Global hyperbolicity is stable in the interval topology\footnote{This archive version includes a detailed discussion of Geroch’s original argument (see Sect. 3), which has been omitted in the published version: J. Math. Phys. 52 (2011) 112504.}

J.J. Benavides Navarro\footnote{Dipartimento di Matematica “U. Dini”, Università degli Studi di Firenze, Viale Morgagni 67/A, I-50134 Firenze, Italy. E-mail: navarro@math.unifi.it} and E. Minguzzi\footnote{Dipartimento di Matematica Applicata “G. Sansone”, Università degli Studi di Firenze, Via S. Marta 3, I-50139 Firenze, Italy. E-mail: ettore.minguzzi@unifi.it}

We prove that global hyperbolicity is stable in the interval topology on the spacetime metrics. We also prove that every globally hyperbolic spacetime admits a Cauchy hypersurface which remains Cauchy under small perturbations of the spacetime metric. Moreover, we prove that if the spacetime admits a complete timelike Killing field, then the light cones can be widened preserving both global hyperbolicity and the Killing property of the field.

1 Introduction

This work is devoted to the proof of the stability of the globally hyperbolic property of spacetimes, where the word stability refers to a particular topology on the space of spacetime metrics. We stress that we are not going to consider the problem of stability of a property under evolution, indeed in this work we do not impose any evolution equation, such as Einstein’s equations, on the spacetime. All the results of this work will concern instead the stability of a property of spacetime where the latter is regarded as a whole.

Of course the interest of physicists for global hyperbolicity comes from the fact that this property assures the predictability of the spacetime evolution from initial data on a spacelike hypersurface. It is a property which allows one to regard the spacetime geometrodynamics as deterministic, in analogy with other field theories. Its stability, in any considered form, is therefore particularly welcomed as it shows that it could be realized in nature. As mentioned, we shall consider only the problem of stability with respect to spacetime metric perturbations rather than Cauchy data perturbations.

In a sense this last problem, at least for global hyperbolicity, makes less sense because, according to the Choquet-Bruhat–Geroch theorem, for every vacuum initial data (given by a Riemannian space and an extrinsic curvature tensor field \((S, h, K)\)), there exists a unique, up to isometric diffeomorphism, vacuum development \((M, g)\), which is globally hyperbolic and inextendible in the class...
of globally hyperbolic Lorentzian manifolds [8]. Thus, perturbing the initial conditions would still lead, after evolution, to a globally hyperbolic spacetime which is still inextendible in the class of globally hyperbolic spacetimes. In this evolutionary sense global hyperbolicity is trivially stable. Of course, if one pairs global hyperbolicity with the inextendibility property (not restricting to the globally hyperbolic class) one obtains a stability problem which is known as Strong Cosmic Censorship Problem [19, 20, 16].

Coming to our framework, the topology on (conformal classes of) spacetime metrics to which we shall refer to is Geroch’s interval topology [5]. In this topology an open set is obtained by giving, continuously and for each spacetime point, two light cones one inside the other and by considering all the metrics whose light cones respect those bounds (see next section). If a metric stays in the open set then perturbing its coefficients in a coordinate chart would keep the metric inside the open set, provided the perturbation is sufficiently small, because the light cone on the tangent space depends continuously on the metric coefficients (indeed the $C^0$ topology on metrics passes to the quotient of conformal classes to Geroch’s interval topology [11]).

Actually, Geroch’s interval topology is one of the coarsest topologies that can be given on the space of (conformal classes of) metrics [11] and hence the stability in this topology is particularly strong. There is only one other important topology that has been used in the literature and which is coarser than Geroch’s interval topology: the compact-open topology [8, Sect. 6.4,7.6] [7]. In this topology a base open set of the topology is obtained as above but the metric light cones are bounded only inside a compact set of spacetime. Nevertheless, it is obvious that a property such as global hyperbolicity cannot be stable in this topology indeed, already for Minkowski $3+1$ spacetime, no matter how large a compact set $C$ is, and how much we constraint the metric inside the compact set, it is possible to open the cones sufficiently far from $C$ so as to produce closed causal curves and hence spoil global hyperbolicity.

An argument of proof was produced by Geroch [5], but, as we shall explain in detail in Sect. 3 that argument does not really work without introducing some non trivial amendments. Perhaps for this reason, this result is not included in any textbook devoted to mathematical relativity.

In the next section we give a direct and strictly topological proof of the stability of global hyperbolicity which, contrary to Geroch’s argument [5], does not use the concept of Cauchy hypersurface or the topological splitting. As a corollary of this result we obtain that every globally hyperbolic spacetime admits a Cauchy hypersurface that remains Cauchy for small perturbations of the spacetime metric (remark 2.9).

We also prove that if the spacetime admits a complete timelike Killing field, then it is possible to enlarge the light cones while keeping global hyperbolicity and stationarity in such a way that the timelike Killing field is left unchanged.

We refer the reader to [15] for most of the conventions used in this work. In particular, we denote with $(M, g)$ a $C^r$ spacetime (connected, Hausdorff, time-oriented Lorentzian and hence paracompact manifold), $r \in 3, \ldots, \infty$ of arbitrary dimension $n \geq 2$ and signature $(-, +, \ldots, +)$. On $M \times M$ the usual
The product topology is defined. The subset symbol \( \subset \) is reflexive, thus \( X \subset X \). We use the dot to denote the boundary of a set, e.g. \( \dot{X} \). With \( J^+_g \) we specify the causal relation referring to metric \( g \). With \( x < y \) we mean that there is a future directed causal curve joining \( x \) and \( y \) and we write \( x \leq y \) (also denoted \( (x, y) \in J^+ \)) if \( x < y \) or \( x = y \). Given two metrics \( g, g' \) we write \( g < g' \) if, for every \( p \in M \), the causal cone of \( g \) in \( TM_p \) is contained in the timelike cone of \( g' \), and we write \( g \leq g' \) if for every \( p \in M \), the causal cone of \( g \) in \( TM_p \) is contained in the causal cone of \( g' \).

2 The proof

Let us recall that a spacetime is non-total imprisoning if no inextendible causal curve is entirely contained in a compact set. A globally hyperbolic spacetime \((M, g)\) is a causal spacetime such that for every \( p, q \in M \), \( J^+(p) \cap J^-(q) \) is compact \([4, 15]\). A spacetime is globally hyperbolic if and only if it admits a Cauchy hypersurface, namely a (closed) acausal set intersected by any inextendible causal curve \([8, 18]\). A useful alternative definition is \([13]\): a spacetime is globally hyperbolic if it is non-total imprisoning and for every \( p, q \in M \), \( J^+(p) \cap J^-(q) \) has compact closure.

A stably causal spacetime \((M, g)\) is a spacetime for which there is a metric \( g' > g \) such that \((M, g')\) is causal. Global hyperbolicity implies stable causality which implies non-total imprisonment \([8, 15]\). The intervals \( (g, \mathcal{I}) = \{g : g < g < \mathcal{I}\} \) form a base for Geroch’s interval topology on the space of conformal classes \( \text{Con}(M) \) (i.e. equivalence classes of metrics which are identical up to a positive conformal factor). This topology is equivalent to that induced from Whitney’s fine \( C^0 \) topology on the space of metrics \( \text{Lor}(M) \) (see \([11]\)).

In general, a conformally invariant property \( P \) of \((M, g)\) is said to be stable in the interval topology if there is an interval \( (g, \mathcal{I}) \supseteq g \) of metrics that share it. Since no mention is made of the differentiability of the metrics contained the intervals this topology is quite coarse and the stability with respect to it is a strong form of stability.

Some properties are inherited by spacetimes obtained by narrowing the light cones, that is: if \((M, g)\) satisfies \( P \) and \( g'' < g \) then \((M, g'')\) satisfies \( P \) (see \([12\ Sect. 2]\)). One such such property is global hyperbolicity as it is evident from the second definition given above. In this case in order to prove the stability of the property we need only to show that there is \( g' > g \) which satisfies the property. In particular, we shall look for a continuous \( g' \). Of course, given a continuous \( g' > g \) which does the job it is not difficult to find another metric in the interval \((g, g')\) with the same degree of differentiability of \( g \). Ultimately, this is the reason why, in all the considerations of stability, the differentiability of the metrics will play no significative role.

The next result is well known, we include the proof for the sake of completeness.

Lemma 2.1. Let \((M, g)\) be a causal spacetime, then it is globally hyperbolic if and only if for every compact set \( K \), \( J^+(K) \cap J^-(K) \) is compact.
Proof. One direction is trivial. For the other direction we have to prove that $J^+(K) \cap J^-(K)$ is compact. Let us consider an arbitrary sequence $p_n \in J^+(K) \cap J^-(K)$, then there are two sequences $r_n, q_n \in K$ such that $r_n < p_n < q_n$. As a first step let us show that $p_n$ is contained in a compact set, which implies that $J^+(K) \cap J^-(K)$ has compact closure. If not we could pass to a subsequence denoted in the same way such that $p_n \to +\infty$ (i.e. escapes every compact set) while $r_n \to r$, $q_n \to q$ for some $r, q \in K$. But then, taking $r' \ll r$ and $q' \gg q$, we have since $I^+$ is open and $I^+ \subset J^+$, $p_n \in J^+(r') \cap J^-(q')$ which is compact, a contradiction. We have therefore shown that there is $p \in M$ such that $p_n \to p$ and it remains to prove that $p \in J^+(K) \cap J^-(K)$. Again passing to a subsequence if necessary we can assume $r_n \to r$ and $q_n \to q$ for some $r, q \in K$. Since $J^+$ is closed (because global hyperbolicity implies causal simplicity) we conclude $r \leq p \leq q$ and hence the thesis.

The next lemma is [12] Lemma 3.2.

**Lemma 2.2.** If $g \leq g'$ then $J^-_g \subset J^-_{g'}$.

**Lemma 2.3.** In a globally hyperbolic spacetime $(M, g)$, if $C$ is a compact set and $\hat{g} > g$ then $J^+(C) = \bigcap_{g < g' < \hat{g}} J^+_{g'}(C)$ (and analogously $J^-(C) = \bigcap_{g < g' < \hat{g}} J^-_{g'}(C)$).

Proof. Let us prove as a first step that $\bigcap_{g < g' < \hat{g}} J^+_{g'}(C) = \bigcap_{g < g' < \hat{g}} J^-_{g'}(C)$. The inclusion $\subset$ is obvious. For the other inclusion let us prove that for every $\hat{g}$, $g < \hat{g} < \hat{g}$, $\bigcap_{g < g' < \hat{g}} J^+_{g'}(C) \subset J^+_{\hat{g}}(C)$, the first step will follow from the arbitrariness of $\hat{g}$. Indeed, let $q \in \bigcap_{g < g' < \hat{g}} J^+_{g'}(C)$ and take $\hat{g}$, $g < \hat{g} < \hat{g}$, such that $\hat{g} < \hat{g}$. Since $q \in J^+_{\hat{g}}(C)$ there is a sequence $q_n \in J^+_{\hat{g}}(C)$ such that $q_n \to q$. Thus there are $p_n \in C$ such that $(p_n, q_n) \in J^+_{\hat{g}}$, and since $C$ is compact we can assume without loss of generality that $p_n \to p \in C$. We conclude that $(p, q) \in J^+_{\hat{g}}$. By lemma 2.2 we have $J^-_{\hat{g}} \subset J^+_{\hat{g}}$ and hence $q \in J^+_{\hat{g}}(p) \subset J^+_{\hat{g}}(C)$, or, due to the arbitrariness of $q$, $\bigcap_{g < g' < \hat{g}} J^+_{g'}(C) \subset J^+_{\hat{g}}(C)$, which completes the first step.

It remains to prove $J^+(C) = \bigcap_{g < g' < \hat{g}} J^+_{g'}(C)$. The inclusion $\subset$ is trivial. For the other inclusion let $q \in \bigcap_{g < g' < \hat{g}} J^+_{g'}(C)$. This means that for every $g' \leq \hat{g}$, $J^+_{\hat{g}}(q) \cap C \neq \emptyset$ and in particular $J^+_{\hat{g}}(q) \cap C \neq \emptyset$. However (Lemma 3.3 of [12] or use [12] Remark 3.8), $J^-_{\hat{g}}(q) = \bigcap_{g < g' < \hat{g}} J^-_{g'}(q)$, thus $J^-_{\hat{g}}(q) \cap C = \bigcap_{g < g' < \hat{g}} (J^-_{g'}(q) \cap C) \neq \emptyset$ where $J^-_{\hat{g}} := \bigcap_{g > \hat{g}} J^-_{g}$ is the Seifert relation [21] [12]. Here the last equality follows from the fact that we are taking the intersection of a family of compact sets which satisfies the finite intersection property (the finite intersection property follows from the fact that given a finite family of metrics $> g$ there is one with the same property and cones narrower than any element of the family). In a globally hyperbolic spacetime $J^-_{\hat{g}} = J^+$ (this result follows from [9] Theorem 2.1), or one can use the stronger results [14] that in a stably causal spacetime $J^-_{\hat{g}}$ is the smallest closed and transitive relation that contains $J^+$ and hence coincides with $J^+$ in globally hyperbolic spacetimes.
See also the appendix.). Thus $J_S^- (q) = J^- (q)$ from which $q \in J^+ (C)$ and the thesis follows.

Lemma 2.4. A globally hyperbolic spacetime $(M, g)$ admits a sequence of compact sets $K_n$ with the properties (actually (i) follows from (ii))

(i) $\bigcup_{n=0}^{\infty} K_n = M$,

(ii) If $C$ is a compact set then there is some $n \geq 0$ such that $C \subset K_n$,

(iii) For $n \geq 0$, $J^+ (K_n) \cap J^- (K_n) \subset \text{Int} K_{n+1}$, 

(iv) There is some metric $g_0 > g$ such that $(M, g_0)$ is stably causal and \( \overline{J_{g_0}^+(K_0)} \cap \text{Int}K_1 \)

Proof. Since $M$ is a second-countable manifold it admits a complete Riemannian metric $h$. Chosen a point $w \in M$ the closed balls $B(w, r)$ of $h$-radius $r \geq 0$ are compact by the Hopf-Rinow theorem. Since the Riemannian distance is continuous it reaches a maximum over a compact set, thus any compact set $C$ is contained in some ball $B(w, r)$ for sufficiently large $r$. Let $K_0 = \{ w \}$ and $r_0 = 1$ and define inductively $K_{n+1} = J^+ (B(w, r_n)) \cap J^- (B(w, r_n))$ and $r_{n+1}$ to be larger than $r_n + 1$ and such that $K_{n+1} \subset \text{Int} B(w, r_{n+1})$. Clearly for $n \geq 1$, $J^+ (K_n) \cap J^- (K_n) \subset J^+ (J^+(B(w, r_{n-1}))) \cap J^- (J^-(B(w, r_{n-1}))) \subset K_n$ thus $J^+ (K_n) \cap J^- (K_n) = K_n$. For $n = 0$ the last equality follows by causality using $K_0 = \{ w \}$. Thus for $n \geq 0$, $J^+ (K_n) \cap J^- (K_n) = K_n \subset \text{Int} B(w, r_n) \subset \text{Int} K_{n+1}$. Furthermore since $r_{n+1} \geq r_n + 1$ property (ii) is satisfied. Finally, global hyperbolicity implies stable causality and stable causality is stable in the interval topology [12 Lemma 2.2] thus there is $g_0 > g$ such that $(M, g_0)$ is stably causal; furthermore $\overline{J_{g_0}^+(K_0)} = J_{g_0}^+(w) = J_{g_0}^+(w)$, where we have used the closure of $J^+$ in a globally hyperbolic spacetime. By causality $\overline{J_{g_0}^+(K_0)} \cap \overline{J_{g_0}^-(K_0)} = \{ w \} \subset \text{Int} B(w, 1) \subset \text{Int} K_1$.

Lemma 2.5. Given $n \geq 0$ assume there is some metric $g_n > g$ such that $\overline{J_{g_n}^+(K_n)} \cap \overline{J_{g_n}^-(K_n)} \subset \text{Int} K_{n+1}$ then there is $g_{n+1} > g$ such that $g_{n+1} \leq g_n$, $g_{n+1} = g_n$ on $K_n$, and $\overline{J_{g_{n+1}}^+(K_{n+1})} \cap \overline{J_{g_{n+1}}^-(K_{n+1})} \subset \text{Int} K_{n+2}$.

Proof. By the properties satisfied by the set $K_n$ and by lemma 2.3 we have

$$\emptyset = J^+ (K_{n+1}) \cap J^- (K_{n+1}) \cap K_{n+2} = \bigcap_{g < g' < g_n} \left( J_{g'}^+ (K_{n+1}) \cap J_{g'}^- (K_{n+1}) \cap K_{n+2} \right).$$

If the compact sets on the right-hand side were non-empty they would satisfy the finite intersection property, and thus also the intersection would be non-empty. The contradiction proves that there is a metric $g_{n+1}$, $g < g_{n+1} < g_n$, such that $\overline{J_{g_{n+1}}^+ (K_{n+1})} \cap \overline{J_{g_{n+1}}^- (K_{n+1})} \cap K_{n+2} = \emptyset$. 

5
This equation implies that no $g'_{n+1}$-causal curve from $K_{n+1}$ can reach $\dot{K}_{n+2}$ and end at $K_{n+1}$ otherwise there would be some point in $J^+_{g'_{n+1}}(K_{n+1}) \cap J^-_{g'_{n+1}}(K_{n+1}) \cap \dot{K}_{n+2}$. We conclude that $J^+_{g'_{n+1}}(K_{n+1}) \cap J^-_{g'_{n+1}}(K_{n+1}) \subset \text{Int} K_{n+2}$. Redefining $g'_{n+1}$ to be in the interval $(g, g'_{n+1})$ and using lemma 2.2 one obtains $J^+_{g'_{n+1}}(K_{n+1}) \cap J^-_{g'_{n+1}}(K_{n+1}) \subset \text{Int} K_{n+2}$.

We can now widen $g_{n+1}$ on $\text{Int} K_{n+1}$ so as to make it equal to $g_n$ on $K_n$ by defining $g_{n+1} = \chi g_{n+1} + (1 - \chi) g_n$ where, $\chi : M \rightarrow [0, 1]$, is a continuous function such that $\chi = 0$ on $K_n$ and $\chi = 1$ on $M \setminus \text{Int} K_{n+1}$. With this definition $g_{n+1} \leq g_n$ and we still have $J^+_{g_{n+1}}(K_{n+1}) \cap J^-_{g_{n+1}}(K_{n+1}) \subset \text{Int} K_{n+2}$ because the metric has been widened only inside $\text{Int} K_{n+1}$ so that $J^+_{g_{n+1}}(K_{n+1}) = J^+_{g'_{n+1}}(K_{n+1})$.

\[\square\]

**Theorem 2.6.** Global hyperbolicity is stable in the interval topology.

**Proof.** We have to prove that on the globally hyperbolic spacetime $(M, g)$ there is some $g' > g$ such that $(M, g')$ is globally hyperbolic.

In the globally hyperbolic spacetime $(M, g)$ there is a sequence of compact sets $K_n$ as in lemma 2.4 and a sequence of metrics $g_n > g$, such that $g_{n+1} \leq g_n$ and $J^+_{g_n}(K_n) \cap J^-_{g_n}(K_n) \subset \text{Int} K_{n+1}$. Indeed, condition (iv) in lemma 2.4 allows us to define inductively the sequence $g_n$ thanks to lemma 2.3. We then define $g'(p) = g_n(p)$ if $p \in K_n$ so that $g' > g$. The fact that $g_{n+1} = g_n$ on $K_n$ proves that $g'$ is continuous. Furthermore, since for every $i$, $g_{i+1} \leq g_i$, we have $g' \leq g_i$ for every $i$. The fact that $g' \leq g_0$ proves that $(M, g')$ is stably causal (see lemma 2.3 point (iv)) and hence non-total imprisoning.

In particular, $J^+_{g'}(K_n) \cap J^-_{g'}(K_n) \subset J^+_{g_n}(K_n) \cap J^-_{g_n}(K_n) \subset \text{Int} K_{n+1}$, thus for every pair $p, q \in M$ there is some $K_n$ which contains $p$ and $q$ thus the set $J^+_{g'}(p) \cap J^-_{g'}(q) \subset J^+_{g_n}(K_n) \cap J^-_{g_n}(K_n) \subset \text{Int} K_{n+1}$ has compact closure. We conclude that $(M, g')$ is globally hyperbolic.

\[\square\]

A time function $t : M \rightarrow \mathbb{R}$ is a continuous function such that $x < y \Rightarrow t(x) < t(y)$. A Cauchy hypersurface $S$ is a closed acausal set which is intersected by any inextendible causal curve. Global hyperbolicity is equivalent to the existence of a Cauchy hypersurface [8].

We recall Geroch’s topological splitting of globally hyperbolic spacetimes [5] [8] Prop. 6.6.8] which we state in a slightly more detailed form (see remark 2.8).

**Theorem 2.7.** Let $(M, g)$ be globally hyperbolic then there is a smooth manifold $S$, a smooth projection $\pi : M \rightarrow S$, a time function $t : M \rightarrow \mathbb{R}$, such that $\phi := (t \times \pi) : M \rightarrow \mathbb{R} \times S$ is a homeomorphism with the property that each hypersurface $S_a = \phi^{-1}(\{a\} \times S)$ is $C^1$- and Cauchy, and the fibers $\pi^{-1}(s)$, $s \in S$, are $(C^1)$ timelike curves. Furthermore, given a smooth timelike vector field $v$, the fibers $\pi^{-1}(s)$ can be chosen to be the integral lines of this field.

**Remark 2.8.** The last statement, the fact that the projection $\pi$ is actually smooth in Geroch’s splitting, and the fact that $S$ is a smooth manifold are
It suffices to multiply the smooth timelike vector field $v$ by a smooth spacetime function which makes it complete. The fact that the quotient manifold is a smooth manifold and that $\pi$ is smooth is then a standard result from manifold theory [10, Theor. 9.16] as the integral lines of this field do not pass arbitrarily close to themselves (because strong causality holds). We shall need this fact only in the stationary case where it has been already used, for instance in [6]. Finally, we remark that one could use also the fact that $\pi$ induces a homeomorphism from $S_a$ to $S$ in order to define a smooth structure on $S_a$. However, it would not be natural, as the inclusion of $S_a$ (endowed with this smooth structure) in $M$ may be non-smooth.

**Remark 2.9.** Thanks to theorem 2.6 this result can be strengthened so that the hypersurfaces $S_a$ are strictly acausal (namely there is $g' > g$ such that $S_a$ are acausal in $(M, g')$). In order to see this it suffices to apply the previous theorem to $(M, g')$ where $g' > g$ is such that $(M, g')$ is globally hyperbolic. Then, since every Cauchy hypersurface for $(M, g')$ is a Cauchy hypersurface for $(M, g)$, one gets easily the thesis. This observation is important as the possibility of finding a strictly acausal Cauchy hypersurface in a globally hyperbolic spacetime has been one of the main difficulties behind the Cauchy hypersurface smoothability problem [2] which essentially asks to prove that, not only the projection $\pi : M \to S$, but also $t : M \to \mathbb{R}$ can be found smooth.

### 2.1 Stationary and static spacetimes

Let $(M, g)$ be a spacetime admitting a timelike Killing vector field $k$. We recall that the timelike Killing field $k$ is twist-free if, defined the 1-form $\eta = g(\cdot, k)$, we have $d\eta \wedge \eta = 0$. This Frobenius condition is equivalent to the vanishing of the vorticity tensor $w_{ab} = h_c^a h_b^d u_{[cd]}$ where $u = k/\sqrt{-g(k, k)}$ and $h^a_b = \delta^a_b + u^a u_b$ is the projector on the vector space perpendicular to $u$ (the equivalence follows from the fact that $w$ is orthogonal to $u$ thus it vanishes if and only if $\varepsilon_{abcd}w^{ab}u^c$ vanishes).

The next result shows that in a globally hyperbolic spacetime it is possible to widen the light cones preserving both global hyperbolicity and stationarity (or staticity).

**Theorem 2.10.** Let $(M, g)$ be a globally hyperbolic spacetime admitting a complete timelike Killing vector field $k$. There is a smooth function $\alpha : M \to \mathbb{R}$, $L_k \alpha = 0$, $\alpha > 0$, such that defined $g' = g - \alpha g(\cdot, k) \otimes g(\cdot, k)$, we have $g' > g$ and $(M, g')$ is globally hyperbolic. In particular $k$ is timelike and Killing also for $(M, g')$. Finally, if $k$ is also twist-free (hypersurface orthogonality, staticity) then $g'$ is such that the property is preserved in $(M, g')$.

**Proof.** Let $\tilde{g} > g$ be such that $(M, \tilde{g})$ is globally hyperbolic and let us consider Geroch’s splitting of the spacetime $(M, \tilde{g})$ as given in theorem 2.7 $\phi = (t \times \pi) : M \to \mathbb{R} \times S$, where $\pi : M \to S$ is the smooth projection on a smooth quotient manifold such that the fibers are the integral lines of $k$. The Killing field is indeed smooth, that is, it reaches the largest differentiability degree allowed
by the differentiability properties of the spacetime manifold and the metric, because for a Killing field: $k_{\beta;\alpha,\mu} = -R_{\mu\nu\alpha\beta}k^{\nu}$.

Let $K_n$ be a sequence of compact sets on $S$ such that $K_n \subset \text{Int}K_{n+1}$ and $S = \cup_n K_n$. Let $\varphi_r$ be the one-parameter group of diffeomorphisms generated by $\kappa$, and let $A = \{ x \in M : x = \varphi_r(s_0), \ s_0 \in S_0, |r| < 1 \}$ where $S_0$ is the Cauchy hypersurface $t = 0$ in Geroch’s splitting. In other words $A$ is the open set between $\varphi_{-1}(S_0)$ and $\varphi_1(S_0)$. We remark that these latter hypersurfaces need not be Cauchy for $\tilde{g}$. The sets $C_n = \overline{\mathcal{F}} \cap \pi^{-1}(K_n)$ are compact sets, indeed $C_n$ is a closed set contained in $\phi^{-1}([-j, j] \times K_j)$ for sufficiently large $j$, and $\phi = (t \times \pi)$ is a homeomorphism (more in detail: for each $s_0 \in \pi^{-1}(K_n) \cap S_0$ the segment of orbit $\varphi_r(s_0), |r| \leq 1$, is compact because of the completeness of $\kappa$ and thus the function $t$ reaches a maximum and a minimum on it that depend continuously on $s_0$, then one uses compactness of $\pi^{-1}(K_n) \cap S_0$). Let $g_{1/k} = g - \frac{1}{k}g(\cdot, k) \otimes g(\cdot, k)$. By compactness and continuity for each $n \geq 1$ there is some $k(n)$ such that $g_{1/k(n)} < \tilde{g}$ on $C_n$ and we can choose $k(n)$ to be an increasing function. Let $\hat{\alpha} : S \rightarrow (0, +\infty)$ be a smooth function such that $\hat{\alpha} < 1/k(n)$ in $K_n \setminus \text{Int}K_{n-1}$.

Let us define $\alpha = \pi^*\hat{\alpha}$, so that $\alpha : M \rightarrow (0, +\infty)$ is smooth and satisfies $L_k\alpha = 0$. The expression $g' = g - \alpha g(\cdot, k) \otimes g(\cdot, k)$ satisfies $L_k g' = 0$ on $M$ and $g' < \tilde{g}$ on $A$. In particular the spacetime $A$ with the metric induced from $g'$ is globally hyperbolic with Cauchy hypersurface $S_0$.

Now we are going to prove that any past inextendible timelike curve $\gamma$ in $(M, g')$ with a point $p$ in the region $t > 0$ must actually intersect $t = 0$ which, together with the dual statement will provide a proof that $S_0$ is a Cauchy hypersurface for $(M, g')$. Indeed, let $A_r = \varphi_r(A)$, so that $A_r$ is made by the hypersurfaces $\varphi_r(S_0)$, for $r - 1 < r' < r + 1$, then each $(A_r, g'|_{A_r})$ is globally hyperbolic (simply because it is isometric with $A$) with Cauchy hypersurface $\varphi_r(S_0)$ (again by isometry). Let $r > 0$ be such that $p = \varphi_r(s_0)$, with $s_0 \in S_0$, then we can find a finite increasing sequence $r_i \geq 0, i = 0, 1, \ldots, m, r_0 = 0, r_m = r$ such that $r_{i+1} < r_i + 1$. Let us consider $A_{r_{m-1}}$. By construction $p$ belongs to $A_{r_{m-1}}$ and stays in the chronological future of $\varphi_{r_{m-1}}(S_0)$ thus $\gamma$ intersects $\varphi_{r_{m-1}}(S_0)$ in a point which we denote $p_{m-1}$. Again by construction $\varphi_{r_{m-1}}(S_0) \subset A_{r_{m-2}}$ thus $\gamma$ intersects a point $p_{m-2} \in \varphi_{r_{m-2}}(S_0)$. Continuing in this way we conclude, after a finite number of steps, that $\gamma$ intersects $S_0$ which proves that $(M, g')$ is globally hyperbolic.

Finally, $g'(\cdot, k) = (1 - \alpha g(k, k))g(\cdot, k)$ thus since $(1 - \alpha g(k, k)) > 0$ the kernel of the 1-form $g'(\cdot, k)$ coincides with the kernel of the 1-form $g(\cdot, k)$. The local integrability of this kernel is the twist-free condition, hence the thesis.

3 Criticism of Geroch’s argument

In its influential paper [5] “Domains of dependence” after introducing the concept of domain of dependence and global hyperbolicity, Geroch devoted a section to the proof of the stability of global hyperbolicity. In this section we shortly review Geroch’s argument clarifying why and where it needs an amendment.
We start recalling Geroch’s construction of the time function in his topological splitting theorem [15, Sect. 3.7]. Let \((M, g)\) be globally hyperbolic. The first step is the introduction of a suitable finite volume measure on spacetime (the measure \(m\) of a complete Riemannian metric \(h\) on \(M\) would work provided it is conformally rescaled to make \(m(M) = 1\)). The functions on \(M\) defined by \(t^-(p) = m(I^-(p))\) and \(t^+ = -m(I^+(p))\) can be proved to be time functions. The former goes to zero following any past inextendible curve, while the latter goes to zero following any future inextendible curve. The time \(t\) appearing in Geroch’s topological splitting theorem is \(t(p) = \ln(-t^-(p)/t^+(p))\) and is surjective on \(\mathbb{R}\). Geroch proves that every surface \(S_n = t^{-1}(a)\) is a Cauchy hypersurface [8, Prop. 6.6.8] and in order to prove the stability of global hyperbolicity he splits the problem in two parts focusing first in the spacetime region \(D^+(S_0)\) and then in \(D^-(S_0)\). His idea is to show that it is possible to widen the cones preserving the Cauchy property of \(S_0\) on past inextendible curves in \(D^+(S_0)\). To do this he regards \(D^+(S_0)\) has a countable union of compact sets \(K_n\), constructed as cups, whose explicit form is \(K_n = I^-(B_n) \cap I^+(S_0)\) where the hypersurface \(B_n\) is defined by \(B_n = \{p \in M : 1/n = m(I^+(p)) \leq m(I^-(p))\}\). He then proves that for each \(n\) there is some \(g_n > g\) such that \(J_{g_n}^+(K_n) \cap J_{g_n}^-(K_n) = K_n\), namely he proves that the compacts \(K_n\) are stably causally convex. Ultimately, it is the attempt at proving this particularly strong result that causes most problems in Geroch’s strategy (in our proof we needed only the weaker result \(J_{g_n}^+(K_n) \cap J_{g_n}^-(K_n) \subset K_{n+1}\)). From this point the proof is not very detailed but it seems that Geroch has in mind a gluing procedure similar to the one that can be found in our proof.

![Figure 1: The spacetime \(\mathbb{R} \times S^1\) of metric \(ds^2 = -dt^2 + d\theta^2\). The timelike geodesic \(\gamma_1\) maximizes the Lorentzian distance between \(S_{-\ln 2}\) and \(p\) while the timelike geodesic \(\gamma_2\) maximizes the Lorentzian distance between \(p\) and \(S_0\). The set \(A\) in Geroch’s argument is perpendicular to \(\gamma_1\) at \(p\) and intersects \(\Sigma\) in more than one point.](image)
The problem with his argument is that in order to prove the strong result
\[ J^+_g(K_n) \cap J^-_g(K_n) = K_n \]
he has to assume that \( S_0 \) and \( B_n \) are strictly spacelike, namely locally strictly acausal (for each \( p \in S_0 \) there is an open set \( O \ni p \), such that \( S \cap O \) is acausal in \((O, g|_O)\)). He thus follows a path that inverts the logical development of our proofs in which our remark 2.9 follows the main theorem 2.6 on stability. The reader may convince herself that the assumption that the Cauchy hypersurfaces are strictly spacelike is particularly strong, in fact as much as the original result on stability that one wishes to prove. Indeed, Geroch feels the need of giving an argument which would allow one to replace the hypersurfaces \( S_0 \) and \( B_n \) with strictly spacelike ones.

This argument is rather involved and in our opinion does not work. We sketch how, according to Geroch, one would have to replace \( S_0 \). He suggests to consider \( S_{- \ln 2} \) and \( S_0 \) and to replace \( S_0 \) with the hypersurface which is equidistant from those according to the Lorentzian distance (which in a globally hyperbolic spacetime is continuous \([1, 13]\)) \[ \Sigma = \{ p \in D^+(S_{- \ln 2}) \cap D^-(S_0) : d(S_{- \ln 2}, p) = d(p, S_0) \} \]
His proof that \( \Sigma \) is strictly spacelike goes as follows. He takes \( p \in \Sigma \setminus \{S_0 \cup S_{- \ln 2}\} \) and considers the maximal (proper length) timelike geodesic \( \gamma \) connecting \( S_{- \ln 2} \) to \( p \), i.e. \( l(\gamma) = d(S_{- \ln 2}, p) \). Then he takes a normal neighborhood \( O \ni p \) and a point \( q \in \gamma \cap O \), so that \( q \leq p \) and \( d(q, p) = \epsilon > 0 \). The set \( A = \{ r : d(q, r) = \epsilon \} \) contains \( p \) and by the reverse triangle inequality for every \( r \in A \), \( d(S_{- \ln 2}, r) \geq l(\gamma) \). From this fact Geroch claims that it follows that \( \Sigma \) is strictly spacelike. It seems that Geroch is assuming that \( A \) intersects \( \Sigma \) only at \( p \) while this is not the case because \( \Sigma \) is not the locus of points at distance \( l(\gamma) \) from \( S_{- \ln 2} \). As \( p \) runs over \( \Sigma \) the length of the maximal geodesic varies. Of course, if \( A \) were entirely on the same side of \( \Sigma \) then it would be possible to open the cone at \( p \), but this is not the case as figure 1 shows. Furthermore, in order to prove the strict spacelikeness of the hypersurface one would have to prove that the cones can be opened in a whole neighborhood of \( p \), which could also be troublesome as \( \gamma \) might not vary continuously with \( p \).

Geroch’s way of replacing the hypersurfaces could be amended by using the smoothability results contained in [2, 3]. One would have to make this replacement for all the Cauchy hypersurfaces appearing in Geroch’s proof showing that this step does not really affect the argument. Of course the proof would be no more self contained and would lose simplicity. Our proof is instead more topological and does not even use the notion of time function not to mention the topological splitting of globally hyperbolic spacetimes.

4 Conclusions

We have given a topological proof of the stability of global hyperbolicity in Geroch’s interval topology. This result implies that every spacetime admits a Cauchy hypersurface which remains Cauchy for small perturbations of the spacetime metric. Moreover, we have proved that the cones can be enlarged preserving not only global hyperbolicity but also the presence of a complete timelike Killing field, being it twist-free or not.
In our opinion the stability of global hyperbolicity might prove to be important for the investigation of the stability of geodesic singularities and singularity theorems, while its role for predictability issues connected with the cosmic censorship problem seems less clear.

At the mathematical level it seems worthwhile to investigate the analogous problem of the stability of causal simplicity. This problem is harder because the property is not preserved by narrowing the light cones, although, examples seem to suggest that causal simplicity is indeed stable in the interval topology.

Acknowledgments

We thank A. Fathi and A. Siconolfi for some useful discussions on the issue of the stability of global hyperbolicity. Their suggestions have allowed us to simplify the proof. This work has been partially supported by GNFM of INDAM under project Giovani Ricercatori 2009 “Stable and generic properties in relativity”.

Appendix

The Seifert relation is defined by \( J^+_S = \bigcap_{g' > g} J^+_g \). In lemma 2.3 we have used the equivalence \( J^+_S = J^+ \) which holds in globally hyperbolic spacetimes. We are going to give a proof of this fact which is similar to [9, Theorem 2.1], but does not use the geodesic triangulation.

We recall that in a globally hyperbolic spacetime the Lorentzian distance \( d : M \times M \to [0, +\infty) \) is finite, continuous and maximized by a connecting geodesic [1].

**Theorem 4.1.** In a globally hyperbolic spacetime \( J^+_S = J^+ \).

**Proof.** Assume that: “if \( x \ll y \) then there is \( g' > g \) such that \( J^+_{g'}(y) \subset I^+(x) \)”.

From this assumption it follows \( J^+_S(y) \subset I^+(x) \subset J^+(x) \). Taking the limit \( x \to y \) and using \( J^+ = J^+ \), \( J^+_S(y) \subset J^+(y) \) the other inclusion being obvious.

It remains to prove \( x \ll y \Rightarrow \) there is \( g' > g \) such that \( J^+_{g'}(y) \subset I^+(x) \).

Let us define for \( 0 < k < 1 \), \( S_k = \{ w : d(x, w) = kd(x, y) \} \). We have that \( I^+(S_k) = \{ w : d(x, w) > kd(x, y) \} \), and \( S_k = \partial I^+(S_k) \), in particular \( S_k \) is an achronal boundary and \( J^+(S_k) \subset I^+(S_k), S_k \subset I^+(x) \). It is trivial to establish that \( S_k \) is actually acausal. We are going to find a strictly spacelike hypersurface \( \tilde{S} \subset I^+(S_{1/3}) \cap I^-(S_{2/3}) \) (see figure 2). Using the results of [4] it would be trivial.

\( ^1 \) (Note not included in the published version.) This work was first submitted to Class. Quantum Grav. on 12th November 2010 and then rejected on 8th July 2011 (to be then rapidly accepted by JMP). Apparently, the referee was very conservative and did not judge positively the possibility of proving the stability of global hyperbolicity without passing through smoothing arguments. We argued that in fact the right approach should be the opposite, which we advocate in remark 2.9, namely to prove topologically that global hyperbolicity is stable and then to prove, as a last step, that smooth time functions exist. This strategy has been successfully followed in the recent paper by A. Fathi and A. Siconolfi: On smooth time functions, to appear in Math. Proc. Camb. Phil. Soc., see e.g. Eq. (32) of that work.
since it is easy to establish that \( S_{1/3} \) and \( S_{2/3} \) are Cauchy hypersurfaces for the spacetime \( I^+(x) \), and there is always a smooth spacelike hypersurface between two Cauchy hypersurfaces, one in the future of the other \([2] \) Prop. 14]. We shall however give a self contained proof.

For every point \( q \in S_{2/3} \) let \( p(q) \) be the intersection between a (make any choice) maximal geodesic \( \sigma \) connecting \( x \) to \( q \) and \( S_{1/3} \). By the reverse triangle inequality this segment from \( x \) to \( p \) is also maximal and since \( p \in S_{1/3} \), it has length \( d(x, y)/3 \), thus the remaining part of the segment, connecting \( p \) to \( q \) has length \( d(x, y)/3 \). For \( \epsilon, 0 < \epsilon < d(x, y)/3 \) sufficiently small we can find a point \( r(q) \in \sigma \) at a distance \( \epsilon \) from \( q \) so that \( r \) is included in a relatively compact convex neighborhood \( C_q \) of \( q \).

Moreover, keeping \( C_q \), there must be an even smaller \( \epsilon \) such that \( (r \) is at distance \( \epsilon \) from \( q \)\) \( J^+(r) \cap J^-(S_{2/3}) \subset C_q \). Indeed if for every \( \epsilon > 0 \) there is some \( s(\epsilon) \in J^+(r) \cap J^-(S_{2/3}) \cap M \setminus C_q \) then connecting \( r(\epsilon) \) to \( s(\epsilon) \) with a maximal geodesic, using the closure of \( J^+ \) in a globally hyperbolic spacetime and taking the limit \( \epsilon \to 0 \) (which implies \( r(q) \to q \)) we could find a point \( s \in C_q \cap J^-(S_{2/3}) \) in the causal future of \( q \) which is a contradiction with the acausality of \( S_{2/3} \).

![Figure 2: The construction of the strictly spacelike hypersurface in the proof of theorem 4.1.](image)

By the global hyperbolicity of \((M, g)\) as \( S_{2/3} \cap \overline{C_q} \) is compact \( J^+(r) \cap J^-(S_{2/3}) \subset J^+(r) \cap J^-(S_{2/3}) \subset S_{2/3} \cap \overline{C_q} \) and hence \( J^+(r) \cap J^-(S_{2/3}) \) is compact.

Thus for sufficiently small \( \epsilon \) the set \( S_{2/3} \cap I^+(r) \) is non-empty, as it contains \( q \), and open relatively compact in the topology of \( S_{2/3} \cap C_q \) and furthermore \( J^+(r) \cap J^-(S_{2/3}) \) is a compact set contained in \( C_q \). In particular \( A(q) = \{ w : d(r(q), w) = \epsilon/2 \} \cap I^+(S_{2/3}) \subset C_q \) is differentiable with spacelike tangent by a property of the distance in a convex set (Gauss lemma [3] Lemma 4.5.2]). The open sets \( O(q) = \{ w : d(r(q), w) > d(r(q), q)/2 \} \cap S_{2/3} \) cover \( S_{2/3} \). Since \( S \) is \( \sigma \)-compact we can pass to a locally finite countable covering \( O(q_i) \).

(*) The set \( \hat{S} = \partial I^+(\bigcup_i A(q_i)) \) is a gluing of the hypersurfaces \( A(q_i) \). In the convex set \( C_q \), the cones can be opened keeping \( A(q_i) \) spacelike and since
locally only a finite number of $C_q$ intersect, the same is true for $\tilde{S}$. (It must be noted that by gluing the sets $A(q_i)$ instead of summing the associated distance functions from $r_i$ we don’t have to deal with some subtleties related to the fact that the squared Lorentzian distance is not $C^1$ where it vanishes.)

(**) Alternatively, instead of using route (*) we can define the function $h_i(z) = \left[ d(r_i, z)/d(r_i, q_i) \right]^4$ if $z \in J^- (S_{2/3})$ (which implies $z \in C_q$, or $h_i(z) = 0$) and $h_i(z) = +\infty$ if $z \in I^+ (S_{2/3})$. The function $h_i$ is a $C^1$ time function wherever it is finite and furthermore it has a timelike gradient wherever it is different from zero (it is sometimes said that $d^2$ is $C^1$ but this is false at the points where the Lorentzian distance vanishes). In order to check that $h_i$ is $C^1$ it suffices to use normal coordinates at $r$ and \[\text{Lemma 5.9}\] according to which $h_i = [(x_0)^2 - (x_1)^2 \ldots - (x_{n-1})^2]^2$. Defined $t = \sum_i h_i$ the function $t$ is a $C^1$ has (past directed) timelike gradient where it is different from zero, and by construction (see the definition of $O(q)$) it takes on $S_{2/3}$ a value greater than $1/2^4$. Thus at $\tilde{S} = t^{-1} (1/2^5)$ the function $t$ is $C^1$ with a timelike gradient.

For the last step, we find $g' > g$ such that $\tilde{S}$ is locally $g'$-acausal. No $g'$-causal curve issued from $y$ can cross $\tilde{S}$ thus $J^+_y(y) \subset I^+(S_{1/2}) \subset I^+(x).$ \hfill $\Box$

References

[1] Beem, J. K., Ehrlich, P. E., and Easley, K. L.: \textit{Global Lorentzian Geometry}. New York: Marcel Dekker Inc. (1996)

[2] Bernal, A. N. and Sánchez, M.: On smooth Cauchy hypersurfaces and Geroch’s splitting theorem. Commun. Math. Phys. 243, 461–470 (2003)

[3] Bernal, A. N. and Sánchez, M.: Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes. Commun. Math. Phys. 257, 43–50 (2005)

[4] Bernal, A. N. and Sánchez, M.: Globally hyperbolic spacetimes can be defined as ‘causal’ instead of ‘strongly causal’. Class. Quantum Grav. 24, 745–749 (2007)

[5] Geroch, R.: Domain of dependence. J. Math. Phys. 11, 437–449 (1970)

[6] Harris, S. G.: Conformally stationary spacetimes. Class. Quantum Grav. 9, 1823–1827 (1992)

[7] Hawking, S. W.: Stable and generic properties in general relativity. Gen. Relativ. Gravit. 1, 393–400 (1971)

[8] Hawking, S. W. and Ellis, G. F. R.: \textit{The Large Scale Structure of Space-Time}. Cambridge: Cambridge University Press (1973)

[9] Hawking, S. W. and Sachs, R. K.: Causally continuous spacetimes. Commun. Math. Phys. 35, 287–296 (1974)
[10] Lee, J. M.: *Introduction to smooth manifolds*. New York: Springer-Verlag (2003)

[11] Lerner, D. E.: The space of Lorentz metrics. Commun. Math. Phys. **32**, 19–38 (1973)

[12] Minguzzi, E.: The causal ladder and the strength of K-causality. II. Class. Quantum Grav. **25**, 015010 (2008)

[13] Minguzzi, E.: Characterization of some causality conditions through the continuity of the Lorentzian distance. J. Geom. Phys. **59**, 827–833 (2009)

[14] Minguzzi, E.: K-causality coincides with stable causality. Commun. Math. Phys. **290**, 239–248 (2009)

[15] Minguzzi, E. and Sánchez, M.: *The causal hierarchy of spacetimes*, Zurich: Eur. Math. Soc. Publ. House, vol. H. Baum, D. Alekseevsky (eds.), Recent developments in pseudo-Riemannian geometry of ESI Lect. Math. Phys., pages 299–358 (2008). ArXiv:gr-qc/0609119

[16] Moncrief, V. and Eardley, D.: The global existence problem and cosmic censorship in general relativity. Gen. Relativ. Gravit. **13**, 887–892 (1981)

[17] Nomizu, K. and Ozeki, H.: The existence of complete Riemannian metrics. Proc. Amer. Math. Soc. **12**, 889–891 (1961)

[18] O’Neill, B.: *Semi-Riemannian Geometry*. San Diego: Academic Press (1983)

[19] Penrose, R.: Gravitational collapse the role of general relativity. Riv. del Nuovo Cim. Numero Speciale **1**, 252–276 (1969). Reprinted in Gen. Rel. Grav. vol 34 p. 1142-1164 (2002)

[20] Penrose, R.: *Singularities and time-asymmetry*, Cambridge: Cambridge University Press, vol. General relativity: An Einstein centenary survey, pages 581–638 (1979)

[21] Seifert, H.: The causal boundary of space-times. Gen. Relativ. Gravit. **1**, 247–259 (1971)