Interactions of multiple spin-2 fields beyond pairwise couplings

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Thus far, all known ghost-free interactions of multiple spin-2 fields have involved at most pairwise couplings of the fields, which are direct generalizations of bimetric interactions. Here we present a class of spin-2 theories with genuine multi-field interactions, and explicitly demonstrate the absence of ghost instabilities. The construction involves integrating out a nondynamical field in a theory of spin-2 fields with only pairwise ghost-free interactions. The new multivierbein interactions generated are not always expressible in terms of the associated metrics.

INTRODUCTION

Interacting theories for multiple fields with spin 0, 1/2 and 1 are well understood and realized in Nature via the Standard Model of Particle Physics, where the multiplets and their mixings are crucial for the viability of the theory. In contrast, General Relativity is the simplest possible theory of a single spin-2 field. It is a fundamental question whether, in analogy with lower spins, consistent theories of multiple spin-2 fields could exist. Such theories could have profound implications for the understanding of the gravitational force beyond General Relativity, but have not been easy to construct.

In a covariant set up, spin-2 fields have more components than physically needed, and generic theories do not have enough symmetries and constraints to remove the unphysical components. Some of these, if not eliminated, give rise to ghost instabilities, an example being the Boulware-Deser ghost of a massive spin-2 field. A few years ago the ghost-free theory of two interacting spin-2 fields was found \cite{1}, which also fulfills some other important consistency criteria\textsuperscript{3,4}. This generalized previous work on a single massive spin-2 field in a fixed background \cite{6}. The model is formulated in terms of two symmetric rank-two tensors (or “metrics”) $g_{\mu\nu}$ and $f_{\mu\nu}$ interacting through a specific potential $V_{bi}(g, f)$, hence the name “ghost-free bimetric theory”\textsuperscript{5}. For a recent review see \cite{3}.

Theories for more than two spin-2 fields are also strongly restricted by the absence of ghosts. From the analysis of \cite{2}, it is easy to see that certain ghost-free theories can be constructed as straightforward extensions of bimetric theory, by simply adding copies of the bimetric potentials $V_{bi}(g, f)$ for pairs of metrics but without forming loops. An example for four metrics $g_{\mu\nu}^{I}$ would be $V_{bi}(g^{1}, g^{2}) + V_{bi}(g^{1}, g^{3}) + V_{bi}(g^{3}, g^{4})$. So far, these pairwise couplings have been the only known ghost-free interactions of multiple spin-2 fields.

An important class of multi spin-2 theories was constructed in \cite{10}, using antisymmetrized products of the vierbein fields. These appeared to be ghost-free, however, a more detailed analysis in \cite{11} revealed that generically such multivierbein interactions contained ghosts. It was argued that the ghost-free subset consisted only of models where the vierbeins could be traded off for the metrics by virtue of a vierbein symmetrization condition, exactly as in bimetric theory. The only known models of this type are the pairwise interactions described above. Hence the question is if genuine multiple spin-2 interactions beyond the pairwise ones, and beyond the class conjectured in \cite{11}, exist. Here we show that this is indeed the case.

Summary of results. In this work we derive a class of ghost-free interactions for multiple spin-2 fields by integrating out a non-dynamical field in a theory with ghost-free bimetric interactions. The result is an interaction term for $N$ vierbeine $(e_{I})_{\mu}^{A}$ of the form,\textsuperscript{6}

$$S_{\text{multi}} = -M^{4} \int d^{4}x \det \left( \sum_{I=1}^{N} \beta_{I} e_{I} \right), \quad (1)$$

involving a mass scale $M$ and arbitrary dimensionless coefficients $\beta_{I}$, $I = 1, \ldots, N$. The kinetic terms of the vierbein fields have the standard Einstein-Hilbert form. For restricted vierbein configurations the multi-vierbein vertex can be expressed in terms of metrics, but this is not always possible. The interactions can involve up to four different vierbeine in each term and are therefore more general than the pairwise couplings known so far.

GENERATING NEW INTERACTIONS

The starting point is a theory for $(N + 1)$ vierbeine, $u_{\mu}^{A}$ and $(e_{I})_{\mu}^{A}$, $I = 1, \ldots, N$, with ghost-free bimetric interactions. Let us denote the corresponding metrics by $f_{\mu\nu}^{0} = u_{\mu}^{A} \eta_{AB} u_{\nu}^{B}$ and $f_{\mu\nu}^{I} = (e_{I})_{\mu}^{A} \eta_{AB} (e_{I})_{\nu}^{B}$. The action has the following structure,

$$S[u, e_{I}] = \sum_{J=0}^{N} S_{\text{EH}}[f^{J}] + S_{\text{int}}[u, e_{I}]. \quad (2)$$

It includes the Einstein-Hilbert kinetic terms,

$$S_{\text{EH}}[f^{J}] = m_{J}^{2} \int d^{4}x \sqrt{f^{J}} R(f^{J}), \quad (3)$$
where, \( m_J \) are the \((\mathcal{N} + 1)\) Planck masses. \( S_{\text{int}} \) contains the simplest ghost-free bimetric interactions between \( u_I^A \) and each one of the \((e_I)^A \mu \),

\[
S_{\text{int}}[u, e_I] = -2m^4 \int d^4x \det u \left( \beta_0 + \sum_{I=1}^{\mathcal{N}} \beta^I \text{Tr}(u^{-1}e_I) \right).
\]

In addition to the traces \( \text{Tr}(u^{-1}e_I) = u^\mu_A (e_I)^A \mu \), we could also include the remaining ghost-free bimetric interactions, but have chosen not to do so. For brevity, we set \( \beta^I = 1 \) for \( I = 1, \ldots, \mathcal{N} \), by scaling \( e_I \rightarrow e_I/\beta^I \), and redefining the Planck masses \( m_I \) in \( \mathcal{I} \) accordingly.

**Lorentz constraints.** Each vierbein contains 6 Lorentz parameters that drop out of the corresponding metric and hence appear in the action \( \mathcal{I} \) only through the potential terms in \( S_{\text{int}} \). Since these are nondynamical, their equations of motion are constraints. Specifically, these are the antisymmetric parts of the equations of motion for \((e_I)^A \mu \) \( \mathcal{I} \), \( \mathcal{I} \), \( \mathcal{I} \), \( \mathcal{I} \), \( \mathcal{I} \), \( \mathcal{I} \), \( \mathcal{I} \), \( \mathcal{I} \), precisely 6 equations per vierbein,

\[
\frac{\delta S_{\text{int}}}{\delta (e_I)^A \mu} \eta^{AB} (e_I)^{\nu B} - \frac{\delta S_{\text{int}}}{\delta (e_I)^A \mu} \eta^{AB} (e_I)^{\mu B} = 0.
\]

The corresponding equation for \( u^\mu_A \) is a linear combination of \( \mathcal{I} \), due to the overall Lorentz invariance of the action. For the potential in \( \mathcal{I} \), the Lorentz constraints uniquely imply the following symmetrization conditions,

\[
u^A_B \eta_{AB}(e_I)^{B \mu} = u^A_B \eta_{AB}(e_I)^{B \mu} , \quad I = 1, \ldots, \mathcal{N}.
\]

These allow us to express the potential in terms of metrics by replacing \( u^{-1}e_I = \sqrt{f_0^{-1}f_I} \delta \) in addition to \( \det u = \sqrt{\det f_0} \). It is then straightforward to verify that the arguments for the absence of ghost in bimetric theory \( \mathcal{I} \) extend to this case.

**Multi-spin-2 action.** Let us take the limit \( m_0 \rightarrow 0 \) to freeze out the dynamics of the vierbein \( u^A \). We can then eliminate \( u^A \) algebraically using its equation of motion to obtain an action for the remaining \( \mathcal{N} \) vierbeine. Indeed, varying the action \( \mathcal{I} \) with respect to the inverse vierbein \( u^A \) gives an equation with the unique solution

\[
u^A = -\frac{3}{\beta_0} \sum_{I=1}^{\mathcal{N}} (e_I)^A \mu.
\]

Using this to eliminate \( u^A \) in \( \mathcal{I} \), gives the new interactions for the remaining \( \mathcal{N} \) vierbeine,

\[
S_{\text{int}}[e_I] = -M^4 \int d^4x \det (e_1 + e_2 + \ldots + e_\mathcal{N}) , \quad (8)
\]

with \( M^4 = -54m^4\beta_0^{-3} \). Although these are not linear combinations of the pairwise bitemetric potentials, we nevertheless expect them to be ghost-free since they were derived from a ghost-free set up. We will explicitly confirm this below.

The \( \mathcal{N} \) symmetrization constraints \( \mathcal{I} \), with \( u^A \) given by \( \mathcal{I} \), provide a set of constraint equations for the new multi-spin-2 theory. Equivalently, one can derive these from the multi-spin-2 action \( \mathcal{I} \) using \( \mathcal{I} \). In matrix notation these new Lorentz constraints are,

\[
\sum_{I=1}^{\mathcal{N}} e_I^\gamma \eta e_I = \sum_{I=1}^{\mathcal{N}} e_J^\gamma \eta e_J , \quad J = 1, \ldots, \mathcal{N}.
\]

Note that if we sum \( \mathcal{I} \) over \( J \), the resulting equation is identically satisfied. Hence there are \( \mathcal{N} - 1 \) independent matrix relations among the antisymmetric parts of the combinations \((e_I)^A \mu \eta_{AB}(e_I)^{B \nu} \). These can be used to eliminate \( 6(\mathcal{N} - 1) \) non-dynamical components of the vierbeine. Another 6 components are removed by the overall Lorentz invariance of the action. This leaves us with \( 16\mathcal{N} - 6\mathcal{N} = 10\mathcal{N} \) independent components, which is the same as the number of components in \( \mathcal{N} \) symmetric rank-2 tensor fields.

**DIRECT GHOST PROOF**

In the following we will use \( 3 + 1 \) metric variables to demonstrate the existence of the constraints that remove the Boulware-Deser ghosts from the physical spectrum. This choice of variables is convenient for isolating the nondynamical fields in the Einstein-Hilbert actions, but their use requires some justification. In general, \( \mathcal{N} \) Lorentzian metrics may not admit compatible notions of space and time and compatible \( 3 + 1 \) decompositions. However, the vierbeine \( e_I \) are not fully independent since the parent theory gives rise to constraints \( \mathcal{I} \). These, as shown in \( \mathcal{I} \), insure that simultaneous \( 3 + 1 \) decompositions exist for the pair \((u, e_I)\), for each \( I \). Geometrically, the null cone of each \( e_I \) shares some common timelike and spacelike directions with the null cone of \( u \). Therefore, with further mild restrictions on the ranges of metric variables, there always exist large classes of configurations where all \( e_I \) share a common spatial hypersurface with \( u \), and hence, admit simultaneous \( 3 + 1 \) decompositions. In the new multi-vierbein theory, the null cones are further correlated by the equation \( \mathcal{I} \) for \( u \). The ghost proof below assumes such configurations admitting simultaneous \( 3 + 1 \) decompositions. It may be possible to further generalize it to a covariant analysis on the lines of \( \mathcal{I} \).

We decompose a vierbein \( e^\mu_A \) into a gauge-fixed vierbein \( E^A \mu \), rotated by a Lorentz transformation \( \Lambda^A_B \),

\[
e^A \mu = \Lambda^A_B E^B \mu \]

and use the following \( 3 + 1 \) parameterization \( \mathcal{I} \), \( \mathcal{I} \), \( \mathcal{I} \),

\[
E^A \mu = \left( \begin{array}{cc} N & 0 \\ E^\gamma_j & E^{\gamma}_j \end{array} \right),
\]

\[
\Lambda^A_B = \left( \begin{array}{cc} \gamma & v_c \\ v^{\gamma}_a + \frac{1}{1 + \gamma} v^{\alpha}_a v_c \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & \left( \Omega \right)^c_b \end{array} \right).
\]


Here \( N \) is the lapse, \( N^i \) is the shift vector, and \( E^a_i \) is a gauge fixed spatial vierbein with six independent components. The Lorentz transformation has been further decomposed into spatial rotations parameterized by an \( SO(3) \) matrix \( \Omega \) (containing three parameters), and Lorentz boosts parameterized by the 3-dimensional (rescaled) boost vector \( v^a \), with \( \gamma \equiv \sqrt{1 + v_a v^a} \).

In the action, we write \( (N-1) \) of the vierbeine \( (e_I)^A \), using the \( 3+1 \) paratermization given above. The overall Lorentz invariance of the action allows us to take the last vierbein to be of the same form but with \( v^a = 0 \) and \( \Omega = 1 \). Since the Lorentz parameters do not show up in the kinetic terms, they appear without derivatives in the action. Their equations are the Lorentz constraints \( \delta \).

It is easy to show that the potential \( \delta \) is linear in all lapses \( N_I \) and shifts \( N^i_J \) before integrating out the Lorentz fields \( v^a_I \) and \( \Omega_I \); The potential is \( \delta_{u} \), where the matrix \( u \) is given by \( \delta \). The \( N_I \) and \( N^i_J \) appear only in the first column \( u^a_I \). Since \( \delta_{u} \) is antisymmetric in the columns of \( u \), each term in it contains only one factor of \( u^a_I \), hence the linearity. The Einstein-Hilbert terms are also linear in these variables, hence, the entire action has the form,

\[
S = \sum_{I=1}^{N} \int d^4x \left( (\pi_I)^a_a (\dot{E}_I)^a_i - N_I C_I - N^i_I C_i^j \right). \quad (12)
\]

Here, \( (\pi_I)^a_a \) are the canonical momenta conjugate to the \( (E_I)^a_a \), and all other fields are nondynamical. The functions \( C_I \) and \( C_i^j \) contain the \( E_I, \pi_I \), and the Lorentz fields, \( v^a_I \) and \( \Omega_I \), but not the lapses and shifts. We note that so far in this section the considerations, including the form of the action in (12), are not restricted to the model \( \delta \), but also apply to the general class of multivierbein theories introduced in \( \Xi \). However, the difference is crucial for the argument that follows.

The dynamical fields \( (E_I)^a_a \) contain the ghost modes. These must be eliminated by the constraints that arise from the equations of motion for the \( N_I \) and \( N^i_J \). That is, \( C_I = 0 \) and \( C_i^j = 0 \), as well as from equations \( \Xi \) for the Lorentz fields, and the gauge fixing conditions for general covariance. In particular it is necessary that, after solving for the Lorentz fields, \( C_I = 0 \) become equations for the \( E_I \) and \( \pi_I \) alone, remaining independent of the lapses. Then they can eliminate the ghost fields in favour of the remaining dynamical variables. The difficulty is that for the general multivierbein interactions of \( \Omega \), which also have the form (12), the Lorentz constraints \( \Xi \) render the spatial rotations \( \Omega_I \), and hence the \( C_I \), dependent on the \( N_I \) \( \Xi \). Then, \( C_I = 0 \) can be solved for the \( N_I \) rather than eliminate the ghost fields which will remain in the spectrum. We now show that the multivierbein interaction in \( \Xi \) circumvents this problem.

First of all, \( (N-1) \) combinations of the \( N \) vector equations \( C_i^j = 0 \) can be used to determine the \( (N-1) \) boost vectors \( v^a_I \) as \( v^a_I (E, \pi, \Omega) \). The unused combination of the \( C_i^j \), say \( C^i \), are the three constraints of spatial diffeomorphisms. To determine the rotation matrices \( \Omega_I \), consider the Lorentz constraints that for the model \( \Xi \) give the \( (N-1) \) independent matrix equations in \( \Omega \). The spatial parts of these are the \( 3(N-1) \) equations,

\[
\sum_{I=1}^{N} (e_I)^A_i \eta_{AB} (e_J)^B_j = 0, \quad J = 1, \ldots, N - 1. \quad (13)
\]

Crucially, the \( (e_I)^A_i \) do not contain the lapses nor the shifts since from (10) it follows that,

\[
e^A_i = \left( \delta^a_b + \frac{v^a_v^b}{\gamma} \right) \Omega^b_e E^c_i. \quad (14)
\]

Therefore, the constraints (13), along with the solutions for \( v^a_I \) determine the \( 3(N-1) \) parameters of the rotation matrices as \( \Omega_I \). This insures that \( C_I \) depend only on the \( E_I \) and \( \pi_I \). After solving for the \( N_I \) in terms of \( N^i_J \). At this stage, the only other variables in the action are the \( E_I \) and the \( \pi_I \).

Of the remaining constraints, one combination of the \( C_I \), say \( C \), together with the unused \( C^i \), form a set of 4 first class constraints associated with general covariance, as in \( \Pi \). The remaining \( N-1 \) combinations of the \( C_i^j \) are second class constraints. Their preservation in time, \( \dot{C}_I = 0 \), gives another set of \( N-1 \) constraints. Although we have not proven this here, their existence can be argued in the parent theory \( \eta \) where the calculations are very similar to the bimetric case explicitly analysed in \( \Omega \).

The degree of freedom count is now easy. The fields \( (E_I)^a_a \) and the momenta \( (\pi_I)^a_a \) contain \( 2 \times 6N = 12N \) phase space variables, including the \( N \) ghost fields and their \( N \) conjugate momenta. The \( N - 1 \) second class constraints and their associated time preservation conditions eliminate \( 2(N-1) \) ghost fields and ghost momenta. The 4 first class constraints and the associated symmetry eliminate 8 phase space variables, including the last pair of ghost variables, just as in general relativity. The physical phase space thus consists of \( 12N - (2N+6) = 10N - 6 \) variables, corresponding to one massless field (with 4 phase space modes) and \( (N-1) \) massive fields of spin-2 (each with 10 phase space modes).

**EXISTENCE OF METRIC FORMULATIONS**

Ghost-free multivierbein theories with only pairwise interactions can be fully expressed in terms of the corresponding metrics. Here we explore the existence of a metric formulation for the non pairwise interaction
in \([3]\). By extracting a factor of \(\det e_1\), and using \(g_{\mu\nu} \equiv (e_1)^A_{\mu} \eta_{AB}(e_1)^B_{\nu}\), the interaction becomes,

\[
S_{\text{int}} = -M^4 \int d^4x \sqrt{g} \det \left( 1 + g^{-1} \sum_{j=2}^N e_j^T \eta e_j \right). \tag{15}
\]

Defining the antisymmetric matrices \(A_{IJ}\) as,

\[
A_{IJ} \equiv \frac{1}{2} (e_I^T \eta e_J - e_J^T \eta e_I),
\]

the action depends on the linear combination \(\sum_{j=1}^N A_{IJ}\). Also, in terms of these, the Lorentz constraints \((14)\) read,

\[
\sum_{J=1}^N A_{IJ} = 0, \quad I = 1, \ldots, N. \tag{17}
\]

Hence, the antisymmetric parts drop out of \((15)\). For any pair of vierbeine, if we set \(A_{IJ} = 0\), then one could replace \(e_I^T \eta e_J = (g_{IJ})^{-1/2}\), where \(g_{IJ} = e_I^T \eta e_J\). Then a covariant formulation in terms of the metrics would be possible if \(A_{IJ} = 0\) for all \(J\). But the constraints \((17)\) that arise in the theory are weaker. Hence, although the \(A_{IJ}\) drop out of the action, the theory has no equivalent metric formulation. However, metric formulations exist under mild restrictions.

**Metric formulation for three fields.** We now consider metric formulations for the case \(N = 3\), following an analysis carried out in Ref. \([18]\). Let us introduce 3 Lorentz matrices \((L_I)^{AB}_C\) as St"uckelberg fields for the 3 vierbeine \((e_I)^A_{\mu}\), that is, we write every vierbein as a Lorentz rotation of a gauge fixed vierbein \((\bar{e}_I)^A_{\mu}\),

\[
(e_I)^A_{\mu} = (\bar{L}_I)^A_{\mu} (e_I)^{AB}_C \bar{C}^{BC}_{\nu} (e_I)^B_{\nu}. \tag{18}
\]

Defining \((L_I)^{\nu}_{\mu} = (\bar{e}_I)^{\nu}_{\nu} (\bar{L}_I)^{\nu}_{\nu} (\bar{e}_I)^{\nu}_{\nu}\), we can also write,

\[
(e_I)^{\nu}_{\mu} = (\bar{e}_I)^{\nu}_{\nu} (\bar{L}_I)^{\nu}_{\nu} (\bar{e}_I)^{\nu}_{\nu}. \tag{19}
\]

Let us choose the gauge fixed vierbeine such that,

\[
(\bar{e}_I)^A_{\mu} \eta_{AB}(\bar{e}_J)^B_{\nu} = (\bar{e}_I)^A_{\nu} \eta_{AB}(\bar{e}_J)^B_{\nu}, \tag{20}
\]

for \(I, J = 1, 2, 3\). General vierbeine may not be gauge fixed in this way \([19]\), but this is possible for restricted field configurations such that the null cones associated with the \(I\) and \(J\) vierbeine intersect \([3]\). Note that this is a stronger requirements than the existence of simultaneous \(3 + 1\) decompositions assumed for the ghost analysis. Now, the Lorentz constraints \([3]\), provide the following two independent sets of equations for the \(L_I\) which determine two of the three Lorentz matrices,

\[
\begin{align*}
L_I^1 e_I^T \eta e_J L_J - (1 \leftrightarrow 2) = L_I^2 e_I^T \eta e_J L_J - (1 \leftrightarrow 3), \\
L_I^2 e_I^T \eta e_J L_J - (2 \leftrightarrow 3) = L_I^3 e_I^T \eta e_J L_J - (1 \leftrightarrow 3). \tag{21}
\end{align*}
\]

Multiplying the first with \(L_I^{1T}\) from the left and \(L_I^{-1}\) from the right and the second with \(L_J^{-1T}\) from the left and \(L_J^{-1}\) from the right, it is obvious that the covariant solution is \(L_1 L_2^{-1} = L_1 L_3^{-1} = L_2 L_3^{-1} = 1\) which implies \(L_1 = L_2 = L_3\). Then the multi-spin-2 potential becomes,

\[
V = M^4 \det(e_1 + e_2 + e_3), \tag{22}
\]

since the undetermined overall Lorentz matrix drops out due to Lorentz invariance. We can extract a factor of \(\det e_1\) to obtain,

\[
V = M^4 \det(e_1) \det(1 + e_1^{-1} e_2 + e_1^{-1} e_3). \tag{23}
\]

Let us introduce the metrics \(g_{\mu\nu}, f_{\mu\nu}\), and \(h_{\mu\nu}\), for the vierbeine \(e_1, e_2, e_3\), respectively. Due to the symmetrization constraint \((20)\) we have that,

\[
\bar{e}_1^{-1} e_2 = \sqrt{g^{-1} f}, \quad \bar{e}_1^{-1} e_3 = \sqrt{g^{-1} h}, \tag{24}
\]

Then the multi-spin-2 potential can be written in terms of metrics as,

\[
V = M^4 \sqrt{g} \det \left( \mathbb{1} + \sqrt{g^{-1} f} + \sqrt{g^{-1} h} \right). \tag{25}
\]

We could have chosen to extract the determinant of a vierbein other than \(e_1\) and correspondingly obtain,

\[
V = M^4 \sqrt{f} \det \left( \mathbb{1} + \sqrt{f^{-1} g} + \sqrt{f^{-1} h} \right). \tag{26}
\]

The above considerations partially generalize to \(N > 3\). Now the gauge choices \((20)\) cannot be made for all vierbeine since these are \(N(N-1)/2\) gauge conditions, while there are at most \(N\) Lorentz transformations to implement the symmetrizations. The remaining conditions may be imposed by hand on the \(\bar{e}_I\), but this is not necessary. To find a metric formulation similar to, say, \((25)\), it is enough to impose \((20)\) only on the \((\bar{e}_I)^A_{\mu} \eta_{AB}(\bar{e}_J)^B_{\nu}\) which gives \(N - 1\) gauge conditions. Then, with appropriate restrictions, and on choosing a similar solution for the \(L_I\), one obtains an expression which is a direct generalization of \((25)\) to \(N\) metrics.

Finally, it is important to note that while the vierbeine were restricted by hand to obtain a metric formulation, such restrictions are inbuilt in the final multivierbein theory. Hence, the resulting multivierbein theories can be considered in their own right, independent of the starting vierbeine formulations.

**DISCUSSION**

To summarize, we have constructed nontrivial interactions of multiple spin-2 fields, beyond the known pairwise potentials, and have demonstrated the existence of constraints that eliminate the extra ghost modes. The interactions are given in terms of the \(N\) vierbeine \((e_I)^A_{\mu}\)
in equation (11). On expressing the determinant in terms of the wedge products of the one-forms $e_I^A$, one gets,

$$
\sum_{I,J,K,L=1}^N \beta^{IJKL} e_{ABC} e_I^A \wedge e_J^B \wedge e_K^C \wedge e_L^D.
$$

(27)

with $\beta^{IJKL} = \beta^I \beta^J \beta^K \beta^L$. Such interactions with general $\beta^{IJKL}$ were proposed in [10] where the $N = 2$ case reproduces the bimetric theory [2]. However, the only ghost-free cases known so far were the pairwise bimetric interactions [11]. Given the interactions in (27) with arbitrary $\beta^{IJKL}$, a direct analysis that identifies the ghost-free cases is so far not known. But the construction presented here generates a class of genuinely multivierbein ghost-free interactions with up to 4 different vierbeine in one vertex. This trivially extends to $D$ space-time dimensions, in which case the vertices would contain up to $D$ different fields.

The class of theories obtained here can be further generalized. First, note that the interactions (11) cannot be simply added to a theory involving the previously known ghost-free interactions of the $N$ vierbeine $(e_I)_{\mu}^a$. Such a setup would correspond to forbidden loop couplings in the parent theory with vierbein $u^\mu_a$. However, compatible interactions can be constructed by extending the parent theory by additional allowed pairwise couplings as will be discussed in (21). The new terms couple any of the vierbeine $e_I$ to an additional set of $N'$ vierbeine $v_K$ and include the following.

1. One of the $e_I$ can interact with the $v_K$ through the standard pairwise interactions, that is the potential can have the form $V \sim \det(\sum_I e_I) + \sum_K V_{bi}(e_I, v_K)$. Such a setup would correspond to forbidden loop couplings in the parent theory with vierbein $u^\mu_a$. However, compatible interactions can be constructed by extending the parent theory by additional allowed pairwise couplings as will be discussed in (21). The new terms couple any of the vierbeine $e_I$ to an additional set of $N'$ vierbeine $v_K$ and include the following.

2. One of the $e_I$ can interact with the $v_K$ through a determinant interaction, i.e., the potential could be $V \sim \det(\sum_I e_I) + \det(e_I + \sum_K e_K)$. It is straightforward to introduce a standard coupling to matter via any of the dynamical vierbeine $(e_I)_{\mu}^a$, as this will not influence the procedure of integrating out the non-dynamical vierbein $u^\mu_a$. On the other hand, if $u^\mu_a$ couples to matter, then the multi-spin-2 theory for the $(e_I)_{\mu}^a$ will have matter interactions that are heavily modified [21]. At low energies these reduce to the matter coupling suggested in (22, 23).

The interactions for $(N + 1)$ vierbeine that we started with in (11) were not of the most general ghost-free form. It would be interesting to extend our setup to more general interactions and obtain possibly ghost-free multi-spin-2 theories that are different from the one studied here. It is not obvious that the algebraic equations for the non-dynamical vierbein can be solved covariantly for more general parameter choices in the action. In any case, if the linear relation in (7) is lost, the resulting interactions may not have the general form in (25) and (27). It is important to find out if such ghost-free multivierbein interactions, beyond the classes discussed here, could exist.

Our couplings are the first instance of ghost-free spin-2 interactions where the vierbein formulation admits more general configurations than the associated, more restrictive, metric formulation. Hence it is interesting to directly investigate the associated multimetric theories, without recourse to the vierbein formulation. One expects that the extra restrictions on the metrics, already encoded in the multimetric interactions [3], would lead to better causal properties for the theory.

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