SHORT WALK ADVENTURES

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To the memory of Jon Borwein, who convinced us that a short walk can be adventurous

Abstract. We review recent development of short uniform random walks, with a focus on its connection to (zeta) Mahler measures and modular parametrisation of the density functions. Furthermore, we extend available “probabilistic” techniques to cover a variation of random walks and reduce some three-variable Mahler measures, which are conjectured to evaluate in terms of $L$-values of modular forms, to hypergeometric form.

0. Introduction

At some stages of our careers we were approached by Jon Borwein to collaborate on a theme that sounded rather off topic to us, who had interests in number theory, combinatorics and related special functions. Somewhat unexpectedly, the theme has become a remarkable research project with several outcomes (including [9] [10] [11], to list a few), a project which we continue to enjoy after the sudden loss of Jon... This note serves as a summary to our recent discoveries that certain “probabilistic” techniques apply usefully to tackling difficult problems on the border of analysis, number theory and differential equations; in particular, in evaluating multi-variable Mahler measures. Our principal novelties are given in Theorems 1–3; these include hypergeometric reduction of the Mahler measures of the three-variable polynomials

$$1 + x_1 + x_2 + x_3 + x_2x_3 \quad \text{and} \quad (1 + x_1)^2 + x_2 + x_3,$$

as well as the (hypergeometric) factorisation of a related differential operator for the Apéry-like sequence

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}^2, \quad \text{where} \ n = 0, 1, 2, \ldots$$

Echoing Jon’s “a short walk can be beautiful” [8], we add that “a short walk can be adventurous.”

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1. Uniform random walks

An \(N\)-step uniform random walk is a planar walk that starts at the origin and consists of \(N\) steps of length 1 each taken into a uniformly random direction. Let \(X_N\) be the distance to the origin after these \(N\) steps. The \(s\)-th moments \(W_N(s)\) of \(X_N\) can be computed \([11]\) via the formula

\[ W_N(s) = \int \cdots \int_{[0,1]^N} |e^{2\pi i \theta_1} + \cdots + e^{2\pi i \theta_N}|^s \, d\theta_1 \cdots d\theta_N 
= \int \cdots \int_{[0,1]^{N-1}} |1 + e^{2\pi i \theta_1} + \cdots + e^{2\pi i \theta_{N-1}}|^s \, d\theta_1 \cdots d\theta_{N-1}, \]

and are related to the (probability) density function \(p_N(x)\) of \(X_N\) via

\[ W_N(s) = \int_0^\infty x^s p_N(x) \, dx = \int_0^N x^s p_N(x) \, dx. \]

That is, \(p_N(x)\) can then be obtained as the inverse Mellin transform of \(W_N(s-1)\). Finally, note that the even moments \(W_3(2n)\) and \(W_4(2n)\) (which are, clearly, positive integers) can be identified with the odd moments of \(I_0(t)K_0(t)\) and \(I_0(t)K_0(t)^3\) respectively, where \(I_0(t)\) and \(K_0(t)\) denote the modified Bessel functions of the first and second kind. Namely, for \(n = 1, 2, \ldots\) we have \([6]\)

\[ W_3(2n) = \frac{3^{2n+3/2}}{2\pi 2^n n!} \int_0^\infty t^{2n} I_0(t)K_0(t)^2 \, dt \]
and

\[ W_4(2n) = \frac{4^{2n+2}}{\pi^2 n!} \int_0^\infty t^{2n+1} I_0(t)K_0(t)^3 \, dt. \]

2. Zeta Mahler measures

For a non-zero Laurent polynomial \(P(x_1, \ldots, x_N) \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]\), its zeta Mahler measure \([3]\) is defined by

\[ Z(P; s) = \int \cdots \int_{[0,1]^N} |P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_N})|^s \, d\theta_1 \cdots d\theta_N, \]

and its logarithmic Mahler measure is

\[ m(P) = \left. \frac{dZ(P; s)}{ds} \right|_{s=0} = \int \cdots \int_{[0,1]^N} \log |P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_N})| \, d\theta_1 \cdots d\theta_N. \]

A straightforward comparison of the two definitions reveals that

\[ W_N(s) = Z(x_1 + \cdots + x_N; s) = Z(1 + x_1 + \cdots + x_{N-1}; s) \]

and

\[ W_N'(0) = m(x_1 + \cdots + x_N) = m(1 + x_1 + \cdots + x_{N-1}) = \int_0^N p_N(x) \log x \, dx, \quad (1) \]
where the derivative is with respect to \( s \). The latter Mahler measures are known as linear Mahler measures. The evaluations

\[
W'_2(0) = 0, \quad W'_3(0) = L'(\chi_3; -1) = \frac{3\sqrt{3}}{4\pi} L(\chi_3; 2), \quad W'_4(0) = -14\zeta'(-2) = \frac{7\zeta(3)}{2\pi^2},
\]

are known \[24\], while the following conjectural evaluations, due to Rodriguez-Villegas \[13\] and verified to several hundred digits \[5\], remain open:

\[
W'_5(0) ? = -L'(f_3; -1) = 6\left(\frac{\sqrt{15}}{2\pi}\right)^5 L(f_3; 4),
\]

\[
W'_6(0) ? = -8L'(f_4; -1) = 3\left(\frac{\sqrt{6}}{\pi}\right)^6 L(f_4; 5),
\]

are cusp eigenforms of weight 3 and 4, respectively. Here and in what follows, Dedekind’s eta function

\[
\eta(\tau) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2/24}, \quad \text{where } q = e^{2\pi i \tau},
\]

serves as a principal constructor of modular forms and functions. No similar formulae are known for \( W'_N(0) \) when \( N \geq 7 \), though the story continues at a different level — see \[14, 30, 31\] for details.

### 3. Generic two-step random walks

Let \( X_1 \) and \( X_2 \) be two (sufficiently nice, independent) random variables on \([0, \infty)\) with probability density \( p_1(x) \) and \( p_2(x) \), respectively, and let \( \theta_1 \) and \( \theta_2 \) be uniformly distributed on \([0, 1]\). Then \( X = e^{2\pi i \theta_1} X_1 + e^{2\pi i \theta_2} X_2 \) describes a two-step random walk in the plane with a first step of length \( X_1 \) and a second step of length \( X_2 \). As in \[10\] eq. (3-3), an application of the cosine rule shows that the \( s \)-th moment of \( |X| \) is

\[
W(s) = E(|X|^s) = \int_{0}^{\infty} \int_{0}^{\infty} g_s(x, y) p_1(x) p_2(y) \, dx \, dy,
\]

where

\[
g_s(x, y) = \frac{1}{\pi} \int_{0}^{\pi} (x^2 + y^2 + 2xy \cos \theta)^{s/2} \, d\theta.
\]

Observe that

\[
\left. \frac{d g_s(x, y)}{d s} \right|_{s=0} = \frac{1}{\pi} \int_{0}^{\pi} \log \sqrt{x^2 + y^2 + 2xy \cos \theta} \, d\theta = \max\{\log |x|, \log |y|\},
\]

so that, in particular,

**Lemma 1.** We have

\[
W'(0) = E(\log |X|) = \int_{0}^{\infty} \int_{0}^{\infty} p_1(x)p_2(y) \max\{\log x, \log y\} \, dy \, dx.
\]
Alternative equivalent expressions, that will be useful in what follows, include
\[
E(\log |X|) = \int_0^\infty \int_0^x p_1(x)p_2(y) \log x \, dy \, dx + \int_0^\infty \int_x^\infty p_1(x)p_2(y) \log y \, dy \, dx
\]
\[
= E(\log X_1) + \int_0^\infty \int_x^\infty p_1(x)p_2(y)(\log y - \log x) \, dy \, dx
\]
\[
= E(\log X_2) + \int_0^\infty \int_0^x p_1(x)p_2(y)(\log x - \log y) \, dy \, dx. \tag{2}
\]

4. Linear Mahler measures

Let \(N, M\) be integers such that \(N > M > 0\). By decomposing an \(N\)-step random walk into two walks with \(N - M\) and \(M\) steps, and applying Lemma \([1]\) in the form \([2]\), we find that
\[
W'_N(0) = W'_M(0) + \int_0^{N-M} p_{N-M}(x) \left( \int_0^x p_M(y)(\log x - \log y) \, dy \right) \, dx.
\]
This formula, together with known formulae for the densities \([1]\), like \(p_1(x) = \delta(x-1)\) (the Dirac delta function) and \(p_2(x) = 2/(\pi \sqrt{4-x^2})\) for \(0 < x < 2\), allows one to produce new expressions for linear Mahler measures. Indeed, taking \(M = 1\) we get
\[
W'_N(0) = \int_1^{N-1} p_{N-1}(x) \log x \, dx \tag{3}
\]
(which can be also derived using Jensen’s formula), while \(M = 2\) results in
\[
W'_N(0) = \int_2^{N-2} p_{N-2}(x) \log x \, dx + \frac{1}{\pi} \int_0^2 p_{N-2}(x)x \cdot 3F2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{x^2}{4}\right) \, dx \tag{4}
\]
(see also \([20]\, \text{eq. (2.1)})]. Here, and in what follows, the hypergeometric notation
\[
_{m}F_{m-1}\left(\begin{array}{c}a_1, a_2, \ldots, a_m \\ b_2, \ldots, b_m\end{array} \mid z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \cdots (a_m)_n z^n}{(b_2)_n \cdots (b_m)_n \, n!}
\]
is used, where
\[
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1) \cdots (a+n-1), & \text{for } n \geq 1, \\ 1, & \text{for } n = 0, \end{cases}
\]
denotes the Pochhammer symbol (the rising factorial). Note that we deduce \([4]\) from
\[
\int_0^x p_2(y)(\log x - \log y) \, dy = \frac{x}{\pi} \cdot 3F2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{x^2}{4}\right),
\]
which is valid if \(0 \leq x \leq 2\).

Equations \([3]\) and \([4]\) and the formula
\[
p_4(x) = \frac{2\sqrt{16-x^2}}{\pi^2 x} \text{Re} \, 3F2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{(16-x^2)^3}{108x^4}\right)
\]
obtained in [11, Theorem 4.9], provide the formulae

\[
W'_5(0) = \frac{7\zeta(3)}{2\pi^2} - \frac{1}{\pi^2} \int_0^1 \sqrt{16 - x^2} \Re_3 F_2 \left( \left[ \frac{1}{2}, \frac{1}{2}, \frac{3}{6} \right] \bigg| \frac{(16 - x^2)^3}{108x^4} \right) d(\log^2 x)
\]

and

\[
W'_6(0) = \frac{7\zeta(3)}{2\pi^2} - \frac{1}{\pi^2} \int_0^2 \sqrt{16 - x^2} \Re_3 F_2 \left( \left[ \frac{1}{2}, \frac{1}{2}, \frac{3}{6} \right] \bigg| \frac{(16 - x^2)^3}{108x^4} \right) d(\log^2 x) \\
+ \frac{2}{\pi^3} \int_0^2 \sqrt{16 - x^2} \Re_3 F_2 \left( \left[ \frac{1}{2}, \frac{1}{2}, \frac{3}{6} \right] \bigg| \frac{(16 - x^2)^3}{108x^4} \right) \cdot 3 F_2 \left( \left[ \frac{1}{2}, \frac{1}{2}, \frac{3}{6} \right] \bigg| \frac{x^2}{4} \right) dx.
\]

These single integrals can be used to numerically confirm the conjectural evaluations of \(W'_5(0)\) and \(W'_6(0)\).

A similar application of Lemma 11 upon decomposing a 6-step walk into two walks with 3 steps, yields the alternative reduction

\[
W'_6(0) = 2 \int_0^3 p_3(x) \log x \left( \int_0^x p_3(y) \, dy \right) \, dx,
\]

where

\[
p_3(x) = \frac{2\sqrt{3x}}{\pi(3 + x^2)} \cdot 2 F_1 \left( \left[ \frac{1}{3}, \frac{2}{3} \right] \bigg| \frac{x^2(9 - x^2)^2}{(3 + x^2)^3} \right).
\]

We discuss this formula further in Section 5.

Finally, we mention that equation (3) and a modular parametrisation of \(p_4(x)\) (which we indicate in Section 6) were independently cast in [23] to produce a double \(L\)-value expression for \(W'_5(0)\).

5. MODULAR PARAMETRISATION OF \(p_3(x)\) AND RELATED FORMULAE

Note that formula (5) can be written as

\[
W'_6(0) = \int_0^3 \log x \, d(P_3(x)^2) = \log 3 - \int_0^3 P_3(x)^2 \frac{dx}{x},
\]

featuring the cumulative density function

\[
P_3(x) = \int_0^x p_3(y) \, dy.
\]

The related modular parametrisation of \(p_3(x)\) is given by

\[
x = x(\tau) = 3 \frac{\eta(\tau)^2 \eta(6\tau)^4}{\eta(2\tau)^4 \eta(3\tau)^2}: (i\infty, 0) \to (0, 3),
\]

so that

\[
p_3(x) = \frac{2\sqrt{3}}{\pi} \frac{\eta(2\tau)^2 \eta(6\tau)^2}{\eta(\tau) \eta(3\tau)} \cdot \frac{dx}{3\pi i \frac{\eta(\tau)^6 \eta(3\tau)^2 \eta(6\tau)^2}{\eta(2\tau)^6}}
\]

and

\[
P_3(x) = 6i\sqrt{3} \int_{i\infty}^{x} \frac{\eta(\tau)^5 \eta(3\tau) \eta(6\tau)^4}{\eta(2\tau)^4} \, d\tau.
\]
is the anti-derivative of a weight 3 holomorphic Eisenstein series
\[
\frac{\eta(\tau)^5 \eta(3\tau) \eta(6\tau)^4}{\eta(2\tau)^4} = E_{3,\chi_{-3}}(\tau) - 8E_{3,\chi_{-3}}(2\tau),
\]
where
\[
E_{3,\chi_{-3}}(\tau) = \frac{\eta(3\tau)^9}{\eta(\tau)^3} = \sum_{m,n=1}^{\infty} \left( \frac{-3}{m} \right) n^2 q^{mn}, \quad \chi_{-3}(m) = \left( \frac{-3}{m} \right) = \frac{e^{2\pi im/3} - e^{-2\pi im/3}}{i\sqrt{3}}.
\]

Though the anti-derivative \( P_3(x) \),
\[
P_3(x) = \frac{3\sqrt{3}}{\pi} \sum_{m,n=1}^{\infty} \left( \frac{-3}{m} \right) \frac{n q^{mn}}{m} - 4 \sum_{m,n=1}^{\infty} \left( \frac{-3}{m} \right) \frac{n q^{2mn}}{m}
\]

and
\[
= \frac{9i}{\pi} \log \prod_{n=1}^{\infty} \left( \frac{1 - e^{2\pi i/3} q^{2n}}{1 - e^{-2\pi i/3} q^{2n}} \right)^n,
\]

is not considered to be sufficiently "natural", it shows up as the elliptic dilogarithm thanks to Bloch’s formula; see [17] [19] for the details. Note that
\[
E_{3,\chi_{-3}} \left( -\frac{1}{3\tau} \right) = \frac{i\tau^3}{3\sqrt{3}} \tilde{E}_{3,\chi_{-3}}(\tau), \quad \tilde{E}_{3,\chi_{-3}}(\tau) = \frac{\eta(\tau)^9}{\eta(3\tau)^3} = 1 - 9 \sum_{m,n=1}^{\infty} \left( \frac{-3}{m} \right) n^2 q^{mn};
\]

and, in addition, we have
\[
\frac{1}{2\pi i} \frac{dx}{d\tau} = \frac{1}{2} \left( \frac{\eta(\tau)^2 \eta(3\tau)^2}{\eta(2\tau)^2} \right)^2 = \frac{1}{18} \left( E_{1,\chi_{-3}}(\tau) - 4E_{1,\chi_{-3}}(4\tau) \right)^2
\]
\[
= \frac{1}{54\tau^2} \left( E_{1,\chi_{-3}} \left( -\frac{1}{12\tau} \right) - E_{1,\chi_{-3}} \left( -\frac{1}{3\tau} \right) \right)^2,
\]

where
\[
E_{1,\chi_{-3}}(\tau) = 1 + 6 \sum_{m,n=1}^{\infty} \left( \frac{-3}{m} \right) q^{mn}.
\]

6. MODULAR COMPUTATION FOR \( W'_5(0) \) AND \( W'_6(0) \)

As (partly) shown in [11] the density \( p_4(x) \) can be parameterised as follows (we make a shift of \( \tau \) by half):
\[
p_4(x(\tau)) = -\text{Re} \left( \frac{2i(1 + 6\tau + 12\tau^2)}{\pi} p(\tau) \right),
\]

where
\[
p(\tau) = \frac{\eta(2\tau)^4 \eta(6\tau)^4}{\eta(\tau) \eta(3\tau) \eta(4\tau) \eta(12\tau)} \quad \text{and} \quad x(\tau) = \left( \frac{2\eta(\tau) \eta(3\tau) \eta(4\tau) \eta(12\tau)}{\eta(2\tau)^2 \eta(6\tau)^2} \right)^3.
\]

The path for \( \tau \) along the imaginary axis from 0 to \( i/(2\sqrt{3}) \) (or from \( i\infty \) to \( i/(2\sqrt{3}) \)) corresponds to \( x \) ranging from 0 to 2, while the path from \( i/(2\sqrt{3}) \) to \(-1/4 + i/(4\sqrt{3}) \) along the arc centred at 0 corresponds to the real range \((2, 4) \) for \( x \). (The arc admits
the parametrisation \( \tau = e^{i\theta}/(2\sqrt{3}) \), \( 1/2 < \theta < 5/6 \). Note that \( x(i/(2\sqrt{15})) = 1 \) and

\[
p_4(x(\tau)) = \begin{cases} 
-\frac{2i \cdot 6\tau}{\pi} p(\tau), & \text{for } \tau \text{ on the imaginary axis,} \\
-\frac{2i(1 + 6\tau + 12\tau^2)}{\pi} p(\tau), & \text{for } \tau \text{ on the arc,}
\end{cases}
\]

and

\[
-\frac{2i(1 + 6\tau + 12\tau^2)}{\pi} p(\tau) = \frac{2\sqrt{16 - x^2}}{\pi x} \cdot F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{2}; \frac{(16 - x^2)^3}{108x} \right)
\]

(this is a general form of [11, Theorem 4.9]). Formulas (1), (3) and (4) reduce the conjectural evaluations of \( W_5'(0) \) and \( W_6'(0) \) to the following ones:

\[
\frac{7\zeta(3)}{2\pi^2} + \int_0^{1/(2\sqrt{3})} y p(y) \log(x(y)) \, dx(y)
\]

and

\[
\frac{7\zeta(3)}{2\pi^2} + 8\int_0^{1/(2\sqrt{3})} y p(y) \log(x(y)) \, dx(y)
\]

Furthermore, note that the Atkin–Lehner involutions \( w_{12}: \tau \mapsto -1/(12\tau) \) and \( w_6: \tau \mapsto (6\tau - 5)/(12\tau - 6) \) act on the modular function \( x(\tau) \) as follows: \( x(w_{12}\tau) = x(\tau) \) and \( x(w_6\tau) = -8/x(\tau) \), and we also have \( p(w_{12}\tau) = -\tau^2 p(\tau) \). The point \( i/(2\sqrt{3}) \) is fixed by \( w_{12} \). Thus, the change of variable \( y \mapsto 1/(12y) \) leads to

\[
\int_0^{1/(2\sqrt{3})} y p(y) \log(x(y)) \, dx(y) = -\int_{1/(2\sqrt{3})}^{\infty} y p(y) \log(x(y)) \, dx(y).
\]

7. Mahler measures related to a variation of random walk

In [23] the Mahler measures \( m(1 + x_1 + x_2) \) and \( m(1 + x_1 + x_2 + x_3) \) are computed using the modular parametrisations of

\[
\sum_{n=0}^{\infty} W_3(2n) z^n = \sum_{n=0}^{\infty} \text{CT}((1 + x_1 + x_2)(1 + x_1^{-1} + x_2^{-1}))^n z^n
\]

and

\[
\sum_{n=0}^{\infty} W_4(2n) z^n = \sum_{n=0}^{\infty} \text{CT}((1 + x_1 + x_2 + x_3)(1 + x_1^{-1} + x_2^{-1} + x_3^{-1}))^n z^n,
\]

where \( \text{CT}(L) \) denotes the constant term of a Laurent polynomial \( L \in \mathbb{Z}[x_1, x_2, \ldots] \). Note that the Picard–Fuchs linear differential equations for the two generating functions give rise to the ones for the densities \( p_3(x) \) and \( p_4(x) \) together with their explicit hypergeometric and modular expressions (see [11, eq. (3.2) and Remark 4.10]), though it remains unclear whether the latter information can be used to compute \( W_N'(0) \) in [11] for \( N = 3, 4 \). This is itself an interesting question to not only assist in
computing of $W_N'(0)$ for $N > 4$ but also in relation with another famous conjecture of Boyd:

$$m(1 + x_1 + x_2 + x_3 + x_2 x_3)^2 = -2L'(f_2; -1) = \frac{15^2}{4\pi^4} L(f_2; 3) = 0.4839979734 \ldots, \quad (6)$$

where $f_2(\tau) = \eta(\tau) \eta(3\tau) \eta(5\tau) \eta(15\tau)$.

In analogy with the case of linear Mahler measures, we define

$$\widetilde{W}(s) = \int_0^1 |1 + e^{2\pi i \theta_1} + e^{2\pi i \theta_2} + e^{2\pi i \theta_3} + e^{2\pi i (\theta_2 + \theta_3)}|^s \, d\theta_1 \, d\theta_2 \, d\theta_3$$

as the $s$-th moment of a random 5-step walk for which the direction of the final step is completely determined by the two previous steps. Then the even moments

$$\widetilde{W}(2n) = CT\left((1 + x_1 + x_2 + x_3 + x_2 x_3)(1 + x_1^{-1} + x_2^{-1} + x_3^{-1} + (x_2 x_3)^{-1})\right)^n$$

satisfy a rather lengthy recurrence equation, which is equivalent to a Picard–Fuchs differential equation of order 4. The latter splits into the tensor product of two differential equations of order 2 and, with some effort, we obtain the following result.

**Theorem 1.** We have

$$\sum_{n=0}^{\infty} \widetilde{W}(2n) \left(\frac{t}{(4 + t)(1 + 4t)}\right)^n = \frac{(4 + t)(1 + 4t)}{4(1 + 4t + t^2)} \ _2F_1\left(\left\{\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{t(4 + t)}{1 + 4t + t^2}\right\}; \frac{t^2}{1 + 4t + t^2}\right) \cdot \ _2F_1\left(\left\{\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{t}{1 + 4t + t^2}\right\}; \frac{t^2}{1 + 4t + t^2}\right)$$

and, more generally,

$$\frac{b}{(b + t)(1 + bt)} \sum_{n=0}^{\infty} \left(\frac{t}{(b + t)(1 + bt)}\right)^n \sum_{k=0}^{n} \left(\frac{n}{k}\right)^2 \left(\frac{2k}{k}\right)^2 \left(\frac{b}{4}\right)^{2k}$$

$$= \ _2F_1\left(\left\{\frac{1}{2}, \frac{1}{2}\right\}, -t(b + t)\right) \cdot \frac{1}{(1 + bt)^{1/2}} \ _2F_1\left(\left\{\frac{1}{2}, \frac{1}{2}\right\}, -\frac{t^2}{1 + bt}\right)$$

$$= \frac{1}{1 + bt + t^2} \ _2F_1\left(\left\{\frac{1}{2}, \frac{1}{2}\right\}, \frac{t(b + t)}{1 + bt + t^2}\right) \cdot \ _2F_1\left(\left\{\frac{1}{2}, \frac{1}{2}\right\}, \frac{t^2}{1 + bt + t^2}\right).$$

**Proof.** Once a factorisation of this type is written down, it is a computational routine to prove it. In other words, a principal issue is discovering such a formula rather than proving it. Our original discovery of Theorem 1 involved a lot of experimental mathematics; however, we later realised that it is deducible from known formulae.
as follows:
\[
\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}^2 x^k = \sum_{k=0}^{\infty} \binom{2k}{k}^2 x^k \sum_{m=0}^{\infty} \binom{k+m}{k}^2 z^{k+m} \\
= \sum_{k=0}^{\infty} \binom{2k}{k}^2 (xz)^k \, _2F_1\left(\frac{k+1}{2}, \frac{k+1}{2} \mid z\right) \\
= \sum_{k=0}^{\infty} \binom{2k}{k}^2 \frac{(xz)^k}{(1-z)^{k+1}} \, _2F_1\left(-k, \frac{k+1}{2} \mid -\frac{z}{1-z}\right) \\
= \frac{1}{1-z} \sum_{k=0}^{\infty} \binom{2k}{k}^2 \left(\frac{xz}{1-z}\right)^k \cdot P_k\left(\frac{1+z}{1-z}\right),
\]
where \(P_k\) denotes the \(k\)-th Legendre polynomial, and the latter generating function is a particular instance of the Bailey–Brafman formula \([15, 34]\) \(\blacksquare\).

We remark that, using the general Bailey–Brafman formula and its generalisation from \([29]\), the proof above extends to the factorisation of the two-variable generating functions
\[
\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{s_k(1-s)_k}{k!^2} x^k
\]
as well as of
\[
\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2k}{k}^2 x^k \quad \text{and} \quad \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{3k}^2 \frac{(3k)!}{k!^3} x^k,
\]
and even of
\[
\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k}^2 u_k x^k
\]
for an Apéry-like sequence \(u_0, u_1, u_2, \ldots\).

Furthermore, we expect that Theorem 1 can lead to a hypergeometric expression for the density function \(\tilde{p}(x)\) (piecewise analytic, with finite support on the interval \(0 < x < 5\)), which is the inverse Mellin transform of \(\tilde{W}(s-1)\), hence to the Mahler measure evaluation
\[
m(1 + x_1 + x_2 + x_3 + x_2 x_3) = \tilde{W}'(0) = \int_0^\infty \tilde{p}(x) \log x \, dx = \int_0^5 \tilde{p}(x) \log x \, dx.
\]

On the other hand, the reduction technique of Sections 3 and 4 suggests a different approach to computing \(\tilde{W}'(0)\), resulting in the following hypergeometric evaluation of the Mahler measure.

**Theorem 2.** We have
\[
m(1 + x_1 + x_2 + x_3 + x_2 x_3) = -\frac{1}{2\pi} \int_0^1 {}_2F_1\left(\frac{1}{2}, \frac{1}{2} \mid 1 - \frac{x^2}{16}\right) \log x \, dx.
\]
Proof. Define a related density \( \hat{p}(x) \) by
\[
\int_0^4 x^s \hat{p}(x) \, dx = \hat{W}(s) = \int_{[0,1]^2} |1 + e^{2\pi i \theta_2} + e^{2\pi i \theta_3} + e^{2\pi i (\theta_2 + \theta_3)}|^s \, d\theta_2 \, d\theta_3
\]
\[
= W_2(s)^2 = \frac{\Gamma(1 + s)^2}{\Gamma(1 + s/2)^4}.
\]
By an application of the Mellin transform calculus, we find that, for \( 0 < x < 4 \),
\[
\hat{p}(x) = \frac{1}{2\pi} \cdot \, _2F_1\left(\frac{1}{2}, \frac{1}{2} \mid 1 - \frac{x^2}{16}\right).
\]
Then it follows from Lemma 1 that
\[
\hat{W}'(0) = \int_1^4 \hat{p}(x) \log x \, dx = -\int_0^1 \hat{p}(x) \log x \, dx,
\]
where we use the evaluation
\[
\int_0^4 \hat{p}(x) \log x \, dx = m(1 + x_2 + x_3 + x_2x_3) = m(1 + x_2) + m(1 + x_3) = 0. \quad \square
\]

The above proof extends to the general formula
\[
m(1 + bx_1 + x_2 + x_3 + x_2x_3) = \log b \int_0^b \hat{p}(x) \, dx + \int_b^4 \hat{p}(x) \log x \, dx
\]
\[
= \frac{1}{2\pi} \int_0^b \, _2F_1\left(\frac{1}{2}, \frac{1}{2} \mid 1 - \frac{x^2}{16}\right) \log \frac{b}{x} \, dx
\]
for \( 0 < b \leq 4 \). A related computation
\[
m(1 + bx_1 + x_2 + x_3 + x_2x_3) = \log b + \frac{8}{\pi^2} \int_b^4 \arccos\left(\frac{b}{x}\right) \log\left(\frac{x}{2\sqrt{b}}\right) \frac{dx}{\sqrt{16 - x^2}}
\]
valid for \( 0 < b \leq 4 \) was given by J. Wan [27]; he also pointed out that \( m(1 + bx_1 + x_2 + x_3 + x_2x_3) = \log b \) for \( b > 4 \) follows from Jensen’s formula.

The left-hand side of another Mahler measure conjecture [13]
\[
m((1 + x_1)^2 + x_2 + x_3) \equiv -L'(\tilde{f}_2; -1) = \frac{72}{\pi^4} L(\tilde{f}_2; 3) = 0.7025655062 \ldots,
\]
where \( \tilde{f}_2(\tau) = \eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau) \) is a cusp form of level 24, can be treated by a similar reduction, using that the densities for \((1 + x_1)^2\) and \(x_2 + x_3\) are \( p_2(t^{1/2})/(2t^{1/2}) \) on \([0, 4]\) and \( p_2(t) \) on \([0, 2]\), respectively. The final result is the elegant formula
\[
m((1 + x_1)^2 + x_2 + x_3) = \frac{2G}{\pi} + \frac{2}{\pi^2} \int_0^1 \arcsin(1 - x) \arcsin x \frac{dx}{x}, \quad (7)
\]
where \( G \) is Catalan’s constant, and, with some further work, we can express the right-hand side hypergeometrically.
Theorem 3. We have
\[
m((1 + x_1)^2 + x_2 + x_3) = \frac{8\Gamma(\frac{3}{4})^2}{\pi^{3/2}} \sum_{r=0}^{3} \binom{3}{r} F_{4}\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4} \mid \frac{1}{4}\right) + \frac{\Gamma(\frac{1}{4})^2}{54\pi^{5/2}} \sum_{r=0}^{3} \binom{3}{r} F_{4}\left(\frac{\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{5}{4}, \frac{3}{4} \mid \frac{1}{4}\right).
\]

Proof. Notice that, for \(0 < x < 1\),
\[
\arcsin(1 - x) = \frac{\pi}{2} - \arccos(1 - x) = \frac{\pi}{2} - \sqrt{2} x F_{1}\left(\frac{1}{2}, \frac{1}{2} \mid \frac{x}{2}\right),
\]
and that, for \(n > -1/2\),
\[
\int_{0}^{1} x^{n-1/2} \arcsin x \, dx = \frac{\sqrt{\pi}}{2n + 1} \left(\sqrt{\pi} - \frac{\Gamma(n + \frac{3}{4})}{\Gamma(\frac{n}{2} + \frac{3}{4})}\right).
\]
Therefore,
\[
\int_{0}^{1} \arcsin(1 - x) \arcsin x \, dx = \frac{\pi}{2} \int_{0}^{1} \arcsin x \, dx - \pi \sqrt{2} \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{(\frac{3}{2}) n (2n + 1)} \frac{1}{2n} + \sqrt{2} \pi \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{3}{2})}{n! (\frac{3}{2}) n (2n + 1)} \frac{1}{2n}.
\]
From this and (7) we deduce
\[
m((1 + x_1)^2 + x_2 + x_3) = \frac{2G}{\pi} + \frac{\log 2}{2} - \frac{2\sqrt{2}}{\pi} \sum_{r=0}^{3} \binom{3}{r} F_{2}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \mid \frac{1}{2}\right) + \frac{8\sqrt{2} \Gamma(\frac{3}{4})^2}{\pi^{3/2} \Gamma(\frac{1}{4})} \sum_{r=0}^{3} \binom{3}{r} F_{4}\left(\frac{\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4} \mid \frac{1}{4}\right) + \frac{\sqrt{2} \Gamma(\frac{1}{4})^2}{54\pi^{5/2} \Gamma(\frac{3}{4})} \sum_{r=0}^{3} \binom{3}{r} F_{4}\left(\frac{\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{5}{4}, \frac{3}{4} \mid \frac{1}{4}\right).
\]
It remains to use
\[
G + \frac{1}{4} \pi \log 2 = \sqrt{2} F_{2}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \mid \frac{1}{2}\right)
\]
(see [10, Entry 30]) and \(\Gamma(\frac{1}{4}) \Gamma(\frac{3}{4}) = \pi \sqrt{2}\). \(\square\)

8. Conclusion

A goal of this final section is to highlight relevance for and links with other research and open problems.

The (hypergeometric) factorisation in Theorem 11 and similar results outlined after its proof are part of a general phenomenon of arithmetic differential equations of order 4. These are the first instances “beyond modularity” in the sense that arithmetic differential equations of order 2 and 3 are always supplied by modular parametrisation. In order 4, we have to distinguish two particular novel situations (though our knowledge about either is imperfect and incomplete): (the Zariski closure of) the monodromy group is the orthogonal group \(O_4 \simeq O_{2,2}\) of dimension 6 or the symplectic group \(Sp_4\) of dimension 10. The example given in Theorem 11 corresponds to the first (orthogonal) situation: on the level of Lie groups, \(O_{2,2}\) can
be realised as the tensor product of two copies of $SL_2$ (or $GL_2$). There is a
limited amount of further examples of this type [21, 29, 33] though we expect that all
underlying Picard–Fuchs differential equations with such monodromy can be repre-
sented as tensor products of two arithmetic differential equations of order 2. There
is a natural hypergeometric production of such orthogonal cases using Orr-type for-
mulae (see [18, 28]) but there are plenty of other cases coming from classical work
of W.N. Bailey and its recent generalisations [29, 34]. Many such cases, mostly
forecast by Sun [25], are still awaiting their explicit factorisation. Though these
situations do not cover symplectic monodromy instances, they can still be viewed
as an intermediate step between classical modularity and $Sp_4$: the antisymmetric
square of the latter happens to be $O_5 \simeq O_{3,2}$ (see [4]).

More in the direction of three-variable Mahler measure, the conjectural evaluation
in [6] and Theorem 2 brings us to the expectation

$$\frac{1}{2\pi} \int_0^1 2F_1 \left( \frac{3}{4}, \frac{1}{2} \left| \frac{x^2}{16} \right. \right) \log x \, dx \equiv 2L'(f_2; -1). \quad (8)$$

This one highly resembles the evaluation

$$\frac{1}{2} \int_0^1 2F_1 \left( \frac{1}{4}, \frac{1}{2} \left| \frac{x^2}{16} \right. \right) \, dx = \frac{1}{2} \cdot 3F_2 \left( \frac{1}{3}, \frac{1}{2}, \frac{1}{2} \left| \frac{1}{16} \right. \right) = 2L'(f_2; 0) \quad (9)$$

established in [22]. The related modular parametrisation

$$x = x(\tau) = 16 \left( \frac{\eta(\tau)\eta(4\tau)^2}{\eta(2\tau)^4} \right)^4$$

corresponds to

$$1 - \frac{x^2}{16} = \left( \frac{\eta(\tau)^2\eta(4\tau)}{\eta(2\tau)^3} \right)^8,$$

$$F \left( \frac{x^2}{16} \right) = \frac{\eta(2\tau)^{10}}{\eta(\tau)^4\eta(4\tau)^4} \quad \text{and} \quad F \left( 1 - \frac{x^2}{16} \right) = -2i\tau F \left( \frac{x^2}{16} \right),$$

where $F$ denotes the corresponding $2F_1$ hypergeometric series. Note that $x$ ranges
from 0 to 4 when $\tau$ runs from $i\infty$ to 0 along the imaginary axis; however, the point
$\tau_0 = i\, 0.8774376613482\ldots$, at which $x(\tau_0) = 1$, is not a quadratic irrationality.

Furthermore, H. Cohen [16] observes another step in the ladder [9], (8):

$$\frac{6}{\pi^2} \int_0^1 2F_1 \left( \frac{1}{2}, \frac{1}{2} \left| \frac{x^2}{16} \right. \right) \log^2 x \, dx \equiv 2L'(f_2; -2) = \frac{3 \cdot 15^3}{8\pi^6} L(f_2; 4) \quad (10)$$

$$= 1.2165632526\ldots,$$

though not linked to a particular Mahler measure.
The expression in Theorem 3 is somewhat different from the one in Theorem 2 and resembles the hypergeometric evaluation of the $L$-value

$$-L'(\hat{f}_2; -1) = \frac{128}{\pi^4} L(\hat{f}_2; 3)$$

$$= \frac{\Gamma(\frac{1}{4})^2}{6\sqrt{2\pi^{5/2}}} \ _3F_3\left(\frac{1}{4}, \frac{3}{2}, \frac{3}{2} \left| \ 1 \right. \right) + \frac{4\Gamma(\frac{3}{4})^2}{\sqrt{2\pi^{5/2}}} \ _3F_3\left(\frac{1}{4}, \frac{3}{2}, \frac{3}{2} \left| \ 1 \right. \right) + \frac{\Gamma(\frac{3}{4})^2}{2\sqrt{2\pi^{5/2}}} \ _3F_3\left(\frac{1}{4}, \frac{3}{2}, \frac{3}{2} \left| \ 1 \right. \right),$$

where $\hat{f}_2(\tau) = \eta(4\tau)^2\eta(8\tau)^2$ is a cusp form of level 32, obtained in [32, Theorem 3].

Finally, we remark that the integral

$$W'_3(0) = \int_0^3 \log x \, dP_3(x) = \log 3 - \int_0^3 P_3(x) \frac{dx}{x}$$

in the notation of Section 5, with $P_3(x)$ related to eta quotients, is visually linked to the following result in [7] (also discussed in greater generality in [2, 26])

$$\int_0^1 \frac{1}{3} \left(1 - \frac{\eta(\tau)^9}{\eta(3\tau)^3}\right) \frac{dq}{q} = \lim_{q \to 1^-} \sum_{m,n=1}^{\infty} \left(\frac{-3}{n}\right) \frac{n}{m} q^{mn} = L'(\chi_{-3}; -1).$$

However, apart from the fact that the two quantities coincide we could not find a direct link between the two integrals.

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