Accelerated Adomian Decomposition Method for the System of Nonlinear Equations

Prince Singh
Department of Mathematics, Lovely Professional University, Phagwara, Punjab, India
Email: princesingh16092@gmail.com

Abstract. A novel technique i.e. Accelerated Adomian Decomposition method is used for the solution of system of nonlinear equations. This method has been explored by including a new form of Adomian polynomial for handling the nonlinear terms present in the equations. The technique has been tested for some examples and the outcome exhibit reliability and efficiency of the proposed methods.

1. Introduction
In computational mathematics, we deconstruct, dissect and derive techniques for solving problems arising from real-world application. Apart from polynomial equations, there are numerous problems that incorporate the transcendental function. Numerical methods are frequently used to obtain estimated solutions to such problems because it is beyond the realm to obtain precise solutions by analytical methods. This is the reason, iterative and semi-analytical techniques for the solution of such equations come into the picture. The construction of iterative and semi-analytical techniques for solving non-linear equations or systems of nonlinear equations are an attractive task in numerical analysis. In the past few years, iterative techniques have been used in numerous assorted fields such as economics, engineering, physics, dynamic models, and so on.

In 1980, Adomian [1-3] introduced an efficient semi-analytical technique for solving various types of nonlinear equations (like algebraic, transcendental, ordinary and partial differential equations) containing nonlinearities. Recently, Adomian decomposition method (ADM) [4–7] and its modified versions have been successfully employed in a variety of problems. This technique was initially implemented by Adomian for finding the solution of some algebraic and transcendental equations. Subsequently, many authors studied the theoretical and practical aspects of ADM and its modified versions [8–11]. In the present work, we have used the application of Adomian decomposition method for the solution of system of nonlinear equations. Here we used a new form of Adomian polynomial which gives us better approximation to the solution of nonlinear system of algebraic and transcendental equations.

2. Accelerated Adomian Decomposition Method
To elucidate the proposed technique, consider the following system of nonlinear equations.
\[
\begin{align*}
    f_1(x_1, x_2, x_3, \ldots, x_n) &= 0; \\
    f_2(x_1, x_2, x_3, \ldots, x_n) &= 0, \\
    \vdots & \\
    f_n(x_1, x_2, x_3, \ldots, x_n) &= 0.
\end{align*}
\]

(*)

These system of equations (*) can be rewritten as
\[ x_1 = c_1 + N_1(x_2, x_3, ..., x_n); \]
\[ x_2 = c_2 + N_2(x_1, x_3, ..., x_n); \]
\[ \vdots \]
\[ x_n = c_n + N_n(x_1, x_2, ..., x_{n-1}); \]  
(1)

where \( c_i, i = 1, 2, 3, ..., n \) are constants, while \( g_i \) be the nonlinear function. In this method, we assume the solution in the form of infinite series as
\[ x_i = \sum_{k=0}^{\infty} x_{ik} \]  
(2)

Where \( x_{i0} = c_i \) and nonlinear terms can be calculated using new form of Adomian polynomial namely Accelerated Adomian polynomial given by
\[ \tilde{A}_{ik} = N_i(S_k) - \sum_{j=1}^{k-1} \tilde{A}_{ij}, k \geq 1 \]  
(3)

Where \( \tilde{A}_0 = N_1(x_{10}, x_{20}, ..., x_{n0}) \), and \( S_k = (x_{i0} + x_{i1} + x_{i2} + ... + x_{ik}) \) substituting (2) and (3) in equation (1), the given system reduces to
\[ \sum_{k=0}^{\infty} x_{ik} = c_i + \sum_{k=0}^{\infty} \tilde{A}_{ik} \]  
(4)

From eq. (4), we get the values of \( x_i \) as
\[ x_{i0} = c_i \]
\[ x_{i1} = A_0 \]
\[ x_{i2} = \tilde{A}_1 \]
\[ \vdots \]

Hence the solution of the system is given by
\[ x_i = x_{i0} + x_{i1} + x_{i2} + ... \]

3. Condition of convergence of Accelerated ADM

In this method, we assume \( x = \sum_{i=0}^{\infty} x_i \) be the series solution of the nonlinear equation \( f(x) = 0 \), where \( f(x) = 0 \) can be rewritten as \( x = c + N(x) \), where \( N(x) \) represents the nonlinear function of \( x \). \( N(x) = \sum_{i=0}^{\infty} \tilde{A}_i \), where \( \tilde{A}_i \) represents the Accelerated Adomian polynomial.
\[ \sum_{i=0}^{\infty} x_i \] is a contraction mapping, Here \( x_0 = c, x_1 = \tilde{A}_1, x_2 = \tilde{A}_2, ... \)
Consider the partial sum \( s_n = x_1 + x_2 + x_3 + ... + x_n = \sum_{i=0}^{n} \tilde{A}_i = N(s_{n-1}) \)
Let \( s_{n+1} = N(x_0 + s_n) \), where \( s_0 = 0 \).

In general, \( s = N(x_0 + s) \), if \( N \) is a contraction mapping then by fixed point theorem, the partial sum \( s_n \) converges to the unique solution ‘s’.

**Theorem 3.1:** If \( N \) is a contraction mapping, then the sequence \( s_{n+1} = N(x_0 + s_n) \), converges to the solution \( s \). The convergence result has been proved by the author in paper [13].

4. Application of Accelerated ADM

**Example 1:** Consider the following system of equations[9]:
\[ x_1^2 - 10x_1 + x_2^2 + 8 = 0 \]
\[ x_1 x_2^2 + x_1 - 10 x_2 + 8 = 0 \] (5)

On applying Accelerated ADM, we have

\[ x_1 = \sum_{k=0}^{\infty} x_{1k} = \frac{8}{10} + \frac{1}{10} x_1^2 + \frac{1}{10} x_2^2 + \frac{1}{10} x_1^2 \]

\[ = \frac{8}{10} + \frac{1}{10} \sum_{k=0}^{\infty} \tilde{A}_{1k} + \frac{1}{10} \sum_{k=0}^{\infty} \tilde{A}_{11k} \] (6)

where \( \sum_{k=0}^{\infty} \tilde{A}_{1k} = x_1^2 = (\sum_{k=0}^{\infty} x_{1k})^2 \) and \( \sum_{k=0}^{\infty} \tilde{A}_{11k} = x_2^2 = (\sum_{k=0}^{\infty} x_{2k})^2 \)

\[ x_2 = \sum_{k=0}^{\infty} x_{2k} = \frac{8}{10} + \frac{1}{10} x_1 x_2^2 + \frac{1}{10} x_1 \]

\[ = \frac{8}{10} + \frac{1}{10} \sum_{k=0}^{\infty} \tilde{A}_{2k} + \frac{1}{10} \sum_{k=0}^{\infty} \tilde{A}_{21k} \] (7)

where \( \sum_{k=0}^{\infty} \tilde{A}_{2k} = x_1 x_2^2 = (\sum_{k=0}^{\infty} x_{1k})(\sum_{k=0}^{\infty} x_{2k})^2 \) and \( \sum_{k=0}^{\infty} \tilde{A}_{21k} = x_1 = \sum_{k=0}^{\infty} x_{1k} \)

In eq.(6) and (7), \( \tilde{A}_{1k}, \tilde{A}_{11k}, \tilde{A}_{2k} \) and \( \tilde{A}_{21k} \) represents Accelerated Adomian polynomial. From eq.(6) and (7) and using eq.(3) to calculate the Accelerated Adomian polynomial term, we have

\[ \begin{align*}
x_{10} & = 0.8, x_{20} = 0.8 \\
x_{11} & = 0.128, x_{21} = 0.1312 \\
x_{12} & = 0.04483174400, x_{22} = 0.04206998323 \\
x_{13} & = 0.01653386224, x_{23} = 0.01616511203 \\
x_{14} & = 0.006417004814, x_{24} = 0.006358558335 \\
x_{15} & = 0.002536189848, x_{25} = 0.002526909168 \\
x_{16} & = 0.001009636525, x_{26} = 0.001008156241 \\
x_{17} & = 0.0004030840207, x_{27} = 0.0004028474758 \\
\end{align*} \]

Hence the approximate solution of eq.(5) is given by

\[ \begin{align*}
x_1 & = x_{10} + x_{11} + x_{12} + \cdots + x_{17} \approx 0.9997315214 \\
x_2 & = x_{20} + x_{21} + x_{22} + \cdots + x_{27} \approx 0.9997315665 \\
\end{align*} \]

which is closed to the exact solution \( x_1 = 1 \) and \( x_2 = 1 \).

**Example 2:** Consider the following system of equations [9]:

\[ \begin{align*}
15 x_1 + x_2^2 - 4 x_3 & = 13, \\
x_1^2 + 10 x_2 - e^{-x_3} & = 11, \\
x_2^2 - 25 x_3 & = -22
\end{align*} \] (8)

On applying Accelerated ADM on the above system of equations, we have

\[ x_1 = \sum_{k=0}^{\infty} x_{1k} = \frac{13}{15} - \frac{1}{15} x_2^2 + \frac{4}{15} x_3 \]
\[
\sum_{k=0}^{\infty} \tilde{A}_{1k} = x_2^2 = \left( \sum_{k=0}^{\infty} x_{2k} \right)^2
\]
\[
x_2 = \sum_{k=0}^{\infty} x_{2k} = \frac{11}{10} - \frac{1}{10} x_1^2 + \frac{1}{10} e^{-x_3}
\]
\[
= \frac{11}{10} - \frac{1}{10} \sum_{k=0}^{\infty} \tilde{A}_{2k} + \frac{1}{10} \sum_{k=0}^{\infty} \tilde{A}_{21k}
\]

\[
\sum_{k=0}^{\infty} \tilde{A}_{2k} = x_1^2 = \left( \sum_{k=0}^{\infty} x_{1k} \right)^2
\]
\[
x_3 = \sum_{k=0}^{\infty} x_{3k} = \frac{22}{25} + \frac{1}{25} x_2^2
\]
\[
= \frac{22}{25} + \frac{1}{25} \sum_{k=0}^{\infty} \tilde{A}_{3k}
\]

In eq. (9), (10) and (11), \( \tilde{A}_{1k} \), \( \tilde{A}_{11k} \), \( \tilde{A}_{2k} \), \( \tilde{A}_{21k} \) and \( \tilde{A}_{3k} \) represents Accelerated Adomian polynomial. From eq. (9), (10) and (11) and using eq. (3) to calculate the Accelerated Adomian polynomial term, we have

\[
x_{10} = 0.86667, \quad x_{20} = 1.1, \quad x_{30} = 0.88
\]
\[
x_{11} = 0.154, \quad x_{21} = -0.0336328, \quad x_{31} = 0.05324
\]
\[
x_{12} = 0.0190547, \quad x_{22} = -0.0312155, \quad x_{32} = -0.00473569
\]
\[
x_{13} = 0.00311048, \quad x_{23} = -0.003739329, \quad x_{33} = -0.0041360886
\]
\[
x_{14} = -0.00058778606, \quad x_{24} = -0.0004839997, \quad x_{34} = -0.0004790856677
\]
\[
x_{15} = -0.0000612111354, \quad x_{25} = 0.0001415716711, \quad x_{35} = -0.00006175715
\]

Hence the approximate solution of the eq. (8) is given by

\[
x_1 = x_{10} + x_{11} + x_{12} + \cdots + x_{15} \approx 1.042182881855936
\]
\[
x_2 = x_{20} + x_{21} + x_{22} + \cdots + x_{25} \approx 1.031069940785755
\]
\[
x_3 = x_{30} + x_{31} + x_{32} + \cdots + x_{35} \approx 0.92382737537429
\]

which is closed to the exact solution of eq. (8) i.e. exact solution is \( x_1 = 1.04214966, x_2 = 1.03109169 \) and \( x_3 = 0.92384809 \).

**Example 3:** Consider the following system of equations [12]:

\[
10x_1 + \sin(x_1 + x_2) - 1 = 0,
\]
\[
8x_2 - \cos^2(x_3 - x_2) - 1 = 0,
\]
\[
12x_3 + \sin x_3 - 1 = 0.
\]

On applying Accelerated ADM on the above system of equations, we have

\[
x_1 = \sum_{k=0}^{\infty} x_{1k} = \frac{1}{10} - \frac{1}{10} \sin(x_1 + x_2)
\]
\[ = \frac{1}{10} - \frac{1}{10} \sum_{k=0}^{\infty} \bar{A}_{1k} \]  

where \( \sum_{k=0}^{\infty} \bar{A}_{1k} = \sin(x_1 + x_2) = \sin(\sum_{k=0}^{\infty} x_{1k} + \sum_{k=0}^{\infty} x_{2k}) \)

\[ x_2 = \sum_{k=0}^{\infty} x_{2k} = \frac{1}{8} + \frac{1}{8} \cos^2(x_3 - x_2) \]

\[ = \frac{1}{8} + \frac{1}{8} \sum_{k=0}^{\infty} \bar{A}_{2k} \]  

\[ = 10^{-1} - \frac{10}{10} \sum_{k=0}^{\infty} \bar{A}_{1k} \]  

(13)

In eq.(13), (14) and (15), \( \bar{A}_{1k} \), \( \bar{A}_{1k} \), \( \bar{A}_{2k} \), \( \bar{A}_{21k} \), \( \bar{A}_{3k} \) and \( \bar{A}_{3k} \) represents Accelerated Adomian polynomial. From eq.(13), (14) and (15) and using eq.(3) to calculate the Accelerated Adomian polynomial term, we have

\[ x_10 = 0.1, \ x_20 = 0.125, \ x_30 = 0.0833333 \]

\[ x_{11} = -0.022310636, \ x_{21} = 0.124783112, \ x_{31} = -0.0069364097 \]

\[ x_{12} = -0.009854488, \ x_{22} = -0.00350435148, \ x_{32} = 0.0005761905 \]

\[ x_{13} = 0.0012676360, \ x_{23} = 0.0001713570, \ x_{33} = -0.0000478748 \]

\[ x_{14} = -0.001368263, \ x_{24} = -9.108695 \times 10^{-6}, \ x_{34} = 0.00000397776 \]

Hence the approximate solution of the eq.(12) is given by

\[ x_1 = x_{10} + x_{11} + x_{12} + x_{13} + x_{14} \approx 0.06896572457749794 \]

\[ x_2 = x_{20} + x_{21} + x_{22} + x_{23} + x_{24} \approx 0.246441984644066 \]

\[ x_3 = x_{30} + x_{31} + x_{32} + x_{33} + x_{34} \approx 0.07692921713340992 \]

which is closed to the exact solution.

5. Conclusion

Based on the ADM, we have introduced an Accelerated Adomian decomposition method, where we have used a new form of Adomian polynomial. The solution obtained from this method is efficient and gives a better approximation to the solution than the ADM. Only a few terms lead to the solution closed to the exact solution and the most important point is that no initial approximation is required to start the iteration process likewise in case of Newton Raphson and Broydon’s method, in which if initial approximation is not chosen properly i.e. hasn’t closed to the exact one then the method may mislead the result. Hence, we can conclude that the proposed method is more reliable and efficient than other semi analytical and iterative techniques.

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