A MOTIVIC INTEGRAL P-ADIC COHOMOLOGY

ALBERTO MERICI

Abstract. We construct an integral $p$-adic cohomology that compares with rigid cohomology after inverting $p$. Our approach is based on the log-Witt differentials of Hyodo–Kato and log-étale motives of Binda–Park–Østvær. In case $k$ satisfies resolutions of singularities, we moreover prove that it agrees with the “good” integral $p$-adic cohomology of Ertl–Shiho–Sprang: from this we deduce some interesting motivic properties and a Künneth formula for the $p$-adic cohomology of Ertl–Shiho–Sprang.

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1. Introduction

Let $k$ be a perfect field of characteristic $p > 0$ and let $W(k)$ be the ring of Witt vectors of $k$. It is now well established (see [AC21]) that there is no cohomology theory $R\Gamma : \text{Sm}(k) \to D(W(k))^{op}$ satisfying the following three hypothesis:

(i) $R\Gamma(X)$ agrees with the rigid cohomology of $X$ after inverting $p$.
(ii) The complex $R\Gamma(X)$ is bounded and the cohomology groups are finitely generated $W(k)$-modules.
(iii) It satisfies finite étale descent, in the sense that for any Čech hypercover $X_\bullet \to X$ associated to a finite étale Galois cover $X_0 \to X$, the induced morphism $R\Gamma(X) \to R\Gamma(X_\bullet)$ is an equivalence.

In general the different incarnations of integral $p$-adic cohomology theories fail to satisfy (ii) for $X$ that is not smooth and proper, as the $p$-torsion is very rich. In fact, as checked in [AC21], if one assumes (i) and (ii), the failure of (iii) follows from a failure of descent along Artin–Schreier covers of $\mathbb{A}_k^1$: as these the archetype of wild ramification at $\infty$, this leads to think that if one relaxes (iii) by avoiding wild ramification, there is a possibility of obtaining a positive result.

This project was supported by the RCN, project 313472 Equations in Motivic Homotopy.
In [ESS21], with very strong assumptions on resolution of singularities, the authors showed that the integral $p$-adic cohomology theory defined by sending $X \in \text{Sm}(k)$ to the log de Rham–Witt cohomology of any smooth compactification $(\overline{X}, \partial X)$ with log poles on the boundary $\partial X$, is well defined, functorial in $X$, and satisfies (i) and (ii) above, but not (iii).

The goal of this paper is to prove the following result, without any assumption on resolutions of singularities of the base field $k$:

**Theorem 1.1.** There exists an integral $p$-adic cohomology that factors through Voevodsky’s stable $\infty$-category of effective motives

$$
\begin{array}{ccc}
\text{Sm}(k) & \xrightarrow{R\Gamma_p} & \mathcal{D}(W(k))^{op} \\
& & \downarrow \text{DM}^{\text{eff}}(k, \mathbb{Z})
\end{array}
$$

For all $X \in \text{Sm}(k)$, there is a canonical map to crystalline cohomology $R\Gamma_p(X) \to R\Gamma_{\text{crys}}(X)$ that factors through $R\Gamma_{\text{crys}}(\overline{X}, \partial X)$ whenever $X$ admits a smooth compactification $\overline{X}$. Moreover, after inverting $p$, there is an equivalence to rigid cohomology $R\Gamma_p(X)[1/p] \simeq R\Gamma_{\text{rig}}(X)[1/p]$.

In particular the factorization through $\text{DM}^{\text{eff}}(k, \mathbb{Z})$ implies the following descent property:

(iii’) It satisfies Nisnevich descent, in the sense that for any Čech hypercover $X_* \to X$ associated to a Nisnevich cover $X_0 \to X$, the induced morphism $R\Gamma_p(X) \to R\Gamma_p(X_*)$

is an equivalence.

and the following properties:

1. Projective bundle formula: for $E \to X$ a vector bundle of rank $n + 1$ and $P(E)$ the associated projective bundle, there is an equivalence

   $$
   R\Gamma_p(P(E)) \simeq \bigoplus_{i=0}^{n} R\Gamma_p(X(i)[2i]).
   $$

2. (Theorem 6.6) Purity: let $Z \subseteq X$ be smooth closed subset of codimension $d$ and $U = X - Z$. Then there is a fiber sequence

   $$
   R\Gamma_p(Z(d)[2d]) \to R\Gamma_p(X) \to R\Gamma_p(U).
   $$

In case $k$ satisfies resolutions of singularities as in 6.1, we are able to prove that $R\Gamma_p(X)$ agrees with the log de Rham–Witt cohomology of any smooth compactification $(\overline{X}, \partial X)$ with log poles on the boundary $\partial X$. In particular, we deduce (independently from [ESS21] and with milder assumptions on resolutions of singularities), the following:

**Theorem 1.2.** The map $R\Gamma_p(X) \to R\Gamma_{\text{crys}}(\overline{X}, \partial X)/W(k)$

---

1. See Hypotheses 1.5-1.8 of [ESS21]: the assumption concerns strong and embedded resolutions of singularities and weak factorizations, analogous to [Hir64, Main Theorem I and II] and [Wlo03, 0.0.1], or functorial resolutions as in [AT19]

2. Notice that we assume strictly less hypotheses than [ESS21]: in particular, we do not assume embedded resolution and weak factorization of pairs as in Hypotheses 1.8 and 1.9
is an equivalence for any smooth compactification. In particular the cohomology theory
\begin{equation}
X \in \text{Sm}(k) \mapsto R\Gamma_{\text{crys}}(\overline{X}, \partial X)/W(k) \in \mathcal{D}(W(k))
\end{equation}
is well defined (i.e. does not depend on the choice of a compactification) and satisfies (i) and (ii).

As observed in [ESS21, Proposition 2.24], the cohomology theory (1.2.1) does not satisfy (iii). On the other hand, it follows from our construction (see 6.1) that it satisfies finite tame descent in the following sense:

(iii") For any Čech hypercover $X_\bullet \to X$ associated to a finite tame Galois cover $X_0 \to X$, the induced morphism
\[ R\Gamma_p(X) \to R\Gamma_p(X_\bullet) \]
is an equivalence.

Moreover, under these assumptions we can strengthen the motivic properties:

1. (Theorem 6.5) Projective bundle formula: for $E \to X$ a vector bundle of rank $n + 1$ and $P(E)$ the associated projective bundle, there is an equivalence
\[ R\Gamma_p(P(E)) \cong \bigoplus_{i=0}^n R\Gamma_p(X)[i] \]
which, if $X$ is proper, agrees with the projective bundle formula of [Gro85].

2. (Theorem 6.6) Purity: let $Z \subseteq X$ be smooth closed subset of codimension $d$ and $U = X - Z$. Then there is a fiber sequence
\[ R\Gamma_p(Z)[d] \to R\Gamma_p(X) \to R\Gamma_p(U). \]

3. (Theorem 6.7) Künneth: If $k$ satisfies 6.1, then for $X, Y \in \text{Sm}(k)$ there is an equivalence
\[ R\Gamma_p(X)^{\otimes L} \to R\Gamma_p(Y) \cong R\Gamma_p(X \times Y) \]
where $\otimes L$ is Ekedal’s complete-derived product of modules on the Raynaud ring $R(k)$.

We give an overview of our arguments: recall the log motivic categories $\log\mathcal{DA}_{\text{let}}^{\text{eff}}$ and $\log\mathcal{DA}_{\text{eff}}^{\text{let}}$ from [BPO22] (see 2.1). The main technical result is the following:

**Theorem 1.3.** For all $S \in \text{Sm}(k)$, the cohomology of the log de Rham–Witt sheaves $W_m\Lambda_S^d$ is representable in the log motivic category $\log\mathcal{DA}_{\text{let}}^{\text{eff}}(S, \mathbb{Z})$. Moreover, if $S = \text{Spec}(k)$, then the sheaves $W_m\Lambda^d$ have log transfers and their cohomology is representable in $\log\mathcal{DA}_{\text{eff}}^{\text{let}}(k, \mathbb{Z})$.

Once we have this result at our disposal, we deduce that for all $S \in \text{Sm}(k)$, the log crystalline cohomology of [Hyo98]:
\[ R\Gamma_{\text{crys}}(\omega/W(k)) : \mathcal{ISm}(S) \to \mathcal{D}(W(k))^\text{op} \quad X \mapsto \text{holim}_m R\Gamma(X, W_m\Lambda^d) \]
factors through $\log\mathcal{DA}_{\text{let}}^{\text{eff}}(S, \mathbb{Z})$ (resp. in $\log\mathcal{DA}_{\text{eff}}^{\text{let}}(k, \mathbb{Z})$ if $S = \text{Spec}(k)$). Here $\mathcal{ISm}(S)$ is the category of fine and saturated log schemes that are log smooth and separated over $S$ equipped with the trivial log structure (see [BPO22]).

Now, by [Par22, Proposition 2.5.7], there is a fully faithful functor
\[ \mathcal{DM}^{\text{eff}}(k, \mathbb{Z}) \xrightarrow{\omega^\ast} \log\mathcal{DM}^{\text{eff}}(k, \mathbb{Z}) \]
from the category of Voevodsky motives to the category of log motives. We then define $R\Gamma_p$ as the composition
\begin{equation}
\text{Sm}(k) \xrightarrow{M^V} \mathcal{DM}^{\text{eff}}(k, \mathbb{Z}) \xrightarrow{\omega^\ast} \log\mathcal{DM}^{\text{eff}}(k, \mathbb{Z}) \xrightarrow{L_{\text{eff}}} \log\mathcal{DA}_{\text{eff}}^{\text{let}}(k, \mathbb{Z}) \xrightarrow{R\Gamma_{\text{crys}}} \mathcal{D}(W(k))^\text{op}
\end{equation}

3Thanks to [KS10, Theorem 1.1], there is no ambiguity in various notions of the adjective tame.
The canonical maps are then given by the $\mathbb{A}^1$-localization functor $M(X, \text{triv}) \to \omega^* M^{\mathbb{A}^1}(X)$.

The key technical point of the proof of Theorem 1.3 is the transfer structure that is needed in order to exploit [BPØ22, Theorem 8.2.16]. This will follow from a comparison with the sheaf with transfers $\mathcal{L}og(W_m \Omega^i)$, where the functor $\mathcal{L}og$ was constructed in [Sai21] for the reciprocity sheaves of [KSyr16]: the proof will follow from a comparison of two Gysin sequences, after some general properties on $\square$-invariant dNis-sheaves which generalize the work of Morel [Mor12] on $\mathbb{A}^1$-invariant sheaves.

If moreover we assume that $k$ satisfies strong resolutions of singularities as in 6.1, by [BPØ22, Theorem 8.2.16] there is an equivalence

\[(1.3.2) \quad \omega^* M^{\mathbb{A}^1}(X) \simeq M(\overline{X}, \partial X),\]

where $\overline{X}$ is any smooth Cartier compactification of $X$ with $\overline{X} - X$ a simple normal divisor that supports the log structure $\partial X$. This implies that there is a canonical equivalence:

\[R \Gamma_p(X) \simeq R \Gamma_{\text{cris}}(\overline{X}, \partial X)\]

that does not depend on the choice of $\overline{X}, \partial X$, and the canonical map $R \Gamma_p(X) \to R \Gamma_{\text{cris}}(X)$ agrees with the map $R \Gamma_{\text{cris}}(\overline{X}, \partial X) \to R \Gamma_{\text{cris}}(X)$.

In particular, we conclude that $(1.2.1)$ is well defined and satisfies (i) and (ii). Finally, the property $(iii")$ follows from a comparison between tame and log-étale Galois coverings due to Fujisawa and Kato (see Lemma 6.4), and the refined motivic properties follow from the fact that $(1.3.2)$ implies that the functor $\omega^*$ is monoidal.

We remark that resolutions of singularity is only used in the explicit computation of the object $\omega^* M^{\mathbb{A}^1}(X)$ in $\log \mathcal{D} \mathcal{M}^{\text{eff}}(k, \mathbb{Z})$. We believe that in fact one can prove that $R \Gamma_p$ satisfies (i), (ii) and $(iii")$ without any assumption on resolution of singularities, provided that one computes $\omega^* M^{\mathbb{A}^1}(X)$. This is a future work in progress, which we believe to be linked with a suitable definition of a tame motivic homotopy type of $X$, where tame is in the sense of [HS20]. Moreover, we remark that alterations are probably not useful for this computation, as we are not inverting $p$.

In Appendix A, we show a method to compute $h^0_{\square \text{litr}}$ in some special cases, and we deduce (without any assumption on resolution of singularities) a computation of $\pi_* M_{\text{tr}}^*(\mathbb{P}^1)$, which will be useful in Appendix B to deduce from 3.10 a fully faithfulness property, which will prove [BM22, Conjecture 0.2] in the case with transfers and correct the gap in [BM21, Section 7]. This corrects and improves [BM21, Section 7], as the assumption on RS can be avoided.

Acknowledgements. The author would like to thank J. Ayoub for great advice and feedback about some delicate points of Section 3 and Appendix B, for pointing out some mistakes in previous versions of the paper and for suggesting the use of residue maps. He also thanks F. Binda, V. Ertl, D. Park, K. Rülling, S. Saito and P.A.Østvær for many valuable discussions and comments. Part of this paper was written while the author was a visiting fellow at the University of Milan, and he is very thankful for the hospitality and the great work environment.

Notation 1.4. For $X \in \text{Sm}(k)$, we let $X_{\text{Zar}}$ (resp $X_{\text{Nis}}$) be the small Zariski (resp. Nisnevich) site of $X$.

For $F \in \mathcal{PSh}(k)$, we let $F_X$ be the presheaf on $X_{\text{Nis}}$ such that for any étale map $U \to X$,

\[F_X(U) = F(U)\]

If $F \in \mathcal{Shv}_{\text{Zar}}(k)$ (resp. $\mathcal{Shv}_{\text{Nis}}(k)$), then by definition $F_X \in \mathcal{Shv}(X_{\text{Zar}})$ (resp. $\mathcal{Shv}(X_{\text{Nis}})$). Similarly, for $X = (\overline{X}, \partial X) \in \mathcal{ISh}(k)$ and $F \in \mathcal{PSh}(k)$, we let $F_X$ be the presheaf on $\overline{X}$ such that for any étale map $U \to \overline{X}$,

\[F_X(U) = F(U, \partial X|_U)\]
If $F \in \mathbf{Shv}_{\log}^{\text{dNis}}(k)$ (resp. $\mathbf{Shv}_{\log}^{\text{Nis}}(k)$ or $\mathbf{Shv}_{\log}^{\text{Zar}}(k)$), then by definition $F_X \in \mathbf{Shv}(\mathcal{X}_{\text{Zar}})$ (resp. $\mathbf{Shv}(\mathcal{X}_{\text{Nis}})$).

2. Recollections

In this section, we recall the main results on logarithmic motives and reciprocity sheaves.

2.1. Recollections on log motives. We recall the construction of the $\infty$-category of logarithmic motives of [BPØ22] and some properties. The standard reference for log schemes is [Ogu18]. We denote by $\mathbf{Ism}(S)$ the category of fs log smooth log schemes over the log scheme $(S, \text{triv})$. We are typically interested in the case where $S = \text{Spec}(k)$. For $X \in \mathbf{Ism}(S)$, we will denote by $X$ the underlying scheme of $X$, by $X^0$ the open subscheme where the log structure is trivial (we will refer to it as the trivial locus), by $\partial X$ the log structure and by $|\partial X|$ its support, seen as a reduced closed subscheme of $X$.

Let $\mathbf{SmlSm}(S)$ be the full subcategory of $\mathbf{Ism}(S)$ having for objects $X \in \mathbf{Ism}(S)$ such that $X$ is smooth over $S$. By e.g. [BPØ22, A.5.10], then in this case $\partial X$ is supported on a strict normal crossing divisor on $X$ and the log scheme $(X, \partial X)$ is isomorphic to the compactifying log structure associated to the open embedding $X - |\partial X| \hookrightarrow X$. If $D$ is a strict normal crossing divisor on $X$, we will often write $(X, D) \in \mathbf{SmlSm}(S)$ meaning the log scheme with log structure supported on $D$.

A morphism $f : X \to Y$ of fs log schemes is called strict if the log structure on $X$ is the pullback log structure from $Y$. In case both $X$ and $Y$ are objects of $\mathbf{SmlSm}(S)$, this translate to an equality $\partial X = f^*(\partial Y)$ as reduced normal crossing divisors on $X$. If $Z$ is a closed subscheme of $X$, we will often denote by $X - Z \subseteq X$ the strict open immersion $(X - Z, \partial X|_{X - Z}) \hookrightarrow X$.

We denote by $\mathbf{PSh}^{\log}(S, A)$ the category of presheaves of $A$ modules on $\mathbf{Ism}(S)$. It has naturally the structure of closed monoidal category. If $\tau$ is a Grothendieck topology on $\mathbf{Ism}(S)$ (see below), we write $\mathbf{Shv}^{\log}_{\tau}(S, A)$ for the full subcategory of $\mathbf{PSh}^{\log}(S, A)$ consisting of $\tau$-sheaves.

Let $\mathbf{SmlSm}(S)$ be the category of fs log smooth $S$-schemes $(X, \partial X)$ which are essentially smooth over $S$, i.e. a limit $\lim_{\leftarrow i} X_i$ over a cofiltered set $I$, where $X_i \in \mathbf{SmlSm}(S)$ and all transition maps are strict étale (i.e. they are strict maps of log schemes such that the underlying maps $f_{ij} : X_i \to X_j$ are étale). For $X \in \mathbf{SmlSm}(S)$ and $x \in X$, let $\iota : \text{Spec}(\mathcal{O}_{X,x}) \to X$ and $\iota^h : \text{Spec}(\mathcal{O}_{X,x}^h) \to X$, be the canonical morphism. Then we will denote by $X_x$ and $X_x^h$ respectively the localization $(\text{Spec}(\mathcal{O}_{X,x}), \iota^*(\partial X))$ and henselization $(\text{Spec}(\mathcal{O}_{X,x}^h), (\iota^h)^*(\partial X))$: both lie in $\mathbf{SmlSm}(S)$. We frequently allow $F \in \mathbf{PSh}^{\log}(S, A)$ to take values on objects of $\mathbf{SmlSm}(S)$ by setting $F(X) := \lim_{\leftarrow i} F(X_i)$ for $X$ as above.

For $\tau$ a Grothendieck topology on $\mathbf{Sch}(S)$, the strict topology $\tau^s$ on $\mathbf{SmlSm}(S)$ is the Grothendieck topology generated by covers $\{e_i : X_i \to X\}$ such that $e_i : X_i \to X$ is a $\tau$-cover and each $e_i$ is strict. Moreover, recall that a morphism of fs log schemes $f : X \to Y$ is called Kummer-étale if it is exact and log étale, see [BPØ22, Proposition A.8.4]. A typical example is given by $(\mathbb{A}^1_k, 0) \to (\mathbb{A}^1_k, 0)$, $t \to t^n$, where $n$ is coprime to char$(k)$. The Kummer-étale topology is the topology generated by Kummer covers: it is finer that the strict étale. Recall from [BPØ22, 3.1.4] that a cartesian square of fs log schemes

$$
\begin{array}{ccc}
Y' & \xrightarrow{g'} & Y \\
\downarrow f' & & \downarrow f \\
X' & \xrightarrow{g} & X
\end{array}
$$

is a dividing distinguished square if $Y' = X' = \emptyset$ and $f$ is a log modifications, in the sense of F. Kato [Kat21] (see [BPØ22, A.11.9] for more details on log modifications). The collection of dividing distinguished squares forms a cd structure on $\mathbf{SmlSm}(S)$, called the
dividing cd structure. For a Grothendieck topology on \( \text{Sch}(S) \), the dividing topology \( d\tau \) on \( \text{SmlSm}(k) \) is the topology on \( \text{SmlSm}(k) \) generated by the strict topology \( s\tau \) and the dividing cd structure. Moreover, we denote by \( \text{lét} \) the log-étale topology, i.e., topology generated by the Kummer-étale topology and the dividing cd structure. We denote by \( \text{Shv}^{\log}_{d\tau}(S, A) \) (resp. \( \text{Shv}^{\log}_{\text{lét}}(S, A) \), resp. \( \text{Shv}^{\log}_{\text{étr}}(S, A) \)) the subcategory of \( d\tau \)-sheaves (resp. \( \text{ét} \)-sheaves, resp. \( \text{étr} \)-sheaves), with exact sheafification functors \( a_{d\tau} \), (resp. \( a_{\text{ét}} \), resp. \( a_{\text{étr}} \)). By [BPO22, Theorem 1.2.2], for \( \tau \in \{ \text{Nis}, \text{ét}, \} \), \( F \) an \( s\tau \)-sheaf (resp. a k-ét-sheaf) we have

\[
H^i_{d\tau}(X, a_{d\tau} F) \simeq \lim_{\rightarrow Y} H^i_{s\tau}(Y, F) \quad \text{(resp.} H^i_{\text{lét}}(X, a_{\text{lét}} F) \simeq \lim_{\rightarrow Y} H^i_{\text{sét}}(Y, F))
\]

where the colimit runs over all log modifications \( Y \to X \). Following [BPO22], we denote by \( \text{lCor}(k) \) the category of finite log correspondences over \( k \). It is a variant of the Suslin–Voevodsky category of finite correspondences \( \text{Cor}(k) \). It has the same objects as \( \text{SmlSm}(k) \), and morphisms are given by the free abelian subgroup

\[
\text{lCor}(X, Y) \subseteq \text{Cor}(X - \partial X, Y - \partial Y)
\]

generated by elementary correspondences \( V^o \subset (X - \partial X) \times (Y - \partial Y) \) such that the closure \( V \subset X \times Y \) is finite and surjective over (a component of) \( X \) and such that there exists a morphism of log schemes \( V^N \to Y \), where \( V^N \) is the fs log scheme whose underlying scheme is the normalization of \( V \) and whose log structure is given by the inverse image log structure along the composition \( V^N \to X \times Y \to X \). See [BPO22, 2.1] for more details, and for the proof that this definition gives indeed a category.

Additive presheaves (of \( A \)-modules) on the category \( \text{lCor}(k) \) will be called presheaves (of \( A \)-modules) with log transfers. Write \( \text{PSh}^{\text{ltr}}(k, A) \) for the resulting category, and by \( \text{Shv}^{\text{ltr}}_{d\tau}(k, A) \) (resp. \( \text{Shv}^{\text{ltr}}_{\text{lét}}(k, A) \)) the subcategory of \( d\tau \) (resp. \( \text{lét} \)) sheaves with transfers. By [BPO22, Theorem 4.5.7], the sheafification \( a_{d\tau} \) (resp \( a_{\text{lét}} \)) preserves transfers.

Let \( T \in \{ \text{dNis}, \text{ét}, \text{lét} \} \). Let \( \mathcal{D}(\text{Shv}^{\log}_{T}(S, A)) \), resp. \( \mathcal{D}(\text{Shv}^{\text{ltr}}_{T}(k, A)) \), be the derived stable \( \infty \)-category of the Grothendieck abelian category \( \text{Shv}^{\log}_{T}(S, A) \), resp. \( \text{Shv}^{\text{ltr}}_{T}(k, A) \), as in [Lur17, Section 1.3.5]; it is equivalent to the underlying \( \infty \)-category of the model category \( \text{Cpx}(\text{PSh}^{\log}(S, A)) \), resp. \( \text{Cpx}(\text{PSh}^{\text{ltr}}(k, A)) \), with the \( T \)-local model structure used in [BPO22] and [BM21].

Finally (see [BPO22, Section 5.2]), let \( \square := (\mathbb{P}^1, \infty) \).

**Definition 2.1.** The stable \( \infty \)-category \( \text{logDM}^{\text{eff}}_{T}(S, A) \) (resp. \( \text{logDM}^{\text{eff}}_{T}(k, A) \)) is the localization of the stable \( \infty \)-category \( \mathcal{D}(\text{Shv}^{\log}_{T}(S, A)) \) (resp. \( \mathcal{D}(\text{Shv}^{\text{ltr}}_{T}(k, A)) \)) with respect to the class of maps

\[
(a_T A(\square \times X))[n] \to (a_T A(X))[n] \quad \text{(resp.}(a_T A_{\text{ltr}}(\square \times X))[n] \to (a_T A_{\text{ltr}}(X))[n])
\]

for all \( X \in \text{SmlSm}(k) \) and \( n \in \mathbb{Z} \). We let \( L^{\log}_{(T, \square)} \) (resp. \( L^{\text{ltr}}_{(T, \square)} \)) be the localization functor and for \( X \in \text{SmlSm}(k) \), we will let \( M^{\log}_{T}(X) = L^{\log}_{(T, \square)}(a_T A(X)) \) (resp. \( M^{\text{ltr}}_{T}(X) = L^{\text{ltr}}_{(T, \square)}(a_T A_{\text{ltr}}(X)) \)).

If \( T = \text{dNis} \), we will often drop it from the notation. We recall the following result [BM21, Thm. 5.7]:

**Theorem 2.2.** The standard \( t \)-structures of \( \mathcal{D}(\text{Shv}^{\log}_{\text{dNis}}(k, A)) \) and \( \mathcal{D}(\text{Shv}^{\text{ltr}}_{\text{dNis}}(k, A)) \) induce accessible \( t \)-structures on \( \text{logDM}^{\text{eff}}_{T}(k, A) \) and \( \text{logDM}^{\text{eff}}_{T}(k, A) \) compatible with filtered colimits in the sense of [Lur17, Def. 1.3.5.20], called the homotopy \( t \)-structures.

\[^4\text{Notice that this notation conflicts with the notation of [BPO22] where the objects were the same as \text{lSm}(k), although the categories of sheaves are the same in light of [BPO22, Lemma 4.7.2]}\]
We denote by \( \logCI(k, A) \) (resp. \( \logCI^\tr(k, A) \)) its heart, which is then identified with the category of strictly \( \mathbb{N} \)-invariant dNis-sheaves and it is a Grothendieck abelian category. The inclusions

\[
\logCI(k, A) \hookrightarrow \Shv_{dNis}^\log(k, A) \quad \logCI^\tr(k, A) \hookrightarrow \Shv_{dNis}^\tr(k, A)
\]

admit both a left adjoint \( h_0^\tr : F \mapsto \pi_0(L_{(dNis)}(F[0])) \) and a right adjoint \( h_\tr \) (see [BM21, Proposition 5.8]), in particular they are exact.

Recall from [BPO22, (4.3.4)] that the functor \( \omega : X \mapsto X^\circ \) induces an adjunction

\[
\Shv_{d\tr}(k, \Lambda) \xleftarrow{\omega_\sharp} \Shv_{\tr}(k, \Lambda) \quad \text{(2.2.1)}
\]

where for \( Y \in \Sm(k), \omega_\sharp F(Y) = F(Y, \text{triv}) \) and for \( X \in \Sm(k), \omega^* F(X) = F(X - |\partial X|) \).

Moreover, since \( \omega \) is monoidal by construction, \( \omega_\sharp \) is monoidal. It is immediate that \( \omega^* \) and \( \omega_\sharp \) preserve transfers, and induce adjunctions of stable \( \infty \)-categories:

\[
\mathcal{D}(\Shv_{d\tr}(k, \Lambda)) \xleftrightarrow{\omega_*} \mathcal{D}(\Shv_{\tr}(k, \Lambda)) \quad \mathcal{D}(\Shv_{d\tr}^\tr(k, \Lambda)) \xleftrightarrow{\omega_*} \mathcal{D}(\Shv_{\tr}^\tr(k, \Lambda))
\]

As observed in [BM21, Proposition 5.11], the adjunctions above induce:

\[
\log D\mathcal{A}_{\eff}(k, \Lambda) \xrightarrow{L\omega_*} D\mathcal{A}_{\eff}(k, \Lambda) \quad \log D\mathcal{M}_{\eff}(k, \Lambda) \xrightarrow{L\omega_*} D\mathcal{M}_{\eff}(k, \Lambda),
\]

and the functors \( \omega^* \) are \( t \)-exact for the homotopy \( t \)-structures. Moreover, if \( k \) satisfies RS1 and RS2, then \( \omega^* : D\mathcal{M}_{\eff}(k, \Lambda) \to \log D\mathcal{M}_{\eff}(k, \Lambda) \) is fully faithful and exact, and sends the Voevodsky motive \( M^V(X) \) to the log motive of any smooth Cartier compactification \( (\overline{X}, \partial X) \).

2.2. Recollection on reciprocity sheaves. Recall the abelian category of reciprocity sheaves \( \RSC_{\Nis}(k) \subseteq \Shv_{\Nis}^\tr(k) \), defined in equivalent ways in [KSYR16], [KSY21] and [RS21]. It contains Voevodsky’s category \( \HI\tr \) of \( \mathbb{A}^1 \)-invariant sheaves and many other interesting non-\( \mathbb{A}^1 \)-invariant sheaves like \( \Omega^n \) and \( W_m \Omega^n \).

By [Sai21] there is a fully faithful and exact functor

\[
\Log : \RSC_{\Nis}(k, A) \to \logCI^\tr(k, A).
\]

such that if \( F \in \HI\tr \) then \( \Log(F) = \omega^* F \). We recall the construction of the Gysin map in [BRS22], in the special case that will be needed later. Let \( F \in \RSC_{\Nis}(k) \): we put \( \gamma^1 F := \Hom_{\RSC_{\Nis}}(G_m, F) \). Let \( X \in \Sm(k) \) and \( i : D \subseteq X \) a smooth divisor. By [BRS22, Theorem 5.10] there is a cycle class map of sheaves on the small Nisnevich site of \( X \) (see Notation 1.4):

\[
c(D) : \gamma^1 F_X \to R^1\Gamma_D(F_X),
\]

which can be described as follows. Let \( j : X - D \to X \) be the open immersion so that \( R^1\Gamma_D(F_X) \cong j_* F_{X - D} / F_X \). If \( t \) is a local equation of \( D \) in \( X \), then let \( f_1 : X - D \to \mathbb{A}^1 - \{0\} \) be the induced morphism and let \( \Delta : X - D \to X \times (X - D) \) be the composition of the diagonal and \( j \times id \). Let \( a \in \gamma^1 F(X) \) that restricts to \( \tilde{a} \in F(X \times (\mathbb{A}^1 - \{0\})) \) via the surjective map \( \mathbb{Z}_{tr}(\mathbb{A}^1 - \{0\}) \to G_m \). Then by [BRS22, (5.10.3)] we have that \( c(D)(a) \) is the class of \( \Delta^*(\Gamma_i \times id)^* (\tilde{a}) \) in \( j_* F_{X - D} / F_X \).

Assume now that \( i : D \to X \) has a retraction \( q : X \to D \). Then we have a map:

\[
iq \gamma^1 F_D \to i_* q_* \gamma^1 F_X \xrightarrow{iq_*(2.2.2)} i_* q_* R^1\Gamma_D(F_X) = i_* q_* \gamma^1 F_X \cong i_* R^1\gamma^1 F_X = R^1\Gamma_D(F_X).
\]

By composing with the “forget support” map and by shifting \(-1\) one gets a map in the homotopy category \( D(\Shv_{\Nis}(X)) \):

\[
g^{\BRS}_{D/X} : i_* \gamma^1 F_D[-1] \to F_X.
\]
which by [BRS22, Theorem 7.16] corresponds to the extension
\[(2.2.5) \quad 0 \to F_X \to \log(F)(X,D) \to i_* \gamma_! F_D \to 0.\]

### 2.3. Recollection on log de Rham–Witt sheaves and complex

Let \( k \) be a field of characteristic \( p > 0 \) and let \( S \) be a Noetherian \( k \)-scheme and \( W_m(S) \) the sheaf of \( m \)-truncated Witt vectors over \( S \), with Frobenius \( \sigma \).

For \( X \in \text{SmlSm}(S) \) and \( m, q \geq 0 \), we consider \( W_m \Lambda_X^{q/(S, \text{triv})} \) the sheaf on \( X_{zar} \) of logarithmic differentials of [Mat17]. For \( \partial X \) trivial, it agrees with \( W_m \Omega_{X/S}^{q} \) of [Ill79], and if \( q = 0 \) it agrees with the \( m \)-truncated Witt sheaf of \( X \) (in particular, it does not depend on \( \partial X \)). The assignment
\[
X \mapsto \Gamma(X, W_m \Lambda_X^{q/(S, \text{triv})})
\]
defines a \( q \)-sheaf on \( \text{SmlSm}(S) \) by [Mat17, Proposition 3.7], that we will briefly indicate as \( W_m \Lambda_X^{q} \). It comes equipped with usual maps \( F, V \) and \( d \), which make \( \{ W_m \Lambda_X^{*} \}_m \) a universal log \( F \)-\( V \) procomplex. Notice that in [Mat17, 7.2] (and only in that section), this sheaf is indicated as \( W_m \Lambda^{q} \), and \( W_m \Lambda^{q} \) is the sheaf over a base with log structure. We will only work over a base with trivial log structure, so the extra notation \( W_m \Lambda^{q} \) is unnecessary.

Recall the weight filtration on \( W_m \Lambda^{q} \):
\[
P_j W_m \Lambda_X^{q} := \text{Im}(W_m \Lambda_X^{q} \otimes W_m \Omega_{D}^{q-j} \to W_m \Lambda_X^{q})
\]

For \( X \in \text{SmlSm}(S) \), let \( \partial X^{(j)} \) be the disjoint union of the \( j \)-by-\( j \) intersections of the components of \( \partial X \) and let \( \iota^{(j)} : \partial X^{(j)} \to X \) be the inclusion. By convention, we put \( \partial X^{(0)} := \emptyset \). For all \( j \), we have exact sequences of sheaves on \( X_{\text{Nis}} \) (see [Mat17, Lemma 8.4] and [Mok93, 1.4]):
\[(2.2.6) \quad 0 \to P_{j-1} W_m \Lambda_X^{q} \to P_j W_m \Lambda_X^{q} \xrightarrow{\text{Res}} \text{Res} \iota^{(j)}_* W_m \Omega_{D, \partial X^{(j)}}^{q-j} \to 0
\]

In case \( \partial X \) has only one component \( D \), then the sequence above gives an exact sequence of sheaves on \( X_{\text{Nis}} \)
\[(2.2.7) \quad 0 \to W_m \Omega_{D}^{q} \xrightarrow{i_*} W_m \Lambda_X^{q} \to i_* W_m \Omega_{D}^{q-1} \to 0.
\]

The class of this extension gives a map in the homotopy derived category \( D(\text{Shv}(X_{\text{Nis}})) \):
\[(2.2.8) \quad g_{D/X}^{\text{Gros}} : i_* W_m \Omega_{D}^{q-1}[-1] \to W_m \Omega_{X}^{q}.
\]

By [Mok93, 4.6], this map agrees with the Gysin map of [Gro85, II, Prop 3.3.9]: if \( D \) is locally given by the equation \( t = 0 \), then for any local lift \( D_m \leftarrow X_m \) to \( W_m \), for any \( \omega \in W_m \Omega^{q-1} \) and any lift \( \omega' \in \Omega^{q-1}_{m, W_m} \), then \( g_{D/X}^{\text{Gros}} \) maps \( \omega' \) to the image of \( \omega \wedge \text{dlog}(t) \in H^{1}_{D}(\Omega_{X_m/W_m}^{q}). \) By composing with the “forget support” map and by shifting \(-1\) one gets \( \text{(2.2.8)} \).

In general, let \( W\Lambda = \text{“lim”}_m W_m \Lambda^* \) denote the log de Rham–Witt complex and consider the cohomology theory
\[
R\Gamma(\_, W\Lambda_S) : \text{SmlSm}(S) \to D(W(k))^{op} \quad X \mapsto \text{holim}_m R\Gamma(X, W_m \Lambda_S^{*/(S, \text{triv})})
\]

Recall that the Raynaud ring \( R(k) \) of Ekchedal [Eke85] is the graded \( W(k) \)-algebra \( R^0 \oplus R^1 \) generated by elements \( F \) and \( V \) in degree 0 and \( d \) in degree 1, subject to the relations \( FV = VF = p, Fa = a^p F, aV = Va^p, ad = da \) (\( a \in W(k) \)), \( d^2 = 0, FdV = d \). Recall that the category of modules over the Raynaud ring admits a universal tensor product \( \ast \) and a derived-complete tensor product \( \ast_{D} \) in \( D(R(k)) \). We recall that similarly to [Ill83, (2.6.1.2) and (5.1.1)], the dga structure of the log de Rham–Witt complex and the Čech comparison give rise to a product map in \( D(W(k)[d]) \)
\[
R\Gamma(X, W\Lambda_{X/S}) \otimes^L R\Gamma(Y, W\Lambda_{Y/S}) \to R\Gamma(X \times Y, W\Lambda_{X \times Y/S})
\]
which is functorial in $X$ and $Y$. Since $R\Gamma(X \times Y, W\Lambda) \in D(R(k))$ and is derived-complete by construction, the product map induces a map

\[(2.2.9) \quad R\Gamma(X, W\Lambda) \times L R\Gamma(Y, W\Lambda) \to R\Gamma(X \times Y, W\Lambda)\]

which is functorial in $X$ and $Y$ by [Eke85, I. Theorem 6.2]. If $X$ and $Y$ have trivial log structure and $S = \text{Spec}(k)$, the previous map is an equivalence by [Eke85, Corollary 1.2.5]. We will prove in Proposition 4.7 that $(2.2.9)$ is always an equivalence using the motivic properties of the sheaves $W_m\Lambda^n$.

3. Complements on $\square$-local sheaves

In this section, we prove some technical results on $\square$-local sheaves, generalizing the results of [Mor12] on $A^1$-local sheaves. First we show the following immediate result:

**Lemma 3.1.** Let $U \subseteq A^1$ be dense open. Then for all $F \in \log CI(k, A)$ and $i > 0$

$$H^i_{\text{dNis}}(U, F) = 0$$

**Proof.** Let $F_{\square}$ and $F_U$ as in Notation 1.4. For $j : U \subseteq P^1$, we have that $R^i j_* = 0$ for $i > 0$, hence by [BM21, Theorem 5.10] we have a short exact sequence in $\text{Shv}(P^1_{\text{Nis}})$

$$0 \to F_{\square} \to j_* F_U \to \text{coker} \to 0$$

where coker is supported on $P^1 - U$, which has dimension 0, so $H^1_{\text{Nis}}(P^1, \text{coker}) = 0$. Since $F \in \log CI$, $H^1_{\text{Nis}}(P^1, F_{\square}) = H^1_{\text{dNis}}(\square, F) = 0$, hence

$$H^1_{\text{dNis}}(U, F) = H^1_{\text{Nis}}(P^1, j_* F_U) = 0$$

\[\square\]

We give the following definition, extending [Mor12, Definition 1.1]:

**Definition 3.2.** Let $F \in \text{PSh}^{\log}(k, A)$. We say that $F$ is unramified if the following statements hold:

1. For any $X \in \text{SmlSm}(k)$ with irreducible components $\{X_\alpha\}$, the obvious map $F(X) \to \prod \alpha F(X_\alpha)$ is a bijection.
2. For any $X \in \text{SmlSm}(k)$ and any dense strict open subscheme $U \subseteq X$ the restriction map $F(X) \to F(U)$ is injective.
3. For any $X \in \text{SmlSm}(k)$, irreducible with generic point $\eta_X$, the injective map $F(X) \to \bigcap x \in X^{(1)} F(X_x)$ is a bijection of subobjects of $F(\eta_X)$.

In this section, we will show the following result:

**Theorem 3.3.** Let $F \in \log CI(k, A)$. Then $F$ is unramified.

**Remark 3.4.** As observed in [Mor12], if $F$ satisfies (1) and (2), then $F$ satisfies (3) if and only if it satisfies

(3′) For all $Z \subseteq X$ of codimension $\geq 2$, the map $F(X) \to F(X - Z, \partial X|_{X - Z})$ is a bijection.

Notice that every $F \in \log CI(k, A)$ satisfies (1), being a sheaf, and (2) by [BM21, Theorem 5.10]. Moreover, by [Sai21, Section 2], for all $G \in \text{RSC}_{\text{Nis}}$, $Log(G)$ satisfies (3′), hence it is unramified.

The key part of the proof of Theorem 3.3 is the following:

---

5 a small descent argument is needed, see [Ill83, Theorem 5.1.2]
Lemma 3.5. Let $X \in \mathbf{SmlSm}(k)$ such that $X$ is a local scheme of dimension $n \geq 2$ with closed point $x$. Then for all $F \in \logCI(k, A)$ we have an isomorphism

$$F(X) \cong F(X - x)$$

and an injective map:

$$H^1_{\text{dNis}}(X, F) \hookrightarrow H^1_{\text{dNis}}(X - x, F).$$

Proof. Let $(X^h, x^h)$ be the henselization of $(X, x)$ and let $j: (X - x) \hookrightarrow X$ be the strict open immersion. Then we have a strict Nisnevich square:

$$\begin{array}{ccc}
(X^h - x^h) & \xrightarrow{j^h} & X^h \\
\downarrow & & \downarrow \\
(X - x) & \xrightarrow{j} & X,
\end{array}$$

in particular $\text{Cofib}(F(j)) \cong \text{Cofib}(F(j^h))$. Moreover, since $X^h$ is henselian and $F \in \logCI(k, A)$, we have that $H^1_{\text{dNis}}(X^h, F) \cong H^1_{\text{dNis}}(X^h, F) = 0$ by [BPO22, (7.8.1)], so we have a commutative diagram

$$\begin{array}{c}
0 \xrightarrow{} F(X) \xrightarrow{F(j)} F(X - x) \xrightarrow{} \pi_0\text{Cofib}(F(j)) \xrightarrow{} H^1_{\text{dNis}}(X, F) \\
\downarrow \quad \downarrow \quad \downarrow \quad \cong \\
0 \xrightarrow{} F(X^h) \xrightarrow{F(j^h)} F(X^h - x) \xrightarrow{} \text{coker}(F(j^h)) \xrightarrow{} 0.
\end{array}$$

In particular, is enough to show that $\text{coker}(F(j^h)) = 0$, hence we can suppose that $X$ is itself henselian. Let $r$ be the number of components of $\partial X$. As observed in [BM21, Lemma 4.3] and [Sai20, Lemma 6.1], there is a regular sequence $(t_1 \ldots t_n)$ in $\mathcal{O}_X(X)$ and an isomorphism $X \cong \text{Spec}(k(x)[t_1, \ldots, t_n])$ such that $|\partial X|$ has local equation $t_1 \ldots t_r$. If $r < n$, let $X' = (X, \partial X')$ where $|\partial X'|$ has local equation $t_1 \ldots t_{r+1}$, and let $j': X' \to X' - x$ be the open immersion. Let $D_{r+1}$ be the divisor with local equation $t_{r+1} = 0$ on $X$. Then as observed in [BM21, (4.5.8)], the localization sequence of [BPO22, Theorem 7.5.4] induces a commutative diagram where the rows are exact and the columns are injective maps:

$$\begin{array}{c}
0 \xrightarrow{} F(X) \xrightarrow{F(j)} F(X') \xrightarrow{} \pi_1\text{Map}(\text{MT}h(N_{D_{r+1}/X}), F) \xrightarrow{} 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \xrightarrow{} F(X - x) \xrightarrow{F(j')} F(X' - x) \xrightarrow{} \pi_1\text{Map}(\text{MT}h(N_{D_{r+1}/X-x}), F),
\end{array}$$

in particular $\text{coker}(F(j)) \subseteq \text{coker}(F(j'))$, so we can suppose that $r = n$. In this case, there is a strict étale map

$$X \to \boxtimes^n_{k(x)}$$

which induces the following strict Nisnevich square in $\mathbf{SmlSm}(k)$:

$$\begin{array}{c}
(X - x) \xrightarrow{j} X \\
\downarrow \quad \downarrow \\
\boxtimes^n_{k(x)} \xrightarrow{j'} \boxtimes^n_{k(x)}.
\end{array}$$
where $\infty^n$ is the point $(\infty, \ldots, \infty)_{k(x)}$ of $\square^n_{k(x)}$, so we have Cofib($F(j)$) $\simeq$ Cofib($F(j')$).

Since $F \in \logCI(k, A)$, we have that $H^n_{\text{fl}}(\square^n_{k(x)}, F) = 0$, so we have following commutative diagram where the rows are short exact sequences:

$$
\begin{array}{cccccc}
0 & \longrightarrow & F(\square^n_{k(x)}) & \xrightarrow{F(j)} & F(\square^n_{k(x)} - \infty^n) & \longrightarrow & \pi_0\text{Cofib}(F(j')) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \cong & & \\
0 & \longrightarrow & F(X) & \xrightarrow{F(j)} & F(X - x) & \longrightarrow & \pi_0\text{Cofib}(F(j)) & \longrightarrow & 0,
\end{array}
$$

hence it is enough to show that $F(j')$ is an isomorphism. Let $L \in \text{SmlSm}(k)$ be a field. For $Y \in \text{SmlSm}(L)$ and $y \in Y^0$ an $L$-rational point, we let $\text{str}: Y \to (\text{Spec}(L), \text{triv})$ be the structural morphism and $i_y: (\text{Spec}(L), \text{triv}) \to Y$ be the closed immersion of $y$, and $F(Y, i_y)$ be the complement of the induced split:

$$(3.5.1) \quad F(\text{Spec}(L), \text{triv}) \xrightarrow{F(\text{str})} F(Y),$$

In our case, we will fix the inclusion $i_0^n$ of $0^n := (0, \ldots, 0)_{k(x)}$ in $\square^n_{k(x)}$ and its open neighborhoods. To conclude, it is enough to show that $F((\square^n_{k(x)} - \infty^n), i_0^n) = 0$: this would imply that the map $F(j')$ is surjective. Consider the open subschemes $A^1 \times (P^1)^{x_{n-1}}$ and $(P^1) \times A^1 \times (P^1)^{x_{n-2}}$ of $(P^1)^{x_n} - \infty^n$: this induces a commutative square (not a cover!)

$$(3.5.2) \quad \begin{array}{ccc}
\mathbb{A}^2 \times \square^{n-2}_{k(x)} & \xrightarrow{\alpha_1} & \mathbb{A}^1 \times \square^{n-1}_{k(x)} \\
\downarrow & & \downarrow \\
\square \times \mathbb{A}^1 \times \square^{n-2}_{k(x)} & \longrightarrow & \square^{n}_{k(x)} - \infty^n
\end{array}$$

Let $p_i: \mathbb{A}^2 \to \mathbb{A}^1$ denote the $i$-th projection. By $\square$-invariance, we have for $i = 0, 1$:

$$
\begin{array}{ccc}
\overline{h}_0(\mathbb{A}^2 \times \square^{n-2}_{k(x)}) & \xrightarrow{\overline{h}_0^{-\alpha_i+1}} & \overline{h}_0(\square^{n}_{k(x)} \times \mathbb{A}^1 \times \square^{n-1-i}_{k(x)}) \\
\downarrow & & \downarrow \\
\overline{h}_0(\mathbb{A}^2 \times k(x)) & \xrightarrow{\overline{h}_0^{-\beta+i}} & \overline{h}_0(A^1 \times k(x)).
\end{array}
$$

Let $i_0: \text{Spec}(k(x)) \to A^1_{k(x)}$ and $i_{(0,0)}: \text{Spec}(k(x)) \to A^2_{k(x)}$ be the inclusions of the point 0 and (0, 0) respectively: the maps $i_0^n$ clearly factor through $i_0$ and $i_{(0,0)}$ followed by the inclusions of $A^1 \times 0^{n-1}$, $0^n \times A^1 \times 0^{n-2}$ and $A^2 \times 0^{n-2}$, hence (3.5.2) induces the following commutative square:

$$(3.5.3) \quad \begin{array}{ccc}
F((\square^n_{k(x)} - \infty^n), i_0^n) & \xrightarrow{F(\alpha_2)} & F((A^1_{k(x)}), i_0) \\
\downarrow & & \downarrow \\
F((A^1_{k(x)}), i_0) & \xrightarrow{F(\beta+i)} & F((A^2_{k(x)}), i_{(0,0)}).
\end{array}$$
where the maps \( \ell_i \) are injective by purity of \( F \) (see [BM21, Theorem 5.10]). We will conclude by showing that the map \( \ell_1 \) is the zero map. We have the following commutative diagrams:

\[
\begin{array}{ccc}
\text{Spec}(k(x)) & \xrightarrow{i_0} \mathbb{A}^1_{k(x)} & \\
\downarrow \text{str} & \downarrow \text{id} & \\
\text{Spec}(k(x)) \xrightarrow{i_{(0,0)}} \mathbb{A}^2_{k(x)} & \\
\downarrow \text{str} & \downarrow \text{id} & \\
\text{Spec}(k(x)) & \xrightarrow{i_0} \mathbb{A}^1_{k(x)} & \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Spec}(k(x)) & \xrightarrow{i_0} \mathbb{A}^1_{k(x)} & \\
\downarrow \text{str} & \downarrow \text{id} & \\
\text{Spec}(k(x)) \xrightarrow{i_{(0,0)}} \mathbb{A}^2_{k(x)} & \\
\downarrow \text{str} & \downarrow \text{id} & \\
\text{Spec}(k(x)) & \xrightarrow{i_0} \mathbb{A}^1_{k(x)} & \\
\end{array}
\]

which imply that we have a map \( F((id \times i_0)^0): F(\mathbb{A}^2_{k(x)}, i_{(0,0)}) \rightarrow F((\mathbb{A}^1_{k(x)}, i_0) \) such that

\[
F((id \times i_0)^0) \circ F(p_1^0) = id_{F((\mathbb{A}^1_{k(x)}, i_0)} \quad \text{and} \quad F((id \times i_0)^0) \circ F(p_2^0) = 0,
\]

so by the commutativity of (3.5.3) we conclude that

\[
\ell_1 = F((id \times i_0)^0) \circ F(p_1^0) \circ \ell_1 = F((id \times i_0)^0) \circ F(p_2^0) \circ \ell_2 = 0
\]

\[\square\]

**Proof of Theorem 3.3.** We need to show that \( F \) satisfies condition \((3')\) in Remark 3.4: let \( X = (X, \partial X) \in \text{SmISm}(k) \) and \( Z \subseteq X \) of codimension \( \geq 2 \) and let \( \eta_Z \) be its generic point. Recall that we have a strict Zariski square of log schemes

\[
\begin{array}{ccc}
X_{\eta_Z} - \eta_Z & \longrightarrow & X_{\eta_Z} \\
\downarrow & & \downarrow \\
X - Z & \longrightarrow & X
\end{array}
\]

which induces by Lemma 3.5 a commutative diagram

\[
\begin{array}{ccc}
F(X) & \longrightarrow & F(X_{\eta_Z}) \\
\downarrow & & \downarrow \simeq \\
F(X - Z) & \longrightarrow & F(X_{\eta_Z} - \eta_Z)
\end{array}
\]

In particular, for \( \alpha \in F(X - Z) \) there is \( \beta \in F(X_{\eta_Z}) \) such that \( \beta \mapsto \alpha \) in \( F(X_{\eta_Z} - \eta_Z) \), in particular there exists \( U \subseteq X \) open with \( U \cap Z \) dense in \( Z \) and \( \beta' \in F(U) \) such that \( \beta \mapsto \alpha \) in \( F(U - Z) \). Now let \( Z_1 \) be the complement of \( U \cup (X - Z) \) in \( X \): as \( U \cap Z \) is dense in \( Z \), we get that \( \text{codim}(Z_1) > \text{codim}(Z) \). As \( F \) is a sZar sheaf, this implies that there exists \( \alpha_1 \in F(X - Z_1) \) such that \( \alpha_1 \mapsto \alpha \) in \( F(X - Z) \).

By repeating the same argument, we find a chain of strict closed subsets

\[
Z_r \subset \ldots \subset Z \subseteq X
\]

and elements \( \alpha_i \in F(X - Z_i) \) such that \( \alpha_i \mapsto \alpha \) in \( F(X - Z) \). As \( Z \) has finite Krull dimension, \( Z_i = 0 \) for \( i \gg 0 \), which implies that \( F(X) \rightarrow F(X - Z) \) is surjective. This together with [BM21, Theorem 5.8] concludes the proof. \[\square\]

We draw some conclusions from Theorem 3.3. First we show the following interesting Gersten-like property:

**Theorem 3.6.** Let \( X \in \text{SmISm}(k) \) and let \( F \in \text{logCI}(k, A) \). Let \( F_{-1} := \text{Ext}^1(P^1, F) \). Then there is a left exact sequence

\[
0 \rightarrow F(X) \rightarrow F(\eta_X) \rightarrow \left( \prod_{x \in (X^0)^{(1)}} F(A^1_{k(x)}/F(k(x) \oplus F_{-1}(k(x)) \right) \times \left( \prod_{x \in (X^0)^{(0)}} F(A^1_{k(x)}/F(k(x)) \right).
\]
Proof. Since $F$ is unramified by Theorem 3.3, it is enough to show it on $X_x$ for $x \in X^{(1)}$. Let $X^h_x \to X_x$ be the henselization and let $\eta^h_x$ be the generic point of $X^h_x$: this gives a strict Nisnevich square

\[
\begin{array}{ccc}
(\eta^h_x, \text{triv}) & \xrightarrow{j^h} & X^h_x \\
\downarrow & & \downarrow \\
(\eta_X, \text{triv}) & \xrightarrow{j} & X_x
\end{array}
\]

which gives the commutative diagram:

\[
\begin{array}{ccc}
0 & \xrightarrow{} & F(X_x) & \xrightarrow{} & \omega_2 F(\eta_X) & \xrightarrow{} & \pi_0 \text{Cofib}(F(j)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{} & F(X^h_x) & \xrightarrow{} & \omega_2 F(\eta^h_x) & \xrightarrow{} & \text{coker}(F(j^h)) & \xrightarrow{} & 0
\end{array}
\]

(3.6.1)

In particular, it is enough to show that for all $X^h_x \in \text{SmilSm}(k)$ henselian DVR with log structure supported on the closed point and with generic point $\eta^h_x$, we have:

\[
F(\eta^h_x)/F(X^h_x, \text{triv}) \cong F(A^1_{k(x)}/F(k(x)) \oplus F_{-1}(k(x))
\]

and

\[
F(\eta^h_x)/F(X^h_x) \cong F(A^1_{k(x)}/F(k(x)).
\]

(3.6.2)

As in the proof of Lemma 3.5 (see also [BM21, Theorem 4.1, (ii)]), there is an étale map $X^h_x \to \mathbf{P}^1_{k(x)}$ inducing strict Nisnevich squares:

\[
\begin{array}{ccc}
(\eta^h_x, \text{triv}) & \xrightarrow{(e^0)} & (X^h_x, \text{triv}) \\
\downarrow & & \downarrow \\
(A^1_{k(x)}, \text{triv}) & \xrightarrow{(e^0)} & (\mathbf{P}^1_{k(x)}, \text{triv})
\end{array}
\]

(3.6.4)

Since $F \in \log\text{CI}(k, A)$, we have that $F(\mathbf{P}^1_{k(x)}) \cong F(k(x))$, $F(\square_{k(x)}) \cong F(k(x))$ and $H^1_{d\text{Nis}}(\square_{k(x)}, F) = 0$, and by Lemma 3.1 we have $H^1_{d\text{Nis}}(A^1, F) = 0$. This implies that we have short exact sequences

\[
\begin{array}{ccc}
0 & \xrightarrow{} & F(A^1_{k(x)}/F(k(x)) \xrightarrow{} & F(\eta^h_x)/F(X^h_x) & \xrightarrow{} & H^1_{\text{Nis}}(\mathbf{P}^1, \omega_2 F) & \xrightarrow{} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{} & F(A^1_{k(x)}/F(k(x)) \xrightarrow{\cong} & F(\eta^h_x)/F(X^h_x) & \xrightarrow{} & 0
\end{array}
\]

(3.6.5)

This implies that the above sequence splits and concludes the proof.

We recall that there is a functor $\omega_2 : \text{PSh}^{\log}(k, A) \to \text{PSh}(k, A)$ defined as $\omega_2 F(X) = F(X, \text{triv})$.

**Definition 3.7.** A presheaf with residues is the datum of $F \in \text{PSh}(k, A)$ and for all $X \in \text{Sm}(k)$ the spectrum of an henselian DVR with generic point $\eta_X$ and residue field $k(k)$, a map $F(\eta_X) \to F(A^1_{k(x)}/F(k(x))$.

**Remark 3.8.** Let $F \in \log\text{CI}$, then $\omega_2 F$ is naturally a presheaf with residues by (3.6.5). Moreover, for all $X^h_x = (\text{Spec}(k(x))\{t\}, t) \in \text{SmilSm}(k)$ and $X^h_x$ the underlying scheme
with trivial log structure, we get the following variant of (3.6.4):

\[ (\eta^h_e, \text{triv}) \rightarrow (X^h_e, \text{triv}) \]

\[ (A^1_{k(x)} - \{0\}, \text{triv}) \rightarrow (A^1_{k(x)}, \text{triv}) \]

\[ (A^1_{k(x)} - \{0\}, \text{triv}) \rightarrow (A^1_{k(x)}, 0) \]

\[ (P^1_{k(x)} - \{0\}, \text{triv}) \rightarrow (P^1_{k(x)}, \text{triv}) \]

\[ (P^1_{k(x)} - \{0\}, \text{triv}) \rightarrow (P^1_{k(x)}, 0) \]

together with the Nisnevich squares

\[ \pi_0 \text{Cofib}(R\Gamma(P^1_{k(x)}, F) \rightarrow F(A^1_{k(x)})) \rightarrow \frac{F(A^1_{k(x)})}{F(A^1_{k(x)}, 0)} \]

\[ \frac{F(\eta^h_e)}{F(X^h_e)} \rightarrow \frac{F(A^1_{k(x)} - \{0\})}{F(A^1_{k(x)}, \text{triv})} \rightarrow \frac{F(A^1_{k(x)} - \{0\})}{F(A^1_{k(x)}, 0)} \rightarrow \frac{F(\eta^h_e)}{F(X^h_e)} \]

which implies by \( \square \)-invariance that one can construct residues via the map on the bottom. This has the advantage that there is no higher cohomology involved in the construction.

The main application of the construction above is the following:

**Proposition 3.9.** For \( F, G \in \logCI \), any map \( \varphi : \omega_F \rightarrow \omega_G \) of presheaves with residues lifts to a map of log sheaves \( \tilde{\varphi} : F \rightarrow G \). If \( F, G \in \logCI_{\text{tr}} \) and \( \varphi \) is a map of presheaves with transfers, then \( \tilde{\varphi} \) is a map of log presheaves with transfers.

**Proof.** Let \( X \in \Sm\Sm \) and let \( j : (X^0, \text{triv}) \rightarrow X \) be the inclusion of the trivial locus. First we observe that it is enough to show that for all \( X \in \Sm\Sm(k) \) irreducible there is a map \( \varphi_X : F(X) \rightarrow G(X) \) that makes the following diagram commute:

\[ F(X) \xrightarrow{F(j)} \omega \varphi X(X^0) \]

\[ G(X) \xrightarrow{G(j)} \varphi \omega X(X^0). \]

(3.9.1)

Let \( \alpha : Y \rightarrow X \) a map of log schemes and assume that \( \varphi_X \) and \( \varphi_Y \) exist. Consider \( \alpha^0 : Y^0 \rightarrow X^0 \) the restriction to the trivial locus, then since \( \omega \varphi \) is a map of presheaves we have a commutative diagram:

\[ \omega \varphi F(X^0) \xrightarrow{\omega \varphi \alpha^0} \omega \varphi F(Y^0) \]

(3.9.2)

\[ \omega \varphi G(X^0) \xrightarrow{\omega \varphi \alpha^0} \omega \varphi G(Y^0). \]

Since the map \( G(j) \) above is injective by [BM21, Theorem 5.10], it is enough to show that \( G(j) \varphi_Y F(\alpha) = G(j)G(\alpha)\varphi_X \), which follows from the commutativity of (3.9.1) for \( X \) and \( Y \) and of (3.9.2).
We now show the existence of the map \( \varphi_X \) in (3.9.1). Let \( \eta_X \) be the generic point of \( X \); since \( F \) and \( G \) are unramified by Theorem 3.3, by the property (3) in Definition 3.2, it is enough to check that for all \( x \in X \) of codimension 1 we have a map \( \varphi_{X_x} \) that fits in the following commutative diagram:

\[
\begin{array}{ccc}
F(X_x) & \longrightarrow & \omega^1 F(\eta_X) \\
\varphi_X & \downarrow & \varphi_{\eta_X} \\
G(X_x) & \longrightarrow & \omega^1 G(\eta_X)
\end{array}
\]

If \( x \not\in |\partial X| \), \( X_x \) has trivial log structure so \( \varphi_{X_x} \) exists, so assume \( x \in |\partial X| \). As in (3.6.1) it is enough to construct a map \( F(X^h_x) \rightarrow G(X^h_x) \). Since \( \varphi \) preserves residues, we have a commutative diagram:

\[
\begin{array}{ccc}
\omega^1 F(\eta^h_x) & \xrightarrow{\text{Res}(F(X^h_x))} & \omega^1 F(A^1_{k(x)})/\omega^1 F(k(x)) \\
\varphi_{\omega^1 G^h_x} & \downarrow & \varphi_{\text{Res}(\varphi)} \\
\omega^1 G(\eta^h_x) & \xrightarrow{\text{Res}(G(X^h_x))} & \omega^1 G(A^1_{k(x)})/\omega^1 G(k(x)).
\end{array}
\]

On the other hand, by 3.6 we have that \( \ker(\text{Res}(F(X^h_x))) = F(X^h_x) \) and same for \( G \), so \( \varphi_{X_x} \) exists as the map induced on the kernel. If \( \varphi \) is a map of presheaves with transfers, then the commutativity of (3.9.1) and (3.9.2) for \( \alpha \) a log finite correspondence implies that \( \varphi \) extends to a map with transfers. \( \square \)

We are now ready to prove the main theorem of this section. First let us fix some notation: let \( k(x) \) be as before and let \( U \subseteq A^1_{k(x)} \) be an open neighborhood of \( \{0\} \). Then for all \( F \in \logCI \), by [BP02, Theorem 7.5.4.6] there is a fiber sequence

\[
\text{Map}_{\logDA}(MTh(N(0)), F) \rightarrow \text{Map}_{\logDA}(M(U, \text{triv})[-1], F) \rightarrow \text{Map}_{\logDA}(M(U, 0)[-1], F)
\]

compatible with open embeddings \( V \hookrightarrow U \). Recall now that

\[
\text{Map}_{\logDA}(MTh(N(0)), F) \cong \text{R}\Gamma_{\text{Nis}}(\mathcal{P}^1_{k(x), \text{triv}}^0, F) = H^1_{\text{Nis}}(\mathcal{P}^1_{k(x), \omega^1 F}[1])
\]

Let \( i_0: \{0\} \hookrightarrow A^1_{k(x)} \); combining (3.9.3) and (3.9.4) gives a Gysin map in \( \mathcal{D}(\mathbf{Shv}((A^1_{k(x)})_{\text{Zar}})) \)

\[
\text{Gys}^{F}_{U \subseteq U}: i_* H^1(\mathcal{P}^1_{k(x), \omega^1 F}) \rightarrow F_{A^1_{k(x)}[1]},
\]

where \( H^1(\mathcal{P}^1_{k(x), \omega^1 F}) \) on the left hand side is seen as a (constant) sheaf on the small site of \( \{0\} \). Via the isomorphism

\[
\pi_0 \text{Map}_{\mathcal{D}(\mathbf{Shv}((A^1_{k(x)})_{\text{Zar}}))}(i_* H^1(\mathcal{P}^1_{k(x), \omega^1 F}), F_{A^1_{k(x)}[1]}) \cong \text{Ext}^1(\mathbf{Shv}((A^1_{k(x)})_{\text{Zar}}))(i_* H^1(\mathcal{P}^1_{k(x), \omega^1 F}), F_{A^1_{k(x)}[1]}),
\]

the map \( \text{Gys}^{F}_{U \subseteq U} \) above corresponds to the extension:

\[
0 \rightarrow \omega^1 F_{A^1_{k(x)}} \rightarrow \omega^1 F_{(A^1_{k(x)})_0} \rightarrow i_* H^1(\mathcal{P}^1_{k(x), \omega^1 F}) \rightarrow 0
\]

**Theorem 3.10.** Let \( F, G \in \logCI \), and let \( \varphi: \omega^1 F \rightarrow \omega^1 G \) be a map. If the Gysin maps induce a commutative diagram in \( \mathcal{D}(\mathbf{Shv}((A^1_{k(x)})_{\text{Zar}})) \)

\[
\begin{array}{ccc}
i_* H^1(\mathcal{P}^1_{k(x), \omega^1 F}) & \xrightarrow{\text{Gys}^{F}_{U \subseteq U}} & F_{A^1_{k(x)}[1]} \\
i_* H^1(\mathcal{P}^1_{k(x), \varphi}) & \downarrow & \varphi_{A^1_{k(x)}} \\
i_* H^1(\mathcal{P}^1_{k(x), \omega^1 G}) & \xrightarrow{\text{Gys}^{G}_{U \subseteq U}} & G_{A^1_{k(x)}[1]},
\end{array}
\]

then there is a map \( \tilde{\varphi} : F \to G \) that lifts \( \varphi \). If \( F, G \in \log \text{CI}^\text{th} \) and \( \varphi \) is a map of presheaves with transfers, then \( \tilde{\varphi} \) is a map of log presheaves with transfers.

**Proof.** The commutative diagram in the statement gives a morphism \( \tilde{\varphi}_{A^1} \) that fits in the exact sequences of (3.9.6) as follows:

\[
\begin{array}{c}
0 \longrightarrow \omega_2 F_{A^1_{k(z)}} \longrightarrow F_{(A^1_{k(z)}, 0)} \longrightarrow i_* H^1(\mathbf{P}^1_{k(z)}, \omega_2 F) \longrightarrow 0 \\
\downarrow \tilde{\varphi}_{A^1} \downarrow \downarrow \downarrow \downarrow \downarrow \\
0 \longrightarrow \omega_2 G_{A^1_{k(z)}} \longrightarrow G_{(A^1_{k(z)}, 0)} \longrightarrow i_* H^1(\mathbf{P}^1_{k(z)}, \omega_2 G) \longrightarrow 0.
\end{array}
\]

Since the cokernels are supported on 0, by taking sections over \( A^1 - \{0\} \to A^1 \) we have that \( \tilde{\varphi}_{A^1} \) induces a commutative diagram

\[
\begin{array}{c}
F(A^1_{k(z)}, 0) \longrightarrow F(A^1_{k(z)} - \{0\}) \\
\downarrow \tilde{\varphi}_{A^1} \downarrow \downarrow \downarrow \\
G(A^1_{k(z)}, 0) \longrightarrow G(A^1_{k(z)} - \{0\}),
\end{array}
\]

As observed in Remark 3.8, this implies that \( \varphi \) is a map of presheaves with residues, which by Proposition 3.9 gives the desired lift, which is a map of presheaves with log transfers if \( \varphi \) has log transfers by Proposition 3.9.

From Theorem 3.10, we will deduce the transfer structure on \( W_m \Lambda^n \) in Theorem B.3 and the proof of Conjecture [BM22, Conjecture 0.2] in Theorem 3.10.

4. Application to log de Rham–Witt cohomology

Let \( S \) be a Noetherian \( k \)-scheme and let \( W_m \Lambda^g \in \text{Shv}_{\text{Zar}}(S, \mathbb{Z}) \) of [Mat17] (see 2.3). The idea behind the following result has been suggested by F. Binda. We thank him for letting us include it here. Let \( X \in \text{SmlSmm}(S) \). Consider the exact sequence on the weight filtration (2.2.6) for \( X \times (\mathbf{P}^n, \mathbf{P}^{n-1}) \). We have that \( \partial(X \times (\mathbf{P}^n, \mathbf{P}^{n-1}))(j) = \partial X(j) \times \mathbf{P}^n \coprod \partial X(j-1) \times \mathbf{P}^{n-1} \), so if again we let \( i_H : \mathbf{P}^{n-1} \to \mathbf{P}^n \) be the inclusion of a fixed hyperplane, we let \( \alpha(j) = i(j) \times id : \partial X(j) \times \mathbf{P}^n \to X \times \mathbf{P}^n \) and \( \beta(j) = i(j-1) \times i_H : \partial X(j-1) \times \mathbf{P}^{n-1} \to X \times \mathbf{P}^n \), we get an exact sequence:

\[
0 \to P_j W_m \Lambda^g_{X \times (\mathbf{P}^n, \mathbf{P}^{n-1})} \to P_j W_m \Lambda^g_{X \times (\mathbf{P}^n, \mathbf{P}^{n-1})} \to P_{j+1} W_m \Lambda^g_{X \times (\mathbf{P}^n, \mathbf{P}^{n-1})} \to 0
\]

(4.0.1)

**Lemma 4.1.** For \( X \in \text{SmlSmm}(k) \) and \( j \geq 1 \) there is an equivalence

\[
R\pi_* P_j W_m \Lambda^g_{X \times (\mathbf{P}^n, \mathbf{P}^{n-1})} \cong P_j W_m \Lambda^g_{X \times (\mathbf{P}^n, \mathbf{P}^{n-1})} \oplus \bigoplus_{i=1}^{n} t_i^{j-1} W_m \Omega^g_{\partial X(j-1) \times \mathbf{P}^{n-1}}[-i]\text{ in } \mathcal{D}(\text{Shv}(X_{\text{Zar}})).
\]

**Proof.** To ease the notation, let \( \mathbf{P}^n := (\mathbf{P}^n, \mathbf{P}^{n-1}) \).

We prove the statement by induction on \( j \). If \( j = 1 \), then (4.0.1) gives the fiber sequence

\[
R\pi_* W_m \Omega^g_{\partial X} \to R\pi_* P_1 W_m \Lambda^g_{X \times \mathbf{P}^n} \to R(\pi_H)_* W_m \Omega^g_{\mathbf{P}^{n-1}} \oplus R(\pi^2_{\partial X(1)})_* W_m \Omega^g_{\partial X(1)} \oplus R(\pi^2_{\partial X(1)})_* W_m \Omega^g_{\partial X(1)} \oplus R(\pi^2_{\partial X(1)})_* W_m \Omega^g_{\partial X(1)} \oplus R(\pi^2_{\partial X(1)})_* W_m \Omega^g_{\partial X(1)}
\]

where \( \pi^2_{\partial X(1)} : \mathbf{P}^n \to \partial X(1) \) is the projection. Then this gives a map

\[
f : R(\pi_H)_* W_m \Omega^g_{\mathbf{P}^{n-1}}[-1] \oplus R(\pi^2_{\partial X(1)})_* W_m \Omega^g_{\partial X(1)}[-1] \to R\pi_* W_m \Omega^g_{\partial X(1)}
\]
By the projective bundle formula of [Gro85], we have by taking cup products with the first Chern class \( c_1(\mathcal{O}_{\mathbb{P}^n}(H)) = \xi \in H^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) \):

\[
\bigoplus_{i=0}^n W_m \Omega^{q-i}_{\Delta} [-i] \cong R\pi_* W_m \Omega^q_{\mathbb{P}^n}
\]

and similarly by cupping with \( i^*_H(\xi) \in H^1(\mathbb{P}^{n-1}, \Omega^1_{\mathbb{P}^{n-1}}) \):

\[
\bigoplus_{i=0}^{n-1} W_m \Omega^{q-i}_{\Delta} [-i] \cong R(\pi_H)_* W_m \Omega^{q-1}_{\mathbb{P}^{n-1}}
\]

Since by construction \( f(\pi^*_H(a) \cup \xi^j_H) = \pi^*(a) \cup \xi^{j+1} \), it fits in a commutative square:

\[
\begin{array}{ccc}
\bigoplus_{i=0}^n W_m \Omega^{q-1-i}_{\partial X^{(i)}} [-i - 1] & \xrightarrow{\cong} & \bigoplus_{i=0}^n W_m \Omega^{q-i}_{\Delta} [-i] \\
R(\pi^{\partial X^{(1)}})_* W_m \Omega^{q-1}_{\partial X^{(1)}} [-1] & \xrightarrow{\cong} & R(\pi)_* W_m \Omega^q_{\mathbb{P}^n} [-1]
\end{array}
\]

Since \( t \) is induced by a split injection, its cofiber is the same as the cofiber of the induced map

\[
\bigoplus_{i=0}^n W_m \Omega^{q-i}_{\Delta} [-i - 1] \to W_m \Omega^q_{\Delta}
\]

which by (2.2.6) is

\[ P_i W_m \Omega^q_{\Delta} \oplus \bigoplus_{i=1}^n W_m \Omega^{q-i-1}_{\Delta}. \]

This concludes the proof for \( j = 1 \). Let \( j \geq 2 \). We have the fiber sequence

\[
R\pi_* P_{j-1} W_m \Omega^q_{\partial X \times \mathbb{P}^n} \to R\pi_* P_j W_m \Lambda^q_{\partial X \times \mathbb{P}^n} \to R(\pi^q H)^* \Omega^{q-j}_{\partial X^{(1-1)}} \oplus R(\pi_{\partial X^{(j)}})_* W_m \Omega^{q-j}_{\partial X^{(j-1)}}
\]

where \( \pi_{\partial X^{(j)}}_j : \mathbb{P}^n_{\partial X^{(j)}} \to \partial X^{(j)} \) and \( \pi^q H : \mathbb{P}^n_{\partial X^{(j-1)}} \to \partial X^{(j-1)} \) are the projections. By induction hypothesis and projective bundle formula we have a commutative diagram

\[
\begin{array}{ccc}
\bigoplus_{i=0}^n i_*^{(j)} W_m \Omega^{q-j-i}_{\partial X^{(j)}} [-i - 1] & \xrightarrow{\cong} & \bigoplus_{i=0}^n W_m \Omega^{q-j-i}_{\partial X^{(j)}} [-i] \\
R(\pi^q H)^* \Omega^{q-j-1}_{\partial X^{(j-1)}} [-1] & \xrightarrow{\cong} & R(\pi^q H)^* \Omega^{q-j-1}_{\partial X^{(j-1)}} [-1]
\end{array}
\]

\[
P_{j-1} W_m \Omega^q_{\Delta} \oplus \bigoplus_{i=1}^n i_*^{(j-1)} W_m \Omega^{q-(j-1)-i}_{\partial X^{(j-1)}} [-i] \to R\pi_* P_{j-1} W_m \Lambda^q_{\mathbb{P}^n}
\]

Again, we can compute its cofiber as the cofiber of

\[
\bigoplus_{i=0}^n W_m \Omega^{q-j-i}_{\partial X^{(j)}} [-i - 1] \to P_{j-1} W_m \Omega^q_{\Delta}
\]
which again by (2.2.6) is
\[ P_j W_m \Omega^q_{-i} \cong \bigoplus_{i=1}^n W_m \Omega^q_{\partial Y} [-i]. \]

This concludes the proof. \(\square\)

We are ready to show the following:

**Theorem 4.2.** The sheaf \(W_m \Lambda^q\) is \((P^n, P^{n-1})\)-local in the \(s\)\(Zar, s\)\(Nis\) and \(k\)\(ét\) topologies. In particular, it is strictly \(div\)-invariant.

**Proof.** By [Niz08, Proposition 3.27], we have that for all \(Y \in \text{SmlSm}(S)\)
\[ R\Gamma_{s\text{Nis}}(Y, W_m \Lambda^q_{S}) \cong R\Gamma_{s\text{Zar}}(Y, W_m \Lambda^q_{S}) \cong R\Gamma_{k\text{ét}}(Y, W_m \Lambda^q_{S}) \]

hence it is enough to show it only for the \(s\)\(Zar\) topology. Let \(P_j\) again be the weight filtration defined above. We have by definition that \(W_m \Lambda^q \cong P_q W_m \Lambda^q\), so since \(W_m \Lambda^{d-q-i}_{\partial X} = 0\) for \(i \geq 1\), we conclude by Lemma 4.1 that \(R\pi_* W_m \Lambda^q_{X \times \text{Spec}(k \cdot p_{m-1})} \cong W_m \Lambda^q_{X} \). Finally by [BP02, Proposition 7.3.1 and Theorem 7.7.4], any \((s\text{Nis}, (P^n, P^{n-1}))\)-local object is automatically \(div\)-local. \(\square\)

**Remark 4.3.** Theorem 4.2 implies that for every \(m\), any \(s\text{Nis}\) (resp. \(k\)\(ét\))-fibrant replacement of the truncated de Rham–Witt complex \(W_m \Lambda^q_{X}^\bullet\) is \((dNis, \square)\)-fibrant. Hence, the functor
\[ \text{SmlSm}(S) \to \mathcal{D}(W(k))^{\text{op}} \] 

factors through \(\log \mathcal{D} \mathcal{A}^{\text{eff}}(S, \mathbb{Z})\) and \(\log \mathcal{D} \mathcal{A}^{\text{eff}}(S, \mathbb{Z})\) via the functor:
\[ R\Gamma_{\text{crys}, S}: \log \mathcal{D} \mathcal{A}^{\text{eff}}(S, \mathbb{Z}) \to \mathcal{D}(W(k))^{\text{op}} \]
\[ M \mapsto \lim_{m} \text{Map}(M, W_m \Lambda^q_{X/S}^\bullet). \]

We now concentrate to the case where \(S = \text{Spec}(k)\). In this case, since \(W_m \Omega^n \in \text{RSC}_{\text{Nis}}(k)\) by [KSYR16, Appendix B], we have \(\mathcal{L} \text{og}(W_m \Omega^n) \in \log \text{Cl}^{\text{ht}}\). Let \(X \in \text{Sm}(k)\) and \(i: D \subset X\) a smooth divisor, and let \(X = (X, D) \in \text{SmlSm}(k)\). By (2.2.5) there is an exact sequence of \(\text{Shv}(X_{\text{Nis}})\):
\[ (4.3.1) \quad 0 \to W_m \Omega^q_{\Delta} \to \mathcal{L} \text{og}(W_m \Omega^n)_X \to \gamma^1 W_m \Omega^n_{D} \to 0 \]

Moreover, by [BRS22, Theorem 11.8] there is an isomorphism
\[ \varphi^1: W_m \Omega^{n-1} \cong \gamma^1 W_m \Omega^n \]
that behaves as follows: let \(Y \in \text{Sm}(k)\) and let \(\lambda_Y: \gamma^1 W_m \Omega^n(Y) \to W_m \Omega^n(Y \times (A^1 - \{0\}))\) be the map induced by \(\mathbb{Z}_\text{tr}(A^1 - \{0\}) \to G_m\). Recall the map of \(\text{Shv}((A^1 - \{0\})_{\text{Zar}})\)
\[ \text{dlog}: (W_m \mathcal{O}_{A^1 - \{0\}}^\times)^{\times} = \mathcal{O}_{A^1 - \{0\}}^{\times} \xrightarrow{t \mapsto \frac{dt}{t}} Z \Omega^1_{A^1 - \{0\}} \xrightarrow{i} W_m \Omega^1_{A^1 - \{0\}}. \]

Then for \(\omega \in W_m \Omega^{n-1}(Y)\) we have:
\[ (4.3.2) \quad \lambda_Y \varphi^1(\omega) = \omega \cdot \text{dlog}(\mathcal{L}) \quad \text{in} \quad W_m \Omega^n(Y \times A^1), \]
where \(\mathcal{L} \in W_m \mathcal{O}^\times(A^1 - \{0\})\) is the Teichmüller lift of \(t\). Then (4.3.1) translates in the following exact sequence in \(\text{Shv}(X_{\text{Nis}})\):
\[ (4.3.3) \quad 0 \to W_m \Omega^q_{\Delta} \to \mathcal{L} \text{og}(W_m \Omega^n)_X \to i_* W_m \Omega^{n-1}_D \to 0, \]

**Theorem 4.4.** The sheaves \(W_m \Lambda^n\) have log transfers, in particular \(W_m \Lambda^n \in \log \text{Cl}^{\text{ht}}(k, W_m(k))\) and the Frobenius, Verschiebung, differential and restrictions are morphisms of log presheaves with transfers.
Proof. Let $\text{Log}(W_m\Omega^n) \in \mathbf{logCl}_{\text{tr}}^d(k, W_m(k))$. We have that

$$\omega_2 \text{Log}(W_m\Omega^n) \cong W_m\Omega^n \cong \omega_2 W_m\Lambda^n$$

(4.4.1)

We will show that (4.4.1) lifts to an isomorphism $W_m\Lambda^n \cong \text{Log}(W_m\Omega^n)$ in $\mathbf{logCl}(k, W_m(k))$. This will give the transfers structure on $W_m\Lambda^n$ in the following way: for $\alpha \in \text{ICor}(Y, X)$, we have a commutative diagram:

$$W_m\Lambda^n(X) \xrightarrow{\sim} \text{Log}(W_m\Omega^n)(X) \hookrightarrow W_m\Omega^n(X^o)$$

$$\downarrow \alpha^* \quad \downarrow (\alpha^*)^*$$

$$W_m\Lambda^n(Y) \xrightarrow{\sim} \text{Log}(W_m\Omega^n)(Y) \hookrightarrow W_m\Omega^n(Y^o)$$

The morphisms $F, V, d$ and $R$ preserve transfers by [RS21, Lemma 7.7].

We need to study the maps $i_* W_m\Omega_{\mathbf{A}^1_{k(x)}}^{n-1} \rightarrow W_m\Omega_{\mathbf{A}^1_{k(x)}}^{n}$ [1] given by the two Gysin maps in $D(\text{Shv}(\mathbf{A}^1_{k(x)}))$: if the two are the same, then we conclude by 3.10. By [Mok93, Prop. 4.7], the Gysin map of $W_m\Lambda^n$ agrees with the Gysin map of $[\text{Gro}55]$, so it is enough to show that the Gysin maps of $[\text{BRS}22]$ and $[\text{Gro}55]$ are the same. This was stated without proof in [BRS22, 11.10]: a proof was communicated to the author by Kay Rülling and we write it here. By construction both maps factor through the local Gysin maps:

$$g_{0/\mathbf{A}^1_{k(x)}}^{\text{Gros}}: i_* W_m\Omega_{\mathbf{A}^1_{k(x)}}^{n-1} \rightarrow R^1\Gamma_{\{0\}}(W_m\Omega_{\mathbf{A}^1_{k(x)}}^{n})$$

$$i_* W_m\Omega_{\mathbf{A}^1_{k(x)}}^{n-1} \cong i_* (\gamma^1 W_m\Omega^n)_{k(x)} \xrightarrow{\text{g}_{0/\mathbf{A}^1_{k(x)}}^{\text{BRS}}} R^1\Gamma_{\{0\}}(W_m\Omega_{\mathbf{A}^1_{k(x)}}^{n})$$

of (2.2.8) and (2.2.4) composed with $\varphi^1$ of [BRS22, Theorem 11.8], hence it is enough that they agree before the “forget support” map. Let us fix the isomorphism

$$R^1\Gamma_{\{0\}}(W_m\Omega_{\mathbf{A}^1_{k(x)}}^{n}) \cong j_* W_m\Omega_{\mathbf{A}^1_{k(x)} - \{0\}}^{n} \rightarrow W_m\Omega_{\mathbf{A}^1_{k(x)}}^{n}.$$
where \( pr_2 \) is the second projection. By composing with \( \Delta^* \) we conclude that
\[
g_{0/A}^{\BRS} |_{\BRS k(x)}(\omega) = [\Delta^* (pr_2^* (q^* \omega) \wedge \text{dlog}(\mathcal{L} \otimes 1))] = [q^* \omega \cdot \text{dlog}(\mathcal{L})] = g_{0/A}^{\Gros}(\omega).
\]
This concludes the proof. \( \square \)

**Corollary 4.5.** We have that
\[
\text{Ext}^i(\mathcal{Z}_{\text{itr}}(\mathbf{P}^1, \text{triv}), W_m \Lambda^n) \cong \begin{cases} W_m \Lambda^n & \text{if } i = 0 \\ W_m \Lambda^{n-1} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}
\]

*Proof.* The case \( i = 0 \) follows from the \( \mathbf{P}^1 \)-invariance of objects in \( \text{logCI} \). Let \( T \in \text{Sm} \text{Sm}(k) \) be Hensel local and let \( \eta_T \) be its generic point. Then by [BM21, Theorem 5.10] we have
\[
\text{Ext}^i_{\text{Shv}_{\text{dNis}}} (\mathcal{Z}_{\text{itr}}(\mathbf{P}^1, \text{triv}), W_m \Lambda^n)(T) \hookrightarrow H^i(\mathbf{P}^1_{\eta_T}, W_m \Omega^n) = 0 \quad \text{for } i \geq 2.
\]
It remains to prove the case \( i = 1 \). By Proposition A.5 and Lemma B.1, we have that
\[
\omega_2 \text{Ext}^1_{\text{Shv}_{\text{dNis}}} (\mathcal{Z}_{\text{itr}}(\mathbf{P}^1, \text{triv}), W_m \Lambda^n) \cong \text{Hom}_{\text{Shv}_{\text{dNis}}} (G_m, W_m \Omega^n),
\]
and by [BRS22, Theorem 11.8], we have:
\[
\text{Hom}_{\text{Shv}_{\text{dNis}}} (G_m, W_m \Omega^n) \cong \omega_2 W_m \Lambda^{n-1},
\]
so we conclude by B.3 that we have an isomorphism. \( \square \)

Let \( G \in \text{logCI}_{\text{itr}}(k, A) \), \( X \in \text{Sm} \text{Sm} \) and \( \mathcal{E} \to X \) a vector bundle of rank \( n + 1 \). Then if \( k \) satisfies RS1 and RS2, the projective bundle formula of [BP022, Theorem 8.3.5] gives an equivalence
\[
R \Gamma_{\text{dNis}}(\mathbf{P}(\mathcal{E}), G) \cong \bigoplus_{i=1}^n \text{Map}_{\text{logDM}^a}(M(X)[2i], \text{Map}((M(\mathbf{P}^1)^0)^{\times i}, G)),
\]
where the notation \(*\) is as in (3.5.1). In particular, we deduce the projective bundle formula for \( W_m \Lambda^n \), generalizing [Gro85]:
\[
(4.5.1) \quad R \Gamma(\mathbf{P}(\mathcal{E}), W_m \Lambda^d) \cong \bigoplus_{i=1}^n R \Gamma(X, W_m \Lambda^{q-i})[-i], \quad R \Gamma(\mathbf{P}(\mathcal{E}), W_m \Lambda) \cong \bigoplus_{i=1}^n R \Gamma(X, W_m \Lambda^{q-i})[-i].
\]
Moreover, let \( Z \subseteq X \) be a smooth closed subscheme of codimension \( d \) that intersects \( |\partial X| \) transversally. Let \( \rho: \tilde{X} \to X \) be the blow up of \( X \) in \( Z \) and consider the log scheme \( \tilde{X} = (\tilde{X}, \partial X + E_Z) \). Then by [BPO22] there is a fiber sequence
\[
\text{Map}(MTh(N_Z), G) \to R \Gamma(X, G) \to R \Gamma(\tilde{X}, G)
\]
On the other hand, there is a fiber sequence
\[
\mathbf{P}(N_Z) \to \mathbf{P}(N_Z \oplus \mathcal{O}) \to MTh(N_Z)
\]
which by the projective bundle formula above implies that
\[
\text{Map}_{\text{logDM}^a}(MTh(N_Z), G) \cong \text{Map}_{\text{logDM}^a}(M(Z)[2d], \text{Map}((M(\mathbf{P}^1)^0)^{\times d}, G)).
\]
so we get a fiber sequence
\[
(4.5.2) \quad R \Gamma(Z, W_m \Lambda^{n-d})[-d] \to R \Gamma(X, W_m \Lambda^n) \to R \Gamma(\tilde{X}, W_m \Lambda^n)
\]
If \( \partial X \) is trivial and \( Z \) is a divisor, then the Gysin sequence above agrees with (2.2.6) of [Mok93]. In general, it agrees the Gysin sequence of [BRS22, Corollary 11.10 (2)] for a reduced modulus pair.
Remark 4.6. The projective bundle formula and the Gysin sequences above were obtained in [BRS22] without the assumption of resolutions of singularities, and are in fact independent of the results of [BPØ22].

Finally, recall the Raynaud ring $R(k)$ of Ekedahl and the map $\mathcal{D}(R(k))$ (see 2.3):

\[(2.2.9) \quad R\Gamma(X, WA) \overset{\sim}{\longrightarrow} R\Gamma(Y, WA) \to R\Gamma(X \times Y, WA).\]

**Proposition 4.7.** The map (2.2.9) is an equivalence in $\mathcal{D}(R(k))$.

**Proof.** We proceed by induction on the dimension of $X \times Y$ and the number of components of $\partial(X \times Y)$. If $\partial(X \times Y)$ is trivial, then $\partial X$ and $\partial Y$ are trivial so (2.2.9) is an equivalence by [Ill83, Theorem 5.1.2]. If $\dim(X \times Y) = 0$, then $\partial(X \times Y)$ is again trivial so we can invoke [Eke85] again. In general, let $|\partial X| = D_1 + \ldots + D_r$ with $r \geq 1$ and consider $|\partial X| = D_2 + \ldots + D_r$. Let $X^{-} := (X, \partial X^{-})$, so that by [Eke85, I. Theorem 6.2] there is a commutative diagram in $\mathcal{D}(R(k))^6$

\[\begin{array}{ccc}
R\Gamma(X^{-}, WA) \overset{\sim}{\longrightarrow} R\Gamma(Y, WA) & \longrightarrow & R\Gamma(X, WA) \\
\downarrow^{(1)} & & \downarrow^{(2)} \\
R\Gamma(X^{-} \times Y, WA) & \longrightarrow & R\Gamma(X \times Y, WA)
\end{array}\]

By induction hypothesis, (1) is an equivalence, so to show that (2) is an equivalence it is enough to show that the map on the cofibers is an equivalence. By [BPØ22, Theorem 7.5.4] together with (4.5.2) (or by using directly [BRS22, Corollary 11.10]), the cofiber is

\[R\Gamma(D_1, WA)|[-1] \overset{\sim}{\longrightarrow} R\Gamma(Y, WA) \to R\Gamma(D_1 \times Y, WA)|[-1]\]

which is an equivalence by induction on dimension. \qed

Remark 4.8. Let $X \in \text{SmlSm}(k)$. By [Sai21, Corollary 2.4] we have that for all $0 \leq d \leq \dim(X)$, $y \in X^{(d)}$:

\[H^j_y(X, W_mA^n) = 0 \quad \text{if} \quad j \neq d.\]

This implies that the complex

\[\ldots \to \bigoplus_{x \in X^{(d)}} H^d_x(X, W_mA^n) \to \bigoplus_{x \in X^{(d+1)}} H^{d+1}_x(X, W_mA^n) \to \ldots\]

computes $H^d_{\text{dim}^\text{red}}(X, W_mA^n)$ since $W_mA^n$ is strictly div-invariant by Theorem 4.2 and [BPØ22]. This generalizes [Gro85, 5.1] to $W_mA^n$.

Moreover, by [Sai21, Corollary 2.4], for a fixed isomorphism $\varepsilon_y : X^h_y \overset{\sim}{\longrightarrow} \text{Spec}(k(y)\{t_1 \ldots t_d\})$ which sends $|\partial X^h_y|$ to the divisor $t_1^{e_1} \cdot \ldots \cdot t_d^{e_d}$ for some $e_i \in \{0, 1\}$,

\[H^q_y(X, W_mA^n) = \tau^{(e_1 \cdot \ldots \cdot e_d)}W_m\Omega^n\]

where the right hand side is defined recursively as:

\[
\begin{align*}
\tau^{(0)}W_m\Omega^n & := \text{Hom}(\mathbb{Z}_l(\mathbb{A}^1, 0), W_m\Lambda^n), \\
\tau^{(1)}W_m\Omega^n & := \tau^{(0)}W_m\Omega^n / \text{Hom}(\mathbb{Z}_l(\mathbb{P}^1, 0 + \infty)i_1), W_m\Lambda^n), \\
\tau^{(e_1 \cdot \ldots \cdot e_s)}W_m\Omega^n & := \tau^{(e_s)}\tau^{(e_1 \cdot \ldots \cdot e_{s-1})}W_m\Omega^n,
\end{align*}
\]

where $(\cdot, i_1)$ is as in (3.5.1) the complement of the section $i_1 : 1 \to \mathbb{A}^1 \to \mathbb{P}^1$.

---

\[\text{Notice that, even though [Eke85, I. Theorem 6.2] is stated for the homotopy categories, the proof shows that } \overset{\sim}{\longrightarrow} \text{ is indeed homotopy coherent}\]
Theorem 4.4 implies that the truncated de Rham–Witt complex $W_m \Lambda^\bullet$ is a complex of objects in $\log CI(k, A)$, in particular its cohomology sheaves are strictly $\square$-invariant for the dNis topology and since they are coherent sheaves, also for the d\'{e}t and by [Niz08, Proposition 3.27] for the l\'{e}t topology. This similarly to Remark 4.3 implies that the cohomology of the log-de Rham–Witt complex:

\[ \text{SmlSm}(k) \to \mathcal{D}(W(k))^{op} \quad X \mapsto \holim_m R\Gamma(X, W_m \Lambda^\bullet) \]

factors through $\log \mathcal{D}_M^{\text{eff}}(k, W(k))$ and $\log \mathcal{D}_M^{\text{eff}}(k, W(k))$, via the functor:

\[ R\Gamma_{\text{cris}}: \log \mathcal{D}_M^{\text{eff}}(k, W(k)) \to \mathcal{D}(W(k))^{op} \]

\[ M \mapsto \varprojlim_m \text{Map}(M, W_m \Lambda^\bullet). \]

Recall that for $\tau \in \{\text{Nis}, \text{\'{e}t}\}$, there is a composition of localizations

\[ \mathcal{D}(\text{Shv}^{\text{lr}}_{\text{detr}}(k, A)) \xrightarrow{\text{L}_{\text{eff}}} \mathcal{D}(\text{Shv}^{\text{lr}}_{\text{det}}(k, A)) \xrightarrow{L_{(A^{1, \tau})}} \mathcal{D}_M^{\tau_{\text{eff}}}(k, A) \]

such that for all $X \in \text{SmlSm}$,

\[ L_{(A^{1, \tau})} L_{\omega_{\text{A}}} A_{\text{tr}}(X \times \square) \simeq L_{(A^{1, \tau})} A_{\text{tr}}(X \times A^1) \simeq L_{(A^{1, \tau})} A_{\text{tr}}(X^0) \simeq L_{(A^{1, \tau})} L_{\omega_{\text{A}}} A_{\text{tr}}(X^0), \]

hence it induces $L_{\omega_{\text{A}}} \cdot \log \mathcal{D}_M^{\tau_{\text{eff}}}(k, A) \to \mathcal{D}_M^{\tau_{\text{eff}}}(k, A)$ such that $L_{\omega_{\text{A}}} \cdot \mathcal{D}_M^{\tau_{\text{eff}}}(k, A)$.

Moreover, in the case where $\tau = \text{Nis}$, by [Par22, Proposition 2.5.7] it agrees with the $(A^1, \text{triv})$-localization. In particular, $L_{\omega_{\text{A}}} \cdot \mathcal{D}_M^{\tau_{\text{eff}}}(k, A)$ commutes with colimits, hence it has a fully faithful right adjoint right adjoint

\[ R\omega_{\text{A}}^*: \mathcal{D}_M^{\tau_{\text{eff}}}(k, A) \to \log \mathcal{D}_M^{\tau_{\text{eff}}}(k, A). \]

whose image agrees with $(A^1, \text{triv})^{-1} \log \mathcal{D}_M^{\tau_{\text{eff}}}(k, A)$. By composing it with $R\Gamma_{\text{cris}}$ above we get the following cohomology:

\[ (5.0.1) \]

\[ R\Gamma_p: \mathcal{D}_M^{\tau_{\text{eff}}}(k, A) \xrightarrow{\omega_{\text{Nis}}^*} \log \mathcal{D}_M^{\tau_{\text{eff}}}(k, A) \xrightarrow{L_{\text{det}}} \log \mathcal{D}_M^{\tau_{\text{eff}}}(k, A) \xrightarrow{R\Gamma_{\text{cris}}} \mathcal{D}(W(k))^{op} \]

We are now ready to prove Theorem 1.1: the cohomology $R\Gamma_p$ has by design the desired map $R\Gamma_p \to R\Gamma_{\text{cris}}$ which is universal among $(A^1, \text{Nis})$-local complexes equipped with a map $M \to R\Gamma_{\text{cris}}$. It remains to prove the comparison with rigid cohomology, which we now spell out.

Let $W \Omega_k = (\varprojlim_m W_m \Omega[1/p])$: $\text{Cor}_k \to \mathcal{D}(W(k)[1/p])$. By [EM19, Theorem 5.3] we have that $W \Omega_k$ is an $h$-sheaf: this implies that for all $X \in \text{SmlSm}_k$ and all proper hypercovers $X_\bullet \to X$, we have an equivalence

\[ R\Gamma(X, W \Omega_k) \simeq R\Gamma(X_\bullet, W \Omega_k). \]

By [Nak12, §9], for all $X \in \text{Sm}_k$ there is a simplicial object $(\bar{X}_\bullet, D_\bullet)$ in $\text{SmlSm}_k$ and such that $X_\bullet := \bar{X}_\bullet - [D_\bullet]$ is a proper hypercovering of $X$. This is functorial in $X$: by [Nak12, Proposition 9.4] there is a commutative diagram in $\text{Sm}_k$

\[ (5.0.2) \]

\[ \begin{array}{ccc} X' & \xrightarrow{\mathcal{F}} & X_\bullet \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f} & X, \end{array} \]

where the vertical arrows are proper hypercovers coming from simplicial objects $(\bar{X}_\bullet, D_\bullet)$ and $(\bar{X}_\bullet, D_\bullet')$ in $\text{SmlSm}_k$ as before and the map $\mathcal{F}$ comes from maps $\mathcal{F}_n: (\bar{X}_n, D_n) \to (\bar{X}_n, D_n)$ in $\text{SmlSm}_k$. By [Nak12, Corollary 11.7], for $X_\bullet \to X$ as above there is an equivalence

\[ R\Gamma_{\text{rig}}(X) \simeq R\Gamma((\bar{X}_\bullet, D_\bullet), W \Lambda Q). \]
functorial in \( X \). Putting everything together, we have that for any \( f: X' \to X \) there commutative diagram functorial in \( X \):

\[
\begin{align*}
R\Gamma_{\text{rig}}(X) \xrightarrow{\sim} & \quad R\Gamma((X_\bullet, D_\bullet), W\Lambda_Q) \quad \longrightarrow \quad R\Gamma(X_\bullet, W\Omega_Q) \quad \xrightarrow{\sim} \quad R\Gamma(X, W\Omega_Q) \\
\downarrow & \quad \downarrow & \quad \downarrow \\
R\Gamma_{\text{rig}}(X') \xrightarrow{\sim} & \quad R\Gamma((X'_\bullet, D_\bullet), W\Lambda_Q) \quad \longrightarrow \quad R\Gamma(X'_\bullet, W\Omega_Q) \quad \xrightarrow{\sim} \quad R\Gamma(X', W\Omega_Q)
\end{align*}
\]

that implies that we have a map in \( \mathcal{D}(\text{Shv}_{\text{Nis}}^A(k, W(k)[1/p])) \)

\[
f: R\Gamma_{\text{rig}} \to W\Omega_Q.
\]

Since \( R\Gamma_{\text{rig}} \) factors through \( \mathcal{D}\mathcal{M}_{\text{eff}}^A(k, \mathbb{Q}) \), by the universal property the map above factors through \( R\Gamma_p[1/p] \to W\Omega_Q \).

**Theorem 5.1.** The map \( R\Gamma_{\text{rig}} \to R\Gamma_p[1/p] \) is an equivalence: in particular \( R\Gamma_{\text{rig}} \to R\Gamma_{\text{crys}}[1/p] \) is universal among the maps from \((\mathbb{A}^1, \text{Nis})\)-local cohomologies.

**Proof.** Since the map \( R\Gamma_p[1/p] \to W\Omega_Q \) is universal among maps from \( \mathbb{A}^1 \)-local sheaves, it is enough to prove that the map in the statement has a retrac tion. Consider the commutative diagram in \( \mathcal{D}(\text{Shv}_{\text{Nis}}^A(k, W(k)[1/p])) \):

\[
\begin{array}{ccc}
R\Gamma_{\text{rig}} & \xrightarrow{f} & W\Omega_Q \\
\downarrow & & \downarrow \\
R\Gamma_p[1/p] & & \\
\end{array}
\]

If \( X \in \text{Sm}_k \) is proper, then we have the classical equivalence of Berthelot [Ber82]:

\[
R\Gamma_{\text{rig}}(X) \xrightarrow{\sim} R\Gamma(X, W\Lambda_Q) \xrightarrow{\sim} R\Gamma_p(X)[1/p]
\]

giving a splitting \( R\Gamma_p(X)[1/p] \to R\Gamma(X, W\Lambda_Q) \simeq R\Gamma_{\text{rig}}(X) \), functorial in \( X \in \text{Sm}_k \) proper. Since \( R\Gamma_{\text{rig}} = \omega^*_p R\Gamma_{\text{rig}} \) and \( R\Gamma_p[1/p] = \omega^*_p R\Gamma_p[1/p] \) with \( \omega^* R\Gamma_{\text{rig}} \) and \( \omega^* R\Gamma_p[1/p] \in \log\mathcal{D}\mathcal{M}_{\text{eff}} \) by \([BP022]\), and \( W\Omega_Q \simeq \omega^* W\Lambda_Q \), we have by Corollary B.4 below a commutative diagram in \( \log\mathcal{D}\mathcal{M}_{\text{eff}}(k, W(k)[1/p]) \):

\[
\begin{array}{ccc}
\omega^* R\Gamma_{\text{rig}} & \xrightarrow{g} & W\Lambda_Q \\
\downarrow & & \downarrow \\
\omega^* R\Gamma_p[1/p] & & \\
\end{array}
\]

It is enough to show that \( g \) is an equivalence when evaluated on \((X, D)\) with \( X \) proper\(^7\): we can use the trick of the Gysin maps similarly to the proof of [BM21, Theorem 4.4]: let \((X, D)\) be a normal crossing pair supporting an object in \( \text{Sm}_{\text{Nis}} \) with \( D = D_1 + \ldots + D_n \) and let \( D' = D_1 + \ldots + D_{n-1} \). The Gysin sequence in \( \log\mathcal{D}\mathcal{M}_{\text{eff}}(k, A) \)

\[
M(X, D) \to M(X, D') \to M\text{th}(N_{D_n})
\]

\(^7\)Beware that at this point we do not know that \( g \) agrees with the equivalence of [Nak12], as (without the assumption of resolutions of singularities) that does not extend to a map on the big site \( \text{ICor}_k \), only on schemes that have normal crossing compactifications.
imply that we have the following commutative diagram functorial in \((X, D)\), where the vertical maps form fiber sequences:

\[
\begin{array}{ccc}
R\Gamma_{\text{rig}}(X - D') & \longrightarrow & R\Gamma_p(X - D')[1/p] \\
\downarrow & & \downarrow \\
R\Gamma_{\text{rig}}(X - D) & \longrightarrow & R\Gamma_p(X - D)[1/p] \\
\downarrow & & \downarrow \\
R\Gamma_{\text{rig}}(Th(N_{D_n - D \hookrightarrow X - D})) & \longrightarrow & R\Gamma_p(Th(N_{D_n - D' \hookrightarrow X - D'}))[1/p] \\
& & \downarrow \text{g} \\
& & R\Gamma(Th(N_{D_n \hookrightarrow X}), W\Lambda_Q)
\end{array}
\]

By the projective bundle formulas for \(\mathcal{DM}_{\text{eff}}\) (see [MVW06, Theorem 15.12]), the \(\mathbb{P}^1\)-stability of rigid cohomology (see [Ber97, A.8]) and the projective bundle formula for \(W\Lambda_Q\) (see [Mer, (4.5.1)] generalizing [Gro85]), we have that the bottom horizontal map is equivalent (functorially in \((X, D)\) and up to a shift) to:

\[
\begin{array}{ccc}
R\Gamma_{\text{rig}}(D_n - D) & \longrightarrow & R\Gamma_p((D_n - D) \times (\mathbb{P}^1)^0)[1/p] \\
\downarrow \text{g} & & \downarrow \\
R\Gamma_p((D_n - D) \cap D_n), W\Lambda_Q).
\end{array}
\]

Let us assume now that \(X\) is proper: by the usual trick of double induction on the dimension of \(X\) and the number of components of \(D\) (see the proof of [BM21, Theorem 4.4]), we conclude that \(g(X, D)\) is an equivalence for every \(X\) proper functorially in \((X, D)\); this gives a commutative diagram functorial in \((X, D)\):

\[
\begin{array}{ccc}
R\Gamma_{\text{rig}}(X - D) & \cong & W\Lambda_Q(X, D) \\
\downarrow & & \downarrow \\
R\Gamma_p[1/p](X - D)
\end{array}
\]

giving a splitting \(R\Gamma_p(X - D)[1/p] \rightarrow R\Gamma((X, D)W\Lambda_Q) \cong R\Gamma_{\text{rig}}(X - D)\) functorial in \((X, D)\). Finally, since both \(R\Gamma_{\text{rig}}\) and \(R\Gamma_p[1/p]\) are \((\mathbb{A}^1, \text{Nis})\)-local, they satisfy \(h\)-descent by [Voe00, Theorem 4.1.12] (see also [Tsu03] for an independent proof for \(R\Gamma_{\text{rig}}\)), so we conclude by (5.0.2) that the retraction constructed above descends to a retraction

\[
\begin{array}{ccc}
R\Gamma_{\text{rig}}(X) & \cong & R\Gamma_{\text{rig}}(X) \\
\downarrow & & \downarrow \\
R\Gamma_p(X, D)[1/p] & \leftrightarrow & R\Gamma_p(X, D)[1/p]
\end{array}
\]

functorial in \(X\): this concludes the proof.

6. Assuming resolution of singularities

If we assume that \(k\) admits resolution of singularities, then (5.0.1) can be made more explicit.

**Notation 6.1.** We say that \(k\) satisfies resolutions of singularities if the following two properties are satisfied (see [BPØ22, Definition 7.6.3.]):

(RS1) For any integral scheme \(X\) of finite type over \(k\), there is a proper birational morphism \(Y \rightarrow X\) of schemes over \(k\), which is an isomorphism on the smooth locus, such that \(Y\) is smooth over \(k\).
(RS2) Let $f: Y \to X$ be a proper birational morphism of integral schemes over $k$ such that $X$ is smooth over $k$ and let $Z_1, \ldots, Z_r$ be smooth divisors forming a strict normal crossing divisor on $X$. Assume that
$$f^{-1}(X - Z_1 \cup \ldots \cup Z_r) \to X - Z_1 \cup \ldots \cup Z_r$$
is an isomorphism. Then there is a sequence of blow-ups
$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-2}} \ldots \xrightarrow{f_0} X_0 \simeq X$$along smooth centers $W_i \subseteq X_i$ such that
a. the composition $X_n \to X$ factors through $f$,
b. $W_i$ is contained in the preimage of $Z_1 \cup \ldots \cup Z_r$ in $X_i$,
c. $W_i$ has strict normal crossing with the sum of the reduced strict transforms of $Z_1, \ldots, Z_r$, $f_0^{-1}(W_0), \ldots, f_{i-1}^{-1}(W_{i-1})$ in $X_i$.

Recall that by [BPO22, Proposition 8.2.8], for $X \in \text{Sm}(k)$ with smooth Cartier compactification $\overline{X}$, for $M^V(X) \in \mathcal{D}\mathcal{M}_{\text{eff}}(k, \mathbb{Z})$ the Voevodsky motive of $X$, we have
$$R\omega_{\text{Nis}}^* M^V(X) = M(\overline{X}, \partial X)$$where $\partial X$ is the log structure supported on $\overline{X} - X$.

**Proposition 6.2.** If $k$ admits RS1 and RS2, for $X \in \text{Sm}(k)$, we have an equivalence:
$$R\Gamma_p(X) \simeq R\Gamma_{\text{crys}}((\overline{X}, X)/W(k))$$

**Proof.** Since $k$ admits resolutions of singularities, there exists $X \hookrightarrow \overline{X}$ a Cartier compactification and let $\partial X$ be the log structure supported on the divisor $\overline{X} - X$. Then the result follows from the equivalence $\omega^* M^V(X) = M(\overline{X}, \partial X)$ of [BPO22, Proposition 8.2.8]. □

**Remark 6.3.** As observed in [ESS21, Proposition 2.24] there exist $X \in \text{Sm}(k)$ with smooth Cartier compactification $(\overline{X}, \partial X)$ and an étale hypercover $X_* \to X$ with smooth Cartier compactification $(\overline{X}_*, \partial X_*)$ such that
$$H^i_{\text{crys}}((X, \partial X)/W(k)) \neq H^i_{\text{crys}}((X_*, \partial X_*)/W(k)).$$

This by Proposition 6.2 implies that $M_{\text{dR}}(\overline{X}, \partial X) \neq M_{\text{dR}}(\overline{X}_*, \partial X_*)$. In particular, this implies that the functor (5.0.1) does not factor through $\mathcal{D}\mathcal{M}_{\text{eff}}(k, \mathbb{Z})$, so the following diagram is not commutative:
$$\begin{array}{ccc}
\mathcal{D}\mathcal{M}_{\text{eff}}(k, \mathbb{Z}) & \xrightarrow{\omega^*} & \log \mathcal{D}\mathcal{M}_{\text{eff}}(k, \mathbb{Z}) \\
\downarrow_{L_{\text{dR}}} & & \downarrow_{L_{\text{dR}}} \\
\mathcal{D}\mathcal{M}_{\text{eff}}^\text{dR}(k, \mathbb{Z}) & \xrightarrow{\omega_{\text{dR}}} & \log \mathcal{D}\mathcal{M}_{\text{eff}}^\text{dR}(k, \mathbb{Z})
\end{array}$$
as $L_{\text{dR}}(M^V(X)) \simeq L_{\text{dR}} M^V(X_*)$. On the other hand, it commutes with rational coefficients. In fact, in this case, let $K = W(k)[1/p]$ and let $X \in \text{Sm} \text{Sm}(k)$ with $\overline{X}$ proper. By [Nak12, Cor.11.7 1)] have that
$$R\Gamma_{\text{crys}}(X)[1/p] \simeq R\Gamma_{\text{rig}}(X^\circ),$$and rigid cohomology factors through $\mathcal{D}\mathcal{M}_{\text{eff}}^\text{dR}(k, \mathbb{Q})$, there is in fact a commutative diagram
$$\begin{array}{ccc}
\mathcal{D}\mathcal{M}_{\text{eff}}^\text{dR}(k, \mathbb{Q}) & \xrightarrow{L_{\text{dR}} \omega^*} & \log \mathcal{D}\mathcal{M}_{\text{eff}}^\text{dR}(k, \mathbb{Q}) \\
\downarrow \simeq & & \downarrow R\Gamma_{\text{rig}}(\omega) \\
\mathcal{D}\mathcal{M}_{\text{eff}}^\text{dR}(k, \mathbb{Q}) & & \mathcal{D}(K).
\end{array}$$
6.1. Finally, we show that $R\Gamma_p$ satisfies condition (iii") of the introduction. Notice that since $R\Gamma_p$ factors through $\text{log}\mathcal{DM}_{\text{eff}}(k, \mathbb{Z})$, $R\Gamma_p$ satisfies the following descent property: log-(iii) For all $(\mathcal{X}, \partial \mathcal{X}) \in \text{Sm}_{\text{log}}(k)$ with $\mathcal{X}$ proper, for all $(\mathcal{U}, \partial \mathcal{U}) \to (\mathcal{X}, \partial \mathcal{X})$ finite log-étale cover (in particular, strict), let $U \to X$ be the finite étale map on the open subschemes where the log structures is trivial. Then the Čech hypercover $U_* \to X$ induces an equivalence

$$R\Gamma_p(X) \to R\Gamma_p(U_*)$$

The link between finite log étale and tame descent is summarized in the following result:

**Lemma 6.4.** If $k$ satisfies RS1 and RS2, then for any $U \to X$ finite tame Galois cover of $k$-schemes, there exist smooth Cartier compactifications $\mathcal{X}$ and $\mathcal{U}$ with normal crossing complement $\partial \mathcal{X}$ and $\partial \mathcal{U}$ such that the induced map of log schemes $(\mathcal{U}, \partial \mathcal{U}) \to (\mathcal{X}, \partial \mathcal{X})$ is a log étale Galois cover.

**Proof.** Let $U \to X$ be a finite étale map. By resolutions of singularities, there exists $\mathcal{X}'$, a smooth Cartier compactification of $X$. By platifification [RG71] (see also [KMSY21, Theorem 1.6.1]), there exists $\mathcal{X} \to \mathcal{X}'$ proper which is an isomorphism on $X$ such that $U \times _{\mathcal{X}_x} \mathcal{X}' \to \mathcal{X}$ is finite. Again by resolutions of singularities, we can take $\mathcal{X}$ to be a smooth Cartier compactification of $X$, so that $U \times _{\mathcal{X}} \mathcal{X}' \to \mathcal{X}$ is finite étale and $\mathcal{U} := U \times _{\mathcal{X}_x} \mathcal{X}'$ is a smooth Cartier compactification of $U$ via

$$U \cong U \times _{\mathcal{X}_x} \mathcal{X} \hookrightarrow U \times _{\mathcal{X}_x} \mathcal{X}' = \mathcal{U}.$$ 

Let $(\mathcal{U}, \partial \mathcal{U}) \to (\mathcal{X}, \partial \mathcal{X})$. Then by a combination of [KS10, Theorem 1.1] and [Ill02, Theorem 7.6]8, $U \to X$ is a finite tame Galois cover if and only if $(\mathcal{U}, \partial \mathcal{U}) \to (\mathcal{X}, \partial \mathcal{X})$ is a finite log étale cover.

We remark that the counterexample to (iii) given in [ESS21] is not a tame cover.

We finish this section by deducing the following results for the cohomology $R\Gamma_p(\_)$:

**Theorem 6.5** (Projective bundle formula). For $\mathcal{E} \to X$ a vector bundle of rank $n + 1$, we have

$$R\Gamma_p(\mathcal{P}(\mathcal{E})) \simeq \bigoplus_{i=1}^{n} R\Gamma_p(X)[-i]$$

**Proof.** From the projective bundle formula, we have

$$M^V(\mathcal{P}(\mathcal{E})) \simeq \bigoplus_{i=1}^{n} M^V(X)(i)[2i]$$

hence by [BPØ22, Proposition 8.2.8]

$$\omega^*(M^V(X)(i)[2i]) \simeq M(X) \otimes (M(\mathcal{P}^1, \text{triv})^0)^{\otimes i},$$

where $M(\mathcal{P}^1, \text{triv})^0$ is as in (3.5.1) the orthogonal to $0 \to \mathcal{P}^1$. By Corollary 4.5 we have

$$\underline{\text{Map}}((M(\mathcal{P}^1, \text{triv})^0, W_m \Lambda^\bullet) \simeq W_m \Lambda^{\bullet-1}[-1]$$

which recursively gives:

$$\underline{\text{Map}}((M(\mathcal{P}^1, \text{triv})^0)^{\otimes i}, W_m \Lambda^\bullet) \simeq W_m \Lambda^{\bullet-i}[-i].$$

The equivalence above is compatible with $W_m \Lambda^{\bullet-i} \to W_m \Lambda^\bullet$, so by taking the limit over $m$ we conclude the proof.

**Theorem 6.6** (Purity). Let $X \in \text{Sm}(k)$ and let $Z \subseteq X$ be a smooth closed subscheme of codimension $d$. Then for $n \leq d - 1$ we have a fiber sequence:

$$R\Gamma_p(X) \to R\Gamma_p(X - Z) \to R\Gamma_p(Z)[-d]$$

8This is stated without proof: for a sketch one can check [Ill02, 3.3]
Proof. By [MV99] and Proposition 6.2 we have a fiber sequence
\[ R \Gamma_p(X) \to R \Gamma_p(X - Z) \to R \Gamma_p(\omega^*(M(Z)(d)[2d])). \]
As before, by [BP02, Proposition 8.2.8] we have
\[ \text{Map}(\omega^*(M^V(Z)(d)[2d]), W_m \Lambda^*) \simeq \text{Map}(M(Z), W_m \Lambda^{•-d)[-d]). \]
which by taking the homotopy limit over \( m \) gives the result. \( \square \)

Theorem 6.7 (Künneth formula). There is an equivalence in \( \mathcal{D}(R(k)) \):
\[ R \Gamma_p(X) \otimes^L R \Gamma_p(Y) \simeq R \Gamma_p(X \times Y) \]
Proof. It follows from 4.7 and [BP02, Theorem 8.2.8] \( \square \)

APPENDIX A. A METHOD FOR COMPUTING \( h_0^\square \)

In this appendix, we will show a general method to compute \( h_0^\square(\_\_\_\_) \) and we will apply it to compute \( h_i^{\square, \log}(\mathbb{P}^1, \text{triv}) \) and \( h_i^{\square, \text{ltr}}(\mathbb{P}^1, \text{triv}) \) for \( i = 0, 1 \), without any assumption on the perfect field \( k \).

Construction A.1. Let \( s \in \{ \log, \text{ltr} \} \). Let \( A \in \text{Shv}^\delta_{\Delta Nis}(k, A) \) and let \( B \in \text{logCI}^s(k, A) \) with \( f : A \to B \) such that \( \omega_2(f) \) is surjective. Then we have a commutative diagram
\[
\begin{array}{ccc}
\omega_2 A & \longrightarrow & \omega_2 B \\
\downarrow & & \downarrow \\
\omega_2 h_0^s(A) & \longrightarrow & \omega_2 h_0^s(B)
\end{array}
\]
In particular \( \omega_2 \text{coker}(h_0^{\square,s}(f)) = 0 \), and since \( \text{coker}(h_0^{\square,s}(f)) \in \text{logCI}^s(k, A) \), we conclude by [BM21, Theorem 5.10] that \( h_0^{\square,s}(f) \) is surjective.

Let \( E \) (resp. \( E' \)) be the kernel of \( f \) (resp. \( h_0^{\square,s}(f) \)). We fix the notation in the following commutative diagram:
\[
\begin{array}{ccc}
0 & \longrightarrow & E \\
\downarrow L_0 & & \downarrow L \\
0 & \longrightarrow & E' \\
& & \downarrow h_0^{\square,s}(f)
\end{array}
\]
(A.1.1)
Let \( K \) be a function field and for each \( t \in K \) consider \( i_t : (K, \text{triv}) \to \square_K \) given by the inclusion of \( [1 : t] \) in \( \mathbb{P}_K^1 \). Notice that for all \( F \in \text{logCI}^s \) and \( t \in K \), the map \( i_t^* : F(\square_K) \to F(K, \text{triv}) \) is an isomorphism inverse of \( F(K, \text{triv}) \to F(\square_K) \) given by the structural map, so for \( t, t' \in K \), \( i_t^* = i_{t'}^* \). We have the following commutative diagram:
\[
\begin{array}{ccc}
Ker(L_K) & \longrightarrow & A(\square_K) \\
\downarrow & & \downarrow \\
A(K, \text{triv}) & \longrightarrow & A(K, \text{triv})
\end{array}
\]
(A.1.2)
Let us consider the following condition:
\( (\ast) \quad q_K(E(K, \text{triv})) \subseteq \text{Im}(i_0^* - i_1^*) \)

Proposition A.2. In the situation of Construction A.1, if \( (\ast) \) is satisfied for every function field \( K \), then \( h_0^{\square,s}(A) \cong B \).
Proof. For simplicity, we will drop the “s” from the notation: the proof works verbatim with and without transfers. We need to show that for all $g: A \to C$ with $C \in \logCI$ there is a unique map $\overline{g}: B \to C$ that factors $g$ via $f$. Notice that if $\overline{g}, \overline{g}'$ such that $\overline{g} \circ f = \overline{g}' \circ f$, by the universal property of $h^\Gamma_0(A)$ we have $\overline{g} \circ h^\Gamma_0(f) = \overline{g}' \circ h^\Gamma_0(f)$, and since $h^\Gamma_0(f)$ is surjective we get $\overline{g} = \overline{g}'$. We are then left to show that such $\overline{g}$ exists: to do so, we will show that there exists a map $B \to h^\Gamma_0(A)$ that factors $A \to h^\Gamma_0(A)$. The condition $(\ast)$ implies that for all function fields $K$

$$q_K^0(L_0)_K(E(K, \text{triv}))^{(A.1.1)} = L_K q_K E(K, \text{triv}) \subseteq L_K (i_0 - i_1(A(\overline{K})))^{(A.1.2)} = 0.$$ 

Since $q^0_K$ is injective by definition (see (A.1.1)), we have $(L_0)_K = 0$. By [BM21, Theorem 5.10], for all $X \in \text{Sm} \text{Sm}(k)$ with fraction field $K$, $h^\Gamma_0(A)(X) \hookrightarrow h^\Gamma_0(A)(K, \text{triv})$, so $E'(X) \hookrightarrow E'(K, \text{triv})$, which implies that $L_0 = 0$. So, we have that $E = \ker(L)$, hence the map $L$ factors through $A/E$. By the universal property of $L$, for all $C \in \logCI$ and $g: A \to C$, we have a unique map $g'$ such that

$$A \xrightarrow{g} C \xrightarrow{h^\Gamma_0(A)} A/E \xrightarrow{p} h^\Gamma_0(A).$$

Moreover, the map $u: A/E \to B$ induces an isomorphism $\omega_2 A/E \cong \omega_2 B$, so we get a map $\varphi = \omega_2 p \circ (\omega_2 u)^{-1}: \omega_2 B \to \omega_2 h^\Gamma_0(A)$.

Let $X^h_x \in \text{Sm}(k)$ hensel local of Krull dimension 1, with generic point $\eta_x$ and closed point $k(x)$. Since $A/E A^1_{k(x)} = B A^1_{k(x)}$, we have $R\Gamma(A^1_{k(x)}, A/E) = A/E(A^1_{k(x)})$, so the commutative diagrams of (3.6.4) and the split $\text{Spec}(k(x)) \to \overline{k}(x) \xrightarrow{\text{str}} \text{Spec}(k(x))$ give a commutative diagram

$$B(\eta_x) \xleftarrow{u_{\eta_x}} \xrightarrow{\cong} A/E(\eta_x) \xrightarrow{p_{\eta_x}} h^\Gamma_0(A)(\eta_x) \xrightarrow{\cong} B(A^1_{k(x)}/R\Gamma(P^1_{k(x)}, B) \xrightarrow{\cong} A/E(A^1_{k(x)}/R\Gamma(P^1_{k(x)}, A/E) \xrightarrow{\cong} h^\Gamma_0(A)(A^1_{k(x)})/R\Gamma(P^1_{k(x)}, h^\Gamma_0(A)) \xrightarrow{\cong} B(A^1_{k(x)}/B(k(x)) \xrightarrow{u_{A^1_{k(x)}}} \xrightarrow{\cong} A/E(A^1_{k(x)}/A/E(k(x)) \xrightarrow{p_{A^1_{k(x)}}} h^\Gamma_0(A)(A^1_{k(x)})/h^\Gamma_0(A)(k(x))).$$

By taking inverses on the horizontal maps on the big square on the left, we conclude that the map $\varphi$ induces a commutative diagram

$$B(\eta_x) \xrightarrow{\varphi_{\eta_x}} h^\Gamma_0(A)(\eta_x) \xrightarrow{\text{Res}_{h^\Gamma_0(A)(\overline{k}_x)}(X^h_x)} B(A^1_{k(x)}/B(k(x)) \xrightarrow{\varphi_{A^1_{k(x)}}} h^\Gamma_0(A)(A^1_{k(x)})/h^\Gamma_0(A)(k(x))).$$

which implies that $\varphi$ is a map of presheaves with residues as in Definition 3.7. By Proposition 3.9, it lifts uniquely to a map $p': B \to h^\Gamma_0(A)$. We need to show that $p' f = L$: let $X \in \text{Sm} \text{Sm}(k)$ and let $X^\circ$ be the subscheme with trivial log structure. Since the map
$h_0^{\text{tr}}(A)(X) \hookrightarrow h_0^{\text{tr}}(A)(X^\circ, \text{triv})$ is injective by [BM21, Theorem 5.10], with an argument similar to (3.9.2) it is enough to check the commutativity on $(X^\circ, \text{triv})$. which follows automatically by the commutativity of

$$
\begin{array}{ccc}
A/E(X^\circ) & \xrightarrow{p_{X^\circ}} & h_0^{\text{tr}}(A)(X^\circ) \\
\downarrow & & \downarrow p_{X^\circ} \\
B(X^\circ) & & \\
\end{array}
$$

\[ \square \]

**Remark A.3.** We would like to use the method explained in Proposition A.2 to the map considered in [RSY22]:

$$
G_a \otimes_{\text{ltr}} G_m \to \Omega^1 \quad (a, b) \mapsto a \cdot \text{dlog}(b).
$$

We conjecture that this map is an isomorphism, but at the moment we are not able to verify $(\ast)$ in this case. We remark that the methods of [RSY22] do not apply here since $Z_{\text{ltr}}(\mathbb{A}^1)(\mathbb{K}) = Z_{\text{ltr}}(\mathbb{A}^1)(K)$: in order to make it work we need a replacement for the modulus pair $(\mathbb{P}^1, 2\infty)$. This is a work in progress.

For $X \in \text{Sm} \text{Sm}(k)$, we let $X^\circ := X - |\partial X| \in \text{Sm}(k)$ and $h_1^{\text{A}^1,\text{ltr}}(X^\circ)$ the $i$-th Suslin homology sheaf. We apply the proposition above to prove the following:

**Proposition A.4.** Let $X \in \text{Sm} \text{Sm}(k)$ such that $X$ is proper and $h_1^{\text{A}^1,\text{ltr}}(X^\circ) = 0$. Then $h_0^{\text{ltr},\text{triv}}(X) \cong \omega^* h_0^{\text{A}^1,\text{ltr}}(X^\circ)$.

**Proof.** The map $f : Z_{\text{ltr}}(X) \to \omega^* h_0^{\text{A}^1,\text{ltr}}(X^\circ)$ induced by adjunction from the map

$$
\omega_2 Z_{\text{ltr}}(X, D) = Z_{\text{tr}}(X^\circ) \to h_0^{\text{A}^1,\text{ltr}}(X^\circ)
$$

satisfies the fact that $\omega_2 f$ is surjective, hence by Proposition A.2, it is enough to show that $(\ast)$ holds in this case. Notice that we have the following commutative diagram

$$
\begin{array}{ccc}
\text{lCor}(\overline{\mathbb{K}}, X) & \xrightarrow{i_0^* - i_1^*} & Z_{\text{ltr}}(X)(K) \\
\downarrow \iota & & \downarrow \\
\text{Cor}(\mathbb{A}^1_K, X^\circ) & \xrightarrow{(i_0^*)^* - (i_1^*)^*} & Z_{\text{tr}}(X^\circ)(K)
\end{array}
$$

and since $h_1^{\text{A}^1,\text{ltr}}(X^\circ) = 0$ we have that

$$
\text{Im}((i_0^*)^* - (i_1^*)^*) = q_K(E(K, \text{triv})).
$$

Moreover, $\iota$ is a bijection since it is clearly injective and by e.g. [KMSY21, Theorem 1.6.2], every finite correspondence $Z : \mathbb{A}^1_K \to X$ such that $|\partial X|_Z = \text{triv}$ extends to a finite log correspondence $\overline{Z} : \overline{\mathbb{K}} \to X$: this concludes the proof. \[ \square \]

We can now compute the homotopy groups of $M^{\text{ltr}}(\mathbb{P}^1, \text{triv})$. First of all, there is a Nisnevich square

\[ (\mathbb{P}^1, 0 + \infty) \longrightarrow (\mathbb{P}^1, 0) \]

\[ (\mathbb{P}^1, \infty) \longrightarrow (\mathbb{P}^1, \text{triv}), \]

giving a long exact sequence

$$
\begin{array}{ccccccc}
h_1^{\text{ltr}}(\mathbb{P}^1, 0) & \otimes & h_0^{\text{ltr}}(\mathbb{P}^1, \text{triv}) & \to & h_0^{\text{ltr}}(\mathbb{P}^1, 0 + \infty) & \to & h_0^{\text{ltr}}(\mathbb{P}^1, \text{triv}) & \to 0 \\
\end{array}
$$
Since the inclusion of the point \([1 : 1]\) in \(\mathbb{P}^1\) induces equivalences in \(\log DM^{\text{eff}}(k, \mathbb{Z})\):

\[
Z[0] \simeq M^{\text{ltr}}(\text{Spec}(k)) \simeq M^{\text{ltr}}(\mathbb{P}^1, 0) \simeq M^{\text{ltr}}(\mathbb{P}^1, \infty)
\]

we conclude that \(h_0^{\text{ltr}}(\mathbb{P}^1, \text{triv}) \simeq h_0^{\text{ltr}}(\text{Spec}(k)) = \mathbb{Z}\). Moreover, we have that the tautological bundle of \(\mathbb{P}^1\) induces a map \(\mathbb{Z}_{\text{triv}}(\mathbb{P}^1, \text{triv}) \to \omega^* G_m[1]\). Since \(\omega^* G_m \in \log CI^{\text{ltr}}\), the map factors through \(M^{\text{ltr}}(\mathbb{P}^1, \text{ltr})\), and by taking \(\pi_1\) we get a map

\[
(A.4.2) \quad h_1^{\text{ltr}}(\mathbb{P}^1, \text{triv}) \to \omega^* G_m.
\]

**Proposition A.5.** The map \((A.4.2)\) is an isomorphism.

**Proof.** By e.g. [BM21, Proposition 5.12], the map above factors through \(\omega^* h_1^A(\mathbb{P}^1) \to \omega^* G_m\), which is an isomorphism by [MVW06, Lecture 4], hence it is enough to prove that \(h_1^{\text{ltr}}(\mathbb{P}^1, \text{triv}) \cong \omega^* h_1^A(\mathbb{P}^1)\). By the long exact sequences induced by Mayer–Vietoris we have a commutative diagram where the rows are exact sequences:

\[
\begin{array}{c}
0 \longrightarrow h_1^{\text{ltr}}(\mathbb{P}^1, \text{triv}) \longrightarrow h_0^{\text{ltr}}(\mathbb{P}^1, 0 + \infty) \longrightarrow h_0^{\text{ltr}}(\square) \times 2 \\
\downarrow \ast(1) \quad \quad \downarrow \ast(2) \\
0 \longrightarrow \omega^* h_1^A(\mathbb{P}^1) \longrightarrow \omega^* h_0^A(A^1 - \{0\}) \longrightarrow \omega^* h_0^A(A^1) \times 2.
\end{array}
\]

By [MVW06, Lemma 4.1], we have \(h_1^A(A^1 - \{0\}) = 0\), so by Proposition A.4 we conclude that the map \((\ast)\) above is an isomorphism, so \((\ast1)\) is an isomorphism too. This concludes the proof. \(\square\)

**Remark A.6.** One would be tempted to generalize the result above to the case without transfers: we have by [Mor12, Theorem 3.37] that \(h_1^A(\mathbb{P}^1) \cong K^1_M\), where the right hand side is the Milnor–Witt \(K\)-theory sheaf. On the other hand, \((\ast)\) is not verified in this situation, so the result does not immediately follow in this case. We leave this (apparently much harder) computation to a future work.

As an application, we have the following corollary that will be useful in the next appendix:

**Corollary A.7.** For all \(F \in \log CI^{\text{ltr}}\), the map \(M(\mathbb{P}^1) \xrightarrow{\delta} \omega^* G_m[1]\) in \(\log DM^{\text{eff}}(k, \mathbb{Z})\) induces an isomorphism

\[
\text{Hom}_{\text{shv}_{\text{Nis}}}(\omega^* G_m, F) \cong a_{\text{Nis}} H^1(\mathbb{P}^1 \times \omega^k F).
\]

**Proof.** The spectral sequence

\[
\text{Ext}_{\text{shv}_{\text{Nis}}}^1(h_1(\mathbb{P}^1, \text{triv}), F) \Rightarrow \pi_i - j \text{Map}_{\log DM^{\text{eff}}}(M(\mathbb{P}^1, \text{triv}), F)
\]

gives the five-terms exact sequence

\[
\text{Ext}^1(h_0(\mathbb{P}^1, \text{triv}), F) \to \pi_i - j \text{Map}(M(\mathbb{P}^1, \text{triv}), F) \to \text{Hom}(h_1(\mathbb{P}^1, \text{triv}), F) \to \text{Ext}^2(h_0(\mathbb{P}^1, \text{triv}), F).
\]

We have that \(h_0(\mathbb{P}^1, \text{triv}) = \mathbb{Z}\), so for all \(X \in \text{Sm} \text{Sm}(k)\) with generic point \(\eta_X\), by [BM21, Theorem 5.10] we have an injection:

\[
\text{Ext}^i(h_0(\mathbb{P}^1, \text{triv}), F)(\eta_X) \hookrightarrow \text{Ext}^i(h_0(\mathbb{P}^1, \text{triv}), F)(\mathbb{P}^1, \text{triv}), F)(\eta_X) = H^i(\eta_X, F) = 0 \quad \text{for } i \geq 1,
\]

which implies that the map \(\theta\) above is an isomorphism. Moreover, by Proposition A.5 we have that \(h_1(\mathbb{P}^1, \text{triv}) = \omega^* G_m\) and the map \(\theta\) is induced by \(\delta\), so we conclude that for every field \(k(x)\) we have

\[
H^i(\mathbb{P}^1, \omega^k F) = \pi_i - j \text{Map}(M(\mathbb{P}^1, \text{triv}), F)(k(x)) \cong \text{Hom}(\omega^* G_m, F)(k(x))
\]

which again by [BM21, Theorem 5.10] allows us to conclude. \(\square\)
Appendix B. Proof of [BM22, Conjecture 0.2] in the case with transfers

In this appendix, we will deduce the aforementioned conjecture from Theorem 3.10. Let $X \in \text{Sm}_{\text{Shv}}(k)$ and consider the map

$$Z_{\text{tr}}(X^0) \to h_0^{A^1,\text{tr}}(X^0).$$

induced by the Suslin complex of $X - D$. By adjunction, we have a map

$$Z_{\text{tr}}(X) \to \omega^*h_0^{A^1,\text{tr}}(X^0),$$

which since $\omega^*h_0^{A^1}(X^0) \in \log\text{CI}^{\text{tr}}$ induces a map

$$h_0^{A^1,\text{tr}}(X) \to \omega^*h_0^{A^1,\text{tr}}(X^0),$$

which is surjective by purity of $\log\text{CI}^{\text{tr}}$ (see [BM21, Theorem 5.10]). This map is functorial in $X$, which gives for all $F \in \log\text{CI}$ a subsheaf:

$$F_{A^1} \subseteq F: (X, D) \mapsto \text{Hom}_{\log\text{CI}^{\text{tr}}}(\omega^*h_0^{A^1,\text{tr}}(X^0), F).$$

Let us fix $F \in \log\text{CI}^{\text{tr}}$ and $G \in \text{HI}^{\text{tr}}$. Let $f: \omega^*G \to F$. Then by definition we have

$$\omega^*G(X) = G(X^0) = \text{Hom}_{\text{HI}^{\text{tr}}}(h_0^{A^1,\text{tr}}(X^0), G) = \text{Hom}_{\log\text{CI}^{\text{tr}}}(\omega^*h_0^{A^1,\text{tr}}(X^0), \omega^*G),$$

where the last equality follows from the fact that $\omega^*$ is fully faithful. This implies that the image of every map $\omega^*G \to F$ is contained in $F_{A^1}$, which implies that we have:

(B.0.1) $$\text{Hom}(\omega^*G, F_{A^1}) = \text{Hom}(\omega^*G, F)$$

Now, by construction we have $F_{A^1} = \omega^*\omega_2F_{A^1}$, and since $\omega_2F_{A^1}$ is $A^1$-invariant, it is $A^1$-local and $\omega^*\omega_2F_{A^1} \in \log\text{CI}^{\text{tr}}$. Moreover, let $h_0^{A^1}(\omega_2^*F) \subseteq \omega_2^*F$ be the biggest $A^1$-invariant subsheaf of $F$ as in [RS21]; then we have $\omega_2^*F_{A^1} \subseteq h_0^{A^1}(\omega_2^*F)$. We are ready to prove the following:

Lemma B.1. For $F \in \log\text{CI}^{\text{tr}}$, then we have

$$\text{Hom}_{\text{Shv}_{\text{Nis}}}(G_m, \omega_2^*F) \cong \omega_2^*\text{Hom}_{\text{Shv}_{\text{Nis}}}(\omega^*G_m, F)$$

Proof. Let $X \in \text{Sm}(k)$. Then we have

$$\text{Hom}_{\text{Shv}_{\text{Nis}}}(G_m, \omega_2^*F)(X) = \text{Hom}_{\text{Shv}_{\text{Nis}}}(G_m, \text{Hom}(Z_{\text{tr}}(X), \omega_2^*F)) = \text{Hom}_{\text{Shv}_{\text{Nis}}}(G_m, \omega_2^*\text{Hom}(Z_{\text{tr}}(X, \text{triv}), F))$$

and

$$\omega_2^*\text{Hom}_{\text{Shv}_{\text{Nis}}}(\omega^*G_m, F)(X) = \text{Hom}_{\text{Shv}_{\text{Nis}}}(\omega^*G_m, \text{Hom}(Z_{\text{tr}}(X, \text{triv}), F)).$$

Since $\text{Hom}(Z_{\text{tr}}(X, \text{triv}), F) \in \log\text{CI}$ (see e.g. [BM21, Lemma 2.10]), up to replacing $F$ with $\text{Hom}(Z_{\text{tr}}(X, \text{triv}), F)$, we are reduced to prove that

$$\text{Hom}_{\text{Shv}_{\text{Nis}}}(G_m, \omega_2^*F) \cong \text{Hom}_{\text{Shv}_{\text{Nis}}}(\omega^*G_m, F).$$

The inclusions $\omega_2^*F_{A^1} \subseteq h_0^{A^1}(\omega_2^*F) \subseteq \omega_2^*F$ induce

$$H^1(P^1, \omega_2^*F_{A^1}) \to H^1(P^1, h_0^{A^1}(\omega_2^*F)) \xrightarrow{(e)} H^1(P^1, \omega_2^*F).$$

By Corollary A.7 and (B.0.1), the composition above is the isomorphism

$$\text{Hom}(\omega^*G_m, F_{A^1}) \cong \text{Hom}(\omega^*G_m, F),$$

and by [MVW06] the term in the middle is

$$H^1(P^1, h_0^{A^1}(\omega_2^*F)) \cong \text{Hom}(G_m, h_0^{A^1}(\omega_2^*F)) = \text{Hom}(G_m, \omega_2^*F),$$

so to conclude it is enough to show that the map $(e)$ above is injective. Let $Q = \text{coker}(h_0^{A^1}(\omega_2^*F) \to \omega_2^*F)$: by the long exact sequence it is enough to show that the map $Q((P^1), i_1) = 0$, where the notation is as in (3.5.1). Let $\omega_1^\text{CI}$ be the right adjoint of $\omega_2$ as in
[BM21, Definition 7.1]. Since \( h^0_{A_1}(\omega_T F) \in \text{HI} \), we have that \( \omega^\text{CI}_{log} h^0_{A_1}(\omega_T F) = \omega^* h^0_{A_1}(\omega_T F) \): this gives a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & h^0_{A_1}(\omega_T F) & \longrightarrow & \omega^\text{CI}_{log} \omega_T F & \longrightarrow & \omega_T (\text{coker}(\alpha)) & \longrightarrow & 0 \\
0 & \downarrow & \downarrow & \downarrow q & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
& & 0 & \longrightarrow & \omega^* h^0_{A_1}(\omega_T F) & \longrightarrow & \omega_T F & \longrightarrow & Q & \longrightarrow & 0,
\end{array}
\]  

(B.1.1)

where the middle vertical arrow is induced by \( \omega_T(*) \), where (*) is as in

\[
F \rightarrow \omega^\text{CI}_{log} \omega_T F \xrightarrow{(*)} \omega^* \omega_T F.
\]

Since \( \omega_T F \cong \omega_T^* \omega_T F \), we have that \( \omega_T(**) \) above is split surjective, which implies that \( q \) is surjective. We make the following claim:

**Claim B.2.** The map \( \omega_T(\text{coker}(\alpha))(\mathbb{P}^1) \rightarrow Q(\mathbb{P}^1) \) induced by \( q \) in (B.1.1) is surjective.

If Claim B.2 holds, since \( \text{coker}(\alpha) \in \text{logCI}^{\text{triv}} \), we have that \( \text{coker}(\alpha)((\mathbb{P}^1), i_1) = 0 \), which implies that \( Q((\mathbb{P}^1), i_1) = 0 \) and finishes the proof.

**Proof of Claim B.2.** In fact, the diagram (B.1.1) is induced by applying \( \omega_T \) to the following diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \omega^* h^0_{A_1}(\omega_T F) & \xrightarrow{\omega^*_\alpha} & \omega^\text{CI}_{log} \omega_T F & \longrightarrow & \text{coker}(\alpha) & \longrightarrow & 0 \\
0 & \downarrow & \downarrow & \downarrow t & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
& & 0 & \longrightarrow & \omega^* h^0_{A_1}(\omega_T F) & \longrightarrow & \omega^* \omega_T F & \longrightarrow & \omega^* Q & \longrightarrow & 0,
\end{array}
\]

(B.2.1)

where the map \( t \) above is [BM21, (7.11)] applied to \( \omega_T F \). Let \( T = \ker(\omega^\text{CI}_{log} \omega_T F \rightarrow \omega^* \omega_T F) \); by (B.2.1), we have that \( T \cong \ker(q') \), so \( \omega_T T \cong \ker(q) \), with \( q \) as in (B.1.1). Since \( q \) is surjective, by the long exact sequence in cohomology, it is enough to show that \( H^1(\mathbb{P}^1, \omega_T T) = 0 \). We have that \( T(\mathbb{P}^1, 0 + \infty) \) is the kernel of the map:

\[
\omega^\text{CI}_{log} \omega_T F(\mathbb{P}^1, 0 + \infty) = \text{Hom}(h^0_0 \mathbb{CI}^{\text{triv}}(\mathbb{P}^1, 0 + \infty), \omega^* \omega_T F) = \text{Hom}(\omega^* \omega_T h^0_0 \mathbb{CI}^{\text{triv}}(\mathbb{P}^1, 0 + \infty), \omega^* \omega_T F) \\
\longrightarrow \text{Hom}(\omega^* \omega_T F(\mathbb{P}^1, 0 + \infty), \omega^* \omega_T F) = \omega^* \omega_T F(\mathbb{P}^1, 0 + \infty)
\]

By Proposition A.5, the map \( Z_{\text{triv}}(A^1 - \{0\}) \rightarrow \omega_T h^0_0 \mathbb{CI}^{\text{triv}}(\mathbb{P}^1, 0 + \infty) \) agrees with the map \( \lambda \) of [MVW06, Lemma 4.4], which is a surjective map of presheaves by loc cit, which implies that (B.2.2) above is injective, so \( T(\mathbb{P}^1, 0 + \infty) = 0 \). By the usual Mayer–Vietoris sequence induced by (A.4.1), we have an exact sequence:

\[
0 \rightarrow H^1(\mathbb{P}^1, \omega_T F) \rightarrow H^1((\mathbb{P}^1, 0), T) \oplus H^1((\mathbb{P}^1, \infty), T) \xrightarrow{\nabla} H^1((\mathbb{P}^1, 0 + \infty), T) \rightarrow 0,
\]

so it is enough to prove that \( \nabla \) is an isomorphism. Let \( G = \text{coker}(F \rightarrow \omega^\text{CI}_{log} \omega_T F) \); by construction there exists \( U \subseteq \text{coker}(F \rightarrow \omega^* \omega_T F) \) and an exact sequence:

\[
0 \rightarrow T \rightarrow G \rightarrow U \rightarrow 0.
\]

Since \( G \in \text{logCI} \) and \( \omega_T U = 0 \), for \( e \in \{0, \infty\} \) the map \( G(\mathbb{P}^1, e) \rightarrow U(\mathbb{P}^1, e) \) factors through \( U(\text{Spec}(k), \text{triv}) \), which is zero, so we conclude that \( G(\mathbb{P}^1, e) \rightarrow U(\mathbb{P}^1, e) \) is the zero map. Again by the usual Mayer–Vietoris we have an exact sequence:

\[
\begin{array}{cccccc}
G(\mathbb{P}^1, \text{triv}) & \xrightarrow{G(\mathbb{P}^1, 0)} & G(\mathbb{P}^1, 0 + \infty) & \xrightarrow{H^1(\mathbb{P}^1, G)} & \xrightarrow{0} & \xrightarrow{0} \\
0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \xrightarrow{U(\mathbb{P}^1, 0)} & \xrightarrow{U(\mathbb{P}^1, 0 + \infty)} & \xrightarrow{0} & \xrightarrow{H^1(U(\mathbb{P}^1, 0), U)} & \xrightarrow{H^1(U(\mathbb{P}^1, \infty), U)}
\end{array}
\]
so by the five-lemma (**) is the zero map as well. Putting everything together, we have a commutative diagram where the vertical maps are long exact sequences:

\[
\begin{array}{ccc}
G(\mathbb{P}^1, 0) \oplus G(\mathbb{P}^1, \infty) & \rightarrow & G(\mathbb{P}^1, 0 + \infty) \\
\downarrow 0 & & \downarrow 0 \\
U(\mathbb{P}^1, 0) \oplus U(\mathbb{P}^1, \infty) & \xrightarrow{=} & U(\mathbb{P}^1, 0 + \infty) \\
\downarrow (\ast 1) & & \downarrow (\ast 2) \\
H^1((\mathbb{P}^1, 0, T) \oplus H^1((\mathbb{P}^1, \infty), T) & \xrightarrow{\nabla} & H^1((\mathbb{P}^1, 0 + \infty), T) \\
\downarrow 0 & & \downarrow 0 \\
H^1((\mathbb{P}^1, 0), G) \oplus H^1((\mathbb{P}^1, \infty), G) = 0 & \rightarrow & H^1((\mathbb{P}^1, 0 + \infty), G) = 0,
\end{array}
\]

which implies that the maps (\ast 1) and (\ast 2) are isomorphisms, so \(\nabla\) is an isomorphism too and we conclude. \(\square\)

We are ready to settle [BM22, Conjecture 0.2] in the case with transfers:

**Theorem B.3.** Let \(F, G \in \logCl^{tr}\), and let \(\varphi : \omega_q F \to \omega_q G\) be a map in \(\text{Shv}^{tr}_{\text{Nis}}\). Then \(\varphi\) lifts to a map \(F \to G\) in \(\logCl^{tr}\).

**Proof.** In order to use Theorem 3.10, we need to show that the Gysin maps are the same. Let \(U \subseteq A^1_{k(x)}\) be a neighborhood of 0. By unwinding the definitions, we have that the Gysin map of [BP022] is given by the following map in \(\mathcal{D}(\text{Shv}^{tr}_{\text{Nis}}(k, \mathbb{Z}))\):

\[
\frac{\mathbb{Z}_{\text{itr}}(U, \text{triv})}{\mathbb{Z}_{\text{itr}}(U, \text{triv})/\mathbb{Z}_{\text{itr}}(U, 0)} \simeq \frac{\mathbb{Z}_{\text{itr}}(\mathbb{P}^1, \text{triv})/\mathbb{Z}_{\text{itr}}(\mathbb{P}^1, 0)}{\mathbb{Z}_{\text{itr}}(\mathbb{P}^1, \text{triv})/\mathbb{Z}_{\text{itr}}(\mathbb{P}^1, 0)}
\]

where the last equivalence is given by strict Zariski descent. Let \(\delta : \mathbb{Z}_{\text{itr}}(\mathbb{P}^1, \text{triv}) \to \mathbb{G}_m[1]\) be the map in \(\mathcal{D}(\text{Shv}^{tr}_{\text{Nis}}(k, \mathbb{Z}))\) induced by the tautological bundle of \(\mathbb{P}^1\). By adjunction \(\delta\) induces a map \(\bar{\delta} : \mathbb{Z}_{\text{itr}}(\mathbb{P}^1, \text{triv}) \to \omega^* \mathbb{G}_m[1]\) in \(\mathcal{D}(\text{Shv}^{tr}_{\text{Nis}}(k, \mathbb{Z}))\). By the choice of a trivialization on \(\mathbb{P}^1 - \{0\}\) and the \(\boxdot\)-invariance of \(\omega^* \mathbb{G}_m\), we have a factorization of \(\delta\) via

\[
\mathbb{Z}_{\text{itr}}(\mathbb{P}^1, \text{triv})/\mathbb{Z}_{\text{itr}}(\mathbb{P}^1, 0) \xrightarrow{\bar{\delta}} \omega^* \mathbb{G}_m[1],
\]

By applying the adjoint functors \(\omega^!\omega_q\) of [BMS21, (2.13.2)], we have a commutative diagram in \(\mathcal{D}(\text{Shv}^{tr}_{\text{Nis}}(k, \mathbb{Z}))\):

(B.3.1)

\[
\begin{array}{ccc}
\mathbb{Z}_{\text{itr}}(U, \text{triv}) & \rightarrow & \mathbb{Z}_{\text{itr}}(\mathbb{P}^1, \text{triv})/\mathbb{Z}_{\text{itr}}(\mathbb{P}^1 - \{0\}) \times \mathbb{Z}_{\text{itr}}(k(x), \text{triv}) \\
\downarrow & & \downarrow \\
\mathbb{Z}_{\text{itr}}(U, \text{triv}) & \rightarrow & \mathbb{Z}_{\text{itr}}(\mathbb{P}^1, \text{triv})/\mathbb{Z}_{\text{itr}}(\mathbb{P}^1, 0) \times \mathbb{Z}_{\text{itr}}(k(x), \text{triv}) \\
\downarrow & & \downarrow \\
\mathbb{Z}_{\text{itr}}(U, \text{triv}) & \rightarrow & \mathbb{Z}_{\text{itr}}(\mathbb{P}^1, \text{triv})/\mathbb{Z}_{\text{itr}}(\mathbb{P}^1, 0) \times \mathbb{Z}_{\text{itr}}(k(x), \text{triv}) \\
\phi & \rightarrow & \omega^* \mathbb{G}_m \times \mathbb{Z}_{\text{itr}}(k(x), \text{triv})[1]
\end{array}
\]

By Corollary A.7 above, we have an isomorphism

(B.3.2)

\[
\text{Hom}(\omega^* \mathbb{G}_m, F)(k(x)) = H^1(\mathbb{P}^1_{k(x)}, \omega_q F)
\]

Now, the top row of (B.3.1) induces a commutative square

\[
\begin{array}{ccc}
\text{Hom}_{\text{Shv}^{tr}_{\text{Nis}}}(\mathbb{G}_m, \omega_q F)(k(x)) & \rightarrow & F_{A^1_{k(x)}}(U)[1] \\
\downarrow \varphi_{\mathbb{G}_m, k(x)} & & \downarrow \varphi_U[1] \\
\text{Hom}_{\text{Shv}^{tr}_{\text{Nis}}}(\mathbb{G}_m, \omega_q G)(k(x)) & \rightarrow & G_{A^1_{k(x)}}(U)[1].
\end{array}
\]
Combining it with the isomorphisms of Lemma B.1 we get a commutative diagram:

\[
\begin{array}{c}
H^1(P^1_{k(x)}, \omega_t^iF) \xrightarrow{\cong} \text{Hom}(G_m, \omega_t^iF)(k(x)) \rightarrow F_{A^i_{k(x)}}(U)[1] \\
\downarrow \varphi \hspace{1cm} \downarrow \varphi_{G_m,k(x)} \hspace{1cm} \downarrow \varphi_U[1] \\
H^1(P^1_{k(x)}, \omega_t^iG) \xrightarrow{\cong} \text{Hom}(G_m, \omega_t^iG)(k(x)) \rightarrow G_{A^i_{k(x)}}(U)[1],
\end{array}
\]

This concludes the proof by Theorem 3.10. \(\Box\)

**Corollary B.4.** The functor

\[
\log \mathcal{D}_{\text{eff}}(k, A) \xrightarrow{i} \mathcal{D}(\text{Shv}_{\text{dNis}}(k, A)) \xrightarrow{\omega_t} \mathcal{D}(\text{Shv}_{\text{Nis}}(k, A))
\]

is fully faithful.

**Proof.** First of all, \(i\) is a localization with left adjoint \(L_{(\text{dNis}, \mathbb{Z})}\) and a right adjoint \(R_{(\text{dNis}, \mathbb{Z})}\) (this follows from the fact that \(i\) commutes with all filtered colimits, see the proof of [BM21, Theorem 5.7]) and \(\omega_t\) has also a left adjoint \(L\) and a right adjoint \(R\omega^*\) (see [BP02, (4.3.5)]): this implies that the functor in the statement commutes with all limits and colimits. Moreover, by construction \(i\) is \(t\)-exact for the homotopy \(t\)-structure on \(\log \mathcal{D}_{\text{eff}}(k, A)\) and the standard \(t\)-structure on \(\mathcal{D}(\text{Shv}_{\text{dNis}}(k, A))\), and since \(\omega_t\) is the derived functor of an exact functor, it is also \(t\)-exact for the standard \(t\)-structures on the derived categories: this implies that \(\omega_t\) is \(t\)-exact. We recall that \(\omega_t\) is conservative: indeed let \(F \in \log \mathcal{D}_{\text{eff}}(k, A)\) such that \(\omega_tF \simeq 0\), by purity [BM21, Theorem 5.8] and \(t\)-exactness we have that for all \(X \in \text{Sm}_{\text{dNis}}\)

\[
\pi_n^{\log \mathcal{D}_{\text{eff}}(k, A)} F(X) \hookrightarrow \pi_n^{\log \mathcal{D}_{\text{eff}}(k, A)} F(X - |\partial X|) = \pi_n^{\text{Nis}}(\omega_t^iF)(X - |\partial X|) = 0.
\]

Since \(\omega_t^i\) is conservative and preserves limits and colimits, it is faithful. We need to show that it is full: let \(F, G \in \log \mathcal{D}_{\text{eff}}(k, A)\) and let \(f: \omega_t^iF \rightarrow \omega_t^iG\). We have that \(F = \lim_{\leftarrow} \tau_{\leq m} \tau_{\geq n} \log \mathcal{D}_{\text{eff}}(k, A) F\) (and similarly to \(G\)), so it is enough to lift the maps

\[
f_{m,n}: \tau_{\leq m} \tau_{\geq n} \log \mathcal{D}_{\text{eff}}(k, A) M \rightarrow \tau_{\leq m} \tau_{\geq n} \log \mathcal{D}_{\text{eff}}(k, A) M,
\]

We now proceed by induction on the length of \([m, n]\): if \(m = n\), then this is Theorem B.3. In general, consider the fiber sequences

\[
\tau_{\leq m} \log \mathcal{D}_{\text{eff}}(k, A) F \rightarrow \tau_{\leq m} \tau_{\geq n+1} \log \mathcal{D}_{\text{eff}}(k, A) M \rightarrow \tau_{\leq m} \tau_{\geq n} \log \mathcal{D}_{\text{eff}}(k, A) M \rightarrow \tau_{\leq m} \tau_{\geq n+1} \log \mathcal{D}_{\text{eff}}(k, A) M[n]
\]

By induction hypothesis, we have maps \(g_{m,n+1}: \tau_{\leq m} \log \mathcal{D}_{\text{eff}}(k, A) M \rightarrow \tau_{\leq m} \tau_{\geq n+1} \log \mathcal{D}_{\text{eff}}(k, A) M\) and \(g_{n,n}: \tau_{\leq m} \tau_{\geq n} \log \mathcal{D}_{\text{eff}}(k, A) M[n] \rightarrow \tau_{\leq m} \tau_{\geq n+1} \log \mathcal{D}_{\text{eff}}(k, A) M[n]\) such that \(\omega_t^i g_{m,n+1} = f_{m,n+1}\) and \(\omega_t^i g_{n,n} = f_{n,n}\). Then the fiber sequences above give a diagram

\[
\begin{array}{c}
\pi_n^{\log \mathcal{D}_{\text{eff}}(k, A)} M[n-1] \xrightarrow{g_{n,n}[n-1]} \pi_n^{\log \mathcal{D}_{\text{eff}}(k, A)} M \\
\downarrow \pi_n^{\log \mathcal{D}_{\text{eff}}(k, A)} \downarrow \pi_n^{\log \mathcal{D}_{\text{eff}}(k, A)} \\
\pi_n^{\log \mathcal{D}_{\text{eff}}(k, A)} N[n-1] \xrightarrow{g_{m,n}} \pi_n^{\log \mathcal{D}_{\text{eff}}(k, A)} N.
\end{array}
\]

Since \(\omega_t^i\) is commutative by assumption and \(\omega_t^i\) is faithful, we conclude that (B.4.1) is commutative, so it induces a map on the cofibers

\[
g_{m,n}: \tau_{\leq m} \tau_{\geq n} \log \mathcal{D}_{\text{eff}}(k, A) M \rightarrow \tau_{\leq m} \tau_{\geq n} \log \mathcal{D}_{\text{eff}}(k, A) N
\]

such that \(\omega_t^i g_{m,n} = f_{m,n}\). This concludes the proof. \(\Box\)
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