Perturbations of Schwarzschild black holes in
Dynamical Chern-Simons modified gravity

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Dynamical Chern-Simons (DCS) modified gravity is an attractive, yet relatively unexplored, candidate to an alternative theory of gravity. The DCS correction couples a dynamical scalar field to the gravitational field. In this framework, we analyze the perturbation formalism and stability properties of spherically symmetric black holes. Assuming that no background scalar field is present, gravitational perturbations with polar and axial parities decouple. We find no effect of the Chern-Simons coupling on the polar sector, while axial perturbations couple to the Chern-Simons scalar field. The axial sector can develop strong instabilities if the coupling parameter $\beta$, associated to the dynamical coupling of the scalar field, is small enough; this yields a constraint on $\beta$ which is much stronger than the constraints previously known in the literature.

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I. INTRODUCTION

Interferometric and resonant-bar gravitational-wave detectors are now working at or near design sensitivity. It is expected that instruments such as LIGO or any of its advanced versions \cite{1} will soon make the first direct detection of gravitational waves on Earth. The detection and measurement of gravitational waves from compact, massive astrophysical bodies opens a new window into the universe, and it also opens up the exciting possibility of testing Einstein’s field equations in an unprecedented way. From measurements of the inspiralling phase of black hole or neutron star binaries, one can test General Relativity’s prediction for the waveform and among others, bound the mass of the graviton \cite{2, 3, 4}. Gravitational-wave observations of the ringdown phase of the final black hole can lead to tests of the no-hair theorem in General Relativity \cite{5, 6}. Taken together, these measurements also allow for tests of Hawking’s area theorem for black holes \cite{7}.

What if there are corrections to the field equations, and do they leave a measurable imprint on gravitational waves? Unfortunately, this important question depends on several unknowns. It depends on the form of the corrections to the field equations, on the black hole solutions to these modified equations and on which specific process is generating the gravitational wave signal. One of such theories is known as Einstein-Dilatonic-Gauss-Bonnet, a simple example of one-loop corrected four-dimensional effective theory of the heterotic superstrings at low energies \cite{8}. The possibility of astrophysical tests of this theory were studied in Ref. \cite{9}.

Another promising extension of general relativity is Chern-Simons (CS) gravity \cite{10, 11, 12}, in which the Einstein-Hilbert action is modified by adding a parity-violating Chern-Simons term, which couples to gravity via a scalar field. This correction arises in many contexts. Such a term could help to explain several problems of cosmology, from inflation (as discussed by Weinberg \cite{13}) to baryon asymmetry \cite{14, 15, 16}. In most of the moduli space of string theory, a CS correction is required to preserve unitarity; furthermore, duality symmetries induce a CS term in all string theories with a Ramond-Ramond scalar \cite{17}. In loop quantum gravity, it is required to ensure gauge invariance of the Ashtekar variables \cite{18} and it also arises naturally if the Barbero-Immirzi parameter is promoted to a field \cite{19, 20}.

An interesting feature of CS modified gravity is that it has a characteristic observational signature, which could allow one to discriminate an effect of this theory from other phenomena. Indeed, the CS term violates parity, and thus it mainly affects the axial-parity component of the gravitational field. For instance, it yields amplitude birefringence of gravitational waves (the two polarizations travel with the same speed, but one is enhanced, while the other is

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suppressed); on the other hand, the Schwarzschild solution is unaffected by CS modified gravity, and then the solar system tests of general relativity do not put strict bounds on the magnitude of this correction. Another possible signature of CS modified gravity may be found in the polarization of primordial gravitational waves \cite{21}.

Most of the literature on CS modified gravity refers to its non-dynamical formulation, in which the scalar field is a prescribed function. Usually, the so-called canonical prescription $\theta \propto t$ is chosen \cite{22,23,24}. An explicit prescription for the scalar field is necessarily \textit{a priori}, and furthermore it breaks gauge invariance (see the discussion in \cite{23}). For these reasons, we prefer to consider the dynamical Chern-Simons (DCS) modified gravity \cite{26}, where the scalar field is treated as a dynamical field. Note that CS gravity and DCS gravity are inequivalent and independent theories: although the CS gravity action can be obtained as a certain limit of the DCS gravity action, the solutions of CS gravity cannot be obtained from the solutions of DCS gravity \cite{25}. We also remark that CS modified gravity, with the canonical prescription, is retrieved in the case of weak gravitational field, if one only keeps the leading order perturbations of Minkowski spacetime \cite{27}; therefore, CS gravity is appropriate to study, for instance, deviations from general relativity in solar system astrophysical processes, or in the motion of binary pulsars far from coalescence, but DCS gravity is required to describe deviation from general relativity in the strong field regime, like in black hole (BH) perturbation theory or in the coalescence of binary systems.

In this paper we study, in the context of DCS gravity, gravitational perturbations of a spherically symmetric black hole. Perturbations of black holes are interesting for several reasons: since black holes populate the universe in large number, assessing their stability in a given theory is tantamount to testing the theory: if the black hole solution in that specific theory is unstable, they would not be seen. Second because a variety of processes taking place in the vicinities of black holes will be observed by gravitational-wave detectors. The two most important are extreme-mass-ratio-inspirals \cite{28}, and quasi-normal ringing \cite{3}. The first consists, for instance, on a small star orbiting around a supermassive black hole. Such a process can be modeled as a test particle inducing perturbations on a black hole background, and can be tackled with perturbation tools. The second is a universal signal: all or almost all events involving black holes produce a gravitational wave signal which at late times consists of a superposition of the characteristic modes of the black hole, the quasi-normal modes (QNMs). Such a signal will be seen by present or future gravitational wave detectors.

BH perturbations have been studied, in the context of CS gravity, in Ref. \cite{29}, where it was found that polar-parity and axial-parity gravitational perturbations are coupled, and the equations do not allow for generic black hole oscillations. BH perturbations in DCS gravity were also briefly discussed in that work, in the framework of two-parameter perturbation theory: the two parameters were $\epsilon$, describing the magnitude of the gravitational perturbation, and $\tau$, describing the magnitude of the CS scalar field. The authors of \cite{29} found that, if the scalar field has both a background (spherically symmetric) component of order $O(\tau)$ and an oscillating component of order $O(\epsilon)$, polar and axial perturbations are coupled, and the equations are extremely involved. Here, we consider the case in which the background scalar field vanishes, i.e. the scalar field is only generated by gravitational perturbations. Under this condition, we find that polar and axial gravitational perturbations decouple, and only axial parity gravitational perturbations are affected by the CS scalar field. Furthermore we find that, under this assumption, gravitational perturbations and the Chern-Simons scalar field are described by a simple set of equations. Numerical integration of these equations is not an easy task, due to the well-known asymptotic divergence which prevented for many years the numerical computation of quasi-normal modes of the Schwarzschild BH \cite{30,31}.

We find some evidence that in the limit $\beta \to \infty$, where $\beta$ is the coefficient in front of the kinetic CS term of the action (see Eq. 1 and discussion below), the BH does not admit QNMs. Furthermore we find that for $0 < \beta M^4 \lesssim 2\pi$, there is at least one (strongly) unstable mode, and the Schwarzschild DCS solution is then unstable. This seems to impose a strong constraint on the theory, if it is to be compatible with the existence of astrophysical black holes, i.e., $\beta \gtrsim 10^{-2}$ km$^{-4}$. Previous bounds were much weaker: in \cite{25} it was found that observational data from the extreme double pulsar system PSR J0737-3039 imply $\beta \gtrsim 10^{-15}$ km$^{-4}$.

Strictly speaking, our results apply only to non-rotating BHs, since rotating BH have a non-vanishing background scalar field \cite{25,32} and are more complex to handle. Astrophysical BHs are in general rotating (perhaps even rapidly rotating), but it is reasonable to expect that the range of $\beta$ incompatible with the existence of non-rotating BH is also incompatible with (slowly) rotating BH. Therefore, we take the bound $\beta \gtrsim 10^{-2}$ km$^{-4}$ as a (strong) indication of the values we may expect for the parameter $\beta$. Finally, we also want to stress that the bound $\beta \gtrsim 10^{-2}$ km$^{-4}$ does not at all rule out the possibility of an observational signature from DCS gravity. Indeed, we restrict the range of $\beta$, but we do not restrict neither the range of $\alpha$ (i.e., with the $\alpha = 1$ normalization, the amplitude of the scalar field), nor the time derivative of the scalar field amplitude. For instance, in \cite{27} it was found that observational data from the double pulsar J0737-3039 imply a bound on $\dot{\vartheta}$, in the context of CS gravity.

The paper is organized as follows. In Section IV we briefly review DCS modified gravity. In Section III we derive the equations for perturbations of a spherically symmetric black hole in DCS modified gravity. In Section IV we discuss the perturbation equations we have derived, finding some of their solutions. In Section V we draw our conclusions.
II. DCS GRAVITY

Following the notation of [22], the action of dynamical Chern-Simons modified gravity is

$$S = \kappa \int d^4x \sqrt{-g} R + \frac{\alpha}{4} \int d^4x \sqrt{-g} \nabla^\mu \nabla^\nu g_{\mu \nu} - \frac{\beta}{2} \int d^4x \sqrt{-g} \left[ g^{ab} \partial_a \vartheta \partial_b \vartheta + V(\vartheta) \right] + S_{\text{mat}}. \quad (1)$$

Note that there are two parameters $\alpha, \beta$, but one of them can be eliminated by choosing the normalization of the scalar field. For instance, in [26] the normalization of $\vartheta$ is chosen such that $\beta = 1$, and $\alpha$ is the only parameter describing the coupling between the scalar field and the gravitational field. We normalize instead the scalar field by imposing $\alpha = 1$, and use geometrical units $c = G = 1$ so that $\kappa = \frac{1}{16\pi}$. Then,

$$[S] = l^2, \quad [\vartheta] = l^2, \quad [\beta] = l^{-4}. \quad (2)$$

Furthermore, we neglect $V(\vartheta)$, and consider the vacuum solutions ($S_{\text{mat}} = 0$); therefore, the equations of motion are

$$R_{ab} = -16\pi C_{ab} + 8\pi \beta \vartheta_{,a} \vartheta_{,b}$$
$$\Box \vartheta = -\frac{1}{4\beta} \ast RR \quad (3)$$

where

$$C^{ab} = \vartheta_{,a} \epsilon^{cde} \nabla_e R^b_d + \vartheta_{,de} \ast R^{(ab)c}$$
$$\ast RR = \frac{1}{2} R_{abcd} \epsilon^{abef} R^{cde}_f. \quad (5)$$

Equations (3), (4) acquire a particularly simple form in the spherically symmetric case [11, 29], yielding (up to $O(\vartheta^2)$) the Schwarzschild solution; indeed, in the Schwarzschild spacetime $\ast RR = 0$ and, assuming that the scalar field is also spherically symmetric $\vartheta = \vartheta(t, r)$, $C^{ab} = 0$. Then, if we neglect the CS stress-energy tensor (quadratic in the scalar field), Eqs. (3), (4) are satisfied by the Schwarzschild solution.

We mention that in [24, 32] the solution for slowly rotating black holes has been found in DCS modified gravity; the corrections from the general relativistic solutions are of the order $\alpha/\beta, \alpha^2/(\beta \kappa)$, i.e., with our normalizations, of the order $\beta^{-1}$. Furthermore, an observational constraint for $\beta^{-1}$ has been derived in [25], from frame dragging effects in the extreme double pulsar system PSR J0737-3039 A/B:

$$\beta^{-1} \lesssim 10^{15} \text{ km}^4. \quad (7)$$

III. PERTURBATIONS OF A SCHWARZSCHILD BACKGROUND

We now consider perturbations of the spacetime geometry away from a Schwarzschild background. We set $g^{(0)}_{\mu\nu}$ to be the Schwarzschild metric and choose the Regge-Wheeler gauge for perturbations:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu},$$

$$g^{(0)}_{\mu\nu} = \text{diag}(1, f^{-1}, r^2, r^2 \sin^2 \theta) \quad (f(r) \equiv 1 - 2M/r),$$

$$h_{\mu\nu} = \begin{pmatrix}
H^{lm}_{\theta} Y^{lm}_{\theta} & H^{lm}_{\phi} Y^{lm}_{\phi} & h^{lm}_{\theta} S^{lm}_{\theta} & h^{lm}_{\phi} S^{lm}_{\phi}
H^{lm}_{\phi} Y^{lm}_{\phi} & H^{lm}_{\phi} Y^{lm}_{\phi} & h^{lm}_{\phi} S^{lm}_{\phi} & h^{lm}_{\phi} S^{lm}_{\phi}
H^{lm}_{\theta} S^{lm}_{\theta} & H^{lm}_{\phi} S^{lm}_{\phi} & i^2 K^{lm} Y^{lm} & 0
h^{lm}_{\theta} S^{lm}_{\theta} & h^{lm}_{\phi} S^{lm}_{\phi} & i^2 K^{lm} Y^{lm} & 0
\end{pmatrix} e^{-i\omega t}, \quad (8)$$

where $Y^{lm}$ are the scalar spherical harmonics and

$$(S^{lm}_\theta, S^{lm}_\phi) \equiv \left(-\frac{1}{\sin \theta} Y^{lm}_\phi, \sin \theta Y^{lm}_\phi\right). \quad (9)$$

Here, $(H_0, H_1, H_2, K)^{lm}$ are (functions of $r$) describing the polar parity metric perturbations, $(h_0, h_1)^{lm}$ describe the axial parity metric perturbations.

We now make one simplifying, self-consistent assumption, namely that the Chern-Simons scalar field is of the order of $O(\hbar)$. In other words, the background scalar field $\vartheta^{(0)}$, the solution of the homogeneous equation $\Box \vartheta^{(0)} = 0$, 

...
is vanishing: $\vartheta^{(0)} \equiv 0$. The only scalar field present is induced by the perturbations. In this respect the case we are considering is different from that considered in Section VI of Ref. 29. Under this assumption, the harmonic expansion of the scalar field $\vartheta$ is

$$\vartheta = \frac{\Theta}{r} Y^l m e^{-i \omega t}.$$  \hfill (10)

From here onwards, we will drop the $^{lm}$ superscripts.

Replacing expansions (8), (10) into Eqs. (3), (4) and neglecting terms quadratic in the metric perturbations or in the scalar field, we find that perturbations with different parities decouple: polar parity metric perturbations are unaffected by the scalar field. These will not be discussed any further here, since the stability properties and QNMs of these solutions are very well known (see 33 and references therein; for recent reviews see 6, 34). On the other hand, the axial parity metric perturbations are coupled with the scalar field. Indeed, equations (3) yield

$$E_1 = r^3 (-4 M + l(l+1) r) h_0 - r f (2ir^4 \omega h_1 + 192 \pi M \Theta + i r^5 \omega h'_1 - 96 \pi M r \Theta' + r^5 h''_0) = 0 ,$$  \hfill (11)

$$E_2 = -i \omega r^3 (2h_0 - ir \omega h_1 - rh'_0) + r^2 f (l^2 + l - 2) h_1 - 96 \pi i M \omega \Theta = 0 ,$$  \hfill (12)

$$E_3 = ir^3 \omega h_0 + r f (+2 M h_1 + r^2 f h'_1) = 0 .$$  \hfill (13)

We note that these three equations are not all independent, it is easy to show that

$$- \frac{f r^4}{i \omega} \left( E_2 / r^2 \right)' - E_3 + \frac{(l-1)(l+2) r}{i \omega} E_1 = 0 .$$  \hfill (14)

Combining (11), (12) and (13) and defining the Regge-Wheeler master function $Q(r)$ by

$$h_1 = f^{-1} r Q ,$$  \hfill (15)

and

$$r_* \equiv r + 2 M \ln (r/2M - 1)$$  \hfill (16)

we finally get

$$\frac{d^2}{dr_*^2} Q + \left[ \omega^2 - f \left( \frac{l(l+1)}{r^2} - \frac{6M}{r^3} \right) \right] Q = -\frac{96 \pi i M f \omega}{r^3} \Theta ,$$  \hfill (17)

$$\frac{d^2}{dr_*^2} \Theta + \left[ \omega^2 - f \left( \frac{l(l+1)}{r^2} \left( 1 - \frac{576 \pi M^2}{r^6 \beta} \right) + \frac{2M}{r^3} \right) \right] \Theta = -f \frac{(l+2)! 6 M}{(l-2)!} \frac{1}{r^3} \omega \Theta Q .$$  \hfill (18)

Equations (17) and (18) form a system of coupled second order differential equations for the perturbations $Q^{lm}$, $\Theta^{lm}$ (with dimensions $Q^{lm} = 1^0$, $\Theta^{lm} = 1^3$), from which one can completely characterize the axial parity metric perturbations and the scalar field.

IV. INTEGRATION OF THE PERTURBATION EQUATIONS

Despite their apparent simplicity, numerical integration of the perturbation equations (17), (18) is not an easy task. We first note that, as $r_* \to \pm \infty$ (i.e. $r \to r_H \equiv 2M, r \to +\infty$), they reduce to the simple wave equations

$$\left( \frac{d^2}{dr_*^2} + \omega^2 \right) \Theta^{lm} = \left( \frac{d^2}{dr_*^2} + \omega^2 \right) Q^{lm} = 0 ;$$  \hfill (19)

then, every solution $(Q, \Theta)$ of the perturbation equations has the asymptotic form, as $r_* \to \pm \infty$,

$$Q = A_{H, \infty}^{\text{out}, \text{in}} e^{i \omega r_*} + A_{H, \infty}^{\text{in}, \text{in}} e^{-i \omega r_*} ,$$

$$\Theta = B_{H, \infty}^{\text{out}, \text{in}} e^{i \omega r_*} + B_{H, \infty}^{\text{in}, \text{in}} e^{-i \omega r_*} ,$$  \hfill (20)

where $\omega, A_{H, \infty}^{\text{out}, \text{in}}, B_{H, \infty}^{\text{out}, \text{in}}$ are complex numbers. A QNM is a solution of the perturbation equation which satisfies the Sommerfeld boundary conditions

$$A_{H}^{\text{out}} = B_{H}^{\text{out}} = A_{\infty}^{\text{in}} = B_{\infty}^{\text{in}} = 0 ,$$  \hfill (21)
i.e., no radiation outgoing from the horizon, no radiation ingoing from infinity.

By fixing the normalization \( B_0^m = 1 \), and defining \( A_0^m = A_0 \in C \), we have a unique solution of equations (17) and (18); thus we end with two complex conditions \((A_0^m = B_0^m = 0)\) to satisfy, and two complex numbers to determine \((A_0 \text{ and } \omega)\). This naive counting of degrees of freedom tells us that, as in the case of the Schwarzschild BH, the dimension of the space of solutions is zero, allowing for a discrete set of complex values for \(\omega\): the quasinormal modes of the black hole. The difference from the Schwarzschild case is that here there are two coupled differential equations, instead of a single Schroedinger-like equation.

We are not aware of any discussion in the literature regarding the QNMs of coupled systems of ODEs such as the one we have here. The main difficulty in integrating equations (17) and (18) is known: for stable spacetimes \(\text{Im}(\omega) < 0\) and the correct behavior at (say) infinity is \(e^{-i\omega r}\), which is exponentially dominant over the unwanted \(e^{-i\omega r}\). In the case of the Schwarzschild BH, many routes have been explored to circumvent the divergence problem discussed above; the main successful approaches are: a reformulation of the equation as a recurrence relation, which is then expressed as a continued fraction, a WKB approximation of the potential; an analytic continuation to the complex plane of the radial coordinate. It seems quite difficult to extend any of these approaches to a system of two coupled equations like (17) and (18). For instance, if one reformulates our perturbation equations in terms of recurrence series, one finds two coupled series, with ten terms at each order, which do not seem to be expressible in terms of continued fractions.

Therefore, we were not able to perform a full, numerically accurate search for the QNMs of the coupled system (17) and (18). We studied only two limiting regimes: \(\beta M^4 \gg 1\) and \(\beta M^4 \ll 1\). Before proceeding to the analysis of these two cases, we mention that, since \(Q, \Theta\) exponentially diverge in the \(r_+ \to \pm \infty\) limits, it is not at all obvious that equations (17), (18) reduce to the form (19) and then that their solutions, in these limits, have the form (20), as we have assumed at the beginning of this section. Actually, Eq. (20) can be proved without assuming (19), as we show in Appendix A.

A. Very large \(\beta\)

In the case \(\beta M^4 \gg 1\) Eqns. (17), (18) reduce to

\[
\frac{d^2}{dr_*^2} Q^{lm} + \left[ \omega^2 - f \left( \frac{l(l+1)}{r_*^2} - \frac{6M}{r_*^3} \right) \right] Q^{lm} = -f \frac{96\pi i\omega M}{r_*^5} \Theta^{lm} \tag{22}
\]

\[
\frac{d^2}{dr_*^2} \Theta^{lm} + \left[ \omega^2 - f \left( \frac{l(l+1)}{r_*^2} + \frac{2M}{r_*^3} \right) \right] \Theta^{lm} = 0. \tag{23}
\]

Thus, in this limit, the eigenvalue problem for \(\Theta\) is homogeneous, and exactly equivalent to the Schwarzschild case. The solutions of (22), \(\omega^{Q_{NM}}, \Theta^{Q_{NM}}\), are well-known and can be computed accurately using the continued fraction approach. Then, we are left with equation (22), which can be considered as a single differential equation with source. One can build the solution to the inhomogeneous problem (22) by using the Green function method:

\[
Q = \frac{1}{W} \left[ Q_\infty \int_{2M}^r Q_H Sdr + Q_H \int_r^\infty Q_\infty Sdr \right], \tag{24}
\]

where \(Q_H, Q_\infty\) are two independent solutions of the homogeneous version of (22) (evaluated with the \(\omega = \omega^{Q_{NM}}\) eigenvalue of the scalar equation), such that \(Q_\infty \sim e^{i\omega r}\) at infinity and \(Q_H \sim e^{-i\omega r}\) close to the horizon; the quantity \(W\) is the wronskian between \(Q_H, Q_\infty\), and \(S = -\Theta^{Q_{NM}} 96\pi i\omega M/r_*^5\).

Solution (24) can be seen to satisfy the required boundary conditions and indeed is the only solution satisfying such boundary conditions. Unfortunately, the integrals in (24) are not well defined: the wavefunctions \(Q_H, Q_\infty\) and \(S\) diverge exponentially at both the horizon and spatial infinity. This seems to indicate that this solution does not exist, i.e. that DCS Schwarzschild BH do not admit QNM in which both the gravitational perturbations and the scalar field are excited. Only trivial QNM solutions, in which \(\Theta \equiv 0\) and the gravitational field oscillates at the frequencies of ordinary Schwarzschild QNM, seem to be allowed in this limit.

B. Instability regime

The opposite regime concerns very small \(\beta\). As might be anticipated from the previous discussion, the behavior of the solutions of the coupled system are intimately connected with the behavior of the homogeneous problem. Now,
in the case of small $\beta$, the equation for the scalar field is

$$\frac{d^2}{dr^2} \Theta^{lm} + \left[ \omega^2 - V(r) \right] \Theta^{lm} = -f \frac{(l+2)!}{(l-2)!} \frac{6iM}{r^5 \beta} Q^{lm},$$

(25)

with $V = f \left( \frac{l(l+1)}{r^2} (1 - 576\pi M^2 \beta^{-1}/r^6) + 2M/r^3 \right)$. A study of the homogeneous equation (i.e., we set $Q = 0$ in the above), shows that unstable modes are possible since the effective potential is negative in some regions. Since the potential is bounded in $-\infty < r_* < +\infty$ and it vanishes at both extrema, a sufficient condition for existence of a bound state of negative energy is

$$\int_{-\infty}^{+\infty} V(r_*) dr_* = \frac{7\beta (2l^2 + 2l + 1) M^4 - 18l(1+l)\pi}{28\beta M^5} < 0.$$  

(26)

This condition is satisfied whenever

$$\beta < \frac{18l(l+1)\pi}{7(2l^2 + 2l + 1) M^4},$$

(27)

which yields $\beta < 108\pi/(91M^4)$, $216\pi/(175M^4)$ for $l = 2, 3$ respectively. An example of an unstable $l = 2$ mode of the homogeneous equation is shown in Figure 1 for $\beta = 1$. This does not prove the existence of unstable modes for the coupled system, but it is a strong argument in its favor.

Numerical integration of the coupled system (17), (18) shows that unstable modes, indeed, do exist. In our search, we set $B_{\text{in}} = 1$, and look for solutions with $\omega$ purely imaginary; in this case, the boundary conditions (21) imply that the functions $Q, \Theta$ vanish at $r \to \infty$.

A typical unstable mode of the coupled system is shown in Figure 1 for $\beta = 1$. The overall qualitative and quantitative behavior of the scalar field $\Theta$ is in agreement with the analysis of the homogeneous scalar wave equation. The instability timescale increases (i.e. Im $\omega$ decreases) when $\beta$ increases. For instance, $M\omega \sim 0.428i, 0.263i$ for $M^4\beta = 1, 2$ respectively. This was to be expected from the analysis of the homogeneous scalar equation. We were not able to find unstable modes for $\beta M^4$ larger than $\sim 2\pi$. This is again in agreement with the heuristic arguments above, though a more systematic approach to investigate this issue is necessary.

We have then shown that spherically symmetric BH in DCS modified gravity admit unstable modes for $\beta M^4 \lesssim 2\pi$; therefore, since astrophysical black holes exist with masses $M_{\text{astro}} \gtrsim 3M_\odot \simeq 5\ km$, we have the constraint

$$\beta \gtrsim \frac{2\pi}{M_{\text{astro}}} \gtrsim 10^{-2}\ km^{-4},$$

(28)
much stronger than (7). The present analysis does not directly apply to astrophysical BHs, since these are rotating in general. Rotating black holes in this theory require a non-vanishing background scalar field \cite{25,32} and therefore our entire analysis needs to be modified. On the other hand, it is hard to imagine that rotation can drastically alter the instability regime. We thus believe that the bound (28) should be considered as a strong indication on the values that we may expect for $\beta$ in the general case.

We remark that the normalization choice $\alpha = 1$ does not really affect our results. If we do not fix the parameter $\alpha$, some terms in equations (17), (18) get multiplied by $\alpha$ or $\alpha^2$, and the bound (28) becomes

$$\frac{\beta}{\alpha^2} \gtrsim \frac{2\pi}{M_{\text{astro}}^4} \gtrsim 10^{-2}\text{km}^{-4}.$$  \tag{29}

Note that the quantity $\alpha^2/(\beta \kappa)$ is what is called $\xi$ in \cite{25}, where the bound (7) is derived.

V. CONCLUSIONS

We have studied perturbations of black holes in the context of dynamical Chern-Simons theory. Under the assumption that the scalar field is of the order of the gravitational perturbation, we found that axial and polar parity gravitational perturbations are decoupled, and only axial parity perturbations are coupled with the scalar field. The equations describing these perturbations are fairly simple, but their numerical integration is tricky: our attempts to generalize to DCS gravity either the recurrence relation approach of \cite{35,40}, or the Riccati equation approach of \cite{30}, have been unsuccessful. Still, we have been able to extract some interesting information from the perturbation equations.

We found that black holes in this theory can develop strong instabilities. The constraint on the coupling parameters necessary to avoid the instability is stringent. We obtain $\beta M^4 > 2\pi$ for the spacetime to be stable, which translates to $\beta \gtrsim 10^{-2}\text{ km}^{-4}$. Thus, the observation of stellar-mass black holes imposes a constraint on the coupling parameter which is about $10^{13}$ times more stringent than bounds from binary pulsar dragging effects (Yunes and Pretorius \cite{25} obtain $\beta \gtrsim 10^{-15}\text{ km}^{-4}$ from the pulsar PSR J0737-3039 A/B). This bound has been derived under the assumption that the BH is non-rotating, but it should provide a reliable estimate on the range of allowed $\beta$ in the general case.

Furthermore, we found some evidence that in the limit $\beta \to \infty$, there are no non-trivial QNM solutions: the only allowed solutions in this limit seem to be the ordinary Schwarzschild gravitational QNM, with a vanishing scalar field. Much more remains to be done. The coupled system of equations (17) and (18) was not solved in general in the present paper, nor were the exact limits of instability investigated. It would be extremely interesting to study the spectra of this system, either by analytical approximations or by numerical means. A promising approach would consist in doing wave scattering using time-evolution methods \cite{41}, and reading off the quasinormal modes directly from the frequency and decay time of these perturbations.

The formalism developed here can be used to study point particles in the background geometry and to compute the gravitational wave signal generated by extreme-mass-ratio inspirals. Generalization to rotating black holes is a major (rotating black hole solutions are only partially understood \cite{12}), but fundamental task to accomplish.

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APPENDIX A: RECURRENCE RELATIONS

Here we try to generalize the continued fraction method \cite{35,40} to solve Eqs (17), (18). The functions $Q, \Theta$ with the correct asymptotic behavior can be written as

$$Q = \left(\frac{r}{2M} - 1\right)^{-2iM\omega} \left(\frac{r}{2M}\right)^{4iM\omega} e^{i\omega(r-2M)} \Phi,$$

$$\Theta = \left(\frac{r}{2M} - 1\right)^{-2iM\omega} \left(\frac{r}{2M}\right)^{4iM\omega} e^{i\omega(r-2M)} \Sigma,$$  \tag{A1}
where $\Psi, \Sigma$ can be expressed as
\[
\Phi = \sum_{n=0}^{\infty} a_n y^n, \quad \Sigma = \sum_{n=0}^{\infty} b_n y^n
\] (A2)
in terms of the dimensionless variable
\[
y \equiv 1 - \frac{2M}{r}.
\] (A3)
Note that $a_n < 0 = b_n < 0$; we set the overall normalization by imposing $a_0 = 1$, and the (complex) constant $b_0$ is an unknown of the problem.

Replacing these expressions in the perturbation equations (17), (18) we find the recurrence relations
\[
a_n \alpha_n + a_{n-1} \beta_n + a_{n-2} \gamma_n + \lambda \left( b_{n-2} \sigma_1^n + \ldots + b_{n-7} \sigma_6^n \right)
\]
\[
b_n \alpha_n + b_{n-1} \beta_n + b_{n-2} \gamma_n + b_{n-3} \delta_n + \rho \left( a_{n-2} \sigma_1^n + \ldots + a_{n-7} \sigma_6^n \right)
\] (A4)
where
\[
\alpha_n = n(n - 4iM\omega)
\]
\[
\beta_n = -2n^2 + (16iM\omega + 2)n - 8iM\omega + 32M^2\omega^2 - l(l + 1) - 1
\]
\[
\gamma_n = n^2 - (8iM\omega + 2)n + 8iM\omega - 16M^2\omega^2 + 1
\]
\[
\tilde{\beta}_n = \beta_n + 4 + \frac{9\pi l(l + 1)}{\beta M^4}
\]
\[
\tilde{\gamma}_n = \gamma_n - 4 - \frac{54\pi l(l + 1)}{\beta M^4}
\]
\[
\tilde{\delta}_n = \frac{135\pi l(l + 1)}{\beta M^4}
\]
\[
\lambda = \frac{3\pi l\omega}{M^4}
\]
\[
\rho = \frac{(l + 2)!}{(l - 2)!} \frac{3i\omega}{16\beta M^4}
\]
\[
\tilde{\sigma}_n = (1, -5, 10, -10, 5, -1).
\] (A5)
The two complex recurrence relations (A4), depending on the two unknown complex quantities $(\omega, b_0)$, in principle converge only for a discrete set $(\omega, b_0)_j$, corresponding to the QNM; it is not clear how to numerically implement this condition, since it seems not possible to express the recurrence relations (A4) in terms of continued fractions.

APPENDIX B: ASYMPTOTIC BEHAVIOR OF THE PERTURBATION EQUATIONS

We first note that, as $r_+ \to \pm \infty$ (i.e. $r \to r_H \equiv 2M, r \to +\infty$), the perturbation equations (17) and (18) reduce to the simple coupled wave equations
\[
\left( \frac{d^2}{dr^2} + \omega^2 \right) \Theta = -f \frac{(l + 2)!}{r^3 (l - 2)!} 6iM \omega Q,
\] (B1)
\[
\left( \frac{d^2}{dr^2} + \omega^2 \right) Q = -f \frac{96iM \omega}{r^5} \Theta.
\] (B2)
Changing scalar wavefunction
\[
\Theta^{\text{old}} \to \frac{1}{4\omega} \sqrt{\frac{(l + 2)!}{(l - 2)!}} \pi \beta \Theta^{\text{new}}
\] (B3)
we get
\[
\left( \frac{d^2}{dr_+^2} + \omega^2 \right) \phi_{\text{new}} = -\frac{f T}{r^5} Q, \quad \text{(B4)}
\]
\[
\left( \frac{d^2}{dr_+^2} + \omega^2 \right) Q = -\frac{f T}{r^5} \phi_{\text{new}}, \quad \text{(B5)}
\]
with \( T = \sqrt{\frac{(l+2)!}{(l-2)!}} 24iM \sqrt{\pi/\beta} \). Finally, defining \( \Psi^+ \equiv Q + \phi_{\text{new}} \) and \( \Psi^- \equiv Q - \phi_{\text{new}} \), one gets the equivalent system
\[
\left( \frac{d^2}{dr_+^2} + \omega^2 \right) \Psi^+ = -\frac{f T}{r^5} \Psi^+, \quad \text{(B6)}
\]
\[
\left( \frac{d^2}{dr_+^2} + \omega^2 \right) \Psi^- = \frac{f T}{r^5} \Psi^-. \quad \text{(B7)}
\]
Each of the wavefunctions \( \Psi^\pm \) has a simple harmonic behavior at the boundaries and therefore (20) follows.

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