A Derivation of the Fradkin-Shenker Result From Duality: Links to Spin Systems in External Magnetic Fields and Percolation Crossovers

Zohar Nussinov

Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545

(Dated: Received March 23, 2022)

In this article, we illustrate how the qualitative phase diagram of a gauge theory coupled to matter can be directly proved and how rigorous numerical bounds may be established. Our work reaffirms the seminal result of Fradkin and Shenker from another vista. Our main ingredient is the combined use of the self-duality of the three dimensional $Z_2/Z_2$ theory and an extended Lee-Yang theorem. We comment on extensions of these ideas and firmly establish the existence of a sharp crossover line in the two dimensional $Z_2/Z_2$ theory.

PACS numbers:

I. INTRODUCTION

Lattice gauge theories [1] witnessed an accelerated revival in condensed matter physics during the last decade. Their applications are widespread. Amongst others, these include novel theories of liquid crystals [2], the incorporation of Berry phase effects in quantum spin systems [3, 4], and stimulating suggestions for long-distance physics of lightly doped Mott-Hubbard insulators [5]. Further research relating to fundamental questions in gauge theories followed, e.g. [6, 7]. Central to many of these investigations is the behavior of matter fields minimally coupled to gauge fields. Several key results in these theories were noted long ago by Fradkin and Shenker [8] (complemented by treatments in [9, 10]). Perhaps the best known result of [8] is the demonstration that (when matter fields carry the fundamental unit of charge) the Higgs and confinement phases of gauge theories are smoothly connected to each other and are as different as a liquid is from a gas. This result remains one of the cornerstones of our understanding of the phases of gauge theories. Although derived long ago, the physical origin of this effect does not seem to be universally agreed upon.

In the current article, we revisit this old result and rederive it for the original $Z_2/Z_2$ theory investigated in [8]. Our proof relies merely on duality and the Lee-Yang theorem. We further illustrate why similar results are anticipated for other gauge theories. Our derivation highlights the origin of this phenomenon as akin to the absence of phase transitions in spin systems in a magnetic field. Notwithstanding the absence of true non-analyticities, some such spin models display a percolation crossover line [10] at which the surface tension of an oppositely oriented spin cluster vanishes. In this article, we firmly establish the existence of precisely such a sharp percolation crossover line for one of the most trivial $Z_2/Z_2$ theories (the $d = 1+1$ dimensional theory).

II. $Z_2$ MATTER COUPLED TO $Z_2$ GAUGE FIELDS

In matter coupled gauge theories, matter fields ($\{\sigma_i\}$ reside as lattice sites $i$ while gauge fields $U_{ij}$ reside on the links connecting sites $i$ and $j$. The $Z_2$ matter coupled to $Z_2$ gauge field theory ($Z_2/Z_2$ in common notation) is the simplest incarnation of a matter coupled gauge theory. Its action reads

$$S = -\beta \sum_{\langle ij \rangle} \sigma_i U_{ij} \sigma_j - K \sum \prod U_{ijkl}$$

on a hypercubic lattice. Here, the first sum is over all nearest neighbor links $\langle ij \rangle$ in the lattice while the second is the product of the four gauge fields $U_{ij} U_{jk} U_{ki} U_{kl}$ over each minimal plaquette (square) of the lattice. Both matter ($\sigma_i$) and gauge ($U_{ij}$) fields are Ising variables within the $Z_2/Z_2$ theory: $\sigma_i = \pm 1$, $U_{ij} = \pm 1$. A trivial yet fundamental observation is that the quantity $z_{ij} \equiv \sigma_i U_{ij} \sigma_j$, where $i$ and $j$ denote two nearest neighboring lattice sites, is invariant under local $Z_2$ gauge transformations

$$\sigma_i \rightarrow \eta_i \sigma_i, \quad U_{ij} \rightarrow \eta_i U_{ij} \eta_j$$

with the arbitrary on-site $\eta_i = \pm 1$ [11]. The action of Eq. (1) may be trivially written in terms of these gauge invariant bond variables $\{z_{ij}\}$ as

$$S = -\beta \sum_{\text{links}} z_{ij} - K \sum z_{zzzz}.$$ (3)

The matter coupling $\beta$ acts as a magnetic field on the spin variable $z$. On a new lattice whose sites reside on the centers of all bonds, this is none other than a model having 4-spin interactions augmented by a $Z_2$ symmetry breaking (for finite $\beta$) magnetic field. For $\beta > 0$, the link expectation value $\langle z_{ij}\rangle \equiv \langle \sigma_i U_{ij} \sigma_j \rangle \neq 0$. As shown by Wegner [12], three (or 2+1) dimensional variants of the $Z_2/Z_2$ model with couplings ($\beta, K$) are equivalent to the same model at couplings ($\beta^*, K^*$) related via the self-duality relations

$$\exp(-2\beta^*) = \tanh K, \quad \exp(-2K^*) = \tanh \beta.$$ (4)

III. DUALITY AND THE LEE YANG THEOREM

As illustrated above, the matter coupled gauge theory can be re-interpreted as a pure gauge theory with an additional
magnetic field applied. Such an analogy immediately triggers a certain intuition regarding the exclusion of phase transitions in certain systems. In standard spin models governed by the classical action

$$S = -\frac{1}{2} \sum_{ij} J_{ij} s_i s_j - \sum_i h s_i, \quad (5)$$

with $H$ the Hamiltonian no phase transition can occur when a symmetry breaking magnetic field ($h \neq 0$) is applied. It is clear that the local magnetization magnetization ($s$) $\neq 0$ and this goes hand in hand with an analytic free energy.

Lee and Yang [13] proved that, in the thermodynamic limit, the partition function cannot have zeros. This can be shown to imply an analytic free energy for magnetic fields for which $|\text{Im}\{h\}| < |\text{Re}\{h\}|$ (with Im and Re the imaginary and real components respectively). This may be extended to many systems. Its generalization to a pure $\mathbb{Z}_2$ lattice gauge action with a magnetic field applied on each gauge link (Eq.(3)) on a general hyper-cubic lattice of dimension $d$ has been done [14]. However, as Eq.(3) is equivalent to the general matter coupled gauge theory of Eq.(1), this implies that the free energy in the presence of matter coupling ($\beta = 0$) is analytic. Along the $\beta = 0$ line of Eq.(1) (the pure gauge only theory), the value $K = K_c$ at which a confining transition occurs may be inferred from the critical temperature of the three dimensional Ising model. Within the confining transition of the pure gauge theory (the action of Eq.(1) in the absence of matter coupling $\beta = 0$), the Wilson loop $W_C = \langle \prod_{i \in C} U_{ij} \rangle$ for a large loop $C$, changes from an asymptotic perimeter law behavior ($W_c \sim e^{-c_1 l}$ with $l$ the perimeter of $C$ and $c_1$ a constant) for large plaquette couplings ($K > K_c$) to a much more rapidly decaying area law ($W_c \sim e^{-c_2 A}$ with $A$ the area of the minimal surface bounded by $C$ and $c_2$ a constant) for weak couplings $K < K_c$. At $K = K_c$, the free energy is non-analytic. By duality (Eqs.(4), the location of this non-analyticity in $K$ along the $\beta = 0$ axis maps onto the location of non-analyticity associated with the transition within the 3D Ising model ($S_{3D \text{Ising}} = -\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j$) at its critical point $\beta = \beta_c$. Following Eqs.(4), the relation between the two is $\tan \beta_c = \exp(-2K_c)$. This implies that the critical value of $K$ within the pure ($\beta = 0$) 3D Ising gauge theory is $K_c \gtrsim 0.761423$, e.g. [15]. The partition function is non-zero and the free energy is analytic within the region given by Eq.(11) which lies within the confining phase of the three dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ model for small $\beta$. Thus, as pointed out in a seminal paper by Fradkin and Shenker [8], the Higgs ($\lambda$) and the confinement ($\mathbb{Z}_2$) phases are analytically connected. No phase transition need be encountered in going from one phase to the other. Here we explicitly prove this for the three dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ model with explicit rigorous numerical bounds as in Eq.(11).

The bound on a finite region of the phase diagram free of partition function zeros complements the classic works of Marra and Miracle-Sole [16] to show that the small $\beta$, $K$ expansion of the free energy corresponding to Eq.(1) converges if $K$ is sufficiently small irrespective of $\beta$, or if both $\beta$ and $K$ are sufficiently large. It is noteworthy that although duality allowed us to generate stringent bounds, the Lee-Yang theorem itself linked points deep within the confining phase ($\beta, K \rightarrow (0, 0)$) to those in the Higgs phase ($\beta \gg 1, K \gg 1$). The extension of the Lee-Yang theorem to gauge theories other than $\mathbb{Z}_2/\mathbb{Z}_2$ is straightforward albeit technically more involved.

We now examine the much more trivial two dimensional incarnation of the $\mathbb{Z}_2/\mathbb{Z}_2$ theory to illustrate that it displays a single phase. By a duality mapping (see an explicit derivation in the Appendix), it is readily seen that the partition function of the two dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ model at matter coupling $\beta$ and gauge coupling $K$, is equal (up to constants) to the partition function of the two dimensional Ising model (of unit lattice spacing) given by Eq.(5) of nearest neighbor exchange

$$K < \frac{1}{2} \ln \tanh[4 \tanh^{-1}(5 - \sqrt{24})^{1/4}] \quad (11)$$
FIG. 1: The region in the phase diagram of the three dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ for which we prove that the partition function is free of zeros and consequently the free energy is analytic. The horizontal axis is $K$ - the strength of the gauge field and the vertical axis depicts $\beta$ - the strength of the matter coupling. Both axis span the region from 0 to $\infty$. The solid line is the bound attained from the Lee Yang theorem. The region above this curve is free of non-analyticities. Thus the union of both regions is analytic. This analyticities. By duality, the region above dashed line is also free of non-analyticities. The boundaries drawn in Fig. (1) are only bounds.

FIG. 2: A schematic representation of the phase diagram of the $\mathbb{Z}_2/\mathbb{Z}_2$ theory in d=3 space dimensions. This phase diagram was proposed by [3]. The boundaries drawn in Fig. (1) are only bounds. The confinement transition extend deep beyond the line implied by the Lee Yang theorem.

constant

$$J_{ij} = \frac{1}{2} \ln \coth \beta \delta_{i-j,1}$$  \hspace{1cm} (12)

and uniform external magnetic field

$$h = \frac{1}{2} \ln \coth K.$$  \hspace{1cm} (13)

As the two dimensional Ising model in a magnetic field displays (via the Lee-Yang theorem) no phase transitions, the two dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ theory exhibits only a single phase regardless of the strength of the couplings. Notwithstanding, as we report towards the end of this Letter, the existence of a line of weak singularities may be firmly established.

IV. GENERAL CONSIDERATIONS FOR A SINGLE HIGGS-CONFINING PHASE

FIG. 3: The phase diagram above was found by [4] for $O(3)$ matter fields coupled to $\mathbb{Z}_2$ gauge links in the context of liquid crystals. Here, the confining, Higgs, and Coulomb phases of the $\mathbb{Z}_2/\mathbb{Z}_2$ theory becomes three different sharp phases (whose siblings are respectively denoted in the above as “Isotropic”, “Ordered”, and “Topological”). We prove, by employing self-duality of the $\mathbb{Z}_2/\mathbb{Z}_2$ theory that a phase diagram having three phases such as that of $O(3)$ matter coupled to $\mathbb{Z}_2$ gauges shown above is impossible. In the $\mathbb{Z}_2/\mathbb{Z}_2$ theory, phase boundaries may only terminate on the $\beta = 0$ or $K = \infty$ axis, no phase boundary can separate the Higgs and confining phases.

Next, we avoid the use rigorous Lee-Yang bounds and ask ourselves what statements can be made regarding the phase diagram on general principle alone both in the presence and absence of dualities. First, we illustrate that a phase diagram such that shown in Fig. (3) is impossible for the $\mathbb{Z}_2/\mathbb{Z}_2$ theory. The phase diagram depicted in Fig. (3) was proposed for the very different theory of $O(3)$ matter fields coupled to $\mathbb{Z}_2$ by [3] in their beautiful theory of liquid crystals.

The proof of the impossibility of such a phase diagram for a $\mathbb{Z}_2/\mathbb{Z}_2$ theory and the necessity of having a single Higgs-confinement phase is quite straightforward. As the $\mathbb{Z}_2/\mathbb{Z}_2$ theory is self-dual (see Eqs. (4)), the phase diagram must look the same under the duality transformation. The phase boundaries where the partition function vanishes, $Z(\beta, K) = 0$, must be the same as those where $Z(\beta^*, K^*) = 0$. A phase diagram such as Fig. (3) does not satisfy self-duality. A critical line emanating from $(\beta = \beta_1, K = 0)$ immediately implies a line of singularities emanating from $(\beta = \infty, K = -\frac{1}{2} \ln \tanh(\beta_1))$. If $Z(\beta^*, K^*) = 0$ along this line then, as the functional form for $Z(\beta^*, K^*)$ is equivalent to that of $Z(\beta^*, K)$ with merely the coupling constant tuned to different values, $Z(\beta, K)$ must also have a line of zeros emanating from $(\beta = \infty, K = -\frac{1}{2} \ln \tanh(\beta_1))$ and the phase diagram must possess, at least, another line of singularities. The same would apply to a line of singularities starting from $(\beta = \infty, K = K_1)$ which is easily excluded.

Next, we look at the physics of the models in their limiting incarnations. At $K = 0$ the partition function of the $\mathbb{Z}_2/\mathbb{Z}_2$ theory is trivially $Z = (2 \cosh \beta)^N$ with $N$ the number of lattice sites. Here, the system is simply that of free bonds in a magnetic field and no singularities can occur at any value of $\beta = \beta_1$ with $K = 0$. Self-duality then implies that no singularities can occur in the self-dual $\mathbb{Z}_2/\mathbb{Z}_2$ theory at $\beta = \infty$ and any finite value of $K$.
Putting all of the pieces together, by employing self-duality, and the absence of singularities at \( K = 0 \), within the \( \mathbb{Z}_2/\mathbb{Z}_2 \) theory, lines of singularities in the phase diagram can only originate from \( (\beta = 0, K = K_c) \) or from \( (\beta = \beta_c, K = 0) \) (with possibly more than one value of \( K_c \) \( \{K_{c,1}\} \)) and/or \( \beta_c \) or form closed loops or lines of transitions terminating in the bulk. States with \( \beta = 0 \) and \( K < \min \{K_{c,1}\} \) and those with \( K = \infty \) and \( \beta < \max \{\beta_{c,2}\} \) must be analytically connected to each other. In the standard spin \( (K = 0, \beta = \beta_c) \) and gauge models \( (\beta = 0, K = K_c) \) only a single critical value appears. The Higgs and confining phases must, asymptotically, be one and the same. A singularity anywhere along the line \( K = 0^+ \) is excluded in the self-dual theory as that limit corresponds to \( \beta \to \infty \) which is completely ordered \( (z_{ij} = 1) \) and no transitions occur. This proves the celebrated result of [8].

We now examine the situation in general non self-dual theories in which the matter fields \( (\sigma_i) \) are in a subgroup of the gauge group (the group \( G \) such that all links \( U_{ij} \in G \)). (This situation does not encompass theories such as those described by [2], [5].) In such instances, the bond variables \( z_{ij} = \sigma_i U_{ij} \sigma_j \) are elements of \( G \). Similar to Eq. (3), we may parameterize the action in terms of the gauge invariant link variables \( \{z_{ij}\} \). In what follows, we focus for concreteness on \( U(1) \) (or \( O(n = 2) \)) theories. First, we note that along the \( K = 0 \) axis, the pure non-interacting links in the effective magnetic field \( \beta \) (leading to \( Z = (I_{n/2-1}(\beta))^{N_4} \) [with \( I = (n/2-1) \) Bessel function of order \( (n/2-1) \) for \( O(n) \) fields) display no singularities in the free energy. Along the \( \beta = \infty \) axis, irrespective of the value of \( K \), all the gauge invariant bonds \( z_{ij} \) in the \( U(1) \) theory are pinned to 1. No transitions occur as \( K \) is varied along the \( \beta = \infty \) line as all bond variables are already frozen at their maximally magnetized unit value. In fact, increasing \( K \) for finite \( \beta \) can only make this magnetization stronger. The partition function has no dependence on \( K \) along this line. Thus, we see that in general no phase boundaries can traverse the \( \beta = \infty \) or the \( K = 0 \) line even in the absence of self-duality and Lee-Yang results which allow us to make matters more elegant and provide rigorous numerical bounds. Thus, the Higgs and confining phases are one and the same for all of these theories. We must nevertheless mention that in non self dual theories, relying only on the above we cannot immediately exclude a transition boundary ending in the bulk at \( K = 0^+ \). To exclude this for different individual theories, we need to examine the radius of convergence in \( K \).

V. ESTABLISHING NEW PERCOLATION CROSSOVERS BY DUALITY

With all stated thus far, it would appear that the single Higgs-confining phase is one bulk phase and no transitions occur within it. We now illustrate that this is not the case-at least not within the simplest of all matter coupled lattice gauge theories- the two dimensional \( \mathbb{Z}_2/\mathbb{Z}_2 \) theory which we now show to possess a richer phase diagram than anticipated (a single phase). With no matter, as is well known e.g. [1], [7], the pure two dimensional \( \mathbb{Z}_2 \) gauge theory given by the plaquette term of Eq. (11) is equivalent (by a trivial gauge fix, e.g. \( U_{ij} = 1 \) on all horizontal bonds in the plane) to a stack of decoupled one dimensional Ising chains (all of which are horizontal Ising chains formed by the vertical bonds in the gauge alluded to here). As Ising chains are disordered at any finite coupling, the two dimensional \( \mathbb{Z}_2 \) gauge theory is always confining.

Now let us introduce matter coupling (a finite \( \beta \) in Eq. (11)) and consider the following thought experiment: we color every appearance of \( z_{ij} = \pm 1 \) in the two dimensional \( \mathbb{Z}_2/\mathbb{Z}_2 \) theory by one of two colors and ask ourselves whether the bonds of a uniform sign (the \( z_{ij} = 1 \) bonds for \( \beta > 0 \)) percolate, upon a trivial mapping, across the sample and if so whether a transition between a percolative and non-percolative clusters can exist within the single Higgs-confining phase. Although this question is very general, we can make easy progress and establish rigorous results by relying on the exact duality of Eqs. (12, 13) to the well studied
VI. CONCLUSIONS

In conclusion, we illustrate that a phase diagram of a gauge theory coupled to matter can be proved directly and stringent numerical bounds provided. Our methods reaffirm the seminal result of Fradkin and Shenker [8]. We further remarked on extensions of this result. Our results suggest that the existence of a single Higgs-confining phase in some theories (as mandated via a generalized Lee-Yang theorem in the $Z_2/Z_2$ theory and strongly hinted by general considerations in other general instances) can often be viewed as the analogue of the absence of phase transitions in spin systems subjected to an external magnetic field. Similar to such spin systems, we speculate that the locus of gauge and matter couplings $(K, \beta)$ at which a correlated percolation of clusters (given by an effective spin state related to gauge invariant bonds variables $z_{ij} \equiv \sigma_i^\sigma_j U_{ij}$) occurs may constitute an analogue of the Kertesz line known in such spin systems [10]. We establish the validity of this anticipation and the existence of a Kertesz line within a simple gauge theory harboring a single confining phase- the $1+1$ dimensional $Z_2/Z_2$ theory. Possible manifestations of this effect for more physically pertinent higher group gauge fields in $d = 4$ remain a speculation.

VII. ACKNOWLEDGMENTS

I am indebted to Jan Zaanen and Asle Sudbø for many conversations and for another work in progress [22]. This work was partially supported by US DOE under LDRD X1WX and by FOM.

APPENDIX A: DERIVATION OF THE DUALITY OF THE $Z_2/Z_2$ THEORY

The $Z_2/Z_2$ theory of Eq. (3) in $d = 2$ dimensions is dual (via Eqs. (12,13)) to the two dimensional Ising in an external magnetic field of Eq. (5). This duality allowed us to prove the existence of a sharp percolation crossover (a Kertesz [10] line) within the Higgs-confining phase of the simplest of all matter coupled gauge theories- the two dimensional $Z_2/Z_2$ theory. The existence of a duality between the two dimensional Ising model in a magnetic field and the two dimensional $Z_2/Z_2$ theory was noted in [8]. As this duality is pivotal in proving our new percolation crossovers (section(V)) even in this simplest of all matter coupled gauge theories, we explicitly illustrate its derivation below.

In what follows, we employ series expansions (a standard approach for deriving many dualities) in the high and low coupling limits to show that the high and low coupling regimes of the two disparate models (the two dimensional $Z_2/Z_2$ theory of Eq. (4) and the two dimensional Ising model in a magnetic field of Eq. (5)) become one and the same upon a change of variables (the duality transformation). Hereafter, we set in Eq. (5) the exchange constant $J_{ij} = J \delta_{|i-j|,1}$. We start by

\begin{equation}
\ln n_V \sim -2hV - \Gamma V^{(d-1)/d}.
\end{equation}

Here, the surface tension $\Gamma$ vanishes in one phase (phase B of Fig. 4) while it is finite in the other (phase A in Fig. 4) [10]. Equivalently, this crossover may be ascertained via the examination of the radii of expansion [19] in $\mu \equiv e^{-2h}$ (see Eq(5) for the magnetization

\begin{equation}
\langle s \rangle = 1 - 2 \sum_V V L_V(u) \mu V,
\end{equation}

where $u \equiv e^{-2J}$, and $L_V$ is a polynomial in $u$. Although for any finite $h$, the radius of convergence in $\mu$ is finite (as indeed no transitions occur by the Lee-Yang theorem), the radius of convergence increases across the percolation line (appearing as jumps in [13]). For couplings $J$ larger than the percolation threshold $J > J_p$, the radius of convergence $\mu$ is to $\mu = 1$ (i.e. it is convergent for all $h \geq 0$ in Eq(5)) and to a larger value $\mu > 1$ for $J < J_p$ (the surface tension free regime)- up to finite negative values of $h$ [19]. Upon dualizing (Eqs. [12,13]), this implies an identical crossover in the single plaquette expectation value of $\langle z_{ij} z_{jk} z_{ki} \rangle$ (which is none other than the minimal Wilson loop $U_{ij} U_{jk} U_{ki}$) when expanded in powers of $\mu \equiv \tanh K$ for fixed $u \equiv \tanh \beta$. The transition is discerned by the convergence of the single plaquette expectation value up to negative $K$ values.

Taken together, the duality relations of Eqs. [12,13] and the firm results of [19] prove, for the first time, that the two dimensional $Z_2/Z_2$ theory must also exhibit a Kertesz line. A sketch of the original phase diagram of Kertesz [10] and its new gauge theory dual are depicted in Figs. 4,5. Percolation transitions established here for the two dimensional $Z_2/Z_2$ theory and others speculated elsewhere might be linked to infinite Wilson loop like observables [20,21].
expanding the partition function

\[ Z = \sum_{(z_{ij})} \exp[-S], \tag{A1} \]

in a small \( \beta, K \) (“high temperature”) series. In Eq. (A1), the action \( S \) is given by Eq. (3) and the summation in Eq. (A1) spans all gauge invariant bond variables \( (z_{ij} = \pm 1 \) on all nearest neighbor links \( (ij) \)). To attain the low coupling expansion of the partition function \( Z \) of Eq. (A1), we employ the identities

\[ \exp[\beta z] = \cosh \beta (1 + z \tanh \beta), \]
\[ \exp[K z z z z z] = \cosh K (1 + z z z z z \tanh K), \tag{A2} \]

to obtain a polynomial expansion in

\[ x = \tanh K, \quad y = \tanh \beta. \tag{A3} \]

With these elements in tow, the partition function of Eq. (A1) becomes a sum of diagrams. These diagrams (Fig. 6) correspond to drawing closed contours in the plane and counting the number of dual lattice sites (the centers of plaquettes surrounded by 4 gauge links- corresponding to the plaquette terms \( z z z z z \) stemming from the exponentiation of the second term in the action of Eq. (3) which are labeled by the solid rectangles) and the number of bonds \( (z) \), obtained from exponentiation of the first term of Eq. (3), labeled by the crosses residing on the contour boundaries. The sum over all values of \( z_{ij} = \pm 1 \) allows only diagrams containing closed loops in which each bond \( (z_{ij}) \) appears to an even powers (all other diagrams necessarily have at least one bond which appears an odd number of times and therefore leads to zero once the sum over \( z_{ij} = \pm 1 \) is performed). The sum over each bond, \( \sum_{z_{ij} = \pm 1} z_{n_{ij}} = 2 \) for all even \( n_{ij} (n_{ij} = 0, 2) \). All in all, the series for the partition function becomes

\[ Z = 4^N (\cosh K)^N (\cosh \beta)^{2N} \sum_{\text{closed loops}} x^A y^{|C|}, \tag{A4} \]

where \( A \) denotes the net area enclosed by the set of loops \( C \) and, \( |C| \) marks the net perimeter of all closed loops making up the joint contour \( C \).

If we expand the partition function corresponding to the action of Eq. (3) about \( J \to \infty \) (corresponding to flipping the spins \( s_t \) from their infinite coupling (“zero temperature”) ground state value of one \( (h > 0) \)) then we will obtain a polynomial expansion in

\[ \tilde{x} = \exp[-2h], \quad \tilde{y} = \exp[-2J]. \tag{A5} \]

Explicitly, the partition function reads

\[ \tilde{Z} = \tilde{Z}_0 \left[ \sum_{\text{clusters of flipped spins}} (\tilde{x})^A (\tilde{y})^{|C|} \right], \tag{A6} \]

where \( \tilde{Z}_0 \) is the zero temperature (infinite \( h \) and \( J \)) partition function, \( A \) is the net area of all clusters of flipped spins \( (s_t = -1) \), and \( |C| \) is the perimeter of all clusters of flipped spins. In Fig. 7, a simple cluster of flipped spins is shown. The flipped spins are marked by black rectangles (each flipped spin incurs a Boltzmann energy penalty of \( \tilde{x} \)), and the bad energetic bonds that the flipped spins generate along their perimeter marked by a thick dashed line. The bonds of the dual lattice are marked by a thin dashed line.

FIG. 6: A contribution to the low coupling series of the two dimensional \( \mathbb{Z}_2/\mathbb{Z}_2 \) action. The centers of plaquettes are labeled by the solid rectangles. The crosses (x) denote energetic bonds (gauge invariant bond variables \( z \) of text) residing on the perimeter of the contour.

FIG. 7: The corresponding contribution to the strong coupling (“low temperature”) series of the two dimensional Ising model in a magnetic field. The flipped spins are marked by black rectangles with the bad energetic bonds that the flipped spins generate along their perimeter marked by a thick dashed line. The bonds of the dual lattice are marked by a thin dashed line.
sponding to any finite $K$ in the action of Eq. (3) on the square lattice, the radii of expansion of the series derived above are infinite and the duality transformations of Eqs. (12, 13) hold throughout.

[1] J. B. Kogut, Reviews of Modern Physics, 51, 659 (1979)
[2] P. E. Lammert, D. S. Rokhsar, and J. Toner, Phys. Rev. E 52, 1778 (1995); ibid, Phys. Rev. E 52, 1801 (1995)
[3] S. Sachdev, cond-mat/0401041
[4] T. Senthil et al., Science 303, 1490 (2004)
[5] T. Senthil and M. P. A. Fisher, J. Phys. A 10 L119 (2001)
[6] H. Kleinert, F. S. Nogueira, and A. Sudbø, Phys. Rev. Lett. 88, 232001 (2002)
[7] N. Nagaosa and P. A. Lee, Phys Rev B 61 (13), 9166 (2000)
[8] E. Fradkin and S. H. Shenker, Phys. Rev. D 19, 3682 (1979)
[9] T. Banks and E. Rabinovici, Nucl. Phys. B 160, 349 (1979)
[10] W. Janke et al., J. Phys. A: Math Gen. 35, 7575 (2002), J. Kertesz, Physica A 161, 58 (1989)
[11] These meson like variables ($\{z_{ij}\}$) have clear physical meaning. For instance, within higher group incarnations of the action of Eq. (3) wherein $z_{ij} = \sigma_i^* U_{ij} \sigma_j$, e.g. the theory of $U(1)$ matter coupled to $U(1)$ gauge (the $U(1)/U(1)$ theory), $z_{ij}$ is directly linked to the gauge invariant current between lattice sites $i$ and $j$. Within the $U(1)/U(1)$ theory describing a hypercubic array of coupled superconducting grains (whose superconducting order parameter is given by a phase ($\sigma_i = e^{i \theta_i}$)) in an external magnetic field (governed by the $U(1)$ gauge field $A$ or, on the hypercubic lattice with $U_{ij} = \exp[i \int A \cdot d\mathbf{r}]$), these gauge invariant fields $z_{ij} = \exp[i \phi_{ij}]$, with the Josephson current between sites $i$ and $j$ given by $J_{ij} = \sin \phi_{ij}$. By considering the general relation giving the current as the variational derivative of the action with respect to the gauge potential, $J = \frac{\delta S}{\delta A}$, it is readily seen that this identification is far more general.
[12] F. Wegner, J. Math. Phys. 12, 2259 (1971)
[13] C. N. Yang and T. D. Lee, Phys Rev 87, 404 (1952)
[14] F. Dunlop, Colloquia Mathematica Societatis Janos Bolyai (1979)
[15] C. Itzykson and K-M. Drouffe, Statistical Field Theory, Cambridge University Press (1989), see Volume 1 (page 355, in particular)
[16] R. Marra and S. Miracle Sole’, Communications in Mathematical Physics 67, 410 (1979)
[17] A. M. Polyakov, Gauge Fields and Strings, Contemporary Concepts in Physics (volume 3), Harwood Academic Publishers (1987)
[18] C. M. Fortuin and P. W. Kasteleyn, Physica 57, 536 (1972)
[19] J. Adler and D. Stauffer, Physica A 175, 222 (1991)
[20] When the matter fields carry the fundamental unit of charge (which is not the situation in certain of the theories of [21]), the Wilson loop $W_C = \langle \prod_{ij \in C} U_{ij} \rangle$ along any contour $C$ (including the boundaries of the system) can be expressed in terms of the gauge invariant fields defined here as $W_C = \langle \prod_{ij \in C} z_{ij} \rangle$.
[21] A. Vester, J. Lidmar, and T. H. Hansson, Europ. Phys. Lett. 69, 256 (2005); A. Vester and J. Lidmar, cond-mat/0502533; M. B. Hastings and Xiao-Gang Wen, cond-mat/0503554
[22] Z. Nussinov, J. Zaanen, and A. Sudbø, In preparation