Research article

Initial boundary value problems for space-time fractional conformable differential equation

Tingting Guan¹, Guotao Wang¹,²,* and Haiyong Xu³

¹ School of Mathematics and Computer Science, Shanxi Normal University, Linfen, Shanxi 041004, China
² Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan, China
³ School of Mathematics and Statistics, Ningbo University, Ningbo 315212, China

* Correspondence: Email: wgt2512@163.com; Tel: +18935042198.

Abstract: In this paper, the authors study a initial boundary value problems (IBVP) for space-time fractional conformable partial differential equation (PDE). Several inequalities of fractional conformable derivatives at extremum points are presented and proved. Based on these inequalities at extremum points, a new maximum principle for the space-time fractional conformable PDE is demonstrated. Moreover, the maximum principle is employed to prove a new comparison principle and estimation of solutions. Beside that, the uniqueness and continuous dependence of the solution of the space-time fractional conformable PDE are demonstrated.

Keywords: space-time fractional conformable differential equation; maximum principle; comparison theorem; uniqueness and continuous dependence

Mathematics Subject Classification: 26A33, 35K55

1. Introduction

Maximum principle, one of the most useful tools, is applied to study of complex dynamic systems without knowing explicit form of solutions [1–7]. In 2009, maximum principle for a fractional partial differential equation (PDE) was formulated in explicit form by Luchko [3]. In addition, he and his partners [1–5] proved the maximum principle for the generalized time-fractional and multi-terms time-fractional diffusion equations. By using the maximum principle, they also obtained the uniqueness and continuous dependence of solutions for the the generalized time-fractional and multi-terms time-fractional diffusion equations on the initial and boundary conditions. In 2019, Wang, Ren and Baleanu [8] applied maximum principle to investigating initial boundary value problems (IBVP) for Hadamard
fractional differential equations involving fractional Laplace operator and got some existence and uniqueness results. In 2020, Mokhtar and Berikbol [9] proposed maximum principle for the space-time fractional diffusion and pseudo-parabolic equations with Caputo and Riemann-Liouville time-fractional derivatives. Based on the maximum principle, it is proved that the uniqueness and continuous dependence of the solution of IBVP for the nonlinear space-time fractional diffusion and pseudo-parabolic equations. It is provided that maximum principle for time-fractional diffusion equations with singular kernel fractional derivatives [10, 11], non-singular kernel fractional derivatives [6, 7, 12–14] or fractional Laplace operators [8, 15–19]. Extremum principles for fractional differential equations have huge potential application and attract the attention for more and more scholars [20–39].

Jarad et al. [40] introduced the fractional conformable derivatives in the sense of Caputo and Riemann-Liouville and stated their properties. To the best of our knowledge, the mathematical literature on the maximum principles and their applications for Caputo fractional conformable derivatives is rarely mentioned. Inspired by the above works, in this paper we investigate an IBVP to fractional diffusion equations involving fractional Laplace operator and got some existence and uniqueness results. In 2020, Mokhtar and Berikbol [9] proposed maximum principle for the space-time fractional conformable PDE. First, we present several inequalities of Caputo fractional derivatives. Based on the maximum principle, it is proved that the uniqueness and continuous dependence of solutions on the initial and boundary conditions.

The rest of this article is organized as follows: In Section 2, we introduce some definitions about Caputo fractional conformable derivatives and establish several extremum principles. In Section 3, these extremum principles are employed to derive maximum principle. Finally, the maximum principle is applied to obtain estimation of solutions, comparison principle and the uniqueness and continuous dependence of solutions on the initial and boundary conditions.

2. Problem formulation and extreme principles

In this paper, we shall investigate the following space-time Caputo fractional conformable PDE

\[ C^\beta_T D^\epsilon_T z(x,t) - \left[ C^\gamma_a D^\epsilon_x z(x,t) + C^\gamma D^\epsilon_{bx} z(x,t) \right] - a(x,t) z(x,t) = g(x,t), \quad (x,t) \in (a,b) \times (T,T_1]. \] (2.1)

Here \( x \) and \( t \) are the space and time variables, \( a(x,t) \in C^{1,1}([a,b] \times [T, T_1]) \), and \( 0 < \epsilon, \beta < 1, 1 < \gamma < 2 \). \( C^\beta_T D^\epsilon_T \) is the left Caputo fractional conformable derivative of order \( \beta \). \( C^\gamma_a D^\epsilon_x \) and \( C^\gamma D^\epsilon_{bx} \) are the left and right Caputo fractional conformable derivatives of order \( \gamma \). For \( f \in C^m_{\epsilon,T}(T,T_1) \), the left Caputo fractional conformable derivative of order \( \beta \) is defined by

\[ C^\beta_T D^\gamma_x f(t) = \frac{1}{\Gamma(n-\beta)} \int_T^t \left( \frac{(t-s)^{\gamma-1}}{\epsilon} \right)^{n-\beta-1} \frac{mT^\gamma f(s)}{(s-T)^{1-\epsilon}} ds. \] (2.2)

For \( f \in C^m_{\epsilon,a}([a,b]) \) (\( f \in C^m_{\epsilon,b}(b,a]) \)), the left (right) Caputo fractional conformable derivatives of order \( \gamma \) can be written, respectively, as

\[ C^\gamma D^\epsilon_x f(t) = \frac{1}{\Gamma(m-\gamma)} \int_a^t \left( \frac{(x-s)^{\gamma-1}}{\epsilon} \right)^{m-\gamma-1} \frac{mT^\gamma f(s)}{(s-a)^{1-\epsilon}} ds, \] (2.3)

and

\[ C^\gamma D^\epsilon_{bx} f(t) = \frac{(-1)^m}{\Gamma(m-\gamma)} \int_x^b \left( \frac{(b-s)^{\gamma-1}}{\epsilon} \right)^{m-\gamma-1} \frac{mT^\gamma f(s)}{(b-s)^{1-\epsilon}} ds, \] (2.4)
with \( n = [\beta]+1, m = [\gamma]+1, a T^\varepsilon f(t) = (t-a)^{1-\varepsilon} f(t), T^\varepsilon f(t) = (b-t)^{1-\varepsilon} f(t), \frac{\partial^n}{\partial t^n} a T^\varepsilon \cdots a T^\varepsilon = a T^\varepsilon a T^\varepsilon \cdots a T^\varepsilon \), \( m T_b = T_b m T_b \cdots T_b \), \( C_{\varepsilon,T}^n[T,T_1] = \{ f : [T,T_1] \to \mathbb{R} | \int_T^{T_1} m T^\varepsilon_f \in I_c[T,T_1] \}, C_{\varepsilon,a}[a,b] = \{ f : [a,b] \to \mathbb{R} | \int_a^b m T^\varepsilon_f \in I_c[a,b] \} \) and \( C_{\varepsilon,b}[a,b] = \{ f : [a,b] \to \mathbb{R} | \int_a^b m T^\varepsilon_f \in \mathcal{I}(a,b) \} \) (where \( I_c[T,T_1], I_c[a,b] \) and \( \mathcal{I}(a,b) \) are defined in Definition 3.1 in [41]). The detailed information of Caputo fractional conformable derivative, see [40].

Denote

\[
H(\bar{U}) = \{ z(x,t) | z(x,t) \in C^{2,1}((a,b) \times (T,T_1)), z(x,t) \in C([a,b] \times [T,T_1]) \}. \tag{2.5}
\]

For our maximum principle, we make use of the following three Caputo fractional conformable extremum principles.

**Lemma 2.1.** If \( f \in C_{\varepsilon,a}^2((a,b)) \) reaches its maximum at a point \( x_0 \in (a,b) \). Then the inequality

\[
c_{\gamma}^a D_x^\varepsilon f(x_0) \leq 0 \tag{2.6}
\]

holds.

**Proof.** Let

\[
g(x) = f(x_0) - f(x) \geq 0, \quad x \in [a,b]. \tag{2.7}
\]

Concurrently, \( g(x) \in C_{\varepsilon,a}^2([a,b]) \), \( g(x_0) = 0 \) and \( c_{\gamma}^a D_x^\varepsilon g(x) = -c_{\gamma}^a D_x^\varepsilon f(x) \).

By calculation, we notice that

\[
c_{\gamma}^a D_x^\varepsilon g(x_0) = \frac{1}{\Gamma(2-\gamma)} \int_a^{x_0} \left(\frac{(x_0-a)^\varepsilon - (s-a)^\varepsilon}{\varepsilon}\right)^{1-\gamma} \left((s-a)^{1-\varepsilon} g'(s)\right)' \, ds
\]

\[
= \frac{1}{\Gamma(2-\gamma)} \left(\frac{(x_0-a)^\varepsilon - (s-a)^\varepsilon}{\varepsilon}\right)^{1-\gamma} (s-a)^{1-\varepsilon} g'(s) \bigg|_{a}^{x_0} + \frac{1-\gamma}{\Gamma(2-\gamma)} \int_a^{x_0} \left(\frac{(x_0-a)^\varepsilon - (s-a)^\varepsilon}{\varepsilon}\right)^{-\gamma} g'(s) \, ds. \tag{2.8}
\]

Since

\[
\lim_{s \to x_0} \frac{1}{\Gamma(2-\gamma)} \left(\frac{(x_0-a)^\varepsilon - (s-a)^\varepsilon}{\varepsilon}\right)^{1-\gamma} (s-a)^{1-\varepsilon} g'(s)
\]

\[
= \frac{1}{\Gamma(2-\gamma)} \lim_{s \to x_0} \frac{g''(s)(s-a)^{1-\varepsilon} + g'(s)(1-\varepsilon)(s-a)^{-\varepsilon}}{(1-\gamma)(\frac{(x_0-a)^\varepsilon - (s-a)^\varepsilon}{\varepsilon})^{\gamma-1}} (s-a)^{1-\varepsilon}
\]

\[
= 0.
\]

Therefore, the formula (2.8) becomes

\[
c_{\gamma}^a D_x^\varepsilon g(x_0) = \frac{1-\gamma}{\Gamma(2-\gamma)} \int_a^{x_0} \left(\frac{(x_0-a)^\varepsilon - (s-a)^\varepsilon}{\varepsilon}\right)^{-\gamma} g'(s) \, ds \tag{2.9}
\]

\[
= \frac{1-\gamma}{\Gamma(2-\gamma)} \left(\frac{(x_0-a)^\varepsilon - (s-a)^\varepsilon}{\varepsilon}\right)^{-\gamma} g(s) \bigg|_{a}^{x_0} + \frac{\gamma(1-\gamma)}{\Gamma(2-\gamma)} \int_a^{x_0} \left(\frac{(x_0-a)^\varepsilon - (s-a)^\varepsilon}{\varepsilon}\right)^{\gamma-1} (s-a)^{\varepsilon-1} g(s) \, ds. \tag{2.10}
\]
Lemma 2.2. If $f \in C^2_{\epsilon,a}([a, b])$ reaches its maximum at a point $x_0 \in (a, b)$. Then the inequality

$$C_\gamma D^\epsilon_{b,x_0} f(x_0) \leq 0$$

holds.

Proof. Let

$$g(x) = f(x_0) - f(x) \geq 0, \quad x \in [a, b].$$

Concurrently, $g(x) \in C^2_{\epsilon,a}([a, b])$, $g(x_0) = 0$ and $C_\gamma D^\epsilon_{b,a} g(x) = -C_\gamma D^\epsilon_{b,a} f(x)$.

By calculation, we notice that

$$C_\gamma D^\epsilon_{b,x_0} g(x_0) = \frac{1}{\Gamma(2-\gamma)} \int_{x_0}^b \left( \frac{(b-x_0)^\epsilon - (b-s)^\epsilon}{\epsilon} \right)^{1-\gamma} \left( b-s \right)^{1-\epsilon} g'(s) ds$$

$$= \frac{1}{\Gamma(2-\gamma)} \left. \left( \frac{(b-x_0)^\epsilon - (b-s)^\epsilon}{\epsilon} \right)^{1-\gamma} \left( b-s \right)^{1-\epsilon} g'(s) \right|_{x_0}^b$$

$$- \frac{1-\gamma}{\Gamma(2-\gamma)} \int_{x_0}^b \left( \frac{(b-x_0)^\epsilon - (b-s)^\epsilon}{\epsilon} \right)^{1-\gamma} g'(s) ds.$$

Since

$$\lim_{s \to x_0} \frac{1}{\Gamma(2-\gamma)} \left( \frac{(b-x_0)^\epsilon - (b-s)^\epsilon}{\epsilon} \right)^{1-\gamma} \left( b-s \right)^{1-\epsilon} g'(s)$$

$$= \frac{1}{\Gamma(2-\gamma)} \lim_{s \to x_0} \frac{g''(s)(b-s)^{1-\epsilon} - g'(s)(1-\epsilon)(b-s)^{-\epsilon}}{(\gamma-1) \left( \frac{(b-x_0)^\epsilon - (b-s)^\epsilon}{\epsilon} \right)^{\gamma-2} (b-s)^{\epsilon-1}}$$

$$= 0.$$
Therefore, the formula (2.15) becomes
\[
C_{\gamma} D_{b,x_0}^\epsilon g(x_0) = - \frac{1 - \gamma}{\Gamma(2 - \gamma)} \int_{x_0}^b \left( \frac{(b - x_0)^\epsilon - (b - s)^\epsilon}{\epsilon} \right)^{-\gamma} g(s) ds.
\]
\[
= - \frac{1 - \gamma}{\Gamma(2 - \gamma)} \left( \frac{(b - x_0)^\epsilon - (b - s)^\epsilon}{\epsilon} \right)^{-\gamma} g(s) \bigg|_{x_0}^b
\]
\[
- \gamma (1 - \gamma) \frac{1 - \gamma}{\Gamma(2 - \gamma)} \int_{x_0}^b \left( \frac{(b - x_0)^\epsilon - (b - s)^\epsilon}{\epsilon} \right)^{-\gamma - 1} (b - s)^{\epsilon - 1} g(s) ds.
\]
\[\text{(2.17)}\]

Since
\[
\lim_{s \to x_0} \frac{1 - \gamma}{\Gamma(2 - \gamma)} \left( \frac{(b - x_0)^\epsilon - (b - s)^\epsilon}{\epsilon} \right)^{-\gamma} g(s) = - \frac{1 - \gamma}{\gamma \Gamma(2 - \gamma)} \lim_{s \to x_0} \frac{g'(s)(b - s)^{1 - \epsilon}}{(b - x_0)^\epsilon - (b - s)^\epsilon} \]
\[
= 0.
\]
\[\text{(2.18)}\]

Therefore, the formula (2.17) becomes
\[
C_{\gamma} D_{b,x_0}^\epsilon g(x_0) = \frac{\gamma - 1}{\Gamma(2 - \gamma)} \left( \frac{(b - x_0)^\epsilon}{\epsilon} \right)^{-\gamma} g(b)
\]
\[
+ \frac{\gamma (\gamma - 1)}{\Gamma(2 - \gamma)} \int_{x_0}^b \left( \frac{(b - x_0)^\epsilon - (b - s)^\epsilon}{\epsilon} \right)^{-\gamma - 1} (b - s)^{\epsilon - 1} g(s) ds
\]
\[\geq 0.
\]
\[\text{(2.19)}\]

We can obtain \(C_{\gamma} D_{b,x_0}^\epsilon f(x_0) \leq 0\).

The lemma is proved. \(\square\)

Using the same method, it is easy to obtain the following lemmas.

**Lemma 2.3.** If \(f \in C^1_{\epsilon,T}([T, T_1])\) reaches its maximum at a point \(t_0 \in (T, T_1]\). Then the inequality
\[
C_T D_{T_0}^\epsilon f(t_0) \geq 0
\]
holds.

**Lemma 2.4.** If \(f \in C^1_{\epsilon,T}([T, T_1])\) reaches its minimum at a point \(t_0 \in (T, T_1]\). Then the inequality
\[
C_T D_{T_0}^\epsilon f(t_0) \leq 0
\]
holds.

**Lemma 2.5.** If \(f \in C^2_{\epsilon,a}([a, b])\) reaches its minimum at a point \(x_0 \in (a, b)\). Then the inequality
\[
C_a D_{x_0}^\epsilon f(x_0) \geq 0
\]
holds.
Lemma 2.6. If \( f \in C^2_{\varepsilon,b}([a,b]) \) reaches its minimum at a point \( x_0 \in (a,b) \). Then the inequality
\[
\frac{\partial D^\varepsilon_{b\varepsilon}}{\partial x} f(x_0) \geq 0
\]
holds.

Example 2.1
If \( f(x) = -(x - \frac{b + a}{2})^2 \), Lemma 2.1 and 2.2 hold.
If \( f(t) = -(t - \frac{T_1 + T}{2})^2 \), Lemma 2.3 holds.
If \( f(t) = (t - \frac{T_1 + T}{2})^2 \), Lemma 2.4 holds.
If \( f(x) = (x - \frac{b + a}{2})^2 \), Lemma 2.5 and 2.6 hold.

3. Maximum principle

In this section, we shall consider the linear space-time Caputo fractional conformable PDE \((2.1)\) on the initial-boundary conditions:
\[
z(x, T) = \varphi(x), \quad x \in [a, b],
\]
\[
z(a, t) = \mu_1(t), \quad z_b(a, t) + hz(b, t) = \mu_2(t) \quad t \in [T, T_1],
\]
where \( h \) is a given positive constant, \( U = (a,b) \times (T, T_1), \bar{U} = [a,b] \times [T, T_1] \) and \( S = ([a,b] \times \{T\} \cup \{a\} \times [a,b] \cup \{b\} \times [a,b]). \)

Theorem 3.1. Assume \( g(x,t) \leq 0 \), \( \forall (x,t) \in U \). If \( z \in H(\bar{U}) \) satisfies the linear space-time Caputo fractional conformable PDE \((2.1),(3.1)\) and \((3.2)\), then
\[
z(x, t) \leq \max\{\max_{x \in [a,b]} \varphi(x), \max_{t \in [T, T_1]} \mu_1(t), \frac{1}{h} \max_{t \in [T, T_1]} \mu_2(t), 0\}, \quad \forall (x,t) \in \bar{U}
\]
holds.

Proof. Arguing by contradiction, assume that there exists a point \((x_0, t_0) \in U\) satisfies
\[
z(x_0, t_0) > \max\{\max_{x \in [a,b]} \varphi(x), \max_{t \in [T, T_1]} \mu_1(t), \frac{1}{h} \max_{t \in [T, T_1]} \mu_2(t), 0\} = M > 0.
\]
Denote \( \varepsilon = z(x_0, t_0) - M > 0 \) and
\[
w(x, t) = z(x, t) + \frac{\varepsilon T_1 - (t - T)}{2 T_1}, \quad (x, t) \in \bar{U}.
\]
According to the definition of \( w \), we have
\[
w(x, t) \leq z(x, t) + \frac{\varepsilon}{2}, \quad (x, t) \in \bar{U},
\]
\[
w(x_0, t_0) \geq z(x_0, t_0) = \varepsilon + M \geq \varepsilon + z(x, t) \geq \varepsilon + w(x, t) - \frac{\varepsilon}{2}
\geq \frac{\varepsilon}{2} + w(x, t), \quad (x, t) \in S.
\]
The latter property implies that the maximum of \( w \) cannot be attained on \( S \). Let \( w(x_1, t_1) = \max_{(x, t) \in \bar{U}} w(x, t) \), then
\[
w(x_1, t_1) \geq w(x_0, t_0) \geq \epsilon + M > \epsilon.
\]

By Lemma 2.1, 2.2 and 2.3, we know
\[
\begin{cases}
C^\beta_J D^\epsilon_t w(x, t) \bigg|_{(x_1, t_1)} \geq 0, & 0 < \epsilon < 1, 0 < \beta < 1, \\
C^\gamma_J D^\epsilon_t w(x, t) \bigg|_{(x_1, t_1)} \leq 0, & 0 < \epsilon < 1, 1 < \gamma < 2, \\
C^\gamma_J D^\epsilon_{bt} w(x, t) \bigg|_{(x_1, t_1)} \leq 0, & 0 < \epsilon < 1, 1 < \gamma < 2.
\end{cases}
\] (3.5)

By calculation, we can show
\[
C^\beta_J D^\epsilon_t \left( \frac{\epsilon T_1 - (t - T)}{T_1} \right) = -\frac{1}{\Gamma(1 - \beta)} \frac{\epsilon}{2 T_1} \int_T^t \left( \frac{(t - T)^\beta - (s - T)^\beta}{\epsilon} \right) ds.
\] (3.6)

Assume \( u = (\frac{t - T}{T})^\epsilon \), substituting into the formula (3.6), we get
\[
C^\beta_J D^\epsilon_t \left( \frac{\epsilon T_1 - (t - T)}{T_1} \right) = -\frac{1}{\Gamma(1 - \beta)} \frac{\epsilon}{2 T_1} \int_0^1 (t - T)^{1 - \epsilon \beta} (1 - u)^{-\beta} u^{1 - \epsilon} du
\] (3.7)

Applying (3.5) – (3.7), it holds
\[
\begin{align*}
C^\beta_J D^\epsilon_t z(x, t) \bigg|_{(x_1, t_1)} - & \left[ C^\gamma_J D^\epsilon_t z(x, t) + C^\gamma_J D^\epsilon_{bt} z(x, t) \right] \bigg|_{(x_1, t_1)} - a(x_1, t_1)z(x_1, t_1) - g(x_1, t_1) \\
= & \left[ C^\gamma_J D^\epsilon_t w(x, t) + C^\gamma_J D^\epsilon_{bt} w(x, t) \right] \bigg|_{(x_1, t_1)} - a(x_1, t_1)w(x_1, t_1) - \frac{\epsilon T_1 - (t_1 - T)}{T_1} - g(x_1, t_1) \\
\geq & \epsilon^{1 - \epsilon \beta} \frac{\epsilon}{2 T_1} \frac{\Gamma(2 - \epsilon)}{\Gamma(3 - \epsilon - \beta)} - a(x_1, t_1)\epsilon \left( 1 - \frac{T_1 - (t_1 - T)}{2 T_1} \right) > 0,
\end{align*}
\] (3.8)

which is in contradiction with (2.1).

This completes the proof of the theorem. \( \square \)

Similarly, the following minimum principle can be obtained by substituting \(-z\) for \(z\) in the Theorem 3.1.

**Theorem 3.2.** Assume \( g(x, t) \geq 0, \forall (x, t) \in U \). If \( z \in H(\bar{U}) \) satisfies the linear space-time Caputo fractional conformable PDE (2.1), (3.1) and (3.2), then
\[
z(x, t) \geq \min \left\{ \min_{x \in [a, b]} \varphi(x), \min_{t \in [t_1, T]} \mu(t), \frac{1}{H} \min_{t \in [T, t_1]} \mu_2(t) \right\}, \quad \forall (x, t) \in \bar{U}
\] (3.9)
holds.
4. Applications of the maximum principle

**Theorem 4.1.** If \( z(x, t) \in H(\bar{U}) \) is a solution of the Eq (2.1) on initial boundary conditions (3.1) and (3.2), then the inequality

\[
\|z\|_{C(\bar{U})} \leq \max_{x \in [a, b]} \|\varphi(x)\|, \quad \max_{t \in [T, T_1]} \|\mu_1(t)\|, \quad \frac{1}{h} \max_{t \in [T, T_1]} \|\mu_2(t)\| + 2M \frac{\Gamma(2 + \epsilon \beta + \epsilon - \beta)}{\beta \epsilon \Gamma(1 + \epsilon \beta - \epsilon)} (T_1 - T)^\epsilon
\]

(4.1)

holds, where

\[
M = \|g\|_{C(\bar{U})},
\]

(4.2)

**Proof.** Let

\[
w(x, t) = z(x, t) - M \frac{\Gamma(2 + \epsilon \beta + \epsilon - \beta)}{\beta \epsilon \Gamma(1 + \epsilon \beta - \epsilon)} (T_1 - T)^\epsilon, \quad (x, t) \in \bar{U}.
\]

If \( z(x, t) \) is a solution of the Eqs (2.1), (3.1) and (3.2), then \( w(x, t) \) is a solution of the problem (2.1) with

\[
g^*(x, t) = g(x, t) - M \frac{\Gamma(2 + \epsilon \beta + \epsilon - \beta)}{\beta \epsilon \Gamma(1 + \epsilon \beta - \epsilon)} c_D (t - T)^\epsilon,
\]

(4.3)

\[
\mu_1^*(t) = \mu_1(t) - M \frac{\Gamma(2 + \epsilon \beta + \epsilon - \beta)}{\beta \epsilon \Gamma(1 + \epsilon \beta - \epsilon)} (t - T)^\epsilon,
\]

(4.4)

\[
\mu_2^*(t) = \mu_2(t) - hM \frac{\Gamma(2 + \epsilon \beta + \epsilon - \beta)}{\beta \epsilon \Gamma(1 + \epsilon \beta - \epsilon)} (t - T)^\epsilon.
\]

(4.5)

\( g^*(x, t), \mu_1^*(t) \) and \( \mu_2^*(t) \) replace \( g(x, t), \mu_1(t) \) and \( \mu_2(t) \), respectively. Due to \( g^*(x, t) \leq 0 \), by using Theorem 3.1 (Maximum principle), we have

\[
z(x, t) \leq \max_{x \in [a, b]} \|\varphi(x)\|, \quad \max_{t \in [T, T_1]} \|\mu_1(t)\| + M \frac{\Gamma(2 + \epsilon \beta + \epsilon - \beta)}{\beta \epsilon \Gamma(1 + \epsilon \beta - \epsilon)} (T_1 - T)^\epsilon,
\]

(4.6)

\[
\frac{1}{h} \max_{t \in [T, T_1]} \|\mu_2(t)\| + M \frac{\Gamma(2 + \epsilon \beta + \epsilon - \beta)}{\beta \epsilon \Gamma(1 + \epsilon \beta - \epsilon)} (T_1 - T)^\epsilon.
\]

(4.7)

Therefore,

\[
z(x, t) \leq \max_{x \in [a, b]} \|\varphi(x)\|, \quad \max_{t \in [T, T_1]} \|\mu_1(t)\|, \quad \frac{1}{h} \max_{t \in [T, T_1]} \|\mu_2(t)\| + 2M \frac{\Gamma(2 + \epsilon \beta + \epsilon - \beta)}{\beta \epsilon \Gamma(1 + \epsilon \beta - \epsilon)} (T_1 - T)^\epsilon.
\]

(4.8)

In a similar fashion, we get

\[
z(x, t) \geq - \max_{x \in [a, b]} \|\varphi(x)\|, \quad \max_{t \in [T, T_1]} \|\mu_1(t)\|, \quad \frac{1}{h} \max_{t \in [T, T_1]} \|\mu_2(t)\| + 2M \frac{\Gamma(2 + \epsilon \beta + \epsilon - \beta)}{\beta \epsilon \Gamma(1 + \epsilon \beta - \epsilon)} (T_1 - T)^\epsilon.
\]

(4.9)

Combining (4.8) and (4.9), the theorem is proved. \( \Box \)
Theorem 4.2. If \( z(x,t) \) is a solution of the IBVP (2.1), (3.1) and (3.2), \( z(x,t) \) continuously depends on the data given in the problem in the sense that if

\[
\| g - g' \|_{C(\mathcal{U})} \leq \varepsilon, \quad \| \varphi(x) - \varphi'(x) \|_{C([a,b])} \leq \varepsilon_0, \quad \| \mu_1(t) - \mu_1'(t) \|_{C([T,T_1])} \leq \varepsilon_1, \quad \| \mu_2(t) - \mu_2'(t) \|_{C([T,T_1])} \leq \varepsilon_2,
\]

then, the estimate

\[
\| z - z' \|_{C(\mathcal{U})} \leq \max\{\varepsilon_0, \varepsilon_1, \varepsilon_2\} + 2\frac{\Gamma(2 + \epsilon_0) - \epsilon - \beta}{\beta \epsilon \Gamma(1 + \epsilon_0)} (T_1 - T)\epsilon_0
\]

for the corresponding classical solution \( z(x,t) \) and \( z'(x,t) \) true.

The demonstrate process is similar to Theorem 4.1.

Theorem 4.3. Assume \( g(x,t) \leq 0, a(x,t) \leq 0 \) for \( (x,t) \in U, \varphi(x) \leq 0 \) for \( x \in (a,b) \) and \( \mu_1(t) \leq 0, \mu_2(t) \leq 0 \) for \( t \in (T,T_1] \). If \( z \in H(\mathcal{U}) \) is a solution of the IBVP (2.1), (3.1) and (3.2), then

\[
\| z(x,t) \|_{C(\mathcal{U})} \leq 0, (x,t) \in \bar{U}
\]

holds.

Theorem 4.4. Assume \( g(x,t) \geq 0, a(x,t) \geq 0 \) for \( (x,t) \in U, \varphi(x) \geq 0 \) for \( x \in (a,b) \) and \( \mu_1(t) \geq 0, \mu_2(t) \geq 0 \) for \( t \in (T,T_1] \). If \( z \in H(\mathcal{U}) \) satisfies the IBVP (2.1), (3.1) and (3.2), then

\[
\| z(x,t) \|_{C(\mathcal{U})} \geq 0, (x,t) \in \bar{U}
\]

holds.

The conclusion of Theorem 4.3 and Theorem 4.4 can be obtained by Theorem 3.1.

Remark 4.1. Assume \( g(x,t) = a(x,t) = 0 \) for \( (x,t) \in U, \varphi(x) = 0 \) for \( x \in (a,b) \) and \( \mu_1(t) = \mu_2(t) \geq 0 \) for \( t \in (T,T_1] \). If \( z \in H(\mathcal{U}) \) satisfies the IBVP (2.1), (3.1) and (3.2), then

\[
\| z(x,t) \|_{C(\mathcal{U})} = 0, \forall (x,t) \in \bar{U},
\]

holds.

Theorem 4.5. Assume \( \frac{\partial z}{\partial x} + a(x,t) \leq 0, \forall (x,t) \in U, \) then IBVP of the following nonlinear space-time fractional conforable PDE

\[
\begin{align*}
\frac{\partial^\beta}{\partial t^\beta} z(x,t) &- \left[ \frac{\partial^\gamma}{\partial x^\gamma} D^\gamma_x z(x,t) + \frac{\partial^\gamma}{\partial t^\gamma} D^\gamma_t z(x,t) \right] - a(x,t)z(x,t) = G(x,t,z), \quad (x,t) \in U \\
z(x,T) &= \varphi(x), \quad x \in [a,b], \\
z(a,t) &= \mu_1(t), \quad t \in [T,T_1], \\
z_a(b,t) + hz(b,t) &= \mu_2(t), \quad t \in [T,T_1],
\end{align*}
\]

has a unique solution on \( H(\bar{U}) \).
Proof. Suppose \( z_1, z_2 \) are two solutions of IBVP (4.5). Let
\[
z(x, t) = z_1(x, t) - z_2(x, t), \quad \forall (x, t) \in \bar{U},
\]
satisfy the equation
\[
\begin{cases}
\frac{C^\beta}{T^\gamma} \partial_x^\frac{\gamma}{\beta} z(x, t) - \left[ \frac{C^\alpha}{a} \partial_x^\frac{\alpha}{a} z(x, t) + \frac{C^\gamma}{b} \partial_x^\frac{\gamma}{b} z(x, t) \right] - a(x, t) z(x, t) = G(x, t, z_1 - G(x, t, z_2), \quad (x, t) \in U, \\
z(x, T) = 0, \quad x \in [a, b], \\
z(a, t) = 0, \quad t \in [T, T_1], \\
z_x(b, t) + hz(b, t) = 0, \quad t \in [T, T_1].
\end{cases}
\] (4.6)

In view of
\[
G(x, t, z_1) - G(x, t, z_2) = \frac{\partial G}{\partial z}(z^*)(z_1 - z_2).
\] (4.7)

where \( z^* = (1 - \lambda)z_1 + \lambda z_2, \quad 0 < \lambda < 1. \)

Using (4.6) and (4.7), we have that
\[
\begin{cases}
\frac{C^\beta}{T^\gamma} \partial_x^\frac{\gamma}{\beta} z(x, t) - \left[ \frac{C^\alpha}{a} \partial_x^\frac{\alpha}{a} z(x, t) + \frac{C^\gamma}{b} \partial_x^\frac{\gamma}{b} z(x, t) \right] = \left( \frac{\partial G}{\partial z}(z^*) + a(x, t) \right) z(x, t), \quad (x, t) \in U \\
z(x, T) = 0, \quad x \in [a, b], \\
z(a, t) = 0, \quad t \in [T, T_1], \\
z_x(b, t) + hz(b, t) = 0, \quad t \in [T, T_1].
\end{cases}
\] (4.8)

Since \( \frac{\partial G}{\partial z} + a(x, t) \leq 0 \), applying Theorem 4.3, we have
\[
z(x, t) \leq 0, \quad (x, t) \in \bar{U}.
\] (4.9)

By the same way, using Theorem 4.4 to \(-z(x, t)\) we have
\[
z(x, t) \geq 0, \quad (x, t) \in \bar{U}.
\] (4.10)

Combining (4.9) and (4.10), we can get
\[
z(x, t) = 0, \quad \forall (x, t) \in \bar{U}.
\]

Thus, the theorem holds. \( \square \)

Example 4.1
Consider the following space-time Caputo fractional conformable PDE:
\[
\begin{cases}
\frac{C^\beta}{T^\gamma} \partial_x^\frac{\gamma}{\beta} z(x, t) - \left[ \frac{C^\alpha}{a} \partial_x^\frac{\alpha}{a} z(x, t) + \frac{C^\gamma}{b} \partial_x^\frac{\gamma}{b} z(x, t) \right] - a(x, t) z(x, t) = G(x, t, z), \quad (x, t) \in U \\
z(x, T) = \varphi(x), \quad x \in [a, b], \\
z(a, t) = \mu_1(t), \quad t \in [T, T_1], \\
z_x(b, t) + hz(b, t) = \mu_2(t), \quad t \in [T, T_1],
\end{cases}
\] (4.11)

where \( 0 < \lambda < 1, \quad \alpha, \beta \in (0, 1), \quad \gamma \in (1, 2). \)
If \( G(x,t,z) = -x^2 \), \( a(x,t) = -x^2 t^2 \), \( \varphi(x) = -x^2 \), \( \mu_1(t) = 0 \) and \( \mu_2(t) = 0 \), Theorem 3.1, 4.1, 4.2 and 4.3 hold.

If \( G(x,t,z) = x^2 \), \( a(x,t) = x^2 t^2 \), \( \varphi(x) = x^2 \), \( \mu_1(t) = 0 \) and \( \mu_2(t) = 0 \), Theorem 3.2, 4.1, 4.2 and 4.4 hold.

If \( a(x,t) = -t^2 \) and \( G(x,t,z) = \frac{x^2}{z} \). We have \( \frac{\partial G}{\partial u} + a(x,t) = -\frac{x^2}{z^2} - t^2 \leq 0 \), then Theorem 4.5 holds.

**Theorem 4.6.** (Comparison Theorem) Assume \( c(x,t) \geq 0 \), \( d(x,t) \geq 0 \) and \( d(x,t) \geq c(x,t) \) for \( (x,t) \in U \).

If \( (z_1,z_2) \in H(\bar{U}) \times H(\bar{U}) \) satisfies

\[
\begin{align*}
&\left\{\begin{array}{l}
C_b T D_{\bar{z}_1} \xi_1(x,t) - [C_b T D_{\bar{z}_2} \xi_1(x,t) + C_b T D_{\bar{z}_2} \xi_1(x,t)] - c(x,t) \xi_2(x,t) - d(x,t) \xi_1(x,t) \geq 0, \\
(x,t) \in U,
\end{array}\right. \\
&\left\{\begin{array}{l}
C_b T D_{\bar{z}_2} \xi_2(x,t) - [C_b T D_{\bar{z}_2} \xi_2(x,t) + C_b T D_{\bar{z}_2} \xi_2(x,t)] - c(x,t) \xi_2(x,t) - d(x,t) \xi_2(x,t) \geq 0, \\
(x,t) \in U,
\end{array}\right.
\end{align*}
\]

\( z_1(x,T) \geq 0, \quad z_2(x,T) \geq 0, \quad x \in [a,b], \)

\( z_1(a,t) \geq 0, \quad z_2(a,t) \geq 0, \quad t \in [T, T_1], \)

\( (z_1)_x(b,t) + h z_1(b,t) \geq 0, \quad (z_2)_x(b,t) + h z_2(b,t) \geq 0, \quad t \in [T, T_1], \)

then

\( z_1(x,t) \geq 0, z_2(x,t) \geq 0, (x,t) \in \bar{U}, \)

hold.

**Proof.** Denote \( \xi(x,t) = z_1(x,t) + z_2(x,t), \ \forall (x,t) \in \bar{U} \). Then, by (4.12), we have

\[
\begin{align*}
&\left\{\begin{array}{l}
C_b T D_{\bar{z}_1} \xi(x,t) - [C_b T D_{\bar{z}_2} \xi(x,t) + C_b T D_{\bar{z}_2} \xi(x,t)] - c(x,t) \xi(x,t) - d(x,t) \xi(x,t) \geq 0, \\
(x,t) \in U,
\end{array}\right. \\
&\xi(x,T) \geq 0, \quad x \in [a,b], \\
&\xi(a,t) \geq 0, \quad t \in [T, T_1], \\
&\xi_x(b,t) + h \xi(b,t) \geq 0, \quad t \in [T, T_1].
\end{align*}
\]

Thus, by (4.13) and Theorem 4.4, we obtain

\( \xi(x,t) \geq 0, \forall (x,t) \in \bar{U}, \)

that is

\( z_1(x,t) + z_2(x,t) \geq 0, (x,t) \in \bar{U}. \)

Using (4.13) and (4.14), we have that

\[
\begin{align*}
&\left\{\begin{array}{l}
C_b T D_{\bar{z}_1} \xi_1(x,t) - [C_b T D_{\bar{z}_2} \xi_1(x,t) + C_b T D_{\bar{z}_2} \xi_1(x,t)] - (d(x,t) - c(x,t)) \xi_1(x,t) \geq 0, \\
(x,t) \in U,
\end{array}\right. \\
&\xi_1(x,T) \geq 0, \quad x \in [a,b], \\
&\xi_1(a,t) \geq 0, \quad t \in [T, T_1], \\
&(z_1)_x(b,t) + h z_1(b,t) \geq 0, \quad t \in [T, T_1],
\end{align*}
\]
and
\[
\begin{aligned}
\begin{cases}
\frac{c_b}{T} D_{x}^{\epsilon} z_1(x, t) - \left[ \frac{c_y}{a} D_{x}^{\epsilon} z_2(x, t) + c_y D_{x}^{\epsilon} z_2(x, t) \right] - (d(x, t) - c(x, t)) z_2(x, t) \geq 0, & (x, t) \in U, \\
z_2(x, T) \geq 0, & x \in [a, b], \\
z_2(a, t) \geq 0, & t \in [T, T_1], \\
(z_2)_n(b, t) + h z_2(b, t) \geq 0, & t \in [T, T_1].
\end{cases}
\end{aligned}
\] (4.16)

Applying Theorem 4.4 to (4.15) and (4.16), we get
\[
z_1(x, t) \geq 0, z_2(x, t) \geq 0, (x, t) \in \bar{U}.
\]

Thus, the Theorem holds. \(\square\)

Using the same way, the following Theorem holds.

**Theorem 4.7.** Assume \(c(x, t) \leq 0, d(x, t) \leq 0\) and \(c(x, t) \geq d(x, t)\) for \((x, t) \in U\). If \((z_1, z_2) \in H(\bar{U}) \times H(\bar{U})\) satisfies
\[
\begin{aligned}
\begin{cases}
\frac{c_b}{T} D_{x}^{\epsilon} z_1(x, t) - \left[ \frac{c_y}{a} D_{x}^{\epsilon} z_2(x, t) + c_y D_{x}^{\epsilon} z_2(x, t) \right] - c(x, t) z_2(x, t) - d(x, t) z_1(x, t) \leq 0, & (x, t) \in U, \\
z_1(x, T) \leq 0, z_2(x, T) \leq 0, & x \in [a, b], \\
z_1(a, t) \leq 0, z_2(a, t) \leq 0, & t \in [T, T_1], \\
(z_1)_n(b, t) + h z_1(b, t) \leq 0, (z_2)_n(b, t) + h z_2(b, t) \leq 0, & t \in [T, T_1],
\end{cases}
\end{aligned}
\] (4.17)
then
\[
z_1(x, t) \leq 0, z_2(x, t) \leq 0, (x, t) \in \bar{U},
\]
hold.

**Remark 4.2.** Assume \(c(x, t) = d(x, t) = 0\) for \((x, t) \in U\), \(\varphi(x) = \varphi^*(x) = 0\) for \(x \in (a, b)\) and \(\mu_1(t) = \mu_1^*(t) = \mu_2(t) = \mu_2^*(t) = 0\) for \(t \in (T, T_1)\). If \((z_1, z_2) \in H(\bar{U}) \times H(\bar{U})\) satisfies
\[
\begin{aligned}
\begin{cases}
\frac{c_b}{T} D_{x}^{\epsilon} z_1(x, t) - \left[ \frac{c_y}{a} D_{x}^{\epsilon} z_2(x, t) + c_y D_{x}^{\epsilon} z_2(x, t) \right] - c(x, t) z_2(x, t) - d(x, t) z_1(x, t) = 0, & (x, t) \in U, \\
z_1(x, T) = \varphi(x), z_2(x, T) = \varphi^*(x), & x \in [a, b], \\
z_1(a, t) = \mu_1(t), z_2(a, t) = \mu_1^*(t), & t \in [T, T_1], \\
(z_1)_n(b, t) + h z_1(b, t) = \mu_2(t), (z_2)_n(b, t) + h z_2(b, t) = \mu_2^*(t), & t \in [T, T_1],
\end{cases}
\end{aligned}
\] (4.18)
then
\[
z_1(x, t) = 0, z_2(x, t) = 0, \forall (x, t) \in \bar{U},
\]
hold.
Next, we consider the following linear space-time fractional conformable PDE

\[
\begin{cases}
\frac{C^\beta}{T} D_t^\epsilon z_1(x, t) - [C_y D_t^\epsilon z_1(x, t) + C_y D_t^\epsilon z_1(x, t)] - c(x, t)z_2(x, t) - d(x, t)z_1(x, t) = g_1(x, t), \\
\frac{C^\beta}{T} D_t^\epsilon z_2(x, t) - [C_y D_t^\epsilon z_2(x, t) + C_y D_t^\epsilon z_2(x, t)] - c(x, t)z_1(x, t) - d(x, t)z_2(x, t) = g_2(x, t), \\
z_1(x, T) = \varphi(x), \quad z_2(x, T) = \varphi^*(x), \\
z_1(a, t) = \mu_1(t), \quad z_2(a, t) = \mu_1^*(t), \\
(z_1)_s(b, t) + h z_1(b, t) = \mu_2(t), \quad (z_2)_s(b, t) + h z_2(b, t) = \mu_2^*(t), \quad t \in [T, T_1],
\end{cases}
\tag{4.19}
\]

**Theorem 4.8.** If \((z_1, z_2) \in H(\bar{U}) \times H(\bar{U})\) is a solution of the linear space-time fractional conformable PDE (4.19), then

\[
\|z_1\| \leq \frac{1}{2}(M_1 + M_2 + M_3 + M_4), \|z_2\| \leq \frac{1}{2}(M_1 + M_2 + M_3 + M_4), \quad (x, t) \in \bar{U},
\]

hold, where

\[
M_1 = \max_{x \in [a, b]} \max_{t \in [T, T_1]} \|\varphi(x) + \varphi^*(x)\|, \max_{t \in [T, T_1]} \|\mu_1(t) + \mu_1^*(t)\|, \frac{1}{h} \max_{t \in [T, T_1]} \|\mu_2(t) + \mu_2^*(t)\|,
\]

\[
M_2 = \max_{x \in [a, b]} \max_{t \in [T, T_1]} \|\varphi(x) - \varphi^*(x)\|, \max_{t \in [T, T_1]} \|\mu_1(t) - \mu_1^*(t)\|, \frac{1}{h} \max_{t \in [T, T_1]} \|\mu_2(t) - \mu_2^*(t)\|,
\]

\[
M_3 = 2\|g_1 + g_2\|_{C(\bar{U})} \frac{\Gamma(2 + \epsilon \beta - \epsilon - \beta)}{\beta \epsilon \Gamma(1 + \epsilon \beta - \epsilon)} (T_1 - T)^{\epsilon \beta},
\]

\[
M_4 = 2\|g_1 - g_2\|_{C(\bar{U})} \frac{\Gamma(2 + \epsilon \beta - \epsilon - \beta)}{\beta \epsilon \Gamma(1 + \epsilon \beta - \epsilon)} (T_1 - T)^{\epsilon \beta}.
\]

**Proof.** Let \(\xi(x, t) = z_1(x, t) + z_2(x, t), \quad \eta(x, t) = z_1(x, t) - z_2(x, t), \quad \forall(x, t) \in \bar{U}.\) Then, by (4.19), we have

\[
\begin{cases}
\frac{C^\beta}{T} D_t^\epsilon \xi(x, t) - [C_y D_t^\epsilon \xi(x, t) + C_y D_t^\epsilon \xi(x, t)] - (c(x, t) + d(x, t))\xi(x, t) = g_1(x, t) + g_2(x, t), \\
\frac{C^\beta}{T} D_t^\epsilon \eta(x, t) - [C_y D_t^\epsilon \eta(x, t) + C_y D_t^\epsilon \eta(x, t)] - (d(x, t) - c(x, t))\eta(x, t) = g_1(x, t) - g_2(x, t), \\
\xi(x, T) = \varphi(x) + \varphi^*(x), \quad \eta(x, T) = \varphi(x) - \varphi^*(x), \\
\xi(a, t) = \mu_1(t) + \mu_1^*(t), \quad \eta(a, t) = \mu_1(t) - \mu_1^*(t), \\
(\xi)_s(b, t) + h \xi(b, t) = \mu_2(t) + \mu_2^*(t), \quad (\eta)_s(b, t) + h \eta(b, t) = \mu_2(t) - \mu_2^*(t), \quad t \in [T, T_1].
\end{cases}
\tag{4.20}
\]

Thus, by (4.20) and Theorem 4.1, we obtain

\[
\|\xi\|_{C(\bar{U})} \leq M_1 + M_3, \tag{4.21}
\]

and

\[
\|\eta\|_{C(\bar{U})} \leq M_2 + M_4. \tag{4.22}
\]
Using (4.21) and (4.22), we have that
\[ \|z_1\|_{C(\bar{U})} \leq \frac{1}{2}(M_1 + M_2 + M_3 + M_4), \]
and
\[ \|z_2\|_{C(\bar{U})} \leq \frac{1}{2}(M_1 + M_2 + M_3 + M_4). \]

Thus, the Theorem holds. \(\square\)

**Theorem 4.9.** Assume \(c(x, t) \leq 0, d(x, t) \leq 0, d(x, t) < c(x, t), g_1(x, t) \leq 0 \) and \(g_2(x, t) \leq 0\) for \((x, t) \in U\), then IBVP (4.19) has a unique solution on \(H(\bar{U}) \times H(\bar{U})\).

**Proof.** Suppose \((z_{11}, z_{21})\), \((z_{12}, z_{22})\) are two solutions of IBVP (4.19). Let
\[ z_1(x, t) = z_{11}(x, t) - z_{12}(x, t), \quad v(x, t) = z_{21}(x, t) - z_{22}(x, t), \quad \forall (x, t) \in \bar{U}, \]
satisfy the equation
\[
\begin{cases}
\frac{C^\beta}{t} D_t^\epsilon z_1(x, t) - \left[ \frac{C^\beta}{t} D_t^\epsilon z_{11}(x, t) + \frac{C^\gamma}{x} D_x^\epsilon z_{11}(x, t) \right] - c(x, t)z_2(x, t) - d(x, t)z_1(x, t) = 0, \\
\frac{C^\gamma}{x} D_x^\epsilon z_2(x, t) - \left[ \frac{C^\beta}{t} D_t^\epsilon z_{22}(x, t) + \frac{C^\gamma}{x} D_x^\epsilon z_{22}(x, t) \right] - c(x, t)z_1(x, t) - d(x, t)z_2(x, t) = 0,
\end{cases}
\]
\( (x, t) \in U, \quad z_1(x, T) = 0, \quad z_2(x, T) = 0, \quad x \in [a, b], \quad z_1(a, t) = 0, \quad z_2(a, t) = 0, \quad t \in [T, T_1], \)
\[ (z_1)_x(b, t) + h z_1(b, t) = 0, \quad (z_2)_x(b, t) + h z_2(b, t) = 0, \quad t \in [T, T_1]. \]

Applying Theorem 4.7, we get
\[ z_1(x, t) \leq 0, z_2(x, t) \leq 0, (x, t) \in \bar{U}. \]

Similarly, employing Theorem 4.7 to \(-z_1(x, t)\) and \(-z_2(x, t)\) we get
\[ z_1(x, t) \geq 0, z_2(x, t) \geq 0, (x, t) \in \bar{U}. \]

Combining (4.24) and (4.25), we have
\[ z_1(x, t) = 0, z_2(x, t) = 0, \quad \forall (x, t) \in \bar{U}. \]

Thus, the Theorem holds. \(\square\)

**Example 4.2**
Consider the linear space-time Caputo fractional conformable PDE (4.19), if \(g_1(x, t) = x^2 t^2, \quad g_2(x, t) = x^4, \quad c(x, t) = \frac{t^2}{2}, \quad d(x, t) = t^2, \quad \varphi(x) = x^2, \quad \mu_1(t) = 0 \) and \(\mu_2(t) = 0\), Theorem 4.6 and 4.8 hold.

If \(g_1(x, t) = -x^2 t^2, \quad g_2(x, t) = -x^4, \quad c(x, t) = -\frac{t^2}{2}, \quad d(x, t) = -t^2, \quad \varphi(x) = -x^2, \quad \mu_1(t) = 0 \) and \(\mu_2(t) = 0\), Theorem 4.7, 4.8 and 4.9 hold.
5. Conclusion

In this paper, we have proved two extreme principles for the Caputo fractional conformable derivatives. Based on these extreme principles, a maximum principle for the space-time fractional conformable diffusion equation is established. Furthermore, the maximum principle is applied to show a new comparison principle, estimation of solutions and the uniqueness and continuous dependence of the solution for the IBVP to the space-time Caputo fractional conformable equations. Our results are new and contribute significantly to the literature on the topic.

Acknowledgments

We would like to express our gratitude to the anonymous reviewers and editors for their valuable comments and suggestions which led to the improvement of the original manuscript.

Conflict of interest

The authors declare no conflict of interest.

References

1. Y. Luchko, Initial-boundary-value problems for the one-dimensional time-fractional diffusion equation, *Frac. Calc. Appl. Anal.*, 15 (2012), 141–160.
2. Y. Luchko, Initial-boundary-value problem for the generalized multi-term time-fractional diffusion equation, *J. Math. Anal. Appl.*, 374 (2011), 538–548.
3. Y. Luchko, Maximum principle for the generalized time-fractional diffusion equation, *J. Math. Anal. Appl.*, 351 (2009), 218–223.
4. Y. Luchko, Maximum principle and its application for time-fractional diffusion equations, *Frac. Calc. Appl. Anal.*, 14 (2011), 110–124.
5. Y. Luchko, M. Yamamoto, On the maximum principle for a time-fractional diffusion equations, *Frac. Calc. Appl. Anal.*, 20 (2017), 1131–1145.
6. M. Al-Refai, T. Abdeljawad, Analysis of the fractional diffusion equations with fractional derivative of non-singular kernel, *Adv. Diff. Equ.*, 2017 (2017), 1–12.
7. M. Al-Refai, Comparison principle for differential equations involving Caputo fractional derivative with Mittag-Leffler non-singular kernel, *Electron. J. Differ. Eq.*, 2018 (2018), 1–10.
8. G. Wang, X. Ren, D. Baleanu, Maximum principle for Hadamard fractional differential equations involving fractional Laplace operator, *Math. Meth. Appl. Sci.*, 2019 (2019), 1–10.
9. M. Kirane, B. Torebek, Maximum principle for space and time-space fractional partial differential equations, *Mahtematic*, 2020 (2020), 1–24.
10. L. Cao, H. Kong, S. Zeng, Maximum principles for time-fractional Caputo-Katugampola diffusion equations, *J. Nonlinear Sci. Appl.*, 10 (2017), 2257–2267.
11. M. Kirane, B. Torebek, Extremum principle for Hadamard derivatives and its application to nonlinear fractional partial differential equations, *Frac. Calc. Appl. Anal.*, 22 (2019), 358–378.

12. M. Borikhanov, M. Kirane, B. Torebek, Maximum principle and its applications for the nonlinear time-fractional diffusion equations with Cauchy-Dirichlet conditions, *Appl. Math. Lett.*, 81 (2018), 14–20.

13. M. Borikhanov, B. Torebek, Maximum principle and its applications for the subdiffusion equations with Caputo-Fabrizio fractional derivative, *Mate. Zhur.*, 18 (2018), 43–52.

14. L. Zhang, B. Ahmad, G. Wang, Analysis and application for diffusion equations with a new fractional derivative without singular kernel, *Elec. J. Diff. Equa.*, 289 (2017), 1–6.

15. X. Cabré, Y. Sire, Nonlinear equations for fractional Laplacians, I: Regularity, maximum principle and Hamiltonian estimates, *Anal. l’Ins. Henri Poin. C, Anal. Line.*, 31 (2014), 23–53.

16. T. Guan, G. Wang, Maximum principles for the space-time fractional conformable differential system involving the fractional laplace operator, *J. Math.*, 2020 (2020), 1–8.

17. A. Capella, J. Dávila, L. Dupaigne, Y. Sire, Regularity of radial extremal solutions for some nonlocal semilinear equations, *Comm. Part. Diff. Equa.*, 36 (2011), 1353–1384.

18. T. Cheng, C. Huang, C. Li, The maximum principles for fractional Laplacian equations and their applications, *Comm. Cont. Math.*, 19 (2017), 1–12.

19. L. Del Pezzo, A. Quaas, A Hopf’s lemma and a strong minimum principle for the fractional p-Laplacian, *J. Differ. Equations*, 263 (2017), 765–778.

20. R. Agarwal, D. Baleanu, J. Nieto, A survey on fuzzy fractional differential and optimal control nonlocal evolution equations, *J. Comp. Appl. Math.*, 339 (2018), 3–29.

21. J. Nieto, Maximum principles for fractional differential equations derived from Mittag-Leffler functions, *Appl. Math. Lett.*, 23 (2010), 1248–1251.

22. H. Ye, F. Liu, V. Anh, I. Turner, Maximum principle and numerical method for the multi-term time-space Riesz-Caputo fractional differential equations, *Appl. Math. Comp.*, 227 (2014), 531–540.

23. Z. Liu, S. Zeng, Y. Bai, Maximum principles for the multi-term space-time variable-order fractional diffusion equations and their applications, *Frac. Calc. Appl. Anal.*, 19 (2016), 188–211.

24. G. Wang, X. Ren, Z. Bai, W. Hou, Radial symmetry of standing waves for nonlinear fractional Hardy-Schrödinger equation, *Appl. Math. Lett.*, 96 (2019), 131–137.

25. G. Wang, Twin iterative positive solutions of fractional q-difference Schrödinger equations, *Appl. Math. Lett.*, 76 (2018), 103–109.

26. L. Zhang, W. Hou, Standing waves of nonlinear fractional p-Laplacian Schrödinger equation involving logarithmic nonlinearity, *Appl. Math. Lett.*, 102 (2020), 106149.

27. G. Wang, X. Ren, Radial symmetry of standing waves for nonlinear fractional Laplacian Hardy-Schrödinger systems, *Appl. Math. Lett.*, 110 (2020), 106560.

28. L. Zhang, W. Hou, B. Ahmad, G. Wang, Radial symmetry for logarithmic Choquard equation involving a generalized tempered fractional p-Laplacian, *Discrete Contin. Dyn. Syst. Ser. S*, 2018.

29. L. Zhang, X. Nie, A direct method of moving planes for the Logarithmic Laplacian, *Appl. Math. Lett.*, 118 (2021), 107141.
30. L. Zhang, B. Ahmad, G. Wang, X. Ren, Radial symmetry of solution for fractional $p$–Laplacian system, *Nonl. Anal.*, **196** (2020), 111801.

31. J. Korbel, Y. Luchko, Modeling of financial processes with a space-time fractional diffusion equation of varying order, *Frac. Calc. Appl. Anal.*, **19** (2016), 1414–1433.

32. M. Alquran, F. Yousef, F. Alquran, T. Sulaiman, A. Yusuf, Dual-wave solutions for the quadratic-cubic conformable-Caputo time-fractional Klein-Fock-Gordon equation, *Math. Comput. Simulat.*, **185** (2021), 62–76.

33. I. Jaradat, M. Alquran, Q. Katatbeh, F. Yousef, S. Momani, D. Baleanu, An avant-garde handling of temporal-spatial fractional physical models, *Int. J. Nonl. Sci. Numer. Simu.*, **21** (2020), 183–194.

34. I. Jaradat, M. Alquran, F. Yousef, S. Momani, D. Baleanu, On (2+1)-dimensional physical models endowed with decoupled spatial and temporal memory indices, *Eur. Phys. J. Plus.*, **134** (2019), 360.

35. H. Khan, T. Abdeljawad, C. Tunc, A. Alkhazzan, A. Khan, Minkowski’s inequality for the AB-fractional integral operator, *J. Inequal. Appl.*, **2019** (2019), 1–12.

36. H. Khan, C. Tunc, A. Khan, Green functions properties and existence theorems for nonlinear singular-delay-fractional differential equations with p-Laplacian, *Disc. Cont. Dyna. Syst. S.*, **13** (2020), 2475–2487.

37. H. Khan, C. Tunc, D. Baleanu, A. Khan, A. Alkhazzan, Inequalities for n- class of functions using the Saigo fractional integral operator, *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matemticas*, **113** (2019), 2407–2420.

38. H. Khan, C. Tunc, W. Chen, A. Khan, Existence theorems and Hyers-Ulam stability for a class of Hybrid fractional differential equations with p-Laplacian operator, *J. Appl. Anal. Comput.*, **8** (2018), 1211–1226.

39. S. Zeng, S. Migórski, V. Nguyen, Y. Bai, Maximum principles for a class of generalized time-fractional diffusion equations, *Frac. Calc. Appl. Anal.*, **23** (2020), 822–835.

40. F. Jarad, E. Uğurlu, T. Abdeljawad, D. Baleanu, On a new class of fractional operators, *Adv. Differ. Equ.*, **2017** (2017), 1–16.

41. T. Abdeljawad, On conformable fractional calculus, *J. Comput. Appl. Math.*, **279** (2015), 57–66.