A pedagogical discussion of $N = 1$ four-dimensional supergravity in superspace

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Abstract

A short introduction to $N = 1$ supergravity in four dimensions in the superspace approach is given emphasising on all steps to obtain the final Lagrangian. In particular starting from geometrical principles and the introduction of superfields in curved superspace, the action coupling matter and gauge fields to supergravity is derived. This review is based on the book “A supergravity Primer: From geometrical principles to the Final Lagrangian” [1] and on several lectures given at the doctoral school of Strasbourg.

keywords: Supergravity, Superfield, Superspace

1 Introduction

The purpose of this brief review is to provide the conceptual scheme that leads to the four-dimensional $N = 1$ supergravity Lagrangian, from geometrical principles and the definition of a had hoc curved superspace. The technical details will be omitted since we would like to emphasise on basic ideas.

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Among the many different approaches we shall follow the one of Wess and Bagger [2]. An alternative approach can be found in, e.g., the book [3] in which superconformal techniques are used.

There are many references on supergravity, among which we would like to mention the first paper on the subject devoted to pure supergravity [4] and the subsequent developments which include the coupling of matter fields [5]. Due to limited number of pages of this review we will not be able to quote all the important articles that have been key to the understanding of supergravity. For a complete list of references the reader may consult e.g. the books [2, 1, 3].

This review closely follows the book [1] of two of us where the reader will be able to deepen in the technical aspects, given there in great detail. In order to facilitate the reading the title of each section corresponds to the same title of the corresponding chapter of the book. The book [1] can also be supplemented with the book [6] (devoted to a pedagogical introduction – in French – of supersymmetry), these two books resulted from lectures and seminars given during several years at the doctoral school of Strasbourg.

2 Curved superspace

The essential idea to consider a superspace in supergravity is to add fermionic directions enhance the usual bosonic spacetime coordinates. The curved bosonic and fermionic coordinates are covariant under general coordinate transformations. However, in the flat tangent space the bosonic (resp. fermionic) coordinates are the standard vector (resp. left and right handed spinors), transforming accordingly under local Lorentz transformations, thus parametrising the usual superspace of supersymmetry. Then, analogously as in general relativity, two elementary superfields are needed: the supervierbein and the spin-connection, together with their corresponding super-curvature and super-torsion tensors (see Section 2.1). It is then shown that imposing some constraints, the so called torsion constraints, it is possible to reduce the number of elementary superfields and then express all the torsion and curvature tensors in terms of three fundamental superfields (see Section 2.2). Fixing the gauge as done in Section 2.4 enables us to identify the gravity multiplet. The lowest component of the superfields are given in Section 2.5. The superconformal group or Weyl supergroup is also identified in Section 2.3.
2.1 Supervierbein, superconnection, curvature and torsion

The first step is to define two different frames: the Einstein frame and the local tangent frame or Lorentz frame (see A for conventions). In the local tangent frame, a point is parametrised by \( z^M = (x^\mu, \theta^\alpha, \bar{\theta}^\dot{\alpha}) \) and we have covariance under local Lorentz transformations:

\[
x^\mu = x'^\nu \Lambda_\nu^\mu(z), \quad \theta^\alpha = \theta'^\beta \Lambda_\beta^\alpha(z), \quad \bar{\theta}^\dot{\alpha} = \bar{\theta}'^\dot{\beta} \Lambda_{\dot{\beta}}^\dot{\alpha}(z). \tag{2.1}
\]

The matrices \( \Lambda_M^N(z) = (\Lambda_{\mu}^\nu(z), \Lambda_\alpha^\beta(z), \Lambda_{\dot{\alpha}}^{\dot{\beta}}(z)) \) are superfields, corresponding to the vector, left-handed and right-handed spinor representations respectively, depend on the point \( z \). Note that transformations in the tangent space do not mix indices of different nature. Furthermore, the Lorentz generators associated to the corresponding representation obey the relations,

\[
J_{\mu\nu} = \frac{1}{4}(\bar{\sigma}^\nu)^{\dot{\beta}\dot{\alpha}}(\bar{\sigma}_\mu)^{\beta\alpha}(\varepsilon_{\alpha\beta}J_{\dot{\alpha}\dot{\beta}} - \varepsilon_{\dot{\alpha}\dot{\beta}}J_{\alpha\beta}),
\]

\[
J_\alpha^\beta = (\sigma^\nu)_\alpha^{\beta} J_{\mu\nu},
\]

\[
J^{\dot{\alpha}}^{\dot{\beta}} = (\bar{\sigma}^\nu)^{\dot{\alpha}}^{\dot{\beta}} J_{\mu\nu}.
\]

We also introduce the \( z^M \)-conjugated variables \( \partial_M = (\partial_\mu, \partial_\alpha, \partial_{\dot{\alpha}}) \) which together satisfy the graded commutation relations

\[
[\partial_M, z^N]|_{M|N} = \partial_M z^N - (-1)^{|M||N|} z^N \partial_M = \delta_M^N,
\]

where \( |\mu| = 0, |\alpha| = 1, |\dot{\alpha}| = 1 \) is the grading of the index \( M = (\mu, \alpha, \dot{\alpha}) \).

In the Einstein (or curved space) frame, a point is parametrised by \( z^\tilde{M} = (x^\tilde{\mu}, \theta^\tilde{\alpha}, \bar{\theta}^\dot{\tilde{\alpha}}) \) and we have invariance under superdiffeomorphisms

\[
z^\tilde{M} = z^M + \xi^\tilde{M}(z),
\]

where \( \xi^\tilde{M}(z) \) are superfields depending on the point \( z \).

The first dynamical superfield is the supervierbein \( E_M^N(z) \) and its inverse \( E_M^{\tilde{N}}(z) \) which connect components in the Einstein frame to the components in the Lorentz (or flat) frame. Thus e.g. for a vector superfield:

\[
V_M = E_M^N V_N, \quad \tilde{V}_M = E_M^{\tilde{N}} \tilde{V}_{\tilde{N}}.
\]

Considering the metric in the flat tangent space \( \eta_{MN} = (\eta_{\mu\nu}, \epsilon_{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}}) \) we thus have

\[
g_{\tilde{M}\tilde{N}} = (-1)^{|M||N|+|\tilde{N}|} E_{\tilde{N}}^N E_{\tilde{M}}^M \eta_{MN}.
\]

The second dynamical variable is the superconnection \( \Omega_{\tilde{M}MN} \) with symmetry property
\[ \Omega_{MMN} = -(-)^{|M||N|} \Omega_{\tilde{M}\tilde{M}N}, \]

which is in addition Lie algebra valued, \textit{i.e.}, is non-vanishing only when \(M\) and \(N\) are of the same nature. The superconnection has the transformation law

\[ \Omega'_{MMN} = (\Lambda^{-1})^M_P \Omega_{\tilde{M}P} Q^N_N - (\Lambda^{-1})^M_P \partial_{\tilde{M}} \Lambda_{PN}, \]

under a local Lorentz transformation and it enables to define the covariant derivatives

\[ \mathcal{D}_{\tilde{M}} X^M = \partial_{\tilde{M}} X^M + (-)^{|\tilde{M}||N|} X^N \Omega_{MN}^M, \quad \mathcal{D}_M X_M = \partial_M X_M - \Omega_{\tilde{M}MM}^N X_N, \quad \mathcal{D}_N X^M = E_N^M \mathcal{D}_{\tilde{M}} X^M, \quad \mathcal{D}_N X_M = E_N^M \mathcal{D}_{\tilde{M}} X_M. \tag{2.5} \]

The closure of the algebra leads to

\[ [\mathcal{D}_M, \mathcal{D}_N]_{|M||N|} = T_{MP}^Q \mathcal{D}_P - \frac{1}{2} R_{MNQ\beta} \mathcal{D}^\beta, \]

where \(\alpha = (\alpha, \dot{\alpha})\), and we only sum over independent generators say \(J_{\alpha\beta}\) and \(J^{\dot{\alpha}\dot{\beta}}\) (see equation (2.2)). The torsion and curvature fulfill the obvious symmetry properties

\[ T_{MNP} = -(-)^{|M||N|} T_{NMP}, \quad R_{MNPQ} = -(-)^{|M||N|} R_{NMPQ}, \quad R_{MNPQ} = -(-)^{|P||Q|} R_{MNQP}. \tag{2.6} \]

Introducing

\[ T_{PQ}^M = -(-)^{|P||Q|} T_{MNQ}^M, \quad R_{MNS}^Q = -(-)^{|M||N|+|S|} R_{MNS}^Q \]

a little algebra leads to

\[ T_{PQ}^M = -\mathcal{D}_P E_Q^M + (-)^{|P||\tilde{Q}|} \mathcal{D}_Q E_P^M, \quad R_{MNS}^Q = \partial_M \Omega_{NS}^Q - (-)^{|M||\tilde{N}|} \partial_N \Omega_{MS}^Q - \Omega_{\tilde{M}MS}^R \Omega_{\tilde{N}R}^Q + (-)^{|M||\tilde{N}|} \Omega_{\tilde{N}MS}^R \Omega_{\tilde{M}R}^Q. \tag{2.7} \]

\[ \text{2.2 The Bianchi identities} \]

The covariant derivatives satisfy the Bianchi identities

\[ 0 = (-)^{|M_1||M_2|} \left[ \mathcal{D}_{M_1}, \left[ \mathcal{D}_{M_2}, \mathcal{D}_{M_3} \right]_{|M_2||M_3|} |M_1||M_2||M_3| \right] + (-)^{|M_1||M_2|} \left[ \mathcal{D}_{M_1}, \left[ \mathcal{D}_{M_2}, \mathcal{D}_{M_3} \right]_{|M_2||M_3|} |M_1||M_2||M_3| \right] + (-)^{|M_2||M_3|} \left[ \mathcal{D}_{M_2}, \left[ \mathcal{D}_{M_3}, \mathcal{D}_{M_1} \right]_{|M_3||M_1|} |M_1||M_2||M_3| \right] + (-)^{|M_2||M_3|} \left[ \mathcal{D}_{M_2}, \left[ \mathcal{D}_{M_3}, \mathcal{D}_{M_1} \right]_{|M_3||M_1|} |M_1||M_2||M_3| \right], \tag{2.8} \]
Developing the double graded commutators using the Lie algebra valuedness of the curvature tensor and the property \([X, Y Z]|X||Y|Z = |X, Y||X||Y|Z + (-)^{|X||Y|}|Y, X||X||Z||X||Z\)
, leads to two series of identities:

\[
0 = \left( -\right)^{|M_1||M_3|} \left[ D_{M_1} T_{M_2 M_3 S} - T_{M_1 M_2} R T_{R M_3 S} + R_{M_1 M_2 M_3 S} \right] \\
(-)^{|M_2||M_1|} \left[ D_{M_2} T_{M_3 M_1 S} - T_{M_2 M_3} R T_{R M_1 S} + R_{M_2 M_3 M_1 S} \right] \\
(-)^{|M_3||M_2|} \left[ D_{M_3} T_{M_1 M_2 S} - T_{M_3 M_1} R T_{R M_2 S} + R_{M_3 M_1 M_2 S} \right] ,
\]

\[
0 = \left( -\right)^{|M_1||M_3|} T_{M_1 M_2} R R_{R M_3 P Q} - D_{M_1} R_{M_2 M_3 P Q} \\
(-)^{|M_2||M_1|} T_{M_2 M_3} R R_{R M_1 P Q} - D_{M_2} R_{M_3 M_1 P Q} \\
(-)^{|M_3||M_2|} T_{M_3 M_1} R R_{R M_2 P Q} - D_{M_3} R_{M_1 M_2 P Q} .
\]

(2.9)

These Bianchi identities are usual. However, since this geometrical construction leads to a large number of superfields, additional constraints are necessary to reduce the number of physical degrees of freedom. These constraints are also necessary to be able to construct minimal supergravity models, \textit{i.e.}, which realise the supersymmetry algebra in flat limit. It turns out that one possible set of constraints is the following:

\[
T_{\dot{\alpha} \dot{\beta}} = T_{\alpha \beta} = T_{\dot{\mu} \nu} = T_{\mu \nu} = T_{\alpha \mu} = 0, \quad T_{\alpha \dot{\alpha}} = T_{\dot{\alpha} \mu} = -2i\sigma^\mu_{\alpha \dot{\alpha}} .
\]

(2.10)

In particular the constraints (2.10) are compatible with the definition of chiral superfields (Section 3). These series of constraints allow to reduce the Bianchi identities and express all torsion and curvature tensors in terms of fewer independent quantities. Thus we end up with three independent superfields, namely the chiral superfield \(\mathcal{R}\), the real vector superfield \(G_\mu\) and the symmetric chiral superfield \(W_{(\alpha \beta \gamma)}\). To obtain the explicit expressions of all torsion and curvature tensors, we need to perform a lengthy computation that we do not reproduce here (for further details see reference [1]), but we point out key observations to get the final results:

1. The second series of identities in (2.9) are consequences of the first series of identities.

2. Accounting of all symmetries the first series of identities leads to thirteen different equations (in fact thirty but only thirteen are independent).

3. The main idea in order to reduce the Bianchi identities is to convert all vector indices into spinor indices. For instance for a component of the curvature tensor

\[
\sigma^\mu_{\gamma \delta} \sigma^\nu_{\delta \delta} R_{\dot{\alpha} \dot{\beta} \mu \nu} = R_{\dot{\alpha} \dot{\beta} \gamma \delta} = 2\varepsilon_{\gamma \delta} R_{\dot{\alpha} \dot{\beta} \gamma \delta} - 2\varepsilon_{\gamma \delta} R_{\dot{\alpha} \dot{\beta} \gamma \delta} ,
\]
where the second equality is a consequence of the Lie algebra valuedness of the curvature tensor. Similarly for a component of the torsion tensor we get

$$\sigma^\mu_{\gamma\dot{\gamma}} T_{\dot{\alpha}\mu\beta} = T_{\dot{\alpha}\gamma\dot{\gamma}} = \varepsilon_{\dot{\alpha}\dot{\gamma}} \varepsilon_{\gamma\beta} T_{\dot{\alpha}\dot{\gamma}} + \varepsilon_{\dot{\alpha}\dot{\gamma}} T_{(\dot{\alpha}\dot{\gamma})(\gamma\beta)} ,$$

(2.11)

where the new $T$-tensors are symmetric with respect to the exchange of the indices in the parentheses and correspond to the decomposition of $T_{\dot{\alpha}\gamma\dot{\gamma}}$ into irreducible representations of $\mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(1, 3)$.

As stated previously all non-vanishing torsion and curvature tensors can be expressed in terms of the superfields $R, G_\mu$ and $W_{\alpha\beta\gamma}$. The non-vanishing brackets are given by

$$\{ D_\alpha, D_\beta \} = -\frac{1}{2} R_{\alpha\beta\gamma\delta} J^{\gamma\delta} ,$$

$$\{ \overline{D}_\dot{\alpha}, \overline{D}_\dot{\beta} \} = -\frac{1}{2} R_{\dot{\alpha}\dot{\beta}\gamma\delta} J^{\gamma\delta} ,$$

$$\{ D_\alpha, \overline{D}_\dot{\alpha} \} = T_{\alpha\dot{\alpha}}^{\mu\nu} D_\mu - \frac{1}{2} R_{\alpha\dot{\alpha}\beta\gamma} J^{\gamma\beta} - \frac{1}{2} R_{\alpha\dot{\alpha}\beta\gamma} J^{\gamma\beta} ,$$

$$\{ D_\alpha, D_\mu \} = T_{\alpha\mu}^{\beta} D_\beta - T_{\alpha\mu}^{\dot{\beta}} \overline{D}_\dot{\beta} - \frac{1}{2} R_{\alpha\mu\beta\gamma} J^{\gamma\beta} - \frac{1}{2} R_{\alpha\mu\beta\gamma} J^{\gamma\beta} ,$$

$$\{ \overline{D}_\dot{\alpha}, D_\mu \} = T_{\dot{\alpha}\mu}^{\dot{\beta}} D_\dot{\beta} - T_{\dot{\alpha}\mu}^{\beta} \overline{D}_\dot{\beta} - \frac{1}{2} R_{\dot{\alpha}\mu\beta\gamma} J^{\gamma\beta} - \frac{1}{2} R_{\dot{\alpha}\mu\beta\gamma} J^{\gamma\beta} ,$$

$$\{ D_\mu, D_\nu \} = T_{\mu\nu}^{\beta} D_\beta - T_{\mu\nu}^{\dot{\beta}} \overline{D}_\dot{\beta} - \frac{1}{2} R_{\mu\nu\beta\gamma} J^{\gamma\beta} - \frac{1}{2} R_{\mu\nu\beta\gamma} J^{\gamma\beta} ,$$

(2.12)

(see (2.5) for the relationship between derivatives with Einstein and Lorentz indices). We do not give all the expression of the $R$– and $T$– tensors. The reader may refer to Tables 3.2, 3.3 and 3.4 of [1] for explicit expressions and constraints upon the various tensors.

### 2.3 The Weyl supergroup

The torsion constraints (2.10) and the corresponding algebra (2.12) have additional symmetries. Indeed, the automorphism group of the algebra, or equivalently the set of transformations upon the generators $D_M$ and $J_{MN}$ which leave (2.12) invariant is the super-Weyl or superconformal group. This is the analogue of the Weyl group of general relativity corresponding to Weyl rescaling of the metric. We now determine the transformations $\delta J_{\underline{\alpha\beta}}$ and $\delta D_M$ which leave (2.12) invariant. The commutation relations of the Lorentz algebra leads obviously to $\delta J_{\underline{\alpha\beta}} = 0$. Setting

$$\delta D_\alpha = -\Phi D_\alpha + D^{\beta} \Phi J_{\alpha\beta}$$

(2.13)
and similar expressions for $\overline{D}_\alpha, D_\mu$ we obtain after lengthy algebraic manipulations

$$
\delta_\Sigma D_\alpha = \left[ \Sigma - 2\Sigma^\dagger \right] D_\alpha - D^\gamma \Sigma J_{\alpha\gamma}, \\
\delta_\Sigma \overline{D}_\dot{\alpha} = \left[ \Sigma^\dagger - 2\Sigma \right] \overline{D}_{\dot{\alpha}} - \overline{D}^\dot{\gamma} \Sigma J_{\dot{\alpha}\dot{\gamma}}, \\
\delta_\Sigma D_\mu = -\left( \Sigma + \Sigma^\dagger \right) D_\mu - \frac{i}{2} \left[ \overline{D} \Sigma \bar{\sigma}_\mu D + D \Sigma \sigma_\mu \overline{D} \right] - D^\nu (\Sigma + \Sigma^\dagger) J_{\mu\nu}, \\
\delta_\Sigma J_{\alpha\beta} = 0, \\
\delta_\Sigma J_{\dot{\alpha}\dot{\beta}} = 0,
$$

(2.14)

where $\Sigma$ is a chiral superfield, i.e., satisfying $\overline{D}_\alpha \Sigma = 0$.

Using the explicit expressions (not given here) of the torsion and curvature tensors together with the algebra (2.12) we obtain

$$
\delta_\Sigma R = -4\Sigma R - \frac{1}{4} (\overline{D} \cdot D - 8 R) \Sigma^\dagger, \\
\delta_\Sigma G_\mu = -\left[ \Sigma + \Sigma^\dagger \right] G_\mu + i D_\mu \left[ \Sigma - \Sigma^\dagger \right], \\
\delta_\Sigma W_{(\alpha\beta\gamma)} = 0,
$$

(2.15)

It can be seen that the superfield $\delta_\Sigma R$ is chiral as it should (see Section 3.1).

Finally from the definition of covariant derivatives (2.5) we obtain

$$
\delta_\Sigma \Omega_{\bar{M}\beta\gamma} = E_{\bar{M}\bar{\beta}} D_\gamma \Sigma + E_{\bar{M}\bar{\gamma}} D_\beta \Sigma - E_{\bar{M}\nu} \varepsilon_\gamma^\alpha (\sigma^{\nu\rho})_\beta{\alpha} D_\rho \left[ \Sigma + \Sigma^\dagger \right], \\
\delta_\Sigma \Omega_{\bar{M}\beta\dot{\gamma}} = E_{\bar{M}\bar{\beta}} \overline{D}_\dot{\gamma} \Sigma^\dagger + E_{\bar{M}\bar{\dot{\gamma}}} \overline{D}_\beta \Sigma^\dagger - E_{\bar{M}\nu} \varepsilon_\dot{\gamma}^\dot{\alpha} (\bar{\sigma}^{\nu\rho})_\dot{\beta}{\dot{\alpha}} D_\rho \left[ \Sigma + \Sigma^\dagger \right], \\
\delta_\Sigma \Omega_{\bar{M}\nu\rho} = -E_{\bar{M}\nu} D_\rho \left[ \Sigma + \Sigma^\dagger \right] + E_{\bar{M}\rho} D_\nu \left[ \Sigma + \Sigma^\dagger \right] - 2E_{\bar{M}\alpha} (\sigma_{\nu\rho})_\alpha{\beta} D_\beta \Sigma - 2E_{\bar{M}\dot{\alpha}} (\bar{\sigma}_{\nu\rho})_\dot{\alpha}{\dot{\beta}} \overline{D}_\dot{\beta} \Sigma^\dagger,
$$

(2.16)

and

$$
\delta_\Sigma E_{\bar{M}}^\alpha = \left[ 2\Sigma^\dagger - \Sigma \right] E_{\bar{M}}^\alpha + \frac{i}{2} E_{\bar{M}}^\mu (D \Sigma^\dagger \bar{\sigma}_\mu)^\alpha, \\
\delta_\Sigma E_{\bar{M}\dot{\alpha}} = \left[ 2\Sigma - \Sigma^\dagger \right] E_{\bar{M}\dot{\alpha}} + \frac{i}{2} E_{\bar{M}}^\mu (D \Sigma \sigma_\mu)_{\dot{\alpha}}, \\
\delta_\Sigma E_{\bar{M}}^\mu = (\Sigma + \Sigma^\dagger) E_{\bar{M}}^\mu.
$$

(2.17)

### 2.4 Supergravity transformations and gauge fixing conditions

We have expressed all quantities in terms of a few number of superfields. Nonetheless using the large symmetry due to the supergravity algebra many components can be set to zero by means of an appropriated choice of parameters of the symmetry transformations (2.1) and
(2.4). Under a Lorentz transformation (2.1) and a superdiffeomorphism (2.4) we have
\[ V^M(z) \rightarrow V^M(z') = V^N(z)\Lambda_N^M(z), \]
\[ E^M_M(z) \rightarrow E^M_M(z') = \frac{\partial z^N}{\partial z^{M'}(\Lambda^{-1})_P^R(z)\Omega^S_R(z)\Lambda^Q_S(z) - \partial \Lambda^Q_R(z)}. \]
\[ \Omega_{\bar{M}P}^Q(z) \rightarrow \Omega_{\bar{M}P}^Q(z') = \frac{\partial \Lambda^S_R(z)}{\partial \Lambda^Q_R(z)} \Omega^S_R(z) \Lambda^Q_S(z) - \partial \Lambda^Q_R(z). \]

At the infinitesimal level with \( \Lambda^M_N = \delta^M_N + L^M_N \) and with the following redefinition
\[ L^P_Q \rightarrow L^P_Q - \xi^Q\Omega_{\bar{M}P}^Q, \]
we obtain
\[ \delta V^M = -\xi^N\nabla_N V^M + V^N L_N^M, \]
\[ \delta E^M_M = -\nabla_M^E\xi^M + \xi^N\nabla_{\bar{M}M}^N + E^P_M L^P_M, \]
\[ \delta \Omega_{\bar{M}P}^Q = -\xi^N R^N_{\bar{M}P} + \Omega^Q_{\bar{M}P} R^R_Q - L^R_P \Omega^Q_{\bar{M}R} - \partial \Lambda^Q_R. \]

Similarly we have for the superfields \( R \) and \( G^\mu \)
\[ \delta R = -\xi^Q\nabla_{\bar{M}} R, \quad \delta G^\mu = -\xi^Q\nabla_{\bar{M}} G^\mu + G^\mu L^\mu. \]

In the computation below we will see that only the lowest component in the Taylor expansion of the various superfields will be relevant. From now on, given a superfield \( X \) we set \( X|\theta=0 \) for the lowest component of \( X \) in its Grassmann expansion. It can be easily seen, using the huge arbitrariness of the parameters, that some of the lowest components can be fixed to zero. In particular for the superconnection
\[ \Omega_{\bar{M}M}^N = \omega_{\bar{M}M}^N(x), \quad \Omega_{\bar{M}M}^N|\theta=0 = 0, \quad \Omega_{\bar{M}M}^N|\theta=0 = 0, \]
and for the supervierbein and its inverse
\[ E_{\bar{M}}^M(z) = \left( \begin{array}{ccc} e_{\mu}^\mu(x) & \frac{1}{2}\psi_{\mu}^\alpha(x) & \frac{1}{2}\bar{\psi}_{\mu\alpha}(x) \\ 0 & \delta_{\alpha}^\alpha & 0 \\ 0 & 0 & \delta_{\alpha}^\alpha \end{array} \right), \]
\[ E_{\bar{M}}^\alpha(z) = \left( \begin{array}{ccc} e_{\mu}^\alpha(x) & -\frac{1}{2}\bar{\psi}_{\mu}(x) & -\frac{1}{2}\bar{\psi}_{\mu\bar{\alpha}}(x) \\ 0 & \delta_{\alpha}^\alpha & 0 \\ 0 & 0 & \delta_{\alpha}^\alpha \end{array} \right), \]
where \( e_{\mu}^\mu \) is the helicity-two graviton, i.e., the vierbein, whereas the Majorana spinor-vector \( (\psi_{\mu}^\alpha, \bar{\psi}_{\mu\alpha}) \) is the gravitino. Using (2.21) it is important for the sequel to observe the relationships
\[ e_{\mu}^\mu e_{\mu}^\nu = \delta_{\mu}^\nu, \quad e_{\mu}^\nu e_{\mu}^\nu = \delta_{\mu}^\nu, \]
\[ \psi_{\mu}^\alpha = e_{\mu}^\alpha \psi_{\mu}^\alpha \delta_{\alpha}^\alpha, \quad \bar{\psi}_{\mu\bar{\alpha}} = e_{\mu}^{\bar{\alpha}} \bar{\psi}_{\mu\bar{\alpha}} \delta_{\alpha}^\alpha. \]
Finally the lowest order components of $\mathcal{R}$ and $G_\mu$ cannot be eliminated, thus we define
\[
\mathcal{R}(z) = -\frac{1}{6} M(x), \quad G_\mu(z) = -\frac{1}{3} b_\mu(x). \tag{2.23}
\]
Following A we define also $b_{\alpha \dot{\alpha}} = \sigma^\mu_{\alpha \dot{\alpha}} b_\mu$.

Having gauge fixed the superconnection and the supervierbein as above the dynamical fields of supergravity are given by the graviton $e_\mu^\rho$, the gravitino $(\psi_\mu^\alpha, \bar{\psi}_\mu\dot{\alpha})$ and two auxiliary fields, the complex scalar $M$ and the real vector $b_\mu$. We will see later that the connection $\omega_{\mu MN}$ is a composite field and it can be expressed in terms of the graviton and the gravitino.

Of course the gauge fixing conditions (2.20) and (2.21) are not preserved by (2.4) and (2.1). We thus have to restrict the set of transformations. We assume at first
\[
\xi^\mu(z) = 0, \quad \xi^\alpha(z) = \bar{\varepsilon}^\alpha(x), \quad \bar{\xi}_\dot{\alpha}(z) = \bar{\varepsilon}_\dot{\alpha}(x), \quad L_{\alpha \dot{\alpha}}(z) = 0. \tag{2.24}
\]
Then in order to preserve the gauge fixing condition, a supergravity transformation reduces to the following combination of superdiffeomorphism and local Lorentz transformation
\[
\xi^\mu(z) = 2i \left[ \hat{\theta} \sigma^\mu \bar{\varepsilon} - \varepsilon \sigma^\mu \bar{\theta} \right], \quad \xi^\alpha(z) = \bar{\varepsilon}^\alpha, \quad \bar{\xi}_\dot{\alpha}(z) = \bar{\varepsilon}_\dot{\alpha}, \quad L_{\alpha \dot{\alpha}}(z) = \frac{1}{3} \left[ \hat{\alpha} \bar{\varepsilon}_\beta M^\beta + b_\beta \bar{\varepsilon}_\beta \right] + \tilde{\gamma}_\beta \left( 2 \varepsilon_\alpha M^\beta + b_\alpha \bar{\varepsilon}_\beta \right), \tag{2.25}
\]
where the fields $(\varepsilon, \bar{\varepsilon})$, $b$ and $M$ depend only on the spacetime coordinates $x$ and all indices are flat superspace indices. We have also introduced $\tilde{\theta}_\alpha = \theta^\alpha E_\alpha \varepsilon \varepsilon_{\alpha \beta} \neq \theta_\alpha$ as well as $\tilde{\bar{\theta}}{\dot{\alpha}}$.

### 2.5 Lowest order component fields

In the previous subsection we have fixed the lowest component of the dynamical fields of supergravity, say the supervierbein and the superconnection. It is also possible to obtain explicit expressions of the lowest component as well as the lowest component of the covariant derivatives of the superfield $\mathcal{R}, G_\mu$ and $W_{(\alpha \beta \gamma)}$. The key observation to get these expressions relies on several facts. First of all the gauge fixing condition (2.21) and (2.20) give using (2.7)
\[
T^{\mu \rho}_{\mu \rho} = -\partial_\rho e_\rho^\mu + \partial_\rho e_\mu^\rho - \omega_\rho^\mu + \omega_\rho^\mu , \\
T_{\mu \rho \alpha} = -\frac{1}{2} \left( \partial_\rho \bar{\psi}_\mu^\alpha - \partial_\mu \bar{\psi}_\rho^\alpha - \psi_\mu^\beta \omega_\rho^\beta \alpha + \bar{\psi}_\rho^\beta \omega_\mu^\beta \alpha \right) = -\frac{1}{2} \bar{\psi}_{\mu \rho}^\alpha , \tag{2.26}
\]
\[
T_{\mu \rho \dot{\alpha}} = -\frac{1}{2} \left( \partial_\rho \bar{\psi}_{\mu \dot{\alpha}} - \partial_\mu \bar{\psi}_{\rho \dot{\alpha}} - \omega_{\rho \mu} \bar{\psi}_{\dot{\alpha}}^\dot{\beta} \alpha + \bar{\psi}_{\rho \dot{\beta}} \omega_{\mu \dot{\beta}} \dot{\alpha} \right) = -\frac{1}{2} \bar{\psi}_{\mu \rho \dot{\alpha}} , \\
R_{\mu \rho \nu} = \partial_\rho \omega_\mu^\nu - \partial_\nu \omega_\mu^\rho + \omega_\nu^\rho \omega_\mu^\nu - \omega_\nu^\mu \omega_\mu^\nu ,
\]
\[\]
where

\[
\omega_{\mu\nu} = e_\nu^\alpha \omega_{\mu\alpha}, \quad \omega_{\mu\alpha} = e_\nu^\alpha e_\rho^\beta \omega_{\mu\nu},
\]

\[
D_\mu \psi_\alpha = \partial_\mu \psi_\alpha + \psi_\beta \omega_{\beta\alpha}, \quad \psi_{\mu\nu} = D_\mu \psi_\alpha - D_\nu \psi_\mu.
\]  

(2.27)

We also define,

\[
\psi_{\mu\nu} = e_\mu^\bar{\mu} e_\nu^\bar{\nu} \bar{\psi}_{\bar{\mu}\bar{\nu}} \quad (\neq D_\mu \psi_\nu - D_\nu \psi_\mu).
\]

Similar expressions hold for the right-handed counterpart of the gravitino.

The next step is to express curvature and torsion tensors with Einstein indices from curvature and torsion tensors with Lorentz indices and to compare the result with (2.26). For instance we obtain in this way

\[
T_{\mu\nu} = E_\rho^M E_\mu^N T_{MN} = -\frac{i}{2} \left( \psi_\mu \delta_\mu \tilde{\psi}_\nu - \psi_\nu \delta_\nu \tilde{\psi}_\mu \right),
\]

which gives, comparing with the first equation (2.26),

\[
\omega_{\mu\nu} = \frac{1}{2} e_{\mu\nu} (\partial_\mu e_\nu^\mu - \partial_\nu e_\mu^\mu) + \frac{1}{2} e_{\mu\nu} (\partial_\nu e_\mu^\mu - \partial_\mu e_\nu^\mu) - \frac{1}{2} e_{\mu\rho} (\partial_\rho e_\mu^\mu - \partial_\mu e_\rho^\mu)
\]

\[
+i \left[ \frac{1}{4} e_{\mu\nu} \left( \psi_\mu \sigma^\mu \tilde{\psi}_\nu - \psi_\nu \sigma^\nu \tilde{\psi}_\mu \right) - \frac{1}{4} e_{\mu\nu} \left( \psi_\nu \sigma^\nu \tilde{\psi}_\mu - \psi_\mu \sigma^\mu \tilde{\psi}_\nu \right) \right] - \tilde{b}.
\]

Of course \(\omega_{\alpha\beta} = 1/4 \omega_{\mu\nu}(\sigma^\mu \sigma^\nu)_\alpha^\beta \) and \(\omega_{\dot{\alpha}\dot{\beta}} = 1/4 \omega_{\mu\nu}(\tilde{\sigma}^\mu \tilde{\sigma}^\nu)^{\dot{\alpha}}_{\dot{\beta}}\).

Passing to the superfields \(\mathcal{R} \) and \(G_\mu\), we already know \(\mathcal{R} \) and \(G_\mu \) (see (2.23)). Similarly, we can obtain \(W_{(\alpha\beta\gamma)}\) and the lowest component of the covariant derivatives of the three superfields, for which one needs the explicit expressions of the torsion and curvature tensors (given in [1]) appearing in (2.12). For instance we get

\[
D_\delta G_\mu = \frac{1}{3} (\sigma^\mu \tilde{\psi}_\nu)_\delta - \frac{i}{12} \varepsilon^{\mu\nu\rho\sigma} (\sigma_\sigma \tilde{\psi}_\nu)_\delta + \frac{i}{6} \tilde{\psi}_\mu \delta M_\mu + \frac{i}{6} (\sigma^\nu \tilde{\psi}_\nu)_\delta b_\mu.
\]

We do not give \(W_{(\alpha\beta\gamma)}\) since it is not relevant here. For the lowest component of covariant derivatives of \(\mathcal{R}\) we refer to (4.8) where \(\mathcal{R}\) is expanded using the new \(\Theta\)–variables introduced in Section 4. Note that the computation of \(D_\alpha \mathcal{R}\) is not difficult since it is related directly to \(D_\alpha G_\mu\) whilst it is tedious to obtain \(D \cdot \mathcal{D} \mathcal{R}\).

With this information we can deduce the supergravity transformations of the supergravity multiplet \(e_\mu^\mu, (\psi_\mu^\alpha, \tilde{\psi}_{\mu\dot{\alpha}}), M\) and \(b_\mu\):
We observe that, as a gauge field, the transformation of the gravitino involves the covariant derivative:

\[ \delta e_{\mu} = \delta E_{\mu} = -\partial_{\mu} \xi + \xi M T_{\mu} + E_{\mu} L_{\nu} = -i \left( \varepsilon(x) \sigma^\mu \tilde{\psi} - \psi \sigma^\mu \tilde{x} \right) , \]

\[ \delta \tilde{\psi}_\mu = 2\delta E_{\mu} + 2E_{\mu} \tilde{L}_{\nu} = -2D_{\mu} \varepsilon^\alpha + 2\varepsilon M T_{\mu} + 2E_{\mu} \tilde{L}_{\nu} = -2D_{\mu} \varepsilon^\alpha + \frac{i}{3} \varepsilon_{\mu} \left( \varepsilon(x) \sigma \tilde{\sigma} \right)^\alpha M + \frac{i}{3} \varepsilon_{\mu} \left( \varepsilon(x) \sigma \tilde{\sigma} \right)^\alpha b^\alpha - 3\varepsilon_{\mu} b^\alpha \left( \varepsilon \cdot \psi \right) b^\mu , \]

\[ \delta M = -6\delta R = 6\xi M D_{\mu} R = -2(\varepsilon \sigma^\mu \psi_{\mu \nu}) + i(\varepsilon \sigma^\mu \tilde{\psi}_{\mu}) M - i(\varepsilon \cdot \psi) b^\mu , \]

\[ \delta b^\mu = -3\delta G^\mu = 3\xi M D_{\mu} G^\mu - 3G^\mu L_{\nu} = 3\xi^\alpha D_{\alpha} G^\mu + 3\xi \bar{D}^\alpha G^\mu \]

\( \left( 2.28 \right) \)

We observe that, as a gauge field, the transformation of the gravitino involves the covariant derivative (2.27).

### 3 Superfield in curved superspace

In this section we introduce superfields in curved superspace, namely chiral and vector superfields. It is important to emphasise that the torsion constraints (2.10) turn out to be essential to extend the superfields of supersymmetry in the supergravity context. Superfields depend on \( z^M = (x^\mu, \theta^\alpha, \tilde{\theta}^\dot{\alpha}) \), but we have to compute lowest order component of derivatives with Lorentz indices. More precisely, for both chiral or vector superfields, in order to obtain the full supergravity action, we have to compute several lowest order components of derivatives up to the fourth order. This computation turns out to use intensively the algebra (2.12), the lowest components of the corresponding curvature or torsion tensors and the torsion constraints (2.10). The second important feature to be used is the conversion of flat indices into curved indices as will be seen below.

#### 3.1 Chiral superfields

Following the conventional definition in supersymmetry, a chiral superfield \( \Phi \) has three independent components: a complex scalar \( \varphi \), a left-handed spinor \( \chi \) and an auxiliary field \( F \) defined by

\[ \varphi = \Phi , \quad \chi_\alpha = \frac{1}{\sqrt{2}} D_\alpha \Phi , \quad F = -\frac{1}{4} D \cdot D \Phi . \]

(3.1)
Similar expressions hold for the anti-chiral superfield. Observe again that these definitions involve flat index derivatives (2.5) whereas the superfield depends on curved quantities. We shall come back to this point in Section 4.

The components of \( \Phi \) given in terms of its first order derivatives (3.1) is the starting point to compute the higher order derivatives. It is not our purpose to address all order derivatives, which is somehow cumbersome, but we would like to stress on the key steps in these computations. In particular we should obtain, using the conversion of flat indices to curved indices:

\[
D_\mu \Phi = E_\mu \tilde{\alpha} \left( D_\alpha \Phi \right) + E_\mu \tilde{\alpha} \left( D\tilde{\Phi} \right),
\]

and using (3.1), and (2.21), (2.20) obtain

\[
D_\mu \Phi = e_\mu \tilde{\nu} \left( \partial_{\tilde{\nu}} - \frac{\sqrt{2}}{2} \psi_{\tilde{\mu}} \cdot \chi \right) \varphi \equiv \hat{D}_\mu \varphi. \tag{3.2}
\]

In a similar way we deduce

\[
D_\mu D_\alpha \Phi = \sqrt{2} e_\mu \tilde{\nu} \left[ \partial_{\tilde{\nu}} \chi_\alpha - \omega_{\tilde{\mu}\beta} \chi_\beta + \frac{1}{\sqrt{2}} \psi_{\tilde{\mu}} \sigma_\nu \left( \sigma_{\bar{\mu}} \psi_{\tilde{\nu}} \right)_\alpha \hat{D}_\nu \varphi \right] \equiv \sqrt{2} \hat{D}_\mu \chi_\alpha.
\]

The two derivatives \( \hat{D}_\mu \varphi \) and \( \hat{D}_\mu \chi \) are covariant with respect to supergravity transformations.

Another fundamental property that we would like to highlight is the following. Given an anti-chiral superfield \( \Phi^\dagger \) then

\[
\overline{D}_\alpha (\overline{D} \cdot \overline{D} - 8 \mathcal{R}) \Phi^\dagger = 0,
\]

or equivalently the superfield

\[
\Xi = (\overline{D} \cdot \overline{D} - 8 \mathcal{R}) \Phi^\dagger,
\]

is chiral. This property is central to obtain a compact chiral expression of the supergravity action (see Section 4). For the lowest order components of covariant derivatives of \( \Xi \) we refer to (4.5) where \( \Xi \) is expanded using the \( \Theta \)-variables introduced in Section 4.

Finally considering a supergravity transformation (2.25) we obtain the infinitesimal variation of a chiral superfield

\[
\delta \Phi = -\xi^M D_M \Phi = -\xi^\alpha D_\alpha \Phi - \xi^\mu D_\mu \Phi \tag{3.4}
\]

or in components

\[
\delta \phi = \delta \Phi \bigg|_{\varepsilon} = -\varepsilon^\alpha D_\alpha \Phi \bigg|_{\varepsilon} = -\sqrt{2} \varepsilon \cdot \chi,
\]

\[
\delta \chi_\alpha = \frac{1}{\sqrt{2}} \delta (D_\alpha \Phi) \bigg|_{\varepsilon} = \sqrt{2} \left[ \varepsilon_\alpha F - i (\sigma^\mu \varepsilon)_\alpha \hat{D}_\mu \varphi \right], \tag{3.5}
\]

\[
\delta F = \frac{1}{4} \delta (\overline{D} \cdot \overline{D} \Phi) \bigg|_{\varepsilon} = \frac{\sqrt{2}}{3} \varepsilon \cdot \chi M^* - i \sqrt{2} \hat{D}_\mu \chi \sigma^\mu \varepsilon - \frac{\sqrt{2}}{6} b_{\mu \chi} \sigma^\mu \varepsilon.
\]
We observe that differently from in supersymmetry the auxiliary field does not transform as a total derivative.

### 3.2 Vector superfields

Given a Lie algebra $g$ and a representation $\mathfrak{R}$ (see A) a vector superfield is specified by $V = V^a T_a$ and satisfies the reality condition

$$V^\dagger = V .$$

The transformation of a vector superfield under a gauge transformation is analogous as in supersymmetry and is given by

$$e^{2gV} \rightarrow e^{-2ig\Lambda} e^{2gV} e^{2ig\Lambda}, \quad (3.6)$$

where $g$ is the coupling constant and $\Lambda = \Lambda^a T_a$ with $\Lambda^a$ chiral superfields. Further as in supersymmetry the transformation $(3.6)$ enables to select the so-called Wess-Zumino gauge. In this gauge many components of $V$ vanish

$$V\big| = 0 , \quad D_\alpha V\big| = 0 , \quad \overline{D}_\alpha V\big| = 0 , \quad D\cdot V\big| = 0 , \quad \overline{D}\cdot V\big| = 0 . \quad (3.7)$$

We define now, in a supergravity context, the superfield strength tensors

$$W_\alpha = -\frac{1}{4} [\overline{D}\cdot \overline{D} - 8\mathcal{R}] e^{2gV} D_\alpha e^{-2gV} , \quad \overline{W}_{\dot{\alpha}} = \frac{1}{4} [D\cdot D - 8\mathcal{R}^\dagger] e^{-2gV} \overline{D}_{\dot{\alpha}} e^{2gV} \quad (3.8)$$

which is similar to its supersymmetric definition with the substitution $D\cdot D$ ($D$ being the usual covariant derivative in supersymmetry) by $\overline{D}\cdot \overline{D} - 8\mathcal{R}$. The operator $\overline{D}\cdot \overline{D} - 8\mathcal{R}$ ensures that $W_\alpha$ is chiral, i.e., $\overline{D}_{\dot{\alpha}} W_\alpha = 0$ (see (3.3)). The physical degrees of freedom of the vector superfield are given by

$$v_\mu = \frac{1}{4} \sigma_\mu^{\dot{\alpha}\alpha} [D_\alpha , \overline{D}_{\dot{\alpha}}] V\big| , \quad \lambda_\alpha = \frac{i}{2g} W_\alpha\big| , \quad \bar{\lambda}_{\dot{\alpha}} = -\frac{i}{2g} \overline{W}_{\dot{\alpha}}\big| ,$$

$$D = \frac{1}{4g} D^\alpha W_\alpha\big| = \frac{1}{4g} \overline{D}_{\dot{\alpha}} \overline{W}^{\dot{\alpha}}\big| , \quad (3.9)$$

where $v_\mu$ is a real vector, $(\lambda, \bar{\lambda})$ is a Majorana spinor and $D$ is a real auxiliary field. Under the gauge transformation $(3.6)$ we have

$$W_\alpha \rightarrow e^{-2ig\Lambda} W_\alpha e^{2ig\Lambda} . \quad (3.10)$$

It is not our purpose to give all derivatives of the vector superfield, but to point out some that will be relevant in the sequel. In particular from $(3.9)$ we obtain

$$D_\alpha \overline{D}_{\dot{\alpha}} V\big| = - \overline{D}_{\dot{\alpha}} D_\alpha V\big| = v_{a\dot{\alpha}} ,$$

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with the conventions of $A$ to convert the vector index to spinor indices. We then obtain

$$D_\mu D_\alpha \overline{D}_\alpha V = E_\mu \hat{\psi}_\mu D_\mu D_\alpha \overline{D}_\alpha V + E_\mu \hat{\psi}_\mu D_\mu D_\alpha \overline{D}_\alpha V + E_\mu \hat{\psi}_\mu \overline{D}_\mu D_\alpha \overline{D}_\alpha V = e_\mu \hat{\psi}_\mu v_{\alpha\dot{\alpha}} + i \left[ \psi_{\mu\alpha} \bar{\lambda}_\dot{\alpha} + \bar{\psi}_{\mu\dot{\alpha}} \lambda_\alpha \right] - \frac{i}{2} \sigma^{\nu\alpha\dot{\alpha}} \left( \psi_{\nu\sigma} \bar{\psi}_\sigma \right) v_\rho \equiv \hat{D}_\mu v_{\alpha\dot{\alpha}},$$

and

$$\hat{D}_\mu \lambda_\alpha = \frac{i}{2g} \left( D_\mu W_\alpha + i \frac{1}{2} g \sigma_{\mu}^{\beta \dot{\beta}} [D_\beta \overline{D}_\dot{\beta} V, W_\alpha] \right), \quad (3.11)$$

$$= e_\mu \hat{\psi}_\mu \left( D_\mu \lambda_\alpha + i g \left[ v_\mu, \lambda_\alpha \right] \right) - \frac{i}{2} \left( \sigma^{\nu\rho} \psi_\mu \right) \alpha \left( \hat{F}^{\nu\rho} \right) + \frac{i}{2} D_\mu \psi_\alpha,$$

with $\hat{F}_{\mu
u} = \hat{D}_\mu v_\nu - \hat{D}_\nu v_\mu + ig \left[ v_\mu, v_\nu \right]$, or

$$\hat{F}_{\mu
u} = e_\mu \psi_\mu \overline{D}_\nu - \overline{D}_\nu v_\mu + ig \left[ v_\mu, v_\nu \right] - \frac{i}{2} \left[ \lambda \sigma_\mu \psi_\nu + \lambda \sigma_\nu \psi_\mu - \bar{\lambda} \sigma_\nu \bar{\psi}_\mu - \lambda \sigma_\mu \bar{\psi}_\nu \right]$$

$$= F_{\mu
u} - \frac{i}{2} \left[ \bar{\lambda} \sigma_\mu \bar{\psi}_\nu + \lambda \sigma_\nu \bar{\psi}_\mu - \bar{\lambda} \sigma_\nu \psi_\mu - \lambda \sigma_\mu \psi_\nu \right]. \quad (3.12)$$

The quantities we have introduced are covariant with respect to the gauge transformations and to supergravity.

For the lowest order components of $W_\alpha$ we refer to (4.7) where $W_\alpha$ is expanded using the $\Theta$–variables introduced in Section 4.

### 3.3 Gauge interactions of chiral superfields

Let $g$ be a compact real Lie algebra that can be simple, semisimple or even reductive, $i.e.$, can have semisimple and $u(1)$ factors, and let $\mathcal{R}$ be a representation of $g$ not necessarily irreducible. We define the vector superfield $V$ as in Section 3.2 and introduce a chiral superfield in the representation $\mathcal{R}$. The anti-chiral superfield is of course in the representation $\mathcal{R}$ (the complex conjugate representation) with generators $T_\alpha \rightarrow -T^*_\alpha$. Since under a gauge transformation $\Phi \rightarrow e^{-2i g A} \Phi$, as in usual supersymmetry $\Phi^\dagger e^{-2i g V} \Phi$, is gauge invariant. In turn this means that the Lagrangian will involve terms of the form $\Phi^\dagger e^{-2g V}$ or $e^{-2g V} \Phi$. Consequently we have to compute

$$\mathcal{X} = (\overline{D} \cdot D - 8\mathcal{R}) \Phi^\dagger e^{-2g V}, \quad \mathcal{X}^\dagger = (\overline{D} \cdot D - 8\mathcal{R}) e^{-2g V} \Phi,$$

which are respectively chiral and anti-chiral superfields. For the first term, since in the Wess-Zumino gauge $V^3 = 0$ we get

$$\mathcal{X} = \mathcal{X} - 2g (\overline{D} \cdot D - 8\mathcal{R}) \Phi^\dagger V + 2g^2 (\overline{D} \cdot D - 8\mathcal{R}) \Phi^\dagger V^2. \quad (3.14)$$
For the lowest order components of $\mathcal{X}$ we refer to (4.6) where $\mathcal{X}$ is expanded using the $\Theta$--variables introduced in Section 4.

Under a supergravity transformation (2.25) $V$ transforms as

$$\delta V = -\xi^M D_M V,$$

but does not remain in the Wess-Zumino gauge. This means that (2.25) has to be combined with a gauge transformation. We do not give here the precise form of the transformation of the chiral and vector superfields under the combined gauge-supergravity transformations. The interested reader is refereed to [1], Chapter 3. We just mention that all transformations involve now derivatives, covariant with respect to gauge symmetry.

4 General principles to construct invariant actions

In this section we introduce a new set of variables in order to facilitate the expansion of chiral superfields. We then introduce the last ingredient in order to construct invariant actions, given in the superspace language together with their transformation under a super-Weyl rescaling. Kähler transformations are also discussed.

4.1 Introduction of hybrid variables

The superfields in curved superspace introduced in Section 3 depend on $z^\hat{M} = (x^{\hat{\mu}}, \theta^{\hat{\alpha}}, \bar{\theta}^{\dot{\alpha}})$ but their components are defined through flat covariant derivatives. Consequently the field expansion is very complicated. To overcome this difficulty we introduce an alternative set of hybrid variables $z^M = (x^{\hat{\mu}}, \Theta^\alpha, \bar{\Theta}^{\dot{\alpha}})$ where the new $\Theta$--variables carry Lorentz indices whereas $x^{\hat{\mu}}$ carry Einstein indices. The new $\Theta$--variables are chosen in such a way that the expansion of a chiral superfield is defined by its covariant derivatives:

$$\Phi(x, \Theta) = \Phi \left| + \sqrt{2}\Theta \cdot (D\Phi) \right| - \frac{1}{4}\Theta \cdot \Theta (D \cdot D \Phi) \right|$$

$$= \phi + \sqrt{2}\Theta \cdot \chi - \Theta \cdot \Theta F .$$

(4.1)

It is now possible to rewrite the transformation (3.4) using the $\Theta$--variables

$$\delta \Phi = -\eta^M \partial_M \Phi ,$$

(4.2)
where
\[ \eta^{\bar{\mu}} = 2i\Theta\sigma^{\bar{\mu}}\bar{\varepsilon} + \Theta \cdot \Theta \tilde{\psi}_{\bar{\mu}}\tilde{\sigma}^{\bar{\mu}}\bar{\sigma}^{\bar{\nu}} \varepsilon, \]
\[ \eta^{\alpha} = \varepsilon^{\alpha} - i\Theta\sigma^{\mu}\bar{\varepsilon}\psi_{\mu}^{\alpha} + \Theta \cdot \Theta \left[ \frac{1}{3}M^{*}\varepsilon^{\alpha} - i\omega^{\alpha\beta}(\sigma^{\bar{\nu}}\bar{\varepsilon})_{\beta} + \frac{1}{6}b_{\mu}(\bar{\varepsilon}\sigma^{\nu})^{\alpha} - \frac{1}{2}\bar{\psi}_{\nu}^{\alpha}\bar{\psi}_{\mu}\sigma^{\bar{\nu}}\sigma^{\bar{\mu}}\varepsilon \right]. \]

As we have seen in (3.5), the higher order component of a chiral superfield does not transform as a total derivative. In order to define invariant actions we introduce a capacity \( \Delta \) with transformation law
\[ \delta \Delta = -(-)^{|\bar{M}|}\partial_{\bar{M}}(\eta^{\bar{M}}\Delta), \tag{4.3} \]
such that
\[ \delta(\Delta \Phi) = -(-)^{|\bar{M}|}\partial_{\bar{M}}(\eta^{\bar{M}}\Delta \Phi), \tag{4.4} \]
by (4.2). This means that the product of a chiral superfield with a capacity is naturally invariant under a supergravity transformation. We choose now a specific capacity denoted \( \mathcal{E} \) such that \( \mathcal{E} = e \) with \( e = \det(e_{\bar{\mu}}^{\nu}) \). Using (4.3) with the transformation property of the supergravity multiplet (see (2.28)) we deduce step-by-step the components of \( \mathcal{E} \):
\[ \mathcal{E} = e \left[ 1 + i\Theta\sigma^{\bar{\mu}}\tilde{\psi}_{\bar{\mu}} - \Theta \cdot \Theta \left( M^{*} + \tilde{\psi}_{\bar{\mu}}\tilde{\sigma}^{\bar{\nu}}\tilde{\psi}_{\nu} \right) \right], \]
with \( \sigma^{\bar{\mu}} = e_{\mu}^{\bar{\nu}}\sigma^{\mu}, \tilde{\sigma}^{\bar{\nu}} = e_{\bar{\mu}}^{\bar{\nu}}e_{\nu}^{\bar{\rho}}\sigma^{\mu\nu}. \)

We recap now the \( \Theta \)-expansion of the basic fields needed to derive the supergravity action. The anti-chiral superfield corresponding to (4.1) leads to the chiral superfields
\[ \Xi = \Xi \sqrt{\frac{2\Theta}{3}}(D\Xi) - \frac{1}{4}\Theta \cdot \Theta (D \cdot D\Xi) = 4F^{\dagger} + \frac{4}{3}M\phi^{\dagger} + \Theta \left\{-4i\sqrt{2}(\sigma^{\mu}\hat{D}_{\mu}\hat{\chi}) + \frac{2\sqrt{2}}{3}b_{\mu}(\sigma^{\mu}\hat{\chi}) + \phi^{\dagger}\left( \frac{8}{3}(\sigma^{\mu\nu}\psi_{\mu\nu}) - \frac{4i}{3}(\sigma^{\mu}\tilde{\psi}_{\mu})M + \frac{4i}{3}\psi_{\mu}b_{\mu} \right) \right\}, \tag{4.5} \]
\[ -\Theta \cdot \Theta \left\{-4e_{\mu}^{\bar{\nu}}\hat{D}_{\mu}\hat{\phi}^{\dagger} + \frac{8i}{3}b_{\mu}\hat{D}_{\mu}\phi^{\dagger} + 2\sqrt{3}\tilde{\psi}_{\nu}\hat{D}_{\mu}\hat{\chi} + \frac{2\sqrt{2}}{3}\tilde{\chi}\sigma^{\mu\nu}\tilde{\psi}_{\mu\nu} - \frac{8}{3}M^{*}F^{\dagger} + \frac{\sqrt{2}}{3}i\tilde{\chi}\tilde{\psi}_{\mu}b_{\mu} + \frac{2\sqrt{2}}{3}\tilde{\chi}\sigma^{\mu\nu}\tilde{\psi}_{\mu}\tilde{b}_{\nu} + \phi^{\dagger}\left( -\frac{2}{3}e_{\psi}^{\mu}\tilde{e}_{\nu}\tilde{R}_{\tilde{\mu}\tilde{\rho}^{\nu}}^{\mu\nu} \right) - \frac{8}{9}MM^{*} + \frac{4}{9}b_{\mu}b_{\mu} + \frac{4i}{3}e_{\mu}^{\bar{\nu}}\hat{D}_{\mu}\tilde{b}_{\nu} + \frac{2}{3}\tilde{\psi}_{\nu}\cdot\tilde{\sigma}_{\mu}\tilde{\psi}_{\mu}\tilde{b}_{\nu} + \frac{2}{3}\psi_{\nu}\sigma^{\nu}\tilde{\psi}_{\mu}b_{\mu} + \frac{4i}{3}\tilde{\psi}_{\nu}\sigma^{\nu}\tilde{\psi}_{\mu} \right\} \right\), \]
\[ + \frac{1}{6}\varepsilon^{\mu\nu\rho\sigma}(\psi_{\mu}\sigma_{\sigma}\tilde{\psi}_{\nu\rho} + \tilde{\psi}_{\mu}\sigma_{\sigma}\psi_{\nu\rho}) \right\}, \]

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and

\[ \mathcal{X} = \mathcal{X} + \sqrt{2} \Theta \cdot (D \mathcal{X}) - \frac{1}{4} \Theta \cdot \Theta (D \cdot D \mathcal{X}) \]

(4.6)

\[ = \Xi + \Theta \cdot \left[ -4 \sqrt{2} g (\sigma^\mu \chi) v_\mu + 8 i g \phi^\dagger \lambda + 4 i g \phi^\dagger v_\mu (\sigma^\mu \phi^\dagger) \right] \]

\[-\Theta \cdot \Theta \left[ 8 i g v_\mu \hat{D}^\mu \phi^\dagger - 4 \sqrt{2} i g \chi \cdot \phi^\dagger \hat{D}_\mu v^\mu + \frac{8}{3} g \phi^\dagger v_\mu b^\mu - 4 g D \phi^\dagger + 4 g^2 \phi^\dagger v_\mu v^\mu \right]. \]

The superfield strength tensor associated to a vector superfield reads

\[ W_\alpha = -2 g \left[ i \lambda_\alpha + \left[ i (\sigma^{\mu \nu} \Theta)_\alpha (\hat{F}_{\mu \nu}) + \Theta_\alpha D \right] \right] \]

(4.7)

\[-\Theta \cdot \Theta \left[ (\sigma^\mu \hat{D}_\mu \lambda)_\alpha - \frac{i}{2} b_\mu (\sigma^\mu \lambda)_\alpha - \frac{i}{2} M^* \lambda_\alpha \right], \]

and the chiral superfield \( \mathcal{R} \) is given by

\[ \mathcal{R} = \mathcal{R} + \sqrt{2} \Theta \cdot (D \mathcal{R}) - \frac{1}{4} \Theta \cdot \Theta (D \cdot D \mathcal{R}) \]

(4.8)

\[ = \frac{1}{6} \left\{ - M + \Theta \cdot \left[ -2 (\sigma^{\mu \nu} \psi_{\mu \nu}) + i (\sigma^\mu \tilde{\psi}_\mu) M - i \psi_\mu b^\mu \right] - \Theta \cdot \Theta \left[ \frac{1}{2} \epsilon^\mu_\nu \epsilon^\nu_\rho R_{\rho \mu \nu} \right] \right. \]

\[ + \frac{2}{3} M M^* - \frac{1}{3} b_\mu b^\mu - i e_\mu \tilde{e}^\nu R_{\rho \mu \nu} - \frac{1}{2} \psi_\mu \tilde{\psi}^\mu M - \frac{1}{2} \tilde{\psi}_\mu \sigma^\nu \tilde{\psi}_\mu b^\mu \]

\[ - i \tilde{\psi}_\mu \sigma^\nu \psi_{\mu \nu} - \frac{1}{8} \varepsilon^{\mu \nu \rho \sigma} \left[ \psi_\mu \sigma_\sigma \tilde{\psi}_\nu \rho + \tilde{\psi}_\mu \sigma_\sigma \psi_{\nu \rho} \right] \right\}. \]

### 4.2 Super-Weyl rescaling and invariant actions

We are now ready to give the supergravity action in the superspace language. To construct a supergravity Lagrangian we chose a compact Lie group \( \mathfrak{g} \) and a unitary representation \( R \). As in section 3.3 we introduce the corresponding chiral and vector superfields (see A for notations). Next we introduce three gauge invariant functions: (1) the superpotential \( W(\Phi) \) a holomorphic function depending on chiral superfields; (2) the gauge kinetic function \( h_{ab}(\Phi) \) (where \( a, b \) are gauge indices in the adjoint representation of \( \mathfrak{g} \)) a holomorphic function depending on chiral superfields and (3) the Kähler potential \( K(\Phi, \Phi^\dagger) \) a real function depending on \( \Phi \) and \( \Phi^\dagger \).

Following (3.13) in order to have gauge invariant action we have to substitute \( K(\Phi, \Phi^\dagger) \) by \( K(\Phi, \Phi^\dagger e^{-2g V}) \). Using the previous subsection, and in particular (4.4), the action reads

\[ \mathcal{L} = \int d^2 \Theta \mathcal{L} \left\{ \frac{3}{8} (\bar{D} \cdot D - 8 \mathcal{R}) e^{-\frac{1}{2} K(\Phi, \Phi^\dagger e^{-2g V})} + W(\Phi) + \frac{1}{16 g^2} h_{ab}(\Phi) W^a_{\alpha} W^b_{\alpha} \right\} + \text{h.c.} \]

(4.9)
Almost all terms of the previous action can be understood directly from the corresponding supersymmetric action passing from flat superspace to curved superspace except one term which seems to be surprising at a first glance. Indeed, we have to exponentiate the Kähler potential. This exponentiation is fundamental in order to have correctly normalised kinetic terms as we shall see in the next section.

If we consider a super-Weyl transformation with parameter $\Sigma$, from [2.15-2.17] we can deduce the transformation of the supergravity multiplet that we do not give here (see [1], Chapter 5 for details) and deduce the transformation of $\mathcal{E}$:

$$\delta\Sigma\mathcal{E} = 6\Sigma\mathcal{E} + \partial\Theta^\alpha(S^\alpha\mathcal{E}) ,$$

(4.10)

where we have introduced the spinorial chiral superfield

$$S^\alpha = \Theta^\alpha [2\Sigma^\dagger - \Sigma] \mid \Sigma \mid + \Theta^\cdot \Theta^D\alpha \mid \Sigma \mid .$$

(4.11)

Furthermore if we impose that chiral fields have a conformal weight $w = 0$, i.e., $\delta\Sigma\varphi = w\varphi$ with $w = 0$ we have (using (2.14))

$$\delta\Sigma\Phi = -S^\alpha\partial\Theta^\alpha\Phi .$$

(4.12)

Imposing that vector superfields have a vanishing conformal weight too, after some algebraic manipulations, we obtain for a finite transformation that the action (4.9) transforms as

$$\mathcal{L} \rightarrow \frac{3}{8} \int d^2\Theta \mathcal{E}(\overline{\mathcal{D}}\mathcal{D} - 8\mathcal{R})e^{2(\Sigma + \Sigma^\dagger)}\exp\left[ -\frac{1}{3} K(\Phi, \Phi^\dagger)e^{-2gV} \right]$$

$$+ \int d^2\Theta \mathcal{E} e^{0\Sigma} W(\Phi) + \frac{1}{16g^2} \int d^2\Theta \mathcal{E} h(\Phi)_{ab} W^a W^b + \text{h.c.} .$$

(4.13)

Thus only the gauge part of the action is conformal invariant. Finally for further use we give the transformations of physical fields under a finite Weyl transformations:

$$e_\mu^\nu \rightarrow e^{(\Sigma + \Sigma^\dagger)}|e_\mu^\nu|e , e \rightarrow e^{2(\Sigma + \Sigma^\dagger)}|e ,$$

$$\chi^i \rightarrow e^{(\Sigma - 2\Sigma^\dagger)}|\chi^i| , \lambda_a \rightarrow e^{-3\Sigma^\dagger}|\lambda_a| , \bar{\psi}_\mu \rightarrow e^{(2\Sigma - \Sigma^\dagger)}\left(\bar{\psi}_\mu - i\bar{\sigma}_\mu D\Sigma \right) ,$$

$$\hat{F}_{\mu\nu} \rightarrow e^{-2(\Sigma + \Sigma^\dagger)}|\hat{F}_{\mu\nu}| , \nu^a_\mu \rightarrow e^{-(\Sigma + \Sigma^\dagger)}|\nu^a_\mu| , D^a \rightarrow e^{-2(\Sigma + \Sigma^\dagger)}|D^a| .$$

(4.14)

There is another transformation closely related to super-Weyl transformations and called Kähler transformation. A Kähler transformation reads:

$$K(\Phi, \Phi^\dagger) \rightarrow K(\Phi, \Phi^\dagger) + F(\Phi) + F^* (\Phi^\dagger) ,$$

$$W(\Phi) \rightarrow e^{-F(\Phi)} W(\Phi) ,$$

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with $F$ gauge invariant and holomorphic. Considering a Kähler transformation with parameter $F = \ln(W)$ we can recast (4.9) on the form

$$\mathcal{L} = \int d^2\Theta \mathcal{E} \left\{ \frac{3}{8} \left( \overline{\mathcal{D}} \mathcal{D} - 8\mathcal{R} \right) e^{-\frac{1}{4} G(\Phi, \Phi^\dagger e^{-2gV})} + 1 + \frac{1}{16g^2} h(\Phi)_{ab} W^{a\alpha} W^b_{\alpha} \right\} + \text{h.c.},$$

with $G = K + \ln |W|^2$ the generalised Kähler potential.

## 5 Supergravity action

In this section we give the main steps to compute in components the action (4.9). The final result is obtained in two steps. We firstly expand in components the action and eliminate the auxiliary fields. It turns out that the various kinetic terms are not correctly normalised. In order to have a correctly normalised action we have to perform a super-Weyl rescaling. This is perhaps the longest computation in the derivation of the final Lagrangian.

### 5.1 The pure supergravity action

The material introduced so far enables us to obtain easily the pure supergravity action:

$$\mathcal{L}_{\text{pure sugra}} = -3 \int d^2\Theta \mathcal{E} \mathcal{R} + \text{h.c.},$$

$$= \frac{1}{2} e_{\mu} \overline{e}^\nu R_{\mu\nu} \mathcal{E} + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \left[ \psi_{[\mu} \overline{\psi}_{\nu] \rho} \right] - \frac{1}{3} e^{\Phi} \mathcal{M}^* + b_\mu b^\mu,$$

where the first term corresponds to the usual Einstein-Hilbert Lagrangian for gravity, the second term corresponds to the Rarita-Schwinger Lagrangian describing the spin $-3/2$ gravitino, and the last two terms describe the auxiliary fields $M$ and $b_\mu$. The action, by construction, is obviously invariant under the supergravity transformation (2.28).

### 5.2 Coupling matter and gauge sector to supergravity

The first step is to expand the Lagrangian (4.9). As stated previously it is a lengthy computation to obtain the final Lagrangian. The Lagrangian has three parts: (1) one involving the Kähler potential, corresponding to the coupling of matter with gauge interactions and supergravity, (2) one involving the superpotential and (3) one gauge part. We have to Taylor
expand all these terms using the rules of Section 4. The first term is the most involved. We just decompose the superspace kinetic energy $\Omega$:

$$\Omega(\Phi, \Phi^\dagger) = -3e^{-\frac{1}{2}K(\Phi, \Phi^\dagger)} = W_I(\Phi^\dagger)W^I(\Phi),$$

where $W^I$ are holomorphic functions whilst $W_I$ are anti-holomorphic functions and the index $I$ corresponds to the label that we assign to each term in such expansion. As in usual supersymmetry we have

$$W^I(\Phi) = W_I + \sqrt{2}\Theta \cdot (W^I_i \chi^i) - \Theta \cdot \Theta\left[W^I_i F^i + \frac{1}{2}W^I_{ij} \chi^i \cdot \chi^j\right], \quad (5.1)$$

(see $A$ for notations) and all functions depend on the scalar part $\varphi$ of $\Phi$. We then substitute, as in Section 3.3, $\Phi^\dagger$ by $\Phi^\dagger e^{-2gV}$ and compute the product of the superfields $\mathcal{E}(\overline{\mathcal{D}} \cdot \mathcal{D} - 8\mathcal{R})\left(W_I(\Phi^\dagger e^{-2gV})\right)W^I(\Phi)$. Section 4 is used to calculate the components of $(\overline{\mathcal{D}} \cdot \mathcal{D} - 8\mathcal{R})W_I(\Phi^\dagger e^{-2gV})$ using (5.1). Since all superfields have a large number of components, this computation is lengthy, but not too complicated. At this stage $\Omega$ play the rôle of the Kähler potential associated to a Kähler manifold. At the end of the computation all terms involving $\Omega$ and its derivatives regroup to the geometrical terms associated to $\Omega$ along the lines of $A$. The second term – associated to the superpotential – is trivial and the last one – the gauge sector – presents no major difficulties.

Having expanded all terms it is straightforward to eliminate the auxiliary fields $F^i$ (associated to chiral superfields), $D^a$ (associated to gauge multiplets) and $M, b_\mu$ (associated to the gravity multiplet). However it turns out that the final Lagrangian is not correctly normalised, especially for all kinetic terms. For instance the Einstein-Hilbert action takes the form

$$\mathcal{L}_{E.H} = -\frac{1}{6}e_{\mu}^{\tilde{\mu}}e_{\nu}^{\tilde{\nu}}R_{\tilde{\mu}\tilde{\nu}\mu\nu}.$$  

(5.2)

Now it comes the most tedious part of the computation. We just have to consider appropriate super-Weyl rescaling in order to obtain a correctly normalised Lagrangian. This is done in two steps. In a first step a Weyl rescaling of the graviton

$$e_\mu^{\tilde{\mu}} \to \exp(-\lambda)e_\mu^{\mu}, \quad \text{with} \quad \exp(2\lambda) = -\frac{3}{\Omega}, \quad (5.3)$$

recasts (5.2) into the correctly normalised Einstein-Hilbert action, plus additional terms. This corresponds to a superconformal transformation with superfield

$$\Sigma_{\text{dilat}} = \Sigma_{\text{dilat}}^\dagger = \frac{1}{2}\lambda. \quad (5.4)$$

Using (4.14) and Eqs. [2.14, 2.16, 2.17] we deduce the transformation of all fields and all their covariant derivatives. We then perform the corresponding substitution into the Lagrangian.
obtained after expansion. The final Lagrangian is not at all correctly normalised, and in order to diagonalise the gravitino kinetic term the Weyl rescaling (5.3) has to be followed by the gravitino shift

\[ \psi_{\hat{\mu}} \rightarrow \psi_{\hat{\mu}} + \frac{\sqrt{2}i}{2} \Omega^{-1} \Omega^i \sigma_{\hat{\mu}} \bar{\chi}^i, \]
corresponding to the superconformal transformation with superfield

\[ \Sigma_{\text{shift}} = \sqrt{2} \left( -\frac{1}{2} \Omega^{-1} \Omega_i \chi^i \cdot \Theta \right). \] (5.5)

Again, using (4.14) and Eqs. [2.14, 2.16, 2.17] we deduce the transformation of all fields and all their covariant derivatives and perform the corresponding substitution into the Lagrangian. Note however that among the three terms in (4.9), the rescaling and the shift of the gauge part are straightforward since the Lagrangian is conformal invariant (see (4.13)). Note also that after the dilation and the shift the kinetic energy \( \Omega \) is naturally replaced by the Kähler potential \( K \). This means that the scalar fields \( \varphi \) parametrise the Kähler manifold with Kähler potential \( K \) and not \( \Omega \) (see A).

The final Lagrangian has many terms that we shall certainly not give in this short review. The interested reader can refer to [1], Table 6.2 and 6.3 or to [2], Appendix G (with almost the same notations and conventions). The kinetic part of the final Lagrangian contains in a natural manner the Einstein-Hilbert and Rarita-Schwinger Lagrangians for the graviton and gravitino, the Yang-Mills part and its corresponding fermionic counterpart for gauge interactions and the fermionic part together with its scalar counterpart describing the coupling of matter with gauge interactions and supergravity. Covariant derivative are naturally covariant with respect to all transformations. For instance for fermionic fields the covariant derivative reads:

\[
\hat{D}_{\mu} \chi^{i\alpha} = \epsilon_{\mu}^{\hat{\mu}} \left[ \partial_{\hat{\mu}} \chi^{i\alpha} + \chi^{i\beta} \omega_{\hat{\mu}\beta}^{\alpha} + igv_{i}^{a} (T_{a} \chi^{\alpha})^{j} + \Gamma_{j}^{i} k \chi^{\alpha j} \hat{D}_{\mu} \phi^{k} \right. \\
\left. - \frac{1}{4} (K_{j} \hat{D}_{\mu} \phi^{j} - K_{j} \hat{D}_{\mu} \phi^{j}) \chi^{\alpha i} \right],
\]

where \( \hat{D}_{\mu} \phi^{i} = \epsilon_{\mu}^{\hat{\mu}} \left[ \partial_{\hat{\mu}} \phi^{i} + igv_{i} \phi^{i} (T_{a} \phi)^{j} \right] \). The last two terms are associated to the Kähler symmetries of the Kähler manifold. The interacting part contains also some natural terms: a coupling of the gravitino with the golstino fundamental in the study of symmetry breaking, mass terms for fermionic fields, a bunch of four-fermions interacting terms and the scalar potential. The non-gauge part of the latter takes the following expression:

\[ V = e^{K} \left[ D_{i} W K^{i} \cdot \hat{D}^{*} W^{*} - 3 |W|^{2} \right], \]
where the covariant derivative of the superpotential is given by

\[ \mathcal{D}_i W = W_i + K_i W. \]

This means that in supergravity, differently from in supersymmetry, the potential is no longer positive. We encourage the readers to consult the reference [1] in order to guide and facilitate their own computations.

Since the computation of the final Lagrangian is laborious various alternative procedures have been proposed, e.g. using superconformal techniques as in [3]. Essentially, a compensating field is introduced in order to have a conformal invariant action. At the end the artificial superconformal symmetry is broken selecting an appropriate form for the compensating field. The form of the compensating field is directly related to the dilation (5.4) and the gravitino shift (5.5) because they can be combined in an appropriate superconformal transformation.

6 Conclusion

The Lagrangian described in Section 5 is the starting point of many studies. Several applications have been considered in [1] such as the study of the super-Higgs mechanisms and no-scale supergravity, the classification of the so-called soft-supersymmetric breaking terms, some applications in particle physics and cosmology. In the latter case supergravity in a Jordan frame has been studied.

A Conventions

In this appendix we give conventions used throughout this review.

For parametrisation of points in the superspace: Lorentz indices are taken to be untilde \( M = (\mu, \alpha, \dot{\alpha}) \) and Einstein indices are tilde \( \tilde{M} = (\tilde{\mu}, \tilde{\alpha}, \tilde{\dot{\alpha}}) \). Greek letters \( \mu, \nu, \cdots \) are devoted to vector indices whilst spinor indices are taken to be \( \alpha, \beta, \cdots \) for left handed spinors and \( \dot{\alpha}, \dot{\beta}, \cdots \) for right-handed spinors.

The metric is taken to be \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \) for vectors and \( \varepsilon_{12} = \varepsilon_{12} = 1 \) and \( \varepsilon^{12} = \varepsilon^{12} = -1 \) for spinors. Spinorial indices are raised and lowered as follows

\[ \psi_\alpha = \varepsilon_{\alpha\beta} \psi^\beta, \quad \psi^\alpha = \varepsilon^{\alpha\beta} \psi_\beta, \quad \bar{\psi}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\psi}^{\dot{\beta}}, \quad \bar{\psi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\beta}}, \]

and the scalar products of spinors are given by

\[ \psi \cdot \lambda = \psi^\alpha \lambda_\alpha, \quad \bar{\psi} \cdot \bar{\lambda} = \bar{\psi}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}. \]
The Levi-Civita tensor normalisation is $\varepsilon^{0123} = 1$.

The Pauli matrices are given by (with the following spin-index structure)

$$
\sigma^\mu_{a\dot{a}} = (1, \sigma^i), \quad \sigma^{\mu\dot{\alpha} \dot{\beta}} = (1, -\sigma^i),
$$

with $\sigma^i$ the usual Pauli matrices and $\sigma^0$ the two-by-two identity matrix. Finally the Lorentz generators in the spin representations are taken to be

$$(\sigma^{\mu
\nu})^\beta_{\alpha} = \frac{1}{4} \left( \sigma^{a \dot{a} \nu \dot{\beta}} - \sigma^{\nu \dot{a} \dot{a} a} \sigma^{\mu \dot{\alpha} \dot{\beta}} \right), \quad (\bar{\sigma}^{\mu \nu})^\dot{\alpha}_{\dot{\alpha} \beta} = \frac{1}{4} \left( \bar{\sigma}^{\mu \dot{\alpha} \dot{\alpha} \alpha} \sigma^{\nu \dot{\alpha} \dot{a} \dot{a}} - \sigma^{\nu \dot{\alpha} \dot{a} \dot{a}} \sigma^{\mu \dot{a} \dot{a} \alpha} \right).$$

The Pauli matrices enable conversion from spinor indices to vectors index and vice versa

$$v_{a\dot{a}} = \sigma^{\mu}_{a\dot{a}} v_{\mu}, \quad v_{\mu} = \frac{1}{2} \bar{\sigma}^{\mu \dot{\alpha} \dot{\alpha}} v_{a\dot{a}}.$$

Consider a compact real Lie algebra $\mathfrak{g}$ of dimension $n$. We take the physicists conventions for unitary representation $\mathfrak{R}$, i.e., the matrices $T_a, a = 1, \cdots, n$ acting on $\mathfrak{R}$ are hermitian and fulfill

$$[T_a, T_b] = i f_{a b c} T_c,$$

with real structure constants $f_{a b c}$.

Chiral superfields are denoted $\Phi^i$ and anti-chiral superfields $\Phi^{\dagger i}$. From the Kähler potential we deduce the geometric quantities of the corresponding Kähler manifold

Kähler metric : $K^i_{i^*} = \frac{\partial^2 K}{\partial \phi^i \partial \phi_{i^*}^\dagger}, \quad (K^{-1})^{i^*}_{i} \equiv K^i_{i^*},$

Christoffel symbols : $\Gamma^k_{ij} = K^k_{i^*} \frac{\partial^3 K}{\partial \phi^i \partial \phi^j \partial \phi_{k^*}^\dagger}, \quad \Gamma^i_{k^* k^*} = K^i_{k^*} \frac{\partial^3 K}{\partial \phi_{k^*}^\dagger \partial \phi_{k^*}^\dagger \partial \phi^i}$

Curvature tensor : $R^i_{j^* j^*} = \frac{\partial^4 K}{\partial \phi^i \partial \phi^j \partial \phi_{j^*}^\dagger \partial \phi_{j^*}^\dagger} - K^k_{k^*} \Gamma^i_{k^* k^*} \Gamma^k_{j^* j^*}.$

We denote $X_i = \partial_{\phi^i} X$ throughout and similarly for higher order derivatives.

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References

[1] M. Rausch de Traubenberg and M. Valenzuela, A Supergravity Primer. World Scientific, 2020.

[2] J. Wess and J. Bagger, Supersymmetry and Supergravity. Princeton University Press, second ed., 1992.

[3] D. Z. Freedman and A. Van Proeyen, Supergravity. Cambridge Univ. Press, 2012.

[4] D. Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara, “Progress toward a theory of supergravity,” Phys. Rev. D13 (1976) 3214–3218.

[5] E. Cremmer, B. Julia, J. Scherk, S. Ferrara, L. Girardello, and P. van Nieuwenhuizen, “Spontaneous symmetry breaking and Higgs effect in supergravity without cosmological constant,” Nucl. Phys. B147 (1979) 105.

[6] B. Fuks and M. Rausch de Traubenberg, Supersymétrie : exercices avec solutions. Ellipses, 2011. http://editions-ellipses.fr/supersymetrie-exercices-avec-solutions-p-7697.html.