Generating random quantum channels

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ABSTRACT
Several techniques of generating random quantum channels, which act on the set of $d$-dimensional quantum states, are investigated. We present three approaches to the problem of sampling of quantum channels and show that they are mathematically equivalent. We discuss under which conditions they give the uniform Lebesgue measure on the convex set of quantum operations and compare their advantages and computational complexity and demonstrate which of them is particularly suitable for numerical investigations. Additional results focus on the spectral gap and other spectral properties of random quantum channels and their invariant states. We compute the mean values of several quantities characterizing a given quantum channel, including its unitarity, the average output purity, and the 2-norm coherence of a channel, averaged over the entire set of the quantum channels with respect to the uniform measure. An ensemble of classical stochastic matrices obtained due to super-decoherence of random quantum stochastic maps is analyzed, and their spectral properties are studied using the Bloch representation of a classical probability vector.

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I. INTRODUCTION
Quantum dynamics corresponding to classically chaotic systems can be described by suitable ensembles of random matrices. In the case of autonomous quantum systems, one mimics Hamiltonians with the help of ensembles of random Hermitian matrices invariant with respect to certain transformations. Depending on the symmetry properties of the system investigated, one uses orthogonal, unitary, or symplectic ensembles.1 In the case of time-dependent, periodically driven systems, the corresponding unitary evolution operators can be described by one of the three circular ensembles of Dyson.2,3

The problem becomes more complex in a physically important situation of open quantum systems interacting with an environment4,5 and dissipative quantum systems.6–8 The time evolution of the principal system coupled with an ancillary system is non-unitary, and the spectrum of the associated evolution operators is contained in the unit disk. Dynamics of dissipative chaotic quantum systems can be described by the master equation,1,7 while properties of chaotic scattering9 can be explained with help of suitable non-unitary ensembles of random matrices.10,11

A very general scheme of a continuous time evolution of an open quantum system can be described by the celebrated Gorini–Lindblad–Kossakowski–Sudarshan equation.12–14 In the case of discrete dynamics, one uses the notion of quantum operation or quantum channel: a completely positive, trace preserving linear map that sends the set of mixed quantum states into itself.15 The set of quantum operations acting on density matrices of a given size $d$ is convex and compact, and it is easy to show that the flat Lebesgue measure in this set is generated by the Hilbert–Schmidt distance in the set of dynamical matrices that determine the maps.

Such an ensemble of random quantum operations was introduced in Ref. 16. The goal of this work is to introduce and study three families of ensembles of quantum operations, motivated by different structural characterizations of quantum channels. We show that these probability measures agree for specific values of the parameters, and we provide practical algorithms to generate such maps numerically. We argue that the most computationally efficient procedure to generate a random quantum channel is to sample its Kraus operators (see Sec. III B). Furthermore, we also present several results on the properties of random operations: we analyze the spectral properties of a generic...
superoperator and the corresponding invariant state and show that in the limit of a large dimension \( d \), a typical channel is close to the unital completely depolarizing channel. As a closely related subject, we discuss the properties of random stochastic and bistochastic matrices that can be considered as classical analogs of random quantum stochastic maps. The present work can be considered as a complementary to the earlier contribution,\(^{17}\) as it extends the study of random quantum states for random operations. Note also that the techniques of random matrices are frequently used in the theory of quantum information.\(^{18,19}\) In particular, the seminal result of Hastings\(^ {20}\) establishing non-additivity of minimum output entropy of quantum channels was obtained with the use of a certain class of random quantum operations.

This work is organized as follows: In Sec. II, we set the scene and introduce concepts and notation necessary for presenting our results. Section III describes the several equivalent sampling methods for random quantum operations, as well as their relation to the flat measure, while in Sec. IV, the distribution of output states of a random channel is investigated. Next, in Sec. V, we discuss the spectral properties of the superoperator representing a quantum channel. Furthermore, we demonstrate that the Bloch representation of a quantum operation is equivalent with the Fano form of the corresponding bi-partite Jamiołkowski state, so certain properties of a random superoperator are related to the properties of a correlation matrix, which describes correlations between two subsystems a bi-partite random state. Effects of super-decoherence and a relation between random quantum stochastic maps and the classical random stochastic maps are discussed in Sec. VI, while in Sec. VII, the spectral properties of invariant states of random quantum channels are analyzed. Appendices A–E contain two technical lemmas and proofs of propositions formulated in the main body of this work. Appendix F provides a short review of several other ensembles of random stochastic and bistochastic quantum maps.

### II. SETTING THE SCENE

Quantum channels, the most general transformations of quantum states allowed by the axioms of quantum mechanics, are modeled by linear maps between matrix algebras satisfying certain positivity and trace preservation properties. There are several equivalent characterizations of quantum channels, each having its own merit depending on the point of view we want to take. We summarize them in the following theorem [see, e.g., Ref. 21 (Corollary 2.27)]:

**Theorem 1.** A linear map \( \Phi : M_{d_1}(C) \rightarrow M_{d_2}(C) \) is called a quantum channel (or quantum operation) if any of the following equivalent conditions is satisfied:

1. **The map \( \Phi \) is both**
   - completely positive: for all \( n \geq 1 \), \( \Phi \otimes \text{id}_n : M_{d_1n}(C) \rightarrow M_{d_2n}(C) \) is positive (i.e., maps positive semidefinite matrices to positive semidefinite matrices)
   - trace preserving: for all \( X \in M_{d_1}(C) \), \( \text{Tr}(\Phi(X)) = \text{Tr}(X) \).

2. **The map \( \Phi \) admits a Kraus decomposition**

\[
\forall X \in M_{d_1}(C), \quad \Phi(X) = \sum_{i=1}^{r} A_i X A_i^\dagger \tag{1}
\]

for matrices \( A_1, \ldots, A_r \in M_{d_1 \times d_2}(C) \), called Kraus operators. The matrices \( A \) satisfy the identity resolution, \( \sum_{i=1}^{r} A_i^\dagger A_i = 1_{d_1} \), corresponding to the fact that \( \Phi \) is trace preserving.

3. **The map \( \Phi \) admits a Stinespring dilation: there exists, for some positive integer \( n \), an isometry \( V : C^n \rightarrow C^{d_1} \otimes C^n \) such that**

\[
\forall X \in M_{d_1}(C), \quad \Phi(X) = [\text{id}_{d_1} \otimes \text{Tr}_n](VXV^\dagger). \tag{2}
\]

4. **The Choi matrix of \( \Phi \),**

\[
J_\Phi := \sum_{i=1}^{d_1} \Phi(|i\rangle \langle j|) \otimes |i\rangle \langle j| \in M_{d_1}(C) \otimes M_{d_1}(C), \tag{3}
\]

is positive semidefinite and has partial trace

\[
[\text{Tr}_{d_1} \otimes \text{id}_{d_2}](J_\Phi) = 1_{d_1}. \tag{4}
\]

A few remarks are in order regarding the result above. First, it suffices to check the complete positivity condition for \( n = d_1 \). Second, regarding the Kraus decomposition, the smallest positive integer \( r \) for which \( \Phi \) admits a Kraus decomposition (1) is called the *Choi rank* of \( \Phi \) and is denoted by \( r_C(\Phi) \). The Choi rank is a measure of the noisiness of the channel \( \Phi \), varying from \( r = 1 \) for a unitary conjugation \((d_1 = d_2 = d)\)

\[
\Phi_U(X) = UXU^*, \quad \text{where } U \in U(d) \text{ is a unitary matrix}, \tag{5}
\]
to \( r = d_1d_2 \) for the completely depolarizing channel

\[
\Phi_\star(X) = \text{Tr}X \frac{1d_1}{d_2}.
\]

(6)

In the Stinespring dilation formulation, the Hilbert space \( \mathbb{C}^n \) is commonly termed the environment. In the special case, when \( d_1 \) divides \( d_2n \), Eq. (2) can be rewritten as

\[
\Phi(X) = [\text{id}_{d_1} \otimes \text{Tr}_r]\left(U(X \otimes |0\rangle\langle 0|)U^\dagger\right),
\]

(7)

with \( U \in \mathcal{U}(d_2n) \) being a unitary operator and \(|0\rangle\) a \( d_2n/d_1 \)-dimensional unit vector. The isometry \( V \) defining the quantum channel \( \Phi \) is then a truncation of the unitary operator \( U \). The environment size \( n \) can be taken to be \( n = d_1d_2 \) without loss of generality, and the minimal \( n \) for which a decomposition (2) exists is equal to the Choi rank \( r_\mathcal{C}(\Phi) \) of the channel.

The final characterization in Theorem 1 is an instance of the Jamiołkowski–Choi isomorphism: with any linear map \( \Phi : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C}) \), one can associate a matrix \( J_\Phi \in M_{d_2}(\mathbb{C}) \otimes M_{d_1}(\mathbb{C}) \); this isomorphism has the following properties:

- maps preserving self-adjointness are mapped to self-adjoint matrices,
- completely positive maps are mapped to positive semidefinite matrices,
- trace preserving maps are mapped to matrices \( J_\Phi \) satisfying \( [\text{Tr}_{d_1} \otimes \text{id}_{d_2}](J_\Phi) = \mathbb{1}_{d_2} \),
- unital maps (i.e., \( \Phi(\mathbb{1}_{d_1}) = \mathbb{1}_{d_2} \) are mapped to matrices \( J_\Phi \) satisfying \( [\text{id}_{d_1} \otimes \text{Tr}_{d_2}](J_\Phi) = \mathbb{1}_{d_1} \).

The Choi matrix \( J_\Phi \) can be written with the help of the maximally entangled state \( |\Omega\rangle = \frac{1}{\sqrt{d_2}} \sum_{i=1}^{d_2} |ii\rangle \in \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_2} \) as

\[
J_\Phi = d_1[\Phi \otimes \text{id}_{d_1}](|\Omega\rangle\langle\Omega|).
\]

(8)

The rescaled Choi matrix, \( J_\Phi/d_1 \), with unit trace is also called the Jamiołkowski state, which explains the notation used.

Moreover, from the Kraus decomposition \( \Phi(X) = \sum_{i=1}^{d_1} A_iXA_i^\dagger \) of the channel \( \Phi \), one can calculate the Choi matrix using the vectorization notation,

\[
J_\Phi = \sum_{i=1}^{d_1} |A_i\rangle\langle A_i|,
\]

(9)

where \(|A\rangle = \sum_{i=1}^{d_1} A|i\rangle \otimes |i\rangle\) denotes the vector of length \( d_2d_1 \) obtained by reshaping a given matrix \( A \) of size \( d_2 \times d_1 \). Let us discuss now the matrix of \( \Phi \), viewed as a linear map between the vector spaces \( M_{d_1}(\mathbb{C}) \cong \mathbb{C}^{d_1^2} \) and \( M_{d_2}(\mathbb{C}) \cong \mathbb{C}^{d_2^2} \). We also denote by \( \Phi \) this matrix, usually called the superoperator; we have, in terms of the Kraus operators,

\[
\Phi = \sum_{i=1}^{d_1} A_i \otimes A_i^\dagger,
\]

(10)

where bar denotes the complex conjugate. In terms of the Choi matrix, the superoperator reads \( \Phi = J_\Phi^R \), where \( R \) denotes the transformation of matrix reshuffling (or realignment) [see Ref. 15, Chap. 10.2],

\[
(|i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle)^R = |i\rangle \otimes |k\rangle \otimes |j\rangle \otimes |l\rangle.
\]

(11)

The linear map \( \Phi : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C}) \) admits an adjoint \( \Phi^\dagger : M_{d_2}(\mathbb{C}) \to M_{d_1}(\mathbb{C}) \) for the Hilbert–Schmidt scalar product on complex matrices \( \langle A|B \rangle = \text{Tr}(A^\dagger B) \). If \( \Phi \) is a quantum channel, the map \( \Phi^\dagger \) is still completely positive, but the trace preservation property of \( \Phi \) is converted to unitality: \( \Phi^\dagger(\mathbb{1}_{d_2}) = \mathbb{1}_{d_1} \). Quantum channels that are unital are called bistochastic, since they satisfy both normalization conditions; the name is a reference to the classical situation where row- and column-stochastic matrices are called bistochastic or doubly stochastic.

An interested reader can find modern expositions of these results and many developments in monographs such as Ref. 22 (Chap. 8), Ref. 15 (Chaps. 10 and 11), or Ref. 21 (Chap. 2.2).

Some basic facts from the theory of random matrices will be relevant in this paper. For an in-depth introduction, we refer to the reader to the classical textbook or to modern presentations. In this work, we are going to use the following ensembles of random matrices:

- The real Ginibre ensemble consisting of matrices with independent and identically distributed real standard Gaussian entries \( G \).
- The complex Ginibre matrices consisting of matrices \( G \) with independent complex entries distributed according to the standard complex Gaussian distribution. One can write \( G = (G_R + iG_I)/\sqrt{2} \), where \( G_R \) and \( G_I \) are independent real Ginibre matrices. Note that the Ginibre matrices can be rectangular, and they are normalized as \( \text{E}[\text{Tr}(GG^\dagger)] = d_1d_2 \). In the square case \( (d_1 = d_2 = d) \), the spectrum of a normalized complex Ginibre matrix \( G/\sqrt{d} \) covers uniformly the unit disk, a result called the circular law of Girko.
- Gaussian Unitary ensemble (GUE) of Hermitian (self-adjoint) matrices, invariant with respect to the unitary group, contains matrices \( H = (G + G^\dagger)/\sqrt{2} \). In the limit of large matrix dimension \( d \), the spectrum of normalized GUE matrices \( H/\sqrt{d} \) converges to Wigner’s semicircle distribution (Ref. 23, Chap. 2).
III. DISTRIBUTIONS OF RANDOM QUANTUM CHANNELS

Let us denote by $C_{d_1,d_2}$ the set of quantum channels acting on density matrices of order $d_1$ that output density matrices of order $d_2$,

$$C_{d_1,d_2} := \{ \Phi : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C}) : \Phi \text{ is completely positive and trace preserving} \}.$$  

The set $C_{d_1,d_2}$ is compact and convex. In this section, we discuss different natural ways of endowing this convex body with natural probability measures. One natural candidate is the flat measure on this set, induced by the Hilbert–Schmidt (HS) distance, $D_{\text{HS}}(A,B) = (\text{Tr}(A - B)(A - B)^\dagger)^{1/2}$. We shall see that the flat (or Lebesgue) measure is actually a special case of several one-parameter families of probability measures on $C_{d_1,d_2}$. In the case of equal input and output dimensions $d_1 = d_2 = d$, each map $\Phi$ can be represented by a Hermitian matrix $J_\Phi$ of order $d^2$, characterized by $d^2$ real parameters. However, the trace preserving condition, $[\text{Tr}_d \otimes \text{id}_{d_2}]J_\Phi = 1_d$, imposes $d^2$ constraints, so the set $C_d$ can be embedded in a real vector space of dimension $d^2 - d^2$. The volume of the convex set $C_d$ with respect to the Hilbert–Schmidt (flat) measure was estimated for a large dimension $d$, while in the case of one-qubit channels, $d = 2$, an exact result is available.

We next introduce three methods to generate random operations and show their equivalence. The motivation for these families of measures comes from different perspectives on quantum channels provided by Theorem 1. The common idea is that we shall consider the different available. The complex Wishart ensemble $\mathcal{C}^d$ is the unique probability measure invariant with respect to left and right multiplication with fixed unitary matrices. The same strategy has been used in the setting of density matrices (or mixed quantum states) in Ref. 32: the flat measure was estimated for a large dimension $d$, while in the case of one-qubit channels, $d = 2$, an exact result is available.

We shall introduce three families of measures on the set of quantum channels $C_{d_1,d_2}$, starting from the most general ones. We shall conclude by identifying the flat (or Lebesgue) measure as a special case of all of them.

A. Random Choi matrix

Define the set of allowed parameters $M$,

$$\mathcal{M}_{d_1,d_2} := \left\{ \left[ \begin{array}{c} d_1 \\ d_2 \end{array} \right] \mid \left[ \begin{array}{c} d_1 \\ d_2 \end{array} \right] + 1, \ldots, d_1d_2 - 1 \right\} \cup \left[ d_1d_2, +\infty \right).$$

Definition 1. Let $M \in \mathcal{M}_{d_1,d_2}$ be a real number. We define $\mu_{\text{Choi}}^{d_1,d_2,M}$ to be the probability measure of the random quantum channel $\Phi \in C_{d_1,d_2}$, defined as follows:

1. Consider a random complex Wishart matrix $W$ of parameters $(d_1d_2,M)$.
2. Find the positive semidefinite matrix defined by the partial trace, $H := [\text{Tr}_{d_1} \otimes \text{id}_{d_2}]W$.
3. Write the dynamical matrix (or Choi matrix)

$$J := (\mathbb{1}_{d_1} \otimes H^{-1/2})W(\mathbb{1}_{d_2} \otimes H^{-1/2}).$$

4. Reshuffle the Choi matrix $J$ to obtain the superoperator $\Phi = J^R$ [see (11) for the definition of reshuffling]; in other words, $\Phi$ is the unique quantum channel having the Choi matrix $J_\Phi = J$.

Several remarks are in order here. First, note that the condition $M \in \mathcal{M}_{d_1,d_2}$ allows for the existence of the Wishart distribution of $W$. Second, the lower bound on the integer values of $M$, $Md_2 \geq d_1$, implies that the random matrix $H$ is, generically, invertible. Indeed, $W$ follows a Wishart distribution of parameters $(d_1,d_2,M)$, and thus $H$ is also Wishart, with parameters $(d_1,Md_2)$. Hence, with probability one, $H$ is
positive definite, rendering valid the normalization procedure from (14). The random matrix \( J \) is constructed to be positive semidefinite, rendering the corresponding channel \( \Phi \) completely positive; the trace preservation condition follows from (14). Since the matrix \( H \) is generically invertible, the rank of the Choi matrix \( J \) (and thus the Choi rank of \( \Phi \)) is, almost surely,

\[
\text{rk}_C(\Phi) = \min(d_1d_2,M).
\]  

Finally, let us point out that from a computational perspective, the costly operation in the procedure above is the inversion of the \( d_1 \times d_1 \) matrix \( H \), needed to enforce the trace preservation condition.

B. Random Kraus operators

**Definition 2.** Let \( M \) be an integer satisfying \( Md_2 \geq d_1 \). We define \( P_{d_1,d_2,M}^{\text{Kraus}} \) to be the probability measure of the random quantum channel \( \Phi \in \mathcal{C}_{d_1,d_2} \), defined as follows:

1. Generate \( M \) independent \( d_2 \times d_1 \) non-Hermitian matrices \( G_1, \ldots, G_M \) from the complex Ginibre ensemble.
2. Compute the positive semidefinite matrix \( H = \sum_{i=1}^M G_i^\dagger G_i \geq 0 \).
3. Define the set of Kraus operators \( A_i = \sum_{i=1}^M A_i = H^{-1/2}A_iH^{-1/2}, i = 1, \ldots, M \).
4. The channel \( \Phi \) is defined via its Kraus decomposition \( \Phi(\cdot) = \sum_{i=1}^M A_i \cdot A_i^\dagger \).

Let us first justify the validity of the construction. As in the random Choi matrix setting above, the matrix \( \Phi \) has a Wishart distribution of parameters \( (d_1, Md_2) \); hence, it is generically positive definite. The operators \( A_i \) satisfy the condition

\[
\sum_{i=1}^M A_i^\dagger A_i = H^{-1/2}HH^{-1/2} = 1_{d_1},
\]

proving that the completely positive map \( \Phi \) associated with Kraus operators \( A_i \) is trace preserving and forms a legitimate quantum channel. By construction, the channel \( \Phi \) has generically Choi rank given by (15). This can also be seen as a consequence of the following result, showing that the probability measures defined as above correspond to the ones obtained from random Choi matrices, in the case of integer parameter \( M \) (for a proof, see Appendix A):

**Proposition 1.** For all integers \( M \) such that \( Md_2 \geq d_1 \), we have \( P_{d_1,d_2,M}^{\text{Kraus}} = P_{d_1,d_2,M}^{\text{Choi}} \).

Finally, let us point out here that the computational cost of the procedure presented in Definition 2 comes from inverting the matrix \( H \).

C. Environmental form

**Definition 3.** Let \( M \) be an integer satisfying \( Md_2 \geq d_1 \). We define \( P_{d_1,d_2,M}^{\text{Stinespring}} \) to be the probability measure of the random quantum channel \( \Phi \in \mathcal{C}_{d_1,d_2} \), defined as follows:

1. Consider a random Haar isometry \( V : \mathbb{C}^{d_1} \rightarrow \mathbb{C}^{d_1} \otimes \mathbb{C}^M \) embedding the input system Hilbert space isometrically into the tensor product of the output space with an environment \( E \) of dimension \( M \).
2. The channel \( \Phi \) is defined by its Stinespring decomposition

\[
\Phi(\cdot) = [\text{id}_{d_1} \otimes \text{Tr}_M](V \cdot V^\dagger).
\]  

We next sketch a construction involving a more physical unitary evolution, which is, however, less general than the one above. Consider a total Hilbert space \( \mathcal{H} \) admitting two tensor product decompositions

\[
\mathcal{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^K = \mathbb{C}^{d_1} \otimes \mathbb{C}^M,
\]

where \( K \), respectively, \( M \), are the dimensions of two auxiliary systems: an input environment \( E_{\text{in}} = \mathbb{C}^K \), initially in an arbitrary pure state \( |\psi\rangle \in \mathbb{C}^K \), and an output environment \( E_{\text{out}} = \mathbb{C}^M \). Evolve the total system with a unitary transformation \( U \) of size \( D = d_1K = d_2M \), which is assumed to be generated according to the Haar measure on \( U(D) \). The channel \( \Phi \) is then defined as

\[
\Phi(\rho) = [\text{id}_{d_1} \otimes \text{Tr}_M](U(\rho \otimes |\psi\rangle \langle \psi|)U^\dagger).
\]  

Note that the random isometry \( V \) appearing in the first step of construction in Definition 3 can be obtained by truncating the \( d_2M \times d_2M \) Haar-random unitary operator \( U \) to a \( d_2M \times d_1 \) matrix \( V \).
As in the previous cases, the corresponding Choi matrix of the channel has generically rank given by (15). Note that in this case, the sampling procedure defined above has as a computational bottleneck, the sampling of the random Haar isometry V. The following result (proven in Appendix B) shows that environmental form construction, using random isometries, is also a special case of the random Choi sampling procedure defined above has as a computational bottleneck, the sampling of the random Haar isometry V. The following result (proven in Appendix B) shows that environmental form construction, using random isometries, is also a special case of the random Choi sampling procedure defined above has as a computational bottleneck, the sampling of the random Haar isometry V. The following result (proven in Appendix B) shows that environmental form construction, using random isometries, is also a special case of the random Choi sampling procedure defined above has as a computational bottleneck, the sampling of the random Haar isometry V.

Proposition 2. For all integers M such that MD_2 \geq d_1, we have \( \mu_{\text{Stinespring}}^{\text{Choi}} = \\mu_{\text{Choi}}^{\text{Stinespring}} \).

D. The Lebesgue (flat) measure

Finally, a natural probability measure is the (normalized) Lebesgue (or flat) measure on the set of quantum channels. Since the set \( \mathcal{C}_{d_1,d_2} \) is a convex compact set, one can endow it with the probability measure obtained by normalizing the volume (or Hilbert–Schmidt) measure to have total mass 1. We have the following remarkable statement (see Appendix C for a proof).

Proposition 3. The flat (or Lebesgue) measure \( \mu_{d_1,d_2}^{\text{Lebesgue}} \) on the set of quantum channels is a particular case of the constructions in Definitions 1, 2, and 3, obtained for the value \( M = d_1 d_2 \),

\[
\mu_{d_1,d_2}^{\text{Lebesgue}} = \mu_{d_1,d_2}^{\text{Stinespring}} = \mu_{d_1,d_2}^{\text{Kraus}} = \mu_{d_1,d_2}^{\text{Choi}}.
\]

To conclude, we have provided several classes of probability measures on the set of quantum channels \( \mathcal{C}_{d_1,d_2} \), indexed by a real or integer parameter \( M \), which coincide for identical values of \( M \geq d_1 d_2 \). The proof of the equivalence of measures generated by families (a) and (b) is the consequence of the isomorphism defined in (5), while the equivalence with (c) follows from the fact that the Ginibre ensemble induces the Haar measure on the set of unitary matrices, given the transformation \( G \mapsto G(G^\dagger G)^{-1/2} \).

The proposed families can be ordered, from particular to general, as below (see Propositions 1–3). The following relation is the main result of Sec. III

\[
\mu_{d_1,d_2}^{\text{Lebesgue}} \in \{ \mu_{d_1,d_2}^{\text{Stinespring}} \}_{M \in \mathbb{N}, M \nmid d_1} = \{ \mu_{d_1,d_2}^{\text{Kraus}} \}_{M \in \mathbb{N}, M \nmid d_1} = \{ \mu_{d_1,d_2}^{\text{Choi}} \}_{M \in \mathbb{N}, M \nmid d_1}.
\]

Note that each of the above procedures has its advantages. The environmental form (c) has a clear physical interpretation and can be approximated in an experiment, in which a random unitary matrix \( U \) can be approximated as an evolution operator of a quantum chaotic system.\(^1\) On the other hand, it is not suitable for numerical simulations. To see this, let us consider, for the sake of simplicity, the case \( d_1 = d_2 = d \), corresponding to the same input and output system sizes. In order to obtain a distribution, parameterized by \( M \) on the set of quantum channels transforming \( d \) dimensional systems to \( d \) dimensional systems, we need to generate and store a unitary matrix \( U \in \mathcal{U}(Md) \).

This, in turn, involves computing the QR decomposition, which for a matrix of dimension \( n \) has the complexity \( O(n^3) \). In our case, we get at least \( O(M^2 d^3) \) multiplications. For \( M = d^3 \), we get the complexity \( O(d^3) \). Forms (a) and (b) based on Wishart matrices and independent random Kraus operators, respectively, are the easiest to work with in numerical simulations of a generic quantum channel. Both cases (a) and (b) involve computing the inverse of square of a \( d \) dimensional matrix. Aside from this, we have in (b) \( d^2 \) multiplications of \( d \) dimensional matrices. As matrix multiplication in typical implementations has the complexity of \( O(d^3) \), we get that the overall complexity is \( O(d^5) \). Case (a) involves the multiplication of the \( d^2 \) dimensional matrix \( W \); hence, it has the complexity of at least \( O(d^8) \). Note also that there does not exist a simple procedure to sample from a Wishart distribution of parameters \( (d_1 d_2, M) \) for a non-integer \( M \in \mathcal{M}_{d_1,d_2} \). Therefore, for numerical implementations, one can recommend algorithm (b) involving random Kraus operators.

Several other families of probability distributions on the set \( \mathcal{C}_{d_1,d_2} \) of physical and mathematical interest are discussed in Appendix F.

IV. DISTRIBUTION OF OUTPUT STATES OF RANDOM QUANTUM CHANNELS

We consider in this section the output state of a random quantum channel for a given input. We start by recalling the induced measures on the set of density matrices. This one-parameter family of probability measures \( \nu_{d,t} \) has been introduced in Ref. 33 and can be described in two equivalent ways. Let us define, for a given Hilbert space dimension \( d \), the set of admissible parameters

\[
\mathcal{S} := \{ 1, 2, \ldots, d-1 \} \cup [d, \infty),
\]

which is precisely the set of allowed parameters for the complex Wishart distribution. On the one hand, one can consider a complex Wishart matrix \( W \) of parameters \( (d, s) \) (where \( d \) is the size of \( W \) and \( s \in \mathcal{S} \) is a parameter) and normalize its trace,

\[
\frac{W}{\text{Tr} W} \sim \nu_{d,s}.
\]

Equivalently, for the integers \( s \in \mathcal{S} \), one can consider a uniformly distributed vector \( x \) on the unit sphere of \( \mathbb{C}^d \) and take its partial trace with respect to the “environment” \( \mathbb{C}^s \),

\[
[\text{id}_d \otimes \text{Tr}_s]|x\rangle\langle x| \sim \nu_{d,s}.
\]
Remarkably, the uniform measure on the set of $d \times d$ density matrices corresponds to the particular value $s = d$. This fact is to be compared with the situation for quantum channels [see Proposition 3 and Eq. (21)].

Proposition 4. Let $\Phi : M_{d_2}(C) \rightarrow M_{d_1}(C)$ be a random quantum channel having distribution $\mu_{\text{Stinespring}}^{d_1,d_2;M}$ for integer $M \geq d_1/d_2$. Then, for any given fixed pure input state $|\psi\rangle\langle\psi|$, the output state $\Phi(|\psi\rangle\langle\psi|)$ has the distribution $\nu_{d_1,M}$.

Proof. We have, using the Stinespring form of the random quantum channel $\Phi$, 

$$\Phi(|\psi\rangle\langle\psi|) = [id_{d_1} \otimes \text{Tr}_M](V|\psi\rangle\langle\psi|V^*) = [id_{d_1} \otimes \text{Tr}_M]|x\rangle\langle x|,$$  

(25)

where $|x\rangle := V|\psi\rangle$. Since the isometry $V$ is Haar-distributed and the unit vector $\psi$ is fixed, the vector $x$ is uniformly distributed on the unit sphere of $C^{d_1,M}$. The conclusion follows from the environmental description of the induced measures [see Eq. (24)].

From the proposition above, we can infer that the average of the output state (with respect to the randomness in the channel) for a fixed input is the maximally mixed state,

$$\mathbb{E}\Phi(|\psi\rangle\langle\psi|) = \mathbb{E}_{\nu_{d_1,M}} p = \frac{d_1}{d_2}.$$  

(26)

This fact is equally a consequence of the following result:

Proposition 5. The average of a random quantum channel having distribution $\mu_{\text{Stinespring}}^{d_1,d_2;M}$ is the maximally depolarizing channel $\mathbb{E}\Phi = \Phi_*$, with

$$\Phi_* : M_{d_2}(C) \rightarrow M_{d_1}(C)$$

$$X \mapsto \text{Tr}(X)\frac{d_1}{d_2}.$$  

(27)

Proof. The conclusion follows easily from the computation of the average Choi matrix $J_\Phi$ using the Weingarten calculus,36

$$\mathbb{E}J_\Phi = M1_{d_1,d_2} \frac{1}{d_2 M} = \frac{d_1 d_2}{d_2} = J_{\Phi_*}.$$  

(28)

Let us now consider two statistical quantities associated with an arbitrary quantum channel: the average output purity and the unitarity:36

$$p(\Phi) = \mathbb{E}\text{Tr}\Phi(|\psi\rangle\langle\psi|)^2.$$  

$$u(\Phi) = \frac{d_1}{d_1 - 1} \mathbb{E}\text{Tr}\{\Phi(|\psi\rangle\langle\psi|) - \Phi(1_{d_1/d_1})\}^2.$$  

(29)

where the expectation corresponds to the choice of a uniform unit vector $\psi$ on the unit sphere of the input space $C^{d_1}$. Note that for unital channels (satisfying $\Phi(1_{d_1}) = 1_{d_1}$ for $d = d_1 = d_2$), the two quantities above are related by the relation $\frac{d_1 - 1}{d_1} u(\Phi) = p(\Phi) - 1/d$. We compute the averages of these two quantities in the next proposition.

Proposition 6. Let $\Phi : M_{d_2}(C) \rightarrow M_{d_1}(C)$ be a random quantum channel having distribution $\mu_{\text{Stinespring}}^{d_1,d_2;M}$ for an integer $M \geq d_1/d_2$. Then, the expectation values of the average output purity and unitarity read

$$\mathbb{E}p(\Phi) = \frac{d_1 M + M}{d_1 M + 1},$$  

(30)

$$\mathbb{E}u(\Phi) = \frac{M(d_2^2 - 1)}{(d_1 M)^2 - 1}.$$  

(31)

Proof. First of all, note that the expectation over the random pure state $|\psi\rangle$ in the definition of the quantities $p,u$ can be absorbed in the expectation over the random channel $\Phi$. Hence, in the following, we shall assume that $|\psi\rangle$ is some fixed unit vector in $C^{d_1}$. We shall make use of the following result, proven in Lemma 4 in Appendix D:

$$\mathbb{E}\text{Tr}[\Phi(A)\Phi(B)] = \frac{\text{Tr}(A)\text{Tr}(B)d_2(M^2 - 1) + \text{Tr}(AB)M(d_2^2 - 1)}{(d_1 M)^2 - 1}.$$  

(32)

Applying the result above for $A = B = |\psi\rangle\langle\psi|$ gives us Eq. (30). Note that this result could have been obtained directly from Proposition 4 using the formula for the average purity of a random density matrix from [Ref. 33, Eq. (5.11)].
To show (31), we make use again of Lemma 4, this time for \( A = |\psi\rangle\langle\psi|, B = 1_d/d_1 \) and then with \( A = B = 1_d/d_1 \).

Note that in the regime where \( d_2 \to \infty \), the average purity of a random quantum channels scales as \( 1/M \).

**Corollary 7.** The average output purity and the average unitarity of a uniformly distributed random quantum channel \( \Phi \sim \mu_{\text{Lebesgue}}^{d_1,d_2} \) are

\[
\mathbb{E} p(\Phi) = \frac{d}{d^2 - d + 1} \quad \text{and} \quad \mathbb{E} u(\Phi) = \frac{d^2}{d^4 + d^2 + 1}.
\]

(V. SPECTRAL PROPERTIES OF THE SUPEROPERATOR)

In this section, we analyze the spectral properties (singular values and eigenvalues) of generic superoperators \( \Phi \) represented by a non-Hermitian matrix of order \( d^2 \) (we consider the case \( d_1 = d_2 = d \) here). Note that we use the same letter \( \Phi \) to denote the linear map representing a quantum channel \( \Phi : M_d(\mathbb{C}) \to M_d(\mathbb{C}) \) and the corresponding matrix, when seen as an operator on \( \mathbb{C}^d \cong M_d(\mathbb{C}) \). As explained in Sec. III, one obtains the superoperator matrix by reshuffling the Choi (or dynamical) matrix, \( \Phi = \Phi^{\text{Fano}} \).

Before we move on to study random superoperators, let us first recall some general properties of such matrices. If \( \Phi \) is the superoperator of a quantum channel (completely positive and trace preserving linear map), then\(^{37,38}\)

1. the spectrum of \( \Phi \) is contained in the unit disk \( \{ z \in \mathbb{C} : |z| \leq 1 \} \),
2. there exists a Perron–Frobenius eigenvalue \( \lambda_1 = 1 \), and
3. the eigenspace of the eigenvalue \( \lambda_1 = 1 \) contains a positive semidefinite element.

Actually, the structure of the spectrum of a superoperator is much richer (see, e.g., Refs. 39 and 40), but the properties above are the only ones we need in this paper. Of crucial importance is the modulus of the sub-leading eigenvalue \( r = |\lambda_2| \leq 1 \) and the spectral gap \( \gamma := 1 - r \geq 0 \) that determines the convergence of the system to the equilibrium (see Fig. 1).

It was noted in Ref. 16 that the properties of the superoperator \( \Phi \) corresponding to a random operation can be modeled by the real Ginibre ensemble. To explain this fact, it is convenient to use the Bloch vector representation of a map. Any density operator \( \rho \) of size \( d \) can be represented using the generalized Bloch vector,

\[
\rho = \frac{1}{d} \left( 1_d + \sum_{i=1}^{d^2-1} \tau_i \Lambda_i \right),
\]

where \( \Lambda_i \) denotes the vector of three Pauli matrices for \( d = 2 \) and are proportional to eight Gell–Mann matrices for \( d = 3 \), while for a higher dimension, it represents the vector of \( d^2 - 1 \) Hermitian and traceless generators of \( SU(d) \), normalized as \( \text{Tr}(\Lambda_i \Lambda_j) = d \delta_{ij} \). Usually the order of the generators is not relevant, but for the purpose of studying the quantum to classical transition and the effects of super-decoherence and coherification of a channel,\(^{41,42}\) it will be convenient to choose the order \( \sigma_z, \sigma_x, \sigma_y \), and in higher dimensions, first select \( d-1 \) generators \( \Lambda_i \) as diagonal ones (see Sec. VI). Since any density matrix \( \rho \) is Hermitian, all components of the Bloch vector \( \tau \) are real, \( \tau_i = \text{Tr} \Lambda_i \rho \in \mathbb{R} \) for \( i = 1, \ldots, d^2-1 \).

In the case of a state \( \rho_{AB} \) of a bipartite \( d \times d \) system, it is convenient to expand the density matrix in the product basis formed by tensor products of the generators, \( \Lambda_i \otimes \Lambda_j \). It leads to the following *Fano form*\(^{45}\) applicable to any bi-partite state:\(^{45}\)

\[
\rho_{AB} = \frac{1}{d^2} \sum_{i,j=0}^{d^2-1} R_{ij} \Lambda_i \otimes \Lambda_j.
\]

**FIG. 1.** Sketch of the spectrum of a superoperator \( \Phi \) associated with a random channel, which consists of the leading Perron–Frobenius eigenvalue \( \lambda_1 = 1 \) and a bulk forming the Girko disk of radius \( r = 1/d \). The spectral gap reads \( \gamma = 1 - r \).
The expansion coefficients are given by the projection of the state onto the elements of the product basis, \( R_{ij} = \text{Tr}(\rho_{AB}(A_i \otimes A_j)) \). As \( \Lambda_0 = I \), the matrix \( R \) takes the form

\[
\hat{R} = \begin{bmatrix}
1 & a^T \\
b & R
\end{bmatrix}.
\]

(36)

The vectors \( a \) and \( b \) of length \( d^2 - 1 \) represent Bloch vectors \( \tau_A \) and \( \tau_B \) of both partial traces, \( \rho_A = [\text{id}_A \otimes \text{Tr}_B] \rho_{AB} \) and \( \rho_B = [\text{Tr}_A \otimes \text{id}_B] \rho_{AB} \), respectively. These parameters can be thus determined locally, while the square matrix \( R \) of size \( d^2 - 1 \), a truncation of \( \hat{R} \), describes correlations between both subsystems. In the case of a product state, \( \rho_{AB} = \rho_A \otimes \rho_B \), its elements are \( R_{ij} = a_i b_j \), so the state is separable. In general, the real correlation matrix \( R \) is non-symmetric, and for fixed local Bloch vectors \( a \) and \( b \), only a suitable choice of \( R \) assures positivity of the state.\(^{44} \) The norm of the correlation matrix \( R \) can be used to formulate separability criteria—for a sufficiently small norm \( \| R \| \), the state \( \rho_{AB} \) is separable.\(^{45,46} \)

Let \( \vec{r} \) represents an initial state \( \rho \) and \( \vec{r}' \) be the Bloch vector of the image \( \rho' = \Phi(\rho) \), where, for simplicity, we assumed that both dimensions are equal, \( d_1 = d_2 = d \). Any channel \( \Phi \) can be now represented by the action on the Bloch vector

\[
\vec{r}' = Q \vec{r} + \kappa,
\]

(37)

where \( Q \) is a real matrix of size \( d^2 - 1 \), while \( \kappa \) is a translation vector of length \( d^2 - 1 \), which vanishes for unital maps. Hence, the superoperator \( \Phi \) can be represented\(^{37} \) by an asymmetric real matrix of order \( d^2 \),

\[
\Phi = \begin{bmatrix}
1 & 0 \\
\kappa & Q
\end{bmatrix}.
\]

(38)

The above form is convenient for spectral analysis: the spectrum of the superoperator \( \Phi \) consists of the leading eigenvalue \( \lambda_1 = 1 \) and the \( d^2 - 1 \) eigenvalues of the matrix \( Q \), which can be complex. The trace of the real distortion matrix \( Q \) has an operational interpretation as it determines the average fidelity between a random pure state \( |\psi\rangle \langle \psi| \) and its image with respect to map \( \Phi \).\(^{48} \)

Note a similarity between form (38) of an arbitrary operation \( \Phi \) and the matrix (36) appearing in the Fano form of a bipartite state. The vector \( a \) vanishes here due to the trace preserving condition. The observed analogy can be formally explained with use of the Jamiołkowski isomorphism.

**Proposition 8.** Bloch representation (38) of a quantum operation \( \Phi \) is equivalent with the Fano form (36) of the partially transposed Jamiołkowski state (8), \( |J_0 \rangle^T / d \).

**Proof.** Action of the map \( \Phi \) on an arbitrary operator \( X \) can be expressed by the corresponding Choi matrix \( J_0 \) in the following way (Ref. 21, Chap. 2.2):

\[
\Phi(X) = [\text{id}_d \otimes \text{Tr}_d] \left( J_0 (1_d \otimes X^T) \right).
\]

(39)

Matrix elements of the Bloch representation (38) of a channel \( \Phi \) can be written as \( \Phi_{ij} = \frac{1}{d} \text{Tr}(\Lambda_i \Phi(\Lambda_j)) \). Therefore, making use of formula (39), we get

\[
\Phi_{ij} = \frac{1}{d} \text{Tr}(\Lambda_i \Phi(\Lambda_j)) = \frac{1}{d} \text{Tr}(\Lambda_i [\text{id}_d \otimes \text{Tr}_d] \left( J_0 (1 \otimes \Lambda_j^T) \right)) = \frac{1}{d} \text{Tr}(\Lambda_i \otimes 1) J_0 (1 \otimes \Lambda_j^T) = \frac{1}{d} \text{Tr}(\frac{1}{d} J_0^T (\Lambda_i \otimes \Lambda_j) + \hat{R}) (\frac{1}{d} J_0^T / d)_{ij}.
\]

(40)

An alternative proof of the above fact, using Kraus representation, can be formulated using algebraic Lemma 5 stated in Appendix E.

Proposition 8 shows a direct link between the problem of finding restrictions for the correlation matrix \( R \) to assure positivity of the bipartite state \( \rho_{AB} \) in (35) analyzed in Ref. 44 and the question for what matrix \( Q \) appearing in (37) the corresponding map \( \Phi \) is completely positive.\(^{49,50} \) For any trace preserving quantum operation, the vector \( a \) in (36) vanishes, so the second question can be considered as a special case of the first one. In the simplest case of one-qubit operation, \( d = 2 \), conditions for the matrix \( Q \) that imply complete positivity are known,\(^{43–45} \) but for larger dimensions, the problem becomes rather complicated.

However, if a random channel \( \Phi \) of a large dimension \( d \gg 1 \) is generated according to the flat measure, \( \rho_{\text{Lebesgue}} \), these constraints become weaker. In the limit of large dimension \( d \), the first two cumulants of \( \Phi \) are identical to those of the Gaussian distribution, and the higher cumulants can be neglected.\(^{51} \) This yields evidence that the statistical properties of the matrix \( Q \) of order \( d^2 - 1 \), forming the core of the superoperator \( \Phi \), can be described by a random matrix \( G_0 \) from the real Ginibre ensemble. Thus, the spectrum of \( Q \) forms in the complex plane a scaled Ginibre disk.\(^{52} \) Normalization of the Ginibre matrix implies that the eigenvalues concentrate in the disk of radius \( 1/\sqrt{M} \), where \( M \) is the number of random Kraus operators defining the map.\(^{16} \)
In the case $M = d^2$, corresponding to the uniform distribution of channels, the radius of the disk behaves as $r \approx 1/d$, and for large dimension, the distribution in the disk becomes uniform. Since the trace preserving condition assures the leading Perron–Frobenius eigenvalue $\lambda_1 = 1$, the average size of the spectral gap, $y = \lambda_1 - |\lambda_2|$, behaves as $(y) \approx 1 - 1/d$.

Let us also mention that the average number of real eigenvalues of a uniform random superoperator $\Phi$ of size $d^2$ has been numerically observed to behave asymptotically as $\sqrt{2/\pi} \cdot d$, fitting the corresponding fraction for the real Ginibre ensemble. In particular, for large dimensions, the fraction of real eigenvalues decreases as $1/d$.

Rigorous mathematical results on the spectral gap have been obtained in the setting where the number $M$ of Kraus operators is fixed, in relation to the so-called quantum expanders. Hastings started this line of inquiry in Ref. 54, where he showed that random mixed unitary channels with $M$ unitary Kraus operators have a spectral gap of order $y \approx 1 - 1/\sqrt{M}$. Similar results were obtained for random quantum channels coming from random isometries in Refs. 55 and 56.

Let us now discuss singular values of a random superoperator $\Phi$. The leading singular value is equal to unity, which is a consequence of preservation of trace,

$$\langle \chi | \Phi | \chi \rangle = \langle \chi | \chi \rangle = 1,$$

where $|\chi\rangle = |\rho_{av}\rangle/|\rho_{av}|_2$ represents the normalized vector of length $d^2$ corresponding to the invariant state of the map, $\rho_{av} = \Phi(\rho_{av})$. To analyze the remaining singular values, we draw again a parallel between the matrix $\Phi$ and a properly normalized reshuffled Wishart matrix of parameters $(d^2, M)$, $W^R/(dM)$. In the case of Wishart matrices, it has been shown in Ref. 57, Theorem 3.1 that in the regime where $d, M \to \infty$, the singular values of

$$\sqrt{M} \left[ W^R/dM - |\chi\rangle\langle\chi| \right]$$

converge, in moments, to the quarter-circle law,

$$d\mu_{qc} = \frac{\sqrt{1-x^2}}{\pi} 1_{[0,2]}(x)dx,$$

related to the Marčenko–Pastur distribution. This is evidence toward the claim that once the projection on the Perron–Frobenius eigenvalue is removed, the spectral norm of the remaining matrix is of order $1/\sqrt{M}$.

**VI. CLASSICAL MAPS: RANDOM STOCHASTIC MATRICES**

In this section, we describe how classical objects (probability vectors and stochastic maps) arise from quantum objects (density matrices and quantum channels) and compare their probability distributions.

A quantum state $\rho$ subjected to the coarse-graining map, which describes quantum decoherence, produces a classical probability vector, $p = \text{diag}(\rho)$. In the next proposition, we compute the probability distribution on the probability simplex induced by the induced measures on the set of density matrices. Let us recall first that a probability vector $p = (p_1, \ldots, p_d) \in \Delta_d$ has a Dirichlet distribution with parameter $s$,

$$D_s(p_1, \ldots, p_d) = C_s p_1^{s-1} p_2^{s-1} \cdots p_d^{s-1} (1-p_1 - \cdots - p_d-1)^{s-1},$$

with a suitable normalization constant $C_s$. We refer the reader to Ref. 58, Chap. XI.4 for more details on Dirichlet distributions.

**Proposition 9.** Let $\rho \in M_d(\mathbb{C})$ be a random density matrix from the induced ensemble of parameters $(d, s)$: $\rho = [\text{id} \otimes \text{Tr}_1](|\psi\rangle \langle \psi|)$, where $\psi \in \mathbb{C}^d$ is uniformly distributed on the unit sphere. Then, $p = \text{diag}(\rho) \in \Delta_d$ has a Dirichlet distribution $D_s(p_1, \ldots, p_d)$ on the probability simplex.

**Proof.** Recall from Refs. 17 and 32 that random density matrices from the induced ensemble of parameters $(d, s)$ are obtained as

$$\rho = \frac{GG^\dagger}{\text{Tr}(GG^\dagger)},$$

where $G \in M_{d \times d}(\mathbb{C})$ is a random rectangular Ginibre matrix (having i.i.d. standard complex Gaussian entries). In particular, the diagonal entries read

$$p_i = p_{ii} = \frac{h_i}{\sum_{j=1}^d h_j},$$

with $h_i = \sum_{k=1}^d |G_{ik}|^2$. It is easy to see that $2h_i$ has a chi-squared distribution of the parameter $2s$ (taking into account the real and imaginary parts of $G_{ik}$). Hence, $h_i$ are i.i.d. random variables with distribution $\text{Gamma}(s)$. The conclusion follows now from the fact that normalizing i.i.d. Gamma random variables yields the Dirichlet distribution [see Ref. 58, Theorem XI.4.1].
Remark 1. The uniform distribution $D_s(p_1, \ldots, p_d)$ on the probability simplex is obtained by taking the diagonal of random pure states ($s = 1$). The diagonal of uniformly distributed random density matrices ($s = d$) does not yield uniform probability vectors but the distribution $D_d(p_1, \ldots, p_d)$, which is more concentrated toward the “central” point $(1/d, \ldots, 1/d)$ of the probability simplex $\Delta_d$. The same effect occurs for any $s > 1$ (see Fig. 2).

In the same way that one obtains a classical probability distribution from a quantum state, one can obtain a classical Markov map from a quantum channel. Any quantum channel $\Phi$ generates a corresponding classical transition matrix $T$ in two equivalent ways. For any channel $\Phi: M_d(\mathbb{C}) \to M_d(\mathbb{C})$, we associate the column-stochastic matrix

$$M_{d_1 \times d_2}(\mathbb{R}) \ni T_{ji} = \langle j|\Phi(|i\rangle\langle i|)|j\rangle.$$  \hfill (47)

Equivalently, $T$ can be obtained by reshaping the diagonal of the Choi matrix $\Phi = \Phi^H$ of size $d_1 d_2$ into a matrix of size $d_2 \times d_1$. As matrix $T$ is weakly positive, so are all entries of $T$. It is easy to check that if $\Phi$ satisfies the trace preserving condition $[\text{Tr}_{d_2} \otimes \text{id}_{d_1}]\Phi = \text{id}_{d_1}$, the corresponding matrix $T$ is (column-)stochastic, and if $\Phi$ is also unital ($d = d_1 = d_2$), $[\text{id}_{d} \otimes \text{Tr}_{d_2}]\Phi = \text{id}_{d_2}$, the corresponding matrix $T$ is also bistochastic [see, e.g., Ref. 41]. If the quantum map $\Phi$ is written in its Kraus form, $\Phi(\rho) = \sum_k A_k \rho A_k^*$, then the quantum superoperator is represented by a matrix of size $d_1 d_2$ obtained as a sum of Kronecker products $\Phi = \sum_{k=1}^M A_k \otimes A_k$, while the corresponding classical transition matrix of size $d_2 \times d_1$ can be represented by a sum of Hadamard products $T = \sum_{k=1}^M A_k \otimes A_k$. Observe that the trace preserving condition $\sum_{k=1}^M A_k^* A_k = \mathbb{1}_{d_1}$ implies that both $\Phi$ and its transpose $T^T$ are (column-)stochastic, and thus, $T$ is bistochastic.

We have shown above that any quantum operation $\Phi$ determines a certain classical transition matrix, which can appear due to the effects of super-decoherence.41 In the four-index notation, one can write $T_{ji} = \Phi_{j|i} = (J_{j|i})_{ji}$. Thus, any ensemble of random quantum operations induces a certain ensemble of classical stochastic matrices. Let us analyze this distribution for the ensemble of quantum channels from Sec. III defined by the environmental form (c) and random Haar isometries.

Proposition 10. Let $\Phi: M_d(\mathbb{C}) \to M_d(\mathbb{C})$ be a random quantum channel from the ensemble $\mu_{\text{Stinespring}}$, where $M \geq d_1 / d_2$ is a fixed parameter. Consider the induced measure on column-stochastic maps $T \in M_{d_2 \times d_1}(\mathbb{R})$. Then, every column of $T$ has Dirichlet distribution with parameter $M$ on the simplex $\Delta_{d_2}$. However, the columns of $T$ are not independent, with the entries having covariances

$$\text{cov}[T_{ji}, T_{jk}] = \frac{d_2 d_2^2 - 1}{d_2^2 M^2 - d_2} - \frac{1}{d_2^4} < 0 \quad \forall j \in [d_2], \forall i_1 \neq i_2 \in [d_1],$$ \hfill (48)

$$\text{cov}[T_{ji}, T_{jk}] = \frac{d_2 M^2 - 1}{d_2 M^2 - d_2} - \frac{1}{d_2} > 0 \quad \forall j_1 \neq j_2 \in [d_2], \forall i_1 \neq i_2 \in [d_1].$$ \hfill (49)

FIG. 2. Dirichlet distributions ($10^5$ samples) on $\Delta_3$ for (a) $s = 1$ (left, uniform distribution on the simplex) and (b) $s = 9$ (right) with an exemplary translation vector $\vec{r}$. a projection of the shift $\delta = \rho - \rho_s$ on the $(d - 1)$-dimensional simplex $\Delta_d$ used in Eqs. 56 and 57. Panels (c) and (d) show eigenvalues of $10^2$ stochastic matrices of order $d = 3$ with columns sampled from $D_1(T_{ji}, T_{jk}, T_{kj})$ and $D_2(T_{ji}, T_{kj}, T_{kj})$, respectively, corresponding to panels (a) and (b). Red dashed circles have radii $r_1 = 1/\sqrt{3}$ and $r_2 = 1/\sqrt{3}$. Panel (e) shows the spectra of $10^2$ random quantum channels acting on $M_3$ sampled with the Lebesgue measure, and the red dashed circle has radius $r_c = 1/3$. 

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Proof. Let us first prove the result on the distribution of the individual columns of $T$. Using the Stinespring representation of the (random) quantum channel $\Phi$ (17), we have

$$T_{\mu} = \sum_{i=1}^{M} |V_{M(i-1)+k_i}|^2.$$  

(50)

For a fixed column $i$, the distribution of the elements $T_{\mu}$ are obtained by summing successive blocks of size $M$ from the vector $|v_i⟩ = V_i ∈ \mathbb{C}^d_\mu M$, which is uniformly distributed on the unit sphere of the corresponding vector space. We recognize the partial trace operation, and we have

$$T_{\mu} = \rho_{\mu}^{(i)} := [\text{id}_{d_i} \otimes \text{Tr}_M](|v_i⟩⟨v_i|)_{\mu}.$$  

(51)

Using Proposition 9 for the random quantum state $\rho_{\mu}^{(i)}$ having the induced distribution with parameters $(d_2, M)$, we obtain the claim about the individual columns of $T$. The covariance expressions can be readily obtained from spherical integration formulas or using the Weingarten calculus (see Ref. 34 or Ref. 59, Proposition 4.2.3),

$$E|V_{a,i}|^2|V_{a,j}|^2 = \frac{1}{d_2M(d_2M + 1)} \quad \forall a ∈ [d_2M], \forall i_1 ≠ i_2 ∈ [d_1],$$  

(52)

$$E|V_{a,i}|^2|V_{a,j}|^2 = \frac{1}{(d_2M)^2 - 1} \quad \forall a_1 ≠ a_2 ∈ [d_2M], \forall i_1 ≠ i_2 ∈ [d_1].$$  

(53)

Remark 2. The columns of the uniform distribution on the set of column-stochastic matrices of size $d_2 × d_1$ are independent, and therefore, their entries have covariances

$$\text{cov}[T_{j_1,i_1}, T_{j_2,i_2}] = 0 \quad \forall j_1,j_2 ∈ [d_2], \forall i_1 ≠ i_2 ∈ [d_1].$$  

(54)

This shows that the distribution $μ_{\text{Stinespring}}^{d_2,d_1,M}$ cannot yield the uniform distribution on the set of (column-)stochastic matrices for any $M ≥ 1$. Note that for $M = 1$ and $d_1 = d_2 = d$, we recover the distribution on the set of unistochastic matrices (see Refs. 60–62).

Let us point out that it is possible to generate a random uniform $d_2 × d_1$ (column-)stochastic matrix from Gaussian distributions. It is convenient to start with a rectangular matrix $G$ of order $d_2 × d_1$ from the complex Ginibre ensemble, all elements of which are independent complex random Gaussian variables. Then, a random matrix given by the Hadamard product $G ⊙ G$ contains non-negative entries $|G_{ij}|^2$. Renormalizing the matrix according to the sum in each column, we introduce a matrix $T$,

$$T_{\mu} = \frac{|G_{ij}|^2}{\sum_{k=1}^{M} |G_{ki}|^2},$$  

(55)

which is stochastic by construction. Furthermore, as each of its columns forms an independent random vector distributed uniformly in the probability simplex of size $d_2$, in the case $d_1 = d_2 = d$, this construction provides the random matrix distributed uniformly in the set of stochastic matrices. Note, however, that this is not the procedure described in Sec. III used to generate random quantum channels from Gaussian Kraus operators, so the above fact does not contradict Remark 2.

Next, let us briefly discuss the spectral properties of random stochastic matrices generated according to the Lebesgue measure in order to compare them to the results in Sec. V. Due to the classical Frobenius–Perron theorem, any stochastic matrix $T$ has the leading eigenvalue $λ_1 = 1$, which corresponds to the invariant state. Furthermore, it is known that the support of the spectrum of $T$ forms a proper subset of the unit disk described by the bounds of Karpelevich, which for a large dimension covers the entire unit disk.

However, for a random stochastic matrix of a large dimension $d$, the density of eigenvalues in the unit disk is not uniform. In order to analyze this issue, it will be useful to distinguish the uniform probability vector $\tilde{p}_μ = (1/d, \ldots, 1/d)$ and represent any other probability vector $\tilde{p}$ in shifted coordinates, $\tilde{p} = \tilde{p}_μ + \delta$. As the normalization condition implies $\sum_{i=1}^{d} δ_i = 0$, the shift vector $\delta$ has $(d - 1)$ independent variables. Let us denote by $\tilde{η}$ the projection of $\delta$ onto the $(d - 1)$ dimensional real space. Hence, the translation vector $\tilde{η} ∈ \mathbb{R}^{d-1}$ is analogous to the generalized Bloch vector $\tau$ used in the quantum case [see Fig. 2(b)]. In fact, one can treat $\tilde{η}$ as the Bloch representation (34) of a diagonal density matrix,

$$\text{diag}^\dagger(p) = \frac{1}{d^2} \left( \delta + \sum_{j=1}^{d-1} η_j \Lambda_j \right).$$  

(56)

where $\text{diag}^\dagger(p)$ is the diagonal matrix with $p$ on the diagonal, the sum goes now over all $(d - 1)$ diagonal generators $\Lambda_j$ of $\text{SU}(d)$. In such a Bloch representation of the probability vector, $p = p(\tilde{η})$, where $\tilde{η}$ is formed by the first $d - 1$ components of the Bloch vector $\tilde{τ}$, the action of the classical transition matrix $T$ can be represented by an affine transformation of the displacement vector,

$$\tilde{η}′ = G\tilde{η} + \chi,$$  

(57)
analogous to the transformation \((37)\) describing a quantum stochastic map. Here, \(C\) represents a real transformation matrix of size \(d - 1\), and the translation vector \(\tilde{\chi}\) vanishes for any bistochastic matrix. In this particular basis, the stochastic transition matrix reads

\[
T = \begin{bmatrix}
1 & 0 \\
\tilde{\chi} & C \\
\end{bmatrix}.
\]

Thus, the spectrum of the stochastic matrix \(T\) consists of the Frobenius–Perron eigenvalue \(\lambda_1 = 1\) and the remaining \(d - 1\) eigenvalues of a real asymmetric matrix \(C\). For a large dimension, \(d \gg 1\), the constraints implied on a matrix \(C\) by the fact that \(T\) is a random stochastic matrix become weak, so the statistical properties of \(C\) can be approximated by a matrix \(G_R\) from the real Ginibre ensemble. Thus, the spectrum of a random stochastic matrix consists of the leading eigenvalue \(\lambda_1 = 1\) and the scaled disk of complex eigenvalues of Girko.\(^{23,26}\) This simple reasoning is consistent with the earlier results of Chafaï\(^{64}\) and Horvat,\(^{65}\) who analyzed an ensemble of random stochastic matrices and found that the spectrum is concentrated in a disk of radius \(r \sim 1/\sqrt{d}\), so the average spectral gap \(\gamma\) behaves as \(1 - d^{-1/2}\). The circular law for random Markov operators in the general i.i.d. case was obtained in Ref. 66.

Note also that the square matrix \(C\) of size \(d - 1\) can be considered as the classical part of the matrix \(Q\) of size \(d^2 - 1\) representing a quantum channel in Eq. \((38)\), which can be written as

\[
Q = \begin{bmatrix}
C & Q_1 \\
Q_2 & Q_Q \\
\end{bmatrix}.
\]

Here, \(Q_1\) and \(Q_2\) denote rectangular matrices describing the coupling between diagonal and off-diagonal parts of the density matrix, while \(Q_Q\) is a square matrix of size \(d^2 - d\) representing the coupling in the space of coherences. In this notation, expression \((38)\) for a superoperator \(\Phi\) contains the transition matrix \(T\) as its principal block,

\[
\Phi = \begin{bmatrix}
1 & 0 \\
\tilde{\chi} & Q \\
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & C & Q_1 \\
0 & \tilde{\chi} & Q_2 \\
\end{bmatrix} = \begin{bmatrix}
T & Q_1 \\
Q_2 & Q_Q \\
\end{bmatrix},
\]

where the Bloch translation vector \(\kappa\) of length \(d^2 - 1\) is given by concatenation of two vectors, \((\tilde{\chi}, \tilde{\chi}')\), of length \(d - 1\) and \(d^2 - d\), respectively, while \(Q_1\) and \(Q_2\) denote the rectangular matrices \(Q_1, Q_2\), extended accordingly. Note that in the above formula, the sizes of each block in the first and the last matrix are different.

The above representation implies that if the core \(Q\) of the quantum superoperator \(\Phi\) can be approximated by an uncorrelated Ginibre matrix \(G_R\) of size \(d^2 - 1\), its classical block \(C\) of order \(d - 1\) inherits similar properties. Furthermore, the coherence of a quantum channel \(\Phi\) can be characterized by the 2-norm of all off-diagonal parts of the corresponding Choi matrix \(J_\Phi\). In the notation used in Eq. \((60)\), the coherence is given by the sum of squared norms of the “quantum” blocks of the superoperator, \(C_2(\Phi) = \|Q_1\|_2^2 + \|Q_2\|_2^2 + \|Q_Q\|_2^2\), where \(\| \cdot \|_2\) denotes the Hilbert–Schmidt norm.

Using Proposition 11 from Ref. 67, which bounds the asymptotic difference, in terms of infinity norm, between random the Choi matrix \(J_\Phi\) and Wishart matrix, we can conclude that the asymptotic behavior of coherence measures for random channels can be derived from the results for the random quantum states. We consider coherence measures defined for quantum channels as

\[
\begin{align*}
C_2(\Phi) &= \sum_{ij}\| (J_\Phi)_{ij} \|_2^2, \\
C_1(\Phi) &= \sum_{ij}\| (J_\Phi)_{ij} \|, \\
C_r(\Phi) &= S(\text{diag}(J_\Phi)) - S(J_\Phi),
\end{align*}
\]

where \(S\) is an appropriate entropy function. Then, if we take, for example, the random Choi matrix with distribution \(\mu_{d,d}^{\text{Lebesgue}}\), following Ref. 68, we have, for \(d \to \infty\),

\[
\begin{align*}
\frac{1}{d}C_2(\Phi) &\approx 1, \\
\frac{1}{d}C_1(\Phi) &\approx \sqrt{\pi} \frac{1}{2}, \\
\frac{1}{d}C_r(\Phi) &\approx \frac{1}{2},
\end{align*}
\]

For any bistochastic quantum map, the translation vector \(\kappa\) has to vanish. A method to generate random bistochastic maps, based on a variant of the Sinkhorn algorithm,\(^{69}\) consisting in alternating rescaling by partial traces performed on the first and the second subsystem was
used in Ref. 70. However, it remains uncertain whose measure in the set of bistochastic operations is induced by this procedure, as it is known that in the classical case, the flat measure in the space of bistochastic matrices is achieved only in the limit of a large system size \( d \).

Finally, we briefly discuss the properties of the spectrum of random stochastic matrices originating from decohered random Choi matrices. Assume that we have a random channel \( \Phi \) sampled from \( \rho^{decoherent}_{d,d} \). We introduce a decohered channel \( \Phi_b \) for some decoherence parameter \( b \in [0,1] \) as

\[
J_{\Phi_b} = bJ_\Phi + (1-b)\text{diag}(J_\Phi)
\]

so that for \( b = 1 \), one recovers an original random quantum channel, while the case of full decoherence with the classical transition matrix \( T \) at the diagonal of the Choi matrix is obtained for \( b = 0 \). The quantum to classical transition occurs as the parameter \( b \) decreases from one to zero, but the effective transition parameter depends also on the dimension \( d \).

As shown in Proposition 10, the fully decohered uniformly chosen channel \( \Phi \) does not give rise to a uniform distribution on the set of stochastic matrices, as there exists some correlation between the elements of the resulting stochastic matrix \( T \), and its columns are not uniformly sampled probability vectors. In the case of one-qubit random channels, the exact distribution of entries of the stochastic matrix \( T \) obtained by superdecoherence of \( \Phi \) was derived in Ref. 31.

For a uniformly sampled quantum channel \( \Phi \) of an arbitrary dimension \( d \), the distribution of entries of each column of the corresponding classical stochastic matrix \( T = T(\Phi) \) can be approximated by the Dirichlet distribution with the parameter \( s = d^2 \), written \( D_s(T_1, \ldots, T_d) \) on \( \Delta_d \) for \( j = 1, \ldots, d \), as visualized for \( d = 3 \) in Fig. 2(b). The spectra of random stochastic matrices with columns sampled from the uniform and the aforementioned distributions are shown in panels 2(c) and 2(d). The structure of the support of the spectra visible in panel 2c, consistent with the equilateral triangle superimposed with the interval \([-1,1]\) according to the bounds\(^\_\) for \( d = 3 \), for larger dimensions covers the entire disk of radius \( r_c \sim d^{-1/2} \). In Fig. 2(e), we show, for comparison, the spectra of uniformly sampled random quantum channels \( \Phi \) acting on the states of size \( d = 3 \).

This distribution implies that the essential spectrum of such a random stochastic matrix is highly concentrated around the origin. Numerical investigations of a stochastic matrix \( T \) obtained by complete decoherence of a random quantum channel corresponding to Eq. (63) with \( b = 0 \) reveal that the radius of the essential spectrum scales as \( d^{-3/2} \) so that the spectral gap \( \gamma \) behaves as \( 1 - d^{-3/2} \). Figure 3 shows the numerically found behavior of the radius of the essential spectrum of the form \( d^{d/2} \) as a function of the parameter \( b \). Note the rapid transition between the regime \( a = -3/2 \) and \( a = -1 \), which we will discuss in the next paragraph.

Let us first analyze this result in the case of complete decoherence, corresponding to \( b = 0 \). Consider a random variable \( x \), a component of a random probability vector of size \( d \) distributed according to the Dirichlet distribution of order \( s \). Consider the variable \( y = x - 1/d \), satisfying \( \mathbb{E}y = 0 \), for which \( \mathbb{E}y^2 = \text{Var}(x) \sim 1/(sd^2) \). Now, we construct a Ginibre-like matrix \( G \) of size \( d \times d \) filled with independent, identically distributed variables \( y \), which mimics the classical stochastic transition matrix \( T \). Observe that

\[
\mathbb{E}\|\tilde{G}\|_2^2 = d^2\mathbb{E}y^2 = \frac{d^2}{sd^2} = \frac{1}{s}.
\]

![FIG. 3. Numerically found dependence of the exponent \( \alpha \) for the model \( r \sim \alpha d^{d/2} \). The plot was obtained by fitting the model \( r \sim \alpha d^{d/2} \) for \( d = 10, 40 \), \( b \in [0,1] \) with steps of 0.005 and 100 samples for each \( d \) and \( b \).](image)
In consistence with the results discussed before, for small values of $\omega$ we recover the numerically found result $r_{\omega}$ is close to be bistochastic (see Lemma 3 for a quantitative statement). Thus, one can expect that the invariant state $\rho_{\text{inv}}$ of random quantum channels is discussed in Sec. III. The properties of the superoperator associated with these maps were discussed in Sec. V. Here, we will provide an in-depth analysis of the Perron–Frobenius invariant state of these maps.

VII. INVARIANT STATES OF RANDOM QUANTUM CHANNELS

In this section, we will focus on the properties of the invariant state of random quantum maps sampled according to the measures discussed in Sec. III. The properties of the superoperator associated with these maps were discussed in Sec. V. Here, we will provide an in-depth analysis of the Perron–Frobenius invariant state of these maps.

One of the properties characterizing random quantum maps is their ability of mixing quantum states. The image of the maximally mixed state $1_{d}/d_{1}$ under random quantum maps will be concentrated around the maximally mixed state $1_{d}/d_{2}$, so a random stochastic channel is close to be bistochastic (see Lemma 3 for a quantitative statement). Thus, one can expect that the invariant state $\rho_{\text{inv}}$ of a random map $\Phi$ can be found in the neighborhood of the maximally mixed state (see Fig. 5 for the numerical evidence). We formalize this statement in the following theorem:

Hence, the radius of the Girko disk describing the spectrum of $\tilde{\mathcal{G}}$ of size $d$ reads

$$r_{\tilde{\mathcal{G}}} = \sqrt{\frac{\|G\|^2}{d}} = \sqrt{\frac{1}{sd}}.$$  \hfill (65)

For a large dimension $d$, we can assume that a random Ginibre matrix $G$ describes the spectral properties of the core $C$ of the random transition matrix (38) generated according to the Dirichlet distribution $D$. In the case $s = 1$ (flat distribution), we have $r_{\tilde{C}} \sim d^{-1/2}$ in accordance with the results of Chafaï, Horvat, and Bordenave et al. (The case $s = d$ yields $r_{\tilde{C}} \sim d^{-1}$. For the completely decohered case, we have $s = d^{2}$, and we recover the numerically found result $r_{\tilde{\Phi}} \sim d^{-3/2}$. Additionally, a more general case $s = d^{k}$ yields $r_{\tilde{\Phi}} \sim d^{-(k+1)/2}$. During the transition from random quantum channels $\Phi$ to classical stochastic transition matrices $T$, the radius of the essential spectrum scales with dimension as $r \sim d^{2}$, with the exponent decreasing from $-1$ (quantum) to $-3/2$ (classical). This transition is visualized in Fig. 3, which shows how the exponent a changes with the decoherence parameter $b$.

In order to estimate the effective scaling parameter, one can expect that the transition occurs as the size $r_{\Phi}$ of the “quantum disk” related to all $d^{2} - d^{3}$ entries of the superoperator related to off-diagonal elements of the Choi matrix is comparable with the radius $r_{\tilde{\mathcal{G}}}$ of the “classical disk” associated with the stochastic matrix $T$ with $d^{2}$ elements obtained by reshaping the diagonal of $T$. As the norm of the off-diagonal part behaves as $\|\{J_{\Phi}\}_{ij}\|^{2} = \sum_{ij} \{J_{\Phi}\}_{ij}^2 \sim b^2$, we obtain an estimate for the “quantum” Girko disk, $r_{\tilde{\Phi}} = \sqrt{\|\{J_{\Phi}\}_{ij}\|^{2}/d} = b/d$. As discussed above, the radius of the “classical” Girko disk behaves as $r_{\tilde{\Phi}} = d^{-3/2}$. Setting $r_{\tilde{\Phi}} = r_{\tilde{\Phi}}$, we arrive at a relation for the critical value of the decoherence parameter, $b_{\star} = 1/\sqrt{d}$, which implies behavior of the dimension-independent scaling parameter $\beta = b/\sqrt{d}$.

To drive this point further, we present the numerically obtained fraction $p$ of the bulk eigenvalues of a uniformly chosen $\Phi$, contained in the “classical” disk, $\|\lambda\| \leq 1/d^{3/2}$, as a function of the super-decoherence parameter $b$ for a few values of the dimension $d$ [see Fig. 4(a)]. In consistence with the results discussed above, for small values of $b$, this fraction stays constant and then decreases sharply for a critical value $b_{\star}$, which depends on $d$. Figure 4(b) shows the same results as a function of the proposed scaling parameter, $\beta = b/\sqrt{d}$. This reveals that the scaling of the critical value $b_{\star} = 1/\sqrt{d}$ is correct, as all data merge into a single curve. These numerical results were obtained using the QuantumInformation.jl package.

Analogous scaling of the critical value of the transition parameter with the dimension was reported recently while studying superdecoherence of random Lindblad operators.

FIG. 4. Plots of the fraction of eigenvalues $p$ of a randomly chosen channel $\Phi$ contained in the “classical” disk $|\lambda| \leq 1/d^{3/2}$. Panel (a) presents results as a function of the super-decoherence parameter $b$, whereas panel (b) presents them as the function of the parameter $b_{\star} = b/\sqrt{d}$ and confirms the scaling, $b_{\star} \sim 1/\sqrt{d}$. For clarity, only a handful of data are plotted and a curve is added to guide the eye.
Theorem 2. Let $\Phi$ be a random channel sampled according to $\mu_{d,d}^{\text{Lebesgue}}$ with the unique invariant state $\rho_{\text{inv}}$. As $d \to \infty$, the invariant state converges almost surely in the trace norm to the maximally mixed state,

$$
\|\rho_{\text{inv}} - \frac{1}{d} \|_1 = O(1/d).
$$

Before we prove Theorem 2, let us first establish a few lemmas, which we will refer to in the main proof. We first prove that random quantum operations are contraction maps. We examine behavior of the Lipschitz constant (with respect to the Schatten one-norm), which is defined as minimum over constants $L$ satisfying

$$
\|\Phi(\rho - \sigma)\|_1 \leq L\|\rho - \sigma\|_1
$$

for all states $\rho, \sigma$. In order to prove this lemma, we need to introduce the diamond norm of a linear map $\Phi$:

$$
\|\Phi\|_\Diamond = \sup_{X \neq 0} \frac{\|(\Phi \otimes \text{id}_d)(X)\|_1}{\|X\|_1}.
$$

For a Hermiticity-preserving map $\Phi$, it suffices to optimize over pure states (see Ref. 21, Chap. 3.3). We have the following upper bound on the Schatten one-norm Lipschitz constant.

Lemma 1. Let $L_\Phi$ denote the Lipschitz constant of random quantum operation $\Phi$ sampled according to $\mu_{d,d}^{\text{Lebesgue}}$. Then, almost surely, as $d \to \infty$,

$$
L_\Phi \leq \frac{3\sqrt{3}}{2\pi} < 1.
$$

Proof. One can obtain

$$
L_\Phi = \max_{\rho \neq \sigma} \left\| \Phi \left( \frac{\rho - \sigma}{\|\rho - \sigma\|_1} \right) \right\|_1 = \max_{H \in \mathcal{H}} \left\| \Phi(H) \right\|_1
$$

$$
= \frac{1}{2} \max_{\langle x | y \rangle} \| \Phi(\langle x | x \rangle - \langle y | y \rangle) \|_1.
$$

In the next step, we will use the diamond norm, $\| \cdot \|_\Diamond$, to bound this value,

$$
L_\Phi \leq \max_{\langle x | y \rangle} \| \Phi(\langle x | x \rangle - 1/d)\|_1 = \max_{\langle x | y \rangle} \| \Phi(\Phi^* - \Phi)\|_1 \leq \| \Phi - \Phi^* \|_\Diamond,
$$

where $\Phi^*$ denotes maximally depolarizing channel. By Ref. 67, Theorem 16, we have $\| \Phi - \Phi^* \|_\Diamond \leq \frac{3\sqrt{7}}{2\pi} < 1$, proving the claim. \qed
The next lemma gives an upper bound on the distance between the maximally mixed state and its image through a random quantum channel. Note that it is stated in a slightly more general setting.

**Lemma 2.** Let $\Phi$ be a random channel sampled according to $\mu_{d_i,d_2}^{\text{Stinespring}}$, where $M_d \geq 1$ is a sequence of integers satisfying $M_d \sim t d^2$ as $d \to \infty$ for some constant $t > 0$. Then, we have

$$
\mathbb{E}\text{Tr}((\Phi(1_d/d) - 1_d/d)^2) = \frac{(d^2 - 1)(td^2 - 1)}{d^2(t^2d^2 - 1)} \sim \frac{1}{td^2}
$$

$$
\text{Var}( (td^2)\text{Tr}( (\Phi(1_d/d) - 1_d/d)^2 )) = \frac{2}{d^4} + O(d^{-1}).
$$

In particular, almost surely,

$$
\lim_{d \to \infty} (td^2)(\Phi(1_d/d) - 1_d/d) = 1.
$$

**Proof.** The first claim follows from Lemma 4 with the choice $A = B = \frac{1}{d}/d$ (see also Proposition 6). The second claim is proven using the Weingarten calculus to compute fourth moments of $\Phi(1_d/d)$; we provide in the supplementary material a Mathematica notebook that performs this tedious computation using the RTNI package provided in Ref. 35. Finally, the almost sure convergence follows from the Borel–Cantelli lemma.

Now, we are ready to prove Theorem 2.

**Proof of Theorem 2.** Using Lemmas 1 and 2, we have that, almost surely as $d \to \infty$,

$$
\left\| \rho_{\text{avg}} - \frac{1}{d} \right\|_1 = \left\| \Phi(\rho_{\text{avg}}) - \Phi\left(\frac{1}{d}\right) + \Phi\left(\frac{1}{d}\right) - \Phi\left(\frac{1}{d}\right) - \Phi\left(\frac{1}{d}\right) + \ldots - \Phi\left(\frac{1}{d}\right) + \Phi\left(\frac{1}{d}\right) - \frac{1}{d} \right\|_1
$$

$$
\leq 2t_{Q} + \frac{1 - L_{0}}{1 - L_{0}} \left\| \Phi\left(\frac{1}{d}\right) - \frac{1}{d} \right\|_1 \xrightarrow{k \to \infty} \frac{\left\| \Phi\left(\frac{1}{d}\right) - \frac{1}{d} \right\|_1}{1 - L_{0}}
$$

$$
\leq \frac{d^{1/2} \left\| \Phi\left(\frac{1}{d}\right) - \frac{1}{d} \right\|_2}{1 - L_{0}} = O(d^{-1}).
$$

We can also show the following stronger result, which we can only prove in the asymptotical regime $1 \ll d_1 \ll d_2$:

**Lemma 3.** Let $\Phi$ be a random channel sampled according to $\mu_{d_i,d_2}^{\text{Choi}}$, where $t > 0$ is some constant value. In the regime $1 \ll d_1 \ll d_2$, the Hermitian non-unitarity shift matrix $\delta H := \sqrt{t}d_1d_2(\Phi(1_{d_1}/d_1) - 1_{d_1}/d_1)$ converges in moments toward the standard semicircular distribution.

**Proof.** We can write

$$
\sqrt{t}d_1d_2(\Phi(1_{d_1}/d_1) - 1_{d_1}/d_1) = \sqrt{t}d_2\left( [id_{d_2} \otimes Tr_{d_2}] \left( \frac{1}{t d_1^2 d_2} W \right) \right) + \sqrt{t}d_1\left( \frac{1}{t d_1^2 d_2} [id_{d_2} \otimes Tr_{d_2}] W - \frac{1}{d_1} \right),
$$

where we introduced a Wishart matrix $W$ with parameters $(d_1, d_2, t d_1 d_2)$ such that the Choi matrix $\rho_B$ is obtained by a partial normalization of $W$ like in the Definition 1. By Corollary 2.5 in Ref. 74, we know that the matrix $\sqrt{t}d_2\left( [id_{d_2} \otimes Tr_{d_2}] W - \frac{1}{d_1} \right)$ converges to the standard semicircular distribution. To finish the proof, we need to bound the first term of the sum,

$$
\sqrt{t}d_2\left( [id_{d_2} \otimes Tr_{d_2}] \left( \frac{1}{t d_1^2 d_2} W \right) \right) = \sqrt{t}d_2\left( [id_{d_2} \otimes Tr_{d_2}] W \left( \frac{1}{t d_1^2 d_2} \right) \right)
$$

$$
\leq \sqrt{t}d_2\left( [id_{d_2} \otimes Tr_{d_2}] W \right) \left( \frac{1}{t d_1^2 d_2} \right) = O(1/d_1 d_2).
$$

One can note that $[id_{d_2} \otimes Tr_{d_2}] W$ is a Wishart matrix with parameters $(d_2, t d_1^2 d_2)$ and $[Tr_{d_2} \otimes id_{d_2}] W$ is a Wishart matrix with parameters $(d_2, (t d_1^2 d_2)^2)$. According to Theorem 2.7 in Ref. 74, we have that in the limit $d_1, d_2 \to \infty$, the spectrum of the matrix $[Tr_{d_2} \otimes id_{d_2}] W/(t d_1^2 d_2)$ concentrates around one with the convergence rate $\| [Tr_{d_2} \otimes id_{d_2}] W/(t d_1^2 d_2) - \frac{1}{d_1} \|_\infty = O(1/d_1 d_2)$; therefore,

$$
\left\| [Tr_{d_2} \otimes id_{d_2}] W \right\|_\infty = \Omega(d_1^2 d_2).
$$

On the other hand, $\| [id_{d_2} \otimes Tr_{d_2}] W \|_\infty = \Omega(d_1^2 d_2)$ (see Theorem 2.7 in Ref. 74); hence, we can upper bound the norm in Eq. (72) by $O(d_1 d_2)$.

\[\Box\]
Numerical results suggest that a stronger convergence with the rate $O(1/d^4)$ holds in Eq. (72); such a result would imply that the conclusion of the lemma holds under the more natural assumption $d_1, d_2 \to \infty$. We leave this question open.

VIII. CONCLUDING REMARKS

In this work, we analyzed various techniques of generating random quantum channels. On the one hand, we have investigated three natural methods and showed under which choice of parameters they become equivalent and induce the desired flat Lebesgue measure in the set $C_{d,k}$ of all quantum operations acting on density matrices of order $d$. On the other hand, we revealed previously unexplored properties of random channels, analyzing the invariant state and showing that due to the measure concentration phenomenon, it converges asymptotically to the maximally mixed state. Furthermore, we estimated typical deviations of a random stochastic channel from unitarity and showed that, almost surely, in the limit of large input and output dimension and for a particular scaling of the size of the environment, it becomes unital.

The spectrum of the superoperator $\Phi$ representing a random quantum channel of size $d$ consists of the leading Frobenius–Perron eigenvalue $\lambda_1 = 1$ and the remaining $d^2 - 1$ complex eigenvalues, which typically belong to the Girko circular disk of radius $r_\theta \approx d^{-1}$. In the case $d \gg 1$, the distribution of these eigenvalues becomes uniform, which is related to the fact that the core $Q$ of the superoperator matrix can be asymptotically approximated by a real random Ginibre matrix of size $d^2 - 1$.

Any quantum channel $\Phi$ determines a classical stochastic transition matrix $T$ encoded in the diagonal of the corresponding Choi matrix $J_\Phi$. The matrix $T$ of order $d$ can also be considered as a truncation of the superoperator $\Phi$ of size $d^2$. Hence, a given ensemble of random quantum channels induces an ensemble of random classical transition matrices. We investigated related ensembles of random quantum channels and the associated classical counterpart of random stochastic matrices and demonstrated common properties of the spectra of operators used in both setups. Representing a $d$-point probability distribution by its displacement vector $\vec{\eta}$ of length $d - 1$, analogous to the generalized Bloch vector $\vec{r}$, the action of the classical transition matrix $T$ is then represented by a real matrix $C$ of size $d - 1$. As this matrix can be obtained as a truncation of the matrix $Q$ representing the quantum stochastic map $\Phi$ and mimicked by a random Ginibre matrix, the spectral properties of $T$ are analogous and the bulk of the spectrum conforms to the circular law. Note that the radius of the Girko disk for random stochastic matrices of size $d$ distributed uniformly scales as $r_1 \approx d^{-1/2}$, while it behaves like $r_\infty \approx d^{-3/2}$ for transition matrices obtained due to decoherence of random quantum channels.

We also analyzed some quantities characterizing typical quantum channels. In particular, we derived expressions for the average output purity, the average unitarity (33), and the average 2-norm coherence of a random quantum channel acting on a $d$ dimensional state.

We would like to conclude this paper by presenting a list of open questions, especially regarding the connection between random quantum channels and random stochastic matrices. These include the following:

1. Which measure in the space of bistochastic operations induces the flat measure in space of bistochastic matrices?
2. Determine whether for a large matrix size $d$, a random bistochastic operation generated by the Sinkhorn algorithm applied to an initial random stochastic channel $\Phi$ covers uniformly the entire set of bistochastic operations and whether the corresponding random transition matrix $B$ obtained by a super-decoherence of a quantum channel covers uniformly the Birkhoff polytope $B_d$ of bistochastic matrices?
3. Is there a purely classical ensemble reproducing the distribution of (column-)stochastic matrices from Proposition 10?
4. Is a random state $\sigma$ of size $d^2$ asymptotically $\mathcal{H}$-free from its reshuffling $\sigma^R$ in the limit $d \to \infty$? Such a conjecture, analogous to the result of Mingo and Popa concerning a Haar random unitary matrix and its transpose (see also Ref. 77), is supported by numerical results presented in Ref. 78.

SUPPLEMENTARY MATERIAL

See the supplementary material for the Weingarten calculus computed by using the Mathematica and RTNI package.

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APPENDIX A: PROOF OF PROPOSITION 1

Proof. Let $\{A_i = G_i H^{-1/2} M_1\}_{i=1}^M$ be the set of the random Kraus operators defined as in Definition 2. The corresponding channel $\Phi$ has distribution $\mathcal{H}_d \sum_{i}^{M} A_k$ and the Choi matrix $J_\Phi$ can be expressed in the terms of given Kraus operators as
where we have used the vectorized form $|A_i\rangle$ of a matrix $A_i$ and introduced the matrix

$$W := \sum_{i=1}^{M} |G_i\rangle\langle G_i|.$$  

(A2)

Since $|G_i\rangle$ are i.i.d. complex Gaussian vectors, the matrix $W$ has a Wishart distribution of parameters $(d_1d_2, M)$. Moreover, we have

$$H^T = \sum_{i=1}^{M} G_i^* G_i = [\text{Tr}_{d_i} \otimes \text{id}_{d_2}] W,$$

which proves the claim. □

APPENDIX B: PROOF OF PROPOSITION 2

Proof. Let $M$ be an integer such that $M \geq d_1/d_2$. Let us take a random Haar isometry $V: \mathbb{C}^{d_1} \rightarrow \mathbb{C}^{d_1} \otimes \mathbb{C}^{M}$ as introduced in Definition 3. We will use the fact that for a random complex Ginibre matrix $G$ of order $d_2M \times d_1$, the matrix $G(G^\dagger G)^{-1/2}$ is a random Haar isometry. That means, we consider $V = G(G^\dagger G)^{-1/2}$, and the Kraus decomposition of the channel $\Phi$ defined by $V$ is determined by operators $A_i = \mathbb{1}_{d_1} \otimes |i\rangle V$ for $i = 1, \ldots, M$. Once more, we can calculate the Choi matrix $J_\Phi$,

$$J_\Phi = \sum_{i=1}^{M} |1_{d_1} \otimes |i\rangle V\rangle\langle 1_{d_1} \otimes |i\rangle V\langle i| = (1_{d_1} \otimes V^\dagger ) \sum_{i=1}^{M} |1_{d_1} \otimes |i\rangle \langle i| (1_{d_1} \otimes V)$$

(B1)

$$(1_{d_1} \otimes (G^\dagger \tilde{G})^{-1/2} ) (1_{d_1} \otimes G^\dagger ) (|1_{d_1}\rangle \langle 1_{d_1}| ) (1_{d_1} \otimes 1_{M} ) (1_{d_2} \otimes \tilde{G}) (1_{d_2} \otimes (G^\dagger \tilde{G})^{-1/2}).$$

Denote $\tilde{G} = (1_{d_2} \otimes G^\dagger )(|1_{d_2}\rangle ) \otimes 1_{M}$. This matrix turns out to be a random Ginibre matrix of size $d_1d_2 \times M$ due to

$$|i\rangle \otimes (j|\tilde{G}|k) = (|i\rangle \otimes (j|G^\dagger )(|1_{d_1}\rangle ) \otimes |k\rangle ) = |i\rangle \otimes (k|\tilde{G}|j).$$

(B2)

Moreover, the following relation holds:

$$[\text{Tr}_{d_i} \otimes \text{id}_{d_2}](G\tilde{G}^*G) = \sum_{i=1}^{d_2} (|i\rangle \otimes G^\dagger )(|1_{d_1}\rangle ) \langle 1_{d_1}| \otimes 1_{M} ) (|j\rangle \otimes \tilde{G}) = G^\dagger \tilde{G},$$

which implies the claim. □

APPENDIX C: PROOF OF PROPOSITION 3

Proof. The last equality that we will obtain is $\mu_{\text{Lebesgue}}^{\text{Choi}} = c_{\text{Choi}}^{\text{Lebesgue}}$. We use standard calculus methods to obtain the distribution of $J_\Phi$ (see Ref. 16). Let $f_{J_\Phi}(D)$ be the probability density function of the random Choi matrix $J_\Phi$ generated according to Definition 1 at the point $D$,

$$f_{J_\Phi}(D) \propto \int \delta(J_\Phi - D) \exp(-\text{Tr} \tilde{G}^\dagger \tilde{G}) dG$$

$$\propto \int \delta(\mathbb{1}_{d_1} \otimes H^{-1/2} \mathbb{1}_{d_1} \otimes H^{-1/2} - D) \delta(H - [\text{Tr}_{d_1} \otimes \text{id}_{d_2}] (G\tilde{G}^\dagger ) ) \exp(-\text{Tr} H) dH dG$$

$$\propto \int \delta(\sqrt{\text{Tr} \tilde{G}^\dagger \tilde{G}} \sqrt{D} - D) \delta(H - \text{Tr} [\text{Tr}_{d_1} \otimes \text{id}_{d_2}] (\sqrt{\text{Tr} \tilde{G}^\dagger \tilde{G}} \sqrt{D} ) \sqrt{D} ) \exp(-\text{Tr} H) \det H^{d_1,d_2} \det D^{d_1,d_2}$$

$$\propto \int \delta((G^\dagger - 1_{d_1,d_1}) \delta(\mathbb{1}_{d_1} - [\text{Tr}_{d_1} \otimes \text{id}_{d_2}] D) ) \exp(-\text{Tr} H) \det H^{d_1,d_2} \det D^{d_1,d_2}$$

$$\propto \delta(1_{d_1} - [\text{Tr}_{d_1} \otimes \text{id}_{d_2}] D ) \det D^{d_1,d_2}. \quad (C1)$$

In the particular case, $M = d_1d_2$, the exponent vanishes and we arrive at the desired result. □
FIG. 6. The diagram corresponding to $\text{Tr}[\Phi(A)\Phi(B)]$. When computing the expectation of this diagram with respect to the Haar-distributed random isometry $V$ (visualized as a block with two inputs and one output), we use the permutation $\alpha$ to pair the outer (red) wires and the permutation $\beta$ to pair the inner (blue) wires. The Hilbert space dimensions are as follows: (outer) upper wires: $d^2_2$, lower wires: $M$; inner wires: $d^2_1$.

APPENDIX D: AVERAGE TRACE OF PRODUCTS OF OUTPUTS OF RANDOM QUANTUM CHANNELS

We state and prove in this appendix a lemma that might be of independent interest.

Lemma 4. Consider integers $d_1, d_2, M$ such that $M \geq d_1/d_2$, and let $\Phi: M_{d_1}(\mathbb{C}) \rightarrow M_{d_2}(\mathbb{C})$ be a random quantum channel having distribution $\mu_{\text{Stinespring}}^{d_1,d_2,M}$. Then, for any matrices $A, B \in M_{d_1}(\mathbb{C})$, we have

$$E\text{Tr}(\Phi(A)\Phi(B)) = \left(\text{Tr}A\right)\left(\text{Tr}B\right)d^2_2M^2 - 1 + \text{Tr}(AB)M(d^2_2 - 1).$$

(D1)

Proof. The proof is a standard application of the graphical Weingarten calculus introduced in Ref. 79. The left-hand side of Eq. (D1) is represented in diagrammatic notation in Fig. 6. According to Ref. 79, Theorem 4.1, the average trace in the statement can be decomposed as a weighted sum of diagrams,

$$E\text{Tr}(\Phi(A)\Phi(B)) = \sum_{\alpha, \beta \in S_2} D_{\alpha,\beta} W_{\text{g}}^{d_1,d_2,M}(\alpha^{-1}\beta),$$

where the Weingarten function on $S_2$ reads

$$W_{\text{g}}^N((1)(2)) = \frac{1}{N^2 - 1} \quad \text{and} \quad W_{\text{g}}^N((12)) = \frac{-1}{N(N^2 - 1)},$$

(D2)

and the four diagrams obtained as follows: The permutation $\beta$ is used to pair the inner wires connected to the $A$ and $B$ boxes. The contributions of these diagrams are, for the identity permutation and for the transposition, respectively,

$$\beta = (1)(2) : \left(\text{Tr}A\right)\left(\text{Tr}B\right) \quad \text{and} \quad \beta = (12) : \text{Tr}(AB).$$

(D3)

Similarly, for the outer wires, we have the following multiplicative contributions:

$$\alpha = (1)(2) : d^2_2M^2 \quad \text{and} \quad \alpha = (12) : d^2_1M.$$

(D4)

Putting together the four contributions, weighted by the corresponding Weingarten functions, we obtained the announced formula. A derivation using the RTNI software package\(^3\) is provided in the supplementary material.

APPENDIX E: ALGEBRAIC LEMMA AND A Proof of Proposition 8

Before presenting an alternative proof of the proposition, which relies on the Kraus representation of the map, we formulate a useful algebraic fact concerning the trace, tensor product, and reshuffling.

Lemma 5. For any four square matrices $A, B, C,$ and $D$ of the same size, the following relation holds:

$$\text{Tr}((A \otimes B)(C \otimes D))^R = \text{Tr}(ACB^TD^T).$$

(E1)

The above identity can be directly verified by playing with indices, $A_{ab}B_{bc}(C_{cd}D_{de})^{ab} = A_{ab}C_{cd}B_{bc}^{de}D_{de}$, where sums over repeating indices takes place, but it is instructive to contemplate the proof in a form of the following diagram depicted in Fig. 7.
To demonstrate Proposition 8, we evaluate matrix elements (38) of the Bloch representation \( \Phi \) of any map \( \Phi \) expressed by the Kraus operators \( A_a \),

\[
\Phi_{ij} = \frac{1}{d} \text{Tr} (A_i \Phi (A_j)) = \frac{1}{d} \sum_a \text{Tr} (A_i A_a A^\dagger_a A^\dagger_i) = \frac{1}{d} \sum_a \text{Tr} \left( (A_i \otimes A^\dagger_i) (A_a \otimes A^\dagger_a)^R \right)
\]

\[
= \frac{1}{d} \text{Tr} \left( (A_i \otimes A^\dagger_i) \left( \sum_a A_a \otimes A^\dagger_a \right)^R \right) = \frac{1}{d} \text{Tr} \left( (A_i \otimes A^\dagger_i)^T J_\Phi \right)
\]

\[
= \frac{1}{d} \text{Tr} \left( J_\Phi^T (A_i \otimes A^\dagger_i) \right) = \tilde{R} \left( J_\Phi^T / d \right)_{ij}.
\]

In the first line, we applied Lemma 5, while in the second line, expansion (10) of the superoperator \( \Phi \) in terms of the Kraus operators \( A_a \) allowed us to arrive at the required form.

### APPENDIX F: RANDOM QUANTUM CHANNELS DISTRIBUTED ACCORDING TO OTHER MEASURES

In Sec. III, we described several families of probability distributions for quantum channels, all of them having the flat, Hilbert–Schmidt (HS) measure as a special case. Various techniques to generate random maps according to these measures were presented and compared. In this Appendix, a short review of other ensembles of quantum operations will be presented. Some ensembles are defined on a certain subset of the entire set of stochastic maps, for instance, the set of bistochastic maps. However, we shall start with an ensemble of random channels, determined by a general procedure leading to a non-uniform probability measure in the space of quantum operations.

#### 1. Measures induced by ensembles of random density matrices

Any ensemble of random states,\(^{17,33}\) used to generate random Choi matrices, defines by the rescaling (14) an ensemble of random operations. For instance, generating a Wishart matrix according to the Bures measure, \( W = XX^T \) with \( X = (I + U) G \), where \( U \in U(d^2) \) denotes a Haar random unitary matrix, while \( G \) stands for a square complex Ginibre matrix of size \( d^2 \), one obtains the Bures-like measure in the space of quantum maps. A larger family of cognate ensembles can be obtained by applying the generalized Bures distributions,\(^{49}\) while taking a product of \( s \) independent random Ginibre matrices \( X = G_1 G_2 \ldots G_s \), leads to the Fuss–Catalan distribution\(^{17} \) of order \( s \).

#### 2. Random coupling with environment in a generic mixed state

A new family of distributions, closely related to the measures \( P_{d_1,d_2,H}^{\text{Stinespring}} \) from Definition 3, can be obtained by generalizing the channel from Eq. (19) to use a mixed state for the environment. We obtain

\[
\Phi_n (\rho) = [\text{id}_{d_2} \otimes \text{Tr}_{d_1} ] [ U(\rho \otimes \sigma) U^T ].
\]

In the above equation, we need to specify the distribution of the environment state \( \sigma \), which could be deterministic or, e.g., \( \sigma = G G^T / \text{Tr} G G^T \), generated according to the HS measure with the use of a Ginibre matrix \( G \) of size \( d_2^2 \). The interaction unitary \( U \) is a Haar random unitary of order \( d_1 d_2 \).

Then, the Choi matrix \( J_{\Phi_n} = \Phi_n^B \) has full rank, but the probability measure in the space of quantum operations induced in this way is not flat. For \( d_1 = d_2 \) and \( \sigma \) sampled uniformly, the distribution of eigenvalues is given by the free product of two Marchenko–Pastur distributions, equivalent to the Fuss–Catalan distribution of order \( k = 2 \).

The proof of the above fact is based on a conjecture\(^{19} \) that a reshuffled Haar random unitary \( U^R \) behaves asymptotically like a generic Ginibre matrix, proved in Ref. 77. For a channel \( \Phi \in C_{d,d} \) of the form

\[
\Phi (\rho) = [\text{id}_{d} \otimes \text{Tr}_{d} ] [ U(\rho \otimes \sigma) U^T ],
\]
where \( \sigma \in M_d(\mathbb{C}) \) is density matrix from the HS distributions and \( U \) is the Haar unitary matrix of order \( d^2 \), we write

\[
\rho = \sum_{i} [\sigma_i \otimes U_i] (|1_1 \rangle \langle 1_1 |) = \sum_{i} [\sigma_i \otimes U_i] [U \otimes 1_2] (|1_1 \rangle \langle 1_1 |) (|1_2 \rangle \langle 1_2 |) (|1_3 \rangle \langle 1_3 |) (|1_4 \rangle \langle 1_4 |)
\]

\[
= (1 \rho \otimes (|1_1 \rangle \langle 1_1 |) (|1_2 \rangle \otimes 1_2) (|1_3 \rangle \otimes 1_3) (|1_4 \rangle \otimes 1_4)) (\sigma \otimes (1_2 \otimes \otimes 1_4) (|1_4 \rangle \otimes 1_4))
\]

\[
= U^\dagger (1_2 \otimes \sigma) (U^\dagger)^\dagger. \tag{F3}
\]

Hence, due to independence of \( U \) and \( \sigma \), the Choi matrix \( \rho \) has the Fuss–Catalan distribution of order \( k = 2 \).

### 3. Random quantum bistochastic channels

Consider an arbitrary Wishart matrix \( W = XX^\dagger \) of order \( d^2 \) acting on a composite space \( \mathbb{C}_d^k \otimes \mathbb{C}_d^m \), which determines by reshuffling transformation a completely positive map, \( \Phi_0 = W^\dagger \). Rescaling it according to (14), we obtain an legitimate Choi matrix \( J_1 \), which satisfies the partial trace condition \( \text{Tr}_{A|B} J_1 = 1_2 \) and represents a trace preserving map, \( \Phi_1 = J_1 \). To obtain a unital channel, one applies the complementary transformation performed on the output matrix,

\[
J_2 = (E^{-1/2} \otimes 1_2) J_1 (E^{-1/2} \otimes 1_2), \tag{F4}
\]

where \( E = \text{Tr}_{B|A} J_1 \) denotes the dual partial trace that forms a semipositive matrix of size \( d \). By construction, \( J_2 \) represents a dynamical matrix that satisfies the unitality condition of partial trace, \( \text{Tr}_{B|A} J_2 = 1_2 \). Alternative iterations by transformations (14) and (F4) performed on an initial random Wishart matrix \( W \) converge with probability one as this scheme belongs to the class of Sinkhorn algorithms.\(^{80-86}\) This procedure can be considered as alternating projections on manifolds\(^{87}\) or as a special case of the technique\(^ {89}\) to produce a bipartite quantum state with both prescribed partial traces. The limiting matrix \( J_\infty \) satisfies thus both partial trace conditions and represents a quantum map that is both trace preserving and unital. Observe that these both transformations correspond to a pre- and post-processing of the initial map \( \Phi_0 \) by conjugation with positive Hermitian matrices, \( \Phi_2 = \Psi_{E^{-1/2}} \cdot \Phi_0 \cdot \Psi_{E^{-1/2}} \), where \( \Psi_{E^{-1/2}}(\rho) = \rho E \) and \( E > 0 \). Convergence of the iteration procedure is related to the possibility to represent any generic completely positive map \( \Phi \) as a concatenation of a bistochastic operation \( \Phi_0 \) sandwiched between the pre- and post-processing, \( \Phi = \Psi_E \cdot \Phi_0 \cdot \Psi_H \) (see Ref. 86). Fast convergence of the above algorithm provides an efficient method to generate bistochastic random quantum channels,\(^{86}\) analogous to the Sinkhorn method of generating random bistochastic matrices.\(^ {11}\) Taking into account these results, one can expect that the measure induced in this way by the HS measure in the space of stochastic maps will not lead to the HS measure in the subset of bistochastic maps.

### 4. Randomized unitary channels

\[ \rho = \sum_{i=1}^M |p_i U_i \rangle \langle 1| \], with independent Haar random unitary matrices \( U_i \) and various choices of the measure for the random probability vector \( p \) of length \( M \). These operations are bistochastic by construction. For instance, to construct quantum expanders, Hastings used such an ensemble\(^ {84}\) with random unitary matrices \( U_i \) mixed with fixed probabilities, \( p_i = 1/M \). Note that choosing \( U_i \) from a given set of \( M \) fixed (non-random) unitaries, \( U_i \in U(d) \), this model is equivalent to random external fields\(^ {20}\) also called mixed unitary channels, often applied in the theory of quantum information. For \( d = 2 \), every bistochastic channel is unitarily equivalent to a Pauli map—a mixed unitary channel—but for larger systems, \( d \geq 3 \), there exist bistochastic quantum channels that do not belong to this class.\(^ {21}\) However, mixed unitary channels have non-zero volume in the set of bistochastic channels.\(^ {22}\) In more advanced approaches, the independent restriction of matrices \( U_i \) is loosened. Dynamics of open quantum systems defined by correlated, time-independent Hamiltonians were investigated in Refs. 89 and 90.

### 5. Random unistochastic maps

These channels can be considered as a particular case of model (f) if the state of environment of size \( M \) is the maximally mixed state, \( \Phi_U (\rho) = [U \otimes 1_M] [U (\rho \otimes 1_M) U^\dagger] \). Such maps are determined only by a unitary matrix \( U \) that is assumed to be random according to the Haar measure. Unistochastic maps are clearly unital and thus bistochastic for any dimension of the environment. It is convenient to assume that \( M = kd \) and call such maps \( k \)-unistochastic.\(^ {23}\) In the simplest case, \( k = 1 \) so that the size of the environment and the system are equal, and the set of one-qubit unistochastic maps forms a (non-convex) proper subset\(^ {24,25}\) of the regular tetrahedron of one-qubit Pauli channels. For a unistochastic channel \( \Phi_U \) determined by a unitary matrix \( U \) of size \( d^2 \), the corresponding Choi dynamical matrix is given by the reshuffled Wishart matrix \( W_1, \ldots, W_k \) and normalizing them to have unit sum (Ref. 93, Sec. V B). This

### 6. Random POVMs

In Ref. 93, the authors introduce the notion of random positive operator valued measures (POVMs), corresponding to an ensemble of generalized quantum measurements. One way to construct random POVMs is to define POVM elements \( A_1, \ldots, A_k \in M_d(\mathbb{C}) \) by sampling independent, identically distributed random Wishart matrices \( W_1, \ldots, W_k \) and normalizing them to have unit sum (Ref. 93, Sec. V B).
construction is related to the ensembles of quantum channels considered in this work in the following way. If $\Phi$ is sampled according to the Choi ensemble $\mu^{\text{Choi}}_{(d,k,n)}$ from Sec. III A, then the channel $\Psi := \text{diag}\Phi$, where diag is the dephasing operator diag $|j\rangle\langle j| = \delta_{ij}|j\rangle\langle j|$, has the distribution of a random POVM of parameters $(d,k,n)$ (see Ref. 93).

Let us end this discussion by mentioning that one can consider mixtures of the different probability measures discussed in this appendix and of those analyzed in the main text. In the recent work, the authors considered a mixture between a random unitary conjugation $X \mapsto UXU^*$ and an independent random quantum channel as in Definition 2. They studied the spectral properties of the corresponding superoperator as a function of the mixing parameter.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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