SOME OBSERVATIONS ON THE SIMPLEX

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Abstract. We discuss some properties of the set of simplices in \( \mathbb{E}^n \).

1. An inspirational example

Consider triangles in the Euclidean plane \( \mathbb{E}^2 \). The set of congruence classes of triangles is parametrized by their three sidelengths (say \( a, b, c \)), while the set of similarity classes of triangles is parametrized naturally by the triples of angles \( \alpha, \beta, \gamma \).

What can be said about the possible values of the triple \( a, b, c \)? It is clearly necessary that the triangle inequalities be satisfied:

\[
a > 0, \quad b > 0, \quad c > 0,
\]

\[
a < b + c, \quad b < a + c, \quad c < a + b.
\]

Since, given a triple \( T = (a, b, c) \) satisfying the three triangle inequalities we can easily construct the triangle with the sidelengths prescribed by \( T \), it follows that, as parametrized by the lengths of sides, the set of non-degenerate triangles in \( \mathbb{E}^2 \) is a convex cone. A similar result holds for the angles: a triple \( \alpha, \beta, \gamma \) of positive numbers gives the angles of a non-degenerate triangle if and only if

\[
\alpha + \beta + \gamma = \pi.
\]

One can ask whether results of this type extend to simplices of higher dimension, and we shall attempt to give as complete an answer as possible.

2. Some negative results

First, we shall see that things appear to become less simple. Firstly

**Theorem 2.1.** Let \( l_{0,1}, \ldots, l_{n-1,n} \) be positive numbers. In order for these to be the edge-lengths of a non-degenerate simplex \( v_0, \ldots, v_n \) in

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$E^n$, so that $l_{ij} = d(v_i, v_j)$ it is necessary, but not sufficient that for any $v_i, v_j, v_k$ the lengths $l_{i,j}, l_{i,k}, l_{j,k}$ satisfy the requisite triangle inequalities.

Proof. The necessity is obvious. To prove the insufficiency, we construct an example as follows: Let $n = 3$, and let

$$l_{1,2} = l_{1,3} = l_{2,3} = 1,$$

while

$$l_{0,1} = l_{0,2} = l_{0,3} = 1/2 + \epsilon.$$

The reader will verify that for any $\epsilon > 0$ all possible triangle inequalities are satisfied, whereas for a sufficiently small $\epsilon$ no simplex with prescribed edgelengths exists. \(\square\)

Things are even worse than that:

**Theorem 2.2.** Let $l_{i,j}, \ 0 \leq i, j \leq n$ be as in the statement of Theorem 2.1. The set of simplices in $E^n$ (for $n > 2$) is not convex when parametrized by the $l_{i,j}$.

Proof. (The example is due to Peter Frankel of Budapest). Consider two simplices $A$ and $B$ as follows:

$$l_{0,1}(A) = l_{0,2}(A) = l_{0,3}(A) = l_{0,1}(B) = l_{0,2}(B) = l_{0,3}(B) = 1,$$

while

$$l_{1,2}(A) = \epsilon,$$
$$l_{2,3}(A) = \sqrt{2},$$
$$l_{3,1}(A) = \sqrt{2},$$
$$l_{1,2}(B) = \sqrt{2},$$
$$l_{2,3}(B) = \sqrt{2},$$
$$l_{3,1}(B) = \epsilon.$$

It is a simple geometric exercise to show that $A$ and $B$ really and truly exist (for any $\epsilon > 0$) while, for $\epsilon$ small enough, there is no simplex $C$ with $l_{i,j}(C) = l_{i,j}(A) + l_{i,j}(B), \ \forall 0 \leq i < j \leq 3$. (alternatively, the reader can peek ahead for a hint to a non-geometric proof of non-existence of $C$.) \(\square\)

We shall have to resort to less geometric methods to see what is really going on.
3. LINEAR ALGEBRA TO THE RESCUE

Let $\Delta$ be a simplex with vertices $v_0 = 0, v_1, \ldots, v_n$ in $\mathbb{E}^n$. The volume of the simplex $\Delta$ is non-zero if and only if the vectors $v_1, \ldots, v_n$ are linearly independent; in fact, its volume is given by
\begin{equation}
\text{vol}\Delta = \frac{1}{n!} \det V,
\end{equation}
where $V$ is the matrix whose columns are the vectors $v_1, \ldots, v_n$.

We now write the so-called Gram matrix of $\Delta$:
\[ G(\Delta) = V^t V. \]
Equivalently,
\[ G_{ij}(\Delta) = \langle v_i, v_j \rangle. \]

**Theorem 3.1.** An $n \times n$ matrix $G$ is the Gram matrix of a simplex $\Delta$ in $\mathbb{E}^n$ if and only if $G$ is symmetric and positive definite.

**Proof.** First we prove the “if” direction. The symmetry of the inner product in $\mathbb{E}^n$ implies that $G$ is symmetric (alternately: $(V^t V)^t = V^t V$.) And for any vector $x \in \mathbb{E}^n$:
\[ x^t G(\Delta) x = x^t V^t V x = \langle V x, V x \rangle = \|V x\| > 0, \]
if $V$ is non-singular and $x$ is non-zero, hence positive definiteness.

To show the “only if” direction, let $G$ be a positive-definite symmetric matrix. That implies that there is an orthogonal matrix $S$ and a diagonal matrix $D$ all of whose diagonal elements are positive, such that $G = S^t D S$. Write $F = \sqrt{D}$ (the elements of $F$ will be simply the non-negative square roots of the corresponding elements of $D$). Then, it is easy to see that $G = S^t D S = S^t F^t F S = (FS)^t (FS)$. Setting $V = FS$ we obtain our simplex. \hfill \Box

The fact, used in the proof of the above theorem, that a symmetric matrix has an orthogonal matrix of eigenvectors is essentially equivalent to the following

**Theorem 3.2** (Rayleigh-Ritz characterization). The smallest eigenvalue $\lambda_0$ of a symmetric matrix $A$ can be written as
\[ \lambda_0 = \min_{\|x\|=1} \langle Ax, x \rangle. \]
from which we have

**Corollary 3.3.**
\[ \lambda_0(A + B) \geq \lambda_0(A) + \lambda_0(B) \]
(in other words, $\lambda_0$ is a concave function on the set of symmetric matrices.)
Proof. By the Rayleigh-Ritz characterization,
\[ \lambda_0(A + B) = \min_{\|x\|=1} \langle (A + B)x, x \rangle. \]
Denote the unit vector achieving the minimum by \( x_{A+B} \). Since
\[ \langle Ax_{A+B}, x_{A+B} \rangle \geq \min_{\|x\|=1} \langle Ax, x \rangle, \]
and similarly
\[ \langle Bx_{A+B}, x_{A+B} \rangle \geq \min_{\|x\|=1} \langle Bx, x \rangle, \]
the assertion follows.

And the final corollary:

**Corollary 3.4.** The set of positive-definite symmetric matrices is a convex cone in the space of \( n \times n \) symmetric matrices (which can be naturally identified with \( \mathbb{R}^{n(n+1)/2} \)).

**Proof.** This is an immediate consequence of Corollary 3.3 since a matrix is positive-definite if and only if all of its eigenvalues are positive.

Corollary 3.4 can be restated as saying that the set of Gram matrices of simplices is a convex cone. This does not see very useful at the moment, since the entries of the Gram matrices are some obscure scalar products, but this can be easily rectified by observing the

**Polarization identity:**
\[ \langle v, w \rangle = \frac{1}{2} \left( \|v\|^2 + \|w\|^2 - \|v - w\|^2 \right). \]

When applied to the Gram matrix, this shows that
\[ G_{ij}(\Delta) = \frac{1}{2} \left( l_{i,0}^2 + l_{j,0}^2 - l_{i,j}^2 \right), \]
where \( l_{i,j} \) are the edge lengths as before, and \( l_{i,i} = 0 \) for any \( i \). We see that the Gram matrix is a linear (matrix valued) function of the squares of the edge lengths of the simplex \( \Delta \), and hence we can put all of the above together to see that:

**Theorem 3.5.** The set of non-degenerate simplices in \( \mathbb{E}^n \) is a convex cone when parametrized by the squares of the edgelengths.

We have now recovered our convexity of the set of simplices, which, in view of the gloomy Section 2 can already be viewed as a success. But, as they say, Wait! There is MORE!
Remark 3.6. The following is related to the results of this section, and follows from a beautiful theorem of I. J. Schoenberg ([Schoenberg37]): the set of simplices parametrized by the logs of the edge lengths is starshaped with respect to the origin (corresponding to the regular simplex).

4. Volume

In this section we shall use the following simple (as the reader will see momentarily) result:

**Theorem 4.1.** The function $\log \det A$ is a concave function on the cone of positive definite symmetric matrices.

**Proof.** Concavity is equivalent to concavity on all lines, which is in turn equivalent to the statement that

$$
\left. \frac{d^2 \log \det(A + tB)}{dt^2} \right|_{t=0} < 0,
$$

for any positive definite $A$ and any symmetric $B$. To show the inequality (3) we have to be able to differentiate the determinant (or its logarithm directly). A couple of ways of doing that will be shown below, but for now we will accept the following as a fact of life:

$$
\frac{d \log \det(A + tB)}{dt} = \text{tr}((A + tB)^{-1}B).
$$

From which it follows easily (by linearity of trace) that

$$
\left. \frac{d^2 \log \det(A + tB)}{dt^2} \right|_{t=0} = -\text{tr}(A^{-1}BA^{-1}B).
$$

Now, since determinant is invariant under conjugation, we can assume that $A$ is diagonal: $A = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Under that assumption, direct computation shows that

$$
\text{tr}(A^{-1}BA^{-1}B) = \sum_{0 \leq i,j \leq n} \frac{b_{ij}^2}{\lambda_i \lambda_j}.
$$

Since all the summands are nonnegative, and at least one is positive (since $B \neq 0$), the theorem is proved. \qed

We finally obtain the following:

**Theorem 4.2.** The volume of any fixed given face (of any dimension) of simplices parametrized by the squares of the edgelengths is log-concave.
Proof. The result for the top-dimensional face follows from the expression (2) and Theorem 4.1 above. Since the Gram matrix of any given face is a submatrix of the Gram matrix of the whole simplex, the result follows. □

Using the usual inequality theory (see [Gardner02] for details) we obtain the following result of a Brunn-Minkowsky type:

**Corollary 4.3.** The \( n \)-th root of the volume of \( n \)-dimensional simplices parametrized by the squares of the edgelengths is a concave function.

These results can be used to produce hitherto unknown characterizations of the regular simplex. First, an observation:

**Lemma 4.4.** Let \( \Omega \) be a compact convex set in an affine space, \( G \) a group of automorphisms acting on \( \Omega \), and \( f \) a concave function invariant under \( G \) (that is, \( f(G(x)) = f(x), \forall x \in \Omega \)). Let \( y \) be a point where \( f \) achieves its maximum. Then \( y \) is invariant under \( G \) (that is, \( g(y) = y, \forall g \in G \)).

**Proof.** Suppose that there is a \( g \in G \) such that \( z = g(y) \neq y \). Since \( \Omega \) is convex, the segment \( S = [y, z] \) is contained in \( \Omega \). The function \( f \) is concave on \( S \), and so

\[
f \left( \frac{y + z}{2} \right) > \frac{f(y) + f(z)}{2} = f(y),
\]

contradicting the assumption that \( y \) is a maximum of \( f \). □

And now, the applications:

**Theorem 4.5.** Let \( \Delta_a \) be the set of \( n \)-dimensional simplices, such that the sum of the squares of their edgelengths is equal to \( a \). Let

\[
P_k = \prod_{\text{k-dimensional faces } f \text{ of } \Delta \in \Delta_a} V(f).
\]

Then \( P_k \) is maximized when \( \Delta \) is a regular simplex (with edgelengths \( 2a/(n + 1)(n) \)).

**Theorem 4.6.** Let \( \Delta_a \) be the set of \( n \)-dimensional simplices, such that the sum of the squares of their edgelengths is equal to \( a \). Let

\[
S_k = \sum_{\text{k-dimensional faces } f \text{ of } \Delta \in \Delta_a} V^{1/k}(f).
\]

Then \( S_k \) is maximized when \( \Delta \) is a regular simplex (with edgelengths \( 2a/(n + 1)(n) \)).

**Proof.** The Theorems above follow immediately from Theorem 4.1 (Corollary 4.3, respectively) and Lemma 4.4. □
5. How to differentiate the determinant

In this section we demonstrate the truth of the formula \((4)\). Consider the family

\[ M(t) = A + tB. \]

Clearly,

\[
\det M(t) = \det A \det (I + tA^{-1}B).
\]

The logarithmic derivative of \(\det M(t)\) at 0 is thus equal to the derivative of \(\det (I + tA^{-1}B)\) at 0. To simplify notation, write \(C = A^{-1}B\). The \(i\)-th column of \(I + tC\) equals \(e_i + tc_i\), where \(e_i\) is the \(i\)th standard basis vector, and \(c_i\) is the \(i\)th column of \(C\). Using the multilinearity of the determinant, we see that

\[
\det(I + tC) = \det(e_1 + tc_1, \ldots, e_n + tc_n) =
\]

\[
1 + t \sum_{i=1}^{n} \det(e_1, \ldots, c_i, \ldots, e_n) + O(t^2) =
\]

\[
1 + \text{tr}(C) + O(t^2),
\]

showing that

\[
\frac{d\det(1 + tC)}{dt} = \text{tr}C,
\]

which completes the proof.

6. Random facts about matrices

**Definition 6.1.** Let \(M\) be a matrix. The adjugate \(\widehat{M}\) of \(M\) is the matrix of cofactors of \(M\). That is, \(\widehat{M}_{ij} = (-1)^{i+j} \det M^{ij}\), where \(M^{ij}\) is \(M\) with the \(i\)-th row and \(j\)-th column removed. The reason for this definition is

**Theorem 6.2** (Cramer’s rule). *For any \(n \times n\) matrix \(M\) (over any commutative ring)*

\[
M\widehat{M} = \widehat{M}M = (\det M)I(n),
\]

where \(I(n)\) is the \(n \times n\) identity matrix.

We also need

**Definition 6.3.** The outer product of vectors \(v = (v_1, \ldots, v_n)\) and \(w = (w_1, \ldots, w_n)\) is the matrix \(v \otimes w\), defined as follows:

\[
(v \otimes w)_{ij} = v_i w_j.
\]
Remark 6.4. As the notation suggests, the outer product is actually a tensor product, though it would be more correct to write \( v \otimes w^* \). The Dirac notation for the outer product would be \( |v||\langle w| \) while the Dirac notation for the inner product would be \( \langle v|w \rangle \), this possibly explaining the name outer product.

Consider an arbitrary vector \( x = (x_1, \ldots, x_n) \). We see that

\[
[(v \otimes w)x]_k = \sum_{i=1}^{n} v_kw_ix_i = v_k\langle w, x \rangle,
\]

so that

\[
(v \otimes w)x = \langle w, x \rangle v.
\]

We see that \( v \otimes w \) is a multiple of the projection operator onto the subspace spanned by \( v \). In particular, in the case when \( \|v\| = 1 \), the operator \( v \otimes v \) is the orthogonal projection operator onto the subspace spanned by \( v \). Since \( v \otimes w \) is a rank 1 operators all of its eigenvalues are equal to 0. The one (potentially) nonzero eigenvalue equals \( \langle v, w \rangle \).

We now show:

**Theorem 6.5.** Suppose that \( M \) has nullity 1, and the null space of \( M \) is spanned by the vector \( v \), while the null space of \( M^t \) is spanned by the vector \( w \). Then

\[
\widehat{M} = cw \otimes v,
\]

**Proof.** Since \( M \) is singular, we know that \( \det M = 0 \), and so every column of \( \widehat{M} \) is in the null-space of \( M \). so, letting \( m_i \) denote the \( i \)th column of \( \widehat{M} \), we see that

\[
m_i = d_i v_i.
\]

However, \( \widehat{M}^t = (\widehat{M})^t \) so performing the computation on transposes we see that

\[
m^t_i = e_i w_i.
\]

We see that

\[
\widehat{M}_{ij} = d_i v_j = e_j w_i.
\]

Writing \( d_i = g_i w_i \), and \( e_j = h_j v_j \), we see that, for every pair \( i, j \), \( g_i w_i v_j = h_j w_i v_j \). The conclusion follows.

**Theorem 6.6.** The constant \( c \) in the statement of the last theorem equals the product of the nonzero eigenvalues of \( M \) divided by the inner product of \( v \) and \( w \).
Proof. By considering the characteristic polynomial of $M$ we see that the product of the nonzero eigenvalues of $M$ equals the sum of the principal $n - 1$ minors. On the other hand, the principal minors of $M$ equal the diagonal elements of $\hat{M}$, so

$$c \sum_{i=1}^{n} cw_i v_i = \prod_{i=1}^{n-1} \lambda_j.$$  

$\Box$

Remark 6.7. By the discussion following Eq. (6) the product of nonzero eigenvalues of $M$ equals

$$\frac{\det (M + w \otimes v)}{\langle v, w \rangle}.$$  

7. Back to simplices

First we define the dual Gram matrix of a simplex $S$. First, let let $f_i$ be the unit outer normal to the $i$-th face. Then

$$G^*_{ij} = \langle f_i, f_j \rangle.$$  

In other words, the $ij$-th entry of the dual Gram matrix is the cosine of the exterior dihedral angle between the $i$-th and the $j$-th face.

Lemma 7.1. The dual Gram matrix $G^*(\Delta)$ of a Euclidean simplex $\Delta$ is symmetric and positive semi-definite, with exactly one 0 eigenvalue.

Proof. The proof proceeds exactly as the proof of Theorem 3.1 Any $n$ of the vectors $f_0, \ldots, f_n$ are linearly independent, and the corresponding $n \times n$ submatrix of the Gram matrix $G^*$ is positive definite, whence the result.  

The above Lemma makes one wonder what the null space of $G^*$ might be. Happily, there is a complete answer, as follows:

Theorem 7.2. The null space of $G^*$ is generated by the vector $A = (A_0, \ldots, A_n)$, where $A_i$ is the area of the face $f_i$.

Proof. This follows from the divergence theorem, which states (in the polyhedral case) that

$$\sum_{i=0}^{n} A_i f_i = 0.$$  

Taking the dot product of eq. (8) with $f_j$, we see that

$$\sum_{i=0}^{n} A_i \langle f_i | f_j \rangle = 0,$$  

$$\sum_{i=0}^{n} A_i \langle f_i | f_j \rangle = 0,$$  

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$$\sum_{i=0}^{n} A_i \langle f_i | f_j \rangle = 0.$$
which is to say that the $j$-th coordinate of $G^*A$ vanishes. \qed

**Corollary 7.3.** With notation as above,

$$\frac{A_i^2}{A_j^2} = \frac{\hat{M}_{ii}}{\hat{M}_{jj}}$$

**Proof.** Immediate from Theorem 6.5 and Theorem 7.2. \qed

**Remark 7.4.** This result has now been generalized to simplices in $\mathbb{H}^3$ by the Novosibirsk State University Geometry seminar.

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