MIRRORS, FUNCTORIALITY, AND DERIVED GEOMETRY

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Abstract. In this survey, I suggest to approach the problem of functorial properties of quantum cohomology by drawing lessons from several versions of Mirror duality involving deformation spaces.

1. Introduction: a mystery of Quantum Cohomology

1.1. A brief summary. Moduli spaces/stacks $M$ of stable curves of all genera with a finite number of marked points endowed with natural correspondences between them form a (modular) operad $h(M)$: see [KoMan], [BehMan], [Man1], and subsequent works.

This operad acts upon each smooth complete algebraic variety/DM–stack $V$ via correspondences. Thus, in a wide sense of the word, motive/cohomology $h(V)$ of $V$ is endowed with a canonical structure of algebra over $h(M)$. This structure is called Quantum Cohomology (QC) of $V$.

A mystery: unlike motives/cohomology theories, we practically do not understand properties of QC considered as functor of $h(V)$.

A related mystery: self–referentiality of the operad $h(M)$, i.e. its action upon its own components, and interaction of it with operadic structure. This problem was explicitly addressed in [ManSm1], [ManSm2].

Here I suggest to approach this problem using certain constructions traditionally used in one of the contexts of Mirror Symmetry: deformation spaces and their enriched/derived versions.

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The point is that these constructions are intrinsically and richly functorial, so that one approach to our mystery consists in bringing them back from the Looking Glass Land.

1.2. Recollection and notation. Here are somewhat more precise notations and statements. Quantum Cohomology of an irreducible smooth projective manifold $V$ is the system of motivic morphisms

$$I^{V}_{g,n,\beta}: h(V^n) \to h(\overline{M}_{g,n}).$$

Here $\overline{M}_{g,n}$ denotes the moduli DM–stack of stable curves of genus $g$ with $n$ marked points, $h$ denotes the respective motive, and $\beta$ runs over divisor classes of $V$.

This system expresses the canonical action of the motivic modular operad upon every “total” motive $h(V)$: I use here the word ”total” in order to stress that it is not clear at all upon which direct summands of total motives this operad acts.

In the framework of this survey, I am focusing on the case $g = 0, n \geq 3$, and consider only that part of information about this action which is compressed in the genus zero quantum cohomology ring $H_q^*(V)$.

Assume for simplicity that $V$ is defined over a field of characteristic 0, and denote by $H^*(V) := H^*(V, K)$ its cohomology ring with coefficients in a $\mathbb{Q}$–algebra $K$.

As a graded $K$–module, this ring is free of finite rank; let $(\Delta_a), a = 0, 1, \ldots, r,$ be its free graded basis such that $\Delta_0$ is the (dual) fundamental class of $V$, that is, the identity of the local artinian ring $H^*(V)$.

The dual homology module $H_*(V, K)$ can be considered then as the module of linear coordinates on $H^*(V, K)$ with graded coordinates $(x_a)$, dual to $(\Delta_a)$. We replace those $x_a$ for which $\Delta_a \in H^2(V, K)$ by their formal exponents $q_a = e^{x_a}$ and construct the ring of formal series $K_q := K[[q_a; x_b]]$ (Novikov’s ring) where $b$ runs over subscripts for which $\Delta_b \notin H^2(V, K)$.

The genus zero quantum cohomology ring $H_q^*(V)$ is then the free $K_q$–module $K_q \otimes_K H^*(V)$ with graded commutative multiplication which is the deformation of the multiplication in $H^*(V)$ in the following sense:

(i) $\Delta_0$ remains the identity in the deformed ring.

(ii) Modulo the maximal ideal $(q^a, x_b \mid b \neq 0)$, the ring structure of $H_q^*(V)$ is the same as that of $H^*(V)$. In other words, the ring $H_q^*(V)$ is a formal deformation of the ring $H^*(V)$.
(iii) Finally, the deformed (“quantum”) multiplication ◦ itself has the following structure.

One starts with constructing the potential $\Phi \in K_q$ whose coefficients are genus zero Gromov–Witten invariants expressing (appropriately defined virtual) numbers of rational curves in $V$ with marked points restricted by incidence conditions. Then one constructs the third derivatives $\Phi_{abc} := \partial_a \partial_b \partial_c \Phi$ where $\partial_a := \partial / \partial x_a$. And finally one sets

$$\Delta_a \circ \Delta_b := \sum_{cd} \Phi_{abc} g^{cd} \Delta_d$$  

where $(g^{ab})$ is the matrix of the Poincaré duality product upon $H^*(V)$. For more details and examples, cf. [BaMan].

1.3. Mirrors and functoriality. In the main body of the paper, this concrete quantum cohomology deformation of $H^*(V)$ will be considered in the more general context of various deformation theories. These deformation theories, on the one hand, are used in several of the many Mirror Symmetry constructions, and on the other hand, these theories, especially their extended and derived versions, have very rich functorial properties.

Functoriality of deformation spaces is explained in section 2 of this survey: it is achieved by putting deformation theories into the general framework of controlling DGLAs (Differential Graded Lie Algebras).

The section 3 is dedicated to concrete examples of applications of deformation theories in Mirror Symmetry constructions. Finally, sections 4 and 5 considerably extend the framework of controlling DGLAs by introducing derived geometric constructions.

In order to trace back historical roots and subsequent developments of deformation philosophy, I can suggest the sources [Gr], [Ar], [Art1], [Art2], [Dr], [MatY], [Sc], [Mi], [BuMi], and [Ma1]–[Ma3].

Here is a brief introduction into one of the most basic cases and its treatment as a Mirror Symmetry construction.

1.4. Deformations of local artinian rings: the case of Jacobi rings of isolated hypersurface singularities. Let $f = f(x_1, \ldots, x_m)$ be the germ of holomorphic (or formal) function $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$. Assume that $(0)$ is the only critical point of this germ. Let $J(f) := \mathcal{O}_{\mathbb{C}^m, 0} / (\partial f / \partial x_k)$ be its Jacobi algebra.
The number $\mu$, its linear dimension over $\mathbb{C}$, is called the Milnor number of the singularity.

An unfolding (or deformation) of $f$ is a holomorphic germ $F(x_1, \ldots, x_m; t_1, \ldots, t_n)$ at $(\mathbb{C}^{m+n}, 0)$ such that $F(x; 0) = f(x)$. Its base is the germ $M = (\mathbb{C}^n, 0)$, with coordinates $(t_j)$. We will say that a germ of tangent vector field to $(\mathbb{C}^{m+n}, 0)$ is vertical if its projection to the base $(\mathbb{C}^n, 0)$ vanishes.

Such an unfolding is called versal, if any other unfolding can be induced from it by an appropriate morphism of bases, and semiuniversal if it is versal and its base has the minimal dimension. This is the first explicit expression of functoriality that we met in this survey.

For more details about morphisms of unfoldings involved here, see [He], pp. 62–63. A version of this definition is discussed on page 64 of [He].

Here is a criterium for checking (semiuni)versality of an unfolding $F$ (cf. [He], Theorem 5.1, and references therein).

Consider the critical space $C = C_F$ of the map $(F, pr_M) : (\mathbb{C}^m \times M, 0) \to (\mathbb{C} \times M, 0)$ which is defined by the ideal $J(F) := (\partial F/\partial x_k)$. For a germ of tangent vector field $X \in T_{M,0}$, denote by $\tilde{X}$ a lift of $X$ to $(\mathbb{C}^m \times M, 0)$.

Since the difference of any two lifts of the same $X$ must be vertical, its restriction upon $C$ vanishes so that the map

$$T_{M,0} \to pr_{M*}\mathcal{O}_C : \quad X \mapsto \tilde{X}F \mod J(F)$$

is well defined.

1.4.1. Theorem ([He], p. 63). a) An unfolding $F$ of an isolated hypersurface singularity is versal iff the map $X \mapsto \tilde{X}F \mod J(F)$ is surjective.

b) An unfolding $F$ of an isolated hypersurface singularity is semiuniversal iff the map $X \mapsto \tilde{X}F \mod J(F)$ is bijective.

As soon as a semiuniversal unfolding is chosen, we can define a commutative associative $\mathcal{O}_M$–bilinear multiplication $\circ$ on $T_{M,0}$ by simply lifting it from $pr_{M*}\mathcal{O}_C$ i. e., by putting

$$\tilde{X} \circ \tilde{Y} \mod J(F) := \tilde{X}F \cdot \tilde{Y}F \mod J(F). \quad (1.2)$$

1.5. Example: semiuniversal unfolding of the singularity $A_r$ vs. quantum cohomology ring $H^*_q(\mathbb{P}^r)$. One of the versions of mirror symmetry starts
with the observation that if a classical cohomology ring $H^*(V)$ is isomorphic to the Jacobi ring $J(f)$ of an isolated hypersurface singularity, then locally (or formally) near the initial point of semiuniversal unfolding space the formal spectrum of the ring $K_q = K[[q_0;x_0]]$ defined in sec. 1.2 must admit a natural (generally non–unique) map to the germ of this unfolding, such that the relative formal spectrum of $H_q^*(V)$ is induced by the relative spectrum of the critical space $C_f$ defined above.

Moreover, one should then try to constrain a choice of this map by requiring its compatibility with additional structures induced on the unfolding spaces of two mirror sides. The first and most important is the compatibility of the Frobenius–multiplication (1.1) with $F$–multiplication (1.2). Besides, one should try to transfer the canonical grading and flat structure on the quantum cohomology side to the unfolding side where they are initially absent.

All these details were thoroughly studied for many homogeneous spaces of classical Lie/algebraic groups: see a recent report [GoPe], and for physical motivation [BelGeKo]. The simplest example is a projective space $\mathbb{P}^r$ over $\mathbb{C}$.

Its classical cohomology with coefficients in any ring $K$ is the free $K$–module freely generated by $\Delta_a$ where $\Delta_a$ is the dual class of $\mathbb{P}^{r-a} \subset \mathbb{P}^r$. As a ring, $H^*(\mathbb{P}^r, K)$ is thus canonically isomorphic to $K[\Delta_1]/(\Delta_1^{r+1})$, with $\Delta_a \equiv \Delta_1^a \mod \Delta_1^{r+1}$.

On the other side, the germ of the function of one variable $f(x) = x^{r+2}$ has the same Jacobi ring $K[x]/(x^{r+1})$. Its semiuniversal unfolding space is the affine space of coefficients of the polynomial

$$p(t) := x^{r+1} + t_1 x^{r-1} + \cdots + t_{r-1} x + t_r$$

For many further details cf. [Man1], Ch. II, sec. 4; Ch. I, sec. 4. In particular, it is explained, how to introduce additional flat structure and grading that make it compatible with the respective structures upon quantum cohomology of $\mathbb{P}^r$.

**1.6. Homogeneous spaces.** This approach to the quantum cohomology of $\mathbb{P}^r$ and its mirror was (at least partially) extended to more general homogeneous spaces $G/P$. The question for what such spaces their classical cohomology is isomorphic to the Jacobi ring of an isolated singularity, seemingly does not have a direct answer in the literature. Probably, the answer is positive at least for minuscule/cominiscule homogeneous spaces. In any case, the ample known information about explicit descriptions of cohomology rings was used in order to successfully produce also a
description of their quantum cohomology focused more on the flat structure and structure connections than upon $F$–multiplication, cf. [Sa].

For some recent surveys/original results see [ChMPe], [LamTem], and references therein. See also an interesting extension of this method in [GoPe], where the authors construct deformations of Jacobi rings of polynomials, and in the context of homogeneous spaces, apply these constructions to equivariant cohomology and $K$–theory.

1.7. **Extended deformations and derived geometry.** The sections 4 and 5 of this survey will be dedicated to the problem of extending deformation contexts, if the basic theory involving controlling DGLAs is not satisfying enough for discussing mirror phenomena. In particular,

*What to do when $H^*(V)$ is NOT a Jacobi ring?*

For starting steps, see articles [Mi], [SchSt], where the deformation theory of $H^*(M)$ is contained, restricted to those deformations that, for complex compact $M$, deform only the complex structure. In this spirit, finite–dimensional graded Artin rings are also considered.

More general is the suggestion to use *not* the naive deformation spaces, but some higher step of the ladder involving the so called *extended* deformation spaces/functors. We present a survey of several of lower steps where the functoriality of main definitions and constructions is the primary concern. Of course, this means that derived and higher derived versions of all objects involved should be briefly presented. In particular, we must pass

- From operads classifying these objects to complexes/simplicial sets/... of operads to homotopy via model structures on the respective categories.

- From categories to 2–categories to $\infty$–categories ...
- From “affine” objects to gluing to ...  

Much more details are given in the monograph [LoVa]. See also [Pr], [DoShVa1], [DoShVa2], [Lu], [To]. In particular, Toën’s survey is a magnificent introduction to the derived deformation theories.
2. Deformation functors and controlling DGLAs

2.1. Formal deformation philosophy. Let $M$ be a “space” that in the next few paragraphs will embody an intuitive idea of “space of deformations of certain structured object $X$”. Thus, $X$ itself will correspond to a point $x \in M$.

In the formal deformation philosophy, we want to get hold of “infinitesimal neighbourhoods” of $x$ in $M$, or even of “germ of $M$” at $x$ if we can speak about analytic moduli spaces.

Imagine first the simplest case when $M$ (or a neighbourhood of $x$ in $M$) is a scheme defined over a field $K$, and $x$ is a $K$–point of $M$. Then a basis of infinitesimal neighbourhoods of $x$ consists of affine spectra of rings $\mathcal{O}_x/m^n_x$, $n = 1, 2, 3, \ldots$, tautologically embedded into $X$. Here $m_x$ is the maximal ideal of $\mathcal{O}_x$.

In order to construct this basis, or its version, we must understand deformations $X_A$ of $X$ over local Artin $K$–algebras $A$, but now up to an isomorphism over $A$, so we have to consider groupoids of deformations over variable bases, forming a contravariant functor $Art_K \rightarrow Grpd$ which is a basic example of deformation functors.

In fact, our initial view of algebraic geometry adjusted to moduli problems can and ought to be vastly extended, including spectral geometry, various versions of derived geometry etc. This is explained in [Lu], but we will not try to explain it in this short survey, cf. [To].

2.2. Deformation functors. Historically, early abstract theories of deformation functors were developed in [Art1], [Art2], [Gr], [Sch], [SchSt].

Here we start with Deligne’s dense expression controlling DGLA, that was explained in a letter of Pierre Deligne [De]. According to a quotation from [GoMi], Deligne observed that

“in characteristic zero, a deformation problem is controlled by a differential graded Lie algebra, with quasi–isomorphic differential graded Lie algebras giving the same deformation theory.”

A choice of controlling DGLA provides another construction of a functor, and identification of both versions furnishes strong tools for studying deformations.

In the remaining part of this section, we focus on the construction of the (potential) deformation functor from the controlling DGLA, mostly accepting the conventions of [GoMi], and later return to the task of identifying two constructions.
2.3. Maurer–Cartan equation. Let $K$ be a field of characteristic zero and $g = (\oplus_{i \geq 0} g^i, d)$ a DGLA over $K$, with $d : g^i \to g^{i+1}$, $d^2 = 0$. Skew-commutativity and Jacobi identities are also supposed to be graded, with Koszul’s signs.

The set of Maurer–Cartan elements of $g$ is defined as

$$MC(g) := \{ x \in g^1 \mid dx + \frac{1}{2}[x, x] = 0 \}. \quad (2.1)$$

This Maurer-Cartan equation is equivalent to the flatness of the respective connection on $g$ that sends any $f \in g^0$ to $\nabla_x(f) := df + [x, f]$, namely $\nabla_x^2 = 0$.

We now want to identify those elements of $MC(g)$ that are connected by the flow corresponding to this action of $g^0$ (see a more sophisticated version of this identification in the next subsection). In order to do it properly, one can assume as in [GoMi] (p. 48) that $K$ is $\mathbb{C}$ or $\mathbb{R}$, and consider the action of simply connected Lie group with Lie algebra $g^0$.

Another version, not involving restrictions on $K$, assumes instead that $g^0$ is nilpotent, together with its action upon $g^1$, or even that $g$ is nilpotent: see [LoVa], p. 499. Then one can construct the respective nilpotent algebraic group and its action. In each of these cases, the standard formula for the action of one-parametric subgroups is applicable: for $a \in g^0$, $e^{ta}$ sends $x \in MC(g)$ to

$$e^{ta} \cdot x + \frac{id - e^{ta} \cdot ad}{ad a}(da). \quad (2.2)$$

In the most important for us series of examples, we start with arbitrary DGLA $g$ and finite dimensional nilpotent commutative $K$–algebra $m$ (so that $K \oplus m$ is a local Artin algebra with maximal ideal $m$). Then $g \otimes m$ is nilpotent, with grading and $d$ coming from $g$.

2.4. Philosophy of controlling DGLAs. The general scheme is as follows: starting with a chosen deformation problem we construct groupoids and arrows in the following diagram:

Deformation groupoid $\Longrightarrow$ Controlling DGLA $L \Longrightarrow$ Groupoid associated to $L$

and finally establish an equivalence between two groupoids in it.

An explicit construction of the relevant DGLA (first arrow here) requires creative thinking and the study of instructive examples, existing in the literature. The second arrow is somewhat more standardised, and we will start with it.
2.5. Deligne groupoid \( D(g, A) \). Let \( g \) be a DGLA as above and \( A = K \oplus m_A \) an Artin local algebra.

Then we put
\[
\text{Ob } D(g, A) := \text{MC}(g \otimes m_A)
\]
and for \( x, y \in \text{MC}(g \otimes m_A) \)
\[
\text{Hom}(x, y) := \{ a \in g^0 \otimes m_A \mid e^a(x) = y \}.
\]
Finally, the composition of morphisms is defined via (2.2).

An elementary, but important remark is that \( D(g, A) \) is itself a covariant functor of \((g, A)\) considered as a variable object of the categorical product of DGLAs with \( \text{Art}_K \). More precisely ([GoMi], p. 53):

(i) For any homomorphism of DGLAs \( \varphi : g \to h \) there is a natural functor \( \varphi^* : D(g, A) \to D(h, A) \).

(ii) For any homomorphism of Artin local \( K \)-algebras \( \psi : A \to A' \) there is a natural functor \( \psi^* : D(g, A) \to D(g, A') \).

(iii) These functors can be chosen in such a way that for \((\varphi, \psi) : (g, A) \to (h, A')\) we have the equality (and not just an equivalence) of functors \( \psi^* \varphi^* = \varphi^* \psi^* : D(g, A) \to D(h, A') \).

The critically important property of this construction is this: if \( \varphi \) is a quasi–isomorphism of DGLAs, then \( \varphi^* \) is an equivalence of groupoids. Actually, for the construction of \( \text{MC}(g \otimes m_A) \) only \( g^i \) with \( i = 0, 1, 2 \) are essential, so that we have ([GoMi], Theorem 2.4):

2.5.1. Proposition. If \( \varphi \) induces isomorphisms \( H^i(g) \to H^i(h) \) for \( i = 0, 1 \) and a monomorphism for \( i = 2 \), then \( \varphi^* \) is an equivalence of groupoids.

3. Deformations of analytic local rings and mirror phenomena

3.1. Groupoids associated to deformations of analytic local rings. Here we will illustrate on a concrete example both steps involved in realisation of the philosophy briefly sketched in sec. 3.5. above. For a detailed treatment of this example, see [BuMi], sec. 5.

Let \( k \) be a complete normed field of characteristic zero. Denote by \( k\langle z_1, \ldots, z_m \rangle \) the ring of convergent power series in \((z_k)\). An analytic local \( k \)-algebra \( B \) is a
quotient of $k(z_1, \ldots, z_m)$ modulo a (topologically closed) ideal. Denote by $Art_k$ the category of Artin local $k$–algebras.

Now fix an analytic local $k$–algebra $B$.

Below I essentially use intuition and conventions related to the version of definition of a moduli groupoid explained in [Man1], Ch. V, Sec. 3.1 and 3.2, pp. 210–211. One notational difference is that since we deal here with affine schemes and/or their versions, omitting the passage to their (Grothendieck) spectra, arrows in the respective categories are inverted in comparison with those in [Man1]. The adjective “cofibered” below reminds about this.

3.1.1. Definition. The cofibered groupoid $Def(B)$ of the deformations of $B$ consists of the following data:

Category of bases. This is the category $Art_k$. For any object $A$ of this category, we denote by $m_A$ its maximal ideal.

Category of families. One object $(B', \rho)$ of this category $Def(B; A)$ (intuitively, a family over the base which is the spectrum of $A \in Ob Art_k$) consists of a flat $A$–algebra $B'$ and a morphism of $A$–algebras $\rho: B' \to B$ which induces an isomorphism $\bar{\rho}: B'/m_AB' \to B$.

One morphism $(B', \rho_1) \to (B'', \rho_2)$ in $Def(B; A)$ is a homomorphism of $A$–algebras $\varphi: B' \to B''$ which modulo $m_A$ induces the identity morphism of $B$.

Base change functor. Given a morphism $A_1 \to A_2$, the respective base change functor $Def(B; A_1) \to Def(B; A_2)$ is $* \mapsto A_2 \otimes_{A_1} *$ where $*$ stands for respective objects, morphisms and diagrams.

3.1.2. Lemma. All endomorphisms in $Def(B)$ are isomorphisms. Moreover, they are exponentials of nilpotent derivations. ([BuMi], p. 45.)

3.2. Passage to resolutions of $B$. Let $R^\bullet$ be a free graded commutative $k$–algebra, with $R^m = 0$ for $m > 0$, endowed with a differential $\partial$ of degree one and a surjective homomorphism $\varepsilon: R^\bullet \to B$ which is a quasi–isomorphism. Then $(R^\bullet, \partial, \varepsilon)$ is called a multiplicative resolution, or resolvent of $B$ over $k$.

Sometimes it is convenient to work instead with $R_\bullet$ where $R_m = R^{-m}$.

Such resolutions exist and are unique up to homotopy equivalence.

3.2.1. Definition. Let $R$ be a resolution $R^\bullet$ as above. The groupoid $Def(R)$ of deformations of $R$ cofibered over $Art_k$ consists of the following data:
Category of bases remains to be $\text{Art}_k$.

Category of families. One object $(R', \rho)$ of this category $\text{Def}(R; A)$ (intuitively, a family of resolutions over the base which is the spectrum of $A \in \text{Ob Art}_k$) consists again of several components.

The first one is a flat deformation $R'$ of the algebra $R$ over $A$. Since $R$ is free, we may and will henceforth assume that $R' = R \otimes_k A$ so that $R'/m_A R' = R$.

The second component is a differential $\partial'$ of $R'$ deforming $\partial$.

Fact ([BuMi], p. 46). $R'$ is a resolution of $H_0(R')$ by free $A$–modules.

One morphism $\varphi : (R, \partial) \to (R', \partial')$ is a homomorphism of differential graded algebras such that $\varphi \equiv \text{id} \mod m_A$.

Base change functor. It is again $* \mapsto A_2 \otimes_{A_1} *$ where $*$ stands for respective objects, morphisms and diagrams.

3.3. Controlling DGLAs. They will belong to a general class of DGLAs defined in [BuMi], p. 4 in the following way.

Let $V = V^\bullet$ be a non–negatively graded vector space over $k$. An endomorphism $T$ of $V$ of degree $l$ is a linear map $T : V^\bullet \to V^\bullet+l$. The space of such maps is denoted $\text{Hom}^l(V, V)$. Their direct sum is denoted $\text{Hom}(V, V)$. It is a graded Lie (super)algebra with commutator

$$[S, T] := S \circ T - (-1)^{ij} T \circ S$$

for $S$, resp $T$, of degree $i$, resp. $j$.

Now assume that $V$ is a graded commutative algebra. Denote by $\text{Der} V$ the space of its graded derivations over $k$. It is closed wrt $[\cdot, \cdot]$. Start with this algebra, or usually its Lie subalgebra of non–negative degree $\text{Der}^+ V$.

Usually our DGLAs will be $L = \text{Der}^+ V$ endowed with an additional derivation $d : L \to L$ of degree 1 with $d^2 = 0$.

3.4. Groupoids associated to DGLAs. Let $L = (L^\bullet, d)$ be a DGLA. We associate with $L$ its deformation groupoid $\mathcal{C}(L)$ cofibered over $\text{Art}_k$.

3.4.1. Definition. The groupoid $\mathcal{C}(L)$ consists of the following data:

Category of bases. It is $\text{Art}_k$. 

Fact ([BuMi], p. 46). $R'$ is a resolution of $H_0(R')$ by free $A$–modules.

One morphism $\varphi : (R, \partial) \to (R', \partial')$ is a homomorphism of differential graded algebras such that $\varphi \equiv \text{id} \mod m_A$.

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Fact ([BuMi], p. 46). $R'$ is a resolution of $H_0(R')$ by free $A$–modules.

One morphism $\varphi : (R, \partial) \to (R', \partial')$ is a homomorphism of differential graded algebras such that $\varphi \equiv \text{id} \mod m_A$.

Base change functor. It is again $* \mapsto A_2 \otimes_{A_1} *$ where $*$ stands for respective objects, morphisms and diagrams.
**Category of families over $A$: objects.** One object of the category $C(L; A)$ is an element $\eta \in L^1 \otimes_k m_A$ satisfying the equation

$$d\eta + \frac{1}{2}[\eta, \eta] = 0.$$

**Category of families over $A$: morphisms.** In [BuMi], p. 5, morphisms are defined in the following way:

$$\text{Mor} C(L; A) := \exp(L^0 \otimes m_A).$$

Here $\exp(L^0 \otimes m_A)$ is a nilpotent Lie/algebraic group with underlying space $L^0 \otimes m_A$ and Campbell–Hausdorff composition

$$X \cdot Y := \log(\exp(X)\exp(Y)).$$

The morphisms act on objects by the “affine action”: $\lambda \in L^0 \otimes m_A$ sends $\eta \in \text{Ob} C(L; A) \subset L^1 \otimes m_A$ to $\alpha(e^\lambda \cdot \eta)$. The latter element is determined by the formula

$$d\alpha(\lambda) \cdot \eta = [\lambda, \eta] - d\lambda.$$

A slightly more transparent version is given in [R–N], p. 2. Each element $\lambda \in L^0 \otimes m_A$ defines a “vector field” on $L^1 \otimes m_A$ sending $\eta \in L^1 \otimes m_A$ to

$$d\lambda + [\lambda, \eta] \in L^1 \otimes m_A.$$

It is tangent to the Maurer–Cartan locus in the following sense: if $\eta(t)$ is a flow of $\lambda$, that is

$$\frac{d}{dt}\eta(t) = d\lambda + [\lambda, \alpha(t)]$$

with $\eta(0)$ satisfying Maurer–Cartan, the all $\eta(t)$ satisfy Maurer–Cartan.

Then the set of morphisms $\eta_0 \to \eta_1$ is defined as the set of $\lambda \in L^0 \otimes m_A$ such that the flow starting with $\eta_0$ for $t = 0$ produces $\eta_1$ for $t = 1$.

**Base change functor.** It is again induced by $* \mapsto A_2 \otimes_{A_1} *$ where $*$ stands for respective objects, morphisms and diagrams.

### 3.5. Equivalence of deformation groupoids and DGLA groupoids

(see [BuMi], pp. 47–48). Let again $B$ an analytic local $k$-algebra, $R = (R^*, \partial)$ its
resolution as above, \( L = (L^*, d) \) be its tangent complex: the differential graded Lie algebra of graded derivations of \( R \) of non–negative degree, and \( d := \text{ad} \partial \). This means that for \( \eta \in L^i \)

\[
d\eta = \partial \circ \eta - (-1)^i \eta \circ \partial.
\]

We wish now construct an equivalence of groupoids \( p : C(L) \to \text{Def}(R) \). Let \( A \) be a local artinian \( k \)-algebra. For an object of \( C(L; A) \), \( \eta \in L^1 \otimes m_A \), we must first of all define its image as an object of \( \text{Def}(R; A) \). Recall that an object of \( \text{Def}(R; A) \) is represented by a flat differential graded \( A \)-algebra \( (R', \partial' \) ) and a map \( \rho : R' \to R \). In particular, we may and will assume that \( R' = R \otimes A \).

Recall that a morphism in \( \text{Def}(R; A) \) is a homomorphism of graded commutative algebras reducing to identity modulo \( m_A \). Denote by \( \beta \) the canonical isomorphism (\( Hom \) overlooks the differentials)

\[
\beta : \text{Hom}_k(R, R) \otimes A \to \text{Hom}_A(R', R'), \quad \beta(\eta \otimes t) = t(\eta \otimes \text{id}).
\]

Now we can define \( p(\eta) \) for any object \( \eta \) of \( C(L; A) \), that is, \( \eta \in L^1 \otimes_k m_A \) satisfying the Maurer–Cartan equation:

\[
p(\eta) := (R \otimes A, p(\partial) + \beta(\eta)).
\]

Finally, we can define \( p \) on morphisms: for \( \exp(\lambda) \in \text{Mor}C(L; A) \) we put

\[
p(\exp(\lambda)) := \beta(\exp(\lambda)).
\]

**3.5.1. Claim** ([BuMi], p. 47). The functor \( p \) is an equivalence of groupoids.

**Comparison of groupoids** \( h : \text{Def}(R) \to \text{Def}(R) \). This functor is defined on objects \( R' \), resp. morphisms \( \varphi \), by

\[
h(R') := H_0(R'), \quad h(\varphi) := H_0(\varphi).
\]

**3.5.2. Claim** ([BuMi], p. 53). \( h \) induces an isomorphism of functors

\[
h : \text{Iso Def}(R) \to \text{Iso Def}(B)
\]

where Iso are the sets (or small categories, p. 5) of isomorphism classes.
3.6. Examples: Mirror symmetry in the Looking Glass Land. Here we briefly describe a version of Mirror Symmetry in which both sides are Deligne (Maurer–Cartan) groupoids associated with different DGLAs: see [CILaPo], pp. 4–6, [CLOvPo], and [ClPo]. In these examples, the central role is played by an additional structure on the controlling DGLAs which is introduced from the start: namely, they are Differential Graded Gerstenhaber Algebras, or briefly DGAs: see [Po] for a very detailed description.

Let \((h, J)\) be a real Lie algebra with integrable complex structure on it. Starting with this datum, one can define a controlling \(DGA(h, J)\). Let now \(k\) be a real Lie algebra with a symplectic form \(\omega\). It produces another \(DGA(k, \omega)\). Roughly speaking, the origin of these data is the fact that de Rham cohomology of a smooth manifold with an additional structure (complex, symplectic, homogeneous) carries a signature of this structure upon its de Rham complex.

3.6.1. Definition. \((h, J)\) and \((k, \omega)\) form a weak mirror pair iff these two DGAs are quasi–isomorphic.

3.6.2. Proposition. If \(h\) and \(k\) are nilpotent Lie algebras of common finite dimension, then a homomorphism \(DGA(h, J) \to DGA(k, \omega)\) is a quasi–isomorphism iff it is an isomorphism.

In [Po], this is applied to the extended deformations of Kodaira surfaces in the spirit of Merkulov: see [Me1], [Me2], and our Sec. 5 below. Remarkably, it turns out that in this world a Kodaira surface is its own mirror image.

4. Extended deformation functors and controlling \(L_\infty\)–algebras

In the last two sections, we will sketch some extensions of the controlling DGLAs philosophy and constructions to the context of \(\infty\)–resolutions and higher categories.

4.1. \(L_\infty\)–algebras. The notion of \(L_\infty\)–algebra, or homotopy Lie algebra \(g\), involves an infinite sequence of brackets on the \(dg\)–space \(g\):

\[\mu_n : \Lambda^n g \to g[2 - n], \quad n = 1, 2, \ldots \infty\]  

(4.1)

satisfying the relations, for all \(n \geq 2\),

\[\sum_{p + q = n + 1} \sum_{\sigma \in S_{p,q}^{-1}} \text{sgn}(\sigma) (-1)^{(p-1)q}(\mu_p \circ_1 \mu_q)^\sigma = 0.\]  

(4.2)
We use here notations of [LoVa], p. 365, Proposition 10.1.7, plus last line of the page, with $\mu_1 = -d_g$. In particular, $Sh_{p,q}$ denotes the set of unshuffles, cf. [LoVa], p. 16. These conventions agree also with those of [FiMaMar].

4.2. Maurer–Cartan equations for $L_\infty$–algebras. We put for a $L_\infty$–algebra $g$:

$$MC_\infty(g) := \{ x \in g^1 | \sum_{n=1}^{\infty} \frac{\mu_n(x^\otimes n)}{n!} = 0 \}. \quad (4.3)$$

4.3. Homotopies in the set $MC_\infty(g)$. The definition involving (2.2) can also be extended to this context, producing oriented paths between elements of $MC_\infty(g)$: cf. [FiMaMar] and below.

4.4. Deligne $\infty$–groupoids from $L_\infty$–algebras. Generalising sec. 2.5, consider an $L_\infty$–algebra $g$ and an Artin local algebra $A = K \oplus m_A$.

Put

$$\text{Ob } D_\infty(g,A) := MC_\infty(g \otimes m_A) \quad (4.4)$$

and for $x, y \in MC(g \otimes m_A)$

$$\text{Hom}(x,y) := \text{paths from } x \text{ to } y.$$  

Actually, here we must not restrict ourselves by the composition of morphisms: equality between two compositions must be replaced by a path in the space of morphisms, and so on \emph{ad infinitum}. So, as a functor of $A$, we will obtain an $\infty$–groupoid.

We omit here a formal description and instead treat a good motivating example from [FiMaMar].

4.5. Semicosimplicial DGLAs. Consider first the category $\Delta$ whose objects are finite sets

$$[n] := \{0, 1, \ldots, n\}, \ n = 0, 1, 2, \ldots$$

and morphisms are order–preserving injective maps. Denote by

$$\delta_{k,i} : [i-1] \to [i], \ k = 0, \ldots, i$$

be the map with image $\{0, 1, \ldots, i\} \setminus \{k\}$. 
For a category $\mathcal{X}$, call a *semicosimplicial $\mathcal{X}$-object* any covariant functor $\Delta \to \mathcal{X}$.

Thus a semicosimplicial DGLA $g^\Delta$ is an infinite sequence of DGLAs $g_i$, $i = 0, 1, \ldots \}$ connected by the morphisms $d_{k,i} : g_{i-1} \to g_i$ corresponding to $\delta_{k,i}$ and satisfying the same relations as $\delta_{k,i}$.

4.6. *Deligne groupoid* $D(g^\Delta, A)$. The first step of its construction leads to an infinite family consisting of objects $D(g_i, A)$ and respective morphisms, $A$ being fixed.

The next step consists in passing to the homotopical limit. The groupoid itself has as the objects ordered pairs of elements $\lambda, \mu \in (g_0^1 \oplus g_0^0) \otimes m_A$ satisfying the conditions

$$
d\lambda + \frac{1}{2}[\lambda, \lambda] = 0, \quad e^\mu(d_{0,1}\lambda_0) = \lambda_1,
$$

$$
e^{d_{0,2}\mu} e^{-d_{1,2}\mu} e^{d_{2,2}\mu} = 1.
$$

Finally, morphisms from $\lambda_0, \mu_0$ to $\lambda_1, \mu_1$ are those elements $a \in g_0^0 \otimes m_A$ for which

$$
e^a(\lambda_0) = \lambda_1, \quad e^{-\mu_0} e^{-d_{1,1}a} e^{\mu_1} e^{d_{1,1}a} = 1.
$$

4.7. *A deformation problem*. In this subsection, $K$ will denote an algebraically closed field of characteristic zero. Consider a smooth algebraic variety $X$ over $K$ and its (finite) covering by open affine subsets $U := \{U_i\}$. It is well known that any infinitesimal deformation over $A = K \oplus m_A$ of an affine manifold $U$ is trivial, so they form a groupoid with the single (isomorphism class of) object(s) $U \times \text{Spec} A$ and its automorphism group $\exp(\Gamma(U, T_U) \otimes_K m_A)$. Thus (isomorphism classes of) all deformations of $X$ over $A$ can be described as the (noncommutative) cohomology set $H^1(U, \exp(\Gamma(U, T_X) \otimes_K m_A))$.

Extending this remark and building upon earlier work by E. Getzler, V. Hinich et al. ([Ge], [Hi1], [Hi2], [HiSch]) one can show ([FiMaMar]) that the whole Čech complex $C^*(U, T_X)$ has a natural structure of $L_\infty$-algebra and the whole $(\infty, 1)$-groupoid of deformations of $X$ is controlled by this $L_\infty$-algebra.

4.8. *Further developments of the deformation theories and controlling DGLAs*. In Sec. 2 and 3, I have explained the basics of controlling DGLAs philosophy as it was presented by the researchers of the 80s. Bruno Vallette suggested me to include a brief picture of its development sketched in his message to me of July 22, 2017. With his permission, I reproduce below an edited version of his sketch.
The first remark concerns the initial Deligne formulation from [De]. The point is that later, when $\infty$–groupoids were introduced, it became clear that the relevant DGLAs and $L_\infty$–algebras are filtered, and that $\infty$–groupoids are stable only wrt filtered quasi–isomorphisms: see a modern treatment by Dolgushev–Rogers in arXiv:1407.6735 using model category and homotopy arguments.

Furthermore, when Lurie (following the letter by Drinfeld of 1988 published as [Dr]) developed his version of the Deligne philosophy, he started with a generalisation of the notion of a general deformation functor. He then produced an infinity functor from DGLAs, and an infinity functor in the opposite direction which together form an $\infty$–equivalence.

5. $F_\infty$–structures on extended deformation spaces

5.1. $F$–manifolds. We start with description of a class of manifolds whose tangent sheaf is endowed with (commutative, associative) multiplication such as (1.2) in the section 1 above.

Below $M$ denotes a (super)manifold: it can be $C^\infty$, or $An$, or (partly) formal, eventually with odd (anticommuting) coordinates. The ground field is denoted $K$, usually we choose $K = \mathbb{C}$.

The structure sheaf is denoted $\mathcal{O}_M$, the tangent sheaf $\mathcal{T}_M$. The tangent sheaf is a locally free $\mathcal{O}_M$–module; its (super)rank is called the (super)dimension of $M$.

Now start with a linear $K$–(super)space $A$ endowed with $K$–bilinear commutative multiplication and a $K$–bilinear Lie bracket.

The Poisson tensor of such a structure assigns to $a, b, c \in A$ the element

$$P_a(b, c) := [a, bc] - [a, b]c - (-1)^{ab}b[a, c].$$

This definition can be easily extended to sheaves.

For a manifold $M$ as above, $\mathcal{O}_M$ has a natural commutative multiplication, whereas $\mathcal{T}_M$ has a natural Lie structure.

Poisson structure involves introducing additional Lie structure upon $\mathcal{O}_M$, whereas $F$–structure involves introducing additional multiplication upon $\mathcal{T}_M$, satisfying axioms below. Below we compare the axioms and particular cases of these two structures.
POISSON STRUCTURE

\(K\)-bilinear (super)Lie bracket \(\{,\}\) on \(\mathcal{O}_M\) satisfying identity
\[ P_f(g, h) \equiv 0 \]

Equivalently: each local function \(f\) on \(M\) becomes a local vector field \(X_f\) on \(M\):
\[ X_f(g) := \{f, g\} \]

Special case (local):
Symplectic structure in canonical coordinates:
\[ \{f, g\} = \sum_{i=1}^{n} \partial_{q_i} f \partial_{p_i} g - \partial_{q_i} g \partial_{p_i} f \]
No local parameters, but a large symplectomorphism group.

F–STRUCTURE

\(\mathcal{O}_M\)-bilinear associative, commutative unital multiplication \(\circ\) on \(\mathcal{T}_M\) satisfying “\(F\)–identity”:
\[ P_X \circ Y = X \circ P_Y + (-1)^{XY} Y \circ P_X \]

Each local vector field on \(M\) becomes a local function on the spectral cover
\[ \widetilde{M} := \text{Spec}_{\mathcal{O}_M}(\mathcal{T}_M, \circ) \]

Special case (local):
Semisimple \(F\)–structure in Dubrovin’s canonical coordinates \(u^a\):
\[ \partial_a \circ \partial_a = \delta_{ab} \partial_a \]
No local parameters; local automorphisms \(u^a \mapsto u^{\sigma(a)} + e^a\)

In a recent article [Do], there is very interesting description of the operad \(\text{FMan}\), classifying algebras \((A, \circ, [,])\) whose basic operation \(\circ\) is commutative and associative, basic operation \([,]\) is the Lie bracket, and finally, their compatibility is expressed by the \(F\)–identity. Notice that \(F\)–identity is a cubic one in the operadic sense so that the connection between \(\text{FMan}\) and quadratic operads Associativity, Poisson and Pre–Lie operads is quite surprising.

Since among these three operads the last one is less well known, we briefly recall that the Pre–Lie operad classifies pre–Lie algebras, and the latter are defined by binary product whose associator is right symmetric: see [LoVa], Sec 13.4. In [DoShVa2], a version of Deligne groupoid and pre–Lie deformation formalism is developed.

5.2. Geometric meaning of the \(F\)–identity ([HeMaTe]). For any (super)manifold \(M\), consider the sheaf of those functions on the cotangent manifold
$T^*M$ which are polynomial along the fibres of projection $T^*M \to M$: that is, the relative symmetric algebra $\text{Symm}_{\mathcal{O}_M}(\mathcal{T}_M)$.

It is a sheaf of $\mathcal{O}_M$–algebras, multiplication in which we denote $\cdot$.

Consider now a triple $(M, \circ, e)$ where $\circ$ is a commutative associative $\mathcal{O}_M$–bilinear multiplication on $\mathcal{T}_M$, eventually with identity $e$.

There is an obvious homomorphism of $\mathcal{O}_M$–algebras

$$\left( \text{Symm}_{\mathcal{O}_M}(\mathcal{T}_M), \cdot \right) \to (\mathcal{T}_M, \circ)$$

5.2.1. Theorem. The multiplication $\circ$ satisfies the $F$–identity

$$P_{X \circ Y} = X \circ P_Y + (-1)^{XY}Y \circ P_X$$

iff its kernel is stable with respect to the canonical Poisson brackets on $T^*M$.

In other words, $F$–identity is equivalent to the fact that the spectral cover of $M$ considered as a closed subspace of its cotangent bundle is coisotropic of maximal dimension.

NB. The spectral cover $\tilde{M} := \text{Spec}_{\mathcal{O}_M}(\mathcal{T}_M, \circ)$ of $M$ is not necessarily a manifold. Its structure sheaf may have zero divisors and nilpotents.

However, it is a manifold, if the $F$–manifold $M$ is semisimple.

Conversely, an embedded submanifold $N \subset T^*M$ is the spectral cover of some semisimple $F$–structure iff $N$ is Lagrangian.

5.3. Local decomposition theorem. Sum of two $F$–manifolds is defined by:

$$(M_1, \circ_1, e_1) \oplus (M_2, \circ_2, e_2) := (M_1 \times M_2, \circ_1 \circ_2, e_1 \oplus e_2)$$

A manifold is called indecomposable if it cannot be represented as a sum in a nontrivial way.

For any point $x$ of a pure even $F$–manifold $M$, the tangent space $T_xM$ is endowed with the structure of a commutative finite dimensional $K$–algebra. This $K$–algebra can be represented as the direct sum of local $K$–algebras. The decomposition is unique in the following sense: the set of pairwise orthogonal idempotent tangent vectors determining is well defined.
5.3.1. Decomposition Theorem. Every germ \((M, x)\) of a complex analytic \(F\)–manifold decomposes into a direct sum of indecomposable germs such that for each summand, the tangent algebra at \(x\) is a local algebra.

This decomposition is unique in the following sense: the set of pairwise orthogonal idempotent vector fields determining it is well defined.

5.3.2. Comments. (i) If \((T_x M, \circ)\) is semisimple, this theorem is equivalent to the existence (and uniqueness) of Dubrovin’s coordinates.

(ii) A proof of this theorem is based upon interpretation of the basic identity of the \(F\)–structure as integrability condition.

(iii) For \(F\)–manifolds with a compatible flat structure, there exists a considerably more sophisticated operation of tensor product which we omit here.

Furthermore, we have ([He], Theorems 5.3 and 5.6):

5.4. Theorem. (i) The spectral cover space \(\tilde{M}\) of the canonical \(F\)–structure on the germ of the unfolding space of an isolated hypersurface singularity is smooth.

(ii) Conversely, let \(M\) be an irreducible germ of a generically semisimple \(F\)–manifold with the smooth spectral cover \(\tilde{M}\). Then it is (isomorphic to) the germ of the unfolding space of an isolated hypersurface singularity. Moreover, any isomorphism of germs of such unfolding spaces compatible with their \(F\)–structure comes from a stable right equivalence of the germs of the respective singularities.

Recall that the stable right equivalence is generated by adding sums of squares of coordinates and making invertible local analytic coordinate changes.

In view of this result, it would be important to understand the following

5.4.1. Problem. Characterize those varieties \(V\) for which the genus zero quantum cohomology Frobenius spaces \(H_{\text{quant}}^*(V)\) have smooth spectral covers.

Theorem 5.4 above produces for such manifolds a weak version of Landau–Ginzburg model, and thus gives a partial solution of the mirror problem for them.

5.5. From \(F\)–manifolds to Frobenius manifolds. We start with an incomplete description of such a passage and steps involved in it.

A Frobenius manifold is an \(F\)–manifold endowed with a compatible flat structure \(\nabla\), an Euler vector field \(E\) and a (pseudo)–Riemannian metric \(g : S^2(T_M) \to \mathcal{O}_M\) such that
(i) \( g \) is flat, and \( \nabla = \text{the Levi–Civita connection of } g \).

(ii) \( g(X \circ Y, Z) = g(X, Y \circ Z) \).

An Euler field \( E \) is compatible with Frobenius structure if

(iii) \( \text{Lie}_E g = Dg \) for a constant \( D \).

NB. This is only an incomplete version because not all restrictions and compatibility conditions on extra structures are spelled out explicitly below.

One condition of compatibility is stated below in more detail because it will be important for the definition of \( F_\infty \)-structure.

5.6. Compatible flat structures. An (affine) flat structure on a (super) manifold \( M \) is given by any of the following equivalent data:

(i) A torsionless flat connection \( \nabla_0 : T_M \to \Omega^1_M \otimes_{\mathcal{O}_M} T_M \).

(ii) A local system \( T^f_M \subset T_M \) of flat vector fields, which forms a sheaf of (super)commutative Lie algebras of rank \( \text{dim } M \) such that \( T_M = \mathcal{O}_M \otimes T^f_M \).

(iii) An atlas whose transition functions are affine linear.

Assume that \( T_M \) is endowed with an \( \mathcal{O}_M \)-bilinear (super)commutative and associative multiplication \( \circ \), and eventually with unit \( e \).

NB \( F \)-identity is not yet postulated!

5.6.1. Definition ([Man2]). a) A flat structure \( T^f_M \) on \( M \) is called compatible with \( \circ \), if in a neighborhood of any point there exists a vector field \( C \) such that for arbitrary local flat vector fields \( X, Y \) we have

\[ X \circ Y = [X, [Y, C]] \]

\( C \) is called then a local vector potential for \( \circ \).

b) \( T^f_M \) is called compatible with \( (\circ, e) \), if a) holds and moreover, \( e \) is flat.

5.6.2. Proposition. If \( \circ \) admits a compatible flat structure, then it automatically satisfies the \( F \)-identity.

5.7. \( F_\infty \)-manifolds. The \( \infty \)-version of \( F \)-manifolds discussed below generalises only the case of infinitesimally deformed germ of a manifold \( (M, \ast) \) and replaces it by its formal smooth graded dg resolution \( (M, \ast) \) supplied with a smooth degree one vector field \( \partial \) satisfying

\[ [\partial, \partial] = 0, \quad \partial I \subset I^2 \]
where $I$ is the ideal of $\ast$.

The role of $\circ$ on $(\mathcal{M}, \ast, \partial)$ will now be played by a structure of $\mathcal{C}_\infty$–algebra

$$\mu_\bullet = \{\mu_n\}_{n \geq 1} : \bigotimes_{\mathcal{O}_\mathcal{M}} T_{\mathcal{O}_\mathcal{M}} \to T_{\mathcal{O}_\mathcal{M}}$$

The former $F$–identity in this context becomes the first step of the ladder:

$$[\mu_2, \mu_2] = 0$$

where by definition $[\mu_2, \mu_2] : \bigotimes^4_{\mathcal{O}_\mathcal{M}} T_{\mathcal{M}} \to T_{\mathcal{M}}$ is given by

$$[\mu_2, \mu_2](X, Y, Z, W) := \text{the left hand side of } F \text{ – identity}$$

The whole ladder involves a system of “polybrackets” $[[\mu_\bullet, \mu_\bullet]]^\nabla$ depending on the additional choice of a torsion free affine connection $\nabla$ and subsequent passage to the its cohomology class

$$[[\mu_\bullet, \mu_\bullet]] \in H(\bigotimes^\bullet_{\mathcal{O}_\mathcal{M}} T_{\mathcal{M}}^\bullet \otimes_{\mathcal{O}_\mathcal{M}} T_{\mathcal{M}})$$

The $F_\infty$–identity then reads

$$[[\mu_\bullet, \mu_\bullet]] = 0,$$

and it defines a structure of $F_\infty$–manifold upon $(\mathcal{M}, \ast, \partial)$.

5.8. **Theorem.** ([Me2]). (i) The formal dg manifold associated with the Hochschild cohomology of an associative algebra is an $F_\infty$–manifold.

(ii) The formal dg manifold associated with singular cohomology of a compact topological space is an $F_\infty$–manifold.

In [DoShVa1], these results are somewhat generalised and/or strengthened.

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References

[Ar] Arnold, V. I. Normal forms of functions near degenerate critical points, the Weyl groups $A_k, D_k, E_k$ and Lagrangian singularities (Russian). Funkc. Anal. i Prilozhen. vol. 6, no. 4, 1972, pp. 3–25.

[Art1] M. Artin. On solutions of analytic equations. Inv. Math. 5, 1968, pp. 277–291.

[Art2] M. Artin. Lectures on Deformations of Singularities. Lectures on Math. and Phys. Tata Institute 54 (1976).

[BaMan] A. Bayer, Yu. Manin. (Semi)simple exercises in quantum cohomology. In: The Fano Conference Proceedings, ed. by A. Collino, A. Conte, M. Marchisio, Università ‘di Torino, 2004, pp. 143–173. arXiv: math.AG/0103164

[BehMan] K. Behrend, Yu. Manin. Stacks of stable maps and Gromov–Witten invariants. Duke Math. Journ., 85:1, 1996, pp. 1–60.

[BelGeKo] A. Belavin, D. Gepner, Y. Kononov. Flat coordinates for Saito Frobenius manifolds and String theory. arXiv:1510.06970

[BuMi] R.–O. Buchweitz, J. Millson. CR–Geometry and Deformations of Isolated Singularities. Memoirs of AMS, Vol. 125, N. 597, 1997, 96 pp.

[ChMPe] P. E. Chaput, L. Manivel, N. Perrin. Quantum Cohomology of minuscule homogeneous spaces. Transformation Groups, vol. 13, no. 1, 2008, pp. 47–89.

[ClLaPo] R. Cleyton, J. Lauret, Yat Sun Poon. Weak Mirror Symmetry of Lie Algebras. J. Symplectic Geom. vol. 8, no. 1, 2010, pp. 37–55, arXiv:1004.3264. 22pp.

[ClPo] Differential Gerstenhaber Algebras Associated to Nilpotent Algebras. Asian J. Math. vol. 12, no. 2, 2008, pp. 255–249.

[ClOvPo11] R. Cleyton, G. Ovando, Yat Sun Poon. Weak Mirror Symmetry of Complex Symplectic Lie Algebras. J. Geom. Phys. 61(2011), no. 8, 2001, pp. 1553–1563. arXiv:0804.4787.

[De] P. Deligne. Letter to J. J. Millson. April 24, 1986.
[Do] V. Dotsenko. *Algebraic structures of F–manifolds via pre–Lie algebras.* arXiv:1706.07340

[DoShVa1] V. Dotsenko, S. Shadrin, B. Vallette. *De Rham cohomology and homotopy Frobenius manifolds.* arXiv:1203.5077

[DoShVa2] V. Dotsenko, S. Shadrin, B. Vallette. *Pre–Lie deformation theory.* arXiv:1502.03280. 31 pp.

[Dr] V. Drinfeld. *A letter from Kharkov to Moscow.* EMS Surv. Math. Sci., 2014, vol. 1, no. 2, pp. 241–248.

[FiMaMar] D. Fiorenza, M. Manetti, E. Martinengo. *Csimplicial DGLAs in Deformation Theory.* Comm. Algebra 40, 2012, vol. 6, pp. 2243–2260

[Ge] E. Getzler. *Lie theory for nilpotent L∞–algebras.* Annals of Math., 170 (1), 2009, pp. 271–301. arXiv:math/0404003.

[GoMi] W. M. Goldman, J. J. Millson. *The deformation theory of representations of fundamental groups of compact Kähler manifolds.* in: Publ. Math. IHES, tome 67, no. 2, 1988, pp. 43–69.

[GoPe] V. Gorbunov, V. Petrov. *Schubert calculus and singularity theory.* arXiv:1006.1464

[Gr] H. Grauert. *Über die Deformationen isolierter Singularitäten analytischer Mengen.* Inv. Math. 15(1972), 171–198.

[He] C. Hertling. *Frobenius spaces and and moduli spaces for singularities.* Cambridge University Press, 2002, ix+270 pp.

[HeMan] C. Hertling, Yu. Manin. *Unfoldings of meromorphic connections and a construction of Frobenius manifolds.* In: Frobenius manifolds, 113–144, Aspects Math., E36, Friedr. Vieweg, Wiesbaden, 2004.

[Hi1] V. Hinich. *Descent of Deligne groupoids.* IMRN 1997, no. 5, pp.‘223–239 arXiv:algh–geom/9606010 .

[Hi2] V. Hinich. *Deformations of sheaves of algebras,* Advances in Math., vol. 195, 2005, pp. 102–164.

[HiSch] V. Hinich, V. Schechtman. *Deformation Theory and Lie Algebra Homology I,II.* Algebra Colloquium 4:2 (1997), pp. 213–240 and 4:3 (1997), pp. 291–316.

[KoMan] M. Kontsevich, Yu. Manin. *Gromov–Witten classes, quantum cohomology, and enumerative geometry.* Comm. Math. Phys., 164:3 (1994), pp. 525–562
[LamTem] Th. Lam, N. Templier. *The Mirror Conjecture for minuscule flag varieties*. arXiv:1705.00758.

[LoVa] J. L. Loday, B. Vallette. *Algebraic Operads*. Springer, 2012, xxiv + 634 pp.

[Lu] J. Lurie. *Derived Algebraic Geometry X: Formal Moduli Problems*. www.math.harvard.edu/~lurie

[Ma1] M. Manetti. *Deformation theory via differential graded Lie algebras*. arXiv:math/0507284

[Ma2] M. Manetti. *Extended deformation functors*. IMRN no. 14, 2002, pp. 719–756. arXiv:9910071

[Ma3] M. Manetti. *On some formality criteria for DG–Lie algebras*. J. Algebra 438, 2015, pp. 90–118. arXiv:1310.3048

[Man1] Yu. Manin. *Frobenius manifolds, quantum cohomology, and moduli spaces*. AMS Colloquium Publications, vol. 47, Providence, RI, 1999, xiii+303 pp.

[Man2] Yu. Manin. *Manifolds with multiplication on the tangent sheaf*. Rendiconti Mat. Appl., Serie VII, vol.26 (2006), 69–85. arXiv:0502578

[ManSm1] Yu. Manin, M. Smirnov. *On the derived category of \( \mathcal{M}_{0,n} \)*. Izvestiya of Russian Ac. Sci., vol. 77, No 3, 2013, 93–108. arXiv:1201.0265

[ManSm2] Yu. Manin, M. Smirnov. *Towards motivic quantum cohomology of \( \overline{M}_{0,S} \)*. Proc. of the Edinburg Math. Soc., Vol. 57 (ser. II), no 1, 2014, pp. 201–230. Preprint arXiv:1107.4915

[MatY] J. N.Mathers, S. S. T. Yau. *Classification of isolated hypersurface singularities by their moduli algebras*. Inv. Math., vol. 69, 1982, pp. 243–251.

[Me1] S. Merkulov. *Frobenius\( \infty \)-invariants of homotopy Gerstenhaber algebras I*. Duke Math. J. 105, 2000, pp. 411–461.

[Me2] S. Merkulov. *Operads, deformation theory and F–manifolds*. In: Frobenius Manifolds (ed. by C. Hertling, M. Marcolli). Vieweg, 2004, pp. 213–251. arXiv:02100478.

[Mi] J. Millson. *Rational homotopy theory and deformation problems from algebraic geometry*. Proc. ICM 90, Kyoto, Math. Soc. Japan, 1991, pp. 549–558.

[Po] Yat Sun Poon. *Extended Deformation of Kodaira Surfaces*. J. reine angew. Math., 590 (2006), pp. 45–65. arXiv:math/0402440
[Pr] J. P. Pridham. *Unifying derived deformation theories.* Advances in Math., 224 (3), 2010, pp. 772–826.

Corrigendum: Advances in Math., 228, 2011, pp. 2554–2556.

[R-N] D. Robert–Nicoud. *Representing the Deligne–Hinich–Getzler ∞–groupoid.* arXiv:1702.02529, 13 pp.

[Sa] K. Saito. *Primitive forms for universal unfolding of a function with an isolated critical point.* Journ. Fac. Sci. Univ. Tokyo, sec IA, vol. 28, 1981, pp. 775–792.

[Sc] J. Scherk. *A propos d’un théorème de Mather and Yau.* C. R. Ac. Sci. Paris, Sér. I, t. 296, 1983, pp. 513–515.

[Sch] M. Schlessinger. *Functors of Artin rings.* Trans. AMS, vol. 130, 1968, pp. 208–222.

[SchSt] M. Schlessinger, J. Stasheff. *The Lie algebra structure of tangent cohomology and deformation theory.* J. Pure Appl. Algebra, vol. 38, no. 2–3, 1985, pp. 313–322.

[To] B. Toën. *Problèmes de modules formels [d’après V. Drinfeld, V. Hinich, M. Kontsevich, J. Lurie ...].* Sém. Bourbaki, Jan. 16, 2016. https://perso.math.univ-toulouse.fr/btoen/files/2012/04/Bourbaki-Toen-2016-final1.pdf.