Improved effective estimates of Pólya’s Theorem for quadratic forms

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Abstract. Following de Loera and Santos, the Pólya exponent of a \( n \)-ary real form (i.e. homogeneous polynomial in \( n \) variables with real coefficients) \( f \) is the infimum of the upward closed set of nonnegative integers \( m \) such that \( (x_1 + \cdots + x_n)^m f \) strictly has positive coefficients. By a theorem of Pólya, a form assumes only positive values over the standard \( (n-1) \)-simplex in Euclidean \( n \)-space if and only if its Pólya exponent is finite. In this note, we compute an upper bound of the Pólya exponent of a quadratic form \( f \) that assumes only positive values over the standard simplex. Our bound improves a previous upper bound due to de Klerk, Laurent and Parrilo. For example, for the binary quadratic form \( f_κ = \lambda^2 x_1^2 - 2κλ x_1 x_2 + x_2^2 \), which assumes only positive values over the standard simplex whenever \( 0 \leq κ < 1 < λ \), our upper bound of its Pólya’s exponent is \( O(1/λ) \) times that of de Klerk, Laurent and Parrilo’s as \( λ \) tends to infinity.

Following de Loera and Santos \[1\], the Pólya exponent of a real form (i.e. real homogeneous polynomial) \( f \) in \( \mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n] \) is the infimum of the upset of nonnegative integers \( m \in \mathbb{N} := \{0, 1, 2, \ldots \} \) such that \( (x_1 + \cdots + x_n)^m f \) strictly has positive coefficients. Here, a form \( g = \sum_{|\beta|=\ell} b_{\beta} x_{\beta} \in \mathbb{R}[x] \) of degree \( \ell \) is said to strictly have positive coefficients if \( b_{\beta} > 0 \) whenever \( |\beta| = \ell \). As usual \( x_{\beta} := x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n} \) and \( |\beta| := \beta_1 + \beta_2 + \cdots + \beta_n \) for \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}^n \). We introduce the notation \( \mu(f) \) for the Pólya exponent of a form \( f \in \mathbb{R}[x] \). For instance, \( \mu((x_1 + \cdots + x_n)^d) = 0 \) whenever \( d \) is a nonnegative integer and \( \mu(-1) = \infty \).

By a theorem of Pólya \[6\] (reproduced in \[5\] pp. 57-60)), a form assumes only positive values over the \((n-1)\)-simplex in \( \mathbb{R}^n \) whose vertices are \((1,0,\ldots,0)\), \((0,1,\ldots,0)\), \ldots, \((0,\ldots,0,1)\) if and only if its Pólya exponent is finite. Let \( \Delta_n \) denote this \((n-1)\)-simplex, which is referred to as the standard \((n-1)\)-simplex in \( \mathbb{R}^n \) (see e.g. \[8\] p. 7]). As a subset of \( \mathbb{R}^n \),

\begin{equation}
\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1, \ldots, x_n \geq 0, x_1 + \cdots + x_n = 1 \}.
\end{equation}

From the work of de Klerk, Laurent and Parrilo \[2\], an upper bound of the Pólya exponent of forms of general degree \( d \) that assume only positive values over \( \Delta_n \) can be

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obtained. In the case of $d = 2$ (i.e. for quadratic forms), this bound can be formulated explicitly as follows. For a quadratic form $f = \sum_{i,j=1}^{n} a_{ij} x_i x_j \in \mathbb{R}[x]$ where $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, n$, if $f$ assumes only positive values over $\Delta_n$, then it follows from [2, Proof of Theorem 1.1] that

$$
\mu(f) \leq \left[ \frac{\max_{i,j=1,\ldots,n} a_{ij}}{\inf_{t \in \Delta_n} f(t)} \right] - 1.
$$

This bound from the work of de Klerk, Laurent and Parrilo improves a previous upper bound that can similarly be obtained from the work of Powers and Reznick [7, Proof of Theorem 1].

A goal of this note is to further improve this upper bound (2) of the Pólya exponent of quadratic forms that assume only positive values over $\Delta_n$. To state our improved upper bound, given a quadratic form $f = \sum_{i,j=1}^{n} a_{ij} x_i x_j \in \mathbb{R}[x]$ where $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, n$, define an associated quadratic form

$$
\hat{f} := \frac{1}{2} \sum_{i,j=1}^{n} (a_{ii} + a_{jj}) x_i x_j \in \mathbb{R}[x].
$$

Then we have

**Theorem 1.** Let $f \in \mathbb{R}[x]$ be a quadratic form. If $f$ assumes only positive values over $\Delta_n$, then

$$
\mu(f) \leq \sup_{t \in \Delta_n} \left| \frac{\hat{f}(t)}{f(t)} \right| - 1.
$$

Here, as usual, $[s]$ is the floor of $s \in \mathbb{R}$, namely the supremum of the downset of integers less than or equal to $s$.

**Proof.** Observe that $(x_1 + \cdots + x_n)^m f$ has strictly positive coefficients if and only if

$$
[x^{t-(m+2)}][(x_1 + \cdots + x_n)^m f] > 0 \text{ for all } t \in \Delta_n \text{ and } m \in \mathbb{N} \text{ with } t \cdot (m+2) \in \mathbb{N}^n,
$$

where $[x^a]g := b_a$ denotes the coefficient of $x^a$ in a form $g = \sum_a b_a x^a \in \mathbb{R}[x]$. But the following identity holds for all quadratic forms $f \in \mathbb{R}[x]$:

$$
[x^{t-(m+2)}][(x_1 + \cdots + x_n)^m f] = \frac{1}{m+1} \left( \frac{m+2}{t \cdot (m+2)} \right) \left( (m+2) f(t) - \hat{f}(t) \right).
$$

(This identity, being linear in $f$, can be verified by checking on basis quadratic forms: namely $f = x_k^2$ for $k = 1, \ldots, n$ and $f = x_i x_j$ for $1 \leq i < j \leq n$.) Thus $[x^{t-(m+2)}][(x_1 + \cdots + x_n)^m f] > 0$ whenever $m > \hat{f}(t)/f(t) - 2$. Therefore $(x_1 + \cdots + x_n)^m f$ strictly has positive coefficients whenever $m \in \mathbb{N}$ satisfies $m > \sup_{t \in \Delta_n} (\hat{f}(t)/f(t)) - 2$. This implies (4).
Corollary 2. Let $f = \sum_{i,j=1}^{n} a_{ij}x_i x_j \in \mathbb{R}[x]$ be a quadratic form where $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, n$. If $f$ assumes only positive values over $\Delta_n$, then

$$(5) \quad \mu(f) \leq \left[ \frac{\max_{i=1,\ldots,n} a_{ii}}{\inf_{t \in \Delta_n} f(t)} \right] - 1.$$ 

Proof. By Theorem 1, it suffices to show that

$$(6) \quad \sup_{t \in \Delta_n} \left\lfloor \frac{\hat{f}(t)}{f(t)} \right\rfloor \leq \left[ \frac{M(f)}{\inf_{t \in \Delta_n} f(t)} \right], \quad \text{where } M(f) := \max_{i=1,\ldots,n} a_{ii}.$$ 

But $\hat{f}(t) = \frac{1}{2} \sum_{i,j=1}^{n} (a_{ii} + a_{jj}) t_i t_j \leq M(f) \sum_{i,j=1}^{n} t_i t_j = M(f) (t_1 + \cdots + t_n)^2 = M(f)$ whenever $t \in \Delta_n$, since $t_1 + \cdots + t_n = 1$ from (1). Hence

$$\frac{\hat{f}(t)}{f(t)} \leq \frac{M(f)}{f(t)} \leq \frac{M(f)}{\inf_{t \in \Delta_n} f(t)} \quad \text{for all } t \in \Delta_n,$$

so (6) follows by the monotonicity of the floor function.

Since $\max_{i} a_{ii} \leq \max_{i,j} a_{ij}$, this bound (5) in Corollary 2 already improves the upper bound (2) from the work of de Klerk, Laurent and Parrilo – not to mention our bound (4) in Theorem 1.

To illustrate the relative tightness of the upper bounds (4) and (5) over (2), let a parameter $\lambda > 1$ be given and consider a binary quadratic form

$$(7) \quad f_{\lambda} := \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \lambda^2 & -\kappa \lambda \\ -\kappa \lambda & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda^2 x_1^2 - 2\kappa \lambda x_1 x_2 + x_2^2 \in \mathbb{R}[x_1, x_2] \quad (\kappa \in \mathbb{R}).$$

If $0 \leq \kappa < 1$, then $f_{\lambda} = (\lambda x_1 - x_2)^2 + 2(1-\kappa)\lambda x_1 x_2 > 0$ whenever $(x_1, x_2) \in \Delta_2$, hence $f_{\lambda}$ has finite Pólya exponent, from Pólya’s Theorem. Calculations show that

$$\sup_{t \in \Delta_2} \frac{\hat{f}_{\lambda}(t)}{f_{\lambda}(t)} - 1 = \frac{\lambda^2 + 2\kappa \lambda + 1}{2\lambda - 2\kappa \lambda} - \frac{\lambda^2 + 2\kappa \lambda + \kappa^2}{1 - \kappa^2} = \frac{\max\{\lambda^2, 1\}}{\inf_{t \in \Delta_2} f_{\lambda}(t)} - 1 = \frac{\max\{\lambda^2, -\kappa \lambda, 1\}}{\inf_{t \in \Delta_2} f_{\lambda}(t)} - 1.$$

Hence $\sup_{t \in \Delta_2} \left[ \frac{\hat{f}_{\lambda}(t)}{f_{\lambda}(t)} \right] - 1 = \frac{1}{2(1-\kappa)} \cdot \lambda + O_{\kappa}(1)$ which is $O_{\kappa}(1/\lambda)$ times that of $[\max\{\lambda^2, 1\}/\inf_{t \in \Delta_2} f_{\lambda}(t)] - 1 = [\max\{\lambda^2, -\kappa \lambda, 1\}/\inf_{t \in \Delta_2} f_{\lambda}(t)] - 1 = \frac{1}{1-\kappa \lambda} \cdot \lambda^2 + O_{\kappa}(\lambda)$ as $\lambda \to \infty$. Here $O_{\kappa}(\lambda^c)$ indicates a quantity bounded in absolute value by $C(\kappa)\lambda^c$ whenever $\lambda \geq 1$, where $C(\kappa)$ is a constant depending only on $\kappa \in [0,1)$ (for $c \in \mathbb{R}$). Therefore

$$\left( \sup_{t \in \Delta_2} \left[ \frac{\hat{f}_{\lambda}(t)}{f_{\lambda}(t)} \right] - 1 \right) / \left( \left[ \frac{\max\{\lambda^2, -\kappa \lambda, 1\}}{\inf_{t \in \Delta_2} f_{\lambda}(t)} \right] - 1 \right) = \frac{1 + \kappa}{2} \cdot \frac{1}{\lambda} + O_{\kappa}(\frac{1}{\lambda^2}) \text{ as } \lambda \to \infty.$$

From the proof of their effective version of Pólya’s Theorem for a general degree real form that assumes only positive values over the standard simplex $\Delta_n$, de Klerk, Laurent and Parrilo obtained an estimate for rate of convergence of an associated hierarchy of linear programming (LP) approximations to the problem of computing the infimum of a polynomial of fixed degree $d$ over $\Delta_n$ [2]. In the same way, we hope that our proof of Theorem [1] in the case of quadratic forms would be of interest to the estimation of the rate of convergence of the particular hierarchy of LP approximations to the standard quadratic optimization (SQO) problem when $d = 2$ (see e.g. [4] and also [3] on a further specialization of the SQO problem to the maximum stable set problem).

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