Quantum advantage does not survive in the presence of a corrupt source: Optimal strategies in simultaneous move games

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Abstract
Effects of a corrupt source on the dynamics of simultaneous move strategic games are analyzed both for classical and quantum settings. The corruption rate dependent changes in the payoffs and strategies of the players are observed. It is shown that there is a critical corruption rate at which the players lose their quantum advantage, and that the classical strategies are more robust to the corruption in the source. Moreover, it is understood that the information on the corruption rate of the source may help the players choose their optimal strategy for resolving the dilemma and increase their payoffs. The study is carried out in two different corruption scenarios for Prisoner’s Dilemma, Samaritan’s Dilemma, and Battle of Sexes.

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a. Introduction: Classical game theory has a very general scope, encompassing questions and situations that are basic to all of the social sciences [1]. There are three main ingredients of a game which is to be a model for real life situations [1]: The first of these is the rational players (decision makers) who share a common knowledge. The second is the strategy set which contains the feasible actions the players can take, and the third one is the payoff which are given to the players as their profit or benefit when they apply a specific action from their strategy set. When rational players interact in a game, they will not play dominated strategies, but will search for an equilibrium. One of the important concepts in game theory is that of Nash equilibrium (NE) in which each player’s choice of action is the best response to the actions taken by the other players. In an NE, no player can increase his payoff by unilaterally changing her action. While the existence of a unique NE makes it easier for the players to choose their action, the existence of multiple NE’s avoids the sharp decision making process because the players become indifferent between them. In pure strategies, the type and the number of NE’s in a game depend on the game. However, due to von Neumann there is at least one NE when the same game is played with mixed strategies [1,2]. Classical game theory has been successfully tested in decision making processes encountered in real-life situations ranging from economics to international relations. By studying and applying the principles of game theory, one can formulate effective strategies, predict the outcome of strategic situations, select or design the best game to be played, and determine competitor behavior, as well as the optimal strategy.

In recent years, there have been great efforts to apply the quantum mechanical toolbox in the design and analysis of games [3,4,5,6,7,8,9,10,11,12,13,14]. As it was the same in other fields such as communication and computation, quantum mechanics introduced novel effects into game theory, too. It has proved to have the potential to affect our way of thinking when approaching to games and game modelling. Using the physical scheme proposed by Eisert et al. (see Fig.1), it has been shown in several games that the dilemma existing in the original game can be resolved by using the paradigm of quantum mechanics [15]. It has also been shown that when one of the players chooses quantum strategies while the other is restricted to classical ones, the player with quantum strategies can always receive better payoff if they share a maximally entangled state [16].

Quantum systems are easily affected by their environment, and physical schemes are usually far from ideal in practical situations. Therefore, it is important to study whether the advantage of the players arising from the quantum strategies and the shared entanglement survive in the presence of noise or non-ideal components in the physical scheme. In this paper, we consider a corrupt source and analyze its effect on the payoffs and strategies of the players. We search answers for the following two questions: (i) Is there a critical corruption rate above which the players cannot maintain their quantum advantage if they are unaware of the action of the noise on the source, and (ii) How can the players adopt their actions if they have information on the corruption rate of the source.

b. Eisert’s scheme: In this physically realizable scheme the quantum version of a two-player-two-strategy classical game can be played as follows: (a) A referee prepares a maximally entangled state by applying an entangling operator J on a product state $|f\rangle|g\rangle$ where $
\{ f,g \} \in \{0,1\}. The output of this entangler, which reads

\[ \hat{J} |fg \rangle = \frac{1}{\sqrt{2}} \left( |fg \rangle + i(1-f+g)|(1-f)(1-g) \right) , \]

delivered to the players. The players apply quantum operations locally on their qubits, and return the resultant state \( |\phi_{\text{out}} \rangle = (\hat{U}_A \otimes \hat{U}_B) \hat{J} |fg \rangle \) back to the referee. Operators \( \hat{U}_A \) and \( \hat{U}_B \) are restricted to two-parameter SU(2) operators given by

\[ \hat{U} = \begin{pmatrix} e^{i\phi} \cos \theta & \sin \theta \tan \phi \\ -\sin \theta \overline{\cos \theta} & e^{-i\phi} \cos \theta \end{pmatrix}, \tag{1} \]

where \( 0 \leq \phi \leq \pi/2 \) and \( 0 \leq \theta \leq \pi \). (c) The referee, upon receiving this state, applies \( \hat{J}^\dagger \) and then makes a quantum measurement \( \Pi_n = |j\ell \rangle \langle j\ell | \) with \( n = 2j + \ell \) and \( j, \ell \in \{0,1\} \). Then the average payoffs of the players become

\[ \$A = \sum_n a_n \operatorname{Tr}(\Pi_n \hat{J}^\dagger \hat{\rho}_{\text{out}} \hat{J}) \]

\[ \$B = \sum_n b_n \operatorname{Tr}(\Pi_n \hat{J}^\dagger \hat{\rho}_{\text{out}} \hat{J}) \tag{2} \]

where \( \hat{\rho}_{\text{out}} = |\phi_{\text{out}} \rangle \langle \phi_{\text{out}} | \), \( a_n \) and \( b_n \) are the payoffs chosen from the classical payoff matrix when the measurement result is \( n \), and \( P_{j,\ell} \) corresponds to the probability of obtaining \( n \). The classical version of the game can be played using the same scheme if the operations corresponding to the classical pure strategies are chosen as \( \sigma_0 \) and \( i\sigma_y \).

Using this scheme, quantum versions of some dilemmaproducing classical games, such as Prisoner’s Dilemma (PD), Samaritan’s Dilemma (SD) and Battle of Sexes (BoS) whose payoffs matrices are given in Fig.2 have been studied. In these games, it has been understood that if the referee starts with the state \( |fg \rangle = |00 \rangle \) generating the entangled state \( |\Psi \rangle = \frac{1}{\sqrt{2}} \left( |00 \rangle + i|11 \rangle \right) \), the players can resolve their dilemma and receive the highest possible total payoff \( \$A + \$B \). It has also been shown that the dynamics of the games changes when the referee starts with a different initial state. For example, if the referee starts with \( |fg \rangle = |01 \rangle \) in SD, four NE’s emerge with the same constant payoff making a solution to the dilemma impossible.

c. Corrupt Source in Quantum Games: As we have pointed out above, the initial state from which the referee prepares the entangled state is a crucial parameter in Eisert’s scheme. Therefore, any corruption or deviation from the ideality of the source which prepares this state will change the dynamics and outcomes of the game. Consequently, the analysis of situations where the source is corrupt is necessary to shed a light in understanding the game dynamics in the presence of imperfections. We consider the source model shown in Fig. 3. This model includes two identical sources constructed to prepare the states \( |0 \rangle \)’s which are the inputs to the entangler at each run of the game. These sources are not ideal and have a corruption rate, \( r \), that is, they prepare the desired state \( |0 \rangle \) with probability \( (1 - r) \) while preparing the unwanted state \( |1 \rangle \) with probability \( r \). The state prepared by these sources thus can be written as
\( \hat{\rho}_1 = \hat{\rho}_2 = (1-r)|0\rangle\langle 0| + r |1\rangle\langle 1| \). Then the combined state generated and sent to the entangler becomes \( \hat{\rho}_1 \otimes \hat{\rho}_2 = (1-r)^2|00\rangle\langle 00| + r^2|11\rangle\langle 11| + r(1-r)(|01\rangle\langle 01| + |10\rangle\langle 10|) \). This results in a mixture of the four possible maximally entangled states
\[
(1-r)^2|\psi^+\rangle\langle \psi^+| + r^2|\psi^-\rangle\langle \psi^-| + r(1-r)(|\phi^+\rangle\langle \phi^+| + |\phi^-\rangle\langle \phi^-|),
\]
where \(|\psi^+\rangle = |00\rangle \mp i|11\rangle\) and \(|\phi^\mp\rangle = |01\rangle \mp i|10\rangle\). This is the state on which the players will perform their unitary operators.

**Scenario I:** In this scenario, the players Alice and Bob are not aware of the corruption in the source. They assume that the source is ideal and always prepares the initial state \(|fg\rangle = |00\rangle\), and hence that the output state of the entanger is always \(|\psi^+\rangle\). Based on this assumption, they apply the operations that is supposed to resolve their dilemma.

We have analyzed PD, SD and BoS according to this scenario, and compared the payoffs of the players with respect to the corruption rate. The payoffs they receive when they stick to their quantum strategies are compared to the payoffs when they play the game classically. We consider the classical counterparts both with and without the presence of noise in the game. That is, the players use the same physical scheme of the quantum version with and without the corrupt source, and apply their actions by choosing their operators from the set \( \{ \hat{\sigma}_0, i\hat{\sigma}_y \} \).

The results of the analysis according to this scenario are depicted in Figs. 3-5. A remarkable result of this analysis is that with the introduction of the corrupt source, the players’ quantum advantage is no longer preserved if the corruption rate, \( r \), becomes larger than a critical corruption rate \( r^* \). At \( r^* \), the classical and quantum strategies produce equal payoffs. Another interesting result is the existence of a strategy \( U_A = U_B = (\hat{\sigma}_0 + i\hat{\sigma}_y)\sqrt{2} \), where the payoffs of the players become constant independent of corruption rate. This strategy could be attractive for risk avoiding and/or paranoid players.

For PD, which is a symmetric game, the optimal classical strategies deliver the payoffs \( \$A = \$B = 1 \) for the actions \( \hat{U}_A = \hat{U}_B = i\hat{\sigma}_y \). In the quantum version with an uncorrupt source, the players get the optimal payoffs \( \$A = \$B = 3 \) if they adopt the strategies \( \hat{U}_A = \hat{U}_B = i\hat{\sigma}_y \). Hence, the dilemma of the game is resolved and the players receive better payoffs than those obtained with classical strategies. However, as seen in Fig. 4, the payoffs of players with classical and quantum strategies become equal to 2.25 when \( r = r^* = 1/2 \). If \( r \) satisfies \( 0 \leq r < 1/2 \), the quantum version of the game always does better than the classical one. Otherwise, the classical game is better.

When the classical version of PD is played with a corrupt source, we find that with increasing corruption rate, the payoffs for the quantum strategy decrease, those of the classical one increase. That is, if \( r > r^* \), then the players would rather apply their classical strategies than the quantum ones. This can be explained as follows: When the players apply classical operations, the game is played as if there is no entanglement in the scheme. That is, players apply their classical operators \( i\hat{\sigma}_y \) on the state prepared by the source. If the source is ideal, \( r = 0 \), they operate on the \(|00\rangle\) which results in an output state \(|11\rangle\). Referee, upon receiving this output state and making the projective measurement, delivers \( \$A = \$B = 1 \). On the other hand, when \( r = 1 \), the state from the source is \(|11\rangle\) and the output state after the players actions becomes \(|00\rangle\). With this output state, referee delivers them the payoffs \( \$A = \$B = 3 \).

Thus, when the players apply...
the classical operator $i\hat{\sigma}_y$, their payoffs continuously increase from one to three with the increasing corruption rate from $r = 0$ to $r = 1$.

Using a classical mixed strategy in the asymmetric game of SD, the players receive $(\mathcal{S}_A, \mathcal{S}_B) = (-0.2, 1.5)$ at the NE. In this strategy, while Alice chooses from her strategies with equal probabilities, Bob uses a biased randomization where he applies one of his actions, $\hat{\sigma}_0$, with probability 0.2. The most desired solution to the dilemma in the game is to obtain an NE with $(\mathcal{S}_A, \mathcal{S}_B) = (3, 2)$. This is achieved when both players apply $i\hat{\sigma}_z$ to $|\psi^+\rangle$. The dynamics of the payoffs in this game with the corrupt source when the players stick to their operators $i\hat{\sigma}_z$ and its comparison with their classical mixed strategy are depicted in Fig. 6. Since this game is an asymmetric one, the payoffs of the players, in general, are not equal. However, with the corrupt source it is found that their payoffs become equal at $r = 1/7$ and at $r = 1$, where the payoffs are 96/49 and 0, respectively. The critical corruption rate, $r^* = 1/2$, which denotes the transition from the quantum advantage to classical advantage regions, is the same for both players. While for increasing $r$, $\mathcal{S}_B$ monotonously decreases from two to zero, $\mathcal{S}_A$ reaches its minimum of $-0.2$ at $r = 0.8$, where it starts increasing to the value of zero at $r = 1$. It is worth noting that when the players apply their classical mixed strategies in this physical scheme, $\mathcal{S}_B$ is always constant and independent of the corruption rate, whereas $\mathcal{S}_A$ increases linearly as $\mathcal{S}_A = -0.2 + 0.9r$ for $0 \leq r \leq 1$. The payoffs of the players are compared in three cases [4]: Case 1: $\mathcal{S}_A \leq 0$ (insufficient solution), Case 2: $0 < \mathcal{S}_A \leq \mathcal{S}_B$ (weak solution), and Case 3: $0 \leq \mathcal{S}_B < \mathcal{S}_A$ (strong solution). In the corrupt source scenario in quantum strategies, Case 1 is achieved for $0.6 \leq r \leq 1$, Case 2 is achieved for $1/7 \leq r < 0.6$, and finally Case 3 for $r < 1/7$. The remarkable result of this analysis is that although the players using quantum strategies have high potential gains, there is a large potential loss if the source is deviated from an ideal one. The classical strategies are more robust to corruption of the source.

In BoS, which is an asymmetric game, the classical mixed strategies, where Alice and Bob apply $\hat{\sigma}_x$ with probabilities 1/3 and 2/3 or vice versa, the players receive equal payoffs of 2/3. However, the dilemma is not solved due to the existence of two equivalent NE. On the other hand, when the physical scheme with quantum strategies is used the players can reach an NE where their payoffs become $\mathcal{S}_A = 1$ and $\mathcal{S}_B = 2$ if both players apply $i\hat{\sigma}_z$ to the maximally entangled state prepared with an ideal source [10]. The advantage of this quantum strategy to the classical mixed strategy is that in the former $\mathcal{S}_A + \mathcal{S}_B$ is higher than the latter. In the presence of corruption in the source, payoffs of the players change as shown in Fig. 6. With an ideal source, the payoffs reads $(\mathcal{S}_A, \mathcal{S}_B) = (1, 2)$, however for increasing corruption rate while $\mathcal{S}_B$ decreases from two to one, $\mathcal{S}_A$ increases from one to two. With a completely corrupt source, $r = 1$, the payoffs become $(\mathcal{S}_A, \mathcal{S}_B) = (2, 1)$. The reason for this is the same as explained for PD. When the quantum strategies with and without corrupt source are compared to the classical mixed strategy without noise, it is seen that the former ones always give better payoffs to the players. However, when the source becomes noisy (corrupt), classical strategies become more advantageous to quantum ones with increasing corruption rate. The range of corruption rate where classical strategies are better than the quantum strategy if the players stick to their operations $i\hat{\sigma}_y$, are $0.2 < r < 0.5$ and $0.5 < r < 0.8$ for Alice and Bob, respectively. When $r = 1/2$, $\mathcal{S}_A = \mathcal{S}_B$ and these payoffs are equal to the ones received with classical mixed strategies. While $\mathcal{S}_A = \mathcal{S}_B$ independently of $r$ for classical mixed strategies, $\mathcal{S}_A$ and $\mathcal{S}_B$ differ when the players stick to their quantum strategies for $r \neq 1/2$. Another interesting result for this game is that, contrary to PD and SD, the strategy $\hat{U}_A = \hat{U}_B = (\hat{\sigma}_0 + i\hat{\sigma}_y)/\sqrt{2}$ discussed above always gives a constant payoff $(3/4, 3/4)$, which is better than that of the classical mixed strategy.

**Scenario II:** In this scenario, the referee knows the characteristics of the corruption in the source, and informs the players on the corruption rate. Then the question is whether the players can find a unique NE for a known source with corruption rate $r$; and if they can, does this NE resolve their dilemma in the game or not. When the corruption rate is $r = 1/2$, the state shared between the players become $\hat{\rho} = \hat{I}/4$. Then independent of what action they choose, the players receive constant payoffs determined by averaging the payoff entries in the classical game payoff matrices. In this case, the players get
In SD, the payoffs are $A\pi/2$ and $1/2$. For PD, an interesting result is that there is no difference in the payoffs between an ideal source, $\theta = 1$, and a completely corrupt source, $\theta = 0$. That is, the players can resolve the dilemma receiving the best possible payoffs, $\$A = \$B = 3$, in both cases. However, the strategies which lead to a unique NE in these two extreme cases are different: When $r = 0$, the players can resolve the dilemma by applying $U_A = U_B = i\sigma_z$; however when $r = 1$, they have to change their actions to $U_A = U_B = (\sigma_0 + i\sigma_z)/\sqrt{2}$ in order to resolve the dilemma.

For BoS, while an NE is achieved resolving the dilemma with $\$A + \$B = 3$ for both $r = 0$ and $r = 1$, the corruption rate shows its effect in the payoffs and the actions to reach NE's. When $r = 0$, the payoffs are $(\$A, \$B) = (1, 2)$, on the other hand when $r = 1$, payoffs become $(\$A, \$B) = (2, 1)$. As can be seen in Table III, the difference in the strategies is the choice of $\phi_A$ and $\phi_B$; while for $r = 1$ the players should choose $\phi_A = \phi_B = 0$ to arrive at the NE, for $r = 0$ they have an infinite number of choices for $\phi_A$ and $\phi_B$ and any of these choices will work equally well.

The effect of a corrupt source is much stronger for the SD game. In this game, in contrast to the other two, although for $r = 0$ there is a unique NE solving the dilemma, for $r = 1$ the players cannot find a unique NE. There emerges an infinite number of different strategies with equal payoffs $(3, 2)$. The players are indifferent between these strategies and cannot make sharp decisions. Therefore, the dilemma of the game survives, although its nature changes.

When we look at some intermediate values for the corruption rate, we see that corruption rate affects BoS and SD strongly. For example, when $r = 3/4$ in SD, there are infinite number of strategies and NE's which have the same payoffs $(21/16, 15/16)$. These NE's are achieved when the players choose their operators as $\theta_A = \theta_B = 0$ and $\phi_B = -\phi_A + \pi/2$. The same is seen in BoS for $r = 1/4$ which results in a payoff $(15/16, 21/16)$ when the players choose $\theta_A = \theta_B = \pi/2$ and $\phi_B = -\phi_A + \pi/2$. A more detailed analysis carried out for PD with increasing $r$ in steps of 0.1 in the range $[0, 1]$ has revealed that the players can achieve a unique NE where their payoffs and strategies depends on the corruption rate. Therefore, information on the source characteristic might help the players to reorganize their strategies. However, whether providing the players with this kind of information in a game is acceptable or not is an open question.

### Conclusion

This study shows that the strategies to achieve NE's and the corresponding payoffs are strongly dependent on the corruption of the source. In a game with corrupt source, the quantum advantage no longer survives if the corruption rate is above a critical value. The corruption may not only cause the emergence of multiple NE's but may cause a decrease in the player's payoff, as well, even if there is a single NE. If the players are given the characteristics of the source then they can adapt their strategies; otherwise they can either continue...
their best strategy assuming that the source is ideal and take the risk of losing their quantum advantage over the classical or choose a risk-free strategy, which makes their payoff independent of the corruption rate. However, in the case where players know the corruption rate and adjust their strategies, the problem is that for some games there emerge multiple NE’s, therefore the dilemmas in those games survive. This study reveals the importance of the source used in a quantum game.

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[15] The quantum strategy \((\hat{\sigma}_0 - i\hat{\sigma}_y)/\sqrt{2}\) can vanish any classical strategy of the other player provided that there is a shared maximally entangled state.

[16] In BoS, if the source is ideal, that is \(r = 0\), there are multiple NE’s with the same payoff. (The strategy set achieving the NE’s are shown in Table III.) However, the strategies with \((\theta_A = \pi, \phi_A)\) and \((\theta_B = \pi, \phi_B)\), where \(\phi_A\) and \(\phi_B\) are in the range \([0, \pi/2]\), is a focal equilibrium point. This strategy set enables the players to converge to the NE without a concern for the choice of \(\phi_A\) and \(\phi_B\).