MINKOWSKI CONCENTRICITY AND COMPLETE SIMPLICIES

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Abstract. This paper considers the radii functionals (circumradius, inradius, and diameter) as well as the Minkowski asymmetry for general (possibly non-symmetric) gauge bodies. A generalization of the concentricity inequality (which states that the sum of the inradius and circumradius is not greater than the diameter in general Minkowski spaces) for non-symmetric gauge bodies is derived and a strong connection between this new inequality, extremal sets of the generalized Bohnenblust inequality, and completeness of simplices is revealed.

1. Introduction

We call any convex, compact set a (convex) body and denote by $\mathcal{K}^n$ the family of convex bodies in $\mathbb{R}^n$ (excluding singletons). For any $K, C \in \mathcal{K}^n$ the inradius $r(K, C)$ of $K$ w.r.t. $C$ is the largest $\lambda \geq 0$ such that $\lambda C$ is contained in a translation of $K$, while the circumradius $R(K, C)$ of $K$ w.r.t. $C$ is the smallest $\lambda > 0$ such that $\lambda C$ contains a translate of $K$. Defining the length of a segment $[x, y]$ w.r.t. $C$ with endpoints $x, y \in \mathbb{R}^n$ by $2R([x, y], C)$, the diameter $D(K, C)$ of $K$ w.r.t. $C$ is the maximal length of a segment with both endpoints in $K$. For arbitrary $C \in \mathcal{K}^n$ the Minkowski asymmetry $s(C)$ of $C$ is the smallest $\lambda > 0$ such that $-C \subset c + \lambda C$ for some $c \in \mathbb{R}^n$, i.e. $s(C) = R(-C, C)$. Asymmetry functions are often used to extend and unify results, which have natural solutions when restricted to centrally symmetric bodies as well as for the general case (see, e.g. [2], [6], [18], [27], for such results using the Minkowski asymmetry and [12] using other symmetry measures).

The Jung ratio of $K$ w.r.t. $C$ is the quotient $j(K, C) := R(K, C)/D(K, C)$ between the circumradius and the diameter, while the Jung constant $j_C$ of a body $C$ is defined as $j_C := \max\{j(K, C) : K \in \mathcal{K}^n\}$. Both namings honour Jung, who proved that $j_{B^n_2} = \sqrt{\frac{n}{2(n+1)}}$ for the Euclidean ball $B^n_2$ [19].

A general upper bound for the Jung constant of a symmetric body $C$ is given by the inequality of Bohnenblust [3]:

\begin{equation}
 j_C \leq \frac{n}{n+1}.
\end{equation}

In [6 Theorem 4.1] a generalization of Bohnenblust’s inequality (1) for arbitrary $C$ is given, involving the Minkowski asymmetry of $K$ and $C$:

For any $K, C \in \mathcal{K}^n$ it holds

\begin{equation}
 j(K, C) \leq \frac{s(K)(s(C) + 1)}{2(s(K) + 1)}
\end{equation}

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which results into
\[ jC \leq \frac{n(s(C) + 1)}{2(n + 1)}, \]
for the Jung constants (which has already been derived in \[10\]).

Denoting the \(n\)-dimensional Euclidean unit ball by \(B_n^2\), Santaló (in case of \(n = 2\)) \[25\] and later Vrélica \[29, Corollary 1\] showed that \(r(K, B_n^2) + R(K, B_n^2) \leq D(K, B_n^2)\).
The above inequality appears frequently in the literature (see e.g. \[4\] and \[13\]) and has been
generalized to arbitrary Minkowski spaces \[11\] (and even to general Banach spaces \[11, 22\]),
resulting in the **concentricity inequality** (the name pointing out that for the equality case it is
necessary that every incenter is also a circumcenter – see \[7, Lemma 2.4\]): For all \(K, C \in \mathcal{K}^n\) with \(C\) (centrally) symmetric it holds
\[ r(K, C) + R(K, C) \leq D(K, C). \]

Moreover, in \[6, Remark 6.3\] the concentricity inequality has been embedded into a chain of
inequalities for symmetric \(C\), involving (besides others) the Minkowski asymmetry of \(K\) and
the extended Bohnenblust inequality \[2\] with \(s(C) = 1\):
\[ (1 + s(K))r(K, C) \leq r(K, C) + R(K, C) \leq \frac{1 + s(K)}{s(K)} R(K, C) \leq D(K, C). \]

In this paper we give two different generalizations of the inequality chain in \[5\] for non-
symmetric \(C\):

**Theorem 1.1.** Let \(K, C \in \mathcal{K}^n\). Then the two following inequality chains hold true:
\[ (1 + s(K))r(K, -C) \leq r(K, -C) + R(K, C) \]
\[ \leq s(C)r(K, C) + R(K, C) \leq \frac{1}{2} (1 + s(C)) D(K, C) \]
\[ (1 + s(K))r(K, -C) \leq r(K, -C) + R(K, C) \]
\[ \leq \frac{1 + s(K)}{s(K)} R(K, C) \leq \frac{1}{2} (1 + s(C)) D(K, C). \]

Let us remark that for general \(K\) the two chains do not fit together.
One should also recognize that the chains include two possible generalizations of the concentricity
inequality to the non-symmetric case, the **mirrored concentricity inequality**
\[ (1 + s(C))r(K, -C) + R(K, C) \leq \frac{1}{2} (1 + s(C)) D(K, C) \]
and the stronger **generalized concentricity inequality**
\[ s(C)r(K, C) + R(K, C) \leq \frac{1}{2} (1 + s(C)) D(K, C). \]

We will see that proving the generalized concentricity inequality is the main ingredient of
proving Theorem \[14\] and that behind the extremality of this inequality lies a generalized
notion of concentricity for two possibly non-symmetric sets.
Given \(K, C \in \mathcal{K}^n\), we say that \(K\) is **(diametrically) complete** w. r. t. \(C\) if any strict superset of
\(K\) has strictly bigger diameter w. r. t. \(C\) than \(K\). Moreover, we say that \(K^*\) is a **completion**
of \(K\) if \(K^*\) is complete, a superset of \(K\), and has the same diameter w. r. t. \(C\) than \(K\).
The history of the concentricity inequality is closely related to efforts in understanding complete
sets and completions. Indeed it is known that completions always exist and that complete
sets fulfill equality in the concentricity inequality (see \[15\] for the Euclidean case, \[21\] for
arbitrary Minkowski spaces, as well as \[11\] and \[11\, Proposition 6\] for a generalization to Banach
spaces). Actually it is shown in \cite{8} that equality holds for the full chain in \cite{5} if $K$ is complete w.r.t. $C$. Thus for a general $K$ and a completion $K^*$ of $K$ it holds $r(K, C) + R(K, C) \leq r(K^*, C) + R(K^*, C) = D(K^*, C) = D(K, C)$ (cf. \cite{29}) proving this way the validity of the concentricity inequality (cf. also \cite{23}).

It is easy to see that for the two inequality chains in Theorem 1.1 equality between any part and the rightmost for some $K$ and $C$ can also be kept when completing $K$. Moreover, for complete sets the two inequality chains in Theorem 1.1 can be joined into one chain:

**Theorem 1.2.** Let $K, C \in \mathbb{K}^n$ such that $K$ is complete w.r.t. $C$. Then

\[
(1 + s(K))r(K, -C) \leq r(K, -C) + R(K, C) \leq \frac{1 + s(K)}{s(K)}R(K, C)
\]

\[
\leq s(C)r(K, C) + R(K, C) \leq \frac{1}{2}(1 + s(C))D(K, C)
\]

holds true.

The equality case of (1) has been first studied in \cite{20} and extended in \cite[Corollary 2.10]{7}:

**Proposition 1.3.** Let $S, C \in \mathbb{K}^n$ be such that $S$ is an $n$-simplex and $C$ symmetric. Then the following are equivalent:

(i) $S - S \subset D(S, C)C \subset (n+1)((S - c) \cap (-S + c))$, for some $c \in \mathbb{R}^n$,
(ii) $S$ fulfills the full chain \cite{5} with equality,
(iii) $S$ fulfill \cite{11} with equality,
(iv) $S$ fulfill \cite{11} with equality, and
(v) $S$ is complete w.r.t. $C$.

While for symmetric $C$ the full chain \cite{5} is fulfilled with equality whenever $K$ is complete w.r.t. $C$ in the general case there exist examples of $K$ complete w.r.t. $C$, showing that none of the inequalities in \cite{10} must hold with equality. On the one hand for example, if we have $K = C - C$, then we may easily calculate that

\[
(1 + s(K))r(K, -C) < r(K, -C) + R(K, C) < \frac{1 + s(K)}{s(K)}R(K, C)
\]

\[
= s(C)r(K, C) + R(K, C) = \frac{1}{2}(1 + s(C))D(K, C),
\]

whereas on the other hand Example \cite{4.4} will give a pair $K$ complete w.r.t. $C$ such that

\[
(1 + s(K))r(K, -C) = r(K, -C) + R(K, C) = \frac{1 + s(K)}{s(K)}R(K, C)
\]

\[
< s(C)r(K, C) + R(K, C) < \frac{1}{2}(1 + s(C))D(K, C).
\]

In the following we write $K \subseteq C$ (resp. $K =_{t} C$) for any two convex sets $K, C$ to stress that there exists a translation vector $c$ such that $K \subseteq c + C$ (resp. $K = c + C$) and abbreviate by $K \subseteq CO$ that $K \subseteq C$, but $K \not\subseteq_{t} \rho C$ for any $\rho < 1$.

Studying the equality case of \cite{3} results in the following extension of Proposition \cite{13} to arbitrary $C$:

**Theorem 1.4.** Let $S, C \in \mathbb{K}^n$ be such that $S$ is an $n$-simplex. Then the following are equivalent:

(i) $(n+1)/n \ S \subseteq_{t} S - S \subseteq \frac{D(S, C)}{2}(C - C) \subseteq_{t} (s(C) + 1)\frac{D(S, C)}{2}C \subseteq_{t} (n+1)(-S)$, and
(ii) $S$ fulfills the full chain \cite{3} as well as the full chain \cite{7} (and therefore the joint chain \cite{10}) with equality,
(iii) $S$ fulfills \cite{11} with equality,
(iv) \( S \) fulfills (B) with equality,
(v) \( S \) is complete w. r. t. \( C \) and \( R(S,C) = ns(C)r(S,C) \).

Observe that the leftmost inclusion in Part (ii) above is always true (and thus could have been added to the chain of inclusion in Part (i) of Proposition 1.3 too). However, we only mention it here since it holds \((n+1)/n S \subset^{opt} \{ n+1 \}(-S)\).

One should also observe that the chain of inclusion under translations in Part (ii) above becomes a direct chain of inclusions if and only if \( S \) and \( C \) have the origin as a common Minkowski center.

2. Preliminaries

For any \( A \subset \mathbb{R}^n \) we denote by \( \text{bd}(A) \) the boundary of \( A \), by \( \text{conv}(A) \) the convex hull of \( A \) and abbreviate by \([x,y] := \text{conv}\{\{x,y\}\}\) the line segment with endpoints \( x, y \in \mathbb{R}^n \).

For any \( \rho > 0 \) and \( A, B \subset \mathbb{R}^n \) let \( A + B := \{ a + b : a \in A, b \in B \} \) denote the Minkowski sum of \( A \) and \( B \) and \( \rho A := \{ \rho a : a \in A \} \) the \( \rho \)-dilatation of \( A \), abbreviating \( -A := (-1)A \).

The support function of a convex body \( K \subset \mathbb{R}^n \) is the function \( h(K,\cdot) : \mathbb{R}^n \to \mathbb{R}, h(K,a) = \sup_{x \in K} a^T x \) and the normal cone of \( C \subset \mathbb{R}^n \) in a point \( p \in C \) is the set \( N(C,p) = \{ a \in \mathbb{R}^n : a^T p = h(C,a) \} \). We write \( H_{a,b} := \{ x \in \mathbb{R}^n : a^T x = b \} \) for the hyperplane orthogonal to \( a \in \mathbb{R}^n \) with value \( b \in \mathbb{R} \), respectively.

The circumradius of \( K \subset \mathbb{R}^n \) with respect to \( C \subset \mathbb{R}^n \) can be formalized as \( R(K,C) = \inf \{ \rho > 0 : K \subset _\rho C \} \) (with \( R(K,C) = \infty \) if and only if the affine hull of \( K \) is not a subset of the affine hull of \( C \)). Analogously, the inradius is \( r(K,C) = \sup \{ \rho \geq 0 : \rho C \subset K \} \) and we have that \( r(K,C) = R(K,C)^{-1} \).

The following Proposition is taken from [5, Theorem 2.3] and gives a characterization for \( K \subset^{opt} C \). Here and below we use \([n]\) to abbreviate \( \{1,\ldots,n\} \) for some \( n \in \mathbb{N} \).

Proposition 2.1 (Optimal Containment Condition). Let \( K,C \subset \mathbb{R}^n \) and \( K \subset C \). Then the following are equivalent:

(i) \( R(K,C) = 1 \).

(ii) there exist \( i \in \{2,\ldots,n+1\} \), \( p^j \in K \cap \text{bd}(C) \), and \( a^j \in N(C,p^j), j \in [i] \) such that

\[ 0 \in \text{conv}\{\{a^j : j \in [i]\}\} \]

Moreover, if \( C = \mathbb{B}^n_2 \) then (i) and (ii) are also equivalent to

(iii) there exist \( i \in \{2,\ldots,n+1\} \) and \( p^j \in K \cap \text{bd}(C), j \in [i] \), such that

\[ 0 \in \text{conv}\{\{p^j : j \in [i]\}\} \]

The \( s \)-breadth \( b_s(K,C) \) of \( K \) w. r. t. \( C \) in the direction \( s \in \mathbb{R}^n \setminus \{0\} \) is \( b_s(K,C) = 2 \frac{h(K-K,s)}{h(C,C,s)} \).

The \( s \)-breadth does not change under central symmetrization (cf. [6]), i.e.

\[ b_s(K,C) = \frac{1}{2}b_s(K-K,C) = 2b_s(K,C-C) = b_s(K-K,C-C). \]

With help of the \( s \)-breadth the diameter \( D(K,C) = 2 \sup_{x,y \in K} R([x,y],C) \) of \( K \) w. r. t. \( C \) can also be expressed as \( D(K,C) = \sup_{s \in \mathbb{R}^n \setminus \{0\}} b_s(K,C) \) (see [6]). Moreover, denoting the norm induced by the gauge body \( 1/2(C-C) \) by \( \|\cdot\|_{1/2(C-C)} \) we also have \( D(K,C) = \max_{x,y \in K} \|y-x\|_{1/2(C-C)} \).

However, denoting also the gauge function induced by a possibly non-symmetric \( C \) by \( \|\cdot\|_{C} \), the diameter of \( K \) may differ from \( \max_{x,y \in K} \|y-x\|_{C} \). One may think that the length of a segment \([x,y]\) should be defined by the value \( \|y-x\|_{C} \) of the gauge function. However, we think (as it was already formulated in [14], pg. 134) that this measure should be translation invariant and symmetric (since segments as 1-dimensional objects are always symmetric).

All three radii \( r, D, R \) are non-decreasing and homogeneous of degree 1 in the first argument, as well as non-increasing and homogeneous of degree \(-1\) in the second. This means, e. g.,
that \( R(K_1, C) \leq R(K_2, C) \) if \( K_1 \subseteq K_2 \) and \( R(\lambda K, C) = \lambda R(K, C) \), for any \( \lambda \geq 0 \). Moreover, all three radii are continuous (w.r.t. the Hausdorff metric) in both arguments and affine invariant (in the sense that, e.g. \( R(A(K), A(C)) = R(K, C) \), for any \( n \)-dimensional regular affine transformation \( A \)).

The Minkowski asymmetry \( s(K) \) can be formalized as \( s(K) := \inf \{ \rho > 0 : -K \subseteq \rho K \} = R(-K, K) \). If \( -(K - c) \subseteq s(K)(K - c) \) for some \( c \in \mathbb{R}^n \) then \( c \) is called a Minkowski center of \( K \), and if \( c = 0 \) is a Minkowski center of \( K \), then \( K \) is called Minkowski centered. Moreover, we say that \( K \) is Minkowski concentric (or mirrored Minkowski concentric) w.r.t. \( C \), if there exists a Minkowski center \( c \) of \( C \) and a translation \( t \in \mathbb{R}^n \) such that \( r(K, C)(C - c) \subseteq K - t \subseteq R(K, C)(C - c) \) (or \( -r(K, -C)(C - c) \subseteq K - t \subseteq R(K, C)(C - c) \), respectively). Now, consider the case that we can choose \( t \) above to be a Minkowski center of \( K \). Then it is easy to see that the Minkowski concentricity of \( K \) w.r.t. \( C \) is equivalent to the Minkowski concentricity of \( C \) w.r.t. \( K \). In this case we simply say that \( K \) and \( C \) are Minkowski concentric. One should also recognize that if \( K \) is mirrored Minkowski concentric w.r.t. \( C \) with \( t \) a Minkowski center of \( K \), then \(-C \) is mirrored Minkowski concentric w.r.t. \(-K \). The Minkowski concentricities turn out to be necessary conditions for the equality cases of many of the relevant inequalities and inequality chains in this paper.

The Minkowski asymmetry fulfills \( 1 \leq s(K) \leq n \) with equality on the left side if and only if \( K = -K \) and equality on the right side if and only if \( K \) is an \( n \)-simplex (see [21], or [6] for a proof given in english).

In [16] it is shown that \( c \) is a Minkowski center of \( K \) if and only if for all \( s \in \mathbb{R}^n \setminus \{0\} \) it holds

\[
\frac{h(-c - K, -s)}{h(-c - K, s)} \in [s(K)^{-1}, s(K)].
\]

The geometrical meaning of (11) is that any hyperplane with normal \( s \) containing the Minkowski center \( c \) of \( K \), splits \( K \) in two parts with an \( s \)-breadth ratio bounded from above by \( s(K) \). In this sense, the Minkowski center with respect to the directional breadth plays the role that the the centroid plays with respect to the volume (cf. [17]).

A set \( K \) is of constant width w.r.t. \( C \), if \( D(K, C) = b_s(K, C) \) independently of the choice of \( s \in \mathbb{R}^n \setminus \{0\} \). It is well known that \( K \) is of constant width if and only if \( K - K = D(K, C)/2(C - C) \) (see, e.g., [15] (A)).

The following is a trivial but important observation: while some symmetric \( C \) do not admit non-trivial constant width sets (like parallelopetes or if they are indecomposable, like crosspolytopes in dimension 3 or greater), the difference body \( C - C \) is always a non-trivial constant width body w.r.t. \( C \), if \( C \) is non-symmetric.

It is well known that if \( K \) is of constant width w.r.t. \( C \) then it is complete w.r.t. \( C \). However, even though this two properties are equivalent in the Euclidean space and in any planar Minkowski space, they are no longer equivalent for general (symmetric) \( C \) if the dimension of the space is greater than 2 (see [15]).

It is shown in [19] Lemma 3.2 (ii)] that we may symmetrize the gauge body \( C \):

**Proposition 2.2.** Let \( K, C \in \mathcal{K}^n \). Then the following are equivalent:

(i) \( K \) is complete w.r.t. \( C \) and

(ii) \( K \) is complete w.r.t. \( C - C \).

One should recognize that \( K \) is complete w.r.t. \( C \) does not imply that \( K - K \) is complete w.r.t. \( C \). Otherwise Proposition 2.2 would imply that \( K - K \) is complete w.r.t. \( C - C \), which is only possible if \( K - K = \rho(C - C) \) for some \( \rho \in \mathbb{R} \). However, the latter would imply that \( K \) is of constant width w.r.t. \( C \), but as mentioned above, completeness does not always imply constant width.
3. New Inequalities

The following lemma collects a series of easy to obtain inequalities, which are used in the proofs of the main theorems below.

**Lemma 3.1.** Let $K, C \in \mathcal{K}^n$. Then

(a) \( \max \{ s(K), s(C) \} \leq \frac{R(K,C)}{r(K,-C)} \) and \( s(C) = \frac{R(K,C)}{r(K,-C)} \) implies that $K$ is mirrored Minkowski concentric w. r. t. $C$ while \( s(K) = \frac{R(K,C)}{r(K,-C)} \) implies that $C$ is mirrored Minkowski concentric w. r. t. $K$.

(b) \( \frac{s(C)+1}{s(C)} \leq \frac{R(K,C)}{R(K,C-C)} \leq s(C) + 1. \)

(c) \( \frac{s(C)+1}{s(C)} \leq \frac{R(K,C)}{R(K,C-C)} \leq s(C) + 1. \)

(d) \( \min \{ s(K), s(C) \} \geq \max \left\{ \frac{r(K,-C)}{r(K,C)}, \frac{r(K,-C)}{r(K,C)} \right\}. \)

(e) \( s(C) \geq \frac{R(K,C)r(K,C-C)}{R(K,C-C)} + 1. \)

**Proof.** (a) As a direct consequence of the definitions of the inradius and the circumradius we obtain

\[
\frac{r(K,-C)}{R(K,C)}(-K) \subset_t r(K,-C)(-C) \subset_t K \subset_t R(K,C)C.
\]

This implies for the Minkowski asymmetries that \( s(K) = R(-K, K) \leq R(K, C)/r(K, -C) \) as well as \( s(C) = R(-C, C) \leq R(K, C)/r(K, -C) \). Now, let us assume that \( s(K) = R(K, C)/r(K, -C) \) and without loss of generality that $K$ is Minkowski centered. Then we immediately see that there exists a translation $t \in \mathbb{R}^n$ such that $-K \subset^{opt} t + R(K, C)(-C) \subset^{opt} s(K)K$. Hence $C$ is mirrored Minkowski concentric w. r. t. $K$. The remaining part of the claim follows analogously.

(b) From the definitions of circumsradius and Minkowski asymmetry it follows

\[
K \subset_t R(K, C - C)(C - C) \subset_t R(K, C - C)(s(C) + 1)C
\]

as well as

\[
K \subset_t R(K, C)C \subset_t R(K, C)\frac{s(C) + 1}{s(C)}(C - C).
\]

While the first chain of inclusions implies $R(K, C) \leq (s(C) + 1)R(K, C - C)$ the other implies $R(K, C - C) \leq R(K, C)s(C)+1/s(C)$.

(c) This can be shown completely analogous to Part (b) replacing the circumradius by the inradii.

(d) Since $r(K, C) = R(K, C)^{-1}$ it suffices to show $s(C) \geq \max \left\{ \frac{r(K,-C)}{r(K,C)}, \frac{r(K,-C)}{r(K,C)} \right\}$. However, from Part (b) we know \( \frac{R(K,-C)}{R(K,C-C)} \leq s(C) + 1 \) and \( \frac{R(K,-C)}{R(K,C-C)} \geq \frac{s(C)+1}{s(C)} \). Dividing the second by the first inequality we obtain \( \frac{R(K,-C)}{R(K,C)} \leq s(C) \). The part of statement involving the inradius-ratio follows analogously applying both inequalities in Part (c).

(e) The last inequality follows directly from dividing the right inequality in Part (b) by the left inequality in (c).

\[\square\]

**Lemma 3.2.** Let $C \in \mathcal{K}^n$ be Minkowski centered and $r \in [0, 1]$. Then

\[
\frac{h(C, a) + h(rC, -a)}{h(C, a) + h(C, -a)} \in \left[ \frac{1 + s(C)r}{1 + s(C)}, \frac{r + s(C)}{r + s(C)} \right], \quad a \in \mathbb{R}^n \setminus \{0\}.
\]

**Proof.** Defining \( x := \frac{h(C,-a)}{h(C,a)} \) we know from (1) that $x \in [s(C)^{-1}, s(C)]$. Using this $x$ we may rewrite the fraction \( \frac{h(C, a) + h(rC, -a)}{h(C, a) + h(C, -a)} \) dividing both enumerator and denominator by
h(C,a) as f(x) := \frac{1+rx}{1+r}. Since f(x) is a decreasing function for r ≤ 1, we conclude that 
f(x) ∈ [f(s(C)), f(s(C)^{-1})], proving the assertion. □

Proof of Theorem 1.1. The left inequality of both chains (since equal) as well as the middle
inequality of (7) follow directly from Lemma 3.1 (3), the middle inequality of (6) directly
from Lemma 3.1 (3). Moreover, the right inequality of (7) follows from (2). Thus it only
remains to show the right inequality in (6), the generalized concentricity inequality (9). For
the proof we may assume w.l.o.g. that K ⊂^opt C ⊂ −s(C)C, i.e. R(K) = 1 and C is
Minkowski centered. Moreover, let c ∈ C be such that c + r(K,C)C ⊂ K. Since K ⊂^opt C
by Proposition 2.1 there exist p^j ∈ bd(K)∩bd(C), j ∈ [m], as well as a^j ∈ N(C,p^j) \ {0} and
λ_j > 0, j ∈ [m] such that \sum_{j=1}^m λ_j = 1 and \sum_{j=1}^m λ_j a^j = 0. The latter implies the existence
of some j ∈ [m] such that c^T a^j ≤ 0, which we may assume to be 1.

Abbreviating r := r(K,C), we obtain from K' := conv({p^1} ∪ (c+rC)) ⊂ K and h(K',a^j) = (p^1)^T a^j = h(C,a^j) that:
\[
\frac{1}{2} D(K,C) \geq \frac{1}{2} h_a(K,C) = \frac{h(K,a^1) + h(K,-a^1)}{h(C,a^1) + h(C,-a^1)}
\geq \frac{h(K',a^1) + h(K',-a^1)}{h(C,a^1) + h(C,-a^1)} = \frac{h(C,a^1) + h(rC,-a^1) - c^T a^1}{h(C,a^1) + h(C,-a^1)}
\]
and using the assumption c^T a^1 ≤ 0 and Lemma 3.2 this is
\[
\geq \frac{h(C,a^1) + h(rC,-a^1)}{h(C,a^1) + h(C,-a^1)} \geq \frac{1 + rs(C)}{1 + s(C)}.
\]

Let us keep two facts, we immediately obtain from the above proof:

a) equality in the generalized concentricity inequality (9) implies that K is Minkowski con-
centric w. r. t. C.

b) equality in the mirrored concentricity inequality (8) implies that s(C) ≤ s(K) and that
C is mirrored Minkowski centered w. r. t. K (in addition to the Minkowski concentricity
of K and C, which one gets from the induced equality in the generalized concentricity
inequality).

Part (b) directly follows from the equality case s(K) = \frac{R(K,C)}{r(K,C)} in Lemma 3.1 (3). To see
that Part (a) holds, let us assume without loss of generality that C is Minkowski centered,
that r(K,C)C ⊂ K and R(K,C) = 1. Our aim is to show that K ⊂ C, thus proving the
Minkowski concentricity of K w. r. t. C. Let us observe the following: Lemma 3.2 shows that
if p^1 ∈ bd(C) then
\[
\frac{1}{2} D(\text{conv}\{p^1\} ∪ r(K,C)C), C) \geq \frac{s(C)r(K,C) + 1}{s(C) + 1}.
\]
Hence for any ρ > 1 and any y ∈ bd(ρC) it holds
\[
\frac{1}{2} D(\text{conv}\{y\} ∪ r(K,C)C), C) \geq \frac{s(C)r(K,C) + ρ}{s(C) + 1} > \frac{s(C)r(K,C) + 1}{s(C) + 1}.
\]
Thus equality in (9) implies that no such y exists in K and therefore K ⊂ C.

A classical result in Euclidean geometry on simplices says that the circumradius-inradius ratio
of an n-simplex S, is at least n, i.e., R(S,B^2_n) ≥ nr(S,B^2_n) (cf. [28], p. 28 – with equality if
and only if S is regular). This was extended in [6, Theorem 6.1] saying the following:
Proposition 3.3. Let $K, C \in \mathbb{K}^n$. Then
\[
\frac{R(K, C)}{r(K, C)} \geq \max \left\{ \frac{s(K)}{s(C)}, \frac{s(C)}{s(K)} \right\}.
\]

Recognizing that Proposition 3.3 can simply be obtained from dividing inequalities from Lemma 3.1 (ii) and (iii), we easily see that $\frac{R(K, C)}{r(K, C)} = \frac{s(K)}{s(C)}$ implies that $s(K) \geq s(C)$ as well as $C$ is mirrored Minkowski concentric w.r.t. $K$ and vice versa that $\frac{R(K, C)}{r(K, C)} = \frac{s(C)}{s(K)}$ implies that $s(C) \geq s(K)$ as well as $K$ is mirrored Minkowski concentric w.r.t. $C$.

Surely, in general no constant exists bounding the circumradius-inradius ratio from above. However, the family of complete bodies allows such an upper bound:

Lemma 3.4. Let $K, C \in \mathbb{K}^n$. If $K$ is complete w.r.t. $C$ then
\[
\frac{R(K, C)}{r(K, C)} \leq s(K)s(C)
\]

and equality implies that $K$ and $C$ are Minkowski concentric.

Proof. Let $K$ be complete w.r.t. $C$. Then $K$ is complete w.r.t. $C - C$ by Proposition 2.2 and therefore fulfills $R(K, C - C)/r(K, C - C) = s(K)$ by [7, Theorem 1.2]. Now the claimed inequality follows directly using Lemma 3.1 (iv). For the concentricity statement let us assume without loss of generality that $C$ is Minkowski centered. It is shown in [7] that $K$ is complete w.r.t. $C - C$ implies that $r(K, C - C)(C - C) \subseteq K - c \subseteq R(K, C - C)(C - C)$ for some Minkowski center $c$ of $K$, thus $K$ and $C - C$ are Minkowski concentric. However, $\frac{R(K, C)}{r(K, C)} \leq s(K)s(C)$ now implies by using Lemma 3.1 (v) again that
\[
r(K, C)C = \left(1 + \frac{1}{s(C)}\right)r(K, C - C)C \subseteq r(K, C - C)(C - C) \subseteq K - c \subseteq R(K, C - C)(C - C) \subset (1 + s(C))R(K, C - C)C = R(K, C)C,
\]

which shows $K$ and $C$ are Minkowski concentric.

See Example 4.3 below for sets $K, C$ such that $K$ is complete w.r.t. $C$ and $\frac{R(K, C)}{r(K, C)} = s(K)s(C)$ for any prescribed values for $s(K)$ and $s(C)$.

Proof of Theorem 1.2. This theorem now follows directly from combining Theorem 1.1 and Lemma 3.4.

Putting together the concentricity statements after the proof of Theorem 1.1 and from Lemma 3.4 we see that equality in the complete chain of inequalities (10) implies that $K$ and $C$ are Minkowski concentric as well as that $C$ is mirrored Minkowski concentric w.r.t. $K$.

Remark 3.5. For a general pair $K, C \in \mathbb{K}^n$ (not necessarily such that $K$ is complete w.r.t. $C$) it always holds
\[
s(K) + \frac{1}{s(K)}K \subseteq K - K \subseteq \frac{D(K, C)}{2}(C - C) \subseteq \frac{D(K, C)}{2}(s(C) + 1)C.
\]

and the following are equivalent:

(i) all inequalities in the inequality chain (10) are fulfilled with equality and
(ii) $\frac{D(K, C)}{2}(s(C) + 1)C \subseteq (s(K) + 1)(-K)$. 

4. Complete simplices

An \( n \)-simplex \( S \in \mathcal{K}^n \), \( S = \text{conv}\{p^1, \ldots, p^{n+1}\} \), is \textit{equilateral} w. r. t. \( C \in \mathcal{K}^n \) if \( D([p^i, p^j], C) = D(S, C) \) for any \( 1 \leq i < j \leq n + 1 \). The following remark shows that a complete simplex is equilateral.

**Remark 4.1.** Let \( S, C \in \mathcal{K}^n \) be such that \( S = \text{conv}\{p^1, \ldots, p^{n+1}\} \) is an \( n \)-simplex being complete w. r. t. \( C \). Then for every boundary point \( p \) of \( S \) and the vertex \( p^j \) not belonging to the facet of \( S \) to which \( p \) belongs, it holds \( D([p, p^j], C) = D(S, C) \) and it follows that \( S \) is equilateral.

**Proof.** For better readability we write \( d(x, y) := D([x, y], C) = ||x - y||_2/2(C - C) \). Since \( S \) is complete, every boundary point of \( S \) must be an endpoint of a diametrical segment of \( S \) and surely the vertex \( p^j \) not belonging to the facet of \( S \) to which \( p \) belongs may be chosen as the other endpoint. Now, consider the point \( p := \sum_{k \in [n + 1] \setminus \{j\}} \frac{1}{n}p^k \in \text{relint}(\text{conv}\{p^k : k \neq j\}) \), which is an endpoint of a diametrical segment such that the other endpoint must be \( p^j \) implying \( d(p, p^j) = D(S, C) \). Hence

\[
D(S, C) = d(p, p^j) = d\left( \sum_{k \in [n + 1] \setminus \{j\}} \frac{1}{n}p^k, p^j \right) \leq \frac{1}{n} \sum_{k \in [n + 1] \setminus \{j\}} d(p^k, p^j) \leq D(S, C),
\]

which in particular implies that \( d(p^k, p^j) = D(S, C) \) for every \( j \in [n + 1] \) and every \( k \in [n + 1] \setminus \{j\} \).

We now prove Theorem 1.4 by connecting it to the equality cases of (3) by connecting it to the equality cases of (5) and to the notion of completeness.

**Proof of Theorem 1.4.** Since all the statements, which we want to show to be equivalent are invariant w. r. t. a simultaneous affine transformation of \( S \) and \( C \) we may suppose w. l. o. g. that \( S = \text{conv}\{p^1, \ldots, p^{n+1}\} \) with \( ||p^j||_2 = 1, j \in [n + 1] \), is a regular simplex (with respect to the Euclidean norm) centered in 0.

Since (i) and (ii) are due to Remark 3.5 anyway equivalent it suffices to show the following line of indications: \( \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \), and \( \text{(i)} \Rightarrow \text{(v)} \Rightarrow \text{(iv)} \Rightarrow \text{(i)} \).

“\( \text{(iii)} \Rightarrow \text{(iv)} \)” Plugging (3) in (iii) we obtain that \( R(S, C) \leq nr(−S, C) \), which together with Lemma 3.1 (a) implies \( R(S, C) = nr(−S, C) \) and then also equality in (3) as well, which proves (iv).

“\( \text{(iv)} \Rightarrow \text{(i)} \)” (iv) implies optimality in all inclusions in (12). In particular we have

\[
\frac{n + 1}{n}S \subset S - S \subset_{\text{opt}}^t \frac{D(S, C)}{2}(s(C) + 1)C.
\]

Moreover, since the only outer normal of \( S - S \) at the vertex \( (n + 1)/np^i \) of \((n + 1)/nS \) is \( p^i \), \( i \in [n + 1] \), this \( p^i \) is also the (only) outer normal of \( \frac{D(S, C)}{2}(s(C) + 1)C \) at \((n + 1)/np^i \), \( i \in [n + 1] \). Hence

\[
\frac{D(S, C)}{2}(s(C) + 1)C \subset \bigcap_{j=1}^{n+1} \left\{ x \in \mathbb{R}^n : x^T p^i \leq \left( \frac{n + 1}{n} - p^i \right)^T p^i \right\} = (n + 1)(−S).
\]

“\( \text{(ii)} \Rightarrow \text{(iii)} \)” Equality in the full chain (5) implies equality in the generalized concentricity inequality (9), which is part of the chain.

“\( \text{(i)} \Rightarrow \text{(iv)} \)” On the one hand (i) implies by Proposition 1.3 that \( S \) is complete w. r. t. \( C - C \), and thus by Proposition 2.2 also completeness of \( S \) w. r. t. \( C \). On the other hand (ii) implies \( s(C)r(S, C) + R(S, C) = \frac{(n + 1)R(S, C)}{n} \) and therefore \( R(S, C)/r(S, C) = ns(C) \).
Assuming that $S$ is complete w.r.t. $C$ it follows from Proposition 2.2 that $S$ is also complete w.r.t. $C - C$. Now, using the equivalences in Proposition 1.3 we obtain

$$(n + 1)r(S, C - C) = \frac{n + 1}{n}R(S, C - C) = D(S, C - C).$$

Moreover, $R(S, C) = ns(C)r(S, C)$ means equality in Lemma 3.1 (c), which implies equality in Lemma 3.1 (b), thus $R(S, C) = (s(C) + 1)R(S, C - C)$. Finally, since $D(S, C) = 2D(S, C - C) = 2(n + 1)r(S, C - C)$, we conclude

$$\frac{R(S, C)}{D(S, C)} = \frac{(s(C) + 1)R(S, C - C)}{2(n + 1)r(S, C - C)} = \frac{(s(C) + 1)n}{2(n + 1)}.$$

□

**Corollary 4.2.** Let $S, C \in K^n$ be such that $S$ is an $n$-simplex. Then $S$ is complete w.r.t. $C$ if and only if $-S$ is complete w.r.t. $C$,

$$\frac{n}{s(C)} \leq \frac{R(\pm S, C)}{r(\pm S, C)} \leq ns(C),$$

and $S$ (resp. $-S$) attains equality in the right-hand side if and only if $-S$ (resp. $S$) attains equality in the left-hand side if and only if $S$ (resp. $-S$) fulfills any/all conditions of Theorem 1.4.

**Proof.** The equivalence of the completeness of $S$ and $-S$ follows directly from Proposition 2.2, while the inequality chain is just a combination of Proposition 3.3 and Lemma 3.4. Moreover, equality on the right means that Condition (v) of Theorem 1.4 is fulfilled and thus $S$ fulfills all the conditions in Theorem 1.4.

Now, let us suppose that $S$ attains equality on the left-hand side. It implies that the chain of inclusions

$$S \subset R(S, C)C \subset R(S, C)s(C)(-C) \subset R(S, C)s(C)(-S)$$

possesses optimal containment of the first in the last set, and thus of any set in any other including set in the chain. Now, we obtain from the optimal containment of the first in the third set that $R(-S, C) = s(C)R(S, C)$, whereas from the second in the fourth that $r(-S, C) = r(S, C)/s(C)$. Hence $R(-S, C)/r(-S, C) = s(C)^2 R(S, C)/r(S, C) = s(C)n$. Finally, because $-S$ is complete w.r.t. $C$ if and only if $S$ is complete w.r.t. $C$ we obtain that $S$ attains equality on the left-hand side means that $-S$ fulfills Condition (v) of Theorem 1.4.

□

The following example presents a particular family of pairs $S, C \in K^n$, $S$ being an $n$-simplex, which fulfill the conditions of Theorem 1.4.

**Example 4.3.** Let $\lambda \geq \mu \geq 0$ and $S, C \in K^n$ such that $S$ is a Minkowski-centered equilateral (w.r.t. $B_2^n$) $n$-simplex and $\lambda S + \mu (-S) \subset C \subset (\lambda + 2\mu)S \cap (2\mu + \lambda)(-S)$.

Then $S$ and $-S$ are complete w.r.t. $C$, while $-S$ is fulfills the conditions of Theorem 1.4, whereas $S$ does not fulfill these conditions, unless $\lambda \in \{0, \mu\}$.

One should recognize that by construction of $C$ it is quite obvious that $S$ and $C$ are Minkowski concentric and (mutually) mirrored Minkowski concentric.
b) Since necessary Minkowski concentricity conditions between $S$ and $C$ are fulfilled, but still $n/s(C) < R(S,C)/r(S,C) < n s(C)$ is not known to us. Thus it is possible that completeness together with the necessary Minkowski concentricity conditions implies all the properties in Theorem 1.4.

Example 4.4. Let $S$ be a Minkowski centered $n$-simplex, $p \in (n+1)(S \cap (-S)) \setminus (S - S)$ and $C = \text{conv}(\{p\} \cup (S - S))$. Then

(i) both, $S$ and $-S$, are complete w. r. t. $C$ but
(ii) $C$ is not Minkowski concentric w. r. t. $S$ nor $-S$.

Proof. a) Since $C \subset (n+1)(S \cap (-S))$, we have that

$$(S - S) \subset 1/2(C - C) \subset (n+1)(S \cap (-S)) + (n+1)(-(S \cap (-S))) = (n+1)(S \cap (-S)),$$

and thus using Proposition 1.3 together with Proposition 2.2 we see that $S$ and $-S$ are complete w. r. t. $C$ (and also that $D(S,C) = 1$).

b) Since

$$\frac{n+1}{n} S \subset C = \text{conv}(\{p\} \cup (S - S)) \subset (n+1)(S \cap (-S)) \subset (n+1)(-S)$$

as well as

$$\frac{n+1}{n} (-S) \subset C = \text{conv}(\{p\} \cup (S - S)) \subset (n+1)(S \cap (-S)) \subset (n+1)S$$

and because the extreme sides of each chain show optimal containment, all inclusions are optimal, i.e. $C \subset^\text{opt} (n+1)(\pm S) \subset^\text{opt} nC$. Now, since $S - S$ is 0-symmetric, $C = \text{conv}(\{p\} \cup (S - S))$ cannot have 0 as a Minkowski center because of Proposition 2.1. However, as 0 is the only Minkowski center of $\pm S$, we conclude that $C$ is not Minkowski concentric w. r. t. $S$ nor $-S$.

An example of a simplex $S$ and a body $C$ such that $S$ is complete w. r. t. $C$ and all the necessary Minkowski concentricity conditions between $S$ and $C$ are fulfilled, but still $n/s(C) < R(S,C)/r(S,C) < n s(C)$ is not known to us. Thus it is possible that completeness together with the necessary Minkowski concentricity conditions implies all the properties in Theorem 1.4.

For the remainder of this section we consider the situation in the planar case. However, doing so we make use of the following classical result on polytopes in arbitrary dimensions(cf. [26]), where $\text{vol}_{n-1}(\cdot)$ denotes the $(n - 1)$-dimensional volume.
Proposition 4.5. Let \( P \in \mathcal{K}^n \) be a polytope with facets \( F_i = P \cap H_{a^i,c_i} \), for some \( a^i \in \mathbb{R}^n \), \( \|a^i\|_2 = 1 \), \( c_i \in \mathbb{R} \), \( i \in [m] \), and \( m \in \mathbb{N} \). Then
\[
\sum_{i=1}^m \vol_{n-1}(F_i)a^i = 0.
\]

Lemma 4.6. Let \( S, C \in \mathcal{K}^2 \) be such that \( S \) is a Minkowski centered triangle. Then the following are equivalent:

(i) \( S - S = C - C \) (and thus \( S \) is of constant width w. r. t. \( C \))
(ii) \( C = \lambda S + (1 - \lambda)(-S) \), for some \( \lambda \in [0,1] \).

Proof. Surely, (ii) implies \( C - C = \lambda S + (1 - \lambda)(-S) - (\lambda S + (1 - \lambda)(-S)) = S - S \) and therefore (i).
To prove that (i) implies (ii) we may assume w.l.o.g. that \( S \) is an equilateral triangle w. r. t. \( \mathbb{B}_2^2 \). Since \( C - C = S - S \), \( C \) must be a polygon with edges parallel to edges of \( S - S \). Let \( l \) be the length of the edge of \( C \) with outer normal \( (\cos(-\pi/2 + j\pi/3), \sin(-\pi/2 + j\pi/3)) \), \( j \in \{0, \ldots, 5\} \). The length of the edge of \( C - C \) with outer normal \( (0, -1) \) equals \( l_1 + l_4 \), with \( (\sqrt{3}/2, -1/2) \) equals \( l_2 + l_5 \), and with \( (\sqrt{3}/2, 1/2) \) equals \( l_3 + l_6 \), and all of them equal the length of any edge of \( S - S \), hence
\[
1 + l_4 = l_2 + l_5 = l_3 + l_6.
\]
By Proposition 4.5 we have that
\[
\sum_{j=1}^6 l_j(\cos(-\pi/2 + j\pi/3), \sin(-\pi/2 + j\pi/3)) = 0,
\]
which implies that \( (l_1 - l_4)(0, -1) + (l_2 - l_5)(\sqrt{3}/2, -1/2) + (l_3 - l_6)(\sqrt{3}/2, 1/2) = 0 \), and therefore
\[
l_2 - l_5 + l_3 - l_6 = 0 \quad \text{and} \quad l_1 - l_4 + l_2 - l_5 = 0.
\]
The solution to the linear system \(13\) and \(14\) is \( l_1 = l_3 = l_5 \) and \( l_2 = l_4 = l_6 \), and thus \( C = l_1 S + l_2 (-S) \), with \( l_1 + l_2 = 1 \). \( \square \)

Particularizing Theorem 1.4 to the planar case (where completeness and constant width coincide [15]), we see that the situation is much simpler:

Corollary 4.7. Let \( S, C \in \mathcal{K}^2 \) be such that \( S \) is a Minkowski centered triangle. Then the following are equivalent:

(i) \( 3/2S \subset S - S = \frac{D(S,C)}{2}(C - C) \subset (s(C) + 1)\frac{D(S,C)}{2}C \subset 3(-S) \),
(ii) \( 3r(-S,C) = r(-S,C) + R(S,C) = 3/2R(S,C) = s(C)r(S,C) + R(S,C) = (s(C) + 1)\frac{D(S,C)}{2} \),
(iii) \( s(C)r(S,C) + R(S,C) = (s(C) + 1)\frac{D(S,C)}{2} \),
(iv) \( j(S,C) = (s(C) + 1)/3 \),
(v) \( S \) is of constant width w. r. t. \( C \) and \( R(S,C) = 2s(C)r(S,C) \),
(vi) \( S \) is of constant width w. r. t. \( C \) and \( j(S,C) \geq j(-S,C) \), and
(vii) \( C = \lambda S + (1 - \lambda)(-S) \) for some \( \lambda \in [0,1/2] \).

Proof. The equivalences of (i) to (v) follow directly from particularizing Theorem 1.4 to the planar case. Hence it suffices to show (iv) \( \Rightarrow \) (vi) \( \Rightarrow \) (vii) \( \Rightarrow \) (i)-(v).
If (iv) holds, \( S \) attains equality in (14) and therefore \( j(-S,C) \leq j_C = j(S,C) \).
Assuming (vi) to be true, $S$ is of constant width w.r.t. $C$ and therefore $S - S = D(S, C - C)(C - C)$. Now, Lemma 4.6 implies that $D(S, C - C)C = \lambda S + (1 - \lambda)(-S)$, for some $\lambda \in [0, 1]$ and using the computations of the radii of $\pm S$ w.r.t. $C$ in Example 4.3 we see that

$$
\frac{1}{2 - \lambda} = \frac{R(-S, C)}{D(-S, C)} = j(-S, C) \leq j(S, C) = \frac{R(S, C)}{D(S, C)} = \frac{1}{1 + \lambda}
$$

implies $\lambda \in [0, 1/2]$, completing (vii).

Finally, if (vii) holds true, (i)-(v) follow directly from the computations of the radii of $\pm S$ w.r.t. $C$ in Example 4.3.

\[\square\]

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