SPECIAL VALUES OF SHIFTED CONVOLUTION DIRICHLET SERIES

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Abstract. In a recent important paper, Hoffstein and Hulse [14] generalized the notion of Rankin-Selberg convolution \(L\)-functions by defining shifted convolution \(L\)-functions. We investigate symmetrized versions of their functions, and we prove that the generating functions of certain special values are linear combinations of weakly holomorphic quasimodular forms and “mixed mock modular” forms.

1. Introduction and Statement of Results

Suppose that \(f_1(\tau) \in S_{k_1}(\Gamma_0(N))\) and \(f_2(\tau) \in S_{k_2}(\Gamma_0(N))\) are even integer weight cusp forms with \(L\)-functions
\[
L(f_i, s) = \sum_{n=1}^{\infty} \frac{a_i(n)}{n^s}.
\]

In the case of equal weights, Rankin and Selberg [23, 25] independently introduced the so-called Rankin-Selberg convolution
\[
L(f_1 \otimes f_2, s) := \sum_{n=1}^{\infty} \frac{a_1(n) a_2(n)}{n^s},
\]
works which are among the most important contributions to the modern theory of automorphic forms. Later in 1965, Selberg [26] introduced shifted convolution \(L\)-functions, and these series have now played an important role in progress towards Ramanujan-type conjectures for Fourier coefficients and the Lindelöf Hypothesis for automorphic \(L\)-functions inside the critical strip (for example, see [1, 10, 11, 18] and the references therein).

In a recent paper, Hoffstein and Hulse [14] introduced the shifted convolution series
\[
D(f_1, f_2, h; s) := \sum_{n=1}^{\infty} \frac{a_1(n+h) a_2(n)}{n^s}.
\]

When \(k_1 = k_2\) they obtained the meromorphic continuation of these series and certain multiple Dirichlet series which are obtained by additionally summing in \(h\) aspect. Moreover, as an important application they obtain a Burgess-type bound for \(L\)-series associated to modular forms.

Here we study the arithmetic properties of these Dirichlet series, where we additionally allow the weights to be non-equal. For this, it is convenient to consider the derived shifted convolution

\[L(\tau) = \sum_{n=1}^{\infty} \frac{a(n)}{n^{s/2}}.\]
series
\begin{equation}
D^{(\nu)}(f_1, f_2; h; s) := \sum_{n=1}^{\infty} \frac{a_1(n + h)a_2(n)(n + h)^{\mu}}{n^s}.
\end{equation}
Of course, we have that \( D^{(0)}(f_1, f_2; h; s) = D(f_1, f_2; h; s) \).

For each \( \nu \geq 0 \) and each \( h \geq 1 \) we define (see Section 3.4) a symmetrized shifted convolution Dirichlet series \( \hat{D}^{(\nu)}(f_1, f_2, h; s) \) using the \( D^{(\nu)}(f_1, f_2, h; s) \). We consider their special values at \( s = k_1 - 1 \) in \( h \)-aspect, and to this end we construct the \( q \)-series
\begin{equation}
\mathbb{L}^{(\nu)}(f_1, f_2; \tau) := \sum_{h=1}^{\infty} \hat{D}^{(\nu)}(f_1, f_2, h; k_1 - 1)q^h,
\end{equation}
where \( q := e^{2\pi i \tau} \). In the special case where \( \nu = 0 \) and \( k_1 = k_2 \), we have that
\[
\hat{D}(f_1, f_2, h; s) = D(f_1, f_2, h; s) - D(f_2, f_1, -h; s),
\]
which then implies that
\[
\mathbb{L}^{(0)}(f_1, f_2; \tau) = \sum_{h=1}^{\infty} \hat{D}(f_1, f_2, h; k_1 - 1)q^h.
\]
It is natural to ask for a characterization of these functions. For example, consider the case where \( f_1 = f_2 = \Delta \), the unique normalized weight 12 cusp form on \( \text{SL}_2(\mathbb{Z}) \). Then we have that
\[
\mathbb{L}^{(0)}(\Delta, \Delta; \tau) = -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 + 4827.434\ldots q^4 - \ldots
\]
Using the usual Eisenstein series \( E_{2k} = E_{2k}(\tau) \) and Klein’s \( j \)-function, we let
\[
\sum_{n=-1}^{\infty} r(n)q^n := -\Delta(j^2 - 1464j - \alpha^2 + 1464\alpha),
\]
where \( \alpha = 106.10455\ldots \). By letting \( \beta = 2.8402\ldots \), we find that
\[
-\frac{\Delta}{\beta} \left( \frac{65520}{691} + \frac{E_2}{\Delta} - \sum_{n \neq 0} r(n)n^{-11}q^n \right)
= -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 + 4827.434\ldots q^4 - \ldots
\]
It turns out that this \( q \)-series indeed equals \( \mathbb{L}^{(0)}(\Delta, \Delta; \tau) \), and the purpose of this paper is to explain such formulæ.

The \( q \)-series \( \mathbb{L}^{(\nu)}(f_1, f_2; \tau) \) arise from the holomorphic projections of completed mock modular forms. In 1980, Sturm [27] introduced the method of holomorphic projection, and in their celebrated work on Heegner points and derivatives of \( L \)-functions, Gross and Zagier [12] further developed this technique. In unpublished work, Zwegers suggested applying such holomorphic projections to the theory of mock modular forms. Recently, Imamoğlu, Raum, and Richter [15] have obtained general theorems in this direction with applications to Ramanujan’s mock theta functions in mind. This work, as well as the preprint [20] by the first author, are based on unpublished notes of Zagier. Here we continue this theme, and we show that the \( \mathbb{L}^{(\nu)}(f_1, f_2; \tau) \) also arise in this way.
The results we obtain depend on the theory of harmonic Maass forms, certain nonholomorphic modular forms which explain Ramanujan’s enigmatic mock theta functions (see [22, 29, 30] and the references therein). For any $f_1$, we denote by $M_{f_1}$ a harmonic Maass form of weight $2 - k_1$ whose shadow is $f_1$ (see Section 2 for the definition). Such forms always exist. Our main result gives a surprising relation between the $\nu$th Rankin-Cohen bracket (again, see Section 2 for definitions) of $M_{f_1}$ and $f_2$ and the generating function $\mathbb{L}^{(\nu)}(f_1, f_2; \tau)$. These Rankin-Cohen brackets are (completed) mixed mock modular forms (see Section 2).

As a technical condition for our result, we need the notion of $M_{f_1}$ being good for $f_2$. By this we mean that the Rankin-Cohen bracket function $[M_{f_1}^+, f_2]_\nu$, where $M_{f_1}^+$ denotes the holomorphic part of $M_{f_1}$, is holomorphic on the upper-half of the complex plane, and is bounded as one approaches all representatives for all cusps (see Section 3 for details). Our main result relates such Rankin-Cohen bracket functions to $\mathbb{L}^{(\nu)}(f_1, f_2; \tau)$ modulo $\bar{M}_{2\nu-k_1+k_2+2}(\Gamma_0(N))$. Here the space $\bar{M}_k(\Gamma_0(N))$ is given by

$$\bar{M}_k(\Gamma_0(N)) = \begin{cases} M_k(\Gamma_0(N)) & \text{if } k \geq 4, \\ C_2 \oplus M_2(\Gamma_0(N)) & \text{if } k = 2, \end{cases}$$

where $M_k(\Gamma_0(N))$ is the space of weight $k$ holomorphic modular forms on $\Gamma_0(N)$. Special care is required for weight 2 because the Eisenstein series $E_2$ is not a holomorphic modular form; it is a quasimodular form. In general we require the spaces $\bar{M}_k(\Gamma_0(N))$, which are the extension of $\bar{M}_k(\Gamma_0(N))$ by the weight $k$ weakly holomorphic modular forms on $\Gamma_0(N)$. Weakly holomorphic modular forms are those meromorphic modular forms whose poles (if any) are supported at cusps. The space of such forms of weight $k$ on $\Gamma_0(N)$ is denoted by $M_k^!(\Gamma_0(N))$.

**Theorem 1.1.** Assume the notation above. If $0 \leq \nu \leq \frac{k_1 - k_2}{2}$, then

$$\mathbb{L}^{(\nu)}(f_2, f_1; \tau) = -\frac{1}{(k_1 - 2)!} \cdot [M_{f_1}^+, f_2]_\nu + F,$$

where $F \in \bar{M}_{2\nu+2-k_1+k_2}(\Gamma_0(N))$. Moreover, if $M_{f_1}$ is good for $f_2$, then $F \in \bar{M}_{2\nu+2-k_1+k_2}(\Gamma_0(N))$.

Two remarks.

(1) One can derive Theorem 1.1 when $\nu = 0$ and $M_{f_1}$ is good for $f_2$ from Theorem 3.5 of [15].

(2) Given cusp forms $f_1$ and $f_2$, it is not generically true that there is a harmonic Maass form $M_{f_1}$ which is good for $f_2$. To see this, consider the case where $f_1 = f_2 = f$, a normalized Hecke eigenform on $\text{SL}_2(\mathbb{Z})$. For $M_f$ to be good for $f_1$, it is necessary that $M_f^+$ has at most a simple pole at infinity. By the theory of Poincaré series (see Section 2.2), it is clear that most Hecke eigenforms $f$ do not satisfy this condition.

Theorem 1.1 is explicitly computable when $f_1 = f_2$ by making use of the theory of Poincaré series (see Section 2.2 for notation and definitions).

**Corollary 1.2.** Suppose that $k \geq 2$ is even and $m$ is a positive integer. If $P(\tau) := P(m, k, N; \tau) \in S_k(\Gamma_0(N))$ and $Q(\tau) := Q(-m, k, N; \tau) \in H_{2-k}(\Gamma_0(N))$, then

$$\mathbb{L}^{(0)}(P, P; \tau) = \frac{1}{m^{k-1} \cdot (k-1)!} \cdot Q^+ P + F,$$
where $F \in \widetilde{M}_2(\Gamma_0(N))$. Moreover, if $m = 1$, then $F \in \widetilde{M}_2(\Gamma_0(N))$.

**Remark.** In Section 2.2 we shall see that $Q(-m, k, N; \tau)$ has a pole of order $m$ at $i\infty$ while $P(m, k, N; \tau)$ generally only has a simple zero there. This explains the special role of $m = 1$ in the corollary above.

**Example 1.** Here we consider the case where $f_1 = f_2 = \Delta$, the unique normalized cusp form of weight 12 on $\text{SL}_2(\mathbb{Z})$. Using the first $10^7$ coefficients of $\Delta$, one obtains the following numerical approximations for the first few shifted convolution values $\hat{D}(\Delta, \Delta, h; 11)$:

| $h$  | 1      | 2      | 3      | 4      | 5      |
|------|--------|--------|--------|--------|--------|
| $\hat{D}(\Delta, \Delta, h; 11)$ | $-33.383\ldots$ | $266.439\ldots$ | $-1519.218\ldots$ | $4827.434\ldots$ | $-5704.330\ldots$ |

We have that $\Delta(\tau) = \frac{1}{\beta} P(1, 12, 1; \tau)$, where $\beta$ can be described in terms of a Petersson norm, or as an infinite sum of Kloosterman sums weighted by $J$-Bessel functions as follows

$$
\beta := \frac{(4\pi)^{11}}{10!} \cdot \|P(1, 12, 1)\|^2 = 1 + 2\pi \sum_{c=1}^{\infty} \frac{K(1, 1, c)}{c} \cdot J_{11}(4\pi/c) = 2.8402\ldots .
$$

Since $m = 1$ and there are no weight 2 holomorphic modular forms on $\text{SL}_2(\mathbb{Z})$, Corollary 1.2 then implies that

$$
\mathbb{L}(0)(\Delta, \Delta; \tau) = \frac{Q^+(-1, 12, 1; \tau) \cdot \Delta(\tau)}{11! \cdot \beta} - \frac{E_2(\tau)}{\beta} = -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 + 4827.434\ldots q^4 - \ldots
$$

Since the expressions for the coefficients of $Q^+(-1, 12, 1; \tau)$ are rapidly convergent, this $q$-series identity provides an extremely efficient method for computing the values $\hat{D}(\Delta, \Delta, h; 11)$.

**Example 2.** Here we consider the newform $f = f_1 = f_2 = \eta(3\tau)^8 \in S_4(\Gamma_0(9))$. This form has complex multiplication by $\mathbb{Q}(\sqrt{-3})$, and is a multiple of the Poincaré series $P(1, 4, 9; \tau)$ because this space of cusp forms is one-dimensional. Here we show how complex multiplication implies some striking rationality properties for the corresponding special values $\hat{D}(f, f, h; 3)$. Using a computer, one obtains the following numerical approximations for the first few shifted convolution values.

| $h$  | 3      | 6      | 9      | 12     | 15     |
|------|--------|--------|--------|--------|--------|
| $\hat{D}(f, f, h; 3)$ | $-10.7466\ldots$ | $12.7931\ldots$ | $6.4671\ldots$ | $-79.2777\ldots$ | $64.2494\ldots$ |

We note that $\hat{D}(f, f, h; 3) = 0$ whenever $n$ is not a multiple of 3.

Below we define real numbers $\beta, \gamma, \delta$ which are approximately

$$
\beta := \frac{(4\pi)^3}{2} \cdot \|P(1, 4, 9)\|^2 = 1.0468\ldots, \quad \gamma = -0.0796\ldots, \quad \delta = -0.8756\ldots
$$
As we shall see, these real numbers arise naturally from the theory of Petersson inner products.

In the table below we consider the first few values of

\[ T(f; h) := \beta \hat{D}(f, f, h; 3) + 24\beta \sum_{d|h} d - 12\beta \sum_{d|h, d \nmid 3d} d. \]

| \( h \) | 3 | 6 | 9 | 12 |
|-------|---|---|---|----|
| \( T(f; h) \) | \(-8.250... \sim -\frac{33}{4}\) | \(22.391... \sim \frac{2799}{125}\) | \(-8.229... \sim -\frac{32919}{4000}\) | \(-61.992... \sim -\frac{8250771}{133100}\) |

It is indeed true that \( T(f; h) \) is rational for all \( h \). To see this, we note that \( f = \frac{1}{\beta} P(1, 4, 9; \tau) \).

By applying Corollary 1.2 with \( m = 1 \), we obtain a congruence between \( Q^+(-1, 4, 9; \tau)f(\tau)/\beta \) and \( L^0(f, f; \tau) \) modulo \( \tilde{M}_2(\Gamma_0(9)) \). In particular, we obtain the following identity

\[ L^0(f, f; \tau) - \frac{Q^+(-1, 4, 9; \tau)f(\tau)}{\beta} = \gamma \left( 1 - 24 \sum_{n=1}^{\infty} \sigma_1(3n)q^{3n} \right) + \delta \left( 1 + 12 \sum_{n=1}^{\infty} \sum_{d|3n, d \nmid \Delta} dq^{3n} \right). \]

The rationality of \( T(f; h) \) follows from the deep fact that \( Q^+(-1, 4, 9; \tau) \), whose first few coefficients are

\[ Q^+(-1, 4, 9; \tau) = q^{-1} - \frac{1}{4}q^2 + \frac{49}{125}q^5 - \frac{3}{32}q^8 - \ldots \]

are all rational. Bruinier, Rhoades and the second author proved a general theorem (see Theorem 1.3 of [7]) which establishes the algebraicity of mock modular forms whose shadows are CM forms.

**Example 3.** Here we consider the case of the weight 24 Poincaré series

\[ f(\tau) := P(2, 24, 1; \tau) \]

\[ = 0.00001585 \ldots q + 2.45743060 \ldots q^2 - 114.85545780 \ldots q^3 + 2845.49507680 \ldots q^4 - \ldots. \]

We have the following numerical approximations for the first few shifted convolution values \( \hat{D}(f, f, h; 23) \):

| \( h \) | 1 | 2 | 3 | 4 | 5 |
|-------|---|---|---|---|---|
| \( \hat{D}(f, f, h; 23) \) | \(-0.00000629 \ldots \) | \(-0.00092041 \ldots \) | \(0.033927 \ldots \) | \(-0.472079 \ldots \) | \(4.628028 \ldots \) |

We define

\[ F := \frac{1}{223} \cdot (\gamma \cdot \frac{E_4^2 E_6}{\Delta} + \delta \cdot E_2) = \frac{\gamma}{223} \cdot q^{-1} + \ldots \]
where $\gamma = -0.00001585 \ldots$ and $\delta = -2.45743 \ldots$. We have that $F \in \tilde{M}'_2(\Gamma_0(1)) \setminus \tilde{M}_2(\Gamma_0(1))$.

It turns out that

$$L(0)(f, f; \tau) = Q + (-2, 24, 1; \tau) \cdot P(2, 24, 1; \tau) \cdot 23! + F,$$

where $Q = -0.00000629 \ldots q - 0.000920 \ldots q^2 + 0.0339 \ldots q^3 - 0.4720 \ldots q^4 + 4.628 \ldots q^5 + \ldots$.

This example illustrates Corollary 1.2 in a situation where there is no $M_f$ which is good for $f$.

As mentioned above, Theorem 1.1 relies critically on the theory of holomorphic projections, harmonic Maass forms, and the combinatorial properties of Rankin-Cohen brackets. In Section 2 we recall the essential properties of the theory of harmonic Maass forms, Maass-Poincaré series, and Rankin-Cohen brackets. In Section 3.4 we define the relevant Dirichlet series and $q$-series $L^{(w)}(f_1, f_2; \tau)$. To obtain Theorem 1.1 we must modify the existing theory of holomorphic projections to obtain a regularized holomorphic projection. This regularization is required for the general case of Theorem 1.1. We give this in Section 3 by suitably modifying earlier work of Gross and Zagier. We then conclude with the proofs of the main results of this paper.

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2. NUTS AND BOLTS

2.1. Harmonic Maass forms. We recall some basic facts about harmonic Maass forms. These real-analytic modular forms were introduced by Bruinier and Funke in [5]. This theory explains the role of Ramanujan’s enigmatic mock theta functions in the theory of automorphic forms (see [22, 29, 30]).

Throughout, we assume that $k \geq 2$ is even, and we let $\tau = x + iy \in \mathbb{H}$, where $x, y \in \mathbb{R}$. We define the weight $k$ slash operator acting on smooth functions $f : \mathbb{H} \to \mathbb{C}$ by

$$(f|k\gamma)(\tau) := (c\tau + d)^{-k} f\left(\frac{ax + by}{c\tau + d}\right),$$

where $\gamma = \left(\begin{smallmatrix}a & b \\ c & d \end{smallmatrix}\right) \in \text{SL}_2(\mathbb{R})$. The weight $k$ hyperbolic Laplacian is defined by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right).$$

Definition 2.1. A smooth function $f : \mathbb{H} \to \mathbb{C}$ is called a harmonic weak Maass form of weight $2-k$ on $\Gamma_0(N)$ if the following conditions hold:

For convenience, we will usually use the term harmonic Maass form and omit the word “weak.”
(1) \( f \) is invariant under the action of \( \Gamma_0(N) \), i.e. for all \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N) \) we have
\[
(f|_2 \gamma)(\tau) = f(\tau)
\]
for all \( \tau \in \mathbb{H} \).

(2) \( f \) lies in the kernel of the weight \( 2 - k \) hyperbolic Laplacian, i.e.
\[
\Delta_{2-k} f \equiv 0.
\]

(3) \( f \) grows at most linearly exponentially at the cusps of \( \Gamma_0(N) \).

The \( \mathbb{C} \)-vector space of harmonic weak Maass forms of weight \( 2 - k \) on \( \Gamma_0(N) \) is denoted by \( H_{2-k}(\Gamma_0(N)) \).

The following property of these functions is well known (for example, see equations (3.2a) and (3.2b) in [5]).

**Lemma 2.2.** Let \( f \in H_{2-k}(\Gamma_0(N)) \) be a harmonic Maass form. Then there is a canonical splitting
\[
f(\tau) = f^+(\tau) + \frac{(4\pi y)^{1-k}}{k-1} c_f(0) + f^-(\tau),
\]
where for some \( m_0, n_0 \in \mathbb{Z} \) we have the Fourier expansions
\[
f^+(\tau) := \sum_{n=m_0}^{\infty} c_f^+(n) q^n,
\]
and
\[
f^-(\tau) := \sum_{n=n_0 \neq 0}^{\infty} c_f^-(n) q^{n k-1} \Gamma(1-k; 4\pi n y) q^{-n},
\]
where \( \Gamma(\alpha; x) \) denotes the usual incomplete Gamma-function.

The series \( f^+ \) in the above lemma is the **holomorphic part** of \( f \), and it is known as a **mock modular form** when the non-holomorphic part \( \frac{(4\pi y)^{1-k}}{k-1} c_f(0) + f^-(\tau) \) doesn’t vanish. Products of mock modular forms and usual modular forms, as well as linear combinations of such functions, are called **mixed mock modular forms**.

The explanation for the complex conjugation of the coefficients in the non-holomorphic part \( \frac{(4\pi y)^{1-k}}{k-1} c_f(0) + f^-(\tau) \) of \( f \) is given in the following proposition due to Bruinier and Funke (see Proposition 3.2 and Theorem 3.7 in [5]).

**Proposition 2.3.** The operator
\[
\xi_{2-k} : H_{2-k}(\Gamma_0(N)) \to M'_k(\Gamma_0(N)), \quad f \mapsto \xi_{2-k} f := 2iy^{2-k} \frac{\partial f}{\partial \tau}
\]
is well-defined and surjective with kernel \( M'_{2-k}(\Gamma_0(N)) \), the space of weakly holomorphic modular forms of weight \( 2 - k \) on \( \Gamma_0(N) \). Moreover, we have that
\[
(\xi_{2-k} f)(\tau) = -(4\pi)^{k-1} \sum_{n=n_0}^{\infty} c_f^-(n) q^n.
\]
The function $-(4\pi)^{1-k}\xi_{2-k}f$ is called the shadow of the mock modular form $f^+$. For our purpose, we only consider cases, where the shadow of $f^+$ is a cusp form. Since the $\xi$-operator is surjective, the natural question arises on how to construct a harmonic Maass form with prescribed shadow. This can be done using Poincaré series which we recall next.

2.2. Poincaré series. Here we recall the classical cuspidal Poincaré series and the harmonic Maass-Poincaré series. We then give their relationship under the $\xi$-operator described above. In order to do so, let us define the Kloosterman sums by

$$K(m, n, c) := \sum_{v(c)^\times} e\left(\frac{m\overline{v} + n\overline{v}}{c}\right),$$

where $e(\alpha) := e^{2\pi i \alpha}$. The sum over $v$ runs through the primitive residue classes modulo $c$, and $\overline{v}$ denotes the multiplicative inverse of $v$ modulo $c$.

The Fourier expansions of the cuspidal Poincaré series (for example, see [16]) are described in terms of infinite sums of Kloosterman sums weighted by $J$-Bessel functions.

Lemma 2.4. If $k \geq 2$ is even and $m, N \in \mathbb{N}$, then the following are true for the Poincaré series

$$P(m, k, N; \tau) = q^m + \sum_{n=1}^\infty a(m, k, N; n)q^n.$$

1. We have that $P(m, k, N; \tau) \in S_k(\Gamma_0(N))$, and the coefficients are given by

$$a(m, k, N; n) = 2\pi (-1)^{\frac{k}{2}} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \cdot \sum_{c>0 (\text{mod } N)} \frac{K(m, n, c)}{c} \cdot J_{k-1}\left(\frac{4\pi \sqrt{mn}}{c}\right).$$

2. We have that $P(-m, k, N; \tau) \in M_k^!(\Gamma_0(N))$, and for positive integers $n$ we have

$$a(-m, k, N; n) = 2\pi (-1)^{\frac{k}{2}} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \cdot \sum_{c>0 (\text{mod } N)} \frac{K(-m, n, c)}{c} \cdot I_{k-1}\left(\frac{4\pi \sqrt{|mn|}}{c}\right).$$

Now we recall one family of Maass-Poincaré series which have appeared in earlier works (for example, see [3, 4, 9, 13, 21]).

Lemma 2.5. If $k \geq 2$ is even and $m, N \geq 1$, then $Q(-m, k, N; \tau)$ is in $H_{2-k}(\Gamma_0(N))$. Moreover, if $I_{k-1}$ is the usual $I$-Bessel function, then

$$Q(-m, k, N; \tau) = (1-k)\left(\Gamma(k-1, 4\pi my) - \Gamma(k-1)\right)q^{-m} + \sum_{n \in \mathbb{Z}} c_m(n, y)q^n.$$
1) If \( n < 0 \), then
\[
c_m(n, y) = 2\pi i^k (1 - k) \Gamma(k - 1, 4\pi|n|y) \left| \frac{n}{m} \right|^{\frac{1-k}{2}} \times \sum_{\substack{c \equiv 0 \pmod{N} \atop c > 0}} K(-m, n, c) \frac{1}{c} \cdot \frac{J_{k-1} \left( \frac{4\pi \sqrt{|mn|}}{c} \right)}{c}.
\]

2) If \( n > 0 \), then
\[
c_m(n, y) = -2\pi i^k \Gamma(k) \left( \frac{n}{m} \right)^{\frac{1-k}{2}} \sum_{\substack{c \equiv 0 \pmod{N} \atop c > 0}} K(-m, n, c) \frac{1}{c} \cdot I_{k-1} \left( \frac{4\pi \sqrt{|mn|}}{c} \right).
\]

3) If \( n = 0 \), then
\[
c_m(0, y) = -(2\pi i)^k m^{k-1} \sum_{\substack{c \equiv 0 \pmod{N} \atop c > 0}} K(-m, 0, c) \frac{1}{c^k}.
\]

These two families of Poincaré series are closely related (for example, see Theorem 1.1 in [3]).

Lemma 2.6. If \( k \geq 2 \) is even and \( m, N \geq 1 \), then
\[
\xi_{2-k}(Q(-m, k, N; \tau)) = (4\pi)^{k-1} m^{k-1}(k - 1) \cdot P(m, k, N; \tau) \in S_k(\Gamma_0(N)).
\]

Proof. The claim follows easily from the explicit expansions in Lemma 2.4 and 2.5 and the definition of \( \xi_{2-k} \). \qed

We conclude this section with a well-known result about the behaviour of the Maass-Poincaré series at the cusps (see Proposition 3.1 in [6]).

Lemma 2.7. For \( k \geq 2 \) and \( m, N \geq 1 \), the harmonic Maass form \( Q(-m, k, N; \tau) \) grows like \( \exp(2\pi my) \) as \( \tau \) approaches the cusp \( i\infty \) and grows moderately approaching any other cusp of \( \Gamma_0(N) \). The cuspidal Poincaré series \( P(m, k, N; \tau) \) generically has simple zeros in all the cusps.

2.3. Rankin-Cohen Brackets. Rankin-Cohen brackets are a generalization of the usual product of modular forms. They are bilinear differential operators which map modular forms to modular forms. They were introduced independently by Rankin in [24] and Cohen in [8]. A very nice discussion of them may be found in [28].

Definition 2.8. Let \( f, g : \mathbb{H} \to \mathbb{C} \) be smooth functions on the upper half-plane and \( k, \ell \in \mathbb{R} \) be some real numbers, the weights of \( f \) and \( g \). Then for a non-negative integer \( \nu \) we define the \( \nu \)th Rankin-Cohen bracket of \( f \) and \( g \) by
\[
[f, g]_\nu := \frac{1}{(2\pi i)^\nu} \sum_{\mu=0}^{\nu} (-1)^{\mu} \binom{k + \nu - 1}{\nu - \mu} \binom{\ell + \nu - 1}{\mu} \frac{\partial^\mu f}{\partial \tau^\mu} \frac{\partial^{\nu-\mu} g}{\partial \tau}.
\]

The following result (see Theorem 7.1 in [8]) gives the connection of this definition to modular forms.
Theorem 2.9 (Cohen). Let \( f, g \) as in Definition 2.8 with integral weights \( k, \ell \). Then the \( \nu \)th Rankin-Cohen bracket commutes with the slash operator, i.e.
\[
[(f|k\gamma), (g|\ell\gamma)]_\nu = ([f, g]_\nu)|_{k+\ell+2\nu}\gamma
\]
for all \( \gamma \in \text{SL}_2(\mathbb{R}) \). In particular, if \( f \) and \( g \) are modular of their respective weights, then \([f, g]_\nu\) is modular of weight \( 2\nu + k + \ell \).

Remark. The 0th Rankin-Cohen bracket of \( f \) and \( g \) is just their usual product, the first bracket defines a Lie bracket on the graded algebra of real-analytic modular forms, giving it the structure of a so-called Poisson algebra.

Remark. The \( \nu \)th Rankin-Cohen bracket is symmetric in \( f \) and \( g \) if \( \nu \) is even, otherwise it is antisymmetric.

Since the Rankin-Cohen bracket provides a “product” on the algebra of modular forms, it is natural to consider functions of the form \([f^+, g]_\nu\) for \( f \in H_{2-k}(\Gamma_0(N)) \) and \( g \in M_\ell(\Gamma_0(N)) \) to be mixed mock modular forms as well.

An interesting observation, that doesn’t seem to appear in the literature, is the following property of Rankin-Cohen brackets of harmonic Maass forms and holomorphic modular forms.

Lemma 2.10. Let \( f \in H_{2-k}(\Gamma_0(N)) \) and \( g \in M_\ell(\Gamma_0(N)) \) for even, positive integers \( k, \ell \). Then for \( \nu > k-2 \) the Rankin-Cohen bracket \([f, g]_\nu\) is a weakly holomorphic modular form of weight \( 2\nu - k + \ell + 2 \).

Proof. Consider the operators
\[
R_k := 2i \frac{\partial}{\partial \tau} + ky^{-1},
\]
\[
L_k := -2iy^2 \frac{\partial}{\partial \tau},
\]
known as the Maass raising operator, resp. Maass lowering operator. These operators map modular forms of weight \( k \) to modular forms of weight \( k + 2 \) (resp. \( k - 2 \)).

An explicit description of the iterated raising operator \( R_k^n := R_{k+2(n-1)} \circ \ldots \circ R_{k+2} \circ R_k \) for \( n > 0 \), \( R_k^0 := \text{id} \) is given by (see \[19\], Equation (4.15))
\[
R_k^n = \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(k+n)}{\Gamma(k+n-m)} y^{-m} (2i)^{n-m} \frac{\partial^{n-m}}{\partial \tau^{n-m}},
\]
which follows easily by induction on \( n \).

With this one obtains by straightforward calculation that if \( f \in H_{2-k}(\Gamma_0(N)) \) and \( g \in M_\ell(\Gamma_0(N)) \), then we have that
\[
L_{2\nu-k+\ell+2}([f, g]_\nu) = \frac{1}{(4\pi)^\nu} \binom{k-2}{\nu} L_{2-k}(f) R_\ell^\nu(g).
\]
But for \( k \in \mathbb{N} \), this is 0 whenever \( \nu > k - 2 \), thus \([f, g]_\nu\) must be holomorphic. \( \square \)

3. Holomorphic Projection and the Proof of Theorem 1.1

Here we prove Theorem 1.1. We begin by recalling the principle of holomorphic projection.
3.1. **Holomorphic Projection.** The basic idea behind holomorphic projection is the following. Suppose you have a weight $k$ real-analytic modular form $f$ on $\Gamma_0(N)$ with moderate growth at the cusps. Then this function defines a linear functional on the space of weight $k$ holomorphic cusp forms by $g \mapsto \langle g, f \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the Petersson scalar product. But this functional must be given by $\langle \cdot, f \rangle$ for some holomorphic cusp form $f$ of the same weight. This $f$ is essentially the holomorphic projection of $f$. In [27], Sturm introduced this notion, which was used and further developed later for example in the seminal work of Gross and Zagier on Heegner points and derivatives of $L$-function [12]. The technique is also used in order to obtain recurrence relations among Fourier coefficients of mock modular forms, see e.g. [15, 20].

Here we briefly recall this framework for holomorphic projections.

**Definition 3.1.** Let $f : \mathbb{H} \to \mathbb{C}$ be a (not necessarily holomorphic) modular form of weight $k$ on $\Gamma_0(N)$ with a Fourier expansion

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_f(n, y)q^n,$$

where again $y = \Im(\tau)$. For a cusp $\kappa_j$, $j = 1, ..., M$ and $\kappa_1 := i\infty$, of $\Gamma_0(N)$ fix $\gamma_j \in \text{SL}_2(\mathbb{Z})$ with $\gamma_j \kappa_j = i\infty$. Suppose that for some $\varepsilon > 0$ we have

1. $f(\gamma_j^{-1}w) \left( \frac{d}{dw} \right)^{\frac{k}{2}} = c_0^{(j)} + O(\Im(w)^{-\varepsilon})$ for all $j = 1, ..., M$ and $w = \gamma_j \tau$,
2. $a_f(n, y) = O(y^{2-k})$ as $y \to 0$ for all $n > 0$.

Then we define the **holomorphic projection** of $f$ by

$$\left( \pi_{\text{hol}} f \right)(\tau) := \left( \pi_{\text{hol}}^k f \right)(\tau) := c_0 + \sum_{n=1}^{\infty} c(n)q^n,$$

with $c_0 = c_0^{(1)}$ and

$$c(n) = \frac{(4\pi n)^{k-1}}{(k-2)!} \int_0^{\infty} a_f(n, y)e^{-4\pi ny}y^{k-2}dy$$

for $n > 0$.

This operator has several nice properties, some of which are summarized in the following proposition.

**Proposition 3.2.** Let $f$ be as in Definition 3.1. Then the following are true.

1. If $f$ is holomorphic, then $\pi_{\text{hol}} f = f$.
2. The function $\pi_{\text{hol}} f$ lies in the space $\tilde{M}_k(\Gamma_0(N))$.

**Proof.** Claim (1) of this proposition is an easy and elementary calculation, see e.g. Proposition 3.2 of [15], while the first proof of (2) using Poincaré series (and “Hecke’s trick” for the case of weight 2) was given in Proposition 5.1 and Proposition 6.2 in [12]. A different proof which uses spectral methods and the language of vector-valued modular forms is given in Theorem 3.3 in [15]. □

**Proposition 3.2** when combined with the previous results on Rankin-Cohen brackets and harmonic Maass forms is sufficient for proving Theorem 1.1 when $M_{f_1}$ is good for $f_2$. As
mentioned in the introduction, this situation is quite rare. Therefore, we must extend this notion to accommodate the most general cases. To this end we introduce regularized holomorphic projections.

3.2. Regularized holomorphic projections. The classical holomorphic projection is constructed to respect the Petersson inner product on the space of cusp forms. We construct a regularized holomorphic projection by the same recipe. This projection shall respect the regularized Petersson inner product introduced by Borcherds in [2].

**Definition 3.3.** Let \( f : \mathbb{H} \to \mathbb{C} \) be a real-analytic modular form of weight \( k \geq 2 \) with a Fourier expansion

\[
f(\tau) = \sum_{n \in \mathbb{Z}} a_f(n, y)q^n.
\]

Suppose further that for each cusp \( \kappa \) of \( \Gamma_0(N) \) there exists a polynomial \( H_\kappa(X) \in \mathbb{C}[X] \), such that we have

\[
(f|k)_{\kappa}^{-1}(\tau) - H_\kappa(q^{-1}) = y^{-\varepsilon}
\]

for some \( \varepsilon > 0 \) and further suppose that \( a_f(n, y) = O(y^{2-k}) \) as \( y \to 0 \) for all \( n > 0 \). Then we define the regularized holomorphic projection of \( f \) by

\[
(\pi_{reg} f) = H_{i\infty}(q^{-1}) + \sum_{n=1}^{\infty} c(n)q^n,
\]

where

\[
c(n) = \lim_{s \to 0} \left( \frac{4\pi n}{(k-2)!} \right) \int_{0}^{\infty} a_f(n, y)e^{-4\pi ny}y^{k-2-s} dy.
\]  

(3.2)

This operator enjoys a natural analog of Proposition 3.2.

**Proposition 3.4.** Let \( f \) be a function as in Definition 3.3.

1. If \( f \) is holomorphic on \( \mathbb{H} \), then we have \( \pi_{reg} f = f \).
2. The function \( \pi_{reg} f \) lies in the space \( \widetilde{M}_k^!(\Gamma_0(N)) \).
3. If \( f \) satisfies Definition 3.1, then \( \pi_{reg}(f) = \pi_{hol}(f) \).

**Proof.** Claims (1) and (3) are clear. The proof of (2) is a modification of the proofs of Propositions 5.1 and 6.2 in [12] which concern the classical holomorphic projection. Here we indicate how to modify these proofs to this setting.

We begin by recalling the regularized Petersson inner product defined by Borcherds [2]. For \( T > 0 \), we denote by \( \mathcal{F}_T(\text{SL}_2(\mathbb{Z})) \) the truncated fundamental domain of \( \text{SL}_2(\mathbb{Z}) \), i.e.

\[
\mathcal{F}_T(\text{SL}_2(\mathbb{Z})) := \left\{ \tau \in \mathbb{H} : |\tau| \geq 1, |\Re(\tau)| \leq \frac{1}{2}, \Im(\tau) \leq T \right\}.
\]

For a finite index subgroup \( \Gamma \leq \text{SL}_2(\mathbb{Z}) \) we define the truncated fundamental domain of \( \Gamma \) as

\[
\mathcal{F}_T(\Gamma) := \bigcup_{\gamma \in \Gamma \setminus \text{SL}_2(\mathbb{Z})} \gamma \mathcal{F}_T(\text{SL}_2(\mathbb{Z})).
\]
The regularized inner product of \( g \in M_k(\Gamma) \) and \( h \) a weight \( k \) real-analytic modular form on \( \Gamma \) \((k \geq 2 \text{ even})\) with at most linearly exponential growth at the cusps is given as the constant term in the Laurent expansion of the meromorphic continuation of the expression

\[
\langle g, h \rangle^{\text{reg}} := \frac{1}{[\text{SL}_2(\mathbb{Z}) : \Gamma]} \lim_{T \to \infty} \int_{\mathcal{F}_T(\Gamma)} g(\tau) h(\tau) y^{k-2s} \frac{dx dy}{y^2}.
\]

It follows from Corollary 4.2 in [7] that the weakly holomorphic Poincaré series from Section 2.2 together with the Eisenstein series (including the quasimodular Eisenstein series \( E_2 \) in case that \( k = 2 \)) are orthogonal to cusp forms with respect to \( \langle \cdot, \cdot \rangle^{\text{reg}} \). By subtracting a suitable linear combination of these functions one can replace \( f \) by a function that satisfies the growth condition in Proposition 5.1 in [12]. The proof then follows mutatis mutandis as in [12].

We consider this product for \( \Gamma = \Gamma_0(N) \), where \( g \) is the regularized Poincaré series

\[
P_{m,s}(\tau) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} (y^s e^{2\pi i m \tau}) |_{k,\gamma},
\]

where \( m \geq 1 \) and \( \Gamma_\infty := \{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \} \) denotes the stabilizer of the cusp \( i\infty \) in \( \Gamma_0(N) \).

Note that this regularization is necessary to ensure convergence in the case of weight 2, if the weight is at least 4, these converge to the cuspidal Poincaré series in Section 2.2 as \( s \to 0 \). Since the domain of integration, the truncated fundamental domain, is compact and all functions in the integrand are continuous, it is legitimate to replace \( P_{m,s} \) by its definition as a series and interchange summation and integration. This yields

\[
\langle g, h \rangle^{\text{reg}} = \frac{1}{[\text{SL}_2(\mathbb{Z}) : \Gamma]} \lim_{T \to \infty} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \int_{\mathcal{F}_T(\Gamma_0(N))} e^{2\pi i m \tau} h(\frac{a \tau + b}{c \tau + d}) \frac{y^{k-s}}{(c \tau + d)^{2(k-s)}} \frac{dx dy}{y^2}
\]

\[
= \frac{1}{[\text{SL}_2(\mathbb{Z}) : \Gamma]} \lim_{T \to \infty} \int_{\mathcal{F}_T(\Gamma_\infty)} e^{2\pi i m \tau} h(\tau) y^{k-s} \frac{dx dy}{y^2}
\]

\[
= \frac{1}{[\text{SL}_2(\mathbb{Z}) : \Gamma]} \int_{\Gamma_\infty \backslash \mathbb{H}} e^{2\pi i m \tau} h(\tau) y^{k-s} \frac{dx dy}{y^2}.
\]

From here, the proof literally is the same as in [12], so we refer the reader to there. \( \square \)

### 3.3. Holomorphic projections of completed mixed mock modular forms

For the proof of our main result, we need to know the action of the holomorphic projection operator on Rankin-Cohen brackets of harmonic Maass forms and cusp forms. The following result (see Theorem 3.6 in [20]) gives an explicit formula for the Fourier coefficients of such a holomorphic projection. The case of ordinary products, i.e. \( \nu = 0 \), is already contained in Theorem 3.5 in [15]. Before the statement of the proposition, let us define the polynomial

\[
G_{a,b}(X,Y) := \sum_{j=0}^{a-2} (-1)^j \binom{a + b - 3}{a - 2 - j} \binom{j + b - 2}{j} X^{a-2-j} Y^j \in \mathbb{C}[X,Y]
\]

which is homogeneous of degree \( a - 2 \).
Proposition 3.5. Let \( f_1 \in S_{k_1}(\Gamma_0(N)) \) and \( f_2 \in S_{k_2}(\Gamma_0(N)) \) be cusp forms of even weights as in the introduction and let \( M_{f_1} \in H_{2-k_1}(\Gamma_0(N)) \) be a harmonic Maass form with shadow \( f_1 \). Then we have

\[
\pi_{hol}^{reg}(M_{f_1}, f_2)(\tau) = [M_{f_1}^+, f_2]_\nu(\tau) - (k_1 - 2)! \sum_{\mu=0}^{\nu} \binom{\nu - k_1 + 1}{\nu - \mu} \binom{\nu + k_2 - 1}{\mu} \sum_{h=1}^{\infty} q^h \times \left[ \sum_{n=1}^{\infty} a_2(n + h)a_1(n)(n + h)^{-\nu - k_2 + 1}G_{2\nu - k_1 + k_2 + 2,k_1-\mu}(n + h,n) - n^{\mu - k_1 + 1}(n + h)^{\nu - \mu} \right].
\]

Proof. We write

\[
M_{f_1}(\tau) = \sum_{n=1}^{\infty} n^{1-k_1-1}a_1(n)\frac{\Gamma(k_1 - 1)}{\Gamma(k_1 - \mu - 1)} \sum_{n=1}^{\infty} n^{\mu - k_1 + 1}e_{f}(n)\Gamma(k_1 - \mu - 1; 4\pi ny)q^{-n}.
\]

Using this, we find that

\[
[M_{f_1}^+, f_2]_\nu(\tau) = \sum_{h \in \mathbb{Z}} b(h, y)q^h,
\]

where

\[
b(h, y) = \sum_{n=1}^{\infty} \sum_{\mu=0}^{\nu} \binom{1 - k_1 + \nu}{\nu - \mu} \binom{k_2 + \nu - 1}{\mu} \times \frac{\Gamma(k_1 - 1)}{\Gamma(k_1 - \mu - 1)} n^{1-k_1+\mu}a_1(n)\Gamma(k_1 - \mu - 1; 4\pi ny)a_2(n + h)(n + h)^{\nu - \mu}.
\]

Thus the holomorphic projection of this becomes

\[
\pi_{hol}^{reg}(f^+, g)_\nu = \sum_{h=1}^{\infty} b(h)q^h,
\]

with

\[
b(h) = \frac{(4\pi h)^{2\nu - k_1 + k_2 + 1}}{(2\nu - k_1 + k_2)!} \sum_{n=1}^{\infty} \sum_{\mu=0}^{\nu} \binom{1 - k_1 + \nu}{\nu - \mu} \binom{k_2 + \nu - 1}{\mu} \times \frac{\Gamma(k_1 - 1)}{\Gamma(k_1 - \mu - 1)} n^{1-k_1+\mu}a_1(n)a_2(n + h)(n + h)^{\nu - \mu} \times \int_0^{\infty} e^{-4\pi ny}y^{2\nu - k_1 + k_2}dy.
\]
For the evaluation of the integral we may interchange the outer integration and the implicit
one in the definition of the incomplete Gamma function, which after several substitutions of
variables and simplification steps (see Lemma 3.7 in [20]) yields the claim.

\[ \square \]

**Remark.** A necessary condition for convergence is obviously that \( \nu \leq \frac{k_1 - k_2}{2} \).

**Remark.** In the case \( k_1 = k_2 = k \) and \( \nu = 0 \), equation (3.4) simplifies to

\[ \hat{\pi}_{\text{reg}}^e(M_{f_1} \cdot f_2)(\tau) = M_{f_1}^+(\tau) \cdot f_2(\tau) - (k - 2)! \sum_{h=1}^{\infty} \left[ \sum_{n=1}^{\infty} a_2(n+h)a_1(n) \left( \frac{1}{(n+h)^{k-1}} - \frac{1}{n^{k-1}} \right) \right] q^h. \]

### 3.4. The Dirichlet series

Here we define the general Dirichlet series in Theorem 1.1. We recall from the introduction the derived shifted convolution series

\[ D^{(\mu)}(f_1, f_2, h; s) := \sum_{n=1}^{\infty} \frac{a_1(n+h)a_2(n)(n+h)^{\mu}}{n^s}. \]

Obviously, we have that \( D(f_1, f_2, h; s) = D^{(0)}(f_1, f_2, h; s) \). Due to the absence of the symmetry in the \( f_i \), it is natural to consider special values of the symmetrized Dirichlet series (3.5)

\[ \hat{D}(f_1, f_2, h; s) := D(f_1, f_2, h; s) - \sum_{j=0}^{k_2-k_1} (-1)^j \binom{k_2-1}{k_2-k_1-j} \binom{j+k_1-2}{j} D^{(j)}(f_2, f_1, -h; s+j). \]

We note that when \( k_2 < k_1 \) the sum is empty. As we shall see, one important consequence of symmetrizing these functions is the convergence of the special values that we consider here. For the case of equal weights, the above definition simplifies to

\[ \hat{D}(f_1, f_2, h; s) = D(f_1, f_2, h; s) - D(f_2, f_1, -h; s). \]

For integers \( \nu \geq 0 \), we consider the generating function in \( h \)-aspect of the symmetrized shifted convolution special values

\[ \hat{D}^{(\nu)}(f_1, f_2, h; k_1 - 1) := \sum_{\mu=0}^{\nu} \alpha(\nu, k_1, k_2, \mu) D^{(\nu-\mu)}(f_1, f_2, h; k_1 - 1 - \mu) \]

\[ - \sum_{j=0}^{w} \beta(\nu, k_1, k_2, j) D^{(j)}(f_2, f_1, -h; k_1 + j - \nu - 1), \]

where \( w := 2\nu - k_1 + k_2 \). The \( \alpha(\nu, k_1, k_2, \mu) \) and \( \beta(\nu, k_1, k_2, j) \) are integers defined by

\[ \alpha(\nu, k_1, k_2, \mu) := \binom{\nu-k_1+1}{\nu-\mu} \binom{\nu+k_2-1}{\mu} \]

and

\[ \beta(\nu, k_1, k_2, j) := \sum_{\mu=0}^{\nu} \binom{\nu-k_1+1}{\nu-\mu} \binom{\nu+k_2-1}{\mu} \binom{2\nu-\mu+k_2-1}{j} \binom{j+k_1-\mu-2}{j}. \]
Of course we have that \( \hat{D}(f_1, f_2, h; s) = \hat{D}^{(0)}(f_1, f_2, h; s) \). The generating function we study is

\[
\mathbb{L}^{(\nu)}(f_1, f_2; \tau) := \sum_{h=1}^{\infty} \hat{D}^{(\nu)}(f_1, f_2, h; k_1-1)q^h.
\]

In the special case when \( \nu = 0 \), we have that \( \mathbb{L}^{(0)}(f_1, f_2; \tau) = \mathbb{L}(f_1, f_2; \tau) \).

### 3.5. Proof of Theorem 1.1 and Corollary 1.2

We can now prove Theorem 1.1.

**Proof of Theorem 1.1.** If one plugs in the definition of \( G_{a,b} \) in (3.3) into (3.4) and rearrange the sums appropriately, one gets that

\[
\pi_{\text{reg}}([M_{f_1}, f_2]_{\nu})(\tau) = [M_{f_1}^+, f_2]_{\nu}(\tau)
\]

\[
- (k_1 - 2)! \left\{ \sum_{j=0}^{2\nu - k_1 + k_2} \sum_{\mu=0}^{2\nu - k_1 + k_2} \binom{\nu - k_1 + 1}{\nu - \mu} \binom{\nu + k_2 - 1}{\mu} \binom{2\nu - \mu + k_2 - 1}{2\nu - k_1 + k_2 - j} \binom{j + k_1 - \mu - 2}{j} \right. 
\]

\[
\times \left. \sum_{h=1}^{\infty} \left[ \sum_{n=1}^{\infty} a_2(n + h)\overline{a_1(n)n^2} \right] q^h \right. 
\]

\[
- \sum_{\mu=0}^{\nu} \binom{\nu - k_1 + 1}{\nu - \mu} \binom{\nu + k_2 - 1}{\mu} \sum_{h=1}^{\infty} \left[ \sum_{n=1}^{\infty} \frac{a_2(n+h)\overline{a_1(n)(n+h)^{\nu-\mu}}}{n^{k_1-\mu-1}} \right] q^h \right\}.
\]

By definition, this is

\[ [M_{f_1}^+, f_2]_{\nu}(\tau) + (k_1 - 2)! \cdot \mathbb{L}^{(\nu)}(f_2, f_1; \tau) \]

where \( \alpha(\nu, k_1, k_2, \mu) \) and \( \beta(\nu, k_1, k_2, j) \) are defined by (3.7) and (3.8).

By Proposition 3.4 this function lies in the space \( \hat{M}_{2\nu - k_1 + k_2 + 2}(\Gamma_0(N)) \) which yields Theorem 1.1 in the general case. When \( M_{f_1} \) is good for \( f_2 \), then Proposition 3.2 implies that it lives in the space \( \hat{M}_{2\nu - k_1 + k_2 + 2}(\Gamma_0(N)) \).

**Proof of Corollary 1.2.** The formulas stated in the corollary are exactly the same as in Theorem 1.1 keeping in mind that by Lemma 2.6 the shadow of \( Q \) is precisely \((1 - k) \cdot P\), we only have to make sure that \( Q \) is good for \( P \). But this follows immediately from Lemma 2.7 since \( P \) decays exponentially in every cusp of \( \Gamma_0(N) \) and \( Q^+ \) has exactly one simple pole at infinity and grows moderately in the other cusps.

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