Two Uncertainty Principles Related to the Linear Canonical S-Transform

Mawardi Bahri
Department of Mathematics, Hasanuddin University, Makassar 90245, Indonesia
E-mail: mawardibahri@gmail.com

Abstract. In this article we derive uncertainty principles for the linear canonical S-transform. This uncertainty principles are obtained by applying the properties and relation between the linear canonical transform and the linear canonical S-transform.

1. Introduction
The main purpose of this article is to derive uncertainty principles for the linear canonical S-transform. To achieve this, we first introduce several important properties of the linear canonical S-transform like orthogonality relation and reconstruction formula. Based on these properties and the basic connection between the linear canonical transform and linear canonical S-transform, we then develop the uncertainty principle associated with the linear canonical S-transform. In addition, based on logarithmic uncertainty principle for the linear canonical transform, we also establish logarithmic uncertainty principle related to the linear canonical S-transform.

The linear canonical transform (lCT) of a complex function $g$ may be defined as follows [1, 2, 3, 4, 5]:

$$C_M\{g\}(v) = \begin{cases} \int_{\mathbb{R}} g(y) C_M(v, y) dy, & n \neq 0 \\ \sqrt{l} e^{i\frac{g}{2}v^2} g(lv), & n = 0 \end{cases}$$

(1)

where $C_M(v, y)$ is given by

$$C_M(v, y) = \frac{1}{\sqrt{2\pi n}} e^{\frac{i}{2} \left( \frac{m}{n} y^2 - \frac{2}{n} y v + \frac{1}{n} v^2 - \frac{\pi}{2} \right)},$$

and $M = \begin{bmatrix} m & n \\ r & l \end{bmatrix} \in \mathbb{R}^{2 \times 2}$ is a matrix parameter such that $|M| = ml - nr = 1$. Throughout this paper we will assume $n \neq 0$. More detailed definitions and properties related to the ICT can be found in [6, 7, 8, 9].

It is not difficult to verify that the following important property holds

$$C_M^{-1}(v, y) = C_M(v, y) = \frac{1}{\sqrt{2\pi n}} e^{-\frac{i}{2} \left( \frac{m}{n} y^2 - \frac{2}{n} y v + \frac{1}{n} v^2 - \frac{\pi}{2} \right)},$$

and

$$M^{-1} = \begin{bmatrix} m^{-1} & -n^{-1} m^{-1} \\ -r^{-1} & l^{-1} \end{bmatrix} \in \mathbb{R}^{2 \times 2}.$$
where \( \bar{g} \) denotes the complex conjugation of \( g \). The reconstruction formula of the lCT can be obtained via formula

\[
C_M^{-1}\{C_M\{g\}\}(y) = g(y) = \int_{\mathbb{R}} C_M\{g\}(v) C_M^{-1}(v,y) \, dv
\]

\[
= \int_{\mathbb{R}} C_M\{g\}(v) \frac{1}{\sqrt{2\pi n}} e^{-\frac{i}{2}(m'y^2 - \frac{2}{n} yv + \frac{L}{n} y^2 - \frac{\pi}{2})} \, dv.
\]

(2)

The inner product in \( L^2(\mathbb{R}) \) follows the formula

\[
(g, h) = \int_{\mathbb{R}} g(y) \overline{h(y)} \, dy.
\]

(3)

We can write the Parseval’s formula for the lCT in the form

\[
(g, h) = (C_M\{g\}, C_M\{h\}).
\]

(4)

Based on the definition of the linear canonical transform above, we obtain the definition of the linear canonical S-transform (lCST). We discuss the connection between the lCST and the lCT which will be necessary to derive the uncertainty principles associated with the lCST.

The linear canonical S-transform of \( g \in L^2(\mathbb{R}) \) with respect to the non-zero window function \( \phi \) is defined as (compare to [10])

\[
S^M_{\phi} g(u, v) = \int_{\mathbb{R}} g(y) \overline{\phi(y-u,v)} \, C_M(y,v) \, dy
\]

\[
= \frac{1}{\sqrt{2\pi n}} \int_{\mathbb{R}} g(y) \overline{\phi(y-u,v)} e^{\frac{i}{2}(m'y^2 - \frac{2}{n} yv + \frac{L}{n} y^2 - \frac{\pi}{2})} \, dy.
\]

(5)

We see that for \( M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \), the linear canonical S-transform becomes S-transform defined by [11, 12]

\[
S_{\phi} g(u, v) = e^{-i\frac{\pi}{4}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) \overline{\phi(y-u,v)} \, e^{-ivy} \, dy.
\]

(6)

2. Fundamental Properties of Linear Canonical S-Transform

The following properties will be used to prove the uncertainty principles for the linear canonical S-transform, which their proofs are left to the reader.

**Theorem 2.1.** Suppose that \( h_1, h_2 \in L^2(\mathbb{R}) \), then

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} S^M_{\phi} h_1(u,v) \overline{S^M_{\phi} h_2(u,v)} \, dv \, du = \left( h_1 \int_{\mathbb{R}} |\phi(u,v)|^2 \, du, h_2 \right).
\]

(7)

In particular, for \( h_1 = h_2 \) it holds

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} |S^M_{\phi} h_1(u,v)|^2 \, dv \, du = \|h_1\|^2_{L^2(\mathbb{R})} \int_{\mathbb{R}} |\phi(u,v)|^2 \, du.
\]

(8)

**Theorem 2.2.** Assume that \( \phi \) satisfies the following:

\[
\int_{\mathbb{R}} |\phi(u,v)|^2 \, du = K_{\phi}, \quad 0 < K_{\phi} < \infty.
\]

(9)

Then, \( h_1 \in L^2(\mathbb{R}) \) can be obtained using the formula

\[
h_1(y) = \frac{1}{K_{\phi} \sqrt{2\pi n}} \int_{\mathbb{R}} S^M_{\phi} h_1(u,v) \phi(u-y,v) e^{-\frac{i}{2}(m'y^2 - \frac{2}{n} yv + \frac{L}{n} y^2 - \frac{\pi}{2})} \, dv \, du.
\]

(10)
It is significant to observe that throughout the rest of the paper, we will always assume that
\[ \int_{\mathbb{R}} |\phi(u, v)|^2 \, du = K_\phi, \quad 0 < K_\phi < \infty. \]  

(11)

3. Uncertainty Principles for the lCST

In this part, we discuss uncertainty principles for the lCST, which states how a complex function relates to its lCST. We shall see that the principle is extension of the uncertainty principle in the ICT domain to the lCST domain. As a result of the logarithmic uncertainty principle for the ICT [13] we obtain a logarithmic uncertainty principle associated with the ICST.

**Theorem 3.1** (ICT uncertainty principle [6, 7]). Let \( h \in L^2(\mathbb{R}) \) be a complex-valued function. If \( C_M\{h\} \in L^2(\mathbb{R}) \) denotes the linear canonical transform of \( h \), then the following inequality holds
\[ \int_{\mathbb{R}} y^2 |h(y)|^2 \, dy \int_{\mathbb{R}} v^2 |C_M\{h\}(v)|^2 \, dv \geq \frac{n^2}{4} \left( \int_{\mathbb{R}} |h(y)|^2 \, dy \right)^2. \]  

Equation (12) above can be rewritten in the form
\[ \int_{\mathbb{R}} y^2 |C_M^{-1}\{C_M\{h\}\}(y)|^2 \, dy \int_{\mathbb{R}} v^2 |C_M\{h\}(v)|^2 \, dv \geq \frac{n^2}{4} \left( \int_{\mathbb{R}} |h(y)|^2 \, dy \right)^2. \]  

(13)

Or, equivalently,
\[ \int_{\mathbb{R}} y^2 |C_M^{-1}\{C_M\{h\}\}(y)|^2 \, dy \int_{\mathbb{R}} v^2 |C_M\{h\}(v)|^2 \, dv \geq \left( \frac{n}{2} \int_{\mathbb{R}} |C_M\{h\}(v)|^2 \, dv \right)^2. \]  

(14)

**Theorem 3.2** (lCST uncertainty principle). Given a complex window function \( \phi \in L^2(\mathbb{R}) \). If \( S_\phi^M h \in L^2(\mathbb{R}) \) is the linear canonical S-transform of \( h \), then for every \( h \in L^2(\mathbb{R}) \) we have the following inequality:
\[ \left( \int_{\mathbb{R}} \int_{\mathbb{R}} v^2 |S_\phi^M h(u, v)|^2 \, dv \, du \right)^{1/2} \left( \int_{\mathbb{R}} y^2 |h(y)|^2 \, dy \right)^{1/2} \geq \frac{n \sqrt{K_\phi}}{2} \|h\|_{L^2(\mathbb{R})}. \]  

(15)

The following lemma will be useful to prove the theorem mentioned above.

**Lemma 3.3.** If \( S_\phi M h \) is defined by (5), then
\[ \int_{\mathbb{R}} |\phi(u, v)|^2 \, du \int_{\mathbb{R}} y^2 |h(y)|^2 \, dy = \int_{\mathbb{R}} \int_{\mathbb{R}} y^2 |C_M^{-1}\{S_\phi^M h(u, v)\}(y)|^2 \, dy \, du. \]  

(16)

**Proof.** It is easy to verify. 

We are now ready to prove Theorem 3.2.

**Proof.** Since \( C_M\{h\} \in L^2(\mathbb{R}) \) and \( S_\phi M h \in L^2(\mathbb{R}) \), then we may replace \( C_M\{h\} \) by \( S_\phi^M h \) on the both sides of (14) to get
\[ \int_{\mathbb{R}} v^2 |S_\phi^M h(u, v)|^2 \, dv \int_{\mathbb{R}} y^2 |C_M^{-1}\{S_\phi^M h(u, v)\}(y)|^2 \, dy \geq \left( \frac{n}{2} \int_{\mathbb{R}} |S_\phi^M h(u, v)|^2 \, dv \right)^2. \]  

(17)
Lemma 3.5. If

\[ C = \psi(t) - \ln \pi, \quad \psi(t) = \frac{d}{dt} \ln[\Gamma(t)], \]

where \( \Gamma(t) \) is Gamma function and \( S(\mathbb{R}) \) is the Schwartz class on complex function.

Applying Parseval’s formula for the ICT (4) into the right-hand side of (22) we easily get

\[ \int_{\mathbb{R}} \ln |y| |h(y)|^2 dy + \int_{\mathbb{R}} |v| |C_M \{h\}(v)|^2 dv \geq (C + \ln |n|) \int_{\mathbb{R}} |h(y)|^2 dy. \]

In this case \( C \) is given by

\[ \mathbb{R} \int \ln |y||h(y)|^2 dy + \int_{\mathbb{R}} \ln |v||C_M \{h\}(v)|^2 dv \geq (C + \ln |n|) \int_{\mathbb{R}} |h(y)|^2 dy. \]

Similarly, we also have the following result.

Theorem 3.4 (ICT logarithmic uncertainty principle). Let \( h \in S(\mathbb{R}) \) be a complex function. Then we have

\[ \int_{\mathbb{R}} \ln |y| |h(y)|^2 dy + \int_{\mathbb{R}} |v| |C_M \{h\}(v)|^2 dv \geq (C + \ln |n|) \int_{\mathbb{R}} |h(y)|^2 dy. \]

In this case \( C \) is given by

\[ \mathbb{R} \int \ln |y||h(y)|^2 dy + \int_{\mathbb{R}} \ln |v||C_M \{h\}(v)|^2 dv \geq (C + \ln |n|) \int_{\mathbb{R}} |h(y)|^2 dy. \]

Similarly, we also have the following result.

Lemma 3.5. If \( S_\phi^M h \) is defined by (5), then

\[ \mathbb{R} \int \ln |y||h(y)|^2 dy = \int_{\mathbb{R}} |\phi(u, v)|^2 du \int_{\mathbb{R}} \ln |y||h(y)|^2 dy \]

\[ = \int \int \ln |y||C_M^{-1}\{S_\phi^M h(u, v)\}(y)|^2 dy du. \]
We now arrive at an inequality related to the lCST, which is expressed in the following result.

**Theorem 3.6.** For $\phi \in S(\mathbb{R})$, then for every $h \in S(\mathbb{R})$ it satisfies the following:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \ln |v| |S^M_\phi h(u,v)|^2 dv du + K_\phi \int_{\mathbb{R}} \ln |y| |h(y)|^2 dy \geq (C + \ln |n|) K_\phi \|h\|_{L^2(\mathbb{R})}^2.$$  \hfill (26)

**Proof.** It is obvious that $C_M \{h\}$ and $S^M_\phi h$ are all in $S(\mathbb{R})$. This enables us to substitute $C_M \{h\}$ by $S^M_\phi h$ on both sides of (24) and obtain

$$\int_{\mathbb{R}} \ln |v| |S^M_\phi h(u,v)|^2 dv + \int_{\mathbb{R}} \ln |y| |C_M^{-1} \{S^M_\phi h(u,v)\}(y)|^2 dy \geq (C + \ln |n|) \int_{\mathbb{R}} |S^M_\phi h(u,v)|^2 dv. \hfill (27)$$

Now integrating this result with respect to $du$ gives

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \ln |v| |S^M_\phi h(u,v)|^2 dv du + \int_{\mathbb{R}} \int_{\mathbb{R}} \ln |y| |C_M^{-1} \{S^M_\phi h(u,v)\}(y)|^2 dy du \geq (C + \ln |n|) \int_{\mathbb{R}} |S^M_\phi h(u,v)|^2 dv du. \hfill (28)$$

From equations (8) and (25) we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \ln |v| |S^M_\phi h(u,v)|^2 dv du + \int_{\mathbb{R}} |\phi(u,v)|^2 dv du \int_{\mathbb{R}} \ln |y| |h(y)|^2 dy \geq (C + \ln |n|) K_\phi \|h\|_{L^2(\mathbb{R})}^2.$$  

Therefore

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \ln |v| |S^M_\phi h(u,v)|^2 dv du + K_\phi \int_{\mathbb{R}} \ln |y| |h(y)|^2 dy \geq (C + \ln |n|) K_\phi \|h\|_{L^2(\mathbb{R})}^2.$$  

We have established (26). \hfill \square

**4. Conclusion**

The linear canonical S-transform has been presented. We investigated its important properties like Parseval’s formula and inversion formula. Using the properties and the relation between the linear canonical transformation and the linear canonical S-transformation we have derived the uncertainty principle for the the linear canonical S-transformation and logarithmic uncertainty principle. The principles could play a key role in the time-frequency analysis in the the linear canonical S-transform domain.

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