In this paper we present the formulation of relativistic electrodynamics (independent of the reference frame and of the chosen system of coordinates in it) that uses the Faraday bivector field $F$. This formulation with $F$ field is a self-contained, complete and consistent formulation that dispenses with either electric and magnetic fields or the electromagnetic potentials. All physical quantities are defined without reference frames or, when some basis is introduced, every quantity is represented as a coordinate-based geometric quantity comprising both components and a basis. The new, observer independent, expressions for the stress-energy vector $T(n)$ (1-vector), the energy density $U$ (scalar), the Poynting vector $S$ and the momentum density $g$ (1-vectors), the angular momentum density $M$ (bivector) and the Lorentz force $K$ (1-vector) are directly derived from the field equations with $F$. The local conservation laws are also directly derived from the field equations.

Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows and only a kind of union of the two will preserve an independent reality. H. Minkowski

Key words: relativistic electrodynamics, Clifford algebra

1. INTRODUCTION

In the usual Clifford algebra treatments of the relativistic electrodynamics, e.g., with multivectors [1 – 3] (for a more mathematical treatment of the Clifford algebra see also [4]), the field equations expressed in terms of the Faraday bivector field $F$ are written as a single equation using the $F$ field and the gradient operator $\partial$ (1-vector). In order to get the more familiar
form the bivector field $F$ is expressed (in [1,2]) in terms of the sum of a relative vector $E_H$ (corresponds to the three-dimensional (3D) electric field vector $E$) and a relative bivector $\gamma_5 B_H$ ($B_H$ corresponds to the 3D magnetic field vector $B$, and $\gamma_5$ is the (grade-4) pseudoscalar for the standard basis $\{\gamma_\mu\}$) by making a space-time split in the $\gamma_0$ - frame, which depends on the observer velocity $\gamma_0$; the subscript 'H' is for - Hestenes. Both $E_H$ and $B_H$ are, in fact, bivectors. Similarly in [3] $F$ is decomposed in terms of 1-vector $E_J$ and a bivector $B_J$; the subscript 'J' is for - Jancewicz. It is generally accepted in the Clifford algebra formalism (and in the tensor formalism as well) that the usual Maxwell equations with the 3D vectors $E$ and $B$ and the field equations written in terms of $F$ (or the abstract tensor $F^{ab}$ in the tensor formalism) are completely equivalent. Further both in the tensor formalism and in the Clifford algebra formalism it is assumed that the components of the 3D $E$ and $B$ define in a unique way the components of $F$. This means that the 3D $E$ and $B$ and not the $F$ field are considered as primary quantities for the whole electromagnetism. Then in order to get the wave theory of electromagnetism the vector potential $A$ is introduced and $F$ is defined in terms of $A$. In that case the $F$ field appears as the derived quantity from the potentials. Thence in all usual treatments of the electromagnetism, both in the tensor formalism and the Clifford algebra formalism, the theory is presented as that the $F$ field does not have an independent existence but is defined either by the components of the 3D $E$ and $B$ or by the components of the electromagnetic potential $A$.

In all usual Clifford algebra formulations of the classical electromagnetism, e.g., the formulations with Clifford multivectors [1-3], the standard transformations of the 3D $E$ and $B$ (first derived by Lorentz [5] and independently by Einstein [6], and subsequently quoted in almost every textbook and paper on relativistic electrodynamics) are considered to be the Lorentz transformations of these vectors, see [1-3]. The same opinion holds in the tensor formalism, see [7]. These transformations are usually derived, [7] and [1-3], by means of the above mentioned identification, i.e., taking that the components of $F$ and of the transformed $F'$ are determined in the same way by the components of the 3D $E$ and $B$ and of the transformed 3D $E'$ and $B'$ respectively. However recently important results are achieved in the works [8] in the tensor formalism and [9] in the Clifford algebra formalism. Namely in these works it is exactly proved that the standard transformations of the 3D $E$ and $B$ are not relativistically correct transformations in the 4D space-time; they are not the Lorentz transformations of the 3D $E$ and $B$. It is also
proved that the standard identification of the components of the 3D $E$ and $B$ with the components of $F$ is not relativistically correct procedure. Thence, contrary to the general belief, the usual Maxwell equations with the 3D $E$ and $B$ (i.e., with $E_H$, $B_H$, or $E_J$, $B_J$) and the observer independent field equations with the $F$ field are not physically equivalent.

Therefore in this paper we present the formulation of relativistic electrodynamics with the bivector field $F$, i.e., the formulation that deal with well-defined, observer independent, 4D quantities and not with ill-defined quantities, the 3D $E$ and $B$, or $E_H$, $B_H$ (or $E_J$, $B_J$). The presented formulation with the $F$ field is a self-contained, complete and consistent formulation that does not make use either electric and magnetic fields or the electromagnetic potential $A$ (thus dispensing with the need for the gauge conditions). In such formulation the $F$ field is the primary quantity for the whole classical electromagnetism. We also give the new expressions for the observer independent stress-energy vector $T(n)$ (1-vector), the energy density $U$ (scalar, i.e., grade-0 multivector), the Poynting vector $S$ (1-vector), the angular momentum density $M$ (bivector) and the Lorentz force $K$ (1-vector). They are all directly derived from the field equations with $F$. The local charge-current density and local energy-momentum conservation laws are also directly derived from the field equations with $F$ and there is no need to introduce the Lagrangian and the Noether theorem.

2. SHORT REVIEW OF THE INVARIANT SPECIAL RELATIVITY

This formulation of the electromagnetism with the $F$ field exclusively deals with well-defined 4D quantities. Namely in such formulation physical quantities are represented by Clifford multivectors that are defined without reference frames (when no basis has been introduced) or equivalently by a coordinate-based geometric quantity comprising both components and a basis (when some basis has been introduced). Thus these quantities are independent of the chosen inertial frame of reference and of the chosen system of coordinates in it, i.e., they are observer independent quantities. The special relativity that exclusively uses quantities defined without reference frames or, equivalently, the coordinate-based geometric quantities, can be called the invariant special relativity. The reason for this name is that upon the passive Lorentz transformations any coordinate-based geometric quantity remains unchanged. The invariance of some 4D coordinate-based geometric
quantity upon the passive Lorentz transformations reflects the fact that such mathematical, invariant, geometric 4D quantity represents the same physical object for relatively moving observers. It is taken in the invariant special relativity that such 4D geometric quantities are well-defined not only mathematically but also experimentally, as measurable quantities with real physical meaning. Thus they do have an independent physical reality. The invariant special relativity is discussed in [10-12] in the tensor formalism and in [13] in the Clifford algebra formalism. It is explicitly shown in [12] that the true agreement with experiments that test special relativity exists when the theory deals with such well-defined 4D quantities, i.e., the quantities that are invariant upon the passive Lorentz transformations. The principle of relativity is automatically included in such formulation with observer independent quantities, i.e., in the invariant special relativity, whereas in the traditional formulation of special relativity this principle acts as the postulate established outside the mathematical formulation of the theory. The $F$ formulation of the electromagnetism that is presented here is a part of the invariant special relativity that is developed in [10-13]. In [13] we have also presented the Clifford algebra formulations of relativistic electrodynamics with 1-vectors $E$ and $B$, with the real multivector $\Psi = E - e_5 c B$ and with the complex 1-vector $\Psi = E - i c B$ ($i$ is the unit imaginary). These formulations are completely equivalent to the formulation with the $F$ field, but every of them is an independent, consistent and complete formulation. However it is worth noting that in these formulations of electrodynamics in [13] with $E$, $B$, the real and complex $\Psi$ the expressions for the stress-energy vector $T(v)$ and all quantities derived from $T(v)$ are written for the special case when $v$, the velocity of observers who measure $E$ and $B$ fields is $v = cn$, where $n$ is the unit normal to a hypersurface through which the flow of energy-momentum $(T(n))$ is calculated. The more general case with $v \neq n$ will be reported in detail elsewhere. It is important to note that the observer independent quantities introduced here and the field equations written in terms of them are of the same form both in the flat and curved spacetimes.

3. SHORT REVIEW OF GEOMETRIC ALGEBRA

First we provide a brief summary of geometric algebra. We write Clifford vectors in lower case ($a$) and general multivectors (Clifford aggregate) in upper case ($A$). The space of multivectors is graded and multivectors containing elements of a single grade, $r$, are termed homogeneous and written
The geometric (Clifford) product is written by simply juxtaposing multivectors \(AB\). A basic operation on multivectors is the degree projection \(\langle A \rangle_r\) which selects from the multivector \(A\) its \(r\)-vector part (0 = scalar, 1 = vector, 2 = bivector ...). We write the scalar (grade-0) part simply as \(\langle A \rangle\).

The geometric product of a grade-\(r\) multivector \(A_r\) with a grade-\(s\) multivector \(B_s\) decomposes into
\[
A_r B_s = \langle AB \rangle_{r+s} + \langle AB \rangle_{r+s-2} + ... + \langle AB \rangle_{|r-s|}.
\]
The inner and outer (or exterior) products are the lowest-grade and the highest-grade terms respectively of the above series \(A_r \cdot B_s \equiv \langle AB \rangle_{|r-s|}\), and \(A_r \wedge B_s \equiv \langle AB \rangle_{r+s}\). For vectors \(a\) and \(b\) we have \(ab = a \cdot b + a \wedge b\), where \(a \cdot b \equiv (1/2)(ab + ba)\), and \(a \wedge b \equiv (1/2)(ab - ba)\). Reversion is an invariant kind of conjugation, which is defined by \(\tilde{AB} = \tilde{B} \tilde{A}\), \(\tilde{a} = a\), for any vector \(a\), and it reverses the order of vectors in any given expression.

In the treatments, e.g., [1–3], one usually introduces the standard basis. The generators of the spacetime algebra (the Clifford algebra generated by Minkowski spacetime) are taken to be four basis vectors \(\{\gamma_\mu\}, \mu = 0...3\), satisfying \(\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \text{diag}(+ - - -)\). This basis is a right-handed orthonormal frame of vectors in the Minkowski spacetime \(M^4\) with \(\gamma_0\) in the forward light cone. The \(\gamma_k\) (\(k = 1, 2, 3\)) are spacelike vectors. This algebra is often called the Dirac algebra \(D\) and the elements of \(D\) are called \(d\)-numbers. The \(\gamma_\mu\) generate by multiplication a complete basis, the standard basis, for spacetime algebra: \(1, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_\mu \gamma_5, \gamma_5\) (\(2^4 = 16\) independent elements). \(\gamma_5\) is the pseudoscalar for the frame \(\{\gamma_\mu\}\).

We remark that the standard basis corresponds, in fact, to the specific system of coordinates, i.e., the Einstein system of coordinates, of the chosen inertial frame of reference. (In the Einstein system of coordinates the Einstein synchronization [6] of distant clocks and Cartesian space coordinates \(x^i\) are used in the chosen inertial frame of reference.) However different systems of coordinates of an inertial frame of reference are allowed and they are all equivalent in the description of physical phenomena. For example, in [10] two very different, but completely equivalent systems of coordinates, the Einstein system of coordinates and "radio" ("r") system of coordinates, are exposed and exploited throughout the paper. For more detail about the "r" system of coordinates see, e.g., [10] and references therein.

Thence instead of the standard basis \(\{\gamma_\mu\}, \mu = 0...3\), for \(M^4\) we can use some basis \(\{e_\mu\}\) (the metric tensor of \(M^4\) is then defined as \(g_{\mu\nu} = e_\mu \cdot e_\nu\)) and its dual basis \(\{e^\mu\}\), where the set of basis vectors \(e^\mu\) is related to the \(e_\mu\) by the conditions \(e_\mu \cdot e^\nu = \delta_\mu^\nu\). The pseudoscalar \(e_5\) of a frame \(\{e_\mu\}\) is defined by
Then, e.g., the position 1-vector \( x \) can be decomposed in the \( S \) and \( S' \) frames and in the standard basis \( \{ \gamma_{\mu} \} \) and some non-standard basis \( \{ e_{\mu} \} \) as \( x = x^\mu \gamma_{\mu} = x'^\mu \gamma_{\mu'} = \ldots = x''_e e_{\mu'} \). The primed quantities are the Lorentz transforms of the unprimed ones. Similarly any multivector \( A \) can be written as an invariant quantity with the components and a basis, i.e., as a coordinate-based geometric quantity. In such interpretation the Lorentz transformations are considered as passive transformations; both the components and the basis vectors are transformed but the whole geometric quantity remains unchanged. Thus we see that under the passive Lorentz transformations a well-defined quantity on the 4D spacetime, i.e., a coordinate-based geometric quantity, is an invariant quantity. In the usual Clifford algebra formalism, e.g., \([1 \sim 4]\), the Lorentz transformations are considered as active transformations; the components of, e.g., some 1-vector relative to a given inertial frame of reference (with the standard basis \( \{ \gamma_{\mu} \} \) are transformed into the components of a new 1-vector relative to the same frame (the basis \( \{ \gamma_{\mu} \} \) is not changed). (We note that a coordinate-free form for the Lorentz transformations is presented in \([13]\) and it can be used both in an active way, when there is no basis, or in a passive way, when some basis is introduced.)

The next step in the usual treatments, e.g., \([1 \sim 3]\), is the introduction of a space-time split and the relative vectors. Since the usual spacetime algebra deals exclusively with the Einstein system of coordinates it is possible to say that a given inertial frame of reference is completely characterized by a single future-pointing, timelike unit vector \( \gamma_0 \) (\( \gamma_0 \) is tangent to the world line of an observer at rest in the \( \gamma_0 \)-system). By singling out a particular time-like direction \( \gamma_0 \) we can get a unique mapping of spacetime into the even subalgebra of the spacetime algebra (the Pauli subalgebra). For each spacetime point (or event) \( x \) this mapping is specified by

\[
x \gamma_0 = ct + x, \quad ct = x \cdot \gamma_0, \quad x = x \wedge \gamma_0.
\]

To each event \( x \) the equation (1) assigns a unique time and position in the \( \gamma_0 \)-system. The set of all position vectors \( x \) is the 3-dimensional position space of the observer \( \gamma_0 \) and it is designated by \( P^3 = P^3(\gamma_0) = \{ x = x \wedge \gamma_0 \} \). The elements of \( P^3 \) are all spacetime bivectors with \( \gamma_0 \) as a common factor \( (x \wedge \gamma_0) \). They are called the relative vectors (relative to \( \gamma_0 \)) and they will be designated in boldface. Then a standard basis \( \{ \sigma_k; k = 1, 2, 3 \} \) for \( P^3 \), which corresponds to a standard basis \( \{ \gamma_{\mu} \} \) for \( M^4 \) is given as \( \sigma_k = \gamma_k \wedge \gamma_0 = \gamma_k \gamma_0 \). The invariant distance \( x^2 \) then decomposes as \( x^2 = (x \gamma_0)(\gamma_0 x) = (ct - x)(ct + x) = c^2 t^2 - x^2 \). The explicit appearance of \( \gamma_0 \) in (1) imply that the space-time split is observer
dependent, i.e., it is dependent on the chosen inertial frame of reference. It has to be noted that in the Einstein system of coordinates the space-time split of the position 1-vector $x$ gives separately the space and time components of $x$ with their usual meaning, i.e., as in the prerelativistic physics, and (as shown above) in the invariant distance $x^2$ the spatial and temporal parts are also separated. (In the ”r” system of coordinates there is no space-time split and also in $x^2$ the spatial and temporal parts are not separated, see [10].) This does not mean that the Einstein system of coordinates does have some advantages relative to other systems of coordinates and that the quantities in the Einstein system of coordinates are more physical than, e.g., those in the ”r” system of coordinates.

Different systems of coordinates refer to the same inertial frame of reference, say the $S$ frame. But if we consider the geometric quantity, the position 1-vector $x$ in another relatively moving inertial frame of reference $S'$, which is characterized by $\gamma_0'$, then the space-time split in $S'$ and in the Einstein system of coordinates is $x\gamma_0' = ct' + x'$, and this $x\gamma_0'$ is not obtained by the Lorentz transformations (or any other coordinate transformations) from $x\gamma_0$. (The hypersurface $t' = \text{const.}$ is not connected in any way with the hypersurface $t = \text{const.}$.) Thus the customary Clifford algebra approaches to special relativity start with the geometric, i.e., coordinate-free, quantities, e.g., $x, x^2$, etc., which are physically well-defined. However the use of the space-time split introduces in the customary approaches such coordinate-dependent quantities which are not physically well-defined since they cannot be connected by the Lorentz transformations. The main difference between our approach to special relativity (by the use of the Clifford algebra) and the other Clifford algebra approaches is that in our approach, as already said, the physical meaning is attributed, both theoretically and experimentally, only to the geometric 4D quantities, and not to their parts. We consider, in the same way as H. Minkowski (the motto in this paper), that the spatial and the temporal components (e.g., $x$ and $t$, respectively) of some geometric 4D quantity (e.g., $x$) are not physically well-defined quantities. Only their union is physically well-defined and only such quantity does have an independent physical reality.

4. THE $F$ FORMULATION OF ELECTRODYNAMICS

4.1. The Determination of the Electromagnetic Field $F$
We start the exposition of electrodynamics writing the field equations in terms of $F_{1-3}$; an electromagnetic field is represented by a bivector-valued function $F = F(x)$ on spacetime. The source of the field is the electromagnetic current $j$ which is a 1-vector field. Then using that the gradient operator $\partial$ is a 1-vector field equations can be written as a single equation

$$\partial F = j/\varepsilon_0 c, \quad \partial \cdot F + \partial \wedge F = j/\varepsilon_0 c. \quad (2)$$

The trivector part is identically zero in the absence of magnetic charge. Notice that in $[1-3]$ the field equations $(2)$ are considered to encode all of the Maxwell equations. Thus it is assumed in all usual approaches in the Clifford algebra formalism that the field equations $(2)$ and the usual Maxwell equations with the 3D $E$ and $B$ (i.e., with $E_H, B_H$ or $E_J, B_J$) are physically equivalent. However, as already said, it is exactly proved in $[8]$ and $[9]$ that, contrary to the general belief, such equivalence does not exist. Hence the field equations $(2)$ are, in fact, a relativistically correct generalization of the usual Maxwell equations with the 3D $E$ and $B$.

The field bivector $F$ yields the complete description of the electromagnetic field and, in fact, there is no need to introduce either the field vectors or the potentials. For the given sources the Clifford algebra formalism enables one to find in a simple way the electromagnetic field $F$. Namely the gradient operator $\partial$ is invertible and $(2)$ can be solved for

$$F = \partial^{-1}(j/\varepsilon_0 c), \quad (3)$$

see, e.g., $[14]$ and $[1]$ Spacetime Calculus. We briefly repeat the main points related to $(3)$ from these references. The important difference is that for us, as proved in $[8]$ and $[9]$, the field equations $(2)$ are not equivalent to the usual Maxwell equations with the 3D $E$ and $B$. $\partial^{-1}$ is an integral operator which depends on boundary conditions on $F$ and $(3)$ is an integral form of the field equations $(2)$. If the charge-current density $j(x)$ is the sole source of $F$, then $(3)$ provides the unique solution to the field equations $(2)$. By using Gauss’ Theorem an important formula can be found that allows to calculate $F$ at any point $y$ inside $m$-dimensional manifold $\mathcal{M}$ from its derivative $\partial F$ and its values on the boundary $\partial \mathcal{M}$ if a Green’s function $G(y, x)$ is known,

$$F(y) = \int_{\mathcal{M}} G(y, x) \partial F(x) \mid dx \mid - \int_{\partial \mathcal{M}} G(y, x) n^{-1} F(x) \mid d^{m-1} x \mid, \quad (4)$$

$n$ is a unit normal, $n^{-1} = n$ if $n^2 = 1$ or $n^{-1} = -n$ if $n^2 = -1$, and $G(y, x)$ is a solution to the differential equation $\partial_y G(y, x) = \delta^m(y - x)$. $(4)$ is the
relation (4.17) in [14].) If \( \partial F = 0 \) the first term on the right side of (4) vanishes. This general relation can be applied to different examples.

An example is the determination of the expression for the classical Liénard-Wiechert field that is given, e.g., in [14] and [1] Spacetime Calculus. The usual procedure ([14] and [1]) is to utilize the general relation (4), in which all quantities are defined without reference frames (Geometric calculus), and to specify it to the Minkowski spacetime \( (m = 4) \). Then a space-time split is introduced by the relation

\[
ct = x \cdot n = x \cdot \gamma_0
\]

from (1). (\( n \) in (4) is taken to be \( \gamma_0 \) and (5) is the equation for a 1-parameter family of spacelike hyperplanes \( S(t) \) with normal \( \gamma_0 \); \( S(t) \) is a surface of simultaneous \( t \).) Further, for simplicity, \( \mathcal{M} \) is taken to be the entire region between the hyperplanes \( S_1 = S(t_1) \) and \( S_2 = S(t_2) \). We note that the same objections hold for such procedure as those that are mentioned at the end of Sec. 3. for the space-time split. Further, as explained in the preceding section, such procedure with the space-time split can be made only when the metric tensor of spacetime is taken to be the Minkowski metric tensor, i.e., when space and time are separated. For example it would not work for the above mentioned "r" system of coordinates, [10]. We shall not discuss this derivation further but we only quote the result for the classical Liénard-Wiechert field. The charge-current density for a particle with charge \( q \) and world line \( z = z(\tau) \) with proper time \( \tau \) is \( j(x) = q \int_{-\infty}^{\infty} d\tau v \delta^4(x - z(\tau)) \), where \( v = v(\tau) = dz/d\tau \). Then the classical Liénard-Wiechert retarded field for \( q \) (see, e.g., Sec. 5 in [14]) is

\[
F(x) = \frac{(q/4\pi\varepsilon_0)[r \wedge (v/c) + (1/2c^2)r\dot{v}(v/c)r]}{(r \cdot v/c)^3},
\]

where \( r = x - z \) satisfies the light-cone condition \( r^2 = 0 \) and \( z, v, \dot{v} = dv/d\tau \) are all evaluated at the intersection of the backward light cone (with vertex at \( x \)) and world line of that charge \( q \). It is worth noting that from the general expression (4) one can derive not only the retarded interpretation for \( F \) of a charge \( q \) but also the advanced interpretation and the present-time interpretation, i.e., an instantaneous action-at-a-distance interpretation. (This present-time interpretation will be reported elsewhere. In the tensor formalism the expressions for \( F^{ab} \) and the 4-vectors \( E^a \) and \( B^a \) in the present-time interpretation for an uniform and uniformly accelerated motion of a charge \( q \) are given in [15].)
4.2. The Lorentz Force and the Motion of a Charged Particle in the Electromagnetic Field $F$

In the field view of particle-to-particle interaction the electrodynamic interaction between charges is described as two-steps process; first fields are seen as being generated from their particle sources and then the fields so generated are perceived as interacting with some target particle. The description of the first step in the $F$ formulation of electrodynamics is given by the above relations (2), (3), (4) and for a particle with charge $q$ with (6). The second step requires the determination of the Lorentz force in terms of $F$ and its use in Newton’s second law. This will be undertaken below.

In the Clifford algebra formalism one can easily derive the expressions for the stress-energy vector $T(n)$ and the Lorentz force density $K_j$ directly from field equations (2) and from the equation for $\tilde{F}$, the reverse of $F$, $\tilde{F}\tilde{\partial} = \tilde{j}/\varepsilon_0 c$ ($\tilde{\partial}$ differentiates to the left instead of to the right). Indeed, using (2) and from the equation for $\tilde{F}$ one finds

$$T(\partial) = (-\varepsilon_0/2)(F \partial F) = j \cdot F/c = -K_j,$$

(7)

where in $(F \partial F)$ the derivative $\partial$ operates to the left and to the right by the chain rule. The stress-energy vector $T(n)$ [1–3] for the electromagnetic field is then defined in the $F$ formulation as

$$T(n) = T(n(x), x) = - (\varepsilon_0/2) \langle FnF \rangle_1.$$  

(8)

We note that $T(n)$ is a vector-valued linear function on the tangent space at each spacetime point $x$ describing the flow of energy-momentum through a hypersurface with normal $n = n(x)$.

The right hand side of (7) yields the expression for the Lorentz force density $K_j$, $K_j = F \cdot j/c$. This relation shows that the Lorentz force density $K_j$ can be interpreted as the rate of energy-momentum transfer from the source $j$ to the field $F$. The Lorentz force in the $F$ formulation for a charge $q$ is

$$K = (q/c)F \cdot v,$$

where $v$ is the velocity 1-vector of a charge $q$ (it is defined to be the tangent to its world line).

In the approaches [1,2] the Lorentz force is discussed using the spacetime split and the corresponding decomposition of $F$ into the electric and magnetic components. Namely the bivector field $F$ is expressed in terms of the sum of a relative vector $E_H$ and a relative bivector $\gamma_5 B_H$ by making a
space-time split in the $\gamma_0$ - frame

$$F = E_H + c\gamma_5 B_H, \quad E_H = (F \cdot \gamma_0)\gamma_0 = (1/2)(F - \gamma_0 F\gamma_0),$$

$$\gamma_5 B_H = (F \wedge \gamma_0)\gamma_0 = (1/2c)(F + \gamma_0 F\gamma_0), \quad (9)$$

and similarly in [3] $F$ is expressed in the $\gamma_0$ - frame by $E_J$ and $B_J$. However the formulation with $F$ is, as already said, a self-contained and complete formulation of electrodynamics and there is no need for the introduction of $E_H$, $B_H$ or $E_J$, $B_J$ from [1-3], i.e., the usual 3D electric $E$ and magnetic $B$ fields. Besides it is shown in [8,9] that the mentioned decompositions of $F$ from [1-3] lead to the standard transformations of the 3D $E$ and $B$ that are not relativistically correct transformations. (The relativistically correct relations that connect the formulation of electrodynamics with $F$ and the equivalent, but independent, self-contained and complete formulation with well-defined 4D quantities, the electric and magnetic 1-vectors $E$ and $B$, are given in [13].) Thus in the analysis of the motion of a charged particle under the action of the Lorentz force we utilize only those parts of the usual approaches [1-3] that are expressed only in terms of $F$ and not those expressed by $E_H$, $B_H$ or $E_J$, $B_J$. Actually we shall only quote the main results from [1-3] for the motion of a charged particle in a constant electromagnetic field $F$ but without using $E_H$, $B_H$ or $E_J$, $B_J$.

The particle equation of motion, i.e., Newton’s second law is

$$m\dot{v} = qF \cdot v, \quad (10)$$

where $\dot{v} = dv/d\tau$; the overdot denotes differentiation with respect to proper time $\tau$. Usually [1,2] the equation (10) is not solved directly but solving the rotor equation $\dot{R} = (q/2m)FR$ and using the invariant canonical form for $F$, which is $F = f e^{i/2} = f(\cos \varphi + I \sin \varphi)$; this holds for $F^2 \neq 0$, and here $\gamma_5$ is denoted as $I$. Let us consider that $F$ is an uniform electromagnetic field. Then denoting $(q/m)F = \Omega, \Omega_1 = f(q/m)\cos \varphi, \Omega_2 = f(q/m)\sin \varphi,$ and making an invariant decomposition of the initial velocity $v(0)$ into a component $v_1$ in the $f$-plane and a component $v_2$ orthogonal to the $f$-plane, $v(0) = f^{-1}(f \cdot v(0)) + f^{-1}(f \wedge v(0)) = v_1 + v_2$, we get

$$v = e^{(1/2)\Omega_1}v_1 + e^{(1/2)\Omega_2}v_2. \quad (11)$$

As stated in [1], Spacetime Calculus, this is an invariant decomposition of the motion into ”electriclike” and ”magneticlike” components. The particle
history is obtained integrating (11)

\[ x(\tau) - x(0) = (e^{(1/2)\Omega_1 \tau} - 1)\Omega_1^{-1}v_1 + e^{(1/2)\Omega_2 \tau}\Omega_2^{-1}v_2. \]  

(For more details see [1-3].) This result applies for arbitrary initial conditions and arbitrary uniform electromagnetic field \( F \). Of course the special cases can be investigated without introducing the electric and magnetic fields. We shall not discuss here another cases that are considered in [1-3] (e.g., a charge in an electromagnetic plane wave), but we note that they also can be examined exclusively in terms of \( F \) without introducing the electric and magnetic fields.

4.3. The Stress-Energy Vector \( T(n) \) and the Quantities Derived from \( T(n) \)

The most important quantity for the momentum and energy of the electromagnetic field is the observer independent stress-energy vector \( T(n) \). It can be written in the following form

\[ T(n) = -(\varepsilon_0/2) [(F \cdot F)n + 2(F \cdot n) \cdot F]. \]  

(13)

We present a new form for \( T(n) \) writing it as a sum of \( n \)-parallel part \((n-\parallel)\) and \( n \)-orthogonal part \((n-\perp)\)

\[ T(n) = -(\varepsilon_0/2) [(F \cdot F) + 2(F \cdot n)^2] n + 
- \varepsilon_0 [(F \cdot n) \cdot F - (F \cdot n)^2 n]. \]  

(14)

The first term in (14) is \( n-\parallel \) part and it yields the energy density \( U \). Namely using \( T(n) \) and the fact that \( n \cdot T(n) \) is positive for any timelike vector \( n \) we construct the expression for the observer independent energy density \( U \) contained in an electromagnetic field as \( U = n \cdot T(n) = \langle nT(n) \rangle \), (scalar, i.e., grade-0 multivector). Thus in terms of \( F \) and \( U \) becomes

\[ U = -(\varepsilon_0/2) \langle FnFn \rangle = -(\varepsilon_0/2) [(F \cdot F) + 2(F \cdot n)^2]. \]  

(15)

The second term in (14) is \( n-\perp \) part and it is \((1/c)S\), where \( S \) is the observer independent expression for the Poynting vector (1-vector),

\[ S = -\varepsilon_0 c [(F \cdot n) \cdot F - (F \cdot n)^2 n], \]  

(16)
and, as can be seen, \( n \cdot S = 0 \). Thus \( T(n) \) expressed by \( U \) and \( S \) is

\[
T(n) = Un + (1/c)S. \tag{17}
\]

Notice that the decompositions of \( T(n) \), (13), (14) and (17), are all observer independent decompositions. Further the observer independent momentum density \( g \) is defined as \( g = (1/c^2)S \), i.e., \( g \) is \( 1/c \) of the \( n-\perp \) part from (14)

\[
g = -(\varepsilon_0/c) \left[ (F \cdot n) \cdot F - (F \cdot n)^2 n \right]. \tag{18}
\]

From \( T(n) \) (14) one finds also the expression for the observer independent angular-momentum density \( M \)

\[
M = (1/c)T(n) \wedge x = (1/c)U(n \wedge x) + g \wedge x. \tag{19}
\]

It has to be emphasized once again that all these definitions are the definitions of the quantities that are independent of the chosen reference frame and of the chosen system of coordinates in it. As I am aware they are not presented earlier in the literature.

All these quantities can be written in some basis \( \{e_\mu\} \), which does not need to be the standard basis, as coordinate-based geometric quantities. The field bivector \( F \) can be written as \( F = (1/2)F^\alpha_\beta e_\alpha \wedge e_\beta \) where the basis components \( F^\alpha_\beta \) are determined as \( F^\alpha_\beta = e^\beta \cdot (e^\alpha \cdot F) = (e^\beta \wedge e^\alpha) \cdot F \). Then the quantities entering into the expressions for \( T(n) \), \( U \), \( S \) and \( M \) are \( F \cdot F = -(1/2)F^\alpha_\beta F_\alpha^\beta \), \( F \cdot n = F^\alpha_\beta n^\beta e_\alpha \), \( (F \cdot n)^2 = F^\alpha_\beta F_\alpha^\mu n^\mu n^\nu \) and \( (F \cdot n) \cdot F = F^\alpha_\beta F_\alpha^\mu n_\beta n^\nu \). Thence \( T(n) \) (13) becomes

\[
T(n) = -(\varepsilon_0/2) \left[ (1/2)F^\alpha_\beta F_\beta^\alpha n^\rho e_\rho + 2F^\alpha_\beta F_\alpha^\rho n^\rho e_\beta \right], \tag{20}
\]

the energy density \( U \) (15) is

\[
U = -(\varepsilon_0/2) \left[ (1/2)F^\alpha_\beta F_\beta^\alpha + 2F^\alpha_\beta F_\alpha^\rho n^\rho n_\beta \right], \tag{21}
\]

and the Poynting vector \( S \) (16) becomes

\[
S = -\varepsilon_0 c \left[ F^\alpha_\beta F_\alpha^\rho n^\rho e_\beta - F^\alpha_\beta F_\alpha^\rho n^\rho n_\beta n^\lambda e_\lambda \right]. \tag{22}
\]

In some basis \( \{e_\mu\} \) we can write the stress-energy vectors \( T^\mu \) as \( T^\mu = T(e^\mu) = (\varepsilon_0/2)F e^\mu F \). The components of the \( T^\mu \) represent the energy-momentum tensor \( T^{\mu \nu} \) in the \( \{e_\mu\} \) basis \( T^{\mu \nu} = T^\mu \cdot e^\nu = (\varepsilon_0/2) \langle F e^\mu F e^\nu \rangle \), which reduces to familiar tensor form

\[
T^{\mu \nu} = \varepsilon_0 \left[ F^{\mu \alpha} g_{\alpha \beta} F^{\beta \nu} + (1/4)F^\alpha_\beta F_\alpha^\rho g^{\mu \rho} \right]. \tag{23}
\]
In the usual Clifford algebra approach, e.g., [1,2], one again makes the space-time split and considers the energy-momentum density in the \( \gamma_0 \)-system (the standard basis \( \{ \gamma_\mu \} \)) \( T^0 = T(\gamma^0) = T(\gamma_0) \); the split \( T^0 \gamma^0 = T^0 \gamma_0 = T^{00} + T^0 \), separates \( T^0 \) into an energy density \( T^{00} = T^0 \gamma^0 \) and a momentum density \( T^0 = T^0 \gamma^0 \). Then from the expression for \( T^\mu \) and the relations (9) one finds [1,2] the familiar results for the energy density \( T^{00} = (\varepsilon_0 / 2)(E^2 c^2 H + c^2 B^2 H) \) and the Poynting vector \( T^0 = \varepsilon_0 (E^{H} \times c B^{H}) \), where the commutator product \( A \times B \) is defined as \( A \times B \equiv (1/2)(AB - BA) \). However, as already said, the space-time split and the usual electric and magnetic fields \( E^{H} \) and \( B^{H} \) are not only unnecessary but, as shown in [8,9], they are relativistically incorrect.

4.4. The Local Conservation Laws in the \( F \)- Formulation

It is well-known that from the field equations in the \( F \)- formulation \( [2] \) one can derive a set of conserved currents. Thus, for example, in the \( F \)- formulation one derives in the standard way that \( j \) from \( [2] \) is a conserved current. Simply, the vector derivative \( \partial \) is applied to the field equations \( [2] \) which yields

\[
(1/\varepsilon_0 c) \partial \cdot j = \partial \cdot (\partial \cdot F).
\]

Using the identity \( \partial \cdot (\partial \cdot M(x)) \equiv 0 \) (\( M(x) \) is a multivector field) one obtains the local charge conservation law

\[
\partial \cdot j = 0. \tag{24}
\]

In a like manner we find from \( [7] \) that

\[
\partial \cdot T(n) = 0 \tag{25}
\]

for the free fields. This is a local energy-momentum conservation law. In the derivation of \( [7] \) we used the fact that \( T(a) \) is symmetric, i.e., that \( a \cdot T(b) = T(a) \cdot b \). Namely using accents the expression for \( T(\partial) \) \( T(\partial) = (-\varepsilon_0 / 2)(F \partial F) \), where \( \partial \) operates to the left and to the right by the chain rule) can be written as \( T(\partial) = \hat{T}(\hat{\partial}) = (-\varepsilon_0 / 2) (\hat{F} \hat{\partial} F + F \hat{\partial} \hat{F}) = 0 \), since in the absence of sources \( \partial F = \hat{F} \hat{\partial} = 0 \) (the accent denotes the multivector on which the derivative acts). Then from the above mentioned symmetry of \( T \) one finds that \( \hat{T}(\hat{\partial}) \cdot a = \partial \cdot T(a) = 0, \forall \text{ const. } a \), which proves the equation \( [25] \).
Inserting the expression (17) for $T(n)$ into the local energy-momentum conservation law (25) we find

$$(n \cdot \partial)U + (1/c)\partial \cdot S = 0. \quad (26)$$

The relation (26) is the well-known Poynting’s theorem but now completely written in terms of the observer independent quantities. Let us introduce the standard basis $\{\gamma_\mu\}$, i.e., an inertial frame of reference with the Einstein system of coordinates, and in the $\{\gamma_\mu\}$ basis we choose that $n = \gamma_0$, or in the component form it is $n^\mu(1,0,0,0)$. Then the familiar form of Poynting’s theorem is recovered in such coordinate system

$$\frac{\partial U}{\partial t} + \partial_i S^i = 0, \quad i = 1, 2, 3. \quad (27)$$

It is worthwhile to note that although $U$ (15) and $S$ (14), taken separately, are well-defined observer independent quantities, the relations (17), (25) and (26) reveal that only $T(n)$ (17), as a whole quantity, i.e., the combination of $U$ and $S$, enters into a fundamental physical law, the local energy-momentum conservation law (25). Thence one can say that only $T(n)$ (17), as a whole quantity, does have a real physical meaning, or, better to say, a physically correct interpretation. An interesting example that emphasizes this point is the case of an uniformly accelerated charge. In the usual (3D) approach to the electrodynamics ([7]; Jackson, Classical Electrodynamics, Sec. 6.8.) the Poynting vector $S$ is interpreted as an energy flux due to the propagation of fields. In such an interpretation it is not clear how the fields propagate along the axis of motion since for the field points on the axis of motion one finds that $S = 0$ (there is no energy flow) but at the same time $U \neq 0$ (there is an energy density). Our approach reveals that the important quantity is $T(n)$ and not $S$ and $U$ taken separately. $T(n)$ is $\neq 0$ everywhere on the axis of motion and the local energy-momentum conservation law (25) holds everywhere.

In the same way one can derive the local angular momentum conservation law, see [1], Space-Time Calculus.

5. COMPARISON WITH EXPERIMENTS

It is shown in [12] that the usual formulation of special relativity (which deals with the observer dependent quantities, i.e., the Lorentz contraction, the dilatation of time, the use of the 3D $E$ and $B$, etc.) shows only an
"apparent" agreement (not the true one) with the traditional and modern experiments, e.g., the Michelson-Morley type experiments. On the contrary it is shown in [12] that the invariant special relativity from [10] (given in terms of geometric quantities - abstract tensors) is in a complete agreement with all considered experiments. This entails that the same complete agreement holds also for the formulation with geometric quantities - the Clifford multivectors, which is presented in this paper.

In addition we briefly discuss the Trouton-Noble experiment [16] (see also [17]). In the experiment they looked for the turning motion of a charged parallel plate capacitor suspended at rest in the frame of the earth in order to measure the earth's motion through the ether. The explanations, which are given until now (see, e.g., [18]), for the null result of the experiments [16] ([17]) are not relativistically correct, since they use quantities that are not well-defined in 4D spacetime; e.g., the Lorentz contraction, the transformation equations for the usual 3D vectors $E$ and $B$ and for the torque as the 3D vector, the nonelectromagnetic forces of undefined nature, etc.. In our approach the explanation is very simple and natural; the energy density $U$, then $g$ and $M$ and the associated integral quantities are all invariant quantities, which means that their values are the same in the rest frame of the capacitor and in the moving frame. Thus if there is no torque (but now as a geometric, invariant, 4D quantity) in the rest frame then the capacitor cannot appear to be rotating in a uniformly moving frame.

6. DISCUSSION AND CONCLUSIONS

The usual Clifford algebra approach to the relativistic electrodynamics deals with the space-time split and the relative vectors $E_H, B_H$ (from [1,2]) or $E_J, B_J$ (from [3]). The investigation presented in [8,9] and [13] reveals that such approach is not relativistically correct. The usual 3D $E$ and $B$, or the relative vectors $E_H, B_H$ [1,2], or $E_J, B_J$ [3], are not only observer dependent quantities but, as shown in [8,9], their transformation laws are meaningless from the special relativity viewpoint; they have nothing to do with the Lorentz transformations of the well-defined quantities on the 4D spacetime. Here we employ quantities that are independent of the reference frame and of the chosen system of coordinates for that frame. We have presented the formulation of electrodynamics by means of the field bivector $F$. This formulation with the $F$ field is a self-contained, complete and consistent formulation that does not make use either electric and magnetic fields or the electromagnetic po-
tential $A$. It provides complete and consistent description of electromagnetic phenomena in terms of observer independent, thus properly defined quantities on the 4D spacetime. The formulation with the $F$ field is not physically equivalent with the usual Maxwell formulation with the 3D vectors $E$ and $B$, since, as shown in [8,9], the transformation laws for the 3D $E$ and $B$ are not relativistically correct transformations. The observer independent field equations with the $F$ field and the new observer independent expressions for the stress-energy vector $T(n)$, the energy density $U$, the Poynting vector $S$, the momentum density $g$, the angular-momentum density $M$ and the Lorentz force $K$ are presented in this paper. The second quantization procedure, and the whole quantum electrodynamics, will be simply constructed using geometric, invariant, quantities $F$, $T(n)$, $U$, $S$, $g$ and $M$. Note that the standard covariant approaches to quantum electrodynamics, e.g., [19], usually deal with the component form (in the specific, i.e., the Einstein system of coordinates) of the electromagnetic 4-potential $A$ (thus requiring the gauge conditions too) instead of to use the geometric quantity, the observer independent bivector field $F$. Furthermore the standard covariant approaches employ the definitions of the field energy and momentum, which are not well-defined from the special relativity viewpoint. Namely, both the field energy and momentum are defined as integrals over the three-space, that is, over the hypersurface $t = \text{const}$. But the hypersurface $t = \text{const}$. in some reference frame $S$ cannot become (under the Lorentz transformations) the hypersurface $t' = \text{const}$. in a relatively moving reference frame $S'$. This is already examined for the classical electrodynamics (the covariant formulation in the Einstein system of coordinates) by Rohrlich [20] and using the component form of the electric and magnetic 4-vectors $E^\alpha$ and $B^\alpha$ (the tensor formalism) in the first paper in [21]. (The second paper in [21] treats relatively moving systems, e.g., a current-carrying conductor, using the component form of the electric and magnetic 4-vectors $E^\alpha$ and $B^\alpha$.) In this paper the local conservation laws are directly derived from the field equations with the $F$ field and written in an invariant way. The observer independent integral field equations and the observer independent global conservation laws (with the definitions of the invariant field energy and momentum) will be treated elsewhere. Particularly it has to be emphasized that the observer independent approach to the relativistic electrodynamics that is presented in this paper is in a complete agreement with existing experiments that test special relativity, which is not the case with the usual approaches. Furthermore we note that all observer independent quantities introduced here and the field
equations written in terms of them hold in the same form both in the flat and curved spacetimes. The formalism presented here will be the basis for the relativistically correct (without reference frames) formulation of quantum electrodynamics and, more generally, of the quantum field theory.

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