Properties of a polyanalytic functional calculus on the $S$-spectrum

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Abstract
The Fueter mapping theorem gives a constructive way to extend holomorphic functions of one complex variable to monogenic functions, that is, null solutions of the generalized Cauchy–Riemann operator in $\mathbb{R}^4$, denoted by $D$. This theorem is divided in two steps. In the first step, a holomorphic function is extended to a slice hyperholomorphic function. The Cauchy formula for this type of functions is the starting point of the $S$-functional calculus. In the second step, a monogenic function is obtained by applying the Laplace operator in four real variables, namely, $\Delta$, to a slice hyperholomorphic function. The polyanalytic functional calculus, that we study in this paper, is based on the factorization of $\Delta = D\overline{D}$. Instead of applying directly the Laplace operator to a slice hyperholomorphic function, we apply first the operator $D$ and we get a polyanalytic function of order 2, that is, a function that belongs to the kernel of $D^2$. We can represent this type of functions in an integral form and then we can define the polyanalytic functional calculus on $S$-spectrum. The main goal of this paper is to show the principal properties of this functional calculus. In particular, we study a resolvent equation suitable for proving a product rule and generate the Riesz projectors.

KEYWORDS
$F$-functional calculus, $P_2$-functional calculus, polyanalytic functions, $Q$-functional calculus, resolvent equation, Riesz projectors, $S$-spectrum

MSC (2020)
Primary: 47A10, 47A60

1 | INTRODUCTION

The polyanalytic functional calculus of order 2 on the $S$-spectrum was introduced in [26]. Similarly as the harmonic functional calculus (see [15]), it can be seen as an intermediate functional calculus between the $S$-functional calculus and the $F$-functional calculus (see [18]).

To better understand the nature of these functional calculi, we recall that the spaces of functions over which they are defined, are strictly related to the Fueter–Scemapping theorem. This theorem is based on a two-step procedure: The first step extends holomorphic functions of one complex variable to slice hyperholomorphic functions, the second one gives a monogenic function by applying the Laplace operator in four real variables to a slice hyperholomorphic function (see
The Fueter construction can be visualized as follows:

\[
\mathcal{O}(U) \xrightarrow{T_F} SH(\Omega_D) \xrightarrow{\Delta} AM(\Omega_D),
\]

where \(\mathcal{O}(D)\) is the set of holomorphic functions on \(D\), \(SH(\Omega_D)\) is the set of slice hyperholomorphic functions induced on \(\Omega_D\), \(AM(\Omega_D)\) is the set of axially monogenic functions on \(\Omega_D\), \(T_F\) denotes the first linear operator of the Fueter construction, called slice operator. Associated to the spaces of functions \(SH(\Omega_D)\) and \(AM(\Omega_D)\), there are two functional calculi. The first one is the \(S\)-functional calculus and it is based on the Cauchy integral formula for slice hyperholomorphic functions (see Theorem 2.13). By applying the Laplace operator to the slice hyperholomorphic Cauchy formula, we obtain the integral representation of an axially monogenic function. This is the so-called Fueter theorem in integral form (see Theorem 2.16). The \(F\)-functional calculus is based on this theorem. The following diagram illustrates these relations:

\[
\begin{align*}
\text{Slice Cauchy Formula} & \quad \xrightarrow{T_F=\Delta} \quad \text{Fueter theorem in integral form} \\
\mathcal{S} - \text{Functional calculus} & \quad \downarrow \quad \mathcal{F} - \text{Functional calculus}
\end{align*}
\]

To proceed further, we fix the notations. We define the quaternions as follows:

\[
\mathbb{H} = \{ q = q_0 + q_1e_1 + q_2e_2 + q_3e_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R} \},
\]

where the imaginary units satisfy the relations

\[
e_1^2 = e_2^2 = e_3^2 = -1 \quad \text{and} \quad e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1, \quad e_3e_1 = -e_1e_3 = e_2.
\]

Given \(q \in \mathbb{H}\), we call \(\text{Re}(q) : = q_0\) the real part of \(q\) and \(\text{Im}(q) = q_1e_1 + q_2e_2 + q_3e_3\) the imaginary part. The modulus of \(q \in \mathbb{H}\) is given by \(|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}\), the conjugate of \(q\) is defined by \(\overline{q} = q_0 - q\) and we have \(|q| = \sqrt{q\overline{q}}\).

We recall that the Fueter operator \(D\) and its conjugate \(\overline{D}\) are defined as follows:

\[
D : = \partial_{q_0} + \sum_{i=1}^{3} e_i \partial_{q_i} \quad \text{and} \quad \overline{D} : = \partial_{q_0} - \sum_{i=1}^{3} e_i \partial_{q_i}.
\]

The Laplace operator \(\Delta\) can be factorized by the Fueter operator \(D\) and its conjugate \(\overline{D}\): \(T_F^2 = \Delta = DD = \overline{D}D\). As a consequence of the Fueter mapping theorem when we apply \(D\) to a slice hyperholomorphic function, we obtain an axially harmonic function, whereas, when we apply \(\overline{D}\) to a slice hyperholomorphic function, we obtain an axially polyanalytic function of order 2. Thus, we have the following diagrams:

\[
\begin{align*}
\mathcal{O}(D) & \xrightarrow{T_F} SH(\Omega_D) \quad \xrightarrow{\overline{D}} \quad AH(\Omega_D) \quad \xrightarrow{D} \quad AM(\Omega_D), \\
\mathcal{O}(D) & \xrightarrow{T_F} SH(\Omega_D) \quad \xrightarrow{\overline{D}} \quad AP_2(\Omega_D) \quad \xrightarrow{D} \quad AM(\Omega_D),
\end{align*}
\]

where \(AH(\Omega_D)\) is the set of axially harmonic functions and \(AP_2(\Omega_D)\) is the set of axially polyanalytic functions of order 2. Basically, diagrams (1.2) and (1.3) are factorizations of diagram (1.1). These diagrams lead to the definition of fine structure.

**Definition 1.1.** We call fine structure of the spectral theory on the \(S\)-spectrum the set of function spaces and the associated functional calculi induced by a factorization of the operator \(T_{F2}\), in the Fueter extension theorem.

In the Clifford algebra setting, diagrams (1.2) and (1.3) are much more involved, see [16]. This is due to fact that the map \(T_{FS2}\) becomes the so-called Fueter–Sce map \(T_{FS2} = \Delta^{\frac{2}{n+1}}\), where \(n\) is an odd number, see [24].
In the literature, the theory of polyanalytic functions plays an important role for studying some elasticity problems, see \cite{30,31}, and investigating some aspects of the time frequency analysis, see, for instance, \cite{1,2,25}. Recently the polyanalytic functions have been studied in the slice hyperholomorphic setting, see \cite{6,7}, and also a Fueter theorem has been considered in \cite{8}. Moreover, a slice polyanalytic functional calculus has been considered in \cite{5}. For further information about the polyanalytic function theory, see \cite{9}, for more applications about this theory, see \cite{3}.

As for the axially monogenic functions, it is possible to obtain an integral representation of the axially harmonic functions and axially polyanalytic functions of order 2. Based on these integral representations, we define the corresponding functional calculi: the harmonic functional calculus ($Q$-functional calculus) and the polyanalytic functional calculus of order 2 ($P_2$-functional calculus). It is possible to visualize these relations with the following diagrams:

\begin{align*}
SH(U) & \quad AH(U) & \quad AM(U) \\
\downarrow & & \downarrow \\
\text{Slice Cauchy Formula} & \xrightarrow{D} & \text{AH in integral form} & \xrightarrow{F} & \text{Fueter theorem in integral form} & \downarrow \\
\downarrow & & \downarrow & & \downarrow \\
S - \text{functional calculus} & & Q - \text{functional calculus} & & F - \text{functional calculus}
\end{align*}

and

\begin{align*}
\text{SH(U)} & \quad AP_2(U) & \quad AM(U) \\
\downarrow & & \downarrow \\
\text{Slice Cauchy Formula} & \xrightarrow{B} & \text{AP}_2 \text{ in integral form} & \xrightarrow{D} & \text{Fueter theorem in integral form} & \downarrow \\
\downarrow & & \downarrow & & \downarrow \\
S - \text{functional calculus} & & P_2 - \text{functional calculus} & & F - \text{functional calculus}
\end{align*}

(1.4)

Once we have proved good definitions of these functional calculi, a natural field of investigation is the determination of their main properties which are the algebraic properties, the resolvent equation, the Riesz projectors, and the product rule. For further information about the $S$-functional calculus, see \cite{18,23}, whereas for the $F$-functional calculus, see \cite{14,18,20}. The properties of the $Q$-functional calculus are studied in \cite{15}.

The goal of this paper is to investigate the properties of the $P_2$-functional calculus.

### 1.1 Outline of the paper

The paper consists of six sections, the first one being this introduction. In Section 2, we give some basic notions of the $S$-functional calculus, the $F$-functional calculus, the $Q$-functional calculus, and the underlying theory of the slice hyperholomorphic functions. In Section 3, we recall the definition of the $P_2$-functional calculus and we prove some algebraic properties. In Section 4, we show a resolvent equation for the $P_2$-functional calculus, by this fundamental tool, we prove the product rule. In this section, a product rule for the $F$-functional calculus based on the $P_2$-functional calculus and the $Q$-functional calculus is also proved. In Section 5, we study the Riesz projectors for the $P_2$-functional calculus. Finally, in Section 6, we prove a different version of the product rule for the $Q$-functional calculus, based on a new resolvent equation.

### 2 PRELIMINARIES RESULTS ON FUNCTIONS AND OPERATORS

In this section, we recall some basic notions about the slice hyperholomorphic functions. Moreover, we give the definitions and the properties of all the functional calculi that we need: $S$-functional calculus, $F$-functional calculus, and $Q$-functional calculus.
2.1 | Hyperholomorphic functions

Before to introduce the definition of slice hyperholomorphic function, we need some preliminary notations.

Let us denote by $\mathbb{S}$ the unit sphere of purely imaginary quaternions:

$$\mathbb{S} = \{ q = q_1 e_1 + q_2 e_2 + q_3 e_3 \mid q_1^2 + q_2^2 + q_3^2 = 1 \}. $$

Notice that if $J \in \mathbb{S}$, then $J^2 = -1$. Therefore, $J$ is an imaginary unit, and we denote by

$$\mathbb{C}_J = \{ u + J v \mid u, v \in \mathbb{R} \},$$

an isomorphic copy of the complex numbers. Given a nonreal quaternion $q = q_0 + q = q_0 + J q_1$, we set $J_q = q/|q| \in \mathbb{S}$ and we associate to $q$ the $2$-sphere defined by

$$[q] := \{ q_0 + J |q| \mid J \in \mathbb{S} \}. $$

**Definition 2.1.** Let $U \subseteq \mathbb{H}$.

1. We say that $U$ is axially symmetric if, for every $u + Jv \in U$, all the elements $u + Jv$ for $J \in \mathbb{S}$ are contained in $U$.
2. We say that $U$ is a slice domain if $U \cap \mathbb{R} \neq \emptyset$ and if $U \cap \mathbb{C}_J$ is a domain in $\mathbb{C}_J$ for every $J \in \mathbb{S}$.

**Definition 2.2.** An axially symmetric open set $U \subset \mathbb{H}$ is called slice Cauchy domain if $U \cap \mathbb{C}_J$ is a Cauchy domain in $\mathbb{C}_J$ for every $J \in \mathbb{S}$. More precisely, $U$ is a slice Cauchy domain if, for every $J \in \mathbb{S}$, the boundary of $U \cap \mathbb{C}_J$ is the union of a finite number of nonintersecting piecewise continuously differentiable Jordan curves in $\mathbb{C}_J$.

The axially symmetric open sets are the suitable domains for the slice hyperholomorphic functions.

**Definition 2.3 (Slice hyperholomorphic functions).** Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and let

$$U^\prime = \{(u,v) \in \mathbb{R}^2 \mid u + \mathbb{S} v \in U\}.$$

We say that a function $f : U \rightarrow \mathbb{H}$ of the form

$$f(q) = \alpha(u,v) + J\beta(u,v)$$

is left slice hyperholomorphic if $\alpha$ and $\beta$ are $\mathbb{H}$-valued differentiable functions such that

$$\alpha(u,v) = \alpha(u,-v), \quad \beta(u,v) = -\beta(u,-v) \quad \text{for all } (u,v) \in U^\prime,$$

and if $\alpha$ and $\beta$ satisfy the Cauchy–Riemann system

$$\partial_u \alpha(u,v) - \partial_v \beta(u,v) = 0, \quad \partial_v \alpha(u,v) + \partial_u \beta(u,v) = 0.$$

A function is right slice hyperholomorphic if it is of the form

$$f(q) = \alpha(u,v) + J\beta(u,v),$$

where $\alpha, \beta$ satisfy the above conditions.

The set of left (resp. right) slice hyperholomorphic functions on $U$ is denoted by the symbol $SH_L(U)$ (resp. $SH_R(U)$). The subset of intrinsic slice hyperholomorphic functions consists of those slice hyperholomorphic functions such that $\alpha, \beta$ are real-valued functions, and is denoted by $N(U)$.

Another class of hyperholomorphic functions that appears in the Fueter construction is the following.
**Definition 2.4** (Fueter regular functions). Let \( U \subset \mathbb{H} \) be an open set. A real differentiable function \( f : U \to \mathbb{H} \) is called (left) Fueter regular if

\[
Df(q) := \partial_{q_0}f(q) + \sum_{i=1}^{3} e_i \partial_{q_i}f(q) = 0.
\]

A bridge between the slice hyperholomorphic and the monogenic functions is the Fueter theorem, see [24, 28].

**Theorem 2.5** (Fueter mapping theorem). Let \( f_0(z) = \alpha(u, v) + i\beta(u, v) \) be a holomorphic function defined in a domain (open and connected) \( D \) in the upper-half complex plane and let

\[
\Omega_D = \{ q = q_0 + q | (q_0, |q|) \in D \}
\]

be the open set induced by \( D \) in \( \mathbb{H} \). Then, the operator \( T_{F1} \) defined by

\[
f(q) = T_{F1}(f_0) := \alpha(q_0, |q|) + \frac{q}{|q|} \beta(q_0, |q|)
\]

maps the set of holomorphic functions in the set of intrinsic slice hyperholomorphic functions. Moreover, the function

\[
f(q) := T_{F2} \left( \alpha(q_0, |q|) + \frac{q}{|q|} \beta(q_0, |q|) \right),
\]

where \( T_{F2} = \Delta \) and \( \Delta \) is the Laplacian in four real variables \( q_\ell, \ell = 0, 1, 2, 3, \) is in the kernel of the Fueter operator, that is,

\[
Df = 0 \quad \text{on} \quad \Omega_D.
\]

We will consider also polyanalytic Fueter regular functions, see [11].

**Definition 2.6.** Let \( U \subset \mathbb{H} \) be an open set and let \( f : U \to \mathbb{H} \) be a function of class \( C^n(U) \), with \( n \geq 1 \). We say that \( f \) is (left) polyanalytic Fueter regular of order \( n \) on \( U \) if

\[
D^n f(q) = \left( \partial_{q_0} + \sum_{i=1}^{3} e_i \partial_{q_i} \right)^n f(q) = 0.
\]

For this kind of functions, it is possible to give the following characterization, see [11, 12].

**Proposition 2.7.** A function \( f \) is (left) polyanalytic Fueter regular of order \( n \) if and only if it can be decomposed in terms of unique (left) Fueter regular functions \( \phi_0(q), \ldots, \phi_{n-1}(q) \) such that we have

\[
f(q) = \sum_{k=0}^{n-1} x_0^k \phi_k(q).
\]

Now, let us consider \( D \) be a domain in the upper-half complex plane. Let \( \Omega_D \) be an axially symmetric open set in \( \mathbb{R}^4 \) and let \( x = x_0 + \xi = x_0 + r\omega \in \Omega_D \). We say that a function \( f : \Omega_D \to \mathbb{H} \) is of axial type if there exist two quaternionic-valued functions \( A(q_0, r) \) and \( B(q_0, r) \) independent of \( \omega \in \mathbb{S} \) such that

\[
f(q) = A(q_0, r) + \omega B(q_0, r), \quad r > 0.
\]

**Definition 2.8** (axially monogenic function). Let \( f : \Omega_D \subset \mathbb{R}^4 \to \mathbb{H} \) be of axial type and of class \( C^3(\Omega_D) \). Then, the function

\[
\hat{f}(q) := \Delta f(q) \quad \text{on} \quad \Omega_D
\]
is called axially monogenic, since by the Fueter theorem mapping theorem, it satisfies

\[ D\tilde{f}(q) = 0 \quad \text{on} \quad \Omega_D. \]

We denote this set of functions \( \mathcal{AM}(\Omega_D) \).

**Definition 2.9** (axially polyanalytic function of order 2). Let \( f: \Omega_D \subset \mathbb{R}^4 \to \mathbb{H} \) be of axial type and of class \( C^3(\Omega_D) \). If we apply the conjugate Fueter operator \( \overline{D} \) to (2.1), we get

\[ \tilde{f}^0(q) = \overline{D}f(x) \quad \text{on} \quad \Omega_D, \]

which is an axially polyanalytic function of order 2, by the Fueter mapping theorem, that is,

\[ D^2\tilde{f}^0(x) = 0 \quad \text{on} \quad \Omega_D. \]

We denote this set of functions by \( \mathcal{AP}_2(\Omega_D) \).

**Definition 2.10** (Axially harmonic functions). If we apply the Fueter operator to an axial function \( f \) of class \( C^3(\Omega_D) \), we get

\[ \tilde{f}(q) := Df(q) \quad \text{on} \quad \Omega_D, \]

which is an axially harmonic function by the Fueter mapping theorem, that is,

\[ \Delta\tilde{f}(q) = 0 \quad \text{on} \quad \Omega_D. \]

We denote this set of functions by \( \mathcal{AH}(\Omega_D) \).

Now we introduce the slice hyperholomorphic Cauchy kernels.

**Definition 2.11.** Let \( s, q \in \mathbb{H} \) with \( q \notin [s] \), then we define

\[ Q_s(q)^{-1} := (q^2 - 2\text{Re}(s)q + |s|^2)^{-1}, \quad Q_{c,s}(q)^{-1} := (s^2 - 2\text{Re}(q)s + |q|^2)^{-1}, \]

that are called pseudo Cauchy kernel and commutative pseudo Cauchy kernel, respectively.

**Definition 2.12.** Let \( s, q \in \mathbb{H} \) with \( q \notin [s] \), then

1. we say that the left slice hyperholomorphic Cauchy kernel \( S_L^{-1}(s,q) \) is written in form I if

\[ S_L^{-1}(s,q) := Q_s(q)^{-1}(\overline{s} - q), \]

2. we say that the right slice hyperholomorphic Cauchy kernel \( S_R^{-1}(s,q) \) is written in form I if

\[ S_R^{-1}(s,q) := (s - q)Q_s(q)^{-1}, \]

3. we say that the left slice hyperholomorphic Cauchy kernel \( S_L^{-1}(s,q) \) is written in form II if

\[ S_L^{-1}(s,q) := (s - \overline{q})Q_{c,s}(q)^{-1}, \]

4. we say that the right slice hyperholomorphic Cauchy kernel \( S_R^{-1}(s,q) \) is written in form II if

\[ S_R^{-1}(s,q) := Q_{c,s}(q)^{-1}(s - \overline{q}). \]
It is possible to prove that the left (resp. the right) slice Cauchy kernel is left (resp. right) slice hyperholomorphic in \( q \) and right (resp. left) slice hyperholomorphic in \( s \) (see [18, Lemma 2.1.27]). In this paper, unless otherwise specified, we refer to \( S_L^{-1}(s, q) \) and \( S_R^{-1}(s, q) \) written in form II.

We can state the Cauchy formulas for the slice hyperholomorphic functions.

**Theorem 2.13** (The Cauchy formulas for slice hyperholomorphic functions). Let \( U \subset \mathbb{H} \) be a bounded slice Cauchy domain, let \( J \in \mathbb{S} \), and set \( ds_J = ds(-J) \). If \( f \) is a left slice hyperholomorphic function on a set that contains \( \overline{U} \), then

\[
f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}J)} S_L^{-1}(s, q) \, ds_J \, f(s), \quad \text{for any } q \in U. \tag{2.2}\]

If \( f \) is a right slice hyperholomorphic function on a set that contains \( \overline{U} \), then

\[
f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}J)} f(s) \, ds_J \, S_R^{-1}(s, q), \quad \text{for any } q \in U. \tag{2.3}\]

These integrals depend neither on \( U \) nor on the imaginary unit \( J \in \mathbb{S} \).

Moreover, for slice hyperholomorphic functions, hold a version of the Cauchy integral theorem (see [18, Theorem 2.1.21]).

**Theorem 2.14** (Cauchy integral Theorem). Let \( U \subset \mathbb{H} \) be open, let \( J \in \mathbb{S} \), and let \( f \in \mathcal{S}^{L}(U) \) and \( g \in \mathcal{S}^{R}(U) \). Moreover, let \( D_J \subset U \cap \mathbb{C}J \) be an open and bounded subset of the complex plane \( \mathbb{C}J \) with \( \partial D_J \subset U \cap \mathbb{C}J \) such that \( \partial D_J \) is a finite union of piecewise continuously differentiable Jordan curves. Then,

\[
\int_{\partial D_J} g(s)ds_J f(s) = 0,
\]

where \( ds_J = ds(-J) \).

By applying the Fueter map, namely, the Laplace operator \( \Delta \), to the second form of the slice Cauchy kernel, we get the so called \( F \)-kernels [13, 21].

**Definition 2.15.** Let \( q, s \in \mathbb{H} \). We define for \( s \not\in [q] \), the left \( F \)-kernel as

\[
F_L(s, q) := \Delta S_L^{-1}(s, q) = -4(s - q)Q_{c,s}(q)^{-2}, \tag{2.4}\]

and the right \( F \)-kernel as

\[
F_R(s, q) := \Delta S_R^{-1}(s, q) = -4Q_{c,s}(q)^{-2}(s - q). \tag{2.5}\]

It is possible to prove that \( F_L(s, q) \) (resp. \( F_R(s, q) \)) is left (resp. right) Fueter regular in \( q \) and right (resp. left) slice hyperholomorphic in \( s \) (see [18]).

**Theorem 2.16** (The Fueter mapping theorem in integral form). Let \( U \subset \mathbb{H} \) be a slice Cauchy domain, let \( J \in \mathbb{S} \), and set \( ds_J = ds(-J) \).

**(1)** If \( f \) is a left slice hyperholomorphic function on a set \( W \), such that \( \overline{U} \subset W \), then the left Fueter regular function \( \tilde{f}(q) := \Delta f(q) \) admits the integral representation

\[
\tilde{f}(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}J)} F_L(s, q)ds_J \, f(s). \tag{2.6}\]
(2) If $f$ is a right slice hyperholomorphic function on a set $W$, such that $\overline{U} \subset W$, then the right Fueter regular function $\tilde{f}(q) = \Delta f(q)$ admits the integral representation

$$\tilde{f}(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}J)} f(s)ds_J F_R(s,q).$$

(2.7)

The integrals depend neither on $U$ nor on the imaginary unit $J \in \mathbb{S}$.

The main advantage to have the Fueter mapping theorem in integral form is that it is possible to obtain a monogenic function by computing the integral of a suitable slice hyperholomorphic function. By applying only the Fueter operator $D$ to the second form of the slice Cauchy kernel, one gets the pseudo Cauchy kernel (see [15]).

**Theorem 2.17.** Let $s, q \in \mathbb{H}$, be such that $s \not\in [q]$, then

$$DS^{-1}_L(s, q) = -2Q_{c,s}(q)^{-1}$$

and

$$S^{-1}_R(s, q)D = -2Q_{c,s}(q)^{-1}.$$ 

We note that the pseudo Cauchy kernels are harmonic functions in $q$ and, respectively, right and left slice hyperholomorphic in $s$.

By applying the Fueter operator $D$ to the Cauchy formulas in Theorem 2.13, we obtain the following result (see [15]).

**Theorem 2.18 (Integral representation of axially harmonic functions).** Let $W \subset \mathbb{H}$ be an open set. Let $U$ be a slice Cauchy domain such that $\overline{U} \subset W$. Then, for $J \in \mathbb{S}$ and $ds_J = ds(-J)$ we have the following:

1. If $f \in SH_L(W)$, then the function $\tilde{f}(q) = Df(q)$ is harmonic and it admits the following integral representation:

$$\tilde{f}(q) = \frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}J)} Q_{c,s}(q)^{-1}ds_Jf(s), \quad q \in U.$$  

(2.8)

2. If $f \in SH_R(W)$, then the function $\tilde{f}(q) = f(q)D$ is harmonic and it admits the following integral representation:

$$f(q) = \frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}J)} f(s)ds_J Q_{c,s}(q)^{-1}, \quad q \in U.$$  

(2.9)

The integrals depend neither on $U$ nor on the imaginary unit $J \in \mathbb{S}$.

## 2.2 The $S$-functional calculus, the $F$-functional calculus, and the $Q$-functional calculus

Let $X$ be a two-sided quaternionic Banach module of the form $X = X_{\mathbb{R}} \otimes \mathbb{H}$, where $X_{\mathbb{R}}$ is a real Banach space. In this paper, we consider $B(X)$ the Banach space of all bounded right linear operators acting on $X$.

In the sequel, we will consider bounded operators of the form $T = T_0 + T_1e_1 + T_2e_2 + T_3e_3$, with commuting components $T_i$ acting on a real vector space $X_{\mathbb{R}}$, that is, $T_i \in B(X_{\mathbb{R}})$ for $i = 0, 1, 2, 3$. The subset of $B(X)$ given by the operators $T$ with commuting components $T_i$ will be denoted by $BC(X)$.

Now let $T : X \to X$ be a right (or left) linear operator. We give the following.

**Definition 2.19.** Let $T \in B(X)$. For $s \in \mathbb{H}$, we set

$$Q_s(T) := T^2 - 2\text{Re}(s)T + |s|^2 I.$$  

We define the $S$-resolvent set $\rho_S(T)$ of $T$ as

$$\rho_S(T) := \{s \in \mathbb{H} : Q_s(T)^{-1} \in B(X)\},$$  

where $Q_s(T)^{-1}$ is the bounded right inverse of $Q_s(T)$.
and we define the $S$-spectrum $\sigma_S(T)$ of $T$ as

$$\sigma_S(T) := \mathbb{H} \setminus \rho_S(T).$$

For $s \in \rho_S(T)$, the operator $Q_s(T)^{-1}$ is called the pseudo $S$-resolvent operator of $T$ at $s$.

**Theorem 2.20.** Let $T \in \mathcal{B}(X)$ and $s \in \mathbb{H}$ with $\|T\| < |s|$. Then, we have

$$\sum_{n=0}^{\infty} T^n s^{-n-1} = -Q_s(T)^{-1}(T - \bar{s}I)$$

and

$$\sum_{n=0}^{\infty} s^{-n-1} T^n = -(T - \bar{s}I)Q_s(T)^{-1}.$$ 

According to the left or right slice hyperholomorphicity, there exist two different resolvent operators.

**Definition 2.21 (S-resolvent operators).** Let $T \in \mathcal{B}(X)$ and $s \in \rho_S(T)$. Then, the left $S$-resolvent operator is defined as

$$S^{-1}_L(s, T) := -Q_s(T)^{-1}(T - \bar{s}I),$$

and the right $S$-resolvent operator is defined as

$$S^{-1}_R(s, T) := -(T - \bar{s}I)Q_s(T)^{-1}. $$

In order to give the definition of the $S$-functional calculus, we need to introduce some notations. Let $T \in \mathcal{B}(X)$. We denote by $SH_L(\sigma_S(T)), SH_R(\sigma_S(T))$, and $N(\sigma_S(T))$ the sets of all left, right, and intrinsic slice hyperholomorphic functions, respectively, with $\sigma_S(T) \subset \text{dom}(f)$.

**Definition 2.22 (S-functional calculus).** Let $T \in \mathcal{B}(X)$. Let $U$ be a slice Cauchy domain that contains $\sigma_S(T)$ and $\overline{U}$ is contained in the domain of $f$. Set $ds_J = -dsJ$ for $J \in \mathbb{S}$ so we define

$$f(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}J)} S^{-1}_L(s, T) ds_J f(s), \quad \text{for every } f \in SH_L(\sigma_S(T))$$

and

$$f(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}J)} f(s) ds_J S^{-1}_R(s, T), \quad \text{for every } f \in SH_R(\sigma_S(T)).$$

The definition of $S$-functional calculus is well posed since the integrals in (2.10) and (2.11) depend neither on $U$ nor on the imaginary unit $J \in \mathbb{S}$, see [18, 22]. The following resolvent equation for the $S$-functional calculus will be useful (see [4]).

**Theorem 2.23.** Let $T \in \mathcal{B}(X)$ such that it commutes with $T$, then we have

$$S^{-1}_R(s, T)S^{-1}_L(q, T) = \{(S^{-1}_R(s, T) - S^{-1}_L(q, T))q +$$

$$-\bar{\delta}(S^{-1}_R(s, T) - S^{-1}_L(q, T))\}Q_s(q)^{-1},$$

where $Q_s(q) := q^2 - 2s_0q + |s|^2$. 

Now we want to introduce the $F$-functional calculus. Let us consider $T = T_0 + T_1e_1 + T_2e_2 + T_3e_3$ such that $T \in \mathcal{B}(X)$.

**Definition 2.24.** Let $T \in \mathcal{B}(X)$. For $s \in \mathbb{H}$, we set

$$Q_{c,s}(T) = s^2I - s(T + \overline{T}) + TT.$$
where $\overline{T} = T_0 - T_1e_1 - T_2e_2 - T_3e_3$. We define the $F$-resolvent set as

$$\rho_F(T) = \{ s \in \mathbb{H} : Q_{c,s}(T)^{-1} \in B(X) \}.$$ 

Moreover, we define the $F$-spectrum of $T$ as

$$\sigma_F(T) = \mathbb{H} \setminus \rho_F(T).$$

By [21, Proposition 4.14], it turns out that the $F$-spectrum is the commutative version of the $S$-spectrum, that is, we have

$$\sigma_F(T) = \sigma_S(T), \quad T \in BC(X),$$

and consequently $\rho_F(T) = \rho_S(T)$.

For $s \in \rho_S(T)$, the operator $Q_{c,s}(T)^{-1}$ is called the commutative pseudo $S$-resolvent operator of $T$. It is possible to define a commutative version of the $S$-functional calculus (see [17]).

**Theorem 2.25.** Let $T \in BC(X)$ and $s \in \mathbb{H}$ be such that $\|T\| < s$. Then,

$$\sum_{m=0}^{\infty} T^m s^{-1-m} = (sI - \overline{T})Q_{c,s}(T)^{-1}$$

and

$$\sum_{m=0}^{\infty} s^{-1-m}T^m = Q_{c,s}(T)^{-1}(sI - \overline{T}).$$

**Definition 2.26.** Let $T \in BC(X)$ and $s \in \rho_S(T)$. We define the left commutative $S$-resolvent operator as

$$S_L^{-1}(s, T) = (sI - \overline{T})Q_{c,s}(T)^{-1},$$

and the right commutative $S$-resolvent operator as

$$S_R^{-1}(s, T) = Q_{c,s}(T)^{-1}(sI - \overline{T}).$$

For the sake of simplicity, we denote the commutative version of the $S$-resolvent operators with the same symbols used for the noncommutative ones. It is possible to define the $S$-functional calculus as done in Definition 2.22. In the sequel, when we deal with the $S$-resolvent operators, we intend their commutative version. Now we give the definition of the $F$-functional calculus.

**Definition 2.27 ($F$-resolvent operators).** Let $T \in BC(X)$. We define the left $F$-resolvent operator as

$$F_L(s, T) = -4(sI - \overline{T})Q_{c,s}(T)^{-2}, \quad s \in \rho_S(T),$$

and the right $F$-resolvent operator as

$$F_R(s, T) = -4Q_{c,s}(T)^{-2}(sI - \overline{T}), \quad s \in \rho_S(T).$$

With the above definitions and Theorem 2.16, we can recall the $F$-functional calculus. It was first introduced in [21] and then investigated in [14, 17, 20].

**Definition 2.28 (The $F$-functional calculus for bounded operators).** Let $U$ be a slice Cauchy domain that contains $\sigma_S(T)$ and $\overline{U}$ is contained in the domain of $f$. Let $T = T_1e_1 + T_2e_2 + T_3e_3 \in BC(X)$, assume that the operators $T_{\ell}, \ell = 1, 2, 3$ have real spectrum and set $ds_I = ds(-J)$, where $J \in \mathbb{S}$. For any function $f \in SH_L(\sigma_S(T))$, we define

$$\tilde{f}(T) := \frac{1}{2\pi i} \int_{\partial(U \cap C_I)} F_L(s, T) ds_I f(s). \quad (2.13)$$
For any \( f \in SH_R(\sigma_S(T)) \), we define
\[
\tilde{f}(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}J)} f(s) \, ds_j \, F_R(s, T).
\] (2.14)

The definition of the \( F \)-functional calculus is well posed since the integrals in (2.13) and (2.14) depend neither on \( U \) nor on the imaginary unit \( J \in \mathbb{S} \).

Another important tool is to write the commutative pseudo \( S \)-resolvent operator in terms of the \( F \)-resolvents, see [18, Theorem 7.3.1].

**Theorem 2.29.** Let \( T \in BC(X) \) and let \( s \in \rho_S(T) \). The \( F \) resolvent operators satisfy the equations
\[
F_L(s, T)s - TF_L(s, T) = -4Q_{c,s}(T)^{-1}
\] (2.15)
and
\[
sF_R(s, T) - F_R(s, T)T = -4Q_{c,s}(T)^{-1}.
\]

We conclude this section with the definition of the \( Q \)-functional calculus (harmonic functional calculus). This is crucial to get a product rule for the \( F \)-functional calculus (see [15]).

**Definition 2.30 \( (Q \)-functional calculus on the \( S \)-spectrum).** Let \( T \in BC(X) \) and set \( ds_j = ds(-J) \) for \( J \in \mathbb{S} \). For every function \( \tilde{f} = Df \) with \( f \in SH_L(\sigma_S(T)) \), we set
\[
\tilde{f}(T) := -\frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}J)} Q_{c,s}(T)^{-1} \, ds_j \, f(s),
\] (2.16)
where \( U \) is an arbitrary bounded slice Cauchy domain with \( \sigma_S(T) \subset U \) and \( \overline{U} \subset \text{dom}(f) \) and \( J \in \mathbb{S} \) is an arbitrary imaginary unit.

For every function \( \tilde{f} = fD \) with \( f \in SH_R(\sigma_S(T)) \), we set
\[
\tilde{f}(T) := -\frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}J)} f(s) ds_j Q_{c,s}(T)^{-1},
\] (2.17)
where \( U \) and \( J \) are as above.

In the previous definitions of the functional calculi, we have always a right version and a left version. However, if we consider an intrinsic function, the two versions coincide. The proof of the following result is similar to the one in [18, Theorem 3.2.11].

**Theorem 2.31.** Let \( T \in BC(X) \). If \( f \in N(\sigma_S(T)) \), then we have
\[
\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}J)} S^{-1}_L(s, T) \, ds_j \, f(s) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}J)} f(s) \, ds_j \, S^{-1}_R(s, T),
\]
\[
-\frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}J)} Q^{-1}_{c,s}(T) \, ds_j \, f(s) = -\frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}J)} f(s) \, ds_j \, Q^{-1}_{c,s}(T),
\]
\[
\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}J)} F_L(s, T) \, ds_j \, f(s) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}J)} f(s) \, ds_j \, F_R(s, T).
\]

**Theorem 2.32 \( \text{(The product rule for the} \ F \text{-functional calculus).} \)** Let \( T \in BC(X) \) and assume \( f \in N(\sigma_S(T)) \) and \( g \in SH_L(\sigma_S(T)) \), then we have
\[
\Delta(fg)(T) = (\Delta f)(T)g(T) + f(T)(\Delta g)(T) - (Df)(T)(Dg)(T).
\] (2.18)
3 | THE POLYANALYTIC FUNCTIONAL CALCULUS OF ORDER 2 ON THE $S$-SPECTRUM

In [26], as far the authors know, a polyanalytic functional calculus of order 2 is defined for the first time. In this section, we recall the main results of [26] and we will prove further properties for this functional calculus. It is based on applying the conjugate of the Fueter operator to the slice hyperholomorphic Cauchy kernels, as illustrated in diagram (1.2).

**Proposition 3.1.** Let $q, s \in \mathbb{H}$ be such that $x \notin [s]$. Let $S^{-1}_L(s, q)$ and $S^{-1}_R(s, q)$ be the slice hyperholomorphic Cauchy kernels written in form II. Then,

1. the function $\overline{D}S^{-1}_L(s, q)$ is a left polyanalytic function of order 2 in the variable $q$ and right slice hyperholomorphic in $s$ and
2. the function $S^{-1}_R(s, q)\overline{D}$ is a right polyanalytic function of order 2 in the variable $q$ and left slice hyperholomorphic in $s$.

In [26], we have provided direct computations for $\overline{D}S^{-1}_L(s, q)$ and $S^{-1}_R(s, q)\overline{D}$. We define the $P^L_2(s, q)$ and $P^R_2(s, q)$ kernels as follows:

**Theorem 3.2.** Let $q, s \in \mathbb{H}$. For $x \notin [s]$, we define the $P^L_2(s, q)$ kernel as

$$P^L_2(s, q) := \overline{D}S^{-1}_L(s, q) = -F_L(s, q)s + q_0F_L(s, q)$$

and the $P^R_2(s, T)$ as

$$P^R_2(s, q) := S^{-1}_R(s, q)\overline{D} = -sF_L(s, q) + q_0F_L(s, q),$$

where $F_L(s, q)$ and $F_R(s, q)$ are defined in (2.4) and (2.5).

**Theorem 3.3** (Integral representation of axially polyanalytic functions of order 2). Let $W \subset \mathbb{H}$ be an open set. Let $U$ be a slice Cauchy domain such that $\overline{U} \subset W$. Then, for $f \in S_H(L)(W)$ and $ds_J = ds(-J)$, we have the following:

1. if $f \in S_H(L)(W)$, then the function $\tilde{f}^0(q) = f(q)$ is polyanalytic of order 2 and it admits the following integral representation:

$$\tilde{f}^0(q) = -\frac{1}{2\pi} \sum_{k=0}^{1} (-q_0)^k \int_{\partial(U \cap C_J)} F_L(s, q)s^{1-k} \; ds_J \; f(s) \; \forall q \in U; \quad (3.1)$$

2. if $f \in S_H(R)(W)$, then the function $\tilde{f}^0(q) = f(q)\overline{D}$ is polyanalytic of order 2 and it admits the following integral representation:

$$\tilde{f}^0(q) = -\frac{1}{2\pi} \sum_{k=0}^{1} (-q_0)^k \int_{\partial(U \cap C_J)} f(s) \; ds_J \; s^{1-k}F_R(s, q) \; \forall q \in U. \quad (3.2)$$

The integrals depend neither on $U$ nor on the imaginary unit $J \in \mathbb{S}$.

In order to get an expansion in series of the $P^L_2(s, q)$ and $P^R_2(s, q)$, in [26] we use the following result.

**Lemma 3.4.** For $n \geq 1$, we have

$$\overline{D}q^n = 2 \left( nq^{n-1} + \sum_{k=1}^{n} q^{n-k}q^{k-1} \right). \quad (3.3)$$

Moreover,

$$q^n\overline{D} = \overline{D}q^n. \quad (3.4)$$
In this paper, we will show that we can prove Lemma 3.4 by using the following result, see [10, Lemma 1].

**Lemma 3.5.** Let \( n \geq 2 \), then we have
\[
Dq^n = (2n + 2)q^{n-1} + 2q \sum_{k=0}^{n-2} q^{n-k-2} q^k.
\]

Before to show Lemma 3.4, we need an auxiliary result.

**Lemma 3.6.** For \( n \geq 1 \) and \( q \in \mathbb{H} \), we have
\[
Dq^n = Dq^n.
\]

**Proof.** We show the formula by induction on \( n \). For \( n = 1 \), it is trivial. Now, we suppose that it holds for \( n \) and we prove it for \( n + 1 \). Since
\[
q^{n+1} = q^n(q + q) - q^{n-1}|q|^2,
\]
by the inductive hypothesis, we have
\[
Dq^{n+1} = D(q^n)(q + q) + 2q^n - D(q^{n-1})|q|^2 - 2q^n
\]
\[
= D(q^n)(q + q) - D(q^{n-1})|q|^2
\]
\[
= D(q^n)(q + q) - D(q^{n-1})|q|^2
\]
\[
= Dq^{n+1}.
\]

**Proof of Lemma 3.4.** By Lemma 3.6, we can write
\[
Dq^n = Dq^n.
\]
Finally, by Lemma 3.5, we get
\[
Dq^n = Dq^n = (2n + 2)q^{n-1} + 2q \sum_{k=0}^{n-2} q^{n-2-k} q^k
\]
\[
= 2 \left( (n + 1)q^{n-1} + \sum_{k=0}^{n-2} q^{n-2-k} q^k \right)
\]
\[
= 2 \left( (n + 1)q^{n-1} + \sum_{k=2}^{n} q^{n-k} q^{k-1} \right)
\]
\[
= 2 \left( nq^{n-1} + \sum_{k=1}^{n} q^{n-k} q^{k-1} \right).
\]

In [26], we showed that Lemma 3.4 was crucial to get an expansion in series of \( P_L(s, q) \) and \( P_R(s, q) \).

**Lemma 3.7.** For \( q, s \in \mathbb{H} \) such that \( |q| < |s| \), we have
\[
P_L^2(s, q) = 2 \sum_{n=1}^{\infty} \left( nq^{n-1} + \sum_{j=1}^{n} q^{n-j} \bar{q}^{j-1} \right) s^{1-n},
\]
and
\[
P_R^2(s, q) = 2 \sum_{n=1}^{\infty} s^{1-n} \left( nq^{n-1} + \sum_{j=1}^{n} q^{n-j} \bar{q}^{j-1} \right).
\]
Now, we have all the tools to introduce the $P_2$-functional calculus.

**Definition 3.8.** Let $T = T_0 + \sum_{i=1}^{3} e_i T_i \in BC(X)$, $s \in \mathbb{H}$, we define the left $\overline{D}$-kernel operator as

$$P^L_2(s,T) = -F_L(s,T)s + T_0 F_L(s,T)$$

and the right $\overline{D}$-kernel operator as

$$P^R_2(s,T) = -s F_R(s,T) + T_0 F_R(s,T).$$

**Definition 3.9** (Polyanalytic functional calculus of order 2 on the $S$-spectrum). Let $T \in \mathcal{M}(X)$ and set $ds_J = ds(-J)$ for $J \in \mathbb{S}$. For every function $\check{f}^\circ = \overline{D}f$ with $f \in SH_L(\sigma_S(T))$, we set

$$\check{f}^\circ(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}J)} P^L_2(s,T) ds_J f(s), \quad (3.6)$$

where $U$ is an arbitrary bounded slice Cauchy domain with $\sigma_S(T) \subset U$ and $\overline{U} \subset \text{dom}(f)$. Moreover, $J \in \mathbb{S}$ is an arbitrary imaginary unit.

For every function $\check{f}^\circ = f \overline{D}$ with $f \in SH_R(\sigma_S(T))$, we set

$$\check{f}^\circ(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}J)} f(s) ds_J P^R_2(s,T), \quad (3.7)$$

where $U$ and $J$ are as above.

In [26], the authors showed that it is possible to write the right and the left $\overline{D}$-kernels operators in terms of $T$ and $\bar{T}$.

**Proposition 3.10.** Let $T = T_0 + \sum_{i=1}^{3} e_i T_i \in BC(X)$, $s \in \mathbb{H}$ and $\|T\| < |s|$, we can write the left $\overline{D}$-kernel operator as

$$P^L_2(s,T) = 2 \sum_{n=1}^{\infty} \left( n T^{n-1} + \sum_{k=1}^{n} T^{n-k} T^k \right)s^{-n-1} \quad (3.8)$$

and the right $\overline{D}$-kernel operator as

$$P^R_2(s,T) = 2 \sum_{n=1}^{\infty} s^{-n} \left( n T^{n-1} + \sum_{k=1}^{n} T^{n-k} T^k \right). \quad (3.9)$$

Finally, we study the regularity of the $\overline{D}$-kernel operators.

**Lemma 3.11.** Let $T \in BC(X)$. The left (resp. right) $\overline{D}$-resolvent operator $P^L_2(s,T)$ (resp. $P^R_2(s,T)$), is a $B(X)$-valued right (resp. left) slice hyperholomorphic function of the variable $s$ in $\rho_S(T)$.

We recall and prove the following result for the sake of completeness.

**Theorem 3.12.** The $P_2$-functional calculus on the $S$-spectrum is well defined, that is, the integrals in (3.6) and (3.7) depend neither on the imaginary unit $J \in \mathbb{S}$ nor on the slice Cauchy domain $U$.

**Proof.** Here, we show only the case $\check{f}^\circ = \overline{D}f$ with $f \in SH_L(\sigma_S(T))$, since the other one follows by analogous arguments.

Since $P^L_2(s,T)$ is a $B(X)$-valued right slice hyperholomorphic function in $s$ and $f$ is left slice hyperholomorphic, the independence from the set $U$ follows by the Cauchy integral formula, see Theorem 2.13 and Theorem 2.14.

Now, we want to show the independence from the imaginary unit. Let us consider two imaginary units $J, I \in \mathbb{S}$ with $J \neq I$ and two bounded slice Cauchy domains $U_q \cup U_s$ with $\sigma_S(T) \subset U_q, U_q \subset U_s$ and $\overline{U}_s \subset \text{dom}(f)$. Then, every $s \in \partial(U_q \cap \mathbb{C}J)$ belongs to the unbounded slice Cauchy domain $\mathbb{H} \setminus U_q$. Recall that $P^L_2(q,T)$ is right slice hyperholomorphic on $\rho_S(T)$, also
at infinity, since \( \lim_{q \to +\infty} P_2^L(q, T) = 0 \). Thus, the Cauchy formula implies

\[
\begin{align*}
P_2^L(s, T) &= \frac{1}{2\pi} \int_{\partial(U \cap C_I)} P_2^L(q, T)\, dq \, S_R^{-1}(q, s) \\
&= \frac{1}{2\pi} \int_{\partial(U \cap C_I)} P_2^L(q, T)\, dq \, S_L^{-1}(s, q).
\end{align*}
\tag{3.10}
\]

The last equality is due to the fact that \( \partial ((\mathbb{H} \setminus U_q) \cap C_I) = -\partial (U_q \cap C_I) \) and \( S_R^{-1}(q, s) = -S_L^{-1}(s, q) \). By Definition 3.9 and (3.10), we get

\[
f^\circ(T) = \frac{1}{2\pi} \int_{\partial(U \cap C_I)} P_2^L(s, T)\, ds \, f(s)
\]

Due to Fubini’s theorem, we can exchange the order of integration and by the Cauchy formula, we obtain

\[
f^\circ(T) = \frac{1}{2\pi} \int_{\partial(U \cap C_I)} \left( \frac{1}{2\pi} \int_{\partial(U \cap C_I)} P_2^L(q, T)\, dq \, S_L^{-1}(s, q) \right)\, ds \, f(s).
\]

This proves the statement. □

The following result is also important to have a well-posed functional calculus.

**Theorem 3.13.** Let \( U \) be a slice Cauchy domain. If \( f, g \in SH_L(U) \) (resp. \( f, g \in SH_R(U) \)) and \( \overline{D} f = \overline{D} g \) (resp. \( f \overline{D} = g \overline{D} \)), then for any \( T \in BC(X) \) such that \( T = T_0 e_0 + T_1 e_1 + T_2 e_2 \), and assuming that the operators \( T_\ell, \ell = 0, 1, 2 \), have real spectrum, we have

\[
f^\circ(T) = \overline{g}^\circ(T).
\]

In order to prove the previous theorem, we need some auxiliary results. First of all, we have to study the following sets:

\[
(\ker \overline{D})_{SH_L(\Omega)} := \{ f \in SH_L(\Omega) : \overline{D}(f) = 0 \} \quad \text{and} \quad (\ker \overline{D})_{SH_R(\Omega)} := \{ f \in SH_R(\Omega) : (f) \overline{D} = 0 \}.
\]

It is necessary to study these sets because in the hypothesis of Theorem 3.13, we have \( \overline{D}(f - g) = 0 \) (resp. \( (f - g) \overline{D} = 0 \)).

**Theorem 3.14.** Let \( \Omega \) be a connected slice Cauchy domain of \( \mathbb{H} \), then

\[
(\ker \overline{D})_{SH_L(\Omega)} = \{ f \in SH_L(\Omega) : f \equiv \alpha \quad \text{for some} \ \alpha \in \mathbb{H} \}
\]

and

\[
(\ker \overline{D})_{SH_R(\Omega)} = \{ f \in SH_R(\Omega) : f \equiv \alpha \quad \text{for some} \ \alpha \in \mathbb{H} \} = (\ker \overline{D})_{SH_R(\Omega)}.
\]

**Proof.** We prove the result in the case \( f \in SH_L(\Omega) \) since the case \( f \in SH_R(\Omega) \) follows by similar arguments. We proceed by double inclusion. The fact that

\[
(\ker \overline{D})_{SH_L(\Omega)} \supseteq \{ f \in SH_L(\Omega) : f \equiv \alpha \quad \text{for some} \ \alpha \in \mathbb{H} \}
\]
is obvious. The other inclusion can be proved observing that if \( f \in \ker \overline{D}_{\mathcal{SH}_L}(\Omega) \), after a change of variable if needed, there exists \( r > 0 \) such that the function \( f \) can be expanded in a convergent series at the origin

\[
    f(q) = \sum_{k=0}^{\infty} q^k \alpha_k \quad \text{for } (\alpha_k)_{k \in \mathbb{N}_0} \subset \mathbb{H} \quad \text{and for any } q \in B_r(0),
\]

where \( B_r(0) \) is the ball centered at 0 and of radius \( r \). By Lemma 3.4, we have

\[
    0 = \overline{D}f(q) \equiv \sum_{k=1}^{\infty} \overline{D}(q^k)\alpha_k = 2 \sum_{k=1}^{\infty} \left( kq^{k-1} + \sum_{s=1}^{k} q^{k-s}s^{s-1} \right) \alpha_k, \quad \forall q \in B_r(0).
\]  

(3.11)

If we restrict the previous series in (3.11) in a neighborhood \( U \) of 0 of the real line, we get

\[
    0 = \sum_{k=1}^{\infty} q_0^{k-1} \alpha_k \quad \forall q_0 \in U
\]

and this implies

\[
    \alpha_k = 0, \quad \forall k \geq 1.
\]

Thus, \( f(q) \equiv \alpha_0 \) in \( B_r(0) \) and since \( \Omega \) is connected, \( f(q) \equiv \alpha_0 \) for any \( q \in \Omega \).

To define a monogenic functional, McIntosh and collaborators, see [29], had as hypothesis that the component \( T_0 \) of the operator \( T = T_0 + T_1e_1 + T_2e_2 + T_3e_3 \) is zero. However, it is possible to set zero a different component of the operator \( T \). In a polyanalytic functional, calculus is not convenient to have \( T_0 = 0 \), due to the left and right structure of the \( \overline{D} \)-kernel (see Definition 3.8). For this reason, in this work, we impose the last component of the operator \( T \) to be zero, that is, \( T_3 = 0 \).

Lemma 3.15. Let \( T \in BC(X) \) be such that \( T = T_0e_0 + T_1e_1 + T_2e_2 \), and assume that the operators \( T_\ell, \ell = 0, 1, 2 \), have real spectrum. Let \( G \) be a bounded slice Cauchy domain such that \( (\partial G) \cap \sigma_S(T) = \emptyset \). For every \( J \in \mathbb{S} \), we have

\[
    \int_{\partial(\mathcal{G} \cap \mathbb{C}J)} \mathcal{P}_L^R(s, T) ds_J = 0 \quad \text{and} \quad \int_{\partial(\mathcal{G} \cap \mathbb{C}J)} ds_J \mathcal{P}_L^R(s, T) = 0.
\]  

(3.12)

Proof. We prove only the first equality of (3.12), since the other one follows by similar computations. Since \( \Delta(1) = 0 \) and \( \Delta(q) = 0 \), by Theorem 2.16, we have

\[
    \int_{\partial(\mathcal{G} \cap \mathbb{C}J)} F_L(s, q) ds_J = \Delta(1) = 0
\]  

(3.13)

and

\[
    \int_{\partial(\mathcal{G} \cap \mathbb{C}J)} F_L(s, q) ds_J s = \Delta(q) = 0,
\]  

(3.14)

for all \( q \notin \partial G \) and \( J \in \mathbb{S} \). By the monogenic functional calculus of McIntosh and collaborators, see [29], we have

\[
    F_L(s, T) = \int_{\partial \Omega} G(\omega, T) D\omega F_L(s, \omega),
\]

where \( D\omega \) is a suitable differential form, the open set \( \Omega \) contains the left spectrum of \( T \), and \( G(\omega, T) \) is the Fueter resolvent operator. By Definition 3.8, we have

\[
    \int_{\partial(\mathcal{G} \cap \mathbb{C}J)} \mathcal{P}_L^R(s, T) ds_J = \int_{\partial(\mathcal{G} \cap \mathbb{C}J)} (-F_L(s, T)s + T_0F_L(s, T)) ds_J
\]

\[
    = -\left( \int_{\partial(\mathcal{G} \cap \mathbb{C}J)} \int_{\partial \Omega} G(\omega, T) D\omega F_L(s, \omega)s ds_J - T_0 \int_{\partial(\mathcal{G} \cap \mathbb{C}J)} \int_{\partial \Omega} G(\omega, T) D\omega F_L(s, \omega) ds_J \right)
\]
\[ \frac{1}{4} \left( \int_{\partial \Omega} G(\omega, T) D\omega \left( \int_{\partial(G \cap C_J)} F_L(s, \omega) ds_j s \right) - T \int_{\partial \Omega} G(\omega, T) D\omega \left( \int_{\partial(G \cap C_J)} F_L(s, \omega) ds_j \right) \right) = 0, \]

where the second equality is a consequence of the Fubini’s Theorem and the last equality is a consequence of formulas (3.13) and (3.14).

\[ \square \]

**Proof of Theorem 3.13.** We prove the theorem when \( f, g \in SH_L(\Omega) \). The case of \( f, g \in SH_R(\Omega) \) follows by similar arguments. We divide the proof in two cases.

**U is connected**

By definition of the \( P_2 \)-functional calculus on the \( S \)-spectrum, see Definition 3.9, we have

\[ \bar{f}(T) = \frac{1}{2\pi} \int_{\partial(U \cap C_J)} P_{\frac{1}{2}}(s, T) ds_j (f(s) - g(s)). \]

Since \( P_{\frac{1}{2}}(s, T) \) is slice hyperholomorphic in the variable \( s \) by Theorem 2.13, we can change the domain of integration to \( B_r(0) \cap C_J \) for some \( r > 0 \) with \( \|T\| < r \). Moreover, by hypothesis, we have that \( f(s) - g(s) \in ker D_{SH_L(\Omega)} \), thus by Theorem 3.14 and Proposition 3.10, we get

\[ \bar{f}(T) = \frac{1}{2\pi} \int_{\partial(B_r(0) \cap C_J)} P_{\frac{1}{2}}(s, T) ds_j \alpha = 0. \]

**U is not connected**

In this case, we write the set \( U \) in the following way: \( U = \bigcup_{\ell=1}^n U_\ell \), where \( U_\ell \) are the connected components of \( U \). Hence, there exist constants \( \alpha_\ell \in \mathbb{H} \) for \( \ell = 1, \ldots, n \), such that \( f(s) - g(s) = \sum_{\ell=1}^n \chi_{U_\ell}(s) \alpha_\ell \). Thus, we can write

\[ \bar{f}(T) = \sum_{\ell=1}^n \frac{1}{2\pi} \int_{\partial(U_\ell \cap C_J)} P_{\frac{1}{2}}(s, T) ds_j \alpha_\ell. \]

Finally, by Lemma 3.15, we get \( \bar{f}(T) = \bar{g}(T) = 0 \).

**Remark 3.16.** If the set \( U \) in Theorem 3.13 is connected, we can show the result for operators of the following form:

\[ T = T_0 e_0 + T_1 e_1 + T_2 e_2 + T_3 e_3 \]

However, in order to have a well-defined functional calculus also for not connected sets, as it happens for the monogenic functional calculus of McIntosh, we need to annihilate a component of the operator \( T \).

We conclude this section by proving some algebraic properties of the \( P_2 \)-functional calculus.

**Proposition 3.17.** Let \( T \in BC(X) \) be such that \( T = T_0 e_0 + T_1 e_1 + T_2 e_2 \), and assume that the operators \( T_\ell, \ell = 0, 1, 2 \), have real spectrum.

1. If \( \bar{f} = \bar{D} f \) and \( \bar{g} = \bar{D} g \) with \( f, g \in SH_L(\sigma_S(T)) \) and \( \alpha \in \mathbb{H} \), then

\[ (\bar{f}(T) + \bar{g}(T)) = \bar{f}(T) \alpha + \bar{g}(T). \]
(2) If \( \check{f}_\circ = \check{f} \mathcal{D} \) and \( \check{g}_\circ = \check{g} \mathcal{D} \) with \( f, g \in SH_k(\sigma_\delta(T)) \) and \( a \in \mathbb{H} \), then

\[
(af_\circ + ag_\circ)(T) = af_\circ(T) + ag_\circ(T).
\]

**Proof.** The above identities follow immediately from the linearity of the integrals in (3.6), resp. (3.7). \( \square \)

**Proposition 3.18.** Let \( T \in BC(X) \) be such that \( T = T_0e_0 + T_1e_1 + T_2e_2 \), and assume that the operators \( T_\ell, \ell = 0, 1, 2 \), have real spectrum.

(1) If \( \check{f}_\circ = \mathcal{D}f \) with \( f \in SH_L(\sigma_\delta(T)) \) and assume that \( f(q) = \sum_{m=0}^{\infty} q^m a_m \) with \( a_m \in \mathbb{H} \), where this series converges on a ball \( B_r(0) \) with \( \sigma_\delta(T) \subset B_r(0) \). Then,

\[
\check{f}_\circ(T) = \sum_{m=1}^{\infty} \left( mT^{m-1} + \sum_{k=1}^{m} T^{m-k} \bar{T}^{k-1} \right) a_m.
\]

(2) If \( \check{f}_\circ = f \mathcal{D} \) with \( f \in SH_R(\sigma_\delta(T)) \) and assume that \( f(q) = \sum_{m=0}^{\infty} a_m q^m \) with \( a_m \in \mathbb{H} \), where this series converges on a ball \( B_r(0) \) with \( \sigma_\delta(T) \subset B_r(0) \). Then,

\[
\check{f}_\circ(T) = \sum_{m=1}^{\infty} a_m \left( mT^{m-1} + \sum_{k=1}^{m} T^{m-k} \bar{T}^{k-1} \right).
\]

**Proof.** We prove the first assertion since the second one can be proven by following similar arguments. We choose an imaginary unit \( J \in \mathbb{S} \) and a radius \( 0 < R < r \) such that \( \sigma_\delta(T) \subset B_R(0) \). Then, the series expansion of \( f \) converges uniformly on \( \partial(B_R(0) \cap \mathbb{C}J) \), and so

\[
\check{f}_0(T) = \frac{1}{2\pi} \int_{\partial(B_R(0) \cap \mathbb{C}J)} \mathcal{P}_2^L(s, T) ds J \sum_{\ell=0}^{\infty} s^\ell a_\ell = \frac{1}{2\pi} \sum_{\ell=0}^{\infty} \int_{\partial(B_R(0) \cap \mathbb{C}J)} \mathcal{P}_2^L(s, T) ds J s^\ell a_\ell.
\]

By Proposition 3.10, we further obtain

\[
\check{f}_0(T) = \frac{1}{2\pi} \int_{\partial(B_R(0) \cap \mathbb{C}J)} \sum_{m=1}^{\infty} \left( mT^{m-1} + \sum_{k=1}^{m} T^{m-k} \bar{T}^{k-1} \right) s^{-1-m} ds J \sum_{\ell=0}^{\infty} s^\ell a_\ell
\]

\[
= \frac{1}{2\pi} \sum_{m=1}^{\infty} \sum_{\ell=0}^{\infty} \int_{\partial(B_R(0) \cap \mathbb{C}J)} \mathcal{P}_2^L(s, T) ds J s^{-1-m+\ell} a_\ell
\]

\[
= \sum_{m=1}^{\infty} \left( mT^{m-1} + \sum_{k=1}^{m} T^{m-k} \bar{T}^{k-1} \right) a_m.
\]

The last equality is due to the fact that \( \int_{\partial(B_R(0) \cap \mathbb{C}J)} s^{-1-m+\ell} ds J \) is equal to \( 2\pi \) if \( \ell = m \), and 0 otherwise. \( \square \)

**Theorem 3.19.** Let \( T \in BC(X) \). Let \( m \in \mathbb{N}_0 \), and let \( U \subset \mathbb{H} \) be a bounded slice Cauchy domain with \( \sigma_\delta(T) \subset U \). For every \( J \in \mathbb{S} \), we have

\[
P_m^2(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}J)} \mathcal{P}_2^L(s, T) ds J s^{m+1},
\]

where

\[
P_m^2(T) := (m+1)T^m + \sum_{k=0}^{m} T^{m-k} \bar{T}^k.
\]
Proof. We start by considering $U$ to be the ball $B_r(0)$ with $\|T\| < r$. By Proposition 3.10, we know that we can expand the left $\overline{D}$-kernel operator as

$$P^L_2(s, T) = \sum_{n=1}^{+\infty} \left( nT^{n-1} + \sum_{k=1}^{n} T^{n-k}\bar{T}^{k-1} \right) s^{-1-n}$$

for every $s \in \partial B_r(0)$. Since the series converges on $\partial B_r(0)$, we have

$$\frac{1}{2\pi} \int_{\partial(B_r(0) \cap \mathbb{C} J)} P^L_2(s, T) ds_j s^{m+1} = \frac{1}{2\pi} \int_{\partial(B_r(0) \cap \mathbb{C} J)} \left( nT^{n-1} + \sum_{k=1}^{n} T^{n-k}\bar{T}^{k-1} \right) s^{-n+m} ds_j$$

$$= (m+1)T^m + \sum_{k=1}^{m+1} T^{m+1-k}\bar{T}^{k-1}$$

$$= (m+1)T^m + \sum_{k=0}^{m} T^{m-k}T^k = P^2_m(T),$$

where we have used

$$\int_{\partial(B_r(0) \cap \mathbb{C} J)} s^{-n+m} ds_j = \begin{cases} 0 & \text{if } n \neq m + 1 \\ 2\pi & \text{if } n = m + 1. \end{cases}$$

This proves the result for the case $U = B_r(0)$. Now we get the result for an arbitrary bounded Cauchy domain $U$ that contains $\sigma_S(T)$. The operator $P^L_2(s, T)$ is right slice hyperholomorphic and the monomial $s^{m+1}$ is left slice hyperholomorphic on the bounded slice Cauchy domain $B_r(0) \setminus U$. By the Cauchy’s integral theorem (see Theorem 2.14), we get

$$\frac{1}{2\pi} \int_{\partial(B_r(0) \cap \mathbb{C} J)} P^L_2(s, T) ds_j s^{m+1} - \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C} J)} P^L_2(s, T) ds_j s^{m+1}$$

$$= \frac{1}{2\pi} \int_{\partial(B_r(0) \setminus U) \cap \mathbb{C} J)} P^L_2(s, T) ds_j s^{m+1} = 0.$$

Finally we have

$$\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C} J)} P^L_2(s, T) ds_j s^{m+1} = \frac{1}{2\pi} \int_{\partial(B_r(0) \cap \mathbb{C} J)} P^L_2(s, T) ds_j s^{m+1} = P^2_m(T),$$

and this concludes the proof. \qed

Finally, by using the same methodology developed in [18, Theorem 3.2.11], we have the following result:

**Lemma 3.20.** Let $T \in BC(X)$. If $f \in N(\sigma_S(T))$ and $U$ is a bounded slice Cauchy domain such that $\sigma_S(T) \subset U$ and $\overline{U} \subset \text{dom}(f)$, then we have

$$\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C} J)} P^L_2(s, T) ds_j f(s) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C} J)} f(s) ds_j P^R_2(s, T).$$

### 4 Resolvent Equation and Product Rule for the Polyanalytic Functional Calculus

In the Riesz–Dunford functional calculus, the main tool to show the product rule and to study the Riesz projectors is the resolvent equation. In order to recall this equation, we need to introduce some notations. Let $A$ be a complex operator defined on a complex Banach space, we denote by $R(\gamma, A) := (\gamma I - A)^{-1}$, the resolvent operator for the holomorphic
The resolvent equation is given by

\[ R(\lambda, A)R(\mu, A) = \frac{R(\lambda, A) - R(\mu, A)}{\mu - \lambda}, \quad \lambda, \mu \in \rho(A), \]  

(4.1)

where \( \rho(A) \) is the resolvent set of the operator \( A \). The main properties of the resolvent equation are the following:

1. The product of two different resolvent operators \( R(\lambda, A)R(\mu, A) \) is transformed into the difference \( R(\lambda, A) - R(\mu, A) \).
2. The difference of resolvent operators \( R(\lambda, A) - R(\mu, A) \) is entangled with the Cauchy kernel \( 1/(\mu - \lambda) \) of the holomorphic functions.
3. The holomorphicity is maintained both in \( \lambda \) and in \( \mu \in \rho(A) \).

In this section, we want to address the following problem.

**Problem 4.1.** Is it possible to show a resolvent equation for the \( P_2 \)-functional calculus on the \( S \)-spectrum that enjoys similar properties to the holomorphic resolvent equation?

In order to answer to this question, the following result is essential.

**Theorem 4.2.** Let \( T \in \mathcal{B}(X) \). For \( q, s \in \rho_S(T) \), with \( s \notin [q] \), the following equation holds:

\[ S_R^{-1}(s, T)P_2^R(q, T) + P_2^L(s, T)S_L^{-1}(q, T) - 4Q_{c,s}(T)^{-1}P_{c,q}(T)^{-1} \]

\[ = [(P_2^R(s, T) - P_2^L(q, T))q - 4(P_2^R(s, T) - P_2^L(q, T))]Q_s(q)^{-1}, \]  

(4.2)

where \( Q_s(q) := q^2 - 2s_0q + |q|^2 \) and \( T = T_1e_1 + T_2e_2 + T_3e_3 \).

**Proof.** We divide the proof in nine steps.

**Step I.** We multiply the \( S \)-resolvent equation (see (2.12)) on the right by \( 4Q_{c,q}(T)q \) and we get

\[ -S_R^{-1}(s, T)F_L(q, T)q = [(4S_R^{-1}(s, T)Q_{c,q}(T)^{-1} + F_L(q, T)q)q - \bar{s}(4S_R^{-1}(s, T)Q_{c,q}(T)^{-1} + F_L(q, T)q)]Q_s(q)^{-1}. \]  

(4.3)

**Step II.** We multiply the \( S \)-resolvent equation on the right by \( -4T_0Q_{c,q}(T)^{-1} \) and we obtain

\[ S_R^{-1}(s, T)T_0F_L(q, T)q = [(4S_R^{-1}(s, T)Q_{c,q}(T)^{-1} + F_L(q, T)q)q - 4\bar{s}(4S_R^{-1}(s, T)Q_{c,q}(T)^{-1} + F_L(q, T)q)]Q_s(q)^{-1}. \]  

(4.4)

**Step III.** We sum Equations (4.3) and (4.4), we get

\[ S_R^{-1}(s, T)P_2^L(q, T) = [-P_2^L(q, T)q + 4(P_2^R(s, T) - P_2^L(q, T))]Q_s(q)^{-1} + 4[-S_R^{-1}(s, T)(T_0 - qI)Q_{c,q}(T)^{-1}q] \]

\[ + \bar{s}(4S_R^{-1}(s, T)Q_{c,q}(T)^{-1} + F_L(q, T)q)]Q_s(q)^{-1}. \]  

(4.5)

**Step IV.** We multiply the \( S \)-resolvent equation on the left by \( 4Q_{c,s}(T)^{-1}s \) and we get

\[ -sF_R(s, T)S_L^{-1}(q, T) = [(-sF_R(s, T) - 4sQ_{c,s}(T)^{-1}S_L^{-1}(q, T))]q - \bar{s}(-sF_R(s, T) - 4sQ_{c,s}(T)^{-1}S_L^{-1}(q, T))]Q_s(q)^{-1}. \]  

(4.6)

**Step V.** We multiply the \( S \)-resolvent equation on the left by \( -4T_0Q_{c,s}(T)^{-1} \) and we get

\[ T_0F_R(s, T)S_L^{-1}(q, T) = [(T_0 - qI)F_R(s, T) + 4T_0Q_{c,s}(T)^{-1}S_L^{-1}(q, T)]q \]

\[ - 4\bar{s}(4T_0Q_{c,s}(T)^{-1}S_L^{-1}(q, T))]Q_s(q)^{-1}. \]  

(4.7)
Step VI. We sum Equations (4.6) and (4.7), we obtain
\[
P^R_2(s, T)S^{-1}_L(q, T) = [P^R_2(s, T)q - sP^R_2(s, T)]Q_s(q)^{-1} + 4\{Q_{cs}(T)^{-1}(T_0 - sT)S^{-1}_L(q, T)q
- sQ_{cs}(T)^{-1}(T_0 - sT)S^{-1}_L(q, T)\}Q_s(q)^{-1}. \tag{4.8}
\]

Step VII. We sum Equations (4.5) and (4.8), we get
\[
S^{-1}_R(s, T)P^R_2(q, T) + P^R_2(s, T)S^{-1}_L(q, T)
= \left((P^R_2(s, T) - P^R_2(q, T))q - s(P^R_2(s, T) - P^R_2(q, T))\right)Q_s(q)^{-1}
+ 4\{(Q_{cs}(T)^{-1}(T_0 - sT)S^{-1}_L(q, T) - S^{-1}_R(s, T)(T_0 - qI)Q_{cq}(T^{-1}))q
- s(Q_{cs}(T)^{-1}(T_0 - sI)S^{-1}_L(q, T) - S^{-1}_R(s, T)(T_0 - qI)Q_{cq}(T^{-1}))\}Q_s(q)^{-1}. \tag{4.9}
\]

Step VIII. We manipulate the term
\[
4\{(Q_{cs}(T)^{-1}(T_0 - sI)S^{-1}_L(q, T) - S^{-1}_R(s, T)(T_0 - qI)Q_{cq}(T^{-1}))q
- s(Q_{cs}(T)^{-1}(T_0 - sI)S^{-1}_L(q, T) - S^{-1}_R(s, T)(T_0 - qI)Q_{cq}(T^{-1}))\}Q_s(q)^{-1},
\]
which is on the right-hand side of Equation (4.9). This term is the sum of the following two terms:
\[
4T_0\{(Q_{cs}(T)^{-1}S^{-1}_L(q, T) - S^{-1}_R(s, T)Q_{cq}(T^{-1}))q
- s(Q_{cs}(T)^{-1}S^{-1}_L(q, T) - S^{-1}_R(s, T)Q_{cq}(T^{-1}))\}Q_s(q)^{-1} \tag{4.11}
\]
and
\[
4\{(S^{-1}_R(s, T)qQ_{cq}(T)^{-1} - Q_{cs}(T)^{-1}sS^{-1}_L(q, T))q
- s(S^{-1}_R(s, T)qQ_{cq}(T)^{-1} - Q_{cs}(T)^{-1}sS^{-1}_L(q, T))\}Q_s(q)^{-1} \tag{4.12}
\]
First, we focus on the term (4.11). By the definitions of the left and the right \(S\)-resolvent operators, we have
\[
Q_{cs}(T)^{-1}S^{-1}_L(q, T) - S^{-1}_R(s, T)Q_{cq}(T^{-1})
= Q_{cs}(T)^{-1}(qI - T)Q_{cq}(T)^{-1} - Q_{cs}(T)^{-1}(sI - T)Q_{cq}(T)^{-1}
= Q_{cs}(T)^{-1}(q - s)Q_{cq}(T)^{-1}.
\]
Thus, the term (4.11) can be rewritten in the following way:
\[
4T_0\{(Q_{cs}(T)^{-1}S^{-1}_L(q, T) - S^{-1}_R(s, T)Q_{cq}(T^{-1}))q
- s(Q_{cs}(T)^{-1}S^{-1}_L(q, T) - S^{-1}_R(s, T)Q_{cq}(T^{-1}))\}Q_s(q)^{-1}
= 4T_0\{(Q_{cs}(T)^{-1}(q - s)Q_{cq}(T)^{-1})q - s(Q_{cs}(T)^{-1}(q - s)Q_{cq}(T^{-1}))\}Q_s(q)^{-1}
= 4T_0\{Q_{cs}(T)^{-1}(-sq + q^2 + |s|^2 - sq)Q_{cq}(T^{-1})\}Q_s(q)^{-1} = 4T_0\{Q_{cs}(T)^{-1}Q_s(q)Q_{cq}(T^{-1})\}Q_s(q)^{-1}
= 4T_0Q_{cs}(T)^{-1}Q_{cq}(T)^{-1}. \tag{4.13}
\]
Now we focus on the term (4.12). By the definitions of the left and the right \(S\)-resolvent operators, we have
\[
S^{-1}_R(s, T)qQ_{cq}(T)^{-1} - Q_{cs}(T)^{-1}sS^{-1}_L(q, T)
= Q_{cs}(T)^{-1}(sI - T)qQ_{cq}(T)^{-1} - Q_{cs}(T)^{-1}(sI - T)Q_{cq}(T)^{-1}
= -Q_{cs}(T)^{-1}(Tq - sT)Q_{cq}(T)^{-1}.
\]
Thus, the term \((4.12)\) can be rewritten in the following way:

\[
4[(S^{-1}_R(s,T)qQ_{c,q}(T)^{-1} - Q_{c,s}(T)^{-1}sS^{-1}_L(q,T))q - \bar{s}(S^{-1}_R(s,T)qQ_{c,q}(T)^{-1} - Q_{c,s}(T)^{-1}sS^{-1}_L(q,T))]Q_s(q)^{-1}
\]

\[
= -4[(Q_{c,s}(T)^{-1}(Tq - st)Q_{c,q}(T)^{-1})q - \bar{s}(Q_{c,s}(T)^{-1}(Tq - st)Q_{c,q}(T)^{-1})]Q_s(q)^{-1}
\]

\[
= -4Q_{c,s}(T)^{-1}(\bar{q}q^2 - s\bar{t}q - s\bar{t}q + [s]^2T)Q_{c,q}(T)^{-1}Q_s(q)^{-1}
\]

\[
= -4Q_{c,s}(T)^{-1}TQ_s(q)Q_{c,q}(T)^{-1}Q_s(q)^{-1} = -4Q_{c,s}(T)^{-1}TQ_{c,q}(T)^{-1}. \tag{4.14}
\]

In conclusion by \((4.13)\) and \((4.14)\), we can write

\[
4[(Q_{c,s}(T)^{-1}(T_0 - sq)S^{-1}_L(q,T) - S^{-1}_R(s,T)(T_0 - qT)Q_{c,q}(T)^{-1})q
\]

\[
- \bar{s}(Q_{c,s}(T)^{-1}(T_0 - sf)S^{-1}_L(q,T) - S^{-1}_R(s,T)(T_0 - qT)Q_{c,q}(T)^{-1})Q_s(q)^{-1}
\]

\[
= 4T_0Q_{c,s}(T)^{-1}Q_{c,q}(T)^{-1} - 4Q_{c,s}(T)^{-1}TQ_{c,q}(T)^{-1} = 4Q_{c,s}(T)^{-1}TQ_{c,q}(T)^{-1}. \tag{4.15}
\]

**Step IX.** Finally, by \((4.15)\) and \((4.9)\), we get

\[
S^{-1}_R(s,T)P^L_2(q,T) + P^R_2(s,T)S^{-1}_L(q,T) - 4Q_{c,s}(T)^{-1}TQ_{c,q}(T)^{-1}
\]

\[
= [(P^R_2(s,T) - P^L_2(q,T))q - \bar{s}(P^R_2(s,T) - P^L_2(q,T))]Q_s(q)^{-1}. \tag{4.16}
\]

**Lemma 4.3.** Let \(T \in BC(X)\) and let \(s \in \rho_S(T)\). The commutative pseudo \(S\)-resolvent operator satisfies the equations

\[
Q_{c,s}(T)^{-1} = \frac{1}{4}(P^L_2(s,T) + TF_L(s,T)) \tag{4.16}
\]

and

\[
Q_{c,s}(T)^{-1} = \frac{1}{4}(P^R_2(s,T) + FR(s,T)). \tag{4.17}
\]

**Proof.** By Theorem 2.29, we have

\[
4Q_{c,s}(T)^{-1} = -F_L(s,T)s + TF_L(s,T)
\]

\[
= -F_L(s,T)s + T_0F_L(s,T) + TF_L(s,T)
\]

\[
= P^L_2(s,T) + TF_L(s,T).
\]

To prove the other equality in the statement, we can proceed in a similar way. \(\square\)

By means of the previous result, we can write a resolvent equation for the \(P_2\)-functional calculus.

**Theorem 4.4.** Let \(T \in BC(X)\). For \(q, s \in \rho_S(T)\), with \(s \notin \{q\}\), the following equation holds:

\[
S^{-1}_R(s,T)P^L_2(q,T) + P^R_2(s,T)S^{-1}_L(q,T) - \frac{1}{4}(P^R_2(s,T)T^2P^L_2(q,T) + P^R_2(s,T)T^2F_L(q,T)
\]

\[
+ FR(s,T)T^2P^L_2(q,T) + FR(s,T)T^3F_L(q,T))
\]

\[
= [(P^R_2(s,T) - P^L_2(q,T))q - \bar{s}(P^R_2(s,T) - P^L_2(q,T))]Q_s(q)^{-1}. \tag{4.18}
\]

**Proof.** From the identities \((4.16)\) and \((4.17)\), we obtain

\[
4^2Q_{c,s}(T)^{-1}TQ_{c,q}(T)^{-1} = (P^R_2(s,T) + FR(s,T))T(P^L_2(q,T) + TF_L(q,T))
\]

\[
= P^R_2(s,T)TP^L_2(q,T) + P^R_2(s,T)T^2F_L(q,T) + FR(s,T)T^2P^L_2(q,T) + FR(s,T)T^3F_L(q,T).
\]

Replacing this identity in \((4.2)\), we obtain the thesis. \(\square\)
We can write Equation (4.18) in the following way:
\[
S^{-1}_R(s, T)P^I_2(q, T) + P^R_2(s, T)S^{-1}_L(q, T) - \frac{1}{4} \left( P^R_2(s, T) T^2 P^I_2(q, T) + P^R_2(s, T) T^2 F_L(q, T) \right) + F_R(s, T) T^2 F_L(q, T) + F_R(s, T) T^3 F_L(q, T)
\]
\[
= \left[ P^R_2(s, T) - P^I_2(q, T) \right] *_{sl} S^{-1}_L(q, s).
\]

This equation can be considered a resolvent equation for the $P_2$-functional calculus. The main differences and the major similarities with the holomorphic resolvent equation are listed below.

1. Due to the noncommutative setting, there are two different $D$-kernel operators $P^I_2(q, T)$ and $P^R_2(s, T)$, which are right slice hyperholomorphic in $q$ and left slice hyperholomorphic in $s$, respectively.
2. The difference $P^R_2(s, T) - P^I_2(q, T)$ is suitably multiplied by the Cauchy kernel of the slice hyperholomorphic functions.
3. The term $\left[ P^R_2(s, T) - P^I_2(q, T) \right] *_{sl} S^{-1}_L(q, s)$ is equal not only to the product of the $P$-resolvent operators but also to other terms:
   - the $S$-resolvent operators,
   - the $F$-resolvent operators.
4. The resolvent equation preserves the slice hyperholomorphicity on the right in $s$ and on the left in $q$.

As it happens for the holomorphic functional calculus, the resolvent equation is crucial to obtain a product formula. Before to go through this, we need to recall the following technical result, see [18].

**Lemma 4.5.** Let $B \in B(X)$. Let $G$ be an axially symmetric domain and assume $f \in N(G)$. Then, for $p \in G$, we have
\[
\frac{1}{2\pi} \int_{\partial (G \cap \mathbb{C}J)} f(s)ds_J(3B - Bp)(p^2 - 2\xi_0 p + |s|^2)^{-1} = Bf(p).
\]

**Theorem 4.6.** Let $T \in BC(X)$ and assume $f \in N(\sigma_S(T))$. If $g \in SH_L(\sigma_S(T))$, then we have
\[
\overline{D}(fg)(T) = f(T)\overline{D}(g)(T) + (\overline{D}f)(T)g(T) - D(f)(T)\overline{D}(g)(T). \tag{4.19}
\]
If $g \in SH_R(\sigma_S(T))$, then we have
\[
\overline{D}(gf)(T) = g(T)(\overline{D}f)(T) + (\overline{D}g)(T)f(T) - D(g)(T)\overline{D}(f)(T). \tag{4.20}
\]

**Proof.** Let $G_1$ and $G_2$ be two bounded slice Cauchy domains such that they contain the $S$-spectrum and $\overline{G}_1 \subset G_2$ and $\overline{G}_2 \subset \text{dom}(f) \cap \text{dom}(g)$. We choose $p \in \partial (G_1 \cap \mathbb{C}J)$ and $s \in \partial (G_2 \cap \mathbb{C}J)$. For every $J \in \mathbb{S}$, from the definitions of the $P_2$-functional calculus, the $S$-functional calculus, the $Q$-functional calculus, and since $f$ is intrinsic by Theorem 2.31, we get
\[
f(T)(\overline{D}g)(T) + (\overline{D}f)(T)g(T) - D(f)(T)\overline{D}(g)(T)
\]
\[
= \frac{1}{(2\pi)^2} \int_{\partial (G_1 \cap \mathbb{C}J)} f(s)ds_J S^{-1}_R(s, T) \int_{\partial (G_2 \cap \mathbb{C}J)} P^I_2(p, T)dp_J g(p)
\]
\[
+ \frac{1}{(2\pi)^2} \int_{\partial (G_2 \cap \mathbb{C}J)} f(s)ds_J P^R_2(s, T) \int_{\partial (G_1 \cap \mathbb{C}J)} S^{-1}_L(p, T)dp_J g(p)
\]
\[
- \frac{1}{\pi^2} \left( \int_{\partial (G_2 \cap \mathbb{C}J)} f(s)ds_J Q_{c,s}(T)^{-1} \right) \left( \int_{\partial (G_1 \cap \mathbb{C}J)} Q_{c,p}(T)^{-1} dp_J g(p) \right)
\]
\[
= \frac{1}{(2\pi)^2} \int_{\partial (G_2 \cap \mathbb{C}J)} \int_{\partial (G_1 \cap \mathbb{C}J)} f(s)ds_J \left[ S^{-1}_R(s, T)P^I_2(p, T) + P^R_2(s, T)S^{-1}_L(p, T) - 4Q_{c,s}(T)^{-1}Q_{c,p}(T)^{-1} \right] dp_J g(p).
\]
Now from Equation (4.2), we obtain

\[
f(T)\overline{D}g(T) + (\overline{D}f)(T)g(T) - D(f)(T)\overline{T}D(g)(T) = \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap C_J)} \int_{\partial(G_1 \cap C_J)} f(s)ds_j (P_2^R(s, T) - P_2^L(p, T))p
- \overline{s}(P_2^R(s, T) - P_2^L(p, T))Q_s(p)^{-1}dp_jg(p).
\]

Since \(pQ_s(p)^{-1}\) and \(Q_s(p)^{-1}\) are intrinsic slice hyperholomorphic on \(G_1\), by the Cauchy integral formula, we get

\[
\frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap C_J)} \int_{\partial(G_1 \cap C_J)} f(s)ds_j P_2^R(s, T) pQ_s(p)^{-1}dp_jg(p) = 0
\]

and

\[
\frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap C_J)} \int_{\partial(G_1 \cap C_J)} f(s)ds_j \overline{s}P_2^R(s, T) Q_s(p)^{-1}dp_jg(p) = 0.
\]

Therefore, we get

\[
\overline{D}(fg)(T) = f(T)\overline{D}g(T) + (\overline{D}f)(T)g(T) - D(f)(T)\overline{T}D(g)(T) = \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap C_J)} \int_{\partial(G_1 \cap C_J)} f(s)ds_j \overline{s}(P_2^R(s, T) - P_2^L(p, T))Q_s(p)^{-1}dp_jg(p).
\]

By Fubini’s theorem, Lemma 4.5 with \(B := P_2^L(p, T)\), and the definition of the \(P_2\)-functional calculus, we get

\[
f(T)\overline{D}g(T) + (\overline{D}f)(T)g(T) - D(f)(T)\overline{T}D(g)(T) = \frac{1}{2\pi} \int_{\partial(G_1 \cap C_J)} P_2^L(p, T)dp_jf(p)g(p)
= \overline{D}(fg)(T).
\]

In [15], a product rule for the \(F\)- functional calculus is proved, see Theorem 2.18. The formula is obtained in terms of the \(Q\)-functional calculus, that is, the operator \(D\) is involved. In the following result, we show a product rule for the \(F\)-functional calculus in which the \(P_2\)-functional calculus is involved, namely, the operator \(\overline{D}\) plays a role.

**Theorem 4.7.** Let \(T \in BC(X)\) and assume \(f \in N(\sigma_S(T))\) and \(g \in SH_t(\sigma_S(T))\). Then, we have

\[
\Delta(fg)(T) = (\Delta f)(T)g(T) + f(T)(\Delta g)(T) - \frac{1}{4}(\overline{D}f)(T)\overline{T}D(g)(T)
- \frac{1}{4}(\overline{D}f)(T)\overline{T}(\Delta g)(T) - \frac{1}{4}(\Delta f)(T)\overline{T}(\overline{D}g)(T) - \frac{1}{4}(\Delta f)(T)\overline{T}(\overline{D}g)(T).
\]

**Proof.** Let \(G_1\) and \(G_2\) be two bounded slice Cauchy domains like in the proof of Theorem 4.6. Let us consider \(p \in \partial(G_1 \cap C_J)\) and \(s \in \partial(G_2 \cap C_J)\). Then, by the definitions of the \(F\)-functional calculus, the \(S\)-functional calculus, the \(P_2\)-functional calculus, and from the fact that \(f\) is intrinsic, see Theorem 2.31, we get

\[
(\Delta f)(T)g(T) + f(T)(\Delta g)(T) - \frac{1}{4}(\overline{D}f)(T)\overline{T}D(g)(T)
- \frac{1}{4}(\overline{D}f)(T)\overline{T}(\Delta g)(T) - \frac{1}{4}(\Delta f)(T)\overline{T}(\overline{D}g)(T) - \frac{1}{4}(\Delta f)(T)\overline{T}(\overline{D}g)(T)
= \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap C_J)} \int_{\partial(G_1 \cap C_J)} f(s)ds_j F_R(s, T) \int_{\partial(G_1 \cap C_J)} S_L^{-1}(p, T)dp_jg(p).
\]
From Lemma 4.3, we get

\[
\frac{1}{4} (P^R_{2}(s, T) P^L_{2}(p, T) + P^R_{2}(s, T) T F_L(p, T) + F_R(s, T) T P^L_{2}(p, T) + F_R(s, T) T^2 F_L(p, T))
\]

\[
= 4 Q_{c,s}(T)^{-1} Q_{c,p}(T)^{-1}.
\]

Therefore, we obtain

\[
(\Delta f)(T) g(T) + f(T) (\Delta g)(T) - \frac{1}{4} (\overline{D} f)(T) (\overline{D} g)(T)
\]

\[
- \frac{1}{4} (\overline{D} f)(T) T (\Delta g)(T) - \frac{1}{4} (\Delta f)(T) T (\overline{D} g)(T) - \frac{1}{4} (\Delta f)(T) T^2 (\Delta g)(T)
\]

\[
= \frac{1}{(2\pi)^2} \int_{\mathbb{G}(\mathbb{C} \cap \mathbb{C}_J)} \int_{\mathbb{G}(\mathbb{C} \cap \mathbb{C}_J)} f(s) ds_j [F_R(s, T) S^L_{-1}(p, T) + S^R_{-1}(s, T) F_L(p, T)]
\]

\[
- 4 Q_{c,s}(T)^{-1} Q_{c,p}(T)^{-1} |dp_j g(p)|.
\]

Now, by [18, Lemma 7.3.2], we know that

\[
F_R(s, T) S^L_{-1}(p, T) + S^R_{-1}(s, T) F_L(p, T) - 4 Q_{c,s}(T)^{-1} Q_{c,p}(T)^{-1}
\]

\[
= [(F_R(s, T) - F_L(s, T)) p - \delta(F_R(s, T) - F_L(p, T))] Q_{s}(p)^{-1}.
\]

Therefore, we obtain

\[
(\Delta f)(T) g(T) + f(T) (\Delta g)(T) - \frac{1}{4} (\overline{D} f)(T) (\overline{D} g)(T)
\]

\[
- \frac{1}{4} (\overline{D} f)(T) T (\Delta g)(T) - \frac{1}{4} (\Delta f)(T) T (\overline{D} g)(T) - \frac{1}{4} (\Delta f)(T) T^2 (\Delta g)(T)
\]

\[
= \frac{1}{(2\pi)^2} \int_{\mathbb{G}(\mathbb{C} \cap \mathbb{C}_J)} \int_{\mathbb{G}(\mathbb{C} \cap \mathbb{C}_J)} f(s) ds_j [(F_R(s, T) - F_L(s, T)) p - \delta(F_R(s, T) - F_L(p, T))] dp_j g(p).
\]
By using similar arguments of the proof of Theorem 4.6, that is, the linearity of the integrals, the Cauchy integral formula, and Lemma 4.5, we obtain

\[(\Delta f)(T)g(T) + f(T)(\Delta g)(T) - \frac{1}{4}(\overline{\Delta f})(T)\overline{\Delta g}(T) - \frac{1}{4}(\Delta f)(T)\overline{\Delta g}(T) - \frac{1}{4}(\Delta f)(T)\overline{\Delta g}(T)\]

\[\begin{align*}
= & \frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C} J)} F_L(p, T) d p f(p) g(p) \\
= & \Delta(fg)(T). \quad \square
\end{align*}\]

5 | RIEZ PROJECTION FOR THE POLYANALYTIC FUNCTIONAL CALCULUS

In the Riesz–Dunford functional calculus, the resolvent equation (4.1) is used to study the Riesz projectors that are given by

\[P_\Omega = \int_\Omega (\lambda I - A)^{-1} d\lambda,\]

where \(A\) is a complex operator on a complex Banach space and the set \(\Omega\) contains part of the spectrum.

The aim of this section is to investigate the Riesz projectors for the \(P_2\)-functional calculus. Before, we need some auxiliary results.

**Theorem 5.1.** Let \(T \in BC(X)\) with \(s \in \rho_S(T)\), then we have

\[P_2^L(s, T)s - TP_2^L(s, T) = 4(S_L^{-1}(s, T) - TQ_{c,s}(T)^{-1})\]

and

\[sP_2^R(s, T) - P_2^R(s, T)T = 4(S_R^{-1}(s, T) - Q_{c,s}(T)^{-1}T).\]

**Proof.** From the definition of the left \(\overline{D}\)-kernel operator and formula (2.15), we get

\[P_2^L(s, T)s - TP_2^L(s, T) = (-F_L(s, T)s + T_0 F_L(s, T))s - T(-F_L(s, T)s + T_0 F_L(s, T))\]

\[= (-F_L(s, T)s + TF_L(s, T))s + T_0(F_L(s, T)s - TF_L(s, T))\]

\[= 4Q_{c,s}(T)^{-1}s - 4T_0 Q_{c,s}(T)^{-1}\]

\[= 4(s - T_0 + T)Q_{c,s}(T)^{-1} - 4TQ_{c,s}(T)^{-1}\]

\[= 4s^{-1}(s, T) - 4TQ_{c,s}(T)^{-1}.\]

Equation (5.2) follows by similar arguments. \(\square\)

In the following result, we provide a suitable generalization of the previous result.

**Theorem 5.2.** Let \(T \in BC(X)\) with \(s \in \rho_S(T)\) and set

\[\mathcal{A}_m^L(s, T) := 4 \sum_{i=0}^{m-1} T^i S_L^{-1}(s, T)s^{m-i-1}, \quad \mathcal{A}_m^R(s, T) := 4 \sum_{i=0}^{m-1} s^{m-i-1} S_R^{-1}(s, T)T^i,\]

and

\[B_m^L(s, T) := 4 \sum_{i=0}^{m-1} T^i Q_{c,s}(T)^{-1}s^{m-i-1}, \quad B_m^R(s, T) := 4 \sum_{i=0}^{m-1} s^{m-i-1} Q_{c,s}(T)^{-1}T^i.\]
Then for \( m \in \mathbb{N} \), we have the following equation:

\[
P^L_2(s, T)s^m - T^m P^L_2(s, T) = A^L_m(s, T) - B^L_m(s, T).
\]

(5.3)

Similarly,

\[
s^m P^R_2(s, T) - P^R_2(s, T)T^m = A^R_m(s, T) - B^R_m(s, T).
\]

(5.4)

**Proof.** We will show only formula (5.3) because formula (5.4) follows by similar computations. We prove the result by induction on \( m \). If \( m = 1 \), we have by formula (5.1)

\[
P^L_2(s, T)s - T P^L_2(s, T) = 4(S^{-1}_L(s, T) - T Q_{c,s}(T)^{-1})
\]

\[
= A^L_1(s, T) - B^L_1(s, T).
\]

Now, we assume that the equation holds for \( m - 1 \) and we will prove it for \( m \). By inductive hypothesis, we have

\[
T^m P^L_2(s, T) = T T^{m-1} P^L_2(s, T) = T(P^L_2(s, T)s^{m-1} - A^L_{m-1}(s, T) + B^L_{m-1}(s, T))
\]

(5.5)

By using formula (5.1), we obtain

\[
T P^L_2(s, T)s^{m-1} = P^L_2(s, T)s^m - 4S^{-1}_L(s, T)s^{m-1} + 4T Q_{c,s}(T)^{-1}s^{m-1}.
\]

(5.6)

Moreover, we have

\[
T A^L_{m-1}(s, T) = 4 \sum_{i=0}^{m-2} T^{i+1} S^{-1}_L(s, T)s^{m-i-2}
\]

(5.7)

and

\[
T B^L_{m-1}(s, T) = 4 \sum_{i=0}^{m-2} T^{i+1} T Q_{c,s}(T)^{-1}s^{m-i-2}
\]

(5.8)

Eventually, by inserting formulas (5.6), (5.7), and (5.8) in (5.5), we get

\[
T^m P^L_2(s, T) = P^L_2(s, T)s^m - 4S^{-1}_L(s, T)s^{m-1} + 4T Q_{c,s}(T)^{-1}s^{m-1}
\]

\[
- 4 \sum_{\ell=1}^{m-1} T^{\ell} S^{-1}_L(s, T)s^{m-\ell-1} + 4 \sum_{\ell=1}^{m-1} T^{\ell} T Q_{c,s}(T)^{-1}s^{m-\ell-1}
\]

\[
= P^L_2(s, T)s^m - 4 \sum_{\ell=0}^{m-1} T^{\ell} S^{-1}_L(s, T)s^{m-\ell-1} + 4 \sum_{\ell=0}^{m-1} T^{\ell} T Q_{c,s}(T)^{-1}s^{m-\ell-1}
\]

\[
= P^L_2(s, T)s^m - A^L_m(s, T) + B^L_m(s, T).
\]

\[\square\]

**Remark 5.3.** Using the relation \( 2T = T - \hat{T} \), we can also write the term \( B^L_m(s, T) \) of Theorem 5.2 in the following way:

\[
B^L_m(s, T) = 2 \sum_{i=0}^{m-1} T^{i+1} Q_{c,s}(T)^{-1}s^{m-i-1} - 2T \sum_{i=0}^{m-1} T^{i} Q_{c,s}(T)^{-1}s^{m-i-1}.
\]
Remark 5.4. Similar equations, like the one in Theorem 5.1, have been considered for the $S$-functional calculus, $F$-functional calculus, and $Q$-functional calculus, see [15, 23]:

1. $S$-functional calculus: 
   \[ S^{-1}_L(s, T) s - TS^{-1}(s, T) = I. \]

2. $Q$-functional calculus: 
   \[ Q_{c,s}(T)^{-1} s - T Q_{c,s}(T)^{-1} = S^{-1}_L(s, T). \]

3. $F$-functional calculus: 
   \[ F_L(s, T) s - T F_L(s, T) = -4 Q_{c,s}(T)^{-1}. \]

For the sake of simplicity, we have considered only the left resolvent operators.

To study the Riesz projectors, we need the following result, which is possible to show in a similar way as [15, Lemma 5.12].

Lemma 5.5. Let \( T \in BC(X) \) be such that \( T = T_0 + T_1 e_1 + T_2 e_2 \) and assume that the operators \( T_0, T_1, T_2 \) have real spectrum. Let \( G \) be a bounded Cauchy domain such that \( (\partial G) \cap \sigma_S(T) = \emptyset \). For every \( J \in \mathbb{S} \), we have

\[
\int_{\partial(G \cap \mathbb{C} J)} Q^{-1}_{c,s}(T) ds_J = 0.
\]

The interesting symmetries that appear in Equation (4.2) allow to study the Riesz projectors.

Theorem 5.6 (Riesz projectors). Let \( T = T_0 + T_1 e_1 + T_2 e_2 \) and assume that the operators \( T_\ell \), with \( \ell = 0, 1, 2 \) have real spectrum. Let \( \sigma_S(T) = \sigma_1 \cup \sigma_2 \) with \( \text{dist}(\sigma_1, \sigma_2) > 0 \). Let \( G_1, G_2 \subset \mathbb{H} \) be two bounded slice Cauchy domains such that \( \sigma_1 \subset G_1 \), \( \overline{G}_1 \subset G_2 \) and \( \text{dist}(G_2, \sigma_2) > 0 \). Then the operator

\[
\tilde{P}_0 := \frac{1}{8\pi} \int_{\partial(G_1 \cap \mathbb{C} J)} P^R_2(s, T) ds_J = \frac{1}{8\pi} \int_{\partial(G_2 \cap \mathbb{C} J)} p dp_J P^R_2(p, T)
\]

is a projection, that is,

\[
\tilde{P}_2^0 = \tilde{P}_0.
\]

Moreover, we have the following commutative relation with respect the operator \( T \) :

\[
T \tilde{P}_0 = \tilde{P}_0 T. \quad (5.9)
\]

Proof. By Theorem 5.1, we know that

\[
S^{-1}_R(s, T) = \frac{1}{4} (s P^R_2(s, T) - P^L_2(s, T) T) + Q_{c,s}(T)^{-1}. \quad (5.10)
\]

Now, by substituting (5.10) in (4.2), we get

\[
\frac{1}{4} s P^R_2(s, T) P^L_2(p, T) - \frac{1}{4} P^R_2(s, T) T P^L_2(p, T) + Q_{c,s}(T)^{-1} P^L_2(p, T) + P^R_2(s, T) S^{-1}_L(p, T) - 4 Q_{c,s}(T)^{-1} T Q_{c,p}(T) = [(P^R_2(s, T) - P^L_2(p, T)) p - \tilde{s}(P^R_2(s, T) - P^L_2(p, T))] Q_s(p)^{-1}. \quad (5.11)
\]

Now, by substituting (5.11) in (4.2), we get

\[
\frac{1}{4} s P^R_2(s, T) P^L_2(p, T) p - \frac{1}{4} P^R_2(s, T) T P^L_2(p, T) p + Q_{c,s}(T)^{-1} P^L_2(p, T) p + P^R_2(s, T) S^{-1}_L(p, T) p - 4 Q_{c,s}(T)^{-1} T Q_{c,p}(T) p = [(P^R_2(s, T) - P^L_2(p, T)) p - \tilde{s}(P^R_2(s, T) - P^L_2(p, T))] Q_s(p)^{-1} p. \quad (5.12)
\]

Now, we multiply formula (5.12) by \( ds_J \) on the left and we integrate it on \( \partial(G_2 \cap \mathbb{C} J) \) with respect to \( ds_J \). Similarly, we multiply formula (5.12) on the right by \( dp_J \) and we integrate it on \( \partial(G_1 \cap \mathbb{C} J) \) with respect to \( dp_J \). Thus, we obtain...
\[
\frac{1}{4} \int_{\partial(G_2 \cap \mathbb{C}_J)} s ds_j P^R_2(s, T) \int_{\partial(G_1 \cap \mathbb{C}_J)} P^L_2(p, T) dp_j p - \frac{1}{4} \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_j P^R_2(s, T)T \int_{\partial(G_1 \cap \mathbb{C}_J)} P^L_2(p, T) dp_j p \\
- \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_j Q_{c,s}(T)^{-1} \int_{\partial(G_2 \cap \mathbb{C}_J)} P^L_2(p, T) dp_j p + \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_j P^R_2(s, T) \int_{\partial(G_1 \cap \mathbb{C}_J)} S^{-1}_L(p, T) dp_j p + \\
- 4 \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_j Q_{c,s}(T)^{-1} T \int_{\partial(G_2 \cap \mathbb{C}_J)} Q_{c,s}(T)^{-1} dp_j = \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_j \int_{\partial(G_1 \cap \mathbb{C}_J)} [(P^R_2(s, T) - P^L_2(p, T)) p \\
- s(P^R_2(s, T) - P^L_2(p, T))] Q_s(p)^{-1} dp_j p.
\]

By Lemma 3.15 and Lemma 5.5, we have
\[
4(2\pi)^2 P^2_0 = \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_j \int_{\partial(G_1 \cap \mathbb{C}_J)} [(P^R_2(s, T) - P^L_2(p, T)) p \\
- s(P^R_2(s, T) - P^L_2(p, T))] Q_s(p)^{-1} dp_j p. \tag{5.13}
\]

Now, since the functions \( p \mapsto Q_s(p)^{-1} \) and \( p \mapsto Q_s(p)^{-1} \) are slice hyperholomorphic and do not have singularities inside \( \partial(G_1 \cap \mathbb{C}_J) \) by the Cauchy theorem, we get
\[
\int_{\partial(G_1 \cap \mathbb{C}_J)} p Q_s(p)^{-1} dp_j p^2 = \int_{\partial(G_1 \cap \mathbb{C}_J)} Q_s(p) dp_j p = 0. \tag{5.14}
\]

This implies that formula (5.13) can be written as
\[
(P^2_0)^2 = -\frac{1}{4(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_j \int_{\partial(G_1 \cap \mathbb{C}_J)} P^L_2(p, T) Q_s(p)^{-1} p dp_j p \\
+ \frac{1}{4(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_j \int_{\partial(G_1 \cap \mathbb{C}_J)} s P^L_2(p, T) Q_s(p)^{-1} p dp_j p \\
= \frac{1}{4(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} \int_{\partial(G_1 \cap \mathbb{C}_J)} ds_j [s P^L_2(p, T) - P^L_2(p, T) p] Q_s(p)^{-1} dp_j p.
\]

By Fubini’s theorem and Lemma 4.5 with \( B := P^L_2(p, T) \), we get
\[
P^2_0 = \frac{1}{8\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} P^L_2(p, T) dp_j p = P_0.
\]

Now, we want to show the commutativity relation (5.9). By (5.1), we know that
\[
TP^L_2(p, T) = P^L_2(s, T) s - 4(S^{-1}_L(s, T) - T_{Q_{c,s}}(T)^{-1}).
\]

From the definition of Riesz projector, we get
\[
T \tilde{P}_0 = \frac{1}{8\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} P^L_2(s, T) ds_j s - \frac{1}{4(2\pi)^2} \int_{\partial(G_1 \cap \mathbb{C}_J)} S^{-1}_L(s, T) ds_j s + \frac{T}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} Q_{c,s}(T)^{-1} ds_j s.
\]

On the other side, by (5.2), we obtain
\[
P^R_2(s, T) = s P^R_2(s, T) - 4(S^{-1}_R(s, T) - T_{Q_{c,s}}(T)^{-1}).
\]

This together with the definition of Riesz projectors, we get
\[
\tilde{P}_0 T = \frac{1}{8\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} s^2 ds_j P^R_2(s, T) - \frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} s ds_j S^{-1}_R(s, T) + \frac{T}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} ds_j Q_{c,s}(T)^{-1}.
\]

Finally, by Theorem 2.31, we have the following statement.
Remark 5.7. Another way to study the Riesz projectors is to rewrite the projector $\tilde{P}_0$ of Theorem 5.6 in another way. Let us consider the set $G_1$ as in the hypothesis of Theorem 5.6, then by [18, Lemma 7.4.1], we know that

$$\int_{\partial(G_2 \cap C_J)} F_L(s,T)ds_j s = 0.$$ 

This implies that

$$\tilde{P}_0 = -\frac{1}{8\pi} \int_{\partial(G_1 \cap C_J)} F_L(s,T)ds_j s^2 - T_0 \frac{1}{8\pi} \int_{\partial(G_1 \cap C_J)} F_L(s,T)ds_j s = -\frac{1}{8\pi} \int_{\partial(G_1 \cap C_J)} F_L(s,T)ds_j s^2 := \tilde{P}.$$ 

The operator $\tilde{P}$ is the Riesz projector for the $F$-functional calculus, see [18, Theorem 7.4.2], namely, $\tilde{P}^2 = \tilde{P}$. Therefore, also the operator $\tilde{P}_0$ is a projector.

### 6 A NEW RESOLVENT EQUATION FOR THE Q-FUNCTIONAL CALCULUS

In this section we provide a new resolvent equation for the $Q$-functional calculus. In [15], the following equation is proved:

$$sQ_{c,s}(T)^{-1}Q_{c,p}(T)^{-1}p - sQ_{c,s}(T)^{-1}\bar{T}Q_{c,p}(T)^{-1} - Q_{c,s}(T)^{-1}\bar{T}Q_{c,p}(T)^{-1}p + Q_{c,s}(T)^{-1}\bar{T}^2 Q_{c,p}(T)^{-1}$$

$$= \left[(sQ_{c,s}(T)^{-1} - pQ_{c,p}(T)^{-1})p - \bar{s}(sQ_{c,s}(T)^{-1} - pQ_{c,p}(T)^{-1})\right](p^2 - 2s_0 p + |s|^2)^{-1}$$

$$+ \left[\overline{(TQ_{c,p}(T)^{-1} - Q_{c,s}(T)^{-1})p} - \bar{s}\overline{(TQ_{c,p}(T)^{-1} - Q_{c,s}(T)^{-1})\bar{T}}\right](p^2 - 2s_0 p + |s|^2)^{-1}. \quad (6.1)$$

This equation can be considered as a resolvent equation for the $Q$-functional calculus because the left slice hyperholomorphicity is maintained in the variables $s$ as well as the right slice hyperholomorphicity is preserved in $p$. Furthermore in [15], this equation is used to study the Riesz projectors for the $Q$-functional calculus. However, in formula (6.1), the term $Q_{c,s}(T)^{-1} - Q_{c,p}(T)^{-1}$ is not transformed into the product of $Q_{c,s}(T)^{-1}$ and $Q_{c,p}(T)^{-1}$. Therefore, one of the main properties of the resolvent equation is missing.

The goal of this section is to obtain a resolvent equation for the $Q$-functional calculus in which a term of the form

$$[Q_{c,s}(T)^{-1} - Q_{c,p}(T)^{-1}] *_{s, \text{left}} S_L^{-1}(s,q)$$

is transformed into the product of $Q_{c,s}(T)^{-1}$ and $Q_{c,p}(T)^{-1}$ and other terms involving the resolvent operators of the $S$-functional calculus. Moreover, we want to maintain the slice hyperholomorphicity.

In order to achieve this aim, we need to recall a suitable modification of the classic $S$-resolvent equation, see [5, Theorem 6.7].

**Theorem 6.1.** Let $T \in BC(X)$ and $B \in B(X)$ such that it commutes with $T$, then we have

$$S_R^{-1}(s,T)BS_L^{-1}(p,T) = [(S_R^{-1}(s,T)B - BS_L^{-1}(p,T))]p + \bar{s}(S_R^{-1}(s,T)B - BS_L^{-1}(p,T))Q_s(p)^{-1}. \quad (6.2)$$

where $Q_s(p) := p^2 - 2s_0 p + |s|^2$.

**Remark 6.2.** If we consider $B = I$ in (6.2), we get (2.12).

Moreover, in order to obtain a new resolvent equation for the $Q$-functional calculus, it is crucial to write the pseudo $S$-resolvent operator in terms of the $F$-resolvent operators, see Theorem 2.29.

Now, we have all the tools to obtain a new resolvent equation for the $Q$-functional calculus.
Theorem 6.3. Let $T \in BC(X)$. For $s, p \in \rho_S(T)$ with $s \notin [p]$, we have the following equation:

$$Q_{c,s}(T)^{-1}S_L^{-1}(p,T) + S_R^{-1}(s,T)Q_{c,p}(T)^{-1} - 2Q_{c,s}(T)^{-1}TQ_{c,p}(T)^{-1}$$

$$= [(Q_{c,s}(T)^{-1} - Q_{c,p}(T)^{-1})p - \bar{s}(Q_{c,s}(T)^{-1} - Q_{c,p}(T)^{-1})]Q_s(p)^{-1},$$

where $Q_s(p) := p^2 - 2s_0p + |s|^2$ and $T = T_1e_1 + T_2e_2 + T_3e_3$.

Proof. We will show this result in seven steps.

**Step I.** We consider $B = T$ in (6.2) and we multiply it on the right by $4Q_{c,p}(T)^{-1}$, then we get

$$-S_R^{-1}(s,T)TF_L(p,T) = [(4S_R^{-1}(s,T)Q_{c,p}(T)^{-1} + TF_L(p,T))p +$$

$$-\bar{s}(4S_R^{-1}(s,T)Q_{c,p}(T)^{-1} + TF_L(p,T))]Q_s(p)^{-1}. \quad (6.4)$$

**Step II.** We consider $B = I$ in (6.2) and we multiply it on the right by $-4Q_{c,p}(T)^{-1}p$, then we obtain

$$S_R^{-1}(s,T)F_L(s,T)p = [(4S_R^{-1}(s,T)Q_{c,p}(T)^{-1}p - F_L(p,T)p) +$$

$$-\bar{s}(4S_R^{-1}(s,T)Q_{c,p}(T)^{-1}p - F_L(p,T)p)]Q_s(p)^{-1}. \quad (6.5)$$

**Step III.** We substitute $B = T$ in (6.2) and we multiply it on the left by $4Q_{c,s}(T)^{-1}$, then we get

$$-F_R(s,T)TS_L^{-1}(s,T) = [(4S_R^{-1}(s,T)Q_{c,p}(T)^{-1}p - F_L(p,T)p) +$$

$$-\bar{s}(4S_R^{-1}(s,T)Q_{c,p}(T)^{-1}p - F_L(p,T)p)]Q_s(p)^{-1}. \quad (6.6)$$

**Step IV.** We substitute $B = I$ in (6.2) and we multiply it on the left by $-4sQ_{c,s}(T)^{-1}$, then we obtain

$$sF_R(s,T)S_L^{-1}(s,T) = [(4sF_R(s,T) + 4sQ_{c,s}(T)^{-1})S_L^{-1}(p,T)p +$$

$$-\bar{s}(4sF_R(s,T) + 4sQ_{c,s}(T)^{-1})S_L^{-1}(p,T)]Q_s(p)^{-1}. \quad (6.7)$$

**Step V.** We make the sum of formulas (6.4), (6.5), (6.6), (6.7) and by Theorem 2.29, we get

$$-4S_R^{-1}(s,T)Q_{c,p}(T)^{-1} - 4Q_{c,s}(T)^{-1}S_L^{-1}(s,T)$$

$$= [(4Q_{c,p}(T)^{-1} - 4Q_{c,s}(T)^{-1})p - \bar{s}(4Q_{c,p}(T)^{-1} - 4Q_{c,s}(T)^{-1})]Q_s(p)^{-1} +$$

$$+ 4[(S_R^{-1}(s,T)TQ_{c,p}(T)^{-1} - S_R^{-1}(s,T)Q_{c,p}(T)^{-1})p - Q_{c,s}(T)^{-1}TS_L^{-1}(p,T) +$$

$$+ sQ_{c,s}(T)^{-1}S_L^{-1}(p,T)p - \bar{s}(S_R^{-1}(s,T)TQ_{c,p}(T)^{-1} - S_R^{-1}(s,T)Q_{c,p}(T)^{-1})p +$$

$$- Q_{c,s}(T)^{-1}TS_L^{-1}(p,T) + sQ_{c,s}(T)^{-1}S_L^{-1}(p,T))]Q_s(p)^{-1}. \quad (6.8)$$

**Step VI.** We show that

$$[(S_R^{-1}(s,T)TQ_{c,p}(T)^{-1} - Q_{c,s}(T)^{-1}TS_L^{-1}(p,T))p +$$

$$-\bar{s}(S_R^{-1}(s,T)TQ_{c,p}(T)^{-1} - Q_{c,s}(T)^{-1}TS_L^{-1}(p,T))]Q_s(p)^{-1} = -Q_{c,s}(T)^{-1}TQ_{c,p}(T)^{-1}. \quad (6.9)$$

We focus on proving formula (6.9). First of all, we observe that by the definition of the $S$-resolvent operators, we have

$$S_R^{-1}(s,T)TQ_{c,p}(T)^{-1} - Q_{c,s}(T)^{-1}TS_L^{-1}(p,T)$$

$$= Q_{c,s}(T)^{-1}(sI - T)TQ_{c,p}(T)^{-1} - Q_{c,s}(T)^{-1}T(pI - T)Q_{c,p}(T)^{-1}$$

$$= Q_{c,s}(T)^{-1}sTQ_{c,p}(T)^{-1} - Q_{c,s}(T)^{-1}TQ_{c,p}(T)^{-1}.$$

Thus, we get
\[
\begin{align*}
(S_R^{-1}(s,T)Q_{c,p}(T)^{-1} - Q_{c,s}(T)^{-1}TS_L^{-1}(p,T))p &+ \\
-\bar{s}(S_R^{-1}(s,T)Q_{c,p}(T)^{-1} - Q_{c,s}(T)^{-1}TS_L^{-1}(p,T))Q_s(p) & \\
= \left( Q_{c,s}(T)^{-1}sTQ_{c,p}(T)^{-1} - Q_{c,s}(T)^{-1}Tp^2Q_{c,p}(T)^{-1} + \\
- Q_{c,s}(T)^{-1}s^2TQ_{c,p}(T)^{-1} + Q_{c,s}(T)^{-1}sTQ_{c,p}(T)^{-1} \right)Q_s(p) & \\
= -Q_{c,s}(T)^{-1}TQ_c(p)Q_{c,p}(T)^{-1}Q_s(p) & \\
= -Q_{c,s}(T)^{-1}TQ_{c,p}(T)^{-1}.
\end{align*}
\]

**Step VII.** The following equality follows by (4.14):
\[
\begin{align*}
\left( (sQ_{c,s}(T)^{-1}S_L^{-1}(p,T) - S_R^{-1}(s,T)Q_{c,p}(T)^{-1}p) \right)p + \\
-\bar{s}(sQ_{c,s}(T)^{-1}S_L^{-1}(p,T) - S_R^{-1}(s,T)Q_{c,p}(T)^{-1}p)Q_s(p) & = Q_{c,s}(T)^{-1}TQ_{c,p}(T)^{-1}.
\end{align*}
\]

**Step VIII.** We put together (6.9) and (6.10) to obtain
\[
\begin{align*}
\left( (S_R^{-1}(s,T)Q_{c,p}(T)^{-1} - S_R^{-1}(s,T)Q_{c,p}(T)^{-1})p - Q_{c,s}(T)^{-1}TS_L^{-1}(p,T) + \\
+ sQ_{c,s}(T)^{-1}S_L^{-1}(p,T) \right)p - \bar{s}(S_R^{-1}(s,T)Q_{c,p}(T)^{-1} - S_R^{-1}(s,T)Q_{c,p}(T)^{-1}p + \\
- Q_{c,s}(T)^{-1}TS_L^{-1}(p,T) + sQ_{c,s}(T)^{-1}S_L^{-1}(p,T))Q_s(p) & = -Q_{c,s}(T)^{-1}TQ_{c,p}(T)^{-1} + Q_{c,s}(T)^{-1}Q_s(T)^{-1}Q_{c,p}(T)^{-1} & \\
& = -2Q_{c,s}(T)^{-1}TQ_{c,p}(T)^{-1}.
\end{align*}
\]

Finally by putting formula (6.11) in (6.8), we get
\[
S_R^{-1}(s,T)Q_{c,p}(T)^{-1} + Q_{c,s}(T)^{-1}S_L^{-1}(p,T) = \\
\left[ (Q_{c,s}(T)^{-1} - Q_{c,p}(T)^{-1})p - \bar{s}(Q_{c,s}(T)^{-1} - Q_{c,p}(T)^{-1})Q_s(p)^{-1} + 2Q_{c,s}(T)^{-1}TQ_{c,p}(T)^{-1} \right].
\]

This proves the statement.

By using the resolvent equation (6.1), in [15] we show the following product rule for the \(Q\)-functional calculus:
\[
2[D(.f)(g)(T) - \overline{T}D(f.g)(T)] = f(T)D(.g)(T) - f(T)\overline{T}D(g)(T) + \\
+D(f(.))(T)g(T) - D(f)(T)\overline{T}g(T).
\]

However, by using formula (6.3), it is possible to obtain a more interesting and nice formula for the product rule of the \(Q\)-functional calculus.

**Theorem 6.4.** Let \(T \in BC(X)\). We assume that \(f \in N(\sigma_S(T))\) and \(g \in SH_f(\sigma_S(T))\), then we have
\[
D(fg)(T) = f(T)(Dg)(T) + (Df)(T)g(T) + (Df)(T)\overline{T}(Dg)(T).
\]

If \(g \in SH_f(\sigma_S(T))\), then we have
\[
D(gf)(T) = f(T)(Dg)(T) + (Df)(T)g(T) + (Df)(T)\overline{T}(Dg)(T).
\]
Proof. Let $G_1$ and $G_2$ be two bounded slice Cauchy domains as in the proof of Theorem 4.6. Let us consider $p \in \partial(G_1 \cap \mathbb{C} \mathcal{J})$ and $s \in \partial(G_1 \cap \mathbb{C} \mathcal{J})$. By the definitions of the $S$-functional calculus and the $Q$-functional calculus, we get

$$f(T)(Dg)(T) + (Df)(T)g(T) + (Df)(T)\overline{(T)(Dg)(T)} =$$

$$= -\frac{1}{2\pi^2} \int_{\partial(G_2 \cap \mathbb{C} \mathcal{J})} S_L^{-1}(s, T) ds f(s) \int_{\partial(G_1 \cap \mathbb{C} \mathcal{J})} Q_{c,p}(T)^{-1} dp g(p)$$

$$- \frac{1}{2\pi^2} \int_{\partial(G_2 \cap \mathbb{C} \mathcal{J})} Q_{c,s}(T)^{-1} ds f(s) \int_{\partial(G_1 \cap \mathbb{C} \mathcal{J})} S_L^{-1}(p, T) dp g(p)$$

$$+ \frac{1}{\pi^2} \int_{\partial(G_2 \cap \mathbb{C} \mathcal{J})} Q_{c,s}(T)^{-1} ds f(s) \int_{\partial(G_1 \cap \mathbb{C} \mathcal{J})} Q_{c,p}(T)^{-1} dp g(p).$$

Since by hypothesis, the function $f$ is intrinsic, by Theorem 2.31 and by Theorem 6.3, we get

$$f(T)(Dg)(T) + (Df)(T)g(T) + (Df)(T)\overline{(T)(Dg)(T)} =$$

$$= \frac{1}{2\pi^2} \int_{\partial(G_2 \cap \mathbb{C} \mathcal{J})} \int_{\partial(G_1 \cap \mathbb{C} \mathcal{J})} f(s) ds \overline{S_L^{-1}(s, T) Q_{c,p}(T)^{-1} - Q_{c,s}(T)^{-1} S_L^{-1}(p, T)} +$$

$$+ 2 Q_{c,s}(T)^{-1} T Q_{c,p}(T)^{-1} dp g(p)$$

$$= -\frac{1}{2\pi^2} \int_{\partial(G_2 \cap \mathbb{C} \mathcal{J})} \int_{\partial(G_1 \cap \mathbb{C} \mathcal{J})} f(s) ds \left[ Q_{c,s}(T)^{-1} - Q_{c,p}(T)^{-1} \right] p + \delta(\overline{Q_{c,s}(T)^{-1} - Q_{c,p}(T)^{-1}}) \cdot Q_{s}(p)^{-1} dp g(p)$$

$$= -\frac{1}{2\pi^2} \int_{\partial(G_2 \cap \mathbb{C} \mathcal{J})} \int_{\partial(G_1 \cap \mathbb{C} \mathcal{J})} f(s) ds \left[ Q_{c,s}(T)^{-1} p Q_{s}(p)^{-1} - Q_{c,p}(T)^{-1} dp g(p) +$$

$$+ \frac{1}{\pi^2} \int_{\partial(G_2 \cap \mathbb{C} \mathcal{J})} \int_{\partial(G_1 \cap \mathbb{C} \mathcal{J})} f(s) ds \left[ Q_{c,p}(T)^{-1} p Q_{s}(p)^{-1} - Q_{c,s}(T)^{-1} Q_{s}(p)^{-1} dp g(p) +$$

$$+ \frac{1}{\pi^2} \int_{\partial(G_2 \cap \mathbb{C} \mathcal{J})} \int_{\partial(G_1 \cap \mathbb{C} \mathcal{J})} f(s) ds \left[ \delta Q_{c,s}(T)^{-1} Q_{s}(p)^{-1} dp g(p) -$$

$$- \frac{1}{\pi^2} \int_{\partial(G_2 \cap \mathbb{C} \mathcal{J})} \int_{\partial(G_1 \cap \mathbb{C} \mathcal{J})} f(s) ds \left[ \delta Q_{c,p}(T)^{-1} Q_{s}(p)^{-1} dp g(p).$$

Since the maps $p \mapsto p Q_{s}(p)^{-1}$ and $p \mapsto Q_{s}(p)$ are intrinsic slice hyperholomorphic on $G_1$, by the Cauchy integral formula, we get that the first and the third integrals in the above formula are zero. By Lemma 4.5 and the definition of $Q$-functional calculus, we get

$$f(T)(Dg)(T) + (Df)(T)g(T) + (Df)(T)\overline{(T)(Dg)(T)} =$$

$$= -\frac{1}{2\pi^2} \int_{\partial(G_2 \cap \mathbb{C} \mathcal{J})} \int_{\partial(G_1 \cap \mathbb{C} \mathcal{J})} f(s) ds \left[ \delta Q_{c,p}(T)^{-1} Q_{s}(p)^{-1} dp g(p) +$$

$$= -\frac{1}{\pi} \int_{\partial(G_1 \cap \mathbb{C} \mathcal{J})} Q_{c,p}(T)^{-1} dp f(g)(p)$$

$$= -\frac{1}{\pi} \int_{\partial(G_1 \cap \mathbb{C} \mathcal{J})} Q_{c,p}(T)^{-1} dp f(g)(p)$$

$$= D(fg)(T).$$

Formula (6.13) follows by similar computations. \qed

Now, we show an application of Theorem 6.4.
Lemma 6.5. Let \( n \geq 1 \). Then we have
\[
\Delta(q^{n+1}) = (\Delta q^n)q_0 - \overline{D}q^n.
\]

Proof. By replacing the operator \( T \in BC(X) \) with a generic quaternion \( q \in \mathbb{H} \) in Theorem 6.4, we get
\[
D(q^{n+1}) = qD(q^n) + D(q)q^n + (Dq)q(Dq^n)
= qD(q^n) - 2q^n - 2\overline{q}D(q^n).
\]
Since \( D(q^n) = -2 \sum_{k=1}^{n} q^{n-k}q^{k-1} \) is real (see [10, 15, Remark 4.4]), we get that
\[
D(q^{n+1}) = (Dq^n)\overline{q} - 2q^n.
\]
By applying the conjugate Fueter operator \( \overline{D} \) to formula (6.14) and by the Leibnitz formula, we get
\[
\Delta(q^{n+1}) = (\Delta q^n)\overline{q} + 2\overline{Dq^n} - 2Dq^n
= (\Delta q^n)\overline{q} - 2(Dq^n) - 2\overline{D}q^n
= (\Delta q^n)q_0 - \overline{D}q^n - (\Delta q^n)\overline{q} - 2Dq^n - \overline{D}q^n.
\]
In order to prove the statement, we have to show the following equality:
\[
(\Delta q^n)\overline{q} + 2Dq^n + \overline{D}q^n = 0.
\]
By [27, Theorem 3.2] and Lemma 3.4, we have
\[
(\Delta q^n)\overline{q} + 2Dq^n + \overline{D}q^n = -4 \sum_{k=1}^{n-1} (n-k)q^{n-k-1}q_{k-1}\left(\frac{q - \overline{q}}{2}\right)
-4 \sum_{k=1}^{n} q^{n-k}q^{k-1} + 2nq^{n-1} + 2 \sum_{k=1}^{n} q^{n-k}q^{k-1}
= -2 \sum_{k=1}^{n} (n-k)q^{n-k}\overline{q}^{k-1} + 2 \sum_{k=1}^{n-1} (n-k)q^{n-k-1}\overline{q}^{k}
-2 \sum_{k=1}^{n} q^{n-k}\overline{q}^{k-1} + 2nq^{n-1}
= -2 \sum_{k=1}^{n} (n-k)q^{n-k}\overline{q}^{k-1} + 2 \sum_{k=0}^{n-1} (n-k)q^{n-k-1}\overline{q}^{k}
-2 \sum_{k=1}^{n} q^{n-k}\overline{q}^{k-1}
= 0. \quad \Box
\]

Remark 6.6. If in formula (4.21) we replace the operator \( T \) with a generic quaternion \( q \in \mathbb{H} \) and by considering \( f(q) = q^n \) and \( g(q) = q \), we get the same result of Lemma 6.5.

By means of the resolvent equation (6.3), it is also possible to study the Riesz projectors for the \( Q \)-functional calculus.

Theorem 6.7. Let \( T = T_0e_0 + T_1e_1 + T_2e_2 \) and assume that the operators \( T_i, i = 0, 1, 2 \), have real spectrum. Let \( \sigma_\delta(T) = \sigma_1 \cup \sigma_2 \) with \( \text{dist}(\sigma_1, \sigma_2) > 0 \).
Let $G_1, G_2 \subset \mathbb{H}$ be two bounded slice Cauchy domains such that $\sigma_1 \subset G_1, \overline{G}_1 \subset G_2$, and $\text{dist}(G_2, \sigma_2) > 0$. Then, the operator

$$\hat{P} := \frac{1}{2\pi} \int_{\partial(G_2 \cap \mathbb{C}J)} s \, ds_j \, Q_{c,s}(T)^{-1} = \frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}J)} Q_{c,p}(T)^{-1} \, dp_j \, p$$

is a projection, that is,

$$\hat{P}^2 = \hat{P}.$$

**Proof.** From the definition of right $S$-resolvent operator, we have

$$S^{-1}_R(s, T) = sQ_{c,s}(T)^{-1} - Q_{c,s}(T)^{-1}T.$$

By inserting formula (6.16) in Equation (6.3) and by multiplying on the right by $p$, we get

$$Q_{c,s}(T)^{-1}S^{-1}_L(p, T) + sQ_{c,s}(T)^{-1}Q_{c,p}(T)^{-1}p - Q_{c,s}(T)^{-1}TQ_{c,p}(T)^{-1}p$$

$$- 2Q_{c,s}(T)^{-1}TQ_{c,p}(T)^{-1}p = [(Q_{c,s}(T)^{-1} - Q_{c,p}(T)^{-1})p - s(Q_{c,s}(T)^{-1} - Q_{c,p}(T)^{-1}))Q_s(p)^{-1}p].$$

Now, we multiply formula (6.17) by $ds_j$ and we integrate it on $\partial(G_2 \cap \mathbb{C}J)$ with respect to $ds_j$ and if we multiply on the right by $dp_j$ and we integrate on $\partial(G_1 \cap \mathbb{C}J)$ with respect to $dp_j$. Therefore, we get

$$\int_{\partial(G_2 \cap \mathbb{C}J)} ds_j \int_{\partial(G_1 \cap \mathbb{C}J)} [Q_{c,s}(T)^{-1} - Q_{c,p}(T)^{-1})p - s(Q_{c,s}(T)^{-1} - Q_{c,p}(T)^{-1})]Q_s(p)^{-1}dp_j \, p.$$

By Lemma 5.5, we get

$$(2\pi)^2 \hat{P}^2 = \int_{\partial(G_2 \cap \mathbb{C}J)} ds_j \int_{\partial(G_1 \cap \mathbb{C}J)} [(Q_{c,s}(T)^{-1} - Q_{c,p}(T)^{-1})p - s(Q_{c,s}(T)^{-1} - Q_{c,p}(T)^{-1})]Q_s(p)^{-1}dp_j \, p.$$

Now, by (5.14), we have

$$\hat{P}^2 = \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}J)} \int_{\partial(G_1 \cap \mathbb{C}J)} ds_j sQ_{c,p}(T)^{-1} - Q_{c,p}(T)^{-1})p]Q_s(p)^{-1}dp_j \, p.$$

By exchanging the role of the integrals and Lemma 4.5 with $B := Q_{c,p}(T)^{-1}$, we get

$$\hat{P}^2 = \frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}J)} Q_{c,p}(T)^{-1} \, dp_j \, p = \hat{P}. \quad \Box$$

We finish by making a table that sums up all the Riesz projectors in the $S$-functional, $Q$-functional, $P_2$-functional, and $F$-functional calculi. We consider the sets $G_1$ as in the hypothesis of Theorem 6.7 and for the sake of simplicity, we consider only the left case.

| **Resolvent operator** | **Riesz projectors** |
|------------------------|----------------------|
| $S$-functional         | $S^{-1}_L(s, T)$     |
| $Q$-functional         | $Q_{c,s}(T)^{-1}$    |
| $P_2$-functional       | $p^2_2(s, T)$        |
| $F$-functional         | $F_1(s, T)$          |
From this table, it is clear that to have a suitable definition of Riesz projector in the functional calculi based on the $S$-spectrum, we have to suitably integrate the respective resolvent operator multiplied by a monomial of a certain degree.

7 | CONCLUDING REMARKS

In this paper, we show that by applying the conjugate Fueter operator, $\overline{D}$, to a slice hyperholomorphic function, we get, as a consequence of the Fueter theorem, a polyanalytic function of order 2. To this set of functions, we associate a functional calculus. If we consider the Fueter theorem in Clifford algebras in dimension of at least 5, it is possible to study polyanalytic functional calculi of order higher than 2.

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**How to cite this article:** A. De Martino and S. Pinton, *Properties of a polyanalytic functional calculus on the \( S \)-spectrum*, Math. Nachr. **296** (2023), 5190–5226. https://doi.org/10.1002/mana.202200318