AMPLE FILTERS OF INVERTIBLE SHEAVES

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Abstract. Let $X$ be a scheme, proper over a commutative noetherian ring $A$. We introduce the concept of an ample filter of invertible sheaves on $X$ and generalize the most important equivalent criteria for ampleness of an invertible sheaf. We also prove the Theorem of the Base for $X$ and generalize Serre’s Vanishing Theorem. We then generalize results for twisted homogeneous coordinate rings which were previously known only when $X$ was projective over an algebraically closed field. Specifically, we show that the concepts of left and right $\sigma$-ampleness are equivalent and that the associated twisted homogeneous coordinate ring must be noetherian.

1. Introduction

Ample invertible sheaves are central to projective algebraic geometry. Let $A$ be a commutative noetherian ring and let $R$ be a commutative $\mathbb{N}$-graded $A$-algebra, finitely generated in degree one. Then ample invertible sheaves allow a geometric description of $R$, by expressing $R$ as a homogeneous coordinate ring (in sufficiently high degree). Further, via the Serre Correspondence Theorem, there is an equivalence between the category of coherent sheaves on the scheme $\text{Proj } R$ and the category of tails of graded $R$-modules [H Exercise II.5.9].

To prove that Artin-Schelter regular algebras of dimension 3 (generated in degree one) are noetherian, [ATV] studied twisted homogeneous coordinate rings of elliptic curves (over a field). In [AV], a more thorough and general examination of twisted homogenous coordinate rings was undertaken, replacing the elliptic curves of [ATV] with any commutative projective scheme over a field. Such a ring $R$ is called twisted because it depends not only on a projective scheme $X$ and an invertible sheaf $\mathcal{L}$, but also on an automorphism $\sigma$ of $X$, which causes the multiplication in $R$ to be noncommutative. We sometimes denote such a ring as $B(X, \sigma, \mathcal{L})$. When $\mathcal{L}$ is right $\sigma$-ample, the ring $R$ is right noetherian and the category of tails of right $R$-modules is still equivalent to the category of coherent sheaves on $X$. When $\sigma$ is the identity automorphism, the commutative theory is recovered and $\sigma$-ampleness reduces to the usual ampleness. We will review $\sigma$-ample invertible sheaves and twisted homogeneous coordinate rings in §7 of this paper.

A priori, there are separate definitions for right $\sigma$-ampleness and left $\sigma$-ampleness. However, when working with projective schemes over an algebraically closed field, [Ke1] shows that right and left $\sigma$-ampleness are equivalent. In [7] we generalize the results of [AV] and [Ke1] to the case of a scheme $X$ which is proper over a commutative noetherian ring $A$. After summarizing definitions and previously known results, we prove

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Theorem 1.1. Let $A$ be a commutative noetherian ring, let $X$ be a proper scheme over $A$, let $\sigma$ be an automorphism of $X$, and let $\mathcal{L}$ be an invertible sheaf on $X$. Then $\mathcal{L}$ is right $\sigma$-ample if and only if $\mathcal{L}$ is left $\sigma$-ample. If $\mathcal{L}$ is $\sigma$-ample, then the twisted homogeneous coordinate ring $B(X, \sigma, \mathcal{L})$ is noetherian.

To prove some of the results of [AV] and [Ke1], the concept of $\sigma$-ampleness was not general enough. One can define the ampleness of a sequence of invertible sheaves [AV, Definition 3.1]. To study twisted multi-homogeneous coordinate rings, [C] considers the ampleness of a set of invertible sheaves indexed by $\mathbb{N}^n$. To achieve the greatest possible generality, we index invertible sheaves by any filter. A filter $\mathcal{P}$ is a partially ordered set such that:

for all $\alpha, \beta \in \mathcal{P}$, there exists $\gamma \in \mathcal{P}$ such that $\alpha < \gamma$ and $\beta < \gamma$.

Let $X$ be a scheme, proper over $\text{Spec} \ A$, where $A$ is a (commutative) noetherian ring. If a set of invertible sheaves is indexed by a filter, then we will call that set a filter of invertible sheaves. An element of such a filter will be denoted $\mathcal{L}_\alpha$ for $\alpha \in \mathcal{P}$. The indexing filter $\mathcal{P}$ will usually not be named.

Definition 1.2. Let $A$ be a commutative noetherian ring, and let $X$ be a proper scheme over $A$. Let $\mathcal{P}$ be a filter. A filter of invertible sheaves $\{\mathcal{L}_\alpha\}$ on $X$, with $\alpha \in \mathcal{P}$, will be called an ample filter if for all coherent sheaves $\mathcal{F}$ on $X$, there exists $\alpha_0$ such that

$$H^q(X, \mathcal{F} \otimes \mathcal{L}_\alpha) = 0, \quad q > 0, \alpha \geq \alpha_0.$$

If $\mathcal{P} \cong \mathbb{N}$ as filters, then an ample filter $\{\mathcal{L}_\alpha\}$ is called an ample sequence.

Of course if $\mathcal{L}$ is an invertible sheaf, then $\mathcal{L} < \mathcal{L}^2 < \ldots$ is an ample sequence if and only if $\mathcal{L}$ is an ample invertible sheaf. It is well-known that the following conditions are equivalent in the case of an ample invertible sheaf [H, Proposition III.5.3].

Our main result is

Theorem 1.3. Let $X$ be a scheme, proper over a commutative noetherian ring $A$. Let $\{\mathcal{L}_\alpha\}$ be a filter of invertible sheaves. Then the following are equivalent conditions on $\{\mathcal{L}_\alpha\}$:

(A1) The filter $\{\mathcal{L}_\alpha\}$ is an ample filter.

(A2) For all coherent sheaves $\mathcal{F}, \mathcal{G}$ with epimorphism $\mathcal{F} \twoheadrightarrow \mathcal{G}$, there exists $\alpha_0$ such that the natural map

$$H^0(X, \mathcal{F} \otimes \mathcal{L}_\alpha) \rightarrow H^0(X, \mathcal{G} \otimes \mathcal{L}_\alpha)$$

is an epimorphism for $\alpha \geq \alpha_0$.

(A3) For all coherent sheaves $\mathcal{F}$, there exists $\alpha_0$ such that $\mathcal{F} \otimes \mathcal{L}_\alpha$ is generated by global sections for $\alpha \geq \alpha_0$.

(A4) For all invertible sheaves $\mathcal{H}$, there exists $\alpha_0$ such that $\mathcal{H} \otimes \mathcal{L}_\alpha$ is an ample invertible sheaf for $\alpha \geq \alpha_0$.

Note that there is no assumed relationship between the various $\mathcal{L}_\alpha$, other than their being indexed by a filter.

From condition (A4), we see that if $X$ has an ample filter, then $X$ is a projective scheme (see Corollary 6.7). A proper scheme $Y$ is divisorial if $Y$ has a so-called ample family of invertible sheaves [H, Definitions 2.2.4–5]. There exist divisorial schemes which are not projective; hence an ample family does not have an associated ample filter in general. See Remark 6.8 for some known vanishing theorems on divisorial schemes.
Theorem 1.3 is proven in §6. We must first review previously known results involving ampleness and intersection theory in §2. In particular, we review the general definitions of numerical effectiveness and numerical triviality.

We then proceed to prove the Theorem of the Base in §3. This was proven by Kleiman when \( A \) was finitely generated over a field [Kl, p. 334, Proposition 3].

**Theorem 1.4.** (See Thm. 3.6) Let \( X \) be a scheme, proper over a commutative noetherian ring \( A \). Then \( \text{Pic} X \) modulo numerical equivalence is a finitely generated free abelian group.

If two invertible sheaves \( \mathcal{L}, \mathcal{L}' \) are numerically equivalent, then \( \mathcal{L} \) is ample if and only if \( \mathcal{L}' \) is ample. Thus, one may study ampleness via finite dimensional linear algebra.

After preliminary results on ample filters in §4, we prove Serre’s Vanishing Theorem in new generality in §5. The following was proven by Fujita when \( A \) was an algebraically closed field [Fj1, §5].

**Theorem 1.5.** Let \( A \) be a commutative noetherian ring, let \( X \) be a projective scheme over \( A \), and let \( \mathcal{L} \) be an ample invertible sheaf on \( X \). For all coherent sheaves \( \mathcal{F} \), there exists \( m_0 \) such that

\[
H^q(X, \mathcal{F} \otimes \mathcal{L}^m \otimes \mathcal{N}) = 0
\]

for \( m \geq m_0, q > 0 \), and all numerically effective invertible sheaves \( \mathcal{N} \).

Finally, we note that before §7, all rings in this paper are commutative.

2. Ampleness and Intersection Theory

In this section, we review various facts about ampleness on a scheme which is proper over a commutative noetherian ring \( A \) or, more generally, proper over a noetherian scheme \( S \). Most results are “well-known,” but proofs are not always easy to find in the literature. We assume the reader is familiar with various characterizations of ampleness which appear in [H]. We mostly deal with noetherian schemes and proper morphisms, though some of the following propositions are true in more generality than stated. All subschemes are assumed to be closed unless specified otherwise.

**Proposition 2.1.** [EGA, II, 4.4.5] Let \( A \) be a noetherian ring, let \( \pi: X \to \text{Spec} A \) be a proper morphism, and let \( \mathcal{L} \) be an invertible sheaf on \( X \). If \( U \) is an affine open subscheme of \( \text{Spec} A \) and \( \mathcal{L} \) is ample on \( X \), then \( \mathcal{L}|_{\pi^{-1}(U)} \) is ample on \( \pi^{-1}(U) \). Conversely, if \( \{U_\alpha\} \) is an affine open cover of \( \text{Spec} A \) and each \( \mathcal{L}|_{\pi^{-1}(U_\alpha)} \) is ample on \( \pi^{-1}(U_\alpha) \), then \( \mathcal{L} \) is ample on \( X \). \( \square \)

This leads to the following definition of relative ampleness.

**Definition 2.2.** [EGA, II, 4.6.1] Let \( S \) be a noetherian scheme, let \( \pi: X \to S \) be a proper morphism, and let \( \mathcal{L} \) be an invertible sheaf on \( X \). The sheaf \( \mathcal{L} \) is relatively ample for \( \pi \) or \( \pi \)-ample if there exists an affine open cover \( \{U_\alpha\} \) of \( S \) such that \( \mathcal{L}|_{\pi^{-1}(U_\alpha)} \) is ample for each \( \alpha \).

**Proposition 2.3.** Let \( S \) be a noetherian scheme, let \( \pi: X \to S \) be a proper morphism, and let \( \mathcal{L} \) be an invertible sheaf on \( X \). If \( U \) is an open subscheme of \( S \) and \( \mathcal{L} \) is \( \pi \)-ample, then \( \mathcal{L}|_{\pi^{-1}(U)} \) is \( \pi|_{\pi^{-1}(U)} \)-ample. Conversely, if \( \{U_\alpha\} \) is an open cover of \( S \) and each \( \mathcal{L}|_{\pi^{-1}(U_\alpha)} \) is \( \pi|_{\pi^{-1}(U_\alpha)} \)-ample, then \( \mathcal{L} \) is \( \pi \)-ample. \( \square \)
Remark 2.4. If \( L \) is \( \pi \)-ample, then some power \( L^n \) is relatively very ample for \( \pi \) in the sense of Grothendieck \([EGA, II, 4.4.2, 4.6.11]\). Further a \( \pi \)-ample \( L \) exists if and only if \( \pi \) is a projective morphism \([EGA, II, 5.5.2, 5.5.3]\). If \( S \) itself has an ample invertible sheaf (for example when \( S \) is quasi-projective over an affine base), then these definitions of very ample invertible sheaf and projective morphism are the same as those in \([H, p. 103, 120]\), as shown in \([EGA, II, 5.5.4(ii)]\). The definition of Grothendieck is useful because in many proofs it allows an easy reduction to the case of affine \( S \).

To find the connection between ampleness over general base rings and intersection theory, we must examine how ampleness behaves on the fibers of a proper morphism. We use the standard abuse of notation \( X \times_{Spec \ A} Spec \ B = X \times_A B \).

**Proposition 2.5.** \([EGA, III_1, 4.7.1]\) Let \( S \) be a noetherian scheme, let \( \pi: X \to S \) be a proper morphism, and let \( L \) be an invertible sheaf on \( X \). Let \( s \in S \) be a point and \( p: X_s = X \times_S k(s) \to X \) be the natural projection from the fiber. If \( p^*L \) is ample on \( X_s \), then there exists an open neighborhood \( U \) of \( s \in S \) such that \( L|_{\pi^{-1}(U)} \) is \( \pi|_{\pi^{-1}(U)} \)-ample.

**Lemma 2.6.** Let \( S \) be a noetherian scheme and let \( U \) be an open subscheme which contains all closed points of \( S \). Then \( U = S \).

**Proof.** Since \( S \) is noetherian, it is a Zariski space. Thus, the closure of any point of \( S \) must contain a closed point \([H, Exercise II.3.17e]\). So \( S \setminus U \) must be empty. \[\square\]

**Proposition 2.7.** Let \( S \) be a noetherian scheme, let \( \pi: X \to S \) be a proper morphism, and let \( L \) be an invertible sheaf on \( X \). Then \( L \) is \( \pi \)-ample if and only if for every closed point \( s \in S \) and natural projection \( p_s: X_s \to X \), the invertible sheaf \( p_s^*L \) is ample on \( X_s \).

**Proof.** Suppose that \( L \) is \( \pi \)-ample. We choose a closed point \( s \) and an open affine neighborhood \( s \in U \subset S \). Then \( L|_{\pi^{-1}(U)} \) is ample on \( \pi^{-1}(U) = X \times_S U \) by Proposition 2.3. Now since \( s \) is closed, the fiber \( X \times_S k(s) \) is a closed subscheme of \( X \times_S U \). So \( p_s^*L \) is ample \([H, Exercise III.3.7]\), as desired.

Conversely, suppose that \( p_s^*L \) is ample for each closed point \( s \in S \). Then by Proposition 2.5 and Lemma 2.6, there is an open cover \( \{U_a\} \) of \( S \) such that each \( L|_{\pi^{-1}(U_a)} \) is \( \pi|_{\pi^{-1}(U_a)} \)-ample. So \( L \) is \( \pi \)-ample by Proposition 2.3. \(\square\)

We now move to a discussion of the intersection theory outlined in \([Ko, Chapter VI.2]\). We will soon see that the above proposition allows us to form a connection between this intersection theory and relative ampleness over a noetherian base scheme.

Let \( S \) be a noetherian scheme, and let \( \pi: X \to S \) be a proper morphism. Let \( F \) be a coherent sheaf on \( X \) with \( \text{Supp} \ F \) proper over a 0-dimensional subscheme of \( S \) and with \( \dim \text{Supp} \ F = r \). This theory defines intersection numbers \( (L_1, \ldots, L_n, F) \) for the intersection of invertible sheaves \( L_i \) on \( X \) with \( F \), where \( n \geq r \). In the case \( F = O_Y \) for a closed subscheme \( Y \subset X \), we also write the intersection number as \( (L_1, \ldots, L_n, Y) \). If all \( L_i = L \) then we write \( (L^n, Y) \). To avoid confusion between self-intersection numbers and tensor powers, we will always write \( L \otimes \cdots \otimes L \) as \( L^\otimes n \) when it appears in an intersection number. (We use invertible sheaves instead of Cartier divisors since invertible sheaves are more general \([H, Remark II.6.14.1]\).)
Note that \( \pi(\text{Supp}\, \mathcal{F}) \) must be a closed subset of \( S \) since \( \pi \) is proper. According to [Ko Corollary VI.2.3], the intersection number \((\mathcal{L}_1, \ldots, \mathcal{L}_n, \mathcal{F})\) may be calculated from \((\mathcal{L}_1, \ldots, \mathcal{L}_n, Y_i)\), where the \( Y_i \) are the irreducible components of the reduced scheme \( \text{red Supp}\, \mathcal{F} \). Since \( Y_i \) is irreducible and maps to a 0-dimensional subscheme of \( S \), we must have \( \pi(Y_i) \) equal to a closed point \( s \) of \( S \). Since \( Y_i \) is reduced, there is a unique induced proper morphism \( f_{\text{red}}: Y_i \to \text{Spec} \, k(s) \), where \( k(s) \) is the residue field of \( s \). Thus we may better understand this more general intersection theory by studying intersection theory over a field.

The intersection theory of the case \( S = \text{Spec} \, k \), with \( k \) algebraically closed, is the topic of the seminal paper [Kl]. Many of the theorems of that paper are still valid in the case of \( S \) equal to the spectrum of an arbitrary field \( k \); the proofs can either be copied outright or one can pass to \( X \times_k \overline{k} \), where \( \overline{k} \) is an algebraic closure of \( k \). The most important proofs in this more general case are included in [Ko Chapter VI.2]. We will also need the following definitions and lemmas.

**Definition 2.8.** Let \( S \) be a noetherian scheme, and let \( \pi: X \to S \) be a proper morphism. A closed subscheme \( V \subset X \) is \( \pi \)-contracted if \( \pi(V) \) is a 0-dimensional (closed) subscheme of \( S \). If \( S = \text{Spec} \, A \) for a noetherian ring \( A \), then we simply say that such a \( V \) is contracted. (This absolute notation will be justified by Proposition 2.15.)

As stated above, if \( V \) is irreducible and \( \pi \)-contracted, then \( \pi(V) \) is a closed point. Recall that an integral curve is defined to be an integral scheme of dimension 1 which is proper over a field.

**Definition 2.9.** Let \( S \) be a noetherian scheme, and let \( \pi: X \to S \) be a proper morphism. An invertible sheaf \( \mathcal{L} \) on \( X \) is relatively numerically effective for \( \pi \) or \( \pi \)-nef (resp. relatively numerically trivial for \( \pi \) or \( \pi \)-trivial) if \( (\mathcal{L}, C) \geq 0 \) (resp. \( (\mathcal{L}, C) = 0 \)) for all \( \pi \)-contracted integral curves \( C \). If \( S = \text{Spec} \, A \) for a noetherian ring \( A \), then we simply say that such an \( \mathcal{L} \) is numerically effective or nef (resp. numerically trivial). (This absolute notation will be justified by Proposition 2.15.)

Obviously, \( \mathcal{L} \) is \( \pi \)-trivial if and only if \( \mathcal{L} \) is \( \pi \)-nef and minus \( \pi \)-nef (i.e., \( \mathcal{L}^{-1} \) is \( \pi \)-nef). Thus, the following propositions regarding \( \pi \)-nef invertible sheaves have immediate corollaries for \( \pi \)-trivial invertible sheaves, which we will not explicitly prove.

We need to study nef invertible sheaves because of their close connection to ample invertible sheaves, as evidenced by the following propositions. We begin with the Nakai criterion for ampleness. This is well-known when \( X \) is proper over a field [Ko Theorem VI.2.18]. The general case follows via Proposition 2.7.

**Proposition 2.10.** [KM Thm. 1.42] Let \( S \) be a noetherian scheme, let \( \pi: X \to S \) be a proper morphism, and let \( \mathcal{L} \) be an invertible sheaf on \( X \). Then \( \mathcal{L} \) is \( \pi \)-ample if and only if \( (\mathcal{L}^\bullet_{\dim V}, V) > 0 \) for every closed \( \pi \)-contracted integral subscheme \( V \subset X \).

We have a similar proposition for nef invertible sheaves, following from the case of \( X \) proper over a base field [Ko Theorem VI.2.17].

**Proposition 2.11.** [KM Thm. 1.43] Let \( S \) be a noetherian scheme, let \( \pi: X \to S \) be a proper morphism, and let \( \mathcal{L} \) be an invertible sheaf on \( X \). Then \( \mathcal{L} \) is \( \pi \)-nef (resp. \( \pi \)-trivial) if and only if \((\mathcal{L}^\bullet_{\dim V}, V) \geq 0 \) (resp. \( (\mathcal{L}^\bullet_{\dim V}, V) = 0 \)) for every closed \( \pi \)-contracted integral subscheme \( V \subset X \).
In fact, a stronger claim is true, namely

Lemma 2.12. \cite{K} p. 320, Thm. 1] Given the hypotheses of Proposition 2.11, if \( L \) and \( N \) are \( \pi \)-nef, then \( (L^*N^{\dim V-i} \geq 0) \) for every closed \( \pi \)-contracted integral subscheme and \( i = 1, \ldots, \dim V \).

From this we easily see

Proposition 2.13. Let \( X \) be projective over a field \( k \), and let \( L \) be an ample invertible sheaf on \( X \). An invertible sheaf \( N \) is numerically effective if and only if \( L \otimes N^m \) is ample for all \( m \geq 0 \).

Proof. Using Propositions 2.7 and 2.13, we have the desired result by examining ampleness and numerical effectiveness on the fibers over closed points.

Ampleness over an affine base is an absolute notion \cite[Remark II.7.4.1]{H}, and this is also so for numerical effectiveness.

Proposition 2.14. Let \( S \) be a noetherian scheme, let \( \pi: X \to S \) be a proper morphism, and let \( L \) be an invertible sheaf on \( X \). An invertible sheaf \( N \) is \( \pi \)-nef if and only if \( L \otimes N^m \) is \( \pi \)-ample, for all \( m \geq 0 \).

Proof. Let \( A_1, A_2 \) be noetherian rings, and let \( \pi_i: X \to \Spec A_i \) be proper morphisms, \( i = 1, 2 \). Let \( V \) be a closed subscheme of \( X \). Then \( V \) is \( \pi_1 \)-contracted if and only if \( V \) is \( \pi_2 \)-contracted. Thus we may simply refer to such a \( V \) as contracted.

Let \( L \) be an invertible sheaf on \( X \). Then \( L \) is \( \pi_1 \)-nef (resp. \( \pi_1 \)-trivial) if and only if \( L \otimes N^m \) is \( \pi \)-ample (resp. \( \pi \)-trivial). Thus we may simply refer to such an \( L \) as numerically effective (resp. numerically trivial).

Proof. Let \( R = H^0(X, O_X) \). Since \( X \) is proper over \( A_1 \), the ring \( R \) is noetherian. There are natural Stein factorizations of \( \pi_i \) [EGA III, 4.3.1]

\[
X \xrightarrow{\pi'_i} \Spec R \xrightarrow{g_i} \Spec A_i.
\]

The \( \pi'_i \) are both equal to the canonical morphism \( f: X \to \Spec R \). The \( g_i \) are finite morphisms and \( f \) is proper. So a closed subscheme \( V \) is \( \pi_1 \)-contracted if and only if it is \( f \)-contracted.

Thus to prove the claim about \( L \), we may assume the rings \( A_i \) are fields and \( X \) is an integral curve. Then \( L \) is \( \pi \)-nef if and only if \( L^{-1} \) is not ample, and this is an absolute notion.

We now examine the behavior of numerical effectiveness under pull-backs. First we will need the following lemma.

Lemma 2.16. Let \( S \) be a noetherian scheme. Let

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
\pi' \downarrow & & \pi \\
S & \xrightarrow{\pi} & S
\end{array}
\]

be a commutative diagram with proper morphisms \( \pi, \pi' \) and (proper) surjective morphism \( f \). Let \( C \subset X \) be a \( \pi \)-contracted integral curve. Then there exists a \( \pi' \)-contracted integral curve \( C' \subset X' \) such that \( f(C') = C \).
Proof. Note that $f$ is necessarily proper by [H Corollary II.4.8c]. If $\pi(C) = s$, we may replace $X, X', S$ by their fibers over $\text{Spec} \, k(s)$. Thus we are working over a field and this case is as in [Kl p. 303, Lemma 1].

Lemma 2.17. (See [Kl p. 303, Prop. 1]) Let $S$ be a noetherian scheme. Let

$\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
\pi' \downarrow & & \downarrow \pi \\
S & \end{array}$

be a commutative diagram with proper morphisms $\pi, \pi'$ and (proper) morphism $f$. Let $\mathcal{L}$ be an invertible sheaf on $X$.

1. If $\mathcal{L}$ is $\pi$-nef (resp. $\pi$-trivial), then $f^* \mathcal{L}$ is $\pi'$-nef (resp. $\pi'$-trivial).
2. If $f$ is surjective and $f^* \mathcal{L}$ is $\pi'$-nef (resp. $\pi'$-trivial), then $\mathcal{L}$ is $\pi$-nef (resp. $\pi'$-trivial).

Proof. By the definition of numerical effectiveness, we need only examine the behavior of $\mathcal{L}$ on the fibers of $\pi, \pi'$. Thus we may assume $S = \text{Spec} \, k$ for a field $k$.

For the first statement, let $C' \subset X'$ be an integral curve and let $C = f(C')$, giving $C$ the reduced induced structure. We have the Projection Formula $(f^* \mathcal{L}, C') = (\mathcal{L}, f_* \mathcal{O}_{C'})$ [Ko Proposition VI.2.11]. Now

$$(\mathcal{L}, f_* \mathcal{O}_{C'}) = (\dim_{\mathcal{O}_C}(f_* \mathcal{O}_{C'}), C)(\mathcal{L}, C)$$

where $c$ is the generic point of $C$ [Ko Proposition VI.2.7]. If $C$ is a point, then $(\mathcal{L}, C) = 0$ and if $C$ is an integral curve, $(\mathcal{L}, C) \geq 0$. So $f^* \mathcal{L}$ is numerically effective.

For the second statement, let $C \subset X$ be an integral curve. By the previous lemma, there is an integral curve $C' \subset X'$ such that $f(C') = C$. Since $f|_{C'}$ is surjective, $\dim_{\mathcal{O}_C}(f_* \mathcal{O}_{C'})_c$ is positive, so the argument of the previous paragraph is reversible. That is,

$$(\mathcal{L}, C) = (f^* \mathcal{L}, C')/(\dim_{\mathcal{O}_C}(f_* \mathcal{O}_{C'})_c) \geq 0.$$ 

The preceding lemma is often used to reduce to the case of $X$ being projective over a noetherian scheme $S$. Given a proper morphism $\pi: X \to S$, there exists a scheme $X'$ and a morphism $f: X' \to X$ such that $f$ is projective and surjective, $\pi \circ f$ is projective, there exists a dense open subscheme $U \subset X$ such that $f|_{f^{-1}(U)}: f^{-1}(U) \to U$ is an isomorphism, and $f^{-1}(U)$ is dense in $X'$ [EGA II, 5.6.1]. This $f: X' \to X$ is called a Chow cover.

Since [Kl] works over an algebraically closed base field, the next lemma is useful for applying results of that paper to the case of a general field. In essence, it says the properties we are studying are preserved under base change.

Lemma 2.18. Let $g: S' \to S$ be a morphism of noetherian schemes, let $\pi: X \to S$ be a proper morphism, and let $\mathcal{L}$ be an invertible sheaf on $X$. Let $f: X \times_S S' \to X$ and $\pi': X \times_S S' \to S'$ be the natural morphisms.

1. If $\mathcal{L}$ is $\pi$-ample (resp. $\pi$-nef, $\pi$-trivial), then $f^* \mathcal{L}$ is $\pi'$-ample (resp. $\pi'$-nef, $\pi'$-trivial).
Lemma 2.20. Let \( S \) be a noetherian scheme, let \( \pi \colon X \to S \) be a proper morphism, and let \( L \) be an invertible sheaf on \( X \). If \( S_0 \) is a locally closed subscheme of \( S \) and \( L \) is \( \pi \)-ample (resp. \( \pi \)-nef, \( \pi \)-trivial), then \( L \) is \( \pi \)-ample (resp. \( \pi \)-nef, \( \pi \)-trivial).

Proof. For the first claim regarding ampleness, we may assume by Proposition 2.3 that \( S = \text{Spec} \ A, S' = \text{Spec} \ A' \) for noetherian rings \( A, A' \).

Suppose that \( L \) is ample. Then \( L^n \) is very ample for some \( n \). We have the commutative diagram

\[
\begin{array}{ccc}
X \times_A A' & \xrightarrow{i_A'} & \mathbb{P}^m_{A'} \\
\downarrow f & & \downarrow f' \\
X & \xrightarrow{i} & \mathbb{P}^m_A
\end{array}
\]

with \( i, i_A' \) closed immersions. Now

\[
f^* L^n = f^* (i_A'^* O_{\mathbb{P}^m_{A'}}(1)) = i_A'^* (f'^* O_{\mathbb{P}^m_{A'}}(1)) = i_A^* O_{\mathbb{P}^m_A}(1).
\]

Thus \( f^* L^n \) is very ample and \( f^* L \) is ample.

Now suppose that \( g(S') \) contains all closed points of \( S \) and \( f^* L \) is ample. We wish to show \( L \) is ample. Let \( s \in S \) be a closed point and let \( s' \in S' \) be a closed point with \( g(s') = s \). Applying Proposition 2.7 to \( L \) and the first claim of this lemma to \( f^* L \), we may replace \( S \) (resp. \( S', X \)) with Spec \( k(s) \) (resp. Spec \( k(s') \), \( X \times_S k(s) \)). Let \( F \) be a coherent sheaf on \( X \). Now Spec \( k(s') \to \text{Spec} \ k(s) \) is faithfully flat, so we have [II, Proposition III.9.3]

\[
H^q(X, F \otimes L^n) \otimes_{k(s)} k(s^{'}) \cong H^q(X \times_S S', f^* F \otimes f^* L^n) = 0
\]

for \( m \geq m_0, q > 0 \). Thus \( L \) is ample.

For the claims regarding numerical effectiveness, we may replace \( X \) with a Chow cover via Lemma 2.17 and thus assume that \( X \) is projective over \( \text{Spec} \ A \). The claims then follow from Proposition 2.14 and the results above. \( \square \)

Now we may generalize Proposition 2.3 as follows for locally closed subschemes, i.e., a closed subscheme of an open subscheme.

Corollary 2.19. Let \( S \) be a noetherian scheme, let \( \pi \colon X \to S \) be a proper morphism, and let \( L \) be an invertible sheaf on \( X \). If \( S_0 \) is a locally closed subscheme of \( S \) and \( L \) is \( \pi \)-ample (resp. \( \pi \)-nef, \( \pi \)-trivial), then \( L|_{\pi^{-1}(S_0)} \) is \( \pi|_{\pi^{-1}(S_0)} \)-ample (resp. \( \pi|_{\pi^{-1}(S_0)} \)-nef, \( \pi|_{\pi^{-1}(S_0)} \)-trivial). Conversely, if \( \{S_i\} \) is a finite cover of locally closed subschemes of \( S \) and each \( L|_{\pi^{-1}(S_i)} \) is \( \pi|_{\pi^{-1}(S_i)} \)-ample (resp. \( \pi|_{\pi^{-1}(S_i)} \)-nef, \( \pi|_{\pi^{-1}(S_i)} \)-trivial), then \( L \) is \( \pi \)-ample (resp. \( \pi \)-nef, \( \pi \)-trivial).

Proof. For the first claim, take \( S' = S_0 \) in Lemma 2.18. For the second claim, take \( S' = \bigsqcup S_i \) (the disjoint union of the \( S_i \)) in Lemma 2.18. \( \square \)

Lemma 2.20. Let \( S, S' \) be noetherian schemes. Consider the commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
\downarrow \pi' & & \downarrow \pi \\
S' & \xrightarrow{g} & S
\end{array}
\]

with proper morphisms \( \pi, \pi' \). Let \( L' \) be an invertible sheaf on \( X \).

\footnote{The published version incorrectly writes \( S' = \bigsqcup S_i \).}
(1) If \( \mathcal{L} \) is \( \pi \)-nef (resp. \( \pi \)-trivial), then \( f^* \mathcal{L} \) is \( \pi' \)-nef (resp. \( \pi' \)-trivial).

(2) Further, suppose that for every \( \pi \)-contracted integral curve \( C \subset X \), there is a \( \pi' \)-contracted integral curve \( C' \subset X' \) such that \( f(C') = C \). (For instance, suppose that \( g = \text{id}_S \) and \( f \) is proper and surjective.) If \( f^* \mathcal{L} \) is \( \pi' \)-nef (resp. \( \pi' \)-trivial), then \( \mathcal{L} \) is \( \pi \)-nef (resp. \( \pi \)-trivial).

Proof. For the first statement, let \( C' \subset X' \) be an integral curve such that \( \pi'(C') = s' \) is a closed point. Let \( s = g(s') \). By the definition of fibered products, we have the induced commutative diagram

\[
\begin{array}{ccc}
C' & \xrightarrow{f'} & X \times_S k(s') \\
\downarrow & & \downarrow \pi \\
\text{Spec } k(s') & \xrightarrow{\pi} & \text{Spec } k(s) \\
\end{array}
\]

Abusing notation, we replace \( \mathcal{L} \) with its restriction on \( X \times_S k(s) \) and we need only show \( (f')^* (f'')^* \mathcal{L} \) is nef. So assume \( \mathcal{L} \) is nef. By Lemma 2.18 \( (f'')^* \mathcal{L} \) is nef. And finally by Lemma 2.17 \( (f')^* (f'')^* \mathcal{L} \) is nef.

Now assume the extra hypothesis of the latter part of the lemma. Let \( C \subset X \) be any integral curve with \( \pi(C) = s \) a closed point. Let \( C' \subset X' \) be such that \( f(C') = C \) and \( \pi'(C') = s' \) is a closed point. Then we abuse notation again, replacing \( \mathcal{L} \) with its restriction on \( C \) and we have the induced commutative diagram

\[
\begin{array}{ccc}
C' & \xrightarrow{f'} & C \times_{k(s')} k(s') \\
\downarrow & & \downarrow \pi'' \\
\text{Spec } k(s') & \xrightarrow{\pi'} & \text{Spec } k(s) \\
\end{array}
\]

with \( f = f'' \circ f' \). Note that since \( C, C' \) are proper curves over a field, they are projective curves. Since \( \pi'' \circ f' \) is projective and \( \pi'' \) is separated, we have that \( f' \) is projective \( \square \). Thus \( f'(C') \) is closed in \( C \times_{k(s')} k(s') \).

Suppose that \( (f')^* (f'')^* \mathcal{L} \) is nef and that \( \mathcal{L} \) is not nef, so that \( (\mathcal{L}, C) < 0 \). Then \( \mathcal{L} \) is minus ample. By Lemma 2.18 \( (f'')^* \mathcal{L} \) is minus ample on \( C \times_{k(s')} k(s') \) and hence is minus ample on \( f'(C') \), the closed image subscheme. Since \( f = f'' \circ f' \) is surjective and \( C' \) is an integral curve, \( f'(C') \) must also be an integral curve. Since \( f' \) is projective, no closed points of \( C' \) can map to the generic point of \( f'(C') \). The preimage of any closed point of \( f'(C') \) must also be a finite set, since finite sets are the only proper closed subsets of \( C' \). Thus, \( f' \) is quasi-finite and projective, hence finite \( \square \). Thus \( (f')^* (f'')^* \mathcal{L} \) is minus ample on \( C' \) \( \square \). Thus \( (f')^* (f'')^* \mathcal{L} \) is nef. So \( \mathcal{L} \) must be nef, as desired.

3. The Theorem of the Base

Let \( S \) be a noetherian scheme and let \( \pi : X \rightarrow S \) be a proper morphism. Two invertible sheaves \( \mathcal{L}, \mathcal{L}' \) on \( X \) are said to be relatively numerically equivalent if \( (\mathcal{L}, C) = (\mathcal{L}', C) \) for all \( \pi \)-contracted integral curves \( C \subset X \). If \( S \) is affine, we say that \( \mathcal{L} \) and \( \mathcal{L}' \) are numerically equivalent. We denote the equivalence by \( \mathcal{L} \equiv \mathcal{L}' \). Of course \( \mathcal{L} \equiv 0 \) exactly when \( \mathcal{L} \) is relatively numerically trivial. Define \( A^1(X/S) = \text{Pic } X/ \equiv \). When \( S = \text{Spec } A \), we denote \( A^1(X/S) \) as \( A^1(X/A) \) or simply \( A^1(X) \),
as the group depends only on $X$ by Proposition \[2.15\]. The rank of $A^1(X)$ is called the Picard number of $X$ and is denoted $\rho(X)$. Since the intersection numbers are multilinear, the abelian group $A^1(X/S)$ is torsion-free.

The $A^1$ groups of two schemes may be related through the following lemmas, which follow immediately from Lemmas \[2.20\] and \[2.18\]. We first generalize \[KI\] p. 334, Proposition 1.

**Lemma 3.1.** Let $S, S'$ be noetherian schemes. Let

$$
\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
p' \downarrow & & \downarrow p \\
S' & \xrightarrow{g} & S
\end{array}
$$

be a commutative diagram with proper morphisms $\pi, \pi'$. Then the pair $(f/g)$ induces a homomorphism

$$(f/g)^*: A^1(X/S) \rightarrow A^1(X'/S').$$

If for every $\pi$-contracted integral curve $C \subset X$, there exists a $\pi'$-contracted integral curve $C' \subset X'$ such that $f(C') = C$ (for instance if $g = \text{id}_S$ and $f$ is proper and surjective), then $(f/g)^*$ is injective. \hfill \Box

**Lemma 3.2.** Let $g: S' \rightarrow S$ be a morphism of noetherian schemes, and let $\pi: X \rightarrow S$ be a proper morphism. Suppose that $g(S')$ contains all closed points of $S$. Then the natural map $A^1(X/S) \rightarrow A^1(X \times_S S'/S')$ is injective. \hfill \Box

The Theorem of the Base says that $A^1(X/S)$ is finitely generated. When $S = \text{Spec } k$ with $k = \overline{k}$, this is \[KI\] p. 305, Remark 3. The case of an arbitrary base field $k$ can be reduced to the algebraically closed case by Lemma \[3.2\].

In \[KI\] p. 334, Proposition 3, the Theorem of the Base was proven when $S$ is of finite-type over an algebraically closed field. We will follow much of this proof to prove Theorem \[3.6\]. However, some changes must be made since normalization was used. If $S$ is of finite-type over a field, the normalization of $S$ is still a noetherian scheme, but there exists a noetherian (affine) scheme $S$ such that its normalization is not noetherian \[E\] p. 127. We will evade this difficulty via the following lemma. (We will not use the claim regarding smoothness here, but it will be needed in \[5\].)

**Lemma 3.3.** \footnote{This lemma is incorrect. See the corrected Lemma \[E1.3\].} Let $A$ be a noetherian domain with field of fractions $K$, let $X$ be an integral scheme with a projective, surjective morphism $\pi: X \rightarrow \text{Spec } A$, and let $d$ be the dimension of the generic fiber of $\pi$. There exists non-zero $g \in A$, a scheme $X'$, and a projective, surjective morphism $f: X' \rightarrow X \times_A A_g$ such that the composite morphism $\pi': X' \rightarrow \text{Spec } A_g$ is flat, projective, and surjective, and each fiber of $\pi'$ is geometrically integral, with generic fiber dimension $d$. Further, if $K$ is perfect, then one can assume that the morphism $\pi'$ is smooth.

**Proof.** Let $X_0 = X \times_A K$ be the generic fiber of $\pi$. Using alteration of singularities \[D\], we may find a regular integral $K$-scheme $\tilde{X}_0$ with projective, surjective morphism $\tilde{\pi}_0 : \tilde{X}_0 \rightarrow X_0$. Since an alteration is a generically finite morphism, $\dim \tilde{X}_0 = d$.

By \[LI\] Lemma 1.4.11, there exists a finite extension field $K' \supset K$ such that every irreducible component of $(\tilde{X}_0 \times_K K')_{\text{red}}$ is geometrically integral. Since $\tilde{X}_0 \times_K K' \rightarrow \tilde{X}_0$ is surjective and finite, we may choose an irreducible component
Let \( \pi_0^* : X_0^0 \to \text{Spec} K \) be the composite morphism. Then \( \pi_0^* \) is certainly flat and projective. Further, if \( K \) is perfect, then \( \tilde{X}_0 \times_K K' \) is smooth over Spec \( K' \) and hence is a regular (and necessarily reduced) scheme [AK Proposition VII.6.3]. Since two irreducible components of \( \tilde{X}_0 \times_K K' \) cannot intersect [H Remark III.7.9.1], \( X_0^0 \) is a connected component (and hence a regular open subscheme) of \( X_0 \times_K K' \). Since \( X_0^0 \) is algebraic over \( K \), the morphism \( \pi_0^* \) is smooth.

So now \( \pi_0^* \) and \( f_0 \) have all the desired properties of \( \pi' \) and \( f \). Since \( \pi_0^* \) and \( f_0 \) are of finite type, we can find \( g_1 \in A \) and an algebraic \( A_{g_1} \)-scheme \( X'_1 \) with finite type morphisms

\[
\begin{align*}
X'_1 & \xrightarrow{f_1} X \times_A A_{g_1} \\
\quad & \quad \xrightarrow{\pi \times \text{id}_{A_{g_1}}} \\
& \quad \xrightarrow{\pi_1} \text{Spec} A_{g_1}
\end{align*}
\]

such that \( \pi'_1 \times A_{g_1} \text{id}_K \cong \pi_0^* \) and \( f_1 \times A_{g_1} \text{id}_K \cong f_0 \). We may shrunk the base by taking the fibered product of (3.4) with Spec \( A_{g_1} \) for some multiple \( g \) of \( g_1 \); we set \( \pi' = \pi'_1 \times \text{id}_{A_{g_1}} \), \( f = f_1 \times \text{id}_{A_{g_1}} \), \( X' = X'_1 \times A_{g_1} \). By the Theorem of Generic Flatness, we may assume \( \pi' \) is flat [EGA IV, 8.9.4]. If \( \{A_s\} \) is the inductive system of one element localizations of \( A \), then \( K = \lim_s A_s \). So we may shrink the base to assume the resulting morphisms \( \pi' \) and \( f \) are projective and surjective [EGA IV, 8.10.5]. Finally, we may assume all fibers of \( \pi' \) are geometrically integral and (if \( K \) is perfect) smooth over \( k(s) \) [EGA IV, 12.2.4]. So if \( K \) is perfect, then \( \pi' \) is smooth since \( \pi' \) is flat and all fibers are smooth over \( k(s) \) [AK Theorem VII.1.8].

We may now prove Theorem 1.4 which we state in the generality of an arbitrary noetherian base scheme. First, a remark regarding fibers of flat morphisms.

**Remark 3.5.** Let \( \pi : X \to S \) be a flat, projective morphism of noetherian schemes. Let \( s \in S \), and let \( X_s = X \times_S \overline{k(s)} \). Let \( \mathcal{L} \) be an invertible sheaf on \( X \), let \( \mathcal{L}_s = \mathcal{L}|_{X_s} \) and let \( \mathcal{L}_s = \mathcal{L}_s \otimes_{k(s)} \overline{k(s)} \). Since \( \pi \) is flat, the Euler characteristic \( \chi(\mathcal{L}_s) \) is independent of \( s \) [Mil Lecture 7.9, Corollary 3]. Since Spec \( \overline{k(s)} \to \text{Spec} k(s) \) is flat, \( \chi(\mathcal{L}_s) = \chi(\mathcal{L}_s \otimes_{k(s)} \overline{k(s)}) \).

**Theorem 3.6 (Theorem of the Base).** Let \( S \) be a noetherian scheme and let \( \pi : X \to S \) be a proper morphism. The torsion-free abelian group \( A^1(X/S) \) is finitely generated.

**Proof.** Let \( \{U_\alpha\} \) be a finite open affine cover of \( S \). Using Corollary 2.19 there is an induced injective homomorphism

\[
A^1(X/S) \hookrightarrow \oplus_\alpha A^1(\pi^{-1}(U_\alpha)/U_\alpha).
\]

Thus we may assume \( S = \text{Spec} A \) is affine. By Lemma 3.1 we may replace \( X \) with a Chow cover, so we may assume \( X \) is projective over Spec \( A \). Let \( X_i \) be the reduced, irreducible components of \( X \). By Lemma 3.1 there is a natural monomorphism \( A^1(X) \hookrightarrow \oplus A^1(X_i) \), so we may assume \( X \) is an integral scheme. Also \( A^1(X/\text{Spec} A) = A^1(X/(\pi(X))_{\text{red}}) \), so we may assume \( A \) is a domain and \( \pi \) is surjective.
We now proceed by noetherian induction on $S = \text{Spec } A$. If $S = \emptyset$, then $X = \emptyset$ and the theorem is trivial. Now assume that if $Y$ is any scheme which is projective over a proper closed subscheme of $S$, then $A^1(Y)$ is finitely generated. The homomorphism $A^1(X) \to A^1(X \times_A [A/(g)]) \oplus A^1(X \times_A A_g)$ is injective, so we may replace Spec $A$ with any affine open subscheme. Then by Lemma 3.1, we may replace $X$ with the $X'$ of Lemma 3.3 and assume $\pi$ is flat and all fibers are geometrically integral.

Let $\eta$ be the generic point of Spec $A$. We claim that $\phi: A^1(X) \to A^1(X_\eta)$ is a monomorphism. To see this, let $L \in \ker \phi$, let $\mathcal{H}$ be an ample invertible sheaf on $X$, and let $r = \dim X_\eta$. By the definition of numerical triviality, Definition 2.9, we need to show that $L$ is numerically trivial on every closed fiber $X_s$ over $s \in \text{Spec } A$. Equivalently by Lemma 2.18, we need to show $L$ is numerically trivial on each $X_\bar{s} = X \times_A k(s)$.

By Remark 3.5, the intersection numbers $(\mathcal{L}_s^i, \mathcal{H}_s^{r-i})$ are independent of $s$. If $r = 0$, then any invertible sheaf on $X_\bar{s}$ is numerically trivial. If $r = 1$, then $\mathcal{L}_s$ is numerically trivial if and only if $(\mathcal{L}_s) = 0$. Finally, if $r \geq 2$, then $\mathcal{L}_s$ is numerically trivial if and only if

\[(\mathcal{L}_s \mathcal{H}_s^{r-1}) = (\mathcal{L}_s^2 \mathcal{H}_s^{r-2}) = 0\]

by [Kl] p. 306, Corollary 3]. Since $\mathcal{L}_\eta$ is numerically trivial, we have shown that $\mathcal{L}_\bar{s}$ is numerically trivial, independent of $s$, and we are done. \hfill \Box

One of the most important uses of the Theorem of the Base is that it allows the use of cone theory in studying ampleness and numerical effectiveness. Now that we know that $A^1(X/S)$ is finitely generated, we may prove Kleiman’s criterion for ampleness in greater generality than previously known. Since the original proofs rely mainly on the abstract theory of cones in $\mathbb{R}^n$, we may reuse the original proofs in [Kl] p. 323–327. A cone $K$ is a subset of a real finite dimensional vector space $V$ such that for all $a \in \mathbb{R}_{>0}$ we have $aK \subset K$ and $K + K \subset K$. The cone is pointed if $K \cap -K = \{0\}$. The interior of a closed pointed cone $K$, Int $K$, in the Euclidean topology of $V$, is also a cone, possibly empty.

**Lemma 3.7.** [V] p. 1209 Let $K$ be a closed pointed cone in $\mathbb{R}^n$. Then $v \in \text{Int } K$ if and only if, for all $u \in \mathbb{R}^n$, there exists $m_0$ such that $u + mv \in K$ for all $m \geq m_0$. \hfill \Box

Now set $V(X/S) = A^1(X/S) \otimes \mathbb{R}$. From Propositions 2.10 and 2.11, one sees that if $\mathcal{L}, \mathcal{L}'$ are invertible sheaves on $X$ with $\mathcal{L} \equiv \mathcal{L}' \in V(X/S)$, then $\mathcal{L}$ is $\pi$-ample (resp. $\pi$-nef, $\pi$-trivial) if and only if $\mathcal{L}'$ has the same property. So for our purposes, we may use the notation $\mathcal{L}$ to represent an element of Pic $X$ or $V(X/S)$ without confusion.

One can easily show that the cone $K$ generated by $\pi$-nef invertible sheaves is a closed pointed cone. The cone $K^\circ$ generated by $\pi$-ample invertible sheaves is open in the Euclidean topology [Kl] p. 325, Remark 6] and $K^\circ \subset \text{Int } K$ by Lemma 3.7.

We would like to say that $K^\circ = \text{Int } K$ for all proper schemes, but this is not true for some degenerate non-projective cases. For example, in [H] Exercise III.5.9], Pic $X = 0$, so $K^\circ = \emptyset \subset \text{Int } K = K = 0$. We need a suitable generalization of projectivity.

**Definition 3.8.** Let $\pi: X \to S$ be a proper morphism over a noetherian scheme $S$. The scheme $X$ is relatively quasi-divisorial for $\pi$ if for every $\pi$-contracted integral
subscheme \(V\) (which is not a point), there exists an invertible sheaf \(L\) on \(X\) and an effective non-zero Cartier divisor \(H\) on \(V\) such that \(L|_V \cong \mathcal{O}_V(H)\). If \(S\) is affine, then \(X\) is quasi-divisorial. (This absolute notation is justified by Proposition 2.15)

If \(\pi\) is projective, then \(X\) is relatively quasi-divisorial; just take \(L\) to be relatively very ample for \(\pi\). If \(X\) is a regular integral scheme (or more generally \(\mathbb{Q}\)-factorial), then \(X\) is relatively quasi-divisorial \([\text{Ko}, \text{Pf. of Theorem VI.2.19}]\). See Remark 6.8 for the definition of a divisorial scheme.

**Theorem 3.9** (Kleiman’s criterion for ampleness). Let \(\pi : X \to S\) be a proper morphism over a noetherian scheme \(S\) with \(X\) relatively quasi-divisorial for \(\pi\). An invertible sheaf \(L\) on \(X\) is \(\pi\)-ample if and only if \(L \in \text{Int} K\). More generally, \(K^\alpha = \text{Int} K\).

**Proof.** By abstract cone theory, the cone generated by the lattice points \(\text{Int} K \cap A^1(X/S)\) is equal to \(\text{Int} K\) \([\text{Kl}]\ p. 325, \text{Remark 5}]\). So we need only show the first claim. We already know that if \(L\) is \(\pi\)-ample, then \(L \in \text{Int} K\).

Now suppose that \(L \in \text{Int} K\). By Proposition 2.10, we need to show that for every \(\pi\)-contracted integral subscheme \(V\) of \(X\), we have \((L^{\dim V}.V) > 0\). Since \(X\) is relatively quasi-divisorial, there exists \(\mathcal{H} \in \text{Pic} X\) such that \(\mathcal{H} \cong \mathcal{O}_V(H)\) with \(H\) a non-zero effective Cartier divisor. Note that if \(\dim V = 1\), then \((\mathcal{H}.V) = (H) = \dim H^0(H, \mathcal{O}_U) > 0\) \([\text{Kl}]\ p. 296, \text{Proposition 1}]\).

For sufficiently large \(n\), we have \(L^n \otimes \mathcal{H}^{-1} \in K\) by Lemma 3.7. And so, \((L^{\dim V}.V \otimes L^{\otimes n} \otimes \mathcal{H}^{-1}.V) \geq 0\) by Lemma 2.12. Thus

\[
n(L^{\dim V}.V) \geq (L^{\dim V}.V \otimes L^{\otimes n} \otimes \mathcal{H}^{-1}.V) = (L^{\dim V}.V) > 0
\]

by induction. Thus \(L\) is ample. \(\square\)

## 4. Ample Filters I

In this section we will collect a few preliminary propositions regarding ample filters which are well-known in the case of an ample invertible sheaf. The main goal of \([\text{H}]\) will be to prove that a certain filter of invertible sheaves is an ample filter; these propositions will allow for useful reductions in that proof.

**Proposition 4.1.** Let \(X, Y\) be proper over a noetherian ring \(A\). Let \(\{L_\alpha\}\) be a filter of invertible sheaves on \(X\).

1. If \(\{L_\alpha\}\) is an ample filter on \(X\), then \(\{L_\alpha|_Y\}\) is an ample filter on \(Y\) for all closed subschemes \(Y \subset X\).
2. The filter \(\{L_\alpha\}\) is an ample filter on \(X\) if and only if \(\{L_\alpha|_{X_{\text{red}}}\}\) is an ample filter on \(X_{\text{red}}\).
3. Suppose that \(X\) is reduced. The filter \(\{L_\alpha\}\) is an ample filter on \(X\) if and only if for each irreducible component \(X_i\), the filter \(\{L_\alpha|_{X_i}\}\) is an ample filter on \(X_i\).
4. Let \(f : Y \to X\) be a finite morphism. If \(\{L_\alpha\}\) is an ample filter on \(X\), then \(\{f^*L_\alpha\}\) is an ample filter on \(Y\).
5. Let \(f : Y \to X\) be a finite surjective morphism. If \(\{f^*L_\alpha\}\) is an ample filter on \(Y\), then \(\{L_\alpha\}\) is an ample filter on \(X\).

**Proof.** The proof of each item is as in [H Exercise III.5.7]. \(\square\)
For the rest of this section we will switch from the proper case to the projective case. However, this is not a real limitation in regards to ample filters since we will see that if $X$ has an ample filter, then $X$ is projective by Corollary 6.7.

**Proposition 4.5.** Let $X$ be a noetherian scheme, and let $\pi: X \to S$ be a proper morphism, and let $\{L_\alpha\}$ be a filter of invertible sheaves on $X$. If $S_0$ is a locally closed subscheme of $S$ and $\{L_\alpha\}$ is a $\pi$-ample filter, then the filter $\{L_\alpha|_{\pi^{-1}(S_0)}\}$ is a $\pi|_{\pi^{-1}(S_0)}$-ample filter.

**Proof.** The scheme $S_0$ is a closed subscheme of an open subscheme $U$ of $S$. Let $F$ be a coherent sheaf on $X \times_S S_0$. Then $i: X \times_S S_0 \hookrightarrow X \times_S U$ is a closed immersion and $F' = i_*F$ is a coherent sheaf on $X \times_S U$. Further, $v: X \times_S U \hookrightarrow X$ is an open immersion and there exists a coherent sheaf $F''$ on $X$ such that $v^*F'' = F'$ [H Exercise II.5.15]. We have the desired vanishing of higher direct images after tensoring with $i^*v^*L_\alpha = L_\alpha|_{\pi^{-1}(S_0)}$ since $u: U \hookrightarrow S$ is flat [H Proposition III.9.3] and $i$ is affine [H Exercise III.4.1].

The following lemma will be useful in [5] for noetherian induction on $X$. Its proof has similarities to the proof of Proposition 2.5.

**Lemma 4.6.** Let $S$ be a noetherian scheme, let $\pi: X \to S$ be a projective morphism, and let $\{L_\alpha\}$ be a filter of invertible sheaves on $X$. Let $T$ be a closed subscheme of $S$ and suppose that $\{L_\alpha|_{\pi^{-1}(T)}\}$ is a $\pi|_{\pi^{-1}(T)}$-ample filter on $\pi^{-1}(T) = \pi^{-1}(T)$.
Proof. If \( T = \emptyset \), then we may take \( U_\alpha = \emptyset \) for all \( \alpha \). So by noetherian induction, we may assume that for all proper closed subschemes \( V \subseteq T \) and coherent sheaves \( \mathcal{F} \) on \( X \), there exists \( \alpha_0 \) and \( U_\alpha \supset V \) such that

\[
R^q\pi_* (\mathcal{F} \otimes \mathcal{L}_\alpha)|_{U_\alpha} = 0
\]

for \( q > 0 \) and \( \alpha \geq \alpha_0 \).

Let \( q > 0 \) and \( \alpha \geq \alpha_0 \). Further, if \( U \) is an open subscheme of \( S \) and \( W = (U \cap T)_{\text{red}} \), then \( \{ \mathcal{L}_\alpha|_{\pi^{-1}(W)} \} \) is a \( \pi|_{\pi^{-1}(W)} \)-ample filter by Proposition 4.5. So we may replace \( S \) with an open subscheme \( U \) as long as \( U \cap T \neq \emptyset \). Thus, we may assume \( S = \text{Spec} \, A \) and \( T = \text{Spec} \, A/I \).

Let \( B = \text{gr}_j(A) = \oplus_{j=0}^\infty J^j/J^{j+1} \). Then \( \text{Spec} \, B \) is called the normal cone \( C_T S \) to \( T \) in \( S \) [Fl, Appendix B.6.1]. Let \( R \) be a ring of polynomials over \( A/I \) which surjects onto \( B \). Then \( u: \text{Spec} \, R \to \text{Spec} \, A/I \) is flat.

Now consider the commutative diagram

\[
\begin{array}{ccc}
X \times_A R & \xrightarrow{g} & X \times_A A/I \\
\downarrow & & \downarrow \pi \\
\text{Spec} \, R & \xrightarrow{u} & \text{Spec} \, A/I \longrightarrow \text{Spec} \, A.
\end{array}
\]

Let \( \mathcal{H} \) be an ample invertible sheaf on \( X \), and let \( h = i \circ g \). Then by Lemma 2.18 \( h^* \mathcal{H} \) is an ample invertible sheaf. Given \( n > 0 \), there exists \( \alpha_0 \) such that

\[
H^q(X \times_A R, h^*(\mathcal{H}^{-n} \otimes \mathcal{L}_\alpha)) = H^q(X \times_A A/I, i^*(\mathcal{H}^{-n} \otimes \mathcal{L}_\alpha)) \otimes_{A/I} R = 0
\]

for \( \alpha \geq \alpha_0 \), since \( u \) is flat and \( \{i^* \mathcal{L}_\alpha\} \) is an ample filter on \( X \times_A A/I \). Thus \( \{h^* \mathcal{L}_\alpha\} \) is an ample filter on \( X \times_A R \) by Proposition 4.2

Let \( I = \pi^{-1}(\mathcal{I}) \mathcal{O}_X \) be the sheaf of ideals of \( X \times_A A/I \) in \( X \). Then there is a canonical embedding [Fl, Appendix B.6.1]

\[
C_{X \times_A A/I} = \text{Spec} \oplus_{j=0}^\infty \mathcal{I}^j/\mathcal{I}^{j+1} \xrightarrow{i''} X \times_A B = X \times_A C_T S
\]

and a closed embedding \( i'': X \times_A R \to X \times_A R \). Let \( f = h \circ i'' \circ i' \). Then by Proposition 4.1 \( \{f^* \mathcal{L}_\alpha\} \) is an ample filter on \( C_{X \times_A A/I} \). We set \( Y = C_{X \times_A A/I} \).

Let \( \mathcal{F} \) be a coherent sheaf on \( X \), and let \( \mathcal{F}' = \oplus_{j=0}^\infty \mathcal{I}^j \mathcal{F}/\mathcal{I}^{j+1} \mathcal{F} \). Then \( \mathcal{F}' \) is a quasi-coherent \( \mathcal{O}_X \)-module with an obvious \( f_* \mathcal{O}_Y \)-module structure. Since \( f \) is affine, there exists a quasi-coherent \( \mathcal{O}_Y \)-module \( \mathcal{G} \) such that \( f_* \mathcal{G} = \mathcal{F}' \) [H Exercise II.5.17]. But since \( \mathcal{F}' \) is a coherent \( f_* \mathcal{O}_Y \)-module, the \( \mathcal{O}_Y \)-module \( \mathcal{G} \) is coherent. So there exists \( \alpha_1 \) such that

\[
\bigoplus_{j=0}^\infty H^q(X, \mathcal{I}^j \mathcal{F}/\mathcal{I}^{j+1} \mathcal{F} \otimes \mathcal{L}_\alpha) \cong H^q(X, \mathcal{F}' \otimes \mathcal{L}_\alpha) \\
\cong H^q(X, f_* (\mathcal{G} \otimes f^* \mathcal{L}_\alpha)) \\
\cong H^q(Y, \mathcal{G} \otimes f^* \mathcal{L}_\alpha) = 0
\]

for \( q > 0 \) and \( \alpha \geq \alpha_1 \), because \( f \) is affine [H Exercise III.4.1] and \( \{f^* \mathcal{L}_\alpha\} \) is an ample filter.
Thus $H^q(X, \mathcal{I}^j \mathcal{F}/\mathcal{I}^{j+1} \mathcal{F} \otimes \mathcal{L}_\alpha) = 0$ for all $q > 0$, $j \geq 0$ and $\alpha \geq \alpha_1$. Now consider the short exact sequence

$$0 \rightarrow \mathcal{F}/\mathcal{I}^j \mathcal{F} \rightarrow \mathcal{F}/\mathcal{I}^{j+1} \mathcal{F} \rightarrow \mathcal{I}^j \mathcal{F}/\mathcal{I}^{j+1} \mathcal{F} \rightarrow 0$$

which yields an exact sequence

$$H^q(X, \mathcal{F}/\mathcal{I}^j \mathcal{F} \otimes \mathcal{L}_\alpha) \rightarrow H^q(X, \mathcal{F}/\mathcal{I}^{j+1} \mathcal{F} \otimes \mathcal{L}_\alpha) \rightarrow H^q(X, \mathcal{I}^j \mathcal{F}/\mathcal{I}^{j+1} \mathcal{F} \otimes \mathcal{L}_\alpha).$$

The last term is 0 for all $j \geq 0$ and the first term is 0 for $j = 1$. Then by induction, the middle term is 0 for all $q > 0$, $j \geq 0$, and $\alpha \geq \alpha_1$.

Let $\hat{A}$ be the $I$-adic completion of $A$. By [EGA] III 1.4.17, $H^q(X, \mathcal{F} \otimes \mathcal{L}_\alpha) \otimes A \hat{A} \cong \lim_{\leftarrow j} H^q(X, \mathcal{F}/\mathcal{I}^j \mathcal{F} \otimes \mathcal{L}_\alpha) = 0$

for $q > 0$ and $\alpha \geq \alpha_1$. Thus for each $\alpha \geq \alpha_1$, there exists $a_\alpha \in 1 + I$ such that $H^q(X, \mathcal{F} \otimes \mathcal{L}_\alpha) \otimes A a_\alpha = 0$ [AM Theorem 10.17]. Since $a_\alpha \in 1 + I$, the open subscheme $\text{Spec } A a_\alpha$ contains the closed subscheme $\text{Spec } A/I$, as desired. \hfill $\square$

In Proposition 4.6, we generalized the first half of Corollary 2.19. We now generalize the second half, which will allow the use of noetherian induction in §5 to show that a certain filter is an ample filter.

**Proposition 4.7.** Let $S$ be a noetherian scheme, let $\pi: X \rightarrow S$ be a proper morphism, and let $\{ \mathcal{L}_\alpha \}$ be a filter of invertible sheaves on $X$. If $\{ S_i \}$ is a finite cover of locally closed subschemes of $S$ and each $\{ \mathcal{L}_\alpha|_{\pi^{-1}(S_i)} \}$ is a $\pi|_{\pi^{-1}(S_i)}$-ample filter, then $\{ \mathcal{L}_\alpha \}$ is a $\pi$-ample filter.

**Proof.** The schemes $S_i$ are closed subschemes of open subschemes $U_i$ of $S$. Let $\mathcal{F}$ be a coherent sheaf on $X$. Then by Lemma 4.6, for each $i$, there exists $a_i$ and open subschemes $U_{i,\alpha}$ of $U_i$ such that $S_i \subset U_{i,\alpha}$ and [EGA III Corollary 3.8.2]

$$R^q(\pi|_{\pi^{-1}(U_i)*})_*((\mathcal{F} \otimes \mathcal{L}_\alpha)|_{\pi^{-1}(U_i)})|_{U_{i,\alpha}} = R^q\pi_*\mathcal{F} \otimes \mathcal{L}_\alpha)|_{U_{i,\alpha}} = 0$$

for $q > 0$ and $\alpha \geq \alpha_i$. Let $a_\alpha \geq a_i$ for all $i$. Then for $\alpha \geq a_0$, the open subschemes $U_{i,\alpha}$ cover $S$, so $R^q\pi_*\mathcal{F} \otimes \mathcal{L}_\alpha = 0$, as desired. \hfill $\square$

We will also need a theorem for global generation of $\mathcal{F} \otimes \mathcal{L}_\alpha$. We do so through the concept of $m$-regularity, which is most often studied when $X$ is projective over a field. However, most of the proofs are still valid when $X$ is projective over a noetherian ring $A$.

**Definition 4.8.** Let $X$ be a projective scheme over a noetherian ring $A$, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf. A coherent sheaf $\mathcal{F}$ is said to be $m$-regular (with respect to $\mathcal{O}_X(1)$) if $H^q(X, \mathcal{F}(m-q)) = 0$ for $q > 0$.

**Proposition 4.9.** Let $X$ be a projective scheme over a noetherian ring $A$, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf. If a coherent sheaf $\mathcal{F}$ on $X$ is $m$-regular, then for $n \geq m$

1. $\mathcal{F}$ is $n$-regular,
2. The natural map $H^0(X, \mathcal{F}(n)) \otimes_A H^0(X, \mathcal{O}_X(1)) \rightarrow H^0(X, \mathcal{F}(n+1))$ is surjective, and
3. $\mathcal{F}(n)$ is generated by global sections.
Proof. Let \( j : X \hookrightarrow \mathbb{P}^r_A \) be the closed immersion defined by \( \mathcal{O}_X(1) \). Then for all \( n \) and \( q \geq 0 \), \( H^q(X, \mathcal{F} \otimes \mathcal{O}_X(n)) \cong H^q(\mathbb{P}^r, (j_*)\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^r}(n)) \). Thus \( \mathcal{F} \) is \( n \)-regular if and only if \( j_*\mathcal{F} \) is \( n \)-regular. So (1) holds on \( X \) if it holds on \( \mathbb{P}^r \). Further, we have the commutative diagram

\[
H^0(\mathbb{P}^r, j_*\mathcal{F}(n)) \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \longrightarrow H^0(\mathbb{P}^r, j_*\mathcal{F}(n+1))
\]

so if (2) holds on \( \mathbb{P}^r \), it holds on \( X \). So for (1) and (2) we have reduced to the case \( X = \mathbb{P}^r \) and this is [O] Theorem 2.

The proof of (3) proceeds as in [K] p. 307, Proposition 1(iii), keeping in mind the more general situation. Let \( f : X \to \text{Spec} A \) be the structure morphism. A coherent sheaf \( \mathcal{G} \) is generated by global sections if and only if the natural morphism \( f^*f_*\mathcal{G} = f^*H^0(X, \mathcal{G}) \to \mathcal{G} \) is surjective [H] Theorem III.8.8. We have a commutative diagram

\[
f^*H^0(\mathcal{F}(n)) \otimes f^*H^0(\mathcal{O}_X(1)) \longrightarrow f^*H^0(\mathcal{F}(n+1))
\]

\[
f^*H^0(\mathcal{F}(n)) \otimes \mathcal{O}_X(1) \longrightarrow f^*H^0(\mathcal{F}(n+1)).
\]

By (2), \( \alpha_n \) is surjective for \( n \geq m \). Also, there exists \( n_1 \geq m \) such that \( \mathcal{F}(n+1) \) is generated by global section for \( n \geq n_1 \), and so \( \beta_{n+1} \) is also surjective for \( n \geq n_1 \). This implies that \( \beta_n \otimes 1 \) (and hence \( \beta_n \)) is surjective for \( n \geq n_1 \). Descending induction on \( n \) gives that \( \beta_n \) is surjective for \( n \geq m \), as desired. \( \Box \)

Corollary 4.10. Let \( X \) be a projective scheme over a commutative noetherian ring \( A \), let \( \mathcal{O}_X(1) \) be a very ample invertible sheaf on \( X \), let \( \{\mathcal{L}_\alpha\} \) be an ample filter on \( X \), and let \( \mathcal{F} \) be a coherent sheaf on \( X \). There exists \( \alpha_0 \) such that

1. The natural map \( H^0(X, \mathcal{F} \otimes \mathcal{L}_\alpha) \otimes H^0(X, \mathcal{O}_X(1)) \to H^0(X, \mathcal{F} \otimes \mathcal{L}_\alpha \otimes \mathcal{O}_X(1)) \) is surjective and
2. \( \mathcal{F} \otimes \mathcal{L}_\alpha \) is generated by global sections

for \( \alpha \geq \alpha_0 \).

Proof. Find \( \alpha_0 \) such that \( H^q(X, \mathcal{F} \otimes \mathcal{L}_\alpha \otimes \mathcal{O}_X(-q)) = 0 \) for all \( \alpha \geq \alpha_0 \) and \( q > 0 \). (This is possible since cd(\( X \)) is finite.) Then \( \mathcal{F} \otimes \mathcal{L}_\alpha \) is 0-regular for \( \alpha \geq \alpha_0 \). The conclusions then follow from the previous proposition. \( \Box \)

5. Generalization of Serre Vanishing

In this section we will prove Theorem 1.5, a generalization of Serre’s Vanishing Theorem. It will allow us to prove our desired implications in Theorem 1.3. Serre’s Vanishing Theorem says that on a projective scheme with coherent sheaf \( \mathcal{F} \) and ample \( \mathcal{L} \), the higher cohomology of \( \mathcal{F} \otimes \mathcal{L}^m \) vanishes for \( m \) sufficiently large. Our generalization says that for \( m \geq m_0 \), we also have the cohomological vanishing of \( \mathcal{F} \otimes \mathcal{L}^m \otimes \mathcal{N} \), with \( m_0 \) independent of \( \mathcal{N} \), where \( \mathcal{N} \) is numerically effective. This was proven for the case of \( X \) projective over an algebraically closed field in [Fj] §5 and we follow some of that proof.
In essence, we will prove that a certain filter of invertible sheaves is an ample filter. Thus we may use the results of [H] to aid us. Let us precisely define our filter of interest.

**Notation 5.1.** Let $X$ be a projective scheme over a noetherian ring $A$. Let $\mathcal{L}$ be an ample invertible sheaf and let $\Lambda$ be a set of (isomorphism classes of) invertible sheaves on $X$. We will define a filter $(\mathcal{L}, \Lambda)$ as follows. As a set, $(\mathcal{L}, \Lambda)$ is the collection of all invertible sheaves $\mathcal{L}^m \otimes \mathcal{N}$ with $m \geq 0$ and $\mathcal{N} \in \Lambda$. For two elements $\mathcal{H}_i$ of $(\mathcal{L}, \Lambda)$, let $m_i$ be the maximum integer $m$ such that $\mathcal{H}_i \cong \mathcal{L}^m \otimes \mathcal{M}$ for some $\mathcal{M} \in \Lambda$. Then $\mathcal{H}_1 \prec \mathcal{H}_2$ if and only if $m_1 < m_2$. This defines a partial ordering on $(\mathcal{L}, \Lambda)$ which makes $(\mathcal{L}, \Lambda)$ a filter of invertible sheaves.

**Notation 5.2.** Let $A$ be a noetherian domain, let $\pi: X \to \text{Spec } A$ be a projective morphism, let $\mathcal{L}$ be an ample invertible sheaf on $X$, and let $\Lambda$ be a set of (isomorphism classes of) invertible sheaves on $X$. (For a locally closed subscheme $Y \subset X$, let $\Lambda|_Y = \{\mathcal{N}|_Y : \mathcal{N} \in \Lambda\}$.) Let $\mathcal{F}$ be a coherent sheaf on $X$. For $q > 0$, we say that $V^q(\mathcal{F}, \mathcal{L}, \Lambda)$ holds if there exists $m_0 = m(\mathcal{F}, q)$ and a non-empty open subscheme $U = U(\mathcal{F}, q) \subset \text{Spec } A$ such that $R^q\pi_* (\mathcal{F} \otimes \mathcal{L}^m \otimes \mathcal{N})|_U = 0$ for all $m \geq m_0$, $q' \geq q$, $\mathcal{N} \in \Lambda$, and all open subschemes $V \subset U$ such that $\mathcal{N}|_{\pi^{-1}(V)}$ is $\pi|_{\pi^{-1}(V)}$-nef. If $\mathcal{L}$ or $\Lambda$ are clear, we write $V^q(\mathcal{F})$.

In the vanishing conditions $V^q$ above, note that $m_0$ is independent of the particular open subscheme $V \subset U$ and that $\mathcal{N}$ does not need to be nef over all of $\text{Spec } A$. Thus the vanishing $V^q$ is in some sense stronger than the vanishing in Theorem 1.5. This will be necessary to reduce some of our work to the case of $A$ being finitely generated over $\mathbb{Z}$.

We first prove reduction lemmas so we may work with schemes with the nice properties listed in Lemma 3.3. It will be useful to replace $\text{Spec } A$ with certain affine open subschemes. This is possible because of [H Corollary III.8.2]

\[ R^q\pi_* (\mathcal{F})|_U = R^q (\pi|_{\pi^{-1}(U)})_* (\mathcal{F}|_{\pi^{-1}(U)}). \]

Many of our proofs implicitly use (5.3). Our proofs also use descending induction on $q$, as we automatically have $V^q(\mathcal{F})$ for any $\mathcal{F}$ and $q > \text{cd}(X)$.

**Lemma 5.4.** Let $X, A, \mathcal{L}, \Lambda, q$ be as in Notation 5.2. Let

\[ 0 \to K \to \mathcal{F} \to \mathcal{G} \to 0 \]

be a short exact sequence. Then for all $q > 0$,

1. $V^q(K, \mathcal{L}, \Lambda)$ and $V^q(G, \mathcal{L}, \Lambda)$ imply $V^q(\mathcal{F}, \mathcal{L}, \Lambda)$, and we may assume $U(\mathcal{F}, q) = U(K, q) \cap U(\mathcal{G}, q)$.
2. $V^q(F, \mathcal{L}, \Lambda)$ and $V^{q+1}(K, \mathcal{L}, \Lambda)$ imply $V^q(\mathcal{G}, \mathcal{L}, \Lambda)$, and we may assume $U(\mathcal{G}, q) = U(\mathcal{F}, q) \cap U(K, q + 1)$, and
3. $V^q(G, \mathcal{L}, \Lambda)$ and $V^{q+1}(F, \mathcal{L}, \Lambda)$ imply $V^{q+1}(K, \mathcal{L}, \Lambda)$, and we may assume $U(K, q + 1) = U(\mathcal{G}, q) \cap U(\mathcal{F}, q + 1)$.

**Proof.** The proof of each of the three statements is nearly the same. For the first statement, set $U(\mathcal{F}, q) = U(K, q) \cap U(\mathcal{G}, q)$ and $m(\mathcal{F}, q) = \max\{m(K, q), m(\mathcal{G}, q)\}$. Since $\text{Spec } A$ is irreducible, the open set $U(\mathcal{F}, q)$ is non-empty. □

**Lemma 5.5.** Let $X, A, \mathcal{L}, \Lambda, q$ be as in Notation 5.2. Let $\mathcal{F}$ be a coherent sheaf on $X$. If $V^q(\mathcal{O}_X, \mathcal{L}, \Lambda)$ holds, then $V^q(\mathcal{F}, \mathcal{L}, \Lambda)$ holds and one can take $U(\mathcal{F}, q) = U(\mathcal{O}_X, q)$. 


We then have $V(U_m)$ for some large $X$ necessarily all) numerically effective invertible sheaves on $X$.

Let $\mathcal{K}$ be a coherent sheaf on $X$. For any $\mathcal{N} \in \Lambda$, we have that $\mathcal{K}|_{\pi^{-1}(U)}$ is a non-zero homomorphism with $\mathcal{N}|_{\pi^{-1}(U)}$ by Lemma 5.4 and $U(\mathcal{F}, q) = U(O_X, q)$.

We may now see a more direct connection between the vanishing $V^q$ and ample filters as follows.

**Corollary 5.6.** Let $X, A, \mathcal{L}$ be as in Notation 5.2. Let $\Lambda$ be a set of (not necessarily all) numerically effective invertible sheaves on $X$. Then $V^1(O_X, \mathcal{L}, \Lambda)$ holds if and only if there exists an affine open subscheme $U \subset \text{Spec} A$ such that $(\mathcal{L}|_{\pi^{-1}(U)}, \Lambda|_{\pi^{-1}(U)})$ is an ample filter.

**Proof.** Suppose that $V^1(O_X, \mathcal{L}, \Lambda)$ holds. We may take $U = U(O_X, 1)$ to be affine. By Lemma 5.5, for any coherent sheaf $\mathcal{F}$ on $X$, we have $V^1(\mathcal{F})$ and $U(\mathcal{F}, 1) = U$. For any $\mathcal{N} \in \Lambda$, we have that $\mathcal{N}|_{\pi^{-1}(U)}$ is nef, so the definition of $V^1$ gives the vanishing necessary for $(\mathcal{L}|_{\pi^{-1}(U)}, \Lambda|_{\pi^{-1}(U)})$ to be an ample filter.

Now suppose that $(\mathcal{L}|_{\pi^{-1}(U)}, \Lambda|_{\pi^{-1}(U)})$ is an ample filter. By (5.3), we may replace Spec $A$ with $U$ and $X$ with $\pi^{-1}(U)$. Then there exists $m_0$ such that $R^q\pi_*((\mathcal{N} \otimes \mathcal{L})|_{\pi^{-1}(U)}) = 0$ for all $q > 0, m \geq m_0$ and $\mathcal{N} \in \Lambda$. If $V \subset U$ is an open subscheme, we trivially have the vanishing necessary for $V^1(O_X, \mathcal{L}, \Lambda)$.

**Lemma 5.7.** Let $X, A, \mathcal{L}, \Lambda, q$ be as in Notation 5.2 and let $m > 0$. Then $V^q(O_X, \mathcal{L}, \Lambda)$ holds if and only if $V^q(O_X, \mathcal{L}^m, \Lambda)$ holds.

**Proof.** By Lemma 5.5, this is actually a statement about the vanishing $V^q(\mathcal{F}, \mathcal{L}, \Lambda)$ and $V^q(\mathcal{F}, \mathcal{L}^m, \Lambda)$ for all coherent sheaves $\mathcal{F}$. Given a coherent sheaf $\mathcal{F}$, the statement $V^q(\mathcal{F}, \mathcal{L}, \Lambda)$ obviously implies $V^q(\mathcal{F}, \mathcal{L}^m, \Lambda)$. Conversely, the statements $V^q(\mathcal{F} \otimes \mathcal{L}^k, \mathcal{L}^m, \Lambda), k = 0, \ldots, m-1$, imply $V^q(\mathcal{F}, \mathcal{L}, \Lambda)$, as in the proof of [II Proposition II.7.5].

**Lemma 5.8.** Let $X, A, \mathcal{L}, \Lambda, q$ be as in Notation 5.2. If for all reduced, irreducible components $X_i$ of $X$ the statement $V^q(O_{X_i}, \mathcal{L}|_{X_i}, \Lambda|_{X_i})$ holds, then $V^q(O_X, \mathcal{L}, \Lambda)$ holds.

**Proof.** This is a standard reduction to the integral case, as in [II (5.10)] or [II Exercise III.5.7].

**Lemma 5.9.** Let $X, A, \mathcal{L}, \Lambda, q$ be as in Notation 5.2, and let $X$ be integral. Let $\omega$ be a coherent sheaf on $X$ with $\text{Supp} \omega = X$. Suppose that $V^q(\omega, \mathcal{L}, \Lambda)$ holds and for all coherent sheaves $\mathcal{G}$ with $\text{Supp} \mathcal{G} \subset X$, the statement $V^q(\mathcal{G}, \mathcal{L}, \Lambda)$ holds. Then for all coherent sheaves $\mathcal{F}$, the statement $V^q(\mathcal{F}, \mathcal{L}, \Lambda)$ holds.

**Proof.** By Lemma 5.5, it suffices to show $V^q(O_X)$ holds. There exists $m$ sufficiently large so that $\text{Hom}(\omega, \mathcal{L}^m) \cong \text{Hom}(\omega, O_X) \otimes \mathcal{L}^m$ is generated by global sections. Now $\text{Hom}(\omega, O_X) \neq 0$ since it is not zero at the generic point of $X$. Thus there is a non-zero homomorphism $\phi: \omega \to \mathcal{L}^m$. Since $\mathcal{L}^m$ is torsion-free, the sheaf $\mathcal{G} = \text{Coker}(\phi)$ has $\text{Supp} \mathcal{G} \subset X$. \[\text{□}\]
Consider the exact sequences
\begin{align}
0 \to \ker(\phi) & \to \omega \to \im(\phi) \to 0 \\
0 \to \im(\phi) & \to \mathcal{L}^m \to \coker(\phi) \to 0.
\end{align}

By descending induction on \( q \), we may assume \( V^{q+1}(\ker(\phi)) \) holds, thus we have that \( V^q(\im(\phi)) \) holds. Then \( V^q(\coker(\phi)) \) holds. Then obviously \( V^q(\mathcal{O}_X) \) holds.

\[\]

**Lemma 5.11.** Let \( X, A, \mathcal{L}, \Lambda, q \) be as in Notation 5.2. Let \( f : X' \to X \) be a projective, surjective morphism, let \( \mathcal{L}' \) be an ample invertible sheaf on \( X' \), and let \( \mathcal{N} \) be a set of invertible sheaves on \( X' \) such that
\begin{equation}
\{ f^* \mathcal{N} : \mathcal{N} \in \Lambda \} \cup \{ f^* \mathcal{L}^m : m \geq 0 \} \subset \Lambda'.
\end{equation}
If for all coherent sheaves \( \mathcal{F} \) on \( X' \), the statement \( V^q(\mathcal{F}', \mathcal{L}', \mathcal{N}') \) holds, then for all coherent sheaves \( \mathcal{F} \) on \( X \), the statement \( V^q(\mathcal{F}, \mathcal{L}, \Lambda) \) holds.

**Proof.** If \( Y \subset X \) is a closed subscheme, then the hypotheses of the lemma are satisfied by \( X' \times_X Y \to Y \). So by Lemma 5.8 and noetherian induction on \( X \), we may assume that \( X \) is integral and that \( V^q(\mathcal{F}) \) holds for all coherent sheaves \( \mathcal{F} \) on \( X \) with \( \text{Supp} \mathcal{F} \subset X \).

The invertible sheaf \( \mathcal{L}' \) is ample and hence \( f \)-ample, so take \( m_0 \) sufficiently large so that \( R^i f_* (\mathcal{L}'^m) = 0 \) for \( m \geq m_0 \) and \( i > 0 \). Let \( \pi' = \pi \circ f \). Since \( V^q(\mathcal{O}_{X'}, \mathcal{L}', \Lambda') \) holds, we may find an open subscheme \( U \subset \text{Spec} A \) and \( m_1 \geq m_0 \) so that \( R^q \pi'_* (\mathcal{L}'^m \otimes \mathcal{N}')|_V = 0 \) for all \( m \geq m_1, q' \geq q \), open \( V \subset U \), and \( \mathcal{N}' \in \Lambda' \) such that \( \mathcal{N}'|_{\pi'-1(V)} \) is \( \pi'|_{\pi'-1(V)} \)-nef.

If \( \mathcal{N} \in \Lambda \) and \( \mathcal{N}|_{\pi-1(V)} \) is \( \pi|_{\pi-1(V)} \)-nef, then
\begin{equation}
f_* (\mathcal{N}|_{\pi-1(V)}) = (f^* \mathcal{N})|_{\pi'-1(V)}
\end{equation}
is \( \pi'|_{\pi'-1(V)} \)-nef by Lemma 2.17. So
\begin{equation}
R^q \pi_* (f_* (\mathcal{L}'^m) \otimes \mathcal{N})|_V = R^q \pi_* (\mathcal{L}'^{m_1} \otimes f^* \mathcal{L}^m \otimes f^* \mathcal{N})|_V = 0
\end{equation}
for all \( m \geq 0, q' \geq q \), open \( V \subset U \) and \( \mathcal{N} \in \Lambda \) such that \( \mathcal{N}|_{\pi-1(V)} \) is \( \pi|_{\pi-1(V)} \)-nef. Thus \( V^q(f_* (\mathcal{L}'^m)) \) holds. Now \( \text{Supp} f_* (\mathcal{L}'^m) = X \) since \( f \) is surjective. So by Lemma 5.9 we are done. \( \square \)

We may now begin proving vanishing theorems akin to Theorem 1.5. We will first work with a ring \( A \) which is of finite-type over \( \mathbb{Z} \), then approximate a general noetherian ring \( A \) with finitely generated \( \mathbb{Z} \)-subalgebras.

\[\]

**Proposition 5.12.** Let \( A \) be a domain, finitely generated over \( \mathbb{Z} \). Let \( X \) be projective over \( A \), let \( \mathcal{L} \) be an ample invertible sheaf on \( X \), and let \( \Lambda \) be the set of all invertible sheaves on \( X \). Then for all coherent sheaves \( \mathcal{F} \) on \( X \), the statement \( V^1(\mathcal{F}, \mathcal{L}, \Lambda) \) holds.

**Proof.** By Lemma 5.8 we may assume \( X \) is integral. If \( \pi : X \to \text{Spec} A \) is not surjective, then the claim is trivial, as we can take \( U(\mathcal{F}, 1) \) to be disjoint from \( \pi(X) \). So assume \( \pi \) is surjective. Let \( d \) be the dimension of the generic fiber of \( \pi \). If \( d = 0 \), then \( \pi \) is generically finite. In this case we may replace \( \text{Spec} A \) with an affine open subscheme to assume \( \pi \) is finite and hence \( X \) is affine. Hence the proposition is trivial [III.3.5]. So assume that the proposition holds for
all projective $Y \to \text{Spec } A$ with generic fiber dimension $< d$. Note that this implies that $V^q(\mathcal{F})$ holds for all $\mathcal{F}$ with $\text{Supp } \mathcal{F} \subset X$.

By the definition of $V^q$, we may replace $S = \text{Spec } A$ by an affine open subscheme and then replace $X$ by a projective, surjective cover by Lemma 5.11. So if $A$ has characteristic 0, we may assume $V^q$ holds for all $\mathcal{F}$ with $\text{Supp } \mathcal{F} \subset X$. Further, we may again replace $S$ with an affine open subscheme to assume the morphism $S \to \text{Spec } \mathbb{Z}$ is smooth [12 Proposition 6.5] and assume that if $s \in S$ with char $k(s) > 0$, then char $k(s) > d$.

Let $\omega_{X/S} = \wedge^q \Omega_{X/S}^1$. The sheaf $\omega_{X/S}$ is invertible since $\pi$ is smooth, and hence $\omega_{X/S}$ is flat over $S$. Let $V \subset S$ be an open subscheme, let $s \in V$ be closed in $V$, and let $X_s = X \times_S k(s)$ be the fiber. The residue field $k = k(s)$ is finite, hence perfect [12 Proposition 6.4]. Let $W_2(k)$ be the ring of second Witt vectors of $k$ [12 3.9]. The closed immersion $\text{Spec } k(s) \to V$ factors as $\text{Spec } k \to \text{Spec } W_2(k) \to V$, since $V \to \text{Spec } \mathbb{Z}$ is smooth [12 2.2]. Thus $X_s$ lifts to $W_2(k)$, i.e., there is scheme $X_1 = X \times_S W_2(k)$ with a smooth morphism $X_1 \to \text{Spec } W_2(k)$, such that $X_s = X_1 \times_{W_2(k)}$. So the Kodaira Vanishing Theorem holds for $X_s$ [12 Theorem 5.8] and thus

$$H^q(X_s, \omega_{X_s/k(s)} \otimes \mathcal{L}|_{X_s} \otimes \mathcal{N}|_{X_s}) = 0$$

for $q > 0$ and any $\mathcal{N}$ such that $\mathcal{N}|_{\pi^{-1}(V)}$ is $\pi|_{\pi^{-1}(V)}$-nef. Now any open cover of the closed points of $V$ covers all of $V$ by Lemma 2.6 and the invertible sheaf $\omega_{\pi^{-1}(V)/V}$ is flat over $V$, so

$$R^q(\pi|_{\pi^{-1}(V)})_* (\omega_{\pi^{-1}(V)/V} \otimes \mathcal{L}|_{\pi^{-1}(V)} \otimes \mathcal{N}|_{\pi^{-1}(V)}) = R^q\pi_* (\omega_{X/S} \otimes \mathcal{L} \otimes \mathcal{N})|_V = 0$$

for $q > 0$ by [H Theorem III.12.11] and descending induction on $q$. Thus we have that $V^1(\omega_{X/S}, \mathcal{L}, \Lambda)$ holds. So we are done with the characteristic 0 case by Lemma 5.9.

If $A$ has characteristic $p > 0$ (and hence is finitely generated over $K = \mathbb{Z}/p\mathbb{Z}$), then $X$ is quasi-projective over $K$, so we may embed $\tilde{X}$ as an open subscheme of an integral scheme $\tilde{X}$ which is projective over $K$. Using alteration of singularities [D] and Lemma 5.11 we may assume $\tilde{X}$ is a regular integral scheme, projective and smooth over $K$ (since $K$ is perfect).

Let $F: \tilde{X} \to \tilde{X}$ be the absolute Frobenius morphism. Since $K$ is perfect, $F$ is a finite surjective morphism. Since $\tilde{X}$ is regular, $F$ is flat [AK Corollary V.3.6], so there is an exact sequence of locally free sheaves

$$0 \to \mathcal{O}_{\tilde{X}} \xrightarrow{\phi} F_* \mathcal{O}_{\tilde{X}}$$

with the morphism $\phi$ locally given by $a \mapsto a^p$. The morphism $\phi$ remains injective locally at $x \in X$ upon tensoring with any residue field $k(x)$, so the kernel of $\phi$ is also locally free [H Exercise II.5.8]. Let $\omega_{\tilde{X}} = \wedge^{\dim X} \Omega_{\tilde{X}/K}^1$ and $\omega_X = \omega_{\tilde{X}|_X} = \wedge^{\dim X} \Omega_{X/K}^1$ [AK Theorem VII.5.1]. Dualizing via [H Exercises III.6.10, 7.2], there is a short exact sequence of locally free sheaves

$$0 \to K|_X \to F_* \omega_{\tilde{X}} \to \omega_X \to 0.$$

Restricting to $X$ gives

$$0 \to K|_X \to F_* \omega_{\tilde{X}} \to \omega_X \to 0.$$
By descending induction on \( q \), assume \( V^{q+1}(\mathcal{K}|_X) \) and \( V^{q+1}(\omega_X) \) hold. Let
\[
U = U(\mathcal{K}|_X, q + 1) \cap U(\omega_X, q + 1), \quad m_0 = \max\{m(\omega_X, q + 1), m(\mathcal{K}|_X, q + 1)\}.
\]
Then \( R^q\pi_*(\mathcal{K}|_X \otimes \mathcal{L}^m \otimes \mathcal{N})|_V = 0 \) for all \( m \geq m_0, q' > q \), open subschemes \( V \subset U \), and \( \mathcal{N} \in \Lambda \) such that \( \mathcal{N}|_{\pi^{-1}(V)} \) is \( \pi|_{\pi^{-1}(V)} \)-nef. Then since \( F \) is finite,
\[
R^q\pi_*(\omega_X \otimes (\mathcal{L}^m \otimes \mathcal{N})^p)|_V \approx R^q\pi_*(\omega_X \otimes F^*(\mathcal{L}^m \otimes \mathcal{N})^p)|_V \\
\approx R^q\pi_*(F_*\omega_X \otimes (\mathcal{L}^m \otimes \mathcal{N})^p)|_V \rightarrow 0
\]
for all \( e > 0, m \geq m_0 \). But the leftmost expression is 0 for large \( e \). Thus \( R^q\pi_*(\omega_X \otimes \mathcal{L}^m \otimes \mathcal{N})|_V = 0 \) for \( m \geq m_0, q' \geq q \), open subschemes \( V \subset U \), and \( \mathcal{N} \in \Lambda \) such that \( \mathcal{N}|_{\pi^{-1}(V)} \) is \( \pi|_{\pi^{-1}(V)} \)-nef. So \( V^q(\omega_X) \) holds and we are again finished by Lemma 5.9.

Remark 5.13. As [Fj2] Lemma 5.8 shows, it is possible to do the proof for \( \text{char } k = p > 0 \) without assuming \( X \) is regular. The cokernel of \( F_*\omega_X \rightarrow \omega_X \) may not be zero, but it is torsion, so the problem is solved by noetherian induction and exact sequences similar to (5.10). In [Fj2], the regular case was presented to show the main idea of the characteristic \( p > 0 \) proof. However, it is interesting to note that now with alteration of singularities, the proposition can be reduced to the regular case in positive characteristic.

We now move to the case of a general noetherian domain \( A \). In the following proof, we will see why it was necessary in Proposition 5.12 to work with \( \Lambda \) equal to all invertible sheaves, instead of only numerically effective invertible sheaves.

**Proposition 5.14.** Let \( A \) be a noetherian domain, and let \( X \) be projective over \( A \). Let \( \mathcal{L}, \mathcal{H}_1, \ldots, \mathcal{H}_j \) be invertible sheaves on \( X \) for some \( j > 0 \), and let \( \mathcal{L} \) be ample. Let \( \Lambda \) be the set of invertible sheaves on \( X \) which are numerically effective and which are in the subgroup of \( \text{Pic} X \) generated by the isomorphism classes of \( \mathcal{H}_i, i = 1, \ldots, j \). Then \( V^1(\mathcal{O}_X, \mathcal{L}, \Lambda) \) holds.

**Proof.** The proposition is trivial if \( \pi: X \rightarrow \text{Spec } A \) is not surjective, so assume this.

Let \( S = \text{Spec } A \). There is a finitely generated \( \mathbb{Z} \)-subalgebra of \( A \), call it \( A_0 \), a scheme \( X_0 \), a commutative diagram
\[
\begin{array}{ccc}
X_0 \times_{A_0} A & \xrightarrow{f} & X_0 \\
\downarrow \pi & & \downarrow \pi_0 \\
S & \xrightarrow{g} & S_0,
\end{array}
\]
and invertible sheaves \( \mathcal{L}_0, \mathcal{H}_{i,0} \) such that \( S_0 = \text{Spec } A_0, X = X_0 \times_{A_0} A, \) and \( \mathcal{L} \cong f^*\mathcal{L}_0, \mathcal{H}_i \cong f^*\mathcal{H}_{i,0} \) [EGA IV3, 8.9.1]. We may further assume that \( \pi_0 \) is projective and surjective [EGA IV3, 8.10.5] and that \( \mathcal{L}_0 \) is ample [EGA IV3, 8.10.5.2]. By the definition of \( V^1 \), we may replace \( S \) (and hence \( S_0 \)) with an affine open subscheme, so we may also assume \( \pi_0 \) (and hence \( \pi \)) is flat [EGA IV3, 8.9.5]. Let \( \Lambda_0 \) be the set of all invertible sheaves on \( X_0 \). By Proposition 5.12, we know \( V^1(\mathcal{O}_{X_0}, \mathcal{L}_0, \Lambda_0) \) holds, so we may replace \( \text{Spec } A_0 \) with \( U(\mathcal{O}_{X_0}, 1) \).

Since all elements of \( \Lambda \) are numerically effective, it suffices to show that there exists \( m_0 \) which gives the vanishing of \( H^q(X, \mathcal{L}^m \otimes \mathcal{N}) \) for \( q > 0, m \geq m_0 \), and \( \mathcal{N} \in \Lambda \). Let \( \mathcal{N} \in \Lambda \), and let \( \mathcal{N}_0 \in \Lambda_0 \) such that \( \mathcal{N} \cong f^*\mathcal{N}_0 \). Let \( s \in S \), and let
Let $s_0 = g(s)$. Then $(\mathcal{L} \otimes \mathcal{N})|_{X_s}$ is ample, and thus $(\mathcal{L}_0 \otimes \mathcal{N}_0)|_{X_0 \times_{A_0} k(s_0)}$ is ample by Lemma 2.18, since $X_s = X \times_A k(s) = (X_0 \times_{A_0} k(s_0)) \times_{k(s_0)} k(s)$. This is true for every $s_0 \in g(S)$, so there exists an open subscheme $U \subset S_0$ such that $g(S) \subset U$ and $(\mathcal{L}_0 \otimes \mathcal{N}_0)|_{\pi_0^{-1}(U)}$ is $\pi_0|_{\pi_0^{-1}(U)}$-ample by Proposition 2.5.

Let $m_0 = m(\mathcal{O}_{X_0}, 1)+1$, which does not depend on $\mathcal{N}$. Then we have $R^q(\pi_0)_*(\mathcal{L}_0^m \otimes \mathcal{N}_0)|_U = 0$ for $q > 0$ and $m \geq m_0$ by $V^{-1}(\mathcal{O}_{X_0})$. Since $\pi_0$ is flat, for each $s_0 \in g(S)$ we have $H^q(X_0 \times_{A_0} k(s_0), (\mathcal{L}_0^m \otimes \mathcal{N}_0)|_{X_0 \times_{A_0} k(s_0)}) = 0$ for $q > 0, m \geq m_0$, and $\mathcal{N} \in \Lambda$ [H Theorem III.12.11]. The flat base change Spec $k(s) \rightarrow$ Spec $k(s_0)$ gives $H^q(X \times_A k(s), (\mathcal{L}_0^m \otimes \mathcal{N})|_{X \times_A k(s)}) = 0$ [H Proposition III.9.3]. Another application of [H Theorem III.12.11] gives the desired vanishing of $H^q(X, \mathcal{L}_0^m \otimes \mathcal{N})$. \hfill \qed

In view of Theorem 1.14, we have in some sense proven Theorem 1.15 "up to numerical equivalence." We now prove a vanishing theorem for numerically trivial invertible sheaves.

**Proposition 5.15.** Let $A$ be a noetherian domain, let $\pi: X \rightarrow \text{Spec } A$ be a flat, surjective, projective morphism with geometrically integral fibers, and let $\mathcal{L}$ be a very ample invertible sheaf on $X$. Let $\Lambda$ be the set of all numerically trivial invertible sheaves on $X$. Then $(\mathcal{L}, \Lambda)$ is an ample filter.

**Proof.** Let $\mathcal{N}$ be a numerically trivial sheaf on $X$, and let $s \in \text{Spec } A$. Using the notation of Remark 3.5, we see that $f(m) = \chi(\mathcal{L}_s^m \otimes \mathcal{N}_s)$ is a polynomial in $m$. Note that $f(m)$ does not depend on $\mathcal{N}$ since $N_s$ is numerically trivial. By [Kl] p. 312, Theorem 3], there exists $m_0$, which depends only on the coefficients of $f(m)$, such that

$$H^q(X_s, \mathcal{L}_s^m \otimes \mathcal{N}_s) = 0$$

for $m \geq m_0$ and $q > 0$.

But $f(m)$ does not depend on $s$, as noted in Remark 3.5. Then using the flat base change Spec $k[s] \rightarrow$ Spec $k(s)$ and [H Theorem III.12.11], we have that

$$H^q(X, \mathcal{L}_s^m \otimes \mathcal{N}) = 0$$

for $m \geq m_0, q > 0$, and all numerically trivial $\mathcal{N}$. Thus by Proposition 4.2 $(\mathcal{L}, \Lambda)$ is an ample filter. \hfill \qed

Now via the following lemma, we tie together the vanishing in Proposition 5.14 with that in Proposition 5.15.

**Lemma 5.16.** Let $k$ be a field, and let $X$ be an equidimensional scheme, projective over $k$. Let $\Lambda$ be a set of (not necessarily all) numerically trivial invertible sheaves on $X$, containing $\mathcal{O}_X$ and closed under inverses. Let $\mathcal{L}$ be an invertible sheaf on $X$ such that $\mathcal{L} \otimes \mathcal{N}$ is very ample for all $\mathcal{N} \in \Lambda$. Let $V$ be a closed subscheme of $X$, let $t > 0$, and let $r(V, t) = 2^{\dim V - t + 1} - 1$ if $\dim V \geq t - 1$, and let $r(V, t) = 0$ otherwise. Let $\mathcal{H}$ be an invertible sheaf on $X$ such that

$$H^q(V, \mathcal{O}_V \otimes \mathcal{H} \otimes \mathcal{L}^{m-1}) = 0$$

for $q' \geq q$ and $m \geq m_0$. Then

$$H^q(V, \mathcal{O}_V \otimes \mathcal{H} \otimes \mathcal{N} \otimes \mathcal{L}^{m-1+r(V, q')}) = 0$$

for $q' \geq q, m \geq m_0$, and all $\mathcal{N} \in \Lambda$. 

Proof. We proceed by induction on \( \dim V \), the claim being trivial when \( \dim V = 0 \). Let \( q > 0 \). We may assume \( q \leq \dim V \). By descending induction on \( q \), we may also assume \( H^q(V, O_V \otimes H \otimes N \otimes L^{m-r(V,q)}) = 0 \) for \( q' > q \), \( m \geq m_0 \), and \( N' \in \Lambda \).

Since \( L \otimes N \) is very ample, there is an effective Cartier divisor \( D \subset V \) with an exact sequence

\[
0 \rightarrow O_V \otimes H \otimes N^{-1} \otimes L^{m-1} \rightarrow O_V \otimes H \otimes L^m \rightarrow O_D \otimes H \otimes L^m \rightarrow 0.
\]

Tensoring (5.17) with \( L^{r(V,q+1)} \) and examining the related long exact sequence, we have \( H^q(D, O_D \otimes H \otimes L^{m+r(V,q+1)}) = 0 \) for \( q' \geq q \) and \( m \geq m_0 \), since \( N^{-1} \in \Lambda \).

Then by induction on \( \dim V \), we have

\[
H^q(D, O_D \otimes H \otimes L^{m+r(V,q+1)+r(D,q)} \otimes M) = 0
\]

for \( m \geq m_0 \) and any \( M \in \Lambda \). Now \( \dim D = \dim V - 1 \) and

\[
r(V, q+1) + r(D, q) = 2^{\dim V - q} - 1 + 2^{\dim D - q + 1} - 1 = 2^{\dim V - q} - 2 = r(V, q) - 1.
\]

Note that since \( \dim V \geq q \), we have \( r(V, q) - 1 \geq 0 \). Tensoring (5.17) with \( L^{r(V,q)-1} \otimes N \), we have \( H^q(V, O_V \otimes H \otimes L^{m-1+r(V,q)} \otimes N) = 0 \) for \( m \geq m_0 \). Since this holds for any \( N \in \Lambda \), we have proven the lemma.

We may now finally prove Theorem 1.5.

Proof. Let \( S = \text{Spec } A \), and let \( \Lambda \) be the set of nef invertible sheaves on \( X \). We wish to show that \( (L, \Lambda) \) is an ample filter. By Proposition 4.1, we may assume \( X \) is an integral scheme. Let \( \pi: X \rightarrow S \) be the structure morphism. We may replace \( S \) with \( \pi(X) \) and hence assume that \( A \) is a domain and that \( \pi \) is surjective.

The theorem is trivial if \( X = \emptyset \). Note that for a closed subscheme \( Y \) of \( X \), that \( \Lambda|_Y \) is a subset of the set of all numerically effective sheaves on \( Y \). So by noetherian induction on \( X \), we may assume that for any proper closed subscheme \( Y \) of \( X \) (in particular \( Y = \pi^{-1}(T) \) for some proper closed subscheme \( T \subset S \)), that \( (\mathcal{L}|_Y, \Lambda|_Y) \) is an ample filter. So by Proposition 4.1, we need only show that there exists an affine open subscheme \( \overline{U} \subset S \) such that \( (\mathcal{L}|_{\overline{U}}, \Lambda|_{\overline{U}}) \) is an ample filter. Or equivalently by Corollary 5.6, we need to show that \( V^1(O_X, \mathcal{L}, \Lambda) \) holds.

By the definition of \( V^1 \), we may replace \( S \) with any affine open subscheme, and by Lemmas 5.11 and 3.3, we may assume \( \pi \) is flat and has geometrically integral fibers. Let \( d \) be the dimension of the generic fiber of \( \pi \).

Let \( \Theta \) be the set of all numerically trivial invertible sheaves on \( X \). By Proposition 5.13, \( (\mathcal{L}, \Theta) \) is an ample filter. So by Corollary 4.10, there exists \( m \) such that \( \mathcal{L}^m \otimes \mathcal{M} \) is very ample for all \( \mathcal{M} \in \Theta \). By Lemma 5.7, we replace \( \mathcal{L} \) by \( \mathcal{L}^m \).

Choose (finitely many) \( \mathbb{Z} \)-generators \( \mathcal{H}_i, i = 1, \ldots, \rho(X) \), of \( A^1(X) \) and let \( \Sigma \) be the set of all nef invertible sheaves in the subgroup of \( \text{Pic } X \) generated by \( \mathcal{H}_i, i = 1, \ldots, \rho(X) \). Then by Proposition 5.14, we may find an affine open subscheme \( U \subset S \) and \( m_0 \) such that

\[
R^q\pi_*((\mathcal{L}^m \otimes \mathcal{H})|_U) = 0
\]

for \( q > 0 \), \( m \geq m_0 \), and \( \mathcal{H} \in \Sigma \). Since \( \pi \) is flat, for \( s \in U \) we then have [Theorem III.12.11]

\[
H^q(X_s, \mathcal{L}_s^m \otimes \mathcal{H}_s) = 0.
\]
Then by Lemma 5.16 we have
\[ H^q(X, \mathcal{L}_s^{m+2d} \otimes \mathcal{H}_s \otimes \mathcal{M}_s) = 0 \]
for \( q > 0, m \geq m_0, \mathcal{H} \in \Sigma, \) and \( \mathcal{M} \in \Theta. \) Now any \( \mathcal{N} \in \Lambda \) can be written as \( \mathcal{N} \cong \mathcal{H} \otimes \mathcal{M} \) for some \( \mathcal{H} \in \Sigma, \) \( \mathcal{M} \in \Theta. \) So another application of [III Theorem III.12.11] gives
\[ R^2\pi_*(\mathcal{L}_m \otimes \mathcal{N})|_U = 0 \]
for \( q > 0, m \geq m_0 + 2d, \) and \( \mathcal{N} \in \Lambda. \) Thus \( V^1(\mathcal{O}_X, \mathcal{L}, \Lambda) \) holds, as desired. \( \square \)

Theorem 1.5 is the best possible in the following sense.

Proposition 5.18. Let \( A \) be a noetherian ring, let \( X \) be projective over \( A, \) and let \( \mathcal{L} \) be an ample invertible sheaf on \( X. \) Let \( \Lambda \) be a set of invertible sheaves on \( X \) such that if \( \mathcal{N} \in \Lambda, \) then \( \mathcal{N}^j \in \Lambda \) for all \( j > 0. \) Suppose that for all coherent sheaves \( \mathcal{F}, \)
\[ H^q(X, \mathcal{F} \otimes \mathcal{L}^m \otimes \mathcal{N}) = 0 \]
for all \( m \geq m_0, q > 0, \) and all \( \mathcal{N} \in \Lambda. \) Then \( \mathcal{N} \) is numerically effective for all \( \mathcal{N} \in \Lambda. \)

Proof. Suppose that \( \mathcal{N} \in \Lambda \) is not numerically effective. Then there exists a contracted integral curve \( f: C \rightarrow X \) with \( (\mathcal{N}, C) < 0. \) Then by the Riemann-Roch formula, for any fixed \( m, \) we can choose \( j \) sufficiently large so that
\[ H^1(C, \mathcal{O}_C \otimes f^*(\mathcal{L}_m^* \otimes \mathcal{N}^j)) \neq 0. \]
But \( \mathcal{N}^j \in \Lambda, \) so this is a contradiction. Therefore, \( \mathcal{N} \) is numerically effective. \( \square \)

We also have an immediate useful corollary to Theorem 1.5 via Corollary 4.10.

Corollary 5.19. Let \( A \) be a noetherian ring, let \( X \) be a projective scheme over \( A, \) let \( \mathcal{L} \) be an ample invertible sheaf on \( X \) with \( \mathcal{L}^n \) very ample, and let \( \mathcal{F} \) be a coherent sheaf on \( X. \) There exists \( m_0 \) such that
(1) The natural map \( H^0(X, \mathcal{F} \otimes \mathcal{L}^m \otimes \mathcal{N}) \otimes H^0(X, \mathcal{L}^n) \rightarrow H^0(X, \mathcal{F} \otimes \mathcal{L}^m+n \otimes \mathcal{N}) \)
is surjective and
(2) \( \mathcal{F} \otimes \mathcal{L}^m \otimes \mathcal{N} \) is generated by global sections
for \( m \geq m_0 \) and all numerically effective invertible sheaves \( \mathcal{N}. \) \( \square \)

6. Ample filters II

In this section we will prove Theorem 1.3. In [AV, Proposition 3.2], it was first shown that \( (A1) \) implied \( (A3), \) and \( (A3) \) implied \( (A4), \) in the case of an ample sequence and \( X \) projective over a field \( k. \) The proof makes strong use of the projectivity of \( X \) and also requires the vanishing of cohomology in \( (A1): \) the surjective map of global sections in \( (A2) \) would not have been strong enough for this method. That \( (A4) \) implies \( (A1) \) was first noted in [Ke1, Proposition 2.3], in the case of an algebraically closed field and a certain ample sequence.

We begin our proof of Theorem 1.3 by using the results of 5.

Proposition 6.1. Let \( X \) be a scheme, proper over a noetherian ring \( A. \) Let \( \{\mathcal{L}_\alpha\} \)
be a filter of invertible sheaves. If \( \{\mathcal{L}_\alpha\} \) satisfies \( (A4), \) then \( \{\mathcal{L}_\alpha\} \) satisfies \( (A1) \)
and \( (A3). \)
Proof. Let \( \mathcal{H} \) be an ample invertible sheaf on \( X \), and let \( \mathcal{F} \) be a coherent sheaf on \( X \). By Theorem 1.5 and Corollary 5.19, there exists \( m \) such that \( \mathcal{F} \otimes \mathcal{H}^m \otimes \mathcal{N} \) has vanishing higher cohomology and is generated by global sections for any nef invertible sheaf \( \mathcal{N} \). By A4, there exists \( \alpha_0 \) such that \( \mathcal{H}^{-m} \otimes \mathcal{L}_\alpha \) is ample (hence nef) for \( \alpha \geq \alpha_0 \). Thus \( \mathcal{F} \otimes \mathcal{L}_\alpha \) has the desired properties for \( \alpha \geq \alpha_0 \).

The statements A1 – A3 do not immediately imply that \( X \) is projective over \( A \), so we may not assume that \( X \) has ample invertible sheaves. It will be much easier to argue certain sheaves are nef. Thus we need to prove that a fifth statement is projective as it has ample invertible sheaves. It will now be much easier to argue certain sheaves are nef. Thus we need to prove that a fifth statement is projective as it has ample invertible sheaves.

Proposition 6.2. Let \( X \) be a scheme, proper over a noetherian ring \( A \), and let \( \{ \mathcal{L}_\alpha \} \) be a filter of invertible sheaves. Then the following are equivalent:

(A4) For all invertible sheaves \( \mathcal{H} \), there exists \( \alpha_0 \) such that \( \mathcal{H} \otimes \mathcal{L}_\alpha \) is an ample invertible sheaf for \( \alpha \geq \alpha_0 \).

(A5) The scheme \( X \) is quasi-divisorial, and for all invertible sheaves \( \mathcal{H} \), there exists \( \alpha_0 \) such that \( \mathcal{H} \otimes \mathcal{L}_\alpha \) is a numerically effective invertible sheaf for \( \alpha \geq \alpha_0 \).

Proof. A4 \( \implies \) A5 is clear since \( X \) is projective as it has an ample invertible sheaf [H, Remark II.5.16.1]. Hence \( X \) is quasi-divisorial. Further, any ample invertible sheaf is necessarily numerically effective. Thus A5 holds.

Now assume A5. Let \( A^1 = A^1(X) \). Then \( A^1 \) is a finitely generated free abelian group by Theorem 3.6, with rank \( \rho(X) \). Let \( \mathcal{H}_1, \ldots, \mathcal{H}_{\rho(X)} \) be a \( \mathbb{Z} \)-basis for \( A^1 \). For \( v \in A^1 \otimes \mathbb{R} \), we may write

\[
v = a_1 \mathcal{H}_1 + a_2 \mathcal{H}_1^{-1} + \cdots + a_{\rho(X)} \mathcal{H}_{\rho(X)}^{-1}
\]

with \( a_i \geq 0 \) in the additive notation of \( A^1 \otimes \mathbb{R} \).

We choose an arbitrary invertible sheaf \( \mathcal{H} \). Since \( \{ \mathcal{L}_\alpha \} \) is a filter, we may choose \( \alpha_0 \) large enough so that for \( i = 1, \ldots, \rho(X) \) and \( \alpha \geq \alpha_0 \), the invertible sheaves \( \mathcal{H} \otimes \mathcal{L}_\alpha \) and \( \mathcal{H}_i^{-1} \otimes \mathcal{H} \otimes \mathcal{L}_\alpha \) are numerically effective. Set \( a = \sum a_i \). Then

\[
v + a(\mathcal{H} \otimes \mathcal{L}_\alpha) + b(\mathcal{H} \otimes \mathcal{L}_\alpha)
\]

is in \( K \), the cone generated by numerically effective invertible sheaves, for \( \alpha \geq \alpha_0 \) and \( b \geq 0 \). Thus by Lemma 3.7, we have \( \mathcal{H} \otimes \mathcal{L}_\alpha \in \text{Int} K \) and hence \( \mathcal{H} \otimes \mathcal{L}_\alpha \) is ample by Theorem 3.9 as desired.

Lemma 6.3. Let \( X \) be proper over a field \( k \), and let \( \mathcal{L} \) be an invertible sheaf on \( X \). Suppose that the base locus \( B \) of \( \mathcal{L} \), the points at which \( \mathcal{L} \) is not generated by global sections, is zero-dimensional or empty. Then \( \mathcal{L} \) is numerically effective.

Proof. Let \( C \subset X \) be an integral curve. Since \( B \) is zero-dimensional or empty, \( C \) is not a subset of \( B \). Thus there exists \( t \in H^0(X, \mathcal{L}) \) such that \( U_t \cap C \neq \emptyset \), where \( U_t \) is the (open) set of points \( x \in X \) such that the stalk \( t_x \) of \( t \) at \( x \) is not contained in \( m_x \mathcal{L}_x \), where \( m_x \) is the maximal ideal of \( \mathcal{O}_{X,x} \) [H, Lemma II.5.14]. Thus \( (\mathcal{L}, C) \geq 0 \) [F1, Example 12.1.2].

Lemma 6.4. Let \( X \) be proper over a noetherian ring \( A \), and let \( \mathcal{L} \) be an invertible sheaf on \( X \). Suppose that there exists an affine open subscheme \( U \) of \( X \) such that \( \mathcal{L} \) is generated by global sections at all \( x \in X \setminus U \). Then \( \mathcal{L} \) is numerically effective.
Proof. Let $V = X \setminus U$ with the reduced induced closed subscheme structure. We choose a closed point $s \in \text{Spec} \ A$. Then $X_s = X \times_A k(s)$ is a closed subscheme of $X$. For all $x \in X_s$ there is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{L} & \longrightarrow & \mathcal{O}_{X_s} \otimes \mathcal{L} \\
\downarrow & & \downarrow \\
\mathcal{L}_x & \longrightarrow & (\mathcal{O}_{X_s} \otimes \mathcal{L})_x
\end{array}
$$

So $\mathcal{O}_{X_s} \otimes \mathcal{L}$ is generated by global sections for all $x \in V \times_A k(s)$. Thus the base locus $B_x$ of points at which $\mathcal{O}_{X_s} \otimes \mathcal{L}$ is not generated by global sections is contained in the open affine subscheme $U \times_A k(s)$. So $B_s$ is an affine scheme since $B_s$ is a closed subscheme of $U \times_A k(s)$. But since $B_s$ is a closed subscheme of $X_s$, the natural morphism $B_s \to \text{Spec} \ k(s)$ is affine and proper, hence finite \cite[Exercise II.4.6]{EGA}. So $B_s$ is zero-dimensional or empty, and so $\mathcal{L}|_{X_s}$ is numerically effective by Lemma 6.3. Since this is true for every closed point $s \in S$, we must have that $\mathcal{L}$ is numerically effective. \hfill \Box

Proposition 6.5. Let $X$ be a scheme, proper over a noetherian ring $A$. Let $\{\mathcal{L}_\alpha\}$ be a filter of invertible sheaves. If $\{\mathcal{L}_\alpha\}$ satisfies \(A3\), then $\{\mathcal{L}_\alpha\}$ satisfies \(A4\).

Proof. We show \(A5\) holds and thus \(A4\) holds by Proposition 6.2. First, we must show that $X$ is quasi-divisorial. Let $V$ be a closed, contracted, integral subscheme of $X$. There exists $\alpha_0$ such that $\mathcal{L}_\alpha|_V$ is generated by global sections for $\alpha \geq \alpha_0$. Since $V$ is an integral scheme, each $\mathcal{L}_\alpha|_V$ must define an effective Cartier divisor \cite[Proposition II.6.15]{EGA}. Suppose that all these Cartier divisors are zero. Then $\mathcal{L}_\alpha|_V \cong \mathcal{O}_V$ for all $\alpha \geq \alpha_0$ and thus all coherent sheaves $\mathcal{F}$ on $V$ are generated by global sections. But then $V$ is affine and proper over $\text{Spec} \ k(s)$. Hence $V$ is zero-dimensional \cite[Exercise II.4.6]{EGA} and hence $V$ is an integral point. So $X$ is quasi-divisorial.

Given any invertible sheaf $\mathcal{H}$ on $X$, there is $\alpha_1$ such that $\mathcal{H} \otimes \mathcal{L}_\alpha$ is generated by global sections for $\alpha \geq \alpha_1$. A trivial application of Lemma 6.4 shows that $\mathcal{H} \otimes \mathcal{L}_\alpha$ is numerically effective, as desired. \hfill \Box

Proposition 6.6. Let $X$ be a scheme, proper over a noetherian ring $A$. Let $\{\mathcal{L}_\alpha\}$ be a filter of invertible sheaves. If $\{\mathcal{L}_\alpha\}$ satisfies \(A2\), then $\{\mathcal{L}_\alpha\}$ satisfies \(A3\).

Proof. Let $\mathcal{F}$ be a coherent sheaf. If $\text{Supp} \ \mathcal{F} = \emptyset$, then the claim is obvious. So by noetherian induction on $X$, suppose that the proposition holds on all proper closed subschemes $V$ of $X$.

Since \(A3\) is equivalent to \(A5\) (by Propositions 6.1, 6.2, and 6.5), we may show that \(A5\) holds. Let $W$ be a closed, contracted, integral subscheme of $X$ with $\pi(W) = s$, where $\pi$ is the structure morphism. We choose a closed point $x \in W$. Then there exists $\alpha_0$ such that

$$H^0(W, \mathcal{L}_\alpha|_W) \rightarrow H^0(\{x\}, \mathcal{L}_\alpha|_{\{x\}})$$

is surjective for $\alpha \geq \alpha_0$, where $\{x\}$ is the reduced closed subscheme defined by the point $x$. So $\dim_{k(s)} H^0(W, \mathcal{L}_\alpha|_W) \geq 1$. Since $W$ is an integral scheme, each $\mathcal{L}_\alpha|_W$ must define an effective Cartier divisor \cite[Proposition II.6.15]{EGA}. Suppose that all these Cartier divisors are zero. Then $\mathcal{L}_\alpha|_W \cong \mathcal{O}_W$ for all $\alpha \geq \alpha_0$ and one can argue that $\mathcal{O}_W$ is an ample invertible sheaf on $W$, using the proof of \cite[Proposition III.5.3]{EGA}. But then $W$ is affine and proper over $\text{Spec} \ k(s)$. Hence $W$ is
zero-dimensional [H, Exercise II.4.6] and hence \( W \) is an integral point. So \( X \) is quasi-divisorial.

Now let \( U \) be an affine open subscheme of \( X \). Set \( V = X \setminus U \) with the reduced induced subscheme structure. Let \( \mathcal{H} \) be an invertible sheaf on \( X \). Since (A3) holds for \( \{L_\alpha|_V\} \) there is \( \alpha_1 \) such that \( (\mathcal{H} \otimes L_\alpha)|_V \) is generated by global sections for \( \alpha \geq \alpha_1 \).

There exists \( \alpha_2 \geq \alpha_1 \) such that \( H^0(X, \mathcal{H} \otimes L_\alpha) \rightarrow H^0(V, (\mathcal{H} \otimes L_\alpha)|_V) \) is an epimorphism for \( \alpha \geq \alpha_2 \). Thus if \( x \in V \), then the stalk \( \mathcal{H} \otimes L_\alpha \otimes \mathcal{O}_{X,x} \) is generated by \( H^0(X, \mathcal{H} \otimes L_\alpha) \). Nakayama’s Lemma implies \( H^0(X, \mathcal{H} \otimes L_\alpha) \) also generates the stalk \( \mathcal{H} \otimes L_\alpha \otimes \mathcal{O}_{X,x} \). So by Lemma 6.4, \( \mathcal{H} \otimes L_\alpha \) is numerically effective for \( \alpha \geq \alpha_2 \), as we wished to show. Thus (A5) holds for \( \{L_\alpha\} \).

We may now summarize the proof of Theorem 1.3.

**Proof.** (A1) ⇒ (A2): Apply the first statement to get the vanishing of \( H^1(X, \text{Ker}(F \to G) \otimes L_\alpha) \). The desired surjectivity then follows from the natural long exact sequence.

(A2) ⇒ (A3): This is Proposition 6.6.

(A3) ⇒ (A4): This is Proposition 6.5.

(A4) ⇒ (A1): This is Proposition 6.1. □

**Corollary 6.7.** Let \( X \) be proper over a noetherian ring \( A \). Then \( X \) is projective over \( A \) if and only if \( X \) has an ample filter of invertible sheaves.

**Proof.** If \( X \) is projective, it has an ample invertible sheaf \( \mathcal{L} \) and the filter \( \{\mathcal{L}, \mathcal{L}^2, \ldots\} \) is an ample filter. Conversely, if \( X \) has an ample filter, it has an ample invertible sheaf by Theorem 1.3. Thus \( X \) is projective [H, Remark II.5.16.1]. □

**Remark 6.8.** Let \( X \) be a separated noetherian scheme. If \( X \) is covered by affine open complements of Cartier divisors, then \( X \) is called divisorial. All projective schemes and all regular proper integral schemes (over an affine base) are divisorial [Ko, Pf. of Theorem VI.2.19], and any divisorial scheme is quasi-divisorial. The above corollary is not true in the divisorial case if one replaces “ample filter of invertible sheaves” with “ample filter of non-zero torsion-free coherent subsheaves of invertible sheaves.” To be more exact, let \( X \) be a normal, divisorial, proper scheme over an algebraically closed field. Then there exists an invertible sheaf \( \mathcal{L} \) and a sequence of non-zero coherent sheaves of ideals \( \{\mathcal{I}_m\} \) such that for all coherent sheaves \( F \) on \( X \), there exists \( m_0 \) such that \( \text{H}^q(X, F \otimes \mathcal{I}_m \otimes \mathcal{L}^m) = 0 \) for all \( q > 0 \) and \( m \geq m_0 \) [B, Corollary 6]. Recently, it was shown that if \( X \) is divisorial and proper over a noetherian ring \( A \), then there exists a non-zero subsheaf \( \mathcal{K} \) of an invertible sheaf \( \mathcal{H} \) such that for all coherent \( F \), there exists \( m_0 \) such that \( \text{H}^q(X, F \otimes \mathcal{K}^m) = 0 \) for all \( q > 0 \) and \( m \geq m_0 \) [BS, Theorem 5.3].

To conclude this section, we examine other related conditions on a filter of invertible sheaves.

**Proposition 6.9.** Let \( X \) be a scheme, proper over a noetherian ring \( A \). Let \( \{\mathcal{L}_\alpha\} \) be a filter of invertible sheaves. The filter \( \{\mathcal{L}_\alpha\} \) is an ample filter if and only if for all invertible sheaves \( \mathcal{H} \), there exists \( \alpha_0 \) such that \( \mathcal{H} \otimes \mathcal{L}_\alpha \) is a very ample invertible sheaf for \( \text{Spec} \ A \) when \( \alpha \geq \alpha_0 \).
Proof. Suppose that \( \{L_\alpha\} \) is an ample filter. By Corollary 6.7, \( X \) is projective over \( A \) and hence has a very ample invertible sheaf \( \mathcal{O}_X(1) \). Let \( \mathcal{H} \) be an invertible sheaf. For \( \alpha \) sufficiently large, \( \mathcal{O}_X(-1) \otimes \mathcal{H} \otimes L_\alpha \) is generated by global sections. Thus
\[
\mathcal{O}_X(1) \otimes \mathcal{O}_X(-1) \otimes \mathcal{H} \otimes L_\alpha \cong \mathcal{H} \otimes L_\alpha
\]
is very ample [H Exercise II.7.5].

The converse is clear, since any very ample invertible sheaf is ample, so \((A4)\) holds for \( \{L_\alpha\} \).

These propositions regarding ample filters also have a relative form. The proofs are as in [H Theorem III.8.8]. We state a relative form of Theorem 1.3.

**Theorem 6.10.** Let \( S \) be a noetherian scheme, let \( \pi: X \to S \) be a proper morphism, and let \( \{L_\alpha\} \) be a filter of invertible sheaves on \( X \). Then the following are equivalent:

1. For all coherent sheaves \( F \) on \( X \), there exists \( \alpha_0 \) such that \( R^q\pi_* (F \otimes L_\alpha) = 0 \) for all \( q > 0 \) and \( \alpha \geq \alpha_0 \), i.e., \( \{L_\alpha\} \) is a \( \pi \)-ample filter.
2. For all coherent sheaves \( F, G \) on \( X \) with epimorphism \( F \twoheadrightarrow G \), there exists \( \alpha_0 \) such that the natural map
\[
\pi_* (F \otimes L_\alpha) \to \pi_* (G \otimes L_\alpha)
\]
is an epimorphism for \( \alpha \geq \alpha_0 \).
3. For all coherent sheaves \( F \), there exists \( \alpha_0 \) such that the natural morphism
\[
\pi^* \pi_* (F \otimes L_\alpha) \to F \otimes L_\alpha
\]
is an epimorphism for \( \alpha \geq \alpha_0 \).
4. For all invertible sheaves \( \mathcal{H} \) on \( X \), there exists \( \alpha_0 \) such that \( \mathcal{H} \otimes L_\alpha \) is a \( \pi \)-ample invertible sheaf for \( \alpha \geq \alpha_0 \).

\( \square \)

7. Twisted homogeneous coordinate rings

Throughout this section, the scheme \( X \) will be proper over a commutative noetherian ring \( A \). We now generalize the results of [Ke1] to this case. Specifically, we show that left and right \( \sigma \)-ampleness are equivalent in this case and thus the associated twisted homogeneous coordinate ring is left and right noetherian.

First we must briefly review the concept of a noncommutative projective scheme from [AZ]. Let \( R \) be a noncommutative, right noetherian, \( \mathbb{N} \)-graded ring, let \( \text{gr} R \) be the category of finitely generated, graded right \( R \)-modules, let \( \text{tors} R \) be the full subcategory of torsion submodules, and let \( \text{qgr} R \) be the quotient category \( \text{gr} R / \text{tors} R \). Let \( \pi: \text{gr} R \to \text{qgr} R \) be the quotient functor. Then the pair \( \text{proj} R = (\text{qgr} R, \pi R) \) is said to be a noncommutative projective scheme. We will work with rings \( R \) such that \( \text{proj} R \) is equivalent to \( (\text{coh}(X), \mathcal{O}_X) \), where \( \text{coh}(X) \) is the category of coherent sheaves on \( X \). By saying that \( \text{proj} R \cong (\text{coh}(X), \mathcal{O}_X) \) we mean that there is a category equivalence \( f: \text{qgr} R \to \text{coh}(X) \) and \( f(\pi R) \cong \mathcal{O}_X \).

Given an \( A \)-linear abelian category \( \mathcal{C} \), arbitrary object \( \mathcal{O} \), and autoequivalence \( s \), one can define a homogeneous coordinate ring
\[
R = \Gamma(\mathcal{C}, \mathcal{O}, s)_{\geq 0} = \bigoplus_{i=0}^{\infty} \text{Hom}(\mathcal{O}, s^i \mathcal{O})
\]
with multiplication given by composition of homomorphisms. That is, given \( a \in R_m, b \in R_n \), we have \( a \cdot b = s^n(a) \circ b \in \text{Hom}(\mathcal{O}, s^{n+m} \mathcal{O}) \).
Without loss of generality, one may assume \( s \) is a category automorphism, i.e., there is an inverse autoequivalence \( s^{-1} \) [AZ Proposition 4.2]. We then have a concept of ample autoequivalence.

**Definition 7.1.** Let \( \mathcal{C} \) be an \( A \)-linear abelian category. A pair \((\mathcal{O}, s)\), with \( \mathcal{O} \in \mathcal{C} \) and an autoequivalence \( s \) of \( \mathcal{C} \), is **ample** if

1. For all \( M \in \mathcal{C} \), there exist positive integers \( l_1, \ldots, l_p \) and an epimorphism \( \bigoplus_{i=1}^{p} s^{-l_i} \mathcal{O} \to M \). 
2. For all epimorphisms \( M \to N \), there exists \( n_0 \) such that \( \text{Hom}(\mathcal{O}, s^n M) \to \text{Hom}(\mathcal{O}, s^n N) \) is an epimorphism for \( n \geq n_0 \).

For convenience, we denote \( \text{Hom}(\mathcal{O}, M) \) by \( H^0(M) \).

**Proposition 7.2.** [AZ Thm. 4.5] Let \((\mathcal{C}, \mathcal{O}, s)\) be as in Definition 7.1. Suppose that the following conditions hold

1. The object \( \mathcal{O} \) is noetherian.
2. The ring \( R_0 = H^0(\mathcal{O}) \) is right noetherian and \( H^0(\mathcal{M}) \) is a finitely generated \( R_0 \)-module for all \( \mathcal{M} \in \mathcal{C} \).
3. The pair \((\mathcal{O}, s)\) is ample.

Then \( R = \Gamma(\mathcal{C}, \mathcal{O}, s)_{\geq 0} \) is a right noetherian \( A \)-algebra and \( \text{proj } R \cong (\mathcal{C}, \mathcal{O}) \).

Given an automorphism \( \sigma \) of \( \mathcal{X} \) and an invertible sheaf \( \mathcal{L} \), we can define the autoequivalence \( s = \mathcal{L} \sigma \otimes - \) on \( \text{coh}(\mathcal{X}) \). For a coherent sheaf \( \mathcal{F} \), define \( \mathcal{L} \sigma \otimes \mathcal{F} = \mathcal{L} \otimes \sigma^* \mathcal{F} \). These are the only autoequivalences of \( \text{coh}(\mathcal{X}) \) which we will examine, due to the following proposition.

**Proposition 7.3.** ([AZ Cor. 6.9], [AV Prop. 2.15]) Let \( \mathcal{X} \) be proper over a field. Then any autoequivalence \( s \) of \( \text{coh}(\mathcal{X}) \) is naturally isomorphic to \( \mathcal{L} \sigma \otimes - \) for some automorphism \( \sigma \) and invertible sheaf \( \mathcal{L} \).

Denote pull-backs by \( \sigma^* \mathcal{F} = \mathcal{F} \sigma \). We then have
\[
\mathcal{F} \otimes (\mathcal{L}_\sigma)^n = \mathcal{F} \otimes \mathcal{L} \otimes \mathcal{L}_\sigma \otimes \cdots \otimes \mathcal{L}_\sigma^{n-1}
\]
and
\[
s^n \mathcal{F} = (\mathcal{L}_\sigma)^n \otimes \mathcal{F} = \mathcal{L} \otimes \mathcal{L}_\sigma \otimes \cdots \otimes \mathcal{L}_\sigma^{n-1} \otimes \mathcal{F}^n.
\]

**Definition 7.4.** Given an automorphism \( \sigma \) of a scheme \( \mathcal{X} \), an invertible sheaf \( \mathcal{L} \) is **left \( \sigma \)-ample** if for all coherent sheaves \( \mathcal{F} \), there exists \( n_0 \) such that
\[
H^q(X, (\mathcal{L}_\sigma)^n \otimes \mathcal{F}) = 0
\]
for \( q > 0 \) and \( n \geq n_0 \). An invertible sheaf \( \mathcal{L} \) is **right \( \sigma \)-ample** if for all coherent sheaves \( \mathcal{F} \), there exists \( n_0 \) such that
\[
H^q(X, \mathcal{F} \otimes (\mathcal{L}_\sigma)^n) = 0
\]
for \( q > 0 \) and \( n \geq n_0 \).

**Lemma 7.5.** [Ke1 Lem. 2.4] An invertible sheaf \( \mathcal{L} \) is right \( \sigma \)-ample if and only if \( \mathcal{L} \) is left \( \sigma^{-1} \)-ample.

**Lemma 7.6.** The pair \((\mathcal{O}_x, \mathcal{L}_\sigma \otimes -)\) is an ample autoequivalence if and only if \( \mathcal{L} \) is left \( \sigma \)-ample.
Proof. If \( s = \mathcal{L}_\sigma \otimes - \) is an ample autoequivalence, then the sequence of invertible sheaves

\[
\{(\sigma^*)^{-n+1}(\mathcal{O}_X \otimes (\mathcal{L}_\sigma)^n)\} \cong \{\mathcal{O}_X \otimes (\mathcal{L}_{\sigma^{-1}})^n\}
\]

satisfies condition (A2) (and hence condition (A1)) of Theorem 1.3, so \( \mathcal{L} \) is right \( \sigma^{-1} \)-ample. Thus \( \mathcal{L} \) is left \( \sigma \)-ample.

Assuming \( \mathcal{L} \) is left \( \sigma \)-ample, we have that \( \mathcal{L} \) is right \( \sigma^{-1} \)-ample. So condition (A1) (and hence conditions (A2) and (A3)) of Theorem 1.3 hold for the sequence \( \{\mathcal{O}_X \otimes (\mathcal{L}_{\sigma^{-1}})^n\} \). This immediately gives (B2) of the definition of ample autoequivalence. Now because (A3) holds for the sequence \( \{\mathcal{O}_X \otimes (\mathcal{L}_{\sigma^{-1}})^n\} \), given a coherent sheaf \( \mathcal{F} \), we may pull back \( \sigma^n \mathcal{F} \otimes (\mathcal{L}_{\sigma^{-1}})^n \) by \( (\sigma^*)^{n-1} \) and have that for all sufficiently large \( n \), the sheaf

\[
(\mathcal{L}_\sigma)^n \otimes \mathcal{F} = s^n \mathcal{F}
\]

is generated by global sections. Thus choosing one large \( n_0 \), we have some \( p \) so that there is an epimorphism \( \oplus_{i=1}^{p} s^{-n_0} \mathcal{O}_X \rightarrow \mathcal{F} \). Thus (B1) of the definition of ample autoequivalence holds.

The following now follows from Corollary 6.7.

**Corollary 7.7.** Let \( X \) be proper over a commutative noetherian ring \( A \). Suppose that there exists an automorphism \( \sigma \) and an invertible sheaf \( \mathcal{L} \) on \( X \) such that \( \mathcal{L} \) is right \( \sigma \)-ample (or equivalently such that \( \mathcal{L}_\sigma \otimes - \) is an ample autoequivalence). Then \( X \) is projective over \( A \).

Let \( B(X, \sigma, \mathcal{L})^{op} = \Gamma(\text{coh}(X), \mathcal{O}_X, \mathcal{L}_\sigma \otimes -)_{\geq 0} \). The ring \( B(X, \sigma, \mathcal{L}) \) is called a twisted homogeneous coordinate ring for \( X \). Through pull-backs by powers of \( \sigma \), one can show the following.

**Lemma 7.8.** [Ke2] Lem. 2.2.5 The rings \( B(X, \sigma, \mathcal{L}) \) and \( B(X, \sigma^{-1}, \mathcal{L}) \) are opposite rings.

The rings \( B(X, \sigma, \mathcal{L}) \) were extensively studied in [AV], though only when \( A \) was a field. In that paper, the multiplication was defined using left modules instead of the right modules used in [AZ]. Thus we stipulate that \( B(X, \sigma, \mathcal{L}) \) is the opposite ring of \( \Gamma(\text{coh}(X), \mathcal{O}_X, \mathcal{L}_\sigma \otimes -)_{\geq 0} \), in order to keep the multiplication consistent between [AV] and [AZ]. From the previous two lemmas and Proposition 7.2 we now have the following.

**Proposition 7.9.** Let \( X \) be proper over a commutative noetherian ring \( A \). Let \( \sigma \) be an automorphism of \( X \) and let \( \mathcal{L} \) be an invertible sheaf on \( X \). If \( \mathcal{L} \) is left \( \sigma \)-ample, then \( B(X, \sigma, \mathcal{L}) \) is left noetherian. If \( \mathcal{L} \) is right \( \sigma \)-ample, then \( B(X, \sigma, \mathcal{L}) \) is right noetherian.

Fix an automorphism \( \sigma \) of \( X \) and set \( \mathcal{L}_m = \mathcal{O}_X \otimes (\mathcal{L}_\sigma)^m \). For a graded ring \( R = \oplus_{i=0}^{\infty} R_i \), the Veronese subring \( R^{(m)} = \oplus_{i=0}^{\infty} R_{mi} \). If \( R \) is commutative and noetherian, then some Veronese subring of \( R \) is generated in degrees 0 and 1 [Mu2] p. 204, Lemma]; however, there are noncommutative noetherian graded rings such that no Veronese subring is generated in degrees 0 and 1 [SZ] Corollary 3.2].

**Proposition 7.10.** Let \( X \) be proper over a commutative noetherian ring \( A \), and let \( \mathcal{L} \) be \( \sigma \)-ample on \( X \). There exists \( n \) such that \( B(X, \sigma, \mathcal{L})^{(n)} \) is generated in degrees 0 and 1 over \( A \).
Proof. Note that $B(X, σ, ℒ)^{(n)} ≃ B(X, σ^n, ℒ_n)$ and $ℒ$ is $σ$-ample if and only if $ℒ_n$ is $σ^n$-ample [AV Lemma 4.1]. Then by Proposition 6.9 we may replace $ℒ$ by $ℒ_n$ and $B$ by $B^{(n)}$ and assume $ℒ$ is very ample for Spec $A$.

For this proof, it is easiest to use the multiplication defined in [AV], namely the maps

$$H^0(X, ℒ_m) ⊗ H^0(X, ℒ_n) → H^0(X, ℒ_m ⊗ ℒ_n) \rightarrow H^0(X, (σ^*)^m ℒ_n) → H^0(X, ℒ_{m+n}).$$

Now choose $n_0$ so that $H^3(X, ℒ^{−q−1} ⊗ ℒ_n) = 0$ for $q = 1, \ldots, \text{cd}(X)$ and $n \geq n_0$. Then $σ^∗ℒ_{n−1}$ is 0-regular with respect to $ℒ$. By Proposition 4.9, the natural map

$$H^0(X, ℒ) ⊗ H^0(X, σ^∗ℒ_{n−1}) → H^0(X, ℒ_n)$$

is surjective for $n \geq n_0$. So twisting by $(σ^∗)^i$, the natural map

$$(7.11) \quad H^0(X, (σ^∗)^iℒ) ⊗ H^0(X, (σ^∗)^{i+1}ℒ_{n−1}) → H^0(X, (σ^∗)^{i}ℒ_n)$$

is surjective for $n \geq n_0$ and all $i ∈ ℤ$.

Now let $ℓ > 0$. According to (7.11), the maps

$$H^0(X, (σ^∗)^{n_0−j}ℒ) ⊗ H^0(X, (σ^∗)^{n_0−j+1}ℒ_{ℓn_0+j−1}) → H^0(X, (σ^∗)^{n_0−j}ℒ_{ℓn_0+j})$$

are surjective for $j = 1, \ldots, n_0$. Thus the map

$$H^0(X, ℒ_{n_0}) ⊗ H^0(X, (σ^∗)^{n_0}ℒ_{ℓn_0}) → H^0(X, ℒ_{ℓ+1}n_0).$$

is surjective and $B(X, σ, ℒ)^{(n_0)}$ is generated in degrees 0 and 1. \qed

Given an automorphism $σ$ of $X$, let $P_σ$ be the action of $σ$ on $A^1(X)$. Thus $P_σ ∈ GL_p(ℤ)$ for some $p$ by the Theorem of the Base [5.6]. We call $σ$ quasi-unipotent if $P_σ$ is quasi-unipotent, that is, when all eigenvalues of $P_σ$ are roots of unity. The statement $"σ$ is quasi-unipotent" is well-defined.

**Proposition 7.12.** Let $X$ be proper over a noetherian ring $A$ and let $σ$ be an automorphism of $X$. Let $P, P' ∈ GL_p(ℤ)$ be two representations of the action of $σ$ on $A^1(X)$. Then $P$ is quasi-unipotent if and only if $P'$ is quasi-unipotent.

**Proof.** Let $P$ be quasi-unipotent. We may replace $σ$ by $σ^i$ and assume $P = I + N$ for some nilpotent matrix $N$. Then for all invertible sheaves $ℋ$ and contracted integral curves $C$, the function $f(m) = (ℋ^m.C) = (P^mℋ,C)$ is a polynomial.

However, if $P'$ not quasi-unipotent, then $P'$ has an eigenvalue $r$ of absolute value greater than 1 [Keel Lemma 3.1]. (If $X$ is projective, then the cone of nef invertible sheaves has a non-empty interior and we may assume $r$ is real [V Theorem 3.1].) Let $v = a_1ℋ_1 + \cdots + a_pℋ_p ∈ A^1(X) ⊗ ℂ$ be an eigenvector for $r$ where the $ℋ_i$ are invertible sheaves. Then there exists a contracted integral curve $C$ such that

$$|a_1(σ^mℋ_1.C)| + \cdots + |a_p(σ^mℋ_p.C)| ≥ |(σ^m.v.C)| = |r|^m \cdot |(v.C)| > 0.$$ 

Thus not all of the $(σ^mℋ_i.C)$ can be polynomials. Thus we have a contradiction and $P'$ must be quasi-unipotent. \qed

We now can state the following generalization of [Keel Theorem 1.3].

**Theorem 7.13.** Let $X$ be proper over a commutative noetherian ring $A$. Let $σ$ be an automorphism of $X$ and let $ℒ$ be an invertible sheaf on $X$. Then $ℒ$ is right $σ$-ample if and only if $σ$ is quasi-unipotent and

$$ℒ ⊗ ℒ^σ ⊗ \cdots ⊗ ℒ^σ^{m−1}$$
These claims depend only on the behavior of \( \sigma \) rightness.

**Proof.** The proof mostly proceeds as in [Ke1] §3–4, using the fact that the sequence \( \{ \mathcal{O}_X \otimes (L \sigma)^m \} \) is an ample sequence if and only if \( [A] \) holds, and then showing the equivalence of \( [A] \) with the condition above. However, when \( \sigma \) is not quasi-unipotent, one must use the method outlined in [Ke1] Remark 3.5, since the proof of [Ke1] Theorem 3.4 relied on the growth of the dimensions of the graded pieces of \( B(X, \sigma, \mathcal{L}) \).

Now since \( \sigma \) is quasi-unipotent if and only if \( \sigma^{-1} \) is quasi-unipotent, we easily get the following, as proved in [Ke1] §5.

**Theorem 7.14.** Let \( A \) be a commutative noetherian ring, let \( X \) be proper over \( A \), let \( \sigma \) be an automorphism of \( X \), and let \( \mathcal{L} \) be an invertible sheaf on \( X \). Then \( \mathcal{L} \) is right \( \sigma \)-ample if and only if \( \mathcal{L} \) is left \( \sigma \)-ample. Thus we may simply say that such an \( \sigma \) is \( \sigma \)-ample. If \( \mathcal{L} \) is \( \sigma \)-ample, then \( B(X, \sigma, \mathcal{L}) \) is noetherian.

**Remark 7.15.** Now suppose that \( X \) is proper over a field \( k \). Then the claims of [Ke1] §6 regarding the Gel’fand-Kirillov dimension of \( B(X, \sigma, \mathcal{L}) \), with \( \sigma \)-ample \( \mathcal{L} \), are still valid. This is because the proofs rely on a weak Riemann-Roch formula [Fl] Example 18.3.6] which is valid over an arbitrary field.

**Definition 7.16.** [AV Definition 3.2] Let \( k \) be a field and let \( R \) be a finitely \( \mathbb{N} \)-graded right noetherian \( k \)-algebra. That is, \( R = \bigoplus_{i=0}^{\infty} R_i \) and \( \dim_k R_i \) is finite for all \( i \). The ring \( R \) is said to satisfy right \( \chi_j \) if for all finitely generated, graded right \( R \)-modules \( M \) and all \( \ell \leq j \),

\[
\dim_k \operatorname{Ext}^{\ell}(R/R_{>0}, M) < \infty,
\]

where \( \operatorname{Ext} \) is the ungraded \( \operatorname{Ext} \)-group, calculated in the category of all right \( R \)-modules. If \( R \) satisfies right \( \chi_j \) for all \( j \geq 0 \), we say \( R \) satisfies right \( \chi \). Left \( \chi_j \) and left \( \chi \) are defined similarly with left modules.

**Theorem 7.17.** Let \( k \) be a field and let \( R \) be a finitely \( \mathbb{N} \)-graded right noetherian \( k \)-algebra which satisfies right \( \chi_1 \). Suppose that there exists a scheme \( X \), proper over \( k \), such that \( \operatorname{proj} R \cong (\operatorname{coh}(X), \mathcal{O}_X) \). Let \( \rho \) be the Picard number of \( X \). Then

(1) \( X \) is projective over \( k \),
(2) \( R \) is noetherian and satisfies left and right \( \chi \),
(3) There exists \( m \) such that the Veronese subring \( R^{(m)} \) is generated in degrees 0 and 1, and
(4) \( \operatorname{GKdim} R \) is an integer and

\[
\dim X + 1 \leq \operatorname{GKdim} R \leq 2 \left( \left\lfloor \frac{\rho - 1}{2} \right\rfloor (\dim X - 1) + \dim X + 1 \right).
\]

**Proof.** These claims depend only on the behavior of \( R \) in high degree. Thus using [AZ] Theorem 4.5, we may assume \( R = \Gamma(\operatorname{coh}(X), \mathcal{O}_X, s) \geq 0 \) for some autoequivalence \( s \). But by Proposition 7.3, we may assume \( s = \mathcal{L} \sigma \otimes - \) for some invertible sheaf \( \mathcal{L} \) and automorphism \( \sigma \). Thus we may assume \( R^{op} = B(X, \sigma, \mathcal{L}) \).

By Lemma 7.6, the sheaf \( \mathcal{L} \) is \( \sigma \)-ample since \( s \) is an ample autoequivalence. So \( X \) is projective by Corollary 7.7. Also \( R \) is noetherian by Theorem 7.14 and the vanishing higher cohomology of \( s^n \mathcal{F} \) for all coherent sheaves \( \mathcal{F} \) gives that \( R \) satisfies right \( \chi \) [AZ Theorem 7.4]. Since \( \mathcal{L} \) is also \( \sigma^{-1} \)-ample, \( R \) satisfies left \( \chi \) by symmetry. The claim regarding Veronese subrings is Proposition 7.10.
Finally, the claim regarding GK-dimension comes from [Ke1, Theorem 6.1]. We need only explain the bounds. First, $\text{GKdim } B(X, \sigma, L) = \text{GKdim } B(X, \sigma^m, L_m)$ [AV, p. 263], so we may assume that $\sigma$ fixes the irreducible components of $X$. For each irreducible component $X_i$, let $\sigma_i$ be the induced automorphism. Then [Ke1, Proposition 6.11] show that

$$\text{GKdim } B(X, \sigma, L) = \max_{X_i} \text{GKdim } B(X_i, \sigma_i, L|_{X_i}).$$

We may also assume that $P_\sigma$ is unipotent, so write $P_\sigma = I + N$ for some nilpotent matrix $N$ and let $\ell$ be the smallest integer so that $N^{\ell+1} = 0$. If $P_{\sigma_i} = I + N_i$, then $N_i^{\ell+1} = 0$, which can be seen by pulling-back an ample invertible sheaf to $X_i$ and using [Ke1, Lemma 4.4]. So to find the desired bounds, we may assume $X$ is irreducible, hence equidimensional, so the bounds in [Ke1, Theorem 6.1] apply. Now $\ell$ is even [Ke1, Lemma 6.12] and $0 \leq \ell \leq \rho - 1$, so the universal bounds follow. \hfill $\Box$

Theorem 7.17(1) seems to be a fortunate result. It says that we cannot have a noncommutative projective scheme $\text{proj } R = (\text{coh}(X), \mathcal{O}_X)$ coming from a commutative non-projective scheme $X$.

Example 7.18. Theorem 7.17(2) may not be true if a structure sheaf other than $\mathcal{O}_X$ is used. In [SZ, Example 4.3], a coherent sheaf $\mathcal{F}$ and ample autoequivalence $s$ on $\mathbb{P}^1$ is chosen so that $R = \Gamma(\text{coh}(\mathbb{P}^1), \mathcal{F}, s)_{\geq 0}$ is right noetherian and satisfies $\chi_1$, but $R$ is not left noetherian and does not satisfy $\chi_2$.

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Erratum

1. Corrected Lemma

As stated, \cite{Ke2003} Lemma 3.3 is definitely incorrect. Take $A = K$ a field and $X$ a projective scheme over $K$, with $X$ integral but not geometrically irreducible over $K$. Let $X'$ be a scheme surjecting onto $X$. Then $X'$ cannot be geometrically integral over $K$ as the lemma asserts as possible. If $X'$ were geometrically integral over $K$, then $X' \times_K \overline{K}$ has only one irreducible component, while $X \times_K \overline{K}$ has more than one. So $X' \times_K \overline{K} = X \times_K \overline{K}$ is not surjective. But surjectivity is preserved under base change \cite{SP2018} Tag 01S1, yielding a contradiction. Our mistake was treating \textit{geometrically integral} as an absolute notion, rather than a relative one.

Before correcting the lemma, we give a presumably well-known result, due to \cite{AM1969}.

**Lemma E1.1.** Let $A, A'$ be noetherian integral domains with $\phi : A \to A'$ a finite ring homomorphism. Let $S = \text{Spec} \ A, S' = \text{Spec} \ A'$ with induced morphism $g : S' \to S$. Write $S_b = S \setminus \text{V}(b)$ for $b \in A$ (and similarly for $S'_c$ with $c \in A'$). Then for any non-zero $c \in A'$, there exists non-zero $b \in A$ such that $S'_c(b) = g^{-1}(S_b) \subseteq S'_c$.

**Proof.** The claim that $S'_c(b) = g^{-1}(S_b)$ is from \cite{AM1969} Ch. 1, Exercise 21]. For simplicity, we can replace $A$ with $\phi(A)$ and thus assume $A \subseteq A'$.

Choose non-zero $c \in A'$. Since $c$ is integral over $A$, there exists a minimal monic polynomial $f$ for $c$ with non-zero constant term $b \in A$. This gives $b = ec$ for some $e \in A'$. Then $S'_c = S'_c(c) = S'_c \cap S'_c$ \cite{AM1969} Ch. 1, Exercise 17], so $S'_c \subseteq S'_c$. \hfill $\Box$

**Remark E1.2.** For the remainder of this erratum, we will abuse notation by replacing $S_b, S'_c$ with the isomorphic schemes $\text{Spec} \ A_b, \text{Spec} \ A'_b$ respectively.

Here is the corrected lemma.

**Lemma E1.3.** Let $A$ be a noetherian domain, let $X$ be an integral scheme with a proper, surjective morphism $\pi : X \to \text{Spec} \ A$, and let $d$ be the dimension of the generic fiber of $\pi$. Then there exists non-zero $b \in A$, a noetherian domain $A'$, an integral scheme $X'$, and finite-type morphisms $f, g, \pi'$ with commutative diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{f} & X_b \\
\pi' \downarrow & & \downarrow \pi_b \\
\text{Spec} \ A' & \xrightarrow{g} & \text{Spec} \ A_b
\end{array}
$$

where $X_b = X \times_A A_b$ and $\pi_b = \pi \times_A \text{id}_{A_b}$.

The diagram has the following properties:

1. The morphisms $f, g, \pi'$ are projective and surjective.
2. The morphisms $g, \pi_b$ are flat.
3. The morphism $g$ is finite.
4. The morphism $\pi'$ is smooth.
5. Every fiber $X'_s$ of $\pi'$ is geometrically integral over its base field $K'(s)$ and $X'_s$ has dimension $d$.
6. There exists an open subscheme $U \subseteq X_b$ such that $f$ restricted to $f^{-1}(U)$ is finite.
Proof. Let \( K \) be the fraction field of \( A \) and let \( X_0 = X \times_A K \) be the generic fiber of \( \pi \). Via Alteration of Singularities [deJ1996 Theorem 4.1], there exists a finite extension \( K' \) of \( K \), an integral scheme \( X'_0 \), and commutative diagram

\[
\begin{array}{ccc}
X'_0 & \xrightarrow{f_0} & X_0 \\
\pi'_0 \downarrow & & \downarrow \pi_0 \\
\Spec K' & \xrightarrow{g_0} & \Spec K
\end{array}
\]

such that

1. \( f_0 \) is an alteration (that is, a dominant, proper, generically finite morphism),
2. \( g_0 \) is finite,
3. \( g_0 \circ \pi'_0 = \pi_0 \circ f_0 \) is projective,
4. \( X'_0 \) is geometrically irreducible and smooth over \( K' \) [deJ1996 Remark 4.2].

Since \( g_0 \circ \pi'_0 = \pi_0 \circ f_0 \) is projective, we have that \( \pi'_0 \) and \( f_0 \) are projective [Ha1977 Exercise II.4.8]. Since \( f_0 \) is proper (hence closed) and dominant, we have \( f_0 \) surjective. Clearly the other morphisms are surjective. Since \( K \) is a field, the maps \( g_0, \pi_0 \) are trivially flat. Also, since \( f_0 \) is generically finite, \( \dim(X'_0) = \dim(X_0) = d \) [deJ1996 2.20].

Finally, since \( \pi'_0 \) is finite-type, we have that \( \pi'_0 \) smooth over \( K' \) [SP2018 Tag 038X]. Thus \( X'_0 \) is geometrically normal and hence geometrically reduced over \( K' \) [SP2018 Tags 0569, 033K]. So \( X'_0 \) is geometrically integral over \( K' \).

Thus, all the properties of the lemma hold over \( \Spec K \). We need to verify that these properties can be spread out. If \( \{A_b\} \) is the inductive system of one element localizations of \( A \), then \( K = \varprojlim A_b \) (and hence \( \Spec K = \varprojlim \Spec A_b \)). By [Ill1996 Proposition 6.2] and surrounding discussion, there exists non-zero \( b \in A \) which gives a diagram of the form (E1.4) that localizes to diagram (E1.5).

To avoid repetition, for the remainder of the proof we will state assumptions that can be made by replacing \( b \) with a multiple of \( b \), and hence shrinking \( \Spec A_b \), sometimes without explicit mention of the shrinking.

A priori, the morphisms are only of finite-type and the lower left and upper right corners are finite-type \( A_b \)-schemes, say \( S', Y \) respectively. By the discussion in [Ill1996], the choice of \( Y \) is unique up to isomorphism (after sufficiently shrinking \( \Spec A_b \)) so we may assume \( Y = X \times_A A_b \) and the morphism \( X \times_A A_b \to \Spec A_b \) is \( \pi_b = \pi \times_A \text{id}_{A_b} \). Also, we may assume that \( g \) is finite [EGA IV, 8.10.5], so \( S = \Spec A' \) where \( A' \) is a finite \( A_b \)-module, and hence noetherian.

By Generic Freeness [EGA IV, 6.9.2], we may assume that \( A' \) is a free \( A_b \)-module. Then \( A' \) must be an integral domain; if \( A' \) had a zero-divisors \( y, z \) with \( yz = 0 \), then tensoring with \( K \) would annihilate \( y \) or \( z \), contradicting the freeness of \( A' \). Similarly, \( X' \) and \( X_b \) have open affine covers with coordinate rings that are free \( A_b \)-modules and hence integral domains. So \( X' \) is integral. This also shows that \( g \) and \( \pi_b \) are flat, as this is the proof of Generic Flatness in [EGA IV, 6.9.1].

We can also assume that \( f, g, \pi' \) are projective and surjective [EGA IV, 8.10.5].

So we have verified the claims of the lemma through part (3).

By Lemma [E1.1] any open condition for \( \pi' \) can be achieved by shrinking \( \Spec A_b \), and can any condition following from \( \Spec K' = \varprojlim \Spec A'_b \) with \( b \in A \). Therefore, we can assume that \( \pi' \) is smooth [Ill1996 Proposition 6.3], that the fibers of \( \pi' \)
are geometrically integral over their base field \cite[EGA IV, 12.2.4]{ega}, and the fibers of \(\pi'\) have dimension \(d\) \cite[EGA IV, 13.1.5]{ega}. The last claim comes from the upper semicontinuity of fiber dimension when \(\pi'\) is proper. So we have verified parts (4) and (5).

Finally, let \(\nu\) be the generic point of \(X\). Then \(\nu\) is also the generic point of \(X_0\). Since \(f_0\) is generically finite, the set \(f_0^{-1}(\nu) = f^{-1}(\nu)\) is finite. Then \(f\) is generically finite and (6) holds by \cite[Exercise II.3.7]{ha1977}. \(\square\)

Remark E1.6. Note that the corrected lemma holds regardless if the quotient field \(K\) of \(A\) is perfect. However, if \(K\) is perfect, then \(g_0 \circ \pi_0'\) is also smooth \cite[Remark 4.2]{dej1996}. In this case, we can also assume \(g \circ \pi'\) is smooth.

Remark E1.7. The original \cite[Lemma 3.3]{ke2003} did not have a claim similar to (6). While not necessary for the rest of the paper, it relativizes the concept of alteration and perhaps will have a future use.

The original lemma also assumed \(\pi\) projective, but assuming \(\pi\) proper required no extra work. Further, it was only argued that the generic fiber of \(\pi'\) has dimension \(d\).

2. Verifying dependent results

In this section, we verify that theorems relying on \cite[Lemma 3.3]{ke2003} are correct. There are three such theorems, namely \cite[Theorems 1.5, 3.6, Proposition 5.12]{ke2003}. Fortunately, having \(g\) finite and faithfully flat (that is, flat and surjective) is close enough to \(g\) being the identity morphism.

We start with a slight generalization of \cite[Lemma 2.16]{ke2003}. Recall that for a proper \(\pi : X \to S\), a closed subscheme \(V \subseteq X\) is \(\pi\)-contracted if \(\pi(V)\) is 0-dimensional \cite[Definition 2.8]{ke2003}.

**Lemma E2.1.** Let \(S, S'\) be noetherian schemes. Consider the commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
\downarrow{\pi'} & & \downarrow{\pi} \\
S' & \xrightarrow{g} & S
\end{array}
\]

with proper morphisms \(\pi, \pi', (\text{proper}) \text{ surjective } f, \text{ and finite } g\). Let \(C \subseteq X\) be a \(\pi\)-contracted integral curve. Then there exists a \(\pi'\)-contracted integral curve \(C' \subseteq X'\) such that \(f(C') = C\).

**Proof.** By \cite[Lemma 2.16]{ke2003}, there exists a \(g \circ \pi'\)-contracted integral curve \(C'\) such that \(f(C') = C\). Since \(g\) is finite, the set \(g^{-1}(g \circ \pi')(C')\) is finite. Thus \(C'\) must be \(\pi'\)-contracted, as desired. \(\square\)

We immediately have

**Theorem E2.2.** Consider a diagram of schemes as in \cite[(E1.4)]{e1}, satisfying the conclusions of Lemma E1.3. Then the conclusions of \cite[Lemmas 2.20, 3.1]{ke2003} hold. Therefore \cite[Theorem 3.6]{ke2003} is true.

**Proof.** By Lemma E2.1, the results \cite[Lemmas 2.20, 3.1]{ke2003} hold when \(g\) is finite.

The original proof of \cite[Theorem 3.6]{ke2003} uses \cite[Lemma 3.1]{ke2003} to replace \(\pi : X_0 \to \Spec A_0\) with \(g \circ \pi' : X' \to \Spec A_0\). Since \cite[Lemma 3.1]{ke2003} holds
Thus the \( V \) is affine. Thus we may assume that \( M \) is relatively nef \([Ke2003, \text{Corollary 2.19}]\). But over an affine base, nefness is \([Ke2003, \text{Notation 5.2}]\). We repeat it here, with an extra indication of the base ring.

**Notation E2.3.** Let \( A \) be a noetherian domain, let \( \pi : X \to \text{Spec } A \) be a projective morphism, let \( \mathcal{L} \) be an ample invertible sheaf on \( X \), and let \( \Lambda \) be a set of (isomorphism classes of) invertible sheaves on \( X \). (For a locally closed subscheme \( Y \subseteq X \), let \( \Lambda|_Y = \{ \mathcal{N}|_Y : \mathcal{N} \in \Lambda \} \).) Let \( \mathcal{F} \) be a coherent sheaf on \( X \). For \( q > 0 \), we say that \( V^q(A)(\mathcal{F}, \mathcal{L}, \Lambda) \) holds if there exists \( m_0 = m(\mathcal{F}, \mathcal{L}) \) and a non-empty open subscheme \( U = U(\mathcal{F}, \mathcal{L}) \subseteq \text{Spec } A \) such that \( R^s\pi_*(\mathcal{F} \otimes \mathcal{L}^m \otimes \mathcal{N})|_U = 0 \) for all \( m \geq m_0, s \geq q, \mathcal{N} \in \Lambda \), and all open subschemes \( V \subseteq U \) such that \( \mathcal{N}|_{\pi^{-1}(V)} \) is \( \pi|_{\pi^{-1}(V)} \)-nef.

We need to show that the definition of \( V^q(A)(\mathcal{F}) \) does not, in a sense, depend on \( A \).

**Lemma E2.4.** Let \( A, A' \) be noetherian domains, let \( \pi : X \to \text{Spec } A' \) be projective and surjective, and let \( g : \text{Spec } A' \to \text{Spec } A \) be finite and surjective. Let \( \mathcal{L} \) be an ample invertible sheaf on \( X \), and let \( \Lambda \) be a set of invertible sheaves on \( X \). Then for \( q > 0 \), if \( V^q(A)(\mathcal{F}, \mathcal{L}, \Lambda) \), then \( V^q(A')(\mathcal{F}, \mathcal{L}, \Lambda) \).

**Proof.** Suppose \( V^q(A)(\mathcal{F}, \mathcal{L}, \Lambda) \) holds. Then there exists \( m_0 \) and a non-empty open subscheme \( U' \subseteq \text{Spec } A' \) such that \( R^s\pi_*(\mathcal{F} \otimes \mathcal{L}^m \otimes \mathcal{N})|_{U'} = 0 \) for all \( m \geq m_0, s \geq q, \mathcal{N} \in \Lambda \), and all open subschemes \( V' \subseteq U' \) such that \( \mathcal{N}|_{\pi^{-1}(V')} \) is \( \pi|_{\pi^{-1}(V')} \)-nef.

Since \( g \) is surjective and \( A \) is reduced, the associated ring homomorphism \( A \to A' \) is injective \([AM1969, \text{Ch. 1, Exercise 21}]\), so we can assume \( A \subseteq A' \). By Lemma E1.1, we may choose non-zero \( c \in A \) with \( \text{Spec } A'_c \subseteq U' \). Recall that projective, surjective, and finite morphisms are preserved under base change \([SP2018, \text{Tag 02V6, 01S1, 01WL}]\), as is relative nefness \([Ke2003, \text{Lemma 2.18}]\). The vanishing condition is also local. So we can replace \( A, A', X \) with Spec \( A_c, A'_c, f^{-1}(\text{Spec } A'_c) \) respectively. Thus we may assume that \( U' = \text{Spec } A' \).

Let \( m \geq m_0, s \geq q, \mathcal{N} \in \Lambda \). Suppose \( V \subseteq \text{Spec } A \) such that \( \mathcal{N}|_{(g \circ \pi)^{-1}(V)} \) is \( (g \circ \pi)|_{(g \circ \pi)^{-1}(V)} \)-nef. Then \( \mathcal{N}|_{(g \circ \pi)^{-1}(W)} \) is also relatively nef, where \( W \) is any affine open subscheme of \( V \) \([Ke2003, \text{Corollary 2.19}]\). But over an affine base, nefness is an absolute notion \([Ke2003, \text{Proposition 2.15}]\). That is, we can simply say that \( \mathcal{N}|_{(g \circ \pi)^{-1}(W)} = \mathcal{N}|_{\pi^{-1}(g^{-1}(W))} \) is nef. So this invertible sheaf is also \( \pi|_{\pi^{-1}(g^{-1}(W))} \)-nef.

Set
\[
\mathcal{M}' = R^s\pi_*(\mathcal{F} \otimes \mathcal{L}^m \otimes \mathcal{N})|_{g^{-1}(W)}, \quad \mathcal{M} = R^s(g \circ \pi)_*(\mathcal{F} \otimes \mathcal{L}^m \otimes \mathcal{N})|_W.
\]
Thus the \( V^q(A)(\mathcal{F}, \mathcal{L}, \Lambda) \) property gives \( \mathcal{M}' = 0 \). Since \( g \) is finite, the scheme \( g^{-1}(W) \) is affine. Thus \( \mathcal{M}' \) equals the sheafification over \( g^{-1}(W) \) of the abelian group \( M \) \([Ha1977, \text{Proposition III.8.5}]\) where
\[
M = H^s(\pi^{-1}(g^{-1}(W)), (\mathcal{F} \otimes \mathcal{L}^m \otimes \mathcal{N})|_{\pi^{-1}(g^{-1}(W))}).
\]
So \( M = 0 \). However, \( W \) is also affine, so \( \mathcal{M} \) is also the sheafification of \( M = 0 \). Thus \( \mathcal{M} = 0 \).
Since $M = 0$ holds for any affine $W \subseteq V$, we have $R^r(g \circ \pi)_*(F \otimes \mathcal{L}^m \otimes \mathcal{N})|_V = 0$ by definition of right derived functor. Hence $V^q_A(F, \mathcal{L}, \Lambda)$ holds. $\square$

**Theorem E2.5.** Lemma [E1.3] can be used to complete the reductions which help prove [Ke2003] Theorem 1.5, Proposition 5.12.

**Proof.** Let $A$ be a noetherian ring and $\pi : X \to \text{Spec} A$ be a projective morphism. Let $\mathcal{L}$ be an ample invertible sheaf and $\Lambda$ a certain set of invertible sheaves on $X$. The goal of both of [Ke2003] Theorem 1.5, Proposition 5.12 is to show $V^q_A(F, \mathcal{L}, \Lambda)$ holds for all coherent sheaves $F$. Standard reductions and the definition of $V^q_A$ allows one to assume that $\pi$ is surjective and that $X$ and $\text{Spec} A$ are integral. (See the original proofs for details.) Then [Ke2003] Lemma 5.11 and the definition of $V^q_A$ allow the reduction to $g \circ \pi' : X' \to \text{Spec} A_b$ for some $b \in A$, where $\pi'$ has the nice properties of Lemma [E1.3].

Finally, Lemma [E2.4] allows the reduction to $\pi' : X' \to \text{Spec} A'$. The original proofs then proceed as before. $\square$

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