LYAPUNOV FUNCTIONALS AND LOCAL DISSIPATIVITY FOR
THE VORTICITY EQUATION IN $L^p$ AND BESOV SPACES

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Abstract. In this paper we establish the local Lyapunov property of certain
$L^p$ and Besov norm of the vorticity fields. We have resolved in part, certain
open problem posed by Tosio Kato for the three dimensional Navier-Stokes
equation by studying the vorticity equation. The local dissipativity of the
sum of linear and non-linear operators of the vorticity equation is established.
One of the main techniques used here is the Littlewood-Paley analysis.

1. Introduction

Stability and control of a dynamical system is often studied using Lyapunov
functions [12,16,17]. The local Lyapunov property we study in this paper can thus
be of interest to the understanding, control and stabilization of turbulent fields [25].
This property also sheds some light towards the research on global Navier-Stokes
solutions in super-critical spaces (see definitions and examples of these spaces
in [7] and [9]). Weak solutions of the Navier-Stokes equation satisfy the energy
inequality which in turn implies that the $L^2$-norm of velocity decreases in time [15].
This idea was generalized by Tosio Kato [13] to prove that for every solution of
Navier-Stokes equation in $\mathbb{R}^m$ ($m \geq 3$), there exist a large number of Lyapunov
functions, which decrease monotonically in time if the solution have small $L^m(\mathbb{R}^m)$-

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Lp-norm for 2 ≤ p < ∞. However the local dissipativity in Ws,p-norm for s > 0 has remained an open problem.

Cannone and Planchon [10] proved the Lyapunov property for the 3-D Navier-Stokes equation in Besov spaces. In particular they proved that if p, q ≥ 2, \( \frac{2}{p} + \frac{2}{q} > 1 \) and as long as the \( \dot{B}_{s,p}^{-1,\infty} \)-norm of the velocity is small, the \( \dot{B}_{1+3/p,q}^{-1,\infty} \)-norm of velocity decreases in time.

In [14] Koch and Tataru considered the local and global (in time) well-posedness for the incompressible Navier-Stokes equation and proved the existence and uniqueness of global mild solution in \( BMO^{-1} \) provided that the initial solution is small enough in this space. Due to Cannone and Planchon [10], existence of the Lyapunov functions for small \( \dot{B}_{s,p}^{-1,\infty} \)-norm is known but the global solvability of Navier-Stokes equation in this space remains an open problem. Noting the embedding theorem \( \dot{B}_{1}^{-1,\infty} \subset BMO^{-1} \), \( BMO^{-1} \) is thus the largest space of initial data for which global mild solution has been shown to exist.

Recently, P. G. Lemarié-Rieusset [18] has extended the result of Cannone and Planchon [10] to a larger class of initial data. He proved that for initial data \( u_0 \in \dot{B}_{s,q}^{p} \cap BMO^{-1} \) where \( s > -1, p \geq 2, q \geq 1 \) and \( s + \frac{2}{q} > 0 \), there exists a constant \( C_0 > 0 \) independent of \( p \) and \( q \), such that if \( u \) is a Koch-Tataru solution of Navier-Stokes equation and satisfying \( \sup_t \| u(t) \|_{\dot{B}_{s,q}^{-1,\infty}} < C_0 \), then \( t \to \| u(t) \|_{\dot{B}_{s,q}^{-1,\infty}} \) is a Lyapunov function.

Local monotonicity of different type has been used in proving the solvability in unbounded domains for Navier-Stokes in 2-D [20], in 3-D [3] and for modified 2-D Navier-Stokes with artificial compressibility [19]. Local monotonicity has also been useful in Control theory [3]. For extensive theories and applications on dissipative and accretive operators see Barbu [1] and Browder [6].

In this paper, we achieve a partial resolution to the open problems posed by Kato [13] for the Navier-Stokes equation by studying the vorticity equation:

\[
\begin{align*}
\partial_t \omega - \nu \Delta \omega + u \cdot \nabla \omega - \omega \cdot \nabla u &= 0, \quad \text{in } \mathbb{R}^m \times \mathbb{R}_+, \\
\nabla \cdot \omega &= 0, \quad \text{in } \mathbb{R}^m \times \mathbb{R}_+, \\
\omega(x,0) &= \omega_0(x), \quad x \in \mathbb{R}^m.
\end{align*}
\]

To be specific, we have proved that the vorticity equation have a family of Lyapunov functions in \( L^p(\mathbb{R}^m) \) for 2 ≤ p < ∞ and m ≥ 3 provided that the \( L^m \)-norm of the velocity is small enough. We then prove \( \dot{B}_p^{-1+3/p,q} \)-norm of the vorticity is a Lyapunov function for 3-D vorticity equation provided the velocity and the vorticity are small in \( \dot{B}_{s,p}^{-1,\infty} \)-norm and \( \dot{B}_{s,q}^{-2,\infty} \)-norm respectively.

We have also proved the dissipativity of the sum of the linear and nonlinear operators of the vorticity equation [14] in \( L^p \) for 2 ≤ p < ∞, which in part answers the open problem of Kato for the local dissipativity of the Navier-Stokes operators in \( W^{1,p} \)-norm.

In Section 2 and 3 we recall some basic facts concerning Littlewood-Paley decomposition, homogeneous Besov spaces and the Paraprodut rule. The main results are presented in section 4.
2. Some Definitions and Estimates

Definition 2.1. (Duality Map) The mapping $G : X \to 2^{X^*}$ is called the duality mapping of the space $X$ if

$$G(x) = \{ x^* \in X^* ; \langle x, x^* \rangle = \| x \|^2_X = \| x^* \|^2_{X^*}, \forall x \in X \}.$$  

Remark 2.2. The duality map for $L^p$ is given by

$$G(x) = \frac{x | x |^{p-2}}{\| x \|^p}. $$

Definition 2.3. (Dissipative Operator) An operator $A$ is said to be dissipative if

$$\langle Ax - Ay, G(x - y) \rangle \leq 0, \quad \forall x, y \in D(A).$$

An operator $A$ is said to be accretive if $-A$ is dissipative.

See [1] and [6] for extensive theories and applications on nonlinear operators in Banach spaces.

Definition 2.4. (Lyapunov Function) Let $v$ be a solution of the Navier-Stokes equation. Then any function $L(v)(t)$ monotonically decreasing in time is called a Lyapunov function associated to $v$.

The most well-known example is certainly provided by energy [15]

$$E(v)(t) = \frac{1}{2} \| v(t) \|^2_2.$$  

The energy equality for the Navier-Stokes equation yield

$$\frac{d}{dt} E(t) + \nu \| \nabla v(t) \|^2_2 = 0,$$

which proves that $E(t)$ is Lyapunov functional.

Let us now recall two lemmas due to Kato [13].

Lemma 2.5. Let $2 \leq p < \infty$ and $\phi \in W^{1,p}$. Define

$$Q_p(\phi) = \int_{\partial \phi(x) \neq 0} | \phi(x) |^{p-2} | \nabla \phi(x) |^2 \, dx \geq 0. \quad (2.1)$$

Then

$$CQ_p(\phi) \leq -\langle | \phi |^{p-2} \phi, \Delta \phi \rangle < \infty, \quad (2.2)$$

where $C$ denotes a positive constant.

Lemma 2.6. Let $2 \leq p < \infty$ and $\phi \in W^{1,p}$. Then

$$\| \phi \|_{\frac{mp}{m-2}} \leq CQ_p(\phi)^{\frac{1}{p}}. \quad (2.3)$$

Lemma 2.7. Let $u$ be the velocity field obtained from $\omega$ via the Biot-Savart law:

$$u(x) = -\frac{\Gamma(m/2 + 1)}{m(m-2)\pi^{m/2}} \int_{\mathbb{R}^m} \frac{(x-y)}{|x-y|^m} \times \omega(y) \, dy, \quad x \in \mathbb{R}^m, m \geq 3. \quad (2.4)$$

(a) Assume that $1 < p < \infty$. Then for every divergence-free vector field $u$ whose gradient is in $L^p$, there exists a $C > 0$, depending on $p$, such that

$$\| \nabla u \|_p \leq C \| \omega \|_p. \quad (2.5)$$
(b) If \( \omega \in L^1(\mathbb{R}^m) \cap L^p(\mathbb{R}^m), \frac{m}{m-1} < p \leq \infty \), then
\[
\| u \|_{L^p(\mathbb{R}^m)} \leq C(\| \omega \|_{L^1(\mathbb{R}^m)} + \| \omega \|_{L^p(\mathbb{R}^m)}).
\]
(2.6)

Proof. (a) See Theorem 3.1.1 in [11].
(b) The proof is due to Ying and Zhang [28], Lemma 3.3.1. \( \square \)

3. Littlewood-Paley Decomposition and Besov Spaces

In this section, we recall some classical results concerning the homogeneous Besov spaces in terms of the Littlewood-Paley decomposition. Several related embedding relations and inequalities will also be given here. For more details the reader is referred to the books [4], [8], [11], [22], [27] for a comprehensive treatment.

3.1. Littlewood-Paley Decomposition: Let us start with the Littlewood-Paley decomposition in \( \mathbb{R}^3 \). To this end, we take an arbitrary function \( \psi \) in the Schwartz class \( \mathcal{S}(\mathbb{R}^3) \) whose Fourier transform \( \hat{\psi} \) is such that
\[\text{supp} \ \hat{\psi} \subset \{ \xi, \frac{1}{2} \leq |\xi| \leq 2 \}, \tag{3.1}\]
and
\[\forall \xi \neq 0, \sum_{j \geq 0} \hat{\psi}(\frac{\xi}{2^j}) = 1. \]
Let us define \( \varphi \) by
\[\hat{\varphi}(\xi) = 1 - \sum_{j \geq 0} \hat{\psi}(\frac{\xi}{2^j}), \] and hence
\[\text{supp} \ \hat{\varphi} \subset \{ \xi, |\xi| \leq 1 \}. \tag{3.2}\]
For \( j \in \mathbb{Z} \), we write \( \varphi_j(x) = 2^{3j} \varphi(2^j x) \). We denote by \( S_j \) and \( \triangle_j \), the convolution operators with \( \varphi_j \) and \( \psi_j \) respectively. Hence
\[S_j(f) = f \ast \varphi_j, \]
and
\[\triangle_j f = \psi_j \ast f, \text{ where } \psi_j(x) = 2^{3j} \psi(2^j x).\]
Then
\[S_j = \sum_{p < j} \triangle_p \quad \text{and} \quad I = \sum_{j \in \mathbb{Z}} \triangle_j. \]
The dyadic decomposition
\[u = \sum_{j \in \mathbb{Z}} \triangle_j u, \tag{3.3}\]
is called the homogeneous Littlewood-Paley decomposition of \( u \) and converges only in the quotient space \( \mathcal{S}' / \mathcal{P} \) where \( \mathcal{S}' \) is the space of tempered distributions.
and $\mathcal{P}$ is the space of polynomials. Now let us mention here the following quasi-orthogonality properties of the dyadic decomposition \[11] (proposition 2.1.1):

\[
\triangle_p \triangle_q u = 0 \quad \text{if} \quad |p - q| \geq 2,
\]
\[
\triangle_p (S_{q-2} u \triangle q u) = 0 \quad \text{if} \quad |p - q| \geq 4.
\]

3.2. Besov Spaces: Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Then a tempered distribution $f$ belongs to the homogeneous Besov space $\dot{B}^{s,q}_p$ if and only if

\[
\left( \sum_{j \in \mathbb{Z}} 2^{jsq} \| \triangle_j f \|^q_p \right)^{\frac{1}{q}} < \infty
\]

and $f = \sum_{j \in \mathbb{Z}} \triangle_j f$ in $\mathcal{S}' / \mathcal{P}_m$ where $\mathcal{P}_m$ is the space of polynomials of degree $\leq m$ and $m = \lfloor s - \frac{d}{p} \rfloor$, the integer part of $s - \frac{d}{p}$.

Besov space is a quasi-Banach space \[24\]. Here we recall the following standard embedding rules \[27\] (chapter 2.7):

If $s_1 > s_2$ and $p_2 \geq p_1 \geq 1$ such that $s_1 - \frac{d}{p_1} = s_2 - \frac{d}{p_2}$, then

\[
\dot{B}^{s_1,q}_{p_1} \hookrightarrow \dot{B}^{s_2,q}_{p_2}.
\]

Moreover if $q_1 < q_2$ then

\[
\dot{B}^{s,q}_{p,q_1} \hookrightarrow \dot{B}^{s,q}_{p,q_2}.
\]

The above mentioned embeddings are also valid for inhomogeneous Besov spaces. For more embedding theorems and their proofs we refer the readers to \[22\] and \[27\].

Next let us recall the following result from Chapter 3 in Triebel \[27\]:

Lemma 3.1. Let $1 \leq p, q \leq \infty$ and $s < 0$. Then $\forall f \in \dot{B}^{s,q}_p$ we have,

\[
\left( \sum_{j \in \mathbb{Z}} 2^{jsq} \| \triangle_j f \|^q_p \right)^{\frac{1}{q}} < \infty \iff \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \| S_j f \|^q_p \right)^{\frac{1}{q}} < \infty.
\]

Now we recall the following versions of Bernstein inequalities (chapter 3 in \[18\]):

Lemma 3.2. Let $1 \leq p \leq \infty$. Then there exist constants $C_0, C_1, C_2 > 0$ such that

(a) If $f$ has its frequency in a ball $\mathbb{B}(0, \lambda)$ (supp $\mathcal{F}(f) \subset \mathbb{B}(0, \lambda)$) then

\[
\| (\Delta)^{\frac{s}{2}} f \|_{p} \leq C_0 \lambda^{s|s|} \| f \|_{p}.
\]

(b) If $f$ has its frequency in an annulus $C(0, A\lambda, B\lambda)$ (supp $\mathcal{F}(f) \subset \{ \xi, |A\lambda| \leq \xi \leq B\lambda \}$) then

\[
C_1 \lambda^{s|s|} \| f \|_{p} \leq \| (\Delta)^{\frac{s}{2}} f \|_{p} \leq C_2 \lambda^{s|s|} \| f \|_{p}.
\]

Now let us state here the modified Poincaré type inequality given by Planche \[23\].

Lemma 3.3. Let $f \in \mathcal{S}$, the Schwartz space, whose fourier transform is supported outside the ball $\mathbb{B}(0, 1)$. Then for $p \geq 2$,

\[
\int |f|^p \, dx \leq C_p \int |\nabla f|^2 \, dx \int |f|^{p-2} \, dx.
\]
3.3. The Paraproduct rule: Another important tool in Littlewood-Paley analysis is the paraproduct operator introduced by J. M. Bony [5]. The idea of the paraproduct enables us to define a new product between distributions which turns out to be continuous in many functional spaces where the pointwise multiplication does not make sense. This is a powerful tool for the analysis of nonlinear partial differential equations.

Let \( f, g \in \mathcal{S}' \). Then using the formal Littlewood-Paley decomposition,
\[
f = \sum_{j \in \mathbb{Z}} \Delta_j f, \quad g = \sum_{j \in \mathbb{Z}} \Delta_j g.
\]

Hence
\[
f g = \sum_{j,l} \Delta_j f \Delta_l g
\]
\[
= \sum_j \sum_{l < j - 2} \Delta_j f \Delta_l g + \sum_j \sum_{l > j + 2} \Delta_j f \Delta_l g + \sum_j \sum_{|l-j| \leq 2} \Delta_j f \Delta_l g
\]
\[
= \sum_j \Delta_j f S_{j-2} g + \sum_j \Delta_j g S_{j-2} f + \sum_{|l-j| \leq 2} \Delta_j f \Delta_l g.
\]

In other words, the product of two tempered distributions is decomposed into two homogeneous paraproducts, respectively
\[
\hat{\pi}(f,g) = \sum_j \Delta_j f S_{j-2} g \quad \text{and} \quad \hat{\pi}(g,f) = \sum_j \Delta_j g S_{j-2} f,
\]
plus a remainder
\[
R(f,g) = \sum_{|l-j| \leq 2} \Delta_j f \Delta_l g.
\]

\( \hat{\pi} \) is called the homogeneous paraproduct operator and the convergence of the above series holds true in the quotient space \( \mathcal{S}' / \mathcal{P} \). Finally, using the quasi-orthogonality properties from (3.4) and (3.5) and after neglecting some non-diagonal terms for simplicity (since the contributions from these non-diagonal terms are taken care of by the terms which are being considered and hence negligible and also this does not affect the convergence of the paraproducts [7, 9]), we obtain
\[
\Delta_j (fg) = \Delta_j f S_{j-2} g + \Delta_j g S_{j-2} f + \Delta_j \left( \sum_{k > j} \Delta_k f \Delta_k g \right). \tag{3.13}
\]

We refer the readers [7, 11, 21, 24] for extensive studies on paraproducts.

4. Main Results

Theorem 4.1 (Local Lyapunov Property in \( L^p \)). Let \( m \geq 3, 2 \leq p < \infty \). Let \( \omega \) be the solution of the vorticity equation (1.1) such that
\[
u \in C([0, T]; L^m \cap L^p), \quad \nabla u \in L^1_{loc}((0, T); L^p),
\]
and
\[ \omega \in C([0, T]; L^m \cap L^p), \quad \nabla \omega \in L^1_{\text{loc}}((0, T); L^p), \quad \text{for } 0 < T \leq \infty. \]

Then
\[ \partial_t \| \omega(t) \|_p^p \leq -C(\nu - K \| u(t) \|_m) Q_p(\omega(t)), \quad 0 < t < T, \quad (4.1) \]
where $K$ denotes a positive constant depending upon $m$ and $p$.

This implies for small $L^m$-norm $t \to \| \omega(t) \|_{L^p}$ is a Lyapunov function.

**Proof.** Consider,
\[
\begin{align*}
\partial_t \| \omega \|_p^p &= \frac{\partial}{\partial t} \int \omega^p dx = \frac{\partial}{\partial t} \int |\omega|^2 \frac{p}{2} dx \\
&= p \int |\omega|^{p-2} \omega \cdot \frac{\partial \omega}{\partial t} dx = p(\langle |\omega|^{p-2} \omega, \partial_t \omega \rangle) \\
&= p(\langle |\omega|^{p-2} \omega, \nu \Delta \omega - u \cdot \nabla \omega + \omega \cdot \nabla u \rangle) \\
&= \nu p(\langle |\omega|^{p-2} \omega, \Delta \omega \rangle - p(\langle |\omega|^{p-2} \omega, u \cdot \nabla \omega \rangle \\
& \quad + p(\langle |\omega|^{p-2} \omega, \omega \cdot \nabla u \rangle). \tag{4.2}
\end{align*}
\]

Using Lemma 2.5 on the first term of the right hand side, we have from (4.2)
\[
\begin{align*}
\partial_t \| \omega \|_p^p &\leq -C \nu Q_p(\omega) - p(\langle |\omega|^{p-2} \omega, u \cdot \nabla \omega \rangle + p(\langle |\omega|^{p-2} \omega, \omega \cdot \nabla u \rangle) \\
&\leq \langle |\omega|^{p-2} \omega, u \cdot \nabla \omega \rangle + p(\langle |\omega|^{p-2} \omega, \omega \cdot \nabla u \rangle. \tag{4.3}
\end{align*}
\]

Now we need to estimate the second and the third terms of the right hand side of the equation (4.3).

Using the fact that $\text{Div} \ u = 0$, we have
\[
\begin{align*}
u \Delta \omega &= u_i \frac{\partial \omega_j}{\partial x_i} = \frac{\partial}{\partial x_i}(u_i \omega_j) - \omega_j \frac{\partial u_i}{\partial x_i} = \nabla \cdot (u \otimes \omega), \tag{4.4}
\end{align*}
\]
where $\otimes$ represents the tensor product.

Then
\[
\begin{align*}
\langle \langle |\omega|^{p-2} \omega, u \cdot \nabla \omega \rangle \rangle &= \langle \langle |\omega|^{p-2} \omega, \nabla \cdot (u \otimes \omega) \rangle \rangle \\
&= \langle \langle |\omega|^{p-2} \omega, \nabla \otimes \omega \rangle \rangle \leq \langle \langle |\omega|^{p-2} \omega, \nabla \omega \rangle, \ |u \otimes \omega \rangle \rangle. \tag{4.5}
\end{align*}
\]

Notice that $| \nabla \cdot \omega |^{p-2} \omega | \leq C | \omega |^{p-2} | \nabla \omega |$. Hence using this and Hölder’s inequality in (4.5) we have
\[
\begin{align*}
\langle \langle |\omega|^{p-2} \omega, u \cdot \nabla \omega \rangle \rangle &\leq \langle \langle |\omega|^{p-2} \ | \nabla \omega \ |, \ |u \otimes \omega \rangle \rangle \\
&\leq \| |\omega|^{p-2} \ | \nabla \omega \ | \|_q \| u \otimes \omega \|_{q'}, \quad \text{where } \frac{1}{q} + \frac{1}{q'} = 1. \tag{4.6}
\end{align*}
\]

Now
\[
\| |\omega|^{p-2} \ | \nabla \omega \ | \|_q^q = \int |\omega|^{q(p-2)} \ | \nabla \omega \ |^q dx \]
\[
= \int |\omega|^{q(p-2)/2} \ (|\omega|^{p-2} \ | \nabla \omega \ |^2)^{q/2} dx.
\]
Since \( \frac{2-p}{2} + \frac{2}{q} = 1 \), Hölder inequality yields

\[
\| | \omega |^{p-2} | \nabla \omega | \|_q^2 \\
\leq \left[ \left( \int | | \omega |^{q(p-2)/2} |^{2/(2-q)} \, dx \right)^{2/q} \, \left( \int \left( | | \omega |^{p-2} | \nabla \omega |^2 \right)^{q/2} \, dx \right)^{2/q} \right]^{q/2} \\
= \left[ \int | \omega |^{q(p-2)/2} | \nabla \omega |^{(2-q)/2} \, dx \right]^{q/2} \\
= \| \omega \|_{r(p-2)/2} Q_p(\omega)^{q/2}, \text{ where } r = \frac{q(p-2)}{2-q}.
\]

Hence

\[
\| | \omega |^{p-2} | \nabla \omega | \|_q \leq \| \omega \|_{r(p-2)/2} Q_p(\omega)^{1/2}. \tag{4.7}
\]

Again by Hölder,

\[
\| u \otimes \omega \|_{q'} \leq C \| u \|_m \| \omega \|_r, \text{ since } \frac{1}{q'} = \frac{1}{m} + \frac{1}{r}. \tag{4.8}
\]

Now from the relations

\[\frac{1}{q} + \frac{1}{q'} = 1, \ r = \frac{q(p-2)}{2-q} \text{ and } \frac{1}{q'} = \frac{1}{m} + \frac{1}{r},\]

we find that

\[r = \frac{mp}{m-2}. \tag{4.9}\]

Using equations \(4.7\) and \(4.8\) in \(4.6\) we have

\[
| \langle | \omega |^{p-2} \omega, u \cdot \nabla \omega \rangle | \leq C \| \omega \|_{r(p-2)/2} Q_p(\omega)^{1/2} \| u \|_m \| \omega \|_r \\
= C \| u \|_m \| \omega \|_{p/2} Q_p(\omega)^{1/2}.
\]

Applying the Lemma 2.6 in the above equation we obtain

\[
| \langle | \omega |^{p-2} \omega, u \cdot \nabla \omega \rangle | \leq C \| u \|_m Q_p(\omega). \tag{4.10}
\]

The third term in the equation \(4.3\) can be estimated by using the fact that \( \text{Div} \, \omega = 0 \) along with the similar kind of techniques taken to estimate the second term.

Thus we get

\[
| \langle | \omega |^{p-2} \omega, \omega \cdot \nabla u \rangle | \leq C \| u \|_m Q_p(\omega). \tag{4.11}
\]

Combining \(4.10\) and \(4.11\) with \(4.3\) we get the desired result \(4.1\). \(\square\)

**Theorem 4.2** (Local Lyapunov Property in Besov Spaces). *Let the initial data \( \omega_0 \) for the 3-D vorticity equation be in \( \dot{B}^{s,q}_{p} \) where \( s = \frac{3}{p} - 1, \ p, q \geq 2, \) and \( \frac{3}{p} + \frac{2}{q} > 1. \) Then there exist small constants \( \varepsilon_1 > 0 \) and \( \varepsilon_2 > 0 \) such that if the velocity field satisfies \( \sup_t \| u(t) \|_{\dot{B}^{-1,\infty}_{p}} < \varepsilon_1 \) and the vorticity field satisfies \( \sup_t \| \omega(t) \|_{\dot{B}^{-2,\infty}_{p}} < \varepsilon_2, \) then \( t \rightarrow \| \omega(t) \|_{\dot{B}^{s,q}_{p}} \) is a Lyapunov function.*
Proof. Let us consider
\[ F(u, w) = u \cdot \nabla \omega - \omega \cdot \nabla u. \]

Multiply the equation (1.1) by \( \triangle_j \) to get,
\[ \partial_t(\triangle_j \omega) - \nu \triangle(\triangle_j \omega) + \triangle_j(F(u, w)) = 0. \tag{4.12} \]

Now,
\[
\begin{align*}
\partial_t \| \triangle_j \omega \|^p_p &= \frac{\partial}{\partial t} \int | \triangle_j \omega |^p \, dx = \frac{\partial}{\partial t} \int (\triangle_j \omega)^2 \frac{|p/2|}{dx} \\
&= p \int | \triangle_j \omega |^{p-2} \triangle_j \omega \cdot \frac{\partial (\triangle_j \omega)}{\partial t} \, dx \\
&= p \langle | \triangle_j \omega |^{p-2} \triangle_j \omega, \partial_t(\triangle_j \omega) \rangle.
\end{align*}
\]

Hence using (4.12) we have from the above equation
\[
\begin{align*}
\partial_t \| \triangle_j \omega \|^p_p &= p \langle | \triangle_j \omega |^{p-2} \triangle_j \omega, \nu \triangle(\triangle_j \omega) \rangle \\
&= \nu p \langle | \triangle_j \omega |^{p-2} \triangle_j \omega, \triangle_j(F(u, w)) \rangle \\
&\quad - p \langle | \triangle_j \omega |^{p-2} \triangle_j \omega, \partial_t(\triangle_j \omega) \rangle.
\end{align*}
\]

Applying the Lemma 2.5 on the first term on the right hand side of the above equation we obtain
\[
\begin{align*}
\partial_t \| \triangle_j \omega \|^p_p &\leq -\nu p \int | \triangle_j \omega |^{p-2} | \nabla \triangle_j \omega |^2 \, dx \\
&\quad - p \langle | \triangle_j \omega |^{p-2} \triangle_j \omega, \triangle_j(F(u, w)) \rangle.
\end{align*}
\]

Hence,
\[
\begin{align*}
\partial_t \| \triangle_j \omega \|^p_p &+ \nu p \int | \triangle_j \omega |^{p-2} | \nabla \triangle_j \omega |^2 \, dx \\
&\leq -p \int | \triangle_j \omega |^{p-2} \triangle_j \omega \triangle_j(F(u, w)) \, dx,
\end{align*}
\]

which is equivalent of considering the equation
\[
\frac{d}{dt} \| \triangle_j \omega \|^p_p + \tilde{C}_p \nu p 2^{2j} \| \triangle_j \omega \|^p_p \leq p \int | \triangle_j \omega |^{p-1} | \triangle_j(F(u, w)) | \, dx, \tag{4.13}
\]

where \( \tilde{C}_p \) is positive constant depending on \( p \).

Now,
\[
| \triangle_j(F(u, w)) | = | \triangle_j(u \cdot \nabla \omega - \omega \cdot \nabla u) | \\
\leq | \triangle_j(u \cdot \nabla \omega) | + | \triangle_j(\omega \cdot \nabla u) |.
\]
Moreover
\[ u \cdot \nabla \omega = u_i \frac{\partial \omega_j}{\partial x_i} = \frac{\partial}{\partial x_i} (u_i \omega_j) - \omega_j \frac{\partial u_i}{\partial x_i} = \nabla \cdot (u \otimes \omega), \] since \( \text{Div} \ u = 0 \), and similarly \( \omega \cdot \nabla u = \nabla \cdot (\omega \otimes u) \), where \( \otimes \) represents the usual tensor product. Since the terms \( \nabla \cdot (u \otimes \omega) \) and \( \nabla \cdot (\omega \otimes u) \) behave in similar fashion, we have from equation (4.13)
\[ \frac{d}{dt} \| \triangle_j \omega \|_p^p + \tilde{C}_p \nu_p 2^{2j} \| \triangle_j \omega \|_p^p \leq 2p \int | \triangle_j \omega |^{p-1} | \triangle_j \nabla \cdot (u \otimes \omega) | \, dx. \] (4.14)

Now using the paraproduct rule (3.13), we have
\[ \triangle_j \nabla \cdot (u \otimes \omega) = \nabla \triangle_j (u \otimes \omega) = \nabla (\triangle_j u S_j - 2u) + \nabla (\triangle_j (\sum_{k \geq j} \triangle_k u \triangle_k \omega)). \] (4.15)

Using (4.15) in (4.14) we obtain,
\[ \frac{d}{dt} \| \triangle_j \omega \|_p^p + \tilde{C}_p \nu_p 2^{2j} \| \triangle_j \omega \|_p^p \leq 2p \int | \triangle_j \omega |^{p-1} | \nabla (\triangle_j u S_j - 2u) | \, dx \]
\[ + 2p \int | \triangle_j \omega |^{p-1} | \nabla (\triangle_j (\sum_{k \geq j} \triangle_k u \triangle_k \omega)) | \, dx. \] (4.16)

We need to estimate each of the terms on the right hand side of (4.16) separately. First consider the term
\[ \int | \triangle_j \omega |^{p-1} | \nabla (\triangle_j u S_j - 2u) | \, dx, \]
and apply Hölder’s Inequality to get,
\[ \int | \triangle_j \omega |^{p-1} | \nabla (\triangle_j u S_j - 2u) | \, dx \leq \| \triangle_j \omega \|_p^{p-1} \| \nabla (\triangle_j u S_j - 2u) \|_p. \]

With the help of Lemma 3.2 we obtain
\[ \int | \triangle_j \omega |^{p-1} | \nabla (\triangle_j u S_j - 2u) | \, dx \]
\[ \leq C_1 \| \triangle_j \omega \|_p^{p-1} 2^j \| \triangle_j u S_j - 2u \|_p \]
\[ = C_1 \| \triangle_j \omega \|_p^{p-1} 2^j \| (2^j \triangle_j u)(2^{-j}S_j - 2u) \|_p \]
\[ \leq C_1 \| \triangle_j \omega \|_p^{p-1} 2^j \| 2^j \triangle_j u \|_p \sup_{j} (2^{-j} \| S_j - 2u \|_\infty) \]
\[ = C_1 2^{2j} \| \triangle_j \omega \|_p^p \sup_{j} (2^{-j} \| S_j - 2u \|_\infty), \] (4.17)
where \( C_1 \) is positive constant.
Now from Lemma 3.1, for $s = -1$ and $p = q = \infty$, we have
\[ 2^{-j} \parallel \Delta_j u \parallel_{L^\infty} \in l^\infty \iff 2^{-j} \parallel S_j u \parallel_{L^\infty} \in l^\infty \]
\[ \Rightarrow \sup_j 2^{-j} \parallel \Delta_j u \parallel_{l^\infty} \iff \sup_j 2^{-j} \parallel S_j u \parallel_{l^\infty} \]
\[ \Rightarrow \parallel u(x, t) \parallel_{B_{\infty}^{-1}, \infty} \iff \sup_j 2^{-j} \parallel S_j u \parallel_{l^\infty}. \quad (4.18) \]

Then using the conditions assumed in the theorem, we get,
\[ \parallel u(x, t) \parallel_{B_{\infty}^{-1}, \infty} \leq \sup_t \parallel u(x, t) \parallel_{B_{\infty}^{-1}, \infty} \leq \varepsilon_1, \]
\[ \Rightarrow \sup_j 2^{-j} \parallel S_j u \parallel_{l^\infty} \leq \varepsilon_1. \quad (4.19) \]

So finally (4.17) yields
\[ \int | \Delta_j \omega |^{p-1} \parallel \nabla (\Delta_j \omega S_j -2u) \parallel \ dx \leq C_1 \varepsilon_1 2^{2j} \parallel \Delta_j \omega \parallel_p^p. \quad (4.20) \]

Now let us consider the term:
\[ \int | \Delta_j \omega |^{p-1} \parallel \nabla (\Delta_j u S_j -2 \omega) \parallel \ dx. \]

As before Hölder’s Inequality and Lemma 3.2 yield
\[ \int | \Delta_j \omega |^{p-1} \parallel \nabla (\Delta_j u S_j -2 \omega) \parallel \ dx \]
\[ \leq \parallel \Delta_j \omega \parallel_p^{p-1} \parallel \nabla (\Delta_j u S_j -2 \omega) \parallel_p \]
\[ \leq C_2 \parallel \Delta_j \omega \parallel_p^{p-1} 2^j \parallel \Delta_j u S_j -2 \omega \parallel_p \]
\[ = C_2 \parallel \Delta_j \omega \parallel_p^{p-1} 2^j \parallel (2^j \Delta_j u) (2^{-2j} S_j -2 \omega) \parallel_p \]
\[ \leq C_2 \parallel \Delta_j \omega \parallel_p^{p-1} 2^j \parallel 2^j \Delta_j u \parallel_p \sup_j (2^{-2j} \parallel S_j -2 \omega \parallel_{l^\infty}), \quad (4.21) \]

where $C_2$ is positive constant.

From Lemma 3.2, equation (3.11), we obtain
\[ 2^j \parallel \Delta_j u \parallel_p \leq \parallel \nabla \Delta_j u \parallel_p. \quad (4.22) \]

The above equation and (2.5) in Lemma 2.7 yield
\[ \parallel \Delta_j u \parallel_p \leq 2^{-j} \parallel \Delta_j \omega \parallel_p. \quad (4.23) \]

Now applying Lemma 3.1, for $s = -2$ and $p = q = \infty$ and proceeding as before we obtain
\[ \sup_j 2^{-j} \parallel S_j \omega \parallel_{l^\infty} \leq \varepsilon_2. \quad (4.24) \]

Using (4.23) and (4.24) in (4.21) we have
\[ \int | \Delta_j \omega |^{p-1} \parallel \nabla (\Delta_j u S_j -2 \omega) \parallel \ dx \leq C_2 \varepsilon_2 2^{2j} \parallel \Delta_j \omega \parallel_p^p. \quad (4.25) \]
Next we estimate the last term

\[ \int | \triangle j \omega |^{p-1} | \nabla \left( \triangle j \left( \sum_{k \geq j} \triangle k u \triangle k \omega \right) \right) | \, dx \]

\[ \leq \| \triangle j \omega \|_{p}^{p-1} \| \nabla \left( \triangle j \left( \sum_{k \geq j} \triangle k u \triangle k \omega \right) \right) \|_{p} \]

\[ \leq C_3 2^j \| \triangle j \omega \|_{p}^{p-1} \| \triangle j \left( \sum_{k \geq j} \triangle k u \triangle k \omega \right) \|_{p} \]

where \( C_3 \) is positive constant.

Using Young’s Inequality as in [9], we have

\[ \int | \triangle j \omega |^{p-1} | \nabla \left( \triangle j \left( \sum_{k \geq j} \triangle k u \triangle k \omega \right) \right) | \, dx \]

\[ \leq C_p 2^j \| \triangle j \omega \|_{p}^{p-1} \left( \sum_{k \geq j} \| \triangle k u \|_{p} \| \triangle k \omega \|_{p} \right), \] (4.26)

where \( C_p \) is positive constant depending on \( p \).

Now

\[ \| \triangle j \omega \|_{p}^{p-1} = 2^{2j} \left( 2^{-2j} \| \triangle j \omega \|_{p} \right) \| \triangle j \omega \|_{p}^{p-2} \]

\[ \leq 2^{2j} \left( \sup_{j} 2^{-2j} \| \triangle j \omega \|_{\infty} \right) \| \triangle j \omega \|_{p}^{p-2} \]

\[ = 2^{2j} \| \omega(x,t) \|_{B_{\infty}^{2,\infty}} \| \triangle j \omega \|_{p}^{p-2} \]

\[ \leq \varepsilon_2 2^{2j} \| \triangle j \omega \|_{p}^{p-2}. \] (4.27)

Using (4.23) and (4.27) in (4.26) we have,

\[ \int | \triangle j \omega |^{p-1} | \nabla \left( \triangle j \left( \sum_{k \geq j} \triangle k u \triangle k \omega \right) \right) | \, dx \]

\[ \leq C_p \varepsilon_2 2^{2j} \| \triangle j \omega \|_{p}^{p-2} \left( \sum_{k \geq j} \| \triangle k \omega \|_{p}^{2} \right). \] (4.28)

Now combining all results from (4.20), (4.25) and (4.28) and neglecting the constants \( C_1, C_2, C_p, \tilde{C}_p \), we obtain from (4.16)

\[ \frac{d}{dt} \| \triangle j \omega \|_{p}^{p} + \nu p 2^{2j} \| \triangle j \omega \|_{p}^{p} \leq 2p \varepsilon_1 2^{2j} \| \triangle j \omega \|_{p}^{p} + 2p \varepsilon_2 2^{2j} \| \triangle j \omega \|_{p}^{p} \]

\[ + 2p \varepsilon_2 2^{2j} \| \triangle j \omega \|_{p}^{p-2} \left( \sum_{k \geq j} \| \triangle k \omega \|_{p}^{2} \right). \]

Simplifying we get,

\[ \frac{d}{dt} \| \triangle j \omega \|_{p}^{2} + p(\nu - 2 \varepsilon_1 - 2 \varepsilon_2) 2^{2j} \| \triangle j \omega \|_{p}^{2} \leq 2p \varepsilon_2 2^{2j} \left( \sum_{k \geq j} \| \triangle k \omega \|_{p}^{2} \right). \] (4.29)
The rest of the construction is motivated by [10]. Multiplying both sides of (4.29) by $2^{jqs} \| \triangle_j \omega \|_p^{q-2}$, we get

$$\frac{d}{dt} (2^{jqs} \| \triangle_j \omega \|_p^q) + p(\nu - 2 \varepsilon_1 - 2 \varepsilon_2)2^{j(qs+2)} \| \triangle_j \omega \|_p^q \leq 2p\varepsilon_2 2^{j(qs+2)} \| \triangle_j \omega \|_p^{q-2} \left( \sum_{k \geq j} \| \triangle_k \omega \|_p^2 \right).$$

Let $s = \frac{q}{p} - 1$, $p, q \geq 2$, and $\frac{q}{p} + \frac{1}{q} > 1$. Then $r = \frac{q}{q} + s > 0$ or $qs + 2 = rq$. Then

$$\frac{d}{dt} (2^{jqs} \| \triangle_j \omega \|_p^q) + p(\nu - 2 \varepsilon_1 - 2 \varepsilon_2)2^{jrj} \| \triangle_j \omega \|_p^q \leq 2p\varepsilon_2 2^{(q-2)jrj} \| \triangle_j \omega \|_p^{q-2} 2^{2rj} \left( \sum_{k \geq j} \| \triangle_k \omega \|_p^2 \right). \quad (4.30)$$

Let $f_j = 2^{jqs} \| \triangle_j \omega \|_p$ and $g_j = 2^{jr} \| \triangle_j \omega \|_p$. Then taking sum over $j$ of (4.30) we have,

$$\frac{d}{dt} \left( \sum_j f_j^q \right) + p(\nu - 2 \varepsilon_1 - 2 \varepsilon_2) \sum_j q_j^q \leq 2p\varepsilon_2 \sum_j q_j^{q-2} 2^{2rj} \left( \sum_{k \geq j} \| \triangle_k \omega \|_p^2 \right) \quad \sum_{k=1}^{\infty} \sum_{j=1}^{k} g_j^{q-2} 2^{2rj} \| \triangle_k \omega \|_p^2 \leq 2p\varepsilon_2 \sum_{k=1}^{\infty} \sum_{j=1}^{k} \sum_{k=1}^{\infty} g_j^{q-2} 2^{2rj} 2^{-2rk} g_k^2. \quad (4.31)$$

Let us consider

$$\sum_{j=1}^{k} g_j^{q-2} 2^{2rj} = 2^{2rk} h_k^{q-2}.$$ 

Then it is clear that

$$\sum_k h_k^q \leq \sum_j g_j^q. \quad (4.32)$$
Here we will prove a stronger property than (4.35).

Using the definition of Besov Spaces in (3.6), we can write,

\[ \| u \|_{H^1} \leq \| u \|_{L^p}^{1/2} \| \nabla u \|_{L^p}^{1/2}. \]

Hence, in the light of (2.6), we note that if \( \omega \) and \( \tilde{\omega} \) are small in \( L^{1/q} \), then

\[ \langle A(u, \omega), G(\omega) \rangle \leq C(\nu - K \| u \|_m \| \omega \|_m \| \tilde{\omega} \|_m) Q_p(\omega - \tilde{\omega}). \]

Here we will prove a stronger property than (4.35).

**Theorem 4.3** (Local Dissipativity in \( L^p \)). Let \( m \geq 3, 2 \leq p < \infty \). Then if \( (\omega - \tilde{\omega}) \in L^1(\mathbb{R}^m) \cap L^p(\mathbb{R}^m) \), for \( r = \frac{mp}{m-2} \),

\[ \langle A(u, \omega) - A(v, \tilde{\omega}), G(\omega - \tilde{\omega}) \rangle \leq -C(\nu - K \| u \|_m \| \omega \|_m \| \tilde{\omega} \|_m) Q_p(\omega - \tilde{\omega}) \]

\[ -K(\| \omega \|_m + \| \tilde{\omega} \|_m) \| \omega - \tilde{\omega} \|_{L^1}^{1/p'} \]

where \( \frac{1}{p} + \frac{1}{p'} = 1 \).

Hence, in the light of (2.6), we note that if \( \omega \) and \( \tilde{\omega} \) are small in \( L^{1/q} \cap L^m \), then

\[ \langle A(u, \omega) - A(v, \tilde{\omega}), G(\omega - \tilde{\omega}) \rangle \leq 0, \]

which is a local dissipativity property for \( A(\cdot, \cdot) \).
Proof. It is clear that
\[ \langle A(u, \omega) - A(v, \tilde{\omega}), G(\omega - \tilde{\omega}) \rangle \]
\[ = \langle \nu \triangle \omega - u \cdot \nabla \omega + \omega \cdot \nabla u - \nu \triangle \tilde{\omega} + v \cdot \nabla \tilde{\omega} - \tilde{\omega} \cdot \nabla v, G(\omega - \tilde{\omega}) \rangle \]
\[ = \nu \langle \triangle (\omega - \tilde{\omega}), G(\omega - \tilde{\omega}) \rangle - \langle u \cdot \nabla \omega - v \cdot \nabla \tilde{\omega}, G(\omega - \tilde{\omega}) \rangle \]
\[ + \langle \omega \cdot \nabla u - \tilde{\omega} \cdot \nabla v, G(\omega - \tilde{\omega}) \rangle. \] (4.37)

According to Lemma 2.5
\[ \nu \langle \triangle (\omega - \tilde{\omega}), G(\omega - \tilde{\omega}) \rangle \leq -C\nu Q_p(\omega - \tilde{\omega}). \] (4.38)

Now we need to estimate the second and third terms of the right hand side of (4.37).

Notice that
\[ | \langle u \cdot \nabla \omega - v \cdot \nabla \tilde{\omega}, G(\omega - \tilde{\omega}) \rangle | \]
\[ = | \langle (u - v) \cdot \nabla \omega + v \cdot \nabla \omega - v \cdot \nabla \tilde{\omega}, G(\omega - \tilde{\omega}) \rangle | \]
\[ \leq | \langle (u - v) \cdot \nabla \omega, G(\omega - \tilde{\omega}) \rangle | + | \langle v \cdot \nabla (\omega - \tilde{\omega}), G(\omega - \tilde{\omega}) \rangle |. \] (4.39)

Let us denote \( \omega^* = \omega - \tilde{\omega} \). Then with the help of (4.10) we obtain
\[ | \langle v \cdot \nabla (\omega - \tilde{\omega}), G(\omega - \tilde{\omega}) \rangle | \leq C \| v \|_m Q_p(\omega - \tilde{\omega}). \] (4.40)

Since \( \text{Div}(u - v) = 0 \), we have
\[ | \langle (u - v) \cdot \nabla \omega, G(\omega - \tilde{\omega}) \rangle | = | \langle \nabla \cdot ((u - v) \otimes \omega), G(\omega^*) \rangle | \]
\[ = | \langle \nabla \cdot ((u - v) \otimes \omega), \omega^* \otimes |p-2| \omega^* \rangle |. \]

Integrating by parts we get,
\[ | \langle (u - v) \cdot \nabla \omega, G(\omega - \tilde{\omega}) \rangle | = | \langle (u - v) \otimes \omega, \nabla (|p-2| \omega^*) \rangle | \]
\[ \leq | \langle (u - v) \otimes \omega, |p-2| \omega^* \rangle | \]
\[ \leq | \langle (u - v) \otimes \omega, \omega^* \otimes |p-2| \nabla \omega^* \rangle |. \]

Now using the Hölder's inequality we obtain,
\[ | \langle (u - v) \cdot \nabla \omega, G(\omega - \tilde{\omega}) \rangle | \leq \| (u - v) \otimes \omega \|_q' \| \omega^* \|_{p-2} \| \nabla \omega^* \|_q, \] (4.41)

where \( \frac{1}{q} + \frac{1}{q'} = 1 \).

Using (4.17) and Hölder’s inequality one more time, we have
\[ | \langle (u - v) \cdot \nabla \omega, G(\omega - \tilde{\omega}) \rangle | \leq C \| u - v \|_r \| \omega \|_m \| \omega^* \|_{p-2}/2 \| Q_p(\omega^*) \|_{1/2}, \] (4.42)

where \( \frac{1}{q'} = \frac{1}{r} + \frac{1}{m} \) and \( r = \frac{mp}{m-2} \).

Notice if \( K \) is the Biot-Savart kernel then \( u - v = K \ast \omega - K \ast \tilde{\omega} = K \ast (\omega - \tilde{\omega}) = K \ast \omega^* \). Hence using the Lemma 2.7, (2.10) we get from (4.42)
\[ | \langle (u - v) \cdot \nabla \omega, G(\omega - \tilde{\omega}) \rangle | \]
\[ \leq C (\| \omega^* \|_L^1 + \| \omega^* \|_{L^r}) \| \omega \|_m \| \omega^* \|_{p-2}/2 \| Q_p(\omega^*) \|_{1/2} \]
\[ = C \| \omega^* \|_{p/2} \| \omega \|_m \| Q_p(\omega^*) \|_{1/2} \]
\[ + C \| \omega^* \|_{1/1} \| \omega^* \|_{(p-2)/2} \| \omega \|_m \| Q_p(\omega^*) \|_{1/2}. \] (4.43)
With the help of Lemma 2.6, equation (4.39) yields
\[
|\langle (u - v) \cdot \nabla \omega, G(\omega - \tilde{\omega}) \rangle | \\
\leq C \| \omega \|_m Q_p(\omega^*) + C \| \omega^* \|_{L^1} \| \omega \|_m Q_p(\omega^*)^{(p-1)/p} \\
= C \| \omega \|_m Q_p(\omega - \tilde{\omega}) + C \| \omega - \tilde{\omega} \|_{L^1} \| \omega \|_m Q_p(\omega - \tilde{\omega})^{1/p'}.
\]
(4.44)

Thus substituting the results from (4.40) and (4.44) in (4.39) we have
\[
|\langle u \cdot \nabla \omega - v \cdot \nabla \omega, G(\omega - \tilde{\omega}) \rangle | \leq C (\| v \|_m + \| \omega \|_m) Q_p(\omega - \tilde{\omega}) \\
+ C \| \omega - \tilde{\omega} \|_{L^1} \| \omega \|_m Q_p(\omega - \tilde{\omega})^{1/p'},
\]
(4.45)

where \( C \) is a positive constant depending upon \( m \) and \( p \).

Next we estimate the third term of the equation (4.37). We notice that
\[
|\langle \omega \cdot \nabla u - \tilde{\omega} \cdot \nabla v, G(\omega - \tilde{\omega}) \rangle | \\
= |\langle (\omega - \tilde{\omega}) \cdot u, G(\omega - \tilde{\omega}) \rangle | + |\langle \tilde{\omega} \cdot (u - v), G(\omega - \tilde{\omega}) \rangle |.
\]
(4.46)

Here we proceed in the similar way as before to get
\[
|\langle (\omega - \tilde{\omega}) \cdot u, G(\omega - \tilde{\omega}) \rangle | \leq C \| u \|_m Q_p(\omega - \tilde{\omega}),
\]
(4.47)

and
\[
|\langle \tilde{\omega} \cdot (u - v), G(\omega - \tilde{\omega}) \rangle | \leq C \| \tilde{\omega} \|_m Q_p(\omega - \tilde{\omega}) \\
+ C \| \omega - \tilde{\omega} \|_{L^1} \| \tilde{\omega} \|_m Q_p(\omega - \tilde{\omega})^{1/p'}.
\]
(4.48)

Thus (4.46) yields
\[
|\langle \omega \cdot \nabla u - \tilde{\omega} \cdot \nabla v, G(\omega - \tilde{\omega}) \rangle | \leq C (\| u \|_m + \| \tilde{\omega} \|_m) Q_p(\omega - \tilde{\omega}) \\
+ C \| \omega - \tilde{\omega} \|_{L^1} \| \tilde{\omega} \|_m Q_p(\omega - \tilde{\omega})^{1/p'},
\]
(4.49)

where \( C \) is a positive constant depending upon \( m \) and \( p \).

Hence (4.38), (4.45) and (4.49) yield the desired result (4.36) from (4.37). \( \square \)

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