_M(y_1, r/8)).$$

As also $\mu(I_1) \lesssim \nu(2I_1 \cap M)$ by (15), $2I_1 \cap M \subset B_M(y_1, 2r)$, and $\nu$ is doubling, we thus get $\mu(I_1) \approx \nu(B_M(y_1, r))$. Repeating the argument for $I_2$ yields $B_M(y_2, r/8) \subset [v_3, v_4] \cap M$ with $\mu(I_2) \approx \nu(B_M(y_2, r))$. Using Lemma 5.3 we get $\nu(B_M(y_1, r)) \approx \nu(B_M(y_2, r))$ yielding (5.3).

Subcase c2: Suppose $v_2 - v_1 \geq r/2$ and $v_4 - v_3 < r/2$. Now, as in subcase c1, we find $B_M(y_1, r/8) \subset [v_1, v_2] \cap M$ with $\mu(I_1) \approx \nu(B_M(y_1, r))$. On the other hand, letting $I_3$ be the longer of the intervals $[b, v_3]$ and $[v_4, c]$, with the help of (iii), we find $y_2 \in M$ with dist($I_3, y_2$) $\lesssim r$ and $s \approx r$ such that $\mu(I_3) \approx \nu(B_M(y_2, s))$. Again, as $\nu$ is doubling we can use Lemma 5.3 to conclude that $\mu(I_3) \approx \nu(B_M(y_1, r)) \approx \nu(B_M(y_2, r)) \approx \mu(I_2)$ as desired.

Subcase c3: Finally, if both $v_2 - v_1 < r/2$ and $v_4 - v_3 < r/2$, we let $I_3$ be the longer of the intervals $[a, v_1]$, $[v_2, b]$ and $I_4$ the longer of the sub-intervals $[b, v_3]$, $[v_4, c]$. As above, we find $B_M(y_1, s_1)$ and $B_M(y_2, s_2)$ so that $\mu(I_1) \approx \nu(B_M(y_1, s_1)) \approx \nu(B_M(y_2, s_2)) \approx \mu(I_2)$. $\square$

Proof of Theorem 5.1: Suppose first that $C$ is $(\alpha_n)$-porous as a subset of $\mathbb{R}$. The Claim (1) follows from the Lemma 4.1 since $C$ is $(\alpha_n/2)$-porous as a subset of $M$. Indeed, for each $J_{n,i}$, let $x_{n,i} = x_{J_{n,i}}$ and consider $B_{n,i} = B_M(x_{n,i}, |J_{n,i}|)$ and $B_{n,i} = B_M(x_{n,i}, |J_{n,i}|/2)$. Then $B_{n,i} \cap B_{n,j} = \emptyset$ if $(n, i) \neq (l, j)$ and moreover, $|J_{n,i}|/2 \geq (\alpha_n/2)|J_{n,i}|$ for all $n$ and $i$.

To prove the claims (2)–(4), we use Lemma 5.2. Then (3) follows from Theorem 2.2 [10] from the result of Staples and Ward [SW98, Theorem 1.4], and (4) from Theorem 2.1. Finally, (2) follows putting (1)–(4) together. $\square$

Remarks 5.5. a) Our choice to put one isolated point in the middle of each gap is somewhat arbitrary. The Theorem 5.1 (and Lemma 5.2) holds true for many other choices of (collections) of isolated points as well. For instance, instead of choosing the middle point of each $J \in \mathcal{J}$, one could consider a Whitney decomposition $\mathcal{W}_J$ of $J$ and choose all the midpoints of the elements of $\mathcal{W}_J$ to be the collection of isolated points inside $J$. Doubling measures on this kind of Whitney modification sets have been considered in [KW95], [WWW98].

b) In many situations, the technical assumption $5.1$ (used only to prove Lemma 5.3) may be omitted. For the middle interval midpoint sets $M(\alpha_n)$, for instance, the claims (2)–(4) in Theorem 5.1 hold also without this assumption.
Kaufman and Wu [KW95] have posed the following problem: Does there exist a compact set $X \subset \mathbb{R}$ with $X = F_X$ and a doubling measure $\nu$ on $X$ so that $\nu|_{E_X}$ is a doubling measure on $E_X$? Recall that $F_X$ is the set of isolated points of $X$ and $E_X = X \setminus F_X$. The following example yields a positive answer to their question.

Example 5.6. Let $(\alpha_n) \in \ell^1$, $X = M(\alpha_n)$, $C = C(\alpha_n)$, $\mu = \mathcal{L}|_{[0,1]}$, and let $\nu$ be a doubling measure on $X$ given by Lemma 5.2. Then $F_X = \cup_{J \in \mathcal{J}} \{x_J\}$, $E_X = C$, and $X = F_X$. Moreover, it is easy to see that $\nu|_C = \mathcal{L}|_C$ is a doubling measure on $C$ since there exists $c = c(\alpha_n)$ so that $\mathcal{L}(C \cap (x-r,x+r)) > cr$ for all $x \in C$ and $0 < r < 1$.

5.2. On sets with positive Lebesgue measure. To complete the discussion on purely atomic doubling measures, we answer a question posed by Lou, Wen, and Wu in [LWW07]. As observed by Wu [Wu98, Example 1], it is possible to construct compact sets $X \subset [0,1]$ with Hausdorff dimension one so that all doubling measures on $X$ are purely atomic. The examples of Wu [Wu98] and Lou, Wen, and Wu [LWW07] are countable unions of self-similar Cantor sets whose dimensions gets closer and closer to one. Another, more direct way to obtain such a set is given by Theorem 5.1: Choosing $\alpha > 1$ for any sequence $(\alpha_n)$ such that $\lim n \to \infty \frac{\log\left(\prod_{k=1}^{n}(1 - \alpha_k)\right)}{n} = 0$

\begin{equation}
(5.6) \lim_{n \to \infty} \frac{\log\left(\prod_{k=1}^{n}(1 - \alpha_k)\right)}{n} = 0
\end{equation}

will do. Note that (5.6) always holds if $\lim_{n \to \infty} \alpha_n = 0$. It was asked by Lou, Wen, and Wu [LWW07] whether there are compact sets $X \subset \mathbb{R}$ with positive Lebesgue measure so that all doubling measures $\mu$ on $X$ are purely atomic. The answer is negative.

Proposition 5.7. If $X \subset \mathbb{R}$ is compact and $\mathcal{L}(X) > 0$, there are doubling measures on $X$ with nontrivial continuous part.

Proof. The claim is a direct consequence of the results of Vol’berg and Konyagin [VK88], see also [Hei01, §13]. For subsets of $\mathbb{R}^n$, they proved the existence of $n$-homogeneous measures. In our case this gives a constant $c < \infty$ and a measure $\mu$ on $X$ so that $\mu(B(x,\lambda r)) \leq c\lambda \mu(B(x,r))$ for all $x \in X$, $\lambda \geq 1$, and $r > 0$.

Putting $\lambda = 1/r$, it follows that $c\mu(B(x,r)) \geq r$ for all $x \in X$ and $0 < r < 1$. Now we may define $\nu = \mu + \mathcal{L}|_X$. If $c'$ is the doubling constant of $\mu$, it follows that for all $x \in X$ and $0 < r < 1$,

$$
\nu(B(x,2r)) = \mu(B(x,2r)) + \mathcal{L}(X \cap B(x,2r)) \leq \mu(B(x,2r)) + 2r
$$

so $\nu$ is a doubling measure on $X$. As $\mathcal{L}(X) > 0$, it follows that $\nu$ has a nontrivial (absolutely) continuous part. \qed

Remark 5.8. While this paper was in preparation, there has been some independent research on the topics of the last section. Wang and Wen [WW12] have constructed a set $X$ with the same properties as in Example 5.6 and Lou and Wu [LW10] have also observed that Proposition 5.7 follows from the above mentioned result of Vol’berg and Konyagin.

Acknowledgements. We are grateful to Kevin Wildrick for useful discussions.
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