SHORT COMMUNICATION

Stability analysis of PE systems via Steklov’s averaging technique

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SUMMARY

The paper presents a novel approach to justification of asymptotic stability of linear time-varying systems with persistently excited right-hand side. This approach combines the direct Lyapunov method with the Steklov averaging technique; its distinguishing feature is closed-form construction of the Lyapunov functional, along with resultant explicit estimates of the rate of convergency. © 2016 The Authors. International Journal of Adaptive Control and Signal Processing published by John Wiley & Sons, Ltd.

1. INTRODUCTION

It is well known that asymptotic stability of a linear time-varying system with persistently excited (PE) right hand side plays an important role in establishing robust properties of adaptive and identification techniques. This stability has been established a long time ago; see, for example, [1–4]. The theory of converse Lyapunov functions guarantees existence of a Lyapunov function in many situations where stability does hold; however, this theory often offers only an implicit, hardly tractable, and abstract function of such a kind. Meanwhile by itself, a closed form of the strict Lyapunov functional brings substantial benefits in engineering applications [5]; for example, with such a functional at hand, many robustness and stabilization issues can be explicitly treated via standard feedback designs or robustness arguments. To the best knowledge of the authors, no explicit form of the Lyapunov functional that proves stability of the aforementioned PE systems is currently available. This Technical Note is aimed at filling this gap via presenting a novel technique based on the Steklov’s averaging functions and ideas borrowed from recent works on estimation of topological entropy of smooth dynamical systems [6]. This technique has already demonstrated its potential for other stability problems; see, for example [7, 8].

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The discussed technique is based upon replacement of a concerned function \( r(\cdot) \) by its Steklov average with a certain time step \( T > 0 \):

\[
r^T(t) = \frac{1}{T} \int_t^{t+T} r(\tau) d\tau.
\]

The following lemma discloses a useful property of this average, which underlies the main result of the paper, it is reminiscent of some results in \[9\], p. 541 and \[10\] (Lemma 5.4.1 and Eq. (5.4.4)), and is presented for the convenience of the reader.

**Lemma 2.1**

Let \( \Delta \subset \mathbb{R} \) be an interval, and let \( T > 0 \) and \( t_0 \in \Delta_T := \{ t \in \Delta : t + T \in \Delta \} \) be given. For any measurable function \( p(\cdot) : \Delta \to \mathbb{R} \) such that \( |p(\tau)| \leq M \) for almost all \( \tau \in \Delta \), and any \( t \in \Delta_T \), the following inequality holds

\[
|\sigma(t, T)| \leq MT, \quad \text{where} \quad \sigma(t, T) := \int_{t_0}^{t} \left( \frac{1}{T} \int_{t}^{t+T} p(s) ds - p(\tau) \right) d\tau.
\]

**Proof**

Because interchange of \( t \) and \( t_0 \) does not alter the left-hand side of the inequality from (1), it can be assumed without any loss of generality that \( t \geq t_0 \). We observe that

\[
\int_{t_0}^{t} d\tau \int_{t}^{t+T} p(s) ds = \int_{t_0}^{t+T} \left[ \min\{t, s\} - \max\{t_0, s - T\} \right] p(s) ds
\]

\[
= \int_{t_0}^{t} \left[ s - \max\{t_0, s - T\} \right] p(s) ds + \int_{t}^{t+T} \left[ t - \max\{t_0, s - T\} \right] p(s) ds
\]

\[
= \int_{t_0}^{\min\{t, t_0 + T\}} \left[ s - t_0 \right] p(s) ds + \int_{t}^{\min\{t, t_0 + T\}} \left[ t - \max\{t_0, s - T\} \right] p(s) ds
\]

\[
+ \int_{t}^{t+T} \left[ t - \max\{t_0, s - T\} \right] p(s) ds.
\]

Hence

\[
\sigma(t, T) = \int_{t_0}^{\min\{t, t_0 + T\}} \frac{s - t_0}{T} p(s) ds + \int_{\min\{t, t_0 + T\}}^{t} \frac{p(s)}{T} ds
\]

\[
+ \int_{t}^{t+T} \frac{t - \max\{t_0, s - T\}}{T} p(s) ds - \int_{t_0}^{t} \frac{p(s)}{T} ds
\]

\[
= \int_{t_0}^{\min\{t, t_0 + T\}} \frac{s - t_0 - T}{T} p(s) ds + \int_{t}^{t+T} \frac{t - \max\{t_0, s - T\}}{T} p(s) ds.
\]

Finally

\[
|\sigma(t, T)| \leq \int_{t_0}^{\min\{t, t_0 + T\}} \frac{t_0 + T - s}{T} |p(s)| ds + \int_{t}^{t+T} \frac{t - \max\{t_0, s - T\}}{T} |p(s)| ds
\]

\[
\leq \frac{M}{T} \left[ \int_{t_0}^{\min\{t, t_0 + T\}} (t_0 + T - s) ds + \int_{t}^{t+T} (t - \max\{t_0, s - T\}) ds \right]
\]

\[
\leq \frac{M}{T} \left[ \int_{\max\{0, t_0 + T - t\}}^{T} \theta d\theta + \int_{t}^{t+T} \min\{t_0, t + T - s\} ds \right]
\]

\[
= \frac{M}{T} \left[ \int_{\max\{0, t_0 + T - t\}}^{T} \theta d\theta + \int_{0}^{T} \min\{t - t_0, \theta\} d\theta \right] \leq \frac{2M}{T} \int_{0}^{T} \theta d\theta = MT.
\]

Whereas “forward averaging” \( \frac{1}{T} \int_{t}^{t+T} \ldots ds \) is employed in (1), a similar estimate is true for the “backward averaging” as well, as is shown by the following.
Corollary 2.2
Under the assumptions of Lemma 2.1, any $t_T^0, t \in \{ t \in \Delta : t - t \in \Delta \}$ give rise to the estimate

$$|\sigma^-(t, T)| \leq MT,$$

where $\sigma^-(t, T) := \int_{t_0}^t \left( \frac{1}{T} \int_{t - T}^t p(s) ds - p(t) \right) d\tau.$

Proof
Applying Lemma 2.1 to the function $s \in -\Delta := \{ s : -s \in \Delta \} \mapsto p(-s)$ and $t_0 := -t_0^-, t := -t$ yields the estimate $|\sigma(-t, T)| \leq MT$. It remains to note that

$$\sigma(-t, T) = \int_{-t_0}^{-t} \left[ \frac{1}{T} \int_{t - T}^{t + T} p(-s) ds - p(-t) \right] d\tau = \int_{-t_0}^{-t} \left[ \frac{1}{T} \int_{t - T}^{t + T} p(-s) ds - p(\theta) \right] d\theta \bigg|_{\theta = -t} = -\sigma^-(t, T).$$

\qed

In the sequel, we demonstrate that the Steklov’s averaging technique may aid in stability analysis of PE systems, provided that it is properly combined with the concept of averaging function [6].

3. MAIN RESULT
We consider the following LTV system

$$\dot{x} = -p(t)p(t)^T x, \quad x \in \mathbb{R}^n, \quad p(t) \in \mathbb{R}^{n \times m}, \quad t \geq t_0,$$

where the matrix-valued function $p$ has the following properties:

i) **Boundedness:** there exists a constant $M \geq 0$ such that

$$\|p(t)\|^2 \leq M$$

for all $t \geq t_0$.

ii) **Persistency of excitation:** There are constants $\lambda > 0$ and $T > 0$ such that

$$\inf_{t \geq t_0} \lambda(t) \geq \lambda,$$

where $\lambda(t)$ is the smallest eigenvalue of the matrix

$$\tilde{P}(t) := \frac{1}{T} \int_{t}^{t + T} p(s)p(s)^T ds.$$
with

\[ v(s) := \frac{x(s)^T p(s)p(s)^T x(s)}{x(s)^T x(s)}. \]  

(8)

**Remark 3.1**

(i) The function of time \( w(\cdot) \) also depends on a whole piece of the trajectory \( x(\tau), \tau \in [t_0, t + T] \). Hence, it should be viewed as a function of time and a function, that is, a functional. So the candidate (6) is also a functional.

(ii) The expressions (7) and (6) are well-defined only for the solutions that vanish nowhere. The functional (6) is extended by continuity on the zero solution by putting \( V := 0 \).

Now, we show that (6) is really a Lyapunov functional.

**Theorem 3.2**

Suppose that (i) and (ii) hold. Along any solution of (3), the functional (6) is differentiable with respect to time, and the following inequalities are true at any time \( t \geq t_0 \)

\[ k_\pm \|x(t)\|^2 \leq V \leq k_\pm \|x(t)\|^2, \quad \text{where} \quad k_\pm := e^{\pm 2MT} > 0, \]

\[ \dot{V} \leq -k_\pm, \quad \text{where} \quad x := \frac{2\lambda}{(1 + MT)^2} > 0. \]  

(9)

**Remark 3.2**

By abstracting from the specific expressions for \( k_\pm, \lambda \), the conclusion of this theorem comes to a common set of requirements; see for example [13–18] whose satisfaction means that when following the line of the second Lyapunov method, the functional \( V \) can be used instead of a function so that the business basically remains as usual. In particular, this concerns the proof of the exponential stability of (3).

**Proof of Theorem 3.2**

Because the claims are clearly true for the zero solution, we focus on the case where \( x(t) \neq 0 \ \forall t \). The smoothness of \( V \) along this solution and the inequalities in the first row from (9) are immediate from (6)–(7) and Lemma 2.1. It remains to prove the inequality in the second row from (9). Direct differentiation of (6) yields \( \dot{V} = -2v(t)V \), where

\[ \tilde{v}(t) := \frac{1}{T} \int_t^{t+T} v(s)ds. \]  

(10)

Hence, it suffices to show that

\[ \tilde{v}(t) \geq \frac{\lambda}{(1 + MT)^2} \quad \forall t \geq t_0. \]  

(11)

To this end, we first notice that

\[ \frac{d}{dt} x(t)^T x(t) = -2x(t)^T p(t)p(t)^T x(t) = -2\|p(t)^T x(t)\|^2 \leq 0, \]

that is, \( x(t)^T x(t) \) is a non-increasing function of time. So by (8), \( v(s) \geq \frac{x(s)^T p(s)p(s)^T x(s)}{x(t)^T x(t)} \) for \( s \geq t \) and

\[ \tilde{v}(t) \geq \frac{1}{T x(t)^T x(t)} \int_t^{t+T} x(s)^T p(s)p(s)^T x(s)ds. \]  

(12)
Our next step is estimation of the integrand
\[ q(s) := x(s)^T p(s) p(s)^T x(s) = \| p(s)^T x(s) \|^2. \] (13)

In doing so, we use the following elementary inequality
\[ \| \xi + \zeta \|^2 \geq \tau \| \xi \|^2 - \frac{\tau}{1-\tau} \| \zeta \|^2, \quad \xi, \zeta \in \mathbb{R}^m, \tau \in (0, 1), \]
which means that the quadratic form \( \| \xi + \zeta \|^2 - \tau \| \xi \|^2 + \frac{\tau}{1-\tau} \| \zeta \|^2 \) in \( \xi, \zeta \) is positive semi-definite, and may be checked via respective standard tests. By putting
\[ \xi = p(s)^T x(t), \quad \zeta = p(s)^T (x(s) - x(t)), \]
we see that \( q(s) = \| \xi + \zeta \|^2 \) and
\[
q(s) \geq \tau \| p(s)^T x(t) \|^2 - \frac{\tau}{1-\tau} \left( \int_t^s \| p(s) \| \| p(s)^T x(s) \| \| p(s)^T x(s) \| d\zeta \right)^2 \\
\geq \tau \| p(s)^T x(t) \|^2 - \frac{\tau}{1-\tau} \left( \int_t^s \| p(s) \| \| p(s)^T x(s) \| \| p(s)^T x(s) \| d\zeta \right)^2 \\
\geq \tau \frac{\| x(t) \|^2 p(s) p(s)^T x(t) - M^2 \tau (s-t) \int_t^s \| p(s) \|^2 d\zeta \} \int_t^s \| p(s)^T x(s) \|^2 d\zeta \}
\]

Here, (a) is based on the inequalities \( \int g(\zeta) d\zeta \leq \int \| g(\zeta) \| d\zeta \) and \( \| M N \eta \| \leq \| M \| \| N \| \| \eta \| \) \( \forall N \in \mathbb{R}^{m \times n}, M \in \mathbb{R}^{n \times m}, \eta \in \mathbb{R}^m; \) (b) is true by the Cauchy–Schwarz inequality; and (c) holds thanks to (4) and (13). By putting (14) into (12), we get
\[
\tilde{v}(t) \geq \frac{\tau}{x(t)^T x(t)} x(t)^T \left[ \frac{1}{T} \int_t^{t+T} p(s) p(s)^T ds \right] x(t)
\[
- \frac{T^{-1} M^2 \tau}{(1-\tau) x(t)^T x(t)} \int_t^{t+T} (s-t) \int_t^s q(s) d\zeta ds \\
\geq \tau \tilde{\lambda} - \frac{M^2 \tau}{(1-\tau) x(t)^T x(t)} \int_t^{t+T} q(s) d\zeta ds = \tau \tilde{\lambda}
\[
- \frac{M^2 \tau}{(1-\tau) x(t)^T x(t)} \int_t^{t+T} q(s) d\zeta ds \\
\geq \tau \tilde{\lambda} - \frac{M^2 T \tau}{(1-\tau) x(t)^T x(t)} \int_t^{t+T} q(s) ds \geq \tau \tilde{\lambda} - \frac{M^2 T^2 \tau}{(1-\tau)} \tilde{v}(t). \]

Here, (a) holds due to (5) and the estimate \( s-t \leq T \); (b) is true because \( \zeta \in [t, t+T] \Rightarrow \int_t^{t+T} ds \leq T \). It follows that
\[
\tilde{v}(t) \geq \frac{\frac{\tau}{x(t)^T x(t)} x(t)^T \left[ \frac{1}{T} \int_t^{t+T} p(s) p(s)^T ds \right] x(t)}{1-\tau + M^2 T^2 \frac{\tau}{x(t)^T x(t)}} \tilde{\lambda}.
\]

By taking \( \tau = (1 + MT)^{-1} \in (0, 1) \), we arrive at (11) via elementary computation.
In (7), the use of the average from the current time $t$ to $t + T$ permits us to define the Lyapunov functional candidate $V$ on the entire time domain $t \geq t_0$ where the system (3) is defined. This brings a certain technical convenience; however, the price is that the functional is not causal (non-anticipating). Meanwhile, so far as analysis of the asymptotic ($t \to \infty$) system’s behavior is concerned, many available techniques are far from being firm in the need for the “complete” domain of definition $t \geq t_0$. It often suffices that the Lyapunov functional is defined only for large enough $t$, whereas the respective finite “gap” may be “filled”, if necessary, via analysis on a finite time interval, where stability comes to the routine property of regular dependence of the system’s trajectory on the initial data and parameters. By reducing the domain of definition from $[t_0, +\infty)$ to $[t_0 + T, +\infty)$, a causal (non-anticipating) Lyapunov functional can be easily offered, as is shown in the following.

**Corollary 3.4**

Theorem 3.2 remains true for the functional given by (6), where $t \geq t_0 + T$ and

$$w(t) = 2 \int_{t_0}^{t} \left( v(s) - \frac{1}{T} \int_{s-T}^{s} v(\tau)d\tau \right) ds \quad t \geq t_0 + T,$$

provided that (9) is considered for only $t \geq t_0 + T$.

**Proof**

The proof is via retracing the arguments from the first paragraph in the proof of Theorem 3.2, where $t \geq t_0 + T$, the function (10) is replaced by

$$v(t) := \frac{1}{T} \int_{t-T}^{t} v(s)ds = \bar{v}(t-T),$$

and the key property $v(t) \geq \frac{\bar{v}}{(1+MT)^2}$ results from putting $t := t - T$ in (11).

**ACKNOWLEDGEMENTS**

The first author acknowledges partial financial support from the Government of the Russian Federation, Grant 074-U01, and by the Ministry of Education and Science of the Russian Federation (Project 14.Z50.31.0031), (Sec. II) during his stay with the ITMO University. The second author was supported by RSF grant 14-21-00041 and the Saint-Petersburg University (Sec. I).

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