

’t Hooft-Polyakov Monopoles with Non-Abelian Moduli

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Abstract

We extend the Georgi-Glashow model of the t’Hooft-Polyakov monopoles to include additional collective coordinates “orientational isospin moduli.” The low-energy theory of these solitonic solutions can be interpreted as dyons with isospin.
1 Introduction

Magnetic monopole is one of the most venerable constructions in theoretical physics. It dates back to Dirac [1]. Implementation of this construction in field theory is due to ‘t Hooft [2] and Polyakov [3] (for reviews and references see e.g. [4, 5, 6]). Remarkable effects were shown to be associated with the ‘t Hooft-Polyakov monopoles, e.g. the Callan-Rubakov effect of the baryon decay catalysis [7, 8, 9]. In the monopole field $SU(2)_{\text{gauge}}$ doublet fermions acquire integer spin (e.g. [5]). This phenomenon is called “spin from isospin.”

In this paper we will consider an extension of the Georgi-Glashow model [10], in which a global $O(3)$ (“isospin”) symmetry is present in the Lagrangian. A simple model possessing this property and suitable for our purposes was suggested in [11] (see also [12]). Conceptually it was inspired by Witten’s cosmic strings [13]. Our task is to demonstrate that in this model the ‘t Hooft-Polyakov monopole acquires additional collective coordinates, isospin moduli. The overall set of collective coordinates includes three coordinates of the monopole center, a $U(1)$ phase coordinate which, upon quantization, corresponds to the electric charge and produces dyons out of the monopoles, and two extra collective coordinates associated with the $O(3)$ isospin. Quantization of the isospin collective coordinates is straightforward, paralleling that of the spherical quantum top.

2 The model

Our starting point is the well-known Georgi-Glashow model [10]. The gauge group is $SU(2)$ and the matter sector is described by a triplet real field $\phi^a$ (belonging to the adjoint representation). The Lagrangian of this model is

$$\mathcal{L}_{\text{GG}} = -\frac{1}{4g^2}G_{\mu\nu}^{a}G^{\mu\nu,a} + \frac{1}{2}(D_{\mu}\phi^a)(D^\mu\phi^a) - \lambda (\phi^a\phi^a - v^2)^2,$$  
(1)

where

$$D_{\mu}\phi^a = \partial_{\mu}\phi^a + \varepsilon^{abc}A_{\mu}^b\phi^c,$$  
(2)

is the covariant derivative in the adjoint representation, and

$$G_{\mu\nu}^a = \partial_{\mu}A_{\nu}^a - \partial_{\nu}A_{\mu}^a + \varepsilon^{abc}A_{\mu}^bA_{\nu}^c$$  
(3)

This latter isospin results from the global $O(3)$ symmetry, not to be confused with $SU(2)_{\text{gauge}}$. 

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is the non-Abelian field strength. We use the following matrix notation for the fields
\[ \phi = \phi^a \frac{\tau_a}{2}, \]
where \( \tau_a \) denote the standard Pauli matrices. In the Lagrangian, \( v \) is a parameter with dimensions of mass, and \( \lambda \) is a dimensionless coupling constant. As is well known, this model supports the t’Hooft-Polyakov magnetic monopoles \[2, 3\] that are topologically stable as a consequence of the symmetry breaking pattern
\[ SU(2)_{\text{gauge}} \to U(1)_{\text{gauge}}. \] (5)

The breaking (5) is enforced by the non-vanishing vacuum expectation value of the \( \phi \) field, which can always be chosen aligned in the 3-direction in \( SU(2)_{\text{gauge}}, \)
\[ \phi^a_{\text{vac}} = v\delta^3 a. \] (6)

Two components of the gauge field, called \( W^{\pm} \), acquire masses
\[ m_W = gv, \] (7)
whilst the component aligned along the vacuum direction remains massless and plays the role of the \( U(1) \) photon. The mapping between the group space and the coordinate space at infinity can be classified by the second homotopy class,
\[ \pi_2(SU(2)/U(1)) = \mathbb{Z}, \] (8)
which guarantees the topological stability of the monopoles.

Historically, the introduction of non-Abelian moduli on topological defects occurred in a rather advanced settings, mostly involving supersymmetric gauge theories (see [14, 15, 16]). In [11] a much simpler setup was suggested resulting in the occurrence of non-Abelian moduli. We will use this setup to study non-Abelian moduli on the world-line of the t’Hooft-Polyakov monopoles. To this end we will end a term \( L_\chi \) to the Lagrangian (9),
\[ \mathcal{L} = \mathcal{L}_{GG} + \mathcal{L}_\chi \]
where
\[ \mathcal{L}_\chi = \partial\mu\chi^i \partial^\mu\chi_i - \gamma \left[ (-\mu^2 + \phi^a \phi^a)\chi^i\chi_i + \beta(\chi^i\chi^i)^2 \right]. \] (10)
Here \( \chi^i \) is an \( O(3) \) triplet uncharged under the gauge group. The potential term coupling \( \chi^i \) to \( \phi^a \) ensures that, if \( \mu < v \), the \( \chi^i \) field will develop in the monopole core and vanish in the vacuum. The mass of the \( \chi \) field is
\[ m_{\chi}^2 = \gamma(v^2 - \mu^2). \] (11)
The global $O(3)$ symmetry of (10) is obvious.
Now, assuming the standard monopole ansatz for the fields $\phi$ and $A$

$$\phi^a = vn^a H(r), \quad A^a_i = \epsilon^{aij} \frac{1}{r} n^j F(r), \quad (12)$$

with $n^a = x^a/r$, and adding a self-evident ansatz for $\chi^i$,

$$\chi^i = \sqrt{\frac{\mu^2}{2\beta}} \chi(r) \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right), \quad (13)$$

we can reduce the energy functional to

$$E = \frac{4\pi v}{g} \int_0^\infty d\rho \rho^2 \left[ \frac{(F')^2}{\rho^2} + \frac{(2F - F^2)^2}{2\rho^4} + \frac{H^2(1 - F)^2}{\rho^2} + \frac{(H')^2}{2} \right.\left. + \frac{\lambda}{g^2} (H^2 - 1)^2 + \frac{\bar{\mu}^2}{2\beta} \left( (\chi')^2 + \frac{\gamma}{g^2} \left[ (-\bar{\mu}^2 + H^2)\chi^2 + \frac{\bar{\mu}^2}{2} \chi^4 \right] \right) \right] \quad (14)$$

in dimensionless units

$$\rho = gv|x|. \quad (15)$$

Here $\bar{\mu}$ is dimensionless,

$$\bar{\mu} = \mu/v. \quad (16)$$

The energy minimization equations are

$$\left( \rho^2 H' \right)' = H \left[ 2(F - 1)^2 + \frac{4\lambda \rho^2}{g^2} (H^2 - 1) + \frac{\gamma \bar{\mu}^2 \rho^2}{g^2 \beta} \chi^2 \right], \quad (17)$$

$$\rho^2 F'' = (-1 + F) \left( (-2 + F) F + \rho^2 H^2 \right), \quad (18)$$

$$\left( \rho^2 \chi' \right)' = \frac{\gamma \rho^2}{g^2} \chi \left( H^2 + \bar{\mu}^2 (-1 + \chi^2) \right). \quad (19)$$

### 2.1 Vacuum structure

In the vacuum, all derivatives of the profile functions must vanish. Clearly equation (18) imposes the condition that $F = 1$ in the vacuum. This leads to four branches of extrema of the energy, the first is

$$H^I_{\text{vac}} = \chi^I_{\text{vac}} = 0, \quad (20)$$
this is a maximum of the energy functional with \( E_{\text{max}}^I / 4\pi v = \lambda / g^3 \). Then, in the vacuum II,

\[
H_{\text{vac}}^{II} = 1, \quad \chi_{\text{vac}}^{II} = 0,
\]

for which the vacuum energy density vanishes \( E_{\text{vac}}^{II} = 0 \). The third branch is

\[
H_{\text{vac}}^{III} = 0, \quad \chi_{\text{vac}}^{III} = 1,
\]

for which

\[
\frac{E_{\text{vac}}^{III}}{4\pi v} = \frac{1}{g^3} \left( \lambda - \tilde{\mu}^4 \gamma \right).
\]

Finally, the fourth branch is

\[
(H_{\text{vac}}^{IV})^2 = \frac{4\beta \lambda - \gamma \tilde{\mu}^2}{4\beta \lambda - \gamma}, \quad (\chi_{\text{vac}}^{IV})^2 = \frac{4\beta \lambda (1 - \tilde{\mu}^2)}{\tilde{\mu}^2 (\gamma - 4\beta \lambda)}
\]

for which

\[
\frac{E_{\text{vac}}^{IV}}{4\pi v} = \frac{\gamma \lambda (\tilde{\mu}^2 - 1)^2}{g^3 (\gamma - 4\beta \lambda)}.
\]

Hence, when \( \gamma < 4\beta \lambda \) this branch corresponds to the actual vacuum. The branches II and IV meet at \( \tilde{\mu} = 1 \), or when \( \mu = v \). When \( \gamma > 4\beta \lambda \) and \( \lambda > \frac{\tilde{\mu}^4 \gamma}{4\beta} \), the vacuum II is the actual vacuum.

Below we look for solutions in vacuum II.

### 3 Numerical results

Here we present the solutions of (17)-(19) obtained by a second order central finite difference numerical procedure with the accuracy \( O(10^{-4}) \). To launch our numerical procedure we cut-off the radial direction at a large value of \( \rho \), which we will call \( R \). We set \( R = 100 \). As an example we choose the following values of the parameters:

\[
\gamma = 4, \quad \lambda = 0.34, \quad \tilde{\mu} = 0.999, \quad \beta = 2.94, \quad g = 0.6.
\]

Our boundary conditions are

\[
H(0) = F(0) = 0, \quad \chi'(0) = 0,
\]

\[
H(R) = F(R) = 1, \quad \chi(R) = 0.
\]
Figure 1: Non-Abelian monopole profiles for $\gamma = 4$, $\lambda = 0.34$, $\bar{\mu} = 0.999$, $\beta = 2.94$ and $g = 0.6$. The solid curve represents $H$, medium dashed $F$ and thin dashed $\chi$.

The sought after solution is shown in Figure 1. The energy (monopole mass) $M_M$ of this solution is

$$\frac{M_M}{4\pi v} = 2.14$$

Monopole with the isospin.

The mass $M_M$ of the standard t’Hooft-Polyakov monopole at $\lambda = 0.34$ and $g = 0.6$ is

$$\frac{M_M}{4\pi v} = 2.33.$$ (30)

Thus, the energy of the monopole with the nonvanishing $\chi^i$ in the core is lower.

4 Stability

In this section we can check that the solution found above with a nonvanishing $\chi$ in the core is stable. The energy functional for the $\chi$ field is

$$E_\chi = \frac{4\pi v}{g} \int_0^\infty d\rho \rho^2 \left[ (\chi')^2 + \frac{\gamma}{g^2} \left( (-\bar{\mu}^2 + H^2)\chi^2 + \bar{\mu}^2 \chi^4 \right) \right].$$ (31)
Introducing
\[ \psi(\rho) = \rho \chi \] (32)
we can derive a one-dimensional Schrödinger-like equation for the \( \psi \) eigenfunctions,
\[ -\psi'' + \frac{\gamma}{g^2 \rho^2} \left( (-\tilde{\mu}^2 + H_1^2) + 3\tilde{\mu}^2 \chi_1^2 \right) \psi = \epsilon \psi \] (33)
where \( H_1 \) and \( \chi_1 \) are the field profiles shown in Figure 1. Using the same parameters as in (26) we find the lowest-lying mode at \( \epsilon \approx 0.00105 \). The positivity of \( \epsilon \) demonstrates the stability of the numerical solution with a non-vanishing \( \chi \) field in the core.

Since the solution with a non-zero \( \chi \) in the core has lower energy than the vanishing-\( \chi \) solution we must also investigate (meta)stability of the vanishing-\( \chi \) solution. Then, following the argument above, we must solve
\[ -\psi'' + \frac{\gamma}{g^2 \rho^2} (-\tilde{\mu}^2 + H_1^2) \psi = \epsilon \psi . \] (34)
We find that the lowest lying mode is at \( \epsilon \approx 0.00103 \) thus confirming the classical stability of the \( \chi = 0 \) solution.

Thus, we observe two classical solutions: one the standard 't Hooft-Polyakov monopole is a local minimum of the energy functional, while the solution with the nonvanishing \( \chi \) in the core represents the global minimum.

5 Quantization of the collective coordinates

Quantization of the standard t'Hooft-Polyakov monopole involves four moduli: three translations of the monopole center plus a rotation around the vacuum direction in the \( SU(2) \) gauge space. In the adiabatic approximation one finds the Lagrangian
\[ L_{QM} = -M_M + \frac{M_M}{2} (\dot{x}_0^2) + \frac{1}{2} \frac{M_M}{m_W^2} \alpha^2 \] (35)
where \( M_M \) is the mass of the monopole, the vector \( \vec{x}_0 \) represents the monopole center, whilst \( \alpha \) is the collective coordinate related to the remaining \( U(1) \) gauge invariance. The full quantum mechanical Hamiltonian for these moduli is
\[ \hat{H}_0 = M_M + \frac{\hat{p}^2}{2M_M} + \frac{1}{2} \frac{m_W^2}{M_M} \hat{\alpha}^2 , \] (36)
where
\[ \pi_\alpha = \frac{M_M}{m_W^2} \dot{\alpha}, \]  
(37)
is the canonical momentum conjugated to the angular variable \( \alpha \).

To obtain the orientational moduli we parametrize the \( \chi \) field as follows
\[ \chi^i = \sqrt{\frac{\mu^2}{2\beta}} \chi(\rho) S^i(t) \]  
(38)
where \( S^i \) is a unit vector which depends on time. Substituting (38) in (14) we obtain the low-energy action for the orientational moduli,
\[ S_0 = \frac{I_1}{2} \int dt \dot{S}^i \dot{S}^i, \quad S^i S^i = 1, \]  
(39)
where
\[ \frac{I_1 v}{4\pi} = \frac{\tilde{\mu}^2}{g^3 \beta} \int_0^\infty d\rho \rho^2 \chi^2 = 7.81, \]  
(40)
and the overdot denotes a derivative with respect to time. The above action can be easily quantized as follows.

The action (39) describes the motion of a rigid rotating body, symmetric quantum top, at a fixed spherical radius \( r_0 \) (in this case \( r_0 = 1 \)). Its quantization can be carried out in a standard way (see e.g. [17]), if we parametrize \( S^i \) in terms of polar and azimuthal angles,
\[ S^1 = \cos \theta, \quad S^2 = \sin \theta \cos \varphi, \quad S^3 = \sin \theta \sin \varphi. \]  
(41)
Upon quantization we get the following symmetric top Hamiltonian:
\[ \hat{H} = -\frac{1}{2I_1} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right]. \]  
(42)
Its eigenvalues are labelled by a non-negative integer \( s \) and reduce to
\[ E_s = \frac{1}{2I_1} s (s + 1). \]  
(43)
They have degeneracy \( 2s + 1 \). The eigenfunctions are the standard spherical harmonics \( Y_s(\theta, \varphi) \).
Including the conventional 't Hooft-Polyakov moduli from Eq. 36 we arrive at the monopole mass formula

\[ M = M_M + \frac{m_W^2 k^2}{2M_M} + \frac{1}{2I_1} s(s+1), \]  

(44)

where \( k \) and \( s \) are integers. The \( k^2 \) term here corresponds to dyonic excitations associated with non-zero electric charge, while the last term is associated with non-zero global isospin. The \( I_1 \) parameter plays the role of the moment of inertia in the isospace. With our illustrative choice of parameters \( I_1^{-1} \sim (1/60)m_W \) while \( m_W/M_M \sim 1/40 \).

6 Conclusions

In this model we completed the program of constructing the simplest topological defects with non-Abelian moduli. We extended the Georgi-Glashow model of the 'tHooft-Polyakov monopoles to include extra global symmetry \( O(3) \) (we referred to it as isospin) and extra collective coordinates “orientational isospin moduli.” The adiabatic quantization of these solitonic solutions was carried out. The result can be interpreted as a dyon with isospin.

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