Quantum state engineering using weak measurement

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(Dated: November 2, 2021)

State preparation via postselected weak measurement in three wave mixing process is studied. Assuming the signal input mode prepared in a vacuum state, coherent state or squeezed vacuum state, separately, while the idler input prepared in weak coherent state and passing the medium characterized by the second-order nonlinear susceptibility. It is shown that when the single photon is detected at one of the output channels of idler beam’s path, the signal output channel is prepared in single-photon Fock state, single-photon-added coherent state or single-photon-added squeezed vacuum state with very high fidelity, depending upon the input signal states and related controllable parameters. The properties including squeezing, signal amplification, second order correlation and Wigner functions of the weak measurement based output states are also investigated. Our scheme promising to provide alternate new effective method for producing useful nonclassical states in quantum information processing.

I. INTRODUCTION

New state generation and its optimization have significant importance in quantum information processing [1–6]. There have plenty of research works studied various quantum states and proposed the schemes for generating them. The particular interests has been devoted to Fock states [7–9], Schrodinger cat states [10–18], squeezed states [19], photon number states [20–26], binomial states [27–30], and squeezed state excitations [31–34]. Another interesting class of nonclassical states such as photon-added coherent states [35], photon-subtracted or -added squeezed states [36] that have been a subject of interest since they also have potential applications in many related quantum information processing [37–40]. Those states can be produced by repeated applications of photon creation or annihilation operators [41], respectively, on a given states [42–46].

We know that the purposing the feasible schemes to generate specific quantum states and their implementations in the Lab are exciting and challenging tasks to the researchers. In specific quantum state generation processes we usually used the conditional measurement since it useful to control the requested parameters to produce the desired quantum states [47–54]. The weak measurement proposed in 1988 [55] by Aharonov, Albert, and Vaidman is a typical conditional measurement characterized by postselection and weak value. The weak measurement theory have various applications (see [56] and references therein) and it recently widely used to the state optimization problems [57–59]. One of the author of this work studied the state optimization by using weak measurement [60–62] and showed that the postselected weak measurement really can change the inherent properties of the given states. Furthermore, in recent work [63], they purposed a theoretical scheme to amplify the single-photon nonlinearity using weak measurements implemented in cross-Kerr interaction medium characterized by the third-order nonlinear susceptibility $\chi^{(3)}$ and its experimental realization is given in [64]. On the other hand, Shikano and his collaborators [65] studied the generation of phase-squeezed optical pulses with large coherent amplitudes by post-selection of single photon based on the same setup of Ref.[63]. Those results also indicated the potential usefulness of postselected weak measurement in quantum state engineering processes. However, to our knowledge, the specific quantum state generation via weak measurement has not been investigated in detail in any literature, and it is worth to study.

In this paper, we introduce a new scheme to generate some typical nonclassical states such single-photon Fock states, single photon added coherent (SPAC) state and single photon added squeezed vacuum (SPASV) state in three optical wave mixing process via postselected weak measurement [55]. In order to achieve our goal, we consider the signal and idler beams as pointer (measuring system) and measured system, respectively. We assume that initially the measured system prepared in very weak coherent state while the pointer (signal) state prepared in coherent or squeezed vacuum state. The strong pump field is treated as classical and the weak coupling between the pointer and measured system is realized by BBO nonlinear crystal which can generate entanglement between them. By properly choosing the pre- and post-selection states of measured system and detecting one photon in one of the output of idler mode, the output channel of the pointer is prepared in desired state with high purity for controllable parameters. We found that if our input pointer state is prepared in coherent (squeezed vacuum) state, then we can generate SPAC (SPASV) state with very high fidelity accompanied by small successful rate. Our results indicated that in our scheme we also can generate single photon Fock state if the initial pointer state prepared in vacuum state. To further confirm the identities of those generated states we also investigate their related properties such as squeezing, second order correlations and Wigner functions. Interestingly, we found

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that the new generated SPAC state in our scheme have
advantages to increase the signal-to-noise ratio (SNR) in
postselected weak measurement over nonpostselected
case.

This paper is organized as follows. Section. II, presents
the basic scheme for generation of new nonclassical states
in three wave mixing process via postselected weak mea-
surement technique. The generation of SPAC and SPAVS
states and their inherent properties are discussed in Sec-
tonian in interaction picture with
ω







III and Section. IV, respectively. In Section. III, we
also investigate the advantages of postselected weak mea-
surement in signal amplification process over nonpostse-
lected case for SPAC state by adjusting the weak value
of measured system observables. Finally, a summary and
concluding remarks are given in Section. V.

II. MODEL SETUP FOR THE NEW STATE
GENERATION VIA POSTSELECTED WEAK
MEASUREMENT

The Hamiltonian of a three-wave mixing device [66],
under the rotating wave approximation (RWA), neglect-
ing external drive and signal fields, is

\[ H = \hbar \omega_a a^\dagger a + \hbar \omega_i b^\dagger b + \hbar \omega_p c^\dagger c + i \hbar \chi^{(2)} (a^\dagger b^\dagger c - abc^\dagger), \]  

(1)

where \( a, b \) and \( c \) are the annihilation operators of the sig-
nal, idler and pump with frequencies \( \omega_s, \omega_i \) and \( \omega_p \), and \( \chi^{(2)} \) is the coupling strength characterized by a second-
order nonlinear susceptibility of BBO. This Hamiltonian
can describe the process of nondegenerate parametric
down-conversion whereby a photon of the pump field is
converted into two photons, one for each of the modes \( a \)
and \( b \) [66]. Using the parametric approximation, assum-
ing that the pump field to be a strong coherent state of
the form \( |\gamma e^{-i\omega t}| \), then we can rewrite the above Hamil-
tonian in interaction picture with \( \omega_p = \omega_i + \omega_s \) as

\[ H_I = i \hbar \eta (a^\dagger b^\dagger - ab), \]  

(2)

where \( \eta = \gamma \chi^{(2)} \). Further, the above Hamiltonian is
equivalent to

\[ H_I = \hbar q (A \otimes p - B \otimes q), \]  

(3)

if we introduce

\[ B = \frac{i}{\sqrt{2}} (b - b^\dagger), \quad A = \frac{1}{\sqrt{2}} (b^\dagger + b), \]  

(4)

and

\[ q = \frac{1}{\sqrt{2}} (a^\dagger + a), \quad p = \frac{i}{\sqrt{2}} (a^\dagger - a), \]  

(5)

with \([A, B] = i \) and \([q, p] = i \), respectively. The two terms
in Hamiltonian, Eq. (3), are in the forms we usually used
in weak measurement problems [55]. In this work we take
the signal beam with variables \( q \) and \( p \) is pointer, and
idler beam with variables \( A \) and \( B \) is measured system,
respectively.

The schematic setup of our state generation model is
shown in Fig. 1. As we can see from Fig. 1, there have
two Mach–Zehnder interferometers. The signal and idler beams acts as pointer
and measured system, respectively. The preselection state
prepared by weak coherent state \( |\alpha\rangle \) passing through the un-
balanced beam splitter (BS) with deviation \( \epsilon \), and signal beam
initially prepared in some specific states. The BBO crystal
playing the role for realizing the weak interaction between
pointer and measured system. The 50:50 BS in the upper
Mach–Zehnder interferometers takes the role of postselection,
and the desired conditional quantum state is generated in the
output mode of signal beam after we detect one photon by
second photon detector (D2) in idler beam’s path.

![Schematic setup of state generation model](https://example.com/schematic.png)
where $U_{kj}$ is the element of the scattering matrix

$$
U = \begin{pmatrix}
\cos \varphi e^{i\varphi} & \sin \varphi e^{-i\varphi} \\
-\sin \varphi e^{i\varphi} & \cos \varphi e^{-i\varphi}
\end{pmatrix}.
$$

(7)

Here, $T = \cos \varphi e^{i\varphi}$ and $R = \sin \varphi e^{i\varphi}$ are transmittance and reflectance of the beam splitter, respectively. If $\varphi = 0$ and $\varphi = \frac{\pi}{4}$, then it becomes $50:50$ beam splitter.

We assume that initially the measured system (idler beam) prepared in weak coherent state with small amplitude ($\alpha \ll 1$), and signal beam prepared in some specific states such as squeezed vacuum state and coherent state separately. In the upper optical path of our scheme, we assume that the first beam splitter is slightly imbalanced with small deviate $\varepsilon$ to $50:50$ so that the preselection of the measured system can be written as [68]

$$
|\psi_i\rangle = \frac{\alpha}{\sqrt{2}} (1 - \varepsilon)|\rangle t + \frac{i\alpha}{\sqrt{2}} (1 + \varepsilon)|\rangle r.
$$

(8)

Here, the subscripts $t$ and $r$ indicates the transmitted and reflected beams from the beam splitter. Then, the three wave mixing is realized by the nonlinear BBO crystal which play the role to implement the weak measurement process. In this process, the input photon annihilates and produces two new mutually entangled photons. The unitary evolution operator corresponding to the interaction Hamiltonian $H_I = \hbar g (A \otimes p - B \otimes q)$ which can implemented by BBO crystal is

$$
U = \exp \left( -\frac{i}{\hbar} \int_0^t H_I d\tau \right) = \exp (-ig [A \otimes p - B \otimes q]),
$$

(9)

where $g = \eta t$. Actually, this is the squeezing operator can generate the two-mode vacuum squeezed state [66]. Here, $g$ is can be considered as squeezing parameter which depends on pump intensity, the crystal length, and its nonlinear coefficients. Following the experimental work [45], we set $g = 0.105$ throughout this work. We can then rewrite the above unitary evolution operator $U$ as

$$
U \approx I - ig (A \otimes p - B \otimes q).
$$

(10)

If we assume that the initial state of system and pointer are $|\psi_i\rangle$ and $|\phi\rangle$, after the unitary evolution the total system state becomes as

$$
|\Psi\rangle = U|\psi_i\rangle \otimes |\phi\rangle \approx [I - ig (A \otimes p - B \otimes q)] |\psi_i\rangle \otimes |\phi\rangle.
$$

(11)

This is total system state before arrive to the second beam splitters in our model (see Fig. 1). In our scheme the second splitters are 50:50 with 50% transmission and 50% reflection. We take a postselection to the idler beam accomplished by detectors in the upper optical paths. Assume that the second photon detector (D2) is detect one photon and the first photon detector (D1) no click i.e., $|1\rangle_{2d}|0\rangle_{1d}$. This postselection process can be described by

$$
|\psi_f\rangle_{2d} = a_{2d}^\dagger (|0\rangle_r |0\rangle_t \\
= \frac{1}{\sqrt{2}}(|0\rangle_r |1\rangle_t - i|1\rangle_r |0\rangle_t),
$$

(12)

where

$$
a_{2d} = \frac{1}{\sqrt{2}} (a_t + ia_r),
$$

(13a)

$$
a_{1d} = \frac{1}{\sqrt{2}} (ia_t + a_r),
$$

(13b)

are the field operators relations between input and output modes of the beam-splitter transformation. After taking the postselection with the postselected state $|\psi_f\rangle$ onto Eq. (11), we can obtain the non normalized form of the final state of the pointer (signal beam) and it reads as

$$
|\Phi\rangle = \langle \psi_f | \psi_i \rangle [1 - ig (\langle A |_w p - \langle B |_w q) | \phi\rangle
$$

$$
= \langle \psi_f | \psi_i \rangle \left[ 1 - \frac{g\alpha}{\sqrt{2}a} - \frac{g}{\sqrt{2}\alpha\varepsilon} \right] | \phi\rangle,
$$

(14)

where

$$
\langle A |_w = \frac{\langle \psi_f | \hat{A} | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle} = \frac{\alpha}{2} - \frac{1}{2\alpha\varepsilon},
$$

(15)

and

$$
\langle B |_w = \frac{\langle \psi_f | \hat{B} | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle} = \frac{i}{2\alpha\varepsilon} + \frac{\alpha}{2},
$$

(16)

are the weak values of $A$ and $B$, respectively. The probability of finding one photon at D2 and no photon at D1 is

$$
P_s = |\langle \psi_f | \psi_i \rangle|^2 = |\alpha\varepsilon|^2.
$$

(17)

As we can see, the success probability of postselection $P_s$ is depends on the imbalance $\varepsilon$ caused by the little difference between the reflection and transmission coefficients of the beam splitter in the upper interferometer and weak coherent state amplitude $\alpha$ of the idler input state. From the Eqs. (15) and (16), it can be seen that the weak values are generally complex, and can take large values when the pre-selected state $|\psi_i\rangle$ and post-selected states $|\psi_f\rangle$ are almost orthogonal. The magnitudes of weak idler input state amplitude $\alpha$, beam splitter’s deviation $\varepsilon$, and coupling coefficient $g$ are all controllable in optical experiments. Thus, we can manipulate and change the inherent properties of the output signal state $|\Phi\rangle$ by adjusting these parameters. In the remaining parts of the work, we study the new state generation and its verification processes by taking the initial signal input state $|\phi\rangle$ as coherent state and vacuum squeezed state, respectively.
III. GENERATION OF SPAC STATE

In this section, we assume that the initial signal input state is prepared as coherent state which defined as,

$$|\phi\rangle = |\beta\rangle = D(\beta)|0\rangle,$$  

(18)

where $\beta = |\beta|e^{i\vartheta}$ is complex number. For this case, the output state of the signal, i.e., Eq. (14), is reads as

$$|\Theta\rangle = \mathcal{N} \left[ \kappa_1|\beta\rangle - \kappa_2 a^\dagger|\beta\rangle \right].$$  

(19)

Here,

$$\mathcal{N} = \left[ |\kappa_1|^2 + |\kappa_2|^2(1 + |\beta|^2) - 2Re[\kappa_1\kappa_2^\ast \beta] \right]^{-\frac{1}{2}},$$  

(20)

is the normalization constant, $\kappa_1 = 1 - \frac{a\beta}{\sqrt{2}}$ and $\kappa_2 = \frac{\beta}{\sqrt{2}\alpha}$, respectively. It is very clear from Eq. (19) that the output signal state is a superposition of coherent state $|\beta\rangle$ and SPAC state $a^\dagger|\beta\rangle$. As aforementioned, since the all parameters $g$, $\alpha$, $\epsilon$ and $\beta$ are adjustable, the dominance of coherent state $|\beta\rangle$ and SPAC state $a^\dagger|\beta\rangle$ can be completely controlled by adjusting the related parameters. From the Eq. (19), we can see that if $\kappa_2 \gg \kappa_1$ the state $|\Theta\rangle$ reduced to the SPAC state $|1, \beta\rangle = \frac{a^\dagger|\beta\rangle}{\sqrt{1 + |\beta|^2}}$. In next sub sections we extend the discussions about properties of the conditional output state $|\Theta\rangle$.

A. State Distance

In quantum information theory, the quantification of the distance of two quantum states described by density operators $\rho$ and $\sigma$ can be characterized by the quantum fidelity (or the called Uhlmann-Jozsa fidelity) which is defined as

$$F = \left( Tr \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}} \right)^2.$$  

(21)

If both states are pure i.e., $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$, then

$$F = |\langle\psi|\phi\rangle|^2.$$  

(22)

This quantity is indeed a natural candidate for the state distance since it corresponds to the closeness of states in the natural geometry of Hilbert space. If $F = 0$, the states are orthogonal or called totally different (i.e., perfectly distinguishable). If $F = 1$, then the two states are totally same, $|\psi\rangle = |\phi\rangle$.

Here, in order to study the similarity of the output signal state $|\Theta\rangle$ between coherent state $|\beta\rangle$ and normalized SPAC state $|1, \alpha\rangle$, the fidelity between $|\beta\rangle$, $|1, \alpha\rangle$ and $|\Theta\rangle$ are calculated, and the result are given by

$$F_1 = |\langle\beta|\Theta\rangle|^2 = |\mathcal{N}(\kappa_1 - \kappa_2 \beta^\ast)|^2,$$  

(23)

and

$$F_2 = |\langle1, \alpha|\Theta\rangle|^2 = \frac{\mathcal{N}^2|\kappa_1 \beta - \kappa_2(1 + |\beta|^2)|^2}{1 + |\beta|^2},$$  

(24)

respectively. In Fig. 2, we plot the fidelity $F_1$ and $F_2$ as a function of coherent state parameter $|\beta|$ for other fixed system parameters. As showed in Fig. 2, the red dashed line shows the closeness between the output signal state and the SPAC state, and it can be seen that the fidelity of these two states always keeping the constant value ($F = 1$) for all $|\beta|$. The Fig. 2 also indicated that the $F_1$ is increased from zero to unity as $|\beta|$ increasing. It can be seen that when $\alpha$, $\epsilon$ are much less than one and $|\beta|$ is smaller, we can deduce that $\kappa_2 \gg \kappa_1$. Under this condition our generated output signal state is exactly the SPAC state.

B. Second order correlation and Mandel factor

Here, we study the second-order correlation function $g^{(2)}(0)$ and Mandel factor $Q_m$ of our generated signal state $|\Theta\rangle$. The second order correlation function of a single-mode radiation field is defined as

$$g^{(2)}(0) = \frac{\langle a^\dagger a^2 \rangle}{\langle a^\dagger a \rangle^2}.$$  

(25)

Its relations with the Mandel factor $Q_m$ is

$$Q_m = \langle a^\dagger a \rangle \left[ g^{(2)}(0) - 1 \right].$$  

(26)

If $0 \leq g^{(2)}(0) < 1$ and $-1 \leq Q_m < 0$ simultaneously, the corresponding radiation field has sub-Poissonian statistics and more nonclassical. We have remember that the Mandel factor $Q_m$ can never be smaller than $-1$ for any radiation fields, and negative $Q_m$ values, which are equivalent to sub-poisonous statistics, cannot be produced by any classical field.
The second-order correlation function \( g^{(2)}(0) \) and Mandel factor \( Q_m \) of our generated output signal state \( |\Theta\rangle \) are given as \[69\]

\[
g^{(2)}(0) = \frac{\langle \Theta | a^d a^\dagger a a | \Theta \rangle}{\langle \Theta | a^d a | \Theta \rangle^2},
\]

and

\[
Q_m = \langle \Theta | a^d a | \Theta \rangle [g^2(0) - 1],
\]

respectively, with

\[
\langle \Theta | a^d a^2 | \Theta \rangle = |N|^2 \{ [\kappa_1^2 |\beta|^2 - 2\kappa_2^2 \kappa_1 \text{Re}[(2|\beta|^2 |\beta| + |\beta|^4)] + \kappa_2^2 (5|\beta|^2 + |\beta|^6 + 4|\beta|^2) \},
\]

and

\[
\langle \Theta | a^d a | \Theta \rangle = |N|^2 \{ [\kappa_1^2 |\beta|^2 - 2\kappa_2^2 \kappa_1 \text{Re}(\beta + |\beta|^2) + \kappa_2^2 (3|\beta|^2 + |\beta|^4 + 1) \}. \tag{30}
\]

In Fig. 3, we plot \( g^{(2)}(0) \) and \( Q_m \) as functions of coherent state parameter \( \beta \) by fixing other parameters to \( \theta = 0, g = 0.105, \alpha = 0.01 \) and \( \epsilon = 0.1 \). As observed in Fig. 3, \( 0 \leq g^{(2)}(0) < 1 \) and \( -1 \leq Q_m < 0 \) for all plotted regions. This means that our generated signal output field have sub-Poisson statistics which only possessed in nonclassical states. Actually, the curves showed in Fig. 3 are matched well with the corresponding curves of SPAC state \( |1, \alpha \rangle \) \[35\]. Thus, we can further verified that in our scheme we could effectively generate the SPAC state if the initial signal input state is in coherent state with moderate parameter \( \beta \).

C. Winger function

To further verify our claim, in this subsection, we investigate the Wigner function of \( |\Theta\rangle \). A state of a quantum mechanical system is completely described by density matrix of a phase space distribution such as the Wigner function. Every state function has it unique phase space distributions and the Wigner distribution function is the closest quantum analogue of the classical distribution function in phase space. By evaluating the Wigner function we can intuitively determine the strength of corresponding quantum nature, and most importantly the negative value of the Wigner function can prove the nonclassicality of the state. In general, the Wigner function is defined as the two-dimensional Fourier transform of the symmetric order characteristic function, and the Wigner function for the state \( \rho = |\Theta\rangle \langle \Theta | \) can be written as \[66\]

\[
W(z) = \frac{1}{\pi^2} \int_{-\infty}^{+\infty} \exp(\lambda^* z - \lambda z^*) C_N(\lambda) e^{-\frac{\lambda^2}{2}} d\lambda, \tag{31}
\]

where \( C_N(\lambda) \) is the normal ordered characteristic function, and is defined as

\[
C_N(\lambda) = Tr \left[ \rho e^{\lambda a^\dagger} e^{-\lambda a} \right]. \tag{32}
\]

After some calculation we can get the explicit expression of the Wigner function of the state \( |\Theta\rangle \) and it given as

\[
W(z) = \frac{2|N|^2}{\pi} \left\{ [\kappa_1^2 e^{-2|z-\beta|^2} - \kappa_2^2] \left(1 - 2z - |\beta|^2\right) e^{-2|z-\beta|^2} - \text{Re} \left[ \kappa_2 \kappa_1^* (2 \text{Re}[\beta] - z) e^{\frac{1}{2} \left[(z-\beta)^2 + (z-\beta)^2\right]} \right] \right\}. \tag{33}
\]

We can see that this Wigner function consists three parts. The first and second terms corresponded to the Wigner function of coherent state \( |\beta\rangle \) and SPAC state \( |1, \beta\rangle \), respectively, and third term caused by their superposition. In Fig. 4, we plot the Wigner function of the state \( |\Theta\rangle \) for different amplitude \( \beta \). From the Fig. 4, we can see that the negativity of \( W(z) \) vanished gradually with increasing the amplitude \( \beta \). We know that every wave function has its phase space distribution which

\[

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{wigner_function.png}
\caption{(Color online) Second order correlation \( g^{(2)}(0) \) and Mandel factor \( Q_m \) of our generated signal output state \( |\Theta\rangle \) as a function of coherent state parameter \( |\beta| \). (a) \( g^{(2)}(0) \) is varied. (b) \( Q_m \) is varied. Other parameters are the same as Fig. 2.}
\end{figure}

\]

\[

\]
characterized by Wigner function, and it is an unique. This presented phenomena in Fig. 4 is exactly the phase space distribution of SPAC state $|\Theta\rangle$ [35]. Thus, when $\kappa_2 \gg \kappa_1$, the $|\Theta\rangle$ gives us the new type of nonclassical state, i.e., $|1,\beta\rangle$.

D. Signal to noise ratio (SNR)

As shown in our schematic Fig. 1, the new output state $|\Theta\rangle$ of the signal beam is generated after we taking the postselection to the idler beam which accomplished by D1 and D2. If we didn’t take the postselection, then final state of the signal will gives by Eq. (11) after taking a trace to the idler beam with state $|\psi_i\rangle$. However, since in the nonpostselection case will not occur weak value of operators $A$ and $B$ which possess the signal amplification feature, the postselected weak measurement may have advantages over nonpostselected measurement in signal amplification process. To show the usefulness of new generated state $|\Theta\rangle$, here we study the ratio of SNRs between the postselected and nonpostselected weak measurements [69]

$$\chi = \frac{\mathcal{R}_X^p}{\mathcal{R}_X^n}.$$  

Here, $\mathcal{R}_X^p$ represents the SNR of postselected weak measurement defined as

$$\mathcal{R}_X^p = \frac{\sqrt{N \delta q'}}{\sqrt{\langle q^2 \rangle_f - \langle q \rangle_f^2}},$$  

with

$$\delta q' = \langle \Theta|q|\Theta \rangle - \langle \beta|q|\beta \rangle.$$  

Here, $N$ is the total number of measurements, $P_s$ is probability of finding the postselected state for a given pres-elected state and for our scheme it equal to $P_s = |\alpha|^2$, and $NP_s$ is the number of times the system was found in a postselected state $|\psi_f\rangle$. Here, $\langle q \rangle_f$ denotes the expectation value of measuring observable which defined in Eq. (5) under the final state of the pointer (signal beam) $|\Theta\rangle$.

When dealing with nonpostselected measurement, there is no postselection process after the interaction between the system and pointer. Thus, the definition of SNR for nonpostselected weak measurement can be given as

$$\mathcal{R}_X^n = \frac{\sqrt{N \delta q}}{\sqrt{\langle q^2 \rangle_f - \langle q \rangle_f^2}},$$  

with

$$\delta q = \langle \Psi|q|\Psi \rangle - \langle \beta|q|\beta \rangle.$$  

Here, $\langle q \rangle_f$ denotes the expectation value of measuring observable under the final state of the pointer without postselection which can be derived in Eq. (11). In order to evaluate the ration $\chi$ of SNRs, we have to calculate the related quantities and related expressions are given as:

1. The expectation value of $\langle q \rangle_f$ is

$$\langle q \rangle_f = \langle \Phi|q|\Phi \rangle = N|\kappa|^2(|\kappa_1|^2 h_1 + |\kappa_2|^2 h_2 - 2 Re[\kappa_1\kappa_2^* h_3]),$$  

where

$$h_1 = \langle \beta|q|\beta \rangle = \sqrt{2} Re[\beta],$$  

$$h_2 = \langle \beta|aq\alpha^\dagger|\beta \rangle = \sqrt{2}(2 + |\beta|^2) Re[\beta],$$  

$$h_3 = \langle \beta|aq\beta |\beta \rangle = \frac{1}{\sqrt{2}}(1 + |\beta|^2 + \beta^2),$$  

Figure 4. (Color online) The Wigner function of output signal state $|\Theta\rangle$. (a) $|\beta \rangle = 0$, (b) $|\beta \rangle = 1$, (c) $|\beta \rangle = 2$. Other parameters are the same as Fig. 2.
(2) the expectation value of \( \langle q^2 \rangle_f \) is

\[
\langle q^2 \rangle_f = \langle \Phi | q^2 | \Phi \rangle = |N|^2 \{ |\kappa_1|^2 w_1 + |\kappa_2|^2 w_2 - 2 \text{Re}[\kappa_1 \kappa_2^* w_2] \},
\]

where

\[
w_1 = \langle \beta | q^2 | \beta \rangle = \frac{1}{2} \left( 2 \text{Re}[\beta^2] + 2 |\beta|^2 + 1 \right),
\]

\[
w_2 = \langle \beta | a q^2 a^\dagger | \beta \rangle = \frac{1}{2} \left( 3 + 7 |\beta|^2 + 2 |\beta|^4 + 2(3 + |\beta|^2) \text{Re}[\beta^2] \right),
\]

\[
w_3 = \langle \beta | a q^2 a^\dagger | \beta \rangle = \frac{1}{2} \left( 3 |\beta|^2 + 2 |\beta|^3 + |\beta|^* |\beta|^2 + 2 |\beta|^2 |\beta|^2 \right). \tag{42c}
\]

The other quantities also can be obtained, and here we didn’t show all of them. The ratio \( \chi \) is increased and can be more larger then unity with increasing the unbalanced parameter \( \epsilon \) of the beam splitter for not very large \( |\beta| \). We have noticed that the magnitudes of weak values of \( A \) and \( B \), Eq. (15) and Eq. (16) are inverse to \( \epsilon \). Thus, the small weak value is, the better postselected SNR is achieved than nonpostselected one. In a word, it can draw a conclusion that the postselected weak measurement can improve the SNR rather than without post-selection one for smaller weak values.

IV. GENERATION OF SINGLE-PHOTON-ADDED VACUUM SQUEEZED STATE

If assume the initial input state \( |\phi \rangle \) of the signal beam is prepared as SV state \([70]\)

\[
|\phi_1 \rangle = S(\xi) |0\rangle,
\]

with \( S(\xi) = \exp \left( \frac{1}{2} \xi a^2 - \frac{1}{2} \xi^* a^2 \right), \) \( \xi = \eta e^{i\varphi} \). Then, the output state of signal beam, Eq. (14), becomes as

\[
|\Omega \rangle = \chi \left( |\phi_1 \rangle - \lambda_1 a |\phi_1 \rangle - \lambda_2 a^\dagger |\phi_1 \rangle \right). \tag{44}
\]

Here \( \lambda_1 = \frac{2 \eta}{\sqrt{2}}, \lambda_2 = \frac{\eta}{\sqrt{2}}, \) and

\[
\chi^{-2} = 1 + |\lambda_1|^2 \sinh^2 \eta - \text{Re}[\lambda_1 \lambda_2^* e^{i\varphi}] \sinh(2\eta) + |\lambda_2|^2 \cosh^2 \eta
\]

is the normalization constant. In the below discussions, we can neglect the term associated with the coefficient \( \lambda_1 \) since it is too small compared to \( \lambda_2 \) for our allowed parameters. As we can see, the state we prepared by optical modeling \(|\Omega \rangle\) is the superposition of vacuum squeezed (VS) and single-photon-added vacuum squeezed (SPAVS) states. These two states dominance depends on the coefficients \( \lambda_1 \) and their amplitudes can be controlled by beam splitters and BBO crystal in our scheme (see Fig. 1). In this section, by calculating the state distance, squeezing parameter and Wigner function we proved that in allowed parameters region our generated new state \(|\Omega \rangle\) is very distinguished over initial input state \(|\phi_1 \rangle\).

A. STATE DISTANCE

In order to investigate the similarities and differences of the generated state \(|\Omega \rangle\) between two states including SV state and SPAV state, we evaluate the state distances between them.

1. The state distance between \(|\Omega \rangle\) and squeezed vacuum (SV) state \(|\phi_1 \rangle\) is given as

\[
F_1 = |\langle \xi | \Phi \rangle|^2 = |\chi|^2. \tag{46}
\]

2. The state distance between \(|\Omega \rangle\) and PASV state \(|\phi_2 \rangle\) is given as

\[
F_2 = |\langle \phi_2 | \Omega \rangle|^2 = \frac{|\chi|^2}{\cosh \eta} \left( 1 - \frac{1}{2} e^{i\varphi} \lambda_1 |\phi_1 \rangle \right). \tag{47}
\]

In Fig. 6, we plot separately the state distances between \(|\Omega \rangle\) and two states vs the squeezing parameter \( \eta \). As indicated in Fig. 6 (a), when the input idler coherent state is too weak, \( \alpha = 0.01 \), the output signal state \(|\Omega \rangle\) is very different with initial input state and the generated state is totally same with the SPASV state. For \( \alpha = 0.75 \), in very weak squeezing parameter \( \eta \), the output state \(|\Omega \rangle\) are very similar to SV and SPASV states. But, as increasing the squeezing parameter \( \eta \), the similarities between \(|\Omega \rangle\) and SPASV (SV) state is increased (decreased) significantly (see the Fig. 6(b)). Although, the SPASV state only have one photon difference between SV state, it have very different features over SV state. Next we study the squeezing parameter and Wigner functions of the new generated state \(|\Omega \rangle\).
Figure 6. (Color online) The state distance of $|\Omega\rangle$ between SVS and PASVS as a function squeezed state parameter $\eta$ (a) for $\alpha = 0.01$, (b) for $\alpha = 0.75$. Other parameters are the same as Fig. 2.

**B. Squeezing parameter**

As we know, the SV state is an ideal state which possesses very strong squeezing effect. To investigate the squeezing effect of the field quadrature of the generated state $|\Omega\rangle$, in this subsection we study the squeezing parameter of $|\Omega\rangle$. The squeezing parameter of radiation field is defined as

$$S_\phi = (\Delta X_\phi)^2 - \frac{1}{2},$$  \hspace{1cm} (48)

where

$$\hat{X}_\phi = \frac{1}{\sqrt{2}} (ae^{-i\phi} + a^\dagger e^{i\phi}), \quad \phi \in [0, 2\pi],$$  \hspace{1cm} (49)

is the quadrature operator of the field, and $\Delta X_\phi = \sqrt{\langle \hat{X}_\phi^2 \rangle - \langle \hat{X}_\phi \rangle^2}$ is the variance of variable $X_\phi$. The minimum value of $S_\phi$ is $-0.5$ and if $-0.5 \leq S_\phi < 0$ the field is called nonclassical. We can calculate the squeezing parameter of SV state $|\phi_1\rangle$, PASV state $|\phi_2\rangle$ and generated output state $|\Omega\rangle$ easily and their curves can be seen in Fig. 7. We observe from Fig. 7 (a) that when $\alpha = 0.01$ the squeezing parameter of the new generated output signal state $|\Omega\rangle$ exactly same with the squeezing parameter of SPASV state $|\phi_2\rangle$, and it have very good squeezing as initial input state $|\phi_1\rangle$ when the squeezing parameter $\eta$ become larger. Furthermore, as showed in Fig. 7(b), if $\alpha = 0.75$, then the squeezing parameter of $|\Omega\rangle$ is same with the initial input state $|\phi_1\rangle$. Here, we have to mention that in our scheme it is required that the measured system is initially prepared in very weak coherent state. Thus, the $\alpha = 0.75$ case is not our main points.

**C. Winger function of new generated state**

To further confirm similarities between SPASV state $|\phi_2\rangle$ and the new generated state $|\Omega\rangle$, in this subsection we study the Wigner function of $|\Omega\rangle$. The Wigner function for the state $\rho = |\Omega\rangle\langle\Omega|$ can be written as [66]

$$W(z) = \frac{1}{\pi^2} \int_{-\infty}^{+\infty} \exp(\lambda^* z - \lambda z^*) C_W(\lambda) d^2\lambda.$$  \hspace{1cm} (50)

Here, $C_W(\lambda)$ is the characteristic function, and is defined as

$$C_W(\lambda) = Tr \left[ \rho e^{\lambda a^\dagger - \lambda^* a} \right],$$  \hspace{1cm} (51)
and $z = x + ip$ is represent the normalized dimensionless position and momentum observables of the beam in phase space. After some math, we can calculate the explicit expression of Wigner function of the new generated state $|\Omega\rangle$, and it reads as

$$W(z) = \frac{1}{\pi} \exp[-2|\tilde{z}|^2] w_1(z)$$

with

$$w_1(z) = \frac{2}{\pi} \exp[-2|\tilde{z}|^2]$$

$$w_2(z) = \frac{2}{\pi} \sinh^2 \eta \exp[-2|\tilde{z}|^2] (4|\tilde{z}|^2 - 1)$$

$$w_3(z) = \frac{2}{\pi} \cosh^2 \eta \exp[-2|\tilde{z}|^2] (4|\tilde{z}|^2 - 1)$$

$$w_4(z) = \frac{4}{\pi} \mu \sinh \eta e^{-\tau}$$

$$w_5(z) = \frac{4}{\pi} \mu^* \cosh \eta e^{-\tau}$$

$$w_6(z) = \frac{4}{\pi} \sinh 2\eta \exp[-2|\tilde{z}|^2] (4|\tilde{z}|^2 - 1)$$

Here, $\tilde{z} = z \cosh \eta - z^* e^{i\theta} \sinh \eta$, $\tau = 2R^2 |\tilde{z}|(\cosh \eta - \sinh \eta)^2 - 23^2 |\tilde{z}|(\cosh \eta + \sinh \eta)^2$ and $\mu = \mathcal{R}(|\tilde{z}|(\sinh \eta - \cosh \eta) + i3|\tilde{z}|(\sinh \eta + \cosh \eta))$. We can observe that this Wigner function is a real function and its value is bounded $-\frac{2}{\pi} \leq W(\alpha) \leq \frac{2}{\pi}$ in whole phase space. In the derivation of the above Wigner function we have used the identities

$$S(\xi)aS^\dagger(\xi) = a \cosh(\eta) - a^\dagger e^{i\pi} \sinh(\eta),$$

$$S(\xi)a^\dagger S^\dagger(\xi) = a^\dagger \cosh(\eta) - a e^{i\pi} \sinh(\eta).$$

The $w_2(z)$ in the Wigner function (52) is the Wigner function of SV state $|\phi_1\rangle$. Although the SV state $|\phi_1\rangle$ is a nonclassical state, its Wigner function is Gaussian and positive in phase space [36]. It is very clear in Eq. (52) that it contains non-Gaussian terms such as $w_2(z)$, $w_3(z)$ and $w_6(z)$. Thus the Wigner function of our new generated signal state is non-Gaussian in the phase space. We present the plots of the Wigner functions of initial input signal state $|\phi_1\rangle$, new generated output signal state $|\Omega\rangle$ and SPASV state $|\phi_2\rangle$ in phase space in Fig. 8 for different squeezing parameters which we set as $\eta = 0, 1, 2$. Figs. 8 (a)-(c) represents Wigner functions of SV state $|\phi_1\rangle$, and Figs. 8 (d)-(f) represents Wigner functions of new generated state $|\Omega\rangle$, and Figs. 8 (g)-(i) represents Wigner functions of SPASV state $|\phi_2\rangle$, respectively. By comparing the curves of those Wigner functions, we observed that the generated state in our scheme is a typical nonclassical state. It is very clear from Fig. 8 (d)-(f) that as initial input state $|\phi_1\rangle$, the new state $|\Omega\rangle$ has squeezing in one of the quadrature, and there is also some negative regions of the Wigner functions in the phase space. These two features of the new state $|\Omega\rangle$ can show that its nonclassicality. Furthermore, it is proved that our new generated state $|\Omega\rangle$ have exactly same phase space distribution as SPASV state $|\phi_2\rangle$ (see second and third rows of Fig. 8). As indicated in Fig. 8(d), if the input state of the pointer is vacuum, then the output signal state is prepared in single-photon Fock state.

V. CONCLUSION

In summary, we have designed a fully laboratory feasible optical model to successfully prepare nonclassical states such as single-photon Fock state, SPAC state and SPASV state by using postselected weak measurement in three wave mixing process. In our scheme the signal and idler beams are taken as pointer and measured system, respectively, and entanglement between them is realized by BBO crystal which can take the role of weak measurement. In other words, in our study, a nonlinear BBO crystal was chosen to introduce weak interaction between three-wave mixing including pump, idle and signal light. By taking the pre- and post-selections on measured system, the final pointer state is prepared desired nonclassical state which depends on the initial input signal state (initial pointer state). Further, we investigated the properties including squeezing, second order correlation and Winger functions of conditional output states.

We found that if the input signal (pointer) is vacuum state then the output signal state is prepared in single-photon Fock state which is typical quantum state exclusively used in many quantum information processing. We also found that if the input signal state is coherent (squeezed vacuum) state, then the output signal state prepared in SPAC (SPASV) state, respectively, and their purities can be controlled by optical elements easily. Furthermore, we also found that the post-selective measurement characterized by weak values and postselection have a positive effect on the output SNR over non-postselection for coherent state input case.

Our scheme for the preparation of nonclassical states can be implemented in optical Lab and we anticipate that this scheme could provide other effective methods to the generation of other useful nonclassical state such as Schrödinger kitten state [71].

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China (Nos. 11865017, 11664041),
Figure 8. (Color online) Comparing the Winger functions of SV state $|\phi_1\rangle$, SPASV state $|\phi_2\rangle$ and new generated state $|\Omega\rangle$ in phase space. Each column is defined for the different squeezing parameters $\eta$ (set $\eta = 0, 1, 2$), and are ordered accordingly from left to right. Figures (a) to (c) correspond to the Winger functions of initial input SV state $|\phi_1\rangle$, (d) to (f) correspond to the Winger functions of new generated output signal state $|\Omega\rangle$, and (g) to (i) correspond to the Winger functions of SPASV state $|\phi_2\rangle$, respectively. Other parameters are the same as Fig. 2.

the Natural Science Foundation of Xinjiang Uyghur Autonomous Region (Grant No. 2020D01A72) and the Introduction Program of High Level Talents of Xinjiang Ministry of Science.

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