Integrable Quartic Potentials and Coupled KdV Equations

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Abstract

We show a surprising connection between known integrable Hamiltonian systems with quartic potential and the stationary flows of some coupled KdV systems related to fourth order Lax operators. In particular, we present a connection between the Hirota-Satsuma coupled KdV system and (a generalisation of) the 1 : 6 : 1 integrable case quartic potential. A generalisation of the 1 : 6 : 8 case is similarly related to a different (but gauge related) fourth order Lax operator.

We exploit this connection to derive a Lax representation for each of these integrable systems. In this context a canonical transformation is derived through a gauge transformation.

1 Introduction

In \cite{4} a surprising connection was shown between integrable cases of the Hénon-Heiles system and the stationary flows of some known integrable PDEs: the 5th order KdV, Sawada-Kotera and Kaup-Kupershmidt equations, which are integrable through second and third order differential Lax operators. This gave rise to a matrix spectral problem for each of the integrable Hénon-Heiles systems. Thus, integrable cubic potentials are associated with second and third order matrix Lax operators.

In \cite{4} the starting point was the Hénon-Heiles system, using some elementary, ad hoc calculations. An alternative approach is to start with the stationary flow of a known integrable PDE and to find the appropriate Hénon-Heiles coordinates. The most natural way of doing this is through the Hamiltonian structure
of the PDE. As motivation, this is presented for the Hénon-Heiles system in section 2.

In this note we exhibit a similar connection between some stationary flows associated with fourth order Lax operators and generalisations of some integrable Hamiltonian systems with quartic potentials:

\[ H = \frac{1}{2} (p_1^2 + p_2^2) + aq_1^4 + bq_1^2q_2^2 + cq_2^4. \]  

(1)

There are 4 nontrivial cases which are integrable:

1. \( a : b : c = 1 : 2 : 1 \),
2. \( a : b : c = 1 : 12 : 16 \),
3. \( a : b : c = 1 : 6 : 1 \),
4. \( a : b : c = 1 : 6 : 8 \).

Various inverse square terms can be added without destroying complete integrability. Cases (1), (2) and (3) are separable in respectively polar, parabolic and Cartesian coordinates. Case (2) is one of the higher degree polynomial examples given in [3]. Case (3) is much more complicated and has only recently been shown to be separable (in a generalised sense) [3], using a Painlevé expansion.

Our main result is to relate cases (3) and (4) to some coupled KdV equations associated with fourth order Lax operators, which are themselves related through a Miura map. The connection with integrable PDEs gives a Lax representation for these finite dimensional systems, together with similarity and canonical transformation, thus giving an explanation of the separation coordinates found in [3].

2 Stationary Flows and Integrable Hénon-Heiles Systems

It was shown in [4] that the general (not necessarily integrable) Hénon-Heiles system can be related to the stationary flow of:

\[ u_t = (\partial^3 + 8au\partial + 4au_x) \delta_u \left( -\frac{1}{3}bu^3 - \frac{1}{2}u_x^2 \right). \]  

(2)

For this stationary flow, the gradient of the above Hamiltonian is in the kernel of the third order Hamiltonian structure. Thus we may write:

\[ \delta_u \left( -\frac{1}{3}bu^3 - \frac{1}{2}u_x^2 \right) = A, \quad \text{where} \quad (\partial^3 + 8au\partial + 4au_x) A = 0. \]
We now set \( A = \alpha y^2 \) to find:

\[
y(y_{xx} + 2auy)_x + 3y_x(y_{xx} + 2auy) = 0.
\]

Setting \( y_{xx} + 2auy = \gamma \), we solve the first order equation for \( \gamma \) to get \( \gamma = 2ky^{-3} \), where \( k \) is a constant. We now have:

\[
\delta_u \left( -\frac{1}{3}bu^3 - \frac{1}{2}u_x^2 \right) = \alpha y^2,
\]

\[
y_{xx} + 2auy = 2ky^{-3},
\]

which, for \( \alpha = -a \), are Lagrangian, with:

\[
L = \frac{1}{2} (u_x^2 + y_x^2) + \frac{1}{3} bu^3 - aux^2 - ky^{-2}.
\]

The standard Legendre transformation now renders a natural Hamiltonian system, which is just the usual generalisation of the Hénon-Heiles system:

\[
q_1 = u, \quad p_1 = u_x, \quad q_2 = y, \quad p_2 = y_x,
\]

\[
h = \frac{1}{2} (p_1^2 + p_2^2) - \frac{1}{3} bu^3 - aux^2 - ky^{-2}.
\]

Thus the Hamiltonian structure of (2) gave us a natural way of defining some interesting coordinates, giving us the Hénon-Heiles representation of the stationary flow. For the integrable cases it is possible to use the Lax representation of (2) to derive a matrix Lax representation for the corresponding Hénon-Heiles system \([4]\), thus proving the complete integrability of the latter. The gauge equivalence of the Sawada-Kotera and Kaup-Kupershmidt equations leads to a canonical transformation between the corresponding Hénon-Heiles systems \((a/b = -1/6 \text{ and } a/b = -1/16)\).

In this paper, we obtain similar results for some quartic potentials.

### 3 Fourth Order Operators

We start with the self adjoint fourth order operator, which we write in factorised form:

\[
L_1 = (\partial + v_1)(\partial + v_2)(\partial - v_2)(\partial - v_1).
\]  

(3)

This can be written as the product of two second order operators:

\[
L_1 = (\partial^2 + f\partial + f_x + g)(\partial^2 - f\partial + g),
\]

to give the Miura map:

\[
f = v_1 + v_2, \quad g = v_1 v_2 - v_1 x.
\]  

(4)
The Lax equation:
\[ L_{11} = [M_1, L_1] \]
where:
\[ M_1 = 2\partial^3 + \frac{3}{2}(2g - f_x - f^2)\partial + \frac{3}{4}(2g_x - f_{xx} - 2ff_x), \]
gives the following coupled KdV system:
\[ \begin{align*}
    f_t &= -\frac{1}{2}(2f_{xxx} + 3ff_{xx} + 3f_x^2 - 3f^2f_x + 6fg_x + 6gf_x) , \\
    g_t &= \frac{1}{4}(2g_{xxx} + 12gg_x + 6fg_{xx} + 12g_{xx} + 18fg_x + 6f^2g_x \\
    &\quad + 3ff_{xxx} + 3ff_{xx} + 18ff_{xx} - 6f^2f_x - 6f^2f_x). 
\end{align*} \] (5)

**Remark 1** The resulting coupled KdV system is simpler in co-ordinates \( r \) and \( s \), corresponding to the operator:
\[ L_{\text{sym}} = \partial^4 + \partial r \partial + s, \]
but the Miura map and Hamiltonian operator are more complicated, so not well suited to our purposes.

A rotation of factors in (3) leads to another operator:
\[ L_2 = (\partial + v_1)(\partial + v_1)(\partial + v_2)(\partial - v_2), \]
which can be written as:
\[ L_2 = (\partial^2 + u + \varphi)(\partial^2 + u - \varphi), \] (6)
with the Miura map:
\[ u = \frac{1}{2}(v_1x - v_2x - v_1^2 - v_2^2), \quad \varphi = \frac{1}{2}(v_1x + v_2x - v_1^2 + v_2^2). \] (7)

The operator (6) was found in [2] to be the Lax operator for the Hirota-Satsuma system:
\[ \begin{align*}
    u_t &= \frac{1}{2}u_{xxx} + 3uu_x - 6\varphi\varphi_x, \quad \varphi_t = -\varphi_{xxx} - 3u\varphi_x, 
\end{align*} \] (8)
which corresponds to the time evolution operator:
\[ M_2 = 2\partial^3 + 3u\partial + \frac{3}{2}(u_x - 2\varphi_x). \]

The isospectral flows of the general fourth order operator are bi-hamiltonian, but the first Hamiltonian structure does not reduce to the 2–component sub-manifolds corresponding to our two operators. As in the case of our third order
operators, both the $f, g$ and $u, \phi$ hierarchies are related to the same modified hierarchy for the functions $v_1$ and $v_2$, which has Hamiltonian form:

$$\left( \begin{array}{c} v_1 \\ v_2 \end{array} \right)_t = \left( \begin{array}{cc} -\partial & 0 \\ 0 & -\partial \end{array} \right) \left( \begin{array}{c} \delta_{v_1} h \\ \delta_{v_2} h \end{array} \right).$$

The Miura maps (4,7) transport this structure respectively onto:

$$B_f = \left( \begin{array}{cc} -2\partial & 0 \\ \partial^2 - f\partial & \partial^3 + (2f_x + 2g - f^2)\partial + (f_{xx} + g_x - ff_x) \end{array} \right)$$

and:

$$B_u = \left( \begin{array}{cc} \frac{1}{2}\partial^3 + 2u\partial + u_x & 2\phi\partial + \phi_x \\ \frac{1}{2}\partial^3 + 2u\partial + u_x & \frac{1}{2}\partial^3 + 2u\partial + u_x \end{array} \right).$$

These are reductions of the general second Hamiltonian structure associated with the general fourth order Lax operator. With these, equations (5) and (8) are respectively generated by

$$G = -\frac{1}{8} (f_x^2 + f^4 + 4fg_x - 4f^2g - 4g^2)$$

and

$$H = \frac{1}{2} u^2 - \frac{1}{2} \phi^2,$$

both of which are pulled back to the modified Hamiltonian

$$h_{mod} = -\frac{1}{8} (v_2 v_1 + v_1 v_2 + v_1^3 + v_2^3) - \frac{3}{4} (v_1 v_2 + v_1^2 v_2 + v_2^2 v_1 - v_1^2 v_2 - v_1 v_2^2),$$

which evidently generates a coupled MKdV system.

### 3.1 Stationary Flows and Related Quartic Potentials

We now consider the stationary flows:

$$B_f \delta_f G = 0 \quad \text{and} \quad B_u \delta_u H = 0,$$

which we show to be respectively related to generalisations of cases (4) and (8) of the quartic potentials (1).

#### 3.1.1 The 1 : 6 : 8 Potential

The stationary flow in the $(f, g)$ space is given by:

$$A_{xx} - f A_x + B_{xxx} + (2f_x + 2g - f^2)B_x + (f_{xx} + g_x - ff_x)B = 0,$$

where, for the Hamiltonian $G$,

$$A = \delta_f G = \frac{1}{4} (f_{xx} - 2g_x - 2f^3 + 4fg),$$

$$B = \delta_g G = \frac{1}{2} (f_x + f^2 + 2g).$$

Equation (11) gives $A = \frac{1}{2}(K - fB - B_x)$, where $K$ is a constant of integration, and:
\[ B_{xxx} + 2(f_x + 2g - \frac{1}{2}f^2)B_x + (f_{xx} + 2g_x - f f_x)B = 0, \]

which has solution \( B = \alpha y^2 \), where:

\[ y_{xx} + \left( \frac{1}{2}f_x - \frac{1}{4}f^2 + g \right)y = L y^{-3}. \]  

(13)

Using (12) we obtain a formula for \( g \):

\[ g = \alpha y^2 - \frac{1}{2}f^2 - \frac{1}{2}f_x, \]

which, from (11), (13) and the definition of \( K \), gives:

\[ f_{xx} - 2f^3 + 3\alpha y^2 f = K, \]

\[ y_{xx} + (\alpha y^2 - \frac{3}{4}f^2)y = L y^{-3}. \]

When \( \alpha = -\frac{1}{4} \) the above equations are Lagrangian with:

\[ \mathcal{L}_{(f,g)} = \frac{1}{2}(f_x^2 + y_x^2) + \frac{1}{16}(y^4 + 6f^2y^2 + 8f^4) + Kf - \frac{L}{2}y^{-2}. \]

With canonical co-ordinates:

\[ Q_1 = y, \quad Q_2 = f, \quad P_1 = y_x, \quad P_2 = f_x, \]

this gives the Hamiltonian:

\[ h_{(f,g)} = \frac{1}{2}(P_1^2 + P_2^2) - \frac{1}{16}(Q_1^4 + 6Q_1^2Q_2^2 + 8Q_2^4) - KQ_2 + \frac{L}{2}Q_1^{-2}, \]  

(14)

which is one of the integrable generalisations of case (3) of (1) [5].

3.1.2 The 1 : 6 : 1 Potential

The stationary flow in the \((u, \varphi)\) space is given by:

\[ \left( \frac{1}{2} \partial^3 + 2u \partial + u_x \right)A + (2\varphi \partial + \varphi_x)B = 0, \]

\[ (2\varphi \partial + \varphi_x)A + \left( \frac{1}{2} \partial^3 + 2u \partial + u_x \right)B = 0, \]

where, for the simple Hamiltonian \( H \), we have \( A = \delta_u H = u \) and \( B = \delta_\varphi H = -2\varphi \). It is straightforward to find quadratic forms for \( A \) and \( B \):

\[ A = \alpha(\psi^2 + \chi^2), \quad B = \alpha(\chi^2 - \psi^2), \]

where:

\[ \psi_{xx} + (u - \varphi)\psi = l\psi^{-3}, \]

\[ \chi_{xx} + (u + \varphi)\chi = k\chi^{-3}. \]
**Remark 2** The motivation for this representation of $A$ and $B$ is found in the squared eigenfunction representation of $L_2 \theta = z \theta$, when written as a $2 \times 2$, second order differential system.

Upon substituting $u$ and $\varphi$ we get:

\[
\begin{align*}
\psi_{xx} + \frac{1}{2} \alpha (\psi^2 + 3 \chi^2) \psi &= l \psi^{-3}, \\
\chi_{xx} + \frac{1}{2} \alpha (3 \psi^2 + \chi^2) \chi &= k \chi^{-3},
\end{align*}
\]

which is Lagrangian with:

\[
L = \frac{1}{2} (\psi_x^2 + \chi_x^2) - \frac{1}{8} \alpha (\psi^4 + 6 \psi^2 \chi^2 + \chi^4) - \frac{1}{2} (l \psi^{-2} + k \chi^{-2}).
\]

With the canonical co-ordinates:

\[
q_1 = \psi, \quad p_1 = \psi_x, \quad q_2 = \chi, \quad p_2 = \chi_x,
\]

this gives (with $\alpha = 1$) the Hamiltonian:

\[
h_{uf} = \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{8} (q_1^4 + 6 q_1^2 q_2^2 + q_2^4) + \frac{1}{2} (l q_1^{-2} + k q_2^{-2}),
\]

which is one of the integrable generalisations of case (3) of (1) \[5\].

### 3.2 The Lax Representations

The Lax representations $L_{it} = [M_i, L_i]$ for the PDEs can be re-written in zero curvature form:

\[
U_{it} - V_{it} + [U_i, V_i] = 0,
\]

where:

\[
U_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-g & f & 1 & 0 \\
0 & 0 & 0 & 1 \\
z & 0 & -g - f_x & -f
\end{pmatrix}, \quad U_2 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
\varphi - u & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
z & 0 & -\varphi - u & 0
\end{pmatrix},
\]

and $V_i$ given by more complicated formulae. The stationary flows are given by the Lax representations $V_{ix} = [U_i, V_i]$, which can be written in terms of $q_i, p_i$ and $Q_i, P_i$. Whilst $U_i$ are very simple, we need to use the equations of motion generated by $h_{(f,g)}$ and $h_{(u,\varphi)}$ to eliminate second and higher derivatives in $V_i$. When written in this way the $V_i$ are the Lax matrices for the equations of motion and can be used to generate the constants of motion. They are given by:

\[
V_1 = \begin{pmatrix}
\frac{1}{8} Q_2 Q_1^2 + \frac{1}{4} Q_1 P_1 - \frac{1}{2} K & a_{21} & -\frac{1}{8} Q_2 Q_1^2 - \frac{1}{4} Q_1 P_1 - \frac{1}{2} K & P_2 + \frac{1}{4} Q_1^2 + Q_2^2 & 2 \\
-\frac{1}{2} Q_2 Z & a_{33} & 0 & 0 \frac{1}{4} Q_1^2 + Q_2^2 & a_{43} \\
2 z & 0 & 0 & a_{44} & a_{44}
\end{pmatrix},
\]
The matrices

\[
V_2 = \begin{pmatrix}
-2q_1p_1 & 2q_1^2 & 0 & 2 \\
2z - 2p_1^2 - 2lq_1^{-2} & 2q_1p_1 & -(q_1^2 + q_2^2) & 0 \\
0 & 2z & -2q_2p_2 & 2q_2^2 \\
-(q_1^2 + q_2^2) & 0 & 2z - 2p_2^2 - 2kq_2^{-2} & 2q_2p_2
\end{pmatrix},
\]

where:

\[a_{21} = \frac{1}{16}Q_1^2Q_2^2 + \frac{L}{4Q_1^2} + \frac{1}{4}P_1^2 + \frac{1}{4}Q_1Q_2P_1 + 2z,\]

\[a_{33} = -\frac{1}{8}Q_2Q_1^2 + \frac{1}{4}Q_1P_1 + \frac{1}{2}K,\]

\[a_{43} = \frac{1}{16}Q_1^2Q_2^2 + \frac{L}{4Q_1^2} + \frac{1}{4}P_1^2 - \frac{1}{4}Q_1Q_2P_1 + 2z,\]

\[a_{44} = \frac{1}{8}Q_2Q_1^2 - \frac{1}{4}Q_1P_1 + \frac{1}{2}K.\]

The first integrals are given by the characteristic equations:

\[
det(V_1 - \mu I) = \mu^4 + \frac{1}{8}(L - 4K^2)\mu^2 + \det V_1, \quad (16)
\]

\[
det(V_2 - \mu I) = \mu^4 + 4(k + l)\mu^2 + \det V_2, \quad (17)
\]

where

\[
det(V_1) = -16z^3 - 8h_{(f,g)}z^2 - \frac{1}{4}k_{(f,g)}z + \frac{1}{256}(L + 4K^2)^2,
\]

\[
det(V_2) = -16z^3 + 32h_{(u,\varphi)}z^2 - 16k_{(u,\varphi)}z + 16lk.
\]

Here \(h_{(f,g)}\) and \(h_{(u,\varphi)}\) are given by (14) and (15) and:

\[
k_{(f,g)} = P_1^4 + \frac{1}{16}(Q_2^2Q_1^2 + Q_1^2Q_1^4) - \frac{1}{4}(Q_1^2P_1^2 + Q_1^4P_1^2) + \frac{1}{64}Q_1^8 + P_1P_2Q_2Q_3^3 - \frac{3}{2}Q_1^2Q_2^2P_2^2
\]

\[-\frac{1}{2}(Q_1^2Q_2P_1 + LQ_1^2) - 2(Q_1^2K^2 - \frac{P_1L}{Q_1}) - \frac{1}{4}Q_1^2L + (\frac{L^2}{Q_1} - Q_1^2Q_2^2K)
\]

\[+4(Q_1P_1P_2K - 2Q_2P_1^2K - \frac{LKQ_2Q_1}{Q_1}).\]

\[
k_{(u,\varphi)} = \left(p_1p_2 + \frac{1}{2}(q_1^3q_2 + q_1q_2^3)\right)^2 + kp_1^2 + \frac{lk}{q_1q_2} + \frac{kq_1^2}{q_2} + \frac{lq_2^2}{q_1} + \frac{lp_2^2}{q_1}.
\]

The Lax equations for \(V_i\) are known to linearise on the Jacobi variety of the algebraic curves defined by (16) and (17), which gives one of the standard integration procedures (see, for instance, Theorem 1, page 67 of [3]).

### 3.3 Gauge and Canonical Transformations

The matrices \(U_i, V_i\) (in the PDE variables) are related by gauge transformation:

\[
U_2 = AU_1A^{-1} + A_xA^{-1}, \quad V_2 = AV_1A^{-1} + A_tA^{-1},
\]
with \( A \) given by:

\[
A = \begin{pmatrix}
-v_1 & 1 & 0 & 0 \\
-v_1 v_2 & v_2 & 1 & 0 \\
0 & 0 & v_2 & 1 \\
\pm \Gamma & 0 & 0 & -v_1 v_2 & -v_1
\end{pmatrix},
\]

whenever the Miura maps (4) and (7) are satisfied. In terms of \( q_i \) and their derivatives, the Miura maps can be rearranged to give:

\[
\frac{2}{3} (v_{1x} - v_1^2) = q_1^2 + \frac{1}{3} q_2^2 - \frac{2q_{1xx}}{3q_2} + \frac{2}{3} k q_2^{-4} ,
\]

\[
\frac{2}{3} (v_{2x} + v_2^2) = q_2^2 + \frac{1}{3} q_1^2 - \frac{2q_{1xx}}{3q_1} + \frac{2}{3} k q_1^{-4} .
\]

We can then write \( v_1 \) and \( v_2 \) in terms of \( q_1 \) and \( q_2 \):

\[
v_1 = -\frac{q_{2x}}{q_2} + \frac{a}{q_2} \quad \text{and} \quad v_2 = \frac{q_{1x}}{q_1} + \frac{b}{q_1},
\]

where \( a^2 = -k \) and \( b^2 = -l \). We can similarly write \( v_i \) in terms of \( Q_1 \) and \( Q_2 \):

\[
v_1 = \frac{1}{2} Q_2 + \frac{Q_{1x}}{Q_1} + \frac{c}{2 Q_1^2} , \quad \text{and} \quad v_2 = \frac{1}{2} Q_2 - \frac{Q_{1x}}{Q_1} - \frac{c}{2 Q_1^2} ,
\]

where \( c = 8(b - a) \).

We use these formulae, together with the Miura maps (4) and the definitions of the canonical coordinates, to explicitly write down the canonical transformation:

\[
Q_1 = 2 \Upsilon^+, \quad P_1 = \left( -\frac{p_1}{q_1} - \frac{p_2}{q_2} + \frac{a}{q_2} - \frac{b}{q_1} \right) \Upsilon^{1/2} - \frac{c}{4} \Upsilon^{-1/2} ,
\]

\[
Q_2 = \frac{p_1}{q_1} - \frac{p_2}{q_2} + \frac{a}{q_2} + \frac{b}{q_1}, \quad P_2 = q_1^2 - q_2^2 + \frac{p_2^2}{q_2} - \frac{p_1^2}{q_1} - \frac{2 a p_2}{q_2} + \frac{2 b p_1}{q_1} + \frac{a^2}{q_2} - \frac{b^2}{q_1},
\]

and its inverse:

\[
q_1 = \frac{1}{2} \Gamma^{1/2} , \quad p_1 = \frac{1}{2} \left( \frac{1}{2} Q_2 - \frac{P_1}{Q_1} - \frac{c}{2 Q_1^2} \right) \Gamma^{1/2} - \frac{b \Gamma^{-1/2}}{2},
\]

\[
q_2 = \frac{1}{2} \Gamma^{-1/2} , \quad p_2 = -\frac{1}{2} \left( \frac{1}{2} Q_2 + \frac{P_1}{Q_1} + \frac{c}{2 Q_1^2} \right) \Gamma^{1/2} + 2 a \Gamma^{-1/2},
\]

where

\[
\Upsilon = q_1^2 + q_2^2 + \frac{2}{q_1 q_2} \left( p_1 p_2 + \frac{b p_2}{q_1} - \frac{a p_1}{q_2} - \frac{a b}{q_1 q_2} \right) ,
\]

\[
\Gamma_\pm = \pm 2 P_2 + Q_2^2 + \frac{1}{2} \frac{4 P_2^2}{Q_1^2} + \frac{4 Q_2 P_1}{Q_1} + \frac{2 c Q_2}{Q_1} - \frac{4 c P_1}{Q_1} - \frac{c^2}{Q_1}.
\]
The original parameters $k,l,K,L$ are related to $a$ and $b$ by:

$$k = -a^2, \quad l = -b^2, \quad L = -16(b - a)^2, \quad K = -2(a + b).$$

**Remark 3** When $a = b = 0$, the case $1:6:1$ separates in Cartesian co-ordinates and the canonical transformation between cases $1:6:1$ and $1:6:8$, constructed in this paper, gives the separation co-ordinates for $1:6:8$ obtained in [4] by a Painlevé approach. Some sort of Lax representation is given in [6], but this involves the square root of $k_{(f,g)}$ and its derivatives, so does not give a clear derivation of the equations of motion as a consequence of the Lax equations.

**4 Conclusions**

The whole essence of our approach is to give a systematic connection between completely integrable, finite dimensional Hamiltonian systems and integrable (in the soliton theory sense) PDEs. This result is, itself, interesting but, more importantly, gives a systematic construction of a matrix Lax pair for the finite dimensional system and (as a bonus) gives a straightforward construction of canonical transformations (via gauge transformations in the PDE framework).

We just presented the most interesting cases ([3] and [4]) of (generalisations of) Hamiltonian ([1]) in this letter, but the method is quite general. Case ([3]) can be obtained from the KdV hierarchy, since it is just a special case of the Garnier system. Case ([4]) is a special case of a class studied in [3], associated with energy dependent Schrödinger operators. Case ([5]) can also be obtained by considering a fifth order Lax operator ([1]).

A more complete treatment will be presented elsewhere.

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