Abstract. The pion multiplicity distribution is widely believed to reflect the statistical aspects of $\bar{p}p$ annihilation at rest. We try to reproduce it in a grand canonical picture with explicit conservation of electric charge, isospin, total angular momentum, and the parity quantum numbers $P$, $C$, and $G$ via the projection operator formalism. Bose statistics is found to be non-negligible, particularly in fixing the interaction volume. The calculated pion multiplicity distribution for $\langle n_\pi \rangle = 5$ turns out to depend strongly on the conservation of the angular momentum and connected quantum numbers, as well as on the spin state occupation in S-wave annihilation. However, the empirical Gaussian pion multiplicity distribution cannot be reproduced. This calls in question either the statistical ansatz or the rather old data themselves.
1 Introduction

Antinucleon-nucleon annihilation is dominated by nonperturbative QCD effects, and up to now there exists no consistent description of its microscopic dynamics within QCD due to the complexity of the underlying mechanisms. It turns out, however, that a number of global features can be reproduced at least qualitatively by phenomenological approaches (for a review, see [1, 2]). In particular, statistical models in several variations have been applied to the \( \bar{p}p \) annihilation system (for a review of the early works, see [3]), and even the fireball picture has been used motivated by the Maxwell-Boltzmann like form of the pion-energy spectra [4]–[6].

Especially the pion multiplicity distribution has always been interpreted as a strong hint for a statistical behaviour of the \( \bar{p}p \) annihilation system: the data [4] seem to be fitted by a Gaussian distribution

\[
P(n_\pi) = \frac{1}{\sqrt{2\pi}D} \exp \left[ -\frac{(n_\pi - \langle n_\pi \rangle)^2}{2D^2} \right],
\]

with \( D = 0.9 \) and \( \langle n_\pi \rangle = 5 \). Fig. 1 also shows in addition a modified Poissonian distribution, which fits the data equally well,

\[
P(n_\pi) = C \cdot \frac{\lambda^{n_\pi} \cdot \exp(-\lambda)}{(n_\pi)!},
\]

where \( \lambda = 14 \) and \( a = 1.6 \). It is therefore natural to try to reproduce the empirical pion distribution by using a statistical model.

On the other hand, conservation of the characteristic quantum numbers of the \( \bar{p}p \) system, like isospin [7]–[9] or total angular momentum [10]–[13], are known to play an important role and to severely constrain the statistical phase space. In this paper we address the question how these conservation laws influence the pion multiplicity distribution in \( \bar{p}p \) annihilation at rest. Comparison of our calculations with the data is hampered by a serious experimental problem: the occupation of the various spin-states \( J^{PC} \) of the protium atom just before its annihilation is apparently not distributed in a statistical way, but depends on the density of the hydrogen target and cannot be experimentally controlled up to now [14, 15]. S-wave annihilation occurs from either a spin singlet (\( J = 0 \)) or a spin triplet (\( J = 1 \)) state, but with unknown relative occupation. Therefore, we here study systematically the gross features of S-wave \( \bar{p}p \) annihilation (e.g. the pion multiplicity distribution) as a function of the ratio

\[
X_r = \frac{W(0^{+-})}{W(1^{--})} \quad \left( \text{where} \quad W(0^{+-}) + W(1^{--}) = 1 \right),
\]
between the spin singlet and triplet occupation probabilities.

In our approach, we will use a statistical model in connection with the projection operator formalism which allows for the explicit conservation of isospin as well as of the so-called external quantum numbers, i.e. the angular momentum and $P$, $C$, and $G$ parities. Instead of the usual Boltzmann approximation, Bose statistics is employed (for reasons that will be discussed) and all mesons and resonances up to the $a_1(1260)$ are included in our calculations. At a fixed average pion multiplicity of $\langle n_\pi \rangle = 5$ we discuss the dependence of the volume parameter on temperature and then present the pion multiplicity distributions for several values of $X_r$.

2 The Model

It is rather easy to implement baryon number and strangeness conservation into the statistical approach: one builds the total partition function from multi-mesonic states only, each of which has vanishing strangeness $S = 0$. Conservation of the electric charge $Q$ is included automatically once the third component of the isospin is conserved, i.e. $I_3 = 0$, and the constraining effects of $P$, $C$, and $G$ parity conservation can easily be taken into account if the angular momentum is fixed, which together with the total spin is subject to the conservation of the total angular momentum of the system. So, it will be sufficient to set up a procedure which allows for the projection of the total partition function onto the isospin quantum numbers $(I, I_3)$ and the angular momentum $L$.

Figure 1: Pion distribution of $\bar{p}p$ annihilation at rest. The empirical data (histogram) \cite{4} are compared with a Gaussian distribution (full line) and a modified Poissonian (dashed line).
A convenient and, above all, consistent way of treating conservation of both Abelian and non-Abelian quantum numbers in the context of a statistical model is provided by the projection operator formalism. The simplest procedure would be to calculate the grand canonical partition function \( Z_{gc} \) of the produced mesons, which are supposed to behave like free particles in a spherical “box” while the interaction between these mesons is accounted for by resonance production \([16, 17]\). The unphysical multi-meson states which are contained in \( Z_{gc} \) due to quantum number fluctuations then have to be projected out by suitable projection operators. This is a well established method \([18] - [21]\) which has been mostly applied to internal quantum numbers like \( U(1) \)-baryon number or -strangeness and \( SU(2) \)-isospin, but can also be extended to treat “external” quantum number conservation like that of angular momentum \([13]\). Following this line, the trick is not to calculate \( Z_{gc} \) itself but a so-called generating function \( \tilde{Z} \). In the case of isospin and angular momentum conservation (AMC), the generating function can be written as

\[
\tilde{Z}(T, V, \tilde{\alpha}_I, \tilde{\alpha}_L) = \text{tr} \left[ \exp \left( -\beta \hat{H} + i\tilde{\alpha}_I \cdot \vec{I} + i\tilde{\alpha}_L \cdot \vec{L} \right) \right],
\]

with the symmetry group parameters \( \tilde{\alpha}_I \) and \( \tilde{\alpha}_L \) for the \( SU(2) \)-isospin and \( SO(3) \)-rotation group, respectively. It can be easily shown, that this generating function can be expressed in an alternative way, namely

\[
\tilde{Z}(T, V, \tilde{\alpha}_I, \tilde{\alpha}_L) = \sum_{I, L} Z_{I, L}(T, V) \frac{Z_{I, L}(T, V)}{(2I + 1)(2L + 1)} \cdot \chi_I(\tilde{\alpha}_I) \cdot \chi_L(\tilde{\alpha}_L).
\]

Here, \( \chi_I \) and \( \chi_L \) are the group characters \([22]\), and the summation has to be done over all irreducible representations of the symmetry groups under consideration. \( Z_{I, L} \) is a constrained partition function which contains only those multi-meson states which are eigenstates of \( \hat{I}^2 \) and \( \hat{L}^2 \) with eigenvalues \( I(I + 1) \) and \( L(L + 1) \) and which is the actual quantity in demand. In order to project \( \tilde{Z} \) on \( Z_{I, L} \), the orthogonality relation of the group characters has to be wrapped up into a corresponding projection operator.

Let us now write down some explicit expressions for the general case of \( r \) different particle species being indexed by \( j = 1, \ldots, r \). In particular, the particle species are distinguished by their isospin \( t \), third component \( t_3 \) and mass, i.e. the index \( j \) is only a short notation for \( (t, t_3, m) \). Furthermore, two particles are considered to be identical if they correspond in \( j, l \) and their one particle energy \( \epsilon(k) \). Because the geometry of the quantum gas resulting from \( \bar{p}p \) annihilation at rest is supposed to be approximately spherically symmetric, the projection onto angular momentum eigenstates should not prefer a certain direction. Hence, no projection on the third component of the angular momentum is needed, and for this reason we are allowed to replace \( \hat{L} \) by \( \hat{L}_3 \) \([22]\) in (3).
In order to get the generating function of the system, the trace in equation (3) is worked out with eigenstates of $(\hat{I}_2^2, \hat{I}_3)$ and $(\hat{L}_2^2, \hat{L}_3)$, and the occupation number representation is employed (for details, see [13]):

$$\tilde{Z}(T, V, \vec{\alpha}, \omega) = \text{tr} \left[ \exp(-\beta \hat{H} + i\vec{\alpha} \vec{I} + i\omega \hat{L}_3) \right]$$

$$= \prod_{j=1}^{r} \prod_{l=0}^{\infty} \prod_{k_l=0}^{\infty} \left[ (D_{l,\ell,\ell_3}^j(\vec{\alpha}))^{n_{j,l,k_l}} \cdot (\chi_l^{j}(\omega))^{n_{\ell,l,k_l}} \cdot \exp\left(\frac{-\beta \epsilon_{k_l}^j}{n_{j,l,k_l}}\right) \right] , \quad (5)$$

where $\beta = 1/T$ and $\epsilon_{k_l}^j$ is the relativistic energy of a particle of type $j$. The occupation numbers $n_{j,l,k_l}$ denote the number of particles of type $j$ in a state with angular momentum $l$ and magnitude of momentum $k_l$. Note, that the momentum $k$ depends on $l$ and the interaction volume $V$, because the radial component of the angular momentum eigenfunction is a spherical bessel function, $j_l(kr)$, which must vanish at the edge of the spherical “box”.

Usually, at this stage the Boltzmann approximation is invoked. The argument is [3, 13] that Bose statistics makes at most only a ten percent correction in the case of the one pion partition function, and the correction decreases rapidly with increasing particle mass. However, we will show that when going to multi-mesonic states then the error resulting from the Boltzmann approximation can be much larger and Bose statistics must be taken into account. Thus, after some simple transformations and proceeding with Bose statistics, equation (3) can be rewritten in the following form:

$$\tilde{Z}(T, V, \vec{\alpha}, \omega) = \exp \left[ \sum_{n=1}^{\infty} \sum_{j=1}^{r} \frac{1}{n} (D_{l,\ell,\ell_3}^j(\vec{\alpha}))^{n} \cdot \sum_{l=0}^{\infty} z_{n,l}^{j}(T, V) \cdot (\chi_l^{j}(\omega))^{n} \right]$$

$$\quad \quad \quad \quad (6)$$

Due to Bose statistics, more than one particle of the same type can occupy a single level characterized by its angular momentum eigenvalue $l$ and its energy $\epsilon(k_l)$. In the above expression, the partition function for $n$ particles of type $j$, each with the same angular momentum $l$, is given by

$$z_{n,l}^{j}(T, V) \equiv \sum_{k_l=0}^{\infty} \exp\left(-\beta \epsilon_{k_l}^j \cdot n\right) , \quad (7)$$

where the sum is over all energy eigenvalues for the given value of $l$ (obtained as zeroes of the spherical Bessel function, see above). The classical Boltzmann approximation is contained in (3) by considering only the term $n = 1$.

In general, the form of the projection operators depends on the choice of the symmetry group parameters $\vec{\alpha}_I$ and $\vec{\alpha}_L$. For convenience we choose for $SU(2)$-isospin the Euler angles...
(α, β, γ) and for $SO(3)$-angular momentum the unit vector $\vec{n}$ of the rotation axis together with the rotation angle $\omega$. Then, we can write for the isospin projector

$$\hat{P}_{I, I_3 = 0} = \frac{2I + 1}{8\pi^2} \int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \int_0^{2\pi} d\gamma \sin \beta P_I(cos \beta),$$

(8)

where the group character $\chi_I$ was replaced by the matrix element $D^I_{0,0}(\alpha, \beta, \gamma) = P_I(cos(\beta))$, because we want to project onto $(I, I_3 = 0)$ rather than only $I$. In the case of AMC we get

$$\hat{P}_L = \frac{2L + 1}{2\pi^2} \int_0^{\pi} d\vec{n} \int_0^{\pi} d\omega \sin \left(\frac{\omega}{2}\right) \sin \left((2L + 1)\frac{\omega}{2}\right).$$

(9)

The only way to apply these projectors directly to $\tilde{Z}$ of equation (6) is to expand the exponential and the resulting expressions until we have a sum over all possible final multi-meson states contained in the generating function. Because of the huge amount of such states (all mesons up to the $a_1(1260)$ are included in the calculations), the evaluation is done by the computer. In particular the partition functions $z^j_{n,l}(T, V)$ are evaluated numerically. Furthermore, the evaluation remains treatable only up to a total number of $N = 7$ mesons. This cut-off in the partition function can be well justified by looking at the data: the contribution of channels with $N > 7$ is negligible.

3 Average Pion Multiplicity and Multiplicity Distribution

In a first step, we want to adjust the model parameters $T$ and $V$ to the empirical pion expectation value $\langle n_\pi \rangle = 5$. In addition to the multiplicity of the directly produced pions, one must also calculate the multiplicities of all the resonances in the system which then decay according to [23] into “secondary” pions.

In order to extract multiplicities from the total partition function of the system $Z(T, V, B, S, I, \ldots)$, we introduce for every particle type $j$ a chemical potential $\mu_j$, and every partition function $z^j_{n,l}$ has to be multiplied by the factor

$$\langle \lambda^j \rangle^n = \exp \beta \mu_j \cdot n .$$

(10)

Then, the particle multiplicities result from the derivative of the logarithm of the total partition function with respect to the fugacities at $\lambda^j = 1$,

$$\langle N_j \rangle = \partial_{\lambda^j} \ln \left[Z(T, V, B, S, I, \ldots, \lambda^j, \ldots)\right]_{\lambda^j = 1} .$$

(11)
In Fig. 2 the interaction volume of the $\bar{p}p$ annihilation system is plotted against the temperature of the system for a fixed value of $\langle n_\pi \rangle = 5$. The dashed lines correspond to $(I, I_3)$ conservation only. Here, the deviation of Boltzmann statistics in curve (a) from Bose statistics (b) is not very significant. However, if AMC and the conservation of the parity quantum numbers are included in our calculations, then the difference is striking: Bose statistics (d) allows for a rather small and therefore much more realistic volume in a temperature range of $T = [140 \ldots 200]$ MeV whereas the Boltzmann case (c) yields a reasonable volume only for very high temperatures of $T \gtrsim 200$ MeV. This is a significant phenomenological improvement compared to previous studies which were plagued by unrealistically large fireball volumes [3, 4]. This effect is obviously an intricate combination of both Bose statistics and conservation of external quantum numbers: the Boltzmann curves (a) and (c) show opposite behaviour, compared to the Bose curves (b) and (d), when going from the dashed to the full lines, i.e when including AMC and conservation of the parity quantum numbers. Note, that in (c) and (d) we did not distinguish between the protonium spin states, because in our calculations the volume is nearly the same (up to 5 percent) for $0^{-+}$ and $1^{--}$. This means, that in our model the volume is independent of the spin-state ratio $X_r$.

Once the parameters $T$ and $V$ are known, the pion multiplicity distribution can be calculated in a second step. To this end we use the following trick: After expanding the projected partition function into a sum over all the allowed multi-meson channels, we
multiply each channel by

\[ 1 = \sum_{n=0}^{\infty} p(n_\pi) , \]  

(12)

where \( p(n_\pi) \) is the probability (calculated from the data tables [23]) that, after all resonances have decayed, this particular channel yields \( n_\pi \) final pions. Then, in order to be able to project out a particular final state with \( n_\pi \) pions, we multiply each term in the sum on the right hand side of eq. (12) with a Lagrange multiplier \( f(n_\pi) \). Explicitly, we make the replacement

\[ z_{n_1,l_1}^{j_1} \cdots z_{n_r,l_r}^{j_r} \mapsto \sum_{n_\pi=0}^{\infty} z_{n_1,l_1}^{j_1} \cdots z_{n_r,l_r}^{j_r} \cdot p_{n_1,\ldots,n_r}^{j_1,\ldots,j_r}(n_\pi) \cdot f(n_\pi) , \]  

(13)

where \( p_{n_1,\ldots,n_r}^{j_1,\ldots,j_r}(n_\pi) \) is the probability for having \( n_\pi \) final pions produced by a channel containing \( n_1 \) resonances of type \( j_1 \), \( n_2 \) resonances of type \( j_2 \), etc. If the logarithm of the so prepared partition function is derived with respect to \( f(n_\pi) \) (setting \( f(n_\pi) = 1 \) afterwards), we get the total probability of having \( n_\pi \) pions in the final state as

\[ P(n_\pi) = \partial f(n_\pi) \ln \left[ Z(T,V,B,S,I,\ldots,f(n_\pi),\ldots) \right]_{f(n_\pi)=1} . \]  

(14)

The calculated pion distribution for the simplest case of isospin conservation only in the Boltzmann approximation is shown in Fig. 3 for two extreme temperatures, \( T = 100 \text{ MeV} \) and \( T = 200 \text{ MeV} \). The main effect of high temperatures is to give the high mass resonances more weight in the partition function, and the resulting pion distribution should be broader than at low temperatures. Especially for Boltzmann statistics, however, this effect seems to be rather small, see Figure 3. The Bose case is a little bit more sensitive to the temperature, although the effect is also not very drastic (Fig. 4). The shapes of both distributions resemble that of a Poissonian, as it should be in a grand canonical ensemble [24], but this Poissonian is slightly modified by resonance decays and isospin conservation. Compared with the empirical distribution of Fig. 4, all the curves of Figs. 3 and 4 are much too broad. This means that the constraint on the partition function originating from isospin conservation alone is too weak, and that there are still too many open channels broadening the pion distribution.

Now, by additional conservation of “external” quantum numbers a stronger constraint is put on the number of possible multi-meson channels. Due to the weak temperature dependence of the distribution and the large numerical effort, we have restricted the calculations to a single temperature value of \( T = 160 \text{ MeV} \). In order to account for the unknown ratio \( X_r \) of the spin-state occupation of protonium just before annihilation, we have plotted the pion distribution for several values of \( W(0^{+-}) \) in Fig. 4, using Bose
Figure 3: Pion multiplicity distribution with $\langle n_\pi \rangle = 5$, for $T = 100$ MeV and $T = 200$ MeV. The Boltzmann approximation is used and only the isospin and its third component are conserved.

Figure 4: Pion multiplicity distribution with $\langle n_\pi \rangle = 5$, for $T = 100$ MeV and $T = 200$ MeV with Bose statistics. Only isospin $I$ and its third component $I_3$ are conserved.
Figure 5: Pion multiplicity distributions with $\langle n_\pi \rangle = 5$, for $T = 160$ MeV using Bose statistics. In addition to the isospin conservation, also AMC and conservation of $P$, $C$, and $G$ parity are included. Six different values for the spin-state occupation of the protonium are taken into account, which are characterized by (a) $W(0^{-+}) = 0.0$, (b) $W(0^{-+}) = 0.2$, (c) $W(0^{-+}) = 0.4$, (d) $W(0^{-+}) = 0.6$, (e) $W(0^{-+}) = 0.8$, and (f) $W(0^{-+}) = 1.0$. 
statistics. The behaviour of the calculated pion distribution with increasing $W(0^{-+})$ can be summarized as follows. Up to a value of $W(0^{-+}) = 0.6$ (Figs. 5a to 5d), the even numbered $n_\pi$ states dominate strongly, especially the $n_\pi = 4$ and $n_\pi = 6$ states. For $W(0^{-+}) = 0.8$ (Fig. 5e) the states with $n_\pi = 3$ up to $n_\pi = 7$ are rather equally distributed. Finally, in the limit of exclusive occupation of the $0^{-+}$ state, the odd numbered states with $n_\pi = 3, 5,$ and $7$ dominate the distribution, see Figure 5f. For statistical occupation of the spin states according to their combinatoric weight, $W(0^{-+}) = 0.25$, we would have a result close to that of Fig. 5b. All in all, the obvious conclusion is that generally it is very difficult to get a Poissonian or even a Gaussian distribution when the external quantum numbers are conserved. This complete destruction of a smooth distribution by the conservation of external quantum numbers is clearly a consequence of the large fraction of directly produced pions in the partition function, which overwhelms the indirect production through resonance decays due to the larger phase space. We want to emphasize here that this is not an artefact of our grand canonical approach, but should also be observed in microcanonical calculations.

4 Conclusions

We have shown that, in the framework of a statistical model, Bose statistics and AMC together with conservation of the parity quantum numbers are crucial ingredients in the description of the multi-meson channels resulting from $\bar{p}p$ annihilation at rest. In such an approach, the projection operator formalism appears to be a convenient and consistent method in order to implement the conservation of non-Abelian quantum numbers.

Isospin conservation alone yields a smooth but much too broad pion multiplicity distribution, nearly independently of the temperature $T$, and the interaction volume $V$ is unphysically big. Only Bose statistics together with conservation of external quantum numbers provide a reasonable interaction volume of $V = [5\ldots20]$ fm$^3$ in the range of $T = [140\ldots200]$ MeV. The resulting pion distribution, however, shows no resemblance to a Gaussian or Poissonian: either the even or odd numbered $n_\pi$ states are strongly dominating. Thus, even in a statistical approach, once the effects of quantum number conservation are taken into account, the appearance of a smooth, Gaussian multiplicity distribution remains a mystery. That microscopic dynamical effects from the underlying QCD mechanism of annihilation should restore the apparent statistical nature of the empirical multiplicity distribution, is hard to believe.
In light of this dilemma, we suggest that the solution might be buried in the inclusive pion data themselves: the available bubble chamber data are rather old, and several experiments with probably different target densities have been exploited. For this reason, it would be desirable to make a new measurement of the pion multiplicity distribution at various fixed values of the hydrogen density under stable conditions.

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