Abstract

We examine cosmological inflation in a broad family of scalar-tensor models characterized by scalar-dependent non minimal kinetic couplings and Gauss-Bonnet terms. Using a slow roll-approximation, we compute in detail theoretical expectations of observables as spectral indexes, scalar-to-tensor ratio, their running and their running of the running in terms of the parameters which characterize the scalar-tensor model. Hierarchies of consistency equations relating scalar and tensor perturbations and higher order running parameters are presented and examined at the slow roll approximation for the kind of models of interest in this work. From We find detailed expressions for constraints among these parameters. For a specific model, we analyse such quantities and make contact with latest Planck observational data.

Keywords: Inflation with non-minimal kinetic, Gauss-Bonnet terms, Running spectral data.

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1 Introduction

Gravity Einstein theory, general relativity (GR) has repeatedly demonstrated its effectiveness at present epochs of the cosmological evolution at least, even in extreme environments: in the weak and strong field regimes in cosmological epochs down to the end of the radiation dominated era. This is the case for the recent observation of gravitational waves [1] or observations, for the first time presented, consistent with expectations for the shadow of a Kerr black hole (BH) as predicted by standard Einsteins-Hilbert (EH) general relativity [2, 3]. Less is known however about the behaviour of gravity at earlier times, at the beginning of the radiation era or, specially at a, highly probable, inflation era. Hints for modifications to the minimal Einstein theory, defined by adding new terms to the EH action, might appear at these very early times in the history of the Universe.

The cosmic inflation scenario [4, 5] has indeed been favored, albeit strongly constrained, by the latest observational data [6–8] as a suitable pre-Big Bang (BB) scenario for the early universe, providing the circumstantial explanation to flatness, horizon and monopole problems, among others, for the standard hot BB cosmology [9–14]. Inflation provides also, most relevantly for the scope of this work, an account of fluctuations which constitute the seeds for the large scale structure and the observed CMB anisotropies, as well as predicts a nearly scale invariant power spectrum [15–22].

Given the observational CMB data, “bottom-up” methods has been explored to reconstruct the terms of a “true” gravitational lagrangian with a scalar sector [8]. These include Taylor expansions of $V$ (and possibly other functions deforming EH gravity), free-form spline searches, non-parametric techniques or more theoretical approaches where general Lagrangians including “all” the possible terms consistent, only restricted by mild symmetry or technical assumptions, are considered. While these techniques will be more and more important in the future, at present level of resolution “top-down” theoretical approaches are certainly useful. In this analysis specific theoretical models, beyond minimal GR, with a small number of free parameters are explored. These toy models allow us to draw, hopefully, clean conclusions about the potential phenomenological interest of some
or other types of extra terms [23–29].

One of the main findings consistently drawn from Planck data is the, mild but not null, scale dependence of the scalar spectral index $n_S$ and a practically negligible running. The index $n_S$ is different from one at least at $8\sigma$ [8]. This result is consistent with a slow-roll inflation epoch with a natural exit. Constraints on the tensor-to-scalar ratio $r$ are also very stringent.

Planck observations have substantially tightened the constraints on slow-roll inflationary models, ruling out for example hybrid models with $n_S > 1$ and power-law inflation. Monomial potential models with $V(\varphi) \propto \varphi^n$ with $n \geq 2$ and low-scale SUSY models are strongly disfavoured [8, 30]. The aim of this work is to contribute these activities by investigating the slow-roll Scalar-Tensor inflation from running spectral data. Using the slow-roll approximation, we can relate the primordial perturbations to the scalar potential, other non-minimal scalar-dependent parameters appearing in the model and its derivatives. This scenario allows us to directly constrain the parameters defining the model. More precisely, we provide accurate expressions for observable quantities as the scalar and tensorial spectral indexes $n_S, n_T$ and the scalar-to-tensor ratio $r$ as their running counterparts in some specific non-minimal gravitational models. These models include simple but not trivial scalar non-minimal kinetic terms and Gauss-Bonnet terms, both of them some scalar-dependent couplings.

Let us summarize quantitatively the latest Planck results [8] which be relevant for some part of this work. The scalar spectral index $n_S$ is different from one at least at $8\sigma$ with a practically negligible running and running of the running: $n_S = 0.959 \pm 0.006, r < 0.10(95\% CL)$. The Planck 2018 already constraints running and running of running parameters: $dn_S/d\ln k = 0.01 \pm 0.01, d^2n_S/d\ln k^2 = 0.02 \pm 0.01$ (CL $\simeq 68\%$), therefore is becoming theoretically relevant to give expressions for these parameters in models beyond GR. Constraints on the tensor-to-scalar ratio $r$ assuming the consistency relation, $n_T = -r/8$, which is the case for slow-roll inflation driven by a single scalar field with a canonical kinetic term. It is possible to get upper bounds on $r$ without imposing the consistency condition. Deviations of this conditions do indeed occur in the models of inflation based on generic scalar-tensor theories which are to be presented in this work. Finally the Planck observations also constrain the slow-roll parameters. They obtain [8], at % 95 CL, $\epsilon_1 < 0.004 - 0.006, \epsilon_2 = 0.030 \pm 0.006$.

The structure of this work is as follows. In section 2, we present the model: a scalar tensor model with minimal coupling of the scalar field to scalar curvature, non-minimal kinetic coupling of the scalar field to the Einstein’s tensor and coupling of the scalar field to the 4d Gauss-Bonnet invariant. We get the equations of motion in a flat FRW background and define slow-roll parameters. In section 3, we analyse the first and second order scalar and tensorial perturbations for this model and provide expressions for the scalar spectral index and tensor/scalar ratio in terms of the slow-roll parameters. In section 4, we give an explicit model. Section 5 is devoted to further physical discussion.
summary and conclusions.

2 The model and background equations

We investigate generic scalar-tensor models with minimal Einstein-Hilbert sector, non-minimal kinetic coupling of the scalar field to the Einstein’s tensor and coupling of the scalar field to the Gauss-Bonnet (GB) 4-dimensional invariant. The associated action, in the Einstein Frame, can be written as

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} \partial_\mu \partial^\mu \varphi - V(\varphi) + J_1(\varphi)G_{\mu\nu} \partial^\mu \varphi \partial^\nu \varphi - J_2(\varphi)G(R) \right]$$

(2.1)

where $G_{\mu\nu}$ is the Einstein’s tensor, and $G$ is the GB 4-dimensional invariant given by

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}$$

(2.2)

where one has taken $\kappa^2 = M_p^{-2} = 8\pi G = 1$. $\varphi$ is a real scalar field with a potential $V(\varphi)$, which, together with $J_1$ and $J_2$, differentiable functions of $\varphi$, describe the model. $R$ is the Ricci scalar. A possible cosmological constant $\Lambda$ can be included through the scalar potential $V$. The family of models defined by the above action, (2.1), depends on three scalar dependent quantities $V, J_1, J_2$ which should be provided by the UV completion of the theory. As the scope of this work is model-independent, they are left arbitrary, they are chosen with the only restriction of producing suitable inflation observables, matching the observational evidence [6–8].

In what follows, we assume a spatially flat FRW, homogeneous and isotropic, background metric in any of the forms

$$ds^2 = -dt^2 + a(t)^2 d\vec{r}^2$$

(2.3)

or $ds^2 = a(\eta)^2 (-d\eta^2 + d\vec{r}^2)$ where $a$ is a scale factor. The metric and the scalar field equations of motion can be written as follows, using (2.3), [23,29,31]

$$3H^2 = \frac{1}{2} \dot{\varphi}^2 + V + 9H^2 J_1 \dot{\varphi}^2 + 24H^3 J_2,$$  

(2.4)

$$-2\dot{H} = \dot{\varphi}^2 + 6H^2 J_1 \dot{\varphi}^2 - 2 \frac{d}{dt} (HJ_1 \dot{\varphi}^2) + 8H^3 J_2 - \frac{d}{dt} (H^2 J_2),$$  

(2.5)

$$\ddot{\varphi} + 3H \dot{\varphi} + V' = -24H^2 \left( H^2 + \dot{H} \right) J_2' - 6HJ_1 \dot{\varphi}(3H^2 + 2\dot{H})$$  

$$-3H^2 \left( 2J_1 \ddot{\varphi} + J_1' \dot{\varphi}^2 \right)$$  

(2.6)

where the dot represents derivatives with respect the time variable $t$, $H \equiv \dot{a}/a$ is the Hubble parameter and $J_1'$ and $J_2'$ are derivatives with respect $\varphi$. Only two of the previous equations are independent due to the existence of the Bianchi identities. Taking $J_1 = J_2 = 0$, we recover explicitly the standard Friedman equations with a single scalar field

$$3H^2 = \frac{1}{2} \dot{\varphi}^2 + V,$$  

(2.7)

$$-2\dot{H} = \dot{\varphi}^2,$$  

(2.8)

$$\ddot{\varphi} + 3H \dot{\varphi} + V' = 0.$$  

(2.9)
After a simple computation, we get from the field equations (2.4)-(2.6) expressions for $\dot{\varphi}^2$ and $V$ as follows

$$V = H^2 \left( 3 - \epsilon_1 - \frac{5}{2} \Delta_1 - 2k_1 - \frac{1}{2} \Delta_1 (\Delta_2 - \epsilon_1) - \frac{1}{3} k_1 (k_2 - \epsilon_1) \right), \quad (2.10)$$

$$\dot{\varphi}^2 = H^2 \left( 2\epsilon_1 - \Delta_1 - 2k_1 + \Delta_1 (\Delta_2 - \epsilon_1) + \frac{2}{3} k_1 (k_2 - \epsilon_1) \right) \quad (2.11)$$

where we have used the parameter definitions:

$$\epsilon_1 = \left( \frac{i}{H} \right) = -\frac{\dot{H}}{H^2}, \quad (2.12)$$

$$k_1 = 3J_1 \dot{\varphi}^2, \quad (2.13)$$

$$\Delta_1 = 8H J_2. \quad (2.14)$$

For further use, we define also the following parameter hierarchy: for any quantity $X$, in particular for $X = \epsilon_1, k_1, \Delta_1$, we define ($n \geq 1$) (using the number of e-folds variable $N$, $N \equiv \log a$, $dN \equiv -H dt$)

$$X_{n+1} = -\frac{d \log |X_n|}{dN} = \frac{d \log |X_n|}{d \log a}. \quad (2.15)$$

This set of parameters satisfies the following properties, for any $X_n$

$$-\frac{dX_n}{dN} = X_n X_{n+1} = -X'_n \varphi_N, \quad (2.16)$$

$$\frac{d^2X_n}{dN^2} = X_{n+1} (X_{n+1} + X_{n+2}) \quad (2.17)$$

where $X_N \equiv dX/dN$. The main reason for the use of these definitions is the following. In terms of the wavenumber $k$, defined by $k \equiv aH$, we have

$$\frac{dX}{dN} = (\epsilon_1 - 1) \frac{dX}{d \log k} \quad (2.18)$$

and

$$\frac{d^2X}{dN^2} = -\epsilon_1 \epsilon_2 \frac{dX}{d \log k} - (1 - \epsilon_1)^2 \frac{d^2X}{d \log k^2}. \quad (2.19)$$

Or, equivalently,

$$\frac{dX}{d \log k} = \frac{1}{(\epsilon_1 - 1)} \frac{dX}{dN}, \quad (2.20)$$

$$\frac{d^2X}{d \log k^2} = \frac{1}{(\epsilon_1 - 1)^2} \left( \frac{dX}{dN} + \frac{\epsilon_1 \epsilon_2}{(\epsilon_1 - 1)} \frac{d^2X}{dN^2} \right). \quad (2.21)$$

\textsuperscript{1}We use the conventions of the PLANCK collaboration [8].
Then, for small values of the parameters ($\epsilon_1 \to 0$) both derivatives coincide
\[
\frac{dX}{d\log k} \simeq -\frac{dX}{dN}, \\
\frac{d^2 X}{d\log k^2} \simeq -\frac{d^2 X}{dN^2}
\] (2.22)

meanwhile, for $\epsilon_1 \to 1$, we have the relations
\[
\frac{dX}{dN} \simeq 0, \\
\frac{dX}{d\log k} \simeq -\frac{1}{\epsilon_2} \frac{d^2 X}{dN^2}.
\] (2.25)

A quasi-exponential inflationary expansion is provided, in particular, in the slow-roll regime. This regime is defined by the conditions $\epsilon_i, k_i, \Delta_i \ll 1$ (slow-roll conditions). Under this assumption, the field equations (2.4)-(2.6) are greatly simplified. They become
\[
3H^2 - V \simeq 0, \\
\dot{H} + \dot{\varphi}^2 \simeq -6H^2J_1\dot{\varphi}^2 - 8H^3\dot{J}_2 = -H^2(2k_1 + \Delta_1), \\
\dot{\varphi} + \frac{V'}{3H} \simeq -6H^2J_1\dot{\varphi} - 8H^3J'_2.
\] (2.26)

At the slow-roll approximation level, there is a degeneracy of models: different triplets, $(V, J_1, J_2)$, appearing in the action (2.1) provide exactly the same observational results. For the purpose of disentangling the effective dependence on the scalar dependent quantities, it is convenient to rewrite slow-roll field equations (2.27) and (2.28) in a slightly different way. Using the number of e-folds, $N$ as a independent variable,we have
\[
\epsilon_1 = \frac{\dot{\varphi}^2}{2H^2}(1 + 6H^2J_1) + 4H\dot{J}_2, \\
-H\frac{d\varphi}{dN} + \frac{V'}{3H} \simeq -6H^2J_1\dot{\varphi} - 8H^3F'_2
\] (2.29)
\[
\frac{d\varphi}{dN} = \frac{V' + 8J_2'V^2/3V'}{V - 1 + 2VJ_1}.
\] (2.30)

which can be written as
\[
\epsilon_1 = \frac{1}{2} \left( \frac{d\varphi}{dN} \right)^2 (1 + 2VJ_1) - \frac{4}{3} VJ'_2\frac{d\varphi}{dN}, \\
\frac{d\varphi}{dN} = \frac{V' + 8J_2'V^2/3V'}{V - 1 + 2VJ_1}.
\] (2.31)

It follows from such expressions that the model depends only on the scalar combinations
\[
v \equiv V'/V, \quad f_1 \equiv VJ_1, \quad f_2 \equiv VJ'_2.
\] (2.33)

In terms of the reduced variables $v, f_1, f_2$ and the quantity
\[
f_3 \equiv J_2/v,
\]
we can write

\[ \epsilon_1 = \frac{1}{2} \left( \frac{d\varphi}{dN} \right)^2 \left( 1 + 2 f_1 \right) - \frac{4}{3} f_2 \frac{d\varphi}{dN}, \quad (2.34) \]

\[ \frac{d\varphi}{dN} = \frac{v + 8/3 f_3}{1 + 2 f_1}. \quad (2.35) \]

The number of e-folds reads as

\[ N = \int_{\varphi_i}^{\varphi_e} \frac{1 + 2 f_1}{v + 8/3 f_3} d\varphi \quad (2.36) \]

where \( \varphi_i \) and \( \varphi_e \) are the values of the scalar field at the beginning and at the end of the inflation period, respectively. The rest of slow-roll parameters can also be written in terms of \( v, f_1, f_2 \) and \( f_3 \). After a straightforward computation, we find the following relevant expressions

\[ \epsilon_1 = \frac{v^2}{2} \frac{(1 + 8/3 f_3)}{1 + 2 f_1}, \quad (2.37) \]

\[ k_1 = f_1 (\varphi, N)^2 = f_1 v^2 \frac{(1 + 8/3 f_3)^2}{(1 + 2 f_1)^2}, \quad (2.38) \]

\[ \Delta_1 = -\frac{8}{3} f_2 \varphi, N = -\frac{8 f_2 v (1 + 8/3 f_3)}{3} \frac{(1 + 8/3 f_3)}{(1 + 2 f_1)}. \quad (2.39) \]

Higher order parameters, for example \( \epsilon_2, k_2, \Delta_2 \), are easily obtained using Eq (2.16). Successful inflation occurs when \( 0 < \epsilon_1 < 1 \). The value of the quantity \( \epsilon_1(\varphi_i) \to 0 \) depends strongly on the quantity Gaus-Bonnet related quantity \( f_3 \propto J_2 \). Small values \( \epsilon_1(\varphi_i) \approx 0 \) can be reached in different regimes: we have two cases, in the first one, if

\[ \varphi, N \simeq 0, \quad (2.40) \]

then GB-related quantity

\[ f_3^e \simeq -3/8 \quad (2.41) \]

. In a second case when,

\[ f_3^e = -1/2, \quad (2.42) \]

then is required that

\[ \varphi, N \simeq -\frac{v}{3} \frac{1}{1 + 2 f_1}. \quad (2.43) \]

The graceful exit of inflation depends on whether the equation \( \epsilon_1(\varphi_e) = 1 \) can be accomplished for some \( \varphi_e \). The graceful exit relation becomes

\[ \epsilon_1(\varphi_e) = 1 = \frac{v^2 (1 + 8/3 f_3)}{2} \left| \frac{d\varphi}{dN} \right|_{\varphi_e}. \quad (2.44) \]
which is a second degree equation in \( v \), it can be written as
\[
\frac{1}{2} v^2 (1 + 8/3f_3) - (1 + 2f_1) = 0,
\] (2.45)
with the solution
\[
\frac{1}{2} v^2 |_{\varphi_e} = \frac{1 + 2f_1}{(1 + 8/3f_3)}.
\] (2.46)

At this point we get a relation connecting the theoretical quantities in the action and inflation exit. The condition for having a real solution for \( v \), and then a graceful inflation exit, is that
\[
(f_2)^2 \geq -\frac{9}{8} (1 + 2f_1)
\] (2.47)
evaluated at \( \varphi_e \). This becomes a constraint imposed on the, otherwise arbitrary, \( J_1 \) and \( J_2 \) functions:
\[
(V J_2')^2 \geq -9/8 (1 + V J_1)
\] (2.48)
which implies a "uncertainty"-like condition between the potential and kinetical functions evaluated at this moment
\[
V(\varphi_e)J_1(\varphi_e) \geq -\frac{1}{2}.
\] (2.49)

3 First and higher order perturbations. Running Consistency equations

The spectral indexes are defined by [8,23,26,27]
\[
\begin{align*}
  n_S - 1 &= \frac{d \log P_S}{d \log k}, \\
  n_T &= \frac{d \log P_T}{d \log k}.
\end{align*}
\] (3.1)
where \( P_S, P_T \) are the power spectra of the scalar and tensor modes. The running and the running of running of the spectral indexes are given by the following expressions in terms of the slow-roll parameters [26,27,31]
\[
\begin{align*}
  n_S - 1 &= -2\epsilon_1 - \frac{2\epsilon_1\epsilon_2 - \Delta_1\Delta_2}{2\epsilon_1 - \Delta_1}, \\
  n_T &= -2\epsilon_1.
\end{align*}
\] (3.3)

Moreover, the scalar-to-tensor ratio takes the following form [31]
\[
r = 8 \left( \frac{2\epsilon_1 - \Delta_1}{1 - \frac{1}{3}k_1 - \Delta_1} \right) \simeq 8 \left( 1 - \frac{1}{3}k_1 - \Delta_1 \right) (2\epsilon_1 - \Delta_1) \simeq 8 (2\epsilon_1 - \Delta_1) + O^2,
\] (3.5)
where \( O^2 \) are terms quadratic in the slow-roll parameters. In this class of models, as in standard GR single scalar inflation, the parameters \( n_S, n_T \) and \( r \) are not independent variables. The models predict a relation among them which reads as
\[
r + 8n_T \simeq -8\Delta_1 + O^2.
\] (3.6)
This relation differs from the standard inflation relation \( r = -8n_T \) by \( \Delta_1 \), this is a supposedly small quantity, proportional to the GB term

\[
r + 8n_T \simeq -64H\dot{J}_2.
\]  

(3.7)

It turns out that we can express observable quantities like the scalar and tensor power spectrum, the spectrum index and tensor/scalar ratio in terms of \( v, f_1, f_2 \) (evaluated at \( \varphi = \varphi_i \)). For instance, the tensor spectrum index and the tensor/scalar ratio are given by

\[
n_T = -v^2 \frac{(1 + 8f_2/3v)}{(1 + 2f_1)},
\]

(3.8)

\[
r = 8v^2 \frac{(1 + 8f_2/3v)^2}{(1 + 2f_1)}.
\]

(3.9)

meanwhile the scalar spectral index is written as

\[
n_s - 1 = -\frac{9v (I_1 + v^2) + 24f_2 (I_2 + 2v^2) + 64f_2^2 v + 2f_1 v (3v + 8f_2)^2}{3 (1 + 2f_1)^2 (3v + 8f_2)}
\]

(3.10)

where one has

\[
I_1 \equiv -\left(1 + \frac{8f_2}{3v}\right)((2 + 4f_1) v' - 2vf_1') - \frac{1}{3} 8 (1 + 2f_1) v \left(\frac{f_2}{v}\right)';
\]

(3.11)

\[
I_2 \equiv \frac{2}{3} f_2 (3 + 8f_2 v) f_1' - (1 + 2f_1) \left(1 + \frac{8f_2}{3v}\right)(f_2 v' + vf_2') + \frac{8}{3} f_2 v \left(\frac{f_2}{v}\right)'^3.
\]

(3.12)

In concrete models we can invert the relations (3.9)-(3.10) to get the triplet of theoretical values \( v, f_1, f_2 \) in terms of the observable triplet \( (n_s, n_T, r) \), or, in terms of the original variables \( (V, J_1, J_2) \).

One can get further relations between the different spectral parameters and their runnings. For example, the physical scalar spectral index can be related to the scalar ratio, the running of the logarithm of the ratio and the parameter \( n_T \). It is straightforward to check that the equation (3.10) can be written as

\[
n_s - 1 = -2\epsilon_1 + \frac{d}{dN} \log(2\epsilon_1 - \Delta_1),
\]

(3.13)

which, using the expressions (3.5), can be written as

\[
n_s - 1 = n_T + (\epsilon_1 - 1) \frac{d\log r}{d\log k} + O^2
\]

(3.14)

where \( O^2 \) are terms of superior order: \( O^2 = \frac{1}{3}k_2 + \Delta_2 \). From (3.14), when \( \epsilon_1 \simeq 1 \) then we arrive to the constraint

\[
n_s - 1 = n_T + O^2
\]

(3.15)

or

\[
n_s - 1 = -r/8 - \Delta_1 + \Delta_2 + \frac{1}{3}k_2.
\]

(3.16)
Hierarchies of consistency equations relating scalar and tensor perturbations and the lowest levels and higher order running parameters have been presented (i.e. \([26,27]\)). The scale running and the running of the running of the scalar spectral index can be easily computed at the slow roll approximation for the kind of models of interest in this work. From (3.13) and (3.14). They are given by

\[
\frac{dn_S}{d \log a} = -\frac{1}{8} \frac{dr}{d \log a} - \frac{d^2 \log r}{d \log a^2} - \frac{d \Delta_1}{d \log a},
\]

\[
\frac{d^2 n_S}{d \log a^2} = -r \left( \frac{d \log r}{d \log a} \right) - \frac{d^2 \log r}{d \log a^2} - \frac{d \Delta_1}{d \log a},
\]

Using Eqs 2.16) and (2.17), we obtain

\[
\frac{d \Delta_1}{d \log a} = \Delta_1 \Delta_2,
\]

\[
\frac{d^2 \Delta_1}{d \log a^2} = \Delta_1 \Delta_2 (\Delta_2 + \Delta_2).
\]

In terms of the derivatives with respect to the field \(\varphi\), they become as follows

\[
\frac{d \log r}{d \log a} = - (\log r)' \varphi,_{N},
\]

\[
\frac{d^2 \log r}{d \log a^2} = (\log r)'' (\varphi,N)^2 + \frac{1}{2} (\log r)' (\varphi,N)^2)',
\]

together with

\[
\frac{d^3 \log r}{d \log a^3} = - (\log r)''' (\varphi,N)^3 - \left( \frac{3}{2} (\log r)'' ((\varphi,N)^2)' + \frac{1}{2} (\log r)' ((\varphi,N)^2)'' \right) \varphi,_{N}.
\]

### 4 A simple toy model: the monomial model

We provide will provide an explicit model dealing with power-law potentials and scalar couplings. In this case, the expressions can be further worked out. Concretely, we focus our attention to couplings of monomial types parameterized as follows

\[
V = \frac{\lambda}{n} \varphi^n, \quad J_1 = \beta_1' \varphi^{m_1-n}, \quad J_2 = \beta_2' \varphi^{m_2-n}.
\]

where \(\lambda\) is a coupling parameter. The physical results depend only on the combinations \((\beta_1 = \frac{\lambda \beta_1}{n}, \beta_2 = \frac{\lambda \beta_2}{n(m_2-n)}) \quad (m_2 \neq n), \quad \beta_2 = 0 \quad (m_2 = n)\) which are given by

\[
v = \frac{n}{\varphi}, \quad f_1 = \beta_1 \varphi^{m_1}, \quad f_2 = \beta_2 \varphi^{m_2-1}.
\]

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In this way, they effectively depend only on two real parameters \( \beta_1, \beta_2 \) and on the integer \( n \). Taking \( c = \frac{8\beta_2}{3n} \) and \( b = 2\beta_1 \), we get slow-roll parameters

\[
\epsilon_1 = \frac{n^2}{2\varphi^2} \left( 1 + c\varphi^{m_2} \right) \left( 1 + b\varphi^{m_1} \right),
\]

\[
k_1 = \frac{n^2b\varphi^{m_1}}{2\varphi^2} \left( 1 + c\varphi^{m_2} \right)^2 \left( 1 + b\varphi^{m_1} \right),
\]

\[
\Delta_1 = \frac{-cn^2\varphi^{m_2} \left( 1 + c\varphi^{m_2} \right)}{\varphi^2} \left( 1 + b\varphi^{m_1} \right).
\]

From Eq (2.16), we obtain higher order-slow-roll parameters \( \epsilon_1, k_1 \) and \( \Delta_1 \) which are given by

\[
\epsilon_1 = \frac{n}{\varphi^2} \left( \frac{1 + c\varphi^{m_2}}{1 + b\varphi^{m_1}} \right) \left( 1 + b\varphi^{m_1} \right) \left( -m_2c\varphi^{m_2} + 2c\varphi^{m_2} + 2 \right),
\]

\[
k_1 = \frac{n^2}{\varphi^2} \left( 1 + b\varphi^{m_1} \right) \left( 1 + c\varphi^{m_2} \right)^2 \left( 1 + b\varphi^{m_1} \right) \left( 1 + 2m_2 \right),
\]

\[
\Delta_1 = \frac{n}{\varphi^2} \left( \frac{bm_1\varphi^{m_1} \left( 1 + c\varphi^{m_2} \right) \left( 1 + b\varphi^{m_1} \right) \left( 1 + c\varphi^{m_2} - m_2c\varphi^{m_2} \right)}{1 + b\varphi^{m_1} \right. \right.
\]

\[
\]

\[
\]

\[
\]

In this way, the number of e-foldings is

\[
N = \int_{\varphi_{\text{end}}}^{\varphi_*} \frac{\varphi \left( 1 + b\varphi^{m_1} \right)}{n \left( 1 + c\varphi^{m_2} \right)} d\varphi.
\]

The last expression can be integrated in a closed form in order to provide

\[
N - N_0 = \frac{\varphi^2}{2n} {}_2F_1 \left( 1, \frac{2}{m_2}, 1 + \frac{2}{m_2}, -c\varphi^{m_2} \right) \left( 1 + b\varphi^{m_1+2} \right) \left( 2 + m_1 \right) \left( 1 + \frac{m_1 + 2}{m_2}, 1 + \frac{m_1 + 2}{m_2}, -c\varphi^{m_2} \right)
\]

\[
\]

\[
\]

\[
\]

where \( {}_2F_1 \) is the Hypergeometric function. For small Gauss-Bonnet coupling \( c \to 0 \), the functions \( g_1 \) and \( g_2 \) reduce to

\[
g_1 \approx 1 - \frac{2c\varphi^{m_2}}{2 + m_2} + O(c)^2,
\]

\[
g_2 \approx 1 - \frac{(2 + m_1)c\varphi^2}{2 + m_1 + m_2} + O(c)^2.
\]

At the leading order in \( c \) and \( b \), we find

\[
N - N_0 \approx \frac{\varphi^2}{2n} - c \frac{2\varphi^{m_2+2}}{n(2 + m_2)} + b \frac{\varphi^{m_1+2}}{2 + m_1}.
\]
In the large coupling \((c, b \to \infty)\), however, we get

\[
N - N_0 \simeq \frac{b/c}{n(m_1 - m_2 + 1)} \varphi^{m_1 - m_2 + 1}. \tag{4.14}
\]

Now we are in position to study a special subcase. When the Gauss-Bonnet term is of the type \(J_2 \sim 1/\varphi^n\) or \((m_2 = 0)\) corresponding to \(J_2 \sim \varphi^{-1}\), we get the slow-roll parameters

\[
\epsilon_1 = \frac{n^2}{\varphi^2} \frac{(1 + c)}{(1 + b \varphi^{m_1})}, \tag{4.15}
\]

\[
k_1 = \frac{n^2 b \varphi^{m_1}}{2 \varphi^2} \frac{(1 + c)^2}{(1 + b \varphi^{m_1})^2}, \tag{4.16}
\]

\[
\Delta_1 = \frac{-cn^2}{\varphi^2} \frac{(1 + c)}{(1 + b \varphi^{m_1})}. \tag{4.17}
\]

For higher order-slow-roll parameters \(\epsilon_1, k_1\) and \(\Delta_1\) are given by

\[
\epsilon_1 = \frac{n}{\varphi^2} \frac{(1 + c)}{(1 + b \varphi^{m_1})^2}, \tag{4.18}
\]

\[
k_1 = \frac{n}{\varphi^2} \frac{(1 + c)\left(2bm_1 \varphi^{m_1} + \varphi^{m_1}(m_1 - 2) + 2(1 + b \varphi^{m_1})\right)}{(1 + b \varphi^{m_1})^2}, \tag{4.19}
\]

\[
\Delta_1 = \frac{n}{\varphi^2} \frac{(1 + c)\left(bm_1 \varphi^{m_1} + 2(1 + b \varphi^{m_1})\right)}{(1 + b \varphi^{m_1})^2}. \tag{4.20}
\]

For this model, the number of e-foldings takes the following from

\[
N = \int_{\varphi_{\text{end}}}^{\varphi} \frac{\varphi + b \varphi^{m_1}}{n + 1 + c} d\varphi. \tag{4.21}
\]

Integrating this expression, we find

\[
N = \frac{1}{n(1 + c)} \left(\frac{\varphi^2}{2} + \frac{b \varphi^{m_1 + 2}}{m_1 + 2}\right)_{\varphi_{\text{end}}}. \tag{4.22}
\]

Roughly, the scalar spectral index and the tensor-to-scalar ratio are, respectively,

\[
n_s = 1 - \frac{n_1 \varphi^{m_1} (1 + c) + (1 + b \varphi^{m_1})(n(1 + c) + 2c + 2)}{(1 + b \varphi^{m_1})^2}, \tag{4.23}
\]

\[
n_T = \frac{n^2}{\varphi^2} \frac{(1 + c)}{(1 + b \varphi^{m_1})}, \tag{4.24}
\]

\[
r = \frac{8n^2}{\varphi^2} \frac{(1 + c)^2}{(1 + b \varphi^{m_1})}. \tag{4.25}
\]

The running of spectral indexes are given by the following expressions

\[
\frac{dn_s}{d\log k} = \frac{1}{1 - \epsilon_1} \frac{dn_s}{d\log a}. \tag{4.26}
\]
where one has

\[
\frac{dn_S}{d \log a} = -n_S' \varphi, N, \quad (4.27)
\]

\[
\frac{dn_S}{d \log a} = -n^2(1 + c)^2(A_1 + A_2) / A_3.
\]

Here, the terms \( A_i \) take the following form

\[
A_1 = (1 + b \varphi^{m_1}) \left( (4 + 2n)(1 + b \varphi^{m_1}) + b m_1 \varphi^{m_1} (4 + n) \right)
\]

\[
A_2 = b m_1^2 \varphi^{m_1} (-1 + b \varphi^{m_1})
\]

\[
A_3 = \varphi^4 (1 + b \varphi^{m_1})^4.
\]

For the sake of illustration we present some simple numerical results for this toy model. In Fig.1, we present the correlation between the spectral indexes \( r, n_S \) for some parameters chosen to provide results consistent with the Planck observations. In Fig.2 (Left), we illustrate certain variations of the running of the scalar spectral index in terms of \( m_1 \). Similarly, in Fig.2 (Right), we show the tensor spectral index in terms of the same parameter \( m_1 \).

### 5 Concluding and further remarks

In this work, we have investigated the possibility of constraining scalar-tensor models of gravitation beyond GR using CMB observations. For the sake of concreteness we have focused in particular in slow-roll inflation originated in models with minimal coupling to scalar curvature, non-minimal scalar kinetic terms and scalar coupling to the Gauss-Bonnet invariant. After presenting the general approach, we have explored some cases where the potential, kinetic coupling and the Gauss-Bonnet coupling are power monomials of different grades. At the slow-roll approximation, we have found that inflation
only depends on two particular combinations of the potential and the other non-minimal couplings. We have given detailed explicit expressions for the theoretical predictions of observables as $n_S, n_T$ and $r$ and their first and second level runnings. In the framework of such models, we have presented some numerical results about the overall effect of the non-minimal couplings in viable inflation scenarios. In particular, we have studied how the present and future CMB observations have the power for discriminating existing inflation scenarios within the scope of these models. For the models considered here, it has been shown that the Introduction of additional interactions, given by non-minimal, kinetic and Gauss-Bonnet couplings, can lower the tensor-to-scalar ratio to values that are consistent with latest observational constraints, and the problem of large fields in chaotic inflation can be avoided.

The Planck observations have, consistently along the years, significantly restricted the space of inflationary models. The latest observational data disfavor monomial-type models $V \propto \varphi^n$ with $n \geq 2$ in the minimally coupled scalar field. With the Introduction of additional interactions like non-minimal, kinetic couplings and Gauss-Bonnet couplings, it has been revealed that the tensor-to-scalar ratio can be lowered to values that are consistent with latest observational constraints [8, 30].

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