We consider the simplest model of a passive biped walking down a slope given by the equations of switched coupled pendula (McGeer T. 1990 Passive dynamic walking. *Int. J. Robot. Res.* 9, 62–82. (doi:10.1177/027836499000900206)). Following the fundamental work by Garcia (Garcia *et al*. 1998 *J. Biomech. Eng.* 120, 281. (doi:10.1115/1.2798313)), we view the slope of the ground as a small parameter $\gamma \geq 0$. When $\gamma = 0$, the system can be solved in closed form and the existence of a family of cycles (i.e. potential walking cycles) can be computed in closed form. As observed in Garcia *et al*. (Garcia *et al*. 1998 *J. Biomech. Eng.* 120, 281. (doi:10.1115/1.2798313)), the family of cycles disappears when $\gamma$ increases and only isolated asymptotically stable cycles (walking cycles) persist. However, no mathematically complete proofs of the existence and stability of walking cycles have been reported in the literature to date. The present paper proves the existence and stability of a walking cycle (long-period gait cycle, as termed by McGeer) by using the methods of perturbation theory for maps. In particular, we derive a perturbation theorem for the occurrence of stable fixed points from 1-parameter families in two-dimensional maps that can be of independent interest in applied sciences.

1. Introduction

In his celebrated paper [1], McGeer proposed to view the passive bipedal walker of figure 1 as a double pendulum, where the roles of the two links swap each time the heelstrike occurs (i.e. swing leg collides with the ground). Assuming that the masses of the feet are negligible
Figure 1. The double pendulum model of a planar biped.

compared to the weight of the body, the paper Garcia et al. [2] formulated double pendulum model of bipedal walker between heelstrikes as

\[
\begin{align*}
\ddot{\theta} - \sin(\theta - \gamma) &= 0 \\
\ddot{\phi} + \dot{\phi}^2 \sin \phi - \cos(\theta - \gamma) \sin \phi &= 0,
\end{align*}
\]

(1.1)

where $\gamma \geq 0$ is the slope of the ground (figure 1). The length of the legs and the gravity acceleration constants have been removed from equations (1.1) by a suitable time rescaling [2].

When the heelstrike occurs (i.e. when $\phi = 2 \theta$), the stance and swing legs swap their roles and the state vector $(\theta, \dot{\theta}, \phi, \dot{\phi})^T$ jumps as follows:

\[
\begin{pmatrix}
\dot{\theta}(t^+) \\
\dot{\phi}(t^+) \\
\phi(t^+) \\
\dot{\phi}(t^+)
\end{pmatrix} = J(\theta(t))
\begin{pmatrix}
\theta(t^-) \\
\dot{\theta}(t^-) \\
\phi(t^-) \\
\dot{\phi}(t^-)
\end{pmatrix}, \text{ if } \phi(t) = 2 \theta(t),
\]

(1.2)

where

\[
J(\theta) =
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \cos 2\theta & 0 & 0 \\
-2 & 0 & 0 & 0 \\
0 & (1 - \cos 2\theta) \cos 2\theta & 0 & 0
\end{pmatrix}.
\]

Using Newton’s method, McGeer found that a certain linearization of switched system (1.1) and (1.2) admits a limit cycle, whose period is close to $T = 3.8$ for small values of slope $\gamma > 0$. In the same paper [1], McGeer reports experimental results that appear to be in agreement with analytic conclusions. Stable limit cycling of a passive biped has been then observed in various other experiments (including three-dimensional printed passive walkers; e.g. [3–6]).

McGeer’s model stimulated a variety of new studies in robotics lately. Freidovich et al. [7] offered a method (virtual holonomic constraints) that reduces the dimension of (1.1) from 4 to 2. A relevant method that reduces the analytic computations in finding limit cycles of a passive biped walking down a staircase is proposed in Tehrani Safa et al. [8] (where a numeric approach is then used to apply the proposed reduction technique). Kuang-Shen [9], Asano-Harata [10] and Or [11] accounted for possible effects coming from friction and sliding during hill-strike (see also related studies by Ivanov [12]). Limit cycling in more complex passive walking devices were studied in, for example, Sabaapour et al. [13] (inclusion of passive turning) and Li et al. [14] (walker with upper body). An analysis of various bifurcation phenomena in passive walkers (including those related to the increase of the slope) can be found in [14–18]. The structure of the basin of attraction of the walking cycle is investigated by Obayashi et al. [19].

To achieve finest energy efficiency results, the control strategies in more complex robots are typically designed to render the dynamics to that of a reference simple passive walker, in which way the control strategy aims at taking the most from the internal dynamics of the passive walker (whose dynamics does not consume external energy; e.g. [7,20–23]). Some strategies in
this direction employ optimization methods in order to come up with the best reference passive walker (e.g. [24]). Learning the ideas available for the optimization of cart- pendulum swing-up (see [25,26], where graphs of trajectories are available in closed form) might improve the optimality of the choice of a reference walker further. The knowledge of limit cycles of simple passive walkers is therefore in the heart of the design of more complex walkers. The proposed analytic results and formulae serve as a tool for the development of an analytic theory of such design (which does not exist to date).

A justification of the existence of such a limit cycle was offered in Garcia et al. [2], where the change of the variables

\[ \gamma = \delta^{3/2}, \quad \theta(t) = \delta^{1/2} \vartheta(t) \quad \text{and} \quad \phi(t) = \delta^{1/2} \Phi(t) \]  

(1.3)
is proposed to expand (1.1) and (1.2) in the powers of small parameter \( \delta > 0 \) and to investigate the existence of the limit cycle based on the leading-order terms. The paper [2] follows a rather rigorous approach (Fredholm alternative) when establishing necessary conditions for the existence of a cycle in switched system (1.1) and (1.2). However, the proof of sufficiency is not addressed in [2] and the proof of stability is addressed briefly. More comprehensive analysis of stability of a limit cycle to (1.1) and (1.2) is carried out in [16,27–29] via Astrom’s like stability criterion [30] (see [31–34] for more results on stability properties of passive biped). Still, the existence of a limit cycle has been assumed given (i.e. computed numerically) in all the papers cited above and sufficient conditions for such an existence have not been derived and verified.

The goal of the present paper is to provide a tool that is capable of proving the existence of a stable limit cycle in switched system (1.1) and (1.2) with a mathematical level of rigour. To this aim, we establish a perturbation theorem for two-dimensional maps, that allows to investigate bifurcation of asymptotically stable fixed points from a 1-parameter family. Such a theorem did not seem to appear in the earlier literature despite of its crucial importance for reliability of passive walking stability analysis. On a more applied side, the present paper provides a method to derive explicit formulae for the Floquet multipliers of the walking cycle (see corollary 7.1), that can be used to tune the parameters of the passive walker (such as e.g. mass distribution in legs and the joint stiffness) with the purpose of strengthening the stability of the walking cycle. Only experimental and numerical results in this direction are currently available (e.g. [5,35]).

The paper is organized as follows. In the next section, we incorporate the change of the variables (1.3) in switched system (1.1) and (1.2) and obtain a switched system (2.1) and (2.2) with a small parameter \( \delta > 0 \) (which corresponds to a perturbation term). In §3, we follow the idea of Garcia et al. [2] and introduce a two-dimensional Poincare map \( \varphi(\theta, \omega) \mapsto P(\theta, \omega, \delta) \) associated with the perturbed switched system (2.1) and (2.2). In §4, we show that, when \( \delta = 0 \), the Poincare map \( \varphi(\theta, \omega) \mapsto P(\theta, \omega, \delta) \) admits a family of fixed points \( (\varphi(\theta, \omega) = \xi(s)) \), where \( \xi \in C^1(\mathbb{R}, \mathbb{R}^2) \) and \( s \) is a parameter. In this way, the problem of the existence of limit cycles to the perturbed switched system (2.1) and (2.2) reformulates as a problem of bifurcation of asymptotically stable fixed points to the Poincare map \( \varphi(\theta, \omega) \mapsto P(\theta, \omega, \delta) \) from the family \( \varphi(\theta, \omega) = \xi(s) \) as \( \delta \) crosses \( 0 \). The problem obtained is a classical problem of the theory of nonlinear oscillations coming back to Malkin [36] and Melnikov [37, ch. 4, §6], and addressed in [38–45] and other works. In the present paper, we follow references [43,44] to provide (in §5) a concise perturbation theorem (theorem 5.1) on bifurcation of fixed points from families in Poincare maps. This perturbation theorem is then applied to the Poincare map \( \varphi(\theta, \omega) \mapsto P(\theta, \omega, \delta) \) of the passive biped in §§6 and 7. In §8 (Conclusion), we discuss the value of this work to the fields of perturbation theory and robotics. The proof of theorem 5.1 is given in appendix A. Finally, appendix B contains some technical formulae. All symbolic computations have been executed in Wolfram Mathematica, which code is uploaded as electronic supplementary material.
2. Expanding McGeer’s model of passive biped into the powers of the slope of the ground

Incorporating the change of variables (1.3) into the switched system (1.1) and (1.2) and using that

\begin{align*}
\sin \tau &= \tau - \frac{\tau^3}{3!} + \frac{\tau^5}{5!} - \frac{\tau^7}{7!} + \cdots \quad \text{and} \quad \cos \tau &= 1 - \frac{\tau^2}{2!} + \frac{\tau^4}{4!} - \frac{\tau^6}{6!} + \cdots ,
\end{align*}

one gets [2]

\[
\begin{aligned}
\dot{\Theta} - (\Theta - \delta) + \frac{1}{6} \delta \dot{\Theta}^3 + o_1(\delta) &= 0 \\
\dot{\Theta} - \Phi - \dot{\Theta}^2 \Phi + \frac{1}{2} \delta \dot{\Theta} \Phi + \frac{1}{6} \delta \Phi^3 + o_2(\delta) &= 0,
\end{aligned}
\]

(2.1)

and

\[
\begin{pmatrix}
\Theta(t^+) \\
\dot{\Theta}(t^+) \\
\Phi(t^+) \\
\dot{\Phi}(t^+)
\end{pmatrix} =
J(\Theta(t), \delta)
\begin{pmatrix}
\Theta(t^-) \\
\dot{\Theta}(t^-) \\
\Phi(t^-) \\
\dot{\Phi}(t^-)
\end{pmatrix}, \quad \text{if } \Phi(t) = 2\Theta(t),
\]

(2.2)

where

\[
J(\Theta, \delta) =
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 - \frac{1}{2} \delta (2\Theta)^2 + o_3(\delta) & 0 & 0 \\
-2 & 0 & 0 & 0 \\
0 & \left(1 - \frac{1}{2} \delta (2\Theta)^2 + o_3(\delta)\right) & \left(\frac{1}{2} \delta (2\Theta)^2 + o_4(\delta)\right) & 0
\end{pmatrix}
\]

and \(o_i(\delta)\) stay for the remainders (perhaps dependent on \(\Theta\) and \(\Phi\)) such that \(o_i(\delta)/\delta \to 0\) as \(\delta \to 0\) uniformly with respect to \((\Theta, \Phi)\) from any compact set. In particular, the statements of this paper deal sufficiently small values of \(\delta > 0\) only (i.e. with sufficiently small values of the slope and step magnitude; see theorem 5.1 and corollary 7.1).

Note, depending on the context, the variable \(\Theta\) either stays for a function or for an element of \(\mathbb{R}\). Specifically, \(\Theta\) is a function when it stays in a differential equation or when \(\Theta\) comes with arguments. \(\Theta\) is an element of \(\mathbb{R}\) otherwise. Same type of notations is used for the variable \(\dot{\Theta}\).

3. The Poincaré map induced by the heelstrike threshold

To construct the Poincaré map induced by the cross section \(\Phi = 2\Theta\), we will consider the initial condition \((\Theta(0^+), \dot{\Theta}(0^+), \Phi(0^+), \dot{\Phi}(0^+))^T\) given by (2.2). Because of the properties of the matrix \(J(\Theta, \delta)\) any vector \((\Theta(0^+), \dot{\Theta}(0^+), \Phi(0^+), \dot{\Phi}(0^+))^T\) coming from (2.2) has the form

\[
(\Theta(0^+), \dot{\Theta}(0^+), \Phi(0^+), \dot{\Phi}(0^+)) = \left(\theta, \omega, 2\theta, \left(2\delta \theta^2 + o_4(\delta)\right) \omega\right).
\]

(3.1)

Therefore, in order to construct a solution \(t \mapsto (\Theta(t), \dot{\Theta}(t), \Phi(t), \dot{\Phi}(t))\) of system (2.1) that originates from the hyperplane \(\Phi = 2\Theta\) at time \(t = 0\) we have to know only \((\Theta(0), \dot{\Theta}(0))\). The value of \((\Phi(0), \dot{\Phi}(0))\) can be then computed as \((\Phi(0), \dot{\Phi}(0)) = (2\Theta(0), (2\delta \Theta(0)^2 + o_4(\delta)) \dot{\Theta}(0))\). Such an observation allows us to introduce a Poincaré map \(P\) of only two dimensions that

(i) maps an initial condition \((\Theta(0), \dot{\Theta}(0))\) to the point \((\Theta(T^-), \dot{\Theta}(T^-))\) (where \(T > 0\) is the time of the nearest heel-strike), and then

(ii) applies the impact law that maps \((\Theta(T^-), \dot{\Theta}(T^-))\) to \((\Theta(T^+), \dot{\Theta}(T^+))\) according to (2.2).
Observe that \( (\theta, \dot{\theta}) \) correspond to ‘reasonably anthropomorphic gaits’. Also, following Garcia et al. [2], only roots within the \((0, 2\pi)\) correspond to ‘reasonably anthropomorphic gaits’. Also, following Garcia et al. [2], we will stick...
to the second root $T_2$ because it corresponds to a symmetric gait in the following sense: plugging $\omega = \alpha(T_2)\theta$ into the third equation of (4.4) gives approximately

$$-1.5339e^{-1} + 0.0339021e^t + 1.5 \cos t + 0.522601 \sin t = 0,$$

whose only solution on $(0, T_2)$ is $T_2/2$ where one has

$$\Theta \left( \frac{T_2}{2}, \theta, \omega, 0 \right) = \Phi \left( \frac{T_2}{2}, \theta, \omega, 0 \right) = 0. \quad (4.8)$$

Property (4.8) corresponds to the event where the two legs coincide. Though (4.8) formally implies a heel-strike (the third equation of (4.4) holds at $T = T_2/2$), it corresponds to just grazing of the swing leg through the floor and no impact event physically occurs. If the value of $\gamma$ increases, then, formally speaking, an impact occurs at $T = T_2/2$, but we will still ignore the impact coming from $T = T_2/2$ as motivated by the experiments (the actual experimental passive planar walker makes slight swings in the third dimension which rules out the impact at $T = T_2/2$, [3]). In other words, for the reasons just explained and following Garcia et al. [2], we will consider the Poincare map (3.3) with

$$T(\theta, \omega, \delta) \to T_2 \quad \text{as} \quad \delta \to 0$$

which satisfies the first condition of (3.5) even though it ‘slightly’ violates the second condition of (3.5) in the neighbourhood of $T_2/2$.

To summarize, for each $\theta \in (\pi/2, \pi/2)$, the point $\omega = \alpha(T_2)\theta$ satisfies the periodicity condition (4.4). Furthermore, the solution of (4.1) and (4.2) with the initial condition $(\theta, \alpha(T_2)\theta, 2\theta, 0)$ has only tangential collision with the cross section $\Phi = 2\theta$ on the interval $(0, T_2)$. Therefore, for each $\theta \in (\pi/2, \pi/2)$, the point $(\theta, \alpha(T_2)\theta, 2\theta, 0)$ is the initial condition of a $T_2$-periodic cycle of switched system (4.1) and (4.2), which experiences exactly one impact per period. A few sample cycles of this family are shown in figure 2a. Accordingly, for each $\theta \in (\pi/2, \pi/2)$, the point $(\theta, \alpha(T_2)\theta)$ is a fixed point of the Poincare map $(\theta, \omega) \mapsto P(\theta, \omega, 0)$. The line segment $\{(\theta, \alpha(T_2)\theta)\}_{\theta \in (-\pi/2, \pi/2)}$ is, therefore, a family of fixed points of the Poincare map $(\theta, \omega) \mapsto P(\theta, \omega, 0)$. This family will be termed a family of long-period fixed points. It is drawn in figure 2b. The set of points $(\theta, \omega)$ given by (4.5) with $T = T_1$ defines a different line segment, which is shown as dashed in figure 2b (we do not use this second line segment in the paper because it does not feature a property like (4.8) and some different justification is required to use the points of this line segment as fixed points of the Poincare map).

5. Perturbation theorem for two-dimensional Poincare maps

Throughout this section, we consider a general two-dimensional Poincare map $(\theta, \omega) \mapsto P(\theta, \omega, \delta) = (P_1(\theta, \omega, \delta), P_2(\theta, \omega, \delta))^T$ with $P_i$ being scalar functions, under an assumption that the unperturbed Poincare map $(\theta, \omega) \mapsto P(\theta, \omega, 0)$ admits a family of fixed points, i.e. $P(\xi(s), 0) = \xi(s)$ for all $s \in \mathbb{R}$, with $s \mapsto \xi(s)$ being a $C^1$ curve. Note, the latter property implies that $P_{\theta, \omega}(\xi(s), 0)\xi'(s) = \xi'(s)$, where $P_{\theta, \omega}(\xi(s), 0)$ stays for the Jacobian

$$P_{\theta, \omega}(\theta, \omega, \delta) = \begin{pmatrix}
\frac{\partial P_1}{\partial \theta}(\theta, \omega, \delta) & \frac{\partial P_1}{\partial \omega}(\theta, \omega, \delta) \\
\frac{\partial P_2}{\partial \theta}(\theta, \omega, \delta) & \frac{\partial P_2}{\partial \omega}(\theta, \omega, \delta)
\end{pmatrix}.$$

As a consequence, one of the eigenvalues of the matrix $P_{\theta, \omega}(\xi(s), 0)$ always equals 1 for all $s \in \mathbb{R}$. To make the notations less bulky, we will identify $P(\theta, \omega, \delta)$ with $P((\theta, \omega), \delta)$ as it does not seem to cause any confusion.

Fix some $s_0 \in \mathbb{R}$ and put

$$(\theta_0, \omega_0) = \xi(s_0).$$

Denote by $y$ and $\tilde{y}$ the eigenvectors of $P_{\theta, \omega}(\theta_0, \omega_0, 0)$ that correspond to the eigenvalues 1 and $\rho \neq 1$, respectively. We then denote by $z$ and $\tilde{z}$ the eigenvectors of $P_{\theta, \omega}(\theta_0, \omega_0, 0)^T$ that correspond
Figure 2. (a) Sample members of the family of cycles of switched system (4.1) and (4.2). The four initial conditions are: \((\Theta(0), \dot{\Theta}(0), \Phi(0), \dot{\Phi}(0)) = (\theta_0 - 0.6, \alpha(T_2)(\theta_0 - 0.6), 2(\theta_0 - 0.6), 0), \) (top initial condition) \((\Theta(0), \dot{\Theta}(0), \Phi(0), \dot{\Phi}(0)) = (\theta_0 - 0.3, \alpha(T_2)(\theta_0 - 0.3), 2(\theta_0 - 0.3), 0), \) \((\Theta(0), \dot{\Theta}(0), \Phi(0), \dot{\Phi}(0)) = (\theta_0, \alpha(T_2)\theta_0, 2\theta_0, 0), \) \((\Theta(0), \dot{\Theta}(0), \Phi(0), \dot{\Phi}(0)) = (\theta_0 + 0.3, \alpha(T_2)(\theta_0 + 0.3), 2(\theta_0 + 0.3), 0), \) (bottom initial condition) where \(\theta_0\) and \(\lambda(T_2)\) are given by (7.5), (4.5) and (4.7). In particular, the period of each of the cycles shown is exactly \(T_2\), as defined in (4.7). (b) Two families of the initial conditions of the cycles of switched system (4.1) and (4.2). The dashed line segment corresponds to the initial conditions given by (4.5) with \(T = T_1\) and the solid line segment corresponds to \(T = T_2\). In other words, the points of the solid line segment are some representatives of the family of fixed points of the Poincare map \((\theta, \omega) \mapsto P(\theta, \omega, 0)\) given by formula (4.5) with \(T = T_2\). The solid point stays for \((\theta_0, \omega_0)\) that transforms into an asymptotically stable fixed point as \(\delta\) crosses 0 (see corollary 7.1). (Online version in colour.)

Assume that the eigenvector \(z\) of \(P(\theta, \omega)\) that corresponds to the eigenvalue 1 does not depend on \(s_0\), in which case we have

\[ z^T P(\theta, \omega, 0) = 0, \quad \text{for all } s \in \mathbb{R}. \] (5.4)

The following theorem can be derived from the results of Kamenski et al. [43] and Makarenkov-Ortega [44].

**Theorem 5.1.** Let \(P\) be a \(C^3\) function. If, for each \(\delta \in \mathbb{R}\), the Poincare map \((\theta, \omega) \mapsto P(\theta, \omega, \delta)\) admits a fixed point \((\theta_0, \omega_0)\) such that

\[ (\theta_0, \omega_0) \to (\theta_0, \omega_0) \quad \text{as } \delta \to 0, \] (5.5)

then

\[ z^T P(\theta_0, \omega_0, 0) = 0. \] (5.6)

Assume that the eigenvector \(z\) of \(P(\theta, \omega)(\theta_0, \omega_0, 0)^T\) that corresponds to the eigenvalue 1 does not depend on \(s_0\). If, in addition to (5.6), it holds that

\[ z^T (P(\theta_0, \omega_0, 0)y \neq 0, \] (5.7)
then, for all $|\delta|$ sufficiently small, the Poincare map $(\theta, \omega) \mapsto P(\theta, \omega, \delta)$ does indeed have a fixed point $(\theta_0, \omega_0)$ that satisfies (5.5). The fixed point $(\theta_0, \omega_0)$ is asymptotically stable, if the eigenvalue $\rho \neq 1$ of $P(\theta, \omega)(\theta_0, \omega_0, 0)$ satisfies
\[ |\rho| < 1, \] (5.8)
and if (5.7) holds in the stronger sense
\[ z^T(P_\delta)(\theta_0, \omega_0, 0)\gamma < 0. \] (5.9)

In what follows, we use theorem 5.1 in order to prove the existence and stability of a limit cycle to (2.1) and (2.2). The proof of theorem 5.1 is given in A.

6. Stability of the family of long-period fixed points

In this section, we analyse the Poincare map $(\theta, \omega) \mapsto P(\theta, \omega, 0)$ of the reduced switched system (4.1) and (4.2).

As explained in §5, one of the eigenvalues of matrix $P(\theta, \alpha(T_2)\theta, 0)$ is always 1. In this section, we compute the second eigenvalue (named $\rho$) of $P(\theta, \alpha(T_2)\theta, 0)$ and verify condition (5.8) of theorem 5.1. We will see that $\rho$ does not depend on $\theta$, so we write $\rho$ as opposed to $\rho(\theta)$ from the beginning.

In what follows, the following notation will be used:
\[ \tau = (T_2, \theta, \alpha(T_2)\theta, 0) \]
to shorten the formulae.

Differentiating (3.3) with respect to the vector variable $(\theta, \omega)$
\[ P(\theta, \omega)(\theta, \alpha(T_2)\theta, 0) = \Delta_0 \begin{pmatrix} \Theta \\ \Theta \end{pmatrix} (\tau)T_{(\theta, \omega)}(\theta, \alpha(T_2)\theta, 0) + \Delta_0 \begin{pmatrix} \Theta \\ \Theta \end{pmatrix} (\theta, \omega) \]
where
\[ \Delta_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \]

Using formulae (4.3) and (4.7), one gets
\[ \begin{pmatrix} \Theta \\ \Theta \end{pmatrix} (T_2, \theta, \omega, 0) = \frac{\Theta(t(T_2, \theta, \omega, 0)}{\Theta(t(T_2, \theta, \omega, 0)} = \begin{pmatrix} 22.6114\theta + 22.6335\omega \\ 22.6335\omega + 22.6114\omega \end{pmatrix} \]
and so
\[ \begin{pmatrix} \Theta \\ \Theta \end{pmatrix} (\tau) = \theta \begin{pmatrix} -1.0452 \\ -1 \end{pmatrix}. \]

In the same way,
\[ \begin{pmatrix} \Theta \\ \Theta \end{pmatrix} (\theta, \omega) = \begin{pmatrix} \Theta_\theta(\tau) & \Theta_\omega(\tau) \\ \Theta_\theta(\tau) & \Theta_\omega(\tau) \end{pmatrix} = \begin{pmatrix} 22.6335 & 22.6114 \\ 22.6114 & 22.6335 \end{pmatrix}. \]

The formula for the derivative of the implicit function (e.g. [46, Sec. 8.5.4 Theorem 1]) further yields
\[ T_{(\theta, \omega)}(\theta, \alpha(T_2)\theta, 0) = - (F_1(T_2, \theta, \alpha(T_2)\theta))^{-1} F_{(\theta, \omega)}(T_2, \theta, \alpha(T_2)\theta), \] (6.1)
where
\[ F(t, \theta, \omega) = \Phi(t, \theta, \omega, 0) - 2\Theta(t, \theta, \omega, 0). \] (6.2)

Plugging formulae (4.3) and (4.7) into (6.1), the function $T_{(\theta, \omega)}(\theta, \alpha(T_2)\theta, 0)$ computes as
\[ T_{(\theta, \omega)}(\theta, \alpha(T_2)\theta, 0) = \frac{1}{\theta}(16.8032, 16.0765). \]
Combining the above findings together, we finally get

$$P_{(\theta,\omega)}(\theta, \alpha(T_2)\theta, 0) = \begin{pmatrix} -5.07075 & -5.8082 \\ 5.8082 & 6.55701 \end{pmatrix}$$

(6.3)

whose eigenvalues are 1 and

$$\rho = 0.48626,$$  

(6.4)

so that condition (5.8) holds.

7. Bifurcation of isolated fixed points from the family of long-period fixed points when the angle of the slope changes from zero to a small positive value

In this section, we verify the remaining conditions (5.6), (5.7) and (5.9) of theorem 5.1 for the Poincare map \((\theta,\omega) \mapsto P(\theta,\omega,\delta)\) of switched system (2.1) and (2.2).

(a) Computing the first-order expansion of the Poincare map with respect to the slope parameter

Here, we compute \(P_\delta\).

Differentiating (3.3) with respect to \(\delta\), one gets

$$P_\delta(\theta,\omega,0) = \Delta_0 \left( \frac{\partial}{\partial \theta} \right)_t (T(\theta,\omega,0),\theta,\omega,0)T_\delta(\theta,\omega,0) + \Delta_0 \left( \frac{\partial}{\partial \omega} \right)_t (T(\theta,\omega,0),\theta,\omega,0)$$

$$+ \Delta_\delta \left( (\frac{\partial}{\partial \delta})_T (T(\theta,\omega,0),\theta,\omega,0),0 \right).$$

(7.1)

The terms \(\Delta_0\) and \(\left( \frac{\partial}{\partial \theta} \right)_t (\tau)\) were computed in the previous section. For the terms \(\Delta_\delta(\theta,\omega,0)\) and \(\left( \frac{\partial}{\partial \omega} \right)_t (\tau)\), the definition of \(\Delta(\theta,\omega,\delta)\) and formula (4.4) yield

$$\Delta_\delta(\theta,\omega,0) = \begin{pmatrix} 0 \\ -2\theta^2 \omega \end{pmatrix}, \quad \left( \frac{\partial}{\partial \omega} \right)_t (\tau) = \begin{pmatrix} -\theta \\ \alpha(T_2)\omega \end{pmatrix}.$$

To compute \(T_\delta(\theta,\omega,0)\), we can use function \(F\) of the previous section, which gives

$$T_\delta(\theta,\omega,0) = - (F_1(T_2,\theta,\omega))^{-1} F_\delta(T_2,\theta,\omega).$$

(7.2)

So it remains to compute the function \(t \mapsto ((\theta, \Phi)^T)_\delta(t,\theta,\omega,0)\), which can be found as the solution \(t \mapsto (h(t),f(t))^T\) of the \(\delta\)-derivative of the initial-value problem (2.1) and (3.2):

$$\begin{cases}
\ddot{h} - h + 1 + \frac{1}{6} \theta(t,\sigma) = 0, \\
\ddot{h} - \dot{f} + (\dot{\theta}(t,\sigma))^2 \Phi(t,\sigma) + \frac{1}{2} (\dot{\beta}(t,\sigma))^2 \Phi(t,\sigma) + \frac{1}{6} (\Phi(t,\sigma))^3 = 0 \\
\dot{h}(0) = 0, \quad \dot{h}(0) = 0, \quad f(0) = 0, \quad f(0) = 2\theta^2 \omega,
\end{cases}$$

(7.3)

and

where \(\sigma\) is a shortcut for \(\sigma = (\theta,\omega,0)\). After plugging (4.3) into (7.3), we get a system of linear inhomogeneous differential equations, whose solution \(t \mapsto (h(t),f(t))^T\) is given in appendix B. In particular, plugging \(t = T_2\), one gets

$$\left( \frac{\partial}{\partial \delta} \right)_T (T_2,\theta,\omega,0) = \begin{pmatrix} h(T_2) \\ f(T_2) \end{pmatrix}$$

$$= \begin{pmatrix} -21.6335 - 236.869\omega^3 - 717.864\omega^2 \theta - 726.524\omega\theta^2 - 246.471\theta^3 \\ -11.7085 + 669.091\omega^3 + 1793.6\omega^2 \theta + 1582.73\omega\theta^2 + 458.155\theta^3 \end{pmatrix}$$

(7.4)
and
\[
\begin{pmatrix} \theta \\ \phi \end{pmatrix}_\delta (\tau) = \begin{pmatrix} h(T) \\ f(T) \end{pmatrix} = \begin{pmatrix} -21.6335 - 0.871197\theta^3 \\ -11.7085 - 0.697524\theta^3 \end{pmatrix}.
\]

Formula (7.2) then provides
\[
T_\delta(\theta, \omega, 0) = \frac{0.940403 + 34.0548\omega^3 + 96.2296\omega^2\theta + 90.4622\omega\theta^2 + 28.3414\theta^3}{\omega + 0.982912\theta}.
\]

Plugging all the above findings into formula (7.1), we conclude
\[
P_\delta(\theta, \alpha(T), 0) = \begin{pmatrix} 5.85426 + 0.348762\theta^3 \\ -7.51458 + 1.75673\theta^3 \end{pmatrix}.
\]

(b) Computing a fixed point of the family of long-period fixed points that satisfies the necessary condition of the perturbation theorem

Here we compute \((\theta_0, \omega_0)\) that satisfies the necessary condition (5.6).

Computing an eigenvector \(z\) of the transpose of matrix (6.3) for the eigenvalue 1, we get
\[
z = (-0.69131, -0.722559)^T.
\]

Therefore, taking into account the relation (4.7) between \(\theta_0\) and \(\omega_0\), the necessary condition (5.6) takes the form
\[1.38262 - 1.51044(\theta_0)^3 = 0.\]

The solution of this equation is
\[
\theta_0 = 0.970956, \quad (7.5)
\]

which coincides with the finding of Garcia et al. [2] (see the table in [2, p. 15]). The fixed point \((\theta_0, \omega_0) = (\theta_0, \alpha(T)\theta_0)\) is drawn as solid point in figure 2.

(c) Computing linearization the first-order expansion found in §7a

Here we compute \(P_{\delta(\theta, \omega)}\).

Differentiating (7.1) with respect to \((\theta, \omega)\), one gets
\[
P_{\delta(\theta, \omega)}(\theta, \alpha(T), 0) = \Delta_0 \left[ \begin{pmatrix} \theta \\ \phi \end{pmatrix}_\delta (\tau) T_{\delta(\theta, \omega)}(\theta, \alpha(T), 0) + \begin{pmatrix} \theta \\ \phi \end{pmatrix}_{t(\theta, \omega)} (\tau) T_{\delta}(\theta, \alpha(T), 0) \right]
+ \Delta_0 \begin{pmatrix} \theta \\ \phi \end{pmatrix}_\delta (\tau) T_{\delta(\theta, \omega)}(\theta, \alpha(T), 0)
+ \Delta_0 \begin{pmatrix} \theta \\ \phi \end{pmatrix}_\delta (\tau) T_{\delta(\theta, \omega)}(\theta, \alpha(T), 0) + \begin{pmatrix} \theta \\ \phi \end{pmatrix}_{\delta(\theta, \omega)} (\tau)
+ \Delta_0 ((\theta, \phi) T_{\delta(\theta, \omega)}(\theta, \alpha(T), 0), \theta, \omega, 0)
\circ \begin{pmatrix} \theta \\ \phi \end{pmatrix}_\delta (\tau) T_{\delta(\theta, \omega)}(\theta, \alpha(T), 0) + \begin{pmatrix} \theta \\ \phi \end{pmatrix}_{\delta(\theta, \omega)} (\tau).
\]

The terms \(\begin{pmatrix} \theta \\ \phi \end{pmatrix}_\delta (t, \theta, \omega, 0)\) and \(\begin{pmatrix} \theta \\ \phi \end{pmatrix}_{\delta(\theta, \omega)} (t, \theta, \omega, 0)\) come by taking the derivatives of \(\begin{pmatrix} \theta \\ \phi \end{pmatrix}_\delta (t, \theta, \omega, 0)\) with respect to \(t\) and \((\theta, \omega)\). The formulae for \(T_{\delta(\theta, \omega)}(\theta, \alpha(T), 0)\) and \(T_{\delta}(\theta, \omega, 0)\) were
computed in §§6 and 7a. To compute $T_{\delta(\theta, \omega)}$, we just differentiate the formula for $T_{\delta}(\theta, \omega, 0)$ of §7a with respect to $(\theta, \omega)$ obtaining

$$T_{\delta(\theta, \omega)}(\theta, \omega, 0) = \frac{-0.924333 + 62.7568\omega^3 + 180.924\omega^2\theta + 173.941\omega\theta^2 + 55.7142\theta^3}{(\omega + 0.982912\theta)^2},$$

$$= \frac{-0.940403 + 68.1095\omega^3 + 196.648\omega^2\theta + 189.17\omega\theta^2 + 60.575\theta^3}{(\omega + 0.982912\theta)^2}.$$ By analogy with (7.4), we compute

$$\begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}^{(T_2, \theta, \omega, 0)} = \begin{pmatrix} h(T_2) \\ \hat{h}(T_2) \end{pmatrix} = \begin{pmatrix} -22.6114 - 717.864\omega^3 - 2163.65\omega^2\theta - 2175.14\omega\theta^2 - 730.293\theta^3 \\ -22.6335 - 2163.65\omega^3 - 6503.87\omega^2\theta - 6518.19\omega\theta^2 - 2178.91\theta^3 \end{pmatrix}.$$ It remains to find $\Delta_{\delta(\theta, \omega)}(\theta, \omega, 0)$ which computes from (3.4) as

$$\Delta_{\delta(\theta, \omega)}(\theta, \omega, 0) = \begin{pmatrix} 0 & 0 \\ -4\theta\omega & -2\theta^2 \end{pmatrix}.$$ Combining all the findings together, the matrix $P_{\delta(\theta, \omega)}(\theta, \alpha(T_2)\theta, 0)$ finally computes as

$$P_{\delta(\theta, \omega)}(\theta, \alpha(T_2)\theta, 0) = \frac{1}{\theta} \begin{pmatrix} 218.645 - 3.99563\theta^3 & 209.189 - 4.82387\theta^3 \\ -218.652 - 30.1237\theta^3 & -209.195 - 33.8632\theta^3 \end{pmatrix}.$$  

(d) Verifying the stability condition of the perturbation theorem

To verify the stability condition (5.9), it remains to compute the eigenvector $y$ matrix (6.3) which corresponds to the eigenvalue 1 and satisfies the normalization property (5.1) with the vector $z$ of §7b. Such a computation leads to

$$y = \begin{pmatrix} 0.69131 \\ 0 \end{pmatrix},$$ Using the formula for $P_{\delta(\theta, \omega)}(\theta_0, \alpha(T_2)\theta_0, 0)$ of §7c and the value $\theta_0$ given by §7b, we get

$$\lambda_+ = z^T P_{\delta(\theta, \omega)}(\theta_0, \alpha(T_2)\theta_0, 0)y = -2.95323,$$ so that both conditions (5.7) and (5.9) hold.

**Corollary 7.1.** Based on theorem 5.1, we now conclude that, for all $\delta > 0$ sufficiently small, the switched system (2.1) and (2.2) admits an asymptotically stable limit cycle with the initial condition $((\theta_0, \omega_0, 2\theta_0, (2\delta(\theta_0) + \omega_0(\delta)\omega_0)))$, where $(\theta_0, \omega_0) \to (\theta_0, \alpha(T_2)\theta_0)$ as $\delta \to 0$. The cycle experiences exactly one impact per the period $T_\delta > 0$ and $T_\delta = T_2$ as $\delta \to 0$. One of the Floquet multipliers $\tilde{\rho}_\delta$ of this cycle converges to $\rho$ given by (6.4) as $\delta \to 0$. An estimate for the second Floquet multiplier $\rho_\delta$ is given by formulæ (7.6), (A3), (A7). Accordingly, the initial model (1.1) and (1.2) admits a walking cycle obtained from the limit cycle of (2.1) and (2.2) over the change of the variables (1.3) with the same Floquet multipliers $\tilde{\rho}_\delta$ and $\rho_\delta$.

**Remark 7.2.** The theoretic prediction for the Floquet multipliers $\tilde{\rho}_\delta$ and $\rho_\delta$ given by corollary 7.1 matches the (numerics assisted) diagram in Das–Chatterjee [27, Fig. 4B] exactly.

**Remark 7.3.** Figure 3a shows the components $\Theta$ and $\Phi$ of the cycle of the reduced system (4.1) and (4.2) that corresponds to the fixed point $(\theta_0, \omega_0)$. One can see that, if scaled by the factor of $\delta^{1/2} = (0.0092^2)^{1/2} \approx 0.21$ (see the change of the variables (1.3)), the curves of figure 3a match qualitatively and quantitatively the respective components of the asymptotically stable long-period-walking-cycle of the initial switched system (1.1) and (1.2) as simulated in Garcia et al. [2, (fig. 2)]. Furthermore, the shapes of the cycles of figure 2 are in agreement with the shape of
Figure 3. (a) The components $\Theta(t)$ and $\Phi(t)$ of the limit cycle of the reduced system (4.1)–(4.2) with the initial condition $(\theta_0, \omega_0, 2\theta_0, 0)$. (b) Simulation of the dynamics of switched system (2.1) and (2.2) for $\delta = \gamma^{1/3}$, $\gamma = 0.007$, upon dropping the reminders $o_1(\delta)$ and $o_2(\delta)$. The figure shows the solution with the initial condition $(\Theta(0), \dot{\Theta}(0), \Phi(0), \dot{\Phi}(0)) = (\theta_0 + 0.1, \omega_0 + 0.1, 2(\theta_0 + 0.1), 0)$, which appears to converge to a limit cycle even though the initial condition (dotted point) is taken rather far from the limit cycle. The solid point stays for $(\theta_0, \omega_0)$. Based on numeric simulation, the period of the limit cycle estimates as 3.856. (Online version in colour.)

Remark 7.4. Figure 3b shows that the domain of attraction of switched system (2.1) and (2.2) with dropped $o_1(\delta)$ and $o_2(\delta)$ is significantly larger than the domain of attraction of the full switched system (2.1) and (2.2) (see e.g. simulations in Li & Yang [47, Fig. 5]). This observation may serve as a starting point for new control designs that enlarge the domain of attraction of walking cycles of (2.1) and (2.2) by eliminating the $o_1(\delta)$ and $o_2(\delta)$ terms.

8. Conclusion

In this paper, we used the results by Kamenskii et al. [43] and Makarenkov-Ortega [44] in order to offer a perturbation theorem for bifurcation of fixed points from 1-parameter families in two-dimensional maps. The respective perturbation theorem is then applied to prove the existence and stability of a walking cycle in the model of passive walker (1.1) and (1.2) with a small parameter as introduced by Garcia et al. [2]. Along the same lines, the proposed theory can be used for establishing the existence of unstable limit cycles (exactly one of the two inequalities in (5.8) and (5.9) should be reversed to ensure instability).

Since the dynamics of passive walkers constitutes an important reference block of more complex robotics models and control algorithms (see Introduction), we like to think that the proposed analytic formulae (corollary 7.1) will be of further use in the field of robotics.

We also anticipate that the present paper makes it feasible to include the biped model as a motivation (and an application) for a perturbation theory chapter in applied mathematics graduate programs. To facilitate this outreach, we attach our Wolfram Mathematica code (that computes all the quantities of the present paper section-by-section) as electronic supplementary material. Logistically, the biped model topic could, e.g. follow the suspension bridge topic by...
Glover et al. [39], where bifurcation of fixed points from 1-parameter families takes place in one-dimensional maps.

Data accessibility. This article has no additional data.

Competing interests. I declare I have no competing interest.

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Appendix A. Derivation of the perturbation theorem of §5 from the results of Kamenskii et al. [43] and Makarenkov-Ortega [44]

The following two results have been established in Kamenskii et al. [43] and they will play the central role in the perturbation theorem (theorem 5.1) that this section develops. We now reformulate the required results of [43] in the notations of the present paper to avoid confusion.

**Theorem A.1 (two-dimensional version of a combination of [43, Theorem 1] and [43, Remark 2]).** Consider a $C^2$-function $(\theta, \omega, \delta) \mapsto F(\theta, \omega, \delta)$. Assume that $(\theta_0, \omega_0)$ is such that $F(\theta_0, \omega_0, 0) = 0$ and $\det \left| F(\theta, \omega, 0) \right| = 0$. Let $\Pi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear projector invariant with respect to $F(\theta, \omega, 0)$ with $F_{\theta, \omega}(\theta_0, \omega_0, 0)$ invertible on $(I - \Pi)\mathbb{R}^2$. Assume that $\Pi F_{\delta}(\theta_0, \omega_0, 0) = 0$, $\Pi F_{\theta, \omega}(\theta_0, \omega_0, 0)\Pi h_1 \Pi h_2 = 0$ for any $h_1, h_2 \in \mathbb{R}^2$, and that

$$
- \Pi F_{\theta, \omega}(\theta_0, \omega_0, 0)h + \Pi (F_{\delta}(\theta_0, \omega_0, 0))^{-1} F_{\delta}(\theta_0, \omega_0, 0),
$$

where $h = (I - \Pi) \left( F_{\theta, \omega}(\theta_0, \omega_0, 0) \right|_{(I - \Pi)\mathbb{R}^2}^{-1} F_{\delta}(\theta_0, \omega_0, 0) $, (A 1)

is invertible on $\Pi \mathbb{R}^2$. Then, there exists a unique $(\theta_1, \omega_1) \in \mathbb{R}^2$ such that, for all $|\delta| \neq 0$ sufficiently small, one can find $(\theta_{1, \delta}, \omega_{1, \delta}) \in \mathbb{R}^2$ that satisfies both

$$
F(\theta_0 + \delta \theta_{1, \delta}, \omega_0 + \delta \omega_{1, \delta}, \delta) = 0
$$

and

$$(\theta_{1, \delta}, \omega_{1, \delta}) \rightarrow (\theta_1, \omega_1) \quad \text{as} \quad \delta \rightarrow 0.
$$

Note, the property $\det \left| F(\theta, \omega, 0) \right| = 0$ ensures that one of the eigenvalues of $F_{\theta, \omega}(\theta_0 + \delta \theta_{1, \delta}, \omega_0 + \delta \omega_{1, \delta}, \delta)$ converges to 0 as $\delta \rightarrow 0$ (see [43, Remark 2]).

**Theorem A.2 (two-dimensional version of [43, Theorem 2]).** Assume all the conditions of theorem A.1. Let $(\theta_{1, \delta}, \omega_{1, \delta})$ be as given by theorem A.1 and let $\lambda_{\delta}$ be such an eigenvalue of $F_{\theta, \omega}(\theta_0 + \delta \theta_{1, \delta}, \omega_0 + \delta \omega_{1, \delta}, \delta)$ that $\lambda_{\delta} \rightarrow 0$ as $\delta \rightarrow 0$. Denote by $\lambda_{s, \delta} \in \mathbb{R}$ the eigenvalue of the linear map

$$
\Pi F_{\theta, \omega}(\theta_1, \omega_1)^T \left|_{\Pi \mathbb{R}^2} + \Pi (F_{\delta}(\theta_0, \omega_0, 0))^{-1} \right|_{\Pi \mathbb{R}^2}.
$$

Then

$$
\lambda_{\delta} = \delta \lambda_{s, \delta} + o(\delta),
$$

(A 3)

In order to apply theorem A.1 to the Poincare map $(\theta, \omega) \mapsto P(\theta, \omega, \delta)$, we consider

$$
F(\theta, \omega, \delta) = P(\theta, \omega, \delta) - (\theta, \omega)^T
$$

and

$$
\Pi \zeta = z^T \zeta y,
$$

(A 4)

and notice that (5.4) implies

$$
z^T P_{\delta}(\xi(s), 0)y = 0, \quad \text{for all} \quad s \in \mathbb{R},
$$

(A 5)

which allows (as we show in the proof of theorem 5.1), to ignore all the expressions of theorem A.1 that involve the second derivative.
Proof of Theorem 5.1. The necessity part. Here we follow the idea of Makarenkov-Ortega [44, Lemma 2]. Assume that \( P(\theta, \omega, \delta) = (\theta, \omega, \delta)^T, \delta \in \mathbb{R} \), for some family \( \{ (\theta, \omega, \delta) \}_{\delta \in \mathbb{R}} \) satisfying (5.5). We claim that (5.6) holds.

The derivative \( F'(\theta, \omega, \delta) \) of the \( C^1 \) function (A 4) is a \( 2 \times 3 \)-matrix. Observe that \( \text{rank} F'(\xi(s_0), 0) = 1 \). Otherwise the equation \( F(\theta, \omega, \delta) = 0 \) should describe a curve in a small neighbourhood of \( (\xi(s_0), 0) \). However, the set \( \{ (\theta, \omega, \delta) : F(\theta, \omega, \delta) = 0 \} \) contains both the curve \( \{ (\xi(s), 0) \}_{s \in \mathbb{R}} \) and also the set \( \{ (\theta, \omega, \delta) \}_{\delta \in \mathbb{R}} \). Now we know that rank \( F'(\xi(s_0), 0) = 1 \) and it remains to prove that

\[
\text{rank} F'(\xi(s_0), 0) = 2, \quad \text{if} \quad z^T F_\delta(\xi(s_0), 0) \neq 0. \quad (A 6)
\]

By Fredholm alternative for matrices (e.g. [48, Theorem 4.5.3]),

\[
\text{Im} F(\theta, \omega, \delta)(\xi(s_0), 0) = \left( \text{Ker} F(\theta, \omega, \delta)(\xi(s_0), 0) \right)^{\perp}.
\]

Since \( \text{Ker} F(\theta, \omega, \delta)(\xi(s_0), 0) = \text{span}(z) \), we conclude that \( (\text{Ker} F(\theta, \omega, \delta)(\xi(s_0), 0))^{\perp} = \text{span}(\tilde{y}) \), where \( \tilde{y} \) is an eigenvector of \( F(\theta, \omega, \delta)(\xi(s_0), 0) \) that corresponds to the non-zero eigenvalue of \( F(\theta, \omega, \delta)(\xi(s_0), 0) \). Therefore, \( \text{Im} F(\theta, \omega, \delta)(\xi(s_0), 0) = \text{span}(\tilde{y}) \). But \( z^T F_\delta(\xi(s_0), 0) \neq 0 \) implies, see formula (5.3), that the vectors \( \tilde{y} \) and \( F_\delta(\xi(s_0), 0) \) are linearly independent, which completes the proof of (A 6).

The sufficiency part. Here we use theorem A.1. The projector \( \Pi \) defined by (A 4) is invariant with respect to \( F(\theta, \omega, \delta, 0) \) and the projector \( I - \Pi \) is given by, see formula (5.3),

\[
(I - \Pi) \xi = z^T \xi \tilde{y},
\]

so that \( F(\theta, \omega, \delta, 0) \) is invertible on \( (I - \Pi) \mathbb{R}^2 \). The requirement \( \Pi F_\delta(\theta_0, \omega, 0, 0) = 0 \) of theorem A.1 holds by (5.6), and the requirement \( \Pi F(\theta, \omega, \delta, 0) = 0 \) holds by (A 5). The properties (A 4) and (A 5) imply that expression (A 1) is invertible on \( \text{span}(y) \) if and only if (5.7) holds. Therefore, the conclusion of the theorem follows by applying theorem A.1.

The stability part. Assume that conditions (5.8) and (5.9) hold. Let \( \rho_\delta \) be the eigenvalue of \( P(\theta, \omega, \delta, \delta) \) such that

\[
\rho_\delta \to 1 \quad \text{as} \quad \delta \to 0.
\]

We have to show that \( |\rho_\delta| < 1 \) for all \( |\delta| > 0 \) sufficiently small. Observe that

\[
\lambda_\delta = \rho_\delta - 1 \quad (A 7)
\]

is the eigenvalue of \( F(\theta, \omega, \delta, \delta) \). As it was established in the sufficiency part of the proof, expression (A 2) coincides with \( z^T \tilde{y} \). Therefore, condition (5.9) ensures that \( \lambda_* \) of theorem A.2 verifies \( \lambda_* < 0 \) and so Theorem A2 ensures that \( \lambda_\delta < 0 \) for all \( \delta > 0 \) sufficiently small.

The proof of the theorem is complete.

Appendix B. The solution of equation (7.3)

The solution \((h(t), f(t))\) of equation (7.3) is given by

\[
h(t) = \frac{1}{384} e^{-3t} \left[ 384 e^{3t} + (\omega - \theta)^3 - e^{6t} (\omega + \theta)^3 + e^{2t} (-192 + 3 \omega^3 (3 + 4t) + 3 \omega^2 (1 - 4t) - 3 \omega (7 + 4t) \theta^2 + (1 + 12t) \theta^3) + e^{4t} (-192 + 3 \omega^3 (-3 + 4t) - 3 \omega (-7 + 4t) \theta^2 + (1 - 12t) \theta^3 + 3 \omega^2 (\theta + 4t)) \right]
\]
and

\[ f(t) = \frac{1}{7680}e^{-3t}[-1920e^{2t} - 1920e^{4t} - 56e^{2t} \omega^3 + 60e^{2t} \omega^3 - 60e^{4t} \omega^3 + 56e^{6t} \omega^3 + 120e^{2t} \omega^3 t + 120e^{4t} \omega^3 t + 168e^{2t} \omega^3 t + 60e^{2t} \omega^3 \theta + 60e^{4t} \omega^3 \theta + 168e^{6t} \omega^2 \theta \]

\[ - 120e^{2t} \omega^2 t \theta + 120e^{4t} \omega^2 t \theta - 168e^{2t} \omega^2 \theta^2 - 780e^{2t} \omega^2 \theta^2 + 780e^{4t} \omega^2 \theta^2 + 168e^{6t} \omega^2 \theta^2 \]

\[ - 120e^{2t} \omega^2 \theta^2 - 120e^{4t} \omega^2 \theta^2 + 56e^3 + 580e^{2t} \theta^3 + 580e^{4t} \theta^3 + 56e^{6t} \theta^3 \]

\[ + 120e^{2t} \theta^3 - 120e^{4t} \theta^3 + 3e^3 \{- 65(\omega - 3t) (\omega - \theta)^2 + 65e^{4t} (\omega + \theta)^2 (\omega + \theta) \}

\[ + e^{2t}(1280 + 140e^3 t - 921e^2 \omega^2 + 60e^3 \omega \theta^2 - 697e^3 \theta) \cos t + 12e^{2t}((-1 + e^{2t}) \omega^3 \theta

\[ + 13(1 + e^{2t}) \omega^2 \theta + 3(-1 + e^{2t}) \omega^2 \theta - 9(1 + e^{2t}) \omega^3 \theta) \cos(2t) + 45e^{3t} \omega^2 \theta \cos(3t) \]

\[ - 135e^{3t} \omega^3 \cos(3t) - 195e^4 \omega^3 \sin t - 179e^{3t} \omega^3 \sin t - 195e^5 \omega^3 \sin t \]

\[ - 195e^5 \omega^2 \sin t + 195e^5 \omega^2 \sin t + 1260e^{3t} \omega^2 t \sin t + 975e^3 \omega^2 \sin t \]

\[ + 3813e^{3t} \omega^2 \sin t + 975e^5 \omega^2 \sin t - 585e^3 \omega^2 \sin t + 585e^5 \omega^3 \sin t \]

\[ + 540e^{3t} t \omega^3 \sin t - 24e^{2t} \omega^3 \sin(2t) - 24e^{4t} \omega^3 \sin(2t) - 48e^{2t} \omega^2 \theta \sin(2t) \]

\[ + 48e^{4t} \omega^2 \theta \sin(2t) + 288e^{2t} \omega^2 \theta \sin(2t) + 288e^{4t} \omega^2 \theta \sin(2t) - 216e^{2t} \theta^3 \sin(2t) \]

\[ + 216e^{4t} \theta^3 \sin(2t) - 5e^{3t} \omega^3 \sin(3t) + 135e^{3t} \omega^2 \sin(3t) \]
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