Exact calculation of spectral properties of a particle interacting with a one dimensional fermionic system.

H. Castella*, X. Zotos

Institut Romand de Recherche Numérique en Physique des Matériaux, (IRRMA)

PHB-Ecublens, CH-1015 Lausanne, Switzerland

and*

Département de Physique de la Matière Condensée

24, quai E. Ansermet, CH-1211 Genève, Switzerland

Using the Bethe ansatz analysis as was reformulated by Edwards, we calculate the spectral properties of a particle interacting with a bath of fermions in one dimension for the case of equal particle-fermion masses. These are directly related to singularities apparent in optical experiments in one dimensional systems. The orthogonality catastrophe for the case of an infinite particle mass survives in the limit of equal masses. We find that the exponent $\beta$ of the quasiparticle weight, $Z \simeq N^{-\beta}$, is different for the two cases, and proportional to their respective phaseshifts at the Fermi surface; we present a simple physical argument for this difference. We also show that these exponents describe the low energy behavior of the spectral function, for repulsive as well as attractive interaction.

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I. INTRODUCTION

The motion of a particle interacting with a Fermi bath has been the subject of many studies as it relates to a variety of physical phenomena: muon diffusion in metals [1], stability of the fully ferromagnetic state (the Nagaoka problem) [2] and particularly relevant to optical experiments in degenerate semiconductors and metals [3,4], where the hole created by optical excitation plays the role of the particle interacting with conduction electrons.

In the late 60’s the Fermi edge singularities in the X-ray absorption spectra of metals were explained in terms of correlation effects between conduction electrons and the photon created hole; Mahan first predicted the singularities before Nozières and DeDominicis calculated the exact spectrum; these pioneering works gave rise to the so called MND problem [3]. Relying on perturbative arguments valid for a three dimensional system, it was believed that such singularities would disappear in valence band excitations where the mass of the hole is comparable to the conduction electron mass, in contrast to the case of core level excitations, where the mass of the localized hole can be considered infinite [5]. In lower dimensions the situation is different [6] and in fact strong enhancement at threshold was recently observed in photoluminescence on quantum wires [4]. These experiments have raised further interest in optical singularities in low dimensional systems.

This model was also studied in connection with ferromagnetism in single band models and it has recently gained renewed attention in two dimensions because of the possible relevance of the single band Hubbard model to describe the high-Tc superconductors. In this context it describes the dynamics of a single spin flip in a ferromagnetic background and adresses the question of the stability of the Nagaoka state or the limitations of Gutzwiller type variational wave functions used for this problem [2].

In this work we study the spectral properties of the particle in one dimension and for the special case where the mass of the particle is the same as that of the fermions in the bath. For this analysis we use the analogy to McGuire’s solution of the problem of a single spin flip interacting with a ferromagnetic background in a one dimensional continuous system,
considering the particle as the reversed spin [7]. Actually McGuire’s work is a precursor, for the continuum model, of Lieb and Wu’s Bethe Ansatz (BA) exact solution of the Hubbard model in one dimension [8]. McGuire calculated static correlation functions as well as the effective mass of the reversed spin; a lattice version of the latter was presented in Ref. [9]. However, despite the BA solution, an exact calculation of the dynamic quantities has not been possible as it involves the calculation of matrix elements between, difficult to handle, BA wavefunctions for the excited states. In this work we use a new presentation of McGuire’s solution, due to Edwards [10]; the relative simplicity of the wave functions in this case allows us to evaluate directly the spectral weight $Z$ and the spectral function $A(k, \omega)$ in the small momentum $k$ regime. Our results show that the orthogonality catastrophe of Anderson occurs [11]: a quasiparticle description of the excitations in terms of non interacting eigenstates is therefore no more possible; the spectral function is totally incoherent and has the same sort of divergence as in the infinite mass case.

The occurrence of the orthogonality catastrophe and the divergence of the spectral weight were recently predicted within perturbation theory for the one dimensional problem [6]. The edge singularities of the absorption spectrum were also derived for a Tomonaga-Luttinger model and it was claimed that the corresponding exponents would not depend on the mass of the hole [12]. In our work however, we show that the exponents of the spectral function differ in the two extreme limits of infinite mass and equal masses. This seems in contradiction with the latter results; we will discuss the importance of backscattering, which was omitted in the Tomonaga-Luttinger model, to correctly describe the case of an infinite particle mass.

We note that, except for the very special case of one hole in a half filled band with infinite repulsion [13], our work presents the first calculation of dynamical correlation functions using directly the BA wavefunctions. Our calculations agree with previous results by Frahm and Korepin [14] thus giving support to the assumption of conformal invariance of the model used.

In summary we show in this article that the orthogonality catastrophe takes place for the case of equal particle-fermions masses and that the same exponents characterize the diver-
gence of the spectral function at threshold in the small momentum regime. The exponents however differ in the two cases of infinite mass and equal masses. The article is organized as follows. In section 2 we present the model and its exact solution following mainly Edwards. Our analysis is performed for the continuous model and we simply quote the results for the lattice. In section 3, we derive the orthogonality catastrophe for the repulsive interaction. In section 4, we calculate the spectral function for the repulsive and attractive case; in the latter we discuss the influence of the bound state.

II. MODEL

The model Hamiltonian describes $N$ fermions of mass $m$ and one particle of mass $m_h$ interacting via a delta function potential and moving on a line of length $L$ with periodic boundary conditions:

$$H = -\frac{1}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - \frac{1}{2m_h} \frac{\partial^2}{\partial x_0^2} + U \sum_{i=1}^{N} \delta(x_i - x_0).$$ (1)

Throughout this article we use the convention $m = 1$ and $m_h \to m_h/m$. When $m_h \to \infty$ the problem reduces to a single particle problem. The fermions evolve in the static potential created by the hole. The response of the Fermi sea to the sudden appearance of the disturbance was calculated by Nozières and De Dominicis [3]. For a finite mass however, this model gives rise to a full many body problem.

We will present at first the problem quite generally for an arbitrary mass and then focus on the solution for $m_h = 1$ following mostly Edwards presentation. It is convenient to express the problem in the reference frame attached to the particle. The wave functions in the two reference frames are related by a simple translation:

$$\Psi(x_0, \ldots, x_N) = e^{iQx_0} f(x_1 - x_0, \ldots, x_N - x_0).$$ (2)

$Q$ is the total momentum. For our periodic boundary conditions $Q = 2\pi m/L$, $m$ being an integer, and $y_j = x_j - x_0 \in [0, L]$. The Schrödinger equation for $f$ is then:
\[
\left( \frac{1}{2m_h} \left( Q + i \sum_{j=1}^{N} \frac{\partial}{\partial y_j} \right)^2 - \frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial y_j^2} + U \sum_{j=1}^{N} \delta(y_j) \right) f = E f \quad (3)
\]

It describes the motion of \( N \) fermions in a static local potential and interacting via their total momentum. In order to satisfy the equation everywhere and the periodicity, we add the boundary conditions:

\[
f(y_1, \ldots, y_N) \big|_{y_i=0}^L = 0 \\
-\frac{1}{\mu} \frac{\partial}{\partial y_i} f(y_1, \ldots, y_N) \big|_{y_i=0}^L = 2U f(y_1, \ldots, y_i = 0, \ldots, y_N) \quad (4)
\]

where \( \mu = m_h/(1 + m_h) \) is the relative mass. Edwards proposed the following solution in the case \( m_h = 1 \):

\[
f(y_1, \ldots, y_N) = \frac{1}{\sqrt{N!L}} \det(\phi_j(y_l)) \quad (5)
\]

For this wave function, the \( \phi_j \) are normalized and the boundary conditions imply : \( \phi_j(0) = \phi_j(L) \) and \( \phi_j'(0) - \phi_j'(L) = U \phi_j(0) \). This can be achieved if the functions \( \phi_j \) have the following form:

\[
\phi_j(y) = \sum_{t=0}^{N} a_t e^{ik_t y} \quad (6)
\]

where the momenta \( k_j \) are all different and solutions of the BA type of equations:

\[
Lk_j = 2\pi (n_j + 1/2) - 2\arctan(2(k_j - \Lambda)/U), \quad j = 0, \ldots, N
\]

\[
Q = \sum_{j=0}^{N} k_j \quad \text{and} \quad E = \frac{1}{2} \sum_{j=0}^{N} k_j^2 \quad (7)
\]

These are Lieb and Wu’s equations [8] in our special spin sector of a single spin flip from the totally ferromagnetic state and in the continuum limit; for an even total number of particles \( N + 1 \), the \( n_j \) are integers. A similar set of equations was originally derived by McGuire. The eigenstates are specified by the quantum numbers \( n_j, \quad j = 0, \ldots, N \) and \( m; \ \Lambda \) is a real number which ensures that the total momentum is \( Q = 2\pi m/L \) for a given choice of \( n_j \).

We would like to focus on the simplicity of this solution compared to McGuire’s original one. Indeed the latter was written in the static reference frame and defined by pieces in
the different regions $x_j < x_0 < x_{j'}$, corresponding to a given ordering of the particles on the line; moreover it was a superposition of $N + 1$ determinants. In contrast Edwards solution is defined on the compact domain $[0, L]^N$ and it takes the form of a single determinant. However, we point out that the plane waves appearing in (6) are not the usual free states (as $k_l \neq 2\pi n/L$ for $U \neq 0$) and are not periodic on $[0, L]$ unless $U = 0$. But the functions $\phi_j$ have to be periodic.

In order to construct the total wavefunction, we have to first solve the BA equations for a given choice of quantum numbers, form the linear combinations in (6) and then build up the determinant. We have $N + 1$ plane waves and $N \phi_j$’s. Is the total wavefunction uniquely defined by this procedure? In fact there is an additional constraint on the coefficients $a^t_j$ coming from the periodicity requirement:

$$\sum_{t=0}^{N} a^t_j \left( 1 - e^{ik_tL} \right) = 0$$

(8)

This restricts the $\phi_j$’s to a $N$ dimensional subspace where we can make an arbitrary choice of basis. The total wavefunction being a determinant is thus unique.

We can find a physically convenient basis of functions:

$$\phi_j(y) = A_j(e^{i(k_jy+\delta_j)} - \sin(\delta_j)g(y)), \quad j = 1, \ldots, N$$

$$g(y) = \frac{\sum_l e^{i(k_ly+\delta_l)}/\sum_l \sin(\delta_l)}{\sum_l \sin(\delta_l)}$$

(9)

where the phaseshifts are given by the BA equations:

$$Lk_j = 2\pi n_j - 2\delta_j, \quad \delta_j = -\frac{\pi}{2} + \arctan \left( \frac{2k_j - \Lambda}{U} \right)$$

(10)

$g(y)$ is an almost localized function for large $L$ and $N$; it corrects the plane wave around the origin in order to achieve periodicity without affecting its plane wave character almost everywhere. We show in the appendix the following statements:

$$A_j \rightarrow \frac{1}{L}(1 + O(1/L))$$

(11)

$$\int_0^L \phi^*_j(y)\phi_l(y)dy = \delta_{jl} + O(1/L)$$

(12)
\[
X_j^p = \frac{1}{\sqrt{L}} \int_0^L e^{-i\frac{2\pi j}{L} y} \phi_j(y) dy = \frac{\sin(\delta_j)}{(n_j - p - \delta_j/\pi)} + O(\log(L)/L), \quad p \text{ integer} \quad (13)
\]

These last results tell us that the \( \phi_j \) behave like an orthonormal set of single particle scattering states for large systems:

\[
\phi_j(y) \simeq \frac{1}{\sqrt{L}} e^{i(k_j y + \delta_j)}, \quad 0 \ll y \ll L \quad (14)
\]

These are central results in our paper and the overlaps \( X_j^p \) are used in all the subsequent calculations.

III. ORTHOGONALITY CATASTROPHE

Anderson studied the influence of a static potential on a Fermi sea \([11]\); he showed that the ground state, \( |\psi_0\rangle \), of the system in the presence of the potential became orthogonal in the thermodynamic limit to the ground state without potential, \( |\tilde{\psi}_0\rangle \):

\[
Z = |\langle \tilde{\psi}_0 | \psi_0 \rangle|^2 \propto N^{-\beta_+ - \beta_-}, \quad \beta_{\pm} = \left( \frac{\delta_{F\pm}}{\pi} \right)^2 \quad (15)
\]

where \( \delta_{F\pm} \) are the phaseshifts of the electrons at the Fermi surface for the two channels of even and odd parity wave functions, \( N \) the number of fermions and the overlap \( Z \) the spectral weight. This orthogonality catastrophe takes place in the MND problem where the static potential is created by the core hole. Moreover the same exponents \( \beta_{\pm} \) appear in the spectral function of the hole. In our model with \( m_h \to \infty \), the odd states do not feel a potential located at the origin and the odd phaseshift vanishes; only the even parity states contribute to Anderson’s orthogonality catastrophe.

In this section we calculate the spectral weight for our model with equal masses and a repulsive interaction. We show that the orthogonality catastrophe also takes place but with different exponents than for an infinite mass. The spectral weight can be evaluated with the use of our wave function for the ground state and the corresponding overlaps \( X_i^m \):

\[
Z = \left( \int dx_0 \ldots dx_N \tilde{f}^* f \right)^2 = \det(X_i^m)^2 \quad (16)
\]
\( f \) is here the interacting ground state wavefunction in the reference frame of the particle and \( \tilde{f} \) the corresponding wavefunction for the non interacting system. The distribution of momenta for the ground state is given by the following choice of quantum numbers: the \( n_j \) are consecutive integers from \(-(N + 1)/2\) to \((N - 1)/2\) and \( \Lambda = 0 \); the total momentum vanishes. Both \( k \) and \(-k\) appear in the spectrum and it is natural to form even and odd combinations of the interacting single particle wavefunctions \( \phi_j \); they are then combinations of \( \cos (k_jy + \delta_j) \) and \( \sin (k_jy + \delta_j) \) for the positive \( k_j \). We note here that both the even and odd channels have a non vanishing phaseshift in contrast to the infinite mass case. The non interacting states can also be expressed as odd and even functions. The spectral weight factorizes then into two contributions coming from both parities, \( Z^+ \) and \( Z^- \):

\[
\sqrt{Z^\pm} \simeq \left( \prod_l \frac{\sin \delta_l}{\pi} \right) \det \left( \frac{1}{(n - n') - \delta_n/\pi} \right) \quad (17)
\]

The indices \( l, n, n' \) run over integers from 0 to \((N - 1)/2\) corresponding to the positive solutions of the BA equations. Anderson calculated this determinant using an algorithm due to Cauchy; his result reads:

\[
Z^\pm \propto N^{-(\delta_F/\pi)^2} \quad (18)
\]

\( \delta_F \) is the phaseshift at the Fermi surface which can be expressed in terms of the density of states \( n_F = 1/\pi k_F \):

\[
\delta_F = -\arctan \left( \frac{U\pi n_F}{2} \right) \quad (19)
\]

The exponent for the lattice has the same form but with the corresponding density of states. We can now compare our result with the known result for the static impurity. So in the \( m_h = 1 \) case, we have:

\[
Z \propto N^{-\beta^+ - \beta^-} \text{ where } \beta^+ = \beta^- = \left( \frac{\delta_F}{\pi} \right)^2 \quad (20)
\]

In contrast in the infinite mass case:

\[
\beta^+ = \left( \frac{\delta_F}{\pi} \right)^2, \text{ with } \delta_F = -\arctan (U\pi n_F) \text{ and } \beta^- = 0 \quad (21)
\]
For \( m_h \to \infty \), the only contribution to the exponent comes from the even parity, the odd phaseshift vanishing. For \( m_h = 1 \), both parities contribute equally; moreover, the density of possible excitations at the Fermi surface is reduced by a factor of 2. This difference can be understood easily by simple perturbative arguments. For an infinite mass particle, both forward and backward scattering at the Fermi surface involve low energy excitations; for \( m_h = 1 \) however, backward scattering is forbidden because the recoil energy of the particle is of the order of the Fermi energy. So only half of the excitations are allowed. In figure 1 we show the exponent for the two cases of equal masses and infinite mass as a function of the dimensionless parameter \( U n_F \).

IV. SPECTRAL FUNCTION

In this section we use our BA eigenstates to evaluate the low energy part of the spectral function \( A(Q,\omega) \); it is defined as the Fourier Transform of the propagator of the particle \( G_Q(t) \) :

\[
A(Q,\omega) = \frac{1}{\pi} \text{Im} \int_{-\infty}^{\infty} e^{i\omega t} G_Q(t)
\]

\[
G_Q(t) = i \langle T \{ b_Q(t) b_Q^\dagger(0) \} \rangle
\]

(22)

\( b_Q^\dagger \) ( resp. \( b_Q \)) is the creation (annihilation) operator of the particle in a free state of momentum \( Q \). The propagator can be expanded in the eigenstates of the interacting system :

\[
G_Q(t) = \sum_\lambda e^{-i(E_\lambda - \tilde{E}_0)t} |\langle \tilde{\psi}_0(Q) | \psi_\lambda(Q) \rangle|^2 \theta(t)
\]

(23)

|\( \tilde{\psi}_0(Q) \rangle \) is the non interacting ground state of \( N \) fermions with the particle at momentum \( Q \); its energy is \( \tilde{E}_0 \). |\( \psi_\lambda(Q) \rangle \) are the eigenstates of the interacting system with energy \( E_\lambda \) and total momentum \( Q \).

The spectral function describes the photoemission spectrum in the so called intrinsic approximation \[15\]. For an infinite mass particle it has no momentum dependence and is totally incoherent \[16\]: it has no quasiparticle peak because of the vanishing spectral weight and presents a power law singularity :
\[ A(\omega) \propto \frac{1}{|\omega - \omega_0|^{1-\beta^+}} \quad (24) \]

In this section we show that the spectral function for the \( m_h = 1 \) has the same low energy behaviour for momenta close to the bottom of the band \( Q \simeq 0 \); we treat the case \( Q = 0 \) in detail and mention the results for other total momenta. We examine first how the low energy spectrum of the interacting system with \( Q = 0 \) can be described in the language of the non-interacting system.

We are going now to discuss separately the case of a repulsive \( (U > 0) \) and attractive interaction \( (U < 0) \); the difference is the presence of a bound state for the attractive potential.

**A. \( U > 0 \)**

There are two types of excitations: ‘single particle’ excitations where one of the quantum numbers \( n_j \) is changed and collective excitations where \( \Lambda \) is changed. In general, we must combine both types to have an eigenstate of zero total momentum. We begin our discussion with the simplest excited states that are exactly single particle like and then extend our description to the others.

For the states with a symmetric distribution of the half integers \( n_j + 1/2 \) around the origin, we can achieve a zero total momentum with \( \Lambda = 0 \) and the distribution of momenta is symmetric as well. Moreover when \( \Lambda = 0 \), the BA equations are decoupled and the excitations are strictly single particle like, in the sense that the spectrum corresponds to the ground state momentum distribution plus a symmetric momentum distribution of ‘particle-hole’ like excitations.

The situation is more complicated for the states with an asymmetric distribution of quantum numbers because zero total momentum implies \( \Lambda \neq 0 \). This causes a global shift of all the momenta compared to the ground state and these excitations are not strictly single particle like. However the excitation energy and spectral weight can be approximated in order to recover the single particle description.
Let’s take the simplest type of these \( \Lambda \neq 0 \) excitations, namely when one of the \( n_j \) close to the Fermi surface is changed: \( n'_j - n_j = J \). If \( J \) is small compared to \( N \) the excitation energy is small and \( \Lambda \) is small as well. A similar analysis of the spectrum as in Ref. [7] can be performed; the main results are as follows:

\[
L(k'_l - k_l) = \frac{4}{U} \frac{1}{1 + (2k_l/U)^2} \Lambda + \frac{16}{U^2} \frac{k_l}{1 + (2k_l/U)^2} \Lambda^2, \quad l \neq j
\]

\[
\Lambda = -\frac{2\pi J}{L} \frac{\pi}{2 \arctan(2k_F/U)}
\]

In order to evaluate the propagator, we need to know the excitation energy and the spectral weight. The change in energy is:

\[
E_\lambda - E_0 = \frac{2\pi J}{L} + \left( \frac{2\pi J}{L} \right)^2 \frac{1}{m^*} + O \left( \frac{1}{L^2} \right)
\]

\[
1/m^* = \frac{\pi}{2} \left( \arctan(2k_F/U) - \frac{2k_F}{U(1 + (2k_F/U)^2)} \right) / (\arctan(2k_F/U))^2
\]

The excitation energy \( E_\lambda - E_0 \) is a sum of two contributions, the first corresponding to the particle-hole excitation and the second to a rearrangement of the Fermi sea, a backflow term. As pointed out by McGuire the momentum distribution of the particle is centered around \( K = 2\pi J/L \) so that we can interpret the usual backflow as a recoil of the particle to the particle-hole excitation in the Fermi sea; its mass is renormalized to \( m^* \). For low lying excitations \( E_\lambda - E_0 \ll 1/2k_F^2 \) so that \( |2\pi J/L| \ll k_F \). In that case the recoil energy is negligible and the excitation energy is of particle-hole type only.

The shift of all the momenta \( (k_l \rightarrow k'_l) \) influences also the spectral weight; as for the ground state, it is a simple determinant of the individual overlaps \( X^p_l \). Because of the particle-hole excitation \( n_j \rightarrow n'_j \), the corresponding overlap \( X^p_j \) is replaced by \( X^{p'}_j \); this is the usual effect of an excitation in an independent particle problem. In addition, all the other overlaps are influenced by the backflow. Using (25) we find for \( j \neq l \):

\[
\left| \frac{X^p_l - X^{p'}_l}{X^p_l} \right| \approx \left| \frac{2\pi |J|}{Lk_F} \frac{1}{2(n_l - p) - 1} \right| \ll 1 \quad \text{for } k_l/U \rightarrow 0 \quad \text{and } k_l \sim k_F
\]

\[
\left| \frac{X^p_l - X^{p'}_l}{X^p_l} \right| \approx \left| \frac{2\pi |J|}{Lk_F} \right| \ll 1 \quad \text{for } k_l/U \rightarrow \infty
\]

(27)
Thus the spectral weight has also the usual independent particle form: if one particle-hole pair is created around the Fermi surface, only one of the overlaps is significantly altered. All the low lying excitations are additive in the sense that their excitation energy and momentum is a sum of individual contributions. This allows us to generalize our results to all low lying excitations.

In summary we note that the backflow is not important in our problem, neither for the excitation energy nor for the spectral weight. This is simply due to the fact that for a scattering between the particle at the bottom of the band and one electron at the Fermi surface, for small momentum transfer $q$, the recoil energy $E = q^2/2$ is negligible compared to the particle hole energy which is linear in $q$. We stress that this is no more the case when the particle does not lie at the bottom of the band.

Now the propagator can be approximated in the following way:

$$G_0(t) \simeq \sum_{\{n_l\}} e^{i \sum (k_l^2 - \tilde{k}_p^2)(t + i\xi_0^{-1})/2} (\det (X_{n_l}^p))^2$$

$$= \det \left( \sum_{l=-\infty}^{\infty} e^{i(k_l^2 - \tilde{k}_p^2)(t + i\xi_0^{-1})/2} X_l^p X_l^p' \right)$$

(28)

where we introduced an energy cutoff $\xi_0$ corresponding to the range of validity of our single particle description. The momenta $\tilde{k}_p$ are the non interacting one and the $k_l$ are solutions of the BA equations for $\Lambda = 0$ ie when we neglect the backflow term. In this last expression we can again use a basis of definite parity; this was not possible for the exact eigenstates but our description of the low lying excitations allows us to recover this symmetry like in the ground state. The propagator factorizes then in two equal contributions for odd and even parity. The calculation of this determinant was performed by Nozières and Combescot and we only quote here their result [17]:

$$G_0(t) \simeq i e^{-it\omega_0} (i\xi_0 t)^{-\beta^+ - \beta^-} \Theta(t)$$

(29)

where $\beta^+ = \beta^- = (\tilde{\delta}(\epsilon_F)/\pi)^2$ and $\tilde{\delta}$ is a determination of the phaseshift at the Fermi surface. Therefore the spectral function has a divergence at an energy $\omega_0 = -(2/\pi) \int_0^{\epsilon_F} \tilde{\delta}(\epsilon) d\epsilon$ and
the exponent is $1 - \beta^+ - \beta^-$. The determination $\tilde{\delta}$ is a priori unknown, but if we compare the threshold energy $\omega_0$ with the correct energy shift due to the interaction $E_0 - \tilde{E}_0 = -(2/\pi) \int_0^{\epsilon_F} \delta(\epsilon) d\epsilon$ we can fix the determination: $\tilde{\delta} = \delta$. In summary, the spectral function at the bottom of the band has a power law singularity and we recover the exponents $\beta^+$ and $\beta^-$ of the orthogonality catastrophe.

What changes if the momentum is not strictly zero but still much smaller than $k_F$? The energies are simply shifted by a recoil term of the particle $Q^2/2m^*$; the overlaps $X^p_l$ around the Fermi surface are not affected as well and the form (29) of the propagator remains:

$$G(Q,t) \simeq e^{-iQ^2t/2m^*}G_0(t)$$

(30)

The spectral function is then only shifted rigidly by the recoil energy : its exponent is the same. Nevertheless this is true only close to the bottom of the band : the numerical calculations showed that this shape was strongly altered in the vicinity of the Fermi surface [18].

**B. $U < 0$**

What changes for the attractive potential? The main difference comes from the appearance of a bound state. Indeed the BA equations have two imaginary solutions which in the large $L$ limit are [7] :

$$\kappa = \Lambda \pm i\frac{U}{2^2}$$

(31)

They always appear in pair in order to have a real total momentum. The ground state is reached when $\Lambda = 0$; the other momenta are solutions of the BA equations with $U < 0$ and the integers $n_j$ are consecutive from $-(N-1)/2$ to $(N-3)/2$. We would like to note that the phaseshifts do not behave like in 3D : the appearance of a bound state is not characterized by a phaseshift going to $\pi$ at the bottom of the band but it is simply opposite to the phaseshift for the repulsive potential with comparable strength $|U|$. 

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The ground state wave function is simple as well: it looks similar to the repulsive case but with one of the $\phi_j$'s describing a bound state:

$$\phi_0(x) = A_0 \cosh \left( \frac{U}{2} (x - \frac{L}{2}) \right), \quad A_0 \propto e^{-UL/2} \quad (32)$$

All the analysis is similar to the repulsive case; we should simply add one bound state in the single particle like excitation; its overlap with the free states is simply:

$$X_B^m = \frac{\alpha_B}{\epsilon_m - \epsilon_B} \quad (33)$$

B stands for bound state, $\epsilon_B = \kappa^2/2$, $\alpha_B = \cosh(\kappa L/2)\kappa/\sqrt{A_0}$ and $\epsilon_m$ is the free state energy.

Nozières and Combescot studied the influence of the bound state and they showed that the spectral function had two divergences, one at the energy of the true ground state, the other at the energy of the lowest excited state which does not contain any bound state. We first consider the absolute threshold.

We can perform the same analysis of the low lying excitations and end up with an asymptotic form of the propagator which is valid for energies close to the ground state energy:

$$G_0(t) \simeq e^{i\epsilon_B t} \det \left( \sum_{l=\infty}^\infty e^{i(k_l^2 - \tilde{k}_l^2)(t+i\xi_l^{-1})/2} X_l^p X_l^{p'} + e^{i(\epsilon_B - \tilde{k}_l^2/2)} X_B^p X_B^{p'} \right) \quad (34)$$

The threshold takes place at an energy $\omega_0 = 2\epsilon_B - (2/\pi) \int_0^{\epsilon_F} \delta(\epsilon)d\epsilon$ and the exponents are the same as for the repulsive potential.

There is also a secondary threshold; we can find an asymptotic form of the propagator taking into account all the excited states in which the bound state is absent. The threshold takes place at an energy $\omega_0 = 2\epsilon_F - (2/\pi) \int_0^{\epsilon_F} \delta(\epsilon)d\epsilon$ and the exponent is $1 - \beta^+ - \beta^-$ with $\beta^+ = (\delta_F/\pi - 1)^2$ and $\beta^- = (\delta_F/\pi)^2$.

V. CONCLUSIONS

In this article we calculated the exponent of the spectral function in the asymptotic low energy range using the BA wavefunctions. The spectral function has no quasiparticle
peak and its incoherent part has a power law divergence at threshold. It is interesting
to note that this singularity is accompanied by an orthogonality catastrophe with similar
exponents; although such a coincidence was expected from perturbative analyses, no exact
relation between these two quantities exist. These features are reminiscent of the static
impurity problem although the exponents are different. This similarity however does not
mean that the particle is localized in our case; in fact the effective mass is finite fo finite $U$
[7,9]. Care must be taken in distinguishing localization as probed by transport or optical
experiments.

Recently, Ogawa et al [12] calculated the absorption spectrum for a Tomonaga-Luttinger
model; they claimed that the exponents of the singularity did not depend on the mass of
the particle; this was also valid for the spectral function. However our results show that
the exponents differ in the two extreme cases $m_h = \infty$ and $m_h = 1$; in fact the authors
neglected the backscattering and this process becomes relevant in the infinite mass limit as
we already pointed out in section 3. We cannot exclude however that their results might be
correct for any finite mass because the infinite mass problem is very particular due to the
broken translation symmetry. A further analysis of the problem for intermediate masses is
then needed. As no exact solution exist for other masses, we are extending our study of the
spectral weight using Quantum Monte Carlo techniques.

We mention the agreement of our results with the calculation in Ref. [14] using conformal
invariance: we recover their exponents in the presence of a strong magnetic field where the
ground state of the full Hubbard model is almost ferromagnetic. Our results are also in good
qualitative agreement with numerical results on this model [18]; however no quantitative
comparison can be performed because of the small systems studied numerically.

Eventually, we can draw some conclusions concerning the experiments performed on
quantum wires. Although we didn’t calculate the absorption spectrum, we have noted the
similarity of our situation with the core level problem and in the latter, the divergence
of the spectral function is closely related to the edge singularities; thus the divergence of
the spectral function in our model is consistent with the interpretation of the experimental
spectra in terms of excitonic effects. The calculation of the absorption spectrum with the BA solutions is under investigation.

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VII. APPENDIX

We prove here the assertions (11-13) which constitute central results of this paper. Suppose that we have found a set \( \{ k_j \} \) which are solutions of the B.A. equations. We first evaluate the overlap between two plane waves built up with different momenta \( k_1 \) and \( k_2 \).

\[
\int_0^L e^{i((k_1-k_2)x+\delta_1-\delta_2)} dx = e^{i(\delta_1-\delta_2)} \frac{e^{i(k_1-k_2)L} - 1}{i(k_1-k_2)} = -4 \frac{\sin(\delta_1) \sin(\delta_2)}{U} \tag{35}
\]

where we have used the following property of the B.A. solutions:

\[
\frac{1 - e^{i(k_1-k_2)L}}{(1 - e^{ik_1L})(1 - e^{-ik_2L})} = \frac{i(k_1-k_2)}{U} \tag{36}
\]

We turn now to the evaluation of the normalization factor \( A_j \):

\[
A_j^2 = \frac{1}{L} \left\{ 1 + \frac{4 \sin(\delta_j)^2}{LU} - 2 \frac{\sin(\delta_j)}{\sum_l \sin(\delta_l)}^{-1} + \left( N + 1 \right) \frac{\sin(\delta_j)^2}{(\sum_l \sin(\delta_l))^2} + O \left( \frac{1}{L^2} \right) \right\} \tag{37}
\]

The sum \( \sum_l \sin(\delta_l) \) is of order \( L \) and we recover the result (11). In order to gain insight in the scaling behaviour of this quantity, we can evaluate the sum for the ground state:

\[
\sum_l \sin(\delta_l) = -\frac{L}{\pi} \int_0^{k_F} \sin \left( \arctan \left( \frac{U}{2k} \right) \right) dk \]

\[
= -\frac{LU}{2\pi} \log \left( \frac{2k_F}{U} + \sqrt{\left( \frac{2k_F}{U} \right)^2 + 1} \right) \tag{38}
\]

We note that the corrections increase with \( U \) and inversely decrease with the density of fermions. This is generic for all the scaling behaviours we studied in this model.
We perform now the overlap between two functions $\phi_j$ and $\phi_n$ with $j \neq n$:

$$\int_0^L \phi_j^*(x)\phi_n(x)dx = A_jA_n\left\{\frac{4\sin(\delta_j)\sin(\delta_n)}{U} - \frac{L(\sin(\delta_j) + \sin(\delta_n))}{\sum_l\sin(\delta_l)}\right\}$$  \hspace{1cm} (39)

As the normalization factors behave like $O\left(1/\sqrt{L}\right)$ these overlaps decrease as well like $1/L$.

We end up the discussion with the asymptotic form of the Fourier components of our functions $\phi_j$ which are essential in the spectral analysis:

$$\frac{1}{\sqrt{L}} \int_0^L e^{-i\frac{2\pi m}{L}x}\phi_j(x)dx = \frac{A_j\sqrt{L}\sin(\delta_j)}{\pi}\left(\frac{1}{(n_j - m) - \delta_j/\pi}\right)$$

$$- \frac{1}{\sum_l\sin(\delta_l)} \sum_p\frac{\sin(\delta_p)}{(n_p - m) - \delta_p/\pi}$$  \hspace{1cm} (40)

If $m$ is close to the Fermi surface at $n_F = (N - 1)/2$, the last sum gives a logarithmic correction:

$$\sum_l\frac{\sin(\delta_l)}{(n_l - m) - \delta_l/\pi} \simeq \sin(\delta_m)\log(n_F + m)$$  \hspace{1cm} (41)

The correction is then of order $\log N/N$ which decreases slowly and tends to shift the scaling region to larger sizes in any evaluation of the spectral properties for finite systems.
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FIGURES

FIG. 1. Exponent $\beta^+ + \beta^-$ of the orthogonality catastrophe as a function of the dimensionless parameter $Un_F$ where $U$ is the interaction strength and $n_F$ the density of states at the Fermi energy for the equal masses case (solid line) and the infinite mass case (dashed line).