NON-HAMILTONIAN ISOTOPIC LAGRANGIANS ON THE
ONE-POINT BLOW-UP OF $\mathbb{C}P^2$

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ABSTRACT. We show that two Hamiltonian isotopic Lagrangians in $(\mathbb{C}P^2, \omega_{FS})$
induce two Lagrangian submanifolds in the one-point blow-up $(\tilde{\mathbb{C}}P^2, \tilde{\omega}_\rho)$ that are
not Hamiltonian isotopic. Furthermore, we show that for any integer $k > 1$ there are $k$ Hamiltonian
isotopic Lagrangians in $(\mathbb{C}P^2, \omega_{FS})$ that induce $k$ Lagrangian submanifolds in the one-point
blow-up such that no two of them are Hamiltonian isotopic.

1. INTRODUCTION

In symplectic topology, one of many important problems that are still far from un-
derstanding is the classification of embedded Lagrangian submanifolds. Among
the various notions of equivalence between Lagrangian submanifolds is the one of Hamil-
tonian isotopic; two Lagrangian submanifolds are said to be Hamiltonian isotopic if
there is a Hamiltonian diffeomorphism that maps one Lagrangian to the other La-
grangian submanifold. This notion is of particular interest, among other cases, in the
context of Fukaya categories; in this category the objects are the classes of Hamilton-
tonian isotopic Lagrangians. In this note, we show that on the symplectic one-point
blow up the collection of Hamiltonian isotopic Lagrangian submanifolds is larger than
that on the symplectic manifold itself. Needless to say it is what is expected; but the
claim rests on the fact that there are Hamiltonian isotopic Lagrangian submanifolds
on the base manifold that lift to Lagrangian submanifolds on the one-point blow up
which are not Hamiltonian isotopic.

To be more precise, consider $(\mathbb{C}P^2, \omega_{FS})$ with the Hamiltonian circle action $\{\psi_t\}_{0 \leq t \leq 1}$
given by

$$\psi_t([z_0 : z_1 : z_2]) = [z_0 : e^{2\pi it}z_1 : e^{2\pi it^2}z_2].$$

The symplectic form $\omega_{FS}$ is the Fubini-Study symplectic form normalized so that the
area of the line is $\pi$. Consider the Lagrangian submanifolds

$$L_0 := \mathbb{R}P^2 \quad \text{and} \quad L_1 := \psi_{1/2^{N_0}}(\mathbb{R}P^2),$$

for a large fixed $N_0 \in \mathbb{N}$. The Hamiltonian diffeomorphism $\psi_{1/2^{N_0}}$ is of the type of
diffeomorphism that appears in Y.-G. Oh’s computation of $\text{HF}(\mathbb{R}P^2, \mathbb{R}P^2)$ in \cite{oh}. In
Section \S we define a symplectic embedding $\iota : (B^4(1/\sqrt{3}), \omega_0) \to (\mathbb{C}P^2, \omega_{FS})$ whose
image misses \( L_0 \) and \( L_1 \). Here \( B^4(r) \) stands for the closed ball of radius \( r \) in \((\mathbb{R}^4, \omega_0)\). Further, we arrange the symplectic embedding so that \( x_0 := \iota(0) \) lies in a unique holomorphic disk, with respect to the standard integrable complex structure, whose boundary lies in the Lagrangian submanifolds \( L_0 \) and \( L_1 \) and its Maslov index is equal to 1. Remember from [8], that in this case all holomorphic disks with boundary on the Lagrangians and connecting intersection points of \( L_0 \) with \( L_1 \) are part of a holomorphic sphere in \((\mathbb{C}P^2, \omega_{FS})\). Next blow up the point \( x_0 \) in \((\mathbb{C}P^2, \omega_{FS})\) with respect to the symplectic embedding \( \iota \); thus we have \( \pi : (\mathbb{C}P^2, \tilde{\omega}_\rho) \to (\mathbb{C}P^2, \omega_{FS}) \) where the weight of the blow up is such that \( \rho^2 = 1/3 \). The fact that the embedded ball misses the Lagrangians \( L_0 \) and \( L_1 \), ensures that \( \tilde{L}_0 := \pi^{-1}(L_0) \) and \( \tilde{L}_1 := \pi^{-1}(L_1) \) are Lagrangian submanifolds in \((\mathbb{C}P^2, \tilde{\omega}_\rho)\). We claim that such Lagrangians are not Hamiltonian isotopic.

**Theorem 1.1.** If \( L_0 := \mathbb{R}P^2 \subset (\mathbb{C}P^2, \omega_{FS}) \) and \((\mathbb{C}P^2, \tilde{\omega}_\rho)\) is the symplectic one-point blow of \((\mathbb{C}P^2, \omega_{FS})\) of weight \( \rho = 1/\sqrt{3} \), then there exists a Lagrangian submanifold \( L_1 \) such that is Hamiltonian isotopic to \( L_0 \); \( \tilde{L}_0 := \pi^{-1}(L_0) \) and \( \tilde{L}_1 := \pi^{-1}(L_1) \) are monotone Lagrangian submanifolds in \((\mathbb{C}P^2, \tilde{\omega}_\rho)\), and its Lagrangian Floer homology is

\[
(2) \quad \text{HF}(\tilde{L}_0, \tilde{L}_1) \simeq \Lambda\{\text{pt}\}.
\]

In particular, \( \tilde{L}_0 \) and \( \tilde{L}_1 \) are not Hamiltonian isotopic in \((\mathbb{C}P^2, \tilde{\omega}_\rho)\).

The generator in Eq. (2) is the intersection point of \( \tilde{L}_0 \) with \( \tilde{L}_1 \) that when projected to \((\mathbb{C}P^2, \omega_{FS})\) under the blow up map, is not on either end of the holomorphic disk that contains the blown up point. Recall that \( \mathbb{R}P^2 \) and \( \psi_{1/2^N} (\mathbb{R}P^2) \) intersect at precisely three points.

In the above result the Lagrangian \( L_1 \) corresponds to \( \psi_{1/2^N}(\mathbb{R}P^2) \). In fact one can take \( L_1 \) to be any Lagrangian submanifold of the form \( \psi_{1/2^N} (\mathbb{R}P^2) \) for any \( N \in \mathbb{N} \) greater than \( N_0 \). What is important about the Lagrangian submanifold is that it should avoid the image of the symplectic embedding \( \iota : (B^4(1/\sqrt{3}), \omega_0) \to (\mathbb{C}P^2, \omega_{FS}) \) and the time-parameter \( 1/2^N \) should be small in order to apply the results of Y.-G. Oh [8]. Hence for such values of \( N \) and Lagrangian \( L(N) := \psi_{1/2^N} (\mathbb{R}P^2) \), the Lagrangian submanifolds \( \tilde{L}_0 \) and \( \tilde{L}(N) := \pi^{-1}(L(N)) \) are not Hamiltonian isotopic in \((\mathbb{C}P^2, \tilde{\omega}_\rho)\).

Theorem 1.1 claims that there is at least one more class of a Hamiltonian isotopic Lagrangian submanifolds in \((\mathbb{C}P^2, \tilde{\omega}_\rho)\) than in \((\mathbb{C}P^2, \omega_{FS})\). However, is possible to obtain more than one new class of Hamiltonian isotopic Lagrangian in the blow up from a collection of Hamiltonian isotopic Lagrangian submanifolds in \((\mathbb{C}P^2, \omega_{FS})\).
Theorem 1.2. If $k$ is a positive integer and $(\mathbb{C}P^2, \tilde{\omega}_\rho)$ is the symplectic one-point blow up of $(\mathbb{C}P^2, \omega_{\text{FS}})$ of weight $\rho = 1/\sqrt{3}$, then there exist Hamiltonian isotopic Lagrangian submanifolds $L_0 = \mathbb{R}P^2, L_1, \ldots, L_k$ in $(\mathbb{C}P^2, \omega_{\text{FS}})$ that lift to monotone Lagrangian submanifolds $\tilde{L}_0, \tilde{L}_1, \ldots, \tilde{L}_k$ in $(\tilde{\mathbb{C}}P^2, \tilde{\omega}_\rho)$ such that no two of them are Hamiltonian isotopic.

We remark that the above argument cannot be implemented in $(\mathbb{C}P^n, \omega_{\text{FS}})$ for $n > 2$. The reason been that if $(\tilde{\mathbb{C}}P^n, \tilde{\omega}_\rho)$ is monotone, then its weight is larger than $1/\sqrt{2}$; and according to P. Biran [2, Theorem 1.B] the image of every symplectic embedding $\iota: (B^{2n}(\rho), \omega_0) \to (\mathbb{C}P^n, \omega_{\text{FS}})$ necessarily intersects $\mathbb{R}P^n$. Hence, it is not possible to lift of $\mathbb{R}P^n$ to $(\tilde{\mathbb{C}}P^n, \tilde{\omega}_\rho)$ as a Lagrangian submanifold when $n > 2$. Since in order to lift a Lagrangian submanifold to the blow up, the Lagrangian must avoid the embedded ball that is used to define the symplectic one-point blow up.

Of course this is the case if we stick to Lagrangian Floer homology for monotone Lagrangian submanifolds. Instead if one considers the theory of Lagrangian Floer homology developed by K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono [3], it seems that our arguments will still work. The advantage in this setting is that the radius of the embedded ball can be made relatively small, this cannot be done in the monotone case. Further one can consider more than two Lagrangian submanifolds and compare the higher order relations $\mu^k$ on the manifold with the higher order relations on the blown up manifold. No attempt has been made here to verify these assertions.

Another important case that is not cover in the present article is the case when then blown up point lies in the Lagrangian submanifold. For instance A. Rieser [10] studied the case of blowing up a point in a Lagrangian submanifold that correspond to the real part of an anti-symplectic involution.

Beside the fact that it is possible to symplectically embed the ball $B^4(1/\sqrt{3})$ into $(\mathbb{C}P^2, \omega_{\text{FS}})$ without intersecting $L_0 = \mathbb{R}P^2$, there is another feature of $(\mathbb{C}P^2, \omega_{\text{FS}})$ that works in our favor. Namely, the fact that the standard complex structure is regular for the pair $(L_0, L_1)$ of Lagrangian manifolds. See Y.-G. Oh [8, Proposition 4.3]. Hence in order to restate Theorem 1.1 for other closed symplectic manifolds $(M, \omega)$ there must exists a regular almost complex structure $J$ such that $\iota^*J = J_0$ where $J_0$ is the standard complex structure on $\mathbb{C}^n$ and $\iota: (B^{2n}(\rho), \omega_0) \to (M, \omega)$ is the symplectic embedding used to define the monotone one-point blow up $(\tilde{M}, \tilde{\omega}_\rho)$.

Theorem 1.3. Let $(M, \omega)$ be a closed symplectic manifold that is simply connected, and $L_0$ and $L_1$ Hamiltonian isotopic Lagrangian submanifolds that intersect transversally and avoid the image of $\iota: (B^{2n}(\rho), \omega_0) \to (M, \omega)$. If

a) there exists a regular almost complex structure $J$ on $(M, \omega)$ for $(L_0, L_1)$ such that $\iota^*J = J_0$, and

b) $\iota(0)$ lies in a unique $J$-holomorphic disk of Maslov index 1, whose boundary lies in the Lagrangian submanifolds,
then \( \tilde{L}_0 := \pi^{-1}(L_0) \) and \( \tilde{L}_1 := \pi^{-1}(L_1) \) are monotone Lagrangian submanifolds in \((\tilde{M}, \tilde{\omega}_\rho)\) that are not Hamiltonian isotopic.

Theorem 1.1 is a particular case of Theorem 1.3 for the case of \((\mathbb{C}P^2, \omega_{FS})\), \(L_0 = \mathbb{R}P^2\) and \(L_1 = \psi_{1/2\pi}(\mathbb{R}P^2)\). The proof of Theorem 1.1 is structured as follows. In Section 6 we define a symplectic embedding \(\iota: (B^4(1/\sqrt{3}), \omega_0) \to (\mathbb{C}P^2, \omega_{FS})\) that avoids the Lagrangian submanifolds \(L_0\) and \(L_1\) and verify assertion \(b\). Finally, as mentioned above, assertion \(a\) follows from the work of Y.-G. Oh [8]. The proof of Theorem 1.1 appears at the end of Section 6.

The reason why \(HF(L_0, L_1)\) is not isomorphic to \(HF(\tilde{L}_0, \tilde{L}_1)\) is the existence of a unique holomorphic disk \(u\) of Maslov index 1 that contains the blown up point \(x_0 = \iota(0)\) in its interior and whose boundary lies in the Lagrangian submanifolds \(L_0\) and \(L_1\). After blowing up \(x_0\), the holomorphic disk \(u\) induces a holomorphic disk \(\tilde{u}\) in the blow up whose Maslov index is no longer equal to 1. Apart from that issue, the two homologies are isomorphic, even if some holomorphic disk intersects the embedded ball \(\iota(B^{2n}(\rho))\).

**Corollary 1.4.** Assume all the hypothesis of Theorem 1.3 except \(b\). If \(\iota(0)\) does not lie in any J-holomorphic disk of Maslov index 1, then

\[ HF(L_0, L_1) \cong HF(\tilde{L}_0, \tilde{L}_1) \]

as \(A\)-modules.

We say that a Hamiltonian diffeomorphism \(\psi\) on \((M, \omega)\) has a lift to \((\tilde{M}, \tilde{\omega}_\rho)\) if there exists \(\tilde{\psi} \in \text{Ham}(\tilde{M}, \tilde{\omega}_\rho)\) such that \(\pi \circ \tilde{\psi} = \psi \circ \pi\). Then from Theorem 1.1 we have the following result on \((\mathbb{C}P^2, \omega_{FS})\) concerning lifts of Hamiltonian diffeomorphisms.

**Corollary 1.5.** Let \(L_0\) and \(L_1\) be Lagrangian submanifolds in \((\mathbb{C}P^2, \omega_{FS})\) as in Theorem 1.1. If \(\psi \in \text{Ham}(\mathbb{C}P^2, \omega_{FS})\) is such that \(\psi(L_0) = L_1\), then \(\psi\) does not admit a lift to \((\mathbb{C}P^2, \tilde{\omega}_\rho)\).

From this result we see that it is impossible to have a Hamiltonian diffeomorphism \(\psi: (\mathbb{C}P^2, \omega_{FS}) \to (\mathbb{C}P^2, \omega_{FS})\) such that \(\psi(L_0) = L_1\) and \(\text{supp}(\psi) \cap \iota B^4(\rho) = \emptyset\); since such diffeomorphism admits a lift. Furthermore, there are Hamiltonian diffeomorphisms \(\psi\) such that \(\text{supp}(\psi) \cap \iota B^4(\rho) \neq \emptyset\) and admit a lift. For instance if \(\psi\) is such that under the coordinates induced by the embedding \(\iota\) it can be expressed as a unitary matrix on \(\iota B^4(\rho)\). Such Hamiltonian diffeomorphisms were considered by the author in [9]. Hence there are no Hamiltonian diffeomorphisms \(\psi\) such that \(\psi(L_0) = L_1\) and behave in a \(U(n)\)-way in a neighborhood of \(\iota B^4(\rho)\).

The bottom line here is to discard the natural Hamiltonian diffeomorphism that comes to mind if we know that \(L_0\) is Hamiltonian isotopic to \(L_1\) in \((\mathbb{C}P^2, \omega_{FS})\) and we would like to see that \(\tilde{L}_0\) is Hamiltonian isotopic to \(\tilde{L}_1\). That is, if \(H_1: (\mathbb{C}P^2, \omega_{FS}) \to \mathbb{R}\) is a Hamiltonian function of \(\psi = \psi_1\) where \(\psi(L_0) = L_1\), then the Hamiltonian
diffeomorphism $\phi = \phi_1 \in \text{Ham}(\mathbb{C}P^2, \omega_\rho)$ induced by $H_t \circ \pi$ does not necessarily map $\tilde{L}_0$ to $\tilde{L}_1$. It goes without saying that the same holds for the type of symplectic manifolds and Lagrangians considered in Theorem 1.3.

The article is organized as follows. In Sections 2 and 3 we review Lagrangian Floer homology and the symplectic blow up respectively, with the intention to set the notation throughout the article. The way to define a lift of a holomorphic disk to the blow up is discussed in Section 4. Also the relation between the Maslov index of a holomorphic disk and its lift is presented in that section. In Section 5 we show regularity of the almost complex structure on the blow up and prove Theorem 1.3. In the last section we focus on $(\mathbb{C}P^2, \omega_{FS})$; we define the required symplectic embedding in order to perform the blow up and prove Theorem 1.1.

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2. Lagrangian Floer Homology

Throughout this note $(M, \omega)$ will denote a closed symplectic manifold, $J = \{J_t\}_{0 \leq t \leq 1}$ a family of $\omega$-compatible almost complex structures, and $L_0$ and $L_1$ compact connected Lagrangian submanifolds that intersect transversally. Let $\mathcal{X}(L_0, L_1)$ denote the set of intersection points. Then for $p$ and $q$ in $\mathcal{X}(L_0, L_1)$ and $\beta \in \pi_2(M, L_0 \cup L_1)$ denote by $\hat{\mathcal{M}}(p, q, \beta, J)$ the set of smooth maps $u : \mathbb{R} \times [0, 1] \to M$ such that:

- satisfy the boundary conditions
  
  $u(s, 0) \in L_0$, and $u(s, 1) \in L_1$, for all $s \in \mathbb{R}$

  and

  $\lim_{s \to -\infty} u(s, t) = q$ and $\lim_{s \to +\infty} u(s, t) = p$;

- represent the class $\beta$, $[u] = \beta$ and

- are $J$-holomorphic,

  $\overline{\partial}_J(u) := \frac{\partial}{\partial s} u(s, t) + J_t \frac{\partial}{\partial t} u(s, t) = 0$.

The moduli space $\hat{\mathcal{M}}(p, q, \beta, J)$ admits an action of $\mathbb{R}$, given by $r \cdot u(s, t) = u(s - r, t)$. Denote by $\mathcal{M}(p, q, \beta, J)$ the quotient space of the action. Elements of $\mathcal{M}(p, q, \beta, J)$ are called holomorphic strips; they also received the name of holomorphic disks since $\mathbb{R} \times [0, 1]$ is conformally equivalent to the closed disk minus two points on the boundary.

In some cases the space $\hat{\mathcal{M}}(p, q, \beta, J)$ is in fact a smooth manifold. To that end, take into account the linearized operator $D_{\overline{\partial}_J} u$ of $\overline{\partial}_J$ at $u \in \hat{\mathcal{M}}(p, q, \beta, J)$. Then for
integers \( k \) and \( p \) such that \( p > 2 \) and \( k > p/2 \) we have the Sobolev space of vector fields whose \( k \)-weak derivatives exist and lies in \( L_p \), and with boundary restrictions;

\[
W^p_k (u^* TM; L_0, L_1) := \{ \xi \in W^p_k (u^* TM) | \xi (s, 0) \in TL_0, \xi (s, 1) \in TL_1 \text{ for all } s \in \mathbb{R} \}.
\]

There exists \( \mathcal{J}_{\text{reg}}(L_0, L_1) \), a dense subset of \( \omega \)-compatible almost complex structures in \( \mathcal{J}(L_0, L_1) \) such that for \( J \in \mathcal{J}_{\text{reg}}(L_0, L_1) \) the linearized operator

\[
D_{\partial J, u}: W^p_k (u^* TM; L_0, L_1) \to L_p (u^* TM).
\]

is Fredholm and surjective for all \( u \in \hat{\mathcal{M}}(p, q, \beta, J) \). Elements of \( \mathcal{J}_{\text{reg}}(L_0, L_1) \) are called regular. In this case the index of \( D_{\partial J, u} \) equals the Maslov index \( \mu_{L_0, L_1}([u]) \) of the homotopy type of \( u \) in \( \pi_2 (M, L_0 \cup L_1) \). Note that in the case when \( J \) is regular the dimension of the moduli space \( \hat{\mathcal{M}}(p, q, \beta, J) \) is independent of the regular almost complex structure.

Let \( CF(L_0, L_1) \) denote the \( \Lambda \)-vector space generated by the intersection points \( \mathcal{X}(L_0, L_1) \). Here \( \Lambda \) stands for the Novikov field

\[
\Lambda := \left\{ \sum_{j=0}^{\infty} a_j T_j^\lambda | a_j \in \mathbb{Z}_2, \lambda_j \in \mathbb{R}, \lim_{j \to \infty} \lambda_j = \infty \right\}.
\]

In the case when \( [u] \in \mathcal{M}(p, q, \beta, J) \), \( \mu_{L_0, L_1}([u]) = 1 \) and \( J \) is regular the moduli space \( \mathcal{M}(p, q, \beta, J) \) is 0-dimensional and compact, thereby a finite set of points. Set \( \#_{\mathbb{Z}_2} \mathcal{M}(p, q, \beta, J) \) to be the module 2 number of points of \( \mathcal{M}(p, q, \beta, J) \). The Floer differential \( \partial J : CF(L_0, L_1) \to CF(L_0, L_1) \) is defined as

\[
\partial J (p) := \sum_{q \in \mathcal{X}(L_0 \cap L_1), [u]: \text{index}([u]) = 1} \#_{\mathbb{Z}_2} \mathcal{M}(p, q, [u], J) T^\omega([u]) q.
\]

If the Lagrangian submanifolds \( L_0 \) and \( L_1 \) are monotone and the minimal Maslov number of \( L_0 \) and \( L_1 \) is greater than or equal to two, then \( \partial J \circ \partial J = 0 \). In this case the Lagrangian Floer homology of \( (L_0, L_1) \) is defined as

\[
HF(L_0, L_1) := \ker \frac{\partial J}{\im \partial J}.
\]

Is important to note that the homology group \( HF(L_0, L_1) \) does not depend on the regular \( \omega \)-compatible almost complex structure.

The role played by the coefficient field \( \Lambda \) becomes essential in the definition of the differential \( \partial \). In principle the sum in Eq. (3) can be infinite, but by Gromov’s compactness there are only finitely many homotopy classes whose energy is below a determined value. Hence \( \Lambda \) assures that Eq. (3) is well-defined.

For further details in the definition of Lagrangian Floer homology in the monotone case see [7]; and also [1] and [3] for a more broader class of symplectic manifold where Lagrangian Floer homology is defined.
3. Review of the symplectic one-point blow up

The symplectic one-point blow up plays a fundamental role in this note. Hence we review the definitions of the complex and symplectic one-point blow up. To that end, consider the complex blow up of $\mathbb{C}^n$ at the origin $\Phi : \tilde{\mathbb{C}}^n \to \mathbb{C}^n$, where $n > 1$. That is

$$\tilde{\mathbb{C}}^n = \{(z, \ell) \in \mathbb{C}^n \times \mathbb{C}P^{n-1} \mid z \in \ell\}$$

and the blow up map is given by $\Phi(z, \ell) = z$. For $r > 0$, let $L(r) := \Phi^{-1}(\text{int}(B^{2n}(r)))$ where $B^{2n}(r) \subset \mathbb{C}^n$ is the closed ball and $\text{int}(\cdot)$ stands for the interior of the set.

If $(M, J)$ is a complex manifold and $\iota : (\text{int}B^{2n}(r), J_0) \to (M, J)$ is such that $\iota^*J = J_0$ and $x_0 = \iota(0)$, then the complex blow up of $M$ at $x_0$ is defined as

$$\tilde{M} := (M \setminus \{x_0\}) \cup L(r)/\sim$$

where $z = \iota(z') \in \iota(\text{int}(B^{2n}(r))) \setminus \{x_0\}$ is identified with the unique point $(z', \ell_{z'}) \in L(r)$ and $\ell_{z'}$ is the line determined by $z'$. So defined, $\tilde{M}$ carries a unique complex structure $\tilde{J}$ such that the blow up map $\pi : (\tilde{M}, \tilde{J}) \to (M, J)$ is $(\tilde{J}, J)$-holomorphic. The preimage of the blown up point $\pi^{-1}(x_0) = E$ is called the exceptional divisor. Further, the blow up map induces a biholomorphism $\tilde{M} \setminus E \to M \setminus \{x_0\}$.

The next task is to define the symplectic one-point blow up. The symplectic blow up relies on the complex blow up. However there is not a unique symplectic blow up; there is a whole family of symplectic forms on the one-point blow up.

As a first step we look at $(\mathbb{C}^n, \omega_0)$ and define a symplectic structure on $\tilde{\mathbb{C}}^n$. Here $\omega_0$ is the standard symplectic form on euclidean space. For $\rho > 0$, consider the symplectic form

$$\omega(\rho) := \Phi^*(\omega_0) + \rho^2 pr^*(\omega_{FS})$$

on $\tilde{\mathbb{C}}^n$ where $pr : \tilde{\mathbb{C}}^n \to \mathbb{C}P^{n-1}$ is the canonical line bundle and the Fubini-Study form $(\mathbb{C}P^{n-1}, \omega_{FS})$ is normalized so that the area of every line is $\pi$. Note that on the exceptional divisor the symplectic form $\omega(\rho)$ restricts to $\rho^2 \omega_{FS}$. So in $(\tilde{\mathbb{C}}^n, \omega(\rho))$ the area of any line in the exceptional divisor is $\rho^2 \pi$.

Next, the symplectic form $\omega(\rho)$ is perturb in such a way so that in the complement of a neighborhood of the exceptional divisor agrees with the standard symplectic form $\omega_0$. Once this is done, following the definition of the blow up manifold (4) it will be possible to define a symplectic form on $\tilde{M}$.

For $r > \rho$ let $\beta : [0, r] \to [\rho, r]$ be any smooth function such that

$$\beta(s) := \begin{cases} \sqrt{\rho^2 + s^2} & \text{for } 0 \leq s \leq \delta \\ s & \text{for } r - \delta \leq s \leq r. \end{cases}$$
and on the remaining part takes any value as long as $0 < \beta'(s) \leq 1$ for $0 < s \leq r - \delta$. Then $F_\rho : L(r) \setminus E \to \text{int}(B^{2n}(r)) \setminus B^{2n}(\rho)$ defined as

$$F_\rho(z) := \beta(|z|) \frac{z}{|z|}$$

is a diffeomorphism such that $\tilde{\omega}(\rho) := F_\rho^*(\omega_0)$ is a symplectic form. So defined $\tilde{\omega}(\rho)$ is such that

- $\tilde{\omega}(\rho) = \omega_0$ on $\text{int}(L(r) \setminus L(r - \delta))$ and
- $\tilde{\omega}(\rho) = \omega(\rho)$ on $L(\delta)$.

We call $(L(r), \tilde{\omega}(\rho))$ the local model of the symplectic one-point blow up.

In order to define the symplectic blow up of $(M, \omega)$ at $x_0$, it is required a symplectic embedding $\iota : (B^{2n}(\rho), \omega_0) \to (M, \omega)$ and an almost complex structure $J$ on $(M, \omega)$ such that $\iota(0) = x_0$ and $\iota^* J = J_0$. Notice that the symplectic embedding $\iota$ extends to $\text{int}(B^{2n}(r))$ for $r$ such that $r - \rho$ is small.

Finally, using the symplectic embedding as a symplectic chart and the local model $(L(r), \tilde{\omega}(\rho))$ defined above, the symplectic form of weight $\rho$ on $\tilde{M}$ is defined as

$$\tilde{\omega}_\rho := \begin{cases} \omega & \text{on } \pi^{-1}(M \setminus \iota B^{2n}(\sqrt{\rho^2 + \delta^2})) \\ \tilde{\omega}(\rho) & \text{on } L_r. \end{cases}$$

For further details and the dependence of the symplectic blow up on the choices that we made see [1], [5] and [6]. The above observations are summarized in the next proposition.

**Proposition 3.1.** Let $(M, \omega)$ be a symplectic manifold, $\iota : (B(2n), \omega_0) \to (M, \omega)$ a symplectic embedding and $J$ a $\omega$-compatible almost complex structure such that $\iota(0) = x_0$ and $\iota^* J = J_0$. If $\rho < r$, then the symplectic blow up $\pi : (\tilde{M}, \tilde{\omega}_\rho) \to (M, \omega)$ of weight $\rho$ satisfies:

1. $\pi : \tilde{M} \setminus E \to M \setminus \{x_0\}$ is a diffeomorphism,
2. $\pi^*(\omega) = \tilde{\omega}_\rho$ on $\pi^{-1}(M \setminus \iota B^{2n}(r))$, and
3. the area of the line in $E$ is $\rho^2 \pi$.
4. $\tilde{J}$ is $\tilde{\omega}_\rho$-compatible.

From now on assume that $J$ on $(M, \omega)$ satisfies the condition $\iota^* J = J_0$, where $J_0$ is the standard complex structure on $\mathbb{C}^n$. It is well known that this condition induces a unique almost complex structure $\tilde{J}$ on $(\tilde{M}, \tilde{\omega}_\rho)$ such that the blow up map $\pi : (\tilde{M}, \tilde{\omega}_\rho) \to (M, \omega)$ is $(J, \tilde{J})$-holomorphic. In particular $\pi : \tilde{M} \setminus E \to M \setminus \{x_0\}$ is biholomorphic.

4. LAGRANGIAN SUBMANIFOLDS AND HOLOMORPHIC DISKS

4.1. Lift of holomorphic disks. Fix a symplectic embedding $\iota : (B^{2n}(\rho), \omega_0) \to (M, \omega)$ and set $x_0 := \iota(0)$ to be the base point.
Consider $L \subset (M, \omega)$ a Lagrangian submanifold such that $L \cap B^{2n}(\rho)$ is empty. Then by part (2) of Proposition 3.1 it follows that $\tilde{L} := \pi^{-1}(L)$ is a Lagrangian submanifold in $(\tilde{M}, \tilde{\omega}_\rho)$. However if $L$ is such that $L \cap B^{2n}(\rho)$ is not empty, then $\tilde{L}$ is not necessarily a Lagrangian submanifold, even if $x_0 \notin L$. Thus from now on we focus only on Lagrangian submanifolds $L \subset (M, \omega)$ that are disjoint from the embedded ball.

Let $J$ be a $\omega$-compatible almost complex structure on $(M, \omega)$ as in Section 3 and $\tilde{J}$ the unique $\tilde{\omega}_\rho$-compatible almost complex structure on $(\tilde{M}, \tilde{\omega}_\rho)$ such that the blow up map $\pi : (M, \omega) \to (\tilde{M}, \tilde{\omega}_\rho)$ is $(J, \tilde{J})$-holomorphic. Hence $\pi : \tilde{M} \setminus E \to M \setminus \{x_0\}$ is biholomorphic, therefore any $\tilde{J}$-holomorphic disk $\tilde{u} : (D, \partial D) \to (\tilde{M}, \tilde{L})$ projects to a $J$-holomorphic disk $\pi \circ \tilde{u} : (D, \partial D) \to (M, L)$. And vice versa, if $u$ is a $J$-holomorphic disc on $(M, L)$ such that $x_0 \notin u(D)$, then $\tilde{u} := \pi^{-1} \circ u$ is a $\tilde{J}$-holomorphic disc in $(\tilde{M}, \tilde{L})$. Recall that we are assuming that the Lagrangian submanifold $L \subset (M, \omega)$ does not contain the base point $x_0$.

It only remains to analyze the lift of the $J$-holomorphic disk $u : (D, \partial D) \to (M, L)$ in the case when $x_0 \in u(D)$. Since the base point $x_0$ is not on the Lagrangian submanifold $L \subset (M, \omega)$ and the almost complex structure satisfies $i^* J = J_0$, in order to define the lift of $u$ to $(\tilde{M}, \tilde{\omega}_\rho)$ we ignore the submanifold $L$ and consider the case $(M, \omega) = (\mathbb{C}^n, \omega_0)$ blown up at the the origin. Thus $x_0 = 0$ and $u : D \to \mathbb{C}^n$ is holomorphic with respect to the standard complex structure and $0 \in u(D)$.

For let $u : D \to \mathbb{C}^n$ be a non constant holomorphic map such that $u(0) = 0$. Then each non constant component function $u_j : D \to \mathbb{C}$ of $u$ can be written as $u_j = z^{k_j} h_j$ where $k_j$ is the order of the zero of $u_j$ at $0 \in D$. Thus $k_j$ is a positive integer and $h_j$ is holomorphic function that does not vanish at $0$. Except if $u_j \equiv 0$, in this case we set $k_j$ to be equal to $\infty$. Thus the holomorphic map $u$ can be expressed as

$$u(z) = z^k (\hat{h}_1(z), \ldots, \hat{h}_n(z))$$

where $k := \min k_j$ and at least one coordinate function $\hat{h}_j$ does not vanish at $0$ since we assumed that $u$ is non constant. The lift $\tilde{u} : D \to \tilde{\mathbb{C}}^n$ of $u$ is defined as

$$\tilde{u}(z) := (u(z), [\hat{h}_1(z) : \cdots : \hat{h}_n(z)]).$$

So defined $\tilde{u}$, is holomorphic and projects to $u$ under the blow up map. Notice that if $\tilde{u}_0$ is another $\tilde{J}$-holomorphic lift of $u$, then $\tilde{u}$ and $\tilde{u}_0$ agree on $D \setminus \tilde{u}^{-1}(E)$. Since the maps are $\tilde{J}$-holomorphic they agree on all $D$, thus the holomorphic lift of $u$ is unique.

**Remark.** If $\psi : \mathbb{C}^n \to \mathbb{C}^n$ is a biholomorphism such that $\psi(0) = 0$ and $u : D \to \mathbb{C}^n$ is as above then the factorization of $\psi \circ u$ as in Eq. (6) gives the same value of $k$ as that of $u$. Thus $k$ is independent of the coordinate system.

Now that we have defined the lift $u : (D, \partial D) \to (M, L)$ to $(\tilde{M}, \tilde{\omega}_\rho)$, there is one more consideration that needs attention; the behavior of $u$ at the blown up point
x_0. If u : (D, ∂D) → (M, L) is a non constant J-holomorphic disk and z ∈ D \ ∂D is such that u(z) = x_0, then we define the \textit{multiplicity of u at z} has the integer k ∈ {0, 1, ..., ∞} that appears in Eq. (5). Also we define the \textit{multiplicity of u at x_0} as k_1 + · · · + k_r where u^{-1}(x_0) = {z_1, ..., z_r} and the multiplicity of u at z_j is k_j. In the case when x_0 is not in the image of u, we say that u has multiplicity zero at x_0. Recall that since u is J-holomorphic, the preimage of a point under u is a finite set.

\textbf{Proposition 4.1.} Let J be an almost complex structure on (M, ω) as above, such that υ^*J = J_0 and ˜J be the unique almost complex structure on (M, ˜ω_ρ) such that π is (J, J)-holomorphic. If u : (D, ∂D) → (M, L) a non constant J-holomorphic disk, then there exists a unique ˜J-holomorphic map ˜u : (D, ∂D) → (M, ˜L) such that π ◦ ˜u = u. Moreover if u has multiplicity k at x_0, then ˜u · E = k.

\textit{Proof.} It only remains to prove the relation ˜u · E = k. Since the underlying manifold (M, ˜ω_ρ) agrees with the complex blow up the exceptional divisor E is a ˜J-holomorphic submanifold of real codimension two of (M, ˜ω_ρ). Now the multiplicity of u at x_0 is k, therefore ˜u · E = k. □

\textbf{4.2. Monotone Lagrangian on blow ups.} For a Lagrangian submanifold L of (M, ω), there exist two classical morphisms

\[ I_{µ,L} : π_2(M, L) → Z \quad \text{and} \quad I_{ω,L} : π_2(M, L) → R; \]

the Maslov index and symplectic area morphisms respectively. The Lagrangian submanifold is said to be \textit{monotone} if there exists λ > 0 such that I_{µ,L} = λ · I_{ω,L}. The constant λ is called the monotonicity constant of L. As mentioned in Section 2 in order to define Lagrangian Floer homology one restricts to monotone Lagrangians. Thus if L ⊂ (M, ω) is a monotone Lagrangian submanifold we need to guarantee that ˜L is a monotone Lagrangian on the one-point blow up (M, ˜ω_ρ).

If L is a monotone Lagrangian submanifold of (M, ω) with monotonicity constant λ, then (M, ω) is monotone symplectic. That is I_c = (λ/2)L_c. In this case I_c is defined on π_2(M) and the morphism I_c : π_2(M) → Z is given by evaluating at the first Chern class of (M, ω) with respect to any almost complex structure. Here α := λ/2 is the monotonicity constant of (M, ω).

The first Chern classes of (M, ω) and the one-point blow (M, ˜ω_ρ) are related by the equation

\[ c_1(M) = π^*(c_1(M)) − (n − 1)PD_{M}(E), \]

where E is the class of the exceptional divisor. Recall that the underlying manifold for the symplectic blow up is independent of the weight. Thus from Eq. (7) the symplectic one-point blow up (M, ˜ω_ρ) is monotone if and only if

\[ \rho^2 = \frac{n − 1}{απ}. \]
Furthermore if Eq. (8) holds, then \((\widehat{M}, \widehat{\omega}_\rho)\) and \((M, \omega)\) have the same monotonicity constant.

Throughout the paper we make the following assumptions. If the condition of monotonicity on \((M, \omega)\) is required, we will assume that the Gromov width of \((M, \omega)\) is greater than \(n - 1/\alpha\), where \(\alpha\) is its monotonicity constant; and the weight \(\rho\) of the one-point blow up \((\widehat{M}, \widehat{\omega}_\rho)\) is subject to Eq. (8).

The two homotopy long exact sequences of the pairs \((\widehat{M}, \widehat{L})\) and \((M, L)\) are related by the blow up map, in the sense that the diagram

\[
\begin{array}{ccccccccc}
\pi_2(L) & \rightarrow & \pi_2(\widehat{L}) & \rightarrow & \pi_2(\widehat{M}, \widehat{L}) & \rightarrow & \pi_2(\widehat{M}) & \rightarrow & \pi_1(\widehat{M}) & \rightarrow \\
\pi_* & \downarrow & \pi_* & \downarrow & \pi_* & \downarrow & \pi_* & \downarrow & \pi_* & \\
\pi_2(L) & \rightarrow & \pi_2(M) & \rightarrow & \pi_2(M, L) & \rightarrow & \pi_2(M) & \rightarrow & \pi_1(M) & \rightarrow
\end{array}
\]

is commutative.

In the remaining results of this section we will assume that \(M\) is simply connected. Hence the one-point blow is also simply connected and by Hurewicz’s Theorem

\[
\pi_2(M) \simeq H_2(M, \mathbb{Z}) \quad \text{and} \quad \pi_2(\widehat{M}) \simeq H_2(\widehat{M}, \mathbb{Z}).
\]

**Lemma 4.2.** If \(L\) is a Lagrangian submanifold in \((M, \omega)\) that does not intersect the embedded ball \(i(B^{2n}(\rho))\) and \(M\) is simply connected, then the map \(\pi_* : \pi_2(\widehat{M}, \widehat{L}) \rightarrow \pi_2(M, L)\) is surjective and

\[
\ker \{\pi_* : \pi_2(\widehat{M}, \widehat{L}) \rightarrow \pi_2(M, L)\} = \tilde{j}_s \ker \{\pi_* : \pi_2(\widehat{M}) \rightarrow \pi_2(M)\}.
\]

**Proof.** Since \((M, \omega)\) is assumed to be simply connected, then \((\widehat{M}, \widehat{\omega}_\rho)\) is also simply connected and \(\pi_* : \pi_2(\widehat{M}) \rightarrow \pi_2(M)\) is surjective. Therefore from the above diagram, \(\pi_* : \pi_2(\widehat{M}, \widehat{L}) \rightarrow \pi_2(M, L)\) is also surjective.

Now we show that the kernel of \(\pi_* : \pi_2(\widehat{M}, \widehat{L}) \rightarrow \pi_2(M, L)\) is contained in \(\tilde{j}_s \ker \{\pi_* : \pi_2(\widehat{M}) \rightarrow \pi_2(M)\}\); the reverse inclusion follows by the commutativity of the diagram.

For, let \(u \in \pi_2(\widehat{M}, \widehat{L})\) be an element that maps to \(e \in \pi_2(M, L)\). Since \(\pi_1(L) = \pi_1(\widehat{L})\) then \(\tilde{\delta}(u) = e\) and by exactness of the top sequence there is \(w \in \pi_2(\widehat{M})\) such that \(\tilde{j}_s(w) = u\). Note that \(\pi_*(\tilde{j}_s(w)) = e\). Thus by exactness and the fact that \(\pi_2(L) = \pi_2(\widehat{L})\), there is \(w' \in \pi_2(\widehat{L})\) such that \(\pi_*(\tilde{i}_s(w')) = \pi_*(w)\). Therefore \(\tilde{j}_s(w - \tilde{i}_s(w')) = u\) and \(w - \tilde{i}_s(w')\) maps to \(e \in \pi_2(M)\).

If follows from Lemma 4.2 that elements of the kernel of \(\pi_* : \pi_2(\widehat{M}, \widehat{L}) \rightarrow \pi_2(M, L)\) are absolute classes. Therefore any \([\tilde{u}] \in \pi_2(\widehat{M}, \widehat{L})\) can be expressed as \([\tilde{u}] = [u_0 \# w]\) where \([w] \in \text{im} \{\pi_2(\widehat{M}) \rightarrow \pi_2(\widehat{M}, \widehat{L})\}\) and \(u_0 : (D, \partial D) \rightarrow (\widehat{M}, \widehat{L})\) does not intersect the exceptional divisor.
Lemma 4.3. If \( L \) is a Lagrangian submanifold in \((M, \omega)\) that does not intersect the embedded ball \( i(B^{2n}(\rho)) \), \( M \) is simply connected and \([u] \in \pi_2(\widetilde{M}, \widetilde{L})\), then there exist \([w] \in \pi_2(\tilde{M})\) and \(u_0 : (D, \partial D) \to (M, \tilde{L})\) such that \( \pi_*(w) = e\), \( u_0 \) does not intersect the exceptional divisor and

\[
[u] = [u_0 \# (\widetilde{j} \circ w)].
\]

Proof. First assume that \([u] = e\). Then by Lemma 4.2 there exists \([w_0] \in \pi_2(\tilde{M})\) such that \([u] = \widetilde{j}_* [w_0] = \widetilde{j} \circ w_0\]. Let \([v] := \pi_*(w_0)\), thus \(j_* [v] = e\). Then by the commutativity of the above diagram and the fact that \(\pi_2(\widetilde{L}) = \pi_2(L)\), there is \([v_0] \in \pi_2(\widetilde{L})\) such that \(\pi_* \circ \widetilde{j}_* [v_0] = [v]\). Therefore \([w_0] - \widetilde{i}_* [v_0] \in \pi_2(\tilde{M})\) is such that \(\widetilde{j}_* ([w_0] - \widetilde{i}_* [v_0]) = [u]\) and maps to \(e\) under \(\pi_*\). Thus the result holds in this case.

Now for arbitrary \([u] \in \pi_2(\tilde{M}, \tilde{L})\), let \([u_0] := \pi_*[u] \in \pi_2(\tilde{M}, \tilde{L})\). Since \(x_0\) is not in \(L\), there exist a continuous map \(u_0' : (D, \partial D) \to (M, \tilde{L})\) such that \(x_0\) is not in \((u_0'(D))\) and \([u_0] = [u_0']\). In particular, the map \(u_0'\) lifts to a map \((\tilde{M}, \tilde{L})\) that does not intersect the exceptional divisor. Let \(\tilde{u}_0\) be such a map, thus \([\tilde{u}_0] \in \pi_2(\tilde{M}, \tilde{L})\) and \([\pi \circ \tilde{u}_0] = [u_0]\). Notice that \([u] - [\tilde{u}_0]\) maps to \(e\) under \(\pi_*\). Hence there exists \([w] \in \pi_2(\tilde{M})\) such that \(\pi_* [w] = e\) and

\[
[u] = [\tilde{u}_0] + \widetilde{j}_* [w] = [\tilde{u}_0 \# (\widetilde{j} \circ w)].
\]

With these results is now possible to show that the lift to the one-point blow up of a monotone Lagrangian submanifold is also monotone.

Lemma 4.4. Let \( L \) be a monotone Lagrangian submanifold in \((M, \omega)\) and the blow up \((\tilde{M}, \tilde{\omega}_\rho)\) as above. If \((M, \omega)\) is simply connected, then \(\tilde{L}\) is monotone Lagrangian submanifold of \((\tilde{M}, \tilde{\omega}_\rho)\) with the same monotonicity constant as \(L\).

Proof. Let \(\lambda\) and \(\alpha := \lambda/2\) be the monotonicity constants of \(L \subset (M, \omega)\) and \((M, \omega)\) respectively. Recall that the value of \(\rho = \sqrt{\frac{\alpha}{\alpha\pi}}\) is such that \((\tilde{M}, \tilde{\omega}_\rho)\) is monotone with monotonicity constant \(\alpha\).

For \([\tilde{u}]\) in \(\pi_2(\tilde{M}, \tilde{L})\), by Lemma 4.3 we have that \([\tilde{u}] = [u_0 \# (\widetilde{j} \circ w)]\) where \(u_0 : (D, \partial D) \to (\tilde{M}, \tilde{L})\) does not intersect the exceptional divisor and \([w] \in \pi_2(\tilde{M})\) is such that \(\pi_* [w] = e\). Therefore

\[
I_{\tilde{\mu}, \tilde{L}}([\tilde{u}]) = I_{\tilde{\mu}, \tilde{L}}([u_0]) + 2c_1(\tilde{M})([w])
\]

Since \(u_0\) does not intersects the exceptional divisor, we can assume that its image lies in \(\tilde{M} \setminus \pi^{-1}(iB^{2n}(\rho))\). By Proposition 3.1, \((\tilde{M} \setminus \pi^{-1}(iB^{2n}(\rho)), \tilde{\omega}_\rho)\) is symplectomorphic to \((M \setminus iB^{2n}(\rho), \omega)\) under the blow up map. The Maslov index is invariant under symplectic diffeomorphisms, thus \(I_{\tilde{\mu}, \tilde{L}}([u_0]) = I_{\mu, L}([\pi \circ u_0])\) and

\[
I_{\tilde{\mu}, \tilde{L}}([\tilde{u}]) = I_{\mu, L}([\pi \circ u_0]) + 2c_1(\tilde{M})([w]).
\]
Now $c_1(\tilde{M})([w])$ depends on the homology class induced by $[w]$ under the Hurewicz morphism. Furthermore, since $\pi_*[w] = e$ then as a homology class $[w]$ is a purely exceptional class. So if $L_E$ denotes the class of the line in the exceptional divisor, then as homology classes $[w] = \ell[L_E]$ for some $\ell \in \mathbb{Z}$. Similarly for $\tilde{\omega}_\rho([w])$. In this case $\tilde{\omega}_\rho([L_E]) = \pi\rho^2$. Thus

$$c_1(\tilde{M})[L_E] = \pi^*(c_1(M))[L_E] - (n - 1)PD_{\tilde{M}}(E)[L_E] = n - 1$$

and

$$I_{\mu,L}([\pi \circ u_0]) + 2c_1(\tilde{M})([w]) = I_{\mu,L}([\pi \circ u_0]) + 2(n - 1)\ell$$

$$= \lambda I_{\omega}([\pi \circ u_0]) + 2(n - 1)\ell$$

$$= \lambda I_{\omega}([\pi \circ u_0]) + (2\alpha)\pi\rho^2\ell$$

That is, $I_{\mu,L}(\tilde{u}) = \lambda I_{\omega}(\tilde{u})$ for all $\tilde{u} \in \pi_2(\tilde{M}, \tilde{L})$ and $\tilde{L}$ is a monotone Lagrangian submanifold.

Notice from the above proof, that the Maslov index of $\tilde{u} : (D, \partial D) \to (\tilde{M}, \tilde{L})$ can be written in terms of the Maslov index of $\pi \circ \tilde{u}$.

**Lemma 4.5.** Let $(M, \omega)$ and $L$ as in Lemma 4.4. If $\tilde{u} : (D, \partial D) \to (\tilde{M}, \tilde{L})$ is a smooth map, then

$$\mu_{\tilde{L}}[\tilde{u}] = \mu_L[\pi \circ \tilde{u}] + 2(n - 1)\ell$$

for some $\ell \in \mathbb{Z}$.

In the case of a $J$-holomorphic disk, we have a precise description of the integer $\ell$ that appears in the above formula.

**Corollary 4.6.** Let $(M, \omega)$ and $L$ as in Lemma 4.4 and $J$ a $\omega$-compatible almost complex structure on $(M, \omega)$. If $\tilde{u} : (D, \partial D) \to (\tilde{M}, \tilde{L})$ is $\tilde{J}$-holomorphic and $[\tilde{u}] \cdot E = \ell \geq 0$, then

$$\mu_{\tilde{L}}[\tilde{u}] = \mu_L[\pi \circ \tilde{u}] + 2(n - 1)\ell.$$
5. LAGRANGIAN FLOER HOMOLOGY ON THE BLOW UP

Let \((M, \omega)\) be a closed symplectic manifold, and \(L_0\) and \(L_1\) Lagrangian submanifolds that intersect transversely and \(\Sigma(L_j) \geq 3\) for \(j = 0, 1\). For the moment, the Lagrangian submanifolds do not have to be monotone. As above, we assume that they do not intersect the image of the embedded ball \(\iota : (B^{2n}(\rho), \omega_0) \to (M, \omega)\). Finally we also assume that there exists a \(\omega\)-compatible almost complex structure \(J\) in \(\mathcal{J}_{\text{reg}}(L_0, L_1)\) such that \(\iota^* J = J_0\).

**Proposition 5.1.** Let \((M, \omega), L_0, L_1\) and \(\iota : (B^{2n}(\rho), \omega_0) \to (M, \omega)\) as above. If \(J\) be a regular \(\omega\)-compatible almost complex structure for \((L_0, L_1)\) and \(\tilde{J}\) the unique \(\tilde{\omega}_\rho\)-compatible almost complex structure on \((\tilde{M}, \tilde{\omega}_\rho)\) such that \(\pi\) is \((\tilde{J}, J)\)-holomorphic, then \(\tilde{J}\) is regular for \((\tilde{L}_0, \tilde{L}_1)\).

**Proof.** Let \(\tilde{u} : (D, \partial D) \to (\tilde{M}, \tilde{L}_0 \cup \tilde{L}_1)\) be a \(\tilde{J}\)-holomorphic disk that joints the intersection points \(\tilde{p}\) and \(\tilde{q}\). Since the blow up map is holomorphic, \(\pi \circ \tilde{u}\) is a \(J\)-holomorphic disk that joints the intersection points \(p = \pi(\tilde{p})\) and \(q = \pi(\tilde{q})\) and its boundary lies in \(L_0 \cup L_1\), \(\pi \circ \tilde{u}(\cdot, j) \in L_j\) for \(j = 1, 2\). Since \(J\) is regular for \((L_0, L_1)\), then the operator

\[
D_{\tilde{u}; \pi \circ \tilde{u}} : W^p_k((\pi \circ \tilde{u})^* TM; L_0, L_1) \to L_p((\pi \circ \tilde{u})^* TM)
\]

is surjective.

The blow up map induces an operator between the spaces of sections \(L_p(\tilde{u}^* \tilde{T}M)\) and \(L_p((\pi \circ \tilde{u})^* TM)\) as follows. In the case when \(\tilde{u}\) does not intersect the exceptional divisor, the map

\[
\pi^L_{\tilde{u}} : L_p(\tilde{u}^* \tilde{T}M) \to L_p((\pi \circ \tilde{u})^* TM)
\]

is defined as \(\pi^L_{\tilde{u}}(\xi) := \pi_*(\xi)\). Note that it is well defined and surjective. Now in the case when \(\tilde{u}(D) \cap E\) is not empty, then since \(\tilde{u}\) is holomorphic we have that \(\tilde{u}^{-1}(E)\) is a finite set in \(D\). So in this case \(\pi^L_{\tilde{u}}(\xi)\) is defined in the same way as in the previous case on \(D \setminus \tilde{u}^{-1}(E)\) and equal to zero on \(\tilde{u}^{-1}(E)\). Also in this case \(\pi^L_{\tilde{u}}\) is well defined and surjective. That is, for every \(\tilde{u}\) holomorphic disk the map \(\pi^L_{\tilde{u}}\) is surjective. The same reasoning shows that the map

\[
\pi^W_{\tilde{u}} : W^p_k(\tilde{u}^* \tilde{T}M; \tilde{L}_0, \tilde{L}_1) \to W^k_p((\pi \circ \tilde{u})^* TM; L_0, L_1)
\]

defined as \(\pi^W_{\tilde{u}}(\xi) = \pi_*(\xi)\) on \(D \setminus \tilde{u}^{-1}(E)\) and zero on \(\tilde{u}^{-1}(E)\) is well defined and surjective.

Notice that we have a commutative relation

\[
\pi^L_{\tilde{u}} \circ D_{\tilde{u}; \pi \circ \tilde{u}} = D_{\tilde{u}; \pi \circ \tilde{u}} \circ \pi^W_{\tilde{u}}.
\]

Since \(D_{\tilde{u}; \pi \circ \tilde{u}}\) is surjective, then \(D_{\tilde{u}; \pi \circ \tilde{u}}\) is surjective and \(\tilde{J}\) is regular for \((\tilde{L}_0, \tilde{L}_1)\). □
For \( \tilde{p} \) and \( \tilde{q} \) in \( \tilde{L}_0 \cap \tilde{L}_1 \), \( J \) a regular \( \omega \)-compatible almost complex structure on \((M, \omega)\) and \( \beta \in \pi_2(M, \tilde{L}_0 \cup \tilde{L}_1) \) there is a smooth map
\[
\mathcal{M}_\pi : \mathcal{M}(\tilde{p}, \tilde{q}, \beta, \tilde{J}) \to \mathcal{M}(p, q, \pi_*(\beta), J)
\]
induced by the blow up map. This map is not necessarily surjective. For, suppose that \( u \) a \( J \)-holomorphic disk such that \([u] = \pi_*(\beta)\), \( \pi_*(\beta) \in \pi_2(M, L_0 \cup L_1) \) is non trivial and \( u(z_0) = x_0 \) for some \( z_0 \in \text{Int}(D) \). Then by Proposition 4.1 and Corollary 4.6, \( u \) has the unique holomorphic lift \( \tilde{u} \) such that \( \mu_{\tilde{L}_0, \tilde{L}_1}(\tilde{u}) > \mu_{L_0, L_1}(u) \). Hence if the class \( \beta \) does not have an exceptional part, we get that the lift \( \tilde{u} \) does not lie in \( \mathcal{M}(\tilde{p}, \tilde{q}, \beta, \tilde{J}) \). This observation is the idea behind Theorem 1.1 that is, what makes \( \text{HF}(L_0, L_1) \) not isomorphic to \( \text{HF}(\tilde{L}_0, \tilde{L}_1) \) in some cases.

However if we ignore the homotopy class and consider the whole moduli space, then by Propositions 4.1 and 5.1, the map \( \mathcal{M}_\pi \) is surjective.

**Proposition 5.2.** Let \( L_1 \) and \( L_2 \) as in Proposition 5.1. The map
\[
\mathcal{M}_\pi : \mathcal{M}(\tilde{p}, \tilde{q}, \beta, \tilde{J}) \to \mathcal{M}(p, q, J)
\]
given by \( \mathcal{M}_\pi(\tilde{u}) = \pi \circ \tilde{u} \) is surjective.

Now we present the proof of Corollary 1.3. In our opinion, it sets the ground for the proof of Theorem 1.3.

**Proof of Corollary 1.3.** Notice that there is a bijection between \( L_0 \cap L_1 \) and \( \tilde{L}_0 \cap \tilde{L}_1 \). Consequently a \( \Lambda \)-linear isomorphism
\[
\pi_{\text{CF}} : \text{CF}(\tilde{L}_0, \tilde{L}_1) \to \text{CF}(L_0, L_1)
\]
defined on generators as \( \pi_{\text{CF}}(\tilde{p}) = p \). And by Lemma 4.4, \( \tilde{L}_j \) is a monotone Lagrangian submanifold in \((\tilde{M}, \tilde{\omega}_0)\) for \( j = 0, 1 \).

By hypothesis there exists a regular \( \omega \)-compatible almost complex structure \( J \) on \((M, \omega)\) for the pair \((L_0, L_1)\) such that \( \iota^*J = J_0 \). Then by Proposition 5.1 the unique \( \tilde{\omega}_0 \)-compatible almost complex structure \( \tilde{J} \) in \((\tilde{M}, \tilde{\omega})\) is regular for \((\tilde{L}_0, \tilde{L}_1)\).

Fix \( \tilde{p}, \tilde{q} \in \tilde{L}_0 \cap \tilde{L}_1 \). If \( \tilde{u} \) is a \( \tilde{J} \)-holomorphic disk of Maslov index equal to 1 with boundary in \( \tilde{u}(\cdot, j) \in \tilde{L}_j \) for \( j = 1, 2 \), and joining \( \tilde{p} \) and \( \tilde{q} \), then it does not intersect the exceptional divisor. Otherwise, by Corollary 4.6, its Maslov index would be greater than 1. Therefore \( \pi \circ \tilde{u} \) does not goes thru \( x_0 \). Hence it is a \( J \)-holomorphic disk of Maslov index equal to 1 with boundary in \( \pi \circ \tilde{u}(\cdot, j) \in L_j \) for \( j = 1, 2 \) and joining \( p \) and \( q \).

Conversely if \( u \) is a \( J \)-holomorphic disk of Maslov index equal to 1 with boundary in \( u(\cdot, j) \in L_j \) for \( j = 1, 2 \), that joints \( p \) and \( q \), then the lift \( \tilde{u} \) joints \( \tilde{p} \) and \( \tilde{q} \) and if \( \tilde{J} \)-holomorphic. Moreover since \( x_0 \) is not inside \( u \), its lift \( \tilde{u} \) does not meet the exceptional divisor and by Corollary 4.6 it has Maslov index equal to 1.
Thus $\pi_{CF}$ is a chain map, $\pi_{CF} \circ \partial_J = \partial_J \circ \pi_{CF}$, that induces an isomorphism $\pi_{CF} : HF(\tilde{L}_0, \tilde{L}_1) \to HF(L_0, L_1)$. \hfill \QED

Next a minor adaptation in the above proof, gives the proof of Theorem 1.3.

Proof of Theorem 1.3. Let $L_0$ and $L_1$ be Hamiltonian isotopic Lagrangian submanifolds. As in the above proof, $\tilde{L}_0$ and $\tilde{L}_1$ are monotone Lagrangian submanifolds in $(\tilde{M}, \tilde{\omega}_\rho)$ and

$$\pi_{CF} : CF(\tilde{L}_0, \tilde{L}_1) \to CF(L_0, L_1)$$

is a $\Lambda$-linear isomorphism.

However in this case there is no bijection between $\tilde{J}$-holomorphic of Maslov index equal to 1 and $J$-holomorphic of Maslov index equal to 1. By hypothesis there exists one $J$-holomorphic disk $u_0$ that goes thru $x_0$, has boundary in $u_0(\cdot, j) \in L_j$ for $j = 1, 2$ and its Maslov index is 1. Hence it contributes to $\langle p, \partial_J(q) \rangle$. By Corollary 4.6 the Maslov index of its lift $\tilde{u}_0$, which is $\tilde{J}$-holomorphic, is greater than 1. Accordingly $\langle \tilde{p}, \partial_{\tilde{J}}(\tilde{q}) \rangle = \langle p, \partial_J(q) \rangle - 1$ in $\mathbb{Z}_2$.

Since there is only one such $J$-holomorphic disk, as $\Lambda$-modules $HF(L_0, L_1)$ is not isomorphic to $HF(\tilde{L}_0, \tilde{L}_1)$. Since $L_0$ is Hamiltonian isotopic to $L_1$, this means that $HF(\tilde{L}_0, \tilde{L}_1)$ is not isomorphic to $HF(\tilde{L}_0) \otimes \Lambda$ and Theorem 1.3 follows. \hfill \QED

6. THE SYMPLECTIC EMBEDDING OF THE BALL IN $(\mathbb{C}P^2, \omega_{FS})$ AND THE PROOF OF THEOREM 1.1

In this section we fixed the Hamiltonian diffeomorphism $\psi : (\mathbb{C}P^2, \omega_{FS}) \to (\mathbb{C}P^2, \omega)$ that will be used to define the Lagrangian submanifold $L_1$ that is part of the hypothesis of Theorem 1.1. Also we define a symplectic embedding $\iota : (B^4(1/\sqrt{3}), \omega_0) \to (\mathbb{C}P^2, \omega_{FS})$ such that its image is disjoint from the Lagrangian submanifolds $L_0 = \mathbb{R}P^2$ and $L_1 = \psi(\mathbb{R}P^2)$.

To that end consider the symplectic embedding $j : (\text{Int} B^4(1), \omega_0) \to (\mathbb{C}P^2, \omega_{FS})$ given by $j(w_1, w_2) = [\sqrt{1 - |w_1|^2 - |w_2|^2} : w_1 : w_2]$. The image of $j$ is precisely the standard open set $U_0 = \{[1 : z_1 : z_2] | z_1, z_2 \in \mathbb{C} \} \subset \mathbb{C}P^2$ and is holomorphic with respect to the standard complex structures. Its inverse is given by

$$j^{-1}([1 : z_1 : z_2]) = \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}} (z_1, z_2).$$

Now let $\{\psi_t\}_{0 \leq t \leq 1}$ be the Hamiltonian circle action on $(\mathbb{C}P^2, \omega_{FS})$ defined at the Introduction, namely

$$\psi_t([z_0 : z_1 : z_2]) = [z_0 : e^{2\pi it - 1} z_1 : e^{2\pi it - 2} z_2].$$
Thus the fixed point set of the action is \( \{ p_0 := [1 : 0 : 0], p_1 := [0 : 1 : 0], p_2 := [0 : 0 : 1] \} \). The Hamiltonian circle action on \((\mathbb{C}P^2, \omega_{FS})\) pulls under \( j^{-1} \) to a Hamiltonian circle action on \((\text{Int}B^4(1), \omega_0)\) defined as

\[
\phi_t(w_1, w_2) = (e^{2\pi it}w_1, e^{2\pi it}w_2).
\]

Hence the origin is the only fixed point, which corresponds to \( p_0 \). We used this observation to define the desired embedding.

**Proposition 6.1.** There exists a symplectic embedding \( \iota : (B^4(1/\sqrt{3}), \omega_0) \rightarrow (\mathbb{C}P^2, \omega_{FS}) \) and a positive integer \( N_0(\iota) \in \mathbb{N} \) that depends on \( \iota \) such that for \( N \geq N_0 \),

\[
\iota(B^4(1/\sqrt{3})) \cap (\mathbb{R}P^2 \cup \psi_{1/N}(\mathbb{R}P^2)) = \emptyset.
\]

Additionally, if \( J \) is the standard complex structure on \((\mathbb{C}P^2, \omega_{FS})\) then \( \iota^*J = J_0 \).

**Proof.** Since \( \text{Vol}(\text{Int}B^2(1), \omega_0)/2 \) is greater than \( \text{Vol}(B^2(1/\sqrt{3}), \omega_0) \), there exists an area preserving diffeomorphism \( j_1 : (B^2(1/\sqrt{3}), \omega_0) \rightarrow (\text{Int}B^2(1), \omega_0) \) such that

- \( j_1(B^2(1/\sqrt{3})) \cap \{(x_1, 0) | -1 < x_1 < 1 \} = \emptyset \) and
- \( j_1(0, 0) = (0, 1/2) \).

Now let \( j_2 : (B^2(1/\sqrt{3}), \omega_0) \rightarrow (\text{Int}B^2(1), \omega_0) \) be any area preserving diffeomorphism such that \( j_2(0, 0) = (0, 0) \). Moreover, is possible to choose \( j_1 \) and \( j_2 \) to be holomorphic.

Next we define \( \iota : (B^4(1/\sqrt{3}), \omega_0) \rightarrow (\mathbb{C}P^2, \omega_{FS}) \) as \( \iota(w_1, w_2) = j(j_1(w_1), j_2(w_2)) \). So defined \( \iota \) is a symplectic embedding, holomorphic and \( \iota(0, 0) = j(i/2, 0) = [1/\sqrt{2} : i/2 : 0] \). Further since \( j_1 \) misses the \( x_1 \)-axis, \( \iota(B^4(1/\sqrt{3})) \cap \mathbb{R}P^2 \) is empty.

Since \( \psi_0 \) is the identity map and \( \iota(B^4(1/\sqrt{3})) \subset \mathbb{C}P^2 \) is closed, then there exists a large \( N_0 \in \mathbb{N} \) such that for every \( N \geq N_0 \), \( \iota(B^4(1/\sqrt{3})) \cap \psi_{1/N}(\mathbb{R}P^2) \) is also empty. \( \square \)

**Remark.** Pulling back under the symplectic diffeomorphism \( j^{-1} \) the above construction, on \((\text{int}(B^4(1)), \omega_0)\) we have the following: the Lagrangian \( \mathbb{R}P^2 \) corresponds to \((x_1, x_2); (i/2, 0) \) corresponds to the center of the embedded ball and the diffeomorphisms \( \psi_1 \) correspond to

\[
\phi_t(x_1 + iy_1, x_2 + iy_2) = (e^{2\pi it}(x_1 + iy_1), e^{2\pi it}(x_2 + iy_2)).
\]

Recall that \((\mathbb{C}P^2 \setminus \mathbb{C}P^1, \omega_{FS})\) is symplectomorphic to \((\text{Int}(B^4(1)), \omega_0)\). According to Brian [2] Theorem 1.B, any symplectic embedding \( \iota : (B^4(r), \omega_0) \rightarrow (\mathbb{C}P^2, \omega_{FS}) \) such that \( r^2 \geq 1/2 \) intersects \( \mathbb{R}P^2 \). Furthermore the bound on the radius is optimal. That is if \( r^2 < 1/2 \), then there exists a symplectic embedding \( \iota : (B^4(r), \omega_0) \rightarrow (\mathbb{C}P^2, \omega_{FS}) \) whose image does not intersects \( \mathbb{R}P^2 \). Notice that the argument of the proof of Proposition 6.1 can be adapted to give the optimal symplectic embedding established by Brian. In the proof of Proposition 6.1, replace \( 1/\sqrt{3} \) by \( r < 1/\sqrt{2} \).

Fix \( N \in \mathbb{N} \) such that \( 1/2^N \geq N_0 \) as in Proposition 6.1 and set \( L_1 := \psi_{1/2^N}(\mathbb{R}P^2) \). For \( r^2 = 1/3 \), the one-point blow up of \((\mathbb{C}P^2, \omega_{FS})\) is a monotone symplectic manifold, and \( L_0 \) and \( L_1 \) do not intersect the image of the embedded ball \((B^4(1/\sqrt{3}), \omega_0)\).
The standard complex structure $J$ on $(\mathbb{CP}^2, \omega_{FS})$ is regular for the pair of Lagrangians $L_0$ and $L_1$; see Oh [8]. Further, $\iota(0,0) = [1 : i/\sqrt{2} : 0]$ lies in a unique holomorphic disk $u_0$ of Maslov index 1 determined by the intersection points $p_0 := [1 : 0 : 0]$ and $p_1 := [0 : 1 : 0]$. Moreover the Hamiltonian loop $\{\psi_t\}$ is induced by the Hamiltonian function

$$H([z_0 : z_1 : z_2]) := \frac{-\pi}{|z_0|^2 + |z_1|^2 + |z_2|^2} \left\{|z_1|^2 + 2|z_2|^2\right\},$$

thus the $J$-holomorphic disk $u_0$ goes from $p_0$ to $p_1$;

$$\lim_{s \to -\infty} u_0(s, t) = p_0 \quad \text{and} \quad \lim_{s \to +\infty} u_0(s, t) = p_1.$$

**Proof of Theorem 1.1.** Let $\iota: (B^4(1/\sqrt{3}), \omega_0) \to (\mathbb{CP}^2, \omega_{FS})$ be the symplectic embedding defined in Proposition 6.1. With respect to this embedding we blow up $(\mathbb{CP}^2, \omega_{FS})$ at $\iota(0,0) = [1/\sqrt{2} : i/2 : 0]$, to obtain $(\hat{\mathbb{CP}}^2, \hat{\omega}_\rho)$. Note that the value of $\rho$ is such that $\hat{\mathbb{CP}}^2, \hat{\omega}_\rho$ is monotone.

Let $N$ be large as in Proposition 6.1 and define $L_1 := \psi_{1/2N}(L_0)$ so that it avoids the image of $\iota$. Since $L_0$ and $L_1$ are monotone Lagrangian submanifolds, it follows by Lemma 4.4 that $\tilde{L}_0 := \pi^{-1}(L_0)$ and $\tilde{L}_1 := \pi^{-1}(L_1)$ are monotone Lagrangian submanifolds in the blow up.

In order to compute the Lagrangian Floer homology of the pair $(\tilde{L}_0, \tilde{L}_1)$ we need a regular almost complex structure. To that end recall from [8, Prop. 4.3] that the standard complex structure $J$ on $(\mathbb{CP}^2, \omega_{FS})$ is regular for the pair of Lagrangian submanifolds $(L_0, L_1)$. Furthermore, by Proposition 6.1 the symplectic embedding used to define the one-point blow up satisfies $\iota^*J = J_0$. Thus if $\tilde{J}$ is the unique $\tilde{\omega}_\rho$-compatible almost complex structure on $(\hat{\mathbb{CP}}^2, \hat{\omega}_\rho)$ induced by $J$, we have by Proposition 5.1 that $\tilde{J}$ is regular for the pair of Lagrangians $(\tilde{L}_0, \tilde{L}_1)$.

Recall that $L_0$ and $L_1$ intersect transversally and $L_0 \cap L_1 = \{p_0 := [1 : 0 : 0], p_1 := [0 : 1 : 0], p_2 := [0 : 0 : 1]\}$. Additionally by [8, Prop. 4.5] $\partial_j(p_j) = 0$ for $j = 0, 1$ and 2 since there is an even number of of $J$-holomorphic disks joining intersection points with boundary in the Lagrangian submanifolds and of Maslov index equal to 1. Hence the lifted Lagrangians also intersect transversally and $\tilde{L}_0 \cap \tilde{L}_1 = \{\tilde{p}_0, \tilde{p}_1, \tilde{p}_2\}$, where $\pi(\tilde{p}_j) = p_j$. It only remains to compute $\partial_j(\tilde{p}_j)$.

To that end, recall from Section 4 that every $J$-holomorphic disk in $(\mathbb{CP}^2, \omega_{FS})$ lifts to a unique $\tilde{J}$-holomorphic disk in $(\hat{\mathbb{CP}}^2, \hat{\omega}_\rho)$. Further, by Corollary 4.6 the Maslov indices of a disk and its lift agree except in the case when it goes thru the blown up point.

There exists a unique regular $J$-holomorphic sphere, namely

$$\{[1 : z_1 : 0]|z_1 \in \mathbb{C}\} \cup \{[0 : 1 : 0]\},$$
in \((\mathbb{C}P^2, \omega_{\text{FS}})\) thru the point \(\iota(0,0) = [1 : i/\sqrt{2} : 0]\). Further the sphere intersects
the Lagrangian submanifolds and contains the points \(p_0\) and \(p_1\). Hence the slice of
the \(J\)-holomorphic sphere that contains \(\iota(0,0)\) and is delimited by the Lagrangians is
the unique \(J\)-holomorphic disk \(u_0\) of Maslov index 1 determined by the intersection
points \(p_0\) and \(p_1\) and goes thru the blown up point. Thus the Maslov index of the
lifted disk \(\tilde{u}_0\) is 3 and \(\partial_J(\tilde{p}_2) = 0\) since any \(J\)-holomorphic disk that contains \(p_2\) and
has Maslov index 1 does not go thru \(\iota(0,0) = [1 : i/\sqrt{2} : 0]\). Further \(\partial_J\) is not the
zero map.

As mentioned above, the \(J\)-holomorphic disk \(u_0\) goes from \(p_0\) to \(p_1\);
\[
\lim_{s \to -\infty} u_0(s, t) = p_0 \text{ and } \lim_{s \to +\infty} u_0(s, t) = p_1.
\]
This means that
\[
\partial_J(\tilde{p}_0) = 0 \text{ and } \partial_J(\tilde{p}_1) = \tilde{p}_0 T^\alpha
\]
where \(\alpha := \pi(2^{N-1} - 1)/2^N\) is the \(\tilde{\omega}_\rho\)-area of \(\tilde{J}\)-holomorphic disk of Maslov index
1 that connects \(\tilde{p}_0\) to \(\tilde{p}_1\); which is the same as the \(\omega_{\text{FS}}\)-area of the corresponding
\(J\)-holomorphic disk of Maslov index 1 that connects \(p_0\) to \(p_1\). Therefore Theorem 1.1
follows and
\[
\text{HF}(\tilde{L}_0, \tilde{L}_1) = \Lambda\{\tilde{p}_2\}.
\]

7. Proof of Theorem 1.2

Before moving on to the proof of Theorem 1.2 there are some adaptations of some
results of [8] that need to be made.

Let \(\{\psi_t\}_{0 \leq t \leq 1}\) be the Hamiltonian circle action on \((\mathbb{C}P^2, \omega_{\text{FS}})\) defined in the previous
section, where \(\psi_0 = 1\) and \(L_0 = \mathbb{R}P^2\). Then in [8], the Hamiltonian diffeomorphism
used to define the Lagrangian \(L_1\) was \(\psi_{1/2^N}\) for a large fixed \(N\). That is \(L_1 := \psi_{1/2^N}(\mathbb{R}P^2)\).
This particular type of Hamiltonian diffeomorphism was used at two key steps.

a) In [8] Prop. 4.3] to prove the regularity of the standard complex structure
\(J\) for \((L_0, L_1)\). Here the Hamiltonian diffeomorphism was useful in order to
show that if \(u\) is a \(J\)-holomorphic disk such that \(u(\cdot, j) \in L_j\) for \(j = 0, 1\), then
is part of a \(J\)-holomorphic sphere.

b) In [8] Prop. 4.5] to prove that the Floer differential on \(\text{CF}(L_0, L_1)\) is identically
zero. If \(u\) is a \(J\)-holomorphic disk such that \(u(\cdot, j) \in L_j\) for \(j = 0, 1\) and has
Maslov index equal to 1, then there exists a different \(J\)-holomorphic disk with
the same properties. That is, holomorphic disks of Maslov index equal to 1
come in pairs.
These two assertions remain valid if the Hamiltonian diffeomorphism $\psi_{1/2N}$ is replaced by $\psi_{1/(2M)}$ for a large fixed $M$. Here the factor of two in the denominator of the time parameter is to guarantee assertion (b).

The next lemma is a technical result that will be used in the proof of Theorem 1.2. Its proof is a straightforward computation.

**Lemma 7.1.** Let $k$ and $N_0$ be a positive integers. If $A := (2k)! \cdot N_0$ and $t_j := \frac{2j}{A}$ for $0 \leq j \leq k$, then

$$t_j - t_i = \frac{1}{2 \cdot M_{ij}}$$

for $i < j$ and some $M_{ij} \in \mathbb{N}$.

Finally we give the proof of Theorem 1.2. To that end, we recall that the symplectic embedding

$$\iota : (B^4(1/\sqrt{3}),\omega_0) \to (\mathbb{C}P^2,\omega_{FS})$$

defined in Proposition 6.1 is such that $\iota(B^4(1/\sqrt{3})) \cap \mathbb{R}P^2$ is empty and $\iota(0,0) = [1 : i/\sqrt{2} : 0]$. This embedding is used to define the one-point blow up $(\widetilde{\mathbb{C}P^2},\widetilde{\omega}_\rho)$ that appears below.

**Theorem 1.2.** If $k$ is a positive integer and $(\widetilde{\mathbb{C}P^2},\widetilde{\omega}_\rho)$ is the symplectic one-point blow of $(\mathbb{C}P^2,\omega_{FS})$ of weight $\rho = 1/\sqrt{3}$, then there exist Hamiltonian isotopic Lagrangian submanifolds $L_0, L_1, \ldots, L_k$ in $(\mathbb{C}P^2,\omega_{FS})$ that lift to monotone Lagrangian submanifolds $\tilde{L}_0, \tilde{L}_1, \ldots, \tilde{L}_k$ in $(\widetilde{\mathbb{C}P^2},\widetilde{\omega}_\rho)$ such that no two of them are Hamiltonian isotopic.

**Proof.** Set $L_0 := \mathbb{R}P^2$ and let $N_0 \in \mathbb{N}$ be such that $\psi_t(L_0)$ does not intersect $\iota B^4(1/\sqrt{3})$ for all $t \leq 1/N_0$. Now given a positive integer $k$, by the above lemma there are $0 = t_0 < t_1 < \cdots < t_k < 1/N_0$ such that

$$t_s - t_r = \frac{1}{2 \cdot M_{rs}}$$

for $r < s$ and $M_{rs} \in \mathbb{Z}$.

Define the collection of Hamiltonian isotopic Lagrangian submanifolds as $L_r := \psi_{t_r}(L_0)$ for $1 \leq r \leq k$. Since $L_r$ does not intersect the embedded ball, $\bar{L}_0 := \mathbb{R}P^2, \ldots, \bar{L}_k$ are Lagrangian submanifolds in $(\widetilde{\mathbb{C}P^2},\widetilde{\omega}_\rho)$. Further by Lemma 4.4, the Lagrangians $\tilde{L}_0, \ldots, \tilde{L}_k$ are monotone.

Now for any $r < s$, the Lagrangian pair $(L_r, L_s)$ is mapped to $(L_0, \psi_{t_s - t_r}(L_0)) = (L_0, \psi_{1/2M_{rs}}(L_0))$ by $\psi_{-t_r}$. Therefore $L_r$ and $L_s$ intersect transversally at $p_0, p_1$ and $p_2$. Furthermore, since the diffeomorphisms $\{\psi_t\}_{0 \leq t \leq 1}$ are in fact Kähler diffeomorphisms and the standard complex structure $J$ of $(\mathbb{C}P^2,\omega_{FS})$ and is regular for
NON-HAMILTONIAN ISOTOPIC LAGRANGIANS ON THE ONE-POINT BLOW-UP OF $\mathbb{C}P^2$ (L₀, ψ₁/2M_r(L₀)), if follows that $J$ is also regular for $(L_r, L_s)$. Therefore

$$HF(L_r, L_s) \simeq HF(L₀, ψ₁/2M_r(L₀)) \simeq H_*(\mathbb{R}P^2; \mathbb{Z}_2) \otimes \mathbb{Z}_2 \Lambda.$$ 

Finally, following the same argument as in the proof of Theorem 1.1, we claim that $\tilde{L}_r$ is not Hamiltonian isotopic to $\tilde{L}_s$. For, $\psi⁻t_r$ maps the holomorphic sphere

$\{[1: z₁: 0]|z₁ ∈ \mathbb{C}\} \cup \{[0: 1: 0]\},$

to itself. As in the proof of Theorem 1.1, there exists a holomorphic disk $u$ with boundary in $u(\cdot, 0) ∈ L₀$ and $u(\cdot, 1) ∈ ψ₁/2M_r(L₀)$, of Maslov index 1, contains $[1 : e^{2πit_r} \cdot t/\sqrt{2} : 0]$ and is part of the holomorphic sphere mentioned above. Therefore $ψ⁻⁻t_r \circ u$ is a holomorphic disk of Maslov index 1, with boundary in $ψ⁻⁻t_r \circ u(\cdot, 0) ∈ L_r$ and $ψ⁻⁻t_r \circ u(\cdot, 1) ∈ L_s$. Hence the disk $ψ⁻⁻t_r \circ u$ lifts to a holomorphic disk $(ψ⁻⁻t_r \circ u)^⁻⁻$ in $(\widetilde{\mathbb{C}P^2, \tilde{ω}_µ})$ with boundary in $(ψ⁻⁻t_r \circ u)^⁻(\cdot, 0) ∈ \tilde{L}_r$ and $(ψ⁻⁻t_r \circ u)^⁻(\cdot, 1) ∈ \tilde{L}_s$ but with Maslov index equal to 3. Therefore

$$HF(\tilde{L}_r, \tilde{L}_s) \simeq \Lambda\{\text{pt}\}.$$ 

and $\tilde{L}_r$ is not Hamiltonian isotopic to $\tilde{L}_s$ in $(\widetilde{\mathbb{C}P^2, \tilde{ω}_µ})$. 

\[\square\]

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