ON A THEOREM OF HAZRAT AND HOOBLER

BENJAMIN ANTIEAU

(Communicated by Lev Borisov)

Abstract. We use cycle complexes with coefficients in an Azumaya algebra, as developed by Kahn and Levine, to compare the $G$-theory of an Azumaya algebra to the $G$-theory of the base scheme. We obtain a sharper version of a theorem of Hazrat and Hoobler in certain cases.

1. Introduction

Let $K^*_s(X;A)$ be the $K$-theory of left $A$-modules which are locally free and finite rank coherent $O_X$-modules and let $G^*_s(X;A)$ be the $K$-theory of left $A$-modules which are coherent $O_X$-modules.

We prove the following theorem.

Theorem 1.1. Let $X$ be a $d$-dimensional scheme of finite type over a field $k$, and let $A$ be an Azumaya algebra on $X$ of constant degree $n$. Let $B_{A} : G^i(X) \to G^i(X;A)$ and $B_{K}^A : K^i(X) \to K^i(X;A)$ be the homomorphisms induced by the functor $F \mapsto A \otimes O_X F$. Then,

1. the kernel and cokernel of $B_{A} : G^i(X) \to G^i(X;A)$ are torsion groups of exponents dividing $n^{2d+2}$;
2. the kernel and cokernel of $B_{K}^A : K^i(X) \to K^i(X;A)$ are torsion groups of exponents dividing $n^{2d+2}$ if $X$ is regular.

Corollary 1.2. If $A$ is an Azumaya algebra of constant degree $n$ over a scheme $X$ of finite type over a field $k$, then the base extension homomorphism

$$B_A : G^*_s(X) \otimes \mathbb{Z} \left[\frac{1}{n}\right] \to G^*_s(X;A) \otimes \mathbb{Z} \left[\frac{1}{n}\right]$$

is an isomorphism.

The theorem above should be compared to the following two theorems, which motivated us in the first place.

Theorem 1.3 (Hazrat-Millar [9]). If $A$ is an Azumaya algebra of constant degree $n$ which is free over a noetherian affine scheme $X$, then

$$B_{K}^A : K^i(X) \to K^i(X;A)$$

has torsion kernel and cokernel of exponents at most $n^4$. 

Received by the editors April 4, 2011 and, in revised form, November 7, 2011.

2010 Mathematics Subject Classification. Primary 14F22; Secondary 19Dxx.

Key words and phrases. Azumaya algebras, twisted algebraic $K$-theory.

The author was supported in part by the NSF under Grant RTG DMS 0838697.

©2013 American Mathematical Society
Reverts to public domain 28 years from publication
Theorem 1.4 (Hazrat-Hoobler [8]). Let $X$ be a $d$-dimensional noetherian scheme, and let $\mathcal{A}$ be an Azumaya algebra on $X$ of constant degree $n$. Then:

1. the kernel of $B_{\mathcal{A}} : G_i(X) \to G_i(X; \mathcal{A})$ is torsion of exponent dividing $n^{2d(d+1)+2}$, and the cokernel is torsion of exponent dividing $n^{4d+2}$;
2. the kernel of $B^K_{\mathcal{A}} : K_i(X) \to K_i(X; \mathcal{A})$ is torsion of exponent dividing $n^{2d(d+1)+2}$ if $X$ is regular, and the cokernel is torsion of exponent dividing $n^{4d+2}$ in this case;
3. the kernel and cokernel of $B^K_{\mathcal{A}} : K_i(X) \to K_i(X; \mathcal{A})$ are torsion groups of exponent dividing $n^{2d+2}$ if $X$ has an ample line bundle.

Since a degree $n$ Azumaya algebra is locally split by degree $n$ extensions, it is expected that the base extension map

\[ B^K_{\mathcal{A}} : K_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/n] \to K_*(X; \mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/n] \]

should be an isomorphism.

Here is a partial history of results and techniques in this direction.

Wedderburn’s theorem [10] easily implies that $K_0(k) \to K_0(A)$ is injective with cokernel isomorphic to $\mathbb{Z}/m$, where $A \cong M_m(D)$ for a central $k$-division algebra $D$.

Green-Handelman-Roberts [5] proved that the map $B^K_{\mathcal{A}}$ in equation (1) is an isomorphism when $\mathcal{A}$ is a central simple algebra of degree $n$ over a field. They used the Skolem-Noether theorem. That case has also been proven by Hazrat [7] using the fact that $A$ is étale locally a matrix algebra.

The theorem of Hazrat-Millar quoted above uses the opposite algebra. The theorem of Hazrat-Hoobler uses Bass-style stable range arguments and Zariski descent for $G$-theory.

Our result uses twisted versions of Bloch’s cycle complexes. These twisted cycle complexes and the twisted motivic spectral sequence that relates them to $G$-theory are due to Kahn and Levine [11]. It is possible that our result could be extended to essentially smooth schemes over Dedekind rings by a combination of the work of Kahn and Levine [11] and Geisser [4].

The following is an interesting corollary of our approach: there are natural filtrations of length $d$ on $G_i(X)$ and $G_i(X; \mathcal{A})$ coming from [11]. The map $B_{\mathcal{A}} : G_i(X) \to G_i(X; \mathcal{A})$ respects the filtrations. We show that the induced maps on each of the $d+1$ slices have kernel and cokernel groups of exponent at most $n^2$.

It is worth mentioning two related functors on Azumaya algebras with values in abelian groups where the base extension maps are isomorphisms. Dwyer and Friedlander [3, 2.4, 3.1] showed that

\[ K^\text{ét}_*(R; \mathbb{Z}/m) \to K^\text{ét}_*(R; A; \mathbb{Z}/m) \]

is an isomorphism in some cases (all of which are Azumaya algebras over a noetherian ring), where $K^\text{ét}_*$ denotes étale $K$-theory, as, for instance, in Thomason [12]. In this direction, it is possible to show (for instance, in the setting of Antieau [1]) that $K^\text{ét}_*(X; \mathcal{A})$ is an invertible object (in the sense of the Picard group) over $K^\text{ét}_*(X)$ in the category of étale sheaves of $K^\text{ét}_*$-module spectra on a scheme $X$.

Finally, Cortiñas and Weibel [2] proved that the base extension maps induce isomorphisms in Hochschild homology over a field $k$. 
2. Twisted higher Chow groups and twisted $G$-theory

Let $X$ in $\textbf{Sch}/k$ be an integral $k$-scheme of finite type, and let $\mathcal{A}$ be a sheaf of Azumaya algebras on $X$ of rank $n^2$. The degree of $\mathcal{A}$ is defined to be the integer $n$. Let $\mathcal{E}$ be a left $\mathcal{A}$-module which is locally free and finite rank $na$ as an $\mathcal{O}_X$-module. For generalities on Azumaya algebras, which as $\mathcal{O}_X$-modules are always locally free and of finite rank, see [6].

As in Kahn-Levine [11], define the cycle complex of $X$ with coefficients in $\mathcal{A}$ as follows. Let $S^X_{(s)}(t)$ denote the set of closed subsets $W \subseteq X \times_k \Delta^n$ such that $\dim_{k} W \cap X \times_k F \leq s + \dim_{k} F$ for all faces $F$ of $\Delta^n$. Taking inverse images, $S^X_{(s)}(t)$ becomes a simplicial set. Let $X_s(t)$ denote the subset of irreducible $W$ in $S^X_{(s)}(t)$ such that $\dim_{k} W = s + t$. Define, for $t \geq 0$,

$$z_s(X, t; \mathcal{A}) = \bigoplus_{W \in X_s(t)} K_0(k(W); \mathcal{A}).$$

See [11, Definition 5.6.1]. Kahn and Levine show that this actually becomes a complex, $z_s(X, \ast; \mathcal{A})$, and they define the higher Chow groups with coefficients in $\mathcal{A}$ as

$$\text{CH}_s(X, t; \mathcal{A}) = H_t(z_s(X, \ast; \mathcal{A})).$$

There are maps relating the complex $z_r(X, \ast; \mathcal{A})$ to $z_r(X, \ast)$, the untwisted complex that computes Bloch’s higher Chow groups. These are induced by the base-change map $B_{\mathcal{E}}$ and the forgetful map $F$ on $K$-theory:

$$B_{\mathcal{E}}^K : K_0(k(W)) \to K_0(k(W); \mathcal{A}),$$

$$F : K_0(k(W), \mathcal{A}) \to K_0(k(W)).$$

The map $B_{\mathcal{E}}$ takes a $k(W)$-vector space and tensors with $\mathcal{E}_{k(W)}$ to produce a left $\mathcal{A}_{k(W)}$-module. The norm map $F$ simply forgets the $\mathcal{A} \otimes_{k(W)}$-module structure on a vector space. The kernels of both of these maps are zero.

Lemma 2.1. The compositions $F \circ B_{\mathcal{E}}^z$ and $B_{\mathcal{E}}^z \circ F$ are multiplication by $na$ on $z_s(X, t)$ and $z_s(X, t; \mathcal{A})$.

Proof. Indeed, since the rank of $\mathcal{E}$ is $na$ as an $\mathcal{O}_X$-module, this follows immediately. □

Corollary 2.2. The cokernel of $F : z_s(X, t; \mathcal{A}) \to z_s(X, t)$ is a torsion group of exponent bounded above by $n^2$, and $B_{\mathcal{E}}^z : z_s(X, t) \to z_s(X, t; \mathcal{A})$ is a torsion group of exponent bounded above by $na$.

Proof. In the first case, one always has $\text{ind}(\mathcal{A}_{k(W)}) | n$, where $\text{ind}(\mathcal{A}_{k(W)})$ is the degree of the unique division algebra over $k(W)$ such that $\mathcal{A}_{k(W)} \cong M_m(D)$ for some $m$. Similarly,

$$\left( \frac{na}{\text{ind}(\mathcal{A}_{k(W)})^2} \right) | na,$$

so the second statement follows. □
Proposition 2.3. The kernels and cokernels of

\[ B^\text{CH}_E : \text{CH}_s(X, t) \to \text{CH}_s(X, t; A) \]

and of

\[ F : \text{CH}_s(X, t; A) \to \text{CH}_s(X, t) \]

are torsion groups of exponent at most \( na \).

Proof. This follows immediately from Lemma [22].

Here is our main theorem. Theorem [11] follows from it by taking \( E = A \).

Theorem 2.4. Let \( X \) be a \( d \)-dimensional scheme of finite type over a field, and let \( A \) be an Azumaya algebra on \( X \). Then, the kernels and cokernels of

\[ B^\text{CH}_E : G_r(X) \to G_r(X; A) \]

and of

\[ F : G_r(X; A) \to G_r(X) \]

are groups of exponent bounded above by \( (na)^d + 1 \) for all \( r \geq 0 \).

Proof. Kahn and Levine [11] show that there is a convergent spectral sequence

\[ E^p,q_2(A) = \text{CH}_q(X, -p - q; A) \Rightarrow G_{-p-q}(X; A). \]

There is also the motivic spectral sequence

\[ E^p,q_2 = \text{CH}_q(X, -p - q) \Rightarrow G_{-p-q}(X). \]

The functors \( B^\text{CH}_E : G(X) \to G(X; A) \) and \( F : G(X; A) \to G(X) \) are compatible with these spectral sequences and the functors \( B^\text{CH}_E \) and \( F \) on higher Chow groups.

Note that \( E^p,q_2 = E^{p,q}_2(A) = 0 \) whenever \( q < 0, -p < 0, \) or \( q > d \).

We will prove the theorem for the kernel of the functor \( B^\text{CH}_E \). The other cases are entirely similar. On the \( E_{-\infty} \)-page, the composition functor \( F \circ B^\text{CH}_E \) is still multiplication by \( na \), so the kernels and cokernels of \( B^\text{CH}_E \) on \( E_{-\infty} \) are still of exponent at most \( na \). The spectral sequences abut to filtrations \( F^s G_r(X; A) \) and \( F^s G_r(X) \), where

\[ F^{(s/s+1)} G_r(X; A) = F^s G_r(X; A)/F^{s+1} G_r(X; A) \cong E_{-r+s,-s}(A), \]

\[ F^{(s/s+1)} G_r(X) = F^s G_r(X)/F^{s+1} G_r(X) \cong E_{-r+s,-s}. \]

The filtration looks like

\[ 0 = F^0 G_r(X) \subset F^{-1} G_r(X) \subset \cdots \subset F^{-d} G_r(X) = G_r(X). \]

The filtration \( F^s G_r(X) \) is of length \( d \) by the vanishing statements. Let \( z \in G_r(X) \) be in the kernel of \( F \), and let \( \overline{z} \) be the image of \( z \) in \( E_{-r-d; d} \). Then, by hypothesis, \( \overline{z} \) is in the kernel of \( F \), so that \( na \cdot \overline{z} = 0 \). Thus, \( na \cdot z \) is contained in \( F^{-d+1} G_r(X) \). Continuing in this way, we see that \( (na)^d + 1 \cdot z \) is contained in \( F^0 G_r(X) = 0 \), so \( (na)^d + 1 \cdot z = 0 \).

Corollary 2.5. The same result holds for \( K \)-theory when \( X \) is regular.

Corollary 2.6. The maps

\[ B^E_{(s/s+1)} : F^{(s/s+1)} G_r(X) \to F^{(s/s+1)} G_r(X; A), \]

\[ F : F^{(s/s+1)} G_r(X; A) \to F^{(s/s+1)} G_r(X) \]

have torsion kernels and cokernels of exponent at most \( na \).
Proof. This follows from the proof of the theorem. \qed

Corollary 2.7. For any commutative ring \( R \) in which \( na \) is invertible, the maps

\[
\begin{align*}
B_\mathcal{E} : & \; z_\mathcal{E}(X,\ast; R) \to z_\mathcal{E}(X,\ast; A, R), \\
B_\mathcal{F} : & \; \mathcal{G}_\mathcal{F}(X; R) \to \mathcal{G}_\mathcal{F}(X; A; R), \\
F_\mathcal{F} : & \; z_\mathcal{F}(X,\ast; A, R) \to z_\mathcal{F}(X,\ast; R), \\
F_\mathcal{G} : & \; \mathcal{G}_\mathcal{G}(X; A; R) \to \mathcal{G}_\mathcal{G}(X; R)
\end{align*}
\]

are isomorphisms.

It is interesting that this method proves the isomorphisms by means of an isomorphism of cycle complexes, not just a quasi-isomorphism.

ACKNOWLEDGMENTS

The author thanks Christian Haesemeyer, Roozbeh Hazrat, and Ray Hoobler for conversations and the referee, who made several useful suggestions for improving the exposition.

REFERENCES

1. Benjamin Antieau, Cohomological obstruction theory for Brauer classes and the period-index problem, J. K-Theory 8 (2010), no. 3, 419–435. MR2863419
2. G. Cortiñas and C. Weibel, Homology of Azumaya algebras, Proc. Amer. Math. Soc. 121 (1994), no. 1, 53–55. MR1181159 (94g:16009)
3. William G. Dwyer and Eric M. Friedlander, Étale K-theory of Azumaya algebras, Proceedings of the Luminy conference on algebraic K-theory (Luminy, 1983), vol. 34, 1984, pp. 179–191. MR772057 (86j:18009)
4. Thomas Geisser, Motivic cohomology over Dedekind rings, Math. Z. 248 (2004), no. 4, 773–794. MR2103541 (2006c:14026)
5. S. Green, D. Handelman, and P. Roberts, K-theory of finite dimensional division algebras, J. Pure Appl. Algebra 12 (1978), no. 2, 153–158. MR0480698 (58:852)
6. Alexander Grothendieck, Le groupe de Brauer. I. Algèbres d’Azumaya et interprétations diverses, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 46–66. MR0244269 (39:5586a)
7. R. Hazrat, Reduced K-theory of Azumaya algebras, J. Algebra 305 (2006), no. 2, 687–703. MR2266848 (2007i:19002)
8. R. Hazrat and R. Hoobler, K-theory of Azumaya algebras over schemes, arXiv e-prints (2009), 0911.1406. To appear in Communications in Algebra.
9. Roozbeh Hazrat and Judith R. Millar, A note on K-theory of Azumaya algebras, Comm. Algebra 38 (2010), no. 3, 919–926. MR2650377 (2011c:19007)
10. I. N. Herstein, Noncommutative rings, The Carus Mathematical Monographs, No. 15, Mathematical Association of America, 1968. MR0227205 (37:2790)
11. Bruno Kahn and Marc Levine, Motives of Azumaya algebras, J. Inst. Math. Jussieu 9 (2010), no. 3, 481–599. MR2650808 (2011j:19005)
12. Robert W. Thomason, Algebraic K-theory and étale cohomology, Ann. Sci. École Norm. Sup. (4) 18 (1985), no. 3, 437–552. MR826102 (87k:14016)

Department of Mathematics, University of California Los Angeles, 520 Portola Plaza, Los Angeles, California 90095
E-mail address: antieau@math.ucla.edu