VECTOR-VALUED EXTENSIONS
FOR FRACTIONAL INTEGRALS OF LAGUERRE EXPANSIONS

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Abstract. We prove some vector-valued inequalities for fractional integrals defined in the context of two different orthonormal systems of Laguerre functions. Our results are based on estimates of the corresponding kernels with precise control of the parameters involved. As an application, mixed norm estimates for the fractional integrals related to the harmonic oscillator are deduced.

1. Introduction

Consider the fractional integral
\[ I_\sigma f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\sigma}} \, dy, \quad x \in \mathbb{R}^n, \quad 0 < \sigma < n, \]
defined for any function \( f \) for which the above integral is convergent. Then, with an appropriate constant \( c_\sigma \),
\[ (-\Delta)^{-\sigma/2} f = c_\sigma I_\sigma f, \quad f \in \mathcal{S}(\mathbb{R}^n), \]
where \( \Delta \) is the standard Laplacian in \( \mathbb{R}^n \), and the negative power is defined in \( L^2(\mathbb{R}^n, dx) \) by means of the Fourier transform, see [22, Ch. 5].

The classical Hardy-Littlewood-Sobolev inequality (see, e.g., [8, 22]) establishes that
\[ \| I_\sigma f \|_{L^q(\mathbb{R}^n, dx)} \leq C \| f \|_{L^p(\mathbb{R}^n, dx)}, \]
when \( 1 < p < \frac{n}{\sigma} \) and \( \frac{1}{q} = \frac{1}{p} - \frac{\sigma}{n} \). Moreover, in the case \( (p, q) = (1, \frac{n}{n-\sigma}) \) a weak type inequality holds.

A weighted version of the Hardy-Littlewood-Sobolev was given in [23].

Numerous analogues of the Euclidean fractional integral operator were investigated in various settings for the last decades. For instance, B. Muckenhoupt and E. M. Stein analyzed the topic for ultraspherical expansions in [16]. The one dimensional Hermite and Laguerre function expansions have been considered in [12, 13, 14]. Recently, B. Bongioanni and J. L. Torrea [3] obtained estimates for the negative powers of the multidimensional harmonic oscillator. Fractional integrals for the multidimensional Laguerre expansions (or negative powers of the corresponding second order differential operators) have been treated by A. Nowak and K. Stempak in [18, 19]. They analyzed \( L^p - L^q \) estimates (with and without weights) for the expansions related to Laguerre functions of Hermite type and Laguerre functions of convolution type (this nomenclature is used by S. Thangavelu in [24]). Sharp bounds for the kernel of potential operators in the case of the Laguerre functions of convolution type have been obtained very recently in [18]. Also, a complete and exhaustive study of fractional integrals for Jacobi and Fourier-Bessel expansions has been developed in [17]. Moreover, a vector-valued extension in the Jacobi case was done in [5].

The aim of this paper is the extension of \( L^p - L^q \) mapping properties concerning fractional integrals (or negative powers) related to two second order differential operators of Laguerre type. Namely, our
target will be the proof of vector-valued extensions of some results given in [19]. We will deal with vector-valued inequalities of the form

\[ \left\| \left( \sum_{j=0}^{\infty} |T_j f_j|^r \right)^{1/r} \right\|_{L^p(X,d\mu)} \leq C \left\| \left( \sum_{j=0}^{\infty} |f_j|^r \right)^{1/r} \right\|_{L^p(X,d\mu)}, \]

where \( \{T_j\}_{j \geq 0} \) is a sequence of operators defined on a measure space \((X,d\mu)\). We will also consider weighted vector-valued inequalities.

If we denote, respectively, by \((L^H_\alpha)^{-\sigma}\) and \((L_\alpha)^{-\sigma}\) the fractional operators for Laguerre expansions of Hermite type and for Laguerre expansions of convolution type (see Section 2 for definitions), we are interested in the analysis of vector-valued inequalities for the sequences of operators \(\{(L^H_\alpha)^{-\sigma}\}_{j \geq 0}\) and \(\{(L_\alpha+a_j)^{-\sigma}\}_{j \geq 0}\), where \(a\) is a positive real parameter. The meaning of \(\alpha + aj\) will be explained below. For Laguerre expansions of Hermite type we will consider the space \((\mathbb{R}^n_+, dx)\), with \(dx\) the Lebesgue measure and our approach will be similar to [19]. But in our case, we will prove an estimate for the kernels of the operators \((L^H_\alpha+a_j)^{-\sigma}\) independent of the parameter \(j\). Then a general result of vector-valued extensions will do the work. We will also include some potential weights. On the other hand, we will deal with fractional integrals related to Laguerre expansions of convolution type, where the space considered will be \((\mathbb{R}^n_+, d\mu_\alpha)\) with \(d\mu_\alpha = x^{2\alpha+1} dx\). However, the way to prove the corresponding vector-valued inequality will be close to the ideas in [3]. Observe that in [3], the authors consider Hermite expansions. The argument given in [19] to treat the convolution type setting is based on a very useful convexity principle. Unfortunately, we cannot apply the convexity principle here. The reason is the following: the constants appearing in this case involve Gamma functions whose log-convexity makes these constants increase with no control at all. The new estimates given here. The reason is the following: the constants appearing in this case involve Gamma functions whose log-convexity makes these constants increase with no control at all. The new estimates given in [18] for \((L_\alpha)^{-\sigma}\) are not appropriate for our target either: these estimates, applied to \((L_\alpha+a_j)^{-\sigma}\), yield constants depending on the parameter \(j\) and in that case we could not deduce the vector-valued extension.

As an application of our result on Laguerre expansions of convolution type, we will analyze the fractional integral operator related to spherical eigenfunctions of the harmonic oscillator. Observe that the result in [3] deals with eigenfunctions of the harmonic oscillator in cartesian coordinates. In our situation we consider the eigenfunctions obtained by using spherical coordinates (this is the reason for the name spherical eigenfunctions). In the spherical case, the eigenfunctions are products of Laguerre functions and spherical harmonics. We consider the mixed norm spaces \(L^{p,2}(\mathbb{R}^n, r^{n-1} dr dr)\) (see Section 3 for definition). These spaces were first systematically studied by A. Benedek and R. Panzone in [2]. They arise frequently in harmonic analysis when the spherical harmonics are involved, see [21] [22] [23] [1].

The organization of the paper is the following. In Section 2 we introduce some definitions related to Laguerre systems and establish our results about boundedness of the fractional integrals related. Section 3 contains our analysis of the fractional integrals for the harmonic oscillator in mixed norm spaces. In Section 4 and Section 5 we give the proofs of the results given in Section 2. Finally, in Section 6 we show the proofs of some technical results used along the paper.

2. Definitions and main results

Let \(\alpha = (\alpha_1, \ldots, \alpha_n) \in (-1, \infty)^n\) be a multi-index and \(x, y \in \mathbb{R}^n_+, n \geq 1\). The Laguerre function on \(\mathbb{R}^n_+\) is the tensor product

\[ \varphi^\alpha_k(x) = \varphi^\alpha_{x_1}(x_1) \cdots \varphi^\alpha_{x_n}(x_n), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+, \]

where \(\varphi^\alpha_{x_i}(x_i)\) are the one dimensional Laguerre functions

\[ \varphi^\alpha_{x_i}(x_i) = \left( \frac{2\Gamma(k_i + 1)}{\Gamma(k_i + \alpha_i + 1)} \right)^{1/2} L^\alpha_{k_i}(x^2_i)x_i^{\alpha_i+1/2}e^{-x^2_i/2}, \quad x_i > 0, \quad i = 1, \ldots, n. \]
and $L^\alpha_k$ denotes the Laguerre polynomial of degree $k \in \mathbb{N}$ and order $\alpha > -1$, see [15, p. 76]. We consider the differential operator

\begin{equation}
L^H_\alpha = -\Delta + |x|^2 + \sum_{i=1}^{n} \frac{1}{x_i^2} \left( \alpha_i^2 - \frac{1}{4} \right),
\end{equation}

here $| \cdot |$ stands for the Euclidean norm. The operator $L^H_\alpha$ is symmetric and positive in $L^2(\mathbb{R}_+^n, dx)$ and the Laguerre functions $\varphi^\alpha_k(x)$ are eigenfunctions of $(2.1)$. Indeed, $L^H_\alpha \varphi^\alpha_k = (4|k| + 2|\alpha| + 2n)\varphi^\alpha_k$; by $|\alpha|$ and $|k|$ we denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$ (thus $|\alpha|$ may be negative) and the length $|k| = k_1 + \cdots + k_n$. We will refer to $\varphi^\alpha_k$ as *Laguerre functions of convolution type*.

For each $\sigma > 0$, the fractional integrals for expansions in Laguerre functions of Hermite type are given by

\[(L^H_\alpha)^{-\sigma} f = \sum_{m=0}^{\infty} (4m + 2|\alpha| + 2n)^{-\sigma} P_m f\]

where

\[P_m f = \sum_{|k|=m} a^\alpha_k(f) \varphi^\alpha_k, \quad a^\alpha_k(f) = \int_{\mathbb{R}_+^n} f(x) \varphi^\alpha_k(x) \, dx.\]

With these notations, our result related to the fractional integrals for the Laguerre expansions of $(2.1)$.

**Theorem 2.1.** Let $\alpha \in [-1/2, \infty)^n$. Let $a \geq 1$, $\sigma > 0$, $1 < p \leq q < \infty$, $1 \leq r \leq \infty$, $t < n/p'$, $s < n/q$, $t + s \geq 0$.

(i) If $\sigma \geq n/2$, then there exists a constant depending only on $\sigma$ and $\alpha$ such that

\[\left\| \left( \sum_{j=0}^{\infty} |(L^H_{\alpha+j})^{-\sigma}(f_j)|^r \right)^{1/r} \right\|_{L^q(\mathbb{R}_+^n, |x|^{-\alpha} dx)} \leq C \left\| \left( \sum_{j=0}^{\infty} |f_j|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}_+^n, |x|^s dx)}\]

for all $f_j \in L^p(\mathbb{R}_+^n, |x|^s dx)$.

(ii) If $\sigma < n/2$, then the same boundedness holds under the additional condition

\[
\frac{1}{q} \geq \frac{1}{p} - \frac{2\sigma - t - s}{n}.
\]

**Remark 2.2.** In the previous theorem, the sequence $\alpha + aj$ has to be understood as $(\alpha_1 + aj, \ldots, \alpha_n + aj)$. It will be clear after the proof that this sequence can be changed into $(\alpha_1 + a_1(j), \ldots, \alpha_n + a_n(j))$ where $\{a_i(j)\}_{j \geq 0}$, for $i = 1, \ldots, n$, are positive, increasing and unbounded sequences such that $a_i(0) = 0$ and $a_i(1) \geq 1$.

Let us focus on the second setting. We consider now the differential operator given by

\begin{equation}
L_\alpha = -\Delta + |x|^2 - \sum_{i=1}^{n} \frac{2\alpha_i + 1}{x_i} \frac{\partial}{\partial x_i},
\end{equation}

This is a symmetric operator on $\mathbb{R}_+^n$ equipped with the measure

\[d\mu_\alpha(x) = x_1^{2\alpha_1+1} \cdots x_n^{2\alpha_n+1} dx.\]

The Laguerre functions $L^\alpha_k$ are defined by

\[L^\alpha_k(x) = L^\alpha_k(x_1) \cdots L^\alpha_k(x_n), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n,
\]

where $L^\alpha_k$ are the one dimensional Laguerre functions

\[L^\alpha_k(x_i) = \left( \frac{2\Gamma(k_i + 1)}{\Gamma(k_i + \alpha_i + 1)} \right)^{1/2} L^\alpha_k(x_i^2) e^{-x_i^2/2}, \quad x_i > 0, \quad i = 1, \ldots, n.
\]

The functions $L^\alpha_k$ are eigenfunctions of the differential operator $(2.2)$. Indeed, we have $L_\alpha L^\alpha_k = (4|k| + 2|\alpha| + 2n)L^\alpha_k$. We will refer to the functions $L^\alpha_k$ as *Laguerre functions of convolution type*. 
For the expansions of Laguerre functions of convolution type we define the fractional integrals as

$$(L_\alpha)^{-\sigma}f = \sum_{m=0}^{\infty} (4m + 2|\alpha| + 2n)^{-\sigma} \mathcal{P}_m f$$

where $\sigma > 0$ and

$$\mathcal{P}_m f = \sum_{|k|=m} b^m_k(f) \ell_k^m, \quad b^m_k(f) = \int_{\mathbb{R}^n} f(x) \ell_k^m(x) \, d\mu_\alpha(x).$$

Our result in this case is the following.

**Theorem 2.3.** Let $0 < \sigma < \alpha + 1$, $\alpha \geq -1/2$, $a \geq 1$ and $1 \leq p,q,r \leq \infty$. Define $u_j(x) = x^{aj}$, $x \in \mathbb{R}^+$, $j = 0,1,\ldots$. Then there exists a constant depending only on $\sigma$ and $\alpha$ such that

$$\left( \sum_{j=0}^{\infty} |u_j(L_\alpha+aj)^{-\sigma}(u_j^{-1}f_j)|^r \right)^{1/r} \leq C \left( \sum_{j=0}^{\infty} |f_j|^r \right)^{1/r} \text{ for all } f_j \in L^q(\mathbb{R}^+,d\mu_\alpha),$$

and if only if $p$ and $q$ satisfy

$$\frac{1}{p} = \frac{\sigma}{\alpha+1} \leq \frac{1}{q} < \frac{\sigma}{\alpha+1},$$

with exclusion of the cases $p = 1$ and $q = \frac{\sigma+1}{\sigma+1}$, and $p = \frac{\sigma+1}{\sigma+1}$ and $q = \infty$.

**Remark 2.4.** In the case of Theorem 2.3 we deduce from our proof that the sequence $\alpha + aj$ can be changed into $\alpha + a(j)$ where $a(j)_{j \geq 0}$ is a positive, increasing and unbounded sequence such that $a(0) = 0$ and $a(1) \geq 1$.

### 3. An Application of Theorem 2.3

As we commented in the previous section, the results in Theorem 2.1 and Theorem 2.3 are extensions of some known inequalities for fractional integrals for Laguerre expansions. However, the inequality (2.3) appears in a natural way in the study of fractional integrals related to the harmonic oscillator. Indeed, the eigenfunctions of the harmonic oscillator in $\mathbb{R}^n$ verify

$$(-\Delta + |x|^2)\phi = E\phi,$$

where $E$ is the corresponding eigenvalue. There are two complete sets of eigenfunctions for this equation. Using cartesian coordinates, one obtains the functions

$$\phi_k(x) = \prod_{i=1}^{n} h_{k_i}(x_i), \quad k = (k_1, \ldots, k_n) \in \mathbb{N}^n,$$

where $h_{k_i}(x_i) = (\sqrt{\pi}2^{k_i}k_i!)^{-1/2} H_{k_i}(x_i)e^{-x_i^2/2}$, and $H_j$ denote the Hermite polynomials of degree $j \in \mathbb{N}$ (see [12, p. 60]). In this case $E_k = 2|k| + d$. The system of functions $\{\phi_k\}_{k \in \mathbb{N}^n}$ is orthonormal and complete in $L^2(\mathbb{R}^n, dx)$. The fractional integrals for this system have been studied in [3] and [19]. Vector-valued extensions of the results in both papers for sequences of functions $\{f_j(x)\}_{j \in \mathbb{N}}$ with $x \in \mathbb{R}^n$, are trivial.

But the situation is completely different if we analyze the eigenfunctions of the harmonic oscillator by using spherical coordinates. Let $\mathcal{H}_j$ be the space of spherical harmonics of degree $j$ in $n$ variables. Let $\{\mathcal{Y}_{j,\ell}\}_{\ell=1,\ldots,\dim \mathcal{H}_j}$ be an orthonormal basis for $\mathcal{H}_j$ in $L^2(\mathbb{S}^{n-1}, d\sigma)$. Then the eigenfunctions of the harmonic oscillator, see [4], are given by

$$\phi_{m,\ell,j}(x) = \left( \frac{2\Gamma(j+1)}{\Gamma(m-j+n/2)} \right)^{1/2} L_j^{n/2-1+m-2j}(x)\mathcal{Y}_{m-2j,\ell}(x)e^{-r^2/2}, \quad r^2 = x_1^2 + \cdots + x_n^2,$$

where $m \geq 0$, $j = 0, \ldots, \lfloor m/2 \rfloor$, $\ell = 1, \ldots, \dim \mathcal{H}_{m-2j}$, and $L_j^b$ are Laguerre polynomials of order $b$ and degree $j \in \mathbb{N}$. This system is orthonormal and complete in $L^2(\mathbb{R}^n, dx)$ and the eigenvalues are $E_{m,\ell,j} = (n+2m)$. Moreover

$$L^2(\mathbb{R}^n, dx) = \bigoplus_{m=0}^{\infty} \mathcal{J}_m$$
We start analyzing $O_f$ where we have $p$, with exclusion of the cases $\sigma > 0$. For each $\sigma > 0$, we define the fractional integrals for the harmonic oscillator as

$$(-\Delta + |\cdot|^2)^{-\sigma} f = \sum_{m=0}^{\infty} \frac{1}{(n+2m)^\sigma} \text{Proj}_{\mathcal{J}_m} f,$$

where

$$\text{Proj}_{\mathcal{J}_m} f = \sum_{j=0}^{\dim \mathcal{H}_{m-j}} \sum_{\ell=1}^{\dim \mathcal{H}_{m-j}} c_{m,j,\ell}(\phi_m, j, \ell, f) = \int_{\mathbb{R}^n} \phi_m, j, \ell(y) f(y) dy.$$

The most appropriate spaces in order to analyze this kind of operators are the mixed norm spaces, defined as

$$L^{p,2} (\mathbb{R}^n, r^{n-1} \, dr \, d\sigma) = \{ f(x) : \| f \|_{L^{p,2} (\mathbb{R}^n, r^{n-1} \, dr \, d\sigma)} < \infty \},$$

with the obvious modification in the case $p = \infty$. The main characteristic of these spaces is that we consider the $L^2$-norm in the angular part and the $L^p$-norm in the radial part. They are very different from $L^p(\mathbb{R}^n, dx)$; in fact $L^p(\mathbb{R}^n, dx) \subset L^{p,2} (\mathbb{R}^n, r^{n-1} \, dr \, d\sigma)$ for $p > 2$, $L^2(\mathbb{R}^n, dx) = L^{2,2} (\mathbb{R}^n, r^{n-1} \, dr \, d\sigma)$, and $L^{p,2}(\mathbb{R}^n, r^{n-1} \, dr \, d\sigma) \subset L^p(\mathbb{R}^n, dx)$ for $p < 2$. These spaces are suitable when spherical harmonics are involved. Indeed, if a function $f$ on $\mathbb{R}^n$ is expanded in spherical harmonics,

$$f(x) = \sum_{j=0}^{\dim \mathcal{H}_j} \sum_{\ell=1}^{\dim \mathcal{H}_j} f_{j,\ell}(r) \mathcal{Y}_{j,\ell}(\frac{x}{r}),$$

we have

$$\| f \|_{L^{p,2} (\mathbb{R}^n, r^{n-1} \, dr \, d\sigma)} = \left\| \left( \sum_{j=0}^{\dim \mathcal{H}_j} \sum_{\ell=1}^{\dim \mathcal{H}_j} |f_{j,\ell}(r)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^+, d\mu_{n/2-1})}.$$

From this and using Theorem 2.3, we can prove the following result.

**Theorem 3.1.** Let $n \geq 2$, $0 < \sigma < n/2$, and $1 \leq p, q \leq \infty$. Assume that $p$ and $q$ satisfy

$$\frac{1}{p} - \frac{2\sigma}{n} \leq \frac{1}{q} < \frac{1}{p} + \frac{2\sigma}{n},$$

with exclusion of the cases $p = 1$ and $q = \frac{n}{n-2\sigma}$, and $p = \frac{n}{2\sigma}$ and $q = \infty$. Then there exists a constant depending only on $\sigma$ such that

$$\left\| (-\Delta + |\cdot|^2)^{-\sigma} f \right\|_{L^{p,2} (\mathbb{R}^n, r^{n-1} \, dr \, d\sigma)} \leq C \| f \|_{L^{p,2} (\mathbb{R}^n, r^{n-1} \, dr \, d\sigma)}$$

for all $f \in L^{p,2} (\mathbb{R}^n, r^{n-1} \, dr \, d\sigma)$.

**Proof.** Consider the decomposition

$$(-\Delta + |\cdot|^2)^{-\sigma} f = O_1 f + O_2 f,$$

with

$$O_1 f = \sum_{k=0}^{\infty} \frac{1}{(n+4k)} \text{Proj}_{\mathcal{J}_{2k}} f \quad \text{and} \quad O_2 f = \sum_{k=0}^{\infty} \frac{1}{(n+4k+2)} \text{Proj}_{\mathcal{J}_{2k+1}} f.$$

We start analyzing $O_1$. After some elementary algebraic manipulations, we have

$$O_1 f = \sum_{j=0}^{\dim \mathcal{H}_{2j}} \sum_{\ell=1}^{\dim \mathcal{H}_{2j}} \sum_{k=0}^{\infty} \frac{1}{(n+4j+4k)} c_{2j+2k,j,\ell}(\tilde{f})(\theta_{2j+2k,j,\ell}.$$
By (3.1), we can deduce the following identity immediately
\[ c_{2j+2k,\ell}(f)\delta_{2j+2k,\ell}(x) = b_{k}^{n/2-1+2j}(\cdot)^{-2j}f_{2j,\ell}(r^{2}e_{k}^{n/2-1+2j}(r))Y_{2j,\ell}(\frac{x}{r}), \]
where we used the notation in the previous section for Laguerre expansions of convolution type. Then
\[ O_{1}f(x) = \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim H_{j}} r^{2j}(L_{n/2-1+2j})^{-\sigma}(\cdot)^{-2j}f_{2j,\ell}(r)Y_{2j,\ell}(\frac{x}{r}). \]
In a similar way, we conclude that
\[ O_{2}f(x) = \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim H_{j}} r^{2j+1}(L_{n/2+2j})^{-\sigma}(\cdot)^{-2j-1}f_{2j+1,\ell}(r)Y_{2j+1,\ell}(\frac{x}{r}) \]
and
\[ (-\Delta + |\cdot|^{2})^{-\sigma}f(x) = \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim H_{j}} r^{j}(L_{n/2-1+j})^{-\sigma}(\cdot)^{-j}f_{j,\ell}(r)Y_{j,\ell}(\frac{x}{r}). \]
So, the inequality (3.2) is equivalent to
\[ \left\| \left( \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim H_{j}} (r^{j}(L_{n/2-1+j})^{-\sigma}(\cdot)^{-j}f_{j,\ell}(r))^2 \right)^{1/2} \right\|_{L^{p}(\mathbb{R}^{+},d\mu_{n/2-1})} \leq C \left\| \left( \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim H_{j}} (f_{j,\ell}(r))^2 \right)^{1/2} \right\|_{L^{p}(\mathbb{R}^{+},d\mu_{n/2-1})}, \]
which is a consequence of Theorem 2.3.

**Remark 3.2.** We have just proved that Theorem 2.3 is a consequence of the vector-valued estimates in Theorem 2.4. An alternative proof could be obtained by using the boundedness of the fractional integrals given in [3] Theorem 8 and an observation due to Rubio de Francia in [21] Remark (a). This has been pointed out to us by G. Garrigós in a personal communication.

4. Proof of Theorem 2.4

The heat semigroup related to \( L_{\alpha}^{H} \) is initially defined in \( L^{2}(\mathbb{R}^{n},d\mu_{\alpha}) \) as
\[ T_{\alpha,t}^{H}f = \sum_{m=0}^{\infty} e^{-t(4m+2|\alpha|+2n)} \sum_{|k|=m} \langle f, \varphi_{k}^{\alpha} \rangle \varphi_{k}^{\alpha}, \quad t > 0, \]
and by \( \langle f, g \rangle \) we denote \( \int_{\mathbb{R}^{n}} f(x)\overline{g(x)} \, dx \). We can write the heat semigroup \( \{ T_{\alpha,t}^{H} \}_{t>0} \) as an integral operator
\[ T_{\alpha,t}^{H}f(x) = \int_{\mathbb{R}^{n}} G_{\alpha,t}^{H}(x,y)f(y) \, dy. \]
The Laguerre heat kernel is given by
\[ G_{\alpha,t}^{H}(x,y) = \sum_{m=0}^{\infty} e^{-t(4m+2|\alpha|+2n)} \sum_{|k|=m} \varphi_{k}^{\alpha}(x)\varphi_{k}^{\alpha}(y). \]
The explicit expression for Laguerre heat kernel is known and it can be found in [15] (4.17.6):\n\[ G_{\alpha,t}^{H}(x,y) = (\sinh 2t)^{-n} \exp \left( -\frac{1}{2} \coth(2t)(|x|^{2} + |y|^{2}) \right) \prod_{i=1}^{n} (x_{i}y_{i})^{1/2} I_{\nu} \left( \frac{x_{i}y_{i}}{\sinh 2t} \right), \]
with \( I_{\nu} \) denoting the modified Bessel function of the first kind and order \( \nu \), see [15] Chapter 5.
In the limit case \( \nu = -1/2 \), we put \( \pi_{-1/2} = \frac{1}{2} (\delta_{-1} + \delta_{1}) \). Consequently, for \( \alpha \in [-1/2, \infty)^n \), the kernel can be expressed as

\[
G_{\alpha,t}^H(x,y) = (xy)^{\alpha+1/2} \left( \sinh(2t) \right)^{-n-|\alpha|} \int_{[-1,1]^n} \exp \left( -\frac{1}{2} \coth(2t)(|x|^2 + |y|^2) - \sum_{i=1}^n \frac{x_i y_i s_i}{\sinh(2t)} \right) d\Pi_\alpha(s),
\]

where \( 1 = (1, \ldots, 1) \in \mathbb{N}^n \), \( xy = (x_1 y_1, \ldots, x_n y_n) \), \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) and

\[
d\Pi_\alpha(s) = \prod_{i=1}^n \frac{(1 - s_i^2)^{\alpha_i - 1/2}}{2^{\alpha_i} \sqrt{\pi} (\alpha_i + 1/2)} ds_i.
\]

Let

\[
q_{\pm} = q_{\pm}(x, y, s) = |x|^2 + |y|^2 \pm 2 \sum_{i=1}^n x_i y_i s_i.
\]

Meda’s change of variable

\[
t(\xi) = \frac{1}{2} \log \frac{1 + \xi}{1 - \xi}, \quad \xi \in (0, 1),
\]

leads to

\[
G_{\alpha,t(\xi)}^H(x,y) = (xy)^{\alpha+1/2} \left( 1 - \frac{\xi^2}{2} \right)^{n+|\alpha|} \int_{[-1,1]^n} \exp \left( -\frac{1}{4\xi} q_+(x, y, s) - \frac{\xi}{4} q_-(x, y, s) \right) d\Pi_\alpha(s).
\]

The following technical lemma can be found in [19, Lemma 2.1].

**Lemma 4.1.** Let \( a \in \mathbb{R} \) be fixed and \( T > 0 \). Then

\[
\int_0^1 \zeta^{-a} \exp(-T\zeta^{-1}) d\zeta \leq C \exp(-T/2), \quad T \geq 1,
\]

and for \( 0 < T < 1 \)

\[
\int_0^1 \zeta^{-a} \exp(-T\zeta^{-1}) d\zeta \begin{cases} T^{-a+1}, & a > 1, \\ \log(2/T), & a = 1, \\ 1, & a < 1. \end{cases}
\]

**Proposition 4.2.** Let \( \alpha \in [-1/2, \infty)^n \). Then

\[
G_{\alpha,t(\xi)}^H(x,y) \leq C \left( \frac{1 - \xi^2}{\xi} \right)^{n/2} \exp \left( -\frac{|x-y|^2}{4\xi} - \frac{\xi|x+y|^2}{4} \right),
\]

with \( C \) independent of \( \alpha \).

**Proof.** Let \( q_{\pm,i} = q_{\pm,i}(x_i, y_i, s_i) = x_i^2 + y_i^2 \pm 2x_i y_i s_i \), for \( i = 1, \ldots, n \). From this identity and (4.2), it follows that \( q_{\pm}(x, y, s) = \sum_{i=1}^n q_{\pm,i}(x_i, y_i, s_i) \). Observe that

\[
\int_{[-1,1]^n} \exp \left( -\frac{q_+}{4\xi} - \frac{\xi q_-}{4} \right) d\Pi_\alpha(s) = \prod_{i=1}^n \int_{-1}^1 \exp \left( -\frac{q_{+,i}}{4\xi} - \frac{\xi q_{-,i}}{4} \right) d\Pi_\alpha(s_i),
\]
so it suffices to deal with the integral in dimension one. The case \( \alpha_i = -1/2 \) is elementary, so we obtain the estimate for \( \alpha_i > -1/2 \). We write

\[
J := \int_{-1}^{1} \exp \left( -\frac{q_{j,i}}{4\xi} - \frac{\xi q_{j,i}}{4} \right) (1 - s_i^2)^{\alpha_i - 1/2} \, ds_i.
\]

With the change of variable \( s_i = 2u - 1 \) we have

\[
J = 4^{\alpha_i} \exp \left( -\frac{(x_i - y_i)^2}{4\xi} - \frac{\xi (x_i + y_i)^2}{4} \right) \int_0^1 \exp \left( -x_i y_i (\frac{1}{\xi} - \xi) \right) u^{\alpha_i - 1/2} (1 - u)^{\alpha_i - 1/2} \, du.
\]

It is easy to check that

\[
\int_{1/2}^1 \exp \left( -x_i y_i (\frac{1}{\xi} - \xi) \right) u^{\alpha_i - 1/2} (1 - u)^{\alpha_i - 1/2} \, du
\]

\[
\leq \int_0^{1/2} \exp \left( -x_i y_i (\frac{1}{\xi} - \xi) \right) u^{\alpha_i - 1/2} (1 - u)^{\alpha_i - 1/2} \, du
\]

and then

\[
J \leq 4^{\alpha_i + 1/2} \exp \left( -\frac{(x_i - y_i)^2}{4\xi} - \frac{\xi (x_i + y_i)^2}{4} \right) \int_0^{1/2} \exp \left( -x_i y_i (\frac{1}{\xi} - \xi) \right) u^{\alpha_i - 1/2} \, du.
\]

Now, taking \( x_i y_i (\frac{1}{\xi} - \xi) = z \), we get

\[
J \leq \frac{4^{\alpha_i}}{(x_i y_i)^{\alpha_i + 1/2}} \left( \frac{\xi}{1 - \xi^2} \right)^{\alpha_i + 1/2} \exp \left( -\frac{(x_i - y_i)^2}{4\xi} - \frac{\xi (x_i + y_i)^2}{4} \right) \int_0^1 e^{-z} z^{\alpha_i - 1/2} \, dz
\]

\[
\leq \frac{4^{\alpha_i}}{(x_i y_i)^{\alpha_i + 1/2}} \left( \frac{\xi}{1 - \xi^2} \right)^{\alpha_i + 1/2} \exp \left( -\frac{(x_i - y_i)^2}{4\xi} - \frac{\xi (x_i + y_i)^2}{4} \right) \Gamma(\alpha_i + 1/2).
\]

From (4.1), (4.3) and (4.5), the estimate in the proposition follows. \( \square \)

For each \( \sigma > 0 \), we define the potential operator

\[
\mathcal{I}_{\alpha, \sigma}^H f(x) = \int_{\mathbb{R}^n} \mathcal{H}_{\alpha, \sigma}^H(x, y) f(y) \, dy, \quad x \in \mathbb{R}^n,
\]

where

\[
\mathcal{H}_{\alpha, \sigma}^H(x, y) = \frac{1}{\Gamma(\sigma)} \int_0^{\infty} G_{\alpha, t}^H(x, y) t^{\sigma - 1} \, dt, \quad x, y \in \mathbb{R}^n
\]

is the potential kernel.

Define the auxiliary convolution kernel \( K_\sigma(x), x \in \mathbb{R}^n \setminus \{0\} \), by

\[
K_\sigma(x) = \exp(-c|x|^2), \quad |x| \geq 1,
\]

and for \( |x| < 1 \),

\[
K_\sigma(x) = \begin{cases} \frac{1}{|x|^{n-2\sigma}}, & \sigma < n/2, \\ \log \left( \frac{1}{|x|^\sigma} \right), & \sigma = n/2, \\ 1, & \sigma > n/2. \end{cases}
\]

**Proposition 4.3.** Let \( \sigma > 0 \) and \( \alpha \in [-1/2, \infty)^n \). Then

\[
\mathcal{H}_{\alpha, \sigma}^H(x, y) \leq C_\sigma K_\sigma(x - y),
\]

with \( C_\sigma \) independent of \( \alpha \).
Proof. By (4.6), Meda’s change of variable (4.3), (4.4), and Proposition 4.2 we have

\[
\mathcal{H}^H_{\alpha,\sigma}(x,y) \leq \frac{C}{2^{\alpha+1}} \int_0^1 \left( \log \frac{1 + \xi}{1 - \xi} \right)^{\sigma-1} (1 - \xi^{2^{\alpha+1}})^{-\frac{n}{2}} \exp \left( -\frac{1}{8\xi^2} |x-y|^2 \right) d\xi
\]

\[
= \int_{1/2}^1 + \int_{1/2}^1 =: I_1 + I_2.
\]

Concerning \(I_1\), observe that there exists \(C\) such that \(\log \frac{1 + \xi}{1 - \xi} < C\xi\), for \(\xi \in (0,1/2)\). Then, we apply Lemma 4.1 with \(a = -\sigma + 1 + n/2\) and \(T = |x-y|^2/4\), and we obtain

\[
I_1 \leq C\sigma \int_0^{1/2} \xi^{\sigma-1/2} \exp \left( -\frac{1}{8\xi^2} |x-y|^2 \right) d\xi \leq C\sigma \begin{cases} 
\frac{1}{|x-y|^{n-\sigma}}, & \sigma < \frac{n}{2}, \\
\log \frac{1}{|x-y|}, & \sigma = \frac{n}{2}, \\
1, & \sigma > \frac{n}{2},
\end{cases}
\]

for \(|x-y| < 2\), and

\[
I_1 \leq C\sigma \exp(-c|x-y|^2)
\]

for \(|x-y| \geq 2\), which is equivalent to the required bound. Now we deal with \(I_2\). By using that \(\xi^{-n/2} \sim 1\), for \(\xi \in (1/2,1)\), and reverting Meda’s change of variable yield

\[
I_2 \leq C\sigma \exp(-c|x-y|^2) \int_{1/2}^1 \left( \log \frac{1 + \xi}{1 - \xi} \right)^{\sigma-1} (1 - \xi^{2^{\alpha+1}}) d\xi
\]

\[
\leq C\sigma \exp(-c|x-y|^2) \int_{\log 2}^{\infty} \|w\|^{\sigma-1} \exp(-w/2) dw \leq C\sigma \exp(-c|x-y|^2),
\]

which is enough for our purposes. \(\square\)

On the other hand, we will use the following result about vector-valued extensions of bounded operators. This result is a version of [11, Ch. 5, Theorem 1.12] in the setting of \(\ell^r\) spaces.

**Lemma 4.4.** Consider \(L^p = L^p(X, m)\), where \((X, m)\) is a measure space. Let \(T : L^r \to L^q\) be a bounded linear operator which is positive (i.e. \(q(x) \geq 0\) implies \(Ty(x) \geq 0\), \(1 \leq p,q \leq \infty\), with norm \(|T|\). Then \(T\) has an \(\ell^r\)-valued extension for \(1 \leq r \leq \infty\) and

\[
\left\| \left( \sum_j |T_j f_j| \right)^{1/r} \right\|_{L^r} \leq |T| \left\| \left( \sum_j |f_j| \right)^{1/r} \right\|_{L^r}, \quad f_j \in L^p.
\]

**Proof of Theorem 2.2.** First we have to prove that \((L^H_{\alpha})^{-\sigma} = \mathcal{T}^H_{\alpha,\sigma}\), as operators on \(L^2(\mathbb{R}^n_+, dx)\). This is obtained by showing that both operators, being bounded in \(L^2(\mathbb{R}^n_+, dx)\), coincide on the linear span of the functions \(\varphi_k^\sigma\) which is a dense subspace of \(L^2(\mathbb{R}^n_+, dx)\). Indeed, in order to check \((L^H_{\alpha})^{-\sigma} \varphi_k^\sigma = \mathcal{T}^H_{\alpha,\sigma} \varphi_k^\sigma\), we write

\[
\int_{\mathbb{R}^n_+} \mathcal{H}^H_{\alpha,\sigma}(x,y) \varphi_k^\sigma(y) dy = \int_{\mathbb{R}^n_+} \int_0^\infty G^H_{\alpha,\sigma}(x,y) t^{\sigma-1} dt \varphi_k^\sigma(y) dy
\]

\[
= \varphi_k^\sigma(x) \int_0^\infty e^{-t(4|k|+2|\alpha|+2n)} t^{\sigma-1} dt = \Gamma(\sigma)(L^H_{\alpha})^{-\sigma} \varphi_k^\sigma(x).
\]

Application of Fubini’s theorem in the second identity was possible since \(\mathcal{H}^H_{\alpha,\sigma}(x,\cdot) \leq K_\sigma(x,\cdot) \in L^1(\mathbb{R}^n_+, dx)\), for each \(x \in \mathbb{R}^n_+\), and \(\varphi_k^\sigma \in L^\infty(\mathbb{R}^n_+, dx)\).

Now we observe that, by Proposition 4.3, there exists a constant \(C_\sigma\) depending only on \(\sigma\) such that for a nonnegative function \(f\),

\[
\mathcal{T}^H_{\alpha+a,\sigma}(f)(x) \leq C_\sigma \int_{\mathbb{R}^n_+} K_\sigma(x-y) f(y) dy.
\]
By [19, Theorem 2.5] and Lemma 1.3 (note that $K_\sigma(x - y)$ is positive), there exists a constant $C$ depending only on $\sigma, s$ and $t$ such that
\[
\left\| \left( \sum_{j=0}^{\infty} \left| \int_{\mathbb{R}_+^n} K_\sigma(x - y) f(y) \, dy \right|^r \right)^{1/r} \right\|_{L^q(\mathbb{R}_+^n, |x|^{-\alpha} \, dx)} \leq C \left\| \left( \sum_{j=0}^{\infty} |f_j|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}_+^n, |x|^\sigma \, dx)},
\]
for $f_j \in L^p(\mathbb{R}_+^n, |x|^\sigma \, dx)$.

Therefore,
\[
\left\| \left( \sum_{j=0}^{\infty} |T_{\alpha+aj,\sigma}^H f_j|^r \right)^{1/r} \right\|_{L^q(\mathbb{R}_+^n, |x|^{-\alpha} \, dx)} \leq C \left\| \left( \sum_{j=0}^{\infty} |f_j|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}_+^n, |x|^\sigma \, dx)},
\]
and the proof is complete. \qed

5. Proof of Theorem 2.3

Recall the differential operator $L_\alpha$ given in (2.2). The heat semigroup related to $L_\alpha$ is initially defined in $L^2(\mathbb{R}_+^n, d\mu_\alpha)$ as
\[
T_{\alpha,t} f = \sum_{m=0}^{\infty} e^{-t(4m+2|\alpha|+2n)} \sum_{|k|=m} \langle f, \ell_k^\alpha \rangle_{d\mu_\alpha} \ell_k^n, \quad t > 0,
\]
and by $\langle f, g \rangle_{d\mu_\alpha}$ we denote $\int_{\mathbb{R}_+^n} f(x)g(x) \, d\mu_\alpha(x)$. We can write the heat semigroup $\{T_{\alpha,t}\}_{t>0}$ as an integral operator
\[
T_{\alpha,t} f(x) = \int_{\mathbb{R}_+^n} G_{\alpha,t}(x, y)f(y) \, d\mu_\alpha(y).
\]

The Laguerre heat kernel in this case is given by
\[
G_{\alpha,t}(x, y) = \sum_{m=0}^{\infty} e^{-t(4m+2|\alpha|+2n)} \sum_{|k|=m} \ell_k^n(x) \ell_k^n(y).
\]

Observe that
\[
G_{\alpha,t}(x, y) = G_{\alpha,t}^H(x, y)(xy)^{-\alpha-1/2}.
\]

We define the potential operator
\[
\mathcal{L}_{\alpha,\sigma} f(x) = \int_{\mathbb{R}_+^n} \mathcal{H}_{\alpha,\sigma}(x, y)f(y) \, d\mu_\alpha(y), \quad x \in \mathbb{R}_+^n,
\]
where
\[
\mathcal{H}_{\alpha,\sigma}(x, y) = \frac{1}{\Gamma(\sigma)} \int_0^{\infty} G_{\alpha,t}(x, y)t^{\sigma-1} \, dt, \quad x, y \in \mathbb{R}_+^n
\]
is the potential kernel. Due to (5.1),
\[
\mathcal{H}_{\alpha,\sigma}(x, y) = \mathcal{H}_{\alpha,\sigma}^H(x, y)(xy)^{-\alpha-1/2}.
\]

For, $n = 1$, which is the case in which we are interested from now on, by using the estimates in the previous section for $\mathcal{H}_{\alpha,\sigma}^H$ and the relation (5.1), we obtain that
\[
\mathcal{H}_{\alpha+aj,\sigma}(x, y) \leq C_{\alpha} K_\sigma(x - y)(xy)^{-\alpha-aj-1/2}.
\]

If one uses this estimate to attain $L^p - L^q$ inequalities, then it turns out that we need an extra restriction on the parameters. In fact, the condition $\frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1}$ arises in the estimate of $\mathcal{H}_{\alpha+aj,\sigma}(x, y)$ due to the presence of the factor $(xy)^{-\alpha-1/2}$ and the measure $d\mu_\alpha$. So, in order to take out such restriction on $p$ and $q$, one has to get suitable estimates for the kernel. In this way, the proof of Theorem 2.3 is based on the estimates collected in Propositions 5.1 and 5.6.
Proposition 5.1. Let $\alpha \geq -1/2$, $a \geq 1$, $j \in \mathbb{N}$, and $0 < \sigma < \alpha + 1$. Then

$$H_{\alpha+j,\sigma}(x, y) \leq C(xy)^{-\alpha-j}K_{\alpha,\sigma}(x, y), \quad x, y \in \mathbb{R}^+,$$

where $C$ depends on $\alpha$ and $\sigma$, but not on $j$, and

$$K_{\alpha,\sigma}(x, y) = \frac{1}{(x+y)^{2\alpha+1}} \begin{cases} W_{\alpha,\sigma}(x, y), & |x-y| < 1, \\ e^{-c(x-y)^2}, & |x-y| \geq 1, \end{cases}$$

for some constant $c > 0$, with

$$W_{\alpha,\sigma}(x, y) = \begin{cases} |x-y|^{2\sigma-1}, & \sigma < 1/2, \\ \log \left( \frac{x+y}{|x-y|} \right), & \sigma = 1/2, \\ \min \{(x+y)^{2\sigma-1}, 1\}, & \sigma > 1/2. \end{cases}$$

The two following lemmas will provide us the main tools to prove the previous proposition.

Lemma 5.2. Let $c > -1$ and $\ell$ be such that $0 < \sigma < c + \ell$ and $a > 0$. Then

$$\int_0^1 \left( \log \left( \frac{1+\xi}{1-\xi} \right) \right)^{\sigma-1} (1-\xi^2)^c \exp \left( -\frac{a}{4\xi} \right) d\xi \leq C A^4 \Gamma(c + \ell - \sigma),$$

where $C$ is independent of $c$.

Lemma 5.3. Let $\alpha \geq -1/2$, $\lambda \in \mathbb{R}$, $b \geq 1$, $0 < B < A$, and

$$I_{\alpha, b}^\lambda = \int_0^1 \frac{(1-s)^{\alpha+b-1/2}}{(A- Bs)^{\alpha+b+\lambda+1/2}} ds.$$

Then, for $\lambda > 0$,

$$I_{\alpha, b}^\lambda \leq \frac{\Gamma(b+\lambda)}{\Gamma(b+\lambda+\lambda+1/2)} \frac{1}{A^{\alpha+1/2}} \frac{1}{B^\lambda};$$

for $\lambda = 0$,

$$I_{\alpha, b}^0 \leq \frac{1}{A^{\alpha+1/2}} \frac{1}{B^\lambda} \log \left( \frac{A}{A-B} \right);$$

and, for $\lambda < 0$,

$$I_{\alpha, b}^\lambda \leq \frac{1}{A^{\alpha+\lambda+1/2}} \frac{1}{B^\lambda}.$$

Additionally, for the case $|x-y| < 1$, $x + y > 2$, and $\sigma > 1/2$ we will use the following result.

Lemma 5.4. Let $\alpha \geq -1/2$, $a \geq 1$, and $j \in \mathbb{N}$. Then

$$G_{\alpha+j,\ell}(x, y) \leq \frac{(1-\xi^2)^{\ell/2}}{\xi^{1/2}} (xy)^{-\alpha-j-\ell-1/2}, \quad x, y \in \mathbb{R}^+,$$

with $C$ a constant independent of $j$.

Also, in the case $|x-y| \geq 1$, the result will follow from an appropriate estimate of the heat kernel.

Lemma 5.5. Let $\alpha \geq -1/2$, $a \geq 1$, and $j \in \mathbb{N}$. Then

$$G_{\alpha+j,\ell}(x, y) \leq \frac{\Gamma(\alpha)}{\Gamma(\alpha + aj + 1/2)} \exp \left( -\frac{(x-y)^2}{8\xi} \right) \frac{(1-\xi^2)^{\alpha+1}}{\xi^{\alpha+1}} |x^2 - y^2|^{-2(\alpha+1)} (xy)^{-aj}, \quad x, y \in \mathbb{R}^+,$$

with $C$ a constant independent of $j$.

The previous lemmas are rather technical and their proofs will be given in the last section.
Indeed, for $\sigma < 1$, by [5.3], (5.1), and Lemma 5.2 with $c = \alpha + aj$, $\ell = 1$ and $a = q+$, and the change of variable $s = 1 - 2u$, we have

$$
\mathcal{H}_{\alpha + aj, \sigma}(x, y) \leq C \Gamma(\alpha + aj + 1 - \sigma) \int_{-1}^{1} \frac{(1 - s^2)^{\alpha + aj - 1/2}}{q^{\alpha + aj + 1 - \sigma} s} ds
$$

$$
= C 4^{\alpha + aj} \Gamma(\alpha + aj + 1 - \sigma) \int_{0}^{1} \frac{(1 - u)^{\alpha + aj - 1/2} u^{\alpha + aj - 1/2}}{(x + y)^{\alpha + aj - 1/2} (4xy)^{\alpha + aj + 1 - \sigma}} du
$$

$$
\leq C 4^{\alpha + aj} \Gamma(\alpha + aj + 1 - \sigma) \int_{0}^{1} \frac{(1 - u)^{\alpha + aj - 1/2}}{(x + y)^{\alpha + aj - 1/2} (4xy)^{\alpha + aj + 1 - \sigma}} du.
$$

Finally, we conclude by using Lemma 5.3 with $b = aj$, $\lambda = 1/2 - \sigma$, $A = (x + y)^2$, and $B = 4xy$. Indeed, for $\sigma < 1/2$,

$$
\mathcal{H}_{\alpha + aj, \sigma}(x, y) \leq C 4^{\alpha + aj} \Gamma(\alpha + aj + 1 - \sigma) \Gamma(aj) \frac{1}{(x + y)^{2\alpha + 1}} \frac{1}{xy^{2j}} \frac{1}{|x - y|^{1 - 2\sigma}}.
$$

for $\sigma = 1/2$

$$
\mathcal{H}_{\alpha + aj, \sigma}(x, y) \leq C \frac{1}{(x + y)^{2\alpha + 1}} \frac{1}{xy^{2j}} \log \left( \frac{x + y}{|x - y|} \right);
$$

and, for $\sigma > 1/2$,

$$
\mathcal{H}_{\alpha + aj, \sigma}(x, y) \leq C \frac{1}{(x + y)^{2\alpha + 2 - 2\sigma}} \frac{1}{xy^{2j}}.
$$

The previous estimate for $\mathcal{H}_{\alpha + aj, \sigma}$, in the case $\sigma > 1/2$, will be used when $\max \{|x, y| \leq 2$ and $|x - y| < 1$. For $\max \{|x, y| > 2$ and $|x - y| < 1$, we obtain a sharper inequality by using Lemma 5.4.

Indeed,

$$
\mathcal{H}_{\alpha + aj, \sigma}(x, y) \leq C (xy)^{-\alpha - \sigma} \int_{0}^{1} \left( \log \frac{1 + \xi}{1 - \xi} \right)^{\sigma - 1} (1 - \xi^{2})^{-1/2} \xi^{\sigma - 1} d\xi
$$

$$
\leq C (xy)^{-\alpha} (x + y)^{- (2\alpha + 1)},
$$

where we have used that in this region $xy \sim x^2 + y^2$ and that the integral is convergent because $\sigma > 1/2$.

To bound the kernel in the case $|x - y| \geq 1$, we use [5.3], Meda’s change of variable (1.3), (1.4), and Lemma [5.3] to obtain that

$$
\mathcal{H}_{\alpha + aj, \sigma}(x, y) \leq C (x + y)^{- (2\alpha + 1)} (xy)^{-\alpha} \exp \left( - \frac{(x - y)^2}{16} \right)
$$

$$
\times \int_{0}^{1} \left( \log \frac{1 + \xi}{1 - \xi} \right)^{\sigma - 1} (1 - \xi^{2})^{\alpha} \xi^{\sigma - 1} \exp \left( - \frac{(x - y)^2}{16\xi} \right) d\xi.
$$

The last integral can be controlled by a constant by applying Lemma 5.2 with $c = \alpha$ and $\ell = 1$, and the condition $|x - y| \geq 1$. Then

$$
\mathcal{H}_{\alpha + aj, \sigma}(x, y) \leq C (x + y)^{- (2\alpha + 1)} (xy)^{-\alpha} \exp \left( - \frac{(x - y)^2}{16} \right)
$$

and the proof is finished.\qed

The next auxiliary result will be used in the proof of Theorem 2.3.

**Proposition 5.6.** Let $\alpha \geq -1/2$, $a \geq 1$, $j \in \mathbb{N} \cup \{0\}$, and $0 < \sigma < \alpha + 1$. Then,

$$
u_{j}(x)(L_{\alpha + aj})^{-\sigma}(u_{j}^{-1}(\cdot))f(x) \leq C \int_{0}^{\infty} f(y)\mathcal{H}_{\alpha, \sigma}(x, y) d\mu_{a}(y),
$$

(5.6)
where $C$ is independent of $j$ and

$$\mathcal{K}_{\alpha,\sigma}(x,y) = (xy)^{-\alpha-1/2}\int_0^1 \left( \log \left( \frac{1 + \xi}{1 - \xi} \right) \right)^{\sigma-1} \xi^{-1/2}(1 - \xi^{-2})^{-1/2} \exp \left( -\frac{(x-y)^2}{4\xi} - \frac{(x+y)^2}{4} \right) d\xi.$$ 

Moreover, for $x > 2$,

$$(5.7) \quad x^{-2\sigma} \int_0^\infty \mathcal{K}_{\alpha,\sigma}(x,y) \, d\mu_\alpha(y) \leq C.$$ 

The proof of the necessity of the condition (2.4) in Theorem 2.3 is an immediate consequence of Theorem 2.2 in [15] where the boundedness of $(L_\alpha)^{-\sigma}$ is characterized. The sufficiency of (2.4) will be obtained from the following result and the Riesz-Thorin interpolation theorem.

**Theorem 5.7.** Let $\alpha \geq -1/2$, $0 < \sigma < \alpha + 1$, $a \geq 1$, and $1 \leq p, q, r \leq \infty$. Then:

a) If $1 - \frac{\sigma}{\alpha + 1} < \frac{1}{r} \leq 1$, there exists a constant $C$ such that

$$\left\| \left( \sum_{j=0}^{\infty} |u_j(L\alpha + aj)^{-\sigma}(w_j^{-1}f_j)|^r \right)^{1/r} \right\|_{L^q(\beta, d\mu_\alpha)} \leq C \left\| \left( \sum_{j=0}^{\infty} |f_j|^r \right)^{1/r} \right\|_{L^p(\beta, d\mu_\alpha)}.$$ 

b) If $\frac{1}{p} < \frac{\sigma}{\alpha + 1}$, there exists a constant $C$ such that

$$\left\| \left( \sum_{j=0}^{\infty} |u_j(L\alpha + aj)^{-\sigma}(w_j^{-1}f_j)|^r \right)^{1/r} \right\|_{L^q(\beta, d\mu_\alpha)} \leq C \left\| \left( \sum_{j=0}^{\infty} |f_j|^r \right)^{1/r} \right\|_{L^p(\beta, d\mu_\alpha)}.$$ 

c) If $\frac{1}{q} < \frac{\sigma}{\alpha + 1}$, there exists a constant $C$ such that

$$\left\| \left( \sum_{j=0}^{\infty} |u_j(L\alpha + aj)^{-\sigma}(w_j^{-1}f_j)|^{1/r} \right) \right\|_{L^q(\beta, d\mu_\alpha)} \leq C \left\| \left( \sum_{j=0}^{\infty} |f_j|^p \right)^{1/r} \right\|_{L^p(\beta, d\mu_\alpha)}.$$ 

d) If $1 - \frac{\sigma}{\alpha + 1} < \frac{1}{p} \leq 1$, there exists a constant $C$ such that

$$\left\| \left( \sum_{j=0}^{\infty} |u_j(L\alpha + aj)^{-\sigma}(w_j^{-1}f_j)|^{1/r} \right) \right\|_{L^q(\beta, d\mu_\alpha)} \leq C \left\| \left( \sum_{j=0}^{\infty} |f_j|^r \right)^{1/r} \right\|_{L^p(\beta, d\mu_\alpha)}.$$ 

e) If $p > 1$, $q < \infty$ and $\frac{1}{p} - \frac{\sigma}{\alpha + 1} = \frac{1}{q}$, there exists a constant $C$ such that

$$\left\| \left( \sum_{j=0}^{\infty} |u_j(L\alpha + aj)^{-\sigma}(w_j^{-1}f_j)|^{1/r} \right) \right\|_{L^q(\beta, d\mu_\alpha)} \leq C \left\| \left( \sum_{j=0}^{\infty} |f_j|^r \right)^{1/r} \right\|_{L^p(\beta, d\mu_\alpha)}.$$ 

The proof of e) is a consequence of a classical result for the Hardy operator (see [10] Theorem 1) and another result about the boundedness of local one-dimensional fractional integrals (see [6] Corollary 5.4). These results are stated below.

**Theorem 5.8.** Let $f$ be a nonnegative function defined on $(0, +\infty)$. Define $F(x) = \int_0^x f(t) \, dt$, if $\gamma > -1$, and $F(x) = \int_x^\infty f(t) \, dt$, if $\gamma < -1$. For $1 \leq p \leq q \leq \infty$, $\gamma \neq -1$, we have

$$\left( \int_0^\infty F(x)^q x^{-1-\gamma(\gamma+1)} \, dx \right)^{1/q} \leq B(p, q, \gamma) \left( \int_0^\infty f(x)^p x^{-\gamma+1} \, dx \right)^{1/p}.$$ 

**Theorem 5.9.** Let $f$ be a nonnegative function defined on $(0, +\infty)$. Let $0 < \beta < 1$ and consider the local fractional integral in $(0, +\infty)$:

$$L_\beta^\text{loc} f(x) = \int_{x/2}^{3x/2} \frac{f(t)}{|x-t|^{1-\beta}} \, dt.$$ 

Then

$$\left( \int_0^\infty L_\beta^\text{loc} f(x)^q x^\sigma \, dx \right)^{1/q} \leq C \left( \int_0^\infty f(x)^p x^b \, dx \right)^{1/p}.$$
holds for \(1 \leq p \leq q < \infty\) if and only if \((b+1)/p - (a+1)/q = \beta\) and either \(1/p - 1/q \leq \beta\) for \(p > 1\), or \(1/p - 1/q < \beta\) for \(p = 1\).

**Proof of Theorem 5.7.** As in the proof of Theorem 2.1 it can be checked easily that \((L_\alpha)^{-\sigma} = \mathcal{I}_{\alpha, \sigma}\) so we omit the details.

In the five parts of the proof we will analyze \(u_j(L_{\alpha+\sigma})^{-\sigma}(u_j^{-1}f_j)\) for \(j \geq 1\). The case \(j = 0\) can be deduced from Theorem 2.2 in [18].

By (5.2) and Proposition 5.1 it is clear that

\[
u_j(x)(L_{\alpha+\sigma})^{-\sigma}(u_j^{-1}f_j)(x) \leq C \int_0^\infty f_j(y)K_{\alpha, \sigma}(x, y) \, d\mu_\alpha(y),
\]

where \(K_{\alpha, \sigma}\) is that in (5.5).

**Proof of a)** The inequality in a) will be deduced from the estimate

\[
\left\| \int_0^\infty f(y)K_{\alpha, \sigma}(x, y) \, d\mu_\alpha(y) \right\|_{L^q(\mathbb{R}_+, d\mu_\alpha)} \leq C \|f\|_{L^1(\mathbb{R}_+, d\mu_\alpha)}
\]

and Lemma 4.3. Applying Minkowski inequality, the previous inequality is a consequence of the estimate

\[
(5.8) \quad \|K_{\alpha, \sigma}(\cdot, y)\|_{L^q(\mathbb{R}_+, d\mu_\alpha)} \leq C, \quad 1 < q < \infty.
\]

To prove the previous bound, we consider the cases \(|x - y| < 1\) and \(|x - y| \geq 1\). For \(|x - y| < 1\) we distinguish three cases:

Case \(\sigma < 1/2\). For \(y \geq 1\), we have

\[
\int_{|x - y| < 1} (K_{\alpha, \sigma}(x, y))^q \, d\mu_\alpha(x) \leq Cy^{(2\alpha+2)(1-q) + 2q \sigma} \int_{|t-1| < 1/y} |1 - t|^{-q(1-2\sigma)} \, dt \sim y^{(2\alpha+1)(1-q)} \leq C.
\]

When \(y < 1\), it is verified that

\[
\int_{|x - y| < 1} (K_{\alpha, \sigma}(x, y))^q \, d\mu_\alpha(x) \leq Cy^{(2\alpha+2)(1-q) + 2q \sigma} \left( \int_0^{1+1/y} \frac{t^{2\alpha+1}}{[1 - t]^{-q(1-2\sigma)}(1 + t)^{q(2\alpha+1)}} \, dt \right)\]

\[
\sim y^{(2\alpha+2)(1-q) + 2q \sigma} + 1 \leq C,
\]

where in the last step we used that \(1 - \frac{\sigma}{q+1} < \frac{1}{q}\).

Case \(\sigma = 1/2\). For \(y \geq 1\), we have

\[
\int_{|x - y| < 1} (K_{\alpha, \sigma}(x, y))^q \, d\mu_\alpha(x) \leq Cy^{(2\alpha+2)(1-q) + q} \int_{|t-1| < 1/y} \left[ \log \left( \frac{t + 1}{|t - 1|} \right) \right]^q \, dt.
\]

And the last integral can be controlled as follows

\[
\int_{|t-1| < 1/y} \left[ \log \left( \frac{t + 1}{|t - 1|} \right) \right]^q \, dt \sim \int_{|t-1| < 1/y} \left( -\log |t - 1| \right)^q \, dt \sim \int_{\log y}^{\infty} s^q e^{-s} \, ds \leq C \left( \frac{\log y}{y} \right)^q,
\]

where in the last step we have used a standard estimate for the incomplete Gamma function.

For \(y < 1\), it holds that

\[
\int_{|x - y| < 1} (K_{\alpha, \sigma}(x, y))^q \, d\mu_\alpha(x) \leq Cy^{(2\alpha+2)(1-q) + q} \int_0^{1+1/y} \frac{t^{2\alpha+1}}{(1 + t)^{q(2\alpha+1)}} \left[ \log \left( \frac{t + 1}{|t - 1|} \right) \right]^q \, dt
\]

\[
\sim y^{(2\alpha+2)(1-q) + q} + 1 \leq C.
\]

Case \(\sigma > 1/2\). For \(y \geq 1\), we have

\[
\int_{|x - y| < 1} (K_{\alpha, \sigma}(x, y))^q \, d\mu_\alpha(x) \leq Cy^{(2\alpha+2)(1-q) + q} \int_{|t-1| < 1/y} \left[ \log \left( \frac{t + 1}{|t - 1|} \right) \right]^q \, dt \sim y^{(2\alpha+1)(1-q)} \leq C.
\]
When \( y < 1 \), it is verified that
\[
\int_{|x-y|<1} (K_{\alpha,\sigma}(x,y))^q \, d\mu_\alpha(x) \leq C y^{(2\alpha+2)(1-q)+2q\sigma} \int_0^{1+1/y} \frac{t^{2\alpha+1}}{(1+t)^{(2\alpha+2-2\sigma)}} \, dt
\sim y^{(2\alpha+2)(1-q)+2q\sigma} + 1 \leq C,
\]
where we use the condition \( 1 - \frac{\sigma}{\alpha+1} < \frac{1}{q} \).

We pass to analyze (5.8) for \(|x-y| \geq 1\). In this region, the inequality \( x^{2\alpha+1} (x+y)^{-q(2\alpha+1)} \leq C \) holds, then
\[
\int_{|x-y| \geq 1} (K_{\alpha,\sigma}(x,y))^q \, d\mu_\alpha(x) \leq C \int_{|x-y| \geq 1} e^{-c(x-y)^2} \, dx \leq C,
\]
and the proof of (5.8) is completed.

**Proof of b.** By using an argument analogous to a), it is enough to prove that
\[
\left\| \int_0^\infty (K_{\alpha,\sigma}(x,y) f(y) \, d\mu_\alpha(y)) \right\|_{L^\infty(\mathbb{R}_+,d\mu_\alpha)} \leq C \| f \|_{L^p(\mathbb{R}_+,d\mu_\alpha)}.
\]

Now, by Hölder inequality, the result will follow from (5.8) and the symmetry of the kernel \( K_{\alpha,\sigma} \), by using the condition \( \frac{1}{p} < \frac{\sigma}{\alpha+1} \).

**Proof of c.** We consider \( x \in (0,2) \) and \( x \geq 2 \) separately. In the first case, by Proposition 5.1 and Lemma 4.4, the inequality is reduced to prove
\[
\left\| \chi_{(0,2)}(x) \int_0^\infty (K_{\alpha,\sigma}(x,y) f(y) \, d\mu_\alpha(y)) \right\|_{L^q(\mathbb{R}_+,d\mu_\alpha)} \leq C \| f \|_{L^\infty(\mathbb{R}_+,d\mu_\alpha)}.
\]

Now, taking into account that
\[
\int_0^\infty (K_{\alpha,\sigma}(x,y) f(y) \, d\mu_\alpha(y)) \leq \| f \|_{L^\infty(\mathbb{R}_+,d\mu_\alpha)} \int_0^\infty (K_{\alpha,\sigma}(x,y) \, d\mu_\alpha(y)),
\]
we will conclude by showing that
\[
\left\| \chi_{(0,2)}(x) \int_0^\infty (K_{\alpha,\sigma}(x,y) \, d\mu_\alpha(y)) \right\|_{L^q(\mathbb{R}_+,d\mu_\alpha)} \leq C,
\]
but this is true by (5.8) with \( q = 1 \).

When \( x \geq 2 \), by (5.6) and Lemma 4.4, it will be enough to prove that
\[
\left\| \chi_{[2,\infty)}(x) \int_0^\infty (K_{\alpha,\sigma}(x,y) f(y) \, d\mu_\alpha(y)) \right\|_{L^q(\mathbb{R}_+,d\mu_\alpha)} \leq C \| f \|_{L^\infty(\mathbb{R}_+,d\mu_\alpha)}.
\]

By applying (5.7), we obtain that
\[
\int_2^\infty \left( \int_0^\infty (K_{\alpha,\sigma}(x,y) f(y) \, d\mu_\alpha(y))^q \right) \, dx \leq \| f \|_{L^\infty(\mathbb{R}_+,d\mu_\alpha)} \int_2^\infty \left( \int_0^\infty (K_{\alpha,\sigma}(x,y) \, d\mu_\alpha(y))^q \right) \, dx \leq C \| f \|_{L^\infty(\mathbb{R}_+,d\mu_\alpha)},
\]
where in the last step we have used the restriction \( \frac{1}{q} < \frac{\sigma}{\alpha+1} \).

**Proof of d.** We distinguish between \( y \in (0,2) \) and \( y \geq 2 \). In the first case, by Proposition 5.1 and Lemma 4.4, the inequality is reduced to prove
\[
\int_0^2 (K_{\alpha,\sigma}(x,y) f(y) \, d\mu_\alpha(y)) \leq C \| f \|_{L^p(\mathbb{R}_+,d\mu_\alpha)},
\]

where we use the condition \( 1 - \frac{\sigma}{\alpha+1} < \frac{1}{q} \).
By Fubini’s theorem and Hölder inequality
\[
\int_0^\infty \int_0^2 f(y)K_{\alpha,\sigma}(x,y)\,d\mu_\alpha(y)\,d\mu_\alpha(x) \leq \|f\|_{L^p(\mathbb{R}^+)} \int_0^\infty \|K_{\alpha,\sigma}(-,y)\|_{L^{p'}(\mathbb{R}^+)}\,d\mu_\alpha(y),
\]
where in the last step we used (5.8).

In the case \(y \geq 2\), by (5.9) and Lemma 4.4 it will be enough to prove that
\[
\left\| \int_2^\infty f(y)K_{\alpha,\sigma}(x,y)\,d\mu_\alpha(y) \right\|_{L^1(\mathbb{R}^+)} \leq C\|f\|_{L^p(\mathbb{R}^+)}.
\]

Applying Fubini’s theorem, Hölder inequality and (5.7), we obtain that
\[
\int_0^\infty \int_2^\infty f(y)K_{\alpha,\sigma}(x,y)\,d\mu_\alpha(y)\,d\mu_\alpha(x) \leq \|f\|_{L^p(\mathbb{R}^+)} \int_0^\infty K_{\alpha,\sigma}(x,y)\,d\mu_\alpha(x) \leq C\|f\|_{L^p(\mathbb{R}^+)},
\]
where in the last step, we used that \(1 - \frac{\sigma}{p+1} < \frac{1}{p} \leq 1\).

Proof of (e). By using Proposition 5.1 we have to distinguish three cases in terms of \(\sigma\).

Case \(\sigma < 1/2\). In this case
\[
u_j(x)(L_{\alpha+a_j})^{-\sigma}(u_j^{-1}(\cdot)f_j)(x)\,d\mu_\alpha(x) \leq C(I_1f(x) + I_2f(x) + I_3f(x)),
\]
where
\[
I_1f(x) = \chi_{[0,2]}(x)x^{-2(\alpha+1)} \int_0^x f(y)\,d\mu_\alpha(y) + \chi_{[0,1]}(x) \int_0^2 y^{2\alpha-1} f(y)\,dy,
\]
\[
I_2f(x) = L_{2\sigma}^\text{loc} f(x), \quad \text{and}
\]
\[
I_3f(x) = \int_{|x-y| \geq 1} \frac{e^{-c(x-y)^2}}{(x+y)^{(2\alpha+1)}} f(y)\,d\mu_\alpha(y).
\]
For \(I_1\) the required bound follows from Theorem 5.8 and for \(I_2\) is an immediate consequence of Theorem 5.9. For \(I_3\), we can prove the boundedness for the pairs \((1/q,1/p) = (1-\sigma/(\alpha+1),1)\) and \((1/q,1/p) = (0,\sigma/(\alpha+1))\) as we did in a) and c). Indeed, the result will follow by applying Minkowski or Hölder inequalities and taking into account that
\[
\int_{|x-y| \geq 1} \frac{e^{-c(x-y)^2}}{(x+y)^{(2\alpha+1)}}\,d\mu_\alpha(y) \leq C,
\]
for any \(r \geq 1\). The complete result is obtained by interpolation.

Case \(\sigma = 1/2\). Now, it is verified that \(1/p - 1/q = 1/(\alpha+1) =: \delta < 1\). Then (since \(-\log t \leq Ct^{1-\delta}\), for \(0 < t < C < 1\))
\[
u_j(x)(L_{\alpha+a_j})^{-\sigma}(u_j^{-1}(\cdot)f_j)(x)\,d\mu_\alpha(x) \leq C(M_1f(x) + M_2f(x) + I_3f(x)),
\]
where
\[
M_1f(x) = \chi_{[0,2]}(x)x^{-2(\alpha+1)} \int_0^x f(y)\,d\mu_\alpha(y) + \chi_{[0,2]}(x) \int_x^2 y^{2\alpha-1} f(y)\,dy,
\]
\[
M_2f(x) = L_{2\sigma}^\text{loc} f(x), \quad \text{and}
\]
\(I_3\) is as in the previous case. In this way, to obtain the estimate we proceed analogously as for \(\sigma < 1/2\) by applying Theorem 5.8 and Theorem 5.9.

Case \(\sigma > 1/2\). Now, we have
\[
u_j(x)(L_{\alpha+a_j})^{-\sigma}(u_j^{-1}(\cdot)f_j)(x)\,d\mu_\alpha(x) \leq C(J_1f(x) + J_2f(x)),
\]
where
\[
J_1f(x) = \chi_{[0,2]}(x)x^{-2(\alpha+1)} \int_0^x f(y)\,d\mu_\alpha(y) + \chi_{[0,2]}(x) \int_x^2 y^{2\alpha-1} f(y)\,dy,
\]
\[
J_2f(x) = \chi_{[0,2]}(x)x^{2(\alpha+1)} \int_2^\infty f(y)\,d\mu_\alpha(y),
\]
and
\[ J_2 f(x) = \int_{x+y \geq 1} \frac{e^{-c(x+y)^2}}{(x+y)^{2a+1}} f(y) \, dp_x(y). \]

The operator \( J_1 \) can be analyzed by using Theorem 5.8 and for \( J_2 \) we proceed analogously as for \( I_3 \) in the case \( \sigma < 1/2 \).

\[ \square \]

6. Proofs of technical results

Proof of Lemma 5.2. First, observe that

\[ \log \frac{1 + \xi}{1 - \xi} \sim \begin{cases} \xi, & \text{for } 0 < \xi \leq 1/2; \\ -\log(1 - \xi^2), & \text{for } 1/2 < \xi < 1. \end{cases} \]

Then, denoting by \( J \) the integral to be estimated, we have

\[ J \leq C \int_0^{1/2} \xi^{\sigma-c-e^{-t}} \exp \left( -\frac{q_+}{4\xi} \right) d\xi + C \int_{1/2}^1 (\log(1 - \xi^2))^{\sigma-1}(1 - \xi^2)^c \xi^{-c-e^{-t}} \exp \left( -\frac{q_+}{4\xi} \right) d\xi \]
\[ =: J_1 + J_2. \]

Now, for \( J_1 \), the change of variable \( s = \frac{q_+}{4\xi} \) produces the required bound. Indeed,

\[ J_1 = \frac{4^{c+\ell-\sigma}}{q_+^{c+\ell-\sigma}} \int_0^\infty e^{-s}(s+1)^{\sigma-1} \, ds \leq C 4^\Gamma(c+\ell-\sigma). \]

In order to control \( J_2 \), we start by using the estimate

\[ t^ce^{-t} \leq \gamma^1e^{-\gamma}, \quad t, \gamma > 0, \]

to deduce that

\[ \xi^{-c-e^{-t}} \exp \left( -\frac{q_+}{4\xi} \right) \leq \frac{4^{c+\ell-\sigma}}{q_+^{c+\ell-\sigma}} \xi^{-\sigma}(c+\ell-\sigma)\xi^{-c-e^{-t}}. \]

Then,

\[ J_2 \leq \frac{4^{c+\ell-\sigma}}{q_+^{c+\ell-\sigma}}(c+\ell-\sigma)\xi^{-\sigma}(c+\ell-\sigma) \int_{1/2}^1 (\log(1 - \xi^2))^{\sigma-1}(1 - \xi^2)^c \xi^{-\sigma} \, d\xi \]
\[ \leq C \frac{4^{c+\ell-\sigma}}{q_+^{c+\ell-\sigma}}(c+\ell-\sigma)\xi^{-\sigma}(c+\ell-\sigma) \int_{1/2}^1 (\log(1 - \xi^2))^{\sigma-1}(1 - \xi^2)^c \, d\xi. \]

Now,

\[ \int_{1/2}^1 (\log(1 - \xi^2))^{\sigma-1}(1 - \xi^2)^c \, d\xi \leq C \int_{1/2}^1 (\log(1 - \xi^2))^{\sigma-1/2}(1 - \xi^2)^c \, d\xi \]
\[ \leq \frac{C}{(c+1)^{\sigma+1/2}} \leq \frac{C}{(c+1)^{1/2}}, \]

after the change of variable \( 1 - \xi^2 = e^{-t} \) in the second inequality. Therefore

\[ J_2 \leq C \frac{4^{c+\ell-\sigma}}{q_+^{c+\ell-\sigma}}(c+\ell-\sigma)c+\ell-\sigma-1/2e^{-\sigma-1/2}. \]

Finally, by Stirling’s approximation, we conclude the bound for \( J_2 \).

\[ \square \]

Proof of Lemma 5.3. With the obvious bound

\[ \left( \frac{1-s}{A - Bs} \right)^{a+1/2} \leq \frac{1}{A^{a+1/2}} \]

we have

\[ I_{a,b}^\lambda \leq \frac{1}{A^{a+1/2}} \int_0^1 \frac{(1-s)^{b-1}}{(A - Bs)^{b+\lambda}} \, ds. \]
Then, the change of variable $1 - s = \frac{A - B}{z}$ gives

\[
I^b_{a,b} \leq \frac{1}{A^{\alpha + 1/2}} \frac{1}{B^{\alpha + 1/2}} (A - B)^{\lambda} \int_0^{\pi/4} \frac{z^{b-1}}{(1 + z)^{b + \lambda}} dz.
\]

Now, for $\lambda > 0$,

\[
\int_0^{\pi/4} \frac{z^{b-1}}{(1 + z)^{b + \lambda}} dz \leq \int_0^{\pi/4} \frac{1}{(1 + z)} dz = \Gamma(b) \Gamma(\lambda) \Gamma(b + \lambda).
\]

In the case $\lambda = 0$, we have

\[
\int_0^{\pi/4} \frac{z^{b-1}}{(1 + z)^{b}} dz \leq \int_0^{\pi/4} \frac{1}{(1 + z)} dz = \log \left( \frac{A}{A - B} \right).
\]

Finally, for $\lambda < 0$,

\[
\int_0^{\pi/4} \frac{z^{b-1}}{(1 + z)^{b + \lambda}} dz \leq C \int_0^{\pi/4} z^{-\lambda - 1} dz = C \left( \frac{A - B}{A} \right)^{\lambda}.
\]

The proof ofLemma 5.4 and Lemma 5.5 follow a same scheme. We provide the proof of Lemma 5.4. The details of the proof of Lemma 5.4 are left to the reader.

**Proof of Lemma 5.4.** Due to the relation (5.1), it suffices to analyze the integral $J$ appearing in the proof of Proposition 4.2 with $\alpha + aj$ instead of $\alpha$, and in one dimension. After the change of variable $s = 2u - 1$ the integral becomes

\[
J = 4^{\alpha + aj} \exp \left( \sum_{\xi = 1}^1 \frac{(x - y)^2}{4 \xi} - \frac{\xi (x + y)^2}{4 \xi} \right) \int_0^{1/2} \exp \left( -xyu \left( \frac{1}{\xi} - \xi \right) \right) u^{\alpha + aj - 1/2} (1 - u)^{\alpha + aj - 1/2} du.
\]

Now, it is easy to check that

\[
J \leq 4^{\alpha + aj + 1/2} \exp \left( \sum_{\xi = 1}^1 \frac{(x - y)^2}{4 \xi} - \frac{\xi (x + y)^2}{4 \xi} \right) \int_0^{1/2} \exp \left( -xyu \left( \frac{1}{\xi} - \xi \right) \right) u^{\alpha + aj - 1/2} (1 - u)^{\alpha + aj - 1/2} du
\]

\[
\leq C 4^{\alpha + aj + 1/2} \exp \left( \sum_{\xi = 1}^1 \frac{(x - y)^2}{4 \xi} - \frac{\xi (x + y)^2}{4 \xi} \right) \int_0^{1/2} \exp \left( -xyu \left( \frac{1}{\xi} - \xi \right) \right) u^{\alpha + aj - 1/2} du.
\]

Since $aj \geq 1$, the change of variable $xyu \left( \frac{1}{\xi} - \xi \right) = z$ gives

\[
J \leq 4^{\alpha + aj + 1/2} \exp \left( \sum_{\xi = 1}^1 \frac{(x - y)^2}{4 \xi} - \frac{\xi (x + y)^2}{4 \xi} \right) \int_0^{1/2} \exp \left( -xyu \left( \frac{1}{\xi} - \xi \right) \right) u^{aj - 1} du
\]

\[
\leq 4^{\alpha + aj + 1/2} \exp \left( \sum_{\xi = 1}^1 \frac{(x - y)^2}{4 \xi} - \frac{\xi (x + y)^2}{4 \xi} \right) \left( xy \right)^{-aj} \left( \frac{\xi}{1 - \xi^2} \right)^{aj} \Gamma(aj)
\]

\[
\leq C 4^{\alpha + aj + 1/2} \exp \left( \sum_{\xi = 1}^1 \frac{(x - y)^2}{8 \xi} \right) \int_{\xi^2 - 2 \xi^2}^{\xi^2 - 2 \xi^2} \left( \frac{\xi}{1 - \xi^2} \right)^{aj} \Gamma(aj)
\]

where in the last step we have used the estimate

\[
\exp \left( \sum_{\xi = 1}^1 \frac{(x - y)^2}{4 \xi} - \frac{\xi (x + y)^2}{4 \xi} \right) \leq C \exp \left( \sum_{\xi = 1}^1 \frac{(x - y)^2}{8 \xi} \right) \exp(-c|y|^2)
\]

\[
\leq C \exp \left( \sum_{\xi = 1}^1 \frac{(x - y)^2}{8 \xi} \right) |x^2 - y^2|^{2\alpha - 1}.
\]

The result follows from (6.1), (6.4) and (6.1).
Proof of Proposition 5.6. From the identity \((L_n)^{-\sigma} = \mathcal{I}_{\alpha,\sigma}\) in \(L^2(\mathbb{R}^+, d\mu_\alpha)\), the estimate in \((5.6)\) can be deduced from \((5.2)\), \((5.3)\), \((5.1)\), \((4.4)\), and Proposition 4.2 (applied in one dimension).

In order to obtain the bound in \((5.7)\), we start considering the case \(|x - y| > x/2\). Proceeding as in the proof of Proposition 4.3 with \(n = 1\), we have that the kernel \(K_{\alpha,\sigma}\) can be estimated by \(e^{(x-y)^2/(xy)}^{\alpha-1/2}\), then
\[
x^{2\sigma} \int_{|x - y| > x/2} K_{\alpha,\sigma}(x, y) \, d\mu_\alpha(y) \leq C \int_{|x - y| > x/2} |x - y|^{2\sigma - \alpha - 1/2} e^{-c(x-y)^2} y^{\alpha+1/2} \, dy.
\]
If \(0 < y < x/2\), we have
\[
\int_{|x - y| > x/2} |x - y|^{2\sigma - \alpha - 1/2} e^{-c(x-y)^2} y^{\alpha+1/2} \, dy \leq C \int_{|x - y| > x/2} |x - y|^{2\sigma} e^{-c(x-y)^2} \, dy.
\]

When \(y > 3x/2\), it is verified that \(|x - y| \sim y\) and so
\[
\int_{|x - y| > x/2} |x - y|^{2\sigma - \alpha - 1/2} e^{-c(x-y)^2} y^{\alpha+1/2} \, dy \leq C \int_0^\infty y^{2\sigma} e^{-cy^2} \, dy \leq C.
\]

In the most delicate region \(|x - y| \leq x/2\) we split the integral in \(K_{\alpha,\sigma}\) into the intervals \((0, 1/2)\) and \([1/2, 1)\). For the second one, by using \((6.1)\), the integral of the kernel is controlled by
\[
x^{2\sigma} e^{-cx^2} \int_{x/2}^{3x/2} e^{-c(x-y)^2} \int_{1/2}^1 (-\log (1 - \xi^2))^{\sigma-1} (1 - \xi^2)^{-1/2} \, d\xi \, dy.
\]
This last integral is bounded because the inner integral is smaller than a constant and \(x^{2\sigma} e^{-cx^2} \leq C\).

In the case \(\xi \in (0, 1/2)\), by using again \((6.1)\) and some changes of variable, we have
\[
x^{2\sigma} \int_{x/2}^{3x/2} \int_0^{1/2} \xi^{\sigma-3/2} \exp \left( - (x-y)^2/4\xi - \frac{\xi x^2}{4} \right) \, d\xi \, dy
\]
\[
= C x \int_{x/2}^{3x/2} \int_0^{x^2/2/4} s^{\sigma-3/2} \exp \left( - (x-y)^2/4s - s/4 \right) \, ds \, dy
\]
\[
= C \int_0^{x^2/2} \int_0^{x^2/2} s^{\sigma-3/2} \exp \left( - z^2/4s - s/4 \right) \, ds \, dz
\]
\[
\leq C \int_0^{\infty} \int_0^{\infty} s^{\sigma-3/2} \exp \left( - z^2/4s - s/4 \right) \, ds \, dz \leq C,
\]
where in the last step we used that
\[
\int_0^\infty \int_0^\infty s^{\sigma-3/2} \exp \left( - z^2/4s - s/4 \right) \, ds \, dz = \int_0^\infty s^{\sigma-1} e^{-s/4} \, ds \int_0^\infty e^{-z^2/4} \, dt = 4^{\sigma} \sqrt{\pi} \Gamma(\sigma).
\]

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