On Minimal Valid Inequalities for Mixed Integer Conic Programs

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August 24, 2014

Abstract

We study mixed integer conic sets involving a general regular (closed, convex, full dimensional, and pointed) cone \( K \) such as the nonnegative orthant, the Lorentz cone or the positive semidefinite cone. In a unified framework, we introduce \( K \)-minimal inequalities and show that under mild assumptions, these inequalities together with the trivial cone-implied inequalities are sufficient to describe the convex hull. We study the properties of \( K \)-minimal inequalities by establishing necessary conditions for an inequality to be \( K \)-minimal. This characterization leads to a broader class of \( K \)-sublinear inequalities, which includes \( K \)-minimal inequalities as a subclass. We establish a close connection between \( K \)-sublinear inequalities and the support functions of sets with a certain structure. This leads to practical ways of showing that a given inequality is \( K \)-sublinear and \( K \)-minimal. We provide examples to show how our framework can be applied.

Our framework generalizes some of the results from the mixed integer linear case. It is well known that the minimal inequalities for mixed integer linear programs are generated by sublinear (positively homogeneous, subadditive and convex) functions that are also piecewise linear. This result is easily recovered by our analysis. So, whenever possible we highlight the connections to the existing literature. However, our study reveals that such a cut generating function view is not possible for the case of general non-polyhedral cones even when the cone involved is the Lorentz cone.

1 Introduction

A Mixed Integer Conic Program (MICP) is an optimization program of the form

\[
\text{Opt} = \inf_{x \in E} \{ \langle c, x \rangle : Ax = b, \ x \in K, \ x \in \mathcal{Z} \}
\]

where \( K \) is a regular cone (full-dimensional, closed, convex and pointed) in a finite dimensional Euclidean space \( E \) with an inner product \( \langle \cdot, \cdot \rangle \); \( c \in E \) is the objective; \( b \in \mathbb{R}^m \) is the right hand side; \( A : E \to \mathbb{R}^m \) is a linear map, and \( \mathcal{Z} \) is a set imposing certain structural restrictions on the variables \( x \). The most common form of these restrictions, for example when \( E = \mathbb{R}^n \), is integrality \( x_i \in \mathcal{Z} \) for all \( i \in I \) where \( I \subset \{1, \ldots, n\} \) is the index set of integer variables. We assume that all of the data involved with MICP, i.e., \( c, b, A \) is rational. Examples of regular cones, \( K \), include the nonnegative orthant, \( \mathbb{R}^n_+ \), the Lorentz cone, \( \mathcal{L}^n \), and the positive semidefinite cone, \( S^n_+ \).

Mixed Integer Linear Programs (MILPs) are a particular case of MICPs with \( K = \mathbb{R}^n_+ \). While MILPs offer an incredible representation power, various optimization problems involving risk constraints and discrete decisions lead to MICPs with conic constraints. These include many applications from decision making.
under uncertainty, i.e., robust optimization and stochastic programming paradigms, such as portfolio optimization with fixed transaction costs in finance (see \cite{37, 55}), stochastic joint location-inventory models \cite{5}. Conic constraints include various specific convex constraints such as linear, convex quadratic, eigenvalue, etc., and hence offer significant representation power (see \cite{18} for a detailed introduction to conic programming and its applications in various domains). Clearly allowing discrete decisions further increases the representation power of MICPs. Moreover, the most powerful relaxations to many linear combinatorial optimization problems are based on conic (in particular semidefinite) relaxations (see \cite{2, 40, 56} for earlier work and \cite{38} for a survey on this topic). Besides, MILPs have been heavily exploited for approximating non-convex nonlinear optimization problems. For a wide range of these problems, MICPs offer tighter relaxations and thus potentially a better overall algorithmic performance. Therefore, MICPs have gained considerable interest and arise in several important applications in many diverse fields.

Currently, cutting plane theory lies at the basis of most efficient algorithms for solving MILPs and has been developed and very well understood in the case of MILPs. While many commercial solvers such as CPLEX \cite{34}, Gurobi \cite{44}, and MOSEK \cite{59} are expanding their features to include solvers for MICPs, the theory and algorithms for solving MICPs are still in their infancy (see \cite{6}). In fact, the most promising approaches to solve MICPs are based on the extension of cutting plane techniques (see \cite{6, 7, 20, 21, 24, 27, 36, 50, 69}) in combination with conic relaxations and branch-and-bound algorithms. Numerical performance of these techniques is still under investigation. Evidence from MILP setup indicates that adding a small yet essential set of strong cutting planes is key to the success of such a procedure. Yet, except very specific and simple cases, it has not yet been possible to theoretically evaluate the strength of the corresponding valid inequalities, i.e., redundancy, domination, etc., for MICPs. This is in sharp contrast to the MILP case, where the related questions have been studied extensively. In particular, the feasible region of an MILP with rational data is a polyhedron and the facial structure of a polyhedron (its faces, and facets) is very well understood. Various ways of proving whether a given linear inequality is a facet or not for an MILP are well established in the literature (see \cite{60}). In addition to this, a new framework is developing rapidly in terms of establishing minimal and extremal inequalities for certain generic semi-infinite relaxations of MILPs (see \cite{30} and references therein). On the other hand, for general MICPs, there is no natural extension of certain important polyhedral notions such as facets. Therefore, establishing such a theoretical framework to measure the strength of cutting planes in the MICP context remains a natural and important question, and our goal in this paper is to address this question.

Our approach in this paper is based on extending the notion of minimal inequalities from the MILP context to a general disjunctive conic framework. In particular, we study the closed convex hull of union of finitely or infinitely many conic sets (in the original space of variables), and characterize properties of minimal inequalities for their descriptions. In this respect, when the cone is taken as the nonnegative orthant, our approach ties back to the cornerstone paper of Johnson \cite{49} as well as the recent work of Conforti et al. \cite{28}. The MILP counterparts of our results and further developments for MILPs were studied extensively in the literature. We demonstrate that these results naturally extend to MICPs, and hence, whenever possible, we highlight these connections.

To the best of our knowledge, the dominance relations among valid inequalities and minimal inequalities have not been defined and studied in the MICP setting. In this paper, we contribute to the literature by establishing properties associated with minimal inequalities in the MICP context, i.e., necessary, and sufficient conditions, as well as practical tools for testing whether a given inequality is minimal or not. Minimal inequalities in fact are directly related to the facial structure of the convex hull of the feasible region of corresponding MICPs, and we demonstrate that such a study can be done in a unified manner for all regular cones $\mathcal{K}$. Our study also reveals that MICPs present new challenges, various useful tools from the
MILP framework such as semi-infinite relaxations or cut generating functions\(^1\) do not extend. Therefore our derivations are based on the actual finite dimensional problem instance, and hence our study does not rely on and differ substantially from the majority of previous literature on minimal inequalities. In a practical cutting plane procedure for MILPs and/or MICPs, one is indeed faced with a finite dimensional problem, and thus we believe that this is not a limitation but rather a contribution to the corresponding MILP literature. In particular, in the linear case, i.e., \( \mathcal{K} = \mathbb{R}^n_+ \), our results partially cover the related results from \([49]\) and \([28]\) showing that minimal inequalities can be properly related to cut generating functions, and these functions are sublinear and piece-wise convex. For non-polyhedral regular cones, we show that there exist extremal inequalities, which cannot be obtained from any cut generating function. Finally, we believe this work is a step forward for the study of MICPs with \( \mathcal{K} = S^n_+ \). Such problems are frequently encountered in the semidefinite relaxations of combinatorial optimization problems. While various recent papers study MICPs with \( \mathcal{K} = L^n_+ \), besides the work of \([27]\), we are not aware of any paper explicitly studying valid inequalities for general MICPs with \( \mathcal{K} = S^n_+ \).

1.1 Preliminaries and Notation

Let \((E, \langle \cdot, \cdot \rangle)\) be a finite dimensional Euclidean space with inner product \(\langle \cdot, \cdot \rangle\). Let \(\mathcal{K} \subset E\) be a regular cone, i.e., full-dimensional, closed, convex and pointed. Note that if every \(K_i \subset E_i\) for \(i = 1, \ldots, k\) is a regular cone, then their direct product \(\tilde{K} = K_1 \times \ldots \times K_k\) is also a regular cone in the Euclidean space \(\tilde{E} = E_1 \times \ldots \times E_k\) with inner product \(\langle \cdot, \cdot \rangle_{\tilde{E}}\), which is the sum of the inner products \(\langle \cdot, \cdot \rangle_{E_i}\). Therefore without loss of generality we only focus on the case with a single regular cone \(K\).

In this paper, given a linear map \(A : E \to \mathbb{R}^m\), a regular cone \(\mathcal{K} \subset E\), and a nonempty set of right hand side vectors, \(B \subseteq \mathbb{R}^m\), we consider the disjunctive union of conic sets defined by \(A, \mathcal{K}\), and the vectors from \(B\), i.e.,

\[
S(A, \mathcal{K}, B) := \{x \in \mathcal{K} : Ax \in B\},
\]

and we are interested in determining valid inequalities for the closed convex hull of \(S(A, \mathcal{K}, B)\). We would like to highlight that we do not impose any structural assumptions on \(B\), i.e., \(B\) is an arbitrary set of vectors, in particular, it can be finite or infinite, or structured such as lattice points or completely unstructured. Without loss of generality, we assume that \(\mathcal{K} \not\subset \{x \in E : Ax \in B\}\) (as otherwise, we have \(S(A, \mathcal{K}, B) = \mathcal{K}\) and \(S(A, \mathcal{K}, B) = \emptyset\), and thus \(B \neq \emptyset\) and there exists \(b \in B\) such that there exists \(x_b \in \mathcal{K}\) satisfying \(Ax_b = b\).

For a given set \(S\), we denote its topological interior with \(\text{int}(S)\), its closure with \(\overline{S}\), its boundary with \(\partial S = \overline{S} \setminus \text{int}(S)\). We use \(\text{conv}(S)\) to denote the convex hull of \(S\), \(\overline{\text{conv}}(S)\) for its closed convex hull, and \(\text{cone}(S)\) to denote the cone generated by the set \(S\). We define the kernel of a linear map \(A : E \to \mathbb{R}^m\), as \(\text{Ker}(A) := \{u \in E : Au = 0\}\) and its image as \(\text{Im}(A) := \{Au : u \in E\}\). We use \(A^*\) to denote the

\(^1\)Informally, a cut generating function generates the coefficient of a variable in a cut using only information of the instance pertaining to this variable, see \([28]\) for an extended discussion.
conjugate linear map \(^2\) given by the identity
\[
y^T A x = \langle A^* y, x \rangle \; \forall (x \in E, y \in \mathbb{R}^m).
\]

We use \(\langle \cdot, \cdot \rangle\) notation for inner product in Euclidean space \(E\), and proceed with usual dot product notation with transpose for the inner product in \(\mathbb{R}^m\).

For a given cone \(K \subseteq E\), we let \(\text{Ext}(K)\) denote the set of its extreme rays, and \(K^*\) to denote its dual cone given by
\[
K^* := \{ y \in E : \langle x, y \rangle \geq 0 \; \forall x \in K \}.
\]

A relation \(a - b \in K\), where \(K\) is a regular cone, is often called conic inequality between \(a\) and \(b\) and is denoted by \(a \succeq_K b\); such a relation indeed preserves the major properties of the usual coordinate-wise vector inequality \(\geq\). We denote the strict conic inequality by \(a \succ_K b\), to indicate that \(a - b \in \text{int}(K)\). In the sequel, we refer to a constraint of the form \(A x - b \in K\) as a conic inequality constraint or simply conic constraint and also use \(A x \succeq_K b\) interchangeably in the same sense.

While our theory is general enough to cover all regular cones, there are three important examples of regular cones common to most MICPs, namely the nonnegative orthant, \(\mathbb{R}^n_+ := \{ x \in \mathbb{R}^n : x_i \geq 0 \; \forall i \}\); the Lorentz cone, \(L^n := \{ x \in \mathbb{R}^n : x_n \geq \sqrt{x_1^2 + \ldots + x_{n-1}^2} \}\); and the positive semidefinite cone \(S^n_+ := \{ X \in \mathbb{R}^{n \times n} : a^T X a \geq 0 \; \forall a \in \mathbb{R}^n, \; X = X^T \}\). These important regular cones are also self-dual, i.e., \(K^* = K\). In the first two cases, the corresponding Euclidean space \(E\) is just \(\mathbb{R}^n\) with dot product as the corresponding inner product. In the last case, \(E\) becomes the space of symmetric \(n \times n\) matrices with Frobenius inner product, \(\langle x, y \rangle = \text{Tr}(x y^T)\).

Notation \(e_i\) is used for the \(i^{th}\) unit vector \(^3\) of \(\mathbb{R}^n\), and \(\text{Id}\) for the identity map in \(E\), i.e., when \(E\) is \(\mathbb{R}^n\), \(\text{Id}\) is just the \(n \times n\) identity matrix, \(I_n\).

1.2 Motivation and Connections to MICPs

While the set of the form given by \(S(A, K, \mathcal{B})\) can be of interest by itself, here we show that it can be seen as a disjunctive conic set and it naturally represents the feasible region of an MICP or used as a relaxation for MICPs.

Let us first consider an example transformation that generalizes the usual disjunctive programming from the polyhedral (linear) case, e.g., [8, 9, 10, 11] to the one with conic constraints that results in the form of a set \(S(A, K, \mathcal{B})\).

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\(^2\) When we consider the standard Euclidean space, i.e., \(E = \mathbb{R}^n\), a linear map \(A : \mathbb{R}^n \to \mathbb{R}^m\) is just an \(m \times n\) real-valued matrix, and its conjugate is given by its transpose, i.e., \(A^* = A^T\).

Suppose that \(E = S^n\), i.e., space of symmetric \(n \times n\) matrices, and let us denote \(\text{Tr}(\cdot)\) for the trace of a matrix, i.e., the sum of its diagonal entries. When \(E = S^n\), it is natural to specify a linear map \(A : S^n \to \mathbb{R}^m\) as a collection \(\{A^1, \ldots, A^m\}\) of \(m\) matrices from \(S^n\) such that
\[
AZ = (\text{Tr}(ZA^1) ; \ldots ; \text{Tr}(ZA^m)) : S^n \to \mathbb{R}^m.
\]

In this case, the conjugate linear map \(A^* : \mathbb{R}^m \to S^n\) is given by
\[
A^* y = \sum_{j=1}^m y_j A^j, \; y = (y_1; \ldots; y_m) \in \mathbb{R}^m.
\]

\(^3\) Throughout the paper, we use Matlab notation to denote vectors and matrices and all vectors are to be understood in column form.
Example 1.1 Suppose that we are given a finite collection of convex sets of the form \( C_i = \{ x \in K : A_i x = b_i \} \) for \( i \in \{ 1, \ldots, \ell \} \), where \( K \subseteq \mathbb{R}^n \) and \( K_i \subseteq \mathbb{R}^{m_i} \) are regular cones, \( A_i \) are \( m_i \times n \) matrices, and \( b_i \in \mathbb{R}^{m_i} \). We immediately observe that \( \bigcup_{i \in \{ 1, \ldots, \ell \}} C_i \) can be represented in the form of \( S(A, K, B) \) as follows:

\[
\begin{cases}
    x \in \mathbb{R}^n : \\
    \begin{pmatrix}
        A_1^T \\
        A_2^T \\
        \vdots \\
        A_{\ell}^T \\
    \end{pmatrix} x \in \\
    \bigcup_{i=1}^{\ell} \begin{pmatrix}
        \{ b_1 \} + K_1 \\
        \mathbb{R}^{m_2} \\
        \vdots \\
        \mathbb{R}^{m_\ell} \\
    \end{pmatrix} \\
    = B
\end{cases},
\]

where \( K \) is a proper cone in the Euclidean space \( \mathbb{R}^n \). We immediately observe that \( \bigcup_{i \in \{ 1, \ldots, \ell \}} C_i \) is the well-known disjunctive set representing the union of polyhedra \([8, 9, 10, 11]\).

Our next set of examples highlight the connection of \( S(A, K, B) \) with the feasible sets of MICP problems and their relaxations.

Example 1.2 Suppose that we are given the following conic optimization problem with integer variables

\[
\text{Opt} = \inf_{x \in \mathbb{R}^n} \left\{ c^T x : \tilde{A} x = b, x \in K, x_i \in \mathbb{Z} \text{ for all } i = 1, \ldots, \ell \right\}. \tag{1}
\]

By defining

\[
A = \begin{bmatrix}
    \tilde{A} \\
    I_n
\end{bmatrix}, \quad \text{and} \quad B = \left\{ \begin{pmatrix}
    b \\
    \mathbb{Z}^\ell \\
    \mathbb{R}^{n-\ell}
\end{pmatrix} \right\},
\]

where \( I_n \) is the \( n \times n \) identity matrix, we can convert this problem into optimizing the same linear function over \( S(A, K, B) \), i.e.,

\[
\text{Opt} = \inf_{x \in \mathbb{R}^n} \left\{ c^T x : Ax \in B, x \in K \right\}. \tag*{\square}
\]

Example 1.3 Let us also consider the following form of conic optimization problem with integer variables (e.g., MICPs studied in [58]):

\[
\text{Opt} := \inf_{y \in \mathbb{R}^n} \left\{ c^T y : \tilde{A} y - b \in \tilde{K}, y_i \in \mathbb{Z} \text{ for all } i = 1, \ldots, \ell \right\}, \tag{2}
\]

where \( \tilde{K} \) is a proper cone in the Euclidean space \( \mathbb{E} \). Then, by introducing new variables \( y^+, y^- \), and setting

\[
x = \begin{pmatrix}
    y^+ \\
    y^-
\end{pmatrix}, \quad K = \mathbb{R}_+^n \times \mathbb{R}_+^n, \quad c = \begin{pmatrix}
    \tilde{c} \\
    -\tilde{c}
\end{pmatrix}, \quad A = \begin{bmatrix}
    \tilde{A} \\
    I_n
\end{bmatrix}, \quad \text{and} \quad B = \left\{ \begin{pmatrix}
    b + \tilde{K} \\
    \mathbb{Z}^\ell \\
    \mathbb{R}^{n-\ell}
\end{pmatrix} \right\},
\]

where \( I_n \) is the \( n \times n \) identity matrix, once again we can precisely convert this problem into our form. \( \square \)

There is an important structural difference in \( S(A, K, B) \) arising in Examples 1.2 and 1.3: While the cone \( K \) in \( S(A, K, B) \) representation in Example 1.2 is pretty general, i.e., it can be any proper cone, the resulting cone in Example 1.3 is a pretty specific one, it is simply the nonnegative orthant. Two important distinctions between a general proper cone and the specific case of nonnegative orthant that will appear
in our discussions later on in section 4 are that the nonnegative orthant is decomposable, i.e., it does not introduce nontrivial correlations among variables, and all of its extreme rays are orthogonal to each other.

We note that in Examples 1.2 and 1.3, we have provided exact representations of the corresponding feasible sets of MICPs in the form of $S(A, K, B)$. Therefore, the convex hull description of the resulting $S(A, K, B)$ is often not easy to characterize. An alternative use of $S(A, K, B)$ is to obtain and study relaxations which are still nontrivial, interesting and useful. For example, we can obtain different relaxations of these MICPs in the form of $S(A, K, B)$ by iteratively adding the integrality requirements of variable $x_i$, i.e., changing the corresponding $\mathbb{R}$ to $\mathbb{Z}$ in the set $B$.

Another option is based on a more practical separation problem. Suppose that we have obtained a feasible solution $\hat{x}$ to the continuous relaxation of MICP in Example 1.2, yet $\hat{x} \notin \text{conv}(S(A, K, B))$. In such a case, we can still introduce a proper relaxation of MICP of Example 1.2 which is of the form $S(A, K, B)$ that will be used to identify valid inequalities that cut off $\hat{x}$. Let $d \in \mathbb{Z}^n$ and $r_0 \in \mathbb{Z}$ be such that $d_i = 0$ for all $i = \ell + 1, \ldots, n$, and $r_0 < \sum_{i=1}^{\ell} d_i x_i < r_0 + 1$, i.e., the split disjunction induced by $\sum_{i=1}^{\ell} d_i x_i \leq r_0$ or $\sum_{i=1}^{\ell} d_i x_i \geq r_0 + 1$ is valid for the feasible solutions of MICP but is violated by the current solution $\hat{x}$. Given such a split disjunction, the question of obtaining cuts separating $\hat{x}$ is equivalent to studying $\text{conv}(S(A, K, B))$ where

$$A = \begin{bmatrix} \tilde{A} \\ d^T \end{bmatrix}, \quad \text{and} \quad B = \left\{ \begin{pmatrix} b \\ r_0 - \mathbb{R}_+ \end{pmatrix} \cup \begin{pmatrix} b \\ r_0 + 1 + \mathbb{R}_+ \end{pmatrix} \right\}.$$ 

Then, every nontrivial inequality defining the closed convex hull of this particular $S(A, K, B)$ leads to a cut for the original MICP separating $\hat{x}$. Clearly, such a split disjunction can be studied in the case of Example 1.3 as well by defining $S(A, K, B)$ with

$$x = \begin{pmatrix} y^+ \\ y^- \end{pmatrix}, \quad K = \mathbb{R}_+^n \times \mathbb{R}_+, \quad A = \begin{bmatrix} \tilde{A} \\ d^T \end{bmatrix}, \quad \text{and} \quad B = \left\{ \begin{pmatrix} b + \tilde{K} \\ r_0 - \mathbb{R}_+ \end{pmatrix} \cup \begin{pmatrix} b + \tilde{K} \\ r_0 + 1 + \mathbb{R}_+ \end{pmatrix} \right\}.$$ 

Note that $\hat{x}$ in our discussion above is not restricted to be an extreme point solution. Considering that $\hat{x}$ is assumed to be obtained from a continuous relaxation of an MICP, it is very unlikely that $\hat{x}$ will be an extreme point solution. Nevertheless, our framework is flexible enough as it allows us to study the separation of an arbitrary point $\hat{x} \notin \text{conv}(S(A, K, B))$. Note that this is a departure from most of the existing MILP literature, where due to the specific structure of their continuous relaxations, and use of simplex algorithm to solve these relaxations, the focus is on separating extreme point solutions. This, in turn via translation of the associated point and set, is cast as separating origin from the convex hull of the feasible solutions in the MILP literature.

When $K = \mathbb{R}_+^n$, relaxations of the above forms have been studied in a number of other contexts, in particular for MILP and complementarity problems. In the MILP setting, i.e., $K = \mathbb{R}_+^n$, clearly the above transformations in these examples are valid. The disjunctive programming work of Balas [10] is closely related when we assume $B$ is finite, nevertheless the questions addressed in the disjunctive programming literature are mostly different. Furthermore, Johnson [49] has studied the set $S(A, \mathbb{R}_+^n, B)$ when $B$ is a finite list under the name of linear programs with multiple right hand side choice. In another closely related recent work, Conforti et al. [28] study $S(A, K, B)$ with $K = \mathbb{R}_+^n$ and possibly an infinite set $B$ such that $B \neq \emptyset$, is closed and $0 \notin B$, and suggest that Gomory’s corner polyhedron [41] as well as some other problems such as linear programs with complementarity restrictions can be viewed in this framework. Both [49] and [28] study characterizations of minimal inequalities, yet [28] views the topic almost entirely through cut generating functions and characterizing the properties of important ones. In addition to this, for example the
set of form \( \{ x \in \mathbb{Z}_{+}^n : \bar{A}x \in \mathbb{Z}^m - \bar{b} \} \), e.g., formulated first in [41], then suggested as an open research avenue in [28], naturally admits a representation in the form of \( S(A, K, B) \) where

\[
K = \mathbb{R}_+^n, \quad A = \begin{bmatrix} \bar{A} \\ I_n \end{bmatrix}, \quad \text{and} \quad B = \left\{ \left( \mathbb{Z}^m - \bar{b} \right) \mathbb{Z}_+^n \right\}.
\]

We finally highlight that since we are not making any particular assumption on \( A, B, K \) beyond the basic ones to avoid trivial cases such as \( S(A, K, B) = \emptyset \) or \( \text{conv}(S(A, K, B)) = K \). In particular, \( B \) can be completely arbitrary, the set \( S(A, K, B) \) offers great flexibility, which can be much beyond the relaxations/representations related to MICPs, e.g., it can be utilized to study relaxations of conic complementarity problems.

### 1.3 Classes of Valid Inequalities and Our Goal

Recall that we are interested in the valid inequalities for the solution set

\[
S(A, K, B) := \{ x \in K : Ax \in B \}.
\]

Given \( \mu \in E \), we define

\[
\vartheta(\mu) := \inf_x \{ \langle \mu, x \rangle : x \in S(A, K, B) \},
\]

as the tightest possible right hand side value for the inequality defined by the vector \( \mu \) still remain valid. Let \( \Pi(A, K, B) \subset E \) be the set of all nonzero vectors \( \mu \in E \) such that \( \vartheta(\mu) \) is finite. Any vector \( \mu \in \Pi(A, K, B) \) and an arbitrary scalar \( \eta_0 \) such that \( \eta_0 \leq \vartheta(\mu) \) where \( \vartheta(\mu) \) is as defined in (3), gives a valid inequality for \( S(A, K, B) \), i.e.,

\[
\langle \mu, x \rangle \geq \eta_0
\]

is satisfied by all \( x \in S(A, K, B) \). For a given vector \( \mu \) and \( \eta_0 \leq \vartheta(\mu) \), we denote the corresponding valid inequality \( \langle \mu; \eta_0 \rangle \) for short hand notation. We say that a valid inequality \( \langle \mu; \eta_0 \rangle \) is right if \( \eta_0 = \vartheta(\mu) \). If both \( \langle \mu; \eta_0 \rangle \) and \( \langle -\mu; -\eta_0 \rangle \) are valid inequalities, then \( \langle \mu, x \rangle = \eta_0 \) holds for all \( x \in S(A, K, B) \), and in this case, we refer to \( \langle \mu; \eta_0 \rangle \) as a valid equation for \( S(A, K, B) \).

Clearly the classification of valid inequalities in terms of their necessity for the description of \( \text{conv}(S(A, K, B)) \) depends on \( A, B \) and \( K \). Nevertheless, from now on we will fix \( A \) and \( B \) as given, and emphasize the classification of valid inequalities based on the cone \( K \) explicitly.

Let \( C(A, K, B) \subset E \times \mathbb{R} \) denote the convex cone of all valid inequalities given by \( \langle \mu; \eta_0 \rangle \). \( C(A, K, B) \), being a convex cone in \( E \times \mathbb{R} \), can always be written as the sum of a linear subspace \( L \) of \( E \times \mathbb{R} \) and a pointed cone \( C \), i.e., \( C(A, K, B) = L + C \). Given \( L \), the largest linear subspace contained in \( C(A, K, B) \), let \( L^\perp \) denote the orthogonal complement of \( L \), then a unique representation for \( C(A, K, B) \) can be obtained as \( C(A, K, B) \cap L^\perp \). A generating set \( (G_L, G_C) \) for a cone \( C(A, K, B) \) is a minimal set of elements \( \langle \mu; \eta_0 \rangle \in C(A, K, B) \) such that \( G_L \subseteq L, G_C \subseteq C \), and

\[
C(A, K, B) = \left\{ \sum_{w \in G_L} \alpha_w w + \sum_{v \in G_C} \lambda_v v : \lambda_v \geq 0 \right\}.
\]

**Remark 1.1** From this definition, it is clear that in a generating set \( (G_L, G_C) \) of \( C(A, K, B) \), without loss of generality, we can assume that each vector from \( G_C \) is orthogonal to every vector in \( G_L \), and all vectors in \( G_L \) are orthogonal to each other.

\[\square\]
Our study of $C(A, K, B)$ will be based on characterizing the properties of the elements of its generating sets. We will refer to the vectors in $G_L$ as generating equalities and the vectors in $G_C$ as generating inequalities of $C(A, K, B)$. An inequality $(\mu; \eta_0) \in C(A, K, B)$ is called an extreme inequality of $C(A, K, B)$, if there exists a generating set for $C(A, K, B)$ including $(\mu; \eta_0)$ as a generating inequality either in $G_L$ or in $G_C$. Note that any non-tight valid inequality, i.e., $(\mu; \eta_0)$ with $\eta_0 < \vartheta(\mu)$ does not belong to a generating set of $C(A, K, B)$.

Clearly the inequalities in generating set $(G_L, G_C)$ of the cone $C(A, K, B)$ are of great importance; they are necessary and sufficient for the description of the closed convex hull of $S(A, K, B)$. It is easy to note that $G_L$ is finite, as a basis of the subspace $L$ can be taken as $G_L$. For nonpolyhedral (nonlinear) cones such as $L^n$ with $n \geq 3$, $G_C$ need not be finite. In fact we provide an example demonstrating this in section 4.

1.4 Outline

The main body of this paper is organized as follows. We start by briefly summarizing the literature in section 2. In section 3, we introduce the class of $K$-minimal inequalities and show that under a mild assumption, this class of inequalities together with $x \in K$ constraint is sufficient to describe $\conv(S(A, K, B))$. We follow this by establishing a number of necessary conditions for $K$-minimality. While in many cases we show that $K$-minimal inequalities are tight, our study also highlights that depending on the structure of $S(A, K, B)$, $K$-minimality does not necessarily imply tightness of the inequality. Furthermore, one of our necessary conditions for $K$-minimality leads us to our next class of valid inequalities, $K$-sublinear inequalities. We study $K$-sublinear inequalities in section 4 and establish a precise relation between $K$-sublinearity and $K$-minimality and show that the set of extreme inequalities in the cone of $K$-sublinear inequalities contains all of the extreme inequalities from the cone of $K$-minimal inequalities. In section 5, we show that every $K$-sublinear inequality is associated with a convex set of certain structure, which we refer as a cut generating set. Moreover, we show that any nonempty cut generating set leads to a valid inequality. Through this connection with structured convex sets, we provide necessary conditions for $K$-sublinearity, as well as sufficient conditions for a valid inequality to be $K$-sublinear and $K$-minimal. In the case of $K = \mathbb{R}_+^n$, our necessary and sufficient conditions for $K$-sublinearity match precisely, and the relation between $K$-sublinear inequalities and the support functions of cut generating sets provides nice connections on to the existing MILP literature, which we highlight in section 5. Furthermore, in an example, we demonstrate that in certain specific cases tight $K$-minimal inequalities associated with the same cut generating set can be grouped together to obtain convex (or conic) inequalities. We finish this section by examining the conic Mixed Integer Rounding (MIR) inequality from [6] in our framework. We provide some characterizations of the linearity space of $C(A, K, B)$ in section 6, and finish by stating some further research questions.

2 Related Literature

The literature on solving MICPs is growing rapidly. On one hand clearly any method for general nonlinear integer programming applies to MICPs as well. A significant body of work has extended known techniques from MILPs to nonlinear integer programs, mostly only involving binary variables. These include Reformulation Linearization Technique (see [68, 66] and references therein), Lift-and-Project and Disjunctive Programming methods [10, 12, 26, 57, 63, 67, 69, 70], and the lattice-free set paradigm [19]. In addition to these, several papers (see [53, 54, 65, 64]) introduce hierarchies of convex (semidefinite programming) relaxations in higher dimensional spaces. These relaxations quickly become impractical due their exponentially growing sizes and the difficulty of projecting them onto the original space of variables. Another
stream of research [1, 22, 36, 61, 71, 72, 73, 74] is on the development of linear outer approximation based branch-and-bound algorithms for nonlinear integer programming. While they have the advantage of fast and easy to solve relaxations, the bounds from these approximations may not be as strong as desired. Moreover, adding too many inequalities that are closer to each other may lead to numerical instability.

Exploiting the conic structure when present, as opposed to general convexity, paves the way for developing algorithms with much better performance. Particularly in the special case of MILPs, this has led to very successful results. Despite the lack of effective warm-start techniques, efficient interior point methods exist for algorithms with much better performance. Particularly in the special case of MILPs, this has led to very successful algorithms. Belotti et al. [22] have introduced conic MIR cuts for a specific set involving \( \mathbb{L}^n \) and then reduced the general case of \( \mathbb{L}^n \) to this simple case in a systematic way. Outer approximation based inequalities are suggested in [36, 74]. Drewes and Pokutta [37] extended the Lift-and-Project paradigm to MICPs with convex quadratic constraints and binary variables. Belotti et al. [16] studied the intersection of a convex set with a two-term disjunction and suggested a new conic cut for MICPs with \( \mathbb{K} = \mathbb{L}^n \). Dadush et al. [35] studied split cuts for ellipsoids, a specific MICP with a bounded feasible region, and independently suggested a valid inequality, which overlaps with the one from [16]. Recently, Andersen and Jensen [3], and Modaresi et al. [57] generalized split and intersection cuts to MICPs with \( \mathbb{L}^n \), and derived closed form conic quadratic expressions for the resulting valid inequalities. Furthermore, closed form expressions for inequalities describing the convex hull of two-term disjunctions on \( \mathbb{L}^n \) and its cross-sections are also derived in [25, 51, 52, 75].

The literature on cutting plane theory for MILP, i.e., when \( \mathbb{K} = \mathbb{R}_+^m \), is extensive (see [31] for a recent survey). Most cutting planes used in MILP can be viewed in the context of Gomory’s corner polyhedron, which dates back to [41]. A particular stream of research studies a related problem based upon building a semi-infinite relaxation of an MILP, i.e., the mixed-integer group problem [42] and introduces the study of minimal functions and extremal functions as convenient ways of examining the properties of functions that generate cut coefficients. In this literature, various papers (see [42, 43, 48] and references therein) have focused on characterizing valid functions, i.e., \( \psi : \mathbb{R}^m \rightarrow \mathbb{R} \) such that the inequality

\[
\sum_{j=1}^{n} \psi(a_j)x_j \geq 1
\]

holds for all feasible solutions \( x \in \mathbb{R}_+^n \) for any possible number of variables and any choice of columns, \( a_j \), corresponding to these variables. In this framework, a valid function \( \psi \) is said to be minimal if there is no valid function \( \psi' \) distinct from \( \psi \) such that \( \psi' \leq \psi \) (the inequality relation between functions is stated as a pointwise relation). As non-minimal valid functions are implied by a combination of minimal valid functions, only minimal valid functions are needed to generate valid inequalities. Gomory and Johnson in [42, 43] studied a single row setting, i.e., \( m = 1 \), and characterized corresponding minimal functions. Johnson [48] extended it further by characterizing minimal functions for \( m \)-row relaxations. Since then many related relaxations have been investigated extensively for deriving valid inequalities from multiple rows of a simplex tableau (see [4, 13, 23, 32] and references therein). We refer the reader to [30] for a recent survey.

Perhaps the papers that are most closely connected to our study in this stream are from the “lattice-free cutting plane” theory. For the MILP case, a number of studies, e.g., [4, 23, 29, 30], establish an intimate connection between minimal functions and maximal (with respect to inclusion) lattice-free (in our context
A B-free convex set is a convex set that does not contain any point from the given set B in its interior. Usually, one is interested in finding a B-free set to generate a valid inequality that cuts off a given point \( \hat{b} \notin B \), therefore one seeks for a B-free convex set that contains \( \hat{b} \) in its interior. In particular, Borozan and Cornuejols \cite{23} established that minimal valid functions for the semi-infinite relaxation of an MILP correspond to maximal lattice-free convex sets, and thus they arise from nonnegative, piecewise linear, positively homogeneous, convex functions. There is a particular relation between these results and the intersection cuts of \cite{9} as well as the multiple right hand side choice systems introduced in \cite{49}. We refer the interested reader to \cite{28, 29, 30, 33} for details and recent results. In the MILP framework, the relaxations of origins described in section 1.2 are studied extensively in the context of “lattice-free cutting plane” theory. Despite the extensive literature available in this area for \( \mathbb{R}^n_+ \), to the best of our knowledge there is no literature on this topic in the general (non-polyhedral) conic case.

3 \( \mathcal{K} \)-Minimal Inequalities for \( \text{conv}(S(A, \mathcal{K}, B)) \)

In this section, we first introduce a relatively small class of valid linear inequalities, and show that this class is nonempty under a mild technical assumption, which is satisfied, for example, when \( \text{conv}(S(A, \mathcal{K}, B)) \) is full dimensional. Under this mild assumption, we establish that this class contains a small yet essential set of nonredundant valid inequalities: along with the \( x \in \mathcal{K} \) constraint, they are sufficient to describe \( \text{conv}(S(A, \mathcal{K}, B)) \). We then, study the properties of inequalities from this class.

We first highlight a trivial class of valid linear inequalities for \( S(A, \mathcal{K}, B) \), which we refer as cone-implied inequalities. These inequalities stem from the observation that \( S(A, \mathcal{K}, B) \subseteq \mathcal{K} \). Using the definition of the dual cone, we note that for any \( \delta \in \mathcal{K}^* \setminus \{0\} \), the inequality \( \langle \delta, x \rangle \geq 0 \) is valid for \( \mathcal{K} \), and thus, it is also valid for \( S(A, \mathcal{K}, B) \). Therefore, \( (\delta; 0) \in C(A, \mathcal{K}, B) \) for any \( \delta \in \mathcal{K}^* \setminus \{0\} \).

On the other hand, unless the restriction \( Ax \in B \) is trivially satisfied by every \( x \in \mathcal{K} \), e.g., \( S(A, \mathcal{K}, B) = \mathcal{K} \), the set of cone-implied inequalities will not be sufficient to fully describe \( \text{conv}(S(A, \mathcal{K}, B)) \). Because the trivial case of \( \text{conv}(S(A, \mathcal{K}, B)) = \mathcal{K} \) is not of interest, from now on, we assume that \( S(A, \mathcal{K}, B) \subseteq \mathcal{K} \), and we study the properties of valid linear inequalities that are non-cone-implied and are needed to obtain a complete description of \( \text{conv}(S(A, \mathcal{K}, B)) \). This leads us to our definition of \( \mathcal{K} \)-minimal inequalities.

**Definition 3.1** A valid linear inequality \( (\mu; \eta_0) \) with \( \mu \neq 0 \) and \( \eta_0 \in \mathbb{R} \) is \( \mathcal{K} \)-minimal (for \( S(A, \mathcal{K}, B) \)) if for all valid inequalities \( (\rho; \rho_0) \) for \( S(A, \mathcal{K}, B) \) satisfying \( \rho \neq \mu \), and \( \rho \preceq_{\mathcal{K}^*} \mu \), we have \( \rho_0 < \eta_0 \).

We next observe that the cone \( \mathcal{K} \), indeed, induces a natural dominance relation among the valid linear inequalities, and \( \mathcal{K} \)-minimality definition is a result of this dominance relation. Let us consider a valid inequality \( (\mu; \eta_0) \) which is not \( \mathcal{K} \)-minimal. Thus, there exists another valid inequality \( (\rho; \rho_0) \) such that \( \rho \neq \mu \), \( \rho \preceq_{\mathcal{K}^*} \mu \), and \( \rho_0 \geq \eta_0 \). But, then the inequality \( (\rho; \rho_0) \) together with \( x \in \mathcal{K} \) constraint implies the inequality \( (\mu; \eta_0) \):

\[
\langle \mu, x \rangle = \langle \rho + (\mu - \rho), x \rangle = \langle \rho, x \rangle + (\mu - \rho, x) \geq \rho_0 + (\mu - \rho, x) \geq \rho_0 \geq \eta_0,
\]

where the first inequality follows from \( x \in \mathcal{K} \) and \( \mu - \rho \in \mathcal{K}^* \). The above relation indicates that when \( x \in \mathcal{K} \) constraint and the linear inequality \( (\rho; \rho_0) \) are included, the inequality \( (\mu; \eta_0) \) is not necessary in the description of \( \text{conv}(S(A, \mathcal{K}, B)) \). The definition of \( \mathcal{K} \)-minimality simply requires an inequality not to be dominated in this fashion: a \( \mathcal{K} \)-minimal inequality \( (\mu; \eta_0) \) cannot be dominated by another inequality, which is the sum of a cone-implied inequality and another valid inequality.
We note that the definition of $\mathcal{K}$-minimality allows for a $\mathcal{K}$-minimal inequality to be implied by the sum of two other non-cone-implied valid inequalities. Therefore, in general $\mathcal{K}$-minimal inequalities need not be extreme. Since characterization of extreme inequalities in general is known to be a much more difficult task, we limit the focus of our study to the properties of $\mathcal{K}$-minimal inequalities.

**Remark 3.1** None of the cone-implied inequalities $\langle \delta; 0 \rangle$ with $\delta \in \mathcal{K}^* \setminus \{0\}$ is $\mathcal{K}$-minimal because we can always write them as the sum of $\frac{1}{4}\langle \delta; 0 \rangle$ and $\frac{1}{4}\langle \delta; 0 \rangle$, which is again a cone-implied inequality. Nevertheless, a cone-implied inequality can be extreme, and thus, necessary in the description of $\mathcal{K}$-minimal inequalities.

**Remark 3.2** In the case of MILP, $\mathcal{K} = \mathbb{R}^n_+$, a minimal inequality is defined as a valid linear inequality $(\mu; \eta_0)$ such that if $\rho \leq \mu$ (where the $\leq$ is interpreted in the component-wise sense) and $\rho \neq \mu$, then $(\rho; \eta_0)$ is not valid, i.e., reducing any $\mu_i$ for $i \in \{1, \ldots, n\}$ will lead to a strict reduction in the right hand side value of the inequality (see [14, 28, 49]). Considering that $\mathbb{R}^n_+$ is a regular and also self-dual cone, we conclude that $\mathcal{K}$-minimality definition is indeed a natural extension of the minimality definition of valid inequalities studied in to the context of MILPs to more general MICPs with regular cones $\mathcal{K}$.

We let $C_m(A, \mathcal{K}, \mathcal{B})$ denote the set of $\mathcal{K}$-minimal valid inequalities. Note that $C_m(A, \mathcal{K}, \mathcal{B})$ is clearly closed under positive scalar multiplication and is thus a cone (but it is not necessarily a convex cone).

The following simple example shows a set $S(A, \mathcal{K}, \mathcal{B})$ together with the $\mathcal{K}$-minimal inequalities describing its convex hull.

**Example 3.1** Let $S(A, \mathcal{K}, \mathcal{B})$ be defined with $\mathcal{K} = \mathcal{L}^3 = \{x \in \mathbb{R}^3 : x_3 \geq \sqrt{x_1^2 + x_2^2}\} = \mathcal{K}^*$, $A = [-1, 0, 1]$ and $\mathcal{B} = \{0, 2\}$, i.e.,

$$S(A, \mathcal{K}, \mathcal{B}) = \{x \in \mathcal{K} : -x_1 + x_3 = 0\} \cup \{x \in \mathcal{K} : -x_1 + x_3 = 2\}.$$ 

Then

$$\text{conv}(S(A, \mathcal{K}, \mathcal{B})) = \{x \in \mathbb{R}^3 : x_3 \geq \sqrt{x_1^2 + x_2^2}, 0 \leq -x_1 + x_3 \leq 2\} = \{x \in \mathbb{R}^3 : \langle x, \delta \rangle \geq 0 \forall \delta \in \text{Ext}(\mathcal{K}^*), -x_1 + x_3 \geq -2\},$$

is closed and thus the cone of valid inequalities is given by

$$C(A, \mathcal{K}, \mathcal{B}) = \text{cone}(\mathcal{K}^* \times \{0\}, \{1; 0; -1; -2\}).$$

The only non-cone-implied extreme inequality in this description is given by $\mu = (1; 0; -1)$ with $\eta_0 = -2 = \vartheta(\mu)$. It is easy to see that this inequality is valid and also necessary for the description of the convex hull. In order to verify that it is in fact $\mathcal{K}$-minimal, consider any $\delta \in \mathcal{K}^* \setminus \{0\}$, and set $\rho = \mu - \delta$. Then $\rho_0$, i.e., the right hand side value for which $\langle \rho, x \rangle \geq \rho_0$ is valid, is given by

$$\rho_0 := \inf_x \{\langle \rho, x \rangle : x \in S(A, \mathcal{K}, \mathcal{B})\}$$

$$\leq \inf_x \{\langle \rho, x \rangle : x \in \mathcal{K}, -x_1 + x_3 = 2\}$$

$$= \inf_x \{x_1 - x_3 - \langle \delta, x \rangle : x \in \mathcal{K}, -x_1 + x_3 = 2\}$$

$$= \inf_x \{-2 - \langle \delta, x \rangle : x \in \mathcal{K}, -x_1 + x_3 = 2\}$$

$$= -2 - \sup_x \{\langle \delta, x \rangle : x \in \mathcal{K}, -x_1 + x_3 = 2\}$$

$$< -2 = \vartheta(\mu),$$
where the strict inequality follows from the fact that $u = (0; 1; 2)$ is in the interior of $K$ and satisfies $-u_1 + u_3 = 2$ (and thus is feasible) to the last optimization problem in the above chain, and moreover for any $\delta \in K^* \setminus \{0\}$, $\langle \delta, u \rangle > 0$. Clearly, all of the other inequalities involved in the description of $\text{conv}(S(A, K, B))$ are of the form $\langle \delta, x \rangle \geq 0$ with $\delta \in \text{Ext}(K^*)$ and hence are not $K$-minimal.

On the other hand, it is important to note that there can be situations where none of the inequalities describing $\text{conv}(S(A, K, B))$ is $K$-minimal even when $\text{conv}(S(A, K, B)) \subseteq K$. To see this, let us consider a slightly modified version of Example 3.1 with a different $B$ set:

**Example 3.2** Let $S(A, K, B)$ be defined with $K = L^3 = \{x \in \mathbb{R}^3 : x_3 \geq \sqrt{x_1^2 + x_2^2}\}$, $A = [-1, 0, 1]$ and $B = \{0\}$. Then

$$\text{conv}(S(A, K, B)) = \{x \in \mathbb{R}^3 : x_3 \geq \sqrt{x_1^2 + x_2^2}, -x_1 + x_3 = 0\} = \{x \in \mathbb{R}^3 : x_1 = x_3, x_2 = 0, x_1, x_3 \geq 0\}.$$

We claim and prove that none of the inequalities in the description of $\text{conv}(S(A, K, B))$ is $K$-minimal. To observe this, let us fix a particular generating set $(G_L, G_C)$ for the cone $C(A, K, B)$. Based on the above representation of $\text{conv}(S(A, K, B))$, we can take for example $G_C = L^3 \times \{0\}$ and $G_L = (\mu; 0)$ where $\mu = (-1; 0; 1)$ with $\eta_0 = 0 = \vartheta(\mu)$. Note that all of the inequalities in $G_C$ as well as one side of the valid equation given by $(\mu; 0)$ are cone-implied (because $\mu \in L^3$), and thus are not $K$-minimal. Moreover the inequality given by $(-\mu; 0)$, e.g., the other side of the valid equation also cannot be $K$-minimal since $\rho = (-0.5; 0; 0.5)$ satisfies $\delta = \mu - \rho = (-0.5; 0; 0.5) \in \text{Ext}(K^*)$ and $(\rho; \eta_0)$ is also valid. In fact, for any valid inequality $(\mu; \eta_0)$ that is in the description of $\text{conv}(S(A, K, B))$, there exists $\tau > 0$ such that we can subtract the vector $\delta = \tau(-1; 0; 1) \in \text{Ext}(K^*)$ from $\mu$, and still obtain $(\mu - \delta; \eta_0)$ as a valid inequality. Finally we note that the generators of $C(A, K, B)$ are uniquely defined up to shifts by the vector $(\mu; 0)$ defining the valid equation. But these shifts do not change the $K$-minimality properties of the inequalities. □

The peculiar situation of Example 3.2 is a result of the fact that $S(A, K, B) \subset \{x \in K : -x_1 + x_3 = 0\}$, i.e., $S(A, K, B)$ is contained in a subspace defined by a cone-implied valid equation. The next proposition formally states that this is precisely the situation in which none of the valid linear inequalities, including the extreme ones, is $K$-minimal.

**Proposition 3.1** Suppose that there exists $\delta \in K^* \setminus \{0\}$ such that $\langle \delta, x \rangle = 0$ for all $x \in S(A, K, B)$, i.e., $(\delta; 0)$ is a valid equation, then $C_m(A, K, B) = \emptyset$.

**Proof.** Let $\delta \in K^* \setminus \{0\}$ be such that $(\delta; 0)$ is a valid equation. Consider any valid inequality $(\mu; \eta_0)$. Note that $(\mu - \delta; \eta_0)$ is also valid because $(-\delta; 0)$ is also valid. But then, $(\mu; \eta_0)$ is not $K$-minimal because $\delta \in K^* \setminus \{0\}$. Given that $(\mu; \eta_0)$ was arbitrary, this implies that there is no $K$-minimal valid inequality under the hypothesis of the proposition. □

Therefore, in the remainder of this paper, we make the following assumption:

**Assumption 1:** For each $\delta \in K^* \setminus \{0\}$, there exists some $x_\delta \in S(A, K, B)$ such that $\langle \delta, x_\delta \rangle > 0$.

Based on Proposition 3.1, Assumption 1 ensures that $C_m(A, K, B) \neq \emptyset$. It is indeed not very restrictive, and is trivially satisfied, for example, when $\text{conv}(S(A, K, B)) \neq K$ and is full-dimensional, e.g., when $\text{Ker}(A) \cap \text{int}(K) \neq \emptyset$ (see Proposition 3.4).
Our main result in this section shows that under Assumption 1, \( K \)-minimal inequalities, along with \( x \in K \) constraint, are sufficient to describe \( \text{conv}(S(A, K, B)) \). In particular, we prove that under Assumption 1, all extreme inequalities are \( K \)-minimal. Due to the previous discussion on the dominance relation among inequalities and \( K \)-minimality, this result is expected. However, to formalize this, we need the following definition. Given two vectors, \( u, v \in C \) where \( C \) is a cone with lineality space \( L \), \( u \) is said to be an \( L \)-multiple of \( v \) if \( u = \tau v + \ell \) for some \( \tau > 0 \), and \( \ell \in L \). From this definition, it is clear that if \( u \) is an \( L \)-multiple of \( v \), then \( v \) is also an \( L \)-multiple of \( u \). Also, we have the following lemma from \([49]\):

**Lemma 3.1** Suppose \( v \) is in a generating set for cone \( C \) and there exists \( v^1, v^2 \in C \) such that \( v = v^1 + v^2 \), then \( v^1, v^2 \) are \( L \)-multiples of \( v \).

Let \( (G_L, G_C) \) be a generating set for the cone \( C(A, K, B) \). Note that whenever the lineality space \( L \) of the cone \( C(A, K, B) \) is nontrivial, the generating valid inequalities are only defined uniquely up to the \( L \)-multiples. We define \( G_C^+ \) to be the vectors from \( G_C \) that are not \( L \)-multiples of any cone-implied inequality, \( \langle \delta; 0 \rangle \) with \( \delta \in K^* \setminus \{0\} \). Note that \( G_C^+ \) is again only uniquely defined up to \( L \)-multiples.

We note that this result is a straightforward extension of the associated result from \([49]\) given in the linear case to our conic case.

**Proposition 3.2** Let \( (G_L, G_C) \) be a generating set for the cone \( C(A, K, B) \). Under Assumption 1, every valid equation in \( G_L \) and every generating valid inequality in \( G_C^+ \) is \( K \)-minimal.

**Proof.** Suppose \( (\mu; \eta_0) \in G_L \cup G_C^+ \) is not \( K \)-minimal. Then there exists a nonzero \( \delta \in K^* \) such that \( (\mu - \delta; \eta_0) \in C(A, K, B) \). Note that \( (\delta; 0) \in C(A, K, B) \), therefore \( (\mu + \delta; \eta_0) \) is valid as well. Then Lemma 3.1 implies that \( (\delta; 0) \) is an \( L \)-multiple of \( (\mu; \eta_0) \). Using the definition of \( G_C^+ \), we get \( (\mu; \eta_0) \in G_L \). Given that \( (\delta; 0) \) is an \( L \)-multiple of \( (\mu; \eta_0) \) and \( G_L \) is uniquely defined up to \( L \)-multiples, we get that \( (\delta; 0) \in G_L \). Hence \( \langle \delta, x \rangle = 0 \) is a valid equation, contradicting to Assumption 1.

Proposition 3.2 immediately implies the following result.

**Corollary 3.1** Suppose that Assumption 1 holds. Then, for any generating set \( (G_L, G_C) \) of \( C(A, K, B) \), \( (G_L, G_C^+) \) generates \( C_m(A, K, B) \).

In the light of Proposition 3.2 and Corollary 3.1, under Assumption 1, we arrive at the conclusion that the set of \( K \)-minimal inequalities, i.e., \( C_m(A, K, B) \), contains all of the non-cone-implied extreme inequalities. In particular, under Assumption 1, we have

\[
\text{conv}(S(A, K, B)) = \{ x \in E : x \in K, \langle \mu; x \rangle = \eta_0 \forall (\mu; \eta_0) \in G_L, \langle \mu; x \rangle \geq \eta_0 \forall (\mu; \eta_0) \in G_C^+ \} = \{ x \in E : x \in K, \langle \mu; x \rangle \geq \eta_0 \forall (\mu; \eta_0) \in C_m(A, K, B) \}.
\]

This motivates us to further study the properties valid inequalities from \( C_m(A, K, B) \).

### 3.1 Necessary Conditions for \( K \)-Minimal Inequalities

Our first proposition states that in certain cases, all \( K \)-minimal inequalities are tight. This also gives us our first necessary condition for \( K \)-minimality.

**Proposition 3.3** Let \( (\mu; \eta_0) \in C_m(A, K, B) \). Then, whenever \( \mu \in K^* \) or \( \mu \in -K^* \), we have \( \eta_0 = \vartheta(\mu) \) where \( \vartheta(\mu) \) is as defined in (3). Furthermore, \( (\mu; \eta_0) \in C_m(A, K, B) \) and \( \mu \in K^* \) (respectively \( \mu \in -K^* \)) implies \( \eta_0 > 0 \) (respectively \( \eta_0 < 0 \)).

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Proof. Consider $(\mu; \eta_0) \in C_m (A, \mathcal{K}, \mathcal{B})$. The validity of $(\mu; \eta_0)$ immediately implies $\eta_0 \leq \vartheta(\mu)$. Assume for contradiction that $\eta_0 < \vartheta(\mu)$. We need to consider only two cases:

(i) $\mu \in \mathcal{K}^* \setminus \{0\}$: Then we should have $\eta_0 > 0$, otherwise it is either a cone-implied inequality or is dominated by a cone-implied inequality, both of which are not possible. Let $\beta = \frac{\eta_0}{\vartheta(\mu)}$, and consider $\rho = \beta \cdot \mu$. Then $(\rho; \eta_0)$ is a valid inequality since $0 < \beta < 1$, $(\mu; \vartheta(\mu)) \in C(A, \mathcal{K}, \mathcal{B})$ and $C(A, \mathcal{K}, \mathcal{B})$ is a cone. But $\mu - \rho = (1 - \beta) \mu \in \mathcal{K}^* \setminus \{0\}$ since $\mu \neq 0$ and $\beta < 1$, this is a contradiction. Thus, we conclude $\eta_0 = \vartheta(\mu) > 0$.

(ii) $-\mu \in \mathcal{K}^* \setminus \{0\}$: Because $(-\mu; 0)$ is trivially valid and we cannot satisfy both $(-\mu; 0)$ and $(\mu; \vartheta(\mu))$ when $\vartheta(\mu) > 0$ unless $\mathcal{S}(A, \mathcal{K}, \mathcal{B}) = \emptyset$. But this is not possible due to our assumptions on $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$, thus, we conclude that $\vartheta(\mu) \leq 0$. Moreover, if $\vartheta(\mu) = 0$, then $\mathcal{S}(A, \mathcal{K}, \mathcal{B}) \subset \{x \in \mathcal{K} : (\mu, x) = 0\}$, which contradicts to Assumption 1. Hence, we conclude that $\eta_0 < \vartheta(\mu) < 0$. Once again let $\beta = \frac{\eta_0}{\vartheta(\mu)}$, and consider $\rho = \beta \cdot \mu$. Then $(\rho; \eta_0)$ is a valid inequality since $\beta > 1$, $(\mu; \vartheta(\mu)) \in C(A, \mathcal{K}, \mathcal{B})$ and $C(A, \mathcal{K}, \mathcal{B})$ is a cone. But $\mu - \rho = (1 - \beta) \mu \in \mathcal{K}^* \setminus \{0\}$ since $\mu \in -\mathcal{K}^* \setminus \{0\}$ and $\beta > 1$. But, this is a contradiction to the $\mathcal{K}$-minimality of $(\mu; \eta_0)$. Hence, we conclude that $\eta_0 = \vartheta(\mu) < 0$.

Clearly, Proposition 3.3 does not cover all possible cases for $\mu$. As we will see later on, it is possible to have $\mu \notin \{\mathcal{K}^*, -\mathcal{K}^*\}$ leading to a $\mathcal{K}$-minimal inequality. While one is naturally inclined to believe that a $\mathcal{K}$-minimal inequality $(\mu; \eta_0)$ is always tight, i.e., $\eta_0 = \vartheta(\mu)$, the following example shows that this is not necessarily true.

Example 3.3 Consider the following solution set $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ defined with $A = [-1, 1]$, $B = \{-2, 1\}$ and $\mathcal{K} = \mathbb{R}_+^2$. First, note that Assumption 1 holds because $\{(0; 1), (2; 0)\} \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$, and $\text{conv}(\mathcal{S}(A, \mathcal{K}, \mathcal{B})) = \text{conv}(\mathcal{S}(A, \mathcal{K}, \mathcal{B})) \neq \mathbb{R}_+^2$. Thus, $\mathcal{K}$-minimal inequalities exist, and are sufficient to describe $\text{conv}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$ together with nonnegativity restrictions. In fact,

$$\text{conv}(\mathcal{S}(A, \mathcal{K}, \mathcal{B})) = \{x \in \mathbb{R}^2 : -x_1 + x_2 \geq -2, x_1 - x_2 \geq -1, x_1 + 2x_2 \geq 2, x_1, x_2 \geq 0\},$$
and one can easily show that each of the nontrivial inequalities in this description is in fact $\mathcal{K}$-minimal.

![Figure 1: Convex hull of $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ for Example 3.3](image)

Now, let us consider the valid inequality given by $(\mu; \eta_0) = (1; -1; -2)$. Note that $\vartheta(\mu) = 1$, therefore $(\mu; \eta_0)$ is not tight and is clearly dominated by $x_1 - x_2 \geq -1$. We will show that $(\mu; \eta_0)$ is $\mathcal{K}$-minimal.
regardless of the fact that it is not tight. We note that, in this example, \( \mathcal{K} \)-minimality is the same as the usual minimality used in the usual MILP literature, e.g., [14, 28, 49].

Suppose that \((\mu; \eta_0)\) is not \( \mathcal{K} \)-minimal, then there exists \( \rho = \mu - \delta \) with \( \delta \in \mathcal{K}^* = \mathbb{R}_+^2 \), and \( \delta \neq 0 \) such that \((\rho; \eta_0)\) is a valid inequality. This implies

\[
-2 = \eta_0 \leq \inf_{x} \{ \langle \rho, x \rangle : x \in S(A, \mathcal{K}, \mathcal{B}) \} = \min_{x} \{ \langle \rho, x \rangle : x \in \text{conv}(S(A, \mathcal{K}, \mathcal{B})) \}
\]

\[
= \min_{x} \{ \langle \rho, x \rangle : -x_1 + x_2 \geq -2, x_1 - x_2 \geq -1, x_1 + 2x_2 \geq 2, x_1, x_2 \geq 0 \}
\]

\[
= \max_{\lambda} \{-2\lambda_1 - \lambda_2 + 2\lambda_3 : -\lambda_1 + \lambda_2 + \lambda_3 \leq \rho_1, \lambda_1 - \lambda_2 + 2\lambda_3 \leq \rho_2, \lambda \in \mathbb{R}_+^3\}
\]

\[
= \max_{\lambda} \{-2\lambda_1 - \lambda_2 + 2\lambda_3 : -\lambda_1 + \lambda_2 + \lambda_3 \leq 1 - \delta_1, \lambda_1 - \lambda_2 + 2\lambda_3 \leq -1 - \delta_2, \lambda \in \mathbb{R}_+^3\},
\]

where the third equation follows from strong duality (clearly the primal problem is feasible), and the fourth equation follows from the definition of \( \rho = \mu - \delta \). On the other hand, the following system

\[
\lambda \geq 0,
\lambda_1 - \lambda_2 - \lambda_3 \geq \delta_1 - 1
-\lambda_1 + \lambda_2 - 2\lambda_3 \geq 1 + \delta_2,
\]

implies that \( 0 \geq -3\lambda_3 \geq \delta_1 + \delta_2 \). Considering that \( \delta \in \mathbb{R}_+^2 \), this leads to \( \delta_1 = \delta_2 = 0 \), which is a contradiction to \( \delta \neq 0 \). Therefore, we conclude that \((\mu; \eta_0) = (1; -1; -2) \in C_m(A, \mathcal{K}, \mathcal{B})\) yet \( \eta_0 \neq \vartheta(\mu) \).

**Remark 3.3** We also note that this issue of non-tightness of some \( \mathcal{K} \)-minimal inequalities is independent of whether the inequalities separate the origin or not. For example, in a variation of Example 3.3 given by \( A = [-1, 1], \mathcal{B} = \{-2, -1\} \) and \( \mathcal{K} = \mathbb{R}_+^2 \), we have the valid inequality given by \((\mu; \eta_0) = (1; -1; \frac{1}{2})\) is \( \mathcal{K} \)-minimal due to the same reasoning, and also separates the origin from the convex hull. Yet it has \( \vartheta(\mu) = 1 \) and hence \((\mu; \eta_0)\) is not tight.

In fact, we can generalize the situation of Example 3.3, and prove the following proposition, which states that under a special condition, \( \text{Ker}(A) \cap \text{int}(\mathcal{K}) \neq \emptyset \), any valid inequality with \( \mu \in \text{Im}(A^*) \) is a \( \mathcal{K} \)-minimal inequality.

**Proposition 3.4** Suppose \( \text{Ker}(A) \cap \text{int}(\mathcal{K}) \neq \emptyset \). Then for any \( \mu \in \text{Im}(A^*) \) and any \( -\infty < \eta_0 \leq \vartheta(\mu) \) where \( \vartheta(\mu) \) is defined by (3), we have \((\mu; \eta_0) \in C_m(A, \mathcal{K}, \mathcal{B})\).

**Proof.** Consider \( d \in \text{Ker}(A) \cap \text{int}(\mathcal{K}) \neq \emptyset \), note that \( d \neq 0 \). For any \( b \in \mathcal{B} \), define the set \( S_b := \{ x \in E : Ax = b, x \in \mathcal{K} \} \), and let \( \widehat{\mathcal{B}} := \{ b \in \mathcal{B} : S_b \neq \emptyset \} \). Because \( S(A, \mathcal{K}, \mathcal{B}) \neq \emptyset \), we have \( \widehat{\mathcal{B}} \neq \emptyset \). For any \( b \in \widehat{\mathcal{B}} \), let \( x_b \in S_b \), then \( P_b := \{ x_b + \tau d : \tau \geq 0 \} \subseteq S_b \) holds. Moreover, \( P_b \cap \text{int}(\mathcal{K}) \neq \emptyset \) for any \( b \in \widehat{\mathcal{B}} \neq \emptyset \) and thus \textbf{Assumption 1} holds here.

Assume that the statement is not true, i.e., there exists \( \mu \in \text{Im}(A^*) \) together with \( \eta_0 \leq \vartheta(\mu) \), such that \((\mu; \eta_0) \notin C_m(A, \mathcal{K}, \mathcal{B})\). Then, there exists \( \delta \in \mathcal{K}^* \setminus \{0\} \) such that \((\mu - \delta; \eta_0) \in C(A, \mathcal{K}, \mathcal{B})\), which implies

\[
\begin{align*}
-\infty < \eta_0 & \leq \inf_{x} \{ \langle \mu - \delta, x \rangle : x \in S(A, \mathcal{K}, \mathcal{B}) \} \\
& \leq \inf_{b \in \mathcal{B}} \inf_{x} \{ \langle \mu - \delta, x \rangle : Ax = b, x \in \mathcal{K} \} \\
& \leq \inf_{b \in \mathcal{B}} \inf_{x} \{ \langle \mu - \delta, x \rangle : x \in P_b \} \\
& \leq \inf_{b \in \mathcal{B}} \left[ \langle \mu - \delta, x_b \rangle + \inf_{\tau \in \mathbb{R}} \{ \langle \mu - \delta, \tau d \rangle : \tau \geq 0 \} \right].
\end{align*}
\]
Also, note that \( \inf_{\tau} \{ \langle \mu - \delta, \tau d \rangle : \tau \geq 0 \} = -\infty \) when \( \langle \mu - \delta, d \rangle < 0 \). But \( \langle \mu - \delta, d \rangle < 0 \) is impossible since it would have implied \( -\infty < \eta_0 \leq -\infty \). Therefore, we conclude that \( \langle \mu - \delta, d \rangle \geq 0 \).

Finally \( \mu \in \text{Im}(A^*) \) implies that there exists \( \lambda \) such that \( \mu = A^*\lambda \). Taking this into account, we arrive at

\[
0 \leq \langle \mu - \delta, d \rangle = \langle A^*\lambda, d \rangle - \langle \delta, d \rangle = \lambda^T (Ad) - \langle \delta, d \rangle = -\langle \delta, d \rangle,
\]

where we used the fact that \( d \in \text{Ker}(A) \). But \( d \in \text{int}(\mathcal{K}) \) and \( \delta \in \mathcal{K}^* \setminus \{0\} \) implies that \( \langle \delta, d \rangle > 0 \), which is a contradiction. \( \square \)

Example 3.3 and Proposition 3.4 indicate a weakness of the \( \mathcal{K} \)-minimality definition. To address this, we should consider only tight \( \mathcal{K} \)-minimal inequalities, e.g., \( (\mu; \eta_0) \in C_m(A, \mathcal{K}, \mathcal{B}) \) where \( \eta_0 \) cannot be increased without making the current inequality invalid, i.e., \( \eta_0 = \vartheta(\mu) \). While we can include a tightness requirement in our \( \mathcal{K} \)-minimality definition, we note that tightness has a direct characterization through \( \vartheta(\mu) \), and also to remain consistent with the original minimality definition for \( \mathcal{K} = \mathbb{R}_+^n \), we opt to work with our original \( \mathcal{K} \)-minimality definition. As will be clear from the rest of the paper, tightness considerations will make minimal change in our analysis.

We next state a proposition which identifies a key necessary condition for \( \mathcal{K} \)-minimality via a certain non-expansiveness property. The following set of linear maps will be of importance for this result.

\[
\mathcal{F}_\mathcal{K} := \{ (Z : E \to E) : Z \text{ is linear, and } Z^*v \in \mathcal{K} \forall v \in \mathcal{K} \},
\]

where \( Z^* \) denotes the conjugate linear map of \( Z \).

**Proposition 3.5** Let \( (\mu; \eta_0) \in C(A, \mathcal{K}, \mathcal{B}) \) and suppose that there exists a linear map \( Z \in \mathcal{F}_\mathcal{K} \) such that \( AZ^* = A \), and \( \mu - Z\mu \in \mathcal{K}^* \setminus \{0\} \), then \( (\mu; \eta_0) \notin C_m(A, \mathcal{K}, \mathcal{B}) \).

**Proof.** Let \( (\mu; \eta_0) \in C(A, \mathcal{K}, \mathcal{B}) \) and \( Z \) be a linear map as described in the proposition. Since \( Z \in \mathcal{F}_\mathcal{K} \), for any \( x \in \mathcal{K} \), we have \( Z^*x \in \mathcal{K} \). Moreover, \( AZ^*x = Ax \) due to \( AZ^* = A \), and thus for any \( x \in S(A, \mathcal{K}, \mathcal{B}) \), \( AZ^*x = Ax \in \mathcal{B} \). Therefore, we have \( Z^*x \in S(A, \mathcal{K}, \mathcal{B}) \) for any \( x \in S(A, \mathcal{K}, \mathcal{B}) \). Now, let \( \delta = \mu - Z\mu \), then \( \delta \in \mathcal{K}^* \setminus \{0\} \) by the premise of the proposition. Define \( \rho := \mu - \delta \), then, for any \( x \in S(A, \mathcal{K}, \mathcal{B}) \) we have

\[
\langle \rho, x \rangle = \langle \mu - \delta, x \rangle = \langle Z\mu, x \rangle = \langle \mu, Z^*x \rangle \geq \eta_0,
\]

where the last inequality follows from the fact that \( Z^*x \in S(A, \mathcal{K}, \mathcal{B}) \) and \( (\mu; \eta_0) \in C(A, \mathcal{K}, \mathcal{B}) \). Hence, we get

\[
\inf_x \{ \langle \rho, x \rangle : Ax \in \mathcal{B}, x \in \mathcal{K} \} \geq \eta_0,
\]

which implies that \( (\mu; \eta_0) \notin C_m(A, \mathcal{K}, \mathcal{B}) \). \( \square \)

Proposition 3.5 states an involved necessary condition for a valid inequality to be \( \mathcal{K} \)-minimal. It states that \( (\mu; \eta_0) \) is a \( \mathcal{K} \)-minimal inequality only if the following holds:

\[
(\text{Id} - Z)\mu \notin \mathcal{K}^* \setminus \{0\} \forall Z \in \mathcal{F}_\mathcal{K} \text{ such that } AZ^* = A.
\]

\( ^4 \) Here, for a linear map \( Z : E \to E \), we use \( Z^* \) to denote its conjugate map given by the identity

\[
\langle x, Zv \rangle = \langle Z^*x, v \rangle \forall (x \in E, v \in E).
\]
Based on this result, the set $\mathcal{F}_K$ has certain importance. In fact $\mathcal{F}_K$ is the cone of $K^* - K^*$ positive maps, which also appear in applications of robust optimization, quantum physics, etc. (see [17]). When $K = \mathbb{R}_+^n$, $\mathcal{F}_K = \{Z \in \mathbb{R}^{n \times n} : Z_{ij} \geq 0 \forall i, j\}$. However, in general, the description of $\mathcal{F}_K$ can be quite nontrivial for different cones $K$. In fact, in [17], it is shown that deciding whether a given linear map takes $S^n_+$ to itself is an NP-Hard optimization problem. In another particular case of interest, when $K = L^n$, a quite nontrivial explicit description of $\mathcal{F}_K$ via a semidefinite representation is given by Hildebrand in [45, 46]. Due to the general difficulty of characterizing $\mathcal{F}_K$, and thus testing the necessary condition of $K$-minimality given in Proposition 3.5, in the next section, we study a relaxed version of the condition from Proposition 3.5 and focus on a larger class of valid inequalities, which subsumes the class of $K$-minimal inequalities. We refer to this larger class as $K$-sublinear inequalities.

4 $K$-Sublinear Inequalities

Definition 4.1 An inequality $(\mu; \eta_0)$ with $\mu \neq 0$ and $\eta_0 \in \mathbb{R}$ is $K$-sublinear if it satisfies both of the following conditions:

(A.1) $\quad 0 \leq \langle \mu, u \rangle$ for all $u$ such that $Au = 0$ and $(\alpha, v)u + v \in K \forall v \in \text{Ext}(K)$ holds for some $\alpha \in \text{Ext}(K^*)$,

(A.2) $\quad \eta_0 \leq \theta(\mu), \text{ i.e., } \eta_0 \leq \langle \mu, x \rangle$ for all $x \in S(A, K, B)$.

It can be easily verified that the set of $(\mu; \eta_0)$ satisfying conditions (A.1)-(A.2) in fact leads to a convex cone in the space $E \times \mathbb{R}$. We denote this cone of $K$-sublinear inequalities with $C_\mu(A, K, B)$.

Condition (A.2) is simply included to ensure the validity of a given inequality, and thus it is satisfied by every valid inequality. On the other hand, condition (A.1) is not very intuitive. The main role of condition (A.1) is to ensure the necessary non-expansivity condition for $K$-minimality established in Proposition 3.5.

There is a particular and simple case of (A.1) that is of interest and deserves a separate treatment. Let $(\mu; \eta_0)$ satisfy (A.1), then $(\mu; \eta_0)$ also satisfies the following condition:

(A.0) $\quad 0 \leq \langle \mu, u \rangle$ for all $u \in K$ such that $Au = 0$.

In order to see that in fact (A.0) is a special case of (A.1), consider any $u \in K \cap \text{Ker}(A)$. Then, for any $\alpha \in \text{Ext}(K^*)$, we have $\langle \alpha, v \rangle \geq 0$ for all $v \in \text{Ext}(K)$, and because $u \in K$ and $K$ is a cone, the requirement of condition (A.1) on $u$, is automatically satisfied for any such $u \in K \cap \text{Ker}(A)$. While (A.1) already implies (A.0), treating (A.0) separately seems to be handy as some of our results depend solely on conditions (A.0) and (A.2). Note also that condition (A.0) is precisely equivalent to

(A.0) $\quad \mu \in (K \cap \text{Ker}(A))^* = K^* + (\text{Ker}(A))^* = K^* + \text{Im}(A^*)$,

where the last equation follows from the facts that $\text{Ker}(A)^* = (\text{Ker}(A))^\perp = \text{Im}(A^*)$ and $K^* + \text{Im}(A^*)$ is closed whenever $K$ is closed (see, e.g., Corollary 16.4.2 in [62]).

While condition (A.0) is not as strong as (A.1), we have the following result establishing that condition (A.0) is necessary for any non-trivial valid inequality.

Proposition 4.1 Suppose $\mu \in \Pi(A, K, B)$, then $\mu$ satisfies condition (A.0).

Proof. Suppose condition (A.0) is violated by some $\mu \in \Pi(A, K, B)$. Then, there exists $u \in K$ such that $Au = 0$ and $\langle \mu, u \rangle < 0$. Note that for any $\beta > 0$ and $x \in S(A, K, B)$, $x + \beta u \in K$ and $A(x + \beta u) = Ax \in B$, hence $x + \beta u \in S(A, K, B)$. On the other hand, the term,

$$\langle \mu, x + \beta u \rangle = \langle \mu, x \rangle + \beta \langle \mu, u \rangle,$$
can be made arbitrarily small by increasing $\beta$, which implies $\vartheta(\mu) = -\infty$ where $\vartheta(\mu)$ is as defined in (3). But then this is a contradiction since we assumed $\mu \in \Pi(A, K, B)$, and so $\vartheta(\mu) \neq -\infty$. \hfill \Box

Our next theorem simply states that every $K$-minimal inequality is also $K$-sublinear.

**Theorem 4.1** If $(\mu; \eta_0) \in C_m(A, K, B)$, then $(\mu; \eta_0) \in C_a(A, K, B)$.

**Proof.** Consider any $K$-minimal inequality $(\mu; \eta_0)$. Since $(\mu; \eta_0) \in C_m(A, K, B)$, then $(\mu; \eta_0)$ is valid for $S(A, K, B)$ and hence condition (A.2) for $K$-sublinearity is automatically satisfied.

Assume for contradiction that condition (A.1) is violated by $(\mu; \eta_0) \in C_m(A, K, B)$, i.e., there exists an $\alpha \in \text{Ext}(K^*)$ and $u$ such that $Au = 0$, $\langle \mu, u \rangle < 0$ and $\langle \alpha, v \rangle u + v \in K \ \forall v \in \text{Ext}(K)$ holds. Based on $u$ and $\alpha$, let us define a linear map $Z : E \rightarrow E$ as

$$Zx = \langle x, u \rangle \alpha + x \ \text{for any} \ x \in E.$$ 

Note that $A : E \rightarrow \mathbb{R}^m$ and thus its conjugate $A^* : \mathbb{R}^m \rightarrow E$. We let $A^*e_i =: A^i \in E$ for $i = 1, \ldots, m$, where $e_i$ is the $i$-th unit vector in $\mathbb{R}^m$. This way, for all $i = 1, \ldots, m$, we have $ZA^*e_i = \langle A^i, u \rangle + A^i = A^i$ where we have used $u \in \text{Ker}(A)$ implies $\langle A^i, u \rangle = 0$. Therefore, we have $ZA^* = A^*$. Also, since $A : E \rightarrow \mathbb{R}^m$ and $Z : E \rightarrow E$ are linear maps, we have $ZA^*$ is a linear map and its conjugate is given by $AZ^* = A$ as desired.

Moreover, for all $w \in K^*$ and $v \in \text{Ext}(K)$, we have

$$\langle Zw, v \rangle = \langle (\langle w, u \rangle \alpha + w), v \rangle = \langle w, u \rangle \langle \alpha, v \rangle + \langle w, \underline{\alpha, v}u + v \rangle \geq 0.$$

As any $v \in K$ can be written as a convex combination of points from $\text{Ext}(K)$, we conclude that $Z \in F_K$. Finally by recalling that $\alpha \in K^*$ and is nonzero, we get

$$\mu - Z\mu = -\langle \mu, u \rangle \alpha \in K^* \setminus \{0\},$$

which is a contradiction to the necessary condition for $K$-minimality given in Proposition 3.5. \hfill \Box

The proof of Theorem 4.1 reveals that a specific case of the condition from Proposition 3.5 underlies the definition of $K$-sublinear inequalities, in particular condition (A.1). Next, we show that condition (A.1) further simplifies in the case of $K = \mathbb{R}^n_+$, and conditions (A.0)-(A.2) underlie the definition of subadditive inequalities from [49] in the MILP case.

**Remark 4.1** When the cone $K$ is known and simple enough, i.e., it has finitely many and orthogonal to each other extreme rays, the interesting cases of condition (A.1), i.e., the ones that are not covered by condition (A.0), can be simplified.

Suppose $K = \mathbb{R}^n_+$, then the extreme rays of $K$ as well as $K^*$ are just the unit vectors, $e_i$. Let us first consider the case of $\alpha = e_i$. In such a case, for any $v \in \text{Ext}(K)$, we have the requirement on $u$ as

$$v_i u + v \in K \ \forall v \in \text{Ext}(K) := \{e_1, \ldots, e_n\}.$$

Considering that we only have unit vectors as the extreme rays, we see that this requirement may have an effect only for the extreme rays with a nonzero $v_i$ value, which is just the case of $v = e_i$. And hence we can equivalently decompose condition (A.1) into $n$ easier to deal with pieces and rephrase it as follows:

(A.1i) $0 \leq \langle \mu, u \rangle$ for all $u$ such that $Au = 0$ and $u + e_i \in \mathbb{R}^n_+$. 

Let $a_i$ denote the $i^{th}$ column of the matrix $A$. By a change of variables, one easily notices that this requirement is equivalent to the following relation:

$$ (A_i^j) \quad \mu_i \leq \langle \mu, w \rangle \text{ for all } w \in \mathbb{R}^n \text{ such that } Aw = a_i, \text{ and for all } i = 1, \ldots, n. $$

In fact, in the simplest case of $K = \mathbb{R}^n_+$, this refinement of condition (A.1), i.e., (A.I), together with the conditions (A.0) and (A.2) is defined as the class of subadditive valid inequalities in [49]. Moreover, suppose that there is a function $\sigma(\cdot)$ that generates the cut coefficients, i.e., $\mu_i = \sigma(a_i)$, then the condition (A.I) above precisely represents the subadditivity property of the function $\sigma(\cdot)$ over the columns of $A$. □

Under Assumption 1, there is a precise relation between the generators of the cones of $K$-sublinear inequalities and $K$-minimal inequalities. We state this in Theorem 4.2 below. Note that this is an extension of the corresponding result from [49] to the conic case, and most of the original proof remains intact but we include the simplified proof below for completeness.

**Theorem 4.2** Suppose that Assumption 1 holds. Then, any generating set of $C_A(A, K, B)$ is of form $(G_L, G_C)$ where $G_a \supseteq G_C^+$ and $(G_L, G_C)$ is a generating set of $C(A, K, B)$. Moreover, if $(\mu; \eta_0) \in G_a \setminus G_C^+$, then $(\mu; \eta_0)$ is not $K$-minimal.

**Proof.** Based on Remark 1.1, let $(G_L, G_C)$ be a generating set of $C(A, K, B)$ such that each vector in $G_C$ is orthogonal to every vector in $G_L$, and all vectors in $G_L$ are orthogonal to each other. Let $(G_L, G_a)$ be a generating set of $C_A(A, K, B)$ such that each vector in $G_a$ is orthogonal to every vector in $G_L$. Note that by Theorem 4.1, we have $C_m(A, K, B) \subseteq C_A(A, K, B) \subseteq C(A, K, B)$.

Under Assumption 1, using Corollary 3.1, we have $C_m(A, K, B)$ has a generating set of the form $(G_L, G_C^+).$ Hence, we conclude that the subspace spanned by $G_L$ both simultaneously contains, and is contained in, the subspace generated by $G_L$. This implies we can take $G_L = G_L$.

Let $Q$ be the orthogonal complement to the subspace generated by $G_L$ and define $C' = C(A, K, B) \cap Q$, $C'_m = C_m(A, K, B) \cap Q$ and $C'_a = C_a(A, K, B) \cap Q$. Then, $C' = \text{cone}(G_C^+)$, and under Assumption 1, $C'_m = \text{cone}(G_C^+)^\perp$. Also, $C'_a$, $C'_m$ and $C'_a$ are pointed cones and satisfy $C'_m \subseteq C'_a \subseteq C'$. Given that the elements of $G_C^+$ are extreme in both $C'$ and $C'_m$, they remain extreme in $C'_a$ as well. Therefore, $G_C^+ \subseteq G_a$.

Finally, consider any $(\mu; \eta_0) \in G_a \setminus G_C^+$. We need to show that $(\mu; \eta_0) \notin C_m(A, K, B)$ . Suppose not, then $(\mu; \eta_0) \in C_m(A, K, B)$ but not in $G_C^+$, which implies that $(\mu; \eta_0)$ is not extreme in $C_m(A, K, B)$. Noting $C_m(A, K, B) \subseteq C_A(A, K, B)$, we conclude that $(\mu; \eta_0)$ is not extreme in $C_A(A, K, B)$ as well. But this is a contradiction to the fact that $(\mu; \eta_0) \in G_a$ and $(G_L, G_a)$ is a generating set for $C(A, K, B)$. Therefore we conclude that for any $(\mu; \eta_0) \in G_a \setminus G_C^+$, $(\mu; \eta_0) \notin C_m(A, K, B)$. □

The above theorem implicitly describes a way of obtaining all of the nontrivial extreme valid inequalities of $C(A, K, B)$: first identify a generating set $(G_L, G_a)$ for $C_A(A, K, B)$ and then test its elements for $K$-minimality to identify $G_C^+$. On one hand, this is good news, as we seem to have a better algebraic handle on $C_A(A, K, B)$ via the conditions given by (A.0)-(A.2). On the other hand, testing these conditions as stated in (A.0)-(A.2), seems to be a nontrivial task. Moreover, we need to establish further ways of characterizing $K$-minimality. Both of these tasks constitute our next section. As we proceed, we also demonstrate how to apply our framework via a few examples.
5 Relations to Support Functions and Cut Generating Sets

In this section, we relate our characterization of $K$-sublinear inequalities to the support functions of sets with certain structure. Recall that a support function of a nonempty set $D \subseteq \mathbb{R}^m$ is defined as

$$\sigma_D(z) := \sup_{\lambda} \{ z^T \lambda : \lambda \in D \} \text{ for any } z \in \mathbb{R}^m.$$ 

For any nonempty set $D$, it is well known that its support function, $\sigma_D(\cdot)$, satisfies the following properties (see [47, 62] for an extended exposure to the topic):

(S.1) $\sigma_D(0) = 0$ (nonnegative),
(S.2) $\sigma_D(z_1 + z_2) \leq \sigma_D(z_1) + \sigma_D(z_2)$ (subadditive),
(S.3) $\sigma_D(\beta z) = \beta \sigma_D(z)$ $\forall \beta > 0$ and for all $z \in \mathbb{R}^m$ (positively homogeneous).

In particular, support functions are positively homogeneous and subadditive, therefore sublinear and thus convex.

There is a particular connection between $K$-sublinear inequalities and support functions of convex sets with certain structure. This connection leads to a cut generating set point of view as well as a number of necessary conditions for $K$-sublinearity. We state this connection in a series of results as follows:

**Theorem 5.1** Consider any $\mu \in E$ satisfying condition (A.0), and define

$$D_\mu = \{ \lambda \in \mathbb{R}^m : A^* \lambda \preceq_{K^*} \mu \}.$$ 

Then, $D_\mu \neq \emptyset$, $\sigma_{D_\mu}(0) = 0$ and $\sigma_{D_\mu}(Az) \leq \langle \mu, z \rangle$ for all $z \in K$.

**Proof.** Since $\mu$ satisfies condition (A.0), we have $\mu \in K^* + \text{Im}(A^*)$, which trivially implies the nonemptyness of $D_\mu$. Given that $\sigma_{D_\mu}(\cdot)$ is the support function of $D_\mu$ and $D_\mu \neq \emptyset$, we have $\sigma_{D_\mu}(0) = 0$.

Finally, for any $z \in K$, we have

$$\sigma_{D_\mu}(Az) = \sup_{\lambda} \{ \lambda^T Az : \lambda \in D_\mu \} = \sup_{\lambda} \{ \langle z, A^* \lambda \rangle : A^* \lambda \preceq_{K^*} \mu \} \leq \sup_{\lambda} \{ \langle z, \mu \rangle : A^* \lambda \preceq_{K^*} \mu \} = \langle z, \mu \rangle,$$

where the last inequality follows from the fact that $z \in K$ and for any $\lambda \in D_\mu$, we have $\mu - A^* \lambda \in K^*$, implying $\langle \mu - A^* \lambda, z \rangle \geq 0$. Therefore, we arrive at $\sigma_{D_\mu}(Az) \leq \langle \mu, z \rangle$. □

We note that every non-trivial valid linear inequality satisfies $\mu \in \text{Im}(A^*) + K^*$ (cf. Proposition 4.1). This, clearly implies that any $\mu$ such that $\mu \notin \text{Im}(A^*) + K^*$ is redundant in the description of $S(A, K, B)$. Furthermore, given a vector $\mu \in \text{Im}(A^*) + K^*$, based on Theorem 5.1, we can use the support function of the corresponding set $D_\mu$, and easily obtain a condition on the right hand side value, $\eta_0$, to ensure the validity of the inequality denoted by $(\mu; \eta_0)$ as follows:

**Proposition 5.1** Suppose $\mu \in E$ that satisfies condition (A.0), then the inequality given by $(\mu; \eta_0)$ with $\eta_0 \leq \inf_{b \in B} \sigma_{D_\mu}(b)$ is valid for $S(A, K, B)$. 

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Proof. From the condition on \( \mu \) and by Theorem 5.1, we immediately have \( D_\mu \neq \emptyset \) and \( \sigma_{D_\mu}(Az) \leq \langle \mu, z \rangle \).

In order to prove that \((\mu; \eta_0)\) is a valid inequality, we need to prove \( \eta_0 \leq \vartheta(\mu) \) where \( \vartheta(\mu) \) is as defined in (3). But this simply follows from the associated definitions. Let \( \widehat{B} := \{ b \in B : \exists x \text{ s.t. } Ax = b, x \in K \} \). Then,

\[
\eta_0 \leq \inf_{b \in \widehat{B}} \sigma_{D_\mu}(b) \leq \inf_{b \in B} \sigma_{D_\mu}(b) = \inf_{b \in \widehat{B}} \left\{ \sigma_{D_\mu}(Ax) : Ax = b, b \in \widehat{B} \right\} \leq \inf_x \left\{ \sigma_{D_\mu}(Ax) : x \in K, Ax \in \widehat{B} \right\} \leq \inf_x \left\{ \langle \mu, x \rangle : x \in K, Ax \in \widehat{B} \right\} = \inf_x \left\{ \langle \mu, x \rangle : x \in K, Ax \in B \right\} = \vartheta(\mu),
\]

where the last inequality follows from the fact that for all \( z \in K \), we have \( \sigma_{D_\mu}(Az) \leq \langle \mu, z \rangle \). \( \square \)

In addition to this, in certain specific cases, we can provide a more precise relation between \( \vartheta(\mu) \) and \( \inf_{b \in B} \sigma_{D_\mu}(b) \).

Corollary 5.1 Suppose \( \text{Ker}(A) \cap \text{int}(K) \neq \emptyset \). Then, for any \( \mu \in E \) satisfying condition (A.0), we have \( \vartheta(\mu) = \inf_{b \in B} \sigma_{D_\mu}(b) \).

Proof. By Proposition 5.1, we also have \( \inf_{b \in B} \sigma_{D_\mu}(b) \leq \vartheta(\mu) \). Moreover,

\[
\inf_{b \in \widehat{B}} \sigma_{D_\mu}(b) = \inf_{b \in B} \left( \sup_{\lambda \in \mathbb{R}^m} \{ b^T \lambda : A^* \lambda \preceq_{K^*} \mu \} \right) = \inf_{b \in B} \inf_{x} \left\{ \langle \mu, x \rangle : x \in K, Ax = b \right\} \geq \vartheta(\mu),
\]

where the last inequality follows from strong conic duality due to the facts that \( \text{Ker}(A) \cap \text{int}(K) \neq \emptyset \) and \( b \in B \), and the definition of \( \vartheta(\mu) \) in (3). Thus, we obtain \( \inf_{b \in B} \sigma_{D_\mu}(b) = \vartheta(\mu) \). \( \square \)

Due to the common structure \( D_\mu \) of the sets that are used in generating valid inequalities, we justifiably refer to the sets of this form as cut generating sets. In particular, Proposition 5.1 highlights a view based on cut generating sets to generate valid linear inequalities that describe \( \text{conv}(S(A, K, B)) \), as opposed to the view based on cut generating function\(^4\) of [28].

While it is tempting to study cut generating functions of [28] for general regular cones, unlike cut generating sets, we provide an examples \( S(A, K, B) \) with \( K = L^3 \), where large classes of extreme inequalities cannot be generated by any cut generating function (see Remark 5.3). In particular, this discrepancy between cut generating set versus function point of view arises from the fact that for given a vector \( \mu \), there is a unique \( D_\mu \) set associated with it, yet, it is possible that several different vectors generate the same set \( D = D_\mu \). We demonstrate this phenomena in Example 5.2. In the case of several vectors \( \mu \) generating the same set \( D = D_\mu \), one needs to consider all such vectors leading to \( D \) to generate valid linear inequalities for the description of \( \text{conv}(S(A, K, B)) \).

\(^4\)This view is based on studying the properties of functions that generate cut coefficients. See [28] for precise definition and further details.
5.1 Necessary Conditions for $K$-Sublinearity

We next establish a number of necessary conditions for $K$-sublinearity via cut generating sets and their support functions.

**Proposition 5.2** Suppose $\mu \in E$ satisfies condition (A.0). For any given $z \in K$, define

$$\perp_z := \{ \gamma \in K^* : \langle \gamma, z \rangle = 0 \}.$$  \hfill (5)

Then, for all $z \in K$ such that $\perp_z \cap (\mu - \text{Im}(A^*)) \neq \emptyset$, we have $\sigma_{D_\mu}(Az) = \langle \mu, z \rangle$ where $D_\mu$ is defined by (4).

**Proof.** Consider any $z \in K$, then we have

$$\sigma_{D_\mu}(Az) = \sup_{\lambda \in \mathbb{R}^m} \{ \lambda^T Az : \lambda \in D_\mu \} = \sup_{\gamma \in E, \lambda \in \mathbb{R}^m} \{ \langle z, A^* \lambda \rangle : A^* \lambda = \mu - \gamma, \gamma \in K^* \} = \langle z, \mu \rangle - \inf_{\gamma \in E} \{ \langle z, \gamma \rangle : \gamma \in \mu - \text{Im}(A^*), \gamma \in K^* \} = \langle z, \mu \rangle$$

where the last equation follows from the fact that $\langle z, \gamma \rangle \geq 0$ for all $z \in K$ and $\gamma \in K^*$, and there exists $\bar{\gamma} \in \perp_z \cap (\mu - \text{Im}(A^*))$, i.e., $\bar{\gamma} \in K^* \cap (\mu - \text{Im}(A^*))$ and $\langle \mu, \bar{\gamma} \rangle = 0$. \hfill \Box

We note that for $\mu \in \partial(K^*) + \text{Im}(A^*)$, we have $\partial(K^*) \cap (\mu - \text{Im}(A^*)) \neq \emptyset$, and thus there exists $z \in \partial K$ such that $\perp_z \cap (\mu - \text{Im}(A^*)) \neq \emptyset$. In particular, for $\mu \in \text{Im}(A^*)$, we have $0 \in K^* \cap (\mu - \text{Im}(A^*))$. Therefore, taking into account condition (A.0) and Theorem 5.1, we have the following corollary of Proposition 5.2:

**Corollary 5.2** For any $\mu \in \partial(K^*) + \text{Im}(A^*)$, we have $D_\mu \neq \emptyset$ and $\sigma_{D_\mu}(Az) = \langle \mu, z \rangle$ holds for at least one $z \in \text{Ext}(K)$ where $D_\mu$ is defined as in (4). Moreover, for any $\mu \in \text{Im}(A^*)$, we have $\sigma_{D_\mu}(Az) = \langle \mu, z \rangle$ for all $z \in K$.

We illustrate this necessary condition for $K$-sublinearity via the following simple example.

**Example 5.1** Let $S(A, K, B)$ be defined with $K = L^3 = \{ x \in \mathbb{R}^3 : x_3 \geq \sqrt{x_1^2 + x_2^2} \}$, $A = [0, 0, 1]$ and $B = \{1, 2\}$ . Then $\text{conv}(S(A, K, B)) = \{ x \in \mathbb{R}^3 : x_3 \geq \sqrt{x_1^2 + x_2^2}, 1 \leq x_3 \leq 2 \}$ (see Figure 2).

![Figure 2: The set $S(A, K, B)$ corresponding to Example 5.1](image-url)
We claim that the inequalities given by \( \mu^{(1)} = (0; 0; 1) \) with \( \eta_0^{(1)} = 1 \) and \( \mu^{(2)} = (0; 0; -1) \) with \( \eta_0^{(2)} = -2 \) are both \( K \)-sublinear (in fact also \( K \)-minimal). Here, we show that the necessary conditions for \( K \)-sublinearity established so far are satisfied. We will revisit this example after establishing sufficient conditions for \( K \)-sublinearity and \( K \)-minimality.

Clearly both \( \mu^{(1)} \), \( \mu^{(2)} \) \( \in \text{Im}(A^*) \) and \( \eta_0^{(i)} = \vartheta(\mu^{(i)}) \). Also, one can easily check that for the corresponding sets \( D_{\mu^{(i)}} \), \( i = 1, 2 \) associated with these inequalities are given by

\[
D_{\mu^{(1)}} = \{ \lambda : \exists \gamma \in K^* \text{ s.t. } \gamma_1 = 0; \gamma_2 = 0; \lambda + \gamma_3 = 1 \} = \{ \lambda : \lambda \leq -1 \}, \\
D_{\mu^{(2)}} = \{ \lambda : \lambda \leq -1 \},
\]

and are nonempty. Moreover, we have \( \sigma_{D_{\mu^{(i)}}}(Az) = \sigma_{D_{\mu^{(i)}}}(z_3) = \langle \mu^{(i)}, z \rangle \) for all \( z \in K \) and \( \inf_{b \in B} \sigma_{D_{\mu^{(i)}}}(b) = \eta_0^{(i)} \) for \( i = 1, 2 \).

When \( K = \mathbb{R}_{+}^n \), using Remark 4.1 the relationship between \( K \)-sublinearity and the support functions of cut generating sets can be further enhanced.

**Proposition 5.3** Suppose that \( K = \mathbb{R}_{+}^n \). Let \( \mu \in E = \mathbb{R}^n \) satisfy conditions (A.0)-(A.1i) for all \( i = 1, \ldots, n \). Then \( \perp_{e_i} \cap (\mu - \text{Im}(A^*)) \neq \emptyset \), and thus \( \sigma_{D_{\mu}}(a_i) = \mu_i \) for all \( i = 1, \ldots, n \) where \( a_i \) is the \( i \)-th column of the matrix \( A \). Moreover, \( \inf_{b \in B} \sigma_{D_{\mu}}(b) = \vartheta(\mu) \).

**Proof.** Suppose that the statement is not true. Then, there exist \( i \) such that \( \perp_{e_i} \cap (\mu - \text{Im}(A^*)) = \emptyset \). Note that \( \perp_{e_i} = \{ \gamma \in \mathbb{R}^n_{+} : \gamma_i = 0 \} = \text{cone}\{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n \} \). Therefore, we arrive at the following system of linear inequalities in \( \gamma, \lambda \) being infeasible:

\[
\gamma + A^* \lambda = \mu, \\
\gamma_j \geq 0 \quad \forall j \neq i, \\
\gamma_i = 0.
\]

Using Farkas’ Lemma, we conclude that \( \exists u, v \) such that \( u + v = 0, v_j \geq 0 \) for all \( j \neq i, Au = 0 \) and \( \langle u, \mu \rangle \geq 1 \). By eliminating \( u \), this implies that \( \exists v \) such that \( v_j \geq 0 \) for all \( j \neq i, Av = 0 \) and \( \langle v, \mu \rangle \leq -1 \). Hence, if \( v_i < -1 \), we can scale \( v \) so that \( v_i \geq -1 \), and arrive at the conclusion that there exists \( v \) such that \( v + e_i \in \mathbb{R}_{+}^n = K, Av = 0 \) and \( \langle v, \mu \rangle < 0 \), which is a contradiction to the condition (A.1i).

By noting that the conditions (A.0)-(A.1i) are necessary for the \( K \)-sublinearity (and also \( K \)-minimality) of \( (\mu; \eta_0) \), we conclude that for all \( i \), we have \( \perp_{e_i} \cap (\mu - \text{Im}(A^*)) \neq \emptyset \), and \( \sigma_{D_{\mu}}(a_i) = \mu_i \) for all \( i = 1, \ldots, n \) follows from Proposition 5.2.

Finally, note that

\[
\inf_{b \in B} \sigma_{D_{\mu}}(b) = \inf_{b \in B} \sup_{\lambda \in \mathbb{R}^n} \{ b^T \lambda : A^* \lambda \leq \mu \} \\
= \inf_{b \in B} \inf_{x \in \mathbb{R}_{+}^n} \{ \mu^T x : Ax = b \} = \vartheta(\mu),
\]

where the second equation follows from the fact that conditions (A.0) implies \( \mu \in \mathbb{R}_{+}^n + \text{Im}(A^*) \), and thus the inner linear optimization problem is feasible, and so strong linear programming duality holds. Thus, we have equality relations throughout implying \( \inf_{b \in B} \sigma_{D_{\mu}}(b) = \vartheta(\mu) \). 

□
Remark 5.1 In fact, for $K = \mathbb{R}^n_+$, the result of Proposition 5.3, together with the basic facts on support functions, relates back nicely to the view based on cut generating functions and lattice-free sets. Let $(\mu; \eta_0)$ be a $K$-sublinear inequality, then based on Proposition 5.3 the value of the support function $\sigma_{D_\mu}(\cdot)$ evaluated at the vector $a_i$, $i^{th}$ column of $A$, the data corresponding to variable $x_i$, precisely matches with the corresponding cut coefficients of $x_i$, i.e., $\mu_i = \sigma_{D_\mu}(a_i)$, and the tightest possible right hand side value is $\vartheta(\mu) = \inf_{b \in B} \sigma_{D_\mu}(b) \geq \eta_0$. Another way to state this is that every $K$-sublinear inequality (its coefficient vector, and the tightest possible right hand side value) is generated by the support function $\sigma_{D_\mu}(\cdot)$. Furthermore, this sublinear function has a very specific form, i.e., it is the support function of the sets of form $\{r \in \mathbb{R}^m : \lambda^T r \leq 1 \forall \lambda \in D_\mu\}$.

Thus, the support functions of these sets are automatically sublinear (subadditive and positively homogeneous), and in fact piecewise linear and convex.

Under Assumption 1, this observation together with the fact that $K$-sublinear inequalities along with $x \in K$ constraint are sufficient to describe $\mathbf{conv}(S(A,K,B))$ provides a simple and intuitive explanation of the well-known strong functional dual for MILPs, e.g., all cutting planes for MILPs are generated by nondecreasing subadditive convex functions (cf., [60]). When $K = \mathbb{R}^n_+$, and $B$ is a finite set, this connection was already established previously in [49]. In fact, for this specific case the work of Johnson in [49] goes further by showing that in order to verify $K$-sublinearity of an inequality, only finitely many of the requirements $(A.0)$, $(A.1)$, and $(A.2)$ (those satisfying a minimal linear dependence condition) are needed to be verified.

From an alternative viewpoint, the seminal work of Balas [8] initiated the study of functions that generate cuts, in particular, the use of gauge functions of lattice-free sets. This view continues to attract a lot of attention in the MILP literature. Several papers in this literature have studied for example the relation between $\mathbb{R}^n_+$-minimal cuts (or minimal cut generating functions) and maximal lattice-free (or $B$-free sets in our context). In many cases, e.g., when the sufficiency of nonnegative functions is known, for every minimal cut generating function $\psi(\cdot)$, the corresponding set $\{r \in \mathbb{R}^m : \psi(r) \leq 1\}$ is a maximal lattice-free set, and vice-versa. Our study provides an alternative view on the same topic based on support functions as follows:

First, we note that the sets underlying gauge functions and support functions are nicely related via polarity. To observe this, let us consider the polar set of $D_\mu$ given by

$$D_\mu^o := \{r \in \mathbb{R}^m : \lambda^T r \leq 1 \forall \lambda \in D_\mu\}.$$ $D_\mu^o$ is a closed convex set containing the origin, and the (Minkowski) gauge function of $D_\mu^o$, $\gamma_{D_\mu^o}(\cdot)$, is defined as

$$\gamma_{D_\mu^o}(r) = \inf_{t > 0} \{t : r \in t D_\mu^o\}.$$ Note that $\gamma_{D_\mu^o}(\cdot)$ is a nonnegative, closed, and sublinear function, and when $0 \notin \text{int}(D_\mu^o)$, $\gamma_{D_\mu^o}(\cdot)$ can take the value of $+\infty$. Moreover, by Theorem C.1.2.5 in [47] we have $D_\mu^o = \{r \in \mathbb{R}^m : \gamma_{D_\mu^o}(r) \leq 1\}$, i.e., the gauge function $\gamma_{D_\mu^o}(\cdot)$ represents the set $D_\mu^o$. For a given sublinear function there is a unique set associated with it in this manner. However, there can be other sublinear functions $\psi(\cdot)$ representing the same set $D_\mu^o$, i.e., $D_\mu^o = \{r \in \mathbb{R}^m : \psi(r) \leq 1\}$. Using the positive homogeneity of sublinear functions, for any sublinear function $\psi(\cdot)$ such that $D_\mu^o = \{r \in \mathbb{R}^m : \psi(r) \leq 1\}$, we have $\gamma_{D_\mu^o}(r) = \psi(r)$ for every $r$ satisfying $\psi(r) > 0$. And, in order to obtain strong valid inequalities, one is interested in the smallest possible such sublinear function $\psi(\cdot)$ representing $D_\mu^o$. It is also well-known that whenever $Q$ is a closed convex set containing the origin, the support function of $Q$ is precisely the gauge function $\gamma_Q$ (see Corollary C.3.2.5 in [47]). For any
\( \mu \in \Pi(A, K, B) \), the set \( D_\mu \) is always closed and convex, yet, we are not always guaranteed to have \( 0 \in D_\mu \). However, whenever \( 0 \in D_\mu \), we conclude \( D_\mu^o = \{ r \in \mathbb{R}^m : \gamma_{D_\mu}(r) \leq 1 \} = \{ r \in \mathbb{R}^m : \sigma_{D_\mu}(r) \leq 1 \} \).

In addition to this, based on the given \( K \)-sublinear inequality \((\mu; \eta_0)\), let us also define the set

\[ V_\mu := \{ r \in \mathbb{R}^m : \sigma_{D_\mu}(r) \leq \eta_0 \}. \]

Note that \( V_\mu \) is a closed convex set since \( \sigma_{D_\mu}(\cdot) \) is a sublinear function. Because \( \vartheta(\mu) = \inf_{b \in B} \sigma_{D_\mu}(b) \geq \eta_0 \) when \( K = \mathbb{R}^n_+ \), we immediately observe that \( B \cap \text{int}(V_\mu) = \emptyset \) (in fact, we have something slightly stronger, i.e., the relative interior of \( V_\mu \) does not contain any points from \( B \)). Moreover, whenever \( \eta_0 > 0 \), the inequality \((\mu; \eta_0)\) separates origin from \( \text{conv}(S(A, K, B)) \), and \( 0 \in V_\mu \). Let us for a moment focus on the case of \( \eta_0 > 0 \), and without loss of generality assume that \( \eta_0 = 1 \). Then, under the assumption that \( 0 \in D_\mu \), we immediately observe that \( V_\mu = D_\mu^o \) and conclude that the polar of the set \( D_\mu \) is a \( B \)-free (lattice-free) set.

This case of \( \eta_0 > 0 \), for example, corresponds to the setup in [28], where the authors also consider \( S(A, K, B) \) with \( K = \mathbb{R}^n_+ \), without making any assumptions on the linear map \( A \), but under the additional assumption that \( B \) is a non-empty, closed set such that \( 0 \notin B \), and they are only interested in cuts \( \mu^T x \geq \eta_0 \) that separate the origin from \( \text{conv}(S(A, K, B)) \). In fact, the focus in [28] is on identifying the properties of finite-valued sublinear functions that generate the cut coefficients of the corresponding cuts, and establishing the relations of these functions with \( B \)-free sets. Under this assumption on \( B \), it is easily seen that \( 0 \notin S(A, K, B) \) (see Lemma 2.1 in [28]) and thus, without loss of generality we can assume that the cuts that separate \( 0 \) from \( \text{conv}(S(A, K, B)) \) have the form \((\mu; 1)\), i.e., their right hand side value, \( \eta_0 \), is 1. Clearly, this setup is a special case of our set \( S(A, K, B) \) and the corresponding cuts separating the origin are a subset of inequalities with vectors from \( C(A, K, B) \). Furthermore, under Assumption 1, whenever these cuts are necessary, they will be \( K \)-minimal and thus \( K \)-sublinear as well, and hence the support functions of the corresponding sets \( D_\mu \) do have a direct relation with the corresponding cut-generating functions of interest from [28].

We note that our support functions \( \sigma_{D_\mu}(\cdot) \) are not always guaranteed to be finite-valued, and thus our results do not immediately lead to the same conclusions from [28]. We believe that it is not necessary to require a function to be finite-valued everywhere in order to use it to generate cuts for a given problem instance. In particular, the functions that are not finite-valued everywhere, such as the support functions we are considering here, can still be meaningful and interesting in terms of generating valid inequalities or understanding their minimality, sublinearity properties. This being said, given a problem instance \( A, B \) and \( K = \mathbb{R}^n_+ \), in certain cases, we can appropriately bound the set \( D_\mu \) to obtain a set \( D \) ensuring \( D \notin \emptyset \), \( \inf_{b \in B} \sigma_{D}(b) = \eta_0 \) and \( \sigma_{D}(a_i) = \mu_i \) for all \( i = 1, \ldots, n \), i.e., the support function of \( D \) is finite-valued everywhere and generates the same inequality \((\mu; \eta_0)\) for the given instance defining \( S(A, K, B) \). Let us for example consider Example 6.1 of [28] where \( A \) is the \( 2 \times 2 \) identity matrix, \( B = (0; 1) \cup \{ Z; -1 \} \) and \( K = \mathbb{R}^2_+ \), which leads to \( S(A, K, B) = \text{conv}(S(A, K, B)) = \{(0; 1)\} \). We note that this particular \( S(A, K, B) \) violates our Assumption 1, and therefore there are no \( \mathbb{R}^2_+ \)-minimal inequalities. On other hand, existence of \( \mathbb{R}^2_+ \)-sublinear inequalities is not based on Assumption 1, and indeed we show that the particular inequality \((\mu; \eta_0) = (-1; 1; 1)\) considered in [28] is \( \mathbb{R}^2_+ \)-sublinear. It is easy to see that the sufficiency conditions for \( K \)-sublinearity established in Proposition 5.5 are satisfied for this inequality. Indeed, the corresponding \( D_\mu = \{ (\lambda_1; \lambda_2) \in \mathbb{R}^2 : \lambda_1 \leq -1, \lambda_2 \leq 1 \} \), and \( \sigma_{D_\mu}(A e_1) = \sigma_{D_\mu}(\mathbb{1}) = -1 = \mu_1 = \mu^T e_1 \) and \( \sigma_{D_\mu}(A e_2) = \sigma_{D_\mu}(\mathbb{1}) = 1 = \mu_2 = \mu^T e_2 \) and clearly \( e_1 + e_2 \in \text{int}(\mathbb{R}^2_+) \). Furthermore, \( \inf_{b \in B} \sigma_{D_\mu}(b) = 1 = \eta_0 \), proving that \((\mu; \eta_0) = (-1; 1; 1)\) is a tight \( \mathbb{R}^2_+ \)-sublinear inequality for this particular \( \text{conv}(S(A, K, B)) \). On the other hand, the support function corresponding to this inequality is not finite valued everywhere. Indeed, if we try to bound \( D_\mu \) to obtain \( D \subseteq D_\mu \), we will easily notice that
while we can ensure $\sigma_{\hat{D}}(Ae_i) = \mu^T e_i$ for $i = 1, 2$, we can not ensure that $\vartheta(\mu) = \inf_{b \in B} \sigma_{\tilde{D}}(b) = 1$. It was conjectured in [28] and later on proved in [33], that under the following “containment” assumption, the cone generated by the columns of $A$ contains the set $B$, one can ensure the existence of finite-valued cut generating functions corresponding to every cut separating the origin from $S(A, K, B)$. The reader familiar to these results would immediately notice the duality relation between the support functions we study here and the functions used in the sufficiency proof of cut generating functions in [33]. We finish our discussion of this example, by examining a slight variant of it obtained from setting $\tilde{B} = (0; 1) \cup \{(Z^-; -1)\}$. Note that we still have $S(A, K, B) = S(A, \mathbb{R}^2_+, \tilde{B})$. Moreover, $S(A, \mathbb{R}^2_+, \tilde{B})$ still violates the containment assumption used in the sufficiency proof of [33]. But, we can point out a proper cut generating function for the inequality given by $(\mu; \eta_0) = (-1; 1; 1)$. Indeed, one can easily check that the support function of the set $\tilde{D} := \{(\lambda_1; \lambda_2) \in \mathbb{R}^2 : \lambda_1 = -1, -1 \leq \lambda_2 \leq 1\}$ obtained from bounding $D_\mu$ will do the job.

Finally, we note that $0 \in D_\mu$ is not always guaranteed. But, by taking the polar of $D_\mu^o$, we obtain $D_\mu^{oo} := (D_\mu^o)^o$, a closed convex set containing the origin. In addition to this, we always have $D_\mu \subseteq D_\mu^{oo}$ and so $\sigma_{D_\mu}(r) \leq \sigma_{D_\mu^{oo}}(r) = \gamma_{D_\mu}(r)$, where the last equation follows from Proposition C.3.2.4 in [47]. In general $\sigma_{D_\mu}(r)$ and $\gamma_{D_\mu}(r)$ may differ quite significantly, i.e., a support function can take negative values while a gauge function cannot. To address this issue of generating negative coefficients in cuts, in [15] the following subset of the relative boundary of $D_\mu^{oo}$ was considered:

$$\tilde{D}_\mu^{oo} := \{\lambda \in D_\mu^{oo} : \exists r \in D_\mu^o \text{ s.t. } \lambda^T r = 1\}.$$  

Under the assumption $0 \in \text{int}(D_\mu)$ (which does not necessarily hold in our setup), it was shown in [15] that among the sublinear functions $\psi(\cdot)$ satisfying $D_\mu^o = \{r \in \mathbb{R}^m : \psi(r) \leq 1\}$, we have the following relation $\sigma_{D_\mu^{oo}}(r) \leq \psi(r) \leq \gamma_{D_\mu}(r)$. Clearly $\sigma_{D_\mu}(r) \leq \sigma_{D_\mu^{oo}}(r)$ holds for all $r$, and studying the cases when we have $\sigma_{D_\mu}(r) = \sigma_{D_\mu^{oo}}(r)$ and $\sigma_{D_\mu}(r) = \gamma_{D_\mu}(r)$ with or without assumption $0 \in \text{int}(D_\mu)$ is of independent interest for understanding the minimality of these support functions $\sigma_{D_\mu}(\cdot)$.

Motivated by the positive result of Proposition 5.3 in the case of $K = \mathbb{R}^n$, one is inclined to think that a similar result will hold for general proper cones $K$, which unfortunately is not true. On the other hand, in the next proposition, we state that when $(\mu; \eta_0) \in C_\alpha(A, K, B)$, even if $\mu \notin \text{Im}(A^*)$, there exists $z \in \text{Ext}(K)$ such that $\sigma_{D_\mu}(A z) = (\mu, z)$. Before we prove Proposition 5.4, we need Lemma 5.1, and the following simple observation, which is needed in the proof of Lemma 5.1.

**Remark 5.2** For any two sets $U$ and $V$ that are independent of each other, we have

$$\inf_{u \in U} \inf_{v \in V} \langle u, v \rangle = \inf_{v \in V} \inf_{u \in U} \langle u, v \rangle.$$  

**Proof.** Let us consider a given $\bar{u} \in U$. Then, for any $v \in V$, we have $\inf_{u \in U} \langle u, v \rangle \leq \langle \bar{u}, v \rangle$, and by taking the infimum of both sides of this inequality over $v \in V$, we obtain $\inf_{v \in V} \inf_{u \in U} \langle u, v \rangle \leq \inf_{v \in V} \langle \bar{u}, v \rangle$ holds for any $\bar{u} \in U$. Now, by taking the infimum of this inequality over $\bar{u} \in U$, and noting that the left hand side is simply a constant, we arrive at $\inf_{v \in V} \inf_{u \in U} \langle u, v \rangle \leq \inf_{u \in U} \inf_{v \in V} \langle u, \bar{v} \rangle = \inf_{u \in U} \inf_{v \in V} \langle u, v \rangle$. To see that the reverse inequality also holds, we can start by considering a given $\bar{v} \in V$, and repeat the same reasoning by interchanging roles of $u$ and $v$.  

**Lemma 5.1** Suppose that $\mu \in E$ satisfies condition (A.0), and $\perp_z \cap (\mu - \text{Im}(A^*)) = \emptyset$ holds for all $z \in \text{Ext}(K)$ where $\perp_z$ is as defined by (5). Then, there exists $\tilde{\gamma} \in \text{int}(K^*) \cap (\mu - \text{Im}(A^*))$. Moreover, $\inf_{b \in B} \sigma_{D_\mu}(b) = \vartheta(\mu)$ where $D_\mu$ is defined by (4) and $\vartheta(\mu)$ is defined by (3).
Proof. First note that since \( \mu \) satisfies condition (A.0), by Theorem 5.1, the set \( D_\mu \) defined in (4) is nonempty, which implies that \( \{ \gamma : \exists \lambda s.t. \gamma + A^* \lambda = \mu, \gamma \in K^* \} \neq \emptyset \).

In addition to this, \( 0 \in \bigcap \xi \in \text{Ext}(K) \perp _z \) and therefore together with the assumption of the lemma that \( \perp _z \cap (\mu - \text{Im}(A^*)) = \emptyset \), we conclude that \( 0 \notin \mu - \text{Im}(A^*) \). Moreover, by rephrasing the statement of lemma and definition of \( \perp _z \), we get

\[
0 < \inf _{z \in \text{Ext}(K)} \inf _{\gamma \in E, \lambda \in \mathbb{R} ^m} \{ \langle \gamma, z \rangle : \gamma + A^* \lambda = \mu, \gamma \in K^* \} = \inf _{\gamma \in E, \lambda \in \mathbb{R} ^m} \left\{ \inf _{z \in \text{Ext}(K)} \{ \langle \gamma, z \rangle : z \in \text{Ext}(K) \} : \gamma + A^* \lambda = \mu, \gamma \in K^* \right\},
\]

where the last equation follows from Remark 5.2 where we take \( U = \text{Ext}(K) \times 0 \subseteq E \times \mathbb{R} ^m \) and \( V = \{(\gamma, \lambda) : \gamma + A^* \lambda = \mu, \gamma \in K^* \} \subseteq \partial K^* \). This together with the above inequality implies that there exists \( \bar{\gamma} \in \partial K^* \) such that \( \langle \bar{\gamma}, z \rangle > 0 \) for all \( z \in \text{Ext}(K) \), which implies that \( \langle \bar{\gamma}, z \rangle > 0 \) for all \( z \in K \setminus \{0\} \). Since \( K^* \) is a closed convex cone, \( \langle \bar{\gamma}, z \rangle > 0 \) for all \( z \in K \setminus \{0\} \) implies that \( \bar{\gamma} \in \text{int}(K^*) \), which is a contradiction. Thus, we conclude that there exists \( \bar{\gamma} \neq 0 \) such that \( \bar{\gamma} \in \text{int}(K^*) \cap (\mu - \text{Im}(A^*)) \).

To finish the proof note that

\[
\vartheta(\mu) := \inf _x \{ \langle \mu, x \rangle : x \in S(A, K, B) \} = \inf _{b \in B} \inf _x \{ \langle \mu, x \rangle : Ax = b, x \in K \} = \inf _{b \in B} \sup _{\lambda \in \mathbb{R} ^m, \gamma \in E} \{ b^T \gamma : A^* \lambda + \gamma = \mu, \gamma \in K^* \} = \inf _{b \in B} \sigma _{D_\mu}(b),
\]

where the third equality follows from strong conic duality, which holds due to the existence of a strictly feasible solution \( \bar{\gamma} \in \text{int}(K^*) \). Therefore, we have \( \vartheta(\mu) = \inf _{b \in B} \sigma _{D_\mu}(b) \).

\( \square \)

Proposition 5.4 Suppose that \( \mu \in \Pi(A, K, B), \) and \( \perp _z \cap (\mu - \text{Im}(A^*)) = \emptyset \) holds for all \( z \in \text{Ext}(K) \) where \( \perp _z \) is as defined by (5). Then, there exists at least one \( z \in \text{Ext}(K) \) such that \( \sigma _{D_\mu}(Az) = \langle \mu, z \rangle \) where \( D_\mu \) is defined by (4).

Proof. Assume for contradiction that \( \sigma _{D_\mu}(Az) < \langle \mu, z \rangle \) for all \( z \in \text{Ext}(K) \). Then, by Lemma 5.1, we conclude that there exists \( \bar{\gamma} \in \text{int}(K^*) \cap (\mu - \text{Im}(A^*)) \) and \( \inf _{b \in B} \sigma _{D_\mu}(b) = \vartheta(\mu) \). Note that due to weak conic duality and the existence of such \( \bar{\gamma} \), we have for all \( b \)

\[
\inf _x \{ \langle \mu, x \rangle : Ax = b, x \in K \} \geq \sigma _{D_\mu}(b) = \sup _{\lambda \in \mathbb{R} ^m, \gamma \in E} \{ b^T \gamma : A^* \lambda + \gamma = \mu, \gamma \in K^* \} > -\infty.
\]

For any \( b \in B \), define \( S_b := \{ x \in K : Ax = b \} \), and let \( \widehat{B} := \{ b \in B : S_b \neq \emptyset \} \). Since \( S(A, K, B) \neq \emptyset \), \( \widehat{B} \neq \emptyset \). Then, for any \( b \in \widehat{B} \), \( x_b \in S_b \) leads to an upper bound on \( \sigma _{D_\mu}(b) \), i.e., \( \sigma _{D_\mu}(b) \leq \langle \mu, x_b \rangle \). Therefore, for any \( b \in \widehat{B} \), the conic optimization problem defining \( \sigma _{D_\mu}(b) \) is bounded above and is strictly feasible, and so we have strong conic duality and the dual problem given by the \( \inf _x \) above is solvable. Consider any \( b \in \widehat{B} \), let \( \bar{x}_b \) be the corresponding optimal dual solution, i.e., \( \bar{x}_b \in S_b \) and \( \langle \mu, \bar{x}_b \rangle = \sigma _{D_\mu}(b) \). Because \( \bar{x}_b \in \text{Ext}(K) \), there exists \( z^1, \ldots, z^\ell \in \text{Ext}(K) \) with \( \ell \leq n \) such that \( \bar{x}_b = \sum _{i=1}^\ell z^i \), which leads to

\[
\langle \mu, \bar{x}_b \rangle = \sigma _{D_\mu}(b) = \sigma _{D_\mu}(Az_b) \leq \sum _{i=1}^\ell \sigma _{D_\mu}(Az^i) \leq \sum _{i=1}^\ell \langle \mu, z^i \rangle = \langle \mu, \bar{x}_b \rangle,
\]

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where the inequality (*) follows from the fact that $\sigma_{D_\mu}(\cdot)$ is a support function, and thus is subadditive, and (***) follows from the assumption that $\sigma_{D_\mu}(A) < \langle \mu, z \rangle$ for all $z \in \text{Ext}(K)$. But this is a contradiction, hence we conclude that there exists $z \in \text{Ext}(K)$ such that $\sigma_{D_\mu}(Az) = \langle \mu, z \rangle$. \hfill \Box

To summarize whenever $\mu \in \Pi(A, K, B)$, Propositions 5.2 and 5.4 together cover all possible cases and indicate that for a $K$-sublinear inequality, there exists at least one $z \in \text{Ext}(K)$ such that $\sigma_{D_\mu}(Az) = \langle \mu, z \rangle$. While one is inclined to think that the above property of support functions of $D_\mu$ can only be valid for a $K$-minimal inequality, the following simple example shows that this property also holds for valid inequalities, which are $K$-sublinear but not $K$-minimal.

**Example 5.2** Consider set $S(A, K, B)$ with $K = \mathcal{L}^3 = \{ x \in \mathbb{R}^3 : x_3 \geq \sqrt{x_1^2 + x_2^2} \}$. $A = [1, 0, 0]$ and $B = \{-1, 1\}$. In this case, $\text{conv}(S(A, K, B)) = \{ x \in \mathbb{R}^3 : x_3 \geq \sqrt{1 + x_2^2}, -1 \leq x_1 \leq 1 \}$ (see Figure 3).

![Figure 3: Convex hull of $S(A, K, B)$ corresponding to Example 5.2](image)

Note that the above description of the convex hull of $S(A, K, B)$ involves the following inequalities:

(a) $\mu^{(+)i} = (1; 0; 0)$ with $\eta_0^{(+)i} = -1$ and $\mu^{(-)i} = (-1; 0; 0)$ with $\eta_0^{(-)i} = -1$;

(b) $\mu^{(t)0} = (0; t; \sqrt{t^2 + 1})$ with $\eta_0^{(t)0} = 1$ for all $t \in \mathbb{R}$.

Here, we show that these inequalities satisfy the necessary conditions for $K$-sublinear inequalities; later on we will in fact show that all of these inequalities are $K$-minimal.

In the first case (a), it is easily seen that the corresponding sets associated with these inequalities $\mu^{(+)i}, \mu^{(-)i}$ are given by

\[ D_{\mu^{(+)i}} = \{ \lambda : \exists \gamma \in K^* \text{ s.t. } \lambda + \gamma_1 = 1; \gamma_2 = 0; \gamma_3 = 0 \} = \{ \lambda : \lambda = 1 \}, \]

\[ D_{\mu^{(-)i}} = \{ \lambda : \lambda = -1 \}. \]

Also, both $\mu^{(+)i}, \mu^{(-)i} \in \text{Im}(A^*)$, and thus Corollary 5.2 implies, $\sigma_{D_{\mu^{(+)i}}}(Az) = \sigma_{D_{\mu^{(-)i}}}(z_1) = \langle \mu^{(t)0}, z \rangle$ for all $z \in K$ for $i \in \{+, -\}$. In addition to this, $\inf_{b \in B} \sigma_{D_{\mu^{(t)0}}}(b) = -1 = \eta_0^{(t)0} i \in \{+, -\}$.

In the second case (b), for any given $t$, we have the associated sets $D_{\mu^{(t)0}}$ given by

\[ D_{\mu^{(t)0}} = \{ \lambda : \exists \gamma \in K^* \text{ s.t. } \lambda + \gamma_1 = 0; \gamma_2 = t; \gamma_3 = \sqrt{t^2 + 1} \} = \{ \lambda : -1 \leq \lambda \leq 1 \}. \]
Moreover, for all $t$, by considering $z^t \in \{(1; -t; \sqrt{t^2 + 1}), (-1; -t; \sqrt{t^2 + 1}) \} \subset \text{Ext}(K)$, we have $\langle \mu^{(t)}, z^t \rangle = 1$ and $\sigma_{D_{\mu^{(t)}}}(Az^t) = \sigma_{D_{\mu^{(t)}}}(z^t_1) = \sigma_{D_{\mu^{(t)}}}(1) = 1$, proving $\langle \mu^{(t)}, z^t \rangle = \sigma_{D_{\mu^{(t)}}}(Az^t)$. Additionally, $\sigma_{D_{\mu^{(t)}}}(1) = 1 = \sigma_{D_{\mu^{(t)}}}(-1)$ implying $\inf_{b \in B} \sigma_{D_{\mu^{(t)}}}(b) = 1 = \eta_0^{(t)}$ for all $t$.

Let us also consider another valid inequality $(\nu; \nu_0)$ given by $\nu = (0; 1; 2)$ and $\nu_0 = 1$. Note that the associated $D_\nu$ set is given by

$$D_\nu = \left\{ \lambda : -\sqrt{3} \leq \lambda \leq \sqrt{3} \right\},$$

and is nonempty. Furthermore for any $z_\nu \in \left\{ \left( \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}; \frac{2}{3} \right), \left( \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}; \frac{2}{3} \right) \right\} \subset \text{Ext}(K)$ we have $\sigma_{D_\nu}(Az_\nu) = \sigma_{D_\nu}(\pm \frac{1}{\sqrt{3}}) = 1 = \langle \nu, z_\nu \rangle$. Also $\inf_{b \in B} \sigma_{D_\nu}(b) = \sqrt{3} > 1 = \nu_0$. Therefore, in terms of the necessary conditions established so far for $K$-sublinearity, there seems to be no difference between $(\nu; \nu_0)$ and the previous inequalities from above. When we revisit this example, we will show that while $(\nu; \nu_0) \in C_\alpha(A, K, B)$, $(\nu; \nu_0)$ is not $K$-minimal. In fact, we can easily show that $(\nu; \nu_0)$ is dominated by $\mu^{(1)} = (0; 1; \sqrt{2})$ and $\eta^{(1)} = 1$. And since $\delta = \nu - \mu^{(1)} = (0; 0; 2 - \sqrt{2})$ is in $K^* \setminus \{0\}$, we conclude that $(\nu; \nu_0) \notin C_m(A, K, B)$.

\[\square\]

### 5.2 Sufficient Conditions for $K$-Sublinearity and $K$-Minimality

Given any $(\mu; \eta_0)$ satisfying condition (A.0), we can easily test it for $K$-sublinearity with the help of the following proposition.

**Proposition 5.5** Let $(\mu; \eta_0)$ be such that $\mu$ satisfies condition (A.0) and $\eta_0 \leq \inf_{b \in B} \sigma_{D_{\eta_0}}(b)$ (or it is known that $(\mu; \eta_0) \in C(A, K, B)$). In addition to this, suppose that there exists $x^i \in \text{Ext}(K)$ such that $\sigma_{D_{\nu}}(Ax^i) = \langle \mu, x^i \rangle$ for all $i \in I$ and $\sum_{i \in I} x^i \in \text{int}(K)$, then $(\mu; \eta_0) \in C_\alpha(A, K, B)$.

**Proof.** If $\eta_0 \leq \inf_{b \in B} \sigma_{D_{\nu}}(b)$, then by Proposition 5.1, we have $(\mu; \eta_0) \in C(A, K, B)$, which automatically implies that condition (A.2) is satisfied. To verify condition (A.1), consider any $\alpha \in \text{Ext}(K^*)$ and $u$ such that $Au = 0$ and $\langle \alpha, v \rangle u + v \in K \forall v \in \text{Ext}(K)$. Let $V_\alpha = \{ v \in \text{Ext}(K) : \langle \alpha, v \rangle = 1 \}$, it is clear that $\langle u + v, \gamma \rangle \geq 0$ holds for all $v \in V_\alpha$ and $\gamma \in K^*$. Also, since $\mu \in K^* + \text{Im}(A^*), \text{there exists } \lambda \in K^* \text{ satisfying } A^*\lambda + \gamma = \mu$. In fact, for any such $\lambda, \gamma$, we have

$$\langle \mu, u \rangle = \langle A^*\lambda + \gamma, u \rangle = \langle \lambda, Au \rangle + \langle \gamma, u \rangle \geq \langle \gamma, -v \rangle \quad \text{for all } v \in V_\alpha.
$$

Note that $\langle \gamma, -v \rangle \leq 0$ for all $\gamma \in K^*$ and $v \in V_\alpha \subset K$. In order to finish the proof, all we need to show is that there exists $\tilde{v} \in V_\alpha$ such that $\langle \tilde{\gamma}, \tilde{v} \rangle = 0$. Clearly when $\mu \in \text{Im}(A^*)$, we can take $\tilde{\gamma} = 0$, and hence conclude that $\langle \mu, u \rangle \geq -\langle \tilde{\gamma}, \tilde{v} \rangle = 0$ holds for all such $u$. In the more general case, we have

$$\inf_{\gamma, \lambda} \left\{ \inf_{v} \langle (\gamma, v) : v \in V_\alpha \rangle : A^*\lambda + \gamma = \mu, \gamma \in K^* \right\}$$

$$= \inf_{v} \left\{ \inf_{\gamma, \lambda} \left\{ (\mu - A^*\lambda, v) : A^*\lambda + \gamma = \mu, \gamma \in K^* \right\} : v \in V_\alpha \right\}$$

$$= \inf_{v} \left\{ \langle \mu, v \rangle - \sup_{\gamma, \lambda} \left\{ \lambda^T(Av) : A^*\lambda + \gamma = \mu, \gamma \in K^* \right\} : v \in V_\alpha \right\}$$

$$\subseteq \sigma_{D_{\mu}}(Av)$$

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Since there exists \( x_i \in \text{Ext}(K) \) such that \( \sigma_{D_\mu}(Ax_i) = \langle \mu, x_i \rangle \) for all \( i \in I \) and \( \sum_{i \in I} x_i \in \text{int}(K) \), for any \( \alpha \in \text{Ext}(K^*) \), at least one of these \( x_i \)'s will be in \( V_\alpha \). Otherwise, we have \( \langle \alpha, x_i \rangle = 0 \) for all \( i \in I \), and thus \( \langle \alpha, \sum_{i \in I} x_i \rangle = 0 \), which is not possible since \( \sum_{i \in I} x_i \in \text{int}(K) \) and \( \alpha \in \text{Ext}(K^*) \). Thus, we conclude that the above infimum is zero. This gives us the desired conclusion that \( \langle \mu, u \rangle \geq 0 \), which proves that the condition (A.1) is also satisfied.

When \( K = \mathbb{R}^*_+ \), Proposition 5.3 together with Theorem 5.1 implies that the conditions stated in Proposition 5.5 are necessary and sufficient for \( K \)-sublinearity. For general regular cones \( K \), considering the results obtained from Theorem 5.1, and Propositions 5.2 and 5.4, we conclude that the conditions stated in Proposition 5.5 are almost necessary. This is up to the fact that we can prove the existence of at least one \( x \in \text{Ext}(K) \) satisfying \( \sigma_{D_\mu}(Ax) = \langle \mu, x \rangle \) when \( (\mu; \eta_0) \in C_\alpha(A, K, B) \), yet the sufficient condition in Proposition 5.5 requires a number of such extreme rays summing up to an interior point of \( K \). We next provide an example highlighting that we cannot close this gap for general regular cones \( K \) other than the nonnegative orthant, i.e., there exists \( K \)-sublinear inequalities that satisfy only the necessary conditions from Propositions 5.2 and 5.4 but not the sufficient condition of Proposition 5.5.

**Example 5.3** Consider set \( S(A, K, B) \) with \( K = L^3 = \{ x \in \mathbb{R}^3 : x_3 \geq \sqrt{x_1^2 + x_2^2} \} \), \( A = [0, 1, 1] \) and \( B = \{-1, 1\} \). In this case, \( \text{conv}(S(A, K, B)) = \{ x \in L^3 : x_2 + x_3 = 1 \} \). Let us consider the following valid inequality \( \langle \mu; \eta_0 \rangle \) given by \( \mu = (0; 0; 1) \) and \( \eta_0 = \theta(\mu) + \frac{1}{2} \). Here, we first show that there is precisely a single ray \( x \in \text{Ext}(K) \) such that \( \sigma_{D_\mu}(Ax) = \langle \mu, x \rangle \), yet the inequality \( \langle \mu; \eta_0 \rangle \) is a \( K \)-sublinear inequality.

The cut generating set associated with \( \mu \) is \( D_\mu = \{ \lambda \in \mathbb{R} : |\lambda| + \lambda \leq 1 \} \). Consider any \( z \in \text{Ext}(K) = \text{Ext}(L^3) \), without loss of generality let us assume that \( z \) is normalized to have \( z_3 = 1 \). Then, we have

\[
\langle \mu, z \rangle = \sigma_{D_\mu}(Az) \iff z_3 = \sup_{\lambda \in \mathbb{R}} \{ \langle z_2 + z_3 \rangle \lambda : |\lambda| + \lambda \leq 1 \} \iff z_3 = \frac{1}{2}(z_2 + z_3),
\]

and so \( z_2 = z_3 = 1 \), by noting \( z \in \text{Ext}(L^3) \), we get \( z_1 = 0 \). Thus, we conclude that there is a unique extreme ray of \( L^3 \), in particular \( z = (0; 1; 1) \) that satisfies \( \langle \mu, z \rangle = \sigma_{D_\mu}(Az) \).

Let us now prove that \( \langle \mu; \eta_0 \rangle \) is indeed \( K \)-sublinear. The conditions (A.0) and (A.2) are easily verified. In order to verify condition (A.1), we need to verify that for any \( \alpha \in \text{Ext}(K^*) \),

\[
0 \leq \langle \mu, u \rangle \text{ for all } u \in E \text{ such that } Au = 0 \text{ and } \langle \alpha, v \rangle u + v \in K \quad \forall v \in \text{Ext}(K),
\]

holds. Let \( \alpha \in \text{Ext}(K^*) \) be given. For any \( v \in \text{Ext}(K) \) if \( \langle \alpha, v \rangle = 0 \), then we automatically have \( \langle \alpha, v \rangle u + v \in K \). And if \( \langle \alpha, v \rangle \geq 0 \), then we can normalize \( v \) to assume that \( \langle \alpha, v \rangle = 1 \). So let us define \( V_\alpha := \{ v \in \text{Ext}(K) : \langle \alpha, v \rangle = 1 \} \). Thus, we can state above requirement as

\[
0 \leq \langle \mu, u \rangle \text{ for all } u \in E \text{ such that } Au = 0 \text{ and } u + v \in K \quad \forall v \in V_\alpha,
\]

which becomes in our particular case as

\[
0 \leq u_3 \text{ for all } u \in \mathbb{R}^3 \text{ such that } u_3 = -u_2 \text{ and } u + v \in L^3 \quad \forall v \in V_\alpha.
\]

Now notice that \( u_3 = -u_2, u + v \in L^3 \) and \( v \in \text{Ext}(L^3) \) implies \( u_3 + v_3 \geq 0 \) and \( u_3^2 + (v_1^2 + v_2^2) + 2u_3v_3 \geq u_3^2 + v_2^2 + 2u_3v_2 + u_1^2 + v_1^2 + 2u_1v_1 \), which is equivalent to \( 2u_3(v_2 + v_3) \geq u_1^2 + 2u_1v_1 \). Now suppose that \( \alpha_1 = 0 \), then \( \tilde{v} = (\frac{1}{\alpha_2}; 0; \frac{1}{\alpha_3}) \in V_\alpha \) and \( \tilde{v} = (\frac{1}{\alpha_2}; 0; \frac{1}{\alpha_3}) \in V_\alpha \). In this case, using these particular \( \tilde{v} \) and \( \tilde{v} \), we conclude \( u_3 \geq \max \left\{ \frac{u_1^2 + 2u_1v_1}{2(v_2 + v_3)}, \frac{u_2^2 + 2u_2v_2}{2(v_2 + v_3)} \right\} \geq \frac{u_2^2 + 2u_2v_2}{2v_3} \geq 0. \) Moreover, when \( \alpha_1 \neq 0, \) we
have \(\alpha_2 + \alpha_3 > 0\) (since \(\alpha \in \text{Ext}(L^3)\)) and considering \(\hat{\nu} = (0; \frac{1}{2(\alpha_2 + \alpha_3)}; \frac{1}{2(\alpha_2 + \alpha_3)}) \in \mathcal{V}_\alpha\), we once again conclude that \(u_3 \geq 0\). Note that this is precisely what we needed to show in order to prove that \((\mu; \eta_0)\) is \(K\)-sublinear.

In addition to Proposition 5.5, under Assumption 1, we can state a sufficient condition for \(K\)-minimality as follows:

**Proposition 5.6** Suppose that Assumption 1 holds and \((\mu; \eta_0) \in C_a(A, K, \mathcal{B})\) is such that \(-\infty < \eta_0 = \inf_{b \in B} \sigma_{D_\mu}(b)\). Let \(\hat{B} = \{b \in B : \sigma_{D_\mu}(b) \leq \eta_0\}\). Then, if there exists \(b^i \in \hat{B}\) and \(x^i \in K\) such that \(\sum_i x^i \in \text{int}(K)\), \(Ax^i = b^i\) and \(\langle \mu, x^i \rangle = \eta_0\), then \((\mu; \eta_0) \in C_m(A, K, \mathcal{B})\).

**Proof.** Suppose \((\mu; \eta_0) \in C_a(A, K, \mathcal{B})\) is such that \(\eta_0 = \inf_{b \in B} \sigma_{D_\mu}(b)\). Assume for contradiction that \((\mu; \eta_0) \notin C_m(A, K, \mathcal{B})\), i.e., \(\exists \delta \in K^* \setminus \{0\}\) such that \((\mu - \delta; \eta_0) \in C(A, K, \mathcal{B})\).

Suppose the premise of the proposition holds for some \(b^i \in \hat{B}\) and \(x^i \in K\) such that \(\sum_i x^i \in \text{int}(K)\), \(Ax^i = b^i\) and \(\langle \mu, x^i \rangle = \eta_0\). Note that for \(\beta_i > 0\) with \(\sum_i \beta_i = 1\), we have \(\bar{x} := \sum_i \beta_i x^i \in \text{int}(K)\) and moreover, by definition, \(\bar{x} \in \text{conv}(S(A, K, \mathcal{B}))\), and \(\langle \mu, \bar{x} \rangle = \eta_0\). Since any valid inequality for \(S(A, K, \mathcal{B})\), in particular \((\mu - \delta; \eta_0)\) has to be valid for \(\text{conv}(S(A, K, \mathcal{B}))\), we arrive at the contradiction

\[\eta_0 \leq \langle \mu - \delta, \bar{x} \rangle < \eta_0,\]

where the last inequality follows from \(\bar{x} \in \text{int}(K)\) and \(\delta \in K^* \setminus \{0\}\) implying \(\langle \delta, \bar{x} \rangle > 0\) together with \(\langle \mu, \bar{x} \rangle = \eta_0\). \(\square\)

In particular, Proposition 5.6 states that a \(K\)-sublinear inequality is also \(K\)-minimal whenever the inequality is tight at a point at the intersection of \(\text{int}(K)\) and \(\text{conv}(S(A, K, \mathcal{B}))\). In the MILP case, clearly, this resembles a sufficient condition for an inequality to be facet defining. Nonetheless, our minimality notion in general is much weaker. In the MILP case, all of the facets are necessary and sufficient; yet in general, one does not need all of the \(K\)-minimal inequalities for the description of \(\text{conv}(S(A, K, \mathcal{B}))\), only a generating set for \(C_m(A, K, \mathcal{B})\) together along with \(x \in K\) constraint is needed.

An immediate implication of Proposition 5.5 together with Corollary 5.2 is as follows:

**Corollary 5.3** For any \(\mu \in \text{Im}(A^*)\) and \(\eta_0 \leq \vartheta(\mu)\), we have \((\mu; \eta_0) \in C_a(A, K, \mathcal{B})\).

Note that we have already seen in Proposition 3.4 that when \(\text{Ker}(A) \cap \text{int}(K) \neq \emptyset\), then any \(\mu \in \text{Im}(A^*)\) and any \(-\infty < \eta_0 \leq \vartheta(\mu)\) leads to a \(K\)-minimal inequality \((\mu; \eta_0)\). Corollary 5.3 complements this result by showing that valid inequalities \((\mu; \eta_0)\) with \(\mu \in \text{Im}(A^*)\) are always \(K\)-sublinear regardless of the requirement \(\text{Ker}(A) \cap \text{int}(K) \neq \emptyset\). Indeed, when \(\text{Ker}(A) \cap \text{int}(K) \neq \emptyset\), it is easy to see that the additional \(K\)-minimality requirements of Proposition 5.6 are trivially satisfied by \((\mu; \vartheta(\mu))\).

**Proposition 5.7** Let \((\mu; \eta_0)\) be a \(K\)-minimal inequality such that \(\mu \in \text{int}(K^*)\). Then \(\eta_0 = \vartheta(\mu) = \inf_{b \in B} \sigma_{D_\mu}(b)\).

**Proof.** Since \((\mu; \eta_0) \in C_m(A, K, \mathcal{B})\) and \(\mu \in K^*\), by Proposition 3.3, we have

\[\eta_0 = \vartheta(\mu) = \inf_x \{\langle \mu, x \rangle : x \in S(A, K, \mathcal{B})\} .\]

Moreover, \((\mu; \eta_0) \in C_m(A, K, \mathcal{B})\), thus \((\mu; \eta_0)\) is a \(K\)-sublinear valid inequality and therefore \(D_\mu\) as defined in (4) is nonempty. Also, by Proposition 5.1, \(\mu \in K^*\) implies that \(\inf_{b \in B} \sigma_{D_\mu}(b) \leq \vartheta(\mu)\). Assume for
We have reached a contradiction, therefore we have
where the second equality follows from strong conic duality, which holds due to the fact that \( \mu \in \text{int}(K^*) \), and the last inequality follows from the definition of \( \vartheta(\mu) \) and the fact that infimum is over \( b \in B \). Hence, we have reached a contradiction, therefore we have \( \eta_0 = \vartheta(\mu) = \inf_{b \in B} \sigma_{D_\mu}(b) \). \( \square \)

To demonstrate the proper uses of Propositions 5.5, 5.6 and 5.7, let us return to our previous examples.

**Example 5.1 (cont.)** First note that the convex hull of \( S(A, K, B) \) is full dimensional. To see this, one can demonstrate the existence of \( n + 1 \) affinely independent points from \( S(A, K, B) \subseteq \mathbb{R}^n \) where \( n = 3 \). Thus there is no valid equation for \( S(A, K, B) \) implying that the lineality space of \( C(A, K, B) \) is just the zero vector. Moreover, \( \hat{z} = (0; 0; 1) \in \text{int}(K) \cap S(A, K, B) \) and hence Assumption 1 is satisfied.

We claim that \( (\mu^{(i)}; \eta^{(i)}_0) \) with \( i = 1, 2 \) are both \( K \)-minimal inequalities. We have already seen that for \( i = 1, 2, \mu^{(i)} \in \text{Im}(A^*) \), the sets \( D_{\mu^{(i)}} \) associated with them are nonempty, there are tight extreme points, i.e., \( \sigma_{D_{\mu^{(i)}}}(Az^{(i)}) = \langle \mu^{(i)}, z^{(i)} \rangle \) satisfying the requirement of Proposition 5.5, and \( \inf_{b \in B} \sigma_{D_{\mu^{(i)}}}(b) = \eta^{(i)}_0 \) holds, and hence by Proposition 5.5 \( (\mu^{(i)}; \eta^{(i)}_0) \) belong to \( C_\mu(A, K, B) \). Finally for \( i = 1, 2 \), the points \( z^{(i)} = (0; 0; t) \) satisfies \( z^{(i)} \in \text{int}(K) \cap S(A, K, B) \) and \( \langle \mu^{(i)}, z^{(i)} \rangle = \eta^{(i)}_0 \), and therefore, using Proposition 5.6, we conclude that these inequalities are also \( K \)-minimal. \( \square \)

**Example 5.2 (cont.)** Once again the convex hull of \( S(A, K, B) \) is full dimensional and thus there is no valid equation for \( S(A, K, B) \) implying that the lineality space of \( C(A, K, B) \) is just the zero vector. Moreover, \( \hat{z} = (1; 0; 2) \in \text{int}(K) \cap S(A, K, B) \) and hence Assumption 1 is satisfied.

We claim that

(a) \( \mu^{(+)\ast} = (1; 0; 0) \) with \( \eta^{(+)\ast}_0 = -1 \) and \( \mu^{(-)} = (-1; 0; 0) \) with \( \eta^{(-)}_0 = -1 \);

(b) \( \mu^{(t)} = (0; t; \sqrt{t^2 + 1}) \) with \( \eta^{(t)}_0 = 1 \) for all \( t \in \mathbb{R} \).

are all \( K \)-minimal inequalities. We have already seen that the associated sets \( D_{\mu^{(i)}} \) are nonempty, \( \inf_{b \in B} \sigma_{D_{\mu^{(i)}}}(b) = \eta^{(i)}_0 \) holds and there are tight extreme points, i.e., \( \sigma_{D_{\mu^{(i)}}}(Az^{(i)}) = \langle \mu^{(i)}, z^{(i)} \rangle \) satisfying the requirement of Proposition 5.5, and hence all of them are in \( C_\mu(A, K, B) \) by Proposition 5.5. Moreover, in case (a), by considering the points \( z^{(+)\ast} = (1; 0; 2) \in \text{int}(K) \cap S(A, K, B) \) and \( z^{(-)} = (-1; 0; 2) \in \text{int}(K) \cap S(A, K, B) \), we get \( \langle \mu^{(i)}, z^{(i)} \rangle = \eta^{(i)}_0 \) holds for all \( i \in \{+, -\} \). Therefore, using Proposition 5.6, we conclude that these inequalities are also \( K \)-minimal. In case (b), for any \( t \in \mathbb{R} \), consider \( z^{(t)}_+ = (1; -t; \sqrt{t^2 + 1}) \in K \cap S(A, K, B) \) and \( z^{(t)}_- = (-1; -t; \sqrt{t^2 + 1}) \in K \cap S(A, K, B) \). Note that we have \( \langle \mu^{(i)}, z^{(t)}_+ \rangle = \eta^{(t)}_0 = \langle \mu^{(i)}, z^{(t)}_- \rangle \) for all \( t \in \mathbb{R} \), and hence \( z^{(t)} := \frac{1}{2}(z^{(t)}_+ + z^{(t)}_-) = (0; -t; \sqrt{t^2 + 1}) \in \text{int}(K) \cap \text{conv}(S(A, K, B)) \). Therefore, using Proposition 5.6, we conclude that \( (\mu^{(t)}; \eta^{(t)}_0) \in C_\mu(A, K, B) \) for all \( t \in \mathbb{R} \).

We can also show that the system of infinitely many linear inequalities corresponding to \( (\mu^{(t)}; \eta^{(t)}_0) = (0; t; \sqrt{t^2 + 1}; 1) \) for all \( t \in \mathbb{R} \) indeed has a compact conic representation as follows: For all \( x \in S(A, K, B) \),
we have

\[
1 \leq 0x_1 + tx_2 + \sqrt{t^2 + 1} x_3 \quad \forall t \in \mathbb{R} \\
\iff 1 \leq \inf_t \{tx_1 + tx_2 + \sqrt{t^2 + 1} x_3 : t \in \mathbb{R}\} \\
\iff 1 \leq \inf_{t, \tau} \{tx_2 + \tau x_3 : t, \tau \in \mathbb{R}, \tau \geq \sqrt{t^2 + 1}\} \\
\iff 1 \leq \inf_{t, \tau} \{tx_2 + \tau x_3 : t \in \mathbb{R}, \tau \geq \sqrt{t^2 + 1}\} \\
\iff 1 \leq \inf_{t, \tau} \{tx_2 + \tau x_3 : t \in \mathbb{R}, (1; t; \tau) \in \mathcal{L}^3\} \\
\iff 1 \leq \sup_{\alpha} \{-\alpha_1 : \alpha_2 = x_2, \alpha_3 = x_3, (\alpha_1; \alpha_2; \alpha_3) \in \mathcal{L}^3\} \quad \text{due to (*)} \\
\iff (-1; x_2; x_3) \in \mathcal{L}^3,
\]

where (*) is due to the fact that the primal conic problem is strictly feasible and hence strong duality applies here. Note that we have arrived at \( x_3 \geq \sqrt{1 + x_2^2} \), constraint, which a cylinder in \( \mathbb{R}^3 \), hence a particular conic inequality. The validity of \( x_3 \geq \sqrt{1 + x_2^2} \) for all \( x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B}) \) follows from its derivation. Moreover, this conic inequality represents all of the constraints \( \mu(t); \eta(t) \) for all \( t \in \mathbb{R} \). Thus, in this example, \( x_3 \geq \sqrt{1 + x_2^2} \) along with \( x \in \mathcal{L}^3 \) constraint, completely describes \( \text{conv}(\mathcal{S}(A, \mathcal{K}, \mathcal{B})) \).

Finally note that we have seen the valid inequality \( \langle \nu; \nu_0 \rangle \) given by \( \nu = (0; 1; 2) \) and \( \nu_0 = 1 \) has an associated \( D_\nu \) set which is nonempty and there are tight extreme points, i.e., \( \sigma_{D_\nu}(A_{z(\nu)}) = \langle \nu, z(\nu) \rangle \) satisfying the requirement of Proposition 5.5 and \( \nu_0 = 1 < \sqrt{3} = \inf_{b \in \mathcal{B}} \sigma_{D_\nu}(b) \), hence by Proposition 5.5 \( \langle \nu; \nu_0 \rangle \in C_a(A, \mathcal{K}, \mathcal{B}) \). While \( \sigma_{D_\nu}(A_{z_\nu}) = \langle \nu, z_\nu \rangle = \nu_0 = 1 \) holds for any (and only) \( z_\nu \in \{(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{2}{3}), (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{2}{3})\} \subseteq \text{Ext}(\mathcal{K}) \) and the mid point of these two points is in the interior of \( \mathcal{K} \), this mid point is not in \( \text{conv}(\mathcal{S}(A, \mathcal{K}, \mathcal{B})) \), i.e., the condition stated in Proposition 5.6 fails. In fact, this can be easily seen via Proposition 5.7. Note that \( \nu \in \text{int}(\mathcal{K}^*) \) fails the necessary condition for \( \mathcal{K} \)-minimality given in Proposition 5.7, i.e., \( \inf_{b \in \mathcal{B}} \sigma_{D_\nu}(b) = \sigma_{D_\nu}(1) = \sigma_{D_\nu}(-1) = \sqrt{3} > 1 = \nu_0 \). Hence, we conclude that \( \langle \nu; \nu_0 \rangle \notin C_m(A, \mathcal{K}, \mathcal{B}) \).

We note as well that any valid conic inequality where at least one of the associated linear inequalities is not \( \mathcal{K} \)-minimal, will not be completely necessary for the description of \( \text{conv}(\mathcal{S}(A, \mathcal{K}, \mathcal{B})) \). For instance, in Example 5.2, \( x_3 \geq \sqrt{1 + \frac{1}{2} x_2^2} \) is a valid conic inequality but it is not necessary.

This example suggests a technique for grouping all of the tight minimal inequalities associated with the same cut generating set to derive closed form expressions of equivalent convex cuts. This approach is further exploited in [51, 52] for analyzing the specific case of two-term disjunctions on \( \mathcal{L}^n \) based on the characterization of tight \( \mathcal{K} \)-minimal inequalities, and the questions around the structure of resulting convex inequalities.

**Remark 5.3** When \( \mathcal{K} = \mathbb{R}^n_+ \), e.g., for the set \( \mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}) \), there is a strong connection between \( \mathcal{K} \)-minimal inequalities, cut generating functions [30], and the strong (functional) dual for MILPs [60].

As we briefly discuss in Remark 5.1, it is recently shown in [33] that for \( \mathcal{K} = \mathbb{R}^n_+ \), when \( \{0\} \notin \mathcal{B} \) and \( \mathcal{B} \) is contained in the convex cone generated by \( \{a_1, \ldots, a_n\} \) where \( a_i \) is the \( i \)th column of the linear map \( A \), for any \( \mathcal{K} \)-minimal inequality separating 0 from \( \text{conv}(\mathcal{S}(A, \mathcal{K}, \mathcal{B})) \), there exists a corresponding finite-valued cut generating function \( \rho : \mathbb{R}^n \to \mathbb{R} \), which maps each \( a_i \) to a cut coefficient \( \mu_i \). This states that under a mild requirement, in order to generate all valid inequalities for \( \mathcal{S}(A, \mathcal{K}, \mathcal{B}) \), it is sufficient to consider all valid inequalities obtained from cut generating functions. Besides, following our discussion in
Remark 5.1, if we allow the function to be extended valued, under Assumption 1, we only need to generate cuts that are K-minimal and for every K-minimal inequality, there exists a corresponding cut generating set, whose support function precisely generates the cut coefficients and right hand side value (cf. Proposition 5.3).

On the other hand, the situation seems to be much more complex for other (non-polyhedral) regular cones \( \mathcal{K} \). Example 5.2 reveals an important fact: Unlike the case with \( \mathcal{K} = \mathbb{R}^n_+ \), unless we make further structural assumptions, for general \( \mathcal{S}(A, \mathcal{K}, \mathcal{B}) \) with a non-polyhedral regular cone \( \mathcal{K} \), even when \( \mathcal{K} = \mathcal{L}^3 \), there are extreme (and also tight, K-minimal) valid linear inequalities such that there is no function that will precisely generate the vector defining the inequality. In particular, we have seen in this example that, for together with the original inequality in \( \mathcal{A} \), and it is shown that when \( b = [b] + f \) with \( f \in (0, 1) \), the following valid inequality

\[
(1 - 2f)(x - [b]) + f \leq t + y + w
\]

together with the original inequality in \( \mathcal{S}_0 \) gives \( \text{conv}(\mathcal{S}_0) \).

5.3 Connections to Conic Mixed Integer Rounding Cuts

Remark 5.4 In the simple case of the polyhedral cone \( \mathcal{K} = \mathcal{L}^2 = \{ x \in \mathbb{R}^2 : x_2 \geq |x_1| \} \), there are only two extreme rays \( \alpha^{(1)} = (1; 1) \) and \( \alpha^{(2)} = (-1; 1) \). These extreme rays are orthogonal to each other, and hence condition (A.1) reduces to

\[
(A.1(i)) \quad 0 \leq \sum_{i=1}^n \mu(a_i)u_i \quad \text{for all } u \text{ such that } Au = 0 \text{ and } u + \alpha^{(i)} \in \mathcal{L}^2 \quad \text{for } i = 1, 2,
\]

where \( a_i \) denotes the \( i \)-th column of \( A \). Following the same reasoning as in Proposition 5.3, one can easily deduce that for any \( \mathcal{K} \)-minimal valid inequality \( (\mu; \eta_0) \) and any extreme ray \( z \) of \( \mathcal{K} = \mathcal{L}^2 \), we have \( \sigma_{D\mu}(Az) = \langle \mu, z \rangle \).

Example 5.4 Using Proposition 5.3 and Remark 5.4, we can now analyze the conic mixed integer rounding cuts introduced in [6]. In particular, in [6], the following simple mixed integer set is studied

\[
\mathcal{S}_0 := \{ (x, y, w, t) \in \mathbb{Z} \times \mathbb{R}^3_+ : |x + y - w - b| \leq t \}, \quad (6)
\]

and it is shown that when \( b = [b] + f \) with \( f \in (0, 1) \), the following valid inequality

\[
(1 - 2f)(x - [b]) + f \leq t + y + w
\]

together with the original inequality in \( \mathcal{S}_0 \) gives \( \text{conv}(\mathcal{S}_0) \).
Here we will prove that (7) is in fact a $K$-minimal inequality. The first step in this analysis is to transform $S_0$ into our normal form as

$$S := \left\{(y, w, t, \gamma) \in \mathbb{R}^3_+ \times \mathcal{L}^2 : \left[\begin{array}{c} y-w \\ t \end{array}\right] - \gamma = \left[\begin{array}{c} b-x \\ 0 \end{array}\right]\right\},$$

(8)

which leads to $K = \mathbb{R}^3_+ \times \mathcal{L}^2$, which is a closed convex pointed cone with nonempty interior, and

$$A = \left[\begin{array}{cccc} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \text{and} \quad B = \left\{\left[\begin{array}{c} f \\ 0 \end{array}\right], \left[\begin{array}{c} 1+f \\ 0 \end{array}\right], \ldots, \left[\begin{array}{c} f-1 \\ 0 \end{array}\right], \left[\begin{array}{c} f-2 \\ 0 \end{array}\right], \ldots\right\}.$$

Before we proceed first note that Assumption 1 is satisfied, i.e., for any $\epsilon_1, \epsilon_2 > 0$, $(y; w; t; \gamma_1; \gamma_2) = (f + \epsilon_1; \epsilon_1; \epsilon_2; 0; \epsilon_2) \in \text{int}(K)$ and also in $S(A, K, B)$, therefore $K$-minimal inequalities exist. However, $S(A, K, B)$ is not full dimensional, $t - \gamma_2 = 0$ is a valid equation. The set $D_e$ corresponding to this valid equation is just $D_e = \{(\lambda_1, \lambda_2) : \lambda_1 = 1, \lambda_2 = 0\} = \{(1, 0)\}$. The point $\bar{z}$ defined in the rest of this example works for this valid equation as well. Thus, the valid equation $t - \gamma_2 = 0$ satisfies the necessary condition for $K$-minimality.

We can use the results of Section 5 to check whether the inequality (7) satisfies the conditions of $K$-minimal inequalities. Using the first equation in (8), we get $y - w - \gamma_1 = b - x$, which implies that $x - [b] = -y + w + \gamma_1 + f$. By substituting $x - [b]$ with $-y + w + \gamma_1 + f$, in (7), we can rewrite it in terms of the variables in our representation as follows:

$$(1 - 2f)(-y + w + \gamma_1 + f) + f \leq t + y + w$$

$$(2 - 2f)y + 2fw + t + (2f - 1)\gamma_1 + 0\gamma_2 \geq f(2 - 2f),$$

which means, $\eta_0 = f(2 - 2f)$, $\mu_1 = 2 - 2f$, $\mu_2 = 2f$, $\mu_3 = 1$, $\mu_4 = 2f - 1$ and $\mu_5 = 0$ in our usual notation. The necessary conditions for $K$-sublinearity state that for $D_\mu$ given by (4), we should have $D_\mu \neq \emptyset$, and $\sigma_{D_\mu}(Az) = \langle \mu, z \rangle$ for all $z \in \text{Ext}(K)$ (since all of the extreme rays of $K$ are orthogonal to each other).

In our specific case, we have

$$D_\mu = \{\lambda \in \mathbb{R}^2 : \exists \gamma \in K^* \text{ such that } A^*\lambda + \gamma = \mu\}$$

$$= \left\{\lambda \in \mathbb{R}^2 : \lambda_1 \leq \mu_1, -\lambda_1 \leq \mu_2, \lambda_2 \leq \mu_3, \left[\begin{array}{c} -\lambda_1 \\ -\lambda_2 \end{array}\right] \leq r^2 \left[\begin{array}{c} \mu_4 \\ \mu_5 \end{array}\right]\right\}$$

$$= \{\lambda \in \mathbb{R}^2 : \lambda_1 \leq 2 - 2f, -\lambda_1 \leq 2f, \lambda_2 \leq 1, |2f - 1 + \lambda_1| \leq \lambda_2\},$$

and the corresponding feasible region of $D_\mu$ is plotted in Figure 4.

As $f \in (0, 1)$, we have $D_\mu \neq \emptyset$, proving that $(\mu; \eta_0)$ is $K$-sublinear. Also the extreme rays of $K$ are precisely $\text{Ext}(K) = \{e_1, e_2, e_3, -e_4 + e_5, e_4 + e_5\}$ where $e_i$ stands for the $i^{th}$ unit vector in $\mathbb{R}^5$. Moreover,

$$\sigma_{D_\mu}(Ae_1) = \sigma_{D_\mu}(a_1) = 2 - 2f = \mu_1 = \langle \mu, e_1 \rangle$$

where $a_i$ denotes the $i^{th}$ column of the matrix $A$. Similarly we can show that $\sigma_{D_\mu}(Ae_i) = \mu_i = \langle \mu, e_i \rangle$ for $i = 1, \ldots, 3$. Moreover, we have $\sigma_{D_\mu}(A(-e_4 + e_5)) = \sigma_{D_\mu}(\left[\begin{array}{c} 1 \\ -1 \end{array}\right]) = 1 - 2f = -\mu_4 + \mu_5 = \langle \mu, (-e_4 + e_5) \rangle$ and $\sigma_{D_\mu}(A(e_4 + e_5)) = \sigma_{D_\mu}(\left[\begin{array}{c} -1 \\ -1 \end{array}\right]) = 2f - 1 = \mu_4 + \mu_5 = \langle \mu, (e_4 + e_5) \rangle$. 

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Finally note that $\sigma_{D_\mu}(b_1^+) = f \cdot \sigma_{D_\mu}(e_1) = f(2 - 2f)$ and for $i = 1, 2, \ldots$, we have $\sigma_{D_\mu}(b_i^{+1}) = (f + i)(2 - 2f) = (2 - 2f)i + 2f - 2f^2$. Considering $f \in (0, 1)$, we conclude $\sigma_{D_\mu}(b_1^+) < \sigma_{D_\mu}(b_2^+) < \ldots$ holds. Similarly $\sigma_{D_\mu}(b_1^-) = (1 - f)\sigma_{D_\mu}(-e_1) = (1 - f)(-2f) = 2f(f - 1)$ and for $i = 1, 2, \ldots$, we have $\sigma_{D_\mu}(b_i^-) = (f - i)(-2f) = 2fi - 2f^2$, which implies $\sigma_{D_\mu}(b_1^-) < \sigma_{D_\mu}(b_2^-) < \ldots$ holds, and hence

$$\inf_{b \in B} \sigma_{D_\mu}(b) = \min \{\sigma_{D_\mu}(b_1^+), \sigma_{D_\mu}(b_1^-)\} = f(2 - 2f) = \eta_0.$$

Finally consider the following set of points

$$\{z_1 := (f; 0, 0, 0, 0), z_2 := (0, 1 - f; 0, 0, 0), z_3 := (0; 0, f; -f; f), z_4 := (0; 0, 1 - f; 1 - f; 1 - f)\}.$$

Given $f \in (0, 1)$, one can easily see that for $i = 1, \ldots, 4$, we have $z_i \in S(A, K, B)$ and $\langle \mu, z_i \rangle = \eta_0 = 2f - 2f^2$. Moreover $\bar{z} := \frac{1}{4} \sum_{i=1}^{4} z_i$ is in the interior of $K = \mathbb{R}^3_+ \times \mathcal{L}^2$. Therefore, using Proposition 5.6, we have shown that the valid inequality given by $\langle \mu; \eta_0 \rangle = (2 - 2f; 2f; 1; 2f - 1; 0; 2f - 2f^2)$, which is equivalent to (7), is a $K$-minimal inequality. \hfill $\Box$

### 6 Characterization of Valid Equations

Our results with regard to the existence of $K$-minimal inequalities was based on Assumption 1, i.e., we assume that for all $\delta \in K^* \setminus \{0\}$, there exists $z_\delta \in S(A, K, B)$ such that $\langle \delta, z_\delta \rangle > 0$. Under a stronger assumption, Assumption 2 stated below, we can show that all valid equations $(\mu; \eta_0)$ satisfy $\mu \in \text{Im}(A^*)$.

**Assumption 2:** There exists $\hat{z} \in S(A, K, B)$ such that $\hat{z} \in \text{int}(K)$ and $A\hat{z} = \hat{b}$ for some $\hat{b} \in B$.

Under Assumption 2, we can provide the following precise characterization of the valid equations.

**Theorem 6.1** Suppose that Assumption 2 holds. Then $(\mu; \eta_0)$ is a valid equation if and only if there exists some $\bar{\lambda} \in \mathbb{R}^m$ such that

$$A^*\bar{\lambda} = \mu \quad \text{and} \quad b^T\bar{\lambda} = \eta_0 = \vartheta(\mu) \quad \text{for all } b \in B.$$

**Proof.**


It is easy to see that the condition in Theorem 6.1 is sufficient. Suppose there exists \( \tilde{\lambda} \in \mathbb{R}^m \) such that
\[
A^* \tilde{\lambda} = \mu \quad \text{and} \quad b^T \tilde{\lambda} = \eta_0,
\]
for all \( b \in \mathcal{B} \). Then for any \( z \in \mathcal{S}(A,K,B) \) we have
\[
\langle \mu, z \rangle = \langle A^* \tilde{\lambda}, z \rangle = \tilde{\lambda}^T Az = \tilde{\lambda}^T b = \eta_0 = \vartheta(\mu),
\]
where the third equation follows from the fact that \( z \in \mathcal{S}(A,K,B) \) and hence \( b \in \mathcal{B} \), proving that \((\mu; \eta_0)\) is a valid equation.

To prove the necessity of the condition, suppose that \((\mu; \eta_0)\) is a valid equation. Then clearly \( \eta_0 = \vartheta(\mu) \). Let \( \hat{b} \) and \( \hat{z} \) be as described in Assumption 2 preceding the theorem, and consider
\[
\inf_z \left\{ \langle \mu, z \rangle : Az = \hat{b}, \ z \in \mathcal{K} \right\}.
\]
Since there exists \( \hat{z} \in \text{int}(\mathcal{K}) \) satisfying \( A\hat{z} = \hat{b} \), this problem is strictly feasible, moreover the solution set of this problem is contained in \( \mathcal{S}(A,K,B) \) and thus \((\mu; \vartheta(\mu))\) being a valid equation implies that its optimum value is equal to \( \vartheta(\mu) \). Using strong conic duality we get
\[
\vartheta(\mu) = \sup_{\lambda \in \mathbb{R}^m} \left\{ \hat{b}^T \lambda : A^* \lambda \preceq_{K^*} \mu \right\},
\]
which implies the existence of an optimal solution \( \tilde{\lambda} \) satisfying
\[
A^* \tilde{\lambda} \preceq_{K^*} \mu \quad \text{and} \quad \hat{b}^T \tilde{\lambda} = \vartheta(\mu).
\]
Note that any feasible solution to the primal problem is optimal including the strictly feasible solution \( \hat{z} \). Therefore, using the complementary slackness of conic duality we have
\[
\langle \hat{z}, \mu - A^* \tilde{\lambda} \rangle = 0.
\]
Because \( \hat{z} \in \text{int}(\mathcal{K}) \), above equation is possible if and only if \( A^* \tilde{\lambda} = \mu \). So now we have established that there exists \( \tilde{\lambda} \) satisfying \( A^* \tilde{\lambda} = \mu \) and \( \hat{b}^T \tilde{\lambda} = \vartheta(\mu) \). Now consider any \( b \in \mathcal{B} \), then we have
\[
\vartheta(\mu) = \langle \mu, z_b \rangle \geq \inf_z \{ \langle \mu, z \rangle : Az = b, \ z \in \mathcal{K} \} \geq \sup_{\lambda \in \mathbb{R}^m} \left\{ \hat{b}^T \lambda : A^* \lambda \preceq_{K^*} \mu \right\} \geq \hat{b}^T \tilde{\lambda}. \quad (9)
\]
Moreover
\[
- \vartheta(\mu) = \langle -\mu, z_b \rangle \geq \inf_z \{ \langle -\mu, z \rangle : Az = b, \ z \in \mathcal{K} \} \geq \sup_{\lambda \in \mathbb{R}^m} \left\{ \hat{b}^T \lambda : A^* \lambda \preceq_{K^*} -\mu \right\} \geq -\hat{b}^T \tilde{\lambda}, \quad (10)
\]
where the second inequality follows from weak duality and the last inequality follows from the fact that \(-\tilde{\lambda}\) is a feasible solution to the dual. By combining (9) and (10), we get \( \vartheta(\mu) = \hat{b}^T \tilde{\lambda} \), which completes the proof.

In addition to the above characterization, we can relate each valid equation with its corresponding cut generating set \( D_\mu \) given by (4) as follows:
Corollary 6.1 Suppose that Assumption 2 holds. Then, for any valid equation \((\mu ; \eta_0)\), there exists \(\lambda_\mu\) satisfying \(D_\mu = \{ \lambda : A^* \lambda \preceq_{\mathcal{K}^*} \mu \} = \lambda_\mu + \{ \lambda : A^* \lambda \preceq_{\mathcal{K}^*} 0 \}\) and \(\eta_0 = \inf_{b \in B} \sigma_{D_\mu}(b) = \sup_{b \in B} \sigma_{D_\mu}(b)\).

Proof. Suppose \((\mu ; \eta_0)\) is a valid equation, then by Theorem 6.1, there exists \(\bar{\lambda} =: \lambda_\mu\) such that \(\mu = A^* \bar{\lambda}\) and \(\partial(\mu) = \bar{b}^T \bar{\lambda}\) for all \(b \in B\). Thus, we have
\[
D_\mu = \{ \lambda : A^* \lambda \preceq_{\mathcal{K}^*} A^* \bar{\lambda} \} = \{ \bar{\lambda} + \lambda : A^* \lambda \preceq_{\mathcal{K}^*} 0 \},
\]
and
\[
\inf_{b \in B} \sigma_{D_\mu}(b) = \inf_{b \in B} \sup_{\lambda \in \mathbb{R}^m} \{ b^T (\bar{\lambda} + \lambda) : A^* \lambda \preceq_{\mathcal{K}^*} 0 \} = \inf_{b \in B} \left[ \frac{b^T \bar{\lambda}}{\eta_0} + \sup_{\lambda \in \{0, +\infty\}} \{ b^T \lambda : A^* \lambda \preceq_{\mathcal{K}^*} 0 \} \right] = \eta_0,
\]
where the last equation follows from the fact that \(\eta_0 \in \mathbb{R}\). Similarly, we can show that \(\eta_0 = \sup_{b \in B} \sigma_{D_\mu}(b)\). \(\square\)

When \(\mathcal{K} = \mathbb{R}^n_+\) (or any cone where each pair of its extreme rays is orthogonal), we note that the statement of Corollary 6.1 gives a complete characterization of valid equations.

7 Conclusions and Further Research

Disjunctive programming and Gomory’s corner polyhedron serve as major tools in the MILP framework, as most cutting planes used in MILP can be viewed in these contexts. Therefore, the generalization of these to the conic setups such as those arising in MICPs offered by \(\mathcal{S}(A, \mathcal{K}, \mathcal{B})\) suggests the potential of \(\mathcal{S}(A, \mathcal{K}, \mathcal{B})\) to serve as a fundamental equivalent relaxation for MICPs. In particular, by studying the structure of \(\mathcal{S}(A, \mathcal{K}, \mathcal{B})\), we can design better cutting planes for MICPs. In the MILP literature, a natural first step in studying the corner polyhedron has been investigating the associated semi-infinite relaxation. Such an alternative seems to be meaningful only when the associated cone is the nonnegative orthant. To the best of our knowledge, the extensions of other well-known regular cones such as the Lorentz cone and the cone of positive semidefinite matrices to the infinite dimensional cases are not well defined. Moreover, in the practical cutting plane procedures to solve MILPs (and also MICPs), one is indeed faced with a problem in a finite dimensional space. Therefore, we chose to follow a different path and focus on a finite dimensional case defined by a given particular problem instance.

We introduce the class of \(\mathcal{K}\)-minimal valid inequalities in the general disjunctive conic programming context and show that this class contains a small yet essential set of nonredundant inequalities. In particular, under a mild technical assumption, we establish that the class of \(\mathcal{K}\)-minimal inequalities together with the original \(x \in \mathcal{K}\) constraint are sufficient to describe \(\text{conv}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))\). This indicates that \(\mathcal{K}\)-minimal inequalities are of great interest, and an efficient cutting plane procedure for solving MICPs should at the least aim at separating inequalities from this class. Nevertheless, the definition of \(\mathcal{K}\)-minimality reveals little about the structure of \(\mathcal{K}\)-minimal inequalities. To address this issue, we show that the class of \(\mathcal{K}\)-minimal inequalities is contained in a slightly larger class of so-called \(\mathcal{K}\)-sublinear inequalities. We establish a close connection between \(\mathcal{K}\)-sublinear inequalities for MICPs and the support functions of convex sets with certain structure. Using this connection, we show that when \(\mathcal{K} = \mathbb{R}^n_+\), \(\mathcal{K}\)-minimal inequalities for MILPs are generated by sublinear (positively homogeneous, subadditive and convex) functions that are also piecewise linear. Thus, our results naturally capture some of the earlier results from MILP setup, and generalize them
to the conic setup. Furthermore, this connection with support functions also led us establish practical ways of showing that an inequality is \(K\)-sublinear and \(K\)-minimal. To the best of our knowledge, these sufficient conditions for \(K\)-sublinearity and \(K\)-minimality of the valid inequalities are new even in the MILP setup.

In this work, we have shed some light on the structure of \(K\)-minimal and \(K\)-sublinear valid inequalities for solution sets of the form \(S(A, K, B)\) involving a regular cone \(K\). However, many questions remain open when we start considering regular cones other than \(\mathbb{R}^n_+\). In particular, we find the following questions of interest:

- **[Characterization of extreme valid inequalities]** Under a mild technical assumption, e.g., Assumption 1, we have shown that all extreme inequalities are \(K\)-minimal. However, not every \(K\)-minimal inequality is extreme (see, e.g., Example 3.3 and Proposition 3.4). Further characterizations of extreme inequalities beyond \(K\)-minimality are of great interest and importance.

- **[Finiteness of the \(K\)-minimal conic inequalities]** When \(K = \mathbb{R}^n_+\) and \(B\) is finite, Johnson [49] proved that the cone of \(K\)-minimal inequalities is finitely generated, i.e., \(G_C\) is finite. Note that \(G_L\) is always finite. For general regular cones, e.g., \(L^n, S^n, L^n_+\), expecting the convex hull description of \(S(A, K, B)\) to be given by finitely many linear inequalities is too much, and against the inherent nonlinear nature of these cones. Even for \(L^3\), there are examples showing that this is not possible, i.e., there exists \(S(A, K, B)\) that requires infinitely many \(K\)-minimal and also extremal linear inequalities (see Example 5.2). On the other hand, in that example, it is clear that the description of \(\text{conv}(S(A, K, B))\) only involves two linear inequalities and one conic inequality involving \(L^3\). While the \(K\)-minimality notion is seemingly defined for linear inequalities, we can immediately extend it to a conic inequality by saying that a conic inequality is \(K\)-minimal if the associated (possibly infinite) set of linear inequalities are all \(K\)-minimal. We believe that instead of focusing on the finiteness of linear inequalities describing \(\text{conv}(S(A, K, B))\), it is more natural and relevant to focus on the finiteness of conic inequalities (of the same type of \(K\)) describing \(\text{conv}(S(A, K, B))\). Therefore, we wonder what can be said in terms of the number of \(K\)-minimal conic inequalities required in the description of \(\text{conv}(S(A, K, B))\). Is it a finite number when \(B\) is finite? Is it finite regardless of the size of \(B\)? Or, can we at least identify the cases where it is finite?

- **[Relations with other valid inequalities for MICPs]** We showed that conic MIR inequalities introduced in [6] can be interpreted in this framework. Moreover, the connection between tight \(K\)-minimal inequalities and explicit expressions for conic cuts obtained by considering a two-term disjunction on \(L^n\) and/or its cross-sections is described in the recent series of papers [51, 52, 75]. These derivations nicely relate back to other recently developed valid inequalities for MICPs based on split or disjunctive arguments as in [3, 16, 35, 57]. It will be nice to understand the relation of our framework and other recent literature in more general setups involving nonconvex quadratic sets such as those considered in [19, 25, 57].

**Acknowledgements**

The author wishes to express her gratitude to the Associate Editor and anonymous referees for their constructive feedback, which led to substantial improvements of the presentation of the material in this paper. This work was first presented at UC Davis during Mixed Integer Programming Workshop in July 2012.
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