Ricci-flat Metrics with $U(1)$ Action and the Dirichlet Boundary-value Problem in Riemannian Quantum Gravity and Isoperimetric Inequalities

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Abstract

The Dirichlet boundary-value problem and isoperimetric inequalities for positive definite regular solutions of the vacuum Einstein equations are studied in arbitrary dimensions for the class of metrics with boundaries admitting a $U(1)$ action. In the case of trivial bundles, apart from the flat space solution with periodic identification, such solutions include the Euclideanised Schwarzschild metrics with an arbitrary compact Einstein-manifold as the base, whereas for non-trivial bundles the regular solutions include the Taub-Nut metric with a $\mathbb{C}P^n$ base and the Taub-Bolt and the Euguchi-Hanson metrics with an arbitrary Einstein-Kähler base. We show that in the case of non-trivial bundles Taub-Bolt infillings are double-valued whereas Taub-Nut and Eguchi-Hanson infillings are unique. In the case of trivial bundles, there are two Schwarzschild infillings in arbitrary dimensions. The condition of whether a particular type of filling in is possible can be expressed as a limitation on squashing through a functional dependence on dimension in each case. The case of the Eguchi-Hanson metric is solved in arbitrary dimension. The Taub-Nut and the Taub-Bolt are solved in four dimensions and methods for arbitrary dimension are delineated. For the case of Schwarzschild, analytic formulae for the two infilling black hole masses in arbitrary dimension have been obtained. This should facilitate the study of black hole dynamics/thermodynamics in higher dimensions. We found that all infilling solutions are convex. Thus convexity of the boundary does not guarantee uniqueness of the infilling. Isoperimetric inequalities involving the volume of the boundary and the volume of the infilling solutions are then investigated. In particular, the analogues of Minkowski’s celebrated inequality in flat space are found and discussed providing insight into the geometric nature of these Ricci-flat spaces.

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1 Introduction

In Riemannian quantum gravity and in the search for holographic dualities relating the bulk gravitational physics to boundary gauge theories one often encounters the Dirichlet problem of finding one or more compact \((n+1)\)-dimensional Riemannian manifolds \((\mathcal{M}, g_{\mu\nu})\) with a given \(n\)-dimensional closed manifold \((\Sigma, h_{ij})\) as the boundary such that \(g_{\mu\nu}\) satisfies the Einstein equation with a cosmological constant term or appropriate matter fields and such that the metric \(h_{ij}\) is induced on \(\partial \mathcal{M} \equiv \Sigma\). A classical solution \((\mathcal{M}, g_{\mu\nu})\) is then referred to as an infilling geometry for the boundary \(\Sigma\). Such solutions provide semi-classical approximations to the path integral and are the starting point for quantum computations.

For a given boundary with a metric finding all infilling solutions, however, is a formidable complex task even in the case of pure gravity. However, certain simplifying features usually arise from physical principles which make the problem more tractable. One often assumes \(h_{ij}\) to have a high degree of symmetry and invariant under the action of some Lie group \(G\) and/or to have other simplifying features. Also, possible infilling geometries are often restricted to the class of cohomogeneity one manifolds under the proper action of \(G\), i.e., \(\dim(\mathcal{M}/G) = 1\). The principal orbits of such a solution will then share the topology and symmetry of \(\Sigma\). However, one can consider metric which are cohomogeneity one under the group action of \(G' \subseteq G\) provided \(G'\) expands to \(G\) on \(\Sigma\). For example one can fill in an \(SU(2) \times SU(2)\)-invariant \(S^3\) boundary with biaxial Bianchi-IX metrics whose principal orbits are in general \(SU(2) \times U(1)\)-invariant. With only a cosmological constant term and in the absence of matter fields the assumption of cohomogeneity in either case reduces the Dirichlet problem to a set of ordinary differential equations arising from the Einstein equation to be solved subject to the boundary data given by the specification of \((\Sigma, h_{ij})\) and the condition of regularity in the interior. In many cases (for example, when \(G\) or \(G'\) has a sufficient degree of symmetry) the general solution of the set of ordinary differential equations and the manifolds over which they can be extended, completely or partially, are known in advance. The problem is then equivalent to the problem of the isometric embedding of a given manifold \(\Sigma\) into known manifolds subject to the condition of regularity for the part(s) of the manifolds enclosed by \(\Sigma\).

In this paper we study the Dirichlet problem for boundaries which are \(S^1\)-bundles over some compact manifolds. In general relativity such boundaries often arise in gravitational thermodynamics. The classic example is that of the trivial bundle \(\Sigma \equiv S^1 \times S^2\). Manifolds with complete Ricci-flat metrics admitting such boundaries are known; they are the Euclideanised Schwarzschild metric and the flat metric with periodic identification. It is known from the work of York \[34\] that there are in general two or no Schwarzschild solutions depending on whether the squashing (the ratio of the radius of the \(S^1\)-fibre to that of the \(S^2\)-base) is below or above a critical value. When such solutions exist, the solution of the boundary-value problem is given by finding the 4-geometries by solving for the masses of the two black holes as functions of the two radii \[34\]. Among other results presented in this paper, we will show that it is possible to find analytic solutions of the infilling Schwarzschild geometries in arbitrary dimension by using methods not very well-known in the physics community. York's results in 4-dimension extend readily to higher dimensions.

In the case of non-trivial bundles, the simplest example arises in quantum cosmology in which the boundary is a compact \(S^3\), i.e., a non-trivial \(S^1\) bundle over \(S^2\). In the case of zero cosmological
constant, regular 4-metrics admitting such an $S^3$ boundary are the Taub-Nut \cite{23} and Taub-Bolt \cite{25} metrics having zero and two-dimensional (regular) fixed point sets of the $U(1)$ action respectively. These metrics are therefore topologically distinct although their principal orbits share the same topology and symmetry. As we will see later in the paper such an $S^3$-boundary can be filled in with a unique Taub-Nut solution and two Taub-Bolt solutions in general. However, in either case the boundary has to satisfy certain inequalities. Another regular metric with non-trivial $S^1$-bundle boundaries (which are not topologically $S^3$) is the Eguchi-Hanson metric in which case the periodicity of the $S^1$-fibre is half of that in the case of an $S^3$ boundary (hence the boundary is topologically $S^3/\mathbb{Z}_2$). This metric also has a singular orbit, i.e., an $S^2$ bolt. As we will see below such a boundary can be filled in with a unique (or no) Eguchi-Hanson solution depending on its geometric data.

Regular cohomogeneity one Ricci-flat metrics in higher dimensions with principal orbits that admit circle actions, i.e., metrics which provide the generalisations to the four dimensional metrics above, are known \cite{5, 26}. We will discuss them and the conditions for their regularity in detail in Section 2 after describing the four dimensional cases first. As we will see, possibilities proliferate as one goes higher in dimension. Naturally to know how the 4-dimensional picture changes in higher dimensions one seeks a method which avoids details coming from dimensionality. The existence and non existence of infilling solutions and, more importantly, the number of infilling solutions as the boundary data is varied will be discussed in Section 3. As will be shown, despite the form of the metrics being rather complicated functions of the radial coordinate (i.e., the coordinate parametrizing the orbit spaces), it is possible to treat the Taub-Nut and the Taub-Bolt metrics generically. In the case of trivial bundles we have been able to solve for the infilling Schwarzschild geometries in arbitrary dimension. This is described in Section 4. It is possible to find the infilling Eguchi-Hanson metrics as well. However, the explicit solutions for the Taub-Nut and Taub-Bolt infilling metrics can only be found in lower dimensions using ordinary algebraic methods. The higher dimensional solutions are discussed in Section 5 and can be solved using insights provided by the 4-dimensional solutions.

Two classic issues in Riemannian geometry which the Dirichlet boundary-value problem above connects us to are discussed in Section 6. One of them is the question of convexity (i.e., whether the second fundamental form of the boundary which is determined by the infilling geometry has positive eigenvalues or not) of the boundary and its possible ramifications for quantum gravity. The other issue is given by Minkowski’s celebrated isoperimetric inequality which in ordinary language tells us that in flat space for a given surface area the greatest volume enclosed is that of a sphere. We find analogues of Minkowski’s inequalities for all of the above spaces and discuss them in detail.

2 Ricci-flat metrics admitting boundaries with $U(1)$ action

2.1 Four Dimensions

In four dimensions, all Ricci-flat metrics that admit circle actions can be obtained as special cases of the Taub-NUT metric. The Taub-NUT metric is a two-parameter Ricci-flat metric and is invariant under the group action of $G \equiv SU(2) \times U(1)$, i.e., biaxial Bianchi-IX type. The Euclidean metric
is usually written in the following coordinates:
\[
ds^2 = \left( \frac{r^2 - L^2}{\Delta} \right) dr^2 + 4L^2 \left( \frac{\Delta}{r^2 - L^2} \right) (d\psi + \cos \theta d\phi)^2 + (r^2 - L^2)(d\theta^2 + \sin^2 \theta d\phi^2) \tag{2.1}
\]
where \( \Delta = (r^2 - 2Mr + L^2) \) and \( L \leq r < \infty \) and \( M \) are two parameters. \( \theta \) and \( \phi \) are the usual coordinates on \( S^2 \) and \( 0 \leq \psi < 4\pi/k \), \( k \in \mathbb{Z} \), is the coordinate parametrizing the \( S^1 \) fibre. For \( k = 1 \) the period of \( \psi \) is \( 4\pi \) and hence the surfaces of constant \( r \) are topologically \( S^3 \).

The general form of the metric (2.1), however, is only valid for a coordinate patch for which \( \Delta \neq 0 \). In general, \( \Delta \) will have two roots:
\[
r_{\pm} = M \pm \sqrt{M^2 - L^2}. \tag{2.2}
\]
At the roots the metric degenerates to that of a round \( S^2 \), and each such root therefore corresponds to a two-dimensional set of fixed points of the Killing vector field \( \partial / \partial \psi \) and hence are singular orbits. However, for \( M = L \) the roots coincide, i.e., \( r_{\pm} = L \) in which case the fixed-point set is zero-dimensional as the two-sphere then collapses to a point. Such two- and zero-dimensional fixed point sets have been given the names “bolts” and “nuts” respectively [19].

For a bolt to be a regular point of the metric, the metric has to “close” smoothly near it, such that the subspace of \( (r, \psi) \) has the metric of \( \mathbb{E}^2 \). This can happen provided one gives \( \psi \) the appropriate period which is equivalent to imposing the following condition [25]:
\[
\frac{d}{dr} \left( \frac{\Delta}{r^2 - L^2} \right)_{r=r_{\text{root}}} = \frac{1}{2kL}. \tag{2.3}
\]
For a nut the metric (2.1) is regular for \( k = 1 \) and approaches the 4-dimensional flat metric near it (see below). The coordinate \( r \) ranges continuously from the nut or bolt till, in principle, another root of \( \Delta \) or to infinity. In the latter case the metric can be defined over a complete manifold \((\tilde{M}, g_{\mu\nu})\) often called an instanton.

**Self-dual Taub-NUT**

Setting \( L = M \) in (2.1), one obtains Hawking’s Taub-NUT instanton [23]:
\[
ds^2 = \left( \frac{r + L}{r - L} \right) dr^2 + 4L^2 \left( \frac{r - L}{r + L} \right) (d\psi + \cos \theta d\phi)^2 + (r^2 - L^2)(d\theta^2 + \sin^2 \theta d\phi^2). \tag{2.4}
\]
The Riemann curvature tensor of the metric is self-dual – \( R_{\mu\nu\rho\sigma} = \frac{1}{2} \epsilon_{\mu\nu\kappa\lambda} R^{\kappa\eta}_{\rho\sigma} \). As already mentioned, the metric has a nut at \( r = L \) and is regular for \( k = 1 \), i.e., if \( \psi \) has a period of \( 4\pi \); the level-surfaces of the regular metric are therefore topologically \( S^3 \) with a biaxial Bianchi-IX metric on it – a property that makes the metric interesting for cosmology. Another interesting property of the metric is that it is Kähler. In the mathematical literature, because of its many special properties, this metric appears in many different contexts. The metric is asymptotically flat and the complete metric, \( 0 \leq r < \infty \), i.e., the self-dual Taub-NUT or Taub-Nut instanton has the topology of \( \mathbb{R}^4 \). To avoid confusion due to divergent conventions in the literature, we will refer to this metric (and its higher dimensional generalisations to be described in the next section) as Taub-Nut metrics to distinguish them from the Taub-Bolt metrics that have two-dimensional regular fixed-point sets and will reserve the word Taub-NUT for the whole two-parameter family which includes other regular and singular metrics.
Taub-Bolt Metric

The Taub-Bolt [25] metric is the only other regular metric for \( k = 1 \), i.e., has level surfaces that are squashed spheres:

\[
ds^2 = \left( \frac{r^2 - L^2}{r^2 - 2.5Lr + L^2} \right) dr^2 + 4L^2 \left( \frac{r^2 - 2.5Lr + L^2}{r^2 - L^2} \right) (d\psi + \cos \theta d\phi)^2 + (r^2 - L^2)(d\theta^2 + \sin^2 \theta d\phi^2). \tag{2.5}\]

Here \( r \) ranges from \( 2L \) to infinity. The two-dimensional fixed-point set of the Killing vector \( \partial/\partial \psi \) is a regular bolt as one can check from (2.3). This is not self-dual unlike the Taub-Nut metric although it is also asymptotically flat. Due to the bolt, the complete metric has a different topology and is defined over a manifold of topology \( \mathbb{C}P^2 - \{0\} \), i.e., of \( \mathbb{C}P^2 \) with its nut removed.

\( k = 0 \): the Schwarzschild Solution

As first observed by Page [25], for the degenerate case \( k = 0 \), one can obtain the Schwarzschild metric by taking the limit \( k \to 0 \) and \( L \to 0 \) while keeping \( r^+ \) fixed [25]:

\[
ds^2 = \left( 1 - \frac{2m}{r} \right) dt^2 + \left( 1 - \frac{2m}{r} \right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \tag{2.6}\]

Here \( t \in [0, \infty) \) and replaces the \( \psi \) coordinate in the previous two examples. The metric has a bolt singularity at \( r = 2m \) which can be made regular by identifying the coordinate \( t \) with a period of \( 8\pi m \). The radial coordinate \( r \) has the range \([2m, \infty)\) and constant \( r \) slices of the regular metric have the trivial product topology of \( S^1 \times S^2 \). The four-metric therefore has the topology of \( \mathbb{R}^2 \times S^2 \).

\( k = 2 \): the Eguchi-Hanson Metric

The only other regular metric in this family is the Eguchi-Hanson metric [16]. It is obtained by defining \( R^2 = 4(r^2 - L^2) \) and then taking the limit \( L \to \infty \) while keeping \( a^2 = 4(r^2_+ - L^2) \) constant [25]:

\[
ds^2 = \left( 1 - \frac{a^4}{R^4} \right)^{-1} dR^2 + \frac{1}{4} R^2 \left( 1 - \frac{a^4}{R^4} \right) (d\psi + \cos \theta d\phi)^2 + \frac{1}{4} R^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{2.7}\]

The metric is self-dual. However, since \( \psi \) has period \( 2\pi \), the level surfaces are \( S^3/\mathbb{Z}_2 \), i.e., \( \mathbb{R}P^3 \) and hence the metric is asymptotically locally Euclidean and asymptotically looks like \( \mathbb{R}^4/\mathbb{Z}_2 \). The complete metric has the topology of \( T^*(\mathbb{C}P^1) \).

2.2 Higher Dimensional Generalisations

In this section we briefly review the possible generalisations of the four-dimensional metrics discussed above. All of the above metrics are radial extensions of \( U(1) \) bundles fibred over \( S^2 \) – a compact Einstein manifold. The bundles are non-trivial except in the case of Schwarzschild. In higher dimensions we therefore seek Ricci-flat metrics with similar Hopf bundle structure that would reduce to the above metrics in four dimensions. The case of four dimensions is special in that all such metrics are obtained as special cases of the Taub-NUT metric as we have seen in the
previous section. However, this fortuitous situation cannot exist in principle in higher dimensions as cohomogeneity one metrics with principal orbits that are the non-trivial $S^1$ bundles (i.e., the proposed generalisation of the Taub-NUT family) and satisfying the vacuum Einstein equation can only exist in even dimensions whereas the Schwarzschild metric can be generalised in arbitrary dimensions (see below). The higher dimensional generalisation of the Taub-NUT family requires the base manifold to be Kähler (complex projective space for Taub-Nut solutions, see below) and hence none of the metrics will be asymptotically flat or Euclidean, locally or globally. With Schwarzschild in higher dimensions we have more choices for the base manifold than a sphere with the usual round metric on it due to the proliferation of compact Einstein manifolds/metrics in dimensions greater than two. Any compact Einstein manifold would suffice as the base space in this case with regularity at the bolt being achieved only through imposing correct periodicity on the fibre-coordinate. Therefore the Schwarzschild metrics obtained as special cases of higher dimensional Taub-NUT metrics form a subclass of all the possible Schwarzschild metrics in that dimension. Since $S^2$ is the only compact Einstein manifold in two dimensions and is isomorphic to $CP^1$, in four dimensions such a coincidence is possible.

Schwarzschild metric

For the Schwarzschild metric in $(n+1)$ dimensions, one simply replaces the trivial bundle $S^1 \times S^2$ by $S^1 \times M_{n-1}$:

$$ds^2 = \left( 1 - \frac{\mu}{r^{n-2}} \right) dt^2 + \left( 1 - \frac{\mu}{r^{n-2}} \right)^{-1} dr^2 + r^2 ds^2_{n-1},$$

(2.8)

where $\mu$ gives the black hole mass $m$ which for $M_{n-1} \equiv S^{n-1}$ is

$$\mu = \frac{16\pi G m}{(n-1)\text{Vol}(S^{n-1})}.$$  

(2.9)

The bolt singularity at $r^{n-2} = \mu$ can be removed by periodically identifying the coordinate $t$ with a period

$$\beta_r = \frac{4\pi}{n-2} \mu^{\frac{1}{n-2}}.$$  

(2.10)

The coordinate $r$ then takes values from $\mu^{\frac{1}{n-3}}$ to infinity and defines a complete metric over a manifold with $\mathbb{R}^2 \times M_{n-1}$ topology possessing an $(n-2)$-dimensional fixed point set of the Killing vector $d/dt$, i.e., a bolt. For $M_{n-1} \equiv S^{n-1}$ the metric is asymptotically Euclidean and in general for other choices of base manifolds are asymptotically conical with the special case $M_{n-1} \equiv S^{n-1}/\Gamma$, where $\Gamma$ is a discrete subgroup of $SO(n+1)$, for which it is asymptotically locally Euclidean. For a recent discussion on various possibilities of base spaces for Schwarzschild metrics in various dimensions and their ramifications, see [18].

The Taub-NUT Family

The generalised Ricci-flat metrics with principal orbits which are non-trivial $S^1$ bundles were constructed in [5] and independently in [26]. Recently they have been discussed in [31] and in [3]. Such a metric is obtained by adding a radial coordinate to the metric on the $U(1)$ bundle over a compact homogeneous Kähler manifold $M$ of $n$ complex dimensions endowed with an Einstein-Kähler metric
\( R_{ij} = \lambda g_{ij} \) and subjecting the \((2n + 2)\)-dimensional metric to Ricci-flatness. The metric on the bundle is
\[
d s_{\text{bundle}}^2 = R^2 \omega \otimes \omega + d s_M^2, \tag{2.11}\]
where
\[
\omega = d \tau + A \tag{2.12}\]
is a connection on the bundle such that \( dA \) is the the Kähler form on \( M \) with \( \tau \) parametrizing the \( S^1 \) fibre. The quantity \( R^2 > 0 \) is some function on \( M \). The bundle is invariant under the group action \( U(1) \times H \) where \( H \) is the symmetry of the base. However, this is not necessarily the maximal symmetry, as we will discuss later.

The general Ansatz for the \((2n+2)\) dimensional space is then taken by adding a radial coordinate \( r \):
\[
d s^2 = \gamma(r)^2 dr^2 + \beta(r)^2 \omega \otimes \omega + \alpha(r)^2 d s_M^2. \tag{2.13}\]
The Kählerian choice of \( M \) renders the vacuum Einstein equations in the simple form:
\[
2n \beta \left( \frac{\alpha'}{\gamma} \right)' + \alpha \left( \frac{\beta'}{\gamma} \right)' = 0, \tag{2.14}\]
\[
2n \beta \left( \frac{\beta' \gamma - \beta' \alpha'}{\alpha^3} - \frac{\beta' \alpha'}{\beta \gamma} \right) = \alpha \left( \frac{\beta'}{\gamma} \right)' = 0, \tag{2.15}\]
\[
(2n - 1) \left( \frac{\alpha'}{\alpha \gamma} \right)^2 + \frac{1}{\gamma^2} \left( \frac{\alpha' \beta'}{\alpha \beta} \right) + \left( \frac{\alpha'}{\gamma} \right)' + \frac{2 \beta^2}{\alpha^4} - \frac{\lambda}{\alpha^2} = 0. \tag{2.16}\]
Adding the first two equations and choosing the coordinate gauge in the form
\[
\gamma \beta = c L \geq 0, \tag{2.17}\]
one finds
\[
\alpha^2 = c (r^2 - L^2). \tag{2.18}\]
With this explicit form of \( \alpha(r) \), \( \beta(r) \) is given by the integral:
\[
\beta^2 = c \lambda r L^2 (r^2 - L^2)^{-n} \left( \int_L^r \frac{(s^2 - L^2)^n}{s^2} ds - C \right), \tag{2.19}\]
where \( C \) is an integration constant. Note that the gauge (2.17) is slightly superfluous. Looking at (2.13), it is easy to see that \( c \) appears as an overall multiplicative factor. We could therefore have absorbed \( c \) into \( L \). However, writing the gauge in this form would help us recover the Taub-Nut and Taub-Bolt metrics in their familiar forms by just setting \( c \) to a constant value and setting the correct value for \( \lambda \) without having to redefine \( L \), as we will see below.

The above considerations are local and do not prescribe any periodicity on \( \tau \). As shown in [26], to extend the above metric globally over a manifold, as we do next, \( \tau \) is required to have a period of
\[
\Delta \tau = \frac{4 \pi p}{| \lambda | k}, \tag{2.20}\]
\footnote{Note that the third term of this equation has a typo in [5].}
where $k$ is a positive integer unrestricted as yet, and $p$ is a non-negative integer such that the first Chern class of the tangent bundle on $H_2(M, \mathbb{Z})$ is divisible by $p$. One can now obtain complete positive definite metrics provided one removes the possible singularities arising from the fixed-points of the Killing vector $\partial/\partial \tau$.

**Taub-Nut**

The fixed point set of the Killing vector $\partial/\partial \tau$ would be zero dimensional if both $\alpha$ and $\beta$ go to zero at $r = L$. This requires setting $C = 0$ and $\tilde{r} = L$ in (2.19). As discussed in [5, 26], this will be a regular nut provided the metric near the nut approaches the flat metric. This is only possible if one choose the base manifold to be $\mathbb{C}P^n$ with the Fubini-Study metric on it. The principal orbits are then spheres via the standard Hopf fibration of $S^{2n+1}$ and hence near $r = L$ the metric approaches the flat metric on $\mathbb{R}^{2n+2}$, i.e., $d\rho^2 + \rho^2 d\Omega^{2n+1}$. Note that for any other choice of compact Kähler base space the nut-singularity cannot be removed. The period of $\tau$ is $4\pi(n+1)/\lambda$ as $p = (n+1)$ for $\mathbb{C}P^n$.

By setting $\lambda = 2$ and $c = 2$, note that the four dimensional self-dual Taub-Nut metric Eq. (2.4) can be reproduced. In higher dimensions the solutions are higher order polynomials in $r$:

$$\beta_6^2 = \frac{4L^2}{3}(r - L)(r + 3L)^2,$$

(2.22)

$$\beta_8^2 = \frac{4L^2}{5}(r - L)(r^2 + 4rL + 5L^2),$$

(2.23)

$$\beta_{10}^2 = \frac{4L^2}{35}(r - L)(5r^3 + 25r^2L + 47rL^2 + 35L^3)/(r + L)^4.$$

(2.24)

$$\beta_{12}^2 = \frac{4L^2}{63}(r - L)(7r^4 + 42r^3L + 102r^2L^2 + 122rL^3 + 63L^4)/(r + L)^5.$$

(2.25)

One can integrate

$$\beta^2 = c\lambda rL^2(r^2 - L^2)^{-n} \left( \int_L^r \frac{(s^2 - L^2)^n}{s^2} ds \right),$$

(2.26)

to obtain

$$\beta^2 = \frac{c\lambda rL^2}{(r^2 - L^2)^n} \left( \frac{L^2}{\sqrt{\pi}} \Gamma(n + 1) \Gamma\left( \frac{1}{2} - n \right) + \left( \frac{1}{r^2} \right)^{\frac{1}{2} - n} \frac{1}{2n - 1} _2F_1\left[ \frac{1}{2} - n, -n, \frac{3}{2} - n, \frac{L^2}{r^2} \right] \right).$$

(2.27)

However, this expression, while exact, consists of two terms coming from the two limits of the integral (in which the value of the integral at the lower limit has been simplified using Gamma functions) and hence is not very useful or illuminating for practical purposes. However, it is possible to express $\beta^2$ as a single expression which captures the simple product form of Eqs. (2.22) - (2.25). For this we will have to wait until the next section where its utility will also be demonstrated.

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2Note that with this choice, the Fubini-Study metric on $\mathbb{C}P^1$ is

$$ds_{\mathbb{C}P^1}^2 = \frac{1}{2}(d\theta^2 + \sin \theta^2 d\phi^2).$$

(2.21)
For $r \geq L$ the integral $\int_{L}^{r} \frac{(s^2-L^2)^n}{s^2} \, ds$ is a monotonically increasing function of $r$ starting from zero. Any positive value of $C$ can therefore be matched with a unique $r = r_b$ 31. Therefore, for $C > 0$, we can replace Eq. (2.19) with

$$\beta^2 = c\lambda rL^2(r^2 - L^2)^{-n} \left( \int_{r_b}^{r} \frac{(s^2 - L^2)^n}{s^2} \, ds \right). \tag{2.28}$$

This automatically guarantees that $r_b > L$.

The $2n$ dimensional set of fixed point of $\partial/\partial \tau$ at $r = r_b$ forms a bolt – a singular orbit. However, this will be a regular point of the metric provided the $(\tau, r)$ sub-plane looks like an $E^2$ at $r = r_b$. We now expand (2.28) in powers of $(r - r_b)$:

$$\beta^2 = \frac{c\lambda L^2}{r_b} (r - r_b) + \text{higher order terms.} \tag{2.29}$$

Recalling that $\tau$ should have a period $4\pi p/|\lambda| k$, the sub-space of $(\tau, r)$ would be flat if the bolt is located at

$$r_b = pL/k. \tag{2.30}$$

Obviously this requires $k (\in \mathbb{Z})$ to be less than $p$. Therefore there are $(p - 1)$ bolt type solutions for any given Einstein-Kähler base.

For $\mathbb{C}P^n$ (therefore $k > (n + 1)$)

$$r_b = (n + 1)L/k. \tag{2.31}$$

Note that only for $k = 1$ is the asymptotic behaviour of the solution similar to that of the Taub-Nut solution. Other $(n - 1)$ bolt solutions, which do not have any analogues in 4-dimensions, though regular, do not have the same asymptotic behaviour. For $k = 1$, the bolt appears at $r_b = 2L$ in four dimensions, $r_b = 3L$ in six dimensions, i.e., at $r_b = (n + 1)L$ in $(2n + 2)$ dimensions. We here mention explicit solutions for $k = 1$, with subscripts indicating the dimensions as before:

$$\beta_6^2 = \frac{4}{3} \frac{(r - 3L)(r + L)}{(r - L)^2}, \tag{2.32}$$

$$\beta_8^2 = \frac{1}{5} \frac{(r - 4L)(4r^5 + 16r^4L + 44r^3L^2 + 176r^2L^3 + 764rL^4 - 5L^5)}{(r^2 - L^2)^3}, \tag{2.33}$$

$$\beta_{10}^2 = \frac{4}{35} \frac{(r - 5L)(5r^7 + 25r^6L + 97r^5L^2 + 485r^4L^3 + 2495r^3L^4 + 12475r^2L^5 + 62235L^6 + 7L^7)}{(r^2 - L^2)^4}. \tag{2.34}$$

In the rest of the paper, unless otherwise stated, we will use Taub-Bolt to mean any solution for arbitrary value of $k$ (which is, however, less than $p$). Also, it is important to note that for the Taub-Nut solutions we require the total $(2n + 2)$-dimensional space to approach flatness at the nut which is only possible if $\mathbb{C}P^n$ is the base. However, in the case of bolt-type solutions, at a bolt the metric is the product of flat $E^2$ and an Einstein manifold of constant radius and hence are regular Ricci-flat solutions for any choice of Einstein-Kähler base. The periodicity of the fibre-coordinate has to be adjusted in this case depending on the value of $p$ which in turn will determine the locations of the bolt in each of the $(p - 1)$ bolt solutions. In general the periodicities are different.
for different choices of base manifolds and the explicit form of $\beta(r)^2$ will be different in each case and for each possible value of $k$. However, Taub-Bolt would be used for all of them in the same way Schwarzschild is used generically irrespective of the choice of base.

Apart from the $(p - 1)$ regular bolt solutions, another regular solution exists for the degenerate case of $k = p$ by taking limits in the same way we have obtained the Eguchi-Hanson metric in the case of four dimensions (see below). Also at the limit $k = 0$ one obtains the Schwarzschild metric. However, these are only a subset of Schwarzschild metrics in even dimensions. The Taub-NUT family being even dimensional cannot offer any Schwarzschild metric in odd dimensions. Therefore one obtains only the sub-class of Schwarzschild metrics with Einstein-Kähler base in any even dimension from the Taub-NUT family and any odd dimensional Schwarzschild metric is precluded automatically.

**Eguchi-Hanson**

For the degenerate case of $k = p$, the $S^1$-fibre has a period of $\frac{4\pi}{n}$. Metrics obtained by taking limits identical to four dimensional case, as described in Sec. 2 (i.e., by defining the coordinate $R^2 = \lambda(r^2 - L^2)$ and taking $r_b$ and $L$ both to infinity while keeping $\lambda(r_b^2 - L^2) = a^2$ a finite constant) are regular and give the generalisations to the Eguchi-Hanson metric. They can most conveniently be written by choosing $\lambda = 2(n + 1)$. The $(2n + 2)$-dimensional metric then has the succinct form:

$$ds^2 = \left(1 - \frac{a^{2n+2}}{r^{2n+2}}\right)^{-1}dr^2 + r^2 \left(1 - \frac{a^{2n+2}}{r^{2n+2}}\right) (d\psi + A)^2 + r^2 ds_M^2.$$  \hspace{1cm} (2.35)

The bolt at $r = a$ is regular with $\psi$ having a period of $\frac{4\pi}{n+1}$ for which the principal orbits are $S^{2n+1}/\mathbb{Z}_{n+1}$ if one chooses $\mathbb{C}P^n$ with the Fubini-Study metric as the base. The metric (2.35) is then Einstein-Kähler and was first found by Calabi by abstract geometric methods \[12\] and was later found by directly solving the Monge-Ampère equation in \[17\]. However, $M$ can be any Einstein-Kähler manifold (see \[7\] and \[26\]).

### 3 The Dirichlet Problem: Uniqueness and Non-uniqueness

In the Dirichlet problem one seeks to obtain non-singular solutions $(\mathcal{M}, g_{\mu\nu})$ which fill in a given boundary $(\Sigma, h_{ij})$ such that $\partial \mathcal{M} = \Sigma$ and $g_{\mu\nu}|_{\partial \mathcal{M}} = h_{ij}$. In our case a boundary is an $S^1$ bundle over a compact Einstein manifold with $h_{ij}$ having the form

$$ds_\Sigma^2 = \alpha^2 ds_M^2 + \beta^2 \omega \otimes \omega,$$  \hspace{1cm} (3.1)

where $\alpha$ and $\beta$ – the radii of the base manifold and the $S^1$-fibre respectively – are known quantities and constitute what we will be referring to as the boundary data. The boundary $\Sigma$ is invariant under the group $G \equiv U(1) \times H$ where $H$ is the symmetry group of the base manifold $\mathcal{M}$. However, $G$ is not necessarily the maximal symmetry of $\Sigma$. For example when the boundary is an $S^1$ bundle over $S^2$, $G$ may enlarge from $U(1) \times SU(2)$ to $SU(2) \times SU(2) \sim SO(4)$ depending on the values of $\alpha$ and $\beta$ and the periodicity of the fibre-coordinate. The same is true for the higher dimensional metrics discussed above.
In the Dirichlet problem we seek to find infilling metrics as functions of $\alpha$ and $\beta$. Since we know the Ricci-flat metrics that have principal orbits sharing the topology and symmetry of $\Sigma$, our problem is equivalent to the isometric embedding problem of $\Sigma$ into known manifolds $(\tilde{M}, g_{\mu\nu})$. When such an embedding is possible, compact part(s) of $(\tilde{M}, g_{\mu\nu})$ cut by the hypersurface $(\Sigma, h_{ij})$ constitute a solution to the Dirichlet problem.

3.1 Schwarzschild Metrics

For a boundary $\Sigma \equiv S^1 \times S^2$, the pair $(\alpha, \beta)$ constitutes the canonical boundary data with the interpretation that $\alpha$ represents the radius of a spherical cavity immersed in a thermal bath with temperature $T = \frac{1}{2\pi\beta}$. It is known from the work of York and others [11,34] that for such canonical boundary data, apart from the obvious infilling flat-space solution with proper identification, there are in general two black hole solutions distinguished by their masses which become degenerate at a certain value of the squashing, i.e., the ratio of the two radii $\frac{\beta}{\alpha}$. This can be seen in the following way. First rewrite the Schwarzschild metric (2.6) in the following form:

$$ds^2 = \left(1 - \frac{2m}{r}\right)64\pi^2 m^2 d\tau^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where $t = 8\pi\tau$ such that $\tau$ has unit period. With this definition one can simply read off the proper length – alternatively the radius – of the $S^1$ fibre and that of the $S^2$ base. They are:

$$\alpha^2 = r^2$$

and

$$\beta^2 = 16m^2\left(1 - \frac{2m}{r}\right)$$

It is easy to see that for a given $(\alpha, \beta)$, $r$ is uniquely determined whereas $m$ is given by the positive solutions of the following equation:

$$m^3 - \frac{1}{2} \alpha m^2 + \frac{1}{32} \alpha \beta^2 = 0.$$  

By solving this equation for $m$, the two Schwarzschild infilling geometries are found\(^3\). There are in general two positive roots of Eq.(3.5) provided $\frac{\beta^2}{\alpha^2} \leq \frac{16}{27}$. When the equality holds the two solutions become degenerate and beyond this value of squashing they turn complex. The remaining root of Eq.(3.5) is always negative. Therefore the two infilling solutions appear and disappear in pairs as the boundary data is varied. The explicit solutions are mentioned in [34].

Higher Dimensions

It is not difficult to see that in arbitrary dimensions for $\Sigma \equiv S^1 \times M_{n-1}$, there will in general be two Schwarzschild solutions or no solution. Simply note that the ratio of the two radii as a function of $\rho \equiv r/\mu^{\frac{n-2}{2}} \in [1, \infty)$ is:

$$\frac{\beta^2}{\alpha^2} = \frac{16}{n-2} \rho^{-2} \left(1 - \frac{1}{\rho^{n-2}}\right).$$

\(^3\)Note that the boundary data $\beta$ is the radius of the $S^1$ fibre and not the proper length $\beta_r$, though they are related: $\beta_r = 2\pi\beta$
For any $n$ the general behaviour is the following: $\frac{\beta^2}{\alpha^2}$ starts from zero and grows monotonically to a maximum value and then decreases and approaches zero asymptotically at infinity, as in Figure 1. The maximal value of the squashing depends on the number of dimensions only and can be found by differentiating Eq. (3.6). The condition on the admissible boundary data in $(n+1)$ dimensions therefore reads:

$$\frac{\beta^2}{\alpha^2} \leq \left(\frac{1}{2}\right)^{(n-2)^{-1}} \left(\frac{1}{2n}ight)^{(n-2)^{-1}} - \left(\frac{1}{2n}ight)^{(n-2)^{-1}}.$$  

(3.7)

Any squashing satisfying the inequality will occur for two values of $\rho$ corresponding to two distinct values of $\mu$. If the squashing of a given data $(\alpha, \beta)$ exceeds this value, there will be no positive solution for $\mu$ and hence no real infilling Schwarzschild solution can be found with positive mass. At the equality of (3.7), the two solutions are degenerate as in four dimensions.

Figure 1: Schwarzschild solutions: below a critical-value of the squashing $(\frac{\beta^2}{\alpha^2})$ there are always two distinct infilling Schwarzschild solutions which become degenerate at the critical value. The critical value and the exact shape of the curve depend on dimension.

### 3.2 Eguchi-Hanson in Arbitrary Dimension

For the Eguchi-Hanson metric (2.35) in $(2n+2)$ dimensions, it is fairly straightforward to check that the infilling solutions are unique. The squashing

$$\frac{\beta^2}{\alpha^2} = 1 - \left(\frac{1}{\rho}\right)^{2n+2}$$  

(3.8)

as a function of $\rho \equiv \frac{\xi}{\alpha} \in [0, \infty)$, increases monotonically from zero and approaches unity as $\rho \to \infty$. For a given boundary data $(\alpha, \beta)$, the infilling solution is given trivially by

$$r = \alpha$$  

(3.9)
and
\[ a = \alpha^{2n+2} \left( \frac{\alpha^2 - \beta^2}{\alpha^2} \right)^{\frac{1}{2n+2}}. \tag{3.10} \]
Solutions will exist for any data \((\alpha, \beta)\) provided \(\alpha > \beta\) which is apparent from the form of the metric (2.35).

### 3.3 Taub-Nut and Taub-Bolt metrics

#### 3.3.1 Four Dimensions

For a boundary \(\Sigma\) which is a twisted \(S^1\) bundle over \(S^2\) and endowed with the metric
\[ ds^2 = \beta^2(d\psi + \cos \theta d\phi)^2 + \alpha^2(d\theta^2 + \sin \theta d\phi^2), \tag{3.11} \]
where \(\psi\) has a period \(4\pi\), a Taub-Nut infilling would be possible if the following system of equations admits real solutions for \(r\) and \(L\):
\[ \alpha^2 - r^2 + L^2 = 0, \tag{3.12} \]
and
\[ \beta^2 - 4L^2 \frac{r - L}{r + L} = 0. \tag{3.13} \]
This has been solved in detail in [2] as a special case of self-dual Taub-NUT-(anti)de Sitter solutions. The important observation is that this system admits the discrete symmetry \((r, L) \leftrightarrow (-r, -L)\), which inspires one to make the following substitution:
\[ x = r + L, \quad y = r - L. \tag{3.14} \]
Eqs. (3.12)–(3.13) then transform into the following two bivariate equations (for \(x\) and \(y\)):
\[ xy - \alpha^2 = 0 \tag{3.15} \]
and
\[ yx^2 - 2xy^2 + y^3 - \beta^2 x = 0. \tag{3.16} \]
The discrete symmetry, \((r, L) \leftrightarrow (-r, -L)\) is now preserved in \((x, y) \leftrightarrow (-x, -y)\). Substituting for \(x\) (\(y\)) one can obtain a univariate equation in \(y \) (\(x\)):
\[ y^4 - 2\alpha^2 y^2 + \alpha^2(\alpha^2 - \beta^2) = 0. \tag{3.17} \]
This is quadratic in \(y^2\). The four solutions for \(y\) therefore will appear in pairs with opposite signs. Since \(\alpha^2\) is positive, this implies that the four corresponding solutions of \((x, y)\) are of the form \((x_1, y_1), (x_2, y_2)\) and \((-x_1, -y_1), (-x_2, -y_2)\). Since the set of solutions have the symmetry \((x_1, y_1) \to (-x_1, -y_1)\), by applying the transformation \((x, y) \to (-x, -y)\), we would obtain no new solutions for \((x, y)\) (hence for \((r, L)\)) and would reproduce the same set. Therefore there are four points in the \(C^2\) plane where the two polynomials (3.15)–(3.16) meet, i.e., four solutions for \((r, L)\) which are related by the reflection symmetry \((r, L) \leftrightarrow (-r, -L)\).
For convenience, let us rewrite \( y^2 = z \). Then the solutions are

\[
\begin{align*}
    z_1 &= \alpha^2 - \alpha \beta, \\
    z_2 &= \alpha^2 + \alpha \beta
\end{align*}
\]  

(3.18)

If \( \beta > \alpha \) then \( z_1 \) is negative which means that \( y \) will be imaginary and so will \( x \) by virtue of equation Eq.(3.15). For either of \( z_1 \) or \( z_2 \) to give a real solution, we must have \( r > L \). For this to happen, one can show trivially, using Eq.(3.14), that one requires

\[
\alpha^2 > z_i.
\]  

(3.19)

This can only be satisfied by \( z_1 \). Therefore for any boundary data \( \alpha > \beta \), one can fill in with a unique Taub-Nut metric. For the same boundary, \((\Sigma, h_{ij})\), Taub-Bolt infilling solutions are possible if the following system can admit real solutions for \( r \) and \( L \) such that \( r > 2L \):

\[
\alpha^2 - r^2 + L^2 = 0,
\]  

(3.20)

and

\[
\beta^2 - 4L^2 \frac{r^2 - 2.5Lr + L^2}{r^2 - L^2} = 0.
\]  

(3.21)

However, in this case the solution is not unique. This can be seen by noting the behaviour of the squashing as a function of the variable \( \rho = r/L \):

\[
\frac{\beta^2}{\alpha^2} = 4 \frac{\rho^2 - 2.5\rho + 1}{(\rho^2 - 1)^2}
\]  

(3.22)

At \( \rho = 2 \), this is zero and as \( \rho \) is increased it increases to reach a maximum and then decreases to reach zero at infinity. The maximum value of \((\text{squashing})^2\) is \(\left(\frac{3}{8} \cdot \frac{3^{2/3}}{\sqrt{3}} - \frac{9}{8} \cdot \sqrt{3} + 1\right) \sim 0.1575\) and occurs approximately at \( \rho \sim 2.851708133 \). For any boundary data \((\alpha, \beta)\) for which squashing is below this limit, the \( \Sigma \) can be filled in with two Taub-Bolt solutions. Note that Eq.(3.22) can be solved exactly for \( \rho \) giving exact infilling geometries as functions of the boundary data \((\alpha, \beta)\). We do not, however, write the solutions here as they are unwieldy and not particularly illuminating.

One must have noted a fundamental difference between the Taub-Nut/Bolt, Eguchi-Hanson cases and the Schwarzschild case. In the Taub-Nut/Bolt and Eguchi-Hanson cases the \( S^1 \) fibre of the boundary \( \Sigma \) is required to have a prescribed period in order to afford regular infilling solutions. Any boundary with a different periodicity of the fibre-coordinate hence cannot be filled in with regular Taub-Nut/Bolt solutions irrespective of the value of boundary data. In the case of Schwarzschild, however, the periodicity of the \( S^1 \)-fibre is determined by the masses of the infilling black holes. Because of the product topology of the boundary, the periodicity of the \( S^1 \)-fibre of the boundary in this case is “arbitrary” in the sense that one can always redefine the coordinate parametrizing the fibre as we did in Section 3.1 and it is meaningful to talk about its periodicity only after the solutions have been found.

### 3.3.2 Higher Dimensions

One can similarly treat the filling in problem in higher dimensions with Taub-Nut and Taub-Bolt metrics and try to solve them algebraically for \( r \) and \( L \) using the explicit forms of \( \alpha(r) \) and \( \beta(r) \)
It is possible to reduce the problem to a one-variable one by treating the squashing as a function of the variable \( r \) as we have done for the Taub-Bolt and Schwarzschild solutions (which equally could have been adopted for the Taub-Nut in Section 3.3.1). However, the corresponding equations soon become too difficult to tackle algebraically with ordinary methods. We will return to the issue of explicit solutions later in Section 5. For the present purpose, i.e., to see whether infilling solutions are unique or non-unique and for what ranges of the boundary data they exist, we need to take a more general approach and avoid a case-by-case study as below. This enables us to make the following statement for Taub-Nut and Taub-Bolt infilling geometries in arbitrary dimensions.

**Theorem:** For a non-trivial \( S^1 \)-bundle over a compact Kähler manifold of \( n \) complex dimensions with metric (3.1), possible Taub-Nut and Eguchi-Hanson infillings are unique and possible Taub-Bolt infilling, irrespective of the periodicity of coordinate parametrizing the \( S^1 \)-fibre, is double-valued.

**Proof:** We have already shown that the Eguchi-Hanson infilling is unique and exists for any boundary data \((\alpha, \beta)\) provided \( \alpha > \beta \).

For the case of Taub-Nut and Taub-Bolt, perhaps the most straightforward way is to use a combination of polynomials and differential equations. Denote the ratio \( \beta(r)^2 / (\alpha(r)^2) \) by \( S(r) \):

\[
S(r) = \frac{2rL^2}{(r^2 - L^2)^{n+1}} \int_{r_b}^{r} \frac{(s^2 - L^2)^n}{s^2} ds,
\]

where we have set \( \lambda = 2 \) without any loss of generality (see Comment 2 below). Recall that \( r_b = L \) for Taub-Nut in arbitrary dimension and that \( r_b = pL/k \) for regular Taub-Bolt solutions where \( k \) is an integer and less than \( p \).

Rescaling \( r \) by \( L \), one obtains

\[
S(\rho) = \frac{2\rho}{(\rho^2 - 1)^{n+1}} \int_{\rho_b'}^{\rho} \frac{(s^2 - 1)^n}{s^2} ds,
\]

where \( \rho_b' \equiv r_b/L \) and \( \rho = r/L \). We now obtain a first order differential equation for \( S(\rho) \) from (3.24):

\[
\rho \left( \rho^2 - 1 \right) S'(\rho) + \left( \rho^2(2n + 1) + 1 \right) S(\rho) - 2 = 0,
\]

where prime denotes differentiation with respect to \( \rho \). Note that this equation is inhomogeneous, first order and linear and its solution is unique for any boundary data \((S(\rho), \rho)\). More importantly, it is *non-autonomous*, unlike the Einstein equations we started with. This latter property, though not desirable in most cases, will greatly facilitate our understanding of the boundary-value problem under consideration.

The apparent singular point \( \rho = 1 \) of Eq. (3.25) is a regular singular point as one can guess. We will discuss it further when we deal with the nut-case. For the purpose of exposition we start with the Bolt case.
Bolt case

At a bolt, i.e., at $\rho = r_b'$ (where $r_b' > 1$), $S(\rho)$ is zero trivially as $\beta = 0$ ($\alpha$ is nonzero). (The integral
\[
\int_{r_b'}^{\rho} \frac{(s^2 - 1)\alpha}{s^2} ds = F[-n, -\frac{1}{2}, -\frac{1}{2}, \rho^2] - F[-n, -\frac{1}{2}, -\frac{1}{2}, r_b'^2]
\] (3.26)
has a factor $(\rho - r_b')$ for arbitrary $n$, as can be checked by expanding and factoring the (finite) hypergeometric series making the squashing zero at the bolt.)

Since $S(\rho)$ is zero at $\rho = r_b'$, Eq.(3.25) implies immediately that $S'(\rho) > 0$ and hence $S(\rho)$ will grow. It will continue to grow monotonically till it reaches the value where $(\rho^2(2n + 1) + 1) S(\rho) - 2 = 0$ – which is an extremum. The second derivative at the extremum (or at any extremum),
\[
S''(\rho) = -\frac{4(2n + 1)}{(\rho^2 - 1)(2\rho^2n + \rho^2 + 1)},
\] (3.27)
is negative. So $S(\rho)$ starts decreasing and we have a “hump”.

Since the second derivative at any extremum is negative for any $\rho > 1$ irrespective of the initial data, a minimum can never occur and hence $S(\rho)$ will decrease monotonically, i.e., $S'(\rho)$ will always be negative after the maximum. This in turn implies that $S(\rho)$ can never be negative, i.e., the curve cannot cross the $\rho$-axis because that would violate Eq.(3.25). (This physically corresponds to the obvious fact that $\alpha(r)$ and $\beta(r)$ are positive.)

![Figure 2: Taub-Bolt solutions: below a critical-value of the squashing ($\frac{\beta^2}{\alpha^2}$) there are always two distinct infilling Taub-Bolt solutions which become degenerate at the critical value.](image)

As $\rho \to \infty$, the only possibility for $S(\rho)$ is therefore to be asymptotic to some constant value. For $\rho$ large Eq.(3.25) can be approximated by:
\[
S'(\rho) + \frac{(2n + 1)}{\rho} S(\rho) - \frac{2}{\rho^2} = 0,
\] (3.28)
Figure 3: Taub-Nut and Taub-Bolt solutions: any boundary Σ which can be filled in with Taub-Bolt solutions can necessarily be filled in with a Taub-Nut solution (assuming $\mathcal{P}^n$ and $k = 1$). This is comparable to the case of Schwarzschild and hot flat space in the case $\Sigma$ is a $S^1 \times S^n$.

giving

$$S(\rho) = \frac{2}{(2n-1)^\rho^2} + \frac{c}{\rho^{2n+1}}.$$  

(3.29)

For all $n$ and any value of $c$ (which depends on the initial data) this approaches zero asymptotically, i.e., $S(\rho)$ is an asymptote to the $\rho$-axis. Therefore for any boundary data $(\alpha, \beta)$ for which $\left(\frac{\beta}{\alpha}\right) < \left(\frac{\beta}{\alpha}\right)_{max}$ there will be precisely two infilling Taub-Bolt metrics. Physically, as $r \to \infty$, $\beta$ stabilizes to a constant value while $\alpha$ continues to grow linearly. Their ratio therefore approaches zero asymptotically at infinity.

**Nut case**

Physically $\alpha$ and $\beta$ have similar power law behaviour near the nut and hence $S(\rho)$ approaches a constant value. This can be seen by redefining $s = y + 1$ and hence $S(\rho)$ approaches a constant value. One finds that $S(\rho)_{\rho=1} = \frac{1}{n+1}$ precisely. In fact, using this substitution one can show:

$$\int_1^\rho \frac{(s^2 - 1)^n}{s^2} ds = \int_0^{\rho - 1} \frac{(y^2 + 2y)^n}{(y + 1)^2} dy = \frac{2^n(\rho - 1)^n + 1}{n + 1} \text{AppellF}_1[n + 1, 2, -n, n + 2, 1 - \rho, \frac{1 - \rho}{2}]$$  

(3.30)

where AppellF$_1$ is the Appell hypergeometric function of two variables, here $1 - \rho$ and $\frac{1 - \rho}{2}$, and is equal to unity for $\rho = 1$. Therefore, for the nut case in general one has

$$S(\rho) = \frac{2^{n+1}}{(n + 1)(1 + \rho)^{n+1}} \text{AppellF}_1[n + 1, 2, -n, n + 2, 1 - \rho, \frac{1 - \rho}{2}],$$

(3.31)

which clearly has the value $\frac{1}{n+1}$ at the nut.
At the nut (\( \rho = 1 \)) one therefore has
\[
\rho (\rho^2 - 1) S'(\rho) = 0,
\]
(3.32)
One can verify by direct differentiation of Eq. (3.31) that \( S'(\rho) \) has a factor \((\rho - 1)\) in the denominator and the numerator vanishes smoothly rendering the quantity \( \rho (\rho^2 - 1) S'(\rho) \) smooth at the nut and that it is negative in general near the nut. In fact it is easier to see it in the following way: at \( \rho = 1 + \epsilon \), where \( \epsilon \) is an arbitrarily small positive quantity, \( \rho (\rho^2 - 1) S'(\rho) \) cannot be positive (implying that \( S'(\rho) \) cannot be positive) because this would then mean an increase in \( S(\rho) \) which is not possible since this would make the left hand side of Eq. (3.25) negative definite. Therefore \( S'(\rho) \) has to be negative at \( \rho = 1 \) and \( S(\rho) \) must decrease. Employing the same argument used for the bolt case \( S(\rho) \) cannot have a minimum and will decrease monotonically to approach zero asymptotically. Therefore for any \( \frac{\beta^2}{\alpha^2} < \frac{1}{(1+n)} \) there will be an unique infilling Taub-Nut metric. □

Note that, for the Taub-Nut solution, \( r'_b = 1 \). For the bolt solutions, no assumption has been made about the periodicity of \( \tau \) as the lower limit \( r'_b \) of the integral (i.e., the initial value of \( \rho \)) has been kept arbitrary. The only property used was that \( r'_b > 1 \) which is a necessary condition for the existence of bolts. The infilling bolts solutions will therefore be regular solutions if the fibre-coordinate has a period \( \frac{4\pi p}{k} \). The periodicity of the \( \tau \) coordinate and the base manifold is determined by specifying the boundary \((\Sigma, h_{ij})\). If the periodicity of the fibre-coordinate is such that \( r'_b \neq p/k \), but \( r'_b > 1 \), \( \Sigma \) can still be filled in with two positive definite bolt solutions although they will be singular at the bolt. Our analysis above applies to such possibilities as well. Also note that we do not require the base manifold to be \( \mathbb{C}P^n \) for the Taub-Nut case, although otherwise the solutions will be singular at the nut.

**Note 1** The flow of Eq. (3.25) is similar for all values of \( k \) though they correspond to different geometries. Even when \( r'_b \neq p/k \), the pattern is unchanged as long as \( r'_b > 1 \), though the corresponding solutions are singular at their bolt. However, as one varies \( r'_b (> 1) \) and hits \( r'_b = 1 \) the flow-pattern changes abruptly. This singular, sudden shift in the flow of Eq. (3.25) at \( r'_b = 1 \) encodes the change in the topological character of the infilling solutions.

**Note 2** We have found a new closed form expression for \( \beta(r) \) for the Taub-Nut through the Appell hypergeometric functions (cf Eq. (3.31)):
\[
\beta(r)_{\text{Nut}}^2 = \frac{\lambda c}{(n+1)(r+L)^n} \text{AppellF}_{1}[n+1,2,-n,n+2,1-r,L-r,\frac{L-r}{2L}].
\]
(3.33)
Note that this form explicitly shows how the ratio of \( \beta(r) \) and \( \alpha(r) \) is non-zero at the nut, though they are separately zero. To our knowledge, the above form (3.33) has not been found before. In terms of ordinary hypergeometric functions the closed form expression for \( \beta(r)_{\text{Bolt}}^2 \) consists of two terms, i.e., the difference of the function at the two limits of the integral (2.29) and hence is not particularly illuminating.

**Comment 1** Note that the boundary data for which there is an infilling Taub-Nut solution is restricted by the squashing being less than or equal to \( \sqrt{\frac{1}{n+1}} \); beyond this there will be no nut solution. By setting \( S'(\rho) = 0 \) the upper bound on the squashing \( S_{\text{max}} \) and the value of \( \rho \) at which
it occurs in the case of Taub-Bolt can be found by solving Eq. (3.24) and Eq. (3.25) simultaneously. Eliminating $S_{\text{max}}$, for example, one needs to solve the following equation for $\rho$:

\[
\left(\rho^2(2n+1)+1\right)\frac{2\rho}{(\rho^2-1)^{n+1}}\int_{\rho/k}^{\rho} \frac{(s^2-1)^n}{s^2}ds - 2 = 0,
\]

(3.34)

which, unfortunately, does not provide a simple expression for $\rho$ and consequently for $S_{\text{max}}$. However, following the previous discussions, we know that there will be a unique positive solution to Eq. (3.34) which one can find for specific $n$ and $k$ numerically.

**Comment 2** For a given boundary metric one can choose the cosmological constant $\lambda$ for the base manifold arbitrarily. The above calculations were carried out for $\lambda = 2$. The statements on the limits on squashing for the Taub-Nut and the Taub-Bolt are to be understood in this light. Obviously the choice of $\lambda$ does not affect the above arguments and one can always convert quantities from one choice of $\lambda$ to another by basic algebraic manipulations. The limits of squashing for $\lambda'$, for example, are obtained from those for $\lambda$ by multiplying by $\lambda'/\lambda$.

### 3.4 On-shell action

Before closing this section note that the on-shell action for any infilling solution $(\mathcal{M}, g_{\mu\nu})$ is [20]:

\[
I_E = -\frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^n x \sqrt{h} K
\]

(3.35)

where $K$ is the trace of the second fundamental form $K_{ij} = \frac{1}{2} \frac{\partial h_{ij}}{\partial n}$ and $h_{ij}$ is the metric of the boundary and $\hat{n}$ is the outward unit normal\(^4\). The bulk contribution vanishes so only the boundary term contributes. Thus the boundary term plays a vital role in the Euclidean approach to quantum gravity as was noted in [20]. An analogous boundary term first appeared in the $3+1$ formulation in [35] and also in [28, 29]. For a detailed discussion of the structure of the boundary term see [14].

### 4 Black Holes in a Cavity of Arbitrary Dimension

In this section we discuss finding solutions to the infilling geometries for Schwarzschild in arbitrary dimension for arbitrary boundary data $(\alpha, \beta)$. As mentioned earlier, the periodic boundary conditions have the natural thermodynamic interpretation of a spherical cavity immersed in a heat bath with temperature proportional to the inverse of the radius of the fibre. As we will see below, it is possible to find exact solutions for the two infilling Schwarzschild geometries as analytic functions of the boundary data, or their squashing to be more specific, in all dimensions.

To find the infilling geometries in $(n+1)$ dimensions, we need to solve the analogue of Eq. (3.35) obtained via the method used for four dimensions:

\[
C^n - \alpha^{(n-2)} C^2 + \frac{1}{4} \alpha^{(n-2)} (n-2)^2 \beta^2 = 0,
\]

(4.1)

\(^4\)Note that in this convention the outward normal on the boundary is positive.
where \( \mu \) has been replaced by \( C^{n-2} \) for notational convenience. As this is an equation in \( C^2 \) in odd dimensions, exact Schwarzschild infilling solutions can be found immediately for dimension five:

\[
C_\pm = \frac{1}{\sqrt{2}} \sqrt{\alpha (\alpha \pm \sqrt{\alpha^2 - 4\beta^2})}
\]

(4.2)

giving the two masses

\[
M_\pm = \frac{3}{16} \pi \alpha (\alpha \pm \sqrt{\alpha^2 - 4\beta^2}).
\]

(4.3)

The condition on squashing is then

\[
\frac{\beta^2}{\alpha^2} \leq \frac{1}{2},
\]

(4.4)

and is in agreement with Eq.(3.7).

Similarly, explicit solutions can be found in seven and nine dimensions as functions of the boundary data by means of ordinary methods. In the case of four dimensions, we already know the solutions from the work of York [34]. In dimensions other than these, the degree of (4.2) is five and above. Explicit solutions are therefore not obtainable in general in terms of radicals, as we know from Galois theory. However, as will be shown below, exact analytic solutions are still possible. They involve techniques less well-known to the physics community and hence probably were not found before.

For convenience, we rewrite Eq.(4.1) in the following form:

\[
x^n - x^2 + p = 0,
\]

(4.5)

where \( x \equiv C/\alpha \) and \( p = \frac{(n-2)^2 \beta^2}{4 \alpha^2} \). Note that \( p \) can be positive only for \( 0 \leq x \leq 1 \). This is because \( x \) here is just the inverse of \( \rho \) in Eq.(3.7).

By the fundamental theorem of algebra, Eq.(4.5) will always admit \( n \) complex solutions. As we have seen, two of these will be positive as long as the squashing does not exceed the value given by Eq.(3.7) which translates into the following requirement for Eq.(4.5):

\[
p \leq \left( \frac{2}{n} \right)^\frac{2}{n-2} \left( 1 - \frac{2}{n} \right).
\]

(4.6)

The existence of two positive roots can be reproduced from purely algebraic arguments. For \( n \) odd, there will be at least one real root which will be negative as the product of all roots, \((-1)^n p\), is negative. Remembering Descartes rule of changes of sign (see, for example, [6]), there will be either two other positive roots (double roots counted twice) or no positive roots – meaning all other roots are complex. For \( n \) even there can be up to two positive and two negative roots while the rest are complex; the positivity of \((-1)^n p\) in this case excludes the possibility of having one positive root and one negative root. So there will be either two positive solutions or no positive solutions for \( C \) depending on the value of \( p \).

### 4.1 Solutions

The problem at hand is to find the \( n \) solutions of Eq.(4.5) and identify the two positive roots which are bound to exist for \( p \) obeying (4.6). Things are much simpler than the general \( n \)-th degree equations as Eq.(4.5) belongs to the class of equations

\[
x^n - ax^s + b = 0 \ (n > s),
\]

(4.7)
known commonly as trinomial equations. Trinomial equations have a long history, and was studied by mathematicians starting from Lambert to Ramanujan in both their general and restricted forms in parallel with the general equation of degree \( n \). An elaborate historical account can be found in [9]. A major success came with the work of Birkeland in 1927 which showed that the \( n \)-roots of the general equation of degree \( n \) can be expressed as linear combinations of higher order hypergeometric functions of several variables [10]. As a special case to the general equation of degree \( n \), he showed that the \( n \)-roots of the general equation of degree \( n \) with arbitrary complex coefficients can be solved exactly in terms of hypergeometric functions of order \( n \) of one variable. We have discussed the general solutions in detail in the Appendix. As far as we know, there is no substantial reference to the general solutions of trinomial equations in the literature available in English.

As in the Appendix, the general solutions of (4.7) are given in terms of the variable
\[
\zeta = \frac{n^n}{s^n(n-s)^{n-s}a^n}. \tag{4.8}
\]
The solutions fall into two separate sectors corresponding to \(|\zeta| < 1\) and \(|\zeta| > 1\). The two sectors have different analytic forms of solutions in terms of hypergeometric functions. In our case, \( a = 1, s = 2, b = p \) and hence
\[
\zeta = \frac{n^n}{4(n-2)^{n-2} b^{n-2}}. \tag{4.9}
\]
Since \( p = \frac{(n-2)^2 \beta^2}{4 \alpha^2} \), \( \zeta \) is a quadratic function of squashing.

Recalling that we are interested only in the positive roots of Eq. (4.5), which exist only for values of \( p \) given by (4.6), we need not consider \( \zeta \) corresponding to values exceeding the bound of (4.6). Now \( \zeta \), as a function of \( p \), increases monotonically from zero (for \( p = 0 \)) to its maximal value corresponding to the equality of (4.6). It is straightforward to check that this maximal value of \( \zeta \) is identically unity. We therefore do not need to consider the set of solutions corresponding to \(|\zeta| > 1\) to find the two positive roots at all. Each of the positive roots of (4.5) (and other roots within this bound) can therefore be given by a single analytic expression. This is not specific to Eq.(4.5), however, as \( \zeta \) will be equal to unity when the general trinomial equation \( x^n - ax^s + b = 0 \) has equal roots. The condition for this reads
\[
\frac{s^s(n-s)^{n-s}}{n^n} a^n = b^{n-s}. \tag{4.10}
\]
In our case, the two positive roots become equal at
\[
p = \left(\frac{2}{n}\right)^{\frac{2}{n-2}} \left(1 - \frac{2}{n}\right)
\]
which is precisely the condition above as one can check comparing it with (4.9).

### 4.1.1 Analytic Solutions

The \( n - 2 \) roots \( x_i \)'s of Eq. (4.5) are found from Eq. (A.4) by setting \( \gamma = 1 \) and \( a = 1 \) and \( b = p \):
\[
x_i = \left(e^{\frac{2\pi i}{n-2}}\right)^i \left(F_0(\zeta) + \frac{1}{n-2} \sum_{\kappa=1}^{n-3} \left(e^{\frac{2\pi i\kappa}{n-2}}\right)^{-i\kappa} \mu_\kappa (-p)^\kappa F_\kappa(\zeta)\right) \quad (i = 1, 2, ..., n-2) \tag{4.11}
\]
in which
\[
\mu_\kappa = \frac{1}{\kappa (\kappa - 1)! \left(\frac{1-2\kappa}{n-2} - 1\right)! \left(\frac{1-2\kappa}{n-2} - \kappa\right)!}. \tag{4.12}
\]
The remaining two roots are found from Eq. (A.11):
\[ x_{n-2+i} = \sqrt{p} e^{i\pi \sqrt{-1}} \left( \phi_0(\zeta) + \frac{1}{2} e^{i n\pi \sqrt{-1} p_{n-2}} \phi_1(\zeta) \right) (i = 1, 2). \] (4.13)

The arguments of the function
\[ F_\kappa(\zeta) = F \left( a_{1,\kappa}, \ldots, a_{n-1,\kappa}, a_{n,\kappa}, b_{1,\kappa}, \ldots, b_{n-1,\kappa}, \zeta \right) \] (4.14)
are given by
\[
\begin{align*}
a_{i,\kappa} &= \frac{\kappa}{n-2} + \frac{n-i}{n} - \frac{1}{n(n-2)} (i = 1, 2, \ldots, n), \\
b_{i,\kappa} &= \frac{\kappa}{n-2} + \frac{3-i}{2} - \frac{1}{2(n-2)} (i = 1, 2), \\
b_{i,\kappa} &= \frac{\kappa}{n-2} + \frac{i-2}{n-2} + \frac{\delta_i}{n-2} (i = 3, \ldots, n-1),
\end{align*}
\] (4.15)
where
\[ \delta_i = 0, \text{ when } i < n - \kappa, \quad \delta_i = 1, \text{ when } i \geq n - \kappa. \] (4.16)

The arguments of the function
\[ \phi_\kappa(\zeta) = F \left( d_{1,\kappa}, \ldots, d_{n-1,\kappa}, d_{n,\kappa}, e_{1,\kappa}, \ldots, e_{n-1,\kappa}, \zeta \right) \] (4.17)
are given by
\[
\begin{align*}
d_{i,\kappa} &= \frac{\kappa}{2} + \frac{i-1}{2} + \frac{1}{2n} (i = 1, 2, \ldots, n), \\
e_{i,\kappa} &= \frac{\kappa}{2} + \frac{i}{n-2} + \frac{1}{2(n-2)} (i = 1, 2, \ldots, n-2), \\
e_{i,\kappa} &= \frac{\kappa}{2} + 1 + \frac{i-n}{2} + \frac{\delta_i}{2} (i = n-1),
\end{align*}
\] (4.18)
and \( \delta_i \) is given through (4.16).

**The two masses**

It is not difficult to single out the two positive roots from (4.11) and (4.13). They are simply the \((n - 2)\)-th and \(n\)-th roots. The masses of the two black hole solutions are therefore given by the following simple expressions:
\[ x_+ = F_0(\zeta) + \frac{1}{n-2} \sum_{\kappa=1}^{n-3} \mu_\kappa (-p)^\kappa F_\kappa(\zeta) \] (4.19)
and
\[ x_- = \sqrt{p} \left( \phi_0(\zeta) + \frac{1}{2} p_{n-2} \phi_1(\zeta) \right). \] (4.20)

The expressions (4.19) and (4.20) converge for \( \zeta < 1 \). For the \( \zeta = 1 \) case, corresponding to \( p = (\frac{2}{n})^{\frac{n-2}{2}} (1 - \frac{2}{n}) \), the double-root is much simpler:
\[ x = \left( \frac{2}{n} \right)^{\frac{1}{n-2}}. \] (4.21)

The solutions (4.19) and (4.20) gives us the masses of the two infilling black hole solutions as analytic functions of the boundary data. This enables us to find the corresponding Euclidean
actions and other thermodynamics quantities as analytic functions of the boundary data and sets
the ground for any future study in higher dimensional black holes in a thermal cavity. Note that,
the smaller mass (4.19) will be less than the value of \( x \) given by (4.21) for which the specific heat
capacity
\[
C_A = 4\pi(n - 2) C^{n-1} (1 - x^{n-2}) \left( \frac{n}{2} x^{n-2} - 1 \right)^{-1}
\]
is negative and hence the black hole solution is thermodynamically unstable. The corresponding
negative mode was found numerically in [21]. This solution is therefore an instanton. However, the
larger mass black hole solution is locally thermodynamically stable.

4.1.2 Action and Free energy

Note that Eq.(4.6) gives the critical temperature \( T_c \) – above which the two black hole solutions
exist – to be inversely proportional to the cavity-radius:
\[
T_c = \frac{1}{4\pi} \left( \frac{n}{2} \right)^{\frac{1}{n-1}} \sqrt{n(n-2)} \frac{1}{\alpha}.
\]
Recall that any cavity can be filled in with a unique hot flat space
\[
ds^2 = d\tau^2 + dr^2 + r^2 d\Omega_{n-1}^2
\]
for any temperature \( T \) by giving \( \tau \) the period \( \frac{1}{T} \). For \( T < T_c \), the only classical solution within the
cavity is hot flat space whereas for temperature equal or above \( T_c \) the two black hole solutions add
to the list. For either of the two black hole solutions the Euclidean action (3.35) in the ‘background’
of the periodically identified flat space is given by:
\[
I_E = \frac{1}{8G} \frac{\alpha^{n-1} x \left( n x^{n-2} + 2(n-1) \left( \sqrt{1-x^{n-2}} - 1 \right) \right)}{(n-2)}.
\]
The flat space action has been subtracted so as to make the action zero for zero mass. The actions
for the two black holes can be found by direct substitution of the two positive roots of Eq.(4.5)
in Eq.(4.25). The action is zero for \( x = 0 \) (by virtue of the above subtraction) and it increases
monotonically to reach a maximum and then decreases monotonically to the minimum value
\[
I_{E_{\text{min}}} = \frac{\alpha^{n-1}(1-3n)}{8G(n-2)}
\]
corresponding to \( x = 1 \). This is strictly negative and hence the action passes through zero for some
non-zero value of \( x \). It is fairly straightforward to work out that this zero occurs for
\[
x = \left( \frac{4(n-1)}{n^2} \right)^{\frac{1}{(n-2)}}.
\]
Therefore for
\[
1 > x > \left( \frac{4(n-1)}{n^2} \right)^{\frac{1}{(n-2)}},
\]
the action will always be negative. Note that the values of \( x \) within this range is always greater
than the value of \( x \) given by (4.21) and hence the action will be negative only for the larger mass
black hole and for a restricted range, i.e., if the temperature is sufficiently high enough. Since the free energy is \( F = \beta I_E \), the larger-mass black hole nucleated spontaneously from hot flat space for any temperature above this will be globally thermodynamically stable. For a given cavity of radius \( \alpha \), this temperature can be found using Eq. (4.29):

\[
T_s = \frac{1}{4\pi} \frac{n-2}{\sqrt{\left(\frac{4(n-1)}{n^2}\right)^{\frac{2}{(n-2)}} - \left(\frac{4(n-1)}{n^2}\right)^{\frac{n}{(n-2)}}}} \frac{1}{\alpha}.
\]  

(4.29)

4.1.3 Series Expansions

The expressions (4.19) and (4.20) are exact and can be used for precise calculation using the known properties of the higher order hypergeometric functions (see, for example, [4, 32]). However, in many situations, approximations via series expansions in powers of the parameter are useful along with the exact solutions. Obtaining such series for (4.19) and (4.20) is nontrivial through their direct expansions. However, such series can be obtained directly from (4.7) by use of Lagrange expansion. If

\[
y = a + h\phi(y),
\]

(4.30)

where \( \phi(y) \) a is function of \( y \), the expansion of any function \( f(y) \) of \( y \) is given by

\[
f(y) = f(a) + h(\phi f') + \frac{k^2}{2!}(\phi^2 f')' + \frac{k^3}{3!}(\phi^3 f')'' + \ldots.
\]

(4.31)

There are three fundamental power series for all roots of a general trinomial equations [15], each of them precisely corresponding to the three analytic expressions described in the Appendix, two for \( \zeta < 1 \), and one for \( \zeta > 1 \). Since our solutions lie within the bound \( \zeta \leq 1 \), only two of them will be of relevance. We will not discuss the general expansion for the general trinomial equation (4.7) – interested readers are referred to [15]. However, the method discussed below is the same for any trinomial equation.

Note that for very small \( p \), Eq. (4.5) has two roots approximately lying on a circle of radius \( \sqrt{p} \) and the other \( (n-2) \) roots approximately on a circle of radius 1 in the \( \mathbb{C} \)-plane. Following [15], we rewrite Eq. (4.5) as

\[
y = 1 - p \frac{1}{n-2},
\]

(4.32)

wherein \( y \equiv x^{n-2} \). The Lagrange expansion (4.31) of the function \( f(y) = y^{\frac{1}{n-2}} \) then gives us the required series for (4.19):

\[
x_+ = 1 - \frac{p^1}{n! (n-2)} - \frac{(n+1)p^2}{2! (n-2)^2} - \frac{(n+3)(2n+1)p^3}{3! (n-2)^3} - \frac{(n+5)(2n+3)(3n+1)p^4}{4! (n-2)^4} - \ldots.
\]

(4.33)

Similarly defining \( y \equiv x^2 \) and rewriting Eq. (4.5) as

\[
y = p + y^{\frac{n}{2}},
\]

(4.34)

the Lagrange expansion of \( f(y) = y^{\frac{1}{2}} \) gives us the series for (4.20):

\[
x_- = p^{\frac{1}{2}} + \frac{p^{n-1}}{1! 2^{\frac{n-3}{2}}} + \frac{(2n-1)p^n}{2! 2^{n-1}} + \frac{(3n-1)(3n-3)p^{3n-5}}{3! 2^3} + \frac{(4n-1)(4n-3)(4n-5)p^{4n-7}}{4! 2^4} + \ldots
\]

(4.35)
The other complex/negative roots are obtained by multiplying \((4.33)\) by the \((n - 1)\) roots of \(z^n - 1 = 0\) and multiplying \((4.35)\) by the other root of \(z^2 - 1 = 0\), i.e., by \(-1\) respectively.

Note that as one takes the boundary to infinity keeping the temperature finite, the stable solution disappears and the instanton solution \(x_\text{-}\) survives thus making hot space unstable to the nucleation of a black hole with negative specific heat as was seen in [22]. This happens for any non-zero temperature. Because this solution has a negative specific heat it will be in unstable equilibrium with its thermal environment. Only within a finite cavity is it therefore possible to have a black hole which is thermodynamically stable. However, the situation is different in the presence of a negative cosmological constant as was shown in [8] for four dimensions. Work on higher dimensions are under progress and will be reported in a forthcoming publication.

5 Solving Taub-Nut and Taub-Bolt in Arbitrary Dimension

In this section, we briefly discuss possible analytic solutions for Taub-Nut and Taub-Bolt infilling geometries in higher dimensions. (We have already mentioned the explicit solutions in the case of Eguchi-Hanson metric in arbitrary dimension in the course of proving uniqueness.) We have found solutions for the Taub-Nut and Taub-Bolt solutions in four dimensions although in the case of the latter we did not write down the explicit form as it is not particularly illuminating. Note that higher dimensional Taub-Nut metrics all have the same symmetry \((r,L) \leftrightarrow (-r,-L)\). With the same substitutions used in four dimensions, namely,

\[
\begin{align*}
  x &= r + L, \\
  y &= r - L,
\end{align*}
\]

one can reduce the problem to the study of a univariate equation – a cubic in six dimensions, quartic in eight dimensions, i.e., an equation of degree \((n + 1)\) in \((2n + 2)\) dimensions. The boundary-value problem for Taub-Nut therefore can be solved exactly in up to eight dimensions using radicals. We do not, however, write them here for lack of space. For dimensions ten and above the relevant equations will be quintic and above, and, as already mentioned, solutions in terms of radicals are not possible in general. As discussed in the previous section, one can solve the general equation of degree \(n\) in terms of the higher order hypergeometric functions\(^5\). However, unlike the case of Schwarzschild, where we had a trinomial equation, these will be higher order hypergeometric functions of several variables. Although it is possible in principle to work them out explicitly, the solutions would not be illuminating as in the case of Schwarzschild solutions and therefore we do not attempt to do it here. Rather it is much easier to treat them numerically. A numerical treatment is rather straightforward as we can treat everything as a function of squashing only – a simplification resulting from the vanishing of the cosmological constant term.

6 Convexity of the Solutions and Isoperimetric Inequalities

Often the condition of convexity is applied to eliminate degenerate infilling solutions for a given boundary. For example, an \(S^3\) of 3-radius \(r\) embedded in an \(S^4\) of 4-radius greater than \(r\) divides

\(^5\)Recently this has been done using \(4\)-hypergeometric functions [33].
the $S^4$ into two unequal hemispheres both of which are infilling solutions \textit{a priori}. However, the smaller hemisphere is the one which is convex, i.e., the eigenvalues of the second fundamental form $K_{ij} = \frac{1}{2} \frac{\partial h_{ij}}{\partial n}$ are positive (using the convention that the outward normal $n$ is positive) whereas for the larger hemisphere it is the opposite. Discarding the latter leaves us with one unique infilling solution.

It is fairly straightforward to check explicitly that $K_{ij}$ has positive eigenvalues for both of the Schwarzschild solutions and for the unique Eguchi-Hanson solution. For the Taub-Nut or Taub-Bolt, the second fundamental form can easily be computed in the orthonormal frame:

$$
K_{\hat{1}\hat{1}} = \frac{1}{2} \frac{\partial h_{\hat{1}\hat{1}}}{\partial n} = \frac{\alpha}{L},
$$
$$
K_{\hat{i}\hat{i}} = \frac{1}{2} \frac{\partial h_{\hat{i}\hat{i}}}{\partial n} = \frac{\beta L}{4} (\beta^2)^\prime,
$$

where $\hat{i} > 1$. Since the scale factors, $\alpha(r)$ and $\beta(r)$ both increase monotonically (for any value of $L$) with $r$, both $K_{\hat{1}\hat{1}}$ and $K_{\hat{i}\hat{i}}$ are strictly positive. Therefore the unique infilling Taub-Nut solution and the two Taub-Bolt infilling solutions are all convex without imposing any further restrictions on the boundary data than those needed for the solutions to exist.

### 6.1 Lower Bound to the Action

Since all the solutions are convex, it follows immediately from a theorem by Reilly \[30\] that the following inequality holds

$$
\frac{A^2}{V^2} > \frac{2n + 2}{2n + 1} \int_{\Sigma} K dA \tag{6.2}
$$

where $A$ and $dA$ are the volume and the volume element of the boundary $\Sigma$ and $V$ is the volume of the $(2n + 2)$-dimensional manifold with the boundary $\Sigma$. Since the boundaries above are convex for any infillings the right hand side of (6.2) is positive for all infilling solutions above. Note that this term is proportional to the Euclidean action (4.25) with a negative proportionality constant and hence the action is negative for all infillings. The inequality (6.2) therefore provides a lower bound for the action of any infilling solution.

### 6.2 Minkowski’s Inequality

For a $(d-1)$-dimensional closed surface $\Sigma$ in $\mathbb{E}^d$, one has the following inequality due to Minkowski (see, for example, \[24\]):

$$
\frac{A^d}{V^{d-1}} \geq d^{d-1} \text{Vol}(S^{d-1}) \tag{6.3}
$$

where $A$ is the $(d-1)$-volume of $\Sigma$, often referred to as “area”, and $V$ is the volume of the $\mathbb{E}^d$ enclosed by it. This states that for a close surface of constant area the greatest volume enclosed is that of a sphere (for which the equality holds). In three dimensions, this gives the celebrated formula

$$
\frac{A^3}{V^2} \geq 36\pi. \tag{6.4}
$$

Naturally, one therefore seeks the Ricci-flat counterparts to this flat-space inequality. This is what we do in the following section with our Ricci-flat metrics which admit a $U(1)$ action. It is not obvious whether such inequalities would obey the bound \[6.3\] in general or under special conditions. As
we will see, the machinery we developed in the preceding sections for finding uniqueness or non-uniqueness of the infilling solutions has already set the ground for such an investigation. As before, we deal with four dimensions first before going to higher dimensions. For the sake of convenience we adopt the terminology of the flat space by denoting the codimension-one volume of Σ as $A$ and often refer to it as the “area”. The term volume and $V$ will be reserved for the infilling solutions.

### 6.2.1 Schwarzschild Solutions

**Four Dimension**

For the Euclidean Schwarzschild metric Eq.(2.6) the area of the 3-surface Σ at radius $r$ is

$$A = r^2 \int_0^{8\pi M} \left(1 - \frac{2M}{r}\right)^{\frac{3}{2}} dt \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi = 32\pi^2 M r^2 \left(1 - \frac{2M}{r}\right)^{\frac{3}{2}}. \quad (6.5)$$

The volume $V$ of the Euclidean Schwarzschild metric bounded by Σ is:

$$V = \int_0^{8\pi M} dt \int_{2M}^{r} s^2 ds \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi = \frac{32}{3}\pi^2 M r^3 \left(1 - \left(\frac{2M}{r}\right)^3\right) \quad (6.6)$$

The dimensionless ratio is therefore

$$S \equiv \frac{A^4}{V^3} = 32 \times 27 \pi^2 M r \left(1 - \frac{2M}{r}\right)^2 \left(1 - \left(\frac{2M}{r}\right)^3\right)^3. \quad (6.7)$$

Rewriting $\frac{2M}{r} \equiv x$, as we did in Section 4, we obtain

$$S(x) = 16 \times 27 \pi^2 \frac{x(1-x)^2}{(1-x^3)^3} \quad (6.8)$$

Recall that for a given hypersurface Σ, there are in general two Euclidean Schwarzschild infilling metrics which are given by the two positive solutions for $x$ satisfying the following algebraic equation

$$x^3 - x^2 + p = 0, \quad (6.9)$$

where $p = \frac{1}{4} \beta^2$. This gives two black hole solutions for a given $(\alpha, \beta)$ which determines the area $A$ of Σ. As we have shown, for the smaller-mass black hole, which is an instanton, $x_- \in [0, \frac{2}{3}]$ and for the larger-mass black hole which is thermodynamically/dynamically stable $x_+ \in [\frac{2}{3}, 1]$.

It is easy to check that $S(x)$ is a monotonically increasing function of $x$ in the interval $[0, 1)$ and blows up at $x = 1$. Therefore the isoperimetric inequality for the lower-mass unstable Schwarzschild infilling solution is

$$S(x_-) \leq S\left(\frac{2}{3}\right), \quad (6.10)$$

i.e.,

$$\frac{A^4}{V^3} \leq 32 \times \left(\frac{27}{19}\right)^3 \pi^2. \quad (6.11)$$

For the larger mass, stable infilling solution this is

$$S(x_+) \geq S\left(\frac{2}{3}\right), \quad (6.12)$$
\[ A^4 \geq 32 \times \left( \frac{27}{19} \right)^3 \pi^2. \]  

(6.13)

**Arbitrary Dimension**

For convenience rewrite the Euclidean Schwarzschild metric in \((n + 1)\) dimensions as

\[
ds^2 = \left( 1 - \left( \frac{C}{r} \right)^{n-2} \right) dt^2 + \left( 1 - \left( \frac{C}{r} \right)^{n-2} \right)^{-1} dr^2 + r^2 ds^2_M. \tag{6.14}
\]

This metric is regular provided \(t\) has a period of \(\frac{4\pi}{n-2} C\). The area of a constant \(r\)-slice \(\Sigma\) is given by

\[
A = \text{Vol}(M) \frac{4\pi C}{(n-2)} r^{n-1} \left( 1 - \left( \frac{C}{r} \right)^{n-2} \right)^\frac{1}{2}, \tag{6.15}
\]

and the volume enclosed by it is

\[
V = \int_0^{\frac{2\pi}{n-2}} dt \int_C^r s^{n-1} ds \text{ Vol}(M) = \text{Vol}(M) \frac{4\pi C}{n(n-2)} (r^n - C^n) \tag{6.16}
\]

giving the ratio \(S(x)\) which is therefore

\[
S(x) \equiv \frac{A^{n+1}}{V^n} = \frac{4\pi n^n}{(n-2)} \text{ Vol}(M) \frac{x(1-x^{n-2})^{n+1}}{(1-x^n)^n}, \tag{6.17}
\]

where \(x \equiv \frac{C}{r}\) as before. Recall that in \((n + 1)\) dimensions \(x\) is the two positive roots of the trinomial equation

\[
x^n - x^2 + p = 0, \tag{6.18}
\]

where \(p = \frac{(n-2)^2 \beta^2}{\alpha^2}\). We found that for the smaller-mass solution \(x_- \in [0, \left( \frac{2}{n} \right)^{\frac{1}{n-2}}]\) and for the larger-mass stable solution \(x_+ \in \left[ \left( \frac{2}{n} \right)^{\frac{1}{n-2}}, 1 \right]\).

It can easily be seen from (6.17) that \(S(x)\) increases monotonically for \(x\) within \([0, 1]\) and blows up at 1. Therefore the isoperimetric inequalities for the two infilling Schwarzschild black holes in \((n + 1)\) dimensions are:

\[
\frac{A^{n+1}}{V^n} \leq \frac{4\pi}{(n-2)} \text{ Vol}(M) \frac{2^{\frac{1}{n-2}} \left( \frac{1}{n} \right)^{\frac{n}{2}} (n-2)^{\frac{n+1}{2}}}{\left( 1 - \left( \frac{2}{n} \right)^{\frac{1}{n-2}} \right)^n}, \tag{6.19}
\]

for the smaller-mass unstable solution and

\[
\frac{A^{n+1}}{V^n} \geq \frac{4\pi}{(n-2)} \text{ Vol}(M) \frac{2^{\frac{1}{n-2}} \left( \frac{1}{n} \right)^{\frac{n}{2}} (n-2)^{\frac{n+1}{2}}}{\left( 1 - \left( \frac{2}{n} \right)^{\frac{1}{n-2}} \right)^n}, \tag{6.20}
\]

for the larger mass stable solution. Note that in the convention used here the metric on the \((n - 1)\) dimensional base manifold satisfies the Einstein equation with a cosmological constant term of \((n - 2)\) and hence for \(\text{Vol}(M) \equiv S^{n-1}\), with the canonical round metric on it, the volume is that of the “unit” \((n - 1)\)-dimensional sphere and is equal to \(2\pi^{\frac{n}{2}} / \Gamma(\frac{n}{2})\). With this choice of base manifold it is easy to see that Minkowski’s inequality (6.3) is always true for the stable larger-mass Schwarzschild solution in any dimension. However, it does not hold in general for the lower-mass, unstable (instanton) solution.

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6.2.2 Taub-Nut

Four dimension

For the Taub-Nut metric (2.4) the area $A$ of a constant $r$-slice is

$$A = 32\pi^2 L (r - L)^{\frac{3}{2}} (r + L)^{\frac{1}{2}}$$  \hspace{1cm} (6.21)

and the volume enclosed within it is

$$V = 2L \int_L^r (s^2 - L^2) ds \int d\psi \wedge \sin \theta \, d\theta \wedge d\phi = \frac{32}{3} \pi^2 L (r - L)^2 (r + 2L).$$  \hspace{1cm} (6.22)

Therefore

$$S(r) \equiv \frac{A^4}{V^3} = 32 \times 27 \pi^2 \frac{(r + L)^2}{(r + 2L)^3}. \hspace{1cm} (6.23)$$

Rewriting $\frac{r}{L} \equiv \rho$ we obtain

$$S(\rho) = 32 \times 27 \pi^2 \frac{(\rho + 1)^2}{(\rho + 2)^3}. \hspace{1cm} (6.24)$$

This is a monotonically decreasing function of $\rho \geq 1$. Therefore for the Taub-Nut space the isoperimetric inequality is

$$\frac{A^4}{V^3} \leq 128 \pi^2.$$  \hspace{1cm} (6.25)

Obviously this does not obey Minkowski’s inequality. In fact this is exactly the opposite of Minkowski’s inequality.

Arbitrary dimension

The higher dimensional Taub-Nut metric has the form

$$ds^2 = \gamma(r)^2 dr^2 + \beta(r)^2 (d\tau + A)^2 + \alpha(r)^2 ds_M^2.$$  \hspace{1cm} (6.26)

In $(2n + 2)$ dimensions the area of the hypersurface $\Sigma$ at $r$ is

$$A = \text{Vol}(M) \beta \alpha^{2n} \int d\tau$$  \hspace{1cm} (6.27)

Following the conventions adopted in Section 2.2, here $\gamma \beta = cL$ and $\alpha^2 = c(r^2 - L^2)$. Recall that $\tau$ has period $\beta \tau = 4\pi (n + 1)/\lambda$ (as $k = 1$ for Taub-Nut), where $\lambda$ is the cosmological constant of the base $\mathbb{C}P^n$. This gives

$$A = \frac{4\pi(n + 1)}{\lambda} \text{Vol}(M) (\sqrt{cL})^{2n+1} \tilde{\beta} (\rho^2 - 1)^n$$  \hspace{1cm} (6.28)

in which $\tilde{\beta} \equiv \beta/\sqrt{cL}$ is a function of $\rho \equiv r/L$. The volume bounded by $\Sigma$ is

$$V \equiv \int d\tau \int_L^r \gamma \beta \alpha^{2n} dr \text{Vol}(M) = (\sqrt{cL})^{2n+2} \times \frac{4\pi(n + 1)}{\lambda} \text{Vol}(M) \int_1^\rho (\rho^2 - 1)^n dp.$$  \hspace{1cm} (6.29)

The ratio is therefore

$$S(\rho) \equiv \frac{A^{2n+2}}{V^{2n+1}} = \frac{4\pi(n + 1)}{\lambda} \text{Vol}(M) \tilde{\beta}^{2n+1} (\rho^2 - 1)^n \left(\frac{(\rho^2 - 1)^n}{\int_1^\rho (s^2 - 1)^n ds}\right)^{2n+1}. \hspace{1cm} (6.30)$$
It is easy to evaluate the integral in the denominator above
\[
\int_1^\rho (s^2 - 1)^n ds = \frac{2^n}{n+1} (\rho - 1)^{n+1} {}_2F_1[1+n, -n, n+2, \frac{1-\rho}{2}]. \tag{6.31}
\]
This gives
\[
\frac{A^{2n+2}}{V^{2n+1}} = 4\pi(n+1) \left( \frac{n+1}{2^n} \right)^2 \Vol(M) \beta^2 \rho^n (\rho^2-1)^n \left( \frac{\rho+1}{\rho-1} \right) \frac{\rho^n (\rho+1)^n}{2F_1[1+n, -n, n+2, \frac{1-\rho}{2}]} \tag{6.32}
\]
Recalling
\[
\beta^2 = \frac{\lambda 2^n \rho (\rho-1)}{(n+1)(1+\rho)^n} \text{Appell} {}_1F_1[n+1, 2, -n, n+2, 1-\rho, \frac{1-\rho}{2}], \tag{6.33}
\]
we obtain after simplification
\[
\frac{A^{2n+2}}{V^{2n+1}} = 4\pi(n+1) \lambda^n \left( \frac{n+1}{2^n} \right)^n \Vol(M) \rho^n (\rho+1)^n \left( \frac{\rho+1}{\rho-1} \right)^{n+1} \frac{\rho^n (\rho+1)^n}{2F_1[1+n, -n, n+2, \frac{1-\rho}{2}]} \tag{6.34}
\]
It is easy to check that the fraction involving the two hypergeometric functions is equal to unity at \( \rho = 1 \) and monotonically decreases with \( \rho \). It falls faster than \( \rho^{n+1} (\rho+1)^{n+1} \) and falls more sharply with increasing (integer) values of \( n \). The isoperimetric inequality is therefore
\[
\frac{A^{2n+2}}{V^{2n+1}} \leq 4\pi(n+1) \lambda^n \left( \frac{n+1}{2^n} \right)^n \Vol(M) \rho^n (\rho+1)^n \tag{6.35}
\]
The volume of \( \mathcal{C}^n \) with the Fubini-Study metric satisfying the Einstein equations with cosmological constant \( 2(n+1) \) is \( \Vol(S^{2n+1})/2\pi(\equiv \pi^n/n!) \) and hence
\[
\frac{A^{2n+2}}{V^{2n+1}} \leq 2^{2n+1} (n+1)^{2n+1} \Vol(S^{2n+1}) \tag{6.36}
\]
which is exactly the opposite of Minkowski’s inequality \( \text{(6.3)} \) in \( (2n+2) \) dimensions:
\[
\frac{A^{2n+2}}{V^{2n+1}} \geq 2^{2n+1} (n+1)^{2n+1} \Vol(S^{2n+1}) \tag{6.37}
\]
Therefore for the Taub-Nut metric in arbitrary dimension the analogue of the Minkowski’s inequality is exactly the opposite of the Minkowski’s inequality in flat space. Since the Taub-Nut metrics approach flatness near the nut, the equality of \( \text{(6.36)} \) coincides with that of \( \text{(6.37)} \) in flat space.

6.2.3 Taub-Bolt

Four dimension

For the Taub-Bolt metric in four dimensions \( \text{(2.5)} \) \( \psi \) has a period of \( 4\pi \) and hence
\[
A = 32\pi^2 L \left( r^2 - 2.5 L r + L^2 \right)^{\frac{1}{2}} \left( r^2 - L^2 \right)^{\frac{1}{2}} \tag{6.38}
\]
and
\[
V = 2L \int_{L}^{r} (s^2 - L^2) \, ds \int \sin \theta \, d\psi \wedge d\theta \wedge d\phi = \frac{32}{3} \pi^2 L (r - L)^2 (r + 2L).
\] (6.39)
giving
\[
S(\rho) \equiv \frac{A^4}{V^3} = 32 \times 27 \pi^2 \frac{(\rho^2 - 2.5\rho + 1)^2 (\rho^2 - 1)^2}{(\rho - 1)^6 (\rho + 2)^3}.
\] (6.40)
where \( \rho = r/L \) as before. At \( \rho = 2 \), \( S(\rho) \) is zero and increases monotonically to a maximum and then decreases monotonically to zero. The maximum value is approximately \( 64.69449106 \pi^2 \) and occurs at \( \rho = 5.279392752 \).

Recall that the two infilling Taub-Bolt solutions are separated at \( \rho \sim 2.851708133 \), i.e., for a given boundary data below a certain squashing there will be two solutions with \( \hat{\rho} \in (2, 2.851708133) \) and another solution with \( \hat{\rho} \in (2.851708133, \infty) \). For a given boundary, therefore, the smaller-\( \rho \) solution (corresponding to larger \( L \)) will satisfy the following inequality
\[
\frac{A^4}{V^3} \leq S(2.851708133) \sim 38.29964761 \pi^2.
\] (6.41)
However, for the larger-\( \rho \) (corresponding to smaller value of \( L \)) solution the inequality is
\[
\frac{A^4}{V^3} < 64.69449106 \pi^2 \quad \text{(approximately)}.
\] (6.42)

Minkowski’s inequality,
\[
\frac{A^4}{V^3} \geq 128 \pi^2,
\] (6.43)
is not satisfied by either of the solutions under any circumstances.

**Arbitrary dimension**

For the \((2n + 2)\)-dimensional Taub-Bolt solution
\[
ds^2 = \gamma(r)^2 dr^2 + \beta(r)^2 (d\tau + A)^2 + \alpha(r)^2 ds_M^2
\] (6.44)
the area of a hypersurface \( \Sigma \) at constant \( r \) is
\[
A = \text{Vol}(M) \beta \alpha^{2n} \int d\tau = \frac{4\pi p}{\lambda k} \text{Vol}(M) (\sqrt{c}L)^{2n+1} \tilde{\beta} (\rho^2 - 1)^n
\] (6.45)
in which \( \tilde{\beta} \equiv \beta/\sqrt{c}L \) is a function of \( \rho = r/L \). Since \( \beta_\tau = 4\pi p/k \lambda \left( k < p \right) \), the volume enclosed within \( \Sigma \) is
\[
V = \int d\tau \int_{pL/k}^{r} \gamma \beta \alpha^{2n} dr \text{Vol}(M) = (\sqrt{c}L)^{2n+2} \times \frac{4\pi p}{\lambda k} \text{Vol}(M) \int_{pL/k}^{\rho} (\rho^2 - 1)^n d\rho.
\] (6.46)
Here, as in the case of Taub-Nut, we are using the conventions of Section 2.2.: \( \gamma \beta = cL \) and \( \alpha^2 = c(r^2 - L^2) \). The ratio is therefore
\[
\frac{A^{2n+2}}{V^{2n+1}} = \frac{4\pi p}{\lambda k} \text{Vol}(M) \tilde{\beta}^{2n+1} (\rho^2 - 1)^n \left( \int_{pL/k}^{r} (s^2 - 1)^nds \right)^{2n+1}
\] (6.47)
The integral in the denominator above can be evaluated. However, unlike Eq. (6.31) the resulting form is not very illuminating. Following the example of the 4-dimensional Taub-Bolt above, we know that a closed form expression does not exist. This is because the equality of the isoperimetric inequality in this case does not lie at the bolt. However, note that the behaviour does not change in higher dimensions. At the bolt, $\tilde{\beta}$ is zero while others are non-zero and therefore $A^{2n+2}/V^{2n+1}$ is zero unlike in the nut case. As $\rho$ is increased $A^{2n+2}/V^{2n+1}$ will increase to a certain value and then will start decreasing and approach zero monotonically. The value of $\rho$ at which the hump of $A^{2n+2}/V^{2n+1}$ occurs is greater than the value of $\rho$ for which the hump of $\frac{\beta(\rho)}{a(\rho)}$ occurs. This has been shown explicitly above for four dimensions leading to two different inequalities for the two solutions. Note that compared to the Minkowski’s inequality these two inequalities are opposite in type. This happens also for Schwarzschild which also possesses bolts – however, in that case explicit inequalities are obtainable. Finally, note that for different Taub-Bolt solutions (corresponding to values of $k \neq 1$) the corresponding inequalities are not the ones obtained by dividing the inequalities corresponding to $k = 1$ by $k$. This is because the locations of the bolts are different for different $k$’s. They need to be treated separately should one look for the isoperimetric inequalities of a certain type albeit they can only be found numerically.

6.2.4 Eguchi-Hanson

Four dimension

For the Eguchi-Hanson metric (2.7) the 3-volume of a constant $R$-slice hypersurface is

$$A = \pi^2 R^3 \left(1 - \frac{a^4}{R^4}\right)^{\frac{1}{2}},$$

and the 4-volume enclosed within it

$$V = \frac{1}{8} \int_a^R s^3 \, ds \, d\psi \wedge \sin \theta \, d\theta \wedge d\phi = \pi^2 \int_a^R s^3 \, ds = \frac{\pi^2}{4} (R^4 - a^4),$$

giving

$$\frac{A^4}{V^3} = 64 \pi^2 \frac{1}{\left(1 - \frac{a^4}{R^4}\right)}.$$

The isoperimetric inequality is then

$$\frac{A^4}{V^3} \geq 64 \pi^2.$$ (6.51)

This sits precisely in the middle of the extremes of flat space and Taub-Nut. Note that all three spaces are self-dual. It would be interesting to investigate the isoperimetric inequalities in other self-dual spaces in four dimensions which, however, is beyond the scope of this paper.

Arbitrary dimension

Recall that the Eguchi-Hanson metric (2.35) was written using the convention that $\lambda = 2(n + 1)$ and hence $\tau$ has a period of $\frac{2\pi}{(n+1)}$. The area of the constant-$r$ hypersurface and the volume enclosed
by it are respectively

\[ A = r^{2n+1} \sqrt{1 - \frac{a^{2n+2}}{r^{2n+2}}} \frac{2\pi}{(n+1)} \text{Vol}(M) \]  

(6.52)

and

\[ V = \frac{1}{2(n+1)} r^{2n+2} \left(1 - \frac{a^{2n+2}}{r^{2n+2}}\right) \frac{2\pi}{(n+1)} \text{Vol}(M). \]  

(6.53)

and hence

\[ \frac{A^{2n+2}}{V^{2n+1}} = \frac{2\pi}{(n+1)} \text{Vol}(M) \frac{2^{n+1} (n+1)^{2n+1} \frac{1}{\left(1 - \frac{a^{2n+2}}{r^{2n+2}}\right)^n}}{2^{n+1} (n+1)^{2n+1}} \]  

(6.54)

giving the isoperimetric inequality for the Eguchi-Hanson metric to be

\[ \frac{A^{2n+2}}{V^{2n+1}} \geq 2^{n+1} (n+1)^{2n} \text{Vol}(M). \]  

(6.55)

For \( M \equiv \mathbb{C}P^n \) which in this convention has the volume of the unit sphere \( S^{2n+1} \) divided by \( 2\pi \), the isoperimetric inequality (6.55) reads

\[ \frac{A^{2n+2}}{V^{2n+1}} \geq 2^{n+1} (n+1)^{2n} \text{Vol}(S^{2n+1}). \]  

(6.56)

This is just \( \frac{1}{n+1} \) times the \((2n + 2)\) dimensional Minkowski’s inequality (6.37) in flat space. This is therefore always somewhere in the middle of flat space and Taub-Nut. Note that this inequality illustrates another fundamental difference between the Eguchi-Hanson case and the Schwarzschild and Taub-Bolt cases than found in terms of number of infilling solutions. In the latter two cases \( S(\rho) \) is zero at the bolt whereas in this case \( S(\rho) \) blows up at the bolt. The lower bound of (6.56) comes from \( \rho \to \infty \). This explains (6.56) immediately via the periodicity of the fibre.

7 Conclusion

In this paper we studied the Dirichlet problem for cohomogeneity one Ricci-flat metrics whose principal orbits are \( S^1 \) bundles over compact Einstein spaces. We then investigated the subsequent isoperimetric inequalities. In the case of trivial bundles the Ricci-flat solutions are the Schwarzschild metrics for arbitrary choices of the compact Einstein base. In the case of non-trivial bundles the base spaces are required to be Einstein-Kähler and solutions exist only in even dimensions. The resulting Ricci-flat solutions can be classified using their 4-dimensional analogues: the Eguchi-Hanson, the Taub-Nut and the Taub-Bolts. All of these metrics can be topologically classified according to the presence and absence of singular orbits, i.e., bolts or nuts. With the correct periodicity of the \( S^1 \)-fibre, bolts and nuts can be made regular and hence can be included into the complete metric. In the case of Taub-Nut one further requires the Einstein-Kähler base to be complex projective space.

When the boundary \( \Sigma \) is a non-trivial bundle of dimension three, it is possible to find explicit 4-dimensional Taub-Nut and Taub-Bolt solutions exactly by treating the problem as a system of two bivariate equations. However, these equations become rather complicated as one goes higher in the ladder of dimensionality making a case-by-case study rather difficult and impossible analytically. This problem can be circumvented by using a general approach involving differential equations
and polynomials which together provide a unified way of treating the Taub-Nut and all Taub-Bolts (including those for which \( p < k < 1 \)) in arbitrary dimension. This method also makes their topologically different characters rather distinct. Such an approach is possible because Ricci-flatness effectively reduce the problem to the one-variable problem of squashing, i.e., involving the ratio of the two radii of \( \Sigma \) rather than their separate absolute values. The Taub-Nut infilling solution is unique and the Taub-Bolt infilling solution is double-valued in all dimensions and do not depend on details like the choice of base and the periodicity of the \( S^1 \) fibre. The upper limit on the squashing for the boundary \( \Sigma \) to admit a Taub-Nut infilling is given by a simple function of dimension. In the case of Taub-Bolts, however, we are not so lucky. Upper limits on squashing, although a function of dimension only need be worked out for each dimension (and for each possible values of \( k \)) separately which is a straightforward task however. In the case that \( \Sigma \) is a trivial bundle the infilling Schwarzschild geometry is double-valued in arbitrary dimension and does not depend on the choice of the base manifold. The upper limit on the squashing for \( \Sigma \) to admit Schwarzschild solutions is a simple function of dimension. One may therefore guess that the origin of non-uniqueness is geometric and is related to the presence of a singular orbit of the group action, i.e., a bolt. This is indeed the case. However, the presence of a bolt acts as a necessary condition only. It is not a sufficient condition as we found that the possible Eguchi-Hanson infilling geometry in arbitrary dimension is unique despite the presence of a bolt. It is worth recalling that the Eguchi-Hanson metric in four dimensions has a self-dual Riemann tensor. Hence the set of ordinary differential equations arising from the Einstein equations is reduced to a first order set and the various scale factors (here the two radii of the evolving hypersurface) fix the values of their derivatives uniquely. They, however, are required to satisfy the constraint equation. The Einstein equation then guarantees a unique evolution (whether it is regular or non-regular at the origin is an \textit{a posteriori} issue). This is also true for the Taub-Nut in four dimensions which is self-dual as well. In the presence of a cosmological constant the Taub-Nut in four dimensions becomes Taub-Nut-(anti)de Sitter which obviously does not have a self-dual Riemann tensor. However, it has a self-dual Weyl tensor which reduces the system to first order and thus the above comments apply. The condition on the boundary data to have a self-dual Taub-Nut-AdS solution has been discussed in detail in \cite{2}. The infilling geometries were found as exact analytic functions of the boundary data despite the presence of a cosmological constant which involves the two radii rather than their squashing. In the case of Taub-Bolt-AdS the number of solutions can be as high as ten which has been shown in \cite{1}.

In the case of trivial bundles, infilling Schwarzschild solutions were obtained by finding the two masses of the black holes as analytic functions of the two radii (effectively their squashing). To our knowledge this is the first study of this kind of higher dimensional Schwarzschild. With the relatively recent work on the negative mode of higher dimensional Schwarzschild in a finite cavity \cite{21}, this provides a straightforward generalisation of the results obtained in four dimensions. Since we have obtained the two masses of the black holes as exact, analytic functions of the cavity radius and temperature, our study provides a basis for further exact semi-classical computations of black hole thermodynamics and dynamics as all classical quantities can be evaluated from the geometry exactly as functions of the cavity radius and its temperature. As one would expect, the introduction of a cosmological constant would also make the problem rather non-trivial. It is indeed the case. However, it has been shown in \cite{3} that in four dimensions there are two infilling Schwarzschild-AdS
solutions in a finite isothermal cavity with positive and negative specific heats. It remains to see how this picture changes in higher dimensions and whether explicit solutions for infilling geometries are possible. These will be reported elsewhere.

We show that for all infilling solutions above, including those occurring in pairs, the boundary is necessarily convex. This answers the question posed by one of the authors at the Samos Meeting on Cosmology, Geometry and Relativity, 1998 [13] of whether convexity can be applied as a criteria for selecting solutions if the boundary admits more than one infilling geometry (of the same type). We show that convexity cannot play such a role for Ricci-flat spaces admitting $U(1)$ actions in general. Using a relatively recent theorem by Reilly [30], we find that convexity gives a lower bound of the Euclidean action for each infilling solution through the $n$-volume of $\Sigma$ and the $(n+1)$-volume of the infilling solutions $M$.

Finally we have discussed the analogues of Minkowski’s inequality and found some interesting results. They can be grouped into two classes. In one class we have the Taub-Nut and the Eguchi-Hanson spaces. We found that the analogue of (flat-space’s) Minkowski’s inequality for the Taub-Nut in arbitrary dimension is just the opposite of Minkowski’s inequality in that dimension. The Eguchi-Hanson is found to lie in the middle in that its inequality has the same sense as that of flat-space Minkowski and is $\frac{1}{n+1}$ times the latter in $(2n+2)$ dimensions. We have explained why this happens. The other class consists of the Schwarzschild and the Taub-Bolt. In either case, the two infilling geometries, although topologically equivalent, are strikingly dissimilar in their isoperimetric inequalities. We have been able to find the inequalities explicitly in the case of the Schwarzschild solutions. The two inequalities are “connected” in that one is given by being equal or greater than some value and the other by the opposite of this inequality. Interestingly the isoperimetric inequality for the larger mass stable black hole solution which has a positive specific heat is always within Minkowski’s bound. In gravitational thermodynamics hot flat space is taken as the background for the Schwarzschild calculations and hence this observation may have some important thermodynamic consequences. For the Taub-Bolt the nature of the polynomials involved prohibits one to obtain the inequalities exactly, and also the two inequalities are not “connected”. They both, however, have the same sense in that they are given by being equal or less than some values. We have obtained analytic expressions in a form from which it is straightforward to find the approximate inequalities (for any choice of base manifold $M$ and for any value of the periodicity $k$) numerically should one need to know them. Note that both of the inequalities obey the bound provided by their nut counterpart, i.e., by the Taub-Nut. Therefore the above comments about the flat space and the Schwarzschild equally apply for the Taub-Nut and Taub-Bolt solutions.

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A Trinomial Equations

In this appendix we describe the solution to the general trinomial equation following Birkeland [10].

We write the generic trinomial equation in the form

$$x^n - ax^s + b = 0 \quad (A.1)$$

where $a$ and $b$ are coefficients, in general complex, and $n > s$. Define

$$\zeta = \frac{n^n b^{n-s}}{s^n (n-s)^{n-s} a^n}. \quad (A.2)$$

Two possibilities can occur depending on whether $|\zeta| < 1$ or $|\zeta| > 1$.

$|\zeta| < 1$

Define

$$x = za^{\frac{1}{n-s}}, \quad l = -ba^{\frac{n}{n-s}} \quad (A.3)$$

The $n - s$ roots are given by

$$x_i^\gamma = a^{\frac{\gamma}{n-s}} e^{i\gamma} \left( F_0(\zeta) + \frac{\gamma}{n-s} \sum_{\kappa=1}^{n-s-1} e^{-i\kappa\mu} \mu_\kappa l^{\kappa} F_\kappa(\zeta) \right) \quad (i = 1, 2, \ldots, n - s) \quad (A.4)$$

where $\gamma$ is an arbitrary constant and

$$\epsilon = e^{\frac{2\pi}{n-s} \sqrt{-1}}, \quad \mu_1 = 1, \quad \mu_\kappa = \frac{1}{\kappa} \left( \frac{\gamma-\kappa s}{n-s} - 1 \right) \quad (A.5)$$

and

$$F_\kappa(\zeta) = F \left( \begin{array}{c} a_{1,\kappa}, \ldots, a_{n-1,\kappa}, \ a_{n,\kappa} \\ b_{1,\kappa}, \ldots, b_{n-1,\kappa}, \ \zeta \end{array} \right) \quad (A.6)$$

are the higher order hypergeometric functions of order $n$ in general$^6$:

$$F_\kappa(\zeta) = \sum_{\mu} \left( \frac{a_{1,\kappa}}{(b_{1,\kappa})_\mu} \cdots \frac{a_{n,\kappa}}{(b_{n,\kappa})_\mu} \right) \zeta^\mu \quad (A.7)$$

in which the Pochhammer symbol has been used: $(a)_\mu$ stands for $a(a + 1)(a + 2) \ldots (a + \mu - 1)$. Also here

$$a_{i,\kappa} = \frac{\kappa}{n-s} + \frac{i}{n} - \frac{\gamma}{n(n-s)} \quad (i = 1, 2, \ldots, n),$$

$$b_{i,\kappa} = \frac{\kappa}{n-s} + \frac{i+1}{s} - \frac{\gamma}{s(n-s)} \quad (i = 1, 2, \ldots, s),$$

$$b_{i,\kappa} = \frac{\kappa}{n-s} + \frac{1-s}{n-s} + \frac{\delta_i}{n-s} \quad (i = s + 1, \ldots, n - 1) \quad (A.8)$$

and

$$\delta_i = 0, \text{ when } i < n - \kappa, \quad \delta_i = 1, \text{ when } i \geq n - \kappa. \quad (A.9)$$

$^6$When $\zeta = 1$, $F_\kappa(\zeta)$’s reduce to hypergeometric functions of order $n - 1.$
Now define

\[ x = z(-l)^{\frac{2}{s}a^n - s}, \quad \lambda = (-l)^{\frac{n-s}{s}} = b^{\frac{n-s}{s}} g^{-\frac{a}{s}}. \]  

(A.10)

Under the condition \(|\zeta| < 1\), the remaining \(s\) roots are

\[ x_{n-s+i}^{\gamma} = (-l)^{\frac{2}{s}g^{\frac{n-s}{s}}\delta^{i\gamma}} \left[ \phi_0(\zeta) + \sum_{\kappa=1}^{s-1} \gamma^{i\kappa} \Delta_\kappa \lambda^\kappa \phi_\kappa(\zeta) \right] \]  

(i = 1, 2, ..., s),

(A.11)

where

\[ \delta = e^{\frac{2\pi}{s} \sqrt{-1}}, \quad \Delta_1 = 1, \quad \Delta_\kappa = \frac{1}{\kappa} \left( \frac{\gamma^{+n}\kappa}{s} - 1 \right) \]  

(A.12)

and

\[ \phi_\kappa(\zeta) = F \left( \frac{d_{1,\kappa}, \ldots, d_{n-1,\kappa}, d_{n,\kappa}}{e_{1,\kappa}, \ldots, e_{n-1,\kappa}, \zeta} \right) \]  

(A.13)

in which

\[ d_{i,\kappa} = \frac{\kappa}{s} + \frac{i}{s} + \frac{\gamma}{s n} \quad (i = 1, 2, \ldots, n), \]

\[ e_{i,\kappa} = \frac{\kappa}{s} + \frac{i}{s} + \frac{\gamma}{s(n-s)} \quad (i = 1, 2, \ldots, n-s), \]  

(A.14)

and \(\delta_i\) is the same as before.

The expansions (A.11) and (A.12) converge for \(|\zeta| < 1\). They will diverge for \(|\zeta| > 1\) and hence the letter is treated separately.

\(|\zeta| > 1\)

The roots for \(|\zeta| > 1\) are given by higher order hypergeometric function of the variable \(\frac{1}{\zeta}\). Defining

\[ x = z(-b)^{\frac{1}{s}}, \quad l_1 = \rho = g(-b)^{-\frac{n-s}{n}}, \]  

(A.15)

all of the \(n\) roots are given by

\[ x_{1}^{\gamma} = \beta^2 \nu \psi^\gamma \left[ \psi_0 \left( \frac{1}{\zeta} \right) + \frac{\gamma}{n} \sum_{\kappa=1}^{n-1} \nu^{i\kappa} \theta_\kappa \rho^\kappa \psi_\kappa \left( \frac{1}{\zeta} \right) \right] \quad (i = 1, 2, \ldots, n), \]  

(A.16)

in which

\[ \nu = e^{\frac{2\pi}{n} \sqrt{-1}}, \quad \theta_1 = 1, \quad \theta_\kappa = \left( \frac{\gamma^{+n}\kappa}{\kappa n} - 1 \right), \]  

(A.17)

and \(\gamma\) is an arbitrary constant as before.

The \(\psi\)’s here are hypergeometric functions of order \(n\) with the explicit form:

\[ \psi_\kappa \left( \frac{1}{\zeta} \right) = F \left( \frac{g_{1,\kappa}, \ldots, g_{n-1,\kappa}, d_{n,\kappa}}{h_{1,\kappa}, \ldots, h_{n-1,\kappa}, \zeta} \right) \]  

(A.18)

in which

\[ g_{i,\kappa} = \frac{\kappa}{s} + \frac{i}{s} + \frac{\gamma}{s n} \quad (i = 1, 2, \ldots, s), \]

\[ g_{i,\kappa} = \frac{\kappa}{s} + \frac{n-i}{n-s} - \frac{\gamma}{n(n-s)} \quad (i = 1, 2, \ldots, n), \]  

(A.19)

\[ h_{i,\kappa} = \frac{\kappa+i}{n} + \frac{\delta_i}{n} \quad (i = 1, 2, \ldots, n-1), \]

where \(\delta_i\) is the same as before.
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