Energy States of Colored Particle in a Chromomagnetic Field

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Abstract

The unitary transformation, which diagonalizes squared Dirac equation in a constant chromomagnetic field is found. Applying this transformation, we find the eigenfunctions of diagonalized Hamiltonian, that describe the states with definite value of energy and call them energy states. It is pointed out that, the energy states are determined by the color interaction term of the particle with the background chromofield and this term is responsible for the splitting of the energy spectrum. We construct supercharge operators for the diagonal Hamiltonian, that ensure the superpartner property of the energy states.

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Introduction

Classical color field configurations are important for the study of theoretical problems of non-abelian charged particles. These problems are connected with the QCD problems [1-3, 11-17] and their study gives a valuable information on different effects in QCD.

Because color magnetic fields have special significance for the vacuum state of QCD [2,1], in Refs. [4] and [5] it was considered motion of colored particle in such fields. For giving a constant and homogenous color background it was applied the constant vector potentials introduced in [6] and the squared Dirac equation was solved for this problem. Similarity of this motion to the motion of an electron in an ordinary magnetic field is that in both cases we have circular orbits for the motion in the uniform (chromo)magnetic field and well-known $s p d f \ldots$ orbitals in a background field having spherical symmetry. But, the energy spectrum in the non-abelian problem differs from the one in the abelian case and in quantized spectrum case it does not look like Landau levels. This is connected with the matrix structure of the color interaction and color matrix structure of the Quantum Mechanics of non-abelian charged particles entirely.

In addition to non-diagonal spin matrices the Dirac equation for these particles contains the non-diagonal color matrices. These matrices mix different color and spin components of the wave function in the equations for these components, which are obtained from the squared Dirac equation. As a result of such mixing, we cannot write an eigenvalue equation separately for each color state and for each spin state in general case of the background field. Therefore, we are not able to correspond the known wave functions to the spectrum and to determine thus the energy of each state with definite value of the spin and color spin. We have not as well a wave function for description of the states with definite energy. The way out of this situation is to diagonalize the squared Dirac operator in the color spin or combined spin-color spin spaces. For this we should find an unitary transformation diagonalizing this operator. The wave function of the particle, as a matrix in these spaces, also will transform under this transformation. New transformed states will obey the eigenvalue equation for the diagonalized squared Dirac operator. We aim to find this transformation and by means of it to construct the states -eigenfunctions of the diagonalized squared Dirac equation for the considered case of the constant chromomagnetic background.

Another property of problems in Quantum Mechanics - supersymmetry in Dirac equation was considered before for this kind of chromomagnetic background field as well [13, 12, 27]. It is reasonable to reconsider the supersymmetry for the diagonalized Hamiltonian and to construct the supercharge operators for the diagonal representation. Knowing the states with definite energy gives us possibility to study a question of superpartner states in this supersymmetry and treat thus supersymmetry as a origin of spectrum degeneration. We suppose, the new supercharge operators will ensure the superpartnership of the energy states.

1 Axial chromomagnetic field

The Dirac equation for a colored particle in an external color field is obtained from one for a free particle by the momentum shift:
\[ (\gamma^\mu P_\mu - M) \psi = 0, \]  
(1.1)

where \( P_\mu = p_\mu + g A_\mu = p_\mu + g A^a_\mu \lambda^a / 2 \); the \( \lambda^a \) are Gell-Mann matrices describing particle’s color spin and within the \( SU_c(3) \) color symmetry group, \( g \) is the color interaction constant and the color index \( a \) runs \( a = 1, 8 \). Written for the Majorana spinors \( \phi \) and \( \chi \) the equation (1.1) has the well-known form

\[ (\sigma^i P_i)^2 \psi = \left( -\frac{\partial^2}{\partial t^2} + M^2 \right) \psi, \]  
(1.2)

where the Pauli matrices \( \sigma^i \) describe the particle’s spin. Hereafter \( \psi \) means \( \phi \) or \( \chi \). The spinors \( \phi \) and \( \chi \) have two components corresponding to the two spin states of a particle \( \psi = \left( \psi_+ \right) \). Each component of \( \psi \) transforms under the fundamental representation of the color group \( SU_c(3) \) and has three color components describing color states of a particle corresponding to the three eigenvalues of color spin \( \lambda^3 \)

\[ \psi_\pm = \begin{pmatrix} \psi_\pm (\lambda^3 = +1) \\ \psi_\pm (\lambda^3 = -1) \\ \psi_\pm (\lambda^3 = 0) \end{pmatrix} \]  
(1.3)

We are going to continue the study of the motion in the chromomagnetic field started in [4], where we applied the constant vector potentials introduced in [6]. Remind here, that for giving an axial chromomagnetic field by the constant vector potential, components of last one are choosen as below

\[ A_1^a = \sqrt{\tau} \delta_{1a}, \quad A_2^a = \sqrt{\tau} \delta_{2a}, \quad A_3^a = 0, \quad A_0^a = 0, \]  
(1.4)

where \( \tau \) is a constant and \( \delta_{\mu a} \) is the Kronecker symbol.

For this \( A_\mu^a \) all components of the field strength tenzor \( F_{\mu\nu}^c = g f^{abc} A_\mu^a A_\nu^b \) are equal to zero, except for

\[ F_{12}^3 = g \tau = H_z, \]  
(1.5)

and (1.4) gives a constant chromomagnetic field directed along the third axes of the ordinary and color spaces. Here \( f^{abc} \) are the structure constants of the \( SU_c(3) \) group.

Setting (1.4) and \( i \partial \psi / \partial t = E \psi \) in (1.2) we obtain the following two equations for the \( \psi_\pm \), which differ from each other by the sign of the last term in the left hand side:

\[ \left[ p^2 + \frac{1}{2} g^2 \tau I_2 + g \tau^{1/2} \left( p_1 \lambda^1 + p_2 \lambda^2 \mp \frac{1}{2} g \tau^{1/2} \lambda^3 \right) \right] \psi_\pm = (E^2 - M^2) \psi_\pm. \]  
(1.6)

Here \( I_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) is the color matrix. Because field (1.5) is directed along the third axis, \( \psi_\pm \) describe the spin states with the up and down projections of \( \sigma^3 \) and Hamiltonians defined as \( H_\pm \psi_\pm = E^2 \psi_\pm \) correspond to these spin states. Because of non-diagonal \( \lambda^1 \) and
\( \lambda^2 \) matrices \( H_\pm \) have not diagonal color structure and so, we cannot write the eigenvalue equation

\[
H_\pm \psi_\pm^{(i)} = E^2 \psi_\pm^{(i)}
\]

for pure color states \( \psi_\pm^{(i)} \) except for the states \( \psi_\pm^{(3)} \). The explicit matrix form of the general Hamiltonian in the combined color and spin spaces is:

\[
H = \begin{pmatrix}
\mathcal{P}^2 & G p_- & 0 & 0 & 0 & 0 \\
G p_+ & \mathcal{P}^2 + G^2 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathcal{P}^2 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathcal{P}^2 + G^2 & G p_- & 0 \\
0 & 0 & 0 & G p_+ & \mathcal{P}^2 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathcal{P}^2 \\
\end{pmatrix}.
\tag{1.7}
\]

For brevity we have introduced notations \( \mathcal{P}^2 = \mathcal{P}^2 + M^2, \mathcal{G} = g \tau^{1/2}, p_\pm = p_1 \pm ip_2 \). Non-diagonality of (1.7) leads to mixing of the different color states \( \psi_\pm^{(i)} \) in the differential equations obtained from (1.6). But equation for the pure states \( \psi_\pm^{(1),(2)} \) obtained from the (1.7) has the same form for all the color and spin states \([4]\). In the cylindrical coordinates the common equation for \( \psi_\pm^{(1),(2)} \) has a solution expressed by the Bessel function \( J_m(x) \):

\[
\psi_\pm^{(i)}(r) = \sum_{m=-\infty}^{+\infty} \frac{1}{2\pi} e^{im\varphi} \exp(ip_3z) J_m(p_\perp r) \xi_\pm^{(i)}.
\tag{1.8}
\]

Here \( m \) is the chromomagnetic quantum number and \( \xi_\pm^{(i)} \) includes the spin and color spin parts of wave function. Similarly to the spin part of wave function, \( \xi_\pm^{(i)} \) can be chosen as following:

\[
\xi_+^{(i)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_-^{(i)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

\[
\zeta^{(1)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \zeta^{(2)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \zeta^{(3)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Such chosen \( \xi \) obeys the normalizing condition below:

\[
|\xi_+^{(i)}|^2 + |\xi_-^{(i)}|^2 = |\zeta^{(1)}|^2 + |\zeta^{(2)}|^2 + |\zeta^{(3)}|^2 = 1.
\]

Let us remind, that inspite of the same \( r \)-dependence of all these states, the different \( \psi_\pm^{(i)}(r) \) transforms differently under transformations in the spin and color spaces. This solution is similar to one for relativistic motion of electron in an axial magnetic field \([7]\) and in a classical picture gives motion on a circular orbits \([4]\). The energy spectrum for (1.6) was found by solving determinant equation obtained from it and has three continous branches in the case of infinite motion \([10-11]\):

\[
E_{1,2}^2 = (\mathcal{P}_\perp + G/2)^2 + p_3^2 + M^2, \quad E_3^2 = \mathcal{P}^2,
\tag{1.9}
\]

where \( \mathcal{P}_\perp = \sqrt{p_\perp^2 + G^2/4}, p_\perp^2 = p_1^2 + p_2^2 \). Equation for the state \( \psi_\pm^{(3)}(r) \) is the equation for the free particle \( \mathcal{P}^2 \psi_\pm^{(3)}(r) = (\mathcal{P}^2 + M^2) \psi_\pm^{(3)}(r) \) and solution (1.8) can be regarded to it as well.
For the motion limited by cylinder with the radius \( r_0 \), the quantized spectrum has the form [4]:

\[
\left( E^{(N)}_m \right)_{1,2}^2 = \left( \sqrt{\left( \frac{\alpha_m^{(N)}}{r_0} \right)^2 + \frac{G^2}{4} + \frac{G}{2}} \right)^2 + p_3^2 + M^2, \quad \left( E^{(N)}_m \right)_{3}^2 = \left( \frac{\alpha_m^{(N)}}{r_0} \right)^2 + p_3^2 + M^2.
\]

Here \( \alpha_m^{(N)} \) are Bessel function’s zeros and \( N \) is the radial quantum number. The finite motion solution is connected with one for the infinite motion by the replacement \( p_\perp = \alpha_m^{(N)}/r_0 \) in (1.8) and for the wave function we do not sum over the \( m \) inasmuch as quantized energy levels are determined by this quantum number:

\[
\psi_{\pm}^{(i)}(r) = \frac{1}{2\pi} e^{im\phi} \sin \left( p_3 z \right) J_m \left( \frac{\alpha_m^{(N)}}{r_0} \xi_{\pm}^{(i)} \right).
\]

Hereafter, under \( \mathbf{p} \) and under spectrum \( E_1^2 \) we shall mean their both continuous and quantized values. The spectrum found in such a way, is not determined by the value of the projection of the color spin operator \( \lambda^3 \) onto the field (1.5). This means, that states \( \psi_{\pm}^{(1),(2)} \) have not get energy from the branches \( E_1 \) or \( E_2 \) definitly. They can get energy from the both branches of the energy spectrum (1.9), but with the different probability. It arise question: what are the wave functions of states having definite energy from the branch \( E_1 \) or from the branch \( E_2 \)? It is clear that, for such states we would be able to write the eigenvalue equation with the spectrum branches (1.9) and this equation can be written only for the diagonal matrix form of the Hamiltonian. So, we need the diagonal form of the Hamiltonian (1.7) in the combined spin-color spin space, in order to write eigenvalue equation by it and then to determine its eigenfunctions. Since Hamiltonian is the hermitian matrix, it has diagonal form in the basis of its eigenfunctions and this diagonal form is unique. We can find the diagonal form of \( H' \) for the Hamiltonian (1.7) and then its diagonal elements will correspond to the spectrum branches from (1.9). By \( H' \) we will be able to write an eigenvalue equation with the eigenvalues from (1.9):

\[
H'\Psi' = E_k^2\Psi'.
\]

Here \( \Psi' \) is the eigenfunction of \( H' \), which is different from \( \psi_{\pm}^{(1),(2)} \). In order to get the diagonal \( H' \) from the non-diagonal Hamiltonian \( H \) (1.7), we should make some transformation \( U \) in the combined spin-color spin space. The wave functions \( \psi_{\pm}^{(i)} \) will be transformed on this transformation as well. More precisely, the Hamiltonian (1.7) will get the diagonal form under some \( U \) transformation of the basis vectors of the combined spin-color spin space. The basis vectors in the combined space are the solutions \( \psi_{\pm}^{(i)} \) (their \( \xi_{\pm}^{(i)} \) part), since the eigenfunctions of the spin and color spin operators \( \sigma^i \) and \( \lambda^a \) are these functions. Under \( U \) transformation of the combined space, the basis vectors, i.e. \( \psi_{\pm}^{(i)} \),
transform according to the rule:

\[ \Psi' = U \Psi, \quad \Psi = \begin{pmatrix} \psi^{(1)}_+ \\ \psi^{(2)}_+ \\ \psi^{(3)}_+ \\ \psi^{(1)}_- \\ \psi^{(2)}_- \\ \psi^{(3)}_- \end{pmatrix}. \tag{1.11} \]

Components of the new basis vector \( \Psi' \) will not be the states with the definite value of projection of spin or color spin, and will be some superposition of \( \psi^{(i)}_\pm \). As we stated above, if \( H' \) is diagonal, then basis vectors of this space are the eigenvectors of this Hamiltonian. So, these components are the states with the definite value of the energy, because they are eigenfunctions of \( H' \) with the eigenvalue \( E_k^2 \) from (1.9). We call the components of \( \Psi' \) the energy states because of describing the states having definite energy from the branches \( E_k \). Since \( H' \) is unique, the transformation matrix \( U \) and the basis vector \( \Psi' \) are unique as well.

Under transformation (1.11) the Hamiltonian (1.7), as any matrix in this space, transforms by similarity transformation:

\[ U^{-1} H U = H'. \tag{1.12} \]

The difficulty of determining of \( H' \) is that, we have no explicit form of either \( U \) or \( H' \); two of three matrices in (1.12) are unknown. But fortunately, it turns out possible to find both of these matrices, relying on their properties. Firstly, from the hermicity of \( H' \) we can conclude that, the transformation matrix \( U \) in addition to uniqueness, should be unitary:

\[ H'^\dagger = U^\dagger H^\dagger U^{-1\dagger} = H' = U^{-1\dagger} H U, \]

\[ U^{-1\dagger} = U \Rightarrow UU^\dagger = 1. \]

Equation (1.12) is the basic relation between two Hamiltonian matrices \( H \) and \( H' \). Multiplied from the right hand side by \( U \) it has got more useful form for solving, since contains only the linear relations between the elements \( u_{ij} \) of \( U \) matrix:

\[ HU = UH'. \tag{1.13} \]

Another important point for the finding \( H' \) and \( U \) is that, the unitary transformation does not change determinant and trace of a matrix, i.e. \( H \) and \( H' \) have the same trace and the same determinant:

\[ \det H' = \det U^{-1} \det H \det U = \det H, \quad \text{Tr} H = \text{Tr} H'. \tag{1.14} \]

The matrices \( H, H' \) and \( U \) are matrices of dimensionality \( 6 \times 6 \). Due to quasidiagonal form of (1.7) we can separately consider two Hamiltonians \( H_\pm \) having dimensions \( 3 \times 3 \),

*We have ignored, the wave functions in Quantum Mechanics are defined up to phase factor. We shall run into this circumstance later.
instead of one with dimension $6 \times 6$. Of course, equations (1.13) and (1.14) will hold for both of them, but with different matrices $U_{\pm}$ correspondingly. The matrices $U_{\pm}$ belong to the color symmetry group $SU_c(3)$, so they obey unimodularity condition as well. Solving equations obtained from (1.13) jointly with the equations of unitarity and unimodularity we find the $U_{\pm}$ matrices explicitly:

$$
U_{\pm} = \left( \frac{1}{2P_{\perp}} \right)^{1/2} \begin{pmatrix}
(P_{\perp} + G/2)^{1/2} e^{i\alpha} & \pm p_- (P_{\perp} + G/2)^{-1/2} e^{-i\alpha} & 0 \\
\mp p_+ (P_{\perp} + G/2)^{-1/2} e^{i\alpha} & (P_{\perp} + G/2)^{1/2} e^{-i\alpha} & 0 \\
0 & 0 & (2P_{\perp})^{1/2}
\end{pmatrix}
$$

(1.15)

and establish the diagonal elements of $H'_{\pm}$, which correspondingly equal to

$$
h'_{11} = p^2 - GP_{\perp} + G^2/2, \quad h'_{22} = p^2 + GP_{\perp} + G^2/2, \quad h'_{33} = p^2
$$

(1.16)

and

$$
h''_{11} = p^2 + GP_{\perp} + G^2/2, \quad h''_{22} = p^2 - GP_{\perp} + G^2/2, \quad h''_{33} = p^2.
$$

(1.17)

According to (1.12) the matrix

$$
U = \begin{pmatrix}
U_+ & 0 \\
0 & U_-
\end{pmatrix}
$$

will reduce the Hamiltonian (1.7) to its following diagonal form:

$$
H' = \begin{pmatrix}
 h'_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & h'_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & h'_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & h''_{11} & 0 & 0 \\
0 & 0 & 0 & 0 & h''_{22} & 0 \\
0 & 0 & 0 & 0 & 0 & h''_{33}
\end{pmatrix}
$$

(1.18)

Inspite $U$ matrix is the transformation of 6-dimensional spin-color spin space, indeed it transforms the color space, because it has quasidiagonal form and does not mix spin indices. Comparing the explicit forms of the diagonal elements (1.16) and (1.17) with the energy branches (1.9) it can be easily established the correspondence between them

$$
h'_{22}, \quad h''_{11} \rightarrow E_1^2
\quad h'_{11}, \quad h''_{22} \rightarrow E_2^2
\quad h'_{33}, \quad h''_{33} \rightarrow E_3^2.
$$

(1.19)

According to (1.11) we construct the wave function $\Psi'$, which will be an eigenfunction of the Hamiltonian (1.18):

$$
\Psi' = U \begin{pmatrix}
\psi_1^{(1)} \\
\psi_2^{(2)} \\
\psi_3^{(3)} \\
\psi_1^{(1)} \\
\psi_2^{(2)} \\
\psi_3^{(3)}
\end{pmatrix} = \begin{pmatrix}
\psi_+^{(+)} \\
\psi_+^{(-)} \\
\psi_+^{(+)} \\
\psi_-^{(+)} \\
\psi_-^{(-)} \\
\psi_-^{(+)}
\end{pmatrix}
$$

(1.20)

\footnote{Details of this finding is described in Ref. [29] or in the next section for the case considered there.}
Due to the correspondence (1.19) the desired eigenvalue equation for \( H' \) and \( \Psi' \) (1.10) can be written in the following explicit form:

\[
\begin{pmatrix}
  h'_{11} & 0 & 0 & 0 & 0 & 0 \\
  0 & h'_{22} & 0 & 0 & 0 & 0 \\
  0 & 0 & h'_{33} & 0 & 0 & 0 \\
  0 & 0 & 0 & h''_{11} & 0 & 0 \\
  0 & 0 & 0 & 0 & h''_{22} & 0 \\
  0 & 0 & 0 & 0 & 0 & h''_{33}
\end{pmatrix}
\begin{pmatrix}
  \psi_+^{(\pm)} \\
  \psi_-^{(\pm)} \\
  \psi_+^{(3)} \\
  \psi_-^{(3)} \\
  \psi_+^{(-)} \\
  \psi_-^{(-)}
\end{pmatrix}
= \begin{pmatrix}
  E_2^2 \psi_+^{(\pm)} \\
  E_2^2 \psi_-^{(\pm)} \\
  E_3^2 \psi_+^{(3)} \\
  E_3^2 \psi_-^{(3)} \\
  E_2^2 \psi_+^{(-)} \\
  E_2^2 \psi_-^{(-)}
\end{pmatrix}.
\]

The correspondence between the energy branches \( E_k \) and the components \( \psi_\pm^{(\pm)} \), which we have called the energy states, can be found from the right hand side of the last equality:

\[
\begin{align*}
E_1^2 & \rightarrow \psi_+^{(+)} , \; \psi_-^{(+)} \\
E_2^2 & \rightarrow \psi_+^{(-)} , \; \psi_-^{(-)} . \\
E_3^2 & \rightarrow \psi_+^{(3)} , \; \psi_-^{(3)} .
\end{align*}
\] (1.21)

Having set in (1.20) the explicit form of the matrix \( \mathcal{U} \) we find the wave functions of states having energy \( E_k \) as a superposition of spin-color spin states \( \psi_\pm^{(a)} \) of non-transformed space:

\[
\psi_\pm^{(+)} = u_{11} \psi_\pm^{(1)} + u_{12} \psi_\pm^{(2)} = (2 \hat{\mathcal{P}}_\perp)^{-1/2} \left( \hat{\mathcal{P}}_\perp + \mathcal{G}/2 \right)^{1/2} e^{i\alpha} \psi_\pm^{(1)}
+ \hat{\mathcal{P}}_- (2 \hat{\mathcal{P}}_\perp)^{-1/2} \left( \hat{\mathcal{P}}_\perp + \mathcal{G}/2 \right)^{-1/2} e^{-i\alpha} \psi_\pm^{(2)},
\]

\[
\psi_\pm^{(-)} = \psi_-^{(+)} = u_{21} \psi_\pm^{(1)} + u_{22} \psi_\pm^{(2)} = \mp \hat{\mathcal{P}}_+ (2 \hat{\mathcal{P}}_\perp)^{-1/2} \left( \hat{\mathcal{P}}_\perp + \mathcal{G}/2 \right)^{-1/2} e^{i\alpha} \psi_\pm^{(1)}
+ (2 \hat{\mathcal{P}}_\perp)^{-1/2} \left( \hat{\mathcal{P}}_\perp + \mathcal{G}/2 \right)^{1/2} e^{-i\alpha} \psi_\pm^{(2)}.
\]

Having written the momentum operators \( \hat{\mathcal{P}}_\pm \) in the polar coordinates \( r \) and \( \varphi \)

\[
\hat{\mathcal{P}}_1 \pm i \hat{\mathcal{P}}_2 = -ie^{\pm i\varphi} \left( \frac{\partial}{\partial r} \pm \frac{i}{r} \frac{\partial}{\partial \varphi} \right)
\]

and applying the recurrent formula for differentiation of the Bessel functions \( J_m(x) \)

\[
J'_m(x) = \frac{m}{x} J_m(x) - J_{m+1}(x) = J_{m-1}(x) - \frac{m}{x} J_m(x)
\]

we easily establish action of these operators on Bessel functions:

\[
(\hat{\mathcal{P}}_1 \pm i \hat{\mathcal{P}}_2) J_m(p_\perp r) e^{im\varphi} = \pm ip_\perp J_{m\pm 1}(p_\perp r) e^{i(m\pm 1)\varphi},
\] (1.22)

\[
\hat{\mathcal{P}}_\perp^2 J_m(p_\perp r) e^{im\varphi} = (\hat{\mathcal{P}}_1 \pm i \hat{\mathcal{P}}_2) (\hat{\mathcal{P}}_1 \mp i \hat{\mathcal{P}}_2) J_m(p_\perp r) e^{im\varphi} = p_\perp^2 J_m(p_\perp r) e^{im\varphi},
\]

\[
\hat{\mathcal{P}}_\perp \psi_\pm^{(i)} = \sqrt{\hat{\mathcal{P}}_1^2 + g^2 r^2/4} \psi_\pm^{(i)} = \mathcal{P}_\perp \psi_\pm^{(i)}.
\]

As is seen, action of operators \( \hat{\mathcal{P}}_1 \pm i \hat{\mathcal{P}}_2 \) shift the state with \( m \) to the state with \( m \pm 1 \). This means that, in the quantized spectrum case, when the transverse momentum \( p_\perp \) gets
values determined by the chromomagnetic quantum number $m : p_\perp = \alpha_{m}^{(N)}/r_0$ and we do not sum over this quantum number, the energy states will be superposition of the states with the different values of $m$:

$$\psi_{\pm}^{(\pm)} = \frac{\sin (p_3 z)}{\sqrt{2\mathcal{P}_\perp}} e^{im\varphi} \left[ (\mathcal{P}_\perp + \mathcal{G}/2)^{1/2} e^{i\alpha} J_m (p_\perp r) \zeta_\pm^{(1)} \right. $$

$$-i\mathcal{P}_\perp (\mathcal{P}_\perp + \mathcal{G}/2)^{-1/2} e^{-i\alpha} J_{m-1} (p_\perp r) e^{-i\varphi} \zeta_\pm^{(2)}] ,$$

$$\psi_{\pm}^{(-)} = \frac{\sin (p_3 z)}{\sqrt{2\mathcal{P}_\perp}} \left[ (\mathcal{P}_\perp + \mathcal{G}/2)^{-1/2} e^{i\alpha} J_{m+1} (p_\perp r) e^{i\varphi} \zeta_\pm^{(1)} \right.$$ 

$$+ (\mathcal{P}_\perp + \mathcal{G}/2)^{1/2} e^{-i\alpha} J_m (p_\perp r) \zeta_\pm^{(2)}] .$$

In the continuous spectrum case, the transverse momentum $p_\perp$ and the spectrum does not depend on $m$ and we can sum over this quantum number in energy states, as we made in (1.8). This just leads to replacement of the momentum operators by their eigenvalues and then the energy states have got the form:

$$\psi_{\pm}^{(\pm)} = \frac{1}{\sqrt{2\mathcal{P}_\perp}} (\mathcal{P}_\perp + \mathcal{G}/2)^{1/2} e^{i\alpha} \psi_{\pm}^{(1)} + \frac{i\mathcal{P}_\perp}{\sqrt{2\mathcal{P}_\perp}} (\mathcal{P}_\perp + \mathcal{G}/2)^{-1/2} e^{-i\alpha} \psi_{\pm}^{(2)},$$

$$\psi_{\pm}^{(-)} = \mp \frac{i\mathcal{P}_\perp}{\sqrt{2\mathcal{P}_\perp}} (\mathcal{P}_\perp + \mathcal{G}/2)^{-1/2} e^{i\alpha} \psi_{\pm}^{(1)} + \frac{1}{\sqrt{2\mathcal{P}_\perp}} (\mathcal{P}_\perp + \mathcal{G}/2)^{1/2} e^{-i\alpha} \psi_{\pm}^{(2)}. \quad (1.23)$$

It can be found from (1.23) that, all energy states have wave functions with same module:

$$|\psi_{\pm}^{(\pm)}|^2 = \frac{\mathcal{P}_\perp + \mathcal{G}/2}{2\mathcal{P}_\perp} |\psi_{\pm}^{(1)}|^2 + \frac{\mathcal{P}_\perp + \mathcal{G}/2}{2\mathcal{P}_\perp} |\psi_{\pm}^{(2)}|^2$$

$$= \frac{1}{6} \left[ \sum_{m=-\infty}^{+\infty} \frac{1}{2\pi} e^{im\varphi} \exp (ip_3 z) J_m (p_\perp r) \right]^2 = F^2(r). \quad (1.24)$$

This means, that a distribution of particles on the states with energy $E_2$ or $E_1$ are same and does not depend on spin or color quantum numbers. This is a result of invariance of distribution with respect to transformation (1.11):

$$|\psi_{\pm}^{(\pm)}|^2 = |\psi_{\pm}^{(-)}|^2 = |\psi_{\pm}^{(1)}|^2 + |\psi_{\pm}^{(2)}|^2 .$$

As is known, eigenfunctions of any conserved quantity are the eigenfunctions of Hamiltonian as well. At now we wish to clear what is the conserved operator in 6-dimensional combined spin-color spin space, eigenfunctions of which are the found energy states $\psi_{\pm}^{(\pm)}$? Of course, sought operator determines the branches of spectrum and will commute with $H'$. In order to find this operator, let us divide Hamiltonian (1.7) into diagonal part $\mathbf{p}^2 + \frac{1}{2} g^2 \tau I_2 + M^2$, which does not changes under transformation (1.12), and non-diagonal part [13], [25]

$$\left( \sigma^i \mathcal{P}_i \right)^2 - \left( \mathbf{p}^2 + \frac{1}{2} g^2 \tau I_2 \right) = g\tau^{1/2} \left( p_1 \lambda^1 + p_2 \lambda^2 - \frac{1}{2} g\tau^{1/2} \sigma^3 \lambda^3 \right) . \quad (1.25)$$
which becomes diagonal under this transformation:

\[
U^{-1}g_{\tau}^{1/2} \left( p_1 \lambda^1 + p_2 \lambda^2 - \frac{1}{2} g_{\tau}^{1/2} \sigma^3 \lambda^3 \right) U = 
\]

\[
\begin{pmatrix}
U^{-1} & 0 \\
0 & U^{-1}
\end{pmatrix} \begin{pmatrix}
(\lambda^b I^b) & 0 \\
0 & (\lambda^b I^b)
\end{pmatrix} \begin{pmatrix}
U & 0 \\
0 & U
\end{pmatrix} = \begin{pmatrix}
(\lambda^b I^b)^r & 0 \\
0 & (\lambda^b I^b)^r
\end{pmatrix}.
\]

(1.26)

This non-diagonal term is the interaction term of the particle with the chromomagnetic background. Here the two operators \( (\lambda^b I^b)_{\pm} = g_{\tau}^{1/2} \left( p_1 \lambda^1 + p_2 \lambda^2 \pm \frac{1}{2} g_{\tau}^{1/2} \lambda^3 \right) \) correspond to the two different spin states of the particle and in the transformed spin-color spin space these operators has got a diagonal form:

\[
(\lambda^b I^b)^r_{\pm} = \begin{pmatrix}
-G\mathcal{P}_{\perp} & 0 & 0 \\
0 & G\mathcal{P}_{\perp} & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad (\lambda^b I^b)^r_{\mp} = \begin{pmatrix}
G\mathcal{P}_{\perp} & 0 & 0 \\
0 & -G\mathcal{P}_{\perp} & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

As is seen from (1.9) and (1.26), the term with \( \pm \) sign in the energy spectrum was appeared by this term, which can be written as the scalar product of two color vectors \( \lambda^b \) and \( I^b \):

\[
(\lambda^b I^b) = g\lambda^b \left( A^b_j p_j - \frac{g}{4} \epsilon_{ijk} f^{abc} A^a_i \sigma^3 A^c_j \right) = 2g \left( A^b_j p_j - \frac{1}{2} F_{12} \sigma^3 \right).
\]

(1.27)

Here \( F_{12} = F_3^3 \lambda^3 / 2 \). This product is the projection of the color spin operator \( \lambda^b \) onto the color vector \( I^b \) and in the rotated color space this projection has got the diagonal form with the three eigenvalues \( \pm G\mathcal{P}_{\perp}, 0 \). Three different branches of energy spectrum correspond to these three values of \( (\lambda^b I^b) \) projection. Since operator \( \sigma^3 / 2 \) commute with the general Hamiltonian (1.7) and consequently, is the conserved quantity, this operator describes the spin of the particle in the field (1.4). Two Hamiltonians \( H_{\pm} \) correspond to the two eigenvalues of this operator and each one in the diagonalized form has three different eigenvalues \( E_1, E_2, E_3 \). Three different eigenfunctions \( \psi^{(\pm),(3)}_{\pm} \) correspond to these eigenvalues of the spectrum. But the operator \( \lambda^3 \) does not commute with (1.7) and consequently, is not conserved operator inspite of the field (1.4) directed along the third axis in the color space as well. This is a reason, why \( \lambda^3 \) does not determine the branches of the energy spectrum (1.9) and the projection of this operator onto the chromomagnetic field is not suitable quantity for the description of color states. Projection of \( \overrightarrow{X} \) operator onto the color vector \( \overrightarrow{I} \), i.e. the scalar product (1.27) commute with the Hamiltonian (1.7), i.e. is a conserved quantity. As we see from this analysis, the term with \( \pm \) sign in the energy spectra (1.9) coming from the \( (\lambda^b I^b)_{\pm} \) term in the Hamiltonians (1.6) and, in fact, results to splitting of the energy spectrum into three branches, which correspond to the eigenvalues of this projection. The wave functions \( \psi^{(\pm)}_{\pm} \) describing states with the energies from these branches of spectra, are eigenfunctions of the operator (1.27) as well. Though the two different operators \( (\lambda^b I^b)_{\pm} \) and \( (\lambda^b I^b)_{\pm} \) correspond to the two different eigenvalues of the spin operator, the different eigenvalues of these operators coincide:

\[
(\lambda^b I^b)^{(+)}_{\pm} = (\lambda^b I^b)^{(-)}_{\pm} = -G\mathcal{P}_{\perp}, \quad (\lambda^b I^b)^{(3)}_{\pm} = (\lambda^b I^b)^{(3)}_{\mp} = G\mathcal{P}_{\perp}, \quad (\lambda^b I^b)^{(3)}_{\pm} = (\lambda^b I^b)^{(3)}_{\mp} = 0.
\]

Under vector here we mean eight component quantity in the color space.
Consequently, the wave functions of the states with the same spin projection, but with the different value of $(\lambda^b I^b)^r$ describe the different energy branches: $\psi_+^{(+)} \to E_2$, $\psi_+^{(-)} \to E_1$ and $\psi_-^{(+)} \to E_1$, $\psi_-^{(-)} \to E_2$ and wave functions of the states with the same projection $(\lambda^b I^b)^r$, but with the different spin projection describe the different energy branches as well: $\psi_+^{(+)} \to E_2$, $\psi_+^{(+)} \to E_1$ and $\psi_-^{(+)} \to E_1$, $\psi_-^{(-)} \to E_2$. Since this operator has three eigenvalues instead of six, it arise twofold degeneracy, which cannot be classified either spin degeneracy or degeneracy on eigenvalues of projection $(\lambda^b I^b)^r$. In the transformed space, where the operator $(\lambda^b I^b)$ has got its diagonal form:

$$
(\lambda^b I^b)^r = 
\begin{pmatrix}
-GP_\perp & 0 & 0 & 0 & 0 & 0 \\
0 & GP_\perp & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & GP_\perp & 0 & 0 \\
0 & 0 & 0 & 0 & -GP_\perp & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

(1.28)

it can be written as the product of two operators $-\sigma^3 (\lambda^b I^b)^r_+$ and easily found that degeneracy of the energy spectrum is degeneracy on this product. Having expressed the spectrum branches in terms of the projections of $\sigma^3$ and $(\lambda^b I^b)^r_+$ we can establish the values of these projections, on which these branches coincide:

$$
E \left( \sigma^3 = 1; \ (\lambda^b I^b)^r_+ = -GP_\perp \right) = E \left( \sigma^3 = -1; \ (\lambda^b I^b)^r_+ = GP_\perp \right) = E_2,
$$

$$
E \left( \sigma^3 = 1; \ (\lambda^b I^b)^r_+ = GP_\perp \right) = E \left( \sigma^3 = -1; \ (\lambda^b I^b)^r_+ = -GP_\perp \right) = E_1.
$$

Thus we conclude, that this is degeneracy on the quantity (1.27), which include spin, color spin and momentum, inspite there is no degeneracy on this quantities separately. In the result of this degeneracy the states having different quantum numbers $\sigma^3$ and $(\lambda^b I^b)^r_+$ have got the same energy: $\psi_+^{(+)}$, $\psi_-^{(-)} \to E_2$ and $\psi_-^{(+)}$, $\psi_+^{(-)} \to E_1$. After this analysis we come to idea to write the energy states as eigenfunctions of $(\lambda^b I^b)^r_\pm$ operator, introducing unit eigenvectors $\zeta^{(\pm)}$ of these operators:

$$
(\lambda^b I^b)^r_\pm \zeta^{(\pm)} = \pm GP_\perp \lambda^3 \zeta^{(\pm)} = \pm GP_\perp (\pm \zeta^{(\pm)}), \quad (\lambda^b I^b)^r_+ \zeta^{(3)} = GP_\perp \lambda^3 \zeta^{(3)} = 0,
$$

(1.29)

which are basis vectors of the transformed color space as well:

$$
\zeta^{(+)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \zeta^{(-)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \zeta^{(3)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
$$

Then, the energy states can be expressed on this basis as following:

$$
\psi^{(\pm)} = \zeta^{(\pm)} \xi F(r).
$$

(1.30)

As we noted, the term $(\lambda^b I^b)$ describes the interaction of the particle with the external field. This interaction occur by means of chromomagnetic moment of the particle due to
its spin, which is included in the last term in (1.27) and by the chromomagnetic moment acquired due to orbital moment, which we suppose, hidden in the first term of (1.27). In the abelian theory, in motion of electron in the axial magnetic field given by vector potential \( A_\mu = (0, -\frac{1}{2}yH_z, \frac{1}{2}xH_z, 0, ) \) the term \( A_j p_j \) in the Hamiltonian is written as the interaction term of the magnetic moment of electron due to orbital moment \( L_z \) with the magnetic field \( H_z \) [7]

\[
\frac{e}{m_0 c} A_j p_j = -\frac{eH_z}{2m_0 c} L_z = -\frac{eH_z}{2m_0 c} \hbar.
\]

This term eliminates the degeneracy of energy levels on magnetic quantum number \( m \) and leads to physical effect of splitting of energy spectrum levels in the magnetic field. In our non-abelian Landau problem the term \( g \lambda^b A^b_j p_j = 2g A_j p_j \) cannot be written proportional to the orbital moment. It has color matrix structure

\[
\lambda^b A^b_j p_j = G \begin{pmatrix} 0 & p_- & 0 \\ p_+ & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

and acting of the \( p_\pm \) on the wave functions \( e^{im\varphi} J_m (p_\perp r) \) is given by (1.22). In quantized spectrum case \( p_\perp = \alpha_m^{(N)}/r_0 \) and for this interaction term we have following eigenvalues:

\[
\lambda^b A^b_j p_j e^{im\varphi} J_m (p_\perp r) \begin{pmatrix} \xi_\pm^{(1)} \\
\xi_\pm^{(2)} \\
\xi_\pm^{(3)} \end{pmatrix} = G \frac{\alpha_m^{(N)}}{r_0} \begin{pmatrix} -e^{i(m-1)m\varphi} J_{m-1} (\alpha_m^{(N)} r_0) \xi_\pm^{(1)} \\
e^{i(m+1)m\varphi} J_{m+1} (\alpha_m^{(N)} r_0) \xi_\pm^{(1)} \\ 0 \end{pmatrix}.
\]

As is seen, the term \( g \lambda^b A^b_j p_j \) in Hamiltonian does not lead to appearance of the terms with factor of chromomagnetic quantum number \( m \) and shifts the state with \( m \) to the states with \( m \pm 1 \). But this term contains \( \alpha_m^{(N)} \), which depends on \( m \) and on \( N \). This means, that the term \( g \lambda^b A^b_j p_j \) splits the spectrum into series defined by \( N \) and levels in series are defined by \( m \). Let us remind, that the radii \( a_m^{(N)} \) of orbits, in which rotates the particle in this field, is also determined by \( \alpha_m^{(N)} \), i.e. by both quantum numbers \( m \) and \( N \) [4]: \( a_m^{(N)} = mr_0/\alpha_m^{(N)} \). Thus we conclude, that the interaction term \( g A_j p_j \) in non-abelian theory, as a result of the interaction of chromomagnetic moment due to the orbital moment with the chromomagnetic background, eliminates degeneracy on \( m \) and \( N \) and split the energy levels in the quantized spectrum case. Beside splitting, this interaction shifts particle from the state \( m \) to the states \( m \pm 1 \) and consequently, from the orbit \( a_m^{(N)} \) to the orbits \( a_m^{(N)} \) in dependence of projection of \( \lambda^3 \) onto the chromomagnetic field. In the transformed color space, according to (1.39) this interaction term splits spectrum into branches, and each branch contains factor \( G P_\perp = G \sqrt{p_\perp^2 + G^2/4} = G \sqrt{(\alpha_m^{(N)}/r_0)^2 + G^2/4} \), which splits spectrum branches into series.

### 2 Spherical background

Let us consider the case of constant spherical chromomagnetic field defined by the constant vector potential:

\[
A_1^a = \sqrt{\tau} \delta_{1a}, \quad A_2^a = \sqrt{\tau} \delta_{2a}, \quad A_3^a = \sqrt{\tau} \delta_{3a}, \quad A_0^a = 0.
\]
This field has the strength tensor with the following non-zero components

\[ F_{23}^1 = F_{31}^2 = F_{12}^3 = H_x^1 = -H_y^2 = H_z^3 = g \tau \]  

(2.2)

and possesses spherical symmetry in ordinary space and in subspace of first three coordinates in the color space. Squared Dirac equation (1.2) in the field (2.1) has got the following explicit form:

\[
\left( p^2 + M^2 + \frac{3g^2\tau}{4} + g\tau^{1/2}\lambda^a p^a - \frac{g^2\tau}{2} \sigma^a \lambda^a \right) \Psi = E^2 \Psi.
\]

(2.3)

The spinor \( \Psi \) has two components \( \psi_+ \) and \( \psi_- \), and each of them transforms under fundamental representation of \( SU_c(3) \) color group. Because the external field (2.2) has the three non-zero components in the both spaces, in this case the spin and color spin states of the particle cannot be described by the projection of the \( \sigma^3 \) and \( \lambda^3 \) operators. So, the components \( \psi_\pm^{(a)} \) are not eigenfunctions of these operators and do not describe the states with definite value of their projection onto the field.

Equation (2.3) contains the non-diagonal \( \sigma^a \) and \( \lambda^a \) matrices, which make it non-diagonal. As a result, this equation does not split into independent ones for the components \( \psi_\pm^{(a)} \). From (2.3) it can be obtained the determinant equation, that is same for all states \( \psi_j^{(i)} \) \((i = 1, 2)\) and has following solution [4]:

\[
\psi_j^{(i)}(r) = \frac{C_j}{p} J_{l+1/2}(p r) Y_i^m(\theta, \varphi) \xi_j^{(i)},
\]

(2.4)

where \( J_{l+1/2}(p r) \) is the first kind Bessel function, \( Y_i^m(\theta, \varphi) \) are spheric functions, \( C_j \) are normalizing constants \((C_j = |p|)\) and \( \xi_j^{(i)} \) are the spin and color spin parts of the wave function. Solving (2.3) for the energy gives the following branches of continuous spectrum in the infinite motion case [10, 11]:

\[
E_{1,2}^2 = \left( \sqrt{p^2 + G^2} + \frac{G}{2} \right)^2 + M^2, \quad E_{3,4}^2 = \left( \sqrt{p^2 + G^2} - \frac{G}{2} \right)^2 + M^2, \quad E_5^2 = p^2 + M^2,
\]

and the quantized spectrum branches in the case of the finite motion inside sphere [4]:

\[
E_{1,2}^2 = \left( \frac{\alpha_i(N)}{r_0} \mp \frac{G}{2} \right)^2 + M^2, \quad E_{3,4}^2 = \left( \sqrt{\left( \frac{\alpha_i(N)}{r_0} \right)^2 + G^2} \mp \frac{G}{2} \right)^2 + M^2, \quad E_5^2 = \left( \frac{\alpha_i(N)}{r_0} \right)^2 + M^2.
\]

(2.5)

Here \( \alpha_i(N) \) are zeros of \( J_{l+1/2}(p r) \), \( r_0 \) is the radius of sphere and \( l \) is the orbital quantum number. Quantized and continuous spectra are interrelated by means of replacement \( |p| = \alpha_i(N)/r_0 \), which is same to quantization of momentum in standing waves. Motion in this case takes place on \( s p d f \ldots \) orbitals. As we observe, the additional non-zero components of the field strength tensor \( F_{23}^1 \) and \( F_{31}^2 \) apparently break the remained color and spin symmetry in Hamiltonian. In the result of this breakdown the spectrum splits into four branches as distinct from the axial background field case, i.e. lead to elimination of the degeneracy of the energy spectrum. From the spectrum (2.5) is seen also, that
energy $E_5$ does not contain the interaction term with the external field and so, does not split into branches. The state $\psi^{(3)}_j(r)$ like to state of the spinless and colorless particle. But solving (2.3) by means of its determinant leads to the absence of correspondence between the branches of spectrum (2.5) and the solutions (2.4). Consequently, it is reasonable to find the wave functions, which will describe the states having definite energy from these branches, i.e. to find energy states in the field (2.1).

Let us write the explicit matrix form of Hamiltonian for (2.3) in the spin-color spin space:

$$H = \begin{pmatrix} h_{11} & Gp_+ & 0 & 0 & 0 \\ Gp_+ & h_{22} & 0 & -G^2 & 0 \\ 0 & 0 & h_{33} & 0 & 0 \\ 0 & -G^2 & 0 & h_{44} & Gp_+ \\ 0 & 0 & 0 & Gp_+ & h_{55} \\ 0 & 0 & 0 & 0 & h_{66} \end{pmatrix}. \quad (2.6)$$

Diagonal elements $h_{ii}$ have the expressions:

$$h_{11} = P^2 + G^2/4 + Gp \ ,$$
$$h_{22} = P^2 + 5G^2/4 - Gp \ ,$$
$$h_{33} = P^2 \ ,$$
$$h_{44} = P^2 + 5G^2/4 + Gp \ ,$$
$$h_{55} = P^2 + G^2/4 - Gp \ ,$$
$$h_{66} = P^2 \quad (2.7)$$

where $P^2 = p^2 + M^2$. The Hamiltonian (2.6) has not quasidiagonal form and so, we cannot diagonalize separately two matrices $H \pm$ with dimension $3 \times 3$ instead of one with dimension $6 \times 6$. In order to diagonalize (2.6) we should make transformation in the combined space of spin and color spin. We can find this transformation and diagonal form of (2.6) applying the method used in the previous section.

As was asserted in the previous section, the diagonal form of $H'$ is unique and transformation $U$ transforming $H$ into $H'$ is unique and unitary due hermicity of $H$. Relying on these two properties, we can find the matrices $H'$ and $U$ simultaneously. From the beginning we find the determinant of this matrix, that equals to

$$\det H = (P^2)^2 \left( (P^2 + G^2/4)^2 - G^2p^2 \right) \left( (P^2 + 5G^2/4)^2 - G^2p'^2 \right) = \det H' \quad (2.8)$$

and can be written as a product of six factors $f_i$:

$$f_{1.2} = P^2, \ f_{3.4} = P^2 + G^2/4 \pm Gp, \ f_{5.6} = P^2 + 5G^2/4 \pm Gp',$$

where $p = |\mathbf{p}| = \sqrt{P^2}$, $p' = \sqrt{P^2 + G^2}$. Sum of these factors equals to the trace of the Hamiltonian (2.6):

$$\sum_i f_i = 6P^2 + 3G^2 = \text{Tr}H = \sum_i h_{ii} = \sum_i h'_{ii}.$$
According to the invariance of the trace and determinant of the matrix under the unitary transformation, we may suppose that, the factors \( f_i \) are the diagonal elements of diagonalized Hamiltonian \( H' \). But this is not enough in order to find the explicit form of \( H' \); we do not know the place of each of factors \( f_i \) along the diagonal of \( H' \) and have no rule to determine this place. At first, we can assume some identification of factors \( f_i \) with the diagonal elements of \( H' \) and then write the equation (1.13) explicitly for this assumption with unknown \( \mathcal{U} \). Equations for \( u_{ij} \) obtained from (1.13) together with the equations obtained from unitarity condition will be solved regularly, if diagonal elements were identified properly. If the factors \( f_i \) were corresponded to the diagonal elements \( h'_{ii} \) incorrectly, then solving (1.13) will lead to mathematical nonsense due to uniqueness of \( H' \). So we make the following identification:

\[
\begin{align*}
    h'_{11} &\equiv f_4 = \mathcal{P}^2 + \frac{g^2 \tau}{4} - \mathcal{G} \rho \\
    h'_{22} &\equiv f_6 = \mathcal{P}^2 + 5g^2 \tau/4 - \mathcal{G} \rho' \\
    h'_{33} &\equiv f_1 = \mathcal{P}^2 \\
    h'_{44} &\equiv f_5 = \mathcal{P}^2 + 5g^2 \tau/4 + \mathcal{G} \rho' \\
    h'_{55} &\equiv f_3 = \mathcal{P}^2 + \frac{g^2 \tau}{4} + \mathcal{G} \rho \\
    h'_{66} &\equiv f_2 = \mathcal{P}^2
\end{align*}
\] (2.9)

Writing (1.13) for this choice, we get systems of linear equations relating elements \( u_{ij} \) of transformation matrix \( \mathcal{U} \), which are simplified into following form:

\[
\begin{align*}
    u_{11} &= u_{21} p_- / (p - p_3) & u_{12} &= -u_{22} p_- / (p' + p_3 - \mathcal{G}) \\
    u_{41} &= u_{21} & u_{42} &= -u_{22} (p' - p_3 - \mathcal{G}) / (p' + p_3 - \mathcal{G}) \\
    u_{51} &= u_{21} p_+ / (p + p_3) & u_{52} &= -u_{22} p_+ / (p' + p_3 - \mathcal{G})
\end{align*}
\]

\[
\begin{align*}
    u_{14} &= u_{24} p_- / (p' - p_3 + \mathcal{G}) & u_{15} &= -u_{25} p_- / (p + p_3) \\
    u_{44} &= -u_{24} (p' + p_3 + \mathcal{G}) / (p' - p_3 + \mathcal{G}) & u_{45} &= u_{25} \\
    u_{54} &= -u_{24} p_+ / (p' - p_3 + \mathcal{G}) & u_{55} &= -u_{25} p_+ / (p - p_3)
\end{align*}
\] (2.10)

All other \( u_{ij} \) are zero except for \( u_{33}, u_{36}, u_{63} \) and \( u_{66} \) elements.

All other elements \( u_{ij} \) in (2.10) are expressed in terms of \( u_{2j} \). Equations (2.10) relates 32 unknowns by 24 relations. So, in order to find all \( u_{ij} \) we need in additional relations. For this purpose we can use relations of unitarity condition, which are 6 equations \( \sum_{j=1}^{6} u_{ij} u^*_{ij} = 1 \) and 30 ones \( \sum_{j=1}^{6} u_{ij} u^*_{kj} = 0 \), but these relations are nonlinear. Besides, some of the unitarity relations coincide with another ones after taking into account relations of (2.10) in them. Thus, taking into account the relations of (2.10) in the unitary relations and solving them we find only the module of \( u_{2i} \):

\[
\begin{align*}
    u_{21} u^*_{21} &= \frac{p_1^2}{4p'^2}, & u_{22} u^*_{22} &= \frac{(p' + p_3 - \mathcal{G})^2}{4p' (p' - \mathcal{G})},
    \end{align*}
\]

\[
\begin{align*}
    u_{24} u^*_{24} &= \frac{(p' + p_3 - \mathcal{G})^2}{4p' (p' + \mathcal{G})}, & u_{25} u^*_{25} &= \frac{p_1^2}{4p^2}.
\end{align*}
\] (2.11)
The arguments of $u_{2i}$ remain unknown and we parametrize them introducing free angles $\alpha, \beta, \gamma$ and $\delta$ as below:

$$u_{21} = \frac{p_+}{2p} e^{i\alpha}, \quad u_{22} = \frac{p' + p_3 - G}{[4p' (p' - G)]^{1/2}} e^{i\beta},$$

$$u_{24} = \frac{p' + p_3 - G}{[4p' (p' + G)]^{1/2}} e^{i\gamma}, \quad u_{25} = \frac{p_+}{2p} e^{i\delta}.$$  \hspace{1cm} (2.12)

In the system (2.10) there is no equation relating these four elements and so, we cannot express one angle in terms of another one\(^\S\). As elements $u_{3j}$, $u_{6j}$ and $u_{j6}$ equal to zero, the equation (1.13) gives trivial relations for the elements $u_{33}, u_{36}, u_{63}$ and $u_{66}$ and does not relate them with another $u_{ij}$ in (2.10). The relations for these elements, obtained from the unitarity condition, consist in the following ones:

$$\begin{cases} 
    u_{33} u_{33}^* + u_{36} u_{36}^* = 1 \\
    u_{63} u_{63}^* + u_{66} u_{66}^* = 1 \\
    u_{33} u_{63}^* + u_{36} u_{66}^* = 0
\end{cases} \hspace{1cm} (2.13)$$

The relations in (2.13) are nonlinear and their number less than the number of unknowns. Consequently, we get one more free parameter introducing angle $\theta$ in order to parametrize elements in (2.13) as below:

$$u_{33} = \cos \theta, \quad u_{36} = \sin \theta, \quad u_{63} = \sin \theta, \quad u_{66} = -\cos \theta.$$  

It is reasonable to assume, that under $U$ transformation the states $\psi^{(3)}_\pm$ do not change, since these states like colorless states ($\lambda^3 = 0$). Under this assumption we can fix parameter $\theta = 0$ and then $u_{33} = -u_{66} = 1, u_{36} = u_{63} = 0$.

As has been expressed before, the Hamiltonian (2.6) owing to non-zero $h_{42}$ and $h_{24}$ elements has not a quasidiagonal form and mixes different spin states. So, we cannot demand the unimodularity of any block of the $U$ matrix, as we made in the previous case. We also have no additional conditions to cut down the number of free parameters ($\alpha, \beta, \gamma, \delta$) or fix them. Thus, the $U$ matrix contains four free parameters and has the following explicit form:

$$U = \begin{pmatrix}
    \frac{p_- P_-}{P_3} & -p_-/P_1 & 0 & p_-/P_2 & p_- P_-'/P_4 & 0 \\
    \frac{P_-}{P_3} & (p' + p_3 - G)/P_1 & 0 & (p' - p_3 + G)/P_2 & P_-'/P_4 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    \frac{P_-}{P_3} & (p' - p_3 - G)/P_1 & 0 & -(p' + p_3 + G)/P_2 & P_-'/P_4 & 0 \\
    \frac{p_+ P_+}{P_4} & p_+/P_1 & 0 & -p_+/P_2 & p_+ P_+'/P_3 & 0 \\
    0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}, \hspace{1cm} (2.14)$$

where $P_1 = [4p' (p' - G)]^{1/2} e^{-i\beta}, P_2 = [4p' (p' + G)]^{1/2} e^{-i\gamma}, P_{3, 4} = \pm p_3, P_\perp = (p_\perp/2p) e^{i\alpha}, P_\perp' = -(p_\perp/2p) e^{i\delta}$. Transforming the basis vectors $\psi^{(i)}_j$ under the $U$ transformation, we

\(^\S\)This was made in the previous case.
can construct the energy states according to (1.20). Multiplying the \( \mathcal{U} \) matrix (2.14) by the column of solutions (2.4) we find the energy states:

\[
\psi_+^{(+)} = \frac{p-p_1}{P_3} \psi_1^{(1)}(\mathbf{r}) - \frac{p-p_1}{P_2} \psi_1^{(2)}(\mathbf{r}) + \frac{p-p_1}{P_4} \psi_2^{(1)}(\mathbf{r}) + \frac{p-p_2}{P_4} \psi_2^{(2)}(\mathbf{r}),
\]

\[
\psi_+^{(-)} = \frac{p-p_1}{P_3} \psi_1^{(1)}(\mathbf{r}) + \frac{p+p_1+G}{P_1} \psi_1^{(2)}(\mathbf{r}) + \frac{p+p_1-G}{P_2} \psi_2^{(1)}(\mathbf{r}) + \frac{p+p_2}{P_4} \psi_2^{(2)}(\mathbf{r}),
\]

\[
\psi_-^{(+)} = \frac{p+p_1}{P_3} \psi_1^{(1)}(\mathbf{r}) + \frac{p-p_1+G}{P_1} \psi_1^{(2)}(\mathbf{r}) - \frac{p+p_1-G}{P_2} \psi_2^{(1)}(\mathbf{r}) + \frac{p+p_2}{P_4} \psi_2^{(2)}(\mathbf{r}),
\]

\[
\psi_-^{(-)} = \frac{p-p_1}{P_3} \psi_1^{(1)}(\mathbf{r}) + \frac{p+p_1-G}{P_1} \psi_1^{(2)}(\mathbf{r}) - \frac{p+p_1+G}{P_2} \psi_2^{(1)}(\mathbf{r}) + \frac{p-p_2}{P_4} \psi_2^{(2)}(\mathbf{r}),
\] (2.15)

We can replace the all momentum operators in (2.15) by their eigenvalues. This does not lead to any changes in (2.15). Remark, the transformation \( \mathcal{U} \) acts on six dimensional combined space of spin and color spin, so the up and down \( \pm \) signs in \( \psi_\pm^{(\pm)} \) in (2.15) are just only notations and have no meanings of projection signs.

Comparing the spectrum branches (2.4) with the diagonal elements of \( H' \) (2.9), we write down obvious correspondence between them:

\[
E_1 \rightarrow h'_{11}, \\
E_2 \rightarrow h'_{55}, \\
E_3 \rightarrow h'_{22}, \\
E_4 \rightarrow h'_{44}.
\]

Using this correspondence we can write the eigenvalue equation (1.10) for \( H' \) by eigenfunctions \( \psi_\pm^{(\pm)} \) (2.15) and spectrum (2.5):

\[
\begin{pmatrix}
  h'_{11} & 0 & 0 & 0 & 0 & 0 \\
  0 & h'_{22} & 0 & 0 & 0 & 0 \\
  0 & 0 & h'_{33} & 0 & 0 & 0 \\
  0 & 0 & 0 & h'_{44} & 0 & 0 \\
  0 & 0 & 0 & 0 & h'_{55} & 0 \\
  0 & 0 & 0 & 0 & 0 & h'_{66}
\end{pmatrix}
\begin{pmatrix}
  \psi_+^{(+)} \\
  \psi_+^{(-)} \\
  \psi_+^{(3)} \\
  \psi_-^{(+)} \\
  \psi_-^{(-)} \\
  \psi_-^{(3)}
\end{pmatrix} = 
\begin{pmatrix}
  E_1^2 \psi_+^{(+)} \\
  E_2^2 \psi_+^{(-)} \\
  E_3^2 \psi_+^{(3)} \\
  E_4^2 \psi_-^{(+)} \\
  E_5^2 \psi_-^{(-)} \\
  E_6^2 \psi_-^{(3)}
\end{pmatrix}.
\]

Thus, we find the wave functions of colored particle describing the states having energy from the spectrum (2.4). This can be written in the following correspondence between the states (2.15) and branches of the energy spectrum (2.4):

\[
E_1 \rightarrow \psi_+^{(+)}, \\
E_2 \rightarrow \psi_-^{(-)}, \\
E_3 \rightarrow \psi_+^{(-)}, \\
E_4 \rightarrow \psi_-^{(+)}.
\]
Consider now the operator, which determines the branches of spectra (2.4), i.e. separate the term in Hamiltonian describing the interaction of the particle with the external field. The non-diagonal part of Hamiltonian (2.6)

\[ g^{1/2} \lambda^b \left( p^b - \frac{g^{1/2}}{2} \sigma^b \right) = g \lambda^b \left( A^b_j p_j - \frac{g}{4} \epsilon_{ijk} f^{acb} A^c_i A^b_j \sigma^k \right) = \left( \lambda^b I^b \right) \]  

under the transformation (1.12) with the \( U \) matrix (2.14) transforms into its diagonal form:

\[ \left( \lambda^b I^b \right)^r = U^{-1} \left( \lambda^b I^b \right) U = \begin{pmatrix} -Gp & 0 & 0 & 0 & 0 & 0 \\ 0 & -Gp' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Gp' & 0 & 0 \\ 0 & 0 & 0 & 0 & Gp & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]  

(2.17)

Here \( \epsilon_{ijk} \) is the unit antisymmetric tensor. As is seen from (2.17), in this case the diagonal form of operator (2.17) has five different eigenvalues, each one of which corresponds to the one branch of energy spectrum (2.4). In non-transformed space this operator can also be written in terms of the interactions of chromomagnetic moments with the chromomagnetic background:

\[ \left( \lambda^b I^b \right) = g \lambda^b \left( A^b_j p_j - \frac{1}{2} \epsilon_{ijk} F^b_{ij} \sigma^k \right) = 2g \left( A^b_j p_j - \frac{1}{2} \epsilon_{ijk} F^b_{ij} \sigma^k \right) \]  

(2.18)

Second term in (2.18) describes the interaction of the particle with the external field due to its spin and first one hides the interaction with the external field due to orbital moment. In transformed space these two kind interactions are joined in the eigenvalues \( \pm Gp, \pm Gp' \) and 0 and cannot be separated. The operator \( \left( \lambda^b I^b \right) \) commutes with the Hamiltonian (2.6) and the quantity corresponding to it is conserved. Thus we conclude, that in external fields, given by the noncommuting potentials (1.4) and (2.1), the projection \( \left( \lambda^b I^b \right) \) describes the interaction of the chromomagnetic moment of the particle with the external chromomagnetic field and cause the splitting of the energy spectrum into the branches and levels. States with definite energy in these fields are determined by this projection instead of projection of spin and color spin onto the field, i.e. are defined in that basis, in which this projection has got the diagonal form.

3 Superpartner states

As we know [12,13], the equation (1.2) in the field (1.5) posseses supersymmetry, i.e. for the motion with \( p_3 = 0 \) the Hamiltonian \( H_1 = H - M^2 \) and two operators \( Q_\pm = P_\pm a_\pm \), which named the supercharge operators, form supersymmetry algebra in Quantum Mechanics [19-24]:

\[ \{ Q_+, Q_- \} = H_1, \ [ Q_\pm, H_1 ] = 0, \ Q_+^2 = Q_-^2 = 0. \]  

(3.1)

Here the brace denotes anticommutator and the square bracket denotes commutator. Defined by formulae
$P_{\pm} = P_1 \pm iP_2, \ a_{\pm} = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$

(3.2)

the operators $P_{\pm}$ and $a_{\pm}$ obey the following commutation and anticommutation relations for the annihilation and creation operators of bosonic and fermionic states, respectively:

$[P_+, P_-] = \lambda^3 g H_z^3, \ \{a_+, a_-\} = 1.$

(3.3)

Let us remark, that the operators for creation and annihilation of the bosonic states interchange roles for the state having value $\lambda^3 = 1$ with the state having value $\lambda^3 = -1$ [13], that can be seen from (3.3). Action of the fermionic operators $a_{\pm}$ turns over the spin of particle:

$$a_+ \xi_- = \left( \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \end{array} \right), \quad a_- \xi_+ = \left( \begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array} \right) = \xi_+, \quad a_+ \xi_+ = 0;$$

$$a_- \xi_+ = \left( \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \end{array} \right), \quad a_- \xi_- = 0. \quad (3.4)$$

Since the action of $a_+$ on $\left( \begin{array}{c} \psi_+ \\ \psi_- \end{array} \right)$ gives $\psi_-$ and $a_-$ gives $\psi_+$, the bosonic creation operator $P_+$ acts only on the $\psi_+$ state and the annihilation operator $P_-$ only on the $\psi_-$ state:

$$P_+ \left( \begin{array}{c} \psi_+^{(1)} \\ \psi_+^{(2)} \\ \psi_+^{(3)} \end{array} \right) = \left( \begin{array}{ccc} p_+ & G & 0 \\ 0 & p_+ & 0 \\ 0 & 0 & p_+ \end{array} \right) \left( \begin{array}{c} \psi_+^{(1)} \\ \psi_+^{(2)} \\ \psi_+^{(3)} \end{array} \right) = \left( \begin{array}{c} p_+ \psi_+^{(1)} + G \psi_+^{(2)} \\ p_+ \psi_+^{(2)} \\ p_+ \psi_+^{(3)} \end{array} \right),$$

$$P_- \left( \begin{array}{c} \psi_-^{(1)} \\ \psi_-^{(2)} \\ \psi_-^{(3)} \end{array} \right) = \left( \begin{array}{ccc} p_- & 0 & 0 \\ G & p_- & 0 \\ 0 & 0 & p_- \end{array} \right) \left( \begin{array}{c} \psi_-^{(1)} \\ \psi_-^{(2)} \\ \psi_-^{(3)} \end{array} \right) = \left( \begin{array}{c} p_- \psi_-^{(1)} \\ G \psi_-^{(1)} + p_- \psi_-^{(2)} \\ p_- \psi_-^{(3)} \end{array} \right). \quad (3.5)$$

Using the supercharge operators it is easy to observe the spin diagonal form of the Hamiltonian $H_1$:

$$H_1 \left( \begin{array}{c} \psi_+ \\ \psi_- \end{array} \right) = \left( \begin{array}{cc} P_- P_+ & 0 \\ 0 & P_+ P_- \end{array} \right) \left( \begin{array}{c} \psi_+ \\ \psi_- \end{array} \right). \quad (3.6)$$

Note, that action of the bosonic creation and annihilation operators $P_{\pm}$ mix the different color states $\psi^{(1)}$ and $\psi^{(2)}$. In Supersymmetric Quantum Mechanics each quantum mechanical state of particle is labeled by fermionic and bosonic quantum numbers [19]-[21]. Superpartner states in this theory are called the states having the same energy, but different fermionic and bosonic quantum numbers. Action of the supercharge operators $Q_{\pm}$ change the fermionic and bosonic quantum numbers and convert the superpartner states into each other.

In the first section we have established that energy states having different spin projection, namely the states $\psi^{(--)}_+$ and $\psi^{(---)}_-$ and the states $\psi^{(++)}_+$ and $\psi^{(--)}_-$ have same energies. So, we have a reason to look for superpartner states among the states $\psi^{(\pm)}_+$ and $\psi^{(\pm)}_-$. Since these states differ by the spin projection we can assert that they are turned into each other under the action of the fermionic operators $a_{\pm}$. If we find a couple of mutually
operators

But, we see from the explicit form of $\psi(\pm)$, i.e. acts by the rule $P_\pm \psi(\pm) \rightarrow \psi(\mp)$, then we can construct supercharge operators $Q_\pm$ by means of these operators, that will obey the supersymmetry algebra (3.1) and will map $\psi(\pm)$ into each other:

$$Q_+\psi_- = q_1\psi_1, \quad Q_-\psi_+ = q_2\psi_2;$$
$$Q_-\psi_+ = q'_1\psi'_1, \quad Q_+\psi_- = q'_2\psi'_2. \quad (3.7)$$

The states $\psi(\pm)$ and $\psi'(\pm)$ and the states $\psi'(\pm)$ and $\psi'(\mp)$ will be superpartner states in the framework of this supersymmetry. So, for the superpartnership of these states it is enough to find suitable bosonic operators.

Under the $U$ transformation the operators $P_\pm$ transform as well:

$$U^{-1}_\pm H_\pm U_\pm = U^{-1}_\pm P_\pm U_\pm U^{-1}_\pm P_\pm U_\pm = P'_\pm P'_\pm. \quad (3.8)$$

Applying $U_\pm$ transformation we find $P'_\pm$ operators in the new basis:

$$P'_\pm = \frac{1}{2p_\perp} \begin{pmatrix} 2p_+ (P_\perp + G/2) & G (P_\perp + G/2) e^{-2i\alpha} & 0 \\ -Gp_\perp^2 (P_\perp + G/2)^{-1} e^{2i\alpha} & 2p_+ (P_\perp - G/2) & 0 \\ 0 & 0 & 2p_\perp p_+ \end{pmatrix},$$

$$P'_- = \frac{1}{2p_\perp} \begin{pmatrix} 2p_- (P_\perp + G/2) & -Gp_\perp^2 (P_\perp + G/2)^{-1} e^{-2i\alpha} & 0 \\ G (P_\perp + G/2) e^{2i\alpha} & 2p_- (P_\perp - G/2) & 0 \\ 0 & 0 & 2p_\perp p_- \end{pmatrix}. \quad (3.9)$$

Product of transformed bosonic operators are diagonalized Hamiltonians $H'_\pm$ (1.17) and (1.26) with $M^2 = 0$ and $p_3 = 0$:

$$P'_- P'_+ = H'_+, \quad P'_+ P'_- = H'_-. \quad (3.10)$$

It is clear that, the new supercharge operators $Q'_\pm = P'_\pm a_\pm$ will obey the supersymmetry algebra (3.1) with the diagonalized Hamiltonian $H'$:

$$\{Q'_+, Q'_-\} = H'_+, \quad [Q'_+, H'_-] = 0, \quad Q'_+^2 = Q'_-^2 = 0. \quad (3.11)$$

But, we see from the explicit form of $P'_\pm$ the action of these operators on $\Psi' = \begin{pmatrix} \psi'(\pm) \\ \psi'(\mp) \end{pmatrix}$ does not maps the states $\psi(\pm)$ and $\psi'(\mp)$ into each other and mix these states. This means that, in the result of action of these operators we get superposition of the $\psi'(\pm)$ states, i.e. states having uncertain energy. So, the operators $P'_\pm$ are unusable to build supersymmetry algebra, in which the states $\psi(\pm)$ and $\psi'(\mp)$ would be turned out superpartners. But, fortunately, it can be constructed the another mutually hermitian conjugate couple of $P''_\pm$ operators

$$P''_\pm = P_\perp \lambda^1 + i\lambda^2 G/2 + p_+ (I_3 - I_2) = \begin{pmatrix} 0 & \mathcal{P}_\perp + G/2 & 0 \\ \mathcal{P}_\perp - G/2 & 0 & 0 \\ 0 & 0 & p_+ \end{pmatrix},$$
\( P''_n = P_\perp x^1 - i \lambda^2 G/2 + p_\perp (I_3 - I_2) = \begin{pmatrix} 0 & P_\perp - G/2 & 0 \\ P_\perp + G/2 & 0 & 0 \\ 0 & 0 & p_\perp \end{pmatrix}, \)  \hspace{1cm} (3.12)

which obey commutation relation \([P''_n, P''_m] = 2\lambda^2 G P_\perp.\) Product of these operators also gives \(H'_\pm: \)

\[ P''_n P''_m = H'_+, \quad P''_+ P''_- = H'_- \]

and so, the supercharge operators constructed using \(Q'_\pm = P''_\pm a_\pm\) also obey the supersymmetry algebra (3.11). Action of the \(P''_\pm\) operators on \(\Psi\) change the upper index of the components of this wave function:

\[ P''_+ \begin{pmatrix} \psi^{(+)} \\ \psi^{(-)} \\ \psi^{(3)} \end{pmatrix} = \begin{pmatrix} (P_\perp + G/2) \psi^{(-)} \\ (P_\perp - G/2) \psi^{(+)} \\ p_+ \psi^{(3)} \end{pmatrix}, \]

\[ P''_- \begin{pmatrix} \psi^{(-)} \\ \psi^{(+)} \\ \psi^{(3)} \end{pmatrix} = \begin{pmatrix} (P_\perp - G/2) \psi^{(+)} \\ (P_\perp + G/2) \psi^{(-)} \\ p_- \psi^{(3)} \end{pmatrix}. \]  \hspace{1cm} (3.13)

Action of these operators on \(\zeta^{(\pm)}\) is given by the formulae:

\[ P''_+ \zeta^{(+)} = (P_\perp - G/2) \zeta^{(-)}, \quad P''_- \zeta^{(-)} = (P_\perp + G/2) \zeta^{(+)}; \]

\[ P''_+ \zeta^{(+)} = (P_\perp + G/2) \zeta^{(-)}, \quad P''_- \zeta^{(-)} = (P_\perp - G/2) \zeta^{(+)}; \]

\[ P''_{\pm} \zeta^{(3)} = p_{\pm} \zeta^{(3)}. \]  \hspace{1cm} (3.14)

So, this action changes the orientation of the \(I^b\) vector in the transformed color space. The operators \(b_{\pm} = P''_{\pm} / (2 G P_\perp)^{1/2}\) obey the Heisenberg-Weil algebra

\[ [b_+, b_-] = 1 \]

for the state \(\zeta^{(+)}\), i.e. are the operators of creation and annihilation, respectively. For the state \(\zeta^{(-)}\) these operators obey the commutation relation

\[ [b_-, b_+] = 1, \]

that again means interchanging the roles of creation and annihilation operators of \(\zeta^{(+)}\) state.

We can write the action of \(Q'_\pm\) operators on \(\psi_{\pm}^{(\pm)}\) using (3.14), (3.4) and (1.40) formulae:

\[ Q'_+ \psi_{\pm}^{(\pm)} = P''_+ a_\pm \xi_\pm \zeta^{(\pm)} F(r) = (P_\perp + G/2) \psi_{\pm}^{(\pm)}, \]

\[ Q'_+ \psi_{\pm}^{(-)} = P''_+ a_\pm \xi_\pm \zeta^{(-)} F(r) = (P_\perp - G/2) \psi_{\pm}^{(+)}, \]

\[ Q'_- \psi_{\pm}^{(+)} = P''_- a_\pm \xi_\pm \zeta^{(+)} F(r) = (P_\perp + G/2) \psi_{\pm}^{(-)}, \]

\[ Q'_- \psi_{\pm}^{(-)} = P''_- a_\pm \xi_\pm \zeta^{(-)} F(r) = (P_\perp - G/2) \psi_{\pm}^{(-)}, \]  \hspace{1cm} (3.15)

\[ Q'_+ \psi_{\pm}^{(\pm)} = Q'_+ \psi_{\pm}^{(+)} = Q'_- \psi_{\pm}^{(\pm)} = Q'_- \psi_{\pm}^{(+)} = 0; \]
\[ Q_- \psi^{(3)}_+ = p_+ \psi^{(3)}_+, \quad Q'_+ \psi^{(3)}_- = p_- \psi^{(3)}_+. \]

This action simultaneously changes the signs of the spin and \((I^b \lambda^b)\) projection so that the energy of the state remains unchanged. The eigenvalues \(q_i\) of the supercharge operators follow from these actions:

\[ q_1 = q'_1 = \mathcal{P}_\perp + \mathcal{G}/2, \quad q_2 = q'_2 = \mathcal{P}_\perp - \mathcal{G}/2. \]

If remind the relationship between the degree \(n\) of the degeneracy of the spectrum and the number \(N\) of the supercharge operators \(Q'_\pm\):

\[ n = 2^{[N/2]}, \]

where \([N/2]\) denotes integer part of \(N/2\), in this choice of the supercharge operators it is easy to explain the twofold degeneracy of the energy spectrum, which we have discussed in first section, as a result of supersymmetry in the Hamiltonian.\(^\dagger\) Thus, action of supercharge operators (3.15) ensure the superpartnership of the energy states \(\psi^{(-)}_+\) with \(\psi^{(+)}_-\) and \(\psi^{(+)}_+\) with \(\psi^{(-)}_-\).

Using (3.11) we can divide the Hamiltonian \(H'_1\) into bosonic \(H_B\) and fermionic \(H_F\) parts:

\[
H'_1 = \{Q'_+, Q'_-\} = 2\mathcal{G}\mathcal{P}_\perp b_+ b_- - 2\lambda^3 \mathcal{G}\mathcal{P}_\perp a_+ a_- = 2\mathcal{G}\mathcal{P}_\perp \left(b_+ b_- + \frac{1}{2}\right) + 2\mathcal{G}\mathcal{P}_\perp \left(-\lambda^3\right) a_+ a_- - \frac{1}{2} = H_B + H_F.
\]

(3.16)

The appearance of \((-\lambda^3)\) factor in the fermionic part is connected with different commutation rules for the bosonic operators of \(\zeta^{(-)}\) and \(\zeta^{(+)}\) states. Actually, \(H_B\) also contains the term proportional to \(\lambda^3\). These two Hamiltonians commute \(^\ddagger\) [\(H_B, H_F\) = 0] and can be considered as Hamiltonians of two independent oscillators having the same frequency \(\omega = (2\mathcal{G}\mathcal{P}_\perp)^{1/2}\):

\[
H_B = \omega^2 \left(b_+ b_- + \frac{1}{2}\right), \quad E_B = \omega^2 \left(n_B + \frac{1}{2}\right);
\]

\[
H_F = \omega^2 \left(-\lambda^3\right) a_+ a_- - \frac{1}{2}, \quad E_B = \omega^2 \left(-\lambda^3\right) n_F - \frac{1}{2}.
\]

(3.17)

In respect to the supersymmetry each quantum mechanical state of the particle with definite energy is described by the bosonic and fermionic quantum numbers \(n_B\) and \(n_F\), which accept values [19]: \(n_B = 0, 1, 2, 3...; n_F = 0, 1\). As is seen from (3.17) and (3.16), this takes place for \(H'_1\) and its eigenstates \(\psi^{(\pm)}_\pm\) as well. Action of the supercharge operators change these quantum numbers as following:

\[
Q_+ (n_B, n_F) = (n_B - 1, n_F + 1), \quad Q_+ (n_B - 1, n_F + 1) = 0;
\]

\[
Q_- (n_B - 1, n_F + 1) = (n_B, n_F), \quad Q_- (n_B, n_F) = 0,
\]

(3.18)

but total energy of bosonic and fermionic oscillators remain unchanged on this action. If we label the states \(\psi^{(\pm)}_\pm\) by these quantum numbers as below:

\[
\psi^{(-)}_-, \psi^{(+)}_- \rightarrow (n_B, n_F); \quad \psi^{(-)}_+, \psi^{(+)}_+ \rightarrow (n_B - 1, n_F + 1)
\]

the two acting rules of the supercharge operators (3.18) and (3.15) will agree.

\(^\dagger\) In [13,12,27] this degeneracy was related with the supersymmetry, but appropriate supercharge operators responsible for this degeneracy was not found, since the energy states was not determined.

\(^\ddagger\) \(H_B\) and \(H_F\) does not commute in another basis [27].
4 Discussion

The non-diagonal generators of color group make the equations in Yang-Mills Quantum Mechanics non-diagonal as well and we decided to diagonalize the equation of motion for the case. Relying on hermicity of the Hamiltonian, we find the unitary transformation, which diagonalizes this equation. Of course, this transformation transforms the color structure of Hilbert space and the transformed basis vectors are the eigenfunctions of the diagonal Hamiltonian. This enabled us to establish the correspondence between the eigenvalues of the Hamiltonian and its eigenvectors.

The diagonal form of the Hamiltonian turned up useful for the study of supersymmetry in the considered case. It allowed to construct the supercharge operators mapping eigenvectors corresponding to the same eigenvalues each into others. Superpartner property of the eigenvectors was easily revealed on this formulation of supersymmetry. Also it became possible to divide the diagonal Hamiltonian into two commuting parts corresponding to the two oscillators of bosonic and fermionic states of supersymmetry.

The number of papers [14] have been devoted to the study of supersymmetric Yang-Mills Quantum Mechanics in connection with the conjecture of Banks, Fischler, Shenker and Susskind concerning the equivalence of M-theory and the $D = 10$ supersymmetric Yang-Mills quantum mechanics [26]. We hope the study of example of supersymmetry, which we have made here, will be useful for the further study of supersymmetry in Yang-Mills Quantum Mechanics in connection with the BFSS conjecture.

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