LARGE SPECTRAL GAP INDUCED BY SMALL DELAY AND ITS APPLICATION TO REDUCTION

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Abstract. We consider general linear neutral differential equations with small delays in the view of pseudo exponential dichotomy. For the autonomous case, we first count the eigenvalues in a certain half plane, which generalized the previous works on serval special retarded differential equations. We next establish the existence of a pseudo exponential dichotomy for the nonautonomous case, and prove that the corresponding spectral gap approaches infinity as the delay tends to zero. The proof for this large spectral gap induced by small delay is owing to exact bounds and exponents for pseudo exponential dichotomy. Then based on above results, we give an invariant manifold reduction theorem for nonlinear neutral differential equations with small delays. Finally, our results are applied to a concrete example.

1. Introduction. Effects of small delays on the dynamical behaviors of differential equations have attracted many attentions in the past few decades. Many works have devoted to investigating the stability, asymptotic behaviors and nonlinear waves for retarded (partial) differential equations with small delays (see, for instance, [1, 4, 7, 13, 14, 15, 16, 23, 31, 39, 41] and references therein). For the study of dynamical behaviors of neutral differential equations with small delays, as far as we know, it remains poorly understood except for [20]. This paper is mainly concerned with pseudo exponential dichotomy and finite-dimensional reduction of neutral differential equations with small delays.

It is well known that exponential dichotomy is one of important properties of dynamical systems. This concept dates back to Perron [38] and it is widely used in studying structural stability, invariant manifolds and so on (see, e.g. [9, 10, 26, 32, 36, 42] and references therein). As the essentially same as exponential dichotomy,
the concept of pseudo exponential dichotomy develops the classical one and plays an
useful role in describing the pseudo hyperbolicity of differential equations ([28, 33,
44]). Let the two-parameter family \{T(t,s) : t \geq s\} of bounded linear operators on
a Banach space \(B\) be a semigroup and be strongly continuous in \(t\) and \(s\). It is said
that \(\{T(t,s) : t \geq s\}\) admits a \textit{pseudo exponential dichotomy} with the exponents
\(\beta < \alpha\) and the bound \(K\) on an interval \(J(\subset \mathbb{R})\) if for each \(s \in J\) there exists a
projection \(P(s)\) on \(B\) which is strongly continuous in \(s\) and constants \(\alpha, \beta \in \mathbb{R},\)
\(\beta < \alpha\) and \(K > 0\) such that

(a) \(T(t,s)P(s) = P(t)T(t,s)\) for \(t \geq s\) in \(J\);

(b) \(T(t,s)|_{\mathcal{R}(P(s))}\) is an isomorphism from \(\mathcal{R}(P(s))\) onto \(\mathcal{R}(P(t))\), where \(\mathcal{R}(P(s))\)
is the range of \(P(s)\). The inverse of \(T(t,s)|_{\mathcal{R}(P(s))}\) is denoted by \(T(s,t) : \mathcal{R}(P(t)) \to \mathcal{R}(P(s))\);

(c) \(|T(s,t)P(t)\phi| \leq Ke^{-\alpha(t-s)}|P(t)\phi|\) for \(t \geq s\) in \(J\) and \(\phi \in \mathcal{B}\);

(d) \(|T(t,s)Q(s)\phi| \leq Ke^{\beta(t-s)}|Q(s)\phi|\) for \(t \geq s\) in \(J\) and \(\phi \in \mathcal{B}\), where \(Q(s) = I - P(s)\) and \(I\) is the identity.

For convenience, we call \(\alpha, \beta\) an upper exponent and a lower exponent of pseudo
exponential dichotomy, respectively. \textit{The spectral gap} for this dichotomy is given by
the following difference:

\[
\text{The spectral gap} = \sup \left\{ \alpha \in \mathbb{R} : \sup_{t \geq s} |T(s,t)P(t)|e^{\alpha(t-s)} < +\infty \right\} \\
- \inf \left\{ \beta \in \mathbb{R} : \sup_{t \geq s} |T(t,s)Q(s)|e^{-\beta(t-s)} < +\infty \right\}.
\]

The notation \(|\cdot|\) is always used to denote norms in different spaces, but no confusion
should arise.

In this paper, we study the pseudo exponential dichotomy for the following linear
neutral differential equation with small delay \(r > 0\) on \(\mathbb{R}^n\)

\[
\frac{d}{dt} M(t)x_t = L(t)x_t, \tag{1}
\]

where for each \(t \in \mathbb{R}\), the operators \(M(t)\) and \(L(t) : C[-r,0] \to \mathbb{R}^n\) are given by

\[
M(t)\phi = \phi(0) - \int_{-r}^{0} d\mu(t,\theta)\phi(\theta), \quad L(t)\phi = \int_{-r}^{0} d\eta(t,\theta)\phi(\theta), \quad \forall \phi \in C[-r,0]. \tag{2}
\]

Here the kernels \(\mu = (\mu_{ij})\) and \(\eta = (\eta_{ij})\) are \(n \times n\) matrix valued functions on
\(\mathbb{R} \times \mathbb{R}\) and \(C[-r,0]\) denotes a Banach space consisting of all continuous functions
from \([-r,0]\) to \(\mathbb{R}^n\) with the supremum norm.

To guarantee that the existence of solutions of equation (1), we always assume
that: (I) both the kernels \(\mu\) and \(\eta\) are continuous from the left in \(\theta\) on \((-r,0)\),
measurable in \((t,\theta)\), normalized so that

\[
\mu(t,\theta) = 0, \quad \eta(t,\theta) = 0, \quad \forall \theta \geq 0,
\]

\[
\mu(t,\theta) = \mu(t,-r), \quad \eta(t,\theta) = \eta(t,-r), \quad \forall \theta \leq -r;
\]

(II) the kernel \(\mu\) has bounded variation in \(\theta\) on \([-r,0]\) uniformly in \(t\), the kernel
\(\eta\) has bounded variation in \(\theta\) on \([-r,0]\) for each \(t \in \mathbb{R}\); (III) both the mappings
\(t \to M(t)\phi, \ t \to L(t)\phi\) are continuous for each \(\phi \in C[-r,0]\); (IV) the kernel \(\mu\) is
uniformly nonatomic at zero, that is, for every \(\epsilon > 0\), there exists a \(\delta > 0\) such that
\(\text{Var}_{[-\delta,0]} \mu(t,\cdot) < \epsilon\) for all \(t \in \mathbb{R}\), where \(\text{Var}_{[-\delta,0]} \mu(t,\cdot)\) is the total variation of \(\mu(t,\cdot)\)
on \([-\delta,0]\). Then, by [22, Theorem 1.1, p.256] equation (1) has a unique solution
where the operators \( M \) and \( L \) satisfy the following hypothesis:

\((\text{H}1)\) for each \( t \in \mathbb{R} \) and each \( r \in \mathbb{R}^+ \), the total variations of \( \mu(t, \cdot) \) and \( \eta(t, \cdot) \) on \([-r, 0]\) satisfy

\[
K_1 := \sup_{t \in \mathbb{R}} \text{Var}_{[-r, 0]} \mu(t, \cdot) < 1, \quad K_2 := \sup_{t \in \mathbb{R}} \text{Var}_{[-r, 0]} \eta(t, \cdot) < +\infty.
\]

Specially, when the operator \( M(t) \) satisfies \( M(t)\phi = \phi(0) \) for \( \phi \in C[-r, 0] \), equation (1) is reduced to retarded differential equation (see [22]).

We further assume that the kernels \( \mu \) and \( \eta \) satisfy the following hypothesis:

\((\text{H}1)\) for each \( t \in \mathbb{R} \) and each \( r \in \mathbb{R}^+ \), the total variations of \( \mu(t, \cdot) \) and \( \eta(t, \cdot) \) on \([-r, 0]\) satisfy

\[
K_1 := \sup_{t \in \mathbb{R}} \text{Var}_{[-r, 0]} \mu(t, \cdot) < 1, \quad K_2 := \sup_{t \in \mathbb{R}} \text{Var}_{[-r, 0]} \eta(t, \cdot) < +\infty.
\]

Specially, when the operator \( M(t) \) satisfies \( M(t)\phi = \phi(0) \) for \( \phi \in C[-r, 0] \), equation (1) is reduced to retarded differential equation (see [22]).

For the strongly continuous semigroup \( \{ T(t) : t \geq 0 \} \) generalized by equation (4), there is a close relation between the dichotomy and the spectrum of \( T(t) \). Suppose that \( T(t) \) has some spectral gaps, it is trivial to give the existence of pseudo exponential dichotomies (see [8]). However, it is difficult to obtain the spectrum of \( T(t) \). Generally, to solve this problem, one can apply the spectrum of its infinitesimal generator to obtain the eigenvalues (i.e. point spectrum) and the residual spectrum of \( T(t) \) in terms of the Spectral Mapping Theorem ([8, 37]). But there are no general principles to give the continuous spectrum of \( T(t) \). Greiner and Schwarz ([18]) and Henry ([27]) developed the so-called Weak Spectral Mapping Theorem to analyze the continuous spectrum of the solution semigroups for neutral equations with the kernels satisfying certain jump conditions. Later, based on the growth bound in terms of the resolvent, Kaashoek and Verduyn Lunel ([29]) derived necessary and sufficient conditions for the existence of the exponential dichotomy. In this paper, we will only assume that the kernels for neutral differential equations satisfy the well-posed conditions, and prove the existence of a pseudo exponential dichotomy for linear autonomous neutral differential equations with small delays.

Motivated by the Spectral Mapping Theorem and the residual spectrum of \( T(t) \) for equation (4) is empty ([22, 25]), to prove that equation (4) admits a pseudo exponential dichotomy, we just only need to analyze the eigenvalues of \( T(t) \) and overcome the difficulty caused by the continuous spectrum. In the past several decades, many efforts have been made to analyze the eigenvalues of several special linear autonomous retarded differential equations with small delays. For example, Driver ([14]) studied a planar system and proved that there are two eigenvalues in the open disk \( \{ \lambda \in \mathbb{C} : |\lambda| < 1/r \} \); Arino and Pituk ([1]) proved for equation \( \dot{x}(t) = Ax(t-r) \) that there exist \( n \) eigenvalues (counted by multiplicity) in the disk \( \{ \lambda \in \mathbb{C} : |\lambda| < 1/r \} \), where \( A \) is an \( n \times n \) real matrix; Faria and Huang in [16, Theorem 3.1] obtained the similar result for linear autonomous retarded differential equations in \( \mathbb{R}^n \) with infinitely many delays. An interesting question is:
• Are there exactly $n$ eigenvalues (counted by multiplicity) in $\{\lambda \in \mathbb{C} : |\lambda| < 1/r\}$ for the general linear autonomous neutral differential equation (4) in $\mathbb{R}^n$ with small delay?

The answer to this question is affirmative. We extend these previous works to more general neutral differential equations by the Argument Principle ([40, Theorem 10.43, p.225]) and Theorem 9.17.4 in [12, p.248], which shows the zeros of analytic functions are continuous with respect to parameters.

Then we discuss pseudo exponential dichotomy for linear nonautonomous neutral differential equation (1). In general, for the semigroup generalized by a linear nonautonomous system, it is difficult to establish the existence of dichotomy, besides several criteria by using such as the admissibility (see [2, 10, 35]), Lyapunov function (see [10]) and so on. In this article, we will not only prove the existence of a pseudo exponential dichotomy for equation (1) with small delays, but also give the accurate estimations of the upper exponent, lower exponent and the bound of pseudo exponential dichotomy, and show the representation of the projection based on the formal adjoint equation. It is well-known that these constants are useful in studying local invariant manifolds (see [3, 5, 26]). However, to the best of our knowledge, there are little works contributing to estimating these constants.

Finally, we prove that the spectral gap for the obtained dichotomy approaches infinity as the delay tends to zero, and apply this result to discuss the invariant manifold reduction for the nonlinear neutral differential equations. As we know, the theory of invariant manifolds is one of effective tools to deal with the problem of finite-dimensional reduction, such as center manifolds, inertial manifolds and so on (see e.g. [5, 17, 26] and references therein). There are two classical methods taken to establish the existence and smoothness of invariant manifolds, one is the Hadamard’s graph transformation method ([21]), the other is the Lyapunov-Perron method ([34, 38]). There also exist many extensive results on the existence and smoothness of invariant manifolds, see, for instance, [3, 9, 26, 28, 30].

However, there are few references giving the size of local invariant manifolds. In the current paper, based on the accurate estimates for the pseudo exponential dichotomy, we obtain the existence and the size of invariant manifolds for nonlinear neutral differential equations with small delays, and show that large spectral gap is induced by small delay. We also illustrate the usefulness of the above observations by using a concrete example. Moreover, it is worth noting that in the example, center manifold theory given by [19, 22] can guarantee the existence of local center manifold near the equilibrium, but fails to determine whether a certain periodic solution lies on the local center manifold. And our results solve this problem.

This paper is organized as follows. In section 2 we first give the general properties of pseudo exponential dichotomy for the semigroups on Banach spaces, and then introduce the formal adjoint equation and a bilinear form for linear neutral differential equations. In section 3 we count the eigenvalues for linear autonomous neutral differential equations in a certain half plane. Section 4 is devoted to studying the existence of a pseudo exponential dichotomy for linear neutral differential equations with small delays. We give the accurate estimates of the exponents and the bound, and further show the representation of the projection based on the formal adjoint equation. In the final section, we prove that the spectral gap can be sufficiently large as the delay is small enough, and derive an invariant manifold reduction theorem for nonlinear neutral differential equations with small delays. Additionally, we also provide a concrete example as an application.
2. Preliminaries. In this section, we introduce some basics concepts on pseudo exponential dichotomy and the formal adjoint equation associate with neutral differential equations as preparations.

Assume that the two-parameter family \( \{ T(t, s) : t \geq s \} \) of bounded linear operators on a Banach space \( \mathcal{B} \) admits a pseudo exponential dichotomy with the exponents \( \beta < \alpha \) and the bound \( K \) on an interval \( J(\subset \mathbb{R}) \). Let \( \mathcal{B}^* \) denote the dual space of \( \mathcal{B} \), that is, the space of all bounded linear functionals on \( \mathcal{B} \). Set \( T^*(s, t) := T(t, s)^* \) for \( t \geq s \). Then the two-parameter family \( \{ T^*(s, t) : t \geq s \} \) of bounded linear operators on \( \mathcal{B}^* \) is a backward evolutionary system on \( \mathcal{B}^* \), that is, \( T^*(s, s) = I \) and \( T^*(s, \sigma)T^*(\sigma, t) = T^*(s, t) \) for \( t \geq \sigma \geq s \), and we see that \( \{ T^*(s, t) : t \geq s \} \) is weak* continuous in \( t \) and \( s \).

Like the definition of a pseudo exponential dichotomy for \( \{ T(t, s) : t \geq s \} \), the backward evolutionary system \( \{ T^*(s, t) : t \geq s \} \) is said to admit a pseudo exponential dichotomy with exponents \( \beta < \alpha \) and bound \( K \) on an interval \( J(\subset \mathbb{R}) \) if for each \( s \in J \) there exists a projection \( P^*(s) \) on \( \mathcal{B}^* \) which is weak* continuous in \( s \) and constants \( \alpha, \beta \in \mathbb{R} \), \( \beta < \alpha \) and \( K > 0 \) such that the following properties hold:

(a) \( P^*(s)T^*(s, t) = T^*(s, t)P^*(t) \) for \( t \geq s \) in \( J \);
(b) \( T^*(s, t)|_{\mathcal{R}(P^*(t))} \) is an isomorphism from \( \mathcal{R}(P^*(t)) \) onto \( \mathcal{R}(P^*(s)) \). The inverse of \( T^*(s, t)|_{\mathcal{R}(P^*(t))} \) is denoted by \( T^*(t, s) : \mathcal{R}(P^*(s)) \to \mathcal{R}(P^*(t)) \);
(c) \( |T^*(t, s)P^*(s)\phi^*| \leq Ke^{-\alpha(t-s)}|P^*(s)\phi^*| \) for \( t \geq s \) in \( J \) and \( \phi^* \in \mathcal{B}^* \);
(d) \( |T^*(s, t)Q^*(t)\phi^*| \leq Ke^{\beta(t-s)}|Q^*(t)\phi^*| \) for \( t \geq s \) in \( J \) and \( \phi^* \in \mathcal{B}^* \).

The following two lemmas show the relation between the pseudo exponential dichotomies of \( \{ T(t, s) : t \geq s \} \) and \( \{ T^*(s, t) : t \geq s \} \). Their proofs follow from standard methods (see, for instance, [32]). For brevity the details are omitted.

Lemma 2.1. Let \( \mathcal{B} \) be a Banach space and \( \mathcal{B}^* \) denote its dual space. Then the two-parameter family \( \{ T(t, s) : t \geq s \} \) defined on \( \mathcal{B} \) admits a pseudo exponential dichotomy with the exponents \( \beta < \alpha \) and the bound \( K \) on an interval \( J(\subset \mathbb{R}) \) if and only if the backward evolutionary system \( \{ T^*(s, t) : t \geq s \} \) defined on \( \mathcal{B}^* \) admits a pseudo exponential dichotomy with the exponents \( \beta < \alpha \) and the bound \( K \) on an interval \( J(\subset \mathbb{R}) \).

Lemma 2.2. Suppose that the two-parameter family \( \{ T(t, s) : t \geq s \} \) admits a pseudo exponential dichotomy with the projection \( P(s) \) and \( \mathcal{R}(P(s)) \) is a finite dimensional space with \( \dim \mathcal{R}(P(s)) = m \). Let the sequence \( \{ u_i(s) \}_{i=1}^m \subset \mathcal{B} \) be the basis of \( \mathcal{R}(P(s)) \). Then there exists a sequence \( \{ u_i^*(s) \}_{i=1}^m \subset \mathcal{B}^* \) such that \( u_i^*(s) \) satisfies

\[
\langle u_i^*(s), u_j(s) \rangle = \delta_{i,j},
\]

where \( \delta_{i,j} \) is the Kronecker delta. Furthermore, the projection \( P^*(s) \) satisfies that

\[
\langle P^*(s)v, u \rangle = \sum_{i=1}^m \langle v, u_i(s) \rangle \langle u_i^*(s), u \rangle
\]

for each \( v \in \mathcal{B}^* \) and \( u \in \mathcal{B} \).

In general, it is not easy to derive the representation for \( \{ T^*(s, t) : t \geq s \} \) associated with a certain evolutionary system. But one luckily obtain it for equation (1). Recall that the solution semigroup \( \{ T(t, s) : t \geq s \} \) for equation (1) is defined by (3), then \( T^*(s, t) = T(t, s)^* \) for \( t \geq s \) is given in the following way (see [22, Section 9.1]). Let \( \mathbb{R}^n \) be the row \( n \)-vectors with real entries, and let \( B_0 \) denote
the dual space of $C[-r,0]$, that is, the Banach space of row-valued functions $\psi : (-\infty,0] \rightarrow \mathbb{R}^n$ which are constants on $(-\infty,-r]$, of bounded variation on $[-r,0]$, vanishing at zero and continuous from the left on $(-r,0)$ with the norm $\text{Var}_{[-r,0]}\psi$. One considers the formal adjoint equation of (1), seeing [22, p.260],

$$y(s) + \int_s^t g(\tau)(d_\mu(\tau, s - \tau) + \eta(\tau, s - \tau)d\tau) = g^t(s), \quad s \leq t, \quad (6)$$

where $g^t(s) = g(s - t)$ and $g \in B_0$. Equation (6) has a unique solution $y(\cdot; t, g)$ defined and locally of bounded variation on $(-\infty,t]$. Then $T^*(\sigma,t)$ is given by

$$T^*(\sigma,t)g = \tilde{g}, \quad \sigma \leq t, \quad (7)$$

where the function $\tilde{g} = \mathcal{F}(\sigma)y_{\tilde{r}}$ and for each function $\psi$ of bounded variation on $[-r,0]$, the function $\mathcal{F}(\sigma)\psi$ is defined by the following way: $(\mathcal{F}(\sigma)\psi)(\theta) = 0$ for $\theta = 0$; $(\mathcal{F}(\sigma)\psi)(\theta) = \psi(\theta) + \int_0^\theta \psi(\alpha)(d_\mu(\sigma + \alpha, \theta - \alpha) + \eta(\sigma + \alpha, \theta - \alpha)d\alpha)$ for $-r < \theta < 0$; $(\mathcal{F}(\sigma)\psi)(\theta) = (\mathcal{F}(\sigma)\psi)(-r)$ for $\theta < -r$.

Moreover, to study the representation for the projection of pseudo exponential dichotomy, we need calculate the following bilinear form

$$(\psi, \phi, t) = \psi(0)\phi(0) + \int_0^r \psi(\alpha) \int_{-r}^0 d\eta(t + \alpha, \theta - \alpha)\phi(\theta)d\alpha \quad + \int_0^r \psi(\alpha)d_\alpha \int_{-r}^0 d\mu(t + \alpha, \theta - \alpha)\phi(\theta), \quad (8)$$

where $\psi : [0,r] \rightarrow \mathbb{R}^n$ is of bounded variation and $\phi \in C[-r,0]$.

The following can be proved by the same method as employed in [23, 24]. For the reader’s convenience, we sketch a proof in Appendix A.

**Lemma 2.3.** Let the function $\psi : (-\infty,r] \rightarrow \mathbb{R}^n$ be of bounded variation on $[0,r]$ and be constant on $(-\infty,0]$, and $\phi$ be in $C[-r,0]$. Let the $r$-shift operator $S$ acting on $\psi$ be in the form $S\psi(\theta) = \psi(r + \theta)$ for $\theta \in (-\infty,0]$. Then

$$(\psi,\phi,t) = -\langle \mathcal{F}(t + r)S\psi, T(t + r, t)\phi \rangle. \quad (9)$$

In the final of this section, we give the representation of the projection of pseudo exponential dichotomy.

**Lemma 2.4.** Suppose that $\{T(t,t_0) : t \geq t_0\}$ defined by (3) admits a pseudo exponential dichotomy with the projection $\mathcal{P}(t_0)$ and $\mathcal{R}(\mathcal{P}(t_0))$ is a finite dimensional space with $\text{dim} \mathcal{R}(\mathcal{P}(t_0)) = m$. Let $\{\phi^i\}_{i=1}^m \subset C[-r,0]$ be the basis of $\mathcal{R}(\mathcal{P}(t_0))$. Then there exists a $m \times n$ matrix valued function $Y : \mathbb{R} \rightarrow \mathbb{R}^{m \times n}$ locally of bounded variation such that for each $t \in \mathbb{R}$,

$$I = Y(t)X(t) + \int_0^r Y(t + \alpha) \int_{-r}^0 d\eta(t + \alpha, \theta - \alpha)X(t + \theta)d\alpha \quad + \int_0^r Y(t + \alpha)d_\alpha \int_{-r}^0 d\mu(t + \alpha, \theta - \alpha)X(t + \theta), \quad (10)$$

where $I$ is $m \times m$ identity matrix, $X = (x^1,...,x^m)$ and $x^i(t) = x(t,t_0,\phi^i)$. Furthermore, the projection $\mathcal{P}(t_0)$ is given by

$$\mathcal{P}(t_0)\varphi = X_{t_0}(Y_{t_0},\varphi,t_0), \quad \forall \varphi \in C[-r,0], \quad (11)$$

where $Y_{t_0}(\theta) := Y(t_0 + \theta)$ for $\theta \in [0,r]$ and $(Y_{t_0},\varphi,t_0)$ is defined by (8).
Proof. For each \(\phi^i, 1 \leq i \leq m\), by Lemma 2.2 we see that there exists a \(\psi \in B_0\) such that \(P^\ast(t_0)\psi = \psi\), \(\langle \psi, \phi^j \rangle = -1\) and \(\langle \psi, \phi^i \rangle = 0\) for \(j \neq i\). From Lemma 2.1 it follows that \(T^\ast(t_0) : \mathcal{R}(P^\ast(t)) \to \mathcal{R}(P^\ast(t_0))\) is an isomorphism. Then one can define \(g^i(s) = (T^\ast(t, t_0)\psi)(s - t)\) for each \(t \in \mathbb{R}\) and \(s \leq t\). Let the function \(y : \mathbb{R} \to \mathbb{R}^n\) locally of bounded variation be given by \(y(s) = y(s, t, g^i)\) for \(s \in \mathbb{R}\) and \(t \geq s\). We claim that this function \(y\) is well-defined on \(\mathbb{R}\), that is, for each \(t_1 \leq t_2\) we have \(y(s, t_1, g^{i_1}) = y(s, t_2, g^{i_2})\) for \(s < t_1\), where \(y(s, t_i, g^{i_i})\) satisfy that for \(s \leq t_1\) and \(i = 1, 2\),

\[
y(s, t_i, g^{i_i}) + \int_s^{t_i} y(\tau, t_i, g^{i_i})(d_\tau \mu(\tau, s - \tau) + \eta(\tau, s - \tau)d\tau) = g^{i_i}(s) . \tag{12}
\]

Set \(g_i(s - t_1) = g^{i_i}(s)\) for \(s \leq t_1\). Then

\[
g_1 = T^\ast(t_1, t_0)\psi = T^\ast(t_1, t_2)T^\ast(t_2, t_0)\psi = T^\ast(t_1, t_2)g_2.
\]

Along with (7) and (12), we have \(g_1(\theta) = (\mathcal{F}(t_1)g_{i_1}(\cdot, t_2, g^{i_2}))(\theta)\) for \(\theta \leq 0\), which implies that \(y(\cdot, t_2, g^{i_2})\), \(s \in [t_1 - r, t_1]\) satisfies the equation

\[
y(s) + \int_s^{t_1} y(\tau)(d_\tau \mu(\tau, s - \tau) + \eta(\tau, s - \tau)d\tau) = g^{i_1}(s) . \tag{13}
\]

Meanwhile, for \(s \leq t_1 - r \leq t_2 - r\), noting \(g^{i_1} \in B_0\), and \(\mu(\tau, s - \tau)\) and \(\eta(\tau, s - \tau)\) are constants for \(t_1 \leq \tau \leq t_2\), we have

\[
y(s, t_2, g^{i_2}) + \int_s^{t_1} y(\tau, t_2, g^{i_2})(d_\tau \mu(\tau, s - \tau) + \eta(\tau, s - \tau)d\tau)
= y(s, t_2, g^{i_2}) + \int_s^{t_2} y(\tau, t_2, g^{i_2})(d_\tau \mu(\tau, s - \tau) + \eta(\tau, s - \tau)d\tau)
- \int_s^{t_2} y(\tau, t_2, g^{i_2})(d_\tau \mu(\tau, s - \tau) + \eta(\tau, s - \tau)d\tau)
= g^{i_2}(t_1 - r) - \int_s^{t_2} y(\tau, t_2, g^{i_2})(d_\tau \mu(\tau, t_1 - r - \tau) + \eta(\tau, t_1 - r - \tau)d\tau)
= y(t_1 - r, t_2, g^{i_2}) + \int_{t_1 - r}^{t_2} y(\tau, t_2, g^{i_2})(d_\tau \mu(\tau, t_1 - r - \tau) + \eta(\tau, t_1 - r - \tau)d\tau)
- \int_{t_1 - r}^{t_2} y(\tau, t_2, g^{i_2})(d_\tau \mu(\tau, t_1 - r - \tau) + \eta(\tau, t_1 - r - \tau)d\tau)
= y(t_1 - r, t_2, g^{i_2}) + \int_{t_1 - r}^{t_1} y(\tau, t_2, g^{i_2})(d_\tau \mu(\tau, t_1 - r - \tau) + \eta(\tau, t_1 - r - \tau)d\tau),
\]

together with the fact that \(g^{i_1}(t_1 - r) = (\mathcal{F}(t_1)g_{i_1}(\cdot, t_2, g^{i_2}))(-r)\), that is,

\[
g^{i_1}(t_1 - r) = y(t_1 - r, t_2, g^{i_2}) + \int_0^{t_1} y(t_1 + r, t, t_2, g^{i_2})(d_\tau \mu(t_1 + r, -r - \tau)
+ \eta(t_1 + r, -r - \tau)d\tau)
= y(t_1 - r, t_2, g^{i_2}) + \int_{t_1 - r}^{t_1} y(\tau, t_2, g^{i_2})(d_\tau \mu(\tau, t_1 - r - \tau)
+ \eta(\tau, t_1 - r - \tau)d\tau),
\]
yields that \(y(\cdot, t_2, g^{i_2})\) satisfies (13) for \(s \in (-\infty, t_1 - r]\). By the uniqueness of solution for equation (13), we see that \(y(s, t_1, g^{i_1}) = y(s, t_2, g^{i_2})\) for \(s < t_1\). Thus the claim is true. Next we define the \(m \times n\) matrix valued function \(Y : \mathbb{R} \to \mathbb{R}^{m \times n}\).
locally of bounded variation such that its $i$-th row vector equals to $y$ corresponding to $\phi^i$. Let $y^i = y(t + \theta)$ for $\theta \in [0, r]$. By Lemma 2.3 and the fact that $F(t + r)Sy^i = T^*(t + r, t_0)\psi$, we have

$$
(y^i, x_i, t) = - \langle F(t + r)Sy^i, T(t + r, t)x_i \rangle
= - \langle F(t + r)Sy^i, T(t + r, t)T(t, t_0)\phi \rangle
= - \langle T^*(t_0, t + r)F(t + r)Sy^i, \phi \rangle = -\langle \psi, \phi \rangle,
$$

which implies $(Y^t, X_i, t) = I$, together with (8), yields that (10) holds. By (14), we find that (11) is true. Therefore, the proof is complete. \hfill \Box

3. **Spectral analysis.** In this section, we investigate the location of eigenvalues for linear autonomous neutral differential equation (4) with small delay.

To emphasize that the kernels $\eta$ and $\mu$ associated with linear autonomous equation (4) are dependent upon the delay $r$, in the following section, we denote by $\eta(\cdot) = \eta(\cdot, r)$ and $\mu(\cdot) = \mu(\cdot, r)$, respectively. Then the operators $M$ and $L$ given in (5) can be rewritten as

$$
M\phi = \phi(0) - \int_{-r}^{0} d\mu(\theta, r)\phi(\theta), \quad L\phi = \int_{-r}^{0} d\eta(\theta, r)\phi(\theta), \quad \forall \phi \in C[-r, 0].
$$

We further assume that the kernels $\mu$ and $\eta$ satisfy the following:

**H2** $K_1 := \sup_{t \in \mathbb{R}^+} \text{Var}_{[-r, 0]} \mu(\cdot, r) < 1, \quad K_2 := \sup_{t \in \mathbb{R}^+} \text{Var}_{[-r, 0]} \eta(\cdot, r) < +\infty.$

In this section we still use the notations $K_i$ as in the hypothesis (H1) without any confusion. The bilinear form (8) corresponding to linear autonomous equation (4) can be converted into

$$
(\psi, \phi) = \psi(0)\phi(0) + \int_{0}^{r} \psi(\alpha) \int_{-r}^{0} d\eta(\theta - \alpha, r)\phi(\theta) d\alpha
+ \int_{0}^{r} \psi(\alpha) d\alpha \int_{-r}^{0} d\eta(\theta - \alpha, r)\phi(\theta),
= \psi(0)\phi(0) + \int_{-r}^{0} \psi(\theta - \alpha) d\eta(\alpha, r)\phi(\theta) d\theta
- \int_{-r}^{0} d\theta \int_{-r}^{0} \psi(\theta - \alpha) d\mu(\alpha, r)\phi(\theta),
$$

where $\psi : [0, r] \to \mathbb{R}^n$ is of bounded variation and $\phi \in C[-r, 0]$. Recall that the solution operator $T(t) : C[-r, 0] \to C[-r, 0]$ of equation (4) is given by

$$
T(t)\phi := x_t(\cdot, \phi), \quad \forall t \geq 0, \quad \forall \phi \in C[-r, 0],
$$

where $x_t(\cdot, \phi)$ is the solution of equation (4) with initial value $x_0 = \phi$, and is a semigroup for $t \geq 0$. Moreover, one can verify that the family of the operators $\{T(t) : t \geq 0\}$ is strongly continuous. We use the notation $\mathcal{A}$ to denote its infinitesimal generator.

To get the eigenvalues for equation (4), by [22, Lemma 2.1, p.263] we need to find all roots of the characteristic equation $h(\lambda, r) := \det \Delta(\lambda, r) = 0$, where

$$
\Delta(\lambda, r) := \lambda \Delta_0(\lambda, r) - \int_{-r}^{0} e^{\lambda \theta} d\eta(\theta, r), \quad \Delta_0(\lambda, r) := I - \int_{-r}^{0} e^{\lambda \theta} d\mu(\theta, r).
$$
As shown in [43, Proposition 2.2], using the convolution property of the Laplace transform ([11, Corollary 2.5, p.451]), we get

\[ h(\lambda, r) = h(\lambda, r)\lambda^n + \sum_{j=1}^{n} \int_{-jr}^{0} e^{\lambda \theta} d\eta_j(\theta, r)\lambda^{n-j}, \quad (17) \]

\[ h(\lambda, r) = \det \Delta_\theta(\lambda, r) = 1 + \sum_{j=1}^{n} \int_{-jr}^{0} e^{\lambda \theta} d\mu_j(\theta, r), \]

where \( \mu_j \) is determined by \( \mu_{ij} \), \( \eta_j \) is determined by \( \mu_{ij} \) and \( \eta_{ij} \), and they are functions of bounded variation defined on \( \mathbb{R} \) with the properties:

\[ \mu_j(\theta, r) = 0, \quad \eta_j(\theta, r) = 0, \quad \forall \theta \geq 0, \]

\[ \mu_j(\theta, r) = \mu_j(-jr, r), \quad \eta_j(\theta, r) = \eta_j(-jr, r), \quad \forall \theta \leq -jr. \]

It is straightforward to show that the partial derivative of \( h(\lambda, r) \) on \( \lambda \) is

\[ h_\lambda(\lambda, r) = h(\lambda, r)\lambda^n + \left( n h(\lambda, r) + \int_{-r}^{0} \theta e^{\lambda \theta} d\eta_1(\theta, r) \right) \lambda^{n-1} + \sum_{j=1}^{n-1} \left( (n-j) \int_{-jr}^{0} e^{\lambda \theta} d\eta_j(\theta, r) + \int_{-jr}^{0} \theta e^{\lambda \theta} d\eta_{j+1}(\theta, r) \right) \lambda^{n-j}. \]

To obtain the location of zeros of the function \( h(\cdot, r) \), we need study the monotonicity of two analytic functions, which plays an essential role in the sequent discussion.

**Lemma 3.1.** Let the functions \( \Phi, \Psi : \mathbb{R} \rightarrow \mathbb{R} \) be given by

\[ \Phi(\lambda) = \lambda e^{-\lambda} - K_1 \lambda, \quad \Psi(\lambda) = K_1 \lambda e^{\lambda} + K_2 e^{r\lambda} - \lambda, \quad \forall \lambda \in \mathbb{R}, \]

where the constants \( 0 \leq K_1 < 1, K_2 > 0 \) and \( r > 0 \). Then there exists a constant \( \lambda_0 \in (0, 1) \), which is the unique real zeros of \( \Phi' \), such that \( \Phi(\lambda_0) = \sup_{\lambda \in \mathbb{R}} \Phi(\lambda) \).

Let the constant

\[ r_0 = \frac{\Phi(\lambda_0)}{K_2}. \quad (18) \]

Then the equation \( \Phi(\lambda) = K_2 r \) for each \( r \in (0, r_0) \) has exactly two positive real roots \( \lambda_1(r) \) and \( \lambda_2(r) \), which are satisfying

\[ 0 < \lambda_1(r) < \frac{r\lambda_0}{r_0} < \lambda_0 < \lambda_2(r) < +\infty, \quad (19) \]

and \( \Psi(\lambda) < 0 \) for \( \lambda \in (\lambda_1(r)/r, \lambda_2(r)/r) \) and \( \Psi(\lambda) \geq 0 \) for \( \lambda \in \mathbb{R} \setminus (\lambda_1(r)/r, \lambda_2(r)/r) \). Furthermore, if \( r \) decreases to 0, then \( \lambda_1(r) \) and \( \lambda_1(r)/r \) decrease to 0 and \( K_2/(1-K_1) \), \( \lambda_2(r) \) and \( \lambda_2(r)/r \) increase to \( -\ln K_1 \) and \( +\infty \), respectively.

**Proof.** The first conclusion follows from a straightforward computation. Since \( 0 < K_2 r \leq \Phi(\lambda_0) \) for any \( r \in (0, r_0) \), by the the monotonicity of \( \Phi \), there are exactly two positive real roots \( \lambda_1(r) \) and \( \lambda_2(r) \) of equation \( \Phi(\lambda) = K_2 r \) such that

\[ 0 < \lambda_1(r) < \lambda_0 < \lambda_2(r) < +\infty. \quad (20) \]

For any \( r \in (0, r_0) \), we investigate that

\[ \Phi\left( \frac{r\lambda_0}{r_0} \right) = \frac{r\lambda_0}{r_0} e^{-\lambda_0} - K_1 \frac{r\lambda_0}{r_0} e^{-\lambda_0} - K_1 \frac{r\lambda_0}{r_0} = \frac{r}{r_0} \Phi(\lambda_0) = K_2 r, \]
which implies that \( r \lambda_0 / r_0 \in (\lambda_1(r), \lambda_2(r)) \). Along with inequality (20) and \( r \lambda_0 / r_0 < \lambda_0 \), inequality (19) holds.

Next we rewrite \( \Psi(\lambda) \) as

\[
\Psi(\lambda) = -(\Phi(r\lambda) - K_2 r) e^{r\lambda}/r.
\]

Clearly, \( \Psi(\lambda) < 0 \) if \( \Phi(r\lambda) > K_2 r \) and \( \Psi(\lambda) \geq 0 \) if \( \Phi(r\lambda) \leq K_2 r \). It means that \( \Psi(\lambda) < 0 \) for \( \lambda \in (\lambda_1(r)/r, \lambda_2(r)/r) \) and \( \Psi(\lambda) \geq 0 \) for \( \lambda \in \mathbb{R}/(\lambda_1(r)/r, \lambda_2(r)/r) \).

Moreover, by the monotonicity of \( \Phi \), we observe that \( \lambda_1(r) \) decreases to 0 and \( \lambda_2(r) \) increases to \(-\ln K_1 \) as the delay \( r \) decreases to 0. Recall that

\[
\lambda_1(r)/r = K_2/(e^{-\lambda_1(r)} - K_1)
\]

and \(-\ln K_1 > 0 \). Then we have that \( \lambda_1(r)/r \) decreases to \( K_2/(1 - K_1) \) and \( \lambda_2(r)/r \) increases to \(+\infty\) as \( r \) decreases to 0. Therefore, the proof is complete.

We remark that without confusion, in what follows we always use the same notations \( \lambda_0, \lambda_1(r), \lambda_2(r) \) and \( r_0 \) as in Lemma 3.1. Next we count the eigenvalues of equation (4) with small delay \( r \) in the half plane \( \text{Re} \lambda \geq -\lambda_0/r \). For convenience, let \( B(d) = \{ \lambda \in \mathbb{C} : |\lambda| < d \} \) and \( \Gamma(d) = \{ \lambda \in \mathbb{C} : |\lambda| = d \} \) for a constant \( d > 0 \).

**Lemma 3.2.** Suppose that equation (4) satisfies assumption (H2) and \( r \in (0, r_0) \), then the roots of \( h(\lambda, r) = 0 \) with \( \text{Re} \lambda \geq -\lambda_0/r \) are contained in the set \( B(\lambda_0/r_0) \cap \{ \lambda \in \mathbb{C} : -\lambda_1(r)/r \leq \text{Re} \lambda \leq \lambda_1(r)/r \} \).

**Proof.** Assume that \( \lambda \) is a root of \( h(\lambda, r) = 0 \) with \( \text{Re} \lambda \geq -\lambda_0/r \). By [22, Lemma 2.1, p.263], there is a nonzero vector \( \xi \in \mathbb{C}^n \) such that \( \Delta(\lambda, r)\xi = 0 \). Then we have

\[
|\lambda| = |\lambda| \int_{-r}^{0} e^{\lambda t} d\mu(\theta, r)\xi + \int_{-r}^{0} e^{\lambda t} d\eta(\theta, r)\xi = K_1 \max\{1, e^{-r\text{Re} \lambda}\}|\lambda| + K_2 \max\{1, e^{-r\text{Re} \lambda}\}|\xi|,
\]

which together with \( \text{Re} \lambda \geq -\lambda_0/r \) implies that \( |\lambda| \leq K_1 e^{\lambda_0} |\lambda| + K_2 e^{\lambda_0} \). By using (18), we further get that

\[
|\lambda| \leq \frac{K_2 e^{\lambda_0}}{1 - K_1 e^{\lambda_0}} = \frac{\lambda_0}{r_0}.
\]

Thus, all roots of \( h(\lambda, r) = 0 \) with \( \text{Re} \lambda \geq -\lambda_0/r \) are in \( \{ \lambda \in \mathbb{C} : |\lambda| \leq \lambda_0/r_0 \} \).

Next we claim that

\[
-\lambda_0/r < -\lambda_1(r)/r \leq \text{Re} \lambda \leq \lambda_1(r)/r < \lambda_0/r.
\]

In fact, by (21) we observe that

\[
(1 - K_1 e^{-r|\text{Re} \lambda|})|\text{Re} \lambda| \leq (1 - K_1 e^{-r|\text{Re} \lambda|})|\lambda| \leq K_2 e^{-r|\text{Re} \lambda|}.
\]

Then \( \Psi(|\text{Re} \lambda|) \geq 0 \). Using Lemma 3.1, we get that \( |\text{Re} \lambda| \in \mathbb{R} \setminus (\lambda_1(r)/r, \lambda_2(r)/r) \). On the other hand, by (22) and \( \lambda_0/r_0 < \lambda_0/r \), we have \( 0 \leq |\text{Re} \lambda| \leq \lambda_0/r \). Thus claim (23) is true.

Finally, we show that \( \lambda \in B(\lambda_0/r_0) \). In fact, if \( \text{Re} \lambda \geq 0 \), (21) implies

\[
|\lambda| \leq \frac{K_2}{1 - K_1} < \frac{K_2 e^{\lambda_0}}{1 - K_1 e^{\lambda_0}} = \frac{\lambda_0}{r_0}.
\]

If \( \text{Re} \lambda \leq 0 \), by (21) and (23) we have

\[
|\lambda| \leq K_1 e^{-r|\text{Re} \lambda|} |\lambda| + K_2 e^{-|\text{Re} \lambda|} < |\lambda|K_1 e^{\lambda_0} + K_2 e^{\lambda_0},
\]
which implies that $|\lambda| < K_2 e^{\lambda_0} / (1 - K_1 e^{\lambda_0}) = \lambda_0 / r_0$. Therefore, $\lambda \in B(\lambda_0 / r_0)$. Along with (23), the proof is complete. \qed

To count the eigenvalues of equation (4), we first make some restrictions on the kernels $\mu$ and $\eta$ and consider a special case.

**Lemma 3.3.** Let the kernels $\mu$ and $\eta$ in (15) satisfy assumption (H2) and

$$
\mu(\theta, r) = \mu(\theta / r, 1), \quad \eta(\theta, r) = \eta(\theta / r, 1) \quad \text{for each } r > 0 \text{ and } \theta \in \mathbb{R}.
$$

Then the number of roots of $h(\lambda, r) = 0$ in the set $B(\lambda_0 / r_0)$ (counted by multiplicity) is a constant for $r \in (0, r_0)$.

Proof. By (24), we have

$$
\int_{-j}^{0} e^{\lambda \theta} d\mu_j(\theta, r) = \int_{-j}^{0} e^{\lambda \theta} d\mu_j(\theta / r, 1) = \int_{-j}^{0} e^{r \lambda \theta} d\mu_j(\theta, 1),
$$

$$
\int_{-j}^{0} e^{\lambda \theta} d\eta_j(\theta, r) = \int_{-j}^{0} e^{\lambda \theta} d\eta_j(\theta / r, 1) = \int_{-j}^{0} e^{r \lambda \theta} d\eta_j(\theta, 1).
$$

Then one can obtain

$$
h(\lambda, r) = h(\lambda, r) \lambda^n + \sum_{j=1}^{n} \int_{-j}^{0} e^{r \lambda \theta} d\eta_j(\theta, 1) \lambda^{n-j},
$$

$$
h(\lambda, r) = 1 + \sum_{j=1}^{n} \int_{-j}^{0} e^{r \lambda \theta} d\mu_j(\theta, 1).
$$

Clearly, $h(\lambda, r)$ is analytic for $(\lambda, r) \in \mathbb{C} \times (0, r_0)$. From Lemma 3.2 it follows that for each $r \in (0, r_0)$, there is no zero of $h(\cdot, r)$ on the set $\Gamma(\lambda_0 / r_0)$, which is the boundary of $B(\lambda_0 / r_0)$. Note that $B(\lambda_0 / r_0)$ is an open bounded subset of $\mathbb{C}$ and its closure is compact in $\mathbb{C}$. By [12, Theorem 9.17.4, p.248], for each $\alpha \in (0, r_0)$ there exists a neighborhood $U_\alpha \in (0, r_0)$ of $\alpha$ such that for all $r \in U_\alpha$ the number of roots of $h(\lambda, r) = 0$ in $B(\lambda_0 / r_0)$, counted by multiplicity, is a constant. Take each $r_1, r_2 \in (0, r_0)$ with $r_1 \neq r_2$. Since $[r_1, r_2]$ is a compact set in $\mathbb{R}$, there exist a finite open covering $U_{\alpha_1}, \cdots, U_{\alpha_m}$, satisfying $U_{\alpha_j} \cap U_{\alpha_{j+1}} \neq \emptyset$ for each $1 \leq j \leq m - 1$, such that $[r_1, r_2] \subset \bigcup_{j=1}^{m} U_{\alpha_j} \subset (0, r_0)$. Take a constant $\beta_j$ in $U_{\alpha_j} \cap U_{\alpha_{j+1}}$ for each $1 \leq j \leq m - 1$. Then by the property of $U_{\alpha_j}$, the functions $h(\lambda, r_1), h(\lambda, r_2)$ and $h(\lambda, \beta_j)$, $1 \leq j \leq m - 1$, have the same number of zeros in $B(\lambda_0 / r_0)$, counted by multiplicity. Thus, the proof is complete. \qed

The following lemma indicates the number of eigenvalues in the set $B(\lambda_0 / r_0)$.

**Lemma 3.4.** Let the kernels $\mu$ and $\eta$ in (15) satisfy assumptions (H2) and (24). Then there exist exactly $n$ roots of $h(\lambda, r) = 0$ in the set $B(\lambda_0 / r_0)$ (counted by multiplicity) for $r \in (0, r_0)$.

Proof. We first prove

$$
h(\lambda, r) \neq 0, \quad \forall r \in (0, r_0), \forall |\lambda| \leq \lambda_0 / r.
$$

(27)

Otherwise, suppose that there exist $r \in (0, r_0)$ and $\lambda$ with $|\lambda| \leq \lambda_0 / r$ such that $h(\lambda, r) = 0$. Then there is a nonzero vector $\xi \in \mathbb{C}^n$ such that $\Delta_0(\lambda, r) \xi = 0$. Along with (24), we have

$$
|\xi| = |\int_{-r}^{0} e^{\lambda \theta} d\mu(\theta, r) \xi| \leq K_1 e^{\lambda_0} |\xi|.
$$
Then $K_1e^{\lambda_0} \geq 1$, which is the contradiction with $1 - K_1e^{\lambda_0} = K_2r_0e^{\lambda_0}/\lambda_0 > 0$, where the equality is from (18). Thus (27) holds. Next we claim that there exists a constant $\delta_0 > 0$, which is independent of $r$, such that $h(\lambda, r) \geq \delta_0 > 0$ for each $r \in (0, r_0)$ and $\lambda \in \Gamma(\lambda_0/r)$. In fact,
\[
\Delta_0\left(\frac{\lambda_0e^{i\omega}}{r}, r\right) = I - \int_{-1}^{0} e^{\lambda_0e^{i\omega}}d\mu(\theta, 1),
\]
which is independent of $r$ and continuous in $\omega \in [-\pi, \pi]$, so is $h_0e^{i\omega}/r, r)$. Note that for each $\lambda \in \Gamma(\lambda_0/r)$, there is a $\omega \in (-\pi, \pi]$ such that $\lambda = \lambda_0e^{i\omega}/r$. Then by the compactness of $[-\pi, \pi]$ in $\mathbb{R}$, the claim is true.

On the other hand, note that the kernels $\mu$ and $\eta$ satisfy the assumption (24), by (25) and (26) we have that for all $1 \leq j \leq n$,
\[
\int_{-j}^{0} \theta e^{\lambda\theta}d\mu_j(\theta, r) = \int_{-j}^{0} \theta e^{\lambda\theta}d\mu_j(\theta, 1), \quad \int_{-j}^{0} e^{\lambda\theta}d\eta_j(\theta, r) = \int_{-j}^{0} e^{\lambda\theta}d\eta_j(\theta, 1). \tag{28}
\]
Since $\mu_j(\cdot, 1)$ and $\eta_j(\cdot, 1)$ are independent of $r$, and $|r\lambda| = \lambda_0$ for $\lambda \in \Gamma(\lambda_0/r)$, then by (28) one can derive that there exists a positive constant $C$, which is independent of $r$, such that for $\lambda \in \Gamma(\lambda_0/r)$ and $1 \leq j \leq n$,
\[
|h_\lambda(\lambda, r)| = |\sum_{j=1}^{n} \int_{-j}^{0} \theta e^{\lambda\theta}d\mu_j(\theta, 1)| \leq Cr.
\]
Thus as $r \to 0^+$, we obtain
\[
h(\lambda, r) = \lambda^n h(\lambda, r)(1 + O(r)), \tag{29}
\]
\[
h_\lambda(\lambda, r) = \lambda^n(h_\lambda(\lambda, r) + nh(\lambda, r)\lambda^{-1} + O(r^2)), \tag{30}
\]
where we use the fact that for $\lambda \in \Gamma(\lambda_0/r)$, $\lambda^{-1} = O(r^j)$ as $r \to 0^+$, $1 \leq j \leq n$. Together with Lemma 3.2, for each $r \in (0, r_0)$ and $\lambda_0/r_0 \leq |\lambda| \leq \lambda_0/r$ we have $h(\lambda, r) \neq 0$. Thus by the Argument Principle, the number of roots of $h(\lambda, r) = 0$ in the set $B(\lambda_0/r_0)$ (counted by multiplicity) is equal to
\[
\frac{1}{2\pi i} \oint_{\Gamma(\lambda_0/r_0)} \frac{h_\lambda(\lambda, r)}{h(\lambda, r)}d\lambda = \frac{1}{2\pi i} \oint_{\Gamma(\lambda_0/r)} \frac{h_\lambda(\lambda, r)}{h(\lambda, r)}d\lambda.
\]
Using the fact that $|h(\lambda, r)| \geq \delta_0 > 0$ and $|h_\lambda(\lambda, r)| \leq Cr$ for each $r \in (0, r_0)$ and $\lambda \in \Gamma(\lambda_0/r)$, by (29) and (30) we further get that
\[
\frac{1}{2\pi i} \oint_{\Gamma(\lambda_0/r_0)} h_\lambda(\lambda, r)\frac{d\lambda}{h(\lambda, r)} = \frac{1}{2\pi i} \oint_{\Gamma(\lambda_0/r)} \left(\frac{h_\lambda(\lambda, r)}{h(\lambda, r)} + \frac{n}{\lambda} + O(r^2)\right)d\lambda.
\]
From (27), we clearly see that $h$ has no zeros with $|\lambda| \leq \lambda_0/r$, which implies that
\[
\frac{1}{2\pi i} \oint_{\Gamma(\lambda_0/r_0)} h_\lambda(\lambda, r)\frac{d\lambda}{h(\lambda, r)} = 0.
\]
Finally, applying Lemma 3.3 and the fact that $\frac{1}{2\pi i} \oint_{\Gamma(\lambda_0/r)} \frac{h_\lambda(\lambda, r)}{h(\lambda, r)}d\lambda = n$, we have
\[
\frac{1}{2\pi i} \oint_{\Gamma(\lambda_0/r_0)} h_\lambda(\lambda, r)\frac{d\lambda}{h(\lambda, r)} = \lim_{r \to 0^+} \frac{1}{2\pi i} \oint_{\Gamma(\lambda_0/r)} h_\lambda(\lambda, r)\frac{d\lambda}{h(\lambda, r)} = n.
\]
Then the proof is complete. \qed

The following theorem gives the location of eigenvalues for linear autonomous equation (4).
Theorem 3.5. Suppose that equation (4) satisfies assumption (H2) and \( r \in (0, r_0) \). Then \( h(\lambda, r) = 0 \) has exactly \( n \) roots with \( \text{Re}\lambda \geq -\lambda_0/r \) (counted by multiplicity). Furthermore, these roots are contained in the set \( B(\lambda_0/r_0) \cap \{ \lambda \in \mathbb{C} : -\lambda_1(r)/r \leq \text{Re}\lambda \leq \lambda_1(r)/r \} \).

Proof. Taking any \( \tilde{r} \in (0, r_0) \), we define two functions of bounded variation on \( \mathbb{R} \):

\[
\bar{\mu}(\theta, r) := \mu\left(\frac{\tilde{r}\theta}{r}, \tilde{r}\right), \quad \bar{\eta}(\theta, r) := \eta\left(\frac{\tilde{r}\theta}{r}, \tilde{r}\right), \quad \forall \theta \in \mathbb{R}.
\]

(31)

Clearly, these two functions satisfy that

\[
\bar{\mu}(\theta, r) = \bar{\mu}\left(\frac{\tilde{r}\theta}{r}, \tilde{r}\right) = \bar{\mu}(\theta, 1), \quad \bar{\eta}(\theta, r) = \bar{\eta}\left(\frac{\tilde{r}\theta}{r}, \tilde{r}\right) = \bar{\eta}(\theta, 1), \quad \forall \theta \in \mathbb{R}.
\]

We note that

\[
\begin{align*}
\int_{-r}^{0} d\bar{\mu}(\theta; r) &= \int_{-\tilde{r}}^{0} d\mu\left(\frac{\tilde{r}\theta}{r}, \tilde{r}\right) = \int_{-\tilde{r}}^{0} d\mu(\theta; \tilde{r}), \\
\int_{-r}^{0} d\bar{\eta}(\theta; r) &= \int_{-\tilde{r}}^{0} d\eta\left(\frac{\tilde{r}\theta}{r}, \tilde{r}\right) = \int_{-\tilde{r}}^{0} d\eta(\theta; \tilde{r}).
\end{align*}
\]

Then \( \bar{\mu} \) and \( \bar{\eta} \) satisfy (H2). Consider the following neutral differential equation

\[
\frac{d}{dt} \left\{ x(t) - \int_{-r}^{0} d\bar{\mu}(\theta; r)x_t \right\} = -\int_{-r}^{0} d\bar{\eta}(\theta; r)x_t, \quad (32)
\]

with \( r \in (0, r_0) \). Then by Lemma 3.4 there exist exactly \( n \) roots of equation \( \tilde{h}(\lambda, r) := \det \bar{\Delta}(\lambda, r) = 0 \) in \( B(\lambda_0/r_0) \) for \( r \in (0, r_0) \), where

\[
\bar{\Delta}(\lambda, r) := \lambda \left( I - \int_{-r}^{0} e^{\lambda \theta} d\bar{\mu}(\theta, r) \right) - \int_{-r}^{0} e^{\lambda \theta} d\bar{\eta}(\theta, r).
\]

In particular, taking \( r = \tilde{r} \), from the definition (31) it follows that \( \bar{\mu}(\theta; \tilde{r}) = \mu(\theta; \tilde{r}) \) and \( \bar{\eta}(\theta; \tilde{r}) = \eta(\theta; \tilde{r}) \), which implies \( h(\lambda, \tilde{r}) = \tilde{h}(\lambda, \tilde{r}) \). Thus both \( h(\lambda, \tilde{r}) \) and \( \tilde{h}(\lambda, \tilde{r}) \) have exactly \( n \) zeros in \( B(\lambda_0/r_0) \) by Lemma 3.4. Moreover, by Lemma 3.2 we see that the roots of \( h(\lambda, r) = 0 \) with \( \text{Re}\lambda \geq -\lambda_0/r \) are contained in the set \( B(\lambda_0/r_0) \cap \{ \lambda \in \mathbb{C} : -\lambda_1(r)/r \leq \text{Re}\lambda \leq \lambda_1(r)/r \} \). Therefore, the proof is complete. \( \Box \)

As a direct consequence of Theorem 3.5, the following shows the location of eigenvalues for linear autonomous retarded differential equations with small delays.

Corollary 1. Consider the linear autonomous retarded differential equation

\[
\dot{x}(t) = Lx_t, \quad (33)
\]

where the operator \( L \) is defined in (5) and satisfies assumption (H2). Then the characteristic equation

\[
\det \Delta(\lambda, r) = 0, \quad \Delta(\lambda, r) := \lambda I - \int_{-r}^{0} e^{\lambda \theta} d\eta(\theta, r),
\]

has exactly \( n \) roots with \( \text{Re}\lambda \geq -1/r \) (counted by multiplicity). Furthermore, these roots are contained in the set \( B(K_2e) \cap \{ \lambda \in \mathbb{C} : -\lambda_1(r)/r \leq \text{Re}\lambda \leq \lambda_1(r)/r \} \).

Proof. Taking \( K_1 = 0 \) in Lemma 3.1, we find that \( \lambda_0 = 1 \) and \( r_0 = 1/(K_2e) \). By Theorem 3.5, Corollary 1 is established. \( \Box \)
We remark that counting the eigenvalues of linear autonomous retarded differential equations with small delays has been investigated in [14, p.154] for a planar system, in [1] for equation \( \dot{x}(t) = Ax(t-r) \), where \( A \) is an \( n \times n \) real matrix, and in [16, Theorem 3.1] for linear retarded equation with infinitely many delays. However, Theorem 3.5 and Corollary 1 generalize these results to general linear autonomous neutral equations and retarded equations, respectively.

In the end of this section, we introduce a splitting of the space \( C[-r,0] \) induced by a finite number of eigenvalues of equation (4). Under the conditions of Theorem 3.5, equation (4) has exactly \( n \) eigenvalues denoted by \( \Lambda := \{\lambda_1, ..., \lambda_n\} \) in the half plane \( \text{Re}\lambda \geq -\lambda_0/r \). As done in [22, Section 9.2], let \( \Phi_\Lambda = (\phi_{\lambda_1}, ..., \phi_{\lambda_n}) \) be the basis of the linear extension of the generalized eigenspaces of \( \lambda_i \)'s and \( \Psi_\Lambda \) be the basis of the linear extension of the corresponding generalized eigenspaces of the formal adjoint equation:

\[
\dot{x}(t) - \int_{-r}^{0} \dot{x}(t-\theta)d\mu(\theta,r) = -\int_{-r}^{0} x(t-\theta)d\eta(\theta,r).
\]

In particular, we can choose \( \Psi_\Lambda \) such that \( (\Psi_\Lambda, \Phi_\Lambda) = I \), where \( (\cdot, \cdot) \) is defined by (16). Then we can define a projection \( P_\Lambda \) on the space \( C[-r,0] \) of the form

\[
P_\Lambda \phi = \Phi_\Lambda (\Psi_\Lambda, \phi), \quad \forall \phi \in C[-r,0].
\]

Unlike the case of retarded equations (see [22, Theorem 6.1, p.214]), the fact that equation (4) has exactly \( n \) eigenvalues in the half plane \( \text{Re}\lambda \geq -\lambda_0/r \) can not yield the existence of a pseudo exponential dichotomy for neutral equation (4). This is because the spectrum of \( T(t) \) contains not only the point spectrum determined by the eigenvalues of equation (4), but also the continuous spectrum which can not be obtained by a general theory (see the remarks in the section 4 of [25, p.116]). We will overcome this obstacle and establish the existence of a pseudo exponential dichotomy in the next section.

4. Existence of a pseudo exponential dichotomy. In this section, we establish the existence of a pseudo exponential dichotomy for linear neutral differential equation (1). Furthermore, we give the explicit expressions of the bound and the exponents associated with this dichotomy.

Assume that the kernels \( \mu \) and \( \eta \) in equation (1) satisfy the hypothesis (H1). We start with the exponential bound on the solutions of equation (1).

**Lemma 4.1.** Let \( x(\cdot, t_0, \phi) \) be the solution of equation (1) with initial value \( x_{t_0} = \phi \). Suppose that assumption (H1) holds. Then \( x(\cdot, t_0, \phi) \) satisfies

\[
|x_t| \leq \frac{1 + 2K_1}{1 - K_1} \exp(\omega(r)(t-t_0))|\phi|, \quad \forall t \geq t_0,
\]

where the constant \( \omega(r) \) is in the form

\[
\omega(r) = \frac{K_2}{1 - K_1} - \frac{1}{r} \ln \left( \frac{1 - K_1}{1 + 2K_1} \right).
\]

**Proof.** Clearly, the solution \( x \) of equation (1) satisfies the following integral equation

\[
x(t) = M(t_0)\phi + \int_{-r}^{0} \mu(t, \theta)x(t + \theta) + \int_{t_0}^{t} L(s)x_s ds, \quad \forall t \geq t_0.
\]
For each \( t \geq t_0 \), let \( y(t) = |x_t| = \sup_{-r \leq \theta \leq 0} |x(t + \theta)| \). Then for \( t_0 \leq t \leq t_0 + r \),
\[
|x(t)| \leq (1 + K_1)|\phi| + K_1y(t) + K_2 \int_{t_0}^{t} y(s)ds.
\]
Note that for each \( t_0 \leq t \leq t_0 + r \), \( |x(\tau)| \leq |\phi| \) for \( \tau \in [t - r, t_0] \) and
\[
|x(\tau)| \leq (1 + K_1)|\phi| + K_1y(\tau) + K_2 \int_{t_0}^{\tau} y(s)ds
\]
\[
\leq (1 + 2K_1)|\phi| + K_1y(t) + K_2 \int_{t_0}^{t} y(s)ds
\]
for \( \tau \in [t_0, t] \). Then we have that
\[
y(t) \leq (1 + 2K_1)|\phi| + K_1y(t) + K_2 \int_{t_0}^{t} y(s)ds, \ \forall t_0 \leq t \leq t_0 + r.
\]
Using the fact that \( 0 < K_1 < 1 \), by Gronwall’s Inequality, we get that
\[
y(t) \leq \frac{1 + 2K_1}{1 - K_1} \exp\left(\frac{K_2(t - t_0)}{1 - K_1}\right)|\phi|, \ \forall t_0 \leq t \leq t_0 + r.
\]
Moreover, in the similar way, we have
\[
y(t) \leq \frac{1 + 2K_1}{1 - K_1} \exp\left(\frac{K_2(t - s)}{1 - K_1}\right)|y(s)|, \ \forall t_0 \leq s \leq t \leq s + r. \tag{36}
\]
Next we prove that (35) holds on \( t_0 \leq t \leq t_0 + nr \) by induction. By the above analysis, along with \( \omega(r) K_2/(1 - K_1) \) we know that (35) holds for \( n = 1 \). Assume that (35) holds for \( t_0 \leq t \leq t_0 + nr \). Then by (36), for \( t_0 + nr \leq t \leq t_0 + (n + 1)r \) we have that
\[
y(t) \leq \frac{1 + 2K_1}{1 - K_1} \exp\left(\frac{K_2(t - s)}{1 - K_1}\right)|y(s)| + \frac{1 + 2K_1}{1 - K_1} \exp\left(\frac{K_2(t - n)}{1 - K_1}\right)e^{\omega(r)(t - t_0 + nr)}|\phi|
\]
\[
= \frac{1 + 2K_1}{1 - K_1} e^{\omega(r)(t - t_0)}|\phi|.
\]
Then the proof is complete. \( \square \)

For each given \( \nu \geq 0 \) and \( t_0 \in \mathbb{R} \), let \( BC^\nu(\mathbb{T}, \mathbb{R}^n) \) denote the set of all continuous functions \( x : \mathbb{T} \rightarrow \mathbb{R}^n \) with
\[
|x|_\nu := \sup_{t \in \mathbb{T}} e^{-\nu|t - t_0|}|x(t)| < +\infty,
\]
where \( \mathbb{T} = \mathbb{R}, [t_0, +\infty) \) or \( (-\infty, t_0] \). Clearly, \( BC^\nu(\mathbb{T}, \mathbb{R}^n) \) is a Banach space equipped with the norm \( |\cdot|_\nu \), and if \( x \in BC^{\nu_1}(\mathbb{T}, \mathbb{R}^n) \) and \( 0 \leq \nu_1 < \nu_2 \), then we can obtain that \( |x|_{\nu_1} \geq |x|_{\nu_2} \).

The following theorem gives the existence of special solutions for equation (1) with small delay.

**Theorem 4.2.** Let \( r \in (0, r_0) \) and \( \nu \in (\lambda_1(r)/r, \lambda_2(r)/r) \). For each \( t_0 \in \mathbb{R} \) and \( \xi \in \mathbb{R}^n \), equation (1) has a unique solution \( x(\cdot; t_0, \xi) \) in \( BC^\nu(\mathbb{R}, \mathbb{R}^n) \) satisfying \( M(t_0)x_{t_0} = \xi \) and
\[
|x|_\nu \leq \frac{\nu}{K_2er^\nu}|\xi|, \tag{37}
\]
Moreover, the solution \( x(\cdot; t_0, \xi) \) is also a unique solution of equation (1) in the space \( BC^\nu((\infty, t_0], \mathbb{R}^n) \) with \( M(t_0)x_{t_0} = \xi \). Additionally, assume that \( 0 < r < -\frac{\lambda_2}{\lambda_1} \ln(2K_1) \) and \( 0 < K_1 < 1/2 \). Then there exists a family of inverse linear maps \( \Omega(t) : \mathbb{R}^n \to \mathbb{R}^n, t \in \mathbb{R} \) such that for each \( t_0 \in \mathbb{R}, \xi \in \mathbb{R}^n \) and \( \nu \in (\lambda_1(r)/r, K_2/K_1) \), equation (1) has a unique solution \( x \in BC^\nu(\mathbb{R}, \mathbb{R}^n) \) satisfying \( x(t_0) = \xi \) and \( M(t_0)x_{t_0} = \Omega(t_0)\xi \).

**Proof.** Fixing \( t_0 \in \mathbb{R} \) and \( \xi \in \mathbb{R}^n \), let

\[
\mathcal{B} := \{ x \in C(\mathbb{R}, \mathbb{R}^n) : \| x \| \nu \leq \nu |\xi|/K_2 e^{\nu r} \},
\]

which is a closed subset of the Banach space \( BC^\nu(\mathbb{R}, \mathbb{R}^n) \). Next we define the operator \( \mathcal{T} \) on \( \mathcal{B} \):

\[
(\mathcal{T}x)(t) = \xi + \int_{t_0}^{t} d\mu(t, \theta)x(t + \theta) + \int_{t_0}^{t} L(s)x_s ds, \quad \forall t \in \mathbb{R}. \tag{38}
\]

Clearly, \( \mathcal{T}x \in C(\mathbb{R}, \mathbb{R}^n) \) and for each \( t \in \mathbb{R} \),

\[
| (\mathcal{T}x)(t) | \leq |\xi| + \frac{K_1 e^{\nu |x|}}{\nu} (e^{\nu |t-t_0|} - 1) + \frac{K_2 e^{\nu |x|}}{\nu} (e^{\nu |t-t_0|} - 1) |\xi|/K_2 e^{\nu r}.
\]

Recall that Lemma 3.1, for \( \nu \in (\lambda_1(r)/r, \lambda_2(r)/r) \) we have \( K_1 e^{\nu r} + K_2 e^{\nu r}/\nu < 1 \), which yields that \( \mathcal{T} \) maps \( \mathcal{B} \) to itself. Moreover, for each \( x, y \in \mathcal{B} \), we have

\[
| (\mathcal{T}x - \mathcal{T}y)(t) | \leq K_1 e^{\nu r} |x - y| e^{\nu |t-t_0|} + \frac{K_2 e^{\nu r} |x - y|}{\nu} (e^{\nu |t-t_0|} - 1) |\xi|/K_2 e^{\nu r}.
\]

Using the fact that \( K_1 e^{\nu r} + K_2 e^{\nu r}/\nu < 1 \) again, we find that \( \mathcal{T} \) is a contraction on \( \mathcal{B} \). By the Contraction Mapping Principle, the operator \( \mathcal{T} \) has a unique fixed point in \( \mathcal{B} \). We denote by this fixed point \( x(\cdot; t_0, \xi) \). Then,

\[
x(t_0; t_0, \xi) = \xi + \int_{t_0}^{t} d\mu(t, \theta)x(t_0 + \theta; t_0, \xi),
\]

and \( M(t_0)x_{t_0} = \xi \). Thus the first result holds.

Similarly, we consider the operator \( \mathcal{T} \) on Banach space \( BC^\nu((\infty, t_0], \mathbb{R}^n) \). By the same way, we find that \( \mathcal{T} \) is also a contraction map. Then \( \mathcal{T} \) has a unique fixed point in \( BC^\nu((\infty, t_0], \mathbb{R}^n) \), denoted by \( \bar{x} \). Note that the restriction of \( x(\cdot; t_0, \xi) \) on \((\infty, t_0] \) is in \( BC^\nu((\infty, t_0], \mathbb{R}^n) \). Then by the uniqueness of fixed point, we have \( \bar{x}(t) = x(t; t_0, \xi) \) for \( t \leq t_0 \). Thus the second result is true.

To prove the final result, we define the following maps

\[
\mathcal{T} : \mathbb{R}^n \to BC^\nu(\mathbb{R}, \mathbb{R}^n), \quad \xi \mapsto x(\cdot; t_0, \xi),
\]

\[
\Omega(t_0) : \mathbb{R}^n \to \mathbb{R}^n, \quad \xi \mapsto \xi + \int_{t_0}^{t} d\mu(t, \theta)x(t_0 + \theta; t_0, \xi) = x(t_0; t_0, \xi).
\]
It is easy to verify that the fixed point \( x(\cdot; t_0, \xi) \) of the map \( \mathcal{T} \) is linear in \( \xi \). Then \( \mathcal{T} \) and \( \Omega(t_0) \) are also linear in \( \xi \). Since \( 0 < K_1 < 1/2 \), by a direct computation we have \( \Phi(-\ln(2K_1)) = -K_1 \ln(2K_1) > 0 \). Then for \( 0 < r < -\frac{K_1}{2} \ln(2K_1) \), we have \( \lambda_1(r) < -\ln(2K_1) < \lambda_2(r) \), which implies \( K_1 e^{\lambda_1(r)} < 1/2 < K_1 e^{\lambda_2(r)} \). Since \( K_1 e^{\lambda_i(r)} + K_2 r e^{\lambda_i(r)} / \lambda_i(r) = 1 \) for \( i = 1, 2 \), then

\[
\frac{K_1 e^{\lambda_1(r)}}{K_2 r} = \frac{K_1 e^{\lambda_2(r)}}{K_2 e^{\lambda_2(r)} / \lambda_2(r)} < \frac{K_1 e^{\lambda_2(r)}}{K_2 / \lambda_2(r)} = \frac{K_1 \lambda_2(r)}{K_2 r}.
\]

This yields \( \lambda_1(r) / r < K_2 / K_1 \). Fixing \( v \in (\lambda_1(r) / r, K_2 / K_1) \), by (37) we obtain

\[
|\Omega(t_0)\xi| \geq |\xi| - \frac{v}{K_2 e^{r/\nu}} K_1 e^{\nu r} |\xi| = (1 - \frac{K_1 v}{K_2}) |\xi|,
\]

which implies that \( \Omega(t_0) \) is an injective map. Recall that \( \Omega(t_0) \) is a linear map from \( \mathbb{R}^n \) to \( \mathbb{R}^n \). Then \( \Omega(t_0) \) is an inverse map. Thus, for each \( \zeta \in \mathbb{R}^n \), there exists a unique solution \( x(\cdot; t_0, (\Omega(t))^{-1}\zeta) \) in \( BC^v(\mathbb{R}, \mathbb{R}^n) \) satisfying \( M(t_0) x_{t_0} = (\Omega(t))^{-1}\zeta \) and \( x(t_0) = \Omega(t) \Omega(t)^{-1}\zeta = \zeta \). Therefore, the proof is complete. □

The next lemma shows that equation (1) with small delay has a matrix solution, which looks like the fundamental matrix solution for ordinary differential equations. For each \( t \geq t_0 \) and the solution \( x(\cdot; t_0, \phi) \) of equation (1), we define the mapping \( \Theta(t) \) from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) by

\[
\Theta(t) x(t) = M(t)x_t.
\]  

(39)

Clearly, \( \Theta(\cdot) x(\cdot) \) is continuously differential on \([t_0, +\infty)\).

**Lemma 4.3.** Let \( r \in (0, r_0) \) and \( \nu \in (\lambda_1(r) / r, \lambda_2(r) / r) \). Then there exists an \( n \times n \) matrix valued function \( X(\cdot, s) \) defined on \( \mathbb{R} \times \mathbb{R} \) such that for each \( s, t, u \in \mathbb{R} \),

(i) each column of \( X(\cdot, s) \) is a solution of equation (1);

(ii) \( \Theta(s) X(s, s) = I \), \( X(t, u) \Theta(u) X(u, s) = X(t, s) \) and \( \Theta(t) X(t, s) \) is an inverse matrix with the inverse \( (\Theta(t) X(t, s))^{-1} = \Theta(s) X(s, t) \);

(iii) \( |X(t, s)| \leq e^{\nu \ln(e^{t-s})} \) and \( |\Theta(t) X(t, s)| \leq e^{\nu \ln(e^{t-s})} \).

If equation (1) satisfies the following additional conditions:

\[
0 < r < -K_1 \ln(2K_1) / K_2, \quad 0 < K_1 < 1/2, \quad \nu \in (\lambda_1(r) / r, K_2 / K_1), \quad (40)
\]

then there exists a unique \( n \times n \) matrix valued function \( Y(\cdot, s) \) defined on \( \mathbb{R} \times \mathbb{R} \) such that for each \( s, t, u \in \mathbb{R} \),

(iv) \( X(t, s) \zeta \) for \( \zeta \in \mathbb{R}^n \) is a solution of equation (1) with \( x(s) = \zeta \);

(v) \( Y(s, s) = I \), \( Y(t, u) Y(u, s) = Y(t, s) \) and \( Y(t, s) \) is an inverse matrix with the inverse \( Y(t, s)^{-1} = Y(s, t) \);

(vi) \( |Y(t, s)| \leq e^{\nu \ln(e^{t-s})} \).

**Proof.** By Theorem 4.2, equation (1) has a unique solution \( x(\cdot; s, \xi) \in BC^v(\mathbb{R}, \mathbb{R}^n) \) satisfying \( M(s) x_s = \xi \), \( x(t, s) \) is linear in \( \xi \) and \( \sup_{t \in \mathbb{R}} e^{-\nu |t-s|} |x(t, s, \xi)| < +\infty \). We define \( n \times n \) matrix valued function \( X \) by

\[
X(t, s) \xi = x(t, s, \xi), \quad \forall s, t \in \mathbb{R}, \quad \xi \in \mathbb{R}^n.
\]

For each \( 1 \leq j \leq n \), let \( e^j \) be the \( j \)th column of the \( n \times n \) identity matrix \( I \). Obviously, \( X(t, s)e^j \) are solutions of equation (1). Thus claim (i) holds.

For claim (ii), by (39) we observe that \( \Theta(s) X(s, s) \xi = \Theta(s) x(s) = M(s) x_s = \xi \), which implies that \( \Theta(s) X(s, s) = I \) holds. Using the definition of \( X \), we have

\[
X(t, u) \Theta(u) X(u, s) \xi = X(t, u) \Theta(u) x(u; s, \xi) = x(t; u, \Theta(u) x(u; s, \xi)).
\]

(41)
Since both \(x(t; u, \Theta(u)x(u; s, \xi))\) and \(x(u; s, \xi)\) are solutions of equation (1) satisfying the condition \(M(u)x_u = \Theta(u)x(u; s, \xi)\). Theorem 4.2 yields that
\[
x(t; u, \Theta(u)x(u; s, \xi)) = x(t; s, \xi),
\]
which implies that \(X(t, u)\Theta(u)X(u, s) = X(t, s)\) holds. Taking \(u = s\) and \(s = t\) in (41), by (42) we get that
\[
\Theta(t)X(t, s)\Theta(s)X(s, t) = \Theta(t)X(t; s, \Theta(s)x(s; t, \xi)) = \Theta(t)x(t; t, \xi) = \xi.
\]
Claim (iii) holds.

For claim (iii), the first estimate is from (37). Since \(X(t, s)\xi\) is a fixed point of the operator \(T\), we have
\[
\Theta(t)X(t, s)\xi = \xi + \int_s^t L(\tau)X(\tau + \cdot, s)\xi d\tau
\]
which together with the first estimate in (iii) yields that
\[
|\Theta(t)X(t, s)\xi| \leq |\xi| + \int_s^t K_2 \frac{\nu}{K_2 e^{\nu t}}|\Theta(\tau)X(\tau, s)\xi| d\tau = |\xi| + \nu \int_s^t |\Theta(\tau)X(\tau, s)\xi| d\tau.
\]
Applying Gronwall's Inequality, we find the second estimate in (iii) holds.

Assume that (40) holds. Theorem 4.2 yields that \(\Omega(t)\) is inverse for each \(t \in \mathbb{R}\). Then for each \(s, t \in \mathbb{R}\) and \(\zeta \in \mathbb{R}^n\), let \(Y(t, s)\zeta = x(t; s, \Omega(s)^{-1}\zeta)\). By the similar argument as above, we get that claims (iv)-(vi) hold. This completes the proof. \(\square\)

For each given \(\phi \in C[-r, 0]\), let \(x(\cdot, t_0, \phi)\) denote the solution of equation (1) with \(x_{t_0} = \phi\). For simplicity, we also use \(x(t) = x(\cdot, t_0, \phi)\). Let \(\mathcal{E}\) denote the set of continuous functions \(y: \mathbb{R} \to \mathbb{R}^n\) satisfying
\[
|y - x|_{\mathcal{E}, 1} := \sup_{t \geq t_0 - r} |y(t) - x(t)| e^{\gamma(t - t_0)} < +\infty,
\]
\[
|y|_{\mathcal{E}, 2} := \sup_{t \leq t_0} |y(t)| e^{\nu(t - t_0)} < +\infty,
\]
where the constants \(\nu, \gamma \in (\lambda_1(r)/r, \lambda_2(r)/r)\) and \(\nu < \gamma\), which will be fixed according to our need. We define a map \(d: \mathcal{E} \times \mathcal{E} \to \mathbb{R}\) by
\[
d(y^1, y^2) := \max \left\{ \sup_{t \geq t_0 - r} |y^1(t) - y^2(t)| e^{\gamma(t - t_0)}, \sup_{t \leq t_0} |y^1(t) - y^2(t)| e^{\nu(t - t_0)} \right\}.
\]
Clearly, the map \(d\) is well defined and induces a metric for the set \(\mathcal{E}\).

We define the Banach spaces
\[
C_b([t_0 - r, +\infty), \mathbb{R}^n) := \left\{ f \in C([t_0 - r, +\infty), \mathbb{R}^n) : \sup_{t \geq t_0 - r} |f(t)| < +\infty \right\}
\]
with the supremum norm and
\[
C_b(( -\infty, t_0], \mathbb{R}^n) := \left\{ f \in C(( -\infty, t_0], \mathbb{R}^n) : \sup_{t \leq t_0} |f(t)| < +\infty \right\}
\]
with the supremum norm, respectively. Next we prove that

Lemma 4.4. The metric space \((\mathcal{E}, d)\) is complete.
Proof. Let \( \{y_m\}_{m=1}^{+\infty} \) be a Cauchy sequence in \((\mathcal{E},d)\). Then for each \( \epsilon > 0 \), there is a positive integer \( N(\epsilon) \) such that for each \( m, m' \geq N(\epsilon) \), \( d(y_{m'}, y_m) < \epsilon \), i.e.,

\[
\max\left\{ \sup_{t \geq t_0 - r} |y_{m'}(t) - y_m(t)| e^{\gamma(t-t_0)}, \sup_{t \leq t_0} |y_{m'}(t) - y_m(t)| e^{\nu(t-t_0)} \right\} < \epsilon.
\]

This implies that \( \{y_m(t)e^{\gamma(t-t_0)} : t \in [t_0 - r, +\infty]\}_{m=1}^{+\infty} \) and \( \{y_m(t)e^{\nu(t-t_0)} : t \in (-\infty, t_0]\}_{m=1}^{+\infty} \) are Cauchy sequences in \( C_b([t_0 - r, +\infty), \mathbb{R}^n) \) and \( C_b((-\infty, t_0], \mathbb{R}^n) \), respectively. By the completeness of Banach spaces, there exist the functions \( \tilde{g}_1(t) \in C_b([t_0 - r, +\infty), \mathbb{R}^n) \) and \( \tilde{g}_2(t) \in C_b((-\infty, t_0], \mathbb{R}^n) \) such that \( \tilde{g}_1(t) \) and \( \tilde{g}_2(t) \) are the limits of the sequences \( \{y_m(t)e^{\gamma(t-t_0)} : t \in [t_0 - r, +\infty]\}_{m=1}^{+\infty} \) in \( C_b([t_0 - r, +\infty), \mathbb{R}^n) \) and \( \{y_m(t)e^{\nu(t-t_0)} : t \in (-\infty, t_0]\}_{m=1}^{+\infty} \) in \( C_b((-\infty, t_0], \mathbb{R}^n) \), respectively. Then for each \( t \in [t_0 - r, t_0] \) and \( m > N(\epsilon) \), we have \( |\tilde{g}_1(t) - y_m(t)e^{\gamma(t-t_0)}| < \epsilon \) and \( |\tilde{g}_2(t) - y_m(t)e^{\nu(t-t_0)}| < \epsilon \). Thus we get that

\[
|\tilde{g}_1(t)e^{-\gamma(t-t_0)} - \tilde{g}_2(t)e^{-\nu(t-t_0)}| \leq |\tilde{g}_1(t)e^{-\gamma(t-t_0)} - y_m(t)| + |\tilde{g}_2(t)e^{-\nu(t-t_0)} - y_m(t)| < (e^{r\gamma} + e^{r\nu})\epsilon.
\]

This implies that

\[
\tilde{g}_1(t)e^{-\gamma(t-t_0)} = \tilde{g}_2(t)e^{-\nu(t-t_0)} \quad \text{for} \quad t \in [t_0 - r, t_0]. \tag{43}
\]

Let \( g : \mathbb{R} \to \mathbb{R}^n \) be the function given by \( g(t) = \tilde{g}_1(t)e^{-\gamma(t-t_0)} \) for \( t \geq t_0 \) and \( g(t) = \tilde{g}_2(t)e^{-\nu(t-t_0)} \) for \( t \leq t_0 \). By (43), we clearly see that \( g \) is continuous on \( \mathbb{R} \). Moreover, for sufficiently large \( m \),

\[
\sup_{t \geq t_0 - r} |g(t) - x(t)| e^{\gamma(t-t_0)} \leq \sup_{t \geq t_0 - r} |g(t) - y_m(t)| e^{\gamma(t-t_0)} + \sup_{t \geq t_0 - r} |y_m(t) - x(t)| e^{\gamma(t-t_0)}
\]

\[
= \sup_{t \geq t_0 - r} |\tilde{g}_1(t) - y_m(t)| e^{\gamma(t-t_0)} + \sup_{t \geq t_0 - r} |y_m(t) - x(t)| e^{\gamma(t-t_0)} < +\infty,
\]

\[
\sup_{t \leq t_0} |g(t)| e^{\nu(t-t_0)} \leq \sup_{t \leq t_0} |g(t) - y_m(t)| e^{\nu(t-t_0)} + \sup_{t \leq t_0} |y_m(t) - x(t)| e^{\nu(t-t_0)}
\]

\[
= \sup_{t \leq t_0} |\tilde{g}_2(t) - y_m(t)| e^{\nu(t-t_0)} + \sup_{t \leq t_0} |y_m(t) - x(t)| e^{\nu(t-t_0)} < +\infty.
\]

It then follows that \( g \in \mathcal{E} \). Additionally, we observe that as \( m \to +\infty \),

\[
\max\left\{ \sup_{t \geq t_0 - r} |g(t) - y_m(t)| e^{\gamma(t-t_0)}, \sup_{t \leq t_0} |g(t) - y_m(t)| e^{\nu(t-t_0)} \right\}
\]

\[
= \max\left\{ \sup_{t \geq t_0 - r} |\tilde{g}_1(t) - y_m(t)| e^{\gamma(t-t_0)}, \sup_{t \leq t_0} |\tilde{g}_2(t) - y_m(t)| e^{\nu(t-t_0)} \right\} \to 0.
\]

Thus \( d(g, y_m) \to 0 \) as \( m \to +\infty \). Then the proof is complete. \( \square \)

Next we fix the parameters \( \gamma \) and \( \nu \) associated with the space \((\mathcal{E},d)\).

**Lemma 4.5.** Let \( r \in (0, r_0) \) and

\[
g(\gamma, \nu) := \frac{K_2\gamma e^{r\gamma} + (1 + K_1)\nu\gamma}{K_2(\gamma e^{r\nu} - \nu e^{r\gamma})}. \tag{44}
\]

Then there exist constants \( \nu_1 \) and \( \gamma_1 \) with \( \lambda_1(r)/r < \nu_1 < \gamma_1 < \lambda_2(r)/r \) such that \( \gamma_1 e^{r\nu_1} - \nu_1 e^{\gamma_1} > 0 \) and \( \tilde{q}_1 := g(\gamma_1, \nu_1) > 0 \).
Proof. For any $\gamma_1 \in (\lambda_1(r)/r, \lambda_2(r)/r)$, by Lemma 3.1 we have

$$K_1 e^{\lambda_1(r)} + \frac{K_2 e^{\lambda_2(r)}}{(\lambda_1(r)/r)} = 1, \quad K_1 e^{\gamma_1} + \frac{K_2 e^{\gamma_1}}{\gamma_1} < 1.$$  

Next we consider the auxiliary function $\Lambda(\nu) := \nu e^{\gamma_1}/\gamma_1 e^{\nu}$. Note that

$$\Lambda\left(\frac{\lambda_1(r)}{r}\right) = \frac{K_2(\lambda_1(r)/r) e^{\gamma_1}}{K_2 e^{\lambda_1(r)}} < \frac{(\lambda_1(r)/r)}{K_2 e^{\lambda_1(r)}} (1 - K_1 e^{\gamma_1}) < \frac{(\lambda_1(r)/r)}{K_2 e^{\lambda_1(r)}} (1 - K_1 e^{\lambda_1(r)}) = 1.$$  

By the continuity of $\Lambda(\nu)$, we can choose the constant $\nu_1 \in (\lambda_1(r)/r, \gamma_1)$ such that

$$\gamma_1 e^{\gamma_1} - \nu_1 e^{\gamma_1} = \gamma_1 e^{\gamma_1} (1 - \Lambda(\nu_1)) > 0,$$  

which implies that $\varrho_1 = \varrho(\gamma_1, \nu_1) > 0$. Therefore, the proof is now complete. \qed

Remark 1. From the proof of Lemma 4.5 we observe that the constants $\gamma_1$ and $\nu_1$ are dependent of $r$, but no confusion should arise, we use $\gamma_1$ and $\nu_1$ instead of $\gamma_1(r)$ and $\nu_1(r)$. Moreover, the constants $\gamma_1$ and $\nu_1$ can be chosen such that both $|\gamma_1 - \lambda_2(r)/r|$ and $|\nu_1 - \lambda_1(r)/r|$ are sufficiently small.

Moreover, to obtain the existence of a pseudo exponential dichotomy and give the explicit expressions of the bounds and exponents associated with this dichotomy, a technical lemma is given as follows.

Lemma 4.6. Let $r \in (0, r_0)$, the constants $\gamma_1, \nu_1$ and $\varrho_1$ be defined as in Lemma 4.5, and $x(\cdot, t_0, \phi)$ be the solution of equation (1) with initial value $x_{t_0} = \phi$. Then there exists a unique vector $l(\phi) \in \mathbb{R}^n$ such that

$$\sup_{t \geq t_0 - r} |x(t, t_0, \phi) - X(t, t_0) l(\phi)| e^{\gamma_1(t-t_0)} < +\infty.$$  

Furthermore, if $l(\phi) = 0$ for some $\phi \in C[-r, 0]$, we have that

$$|x(t, t_0, \phi)| \leq \varrho_1 e^{-\gamma_1(t-t_0)}|\phi|, \quad \forall t \geq t_0. \quad (46)$$  

Proof. Let $x(\cdot, t_0, \phi)$ be the solution of equation (1) with initial value $x_{t_0} = \phi$ and $(E, d)$ be the complete metric space with $\varrho = \nu_1$ and $\gamma = \gamma_1$. For each $y \in E$, we define an operator $F$ in the following way:

$$F(y)(t) = M(t)x_t + \int_{t}^{t_0} d\mu(t, \theta) y_t - \int_{t}^{+\infty} L(s)(y_s - x_s)ds, \quad (47)$$  

for $t \geq t_0$ and

$$F(y)(t) = M(t_0)x_{t_0} - \int_{t_0}^{+\infty} L(s)(y_s - x_s)ds + \int_{t}^{0} d\mu(t, \theta) y_t + \int_{t_0}^{t} L(s)y_s ds, \quad (48)$$  

for $t < t_0$. We note that for any $t_2 \geq t_1 \geq t_0$,  

$$|\int_{t_1}^{t_2} L(s)(y_s - x_s)ds| \leq K_2 \int_{t_1}^{t_2} |y_s - x_s|ds \leq K_2 |y - x|_{E, 1} \int_{t_1}^{t_2} e^{-\gamma_1(s-t_0-r)}ds \leq K_2 |y - x|_{E, 1} e^{-\gamma_1(t_1-t_0-r)}/\gamma_1,$$

which together with the Cauchy Convergence Principle implies that the integral

$$\int_{t}^{+\infty} L(s)(y_s - x_s)ds, \quad \forall y \in E \text{ and } t \geq t_0,$$  

...
Combining (49 - 52), we get that which implies that is well defined, so is $F$. Moreover, since $x$ and $y$ are continuous functions on $[t_0 - r, +\infty)$ and $\mathbb{R}$, respectively, then $F(y)(t_0) = F(y)(t_0 +)$ and

$$F(y)(t_0 +) = M(t_0)x_{t_0} + \int_{-r}^{0} d\mu(t, \theta)(y_t - x_t) - \int_{t_0}^{+\infty} L(s)(y_s - x_s)ds = F(y)(t_0 -).$$

Thus $F(y)$ is continuous at $t = t_0$, implying that $F(y)$ is continuous on $\mathbb{R}$. Next we prove that $F$ maps $\mathcal{E}$ to itself. In fact, for any $t \geq t_0$, we observe that

$$|F(y)(t) - x(t)| = \left| \int_{-r}^{0} d\mu(t, \theta)(y_t - x_t) - \int_{t_0}^{+\infty} L(s)(y_s - x_s)ds \right|$$

$$\leq K_1 e^r \|y - x\|_{\mathcal{E}, 1} e^{-\gamma_1 (t-t_0)} + \frac{K_2 e^{r\gamma_1}}{\nu_1} |y - x|_{\mathcal{E}, 1} e^{-\gamma_1 (t-t_0)}$$

$$\leq \left( (K_1 e^{r\gamma_1} + \frac{K_2 e^{r\gamma_1}}{\nu_1}) |y - x|_{\mathcal{E}, 1} e^{-\gamma_1 (t-t_0)} \right).$$

(49)

Set $\zeta := M(t_0)x_{t_0} - \int_{t_0}^{+\infty} L(s)(y_s - x_s)ds$. By a direct calculation, we get that

$$|\zeta| \leq (1 + K_1) |\phi| + \frac{K_2 e^{r\gamma_1}}{\nu_1} |y - x|_{\mathcal{E}, 1}.$$  

(50)

Then we have that for $t \leq t_0$,

$$|F(y)(t)| \leq |\zeta| + K_1 e^{r\nu_1} |y|_{\mathcal{E}, 2} e^{-\nu_1 (t-t_0)} + \frac{K_2 e^{r\nu_1}}{\nu_1} |y|_{\mathcal{E}, 2} (e^{-\nu_1 (t-t_0)} - 1),$$

(51)

and for $t_0 - r \leq t \leq t_0$,

$$|F(y)(t) - x(t)| e^{\gamma_1 (t-t_0)} \leq \left( |\phi| + |\zeta| - \frac{K_2 e^{r\nu_1}}{\nu_1} |y|_{\mathcal{E}, 2} \right) e^{\gamma_1 (t-t_0)} + K_1 e^{r\gamma_1} |y|_{\mathcal{E}, 2} e^{(\gamma_1 - \nu_1)(t-t_0)}.$$  

(52)

Combining (49 - 52), we get that $F(y) \in \mathcal{E}$.

To verify that $F$ is a contraction, for any $y, z \in \mathcal{E}$ and $t \geq t_0$ we note that

$$|F(y)(t) - F(z)(t)| = | \int_{-r}^{0} d\mu(t, \theta)(y_t - z_t) - \int_{t_0}^{+\infty} L(s)(y_s - z_s)ds |$$

$$\leq K_1 e^{r\gamma_1} d(y, z) e^{-\gamma_1 (t-t_0)} + \frac{K_2 e^{r\gamma_1}}{\gamma_1} d(y, z) e^{-\gamma_1 (t-t_0)}$$

$$\leq (K_1 e^{r\gamma_1} + \frac{K_2 e^{r\gamma_1}}{\gamma_1}) d(y, z) e^{-\gamma_1 (t-t_0)},$$

which implies that

$$\sup_{t \geq t_0} |F(y)(t) - F(z)(t)| e^{\gamma_1 (t-t_0)} \leq (K_1 e^{r\gamma_1} + \frac{K_2 e^{r\gamma_1}}{\gamma_1}) d(y, z).$$

For $t \leq t_0$,

$$|F(y)(t) - F(z)(t)|$$

$$= | \int_{t_0}^{+\infty} L(s)(z_s - y_s)ds + \int_{-r}^{0} d\mu(t, \theta)(y_t - z_t) + \int_{t_0}^{t} L(s)(y_s - z_s)ds |$$

$$\leq \frac{K_2 e^{r\gamma_1}}{\gamma_1} d(y, z) + K_1 e^{r\nu_1} d(y, z) e^{-\nu_1 (t-t_0)} + \frac{K_2 e^{r\nu_1}}{\nu_1} d(y, z) (e^{-\nu_1 (t-t_0)} - 1)$$

$$\leq (\frac{K_2 e^{r\gamma_1}}{\gamma_1} - \frac{K_2 e^{r\nu_1}}{\nu_1}) d(y, z) + (K_1 e^{r\nu_1} + \frac{K_2 e^{r\nu_1}}{\nu_1}) d(y, z) e^{-\nu_1 (t-t_0)}.$$
It then follows from (45) that
\[
\sup_{t \leq t_0} | \mathcal{F}(y)(t) - \mathcal{F}(z)(t) e^{\nu_1(t-t_0)} | \leq (K_1 e^{r_{\nu_1}} + \frac{K_2 e^{\gamma_1}}{\nu_1}) d(y, z),
\]
which together with the fact that $\nu_1 < \gamma_1$ implies that for $t_0 - r \leq t \leq t_0$,
\[
\sup_{t_0 - r \leq t \leq t_0} | \mathcal{F}(y)(t) - \mathcal{F}(z)(t) e^{\gamma_1(t-t_0)} | \leq \sup_{t_0 - r \leq t \leq t_0} | \mathcal{F}(y)(t) - \mathcal{F}(z)(t) e^{\nu_1(t-t_0)} | \\
\leq (K_1 e^{r_{\nu_1}} + \frac{K_2 e^{\gamma_1}}{\nu_1}) d(y, z).
\]
Thus, $\mathcal{F}$ is a contraction. By the Contraction Mapping Principle, the operator $\mathcal{F}$ has a unique fixed point $\bar{y}$ in $(\mathcal{E}, d)$, implying that
\[
\sup_{t \geq t_0 - r} | y(t, t_0, \phi) - \bar{y}(t) e^{\gamma_1(t-t_0)} | < +\infty, \quad \sup_{t \leq t_0} | \bar{y}(t) e^{\nu_1(t-t_0)} | < +\infty.
\]
Moreover, in view of (47) and (48) we can check that $\bar{y}$ satisfies equation (1). By Theorem 4.2, there exists a unique vector $l(\phi) \in \mathbb{R}^n$ such that $\bar{y}(t) = X(t, t_0) l(\phi)$. Therefore, the first result holds.

Define
\[
Q := \{ y \in \mathcal{E} : | y - x|_{E, 1} \leq \varrho_1 | \phi |, \quad | y|_{E, 2} \leq \varrho_2 | \phi | \}.
\]
Clearly, $Q$ is a closed subset of the complete metric space $(\mathcal{E}, d)$. For any $y \in Q$, by (50) and (51) we see that for $t \leq t_0$,
\[
| \mathcal{F}(y)(t) e^{\nu_1(t-t_0)} | \leq \frac{K_2 e^{\gamma_1}}{\nu_1} \varrho_2 | \phi | e^{\nu_1(t-t_0)} + K_1 e^{r_{\nu_1}} \varrho_2 | \phi | + \frac{K_2 e^{\gamma_1}}{\nu_1} \varrho_2 | \phi | (1 - e^{\nu_1(t-t_0)}) \\
\leq (K_1 e^{r_{\nu_1}} + \frac{K_2 e^{\gamma_1}}{\nu_1}) \varrho_2 | \phi | \leq \varrho_2 | \phi |.
\]

By (49) we see that for $t \geq t_0$,
\[
| \mathcal{F}(y)(t) - x(t) e^{\gamma_1(t-t_0)} | \leq (K_1 e^{r_{\gamma_1}} + \frac{K_2 e^{\gamma_1}}{\gamma_1}) | y - x|_{E, 1} \leq \varrho_1 | \phi |.
\]
Following (53) we see that for $t_0 - r \leq t < t_0$,
\[
| \mathcal{F}(y)(t) - x(t) e^{\gamma_1(t-t_0)} | \leq | \phi | e^{\gamma_1(t-t_0)} + \varrho_2 | \phi | e^{(\gamma_1 - \nu_1)(t-t_0)} \\
\leq (1 + \varrho_2) | \phi | = \varrho_1 | \phi |.
\]
Thus $\mathcal{F}(y) \in Q$ for any $y \in Q$. Recall that the operator $\mathcal{F}$ is a contraction on $(\mathcal{E}, d)$ and has a unique fixed point $\bar{y}$ in $(\mathcal{E}, d)$. Then $\bar{y}$ is also the unique fixed point of $\mathcal{F}$ on the set $Q$. Thus, we get that
\[
\sup_{t \geq t_0 - r} | x(t) - X(t, t_0) l(\phi) e^{\gamma_1(t-t_0)} | = \sup_{t \geq t_0 - r} | x(t) - \bar{y}(t) e^{\gamma_1(t-t_0)} | \leq \varrho_1.
\]

Particularly, if $l(\phi) = 0 \in \mathbb{R}^n$, (46) holds. Therefore, the proof is now complete. □

Now we give the main results in this section.
Theorem 4.7. Suppose that the delay \( r \) for equation (1) satisfies \( r \in (0, r_0) \). Then there exists a projection \( P(t_0) \) such that the solution operators \( T(t, t_0) \), \( t \geq t_0 \), associated with equation (1), admits a pseudo exponential dichotomy:

\[
|T(t, t_0)P(t_0)\phi| \leq \kappa_1(t)e^{\nu_1(t-t_0)}|P(t_0)\phi|, \quad \forall t \leq t_0, \tag{54}
\]

\[
|T(t, t_0)Q(t_0)\phi| \leq \kappa_2(r)e^{-\gamma_1(t-t_0)}|Q(t_0)\phi|, \quad \forall t \geq t_0, \tag{55}
\]

where \( \phi \in C[-r, 0] \), \( Q(t_0) = I - P(t_0) \),

\[
\kappa_1(r) = (1 + K_1)\nu_1, \quad \kappa_2(r) = \frac{(K_2e^{\nu_1} + (1 + K_1)\nu_1)\gamma_1 e^{\nu_1}}{K_2(\gamma_1 e^{\nu_1} - \nu_1 e^{\gamma_1})},
\]

and \( \nu_1 \) and \( \gamma_1 \) defined in Lemma 4.5 satisfy \( \lambda_1(r)/r < \nu_1 < \gamma_1 < \lambda_2(r)/r \) and \( \gamma_1 e^{\nu_1} - \nu_1 e^{\gamma_1} > 0 \). Furthermore, there exists an \( n \times n \) matrix valued function \( Y(\cdot, t_0) : \mathbb{R} \rightarrow \mathbb{R}^{n \times n} \) locally of bounded variation such that for any \( t \in \mathbb{R} \),

\[
I = Y(t, t_0)X(t, t_0) + \int_0^t Y(t + \alpha, t_0)\int_{-r}^0 d\eta(t + \alpha, \theta - \alpha)X(t + \theta, t_0)d\alpha
+ \int_0^t Y(t + \alpha, t_0)d\alpha \int_{-r}^0 d\mu(t + \alpha, \theta - \alpha)X(t + \theta, t_0),
\]

and the projection \( P(t_0) \) is given by

\[
P(t_0)\phi = X_{t_0}(\cdot, t_0)(Y^t_0, \phi, t_0), \quad \forall \phi \in C[-r, 0],
\]

where \( Y^t_0(\theta) = Y(t_0 + \theta) \) for \( \theta \in [0, r] \) and \( (Y^t_0, \phi, t_0) \) is defined by (8).

Proof. By Lemma 4.5, we first note that the constants \( \nu_1 \) and \( \gamma_1 \) can be chosen such that \( \lambda_1(r)/r < \nu_1 < \gamma_1 < \lambda_2(r)/r \) and \( \gamma_1 e^{\nu_1} - \nu_1 e^{\gamma_1} > 0 \). Let the operator \( P(t_0) : C[-r, 0] \rightarrow C[-r, 0] \) be given by

\[
(P(t_0)\phi)(\theta) = X(t_0 + \theta, t_0)l(\phi), \quad \forall \phi \in C[-r, 0], \quad \forall \theta \in [-r, 0],
\]

where \( l(\phi) \) is defined as in Lemma 4.6. We claim that \( P(t_0) \) is a projection on \( C[-r, 0] \). For any \( \phi_1 \) and \( \phi_2 \) in \( C[-r, 0] \), by Lemma 4.6 we have

\[
\sup_{t \geq t_0 - r} |x(t, t_0, \phi_1) - X(t, t_0)l(\phi_1)|e^{\gamma_1(t-t_0)} < +\infty, \quad i = 1, 2,
\]

then by linearity of equation (1),

\[
\sup_{t \geq t_0 - r} |x(t, t_0, \phi_1 + \phi_2) - X(t, t_0)(l(\phi_1) + l(\phi_2))|e^{\gamma_1(t-t_0)}
= \sup_{t \geq t_0 - r} |x(t, t_0, \phi_1) + x(t, t_0, \phi_2) - X(t, t_0)(l(\phi_1) + l(\phi_2))|e^{\gamma_1(t-t_0)}
\leq \sup_{t \geq t_0 - r} |x(t, t_0, \phi_1) - X(t, t_0)l(\phi_1)|e^{\gamma_1(t-t_0)}
+ \sup_{t \geq t_0 - r} |x(t, t_0, \phi_2) - X(t, t_0)l(\phi_2)|e^{\gamma_1(t-t_0)} < +\infty.
\]

Thus by the uniqueness stated in Lemma 4.6, we have \( l(\phi_1 + \phi_2) = l(\phi_1) + l(\phi_2) \), which implies the operator \( P(t_0) \) is linear on \( C[-r, 0] \). By using the uniqueness stated in Lemma 4.6 again, we see \( P(t_0)X_{t_0}(l(\phi)) = X_{t_0}l(\phi) \), which yields \( (P(t_0))^2 \phi = P(t_0)X_{t_0}l(\phi) = X_{t_0}l(\phi) = P(t_0)\phi \). Thus the claim is proved.
For any \( s \geq t \geq t_0 \), using result (ii) in Lemma 4.3 yields
\[
\sup_{s \geq t-r} |x(s, t, x_t) - X(s, t)\Theta(t)X(t, t_0)l(\phi)|e^{\gamma_1(s-t)} = \sup_{s \geq t-r} |x(s, t_0, \phi) - X(s, t_0)l(\phi)|e^{\gamma_1(s-t)} \leq \sup_{s \geq t-r} |x(s, t_0, \phi) - X(s, t_0)l(\phi)|e^{\gamma_1(s-t_0)} \leq \sup_{s \geq t_0-r} |x(s, t_0, \phi) - X(s, t_0)l(\phi)|e^{\gamma_1(s-t_0)} < +\infty.
\]

Then by Lemma 4.3 and Lemma 4.6 we have \( l(x_t) = \Theta(t)X(t, t_0)l(\phi) \), which implies
\[
T(t, t_0)P(t_0)\phi = T(t, t_0)X_{t_0}l(\phi) = X_t(\cdot, t_0)l(\phi) = X_t(\cdot, t)\Theta(t)X(t, t_0)l(\phi) = X_t(\cdot, t)l(x_t) = P(t)T(t, t_0)\phi.
\]

Note that \( \mathcal{R}(P(t_0)) \) is expanded by the columns of \( X_{t_0} \), then \( T(t, t_0)|\mathcal{R}(P(t_0)) \) is an isomorphism from \( \mathcal{R}(P(t_0)) \) onto \( \mathcal{R}(P(t)) \) by the invertibility of \( \Theta(t)X(t, t_0) \) for \( t \geq t_0 \).

For any \( \phi \in \mathcal{R}(P(t_0)) \) with \( |\phi| \leq 1 \) and \( t \leq t_0 \), using (37) yields
\[
|T(t, t_0)\phi| = |x_t(\cdot, t_0, M(t_0)\phi)| \leq \frac{\nu_1}{K_2} e^{\nu_1(t_0-t)} |M(t_0)\phi| \leq \frac{(1 + K_1)\nu_1}{K_2} e^{\nu_1(t_0-t)}.
\]

Thus (54) holds. For any \( \phi \in \mathcal{R}(Q(t_0)) \) with \( |\phi| \leq 1 \), then \( l(\phi) = 0 \) and by (46),
\[
|T(t, t_0)\phi| = |x_t(\cdot, t_0, \phi)| \leq e^{\gamma_1}\varrho_1(\gamma_1, \nu_1)e^{-\gamma_1(t-t_0)}, \quad t \geq t_0.
\]

Thus (55) holds. The last statement is a direct corollary of Lemma 2.4. Therefore, the proof is now complete.

Finally, as a direct consequence of Theorem 4.7, we establish the existence of a pseudo exponential dichotomy for linear autonomous equation (4) with small delay.

**Theorem 4.8.** Suppose that linear autonomous neutral differential equation (4) with the delay \( r \in (0, r_0) \) satisfies the hypothesis (H2). Then \( \{T(t) : t \geq 0\} \) admits a pseudo exponential dichotomy with the projection \( P = P_\Lambda \) and for each \( \phi \in C[-r, 0] \),
\[
|T(-t)P\phi| \leq \kappa_1(r)e^{\nu_1 t}|P\phi|, \quad \forall t \geq 0, \quad (59)
\]
\[
|T(t)(I - P)\phi| \leq \kappa_2(r)e^{-\gamma_1 t}|(I - P)\phi|, \quad \forall t \geq 0, \quad (60)
\]
where the constants \( \nu_1 \) and \( \gamma_1 \) satisfy \( \lambda_1(r)/r < \nu_1 < \gamma_1 < \lambda_2(r)/r \) and \( \gamma_1 e^{\nu_1} - \nu_1 e^{\gamma_1} > 0 \), and \( \kappa_1(r), \kappa_2(r) \) are in the form
\[
\kappa_1(r) = \frac{(1 + K_1)\nu_1}{K_2}, \quad \kappa_2(r) = \frac{(K_2e^{\nu_1} + (1 + K_1)\nu_1)\gamma_1 e^{\gamma_1}}{K_2(\gamma_1 e^{\nu_1} - \nu_1 e^{\gamma_1})}.
\]

**Proof.** By Theorem 4.7, the semigroup \( \{T(t) : t \geq 0\} \) of equation (4) with \( r \in (0, r_0) \) admits a pseudo exponential dichotomy with exponents \( -\nu_1 \) and \( -\gamma_1 \), and bound \( \kappa \) on \([0, +\infty)\), furthermore, the projection \( P \) of this pseudo exponential dichotomy satisfies \( \dim\mathcal{R}(P) = n \). In order to prove Theorem 4.8, by Theorem 4.7 we see that it is only necessary to prove \( P = P_\Lambda \), where \( P_\Lambda \) is given by (34). By (54) and (55), the spectrum of \( T(t), t > 0, \) is in the set \( \{\lambda \in \mathbb{C} : \Re \lambda \geq e^{-\nu_1 t} \text{ or } \Re \lambda \leq e^{-\gamma_1 t}\} \).

Thus by the Spectral Mapping Theorems (see [37, Section 2.2]), the spectrum of the generator \( A \) of the semigroup \( \{T(t) : t \geq 0\} \) are all in the set \( \{\lambda \in \mathbb{C} : \Re \lambda \geq e^{-\nu_1 t} \text{ or } \Re \lambda \leq e^{-\gamma_1 t}\} \).
More precisely, we consider the following nonlinear neutral differential equation
\[ \frac{d}{dt}Mx_t = Lx_t + F(x_t), \quad \forall t \geq 0, \forall x \in \mathbb{R}^n, \]  
(61)
where \( M \) and \( L \) are the linear operators, which are given by (15) and satisfy assumption (H2), and the nonlinear term \( F: C[-r,0] \rightarrow \mathbb{R}^n \) satisfies the assumption: (H3) The map \( F \) is continuous, \( F(0) = 0 \) and there exists a continuous non-decreasing function \( \Upsilon: \mathbb{R} \rightarrow \mathbb{R} \) with \( \Upsilon(0) = 0 \) such that for any \( |\phi_1|, |\phi_2| \leq \delta \),
\[ |F(\phi_1) - F(\phi_2)| \leq \Upsilon(\delta)|\phi_1 - \phi_2|. \]

Let \( \kappa_1(r), \kappa_2(r) \) and \( P \) be given as in Theorem 4.8. As a preparation, we first give the upper bound estimations for the norms of projections \( P \) and \( I - P \).

**Lemma 5.1.** Let \( p_+ = |P| \) and \( p_- = |I - P| \). Then we have
\[ p_+ \leq p(r) := \frac{6e\varpi(r)(\kappa(r))^{2\varpi(r)-1}}{1 - K_1}, \]
(62)
where \( \varpi(r) = \frac{\omega(r) + \gamma_1}{\gamma_1 - \kappa_1} \), \( \kappa(r) = \max\{\kappa_1(r), \kappa_2(r)\} \), and \( \nu_1, \gamma_1 \) are defined as in Lemma 4.5.

**Proof.** By (59), for any \( \tau > 0 \) and \( \phi \in C[-r,0] \) we have that
\[ |P\phi| = |T(-\tau)PT(\tau)\phi| \leq \kappa_1(r)e^{\nu_1\tau}PT(\tau)\phi \leq \kappa_1(r)e^{\nu_1\tau}|T(\tau)P\phi|, \]
which means that
\[ |T(\tau)P| \geq (\kappa_1(r))^{-1}e^{-\nu_1\tau}p_. \]
(63)
By (60) we find that
\[ |T(\tau)(I - P)| \leq \kappa_2(r)e^{-\gamma_1\tau}p_- \]
(64)
Denote
\[ f(\tau) := (\kappa_1(r))^{-1}e^{-\nu_1\tau} - \kappa_2(r)e^{-\gamma_1\tau} \quad \text{for} \quad \tau > 0. \]
Since \( -\nu_1 > -\gamma_1 \), then \( f(\tau) > 0 \) for \( \tau > \tau_0 := (\gamma_1 - \nu_1)^{-1}\ln(\kappa_1(r)\kappa_2(r)) \). Along with the fact
\[ |p_+ - p_-| = ||P| - |I - P|| \leq |P + I - P| = 1, \]
by (63), (64) and Lemma 4.1, we have that for \( \tau > \tau_0 \),
\begin{align*}
  f(\tau) & \leq p_-^{-1}|T(\tau)P| - p_+^{-1}|T(\tau)(I - P)| \leq |p_-^{-1}T(\tau)P + p_+^{-1}T(\tau)(I - P)| \\
  & \leq \frac{3e^{\varpi(r)\tau}}{1 - K_1}|p_-^{-1}P + p_+^{-1}(I - P)| \leq \frac{3e^{\varpi(r)\tau}}{1 - K_1}(p_-^{-1} + |p_-^{-1} - p_+^{-1}|p_+) \\
  & = \frac{3e^{\varpi(r)\tau}}{1 - K_1}p_-^{-1}|1 + |p_- - p_+|| \leq \frac{6e^{\varpi(r)\tau}}{1 - K_1}p_-^{-1}. \quad (65)
\end{align*}
Thus \( p_- \leq g(\tau) := 6e^{\omega(r)\tau}((1 - K_1)f(\tau))^{-1} \) for \( \tau > \tau_0 \). Similarly, \( p_+ \leq g(\tau) \) for \( \tau > \tau_0 \). On the other hand, observe that

\[
g'(\tau) = \frac{6}{1 - K_1} \frac{f'(\tau)e^{\omega(r)\tau} - \omega(\tau)e^{\omega(r)\tau}f(\tau)}{f^2(\tau)}
\]

and \( f'(\tau)e^{\omega(r)\tau} - \omega(\tau)e^{\omega(r)\tau}f(\tau) = 0 \) has a unique root

\[
\bar{\tau} = \frac{1}{\gamma_1 - \nu_1} \ln \frac{\kappa_1(r)\kappa_2(r)(\omega(r) + \gamma_1)}{\omega(r) + \nu_1} > \tau_0,
\]

implying that \( g(\bar{\tau}) \) is the minimum value of the function \( g \) on \([\tau_0, +\infty)\). By a simply calculation, we find that

\[
g(\bar{\tau}) = \frac{6\kappa_1(r)\varpi(r)}{1 - K_1} \left(1 + \frac{1}{\varpi(r) - 1}\right)^{-\varpi(r)-1} (\kappa_1(r)\kappa_2(r))^{2\varpi(r)-2} \geq p_\pm. \tag{66}
\]

Finally, noting that \( \varpi(r) > 1, \ (1 + (\varpi(r) - 1)^{-1})^{-\varpi(r)-1} \leq e \) and \( 1 \leq \kappa_i(r) \leq \kappa(r) \), (62) is established. Therefore, the proof is now complete. \( \square \)

Next we prove the existence of invariant manifolds for nonlinear equation (61) with small delay.

**Theorem 5.2.** Let the constants \( k(r) \) and \( p(r) \) be given by Lemma 5.1. Suppose that equation (61) satisfies (H3). Then for each \( r \in (0, r_0) \), equation (61) has local invariant manifolds

\[
W_r^+ := \{ \phi \in C[-r, 0] : |\phi| \leq \delta(r), \phi = \varphi + w_+^r(\varphi), \varphi \in R(P) \cap \{ \varphi \in C[-r, 0] : |\varphi| \leq 2\delta(r) \} \},
\]

\[
W_r^- := \{ \phi \in C[-r, 0] : |\phi| \leq \delta(r), \phi = \varphi + w_-^r(\varphi), \varphi \in R(I - P) \cap \{ \varphi \in C[-r, 0] : |\varphi| \leq 2\delta(r) \} \},
\]

where \( w_\pm^r \) are lipschitz maps with \( w_\pm^r(0) = 0 \), the projection \( P \) is defined as in Theorem 4.8, the constants \( \delta(r) \) and \( \mathcal{L}(r) \) are respectively given by

\[
\delta(r) = \sup \{ \delta : 5Y(2\delta) \leq \mathcal{L}(r) \}, \quad \mathcal{L}(r) = \frac{\gamma_1 - \nu_1}{4(\kappa(r))^2 p(r)}, \tag{67}
\]

and \( \nu_1, \gamma_1 \) are defined as in Lemma 4.5. Furthermore, \( W_r^+ \) and \( W_r^- \) are respectively tangent to \( R(P) \) and \( R(I - P) \) at the origin, and for any \( \vartheta \in (\nu_1, \gamma_1) \),

\[
W_r^+ := \{ \phi \in C[-r, 0] : |\phi| \leq \delta(r), T(t)\phi \text{ exists for } t \leq 0, e^{\vartheta t}(T(t)\phi) \rightarrow 0 \text{ as } t \rightarrow -\infty \},
\]

\[
W_r^- := \{ \phi \in C[-r, 0] : |\phi| \leq \delta(r), e^{\vartheta t}(T(t)\phi) \rightarrow 0 \text{ as } t \rightarrow +\infty \}.
\]

**Proof.** To prove this theorem, consider a modification of equation (61):

\[
\frac{d}{dt} Mx_t = Lx_t + F_{\delta(r)}(x_t), \quad \forall t \geq 0, \forall x \in \mathbb{R}^n,
\]

where \( F_{\delta(r)}(\phi) = F(\chi\left(\frac{|\phi|}{\delta(r)}\right)\phi) \) for each \( \phi \in C[-r, 0] \) and the cut-off function \( \chi : [0, +\infty) \rightarrow [0, 1] \) satisfies the following properties: (i') \( \chi \in C^\infty \); (ii') \( \chi(s) = 1 \) for \( s \in [0, 1] \) and \( \chi(s) = 0 \) for \( s \in [2, +\infty) \); (iii') the derivative of order 1 of \( \chi \) is less than 2, that is, \( \chi'(s) \leq 2 \) for \( s \in [0, +\infty) \). By the basic calculation, we find that \( F_{\delta(r)} \) satisfies the following properties: (i'') \( F_{\delta(r)}(\phi) = F(\phi) \) for \( |\phi| \leq \delta(r) \); (ii'')}
Moreover, by (67) and Lemma 5.1 we clearly see that
\[ |F_{\delta_1}(\phi) - F_{\delta_2}(\phi)| \leq 5\Upsilon(2\delta(r))|\phi_1 - \phi_2|. \]
Furthermore, by Theorem 5.1, we obtain the following theorem on the finite-dimensional reduction and large spectral gap induced by small delay.

By Theorem 5.2, we obtain the following theorem on the finite-dimensional reduction and large spectral gap induced by small delay.

**Theorem 5.3.** Suppose that equation (61) satisfies the hypothesis (H3). Then as the delay \( r \) tends to zero, the spectral gap in Theorem 4.8, and the constant \( L(r) \) defined in (67) both approach infinity. In addition, suppose that there exists a family of orbits \( \Sigma(r) := \{x(t,r) : t \in \mathbb{R}\} \) for \( r \in (0, r_0) \), which are continuous in \( r \) and uniformly bounded for \( r \in (0, r_0) \). Then there exists a constant \( \tilde{r} \in (0, r_0) \) such that \( \Sigma(r) \) lies on \( W^s_r \) for each \( r \in (0, \tilde{r}) \).

**Proof.** Since the spectral gap in Theorem 4.8 is not less than \( \gamma_1 - \nu_1 \), and \( \kappa(r) = \max\{\kappa_1(r), \kappa_2(r)\} \geq 1 \) and \( p(r) \geq 1 \), then the spectral gap is not less than \( L(r) \).

Thus, to prove the first statement, it is only necessary to prove the claim:

\[ L(r) \to +\infty, \quad \text{as } r \to 0^+. \]

Redefine \( \nu_1 = \lambda_1(r)/r \) and \( \gamma_1 = \lambda_2(r)/r \) in \( \kappa(r), \kappa_1(r), \kappa_2(r) \), \( \varpi(r) \) and \( p(r) \), and then denoted by \( \kappa^0(r), \kappa^1_1(r), \kappa^1_2(r), \varpi^0(r) \) and \( p^0(r) \). By Lemma 3.1 we have the following limits:

\[ \lambda_1(r) \to 0, \quad \lambda_1(r)/r \to K_2/(1 - K_1), \quad \lambda_2(r) \to -\ln K_1, \quad \text{as } r \to 0^+. \]

Then, as \( r \to 0^+ \), we have that

\[ \kappa^0_1(r) \to \frac{1 + K_1}{1 - K_1}, \]
\[ \kappa^0_2(r) \to \frac{2}{K_1(1 - K_1)}, \]
\[ \varpi^0(r) \to \frac{\ln (K_1(1 - K_1)/(1 + 2K_1))}{\ln K_1}. \]

We further find that as \( r \to 0^+ \),

\[ 4(\kappa^0(r))^2p^0(r) \to \frac{24e\ln(K_1(1 - K_1)/(1 + 2K_1))}{(1 - K_1)\ln K_1} \]
\[ \times \exp \left\{ \frac{\ln((K_1)^3(1 - K_1)^2/(1 + 2K_1)^2)}{\ln K_1} \ln \left( \frac{2}{K_1(1 - K_1)} \right) \right\} > 0. \]

Noting the fact that \( \lambda_2(r)/r - \lambda_1(r)/r \to +\infty \) as \( r \to 0^+ \), we have that

\[ \frac{\lambda_2(r)/r - \lambda_1(r)/r}{4(\kappa^0(r))^2p^0(r)} \to +\infty, \quad \text{as } r \to 0^+. \]

By Remark 1 we learn that the constants \( \gamma_1 \) and \( \nu_1 \) can be appropriately chosen such that both \( |\gamma_1 - \lambda_2(r)/r| \) and \( |\nu_1 - \lambda_1(r)/r| \) are sufficiently small. Without loss of generality, we assume that the following inequalities hold:

\[ \frac{\gamma_1 - \nu_1}{\lambda_2(r)/r - \lambda_1(r)/r} > \frac{1}{\sqrt{2}}, \quad \frac{(\kappa^0(r))^2p^0(r)}{(\kappa(r))^2p(r)} > \frac{1}{\sqrt{2}}, \quad \forall r \in (0, r_0). \]
Thus we get that
\[
\mathcal{L}(r) = \frac{\gamma_1 - \nu_1}{\lambda_2(r)/r - \lambda_1(r)/r} \frac{\lambda_2(r)/r - \lambda_1(r)/r}{4(\kappa_0(r))^2p_0(r)} > \frac{\lambda_2(r)/r - \lambda_1(r)/r}{2 \cdot 4(\kappa_0(r))^2p_0(r)},
\]
which implies (68). Then the first statement is proved.

Suppose that the orbits \(\Sigma(r)\) are uniformly bounded for all \(r \in (0, r_0)\). Then there exists a constant \(\delta > 0\) such that \(\Sigma(r) \subset \{x \in \mathbb{R}^n : |x| \leq \delta\}\) for \(r \in (0, r_0)\). In view of (68), we always may choose a constant \(\tilde{r} \in (0, r_0)\) such that \(5\gamma(2\delta) \leq \mathcal{L}(r)\) for \(r \in (0, \tilde{r}]\). It then follows from (67) that \(\delta \leq \delta(r)\) for \(r \in (0, \tilde{r}]\). Taking any section \(\phi \in C[-\tilde{r}, 0]\) on \(\Sigma(r)\), we directly obtain that \(|\phi| \leq 2\delta(r)\) and
\[
e^{\delta t}|T(t)|\phi| \leq \delta(r)e^{\delta t} \to 0, \quad \text{as } t \to -\infty,
\]
where the constant \(\delta \in (\nu_1, \gamma_1)\). Along with Theorem 5.2, we find that \(\Sigma(r) \subset W^r_r\). Thus, the proof is now complete. \(\square\)

In the end, as an application of Theorem 5.3, we consider the following example.

**Example 1.** Consider the following nonlinear autonomous neutral differential equation:
\[
\begin{pmatrix}
\dot{x}(t) \\
\dot{y}(t)
\end{pmatrix} + k \begin{pmatrix}
\dot{x}(t - r) \\
\dot{y}(t - r)
\end{pmatrix} = k \begin{pmatrix}
\sin r & -\cos r \\
\cos r & \sin r
\end{pmatrix} \begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix} + \begin{pmatrix}
-(x^2 + y^2)y \\
(x^2 + y^2)x
\end{pmatrix}, \quad (69)
\]

where \((x, y)^T \in \mathbb{R}^2, 0 < k < 1\) and \(r > 0\). Clearly, equation (69) satisfies assumption (H3). Here \(K_1 = K_2 = k\). By a direct computation, we can check that for any \(r > 0\), equation (69) has a periodic orbit
\[
\Sigma = \{(x(t), y(t))^T \in \mathbb{R}^2 : x(t) = \cos t, y(t) = \sin t, t \in \mathbb{R}\},
\]

and the equilibrium point \((0, 0)^T\). We observe that the linearization of equation (69) at \((0, 0)^T\) is
\[
\begin{pmatrix}
\dot{x}(t) \\
\dot{y}(t)
\end{pmatrix} + k \begin{pmatrix}
\dot{x}(t - r) \\
\dot{y}(t - r)
\end{pmatrix} = k \begin{pmatrix}
\sin r & -\cos r \\
\cos r & \sin r
\end{pmatrix} \begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix}, \quad (70)
\]

For sufficiently small \(r\), equation (70) has the eigenvalues
\[
\omega_{1,2}(r) = \pm i \frac{k \sqrt{k^2 - (1 - k^2)\tan^2 r}}{(1 - k)\tan r}.
\]

In [22, Theorem 2.1, p.314], it shows that equation (69) has a local center manifold. Furthermore, we are also interested in the following question:

- **Does the periodic solution \(\Sigma\) lie on the center manifold?**

The answer to this question is affirmative. In fact, by Theorem 5.3, there exists a constant \(\tilde{r} \in (0, r_0)\) such that \(\Sigma \subset W^r_r\) for \(r \in (0, \tilde{r}]\), where \(W^r_r\) is given by the graph of a Lipschitz continuous map defined on the linear extension of \(\phi_1 = \sin(\omega_1(r)\theta)\) and \(\phi_2 = \cos(\omega_1(r)\theta)\) for \(\theta \in [-r, 0]\). Thus \(\Sigma\) lies on the center manifold of equation (69) with delay \(r \in (0, \tilde{r}]\).

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Appendix: Proof of Lemma 2.3. Let \( x \) be the solution of linear neutral differential equation (1) with \( x_i = \phi \). Then by using [6, Theorem 6.1.1, p.88], [6, Theorem 6.1.6, p.89] and several variable transformations, we have

\[
- \langle \mathcal{F}(r + x)S\psi, T(r + x) \phi \rangle 
\]

\[
= - \int_{-r}^{0} d_\theta \langle \mathcal{F}(r + x)S\psi(\theta) \rangle x(t + r + \theta) 
\]

\[
- \langle \mathcal{F}(r + x)S\psi(0) - \mathcal{F}(r + x)S\psi(0-) \rangle x(t + r) 
\]

\[
= \psi(x)x(t + r) - \int_{-r}^{0} \left\{ d_\theta \psi(t + \theta) + d_\theta \left( \int_{\theta}^{0} \psi(t + x) d_\mu(t + x + \theta - \alpha) \right) \right\} x(t + \theta) 
\]

where the last equality is from the fact that \( \mu(t + x, \theta - \alpha) = 0 \) and \( \eta(t + x, \theta - \alpha) = 0 \) for \( 0 \leq \alpha \leq \theta \). Again, using the facts that \( \eta(t, \theta) = 0 \) for \( \theta \geq 0 \) and \( \eta(t, \theta) = \eta(t, -r) \) for \( \theta \leq -r \), for \( \alpha \in [0, r] \) we observe that

\[
\frac{d}{d\alpha} M(t + \alpha)x_{t+\alpha} = \int_{-r}^{0} d_\theta \eta(t + x, \theta) x(t + x + \alpha) 
\]

\[
= \int_{-r}^{0} \eta(t + x, \theta - \alpha) x(t + \theta) 
\]

\[
= \int_{-r}^{0} d_\theta \eta(t + x, \theta - \alpha) x(t + \theta) + \int_{0}^{r} d_\theta \eta(t + x, \theta - \alpha) x(t + \theta). 
\]

Then we have

\[
\int_{0}^{r} d_\theta \left( \int_{0}^{r} \psi(x) (d_\alpha \mu(t + x, \theta - \alpha) + \eta(t + x, \theta - \alpha) d\alpha) \right) x(t + \theta) 
\]

\[
= \int_{0}^{r} \psi(x) d_\alpha \int_{0}^{r} d_\theta \mu(t + x, \theta - \alpha) x(t + \theta) 
\]

\[
+ \int_{0}^{r} \psi(x) \left( \frac{d}{d\alpha} M(t + x)x_{t+\alpha} - \int_{-r}^{0} d_\theta \eta(t + x, \theta - \alpha) x(t + \theta) \right) d\alpha. 
\]

Applying integration by parts formula [6, Theorem 6.2.2, p.100] yields

\[
\int_{0}^{r} d_\theta \psi(\theta) x(t + \theta) 
\]
\[
\begin{align*}
&= \int_0^r \rho \psi(\theta) M(t + \theta)x(t + \theta) + \int_0^r \rho \psi(\theta) \int_{-\tau}^0 d_\tau \mu(t + \theta, \tau)x(t + \theta + \tau) \\
&= \psi(r) M(t + r)x(t + r) - \psi(0) M(t)x(t) - \int_0^r \psi(\theta) \left( \frac{d}{d\theta} M(t + \theta)x(t + \theta) \right) d\theta \\
&\quad + \int_0^r \rho \psi(\theta) \int_{-\tau}^0 d_\tau \mu(t + \theta, \tau)x(t + \theta + \tau),
\end{align*}
\]

where
\[
\begin{align*}
&\int_0^r \rho \psi(\theta) \int_{-\tau}^0 d_\tau \mu(t + \theta, \tau)x(t + \theta + \theta) \\
&\quad - \int_0^r \rho \psi(\theta) d_\theta \int_{-\tau}^0 d_\tau \mu(t + \theta, \tau)x(t + \theta + \theta) \\
&\quad - \psi(0) \int_{-\tau}^0 d_\tau \mu(t, \tau)x(t + \tau) + \psi(r) \int_{-\tau}^0 d_\tau \mu(t + r, \tau)x(t + r + \tau) \\
&\quad - \int_0^r \rho \psi(\theta) d_\theta \int_{-\tau}^0 d_\tau \mu(t + \theta, \tau - \theta)x(t + \tau) \\
&\quad - \int_0^r \rho \psi(\theta) d_\theta \int_0^x d_\mu(t + \theta, \tau - \theta)x(t + \tau) \\
&\quad - \psi(0) \int_{-\tau}^0 d_\tau \mu(t, \tau)x(t + \tau) + \psi(r) \int_{-\tau}^0 d_\tau \mu(t + r, \tau)x(t + r + \tau).
\end{align*}
\]

By (A.1 - A.3), we obtain
\[
-(F(t + r)S\psi, T(t + r, t)\phi) = \psi(0)x(t) + \int_0^r \psi(\alpha) \int_{-\tau}^0 d\eta(t + \alpha, \theta - \alpha)x(t + \theta) d\alpha \\
&\quad + \int_0^r \psi(\alpha) d\alpha \int_{-\tau}^0 d_\eta(t + \alpha, \theta - \alpha)x(t + \theta),
\]

together with (8), yields that (9) holds. Therefore, the proof is now complete. \(\square\)

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