SUPER JACK-LAURENT POLYNOMIALS

A.N. SERGEEV

Dedicated to A.A. Kirillov
on the occasion of his 81th birthday

Abstract. Let $D_{n,m}$ be the algebra of quantum integrals of the deformed Calogero-Moser-Sutherland problem corresponding to the root system of the Lie superalgebra $gl(n,m)$. The algebra $D_{n,m}$ acts naturally on the quasi-invariant Laurent polynomials and we investigate the corresponding spectral decomposition. Even for general value of the parameter $k$ the spectral decomposition is not multiplicity free and we prove that the image of the algebra $D_{n,m}$ in the algebra of endomorphisms of the generalised eigenspace is $k[[\varepsilon]] \otimes R$ where $k[[\varepsilon]]$ is the algebra of dual numbers. The corresponding representation is the regular representation of the algebra $k[[\varepsilon]] \otimes R$.

Contents

1. Introduction 1
2. Quasi-invariants and quasi-homomorphisms 4
3. Algebra of deformed CMS integrals and spectral decomposition 9
4. Description of equivalence classes 14
5. Weights and bipartitions 22
6. Action on the generalised eigenspace 25
7. Concluding remarks 28
8. Acknowledgements 28
References 28

1. Introduction

It is known that for any root system of a semi-simple Lie algebra one can construct the quantum (trigonometric) Calogero - Moser - Sutherland operator (see for example [8]). In particular if the root system is of the type $A_{n-1}$ then we have the following operator

$$L_2^{(n)} = \sum_{i=1}^{n} \left( x_i \frac{\partial}{\partial x_i} \right)^2 - k \sum_{i<j}^{n} \frac{x_i + x_j}{x_i - x_j} \left( x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right)$$

(1)

where $k$ is a complex parameter which we suppose is not rational number. We are considering only trigonometric version of such an operator since it

...
has close connections with representation theory of the corresponding Lie algebra.

This operator is integrable in the sense that there are enough differential operators commuting with it. Let us denote by $\mathcal{D}_n$ the algebra consisting of all such operators (sometimes called integrals). A natural area of the action of the algebra $\mathcal{D}_n$ is the algebra of the symmetric Laurent polynomials $\Lambda^\pm_n = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n}$. There is the basis in this algebra consisting of joint eigenfunctions. These eigenfunctions are called Jack polynomials. The Jack polynomials has many applications in the combinatorics, representation theory and mathematical physics.

In this paper we investigate the natural generalisation of the operator (1) to the case of the "deformed" root system $A_{n-1,m-1}$. Instead of the symmetric group it is natural to consider the group $G$ generated by reflections with respect to a deformed bilinear form (see section 2 for details). This group is well defined for all $k \in \mathbb{C}$ except $k = -1$. Actually the groupoid from the paper [14] can be naturally described using the group $G$.

The corresponding operator has the following form

$$\mathcal{L}_2^{(n,m)} = \sum_{i=1}^{n} \left( x_i \frac{\partial}{\partial x_i} \right)^2 + k \sum_{j=n+1}^{n+m} \left( x_j \frac{\partial}{\partial x_j} \right)^2 - k \sum_{i<j}^{n} \frac{x_i + x_j}{x_i - x_j} \left( x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right)$$

$$+ \sum_{n<i<j}^{m+n} \frac{x_i + x_j}{x_i - x_j} \left( x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right) - \sum_{i=1}^{n} \sum_{j=n+1}^{n+m} \frac{x_i + x_j}{x_i - x_j} \left( x_i \frac{\partial}{\partial x_i} - k x_j \frac{\partial}{\partial x_j} \right)$$

where as before $k$ is not rational number.

This operator has only partial symmetry with respect to the group $G$, in other words the operator is only symmetric with respect to the subgroup $S_n \times S_m$ but it is integrable and we will denote by $\mathcal{D}_{n,m}$ the corresponding algebra of integrals. A natural area of the action of the algebra $\mathcal{D}_{n,m}$ is the algebra of the quasi-invariant Laurent polynomials

$$\Lambda^\pm_{n,m} = \{ f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n \times S_m} | s_\alpha f - f \in (\alpha^2), \alpha \in R_1 \}$$

where $x_i = e^{i}, i = 1, \ldots, n + m$ with a natural action of the group $G : g e^{x} = e^{g x}$ and $R_1$ is the set of odd roots (see for details section 2).

Let us point out the main difficulties in the deformed case. The first one is that we need to use quasi-invariants and also quasi-homomorphisms in order to prove that the algebra $\Lambda^\pm_{n,m}$ is preserved by the algebra $\mathcal{D}_{n,m}$. Actually we interpret the Moser matrix as a linear operator on the vector space of quasi-homomorphisms.

The second one is that it turns out that even for general value of $k$ there is no any eigenbasis in the algebra $\Lambda^\pm_{n,m}$. Therefore in this situation we can only use the decomposition into generalised eigenspaces.

The third main difference with the classical case is that in order to describe this decomposition in terms of the weights and deformed root system we need to use some geometric language which actually close to the language used by S. Kerov [4] for Young diagrams.
In general there is no canonical form for more then one commuting operator. But in our case we are able to describe explicitly the action of the algebra $D_{n,m}$ on the generalised eigenspace. In order to prove the corresponding fact we need to use the infinite dimensional version of the trigonometric CMS operator. And it turns out that in this situation the geometric language of weights and geometric interpretation of Young diagrams are quite convenient. In particularly we give a geometric interpretation of the correspondence between bi-partitions from $(n,m)$ cross (see Definition 5.1 section 5) and the dominant weights of the Lie superalgebra $gl(n,m)$ which was known before in the algebraic form [6, 7]. One of the main result of the paper can be formulated in the following way:

**Theorem 1.1.** Let $k$ be not rational number (in particular $k$ assumed to be non-zero). Then:
1) $\Lambda_{n,m}^\pm$ as a module over the algebra $D_{n,m}$ can be decomposed into direct sum of generalised eigenspaces

$$\Lambda_{n,m}^\pm = \bigoplus_{\chi \in X_{reg}^+(n,m)} \Lambda_{n,m}^\pm(\chi),$$

where $X_{reg}^+(n,m)$ is defined in section 4 definition 4.12. 

2) The dimension of the space $\Lambda_{n,m}^\pm(\chi)$ is equal to $2^r$ where $r$ is the number of odd positive roots $\alpha$ such that $(\chi + \rho, \alpha) + \frac{1}{2}(\alpha, \alpha) = 0$ and $\rho$ is defined in section 3 definition 3.3. 

3) The algebra $\Lambda_{n,m}^\pm$ is generated by the deformed power sums

$$p_s(x_1, \ldots, x_{n+m}) = x_1^s + \cdots + x_n^s + \frac{1}{k}(x_{n+1}^s + \cdots + x_{m+n}^s), \ s = \pm 1, \pm 2, \ldots$$

4) If $k$ is not algebraic number then the image of the algebra $D_{n,m}$ in the algebra $End(\Lambda_{n,m}^\pm(\chi))$ is isomorphic to $\mathbb{C}[\varepsilon]^{\otimes r}$, where $\mathbb{C}[\varepsilon]$ with $\varepsilon^2 = 0$ is the algebra of dual numbers and $\Lambda_{n,m}^\pm(\chi)$ is the regular representation relative to this action.

The paper is organised in the following way. In the section 2 we define the group $G$ and the notion of the quasi-homomorphism. Then we prove the main result of this section which states that the Moser matrix is actually a linear operator on the space of quasi-homomorphisms.

In the section 3 we reproduced one of the results of the paper [13] about spectral decomposition. We give a more conceptual and technically more clear proof of some of the key steps. We also show that the image of the algebra of integrals under the Harish-Chandra homomorphism is the algebra of quasi-invariants with respect to the group $G$.

In section 4 we describe the equivalence classes in the spectral decomposition using some geometric interpretation of the dominant weights and then we translate these results into language of deformed root systems.
In section 5 we give a geometric interpretation of the correspondence between dominant weights and bi-partitions which was known before in the algebraic form.

In section 6 we explicitly describe the action of the algebra of integrals in the generalised eigenspace, using some of the main results about the infinite dimensional CMS operator. In the paper [11] it was introduced and studied a Laurent version of Jack symmetric functions - Jack–Laurent symmetric functions $P_{\lambda,\mu}$ as certain elements of $\Lambda^\pm$ depending on complex parameters $k, p_0$ and labelled by pairs of the partitions $\lambda$ and $\mu$. Here $\Lambda^\pm$ is freely generated by $p_i$ with $i \in \mathbb{Z} \setminus \{0\}$ being both positive and negative. The usual Jack symmetric functions $P_\lambda$ are particular cases of $P_{\lambda,\mu}$ corresponding to empty second partition $\mu$. In the paper [11] it was proved the existence of $P_{\lambda,\mu}$ for all $k \notin \mathbb{Q}$ and $p_0 \neq n + k^{-1}m, m,n \in \mathbb{Z}_{>0}$. The special case $p_0 = n + k^{-1}m, m,n \in \mathbb{Z}_{>0}$ was studied in the paper [15].

2. QUASI-INVARIENTS AND QUASI-HOMOMORPHISMS

Let $I = \{1, \ldots, n+m\}$ be a set of indices with the parity function $p(i) = 0$ if $1 \leq i \leq n$ and $p(i) = 1$ if $n < i \leq n+m$ and we suppose that $0,1 \in \mathbb{Z}_2$. Let also $V$ be a vector space of dimension $n+m$ with a basis $\varepsilon_1, \ldots, \varepsilon_{n+m}$ and a bilinear symmetric form $(\varepsilon_i, \varepsilon_j) = \delta_{ij} k^{p(i)}$. We always suppose that $k$ is not rational. Let us also denote by $R, R^+$ the set of roots and positive roots

$$R = \{\varepsilon_i - \varepsilon_j \mid i, j \in I; i \neq j\}, \quad R^+ = \{\varepsilon_i - \varepsilon_j \mid i, j \in I; i < j\}$$

The set $R$ can be naturally represented as a disjoint union $R = R_0 \cup R_1$ of the even and odd roots, where the parity of the root $\varepsilon_i - \varepsilon_j$ is $p(i) + p(j)$.

Definition 2.1. Let $G$ be the group generated by reflections $s_\alpha, \alpha \in R$ where

$$s_\alpha(v) = v - \frac{2(v, \alpha)}{(\alpha, \alpha)} \alpha$$

Remark 2.2. Let us note, that if $k \neq -1$ then $(\alpha, \alpha) = 1+k \neq 0$ for any odd root $\alpha$ and the definition makes sense. Besides if $k = 1$ then $G = S_{n+m}$ is the symmetric group. We can also construct a groupoid which set of objects is $R_1$ and the set of morphism from $\alpha$ to $\beta$ is $w \in S_n \times S_n \subset G$ such that $w(\alpha) = \beta$ or $s_\beta w$ where $w(\alpha) = -\beta$. It is easy to see that if we add to this groupoid group $S_n \times S_m$ as one point groupoid then we get the same groupoid as in [14]. This groupoid is important when $k = -1$, because the group $G$ is not well defined then, but groupoid still makes sense.

Definition 2.3. We will denote by $\mathbb{C}[[V]]$ the algebra of formal power series in $\varepsilon_1, \ldots, \varepsilon_{n+m}$ and by $S(V)$ its sub-algebra spanned by $e^v$ where $v \in V$ and the vector $v$ has integer coordinates with respect to the basis $\varepsilon_1, \ldots, \varepsilon_{n+m}$.

Since the group $G$ acts on $V$ it therefore naturally acts on $\mathbb{C}[[V]]$. In particular $g(e^v) = e^{gv}$ for $g \in G$. It is easy to see that the subgroup generated by $s_\alpha, \alpha \in R_0$ is the product $S_n \times S_m$ of two symmetric groups.
Definition 2.4. An element $f \in \mathbb{C}[[V]]$ is called quasi-invariant for the group $G$ if it is invariant with respect to $S_n \times S_m$ and for each $\alpha \in R_1$ we have $s_\alpha f - f \in (\alpha^2)$ where $(\alpha^2)$ means the ideal generated by $\alpha^2$.

We also need a more general definition.

Definition 2.5. A linear map $\varphi : V \rightarrow \mathbb{C}[[V]]$ is called quasi-homomorphism if it commutes with the action of the subgroup $S_n \times S_m$ and for $\alpha \in R_1$ and $v \in V$ we have

$$s_\alpha \varphi(v) - \varphi(s_\alpha v) \in (\alpha^2)$$

We should mention that the notion of quasi-invariant was introduced by O.Chalykh and A. Veselov (see [2]) and this notion is related to the notion of quasi-homomorphism. Namely, if $s_\alpha v = v$ and $\varphi$ is a quasi-homomorphism, then $\varphi(v)$ is a quasi-invariant with respect to $s_\alpha$. The above definitions can be given in infinitesimal form and they will also work for $k = -1$.

Lemma 2.6. 1) An element $f \in \mathbb{C}[[V]]$ is a quasi-invariant if and only if

$$\partial_\alpha f \in (\alpha)$$

2) A linear map $\varphi : V \rightarrow \mathbb{C}[[V]]$ is a quasi-homomorphism if and only if

$$\partial_\alpha \varphi(v) - \frac{(\alpha,v)\varphi(\alpha)}{\alpha} \in (\alpha)$$

3) An element $f \in \hat{S}(V)$ is a quasi-invariant if and only if $\partial_\alpha f \in (e^\alpha - 1)$.

4) A linear map $\varphi : V \rightarrow \hat{S}(V)$ is a quasi-homomorphism if and only if

$$\partial_\alpha \varphi(v) - \frac{(\alpha,v)\varphi(\alpha)}{e^\alpha - 1} \in (e^\alpha - 1)$$

Proof. The first statement is a particular case of the second one. So, it is enough to prove the latter statement. It is easy to see that for any $f \in \mathbb{C}[[V]]$ we have the following equality

$$s_\alpha f = f - \frac{2}{(\alpha,\alpha)}(\partial_\alpha f)\alpha + f_1 \alpha^2, f_1 \in \mathbb{C}[[V]]$$

Therefore

$$s_\alpha \varphi(v) - \varphi(s_\alpha v) = \varphi(v) - \frac{2}{(\alpha,\alpha)}\partial_\alpha \varphi(v)\alpha - \varphi(v) + \frac{2(\alpha,v)}{(\alpha,\alpha)}\varphi(\alpha) + f_1 \alpha^2$$

$$= -\frac{2}{(\alpha,\alpha)}[\partial_\alpha \varphi(v)\alpha - (\alpha,v)\varphi(\alpha)] + f_1 \alpha^2$$

Now, let us prove the third statement. It is clear that the condition $\partial_\alpha f \in (e^\alpha - 1)$ implies the condition $\partial_\alpha f \in (\alpha)$. Let us prove the converse statement. We can assume that $n = m = 1$. Let $f \in \hat{S}(V)$ and $\partial_\alpha f \in (\alpha)$. Consider homomorphism

$$\psi : \mathbb{C}[[\epsilon_1,\epsilon_2]] \rightarrow \mathbb{C}[[t]], \psi(\epsilon_1) = \psi(\epsilon_2) = t$$

We can write

$$f = \sum_{\lambda_1,\lambda_2 \in \mathbb{Z}} c_{\lambda_1,\lambda_2} e^{\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2}$$
From the condition \( \partial_\alpha f \in (\alpha) \) it follows that
\[
\psi(f) = \sum_{\lambda_1, \lambda_2 \in \mathbb{Z}} c_{\lambda_1, \lambda_2} e^{(\lambda_1 + \lambda_2)t} = 0
\]

Therefore
\[
f = \sum_{\lambda_1, \lambda_2 \in \mathbb{Z}} c_{\lambda_1, \lambda_2} e^{\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2} = \sum_{\lambda_1, \lambda_2 \in \mathbb{Z}} c_{\lambda_1, \lambda_2} \left(e^{\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2} - e^{(\lambda_1 + \lambda_2) \varepsilon_1}\right)
\]
\[
= \sum_{\lambda_1, \lambda_2 \in \mathbb{Z}} c_{\lambda_1, \lambda_2} e^{\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2} \left(1 - e^{\lambda_2 (\varepsilon_1 - \varepsilon_2)}\right) \in (e^\alpha - 1)
\]

and the third statement follows.

Now, let us prove the fourth statement. It is easy to see that the formula (3) is equivalent to the following two conditions:

a) if \((\alpha, v) = 0\), then \(\partial_\alpha \varphi(v) \in (\alpha)\);
b) \(\varphi(\alpha) \in (\alpha)\).

Indeed, it is easy to see that the conditions a) and b) follow from the formula (3) if \(k \neq -1\). Let us prove now that the conditions a) and b) imply the formula (3). Since the formula (3) is linear with respect to \(v\) we need to check it in the cases a) and b). In the case \((\alpha, v) = 0\) the condition a) is equivalent to the formula (3). In the case b) formula (3) has the form
\[
\partial_\alpha \varphi(\alpha) - \frac{(\alpha, \alpha) \varphi(\alpha)}{\alpha} = \partial_\alpha \left(\frac{\varphi(\alpha)}{\alpha}\right) \alpha.
\]

Therefore if \(v = \alpha\) the formula (3) follows from the condition b). Now using the same arguments as in the proof of the third statement it is not difficult to deduce that in the case \(\varphi(V) \subset \hat{S}(V)\) these two conditions are equivalent to another two:

a') if \((\alpha, v) = 0\), then \(\partial_\alpha \varphi(v) \in (e^\alpha - 1)\)
b') \(\varphi(\alpha) \in (e^\alpha - 1)\).

It is also easy to check that the conditions a') and b') are equivalent to the formula (3) in the Lemma.

Let us denote by \(x_i = e^{\varepsilon_i}, \ i = 1, \ldots, n+m\). Then the algebra \(\hat{S}(V)\) can be identified with the algebra of Laurent polynomials \(\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\). Now introduce the following quantum Moser matrix as \((n+m) \times (n+m)\)-matrix \(L\) by the formulæ
\[
L_{ii} = \partial_{\varepsilon_i} - \sum_{j \neq i} k^{1-p(j)} \frac{x_i}{x_i - x_j}, \quad L_{ij} = k^{1-p(j)} \frac{x_i}{x_i - x_j}, i \neq j
\]

If \(\varphi : V \to \hat{S}(V)\) is a linear map, then we can define the map \(\psi\) by the following formulæ
\[
\begin{pmatrix}
\psi(\varepsilon_1) \\
\vdots \\
\psi(\varepsilon_{n+m})
\end{pmatrix} =
\begin{pmatrix}
L_{1,1} & \cdots & L_{1,n+m} \\
L_{2,1} & \cdots & L_{2,n+m} \\
\cdots & \cdots & \cdots \\
L_{n+m,1} & \cdots & L_{n+m,n+m}
\end{pmatrix}
\begin{pmatrix}
\varphi(\varepsilon_1) \\
\vdots \\
\varphi(\varepsilon_{n+m})
\end{pmatrix}
\]
Theorem 2.7. Let \( \varphi : V \to \hat{S}(V) \) be a quasi-homomorphism. Then \( \psi \) has a unique linear extension to the whole space \( V \), and this extension is a quasi-homomorphism to \( \hat{S}(V) \).

Proof. We will prove the theorem in several steps. First we will prove it in the case \( n = m = 1 \). In this case we have

\[
L = \left( \partial_{\varepsilon_1} - \frac{x_1}{x_2 - x_1} k \partial_{\varepsilon_2} - \frac{x_1}{x_2 - x_1} \right)
\]

We also have the following equalities

\[
\partial_{\varepsilon_1} = \frac{k}{1 + k} \partial_{v^+} + \frac{1}{1 + k} \partial_{\alpha}, \quad \partial_{\varepsilon_2} = \frac{k}{1 + k} \partial_{v^+} - \frac{k}{1 + k} \partial_{\alpha}
\]

where \( v^+ = \varepsilon_1 + \frac{1}{k} \varepsilon_2 \). Therefore we have the following equalities

\[
\begin{align*}
\psi(\varepsilon_1) &= \frac{1}{1 + k} \partial_{\alpha} \varphi(\varepsilon_1) - \frac{x_1}{x_1 - x_2} \varphi(\alpha) + \frac{k}{1 + k} \partial_{v^+} \varphi(\varepsilon_1) \\
\psi(\varepsilon_2) &= -\frac{k}{1 + k} \partial_{\alpha} \varphi(\varepsilon_2) - \frac{k x_2}{x_1 - x_2} \varphi(\alpha) + \frac{k}{1 + k} \partial_{v^+} \varphi(\varepsilon_2)
\end{align*}
\]

It is easy to check that the correspondence

\[
\varepsilon_1 \mapsto \partial_{v^+} \varphi(\varepsilon_1), \quad \varepsilon_2 \mapsto \partial_{v^+} \varphi(\varepsilon_2)
\]

is a quasi-homomorphism. So we need to prove that the formulae

\[
\begin{align*}
\psi(\varepsilon_1) &= \frac{1}{1 + k} \partial_{\alpha} \varphi(\varepsilon_1) - \frac{x_1}{x_1 - x_2} \varphi(\alpha) + \frac{k}{1 + k} \partial_{v^+} \varphi(\varepsilon_1) \\
\psi(\varepsilon_2) &= -\frac{k}{1 + k} \partial_{\alpha} \varphi(\varepsilon_2) - \frac{k x_2}{x_1 - x_2} \varphi(\alpha) + \frac{k}{1 + k} \partial_{v^+} \varphi(\varepsilon_2)
\end{align*}
\]

give a quasi-homomorphism. Let us check first that \( \psi(\alpha) \in (x_1 - x_2) \). And the last expression can be simplified to the form \( -\varphi(\alpha) \) and therefore belongs to the ideal \( (x_1 - x_2) \). So \( \psi(\alpha) \in (x_1 - x_2) \).

And we only need to prove that \( \partial_{\alpha} \psi(v^+) \in (x_1 - x_2) \). We have

\[
\psi(v^+) = \psi(\varepsilon_1) + \frac{1}{k} \psi(\varepsilon_2) = \frac{1}{1 + k} \partial_{\alpha} \varphi(\varepsilon_1) - \frac{x_1}{x_1 - x_2} \varphi(\alpha)
\]

\[
+ \frac{1}{k} \left( -\frac{k}{1 + k} \partial_{\alpha} \varphi(\varepsilon_2) - \frac{k x_2}{x_1 - x_2} \varphi(\alpha) \right) = \frac{1}{1 + k} \partial_{\alpha} \varphi(\alpha) - \frac{x_1 + x_2}{x_1 - x_2} \varphi(\alpha)
\]

We know, that \( \varphi(\alpha) = f(x_1 - x_2) \). Therefore the previous formula can be rewritten in the form

\[
\psi(v^+) = \frac{1}{1 + k} \left[ \partial_{\alpha} f(x_1 - x_2) - (k x_1 + x_2) f \right]
\]
And it is easy to verify that
\[ \partial_a \psi(v^+) = \frac{1}{1 + k} \left[ \partial^2 f + (1 - k) \partial_a f \right] (x_1 - x_2) \in (x_1 - x_2) \]

So the case \( n = m = 1 \) is completely proved.

Now let us proceed to the general case. It is easy to check that matrix elements of \( L \) satisfy the relations \( L_{\sigma(i),\sigma(j)} = L_{ij} \) for \( \sigma \in S_n \times S_m \). Therefore by Lemma 2.6 we only need to prove that if \( \alpha \in R_1 \), then \( \psi(\alpha) \in (e^\alpha - 1) \) and \( \partial_\alpha(\psi(v)) \in (e^\alpha - 1) \) for \( (\alpha, v) = 0 \). Let us take \( \alpha = \varepsilon_i - \varepsilon_j \), where \( 1 \leq i \leq n, n + 1 \leq j \leq n + m \) and prove that \( \psi(\alpha) \in (x_i - x_j) \). We have
\[
\psi(\varepsilon_i) = \partial_{\varepsilon_i} \varphi(\varepsilon_i) - \sum_{r \neq i} \frac{k^{1-p(r)}x_i}{x_i - x_r} (\varphi(\varepsilon_i) - \varphi(\varepsilon_r))
\]

\[
= \partial_{\varepsilon_i} \varphi(\varepsilon_i) - \frac{x_i}{x_i - x_j} \varphi(\alpha) - \sum_{r \neq i,j} \frac{k^{1-p(r)}x_i}{x_i - x_r} (\varphi(\varepsilon_i) - \varphi(\varepsilon_r))
\]

In the same way we have
\[
\psi(\varepsilon_j) = \partial_{\varepsilon_j} \varphi(\varepsilon_j) - \frac{kx_j}{x_i - x_j} \varphi(\alpha) - \sum_{r \neq i,j} \frac{k^{1-p(r)}x_j}{x_i - x_r} (\varphi(\varepsilon_i) - \varphi(\varepsilon_r))
\]

Therefore
\[
\psi(\alpha) = \partial_{\varepsilon_i} \varphi(\varepsilon_i) - \partial_{\varepsilon_j} \varphi(\varepsilon_j) - \frac{x_i - kx_j}{x_i - x_j} \varphi(\alpha)
\]

\[+ \sum_{r \neq i,j} \frac{k^{1-p(r)}x_i}{x_i - x_r} (\varphi(\varepsilon_i) - \varphi(\varepsilon_r)) - \sum_{r \neq i,j} \frac{k^{1-p(r)}x_j}{x_j - x_r} (\varphi(\varepsilon_j) - \varphi(\varepsilon_r)) \]

And it is easy to check the following identity
\[
\frac{x_i \varphi(\varepsilon_i - \varepsilon_r)}{x_i - x_j} - \frac{x_j \varphi(\varepsilon_j - \varepsilon_r)}{x_j - x_r} = \frac{x_i \varphi(\alpha)}{x_i - x_r} - \frac{x_r (x_i - x_j) \varphi(\varepsilon_j - \varepsilon_r)}{(x_i - x_r)(x_j - x_r)}
\]

So we have proved the condition b*) for quasi-homomorphism. Let us prove the condition a*). We need to consider two different cases: first \( v = \varepsilon_i + \frac{1}{k} \varepsilon_j \) and the second one \( v = \varepsilon_s, s \neq i, j \).

In the first case we have
\[
\psi(\varepsilon_i + \frac{1}{k} \varepsilon_j) = \partial_{\varepsilon_i} \varphi(\varepsilon_i) + \partial_{\varepsilon_j} \varphi(\varepsilon_j) - \frac{x_i + x_j}{x_i - x_j} \varphi(\alpha)
\]

\[+ \sum_{r \neq i,j} \frac{k^{1-p(r)}x_i}{x_i - x_r} (\varphi(\varepsilon_i) - \varphi(\varepsilon_r)) + \frac{k^{1-p(r)}x_j}{x_j - x_r} (\varphi(\varepsilon_j) - \varphi(\varepsilon_r)) \]

Since the case \( n = m = 1 \) has been already considered, we only need to prove that
\[
\partial_{\alpha} \left( \frac{kx_i}{x_i - x_r} (\varphi(\varepsilon_i) - \varphi(\varepsilon_r)) + \frac{x_j}{x_j - x_r} (\varphi(\varepsilon_j) - \varphi(\varepsilon_r)) \right) \in (x_i - x_j)
\]
We have
\[
\frac{1}{x_i - x_r} \left( \varphi(\varepsilon_i) - \varphi(\varepsilon_r) \right) + \frac{1}{x_j - x_r} \left( \varphi(\varepsilon_j) - \varphi(\varepsilon_r) \right) = \left( \frac{1}{x_i - x_r} + \frac{1}{x_j - x_r} \right) \varphi(\varepsilon_r) - \left( \frac{1}{x_i - x_r} \varphi(\varepsilon_i) + \frac{1}{x_j - x_r} \varphi(\varepsilon_j) \right)
\]
Further we have
\[
\partial_\alpha \left[ \left( \frac{1}{x_i - x_r} + \frac{1}{x_j - x_r} \right) \varphi(\varepsilon_r) \right] = \left( \frac{1}{x_i - x_r} + \frac{1}{x_j - x_r} \right) \varphi(\varepsilon_r)
\]
\[
+k \left( \frac{x_i}{x_i - x_j} - \frac{x_i^2}{(x_i - x_j)^2} \right) - \frac{x_j}{x_i - x_j} + \frac{x_j^2}{(x_i - x_j)^2} \in (x_i - x_j)
\]
since \( \partial_\alpha \varphi(\varepsilon_r) \in (x_i - x_r) \).

In the second case we have
\[
\psi(\varepsilon) = \partial_\varepsilon \varphi(\varepsilon) - \sum_{r \neq s, i} k_{1-p(r)} x_s \left( \varphi(\varepsilon_s) - \varphi(\varepsilon_r) \right)
\]
\[
- \frac{k x_s}{x_s - x_i} \left( \varphi(\varepsilon_s) - \varphi(\varepsilon_i) \right) - \frac{x_s}{x_s - x_j} \left( \varphi(\varepsilon_s) - \varphi(\varepsilon_j) \right)
\]
It is easy to see that \( \partial_\alpha (\partial_\varepsilon \varphi(\varepsilon)) \in (x_i - x_j) \). So we only need to prove the same for the last two summands. But we have
\[
\frac{k x_s}{x_s - x_i} \left( \varphi(\varepsilon_s) - \varphi(\varepsilon_i) \right) + \frac{x_s}{x_s - x_j} \left( \varphi(\varepsilon_s) - \varphi(\varepsilon_j) \right)
\]
\[
= \left[ \frac{k}{x_s - x_i} + \frac{1}{x_s - x_j} \right] x_s \varphi(\varepsilon_s) - \left[ \frac{k}{x_s - x_i} \varphi(\varepsilon_i) + \frac{1}{x_s - x_j} \varphi(\varepsilon_j) \right] x_s
\]
and
\[
\partial_\alpha \left[ \frac{k}{x_s - x_i} + \frac{1}{x_s - x_j} \right] = \left[ \frac{k x_i}{x_s - x_i} - \frac{k x_j}{x_s - x_j} \right] \in (x_i - x_j)
\]
and
\[
\partial_\alpha \left[ \frac{k}{x_s - x_i} \varphi(\varepsilon_i) + \frac{1}{x_s - x_j} \varphi(\varepsilon_j) \right] = \frac{k x_s (x_i - x_j)}{(x_s - x_i) (x_s - x_j)} \varphi(\varepsilon_i) + \frac{k x_j (x_i - x_j)}{(x_s - x_i) (x_s - x_j)} \varphi(\varepsilon_j)
\]
The case \( s > m \) can be considered in the same manner.

3. Algebra of deformed CMS integrals and spectral decomposition

Let us define CMS integrals \( \mathcal{L}_r, r = 1, 2, \ldots \) by the following formulae
\[
\mathcal{L}_r = e^* L^r e.
\]
where \( e^* \) is a row such that \( e_i^* = k_{1-p(i)}, i = 1, \ldots, m+n \) and \( e \) is a column such that \( e_i = 1, i = 1, \ldots, n+m \) and \( L \) is the Moser matrix given by (5).

**Theorem 3.1.** The operators \( \mathcal{L}_r \) are quantum integrals of the deformed CMS system:
\[
[\mathcal{L}_r, \mathcal{L}_2] = 0.
\]
For the proof of the Theorem see [13] Theorem 2.1.

**Definition 3.2.** Let us denote by $D_{n,m}$ the algebra generated by operators $\mathcal{L}_r$, $r = 1, 2, \ldots$

Define now the Harish-Chandra homomorphism $\phi_{n,m} : D_{n,m} \to S(V^*)$

by the conditions:

$$\phi_{n,m}(\partial_{\varepsilon_i}) = \frac{n^p(i)}{2} \varepsilon_i^*, \quad \phi_{n,m}\left(\frac{x_i}{x_i - x_j}\right) = 1,$$

if $i < j$.

where $\varepsilon_i^*, i = 1, \ldots, n + m$ is the basis dual to the basis $\varepsilon_i, i = 1, \ldots, n + m$.

**Definition 3.3.** Let $\rho \in V$ be the following deformed analogue of the Weyl vector

$$\rho = \frac{1}{2} \sum_{i=1}^{n} (k(2i - n - 1) - m) \varepsilon_i + \frac{1}{2} \sum_{j=1}^{m} (k^{-1}(2j - m - 1) + n) \varepsilon_{j+n} \quad (8)$$

Let us define an affine action of the group $G$ on the space $V$. Namely

$$g \circ v = g(v + \rho) - \rho \quad (9)$$

It is easy to see that under this affine action the vector $-\rho$ is fixed. Besides, for any root $\alpha$ from $R$ the corresponding reflection $s_\alpha$ is the reflection with respect to the hyperplane $\langle v + \rho, \alpha \rangle = 0$. Indeed

$$s_\alpha \circ (v) = s_\alpha(v + \rho) - \rho = v - 2\left(\frac{(v + \rho, \alpha)}{(\alpha, \alpha)}\right)\alpha$$

So we see if $\langle v + \rho, \alpha \rangle = 0$ then $s_\alpha \circ (v) = v$ and $\alpha$ is orthogonal to hyperplane $\langle v + \rho, \alpha \rangle = 0$.

Let us define for any $\alpha \in R_1^+$ the following linear functions on $V$

$$l_\alpha^+(v) = (v + \rho, \alpha) - \frac{1}{2} (\alpha, \alpha), \quad l_\alpha^-(v) = (v + \rho, \alpha) + \frac{1}{2} (\alpha, \alpha)$$

**Definition 3.4.** A polynomial $f \in S(V^*)$ is called a quasi-invariant with respect to the affine action of the group $G$ if it satisfies the following conditions

1) $f(s_\alpha \circ v) = f(v)$ if $\alpha \in R_0$.

2) $f(s_\alpha \circ v) - f(v) \in (l_\alpha^+ l_\alpha^-)$ if $\alpha \in R_1^+$.

**Theorem 3.5.** If $k$ is not rational then Harish-Chandra homomorphism is injective and its image is the sub-algebra $\Lambda_{n,m}^* \subset S(V^*)$ consisting of polynomials which are quasi-invariant with respect to the affine action of the group $G$.

**Proof.** Let us show that this Theorem is actually a reformulation of the Theorem 2.2 from [13].
Take any $\alpha \in R^+_1$ and set $h(v) = f(s_\alpha \circ v) - f(v)$ and suppose that $f$ satisfies the conditions

$$f(w(v + \rho) - \rho) = f(v), \quad w \in S_n \times S_m$$

and for every $\alpha \in R^+_1$ $f(v - \alpha) = f(v)$ on the hyperplane $(v + \rho, \alpha) = \frac{1}{2}(1 + k)$.

If $l^+_\alpha(v) = 0$, then $s_\alpha \circ (v) = v - \alpha$, therefore $h(v) = 0$. Therefore $h \in (l^+_\alpha)$. It is easy to see that $h(s_\alpha \circ v) = -h(v)$ and $l^+_\alpha(s_\alpha \circ v) = l^+_\alpha(v)$. Therefore $h \in (l^-_\alpha)$ and the conditions of Theorem 2.2 from [13] imply the conditions of the present Theorem.

Now let us prove the opposite statement. So, let $h \in (l^-_\alpha l^+_\beta)$ and $l^+_\alpha(v) = 0$. Then as we have already seen $s_\alpha \circ (v) = v - \alpha$, therefore $0 = h(v) = f(v - \alpha) - f(v)$. □

**Corollary 3.6.** Operators $L_r$ commute with each other.

Now we are going to investigate the action of the algebra of integrals on the space of quasi-invariants.

**Definition 3.7.** Denote by $\Lambda_{n,m}^\pm$ the subalgebra of quasi-invariants in the algebra $\hat{S}(V)$

$$\Lambda_{n,m}^\pm = \{ f \in \hat{S}(V)^{S_n \times S_m} \mid \partial_\alpha f \in (e^\alpha - 1), \alpha \in R_1 \}$$

It is easy to check that the algebra $\Lambda_{n,m}^\pm$ can be identified with the algebra of $S_n \times S_m$-invariant Laurent polynomials $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_{m+n}^{\pm 1}]$ satisfying the conditions

$$x_i \frac{\partial f}{\partial x_i} - k x_j \frac{\partial f}{\partial x_j} = 0, \quad 1 \leq i \leq n, \quad n + 1 \leq j \leq n + m$$

(10)

on the hyperplane $x_i = x_j$ for all $i = 1, \ldots, n, \quad j = 1, \ldots, m$.

For any Laurent polynomial

$$f = \sum_{\mu \in X(n,m)} c_{\mu} x^\mu, \quad X(n,m) = \mathbb{Z}^n \oplus \mathbb{Z}^m$$

consider the set $M(f)$ consisting of $\mu$ such that $c_{\mu} \neq 0$ and define the support $S(f)$ as the intersection of the convex hull of $M(f)$ with $X(n,m)$.

**Theorem 3.8.** The operators $L_r$ for all $r = 1, 2, \ldots$ map the algebra $\Lambda_{n,m}^\pm$ to itself and preserve the support: for any $D \in D_{n,m}$ and $f \in \Lambda_{n,m}^\pm$

$$S(Df) \subseteq S(f).$$

**Proof.** Let us prove the first statement. Let $f \in \Lambda_{n,m}^\pm$. Consider the following quasi-homomorphism

$$\varphi : V \rightarrow \mathbb{C}[x_1^{\pm 1}, \ldots, x_{m+n}^{\pm 1}], \varphi(\varepsilon_i) = f, \quad i = 1, \ldots, n + m$$

Therefore by the Theorem 2.7 $\psi_r = L_r \varphi$ is a quasi-homomorphism and

$$e^s L_r f = \psi_r(v), v = \varepsilon_1 + \cdots + \varepsilon_n + \frac{1}{k}(\varepsilon_{n+1} \cdots + \varepsilon_{n+m})$$
Therefore $e^r L^r f = L_r(f)$ is a quasi-invariant since the vector $v$ is invariant with respect to the group $G$.

Now let prove the second statement. Consider the quasi-homomorphism $\varphi$ such that $\varphi(\varepsilon_i) = f_i$ where $S(f_i) \subset S(f)$ and $f$ is any quasi-invariant. Therefore by Theorem [2.7] for any $i, j$

$$g_{ij} = \frac{x_i}{x_i - x_j} (f_i - f_j)$$

is a polynomial. Since

$$f_i - f_j = \left(1 - \frac{x_j}{x_i}\right) g_{ij}(x)$$

and the support of a product of two Laurent polynomials is the Minkowski sum of the supports of the factors this implies that $S(g_{ij}) \subseteq S(f_i - f_j) \subseteq S(f)$. Therefore if $\psi = L\varphi$, then $S(\psi(\varepsilon_i)) \subset S(f)$. Now let $\varphi$ be the homomorphism such that $\varphi(\varepsilon_i) = f$. Then by induction on $r$

$$S(L_r(f)) = S(e^r L^r f) \subset S(f)$$

□

Now we are going to investigate the spectral decomposition of the action of the algebra of CMS integrals $D_{n,m}$ on $\Lambda_{n,m}^\pm$.

We will need the following partial order on the set of integral weights $\chi \in X(n,m)$: we say that $\tilde{\chi} \preceq \chi$ if and only if

$$\tilde{\chi}_1 \leq \chi_1, \tilde{\chi}_1 + \tilde{\chi}_2 \leq \chi_1 + \chi_2, \ldots, \tilde{\chi}_1 + \cdots + \tilde{\chi}_{n+m} \leq \chi_1 + \cdots + \chi_{n+m} \quad (11)$$

In the following Proposition we are considering the set $X(n,m)$ as a subset of the vector space $V$ with respect to the following inclusion

$$\chi = (\chi_1, \ldots, \chi_{n+m}) \mapsto \sum_{i=1}^{n+m} \chi_i \varepsilon_i$$

**Proposition 3.9.** Let $f \in \Lambda_{n,m}^\pm$ and $\chi$ be a maximal element of $M(f)$ with respect to partial order.

1) Then for any $D \in D_{n,m}$ there is no $\tilde{\chi}$ in the $M(D(f))$, $\tilde{\chi} \neq \chi$ such that $\chi \preceq \tilde{\chi}$. The coefficient at $x^\chi$ in $D(f)$ is $\phi_{n,m}(D)(\chi)c_\chi$, where $c_\chi$ is the coefficient at $x^\chi$ in $f$.

2) If $\chi$ is the only maximal element of $M(f)$ then $\tilde{\chi} \preceq \chi$ for any $\tilde{\chi}$ from $M(D(f))$.

**Proof.** Let us prove the first statement. Let $\psi$ be a quasi-homomorphism and $\tilde{\psi} = L\psi$. We are going to prove that if for any $\tilde{\chi} \in M(\psi(\varepsilon_i))$, $i = 1,\ldots,n+m$ the inequality $\tilde{\chi} \succ \chi$ is impossible, then the same is true for $\psi$ instead of $\psi$. If we set

$$g_{ij} = \frac{x_i}{x_i - x_j} (\psi(\varepsilon_i) - \psi(\varepsilon_j))$$
then it is easy to verify that if $\tilde{\chi} \in M(g_{ij})$ then the inequality $\tilde{\chi} > \chi$ is impossible. Since

$$\tilde{\psi}(\varepsilon_i) = \partial_{\varepsilon_i} \psi(\varepsilon_i) + \sum_{j \neq i} g_{ij}$$  \hspace{1cm} (12)$$

the same statement is true for $\tilde{\psi}(\varepsilon_i)$.

Now let us define a functional on the space of Laurent polynomials by the formula $l_{\chi}(f) = c_{\chi}$, where $c_{\chi}$ is the coefficient at $x^\chi$ in $f$. Let us prove that $l_{\chi}(\tilde{\psi}) = \phi_{n,m}(L) l_{\chi}(\psi)$. Since $l_{\chi}$ is a linear functional it is enough to prove that for every summand in the sum (12). But

$$l_{\chi}(\partial_{\varepsilon_i} f) = k^{p(i)}_{\chi} l_{\chi}(f) = k^{p(i)}_{\chi} \varepsilon^*_i(\chi) l_{\chi}(f) = \phi_{n,m}(\partial_{\varepsilon_i}(\chi) l_{\chi}(f)).$$

Since for any $\tilde{\chi} \in M(\psi(\varepsilon_i)) \cup M(\psi(\varepsilon_i))$ the inequality $\tilde{\chi} > \chi$ is impossible, therefore for $i < j$ we have

$$l_{\chi}(g_{ij}) = \begin{cases} l_{\chi}(\psi(\varepsilon_i) - \psi(\varepsilon_j)), & i < j \\ 0, & i > j \end{cases}$$

This proves the first part. The proof of the second part is similar. \hfill \Box

Let $\theta : \mathcal{D}_{n,m} \to \mathbb{C}$ be a homomorphism and define the corresponding generalised eigenspace $\Lambda_{n,m}^\pm(\theta)$ as the set of all $f \in \Lambda_{n,m}^\pm$ such that for every $D \in \mathcal{D}_{n,m}$ there exists $N \in \mathbb{N}$ such that $(D - \theta(D))^N(f) = 0$. If the dimension of $\Lambda_{n,m}^\pm(\theta)$ is finite then such $N$ can be chosen independent on $f$.

**Proposition 3.10.** Algebra $\Lambda_{n,m}^\pm$ as a module over the algebra $\mathcal{D}_{n,m}$ can be decomposed into direct sum of generalised eigenspaces

$$\Lambda_{n,m}^\pm = \bigoplus_{\theta} \Lambda_{n,m}^\pm(\theta),$$

where the sum is taken over the set of some homomorphisms $\theta$ (explicitly described below).

**Proof.** Let $f \in \Lambda_{n,m}^\pm$ and define a vector space

$$W(f) = \{ g \in \Lambda_{n,m}^\pm \mid S(g) \subseteq S(f) \}.$$  

By Theorem 3.8 $W(f)$ is a finite dimensional module over $\mathcal{D}_{n,m}$. Since the proposition is true for all finite-dimensional modules, the claim follows. \hfill \Box

Now we describe all homomorphisms $\theta$ such that $\Lambda_{n,m}^\pm(\theta) \neq 0$. We say that the integral weight $\chi \in X(n,m)$ dominant if

$$\chi_1 \geq \chi_2 \geq \cdots \geq \chi_n, \quad \chi_{n+1} \geq \chi_{n+2} \geq \cdots \geq \chi_{n+m}.$$  

The set of dominant weights is denoted by $X^+(n,m)$.

For every $\chi \in X^+(n,m)$ we define the homomorphism $\theta_\chi : \mathcal{D}_{n,m} \to \mathbb{C}$ by

$$\theta_\chi(D) = \phi_{n,m}(D)(\chi), \quad D \in \mathcal{D}_{n,m}$$

where $\phi_{n,m}$ is the Harish-Chandra homomorphism.
Proposition 3.11. 1) For any $\chi \in X^+(n,m)$ there exists $\theta$ and $f_\chi \in \Lambda^\pm_{n,m}(\theta)$, which has the only maximal term $x^\chi$.

2) $\Lambda^\pm_{n,m}(\theta) \neq 0$ if and only if there exists $\chi \in X(n,m)^+$ such that $\theta = \theta_\chi$.

3) If $\Lambda^\pm_{n,m}(\theta)$ is finite dimensional then its dimension is equal to the number of $\chi \in X^+(n,m)$ such that $\theta_\chi = \theta$.

For the proof see [13].

Corollary 3.12. The set of homomorphisms in Proposition 3.10 consists of $\theta = \theta_\chi$, $\chi \in X^+(n,m)$.

4. Description of equivalence classes

Following the last statement of the Proposition 3.11 let us introduce the following definition.

Definition 4.1. Two weights $\chi, \tilde{\chi} \in X^+(n,m)$ are called equivalent if $\theta_\chi = \theta_{\tilde{\chi}}$.

In this section we are going to investigate further for non rational $k$ this equivalence relation. In particular we will describe explicitly the equivalence classes. It is easy to see from Theorem 3.5 that two weights $\chi, \tilde{\chi} \in X^+(n,m)$ are equivalent if and only if $f(\chi) = f(\tilde{\chi})$ for any $f \in \Lambda^*_{n,m}$. Let denote by $E(\chi)$ the equivalence class containing $\chi$. In the paper [14] we described equivalence classes without any assumptions on the weight $\chi$. But in the case when $\chi \in X^+(n,m)$ and $k$ is not rational we can say much more. It turns out that in such a case it is more convenient to use a geometric language of polygonal lines, rather then the one of the Young diagrams.

Definition 4.2. Let $a_1, \ldots, a_n$ be any non-increasing sequence of integers. Consider $2n$ points on the plane

$M_1 = (a_1,0), M_2 = (a_1,1), \ldots, M_{2n-1} = (a_n,n-1), M_{2n} = (a_n,n)$

and two additional points “at infinity”,

$M_0 = (+\infty,0), M_{2n+1} = (-\infty,n)$.

Let us denote by $\Gamma_a$ the polygonal line

$\Gamma_a = \bigcup_{i=0}^{2n}[M_i, M_{i+1}]$

consisting of the segments $[M_i, M_{i+1}]$. The segments $[M_0, M_1], [M_{2n}, M_{2n+1}]$ mean the corresponding half lines.

Let us also denote by $\tau_n$ and $\omega$ the following transformations of $\mathbb{R}^2$

$\tau_n(x,y) = (x,y+n), \quad \omega(x,y) = (y,x)$

Let $\chi \in X^+(n,m)$. Then we can write $\chi = (a_1, \ldots, a_n \mid b_1, \ldots, b_m)$ where $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are two sequences of non increasing integers. Then
we can define two polygonal lines: $\Gamma_a$ and $\hat{\Gamma}_b = \tau_n \circ \omega(\Gamma_b)$. It easy to check that the polygonal line $\hat{\Gamma}_b$ can be described in the following way

$$\hat{\Gamma}_b = \bigcup_{j=0}^{2m} [N_j, N_{j+1}]$$

where

$N_1 = (0, n+b_1), N_2 = (1, n+b_1), \ldots, N_{2m-1} = (m-1, n+b_m), N_{2m} = (m, n+b_m)$

and

$N_0 = (0, +\infty), N_{2m+1} = (m, -\infty)$.

**Definition 4.3.** Denote by $D^a_\alpha$ the part of the plane $\mathbb{R}^2$ bounded by the lines $\Gamma_a$, $y = n$, $x = 0$. Denote by $D^b_\alpha$ the part of the plane bounded by the lines $\Gamma_b$, $y = 0$, $x = 0$. For $\chi = (a_1, \ldots, a_n \mid b_1, \ldots, b_m)$ define

$$\hat{\Gamma}_b^- = \tau_n \circ \omega(\Gamma_b^-), \quad \hat{\Gamma}_b^+ = \tau_n \circ \omega(\Gamma_b^+)$$

Now let us set

$$b_r(x_1, \ldots, x_n, k, h) = \sum_{i=1}^{n} [B_r(x_i + k(i - 1) + h) - B_r(k(i - 1) + h)]$$

where $B_r(x)$ are the Bernoulli polynomials and $k, h$ are complex numbers.

**Proposition 4.4.** For any $r \in \mathbb{Z}_{>0}$ the polynomials

$$b_r^{(n,m)}(\xi) = b_r(\xi_1, \ldots, \xi_n, k, 0) + k^{r-1}b_r(\xi_{n+1}, \ldots, \xi_{n+m}, k^{-1}, n)$$

belong to the algebra $\Lambda_n^{n,m}$ and generate it.

**Proof.** We see that $b_r^{(n,m)}(\xi) = f_r(\xi + \rho + v)$, where

$$v = \frac{1}{2} (kn + m - k) \sum_{i=1}^{n} \varepsilon_i + \frac{1}{2} (n + k^{-1}m - k^{-1}) \sum_{j=1}^{m} \varepsilon_{n+j}$$

and $f_r$ is symmetric with respect to $S_n \times S_m$. Therefore for any even root $\alpha$ we have

$$b_r^{(n,m)}(s_\alpha \circ \xi) = b_r^{(n,m)}(s_\alpha(\xi + \rho) - \rho) = f_r(s_\alpha(\xi + \rho) + v) = f_r(s_\alpha(\xi + \rho + v))$$

$$= f_r(\xi + \rho + v) = b_r(\xi)$$

Now let $\alpha = \varepsilon_i - \varepsilon_{n+j}$ be an odd root and $(\xi + \rho, \alpha) = \frac{1}{2}(\alpha, \alpha)$, so we have

$$b_r^{(n,m)}(\xi) - b_r^{(n,m)}(\xi - \alpha) = B_r(\xi_i + k(i - 1)) + k^{r-1}(\xi_{n+j} + k^{-1}(j - 1)) - B_r(\xi_i - 1 + k(i - 1)) + k^{r-1}B_r(\xi_{n+j} + 1 + k^{-1}(j - 1) + n)$$

Since the Bernoulli polynomials have the property $B_r(x+1) - B_r(x) = r x^{r-1}$ then we have

$$b_r^{(n,m)}(\xi) - b_r^{(n,m)}(\xi - \alpha) = r(\xi_i - 1 + k(i - 1))^{r-1} - r k^{r-1}(\xi_{n+j} + k^{-1}(j - 1) + n)^{r-1}$$

But it is not difficult to verify, that condition

$$\xi_i - 1 + k(i - 1) = k(\xi_{n+j} + k^{-1}(j - 1) + n)$$
is equivalent to the condition \((\xi + \rho, \alpha) = \frac{1}{2}(\alpha, \alpha)\). Therefore the polynomials \(b_r^{(n,m)}(\xi)\) belong to the algebra \(\Lambda^*_{n,m}\). The fact that they generate this algebra has been proved in [10].

\[\square\]

**Corollary 4.5.** Let \(\chi, \tilde{\chi} \in X^+(n,m)\). Then they are equivalent if and only if

\[b_r^{(n,m)}(\chi) = b_r^{(n,m)}(\tilde{\chi}), \ r = 1, 2, \ldots.\]

**Definition 4.6.** Let \(\square\) be a unit square on the plane such that all its vertices have integer coordinates. Then we denote \(c_k(\square) = x + ky\), where \(k\) is a complex number and \((x, y)\) are the coordinates of the left lower vertex.

The next lemma gives some geometric formula for the value of \(b_r^{n,m}(\chi)\).

**Lemma 4.7.** Let \(\chi \in X^+(n,m)\) then

\[b_r^{(n,m)}(\chi) = r \left[ \sum_{\square \in D^+_n \cup \hat{D}^+_n} c_k(\square)^r - \sum_{\square \in D^-_n \cup \hat{D}^-_n} c_k(\square)^r \right] \]

**Proof.** From the definition of the Bernoulli polynomials we have the following formulae for integer \(z\)

\[B_r(z + h) - B_r(h) = \begin{cases} \sum_{j=1}^{z} (j - 1 + h)^{r-1}, & z > 0 \\ \sum_{j=1}^{-z} (j + h)^{r-1}, & z < 0 \end{cases} \]

Therefore

\[\sum_{a_i > 0} B_r(a_i + k(i - 1) + h)^{r-1} - B_r(k(i - 1) + h)^{r-1} = r \sum_{\square \in D^+_n} (c_k(\square) + h)^{r-1} \]

and

\[\sum_{a_i < 0} B_r(a_i + k(i - 1) + h)^{r-1} - B_r(k(i - 1) + h)^{r-1} = -r \sum_{\square \in D^-_n} (c_k(\square) + h)^{r-1} \]

Further we have

\[k^{r-1} \sum_{j=1}^{m} B_r(b_i + k^{-1}(j - 1) + n) - B_r(k^{-1}(j - 1) + n) = \]

\[rk^{r-1} \sum_{\square \in D^+_n} (c_{k^{-1}}(\square) + n)^{r-1} - rk^{r-1} \sum_{\square \in D^-_n} (c_{k^{-1}}(\square) + n)^{r-1} \]

and we only need to show that

\[k^{r-1}(c_{k^{-1}}(\square) + n) = c_k(\tau_n \circ \omega(\square)) \]

But it easy follows from the direct calculations and the fact that if \((x, y)\) is the left lower vertex of a square \(\square\) then \((y, x + n)\) is the left lower vertex of the square \(\tau_n \circ \omega(\square)\). \[\square\]
Corollary 4.8. Let us set
\[ D^+_\chi = D^+_a \cup D^+_b, \quad D^-_\chi = D^-_a \cup D^-_b. \]
Then the following formulae hold true:
\[ b^{(n,m)}_{(r)}(\chi) = r \left[ \sum_{\square \in D^+_\chi \setminus D^-_\chi} c_k(\square)^{r-1} - \sum_{\square \in D^-_\chi \setminus D^+_\chi} c_k(\square)^{r-1} \right], \quad r = 1, 2, \ldots \]

Theorem 4.9. Suppose \( k \not\in \mathbb{Q} \) and
\[ \chi = (a_1, \ldots, a_n | b_1, \ldots, b_m), \quad \tilde{\chi} = (\tilde{a}_1, \ldots, \tilde{a}_n | \tilde{b}_1, \ldots, \tilde{b}_m). \]
Then the conditions \( b^{(n,m)}_{(r)}(\chi) = b^{(n,m)}_{(r)}(\tilde{\chi}) \) for \( k = 1, 2, \ldots \) are equivalent to the condition
\[ \Gamma_a \cup \hat{\Gamma}_b = \Gamma_{\tilde{a}} \cup \hat{\Gamma}_{\tilde{b}}. \]

Proof. It is easy to see that equality \( \Gamma_a \cup \hat{\Gamma}_b = \Gamma_{\tilde{a}} \cup \hat{\Gamma}_{\tilde{b}} \) is equivalent to the equalities
\[ D^+_\chi \setminus D^-_\chi = D^+_\tilde{\chi} \setminus D^-_\tilde{\chi}, \quad D^-_\chi \setminus D^+_\chi = D^-_\tilde{\chi} \setminus D^+_\tilde{\chi} \]
So since \( k \) is not a rational number the last two equalities are equivalent to the statement that two sequences
\[ (c_k(\square))_{\square \in D^+_\chi \setminus D^-_\chi}, \quad (c_k(\square))_{\square \in D^+_\tilde{\chi} \setminus D^-_\tilde{\chi}} \]
coincide up to a permutation. And finally by Corollary 4.8 this is equivalent to the conditions \( b^{(n,m)}_{(r)}(\chi) = b^{(n,m)}_{(r)}(\tilde{\chi}), k = 1, 2, \ldots \). \( \square \)

Let \( \chi \in X^+(n,m) \). Consider the decomposition of the intersection \( \Gamma_a \cap \hat{\Gamma}_b \) into connected components
\[ \Gamma_a \cap \hat{\Gamma}_b = \gamma_1 \cup \cdots \cup \gamma_{r+1} \]

And let \( P_i, Q_i \) be the boundary points of the line \( \gamma_i \) (if \( \gamma_i \) is one point then \( P_i = Q_i \)). We suppose that
\[ P_1 \geq Q_1 \geq P_2 \geq Q_2 \geq \cdots \geq P_{r+1} \geq Q_{r+1} \]

and we use here the total order on the points such that \( P = (x,y) \) if and only if \( y > \tilde{y} \) or \( y = \tilde{y} \) and \( x < \tilde{x} \). For every \( i = 1, \ldots, r \) there are exactly two ways to get from \( Q_i \) to \( P_{i+1} \) along the line \( \Gamma_a \cup \hat{\Gamma}_b \). Let us denote the lower way by \( L_i \) and the upper way by \( U_i \). Denote by \( \nu_i \) the part of the plane bounded by \( L_i \) and \( U_i \). Set \( \nu = \bigcup_{i=1}^r \nu_i \).

Corollary 4.10. Let \( \chi \in X^+(n,m) \) then:
1) If \( \tilde{\chi} \in E(\chi) \) then \( \Gamma_{\tilde{a}} \) can be obtained from \( \Gamma_a \) by replacing one of the two possible ways from \( Q_i \) to \( P_{i+1} \) by the other one for some of the indices \( i = 1, \ldots, r \).
2) The number of elements in \( E(\chi) \) is equal to \( 2^r \).
3) Every equivalence class contains a unique weight \( \chi_{\min} \) such that the corresponding line \( \Gamma_{\chi_{\min}} \) contains all the lower paths between the points \( P_i \) and \( Q_{i+1} \) for each \( i = 1, \ldots, r \).
Proof. This corollary easily follows from geometric considerations. □

Now we are going to reformulate the previous results in terms of the weights, odd roots and the deformed scalar product. Let us define the bijection \( \eta : R^+_1 \rightarrow Q_{m,n} \) from the set of odd positive roots to the set of unit squares with integer coordinates contained in the rectangle \([0, m] \times [0, n]\) by the following rule: \( \eta(\varepsilon_i - \varepsilon_{n+j}) \) is the square with the upper right vertex \((j, i)\). The following theorem is the main result of this section.

**Theorem 4.11.** Let \( E \) be an equivalence class and \( \chi = \chi_{\text{min}} \in E \). Then the following statements hold true

1) We have \(| E | = 2^r\), then \( r \) is equal to the number of the odd positive roots \( \alpha \) such that

\[
(\chi + \rho, \alpha) + \frac{1}{2}(\alpha, \alpha) = 0
\]

2) Let us denote by \( R(\chi) \) the set of all \( \alpha \in R^+_1 \) such that there exist a sequence of odd positive roots \( \alpha_1, \ldots, \alpha_N \) satisfying the following conditions

\[
(\chi + \rho + \alpha_1 + \cdots + \alpha_{i-1}, \alpha_i) + \frac{1}{2}(\alpha_i, \alpha_i) = 0, \ i = 1, \ldots, N
\]

Then \( \eta(R(\chi)) = \nu \).

3) If

\[
R(\chi) = R^{(1)} \cup \cdots \cup R^{(s)}
\]

is the decomposition into orthogonal components, then \( s = r \) and

\[
\eta(R(\chi)) = \eta(R^{(1)}) \cup \cdots \cup \eta(R^{(s)})
\]

is the decomposition into connected components.

4) Set \( \beta_t = \sum_{\alpha \in R^{(t)}} \alpha \). Then every weight \( \tilde{\chi} \) from \( E(\chi) \) can be written as

\[
\tilde{\chi} = \chi + \vartheta_1 \beta_1 + \cdots + \vartheta_r \beta_r, \ \vartheta_t \in \{0, 1\}, \ t = 1, \ldots, r
\]

5) There exist pairwise commuting elements \( g_1, \ldots, g_r \) of the group \( G \) such that any element \( \tilde{\chi} \in E(\chi) \) can be written as

\[
\tilde{\chi} = g_1^{\vartheta_1} \circ g_2^{\vartheta_2} \circ \cdots \circ g_r^{\vartheta_r}(\chi_{\text{min}})
\]

6) Let \( \tilde{\chi} \in E(\chi) \). Then

\[
\prod_{\alpha \in R^+_1} [(\tilde{\chi} + \rho, \alpha) - \frac{1}{2}(\alpha, \alpha)] \neq 0
\]

if and only if \( \tilde{\chi} = \chi_{\text{min}} \).

**Proof.** Let us prove the first statement. It is easy to check that for \( \alpha = \varepsilon_i - \varepsilon_{n+j} \) we have

\[
(\chi + \rho, \alpha) + \frac{1}{2}(\alpha, \alpha) = \chi_i - j + 1 - k(\chi_{n+j} + n - i)
\]

Therefore the condition \( (\chi + \rho, \alpha) + \frac{1}{2}(\alpha, \alpha) = 0 \) is equivalent to the conditions

\[
\chi_i - j + 1 = 0, \ \chi_{n+j} + n - i = 0
\]
But it is easy to check that the last two conditions are equivalent to the condition $M_{2i} = N_{2j-1}$. Since $\chi = \chi_{\text{min}}$ then there exists $1 \leq s \leq r$ such that $M_{2i} = N_{2j-1} = Q_s$ and such $s$ is unique. And it is easy to check that this correspondence is a bijection between the points $Q_s, 1 \leq s \leq r$ and the set of $\alpha \in R_1^+$ such that $(\chi + \rho, \alpha) + \frac{1}{2}(\alpha, \alpha) = 0$. Thus we proved the first statement.

Let us prove the second statement. Let $\square \in Q_{n,m}$ be the square with the upper right vertex $(j, i)$. Then we set

$$f(\square) = a_i - j + 1, \quad g(\square) = b_j + n - i$$

It is easy to check the following statements.

a) If $\square$ is located on the right of $\Gamma_a$ then $-f(\square)$ is equal to the number of cells between the cell $\square$ and the line $\Gamma_a$ with the same coordinate $i$; if $\square$ is located on the left of $\Gamma_a$ then $f(\square) - 1$ is equal to the number of cells between the cell $\square$ and the line $\Gamma_a$ with the same coordinate $i$.

b) If $\square$ is located below $\hat{\Gamma}_b$ then $g(\square)$ is equal to the number of cells between the cell $\square$ and the line $\hat{\Gamma}_b$ with the same coordinate $j$; if $\square$ is located above $\hat{\Gamma}_b$ then $-1 - g(\square)$ is equal to the number of cells between the cell $\square$ and the line $\hat{\Gamma}_b$ with the same coordinate $j$.

Let us first prove that $R(\chi) \supset \eta^{-1}(\nu)$. Let $\alpha = \varepsilon_i - \varepsilon_{n+j}$ and $\eta(\alpha) \in \nu$. Let $\nu_t$ be the connected component containing $\alpha$. Let $\{\square_1, \ldots, \square_N\}$ be a sequence of cells located on the left of or above the cell $\eta(\alpha)$ in the component $\nu_t$. Then it is easy to verify that

$$(\eta(\square_1) + \cdots + \eta(\square_N), \alpha) = A + kB$$

where $A$ is the number of of cells between the cell $\eta(\alpha)$ and the line $\Gamma_a$ with the same coordinate $i$ and $B$ is the number of cells between the cell $\eta(\alpha)$ and the line $\hat{\Gamma}_b$ with the same coordinate $j$. Therefore we have

$$(\chi + \rho + \alpha_1 + \cdots + \alpha_N, \alpha) + \frac{1}{2}(\alpha, \alpha) = 0,$$

and we have proved the inclusion $R(\chi) \supset \eta^{-1}(\nu)$. Let us prove the opposite inclusion. Suppose that the conditions [ ] are fulfilled. Then

$$\alpha_1 + \cdots + \alpha_N, \alpha = -(\chi + \rho, \alpha) - \frac{1}{2}(\alpha, \alpha) = -f(\eta(\alpha)) + kg(\eta(\alpha))$$

Therefore $f(\eta(\alpha)) \leq 0$ and $g(\eta(\alpha)) \geq 0$. So $\eta(\alpha) \in \nu$ and the second statement is proved.

To prove the third statement let us consider the decomposition of $\nu$ into connected components $\nu = \nu_1 \cup \nu_2 \cdots \cup \nu_r$. Then

$$R(\chi) = \eta^{-1}(\nu_1) \cup \eta^{-1}(\nu_2) \cdots \cup \eta^{-1}(\nu_r)$$

and it is easy to see that $\eta^{-1}(\nu_t) \perp \eta^{-1}(\nu_s)$ for $t \neq s$. Since the set $\eta^{-1}(\nu_t)$ (as a set of cells) is connected it can not be represented as a disjoint union of orthogonal subsets. And the third statement is proved.
Now let us prove the fourth statement. Let $\tilde{\chi} \in E(\chi)$. Then according to the Corollary 4.10 there exist a subset $D \subset \{1, 2, \ldots, r\}$ such that $\Gamma_a$ can be obtained from $\Gamma_a$ by replacing $L_i$ by $U_i$ and $\tilde{\Gamma}_b$ can be obtained from $\tilde{\Gamma}_b$ by replacing $U_i$ by $L_i$ for $i \in D$. It is enough to consider the case when $D = \{t\}$ consists of one element. Let $\nu_t$ be the corresponding connected component. Then we can define two sets

$$I_t = \{i \in \{1, \ldots, n\} \mid \exists j, \varepsilon_i - \varepsilon_{n+j} \in \eta^{-1}(\nu_t)\}$$

and

$$J_t = \{j \in \{1, \ldots, n\} \mid \exists i, \varepsilon_i - \varepsilon_{n+j} \in \eta^{-1}(\nu_t)\}$$

Since the connected component $\nu_t$ is a skew Young diagram it has rows and columns. For $i \in I_t$ let us denote by $d_i$ the length of the row of the skew diagram $\nu_t$ which contains box $\eta(\varepsilon_i - \varepsilon_{n+j})$ for some $j$. Similarly for $j \in I_t$ let us denote by $\tilde{d}_j$ the length of the column of the skew diagram $\nu_t$ which contains box $\eta(\varepsilon_i - \varepsilon_{n+j})$ for some $i$. Then it is not difficult to verify the following formulae

$$\tilde{a}_i = \begin{cases} a_i, & i \notin I_t \\ a_i + d_i, & i \in I_t, \end{cases} \quad 1 \leq i \leq n \quad \tilde{b}_j = \begin{cases} b_j, & j \notin J_t \\ b_j - \tilde{d}_j, & j \in J_t, \end{cases} \quad 1 \leq j \leq m$$

Therefore

$$\tilde{\chi} = \chi + \sum_{i=1}^{p} d_i \varepsilon_i - \sum_{j=1}^{q} \tilde{d}_j \delta_j$$

But

$$\sum_{i=1}^{p} d_i \varepsilon_i - \sum_{j=1}^{q} \tilde{d}_j \delta_j = \sum_{\square \in \nu_t} \eta^{-1}(\square) = \beta_t$$

and we proved the fourth statement.

Let us prove the fifth statement. Let

$$R(s) = \{\alpha_1, \ldots, \alpha_N\}$$

be one of the orthogonal components of $R(\chi)$. Suppose that

$$(\chi + \rho + \alpha_1 + \cdots + \alpha_{i-1}, \alpha_i) + \frac{1}{2}(\alpha_i, \alpha_i) = 0, \quad i = 1, \ldots, N$$

Set $g_s = s_{\alpha_N} \circ \cdots \circ s_{\alpha_1}$. Then by using the previous relations, it is easy to check that $g(\chi) = \chi + \beta_s$. Since $R(s) \perp R(t)$ for $s \neq t$, the elements $g_s$ and $g_t$ commute.

Let us prove the sixth statement. We will prove first that $(\chi_{\min} + \rho, \alpha) - \frac{1}{2}(\alpha, \alpha) \neq 0$ for any $\alpha \in R_1$. If $\alpha = \varepsilon_i - \varepsilon_{n+j}$, and

$$(\chi_{\min} + \rho, \alpha) - \frac{1}{2}(\alpha, \alpha) = a_i - j - k(b_j + n - i + 1) = 0$$

then $M_{2i-1} = N_{2j}$ is the intersection point of the lines $\Gamma_a$ and $\tilde{\Gamma}_b$. But this is impossible since $\Gamma_a$ is located on the left of the line $\tilde{\Gamma}_b$. Now let

$$\tilde{\chi} = \chi_{\min} + \vartheta_1 \beta_1 + \cdots + \vartheta_r \beta_r.$$
Then there is \( s \) such that \( \vartheta_s = 1 \). Let \( Q_s = (ji - 1) \) be the intersection point of the lines \( \Gamma_a \) and \( \tilde{\Gamma}_b \). Since \( \vartheta_s = 1 \) the line \( \tilde{\Gamma}_b \) contains also point \( (j - 1, i - 1) \). Therefore \( (ji - 1) = N_{2j} \) and since the line \( \Gamma_a \) contains point \( (j, i) \) we have \( (j, i - 1) = M_{2i-1} \). Therefore

\[
(\tilde{\chi} + \rho, \alpha) - \frac{1}{2}(\alpha, \alpha) = 0
\]

and the theorem is proved. \( \square \)

**Definition 4.12.** Let us denote by \( X^+_{reg}(n, m) \) the set of weights \( \chi \) from \( X^+(n, m) \) such that

\[
(\chi + \rho, \alpha) - \frac{1}{2}(\alpha, \alpha) \neq 0 \quad \text{for any positive odd root} \ \alpha.
\]

For brevity we will write \( \Lambda^\pm_{n,m}(\chi) \) instead of \( \Lambda^\pm_{n,m}(\theta \chi) \). The next proposition is a description of the spectral decomposition in terms of the root system.

**Proposition 4.13.** The following statements hold true:

1) \( \Lambda^\pm_{n,m} \) as a module over the algebra \( D_{n,m} \) can be decomposed into a direct sum of generalised eigenspaces

\[
\Lambda^\pm_{n,m} = \bigoplus_{\chi \in X^+_{reg}(n, m)} \Lambda^\pm_{n,m}(\chi), \tag{15}
\]

where \( \Lambda^\pm_{n,m}(\chi) \) is the generalised eigen-space corresponding to the homomorphism \( \theta \chi \).

2) The dimension of the space \( \Lambda^\pm_{n,m}(\chi) \) is equal to \( 2^r \) where \( r \) is the number of the odd positive roots such that \( (\chi + \rho, \alpha) + \frac{1}{2}(\alpha, \alpha) = 0 \).

3) Algebra \( \Lambda^\pm_{n,m} \) is generated by the deformed power sums

\[
p_s(x_1, \ldots, x_{n+m}) = x_1^s + \cdots + x_n^s + \frac{1}{k}(x_{n+1}^s + \cdots + x_{n+m}^s), \quad s = \pm 1, \pm 2, \ldots
\]

**Proof.** The first two statements follow from Proposition 3.10, Proposition 3.11 and the Theorem 4.11.

So let us prove the third statement. Let us denote by \( \hat{\Lambda}^\pm_{n,m} \) the subalgebra in \( \Lambda^\pm_{n,m} \) which is generated by the deformed power sums and

\[
\Lambda^\pm_{n,m} = \Lambda^\pm_{n,m} \cap \mathbb{C}[x_1, \ldots, x_{n+m}], \quad \Lambda^\pm_{n,m} = \Lambda^\pm_{n,m} \cap \mathbb{C}[x_1^{-1}, \ldots, x_{n+1}^{-1}]
\]

We have already proved that subspace \( \Lambda^\pm_{n,m}(\chi) \) is finite dimensional. Let us prove that \( \Lambda^\pm_{n,m}(\chi) \subset \hat{\Lambda}^\pm_{n,m} \). Let \( \chi = (a_1, \ldots, a_n \mid b_1, \ldots, b_m) \in X^+(n, m) \) and \( c, d \) are integers such that

\[
\lambda_i = a_i - 2m - c \geq 0, \quad i = 1, \ldots, n, \quad \mu_j = b_j + 2n - d \geq 0, \quad j = 1, \ldots, m
\]

In other words \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and \( \mu = (\mu_1, \ldots, \mu_m) \) are partitions. Consider the polynomials

\[
g_1(x_1, \ldots, x_{n+m}) = \prod_{i,j} (x_i - x_{i+j})^2 s_{\lambda}(x_1, \ldots, x_n) s_{\mu}(x_{n+1}, \ldots, x_{n+m})
\]
and 
\[ g_2(x_1, \ldots, x_{n+m}) = \prod_{i,j}(x_i^{-1} - x_{n+j}^{-1})^2(x_1 \ldots x_n)^c(x_{n+1} \ldots, x_{n+m})^d \]
where \( s_\lambda(x_1, \ldots, x_n), s_\mu(x_{n+1} \ldots, x_{n+m}) \) are Schur polynomials. Since the highest terms of these Schur polynomials have weights \( \lambda \) and \( \mu \) the highest term of the product 
\[ g(x_1, \ldots, x_{n+m}) = g_1(x_1, \ldots, x_{n+m})g_2(x_1, \ldots, x_{n+m}) \]
has the weight \( \chi \). It is easy to verify that \( g_1(x_1, \ldots, x_{n+m}) \in \Lambda_{n,m}^+ \) and \( g_2(x_1, \ldots, x_{n+m}) \in \Lambda_{n,m}^- \). But by Theorem 2 from \[9\] the algebra \( \Lambda_{n,m}^+ \) is generated by the positive power sums, hence the algebra \( \Lambda_{n,m}^- \) is generated by the negative power sums. So \( g(x_1, \ldots, x_{n+m}) \in \Lambda_{n,m}^\pm \). Since the integrals preserves the algebra \( \tilde{\Lambda}_{n,m}^\pm \) (see \[12\] formula \( 46, 47 \)) one can define \( \tilde{\Lambda}_{n,m}^\pm(\chi) \) in the same way as \( \Lambda_{n,m}^\pm(\chi) \) was defined and the same arguments as in Proposition \[3.11\] show that there exists \( f_\chi \in \tilde{\Lambda}_{n,m}^\pm(\chi) \) with the only maximal weight \( \chi \). Hence \( f_\chi, \tilde{\chi} \in E(\chi) \) make a basis in \( \tilde{\Lambda}_{n,m}^\pm(\chi) \). Therefore 
\( \tilde{\Lambda}_{n,m}^\pm(\chi) = \Lambda_{n,m}^\pm(\chi) \). \( \square \)

5. Weights and bipartitions

By a bipartition we will mean a pair of partitions \((\lambda, \mu)\). We will denote by \( H(n, m) \) the set of partitions \( \lambda \) such that \( \lambda_{n+1} \leq m \), or \( \lambda_{m+1} \leq n \). The following definition is not standard, but one can easily check that it is equivalent to the usual definition \[17\].

**Definition 5.1.** We say that the bipartition \((\lambda, \mu)\) is contained in the \((n, m)\) cross if there are nonnegative integers \( p, q, r, s \) such that \( p + q = n \), \( r + s = m \) and \( \lambda \in H(p, r), \mu \in H(q, s) \). We will denote the set of all such bipartitions by \( Cr(n, m) \).

The main aim of this section is to give a geometric description of a bijection between the set \( Cr(n, m) \) and the set \( X^+(n, m) \).

**Definition 5.2.** Let \((\lambda, \mu) \in Cr(n, m)\). Set 
\[ H(\lambda, \mu) = \{(i, j) \mid \lambda \in H(i, m - j), \mu \in H(n - i, j)\} \]
and define 
\[ p = \max\{i \mid (i, j) \in H(\lambda, \mu)\}, \ s = \max\{j \mid (p, j) \in H(\lambda, \mu)\}. \]
We will call the pair \((p, s)\) the extremal one.

There is an easy way to find the extremal pair for a given bipartition \((\lambda, \mu) \in Cr(n, m)\).

**Definition 5.3.** Let \((\lambda, \mu) \in Cr(n, m)\). Then we set \( F(\lambda, \mu) = \tilde{\lambda}, \tilde{\mu} \) where 
\[ \tilde{\lambda}_i = \lambda_i - \lambda_{n+1}, 1 \leq i \leq n + 1, \quad \tilde{\mu}_j' = \mu'_j - \mu'_{m+1}, 1 \leq j \leq m + 1 \]
and we also set \( \tilde{n} = n - \mu'_{m+1}, \tilde{m} = m - \lambda_{n+1} \).
Lemma 5.4. The map $F$ has the following properties:

1) If $(\lambda, \mu) \in Cr(n, m)$ then $F(\lambda, \mu) \in Cr(\tilde{n}, \tilde{m})$.

2) If the pair $(p, s)$ is extremal for $(\lambda, \mu)$ then it is also extremal for $F(\lambda, \mu)$.

3) $F(\lambda, \mu) = (\lambda, \mu)$ if and only if $\lambda \in H(n, 0), \mu \in H(0, m)$. The extremal pair in this case is $(n, m)$.

Proof. Let us prove the first statement. We have

$$\tilde{\lambda}_{p+1} = \lambda_{p+1} - \lambda_{n+1} \leq m - s - \lambda_{n+1} = \tilde{m} - s$$

$$\tilde{\mu}'_{s+1} = \mu'_{s+1} - \mu'_{m+1} \leq n - p - \mu'_{m+1} = \tilde{m} - p$$

So we see that $\tilde{\lambda} \in H(p, \tilde{n} - s)$ and $\tilde{\mu} \in H(n - p, s)$, or equivalently this means that $(\tilde{\lambda}, \tilde{\mu}) \in Cr(\tilde{n}, \tilde{m})$. This we proved the first statement.

Now let us prove the second statement. Let $(\tilde{p}, \tilde{s}) \in H(\tilde{\lambda}, \tilde{\mu})$. Then by definition we have:

$$\tilde{\lambda}_{\tilde{p}+1} \leq \tilde{m} - \tilde{s}, \quad \tilde{\mu}'_{\tilde{s}+1} \leq \tilde{n} - \tilde{p}$$

Therefore

$$\lambda_{\tilde{p}+1} \leq m - \tilde{s}, \quad \mu'_{\tilde{s}+1} \leq n - \tilde{p}$$

and $\tilde{p} \leq p$ and $\tilde{s} \leq s$. This proves the second statement. The third statement easily follows from the definition. □

There exists a natural map $\pi : Cr(n, m) \to X^+(n, m)$ (see for example [7]) and we are going to give a geometric interpretation of this map.

Definition 5.5. Let $(\lambda, \mu) \in Cr(n, m)$ and $(p, s)$ be corresponding extremal pair. Then we set

$$\pi(\lambda, \mu) = (\lambda_1, \ldots, \lambda_p, m - \mu_q, \ldots, m - \mu_1 | \lambda'_1 - n, \ldots, \lambda'_r - n, -\mu'_s, \ldots, -\mu'_1)$$

It is not difficult to verify that $\pi(\lambda, \mu) \in X^+_{n,m}$ for any $(\lambda, \mu) \in Cr(n, m)$.

Below we are going to prove some properties of the map $\pi$ by using our geometric interpretation of the set $X^+(n, m)$ and a geometric interpretation of bipartitions, in the same way as S. Kerov did in his paper [4] for partitions.

Definition 5.6. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition. Consider the following points on the plane:

$$M_0 = (+\infty, 0), \ M_1 = (\lambda_1, 0), \ M_2 = (\lambda_1, 1), \ M_3 = (\lambda_2, 1), \ M_4 = (\lambda_2, 2), \ldots$$

Let us denote by $Y_\lambda$ the polygonal line

$$Y_\lambda = \bigcup_{i \geq 0} [M_i, M_{i+1}]$$

consisting of the segments $[M_i, M_{i+1}]$. The segment $[M_0, M_1]$ corresponds to the half line.
Remark 5.7. The same polygonal line can be defined by using the conjugate partition. Consider the points

\[ N_0 = (0, +\infty), \ N_1 = (0, \lambda_1), \ N_2 = (1, \lambda_1'), \ N_3 = (1, \lambda_2), \ N_4 = (2, \lambda_2'), \ldots. \]

It is easy to check that

\[ Y_{\lambda} = \bigcup_{j \geq 0} [N_j, N_{j+1}] \]

Definition 5.8. Let \( n, m \) be nonnegative integers. Define a bijective map

\[ \vartheta_{n,m} : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad \vartheta_{n,m}(x, y) = (n - x, m - y) \]

So for any bipartition \((\lambda, \mu)\) we can define a pair of polygonal lines

\[ Y_{\lambda,\mu} = (Y_{\lambda}, \vartheta_{n,m}(Y_{\mu})) \]

We can now reformulate the definition of the set \( Cr(n, m) \) in the following way.

Lemma 5.9. A bipartition \((\lambda, \mu)\) belongs to \( Cr(n, m) \) if and only if the polygonal lines \( Y_{\lambda} \) and \( \vartheta_{n,m}(Y_{\mu}) \) have at least one intersection point.

Proof. Let \((\lambda, \mu) \in Cr(n, m)\) and \((p, s)\) the extremal pair for them. Let us prove that the point \( M = (r, p) \) is an intersection point. First let us prove that \( M \in Y_{\lambda} \). Since \([M_{2p+1}, M_{2p}] \subset Y_{\lambda}\) we only need to prove that \( M \in [M_{2p+1}, M_{2p}] \). By definition \( \lambda_{p+1} \leq r \). Suppose that \( \lambda_p > r \). Then \( \lambda_{p+1} \leq \lambda_p \leq r - 1 \). Therefore \( \lambda \in H(p, r - 1) \) and \( \mu \in H(q, s) \subset H(q, s + 1) \). But this contradicts the condition that the pair \((p, s)\) is the extremal one. So \( \lambda_p \geq r \) and \( M \in Y_{\lambda} \).

In the same way we can prove that \( \mu'_{q+1} \leq s \leq \mu'_{q} \). Therefore point \( N = (s, q) \in [N_{2s+1}, N_{2s}] \subset Y_{\mu} \) and \( \vartheta_{n,m}(N) = M \in \vartheta_{n,m}(Y_{\mu}) \).

Now let us prove the converse statement. Let \( M = (\tilde{r}, \tilde{p}) \in Y_{\lambda} \cap \vartheta(Y_{\mu}) \). Then it is easy to check that

\[ Y_{\lambda} \in H(\tilde{p}, \tilde{r}), \quad Y_{\mu} \in H(n - \tilde{p}, m - \tilde{r}) \]

Therefore \((\lambda, \mu) \in Cr(n, m)\). The lemma is proved.

Corollary 5.10. Let \((\lambda, \mu) \in Cr(n, m)\) and \((p, s)\) be the corresponding extremal pair. Then the point \( M = (r, p) \) is the maximal of the intersection points of the lines \( Y_{\lambda} \) and \( \vartheta_{n,m}(Y_{\mu}) \).

Let us now define a map \( \sigma : Cr(n, m) \rightarrow X^+(n, m) \). Let \((\lambda, \mu) \in Cr(n, m)\) and \( Y_{\lambda}, \vartheta(Y_{\mu}) \) be the corresponding lines. Let \( M \) be the maximal intersection point of these lines. We have two disjoint unions

\[ Y_{\lambda} = Y_{\lambda}^L \cup \{M\} \cup Y_{\lambda}^R, \quad \vartheta_{n,m}(Y_{\mu}) = \vartheta_{n,m}(Y_{\mu})^L \cup \{M\} \cup \vartheta_{n,m}(Y_{\mu})^R \]

where \( Y_{\lambda}^L \) means the set of points on the line \( Y_{\lambda} \) which are strictly less than \( M \), and \( Y_{\lambda}^R \) means the set of points on the line \( Y_{\lambda} \) which are strictly greater than \( M \). The same for the line \( \vartheta_{n,m}(Y_{\mu}) \) instead of \( Y_{\lambda} \).
Definition 5.11. Let us define the following two sets as disjoint unions

\[ \Gamma_a = Y_a^L \cup \{M\} \cup \partial_{n,m}(Y_\mu)^R, \quad \Gamma_b = Y_b^R \cup \{M\} \cup \partial_{n,m}(Y_\mu)^L \]

and \( \pi(\lambda, \mu) = (a, b) \).

Proposition 5.12. We have the following equality

\[ \pi(\lambda, \mu) = \sigma(\lambda, \mu) \]

Proof. It easily follows from the definition of the map \( \sigma \).

6. Action on the generalised eigenspace

In order to describe the action of the algebra \( D_{n,m} \) on the subspaces \( \Lambda_{n,m}(\chi) \) we will use the infinite dimensional version of the CMS operators investigated in \([15]\). But first we will prove a more general abstract result.

Let \( \varphi : A \rightarrow B \) be a surjective homomorphism of commutative algebras. Let \( U, V \) be modules over the algebras \( A, B \) respectively. Let \( \psi : U \rightarrow V \) be a linear surjective map such that \( \psi(au) = \varphi(a)\psi(u) \). Suppose that both modules split into direct sums of generalised finite dimensional eigenspaces

\[ U = \bigoplus_{\chi \in S} U(\chi), \quad V = \bigoplus_{\tilde{\chi} \in T} V(\tilde{\chi}) \]

where \( S \subset A^* \), \( T \subset B^* \) are some sets of homomorphisms.

The homomorphism \( \varphi \) induces a map \( \varphi^* : T \rightarrow S \) such that \( \varphi^*(\tilde{\chi}) = \tilde{\chi} \circ \varphi \).

Suppose we have a basis of \( e_i, i \in I \) in \( U \) and a basis of \( f_j, j \in J \) in \( V \) such that \( e_i \in U(\chi_i) \) and \( f_j \in V(\tilde{\chi}_j) \) for some \( \chi_i \in S \) and \( \tilde{\chi}_j \in T \). Suppose there exists an injective map \( \gamma : J \rightarrow I \) such that \( \varphi^*(\tilde{\chi}_j) = \chi_{\gamma(j)} \).

Lemma 6.1. Suppose that all the previous conditions are fulfilled. Then the set of vectors \( \{e_i \mid i \notin \gamma\} \) makes a basis of the space \( \ker \psi \) and if \( \chi = \varphi^* \tilde{\chi} \) then the restriction of \( \psi \) to \( U(\chi) \) is isomorphism onto \( V(\tilde{\chi}) \).

Proof. Let \( \chi \in T \). If \( \psi(U(\chi)) \neq 0 \) then there exists \( \tilde{\chi} \) such that \( \tilde{\chi} = \chi \circ \varphi \). Therefore \( \chi \in \text{im} \varphi^* \) and since \( \psi \) is surjection, then the restriction map \( \psi : U(\chi) \rightarrow V(\tilde{\chi}) \) is also a surjection. So we see that if \( \chi \notin \text{im} \varphi^* \) then \( \psi(U(\chi)) = 0 \). Therefore \( e_i \in \ker \psi \) for \( i \notin \gamma \). Let us prove now that if \( \chi = \varphi^* \tilde{\chi} \) then \( \psi : U(\chi) \rightarrow V(\tilde{\chi}) \) is isomorphism. We have already seen that this is a surjective map. Therefore it is enough to prove that \( \dim U(\chi) = \dim V(\tilde{\chi}) \). Let us set

\[ I_\chi = \{i \mid \chi_i = \chi\}, \quad J_\tilde{\chi} = \{j \mid \tilde{\chi}_j = \tilde{\chi}\} \]

From the conditions of the present lemma it follows that \( \dim U(\chi) = |I_\chi| \) and \( \dim V(\tilde{\chi}) = |J_\tilde{\chi}| \). So it is enough to prove that \( |I_\chi| = |J_\tilde{\chi}| \). It is easy to see that we only need to prove that \( |I_\chi| = |J_\tilde{\chi}| \). Let \( j \in J_\tilde{\chi} \) then \( \chi_{\gamma(j)} = \varphi^*(\tilde{\chi}_j) = \tilde{\chi} \). Conversely let \( i \in I_\chi \) and \( i = \gamma(j) \), then \( \varphi^*(\tilde{\chi}_j) = \chi_{\gamma(j)} = \chi_i = \chi \). Since \( \varphi^* \) is injective map we get \( \tilde{\chi}_j = \tilde{\chi} \) and therefore \( j \in J_\tilde{\chi} \).
Theorem 6.2. Let $k$ be not an algebraic number and $\dim \Lambda_{n,m}^\pm(\chi) = 2^r$. Then the image of the algebra $D_{n,m}$ in the algebra $\text{End}(\Lambda_{n,m}^\pm(\chi))$ is isomorphic to $\mathbb{C}[\varepsilon]^{\otimes r}$, where $\mathbb{C}[\varepsilon]$ is the algebra of dual numbers and the space $\Lambda_{n,m}^\pm(\chi)$ is the regular representation with respect to this action.

Proof. We will use Lemma 6.1. Let $A = D$ be the algebra of infinite integrals with $p_0 = n + k^{-1}m$ (see [11]) and $B = D_{n,m}$. Let $V = \Lambda^\pm$ be the algebra of Laurent symmetric functions and $U = \Lambda_{n,m}^\pm$. Let $I$ be the set of all bipartitions and $J = X^+(n, m)$. The basis in the space $V$ is the set of $Q_{\lambda,\mu}$ constructed in [15]. The basis $f_\chi$ in the space $U$ the one was constructed in Proposition 3.11. Besides we have the following maps:

$$\psi : \Lambda^\pm \rightarrow \Lambda_{n,m}^\pm, \quad \psi(p_r) = x_1^{s_1} \cdots x_n^{s_n} + \frac{1}{k}(x_{n+1}^{s_{n+1}} + \cdots + x_{n+m}^{s_{n+m}}), \quad s = \pm 1, \pm 2 \ldots$$

and the map $\varphi$ comes from the following commutative diagram [11]

$$\begin{array}{ccc}
\Lambda^\pm & \xrightarrow{D} & \Lambda^\pm \\
\downarrow \psi & & \downarrow \psi \\
\Lambda_{n,m}^\pm & \xrightarrow{\varphi(D)} & \Lambda_{n,m}^\pm
\end{array}$$

where $D \in \mathcal{D}$. The set $S$ is defined by

$$S = \{ \theta_{\lambda,\mu} | \theta_{\lambda,\mu}(D) = \phi(D)(\lambda, \mu) \}$$

where $\phi : \mathcal{D} \rightarrow \Lambda^*$ is the Harish-Chandra homomorphism from [11], $\Lambda^*$ is the algebra of shifted symmetric functions and for $f \in \Lambda^*$ the value $f(\lambda, \mu)$ is defined in [11], page 79, formula (56). The set $T$ is the set of all $\theta_\chi$ where $\chi \in X_{n,m}^+$. The corresponding map $\gamma$ is the inverse map to $\pi$:

$$\gamma : J \rightarrow I, \quad \gamma(\chi) = \pi^{-1}(\chi)$$

So in order to prove the Theorem we only need to prove the equality

$$\theta_{\pi(\lambda, \mu)} \circ \varphi = \theta_{\lambda, \mu} \quad \text{for} \quad (\lambda, \mu) \in C_\tau(n, m)$$

But from the commutative diagram

$$\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\phi} & \Lambda^* \\
\downarrow \varphi & & \downarrow \varphi^* \\
\mathcal{D}_{n,m} & \xrightarrow{\phi_{n,m}} & \Lambda_{n,m}^*
\end{array}$$

we see that enough to prove the equality $f(\lambda, \mu) = \varphi^*(f(\pi(\lambda, \mu)))$ where $f \in \Lambda^*$.

Let us identify Young diagram $\lambda$ with the part of the plane bounded by the lines $x = 0$, $y = 0$, $Y_\lambda$. Let us recall that for any unit square such that all its vertices have integer coordinates we defined the function $c_k(\square) = x + ky$ where $(x, y)$ are the coordinates of the left lower vertex of the square.
Lemma 6.3. Let \( f \in \Lambda^* \) and \( (\lambda, \mu) \in Cr(n, m) \). Then
\[
f(\lambda, \mu) = \varphi^w(f)(\pi(\lambda, \mu))
\]

Proof. It is enough to prove the above equality for the shifted symmetric function corresponding to Bernoulli polynomials
\[
b_r^\infty(x) = \sum_{i \geq 1} [B_r(x_i + k(i - 1)) - B_r(k(i - 1))] \quad r, 1, 2, \ldots
\]
But in this case (see [11] Lemma 5.8) we have
\[
b_r^\infty(\lambda, \mu) = r \sum_{\square \in \lambda} c_k(\square)^{r-1} + r(-1)^r \sum_{\square \in \mu} (c_k(\square) + 1 + k - kn - m)^{r-1}
\]
\[
= r \sum_{\square \in \lambda} c_k(\square)^{r-1} - r \sum_{\square \in \theta_{n,m}(\mu) \setminus \Delta} c_k(\square)^{r-1}
\]
Let us denote by \( \Delta \) the part of the plane bounded by the lines \( x = 0, \ y = n, \ Y_\lambda, \ \vartheta(Y_\mu) \). Since \( \lambda \) and \( \vartheta_{n,m}(\mu) \) both contain \( \Delta \), we have
\[
b_r^\infty(\lambda, \mu) = r \sum_{\square \in \lambda \setminus \Delta} c_k(\square)^{r-1} - r \sum_{\square \in \theta_{n,m}(\mu) \setminus \Delta} c_k(\square)^{r-1} =
\]
\[
= r \sum_{\square \in D_+^\lambda \cup D_+^\mu} c_k(\square)^{r-1} - r \sum_{\square \in D_-^\lambda \cup D_-^\mu} c_k(\square)^{r-1} = b_r^{(n,m)}(\pi(\lambda, \mu))
\]
We only need to prove that \( \varphi^w(b_r^\infty) = b_r^{(n,m)} \). In general Jack-Laurent symmetric functions \( P_{\lambda,\mu} \) are eigenfunctions of the algebra \( D \). In our case \( p_0 = n + k^{-1}m \) so not all the Jack-Laurent symmetric functions are well defined. But if we take \( \lambda = (\lambda, \emptyset) \) a bipartition with the empty second part then the corresponding Jack-Laurent symmetric function \( P_{\lambda,\emptyset} \) does not depend on \( p_0 \) therefore it is well defined and it is simply the Jack symmetric function \( P_{\lambda} \) corresponding to the partition \( \lambda \). Therefore by definition we have
\[
\phi_{n,m}(D)(\chi)\psi(P_{\lambda}) = \psi(D)(\lambda, \emptyset)\psi(P_{\lambda}), \ D \in D.
\]
For \( \lambda \) with \( \lambda_n \geq m \) the highest weight of the polynomial \( \psi(P_{\lambda}) \) is
\[
\pi(\lambda, 0) = (\lambda_1, \ldots, \lambda_n | \lambda'_1 - n, \ldots, \lambda'_m - n)
\]
(see [10]). So if we take \( D \) such that \( \phi(D) = b_r^\infty \) then by definition
\[
\varphi^w(b_r^\infty)((\lambda_1, \ldots, \lambda_n|\lambda'_1 - n, \ldots, \lambda'_m - n)) = b_r^{(n,m)}((\lambda_1, \ldots, \lambda_n|\lambda'_1 - n, \ldots, \lambda'_m - n))
\]
for all \( \lambda \) with \( \lambda_n \geq m \). Therefore \( \varphi^w(b_r^\infty) = b_r^{(n,m)} \). \( \square \)

Thus we have proved the Lemma and the Theorem. \( \square \)

The parameter \( p_0 \) plays a special role in the theory of Jack-Laurent symmetric function. This parameter can be considered as a natural generalisation of the dimension. If \( p_0 = n + k^{-1}m \) then as it was proved in the Theorem [6.2] the infinite dimensional integrable system has a finite dimensional quotient. The next corollary of the Theorem [6.2] shows that the kernel of the
homomorphism $\psi$ can be described in terms of the Jack-Laurent polynomials which are well defined at $p_0 = n + k^{-1}m$ and eigenfunctions of the algebra $D_{n,m}$ can be constructed in terms of Jack-Laurent symmetric functions as well.

**Corollary 6.4.** 1) Let $k$ be not a rational number and $(\lambda, \mu) \notin Cr(n, m)$. Then the Jack-Laurent polynomial $P_{\lambda, \mu}$ is well defined at $p_0 = n + k^{-1}m$ and the linear span of all such $P_{\lambda, \mu}$ is the kernel of the homomorphism $\psi$.

2) Let $\chi \in X_{\text{reg}}^{+}(n, m)$. Then generalised eigenspace $\Lambda_{n,m}^{\pm}(\chi)$ contains the only eigenfunction $J_\chi$ of the algebra $D_{n,m}$. The Jack-Laurent symmetric function $P_{\pi^{-1}(\chi)}$ is well defined at $p_0 = n + k^{-1}m$ and $J_\chi = d\psi(P_{\pi^{-1}(\chi)})$, where $d \in \mathbb{C}$ is a constant.

**Proof.** This corollary follows directly from Lemma 6.1 from the proof of the Theorem 6.2 and the Theorem 3.6 from [15].

7. Concluding remarks

We have shown that the spectral decomposition of the algebra of quasi-invariant Laurent polynomials under the action of the algebra of deformed integrals has a nice description, despite the fact that this decomposition is not multiplicity free.

One of the main steps in the proof is using an infinite dimensional version of the CMS problem to describe the algebra of endomorphisms of a generalised eigenspace. It would be good to prove this fact directly, without appealing to the infinite dimensional version. The Moser matrix as a linear operator on the quasi-homomorphisms seems to be a key ingredient in such a proof.

It also looks like that the group $G$ generated by reflections with respect to the deformed inner product should play an important role in the whole theory of deformed CMS systems and of the corresponding groupoids.

In the paper [3] a natural generalisations of the algebra $\Lambda_{n,m}^{+}$ were investigated. It would be interesting to investigate the same sort of generalisations of the algebras $\Lambda_{n,m}^{\pm}$ as well.

8. Acknowledgements

This work has been funded by Russian Ministry of Education and Science (grant 1.492.2016/1.4) and partially by the Russian Academic Excellence Project ‘5-100’.

References

[1] C.J. Cummins and R.C. King, J. Composite Young-Diagrams, Supercharacters Of $U(M/N)$ And Modification Rules. Phys. A 20, 3121 (1987).
[2] O. A. Chalykh and A. P. Veselov, Commutative rings of partial differential operators and Lie algebras. Comm. Math. Phys. 126(3) (1990), 597611.
[3] P. Etingof, E. Rains (with an appendix by Misha Feigin) On Cohen-Macaulayness of Algebras Generated by Generalized Power Sums. Commun. Math. Phys. 347, 163182 (2016)

[4] S.V. Kerov Anisotropic Young Diagrams and Jack Symmetric Functions. Functional Analysis and Its Applications, Vol. 34, No. 1, 2000

[5] I. G. Macdonald Symmetric functions and Hall polynomials. 2nd edition, Oxford Univ. Press, 1995.

[6] E.M. Moens, J. Van der Jeugt A character formula for atypical critical $gl(m|n)$ representations labelled by composite partitions. J. Phys. A: Math.Gen. 37 (2004), 12019-12039.

[7] E.M. Moens and J. Van der Jeugt, Composite supersymmetric $S$ functions and characters of $gl(m|n)$ representations. Proceedings of the VI International Workshop on Lie Theory and its Applications in Physics, ed. H.-D. Doebner and V.K. Dobrev, Heron Press Ltd, Sofia (2006), 251–268

[8] M.A. Olshanetsky, A.M. Perelomov Quantum integrable systems related to Lie algebras. Phys. Rep. 94 (1983), 313-404.

[9] A.N. Sergeev, A.P. Veselov Deformed quantum Calogero-Moser problems and Lie superalgebras. Comm. Math. Phys. 245 (2004), no. 2, 249–278.

[10] A.N. Sergeev, A.P. Veselov Generalised discriminants, deformed Calogero-Moser-Sutherland operators and super-Jack polynomials. Adv. Math. 192 (2005), no. 2, 341–375.

[11] A.N. Sergeev, A.P. Veselov Jack - Laurent symmetric functions. Proc. London Math. Soc. (3) 111 (2015) 63–92

[12] A. N. Sergeev and A. P. Veselov (2015) Dunkl Operators at Infinity and Calogero-Moser Systems. International Mathematics Research Notices, Vol. 2015, No. 21, pp. 10959–10986.

[13] A.N.Sergeev, A.P. Veselov Symmetric Lie superalgebras and deformed quantum Calogero-Moser problems. Adv. Math. 304 (2017), 728-768.

[14] A.N. Sergeev, A.P. Veselov Orbits and invariants of Super Weyl Groupoid. International Mathematical Research Notes, vol. 2017, No. 20, pp. 6149-6167.

[15] A.N. Sergeev, A.P. Veselov Jack - Laurent symmetric functions for special values of the parameters Glasgow Math. J. 58 (2016) 599–616

[16] R. Stanley Some combinatorial properties of Jack symmetric functions. Adv. Math. 77 (1989), no. 1, 76–115.

Department of Mathematics, Saratov State University, Astrakanskaya 83, Saratov 410012 and National Research University Higher School of Economics, Russian Federation.

E-mail address: SergeevAN@info.sgu.ru, a sergeev@hse.ru