PARKING FUNCTIONS AND TRIANGULATION OF THE ASSOCIAHEDRON

JEAN-LOUIS LODAY

Abstract. We show that a minimal triangulation of the associahedron (Stasheff polytope) of dimension \( n \) is made of \((n + 1)^{n-1}\) simplices. We construct a natural bijection with the set of parking functions from a new interpretation of parking functions in terms of shuffles.

Introduction

The Stasheff polytope, also known as the associahedron, is a polytope which comes naturally with a poset structure on the set of vertices (Tamari poset), hence a natural orientation on each edge. We decompose this polytope into a union of oriented simplices, the orientation being compatible with the poset structure. This construction defines the associahedron as the geometric realization of a simplicial set. In dimension \( n \) the number of (non-degenerate) simplices is \((n + 1)^{n-1}\).

A parking function is a permutation of a sequence of integers \( i_1 \leq \cdots \leq i_n \) such that \( 1 \leq i_k \leq k \) for any \( k \). For fixed \( n \) the number of parking functions is \((n + 1)^{n-1}\). We show that the set \( PF_n \) of parking functions of length \( n \) admits the following inductive description:

\[
PF_n = \bigcup_{p+q=n-1 \atop p \geq 0, q \geq 0} \{1, \ldots, p + 1\} \times Sh(p, q) \times PF_p \times PF_q
\]

where \( Sh(p, q) \) is the set of \((p, q)\)-shuffles. From this bijection we deduce a natural bijection between the top dimensional simplices of the associahedron and the parking functions.

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In the last section we investigate a similar triangulation of the permutohedron.

Thanks to Andrè Joyal for mentioning to me Abel’s formula during the Street’s fest.

1. **Associahedron**

1.1. **Planar trees, Stasheff complex.** The associahedron can be constructed as a cellular complex as follows (cf. for instance [BV]).

Let \( Y_n \) be the set of planar binary rooted trees with \( n \) internal vertices:

\[
Y_0 = \{ \mid \}, \quad Y_1 = \{ \begin{array}{c} \diagup \end{array} \}, \quad Y_2 = \{ \begin{array}{c} \diagup \diagup \\ \diagup \end{array}, \quad \begin{array}{c} \diagup \\ \diagup \end{array} \}, \\
Y_3 = \{ \begin{array}{c} \begin{array}{c} \diagup \\ \diagup \\ \diagup \end{array}, \quad \begin{array}{c} \diagup \\ \diagup \\ \diagup \end{array}, \quad \begin{array}{c} \begin{array}{c} \diagup \\ \diagup \\ \diagup \end{array}, \quad \begin{array}{c} \begin{array}{c} \diagup \\ \diagup \\ \diagup \end{array}, \quad \begin{array}{c} \begin{array}{c} \diagup \\ \diagup \\ \diagup \end{array} \end{array} \end{array} \}
\]

Observe that \( t \in Y_{n+1} \) has \( n \) internal edges. For each \( t \in Y_{n+1} \) we take a copy of the cube \( I^n \) (where \( I = [0,1] \) is the interval) which we denote by \( I^n_t \). Then the associahedron of dimension \( n \) is the quotient

\[
\mathcal{K}^n := \bigsqcup_t I^n_t / \sim
\]

where the equivalence relation is as follows. We think of an element \( \tau = (t; \lambda_1, \ldots, \lambda_n) \in I^n_t \) as a tree of type \( t \) where the \( \lambda_i \)'s are the lengths of the internal edges. If some of the \( \lambda_i \)'s are 0, then the geometric tree determined by \( \tau \) is not binary anymore (since some of its internal edges have been shrinked to a point). We denote the new tree by \( \bar{\tau} \). For instance, if none of the \( \lambda_i \)'s is zero, then \( \bar{\tau} = t \); if all the \( \lambda_i \)'s are zero, then the tree \( \bar{\tau} \) is the corolla (only one vertex). The equivalence relation \( \tau \sim \tau' \) is defined by the following two conditions:

- \( \bar{\tau} = \bar{\tau'} \),
- the lengths of the nonzero-length edges of \( \tau \) are the same as those of \( \tau' \).
Hence $K^n$ is obtained as a cubical realization:

\[
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{triangle.png}}
\end{array}
\]

Since a cube can be decomposed into simplices, we can get a simplicial decomposition of $K^n$. However our aim is to construct a minimal simplicialization. It was shown by Stasheff in [Sta] that $K^n$ is homeomorphic to a ball. In fact this Stasheff complex can be realized as a polytope (cf. [Lee],[GKZ],[SS],[Lod2]). One way to construct it is recalled in the following section (taken from [Lod2]).

1.2. **Associahedron.** To any tree $t \in Y_n, n > 0$, we associate a point $M(t) \in \mathbb{R}^n$ with integral coordinates as follows. Let us number the leaves of $t$ from left to right by $0, 1, \ldots, n$. So one can number the vertices from 1 to $n$ (the vertex number $i$ is in between the leaves $i-1$ and $i$). We consider the subtree generated by the $i$th vertex. Let $a_i$ be the number of offspring leaves on the left side of the vertex $i$ and let $b_i$ be the number of offspring leaves on the right side. Observe that these numbers depend only on the subtree determined by the vertex $i$. We define:

\[
M(t) := (a_1b_1, \ldots, a_ib_i, \ldots, a_nb_n) \in \mathbb{R}^n.
\]

In low dimension we get:

\[
M\left( \begin{array}{c} \text{\includegraphics[height=0.5cm]{triangle1.png}} \end{array} \right) = (1 \times 1) = (1),
\]

\[
M\left( \begin{array}{c} \text{\includegraphics[height=0.5cm]{triangle2.png}} \end{array} \right) = (1 \times 1, 2 \times 1) = (1, 2),
\]

\[
M\left( \begin{array}{c} \text{\includegraphics[height=0.5cm]{triangle3.png}} \end{array} \right) = (1 \times 2, 1 \times 1) = (2, 1),
\]

\[
M\left( \begin{array}{c} \text{\includegraphics[height=0.5cm]{triangle4.png}} \end{array} \right) = (1 \times 1, 2 \times 1, 3 \times 1) = (1, 2, 3),
\]

\[
M\left( \begin{array}{c} \text{\includegraphics[height=0.5cm]{triangle5.png}} \end{array} \right) = (1 \times 1, 2 \times 2, 1 \times 1) = (1, 4, 1).
\]
The planar binary trees with \( n \) internal vertices are in bijection with the parenthesizings of the word \( x_0x_1 \cdots x_{n+1} \). For the tree corresponding to the parenthesizing \(((x_0x_1)x_2)(x_3x_4)\) one gets the following point \((1 \times 1, 2 \times 1, 3 \times 2, 1 \times 1) = (1, 2, 6, 1)\).

Denote by \( H_n \) the hyperplane of \( \mathbb{R}^n \) whose equation is
\[
x_1 + \cdots + x_n = \frac{n(n + 1)}{2}.
\]

One can show that for any tree \( t \in Y_n \) the point \( M(t) \) belongs to the hyperplane \( H_n \).

Then by \[\text{Lod2}\] the associahedron or Stasheff polytope \( K^{n-1} \) is the convex hull of the points \( M(t) \) in the hyperplane \( H_n \) for \( t \in Y_n \).

\[
\begin{array}{c}
(1) \\
\bullet
\end{array}
\quad
\begin{array}{ccc}
(1, 2) & (2, 1) & (2, 1, 3) \\
\begin{array}{c}
(3, 1, 2) \\
(3, 2, 1)
\end{array}
\end{array}
\quad
\begin{array}{c}
(1, 4, 1)
\end{array}
\]

\( K^0 \quad K^1 \quad K^2 \)

1.3. **Order structure.** Let us recall that, on the set \( Y_n \), there is a partial order known as the Tamari order. It is induced by the order \( \begin{array}{c} \nearrow \searrow \end{array} \rightarrow \begin{array}{c} \nearrow \searrow \end{array} \) on \( Y_2 \) as follows. There is a covering relation \( t \rightarrow t' \) between two elements of \( Y_n \) if \( t' \) can be obtained from \( t \) by replacing locally a subtree of the form \( \begin{array}{c} \nearrow \searrow \end{array} \) by a subtree of the form \( \begin{array}{c} \nearrow \searrow \end{array} \). In low dimension the covering relations induce the following order on the vertices:
Our aim is to triangulate (we mean simplicialize) the associahedron $K^n$ by (oriented) $n$-simplices, so that the oriented edges of the simplices are coherent with the Tamari order. We observe immediately that there are two choices for $K^2$:

We choose the first one and we will construct a triangulation for $K^n$ consistent with this choice.

1.4. **Boundary of $K^n$.** The cells of the associahedron are in bijection with the planar rooted trees (see for instance [LR2]). The vertices correspond to the binary trees and the big cell corresponds to the corolla. The boundary of $K^n$, denoted $\partial K^n$, is made of cells of the form $K^p \times K^q$, $p + q = n - 1$. They are in bijection with the trees $\gamma(a; p, q)$ with two vertices:

Here $p + 2$ is the number of outgoing edges at the root, $q + 2$ is the number of outgoing edges (leaves) at the other vertex, and $a$ is the index of the only edge which is not a leaf, so $0 \leq a \leq p + 1$. By convention we index the edges at the root from 0 to $p + 1$ from right to left.

Let us denote by $S$ (like South pole) the vertex of $K^n$ with coordinates $(n, n - 1, \ldots, 1)$, which corresponds to the right comb.

1.5. **Proposition.** The associahedron $K^n$ is the cone with vertex $S$ over the union of the cells $\gamma(a; p, q), a \geq 1$, in $\partial K^n$.

**Proof.** Since $K^n$ is a ball, $\partial K^n$ is a sphere. The $n$-cells of $\partial K^n$ which contain $S$ are such that $a = 0$, because the tree of such a cell is obtained from the right comb (by collapsing $n - 2$ edges). The other ones, for which $a \geq 1$, form a ball of dimension $n - 1$ and obviously $K^n$ can be viewed as a cone with vertex $S$ over this $(n - 1)$-dimensional ball. □
1.6. **Theorem.** The associahedron $K^n$ can be constructed out of $K^{n-1}$ as follows:

(a) start with $K^{n-1}$,
(b) “fatten” $K^{n-1}$ by replacing its boundary faces $\gamma(a-1; p-1, q)$ (of the form $K^{p-1} \times K^{q}$), $p+q = n-1$, by $\gamma(a; p, q)$ (of the form $K^p \times K^q$),
(c) take the cone over the fat-$K^{n-1}$.

**Proof.** From Proposition 1.5 it suffices to check that, in $K^n$, the union of the faces $\gamma(a; p, q)$, $a \geq 1$, is precisely fat-$K^{n-1}$. Indeed, $\gamma(1; 0, n-1)$ corresponds to $K^{n-1}$ and all the other faces $\gamma(a; p, q)$, $a \geq 1, p \geq 1$, come from the cells $\gamma(a-1; p-1, q)$ of $\partial K^{n-1}$ by fattening. \(\square\)

1.7. **Examples.** $n = 1$:

- $K^1$
- $\triangledown$
- fattened $K^1$
- Cone over fat-$K^1 = K^2$

Example $n = 2$:

- $K^2$
- $\triangledown$
- fattened $K^2$
- Cone over fat-$K^2 = K^3$
2. Triangulation of the associahedron

From the construction of $K^n$ out of $K^{n-1}$ performed in the preceding section, it is clear that one can triangulate $K^n$ by induction.

2.1. **Product of simplices.** Let us recall that, if $\Delta^n$ denotes the standard simplex, then, in the triangulation of $\Delta^p \times \Delta^q$, the simplices are indexed by the $(p, q)$-shuffles.

For instance the triangulation of $\Delta^1 \times \Delta^1$ is:

\[
\begin{array}{c}
(0, 1) \\
21 \\
(0, 0) \\
\end{array}
\begin{array}{c}
(1, 1) \\
12 \\
(1, 0) \\
\end{array}
\]

The triangulation of $\Delta^2 \times \Delta^1$ (a prism) is made of three tetrahedrons:

| shuffle | vertices |
|---------|----------|
| 123     | (0, 0), (1, 0), (2, 0), (2, 1) |
| 132     | (0, 0), (1, 0), (1, 1), (2, 1) |
| 312     | (0, 0), (0, 1), (1, 1), (2, 1) |

Here $(i, j)$ stands for the point of $\Delta^2 \times \Delta^1$ which is the $i$th vertex of $\Delta^2$ times the $j$th vertex of $\Delta^1$.

2.2. **Theorem.** The associahedron $K^n$ admits a triangulation by $(n + 1)^{n-1}$ simplices, whose orientation is compatible with the Tamari order on the set of vertices. An $n$-simplex of this triangulation is a cone over an $(n - 1)$-simplex determined by the following choices. Choose either

- an $(n - 1)$-simplex of $K^{n-1}$, or
- in the fattened cell $\gamma(a; p, q)$ isomorphic to $K^p \times K^q$ choose a $p$-simplex of $K^p$, a $q$-simplex of $K^q$ and a $(p, q)$-shuffle.

**Proof.** The proof of the second assertion follows from Theorem [1.6](#) and the fact that the triangulation of $\Delta^p \times \Delta^q$ is indexed by the $(p, q)$-shuffles.

From this description of the triangulation we can count the number $d_n$ of top dimensional simplices of $K^n$ by induction. Let us suppose that $d_p = (p + 1)^{p-1}$ for $p < n$, and recall that the number of $(p, q)$-shuffles is the binomial coefficient $\binom{p+q}{q}$. If $p$ is fixed, then $a$ can take the values $1, \ldots, p+1$. Hence there are $p+1$ cells of the form $K^p \times K^q, p+q = n-1$. 
We get

\[ d_n = \sum_{p=0}^{n-1} (p+1) \binom{n-1}{p} d_p d_{n-p-1} \]

\[ = \sum_{p=0}^{n-1} \binom{n-1}{p} (p+1)^p (n-p)^{n-p-2} \]

\[ = (n+1)^{n-1}. \]

The last equality is a particular case of Abel’s formula, cf. [R],

\[ x^{-1}(x+y+n)^n = \sum_{k=0}^{n} \binom{n}{k} (x+k)^{k-1}(y+n-k)^{n-k} \]

for \( x = y = 1 \) and \( k = n-p \). Observe that, at each step of the construction of \( K^n \) out of \( K^{n-1} \), the orientation of the simplices coincides with the orientation of the edges given by the Tamari poset order. \( \square \)

**Example:** triangulation of fat-\( K^2 \) giving rise to the triangulation (by tetrahedrons) of \( K^3 \):
3. Parking functions

A parking function is a sequence of integers \((i_1, \ldots, i_n)\) such that the associated ordered sequence \(j_1 \leq \ldots \leq j_n\) satisfies the following conditions: \(1 \leq j_k \leq k\) for any \(k\). For instance there is only one parking function of length one: \((1)\), there are three parking functions of length two: \((1, 2), (2, 1), (1, 1)\). There are sixteen parking functions of length three: the permutations of \((1, 2, 3), (1, 1, 3), (1, 2, 2), (1, 1, 2), (1, 1, 1)\) (remark that \(6+3+3+3+1=16\)). It is well-known, cf. for instance [NT], that there are \((n+1)^{n-1}\) parking functions of length \(n\). We denote by \(PF_n\) the set of parking functions of length \(n\). We denote by \(Sh(p, q)\) the set of permutations which are \((p, q)\)-shuffles.

3.1. Theorem. For any \(n\) there is a bijection

\[
\pi: \bigcup_{p+q=n-1, p \geq 0, q \geq 0} \{1, \ldots, p+1\} \times Sh(p, q) \times PF_p \times PF_q \longrightarrow PF_n
\]

given by \(\pi(a, \theta; f, g) = (a, \theta*(f_1, \ldots, f_p, p+1+g_1, \ldots, p+1+g_q))\).

Proof. Let \(a \in \{1, \ldots, p+1\}, \theta \in Sh(p, q), f \in PF_p\) and \(g \in PF_q\).

Let us first show that the sequence

\[
x := (a, \theta*(f_1, \ldots, f_p, p+1+g_1, \ldots, p+1+g_q))
\]

is a parking function. Let \((\phi_1, \ldots, \phi_p), (\psi_1, \ldots, \psi_q)\), be the sequence \(f\), resp. \(g\), put in increasing order. In the sequence \(x\) put in increasing order we first find the sequence \(f\) with the number \(a\) inserted, then the sequence \(\psi\). Since the sequence \(\phi\) has \(p\) elements and \(1 \leq a \leq p+1\) the expected inequality is true for \(a\). It is also true for all the elements of \(\phi\) since \(\phi_j\) is either at the place \(j\) or at the place \(j+1\). The expected inequality is true for all the elements of the sequence \(p+1+\psi\) since \(p+1+\psi_j\) is at the place \(p+1+j\). Hence \(x\) is a parking function.

Let us now construct a map in the other direction. Let

\[
a = (a = a_1, a_2, \ldots, a_n)
\]

be a parking function, referred to as the original sequence. Let \(\underline{x} = (x_1, \ldots, x_n)\) be the ordered sequence where \(x_j = a\). Let \(k\) be the smallest integer such that \(k > j\) and \(x_k = k\). Then we put \(p+2 = k\). It may happen that there is no such integer. In that case we put \(p+2 = n+1\), that is \(p+1 = n\) and so \(q = 0\). With these choices there exists \(\sigma \in S_{k-2}, \sigma' \in S_{n-k+1}\) and \(\theta \in Sh(k-2, n-k+1)\) such that \((a, \theta*(\sigma \times \sigma')(x_1, \cdots, x_{j-1}, x_j, \cdots, x_k, \cdots, x_n))\) is the original sequence \(a\).
Example: \( a = (3, 6, 1, 7, 2, 1, 3, 6) \). Then we get \( x = (1, 1, 2, 3, 3, 6, 6, 7) \) (where \( a \) has been underlined), \( j = 4 \) and \( k = 6 \), therefore \( a = 3, p = 4, q = 3 \). The two parking functions are \((1, 2, 1, 3)\) and \((1, 2, 1)\) and the \((4, 3)\)-shuffle is the permutation whose action on \( u_1u_2u_3u_4v_1v_2v_3 \) gives \( v_1u_1v_2u_2u_3u_4v_3 \).

Hence we have constructed a map from \( PF_n \) to
\[
\bigcup_{p+q=n-1\atop p\geq 0, q\geq 0} \{1, \ldots, p+1\} \times Sh(p,q) \times PF_p \times PF_q.
\]

In order to show that it is the inverse of the previous map, it is sufficient to verify that our algorithm gives \( k - 2 = p \) when we start with a parking function of the form \((a, sh(pf(1, \ldots, p), pf(p+2, \ldots, p+1+q)))\). First, in the ordered sequence of a parking function the first element is always 1, hence \( p + 2 \) is the smallest element in \( pf(p+2, \ldots, p+1+q) \), or \( q = 0 \) and \( p = n + 1 \). Second, we know that \( a \leq p + 1 \), so in the ordered sequence \( a = x_j \) appears before \( p + 2 = k \), hence \( p + 2 \) is at the place \( p + 2 \), whence \( k > j \) and we are done. \( \square \)

3.2. Remark. As a Corollary we get from Abel’s formula (cf. the proof of Theorem 2.2) the well-known result:
\[ \#PF_n = (n + 1)^{n-1}. \]

3.3. Examples.

| \( n \) | \( a \) | \( p \) | \( q \) | Parking Functions |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | (1) |
| 2 | 1 | 1 | 0 | (1,1) |
|  | 0 | 1 | (1,2) |
|  | 2 | 1 | 0 | (2,1) |
| 3 | 1 | 2 | 0 | (1,1,1) (1,1,2) (1,2,1) |
|  | 1 | 1 | (1,1,3) (1,3,1) |
|  | 0 | 2 | (1,2,2) (1,2,3) (1,3,2) |
|  | 2 | 0 | (2,1,1) (2,1,2) (2,2,1) |
|  | 2 | 1 | (2,1,3) (2,3,1) |
|  | 3 | 2 | (3,1,1) (3,1,2) (3,2,1) |

In the following statement we use Theorem 2.2.

3.4. Theorem. Let \( \sigma \) be a simplex of \( K^n \) determined either by
- a simplex \( \omega \) of \( K^{n-1} \), or by
- a triple \( (a, p, q) \) (determining a face \( K^p \times K^q \), where \( 1 \leq a \leq p+1 \), \( p + q = n - 1 \), and a simplex \( \alpha \) of \( K^p \), a simplex \( \beta \) of \( K^q \) and a \((p, q)\)-shuffle \( \theta \).

The map \( \Phi_n \), which assigns to \( \sigma \) the parking function
\[ \Phi_n(\sigma) := (1, 1 + \Phi_{n-1}(\theta)) \] in the first case,
\[ \Phi_n(\sigma) := (a, \theta_*(\Phi_p(\alpha), p + 1 + \Phi_q(\beta))) \text{ in the second one,} \]
is a bijection from the set of \(n\)-simplices of \(K^n\) to the set \(PF_n\) of parking functions.

**Proof.** We work by induction on \(n\). For \(n = 1\) there is no choice: \(\Phi(1\text{-cell}) = (1)\). In the description of the triangulation of \(K^n\) given in Theorem 2.2 we have constructed a bijection between the set of \(n\)-simplices of \(K^n\) and the set \(\bigcup_{p+q=n-1} \{1, \ldots, p+1\} \times Sh(p, q) \times PF_p \times PF_q\). By Theorem 3.1 this set is in bijection with \(PF_n\). It is immediate to check that the composite of the two bijections is the map \(\Phi_n\) described in the statement of the Theorem. \(\square\)

3.5. **Examples.**
3.6. **Remark.** Another way of stating Theorem 2.2 and Theorem 3.4 is to say that we have constructed a simplicial set $K^n$ such that $K^n_0 = Y_n$, $K^n_0 / \{\text{degenerate elements}\} = PF_n$, and such that the geometric realization is $|K^n| = K^n$.

3.7. **Associahedron and cube.** It is well-known that for the cube $I^n$ the triangulation is indexed by the permutations, elements of the symmetric group $S_n$. In [Lod2] we observed that the associahedron is contained in a certain cube. Locally around the North Pole (vertex with coordinates $(1, 2, \ldots, n)$), the triangulation of the cube and the triangulation of the associahedron coincide. Our indexing of the simplices of the associahedron is such that the two indexings also coincide.

3.8. **Other relationship between parking functions and associahedron.** There are other links between parking functions and associahedron which are treated in [PS] by Pitman and Stanley, in [Po] by Postnikov and by Hivert (personal communication). It would be interesting to compare all of them.

4. **Triangulation of the permutohedron**

In this section we briefly indicate how to simplicialize the permutohedron along the same line as the associahedron. So far we do not know of a nice combinatorial object playing the role of the parking functions.

4.1. **Permutohedron.** Let us recall that the *permutohedron* $P^{n-1}$ is the convex hull of the points $M(\sigma) = (\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^n$, where $\sigma \in S_n$ is a permutation.

\[ P^0 \quad P^1 \quad P^2 \quad P^3 \]

The weak Bruhat order on $S_n$ is a partial order whose covering relations are in one to one correspondence with the edges of $P^{n-1}$.

It is helpful to replace the permutations by *planar binary trees with levels*. See for instance [LR1] for a discussion of this framework. Under
this remplacement the faces of the permutohedron $P^n$ are labelled by the planar leveled trees with $n + 2$ leaves which have only two levels. Let us denote by $[p]$ the totally ordered set $\{0, \ldots, p\}$. A planar leveled tree with $n + 2$ leaves and two levels is completely determined by the arity of the root, let us say $p + 1$, and an ordered surjective map $f : [n + 1] \to [p + 1], 0 \leq p \leq n - 1$. We denote the corresponding face by $\gamma(p; f)$.

Example of faces of $P^2$: 

\[
\begin{align*}
\gamma(0; f) &= \begin{array}{c}
\text{\hspace{1cm}}
\end{array}
\end{align*}
\]

where the image of $f$ is $(0,0,0,1)$,

\[
\begin{align*}
\gamma(0; f) &= \begin{array}{c}
\text{\hspace{1cm}}
\end{array}
\end{align*}
\]

where the image of $f$ is $(0,0,1,1)$,

\[
\begin{align*}
\gamma(1; f) &= \begin{array}{c}
\text{\hspace{1cm}}
\end{array}
\end{align*}
\]

where the image of $f$ is $(0,0,1,2)$.

The vertex $M(n+1,n,\ldots,1)$, whose corresponding tree is the right comb, is called the South pole. For a given $p$ the face corresponding to the surjective map $f_0$, whose image is $(0,1,\ldots,p,p,\ldots,p)$, contains the South pole. We denote by $SM(n,p)$ the set of ordered surjective maps from $[n+1]$ to $[p+1]$ minus the map $f_0$. For instance $SM(4,1)$ has $10 - 1 = 9$ elements.

4.2. **Triangulation of $P^n$.** We construct a simplicialization of $P^n$ by induction as follows. For $n = 0$, the space $P^0$ is a point (0-simplex). For $n = 1$, since $P^1$ is the interval (1-simplex). The permutohedron $P^n$ is the cone over the South pole with basis the union of the faces which do not contain the South pole, that is the faces whose ordered surjective map $f : [n + 1] \to [p + 1]$ is not the map $f_0$ given by $(0,1,\ldots,p+1,p+1,\ldots,p+1)$. Therefore an $n$-simplex of the triangulation of $P^n$ is the cone over the South pole $S$ with basis an $n - 1$-simplex of the triangulation of the union of faces not containing $S$. Such an $(n-1)$-simplex is completely determined by the following choices

- a face $\gamma(p; f)$ not containing $S$ (i.e. $f \neq f_0$),
- a top dimensional simplex in $P^p$,
- a top dimensional simplex in $P^{n-p-1}$,
- a shuffle in $Sh(p, n - p - 1)$. 
From this choice, it is clear by induction that the orientation of the simplices are compatible with the orientation of the edges induced by the weak Bruhat order.

Example: triangulation of $\mathcal{P}^2$:

\[
\begin{array}{cccccccc}
& & & & & & & \\
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& & & & & & & \\
& & & & & & & \\
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& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\end{array}
\]

In conclusion we have proved the following result.

4.3. Theorem. The set $ZP_n$ of top dimensional simplices ($n$-simplices) of the permutohedron $\mathcal{P}^n$ satisfies the following recursive formula

\[
ZP_n = \bigcup_{p=0}^{n-1} SM(n, p) \times Sh(p, n-p-1) \times ZP_p \times ZP_{n-p-1},
\]

where $SM(n, p) = \binom{n+1}{p+1} - 1$, and $Sh(p, n-p-1) = \binom{n-1}{p}$.

\[\Box\]

The number of top simplices is as follows in low dimension:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| $\#ZP_n$ | 1 | 1 | 4 | 34 | 488 | 10512 | 316224 | 12649104 | 649094752 |

4.4. Permutohedron analogue of parking functions. It would be interesting to find a sequence of combinatorial objects analogue of the parking functions, that is satisfying the inductive relation of Theorem 4.3.

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Institut de Recherche Mathématique Avancée, CNRS et Université Louis Pasteur, 7 rue R. Descartes, 67084 Strasbourg Cedex, France
E-mail address: loday@math.u-strasbg.fr