Abstract

We establish new functional versions of the Blaschke-Santaló inequality on the volume product of a convex body which generalize to the non-symmetric setting an inequality of K. Ball [2] and we give a simple proof of the case of equality. As a corollary, we get some inequalities for log-concave functions and Legendre transforms which extend the recent result of Artstein, Klartag and Milman [1], with its equality case.

Université de Marne la Vallée, Laboratoire d’Analyse et de Mathématiques Appliquées (UMR 8050) Cité Descartes - 5, Bd Descartes Champs-sur-Marne 77454 Marne la Vallée Cedex 2, France

Email: Matthieu.Fradelizi@univ-mlv.fr, Mathieu.Meyer@univ-mlv.fr

Fax: 33 1 60 95 75 45
1 Introduction

For a Borel subset $K$ of $\mathbb{R}^n$ and a point $z \in \mathbb{R}^n$, the polar body $K^*\circ z$ of $K$ with respect to $z$ is the convex set defined by:

$$K^*\circ z = \{y \in \mathbb{R}^n; \langle y - z, x - z \rangle \leq 1 \text{ for every } x \in K\}.$$  

Here $\mathbb{R}^n$ is endowed with the canonical scalar product $\langle \cdot , \cdot \rangle$ and the associated Euclidean norm $| \cdot |$. For $z = 0$, we simply write $K^\circ$ instead of $K^*\circ 0$. Denote by $|K|$ the Lebesgue measure of a Borel subset $A$ of $\mathbb{R}^n$. The Santaló point $s(K)$ of $K$ is a point for which

$$|K^*s(K)| = \min_z |K^*\circ z|.$$  

If $K$ is bounded and not contained in a hyperplane, its Santaló point $z$ is characterized by the property that it is the center of mass of $K^*\circ z$. The inequality of Blaschke-Santaló (Blaschke [4], Santaló [19]) states that

$$|K| \cdot |K^*s(K)| \leq v_n^2 := |B_n^2|,$$

where $B_n^2 = \{x \in \mathbb{R}^n; |x| \leq 1\}$ is the Euclidean ball.

We shall prove here new functional versions of the Blaschke-Santaló inequality and give applications which extend the theorem of Ball [2] as well as the recent result of Artstein, Klartag and Milman [1]. Notice that Lutwak and Zhang [15] and Lutwak, Yang and Zhang [14] gave other very different functional forms of the Blaschke-Santaló inequality and recently Klartag and Milman [13], Klartag [12] and Colesanti [6] also established functional forms of some other geometric inequalities.

The first main result of this paper generalizes with a new proof an inequality of K. Ball [2]; it treats the case of "centered" functions:

**Proposition** Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ and $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}_+$ be measurable functions such that

$$f_1(x)f_2(y) \leq \rho^2(\langle x, y \rangle) \text{ for every } x, y \in \mathbb{R}^n \text{ satisfying } \langle x, y \rangle > 0.$$  

If the star shaped set $K_1 = \{x \in \mathbb{R}^n; \int_0^{+\infty} r^{n-1}f_1(rx)dr \geq 1\}$ is centrally symmetric (which holds if $f_1$ is even), or is a convex body with center of mass at the origin, then

$$\int_{\mathbb{R}^n} f_1(x)dx \int_{\mathbb{R}^n} f_2(y)dy \leq \left( \int_{\mathbb{R}^n} \rho(|x|^2)dx \right)^2.$$  

The idea is to attach bodies $K_1$ and $K_2$ to the functions $f_1$ and $f_2$. From the duality relation on the $f_j$'s, we deduce, using the Prékopa-Leindler inequality for the geometric mean, that the sets $K_j$'s satisfy the inclusion $K_2 \subset c_n(\rho)K_1^{\circ}$. 


for some constant $c_n(r)$. Then the result follows from the Blaschke-Santaló inequality for sets.

As an application of this proposition, we treat the case of "non centered" functions:

**Theorem** Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be measurable and $f : \mathbb{R}^n \to \mathbb{R}_+$ be a log-concave function such that $0 < \int f < +\infty$. Then there exists $z \in \mathbb{R}^n$ with the following property: for any measurable function $g : \mathbb{R}^n \to \mathbb{R}_+$ satisfying

$$f(x)g(y) \leq \rho^2(\langle x - z, y - z \rangle)$$

for every $x, y \in \mathbb{R}^n$ with $\langle x - z, y - z \rangle > 0$, one has

$$\int_{\mathbb{R}^n} f(x)dx \int_{\mathbb{R}^n} g(y)dy \leq \left( \int_{\mathbb{R}^n} \rho(|x|^2)dx \right)^2.$$

In the proof, we attach, for every $z \in \mathbb{R}^n$, the convex body

$$K_z = \left\{ x \in \mathbb{R}^n; \int_0^{+\infty} f(z + rx)r^{n-1}dr \geq 1 \right\}$$

and show that there exists $z_0 \in \mathbb{R}^n$ such that the center of mass of $K_{z_0}$ is at the origin. Then the result follows from the preceding proposition. The existence of such a $z_0$ is proved using Brouwer’s fixed point theorem.

The main consequence of this theorem is the following generalization of the results of Artstein, Klartag and Milman [1] (who considered only the cases $\rho(t) = e^{-t}$ and $\rho(t) = (1 - t)^m_t$) for the Legendre transform $L_z\phi$ of a convex function $\phi$.

**Theorem** Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be a log-concave non-increasing function and let $\phi$ be a convex function such that $0 < \int_{\mathbb{R}^n} \rho(\phi(x))dx < +\infty$. Then for some $z \in \mathbb{R}^n$, one has

$$\int_{\mathbb{R}^n} \rho(\phi(x))dx \int_{\mathbb{R}^n} \rho(L_z\phi(y))dy \leq \left( \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{2}\right)dx \right)^2.$$

In all these functional forms of Blaschke-Santaló inequality, we determine the equality cases and establish some geometric corollaries. In particular we investigate the following question:

What are the Borel measures $\mu$ on $\mathbb{R}^n$ and the sets $K$ in $\mathbb{R}^n$ which satisfy a Blaschke-Santaló type inequality

$$\mu(K) \cdot \mu(K^\circ) \leq \mu(B_n^2)^2 ?$$

Cordero-Erausquin [7] proved such an inequality in $\mathbb{C}^n$ for plurisubharmonic measures and $\mathbb{C}$-symmetric pseudo-convex sets, using complex interpolation. He
also remarked that it holds for the Gaussian measure in $\mathbb{R}^n$ and asked whether it still holds for any symmetric log-concave measures $\mu$ and any symmetric convex body $K$ in $\mathbb{R}^n$. Klartag also established this inequality for a special class of measures in [12]. As corollaries of our functional inequalities, we get that this inequality holds:
- for any unconditional log-concave measure $\mu$ and unconditional measurable set $K$
- for any rotation invariant log-concave measure $\mu$ and any centrally symmetric measurable set $K$.
And we determine the equality cases.

The paper is organized in the following way. In section 2, we treat the case of unconditional functions and sets, where one can apply a multiplicative version of the Prékopa-Leindler inequality. In Section 3, we prove the proposition stated above concerning the case of "centered" functions. Section 4 is devoted to the proof of our theorem on general (not centered) functions. In Section 5, we prove the consequences for Legendre transforms of convex functions.

It should be observed that the main difficulty when working with Santaló type inequalities for non-symmetric bodies or functions is to find a good center. If $G(K)$ is the center of mass of $K$ ($G(K) = \int_K x dx/|K|$), one has as well

$$|K| \cdot |K^{*G(K)}| \leq v_n^2,$$

because Blaschke-Santaló inequality can be applied to $K^{*G(K)}$. But if $K$ is centrally symmetric, the situation is simpler: $\min_z |K^*z|$ is reached at 0, and then $|K| \cdot |K^0| \leq |B_2^n|^2$. We shall also make use of the equality case in Blaschke-Santaló inequality: there is equality if and only if $K$ is an ellipsoid. At the end of the paper, we give a new and elementary proof of this result.

### 2 An inequality for unconditional functions

We say that a function $\varphi : \mathbb{R}^n \mapsto \mathbb{R}$ is unconditional if

$$\varphi(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) = \varphi(x_1, \ldots, x_n)$$

for every $(\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$ and every $(x_1, \ldots, x_n) \in \mathbb{R}^n$. In the same way, a subset $K$ in $\mathbb{R}^n$ is unconditional if its characteristic function $\chi_K$ is unconditional. Observe that an unconditional convex function $W : \mathbb{R}^n \mapsto \mathbb{R}$ is minimal at 0 and is moreover increasing, in the sense that $W(x) \leq W(y)$ whenever $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ satisfy $|x_i| \leq |y_i|$, $1 \leq i \leq n$.

In particular, if $W$ is unconditional and convex, one has

$$W(\sqrt{x_1 y_1}, \ldots, \sqrt{x_n y_n}) \leq W\left(\frac{x + y}{2}\right) \leq \frac{W(x) + W(y)}{2},$$

for all $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}_+^n$. 


The next proposition is a form of Prékopa-Leindler inequality for the geometric mean due to Borell (5), Ball (3), Uhrin (20). This result is well known and follows from the usual Prékopa-Leindler inequality. We prove it here for the convenience of the reader. As we shall see in the corollary, this proposition gives a first functional form of Blaschke-Santaló inequality.

**Proposition 1 (Prékopa-Leindler inequality for the geometric mean)**

Let $f_1, f_2, f_3 : \mathbb{R}^n \to \mathbb{R}_+$ be unconditional measurable functions such that

$$f_1(x_1, \ldots, x_n) f_2(y_1, \ldots, y_n) \leq f_3(\sqrt{x_1 y_1}, \ldots, \sqrt{x_n y_n})^2$$

for every $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n) \in \mathbb{R}_+^n$. Then

$$\int_{\mathbb{R}^n} f_1(x) dx \int_{\mathbb{R}^n} f_2(y) dy \leq \left( \int_{\mathbb{R}^n} f_3(z) dz \right)^2$$

with equality if and only if there exists a continuous function $\tilde{f}_3 : \mathbb{R}_+ \to \mathbb{R}_+$ such that the following two conditions hold:

a. $f_3 = \tilde{f}_3$ a.e. and $\tilde{f}_3(x_1, \ldots, x_n) \tilde{f}_3(y_1, \ldots, y_n) \leq \tilde{f}_3(\sqrt{x_1 y_1}, \ldots, \sqrt{x_n y_n})^2$

b. for some $c_1, \ldots, c_n > 0$ and $d > 0$, one has

$$f_1(x_1, \ldots, x_n) = d \tilde{f}_3(c_1 x_1, \ldots, c_n x_n) \quad \text{and} \quad f_2(x) = \frac{1}{d} \tilde{f}_3\left(\frac{x_1}{c_1}, \ldots, \frac{x_n}{c_n}\right) \quad \text{a.e.}$$

**Proof:** Since the $f_j$ are unconditional, one has $\int_{\mathbb{R}^n} f_j = 2^n \int_{\mathbb{R}_+^n} f_j$, $j = 1, 2, 3$.

For $(t_1, \ldots, t_n) \in \mathbb{R}^n$, we define

$$g_j(t_1, \ldots, t_n) = f_j(e^{t_1}, \ldots, e^{t_n}) e^{\sum_{i=1}^n t_i}.$$  

We get

$$\int_{\mathbb{R}_+^n} f_j = \int_{\mathbb{R}^n} g_j$$

and for every $s, t \in \mathbb{R}^n$,

$$g_1(s) g_2(t) \leq g_3\left(\frac{s + t}{2}\right)^2.$$ 

Hence the result follows from Prékopa-Leindler inequality. For the equality case, see [9].

As a corollary, we get the following generalized form of Blaschke-Santaló inequality for unconditional sets, together with its case of equality.

**Corollary 2** Let $W : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an unconditional convex function and let $\mu$ be the Borel measure on $\mathbb{R}^n$ with density $e^{-W(x)}$ with respect to the Lebesgue measure. Then one has

$$\mu(K) \mu(K^c) \leq \mu(B_2^n)^2,$$
for every unconditional measurable set $K \subset \mathbb{R}^n$.

If moreover the support of $\mu$ is $\mathbb{R}^n$, there is equality if and only if there exists a diagonal matrix $T$, with diagonal entries $(t_1, \ldots, t_n) \in \mathbb{R}_+$ such that:

- $K = T(B^n_2)$
- $W(x) = W(Px)$, for every $x \in K \cup K^0 \cup B^n_2$, where $P$ is the orthogonal projection on the subspace spanned by the $(e_i)_{i \in I}$ and $I = \{i; 1 \leq i \leq n, t_i = 1\}$.

Proof:

A. The inequality.

We apply Proposition 1 to

$$f_1(x) = e^{-W(x)}\chi_K(x), \ f_2(x) = e^{-W(x)}\chi_{K^c}(x), f_3(x) = e^{-W(x)}\chi_{B^n_2}(x).$$

The hypotheses are satisfied since for all $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, one has

$$\chi_K(x)\chi_{K^c}(y) \leq \chi_{B^n_2}(\sqrt{x_1^2y_1}, \ldots, \sqrt{x_n^2y_n})$$

and

$$W(\sqrt{x_1^2y_1}, \ldots, \sqrt{x_n^2y_n}) \leq W\left(\frac{x + y}{2}\right) \leq \frac{W(x) + W(y)}{2}. \quad (1)$$

as explained at the beginning of this section. This gives the inequality.

B. The case of equality.

Assume that the support of $\mu$ is $\mathbb{R}^n$ (hence $W(x) < +\infty$, for every $x \in \mathbb{R}^n$) and that there is equality in the preceding inequality. From the equality case in Proposition 1 there exists $t_1, \ldots, t_n > 0$ and $d > 0$, such that if we denote by $T$ the diagonal matrix with diagonal entries $(t_1, \ldots, t_n)$, then

$$e^{-W(x)}\chi_K(x) = d e^{-W(Tx)}\chi_{B^n_2}(Tx)$$

and

$$e^{-W(x)}\chi_{K^c}(x) = \frac{1}{d} e^{-W(T^{-1}x)}\chi_{B^n_2}(T^{-1}x).$$

We get $K = T^{-1}(B^n_2)$ and $K^0 = T(B^n_2)$. Taking $x = 0$ gives $d = 1$ so that

$$W(x) = W(Tx) = W(T^{-1}x) \quad \text{for every $x \in B^n_2$.}$$

Let $S = \frac{T \cdot T^{-1}}{2}$ be the diagonal matrix with diagonal entries $s_i = \frac{1}{2} \left( t_i + \frac{1}{t_i} \right)$, $1 \leq i \leq n$. One has $s_i > 1$ for all $i \notin I := \{j : t_j = 1\}$ hence $\lim_{k \to +\infty} S^{-k}(x) = Px$, for all $x \in \mathbb{R}^n$. Using the inequalities (1) for $Tx$ and $T^{-1}x$, we get

$$W(x) \leq W\left(\frac{Tx + T^{-1}x}{2}\right) \leq \frac{W(Tx) + W(T^{-1}x)}{2} = W(x).$$

Hence $W(Sx) = W(x)$ for every $x \in B^n_2$. The result follows from the continuity of $W$. $\square$
Remarks:
1) Actually the proof shows that the inequality of Corollary 2 still holds true when the hypothesis that $W$ is convex is replaced with the weaker hypothesis that $\langle t_1, \ldots, t_n \rangle \mapsto W(e^{t_1}, \ldots, e^{t_n})$ is convex on $\mathbb{R}^n$.
2) The Prékopa-Leindler inequality for the geometric mean was also used in [8] to prove that if $K$ is an unconditional convex body and $\mu$ has an unconditional log-concave density with respect to the Lebesgue measure, then $t \mapsto \mu(e^t K)$ is a log-concave function.

3 The Blaschke Santaló inequality for centered functions.

In the next result, we generalize with a new proof an inequality obtained by K. Ball [2] in the special case of even functions, and we characterize the case of equality.

Proposition 3 Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ and $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}^+$ be measurable functions such that

$$f_1(x)f_2(y) \leq \rho^2((x,y)) \text{ for every } x, y \in \mathbb{R}^n \text{ satisfying } \langle x, y \rangle > 0 .$$

If the star shaped set $K_1 = \{ x \in \mathbb{R}^n ; \int_0^{\infty} r^{n-1} f_1(rx)dr \geq 1 \}$ is centrally symmetric (which holds if $f_1$ is even), or if $K_1$ is a convex body with center of mass at the origin, then

$$\int_{\mathbb{R}^n} f_1(x)dx \int_{\mathbb{R}^n} f_2(y)dy \leq \left( \int_{\mathbb{R}^n} \rho(|x|^2)dx \right)^2$$

with equality if and only if for some continuous function $\tilde{\rho} : \mathbb{R}^+ \to \mathbb{R}^+$ one has

a. $\rho = \tilde{\rho}$ a.e., $\sqrt{s}\rho(s)\tilde{\rho}(t) \leq \tilde{\rho}(\sqrt{st})$ for every $s, t \geq 0$ and if $n \geq 2$, $\tilde{\rho}(0) > 0$ or $\tilde{\rho}$ is the null function.

b. For some positive definite $[n \times n]$ matrix $T$ and for some $d > 0$, one has

$$f_1(x) = d\tilde{\rho}(|Tx|^2) \quad \text{and} \quad f_2(x) = \frac{1}{d}\tilde{\rho}(|T^{-1}x|^2) \quad \text{a.e.}$$

Proof:
A. The inequality.

Let $x_1, x_2 \in \mathbb{R}^n$ satisfying $\langle x_1, x_2 \rangle > 0$. We define $g_j : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$g_j(s) = s^{n-1}f_j(sx_j), \quad j = 1, 2 \quad \text{and} \quad g_3(u) = u^{n-1}\rho(u^2\langle x_1, x_2 \rangle).$$
Then by hypothesis, one has \( g_1(s)g_2(t) \leq (st)^{n-1} \rho^2(st\langle x_1, x_2 \rangle) = g_3^2(\sqrt{st}) \). It follows from Proposition 1 (\( n = 1 \)) that

\[
\int_{\mathbb{R}^+} s^{n-1} f_1(xs_1)ds \int_{\mathbb{R}^+} t^{n-1} f_2(ts_2)dt \leq \left( \int_{\mathbb{R}^+} u^{n-1} \rho \left( (u^2\langle x_1, x_2 \rangle) \right) du \right)^2
\]

\[
= \frac{1}{\langle x_1, x_2 \rangle^n} \left( \int_{\mathbb{R}^+} r^{n-1} \rho(r^2)dr \right)^2 \leq \frac{c_n(\rho)^n}{\langle x_1, x_2 \rangle^n}.
\]

where \( c_n(\rho) := \left( \int_{\mathbb{R}^+} r^{n-1} \rho(r^2)dr \right)^{\frac{1}{n}} \). For \( j = 1, 2 \), we define

\[
K_j = \{ x \in \mathbb{R}^n; \int_{\mathbb{R}^+} r^{n-1} f_j(rx)dr \geq 1 \}.
\]

The sets \( K_1 \) and \( K_2 \) are starshaped with respect to the origin. Denote their gauge by \( \| \cdot \|_{K_j}, j = 1, 2 \). One has

\[
\| x \|_{K_j} = \inf\{ \lambda > 0; \ x \in \lambda K_j \} = \left( \int_{\mathbb{R}^+} r^{n-1} f_j(rx)dr \right)^{-\frac{1}{n}} \text{ for all } x \in \mathbb{R}^n.
\]

The preceding inequality may be read as follows: for every \( x_1, x_2 \in \mathbb{R}^n \) such that \( \langle x_1, x_2 \rangle > 0 \), one has

\[
\langle x_1, x_2 \rangle \leq c_n(\rho)\| x_1 \|_{K_1} \| x_2 \|_{K_2}.
\]

This means that

\[
K_2 \subset c_n(\rho)K_1^\circ.
\]

Under our hypotheses, either \( K_1 \) is centrally symmetric, so its closed convex hull is also centrally symmetric and has its center of mass at the origin, or \( K_1 \) is itself a convex body with center of mass at the origin. In both cases, the origin is actually the Santaló point of \( K_1^\circ \), and it follows from Blaschke-Santaló inequality that \( |K_1| |K_1^\circ| \leq v_n^2 \). We get thus

\[
|K_1| |K_2| \leq c_n(\rho)^n|K_1| |K_1^\circ| \leq c_n(\rho)^n v_n^2.
\]

Integrating in polar coordinates for \( j = 1, 2 \), one has

\[
\int_{\mathbb{R}^n} f_j(x)dx = nv_n \int_{S^{n-1}} \int_{\mathbb{R}^+} s^{n-1} f_j(su)dsd\sigma(u) = nv_n \int_{S^{n-1}} \frac{d\sigma(u)}{\| u \|_{K_j}^{n-1}} = n|K_j|,
\]

where \( \sigma \) denotes the rotation invariant probability on the unit sphere \( S^{n-1} := \{ u \in \mathbb{R}^n : |u| = 1 \} \). Thus

\[
\int_{\mathbb{R}^n} f_1(x)dx \int_{\mathbb{R}^n} f_2(y)dy = n^2|K_1||K_2| \leq (nv_n)^2 c_n(\rho)^n = \left( \int_{\mathbb{R}^n} \rho(\| x \|)^2dx \right)^2.
\]
B. The case of equality.

Assume now that there is equality. By the case of equality of Blaschke-Santaló inequality, $K_1$ is an ellipsoid centered at the origin and $K_2 = c_n(\rho)K_1^n$. We may and do assume that $K_1 = B_2^n$. For every $x \in S^{n-1}$, one has $\langle x, x \rangle = 1 = c_n(\rho)\|x\|_{K_1}\|x\|_{K_2}$, which means that there is equality in (2) for $x_1 = x_2 = x$.

From the equality case of Proposition $\mathbb{H}(n = 1)$, it follows that there exists a continuous function $\hat{\rho} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

- $\rho = \hat{\rho}$ a.e., $\sqrt{\hat{\rho}(s)\hat{\rho}(t)} \leq \hat{\rho}(\sqrt{st})$ for every $s, t \geq 0$
- for every $x \in S^{n-1}$, there exists $c = c(x) > 0, d = d(x) > 0$ such that

$$g_1(s) = dg_3(cs) \text{ and } g_2(s) = \frac{1}{d}g_3\left(\frac{s}{c}\right) \text{ for a.e. } s \geq 0.$$ 

Let us prove that $c$ and $d$ are constant functions. Since

$$1 = |x|^{-n} = \|x\|^{-n}_{K_1} = \int_{\mathbb{R}^+} g_1(s)ds = \frac{d(x)}{c(x)} \int_{\mathbb{R}^+} g_3(u)du = (c_n(\rho))^{\frac{n}{2}} \frac{d(x)}{c(x)} ,$$

we have $d(x) = \frac{c(x)}{c_n(\rho)}c_n(\rho)$. Hence for a.e. $s \geq 0$

$$f_1(sx) = \left(\frac{c(x)}{\sqrt{c_n(\rho)}}\right)^n \hat{\rho}(c(x)^2 s^2), \ f_2(sx) = \left(\frac{\sqrt{c_n(\rho)}}{c(x)}\right)^n \hat{\rho}\left(\frac{s^2}{c(x)^2}\right).$$

By the hypotheses, for every $x, y \in S^{n-1}$ satisfying $\langle x, y \rangle > 0$ and $s, t \geq 0$

$$\left(\frac{c(x)}{c(y)}\right)^n \hat{\rho}(c(x)^2 s^2)\hat{\rho}\left(\frac{t^2}{c(y)^2}\right) \leq \hat{\rho}^2(st\langle x, y \rangle).$$

If $\hat{\rho}(0) \neq 0$, we take $s = t = 0$, simplify and get $c(x) \leq c(y)$, for any $x, y \in S^{n-1}$. Therefore $c$ is a constant function.

If $\hat{\rho}(0) = 0$ and $n \geq 2$, we take $x, y \in S^{n-1}$ with $\langle x, y \rangle = 0$ (this is possible since $\hat{\rho}$ is continuous), we get that $\hat{\rho}$ is the null function. \hfill \square

Remarks:

1) We did not follow here the more natural proof given by K. Ball in the even case. For sake of completeness, we outline his proof in the case where $\rho$ is non-increasing. Setting for $t > 0, i = 1, 2, p_i(t) = \{f_i > t\},$ one has

$$\int f_i = \int_0^{+\infty} p_i(t)dt.$$ The hypothesis on $f_1$ and $f_2$ gives that for every $s, t > 0$, one has $\{f_2 > t\} \subset \rho^{-1}(\sqrt{st})\{f_1 > s\}^2$. Now, the fact that $f_1$ is even implies that its level sets are centrally symmetric and this allows to apply Blaschke-Santaló inequality to get for all $s, t > 0$,

$$p_1(s)p_2(t) \leq \left(\rho^{-1}(\sqrt{st})\right)^n v_n^2,$$

and the result follows from Proposition $\mathbb{H}$ applied in dimension 1.
2) The idea of attaching a convex set of the form of $K_1$ to a log-concave function $f_1$ to prove a functional inequality was originally used by K. Ball in [3] and is also used by Klartag and Milman in [10].

3) There are many ways to recover the usual Blaschke-Santaló inequality for symmetric sets from Proposition 3. As noticed by K. Ball in [2], the more natural is to apply it to $f_1 = \chi_K$, $f_2 = \chi_{K^c}$ and $\rho = \chi_{[0,1]}$. But more generally, we get the same result by applying it to $f_1(x) = \rho(\|x\|_K^{-2})$, $f_2(y) = \rho(\|y\|_K^{-2})$ and any function $\rho$ such that $t \mapsto \rho(e^t)$ is log-concave and non-increasing on $\mathbb{R}$. This was noticed by Artstein, Klartag and Milman [1] in the case when $\rho(t) = e^{-t}$.

4) Let $K$ be a convex body whose center of mass is at the origin. If we set $f_1 = \chi_K$, $f_2 = \chi_{K^c}$ and $\rho = \chi_{[0,1]}$, we get $K_1 = K/n^{1/n}$ so that center of mass of $K_1$ is at the origin. Hence Proposition 3 also permits to recover the general Blaschke-Santaló inequality for convex sets.

As a corollary of Proposition 3, let us prove a generalized form of Blaschke-Santaló inequality for symmetric sets and some class of rotation invariant measures. This inequality is known for the Lebesgue measure and the Gaussian measure (see [7]); and also for a special class of measures (see [12]). It was asked in [4] whether it holds for any symmetric log-concave measure. We also give here a partial answer:

**Corollary 4** Let $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a non-increasing function which satisfies that $t \mapsto h(e^t)$ is log-concave on $\mathbb{R}$. Let $\mu$ be the rotation invariant measure on $\mathbb{R}^n$, with density $h(|x|)$ with respect to the Lebesgue measure. Then, for every centrally symmetric measurable set $K \subset \mathbb{R}^n$, one has

$$
\mu(K) \mu(K^c) \leq \mu(B_2^n)^2.
$$

If moreover, the support of $\mu$ is $\mathbb{R}^n$, there is equality if and only if
- either $K = B_2^n$
- or $K = T(B_2^n)$ for some positive definite matrix $T \neq I$ and $h$ is constant on $[0, \max(\|T\|, \|T^{-1}\|)]$, where $\|T\| = \max_{|x|=1} |Tx|$.

**Proof:**

**A. The inequality.**

We apply Proposition 3 to

$$
f_1(x) = h(|x|)\chi_K(x), \quad f_2(y) = h(|y|)\chi_{K^c}(y) \quad \text{and} \quad \rho(t) = h(\sqrt{t})\chi_{[0,1]}(t).
$$

The hypotheses are satisfied since for all $x, y \in \mathbb{R}^n$ such that $\langle x, y \rangle > 0$, one has

$$
f_1(x)f_2(y) \leq h^2(\sqrt{|x||y|})\chi_{[0,1]}(\langle x, y \rangle) \leq h^2(\sqrt{\langle x, y \rangle})\chi_{[0,1]}(\langle x, y \rangle) = \rho^2(\langle x, y \rangle)
$$

and $f_1$ is even. We get thus

$$
\int f_1(x)dx \int f_2(y)dy = \mu(K) \mu(K^c) \leq \left(\int_{\mathbb{R}^n} \rho(|x|)dx\right)^2 = \mu(B_2^n)^2.
$$
B. The case of equality.

Assume that the support of $\mu$ is $\mathbb{R}^n$ (hence $h > 0$) and that there is equality. It follows from Proposition 3 that for some positive matrix $T$ and for some $d > 0$, one has

$$f_1(x) = d\rho(|Tx|^2) \quad \text{and} \quad f_2(y) = \frac{1}{d}\rho(|T^{-1}y|^2)$$

for all $x, y \in \mathbb{R}^n$.

This gives

$$h(|x|)\chi_K(x) = dh(|Tx|)\chi_{[0,1]}(|Tx|)$$

and

$$h(|y|)\chi_{K^o}(y) = \frac{1}{d}h(|T^{-1}y|)\chi_{[0,1]}(|T^{-1}y|).$$

Hence $K = T^{-1}(B^n_2)$, $K^o = T(B^n_2)$ and $h(|Tz|) = h(|z|) = h(|T^{-1}z|)$ for every $z \in B^n_2$. If $K \neq B^n_2$, one has $\max(||T||, ||T^{-1}||) > 1$. We may assume that $||T|| > 1$. Let $z_0 \in S^{n-1}$ satisfying $|Tz_0| = ||T|| = \lambda \in [0, ||T||]$. Applying the previous equality to $z = \lambda z_0/||T||$, we get $h(\lambda) = h(|Tz_0|) = h(|z|) = h(\lambda/||T||)$.

From the continuity of $h$, $h(\lambda) = h(\lambda/||T||^n) = h(0)$.  

4 The general case

We are now in position to prove the following theorem.

**Theorem 5** Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be measurable and $f : \mathbb{R}^n \to \mathbb{R}_+$ be a log-concave function such that $0 < \int f < +\infty$. Then there exists $z \in \mathbb{R}^n$ such that for any measurable function $g : \mathbb{R}^n \to \mathbb{R}_+$ satisfying

$$f(x)g(y) \leq \rho^2(\langle x - z, y - z \rangle)$$

for every $x, y \in \mathbb{R}^n$ such that $\langle x - z, y - z \rangle > 0$, one has

$$\int_{\mathbb{R}^n} f(x)dx \int_{\mathbb{R}^n} g(y)dy \leq \left(\int_{\mathbb{R}^n} \rho(|x|^2)dx\right)^2$$

with equality if and only if the following two conditions hold:

a. For some positive definite $[n \times n]$ matrix $T$, some $z \in \mathbb{R}^n$ and some $d > 0$,

$$f(x) = d\rho(|T(x - z)|^2) \quad \text{and} \quad g(x) = \frac{1}{d}\rho(|T^{-1}(x - z)|^2) \quad \text{a.e.}$$

b. $\sqrt{\rho(s)}\rho(t) \leq \rho(\sqrt{st})$ a.e.

Proof:
For every $z \in \mathbb{R}^n$ let
\[
K_z = \left\{ x \in \mathbb{R}^n; \int_0^{+\infty} f(z + rx)r^{n-1}dr \geq 1 \right\}.
\]

Since $f$ is log-concave, it follows from Ball [3] that for every $z \in \mathbb{R}^n$, the set $K_z$ is a convex body. If we can prove that there exists $z_0 \in \mathbb{R}^n$ such that the center of mass of $K_{z_0}$ is at the origin, we get the result from proposition 3 applied to $f_1(x) = f(x + z_0)$ and $f_2(x) = g(x + z_0)$.

This will be done in the following two lemmas, using Brouwer's fixed point theorem.

**Lemma 6** Let $n \geq 2$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a log-concave function such that $0 < \int f < +\infty$. For $z, x \in \mathbb{R}^n$, define $r_z(x) = \left(\int_0^{+\infty} f(z + rx)r^{n-1}dr\right)^{\frac{1}{n}}$. One has then
1) For all $\varepsilon > 0$ and $\alpha < 1$, there exists $M > 0$ such that $r_z(u) \leq \varepsilon$ whenever $u \in S^{n-1}$ and $z \in \mathbb{R}^n$ satisfy $\langle u, z \rangle \geq -\alpha|z|$ and $|z| \geq M$.
2) $r_z\left(-\frac{z}{|z|}\right) \rightarrow +\infty$ when $|z| \rightarrow +\infty$.

**Proof:**

From the hypotheses on $f$, it is easy to see that for some $a, b, c, d > 0$, one has
\[
a \chi_{B_2^d}(x) \leq f(x) \leq d e^{-c|x|} \text{ for every } x \in \mathbb{R}^n.
\]

1) If $u \in S^{n-1}$ and $z \in \mathbb{R}^n$ satisfy $-\langle u, z \rangle \leq \alpha|z|$, then for every $r \geq 0$
\[
|z + ru|^2 \geq |z|^2 - 2\alpha|z|r + r^2 \geq (1 - \alpha)(|z|^2 + r^2) \geq \frac{1 - \alpha}{2}(|z| + r)^2.
\]
It follows that
\[
r_z(u)^n \leq de^{-c\sqrt{\frac{1}{|z|}}} \int_0^{+\infty} r^{n-1}e^{-c\sqrt{\frac{r}{|z|}}}dr \rightarrow 0 \text{ when } |z| \rightarrow +\infty.
\]

2) Let $u = -\frac{z}{|z|}$. Then
\[
r_z(u)^n = \int_0^{+\infty} r^{n-1}f((r - |z|)u)dr \geq a \int_0^{+\infty} r^{n-1}\chi_{[-b,b]}(r - |z|)dr
\]

Thus, for $|z| > b$, one has $r_z(u)^n \geq \frac{2}{n}((|z| + b)^n - (|z| - b)^n) \rightarrow +\infty$ when $|z| \rightarrow +\infty$.

As we have already seen, for every $z \in \mathbb{R}^n$, the set $K_z$ is a convex body. Moreover notice that under our hypotheses, the origin is in the interior of $K_z$ and $r_z$ is the radial function of $K_z \ (r_z(u) = \max\{\lambda > 0; \lambda u \in K_z\}$, for every $u \in S^{n-1}$.)
Hence part 1) of the preceding lemma means that for all $\varepsilon > 0$ and $\alpha < 1$, there exists $M > 0$ such that for every $|z| \geq M$,

$$\{ x \in K_z : \langle x, z \rangle \geq -\alpha |z| \} \subset \varepsilon B_2^n .$$

**Lemma 7** Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a log-concave function such that $0 < \int f < +\infty$. For every $z \in \mathbb{R}^n$ let

$$K_z = \left\{ x \in \mathbb{R}^n : \int_0^{+\infty} f(z + rx)r^{n-1}dr \geq 1 \right\} .$$

Then there exists $z_0 \in \mathbb{R}^n$ such that the convex body $K_{z_0}$ has its center of mass at the origin.

**Proof:**

Notice first that for $n = 1$, the result is easy, one chooses the unique point $z_0 \in \mathbb{R}$ such that

$$\int_{z_0}^{+\infty} f(r)dr = \int_{-\infty}^{z_0} f(r)dr ,$$

then $K_{z_0}$ is a symmetric interval. We assume from now on that $n \geq 2$. It is clear that $z \mapsto K_z$ is continuous for the Hausdorff distance, so that if $G(z)$ is the centre of mass of $K_z$, then $G : \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

**A.** We first show that

$$|G(z)| \to +\infty \quad \text{and} \quad \frac{G(z)}{|G(z)|} \quad \text{for all} \quad |z| \to +\infty .$$

Let $h_{K_z}$ be the support function of $K_z$ i.e.

$$h_{K_z}(y) = \max_{x \in K_z} \langle x, y \rangle \quad \text{for every} \quad y \in \mathbb{R}^n .$$

It is well known that one has, for all $u \in S^{n-1}$,

$$-h_{K_z}(-u) + \frac{h_{K_z}(u) + h_{K_z}(-u)}{n+1} \leq \langle G(z), u \rangle \leq h_{K_z}(u) - \frac{h_{K_z}(u) + h_{K_z}(-u)}{n+1} .$$

By part 1) of **Lemma 6** applied with $\alpha = 0$, for every $\varepsilon > 0$, there exists $M > 0$ such that

$$\{ x \in K_z : \langle x, z \rangle \geq 0 \} \subset \varepsilon B_2^n , \quad \text{for all} \quad |z| \geq M .$$

Moreover $K_z$ contains the origin, hence

$$h_{K_z}\left(\frac{z}{|z|}\right) = \max \left\{ \frac{z}{|z|}, v \in K_z, \langle z, v \rangle \geq 0 \right\} \to 0 ,$$
when \( |z| \to +\infty \). By part 2) of Lemma 6

\[
h_{K_z} \left( \frac{z}{|z|} \right) \geq r_z \left( \frac{z}{|z|} \right) \to +\infty.
\]

It follows that \( \langle G(z), \frac{z}{|z|} \rangle \to -\infty \), and thus that \( |G(z)| \to +\infty \) when \( |z| \to +\infty \).

But since \( K_z \) is a convex body, \( G(z) \in K_z \), and thus \( |G(z)| \leq r_z \left( \frac{G(z)}{|G(z)|} \right) \). Since \( |G(z)| \to +\infty \), one has \( r_z \left( \frac{G(z)}{|G(z)|} \right) \to +\infty \) when \( |z| \to +\infty \). It follows again from part 1) of Lemma 6 that for every \( \alpha < 1 \), there exists \( M > 0 \) such that if \( |z| > M \), then

\[
\langle G(z), \frac{z}{|z|} \rangle \leq -\alpha |z|.
\]

This means that

\[
\langle G(z), \frac{z}{|G(z)|} \rangle \to -1 \text{ when } |z| \to +\infty.
\]

**B.** Let us prove that there exists \( z_0 \in \mathbb{R}^n \) such that \( G(z_0) = 0 \):

Suppose that \( G \) does not vanish. Let \( C_2^n = \{ x \in \mathbb{R}^n : |x| < 1 \} \) be the open Euclidean unit ball, and define \( z : C_2^n \to \mathbb{R}^n \) by

\[
z(x) := \frac{x}{1 - |x|}.
\]

Define also \( F : B_2^n \to S^{n-1} \) by

\[
F(x) = \frac{G(z(x))}{|G(z(x))|} \text{ for } x \in C_2^n, \text{ and } F(u) = -u \text{ for } u \in S^{n-1}.
\]

Let us prove that \( F \) is continuous on \( B_2^n \): It is clear that \( F \) is continuous on \( C_2^n \). Let \( u \in S^{n-1} \). If \( x \to u \), then \( |z(x)| \to +\infty \) and \( \frac{z(x)}{|z(x)|} = \frac{x}{|x|} \to u \). Whence by **A.**,

\[
\langle \frac{z(x)}{|z(x)|}, \frac{G(z(x))}{|G(z(x))|} \rangle \to -1,
\]

which implies that

\[
F(x) = \frac{G(z(x))}{|G(z(x))|} \to -u.
\]

Thus \( F : B_2^n \to S^{n-1} \) is continuous and satisfies \( F(u) = -u \) for every \( u \in S^{n-1} \). To conclude, we define \( Q : B_2^n \to B_2^n \), by

\[
Q(x) = \frac{x + F(x)}{2} \text{ for every } x \in B_2^n.
\]

Then \( Q \) is continuous, but has no fixed point, which contradicts Brouwer fixed point theorem. \( \square \)
Remark:
Theorem 5 can be generalized in the following way: given \( h : (0, +\infty) \to (0, +\infty) \)
such that \( t \mapsto h(e^t) \) is log-concave and \( h(r)^{n-1} \to +\infty \) when \( r \to +\infty \), let \( \mu \)
be the measure on \( \mathbb{R}^n \) with density \( h(|x|) \). Let \( \rho : \mathbb{R}_+ \to \mathbb{R}_+ \)
be measurable and \( f : \mathbb{R}^n \to \mathbb{R}_+ \) be a log-concave function such that \( 0 < \int f d\mu < +\infty \).
Then there exists \( z \in \mathbb{R}^n \) such that for any measurable function \( g : \mathbb{R}^n \to \mathbb{R}_+ \)
satisfying
\[
f(x)g(y) \leq \rho^2(\langle x-z, y-z \rangle)
\]
for every \( x, y \in \mathbb{R}^n \) such that \( \langle x-z, y-z \rangle > 0 \), one has
\[
\int_{\mathbb{R}^n} f(x) d\mu(x) \int_{\mathbb{R}^n} g(y) d\mu(y) \leq \left( \int_{\mathbb{R}^n} \rho(|x|) \, dx \right)^2.
\]

5 Consequences on Legendre transform

Given a function \( \phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) and \( z \in \mathbb{R}^n \), we recall that the Legendre
transform \( L_z \phi \) of \( \phi \) with respect to \( z \in \mathbb{R}^n \) is defined by
\[
L_z \phi(y) = \sup_{x} (\langle x-z, y-z \rangle - \phi(x)) \quad \text{for all } y \in \mathbb{R}^n.
\]

For \( z = 0 \), we use the notation \( L := L_0 \). Observe that \( L \phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \)
is convex and that by a classical separation argument, \( L(L \phi) = \phi \), whenever \( \phi \)
is itself convex and \( \phi(z) < +\infty \). Notice also that the function \( \phi(x) = |x|^2/2 \)
is the unique function which satisfies \( L \phi = \phi \). As a consequence of Theorem 5 we get the following theorem which generalizes the results of Artstein, Klartag
and Milman \[1\] who considered only the cases \( \rho(t) = e^{-t} \) and \( \rho(t) = (1-t)^n_+ \).

Theorem 8 Let \( \rho : \mathbb{R}_+ \to \mathbb{R}_+ \) be a log-concave non-increasing function and let \( \phi \) be
a convex function such that \( 0 < \int_{\mathbb{R}^n} \rho(\phi(x)) \, dx < +\infty \). Then for some \( z \in \mathbb{R}^n \), one has
\[
\int_{\mathbb{R}^n} \rho(\phi(x)) \, dx \int_{\mathbb{R}^n} \rho(L_z \phi(y)) \, dy \leq \left( \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{2} \right) \, dx \right)^2.
\]
If \( \rho \) is decreasing, there is equality if and only if for some positive definite matrix \( T : \mathbb{R}^n \to \mathbb{R}^n \) and some \( c \in \mathbb{R} \), one has
\[
\phi(x) = \frac{|T(x+z)|^2}{2} + c, \quad \text{for all } x \in \mathbb{R}^n,
\]
and moreover either \( c = 0 \) or \( \rho(t) = e^{at+b} \) for some \( a < 0 \), some \( b \in \mathbb{R} \), and all \( t \in [-|c|, +\infty) \).
Proof.

A. The inequality.

We apply Theorem 5 to the log-concave function $f := \rho \circ \phi$ to get a convenient $z \in \mathbb{R}^n$. By the definition of $L_z$ and the fact that $\rho$ is log-concave and non-increasing, one has for every $x, y \in \mathbb{R}^n$ such that $\langle x - z, y - z \rangle > 0$,

$$\rho(\phi(x)) \rho(\phi(y)) \leq \rho^2 \left( \frac{\phi(x) + L_z \phi(y)}{2} \right) \leq \rho^2 \left( \frac{\langle x - z, y - z \rangle}{2} \right).$$

Setting $g(y) = \rho(\phi(y))$, we may apply Theorem 5, to get the inequality.

B. The case of equality.

We may assume that $z = 0$. Set $\psi = L\phi$. If there is equality, we get from Theorem 5 that for some positive definite matrix $T : \mathbb{R}^n \to \mathbb{R}^n$ and some $d > 0$,

$$\frac{1}{d} \rho(\phi(\|T^{-1}x\|)) = d \rho(\psi(\|Tx\|)) = d \rho \left( \frac{|x|^2}{2} \right),$$

for every $x \in \mathbb{R}^n$. Since $\rho$ is log-concave and decreasing one has

$$\rho \left( \frac{|x|^2}{2} \right) = \sqrt{\rho(\phi(\|T^{-1}x\|)) \rho(\psi(Tx))} \leq \rho \left( \frac{\phi(T^{-1}x) + \psi(Tx)}{2} \right) \leq \rho \left( \frac{\langle T^{-1}x, Tx \rangle}{2} \right) = \rho \left( \frac{|x|^2}{2} \right).$$

Since $\rho$ is decreasing, we get $\phi(T^{-1}x) + \psi(Tx) = |x|^2$ for all $x \in \mathbb{R}^n$. Thus

$$|x|^2 - \phi(T^{-1}x) = \psi(Tx) = \sup_y \left( \langle Tx, y \rangle - \phi(y) \right) = \sup_w \left( \langle x, w \rangle - \phi(T^{-1}w) \right).$$

We get $\phi(T^{-1}x) - \phi(T^{-1}w) \leq |x|^2 - \langle x, w \rangle$, for every $w, x \in \mathbb{R}^n$, Setting $C(x) = \phi(T^{-1}x) - \frac{|x|^2}{2}$, it follows that

$$|C(x) - C(w)| \leq \frac{|x - w|^2}{2} \text{ for all } x, w \in \mathbb{R}^n.$$

It is easy then to conclude that $C$ is actually constant, and this gives that for some $c > 0$, one has

$$\phi(x) = \frac{|Tx|^2}{2} + c \quad \text{and} \quad \psi(x) = \frac{|T^{-1}x|^2}{2} - c.$$

This implies that $\rho$ satisfies

$$\rho \left( \frac{|x|^2}{2} \right)^2 = \rho \left( \frac{|x|^2}{2} + c \right) \rho \left( \frac{|x|^2}{2} - c \right)$$

and using again the log-concavity of $\rho$, either $c = 0$ or $\log(\rho)$ is affine on $[-|c|, +\infty)$. \qed
Remarks:
1) The cases when $\rho(t) = e^{-t}$ or $\rho(t) = (1 - t)^m$ of Theorem 8 were proved by Artstein, Klartag and Milman in [1] by applying the Blaschke-Santalo inequality for sets to a sequence of convex bodies $(K_s(\phi))_{s \in \mathbb{N}}$ in $\mathbb{R}^{n+s}$ and by letting $s \to +\infty$. The use of this sequence makes the case of equality much more difficult than in our proof.

2) In the case when the function $\rho$ is strictly convex (for example if $\rho(t) = e^{-t}$), then
\[
\min_z \int_{\mathbb{R}^n} \rho(L_z \phi(y)) \, dy = \min_z \int_{\mathbb{R}^n} \rho(\mathcal{L} \phi(y) - \langle z, y \rangle) \, dy
\]
is reached at a unique point $z_0$ which satisfies
\[
z_0 = \int_{\mathbb{R}^n} y \rho'(L_{z_0} \phi(y)) \, dy / \int_{\mathbb{R}^n} \rho'(L_{z_0} \phi(y)) \, dy.
\]
It follows that the inequality of Theorem 8 is also valid at this point $z = z_0$.

3) Actually, it is also possible to prove Theorem 8 by following step by step the method used by Meyer and Pajor ([16]) for proving Blaschke-Santaló inequality for convex bodies. The idea is to prove that the quantity
\[
\min_z \int_{\mathbb{R}^n} \rho(L_z \phi(x)) \, dx
\]
increases if we apply to the epigraph $E_{\phi} := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : \phi(x) \leq t\}$ of the function $\phi$ a well chosen Steiner symmetrisation to get a function $\tilde{\phi}$ which is symmetric with respect to the symmetrisation hyperplane. After $n$ symmetrisations with respect to mutually orthogonal hyperplanes, the function is unconditional and the result follows from the application of the Prékopa-Leindler inequality for the geometric mean (Theorem 1). However, this proof is much longer, and seems to require some additionally hypotheses on the function $\rho$, namely that $\rho$ is convex and decreasing and that $-\rho'$ is log-concave.

4) Shortcut for the proof of the equality case in Blaschke-Santaló inequality.

There exists different proofs of the equality case for Blaschke-Santalo’s inequality. It was first proved in the centrally symmetric case by Saint-Raymond [18], using a tricky lemma for functions of one variable, then in the general case by Petty [17] with some involved arguments of PDE (see also D. Hug [11]). A simpler proof together with a stronger inequality was then given by Meyer and Pajor [16] using the Steiner symmetrization, a result of [10] and finally the lemma of Saint-Raymond.

In fact, one can give the following simpler argument.

a. If $K$ is unconditional with maximal volume product, we have seen that the case of equality follows easily from the equality case in the one-dimensional Prékopa-Leindler inequality.

b. Suppose now that $K$ has maximal volume product and is centrally symmetric. Then for every $u \in S^{n-1}$, after $n$ Steiner symmetrizations with respect to
pairwise orthogonal hyperplanes, the last one being with respect to \( \{u\} \), we get from \( K \) an unconditional body with maximal volume product (recall that a Steiner symmetrization does not decrease volume product), and thus by a. an ellipsoid. To conclude that \( K \) is itself an ellipsoid, we use the following elementary lemma, where for \( v \in \mathbb{S}^{n-1} \), we denote by \( S_v K \) the Steiner symmetral of \( K \) with respect to the hyperplane \( v^\perp := \{ x \in \mathbb{R}^n \mid \langle x, v \rangle = 0 \} \).

**Lemma.** Let \( K \) be a centrally symmetric convex body. Then \( K \) is an ellipsoid if and only if for every orthonormal basis \((u_1, \ldots, u_n)\) of \( \mathbb{R}^n \), \( S_{u_n} S_{u_{n-1}} \cdots S_{u_1} K \) is an ellipsoid.

**Proof:** The "only if" part is well known. For the "if" part, fix \( u \in \mathbb{S}^{n-1} \), and \((u_1, \ldots, u_n)\) be an orthonormal basis such that \( u = u_n \). Let \( L = S_{u_{n-1}} \cdots S_{u_1} K \). Then \( L \) is centrally symmetric (since \( K \) is), and symmetric with respect to the \((n-1)\) pairwise orthogonal hyperplanes \( u_i^\perp \), \( 1 \leq i \leq n-1 \). It follows that \( L \) is also symmetric with respect to \( u_n^\perp \), so that \( L = S_{u_n} L = S_{u_n} S_{u_{n-1}} \cdots S_{u_1} K \) is an ellipsoid. Thus for some \( a_1, \ldots, a_n > 0 \) one has

\[
L = \left\{ x = x_1 u_1 + \cdots + x_n u_n; \sum_{i=1}^{n} \frac{x_i^2}{a_i^2} \leq 1 \right\} .
\]

Let \( h_K(u) := \max\{ \langle x, u \rangle; x \in K \} \). It is easy to see that whenever \( v \in \mathbb{S}^{n-1} \) satisfy \( \langle v, u \rangle = 0 \), then

\[
h_K(u) = h_{S_v K}(u) \quad \text{and} \quad \int_K \langle x, u \rangle^2 dx = \int_{S_v K} \langle x, u \rangle^2 dx .
\]

It follows that \( a_n = h_L(u_n) = h_K(u_n) \) and

\[
\int_K \langle x, u_n \rangle^2 dx = \int_L \langle x, u_n \rangle^2 dx = \frac{v_n}{n+2} \cdot a_1 \cdots a_n \cdot a_n^2 .
\]

Since \( |L| = |K| \), one has \( v_n a_1 \cdots a_n = |K| \). Thus

\[
h_K(u)^2 = \frac{n+2}{|K|} \int_K \langle x, u \rangle^2 dx \quad \text{for every} \quad u \in \mathbb{S}^{n-1} .
\]

It follows that \( K^\circ \) and thus \( K \) is an ellipsoid.

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