Two-sided bounds on the rate of convergence for continuous-time finite inhomogeneous Markov chains

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Abstract

We suggest an approach to obtaining general two-sided bounds on the rate of convergence in terms of special “weighted” norms related to total variation. Some important classes of continuous-time Markov chains are considered: birth-death-catastrophes processes, queueing models with batch arrivals and group services, chains with absorption in zero.

Keywords: continuous-time Markov chains, inhomogeneous Markov chains, ergodicity bounds, special norms

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1. Introduction

The problem of finding sharp bounds on the rate of convergence to the limiting characteristics for Markov chains is very important for at least two following reasons:

(i) Beginning from what point can be the exact characteristics of the process be replaced by the limiting characteristics that are much easier to calculate? To find the bounds on this time moment we need sharp upper (or both upper and lower exact) bounds on the rate of convergence.

(ii) Perturbation bounds play a significant role in applications. It is well known that even for homogeneous chains the best bounds require the corresponding best bounds on the rate of convergence (Kartashov, 1985, 1996; Liu, 2012; Mitrophanov, 2003, 2004). Bounds for general inhomogeneous
Markov chains are also based on the estimates of the rate of convergence in the special weighted norms (Zeifman and Korolev, 2014a).

In this note we deal with the weak ergodicity of finite inhomogeneous continuous-time Markov chains and formulate an algorithm for obtaining sharp upper and lower bounds on the rate of convergence in the special weighted norms related to total variation. Our general approach is closely connected with the notion of logarithmic norm and the corresponding bounds for the Cauchy matrix (Zeifman, 1985, 1995; Granovsky and Zeifman, 1997, 2004; Zeifman et al, 2006, 2013; 2014b), and allows to obtain general upper and lower bounds on the rate of convergence for a countable state space and general initial conditions. On the other hand, if the state space of the chain is finite, then the desired bounds can be obtained in a simple way without this special technique.

Section 2 contains preliminary material. The general result is presented is Section 3. In Sections 4-6 three important classes of continuous-time Markov chains are considered for which it is possible to obtain exact convergence rate estimates.

2. Preliminaries

Let $X = X(t)$, $t \geq 0$ be an inhomogeneous finite continuous-time Markov chain. Let $p_{ij}(s, t) = \Pr\{X(t) = j | X(s) = i\}$, $0 \leq i, j \leq S$, $0 \leq s \leq t$, be the transition probabilities for $X = X(t)$, $p_i(t) = \Pr\{X(t) = i\}$ be its state probabilities, and $p(t) = (p_0(t), p_1(t), \ldots, p_S(t))^T$ be the corresponding probability distribution. Throughout the paper we assume that (in the inhomogeneous case) all intensity functions are locally integrable on $[0, \infty)$.

Let $a_{ij}(t) = q_{ji}(t)$ for $j \neq i$ and let $a_{ii}(t) = -\sum_{j \neq i} a_{ji}(t) = -\sum_{j \neq i} q_{ij}(t)$. The probabilistic dynamics of the process is represented by the forward Kolmogorov system

$$\frac{dp}{dt} = A(t)p(t),$$

where $A(t)$ is the transposed intensity matrix of the process.
Throughout the paper by $\| \cdot \|$ we denote the $l_1$-norm, i.e. $\|x\| = \sum |x_i|$, and $\|B\| = \sup_j \sum_i |b_{ij}|$ for $B = (b_{ij})_{i,j=0}^n$. Let $\Omega$ be the set all stochastic vectors, i.e., vectors with nonnegative coordinates and unit $l_1$-norm. It is well known that the Cauchy problem for differential equation (2) has a unique solution for an arbitrary initial condition, and $p(s) \in \Omega$ implies $p(t) \in \Omega$ for $t \geq s \geq 0$. Hence, we can put $p_0(t) = 1 - \sum_{i \geq 1} p_i(t)$, and obtain from (2) the equation (see detailed discussion in (Granovsky and Zeifman, 2004; Zeifman et al, 2006))

$$\frac{dz}{dt} = B(t)z(t) + f(t),$$

where $f(t) = (a_{10}, a_{20}, \cdots, a_{S0})^T$, $z(t) = (p_1, p_2, \cdots, p_S)^T$,

$$B = \begin{pmatrix} a_{11} - a_{10} & a_{12} - a_{10} & \cdots & a_{1S} - a_{10} \\ a_{21} - a_{20} & a_{22} - a_{20} & \cdots & a_{2S} - a_{20} \\ a_{31} - a_{30} & a_{32} - a_{30} & \cdots & a_{3S} - a_{30} \\ \vdots \\ a_{S1} - a_{S0} & a_{S2} - a_{S0} & \cdots & a_{SS} - a_{S0} \end{pmatrix}. \quad (4)$$

Recall that an inhomogeneous Markov chain $X(t)$ is called weakly ergodic, if $\|p^*(t) - p^{**}(t)\| \to 0$ as $t \to \infty$ for any initial conditions $p^*(0)$, $p^{**}(0)$, where $p^*$, $p^{**}$ are two distributions of state probabilities for $X(t)$. A matrix $H$ is called essentially nonnegative if all its off-diagonal elements are nonnegative.

3. General bounds

Let $D$ be a nonnegative matrix such that

(i) $\|z\|_D = \|Dz\| \geq d\|z\|$ for some positive $d$ and

(ii) $DB(t)D^{-1}$ is essentially nonnegative for any $t \geq 0$.

It is worth noting that

$$p(t) = \left(1 - \sum_{i=1}^S z_i(t) \right),$$

and hence $z(t)$ completely determines the state probabilities for $X(t)$.

Let $DB(t)D^{-1} = H(t) = (h_{ij}(t))_{i,j=1}^S$. Consider the quantities

$$h^*(t) = \max_j \sum_i r_{ij}, \quad h_*(t) = \min_j \sum_i r_{ij} \quad (5)$$

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**Theorem 1.** Let \( X(t) \) be a given finite inhomogeneous Markov chain, and let there exist a matrix \( D \) satisfying (i)–(ii). Then the following bounds hold:

\[
\|z^*(t) - z^{**}(t)\|_{1D} \leq e^{\int_0^t h^*(\tau)\,d\tau} \|z^*(0) - z^{**}(0)\|_{1D},
\]

for any corresponding initial conditions \( p^*(0), p^{**}(0) \), and

\[
\|z^*(t) - z^{**}(t)\|_{1D} \geq e^{\int_0^t h^*(\tau)\,d\tau} \|z^*(0) - z^{**}(0)\|_{1D},
\]

if the initial conditions are such that \( D(z^*(0) - z^{**}(0)) \geq 0 \).

**Proof.** Put \( v(t) = z^*(t) - z^{**}(t) \), and let \( x(t) = Dv(t) \). Then

\[
\frac{dx}{dt} = H(t)x.
\]

First, let \( x(0) \geq 0 \). Then \( x(t) \geq 0 \) for any \( t \geq 0 \), hence \( \|x(t)\| = \sum_i x_i(t) \).

From (3) we obtain that

\[
\frac{d\|x\|}{dt} = \frac{d\sum_i x_i}{dt} = \sum_i \left( \sum_j h_{ij} x_j \right) = \sum_j \left( \sum_i h_{ij} \right) x_j \leq h^*(t) \sum_j x_j = h^*(t)\|x\|.
\]

Then \( \|x(t)\| \leq e^{\int_0^t h^*(\tau)\,d\tau} \|x(0)\| \), if \( x(0) \geq 0 \). Let now \( x(0) \) be arbitrary. Put

\[
x^+_i(0) = \max(x_i(0), 0), \quad x^+_i(0) = (x^+_1(0), \ldots, x^+_n(0))^T,
\]

and \( x^-_i(0) = x(0) - x^+_i(0) \). Then \( x^+_i(0) \geq 0, \quad x^-_i(0) \geq 0, \quad x(0) = x^+_i(0) - x^-_i(0) \), and, moreover, \( \|x(0)\| = \|x^+_i(0)\| + \|x^-_i(0)\| \). Therefore,

\[
\|x(t)\| = \|x^+_i(t) - x^-_i(t)\| \leq \|x^+_i(t)\| + \|x^-_i(t)\| \leq e^{\int_0^t h^*(\tau)\,d\tau} \left( \|x^+_i(0)\| + \|x^-_i(0)\| \right) = e^{\int_0^t h^*(\tau)\,d\tau} \|x(0)\|.
\]

On the other hand, if \( x(0) \geq 0 \), then

\[
\frac{d\|x\|}{dt} = \sum_j \left( \sum_i h_{ij} \right) x_j \geq h^*_i(t) \sum_j x_j = h^*_i(t)\|x\|,
\]

and \( \|x(t)\| \geq e^{\int_0^t h^*_i(\tau)\,d\tau} \|x(0)\| \).

**Remark 1.** If we can find \( D \) such that \( h^*_i(t) = h^*_i(t) \), then Theorem 1 gives sharp bound on the rate of convergence.
Remark 2. Let $X(t)$ be a homogeneous Markov chain. Then the corresponding spectral gap, or decay parameter,

$$
\beta := \sup \{ a > 0 : \| p(t) - \pi \| = O(e^{-at}) \text{ as } t \to \infty \text{ for all } \mu(0) \},
$$

satisfies the inequality $h_s \leq \beta \leq h^*$. This bound makes it possible to find the asymptotic behavior of the spectral gap as the number of states tends to infinity, see, for instance (Granovsky and Zeifman, 1997, 2000, 2005; Van Doorn et al, 2010).

4. Birth-death-catastrophe process

Let $X(t)$ be a birth-death-catastrophe process (BDPC) on the finite state space $\{0, \ldots, S\}$ and let $\lambda_n(t)$, $\mu_{n+1}(t)$, and $\xi_{n+1}(t)$, $n = 0, \ldots, S-1$ be the corresponding birth, death and catastrophe intensities. Then

$$
a_{ij}(t) = \begin{cases} 
\lambda_{i-1}(t), & \text{if } j = i-1, \\
\mu_{i+1}(t), & \text{if } j = i+1 > 1, \\
-(\lambda_i(t) + \mu_i(t) + \xi_i(t)), & \text{if } j = i, \\
\xi_j(t), & \text{if } i = 0, j > 1, \\
\mu_1(t) + \xi_1(t), & \text{if } i = 0, j = 1, \\
0, & \text{otherwise}
\end{cases}
$$

are the transposed intensities of the process. A detailed study of this class of processes can be found in (Zeifman et al, 2013). Then the assumptions (i) and (ii) of Section 3 are fulfilled for the upper triangular matrix

$$
D = \begin{pmatrix}
d_1 & d_1 & d_1 & \cdots & d_1 \\
0 & d_2 & d_2 & \cdots & d_2 \\
0 & 0 & d_3 & \cdots & d_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & d_S
\end{pmatrix}.
$$

Let $\alpha_k(t) = -\sum_i h_{ik}(t)$. Then

$$
\alpha_k(t) = \lambda_k(t) + \mu_{k+1}(t) + \xi_{k+1}(t) - \frac{d_{k+1}}{d_k}\lambda_{k+1}(t) - \frac{d_{k-1}}{d_k}\mu_k(t), \quad k = 0, \ldots, S-1,
$$

\[\tag{11}\]
and

\[ h^*(t) = -\beta^*(t) = -\min_{0 \leq k \leq S-1} \alpha_k(t), \quad h_*(t) = -\beta_*(t) = -\max_{0 \leq k \leq S-1} \alpha_k(t). \]  \tag{12}

Hence, the following statement holds.

**Theorem 2.** Let \( X(t) \) be a finite BDPC, and let there exist positive numbers \( \{d_i\} \) such that

\[ \int_0^\infty \beta^*(t) \, dt = +\infty. \]  \tag{13}

Then \( X(t) \) is weakly ergodic and the following bounds hold:

\[ \|z^*(t) - z^{**}(t)\|_D \leq e^{-\int_0^t \beta^*(\tau) \, d\tau} \|z^*(0) - z^{**}(0)\|_D, \]  \tag{14}

for any corresponding initial conditions \( p^*(0), p^{**}(0) \), and

\[ \|z^*(t) - z^{**}(t)\|_D \geq e^{-\int_0^t \beta_*(\tau) \, d\tau} \|z^*(0) - z^{**}(0)\|_D, \]  \tag{15}

if the initial conditions are such that \( D(z^*(0) - z^{**}(0)) \geq 0 \).

**Example 1.** Let \( \lambda_n(t) = \lambda(t), \mu_{n+1}(t) = (n+1)\mu(t), \) for \( n = 0, \ldots, S-1, \) \( \xi_S(t) \equiv 0, \) and \( \xi_{n+1}(t) = \lambda(t), \) if \( n = 0, \ldots, S-2. \) Put all \( d_i = 1. \) Then in \( (\text{17}) \) all \( \alpha_k(t) = \lambda(t) + \mu(t), \) and Theorem 2 gives the inequality

\[ \|z^*(t) - z^{**}(t)\|_D \leq e^{-\int_0^t (\lambda(\tau) + \mu(\tau)) \, d\tau} \|z^*(0) - z^{**}(0)\|_D, \]  \tag{16}

for any initial conditions. Moreover, if the initial conditions are such that \( D(z^*(0) - z^{**}(0)) \geq 0, \) then

\[ \|z^*(t) - z^{**}(t)\|_D = e^{-\int_0^t (\lambda(\tau) + \mu(\tau)) \, d\tau} \|z^*(0) - z^{**}(0)\|_D. \]  \tag{17}

**Remark 3.** A finite birth-death process with constant rates of birth \( \lambda_k(t) = a \) and \( \mu_{k+1}(t) = b \) was considered in (Granovsky and Zeifman. 1997), where the corresponding \( D \) and sharp \( \beta_* = \beta^* = a+b-2\sqrt{ab}\cos \frac{\pi}{S+1} \to (\sqrt{a}-\sqrt{b})^2 \) as \( S \to \infty \) were obtained.
5. SZK model

Let $X = X(t), t \geq 0$, be a Markov chain with intensities $q_{i,i+k}(t) = \lambda_k(t), q_{i,i-k}(t) = \mu_k(t)$ for any $k > 0$ (SZK model, see Satin et al, 2013). In other words, in terms of queueing theory, we suppose that the arrival rates $\lambda_k(t)$ and the service rates $\mu_k(t)$ do not depend on the length of the queue. In addition, we assume that $\lambda_{k+1}(t) \leq \lambda_k(t)$ and $\mu_{k+1}(t) \leq \mu_k(t)$ for any $k$ and almost all $t \geq 0$. Then

$$A(t) = \begin{pmatrix}
a_{00}(t) & \mu_1(t) & \mu_2(t) & \mu_3(t) & \cdots & \mu_S(t) \\
\lambda_1(t) & a_{11}(t) & \mu_1(t) & \mu_2(t) & \cdots & \mu_S-1(t) \\
\lambda_2(t) & \lambda_1(t) & a_{22}(t) & \mu_1(t) & \cdots & \mu_S-2(t) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda_S(t) & \lambda_S-1(t) & \lambda_S-2(t) & \cdots & \cdots & a_{SS}(t)
\end{pmatrix}, \quad (18)$$

where $a_{ii}(t) = -\sum_{k=1}^{S} \mu_k(t) - \sum_{k=1}^{S-i} \lambda_k(t)$ is the transposed intensity matrix of the process. This class of processes was introduced and studied in our previous papers (Satin et al, 2013, Zeifman et al, 2014b). Then the assumptions (i) and (ii) of Section 3 are fulfilled for upper triangular matrix (10). In this case,

$$DB^{-1}D^{-1} =$$

$$\begin{pmatrix}
a_{11} - \lambda_S & (\mu_1 - \mu_2) \frac{d_2}{d_2} & (\mu_2 - \mu_3) \frac{d_3}{d_2} & \cdots & (\mu_S-1 - \mu_S) \frac{d_S}{d_S} \\
(\lambda_1 - \lambda_S) \frac{d_2}{d_1} & a_{22} - \lambda_S-1 & (\mu_1 - \mu_3) \frac{d_3}{d_2} & \cdots & (\mu_S-2 - \mu_S) \frac{d_S}{d_S} \\
(\lambda_2 - \lambda_S) \frac{d_2}{d_1} & (\lambda_1 - \lambda_S-1) \frac{d_3}{d_2} & a_{33} - \lambda_S-2 & \cdots & (\mu_S-3 - \mu_S) \frac{d_S}{d_S} \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
(\lambda_{S-1} - \lambda_S) \frac{d_2}{d_1} & (\lambda_{S-2} - \lambda_S-1) \frac{d_S}{d_2} & (\lambda_{S-3} - \lambda_S-2) \frac{d_S}{d_3} & \cdots & a_{SS} - \lambda_S
\end{pmatrix}, \quad (19)$$

and $-\alpha_k(t) = \sum_i h_{ik}(t)$ equals to the sum of elements of the $k-$th column of $DB(t)D^{-1}$. Hence, the following statement holds.

**Theorem 3.** Let $X(t)$ be a finite SZK model, and let there exist positive numbers $\{d_i\}$ such that

$$\int_0^\infty \beta^*(t) \, dt = +\infty. \quad (20)$$

Then $X(t)$ is weakly ergodic and the following bounds hold:

$$\|z^*(t) - z^{**}(t)\|_1D \leq e^{-\int_0^t \beta^*(\tau) \, d\tau} \|z^*(0) - z^{**}(0)\|_1D, \quad (21)$$
for any corresponding initial conditions $\mathbf{p}^*(0)$, $\mathbf{p}^{**}(0)$, and
\[
\|z^*(t) - z^{**}(t)\|_D \geq e^{-\int_0^t \beta(\tau) \, d\tau} \|z^*(0) - z^{**}(0)\|_D, \tag{22}
\]
if the initial conditions are such that $D(z^*(0) - z^{**}(0)) \geq 0$.

**Example 2.** Let $\lambda_k(t) = \frac{\lambda(t)}{k}$, $\mu_k = \frac{\mu(t)}{k}$. Let $X(t)$ be a queue-length process for the corresponding queueing model with batch arrivals and group services. Let $d_k = 1$, $k \geq 1$. Then $\alpha_k(t) = \lambda(t) + \mu(t)$, $k \geq 1$, and Theorem 3 gives the inequality
\[
\|z^*(t) - z^{**}(t)\|_D \leq e^{-\int_0^t (\lambda(\tau) + \mu(\tau)) \, d\tau} \|z^*(0) - z^{**}(0)\|_D, \tag{23}
\]
for any initial conditions. Moreover, if the initial conditions are such that $D(z^*(0) - z^{**}(0)) \geq 0$, then
\[
\|z^*(t) - z^{**}(t)\|_D = e^{-\int_0^t (\lambda(\tau) + \mu(\tau)) \, d\tau} \|z^*(0) - z^{**}(0)\|_D. \tag{24}
\]

6. Markov chain with absorption in zero

Let now $X(t)$, $t \geq 0$ be a Markov chain with absorption in zero, i.e., let $q_{00}(t) \equiv 0$. Then
\[
B = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1S} \\
a_{21} & a_{22} & \cdots & a_{2S} \\
a_{31} & a_{32} & \cdots & a_{3S} \\
\cdots \\
a_{S1} & a_{S2} & \cdots & a_{SS}
\end{pmatrix} \tag{25}
\]
is essentially nonnegative itself. Hence, the assumptions (i) and (ii) of Section 3 are fulfilled for the diagonal matrix
\[
D = \begin{pmatrix}
d_1 & 0 & 0 & \cdots & 0 \\
0 & d_2 & 0 & \cdots & 0 \\
0 & 0 & d_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & d_S
\end{pmatrix}. \tag{26}
\]
If $\alpha_k(t) = -\sum_i h_{ik}(t)$, then
\[
\alpha_k(t) = a_{kk}(t) - \sum_{i \neq k} \frac{d_i}{d_k} a_{ik}(t), \quad k = 1, \ldots, S, \tag{27}
\]
and
\[ h^*(t) = -\beta^*(t) = -\min_{1 \leq k \leq S} \alpha_k(t), \quad h_*(t) = -\beta_*(t) = -\max_{1 \leq k \leq S} \alpha_k(t). \tag{28} \]

Hence, the following statement holds.

**Theorem 4.** Let \( X(t) \) be a finite a Markov chain with absorption in zero, and let there exist positive numbers \( \{d_i\} \) such that
\[ \int_0^\infty \beta^*(t) \, dt = +\infty. \tag{29} \]

Then \( X(t) \) is weakly ergodic and the following bounds hold:
\[ \|z^*(t) - z^{**}(t)\|_{1D} \leq e^{-\int_0^t \beta^*(\tau) \, d\tau} \|z^*(0) - z^{**}(0)\|_{1D}, \tag{30} \]
for any corresponding initial conditions \( p^*(0), p^{**}(0) \), and
\[ \|z^*(t) - z^{**}(t)\|_{1D} \geq e^{-\int_0^t \beta_*(\tau) \, d\tau} \|z^*(0) - z^{**}(0)\|_{1D}, \tag{31} \]
if the initial conditions are such that \( z^*(0) - z^{**}(0) \geq 0 \).

**Example 3.** Let now \( X(t) \) be a BDP with birth rates \( \lambda_0(t) \equiv 0, \lambda_k(t) = 2\phi(t), 1 \leq k < S \), and death rates \( \mu_1(t) = 3\phi(t), \mu_k(t) = 6\phi(t), 1 < k < S, \mu_S(t) = 2\phi(t) \). Let \( d_k = 2^{k-1}, 1 \leq k \leq S \). Then \( \alpha_k(t) = \phi(t), 1 \leq k \leq S \), and Theorem 4 gives the inequality
\[ \|z^*(t) - z^{**}(t)\|_{1D} \leq e^{-\int_0^t \phi(\tau) \, d\tau} \|z^*(0) - z^{**}(0)\|_{1D}, \tag{32} \]
for any initial conditions. Moreover, if the initial conditions are such that \( z^*(0) - z^{**}(0) \geq 0 \), that
\[ \|z^*(t) - z^{**}(t)\|_{1D} = e^{-\int_0^t \phi(\tau) \, d\tau} \|z^*(0) - z^{**}(0)\|_{1D}. \tag{33} \]

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