Rational points of the group of components of a Néron model

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Let $A_K$ be an abelian variety over a discrete valuation field $K$. Let $A$ be the Néron model of $A_K$ over the ring of integers $O_K$ of $K$ and $A_k$ its special fibre over the residue field $k$ of $O_K$. Denote by $A^0$ and $A^0_k$ the corresponding identity components. Then we have an exact sequence

$$0 \to A^0_k \to A_k \to \phi_A \to 0,$$

where $\phi_A$ is a finite étale group scheme over $k$. The latter is called the group of components of $A$. The group of rational points $\phi_A(k)$ counts the number of connected components of the special fibre $A_k$ which are geometrically connected.

In this paper we are interested in “computing” this group and the image of $A_K(K) \to \phi_A(k)$. The starting point of this work is an e-mail of E. Schaefer to the second author. He convinced us of the interest in computing $\phi_A(k)$.

This paper is organized as follows. Section 1 deals with the case where $A_K$ is the Jacobian of a curve $X_K$ over $K$. Let $X$ be a regular model of $X_K$ over $O_K$. Then a modified intersection matrix gives an explicit subgroup of $\phi_A(k)$ and the quotient can be controlled by some cohomology groups. The main result of this section is Theorem 1.11 which determines $\phi_A(k)$ when $k$ is finite.

In section 2, we put together some classical results and general remarks about the canonical map $A_K(K) \to \phi_A(k)$.

In sections 3 and 4, we assume that $K$ is complete. First we consider algebraic tori $T_K$ with multiplicative reduction (so $T_K$ is not an abelian variety in this section). Let $T$ be the Néron model of $T_K$. We show in 3.2 that $\phi_T(k)$ coincides with $\phi_{T_G}(k)$, where $T_{G,K}$ is the biggest split subtorus of $T_K$, and that $T(O_K)/T^0(O_K) \to \phi_T(k)$ is an isomorphism. If $T_K$ does not admit multiplicative reduction, the same constructions lead to subgroups of finite index; cf. 3.3.

Finally, we add some results on abelian varieties $A_K$ with semi-stable reduction, which are more or less known. When the toric part of $A_k$ is split, then $\phi_A$ is constant; cf. 4.2. In general, using data coming from the rigid uniformization of $A_K$, we are able to interpret the image of $A_K(K) \to \phi_A(k)$; see 4.4.

Throughout this paper, we fix a separable closure $k^s$ of $k$, and we denote by $G$ the absolute Galois group $\text{Gal}(k^s/k)$ of $k$.

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1 Component groups of Jacobians

In this section, we fix a connected, proper, flat and regular curve $X$ over $\mathcal{O}_K$. Let us start with some notations. Let $\Gamma_i$, $i \in I$, be the irreducible components of the special fibre $X_k$. Denote by $\mathbb{Z}^I$ the free $\mathbb{Z}$-module generated by the $\Gamma_i$'s. It can be identified canonically with the group of Weil divisors on $X$ with support in $X_k$. We denote by $d_i$ the multiplicity of $\Gamma_i$ in $X_k$, $e_i$ its geometric multiplicity (see [2], Def. 9.1.3), and let $r_i = [k(\Gamma_i) \cap k^s : k]$. The integer $r_i$ is also the number of irreducible components of $(\Gamma_i)_{k^s}$. For two divisors $V_1, V_2$ on $X$, such that at least one of them, say $V_1$, is vertical (i.e. contained in $X_k$), we denote by $V_1 \cdot V_2$ their intersection number $\deg_k \mathcal{O}_X(V_2)|_{V_1}$. When it is necessary to refer to the ground field $k$, we denote this number by $\langle V_1, V_2 \rangle_k$.

Now let us define two homomorphisms of $\mathbb{Z}$-modules which are essential for the computing of $\phi_A(k)$. First, $\alpha : \mathbb{Z}^I \to \mathbb{Z}^I$ is defined by

$$\alpha(V) = \sum_i r_i^{-1} e_i^{-1} \langle V, \Gamma_i \rangle_k \Gamma_i$$

for any $V \in \mathbb{Z}^I$ (see Lemma 1.2 which shows that $\alpha$ really takes values in $\mathbb{Z}^I$). Define $\beta : \mathbb{Z}^I \to \mathbb{Z}$ by $\beta(\Gamma_i) = r_i d_i e_i$. Note that $\alpha$ can be defined more canonically as a map $\mathbb{Z}^I \to (\mathbb{Z}^I)'$ using a suitable (not necessarily symmetric) bilinear form. But for our purpose, this seems not be useful.

Let $\mathcal{O}_{K}^{sh}$ denote a strict henselization of $\mathcal{O}_K$. The residue field of $\mathcal{O}_{K}^{sh}$ is $k^s$. The base change $X \times \text{Spec} \mathcal{O}_{K}^{sh} \to \text{Spec} \mathcal{O}_{K}^{sh}$ gives rise to a regular surface with special fibre $X_{k^s}$. Let $\mathcal{T}$ be a set indexing the irreducible components of $X_{k^s}$. We can define similarly $\overline{\alpha} : \mathbb{Z}^\mathcal{T} \to \mathbb{Z}^\mathcal{T}$ and $\overline{\beta} : \mathbb{Z}^\mathcal{T} \to \mathbb{Z}$. The Galois group $G$ acts on $\mathbb{Z}^\mathcal{T}$ via its action on $X_{k^s}$. Moreover, it is not hard to check that the action of $G$ commutes with $\overline{\alpha}$ and $\overline{\beta}$. Note that since $\phi_A$ is étale over $k$, $\phi_A(k^s) = \phi_A(k^\text{alg})$.

**Theorem 1.1** (Raynaud) Let $X$ be a proper flat and regular curve over $\mathcal{O}_K$, with geometrically irreducible generic fibre. Assume further that either $k$ is perfect or $X$ has an étale quasi-section. Let $A$ be the Néron model of the Jacobian of $X_K$. Then there exists a canonical exact sequence of $G$-modules

$$0 \to \text{Im} \overline{\alpha} \to \text{Ker} \overline{\beta} \to \phi_A(k^s) \to 0$$

(1)

**Proof.** The existence and exactness of the complex as abstract groups are proved in [2], Theorem 9.6.1. Let us just explain quickly why the map $\text{Ker} \overline{\beta} \to \phi_A(k^s)$ commutes with the natural action of $G$ on both sides. To do this, let us go back to the construction of the map $\text{Ker} \overline{\beta} \to \phi_A(k^s)$ as done in [3], Lemma 9.5.9.

Let $P$ be the open subfunctor of $\text{Pic}_X/\mathcal{O}_K$ corresponding to line bundles of total degree 0, then the Néron model $A$ is a quotient of $P$ ([4], Theorem 9.5.4). Since our assertion concerns only the special fibre and since the formation of Néron models commutes with étale base change, we can replace $\mathcal{O}_K$ by a henselization and thus assume that $\mathcal{O}_K$ is henselian. Then $\mathcal{O}_{K}^{sh}$ is Galois over $\mathcal{O}_K$ with group $G$. Let $S = \text{Spec} \mathcal{O}_{K}^{sh}$, $Y = X \times S$, and let $\mathbb{Z}^\mathcal{T}$ be identified with...
For this purpose, we need some informations on the action of $\rho|_{P(S)}$. This part can also be proved using the projection formula for $\rho|_{P(S)}$. Before going back to groups of components, let us derive the following consequence.

**Corollary 1.3** Let $Y$ be a regular, proper scheme over a discrete valuation ring $O_K$. Let $\Gamma_i$, $i \in I$, be the irreducible components of the special fibre $Y_k$. Denote by $d_i$ the multiplicity of $\Gamma_i$ in $Y_k$ and set $r_i = [k^s \cap k(\Gamma_i) : k]$. Then for any closed point $P$ of the generic fibre $Y_K$, the degree $[K(P) : K]$ is divisible by $\gcd\{r_i d_i \mid i \in I\}$.

**Proof.** Let $C := \overline{\{P\}}$ be the Zariski closure of $\{P\}$ in $Y$. Then we have

$$\langle \Gamma, C \rangle_k = [k' \cap k(\Gamma) : k] \langle \Gamma', p^* C \rangle_{k'}$$

where $p$ is the canonical projection $Y_{O_{K'}} \to Y$. Moreover, if $Y$ has dimension 2, then the geometric multiplicity $e$ of $\Gamma$ divides $\langle \Gamma', p^* C \rangle_{k'}$. 

Before going back to groups of components, let us derive the following consequence.
\[ [K(P) : K] = \langle Y_k, C \rangle_k = \sum_{i \in I} d_i \langle \Gamma_i, C \rangle_k \]

The multiplicity \( r_i \) can be computed on a finite Galois extension \( k'/k \) instead of \( k^s/k \). Enlarging \( k' \) if necessary, the extension \( k'/k \) can be lifted to an étale Galois extension \( \mathcal{O}_{k'}/\mathcal{O}_K \) (here ‘lift’ means that \( k' \) is the residue field of the localization of \( \mathcal{O}_{k'} \) at some maximal ideal). According to Lemma 1.2, this implies that \( r_i \) divides \( \langle \Gamma_i, C \rangle_k \). Thus the corollary is proved. \( \square \)

**Remark 1.4** This corollary confirms a prediction of Colliot-Thélène and Saito ([6], Remarque 3.2 (a)). Actually, let \( I_3 = \gcd\{r_id_i \mid i \in I\} \), and let \( I_2 \) be the g.c.d of \( [K(P) : K] \), where \( P \) varies over the closed points of \( Y_K \) (see [6], Théorème 3.1). Then Corollary 1.3 is just the divisibility relation \( I_3 \mid I_2 \). We understood that in a forthcoming preprint, they will prove that \( I_1 = I_2 = I_3 \) for \( p \)-adic fields. We think that Corollary 1.3 should still hold if one replaces \( r_i \) by \( [k^{alg} \cap k(\Gamma_i) : k] = r_i e_i \).

**Corollary 1.5** Let \( X \) be a connected, proper, flat and regular curve over \( \mathcal{O}_K \). Let \( g \) be the genus of the generic fibre \( X_K \) and let \( d' = \gcd\{r_id_i \mid i \in I\} \). Then \( d' \mid 2g - 2 \).

**Proof.** Let \( \mathcal{O}_{k'} \) be a finite étale Galois extension of \( \mathcal{O}_K \) with a residue field \( k' \) containing \( k^s \cap k(\Gamma_i) \) for all \( i \in I \) (see the proof of Corollary 1.3). Denote by \( p : X_{\mathcal{O}_{k'}} \rightarrow X \) the projection. Let \( \omega_{X/\mathcal{O}_K} \) be the relative dualizing sheaf of \( X \). Consider the divisor \( V := \sum_{i \in I} \frac{r_id_i}{d} \Gamma_{i,0} \) on \( X_{\mathcal{O}_{k'}} \), where \( \Gamma_{i,0} \) is an irreducible component of \( (\Gamma_i)_{k'} \). Then we have
\[
\langle p^*\omega_{X/\mathcal{O}_K}, V \rangle_{k'} = \sum_{i \in I} \frac{d_i}{d'} \langle \omega_{X/\mathcal{O}_K}, \Gamma_i \rangle_{k'} = \frac{1}{d'} \langle \omega_{X/\mathcal{O}_K}, X_k \rangle_k = \frac{2g - 2}{d'}
\]
This proves the corollary. \( \square \)

**Remark 1.6** It is known that \( d \mid g - 1 \) (apply the adjunction formula to \( \frac{1}{g}X_k \)). It should be noticed that, to the contrary, \( d' \) does not divide \( g - 1 \) in general.

Now let us return to Galois action. Consider the natural injective map \( \lambda : \mathbb{Z}^I \rightarrow \mathbb{Z}^I \) which sends \( \Gamma \) to \( \Gamma^* \).

**Proposition 1.7** Let \( X \) be a proper flat and regular curve over \( \mathcal{O}_K \) with geometrically connected generic fibre. Then we have \( (\mathbb{Z}^I)^G = \mathbb{Z}^I \). Let \( d = \gcd\{d_i \mid i \in I\} \) and \( V_0 := \frac{1}{d}X_k \). Then we have the following commutative diagram of complexes:

\[
\begin{array}{ccc}
0 & \longrightarrow & V_0\mathbb{Z} \quad \longrightarrow \quad \mathbb{Z}^I \quad \longrightarrow \quad \mathbb{Z}^I \quad \longrightarrow \quad \mathbb{Z} \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & V_0\mathbb{Z} \quad \longrightarrow \quad \mathbb{Z}^I \quad \longrightarrow \quad \mathbb{Z}^I \quad \longrightarrow \quad \mathbb{Z} \\
\end{array}
\]
The only non-trivial point is to check that $\overline{\pi} \circ \lambda = \lambda \circ \alpha$. Let $V \in \mathbb{Z}^I$ be considered as a Weil divisor on $X$. Then $\alpha(V) = \sum_i r_i^{-1} e_i^{-1} \langle V, \Gamma_i \rangle_k \Gamma_i$. Denote by $\Gamma_{ij}$ the irreducible components of $X_{k^s}$ lying over $\Gamma_i$. Then $\lambda(\alpha(V)) = \sum_{i,j} r_i^{-1} e_i^{-1} \langle V, \Gamma_i \rangle_k \Gamma_{ij}$. By Lemma 1.2, $\lambda(\alpha(V)) = \sum_{i,j} e_i^{-1} \langle V, \Gamma_{ij} \rangle_{k^s} \Gamma_{ij}$. (As in the proof of Corollary 1.3, one can reduce to a finite Galois extension before applying Lemma 1.2). Thus $\lambda(\alpha(V)) = \alpha(V_{k^s})$. $\square$

**Corollary 1.8** If $r_i = 1$ (i.e. $\Gamma_i$ is geometrically irreducible) for all $i$, then $\phi_A$ is a constant algebraic group (or equivalently, $\phi_A(k) = \phi_A(k^s)$).

**Remark 1.9** It is known that for modular curves $X_0(N)$ over $\mathbb{Q}$, the multiplicities $r_i$ are equal to 1 (at least when $N$ is square-free). Thus the component group of the Jacobian $J_0(N)$ is constant. This fact was stated in [10], §1.

**Corollary 1.10** We have a canonical exact sequence of groups

$$0 \to \text{Im} \, \alpha \to \text{Ker} \, \beta \to \phi_A(k) \to H^1(G, \text{Im} \, \overline{\pi}) \to H^1(G, \text{Ker} \, \overline{\beta})$$

(2)

**Proof.** It is clear that $(\text{Ker} \, \overline{\beta})^G = \text{Ker} \, \beta$. Let us show that $(\text{Im} \, \overline{\pi})^G = \text{Im} \, \alpha$. Consider the exact sequence $0 \to V_0 \mathbb{Z} \to \mathbb{Z}^I \to \text{Im} \, \overline{\pi} \to 0$, where $V_0$ is defined in the statement of Proposition 1.7, and take the long exact sequence of cohomology. It is enough to see that $H^1(G, V_0 \mathbb{Z}) = 0$. This follows immediately from the facts that $G$ acts trivially on $V_0 \mathbb{Z}$, $G$ is profinite and that $V_0 \mathbb{Z}$ has no torsion. Now we get the corollary just by taking Galois cohomology of the exact sequence (1) of Theorem 1.1. $\square$

**Theorem 1.11** Let $X$ be a proper flat and regular curve over $\mathcal{O}_K$ with geometrically irreducible generic fibre $X_K$. Let $d = \gcd\{d_i \mid i \in I\}$ and $d' = \gcd\{r_i d_i \mid i \in I\}$. Assume that $\text{Gal}(k'/k)$ is procyclic (i.e. any finite extension $k'/k$ is cyclic) and that either $k$ is perfect, or $X$ has an étale quasi-section. Let $A$ be the Néron model of the Jacobian of $X_K$. Then we have an exact sequence

$$0 \to \text{Ker} \, \beta / \text{Im} \, \alpha \to \phi_A(k) \to qd \mathbb{Z} / d' \mathbb{Z} \to 0$$

with $q = 1$ if $d' \mid g - 1$ and $q = 2$ otherwise.

**Remark 1.12** The group $\text{Ker} \, \beta / \text{Im} \, \alpha$ can be determined by means of elementary divisors of the matrix $(e_j^{-1} r_j^{-1} \langle \Gamma_i, \Gamma_j \rangle_k)_{i,j \in I}$ as in [2], Corollary 9.6.3.

The remainder of the section is devoted to the proof of Theorem 1.11.

**Lemma 1.13** Let $G'$ be a finite solvable group acting on a finite set $J$. Let $\mathbb{Z}^J$ be endowed with the natural action of $G'$. Then $H^1(G', \mathbb{Z}^J) = 0$. 


Proof. First assume that \( G' \) has prime order. Then \( \mathbb{Z}^J \) is a direct sum of free \( G' \)-modules and of (free) \( \mathbb{Z} \)-modules with trivial action of \( G' \). Thus \( H^1(G', \mathbb{Z}^J) = 0. \) The general case is easily derived by induction. Note that the lemma is true for any finite group \( G' \) due to Shapira’s lemma (see [5], page 73).

Let \( k'/k \) be a finite Galois extension containing \( k^s \cap k(\Gamma_i) \) for all \( i \in I \). Then the components of \( X_{k'} \) are geometrically irreducible. Thus the exact sequences (1) and (2) can be determined over \( k' \) (Corollary [13]). For simplicity, in the rest of the proof, we denote by \( G \) the group \( \text{Gal}(k'/k) \). Since \( G \) is cyclic, we can determine explicitly each group of this exact sequence. Let us recall some notations and results of [12], VIII, §4. Fix a generator \( \sigma \) of \( G \). Let \( m = |G|, N = \sum_{0 \leq j \leq m-1} \sigma^j \) and \( D = \sigma - 1. \) Recall that for any \( G \)-module \( M \) we have the isomorphisms

\[
H^1(G, M) \simeq N M / DM, \quad H^2(G, M) \simeq M^G / NM
\]

Moreover, if \( 0 \to M' \to M \to M'' \to 0 \) is an exact sequence of \( G \)-modules, then the transition homomorphisms

\[
\delta_1 : N M'' / DM'' \to M^G / NM', \quad \delta_0 : M''^G \to N M'/DM'
\]

are given by

\[
\delta_1([x]) = [Ny], \quad \delta_0([x]) = [Dy]
\]

(3)

if \( y \in M \) is in the preimage of \( x \in M'' \).

Lemma 1.14 Recall that \( V_0 = \frac{1}{d} X_k \). The following properties hold:

(i) The map \( H^1(G, \text{Im } \overline{\alpha}) \to \frac{md}{d} \mathbb{V}_0 \mathbb{Z} / m \mathbb{V}_0 \mathbb{Z} \) defined by \( [\overline{\alpha}(V)] \mapsto [N(V)] \) is an isomorphism.

(ii) Let \( U \in \mathbb{Z}^T \). Then \( DU \in N \text{Ker } \overline{\beta}, \) and the map \( [DU] \mapsto [\overline{\beta}(U)] \) induces an isomorphism \( H^1(G, \text{Ker } \overline{\beta}) \simeq d \mathbb{Z} / d' \mathbb{Z}. \)

Proof. (i) We have an exact sequence

\[
0 \to H^1(G, \text{Im } \overline{\alpha}) \to H^2(G, V_0 \mathbb{Z}) = V_0 \mathbb{Z} / m V_0 \mathbb{Z} \to H^2(G, \mathbb{Z}^T)
\]

Let \( J_i \) denote the set of irreducible components of \( (\Gamma_i)_{k'}. \) Then

\[
H^2(G, \mathbb{Z}^T) = \bigoplus_{i \in I} H^2(G, \mathbb{Z}^T_i) = \bigoplus_{i \in I} \Gamma_i \mathbb{Z} / m r^{-1} \Gamma_i \mathbb{Z}
\]

and the homomorphism \( H^2(G, V_0 \mathbb{Z}) \to H^2(G, \mathbb{Z}^T) \) sends \( [V_0]\) to \( ([d_i d^{-1} \Gamma_i]), \)

Then it is not hard to check (i) using the definition of \( \delta_i. \)

(ii) We have the exact sequence \( 0 \to \text{Ker } \overline{\beta} \to \mathbb{Z}^T \to d \mathbb{Z} \to 0. \) Taking Galois cohomology we get

\[
0 \to \text{Im } \beta = d' \mathbb{Z} \to d \mathbb{Z} \to H^1(G, \text{Ker } \overline{\beta}) \to H^1(G, \mathbb{Z}^T) = 0
\]

(Lemma [1.13]).
Proof of Theorem $\[1.1\]$. Let us first describe the map $\psi : H^1(G, \text{Im} \bar{\alpha}) \to H^1(G, \text{Ker} \bar{\beta})$ in the exact sequence $\[2\]$. One should notice that while these groups are isomorphic, $\psi$ is not an isomorphism in general. Let $L : DZ^T \to \mathbb{Z}^T$ be a section of $D : \mathbb{Z}^T \to D\mathbb{Z}^T$. Let $\bar{\alpha}(V) \in N \text{Im} \bar{\alpha}$. Since $H^1(G, \mathbb{Z}^T) = 0$, one has $\bar{\alpha}(V) \in D\mathbb{Z}^T$, and thus $\bar{\alpha}(V) = D(L \circ \bar{\alpha}(V))$. Hence using Lemma $\[1.14\] (ii), we see that $\psi$ is given by the formula

$$\psi(\bar{\alpha}(V)) = [\bar{\beta}(L \circ \bar{\alpha}(V))] \in H^1(G, \text{Ker} \bar{\beta}) \simeq d\mathbb{Z}/d'\mathbb{Z}.$$ 

Fix for each $i \in I$ an irreducible component $\Gamma_{i,0}$ of $(\Gamma_i)_{k'}$, and put $\Gamma_{i,j} := \sigma^j(\Gamma_{i,0})$. Let $V_i := \sum_i \frac{rd_i}{d} \Gamma_{i,0}$. Since $N(V_i) = \frac{md}{d'} V_0$, Lemma $\[1.14\] (i) implies that $H^1(G, \text{Im} \bar{\alpha}) = [\bar{\alpha}(V_i)]d\mathbb{Z}/d'\mathbb{Z}$. Put $n := \bar{\beta}(L \circ \bar{\alpha}(V_i)) \in \mathbb{Z}$. Then $\ker \psi$ is generated by $q[\bar{\alpha}(V_i)]$, where $q$ is the smallest positive integer such that $d' \mid qn$. Using Corollary $\[1.3\]$, we see that to prove the theorem, it is enough to show that $n \equiv g - 1 \mod d'$. 

Now let us construct a section of $D : \mathbb{Z}^T \to D\mathbb{Z}^T$. Since the set

$$\{\Gamma_{i,0}, D\Gamma_{i,j} \mid i \in I, 0 \leq j \leq r_i - 2\}$$

form a basis of $\mathbb{Z}^T$, we have a well-defined $\mathbb{Z}$-linear map $L' : \mathbb{Z}^T \to \mathbb{Z}^T$ given by $L'(\Gamma_{i,0}) = 0$, $L'(D\Gamma_{i,j}) = \Gamma_{i,j}$ for any $i \in I$ and $0 \leq j \leq r_i - 2$. By construction it is clear that $L := L'|_{D\mathbb{Z}^T}$ is a section of $D : \mathbb{Z}^T \to D\mathbb{Z}^T$. Replacing $D\Gamma_{i,j}$ by $\Gamma_{i,j+1} - \Gamma_{i,j}$, we see that $L'(\Gamma_{i,j}) = \sum_{0 \leq l \leq j - 1} \Gamma_{i,l}$ for any $i \in I$ and $0 \leq j \leq r_i - 1$.

Let us compute the integer $n$. Applying the definitions of $\bar{\alpha}$ and $L$, we get

$$n = \sum_{i \in I, 0 \leq j \leq r_i - 1} e_i^{-1}(V_i \cdot \Gamma_{i,j}) \bar{\beta} \circ L'(\Gamma_{i,j}) = \sum_{i,j} j d_i(V_i \cdot \Gamma_{i,j}) = \sum_i d_i(V_i \cdot U_i), \quad (4)$$

where $U_i := \sum_{0 \leq j \leq r_i - 1} j \Gamma_{i,j}$. Consider $W_i := \sum_{0 \leq j \leq m - 1} j \Gamma_{i,j}$. Since $\Gamma_{i,j} = \Gamma_{i,j'}$ if $j \equiv j'$ modulo $r_i$, we have (put $a = mr_i^{-1}$)

$$W_i = \sum_{0 \leq l \leq a - 1} \sum_{0 \leq h \leq r_i - 1} (lr_i + h) \Gamma_{i,h} = \frac{a(a - 1)}{2} r_i \Gamma_i + a U_i.$$ 

Since $N(V_i) = \frac{md}{d'} V_0 \in V_0 \mathbb{Q}$, we see that $V_i \cdot \Gamma_i = m^{-1}(N(V_i) \cdot \Gamma_i) = 0$. So replacing in the equality $\[4\]$ the divisor $U_i$ by $r_i m^{-1} W_i$, and then $V_i$ by its definition, we get

$$n = \sum_{i \in I} \frac{r_i d_i r_i d_i}{md'} \sum_{1 \leq j \leq m - 1} j (\Gamma_{i,j} \cdot \Gamma_{i,0})$$

On the other hand, $\Gamma_{i,j} \cdot \Gamma_{i,0} = \sigma^{m-j}(\Gamma_{i,j}) \cdot \sigma^{m-j}(\Gamma_{i,0}) = \Gamma_{i,0} \cdot \Gamma_{i,m-j}$. So

$$n = \sum_{i \in I} \frac{r_i d_i r_i d_i}{md'} \sum_{1 \leq j \leq m - 1} (m - j) (\Gamma_{i,0} \cdot \Gamma_{i,j})$$

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Adding these two equalities leads to

\[ 2n = \sum_{i,j \in I} r_i d_r j l d_i (mr_i^{-1} \Gamma_i - \Gamma_{i,0}) \cdot \Gamma_{l,0} = \left( \sum_{i \in I} mr_i d_i d' X_k, \cdot \Gamma_{l,0} \right) - d' V^2 = -d' V^2 \]

Using the adjunction formula, and with the notations of the proof of Corollary \[1.5\], we see that \( V^2 \) is congruent to \( \langle p^* \omega_X / \mathcal{O}_K, V \rangle \mod 2 \). The latter is equal to \( \frac{1}{2}(2g-2) \) as calculated in the proof of Corollary \[1.5\]. This achieves the proof of Theorem \[1.1\]. \( \square \)

**Remark 1.15** Let \( X_k \) be an elliptic curve over \( K \). Let \( X \) be its minimal regular model over \( \mathcal{O}_K \). One can apply Theorem \[1.1\] and Remark \[1.12\] to compute \( \phi_A(k) \). But one can also determine directly \( \phi_A(k) \) as a subset of \( \phi_A(k^*) \) using the fact that the Néron model \( A \) of \( X_k \) is the smooth locus of \( X \).

**Example 1.16** Assume that \( k \) is perfect and \( \text{char}(k) \neq 2 \). Let \( a, b \in \mathcal{O}_K \) be invertible and such that the class \( \bar{a} \in k \) is not a square. Let \( n \geq 1 \) be an integer. Consider the elliptic curve \( A^t \) given by the equation

\[ y^2 = (x^2 - b\pi^{2n})(x + a) \]

where \( \pi \) is a uniformizing element of \( \mathcal{O}_K \). Then the minimal regular model of \( A^t \) over \( \mathcal{O}_K \) consists of a projective line \( \Gamma_1 \) over \( k \), followed by a chain of \( n-1 \) projective lines over \( k(\sqrt{\bar{a}}) \), and ends with the conic \( \Gamma_2n \) given by the equation \( \pi = (u^2 - b^2)\bar{a} \). Thus \( \phi_A(k^*) = \mathbb{Z}/2n\mathbb{Z}, \phi_A(k) = \mathbb{Z}/2\mathbb{Z} \), and \( A_k(k)/A^t_k(k) = \mathbb{Z}/2\mathbb{Z} \) or 0 depending on \( \Gamma_{2n} \) has a rational point or not. This shows that one cannot expect a good control of the order of \( \phi_A(k^*)/\phi_A(k) \).

**Example 1.17** Assume \( \text{char}(k) \neq 2 \). Let \( g \geq 1 \), let \( X_K \) be the hyperelliptic curve defined by an equation \( y^2 = a_0 \prod_{1 \leq i \leq g+1} (x - a_i)^2 + \pi \), where \( a_i \in \mathcal{O}_K \) are such that their images \( \bar{a}_i \in k \) are pairwise distinct and \( \bar{a}_0 \) is not a square. Finally \( \pi \) is a uniformizing element of \( \mathcal{O}_K \). Let \( X \) be the minimal regular model of \( X_K \) over \( \mathcal{O}_K \). Then \( X_k \) is integral with \( g + 1 \) ordinary double points. Over \( k' = k[\sqrt{a_0}] \), \( X_{k'} \) splits into two components isomorphic to \( \mathbb{P}^1_{k'} \), intersecting transversally at \( g + 1 \) points. Thus using Theorems \[1.1\] \[1.1\] and Remark \[1.12\], we see that \( \phi_A(k^*) = \mathbb{Z}/(g + 1)\mathbb{Z} \) and \( \phi_A(k) = 0 \).

## 2 The homomorphism \( A_K(K) \to \phi_A(k) \)

In this section, \( A_K \) is an abelian variety over \( K \). Let \( A \) be the Néron model of \( A_K \) over \( \mathcal{O}_K \). We would like to discuss some relationships between \( A(K) \) and \( \phi_A(k) \). By the properties of Néron models, \( A(\mathcal{O}_K) = A_K(K) \). The specialization map gives rise to a homomorphism of groups \( A_K(K) \to A_k(k) \). The second
group maps canonically to \( \phi_A(k) \). In general, the map \( A_k(k) \rightarrow \phi_A(k) \) is not surjective. The reason is that \( \phi_A(k) \) counts the number of geometrically connected components of \( A_k \), while the image of \( A_k(k) \) in \( \phi_A(k) \) (which is isomorphic to \( A_k(k)/A_k^0(k) \)) parameterizes the components with rational points. Each geometrically connected component is a torsor under \( A_k^0 \). But such a torsor may be non-trivial (that is, without rational point).

**Lemma 2.1** Let \( A_K \) be an abelian variety over \( K \).

(i) If \( K \) is henselian (e.g. complete), then \( A_K(K) \rightarrow A_k(k) \) is surjective.

(ii) If \( k \) is finite, or if \( A_k^0 \) is an extension of a unipotent group by a split torus with \( k \) perfect, then \( A_k(k) \rightarrow \phi_A(k) \) is surjective.

**Proof.** (i) Since \( K \) is henselian and \( A \) is smooth, the map \( A(O_K) \rightarrow A_k(k) \) is surjective (see for instance [2], Prop. 2.3.5).

(ii) Let \( k'/k \) be a finite Galois extension of \( k \) such that \( A_k(k') \rightarrow \phi_A(k') \) is surjective (such an extension exists because \( \phi_A \) is finite). Then it is enough to show that \( H^1(\text{Gal}(k'/k), A_k^0(k')) = 0 \). The case \( k \) finite is a theorem of Lang ([8], Theorem 2). The remaining case is Hilbert’s 90th Theorem (see [12], Chap. X, §1) with induction on the dimension of \( A_k^0 \). \( \square \)

### 3 Algebraic tori

In this section we consider an algebraic torus \( T_K \) over \( K \), its Néron model \( T \) over the ring of integers \( \mathcal{O}_K \) of \( K \), and the associated component group \( \phi_T \). As the formation of Néron models is compatible with passing from \( K \) to its completion by [2], 10.1.3, \( \phi_T \) remains unchanged under this process, and we will assume in the following that \( \mathcal{O}_K \) and \( K \) are *complete*. Writing \( \mathcal{O}_K^{sh} \) for a strict henselization of \( \mathcal{O}_K \) and \( K^{sh} \) for the field of fractions of \( \mathcal{O}_K^{sh} \), we know then that the extension \( K^{sh}/K \) is Galois. The attached Galois group \( G \) is canonically identified with the one of \( k^{s}/k \), the residue extension of \( K^{sh}/K \).

Let us first assume that \( T_K \) has multiplicative reduction, so that the identity component \( T_k^0 \) of the special fibre \( T_k \) is a torus. Then \( T_K \) splits over \( K^{sh} \), and we can view the group of characters \( X \) of \( T_K \) as a \( G \)-module. It is well-known that in this case we have an isomorphism of \( G \)-modules

\[ \phi_T \simeq \text{Hom}(X, \mathbb{Z}); \]

see for example [13], 1.1. In particular, if \( T_K \) is split over \( K \), the action of \( G \) on \( X \) is trivial, and \( \phi_T \) is isomorphic to the constant group \( \mathbb{Z}^d \) with \( d = \dim T_K \).

**Lemma 3.1** Let \( X_G \) be the biggest \( \mathbb{Z} \)-free quotient of \( X \) which is fixed by \( G \). Then the projection \( X \rightarrow X_G \) gives rise to an isomorphism

\[ \text{Hom}(X_G, \mathbb{Z}) \rightarrow \text{Hom}(X, \mathbb{Z})^G \]

of groups which canonically can be identified with \( \phi_T(k) \), the group of \( k \)-rational points of \( \phi_T \).
Proof. The epimorphism $X \rightarrow X_G$ induces injections
\[ \text{Hom}(X_G, \mathbb{Z}) \hookrightarrow \text{Hom}(X, \mathbb{Z})^G \hookrightarrow \text{Hom}(X, \mathbb{Z}), \]
and we have to show that the left injection is, in fact, a bijection. To do this, consider a $G$-morphism $f: X \rightarrow \mathbb{Z}$ which is fixed by $G$. Then $f$ factors through a $G$-morphism $X/W \rightarrow \mathbb{Z}$ where $W \subset X$ is the submodule generated by all elements of type $x - \sigma(x)$ with $x \in X$ and $\sigma \in G$. As $X_G$ is obtained from $X/W$ by dividing out its torsion part and as $\mathbb{Z}$ is torsion-free, we see that $f$ must factor through $X_G$. Hence, the map $\text{Hom}(X_G, \mathbb{Z}) \hookrightarrow \text{Hom}(X, \mathbb{Z})^G$ is bijective, as claimed. □

Now let $T_{G,K}$ be the torus with group of characters $X_G$. The projection $X \rightarrow X_G$ defines $T_{G,K}$ as the biggest subtorus of $T_K$ which is split over $K$, and we can identify the associated morphism $\text{Hom}(X_G, \mathbb{Z}) \rightarrow \text{Hom}(X, \mathbb{Z})$ with the corresponding morphism of component groups $\phi_{T_G} \rightarrow \phi_T$. Thereby we can conclude from 3.1:

**Proposition 3.2** Let $T_K$ be a torus with multiplicative reduction, and let $T_{G,K}$ be the biggest subtorus which is split over $K$. Assume that $K$ is complete. Then the injection $T_{G,K} \hookrightarrow T_K$ and the associated morphism of Néron models $T_G \rightarrow T$ induce a monomorphism of component groups $\phi_{T_G} \rightarrow \phi_T$ and an isomorphism $\phi_{T_G}(k) \approx \phi_T(k)$ between groups of $k$-rational points.

Furthermore, the canonical map $T_K(K) \rightarrow \phi_T(k)$ is surjective, as the same is true for the split torus $T_{G,K}$.

What can be said if, in the situation of 3.2, $T_K$ does not have multiplicative reduction? In this case we can still view the group of characters $X$ of $T_K$ as a Galois module under the absolute Galois group of $K$. Similarly as above, we can use the inertia group $I$ and look at the biggest subtorus $T_{I,K} \subset T_K$ which splits over the maximal unramified extension $K^{sh}$ of $K$. We get an exact sequence of tori
\[ 0 \rightarrow T_{I,K} \rightarrow T_K \rightarrow \tilde{T}_K \rightarrow 0 \]
with a torus $\tilde{T}_K$ such that $\tilde{T}_K \otimes_K K^{sh}$ does not admit a subgroup of type $\mathbb{G}_m$. The Néron model $\tilde{T}$ of $\tilde{T}_K$ is quasi-compact by [3], 10.2.1, and, hence, the component group $\phi_{\tilde{T}}$ must be finite.

We view now Néron models as sheaves with respect to the étale (or smooth) topology on $\mathcal{O}_K$. Then the above exact sequence of tori induces a sequence of Néron models
\[ 0 \rightarrow T_I \rightarrow T \rightarrow \tilde{T} \rightarrow 0 \]
which is exact by [4], 4.2. Furthermore, using the right exactness of the formation of component groups, see [4], 4.10, in conjunction with the facts that $T_{I,K}$ has multiplicative reduction and, hence, that the component group $\phi_{T_I}$ cannot have torsion, we get an exact sequence of component groups
\[ 0 \rightarrow \phi_{T_I} \rightarrow \phi_T \rightarrow \phi_{\tilde{T}} \rightarrow 0. \]
Restriction to $k$-rational points preserves the exactness,

$$0 \rightarrow \phi_{T_I}(k) \rightarrow \phi_T(k) \rightarrow \phi_{\tilde{T}}(k) \rightarrow 0,$$

as $H^1(G',\mathbb{Z}^d) = \text{Hom}(G',\mathbb{Z}^d) = 0$ for any finite group $G'$ acting trivially on $\mathbb{Z}^d$.

Now, taking into account that $T_{I,K}$ has multiplicative reduction and that $\phi_{\tilde{T}}(k)$ is finite, we can conclude from 3.2:

**Corollary 3.3** Let $T_K$ be an algebraic torus, let $T_{G,K}$ be the biggest subtorus which is split over $K$, and let $\tilde{T}_K$ be defined as above. Assume that $K$ is complete. Then the canonical sequence

$$0 \rightarrow \phi_{T_G}(k) \rightarrow \phi_T(k) \rightarrow \phi_{\tilde{T}}(k) \rightarrow 0,$$

is exact with $\phi_{T_G}(k)$ being free and $\phi_{\tilde{T}}(k)$ finite.

In particular, the image of $T_{G,K}(K)$ is of finite index in $\phi_T(k)$, and the same is true for the image of $T_K(K)$.

## 4 Abelian varieties with semi-stable reduction

Let $A_K$ be an abelian variety over the base field $K$, which is assumed to be complete. We will view $A_K$ as a rigid $K$-group and use its uniformization in the sense of rigid geometry; cf. [11] and [4], Sect. 1. So $A_K$ can be expressed as a quotient $E_K/M_K$ of rigid $K$-groups with the following properties:

(i) $E_K$ is a semi-abelian variety sitting in a short exact sequence

$$0 \rightarrow T_K \rightarrow E_K \rightarrow B_K \rightarrow 0,$$

where $T_K$ is an algebraic torus and $B_K$ an abelian variety with potentially good reduction.

(ii) $M_K$ is a lattice in $E_K$ of maximal rank; i.e., a closed analytic subgroup of $E_K$ which, after finite separable extension of $K$, becomes isomorphic to the constant group $\mathbb{Z}^d$ with $d = \dim T_K$.

Let $A$ be the Néron model of $A_K$ and $A^0$ its identity component. Recall that $A_K$ is said to have semi-stable reduction if the special fibre $A^0_k$ of $A^0$ is semi-abelian. Furthermore, let us talk about a split semi-stable reduction if the toric part of $A^0_k$ is split over $k$. The property of semi-stable reduction is reflected on the uniformization of $A_K$ in the following way:

**Proposition 4.1** The abelian variety $A_K$ has semi-stable (resp. split semi-stable) reduction over $K$ if and only if the following hold:

(i) The torus $T_K$ splits over a finite unramified extension of $K$ (resp. over $K$).

(ii) The abelian variety $B_K$ has good reduction over $K$.

If the above conditions are satisfied with $T_K$ being split over $K$, the same is true for the lattice $M_K \subset E_K$; i.e., $M_K$ is then isomorphic to the constant $K$-group $\mathbb{Z}^d$, where $d = \dim T_K$. 

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Proof. As any abelian variety with semi-stable reduction acquires split semi-stable reduction over a finite unramified extension of $K$, we need only to consider the case of split semi-stable reduction. So assume that $A_K$ has split semi-stable reduction. Then we have an exact sequence

$$0 \to T_k \to A^0_k \to B_k \to 0,$$

where $T_k$ is a split torus and $B_k$ an abelian variety over $k$. Let $A$ be the formal completion of $A$ along $A_k$ and $A^0$ its identity component. Using the infinitesimal lifting property of tori, see [7], exp. IX, 3.6, and working in terms of formal Néron models in the sense of [3], we see that $T_k$ lifts to a split formal subgroup torus $T \subset A^0$ such that the quotient $B = A^0 / T$ is a formal abelian scheme lifting $B_k$. The theory of uniformizations, as explained for example in [1], Sect. 1, says now that the exact sequence

$$0 \to T \to A^0 \to B \to 0,$$

coincides with the one obtained from

$$0 \to T_k \to E_k \to B_k \to 0$$

by passing to identity components of associated formal Néron models. As the group of characters of $T_k$ coincides with the one of $T$, we see that $T_K$ is a split torus. Furthermore, $B$ is algebraizable with generic fibre $B_K$ and, thus, $B_K$ has good reduction over $K$.

Let us show that in this situation $M_K$ will be constant. Indeed, writing $K^s$ for a separable closure of $K$, we choose free generators of the group of characters of $T_K$ and look at the associated “valuation”

$$\nu: E_K(K^s) \longrightarrow |K^s|^d \longrightarrow \log \longrightarrow \mathbb{R}^d,$$

where $d = \dim T_K$. One knows that $M_K$ being a lattice (of maximal rank) in $E_K$ means that $M_K$ is of dimension zero and that $M_K(K^s)$ is mapped bijectively under $\nu$ onto a lattice (of maximal rank) in $\mathbb{R}^d$.

Now let us look at the action of the absolute Galois group $G_K := \text{Gal}(K^s/K)$ of $K$ on $M_K(K^s)$ and show that $M_K$ is constant. As $K$ is complete, the action of $G_K$ is trivial on $|K^s|$. Hence, it respects the map $\nu$. Therefore $\nu$ can only be injective if the action of $G_K$ on $M_K(K^s)$ is trivial. However, then all points of $M_K$ must be rational, and $M_K$ is constant.

The converse, that conditions (i) and (ii) imply semi-stable reduction for $A_K$, follows from [4], 5.1. \(\square\)

Let us consider now an abelian variety $A_K$ with semi-abelian reduction and with uniformization given by the exact sequence

$$0 \to M_K \to E_K \to A_K \to 0.$$

Then, by [4, 5], $M_K$ becomes constant over an unramified extension of $K$, and the associated sequence of formal Néron models
0 \rightarrow \mathcal{M} \rightarrow \mathcal{E} \rightarrow \mathcal{A} \rightarrow 0

is exact due to [4], 4.4. As the component group \( \phi_M \) is torsion-free, and as the formation of component groups is right-exact, see [4], 4.10, the induced sequence

\[ (*) \quad 0 \rightarrow \phi_M \rightarrow \phi_E \rightarrow \phi_A \rightarrow 0 \]

is exact, so that \( \phi_A \) may be identified with the quotient \( \phi_E / \phi_M \). Thus, if we view the objects of the latter sequence as Galois modules under \( G = \text{Gal}(K^{ab}/K) \) and apply Galois cohomology, we see:

**Lemma 4.2** As before, let \( A_K \) be an abelian variety with semi-stable reduction. Then the uniformization of \( A_K \), in particular, the above sequence \((*)\), gives rise to an exact sequence

\[ 0 \rightarrow \phi_M(k) \rightarrow \phi_E(k) \rightarrow \phi_A(k) \rightarrow H^1(G, M_K) \rightarrow \ldots \]

If \( A_K \) has split semi-stable reduction, \( M_K \) is constant and, hence, \( H^1(G, M_K) \) is trivial.

Thus, the quotient \( \phi_E(k)/\phi_M(k) \) may be viewed as a subgroup of the group of \( k \)-rational points of \( \phi_A \), and it coincides with \( \phi_A(k) \) in the case of split semi-stable reduction.

Let \( T_K \) be the toric and \( B_K \) the abelian part of \( E_K \). Then we have an exact sequence

\[ 0 \rightarrow T_K \rightarrow E_K \rightarrow B_K \rightarrow 0 \]

of algebraic \( K \)-groups and, associated to it, a sequence of Néron models

\[ 0 \rightarrow T \rightarrow E \rightarrow B \rightarrow 0. \]

In terms of sheaves for the étale (or smooth) topology on \( \mathcal{O}_K \), the latter is exact due to [4], 4.2, as \( A_K \) having semi-abelian reduction implies that \( T_K \) splits over an unramified extension of \( K \); use [4], 1.1 and [4], 5.1. Similarly as before, we get an exact sequence of component groups

\[ 0 \rightarrow \phi_T \rightarrow \phi_E \rightarrow \phi_B \rightarrow 0, \]

where \( \phi_B \) is trivial, since \( B_K \) has good reduction. Thus, the morphism \( T \rightarrow E \) induces an isomorphism \( \phi_T \rightarrow \phi_E \) and, using the above exact sequence \((*)\), we can view \( \phi_A = \phi_E / \phi_M \) as a quotient \( \phi_T / \phi_M \), although the morphism \( M \rightarrow E \) might not factor through \( T \).

**Proposition 4.3** Let \( A_K \) be an abelian variety with split semi-stable reduction; i.e., we assume that the identity component \( A^0_k \) of the special fibre of the Néron model \( A \) of \( A_K \) is extension of an abelian variety by a split algebraic torus. Then:

(i) The component group \( \phi_A \) is constant (also valid if \( K \) is not necessarily complete).

(ii) The canonical map \( A_K(K) \rightarrow \phi_A(k) \) is surjective.
Proof. It follows from [3, I] that $M_K$ is constant and that $T_K$ is split. Thus, the $k$-groups $\phi_T$ and $\phi_M$ are constant, and so is their quotient $\phi_A$. If $K$ is not complete, we may pass to the completion of $\overline{K}$ without changing the reduction of $A_K$ and its type. This establishes assertion (i). Furthermore, assertion (ii) is due to the fact that the map $T_K(K) \to \phi_A(k)$ is surjective, as $T_K$ is a split torus. \hfill \square

If the semi-stable reduction of $A_K$ is not necessarily split, the quotient $\phi_T(k)/\phi_M(k)$ will, in general, be a proper subgroup of $\phi_A(k)$; its index is controlled by the cohomology group $H^1(G, M_K)$. To make this subgroup more explicit, let $X$ be the group of characters of the toric part $T_K$ of $E_K$. As is explained in [3], Sect. 3 or [4], Sect. 5, we can evaluate characters of $X$ semi-stable reduction. $H$ injectively into $\phi$ in the image of $A_K$, controlled by the cohomology group $A$ of $X$, and its type. This establishes assertion (i). Furthermore, assertion (ii) is due to the fact that the map $T_K(K) \to \phi_A(k)$ is surjective, as $T_K$ is a split torus.

Proposition 4.4 Let $A_K$ be an abelian variety with semi-stable reduction and with uniformization $A_K = E_K/M_K$. Let $X$ be the group of characters of the toric part of $E_K$. Then $\Sigma = \text{Hom}(X, \mathbb{Z})/M_K^\Sigma$ is a subgroup of $\phi_A(k)$, contained in the image of $A_K(K) \to \phi_A(k)$, such that the quotient $\phi_A(k)/\Sigma$ is mapped injectively into $H^1(G, M_K)$. Furthermore, $\Sigma$ coincides with $\phi_A(k)$ if $A_K$ has split semi-stable reduction.

If the abelian variety does not admit semi-stable reduction, we still have maps

$$\phi_{T_G} \to \phi_{T_I} \to \phi_T \to \phi_E \to \phi_A,$$

where $T_{G,K}$ stands for the maximal subtorus of $T_K$ which is split over $K$ and, likewise, $T_{I,K}$ for the maximal subtorus of $T_K$ which splits over $K^{sh}$. The image in $\phi_A$ of each of these groups gives rise to a subgroup of $\phi_A$, and we thereby get a filtration of $\phi_A$. Up to the term $\phi_{T_G}$, this filtration was dealt with in [4],...
Sect. 5; it goes back to Lorenzini [9]. Subsequent factors of the filtration are controlled by suitable first cohomology groups or by the component group of $B_K$; cf. [4], 5.5. So, to make general statements about $k$-rational points seems to be a little bit out of reach. However, the groups $\phi_{T_0}$ and $\phi_{T_2}$ are accessible, and this leads to the understanding of rational components in the semi-stable reduction case.

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