Explicit solutions of $G$-heat equation with a class of initial conditions by $G$-Brownian motion

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Abstract. We obtain the viscosity solution of $G$-heat equation with the initial condition $\phi(x) = x^n$ for each integer $n \geq 1$ using the method of $G$-Brownian motion.

Keywords: $G$-heat equation, sublinear expectation, $G$-normal distribution, $G$-expectation, $G$-Brownian motion.

1 Introduction
The notions of $G$-normal distribution, $G$-expectation and $G$-Brownian motion were firstly introduced by Peng (see [10] and [11]) via the following $G$-heat equation:

$$\frac{\partial u}{\partial t} - G(\frac{\partial^2 u}{\partial x^2}) = 0, \quad u|_{t=0} = \phi, \quad (t, x) \in [0, \infty) \times \mathbb{R},$$

where $G(a) = \frac{1}{2}(a^+ - \sigma^2 a^-)$. Here $\sigma \in [0, 1]$ is a fixed constant and $a^+ = \max\{0, a\}, a^- = (-a)^+$. This equation is also the Barenblatt equation except the case $\sigma = 0$ (see [1], [2] and [6]).

Under the sublinear framework, Peng gave many important notions corresponding to linear case and obtained many important properties of $G$-Brownian motion (see [13]). Recently, Peng developed the law of large numbers and central limit theorem under sublinear expectations, which indicate that $G$-normal distribution plays the same important role in the theory of sublinear expectations as normal distribution in the linear expectations (see [12] and [14]).

Since the importance of $G$-normal distribution and $G$-Brownian motion, Peng proposed the problem of how to calculate the $G$-expectation of $\phi(B_t)$? For convex or concave $\phi$, Peng gave the formula in [10] and [11]. But for neither convex nor concave $\phi$, how to calculate? In particular, the calculation of $B_t^{2n+1}$, for each integer $n \geq 1$.

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In this paper, we give the relation between the solution of $G$-heat equation \(1\) with the initial condition $\phi(x) = x^n$ for each integer $n \geq 1$ and the solution of ordinary differential equation (see Section 3), then we can get the solution of $G$-heat equation \(1\) by solving ordinary differential equation. In particular, we get the $G$-expectation of $B^{2n+1}_t$ for each integer $n \geq 1$. In fact, we also get the solution of the Barenblatt equation with the same initial condition. Finally, we point out the application of our result in mathematical finance.

This paper is organized as follows: in Section 2, we recall briefly the notions of $G$-normal distribution, $G$-expectation and $G$-Brownian motion. Also, we prove that the $G$-expectation of $B^{2n+1}_t$ is strictly bigger than 0 which is different from linear case. The main theorem in which we get the solution of $G$-heat equation (1) with the initial condition $\phi(x) = x^n$ for each integer $n \geq 1$ is stated and proved in Section 3. In Section 4, we consider the application of our result in mathematical finance.

2 $G$-Brownian Motions under $G$-expectations

Let $\Omega = C_0(\mathbb{R}^+)$ be the space of all $\mathbb{R}$-valued continuous paths $(\omega_t)_{t \in \mathbb{R}^+}$, with $\omega_0 = 0$, equipped with the distance

$$
\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} \left( \max_{t \in [0, i]} |\omega^1_t - \omega^2_t| \right) \wedge 1.
$$

The corresponding canonical process $B_t(\omega) = \omega_t$, $t \in [0, \infty)$, for $\omega \in \Omega$. We denote $C_{t, \text{Lip}}(\mathbb{R}^n)$ the linear space of (local Lipschitz) functions $\phi$ satisfying

$$
|\phi(x) - \phi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \quad \forall x, y \in \mathbb{R}^n,
$$

for some $C > 0$, $m \in \mathbb{N}$ depending on $\phi$.

For each fixed $T \in [0, \infty)$, we set

$$
L_{ip}^0(\mathcal{F}_T) := \{ \phi(B_{t_1}, \cdots, B_{t_n}) : \forall n \in \mathbb{N}, t_1, \cdots, t_n \in [0, T], \forall \phi \in C_{t, \text{Lip}}(\mathbb{R}^n) \}.
$$

It is clear that $L_{ip}^0(\mathcal{F}_t) \subseteq L_{ip}^0(\mathcal{F}_T)$, for $t \leq T$. We also set

$$
L_{ip}^0(\mathcal{F}) := \bigcup_{n=1}^{\infty} L_{ip}^0(\mathcal{F}_n).
$$

Following \[10\] and \[11\], we can construct a consistent sublinear expectation called $G$-expectation $\hat{\mathbb{E}}[\cdot] : L_{ip}^0(\mathcal{F}) \mapsto \mathbb{R}$ satisfying, for each $X, Y \in L_{ip}^0(\mathcal{F})$,

(a) **Monotonicity:** if $X \geq Y$, then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$.

(b) **Constant preserving:** $\hat{\mathbb{E}}[c] = c$, $\forall c \in \mathbb{R}$.

(c) **Sub-additivity:** $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$. 

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(d) **Positive homogeneity**: \( \hat{E}[\lambda X] = \lambda \hat{E}[X], \forall \lambda \geq 0. \)

Under \( G \)-expectation \( \hat{E}[\cdot] \), the canonical process \( \{B_t : t \geq 0\} \) is called \( G \)-Brownian motion, and the distribution of \( B_1 \) is said to be the \( G \)-normal distribution, i.e., for each \( \phi \in C_{l,Lip}(\mathbb{R}) \), the function

\[
u(t,x) := \hat{E}[\phi(x + \sqrt{t}B_1)], \quad (t,x) \in [0, \infty) \times \mathbb{R}
\]

is the viscosity solution (see [3]) of \( G \)-heat equation (1). Moreover, \( G \)-Brownian motion has independent and stationary increments, i.e., for each \( n \in \mathbb{N}, \phi \in C_{l,Lip}(\mathbb{R}^n), s,t \geq 0 \) and \( t_1, \ldots, t_{n-1} \in [0,t], \) we have

\[
\hat{E}[\phi(B_{t_1}, \ldots, B_{t_{n-1}}, B_{t+s} - B_t)] = \hat{E}[\phi(B_{t_1}, \ldots, B_{t_{n-1}})],
\]

where \( \varphi(x_1, \ldots, x_{n-1}) = \hat{E}[\phi(x_1, \ldots, x_{n-1}, \sqrt{s}B_1)]. \) Specially, for each \( \phi \in C_{l,Lip}(\mathbb{R}), \hat{E}[\phi(B_t)] = \hat{E}[\phi(\sqrt{t}B_1)], \forall t > 0 \). In particular, for each integer \( n \geq 1, \hat{E}[B_1^{2n+1}] = t^{n+\frac{1}{2}}\hat{E}[B_1^{2n+1}]. \)

**Remark 1** For \( \sigma \in (0,1], \) \( G \)-heat equation (1) is a uniformly parabolic PDE and \( G \) is a convex function, then it has a unique \( C^{1,2} \) solution (see [7] and [10]).

In the following, \( P^W \) denotes Wiener probability measure on \( \Omega, E^W \) always denotes the linear expectation with respect to \( P^W \). Under \( P^W \), the canonical process \( \{B_t : t \geq 0\} \) is the classical standard Brownian motion.

**Lemma 2** For each fixed \( \sigma \in [0,1), \) we have

(i) For each \( \phi \in C_{l,Lip}(\mathbb{R}), \hat{E}[\phi(B_t)] \geq \sup_{\sigma \leq \nu \leq 1} E^W[\phi(\nu B_t)]. \)

(ii) For each integer \( n \geq 1, \hat{E}[B_1^{2n+1}] > 0. \)

**Proof.** Noting that \( u(t,x) := \hat{E}[\phi(x + B_t)] \) and \( u^\nu(t,x) := E^W[\phi(x + \nu B_t)] \) are respective the viscosity solution of \( G \)-heat equation (1) and \( u^\nu_t - \frac{1}{2}\nu^2 u^\nu_{xx} = 0 \) with the same initial condition, by comparison theorem for parabolic partial differential equations (see [3]), we get (i). It follows from (2) that

\[
\hat{E}[B_1^{2n+1}] = \hat{E}[(B_{\frac{1}{n}} + B_1 - B_{\frac{1}{n}})^{2n+1}] = \hat{E}[\phi(B_{\frac{1}{n}})],
\]

where \( \phi(x) = \hat{E}[(x + B_{\frac{1}{n}})^{2n+1}]. \) As a consequence of (i),

\[
\phi(x) \geq \sup_{\sigma \leq \nu \leq 1} E^W[(x + \nu B_{\frac{1}{n}})^{2n+1}]
\]

\[
\geq \sum_{i=0}^{n} \binom{2n+1}{2i} x^{2(n-i)+1} E^W[B_{\frac{1}{n}}^{2i}] + \frac{n(2n+1)}{2}(1 - \sigma^2)(x^-)^{2n-1}.
\]

Therefore,

\[
\hat{E}[B_1^{2n+1}] = \hat{E}[\phi(B_{\frac{1}{n}})] \geq E^W[\phi(B_{\frac{1}{n}})] \geq \frac{n(2n+1)}{2}(1 - \sigma^2)E^W[(B_{\frac{1}{n}})^{2n-1}].
\]

As \( \sigma < 1, \) we get \( \hat{E}[B_1^{2n+1}] > 0 \) for each integer \( n \geq 1. \)
In this Section, we discuss the solution of G-heat equation with the initial condition \( \phi(x) = x^n \) for each integer \( n \geq 1 \). Noting that for convex (resp. concave) \( \phi \), \( u(t,x) := E^W[\phi(x + B_t)] \) (resp. \( u(t,x) := E^W[\phi(x + \sigma B_t)] \)) is the solution of G-heat equation. Consequently, we only consider the following G-heat equation:

\[
\frac{\partial u}{\partial t} - \frac{1}{2}(\frac{\partial^2 u}{\partial x^2})^+ - \sigma^2(\frac{\partial^2 u}{\partial x^2})^- = 0, \quad u(0,x) = x^{2n+1},
\]

where \( \sigma \in [0,1] \), \( (t,x) \in [0,\infty) \times \mathbb{R} \). For \( \sigma = 1 \), G-heat equation is classical heat equation, then \( u(t,x) := E^W[(x+1)^{2n+1}] \) is the unique solution. In order to give the solution of G-heat equation with \( \sigma \in [0,1] \), we define for each integer \( n \geq 0 \),

\[
\begin{align*}
g_n(x) &= \sum_{i=0}^{n} \frac{(2n+1)!}{(2(n-i))!(2i+1)!} x^{2i+1}, \\
h_n(x) &= \sum_{i=0}^{n} \frac{(n+i)!(n-i)!}{(2(n-i))!(2i)!} \left( \binom{2n+1}{0} + \cdots + \binom{2n+1}{n-i} \right) x^{2i},
\end{align*}
\]

with the convention that \( 0! = 1 \) and \( 0!! = 1 \). Now, we give our main Theorem.

**Theorem 5** For each integer \( n \geq 1 \), \( g_n(\cdot) \) and \( h_n(\cdot) \) defined in (4). The following statements hold.

(i) For each fixed \( \sigma \in (0,1) \), we define

\[
P^n_\sigma(x) = \begin{cases}
g_n(x) + \frac{b_n}{(2n)!} h_n(x) \exp(-\frac{x^2}{2}) - g_n(x) \int_{-\infty}^{\infty} \exp(-\frac{y^2}{2}) \, dy, & x \geq c_n \\
\sigma^{2n+1} g_n(\frac{x}{\sigma}) + \frac{d_n}{(2n)!} h_n(\frac{x}{\sigma}) \exp(-\frac{x^2}{2\sigma^2}) + g_n(\frac{x}{\sigma}) \int_{-\infty}^{\infty} \exp(-\frac{y^2}{2\sigma^2}) \, dy, & x < c_n
\end{cases}
\]
where \( c_n, k_n \) and \( d_n \) are constants such that

\[
\begin{align*}
\begin{cases}
  h_{n-1}(c_n) + g_{n-1}(c_n) \exp(\frac{c_n^2}{2\sigma^2}) \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt \\
  = \sigma^{2n}[h_{n-1}(c_n) - g_{n-1}(c_n) \exp(\frac{c_n^2}{2}) \int_{c_n}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt] \\
  k_n = -\frac{(2n)!g_{n-1}(c_n)}{h_{n-1}(c_n) \int_{c_n}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt} \\
  d_n = \frac{\sigma g_{n-1}(c_n)}{g_{n-1}(c_n)} k_n \exp\left(\frac{1 - \sigma^2}{2\sigma^2 c_n^2}\right) \\
  c_n < 0
\end{cases}
\end{align*}
\]

Then \( u_n(t, x) := t^{n+\frac{1}{2}} P_n(x) \sqrt{t} \) is the unique solution of G-heat equation (3).

(ii) For \( \sigma = 0 \), we define

\[
P_n(x) = \begin{cases}
  g_n(x) + \frac{k_n}{(2n)!}\left[ h_n(x) \exp\left(-\frac{x^2}{2}\right) - g_n(x) \int_x^{\infty} \exp\left(-\frac{t^2}{2}\right) dt \right], & x \geq c_n \\
  x^{2n+1}, & x < c_n
\end{cases}
\]

where \( c_n, k_n \) are constants such that

\[
\begin{align*}
\begin{cases}
  (2n-1)! = c_n^{2n}[h_{n-1}(c_n) - g_{n-1}(c_n) \exp(\frac{c_n^2}{2}) \int_{c_n}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt] \\
  k_n = -\frac{2n}{(2n-1)!g_{n-1}(c_n)c_n^{2n} \exp(\frac{c_n^2}{2})} \\
  c_n < 0
\end{cases}
\end{align*}
\]

Then \( u_n(t, x) := t^{n+\frac{1}{2}} P_n(x) \sqrt{t} \) is the unique viscosity solution of G-heat equation (3).

In order to prove the main Theorem, we need the following Lemmas. The first Lemma gives the relation between the solution of G-heat equation (3) and the solution of the following ordinary differential equation (ODE for short):

\[
(y')^+ - \sigma^2 (y'')^- + xy' - (2n+1)y = 0, \quad \lim_{t \to 0^+} t^{n+\frac{1}{2}} y(\frac{x}{\sqrt{t}}) = x^{2n+1}. \tag{9}
\]

**Lemma 6** Let \( \sigma \in (0, 1] \) be a fixed constant. The following statements hold.

(i) If \( u \) is the solution of G-heat equation (3), then \( P(x) := u(1, x) \) is the \( C^2 \) solution of ODE (9).

(ii) If \( P \) is the \( C^2 \) solution of ODE (9), then \( u(t, x) := t^{n+\frac{1}{2}} P(x) \sqrt{t} \) is the solution of G-heat equation (3).
**Proof.** For (i), noting that $B_1$ is $G$-normal distributed, then for $t > 0$,

$$u(t, x) = \mathbb{E}[(x + \sqrt{t}B_1)^{2n+1}] = t^{n+\frac{1}{2}}\mathbb{E}[(\frac{x}{\sqrt{t}} + B_1)^{2n+1}] = t^{n+\frac{1}{2}}u(1, \frac{x}{\sqrt{t}}).$$

Consequently, $u(t, x) = t^{n+\frac{1}{2}}P(\frac{x}{\sqrt{t}})$ for $t > 0$. Also $u(0, x) = x^{2n+1}$, we get

$$\lim_{t \to 0^+} t^{n+\frac{1}{2}}P(\frac{x}{\sqrt{t}}) = x^{2n+1}.$$  

According to Remark 1, we have $P \in C^2$. It is easy to verify that, for $t > 0$,

$$\frac{\partial u}{\partial t} = t^{n-\frac{1}{2}}[(n + \frac{1}{2})P(\frac{x}{\sqrt{t}}) - \frac{x}{2\sqrt{t}}P'(\frac{x}{\sqrt{t}})], \quad \frac{\partial^2 u}{\partial x^2} = t^{n-\frac{1}{2}}P''(\frac{x}{\sqrt{t}}).$$  

(10)

Substituting (10) into $G$-heat equation (3), we conclude the result. The proof for (ii) is similar. □

**Remark 7** Let $u^\sigma$ denote the viscosity solution of $G$-heat equation (3) with $\sigma \in [0, 1]$, then for each fixed $(t, x) \in (0, \infty) \times \mathbb{R}$, $\phi(\sigma) := u^\sigma(t, x), \sigma \in [0, 1]$, is a continuous and decreasing function (see [3] and [10]).

In the following, we first solve ODE (9) for $\sigma \in (0, 1]$, then by Lemma (6) we get the corresponding solution of $G$-heat equation (3). By the above Remark, taking $\sigma \downarrow 0$, we get the solution of $G$-heat equation (3) with $\sigma = 0$. For this, we give the following Lemmas.

**Lemma 8** Let $\sigma \neq 0$, $\alpha > 0$, if $\varphi(\cdot)$ is a solution of ODE $y'' + xy' - \alpha y = 0$, then $\psi(x) := \varphi(\frac{x}{\sigma})$ is a solution of ODE $\sigma^2y'' + xy' - \alpha y = 0$.

**Proof.** It is easy to verify the result. □

In the following, we use $g_n(\cdot)$ and $h_n(\cdot)$ defined in (4).

**Lemma 9** The general solution of ODE $y'' + xy' - (2n + 1)y = 0$ is

$$y(x) = \lambda_1g_n(x) + \lambda_2[h_n(x)\exp(-\frac{x^2}{2}) - g_n(x)\int_x^\infty \exp(-\frac{t^2}{2})dt],$$

where $\lambda_1$ and $\lambda_2$ are arbitrary constants.

**Proof.** Applying Lemma 6 with $\sigma = 1$, we get $\bar{y}_n(x) = E^W[(x + B_1)^{2n+1}]$ and $\hat{y}_n(x) = \sqrt{2\pi}E^W[((x + B_1)^{-2n+1}]$ are two linear independent solutions of ODE $y'' + xy' - (2n + 1)y = 0$. It is easy to check that $\bar{y}_n(x) = g_n(x)$ and

$$\hat{y}_n(x) = h_n(x)\exp(-\frac{x^2}{2}) - g_n(x)\int_x^\infty \exp(-\frac{t^2}{2})dt,$$

which completes the proof. □
Lemma 10 Let $x > 0$, then for each integer $n \geq 1$, we have
\[
\frac{h_n'(x) + g_n(x)}{g_n(x) + xg_n(x)} \exp\left(-\frac{x^2}{2}\right) \leq \int_x^\infty \exp\left(-\frac{t^2}{2}\right) dt \leq \frac{h_n(x)}{g_n(x)} \exp\left(-\frac{x^2}{2}\right). \tag{11}
\]

Proof. We define $m_n(x) = \sqrt{2\pi} E^W[((x + B_1)^{-2n+1})$, then by the proof of the above Lemma, we know
\[
m_n(x) = h_n(x) \exp\left(-\frac{x^2}{2}\right) - g_n(x) \int_x^\infty \exp\left(-\frac{t^2}{2}\right) dt.
\]
Noting $m_n(x) \geq 0$, the right-hand side inequality of (11) holds. We also observe that $m_n(\cdot)$ is a decreasing function, then $m_n'(x) \leq 0$, which yields the left-hand side inequality of (11). □

Remark 11 It is easy to verify that, for each fixed $x > 0$, as $n \to \infty$,
\[
\frac{h_n'(x) + g_n(x)}{g_n(x) + xg_n(x)} \exp\left(-\frac{x^2}{2}\right) \int_x^\infty \exp\left(-\frac{t^2}{2}\right) dt,
\]
\[
\frac{h_n(x)}{g_n(x)} \exp\left(-\frac{x^2}{2}\right) \int_x^\infty \exp\left(-\frac{t^2}{2}\right) dt.
\]
Furthermore, we can also prove that
\[
\int_x^\infty \exp\left(-\frac{t^2}{2}\right) dt - \frac{h_n'(x) + g_n(x)}{g_n(x) + xg_n(x)} \exp\left(-\frac{x^2}{2}\right) \leq \min_{1 \leq k \leq n} \left\{ \frac{\sqrt{2\pi}(2k)!}{x^{2k+2k}k!} \right\},
\]
\[
\frac{h_n(x)}{g_n(x)} \exp\left(-\frac{x^2}{2}\right) - \int_x^\infty \exp\left(-\frac{t^2}{2}\right) dt \leq \min_{1 \leq k \leq n-1} \left\{ \frac{\sqrt{2\pi}(2k+1)!}{2x^{2k+2k+1}(2n)!} \right\}.
\]

Lemma 12 Let $\sigma \in (0, 1)$ be a fixed constant, and we define for $n \geq 1$,
\[
f_n(x) := h_n^{-1}\left( \frac{x}{\sigma} \right) + g_n^{-1}\left( \frac{x}{\sigma} \right) \exp\left( \frac{x^2}{2\sigma^2} \right) \int_{-\infty}^\frac{x}{\sigma} \exp\left( -\frac{t^2}{2} \right) dt - \sigma^2 h_n^{-1}(x)
\]
\[
+ \sigma^2 g_n^{-1}(x) \exp\left( \frac{x^2}{2} \right) \int_x^\infty \exp\left( -\frac{t^2}{2} \right) dt,
\]
then there exists a unique $x_0 < 0$ such that $f_n(x_0) = 0$.

Proof. It is easy to verify that
\[
f_n(0) = (1 - \sigma^2)(2n-1)!! > 0, \quad f_n(-\infty) = -\infty,
\]
\[
f_n'(x) = \frac{1}{\sigma} \left( g_n'(x) + g_n^{-1}(x) \right) \exp\left( \frac{x^2}{2\sigma^2} \right) \int_{\frac{x}{\sigma}}^\infty \exp\left( -\frac{t^2}{2} \right) dt
\]
\[
+ \frac{1}{\sigma} \left( h_n'(x) + g_n^{-1}(x) \right) - \sigma^2 (h_n'(x) + g_n^{-1}(x))
\]
\[
+ \sigma^2 (g_n'(x) + xg_n^{-1}(x)) \exp\left( \frac{x^2}{2} \right) \int_x^\infty \exp\left( -\frac{t^2}{2} \right) dt.
\]
As \( x < 0 \),

\[
f'_n(x) > \frac{1}{\sigma}(g'_n(x) + \frac{x}{\sigma}g_n(x)) \exp\left(\frac{x^2}{2\sigma^2}\right) \int_{-\infty}^{\frac{x}{\sigma}} \exp\left(\frac{-t^2}{2}\right) dt
\]

\[
+ \frac{1}{\sigma}(h'_n(x) + g_n(x)).
\]

Applying Lemma 10, we get \( f'_n(x) > 0 \) for \( x < 0 \), which completes the proof. \( \Box \)

**Lemma 13** For each integer \( n \geq 1 \), we have

\[
\lim_{x \to -\infty} x^{2n} [h_{n-1}(x) + g_{n-1}(x) \exp\left(\frac{x^2}{2}\right) \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt] = (2n - 1)!.
\]

**Proof.** It is easy to prove that

\[
h_{n-1}(x)g_n(x) - g_{n-1}(x)h_n(x) = (2n - 1)!x,
\]

\[
h_{n-1}(x)g'_n(x) + xh_{n-1}(x)g_n(x) - h'_n(x)g_{n-1}(x) - g'^2_n(x) = (2n - 1)!
\]

then for \( x < 0 \), by Lemma 10, we obtain

\[
x^{2n} [h_{n-1}(x) + g_{n-1}(x) \exp\left(\frac{x^2}{2}\right) \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt] \geq \frac{(2n - 1)!x^{2n+1}}{g_n(x)},
\]

\[
x^{2n} [h_{n-1}(x) + g_{n-1}(x) \exp\left(\frac{x^2}{2}\right) \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt] \leq \frac{(2n - 1)!x^{2n}}{g_{n-1}(x) + xg_{n-1}(x)}.
\]

Taking \( x \to -\infty \) yields the result. \( \Box \)

**Proof of Theorem 5.** For each fixed \( \sigma \in (0, 1) \), we denote by \( P^\sigma_n(\cdot) \) the \( C^2 \) solution of ODE (9). We assume that there exists a constant \( c_n < 0 \) such that

\[
\left(\frac{d^2}{dx^2}P^\sigma_n\right)(x) \geq 0 \text{ for } x \geq c_n; \quad \left(\frac{d^2}{dx^2}P^\sigma_n\right)(x) \leq 0 \text{ for } x \leq c_n.
\]

Under this assumption, by Lemma 8, Lemma 9 and \( \lim_{t \to 0^+} t^{n+\frac{1}{2}} P^\sigma_n\left(\frac{t^n}{\sqrt{\sigma}}\right) = x^{2n+1} \), it follows that \( P^\sigma_n(\cdot) \) has an expression as in (5), where \( c_n, k_n \) and \( d_n \) are undetermined constants. Note that \( P^\sigma_n \in C^2 \), we have

\[
\lim_{x \to c_n^-} P^\sigma_n(x) = \lim_{x \to c_n^-} \frac{d^2}{dx^2}P^\sigma_n(x) = 0, \quad \lim_{x \to c_n^+} \frac{d^2}{dx^2}P^\sigma_n(x) = 0.
\]

This yields (4) by direct verification. Applying Lemma 12, we know that there exists a unique \( c_n < 0 \) satisfying the first equation in (9), then \( k_n \) and \( d_n \) are also unique. It is easy to verify that \( P^\sigma_n \) determined by (5) and (6) belongs to
According to Remark 7, we get
\[
\frac{1}{(2n+1)(2n)} \left( \frac{d^2}{dx^2} P_n^\sigma \right)(x)
= g_{n-1}(x) + \frac{k_n}{(2n)!} (h_{n-1}(x) \exp(-\frac{x^2}{2}) - g_{n-1}(x) \int_x^\infty \exp(-\frac{t^2}{2}) dt)
\]
\[= E^W[(x + B_1)^{2n-1}] - \frac{E^W[((x + B_1)^{2n-1})]}{E^W[((c_n + B_1)^{-2n-1})]} E^W[(c_n + B_1)^{2n-1}]
\]
\[= E^W[((x + B_1)^{2n-1})] - \frac{E^W[((c_n + B_1)^{-2n-1})]}{E^W[((c_n + B_1)^{-2n-1})]} E^W[(c_n + B_1)^{2n-1}].
\]
It is easy to see that, for \( x \geq c_n \),
\[
E^W[((c_n + B_1)^{-2n-1})] \geq E^W[((x + B_1)^{-2n-1})],
\]
\[E^W[((x + B_1)^{2n-1})] \geq E^W[((c_n + B_1)^{2n-1})],
\]
therefore, we get \( \left( \frac{d^2}{dx^2} P_n^\sigma \right)(x) \geq 0 \) for \( x \geq c_n \). Similarly, we can prove that \( \left( \frac{d^2}{dx^2} P_n^\sigma \right)(x) \leq 0 \) for \( x \leq c_n \). Thus, this \( P_n^\sigma \) is indeed the \( C^2 \) solution of ODE. By Lemma 6, \( u_n^\sigma(t,x) := t^{n+\frac{k}{2}} P_n^\sigma \left( \frac{x}{\sqrt{t}} \right) \) is the solution of \( G \)-heat equation, and then the proof of part (i) is complete. We now prove part (ii). We define
\[l_n(x) = \int_x^\infty \exp(-\frac{t^2}{2}) dt - \frac{h_{n-1}(x)}{g_{n-1}(x)} \exp(-\frac{x^2}{2}), \quad x < 0.
\]
It is easy to prove that
\[\lim_{x \to 0} l_n(x) = +\infty; \quad l_n'(x) = \frac{(2n-1)!}{g_{n-1}(x)} \exp(-\frac{x^2}{2}) > 0, \quad x < 0.
\]
According to Remark 7, \( u^\sigma(1,0) = k_n(\sigma) = \frac{(2n)!}{l_n(c_n(\sigma))} \downarrow 0 \) as \( \sigma \uparrow 0 \), then by \(\square\) we get \( c_n(\sigma) \uparrow 0 \) as \( \sigma \uparrow 1 \). Hence, \( \frac{c_n(\sigma)}{\sigma} \downarrow -\infty \) as \( \sigma \downarrow 0 \). Taking \( \sigma \downarrow 0 \), by Lemma 13, we obtain (ii). \(\square\)

**Remark 14** For each fixed \( \sigma \in (0,1) \), \( \hat{E}[B_t^{2n+1}] = u_n^\sigma(t,0) = k_n(t^{n+\frac{k}{2}} \), where \( k_n \) is determined by \(\square\). In order to get \( k_n \), we solve the first equation in \(\square\) of \( c_n \), then \( k_n \) is uniquely determined by \( c_n \). Specially, \( \widehat{E}[B^2_t] = k t^\frac{k}{2} \), where \( k \) is determined by the following equations:

\[
k = \frac{2c}{\exp(-\frac{c^2}{2}) - c \int_c^\infty \exp(-\frac{t^2}{2}) dt}
\]
\[c < 0
\]
\[1 + \frac{c}{\sigma} \exp(\frac{c^2}{2\sigma^2}) \int_{-\infty}^\infty \exp(-\frac{t^2}{2}) dt
\]
\[= \sigma^2 \left[ 1 - c \exp(\frac{c^2}{2}) \int_c^\infty \exp(-\frac{t^2}{2}) dt \right]
\]

(13)
For $\sigma = 0$, $\hat{E}[B^{2n+1}_t] = u_n(t,0) = k_n t^{n+\frac{1}{2}}$, where $k_n$ is determined by (8). In particular, $\hat{E}[B^3_t] = k t^\frac{3}{2}$, where $k$ satisfies the following equations:

\[
\begin{cases}
1 = c^2 - c^3 \exp\left(\frac{c^2}{2}\right) \int_{-\infty}^\infty \exp\left(-\frac{t^2}{2}\right) dt \\
k = -2c^3 \exp\left(\frac{c^2}{2}\right) \\
c < 0
\end{cases}
\]

(14)

In fact, for each fixed $\sigma \in [0,1]$, we can get $\hat{E}[B^3_t + lB^{2}_t]$ for each $l \in \mathbb{R}$ by the corresponding solution.

**Remark 15** Our method can be extended to the functions $\phi(x) = l_1(x^+)^\alpha + l_2(x^-)^\alpha$, where $\alpha > 0$ is any given number, $l_1, l_2 \in \mathbb{R}$. In fact, this is the only type of functions satisfying $\phi(\lambda x) = \lambda^\alpha \phi(x), \forall \lambda \geq 0, x \in \mathbb{R}$. It is easy to obtain the related Lemma 6, then we get a method to solve G-heat equation (1) with this kind of initial conditions. Specially, for each $\sigma \in [0,1]$, let $u_n^\sigma$ denote the solution of G-heat equation (3), then $\hat{u}_n^\sigma(t,x) := u_n^\sigma(t,-x)$ is the solution of G-heat equation (1) with the initial condition $\phi(x) = -x^{2n+1}$.

**4 Application to Mathematical Finance**

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(B_t)_{t \geq 0}$ be a 1-dimensional standard Brownian motion in this space. We denote by $(\mathcal{F}_t)_{t \geq 0}$ the natural filtration generated by $B$, i.e.,

$\mathcal{F}_t := \sigma\{B_s : s \leq t\} \vee \mathcal{N},$

where $\mathcal{N}$ denotes the set of all $P$-null subsets in $\mathcal{F}$. For given $\sigma \in [0,1]$, we denote by $\mathcal{U}_\sigma$ the space of all $\mathcal{F}_t$-adapted controls with values in $[\sigma,1]$. For fixed $T > 0$, let $(S_t)_{t \leq T}$ be the price of a stock and let $(\log S_t)_{t \leq T}$ satisfy the following stochastic differential equation

$d\log S_t = \mu dt + \sigma_t dB_t, \quad S_0 = 1,$

where $\mu$ is a constant, $\sigma_t \in \mathcal{U}_\sigma$. In mathematical finance, we often need to calculate (see [8])

$$\sup_{\sigma \in \mathcal{U}_\sigma} E_P[\varphi(S_T)].$$

Specially, for $\varphi(S_T) = (\log S_T)^n$, $n \geq 1$, according to Remark 4,

$$\sup_{\sigma \in \mathcal{U}_\sigma} E_P[(\log S_T)^n] = \hat{E}[(\mu T + B_T)^n].$$

Therefore, by Theorem 5, we can get $\sup_{\sigma \in \mathcal{U}_\sigma} E_P[(\log S_T)^n]$. 

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