Non-critical NSR string field theory
and
discrete states interaction in 2D supergravity.

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Abstract

String field theory for the non-critical NSR string is described. In particular it gives string field theory for the 2D super-gravity coupled to a \( c = 1 \) matter field. For this purpose double-step pictures changing operators for the non-critical NSR string are constructed. Analogues of the critical supersymmetry transformations are written for \( D < 10 \), they form a closed on-shell algebra, however their action on vertices is defined only for discrete value of the Liouville momentum. For \( D=2 \) this means that spinor massless field has its superpartner in the NS sector only if its momentum is fixed.

Starting from string field theory we calculate string amplitudes. These amplitudes for \( D=2 \) have poles which are related with discrete set of primary fields, namely 2R→2R amplitude has poles corresponding to the \( n \)-level NS excitations with discrete momenta \( p_1 = n, \quad p_2 = -1 \pm (n + 1) \).

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1 Introduction

The goal of this paper is to present a string field theory associated with the continuum super-Liouville theory. We will use the free field representation [1, 2] and the modified Witten string field theory [3, 4, 5].

String field theory is supposed to be background independent and, therefore, to give a framework for discussing nonperturbative effects. Considerable progress in understanding non-perturbative effects was achieved for 2-dimensional gravity coupled to a $c = 1$ matter within the frameworks of two approaches to this problem: matrix models [6] and the continuum Liouville theory [7, 8, 9]. It is thus of interest to construct a string field theory in the context of these models as well as their super generalizations. For bosonic case the string field theory associated with matrix models was constructed in [10] and with the continuum Liouville theory in [12]. String field theory for $D < 26$ non-critical bosonic string was presented in [13].

2D string theories are characterized by the existence of a ring of discrete primary fields [14]. Discrete states have found in matrix model calculations [15, 16] and continuum calculations of tachyon amplitudes [7, 12, 17, 18] and are related with non-trivial cohomology of the BRST charge [19, 20, 21].

One of the outstanding nonperturbative problems of string theory is that of supersymmetry breaking. It is therefore of great interest to formulate lower dimensional model of superstrings. Supermatrix models are discussed in [22].

We investigate string states in both NS sector and R sector in the context of string field theory. According to rough estimations, in the NSR string after the GSO projection there is only one degree of freedom which corresponds to massless spinor field. However a host of extra discrete states appears in the careful analysis of the quantization procedure in the light-cone gauge. This analysis shows that in the NS sector there are only discrete states and in the R sector there are massless spinor field and discrete states. So one gets an asymmetrical spectrum (the massless spinor and 2D vector field cannot form a supermultiplet) and an operator of supersymmetry does not exist. Nevertheless analogues of the critical supersymmetry transformations can be written for $D < 10$, they form a closed on-shell algebra, however their action on vertices are defined only for discrete value of the Liouville momentum. In this sense spinor state with fixed value of momentum $|R; p_1 = 0, p_2 = -1>$ has its superpartner in the bosonic sector, it is the discrete state $|NS; p_1 = 0, p_2 = 0>$. In the bosonic sector we have tachyon field (massless field after the redefinition of momentum) however this field should be excluded after the GSO projection. It would be interesting to find the corresponding discrete set of states in the matrix model in the super case.

String field theory provides a systematic method for calculation of states interaction. In particular, the string field action gives an action for the massless spinor field, which is a fermionic analogue of the Das-Jevicki-Polchinski (DJP) action. One cannot say that this action is a superpartner of the DJP action since the DJP action describes the tachyon action and the tachyon should be excluded after the GSO projection.

String field theory gives also a regular method for calculation of amplitudes. We construct 4-points amplitudes in the different sectors. RR→RR amplitude has poles which correspond to the on-shell NS discrete states with momentum-energy $(p_1, p_2)$ equal to $p_1 = n, p_2 = -1 \pm (n + 1), \quad (n = 0, 1, 2, \ldots)$. 


The paper is organized as follows. First, we review some well-known facts about non-critical NSR string. Then we present an explicit formula for picture-changing operators. These operators will allow us to examine string states in different pictures. For completeness we briefly present the light-cone analysis of spectrum and in more details we investigate questions connected with supersymmetry. We then present string field action and discuss how fermionic analogue of the Das-Jevicki-Polchinski action can be extracted from it. We end with the derivation of general formulae for N-point on-shell spinor amplitudes and the detail discussion of properties of the 4-point on-shell amplitudes.

2 Non-critical NSR string.

2.1 Notations

In the free field representation \cite{1, 2} the first quantized non-critical NSR string action in the conformal gauge has the form

\[
S = -\frac{1}{8\pi} \int d^2\zeta \sqrt{\hat{g}} (\hat{g}^{\alpha\beta} \partial_\alpha X_\mu \partial_\beta X_\mu + i \hat{R} Q_\mu X_\mu - \frac{i}{2} \bar{\psi}_\mu \gamma^\alpha D_\alpha \psi_\mu + \text{ghosts})
\]  

Here \( \zeta = (\zeta_1, \zeta_2) \) are world-sheet coordinates, \( \hat{g} \) is a background world-sheet metric with curvature \( \hat{R} \), \( Q_\mu = (0, ..., 0, Q) \) is a background charge, \( D_\alpha \) is covariant spin derivative; \( X_\mu \) and \( \psi_\mu (\mu = 1, 2, ..., D) \) are \( D \)-dimensional matter fields \( X_\mu = (X_i, \varphi) \), and \( \psi_\mu = (\psi_i, \psi) \), \((i = 1, ..., D - 1)\), where \( X_i \) and \( \psi_i \) are embedding bosonic and fermionic coordinates and \( \varphi \) and \( \psi \) are the superLiouville modes. Signature of the \( D \)-dimensional metric is chosen as \((+ + \cdots + -)\).

The conformal energy-momentum tensor for such theory reads

\[
T(z) = T^X(z) + T^\psi(z) + T^{bc}(z) + T^{b\gamma}(z),
\]

where \( T^X \) and \( T^\psi \) are energy-momentum tensors for \( X_\mu \) and \( \psi_\mu \) fields

\[
T^X = -\frac{1}{2} \partial X_\mu \partial X_\mu - \frac{i}{2} Q_\mu \partial^2 X_\mu,
\]

\[
T^\psi = \frac{1}{2} \bar{\psi}_\mu \partial \psi_\mu
\]

and \( T^{bc} \) and \( T^{b\gamma} \) are energy-momentum tensors of the spin \((2,-1)\) \((bc)\)-ghosts and the spin \((3/2,-1/2)\) \((b\gamma)\)-ghosts,

\[
T^{bc} = -2b \partial c - \partial bc,
\]

\[
T^{b\gamma} = -\frac{3}{2} \beta \partial \gamma - \frac{1}{2} \partial \beta \gamma.
\]

In all that follows we shall use the standard bosonization of the superconformal ghosts \( \beta \) and \( \gamma \) (see \cite{23})

\[
\gamma = \eta e^\phi, \quad \beta = e^{-\phi} \partial \xi.
\]

The energy-momentum tensors for fields \( \phi \) and \( \eta \xi \) have the forms

\[
T^\phi = -\frac{1}{2} \partial \phi \partial \phi - \partial^2 \phi
\]
\[
T_{\kappa\lambda} = \partial \xi \eta. \tag{9}
\]

It is useful to write expressions for generators of the superconformal transformations. We have
\[
F^{X\psi} = -\frac{1}{2} (\psi_{\mu} \partial X_{\mu} + iQ_{\mu} \partial \psi_{\mu}) \tag{10}
\]
and
\[
F^{bc,\beta\gamma} = -c \partial \beta - \frac{3}{2} \partial c\beta + \frac{1}{2} \gamma b \tag{11}
\]
for \(x\psi\)- and \(bc\beta\gamma\)- systems respectively. Superconformal invariance relates the background charge \(Q_{\mu}\) to the dimension \(D\) of the target space as
\[
D - 2Q^2 - 10 = 0 \tag{12}
\]

The BRST charge has the form
\[
Q_{BRST} = \oint \frac{dz}{2\pi i} j(z),
\]
\[
\begin{aligned}
j(z) &= c(T^X + T^\psi + \frac{1}{2}T^{bc} + T^{\xi\eta} + T^\phi)(z) - \eta e^\phi F^{X\psi}(z) + \frac{1}{4} b \partial \eta e^{2\phi}(z).
\end{aligned}
\tag{13}
\]

### 2.2 Picture-changing operators

Now let us write the expressions for the picture-changing operators for the non-critical case. The first pair of picture-changing operators \(X(z)\) and \(Y(z)\) can be constructed in standard manner as for the critical case \([3, 24]\). The operator \(X(z)\) is represented as a BRST commutator in a large superconformal ghost algebra (\(\xi\)-algebra) and has the form
\[
X(z) = \{Q_{BRST}, \xi(z)\} =
\]
\[
= \frac{1}{2} e^\phi (\psi \cdot \partial X + iQ \cdot \partial \psi)(z) + c \partial \xi(z) + \frac{1}{4} b \partial \eta e^{2\phi}(z) + \frac{1}{4} \partial (b \eta e^{2\phi})(z). \tag{14}
\]

The inverse operator \(Y(z)\) has the same form as in the critical case
\[
Y(z) = 4c \partial \xi e^{-2\phi}(z) \tag{15}
\]

The double-steep picture-changing operators \(W(z)\) and \(Z(z)\) \([4]\)
\[
Z(z)W(z) = 1, \ Y(z)W(z) = X(z), \ X(z)Z(z) = Y(z), \tag{16}
\]

which suppose to be BRST invariant primary conformal fields with zero conformal dimension can also be found explicitly
\[
W(z) =: X^2 : (z) =
\]
\[
- \frac{19}{192} \{Q_{BRST}, \partial b \partial e^{2\phi}(z)\} - \frac{5}{48} \{Q_{BRST}, \partial^2 b e^{2\phi}(z)\} - \frac{1}{96} \{Q_{BRST}, b \partial^2 e^{2\phi}(z)\}, \tag{17}
\]
\[
Z(z) = -4e^{-2\phi}(z) - \frac{5}{8} c \partial \xi e^{-3\phi}(\psi \cdot \partial X + iQ \cdot \partial \psi)(z). \tag{18}
\]

The properties \([16]\) can be easy verified by OPE technique.
3 String states.

3.1 NS sector.

In this section we consider string states in the bosonic sector of the non-critical NSR string theory. As in the critical case such states can be constructed for different choices of superconformal ghost vacuum (different pictures).

Let us remind that the superconformal ghost vacuum $|\nu\rangle$ in $\nu$-picture is connected with $SL_2$-invariant vacuum $|0\rangle$ as

$$|\nu\rangle = e^{\nu \phi(0)} |0\rangle .$$

There are two traditional pictures ($\nu = -1$ and $\nu = 0$) in which NS states are constructed.

For the critical theory the GSO condition is imposed on string states in both NS and R sectors. This condition excludes one half of states and leads to the theory with space-time supersymmetry. We must impose the GSO condition in the non-critical case also in order to remove boson states with odd fermion number. Vertex operators corresponding to the lowest level states in $\nu = -1$ and $\nu = 0$ pictures have the form

$$O_{-1}(z) = \zeta_\mu ce^{-\phi \psi_\mu} e^{ikX(z)},$$

$$O_0(z) = \zeta_\mu \left[ \frac{1}{2} c(\partial X_\mu + ik \cdot \psi \psi_\mu) - \frac{1}{4} \eta e^{\phi \psi_\mu} \right] e^{ikX(z)},$$

where polarization vector $\zeta_\mu(k)$ satisfies the on-shell conditions

$$(k^2 + k \cdot Q)\zeta_\mu = 0,$$

$$(k + Q) \cdot \zeta = 0.$$

It is not difficult to verify that $O_{-1}(z)$ and $O_0(z)$ are BRST invariant primary conformal fields with zero conformal dimension.

Let us note that the lowest level vertex operators (20) and (21) are nothing but the on-shell NS string field theory states in the Siegel gauge. In field theory the string states appear as a result of decomposition of string field. In the NSR string field theory for the free case one can work in one of these pictures. For the free theory these pictures are equivalent and lead to the same on-shell conditions on the component space-time fields [29]. However at the presence of an interaction the different choices of pictures lead to different theories. It is known that the NS string field theory in the $(−1)$-picture leads to divergent tree level amplitudes [25] and one can construct the Chern-Simons-like superstring field theory only in the $(0)$-picture [4]. In the first quantized approach it is more suitable to deal with vertex operators in the $(−1)$-picture and put the necessary number of the picture-changing operators on external states.

Let us consider decomposition of the NS string fields in both pictures. Having in mind the GSO condition and using the correspondence between states and vertex operators we write

$$|A_\nu\rangle = A_\nu(0)|0\rangle ,$$

(24)
where

\[ A_{-1}(z) = \int dk[A_\mu(k)ce^{-\phi}\psi_\mu + B(k)\partial cc\partial \xi e^{-2\phi} + \text{higher level terms }]e^{ikX(z)}, \]

\[ A_0(z) = \int dk[\Phi(k)c + \frac{1}{2}A_\mu(k)\partial X_\mu - \frac{1}{4}B_\mu(k)\eta e^\phi\psi_\mu + \frac{1}{2}F_\mu(\psi_\mu\psi_\nu + B(k)\partial c + F(k)\partial \phi + \text{higher level terms }]e^{ikX(z)}. \]

It is important to stress that off-shell string fields (25) and (26) cannot be transformed in each other. However the mass-shell equation \( \{ Q_{BRST}, A_\mu(z) \} = 0 \) leads to the same conditions on the component fields for any picture:

\[ (k^2 + k \cdot Q)A_\mu - ik_\mu B = 0, \quad i(k + Q) \cdot A + B = 0, \]

\[ A_\mu = B_\mu, \quad F_{\mu\nu} = \frac{1}{2}ik_{[\mu}A_{\nu]}, \quad F = 0, \quad \Phi = 0. \] (27)

The higher order excitation can be considered in the similar way, however the expressions for vertex operators are more complicate.

We are going make some comments about the specific case \( D = 2 \). It is instructive to recall the situation for the 2D bosonic string. Naively one can expect that in \( D = 2 \) case after gauge fixing survives only the tachyonic state. However on mass-shell survive also the states with discrete value of the momentum [7, 19, 12, 20, 21]. These states appear as a poles in the correlation functions [8, 17] and in the scattering amplitudes [4, 12, 18] and are related with non-trivial cogomology of BRST charge [13, 20, 21]. Let us sketch the appearance of these discreet states in light-cone gauge.

As usual [23] one introduces

\[ p^\pm = \frac{1}{\sqrt{2}}(p_1 \pm p_2), \quad \alpha^\pm = \frac{1}{\sqrt{2}}(\alpha_{n1} \pm \alpha_{n2}). \] (28)

The mode expansion of the Virassoro operators corresponding to the momentum-energy tensor \( T(z) \) [2] has the form

\[ L_m = P^+(m)\alpha^-_m + P^-(m)\alpha^+_m + \sum_{k \neq 0, m} \alpha^+_{m-k}\alpha^-_k \] (29)

for \( m \neq 0 \), where

\[ P^+(m) = p^+ + \frac{Q}{2\sqrt{2}}(m + 1) \] (30)

\[ P^-(m) = p^- - \frac{Q}{2\sqrt{2}}(m + 1) \] (31)

and

\[ L_0 = p^+ p^- - \frac{Q}{2\sqrt{2}}(p^+ - p^-) + \sum_{k \neq 0} \alpha^+_{-k}\alpha^-_k. \] (32)

As in the usual case (\( Q = 0, \ D = 26 \)), to perform the quantization in the light-cone gauge one has to fix \( \alpha^+_m = 0 \) for \( m = \pm 1, 2, ... \), then solve the constraints

\[ L_m = 0, \quad m \neq 0 \] (33)
For \( D=2 \) case there is some subtlety in applying this procedure. The solution of the constraints (33) becomes dependent of the values of momentum and energy. If \( P^+(m) \) are not equal to zero for all \( m \neq 0 \), then after fixing \( \alpha^+ = 0 \) the constraints (33) imply \( \alpha^- = 0 \) for all \( m \neq 0 \). In this case one gets the single lowest scalar state which corresponds to the so-called tachyon.

If there is such \( m = m_0 > 0 \) that \( P^+(m_0) = 0 \) one can again fix \( \alpha^-_{m_0} = 0 \) and constraints (33) allow \( \alpha^-_{m_0} \neq 0 \). To perform the quantization we have to assume that the corresponding conjugated variables \( \alpha^+_{-m_0} \) is also non-zero, \( [\alpha^+_{-m_0}, \alpha^+_{m_0}] = m_0 \). The constraints \( L_m \vert \text{phys. states} > = 0 \) for \( m \neq m_0 \) are trivially satisfied. One can guarantee the condition

\[
L_{m_0} \vert \text{phys. states} > = [P^+(m_0)\alpha^-_{m_0} + P^-(m_0)\alpha^+_{m_0}] \vert \text{phys. states} > = 0 \tag{34}
\]
on states of the form

\[
\vert \text{phys. states} > = (\alpha^+_{-m_0})^k \vert p > \tag{35}
\]
with \( p \) such that the condition

\[
P^+(m_0) = p^+ + \frac{Q}{2\sqrt{2}}(m_0 + 1) = 0 \tag{36}
\]
is satisfied. For bosonic case \( Q = 2\sqrt{2} \). The equation

\[
(L_0 - 1) \vert \text{phys. states} > = 0 \tag{37}
\]
with \( L_0 \) being (32) fixes the values of \( p^- \) in (33)

\[
p^- = k + 1. \tag{38}
\]
So, the momentum and energy in (34) take the following values:

\[
p^{(I)}_1 = -\frac{1}{\sqrt{2}}(m_0 - k), \quad p^{(I)}_2 = -\sqrt{2} - \frac{1}{\sqrt{2}}(m_0 + k). \tag{39}
\]

We labelled this series of states by upper index \((I)\).

The case when \( P^-(m_0) = 0 \) for some \( m_0 \) can be considered analogously. The condition (34) holds for all \( m > 0 \) on the states

\[
\vert \text{phys. states} > = (\alpha^-_{-m_0})^k \vert p > \tag{40}
\]
Equation (37) gives momenta of the states (40)

\[
p^{(II)}_1 = \frac{1}{\sqrt{2}}(m_0 - k), \quad p^{(II)}_2 = -\sqrt{2} - \frac{1}{\sqrt{2}}(m_0 + k). \tag{41}
\]
Comparing (39) and (41) one can see that these series have the same energy and opposite momentum. So, they describe the right and left moving excitations with the same dispersion. It is amusing to note that for the values of \( p \) as in equations (39) \( P^-(k) = 0 \) holds.

Equation (34) is enough for vanishing of matrix elements of \( L_{-m_0} \). To see this, let us remind the hermitian conjugation rules for operators \( p \) and \( \alpha^\pm_m \): \( p^*_1 = p_1, \quad p^*_2 = -p_2 - Q \),
holds. Taking into account (39) and (41) for dual states we have
\[ p \rightarrow p' \]

Here \( Q \) or \( M \) case due to superconformal invariance one can assume that all components \( \alpha_+ \) or \( \alpha_- \) are equal to zero. In light-cone notations \( \psi_s^+ = \sqrt{2}(\psi_{1s} \pm \psi_{2s}) \) one can write the super Virasoro generators in the form

\[
L_m = P^+(m)\alpha_m^+ + P^-(m)\alpha_m^- + \sum_{k \neq 0, m} \alpha_{m-k}^+ \alpha_k^- + s(s + \frac{1}{2})\psi_{m-s}^+ \psi_s^-
\]

\[
F_s = \frac{1}{2}[K^+(s)\psi_s^- + K^-(s)\psi_s^+ + \sum_{m \neq 0} (\psi_{s-m}^+ \alpha_m^- + \psi_{s-m}^- \alpha_m^+)]
\]

Here \( Q = 2, P^\pm(m) \) are defined by (30), (31) and

\[
K^\pm(s) = p^\pm \pm \frac{Q}{\sqrt{2}}(s + \frac{1}{2}).
\]

If both \( P^\pm(m) \) and \( K^\pm(s) \) are non-zero for all \( m > 0 \) and \( s > 0 \) then in the light-cone gauge \( \alpha_m^+ = \psi_s^+ = 0 \) one has only trivial solution of constraint equations \( L_m = F_s = 0 \):

\[
\alpha_m^- = 0, \quad m \neq 0; \quad \psi_s^- = 0
\]

The solution (47) corresponds to a string state, which contains only the lowest level scalar field.

However, if there is a such \( m_0 > 0 \) that \( P^+(m_0) = 0 \) then we have a non-trivial solution \( \alpha_{m_0}^+ \neq 0 \). For \( P^-(m_0) = 0 \) then we have a non-trivial solution \( \alpha_{m_0}^- \neq 0 \). Analogously, if there is a such \( s = s_0 \) that

\[
K^\pm(s_0) = p^\pm \pm \frac{Q}{\sqrt{2}}(s_0 + \frac{1}{2}) = 0
\]

then we have \( \psi_{s_0}^\pm \neq 0 \).

The \( m_0 \)- and \( s_0 \)-modes produce the following excitations:

\[
(\alpha_{-m_0}^\pm)^k|p>,
\]

\[
\psi_{s_0}^\pm(\alpha_{-m_0}^\pm)^k|p>,
\]

The states (50) assume that the conditions (30) (or (31)) and (48) hold simultaneously, hence here \( s_0 \) and \( m_0 \) should be related as

\[
\frac{1}{2}(m_0 + 1) = s_0 + \frac{1}{2} \quad \text{or} \quad m_0 = 2s_0
\]
Note that in the traditional (-1)-picture the states \((\mathbb{R})\) and lowest level scalar state must be exclude by the GSO projection, since they describe the states with incorrect statistic. In (0)-picture only the states of form \((\mathbb{R})\) survive after the GSO projection. Remind also that in our notations string fields in the NS sector assume to be fermionic.

The mass-shell equation

\[(L_0 - \frac{1}{2})|\text{phys. states}> = 0\]  

for physical states \((50)\) gives the value of \(p^- (p^+):\)

\[p^\pm = \pm \sqrt{2}(1 + k).\]  

So, one can see that in the NS sector after the GSO projection on the \(n\)-mass level \((n = m_0k + s_0 - \frac{1}{2} = m_0(k + \frac{1}{2}) - \frac{1}{2})\) only states with discrete momenta

\[p^{(I)}_1 = \frac{1}{2} + \frac{1}{2}(2k - m_0), \quad p^{(I)}_2 = -\frac{3}{2} - \frac{1}{2}(2k + m_0);\]  

\[p^{(II)}_1 = -\frac{1}{2} - \frac{1}{2}(2k - m_0), \quad p^{(II)}_2 = -\frac{3}{2} - \frac{1}{2}(2k + m_0)\]  

survive.

As for bosonic case we must add to the states \((50)\) the dual states

\[<p'_1, p'_2 | (\alpha^{\pm}_{m_0})^k \psi_{s_0}^\mp\]

with

\[p'^{(I)}_1 = \frac{1}{2} + \frac{1}{2}(2k - m_0), \quad p'^{(I)}_2 = -\frac{3}{2} + \frac{1}{2}(2k + m_0);\]  

\[p'^{(II)}_1 = -\frac{1}{2} - \frac{1}{2}(2k - m_0), \quad p'^{(II)}_2 = -\frac{3}{2} + \frac{1}{2}(2k + m_0).\]

### 3.2 R sector

Now we consider string states in the fermionic sector of the non-critical NSR string. As in case of the critical string such states are describes by spin field vertex operators. Such operators are expressed for even \(D\) in terms of anticommuting world-sheet fields \(\psi_\mu(z)\) when \(\psi_\mu(z)\) is represented in bosonized form (see [26])

\[\psi_{2j-1} \pm i\psi_{2j} = e^{\pm \varphi_j} c_j.\]  

Here \(\varphi_j(z)\) \((j = 1, ..., \frac{D}{2})\) are free bosons and \(c_j\) are co-cycle operators, which are, roughly speaking, Jordan-Wigner factors necessary to ensure that different fermions, when written in bosonized form, anticommute. The spin fields are represented as

\[S_\alpha(z) = e^{\lambda_\alpha \cdot \varphi} c_\alpha,\]

where \(\lambda_\alpha\) denotes an \(\frac{D}{2}\)-component spinor weight of form \(\lambda_\alpha = \frac{1}{2}(\pm, ..., \pm)\). There are \(2^\frac{D}{2}\) possibilities, the correct number for \(SO(D)\)-spinor. One can decompose spin field \(S_\alpha\) on
two chirality components, which corresponds to two irreducible representation of \( SO(D) \): its positive (no dot on \( \alpha \) ) and negative (dot on \( \alpha \) ) chirality components.

Spin fields are operators with the conformal dimension \( d(S_\alpha) = \frac{D}{16} \). As in the critical \( D = 10 \) case we can consider two chiral component of \( S_\alpha \) multiplied by superghost factor \( e^{-\frac{\phi}{2}} \)

\[
S_\alpha e^{\pm \frac{\phi}{2}} \quad \text{and} \quad S_\alpha e^{\pm \frac{\phi}{2}}.
\]

One can consider \( S_\alpha e^{-\frac{\phi}{2}} \) as a candidate on fermion vertex. Conformal dimension of this operator is

\[
d = \frac{D}{16} + \frac{3}{8} = \frac{1}{8} Q^2 + 1.
\]

To obtain the operator with conformal dimension one we need to multiply it by operator with \( d = \frac{1}{8} Q^2 \). This may be done by multiplying \( S_\alpha e^{\frac{\phi}{2}} \) on \( e^{i q X} \) with \( q \) satisfying

\[
\frac{1}{2} q (q + Q) = -\frac{1}{8} Q^2,
\]

\[
(q + \frac{1}{2} Q)^2 = 0.
\]

The BRST invariant spin operator which corresponds to state at lowest mass level has the form

\[
O_{\frac{1}{2}}(z) = \zeta^\alpha ce^{-\frac{\phi}{2}} S_\alpha e^{i q X}(z)
\]

and corresponds to \( \nu = -\frac{1}{2} \) picture. The condition \( \{ Q_{BRST}, O_{\frac{1}{2}} \} = 0 \) implies the mass-shell condition

\[
(q + \frac{1}{2} Q) \gamma_\mu^\alpha \gamma_\beta \zeta^\beta = 0
\]

on polarization spinor \( \zeta^\alpha \). Eq. (63) is the Dirac-like equation for the lowest level R string state.

Acting on \( O_{\frac{1}{2}} \) by picture-changing operator \( X(z) \) one gets the BRST invariant lowest level spin operator in \( \frac{1}{2} \)-picture.

Shifting \( q \to q - \frac{1}{2} Q \) we get massless spinor equation. Hence, at the lowest level we have the massless spinor field. At higher levels there are spin-vector fields. Let us examine these higher excitations for \( D = 2 \) case. As in the case of the bosonic string and the NS sector naively one can expect that these spin-vector states can be removed by gauge transformation. However, these states can be removed only if their momentum are not equal to some special values and once again we get a set of discrete states.

Since we are interesting about spin-vector fields with fermionic statistic we assume that only some \( \alpha \)-operators survive after gauge fixing and solving the constraints equation, i.e. the situation is similar to the case of bosonic string. Physical space is spanned by

\[
(\alpha_{-m_0}^+)^k|\beta,p>, \quad \text{or} \quad (\alpha_{-m_0}^-)^k|\beta,p>
\]

(here \( \beta \) in \( |\beta,p> \) is spinor index) with the condition (66) on \( p^+ \), or \( P^- (m_0) = 0 \) on \( p^- \).

The equation \( L_0 |\text{phys. states} > = 0 \) implies

\[
p^+ p^- - \frac{Q}{2\sqrt{2}} (p^+ - p^-) - \frac{Q^2}{8} = -km_0
\]
So, momenta of the states (64) are given by
\[ p_1^{(I)} = \frac{1}{2} (2k - m_0), \quad p_2^{(I)} = -1 - \frac{1}{2} (2k + m_0); \]  
\[ p_1^{(II)} = -\frac{1}{2} (2k - m_0), \quad p_2^{(II)} = -1 - \frac{1}{2} (2k + m_0). \]  
\[ (66) \]
\[ (67) \]
Hence, in the R sector we have massless spinor fields as well as a set of spin-vector field living only with fixed values of momentum and energy.

Series of discrete states dual to (64) are
\[ p_1'^{(I)} = \frac{1}{2} (2k - m_0), \quad p_2'^{(I)} = -1 + \frac{1}{2} (2k + m_0). \]  
\[ p_1'^{(II)} = -\frac{1}{2} (2k - m_0), \quad p_2'^{(II)} = -1 + \frac{1}{2} (2k + m_0). \]  
\[ (68) \]
\[ (69) \]

4 Supersymmetry

We are going to examine the existence of supersymmetry in the non-critical NSR string theory. Remind that in the critical theory supersymmetry takes place after the GSO condition is imposed. Then at each mass level the number and the masses of states in the NS sector are equal to the number and the masses of states in in the R sector. The situation in the non-critical theory is different. Here equal level states in the NS and R sectors have different masses. Consider for example the lowest level states in both sectors. After redefinition of momentum
\[ p = k + \frac{1}{2} Q \]  
the mass-shell conditions on polarization vector \( \zeta_\mu \) and spinor \( \zeta_\alpha \) look as following
\[ (p^2 - \frac{1}{4} Q^2) \zeta_\mu = (p + \frac{1}{2} Q) \cdot \zeta = 0, \]  
\[ (71) \]
\[ p^2 \zeta_\alpha = (\hat{p} \zeta)_\alpha = 0. \]  
\[ (72) \]
Hence the lowest excitation in the NS sector is massive while the lowest excitation in the R sector is massless. At \( n \) level one has the following mass-shell condition
\[ p^2 + 2n - \frac{1}{4} Q^2 = 0 \]  
\[ (73) \]
in the NS sector and
\[ p^2 + 2n = 0 \]  
\[ (74) \]
in the R sector.

Hence, it is impossible to construct a linear local transformation between the NS and R sectors which acts separately at each level. So, an analog of the critical NSR supersymmetry transformation in the non-critical NSR theory does not exist.

This can be also clarified by following. Let us consider the fermion vertex with positive chirality components which is obtained by the GSO projection. As in the critical theory such
vertex can be chosen as a candidate for NS-R-symmetry operator. The fermion vertex with zero momentum has fractional conformal dimension and cannot be used as a such operator. In order to obtain a spin operator with conformal dimension one it is necessary to take a fermion vertex with momentum \( q = -\frac{1}{2}Q \). Then a candidate for NS-R-symmetry operator can be written as

\[
\Sigma_{-\frac{1}{2}} = \oint \frac{dz}{2\pi i} \Theta^\alpha e^{-\frac{\phi}{2}} S_\alpha e^{iqX}(z).
\] (75)

and

\[
[Q_{BRST}, \Sigma_{-\frac{1}{2}}] = 0.
\] (76)

The fact that momentum of the fermionic vertex is non-zero has rather non-trivial consequences. First, this means that NS-R transformations is given by a non-local operator that can be expected by comparing (71) and (72). Secondly, the transformation (75) has meaning on R vertices with the momenta being a subject of some restrictions. In particular, for the lowest level R vertex

\[
O_{-\frac{1}{2}}(z) = \zeta^\alpha c e^{-\frac{\phi}{2}} S_\alpha e^{ikX}(z)
\] (77)

this restriction has the form

\[
k \cdot Q = -\frac{1}{2}Q^2.
\] (78)

Indeed, acting by (77) on the vertex (77) we get

\[
[\Sigma_{-\frac{1}{2}}, O_{-\frac{1}{2}}(z)] =
\]

\[
= \oint \frac{dw}{2\pi i} \Theta^\alpha \zeta^\beta (ce^{-\phi} \psi_{\alpha\beta} e^{ikX}) (w) (w - z)^{-\frac{D}{8} + \frac{1}{4} + kq} + \text{less singular terms .}
\] (79)

In order to get a meaningful expression similar to the NS vertex we must have the first order pole in the first term in (79), so, taking into account (12), we must set

\[
q \cdot k = \frac{1}{4}Q^2
\] (80)

This equation fixes the Liouville component \( k_D \) of momentum \( k \) in the R sector

\[
k_D = -\frac{1}{2}Q.
\]

The expression (79) with condition (80) being imposed, becomes the NS vertex in (-1)-picture. The momentum in (79) after redefinition \( p = k + q + \frac{1}{2}Q \) satisfies the equation

\[
p^2 = \frac{1}{4}Q^2.
\] (81)

In D=2 case this means that the transformation (75) relates the NS lowest level excitation and the R lowest level excitation with fixed value of the momentum. All the 0-level R states (except the state with \( p_2 = -\frac{1}{2}Q \)) have not their own partners in the NS sector.

Acting by the operator (75) on the lowest level NS vertex in the (0)-picture (21) we see that in order to reproduce the structure of the R vertex we must demand

\[
k \cdot Q = 0 \quad \text{or} \quad k_D = 0
\] (82)
on states in the NS sector. The operator (73) together with their picture-changing version

\[ \Sigma_1^\pm = \{ Q_{BRST}, \int \frac{dz}{2\pi i} \Theta^\alpha \xi e^{-\frac{\alpha}{2}^2} S_\alpha e^{iqX}(z) \} \]  

form the following algebra

\[ [\Sigma_1^+, \Sigma_1^-] = \Theta^\alpha \Theta^\beta \frac{1}{2} \int \frac{dz}{2\pi i} (\partial \hat{X} - iQ \cdot \psi \hat{\psi})_{\alpha\beta} e^{-iQX}. \]  

(84)

To summarize the above discussion, we saw that although the non-critical analogs of the SUSY transformations form the closed algebra being the natural generalization of the algebra of supersymmetry, they have sense on vertices with fixed "energy".

5 Non-critical NSR string field action

Now we can write a NSR string field theory action for the non-critical theory. In fact, all of ingredients of the modified Witten version of open superstring field theory, including an associative * product can be constructed for non-critical case. The form of action is the same as for the critical NSR superstring [4]

\[ S_{NSR} = < (IA)(\infty)Z(i)Q_{BRST}A(0) > + \frac{2}{3} < Z(z_0)(hA)(z_1)(hA)(z_2)(hA)(z_3) > + \]

\[ + < (I\Psi)(\infty)Y(i)Q_{BPST}\Psi(0) > + 2 < Y(z_0)(h\Psi)(z_1)(hA)(z_2)(h\Psi)(z_3) > . \]  

(85)

Here I is the map \( z \rightarrow z' = -\frac{1}{\pi} \) from the inside of unit disk to the outside and \( h \) is the map from interaction three-string Witten’s configuration to upper half-plane. The NS string field A and the R string field \( \Psi \) are built under vacuum with picture \( \nu = 0 \) and \( \nu = -\frac{1}{2} \) respectively.

The action (85) is gauge invariant. The form of gauge transformation is the same as for the critical NSR superstring theory. However in our case the theory is not supersymmetric in the usual sense. As it was discussed in section 4 the absence of supersymmetry takes place due to the fact that equal level excitations in the NS and R sectors have different masses.

Let us make some comments about an action for local fields which can be extracted from the action (85). The free action for the lowest right (left) excitation \( \psi_{\pm}(x) \) in the 2D R sector which comes from the first term of the second line in (85) has the form

\[ S_{0,\psi} = \int d^Dx e^{-2ix^\mu \psi_{\pm}(x)(i\partial_1 \pm i\partial_2 \mp 1)\psi_{\pm}(x)}. \]  

(86)

As to the free action for lowest lying fields in the NS sector there the situation is more complicate since there are auxiliary fields (compare with the critical case [29]).

The string field action (85) contains an interaction between massless spinor and higher level fields. Since the action (85) does not contain the terms describing \( \psi^4 \) interaction, \( \psi^4 \) term comes from the interaction between massless spinor and others fields. Contributions to \( \psi^4 \) term can been calculated using off-shell conformal methods or level truncation approximation [30]. For example the \( g^2 \)-order contributions can be written schematically as

\[ S_{int,\psi} = g^2 \int Y \Psi * \Psi * \frac{b_0}{L} X(\Psi * \Psi) - g^2 \int Y \Psi * \Psi * \frac{b_0}{L} W \frac{b_0}{L} Y Q(\Psi * \Psi). \]  

(87)
6 Tree amplitudes

We shall employ the formalism of string field theory to get string amplitudes. The Liouville degree of freedom make the Witten vertex BRST invariant, and the \(*\) product associative. These features will then guarantee duality and tree-level unitarity (factorisation).

Let us note that principles of calculation of tree amplitudes are the same as in the critical superstring field theory. Remind the main steps of this calculations [27, 28]. To invert the kinetic operator we must fix the gauge. A convenient choice is the Siegel gauge

\[ b_0 A = b_0 \Psi = 0 \]

where \( b_0 \) is the zero mode of \( b(z) \). The propagators in this gauge are

\[ \Delta_{NS} = \frac{b_0}{L} Q_{BRST} W(i) \frac{b_0}{L} \]

in the NS sector and

\[ \Delta_R = \frac{b_0}{L} Q_{BRST} X(i) \frac{b_0}{L} \]

in the R sector. With each Feynman graph a string configuration is associated. External string are semi-infinite rectangular strips of width \( \pi \). The effect of \( \frac{1}{L} = \int_0^\infty d\tau e^{-\tau L} \) in propagators is to introduce a world-sheet strip of width \( \pi \) and length \( \tau \). The interaction glues the strip end together in pairwis manner.

It may be shown that contribution to a particular Feynman graph is

\[ A_N = (\prod_{i=1}^{N-3} \int d\tau_i) \prod_{r=1}^{N} O^{(r)}(w_r) \prod_{i=1}^{N-3} \int dw'_i b(w'_i) \prod_j Z(w''_j) > R_r, \quad (88) \]

Where \( Z_j(w''_j) \) are appropriate picture-changing operator insertions. The correlation function (88) is considered on the string configuration \( R_\tau \) described above.

Analysis of field theory tree level amplitudes (88) for the non-critical case is completely the same as for the critical theory [4]. The result is that all tree-level amplitudes are finite and can be presented in the first quantized language.

6.1 Four-fermion amplitude

The amplitude for four lowest level states in the R sector can be represented as

\[ A_{4F} = \int_{z_4}^{z_2} dz_3 < O^{(1)}_{-\frac{1}{2}}(z_1) O^{(2)}_{-\frac{1}{2}}(z_2) V^{(3)}_{-\frac{1}{2}}(z_3) O^{(4)}_{-\frac{1}{2}}(z_4) > \quad (89) \]

Here \( O_{-\frac{1}{2}}(z) \) are vertex operators in \((-\frac{1}{2})\)-picture which have the following form

\[ O^{(r)}_{-\frac{1}{2}}(z_r) = eV^{(r)}_{-\frac{1}{2}}(z_r) = \zeta^{(r)\alpha} c e^{-\frac{\phi}{2}} S_\alpha e^{i k^{(r)} X(z_r)}. \quad (90) \]

The polarization spinors \( \zeta^{(r)\alpha} \) satisfy the mass-shell condition

\[ (k^{(r)} + \frac{1}{2}Q)^2 \zeta^{(r)\alpha} = (k^{(r)} + \frac{1}{2}Q)^\alpha \zeta^{(r)\beta} = 0 \quad (91) \]
The points $z_r$ are arbitrary and lie on real axis so that $z_1 > z_2 > z_3 > z_4$. Usual choice is $z_1 = \infty$, $z_2 = 1$, $z_4 = 0$. It is not difficult to verify that the correlation function (89) may be represented in the form

$$A_{4F} = \int_0^1 dx x^{k(4)-\frac{1}{2}Q^2-1} (1-x)^{k(2)-\frac{1}{2}Q^2-1} K(\zeta^{(r)}, k^{(r)}, x),$$

(92)

where function $K$ is depended on polarization spinors $\zeta^{(r)}$, external momentums $k^{(r)}$ and is linear under $x$.

One can introduce $s$- and $t$- channel variables

$$s = (k^{(1)} + k^{(2)}) + \frac{1}{2}Q^2, \quad t = (k^{(2)} + k^{(3)}) + \frac{1}{2}Q^2.$$

(93)

On mass-shell $s$ and $t$ variables can be written as

$$s = 2k^{(1)}k^{(2)} - \frac{1}{4}Q^2, \quad t = 2k^{(2)}k^{(3)} - \frac{1}{4}Q^2$$

(94)

Taking into account the relation $k^{(1)}k^{(2)} = k^{(3)}k^{(4)}$ one can represent the amplitude (92) in the form similar to the well known form of fermionic 4-point amplitude

$$A_{4F} = \int_0^1 dx x^{\frac{1}{2}s - \frac{1}{8}Q^2 - 1} (1-x)^{\frac{1}{2}t - \frac{1}{8}Q^2 - 1} K(\zeta^{(r)}, k^{(r)}, x)$$

(95)

To study a pole structure of $A_{4F}$ we set simply $K = 1$. Then

$$A_{4F} \sim \frac{\Gamma\left(\frac{1}{2}s - \frac{1}{8}Q^2\right)\Gamma\left(\frac{1}{2}t - \frac{1}{8}Q^2\right)}{\Gamma\left(\frac{1}{2}s + \frac{1}{2}t - \frac{1}{4}Q^2\right)}$$

(96)

The four-fermion amplitude has $s$- and $t$- poles which correspond to the excitations in the NS sector:

$$s = \frac{1}{4}Q^2 - 2n, \quad t = \frac{1}{4}Q^2 - 2n, \quad (n = 0, 1, 2, \ldots)$$

(97)

For $D = 2$ case due to the special kinematical relations this amplitude can be presented as a function of the individual external momentum. Indeed, the mass-shell condition for external states,

$$k^{(r)}_2 = -1 + e_r k^{(r)}, \quad e_r = \pm 1$$

(98)

and the momentum-energy conservation low

$$\sum_{r=1}^4 k^{(r)}_1 = 0, \quad \sum_{r=1}^4 k^{(r)}_2 = -Q$$

(99)

in the particular case $e_i = 1$, $i = 1, 2, 3$, $e_4 = -1$, (three right-moving and one left-moving fermions) give

$$k^{(4)}_1 = -1, \quad k^{(1)}_1 + k^{(2)}_1 + k^{(3)}_1 = 1,$$

and

$$s = -1 - 2(k^{(1)}_1 + k^{(2)}_1), \quad t = -1 - 2(k^{(2)}_1 + k^{(3)}_1),$$

(100)
and therefore

\[ A^{4F} = \frac{\Gamma(-2 + k_1^{(3)})\Gamma(-2 + k_1^{(1)})}{\Gamma(-3 - k_1^{(2)})} \]  

(100)

This amplitude has the s-poles which correspond to the state with fixed momentum and energy

\[ k_1 = n, \quad k_2 = -1 \pm (n + 1), \quad n = 0, 1, \ldots \]  

(101)

Comparing (101) with formulas (55) and (58) for \( k = 0 \) one can see that these poles correspond to a part of the set of discrete states in the NS sector:

\[ \psi_{-m + \frac{1}{2}}^-(k_1, k_2) > \quad \text{and} \quad \psi_{-m + \frac{1}{2}}^+(k_1, -k_2 - Q) > . \]  

(102)

Note that all scattering amplitudes for the fermions of the same chirality (i.e. only right-moving or left-moving) are forbidden by the momentum-energy conservation law. Therefore after the GSO projection we left with trivial \( N \)-point fermionic amplitudes.

### 6.2 Two-boson-two-fermion amplitude

To calculate the scattering amplitude of two lowest level bosons and two lowest level fermions we consider the following four vertex correlation function

\[ A_{2B2F} = \int_{z_4}^{z_2} dz_3 < O_0^{(1)}(z_1)O_{-1}^{(2)}(z_2)V_{-1}^{(3)}(z_3)O_{-1}^{(4)}(z_4) > \]  

(103)

where

\[ O_{-1}^{(2)}(z_2) = \zeta_\mu^{(2)} c e^{-\phi} \psi_\mu e^{ik^{(2)}X}(z_2) \]  

(104)

\[ O_0^{(1)}(z_1) = \zeta_\mu^{(1)} \left[ \frac{1}{2} c (\partial X_\mu + i k^{(1)} \cdot \psi_\mu) - \frac{1}{4} \eta e^{\phi} \psi_\mu \right] e^{ik^{(1)}X}(z_1), \]  

(105)

\[ O_{-1}^{(4)}(z_4) = c V_{-1}^{(3)}(z_3) \]  

is given by (74). The polarization spinors \( \zeta^{(j)\alpha}, j = 3, 4 \) and the polarization vectors \( \zeta^{(i)}_\mu, i = 1, 2 \) satisfy to the mass-shell conditions (91) and

\[ ((k^{(i)} + \frac{1}{2} Q)^2 - \frac{1}{4} Q^2) \zeta^{(i)}_\mu = (k^{(i)} + Q) \mu \zeta^{(i)}_\mu = 0 \]  

(106)

respectively.

After the OPE calculations one gets

\[ A_{2B2F} = \int_0^1 dx x^{k^{(3)}k^{(4)} - \frac{1}{2} Q^2 - 1} (1 - x)^{k^{(2)}k^{(3)} - 1} K(\zeta^{(r)}, k^{(r)}, x) \]  

(107)

The on-shell amplitude (107) being written in the \( s \)- and \( t \)- channel variables

\[ s = 2k^{(1)}k^{(2)} + \frac{1}{4} Q^2, \quad t = 2k^{(2)}k^{(3)}. \]  

(108)

due to the kinematical relation \( k^{(1)}k^{(2)} = k^{(3)}k^{(4)} - \frac{1}{4} Q^2 \), has the form

\[ A_{2B2F} = \int_0^1 dx x^{\frac{1}{2} s - \frac{1}{2} Q^2 - 1} (1 - x)^{\frac{3}{2} t - 1} K(\zeta^{(r)}, k^{(r)}, x) \sim \]
\[
\Gamma\left(\frac{1}{2}s - \frac{1}{8}Q^2\right)\Gamma\left(\frac{1}{2}t\right)
\]
\[
\Gamma\left(\frac{1}{2}s + \frac{1}{2}t - \frac{1}{8}Q^2\right)
\]
\(\sim\) (109)

This amplitude has \(s\)- and \(t\) poles at

\[s = \frac{1}{4}Q^2 - 2n, \quad t = -2n, \quad (n = 0, 1, 2, \ldots)\] (110)

For the D=2 case the amplitude (110) does not contain poles connected with discrete states (compare with 2-vector-2-tachyon scattering amplitude for the bosonic case [12]).

### 6.3 Four boson amplitude

Four boson scattering amplitude

\[A_{4B} = \int_{z_1}^{z_2} dz_3 < O^{(1)}_0(z_1)O^{(2)}_{-1}(z_2)V^{(3)}_{-1}(z_3)O^{(4)}_{0}(z_4) >\] (111)

can be present in the form

\[A_{4B} = \int_0^1 dx k^{(3)}k^{(4)}k^{(2)}k^{(3)}\frac{1}{4}Q^2 - 1(1 - x)k^{(2)}k^{(3)}\frac{1}{4}Q^2 - 1 K(\zeta^{(r)}, k^{(r)}, x)\] (112)

In \(s\)- and \(t\)-variables this amplitude can be rewritten as

\[A_{4B} = \int_0^1 dx \frac{1}{2}s - \frac{1}{8}Q^2 - 1(1 - x)\frac{1}{2}t - \frac{1}{8}Q^2 - 1 K(\zeta^{(r)}, k^{(r)}, x).\]

Dropping the kinematical factor \(K\) one get

\[A_{4B} \sim \frac{\Gamma\left(\frac{1}{2}s - \frac{1}{8}Q^2\right)\Gamma\left(\frac{1}{2}t - \frac{1}{8}Q^2\right)}{\Gamma\left(\frac{1}{2}s + \frac{1}{2}t - \frac{1}{4}Q^2\right)}\] (113)

This amplitude has \(s\)- and \(t\)-poles at

\[s = \frac{1}{4}Q^2 - 2n, \quad t = \frac{1}{4}Q^2 - 2n, \quad (n = 0, 1, 2, \ldots)\] (114)
A APPENDIX. OPE for spin fields for non-critical dimension

The operator product expansion of two $S_\alpha$ can be written as

$$S_\alpha(z)S_\beta(w) = \frac{1}{(z-w)^{D/2}} C_{\alpha\beta} +$$

$$+ \frac{1}{(z-w)^{D/2}} \gamma_{\mu}^{\alpha} \psi_\mu(w) + \frac{1}{(z-w)^{D/2} - 1/2} \gamma_{\mu}^{\alpha} \gamma_{\nu}^{\beta} \gamma_{\nu}^{\beta} \psi_\mu(w) + \ldots. \quad (115)$$

The coefficients $c_{\alpha\beta}$ and $\gamma_{\alpha\beta}$ are defined by (115). The indices $\alpha$ and $\dot{\alpha}$ correspond to $\lambda_\alpha$ with even and odd number of minuses respectively. Situation will be different in dependences of would be $\frac{D}{2}$ even or not.

If $\frac{D}{2}$ is even, the leading singularity in (115) will appear when indices $\alpha$ and $\beta$ correspond to same chirality. Then we have

$$S_\alpha(z)S_\beta(w) = \frac{1}{(z-w)^{D/2}} C_{\alpha\beta} + O((z-w)^{D/2+1}),$$

$$S_\alpha(z)S_{\dot{\beta}}(w) = \frac{1}{(z-w)^{D/2} - 1/2} \gamma_{\mu}^{\alpha} \psi_\mu(w) + O((z-w)^{D/2+1/2}). \quad (116)$$

In this case the $2^{\frac{D}{2}-1}$ operators $S_\alpha$ and $2^{\frac{D}{2}-1}$ operators $S_{\dot{\alpha}}$ are transformed are the two different spinor representation of $SO(D)$.

When $\frac{D}{2}$ is odd, we have

$$S_\alpha(z)S_\beta(w) = \frac{1}{(z-w)^{D/2}} C_{\alpha\dot{\beta}} + O((z-w)^{D/2+1}),$$

$$S_\alpha(z)S_{\dot{\beta}}(w) = \frac{1}{(z-w)^{D/2} - 1/2} \gamma_{\mu}^{\alpha} \psi_\mu(w) + O((z-w)^{D/2+1/2}) \quad (117)$$

and $S_\alpha$ and $S_{\dot{\alpha}}$ are transformed as isomorphic representations.

Formulas (116) and (117) define the charge-conjugation matrix $C$, which reduces to the spinor metric. For even $\frac{D}{2}$ $C$ split into two spinor metrics $C_{\alpha\beta}$ and $C_{\dot{\alpha}\dot{\beta}}$. For odd $\frac{D}{2}$ we have $C_{\alpha\dot{\beta}} = -C_{\dot{\beta}\alpha}$. We remind that in chiral basis the gamma matrices have one upper and one lower index, one dotted and one undotted. For any $\frac{D}{2}$ indices may be raised and lowered with the appropriate spinor metric.
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