A 3+1 covariant suite of Numerical Relativity Evolution Systems

C. Bona, T. Ledvinka and C. Palenzuela

Departament de Física, Universitat de les Illes Balears, Ctra de Valldemossa km 7.5, 07071 Palma de Mallorca, Spain

\textsuperscript{1} Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 180 00 Prague 8, Czech Republic

A suite of three evolution systems is presented in the framework of the 3+1 formalism. The first one is of second order in space derivatives and has the same causal structure of the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) system for a suitable choice of parameters. The second one is the standard first order version of the first one and has the same causal structure of the Bona-Masso system for a given parameter choice. The third one is obtained from the second one by reducing the space of variables in such a way that the only modes that propagate with zero characteristic speed are the trivial ones. This last system has the same structure of the ADM system by introducing a number of arbitrary parameters. The correspondence between both sets of parameters is explicitly given. The fact that the suite started with a system in which all the dynamical variables behave as tensors (contrary to what happens with BSSN system) allows one to keep the same parametrization when passing from one system to the next in the suite. The direct relationship between each parameter and a particular characteristic speed, which is quite evident in the second and the third systems, is a direct consequence of the manifest 3+1 covariance of the approach.

I. INTRODUCTION

In a recent article \[1\], the KST paper, a wide class of hyperbolic first order formalisms has been studied with a view to Numerical Relativity applications. All these formalisms do use the well known 3 + 1 decomposition of spacetime

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij} \left( dx^i + \beta^i dt \right) \left( dx^j + \beta^j dt \right)$$ \quad i, j = 1, 2, 3 \quad (1)

which allows one to use Einstein’s field equations as an evolution system for the metric (1), namely

$$\left( \partial_t - L_{\beta} \right) \gamma_{ij} = -2 \alpha \ K_{ij}$$ \quad (2)

$$\left( \partial_t - L_{\beta} \right) K_{ij} = -\nabla_i \alpha_j$$ \quad (3)

$$+ \alpha \left[ \gamma^{ij} R_{ij} - 2K^2_{ij} + tr K K_{ij} - \tilde{R}_{ij} \right]$$

(ADM system). The KST paper describes many ways of obtaining a first order version of the second order ADM system by introducing a number of arbitrary parameters. Different parameter choices correspond to different ways of using the energy and momentum constraints

$$R - tr(K^2) + (tr K)^2 = 2 \tau$$ \quad (4)

$$\nabla_i K^k_i - \partial_i (tr K) = S_i$$ \quad (5)

to modify the structure of the resulting systems. The goal is to get well posed systems, so that existence, unicity and stability of the solutions can be ensured. In this sense, the KST paper can be understood as a generalization of previous works by other groups \[3–5\], which have in common the fact that the only independent quantities are the metric coefficients and their first derivatives.

A similar approach can be used to generalise the hyperbolic first order formalisms proposed by the Palma group \[6,7\] as suggested in Ref. \[8\]. The main difference with the formalisms studied in the KST paper is that in this case there are three supplementary variables, independent of the metric derivatives, whose evolution equations are obtained by reversing the order of space and time derivatives in the momentum constraint (3). This use of the momentum constraint to evolve three supplementary variables can also be found in another context \[9,10\], where a second order system of the form (1) is considered (BSSN system).

To summarize, we have different formalisms in which the constraints (4, 5) are used to modify the structure of the ADM evolution system. The momentum constraint is used in some cases to evolve supplementary quantities, either keeping the system to be of second order \[10\] or getting first order versions \[6,7\]. But the same momentum constraint is used instead in other cases to modify the first order versions without introducing extra quantities \[3–5\].

The purpose of the present article is to point out the strong relationship among all these formalisms from the theoretical point of view. With this aim, we propose a new covariant framework in which the momentum constraint is used to evolve a vector quantity, so that the properties of the resulting systems are independent of the choice of space coordinates.

The paper is organised as follows: In Section II, we introduce the vector quantity $Z_i$, which will replace the corresponding non-vector supplementary quantities in (2, 3). The original ADM system is then generalised by making use of the vector $Z_i$ and the energy constraint (4).

In Section III, the resulting second order system (which will be hereafter referred to as system A) is compared with the BSSN system (4, 5).

In Section IV, we obtain a first order version of system A (which will be hereafter referred to as system B), which is compared with the formalisms presented in (6, 7).

Finally, in section V, system B is further modified in order to be compared with the formalisms discussed in the KST paper.
II. INTEGRATING THE MOMENTUM CONSTRAINT

Let us define the vector $Z^i$ as follows:

$$(\partial_t - \mathcal{L}_{\beta})Z_i = \alpha \left[ \nabla_j \left( K^j_{\gamma} - \delta^j_{\gamma} trK \right) - S_i \right]$$

(6)

so that it is clear that $Z_i$ vanishes for physical solutions, which can be defined as the ones verifying both the energy and momentum constraints (3, 4). The "zero" vector $Z_i$ can then be considered as a good covariant measure of the deviation of any solution $(\alpha, \gamma_{ij}, K_{ij})$ of the ADM system from the submanifold of physical solutions, as far as the momentum constraint is concerned. This means that the cumulative effect of the differential constraint (3) to implement covariant quasilinear modifications of these quantities can be also considered as supplementary dynamics, can be also considered as supplementary dynamical variables. This is the kind of approach proposed in Ref. [11], where a "l-system" was proposed containing as much as 40 supplementary quantities to be evolved (as they do) the same submanifold of physical solutions. In this context, it is natural to use system A as a framework to compare different formalisms that have been proposed with a view to Numerical Relativity applications, where the causal structure of the evolution system is important to ensure the well posedness of the problem, namely the existence, unicity and stability of the solutions.

We can also take advantage of the covariance of system A under coordinate transformations of the generic form:

$$t' = f(t)$$

$$x' = g(x^i, t)$$

(3+1 covariance) so that we can choose $\beta^t = 0$ (normal coordinates) in what follows without any loss of generality.

For instance, if we introduce "damping terms" to modify eq. (6) (as proposed in [11] with a view to Numerical Relativity applications), we are not affecting the principal part (damping terms are algebraic in $Z_i$), so that the resulting "damped" system would be quasiequivalent to system A. Also, if we introduce additional terms of the form $Z_i Z_j$ or $Z^2 \gamma_{ij}$ in (8), the resulting modified system would be quasiequivalent to system A as well.

On the contrary, different values of the parameters $\mu, \nu, n$ in (8) do correspond to different forms of the principal part, so that they would lead to different evolution systems which are not quasiequivalent, even if they have (as they do) the same submanifold of physical solutions.

III. COMPARING WITH THE BSSN SYSTEM

Let us consider here the system introduced in by Shibata and Nakamura [3] and Baumgarte and Shapiro [10] (BSSN system). The metric coefficients $\gamma_{ij}$ are expressed in terms of a conformal metric:

$$\tilde{\gamma}_{ij} = e^{-4 \phi} \gamma_{ij}$$

(10)

with unit determinant, so that

$$e^4 \phi = \gamma^{1/3} = [det(\gamma_{ij})]^{1/3}$$

(11)

The second fundamental form $K_{ij}$ is then decomposed into its trace and trace-free components, namely

$$K = \gamma^{ij} K_{ij}$$

$$\tilde{A}_{ij} = e^{-4 \phi} \left( K_{ij} - \frac{1}{3} K \gamma_{ij} \right)$$

(12)

(13)

so that the principal part of their evolution equations can be written as [10]:

$$\partial_t K = -\gamma^{ij} \nabla_i \alpha_j + ...$$

(14)

$$\partial_t \tilde{A}_{ij} = e^{-4 \phi} \left( -\tilde{\gamma} \nabla_i \alpha_j + \alpha R_{ij} \right)_{\text{TF}} + ...$$

(15)

where the superscript TF denotes the trace-free part of a tensor.
The principal part of the Ricci tensor in \([15]\) is decomposed as follows \([10]\):

\[ R_{ij} = -2 \partial^2_{ij}\phi - \frac{1}{2} \bar{\gamma}_{rs} \bar{\gamma}_{ij,rs} + \bar{\gamma}_{k(i} \partial_{j)} \bar{\Gamma}^k + ... \]  

(16)

where the “conformal connection” quantities \(\bar{\Gamma}^i\) can be defined as:

\[ \bar{\Gamma}^i = -\bar{\gamma}_{ij} \]

(17)

Up to now we have only performed a transformation of variables of the original ADM system. But notice that the system \([14, 13]\) contains seven evolution equations, compared with the six equations in \([3]\). The extra quantity is the trace of \(\bar{A}_{ij}\), which is supposed to vanish by construction. This opens the way to two different approaches:

a) to consider \(\text{tr}(\bar{A})\) as an independent variable, which can take non-zero values during numerical evolution. The equation

\[ \text{tr}(\bar{A}) = 0 \]  

(18)

is to be considered as an additional constraint which can be used to monitor the accuracy of the simulation (free evolution approach).

b) to enforce the vanishing of \(\text{tr}(\bar{A})\), by removing systematically any contribution that could arise from truncation errors in the discretization of equation \([13]\), so that \(\text{tr}(\bar{A})\) is not considered as a new dynamical variable (constrained evolution approach).

In what follows, we will adopt the second approach, so that the system \([14, 13]\) is then quasi-equivalent to the ADM system.

The BSSN system does consider the \(\bar{\Gamma}^i\) as independent dynamical variables, so that the full set of independent BSSN variables is given by

\[ u = \{\alpha, \phi, \bar{\gamma}_{ij}, K, \bar{A}_{ij}, \bar{\Gamma}^i\} \]

(19)

where we must remember that both the trace of \(\bar{A}_{ij}\) and the determinant of \(\bar{\gamma}_{ij}\) do have fixed, non-dynamical values. An evolution equation for the \(\bar{\Gamma}^i\) could be derived \([10]\) by permuting a time derivative with the space derivative in \([17]\):

\[ \partial_t \bar{\Gamma}^i = -2 \alpha \bar{A}^i_{j,j} + ... \]  

(20)

But the BSSN system uses instead the momentum constraint \([3]\) to transform eq. \([20]\) into

\[ \partial_t \bar{\Gamma}^i = -\frac{4}{3} \alpha \bar{\gamma}^{ij} K_{j,j} + ... \]  

(21)

Now we are in position to compare the BSSN system with the one introduced in the previous section (system A), with a set of variables given by

\[ u = \{\alpha, \gamma_{ij}, K_{ij}, Z_i\} \]  

(22)

The conformal decomposition \([10,11,12,13]\) can be easily performed on system A as well. The key point is then the passage from \([20]\) to the BSSN evolution equation \([21]\). In the framework of system A, the momentum constraint is just the right-hand-side of the evolution equation for \(Z_i\). It follows that the use of the momentum constraint to transform (the principal part of) the right-hand-side of \([21]\) into \([20]\) does correspond to the following transformation of \(\bar{\Gamma}^i\):

\[ \bar{\Gamma}_i = -\bar{\gamma}_{ik} \bar{\gamma}^{kj} + 2 Z_i \]  

(23)

which is obviously consistent with the previous expression \([17]\) if we remember that \(Z_i = 0\) for the physical solutions.

Now we can compare easily \([14]\) with the trace-free part of \([8]\): it follows that they coincide if and only if \(\mu = 2\). The same comparison can be performed again between \([14]\) and the trace part of \([8]\): it follows that they coincide if and only if \(\nu = n = 4/3\).

To summarize, we have shown then that the BSSN system is quasi-equivalent to system A for the parameter choice

\[ \mu = 2, \ \nu = n = 4/3 \]

(24)

provided that the “conformal connection” quantities \(\bar{\Gamma}^i\) in the BSSN system are related with the vector \(Z_i\) by equation \([23]\).

IV. A FIRST ORDER EVOLUTION SYSTEM

A first order version of the system A can be obtained in the standard way by considering the first space derivatives

\[ A_k = \partial_k (\ln \alpha) \]

(25)

\[ D_{kij} = \frac{1}{2} \partial_k \gamma_{ij} \]

(26)

as independent dynamical quantities whose evolution equations are given by

\[ \partial_t A_k + \partial_k (\alpha \ Q) = 0 \]

(27)

\[ \partial_t D_{kij} + \partial_k (\alpha \ K_{ij}) = 0 \]

(28)

where the quantity \(Q\), defined as

\[ \partial_t \ln \alpha = -\alpha \ Q \]  

(29)

can be related with \(\text{tr} K\) in order to fix the time coordinate gauge, namely

\[ Q = f \ \text{tr} K \]  

(30)

where \(f\) is an arbitrary function of the lapse \(\alpha\).

This allows to transform system A into a first order evolution system for the set of variables
the following transformation of the right-hand-side of (37) into (38) does correspond to the momentum constraint to transform (the principal part of) the Bona-Masso evolution equation (38). The use of the momentum is quasiequivalent to system B for the parameter choice (41), provided that the supplementary quantities \( V_i \) in the Bona-Masso system are related with the vector \( Z_i \) by equation (40). As a consequence of this fact, system B inherits (for the parameter choice (41)) the causal structure of the Bona-Masso system, which is known to be strongly hyperbolic (real characteristic speeds and a complete set of eigenfields) if and only if \( f > 0 \).

V. REDUCING VARIABLES SPACE

System B has two sets of non-trivial standing eigenmodes (the ones with zero characteristic speed in normal coordinates). The first one is related with gauge evolution, namely

\[ u = \{ \gamma_{ij}, K_{ij}, A_i, D_{kij}, V_i \} \]  

where the quantities \( V_i \) can be defined as

\[ V_i = D_i - D_{ji}^j \]  

An evolution equation for the \( V_i \) could be derived by combining (36) with the evolution equations (28) for the \( D_{kij} \)

\[ \partial_t V_i = \alpha \partial_j (K_{ij} - trK \delta_{ij}) + ... \]  

but the Bona-Masso system uses instead the momentum constraint to transform (27) into an evolution equation with a trivial principal part, namely

\[ \partial_t V_i = ... \]  

The remaining evolution equations in the Bona-Masso system are identical to those in system B, with the replacement of (33) by

\[ \lambda_{ij} = D_{kij} + \frac{1}{2} \delta^k_{ij}(A_j + 2 D_j - D_{ij}^r - \mu Z_i) \]

\[ + \frac{1}{2} \delta_{ij}(A_i + D_i - 2 D_{ri}^r - \mu Z_i) \]

\[ - \frac{n}{2} (D^k - D^r_{kr}) \gamma_{ij} + \frac{1}{2} \sigma^k \gamma_{ij} \]

and

\[ D_k = \gamma^{ij} D_{kij} \]  

This first order system will be called system B in what follows.

System B is very similar to the Bona-Masso system \[ \text{[6,7]} \] (we will follow here the notation of Ref. \[ 7 \] to avoid confusion), which is expressed in terms of the set of variables

\[ u = \{ \alpha, \gamma_{ij}, K_{ij}, A_i, D_{kij}, V_i \} \]  

where we have noted

\[ \lambda_{ij} = D_{kij} + \frac{1}{2} \delta^k_{ij}(A_j + 2 D_j - D_{ij}^r - \mu Z_i) \]  

which is again consistent with (33) because \( Z_i \) vanishes for physical solutions.

Now we can substitute (40) into (39) and compare with (38): it follows that they coincide if and only if

\[ \mu = 2, \; \nu = n \]  

To summarize, we have shown that the Bona-Masso system is quasiequivalent to system B for the parameter choice (41), that the supplementary quantities \( V_i \) in the Bona-Masso system are related with the vector \( Z_i \) by equation (40). As a consequence of this fact, system B inherits (for the parameter choice (41)) the causal structure of the Bona-Masso system, which is known to be strongly hyperbolic (real characteristic speeds and a complete set of eigenfields) if and only if \( f > 0 \).
In doing this, we have restricted ourselves to the \( f = \text{constant} \) case, in spite of the fact that this case is known \[1\] to be prone to numerical gauge instabilities if \( f \neq 1 \). We do that with a view to mimic the KST system, where \( \sigma \) is defined to be a constant and, as a consequence of it, only examples with \( \sigma = 1/2 \) (\( f = 1 \)) are provided. In contrast, system B would allow more general gauge choices, like the "1+log" one (\( f = 2/\alpha \)) which is frequently used in Numerical Relativity.

Disposing of the supplementary variables \( Z_i \) in \[47\] is not so easy. We will take as a guide the KST evolution equations corresponding ones in system B: we need just to replace \[33\] by

\[
2 \lambda_{ij} = d^k_{ij} + \frac{1}{2} \delta^k_{ij}[(1 + 2 \sigma) d_j - 2 d_{rj}']\]

\[
+ \frac{1}{2} \delta^k_{ij}[(1 + 2 \sigma) d_i - 2 d_{ri}'] - \frac{n}{2}(d_k - d^k r) \gamma_{ij}
\]

where we have taken \( \gamma = -1, \gamma = -n/2 \) in the original KST expression to reproduce \[33\] more closely.

The main difference between system B and the KST one comes instead from the evolution equations for \( d_{kij} \), namely

\[
\partial_t d_{kij} = -2 \alpha \partial_k K_{ij} + \chi \alpha \gamma_{ij}(\partial_k K'r_k - \partial_k trK) \]

\[
+ \frac{\eta}{2} \alpha[\gamma_{ji}(\partial_k K'r_j - \partial_j trK) + \gamma_{kj}(\partial_k K'r_i - \partial_i trK)] + ...
\]

where \( \eta, \chi \) are arbitrary parameters. These equations are obtained by using the momentum constraint \[1\] to modify the standard equations \[28\] which were used in system B. The key point again is to realize that the use of the momentum constraint to transform (the principal part of) the right-hand-side of \[28\] into that of \[41\] amounts, allowing for \[43\], to modify the relationship \[44\] between \( D_{kij} \) and \( d_{kij} \) in the following way:

\[
2 D_{kij} = d_{kij} - \eta \gamma_{k(i} Z_{j)} - \chi Z_k \gamma_{ij}
\]

which is consistent again with \[29\] because \( Z_i \) vanishes for physical solutions. A straightforward substitution of \[44\] and \[43\] into \[43\] shows that the vector \( Z_i \) disappears from the principal part if we just relate the free parameters in system B with those of the KST system as follows:

\[
f = 2 \sigma
\]

\[
\nu = \chi - n(\chi - \eta/2)
\]

\[
\mu = \eta - \chi/2 - \sigma(\eta + 3 \chi)
\]

so that the Einstein-Christoffel system \[3\] (corresponding to \( \sigma = 1/2, \eta = 4, \gamma = \chi = 0 \) in the KST paper) is recovered when

\[
f = 1, \mu = 2, \nu = n = 0
\]

To summarize, we have shown here that the KST systems are quasiequivalent (in the case \( \zeta = -1 \)) to a first order system which can be obtained by reducing dynamical variables in system B to the minimal set \[43\]. The reduced system will be called system B’ in what follows. The four parameters of system B’ can be expressed in terms of those of the KST system by \[51\], where we must remember that \( n = -2 \gamma \).

VI. CONCLUDING REMARKS

The evolution systems A, B and B’ presented in this paper are closely related in a transparent way, up to the point that we have been able to keep the same set of parameters \( f, \mu, \nu, n \) in the three cases. The fact that, for suitable parameter choices, they are quasiequivalent respectively to the BSSN \[9,10\], Bona-Masso \[6,7\] and Einstein-Christoffel \[5\] systems points out the strong relationship between these systems, proposed independently in different contexts by different authors, all of them extremely useful in Numerical Relativity applications.

There are at least two specific points in which the suite A, B, B’ can improve our understanding of the previous systems. The first one is that the supplementary quantities \( Z_i \) in systems A and B are three-dimensional vectors, contrary to what happens with their counterparts \( \Gamma^i, V_i \) respectively, which do not have tensor behaviour. Then, systems A and B can be used to build up covariant counterparts of the second order BSSN \[9,10\] and first order Bona-Masso \[6,7\] systems, respectively.

The second point can be seen when analysing the list of non-trivial characteristic speeds in the KST systems, namely

\[
\{ \pm 1, \pm c_1, \pm c_2, \pm c_3 \}
\]

where in our case \( \zeta = -1 \) we have \[1\]

\[
c_1^2 = 2 \sigma
\]

\[
c_2^2 = \frac{1}{4}(2 \eta - 2 \eta \sigma - \chi - 6 \chi \sigma)
\]

\[
c_3^2 = 1 + 2 \gamma - \gamma \eta + \chi + 2 \gamma \chi
\]

which can be translated, allowing for \[54\] into

\[
c_1^2 = f, \quad c_2^2 = \mu/2, \quad c_3^2 = 1 + \nu - n
\]

where we have used \( n = -2 \gamma \).

Expression \[54\] allows to decouple the parameter set in a way such that every parameter is related with just one characteristic speed, making the physical interpretation more transparent: the gauge parameter \( f \) is associated only with \( c_1 \) (gauge speed), the parameter \( \mu \) (which appears in the terms affecting to the longitudinal part of \( K_{ij} \)) is associated only with \( c_2 \), whereas the two parameters \( \nu \) and \( n \) appearing in the transverse trace terms are related with \( c_3 \) (notice that there is some degree of redundancy between \( \nu \) and \( n \)).
It is not surprising then that in all the particular sets of parameters corresponding to previous works \[4–10\], one has

\[
\mu = 2, \quad \nu = n
\]  

which amounts to the “physical speed” requirement for the degrees of freedom not related with the gauge \(c_2^2 = c_3^2 = 1\). We are aware that in the case of the BSSN system we can not speak properly about characteristic speeds, but the close relationship between systems A and B can bring some light even in this case (see for instance Ref. \[13\] for a similar approach).

The question arises of whether systems A, B, B’ will perform any better than their well known counterparts (BSSN, Bona-Masso, KST) in numerical simulations. We have done preliminary tests, comparing system B against the Bona-Masso one in black-hole spacetimes. Our results indicate that the Bona-Masso code performs better, because of the fact that the additional quantities \(V_i\) evolve with an ordinary differential equation \(\dot{\Gamma}^i\), avoiding the truncation errors inherent to the discretization of the partial derivatives that appear instead in the evolution equation \(\dot{Z}_i\) for \(Z_i\). The fact that \(Z_i\) is a true tridimensional vector does not help very much in that context. Something similar happens when comparing system A with BSSN: the evolution equation \(\dot{\Gamma}^i\) for the \(\dot{\Gamma}^i\) is still simpler than \(\dot{Z}_i\). In our opinion, the BSSN, Bona-Masso and KST systems are the state of the art (each one in its class) and will be difficult to beat by any other quasiequivalent system. One would need to provide something more different, modifying even the causal structure of the system (which is given by the principal part), in order to improve in a significant way the performance (stability and accuracy) of the current numerical simulations.

Acknowledgements: This work has been supported by the EU Programme ‘Improving the Human Research Potential and the Socio-Economic Knowledge Base’ (Research Training Network Contract (HPRN-CT-2000-00137), by the Spanish Ministerio de Ciencia y Tecnologia through the research grant number BFM2001-0988 and by a grant from the Conselleria d’Innovacio i Energia of the Govern de les Illes Balears.

[1] L. E. Kidder, M. A. Scheel and S. A. Teukolsky, Phys. Rev. D64, 064017 (2001).
[2] F. John, Partial Differential equations, (4th edition), Springer-Verlag, New York (1982).
[3] S. Frittelli and O. A. Reula, Commun. Math. Phys. 166 221 (1994).
[4] S. Frittelli and O. A. Reula, Phys. Rev. Lett. 76 4667 (1996).
[5] A. Anderson and J. W. York, Jr., Phys. Rev. Lett. 82 4384 (1999).
[6] C. Bona and J. Massó, Phys. Rev. Lett. 68 1097 (1992).
[7] C. Bona, J. Massó, E. Seidel and J. Stela, Phys. Rev. Lett. 75 600 (1995).
[8] C. Bona in Hyperbolic problems: theory, numerics, applications, Int. Series of Numerical Mathematics, Vol.129, Birkhuser (1999).
[9] M. Shibata and T. Nakamura, Phys. Rev. D52 5428 (1995).
[10] T. W. Baumgarte and S. L. Shapiro, Phys. Rev. D59 024007 (1999).
[11] S. Frittelli and O. A. Reula, J. Math. Phys. 40 5143 (1999).
[12] M. Alcubierre and J. Massó, Phys. Rev. D57 4511 (1998).
[13] M. Alcubierre, B. Brugmann, M. Miller and W. M. Suen Phys. Rev. D60 064017 (1999).