Disappearance of the Measurement Paradox
in a Metaplectic Extension of Quantum Dynamics

Daniel I. Fivel
Department of Physics, University of Maryland
College Park, Md.

November 20, 2003

Abstract

It is shown that Schrödinger dynamics can be embedded in a larger dynamical theory which extends its symmetry group from the unitary group to the full metaplectic group, i.e. the group of linear canonical transformations. Among the newly admitted non-unitary processes are analogues of the classical measurement process which makes it possible to treat the wave-function as an objective property of the quantum mechanical system on the same footing as the phase-space coordinates of a classical system. The notion of “observables” that in general have values only when measured can then be dispensed with, and the measurement paradox disappears.

Schrödinger dynamics is restricted to unitary transformations. The measurement paradox arises from the inadequacy of such transformations to account for measurement processes. Classical dynamics, on the other hand, has no measurement paradox. The task of classical measurement, which is to assign phase space coordinates to an object system, can be performed through classically describable interactions between object systems and measuring devices. The simplest such interactions (see below) are one-parameter subgroups of the so-called metaplectic\textsuperscript{1} group $\mathcal{M}$ which is the group of lin-
ear canonical transformations, i.e. linear transformations of phase space that leave the Poisson bracket invariant. However, they lie outside of its unitary subgroup $\mathcal{M}_o$ and hence have no analogue in Schrödinger dynamics. Thus while one can construct a closed universe governed by classical mechanics, it is not possible to construct a closed universe governed by the Schrödinger equation.

In this paper it will be shown that Schrödinger dynamics can be embedded in a larger system which extends the available transformations from $\mathcal{M}_o$ to $\mathcal{M}$. A quantum mechanical analogue of the classical measurement process then appears, and the measurement paradox disappears. The resulting theory reproduces the standard predictions within a closed universe.

Let us begin by formulating Schrödinger dynamics in a way that closely resembles classical dynamics. Let $\Psi$ be a separable Hilbert space of wave-functions, the components of which in some basis may be written $\psi_j = (p_j + i q_j)/\sqrt{2}$. The transformations of Schrödinger dynamics are one parameter subgroups of the unitary group which we can write in the form

$$\psi \to \psi_t = e^{-itH} \psi,$$  \hspace{1cm} (1a)$$

where $H$ is a hermitian matrix. Now compare this kind of transformation of $\psi$ with what we would obtain if $\Psi$ were a phase-space instead of a Hilbert space. We can pretend that the $q_j$’s and $p_j$’s are coordinates and momenta and subject $\psi$ to one-parameter sub-groups of the group of canonical transformations. These are generated by $C_\infty$ functions $\Gamma$ of the $p_j$’s and $q_j$’s or equivalently of the $\psi_j$’s and $\psi^*_j$’s by the rule

$$\psi \to \psi_t = e^{-itAd\Gamma} \psi,$$ \hspace{1cm} (1b)$$
where $Ad$ acts by the Poisson bracket:

$$AdA \cdot B = i \sum_k \left( \frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial q_k} \right) = \sum_k \left( \frac{\partial A}{\partial \psi_k^*} \frac{\partial B}{\partial \psi_k} - \frac{\partial A}{\partial \psi_k} \frac{\partial B}{\partial \psi_k^*} \right).$$

(1c)

(For notational convenience this definition of the Poisson bracket differs from the usual one by a factor of $i$.)

The transformations of the metaplectic group $\mathcal{M}$ are generated by quadratic forms $\Gamma(\psi^*, \psi)$ in the components of $\psi$ and $\psi^*$. They transform components of $\psi$ into linear combinations of components of both $\psi$ and $\psi^*$ in general. The subgroup $\mathcal{M}_o$ of $\mathcal{M}$ has generators of the form

$$\Gamma = \mathcal{H}(\psi^*, \psi) = \sum_{jk} \psi_j^* H_{jk} \psi_k,$$

(2)

where $H$ is a hermitian matrix. These transformations transform components of $\psi$ linearly without mixing in components of $\psi^*$. One then verifies that

$$e^{-itAdH} \psi_j = (e^{-itH} \cdot \psi)_j.$$

(3)

Comparing this with (1a) we see that all of Schrödinger dynamics can be described by one-parameter sub-groups of $\mathcal{M}_o$ acting on $\psi$ when we treat it as a point in a phase-space rather than a vector in a Hilbert space. This way of representing Schrödinger dynamics is not restricted to wave-functions with discrete indices. For example a spatial wave-function $\psi(x)$ can be treated like a classical field with sums over indices replaced by integrals and the Poisson bracket defined with variational derivatives.

The one-parameter subgroups of $\mathcal{M}$ that are outside of $\mathcal{M}_o$ have no counterpart in Schrödinger dynamics. They have generators of the form

$$\Gamma = W = \sum_{jk} \psi_j^* W_{jk} \psi_k^* + \text{complex conjugate}.$$

(4)
These transformations transform components of $\psi$ and $\psi^*$ among one another. They are therefore linear only on real linear combinations. We shall first show that processes of this kind describe what happens in the simplest kind of classical measurement. This will motivate us to extend quantum dynamics so that $\psi$ can transform under the analogue of the full metaplectic group thereby providing a way to incorporate quantum measurement into the theory.

In the simplest classical measurement the object system and measuring device will each have a single complex degree of freedom with phase-space coordinates $\lambda = (p + iq)/\sqrt{2}$ and $\mu = (P + iQ)/\sqrt{2}$ respectively. The task of the measurement process is to resolve distinct object states and thereby assign values of $\lambda$ to them. We assume that we know how to do this for states of the measuring device provided that $|\mu|$ is sufficiently large (macroscopic). We therefore seek an interaction between object and device which, when strong enough, makes orbits of $\mu$ for distinct initial $\lambda$ diverge to any desired extent. Moreover it must do so in a time that can be made arbitrarily small so that other dynamical processes that might be present can be neglected.

Let us see how this can be accomplished. Consider the orbit of $\psi = (\lambda, \mu)$ under the one-parameter metaplectic subgroup generated by

$$W = \eta(\lambda\mu)^* + \eta^*(\lambda\mu),$$

where $\eta$ is a complex parameter. From (1b) we obtain:

$$\psi \to \psi_t = e^{-itAdW} \psi = (\lambda_t, \mu_t),$$

$$\lambda_t = \lambda \cosh(|\eta|t) + \mu^* e^{i \arg \eta} \sinh(|\eta|t),$$

$$\mu_t = \mu \cosh(|\eta|t) + \lambda^* e^{i \arg \eta} \sinh(|\eta|t).$$
To use this process to perform the task of assigning values to the object coordinates, choose the initial device coordinate (the “ready” state) to be $\mu = 0$ so that

$$\lambda_t = \lambda \cosh(|\eta| t), \quad \mu_t = \lambda^* e^{i \arg \eta} \sinh(|\eta| t), \quad (8a)$$

$$|\psi_t|^2 = |\lambda_t^2 + |\mu_t|^2| = |\lambda|^2 \cosh(2|\eta| t). \quad (8b)$$

Comparing the $\mu$ orbits for two initial choices $\lambda^{(1)}$ and $\lambda^{(2)}$ of $\lambda$ we have

$$|\mu_t^{(1)} - \mu_t^{(2)}| = |\lambda^{(1)} - \lambda^{(2)}| \sinh(|\eta| t). \quad (8c)$$

Thus no matter how close the initial values $\lambda^{(1)}$ and $\lambda^{(2)}$ might be, we can choose $|\eta|$ sufficiently large that the distance between the device coordinates at time $t$ becomes as large as we please as quickly as we please. Thus $\mu_t$ becomes a *macroscopic pointer* from which (8a) gives the initial value of $\lambda$ if $\eta$ is known. Since the process is a group, there is an inverse by which the initial $\lambda$ can be restored after the determination of $\mu_t$. Moreover, by choosing a sufficiently large $|\eta|$, this can be done so rapidly that any other dynamical processes that may be acting can be ignored. Thus the availability of non-unitary metaplectic processes makes the phase space coordinates a determinate property of a classical system.

Let us contrast this with quantum mechanics. In the Dirac formulation we have the projective map from $\psi \in \Psi$ (excluding $\psi$ with $|\psi| = 0$) to unit state vectors

$$\psi \rightarrow |\hat{\psi}\rangle \equiv \psi/|\psi|. \quad (9)$$

There is a non-vanishing probability that a system with state vector $|\hat{\psi}\rangle$ will pass a filter for a system with a different state vector $|\hat{\phi}\rangle$ unless the two vectors are orthogonal. Thus only orthogonal states can be perfectly resolved. To deal with this, the orthodox (Dirac-von Neumann) theory introduces
the notion of “observables” defined by hermitian operators. An observable
does not have a definite value until it is measured unless the system is in
an eigenstate. The problems that arise from this interpretation, e.g. the
unexplained collapse mechanism and the existence of grotesque macroscopic
superpositions (“Schrödinger cats”), are familiar\textsuperscript{2,3}. What is needed is, as
J.S. Bell\textsuperscript{4} put it, a theory of “be-ables” rather than “observables” in which
objective properties are assigned to systems.

To treat $\psi$ as an assignable property of a classical system we made use of the
metaplectic transformation that caused phase space distances between dis-
tinct $\psi$’s to become arbitrarily large. Such transformations have no quantum
counterpart if we use the Dirac map (9) because no change in the state vector
$|\psi\rangle$ occurs when $\psi$ is multiplied by a scale factor $\tau > 0$. The justification
for (9) is that since $|R\psi| = |\psi|$ for unitary transformations,

$$|R\psi\rangle = R|\psi\rangle \quad (10)$$

which insures that Schrödinger dynamics, which acts on wave-functions and
implements the superposition principle, transfers properly to state vectors.
Dirac\textsuperscript{5} defends the inability of (9) to represent processes that scale the
wave-function by asserting that the norm $|\psi|$ has no physical meaning. Our
sought-after extension of dynamics to include non-unitary processes will give
$|\psi|$ physical meaning and thereby justify modifying (9). Thus our first task
is to show that (9) can be replaced by a map that preserves (10) for unitary
transformations while permitting the representation of processes in which
the norm changes.

For each $\psi$ in the Hilbert space $\Psi$ let there correspond a unitary operator
$U(\psi)$ on a Hilbert space $\mathcal{H}$ such that $U(0) = I$. Let $|0\rangle$ be a distinguished
unit vector of $\mathcal{H}$ which will be called the “vacuum”. The map

$$\psi \rightarrow |\psi\rangle \equiv U(\psi)|0\rangle \quad (11)$$
will then define a unit vector associated with $\psi$. One may think of $\psi$ as the “instructions” for creating the state from the vacuum.

Let $\mathcal{F}$ be the set of unit vectors in $\mathfrak{h}$ that correspond to some $\psi \in \Psi$. Unlike Dirac kinematics, not every unit vector in $\mathfrak{h}$ is necessarily a member of $\mathcal{F}$. The set $\mathcal{F}$ will be a manifold, but, since linear combinations of unitary operators are not in general unitary, it will not be a linear space. Hence there must be a constraint on the form of $U(\psi)$ to insure that the superposition principle, which holds in the Hilbert space $\Psi$, is properly implemented in $\mathcal{F}$. Intuitively the superposition principle says that we should be able to make the state $|\psi^{(1)} + \psi^{(2)}\rangle$ from the operations used to make $|\psi^{(1)}\rangle$ and $|\psi^{(2)}\rangle$. This suggests that $U(\psi)$ be required to satisfy

$$U(\psi^{(1)} + \psi^{(2)}) = e^{i\theta_{12}}U(\psi^{(1)})U(\psi^{(2)}),$$

in which the phase $\theta_{12}$ may depend on the $\psi$’s and their order.

Equation (12) is the defining relation for the Weyl-Heisenberg group. We shall take it to be the fundamental condition defining the symmetry of the theory, i.e. the group of allowed transformations. A transformation $\psi \rightarrow g\psi$, where $g$ is not necessarily linear, will be allowed if (12) remains valid (with the same phase) if $\psi$ is replaced by $g\psi$. It suffices for this that there exist a unitary operator $V_g$ on $\mathfrak{h}$ such that

$$U(g\psi) = V_gU(\psi)V_g^\dagger.$$  

Below we will explicitly construct $V_g$’s for one-parameter subgroups of the metaplectic group, thereby showing that (12) is consistent with an extension of dynamics to that group.

The Stone-von Neumann theorem tells us that all of the representations of (12) are obtained up to unitary equivalence (13) by tensoring Fock representations together. Fock representations are obtained by choosing a basis and
exponentiating the Heisenberg algebra, i.e.

\[ U(\psi) = e^{\psi \cdot a^\dagger - \psi^* \cdot a} \quad (14a) \]

in which

\[ \psi \cdot a^\dagger = \psi_1 a_1^\dagger + \psi_2 a_2^\dagger + \cdots, \quad (14b) \]

where the \( a \)'s and their adjoints are operators on \( \mathfrak{h} \) such that:

\[ [a_i, a_j] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij} I. \quad (15a) \]

One then verifies that (12) holds with

\[ \theta_{12} = \Im (\psi^{(1)*} \cdot \psi^{(2)}), \quad (15ba) \]

To define the state \( |\psi\rangle \) we must specify the vacuum state \( |0\rangle \) which we take to be the tensor product of the states annihilated by the \( a_j \)'s so that

\[ a_j |0\rangle = 0 \quad \forall j. \quad (16a) \]

It then follows that

\[ \langle 0|\psi \rangle = e^{-|\psi|^2/2}, \quad (16b) \]

whence from (12)

\[ \langle \psi^{(1)}|\psi^{(2)} \rangle = e^{i\theta_{12}} e^{-|\psi^{(1)} - \psi^{(2)}|^2/2}. \quad (16c) \]

Equation (16c) relates the geometry of \( \mathcal{F} \) to the geometry of \( \Psi \). It shows that the effect on \( \mathcal{F} \) of successive dilation of the phase space \( \Psi \) is to make all of its vectors approach mutual orthogonality. Observe that since not all unit vectors of \( \mathfrak{h} \) are in \( \mathcal{F} \) there is “room” for the vectors of \( \mathcal{F} \) to approach mutual orthogonality.

The appearance of bose operators in (14a) does not mean that this formalism applies only to systems with bose statistics. For fermionic systems we
need only restrict the Hilbert space $\Psi$ be a space of anti-symmetric wavefunctions. Also, as remarked earlier, one is not restricted to discrete indices. For example to construct $U(\psi)$ for states described by spatial wave-functions $\psi(x)$ we understand $\psi^* \cdot a$ to mean an integral over $x$ with operators $a(x)$ satisfying $[a(x), a(y)^\dagger] = \delta(x - y)I$.

We can see quite simply why the metaplectic group preserves (12): Since in general its elements transform components of $\psi$ linearly into components of both $\psi$ and $\psi^*$, they act linearly on real linear combinations such as the one that appears in the argument of $U$ on the left side of (12). Moreover one readily verifies that to leave the Poisson bracket invariant the coefficients $\alpha, \beta$ in $\psi \to \alpha \psi + \beta \psi^*$ must also leave the imaginary part of the scalar product $\theta_{12}$ (16a) invariant. The unitary subgroup $\mathcal{M}_o$ leaves the scalar product itself invariant.

We can explicitly construct the transformations $V_t$ that implement one-parameter subgroups $M_t$ of $\mathcal{M}$ as follows: Let $M_t$ act on $\psi$ by (1b) with the quadratic generator $\Gamma(\psi^*, \psi)$. The algebraic relationship between commutator brackets and Poisson brackets is such that we have the identity:

\[
U(M_t \psi) = V_t U(\psi) V_t^\dagger, \tag{17a}
\]

\[
V_t = e^{-it: \Gamma(a, a^\dagger):}. \tag{17b}
\]

Here $\Gamma(a, a^\dagger)$ is obtained by substituting $a$ for $\psi^*$ and $a^\dagger$ for $\psi$ in the quadratic form $\Gamma(\psi, \psi^*)$. The colons indicate normal ordering (putting $a^\dagger$’s to the left of $a$’s). What makes (17) work is that the transformation of (14a) by $V$’s of the form (17b) transforms the bose operators in the exponent by

\[
a \to V a V^\dagger = A a + B a^\dagger, \tag{18}
\]

in which the matrices $A, B$ are such that the Heisenberg algebra structure is preserved$^1$. 

9
We shall be interested in the orbit of the quantum state corresponding to the orbit of \( \psi \) under \( M_t \). We have
\[
|\psi_t\rangle = U_t(\psi)|0\rangle, \quad U_t(\psi) \equiv U(M_t\psi).
\] (19)

From (17) we have the generalized Schrödinger equation
\[
\partial_t U_t(\psi) = -i Ad \Gamma U_t(\psi),
\] (20)
where \( Ad \) acts by the commutator, i.e.
\[
AdA \cdot B \equiv [A, B].
\] (21)

It reduces to the usual Schrödinger equation when \( \Gamma \) is a generator of the unitary subgroup \( M_o \). To see this observe that corresponding to the generator (2) we will have, according to (17b), an operator \( \Gamma \) of the form
\[
\mathcal{H} = \sum_{jk} a_j^\dagger H_{jk} a_k,
\] (22)
which annihilates \( |0\rangle \), so that \( e^{-itH} \) leaves \( |0\rangle \) invariant. Hence applying (17a) to \( |0\rangle \) we obtain
\[
|e^{-iHt}\psi\rangle = e^{-itH}|\psi\rangle.
\] (23)
Thus the property (10) of the Dirac map needed for the implementation of Schrödinger dynamics is preserved. Indeed we obtain the usual Schrödinger equation
\[
\partial_t |\psi_t\rangle = -i \mathcal{H}|\psi_t\rangle
\] (24)
when transformations are restricted to the unitary subgroup \( M_o \).

We are now able to construct the quantum mechanical analogue of the classical measurement process described in equations 6-8. The generalization of (5) for a multi-dimensional phase-space is
\[
W = \sum_j \eta_j (\lambda_j \mu_j)^* + \text{complex conjugate}.
\] (25)
Assuming again that the ready state of the device corresponds to \( \mu = 0 \) the orbit will be \( |\psi_t\rangle \) with \( \psi_t = (\lambda_t, \mu_t) \) where

\[
\lambda_{jt} = \lambda_j \cosh(|\eta_j| t), \quad \mu_{jt} = \lambda_j^* e^{i \text{arg } \eta_j} \sinh(|\eta_j| t).
\]

(26)

Suppose that we let \( |\eta_j| \) have the same value \( |\eta| \) for all \( j \). If \( \psi^1_t \) and \( \psi^2_t \) are the orbits corresponding to different initial values \( \lambda^1 \) and \( \lambda^2 \) of the object system we have from (16c)

\[
|\langle \psi^1_t | \psi^2_t \rangle|^2 = e^{-|\lambda^1 - \lambda^2|^2 \cosh(2|\eta| t)}
\]

(27)

which tends to zero in a super-rapid way since a hyperbolic function appears in the exponent. Given any lattice on the object phase space, no matter how fine, and any \( \epsilon > 0 \), we can choose a sufficiently large \( |\eta| \) that all scalar products (27) between states corresponding to distinct lattice points become smaller than \( \epsilon \) at a time \( t \) which can be made arbitrarily small. Thus the effect of the interaction is to make the set of state vectors associated with distinct \( \lambda \)'s on the lattice become mutually orthogonal to any desired approximation. The set thus becomes classical in the sense that all propositions have only yes-no answers except for fluctuations of negligible probability. Thus as in the classical case we can in principle read the coordinates \( \mu_{jt} \) for all \( j \) of the device and thereby deduce \( \lambda_j \) from the relation

\[
\mu_{jt} = e^{i \eta_j} \lambda_j^* \sinh(|\eta| t)
\]

(28a)

provided that \( |\eta| \) and \( \text{arg } \eta_j \) are known. If we do not know \( |\eta| \) or the phases of \( \eta_j \) we can nonetheless determine the ratios

\[
|\lambda_j / \lambda_k|^2 = |\mu_{jt} / \mu_{kt}|^2.
\]

(28b)

Being a group, the process generated by \( \mathcal{W} \) is reversible, so we can assign initial \( \lambda \)-values to object states and return to those states in a short enough
time that the effect of other dynamical processes that might be present can be neglected.

We have now established that the metaplectic extension of Schrödinger dynamics introduces non-unitary processes by which the wave-function of an object system becomes a determinate property of the system on the same footing as the phase-space coordinates of a classical system. In the extended theory the norm $|\psi|$ has physical meaning which we explore next.

Let us first observe that there is a sense in which the Dirac map (9) is the limiting form of (11) for $|\psi| \to 0$. To see this note that when (9) is used there can be no state corresponding to $\psi \equiv 0$ whereas with (11) this corresponds to the vacuum $|0\rangle$. We therefore examine the normalized projection $|\psi_\perp\rangle$ of $|\psi\rangle$ in the direction orthogonal to the vacuum. From (16b) one obtains

$$|\psi_\perp\rangle = (|\psi\rangle - e^{-|\psi|^2/2}|0\rangle)/(1 - e^{-|\psi|^2})^{1/2}. \quad (29)$$

From (11) and (14a) we have

$$|\psi\rangle \equiv \lim_{|\psi| \to 0} |\psi_\perp\rangle = (\hat{\psi} \cdot a^\dagger)|0\rangle, \quad \hat{\psi} \equiv |\psi|.$$

Thus

$$\langle \hat{\phi} | \hat{\psi} \rangle = \hat{\phi}^* \cdot \hat{\psi}, \quad (31)$$

which identifies the states $|\hat{\psi}\rangle$ as the Dirac states defined by (9). Hence

$$|\psi_\perp\rangle \to \begin{cases} |\hat{\psi}\rangle, & |\psi| \to 0 \\ |\psi\rangle, & |\psi| \to \infty \end{cases}, \quad (32)$$

so that $|\psi_\perp\rangle$ interpolates between states with small $|\psi|$ that act quantum mechanically and states with large $|\psi|$ that act classically. Thus the norm $|\psi|$ is a measure of the “classicality” of the system.

It follows from (12) that for any integer $n$

$$U(\psi) = (U(\psi/n))^n \quad (33)$$
so that the states with large norm can be created by repeated application of the creation operator $U$ for a state with an arbitrarily small norm. This suggests that $U(\psi)$ can be interpreted as the creator of a “beam” of copies of the quantum state defined by $\hat{\psi}$, with an intensity given by some monotonically increasing function $I(|\psi|)$ of the norm. We can deduce this function as follows: If $\psi^a, \psi^b$ belong to orthogonal subspaces of $\Psi$ so that $U(\psi^a)$ and $U(\psi^b)$ commute, there will be no interference when the beams they create are combined. Hence the intensities simply add. Thus from (12)

$$I(|\psi^a + \psi^b|) = I(|\psi^a|) + I(|\psi^b|),$$

whence except for an arbitrary choice of scale we must set

$$I(|\psi|) = |\psi|^2.$$

In the standard description of quantum measurements each complete set of commuting observables defines a basis in $\Psi$ considered as a Hilbert space, namely the basis in which they are all diagonal. This determines a decomposition

$$\psi = \sum_j \psi^j$$

in which $\psi^j$ is the projection of $\psi$ on the one-dimensional subspace determined by the $j$'th basis vector. It follows from (12) that the operators $U(\psi^j)$ mutually commute and that

$$U(\psi) = \prod_j U(\psi^j).$$

Thus each complete set of commuting observables defines a factorization of $U(\psi)$ in which the factors represent non-interacting beams. The intensity of the beam created by $U(\psi)$ will be the sum of the intensities of the constituent beams.
Corresponding to the factorization (36) there will be a factorization of the vacuum state into a tensor product of states \(|0, j\rangle\) annihilated by \(a_j\). Defining

\[ N_j = a_j^\dagger a_j, \quad N = \sum_j N_j, \quad (38) \]

we see that the \(N_j\)'s form a complete set of commuting observables. From (14,15)

\[ a_j |\psi\rangle = \psi_j |\psi\rangle, \quad (39) \]

whence we obtain the expectation value

\[ \overline{N}_k \equiv \langle \psi_\perp |N_k|\psi_\perp \rangle = |\psi_k|^2 (1 - e^{-|\psi|^2})^{-1} \rightarrow \begin{cases} |\psi_k|^2, \quad |\psi| \rightarrow 0, \\ |\psi_k|^2, \quad |\psi| \rightarrow \infty. \end{cases} \quad (40) \]

Thus in the quantum limit \(\overline{N}_k\) can be interpreted as the probability of a copy being in the \(k\)'th subbeam whereas in the classical limit it coincides with the intensity of that subbeam.

Let \(\Delta N_k\) be the dispersion of \(N_k\) in the state \(|\psi_\perp\rangle\). Then one verifies that

\[ \Delta N_k / \overline{N}_k \rightarrow 1 / |\psi_k| \quad \text{for} \quad |\psi| \rightarrow \infty. \quad (41) \]

This goes to zero for any \(k\) for which \(\widetilde{\psi}_k \neq 0\). Thus there is a sharp value of \(N_k\), in the \(k\)'th subbeam, namely the intensity \(|\psi_k|^2\) of the subbeam. Ratios of these intensities then give the ratios \(|\widetilde{\psi}_k / \widetilde{\psi}_j|\). As we saw in (28b) it is these ratios that we can obtain in the quantum analogue of the classical measurement process when we cannot control the amplification parameter \(\eta\). Thus if we knew how to implement the amplification process generated by \(W\) but could not control \(\eta\), the information we would obtain about the state would be identical to the information obtainable by comparing subbeam intensities when the intensities are large. This is precisely what we predict from quantum measurements as they are described in the orthodox formulation.
The operator $N$ commutes with all generators of the form (22) and therefore defines an observable that is constant for all Schrödinger processes. The expectation values of $N$ and $N^2$ in $|\psi\rangle$ are given by

\[
\overline{N} = |\psi|^2 (1 - e^{-|\psi|^2})^{-1}, \quad \overline{N^2} = (|\psi|^4 + |\psi|^2)(1 - e^{-|\psi|^2})^{-1}.
\] (42)

Note that

\[
\overline{N} \to \begin{cases} 
1, & |\psi| \to 0, \\
|\psi|^2, & |\psi| \to \infty.
\end{cases}
\] (43)

Thus if we interpret $N$ as a “counter” for the number of copies in the beam, the quantum limit has one copy while the classical limit has a large number indicated by $|\psi|^2$. If $\Delta N$ is the dispersion we then find from (42) that $\Delta N/\overline{N}$ tends to zero both for $|\psi| \to 0$ and $|\psi| \to \infty$. It has a maximum of $\approx 0.55$ at $|\psi| \approx 1.8$. Thus this ratio provides a “marker” for the transition from the low intensity quantum mechanical regime to the high intensity classical regime.

We have now shown that Schrödinger dynamics can be embedded in a larger framework that enjoys the full metaplectic group as its symmetry group and that the extension incorporates a quantum analogue of the classical measurement process. The wave-function $\psi$ becomes a determinate property of systems such that those with small $|\psi|$ behave quantum mechanically and those with large $|\psi|$ behave classically. In the latter case $|\psi|^2$ is the intensity of a beam consisting of copies of the state described by the unit vector $\hat{\psi}$.

The processes of the extended theory obey a generalized Schrödinger equation which reduces to the usual Schrödinger equation for unitary processes, so that the predictions of the standard theory are unaltered. It is no longer necessary to formulate the theory in terms of observables which only have values when they are measured. Hence the measurement paradox disappears, and it is possible to construct a closed universe governed by the generalized Schrödinger equation.
References

1. G.B. Folland (1989), *Harmonic Analysis in Phase Space*, (Princeton: Princeton University Press).

2. J. Bub (1997), *Interpreting the Quantum World*, (Cambridge: Cambridge University Press).

3. A.J. Leggett (1987), *Quantum Implications*, B.J. Hiley and F. David Peat, ed. Chap. 5, (London and N.Y: Rutledge).

4. J.S. Bell (1987), *Speakable and Unspeakable in Quantum Mechanics* (Cambridge: Cambridge University Press).

5. P.A.M. Dirac (1958), *The Principles of Quantum Mechanics*, p.17, (Oxford: Clarendon Press).

6. A. Perelomov (1986), *Generalized Coherent States and Their Applications*, Sec. 1.1, (Berlin: Springer-Verlag).