Locality and causality in perturbative AQFT

Kasia Rejzner

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Abstract

In this paper we introduce a notion of a group with causality, which is a natural generalization of a locality group, introduced by P. Clavier, L. Guo, S. Paycha, and B. Zhang. We also propose a generalization of the Hammerstein property, which characterizes a class of maps that intuitively would be described as local. All these abstract structures are then illustrated by concrete examples from classical and quantum field theory.

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1 Introduction

In this article we show how the different meanings of the term “locality” appearing in the quantum field theory (QFT) context can be described by properties of a common mathematical structure that we call group with causality. This is a generalization of
the ideas proposed in [CGPZ18a] and applied to renormalization in Euclidean QFT in [CGPZ18b].

It was argued in [BDF09] that the notion of locality for smooth functionals on the space of field configurations is captured by the following property:

\[ F(\varphi_1 + \varphi_2 + \varphi_3) = F(\varphi_1 + \varphi_2) + F(\varphi_2 + \varphi_3) - F(\varphi_2), \]  

(1)

if \( \text{supp}(\varphi_1) \cap \text{supp}(\varphi_3) = \emptyset \), where \( \varphi_1, \varphi_2, \varphi_3 \in \mathcal{C}_\infty(M, \mathbb{R}) \), \( F \in \mathcal{C}_\infty(\mathcal{C}_\infty(M, \mathbb{R}), \mathbb{C}) \), and \( M \) is a Lorentzian manifold (here we typically assume \( M \) to be globally hyperbolic). In [BDF09] this property was termed Additivity, as it can be seen as generalization of linearity. Indeed, it is clearly satisfied for \( F \) linear. In mathematics, this property has been known as Hammerstein property [Bat73].

For \( F \) with \( F(0) = 0 \) additivity implies that

\[ F(\varphi_1 + \varphi_3) = F(\varphi_1) + F(\varphi_3), \]  

(2)

if \( \text{supp}(\varphi_1) \cap \text{supp}(\varphi_3) = \emptyset \). This condition is called partial additivity in [BDLGR18], where it is also shown (by a counter example) that it is strictly weaker than (1). However, for polynomial functionals (1) and (2) are equivalent, as shown in [BDF09].

In [BDLGR18] it was shown, following the sketch of the proof given in [BFR12], that (1) together with an additional regularity condition that we will recall in section 5.1 is equivalent to saying that \( F \) can be written as an integral of some smooth function on the jet space.

In a seemingly different context, the term locality is used in AQFT to express the fact that local algebras \( \mathfrak{A}(O_1) \) and \( \mathfrak{A}(O_2) \), assigned to bounded regions \( O_1, O_2 \subset \mathbb{M} \) of Minkowski spacetime \( \mathbb{M} \), commute if \( O_1 \) is spacelike to \( O_2 \), i.e.

\[ [\mathfrak{A}(O_1), \mathfrak{A}(O_2)] = \{0\}. \]

This property is also called Einstein causality. We will show here that these two notions of locality can be brought together in perturbative AQFT (pAQFT), when applied to formal S-matrices.

We claim that the physical notion of locality is well captured by what we refer to as the generalized Hammerstein property.

## 2 Locality and causality structures

In [CGPZ18a] the authors introduce an abstract notion of locality captured by the definition of the locality set:

**Definition 2.1.** (a) A locality set is a pair \((X, \mathcal{T})\) where \( X \) is a set and \( \mathcal{T} \) is a symmetric binary relation on \( X \). For \( x_1, x_2 \in X \), denote \( x_1 \mathcal{T} x_2 \) if \((x_1, x_2) \in \mathcal{T}\).

(b) For any subset \( U \subset X \), let

\[ U^\mathcal{T} := \{ x \in X | \{ x \} \times U \subset \mathcal{T} \} \]

denote the polar subset of \( U \).
c) For integers $k \geq 2$, denote
\[ X^T_k \equiv X \times_\perp \cdots \times_\perp X \equiv \{(x_1, \ldots, x_k) \in X^k | x_i \perp x_j \text{ for all } 1 \leq i \neq j \leq k\} \].

(d) We call two subsets $A$ and $B$ of a locality set $(X, \perp)$, independent, if $A \times B \subset \perp$.

An example of such relation is disjointness of sets. In particular, the primary example we are interested in is the power set $\mathcal{P}(M)$, of a manifold $M$ equipped with the disjointness of sets relation $\perp$.

If $M$ is an oriented, time-oriented Lorentzian spacetime, the notion of disjointness of sets can be refined with the use of the causal structure. A curve $\gamma : \mathbb{R} \supset I \to M$ with a tangent vector $\dot{\gamma}$ is:

- spacelike if $g(\dot{\gamma}, \dot{\gamma}) < 0$,
- timelike if $g(\dot{\gamma}, \dot{\gamma}) > 0$,
- lightlike if $g(\dot{\gamma}, \dot{\gamma}) = 0$,
- causal if $g(\dot{\gamma}, \dot{\gamma}) \geq 0$.

A causal/lightlike/timelike curve $\gamma$ is called future-pointing if $g(u, \dot{\gamma}) > 0$, where $u$ is the time orientation (see footnote).

**Definition 2.2.** Let $x \in M$, where $M$ is a Lorentzian oriented, time-oriented spacetime. We define $J^\pm(x)$, the future/past of $x$, as the set of all points $y \in M$ such that there exists a future/past pointing causal curve from $x$ to $y$.

We are now ready to introduce the causality relation on subsets of $M$ (see e.g. [BPS19]).

**Definition 2.3.** Let $\mathcal{O}_1, \mathcal{O}_2 \subset M$, where $M$ is a Lorentzian manifold. We say $\mathcal{O}_1$ “is not later than” $\mathcal{O}_2$, i.e. $\mathcal{O}_1 \preceq \mathcal{O}_2$ if and only if $\mathcal{O}_1 \cap J^+(\mathcal{O}_2) = \emptyset$.

This relation is crucial for formulation of the causal factorization property of formal S-matrices (we will come to this in Section 4). Clearly, this is not a symmetric relation, but it can be symmetrized and we obtain:

**Observation 2.4.** Symmetrization of $\preceq$ is the relation of being spacelike, $\times$, i.e.
\[ \mathcal{O}_1 \times \mathcal{O}_2 \Leftrightarrow \mathcal{O}_1 \preceq \mathcal{O}_2 \cap \mathcal{O}_2 \preceq \mathcal{O}_1. \]

There are two features of $\sim$ that we want to single out as important: each subset $\mathcal{O} \subset M$ has non-zero intersection with both the past and the future of itself, so $\mathcal{O} \sim \mathcal{O}$. There exist subsets for which the relation is non symmetric, i.e. $\mathcal{O}_1 \preceq \mathcal{O}_2$, but $\mathcal{O}_2 \not\preceq \mathcal{O}_1$ (existence of “preferred direction”).

Motivated by the above example we define:

---

$^1M$ is equipped with a smooth Lorentzian metric $g$, a nondegenerate volume form and a smooth vector field $u$ on $M$ such that for every $p \in M$, $g(u, u) > 0$.
Definition 2.5. (a) A causality set is $(X, \vdash)$, where $X$ is a set and $\vdash$ a relation such that for all $x \in X$ $x \sim \vdash x$ (the negation of $\vdash$ is reflexive) and $\vdash$ is not symmetric (i.e. there exist $x, y \in X$ such that $x \vdash y$ and $y \sim \vdash x$).

(b) For any subset $U \subset X$, let

$$^\uparrow U := \{ x \in X | x \vdash y, \text{ for all } y \in U \} ,$$

$$U^\downarrow := \{ x \in X | y \vdash x, \text{ for all } y \in U \} .$$

Observation 2.6. Given a causality set $(X, \vdash)$ we obtain a locality set by symmetrizing $\vdash$.

In [CGPZ18a], the authors introduce the notion of a locality group, which is a set $(G, \top)$ together with a product law $m_G$ defined on $\top$, for which the product is compatible with the locality relation on $G$ and the other group properties (associativity, existence of the unit, inverse) hold in a restricted sense, again compatible with $\top$. This structure is an analogue of a partial algebra [KS75].

However, in our context, we will need to equip a locality set $(G, \top)$ with a full group structure (without restricting to $\top$), compatible with $\top$. To make a distinction to [CGPZ18a] we will call this a group with locality.

Definition 2.7. A group with locality $(G, \top, m_G, 1_G)$ is a group $(G, m_G, 1_G)$ and a locality set $(G, \top)$, for which the product law $m_G$ is compatible with $\top$, i.e.

$$m_G((U^\top \times U^\top) \cap \top) \subset U^\top$$

and such that $\{1_G\}^\top = G$.

Let $M$ be a Lorentzian manifold. The set $\mathcal{E}(M) \doteq \mathcal{C}^\infty(M, \mathbb{R})$ together with the disjointness of supports relation $\top$ and with addition $(f, g) \mapsto f + g$ forms a group with locality $(\mathcal{E}(M), \top, +, 0)$. We can formulate the notion of partial additivity in terms of morphisms of groups with locality. We recall after [CGPZ18a]:

Definition 2.8. A locality map from a locality set $(X, \top_X)$ to a locality set $(Y, \top_Y)$ is a map $\phi : X \to Y$ such that $(\phi \times \phi)(\top_X) \subset \top_Y$.

With a slight modification of the definition of locality morphisms from [CGPZ18a], we introduce

Definition 2.9. Let $(X, \top_X, \cdot_X, 1_X)$ and $(Y, \top_Y, \cdot_Y, 1_Y)$ be groups with locality. A map $\phi : X \to Y$ is called a morphism of groups with locality, if it

1. is a locality map;
2. is locality multiplicative: for $(a, b) \in \top_X$ we have $\phi(a \cdot_X b) = \phi(a) \cdot_Y \phi(b)$,
3. preserves the unit $\phi(1_X) = 1_Y$.

We equip $\mathbb{R}$ with the trivial locality relation $\top_\mathbb{R} = \mathbb{R} \times \mathbb{R}$ and the locality group structure $(\mathbb{R}, \top_\mathbb{R}, +, 0)$. 

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Observation 2.10. A functional $F : E(M) \to \mathbb{R}$ is partially additive (2) and satisfies $F(0) = 0$, if and only if $F$ is a morphism with locality between the groups $(E(M), \top, +, 0)$ and $(\mathbb{R}, \top_\mathbb{R}, +, 0)$.

Similar to a group with locality, we can also define a group with causality. This notion will be crucial in section 4.

Definition 2.11. A group with causality $(G, \cdot_G, m_G, 1_G)$ is a group $(G, m_G, 1_G)$ and a causality set $(G, \cdot)$, for which the product law $m_G$ is compatible with $\cdot$, i.e.

- if $x_1 \cdot y$ and $x_2 \cdot y$, then $m_G(x_1, x_2) \cdot y$.
- if $y \cdot x_1$ and $y \cdot x_2$, then $y \cdot m_G(x_1, x_2)$.
- $(1_G)^\cdot = G$ and $\cdot (1_G) = G$.

Definition 2.12. Let $(X, \cdot_X, \cdot_X^\cdot, 1_X)$ and $(Y, \cdot_Y, \cdot_Y^\cdot, 1_Y)$ be groups with causality.

A map $\phi : X \to Y$ is called a morphism of groups with causality, if it

1. is a causality map (i.e. if $a \cdot_X b$ then $\phi(a) \cdot_Y \phi(b)$);
2. is causality multiplicative: for $a \cdot_X b$ we have $\phi(a \cdot_X b) = \phi(a) \cdot_Y \phi(b)$,
3. preserves the unit $\phi(1_X) = 1_Y$.

A notion stronger than being locality or causality multiplicative is provided by the Hammerstein property. Here we state it for groups, but one can also introduce it in other contexts. First we need one more definition.

Definition 2.13. Let $(X, \cdot_X, 1_X), (Y, \cdot_Y, 1_Y)$ be groups and let $\phi : X \to Y$. Given $a \in X$, we define the map $\phi_a : X \to Y$ by

$$\phi_a(b) = \phi(a)^{-1} \cdot_Y \phi(a \cdot_X b),$$

where $b \in X$.

Remark 2.14. For non-commutative groups one can consider variants of this definition, where the order of terms is interchanged. Our choice of convention is motivated by the one used in QFT for the definition of the relative $S$-matrix (see [BDF09] and a brief discussion at the end of Section 5.2.3)

Definition 2.15 (Generalized Hammerstein property). Consider a group $(X, \cdot_X, 1_X)$ with causality/locality specified by the relation $\cdot_X / \top_X$, respectively. Let $(Y, \cdot_Y, 1_Y)$ be another group. A map $\phi : X \to Y$ satisfies the generalized Hammerstein property, if $\phi_a : X \to Y$ (as given in Definition 2.13) is causality/locality multiplicative for all $a \in X$.

Remark 2.16. For commutative $(X, \cdot_X, 1_X)$, Definition 2.15 is equivalent to the condition that for $a \cdot_X b$ we have

$$\phi(a \cdot_X c \cdot_X b) = \phi(a \cdot_X c) \cdot_Y \phi(c)^{-1} \cdot_Y \phi(c \cdot_X b),$$

which is now more readily recognized as the Hammerstein property with non-commutative target.
We believe that this property in some sense singles out structures that we intuitively describe as local. In this note we show that it features in the definition of:

- Local functionals,
- Local Haag-Kastler nets,
- Local $S$-matrices,
- Local renormalization maps.

### 3 Haag-Kastler axioms

In the Haag-Kastler axiomatic approach to quantum field theory [HK64], one describes a model by assigning algebras $\mathfrak{A}(\mathcal{O})$ (originally, $C^*$ or von Neumann algebras) to bounded regions $\mathcal{O}$ in Minkowski spacetime $M$. More generally, in locally covariant QFT [BFV03, HW01, HW02], one can work with causally convex relatively compact subsets of a globally hyperbolic spacetime $M$.

This assignment of algebras to regions (the net of algebras) has to fulfill several physical requirements, among them:

**HK1 Isotony**: if $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$.

**HK2 Locality (Einstein causality)**: if $\mathcal{O}_1 \times \mathcal{O}_2$, then $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)]_{\mathfrak{A}(\mathcal{O})} = \{0\}$, where $\mathcal{O}$ is any causally convex relatively compact region containing both $\mathcal{O}_1$ and $\mathcal{O}_2$.

**HK3 Time-slice axiom**: if $\mathcal{N} \subset \mathcal{O}$ is a neighborhood of a Cauchy surface of $\mathcal{O}$, then $\mathfrak{A}(\mathcal{N}) = \mathfrak{A}(\mathcal{O})$.

In perturbative AQFT (see e.g. [BDF09, DF01, BF00, Rej16]), algebras $\mathfrak{A}(\mathcal{O})$ are considered formal power series with coefficients in topological star algebras. In the latter approach, one can construct the local algebras $\mathfrak{A}(\mathcal{O})$ using the concept of formal $S$-matrices.

### 4 Formal $S$-matrices

Following [FR15, BF19], we review the construction of the net of algebras satisfying HK1-HK3, using formal $S$-matrices.

**Definition 4.1 (Generalized local $S$-matrix).** Let $(G, \lhd, +, 0)$ be a group with causality and $\mathfrak{A}$ a unital topological $^*$-algebra, with $U(\mathfrak{A}) \subset \mathfrak{A}$ denoting its group of unitary elements. A map $S : G \to U(\mathfrak{A})$ is a generalized local $S$-matrix on $(G, \lhd, +, 0)$ (or labeled by the elements of $G$) if it fulfills the following axioms:

**S1 Identity preserving**: $S(0) = 1$. 
**S2 Locality:** $S$ satisfies the Hammerstein property (Def. 2.15), i.e. $f_1 \vdash f_2$ implies that

$$S(f_1 + f + f_2) = S(f_1 + f)S(f)^{-1}S(f_2),$$

where $f_1, f, f_2 \in G$.

Let $\mathcal{S}_G$ denote the space of all generalized local $S$-matrices for the given group $G$ with causality.

There are some further physically motivated axioms that one can impose, related to the dynamics. We will discuss this in Section 5.2, following the approach of [BF19].

**Remark 4.2.** Given a group with causality $(G, \vdash, +, 0)$ it is easy to obtain a generalized local $S$-matrix by setting $\mathfrak{A}$ to be $\mathfrak{A}_G$, the group algebra over $\mathbb{C}$ of the free group generated by elements $S(f)$ (these are now formal generators), $f \in G$, modulo relations $S^2$ and $S_1$ (see e.g. [BF19]).

Note that Definition 4.1 implies in particular that $S$ is unit preserving and causality multiplicative, since for $f_1 \vdash f_3$ we have

$$S(f_1 + f_3) = S(f_1)S(f_3).$$

By symmetrizing $\vdash$, we obtain from $(G, \vdash, +, 0)$ a group $(G, \triangleright, +, 0)$ with locality relation $\triangleright$ defined by: $f_1 \triangleright f_2$ if both $f_1 \vdash f_2$ and $f_2 \vdash f_1$. It follows that a generalized local $S$-matrix $S : (G, \triangleright, +, 0) \to U(\mathfrak{A})$ is locality multiplicative, i.e. if $f_1 \triangleright f_3$ then

$$S(f_1 + f_3) = S(f_1)S(f_3) = S(f_3)S(f_1),$$

hence

$$[S(f_1), S(f_3)] = 0.$$

Consider $(\mathcal{D}(M), \preceq, +, 0)$, the space of test functions $\mathcal{D}(M) = \mathcal{C}_c^\infty(M, \mathbb{R})$ with the additive group structure and the causality relation $\preceq$ of being not in the future. Clearly, if $S$ is a formal $S$-matrix on $(\mathcal{D}(M), \preceq, +, 0)$ and we define $\mathfrak{A}(\mathcal{O})$ as the algebra generated by $S(f)$, where $\text{supp}(f) \subset \mathcal{O}$, then the net $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ satisfies HK2.

This is how one can connect the abstract notion of locality expressed in terms of the Hammerstein property to the locality property HK2 in terms of Haag-Kastler axioms.

### 4.1 Renormalization group

#### 4.1.1 Abstract definition

**Definition 4.3** (Renormalization group). Let $(G, \vdash, +, 0)$ be a group with causality. The renormalization group $\mathcal{R}_G$ for $G$ is the group of maps $Z : G \to G$, which are:

**Z1 Identity preserving:** $Z(0) = 0$.

**Z2 Causality preserving:** $Z_f$ (as in Def. 2.13) is a causality preserving map for all $f \in G$, i.e.: $f_1 \vdash f_2$ implies that

$$(Z(f_1 + f) - Z(f)) \vdash (Z(f_2 + f) - Z(f))$$

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Z₃ Local: \( Z \) satisfies the Hammerstein property (Def. 2.15), i.e. \( f_1 \vdash f_2 \) implies that
\[
Z(f_1 + f + f_2) = Z(f_1 + f) - Z(f) + Z(f_2 + f),
\]
where \( f_1, f_2, f \in G \).

Clearly, if \( Z \in \mathcal{R}_G \), and \( S \in \mathcal{S}_G \), then \( S \circ Z \) is also an S-matrix, hence:

**Proposition 4.4.** \( \mathcal{S}_G \) is an \( \mathcal{R}_G \)-module.

**Proof.** The only non-trivial check is the Locality \( S \). For \( f_1 \vdash f_2 \) we have
\[
\tilde{S}(f_1 + f + f_2) = S \circ Z(f_1 + f + f_2) = S(Z(f_1) + Z(f_2) + Z(f))
\]
\[
= S(Z(f_1) + Z(f))S(Z(f))^{-1}S(Z(f_2) + Z(f)) = \tilde{S}(f_1)\tilde{S}(f)^{-1}\tilde{S}(f_2).
\]

where in the second step we used \( Z \) while in the third step we used \( Z \) together with \( S \).

One can now ask the question whether for given \( S, \tilde{S} \in \mathcal{S}_G \), there exists a \( Z \in \mathcal{R}_G \) such that \( \tilde{S} = S \circ Z \). This is more tricky to show and has been proven only under some additional assumptions, e.g. in the perturbative setting of \([BDF09]\). It would be interesting to investigate this problem for a finite dimensional Lie group \( G \) with additional requirement that elements of \( \mathcal{S}_G \) and \( \mathcal{R}_G \) are analytic maps.

### 4.1.2 Perturbative renormalization group

Replace \( G \) with \( \lambda G[[\lambda]] \) (where \( \lambda \) is interpreted as the coupling constant). We write the generalized local S-matrices as
\[
S(\lambda f) = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} T_n(f \otimes^n),
\]
where \( f \in G \) and call the multilinear maps \( T_n : G^{\otimes n} \to \mathfrak{A} \), the \( n \)-fold time-ordered products (they extend to maps on \( \lambda G[[\lambda]] \) in the obvious way). We require in addition to the previous axioms for local S-matrices that

**S3** The map
\[
T_1 : G \to \mathfrak{A},
\]
(physically interpreted as the quantization map) is invertible.

**S4** Causality preserving: \( T_1^{-1} \circ S_f \) (with the notations of Def. 2.13) is a causality preserving map for all \( f \in G \), i.e.: \( f_1 \vdash f_2 \) implies that
\[
T_1^{-1}(S_f(f_1)) \vdash T_1^{-1}(S_f(f_1)).
\]

From renormalization group elements, we require in addition that:

**Z4** \( Z = \text{id} + O(\lambda) \).
They are then given in terms of formal power series, so that

\[ Z(\lambda f) = f + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} Z_n(f^{\otimes n}), \]

where \( f \in G \).

With these caveats, the following theorem (main theorem of renormalization) has been proven in [BDF09] in the context of QFT:

**Theorem 4.5.** Let \( \lambda G[[\lambda]] \) be a group with causality. Given two generalized S-matrices (in the sense of definition 4.1), \( S, \tilde{S} \in \mathcal{S}_G \), there exists a unique element of the renormalization group \( Z \in \mathcal{R}_G \), such that

\[ \tilde{S} = S \circ Z. \]

Also the converse holds.

**Proof.** We follow directly the inductive proof of [BDF09]. Given \( \tilde{S} \) and \( S \), we want to construct \( Z \) as a formal power series:

\[ Z(f) = f + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} Z_n(f^{\otimes n}), \]

Assume that we have constructed \( Z \) as a formal power series up to order \( N \), i.e. that we have a family of maps

\[ Z_k : G^{\otimes k} \to G, \quad k \leq N \]

such that

\[ \tilde{S}(f) = S \circ Z^N(f) + \mathcal{O}(\lambda^{N+1}), \]

where

\[ Z^N(f) = f + \sum_{k=1}^{N} \frac{\lambda^n}{n!} Z_k(f^{\otimes k}) \]

and \( Z^N \in \mathcal{R}_G \). Clearly, \( \tilde{S}^N = S \circ Z^N \in \mathcal{S}_G \). We can now define

\[ Z_{N+1} = \tilde{T}^{-1}_1(\tilde{T}_{N+1} - \tilde{T}^N_{N+1}), \]

where \( \tilde{T}_{N+1} \) and \( \tilde{T}^N_{N+1} \) are coefficients in expansion of \( \tilde{S} \) and \( \tilde{S}^N \) respectively. We need to check that

\[ Z^{N+1} = Z^N + \frac{\lambda^{N+1}}{(N+1)!} Z_{N+1} \]

is an element of \( \mathcal{R}_G \). The only non-trivial properties to check are \( \mathcal{Z}2 \) and \( \mathcal{Z}3 \). These, however, are easy to verify using \( \mathcal{S}2 \) and \( \mathcal{S}4 \). \( \square \)
5 Local functionals and renormalization

5.1 Smooth local functionals

As stated in the introduction, local functionals can be characterized by requiring a regularity condition together with the Hammerstein property. For concreteness, we fix a globally hyperbolic spacetime $M$ and we will focus on the example of the real scalar field, i.e., we start with the classical field configuration space $\mathcal{E} \equiv C^\infty(M, \mathbb{R})$. For future reference, denote $\mathcal{D} \equiv C^\infty_c(M, \mathbb{R})$, the space of smooth functions on $M$ with compact support. Classical observables are smooth functionals of $\mathcal{E}$, i.e., elements of $C^\infty(\mathcal{E}, \mathbb{R})$. Smoothness is understood in the sense of Bastiani [Bas64, Ham82, Mil84, Nee06]:

**Definition 5.1.** Let $X$ and $Y$ be topological vector spaces, $U \subseteq X$ an open set and $F : U \rightarrow Y$ a map. The derivative of $f$ at $x \in U$ in the direction of $h \in X$ is defined as
\[
\langle F^{(1)}(x), h \rangle \doteq \lim_{t \rightarrow 0} t \frac{1}{t} (F(x + th) - F(x))
\]
whenever the limit exists. The function $f$ is called differentiable at $x \in U$ if $\langle F^{(1)}(x), h \rangle$ exists for all $h \in X$. It is called continuously differentiable if it is differentiable at all points of $U$ and $F^{(1)} : U \times X \rightarrow Y, (x, h) \mapsto F^{(1)}(x)(h)$ is a continuous map. It is called a $C^1$-map if it is continuous and continuously differentiable. Higher derivatives are defined by
\[
\langle F^{(k)}(x), v_1 \otimes \cdots \otimes v_k \rangle \doteq \frac{\partial^k}{\partial t_1 \cdots \partial t_k} F(x + t_1 v_1 + \cdots + t_k v_k) \Bigg|_{t_1 = \cdots = t_k = 0},
\]
and $f$ is $C^k$ if $f^{(k)}$ is jointly continuous as a map $U \times \mathbb{R}^k \rightarrow Y$. We say that $f$ is smooth if it is $C^k$ for all $k \in \mathbb{N}$.

The following definition of locality has been proposed in [BDF09, BFR12] and refined in [BDLGR18].

**Definition 5.2.** Let $\mathcal{X}$ and $\mathcal{Y}$ be topological vector spaces, $U \subseteq \mathcal{X}$ an open set and $F : U \rightarrow \mathcal{Y}$ a map. The derivative of $f$ at $x \in U$ in the direction of $h \in \mathcal{X}$ is defined as
\[
\langle F^{(1)}(x), h \rangle \doteq \lim_{t \rightarrow 0} t \frac{1}{t} (F(x + th) - F(x))
\]
whenever the limit exists. The function $f$ is called differentiable at $x \in U$ if $\langle F^{(1)}(x), h \rangle$ exists for all $h \in \mathcal{X}$. It is called continuously differentiable if it is differentiable at all points of $U$ and $F^{(1)} : U \times \mathcal{X} \rightarrow \mathcal{Y}, (x, h) \mapsto F^{(1)}(x)(h)$ is a continuous map. It is called a $C^1$-map if it is continuous and continuously differentiable. Higher derivatives are defined by
\[
\langle F^{(k)}(x), v_1 \otimes \cdots \otimes v_k \rangle \doteq \frac{\partial^k}{\partial t_1 \cdots \partial t_k} F(x + t_1 v_1 + \cdots + t_k v_k) \Bigg|_{t_1 = \cdots = t_k = 0},
\]
and $f$ is $C^k$ if $f^{(k)}$ is jointly continuous as a map $U \times \mathcal{X}^k \rightarrow \mathcal{Y}$. We say that $f$ is smooth if it is $C^k$ for all $k \in \mathbb{N}$.

The following result relates the above formulation to the more “traditional” notion of locality used in physics.

**Theorem 5.3** (VI.3 in [BDLGR18]). Let $U \subseteq \mathcal{E}$ and $F : U \rightarrow \mathbb{R}$ be Bastiani smooth. Then, $F$ is local in the sense of definition 5.2 if and only if for every $\varphi \in U$, there is a neighborhood $V$ of $\varphi$, an integer $k$, an open subset $V \subseteq J^k M$ and a smooth function $F$ defined on $V$ such that $F^* = F$.\footnote{We want the observables to be real-valued, since in the quantum theory the involution is defined as the complex conjugation and we want the observables to be “self-adjoint” in the sense that $F^* = F$.}
function $f \in C^\infty(V)$ such that $x \in M \mapsto f(j^k_x \psi)$ is supported in a compact subset $K \subset M$ and
\[ F(\varphi + \psi) = F(\varphi) + \int_M f(j^k_x \psi) \, dx, \]
whenever $\varphi + \psi \in V$ and where $j^k_x \psi$ denotes the $k$-jet of $\psi$ at $x$.

Here $J^k M$ denotes the bundle of $k$-jets and $j^k_x(\varphi)$ is the $k$-th jet of $\varphi$ as a point $x \in M$.\(^3\)

Clearly, the space of local functionals $\mathcal{F}_{\text{loc}}$ forms a group with addition. In order to equip it with causality or locality structure, we will also need a notion of *spacetime support*:

**Definition 5.4.** For $F \in C^\infty(\mathcal{E}, \mathbb{R})$, its support is defined by
\[ \text{supp} F = \{ x \in M \mid \forall \text{open } U \ni x \exists \varphi, \psi \in \mathcal{E}, \text{supp } \psi \subset U, F(\varphi + \psi) \neq F(\varphi) \}. \]

### 5.2 Local $S$-matrices in the functional approach

Using local functionals one can construct a concrete realization of generalized local $S$-matrices from definition 4.1 that is relevant for building QFT models. In this section we summarize how both classical and quantum dynamics can be nicely described using maps satisfying the Hammerstein property.

#### 5.2.1 Classical dynamics

We introduce generalized Lagrangians, following the approach of [BDF09].

**Definition 5.5.** A generalized Lagrangian is a smooth map $L$ from $(\mathcal{D}, \tau, +, 0)$ to $(\mathcal{F}_{\text{loc}}, \tau, +, 0)$ (here $\tau$ is disjointness of supports), which is support preserving
\[ \text{supp}(L(f)) \subset \text{supp } f. \]

(in particular it is a *locality map*) and satisfies the Hammerstein property, i.e.
\[ L(f_1 + f_2) = L(f_1 + f) - L(f) + L(f_2 + f), \]
if $f_1 \perp f_2$. Let $\mathcal{L}$ denote the space of generalized Lagrangians.

Following [BF19], we introduce some notation.

**Definition 5.6.** Let $L \in \mathcal{L}$, $\varphi \in \mathcal{E}$. We define a functional $\delta L(\varphi) : \mathcal{D} \times \mathcal{E} \to \mathbb{R}$ by
\[ \delta L(\psi)[\varphi] = L(f)[\varphi + \psi] - L(f)[\varphi], \]
where $\varphi \in \mathcal{E}$, $\psi \in \mathcal{D}$ and $f \equiv 1$ on $\text{supp } \psi$ (the map $\delta L(\psi)[\varphi]$ thus defined does not depend on the particular the choice of $f$).

\(^3\)Recall that $j^k_x(\varphi)$ is the equivalence class of $\varphi$ in the quotient $C^\infty(M)/I^{k+1}_x \cong J^k_x(M)$, with the understanding that $I^{k+1}_x$ coincides with the ideal of smooth functions on $M$ whose $k + 1$ order derivatives vanish at the point $x$. $J^k_x(M)$ is the disjoint union $\bigsqcup_{x \in M} J^k_x(M)$.
The above definition can be turned into a difference quotient and we can use it to introduce the Euler-Lagrange derivative of $L$.

**Definition 5.7.** The Euler-Lagrange derivative of $L$ is a map $dL : E \times D \to \mathbb{R}$ defined by

$$
\langle dL(\varphi), \psi \rangle = \lim_{t \to 0} \frac{1}{t} \delta L(t\psi)[\varphi],
$$

where $\psi \in D$, $\varphi \in E$.

Note that $dL$ can be seen as a 1-form on $E$ (i.e. as a map from $E$ to $D'$). The zero locus of $dL$ is the space of solutions to the equations of motion, i.e. $\varphi \in E$ is a solution if

$$
dL(\varphi) \equiv 0
$$
as an element of $D'$.

For the free scalar field the Lagrangian is

$$
L(f)[\varphi] = \frac{1}{2} \int_M \left( \nabla_\nu \varphi \nabla^\nu \varphi - m^2 \varphi^2 \right) f d\mu_g,
$$

where $\mu_g$ is the invariant measure associated with the metric $g$ of $M$. The equation of motion is

$$
dL(\varphi) = P\varphi = 0,
$$

where $P = -(\Box + m^2)$ is (minus) the Klein-Gordon operator. On a globally hyperbolic spacetime $M$, $P$ admits retarded and advanced Green’s functions $\Delta^R$, $\Delta^A$.

These are distinguished by the properties that

$$
P \circ \Delta^{R/A} = \text{id}_D \quad \Delta^{R/A} \circ (P\big|_D) = \text{id}_D
$$

and

$$
\text{supp}(\Delta^R) \subset \{(x,y) \in M^2 | y \in J^-(x)\},
$$

$$
\text{supp}(\Delta^A) \subset \{(x,y) \in M^2 | y \in J^+(x)\}.
$$

This is illustrated on the diagram below.
5.2.2 Quantum dynamics

Following the approach of [BF19], we can now define the dynamical \( S \)-matrix. Firstly, note that \( G = (\mathcal{F}_{\text{loc}}, \lesssim, +, 0) \) is a group with causality, if we define \( \lesssim \) as the “not in the future of” relation on supports of functionals.

Let \( L \in \mathcal{L} \) (this is interpreted as the Lagrangian of the theory). We then define \( \mathfrak{A}_L \) using the group algebra over \( \mathbb{C} \) of the free group generated by elements \( S(F) \), \( F \in G \), modulo relations \( S^2 \), \( S^1 \) and the following relation proposed by [BF19] that encodes the dynamics:

\[
S(F)S(\delta L(\varphi)) = S(F^\varphi + \delta L(\varphi)) = S(\delta L(\varphi))S(F).
\] (S5)

where \( F^\varphi(\psi) = F(\varphi + \psi) \), \( \varphi, \psi \in \mathcal{E} \) and \( \delta L \) is given in Def. 5.6.

Physically, this relation can be interpreted as the Schwinger-Dyson equation on the level of formal \( S \)-matrices.

The construction of the dynamical formal \( S \)-matrix presented here allows one to construct interacting nets of algebras satisfying the axioms HK1-HK3, starting from the given Lagrangian (see [BF19] for details). This, however, is not sufficient to have a complete physical description of the system, since one still needs to identify physically relevant states. The existence of such states (e.g. the vacuum state or thermal states on Minkowski spacetime) for a given theory has not been established yet, but a perturbative procedure is known, starting from states of the free theory.

5.2.3 Concrete models and states

In this section we outline another construction of the local \( S \)-matrix starting from a Lagrangian, which is closely related to that of Section 5.2.2. The advantage is that it is more explicit and it comes with an obvious prescription how to define states, perturbatively, for the interacting theory. For the purpose of this review, we will treat only compactly supported interactions. For the discussion of adiabatic limit, see for example [BDF09, FR15].

We work in the functional formalism. First we define the Pauli-Jordan (commutator) function as the difference of the retarded and advanced Green functions:

\[
\Delta = \Delta^R - \Delta^A.
\]

This gives us the Poisson bracket of \( F, G \in \mathcal{F}_{\text{loc}} \) by

\[
\{F, G\} = \langle F^{(1)}, \Delta G^{(1)} \rangle
\]

where \( \langle \cdot, \cdot \rangle \) denotes the dual pairing between \( F^{(1)}(\varphi) \in \mathcal{D} \) and \( \Delta G^{(1)}(\varphi) \in \mathcal{E} \) given by the integration over \( M \) with the invariant volume measure \( d\mu_g \) induced by the metric. This pairing can be extended also to distributions (as we will often do in the formulas that follow), as long as their wavefront sets satisfy the Hörmander criterion [Hör03].

The \( \ast \)-product (deformation of the pointwise product) is defined by

\[
(F \ast G)(\varphi) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}(\varphi), W^{\otimes n} G^{(n)}(\varphi) \right\rangle,
\]

where \( \hbar \) is the reduced Planck constant.
where $W$ is the 2-point function of some quasifree Hadamard state\(^4\) and it differs from \(\frac{i}{2}\Delta\) by a symmetric bidistribution:

$$W = \frac{i}{2}\Delta + H.$$  

Without going into technical details, we note that $W$ has the following crucial properties:

(H 1) $\Delta = 2\text{Im}(W)$.

(H 2) $W$ is a distributional bisolution to the field equation, i.e. \(\langle W, Pf \otimes g \rangle = 0\) and \(\langle W, f \otimes Pg \rangle = 0\) for all $f, g \in D_c$.

(H 3) $W$ is of positive type, meaning that \(\langle W, \bar{f} \otimes f \rangle \geq 0\), where $\bar{f}$ is the complex conjugate of $f \in D_C$.

Now define the corresponding Feynman propagator by

$$\Delta^F = \frac{i}{2}(\Delta^A + \Delta^R) + H$$

For $F_1, \ldots, F_n \in \mathcal{T}_\text{loc}$ such that $\text{supp } F_i \cap \text{supp } F_j = \emptyset$ for every pair $i, j \in \{1, \ldots, n\}$, the $n$-fold time-ordered product is defined by

$$\mathcal{T}_n(F_1, \ldots, F_n) = m_n \circ e^{\sum_{1 \leq i < j \leq n} D^F_{ij}}(F_1 \otimes \cdots \otimes F_n),$$

where

$$D^F_{ij} = \left\langle \Delta^F, \frac{\delta}{\delta \varphi_i} \otimes \frac{\delta}{\delta \varphi_j} \right\rangle$$

and $m_n$ is the pullback through the diagonal map $E \to E^\otimes n$, $\varphi \mapsto \varphi \otimes \cdots \otimes \varphi$.

**Remark 5.8.** The family of $n$-fold time-ordered products defined above can be used to construct a certain locality algebra in the sense of [CGPZ18a] (as the product is not always defined), namely $(\mathcal{T}(\mathcal{F}_{\text{ml}}^{\text{pdsl}}[\hbar]), \triangledown, \cdot, +, 0, 1)$, where:

- $\mathcal{F}_{\text{ml}}^{\text{pdsl}}$ is the space of functionals that arise as finite sums of local functionals and products of local functionals with $F(0) = 0$ and pairwise disjoint supports.
- The locality relation $\triangledown$ is the disjointness of supports of functionals.
- The map $\mathcal{T}$ is defined by

$$\mathcal{T} = \bigoplus_{n=0}^{\infty} \mathcal{T}_n \circ \beta,$$

where $\beta$ is the inverse of multiplication on multilocal functionals (as defined in [FR12]).

\(^4\)On Minkowski spacetime this is just the Wightman vacuum 2-point function. In general, a quasifree Hadamard state is distinguished by the fact that its 2-point function has a singularity structure (i.e. wavefront set) which resembles that of the Minkowski vacuum. In particular, the wavefront set of $W$ is such all the pairings between $W^\otimes n$ and derivatives of $F$ and $G$ are well defined. See [BF00] for details.
• The product is

\[ F \cdot_G G = \mathcal{T}(\mathcal{T}^{-1} F \cdot \mathcal{T}^{-1} G) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \langle F^{(n)}, (\Delta^F)^{\otimes n} G^{(n)} \rangle. \]

In order to obtain the local $S$-matrix we need to extend the domain of definition of $\mathcal{T}_n$ to tensor products of arbitrary local functionals.

We set $G \equiv \mathcal{T}_{\text{loc}}$ and $F \vdash G$ iff $\text{supp}(F) \subseteq \text{supp}(G)$. As for $\mathfrak{A}$, we will consider the space of Laurent series in the formal parameter $\hbar$ with coefficients in formal power series in $\lambda$, i.e.

\[ \mathfrak{A} = (\mathcal{F}_\mu[[\lambda]]((\hbar))), \ast_H, \]

where $\mathcal{F}_\mu$ is the space of microcausal functionals on $M$ defined by

**Definition 5.9.** A functional $F \in \mathcal{C}^\infty(\mathcal{E}, \mathbb{R})$ is called *microcausal* if it is compactly supported and satisfies

\[ \text{WF}(F^{(n)}(\varphi)) \subset \Xi_n, \quad \forall n \in \mathbb{N}, \forall \varphi \in \mathcal{E}, \quad (8) \]

where $\Xi_n$ is an open cone defined as

\[ \Xi_n = T^* M^n \setminus \{(x_1, \ldots, x_n; k_1, \ldots, k_n) | (k_1, \ldots, k_n) \in (\overline{T}^+ \cup \overline{T}^-)(x_1, \ldots, x_n)\}, \quad (9) \]

where $(\overline{T}_\pm)_x$ is the closed future/past lightcone understood as a conic subset of $T^*_x M$.

For $\mathcal{S}$ given in terms of $\mathcal{T}_n$ to satisfy $S2$, we need to require that

\[ \mathcal{T}_n(V_1, \ldots, V_n) = \mathcal{T}_k(V_1, \ldots, V_k) \ast \mathcal{T}_k(V_{k+1}, \ldots, V_n), \quad (T1) \]

if $\text{supp}V_{k+1}, \ldots, \text{supp}V_n$ are not later than $\text{supp}V_1, \ldots, \text{supp}V_k$. This is the key property (called *causal factorisation*) for setting up the inductive procedure of Epstein and Glaser [EG73] ([EG73] and a comprehensive review [Dütt19]). This procedure allows one to extend $\mathcal{T}_n$ to the full domain $\mathcal{F}^{\otimes n}_{\text{loc}}$, so for $V \in \mathcal{F}_{\text{loc}}$, we define

\[ \mathcal{S}(\lambda V) = e^{i\lambda V / \hbar} = \sum_{n=0}^{\infty} \left( \frac{i\lambda}{\hbar} \right)^n \frac{1}{n!} \mathcal{T}_n(V^{\otimes n}), \]

The ambiguity of extensions of $\mathcal{T}_n$ is governed by the renormalization group $\mathcal{R}_G$ (as two $S$-matrices have to differ by $Z \in \mathcal{R}_G$ by virtue of Theorem 4.5).

**Remark 5.10.** Analogously to the construction in Remark 5.8, given renormalized $\mathcal{T}_n$, one constructs the algebra $(\mathcal{T}(\mathcal{F}_\text{ml}[[\hbar]]), \cdot, +, 0, 1)$, where $\mathcal{F}_\text{ml}$ is analogous to $\mathfrak{S}_\text{ml}$, but without the support restriction and $\mathcal{T}$ is the same as in (7) (see [FR12]).

States on $\mathfrak{A}$ can be defined using evaluation functionals. For example, the evaluation at $\varphi = 0$ corresponds to the expectation value in the quasifree Hadamard state, whose 2-point function $W$ has been used to define $\ast$.

In the next step one introduces the interacting fields. Given interaction $V \in \mathcal{F}_{\text{loc}}$ and $F \in \mathcal{F}_{\text{loc}}$, we introduce the *relative $S$-matrix* using Def. 2.13, i.e.

\[ S_{\lambda V}(\mu F) \equiv \mathcal{S}(\lambda V)^{-1} \ast \mathcal{S}(\lambda V + \mu F), \]

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understood as a formal power series in both $\lambda$ and $\mu$ and a Laurent series in $\hbar$.

The formal $S$-matrix is a generating functional for the interacting fields. We define the interacting observable corresponding to $F$ by

$$F_{\text{int}} = \left. -i\hbar \frac{d}{d\mu}S_{\lambda V}(\mu F) \right|_{\mu=0}.$$  

Correlation functions of interacting observables corresponding to $F_1, \ldots, F_n$ can then be easily computed by means of

$$\omega_{\text{int}}(F_1, \ldots, F_n) = \left. (F_{1\text{int}} \ast \cdots \ast F_{n\text{int}}) \right|_{\varphi=0}.$$  

6 Conclusions

In this paper we have introduced the abstract notion of a group with causality and we have shown how various constructions in Lorentzian QFT can be formulated using that notion. We have also shown that the intuitive notion of “locality” in classical and quantum field theory is nicely captured by the appropriate generalization of the Hammerstein property. We hope that further investigation of structures with that property will lead to a better understanding of locality in physics and in mathematics.

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