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Amplification Induced by White Noise

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We investigate the amplification of a field induced by white noise. In the present study, we study a stochastic equation which has two parameters, the energy $\omega(\vec{k})$ of a free particle and the coupling strength $D$ between the field and the white noise, where the quantity $\vec{k}$ represents the momentum of the free particle. This equation is reduced to an equation with one parameter $\alpha(\vec{k})$, which is defined as $\alpha(\vec{k}) = D \left( \frac{\omega(\vec{k})}{2} \right)^{3/2}$. We obtain an expression of the exponent statistically averaged over a unit time and derive an approximate expression of it. In addition, the exponent is obtained numerically by solving the stochastic equation. We find that the amplification increases with $\alpha(\vec{k})$. This indicates that when the energy $\omega(\vec{k})$ is equal to $p^2 + \vec{k}^2$, white noise can amplify the fields for soft modes if the mass $m$ of the field is sufficiently small and if the strength of the coupling between the field and white noise is sufficiently strong. We show that the $\alpha(\vec{k})$ dependence of the exponent statistically averaged is qualitatively similar to that of the exponent obtained by solving the stochastic equation numerically, and that for small values of $\alpha(\vec{k})$ these two exponents are quantitatively similar.

§1. Introduction

Over the past few decades, a considerable number of studies have been carried out on phenomena induced by noise, and it has come to be understood that noise plays important roles in many branches of physics. Well-known phenomena in which noise plays a prominent role include stochastic resonances,\(^1\)-\(^4\) stochastic synchronization,\(^5\),\(^6\) noise induced propagation,\(^7\) phase transition induced by multiplicative noise,\(^8\) noise enhanced phase locking,\(^9\) and coherence resonance.\(^10\) These examples indicate that noise plays essential roles in some systems, and an understanding of noise is important to explain many physical phenomena. One phenomenon in which noise may be important is particle production that thermalizes a system, as studied in the context of the early universe and heavy ion collisions. The number of particles produced may be enhanced by noise, because noise can supply energy. With this in mind, it is important to investigate the amplification of a field by noise.

The amplification of a field has been investigated in terms of particle production, because the produced particles affect the time evolution of the system. One mechanism of this amplification is spinodal decomposition,\(^11\) in which the field is amplified through the roll-down from the maximum of the potential (false vacuum) to the minima (true vacuum), because the mass squared is negative near the maximum of the potential. Another mechanism is parametric resonance,\(^12\)-\(^25\) in which an oscillating field amplifies other fields. The effect of noise on parametric resonance

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has been studied,¹⁴,¹⁵ and it has been found that noise modifies the amplification quantitatively. For example, noise shifts enhanced modes.¹⁶

These mechanisms are well known as those of the amplification of a field through a phase transition. Amplification through spinodal decomposition occurs at the beginning of the phase transition and that through parametric resonance occurs at the end of the phase transition. Particles are not produced by parametric resonance if no oscillating field exists, because no parametric resonance occurs in the absence of an oscillating field. However, another mechanism of amplification may act even in such a case if noise exists. Phenomenological equations for such processes are similar to the equation for the motion of a pendulum.

Oscillators with a randomly varying mass and/or friction coefficient have been studied for the purpose of understanding the effects of noise. In Ref. 26), the equations of motion were described using amplitude-angle variables, and the condition for the growth of the amplitude was derived. In a similar manner, the phase transition was investigated for a pendulum with a randomly vibrating suspension axis.²⁷ Recently, Mallick and Marcq investigated a nonlinear oscillator with white and colored noise.²⁸–³⁰ In their studies, the equations of motion were described using energy-angle (or action-angle) variables. The power exponents of the time development were obtained by solving a (effective) Fokker-Planck equation that describes the evolution of a slow variable. In these studies, the Fokker-Planck equation is used as the main tool.

In this paper, we investigate the effect of noise on the amplification of the fields by studying an equation with white noise. We attempt to derive an approximate expression for the exponent and to obtain the exponents for various values of the parameter. The role of physical parameters in the amplification is clearly shown by evaluating the exponents in the case that white noise exists. In §2, the basic equation employed in the present study is introduced. This equation describes a harmonic oscillator with varying mass. An approximate expression for the exponent is derived by averaging the term with respect to white noise statistically and by using the steepest descent method, without using the Fokker-Planck equation (although, we acknowledge that the Fokker-Planck equation is a powerful tool). In §3, we report the results of the numerical evaluation of the stochastic equation introduced in §2. We obtained many trajectories as numerical solutions, and we extracted the exponents by averaging these trajectories. We compare the values of the exponents obtained from the expression derived in §2 with those of the exponents obtained by solving the stochastic equation. Section 4 is devoted to conclusions.

§2. Amplification induced by white noise in a scalar field theory

2.1. Equation of motion

To begin, we introduce the parameters and variables used in the following investigation. The quantity $\omega(\vec{k})$ represents the energy with momentum $\vec{k}$. The function $n(t)$ is white noise, which has the properties

$$\langle n(t) \rangle = 0,$$

(2-1a)
where \( \langle \cdot \cdot \cdot \rangle \) represents the statistical average. The starting point of this study is the following equation, which describes the motion of the field \( \phi \) with momentum \( \vec{k} \) near the bottom of the potential with white noise:

\[
\frac{d^2 \phi}{dt^2} + \left( \left( \omega(\vec{k}) \right)^2 + Dn(t) \right) \phi = 0.
\]  

(2.2)

Here, the parameter \( D \) is the coupling strength between the noise and the field. Without loss of generality, we can assume that this parameter is positive or zero. No oscillating term appears in Eq. (2.2), while an oscillating term does appear in the coefficient of \( \phi \) in Refs. 14) and 15). Equation (2.2) approximately describes a system with an oscillating field of quite small amplitude or without an oscillating field.\(^{17}\) We regard Eq. (2.2) as the basic equation in this study. It is not our aim to derive Eq. (2.2). The following investigation is meaningful if Eq. (2.2) is meaningful.

In order to represent Eq. (2.2) by dimensionless variables, we rewrite Eq. (2.2) in terms of the new variable \( z \), given by \( z = \omega(\vec{k}) t \). This gives

\[
\frac{d^2 \phi}{dz^2} + \left( 1 + \alpha(\vec{k}) r(z; \vec{k}) \right) \phi = 0,
\]  

(2.3)

where \( \alpha(\vec{k}) = D \left( \omega(\vec{k}) \right)^{-3/2} \) and \( r(z; \vec{k}) = \left( \omega(\vec{k}) \right)^{-1/2} n(z/\omega(\vec{k})) \), which has the property

\[
\langle r(z; \vec{k}) r(z'; \vec{k}) \rangle = \delta(z - z').
\]  

(2.4)

The momentum \( \vec{k} \) does not play an essential role in the calculation of the amplification from Eqs. (2.3) and (2.4), except for the momentum dependence of the field amplification. The amplification is determined by only the value of \( \alpha(\vec{k}) \). Equation (2.3) is rewritten by introducing \( p_\phi = d\phi/dz \) and \( dW = r(z; \vec{k}) dz \), because it is not easy to treat Eq. (2.3) numerically. It should be noted that \( dW \) is a Wiener process. Therefore, Eq. (2.3) can be rewritten as

\[
d\phi = p_\phi \, dz, \tag{2.5a}
\]
\[
dp_\phi = -\phi \, dz - \alpha(\vec{k}) \, \phi \circ dW. \tag{2.5b}
\]

We regard Eqs. (2.5a) and (2.5b) as Stratonovich equations. (Here, \( \circ \) represents the Stratonovich product.) A Stratonovich equation is easily converted into an Itô equation. The Itô equations corresponding to Eqs. (2.5a) and (2.5b) take the same form, that is,

\[
d\phi = p_\phi \, dz, \tag{2.6a}
\]
\[
dp_\phi = -\phi \, dz - \alpha(\vec{k}) \, \phi \, dW. \tag{2.6b}
\]

We have solved Eqs. (2.6a) and (2.6b) numerically and we present the results in the next section.
2.2. Estimation of the exponents

Equation (2.3) describes the motion of a harmonic oscillator when \( \alpha(\vec{k}) \) is zero. Therefore, the solution is a sine function for \( \alpha(\vec{k}) = 0 \). Hereafter, we omit the argument \( \vec{k} \) of \( \alpha(\vec{k}) \) when there is no chance of confusion. The coefficient \( \left( 1 + \alpha(\vec{k}) r(z; \vec{k}) \right) \) can be negative for \( \alpha \neq 0 \), because \( r(z; \vec{k}) \) is a random variable with zero mean. \( \alpha \) is not negative, because \( D \) is larger than or equal to zero.) In this subsection, we estimate the magnitude of the exponent \( \Delta g \), which describes the increase of the amplitude \( A(z) \) of the oscillator over a time interval \( \Delta z \):

\[
A(z + \Delta z) = e^{\Delta g} A(z).
\]

The amplitude does not increase if the coefficient of \( \phi \) in Eq. (2.3) is positive. Therefore we consider only the case in which this coefficient is negative. Then the exponent \( \Delta g \) for an extremely short time step \( \Delta z \), in which the random variable \( r(z; \vec{k}) \) is constant, is given by

\[
\Delta g = \Delta z \Theta \left( -1 - \alpha(\vec{k}) r(z; \vec{k}) \right) \left( -1 - \alpha(\vec{k}) r(z; \vec{k}) \right)^{1/2},
\]

where \( \Theta(x) \) is a step function that is 1 for \( x > 0 \) and 0 for \( x < 0 \). If the time step \( \Delta z \) is sufficiently large compared with the length of the time interval over which the random variable \( r(z; \vec{k}) \) is constant, the statistical average of \( \Delta g \) should be taken. In this sense, the width \( \Delta z \) represents the resolution of the observation. To make this fact clear, we divide the time step \( \Delta z \) into \( N \), labeled as \( i = 1, \cdots, N \), regions over which the random noise \( r(z; \vec{k}) \) is constant. When the amplification of the field is given by \( \phi(z + (\Delta z)/N) = \exp(\beta_i(\Delta z)/N)\phi(z_{i-1}) \) for the region \( i \), the amplification of the field in \( \Delta z \) is given by

\[
\phi(z + \Delta z) = \prod_{i=1}^{N} \exp \left( \beta_i \frac{(\Delta z)}{N} \right) \phi(z) = \exp \left( \frac{1}{N} \sum_{i=1}^{N} \beta_i(\Delta z) \right) \phi(z). \tag{2.8}
\]

The quantity \( \beta_i(\Delta z) \) is the exponent in the case that the random noise is constant over the time step \( \Delta z \). Equation (2.8) implies that the exponent in \( \Delta z \) is the statistical average of the realized exponents \( \beta_i(\Delta z) \) when the noise \( r(z; \vec{k}) \) varies frequently over \( \Delta z \). If \( \beta_j \) is purely imaginary, it does not affect the amplification. This is the reason that the step function is included in Eq. (2.7), as stated above. Therefore we take the statistical average, because the Gaussian white noise varies over very small time intervals. As in the previous subsection, we define \( \Delta W = r(z; \vec{k})\Delta z \). The probability distribution function \( P(\Delta W) \) for Wiener process \( \Delta W \) is a Gaussian distribution with zero mean and variance \( \Delta z \):

\[
P(\Delta W) = \frac{1}{\sqrt{2\pi \Delta z}} \exp \left( -\frac{(\Delta W)^2}{2\Delta z} \right). \tag{2.9}
\]

We estimate the amplification by \( \exp [(\Delta g)_{st}] \), where \( (\Delta g)_{st} \) is defined by

\[
(\Delta g)_{st} = \int_{-\infty}^{\infty} d(\Delta W) P(\Delta W) \Delta g. \tag{2.10}
\]
The exponent \((\Delta g)_{st}\) with Eq. (2.9) can be rewritten as

\[
(\Delta g)_{st} = \frac{\alpha^{1/2}(\Delta z)^{3/4}}{(2\pi)^{1/2}} \exp \left( -\frac{\Delta z}{2\alpha^2} \right) \int_0^\infty dt \ t^{1/2} \exp \left( -\frac{t^2}{2} - \frac{(\Delta z)^{1/2}}{\alpha} t \right) .
\] (2.11)

Equation (2.11) is easily evaluated for \(\alpha \gg (\Delta z)^{1/2}\) and \(\alpha \ll (\Delta z)^{1/2}\). The exponent \((\Delta g)_{st}\) for \(\alpha \gg (\Delta z)^{1/2}\) is given by

\[
(\Delta g)_{st} \sim \frac{\Gamma(3/4)}{2^{3/4}\pi^{1/2}} \alpha^{1/2}(\Delta z)^{3/4} \sim 0.411\alpha^{1/2}(\Delta z)^{3/4} .
\] (2.12)

The exponent \((\Delta g)_{st}\) for \(\alpha \ll (\Delta z)^{1/2}\) is given by

\[
(\Delta g)_{st} \sim \frac{\alpha^2}{2^{3/2}} \exp \left( -\frac{\Delta z}{2\alpha^2} \right) \sim 0.354\alpha^2 \exp \left( -\frac{\Delta z}{2\alpha^2} \right) .
\] (2.13)

We next introduce an arbitrary integer \(n\) satisfying \(n \geq 1\) in order to evaluate Eq. (2.11) approximately for other values of \(\alpha\). Rewriting Eq. (2.11) in terms of the new variable \(x\) given by \(t^{1/2} = x^n\) and using the method of steepest descent, we finally obtain

\[
(\Delta g)_{st} \sim n\alpha^{1/2}(\Delta z)^{3/4} \left[ f_n^{(2)} \left( x_M^{(n)} \right) \right]^{-1/2} \exp \left( -\frac{\Delta z}{2\alpha^2} - f_n \left( x_M^{(n)} \right) \right)
\times \text{erfc} \left( -2^{-1/2}x_M^{(n)} \left[ f_n^{(2)} \left( x_M^{(n)} \right) \right]^{1/2} \right),
\] (2.14a)

\[
f_n(x) = \frac{1}{2} x^{4n} + \frac{(\Delta z)^{1/2}}{\alpha} x^{2n} + (1 - 3n) \ln x ,
\] (2.14b)

\[
x_M^{(n)} = \left\{ \frac{1}{2} \left[ -\frac{(\Delta z)^{1/2}}{\alpha} + \sqrt{\left[ \frac{(\Delta z)^{1/2}}{\alpha} \right]^2 + \left( 6 - \frac{2}{n} \right)} \right] \right\} \frac{1}{\pi^n} ,
\] (2.14c)

where \(f_n^{(2)}(x)\) is the second-order derivative of \(f_n(x)\) with respect to \(x\), and the error function is defined as

\[
\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty dy \ \exp \left( -y^2 \right) .
\] (2.15)

For example, \((\Delta g)_{st}\) from Eq. (2.14a) for \(\Delta z \ll \alpha^2\) is approximately evaluated as

\[
(\Delta g)_{st} \sim 0.428\alpha^{1/2}(\Delta z)^{3/4} \quad \text{for} \ n = 1 \quad \text{and} \quad (\Delta g)_{st} \sim 0.400\alpha^{1/2}(\Delta z)^{3/4} \quad \text{for} \ n = 2.
\]

The quantity \((\Delta g)_{st}\) can be evaluated directly from Eq. (2.11) for \(\Delta z \ll \alpha^2\), in which case we obtain \((\Delta g)_{st} \sim 0.411\alpha^{1/2}(\Delta z)^{3/4}\). Therefore, it is conjectured that Eq. (2.14a) gives an approximate value of Eq. (2.11). Note that Eq. (2.11) is directly described by special functions, namely the gamma function and the parabolic cylinder function. Equations (2.12) and (2.13) are derived again using asymptotic expressions of the parabolic cylinder function. Our purpose is to evaluate the exponent over the unit time, \(\Delta z = 1\). We cannot take \(\Delta z \ll 1\), because the exponent \((\Delta g)_{st}\) is calculated statistically. For this reason, we set \(\Delta z = 1\) in the following. In the next section, we compare Eq. (2.14a) with the exponent obtained from the numerical solutions for the equation of motion.
In this section, we report the results of numerical calculation of Eqs. (2.6a) and (2.6b) for various values of $\alpha$. The initial conditions are $\phi(0) = 1$ and $d\phi(z)/dz|_{z=0} = 0$ in the presently considered calculations. The equations were solved numerically from $z = 0$ to $z = 500$. The time step in $z$ was set to 0.05. We applied the Euler-Maruyama scheme to the stochastic equations, taking into consideration the symplectic structure of the equations in the noiseless case in order to avoid the instability arising in the Euler scheme. This procedure is as follows. The time $z$ is divided into many small regions to solve the equations numerically. We introduce $\phi_n$ and $\phi_{n+1}$, which are the values of $\phi$ at the beginning and the end of a divided small region $n$, respectively. In the ordinary manner, the right-hand side of Eq. (2.6b) is evaluated with $\phi_n$, because Eq. (2.6b) is the Itô equation. However, taking into consideration the symplectic structure, we evaluate the right-hand side of Eq. (2.6b) with $\phi_{n+1}$ by converting Eq. (2.6b) into the equation evaluated at the end of the divided small region. Therefore, the scheme used in this paper is simply the symplectic-Euler scheme in the noiseless case. It was found that the amplitude remains essentially constant for a long time when no noise exists.

Figure 1 displays the time evolution of $\phi(z)$ for $\alpha = 0.6$. Apparently, the field is amplified due to the influence of the white noise. One trajectory can be calculated when the sequence of the noise is given. Therefore, we may have found the amplification unexpectedly in the cases where the amplification rarely occurs. For this
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Fig. 3. The exponents over a unit time, \( \Delta g|_{\Delta z=1} \) and \( (\Delta g)_{st}|_{\Delta z=1} \): (a) the narrow region \((\alpha \leq 3)\); (b) the wide region \((\alpha \leq 20)\). The symbols “x” represent the data points for \( \Delta g|_{\Delta z=1} \) obtained from \( \bar{\phi}(z) \), and the thick curve is a guide to the eye. The dotted curve represents \( (\Delta g)_{st}|_{\Delta z=1} \) obtained from Eq. (2.11), and the dashed curve represents \( (\Delta g)_{st}|_{\Delta z=1} \) obtained from Eq. (2.14a) with \( n=2 \).

reason, we should calculate many trajectories and take their average. We calculated the mean value of the trajectories of the field \( \phi^{(j)}_i(z) \), where the subscript \( i \) indicates the batch and the superscript \( (j) \) indicates the trajectory in a certain batch. One batch contains 500 samples (trajectories), and the mean value \( M_i(z) \) of these trajectories is calculated. We considered 20 batches, and with them calculated the mean value over 20 batches, \( \bar{\phi}(z) \). The averaged behavior \( \bar{\phi}(z) \) for \( \alpha = 0.6 \) from \( z = 100 \) to 150 is displayed in Fig. 2. The thick curve represents the mean value \( \bar{\phi}(z) \), and the vertical bars indicate the 50\% confidence interval. These bars are depicted every 20 points in the figure. They become wider as the time \( z \) increases because the present process is a Wiener process.

It is found from this figure that the field is amplified by noise. We use \( \bar{\phi}(z) \) for various values of \( \alpha \) in the subsequent evaluation of the exponents.

Figure 3 displays the exponents over a unit time, \( \Delta g|_{\Delta z=1} \), and the approximated exponents over a unit time, \( (\Delta g)_{st}|_{\Delta z=1} \), for various values of \( \alpha \). The exponent \( \Delta g|_{\Delta z=1} \) is estimated from \( \bar{\phi}(z) \) in the region \( 100 \leq z \leq 500 \), in order to decrease the effects of the initial conditions. This estimation is obtained as follows. 1) First, the set \( (z_i, \ln \bar{\phi}(z_i)) \) is determined, where \( z_i \) is the time at which \( \bar{\phi}(z_i) \) is a local maximum and positive. 2) This set is fit with a linear function. The coefficient of the time \( z \) is adopted as \( \Delta g|_{\Delta z=1} \). The exponent \( (\Delta g)_{st}|_{\Delta z=1} \) is calculated in
two ways. One, \( [(\Delta g)_{st}|_{\Delta z=1} \text{ (full)}] \), is obtained using Eq. (2.11), and the other, \( [(\Delta g)_{st}|_{\Delta z=1} \text{ (app. : } n=2)] \), is obtained using Eq. (2.14a) with \( n=2 \). In Figs. 3 (a) and (b), the symbols “×” represent points for \( dg|_{\Delta z=1} \), and the thick curve is a guide to the eye. The dotted curve represents \( (\Delta g)_{st}|_{\Delta z=1} \text{ (full)} \), and the dashed curve represents \( (\Delta g)_{st}|_{\Delta z=1} \text{ (app. : } n=2)] \). It is seen that in both figures, the dashed and dotted curves are quantitatively similar. We find that the exponent \( dg|_{\Delta z=1} \) is close to the exponent \( (\Delta g)_{st}|_{\Delta z=1} \) for small values of \( \alpha \), as in Fig. 3 (a), and the difference between \( dg|_{\Delta z=1} \) and \( (\Delta g)_{st}|_{\Delta z=1} \) increases with \( \alpha \), as in Fig. 3 (b). We thus conclude that the exponent \( dg|_{\Delta z=1} \) can be approximately evaluated with the exponent \( (\Delta g)_{st}|_{\Delta z=1} \) in the region of small values of \( \alpha \). It seems that the numerical results imply that \( dg|_{\Delta z=1} \) is a monotonically increasing function of \( \alpha \). The numerically obtained exponents plotted in Fig. 3 (a) seem to be zero below a certain value of \( \alpha \) which is close to 0.3. However, this is not correct. The \( \alpha \) dependence of the exponents near \( \alpha = 0.3 \) is depicted in the small window in Fig. 3 (a). Apparently, the values of the exponents are not zero. Note that in some cases, the exponents obtained numerically can be negative near \( \alpha = 0.2 \), because these values are derived from the averages of random processes. These values depend on random sequences in the numerical calculation and the region of \( z \) used for the extraction of the exponents. In these figures, the \( \alpha \) dependences of \( (\Delta g)_{st}|_{\Delta z=1} \) and \( dg|_{\Delta z=1} \) are similar. It is inferred that the exponent \( dg|_{\Delta z=1} \) estimated from \( \bar{\phi}(z) \) is equal to or larger than the exponent \( (\Delta g)_{st}|_{\Delta z=1} \) in most cases, because the arithmetic mean of a set of positive real numbers is equal to or larger than their geometric mean. When a field \( \phi_j(z) \) is given by \( \phi_j(z) = \exp(\gamma_j)\phi(0) \) (with the subscript \( j \) indicating the trajectory) and the number of the trajectories is \( M \), the following inequality is satisfied:

\[
|\bar{\phi}(z)| = \frac{1}{M} \sum_{j=1}^{M} \exp(\gamma_j)|\phi(0)| \geq \exp \left( \frac{1}{M} \sum_{j=1}^{M} \gamma_j \right) |\phi(0)|. \tag{3.2}
\]

The exponent of the right-hand side of Eq. (3.2) is the average of the exponents obtained from the trajectories. Assuming that this exponent is equal to the average of the exponents in time [given by Eq. (2.10)], we can conclude that the exponent \( dg|_{\Delta z=1} \) is larger than or equal to the exponent \( (\Delta g)_{st}|_{\Delta z=1} \). We note that rarely in the numerical calculations \( (\Delta g)_{st}|_{\Delta z=1} \) is found to be larger than \( dg|_{\Delta z=1} \). (This happens for small \( \alpha \), because for such values the exponents obtained numerically are negative in some cases.)

§4. Conclusions

We investigated the amplification of a field by white noise. We obtained an expression for the exponent statistically averaged and an expression for the exponent approximated using the method of steepest descent. In addition, the exponents were extracted from the numerical solutions of the stochastic differential equations.

We summarize the results as follows. 1) White noise amplifies the fields, especially for large values of \( \alpha(\vec{k}) \). This fact indicates that the amplification for soft modes
is stronger than that for hard modes when the energy $\omega(\mathbf{k})$ is equal to $\sqrt{m^2 + k^2}$, where $m$ is the mass of the field $\phi$. The amplification decreases with the mass of the amplified field. 2) The expression for the exponent statistically averaged is reliable for estimating the magnitude of the exponent for small values of $\alpha(\overline{k})$. In particular, the expression approximated with the method of steepest descent is easy to use, because this expression contains only well-known functions. These results imply that the soft modes can grow on the vacuum that is located at the bottom of the potential if the coupling between the field and the white noise (i.e. the parameter $D$) is sufficiently strong and if the mass of the field $\phi$ is sufficiently small.

As pointed out in §2.2, the coefficient of $\phi$ in Eq. (2.3) can be negative, due to the influence of noise. The amplification investigated in the present study is caused by the negative coefficient. Similarly, the amplification in spinodal decomposition processes is induced by a negative mass squared. Therefore, soft modes are amplified strongly in both the present process and spinodal decomposition process.

It is conjectured that the stochastic equation used in the present study appears in various fields of physics, for example, in the study of the early universe and chiral symmetry rebreaking. Though it may be possible to have $\alpha(\overline{k}) \leq 1$, the field can be amplified sufficiently if the value of $\alpha(\overline{k})$ is larger than approximately 0.3. [See Fig. 3(a).] The parameter $\alpha(\overline{k})$ decreases as a function of the mass $m$ if the energy $\omega(\overline{k})$ is equal to $\sqrt{m^2 + k^2}$. For this reason, the amplification is strong for shallow potentials.

In the present study, we treated a linear equation with white noise. However, in the more general case, nonlinear terms and colored noise appear. We would like to treat the effects of these terms in future studies.

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