Linear equation systems for structural analysis: imagining resolutions

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Abstract. This work is based on a study into new ways of resolving the equilibrium equation systems for manual analyses of certain structures commonly found in building. It suggests finding solutions based on images that reproduce the operations of current methods, which may inspire the design of others that qualitatively reflect those of other more effective procedures. To date three methods (Gauss, Cholesky & Crout) have been imagined: (i) by “visualising” their operations through the mechanical behaviour of models during the equilibrium phase. These visualisations may help suggest other physical responses that can balance models more quickly and identify with new, more direct numerical methods; (ii) by “geometrising” operations by means of lines sketched freehand. This geometrisation may reveal hidden links between the parts of the calculation of current methods that enable more direct but equally precise new methods to be created. The paper shows four images to reinforce these viewpoints. Two visualise the methods of Gauss-Jordan and Cramer, confirming that the abstract procedures that resolve the systems may be linked to specific mechanical behaviours. The other two geometrisate the resolutions by Gauss and Gauss-Jordan when the stiffness matrices are asymmetric. Their systems could emerge from the analysis of cracked models or from obtaining the equivalent actions in the P-Δ method, in line with a procedure drawn up previously. The paper ends by geometrisating the resolution of a system at different scales and comparing the outcomes with those of numerical methods. The results (i) confirm that geometrisating scalar and vectorial magnitudes for numerical analysis procedures reduces application times if they are calculated freehand; and (ii) point to possible lines of research for developing further graphic methods that can analyse other types of structure directly and accurately.

1. Introduction
This work comes from a research carried out at the School of Architecture in Donostia (University of the Basque Country) to develop new ways of solving the linear equations systems for the manual equilibrium analysis of building structures. The aim is to find new methods by transforming the operations of the current numerical procedures into images because it is believed that they could help to find others that qualitatively reflect the numerical structure of the new procedures. So far, the Gauss, Cholesky and Crout methods have been imagined from two perspectives:

a) Assuming that their operations reproduce certain mechanical behaviours of the model during the equilibrium stage. Considering that the deflection is not governed by the physical response of the model due to the low deformability of the materials, multiple mechanical behaviours may be considered to justify the same deflection, each one giving rise to a numerical way of solving the
equation systems. Thereby, the operations of Gauss [1], Crout (Figures 1c,d) and Cholesky methods have been related to certain structural responses, called "visualisations" here, using Cross's philosophy, and following the results, a possible mechanical behaviour was suggested for simplifying the calculation of the stresses.

b) By reproducing their operations with drawings traced according to certain graphic procedures [2] (Figures 1f,g). This "geometrisation" of operations is inspired by procedures from the past [3,4], and certain philosophies [5,6], the thinking behind which has recently been recovered from technical literature [7]. When freehand drawing is used, sufficiently accurate results are obtained in less time than by conventional numerical calculation carried out manually with the aid of a non-programmable calculator. It is also believed that a unitary picture of all operations could help to find hidden relationships between the various parts of the calculation that enable these procedures to be transformed into more direct but equally precise ones.

\[
\begin{align*}
[A] \cdot \{\theta\} &= \{M\} \\
[L][U] \cdot \{\theta\} &= \{M\} \\
[U] \cdot \{\theta\} &= \{\alpha^\top\}
\end{align*}
\]

Figure 1. images derived from solving an equilibrium equation system: i) of a continuous beam, resolved by Gauss: a) numerical schematice and c) visualisation (\(M_{ai} = \) active moment; \(M_{ri} = \) restrictive moment); resolved by Crout: b) schematice and d) visualisation (\(\alpha_{ai} = \) active rotation); ii) of the portico e): geometrisations by Gauss f,g) and by Crout, h,i).

This paper provides four images that support these perspectives. Two are visualisations that physically interpret the operations of the Gauss-Jordan and Cramer methods; the other two are geometrisations of the operations that resolve systems with asymmetric matrices by the Gauss and the Gauss-Jordan procedures. The geometrisations could be used to obtain graphically the deflection of previously cracked models whose members have different stiffnesses depending on the transmission sense of the moments. They could also be used to obtain the values of the equivalent actions of the P-\(\Delta\) method according to [2]. Although the procedure could be adapted to analyse uncracked models, another method that requires less drawing seems more appropriate. The work starts visualising the resolution of (1) derived from the uncracked model of Figure 1c with EI constant; then the resolution of a generic asymmetric matrix system (8) is qualitatively geometrised; finally, the deflection of a particular case is obtained graphically and compared with the exact one.

\[
EI[A]\cdot\{\theta\} = \{M\}
\]
2. Visualizing the Gauss-Jordan’s operations

Perhaps the Gauss-Jordan method could be considered a modification of the Gauss one by converting (1) into another system $S'$ (Figure 1a) and then transforming $S'$ into $S''$ (2). The first operations are equal to interpreting the deflection as a sum of partial ones (Figure 1c); and the second operations are equal to transforming those deflections into others (Figure 2h) through a process that seems to crack each node differently, following the reverse order of elimination of unknowns used by Gauss. When a node cracks, the nodal action is reduced to match the real rotation with the one of this node, and so the method appears to replace the load states in Figure 1c by others (Figure 2h). A visualisation of (2) is given below.

\[
\begin{bmatrix}
  a & d & 0 \\
  0 & b & e \\
  0 & 0 & c \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  a & d & 0 \\
  0 & b & 0 \\
  0 & 0 & c \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  M^a_A \\
  -M^a_B \\
  M^a_C \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  M^a_A + \frac{d}{b} M^a_B = M^a_B \\
  M^a_B \\
  M^a_C \\
\end{bmatrix}
\]

2.1 By eliminating $e$

The operations in (2) that eliminate $e$ are equivalent to converting the deflections of Figure 1c into those of Figure 2d by breaking down deflection III into two partial deflections (Figures 2a,b). The first one is deflection V and the second one together with II makes deflection IV.

The deflection in Figure 2a is obtained by adding a moment at $B$ to deflection III which cancels $\theta_A$ and $\theta_B$ and increases the bending of the bars attached to $C$. The increase in bending causes these bars to crack, increasing $\theta_C$ to the value in Figure 1c. The forces transmitted by the rotation of this node are still real because the loss of stiffness of the bars attached to $C$ is the same. (3) shows the relationship between the original inertia $I$ and that which results from cracking. $\theta_C'$ represents the stiffness of node $C$ before cracking.

\[
\frac{I}{I_f} = \frac{1}{\theta'_C} \quad (3)
\]

\[a) \quad \frac{e}{d} \quad M^a_C \quad M^a_C \quad M^a_C \quad \rightarrow \quad \begin{array}{c}
\begin{array}{c}
M^a_A \\
M^a_B \\
M^a_C \\
\end{array}
\end{array} \]

\[b) \quad \frac{e}{d} \quad M^a_C \quad M^a_C \quad M^a_C \quad \rightarrow \quad \begin{array}{c}
\begin{array}{c}
M^a_B \\
M^a_B \\
M^a_C \\
\end{array}
\end{array} \]

\[c) \quad \frac{e}{d} \quad M^a_C \quad M^a_C \quad M^a_C \quad \rightarrow \quad \begin{array}{c}
\begin{array}{c}
M^a_C \\
M^a_C \\
M^a_C \\
\end{array}
\end{array} \]

\[d) \quad \frac{e}{d} \quad M^a_C \quad M^a_C \quad M^a_C \quad \rightarrow \quad \begin{array}{c}
\begin{array}{c}
M^a_C \\
M^a_C \\
M^a_C \\
\end{array}
\end{array} \]

\[e) \quad \frac{e}{d} \quad M^a_C \quad M^a_C \quad M^a_C \quad \rightarrow \quad \begin{array}{c}
\begin{array}{c}
M^a_C \\
M^a_C \\
M^a_C \\
\end{array}
\end{array} \]

\[f) \quad \frac{e}{d} \quad M^a_C \quad M^a_C \quad M^a_C \quad \rightarrow \quad \begin{array}{c}
\begin{array}{c}
M^a_C \\
M^a_C \\
M^a_C \\
\end{array}
\end{array} \]

\[g) \quad \frac{e}{d} \quad M^a_C \quad M^a_C \quad M^a_C \quad \rightarrow \quad \begin{array}{c}
\begin{array}{c}
M^a_C \\
M^a_C \\
M^a_C \\
\end{array}
\end{array} \]

\[h) \quad \frac{e}{d} \quad M^a_C \quad M^a_C \quad M^a_C \quad \rightarrow \quad \begin{array}{c}
\begin{array}{c}
M^a_C \\
M^a_C \\
M^a_C \\
\end{array}
\end{array} \]

\[\text{Figure 2. Visualizing the Gauss-Jordan’s operations: a-d) by eliminating e; e-h) by eliminating d.}\]

2.2 By eliminating $d$

The operations in (2) that eliminate $d$ are equivalent to converting the deflections in Figure 2d to those in Figure 2h by means of a process similar to that in section 2.1 that can be followed by observing Figures 2e-h.
3. Visualizing the Cramer operations

Obtaining the deflection of the above beam by solving (1) by Cramer means obtaining the nodal rotations according to (4) by combining certain determinants (5) dependent on $[A]$ and $\{M\}$. A physical meaning for them is suggested below.

$$\theta_i = \frac{|A_i|}{|A|}$$

(4)

$$|A_i| \cdot |A| = |A_a| \cdot |A_b| \cdot |A_c|$$

(5)

Assuming that all the bars cracked in the same way before the actions were applied and assuming that $|A| (>1)$ was the ratio (6) between the inertia $I$ of the uncracked bar and $I_r$ of the same bar when cracked, $\{A_i\}$ could represent the nodal rotations of the cracked model (Figure 3). This interpretation is valid because (7) is fulfilled. It’s Gauss resolution provides $|A_i|$ rotations broken down into partial deflections as in Figure 1d and with these results, the Cramer method obtains the real deflection by rectifying Figure 3 with (4). From this perspective, the Cramer operations seem to obtain the deflection of the cracked model directly without revealing the procedure used because they do not transform (1) as occurs with Gauss-Jordan and with the methods in Figure 1.

On the other hand, comparing this interpretation with that of Gauss-Jordan, it can be seen that the real deflection is always obtained from partial cracked ones. In the first two cases, the deflection is obtained by adding other previously rectified deflections and in the third case it is obtained by rectifying the external actions.

$$|A| = \frac{I}{I_r}$$

(6)

$$EI, [A]|A_i| = \{M\}$$

(7)

4. Geometrizing the Gauss operations

The drawing rules that geometrise the Gauss operations are exposed by graphically resolving system (8) with an asymmetric matrix. It could belong to the model of Figure 4a formed with partially cracked bars. While $\{M\}$ is represented by the column of boxes in Figure 4n, $[A]$ is represented by a stepped area formed in the same way as in [2], whose factors $a_{ij}$ and $a_{ji}$ are arranged as segments as in Figure 4b. The signs of the segments are opposite to those in (8) to facilitate the graphic resolution. The following describes the numerical and graphic transformation of (8) into (11) and then the obtaining of the deflection.

$$[A]\{\theta\} = \{M\} = \begin{pmatrix} a & b & d_1 \\ b & c & e_1 \\ d & e & f \end{pmatrix} \begin{pmatrix} \theta_A \\ \theta_B \\ \theta_C \end{pmatrix} = \begin{pmatrix} M_A \\ 0 \\ 0 \end{pmatrix}$$

(8)

Figure 3. Visualizing the Cramer determinants.
4.1 By eliminating \( b \).
This is carried out by modifying the coefficients of row 2 according to (9). These operations correspond to a drawing on Figures 4b,n carried out in the following steps: i) obtain \( r_{ab} \) by drawing a dashed line from segment \( b \) (Figure 4c); ii) project segments \( b_1 \) and \( d_1 \) with \( r_{ab} \) on the horizontal (Figure 4d); iii) modify certain segments located in the boxes traversed by line 1-2-3, which starts from the box containing \( b \). The segments are parallel to the line when it "leaves" the box containing the segment; iv) project the segment \( M_A \) (Figure 4n) onto the horizontal with \( r_{ab} \) and place the result with its sign in the lower box (Figure 4o).

\[
\begin{pmatrix}
a & b_i & d_i \\
0 & c - \frac{b}{a} b_i & e_i - \frac{b}{a} d_i \\
d & e & f
\end{pmatrix}
\begin{pmatrix}
M_A \\
0 & M_B \\
0 & M_C
\end{pmatrix}
= \begin{pmatrix}
a & b_i & d_i \\
0 & c' & e'_i \\
d & e & f
\end{pmatrix}
\begin{pmatrix}
M_A^u \\
M_B^u \\
M_C^u
\end{pmatrix}
\tag{9}
\]

4.2 By eliminating \( d \).
Carried out numerically in (10) and graphically as before in Figures 4f,g,h,p.

\[
\begin{pmatrix}
a & b_i & d_i \\
0 & c' & e'_i \\
0 & e - \frac{d}{a} b_i = e' - \frac{d}{a} d_i = f'
\end{pmatrix}
\begin{pmatrix}
M_A^u \\
M_B^u \\
\frac{d}{a} M_A
\end{pmatrix}
= \begin{pmatrix}
a & b_i & d_i \\
0 & c' & e'_i \\
0 & e' & f'
\end{pmatrix}
\begin{pmatrix}
M_A^u \\
M_B^u \\
M_C^u
\end{pmatrix}
\tag{10}
\]

4.3 By eliminating \( e' \).
Similarly, (10) is transformed into (11); the associated drawing is described in Figures 4i,j,k,l,q.

\[
\begin{pmatrix}
a & b_i & d_i \\
0 & c' & e'_i \\
0 & 0 & f'' - \frac{e''}{c} e' = f'''
\end{pmatrix}
\begin{pmatrix}
M_A^u \\
M_B^u \\
M_C^u
\end{pmatrix}
= \begin{pmatrix}
a & b_i & d_i \\
0 & c' & e'_i \\
0 & 0 & f''
\end{pmatrix}
\begin{pmatrix}
M_A^u \\
M_B^u \\
M_C^u
\end{pmatrix}
= \begin{bmatrix} A \end{bmatrix}[M^u]
\tag{11}
\]

4.4 By obtaining the deflection.
In (12) the solution of (1) has been obtained as a function of \((1/EI)\) by adding the partial deflections of Figure 4m, and the drawing representing these operations is shown in Figure 5c. This is done according to [2] by combining the segments of Figure 5a with those of Figure 5b.
Figure 4. Geometrizing a system according to Gauss: a) model; i) geometrizing the stepped area b): by eliminating \( b \) (c,d,e), by eliminating \( d \) (f,g,h) and by eliminating \( e' \) (j,k,l); ii) geometrizing \( M_i \) in the active moments of m) (o,p,q).

\[
\begin{align*}
\theta_a &= \frac{M^*_{a} - b \theta_b - c \theta_c}{a} = \frac{\beta_a}{a} \\
\theta_b &= \frac{M^*_{b} - c' \theta_c}{c'} = \frac{\beta_b}{c'} \\
\theta_c &= \frac{M^*_{c} - \theta_a}{f''} = \frac{\beta_c}{f''}
\end{align*}
\]
5. Geometrizing the Gauss-Jordan operations

The procedure transforms (11) into (14) by eliminating the coefficients \( a_{ij} \) located above the main diagonal and converting \( \{M^a\} \) into \( \{M^\alpha\} \) formed by the active moments of new load states applied on the cracked model. The drawings reproducing this transformation have been made on Figure 6a which combines the segments of Figures 5a,b in another way. The process is described below and then the deflection is determined.

5.1 By eliminating \( d_1 \) and \( e'_1 \)

This is achieved with operations that transform \( \{M^a\} \) of (11) into the vector of (13). The operations can be carried out graphically by modifying \( M^a_B \) and \( M^a_A \) with measures that come from projecting the segments \( e'_1 \) and \( d_1 \) with \( r_C \) on the horizontal (Figure 6b).

5.2 By eliminating \( b_1 \)

Similarly, expression (13) becomes (14) and its graphical operations, similar to those above, are shown in Figure 6c.

\[
\begin{bmatrix}
a & b_1 & 0 \\
0 & c' & 0 \\
0 & 0 & f''
\end{bmatrix}
\begin{bmatrix}
M^*_A - d_c \theta_c \\
M^*_C - e'_1 \theta_C \\
M^*_B - e'_1 \theta_C
\end{bmatrix}
= 
\begin{bmatrix}
M^*_A - d_c \theta_c = M^\alpha_C \\
M^*_B - e'_1 \theta_C = M^\alpha_B \\
M^*_C = \beta_C
\end{bmatrix}
\tag{13}
\]

\[
\begin{bmatrix}
a & 0 & 0 \\
0 & c' & 0 \\
0 & 0 & f''
\end{bmatrix}
\begin{bmatrix}
M^*_A - d_c \theta_c - b_1 \theta_B \\
M^*_B - b_1 \theta_B = M^\alpha_A = \beta_A \\
M^*_C = \beta_C
\end{bmatrix}
\tag{14}
\]

5.3 By obtaining the deflection

The rotations \( \theta_i \) expressed as a function of \( (1/ EI) \) are determined as vertical \( s_i \) segments located on the right side of each box \( a_{ii} \) (Figure 6d) with the following steps: i) define a triangle with the horizontal and vertical segments; ii) draw a line \( s \) starting from the upper left vertex and perpendicular to the hypotenuse of the triangle; iii) The vertical segment \( s'_j \) obtained such as \( s'_j \) in Figure 6b is equal to the rotation associated with the box. This is demonstrated with (15) deduced in this Figure.
Figure 6. Geometrizing a deflection by Gauss-Jordan: a) matrix $[A']$ with vector $\{M'\}$; b) elimination of $d_i$ and $e'_i$; c) elimination of $b_i$; d) obtaining the deflection.

6. Example

The triangular frame in Figure 4a formed with three equal bars five metres long is analysed. The derived system (16), obtained assuming that the bars are cracked according to (17) with $I$ being the original inertia, is resolved numerically and geometrically by freehand using the above procedures on a mesh formed with 4 mm side squares. The drawings are made in three sizes to assess the influence of scale on the results. Sizes I and III have not very comfortable dimensions for drawing since the sides of their squares measure 0.8 and 3.6 cm, respectively.

(16) is transformed into (18) graphically (Figures 7a,d,g) on an area representing Figures 4b,n. Drawings reproducing (12) are shown by size in Figures 7b,e,h, and those reproducing (13-15) are in Figures 7c,f,i (the results are as a function of $1/EI$). In all cases the time required is similar, as the greater number of lines in the former is offset by the care taken in drawing the latter (any small deviation in the slopes of the straight lines $s_i$ influences the results very much). It is also observed that, regardless of the procedure, the larger the drawing is, the shorter the calculation time is required.

$$\frac{\beta_A}{a} = \frac{s'_A}{l} = \theta_A$$  \hspace{1cm} (15)
Figure 7. Geometric resolution of an equilibrium equation system: i) at size I: a) transformation and deflection b) by Gauss and c) by Gauss-Jordan; ii) at size II: d) transformation and deflection e) by Gauss and f) by Gauss-Jordan; iii) at size III: g) transformation and deflection h) by Gauss and i) by Gauss-Jordan

| Table 1. Results (as a function of $M_A/El$) |
|---------------------------------------------|
| Exact | Graphical (size I) | Graphical (size II) | Graphical (size III) |
|       | By G   | By GJ  | By G   | By GJ  | By G   | By GJ  |
| $\theta_A$ | 1.37 | 1.25 | 1.37 | 1.18 | 1.18 | 1.12 | 1.21 |
| $\theta_B$ | -0.16 | -0.18 | -0.18 | -0.15 | -0.18 | -0.18 | -0.12 |
| $\theta_C$ | -0.17 | -0.18 | -0.18 | -0.18 | -0.12 | -0.18 | -0.1 |

Comparing the Gauss (Figure 8a) and Gauss-Jordan (Figure 8b) results, it can be seen that the most accurate results are almost always those obtained using size I, although this does not enable conclusions to be drawn; it can also be seen that the Gauss results seem more accurate, probably because it is easier to draw parallel lines than perpendicular lines freehand (Figure 8c).
Figure 8. Visual comparison between results: i) according to sizes a) by Gauss and b) by Gauss-Jordan; ii) according to the mean values obtained by each method.

7. Conclusions
I) The current numerical procedures that resolve the equilibrium equation systems may be linked to mechanical behaviours of the models in the equilibrium phase.

II) Geometrising the scalar and vectorial magnitudes for numerical analysis procedures reduces application times if they are calculated freehand.

III) Possible lines of research are also pointed to for developing further graphic methods that can analyse other types of structure directly and accurately.

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