New Massive Gravity Domain Walls

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ABSTRACT

The properties of the asymptotic $AdS_3$ space-times representing flat domain walls (DW’s) solutions of the New Massive 3D Gravity with scalar matter are studied. Our analysis is based on \textit{Ist} order BPS-like equations involving an appropriate superpotential. The Brown-York boundary stress-tensor is used for the calculation of DW’s tensions as well as of the $CFT_2$ central charges. The holographic renormalization group flows and the phase transitions in specific deformed $CFT_2$ dual to 3D massive gravity model with quadratic superpotential are discussed.
1 Introduction

The new massive gravity (NMG) represents an appropriate “higher derivatives” generalization of 3D Einstein gravity action:

\[
S_{\text{NMG}}(g_{\mu\nu},\sigma;\kappa,\Lambda) = \frac{1}{\kappa^2} \int dx^3 \sqrt{-g} \left( \epsilon R + \frac{1}{m^2} \mathcal{K} - \kappa^2 \left( \frac{1}{2} \nabla^2 \sigma^2 + V(\sigma) \right) \right)
\]

\[
\mathcal{K} = R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2, \quad \kappa^2 = 16\pi G, \quad \epsilon = \pm 1
\]

which unlike its 4D Einstein version is unitary (i.e. ghost-free) under certain restrictions on the matter potential \(V(\sigma)\), on the values of the cosmological constant \(\Lambda = -\frac{\kappa^2}{2} V(\sigma^*)\) and of the new mass parameter \(m^2\) for the both choices of the sign of the \(R\)-term [1]. It is 1-loop UV finite, but power-counting non-renormalizable \(^3\) quite as in the case of 4D Einstein gravity [2],[3]. The NMG vacuum (\(\sigma = \text{const}\)) sector contains two propagating (massive) degrees of freedom (the “graviton” polarizations) and as a result it admits a variety of physically interesting classical solutions - gravitational waves, black holes, etc. [5]. When matter is added the only known exact solutions [6] are certain asymptotically \(dS_3\) geometries that describe Bounce-like evolutions of 3D Universe.

The problem addressed in the present paper concerns the construction of a family of flat static domain walls (DW’s) solutions, i.e. \(\sigma = \sigma(z)\) and

\[
\text{ds}^2 = dz^2 + e^{\varphi(z)}(dx^2 - dt^2)
\]

of the NMG model (1) for polynomial matter potentials \(V(\sigma)\). We are looking for DW’s interpolating between two different \(AdS_3\) vacua \((\sigma^*_A, \Lambda^A_{\text{eff}})\), parametrized by the solutions of the algebraic equations:

\[
V'(\sigma^*_A) = 0, \quad 2\Lambda^A_{\text{eff}} \left( 1 + \frac{\Lambda^A_{\text{eff}}}{4\epsilon m^2} \right) = \epsilon \kappa^2 V(\sigma^*_A)
\]

The study of such DW’s is motivated by their important role in the description of the “holographic” renormalization group (RG) flows [7] and of the corresponding phase transitions in two-dimensional QFT “dual” to 3D massive gravity (1). The generalization of the superpotential method proposed in ref. [6] allows an explicit construction of qualitatively new DW’s relating “old” to the “new” purely NMG vacua. Assuming further that the \(AdS_3/CFT_2\) correspondence [8],[9],[10] takes place for the extended NMG model (1) as well, we investigate the changes induced by the counter-terms \(\mathcal{K}\) (and by the sign factor \(\epsilon\)) on the structure of the corresponding \(QFT_2\’s\) \(\beta\)-function, concerning the “\(m^2\)-corrections” to the central charges, scaling dimensions and to its free energy. The example of the DW’s of NMG model with quadratic superpotential and the phase transitions in its dual perturbed \(CFT_2\) \((\mu CFT_2)\) are studied in some details. An extension of \(d = 3\) NMG’s BPS-like \(I^\text{st}\) order system and related to it superpotential to the case of \(d\text{-dimensional}\) New Massive Gravity for \(d > 3\) is introduced in Sect.7.

2 Superpotential

Although the NMG action (1) involves up to fourth order derivatives of 3D metrics \(g_{\mu\nu}\), the corresponding equations for the DW’s (2) are of second order:

\[
\ddot{\sigma} + \dot{\sigma} \dot{\varphi} - V'(\sigma) = 0
\]

\(^3\)although there exist controversy claims concerning its super-renormalizability [4].
\[ \ddot{\varphi} \left(1 - \frac{\dot{\varphi}^2}{8\epsilon m^2}\right) + \frac{1}{2}\varphi^2 \left(1 - \frac{\dot{\varphi}^2}{16\epsilon m^2}\right) + \epsilon \kappa^2 \left(\frac{1}{2}\dot{\sigma}^2 + V(\sigma)\right) = 0 \]
\[ \varphi^2 \left(1 - \frac{\dot{\varphi}^2}{16\epsilon m^2}\right) + \epsilon \kappa^2 (-\dot{\sigma}^2 + 2V(\sigma)) = 0 \]  

(4)

due to a particular form of the higher derivatives \(K\)-term. A powerful method for construction of analytic non-perturbative solutions of eqs. (4) consists in the introduction of an auxiliary function \(W(\sigma)\) called superpotential\(^4\) [6], [12] such that

\[ \kappa^2 V(\sigma) = 2(W')^2 \left(1 - \frac{\kappa^2 W^2}{2\epsilon m^2}\right)^2 - 2\epsilon \kappa^2 W^2 \left(1 - \frac{\kappa^2 W^2}{4\epsilon m^2}\right) \]
\[ \varphi = -2\epsilon \kappa W, \quad \dot{\sigma} = \frac{2}{\kappa} W' \left(1 - \frac{\kappa^2 W^2}{2\epsilon m^2}\right) \]  

(5)

where \(W'(\sigma) = \frac{dW}{d\sigma}\), \(\dot{\sigma} = \frac{d\sigma}{dz}\) etc. The statement is that for each given \(W(\sigma)\) all the solutions of the first order system (5) are solutions of the eqs. (4) as well. For example, the linear superpotential \(W(\sigma) = B\sigma\ (B = \text{const})\) describes a particular double-well matter potential

\[ V(\sigma) = \frac{\gamma}{4} \left(\sigma^2 - \frac{m_\sigma^2}{2\gamma}\right)^2 - \frac{2\Lambda}{\kappa^2} \]  

(6)

for \(\epsilon m^2 > 0\) and \(\gamma, m_\sigma^2\) and \(\Lambda\) given by

\[ \gamma = \frac{2B^4\kappa^2}{m^2} \left(1 + \frac{B^2}{m^2}\right), \quad m_\sigma^2 = 8\epsilon B^2 \left(1 + \frac{B^2}{m^2}\right), \quad \Lambda = m^2 \]

The corresponding DW’s solutions of eq. (4)

\[ \sigma(z) = \frac{\sqrt{2\epsilon m^2}}{B\kappa} \tanh \left(\frac{\sqrt{2}}{\epsilon m^2}(z - z_0)\right), \quad e^{\varphi(z) + \phi_0} = \left[ \cosh \left(\frac{B^2}{\epsilon m^2}(z - z_0)\right)\right]^{\frac{2\epsilon m^2}{B^2}} \]  

(7)

have as asymptotics at \(z \to \pm \infty\) two very special NMG - vacua with \(\lambda_{BHT} = -\frac{\Lambda}{m^2} = -1\) [1] placed at two degenerate minima

\[ \sigma^\pm = \sigma(z \to \pm \infty) = \pm \frac{\sqrt{2\epsilon m^2}}{B\kappa} \]

of the potential (6) and representing two AdS\(_3\) spaces of equal cosmological constant \(\Lambda_\text{eff}^\pm = -2\epsilon m^2 < 0\).

3 Vacua and Domain Walls

All the constant \(\sigma\) solutions of eqs. (5) are determined by the real roots of the following algebraic equations: (a) \(W'(\sigma_a^\star) = 0\) and (b) \(W^2(\sigma_b^\star) = \frac{2\epsilon m^2}{\kappa^2}\), that describe (a part of) the matter potential \(V(\sigma)\) extrema. Each one of them defines an AdS\(_3\) space (i.e. one vacua solution of eqs. (5))

\[ ds^2 = dz^2 + e^{-2\epsilon \sqrt{\left|\Lambda^4_{\text{eff}}\right|}^2}(dx^2 - dt^2), \quad A = a, b \]

\(^4\)it represents an appropriate \(D = 3\) NMG adapted version of the Low-Zee superpotential [11] introduced in the context of DW’s solutions of \(D = 5\) Gauss-Bonnet improved gravity
of cosmological constant $\Lambda_{\text{eff}}^A = -\kappa^2 W^2(\sigma^*_A)$ as one can see by calculating the values of 3D scalar curvature:

$$R = -2\dddot{\varphi} - \frac{3}{2} \dot{\varphi}^2 \equiv 8\epsilon(W')^2 \left(1 - \frac{\kappa^2 W^2}{2\epsilon m^2}\right) - 6\kappa^2 W^2$$

(8)
i.e. $R_{\text{vac}} = -6\kappa^2 W^2(\sigma^*_A) = 6\Lambda_{\text{eff}}^A$. Hence the variety of admissible vacua of NMG model (1) is defined by the values of the extrema $\sigma^*$ of the matter potential $V(\sigma)$ and by the signs of the parameters $\epsilon$ and $m^2$. For example in the case of quadratic superpotential $W_2(\sigma) = B\sigma^2 + D$ for $B > 0$ and $D \neq 0$ we find one type (a) vacuum $\sigma^* = 0$ of cosmological constant $\Lambda_{\text{eff}}^A = -\kappa^2 D^2$ and for $\epsilon m^2 > 0$ few type (b) vacua given by

$$(\sigma^*_\pm)^2 = \pm \sqrt{\frac{2\epsilon m^2}{\kappa B}} - \frac{D}{B}, \quad (\sigma^*)^2 \leq (\sigma^*_\pm)^2$$

(9)
Depending on the range of values of $D$ (i.e. on the shape of potential $V(\sigma)$) we have: (1) no one type (b) vacuum for $D > \frac{2\epsilon m^2}{\kappa}$; (2) two type (b) vacua $\{\pm|\sigma^*_\pm|\}$ for $-\frac{\sqrt{2\epsilon m^2}}{\kappa} < D < \frac{\sqrt{2\epsilon m^2}}{\kappa}$ and (3) four type (b) vacua $\{\pm|\sigma^*_\pm|, \pm|\sigma^*_\pm|\}$ for $D < -\frac{\sqrt{2\epsilon m^2}}{\kappa}$.

Note that all the type (b) vacua have by construction equal cosmological constants $\Lambda_{\text{eff}}^A = -2\epsilon m^2$. We consider as an example the DW’s one can construct in the region (2) above, characterized by the three vacua $\pm|\sigma^*_\pm|$ and $\sigma^*_A = 0$. Then the two DW’s solutions of eqs. (5) connecting $|\sigma^*_\pm|$ (or $-|\sigma^*_\pm|$) with $\sigma^*_A$ have the following rather implicit form:

$$e^{\varphi - \varphi_0} = \left(\sigma^2 + \sqrt{\frac{2\epsilon m^2}{\kappa B}} - \frac{D\kappa}{2\epsilon m^2}\right)\left((\sigma^*_\pm)^2 - \sigma^2\right)^{-\alpha_+} \left((\sigma^*_\pm)^2 - \sigma^2\right)^{-\alpha_-} = e^{\frac{16\epsilon}{\kappa}(z - z_0)}$$

(10)
where we have denoted:

$$\alpha_+ = \left(1 - \frac{D\kappa}{\sqrt{2\epsilon m^2}}\right)^{-1}, \quad \alpha_- = \left(1 + \frac{D\kappa}{\sqrt{2\epsilon m^2}}\right)^{-1}.$$ Nevertheless one can easily verify that the corresponding asymptotics (at $z \to \pm \infty$) of $\sigma(z)$:

$$\sigma(z) \sim \pm \infty \sim \sigma^*_A - \sigma^0_A e^\pm \sqrt{2\kappa^2 W^2}, \quad \sigma(\infty) = \pm |\sigma^*_\pm|, \quad \sigma(-\infty) = \sigma^*_A = 0$$

$$\Lambda_{\text{eff}}^A = -2\epsilon m^2, \quad \Lambda_{\text{eff}}^A = -\kappa^2 D^2, \quad \Delta_A = 1 + \sqrt{1 - \frac{m^2(\sigma_A)}{\Lambda_{\text{eff}}^A}}, \quad m^2 = V''(\sigma^*_A)$$

(11)
indeed coincide with the vacuum data $(\sigma^*_A, \Lambda_{\text{eff}}^A, \Delta_A)$ that determine the boundary conditions for the DW solutions in region (2). Observe that the scale factor $e^{\varphi(z)}$ has different asymptotic behaviour depending on the sign of $D$: in the case of negative values, i.e. for $D \in \left(-\frac{\sqrt{2\epsilon m^2}}{\kappa}, 0\right)$ and $\epsilon = -1, m^2 \leq 0$, we find that:

$$e^{\varphi} \sim \pm \infty \sim e^{2\sqrt{2\kappa^2 W^2}} \to \infty, \quad e^{\varphi} \sim \pm \infty \sim e^{-2\sqrt{2\kappa^2 W^2}} \to \infty$$

(12)
while for $\epsilon = -1, m^2 < 0$ and considering positive values within the interval $D \in \left(0, \frac{\sqrt{2\epsilon m^2}}{\kappa}\right)$ we have that:

$$e^{\varphi} \sim \pm \infty \sim e^{2\sqrt{2\kappa^2 W^2}} \to \infty, \quad e^{\varphi} \sim \pm \infty \sim e^{-2\sqrt{2\kappa^2 W^2}} \to 0$$

(13)

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As it well known the divergences of the scale factor correspond to $AdS_3$ type of boundaries. The regions of vanishing scale factor (which are not curvature singularities) represent null Cauchy horizons, where the causal description in the Poincare patch terminates. Therefore our DW’s (10) define particular asymptotically $AdS_3 ((a)AdS_3)$ spaces: the one with $D < 0$ (see eqs. (12)) has two different boundaries and the other one with $D > 0$ (see eqs. (13)) has one boundary at $z \to \infty$ and one null horizon at $z \to -\infty$. Let us also mention that the linear superpotential DW’s (7) (and more general all the DW’s relating two type (b) vacua) in the case $\epsilon = -1$, $m^2 < 0$ describe $AdS_3$ spaces of two boundaries, but in this case of equal cosmological constants $\Lambda^b_{eff} = -2em^2$.

4 Unitarity and BF - conditions

The Bergshoeff-Hohm-Townsend (BHT) unitarity conditions [1]:

$$m^2 \left( \Lambda^A_{eff} - 2\epsilon m^2 \right) > 0$$

(14)

together with the Higuchi bound:

$$\Lambda^A_{eff} \leq M^2_{gr}(A) = -\epsilon m^2 + \frac{1}{2}\Lambda^A_{eff}$$

(15)

for the massive spin two field (“graviton”) are result of the requirement of perturbative (1 - loop) unitarity consistency of NMG model (1) for $\sigma = const$. When massive scalar field is also included one have to further impose the Breitenlohner-Freedman (BF) condition [14] which for $D = 3$ reads:

$$\Lambda^A_{eff} \leq m^2_{\sigma}(\sigma^*_A) = V''(\sigma^*_A)$$

(16)

or in its stronger form:

$$\Lambda^A_{eff} \leq m^2_{\sigma}(\sigma^*_A) < 0$$

(17)

It is convenient to parametrize the effective “vacuum masses” $m^2_{\sigma}(\sigma^*_A) = \kappa^2 W^2_A y_A(y_A - 2)$ in terms of the scaling dimensions $\Delta_A = 2 - y_A$ of 2D field $\phi_{\sigma}(x, t)$ “holographically dual” of $\sigma(z)$ [7] ($A = a, b$):

$$y_a = y(\sigma^*_a) = \frac{2\epsilon W''_A}{\kappa^2 W_A} \left( 1 - \frac{\kappa^2 W^2_A}{2\epsilon m^2} \right), \quad y_b = y(\sigma^*_b) = -\frac{4\epsilon(W'_b)^2}{\kappa^2 W^2_b}$$

(18)

Let us consider the case $\epsilon = -1$, $m^2 < 0$. Then the above unitarity conditions (14), (15), (17) take the following simple form

$$0 \leq \frac{\kappa^2 W^2_A}{2\epsilon m^2} \leq 2, \quad 0 \leq y_a < 2$$

(19)

which imposes restrictions on the values of the parameters of NMG model (1). For example, in the case of linear superpotential both vacua satisfy all the unitarity conditions only for specific values of the parameter $B$ such that: $B^2 \leq |m^2|$. In the particular case of quadratic superpotential of only three vacua $|\sigma^*_\pm|$ and $\sigma^*_a = 0$ (i.e. in region (2)) all the vacua are unitary and satisfy the weak BF condition, when the superpotential parameters are restricted as follows: $\kappa^2D^2 < 2em^2$ and $B > 0$. Therefore the corresponding DW’s (10) are interpolating between two unitary vacua. Such DW’s turns out to also have positive tensions $\tau_{DW} > 0$ as it shown in Sect. 6.
5 Domain Walls Tensions

In all the “planar” DW’s (2) of NMG model (1) the scalar matter is uniformly distributed (i.e. $\frac{\partial a}{\partial x} = 0$) along the whole $x$-axis and therefore such DW’s have infinite energy. As it well known [13], an important characteristics of the gravitational properties of such DW’s is given by the values of their energy densities $\epsilon_{DW} = \frac{E_{DW}}{L_x}$ (equals of their tensions $\tau_{DW}$). In the case of (a)AdS$_3$ geometries it is given by [15]:

$$\tau_{DW} = \lim_{L_x \to \infty} \frac{1}{L_x} \sum_{A=\pm} v_A \int_{-L_x/2}^{L_x/2} dx \xi^i T_{ij}^{(A)} \xi^j, \quad i,j = 0,1$$

(20)

where $A = \pm$ denote the two $z \to \pm \infty$ limits $(\partial M)_A$ describing (a)AdS$_3$ boundaries or/and horizons; $v_\pm = \pm 1$ and $\xi^\mu = (0, \xi^i)$ is time-like Killing vector, orthogonal to both $(\partial M)_A$ surfaces and normalized as $\xi_i^A \xi^i_A = -1$. The Brown-York “boundary” stress-tensor $T_{ij}^{(A)}$ [16] is defined as follows:

$$T_{ij}^{(A)} = -\frac{2}{\sqrt{-\gamma^A}} \frac{\delta S_{NMG}^{BY}}{\delta \gamma_{ij}^A} v_A$$

(21)

where $\gamma_{ij}^A$ are the corresponding “boundary/horizon” $\partial M_A$ - metrics:

$$\gamma^A_{ij}(x,t) = \lim_{z \to \pm \infty} \gamma_{ij}(x,t|z), \quad \gamma_{ij}(x,t|z) = e^{\phi(z)} \eta_{ij}, \quad \eta_{ij} = \text{diag}(+, -, -)$$

The main ingredient of the NMG version of the Brown-York formula (20) and (21) is the improved NMG action $S_{NMG}^{BY} = S_{NMG} + S_{gGH}$ with few “boundary” terms $S_{gGH}$ added. They represent an appropriate generalization of the Gibbons-Hawking boundary action to the case of NMG model (1) recently proposed by Hohm and Tonni [17]:

$$S_{gGH} = -\frac{2}{\kappa^2} \sum_{A=\pm} v_A \int_{(\partial M)_A} dx dt \sqrt{-\gamma} \left( \epsilon K - \frac{1}{2} f K + \frac{1}{2} f_{ij} K^{ij} \right)$$

(22)

where $K_{ij}$ is the extrinsic curvature of 2D “boundary” surface $(\partial M)_A$; $f_{\mu\nu}$ is the auxiliary Pauli-Fierz spin two field [1] whose “on-shell” form

$$f_{\mu\nu} = \frac{2}{m^2} \left( R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \right), \quad \mu; \nu = 0,1,2$$

is used in eq. (22); $f = \gamma^{ij} f_{ij}$ and $K = \gamma^{ij} K_{ij}$. In the case of DW’s (2) one can further apply the $I^{st}$ order equations (5) in order to derive the following simple “boundary” form of the improved action $^5$:

$$S_{NMG}^{BY}(DW) = -\frac{2}{\kappa} \sum_{A=\pm} v_A \int_{(\partial M)_A} dx dt \sqrt{-\gamma} W(\sigma) \left( 1 + \frac{\kappa^2 W^2(\sigma)}{2\epsilon m^2} \right)$$

(23)

Then according to the definitions (21) and (22) one easily obtains the corresponding explicit form of the “boundary” stress-tensor for the NMG model with scalar matter:

$$T_{ij}^{(A)}(DW) = -\frac{2}{\kappa} W(\sigma^*_A) \left( 1 + \frac{\kappa^2 W^2(\sigma^*_A)}{2\epsilon m^2} \right) \gamma_{ij}^A$$

(24)

$^5$unlike the case of the Einstein gravity where all the flat DW’s are of BPS type [18], for the 3D NMG DW’s one need to use the $I^{st}$ order eqs.(5) in order to prove that the remaining terms in the bulk action represent a total derivative.
which allows us to calculate the values of the DW’s tensions:

$$\tau_{DW} = \frac{2}{\kappa} \sum_{A=\pm} v_A W_A \left(1 + \frac{\kappa^2 W_A^2}{2\epsilon m^2}\right), \quad W_A = W(\sigma_A^*)$$  \hspace{1cm} (25)

Note that in the $m^2 \to \infty$ limit the above formula reproduces the well know results for a flat DW’s tensions in 3D Einstein gravity obtained by the Israel’s thin wall approximation [13].

6 Boundary counter-terms and central charges

Consider NMG model (1) in the limit of small effective cosmological constant: $L_A \gg G = l_{pl}$, where $|A_{eff}^A|L_A^2 = 1$. The $AdS_3/CFT_2$ correspondence suggests that each of its vacua $(\sigma_A^*, A_{eff}^A, \Delta_A^A)$ determine the main features of certain $CFT_2$, “living” on the corresponding 2-D boundaries/horizons $(\partial M)_A$ of $(a)AdS_3$ space-times (11). As in the case of 3D Einstein Gravity (i.e. the $m^2 \to \infty$ limit of (1)), all the 2D data, namely: central charges, scaling dimensions $\Delta_A(\sigma_A^*) = 2 - y_A$ and the vacuum expectation values

$$< A_{vac} | \hat{T}_{ij}^A(x_+, x_-) | A_{vac} > = -\frac{2}{\sqrt{-\gamma^A}} \frac{\delta S_{ren}^{NMG}(DW)}{\delta \gamma^A(x_+, x_-)} = T_{ij}^A + T_{ij}^{ct,A} = T_{ij}^{ren}(A), \quad x_\pm = x \pm t$$  \hspace{1cm} (26)

$$< A_{vac} | \hat{\Phi}_\sigma(x_+, x_-) | A_{vac} > , \text{ etc. of 2D operators } \hat{T}_{ij}^A \text{ and } \hat{\Phi}_\sigma, \text{ duals of the 3D NMG model fields } \gamma_{ij} \text{ and } \sigma - \text{ can be extracted from NMG classical action (23), appropriately renormalized: } S_{NMG}^{ren} = S_{NMG}^{BY} + S_{NMG}^{ct}, \text{ where}

$$S_{NMG}^{ct} = \frac{2}{\kappa} \sum_{A=\pm} v_A \int_{(\partial M)_A} dx dt \sqrt{\gamma^A} W(\sigma^*) \left(1 + \frac{\kappa^2 W^2(\sigma)}{2\epsilon m^2}\right)$$  \hspace{1cm} (27)

The particular form$^6$ of the boundary counter-terms (27) we have introduced above is a consequence of the condition that the vacua NMG solutions are conjectured to describe the vacua states $|A_{vac} > = |\sigma_A^*, A_{eff}^A >$ of corresponding (UV and IR) $CFT_2$’s, which by definition must have vanishing dimensions $\Delta_A^A = 0$ and energy $E_A^{vac} = 0$ (for planar 2D geometries), i.e.

$$< \hat{T}_{ij}^A(x_+, x_-) | A > = 0 = T_{ij}^A + T_{ij}^{ct,A}$$  \hspace{1cm} (28)

Note that $S^{ct}$ makes NMG action convergent, i.e. we have $S_{NMG}^{ren}(DW) = 0$, thus providing hints that such DW’s are stable.

The central charges $c_A$ of these $CFT_2$’s are given by the normalization constants of the stress-tensor’s 2-point functions

$$< \hat{T}_{\pm \pm}^A(x_\pm) \hat{T}_{\pm \pm}^A(0) > = \frac{c_A}{2x_\pm^4}$$  \hspace{1cm} (29)

or equivalently by the coefficients of the inhomogeneous part of the stress-tensor’s transformation laws:

$$< \delta \xi \hat{T}_{\pm \pm}^A(x_\pm) > = -\frac{c_A}{24\pi} \xi''_\pm$$  \hspace{1cm} (30)

$^6$although all our arguments are based on specific DW’s solutions of NMG model(1) it is expected that as in the Einstein gravity case, this form of the counter-terms is universal, i.e. it cancels the $S^{BY}_{NMG}$ divergences for larger class of solutions.
under infinitesimal 2D transformations: $x'_\pm = x_\pm + \xi_\pm(x_\pm)$. According to the Brown-Henneaux’s observation \[19\] the 3D counterparts of these 2D conformal symmetries are given by special 3D diffeomorphisms that keep invariant the asymptotic form of the $AdS_3$ metrics\(^7\) such that

$$\delta_\xi \gamma_{\pm\pm}(x_+, x_-) = -\frac{L^2}{2} \xi'''_\pm. \quad (31)$$

As a result the corresponding improved Brown-York stress-tensor $T^\text{ren}_{ij}(A)$ (proportional to $\gamma_{ij}$) gets inhomogeneous terms under these transformations:

$$\delta_\xi T^\text{ren}_{\pm\pm}(A) = \frac{L^2}{\kappa} W_A \left( 1 + \frac{\kappa^2 W_A^2}{2\epsilon m^2} \right) \xi''_\pm \quad (32)$$

which allows to calculate the $CFT_2$’s central charges in terms of the NMG vacuum data:

$$c_A = -\frac{3L^2_A}{2G} \kappa W_A \left( 1 + \frac{\kappa^2 W_A^2}{2\epsilon m^2} \right) \quad (33)$$

Consider for example the domain wall solution (10) for quadratic superpotential under the restriction $0 < \kappa D < \sqrt{2\epsilon m^2}$ and $B > 0$, which for $\epsilon = -1$ and $m^2 < 0$ interpolates between two vacua, i.e. $AdS_3$’s of effective cosmological constants $\Lambda_+ = -2\epsilon m^2$ and $\Lambda_a = -\kappa^2 D^2$ as one can see from its asymptotic form (13). Since in this case we have $W(\sigma) > 0$, the following identification $\kappa W_A = -\frac{\epsilon}{L_A}$ takes place. As a consequence the corresponding central charges of the two $CFT$’s representing these vacua get the familiar form \[1\],\[17\]:

$$c_A = -\frac{3\epsilon L_A}{2G} \frac{L_{gr}}{L_A^2} \left( 1 + \frac{L_{gr}^2}{L_A^2} \right), \quad L_{gr} = \frac{1}{2\epsilon m^2} \gg l_{pl}^2 \quad (34)$$

The same central charge formula turns out to be valid in the more general case of DW’s for which the superpotential $W(\sigma)$ does not change its sign between the two vacua $\sigma^*_A$ and for the case of “non-unitary” vacua with $\epsilon = 1$ and $m^2 > 0$ as well. It is worthwhile to mention an interesting fact that the DW’s tensions (25) can be rewritten in terms of the central charges as follows:

$$\tau_{DW}(L_+, L_a) = -\frac{1}{12\pi} \left( \frac{c_+}{L_+^2} - \frac{c_a}{L_a^2} \right) \quad (35)$$

Observe that the condition of positive tensions, i.e. $\tau_{DW}(L_+, L_a) > 0$ requires $|\Lambda_+| > |\Lambda_a|$ which is automatically satisfied in the example discussed above. In the case of ”unitary” BHT -vacua $\epsilon = -1$ and $m^2 < 0$ (i.e. for negative $c_A$’s ) this condition is equivalent to the following restriction on the central charges:

$$\left| \frac{c_a}{c_+} \right| > \frac{L_a^2}{L_+^2} > 1 \quad (36)$$

i.e. we have $c_+ > c_a$. It turns out that such “ordering” of the UV and IR central charges determines the direction of the RG flow in the dual 2D $pCFT_2$ as we are going to show in the next section.
7 Comments on holographic RG flows and NMG’s extensions

The off-critical \((a)AdS_5/pCFT_2\) version of the holographic principle relates certain static DW’s solutions of 3D gravity (Einstein or NMG) with scalar matter to the RG flows in specific deformed (supersymmetric) \(CFT_2\) [7]. These non-conformal \(QFT_2\)’s can be realized as an appropriate perturbations of the ultraviolet (UV) \(CFT_2\) by marginal or/and relevant operators \(\Phi_\sigma\) that break 2D conformal symmetry to the Poincare one:

\[
S_{pCFT_2}^{ren}(\sigma) = S_{CFT_2}^{UV} + \sigma(L_s) \int d^2x \sqrt{-g} \Phi_\sigma(x^i)
\]

The scale-radial duality [20] allows to identify the “running” coupling constant \(\sigma(L_*)\) of \(pCFT_2\) (37) with the scalar field \(\sigma(z)\) and the RG scale \(L_*\) with the scale factor \(e^{\bar{r}(z)}\) as follows: \(L_* = l_p e^{-\bar{r}/2}\). This identification is based on the equivalence of the “radial” evolution equations (5) and the Wilson RG equations for the \(pCFT_2\):

\[
\frac{d\sigma}{dl} = -\beta(\sigma) = \frac{2\epsilon}{\kappa^2} W'(\sigma) \left(1 - \frac{W^2(\sigma)\kappa^2}{2em^2}\right), \quad l = \ln L_*
\]

It is evident that the zeros of the \(\beta\)-function (38) \(\sigma_b^*\) coincide with the NMG vacuum of type \((a)\) (i.e. \(W'(\sigma_a^*) = 0\)) or of the type \((b)\) (i.e. \(W^1(\sigma_\pm) = \frac{2m^2}{\kappa^2}\)). We also realize that the anomalous dimensions \(\Delta_\Phi\) of the operator \(\Phi_\sigma(x^i)\) at each critical point:

\[
y(\sigma^*) = 2 - \Delta_\Phi(\sigma^*) = -\frac{d\beta(\sigma)}{d\sigma}\bigg|_{\sigma=\sigma^*}
\]

are nothing but the parameters \(\Delta_{a,b}^\Delta(\sigma^*)\) and \(y(\sigma_{a,b}^*)\) given by eqs. (18), that determine the asymptotic behaviour at \(z \to \pm\infty\) of the matter 3D bulk gravity field \(\sigma(z)\).

As it well known, when the explicit form of the \(\beta(\sigma)\) - function is given, say by eq. (38), it provides the key ingredient that allows to further derive the free energy and certain thermodynamical characteristics of 2D classical statistical model related (in its thermodynamic limit) to the quantum \(pCFT_2\) in discussion\(^8\). We are interested in the description of the scaling laws, critical exponents and the phase structure of particular \(pCFT_2\) dual to NMG model (1) with quadratic superpotential in the case the range of its parameters \(B\) and \(D\) belongs to the region (2). Following the standard RG methods (see for example [22]) we find that the singular part of the reduced free energy per 2D volume \(F_s(\sigma)\) has the following simple form:

\[
F_s(\sigma) \approx (\sigma^2)^{\frac{1}{y_0}} \left((\sigma_+^*)^2 - \sigma^2\right)^\frac{\nu_+}{y_0} \left(||(\sigma_-^*)^2| + \sigma^2\right)^\frac{\nu_-}{y_0}
\]

The critical exponents \(\nu_A = \frac{1}{y_A}\) related to the correlation length singularities \(\xi_A \approx (\sigma - \sigma^*_A)^{-\frac{1}{\nu_A}}\) at each critical point (i.e.the NMG’s vacua with \(A = 0, \pm\)) are given by:

\[
y_0 = -\frac{4\epsilon B}{D\kappa^2\alpha_+\alpha_-}, \quad y_+ = \frac{8\epsilon B}{\alpha_+\kappa\sqrt{2em^2}}, \quad y_- = -\frac{8\epsilon B}{\alpha_-\kappa\sqrt{2em^2}}
\]

For a particular choice of the “unitary” (for \(\epsilon = -1, m^2 < 0\) DW’s (10) and of the coupling constant \(\sigma\) within the range \(0 < \sigma < \sigma_+^*\) we have that \(y_0 < 0\) (i.e. the IR \(CFT_2\) with \(\Phi_\sigma\) as

\(^8\)indeed we have to consider the euclidean version of NMG such that the corresponding “boundaries/horizons” of \((a)H_3\) are flat euclidean planes or spheres \(S^2\).
irrelevant operator) and 0 < y_+ < 2 (i.e. the UV CFT_2 with Φ_σ representing now a relevant operator). Therefore such DW describes massless RG flow from UV critical point σ_+ to the IR one σ_0 = 0. An important characteristics of all the massless flows is the so called Zamolodchikov’s central function:

\[ C(σ) = -\frac{3}{2GκW(σ)} \left( 1 + \frac{κ^2W^2(σ)}{2em^2} \right) \]  

which at the critical points σ_+ takes the values (34). It represents a natural generalization [23] of the well known result for m^2 → ∞ limit [7],[20]. According to its original 2D definition [21] it is intrinsically related to the β-function:

\[ β(σ) = -\frac{4GεW(σ)}{3κ} \left( \frac{dC(σ)}{dσ} \right) \]  

of the pCFT_2 dual of the NMG model (1). Taking into account the RG equations (38) we realize that:

\[ \frac{dC(σ)}{dl} = -\frac{3}{4GW(σ)} \left( \frac{dσ}{dl} \right)^2 \]  

and therefore when W(σ) > 0 is positive (as in our example) the central function is decreasing during massless flow we are discussing, i.e. we have c_+ > c_a.

Observe that for σ > σ_+ and for σ → ∞ the correlation length remains finite due to the following “resonance” property \( \frac{1}{2y_0} + \frac{1}{y_+} + \frac{1}{y_-} = 0 \), specific for the quadratic superpotential we are studying. Hence this region of the coupling constant space corresponds to the massive phase of the pCFT_2, which is described “holographically” by singular DW metrics giving rise to (a)AdS_3 space-time with naked singularity, as one can see from the generic form (10) of our DW’s solutions.

We have therefore an example of phase transition from massive to massless phase that occurs at the UV critical point σ_+. For the description of such phase transition we need two different NMG solutions having coinciding boundary conditions (σ_+, A^A_{ij}, Δ_A) at their common boundary z → ∞.

We next briefly discuss the possibility to extend our d = 3 superpotential constructions (4) to the case of d > 3 NMG models:

\[ S = \frac{1}{κ^2} \int d^dx√g \left\{ R + \frac{1}{m^2} \left( R^{μν}R_{μν} - \frac{d}{4(d-1)}R^2 \right) - κ^2 \left( \frac{1}{2} |∇σ|^2 + V(σ) \right) \right\} \]  

As in d = 3 case the static flat DW’s solutions of such d-dimensional NMG model are defined by

\[ ds^2 = dz^2 + e^{2β(z)}η_{ij}dx^idx^j, \quad σ = σ(z), \quad β = \frac{1}{√2(d-1)(d-2)}, \quad α = (d-1)β, \]  

that leads us to a system of second order equations of the type (4), but with different d-dependent coefficients. It is then natural to introduce the following generalization of d = 3 NMG superpotential and of the I^st order system (5) for arbitrary d > 3 :

\[ \dot{φ} = -2καW(σ), \quad \dot{σ} = \frac{2}{κ} W'(σ) \left( 1 + \frac{κ^2 (d-4) W^2}{2(d-2) m^2} \right) \]  

\[ V(σ) = 2(W')^2 \left( 1 + \frac{κ^2 (d-4) W^2}{2(d-2) m^2} \right) - 2κ^2α^2W^2 \left( 1 + \frac{κ^2 (d-4) W^2}{4(d-2) m^2} \right) \]  

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which in \(d = 5\) case reproduces the Low-Zee superpotential [11] for the Gauss-Bonnet(GB) extended 5D gravity. It is worthwhile to mention the well known fact (see ref. [2] for example) that for \textit{conformally flat} solutions (i.e. of vanishing \(d > 3\) Weyl tensor as in the case of the DW’s (46)) the action of the Gauss-Bonnet-Einstein gravity becomes identical to the \(d > 3\) NMG’s one (45). Therefore the solutions of the eqs.(48) describe the flat static DW’s of the both models. The form of the eqs.(48) above makes evident that any given superpotential \(W_d(\sigma)\) describe qualitatively different matter potentials \(V_d(\sigma)\) depending on the values of \(d = 3, 4, 5\). Hence the properties of DW’s solutions of corresponding NMG model’s, as well as of the \(\beta_d(\sigma)\)—functions of their \(p\text{CFT}_d\) duals, are expected to be rather different depending on the space-time dimensions. We leave the problem of the identification of these \(QFT_d\) models and of the geometrical NMG’s description of their phase structure to our forthcoming paper [25].

Let us emphasize in conclusion the advantages of the superpotential method in the study of the DW’s properties of the NMG models (45) as well as of the holographic RG flows in their dual \(p\text{CFT}_d\) models. As we have shown on the example of 3D NMG model (1) with quadratic \(W(\sigma)\), the DW’s solutions provide an important information about the phase transitions in its dual 2D model. It is important to note however that although we have recognized many of the ingredients of the \(AdS_3/CFT_2\) correspondence as central charges, scaling dimensions, free energy, etc. the answer to the question of whether and under which conditions such correspondence takes place for the NMG model (1) remains still open. The complete identification and the description of all the properties of the dual \(p\text{CFT}_2\) in terms of the NMG model’s solutions requires better understanding of the apparent “unitarity discrepancy” that relates (1-loop) \textit{unitary} massive 3D gravity to \textit{non-unitary} \(CFT_2\)’s of negative central charges in the approximation of small effective cosmological constants. Negative central charges are known to appear in different contexts in the (supersymmetric) \(CFT_2\)’s. For example, the classical and semi-classical limits \(\hbar \to 0\) of the central charges \(c_q = 1 - 6 \frac{Q^2}{\hbar}\) of the so called minimal \textit{unitary} Virasoro algebra models (as well as of their \(N = 1\) SUSY extensions) are big \textit{negative} numbers [24]. There exist also families of non-unitary 2D models representing interesting statistical mechanical problems, as for example the Lee-Yang “edge singularity” \(CFT_2\) of \(c = -\frac{22}{3}\). We remind these facts just to indicate few directions for further investigations that might result in the exact identification of the 2D QFT’s duals to 3D New Massive gravity models.

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\textbf{References}

[1] E.A. Bergshoeff, O. Hohm and P.K. Townsend.,Phys. Rev. Lett.\textbf{102}, 201301(2009);  
Phy. Rev. \textbf{D79} 124042/2009

[2] E.A. Bergshoeff, O. Hohm and P.K. Townsend,Gravitons in Flatland, arXiv:1007.4561.

[3] S.Deser,Phys. Rev. Lett.\textbf{103}, 101302(2009).

[4] I.Oda, JHEP0905,064 (2009).
[5] G. Clement, Class. Quant. Grav. 26, 105015 (2009), Phys. Rev. Lett. 102, 201301 (2009); E. Ayon-Beato, G. Giribet, M. Hassaine, JHEP 0905, 029 (2009); J. Oliva, D. Tempo and R. Troncoso, JHEP 0907, 011 (2009); H. Ahmedov and A. N. Aliev, The general type N solutions of New Massive Gravity, arxiv:1008.0303.

[6] H. L. C. Louzada, U. Camara dS and G. M. Sotkov, Phys. Lett. 686 B (2010) 268.

[7] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, Adv. Theor. Math. Phys. 3: 363-417 (1999).

[8] J. Maldacena, Adv. Theor. Math. Phys. 2: 231-252 (1998).

[9] E. Witten, Adv. Theor. Math. Phys. 2: 253 (1998).

[10] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. 428 B (1998) 105.

[11] I. Low and A. Zee, Naked Singularity and Gauss-Bonnet term in Brane world scenarios, NPB 585, 395-401 (2000).

[12] D. Z. Friedman, C. Nunez, M. Schnabl, K. Skenderis, Phys. Rev. D 69, 014027 (2004);
M. Cvetic, S. Griffies, S. J. Rey, Nucl. Phys. B 381, 1992301

[13] M. Cvetic and H. H. Soleng, Phys. Rep. 282 (1997) 159 and references therein.

[14] P. Breitenlohner and D. Z. Freedman, PLB 115, 1982, 1197;
Ann. Phys. 144 (1982) 249.

[15] J. D. Brown and J. W. York, Phys. Rev. D 47, 1407 (1993).

[16] V. Balasubramanian and P. Kraus, Commun. Math. Phys. 208 (1999) 413.

[17] O. Hohm and A. Tonni, JHEP 1004:093 (2010).

[18] K. Skenderis and P. K. Townsend, Gravitational stability and Renormalization group flow, PLB 468, 1999, 46

[19] J. D. Brown and M. Henneaux, Commun. Math. Phys. 104, 207 (1986).

[20] J. de Boer, Fortsch. Phys. 49: 339-358 (2001), hep-th/0101026;
E. Verlinde, H. Verlinde and J. de Boer, JHEP 0008:003 (2000), hep-th/9912012.

[21] A. B. Zamolodchikov, Sov. Phys. JETP Lett. 43 (1986) 1731;
Sov. J. Nucl. Phys. 46 (1987), 1090.

[22] G. Mussardo, Statistical Field Theory, Oxford University Press Inc., New York, 2010.

[23] A. Sinha, JHEP 1006:061 (2010).

[24] V. Fateev and S. Lukyanov, Sov. Sci. Rev. A. Phys. Vol. 15 (1990), pp. 1-117 (see pg. 18);
G. Sotkov and M. Stanishkov, NPB 356, (1991), 439.

[25] U. Camara dS, C. P. Constantinidis and G. M. Sotkov, Domain Walls and Holographic RG flows in D-dimensional New Massive Gravity (in preparation).