On near–MDS codes and caps

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Abstract
Several classes of near-MDS sets of PG(3, q) are described. They are obtained either by considering the intersection of an elliptic quadric ovoid and a Suzuki-Tits ovoid of a symplectic polar space W(3, q) or starting from the q + 1 points of a twisted cubic of PG(3, q). As a by-product two classes of complete caps of PG(4, q) of size 2q^2 − q ± √2q + 2 are exhibited.

Keywords Near-MDS code · Cap · Ovoid · Twisted cubic

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1 Introduction

A q-ary linear code C of dimension k and length N is a k-dimensional vector subspace of \( \mathbb{F}_q^N \), whose elements are called codewords. A generator matrix of C is a matrix whose rows form a basis of C as an \( \mathbb{F}_q \)-vector space. The minimum distance of C is \( d = \min\{d(u, 0) \mid u \in C\} \).
C, \ u \neq 0\}, \text{where } d(u, v) \text{, } u, v \in \mathbb{F}_q^N, \text{is the Hamming distance on } \mathbb{F}_q^N, \text{namely the number of different components between } u \text{ and } v. \text{ A vector } u \text{ is } \rho\text{-covered by } v \text{ if } d(u, v) \leq \rho. \text{ The covering radius of a code } C \text{ is the smallest integer } \rho \text{ such that every vector of } \mathbb{F}_q^N \text{ is } \rho\text{-covered by at least one codeword of } C. \text{ A linear code with minimum distance } d \text{ and covering radius } \rho \text{ is said to be an } [N, k, d]_q \text{ } \rho\text{-code. Sometimes } d \text{ and } \rho \text{ are omitted and the notation } [N, k]_q \text{ code is used. The dual of a code } C \text{ is } C^\perp = \{ v \in \mathbb{F}_q^N \mid v \cdot c = 0, \forall c \in C \} \text{ (here } \cdot \text{ is the Euclidean inner product). The dimension of the dual code } C^\perp \text{ or the codimension of } C \text{ is } N - k. \text{ Any matrix which is a generator matrix of } C^\perp \text{ is called a parity check matrix of } C. \text{ If } C \text{ is linear with parity check matrix } M, \text{ its covering radius is the smallest } \rho \text{ such that every } w \in \mathbb{F}_q^{N-k} \text{ can be written as a linear combination of at most } \rho \text{ columns of } M.

Let PG(k − 1, q) be the (k − 1)-dimensional projective space over the finite field \mathbb{F}_q \text{ and let } X_1, X_2, \ldots, X_k \text{ be homogeneous projective coordinates. We denote by } U_i, i = 1, \ldots, k \text{ the point of PG(k − 1, q) having 1 in the } i\text{-th position and 0 elsewhere. An } n\text{-cap of PG(k − 1, q)} \text{ is a set of } n \text{ points no three of which are collinear. An } n\text{-cap of PG(k − 1, q)} \text{ is said to be complete if it is not contained in an } (n + 1)\text{-cap of PG(k − 1, q). By identifying the representatives of the points of a complete } n\text{-cap of PG(k − 1, q)} \text{ with columns of a parity check matrix of a } q\text{-ary linear code it follows that (apart from two sporadic exceptions) complete } n\text{-caps in PG(k − 1, q)} \text{ with } n > k \text{ and non-extendable linear } [n, n - k, 4]_q \text{ 2-codes are equivalent objects, see [11]. One of the main issues is to determine the spectrum of the sizes of complete caps in a given projective space. The interested reader is referred to [13] and references therein for an account on the subject.}

For an [N, k, d]_q \text{ code the so called Singleton bound holds: } d \leq N - k + 1; \text{ the integer } N - k + 1 - d \text{ is known as the Singleton defect of } C. \text{ A code with zero Singleton defect is called maximum distance separable (or MDS for short), whereas a code } C \text{ such that both of } C \text{ and } C^\perp \text{ have Singleton defect one is said to be near-MDS code. In particular, an } [N, k]_q \text{ linear code } C \text{ is a near-MDS code if and only if the the columns of a generator matrix } G \text{ of } C \text{ satisfies the following conditions:}

- any } k - 1 \text{ columns of } G \text{ are linearly independent,}
- there exist } k \text{ linearly dependent columns in } G,
- any } k + 1 \text{ columns of } G \text{ have full rank.}

By considering the columns of } G \text{ as representatives of projective points of PG(k − 1, q), } k \geq 3, \text{ it follows that near-MDS codes are equivalent to subsets } \mathcal{X} \text{ of PG(k − 1, q)} \text{ having the following properties:}

- every } k - 1 \text{ points of } \mathcal{X} \text{ generate a hyperplane in } PG(k - 1, q),
- there exist } k \text{ points in } \mathcal{X} \text{ lying on a hyperplane,}
- every } k + 1 \text{ points of } \mathcal{X} \text{ generate } PG(k - 1, q).

The reader is referred to [8] for more details. Throughout the paper we will refer to a pointset of PG(k − 1, q) satisfying the properties above as an NMDS-set. An NMDS-set is said to be complete if it is maximal with respect to set theoretical inclusion. The size of an NMDS-set of PG(k − 1, q) is at most } 2q + k - 2, \text{ if } q > 3 \text{ and } 2q + k \text{ otherwise [8, Proposition 6.2, Proposition 5.1]. The largest known NMDS-sets arise from elliptic curves and have size } q + \lfloor 2\sqrt[3]{q} \rfloor, \text{ if } q = p^r, r \geq 3 \text{ odd, and } p\lfloor 2\sqrt[3]{q} \rfloor \text{ or } q + \lfloor 2\sqrt[3]{q} \rfloor + 1 \text{ otherwise. The completeness of these sets has been investigated in [12]. Further constructions of NMDS-sets have been provided in [1, 2, 19].}

In this paper we deal with near-MDS sets of dimension 4 and caps of PG(4, q). In Sect. 2 a class of NMDS-sets of PG(3, q), q = 2^{2h+1}, h \geq 1, having size } q + \sqrt[4]{2q} + 1 \text{ is exhibited. It is obtained by looking at the intersection of an elliptic quadric and a Suzuki-Tits ovoid.
of a symplectic polar space $\mathcal{W}(3, q)$. Basing on this result, we also describe two classes of complete caps of $\text{PG}(4, q)$ of size $2q^2 - q \pm \sqrt{2q} + 2$. Our construction arises by considering the union of an elliptic ovoid and a Suzuki-Tits ovoid both embedded in a parabolic quadric $Q(4, q)$. It should be noted that constructions using two distinct elliptic quadrics lying in two distinct hyperplane sections of $\text{PG}(4, q)$ have been done by Tallini in [18]. Examples of caps of size $2q^2 + q + 9$, if $q > 4$ even, or of order $\frac{5}{2}q^2$, if $q$ is odd, or of size $3q^2 + 4$, if $q$ is even and $q \equiv 1 \pmod{3}$, have been provided in [5, 9, 10]. Finally in Sect. 3 we completely determine how many points an NMDS-set containing the $q + 1$ points of a twisted cubic of $\text{PG}(3, q)$ can have.

2 NMDS-sets from ovoids of $\mathcal{W}(3, q)$

In this section we study a class of NMDS-sets of $\text{PG}(3, q)$, $q = 2^{2r+1}$, arising by intersecting an elliptic quadric and a Suzuki-Tits ovoid. Let $\mathcal{W}(3, q)$ be a non-degenerate symplectic polar space of $\text{PG}(3, q)$, i.e. the set of all totally isotropic points and totally isotropic lines (called generators) with respect to a (non-degenerate) alternating bilinear form of the vector space underlying $\text{PG}(3, q)$. Thus, $\mathcal{W}(3, q)$ consists of all the points of $\text{PG}(3, q)$ and of $(q + 1)(q^2 + 1)$ generators. Through every point $P \in \text{PG}(3, q)$ there pass $q + 1$ generators and these lines are coplanar. The plane containing these lines is the polar plane of $P$ with respect to the symplectic polarity defining $\mathcal{W}(3, q)$. The incidence structure $\mathcal{W}(3, q)$ is preserved by the projective symplectic group $\text{PSp}(4, q)$. An ovoid $O$ of $\mathcal{W}(3, q)$ is a set of $q^2 + 1$ points of $\mathcal{W}(3, q)$ such that every generator of $\mathcal{W}(3, q)$ meets $O$ in exactly one point. It is well known that $\mathcal{W}(3, q)$ possesses no ovoids if $q$ is odd, whereas there are two known classes of ovoids if $q$ is even, namely the elliptic quadric preserved by the group $\text{PSL}(2, q^2) \cong \Omega^-(4, q)$ and the Suzuki-Tits ovoid admitting the group $S_\zeta(2^{2r+1})$, [16, Section 7.2]. A plane of $\text{PG}(3, q)$ meets an ovoid of $\mathcal{W}(3, q)$ in one point or $q + 1$ points. The latter pointset is a conic if the ovoid is elliptic or a translation oval (that is not a conic) if the ovoid is of Suzuki-Tits type. See [7] for further details on this topic. Two elliptic quadrics of $\mathcal{W}(3, q)$ meet in a point or in a conic. From [3], two Suzuki-Tits ovoids of $\mathcal{W}(3, q)$ have $1, q + 1, 2q + 1$ or $q \pm \sqrt{2q} + 1$ points in common, whereas an elliptic quadric and a Suzuki-Tits ovoid of $\mathcal{W}(3, q)$ meet in $q \pm \sqrt{2q} + 1$ points. Let $\mathcal{W}(3, q)$, $q = 2^{2r+1}$ and $r \geq 1$, be given by the alternating bilinear form

$$x_1y_4 + x_4y_1 + x_2y_3 + x_3y_2$$

and denote by $\perp$ the associated symplectic polarity of $\text{PG}(3, q)$. Let $T$ be a Suzuki-Tits ovoid of $\mathcal{W}(3, q)$. We may assume w.l.o.g. that

$$T = \{(1, x_2, x_3, x_4) \mid x_2, x_3, x_4 \in \mathbb{F}_q, x_2x_3 + x_2^{q+2} + x_3^2 + x_4 = 0\} \cup \{U_4\},$$

where $x^\sigma = x^{2^{r+1}}$ and hence $x^{\sigma^2} = x^{2^r}$. The Suzuki group $S_\zeta(q) \leq \text{PSp}(4, q)$ leaving $T$ invariant has a 2-transitive action on points of $T$ and has two orbits on points of $\mathcal{W}(3, q)$ [14, Section 16.4].

**Lemma 2.1** Let $E$ be an elliptic quadric of $\mathcal{W}(3, q)$ and let $T$ be a Suzuki-Tits ovoid of $\mathcal{W}(3, q)$. A plane of $\text{PG}(3, q)$ has at most four points in common with $E \cap T$.

**Proof** Let $\pi$ be a plane of $\text{PG}(3, q)$. We only need to consider the case when $|\pi \cap E| = |\pi \cap T| = q + 1$. Since $S_\zeta(q)$ is transitive on these planes we may assume $\pi : X_2 = 0$.  

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Hence $\pi^\perp = U_3$ is the nucleus of the oval $\pi \cap T$. Observe that the conic $\pi \cap E$ has the same nucleus $\pi^\perp = U_3$, due to the fact that $E$ is an ovoid of $\mathcal{W}(3, q)$. Hence, by [15, Corollary 7.12], $\pi \cap E$ is the set of points satisfying the quadratic equation

$$a_{11}X_1^2 + a_{33}X_3^2 + a_{44}X_4^2 + X_1X_4 = 0,$$

for some $a_{11}, a_{33}, a_{44} \in \mathbb{F}_q$, with $a_{33} \neq 0$. Suppose that $U_4$ is not a point of $\pi \cap E$. Then $a_{44} \neq 0$ and the point $P = (1, 0, x, x^\sigma) \in \pi \cap T$ belongs to $\pi \cap E$ if and only if there exists $x \in \mathbb{F}_q$ such that

$$a_{11} + a_{33}x^2 + a_{44}x^{2\sigma} + x^\sigma = 0,$$

that is

$$0 = (a_{11} + a_{33}x^2 + a_{44}x^{2\sigma} + x^\sigma)^{2r} = a_{11}^{2r} + a_{33}^{2r}x^{\sigma} + a_{44}^{2r}x^2 + x.$$

Therefore

$$x^{\sigma} = \left(\frac{a_{44}}{a_{33}}\right)^{2r} x^2 + \frac{1}{a_{33}^{2r}} x + \left(\frac{a_{11}}{a_{33}}\right)^{2r}$$

and substituting in Eq. (2.1), we get that this equation has at most four solutions. If $P = U_4$, then $a_{44} = 0$ and, arguing as above, we get that Eq. (2.1) has at most two solutions, i.e. the plane $\pi$ contains at most three points of $E \cap T$.

**Proposition 2.2** Let $E$ be an elliptic quadric of $\mathcal{W}(3, q)$ and let $T$ be a Suzuki-Tits ovoid of $\mathcal{W}(3, q)$ such that $|E \cap T| = q + \sqrt{2q} + 1$. Then $E \cap T$ is an NMDS-set.

**Proof** It is sufficient to observe that if $|E \cap T| = q + \sqrt{2q} + 1$, then there are planes intersecting $E \cap T$ in four points. Assume on the contrary that every plane intersects $E \cap T$ in at most three points. If $\ell$ is a line having two points in common with $E \cap T$, then the $q + 1$ planes through $\ell$ cover at most $q + 3$ points of $E \cap T$, a contradiction.

**Remark 2.3** Some computations performed with the aid of Magma [6] show that the NMDS-set constructed in the previous proposition can be extended by adding two further points if $q = 8$ and it is complete if $q = 32$.

**Problem 2.4** It is an open problem to determine whether or not the set $E \cap T$ can be obtained by means of $\mathbb{F}_q$-rational points of elliptic curves.

Indeed, assume $q = 8$. In this case, it is possible to choose $E$ in such a way that by projecting $E \cap T$ from one of its points $P$ to a plane $\pi$ with $P \notin \pi$, a pointset of $\pi$ contained in an elliptic cubic curve $C$ of $\pi$ arises. On the other hand, this situation does not occur if $q = 32$.

Let $\omega$ be a primitive element of $\mathbb{F}_q$ whose minimal polynomial is $F(X)$. Consider the elliptic quadric $E$ of $\text{PG}(3, q)$ given by $X_1X_4 + X_2^2 + X_2X_3 + \delta X_3^2 = 0$, for a suitable $\delta \in \mathbb{F}_q$. The projection of $E \cap T$ from $P = (0, 0, 0, 1)$ to $\pi : X_4 = 0$ gives a subset $\mathcal{P}$ of $\pi$.

If $q = 8$ let $F(X) = X^3 + X + 1$ and $\delta = \omega^3$. Then

$$\mathcal{P} = \{(1, \omega^4, 0, 0), (1, 1, \omega^2, 0), (1, \omega^2, \omega, 0), (1, 1, 0, 0), (1, \omega, \omega, 0), (1, \omega^3, \omega^2, 0), (1, \omega^3, 0, 0), (1, \omega^2, \omega^6, 0), (1, \omega^5, 0, 0), (1, \omega^5, \omega^5, 0), (1, 1, 0, 0, 0), (1, 0, 0, 0, 0)\}$$

and $\mathcal{C} = \mathcal{P} \cup \{(0, 0, 1, 0)\}$, where $\mathcal{C}$ consists of the points of $\pi$ satisfying $X_2^2 + X_1^2X_2 + \omega^5X_2X_3 + X_1X_3^2 = 0$. 

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If \( q = 32 \) let \( F(X) = X^5 + X^2 + 1 \) and \( \delta = \omega^5 \). Then

\[
P = \{(1, 0, 0, 0), (1, \omega^{10}, \omega^{20}, 0), (1, \omega^8, \omega^{20}, 0), (1, \omega^{10}, \omega^{19}, 0), (1, \omega^6, \omega^{21}, 0),
\]
\[
(1, \omega^{18}, \omega^{19}, 0), (1, \omega^{24}, \omega^{11}, 0), (1, \omega^{24}, \omega^{8}, 0), (1, \omega^{18}, \omega^{11}, 0), (1, \omega^{25}, \omega^{26}, 0),
\]
\[
(1, \omega^{18}, \omega^{8}, 0), (1, \omega^{19}, \omega^{27}, 0), (1, \omega^{23}, \omega^{25}, 0), (1, \omega^{13}, \omega^{29}, 0), (1, \omega^{11}, \omega^{28}, 0),
\]
\[
(1, \omega^2, \omega^{11}, 0), (1, \omega^{15}, \omega^{23}, 0), (1, \omega^2, \omega^8, 0), (1, \omega^{27}, \omega^{16}, 0), (1, \omega^{23}, \omega^{17}, 0),
\]
\[
(1, 1, \omega^6, 0), (1, \omega^{25}, \omega^{14}, 0), (1, \omega^{13}, \omega^{18}, 0), (1, \omega^{27}, \omega^{10}, 0), (1, 0, \omega^6, 0),
\]
\[
(1, \omega^3, \omega^{20}, 0), (1, \omega^3, \omega^{19}, 0), (1, \omega^{20}, \omega^{30}, 0), (1, \omega^{11}, \omega^{13}, 0), (1, \omega^{16}, \omega^{30}, 0),
\]
\[
(1, \omega^{28}, \omega^{22}, 0), (1, \omega^{21}, \omega^3, 0), (1, \omega^{15}, \omega^5, 0), (1, \omega^6, \omega^{30}, 0),
\]
\[
(1, \omega^{21}, \omega, 0), (1, \omega^{20}, \omega^{21}, 0), (1, 1, 0, 0), (1, \omega^{19}, 1, 0), (1, \omega^{16}, \omega^{21}, 0),
\]
\[
(1, \omega^{28}, \omega^{15}, 0)\}
\]

and

\[
C \cap P = \{(1, 0, 0, 0), (1, \omega^{10}, \omega^{20}, 0), (1, \omega^8, \omega^{20}, 0), (1, \omega^{10}, \omega^{19}, 0), (1, \omega^6, \omega^{21}, 0),
\]
\[
(1, 1, \omega^6, 0), (1, 0, \omega^6, 0), (1, 1, 0, 0), (1, \omega^3, \omega^{19}, 0), (1, \omega^8, \omega^{19}, 0),
\]
\[
(1, \omega^6, \omega^{30}, 0), (1, \omega^3, \omega^{20}, 0)\},
\]

where \( C \) is the cubic curve of \( \pi \) given by \( X_3^3 + \omega^{22}X_1^2X_2 + \omega^{19}X_1^2X_3 + \omega^{10}X_1X_2X_3 + \omega^{13}X_1X_3^2 + X_2X_3^2 = 0 \). Note that \(|C \cap P| = 12\) and hence there is no cubic curve of \( \pi \) containing \( P \).

### 2.1 Complete caps of PG(4, \( q \))

On the basis of the results obtained in Sect. 2, we exhibit two classes of complete caps of PG(4, \( q \)), \( q \) even, starting from two ovoids of a parabolic quadric. Let \( Q(4, q) \) be the parabolic quadric of PG(4, \( q \)) defined by \( X_1X_5 + X_2X_4 + X_3^2 = 0 \). The quadric \( Q(4, q) \) has \( (q + 1)(q^2 + 1) \) points and \( (q + 1)(q^2 + 1) \) lines (or generators). Through every point \( P \in Q(4, q) \) there pass \( q + 1 \) generators that are the lines of a quadratic cone. We will denote by \( t_P \) the three-dimensional projective space containing this cone and we will refer to it as the tangent space to \( Q(4, q) \) at \( P \). The quadric \( Q(4, q) \) has the point \( U_3 \) as a nucleus. If \( R \) is a point of PG(4, \( q \)) not on \( Q(4, q) \), let \( P \) be the unique point in common between \( Q(4, q) \) and the line \( U_3R \); thus the \( q^2 + q + 1 \) lines joining \( R \) with the points of \( t_P \cap Q \) are the lines that are tangent to \( Q(4, q) \) and pass through \( R \). Let PGO(5, \( q \)) denote the group consisting of the projectivities of PG(4, \( q \)) leaving invariant \( Q(4, q) \). An ovoid \( O \) of \( Q(4, q) \) is a set of \( q^2 + 1 \) points of \( Q(4, q) \) such that every generator of \( Q(4, q) \) meets \( O \) in exactly one point. Since \( q \) is even, the ovoids of \( Q(4, q) \) and the ovoids of \( W(3, q) \) are equivalent objects. Indeed, by projecting the points of \( Q(4, q) \) from \( U_3 \) onto a hyperplane \( \Pi \) of PG(4, \( q \)) not containing \( U_3 \), the points and the lines of \( Q(4, q) \) are mapped to the points and the lines of a symplectic polar space \( W(3, q) \) of \( \Pi \). Also, by projecting the conics of \( Q(4, q) \) having \( U_3 \) as a nucleus one gets the lines of \( \Pi \) that are not lines of \( W(3, q) \). It turns out that elliptic quadrics of \( W(3, q) \) correspond to three-dimensional hyperplane sections meeting \( Q(4, q) \) in an elliptic quadric, whereas Suzuki-Tits ovoids of \( W(3, q) \) correspond to ovoids of \( Q(4, q) \) spanning the whole PG(4, \( q \)). If \( T' \) is a Suzuki-Tits ovoid of \( Q(4, q) \), \( q = 2^{2r+1} \), we may assume w.l.o.g. that
$$T' = \left\{(1, x_2, x_3, x_4, x_5) \mid x_2, x_3, x_4, x_5 \in \mathbb{F}_q, x_4 + x_2^{\sigma+1} + x_3^{\sigma} = x_5 + x_2 x_3^{\sigma} + x_2^{\sigma+2} + x_3^2 = 0\right\} \cup \{U_5\},$$

where $x^{\sigma} = x^{2^{r+1}}$ and hence $x^{\sigma^2} = x^2$, see [17]. From the discussion before Lemma 2.1, it follows that the Suzuki group $SZ(q) \leq \text{PGO}(5, q)$ leaving invariant $T'$ has a 2-transitive action on points of $T'$ and has two orbits on points of $Q(4, q)$.

**Lemma 2.5** The group $SZ(q)$ has three orbits $\{U_3\}, O_1, O_2$ on points of $\text{PG}(4, q) \setminus Q(4, q)$ of size $1, (q^2+1)(q-1)$ and $q(q-1)(q^2+1)$, respectively.

**Proof** The group $SZ(q)$ has to fix the nucleus $U_3$. To see that it has two further orbits on points of $\text{PG}(4, q) \setminus Q(4, q)$, note that the subgroup of $SZ(q)$ of order $q - 1$ given by

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & d & 0 & 0 & 0 \\
0 & 0 & d^{\sigma+2} & 0 & 0 \\
0 & 0 & 0 & d^{\sigma+1} & 0 \\
0 & 0 & 0 & 0 & d^{\sigma+2}
\end{pmatrix},
\ d \in \mathbb{F}_q \setminus \{0\},
$$

permutes in a single orbit the $q - 1$ points of both $U_1U_3 \setminus \{U_1, U_3\}$ and $U_2U_3 \setminus \{U_2, U_3\}$. Moreover the group $SZ(q)$ is transitive on points of $T'$ and points of $Q(4, q) \setminus T'$. This concludes the proof. \hfill \Box

**Lemma 2.6** A point of $\text{PG}(4, q) \setminus (Q(4, q) \cup \{U_3\})$ lies on 0 or $q/2$ lines that are secant to $T'$ according as it belongs to $O_1$ or $O_2$, respectively.

**Proof** A line $\ell$ that is secant to $T'$ has $q - 1$ points belonging to $O_2$. Indeed, the plane $\langle \ell, U_3 \rangle$ meets $Q(4, q)$ in a conic that has $U_3$ as a nucleus. By projecting this conic from the nucleus $U_3$ onto a hyperplane $\Pi$ not passing through $U_3$, we get a line $m$ not of $\mathcal{W}(3, q)$ having two points in common with the Suzuki ovoid $T$ of $\Pi$. If there were another point on the conic belonging to $T'$ then the line $m$ would intersect $T$ in at least three points, a contradiction since $T$ is an ovoid of $\Pi$. This means that each line joining $U_3$ with one of the $q - 1$ points of $\ell \setminus T'$ intersects the quadric $Q(4, q)$ at a point not in $T'$. Consider the incidence structure having as pointset $O_2$ and as blocks the $q^2(q^2+1)/2$ lines that are secant to $T'$, where incidence is the natural one. Since $|O_2| = q(q - 1)(q^2 + 1)$ and a block contains $q - 1$ points of $O_2$, it follows that through a point of $O_2$ there pass $\frac{q^2(q^2+1)(q-1)}{2q(q-1)(q^2+1)} = \frac{q}{2}$ lines that are secant to $T'$. \hfill \Box

**Proposition 2.7** Let $O, O'$ be two ovoids of $Q(4, q)$. Then $O \cup O' \cup \{U_3\}$ is a cap of $\text{PG}(4, q)$.

**Proof** Let $B = O \cup O' \cup \{U_3\}$. Every line of $Q(4, q)$ has 0, 1 or 2 points in common with $B$. Since every line through $U_3$ has exactly one point in common with $Q(4, q)$, it follows that $B$ is a cap. \hfill \Box

**Theorem 2.8** Let $E'$ be an elliptic quadric of $Q(4, q)$, $q = 2^{2r+1}$, and let $T'$ be a Suzuki-Tits ovoid of $Q(4, q)$. Thus $E' \cup T' \cup \{U_3\}$ is a complete cap of $\text{PG}(4, q)$ of size $2q^2 - q \pm \sqrt{2q} + 2$.

**Proof** Let $B = E' \cup T' \cup \{U_3\}$ and denote by $\Pi$ the three-dimensional projective space containing $E'$. By the previous proposition $B$ is a cap.

We show that $B$ is complete. Let $P$ be a point of $\text{PG}(4, q) \setminus (\Pi \cup Q(4, q) \cup \{U_3\})$. If $P$ belongs to $O_2$, by Lemma 2.6, through $P$ there pass $q/2$ lines that are secant to $T' \subset B$, whereas if $P \in O_1$ then $U_3P$ meets $T'$ in a point $Q$ belonging to $T'$ and hence $P$ lies on the
line $U_3Q$ that is secant to $B$. Let $P \in Q(4, q) \setminus B$. We claim that there is a generator through $P$ intersecting $B$ in two points. To this end, it is enough to show that not all the $q + 1$ lines of $Q(4, q)$ passing through $P$ meet $E' \cap T'$.

Indeed, let $W(3, q)$ be the symplectic polar space of $\Pi$ obtained by projecting $Q(4, q)$ from $U_3$ onto $\Pi$. Thus $T'$ corresponds to $T$ and the $q + 1$ generators of $Q(4, q)$ containing $P$ are mapped to the $q + 1$ lines of $W(3, q)$ through the point $PU_3 \cap \Pi$. Since these latter lines lie in a plane, by Lemma 2.1 we have that at most four of them have at least one point in common with $E \cap T$. □

3 NMDS-sets containing a twisted cubic of $PG(3, q)$

3.1 Some geometry of plane cubic curves

The following preliminary result is based on [4]. Here and in the sequel we will denote by $A_D$ the set of points of a plane $\pi$ not lying on any of the lines of $\pi$ sharing three points with a cubic curve $D$ of $\pi$. Let $\square_q$ and $\square_q^{-}$ denote the sets of non-zero squares and non-squares of $\mathbb{F}_q$, respectively. Also, we will denote by $\text{Tr}(\cdot)$ the absolute trace from $\mathbb{F}_q$, $q$ even, to $\mathbb{F}_2$.

Lemma 3.1 Let $q \geq 23$ and let

$$D_1 : X_1X_2^3 - X_3^3 = 0,$$

then

$$A_{D_1} = \begin{cases} \{(1, 0, 0), \} & q \equiv 1 \pmod{3}, \\ \{(1, 0, 0), (0, 1, 0), \} & q \equiv -1 \pmod{3}, \\ \{(x, 1, 0) \mid x \in \mathbb{F}_q, x \in \square_q \cup \{0\}\} \cup \{(1, 0, 0), \} & q \equiv 0 \pmod{3}. \end{cases}$$

Let $q \geq 29$ be odd and $s \in \square_q^{-}$. Let

$$D_2 : X_2^2(X_3 - \lambda X_2) - X_1(sX_1 - X_3)^2 = 0, \quad \xi \in \mathbb{F}_q, \text{ s.t. } \xi^3 + 3\lambda\xi^2 + 3s\xi + \lambda s = 0,$$

then

$$A_{D_2} = \begin{cases} \{(1, -\frac{8s\xi}{3\xi^2 + s}, \frac{3s(\xi^2 + 3\lambda)}{3\xi^2 + s})\}, \{(1, 0, s)\}, \quad q \not\equiv -1 \pmod{3}, \\ \{(1, \frac{8\xi}{\xi^2 + 1}, \frac{3(9 - \xi^2)}{\xi^2 + 1}), \{(1, \frac{(\xi + 3)(1 + \xi)}{1 + \xi}, \frac{3\xi(\xi + 3)}{1 + \xi})\}, \{(1, 0, -3)\}, \quad q \equiv -1 \pmod{3} \text{ and } s = -3. \end{cases}$$

Let $q \equiv -1 \pmod{3}$ be odd, $q \geq 29$, and let

$$D_3 : X_2^2(X_3 - \lambda X_2) - X_1(3X_1 + X_3)^2 = 0, \quad F(T) = T^3 + 3\lambda T^2 - 9T - 3\lambda$$

irreducible over $\mathbb{F}_q$, then $A_{D_3} = \{(1, 0, -3)\}$.

Let $q \equiv -1 \pmod{3}$ be odd, $q \geq 29$, and let

$$D_4 : X_2^3 - 27(\lambda - 1)^3X_1^2X_3 - (3\lambda^2 - 3\lambda + 1)X_1X_3^2 - 9(\lambda - 1)(2\lambda^2 - 2\lambda + 1)X_1X_2X_3 - \lambda(3\lambda^2 - 3\lambda + 1)X_2^2X_3 = 0, \quad \lambda \in \mathbb{F}_q \setminus \{1, 1/2\},$$

then

$$A_{D_4} = \left\{\left((3\lambda^2 - 3\lambda + 1)^2, -9(\lambda - 1)^2(3\lambda^2 - 3\lambda + 1), 27(\lambda - 1)^3\right) \mid (1, -3, 0), \right\}.$$
\[(\lambda^2 - 3\lambda^2 - \lambda + 1), 27(\lambda - 1), (\lambda^2 - 3(\lambda - 1)^2, 0)\].

Let \(q \geq 32\) be even and \(\delta \in \mathbb{F}_q\), with \(\text{Tr}(\delta) = 1\). If
\[
D_5 : X_2^3 + \delta(\delta + 1)X_1^2X_2 + \delta X_1^2X_3 + X_1X_2X_3 = 0,
\]
then
\[
A_{D_5} = \begin{cases} 
(1, \delta, \delta + 1), (1, \delta, \delta), & q \equiv 1 \pmod{3}, \\
(0, 1, b), (0, 1, b + 1), (1, \delta, \delta + 1), (1, \delta, \delta), & q \equiv -1 \pmod{3}, b \in \mathbb{F}_q, \\
b^2 + b + \delta + 1 = 0.
\end{cases}
\]

Let \(q \geq 32\) be even, and let
\[
D_6 : (\delta^2 + \delta)X_1^3 + (\lambda + 1)X_2^2 + (\delta + \lambda)X_1^2X_2 + \lambda X_1X_2X_3 + X_1X_2X_3 + X_2X_3 = 0,
\]
\(\lambda \in \mathbb{F}_q\), then
\[
A_{D_6} = \begin{cases} 
(1, 1, 0), q \equiv -1 \pmod{3}, \delta = 1, F(T) = T^3 + \lambda T^2 + (\lambda + 1)T + 1 & \text{irreducible over } \mathbb{F}_q, \\
(\xi^2 + \xi + \delta + 1, \xi^2 + \xi + \delta, (\delta + 1)\xi^2 + \delta\xi + \delta + 1), (1, 1, \delta + 1), & q \equiv 1 \pmod{3}, \xi \in \mathbb{F}_q, \xi^3 + \lambda\xi^2 + (\lambda + 1)\xi + \delta\lambda + \lambda + 1 = 0.
\end{cases}
\]

Let \(q \equiv -1 \pmod{3}, q \geq 32\) be even, and let
\[
D_7 : X_2^3 + X_1X_2^2 + \left(\frac{\lambda + 1}{\lambda}\right)^3 X_1X_3 + \frac{\lambda + 1}{\lambda} X_1X_2X_3 = 0, \quad \lambda \in \mathbb{F}_q \setminus \{0, 1\},
\]
then
\[
A_{D_7} = \begin{cases} 
(0, 1, 0), \left(1, \frac{\lambda + 1}{\lambda}\right)^2, (0, 1, \frac{\lambda + 1}{\lambda}), (1, \frac{\lambda + 1}{\lambda}), (\frac{\lambda + 1}{\lambda})^3, \left(\frac{\lambda + 1}{\lambda}\right)^3.
\end{cases}
\]

**Proof** The plane curve \(D_i, i = 1, \ldots, 7\), is singular and a line through its singular point has at most one further point in common with \(D_i\). Then the singular point of \(D_i\) belongs to \(A_{D_i}\).

The curve \(D_1 : X_1X_2^3 - X_2 = 0\) consists of \(q + 1\) points. The point \((1, 0, 0)\) is a cuspinus and \(D_1\) has either one or \(q\) inflexion points, according as \(q \equiv 0 \pmod{3}\) or \(q \equiv 0 \pmod{3}\). By [4, Proposition 2.1] the number of points of \(\text{PG}(2, q) \setminus D_1\) lying on no line intersecting \(D_1\) in three points is either zero or one or \((q + 1)/2\), according as \(q \equiv 1\) or \(-1\) or \(0\) \(\pmod{3}\). In particular, if \(q \equiv -1 \pmod{3}\) the unique point is \((0, 0, 1)\) and if \(q \equiv 0 \pmod{3}\) the \((q + 1)/2\) points are \((\alpha, 1, 0)\), where \(\alpha\) is zero or belongs to \(\mathbb{Z}_q\).

Assume \(q\) is odd. Let \(s \in \mathbb{N}_q\) and let \(\lambda \in \mathbb{F}_q\) such that \(F(T) = T^3 + 3\lambda T^2 + 3sT + \lambda s\) is reducible over \(\mathbb{F}_q\). The polynomial \(F\) has exactly one root in \(\mathbb{F}_q\) whenever \(q \neq -1 \pmod{3}\); otherwise \(F\) is completely reducible if and only if \((\lambda - i)q - 1\) is a cube in \(\mathbb{F}_q^2\), with \(i \in \mathbb{F}_q^2\).
such that \( i^2 = s \). In this case the three roots of \( F \) are in \( \mathbb{F}_q \), see [15]. Let \( \xi \in \mathbb{F}_q \) such that \( F(\xi) = 0 \).

The curve \( \mathcal{D}_2 : X_3^2 (X_3 - \lambda X_2) - X_1 (s X_1 - X_3)^2 = 0 \) has \( q + 2 \) points. The point \((1, 0, s)\) is an isolated double point and \( \mathcal{D}_2 \) has either one or three inflexion points as \( q \equiv -1 \pmod{3} \) or \( q \equiv -1 \pmod{3} \). The projectivity of \( \text{PG}(2, q) \) associated with the matrix

\[
\begin{pmatrix} -s \xi & s & \xi \\ \frac{3 \xi^2 + s}{\xi^2 - s} & \frac{3 \xi}{\xi^2 - s} & 1 \\ -s & \frac{3 \xi}{\xi^2 - s} & 1 \end{pmatrix}
\]

maps \( \mathcal{D}_2 \) to the cubic curve given by \( X_2 (X_1^2 - s X_2^2) = X_3^3 \). By [4, Proposition 2.3, Proposition 2.7], when \( q \geq 29 \), the number of points of \( \text{PG}(2, q) \setminus \mathcal{D}_2 \) lying on no line intersecting \( \mathcal{D}_2 \) in three points is either one if \( q \not\equiv -1 \pmod{3} \) or three if \( q \equiv -1 \pmod{3} \). In particular, if \( q \not\equiv -1 \pmod{3} \) the unique point is \( (1, -2, \frac{8 s \xi}{2 \xi^2 - 3 \lambda \xi + 3}) \) and if \( q \equiv -1 \pmod{3} \) the three points are \( (1, -2, \frac{8 s \xi}{2 \xi^2 - 3 \lambda \xi + 3}), (1, -2, \frac{3 \lambda \xi}{2 \xi^2 - 3 \lambda \xi + 3}), (1, -2, \frac{3 \lambda \xi}{2 \xi^2 - 3 \lambda \xi + 3}) \).

If \( F \) is irreducible over \( \mathbb{F}_q \), then \( q \equiv -1 \pmod{3} \) and hence we may assume \( s = -3 \). In this case the projectivity of \( \text{PG}(2, q) \) associated with the matrix

\[
\begin{pmatrix} -9 & 3 & -3 \\ -9 & \frac{2 s + 3}{4} & 1 \\ 3 & 3 & 1 \end{pmatrix}
\]

maps \( \mathcal{D}_3 \) to the cubic curve given by \( X_2 (X_1^2 + 3 X_2^2) = X_3^3 + \frac{\lambda^2}{36} X_1 (X_1^2 - 9 X_2^2) \). By [4, Proposition 2.4], if \( q \geq 29 \), every point of \( \text{PG}(2, q) \setminus \mathcal{D}_3 \) lies on at least a line meeting \( \mathcal{D}_3 \) in three points.

Let \( \lambda \in \mathbb{F}_q \setminus \{1, 1/2\} \) and \( \mathcal{D}_4 : X_3^2 - 27 (\lambda - 1)^3 X_1^2 X_2 - (3 \lambda^2 - 3 \lambda + 1) X_1 X_3^2 - 9 (\lambda - 1)(2 \lambda^2 - 2 \lambda + 1) X_1 X_2 X_3 - \lambda (3 \lambda^2 - 3 \lambda + 1) X_2^2 X_3 = 0 \). The projectivity of \( \text{PG}(2, q) \) associated with the matrix

\[
\begin{pmatrix} -27 (\lambda - 1)^3 & -9 (\lambda - 1)^3 & (3 \lambda^2 - 2 \lambda + 1)(3 \lambda^2 - 3 \lambda + 1) \\ \frac{27}{4} (\lambda - 1)^3 & \frac{9}{2} \lambda^2 (\lambda - 1) & \frac{\lambda^2}{4} \\ 27 (\lambda - 1)^3 & 3 (\lambda - 1)(3 \lambda^2 - 2 \lambda + 1) & \lambda (3 \lambda^2 - 3 \lambda + 1) \end{pmatrix}
\]

maps \( \mathcal{D}_4 \) to the cubic curve given by \( X_2 (X_1^2 + 3 X_2^2) = X_3^3 \). In this case, from [4, Proposition 2.3], when \( q \geq 29 \), apart from the singular point \((3 \lambda^2 - 3 \lambda + 1)^2, -9 (\lambda - 1)^2 s (3 \lambda^2 - 3 \lambda + 1), 27 (\lambda - 1)^3 \), \( \mathcal{A}_{\mathcal{D}_4} \) contains the points \((1, -3, 0), (\lambda^2, -3 (\lambda^2 - \lambda + 1), 27 (\lambda - 1)), (\lambda^2, -3 (\lambda^2 - \lambda + 1)^2, 0) \).

Assume \( q \) is even. Let \( \delta \in \mathbb{F}_q \) such that \( \text{Tr}(\delta) = 1 \). The curve \( \mathcal{D}_5 : X_2^3 + \delta (\delta + 1) X_1^2 X_2 + \delta X_1^2 X_3 + X_1 X_3^2 + X_1 X_2 X_3 = 0 \) has \( q + 2 \) points and \((1, \delta, \delta)\) is an isolated double point. If \( q \equiv -1 \pmod{3} \), let \( b \in \mathbb{F}_q \) such that \( b^2 + b + \delta + 1 = 0 \). The projectivity associated with the matrix

\[
\begin{pmatrix} \delta (b + 1) & 0 & b \\ \delta + 1 & 0 & 1 \\ \delta & 1 & 0 \end{pmatrix}
\]

maps \( \mathcal{D}_5 \) to the cubic curve \( X_2 (X_1^2 + X_1 X_3 + X_3^2) + X_1^2 X_3 + X_1 X_3^2 = 0 \). By [4, Proposition 2.5], when \( q \geq 32 \), apart from the singular point, \( \mathcal{A}_{\mathcal{D}_5} \) consists of the points \((0, 1, b)\),
(0, 1, b + 1), (1, δ, δ + 1). If \( q \equiv 1 \pmod{3} \), then the projectivity associated with the matrix

\[
\begin{pmatrix}
\delta^2 & \delta & 0 \\
\delta^2(\delta + 1) & \delta^2 & 0 \\
0 & 1 & 1
\end{pmatrix}
\]

sends \( D_5 \) to the cubic curve \( X_2(X_1^2 + X_1X_3 + \delta X_3^2) + (\delta + 1)X_1^3 + \delta X_2^2X_3 + \delta^2X_1X_3^2 = 0 \) and by [4, Proposition 2.9], when \( q \geq 64 \), \( \mathcal{A}_D \) contains the isolated double point and the point \((1, \delta, \delta + 1)\).

Set \( D_6 : (\delta^2 + \delta + \lambda)X_1^3 + (\lambda + 1)X_2^2 + (\delta + \lambda)X_1X_2X_3 + X_1X_2X_3 + X_1X_2X_3 = 0 \), for some \( \lambda \in \mathbb{F}_q \). If \( q \equiv 1 \pmod{3} \), let \( \xi \in \mathbb{F}_q \) such that \( \xi^3 + (\lambda + 1)\xi^2 + (\delta + \lambda + 1)\xi + \delta\lambda + \lambda + 1 = 0 \). The projectivity of \( \mathcal{P}G(2, q) \) associated with the matrix

\[
\begin{pmatrix}
\xi^2 + \xi(\delta + 1) \\
\xi(\delta(\xi^3 + (\delta + 1)\xi^2 + \delta^2 + \delta + 1)) \\
\xi^2 + \xi^3
\end{pmatrix}
\]

sends \( D_6 \) to the cubic curve \( X_2(X_1^2 + X_1X_3 + \delta X_3^2) + (\delta + 1)X_1^3 + \delta X_2^2X_3 + \delta^2X_1X_3^2 = 0 \). From [4, Proposition 2.9], when \( q \geq 64 \), it follows that \( \mathcal{A}_{D_6} \) consists of the points \((1, 1, \delta + 1)\) and \((\xi^2 + \xi + \delta + 1, \xi^2 + \xi + \delta, (\delta + 1)\xi^2 + \delta\xi + \delta + 1)\). Let \( q \equiv -1 \pmod{3} \) and let \( \delta = 1 \).

If \( F(T) = T^3 + \lambda T^2 + (\lambda + 1)T + 1 \) is reducible over \( \mathbb{F}_q \), let \( \xi \in \mathbb{F}_q \) such that \( F(\xi) = 0 \). Thus the projectivity of \( \mathcal{P}G(2, q) \) associated with the matrix

\[
\begin{pmatrix}
\xi^2 + \xi + 1 \\
\xi(\xi + 1) \\
\xi^2 + \xi + 1
\end{pmatrix}
\]

maps \( D_6 \) to the cubic curve \( X_2(X_1^2 + X_1X_3 + X_3^2) + X_1^2X_3 + X_1X_3^2 = 0 \) and, by [4, Proposition 2.5], when \( q \geq 32 \), \( \mathcal{A}_{D_6} \) consists of the points \((1, \frac{\xi^2 + \xi + 1}{\xi(\xi + 1)}, \frac{\xi^2 + \xi + 1}{\xi^2 + \xi + 1})\), \((1, \frac{\xi^2 + \xi + 1}{\xi(\xi + 1)}, \frac{\xi^2 + \xi + 1}{\xi^2 + \xi + 1})\), \((1, \frac{\xi^2 + \xi + 1}{\xi(\xi + 1)}, \frac{\xi^2 + \xi + 1}{\xi^2 + \xi + 1})\), \((1, 1, 0)\). Assume \( F(T) \) is irreducible over \( \mathbb{F}_q \). The projectivity of \( \mathcal{P}G(2, q) \) associated with the matrix

\[
\begin{pmatrix}
\lambda & \lambda & 0 \\
\lambda + 1 & \lambda & 1 \\
0 & 0 & \lambda
\end{pmatrix}
\]

sends \( D_6 \) to the cubic curve \( X_2(X_1^2 + X_1X_3 + X_3^2) + X_1^2X_3 + X_1X_3^2 + \frac{1}{\xi + 1} (X_3^2 + X_1X_3^2 + X_3^2) = 0 \), if \( \lambda \neq 0 \), whereas \( D_6 \) is sent to the cubic curve \( X_2(X_1^2 + X_1X_3 + X_3^2) + X_1^2X_3 + X_1X_3^2 + X_3^2 = 0 \) by the projectivity associated with the matrix

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]

if \( \lambda = 0 \). In this case, by [4, Proposition 2.8], when \( q \geq 32 \), we get \( \mathcal{A}_{D_6} = \{(1, 1, 0)\} \).
Finally, let \(\lambda \in \mathbb{F}_q \setminus \{0, 1\}\) and \(D_7 : X_2^3 + X_1 X_2^2 + (\frac{\lambda + 1}{\lambda})^3 X_1^2 X_3 + \frac{\lambda + 1}{\lambda} X_1 X_2 X_3 = 0\). The projectivity of PG(2, q) associated with the matrix

\[
\begin{pmatrix}
\left(\frac{\lambda + 1}{\lambda}\right)^3 & \frac{\lambda + 1}{\lambda} & 0 \\
\frac{\lambda + 1}{\lambda} & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

maps \(D_7\) to the cubic curve given by \(X_2(X_1^2 + X_1 X_3 + X_3^2) + X_1^2 X_3 + X_1 X_2 X_3 = 0\). In this case, from [4, Proposition 2.5], when \(q \geq 32\), apart from the singular point \(\left(1, \left(\frac{\lambda + 1}{\lambda}\right)^2, \left(\frac{\lambda + 1}{\lambda}\right)^3\right)\), \(A_{D_7}\) contains the points \((0, 1, 0), \left(1, \left(\frac{\lambda + 1}{\lambda}\right)^2, 0\right)\), \((0, 1, \frac{\lambda + 1}{\lambda})\). □

### 3.2 The NMDS-sets

Let \(C\) be the twisted cubic of PG(3, q) consisting of the \(q + 1\) points \(\{P_t \mid t \in \mathbb{F}_q\} \cup \{(0, 0, 0, 1)\}\), where \(P_t = (1, t, t^2, t^3)\). It is well known that a line of PG(3, q) meets \(C\) in at most 2 points and a plane shares with \(C\) at most 3 points (i.e., \(C\) is a so called \((q + 1)\)-arc). A line of PG(3, q) joining two distinct points of \(C\) is called a real chord and there are \(q(q + 1)/2\) of them. Let \(\hat{C} = \{P_t \mid t \in \mathbb{F}_q^2\} \cup \{(0, 0, 0, 1)\}\) be the twisted cubic of PG(3, \(q^2\)) which extends \(C\) over \(\mathbb{F}_q^2\). The line of PG(3, \(q^2\)) obtained by joining \(P_t\) and \(P_{t^2}\), with \(t \notin \mathbb{F}_q\), meets the canonical Baer subgeometry PG(3, q) in the \(q + 1\) points of a line skew to \(C\). Such a line is called an imaginary chord and they are \(q(q - 1)/2\) in number. Also, for each point \(P\) of \(C\), the line \(\ell_P = \langle P, P'\rangle\), where \(P'\) equals \((0, 1, 2t, 3t^2)\) or \(U_3\) if \(P = P_t\) or \(P = U_4\), respectively, is called the tangent line to \(C\) at \(P\). At each point \(P_t\) (resp. \(U_4\)) of \(C\) there corresponds the osculating plane with equation \(t^3 X_1 - 3t^2 X_2 + 3t X_3 - X_4 = 0\) (resp. \(X_1 = 0\)), meeting \(C\) only at \(P_t\) (resp. \(U_4\)) and containing the tangent line. For more properties and results on \(C\) the reader is referred to [14, Chapter 21].

**Lemma 3.2** [14, Theorem 21.1.9] Every point of PG(3, q) \(\setminus \hat{C}\) lies on exactly one chord or a tangent of \(C\).

Let \(G\) be the group of projectivities of PG(3, q) stabilizing \(C\). Then \(G \simeq \text{PGL}(2, q)\) whenever \(q \geq 5\), and elements of \(G\) are induced by the matrices

\[
\begin{pmatrix}
a^3 & 3a^2b & 3ab^2 & b^3 \\
a^2c & a^2d + 2abc & b^2c + 2abd & b^2d \\
aec & bcd & ad^2 + 2bcd & bd^2 \\
c^3 & 3c^2d & 3cd^2 & d^3
\end{pmatrix}
\]

(3.1)

where \(a, b, c, d \in \mathbb{F}_q\), \(ad - bc \neq 0\).

**Lemma 3.3** [14, Corollary 5, Lemma 21.1.11] The group \(G\) has one or two orbits on points lying on imaginary chords of \(C\) according as \(q \equiv -1 \pmod{3}\) or \(q \equiv -1 \pmod{3}\), respectively. \(G\) has one or two orbits on points of PG(3, q) \(\setminus \hat{C}\) lying on tangent lines to \(C\) according as \(q \equiv 0 \pmod{3}\) or \(q \equiv 0 \pmod{3}\), respectively.

As representatives of \(G\)-orbits on points lying on tangent lines and not on \(C\), we may consider either \(U_2\) or \(U_1 - U_2\) and \(U_2\), according as \(q \equiv 0 \pmod{3}\) or \(q \equiv 0 \pmod{3}\). By [14, Lemma 21.1.11], if \(q \equiv -1 \pmod{3}\), a point lying on an imaginary chord belongs to one of the two orbits according as there are three osculating planes passing through it or none, respectively.
Let \( q \) be odd. Let \( s \) be a fixed element of \( \mathbb{D}_q \) and let \( \ell \) be the line joining \( Q = (1, 0, s, 0) \) and \( Q_1 = (0, 1, 0, s) \). The line \( \ell \) is an imaginary chord, whose extension over \( \mathbb{F}_{q^2} \) intersects \( C \) in the conjugated points \( P_i \) and \( P_{i^2} \), where \( i \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q \) such that \( i^2 = s \). If \( q \not\equiv -1 \) (mod 3), the point \( Q \) belongs to the unique \( G \)-orbit of points lying on imaginary chords. If \( q \equiv -1 \) (mod 3), we may assume \( s = -1 \). In this case through \( Q \) and \( Q_1 \) there pass three osculating planes, whereas if \( \hat{F}(T) = T^3 - 3\lambda T^2 - 9T + 3\lambda \) is irreducible over \( \mathbb{F}_q \) there is no osculating plane through the point \((1, \lambda, s, \lambda s)\). Note that \( \hat{F}(T) \) is irreducible over \( \mathbb{F}_q \) if and only if \((\lambda + i)^{q-1} \neq 1 \) is not a cube in \( \mathbb{F}_{q^2} \). This happens if and only if \( F(T) = T^3 + 3\lambda T^2 - 9T - 3\lambda \) is irreducible over \( \mathbb{F}_q \).

Let \( q \) be even. Let \( \delta \) be a fixed element of \( \mathbb{F}_q \) such that \( \text{Tr}(\delta) = 1 \) and let \( \ell \) be the line joining \( S = (1, 0, \delta, \delta) \) and \( S_1 = (0, 1, 1, \delta + 1) \). The line \( \ell \) is an imaginary chord, whose extension over \( \mathbb{F}_{q^2} \) meets \( C \) in the points \( P_i \) and \( P_{i^2} \), with \( i \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q \) such that \( i^2 + i = \delta \). Therefore, if \( q \not\equiv -1 \) (mod 3), the point \( S_1 \) is a representative of the unique \( G \)-orbit of points lying on imaginary chords. If \( q \equiv -1 \) (mod 3), we may assume \( \delta = 1 \). In this case through \( S_1 \) there pass three osculating planes; through the point \((1, \lambda, \lambda + 1, 1)\) there are three osculating planes or none, according as \( \hat{F}(T) = T^3 + \lambda T^2 + (\lambda + 1)T + 1 \) is reducible or not over \( \mathbb{F}_q \). Note that \( \hat{F}(T) \) is irreducible over \( \mathbb{F}_q \) if and only if \((\lambda + i)^{1-q} \neq 1 \) is not a cube in \( \mathbb{F}_{q^2} \).

Let \( \mathcal{X} \) be the pointset obtained by adding to \( C \) a point on a tangent line to \( C \). We show that \( \mathcal{X} \) is an NMDS-set that either is complete or it can be completed by adding at most one further point.

**Proposition 3.4** The set \( \mathcal{X} = C \cup \{U_2\} \) is an NMDS-set such that

1. \( \mathcal{X} \) is complete if \( q \equiv 1 \) (mod 3) and \( q \geq 23 \);
2. \( \mathcal{X} \cup \{R\} \) is complete, where \( R \in \{ (\alpha, \beta, 1, 0) | \alpha, \beta \in \mathbb{F}_q, \alpha \in \mathbb{D}_q \cup \{0\} \} \) if \( q \equiv 0 \) (mod 3) and \( q \geq 81 \), or \( R \in U_2 \cup \{U_2\} \) if \( q \equiv -1 \) (mod 3) and \( q \geq 23 \).

**Proof** By projecting \( C \) from \( U_2 \) onto the plane \( \pi : X_2 = 0 \), we obtain the \( q + 1 \) points of the cubic curve \( \mathcal{D}_1 : X_1X_2^2 - X_3^3 = 0 \) of \( \pi \). The set \( \mathcal{X} = \mathcal{C} \cup \{U_2\} \) consists of \( q + 2 \) points no three collinear and no five coplanar. Indeed if \( r \) is a line of \( \pi \) meeting \( \mathcal{D}_1 \) in three points, then the plane spanned by \( r \) and \( U_2 \) contains four points of \( \mathcal{X} \). If a point \( R \) of \( \pi \setminus \mathcal{D}_1 \) lies on a line of \( \pi \) intersecting \( \mathcal{D}_1 \) in three points, then no point of the line \( U_2 \mathcal{R} \) can be added to \( \mathcal{X} \) in order to get a larger NMDS-set.

By Lemma 3.1, if \( q \equiv 1 \) (mod 3) and \( q \geq 23 \), we have that \( \mathcal{A}_{\mathcal{D}_1} = \{U_1\} \) and hence \( \mathcal{X} \) is complete. If \( q \equiv -1 \) (mod 3) with \( q \geq 23 \), then \( \mathcal{A}_{\mathcal{D}_1} = \{U_1, U_3\} \). In this case no point of \( U_2 \cup U_3 \) is on a real chord and the result follows. If \( q \equiv 0 \) (mod 3) and \( q \geq 81 \), there are \((q + 1)/2\) points of \( \pi \setminus \mathcal{D}_1 \) lying on no line intersecting \( \mathcal{D}_1 \) in three points. They are \( (\alpha, 0, 1, 0) \), where \( \alpha \) is zero or in \( \mathbb{D}_q \) and they lie on the line \( r : X_4 = 0 \) of \( \pi \). It can be easily checked that there arise \((q + 1)/2\) points of the plane \( r \setminus U_2 \) none of them on a real chord. Hence each of them can be added to \( \mathcal{X} \) in order to get a larger NMDS-set. These \((q + 1)/2\) points are permuted into two orbits under the action of the stabilizer of \( U_2 \) in \( G \), namely \( \{ (\alpha, \beta, 1, 0) | \alpha, \beta \in \mathbb{F}_q, \alpha \in \mathbb{D}_q \} \) and \( U_2 \cup U_3 \setminus \{U_2\} \). Let \( R \) be a point belonging to one of these two orbits.

If \( R = U_3 \), by projecting \( C \) from \( U_3 \) onto the plane \( \pi' : X_3 = 0 \), we obtain the \( q + 1 \) points of the cubic curve \( \mathcal{D}' : X_1^2X_4 - X_3^3 = 0 \) (which is, up to a projectivity, the cubic \( \mathcal{D}_1 \) of Lemma 3.1) of \( \pi' \). Hence, if there were another point, say \( R' \), such that \( \mathcal{C} \cup \{R, R'\} \) is an NMDS-set, then \( R' \in \{ (0, 1, \beta', \alpha') | \alpha', \beta' \in \mathbb{F}_q, \alpha' \in \mathbb{D}_q \} \) and it would belong to a line joining \( U_2 \) and a point \( T = (\alpha, 0, 1, 0) \) for some \( \alpha \in \mathbb{D}_q \). This is a contradiction since the two lines \( U_3 \mathcal{R}' \) and \( U_2 \mathcal{T} \) are skew. Hence \( \mathcal{X} \cup \{U_3\} \) is complete.

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Let $R = (1, 0, s, 0)$. By projecting $C$ from $R$ onto the plane $\pi'' : X_1 = 0$, we obtain the points of the plane cubic curve $D'' : X_2(sX_2 - X_4)^2 - X_3^2X_4 = 0$ of $\pi''$ (which is, up to a projectivity, the cubic curve $D_2$ of Lemma 3.1 with $\lambda = 0$) consisting of the isolated double point $Q_1 = (0, 1, 0, s)$ and of $q + 1$ simple points. In this case $\mathcal{A}_{D''} = \{U_2, Q_1\}$. Therefore if there were another point $R'$ such that $\mathcal{C} \cup \{R, R'\}$ is an NMDS-set, then $R'$ would belong to the intersection between the two lines $\{\lambda, 1, \lambda s, s \mid \lambda \in \mathbb{F}_q\}$ and $U_2T$, where $T = (\alpha, 0, 1, s)$, for some $\alpha \in \mathbb{F}_q$. As before, it gives a contradiction since the two lines are skew. Hence $\mathcal{X} \cup \{R\}$ is complete.

\begin{proposition}
Let $q \geq 81$, $q \equiv 0 \pmod{3}$. The set $\mathcal{X} = \mathcal{C} \cup \{U_1 - U_2\}$ is a complete NMDS-set.
\end{proposition}

\begin{proof}
By projecting $C$ from $U_1 - U_2$ onto the plane $\pi : X_1 = 0$, we obtain the $q + 1$ points of the cubic curve $D : X_2X_4^2 - X_3^2X_4 - X_3^3 = 0$ of $\pi$. The curve $D$ has a cusp, namely $U_2$, and no inflexion points. The set $\mathcal{X} = \mathcal{C} \cup \{U_1 - U_2\}$ consists of $q + 2$ points no three collinear and no five coplanar. By [4, Proposition 2.6] every point of $\pi \setminus D$ lies on a line of $\pi$ intersecting $D$ in three points. Hence $\mathcal{X}$ is complete.
\end{proof}

Next we deal with the case when $\mathcal{X}$ is obtained by adding to $C$ a point on an imaginary chord. It turns out that $\mathcal{X}$ is an NMDS-set that is either complete or it can be completed by adding at most three further points.

\begin{proposition}
Let $q \geq 29$ be odd and let $Q = (1, 0, s, 0)$, $Q_1 = (0, 1, 0, s)$, $Q_2 = (0, 1, 0, 9s)$, $Q_3 = (0, 1, 3, 0)$ and $Q_6 = (0, 1, -3, 0)$. The set $\mathcal{X} = \mathcal{C} \cup \{Q\}$ is an NMDS-set such that

1. $\mathcal{X} \cup \{R\}$ is complete, where $R \in QQ_1 \setminus \{Q\}$ or $R \in QQ_2 \setminus \{Q\}$, if $q \not\equiv -1 \pmod{3}$;
2. either $\mathcal{X} \cup \{R\}$, $R \in QQ_1 \setminus \{Q\}$, is complete, or $\mathcal{X} \cup \{R_1, R_2, R_3\}$ is complete, where $R_1, R_2, R_3$ are three distinguished points belonging to $QQ_2 \setminus \{Q, Q_3\}, QQ_3 \setminus \{Q, Q_6\}$, respectively, if $q \equiv -1 \pmod{3}$.
\end{proposition}

\begin{proof}
By projecting $C$ from $Q$ onto the plane $\pi : X_1 = 0$, we obtain the $q + 1$ points of the cubic curve $D : X_2^2X_4 - X_2(sX_2 - X_4)^2 = 0$ of $\pi$ (which is, up to projectivities, the plane cubic curve $D_2$ with $\lambda = 0$ and $\xi = 0$ of Lemma 3.1). As before, if a point of the line $QR$, where $R \in \pi$, can be added to $\mathcal{X}$ in order to get a larger NMDS-set then no line of $\pi$ meeting $D$ in three points passes through $R$.

Assume $q \not\equiv -1 \pmod{3}$, and $q \geq 29$. By Lemma 3.1, the set $\mathcal{A}_D$ consists of the points $Q_1 = (0, 1, 0, s)$ and $Q_2 = (0, 1, 0, 9s)$. Note that every point of $QQ_2$ lies on an imaginary chord, except $Q_2$ if $q \equiv 0 \pmod{3}$. Hence $R \in (QQ_1 \cup QQ_2) \setminus \{Q\}$ if and only if $\mathcal{X} \cup \{R\}$ is an NMDS-set, for some $R \notin \mathcal{X}$. Let $R \in QQ_1 \setminus \{Q\}$. For $R = Q_1$ by projecting $C$ from $Q_1$ onto the plane $\pi' : X_2 = 0$, we get the $q + 1$ simple points of the cubic curve $D' : X_3^2(X_3 - X_3)^2 - X_4^2X_1 = 0$ of $\pi'$ which is mapped to $X_1(sX_1 - X_4)^2 - X_3^2X_4 = 0$ by the projectivity $X_1' = X_3, X_2' = X_4, X_3' = X_4, X_4' = s^2X_1$ of $\pi'$ (which is again, up to projectivities, the cubic curve $D_2$ of Lemma 3.1). Hence, by Lemma 3.1 the two points of $\mathcal{A}_{D'}$ are $Q_3 = (9, 0, s, 0)$ and $Q$. As before we have that $R' \in (QQ_1 \cup QQ_3) \setminus \{Q_1\}$ if and only if $\mathcal{C} \cup \{Q_1, R'\}$ is a complete $\mathcal{X}$ such that $\mathcal{A}_{D'} = \mathcal{C} \cup \{Q_1, R'\}$ is complete. Similarly, for $R = (1, 1, s, \lambda s), \lambda \in \mathbb{F}_q \setminus \{0\}$, by projecting $C$ from $R$ onto $\pi$, we get the $q + 1$ simple points of the cubic curve $D'' : X_3^2(X_4 - \lambda X_3) - X_2(sX_2 - X_4)^2 = 0$ (which is again, up to projectivities, the cubic curve $D_2$ of Lemma 3.1). In this case the two points of $\mathcal{A}_{D''}$ are $Q_4 = \left(0, 1, -\frac{8s^6}{3s^2+s}, \frac{3s^6(2s^3+3s)}{3s^2+s}\right)$, $Q_1$ and $R' \in (QR_1 \cup QR_4) \setminus \{R\}$ if and
only if $\mathcal{C} \cup \{R, R'\}$ is an NMDS-set, for some $R' \not\in (\mathcal{C} \cup \{R\})$. Again $\mathcal{X} \cup \{R\}$ is complete. We have seen that if $R \in Q Q_1 \setminus \{Q\}$ then there is no point $R'$ of $Q Q_2 \setminus \{Q\}$ such that $\mathcal{X} \cup \{R, R'\}$ is an NMDS-set. This implies that if $R \in Q Q_2 \setminus \{Q\}$ then there is no point $R'$ of $Q Q_1 \setminus \{Q\}$ such that $\mathcal{X} \cup \{R, R'\}$ is an NMDS-set.

Assume $q \equiv -1 \pmod{3}$, $q \geq 29$ and $s = -3$. By Lemma 3.1, the points of $A_{D'}$ are $Q_1 = (0, 1, 0, -3)$, $Q_2 = (0, 1, 0, -27)$, $Q_5 = (0, 1, 3, 0)$ and $Q_6 = (1, -3, 0)$. Some calculations show that the lines $Q Q_i$, $i = 2, 5, 6$, are permuted in a single orbit by the subgroup of $G$ generated by

$$
\begin{pmatrix}
1 & 3 & 3 & 1 \\
-3 & -5 & -1 & 1 \\
9 & 3 & -5 & 1 \\
-27 & 27 & -9 & 1
\end{pmatrix}.
$$

Such a subgroup has order three and fixes $Q$. Moreover the point $(\lambda, 1, 3(1 - \lambda), 0) \in Q Q_5$ is on an imaginary chord if $\lambda \in F_q \setminus \{1, 1/2\}$, otherwise it lies on a tangent line. A point $R$ belongs to $(Q Q_1 \cup Q Q_2 \cup Q Q_3 \cup Q Q_6) \setminus \{Q\}$ if and only if $\mathcal{X} \cup \{R\}$ is an NMDS-set, for some $R \not\in \mathcal{X}$. Let $R \in Q Q_1 \setminus \{Q\}$. Let $D'$ or $D''$ be the cubic curves obtained by projecting $C$ from $R$ onto the plane $\pi'$: $X_2 = 0$ or $\pi$, according as $R$ equals $Q_1$ or $R = (1, \lambda, -3, -3\lambda)$, $\lambda \in F_q \setminus \{0\}$, respectively. Note that $D'$ is projectively equivalent to the cubic curve $D_2$ of Lemma 3.1, whereas $D''$ is equivalent either to $D_3$ or $D_2$ of Lemma 3.1, according as the polynomial $T^3 + 3\lambda T^2 - 9T - 3\lambda$ is irreducible over $F_q$ or not, respectively. By repeating the previous arguments we find that, when $q \geq 29$, $A_{D'}$ consists of the points $Q, (1, 0, -1/3, 0)$, (0, 0, 1, ±3), whereas $A_{D''}$ either consists of the point $Q_1$ or is formed by the points $Q_1$, $Q_2 = (0, 1, 8\xi/2^2 - 1, 3(9 - 3\xi)/2^2 - 1), (0, 1, (\xi + 3\xi)/(1 + \xi), (3\xi(\xi + 3\xi)/(1 + \xi))$. It follows that $\mathcal{X} \cup \{R\}$ is complete.

Let $R \in Q Q_5 \setminus \{Q\}$. Let $D'''$: $X_3^3 - 27(\lambda - 1)X_1^2X_4 - (3\lambda^2 - 3\lambda + 1)X_1X_4^2 - 9(\lambda - 1)(2\lambda^2 - 2\lambda + 1)X_1X_3X_4 - (\lambda^2 - 3\lambda + 1)X_3^2X_4 = 0$ be the cubic curve obtained by projecting $C$ from $R$ onto the plane $\pi'$ (which is, up to projectivities, the cubic curve $D_4$ of Lemma 3.1). By Lemma 3.1, for $q \geq 29$, we have that $A_{D'''}$ consists of the four points:

$$(1, 0, -3, 0),$$
$$(\lambda^2, 0, -3(\lambda - 1)^2, 0),$$
$$(3\lambda^2 - 3\lambda + 1)^2, 0, -9(\lambda - 1)^2(3\lambda^2 - 3\lambda + 1), 27(\lambda - 1)^2),$$
$$(\lambda^2, 0, -3(\lambda^2 - \lambda + 1), 27(\lambda - 1)).$$

It follows that $(\lambda, 1 - \lambda, -3\lambda, 27(\lambda - 1)), (\lambda, 1 - 2\lambda, 3(\lambda - 1), 0)$ are the only points that can be added to $\mathcal{X} \cup \{R\}$ in order to have a complete NMDS-set.

**Proposition 3.7** Let $q \geq 32$ even and let $S = (1, 0, \delta, \delta), S_1 = (0, 1, 1, \delta + 1), S_2 = (1, 0, \delta, \delta + 1), S_4 = (0, 0, 1, 1)$ and $S_5 = (0, 0, 1, 0)$. The set $\mathcal{X} = \mathcal{C} \cup \{S_1\}$ is an NMDS-set such that

1. $\mathcal{X} \cup \{R\}$ is complete, where $R \in S_1 S \setminus \{S_1\}$ or $R \in S_1 S_2 \setminus \{S_1\}$, if $q \equiv 1 \mod{3}$;
2. either $\mathcal{X} \cup \{R\}, R \in S_1 S \setminus \{S_1\}$, is complete, or $\mathcal{X} \cup \{R_1, R_2, R_3\}$ is complete, where $R_1, R_2, R_3$ are three distinguished points belonging to $S_1 S_2 \setminus \{S_1\}, S_1 S_4 \setminus \{S_1\}, S_1 S_5 \setminus \{S_1\}$, respectively, if $q \equiv -1 \mod{3}$.

**Proof** By projecting $C$ from $S_1$ onto the plane $\pi : X_2 = 0$, we obtain the $q + 1$ points of the cubic curve $D : X_3^3 + \delta(\delta + 1)X_1X_3 + \delta X_2^2X_4 + X_1X_4^2 + X_1X_3X_4 = 0$ of $\pi$ (which is, up to projectivities, the cubic curve $D_5$ of Lemma 3.1).
Assume $q \equiv 1 \pmod{3}$. By Lemma 3.1, $A_D = \{S, S_2\}$. Observe that
\[
\frac{1+2\lambda+\lambda^2+\lambda+1}{\lambda^2+\lambda+1}
\]
is a root of $X^2 + \frac{\delta + 2\lambda + 1}{\delta + \lambda^2 + \lambda} + \lambda + 1 + \lambda^2$ and hence $\mathrm{Tr} \left( \frac{\delta + 2\lambda + 1}{\delta + \lambda^2 + \lambda} + \lambda + 1 + \lambda^2 \right) = 1$. It follows that every point of $S_1 S_2$ lies on an imaginary chord and $R \in (S_1 S_2) \setminus \{S_1\}$ if and only if $X \cup \{R\}$ is an NMDS-set, for some $R \not\in X$. Let $R = (1, \lambda, \delta + \lambda, \delta(1 + \lambda) + \lambda) \in SS_1 \setminus \{S_1\}$. By projecting $C$ from $R$ onto the plane $\pi' : X_1 = 0$, we get the $q + 1$ simple points of the cubic curve $D' : (\delta^2 + \delta + \lambda) X_2^2 + (\lambda + 1) X_3^2 + (\delta + \lambda) X_2^2 X_3 + \lambda X_2 X_3^2 + X_2 X_3 X_4 + X_3^2 X_4 = 0$ of $\pi'$ (which is, up to projectivities, the cubic curve $D_0$ of Lemma 3.1). By Lemma 3.1 the two points of $A_{D'}$ are $S_3 = (0, \xi^2 + \xi + \delta + 1, \xi^2 + \xi + \delta, (\delta + 1)\xi^2 + \delta\xi + \delta + 1)$ and $S_4$. As before we have that $R' \in (RS_1 \cup RS_3) \setminus \{R\}$ if and only if $C \cup \{R, R'\}$ is an NMDS-set, for some $R' \not\in (C \cup \{R\})$. It follows that $X \cup \{R\}$ is complete. We have seen that if $R \in S_1 S \setminus \{S_1\}$ then there is no point $R'$ of $S_1 S \setminus \{S_1\}$ such that $X \cup \{R, R'\}$ is an NMDS-set. This implies that if $R \in S_2 S_1 \setminus \{S_1\}$ then there is no point $R'$ of $S_1 S \setminus \{S_1\}$ such that $X \cup \{R, R'\}$ is an NMDS-set.

Assume $q \equiv -1 \pmod{3}$ and $\delta = 1$. By Lemma 3.1, the points of $A_D$ are $S = (1, 0, 1, 1)$, $S_2 = (1, 0, 1, 0)$, $S_4 = (0, 0, 1, 1)$ and $S_5 = (0, 0, 1, 0)$ (in such a case $D$ is projectively equivalent to $D_5$ with $\delta = b = 1$). Some calculations show that the lines $S_1 S_i, i = 2, 4, 5,$ are permuted in a single orbit by the subgroup of $G$ generated by
\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1
\end{pmatrix}.
\]
Such a subgroup has order three and fixes $S_1$. The point $(0, \lambda, \lambda + 1, 0) \in S_1 S_5 \setminus \{S_1\}$ is on an imaginary chord if $\lambda \in \mathbb{F}_q \setminus \{0, 1\}$, otherwise it lies on a tangent line. A point $R$ belongs to $(S_1 S_2 \cup S_2 S_2 \cup S_2 S_1 \cup S_1 S_5) \setminus \{S_1\}$ if and only if $X \cup \{R\}$ is an NMDS-set, for some $R \not\in X$. Let $R \in S_1 S \setminus \{S_1\}$. Let $D'$ be the cubic curve obtained by projecting $C$ from $R$ onto the plane $\pi' : X_1 = 0$ (which is again, up to projectivities, the cubic curve $D_6$ of Lemma 3.1). By repeating the previous arguments and taking Lemma 3.1 into account, we find that, when $q \geq 32$,
\[
A_{D''} \text{ either consists of the point } S_1 \text{ or is formed by the points } (0, 1, \frac{\xi^2 + \xi + 1}{\xi}, \frac{\xi^2 + \xi + 1}{\xi^2 + \xi + 1}), (0, 1, \frac{\xi^2 + \xi + 1}{\xi}, \frac{\xi^2}{\xi^2 + \xi + 1}), (0, 1, \frac{\xi^2 + \xi + 1}{\xi}, \frac{\xi^2}{\xi^2 + \xi + 1}), S_1 ,
\]
as indicated by the polynomial $T^3 + \lambda T^2 + (\lambda + 1)T + 1$ is irreducible over $\mathbb{F}_q$ or not, respectively. It follows that $X \cup \{R\}$ is complete.

Let $R \in S_1 S_5 \setminus \{S_1\}$. Let $D''' : X_3^3 + X_1 X_4^2 + \left(\frac{\lambda + 1}{\lambda}\right)^3 X_2 X_4 + \frac{\lambda + 1}{\lambda} X_1 X_3 X_4 = 0$ be the cubic curve obtained by projecting $C$ from $R$ onto the plane $\pi$ (which is, up to projectivities, the cubic curve $D_7$ of Lemma 3.1). By Lemma 3.1, we have that $A_{D'''}$ consists of the four points:
\[
\left(1, 0, \frac{\lambda + 1}{\lambda}\right)^2, \left(\frac{\lambda + 1}{\lambda}\right)^3, (0, 0, 1, 0), \left(1, 0, \frac{\lambda + 1}{\lambda}\right)^2, 0, \left(0, 0, 1, \frac{\lambda + 1}{\lambda}\right).
\]
It follows that $(\lambda, 1, \lambda + 1, 0), (0, \lambda, 1, \lambda + 1)$ are the only points that can be added to $X \cup \{R\}$ in order to have a complete NMDS-set.

In a similar way, by taking into account the cubic curves $D_3$ and $D_6$ of Lemma 3.1, it can be checked that the following result holds true.

**Proposition 3.8** Let $q \equiv -1 \pmod{3}$ and let $R$ be the point $(1, \lambda, -3, -3\lambda)$, where $\lambda \in \mathbb{F}_q$ is such that $T^3 - 3\lambda T^2 - 9T + 3\lambda$ is irreducible over $\mathbb{F}_q$ if $q \geq 29$ is odd or $(1, \lambda, \lambda + 1, 1)$.
where $\lambda \in \mathbb{F}_q$ is such that $T^3 + \lambda T^2 + (\lambda + 1)T + 1$ is irreducible over $\mathbb{F}_q$ if $q \geq 32$ is even. The set $\mathcal{X} = \mathcal{C} \cup \{R\}$ is a complete NMDS-set.

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**References**

1. Abatangelo V., Larato B.: Near-MDS codes arising from algebraic curves. Discret. Math. 301, 5–19 (2005).
2. Aguglia A., Giuzzi L., Sonnino A.: Near-MDS codes from elliptic curves. Des. Codes Cryptogr. 89, 965–972 (2021).
3. Bagchi B., Sastry N.: Intersection pattern of the classical ovoids in symplectic 3-space of even order. J. Algebra 126, 147–160 (1989).
4. Bartoli D., Marcugini S., Pambianco F.: On the completeness of plane cubic curves over finite fields. Des. Codes Cryptogr. 83, 233–267 (2017).
5. Bierbrauer J., Edel Y.: A family of caps in projective 4-space in odd characteristic. Finite Fields Appl. 6(4), 283–293 (2000).
6. Bosma W.: The Magma algebra system. I. The user language. J. Symbolic Comput. 24, 25–265 (1997).
7. Brown M.R.: Ovoids of $\text{PG}(3, q)$, $q$ even, with a conic section. J. Lond. Math. Soc. 62(2), 569–582 (2000).
8. Dodunekov S., Landgew I.: On near-MDS codes. J. Geom. 54, 30–43 (1995).
9. Edel Y., Bierbrauer J.: A family of caps in projective 4-space in characteristic 2. Congr. Numer. 141, 191–202 (1999).
10. Edel Y., Bierbrauer J.: Caps of order $3q^2$ in affine 4-space in characteristic 2. Finite Fields Appl. 10(2), 168–182 (2004).
11. Gabidulin E.M., Davydov A.A., Tombak L.M.: Linear codes with covering radius 2 and other new covering codes. IEEE Trans. Inform. Theory 37, 219–224 (1991).
12. Giulietti M.: On the Extendibility of Near-MDS Elliptic Codes. Appl. Algebra Eng. Commun. Comput. 15, 1–11 (2004).
13. Giulietti M.: The geometry of covering codes: small complete caps and saturating sets in Galois spaces, *Surveys in combinatorics 2013, 51–90. London Math. Soc. Lecture Note Ser.*, 409, Cambridge Univ. Press, Cambridge, (2013).
14. Hirschfeld J.W.P.: Finite Projective Spaces of Three Dimensions. Oxford Science Publications, Oxford (1985).
15. Hirschfeld J.W.P.: Projective Geometries Over Finite Fields. Oxford Science Publications, Oxford (1998).
16. Hirschfeld J.W.P., Thas J.A.: General Galois Geometries. Monographs in Mathematics. Springer, London (2016).
17. Penttila T., Williams B.: Ovoids of parabolic spaces. Geom. Dedicata 82, 1–19 (2000).
18. Tallini G.: Calotte complete di $S_q$ contenenti due quadriche ellittiche quali sezioni iperpiane. Rend. Mat. Pura Appl. 23, 108–123 (1964).
19. Wang Q., Heng Z.: Near MDS codes from oval polynomials. Discret. Math. 112277, 344 (2021).