Semimartingale detection and goodness-of-fit tests

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Abstract
In quantitative finance, we often fit a parametric semimartingale model to asset prices. To ensure our model is correct, we must then perform goodness-of-fit tests. In this paper, we give a new goodness-of-fit test for volatility-like processes, which is easily applied to a variety of semimartingale models. In each case, we reduce the problem to the detection of a semimartingale observed under noise. In this setting, we then describe a wavelet-thresholding test, which obtains adaptive and near-optimal detection rates.

1 Introduction
In quantitative finance, we often model asset prices as semimartingales; in other words, we assume prices are given by a sum of drift, diffusion and jump processes. As these models can be difficult to fit to data, we often restrict our attention to a parametric class, of which many have been suggested by practitioners. To verify our choice of parametric class, we must then perform goodness-of-fit tests.

As semimartingale models can be quite complex, there are many potential tests to perform. In the following, we will be interested in testing whether models accurately describe processes such as the volatility, covolatility, vol-of-vol or leverage. We will further be looking for tests which can be shown to obtain good rates of detection against a variety of alternatives.

While many goodness-of-fit tests exist in the literature, fewer have been shown to obtain good detection rates. Those tests which do achieve good rates are generally designed for one type of semimartingale model, and one way of measuring performance.

In the following, we will therefore describe a new goodness-of-fit test for volatility-like processes in semimartingales. Our test can easily be applied
to a wide range of models, including stochastic volatility, jumps and microstructure noise, and obtains good detection rates against both local and nonparametric alternatives.

Our method involves reducing any goodness-of-fit test to one of semimartingale detection: given a series of observations, is the series white noise, or does it contain a hidden semimartingale? We will show how this problem can be solved efficiently, obtaining adaptive and near-optimal detection rates.

We now describe in more detail the problems we consider, as well as relevant previous work. Our goal will be to test the goodness-of-fit of a parametric semimartingale model. Many such models have been described, including simple models such as Black-Scholes or Cox-Ingersoll-Ross; Lévy models such as the generalised hyperbolic or CGMY processes; and stochastic volatility models such as the Heston or Bates models. (For definitions, see Cont and Tankov, 2004; Papapantoleon, 2008.)

In the simplest case, where our observations are known to come from a stationary or ergodic diffusion process, a great many authors have described goodness-of-fit tests. We briefly mention some initial work (Aït-Sahalia, 1996; Corradi and White, 1999; Kleinow, 2002) as well as more recent discussion (González-Manteiga and Crujeiras, 2013; Papanicolaou and Giesecke, 2014; Chen et al., 2015).

In a financial setting, however, even if our model is stationary, we may need to test it against non-stationary alternatives. When observations can come from a non-stationary diffusion, goodness-of-fit tests have been described using the integrated volatility (Corradi and White, 1999), estimated residuals (Lee, 2006; Lee and Wee, 2008; Nguyen, 2010) and marginal density (Aït-Sahalia and Park, 2012). Goodness-of-fit tests also exist for regressions between diffusions (Mykland and Zhang, 2006).

In the following, we will be interested in goodness-of-fit tests which not only detect non-stationary alternatives, but also achieve good detection rates. In this setting, Dette and von Lieres und Wilkau (2003) propose a test which can detect misspecification of the volatility at a rate $n^{-1/4}$ in $L^2$ norm (see also Dette et al., 2006; Podolskij and Ziegler, 2008; Papanicolaou and Giesecke, 2014).

A similar test proposed by Dette and Podolskij (2008) detects alternatives in a fixed direction at the faster rate $n^{-1/2}$, although the authors do not give rates in $L^p$. This test can also be applied to more complex models, including stochastic volatility (Vetter, 2012) and microstructure noise (Vetter and Dette, 2012).

In some volatility testing problems, previous work has described tests which achieve optimal detection rates against nonparametric alternatives (Reiß et al., 2014; Bibinger et al., 2015). However, these tests are specific to the problems considered, and do not assess the goodness-of-fit of general models.
In the following, we will therefore describe a new method of goodness-of-fit testing for volatility-like processes. We will show how our approach applies to a wide variety of semimartingale models, including those with jumps, stochastic volatility and microstructure noise. In each case, we will obtain adaptive detection rates, with near-optimal behaviour not only against alternatives in a fixed direction, but also against nonparametric alternatives.

To construct our tests, we will reduce each goodness-of-fit problem to one of semimartingale detection: we will construct a series of observations $Z_i$, which under the null hypothesis are approximately white noise, and then test whether the $Z_i$ contain a hidden semimartingale $S_t$.

For example, suppose we have a semimartingale
\[ dX_t = b_t \, dt + \sqrt{\mu_t} \, dB_t, \]
where $B_t$ is a Brownian motion, $b_t$ and $\mu_t$ are predictable processes, and we make observations $X_{t_i}$, $i = 0, \ldots, n$, where the times $t_i := i/n$. Further suppose we have a model $\mu(t, X_t)$ for the volatility, and wish to test the hypotheses
\[ H_0 : \mu_t = \mu(t, X_t) \quad \text{vs.} \quad H_1 : \mu_t \text{ unrestricted.} \]

To estimate $\mu_t$, we define the realised volatility estimates
\[ Y_i := n(X_{t_{i+1}} - X_{t_i})^2, \quad i = 0, \ldots, n - 1. \]
Since the scaled increments $\sqrt{n}(X_{t_{i+1}} - X_{t_i})$ are approximately $N(0, \mu_{t_i})$, the observations $Y_i$ have approximate mean $\mu_{t_i}$ and variance $2\mu_{t_i}^2$. Under $H_0$, we thus have that the normalised observations
\[ Z_i := (Y_i - \mu(t_i, X_{t_i}))/\sigma(t_i, X_{t_i}), \quad \sigma^2 := 2\mu^2, \]
are approximately white noise.

Under $H_1$, we instead obtain
\[ Z_i = S_{t_i} + \varepsilon_i, \quad (1) \]
where the semimartingale
\[ S_t := (\mu_t - \mu(t, X_t))/\sigma(t, X_t), \]
and the approximately-centred noises
\[ \varepsilon_i := (Y_i - \mu_{t_i})/\sigma(t_i, X_{t_i}). \]
To test our hypotheses, we must therefore test whether the the series $Z_i$ is approximately white noise, or contains a hidden semimartingale $S_t$.

If the noises $\varepsilon_i$ were independent standard Gaussian, independent of $S_t$, we could consider this a standard detection problem in nonparametric
regression. Conditioning on $S_t$, we could take the semimartingale as fixed, and then apply the methods of Ingster and Suslina (2003), for example.

Under suitable assumptions on the process $S_t$, its sample paths would be almost $\frac{1}{2}$-smooth, and we would thus be able to detect a signal $S_t$ at rate $n^{-1/4}$ in supremum norm, up to log terms. Alternatively, if we wished to detect signals $S_t \propto e_t$, for a fixed direction $e_t$, we could do so at a rate $n^{-1/2}$.

In general, however, the signal $S_t$ may depend on past values of the noises $\varepsilon_t$, and vice versa. We will thus not be able to appeal directly to results in nonparametric regression, and will instead need to use arguments developed specifically for the semimartingale setting.

In the following, we will show that testing problems like (1) can be solved with detection rates similar to those of nonparametric regression. We will further show that many semimartingale goodness-of-fit tests can be described in a form like (1), including models with stochastic volatility, jumps or microstructure noise.

Our approach will be similar to wavelet thresholding (Donoho et al., 1995; Hoffmann et al., 2012); essentially, we will reject the null whenever a suitable wavelet-thresholding estimate of $S_t$ is non-zero. While this method is known to work well in the standard nonparametric setting, we will need to prove new results to apply it to settings like (1).

Our proofs will use a Gaussian coupling derived from Skorokhod embeddings. We note that as our results must apply in a general semimartingale setting, we will not be able to use faster-converging couplings, such as the KMT approximation. We will show, however, that under reasonable moment bounds, a Skorokhod embedding will suffice to achieve the desired detection rates.

Indeed, with this construction we will show our tests detect semimartingales $S_t$ at a rate $n^{-1/4}$ in supremum norm, up to log terms, even when $S_t$ contains finite-variation jumps. Furthermore, our tests will simultaneously detect simpler signals at faster rates; for example, we will be able to detect signals $S_t$ in a fixed direction $e_t$ at a rate $n^{-1/2}$ up to logs, without knowledge of the direction $e_t$.

We will finally show that in each case, the rates obtained are near-optimal. Applying our tests to problems like (1), we will thus be able to construct goodness-of-fit tests for a wide variety of semimartingale models, obtaining adaptive and near-optimal detection rates.

The paper will be organised as follows. In Section 2, we give a rigorous description of the problems we consider, and discuss examples. In Section 3, we then construct our tests, and state our theoretical results. In Section 4, we then give empirical results, and in Section 5, proofs.
2 Semimartingale detection problems

We now describe our concept of a semimartingale detection problem. Our setting will include volatility goodness-of-fit problems like (1), as well as many other semimartingale goodness-of-fit tests.

We begin with some examples of the problems we will consider. In each case, we will describe a semimartingale model with a volatility-like process $\mu_t$. We will wish to test the null hypothesis that $\mu_t$ is given by some known function $\mu(\theta_0, t, X_t)$, for an unknown parameter $\theta_0 \in \Theta$, and an estimable covariate process $X_t \in \mathbb{R}^q$; our alternative hypothesis will be that $\mu_t$ is not given by $\mu$.

To test our hypothesis, we will construct $\mathcal{F}_{t+i+1}$-measurable observations $Y_i$, and a variance function $\sigma^2$. Under the null, and conditional on $\mathcal{F}_{t+i}$, the observations $Y_i$ will have approximate mean and variance $\mu(\theta_0, t_i, X_{t_i})$ and $\sigma^2(\theta_0, t_i, X_{t_i})$. To estimate these means and variances, we will further construct estimates $\hat{\theta}$ and $\hat{X}_i$ of the parameters $\theta_0$ and covariates $X_{t_i}$.

We will then be able to estimate the difference between the observations $Y_i$ and their means $\mu$, scaled according to their variances $\sigma^2$; we will reject the null hypothesis when the size of these scaled differences are large. In Section 3, we describe in detail how we perform such tests, as well as giving theoretical results on their performance.

For now, we proceed with some examples of semimartingale goodness-of-fit problems in this form. Let $B_t$ and $B'_t$ be independent Brownian motions, $\lambda(dx,dt)$ be an independent Poisson random measure with intensity $dxdt$, and $b_t$ and $b'_t$ be predictable locally-bounded processes, and $f_t(x)$ be a predictable function with $\int_{\mathbb{R}} 1 \wedge |f_t(x)|^\beta dx$ locally bounded, for some $\beta \in [0,1)$. Further define times $t'_i := i/n^2$.

We then have the following examples.

**Local volatility** We wish to test a model $\mu$ for $\mu_t$ in the process

$$dX_t = b_t dt + \sqrt{\mu_t} dB_t,$$ (2)

making observations $X_{t_i}$, $i = 0, \ldots, n$. We set $\hat{X}_i := X_{t_i}$, and estimate $\mu_{t_i}$ by the realised volatility (Andersen et al., 2001; Barndorff-Nielsen and Shephard, 2002),

$$Y_i := n(X_{t_{i+1}} - X_{t_i})^2.$$

We then define the variance function $\sigma^2 := 2\mu^2$.

**Jumps** We wish to test a model $\mu$ for $\mu_t$ in the process

$$dX_t = b_t dt + \sqrt{\mu_t} dB_t + \int_{\mathbb{R}} f_t(x) \lambda(dx, dt),$$
making observations \(X_{t_i}, i = 0, \ldots, n\). We set \(\tilde{X}_t := X_{t_i}\) and estimate \(\mu_t\) by the truncated realised volatility (Mancini, 2009; Jacod and Reiß, 2014),

\[ Y_i = g_n(\sqrt{n}(X_{t_{i+1}} - X_{t_i})), \quad g_n(x) = x^2 1_{x^2 < \alpha_n}, \]

for any sequence \(\alpha_n > 0\) satisfying

\[ \log(n) = o(\alpha_n), \quad \alpha_n = o(n^\kappa) \text{ for all } \kappa > 0. \quad (3) \]

We then define the variance function \(\sigma^2 := 2\mu^2\).

**Microstructure noise** We wish to test a model \(\mu\) for \(\mu_t\) in the process

\[ dX_{1,t} = b_t \, dt + \sqrt{\mu_t} dB_t. \]

We make observations

\[ \tilde{X}_{1,i} := X_{1,t_i'} + \varepsilon_i, \quad i = 0, \ldots, n^2, \]

where the noises \(\varepsilon_i\) are measurable in the filtrations \(\mathcal{F}_{t_i'} := \bigcap_{s>t_i'} \mathcal{F}_s\), and satisfy

\[
\begin{align*}
\mathbb{E}[\varepsilon_i | \mathcal{F}_{t_i'}] &= 0, \\
\mathbb{E}[\varepsilon_i^2 | \mathcal{F}_{t_i'}] &= X_{2,t_i'}, \\
\mathbb{E}[|\varepsilon_i|^\kappa | \mathcal{F}_{t_i'}] &\leq C,
\end{align*}
\]

for an Itô semimartingale \(X_{2,t}\) with locally-bounded characteristics, and constants \(\kappa > 8, C > 0\). We estimate \(X_{t_i}\) and \(\mu_{t_j}\) by their pre-averaged counterparts (Jacod et al., 2009; Reiß, 2011),

\[
\begin{align*}
\tilde{X}_{1,j} &:= n^{-1} \sum_{i=0}^{n-1} \tilde{X}_{1,nj+i}, \\
\tilde{X}_{2,j} &:= (2n)^{-1} \sum_{i=0}^{n-1} (\tilde{X}_{1,nj+i+1} - \tilde{X}_{1,nj+i})^2, \\
Y_j &:= \pi^2 (2n^{-1}(\sum_{i=0}^{n-1} \cos(\pi(i + 1/2)/n)\tilde{X}_{1,nj+i})^2 - \tilde{X}_{2,j}).
\end{align*}
\]

We then define the variance function \(\sigma^2 := 2(\mu + \pi^2 X_{2,t_i})^2\).

**Stochastic volatility** We wish to test a model \(\mu\) for \(\mu_t\) in the processes

\[
\begin{align*}
dX_{1,t} &= b_t \, dt + \sqrt{X_{2,t}} \, dB_t, \\
dX_{2,t} &= b'_t \, dt + \sqrt{\mu_t} dB'_t,
\end{align*}
\]

making observations \(X_{1,t_i'}, i = 0, \ldots, n^2\). We define volatility estimates

\[ \tilde{X}_{2,i} := n^2 (X_{1,t_{i+1}} - X_{1,t_i})^2, \quad i = 0, \ldots, n^2 - 1, \]
which we use to estimate $X_{t_j}$ and $\mu_{t_j}$ (Barndorff-Nielsen and Veraart, 2009; Vetter, 2012),

$$\tilde{X}_{1,j} := X_{1,t_j},$$  
$$\tilde{X}_{2,j} := n^{-1} \sum_{i=0}^{n-1} \tilde{X}_{2,nj+i},$$  
$$Y_j := 2\pi^2(n^{-1}(\sum_{i=0}^{n-1} \cos(\pi(i + \frac{1}{2})/n)\tilde{X}_{2,nj+i})^2 - \tilde{X}_{2,j}^2).$$

We then define the variance function $\sigma^2 := 2(\mu + 2\pi^2X_{2,t}^2)^2$.

**Others** Many other models, for example including covolatility or leverage, or combining any of the above features, can be described similarly.

For simplicity, we assume in the following that the times $t_i$ are deterministic and uniform; however, models with uneven or random times that are suitably dense and predictable can be addressed in a similar fashion.

To concisely describe these examples, and others, we will state a set of assumptions on the observations $Y_i$, mean and variance functions $\mu$ and $\sigma^2$, parameters $\theta$, covariates $X_{t_i}$, and estimates $\hat{X}_i$. It will be possible to show that the above models all lie within our assumptions, and we may thus work within these assumptions with some generality.

To begin, we define some notation. Let $\| \cdot \|$ denote any finite-dimensional vector norm; write $a = O(b)$ if $\|a\| \leq C\|b\|$, for some universal constant $C$; and write $a = O_p(b)$ if for each $\varepsilon > 0$, the random variables $a$ and $b$ satisfy $P(\|a\| > C\varepsilon\|b\|) \leq \varepsilon$, for universal constants $C_\varepsilon$.

We stress here that the implied constants $C$ and $C_\varepsilon$ are universal; in statements such as $a = O(1)$, we require the supremum $\sup\|a\|$ over all such $a$ to be bounded. Given a function $f : X \to \mathbb{R}$, we also define the supremum norm $\|f\|_\infty := \sup_{x \in X}|f(x)|$.

Our assumptions are then as follows.

**Assumption 1.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$ be a filtered probability space, with adapted unobserved mean, variance and covariate processes $\mu_t \in \mathbb{R}, \sigma_t^2 \geq 0,$ and $X_t \in \mathbb{R}^q$, respectively. For $0 \leq t \leq t + h \leq 1$, letting $W_t$ denote either of the processes $\mu_t$ or $X_t$, we have

$$W_t = O(1),$$  
$$\mathbb{E}[W_{t+h} - W_t \mid \mathcal{F}_t] = O(h),$$  
$$\mathbb{E}[\|W_{t+h} - W_t\|^2 \mid \mathcal{F}_t] = O(h).$$

For $i = 0, \ldots, n - 1$, we have $\mathcal{F}_{t_i+1}$-measurable estimates $\hat{X}_i$ of $X_{t_i}$, satisfying

$$\mathbb{E}[\|\hat{X}_i - X_{t_i}\|^2 \mid \mathcal{F}_{t_i}] = O(n^{-1}),$$  
$$\mathbb{E}[\|\hat{X}_i - X_{t_i}\|^4 \mid \mathcal{F}_{t_i}] = O(n^{-1}).$$

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We also have $\mathcal{F}_{t+1}$-measurable observations $Y_i$, satisfying
\begin{align}
\mathbb{E}[Y_i | \mathcal{F}_t] &= \mu_t + O(n^{-1/2}), \\
\operatorname{Var}[Y_i | \mathcal{F}_t] &= \sigma^2_t + O(n^{-1/4}), \\
\mathbb{E}[|Y_i|^{4+\varepsilon} | \mathcal{F}_t] &= O(1),
\end{align}
for a constant $\varepsilon > 0$.

Under the null hypothesis $H_0$, we suppose our observations $Y_i$ are described by a parametric model,
\[ \mu_t = \mu(\theta_0, t, X_t), \quad \sigma^2_t = \sigma^2(\theta_0, t, X_t), \]
for known functions $\mu, \sigma^2 : \Theta \times [0, 1] \times \mathbb{R}^q \to \mathbb{R}$, and an unknown parameter $\theta_0 \in \Theta$. We suppose that $\Theta \subseteq \mathbb{R}^p$ is closed, and $\sigma^2$ is positive. We also suppose the functions $\mu$ and $\sigma^2$ are locally Lipschitz in $\theta$, continuously differentiable in $t$, and twice continuously differentiable in $X$. Finally, we suppose we have a good estimate $\hat{\theta}$ of $\theta_0$, satisfying
\[ \hat{\theta} - \theta_0 = O_p(n^{-1/2}). \]

Under the alternative hypothesis $H_1$, we instead allow $\mu_t, \sigma_t$ unrestricted, and require only that $\hat{\theta} = O_p(1)$.

To ensure the examples given above lie within Assumption 1, we must require that the parameter space $\Theta \subseteq \mathbb{R}^p$ be closed, and the model function $\mu$ be locally Lipschitz in $\theta$, continuously differentiable in $t$, and twice continuously differentiable in $X_t$. These conditions should all be satisfied for most common models.

We must further require the semimartingales $X_t$ to be bounded, and have bounded characteristics. In general, this assumption may not hold directly; however, we can assume it without loss of generality using standard localisation arguments.

In Section 5.3, we then check that the above examples satisfy our conditions on the processes $\mu_t, \sigma_t$, and $X_t$; estimates $\hat{X}_i$; and observations $Y_i$. Most of these conditions follow from standard results on stochastic processes; where necessary, higher-moment bounds can be proved using our Lemma 1 below.

To satisfy Assumption 1, it remains to choose an estimate $\hat{\theta}$ of $\theta_0$, having error $O_p(n^{-1/2})$ under $H_0$, and being $O_p(1)$ under $H_1$. While our results are agnostic as to the choice of $\theta$, a simple choice is given by the least-squares estimate
\[ \hat{\theta} := \arg \min_{\theta \in \Theta} \sum_{i=0}^{n-1} (Y_i - \mu(\theta, t_i, \hat{X}_i))^2, \]
which can be found by numerical optimisation. Under standard regularity assumptions for nonlinear regression, this estimate $\hat{\theta}$ can be shown to satisfy
our conditions, arguing for example as in Section 5 of Vetter and Dette (2012).

Finally, we note that in the microstructure noise and stochastic volatility models, we need to make \( n^2 + 1 \) observations of the underlying process \( X_t \) to construct the \( n \) estimates \( Y_i \). We may thus expect to achieve the square-root of any convergence rates given below; such behaviour, however, is common to all approaches to these problems in the literature.

We have thus shown that many different semimartingale goodness-of-fit problems can be described by our Assumption 1. Next, we will describe our solutions to these problems.

3 Wavelet detection tests

To state our tests for the problems given by Assumption 1, we first consider the signal function

\[
S_t(\theta) := (\mu_t - \mu(\theta, t, X_t))/\sigma(\theta, t, X_t).
\]

This function measures the distance of the model mean \( \mu \) from the true mean \( \mu_t \), weighted by the model variance \( \sigma^2 \). Under \( H_0 \), we have

\[
S_t(\hat{\theta}) \approx S_t(\theta_0) = 0,
\]

while under \( H_1 \), we can in general expect \(|S_t(\hat{\theta})|\) to be large. We may thus reject \( H_0 \) whenever an estimate of \( S_t(\hat{\theta}) \) is significantly different from zero.

To estimate the signal \( S_t(\theta) \), we will use wavelet methods. Let \( \varphi \) and \( \psi \) be the Haar scaling function and wavelet,

\[
\varphi := 1_{[0,1]}, \quad \psi := 1_{[0,1/2]} - 1_{[1/2,1]},
\]

and for \( j = 0, 1, \ldots, k = 0, \ldots, 2^j - 1 \), define the Haar basis functions

\[
\varphi_{j,k}(t) := 2^{j/2}\varphi(2^j t - k), \quad \psi_{j,k}(t) := 2^{j/2}\psi(2^j t - k).
\]

We can then describe \( S_t(\theta) \) in terms of its scaling and wavelet coefficients

\[
\alpha_{j,k}(\theta) := \int_0^1 \varphi_{j,k}(t)S_t(\theta) \, dt, \quad \beta_{j,k}(\theta) := \int_0^1 \psi_{j,k}(t)S_t(\theta) \, dt.
\]

To estimate these coefficients, we first pick a resolution level \( J \in \mathbb{N}_0 \), so that \( 2^J \) is of order \( n^{1/2} \). We then estimate the scaling coefficients \( \alpha_{J,k}(\theta) \) by

\[
\hat{\alpha}_{J,k}(\theta) := n^{-1} \sum_{i=0}^{n-1} \varphi_{J,k}(t_i)Z_i(\theta),
\]

where the normalised observations

\[
Z_i(\theta) := (Y_i - \mu(\theta, t_i, \hat{X}_i))/\sigma(\theta, t_i, \hat{X}_i).
\]
We note that for fixed \( \theta \), these estimates can be computed in linear time, as each observation \( Y_i \) contributes to only one coefficient \( \widehat{\alpha}_{J,k}(\theta) \).

To estimate the coefficients \( \alpha_{0,0}(\theta) \) and \( \beta_{j,k}(\theta), \ 0 \leq j < J \), we then perform a fast wavelet transform, obtaining estimates

\[
\widehat{\alpha}_{0,0}(\theta) := \sum_l \widehat{\alpha}_{J,l}(\theta) \int_0^1 \varphi_{J,l} \varphi_{0,0}, \quad \widehat{\beta}_{j,k}(\theta) := \sum_l \widehat{\alpha}_{J,l}(\theta) \int_0^1 \varphi_{J,l} \psi_{j,k}.
\]

We note that efficient implementations of this transformation, running in linear time, are widely available.

To test our hypotheses, we will take the maximum size of these estimated coefficients, producing test statistics

\[
\widehat{T}(\theta) := \max_{0 \leq j < J, k} |\widehat{\alpha}_{0,0}(\theta)|, |\widehat{\beta}_{j,k}(\theta)|.
\]

We will show that under \( H_0 \), \( \widehat{T}(\theta) \) is asymptotically Gumbel distributed, while under \( H_1 \), \( \widehat{T}(\theta) \) will tend to be greater.

**Theorem 1.** Let Assumption 1 hold.

(i) Under \( H_0 \),

\[
a_{2,1}^{-1}(n^{1/2}\widehat{T}(\theta) - b_{2,1}) \overset{d}{\to} G
\]

uniformly, where the constants

\[
a_m := (2 \log(m))^{-1/2}, \quad b_m := a^{-1} - \frac{1}{2}a_m \log(\pi \log(m)),
\]

and \( G \) denotes the standard Gumbel distribution.

(ii) Under \( H_1 \),

\[
\widehat{T}(\theta) - T(\theta) = O_p(n^{-1/2} \log(n)^{1/2})
\]

uniformly, where

\[
T(\theta) := \max_{0 \leq j < J, k} |\alpha_{0,0}(\theta)|, |\beta_{j,k}(\theta)|.
\]

We thus obtain that under \( H_0 \), \( \widehat{T}(\theta) \) concentrates around zero at a rate \( n^{-1/2} \log(n)^{1/2} \). Under \( H_1 \), it concentrates at the same rate around the quantity \( T(\theta) \), which measures the size of the signal \( S_t(\theta) \). We can use this result to construct tests of our hypotheses, and prove bounds on their performance; we first note that for some of our bounds, we will require the following assumption.

**Assumption 2.** The processes \( \mu_t \) and \( X_t \) are Itô semimartingales,

\[
\mu_t = \int_0^t (b^\mu_s) ds + (c^\mu_s)^T dB_s + \int_\mathbb{R} f^\mu_s(x) \lambda(dx, ds),
\]

\[
X_{i,t} = \int_0^t (b^X_{i,s}) ds + (c^X_{i,s})^T dB_s + \int_\mathbb{R} f^X_{i,s}(x) \lambda(dx, ds),
\]

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for a Brownian motion $B_s \in \mathbb{R}^{q+1}$, independent Poisson random measure $\lambda(dx,ds)$ having compensator $dxds$, predictable processes $b^{X}_{s}, b^{X}_{t,s}, c^{X}_{s}, c^{X}_{t,s} = O(1)$, and predictable functions $f^{X}_{s}(x), f^{X}_{t,s}(x)$ satisfying $\int_{\mathbb{R}} 1 \wedge |f_{s}(x)| \, dx = O(1)$.

Under Assumption 2, we thus have that $\mu_t$ and $X_t$ are Itô semimartingales, with bounded characteristics and finite-variation jumps. This assumption holds for many common financial models, if necessary after a suitable localisation step. Using this condition, we are now ready to describe our tests, and bound their performance.

**Theorem 2.** Let Assumption 1 hold, and for $\alpha \in (0, 1)$, define the Gumbel quantile

$$q_{n,\alpha} := -a_{2J} \log(-\log(1-\alpha)) + b_{2J},$$

and critical region

$$C_{n,\alpha} := \{n^{1/2} \hat{T}(\theta) > q_{n,\alpha}\}.$$  

(i) Under $H_0$, we have $P[C_{n,\alpha}] \to \alpha$ uniformly.

(ii) Under $H_1$, let $M_n > 0$ be a fixed sequence with $M_n \to \infty$. If $E_n$ is one of the events:

(a) $\|S(\hat{\theta})\|_{\infty} \geq M_n n^{-1/4} \log(n)^{1/2}$, given also Assumption 2; or

(b) $\max_{0 \leq j \leq J, k} 2^{j/2} |\int_{2^{-j-k}}^{2^{-j-k+1}} S_{t}(\hat{\theta}) \, dt| \geq M_n n^{-1/2} \log(n)^{1/2}$;

we have $P[E_n \setminus C_{n,\alpha}] \to 0$ uniformly.

We thus obtain that the test which rejects $H_0$ on the event $C_{n,\alpha}$ is of asymptotic size $\alpha$, and under Assumption 2, can detect signals $S_t(\hat{\theta})$ at the rate $n^{-1/4} \log(n)^{1/2}$ in supremum norm. We further have that, even without Assumption 2, our test can detect a signal whenever the size of its mean over a dyadic interval is large.

In particular, if $S_t(\hat{\theta}) \propto e_t$ for some non-zero deterministic process $e_t$, then $e_t$ must have non-zero integral over some dyadic interval $2^{-j}[k, k+1)$. We deduce that our test can detect signals in the fixed direction $e_t$ at the rate $n^{-1/2} \log(n)^{1/2}$, without prior knowledge of $e_t$.

We can further show that these detection rates are near-optimal.

**Theorem 3.** Let Assumption 1 hold, and $\delta_n > 0$ be a fixed sequence with $\delta_n \to 0$. If $E_n$ is one of the events:

(i) $\|S(\hat{\theta})\|_{\infty} \geq \delta_n n^{-1/4}$, given also Assumption 2; or

(ii) $\max_{k} 2^{j_n/2} |\int_{2^{-j_n-k}}^{2^{-j_n-k+1}} S_{t}(\hat{\theta}) \, dt| \geq \delta_n n^{-1/2}$, for some $j_n = 0, \ldots, J$.  

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then no sequence of critical regions $C_n$ can satisfy
\[ \limsup_n P[C_n] < 1 \]
uniformly over $H_0$, and
\[ P[E_n \setminus C_n] \to 0 \]
uniformly over $H_1$.

We thus conclude that our goodness-of-fit tests achieve the near-optimal detection rate of $n^{-1/4} \log(n)^{1/2}$ against general nonparametric alternatives, in a wide variety of semimartingale models. This result is already a significant improvement over previous work; we note that similar methods do not establish near-optimality for the procedures of Dette and von Lieres und Wilkau (2003), for example, where the corresponding lower bound would be $n^{-1/3}$.

Furthermore, we have shown that our method simultaneously provides near-optimal detection rates against alternatives which are easier to detect, including the case where the signal $S_t(\hat{\theta})$ lies in a fixed direction $e_t$. We may thus achieve good detection rates in a fully nonparametric setting, without sacrificing performance against fixed alternatives.

\section{Finite-sample tests}

We next consider the empirical performance of our tests. As convergence to the Gumbel distribution can be quite slow, in the following, we will consider a bootstrap version of our tests, which will be more accurate in finite samples.

The general procedure is as follows. First, we estimate the parameters $\theta$ from the data, using some estimate $\hat{\theta}$. Next, we simulate many sets of observations $Y_i^{(j)}$ from the null hypothesis, with parameters chosen by $\hat{\theta}$. Any components of the null hypothesis not described by $\theta$, such as drift or jump processes, are set to zero.

For each set of simulated observations $Y_i^{(j)}$, we then compute a parameter estimate $\hat{\theta}^{(j)}$, and statistic $\hat{T}^{(j)}(\hat{\theta}^{(j)})$. Finally, we reject the null hypothesis if the original statistic $T(\hat{\theta})$ is larger than the $(1 - \alpha)$-quantile of the simulated statistics $\hat{T}^{(j)}(\hat{\theta}^{(j)})$.

We now perform some simple Monte Carlo experiments on these tests. We will compare our tests to those of Dette and von Lieres und Wilkau (2003), Dette et al. (2006) and Dette and Podolskij (2008), using the same methodology as Dette and Podolskij. As in that paper, we will generate Monte Carlo observations in the local volatility setting (2). We will then use our tests to evaluate the goodness-of-fit of various parametric models for the volatility.
In each case, we consider receiving \( n = 100, 200 \) or 500 observations, and constructing confidence tests at the \( \alpha = 5\% \) or 10\% level. We then generate 1,000 realisations of simulated data, compare our statistic against 1,000 bootstrap samples in each realisation, and report the proportion of runs in which the null hypothesis is rejected.

In our tests, we set the resolution level \( J := \lfloor \log_2(n)/2 \rfloor \), and use the least-squares parameter estimates \( \hat{\theta} \) given by (7). As the models we consider will be linear in the parameters \( \theta \), we will be able to compute these estimates in closed form, as linear regressions.

Table 1 then gives the observed rejection probabilities of our tests in two models: a constant volatility model, where \( \mu(x, t, \theta) = \theta \); and a proportional volatility model, where \( \mu(x, t, \theta) = \theta x^2 \). In each case, we give results for our tests under a variety of null and alternative hypotheses.

We note the hypotheses tested are the same as in Tables 1–4 of Dette and Podolskij (2008), as well as Table 3 of Dette and von Lieres und Wilkau (2003), and Tables 3.1 and 3.4 of Dette et al. (2006). We may thus directly compare the performance of our tests to those given in previous work.

We find that in both models, our tests have good coverage under the null hypothesis, and reliably reject under the alternative hypothesis. The power of our tests is competitive with previous work under the constant volatility model, and generally improves upon previous work under the proportional volatility model.

We conclude that our tests not only achieve good theoretical detection rates, but also provide strong finite-sample performance. They may thus be recommended for many different goodness-of-fit problems, whether previously discussed in the literature, or newly described by our more general assumptions.

5 Proofs

We now give proofs of our results. In Section 5.1 we will state some technical results, in Section 5.2 give our main proofs, and in Section 5.3 prove our technical results.

5.1 Technical results

We first state the technical results we will require. Our main technical result will be a central limit theorem for martingale difference sequences, bounding the exponential moments of the distance from Gaussian.

Lemma 1. Let \( (\Omega, \mathcal{F}, (\mathcal{F}_j)_{j=0}^n, \mathbb{P}) \) be a filtered probability space, and let \( X_i, i = 0, \ldots, n - 1 \), be \( \mathcal{F}_{i+1} \)-measurable real random variables. Suppose that for
| $n$ | 100 | 200 | 500 |
|-----|-----|-----|-----|
| $\alpha$ | 5% | 10% | 5% | 10% | 5% | 10% |
| Constant volatility, null, $\mu_t = 1$ |
| $b_t = 0$ | 0.048 | 0.105 | 0.056 | 0.101 | 0.035 | 0.089 |
| $b_t = 2$ | 0.055 | 0.114 | 0.057 | 0.103 | 0.044 | 0.084 |
| $b_t = X_t$ | 0.056 | 0.101 | 0.041 | 0.093 | 0.037 | 0.092 |
| $b_t = 2 - X_t$ | 0.048 | 0.095 | 0.052 | 0.105 | 0.051 | 0.100 |
| $b_t = tX_t$ | 0.038 | 0.094 | 0.060 | 0.101 | 0.063 | 0.111 |
| Constant volatility, alternative, $b_t = X_t$ |
| $\sqrt{\mu_t} = 1 + X_t$ | 0.777 | 0.840 | 0.898 | 0.932 | 0.976 | 0.985 |
| $\sqrt{\mu_t} = 1 + \sin 5X_t$ | 0.964 | 0.977 | 0.997 | 0.999 | 1.000 | 1.000 |
| $\sqrt{\mu_t} = 1 + X_t \exp t$ | 0.954 | 0.975 | 0.987 | 0.994 | 0.999 | 0.999 |
| $\sqrt{\mu_t} = 1 + X_t \sin 5t$ | 0.851 | 0.908 | 0.970 | 0.982 | 0.994 | 0.995 |
| $\sqrt{\mu_t} = 1 + tX_t$ | 0.742 | 0.796 | 0.883 | 0.914 | 0.951 | 0.972 |
| Proportional volatility, null, $\mu_t = X_t^2$ |
| $b_t = 0$ | 0.062 | 0.119 | 0.044 | 0.090 | 0.043 | 0.087 |
| $b_t = 2$ | 0.073 | 0.120 | 0.056 | 0.106 | 0.043 | 0.081 |
| $b_t = X_t$ | 0.070 | 0.115 | 0.055 | 0.100 | 0.043 | 0.098 |
| $b_t = 2 - X_t$ | 0.053 | 0.085 | 0.055 | 0.100 | 0.034 | 0.081 |
| $b_t = tX_t$ | 0.070 | 0.106 | 0.062 | 0.123 | 0.045 | 0.106 |
| Proportional volatility, alternative, $b_t = 2 - X_t$ |
| $\mu_t = 1 + X_t^2$ | 0.602 | 0.673 | 0.700 | 0.766 | 0.844 | 0.884 |
| $\mu_t = 1$ | 0.832 | 0.871 | 0.927 | 0.951 | 0.979 | 0.991 |
| $\mu_t = 5|X_t|^{3/2}$ | 0.580 | 0.669 | 0.672 | 0.760 | 0.854 | 0.902 |
| $\mu_t = 5|X_t|$ | 0.896 | 0.932 | 0.963 | 0.974 | 0.995 | 0.998 |
| $\mu_t = (1 + X_t)^2$ | 0.831 | 0.878 | 0.894 | 0.929 | 0.964 | 0.979 |

Table 1: Observed rejection probabilities for bootstrap test.
some $\kappa \geq 1$,

\[
\mathbb{E}[X_i \mid \mathcal{F}_i] = 0, \\
\sum_{i=0}^{n-1} \mathbb{E}[|X_i|^{4\kappa} \mid \mathcal{F}_i] = O(n^{1-2\kappa}).
\]

(i) If also

\[
\mathbb{E}[\sum_{i=0}^{n-1} \mathbb{E}[X_i^2 \mid \mathcal{F}_i] - 1^{2\kappa} \mid \mathcal{F}_0] = O(n^{-\kappa}),
\]

then on a suitably-extended probability space, we have real random variables $\xi$, $\eta$ and $M$, independent of $\mathcal{F}$ given $\mathcal{F}_n$, such that

\[
\sum_{i=0}^{n-1} X_i = \xi + \eta;
\]

$\xi$ is standard Gaussian given $\mathcal{F}_0$; we have

\[
\mathbb{E}[|\eta|^{4\kappa} \mid \mathcal{F}_0] = O(n^{-\kappa});
\]

for $u \in \mathbb{R}$,

\[
\mathbb{E}[\exp(u\eta - \frac{1}{2}u^2 M) \mid \mathcal{F}_0] \leq 1;
\]

and $M \geq 0$ satisfies

\[
\mathbb{E}[M^{2\kappa} \mid \mathcal{F}_0] = O(n^{-\kappa}). \tag{8}
\]

(ii) For random variables $c_i = O(1)$, let $v_c := \sum_{i=0}^{n-1} c_i X_i$. Then on a suitably-extended probability space, we have a constant $A = O(1)$ and real random variable $M$, independent of $\mathcal{F}$ given $\mathcal{F}_n$, such that

\[
\sup_c \mathbb{E}[|v_c|^{4\kappa} \mid \mathcal{F}_0] = O(1);
\]

for $u \in \mathbb{R}$,

\[
\sup_c \mathbb{E}[\exp(u v_c - \frac{1}{2}u^2 (A + M)) \mid \mathcal{F}_0] \leq 1;
\]

and $M \geq 0$ satisfies (8).

We will also need the following result on combining exponential moment bounds.

**Lemma 2.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, with real random variables $(X_i)_{i=0}^{n-1}$ and $M$. Suppose that for $u \in \mathbb{R}$,

\[
\mathbb{E}[\exp(uX_i - \frac{1}{2}u^2 M)] = O(1),
\]

and $M = O_p(r_n)$ for some rate $r_n > 0$. Then

\[
\max_i |X_i| = O_p(r_n^{1/2} \log(n)^{1/2}).
\]
Our next technical result will bound the moments of our observations $Y_i$, and their normalisations $Z_i(\theta)$. The result will be stated using the Hölder spaces $C^s$, defined as follows. Given a function $f : X \to \mathbb{R}$, for suitable $X \subseteq \mathbb{R}^d$, we define the 1-Hölder norm

$$
\|f\|_{C^1} := \|f\|_\infty \vee \sup_{x,y \in X} |f(x) - f(y)|/\|x - y\|,
$$

and the 2-Hölder norm

$$
\|f\|_{C^2} := \begin{cases} 
\|f\|_\infty \vee \max_{i=1}^d \|\nabla f_i\|_{C^1}, & f \text{ is differentiable,} \\
\infty, & \text{otherwise.}
\end{cases}
$$

We also say $f$ is $C^s$ if $\|f\|_{C^s} < \infty$.

**Lemma 3.** Under $H_0$ or $H_1$, suppose the $\hat{X}_i = O(1)$, and $\Theta$ is bounded.

(i) For fixed $i$ and $Y_i$, the variables $Z_i(\theta)$ are $C^1$ functions of $\theta$ and $\hat{X}_i$, with Hölder norm $O(1 + |Y_i|)$.

(ii) The variables $S_t(\theta)$ are $C^1$ functions of $\theta$, $t$, $\mu_t$ and $X_t$, and for fixed $\theta$ and $t$, also $C^2$ functions of $\mu_t$ and $X_t$, both with Hölder norm $O(1)$.

(iii) For $\theta \in \Theta$, we have

$$
\mathbb{E}[Z_i(\theta) \mid \mathcal{F}_{t_i}] = S_{t_i}(\theta) + O(n^{-1/2}),
$$

$$
\mathbb{E}[|Z_i(\theta)|^{4+\varepsilon} \mid \mathcal{F}_{t_i}] = O(1),
$$

and under $H_0$, also

$$
\mathbb{E}[Z_i(\theta_0)^2 \mid \mathcal{F}_{t_i}] = 1 + O(n^{-1/4}).
$$

(iv) Define times

$$
s_k := \left\lceil n2^{-J}k \right\rceil / n, \quad k = 0, \ldots, 2^J.
$$

Then

$$
\max_k n^{-1/2} \sum_{i=n s_k}^{n s_{k+1}} Y_i^2 = O_p(1).
$$

Finally, we will need a result controlling the behaviour of the processes $S_t(\theta)$ under Assumption 2.

**Lemma 4.** Under $H_1$, suppose $\Theta$ is bounded, let $\hat{\Theta}_n \subseteq \Theta$ be a sequence of finite sets, of size $O(n^\kappa)$ for some $\kappa \geq 0$, and let $\delta_n = O(n^{-1/2})$. Given Assumption 2, we have

$$
S_t(\theta) = \tilde{S}_t(\theta) + \overline{S}_t(\theta),
$$

where the processes $\tilde{S}_t(\theta)$ and $\overline{S}_t(\theta)$ are as follows.
(i) We have
\[ \sup_{\theta \in \Theta_n, |s-t| \leq \delta_n} |\tilde{S}_s(\theta) - \tilde{S}_t(\theta)| = O_p(n^{-1/4} \log(n)^{1/2}). \]

(ii) In \( L^2([0,1]) \), let \( P_J f \) denote the orthogonal projection of \( f \) onto the subspace spanned by the scaling functions \( \varphi_{J,k} \), and define the remainder \( R_J f := f - P_J f \). Then
\[ \sup_{\theta \in \Theta_n} \| R_J \tilde{S}(\theta) \|_\infty = O_p(n^{-1/4} \log(n)^{1/2}). \]

(iii) We have a random variable \( N \in \mathbb{N} \), and random times \( 0 = \tau_0 < \cdots < \tau_N = 1 \), such that the processes \( S_t(\theta), \theta \in \Theta_n \), are constant on intervals \( [\tau_i, \tau_{i+1}) \), \( [\tau_{N-1}, \tau_N] \), and
\[ P[\min_i (\tau_{i+1} - \tau_i) < \delta_n] \to 0. \]

5.2 Main proofs

We may now proceed with our main proofs. We first prove Theorem 1, beginning with a lemma controlling the variance of our estimated scaling coefficients \( \hat{\alpha}_{J,k}(\theta) \).

Lemma 5. For \( k = 0, \ldots, 2^J - 1, \theta \in \Theta \), define scaling-coefficient variance terms
\[ \tilde{\alpha}_{J,k}(\theta) := n^{-1} \sum_{i=0}^{n-1} \varphi_{J,k}(t_i)(Z_i(\theta) - \mathbb{E}[Z_i(\theta) | \mathcal{F}_{t_i}])), \]

(i) Under \( H_0 \), suppose the \( \hat{X}_i = O(1) \). Then on a suitably-extended probability space, we have a filtration \( (\mathcal{G}_k)_{k=0}^{2^J} \), and \( \mathcal{G}_{k+1} \)-measurable real random variables \( \xi_k, \eta_k, M_k \), such that
\[ n^{1/2} \tilde{\alpha}_{J,k}(\theta_0) = \xi_k + \eta_k; \]
the variables \( \xi_k \) are standard Gaussian given \( \mathcal{G}_k \);
\[ \mathbb{E}[\exp(u\xi_k - \frac{1}{2}u^2 M_k) | \mathcal{G}_k] \leq 1; \]
and the variables \( M_k \geq 0 \) satisfy
\[ \mathbb{E}[M_k^{2+\varepsilon/2} | \mathcal{G}_k] = O(n^{-1/2+\varepsilon/8}). \]  

(ii) Under \( H_1 \), suppose \( \Theta \) is bounded, and the \( \hat{X}_i = X_i \). We then have constants \( A_k = O(1) \), and on a suitably-extended probability space, a filtration \( (\mathcal{G}_k)_{k=0}^{2^J} \) and real random variables \( M_k \), such that
\[ \sup_{\theta \in \Theta} \mathbb{E}[\exp(u^{1/2} \tilde{\alpha}_{J,k}(\theta) - \frac{1}{2}u^2(A_k + M_k)) | \mathcal{G}_k] \leq 1; \]
the variables \( M_k \geq 0 \) satisfy (10); and the \( \tilde{\alpha}_{J,k}(\theta) \) and \( M_k \) are \( \mathcal{G}_{k+1} \)-measurable.
Proof. We first prove part (i), and argue by induction on \( k \). Let \( G_0 = F_0 \), and suppose that for \( i = 0, \ldots, k - 1 \) we have constructed, on an extended probability space, \( \sigma \)-algebras \( G_{i+1} \), and random variables \( \xi_i, \eta_i, M_i \) satisfying our conditions. We suppose also that \( G_k \) has been chosen to be independent of \( F \) given \( F_{s_k} \), where the times \( s_k \) are given by (9); we note this condition is trivially satisfied for \( G_0 \).

We can then write

\[
n^{1/2} \tilde{\alpha}_{J,k}(\theta_0) = \sum_{i=n s_k}^{n s_{k+1}-1} \zeta_i,
\]

where the \( m := n(s_{k+1} - s_k) \) summands

\[
\zeta_i := n^{-1/2} 2^{J/2}(Z_i(\theta_0) - \mathbb{E}[Z_i(\theta_0) \mid F_{t_i}]).
\]

To compute the moments of the \( \zeta_i \), we may apply Lemma 3(iii), noting that since we are only interested in \( \theta = \theta_0 \), we may assume \( \Theta \) is bounded. We thus have

\[
\mathbb{E}[\zeta_i \mid F_{t_i}, G_k] = 0,
\]

\[
\sum_{i=n s_k}^{n s_{k+1}-1} \mathbb{E}[\zeta_i^2 \mid F_{t_i}, G_k] = 1 + O(m^{-1/2}),
\]

\[
\sum_{i=n s_k}^{n s_{k+1}-1} \mathbb{E}[|\zeta_i|^{4+\varepsilon} \mid F_{t_i}, G_k] = O(m^{-(1+\varepsilon/2)}),
\]

using also that the \( \zeta_i \) are independent of \( G_k \) given \( F_{t_i} \).

We may therefore apply Lemma 1(i) to the variables \( n^{1/2} \tilde{\alpha}_{J,k}(\theta_0) \). On a further-extended probability space, we obtain random variables \( \xi_k, \eta_k, M_k \) satisfying the conditions of part (i), independent of \( F \) given \( G_k \) and \( F_{s_{k+1}} \). Defining \( G_{k+1} \) to be the \( \sigma \)-algebra generated by \( G_k, F_{s_{k+1}}, \xi_k, \eta_k \) and \( M_k \), we deduce that \( G_{k+1} \) satisfies the conditions of our inductive hypothesis. By induction, we conclude that part (i) of our result holds.

To prove part (ii), we argue similarly, noting that the random variables

\[
n^{1/2} \tilde{\alpha}_{J,k}(\theta) = \sum_{i=n s_k}^{n s_{k+1}-1} c_i(\theta) \tilde{\zeta}_i,
\]

where the \( F_{t_{i+1}} \)-measurable summands

\[
\tilde{\zeta}_i := n^{-1/2} 2^{J/2}(Y_i - \mathbb{E}[Y_i \mid F_{t_i}]),
\]

and the \( F_{t_i} \)-measurable coefficients

\[
c_i(\theta) := 1/\sigma(\theta, t_i, X_{t_i}).
\]

As the function \( \sigma \) is continuous and positive, and \( \theta \) and \( X_t \) are bounded, we have the variables \( c_i(\theta) = O(1) \). We may thus apply Lemma 1(ii), producing random variables \( A_k, M_k \) satisfying the conditions of part (ii). The result then follows as before. \( \square \)
We now prove a lemma bounding the variance of our estimated scaling and wavelet coefficients $\hat{\alpha}_{0,0}(\theta), \hat{\beta}_{j,k}(\theta)$.

**Lemma 6.** Suppose the $\tilde{X}_i = O(1)$, and for $j = 0, \ldots, J-1$, $k = 0, \ldots, 2^j - 1$ and $\theta \in \Theta$, define the wavelet-coefficient variance terms

$$
\tilde{\beta}_{j,k}(\theta) := n^{-1} \sum_{i=0}^{n-1} \psi_{j,k}(t_i)(Z_i(\theta) - \mathbb{E}[Z_i(\theta) \mid F_i]).
$$

Similarly define scaling-coefficient variance terms $\tilde{\alpha}_{0,0}(\theta)$ using $\varphi_{0,0}$.

(i) Under $H_0$, suppose $\tilde{\theta} - \theta_0 = O(n^{-1/2})$. Then on a suitably-extended probability space, we have real random variables $\tilde{\xi}_{j,k}, \tilde{n}_{j,k}, \tilde{\upsilon}_{j,k}$ such that

$$
n^{1/2} \tilde{\alpha}_{0,0}(\tilde{\theta}) = \tilde{\xi}_{-1,0} + \tilde{n}_{-1,0} + \tilde{\upsilon}_{-1,0};
$$

$$
n^{1/2} \tilde{\beta}_{j,k}(\tilde{\theta}) = \tilde{\xi}_{j,k} + \tilde{n}_{j,k} + \tilde{\upsilon}_{j,k};
$$

the $\tilde{\xi}_{j,k}$ are independent standard Gaussian; and for some $\varepsilon' > 0$,

$$
\max_{j,k} |\tilde{n}_{j,k}| = O_p(n^{-\varepsilon'}), \quad \max_{j,k} 2^{j/2} |\tilde{\upsilon}_{j,k}| = O_p(1).
$$

(ii) Under $H_1$, suppose $\tilde{\theta} \in \tilde{\Theta}$ is bounded. Then

$$
\sup_{j,k,\theta \in \Theta} |\tilde{\alpha}_{0,0}(\theta)|, |\tilde{\beta}_{j,k}(\theta)| = O_p(n^{-1/2} \log(n))^{1/2}).
$$

**Proof.** We will consider the wavelet-coefficient variance terms $\tilde{\beta}_{j,k}(\theta)$; we note we may include scaling-coefficient variance terms $\tilde{\alpha}_{0,0}(\theta)$ similarly. To prove part (i), we then apply Lemma 5(i). We obtain a filtration $\mathcal{G}_i$, and variables $M_i, \xi_i$ and $\eta_i$ as in the statement of the lemma. Since

$$
\tilde{\beta}_{j,k}(\theta) = \sum_l b_{j,k,l} \tilde{\alpha}_{j,l}(\theta),
$$

where the coefficients

$$
b_{j,k,l} := \int_{0}^{1} \psi_{j,k}(\phi_{j,l}),
$$

we have

$$
n^{1/2} \tilde{\beta}_{j,k}(\tilde{\theta}) = \tilde{\xi}_{j,k} + \tilde{n}_{j,k} + \tilde{\upsilon}_{j,k},
$$

for terms

$$
\tilde{\xi}_{j,k} := \sum_l b_{j,k,l} \xi_l, \quad \tilde{n}_{j,k} := \sum_l b_{j,k,l} \eta_l;
$$

and

$$
\tilde{\upsilon}_{j,k} := n^{1/2} (\tilde{\beta}_{j,k}(\tilde{\theta}) - \tilde{\beta}_{j,k}(\theta_0)).
$$

We first describe the terms $\tilde{\xi}_{j,k}$. Since the $\xi_i$ are jointly centred Gaussian, so are the $\tilde{\xi}_{j,k}$. Furthermore, we have

$$
\text{Cov}[\tilde{\xi}_{j,k}, \tilde{\xi}_{j',k'}] = \sum_l b_{j,k,l} b_{j',k',l} = \int_{0}^{1} \left( \sum_l b_{j,k,l} \phi_{j,l} \right) \left( \sum_l b_{j',k',l} \phi_{j',l} \right) = \int_{0}^{1} \psi_{j,k} \psi_{j',k'} = 1_{(j,k) = (j',k')},
$$

(11)
We deduce that the $\tilde{\xi}_{j,k}$ are independent standard Gaussian.

We next bound the $\tilde{\eta}_{j,k}$. Setting

$$M := \max_l M_l,$$

we have that

$$\mathbb{E}[M^{2+\varepsilon/2}] \leq \sum_l \mathbb{E}[M_l^{2+\varepsilon/2}] = O(n^{-\varepsilon/8}),$$

so $M = O_p(n^{-\varepsilon'})$ for some $\varepsilon' > 0$. Using (11), we also have

$$\mathbb{E}[\exp(u\tilde{\eta}_{j,k} - \frac{1}{2}u^2 M)] \leq \mathbb{E}\left[\prod_l \exp(u b_{j,k,l} \eta_l - \frac{1}{2}u^2 b_{j,k,l}^2 M_l)\right] \leq 1.$$

The desired result follows by applying Lemma 2.

Finally, we control the $\tilde{\upsilon}_{j,k}$. Since we are only interested in $\theta = \theta_0, \hat{\theta}$, we may assume $\Theta$ is bounded. For $\theta, \theta' \in \Theta$, $|\theta - \theta'| = O(n^{-1/2})$, we then have

$$\sup_{j,k,\theta,\theta'} 2^{1/2} |\tilde{\beta}_{j,k}(\theta) - \tilde{\beta}_{j,k}(\theta')|$$

$$= \max_{j,k} O(n^{-3/2} 2^{1/2} \sum_{i=0}^{n-1} \psi_{j,k}(t_i) (1 + |Y_i|),$$

using Lemma 3(i),

$$= O(n^{-1/2}) (1 + \max_k n^{-1/2} \sum_{i=n_{s_k}+1}^{n_{s_k+1}} |Y_i|)$$

$$= O(n^{-1/2}) (1 + (\max_k n^{-1/2} \sum_{i=n_{s_k}+1}^{n_{s_k+1}} Y_i^2)^{1/2}),$$

by Cauchy-Schwarz,

$$= O_p(n^{-1/2}),$$

using Lemma 3(iv) We deduce that

$$\sup_{j,k} 2^{1/2} |\tilde{\upsilon}_{j,k}| = O_p(1).$$

To prove part (ii), we first claim we may assume the $\hat{X}_i = X_{i_t}$ To prove the claim, we define terms

$$Z'_{i}(\theta) := (Y_i - \mu(\theta, t_i, X_{i_t}))/\sigma(\theta, t_i, X_{i_t}),$$

and

$$\tilde{\beta}'_{j,k}(\theta) := n^{-1} \sum_{i=0}^{n-1} \psi_{j,k}(t_i)(Z'_{i}(\theta) - \mathbb{E}[Z'_{i}(\theta) \mid \mathcal{F}_{i}]).$$

We then have

$$\sup_{j,k,\theta \in \Theta} |\tilde{\beta}_{j,k}(\theta) - \tilde{\beta}'_{j,k}(\theta)|$$

$$= O(n^{-1}) \max_{j,k} \sum_{i=0}^{n-1} \psi_{j,k}(t_i) (1 + |Y_i|) \|\hat{X}_i - X_{i_t}\|,$$
using Lemma 3(i),

\[ = O(n^{-1/2})(\max_{j,k} n^{-1}\sum_{i=0}^{n-1} \psi_{j,k}(t_i)(1 + Y_i^2))^{1/2} \]

\[ \times (\sum_{i=0}^{n-1} ||\hat{X}_i - X_{t_i}||^2)^{1/2}, \]

by Cauchy-Schwarz,

\[ = O_p(n^{-1/2})(1 + \max_{k} n^{-1/2} \sum_{i=ns_k}^{n-1} Y_i^2)^{1/2}, \]

since \( E[\sum_{i=0}^{n-1} ||\hat{X}_i - X_{t_i}||^2] = O(1) \),

\[ = O_p(n^{-1/2}), \]

using Lemma 3(iv).

We may thus assume the \( \hat{X}_i = X_{t_i} \), and so apply Lemma 5(ii). On an extended probability space, we obtain a filtration \( G_t \), constants \( A_t = O(1) \), and variables \( M_t \) as in the statement of the lemma. Setting \( M := \max_l (A_l + M_l) \),

we obtain that \( M = O_p(1) \), and

\[ \sup_{\theta \in \Theta} E[\exp(u n^{1/2} \tilde{\beta}_{j,k}(\theta) - \frac{1}{2} u^2 M)] \leq 1, \]

arguing as in part (i). Letting \( \Theta_n \) denote a \( n^{-1/2} \)-net for \( \Theta \subset \mathbb{R}^p \), of size \( O(n^{p/2}) \), we thus have

\[ \max_{j,k,\theta \in \Theta_n} |\tilde{\beta}_{j,k}(\theta)| = O_p(n^{-1/2} \log(n^{p/2})) \]

\[ = O_p(n^{-1/2} \log(n^{1/2})), \]

using Lemma 2.

Next, for any \( \theta \in \Theta \), we have a point \( \hat{\theta} \in \Theta_n \) with \( \theta - \hat{\theta} = O(n^{-1/2}) \). Using (12), we deduce that

\[ \sup_{j,k,\theta \in \Theta} |\tilde{\beta}_{j,k}(\theta) - \tilde{\beta}_{j,k}(\hat{\theta})| = O_p(n^{-1/2}). \]

We conclude that

\[ \sup_{j,k,\theta \in \Theta} |\tilde{\beta}_{j,k}(\theta)| = O_p(n^{-1/2} \log(n^{1/2})). \]

Next, we prove a lemma bounding the bias of our estimated scaling and wavelet coefficients \( \hat{\alpha}_{0,0}(\theta), \tilde{\beta}_{j,k}(\theta) \).

**Lemma 7.** Suppose the \( \hat{X}_i = O(1) \), and for \( j = 0, \ldots, J-1, k = 0, \ldots, 2^J - 1 \) and \( \theta \in \Theta \), define the wavelet-coefficient bias terms

\[ \overline{\beta}_{j,k}(\theta) := n^{-1} \sum_{i=0}^{n-1} \psi_{j,k}(t_i)E[Z_i(\theta) | \mathcal{F}_{t_i}] - \beta_{j,k}(\theta), \]

Similarly define scaling-coefficient bias terms \( \overline{\alpha}_{0,0}(\theta) \) using \( \varphi_{0,0} \).
(i) Under $H_0$, suppose $\hat{\theta} - \theta_0 = O(n^{-1/2})$. Then

$$\max_{j,k} |\bar{\sigma}_{0,0}(\hat{\theta})|, 2^{j/2} |\bar{\beta}_{j,k}(\hat{\theta})| = O_p(n^{-1/2}).$$

(ii) Under $H_1$, suppose $\Theta$ is bounded. Then

$$\sup_{j,k,\theta \in \Theta} |\bar{\sigma}_{0,0}(\theta)|, |\bar{\beta}_{j,k}(\theta)| = O_p(n^{-1/2}).$$

Proof. We will bound the wavelet-coefficient bias terms $\bar{\beta}_{j,k}(\theta)$; we note we may include the scaling-coefficient bias terms $\bar{\sigma}_{0,0}(\theta)$ similarly. For $t \in [0, 1]$, define $t := \lfloor nt \rfloor / n$, and set

$$\bar{\beta}_{j,k}(\theta) := \int_0^1 \psi_{j,k}(t)(S_t(\theta) - S_t) dt.$$ 

In each part (i) and (ii), we will show that $\bar{\beta}_{j,k}(\theta)$ is close to $\bar{\beta}_{j,k}(\theta)$, which is small.

We note that in either part we may assume $\Theta$ is bounded, since in part (i), we are only interested in $\theta = \theta_0, \hat{\theta}$. We then have

$$|\bar{\beta}_{j,k}(\theta) - \bar{\beta}_{j,k}(\theta)| \leq n^{-1} \sum_{i=0}^{n-1} |\psi_{j,k}(t_i)||E[Z_i(\theta) | \mathcal{F}_{t_i}] - S_{t_i}(\theta)|$$

$$+ \int_0^1 |\psi_{j,k}(t) - \psi_{j,k}(\hat{\theta})||S_t(\theta)| dt$$

$$= O(n^{-1/2}2^{-j/2}),$$

using Lemmas 3(ii) and (iii). It thus remains to bound the $\bar{\beta}_{j,k}(\theta)$.

To prove part (i), we note that

$$\bar{\beta}_{j,k}(\theta_0) = \sum_{i=0}^{n-1} \zeta_{i,j,k},$$

where the $\mathcal{F}_{t_{i+1}}$-measurable summands

$$\zeta_{i,j,k} := \int_{t_i}^{t_{i+1}} \psi_{j,k}(t)(S_t(\theta_0) - S_{t_i}(\theta_0)) dt.$$ 

Using Lemma 3(ii) and Taylor’s theorem, we also have that

$$S_t(\theta_0) - S_{t_i}(\theta_0) = c_i (\mu_t - \mu_{t_i}) + d_i^T (X_t - X_{t_i})$$

$$+ O(|\mu_t - \mu_{t_i}|^2 + \|X_t - X_{t_i}\|^2 + n^{-1}),$$

for bounded $\mathcal{F}_{t_i}$-measurable random variables $c_i \in \mathbb{R}, d_i \in \mathbb{R}^q$.

We deduce that

$$E[\zeta_{i,j,k} | \mathcal{F}_{t_i}] = O(n^{-2}2^{j/2}),$$

and similarly

$$\text{Var}[\zeta_{i,j,k} | \mathcal{F}_{t_i}] \leq \text{Var}[\zeta_{i,j,k}^2 | \mathcal{F}_{t_i}] = O(n^{-3}2^j).$$
Furthermore, for fixed $j$ and $k$, we have that all but $O(n^{2-j})$ of the $\zeta_{i,j,k}$ are almost-surely zero. We thus have

$$E[\beta_{j,k}(\theta_0)^2] = O(n^{-2}).$$

We deduce that

$$E[\max_{j,k} \beta_{j,k}(\theta_0)^2] \leq \sum_{j,k} E[\beta_{j,k}(\theta_0)^2] \leq O(n^{-2}) \sum_j 2^j = O(n^{-3/2}),$$

so $\max_{j,k}|\beta_{j,k}(\theta_0)| = O_p(n^{-3/4})$. We also have

$$\beta_{j,k}(\theta_0) - \beta_{j,k}(\hat{\theta}) = O(n^{-1/2}2^{-j/2}),$$

using Lemma 3(ii). We conclude that

$$\max_{j,k,\theta \in \Theta} |\beta_{j,k}(\theta)| = O_p(n^{-1/2}).$$

To prove part (ii), using Lemma 3(ii), we have

$$S_{\hat{\theta}}(\theta) - S_t(\theta) = O(|\mu_{\hat{\theta}} - \mu_t| + \|X_{\hat{\theta}} - X_t\| + n^{-1}).$$

We deduce that

$$\sup_{j,k,\theta \in \Theta} |\beta_{j,k}(\theta)| = O(1) \sup_{j,k} \int_0^1 |\psi_{j,k}(t)|(|\mu_{\hat{\theta}} - \mu_t| + \|X_{\hat{\theta}} - X_t\| + n^{-1}) dt$$

$$= O(1)(\sup_{j,k} \int_0^1 \psi_{j,k}(t) dt)^{1/2} \times (\int_0^1 (|\mu_{\hat{\theta}} - \mu_t|^2 + \|X_{\hat{\theta}} - X_t\|^2 + n^{-2}) dt)^{1/2},$$

by Cauchy-Schwarz,

$$= O_p(n^{-1/2}),$$

since $\int_0^1 \psi_{j,k}(t) dt = 1$, and

$$E[\int_0^1 (|\mu_{\hat{\theta}} - \mu_t|^2 + \|X_{\hat{\theta}} - X_t\|^2 + n^{-2}) dt] = O(n^{-1}).$$

Using (13), we conclude that

$$\sup_{j,k,\theta \in \Theta} |\beta_{j,k}(\theta)| = O_p(n^{-1/2}).$$
We can now prove our limit theorem for the statistic \( \hat{T}(\hat{\theta}) \).

**Proof of Theorem 1.** We first note that our estimated scaling and wavelet coefficients are equivalently given by

\[
\hat{\alpha}_{0,0}(\theta) = n^{-1} \sum_{i=0}^{n-1} \varphi_{0,0}(t) Z_i(\theta), \quad \hat{\beta}_{j,k}(\theta) = n^{-1} \sum_{i=0}^{n-1} \psi_{j,k}(t) Z_i(\theta).
\]

We may thus make the variance-bias decomposition

\[
\hat{\alpha}_{0,0}(\theta) - \alpha_{0,0}(\theta) = \tilde{\alpha}_{0,0}(\theta) + \alpha_{0,0}(\theta),
\]

\[
\hat{\beta}_{j,k}(\theta) - \beta_{j,k}(\theta) = \tilde{\beta}_{j,k}(\theta) + \beta_{j,k}(\theta),
\]

where the terms \( \tilde{\alpha}_{0,0}, \tilde{\beta}_{j,k} \) and \( \beta_{j,k} \) are defined by Lemmas 6 and 7. We will proceed to bound the distribution of \( \hat{T}(\hat{\theta}) \) using these lemmas.

We begin by showing we may assume the estimated covariates \( \hat{X}_i = O(1) \). We note that

\[
E[\max_i \|\hat{X}_i\|^2] \leq E[\sup_t \|X_t\|^2] + \sum_i E[\|\hat{X}_i - X_{i,t}\|^2] = O(1),
\]

so \( \max_i \|\hat{X}_i\| = O_p(1) \). For a constant \( R > 0 \), define the variables

\[
\tilde{X}_i := \begin{cases} 
\hat{X}_i, & \text{if } \|\hat{X}_i\| \leq R, \\
X_{i,t}, & \text{otherwise.}
\end{cases}
\]

Then as \( R \to \infty \), the probability that the \( \hat{X}_i \) and \( \tilde{X}_i \) agree tends to one, uniformly in \( n \). It thus suffices to prove our results replacing the \( \hat{X}_i \) with the \( \tilde{X}_i \); equivalently, we may assume the \( \hat{X}_i = O(1) \).

We now prove part (i). Since \( \hat{\theta} - \theta_0 = O_p(n^{-1/2}) \), we may similarly assume \( \hat{\theta} - \theta_0 = O(n^{-1/2}) \). Let \( J_2 = \lfloor J/2 \rfloor \), and write

\[
\hat{T}(\theta) = \max(\overline{T}(\theta), \overline{T}(\theta)),
\]

where the terms

\[
\overline{T}(\theta) := \max_{0 \leq j < J_2} |\hat{\alpha}_{0,0}(\theta)|, |\hat{\beta}_{j,k}(\theta)|,
\]

\[
\overline{T}(\theta) := \max_{J_2 \leq j < J} |\hat{\beta}_{j,k}(\theta)|.
\]

Under \( H_0 \), using Lemmas 6(i) and 7(i), we can then write

\[
n^{1/2} \overline{T}(\hat{\theta}) = \max_{0 \leq j < J_2} |\tilde{\xi}_{j,k}| + O_p(1),
n^{1/2} \overline{T}(\hat{\theta}) = \max_{J_2 \leq j < J} |\tilde{\xi}_{j,k}| + O_p(n^{-\varepsilon'}),
\]

for some \( \varepsilon' > 0 \), and independent standard Gaussians \( \tilde{\xi}_{j,k} \).
By standard Gumbel limits, we also have
\[ a_{2J}^{-1}(\max_{0 \leq j < J, k} |\tilde{\xi}_{j,k}| - b_{2J}) \xrightarrow{d} G, \]
we note that in the second limit, we may use the constants \( a_{2J} \) and \( b_{2J} \), rather than \( a_{2J-2J} \) and \( b_{2J-2J} \), as the difference is negligible. We deduce that
\[ P[\hat{T}(\hat{\theta}) = \tilde{T}(\hat{\theta})] \rightarrow 1, \]
and so
\[ a_{2J}^{-1}(n^{1/2}\hat{T}(\hat{\theta}) - b_{2J}) = a_{2J}^{-1}(n^{1/2}\tilde{T}(\hat{\theta}) - b_{2J}) + o_p(1) \]
\[ \xrightarrow{d} G. \]

Next, we prove part (ii). As before, since \( \hat{\theta} = O_p(1) \), we may assume \( \hat{\theta} = O(1) \), and hence that \( \Theta \) is bounded. Using Lemmas 6(ii) and 7(ii), we then have
\[ \hat{T}(\hat{\theta}) - T(\hat{\theta}) = O(1) \max_{0 \leq j < J, k} |\hat{\alpha}_{0,0}(\hat{\theta}) - \alpha_{0,0}(\hat{\theta})|, |\hat{\beta}_{j,k}(\hat{\theta}) - \beta_{j,k}(\hat{\theta})| \]
\[ = O_p(n^{-1/2} \log(n)^{1/2}). \]

Finally, we note that the rates of convergence proved depend only upon the bounds assumed on the inputs. They therefore hold uniformly over models satisfying our assumptions.

Next, we will prove our results on test coverage and detection rates.

**Proof of Theorem 2.** We first note that part (i) is immediate from Theorem 1(i). To prove part (ii), we consider separately the cases (a) and (b). In each case, we will prove that with probability tending to one, the event \( E_n \) implies
\[ T(\hat{\theta}) \geq M'_n n^{-1/2} \log(n)^{1/2}, \]
for a fixed sequence \( M'_n \rightarrow \infty \). The result will then follow from Theorem 1(ii).

In case (a), we note that arguing as in Theorem 1, we may assume \( \Theta \) is bounded. Let \( \Theta_n \) be an \( n^{-1/4} \)-net for \( \Theta \), of size \( O(n^{1/4}) \), and \( \hat{\Theta} \) be an element of \( \Theta_n \) satisfying
\[ \hat{\theta} - \hat{\theta} = O(n^{-1/4}). \]

Using Lemma 3(ii), we have
\[ S_t(\hat{\theta}) = S_t(\hat{\Theta}) + O(n^{-1/4}), \]

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so on \( E_n \),

\[
\|S(\hat{\theta})\|_\infty \geq \|S(\hat{\theta})\|_\infty - O(n^{-1/4}) \geq M_n n^{-1/4} \log(n)^{1/2} / 2,
\]

for large \( n \). We may thus assume further that \( \hat{\theta} \in \Theta_n \).

We then apply Lemma 4, obtaining processes \( S_i(\theta), \bar{S}_i(\theta) \), and times \( \tau_i \).

On the event \( E_n \), for some point \( u \in [0, 1] \), we have

\[
|S_u(\hat{\theta})| \geq M_n n^{-1/4} \log(n)^{1/2}.
\]

We thus have \( u \in [\tau_i, \tau_i+1) \) for some \( i < N-1 \), or \( u \in [\tau_i, \tau_{i+1}] \) for \( i = N-1 \).

From Lemma 4(iii), with probability tending to one we also have

\[
\tau_{i+1} - \tau_i \geq 2^{1-J},
\]

and so there exists a point \( v \in [\tau_i + 2^{-J}, \tau_i + 1 - 2^{-J}], |u - v| \leq 2^{-J} \).

We deduce that with probability tending to one,

\[
|a_{0,0}(\hat{\theta})| + \sum_{j=0}^{J-1} 2^{j/2} |\beta_{j,2^{-j}[2^j v]}(\hat{\theta})|
\]

\[
= |a_{0,0}(\hat{\theta})| \varphi_{0,0}(v) + \sum_{0 \leq j < J} |\beta_{j,k}(\hat{\theta})| \psi_{j,k}(v)|
\]

\[
\geq |a_{0,0}(\hat{\theta})| \varphi_{0,0}(v) + \sum_{0 \leq j < J} |\beta_{j,k}(\hat{\theta})| \psi_{j,k}(v)|
\]

\[
= |P_J S_u(\hat{\theta})|,
\]

writing the projection \( P_J \) in terms of the wavelet functions \( \psi_{j,k} \),

\[
\geq |S_u(\hat{\theta})| - |S_u(\hat{\theta}) - \bar{S}_u(\hat{\theta})| - |R_J S_u(\hat{\theta})|
\]

\[
\geq |S_u(\hat{\theta})| - |\bar{S}_u(\hat{\theta}) - \bar{S}_u(\hat{\theta})| - |R_J S_u(\hat{\theta})|
\]

since \( \bar{S}_i(\hat{\theta}) \) is constant within a distance \( 2^{-J} \) of \( v \),

\[
\geq M_n n^{-1/4} \log(n)^{1/2} / 2,
\]

using Lemmas 4(i) and (ii). We deduce that

\[
T(\hat{\theta}) \geq \max(|a_{0,0}(\hat{\theta})|, |\beta_{j,2^{-j}[2^j v]}(\hat{\theta})| : j = 0, \ldots, J-1),
\]

\[
\geq 2^{-(J+3)/2} \left( |a_{0,0}(\hat{\theta})| + \sum_{j=0}^{J-1} 2^{j/2} |\beta_{j,2^{-j}[2^j v]}(\hat{\theta})| \right)
\]

\[
\geq M_n' n^{-1/2} \log(n)^{1/2},
\]

for a sequence \( M'_n \rightarrow \infty \).

In case (b), on the event \( E_n \), we likewise have

\[
|a_{0,0}(\hat{\theta})| + \sum_{j=0}^{j_n-1} 2^{j/2} |\beta_{j,2^{-j}[2^{j} - 2^j k_n]}(\hat{\theta})|
\]

\[
\geq |P_{j_n} S_{2^{-j_n} k_n}(\hat{\theta})|
\]

\[
= 2^{j_n} \int_{2^{-j_n} k_n}^{2^{-j_n}(k_n+1)} S_i(\hat{\theta}) \, dt
\]

\[
\geq M_n 2^{j_n/2} n^{-1/2} \log(n)^{1/2},
\]

for some \( j_n = 0, \ldots, J \) and \( k_n \). The result then follows as in part (i). \( \square \)
Finally, we can prove our lower bound on detection rates.

**Proof of Theorem 3.** In each case (i) and (ii), we will reduce the statement to a known testing inequality. We will consider the model

\[ Y_i := \delta_n^{1/2} n^{-1/2} 2^{j_n} (B_{t_i \vee \tau} - B_\tau) + \varepsilon_i, \]

where \( B_t \) is an adapted Brownian motion, the independent \( \mathcal{F}_{t_i+1} \)-measurable variables \( \varepsilon_i \) are standard Gaussian given \( \mathcal{F}_{t_i} \), \( \tau \in [0,1] \) is to be defined, and in case (i) we set \( j_n := J \). It can be checked that this model satisfies our assumptions.

Under \( H_0 \), we set \( \tau := 1 \), so we have mean and variance functions

\[ \mu := 0, \quad \sigma^2 := 1. \]

Under \( H_1 \), we instead set \( \tau := t_m \), where \( m := \lfloor n(1 - 2^{-j_n}) \rfloor \). We then have

\[ S_t = \delta_n^{1/2} n^{-1/2} 2^{j_n} (B_{t \vee \tau} - B_\tau), \]

so in case (i),

\[ \mathbb{P}[E_n] = \mathbb{P}[\|S\|_\infty \geq \delta_n n^{-1/4}] \to 1. \]

Similarly, in case (ii),

\[ \mathbb{P}[E_n] \geq \mathbb{P}[2^{j_n/2} \int_{1-2^{-j_n}}^1 S_t \, dt \geq \delta_n n^{-1/2}] \to 1. \]

It remains to show that no sequence of critical regions \( C_n \) can satisfy \( \limsup_n \mathbb{P}[C_n] < 1 \) under \( H_0 \), and \( \mathbb{P}[C_n] \to 1 \) under \( H_1 \). We note that under \( H_0 \), we have \( Y \sim N(0, I) \), while under \( H_1 \), \( Y \sim N(0, I + \delta_n \Sigma) \), for a covariance matrix

\[ \Sigma_{k,l} = 0 \lor 2^{2j_n} (k \land l - m)/n^2. \]

As \( \Sigma \) is non-negative definite, and has Frobenius norm \( O(1) \), the result follows from Lemma 2.1 of Munk and Schmidt-Hieber (2010).

5.3 Technical proofs

We now give proofs of our technical results, beginning with a demonstration that our examples satisfy our assumptions.

**Lemma 8.** The examples in Section 2 satisfy the conditions of Assumption 1.

**Proof.** As in Section 2, we may assume our conditions on \( \mu, \sigma \) and \( \hat{\theta} \) are satisfied. It thus remains to establish the conditions (4)–(6) for each of the examples.
Local volatility By standard localisation arguments, we may assume condition (4), as well as that $b_t, \mu_t^{-1} = O(1)$. Condition (5) is trivial. To establish condition (6), we make the decomposition

$$Y_i = (Z_{1,i} + Z_{2,i} + Z_{3,i})^2,$$

where

$$Z_{1,i} := \sqrt{n} \int_{t_i}^{t_{i+1}} \mu_t^{-1} dB_t,$$
$$Z_{2,i} := \sqrt{n} \int_{t_i}^{t_{i+1}} (\sqrt{\mu_t} - \sqrt{\mu_{t_i}}) dB_t,$$
$$Z_{3,i} := \sqrt{n} \int_{t_i}^{t_{i+1}} b_t dt.$$

We then have that, for $p > 0$,

$$Z_{1,i} \mid \mathcal{F}_{t_i} \sim N(0, \mu_{t_i}),$$
$$\mathbb{E}[|Z_{2,i}|^p \mid \mathcal{F}_{t_i}] = O(n^{-p/2}),$$

using Burkholder-Davis-Gundy, and

$$|Z_{3,i}| = O(n^{-1/2}).$$

The desired bounds follow using Cauchy-Schwarz.

**Jumps** By localisation, we may again assume (4), as well as the bounds $b_t, \mu_t^{-1}, \int_{\mathbb{R}} 1 \wedge |f_t(x)|^\beta dx = O(1)$. Condition (5) is again trivial. For (6), we write

$$Y_i = g_n(Z_{1,i} + Z_{2,i} + Z_{3,i} + Z_{4,i} + Z_{5,i}),$$

where

$$Z_{1,i} := \sqrt{n} \int_{t_i}^{t_{i+1}} \sqrt{\mu_t} dB_t,$$
$$Z_{2,i} := \sqrt{n} \int_{t_i}^{t_{i+1}} f_t(x) \lambda(dx, dt),$$
$$Z_{3,i} := \sqrt{n} \int_{t_i}^{t_{i+1}} f_t(x) \lambda(dx, dt),$$
$$Z_{4,i} := \sqrt{n} \int_{t_i}^{t_{i+1}} (\sqrt{\mu_t} - \sqrt{\mu_{t_i}}) dB_t,$$
$$Z_{5,i} := \sqrt{n} \int_{t_i}^{t_{i+1}} b_t dt,$$

and

$$A_t := \{x \in \mathbb{R} : |f_t(x)| \leq n^{-1/2+\delta'}\},$$

for some sufficiently small $\delta' > 0$.

We then have that, for $p = 2, 4, 16,$ and some $\epsilon' > 0$,

$$Z_{1,i} \mid \mathcal{F}_{t_i} \sim N(0, \mu_{t_i}),$$
$$\mathbb{E}[|Z_{2,i}|^p \mid \mathcal{F}_{t_i}] = O(n^{-1/2-\epsilon'}),$$
by repeated application of Burkholder-Davis-Gundy,
\[ \mathbb{E}[Z_1,i Z_2,i \mid \mathcal{F}_t] = 0, \]
by Itô’s Lemma,
\[ \mathbb{E}[|Z_4,i|^p \mid \mathcal{F}_t] = O(n^{-p/2}), \]
by Burkholder-Davis-Gundy, and
\[ |Z_5,i| = O(n^{-1/2}). \]
Letting \( Z_i := Z_{1,i} + Z_{2,i} + Z_{4,i} + Z_{5,i}, \) and \( Y'_i := Z_{2,i} \), we deduce that
\[ \mathbb{E}[Y'_i \mid \mathcal{F}_t] = \mu'_i + O(n^{-1/2}), \]
\[ \mathbb{E}[|Y'_i|^2 \mid \mathcal{F}_t] = \sigma'_2 + \mu'^2_i + O(n^{-1/4}), \]
\[ \mathbb{E}[|Y'_i|^8 \mid \mathcal{F}_t] = O(1). \]
It thus remains to control the difference between \( Y_i \) and \( Y'_i \).
We first note that for any \( p \geq 2 \), small enough \( q > 1 \), and \( \alpha_n \) as in (3),
\[ \mathbb{E}[|Z_i|^p 1_{Z_i^2 > \alpha_n}] = O(n^{-1/2}) + O(1)\mathbb{E}[|Z_i|^p 1_{Z_i^2 > \alpha_n}] \]
\[ = O(n^{-1/2}) + O(1)\mathbb{P}[Z_i^2 > \alpha_n]^{1/q}, \]
using Hölder’s inequality,
\[ = O(n^{-1/2}), \]
using a Gaussian tail bound, Markov’s inequality, and that \( \alpha_n \) grows super-logarithmically. We also have
\[ \mathbb{P}[Z_{3,i} \neq 0 \mid \mathcal{F}_t] = O(n^{-1/2-\delta'}), \]
using our assumptions on \( f_t(x) \). As \( \alpha_n \) grows sub-polynomially, the required bounds follow also for \( Y_i \).

**Microstructure noise** As before, by localisation we may assume condition (4), as well as the boundedness of characteristic processes. To show (5), we first note that
\[ \tilde{X}_{1,j} - X_{1,t_j} = W_{1,j} + W_{2,j}, \]
where
\[ W_{1,j} := n^{-1} \sum_{i=0}^{n-1} \varepsilon_{i,nj+i}, \]
\[ W_{2,j} := n^{-1} \sum_{i=0}^{n-1} (X_{1,t'_n,j+i} - X_{1,t_j}). \]
For $p = 2, 4$, we then have

$$E[|W_{1,j}|^p \mid \mathcal{F}_{t_j}] = O(n^{-p/2}),$$

using Lemma 1(ii), and

$$E[|W_{2,j}|^p \mid \mathcal{F}_{t_j}] = O(n^{-p/2}),$$

using Burkholder-Davis-Gundy. We deduce that

$$E[|\hat{X}_{1,j} - X_{1,t_j}|^p \mid \mathcal{F}_{t_j}] = O(n^{-p/2}).$$

By a similar method, decomposing $\hat{X}_{2,j}$ into a sum of nuisance terms and martingale difference sequences, the same bound holds also for $\hat{X}_{2,j}$.

To show (6), we write

$$Y_j = 2(Z_{1,j} + Z_{2,j} + Z_{3,j})^2 - \pi^2 \hat{X}_{2,j},$$

using integration by parts, where

$$Z_{1,j} := \sqrt{n} \int_{t_j}^{t_j+1} \sin(\pi n(t - t_j)) \sqrt{\mu_t} dB_t$$
$$+ \pi n^{-1/2} \sum_{i=0}^{n-1} \cos(\pi(i + 1/2)/n) \varepsilon_{nj+i},$$

$$Z_{2,j} := \sqrt{n} \int_{t_j}^{t_j+1} \sin(\pi n(t - t_j)) b_t dt,$$

$$Z_{3,j} := \pi(n^{-1/2} \sum_{i=0}^{n-1} \cos(\pi(i + 1/2)/n) X_{1,t'_{nj+i}}$$
$$- n^{3/2} \int_{t_j}^{t_j+1} \cos(\pi n(t - t_j)) X_{1,t} dt).$$

We then have

$$E[2Z_{1,j}^2 \mid \mathcal{F}_{t_j}] = \mu_{t_j} + \pi^2 X_{2,t_j} + O(n^{-1/2}),$$

by direct computation,

$$E[4Z_{1,j}^4 \mid F_{t_j}] = 3(\mu_{t_j} + \pi^2 X_{2,t_j})^2 + O(n^{-1/4}),$$

using Lemma 1(i),

$$E[|Z_{1,j}|^\kappa \mid \mathcal{F}_{t_j}] = O(1),$$

using Lemma 1(ii),

$$|Z_{2,j}| = O(n^{-1/2}),$$

and for $p > 0$,

$$E[|Z_{3,j}|^p \mid \mathcal{F}_{t_j}] = O(n^{-p}),$$

using Lemma 1(ii). The desired results follow.
**Stochastic volatility** Using integration by parts, we can make a decomposition

\[ Y_j = 2(Z_{1,j} + Z_{2,j} + Z_{3,j} + Z_{4,j})^2 - 2\pi^2 \hat{X}_{2,j}^2, \]

where

\[ Z_{1,j} := \sqrt{n} \int_{I_j} \sin(\pi(n(t - t_j))) \sqrt{\mu_t} dB_t' \]

\[ + \pi n^{-1/2} \sum_{i=0}^{n-1} \cos(\pi(i + \frac{1}{2})/n) \times \]

\[ (\hat{X}_{2,nj+i} - \mathbb{E}[\hat{X}_{2,nj+i} | \mathcal{F}_{n'_{nj+i}}]), \]

\[ Z_{2,j} := \sqrt{n} \int_{I_j} \sin(\pi(n(t - t_j))) b_t \ dt \]

\[ Z_{3,j} := \pi(n^{-1/2} \sum_{i=0}^{n-1} \cos(\pi(i + \frac{1}{2})/n) X_{2,t'_{nj+i}} - n^{3/2} \int_{I_j} \cos(\pi(n(t - t_j))) X_{2,t} \ dt), \]

\[ Z_{4,j} := \pi n^{-1/2} \sum_{i=0}^{n-1} \cos(\pi(i + \frac{1}{2})/n) \times \]

\[ (\mathbb{E}[\hat{X}_{2,nj+i} | \mathcal{F}_{n'_{nj+i}}] - X_{2,t'_{nj+i}}). \]

The desired results then follow similarly to the microstructure noise example. \(\square\)

Next, we give some standard exponential moment bounds on stochastic integrals.

**Lemma 9.** Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,\infty]}, \mathbb{P})\) be a filtered probability space, with adapted Brownian motion \(B_t \in \mathbb{R}\), and Poisson random measure \(\lambda(dx,dt)\) having compensator \(dxdt\).

(i) Let \(c_t \in \mathbb{R}\) be a predictable process, and define

\[ \rho_t^2 := \int_0^t c_s^2 ds. \]

If \(\rho_\infty^2 < \infty\) almost surely, then for \(u \in \mathbb{R}\), the stochastic integral

\[ W_t := \int_0^t c_s dB_s \]

satisfies

\[ \mathbb{E}[\exp(uW_\infty - \frac{1}{2}u^2\rho_\infty^2) | \mathcal{F}_0] \leq 1. \]

(ii) In the setting of part (i), if \(\rho_\infty^2 \leq R\) almost surely, we further have

\[ \mathbb{E}[\exp(u\sup_{t \geq 0}|W_t| - u^2 R) | \mathcal{F}_0] = O(1). \]

(iii) Let \(t \geq 0\), and \(f_s(x) \in \mathbb{R}\) be a predictable function, with \(|f_s(x)| \leq 1, \int_0^t \int_\mathbb{R}|f_s(x)| dx \ ds = O(R), \)
for some $R \in (0, 1)$. Then for $u \leq -\log(R)$, the stochastic integral

$$W_t := \int_0^t \int_\mathbb{R} f_s(x) \left( \lambda(dx, ds) - dx ds \right)$$

satisfies

$$\mathbb{E}[\exp(u \sup_{s \in [0,t]} |W_s|) \mid \mathcal{F}_0] = O(1).$$

Proof. We begin with part (i). By localisation and martingale convergence, we have

$$W_t \xrightarrow{a.s.} W_\infty.$$

The Doléans-Dade exponential

$$\mathcal{E}(uW)_t := \exp(uW_t - \frac{1}{2} u^2 \rho_t^2)$$

is a non-negative local martingale, so a supermartingale. Hence by Fatou’s lemma,

$$\mathbb{E}[\mathcal{E}(uW)_\infty \mid \mathcal{F}_0] \leq \liminf_{t \to \infty} \mathbb{E}[\mathcal{E}(uW)_t \mid \mathcal{F}_0] \leq \mathbb{E}[\mathcal{E}(uW)_0 \mid \mathcal{F}_0] = 1.$$

To show part (ii), we have

$$\mathbb{E}[\mathcal{E}(uW)_\infty^2 \mid \mathcal{F}_0] \leq \exp(u^2 R) \mathbb{E}[\mathcal{E}(2uW)_\infty \mid \mathcal{F}_0] \leq \exp(u^2 R),$$

so $\mathcal{E}(uW)_t$ is a true martingale. We deduce that

$$\mathbb{E}[\exp(u \sup_{t \geq 0} |W_t| - u^2 R) \mid \mathcal{F}_0] \leq \exp(-\frac{1}{2} u^2 R) \mathbb{E}[\sup_{t \geq 0} \mathcal{E}(uW)_t + \sup_{t \geq 0} \mathcal{E}(-uW)_t \mid \mathcal{F}_0]$$

$$= O(1) \exp(-\frac{1}{2} u^2 R) \mathbb{E}[\mathcal{E}(uW)_\infty^2 + \mathcal{E}(-uW)_\infty^2 \mid \mathcal{F}_0]^{1/2},$$

using Doob’s martingale inequality,

$$= O(1).$$

We now prove part (iii), noting we may assume $u \geq 0$. Defining the variation martingale

$$M_t := \int_0^t \int_\mathbb{R} |f_s(x)| (\lambda(dx, ds) - dx ds),$$

we then have

$$\sup_{s \in [0,t]} |W_s| \leq M_t + O(R),$$

so it suffices to bound $M_t$. Let $M_t^\vee$ denote the continuous part of $M_t$, and $\Delta M_t$ its jump at time $t$. Then for $v \geq 0$, the Doléans-Dade exponential

$$\mathcal{E}(vM)_t := \exp(vM_t^\vee) \prod_{s \leq t} (1 + v \Delta M_s)$$

is a non-negative local martingale, so a supermartingale.
Furthermore, since \( u \Delta M_s \in [0, -\log(R)] \), we have
\[
\exp(u \Delta M_s) \leq 1 + cu \Delta M_s,
\]
where the constant \( c := (1 - R^{-1})/\log(R) \). We deduce that
\[
\mathbb{E}[\exp(u M_t) \mid \mathcal{F}_0] = O(1)\mathbb{E}[\exp((c - 1)u M_t^c + u M_t) \mid \mathcal{F}_0]
\]
\[
= O(1)\mathbb{E}[\exp(cu M_t^c + u \sum_{s \leq t} \Delta M_s) \mid \mathcal{F}_0]
\]
\[
= O(1)\mathbb{E}[\mathcal{E}(cu M)_t \mid \mathcal{F}_0]
\]
\[
= O(1)\mathcal{E}(cu M)_0
\]
\[
= O(1).
\]

We may now prove our central limit theorem for martingale differences. Our argument uses a Skorokhod embedding, as in Mykland (1995) or Oblój (2004), for example.

**Proof of Lemma 1.** We begin with a Skorokhod embedding, allowing us to consider the variables \( X_i \) as stopped Brownian motions on an extended probability space. Our argument proceeds by induction on a variable \( k = 0, \ldots, n \).

We claim that for \( i = 0, \ldots, k - 1 \), on an extended probability space, we can construct processes \((B_i,t)_{t \in [0,\infty)}\), which are Brownian motions given the \( \sigma \)-algebra \( \widetilde{\mathcal{F}}_i \) generated by \( \mathcal{F}_i \) and \( B_0, \ldots, B_{i-1} \), and are independent of \( \mathcal{F} \) given \( \mathcal{F}_{i+1} \). We further claim we can construct variables \( \tau_i \) which are stopping times in the natural filtrations \( \mathcal{G}_{i,t} \) of the \( B_i,t \), so that \( X_i = B_i,\tau_i \).

For \( k = 0 \), the claim is trivial; we will show that if the claim holds for \( k \), it holds also for \( k + 1 \). By Skorokhod embedding, on a further-extended probability space, we can construct a process \( \widetilde{B}_{k,t} \) which is a Brownian motion given \( \widetilde{\mathcal{F}}_k \), and a variable \( \widetilde{\tau}_k \) which is a stopping time in the natural filtration of \( \widetilde{B}_{k,t} \), such that the variable \( \widetilde{X}_k := \widetilde{B}_{k,\widetilde{\tau}_k} \) is distributed as \( X_k \) given \( \widetilde{\mathcal{F}}_k \).

Since the stopped process \((\widetilde{B}_{k,t \wedge \widetilde{\tau}_k})_{t \in [0,\infty)}\) is continuous and eventually constant, the pair \((\widetilde{B}_{k,t \wedge \widetilde{\tau}_k})_{t \in [0,\infty)}, \widetilde{\tau}_k \) takes values in a Polish space. We can thus define the regular conditional distribution \( \mathbb{Q}_k(x) \) of \((\widetilde{B}_{k,t \wedge \widetilde{\tau}_k})_{t \in [0,\infty)}, \widetilde{\tau}_k \) given \( \widetilde{X}_k = x \) and \( \widetilde{\mathcal{F}}_k \). On a further-extended probability space, we can then generate a pair \((B_{k,t \wedge \tau_k})_{t \in [0,\infty)}, \tau_k \) with distribution \( \mathbb{Q}_k(X_k) \) given \( \mathcal{F}_k \) and \( X_k \), independent of \( \mathcal{F} \) given \( \mathcal{F}_{k+1} \).

We deduce that the triplet \((B_{k,t \wedge \tau_k})_{t \in [0,\infty)}, \tau_k, X_k \) is distributed as the triplet \((\widetilde{B}_{k,t \wedge \widetilde{\tau}_k})_{t \in [0,\infty)}, \widetilde{\tau}_k, \widetilde{X}_k \) given \( \widetilde{\mathcal{F}}_k \), and hence \( B_{k,t \wedge \tau_k} \) and \( \tau_k \) satisfy the conditions of our claim. It remains to define \( B_{k,t} \) for \( t > \tau_k \); we set
\[
B_{k,t + \tau_k} := B_{k,\tau_k} + B_{k,t}, \quad t \geq 0,
\]
for an independent Brownian motion $B'_{k,t}$. We then conclude that $B_{k,t}$ and $\tau_k$ satisfy the conditions of our claim; by induction, the claim thus holds for $k = n$.

Next, we will show we can realise the sums $\sum_{i=0}^{n-1} c_i X_i$ as integrals against a common Brownian motion. Define a process

$$B_t := \sum_{j=0}^{n-2} B_{j,T(j,t) \wedge \tau_j} + B_{n-1,T(n-1,t)},$$

where the variables

$$T(j, t) := 0 \vee (t - \sum_{i=0}^{j-1} \tau_i).$$

We will show that $B_t$ is a Brownian motion with respect to a suitable filtration $G_t$, and that the sums $\sum_{i=0}^{n-1} c_i X_i$ can be written as stochastic integrals against $B_t$.

For fixed $j = 0, \ldots, n - 1$, the $\sigma$-algebras

$$\tilde{G}_{j,t} := \sigma(\tilde{F}_j, G_{j,t})$$

form a filtration in $t \geq 0$, and the variables $T(j, t)$ are $\tilde{G}_{j,t}$-stopping times. For fixed $t \geq 0$, we can thus define the $\sigma$-algebras $\tilde{G}_{j,T(j,t)}$, which form a filtration in $j = 0, \ldots, n - 1$, and the variables

$$j(t) := \max\{j = 0, \ldots, n - 1 : \sum_{i=0}^{j-1} \tau_i \leq t\},$$

which are $\tilde{G}_{j,T(j,t)}$-stopping times.

We can then define the $\sigma$-algebras

$$G_t := \tilde{G}_{j(t),T(j(t),t)},$$

which form a filtration in $t$, and check that the process $B_t$ is a $G_t$-Brownian motion. We conclude that given $F_t$-measurable variables $c_i$, the sums

$$\sum_{i=0}^{n-1} c_i X_i = \int_0^\infty f_c(t) \, dB_t, \quad (14)$$

where the $G_t$-predictable integrands

$$f_c(t) := \sum_{j=0}^{n-1} c_j 1_{(0,\tau_j]}(T(j,t)).$$

In part (i), we consider the case $c_i = 1$, and obtain

$$\sum_{i=0}^{n-1} X_i = B_\nu, \quad \nu := \sum_{i=0}^{n-1} \tau_i.$$

Defining the random variables

$$\xi := B_1, \quad \eta := B_\nu - B_1,$$

we then have $\sum_{i=0}^{n-1} X_i = \xi + \eta$, and $\xi \sim N(0,1)$ given $F_0$. 34
Furthermore, using Burkholder-Davis-Gundy, we have
\[ \mathbb{E}[|\eta|^{4\kappa} \mid \mathcal{F}_0] = O(1) \mathbb{E}[|\nu - 1|^{2\kappa} \mid \mathcal{F}_0], \]
while using Lemma 9(i),
\[ \mathbb{E}[\exp(u\eta - \frac{1}{2} u^2|\nu - 1|) \mid \mathcal{F}_0] \leq 1. \]
It thus remains to bound the distance of \( \nu \) from 1.

For \( j = 0, \ldots, n \), define the \( \tilde{F}_j \)-martingale
\[ V_j := \sum_{i=0}^{j-1} (\tau_i - \mathbb{E}[\tau_i \mid \tilde{F}_i]), \]
and the total mean
\[ \bar{\nu} := \sum_{i=0}^{n-1} \mathbb{E}[\tau_i \mid \tilde{F}_i]. \]
We then have
\[ |\nu - 1| \leq |V_n| + |\bar{\nu} - 1|; \]
we will show that both terms on the right-hand side are small.

We first obtain
\[ \mathbb{E}[|V_n|^{2\kappa} \mid \mathcal{F}_0] = O(1) \mathbb{E}[(\sum_{i=0}^{n-1} |\tau_i - \mathbb{E}[\tau_i \mid \tilde{F}_i]|^2)^\kappa \mid \mathcal{F}_0], \]
by Burkholder-Davis-Gundy,
\[ = O(n^{\kappa-1}) \sum_{i=0}^{n-1} \mathbb{E}[|\tau_i - \mathbb{E}[\tau_i \mid \tilde{F}_i]|^{2\kappa} \mid \mathcal{F}_0], \]
by Jensen’s inequality,
\[ = O(n^{\kappa-1}) \sum_{i=0}^{n-1} \mathbb{E}[|X_i|^{4\kappa} \mid \mathcal{F}_0], \]
by Burkholder-Davis-Gundy and Doob’s martingale inequality,
\[ = O(n^{-\kappa}). \tag{15} \]

We also have that
\[ \bar{\nu} = \sum_{i=0}^{n-1} \mathbb{E}[X_i^2 \mid \tilde{F}_i], \]
by Ito’s isometry,
\[ = \sum_{i=0}^{n-1} \mathbb{E}[X_i^2 \mid \mathcal{F}_i], \]
as the \( B_{j,t} \) are independent of \( \mathcal{F} \) given \( \mathcal{F}_{j+1} \). We deduce that
\[ \mathbb{E}[|\nu - 1|^{2\kappa} \mid \mathcal{F}_0] = O(n^{-\kappa}), \]

as required.

In part (ii), we again apply Burkholder-Davis-Gundy and Lemma 9(i) to the sums (14). We claim that

$$\sup_c \int_0^\infty f^2_c(t) \, dt \leq A + M,$$

for terms $A$ and $M$ as in the statement of the Lemma, so

$$\sup_c \mathbb{E}[|v_c|^{4\kappa} \mid \mathcal{F}_0] = O(1),$$

and

$$\sup_c \mathbb{E}[\exp(uv_c - \frac{1}{2}u^2(A + M)) \mid \mathcal{F}_0] \leq 1.$$

It thus remains to prove the claim.

As before, we have

$$|\nu| \leq |V_n| + |\nu|,$$

and

$$\nu = \sum_{i=0}^{n-1} \mathbb{E}[X_i^2 \mid \mathcal{F}_i] = O(n^{1-1/2\kappa}(\sum_{i=0}^{n-1} \mathbb{E}[|X_i|^{4\kappa} \mid \mathcal{F}_i])^{1/2\kappa},$$

by Jensen’s inequality,

$$= O(1).$$

For any random variables $c_i = O(1)$, we deduce that

$$\int_0^\infty f^2_c(t) \, dt = \sum_{i=0}^{n-1} c_i^2 \tau_i = O(n) = O(1 + |V_n|).$$

The claim thus holds for terms $A = O(1), M = O(|V_n|)$, and we further have that $M$ satisfies (8), using (15).

We next prove our result on combining exponential moment bounds.

Proof of Lemma 2. We first note that by rescaling the $X_i$, we may assume $r_n = 1$. Then on an extended probability space, let $\xi$ be standard Gaussian, independent of $\mathcal{F}$. For any $R > 0$, we have

$$\mathbb{E}[\exp(X_i^2/4R)1_{M \leq R}] = \mathbb{E}[\mathbb{E}[\exp((2R)^{-1/2}X_i\xi)1_{M \leq R} \mid \mathcal{F}]]$$

$$\leq \mathbb{E}[\exp(\xi^2/4)\mathbb{E}[\exp((2R)^{-1/2}\xi X_i - \frac{1}{2}(2R)^{-1}\xi^2 M)1_{M \leq R} \mid \xi]]$$

$$= O(1)\mathbb{E}[\exp(\xi^2/4)]$$

$$= O(1).$$

We deduce that, for any $R > 0$,

$$\mathbb{E}[\max_i \exp(X_i^2/4R)1_{M \leq R}] \leq \sum_i \mathbb{E}[\exp(X_i^2/4R)1_{M \leq R}] = O(n),$$

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so

$$\max_{i} |X_i| 1_{M \leq R} = O_p(R^{1/2} \log(n)^{1/2}).$$

Since $M = O_p(1)$, we conclude that $\max_{i} |X_i| = O_p(\log(n)^{1/2}).$ \hfill \qed

We continue with a proof of our moment bounds on the $Y_i$ and $Z_i(\theta)$.

**Proof of Lemma 3.** To show part (i), we note that the functions $\mu$ and $\sigma^2$ are locally Lipschitz, $\sigma^2$ is positive, and $\theta$, $t_i$ and $\hat{X}_i$ are bounded. We may thus restrict the functions $\mu$ and $\sigma^2$ to a compact set, on which $\mu$ and $\sigma^2$ are $C^1$, and $1/\sigma^2$ is bounded. We deduce that part (i) holds; by a similar argument, part (ii) holds also.

To show part (iii), we then have

$$\mathbb{E}[|Z_i(\theta)|^{4+\varepsilon} \mid \mathcal{F}_{t_i}] = O(1)\mathbb{E}[1 + |Y_i|^{4+\varepsilon} \mid \mathcal{F}_{t_i}] = O(1),$$

and

$$Z_i(\theta) = (Y_i - \mu(\theta, t_i, X_{t_i})) / \sigma(\theta, t_i, X_{t_i}) + \gamma_i,$$

for a term

$$\gamma_i = O(1)(1 + |Y_i|)\|\hat{X}_i - X_{t_i}\|.$$

Using Cauchy-Schwarz, we obtain

$$\mathbb{E}[\gamma_i \mid \mathcal{F}_{t_i}] = O(n^{-1/2}),$$

$$\mathbb{E}[\gamma_i^2 \mid \mathcal{F}_{t_i}] = O(n^{-1/2}).$$

We conclude that

$$\mathbb{E}[Z_i(\theta) \mid \mathcal{F}_{t_i}] = S_{t_i}(\theta) + \mathbb{E}[\gamma_i \mid \mathcal{F}_{t_i}],$$

$$= S_{t_i}(\theta) + O(n^{-1/2}),$$

and under $H_0$, using Cauchy-Schwarz, also

$$\mathbb{E}[Z_i(\theta_0)^2 \mid \mathcal{F}_{t_i}] = 1 + O(1)(\mathbb{E}[\gamma_i^2 \mid \mathcal{F}_{t_i}]^{1/2} + \mathbb{E}[\gamma_i^2 \mid \mathcal{F}_{t_i}]),$$

$$= 1 + O(n^{-1/4}).$$

To show part (iv), we define the random variables

$$R_k := m^{-1} \sum_{i=n_{sk}^k+1}^{n_{sk+1}^k-1} ((Y_i - \mathbb{E}[Y_i \mid \mathcal{F}_{t_i}])^2 - \text{Var}[Y_i \mid \mathcal{F}_{t_i}]),$$

where $m := n(s_{k+1} - s_k)$. $R_k$ is then an average of $m$ terms of a martingale difference sequence, whose conditional variances are bounded. We deduce that

$$\mathbb{E}[R_k^2] = O(m^{-1}) = O(n^{-1/2}).$$
and so
\[
E[(\max_k n^{-1/2} \sum_{i=n_k}^{n_{k+1}-1} Y_i^2)^2] = O(1)E[1 + \max_k R_k^2]
= O(1)(1 + \sum_k E[R_k^2])
= O(1).
\]
The desired result follows.

Finally, we prove our result on the behaviour of the processes $S_t(\theta)$ under Assumption 2.

Proof of Lemma 4. We begin by defining the processes $\tilde{S}_t(\theta)$, $S_t(\theta)$, and times $\tau_i$. We can split the process $\mu_t$ into parts
\[
\mu_t = \tilde{\mu}_t + \overline{\mu}_t,
\]
where $\tilde{\mu}_t$ is a process with jumps of size at most $n^{-1/4} \log(n)^{1/2}$, and $\overline{\mu}_t$ is an orthogonal pure-jump process with jumps of size at least $n^{-1/4} \log(n)^{1/2}$. We can similarly define terms $\tilde{X}_t$, $X_t$.

Let $\tau_1 < \cdots < \tau_{N-1}$ denote the times at which $\overline{\mu}_t$ or $X_t$ jump, and set $\tau_0 := 0$, $\tau_N := 1$. We can then decompose the processes
\[
S_t(\theta) = \tilde{S}_t(\theta) + \overline{S}_t(\theta),
\]
where
\[
\overline{S}_t(\theta) := \sum_{\tau_i \leq t} \Delta S_{\tau_i}(\theta),
\]
letting $\Delta S_t(\theta)$ denote the jump in $S_t(\theta)$ at time $t$, and $\tilde{S}_t(\theta)$ is then defined by (16).

To prove part (i), we first note that the model functions $\mu$ and $\sigma^2$ are continuously differentiable in $t$, twice continuously differentiable in $X$, and $\sigma^2$ is positive. By Itô’s lemma, we can thus write
\[
d\tilde{S}_t(\theta) = \tilde{b}_t(\theta) dt + \tilde{c}_t(\theta)^T dB_t + \int_{\mathbb{R}} \tilde{F}_t(x, \theta) (\lambda(dx, dt) - dx dt),
\]
for integrators $B_t$ and $\lambda(dx, dt)$ given by Assumption 2, predictable processes $\tilde{b}_t(\theta)$, $\tilde{c}_t(\theta)$, and predictable functions $\tilde{F}_t(x, \theta)$. Since $\theta$, $t$, $\mu_t$ and $X_t$ are bounded, we also have $\tilde{b}_t(\theta)$, $\tilde{c}_t(\theta) = O(1)$, $\tilde{F}_t(x, \theta) = O(n^{-1/4} \log(n)^{1/2})$, and $\int_{\mathbb{R}} |\tilde{F}_t(x, \theta)| dx = O(1)$.

To bound the size of changes in $\tilde{S}_t(\theta)$, we will consider the variables
\[
M_k(\theta) := \sup_{t \in I_k} |\tilde{S}_t(\theta) - \tilde{S}_{2^{-J}k}(\theta)|,
\]
where the intervals $I_k := 2^{-J}[k, k+1]$. We have
\[
M_k(\theta) \leq \sum_{i=0}^{q+2} M_{k,i}(\theta),
\]
for
for terms
\[ M_{k,i}(\theta) := \begin{cases} 
\sup_{t \in I_k} \left| \int_{t_2 - J_k}^t \tilde{b}_k(\theta) \, dt \right|, & i = 0, \\
\sup_{t \in I_k} \left| \int_{t_2 - J_k}^t \tilde{c}_{i,t}(\theta) \, dB_{i,t} \right|, & i = 1, \ldots, q + 1, \\
\sup_{t \in I_k} \left| \int_{t_2 - J_k}^t \int_{I_k} \tilde{f}_i(x, \theta) \left( \lambda(dx, dt) - dx \, dt \right) \right|, & i = q + 2.
\]

In each case \( i = 0, \ldots, q + 2 \), we will bound the maximum
\[ \tilde{M}_i := \max_{k, \theta \in \Theta} M_{k,i}(\theta). \]

From the definitions, we have
\[ \tilde{M}_0 = O(n^{-1/2}). \]

For \( i = 1, \ldots, q + 1 \), we use Lemma 9(ii), obtaining that
\[ \mathbb{E}[\exp(un^{1/4} M_{k,i}(\theta) - u^2 R)] = O(1), \]
for all \( u \in \mathbb{R} \), and some fixed \( R > 0 \). Using Lemma 2, we deduce that
\[ \tilde{M}_i = O_p(n^{-1/4} \log(n)^{1/2}). \]

Finally, using Lemma 9(iii), for small enough \( \varepsilon' > 0 \) we have
\[ \mathbb{E}[\exp(\varepsilon' n^{1/4} \log(n)^{1/2} M_{k,q+2}(\theta))] = O(1). \]

We deduce that
\[ \mathbb{E}[\exp(\varepsilon' n^{1/4} \log(n)^{1/2} \tilde{M}_{q+2})] \leq \sum_{k, \theta \in \Theta} \mathbb{E}[\exp(\varepsilon' n^{1/4} \log(n)^{1/2} M_{k,q+2}(\theta))] \]
\[ = O(n^{r+1/2}), \]
and so
\[ \tilde{M}_{q+2} = O_p(n^{-1/4} \log(n)^{1/2}). \]

We conclude that the random variable
\[ M := \max_{k, \theta \in \Theta} M_k(\theta) \leq \sum_{i=0}^{q+2} \tilde{M}_i = O_p(n^{-1/4} \log(n)^{1/2}). \]

Part (i) then follows trivially.

To show part (ii), for \( s, t \in [0,1] \), define the translated processes
\[ \tilde{S}_s^{(t)}(\theta) := \tilde{S}_s(\theta) - \tilde{S}_t(\theta). \]

We then have
\[ R_j \tilde{S}_t(\theta) = R_j \tilde{S}_s^{(t)}(\theta), \]
since wavelets are orthogonal to constant functions,

$$= - P_J \tilde{S}_t^{(t)}(\theta),$$

since $\tilde{S}_t^{(t)}(\theta) = 0$,

$$= - \sum_k \varphi_{J,k}(t) \int_0^1 \varphi_{J,k}(s)(\tilde{S}_s(\theta) - \tilde{S}_t(\theta)) \, ds$$

$$= O(M),$$

using the compact support of $\varphi$. The desired result follows.

To show part (iii), we first note that the processes $\tilde{S}_t(\theta)$ are constant on the intervals $[\tau_i, \tau_{i+1})$ and $[\tau_{N-1}, \tau_N]$. Setting $\tau_i = 1$ for $i > N$, we then have

$$P[\exists i \leq n^{1/4} : \tau_i \leq 1 - \delta_n, \tau_{i+1} < \tau_i + \delta_n]$$

$$\leq \sum_{i=0}^{n^{1/4}} P[\tau_i \leq 1 - \delta_n]P[\tau_{i+1} < \tau_i + \delta_n | \tau_i \leq 1 - \delta_n]$$

$$= o(n^{-1/4}) \sum_{i=0}^{n^{1/4}} P[\tau_i \leq 1 - \delta_n],$$

using Assumption 2, and that $\tau_i$ is a stopping time,

$$= o(1).$$

Similarly, we have

$$P[\exists i : \tau_i \in (1 - \delta_n, 1)], P[N > n^{1/4}] = o(1).$$

The desired result follows.

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