Composition operators on Herz-type Triebel–Lizorkin spaces with application to semilinear parabolic equations

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Abstract
Let \( G : \mathbb{R} \to \mathbb{R} \) be a continuous function. In the first part of this paper, we investigate sufficient conditions on \( G \) such that

\[
\{ G(f) : f \in \dot{K}^\alpha_{p,q} F^s_\beta \} \subset \dot{K}^\alpha_{p,q} F^s_\beta
\]

holds. Here \( \dot{K}^\alpha_{p,q} F^s_\beta \) are Herz-type Triebel–Lizorkin spaces. These spaces unify and generalize many classical function spaces such as Lebesgue spaces of power weights, Sobolev and Triebel–Lizorkin spaces of power weights. In the second part of this paper we will study local and global Cauchy problems for the semilinear parabolic equations

\[
\partial_t u - \Delta u = G(u)
\]

with initial data in Herz-type Triebel–Lizorkin spaces. Our results cover the results obtained with initial data in some known function spaces such us fractional Sobolev spaces. Some limit cases are given.

Keywords Besov spaces · Triebel–Lizorkin spaces · Herz spaces · Nemytzkij operators · Semilinear parabolic equations

Mathematics Subject Classification 46E35 · 47H30 · 35K55

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1 Introduction

Let $G : \mathbb{R} \to \mathbb{R}$ be a function. In this paper we consider the Cauchy problem for semilinear parabolic equations on $\mathbb{R}^n$ of the following form:

$$\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) + G(u(t, x)), \quad (t, x) \in (0, \infty) \times \mathbb{R}^n \quad (1)$$

subject to the initial value condition

$$u(0, x) = u_0(x) \quad \text{on} \quad \mathbb{R}^n.$$  

The most classical examples of such equations are the semilinear heat equations

$$\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) + u|u|^{\mu-1}, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \mu > 1, \quad (2)$$

the Burgers viscous equations

$$\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) + \partial_x(|u|^{\mu}), \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \mu > 1$$

and the Navier–Stokes equation

$$\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) + \mathcal{P}\nabla(u \otimes u), \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \mu > 1,$$

where $\mathcal{P}$ denotes the projector on the divergence free vector field. Let us recall briefly some results on most known function spaces. For Lebesgue space, Weissler [60, 61] studied (2) with singular data in certain Lebesgue spaces $L^p$. In [60] he proved the local existence of (2) with initially data in $L^{pc}$ with $pc = \frac{n(\mu-1)}{2} > 1$ and the solution belongs to $C([0, T), L^p)$, and that $T$ can be taken as infinity for sufficiently small data in $L^{pc}$. Giga [27] proved that the solution belongs to $L^q(0, T), L^p)$ with $\frac{1}{q} = \frac{\mu}{2} - \frac{1}{p}$, $p, q > p_c$ and $q > \mu$.

Weissler [60] proved the local existence of (2) for initial values in $L^p$ with $p > p_c$ and $p \geq \mu$. See [27] for further results.

In case of $1 < p < p_c$ there exist some non-negative initial data in $L^p$ for which there is no non-negative solution for any positive time $T > 0$, see e.g. [13, 61].

Further results, for the well-posedness of the Cauchy problem of (2) can be found in [17, 42, 53, 54, 59].

In the framework of fractional Sobolev spaces, [43] established local well-posedness of problem (1) with some suitable assumptions on $G$ and obtained existence of global small solutions in $H^\sigma_p \cap L^1$. Miao and Zhang [40] establish the local well-posedness and small global well-posedness in Besov spaces $B^{0,2}_{p,2}$. Also, they establish the local well-posedness and small global well-posedness of problem (1) in the critical space $B^{0,2}_{p,2}$.

In [22] the author study the Eq. (1) with
\[ |G(x) - G(y)| \leq |x - y|(|x|^\mu - 1 + |y|^\mu - 1), \quad x, y \in \mathbb{R}, \mu > 1, G(0) = 0 \]  
and initial data in Herz spaces \( \dot{K}_p^\alpha \). Herz spaces play an important role in Harmonic Analysis. After they have been introduced in [30], the theory of these spaces had a remarkable development in part due to its usefulness in applications. For instance, they appear in the characterization of multipliers on Hardy spaces [2], in the summability of Fourier transforms [26] and in regularity theory for elliptic equations in divergence form [41], and in the Navier–Stokes equations [58]. They unify and generalize the classical Lebesgue spaces of power weights. More precisely, if \( \alpha = 0 \) and \( p = q \), then \( \dot{K}_p^\alpha \) coincides with the Lebesgue spaces \( L^p \) and 
\[ \dot{K}_p^\alpha = L^p(\mathbb{R}^n, | \cdot |^{ap}), \quad \text{(Lebesgue space equipped with power weight)}. \]

The aims of the present paper is to study the Eq. (1) in Herz-type Triebel–Lizorkin spaces \( \dot{K}_p^\alpha F_s^\beta \). These spaces unify and generalize the classical Lebesgue spaces of power weights, fractional Sobolev spaces of power weights and Triebel–Lizorkin spaces of power weights. We will assume that \( G \in \text{Lip}_\mu \), see Section 3 for the definition of the spaces \( \text{Lip}_\mu \).

We recall that the solution in the function space \( \dot{K}_p^\alpha F_s^\beta \) of the integral equation
\[ u(t, x) = e^{t \Delta} u_0 + \int_0^t e^{(t-\tau)\Delta} G(u)(\tau, x) d\tau \]  
(4)
is usually defined as the mild solution of the Cauchy problem (1). Under some assumption on \( p, q, \beta, \alpha \) and \( s \) we prove that for all initial data \( u_0 \) in \( \dot{K}_p^\alpha F_s^\beta \) with \( s > \bar{s} = \frac{n}{p} - \frac{2}{\mu - 1} \), there exists a maximal solution \( u \) to (4) in \( C([0, T_0]), \dot{K}_p^\alpha F_s^\beta \) with \( T_0 \geq C \| u_0 \|_{\dot{K}_p^\alpha F_s^\beta} \). If \( \theta < (s - \bar{s})(\mu - 1) \), then we prove that
\[ u - e^{t \Delta} u_0 \in C([0, T_0]), \dot{K}_p^\alpha F^{s+\theta}_\beta. \]  
(5)

Now if \( \theta = (s - \bar{s})(\mu - 1), s > 1 \) with \( G \in \text{Lips}_0 \) and
\[ s_0 = \frac{n}{p} + \alpha, \]  
then we have (5), which was not treated in [43]. Our results cover the corresponding results of [43]. Moreover, we present the limit case
\[ s = 1 + \frac{\mu - 1}{\mu} \left( \frac{n}{p} + \alpha \right) \]  
and the case when \( s > \frac{n}{p} + \alpha \). To study (1) we investigate sufficient conditions on \( G \) such that
\[ \{ G(f) : f \in \dot{K}_p^\alpha F_s^\beta \} \subset \dot{K}_p^\alpha F_s^\beta \]
In Sobolev space, [39] have presented the necessary and sufficient conditions on $G$ such that

$$G(W^1_p(\mathbb{R}^n)) \subset W^1_p(\mathbb{R}^n),$$

except the case $p = n \geq 2$. A complete characterization of this problem in Sobolev spaces has been given by Bourdaud [3, 6]. The surprise result in Sobolev spaces is that under some assumptions there is no non-trivial function $G$ which acts via left composition on such spaces. More precisely, in 1978 Dahlberg [18] proved that

$$G(f) \in W^m_p(\mathbb{R}^n), \quad f \in W^m_p(\mathbb{R}^n), \quad 1 < p < \infty, \quad 2 \leq m < \frac{n}{p}$$

implies $G(t) = ct$ for some $c \in \mathbb{R}$. In the framework of Sobolev spaces with fractional order, $H^s(\mathbb{R})$, $0 < s < 1$, $s \neq 2$, Igari [31] gave the necessary and sufficient conditions on $G$ such that $G(H^s(\mathbb{R})) \subset H^s(\mathbb{R})$. He observed the necessity of local Lipschitz continuity for the first time. See [33] for the Hardy–Sobolev space $F^{1,2}_{2,2}(\mathbb{R}^n)$.

The extension of the above results to Besov and Triebel–Lizorkin spaces is given by Bourdaud [4, 5], Runst [44], and Sickel [49–51]. Further results concerning the composition operators in Besov and Triebel–Lizorkin spaces are given [7, 8, 10, 11, 14, 45]. Recently, Bourdaud and Moussai [9] proved the continuity of the composition operator in $W^m_p(\mathbb{R}^n) \cap W^1_{mp}(\mathbb{R}^n)$ to itself, for every integer $m \geq 2$ and any $1 \leq p < \infty$ and in Sobolev spaces $W^m_p(\mathbb{R}^n)$, with $m \geq 2$ and $1 \leq p < \infty$. The author in [21] gave the necessary and sufficient conditions on $G$ such that

$$G(W^m_p(\mathbb{R}^n, | \cdot |^a)) \subset W^m_p(\mathbb{R}^n, | \cdot |^a), \quad \text{(Sobolev space of power weight)},$$

with some suitable assumptions on $m$, $p$ and $a$.

### 1.1 Notation and conventions

Throughout this paper, we denote by $\mathbb{R}^n$ the $n$-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. The expression $f \lesssim g$ means that $f \leq c g$ for some independent constant $c$ (and non-negative functions $f$ and $g$), and $f \approx g$ means $f \lesssim g \lesssim f$. As usual for any $x \in \mathbb{R}$, $[x]$ stands for the largest integer smaller than or equal to $x$.

For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball in $\mathbb{R}^n$ with center $x$ and radius $r$. By supp$f$ we denote the support of the function $f$, i.e., the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of $E$ and $\chi_E$ denotes its characteristic function. For any $u > 0$, we set $C(u) = \{x \in \mathbb{R}^n : \frac{u}{2} < |x| \leq u\}$. By $c$ we denote generic positive constants, which may have different values at different occurrences.

Given a measurable set $E \subset \mathbb{R}^n$ and $0 < p \leq \infty$, we denote by $L^p(E)$ the space of all functions $f : E \to \mathbb{C}$ equipped with the quasi-norm
$\|f\|_{L^p(E)} = \left( \int_E |f(x)|^p \, dx \right)^{1/p} < \infty$

with $0 < p < \infty$ and

$\|f\|_{L^\infty(E)} = \text{ess-sup}_{x \in E} |f(x)| < \infty$.

If $E = \mathbb{R}^n$, then we put $L^p(\mathbb{R}^n) = L^p$ and $\|f\|_{L^p(\mathbb{R}^n)} = \|f\|_p$.

Let $w$ denote a positive, locally integrable function and $0 < p < \infty$. Then the weighted Lebesgue space $L^p(\mathbb{R}^n, w)$ contains all measurable functions $f$ such that

$\|f\|_{L^p(\mathbb{R}^n, w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty$.

If $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then $p'$ is called the conjugate exponent of $p$.

By $\mathcal{S}(\mathbb{R}^n)$ we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by

$\mathcal{F}(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) \, dx$.

Its inverse is denoted by $\mathcal{F}^{-1}f$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the dual Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ in the usual way.

For $\nu \in \mathbb{Z}$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, let $Q_{\nu,m}$ be the dyadic cube in $\mathbb{R}^n$, $Q_{\nu,m} = \{ (x_1, \ldots, x_n) : m_i \leq 2^\nu x_i < m_i + 1, i = 1, 2, \ldots, n \}$. Also, we set $\chi_{j,m} = \chi_{Q_{j,m}}$, $j \in \mathbb{Z}, m \in \mathbb{Z}^n$.

Recall that $\eta_{R,m}(x) = R^n(1 + |x|)^{-m}$, for any $x \in \mathbb{R}^n$ and $m, R > 0$. Note that $\eta_{R,m} \in L^1(\mathbb{R}^n)$ when $m > n$ and that $\|\eta_{R,m}\|_1 = c_m$ is independent of $R$, where this type of function was introduced in [19, 28].

## 2 Function spaces

In this section we present the Fourier analytical definition of Herz-type Triebel–Lizorkin spaces and we present their basic properties such as Sobolev embeddings. We start by recalling the definition and some properties of Herz spaces. For convenience, we set

$B_k = B(0, 2^k), \quad \bar{B}_k = \{ x \in \mathbb{R}^n : |x| \leq 2^k \}, \quad k \in \mathbb{Z}$

and

$R_k = B_k \setminus B_{k-1}, \quad \chi_k = \chi_{R_k}, \quad k \in \mathbb{Z}$.
Definition 2.1 Let $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$. The homogeneous Herz space $\dot{K}^\alpha_{p,q}$ is defined as the set of all $f \in L^p_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ such that 

$$
\|f\|_{\dot{K}^\alpha_{p,q}} = \left( \sum_{k \in \mathbb{Z}} 2^{kq} \|\chi_k f\|_p^q \right)^{1/q} < \infty
$$

(with the usual modifications when $q = \infty$).

Remark 2.2 Let $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$.

(i) The space $\dot{K}^\alpha_{p,q}$ coincides with the Lebesgue space $L^p(\mathbb{R}^n, |\cdot|^\alpha)$. In addition

$$
\dot{K}^0_{p,p} = L^p.
$$

(ii) Let $0 < q_1 \leq q_2 \leq \infty$. Then

$$
\dot{K}^\alpha_{p,q_1} \subset \dot{K}^\alpha_{p,q_2}.
$$

(iii) The spaces $\dot{K}^\alpha_{p,q}$ are quasi-Banach spaces and if $\min(p,q) \geq 1$ then $\dot{K}^\alpha_{p,q}$ are Banach spaces.

Remark 2.3 A detailed discussion of the properties of Herz spaces may be found in [29] and [38], and references therein.

To present the definition of Herz-type Triebel–Lizorkin spaces, we first need the concept of a smooth dyadic resolution of unity. Let $\varphi$ be a function in $\mathcal{S}(\mathbb{R}^n)$ such that

$$
0 \leq \varphi \leq 1 \quad \text{and} \quad \varphi(x) = \begin{cases} 
1, & \text{if } |x| \leq 1, \\
0, & \text{if } |x| \geq \frac{3}{2}.
\end{cases}
$$

We put $\mathcal{F}\varphi_0 = \varphi$, $\mathcal{F}\varphi_1 = \varphi(\frac{\cdot}{2}) - \varphi$ and $\mathcal{F}\varphi_j = \mathcal{F}\varphi_1(2^{1-j} \cdot)$ for $j = 2, 3, \ldots$. Then $\{\mathcal{F}\varphi_j\}_{j \in \mathbb{N}_0}$ is a smooth dyadic resolution of unity. $\sum_{j=0}^{\infty} \mathcal{F}\varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$. Thus we obtain the Littlewood-Paley decomposition

$$
f = \sum_{j=0}^{\infty} \varphi_j * f
$$

of all $f \in \mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$).

We are now in a position to state the definition of Herz-type Triebel–Lizorkin spaces.

Definition 2.4 Let $\alpha, s \in \mathbb{R}, 0 < p, q < \infty$ and $0 < \beta \leq \infty$. The Herz-type Triebel–Lizorkin space $\dot{K}^\alpha_{p,q} F^s_\beta$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$
\|f\|_{\dot{K}^\alpha_{p,q} F^s_\beta} = \left\| \left( \sum_{j=0}^{\infty} 2^{j\beta} |\varphi_j * f|^\beta \right)^{1/\beta} \right\|_{\dot{K}^\alpha_{p,q}} < \infty,
$$

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with the obvious modification if $\beta = \infty$.

**Remark 2.5** Let $s \in \mathbb{R}, 0 < p, q < \infty, 0 < \beta \leq \infty$ and $\alpha > \frac{n}{p}$. The spaces $K_{p,q}^s F^{s}_{\beta}$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\mathcal{F}\varphi_j\}_{j \in \mathbb{N}_0}$ (in the sense of equivalent quasi-norms). In particular $K_{p,q}^s F^{s}_{\beta}$ are quasi-Banach spaces and if $p, q, \beta \geq 1$, then they are Banach spaces. Further results, concerning, for instance, lifting properties, Fourier multiplier and local means characterizations can be found in [23–25, 62, 65].

Now we give the definition of the spaces $F^{s}_{p,\beta}$.

**Definition 2.6** Let $s \in \mathbb{R}, 0 < p < \infty$ and $0 < \beta \leq \infty$. The Triebel–Lizorkin space $F^{s}_{p,\beta}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$
\|f\|_{F^{s}_{p,\beta}} = \left\| \left( \sum_{j=0}^{\infty} 2^{j\beta} \left| \varphi_j \ast f \right|^\beta \right \|_p < \infty.
$$

The theory of the spaces $F^{s}_{p,\beta}$ has been developed in detail in [46, 55, 56] but has a longer history already including many contributors; we do not want to discuss this here. Clearly, for $s \in \mathbb{R}, 0 < p < \infty$ and $0 < \beta \leq \infty$,

$$
K^0_{p,p} F^{s}_{p,\beta} = F^{s}_{p,\beta}.
$$

Let $w \in \mathcal{A}_\infty$. Muckenhoupt classes, $s \in \mathbb{R}, 0 < \beta \leq \infty$ and $0 < p < \infty$. We define weighted Triebel–Lizorkin space $F^{s}_{p,\beta}(\mathbb{R}^n, w)$ to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$
\|f\|_{F^{s}_{p,\beta}(\mathbb{R}^n, w)} = \left\| \left( \sum_{j=0}^{\infty} 2^{j\beta} \left| \varphi_j \ast f \right|^\beta \right \|_{L^p(\mathbb{R}^n, w)} < \infty.
$$

is finite. In the limiting case $\beta = \infty$ the usual modification is required.

The spaces $F^{s}_{p,\beta}(\mathbb{R}^n, w) = F^{s}_{p,\beta}(w)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\mathcal{F}\varphi_j\}_{j \in \mathbb{N}_0}$ appearing in their definitions. They are quasi-Banach spaces (Banach spaces for $p, q \geq 1$). Moreover, for $w \equiv 1$ we obtain the usual (unweighted) Triebel–Lizorkin spaces. We refer, in particular, to the papers [15, 16, 32] for a comprehensive treatment of weighted function spaces. Let $w_\gamma$ be a power weight, i.e., $w_\gamma(x) = |x|^{\gamma}$ with $\gamma > -n$. Then we have

$$
F^{s}_{p,\beta}(w_\gamma) = K^0_{p,p} F^{s}_{p,\beta},
$$
in the sense of equivalent quasi-norms.

**Definition 2.7** (i) Let $1 < p < \infty, 0 < q < \infty, -\frac{n}{p} < \alpha < n(1 - \frac{1}{q})$ and $s \in \mathbb{R}$. Then the Herz-type Bessel potential space $k^{\alpha,q}_{p,s}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that
\[ \|f\|_{K_\alpha^p} = \| (1 + |\xi|^2)^{\frac{s}{2}} \ast f \|_{K_\alpha^p} < \infty. \]

(ii) Let \( 1 < p < \infty, 0 < q < \infty, -\frac{n}{p} < \alpha < n(1 - \frac{1}{p}) \) and \( m \in \mathbb{N} \). The homogeneous Herz-type Sobolev space \( W_{p,m}^{\alpha,q} \) is the collection of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that

\[ \|f\|_{W_{p,m}^{\alpha,q}} = \sum_{|\beta| \leq m} \left\| \frac{\partial^\beta f}{\partial x^\beta} \right\|_{K_\alpha^p} < \infty, \]

where the derivatives must be understood in the sense of distribution.

In the following, we will present the connection between the Herz-type Triebel–Lizorkin spaces and the Herz-type Bessel potential spaces; see \[37, 64\]. Let \( 1 < p, q < \infty \) and \( -\frac{n}{p} < \alpha < n(1 - \frac{1}{p}) \). If \( f \in \mathbb{R} \), then

\[ K_\alpha^\alpha F_2 = \hat{K}_\alpha^\alpha F_2 \]

with equivalent norms. If \( s = m \in \mathbb{N} \), then

\[ K_\alpha^\alpha F_2^m = W_{p,m}^{\alpha,q} \]

with equivalent norms. In particular

\[ \hat{K}_\alpha^\alpha F_2^m = W_{m}^p(\mathbb{R}^n, | \cdot |^q) \quad \text{(Sobolev spaces of power weights)} \]

and

\[ \hat{K}_\alpha^\alpha F_2^m = W_{m}^p \quad \text{(Sobolev spaces),} \quad \hat{K}_\alpha^\alpha F_2^m = \hat{K}_\alpha^\alpha F_2^m \quad \text{(6)} \]

Let \( 0 < \theta < 1, 0 < p_0, p_1, q_0, q_1 < \infty, 0 < \beta_0, \beta_1 \leq \infty \) and \( \alpha_0, \alpha_1, s_0, s_1 \in \mathbb{R} \). We set

\[ \frac{1}{p} = 1 - \theta \frac{1}{p_0} + \theta \frac{1}{p_1}, \quad \frac{1}{q} = 1 - \theta \frac{1}{q_0} + \theta \frac{1}{q_1}, \quad \frac{1}{\beta} = 1 - \theta \frac{1}{\beta_0} + \theta \frac{1}{\beta_1} \]

and

\[ \alpha = (1 - \theta) \alpha_0 + \theta \alpha_1, \quad s = (1 - \theta) s_0 + \theta s_1. \]

As an immediate consequence of Hölder’s inequality we have the so-called interpolation inequalities:

\[ \|f\|_{K_\alpha^\beta F_\beta} \leq \|f\|_{K_\alpha^{\beta_0} F_0}^{1 - \theta} \|f\|_{K_\alpha^{\beta_1} F_1}^\theta \]

holds for all \( f \in \hat{K}_\alpha^{\alpha_0} F_0^{s_0} \cap \hat{K}_\alpha^{\alpha_1} F_1^{s_1} \).

We collect some embeddings on these functions spaces as obtained in \[24\].

**Theorem 2.8** Let \( \alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}, 0 < s, p, q, r < \infty, 0 < \beta \leq \infty, \alpha_1 > -\frac{n}{q} \) and \( \alpha_2 > -\frac{n}{q} \).

We suppose that
\[ s_1 - \frac{n}{s} - \alpha_1 = s_2 - \frac{n}{q} - \alpha_2. \]

Let \( 0 < q \leq s < \infty \) and \( \alpha_2 \geq \alpha_1 \). The embedding
\[
\dot{K}^{\alpha_2}_{q,s} F^{s_2} \hookrightarrow \dot{K}^{\alpha_1}_{s,p} F^{s_1}
\]
holds if \( 0 < r \leq p < \infty \).

Let \( 0 < p, q < \infty \). For later use, we introduce the following abbreviations:
\[
\sigma_p = n \max \left( \frac{1}{p} - 1, 0 \right) \quad \text{and} \quad \sigma_{p,q} = n \max \left( \frac{1}{p} - 1, \frac{1}{q} - 1, 0 \right).
\]
In the next we shall interpret \( L^1_{loc} \) as the set of regular distributions, see \[20\].

**Theorem 2.9** Let \( 0 < p, q < \infty, 0 < \beta \leq \infty, \alpha > -\frac{n}{p} \) and \( s > \max(\sigma_p, \frac{n}{p} + \alpha - n) \). Then
\[
\dot{K}^{\alpha}_{p,q} F^{s} \subset L^1_{loc}.
\]

For any \( a > 0 \), \( f \in \mathcal{S}'(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \), we denote, Peetre maximal function,
\[
\varphi^{s,a}_j f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\varphi_j * f(y)|}{(1 + 2^{|x - y|}a)} \quad j \in \mathbb{N}_0.
\]
We now present a fundamental characterization of the above spaces, which plays an essential role in this paper, see [62, Theorem 1].

**Theorem 2.10** Let \( s \in \mathbb{R}, 0 < p, q < \infty, 0 < \beta \leq \infty \) and \( \alpha > -\frac{n}{p} \). Let \( a > \frac{n}{\min(p, \beta)} \). Then
\[
\|f\|^{\star}_{\dot{K}^{\alpha}_{p,q} F^{s}} = \left\| \left( \sum_{j=0}^{\infty} 2^{js\beta} (\varphi^{s,a}_j f)^{\beta} \right)^{1/\beta} \right\|_{\dot{K}^{\alpha}_{p,q} F^{s}},
\]
is an equivalent quasi-norm in \( \dot{K}^{\alpha}_{p,q} F^{s} \).

3 Composition operators

Let \( G : \mathbb{R} \to \mathbb{R} \) be a continuous function. To solve (4), we study the action of the nonlinear function \( G \) on Herz-type Triebel–Lizorkin spaces.

In this section we investigate sufficient conditions on \( G \) such that
\[
T_G(\dot{K}^{\alpha}_{p,q} F^{s}) \subset \dot{K}^{\alpha}_{p,q} F^{s},
\]
holds, where $\mathcal{K}_{p,q}^{\alpha} F_\beta^\gamma$ is the real-valued part of $K_{p,q}^{\alpha} F_\beta^\gamma$.

First we need the following lemma, which is basically a consequence of Hardy’s inequality in the sequence Lebesgue space $\ell_q$.

**Lemma 3.1** Let $0 < \alpha < 1$ and $0 < q \leq \infty$. Let $\{ \varepsilon_k \}_{k \in \mathbb{N}_0}$ be a sequences of positive real numbers and denote $\delta_k = \sum_{j=0}^{k} a^{k-j} \varepsilon_j$ and $\eta_k = \sum_{j=k}^{\infty} a^{j-k} \varepsilon_j$, $k \in \mathbb{N}_0$. Then there exists a constant $c > 0$ depending only on $\alpha$ and $q$ such that

$$
\left( \sum_{k=0}^{\infty} \delta_k^\alpha \right)^{1/q} + \left( \sum_{k=0}^{\infty} \eta_k^\alpha \right)^{1/q} \leq c \left( \sum_{k=0}^{\infty} \varepsilon_k^\alpha \right)^{1/q}.
$$

As usual, we put

$$
\mathcal{M}(f)(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(y)| \, dy , \quad f \in L^1_{\text{loc}},
$$

where the supremum is taken over all cubes with sides parallel to the axis and $x \in Q$. Also, we set $\mathcal{M}_a(f) = \left( \mathcal{M}(|f|^a) \right)^{1/a}$, $0 < a < \infty$.

Various important results have been proved in the space $K_{p,q}^{\alpha}$ under some assumptions on $\alpha, p$ and $q$. The conditions $-\frac{n}{p} < \alpha < n(1 - \frac{1}{p})$, $1 < p < \infty$ and $0 < q \leq \infty$ is crucial in the study of the boundedness of classical operators in $K_{p,q}^{\alpha}$ spaces. This fact was first realized by Li and Yang [36] with the proof of the boundedness of the maximal function. Some of our results of this paper are based on the following result, see Tang and Yang [52].

**Lemma 3.2** Let $1 < \beta < \infty$, $1 < p < \infty$ and $0 < q \leq \infty$. If $\{ f_j \}_{j \in \mathbb{N}_0}$ is a sequence of locally integrable functions on $\mathbb{R}^n$ and $-\frac{n}{p} < \alpha < n(1 - \frac{1}{p})$, then

$$
\left\| \left( \sum_{j=0}^{\infty} \left( \mathcal{M}(f_j) \right)^\beta \right)^{1/\beta} \right\|_{K_{p,q}^{\alpha}} \leq c \left\| \left( \sum_{j=0}^{\infty} |f_j|^{\beta} \right)^{1/\beta} \right\|_{K_{p,q}^{\alpha}}.
$$

Let $\mu > 0$ and $f \in L^1_{\text{loc}}$. Define

$$
P_k^\mu(f)(x) = \int_{B_{-k}} |f(x + z) - f(x)|^\mu \, dz , \quad x \in \mathbb{R}^n, k \in \mathbb{Z}.
$$

**Lemma 3.3** Let $0 < p, q < \infty$, $0 < \beta < \infty$, $\alpha > -\frac{\beta}{p}$ and

$$
\max \left( \sigma_{p,\beta}, \frac{n}{p} + \alpha - n \right) < s < \mu.
$$

Then there exists a constant $c > 0$ such that

$$
\left\| \left( \sum_{k=-\infty}^{\infty} 2^{(n+s)k\beta} |P_k^\mu(f)|^{\beta} \right)^{1/\beta} \right\|_{K_{p,q}^{\alpha}} \leq c \left\| f \right\|_{K_{p,q}^{\alpha}}^{\mu} \frac{\alpha}{K_{p,q}^{\alpha}}^{\beta} F_{\beta}^\gamma.
$$
holds for all \( f \in L^\max_{\text{loc}}(\mathbb{R}^n) \) with
\[
f = \sum_{j=0}^{\infty} \phi_j \ast f,
\]
in \( L^\mu_{\text{loc}} \), with the obvious modification if \( \beta = \infty \).

**Proof** We will do the proof in two steps.

**Step 1.** We set \( \Delta f(x) = f(x+y) - f(x) \), \( x, y \in \mathbb{R}^n \). A change of variable yields
\[
2^{(n+\beta)k} f_k^\mu(f)(x) = 2^{sk} \int_{B_0} |\Delta_{2^{-k}} f(x)|^\mu \, dz \lesssim J_{1,k}(f)(x) + J_{2,k}(f)(x)
\]
for all \( x \in \mathbb{R}^n \), where the implicit constant is independent of \( x \) and \( k \),
\[
J_{1,k}(f)(x) = 2^{sk} \int_{B_0} |\sum_{j=0}^{k} \Delta_{2^{-j}}(\phi_j \ast f)(x)|^\mu \, dz
\]
and
\[
J_{2,k}(f)(x) = 2^{sk} \int_{B_0} |\sum_{j=k+1}^{\infty} \Delta_{2^{-j}}(\phi_j \ast f)(x)|^\mu \, dz.
\]

Estimate of \( J_{1,k} \). Let \( \Psi, \Psi_0 \in \mathcal{S}(\mathbb{R}^n) \) be two functions such that \( \mathcal{F} \Psi = 1 \) and \( \mathcal{F} \Psi_0 = 1 \) on \( \text{supp} \phi_1 \) and \( \text{supp} \psi \), respectively. Using the mean value theorem we obtain for any \( x \in \mathbb{R}^n, j \in \mathbb{N}_0 \) and \( |z| \leq 1 \)
\[
|\Delta_{2^{-j}}(\phi_j \ast f)(x)| = |\Delta_{2^{-j}}(\Psi_j \ast \phi_j \ast f)(x)|
\leq 2^{-k} \sup_{|x-y| \leq 2^{-j+1}} \sum_{|\beta|=1} |\mathcal{D}^\beta(\Psi_j \ast \phi_j \ast f)(y)|,
\]
with some positive constant \( c \) independent of \( x, j \) and \( k \), and
\[
\Psi_j(\cdot) = 2^{(j-1)n} \Psi(2^{(j-1)} \cdot) \quad \text{for} \quad j = 1, 2, \ldots
\]

We see that if \( |\beta| = 1 \) and \( \alpha > 0 \)
\[
|\mathcal{D}^\beta(\Psi_j \ast \phi_j \ast f)(y)|
= 2^{(j-1)n} \left| \int_{\mathbb{R}^n} \mathcal{D}^\beta(\Psi_j(2^{j-1}(y-z))) \phi_j \ast f(z) \, dz \right|
\leq 2^{(j-1)(n+1)} \int_{\mathbb{R}^n} \left| (\mathcal{D}^\beta \Psi_j(2^{j-1}(y-z))) \phi_j \ast f(z) \right| \, dz.
\]
The right-hand side in (9) may be estimated as follows:
\[
c 2^{i(n+1)\psi_j^{*,a}(y)} \int_{\mathbb{R}^n} \left| (D^\beta \Psi)(2^{j-1}(y-z)) \right| (1+2^j|y-z|)^a \, dz \\
\leq c 2^j \phi_j^{*,a}(y).
\]

Then we obtain for any \( x \in \mathbb{R}^n, |z| \leq 1 \) and any \( j, k \in \mathbb{N}_0 \)
\[
\left| \Delta_{x,z} \phi_j^{*,a}(f)(x) \right| \leq c 2^{j-k} \sup_{|x-y| \leq 2^{-k}} \phi_j^{*,a}(f(y)) \\
\leq c 2^{j-k} (1+2^{-k})^a \sup_{|x-y| \leq 2^{-k}} \frac{\phi_j^{*,a}(f(y))}{(1+2^j|x-y|)^a} \\
\leq c 2^{j-k} \phi_j^{*,a}(f(x)),
\]
if \( 0 \leq j \leq k, k \in \mathbb{N}_0 \) and \( x \in \mathbb{R}^n \). Therefore
\[
J_{1,k}(f)(x) \lesssim 2^{sk} \left( \sum_{j=0}^k 2^{j-k} \phi_j^{*,a}(f(x)) \right)^\mu,
\]
where the implicit constant is independent of \( x \) and \( k \), and this yields that
\[
\left\| \left( \sum_{k=0}^\infty |J_{1,k}(f)|^\beta \right)^{1/\beta} \right\|_{K_{p,q}^s}
\]
can be estimated by
\[
c \left\| \left( \sum_{k=0}^\infty \left( \sum_{j=0}^k 2^{j-k(1-\frac{\alpha}{n})} 2^{j^2\psi_j^{*,a}f} \right)^{\mu \beta} \right)^{1/\mu \beta} \right\|_{K_{p,q}^{\frac{s}{p},\frac{s}{q}}(F_{\mu\beta})}.
\]
Using Lemma 3.1 the last expression is bounded by
\[
c \left\| \left( \sum_{k=0}^\infty \left( 2^{k^2\beta} \phi_k^{*,a}f \right)^{\mu \beta} \right)^{1/\mu \beta} \right\|_{K_{p,q}^{\frac{s}{p},\frac{s}{q}}(F_{\mu\beta})} \lesssim \|f\|_{K_{p,q}^{\frac{s}{p},\frac{s}{q}}(F_{\mu\beta})}.
\]
where we have used Theorem 2.10.

Estimate of \( J_{2,k} \). We can distinguish two cases as follows:

- **Case 1.** \( \min(p, \beta) > 1 \). Therefore \( s > \max \left( 0, \frac{\alpha}{p} + \alpha - n \right) \). Assume that \( \alpha > n(1 - \frac{1}{p}) \). Let \( 1 - s_{\min(p, \beta)} < \lambda < \min(\frac{np}{n+\alpha p}, \beta) \) be a strict positive real number, which is possible because of
\[
s > \frac{n}{p} + \alpha - n > \frac{np(\frac{n}{p} + \alpha - n)}{\min(p, \beta)(n + \alpha p)} = \frac{n}{\min(p, \beta)} \left( 1 - \frac{np}{n+\alpha p} \right).
\]
Let \( \frac{n}{\mu \min(p, \beta)} < a < \frac{s}{\mu(1-\lambda)} \). Then
\[
\frac{s}{\mu} > a(1 - \lambda).
\] 

(10)

If \(-\frac{n}{p} < \alpha < n(1 - \frac{1}{p})\), then we take \(\lambda = 1\). From this we deduce that for all \(x \in \mathbb{R}^n\),
\[
2^{-sk}J_{2,k}(f)(x)
\]
can be estimated by
\[
c \sum_{j=k+1}^{\infty} 2^{(j-k)p} \int_{B_0} |\Delta_{2^{-k}}(\varphi_j * f)(x)|^{\mu} dz
\]
\[
\leq \sum_{j=k+1}^{\infty} 2^{(j-k)p} \sup_{x \in B_0} |\Delta_{2^{-k}}(\varphi_j * f)(x)|^{\mu(1-\lambda)} \int_{B_0} |\Delta_{2^{-k}}(\varphi_j * f)(x)|^{\mu\lambda} dz
\]
where \(0 < \frac{2p}{\mu} \leq \frac{s}{\mu} - a(1 - \lambda)\) and the positive constant \(c\) is independent of \(k\) and \(x\).

Observe that
\[
\int_{B_0} |\Delta_{2^{-k}}(\varphi_j * f)(x)|^{\mu\lambda} dz
\]
\[
\leq |\varphi_j * f(x)|^{\mu\lambda} + 2^{kn} \int_{|y-x| \leq 2^{-k}} |\varphi_j * f(y)|^{\mu\lambda} dy
\]
\[
\leq |\varphi_j * f(x)|^{\mu\lambda} + M(|\varphi_j * f|^{\mu\lambda})(x).
\]

This estimate combined with
\[
|\Delta_{2^{-k}}(\varphi_j * f)(x)| \leq c 2^{(j-k)p} \varphi_j^{*,a} f(x)
\]
(11)

for any \(x \in \mathbb{R}^n\), \(|z| \leq 1\) and any \(j \geq k + 1\), yield
\[
J_{2,k}(f) \leq J_{2,k,1}(f) + J_{2,k,2}(f),
\]
where
\[
J_{2,k,1}(f) = \sum_{j=k+1}^{\infty} 2^{(j-k)(p+\alpha(1-\lambda)-\mu)} \left(2^{\frac{j\lambda}{\mu}} \varphi_j^{*,a} f\right)^{\mu(1-\lambda)} \left(2^{\frac{j\lambda}{\mu}} \varphi_j * f\right)^{\mu\lambda}
\]
and
\[
J_{2,k,2}(f) = \sum_{j=k+1}^{\infty} 2^{(j-k)(p+\alpha(1-\lambda)-\mu)} \left(2^{\frac{j\lambda}{\mu}} \varphi_j^{*,a} f\right)^{\mu(1-\lambda)} M \left(2^{\frac{j\lambda}{\mu}} |\varphi_j * f|\right)^{\mu\lambda}.
\]

By similarity we estimate only \(J_{2,k,2}(f)\). Using Lemma 3.1 and Hölder’s inequality we get
\[
\left( \sum_{k=0}^{\infty} (J_{2,k,2}(f))^\beta \right)^{1/\beta} \\
\lesssim \left( \sum_{k=0}^{\infty} \left( 2^{k^2} \varphi_k^{*,a} f \right)^{\mu(1-\lambda)} \left( \mathcal{M}(2^{k^2} |\varphi_k * f|) \mu^{\lambda} \right)^{\beta} \right)^{1/\beta} \\
\lesssim \left( \sum_{k=0}^{\infty} \left( 2^{k^2} \varphi_k^{*,a} f \right) \mu^{\beta} \left( \sum_{k=0}^{\infty} \left( \mathcal{M}(2^{k^2} |\varphi_k * f|) \mu^{\lambda} \right)^{\beta/\lambda} \right)^{\lambda/\beta} \right).
\]

Again by Hölder’s inequality
\[
\left\| \left( \sum_{k=0}^{\infty} (J_{2,k,2}(f))^\beta \right)^{1/\beta} \right\|_{K_{p,q}^s},
\]
can be estimated by
\[
c \left\| \left( \sum_{k=0}^{\infty} \left( 2^{k^2} \varphi_k^{*,a} f \right)^{\mu\beta} \right)^{1-\lambda} \right\|_{K_{p,q}^s}^{1-\lambda} \\
\times \left\| \left( \sum_{k=0}^{\infty} \left( \mathcal{M}(2^{k^2} |\varphi_k * f|)^{\mu/\lambda} \right)^{\lambda/\beta} \right)^{1/\beta} \right\|_{K_{p,q}^s}^{\lambda/\beta}
\lesssim \|f\| \left( 1-\lambda \right)^{\mu} \left( \sum_{k=0}^{\infty} \left( 2^{k^2} |\varphi_k * f| \right)^{\mu\beta} \right)^{1/\beta} \left( \sum_{k=0}^{\infty} \left( \mathcal{M}(2^{k^2} |\varphi_k * f|)^{\mu/\lambda} \right)^{\lambda/\beta} \right)^{1/\beta}
\]

where we have used Theorem 2.10 and Lemma 3.2. Obviously we can estimate the last term by
\[
c \|f\| \left( 1-\lambda \right)^{\mu} \left( \sum_{k=0}^{\infty} 2^{k^2} |\varphi_k * f| \right)^{\mu\beta} \left( \sum_{k=0}^{\infty} \left( \mathcal{M}(2^{k^2} |\varphi_k * f|)^{\mu/\lambda} \right)^{\lambda/\beta} \right)^{1/\beta}
\]

- **Case 2.** \( \min(p, \beta) \leq 1 \). If \( \frac{n}{p} < \alpha < n(1 - \frac{1}{p}) \), then \( s > \frac{n}{\min(p, \beta)} - n \). Taking \( \max(0, 1 - \frac{s}{\min(p, \beta)}) < \lambda < \min(1, p, \beta) \). The same arguments as in Case 1 yield the desired estimate. Now assume that \( \alpha \geq n(1 - \frac{1}{p}) \). Therefore
\[
s > \max \left( \frac{n}{\min(p, \beta)} - n, \frac{n}{p} + \alpha - n \right).
\]

Taking \( \max(0, 1 - \frac{s}{\min(p, \beta)}) < \lambda < \min(p, \frac{np}{n+ap}\beta) \). The desired estimate can be done in the same manner as in Case 1.

**Step 2.** We will estimate
\[
\left\| \left( \sum_{k=-\infty}^{-1} 2^{(n+s)k\beta} |f_k^\mu(f)|^{\beta} \right)^{1/\beta} \right\|_{K_{p,q}^s}.
\]

We employ the same notations as in Step 1. Recall that
Define

\[ f = \sum_{j=0}^{\infty} \varphi_j \ast f. \]

As in the estimation of \( J_{2,k} \), we obtain

\[ M_{2,k}(f) \lesssim M_{2,k,1}(f) + M_{2,k,2}(f), \]

where

\[ M_{2,k,1}(f) = 2^{-ka\mu(1-\lambda)} \sum_{j=0}^{\infty} 2^{j(\varepsilon + \alpha\mu(1-\lambda) - \gamma)} \left( 2^{j\mu} \varphi_j \ast f \right)^{\mu(1-\lambda)} \left| 2^{j\mu} \varphi_j \ast f \right|^\mu, \]

and

\[ M_{2,k,2}(f) = 2^{-ka\mu(1-\lambda)} \sum_{j=0}^{\infty} 2^{j(\varepsilon + \alpha\mu(1-\lambda) - \gamma)} \left( 2^{j\mu} \varphi_j \ast f \right)^{\mu(1-\lambda)} \mathcal{M} \left( 2^{j\mu} |\varphi_j \ast f| \right)^\mu, \]

with the help of (11). By similarity we estimate only \( M_{2,k,2} \). Obviously

\[ M_{2,k,2}(f) \lesssim 2^{-ka\mu(1-\lambda)} \sup_{j \in \mathbb{N}_0} \left( \left( 2^{j\mu} \varphi_j \ast f \right)^{\mu(1-\lambda)} \mathcal{M} \left( 2^{j\mu} |\varphi_j \ast f| \right)^\mu \right) \]

and this yields that

\[ \left( \sum_{k=-\infty}^{-1} 2^{\beta k} |M_{2,k,2}|^\beta \right)^{1/\beta} \lesssim \sup_{j \in \mathbb{N}_0} \left( \left( 2^{j\mu} \varphi_j \ast f \right)^{\mu(1-\lambda)} \mathcal{M} \left( 2^{j\mu} |\varphi_j \ast f| \right)^\mu \right). \]

By the same arguments as used in Step 1 we obtain the desired estimate. The proof is complete. \( \square \)

Now we present the case of \( s = \mu \), where the proof is very similar to Lemma 3.3.

**Lemma 3.4** Let \( 0 < p, q < \infty, \alpha > -\frac{n}{p} \) and

\[ \max \left( \sigma_p, \frac{n}{p} + \alpha - n \right) < \mu. \]

Then there exists a positive constant \( c \) such that

\[ \left\| \sup_{k \in \mathbb{Z}} 2^{(n+\mu)k} |F_k(f)| \right\|_{K_p^{\mu}}^{\alpha} \leq c \left\| f \right\|_{K_p^{\mu}}^{\alpha}, \]

holds for all \( f \in L_{\text{loc}}^{\max(1,\mu)} \) with
\[ f = \sum_{j=0}^{\infty} \varphi_j \ast f, \]
in \( L^\mu_{\text{loc}} \).

Using the fact that \( \|f\|_{K_{p,q}^{\mu} F_{\beta}^{s}} \leq \|f\|_{K_{p,q}^{\mu} F_{\beta}^{s}} \|f\|_{\infty}^{\mu-1} \), we immediately arrive at the following results.

**Lemma 3.5** Let \( 0 < p, q < \infty, 0 < \beta \leq \infty, \alpha > -\frac{n}{p} \) and
\[
\max \left( 1, \sigma_{p,\beta}, \frac{n}{p} + \alpha - n \right) < s < \mu.
\]
Then there exists a positive constant \( c \) such that
\[
\left\| \left( \sum_{k=-\infty}^{\infty} 2^{(n+s)k\beta} |f_k^\mu(f)|^\beta \right)^{1/\beta} \right\|_{K_{p,q}^{\mu} F_{\beta}^{s}} \leq c \left\| f \right\|_{K_{p,q}^{\mu} F_{\beta}^{s}} \|f\|_{\infty}^{\mu-1}
\]
holds for all \( f \in K_{p,q}^{\mu} F_{\beta}^{s} \cap L^\infty \).

**Remark 3.6** Corresponding statements to Lemmas 3.3, 3.4 and 3.5 were proved by Runst [44, Lemma 1], with \( \alpha = 0, p = q \) and the case of bounded functions, while with \( \alpha = 0, p = q \) has been given by Sickel in [48, Lemmas 1, 2]. In our proof we have used the ideas of [48, Lemmas 1, 2].

The next two lemmas are used in the proof of our results, see e.g. [12].

**Lemma 3.7** Let \( s \in \mathbb{R}, A, B > 0, 0 < p, q < \infty, 0 < \beta \leq \infty \) and \( \alpha > -\frac{n}{p} \). Let \( \{f_i\}_{i \in \mathbb{N}_0} \) be a sequence of functions such that
\[
\text{supp} \mathcal{F}f_0 \subseteq \{ \xi \in \mathbb{R}^n : |\xi| \leq A \}
\]
and
\[
\text{supp} \mathcal{F}f_i \subseteq \{ \xi \in \mathbb{R}^n : B2^{l+1} \leq |\xi| \leq A2^{l+1} \}.
\]
There exists a constant \( c > 0 \) such that the following inequality
\[
\left\| \sum_{l=0}^{\infty} f_l \right\|_{K_{p,q}^{\mu} F_{\beta}^{s}} \leq c \left\| \left( \sum_{l=0}^{\infty} 2^{l\beta} |f_l|^\beta \right)^{1/\beta} \right\|_{K_{p,q}^{\mu} F_{\beta}^{s}}
\]
holds.
Lemma 3.8 Let $A, B > 0, p, q < \infty, 0 < \beta \leq \infty$ and $\alpha > -\frac{n}{p}$. Let $s > \max(\sigma_p, \frac{n}{p} + \alpha - n)$. Let $\{f_i\}_{i \in \mathbb{N}_0}$ be a sequence of functions such that

$$\text{supp} \mathcal{F}f_i \subseteq \{ \xi \in \mathbb{R}^n : |\xi| \leq A 2^{i+1} \}.$$

Then it holds that

$$\left\| \sum_{i=0}^{\infty} f_i \right\|_{K_p^{a_F} F_p^s} \leq c \left\| \left( \sum_{i=0}^{\infty} 2^{i\beta} |f_i|^\beta \right)^{1/\beta} \right\|_{K_p^{a_F} F_p^s}.$$

Let $G : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. We shall deal with sufficient conditions on $G$ to guarantee an embedding

$$T_G(\dot{K}_p^{a_F} F_p^s) = G(\dot{K}_p^{a_F} F_p^s) \subset \dot{K}_p^{a_F} F_p^s.$$

First we begin with the case where $G$ is polynomial.

Theorem 3.9 Let $0 < p, q < \infty, 0 < \beta \leq \infty, s \geq \frac{n}{p} - \frac{n}{q}, \alpha \geq 0$ and

$$\max \left( 0, \frac{n}{p} + \alpha - \frac{n}{m} \right) < s < \frac{n}{p} + \alpha, \quad m = 2, 3, \ldots \quad (12)$$

We put

$$s_m = s - (m-1) \left( \frac{n}{p} + \alpha - s \right).$$

Then

$$\| f^m \|_{K_p^{a_F} F_p^s} \lesssim \| f \|^m_{K_p^{a_F} F_p^s} \quad (13)$$

holds for all $f \in \dot{K}_p^{a_F} F_p^s$.

Proof We will do the proof into three steps.

Step 1. Preparation. Let $\{\mathcal{F} \varphi_j\}_{j \in \mathbb{N}_0}$ be a partition of unity and $f \in \mathcal{S}'(\mathbb{R}^n)$. We define the convolution operators $\Delta_j$ by the following:

$$\Delta_j f = \varphi_j * f, \quad j \in \mathbb{N} \quad \text{and} \quad \Delta_0 f = \varphi_0 * f = \mathcal{F}^{-1} \psi * f.$$

We define the convolution operators $Q_j, j \in \mathbb{N}_0$ by the following:

$$Q_j f = \mathcal{F}^{-1} \psi_j * f, \quad j \in \mathbb{N}_0,$$

where $\mathcal{F}^{-1} \psi_j = 2^{in} \mathcal{F}^{-1} \psi(2^j \cdot)$ and we see that

$$Q_j f = \sum_{k=0}^{j} \Delta_k f, \quad j \in \mathbb{N}_0.$$

For all $f_i \in \mathcal{S}'(\mathbb{R}^n), i = 1, 2, \ldots, m$ the product $\prod_{i=1}^{m} f_i$ is defined by
\[ \prod_{i=1}^{m} f_i = \lim_{j \to \infty} \prod_{i=1}^{m} Q_j f_i, \]

if the limit on the right-hand side exists in \( S'(\mathbb{R}^n) \). The following decomposition of this product is given in [45, Chapter 4]. We have the following formal decomposition:

\[
\prod_{i=1}^{m} f_i = \sum_{k_1, \ldots, k_m = 0}^{\infty} \prod_{i=1}^{m} (\Delta_k f_i).
\]

The fundamental idea is to split \( \prod_{i=1}^{m} f_i \) into two parts, both of them being always defined. Let \( N \) be a natural number greater than \( 1 + \log_2 3(m - 1) \). Then we have the following decomposition:

\[
\prod_{i=1}^{m} f_i = \sum_{j=0}^{\infty} [Q_{j-N} f_1 \cdot \cdots \cdot Q_{j-N} f_{m-1} \cdot \Delta_j f_m \\
+ \cdots + (\Pi_{i \neq k} Q_{j-N} f_i) \Delta_j f_j + \cdots + \Delta_j f_1 \cdot Q_{j-N} f_2 \cdot \cdots \cdot Q_{j-N} f_m] \\
+ \sum_{j=0}^{\infty} \sum_{k} (\Delta_j f_1) \cdot \cdots \cdot (\Delta_j f_m),
\]

where the \( \sum^j \) is taken over all \( k \in \mathbb{Z}^n_+ \) such that

\[ \max_{\ell=1, \ldots, m} k_1 = k_{k_{m_0}} = j \quad \text{and} \quad \max_{\ell \neq m_0} |\ell - k_{\ell}| < N. \]

Of course, if \( k < 0 \) we put \( \Delta_j f = 0 \). Probably \( \sum^j \) becomes more transparent by restricting to a typical part, which can be taken to be

\[
\left( \prod_{i \in I_1} \Delta f_i \right) \prod_{i \in I_2} Q f_i,
\]

where

\[ I_1, I_2 \subseteq \{1, \ldots, m\}, \quad I_1 \cap I_2 = \emptyset, \quad I_1 \cup I_2 = \{1, \ldots, m\} = I, \quad |I_1| \geq 2. \]

We introduce the following notations

\[
\Pi_{1,k}(f_1, f_2, \ldots, f_m) = \sum_{j=N}^{\infty} \left( \prod_{i \neq k} Q_{j-N} f_i \right) \Delta_j f_k
\]

and

\[
\Pi_{2}(f_1, f_2, \ldots, f_m) = \sum_{j=0}^{\infty} \sum_{i=1}^{m} (\prod_{i=1}^{m} \Delta_k f_i).
\]

The advantage of the above decomposition is based on
Composition operators on Herz-type Triebel–Lizorkin spaces…

\[ \text{supp } \mathcal{F} \left( \prod_{i \neq k}^j Q_{j_i, -N_i} \right) \Delta f_k \subset \{ \xi \in \mathbb{R}^n : 2^j - 1 \leq |\xi| \leq 2^{j+1} \}, \quad j \geq N \]

and

\[ \text{supp } \mathcal{F} \left( \sum_{i=1}^j \left( \prod_{i=1}^m \Delta_k f_i \right) \right) \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2^{j+N-2} \}, \quad j \in \mathbb{N}_0. \]

**Step 2.** We will prove (13). Observe that we need only to estimate

\[ \Pi_1(f, f, \ldots, f) = \sum_{j=N}^{\infty} (Q_{j, -N} f)^{m-1} \Delta f \]

and

\[ \Pi_2(f, f, \ldots, f) = \sum_{j=0}^{\infty} (\Delta_j f)^{|I_1|} (Q_j f)^{|I_2|}. \]

Define

\[ \frac{1}{v} = \frac{1}{p} + (m-1) \left( \frac{1}{p} - \frac{s}{n} \right). \]

Therefore we have the following Sobolev embeddings

\[ \dot{K}_{v, \alpha}^{aM} F_\beta \subset \dot{K}_{p, \alpha}^{aM} F_\beta. \]

Lemma 3.7 gives

\[ \| \Pi_1(f, f, \ldots, f) \|_{K_{v, \alpha}^{aM} F_\beta} \]

can be estimated by

\[ c \left\| \left( \sum_{j=N}^{\infty} |2^{js} (Q_{j, -N} f)^{m-1} \Delta f|^\beta \right)^{\frac{1}{\beta}} \right\|_{K_{v, \alpha}^{aM}} \]

\[ \lesssim \left\| (\sup_{j \geq N} |Q_{j, -N} f|)^{m-1} \left( \sum_{j=N}^{\infty} |2^{js} \Delta f|^\beta \right)^{\frac{1}{\beta}} \right\|_{K_{v, \alpha}^{aM}}. \]

By Hölder’s inequality we estimate the last term by

\[ c \| \sup_{j \geq N} |Q_{j, -N} f| \|_{K_{v, \alpha}^{aM}}^{m-1} \| f \|_{K_{p, \alpha}^{aM} F_\beta}, \]

with \( \frac{1}{b} = \frac{1}{p} - \frac{s}{n} \). Recall that
\[ \| \sup_{j \in \mathbb{N}} |Q_{j^{-a}} f| \|_{K_{b,\infty}^a} \leq \| \sup_{j \in \mathbb{N}} |Q_{j^{-a}} f| \|_{K_{b,b}^a} \leq \| f \|_{F_{b,b}^{p,q}(\mathbb{R}^n,|\cdot|^{ab})} \leq \| f \|_{K_{p,q}^a}^2, \]  
\tag{14} 

see [15, Theorem 1.4], because of \(-\frac{n}{b} < \alpha < n(1 - \frac{1}{b})\). Since, \(s \geq \frac{n}{p} - \frac{n}{q}\), thanks to the embedding

\[ K_{p,q}^a F_s^x \hookrightarrow K_{b,b}^a F_2^s, \]  
\tag{15} 

see Theorem 2.8, we obtain

\[ \| \Pi_1(f, f, \ldots, f) \|_{K_{p,q}^{am} F_s^x} \lesssim \| f \|_{K_{p,q}^{m} F_s^x}. \]

Now we estimate \(\Pi_2(f, f, \ldots, f)\). Define

\[ \frac{1}{u} = \frac{|I_1|}{p} + \frac{|I_2|}{b}, \quad \sigma - \frac{n}{u} - am = s_m - \frac{n}{p} - \alpha. \]

Observe that \(\sigma = |I_1|s\). Hence

\[ K_{u,\frac{|I_1|}{p}}^{am} F_s^{|I_1|} \hookrightarrow K_{p,q}^a F_m^s. \]

From (12) it follows that \(\sigma > \max\left(0, \frac{n}{u} + am - n\right)\). Lemma 3.8 gives

\[ \| \Pi_2(f, f, \ldots, f) \|_{K_{p,q}^{am} F_s^{|I_1|}} \lesssim \left( \sum_{j=0}^{\infty} |2^{j|I_1|} (Q_j f)^{|I_1|} (\Delta_j f)^{|I_1|} \Delta_j f^{|I_1|} \|_{F_s^x} \right) \| f \|_{K_{p,q}^{am} F_s^{|I_1|}} \].

Again, by Hölder’s inequality we estimate the last term by

\[ c \| \sup_{j \in \mathbb{N}} |Q_j f| \|_{K_{b,\infty}^a} \left( \sum_{j=0}^{\infty} |2^{j|I_1|} (\Delta_j f^{|I_1|} \Delta_j f)^{|I_1|} \|_{F_s^x} \right)^{\frac{1}{p}} \| f \|_{K_{p,q}^{m} F_s^x} \lesssim \| f \|_{K_{p,q}^{m} F_s^x}, \]

where we have used (14) and (15).

\[ \square \]

**Theorem 3.10** Let \(0 < p, q < \infty, 0 < \beta \leq \infty, \alpha \geq 0\) and

\[ s > \max\left(0, \frac{n}{p} + \alpha - n\right), \quad m = 2, 3, \ldots \]

Then

\[ \text{Birkhäuser} \]
\[ \|f^m\|_{\dot{K}^a_{p,q} F^s} \lesssim \|f\|_{\dot{K}^a_{p,q} F^s} \|f\|_\infty^{m-1} \]

holds for all \( f \in \dot{K}^a_{p,q} F^s \cap L^\infty \).

**Proof** First, we estimate \( \Pi_1(f, f, \ldots, f) \). Recall that
\[
\sup_{j \in \mathbb{N}_0} |Q_j f| \lesssim \|f\|_\infty \quad \text{and} \quad \sup_{j \in \mathbb{N}_0} |\Delta_j f| \lesssim \|f\|_\infty.
\]
(16)
Lemma 3.7 gives
\[ \|\Pi_1(f, f, \ldots, f)\|_{\dot{K}^a_{p,q} F^s} \]
and can be estimated by
\[
\begin{align*}
&c \left\| \left( \sum_{j=N}^\infty 2^{js} (Q_j f)^{m-1} \Delta_j f \right)^{\frac{1}{\beta}} \right\|_{\dot{K}^a_{p,q}} \\
\lesssim & \left\| \left( \sup_{j \geq N} |Q_j f| \right)^{m-1} \left( \sum_{j=N}^\infty 2^{js} \Delta_j f \right)^{\frac{1}{\beta}} \right\|_{\dot{K}^a_{p,q}} \\
\lesssim & \|f\|_\infty^{m-1} \|f\|_{\dot{K}^a_{p,q} F^s},
\end{align*}
\]
where we used (16). Lemma 3.8 gives
\[ \|\Pi_2(f, f, \ldots, f)\|_{\dot{K}^a_{p,q} F^s} \lesssim \left\| \left( \sum_{j=0}^\infty 2^{js} (Q_j f)^{l_1(1)} (\Delta_j f)^{l_1} \right)^{\frac{1}{\beta}} \right\|_{\dot{K}^a_{p,q}} \]
\[ \lesssim \|f\|_\infty^{m-1} \left\| \left( \sum_{j=0}^\infty 2^{js} \Delta_j f \right)^{\frac{1}{\beta}} \right\|_{\dot{K}^a_{p,q}} \]
\[ \lesssim \|f\|_\infty^{m-1} \|f\|_{\dot{K}^a_{p,q} F^s}, \]
with the help of (17).

\[ \square \]

**Remark 3.11** Theorem 3.9 in the case \( m = 2, p = q \) and \( \alpha = 0 \) is contained in [66] and also in [47]. For \( m > 2, p = q \) and \( \alpha = 0 \) see [48, Remark 17] and [45, p. 291]. We refer the reader to the monograph [45] and the paper [34] for further details, historical remarks and more references on multiplication in Besov and Triebel–Lizorkin spaces.

**Definition 3.12** Let \( \mu > 0 \). Let \( L \in \mathbb{N}_0 \), and let \( 0 < \nu \leq 1 \) such that \( \mu = L + \nu \). The spaces \( \text{Lip}_\mu \) is the collection of all \( f \in C^{L,\text{loc}}(\mathbb{R}) \) such that
\[ f^{(l)}(0) = 0, \quad l = 0, 1, 2, \ldots, L \]
and
Then we put

$$\sup_{t_0, t_1 \in \mathbb{R}} \frac{|f^{(L)}(t_0) - f^{(L)}(t_1)|}{|t_0 - t_1|^\nu} < \infty.$$ 

Remark 3.13 \(\|\cdot\|_{\text{Lip}_\mu}\) defines not a norm, but for simplicity we will use this notation, see [45, p. 295]. A typical example of a function belongs to \(\text{Lip}_\mu\) is \(f(t) = |t|^{\mu}, \mu > 1\). Recall that \(\text{Lip}_\mu\) is not monotone with respect to \(\mu\).

We follow the same notations as in [45, Chapter 5].

Definition 3.14 (i) For \(f \in S' (\mathbb{R}^n)\) we define a distribution \(\tilde{f}\) by

$$\tilde{f}(\varphi) = f(\overline{\varphi}), \quad \varphi \in S(\mathbb{R}^n).$$

(ii) The space of real-valued distributions \(S' (\mathbb{R}^n)\) is defined to be

$$S' (\mathbb{R}^n) = \{f \in S' (\mathbb{R}^n) : \tilde{f} = f\}.$$ 

(iii) Let \(A\) be a complex-valued, quasi-normed distribution space such that \(A \hookrightarrow S' (\mathbb{R}^n)\). Then we define the real-valued part \(\mathcal{A}\) of \(A\) to be the restriction of \(A\) to \(S' (\mathbb{R}^n)\) equipped with the same quasi-norm as \(A\).

Now we are in position to state the first result of this section.

Theorem 3.15 Let \(0 < p, q < \infty, 0 < \beta \leq \infty, \mu > 1, \alpha \geq 0, s \geq \frac{n}{p} - \frac{n}{q}\) and

\[
0 < s < \frac{n}{p} + \alpha.
\]

We put

$$s_\mu = s - (\mu - 1)(\frac{n}{p} + \alpha - s).$$

Let \(G \in \text{Lip}_\mu\) and

$$\max \left(0, \frac{n}{p} + \alpha - n\right) < s_\mu < \mu.$$ (17)

Then

$$\|G(f)\|_{K_{p,q}^\alpha F_{-\nu}} \lesssim \|G\|_{\text{Lip}_\mu} \|f\|_{K_{p,q}^\alpha F_{-\nu}}$$

holds for any \(f \in K_{p,q}^\alpha F_{-\nu}^s\).
Proof We will do the proof in three steps.

Step 1. Preparation. Consider the partition of the unity \( \{ \mathcal{F} \varphi_j \}_{j \in \mathbb{N}_0} \). Let \( f \in K_{p,q}^{a} F^s_\infty \). We set

\[
\frac{1}{b} = \frac{n}{p} + \frac{\alpha - s}{n + \alpha p} \quad \text{and} \quad \alpha_1 = \frac{\alpha p}{b}.
\]

Then

\[
\max(1, p) < b < \infty \quad \text{and} \quad - \frac{n}{b} < \alpha_1 < \min(\alpha, n - \frac{n}{b}).
\]

Hence

\[
K_{p,q}^{a} F^s_\infty \hookrightarrow \dot{K}_{b,r}^{a_1}, \quad \max(1, q, \mu) < r.
\]

Since \( G \in \text{Lip}_\mu, \frac{b}{\mu} > 1 \) and \( \alpha_1 \mu < n - \frac{n \mu}{b} \), we have

\[
G(f) \in \dot{K}_{b,r}^{a_1 - \frac{n}{p} + \frac{n}{b}} \hookrightarrow S'((\mathbb{R}^n))
\]

and so we can interpret \( G \) as a mapping of a subspace of \( S'((\mathbb{R}^n)) \) into \( S'((\mathbb{R}^n)) \). In addition

\[
f = \sum_{j=0}^{\infty} \varphi_j \ast f, \quad \text{in} \quad K_{\mu,r}^{a_1 - \frac{n}{p} + \frac{n}{b}}.
\]

Indeed, let

\[
o_k = \sum_{j=0}^{k} \varphi_j \ast f, \quad k \in \mathbb{N}_0.
\]

Obviously \( \{ o_k \} \) converges to \( f \) in \( S((\mathbb{R}^n)) \) and by the embedding (19) we derive that \( \{ o_k \} \subset \dot{K}_{b,r}^{a_1} \). Furthermore, \( \{ o_k \} \) is a Cauchy sequences in \( \dot{K}_{b,r}^{a_1} \), and hence it converges to \( g \in \dot{K}_{b,r}^{a_1} \). Let us prove that \( f = g \) a.e. Let \( \varphi \in \mathcal{D}(\mathbb{R}^n) \). We write

\[
\langle f - g, \varphi \rangle = \langle f - o_N, \varphi \rangle + \langle g - o_N, \varphi \rangle, \quad N \in \mathbb{N}_0,
\]

Here \( \langle \cdot, \cdot \rangle \) denotes the duality bracket between \( \mathcal{D}'(\mathbb{R}^n) \) and \( \mathcal{D}(\mathbb{R}^n) \). Clearly, the first term tends to zero as \( N \to \infty \), while by Hölder’s inequality there exists a constant \( C > 0 \) independent of \( N \) such that

\[
|\langle g - o_N, \varphi \rangle| \leq C \| g - o_N \|_{K_{b,r}^{a_1}},
\]

which tends to zero as \( N \to \infty \). Therefore \( f = g \) almost everywhere. Consequently, \( f = \sum_{j=0}^{\infty} \varphi_j \ast f \) in \( \dot{K}_{b,r}^{a_1} \). Finally, (20), follows by the embedding \( \dot{K}_{b,r}^{a_1} \hookrightarrow K_{\mu,r}^{a_1 - \frac{n}{p} + \frac{n}{b}} \). We have also
because of \( K_{\mu,r}^{a_1 - \frac{n}{\mu} + \frac{s}{b}} \hookrightarrow L^\mu_{\text{loc}} \). Indeed, let \( B(0, 2^M) \subset \mathbb{R}^n, M \in \mathbb{Z} \). Hölder’s inequality and the fact that \( \alpha_1 - \frac{n}{\mu} + \frac{n}{b} = \frac{n}{p} + \alpha - \frac{n}{\mu} - s < 0 \), see (17), give

\[
\|f\|_{L^\mu(B(0, 2^M))}^\mu = \sum_{i=-\infty}^{M} \|f X_i\|_{\mu}^\mu
\]

\[
\leq C(M) \left( \sum_{i=-\infty}^{M} 2^{i(\alpha_1 - \frac{n}{\mu} + \frac{s}{b})} \|f X_i\|_{\mu}^\mu \right)^{\frac{\mu}{2}}
\]

We put \( \mu = L + \nu \), where \( 0 < \nu \leq 1 \). The function \( G \) has the Taylor expansion

\[
G(t) = \sum_{l=0}^{L-1} \frac{G^{(l)}(z)}{l!} (t - z)^l + R(t, z), \quad t, z \in \mathbb{R},
\]

where

\[
R(t, z) = \frac{1}{L!} \int_{z}^{t} (t - y)^{L-1} G^{(L)}(y) dy.
\]

Since \( f \in K_{\mu,r}^{a_1 - \frac{n}{\mu} + \frac{s}{b}} F^s \) and \( s > \max(0, \frac{n}{p} + \alpha - n) \) there exists a set \( A \) of Lebesgue-measure zero such that \( |f(x)| < \infty \) for all \( x \in \mathbb{R}^n \setminus A \). We can suppose that \( |f(x)| < \infty \) for all \( x \in \mathbb{R}^n \). Therefore

\[
G(f(y)) = \sum_{l=0}^{L-1} \frac{1}{l!} \sum_{j=0}^{l} (-1)^{l-j} C_l^{j} j!(y)(\psi_k * f(x))^{l-j} G^{(j)}(\psi_k * f(x))
\]

\[ + R_k(f(y), \psi_k * f(x)), \]

where, \( x, y \in \mathbb{R}^n \),

\[
\psi_k * f = \sum_{i=0}^{k} \phi_i * f, \quad k \in \mathbb{N}_0
\]

and

\[
R_k(f(y), \psi_k * f(x)) = \frac{1}{L!} \int_{\psi_k * f(x)}^{f(y)} (f(y) - h)^{L-1} G^{(L)}(h) dh.
\]

We put \( K_{j,l} = (-1)^{l-j} C_l^{j} \frac{1}{j!} \), with \( 0 \leq l \leq L - 1, 0 \leq j \leq l \). Consequently
where

\[ H_{k, 1, j, l}(x) = K_{j, l}(\psi_k * f(x))^{-j} G^{(l)}(\psi_k * f(x)) \int_{\mathbb{R}^n} \varphi_k(x - y) f^j(y) dy \]

with \(0 \leq l \leq L - 1, 0 \leq j \leq l\) and

\[ H_{k, 2}(x) = \frac{1}{L!} \int_{\mathbb{R}^n} \varphi_k(x - y) \int_{\psi_k * f(x)}^{f(y)} (f(y) - h)^{L-1} G^{(L)}(h) dh dy. \]

We will estimate each term separately.

**Step 2.** Estimate of \(H_{k, 1, j, l}\). First assume that \(0 < j \leq L - 1\). Recall that

\[ s_i = s - (i - 1) \left( \frac{n}{p} + \alpha - s \right), \quad i > 1 \]

and \(s_i \leq s, i \geq v > 1\). Define

\[ p_1 = \frac{n + ap}{(\mu - j) \left( \frac{n}{p} + \alpha - s \right)} \]

and

\[ p_2 = \frac{n + ap}{s - s_j + \frac{n}{p} + \alpha}, \quad (21) \]

where

\[ s_\mu < s < \min(\mu, s_L), \]

with \(0 < j \leq l, 0 \leq l \leq L - 1\). Since \(s_j - \frac{n}{p} - \alpha = -j(\frac{n}{p} + \alpha - s) < 0\), (21) is well defined. We put \(\tilde{p} = \frac{1}{p_1} + \frac{1}{p_2}\). Hence

\[ \frac{s - n}{p} - \alpha = s_\mu - \frac{n}{p} - \alpha, \quad \tilde{p} < p < \min\left( (\mu - j)p_1, p_2 \right). \]

In addition

\[ s - \frac{n}{p} - \alpha = -\frac{n}{(\mu - j)p_1} - \frac{ap}{(\mu - j)p_1} \quad \text{and} \quad s_j - \frac{n}{p} - \alpha = \frac{s - n}{p_2} - \frac{ap}{p_2}. \]

These choices guarantee the Sobolev embeddings
\[ \dot{K}_{p,q}^{\alpha} F_{\mu}^{x} \subset \dot{K}_{(\mu-j)p_1,1,\infty}^{0} F_{1}^{0}, \quad \dot{K}_{p,q}^{\alpha} F_{\mu}^{x} \subset \dot{K}_{p,q}^{\alpha} F_{\mu}^{x}, \quad 0 < r \leq \infty, \]  

(22)

and

\[ \dot{K}_{p,q}^{\alpha} F_{\mu}^{x} \subset \dot{K}_{p,q}^{\alpha} F_{\mu}^{x} \subset \dot{K}_{p,q}^{R} F_{r}, \quad \text{for any } k \in \mathbb{N}, \]  

(23)

see Theorem 2.8. We will prove that

\[ \left\| \sup_{k \in \mathbb{N}_0} 2^{k^3} |H_{k,1,i,j} + H_{k,2}| \right\|_{K_{p,q}^{\mu}} \lesssim \left\| f \right\|_{K_{p,q}^{\mu}}. \]

By Hölder’s inequality and the fact that

\[ |G^{(l)}(t)| \leq \left\| G \right\|_{\text{Lip}_{\mu}}|t|^{\mu-l}, \quad t \in \mathbb{R}, \quad l = 0, \ldots, L - 1 \]

we obtain that

\[
2^{k^3} \left\| H_{k,1,i,j,l} \right\|_{K_{p,q}^{\mu}} \lesssim \left\| \psi_k * f \right\|_{K_{p,q}^{\mu}}^{l-j} \left\| G^{(l)}(\psi_k * f) \right\|_{K_{p,q}^{\mu}}^{l-j} \lesssim \left\| G \right\|_{\text{Lip}_{\mu}} \left\| \psi_k * f \right\|_{K_{p,q}^{\mu}}^{l-j} \left\| G \right\|_{\text{Lip}_{\mu}} \left\| \psi_k * f \right\|_{K_{p,q}^{\mu}}^{l-j} \lesssim \left\| G \right\|_{\text{Lip}_{\mu}} \left\| f \right\|_{K_{p,q}^{\mu}}^{l-j} \left\| f \right\|_{K_{p,q}^{\mu}}^{l-j} \lesssim \left\| G \right\|_{\text{Lip}_{\mu}} \left\| f \right\|_{K_{p,q}^{\mu}}^{l-j} \left\| f \right\|_{K_{p,q}^{\mu}}^{l-j} \lesssim \left\| G \right\|_{\text{Lip}_{\mu}} \left\| f \right\|_{K_{p,q}^{\mu}}^{l-j} \left\| f \right\|_{K_{p,q}^{\mu}}^{l-j} \]

for any \( k \in \mathbb{N}_0 \), where we have used the embeddings (22) and (23), and Theorem 3.9 with the fact that

\[ s > \max \left( 0, \frac{n}{p} + \alpha - \frac{n}{\mu} \right), \]

see (17). Now we estimate \( H_{k,1,0,j} \). Let us recall some properties of our system \( \{ \mathcal{F} \varphi_k \}_{k \in \mathbb{N}_0} \). It holds

\[
\int_{\mathbb{R}^n} \varphi_k(y) dy = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} \varphi_0(y) dy = c \neq 0 \quad k \in \mathbb{N}.
\]

Therefore we need only to estimate \( H_{0,1,0,l}, 0 \leq l \leq L - 1 \). We have, again by (24),

\[ \left\| H_{0,1,0,l} \right\|_{K_{p,q}^{\mu}} \lesssim \left\| G \right\|_{\text{Lip}_{\mu}} \left\| \varphi_0 * f \right\|_{K_{p,q}^{\mu}}^{\mu} \lesssim \left\| G \right\|_{\text{Lip}_{\mu}} \left\| f \right\|_{K_{p,q}^{\mu}}^{\mu}. \]

Thanks to the embeddings
Composition operators on Herz-type Triebel–Lizorkin spaces...

because of

\[
\tilde{s} - \frac{n + \alpha p}{p} = \mu \left( s - \frac{n}{p} - \alpha \right) \quad \text{and} \quad \tilde{p} \mu > p,
\]

we obtain

\[
\|H_{0,1,0,l}\|_{K_{p,q}^{\tilde{p}p}} \lesssim \|G\|_{Lip} \|f\|_{K_{p,q}^{\tilde{p}p}}^\mu.
\]

**Step 3.** Estimate of $H_{k,2}$. We have

\[
\int_{\psi_k \ast f(x)}^{f(y)} (f(y) - h)^{L-1} G^{(L)}(h)dh
\]

\[
= G^{(L)}(\psi_k \ast f(x)) \left( \frac{(f(y) - \psi_k \ast f(x))^L}{L} \right)
\]

\[
+ \int_{\psi_k \ast f(x)}^{f(y)} (f(y) - h)^{L-1} (G^{(L)}(h) - G^{(L)}(\psi_k \ast f(x)))dh
\]

\[
= H_{k,2,1}(x,y) + H_{k,2,2}(x,y).
\]

The estimation of $H_{k,2,1}$ can be obtained by the same arguments given in Step 2. We estimate $H_{k,2,2}$. Using the fact that

\[
|G^{(L)}(t_0) - G^{(L)}(t_1)| \leq \|G\|_{Lip} |t_0 - t_1| \quad t_0, t_1 \in \mathbb{R},
\]

we obtain

\[
|H_{k,2,2}(x,y)| \lesssim \|G\|_{Lip} |\psi_k \ast f(x) - f(y)| \mu, \quad x, y \in \mathbb{R}^n.
\]

Obviously

\[
|\psi_k \ast f(x) - f(y)| \leq |\psi_k \ast f(x) - f(x)| + |f(x) - f(y)|,
\]

which yields that

\[
\int_{\mathbb{R}^n} |\varphi_k(x-y)| |H_{k,2,2}(x,y)|dy \leq S_{k,1}(f)(x) + S_{k,2}(f)(x),
\]

where

\[
S_{k,1}(f)(x) = \|G\|_{Lip} \int_{\mathbb{R}^n} |\varphi_k(x-y)| |\psi_k \ast f(x) - f(x)| \mu dy
\]

\[
\lesssim \|G\|_{Lip} |\psi_k \ast f(x) - f(x)| \mu
\]

and
First we estimate $S_{k,1}(f)$. Observe that

$$f - \psi_k * f = \sum_{i=k+1}^{\infty} \varphi_i * f, \quad k \in \mathbb{N}_0.$$  

Therefore

$$\left\| \sup_{k \in \mathbb{N}_0} (2^k S_{k,1}(f)) \right\|_{K_{\mu,q}^f} \lesssim \left\| G \right\|_{\text{Lip}_\mu} \sup_{k \in \mathbb{N}_0} 2^{\frac{k^2}{2}} \left( \sum_{i=k+1}^{\infty} \left| \varphi_i * f \right| \right) \left\| \mu \right\|_{K_{\mu,q}^f}^{\frac{\mu}{\rho}}$$

$$\lesssim \left\| G \right\|_{\text{Lip}_\mu} \sup_{k \in \mathbb{N}_0} 2^{\frac{k^2}{2}} \left| \varphi_k * f \right| \left\| \mu \right\|_{K_{\mu,q}^f}^{\frac{\mu}{\rho}}$$

$$\lesssim \left\| G \right\|_{\text{Lip}_\mu} \left\| f \right\|_{K_{\mu,q}^f}^{\frac{\mu}{\rho}}$$

where we used Lemma 3.1. We conclude our desired estimate by the embeddings (24).

Now we estimate $S_{k,2}(f)$. Since $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$, this yields

$$|\varphi_k(z)| \lesssim \eta_{2^k,M}(z), \quad z \in \mathbb{R}^n,$$

where $M$ is an arbitrary positive real number and the implicit constant is independent of $z$ and $k \in \mathbb{N}_0$. By means of this inequality we find

$$\int_{\mathbb{R}^n} |\varphi_k(-z)||f(x) - f(x + z)|^\mu dz$$

$$\lesssim \int_{\mathbb{R}^n} |\varphi_k(-z)||f(x) - f(x + z)|^\mu dz$$

$$+ \sum_{l=0}^{\infty} \int_{\mathbb{R}^n} |\varphi_k(-z)||f(x) - f(x + z)|^\mu dz$$

$$\lesssim 2^k \sum_{l=0}^{\infty} 2^{-lM} \int_{\mathbb{R}^n} |f(x) - f(x + z)|^\mu dz$$

$$\lesssim 2^k \sum_{l=0}^{\infty} 2^{-lM} \mu_{k-l}(f)(x),$$

where the implicit constant is independent of $x$ and $k$. Let $d = \min(1, \tilde{p})$. Taking $M$ large enough such that $M - n - \tilde{s} - 1 > 0$ and using Lemma 3.3, we obtain
\[
\left\| \sup_{k \in \mathbb{N}_0} 2^{k \beta} |S_{k,2}(f)| \right\|_{K_{\alpha/4}^{\frac{d}{\beta}}}^{d_{ap}} 
\leq \left\| G \right\|_{\text{Lip}}^{d} \sum_{l=0}^{\infty} 2^{-lMd} \left\| \sup_{k \in \mathbb{N}_0} \left( 2^{k(n+3)} I_{k-l}^{\beta}(f) \right) \right\|_{K_{\alpha/4}^{d_{ap}}}^{d_{ap}} 
\leq \left\| G \right\|_{\text{Lip}}^{d} \sum_{l=0}^{\infty} 2^{-l(M-n-3)d} \left\| \sup_{i \beta-l} \left( 2^{i(n+3)} I_{i}^{\mu}(f) \right) \right\|_{K_{\alpha/4}^{d_{ap}}}^{d_{ap}} 
\leq \left\| G \right\|_{\text{Lip}}^{d} \left\| f \right\|_{K_{\alpha}^{d_{ap}} F_{\beta}^{s}}^{d_{ap}}.
\]

Our desired estimate follows by the embedding (25). The proof is complete. \qed

From Theorem 3.15 and the fact that \( G(t) = |f|^{\mu} \in \text{Lip}_\mu, \mu > 1 \), we immediately arrive at the following result.

**Corollary 3.16** Under the hypotheses of Theorem 3.15, we have

\[
\left\| |f|^{\mu} \right\|_{K_{\alpha}^{d_{ap}} F_{\beta}^{s}}^{d_{ap}} \leq c \left\| G \right\|_{\text{Lip}_\mu} \left\| f \right\|_{K_{\alpha}^{d_{ap}} F_{\beta}^{s}}^{\mu}
\]

holds for any \( f \in K_{\alpha/4}^{d} \mathbb{L}^{s} \).

**Remark 3.17** The valued \( s_\mu \) in Theorem 3.15 is optimal. Indeed, we put

\[
f_\kappa(x) = \theta(x)|x|^\kappa,
\]

where \( \kappa > 0 \) and \( \theta \) is a smooth cut-off function with \( \text{supp}\theta \subset \{ x : |x| \leq \theta \} \), \( \theta > 0 \) sufficiently small. As in [21] we can prove that \( f_\kappa \in K_{\alpha/4}^{d} F_{\beta}^{s} \) if and only if \( s < \frac{n}{p} + \alpha + \kappa \). Let \( G(x) = |x|^{\mu}, \mu > 1, x \in \mathbb{R} \). Then

\[
G(f_\kappa) \notin K_{\alpha/4}^{d} F_{\beta}^{s'}
\]

if \( d \geq \frac{n}{p} + \alpha + \kappa \mu > s_\mu \).

**Theorem 3.18** Let \( 0 < p, q < \infty, 0 < \beta \leq \infty, \mu > 1, \alpha \geq 0 \) and

\[
\max \left( 0, \frac{n}{p} + \alpha - n \right) < s < \mu.
\]

Let \( G \in \text{Lip}_\mu \). Then

\[
\left\| G(f) \right\|_{K_{\alpha/4}^{d} F_{\beta}^{s}}^{d_{ap}} \leq c \left\| G \right\|_{\text{Lip}_\mu} \left\| f \right\|_{K_{\alpha/4}^{d} F_{\beta}^{s}}^{d_{ap}} \left\| f \right\|_{\infty}^{\mu-1}
\]

holds for any \( f \in K_{\alpha/4}^{d} F_{\beta}^{s} \cap \mathbb{L}^{\infty} \).

**Proof** We employ the notation of Theorem 3.15. We will prove that
\[
\left\| \sup_{k \in \mathbb{N}_0} 2^{ks} \left( H_{k,1,j,l} + H_{k,2} \right) \right\|_{K_{p,q}^s} \lesssim \left\| G \right\|_{\text{Lip}_\mu} \left\| f \right\|_{K_{p,q}^s} \| f \|_\infty^{\mu-1}.
\]

Thanks to (24) and Theorem 3.10 it follows
\[
2^{ks} \| H_{k,1,j,l} \|_{K_{p,q}^s} \lesssim \left\| \left| \psi_k * f \right| \right\|_{L^p(J)} \left( \left\| \psi_k * f \right\|_\infty \right)^{\mu-1} \lesssim \left\| G \right\|_{\text{Lip}_\mu} \left\| f \right\|_{K_{p,q}^s} \| f \|_\infty^{\mu-1} \lesssim \left\| G \right\|_{\text{Lip}_\mu} \left\| f \right\|_{K_{p,q}^s} \| f \|_\infty^{\mu-1}.
\]

where we used \( \| \psi_k * f \|_\infty \lesssim \| f \|_\infty \), by Young’s inequality. Now
\[
\| H_{0,1,0,j} \|_{K_{p,q}^s} \lesssim \left\| G \right\|_{\text{Lip}_\mu} \left\| \varphi_0 * f \right\|_\infty \lesssim \left\| G \right\|_{\text{Lip}_\mu} \left\| f \right\|_{K_{p,q}^s} \| f \|_\infty^{\mu-1} \lesssim \left\| G \right\|_{\text{Lip}_\mu} \left\| f \right\|_{K_{p,q}^s} \| f \|_\infty^{\mu-1}.
\]

Observe that
\[
S_{k,1}(f)(x) \lesssim \left\| G \right\|_{\text{Lip}_\mu} \left\| \psi_k * f(x) - f(x) \right\|_\mu \lesssim \left\| G \right\|_{\text{Lip}_\mu} \left\| f \right\|_\infty^{\mu-1} \left\| \psi_k * f(x) - f(x) \right\|.
\]

Then
\[
\left\| \sup_{k \in \mathbb{N}_0} \left( 2^{ks} S_{k,1}(f) \right) \right\|_{K_{p,q}^s} \lesssim \left\| G \right\|_{\text{Lip}_\mu} \left\| f \right\|_{K_{p,q}^s} \left( \sum_{k \in \mathbb{N}_0} \left\| \varphi_k * f \right\|_\infty \right)^{\mu-1} \lesssim \left\| G \right\|_{\text{Lip}_\mu} \left\| f \right\|_{K_{p,q}^s} \left( \sum_{k \in \mathbb{N}_0} \left\| \varphi_k * f \right\|_{K_{p,q}^s} \right)^{\mu-1} \lesssim \left\| G \right\|_{\text{Lip}_\mu} \left\| f \right\|_{K_{p,q}^s} \left( \sum_{k \in \mathbb{N}_0} \left\| \varphi_k * f \right\|_{K_{p,q}^s} \right)^{\mu-1}.
\]

by Lemma 3.1. Using Lemma 3.3, we obtain
\[
\left\| \sup_{k \in \mathbb{N}_0} 2^{ks} |S_{k,2}(f)| \right\|_{K_{p,q}^s} \lesssim \left\| G \right\|_{\text{Lip}_\mu} \left\| f \right\|_{K_{p,q}^s} \left( \sum_{k \in \mathbb{N}_0} \sum_{l=0}^\infty 2^{-lMd} \left\| \sup_{k \in \mathbb{N}_0} \left( 2^{k(n+s)} \left| 1_{I_k} \right| (f) \right) \right\|_{K_{p,q}^s} \right)^{\mu-1} \lesssim \left\| G \right\|_{\text{Lip}_\mu} \left\| f \right\|_{K_{p,q}^s} \left( \sum_{k \in \mathbb{N}_0} \sum_{l=0}^\infty 2^{-l(M-n-\delta)d} \left\| \sup_{l>l} \left( 2^{(l+n+s)} \left| 1_{I_k} \right| (f) \right) \right\|_{K_{p,q}^s} \right)^{\mu-1} \lesssim \left\| G \right\|_{\text{Lip}_\mu} \left\| f \right\|_{K_{p,q}^s} \left( \sum_{k \in \mathbb{N}_0} \sum_{l=0}^\infty 2^{-l(M-n-\delta)d} \right)^{\mu-1} \lesssim \left\| G \right\|_{\text{Lip}_\mu} \left\| f \right\|_{K_{p,q}^s} \left( \sum_{k \in \mathbb{N}_0} \sum_{l=0}^\infty 2^{-l(M-n-\delta)d} \right)^{\mu-1}.
\]

The desired estimate follows by the fact that
\[
\|f\|_{\mathcal{K}_{p,q}^{\alpha} F_p^s}^\mu \lesssim \|f\|_\infty^{\mu-1} \|f\|_{\mathcal{K}_{p,q}^{\alpha} F_p^s}^\mu.
\]

The proof is completed. \(\square\)

Now we present some limit case.

**Theorem 3.19** Let \(0 < p, q < \infty, \alpha \geq 0, \mu \geq \frac{\frac{n}{q} + \alpha}{\frac{n}{q} + \alpha + 1} \) and
\[
\max \left( 1, \frac{n}{p} + \alpha - n \right) < \mu < \frac{n}{p} + \alpha. \tag{26}
\]

Let \(G \in \text{Lip}_\mu\) and
\[
s = 1 + \frac{\mu - 1}{\mu} \left( \frac{n}{p} + \alpha \right). \tag{27}
\]

Then
\[
\|G(f)\|_{\mathcal{K}_{p,q}^{\alpha} F_p^s}^\mu \leq c \|G\|_{\text{Lip}_\mu} \|f\|_{\mathcal{K}_{p,q}^{\alpha} F_p^s}^\mu
\]
holds for any \(f \in \mathcal{K}_{p,q}^{\alpha} F_p^s\).

**Proof** We employ the notation of the proof of Theorem 3.15. From (26) and (27), we obtain \(\mu < s < \frac{n}{p} + \alpha\). With the help of (26) we get (18), \(\alpha \mu > 1\) and \(\alpha_1 \mu < n - \frac{n\mu}{b}\).

Consequently the embedding (17) holds. We have \(s_y = \mu\) and we will take \(s = s_y\) and \(\bar{p} = p\). The proof is very similar as in Theorem 3.15, but here we use Lemma 3.4 instead of Lemma 3.3. \(\square\)

From Theorem 3.19 and the fact that \(G(t) = |f|^\mu \in \text{Lip}_\mu, \mu > 1\), we get the following result:

**Corollary 3.20** Under the hypotheses of Theorem 3.19, we have
\[
\||f|^\mu\|_{\mathcal{K}_{p,q}^{\alpha} F_p^s}^\mu \leq c \|G\|_{\text{Lip}_\mu} \|f\|_{\mathcal{K}_{p,q}^{\alpha} F_p^s}^\mu
\]
holds for any \(f \in \mathcal{K}_{p,q}^{\alpha} F_p^s\).

**Remark 3.21** Corresponding statements to Theorems 3.15, 3.18 and 3.19 were proved in [45] and [47, Theorem 6] with \(a = 0\), see also [44].
4 Semilinear parabolic equations in Herz–Triebel–Lizorkin spaces

4.1 Heat kernel estimates

Let $t > 0, x \in \mathbb{R}^n$ and $f \in S'(\mathbb{R}^n)$. We put

$$e^{t\Delta} f(x) = \mathcal{F}^{-1}(\exp(-t|\xi|^2)\mathcal{F} f)(x).$$

Recall that

$$g(x) = \mathcal{F}^{-1}(\exp(-t|\xi|^2))(x) = (4\pi t)^{-\frac{n}{2}} \exp(-4t^{-1}|x|^2), \quad x \in \mathbb{R}^n.$$

We will give some key estimates of heat kernel $e^{t\Delta}$ needed in the proofs of the main statements. First, we estimate the heat kernel $e^{t\Delta}$ in Herz-type Triebel–Lizorkin spaces. We follows the arguments of [1] and [57]. We need the so called molecular and wavelet characterizations of Herz-type Triebel–Lizorkin spaces.

**Definition 4.1** Let $K, L \in \mathbb{N}_0$ and $M > 0$. A $K$-times continuous differentiable function $\mu$ is called a $[K, L, M]$-molecule concentrated in $Q_{j,m}$ if for some $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$

$$|D^\gamma \mu(x)| \leq 2^{|\gamma|j}(1 + 2^j |x - 2^{-j}m|)^{-M}, \quad 0 \leq |\gamma| \leq K$$

and

$$\int_{\mathbb{R}^n} x^\gamma \mu(x) dx = 0 \quad \text{if} \quad 0 \leq |\gamma| < L, j \in \mathbb{N}.$$

Notice that for $L = 0$ or $j = 0$ there are no moment conditions on $\mu$. If $\mu$ is a molecule concentrated in $Q_{j,m}$, then it is denoted $\mu_{j,m}$.

We introduce the sequence spaces associated with the function spaces $\dot{K}^a_{p,q} F^s$. Let $\alpha, s \in \mathbb{R}, 0 < p, q < \infty$ and $0 < \beta \leq \infty$. We set

$$\dot{K}^a_{p,q} F^s = \{ \lambda = \{ \lambda_{j,m} \}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \subset \mathbb{C} : \| \lambda \|_{\dot{K}^a_{p,q} F^s} < \infty \},$$

where

$$\| \lambda \|_{\dot{K}^a_{p,q} F^s} = \left\| \left( \sum_{j=0}^\infty \sum_{m \in \mathbb{Z}^n} 2^{j\beta} |\lambda_{j,m}|^\beta \chi_{j,m} \right)^{1/\beta} \right\|_{\dot{K}^a_{p,q}}.$$ Now we come to the molecule decomposition theorem for $\dot{K}^a_{p,q} F^s$ spaces. For the proof, see [24] and [63].

**Theorem 4.2** Let $s \in \mathbb{R}, 0 < p, q < \infty, 0 < \beta \leq \infty$ and $\alpha > -\frac{n}{p}$. Furthermore, let $K, L \in \mathbb{N}_0$ and let $M > 0$ with

$$L > \sigma_{p,\beta} - s, \quad K > s$$

and $M$ large enough.
If \( a_{j,m} \) are \( [K, L, M] \)-molecules concentrated in \( Q_{j,m} \) and 

\[
\lambda = \{ \lambda_{j,m} \}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathcal{K}^a_{p,q,f,s} \,,
\]

then the sum 

\[
f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} a_{j,m}
\]

(28)

converges in \( \mathcal{S}'(\mathbb{R}^n) \) and 

\[
\|f\|_{K^a_{p,q,F,s}} \lesssim \|\lambda\|_{K^a_{p,q,F,s}}.
\]

Let \( J \in \mathbb{N} \) and \( \psi_F, \psi_M \in C^J(\mathbb{R}) \) be real-valued compactly supported Daubechies wavelets with

\[
\mathcal{F}\psi_F(0) = (2\pi)^{-\frac{1}{2}}, \quad \int_{\mathbb{R}} x_l^j \psi_M(x) \, dx = 0, \quad l \in \{0, \ldots, J-1\}
\]

and

\[
\|\psi_F\|_2 = \|\psi_M\|_2 = 1.
\]

We have that 

\[
\{ \psi_F(x-m), 2^j \psi_M(2^j x-m) \}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n}
\]

is an orthonormal basis in \( L^2(\mathbb{R}) \). This orthonormal basis can be generalized to the \( \mathbb{R}^n \) by the usual multiresolution procedure. Let

\[
G = \{ G_1, \ldots, G_n \} \in G^0 = \{ F, M \}^n
\]

which means that \( G_r \) is either \( F \) or \( M \). Let

\[
G = \{ G_1, \ldots, G_n \} \in G^j = \{ F, M \}^{n^*}, \quad j \in \mathbb{N},
\]

where indicates that at least one of the components of \( G \) must be an \( M \). Let

\[
\Psi^j_{G,m}(x) = 2^{\frac{j}{2}} \prod_{r=1}^{n} \psi_{G_r}(2^j x_r - m_r), \quad G \in G^j, m \in \mathbb{Z}^n, x \in \mathbb{R}^n, j \in \mathbb{N}_0.
\]

Then

\[
\Psi = \{ \Psi^j_{G,m} : \quad j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^n \}
\]

is an orthonormal basis in \( L^2(\mathbb{R}^n) \).

Let \( \alpha, \beta, \gamma, \delta \in \mathbb{R}, 0 < \gamma, q < \infty \) and \( 0 < \beta, \delta < \infty \). We set

\[
\mathcal{K}^a_{p,q,F,s} = \{ \lambda = \{\lambda_{j,m}^G\}_{j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^n} \subset \mathbb{C} : \|\lambda\|_{\mathcal{K}^a_{p,q,F,s}} < \infty \},
\]

\( \mathbb{B} \) Birkhäuser
where
\[ \| \hat{\lambda} \|_{K^s_{p,q} f^s} = \left\| \left( \sum_{j=0}^{\infty} \sum_{G \in \mathcal{G}} \sum_{m \in \mathbb{Z}^n} 2^{j\beta} |\lambda_{j,m}^G f| \right)^{1/\beta} \right\|_{K^s_{p,q}}. \]

**Theorem 4.3** Let \( \alpha, s \in \mathbb{R}, 0 < p, q < \infty, 0 < \beta \leq \infty \) and \( \alpha > -\frac{n}{p} \). Let \( \{\Psi^j_{G,m}\} \) be the wavelet system with \( J > \max(\sigma_{p,\beta} - s, s) \). Let \( f \in S'(\mathbb{R}^n) \). Then \( f \in K^s_{p,q} f^s \) if and only if
\[ f = \sum_{j=0}^{\infty} \sum_{G \in \mathcal{G}} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m}^G 2^{-j\beta} \Psi^j_{G,m} \quad \lambda \in K^s_{p,q} f^s \]
with unconditional convergence in \( S'(\mathbb{R}^n) \) and in any space \( K^s_{p,q} f^s \) with \( \sigma < s \). The representation (29) is unique. We have
\[ \lambda_{j,m}^G = \lambda_{j,m}^G(f) = 2^{j/2} \langle f, \Psi^j_{G,m} \rangle \]
and
\[ I : f \mapsto \{ \lambda_{j,m}^G(f) \} \]
is an isomorphic map from \( K^s_{p,q} f^s \) into \( K^s_{p,q} f^s \). In particular, it holds
\[ \|f\|_{K^s_{p,q} f^s} \approx \| \hat{\lambda} \|_{K^s_{p,q} f^s}. \]

For the proof, see again, [24, 63]. To estimate the heat kernel \( e^{t\Delta} \) in Herz-type Triebel–Lizorkin spaces, we need the following lemma.

**Lemma 4.4** Let \( s > 0, \theta \geq 0, 0 < t < T, 0 < p, q < \infty, 0 < \beta \leq \infty \) and \( \alpha > -\frac{n}{p} \). We set
\[ b^j_{G,m}(x,t) = 2^{-j/2} e^{t\Delta} \Psi^j_{G,m}(x). \]
Then there exists \( C > 0 \) such that the functions
\[ b^j_{G,m}(x,t) = C 2^{\theta} t^{\frac{\alpha}{2}} b^j_{G,m}(x,t), \quad j \in \mathbb{N}_0, G \in G^*\m, m \in \mathbb{Z}^n \]
\([K,L,M]\)-molecules for any fixed \( t \) with \( 2^{j/2} \geq 1 \), provided that \( L \leq J, K \leq J, L + n - 1 < M < J + n - \theta \) and \( \theta \leq J = L + 1 \). Assume that
\[ J > \theta + \max(s, \sigma_{p,\beta}). \]
Then, the numbers \( K, L, M \) can be chosen such that for some \( C > 0 \) and any \( t \) with \( 2^{j/2} \geq 1 \), such that (30) are molecules for \( K^s_{p,q} f^{s+\theta}_{p,q} \).
Proof We use the arguments of \cite[Proposition 3.1]{1} and we need only to prove the second part of the Lemma. Let $L = [\sigma_{p,\beta}] + 1$, which yields that $L > \sigma_{p,\beta} - s - \theta$. Since $J > \sigma_{p,\beta}$ it follows that $J \geq L$. Hence

$$
\int_{\mathbb{R}^n} x^\nu b^j_{G,m}(x,t) \, dx = 0, \quad 0 \leq |\nu| < L, \, j \in \mathbb{N}.
$$

Let $M$ large enough be such that $\sigma_{p,\beta} + n < M < J + n - \theta$. Then $M > L + n - 1$ and $\theta < J - \sigma_{p,\beta} < J - L + 1$. Regarding the derivatives of $b^j_{G,m}(x,t)$ we claim $s + \theta < K$.

We present one of the main tools used in this section.

**Lemma 4.5** Let $s > 0, \theta \geq 0, 0 < t < T, 1 < p, q < \infty, 1 < \beta \leq \infty$ and $-\frac{n}{p} < \alpha < n - \frac{n}{p}$. Then there exists a positive constant $C(T) > 0$ independent of $t$ such that

$$
\| e^{t \Delta} f \|_{K^a_{p,q} \tilde{F}^s_{\alpha,\beta}} \leq C(T) t^{-\frac{n}{2}} \| f \|_{K^a_{p,q} \tilde{F}^s_{\alpha,\beta}}
$$

for any $f \in K^a_{p,q} \tilde{F}^s_{\alpha,\beta}$.

**Proof** Let $k \in \mathbb{N}$ be such that $2^{-2k} < \frac{t}{T} \leq 2^{-2(k-1)}$. From Theorem 4.3 we have $f = f_{1,k} + f_{2,k}$, with

$$
f_{1,k} = \sum_{j=0}^{k-1} \sum_{G \in G'} \sum_{m \in \mathbb{Z}^n} \lambda^G_{j,m} 2^{-j^2/2} \Psi^j_{G,m}
$$

and

$$
f_{2,k} = \sum_{j=k}^{\infty} \sum_{G \in G'} \sum_{m \in \mathbb{Z}^n} \lambda^G_{j,m} 2^{-j^2/2} \Psi^j_{G,m},
$$

where $\lambda \in K^a_{p,q} \tilde{F}^s_{\alpha,\beta}$.

Estimate of $f_{1,k}$. We claim that

$$
|e^{t \Delta} (\varphi_j * f_{1,k})(x)| \lesssim \mathcal{M}(\varphi_j * f_{1,k})(x), \quad x \in \mathbb{R}^n, j \in \mathbb{N}_0, \tag{31}
$$

where the implicit constant is independent of $x, k, j$ and $t$. Using the estimate (31) and Lemma 3.2 we obtain
Obviously, the first term of (34) is bounded by \( c \mathcal{M}(\varphi_j * f_{1,\lambda})(x) \). We have
$\eta_{r,t^+,m}^L X_{B^+_{(x,t^+,\frac{1}{2})}} * |\varphi_j * f_{1,k}|(x)$

$= \sum_{i=1}^{\infty} \eta_{r,t^+,m}^L X_{B^+_{(x,t^+,\frac{1}{2})}} * |\varphi_j * f_{1,k}|(x)$

$\leq \sum_{i=1}^{\infty} 2^{-im} \eta_{r,t^+,m}^L X_{B^+_{(x,t^+,\frac{1}{2})}} * |\varphi_j * f_{1,k}|(x)$

$\lesssim \mathcal{M}(\varphi_j * f_{1,k})(x) \sum_{i=1}^{\infty} 2^{i(n-m)}$.

Estimate of $f_{2,k}$. If $j \geq k$, then $2^{l} (\frac{t}{T})^{t^2} > 2^{j-k} \geq 1$, which yields that

$e^{t^2} f_{2,k} = \sum_{j=k}^{\infty} \sum_{G \in G_j} \sum_{m \in \mathbb{Z}^n} 2^{-j^2} \left( \frac{t}{T} \right)^{t^2} \lambda_{j,m} G^2 2^{-j^2} 2^{t^2} \left( \frac{t}{T} \right)^{j^2} e^{j^2} \psi_{G,m}^j$

$= \sum_{j=k}^{\infty} \sum_{G \in G_j} \sum_{m \in \mathbb{Z}^n} \mu_{j,m} G^2 \psi_{G,m}^j$.

where

$C \mu_{j,m} = 2^{-j^2} \left( \frac{t}{T} \right)^{t^2} \lambda_{j,m} G^2 \psi_{G,m}^j$ and $\psi_{G,m}^j = 2^{-j^2} 2^{t^2} \left( \frac{t}{T} \right)^{j^2} \psi_{G,m}^j$.

and $C$ as in (29). Let

$\mu^* = \left\{ 2^{-j^2} \left( \frac{t}{T} \right)^{t^2} \lambda_{j,m} G^2, j \in \mathbb{N}_0, G \in G^*, m \in \mathbb{Z}^n \right\}$.

Again, from Theorem 4.3 we obtain

$\|e^{j^2} f_{2,k}\|_{K^2_{p,\rho}} \lesssim \|\mu^*\|_{K^2_{p,\rho}}$

$= c \left( \sum_{j=k}^{\infty} \sum_{G \in G_j} \sum_{m \in \mathbb{Z}^n} 2^{-j^2} \left| \lambda_{j,m} G^2 \right| \right)^{1/\beta}$

$\lesssim c \left( \sum_{j=k}^{\infty} \sum_{G \in G_j} \sum_{m \in \mathbb{Z}^n} 2^{-j^2} \left| \lambda_{j,m} G^2 \right| \right)^{1/\beta}$

and this completes the proof.

The following lemmas was proved in [22].
Lemma 4.6 Let $\alpha_1, \alpha_2 \in \mathbb{R}, 0 < t < \infty$ and $1 < p, \kappa, q, r < \infty$. We suppose that $1 < q \leq p < \infty$ and $-\frac{n}{p} < \alpha_1 \leq \alpha_2 < n - \frac{n}{q}$. Then there exists a positive constant $C > 0$ independent of $t$ such that

$$\|e^{\Delta f}\|_{\dot{K}^\alpha_{p,\kappa}} \leq Ct^{-\frac{1}{2}}(\frac{q}{p} + a_2 - a_1) \|f\|_{\dot{K}^\delta_{q,\delta}}$$

for any $f \in \dot{K}^\alpha_{q,\delta}$, where

$$\delta = \begin{cases} \ k, \text{ if } \alpha_2 = \alpha_1, \\ \kappa, \text{ if } \alpha_2 > \alpha_1. \end{cases}$$

4.2 The results and their proofs

We look for mild solutions of (1) i.e. for solutions of integral equation

$$u(t, x) = e^{\Delta} u_0(x) + \int_0^t e^{(t-\tau)\Delta} G(u)(\tau, x) d\tau. \quad (35)$$

We set

$$F(u)(t, x) = \int_0^t e^{(t-\tau)\Delta} G(u)(\tau, x) d\tau.$$

We study Cauchy problem for semilinear parabolic equations (1) with initially data in Herz-type Triebel–Lizorkin spaces and will assume that $G$ belongs to $G \in \text{Lip}_\mu$. We set

$$\tilde{s} = \frac{n}{p} + \alpha - \frac{2}{\mu - 1} \quad \text{and} \quad \theta = \frac{s - \tilde{s}}{2}.$$

We now state the existence of mild solutions of (35).

**Theorem 4.7** Let $1 < p, q < \infty$, $1 < \beta \leq \infty$, $\mu > 1$, $0 \leq \alpha < n - \frac{n}{p}$, $s \geq \frac{n}{p} - \frac{n}{q}$ and

$$0 < s < \frac{n}{p} + \alpha.$$

Let $G \in \text{Lip}_\mu$ and

$$0 < \sigma < \mu.$$

(i) For all initial data $u_0$ in $\dot{K}^\alpha_{p,q} F^s_\beta$ with $s > \tilde{s}$, there exists a maximal solution $u$ to (35) in $C([0, T_0), \dot{K}^\alpha_{p,q} F^s_\beta)$ with $T_0 \geq C\|u_0\|_{\dot{K}^\frac{1}{2}_{p,\delta} F^s_\delta}$.

(ii) Let $\theta < 2\theta(\mu - 1) \text{ or } \theta = 2\theta(\mu - 1)$, $s > 1$ and $G \in \text{Lips}_0$ with
We have

\[ u - e^{t\Delta}u_0 \in C([0, T_0), \tilde{K}^\alpha_{p,q} F^{s + \beta}_\beta). \]

**Proof** We will do the proof into two steps. Our arguments are based on [43].

**Step 1.** We prove part (i) of the theorem.

**Substep 1.1.** In this step we prove the existence of a solution to (35). Recall that

\[ F(u)(t, x) = \int_0^t e^{(t-\tau)\Delta} G(u)(\tau, x) d\tau \quad \text{and} \quad \frac{1}{\tilde{p}} = \frac{1}{p} + \frac{\alpha - s}{n}. \]

For simplicity, we consider the spaces

\[ Y = C([0, T), \tilde{K}^\alpha_{p,q} F^s_\beta) \quad \text{and} \quad X = C([0, T), \tilde{K}^0_{p,q}). \]

Further, we consider the sequence of functions

\[ u^0 = e^{t\Delta}u_0 \quad \text{and} \quad u^{i+1} = u^0 + F(u^i), \quad j \in \mathbb{N}. \] (36)

From Lemma 4.6 and Sobolev embedding \( K^\alpha_{p,q} F^s_\beta \hookrightarrow \tilde{K}^0_{p,q} \), see Theorem 2.8, we deduce that

\[ \|u^0\|_{\tilde{K}^\alpha_{p,q}} \lesssim \|u_0\|_{\tilde{K}^\alpha_{p,q}} \lesssim \|u_0\|_{K^\alpha_{p,q} F^s_\beta}. \] (37)

Let \( u, v \in X \). Since \( \frac{\tilde{p}}{\mu} > 1 \), again, by Lemma 4.6 we obtain

\[ \|F(u)(t, \cdot) - F(v)(t, \cdot)\|_{\tilde{K}^\alpha_{p,q}} \]

\[ \leq \int_0^t \|e^{(t-\tau)\Delta} (G(u)(\tau, \cdot) - G(v)(\tau, \cdot))\|_{\tilde{K}^\alpha_{p,q}} d\tau \]

\[ \leq \int_0^t \|e^{(t-\tau)\Delta} (G(u)(\tau, \cdot) - G(v)(\tau, \cdot))\|_{\tilde{K}^\alpha_{p,q}} d\tau \]

\[ \leq C \int_0^t (t - \tau)^{-\frac{n(\mu-1)}{2p}} \|G(u)(\tau, \cdot) - G(v)(\tau, \cdot)\|_{\tilde{K}^\alpha_{p,q}} d\tau, \] (38)

where the second estimate follows by the embedding \( \tilde{K}^0_{p,q} \hookrightarrow \tilde{K}^0_{p,q} \) and the positive constant \( C \) is independent of \( t \). Observe that

\[ |G(u)(\tau, \cdot) - G(v)(\tau, \cdot)| \leq |u - v|(|u|^\mu - 1 + |v|^\mu - 1) \]

and
\[
\frac{\mu}{\tilde{p}} = \frac{1}{\tilde{p}} + \frac{\mu - 1}{\tilde{p}}, \quad \frac{\mu}{q} = \frac{1}{q} + \frac{\mu - 1}{q}.
\]

Therefore, by H"older's inequality
\[
\|G(u)(\tau, \cdot) - G(v)(\tau, \cdot)\|_{K^0_{\tilde{p}, \tilde{q}}} \\
\leq \|u(\tau, \cdot) - v(\tau, \cdot)\|_{K^0_{\tilde{p}, \tilde{q}}} \left( \|u(\tau, \cdot)\|_{K^0_{\tilde{p}, \tilde{q}}}^{\mu - 1} + \|v(\tau, \cdot)\|_{K^0_{\tilde{p}, \tilde{q}}}^{\mu - 1} \right). \tag{39}
\]

Substituting (39) into (38) and then using
\[
\frac{n(\mu - 1)}{2\tilde{p}} = \frac{\mu - 1}{2} \left( \frac{n}{p} + \alpha - s \right) = 1 - \frac{(\mu - 1)(s - \tilde{s})}{2},
\]
this gives
\[
\|F(u) - F(v)\|_X \leq CT \frac{(\mu - 1)(s - \tilde{s})}{2} \|u - v\|_X \left( \|u\|_X^{\mu - 1} + \|v\|_X^{\mu - 1} \right). \tag{40}
\]

In view of (36), (40) and (37), we obtain
\[
\|u^{i+1}\|_X \leq \|u^0\|_X + \|F(u^i)\|_X \\
\leq \|u_0\|_{K^0_{\tilde{p}, \tilde{q}}F^i_{\tilde{p}}} + CT \frac{(\mu - 1)(s - \tilde{s})}{2} \|u^i\|_X
\]
and
\[
\|u^{i+1} - u^i\|_X \leq CT \frac{(\mu - 1)(s - \tilde{s})}{2} \|u^i - u^{i-1}\|_X \left( \|u^i\|_X^{\mu - 1} + \|u^{i-1}\|_X^{\mu - 1} \right).
\]

Let
\[
\bar{F} = \left( \frac{1}{C} \right)^{\frac{2}{(p-1)(r-1)}} \left( \frac{\mu^{-1}}{\mu^{-\mu}} \right)^{\frac{2}{s-1}}.
\]

As in [22, 35], the fixed point argument shows that if
\[
T < \bar{F} 2^{\frac{2}{(p-1)(r-1)}} \left( 1 - \frac{1}{\mu} \right)^{\mu - 1} \|u_0\|_{K^0_{\tilde{p}, \tilde{q}}F^0_{\tilde{p}}}, \tag{41}
\]
then the sequence \( \{u^i\}_i \) converges strongly in \( X \) to a limit \( u \) which is a solution of the integral equation (34).

**Substep 1.2.** In this step we prove that the solution of the integral equation (34) belongs to \( Y \). We employ the notation of Substep 1.1. We claim that
\[
\|u^{i+1}\|_Y \leq \|u_0\|_{K^0_{\tilde{p}, \tilde{q}}F^i_{\tilde{p}}} + CT \frac{(\mu - 1)(s - \tilde{s})}{2} \|u^i\|_Y. \tag{42}
\]

From (41) and (42), the sequence \( \{u^i\}_i \) is bounded. Then we can extract a subsequence \( \{u^j\}_i \) converges weakly to \( \tilde{u} \in Y \). From Step 1, \( \{u^i\}_i \) converges weakly to
Observe that two solutions for the same initial data \( u, v \) in the same cases lead to (35). Let \( u, v \in Y \) be two solutions for the same initial data \( u_0 \). Using the fact that \( u \) and \( v \) solve (35), we obtain

\[
\| u - v \|_X = \| F(u) - F(v) \|_X \leq 2CT^{\frac{\mu - 1 + s - \bar{s}}{2}} A^{\mu - 1} \| u - v \|_X
\]

where

\[
A = \sup_{t \in [0, T]} (\| u(t) \|_{K^{\phi_0}_{\mu q}, p}, \| v(t) \|_{K^{\phi_0}_{\mu q}, p}).
\]

Taking \( T \) small enough such that

\[
2CT^{\frac{\mu - 1 + s - \bar{s}}{2}} A^{\mu - 1} < \frac{1}{2}
\]

we obtain \( u = v \) on \([0, T]\). We iterate this to prove that \( T_0(u) = T_0(v) \) and \( u = v \) on \([0, T_0(u)]\), which ensures the uniqueness of the solution of (35).

**Step 2.** We prove part (ii) of the theorem. We split our considerations into the cases \( \theta < 2\theta(\mu - 1) \) and \( \theta = 2\theta(\mu - 1) \).

- **Case 1.** \( \theta < 2\theta(\mu - 1) \). Let \( u \in Y \) be a solution of (35) with initial data \( u_0 \). Observe that \( 2\theta(\mu - 1) = (\mu - 1)(s - \bar{s}) = 2 - s + s_\mu \). Thanks to Lemma 4.5 and Theorem 3.15 it follows...
\[
\|u - e^{t\Delta}u_0\|_{K_{p,q}^{\alpha+i\theta}} \leq \int_0^t \| e^{(t-\tau)\Delta}(G(u)(\tau, \cdot)) \|_{K_{p,q}^{\alpha+i\theta}} d\tau \\
\leq C \int_0^t (t-\tau)^{-\frac{\alpha-s\mu}{2}} \| G(u)(\tau, \cdot) \|_{K_{p,q}^{\alpha+i\theta}} d\tau \\
\leq CT_0^{1-\frac{\alpha-s\mu}{2}} \| u \|_{^\mu_Y},
\]

since \(1 - \frac{\alpha-s\mu}{2} = -\frac{\theta}{2} + \frac{(\mu-1)(s-3)}{2} > 0\).

*Case 2. \(\theta = 2\theta(\mu-1)\).* Observe that \(s_\mu < \mu\), this gives

\[
\mu > \frac{n}{\mu} + \frac{\alpha}{\mu} = s_0 > 1 \quad \text{and} \quad s = 1 + \frac{s_0 - 1}{s_0} \left(\frac{n}{\mu} + \alpha\right).
\]

In addition

\[
0 < s_\mu < s_0 < \frac{n}{\mu} + \alpha \quad \text{and} \quad s_0 \geq \frac{n}{\mu} + \alpha + 1.
\]

Assume that \(s + \theta = 2 + s_\mu < s_0\). Let \(2 + s_\mu < s_1 < s_0\) and \(0 < \gamma < 1\) be such that \(s + \theta = \gamma s_\mu + (1 - \gamma)s_1\). From interpolation inequality (7), Lemma 4.5, Theorems 3.15 and 3.19 we get

\[
\|u - e^{t\Delta}u_0\|_{K_{p,q}^{\alpha+i\theta}}
\]

can be estimated by

\[
\int_0^t \| e^{(t-\tau)\Delta}(G(u)(\tau, \cdot)) \|_{K_{p,q}^{\alpha+i\theta}} d\tau \\
\leq C \int_0^t \| e^{(t-\tau)\Delta}G(u)(\tau, \cdot) \|_{K_{p,q}^{\alpha+i\theta}}^{1-\gamma} \| G(u)(\tau, \cdot) \|_{K_{p,q}^{\alpha+i\theta}}^{\gamma} d\tau \\
\leq C \int_0^t \| G(u)(\tau, \cdot) \|_{K_{p,q}^{\alpha+i\theta}}^{1-\gamma} \| G(u)(\tau, \cdot) \|_{K_{p,q}^{\alpha+i\theta}}^{\gamma} d\tau \\
\leq CT_0 \| u \|_{^\gamma_Y}^{\gamma} \| u \|_{^\gamma_Y}^{(1-\gamma)s_0}.
\]

Now assume that \(\theta + s = 2 + s_\mu \geq s_0\). Let \(\kappa > 0\) be such that \(2 + s_\mu - s_0 < \kappa < 2\). Let \(0 < \phi < 1\) be such that \(s + \theta = s_\mu + \phi(s_0 + \kappa)\). Again, from interpolation inequality we obtain

\[
\|u - e^{t\Delta}u_0\|_{K_{p,q}^{\alpha+i\theta}}
\]

is bounded by
\[
\int_0^t \| e^{(t-r)\Delta} (G(u)(\tau, \cdot)) \|_{\mathring{K}_p^\alpha F^{s+\delta}_\beta} \, d\tau \\
\leq C \int_0^t \| e^{(t-r)\Delta} G(u)(\tau, \cdot) \|_{k_p^\alpha F^{s+\delta}_\beta}^\circ \| e^{(t-r)\Delta} G(u)(\tau, \cdot) \|_{k_p^\alpha F^{1-\circs}_{\beta}}^{1-\circs} \, d\tau.
\]

Applying Hölder’s inequality, Lemma 4.5, Theorems 3.15 and 3.19, we estimate the last expression by
\[
C \left( \int_0^t \| e^{(t-r)\Delta} G(u)(\tau, \cdot) \|_{k_p^\alpha F^{s+\delta}_\beta}^\circ \, d\tau \right)^{\circ} \\
\times \left( \int_0^t \| G(u)(\tau, \cdot) \|_{k_p^\alpha F^\circ_{\beta}} \, d\tau \right)^{\circ} \\
\times \left( \int_0^t (t-\tau)^{-\frac{s}{\alpha}} \| G(u)(\tau, \cdot) \|_{k_p^\alpha F^{1-\circs}_{\beta}} \, d\tau \right)^{\circ} \\
\leq CT_0^{1+\circs} \| u \|_{Y}^{\circs} \| u \|_{Y}^{(1-\circs)\circs}.
\]

The proof is completed. \(\square\)

Using a combination of the arguments used in the proof of Theorem 4.7 with the help of Theorem 3.19 we get the following result:

**Theorem 4.8** Let \(0 < p, q < \infty, 0 < \alpha < n - \frac{n}{p}, \mu \geq \frac{n + \alpha}{q + n + 1}\) and
\[
1 < \mu < \frac{n}{p} + \alpha.
\]

Let \(G \in \text{Lip}_\mu\) and
\[
s = 1 + \frac{\mu - 1}{\mu} \left( \frac{n}{p} + \alpha \right).
\]

(i) For all initial data \(u_0 \in k_p^\alpha F^s_\beta\) with \(s > \bar{s}\), there exists a maximal solution \(u\) to (35) in \(C([0, T_0], k_p^\alpha F^s_\beta)\) with \(T_0 \geq C \| u_0 \|_{k_p^\alpha F^\circ_{\beta}}^{\frac{1}{\bar{s}}}.
\)

(ii) Let \(0 \leq 2\theta(\mu - 1)\). We have
\[
u - e^{i\Delta} u_0 \in C([0, T_0], k_p^\alpha F^{s+\theta}_\beta).
\]

Let \(s > \frac{n}{p} + \alpha\). Using Theorem 3.18, the embedding \(k_p^\alpha F^s_\beta \hookrightarrow L^{\infty}\), we immediately arrive at the following result. We omit the proof since is essentially similar to the proof of Theorem 4.7.
Theorem 4.9 Let \( 1 < p, q \leq \infty, 1 < \beta < \infty, \mu > 1 \) and \( 0 \leq \alpha < n - \frac{n}{p} \). Let \( G \in \text{Lip}_\mu \) and
\[
\frac{n}{p} + \alpha < s < \mu.
\]

(i) For all initial data \( u_0 \) in \( K^{\alpha}_{p,q} F_s^{\beta} \) with \( s \geq \bar{s} \), there exists a maximal solution \( u \) to (35) in \( C((0, T_0), K^{\alpha}_{p,q} F_s^{\beta}) \) with \( T_0 \geq C\| u_0 \|_{K^{\alpha}_{p,q} F_s^{\beta}}^{\frac{1}{\beta}} \).

(ii) Let \( \theta < 2 \). We have
\[
u - e^{t\Delta} u_0 \in C((0, T_0), K^{\alpha}_{p,q} F_{s+\theta}^{\beta}).
\]

Remark 4.10 Corresponding statements to Theorem 4.7 were proved by Ribaud [43], with \( \theta < 2\theta(\mu - 1) \), \( \alpha = 0 \), \( p = q \) and \( \beta = 2 \), under the assumption
\[
\frac{n}{p} - \frac{n}{\mu p} < s < \min \left( \frac{(1 + \frac{n}{p})(\mu - 1)}{\mu}, \frac{n}{p} \right).
\]

Here we are requiring
\[
\max \left( 0, \frac{n}{p} - \frac{n}{\mu} \right) < s < \min \left( 1 + \frac{\mu - 1}{p}, \frac{n}{p} \right),
\]
which improve (43).

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