SPECTRAL DECONVOLUTION OF UNITARILY INVARIANT MATRIX MODELS

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ABSTRACT. In this paper, we implement a complex analytic method to recover the spectrum of a matrix perturbed by either the addition or the multiplication of a random matrix noise, under the assumption that the distribution of the noise is unitarily invariant. This method, which has been previously introduced by Arizmendi, Tarrago and Vargas, is done in two steps: the first step consists in a fixed point method to compute the Stieltjes transform of the desired distribution in a certain domain, and the second step is a classical deconvolution by a Cauchy distribution, whose parameter depends on the intensity of the noise. We also provide explicit bounds for the mean squared error of the first step.

To Roland Speicher, for his 60th birthday.

1. Introduction

Recovery of data from noisy signal is a recurrent problem in many areas of mathematics (geology, wireless communication, finance, electroencephalography...). From a statistical point of view, this can be seen as the recovery of a probability distribution from a sample of the distribution perturbed by a noise. In the simplest case, the perturbation is a convolution of the original distribution with a distribution representing the noise, and the recovery of the original probability distribution from a sample of the convolved one is named the deconvolution. In [Fan91, Fan92], Fan presented a first general approach to the deconvolution of probability distributions, which allowed to both recover the original data and to get a bound on the accuracy of the recovery. Since this seminal paper, several progresses have been made towards a better understanding of the classical deconvolution of probability measures.

In this paper, we are interested in the broader problem of the recovery of data in a non-commutative setting. Namely, we are given a matrix \(g(A,B)\), which is an algebraic combination of a possibly random matrix \(B\) representing the data we want to recover and a random matrix \(A\) representing the noise, and the goal is to recover the matrix \(B\). Taking \(A\) and \(B\) diagonals and independent with entries of each matrix iid, and considering the case \(g(A,B) = A + B\), yield the classical deconvolution problem. This non-commutative setting has already seen many applications in the simplest cases of the addition and multiplication of matrices, [BBP17, LW04, BABP16]. Yet, the recovery of \(B\) is a complicated process already in those situations and we propose to address issues related to these two cases in the present manuscript.

A first question is the notion of independance. In the classical case, independance is a fundamental hypothesis in the success of the deconvolution, which allows to use the product formula of the Fourier transform for the convolution of measures. In the non-commutative setting, two main approaches of independance are usually considered: either the entries of \(A\) and \(B\) are assumed to be independent and the entries of \(A\) are assumed iid (up to a symmetry if \(A\) is self-adjoint), or the distribution of the noise matrix \(A\) is assumed to be invariant by unitary conjugation. Both hypotheses yield the same results but require different tools. In this paper, we focus on the second hypothesis of a unitarily invariant noise, which has already been studied in [BABP16, BGEM19, LPT1]. Note that in the case of Gaussian matrices with independent entries, the hypothesis of unitarily invariance of the distribution is also satisfied. Our results extend of course to the case of orthogonally invariant noises, up to a numerical constant.

The second question is the scope of the deconvolution process: assuming \(B\) self-adjoint, a perfect recovery of \(B\) would mean the recovery of both its eigenvalues and its eigenbasis. The recovery of the eigenbasis heavily depends on the model. Indeed, in we consider the model \(ABA^*\) where the law of \(A\) is invariant by left multiplication by a Haar unitary, then the law of \(ABA^*\)
is unitarily invariant, which prevents any hope to recover the eigenbasis of $B$. On the contrary, we will show that it is always possible to recover, to some extent, the eigenvalues of $B$, with an accuracy improving when the size of the matrices grows. In some cases, obtaining the spectrum of $B$ is a first step towards the complete recovery of $B$. This is the main approach of [LP11] in the estimation of large covariance matrices, which has led to the successful shrinkage method of [LW04, LW15]. This method has been generalized in [BABP16, BGEM19]. To summarize the above paragraph, we are led to consider the spectral deconvolution of unitarily invariant models.

In the classical deconvolution, the known fact that the Fourier transform of the convolution of two probability measures is the product of the Fourier transform of both original measures has been the starting point of the pioneering work of Fan [Fan91]. Indeed, apart from definition issues, one can see the classical deconvolution as the division of the Fourier transform of the received signal by the Fourier transform of the noise. In the non-commutative setting, there is no close formula describing the spectrum of algebraic combination of finite size matrices, which prevents any hope of concrete formulas in the finite case. However, as the size goes to infinity, the spectral properties of sums and products of independent random matrices is governed by the free probability theory [Voi91]. The spectral distribution of the sum of independent unitarily invariant random matrices is close to the so-called free additive convolution of the spectral distributions of each original matrices, the one of the product is close to the free multiplicative convolution of the spectral distribution. Based on this theory and complex analysis, the subordination method (see [Bia98, Bel05, BB07, Voi00, BMS17]) provides us tools to compute very good approximations of the spectrum of sums and multiplications of independent random matrices in the same flavor as the multiplication of the Fourier transforms in the classical case. In the important case of the computation of large covariance matrices, the subordination method reduces to the Marchenko-Pastur equation, which lies at the heart of the nonlinear shrinkage method [LW04].

In [ATV17], Arizmendi, Vargas and the author developed an approach to the spectral deconvolution by inverting the subordination method. This approach showed promising results on simulations, and the goal of this manuscript is to show theoretically that it successfully achieves the spectral deconvolution of random matrix models in the additive and multiplicative case. We also provide first concentration bounds on the result of the deconvolution, in the vein of Fan’s results on the classical deconvolution [Fan91]. The concentration bounds we get depend on the first six moments of the spectral distribution of $A$ and $B$ in the additive case, and also on the bound of the support of the spectral distribution of $A$ in the multiplicative case.

In his first two papers dealing with deconvolution, Fan already noted that the accuracy of the deconvolution greatly worsens as the noise gets smoother, and improves as the distribution to be recovered gets smoother. This can be seen at the level of the Fourier transform approach. Indeed, the Fourier transform of a smooth noise is rapidly decreasing to zero at infinity and thus the convolution with a smooth noise sets the Fourier transform of the original distribution close to zero for higher modes, acting as a low pass filter. Hence, when the original distribution has non-trivial higher modes, it is thus extremely difficult to recover those higher frequencies in the deconvolution, which translates into a poor concentration bound on the accuracy of the process. When the original distribution is also very smooth, those higher modes do not contribute to the distribution and thus the recovery is still accurate. In the supersmooth case where the Fourier transform of the noise is decreasing exponentially to zero at infinity, the accuracy is logarithmic, except when the original distribution is also supersmooth.

In [BB04], Belinschi and Bercovici proved that the free additive and multiplicative convolutions of probability measures are always analytic, except at some exceptional points. As the spectral deconvolution is close to reversing a free convolution, we should expect the behavior of the spectral convolution to be close to the ultrasmooth case of Fan. Indeed, the method proposed in [ATV17] first builds an estimator $\widehat{C}_B$ of the convolution $C_B$ of the desired distribution by a certain Cauchy distribution, and then achieve the classical deconvolution of $\widehat{C}_B$ by this Cauchy distribution, which is a supersmooth. Therefore, the accuracy of the spectral deconvolution method should be approximately the one of a deconvolution by a Cauchy transform. We propose to measure this accuracy by two main quantities: the parameter of the Cauchy
transform involved in the first step of the deconvolution, and the size of the matrices. We show that the parameter of the Cauchy transform, which gives the range of Fourier modes we can recover, depends mainly on the intensity of the noise, while the precision of the recovery of \( C_B \) depends on the size \( N \) of the model. This is similar to the situation in the classical case \cite{Fan91}. We provide concentration bounds for the recovery of \( C_B \). These concentration bounds depend on the first six moments of the spectral distribution of \( A \) and \( B \) in the additive case, and also on the bound of the support of \( A \) in the multiplicative case.

Let us describe the organization of the manuscript. In Section 2, we explain precisely the models, recall the deconvolution procedure implemented in \cite{ATV17} and states the concentration bounds. This section is self-contained for a reader only interested in an overview of the deconvolution, and in particular the free probabilistic background is postponed to next section. The method for the multiplicative deconvolution has been improved from the one in \cite{ATV17}, and the proof of the improved version is postponed to Appendix A. We also provide simulations to illustrate the deconvolution procedure and to show how the concentration bounds compare to simulated errors. In Section 3, we introduce all necessary background to prove the concentration bounds, and we introduce matricial subordinations of Pastur and Vasilchuk \cite{PV00}, which is the main tool of our study. The proof of the concentration of the Stieltjes transform of the original measure is done in Section 4, 5. These proofs heavily rely on integration formulas and concentration bounds on the unitary groups, which are respectively described in Appendix B and C.

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2. Description of the model and statement of the results

2.1. Notations. In the sequel, \( N \) is a positive number denoting the dimension of the matrices, \( \mathbb{C} \) denotes the field of complex numbers, and \( \mathbb{C}^+ \) denotes the half-space of complex numbers with positive imaginary part. For \( K > 0 \), we denote by \( \mathbb{C}_K \) the half-space of complex numbers with imaginary part larger than \( K \).

When \( X \in \mathcal{M}_N(\mathbb{C}) \) is self-adjoint, we denote by \( X = X^+ + X^- \) the unique decomposition of \( X \) such that \( X^+ \geq 0 \) and \( X^- \leq 0 \). The matrix \( X^+ \) is called the positive part of \( X \) and \( X^- \) its negative part. We recall that the normalized trace \( \text{Tr}(X) \) of \( X \) is equal to \( \frac{1}{N} \sum_{i=1}^{N} X_{ii} \). The resolvent \( G_X \) of \( G \) is defined on \( \mathbb{C}^+ \) by

\[
G_X(z) = (X - z)^{-1}.
\]

When \( X \in \mathcal{M}_N(\mathbb{C}) \), we denote by \( \lambda_1^X, \ldots, \lambda_N^X \) its eigenvalues and by

\[
\mu_X = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i^X}
\]

its spectral distribution. We use the convention to use capital letters to denotes matrices, and corresponding small letter with index \( i \in \mathbb{N} \) to denotes the \( i \)-th moment of the corresponding spectral distribution, when it is defined. For example, if \( X \) is a Hermitian and \( i \in \mathbb{N} \), then

\[
x_i = \text{Tr}(X^i) = \frac{1}{N} \sum_{i=1}^{N} \lambda_i^X,
\]

and

\[
x_0^i = \text{Tr}((X - \text{Tr}(X))^i).
\]

Finally, we write \( x_\infty \) for the norm of \( X \).

When \( \mu \) is a probability distribution on \( \mathbb{R} \) and \( f : \mathbb{R} \to \mathbb{R} \) is a measurable function, we set \( \mu(f) = \int_{\mathbb{R}} f(t) d\mu(t) \) and we write \( \mu(k) \) for the \( k \)-th moment of \( \mu \), when it is well defined. When \( \mu \)
admits moments of order 2, we denote by $\text{Var}(\mu) = \mu(2) - \mu(1)^2$ the variance of $\mu$. The Stieltjes transform of a probability measure $\mu$ is the analytic function defined on $\mathbb{C}^+$ by

$$m_\mu(z) = \int_{\mathbb{R}} \frac{1}{t-z} d\mu(t).$$

In the special case where $\mu = \mu_X$ for some hermitian matrix $X$, we simply write $m_X$ instead of $m_{\mu_X}$.

### 2.2. Unitarily invariant model and reduction of the problem.

The main topic of this paper is the estimation of the spectral density of a matrix which is modified by an additive or multiplicative noise. We fix a hermitian matrix $B = B^* \in \mathcal{M}_N(\mathbb{C})$, the signal matrix. We denote by $\lambda_1, \ldots, \lambda_N$ its eigenvalues and by $\mu_B = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}$ its spectral distribution. Moreover, we consider another hermitian matrix $A \in \mathcal{M}_N(\mathbb{C})$, the noise matrix with $\mu_A$ random satisfying Condition 2.1. Our main assumption is that the noise is unitarily invariant. This is a sufficient condition to ensure asymptotic freeness between the unknown matrix and the noise, see Section 3. In particular, we can assume that $A$ and $B$ are diagonal. Note that we could as well assume orthogonal invariance with the same results, up to a numerical constant.

**Condition 2.1.** There exists a known probability measure $\mu_1$ with moments of order 6 and a constant $C_A > 0$ such that for any Lipschitz functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with Lipschitz constant $K$,

$$\mathbb{E}(|\mu_A(f) - \mu_1(f)|^2) \leq \frac{C_A K^2}{N^2},$$

and there exists a constant $c > 0$ such that

$$|\mu_1(i) - a_i| \leq (1 + \frac{c}{N})^i,$$

for $1 \leq i \leq 6$, where we recall that $a_i = \mu_A(i) = \text{Tr}(A^i)$. Moreover, $\mu_1(1) = 0$ in the additive case and $\mu_1(1) = 1$ in the multiplicative case.

The last condition is a simple scaling to simplify the formula of the manuscript, and the second condition is mostly technical, and can be relaxed at the cost of coarsening the concentration bounds. Indeed, we use several constants involving moment of the unknown distribution $\mu_A$, and the bounding assumption of Condition 2.1 allows to use the moments of $\mu_1$ instead. Finally, the first condition is usually satisfied in most known cases. See [GZ00] for concentration inequalities in the case where $A$ is either Wigner or Wishart. Then, we consider the additive problem

**Problem 2.2** (Additive case). Given $H = B + UAU^*$ with $A, B \in \mathcal{M}_N$ diagonal matrices, $\mu_H(1) = 0$ and $\mu_A$ random satisfying (2.1), reconstruct $\mu_B$.

and the multiplicative one,

**Problem 2.3** (Multiplicative case). Given $M = A^{1/2} UBU^* A^{1/2}$ with $A, B \in \mathcal{M}_N$ diagonal, $\mu_M(1) = 1$, $A \geq 0$ and $\mu_A$ random satisfying (2.1), reconstruct $\mu_B$.

The normalization $\mu_H(1), \mu_M(1) = 0$ can easily be removed and its only role is to simplify the formulas of the manuscript. Note that in the multiplicative case, we could consider the more general model $M = TBT^* \in \mathcal{M}_N'(\mathbb{C})$, where $T \in \mathcal{M}_{N',N}(\mathbb{C})$ is a random matrix with $N' \in \mathbb{N}$ and $B$ is hermitian without the positivity assumption. Writing $A = T^*T$, then the spectral distribution of $M$ is also equal to

$$\mu_TBT^* = \frac{N}{M} \mu_A^{1/2} BA^{1/2} + \frac{M - N}{M} \delta_0,$$

and we can up to a shift by a known constant assume that $T = A^{1/2}$. Hence, in the multiplicative case, we can assume without loss of generality that $M = A^{1/2} BA^{1/2}$ with $A \geq 0$ (not necessarily invertible). Likewise, since the positive part $M^+$ of $M$ is equal to $A^{1/2}B^+A^{1/2}$, and the negative part $M^-$ of $M$ is equal to $A^{1/2}B^-A^{1/2}$, we can directly separate the recovery of $B^+$ and $B^-$ at the level of $M$. Hence, we can assume that $B \geq 0$ and $M = A^{1/2}BA^{1/2}$ with $A \geq 0$. 


2.3. Deconvolution procedure. We now explain the deconvolution procedure leading to an estimator \( \hat{\mu}_B \) of \( \mu_B \). This deconvolution is done in two steps. The first step is to build an estimator \( \hat{C}_B[\eta] \) of the convolution \( C[\eta] = \mu_B \ast \text{Cauchy}(\eta) \) of \( \mu_B \) with a Cauchy distribution \( \text{Cauchy}(\eta) \) or parameter \( \eta \). We recall that

\[
d\text{Cauchy}[\eta](t) = \frac{1}{\pi} \frac{\eta}{t^2 + \eta^2},
\]

for \( t \in \mathbb{R} \). The estimator only exists for \( \eta \) larger that some threshold depending on the moments of the noise (and also of the data in the multiplicative case). Then, the second step is to build an estimator \( \hat{\mu}_B \) of \( \mu_B \) from \( \hat{C}_B[\eta] \) by simply doing the classical deconvolution of \( \hat{C}_B[\eta] \) by the noise \( \text{Cauchy}(\eta) \). The first step is quite new and requires complex analytic tools. In particular, the main aspect of \( C_B[\eta] \) is the Stieltjes inversion formula saying that for \( t \in \mathbb{R} \),

\[
C_B[\eta] = \frac{1}{\pi} \Re m_B(t + i\eta),
\]

where \( m_B \) is the Stieltjes transform of \( \mu_B \) introduced in Section 2.1. Hence, our main purpose will be to construct an estimator of \( \mu_B \) which exists on a certain upper half-plane \( \mathbb{C}_\eta \). In the additive case, we can simply take \( \eta = 2\sqrt{2\Var(\mu_1)} \), while the multiplicative case is more complicated, due to the higher instability of the convolution. Then, the second step consisting in a classical deconvolution by a Cauchy distribution is well-studied in the literature.

**Additive case.** Set \( \sigma_1 = \sqrt{\Var(\mu_1)} \). Then, we have the following convergence result from [ATV17].

**Theorem 2.4.** [ATV17] There exist two analytic functions \( \omega_1, \omega_3 : \mathbb{C}_{2\sqrt{\sigma_1}} \to \mathbb{C}^+ \) such that for all \( z \in \mathbb{C}_{2\sqrt{\sigma_1}} \),

- \( \Im \omega_1(z) \geq \frac{3\pi}{4}, \Im \omega_3(z) \geq \frac{3\pi}{4} \),
- \( \omega_1 + z = \omega_3 - \frac{1}{m_{\mu_1}(\omega_1(z))} = \omega_3 - \frac{1}{m_H(\omega_3(z))} \).

Moreover, setting \( h_{\mu_1}(w) = -w - \frac{1}{m_{\mu_1}(w)} \), \( \omega_3(z) \) is the unique fixed point of the function \( K_{\omega_3}(\omega) = z - h_{\mu_1}(w - \frac{1}{m_H(w)} - z) \) in \( \mathbb{C}_{3\Im(z)/4} \) and we have

\[
\omega_3(z) = \lim_{n \to \infty} K_{\omega_3}^n(w),
\]

for all \( w \in \mathbb{C}_{3/4\Im(z)} \).

An important fact is that the above theorem gives us a concrete way to build the function \( \omega_3 \) by iteration of the map \( K_{\omega_3} \). This iteration converges quickly, since it is a contraction of the considered domain. The latter theorem leads us to the construction of \( \hat{\mu}_B \).

**Definition 2.5.** The additive Cauchy estimator of \( \mu_B \) at \( t \in \mathbb{R} \) is

\[
\hat{C}_B(t) = \frac{1}{\pi} \Im \left[ m_H(\omega_3(t + 2\sqrt{2}\sigma_1i)) \right],
\]

where \( \omega_3 \) is defined in Theorem 2.4.

Let us explain the intuition behind this definition. The functions \( \omega_1, \omega_3 \) are called subdivision functions of the free deconvolution for the following reason : suppose that \( \mu_H = \mu_1 \boxplus \mu_B \) (in the sense of Section 3.2), then \( m_{\mu_B}(z) = m_H(\omega_3(z)) = m_{\mu_1}(\omega_3(z)) \) for all \( z \in \mathbb{C}_{2\sqrt{\sigma_1}} \) (see Section 3.2). We never have \( \mu_H = \mu_1 \boxplus \mu_B \), but by Theorem 3.2 \( \mu_H \simeq \mu_A \boxplus \mu_B \) and by Condition 2.1 \( \mu_A \simeq \mu_1 \); hence we have the approximate free convolution \( \mu_H \simeq \mu_1 \boxplus \mu_B \), and thus \( m_{\mu_B}(z) \simeq m_H(\omega_3) \) on \( \mathbb{C}_{2\sqrt{\sigma_1}} \). Then, taking the imaginary part gives the approximated value of \( \hat{C}_B \).
Multiplicative case. The nice property of the additive case is that the domain on which the fixed point procedure works is relatively well described by $\sigma_1$, which measures the magnitude of $\mu_1$. In the multiplicative case, the fixed point method is not so efficient (see the bound in [ATV17, Proposition 3.4]). We propose here a different approach which yields better results at a cost of increased complexity. In the multiplicative case, we are looking for subordination functions $\omega_1(z)$ and $\omega_3(z)$ satisfying the relations

$$\omega_1 z = \omega_3 \frac{\omega_3 m_H(\omega_3)}{1 + \omega_3 m_H(\omega_3)} = \omega_3 \frac{\omega_1 m_{\mu_1}(\omega_1)}{1 + \omega_1 m_{\mu_1}(\omega_1)}.$$  

Equation (1) is more unstable than in the additive case, and thus requires more moments assumption in order to be solvable on some region $\mathbb{C}_K$. Set

$$\sigma_1^2 = \mu_1(3)\mu_1(1) - \mu_1(2)^2, \quad \sigma_H = h_2 - h_1^2$$

and $\sigma_H = h_3 h_1 - h_2^2$.

Then for $t \geq 2$, define

$$g(\xi) = \xi + \frac{1}{k(\xi)} \left( 1 + \left( \frac{1}{k(\xi)} + \frac{\sigma_H^2 - \sigma_1^2}{k(\xi) \sigma_1} + \frac{\sigma_H^2}{\sigma_1^2} \frac{1}{\sigma_1} \right) \right),$$

where $k(t) = \frac{t + \sqrt{t^2 - 4}}{2}$ is real for $t \geq 2$ and greater than 1, and

$$t(\xi) = \left( \frac{\sigma_1^2}{k(\xi) \sigma_1} + \frac{\sigma_1^2 + \sigma_H^2/2}{k(\xi) \sigma_1^2} \right) \left( 1 + \frac{\sigma_1^2}{\sigma_1^2} \frac{\sigma_H^2 + \sigma_H^2/2}{\sigma_1^2 \sigma_1^2} \right).$$

The function $g$ controls the imaginary part of the multiplicative subordination function $\omega_3(z)$ by the one of $z$ (see Lemma A.3), whereas the function $t$ controls the stability of the subordination equation (1). A quick computation shows that $g'$ is strictly increasing and we write $\xi_g$ for the unique zero of $g'$ in $[2, +\infty]$. Then $g$ is strictly increasing from $[\xi_g, \infty]$ to $[g(\xi_g), \infty]$, and we can define $g^{-1}$ on $[g(\xi_g), \infty]$. Moreover, $t$ is decreasing in $\xi$ and converges to 0 as $\xi$ goes to infinity, and thus we can define $\xi_0$ as

$$\xi_0 = \inf (\xi \geq \xi_g, t(\xi) < 1).$$

**Theorem 2.6.** There exist two analytic functions $\omega_1, \omega_3 : \mathbb{C}_{g(\xi_0)\sigma_1} \to \mathbb{C}^+$ such that

$$\omega_1 z = \omega_3 \frac{\omega_3 m_H(\omega_3)}{1 + \omega_3 m_H(\omega_3)} = \omega_3 \frac{\omega_1 m_{\mu_1}(\omega_1)}{1 + \omega_1 m_{\mu_1}(\omega_1)}$$

for all $z \in \mathbb{C}_{g(\xi_0)\sigma_1}$. Moreover, setting $K_z(w) = -h_{\mu_1} \left( w^2 \frac{m_H(w)}{1 + w m_H(w)} / z' \right) z'$ for $z \in \mathbb{C}^+$ and $w \in \mathbb{C}^+$.

1. If $\Re z < -K$ with $K$ given in Lemma A.7, then

$$\omega_3(z) = \lim_{n \to \infty} K^{\infty}_z(z),$$

2. if $z \in \mathbb{C}_{g(\xi_0)\sigma_1}$, then for all $z' \in \mathbb{C}_K \cap B(z, \mathbb{R}(g^{-1}(\Im z)))$, with $\mathbb{R}(g^{-1}(\Im z)) > 0$ given in (46),

$$\omega_3(z') = \lim_{n \to \infty} K^{\infty}_z(\omega_3(z)).$$

Hence, we can construct $\omega_3$ on $\mathbb{C}_{g(\xi_0)}$ by first applying the first fixed point procedure of the latter theorem for real parts small enough, and then move to increasing real parts with the second fixed point procedure. The proof of this theorem is postponed to Appendix A. We deduce from the latter theorem a definition of $\tilde{C}_B[\eta]$.

**Definition 2.7.** The multiplicative Cauchy estimator of $\mu_B$ at $\eta > g(\xi_0)$ is the function $\tilde{C}_B[\eta]$ whose value at $t \in \mathbb{R}$ is

$$\tilde{C}_B[\eta](t) = \frac{1}{\pi} \Im \left[ \omega_3(t + i\eta) m_H(\omega_3(t + i\eta)) \right],$$

where $\omega_3$, $g$ and $\xi_0$ are defined above.

A intuitive justification of this construction can be given like in the additive case. The main difference with the additive case is that assuming $\mu_3 = \mu_1 \boxdot \mu_2$, the subordination function $\omega_3(z)$ satisfies the more complicated relation $\omega_3(z) m_{\mu_1}(\omega_3(z)) = zm_{\mu_2}(z)$. 
Estimating the distribution $\mu_2$. The last step is to recover $\mu_2$ from $\hat{C}_B[\eta]$, which is a classical deconvolution of $\hat{C}_B[\eta]$ by a Cauchy distribution. This is a classical problem in statistic which has been deeply studied since the pioneering work of Fan [Fan91]. The main aspect of this problem is the supersmooth aspect of the Cauchy distribution. In particular, the convergence of the deconvolution may be very slow depending on the smoothness of the original measure. There are two main situations, which are solved differently:

- the original measure $\mu_B$ is sparse, meaning that it consists of few atoms. In this case, one solves the deconvolution problem by solving the Beurling LASSO problem

$$
\hat{\mu}_2 = \arg \min_{\mu \in M(\mathbb{R})} \| \mu \ast \text{Cauchy}(\eta) - \hat{C}_B[\eta] \|_{L^2}^2 + \lambda \mu(\mathbb{R}),
$$

where $M(\mathbb{R})$ denotes the space of positive measures on $\mathbb{R}$, and $\lambda > 0$ is a parameter to tune depending on $MSE(N)$ (see [DP17] for more information on the choice of $\lambda$). This minimization problem can be solved by a constrained quadratic programming method (see [BV04]). The constraints of the domain on which the minimization is achieved actually enforce the sparsity of the solution.

- the original $\mu_B$ is close to a probability distribution with a density $f$ in $L^2(\mathbb{R})$: in this case, it is better to take a Fourier approach. The convolution of $f$ by a Cauchy distribution $\text{Cauchy}(\eta)$ on $L^2(\mathbb{R})$ is a multiplication of $F(f)$ by the map $\xi \mapsto e^{-\eta|\xi|}$. Hence, a naive estimator of $f$ would be to consider the estimator $\hat{f} = F^{-1}(H_\eta)$, where $H_\eta(\xi) = e^{\eta|\xi|}F(C_B[\eta])$. This estimator does not work properly due to the fast divergence of the map $\xi \mapsto e^{\eta|\xi|}$. A usual way to circumvent this problem is to consider instead the estimator

$$
\hat{f} = F^{-1}(K_\epsilon H_\eta),
$$

where $K_\epsilon$ is a regularizing kernel depending on a parameter $\epsilon$ to choose. For example, one can simply take $k_\epsilon = 1_{[-\epsilon, \epsilon]}$ with $\epsilon$ a function of $\eta$ and $\mathbb{E} \left\| \hat{C}_B[\eta] - C_B[\eta] \right\|_{L^2}^2$. The regularizing kernel allows to reduce the instability in the higher modes of the Fourier transform, at the cost of losing some information on the density to estimate. See [Lac06] for an explicit method to choose $\epsilon$ given $\eta$ and the bound on $\mathbb{E} \left\| \hat{C}_B[\eta] - C_B[\eta] \right\|_{L^2}^2$ that is provided in the next section. Several more advanced techniques (see for example [Huy10] for density with compact support) can also be used for refined results.

2.4. Concentration bounds. Recall that $C_B[\eta] = \mu_B \ast \text{Cauchy}(\eta)$. We now state the concentration bounds for the estimators we constructed before. Our inequalities involve moments of $\mu_1$ and $B$ up to order 6 in the additive case, and also the bound on $A$ in the multiplicative case.

**Theorem 2.8** (Additive case). Suppose that $N^2 \geq C_{\text{threshold}}$, with

$$
C_{\text{threshold}} = \frac{2^2 \max(C_{\text{thres},A}(3\sigma_1/\sqrt{2}), C_{\text{thres},B}(3\sigma_1/\sqrt{2}))}{3^3 \sqrt{2} \sigma_1^3}.
$$

Then,

$$
MSE := \mathbb{E} \left( \| \hat{C}_B - C_B \|_{L^2}^2 \right) \leq \frac{K_1(\eta, N)}{N^2} + \frac{K_2(\eta, N)}{N^3} + \frac{K_3(\eta, N)}{N^4},
$$

with

$$
K_1(\eta, N) = \frac{C_A \sqrt{C_D(N)^2} + 2 \sqrt{2} C_2(\sigma_1)}{4 \sqrt{2} \pi \sigma_1^3} + \frac{C_3(N)^2}{3^2 \sigma_1} \left( 1 + \frac{2 \sigma_1^2 \sigma_1 + a_4}{3^2 \sigma_1^4} \right),
$$

$$
K_2(\eta, N) = \frac{C_1(N)}{2 \pi \sqrt{2} \sigma_1} \left( \sqrt{C_A \sqrt{C_D(N)^2} + 2 \sqrt{2} C_3(N)} \right) + \frac{C_3(N)}{3} \left( 1 + \frac{2 \sigma_1^2 \sigma_1 + a_4}{3^2 \sigma_1^4} \right),
$$

and

$$
K_3(\eta, N) = \frac{C_1(N)^2}{2 \pi \sqrt{2} \sigma_1},
$$

with $C_1(N)$, $C_2(N)$ and $C_3(N)$ respectively given in (31), (32) and (33), and $C_{\text{thres},A}$, $C_{\text{thres},B}$ given in Proposition 4.5.
In the multiplicative case, we have the following concentration bound which holds for any \( \eta > g(\xi_0)\sigma_1 \).

**Theorem 2.9** (Multiplicative case). Let \( \eta = \kappa\sigma_1 \) with \( \kappa > g(\xi_0)\sigma_1 \), and suppose that \( N^2 \geq C_{\text{threshold}} \), with

\[
C_{\text{threshold}} = \max(C_{\text{thres},A}, C_{\text{thres},B}) \left( 1 - \frac{1}{k \circ g^{-1}(\kappa)} \right) \frac{3\kappa}{\xi^3\sigma_1^2}.
\]

Then,

\[
MSE := \mathbb{E}(\|\hat{\mathcal{C}}_B[\eta] - \mathcal{C}_B[\eta]\|_{L^2}) \leq \frac{K_1(\eta)}{N^2} + \frac{K_2(\eta)}{N^3} + \frac{K_3(\eta)}{N^4} + \frac{K_4(\eta)}{N^6},
\]

with

\[
K_1(\eta) = \frac{2}{\pi \kappa \sigma_1} \left( \frac{3^4 C_2(\eta)^2 C_A}{2^4 g^{-1}(\kappa) \xi^4 \sigma_1^4 N^2} + \frac{C_3(\eta)^2 \Delta(\kappa)}{g^{-1}(\kappa) \xi^2 \sigma_1^2} \right),
\]

\[
K_2(\eta) = \frac{2 C_1(\eta)}{\pi \eta} \left( \frac{9 \sqrt{C_A C_2(\eta)}}{4 g^{-1}(\kappa) \xi^2 \sigma_1^2} + \frac{C_3(\eta) \sqrt{\Delta(\kappa)}}{g^{-1}(\kappa) \xi \sigma_1} \right),
\]

\[
K_3(\eta) = \frac{C_1(\eta)^2}{\pi \eta},
\]

and

\[
K_4(\eta) = \frac{2^3 \max(C_{\text{thres},A}, C_{\text{thres},B})^3}{3^{3/2}(\xi \sigma_1)^3} \left( 1 - \frac{1}{k g^{-1}(\kappa)} \right) \left( \kappa \sigma_1 + 1 + \frac{\beta_2}{\kappa \sigma_1} \right)^2
\]

where \( C_1(N), C_2(N) \) and \( C_3(N) \) are respectively given in (36), (37), and (38), \( C_{\text{thres},A}, C_{\text{thres},B} \) are given in Proposition 4.9 and Proposition 4.10 and \( \Delta(\kappa) \) is given in (39).

2.5. **Accuracy of the classical deconvolution.** Concentration properties of the classical deconvolution are already known, in the atomic or in the continuous case. We quickly review some general results in this framework, since we will deeper study this question in a separate paper [JT].

- In the atomic case, the precision of the deconvolution depends on the number \( m \) of atoms and on the minimum separation \( t = \min\{|x - x'|, x, x' \in \text{Supp}(\mu_2)\} \) between atoms. There exist then constants \( C(\eta, m), \Delta(\eta) \) such that for \( t \geq \Delta(\eta) \) (see [DP17, Ben17]),

\[
\mathbb{E}(\|\hat{f} - \mu_2\|_{W_2(\mathbb{R})}) \leq C(\eta, m) \frac{MSE}{mt^{4m-2}},
\]

where \( W_2 \) denotes the Wasserstein distance. Two important remarks have to be done on the limitations of this result. First, the exponent \( m \) in the error term shows that the recovery of \( \mu_2 \) is very inaccurate when \( m \) is large, whence the sparsity hypothesis of the data. This can directly be seen at the level of the deconvolution procedure [4], since the minimization of the \( L^1 \)-norm yields a result with few atoms. More importantly, the threshold \( \Delta \) is up to a constant the inverse of the Nyquist frequency of a low pass filter with a cut-off in the frequency domain around \( \frac{1}{\eta} \). Hence, the resolution of the deconvolution depends dramatically on the imaginary line \( i\eta \) on which the first step of the deconvolution is done. This limit can be overcome when we assume that the signal is clustered around a certain value, see [DDP17] for such results for the recovery of positive measures.

- In the continuous case, Fan already gave in [Fan92] first bounds for the deconvolution by supersmooth noise, when the expected density \( f \) of \( \mu_2 \) is assumed \( C^k \) for some \( k > 0 \). Due to the exponential decay of the Fourier transform of the noise, the rate of convergence is logarithmic. Later, Lacour [Lac06] proved that choosing appropriately the parameter \( \epsilon \) in the deconvolution procedure lead to a power decay in the case where the density is analytic, with an exponent depending on the complex domain on which the density can be analytically extended. This yields the following inequality, from whom the accuracy of the deconvolution can be deduced; suppose that \( d_{W_2}(\mu_2, \mu_f) \leq \delta \), with \( \mu_f \) being a probability distribution with density \( f \). Then,
(1) If $f$ is $C^k$, with $\|f^{(k)}\|_{L^2} \leq K$, then there exists $C(K, \eta)$ such that

$$d_{W_2}(\mu_1, \mu_2) \leq \delta + \frac{C(K, \eta)}{\log (\|\widehat{C}_B[\eta] - C_B[\eta]\|_{L^2}^2 + \frac{\delta^2}{\eta^2})^{k-1/2}}.$$ 

(2) If $f$ can be analytically extended to the complex strip \( \{x + iy, -a < y < a \} \), and $\|f(\cdot + iy)\|_{L^2} \leq K$ for all $-a < y < a$, then there exists $C(a, K, \eta)$ such that

$$d_{W_2}(\mu_1, \mu_2)^2 \leq \delta + C(a, K, \eta) \|\widehat{C}_B[\eta] - C_B[\eta]\|_{L^2}^2 + \frac{\delta^2}{\eta^4}a^{\frac{n}{2n+n-1}},$$

and a mean squared estimate can be deduced from the above bound. Improved bounds also exist when more regularity is assumed (see [Lac06, Theorem 3.1]).

From example, if $\mu_2$ is the discretization of the Gaussian density, so that $\delta \approx \frac{1}{N}$, then $d_{W_2}(\mu_1, \mu_2)$ shrinks almost linearly with $\|\widehat{C}_B[\eta] - \mu_B \ast \text{Cauchy}(\eta)\|_{L^2}$.

2.6. Simulations. We provide here some simulations to show the accuracy and limits of the concentration bounds we found on the mean squared error in Section 2.4. In the additive and multiplicative cases, we take an example, perform the first step of the deconvolution as explained in Section 2.3 and compute the error with $\widehat{C}_B[\eta]$, and then compare this error with the constant we computed according to the formulas in Theorem 2.8 and Theorem 2.9.

Additive case. We consider a data matrix $B$ which is diagonal with iid entries following a real standard Gaussian distribution, and a noise matrix $A$ which follows a GUE distribution (namely, $A = (X + X^*)/\sqrt{2}$, with the entries of $X$ iid following a complex centered distribution with variance $1/N$). Hence, $\mu_A$ is close to a standard semi-circular distribution $\mu_1$ in the sense of Condition 2.1. Then, we consider the additive model $H = B + UAU^*$ (even if the presence of $U$ is redundant, since the distribution of $A$ is already unitarily invariant). We performed the iteration procedure explained in Theorem 2.4 at $\eta = 2\sqrt{2}\sigma_1 = 2\sqrt{2}$. In Figure 1, we show an example of the spectral distribution of $H$, the result of the first step of the deconvolution, and then the result of the deconvolution after the classical deconvolution by a Cauchy distribution (we used a constrained Tychonov method see [Neu88]), and a comparison with $\mu_B$.

![Figure 1](image-url)

**Figure 1.** Histogram of the eigenvalues of $H$, result of the first step of the deconvolution, result of the second step of the deconvolution and comparison with the histogram of $\mu_B$ ($N = 500$).

The result is very accurate, which is not surprising due to the analyticity property of the Gaussian distribution (see the discussion in Section 2.5). Then, we simulate $\sqrt{\hat{MSE}}$ with a sampling of deconvolutions with the size $N$ going from 50 to 2000. The lower bound on $N$ for the validity of Theorem 2.8 is 4, which is directly satisfied. We can then compare the simulated standard deviation to the square root of the bound given in Theorem 2.8. This gives a measure of the error we made on the estimation of the standard deviation, which is displayed in Figure 2. The first diagram is a graph of the estimated square root of $MSE$ and the second one is the graph of the theoretical constant we computed according to $N$. The third graph is a ratio of both quantities according to $N$. 


We see that the error on the bound is better when $N$ is larger. When $N$ is small, the terms $K_2N^{-3}$ and $K_3N^{-4}$ are non negligible, and approximations in the concentration results of the subordination function in Section 4 contribute to this higher ratio. When $N$ gets larger, the only term remaining is $K_1N^{-2}$ and the ratio between the theoretical constant and the estimated error gets better.

**Multiplicative case.** In the multiplicative case, we consider for the data matrix a shifted Wigner matrix $B = (X + X^*)/(2\sqrt{2}) + 1$, with the entries of $X$ iid following a complex centered distribution with variance $1/N$. Hence, $\mu_B$ is close to a semicircular distribution with center 1 and variance $1/4$. Then, we consider a noise matrix $A = YY^*$, with $Y$ a square matrix of size $N$ iid following a complex centered distribution with variance $1/N$. Hence, $\mu_A$ is close to a Marchenko-Pastur distribution $\mu_1$ with parameter 1 in the sense of Condition 2.1. Then, we consider the multiplicative model $M = A^{1/2}UBU^*A^{1/2}$ and we apply the deconvolution procedure explained in Section 2.3. First, we compute $\xi_0 = 3.5$ and then $\eta_0 = g(\xi_0)\tilde{\sigma}_1 = 4.1$. Remark that this constant is quite sharp, since in this example the fixed point procedure converged until $\eta \simeq 3.6$. In Figure 3, we show an example of such a deconvolution, with the histogram of the eigenvalues of $M$, the first and second steps of the deconvolution and a comparison with $\mu_B$. Like in the additive case, the result is accurate thanks to the good analyticity property of the semi-circular distribution.

![Figure 2](image1.png)

**Figure 2.** Simulation of $\sqrt{MSE}$ in the additive case for $N$ from 50 to 2000 (with a sampling of size 100 for each size), theoretical bound on $\sqrt{MSE}$ provided in Theorem 2.8 and ratio of the theoretical bound on the simulated error.

![Figure 3](image2.png)

**Figure 3.** Histogram of the eigenvalues of $M$, result of the first step of the deconvolution, result of the second step of the deconvolution and comparison with the histogram $\mu_B$ ($N = 500$).

Then, we do the same study than in the additive case. The lower bound on $N$ given in Theorem 2.9 is in our case 72, hence we chose to compare the theoretical and simulated deviation for $N$ going from 100 to 2000. This gives the result depicted in Figure 4 (we follow the same convention than in the additive case).

The result is similar to the additive case, with constant which are worse for small $N$. This can be explained by the more complicated study of the multiplicative case, which induce additional approximations.
3. Unitarily invariant model and free convolution

We introduce here necessary background for the proof of the theorems of this manuscript.

3.1. Probability measures, cumulants and analytic transformations. Let \( \mu \) be a probability measure on \( \mathbb{R} \). When \( X \) is a self-adjoint matrix and \( \mu_X \) is corresponding spectral measure, we thus have \( \mu_X(k) = \text{Tr}(X^k) \).

**Free cumulants.** Throughout this manuscript, free probability theory will be present without being really mention. In particular, several quantities involve free cumulants of probability measures and mixed moments of free random variables, which have been introduced by Speicher in \cite{Spe94}. Since we will only use moments of low orders, we won’t develop the general theory of free cumulants and the interested reader should refer to \cite{NS06} for more information on the subject, in particular for having the non crossing partitions picture explaining the formulas below.

The free cumulant of order \( r \) of \( \mu \) is denoted by \( k_r(\mu) \). In this paper, we use only the first three free cumulants, which are the following:

\[
k_1(\mu) = \mu(1), \quad k_2(\mu) = \text{Var}(\mu) = \mu(2) - \mu(1)^2, \quad k_3(\mu) = \mu(3) - 3\mu(2)\mu(1) + 2\mu(1)^3.
\]

If \( \mu, \mu' \) are two probability measures on \( \mathbb{R} \) and \( \vec{k}, \vec{k}' \in \mathbb{N}' \), with \( r > 0 \) we denote by \( m_{\mu,\mu'}(\vec{k}, \vec{k}') \) the mixed moments of \( \mu_1, \mu_2 \) when they are assumed in free position (see \cite{NS06} for more background on free random variables). Once again, we only need the formula for few value of \( \vec{k}, \vec{k}' \), which are as follow:

\[
m_{\mu,\mu'}(k, k') = \mu(k)\mu'(k'),
\]

\[
m_{\mu,\mu'}(k_1 k_2, k'_1 k'_2) = \mu(k_1 + k_2)\mu'(k'_1)\mu'(k'_2) + \mu(k_1)\mu(k_2)\mu'(k'_1 + k'_2) - \mu(k_1)\mu(k_2)\mu'(k'_1)\mu'(k'_2),
\]

and

\[
m_{\mu,\mu'}(k_1 k_2 k_3, 1^3) = \mu'(1)^3 \mu(k_1 + k_2 + k_3) + \mu'(1) \text{Var}(\mu')(\mu(k_1 + k_2)\mu(k_3) + \mu(k_2 + k_3)\mu(k_1) + \mu(k_3 + k_1)\mu(k_2)) + k_3(\mu')\mu(k_1)\mu(k_2)\mu(k_3).
\]

By abuse of notation, we simply write \( k_r(X) \) for \( k_r(\mu_X) \) and \( m_{X,X'}(\vec{k}, \vec{k}') \) for \( m_{\mu_X,\mu_{X'}}(\vec{k}, \vec{k}') \), when \( X, X' \) are self-adjoint matrices.

**Analytic transform of probability distributions.** The Stieltjes transform of a probability distribution \( \mu \) is the analytic function \( m_\mu : \mathbb{C}^+ \to \mathbb{C} \) defined by the formula

\[
m_\mu(z) = \int_{\mathbb{R}} \frac{1}{t-z} d\mu(t), \quad z \in \mathbb{C}^+.
\]
We can recover a distribution from its Stieltjes transform through the Stieltjes Inversion formula, which gives
\[ d\mu(t) = \frac{1}{\pi} \lim_{y \to 0} \Im m_{\mu}(t + iy) \]
in a weak sense. We will mostly explore spectral distributions through their Stieltjes transforms, since the latter have very good analytical properties. The first important property is that \( m_{\mu}(\mathbb{C}^+) \subset \mathbb{C}^+ \). Actually, Nevanlinna’s theory provides a reciprocal result.

**Theorem 3.1.** [MS17, Theorem 3.10] Suppose that \( m: \mathbb{C}^+ \to \mathbb{C}^+ \) is such that
\[ -\text{sym}(iy) \xrightarrow{n \to \infty} 1, \]
then there exists a probability measure \( \rho \) such that \( m = m_{\rho} \).

We will use the following transforms of \( m_{\mu} \), whose given properties are direct consequences of Nevanlinna’s theorem and the expansion at infinity \( m_{\mu}(z) = -\sum_{k=0}^{\infty} \frac{\mu(r)}{z^{k+1}} + o(z^{-(r+2)}) \), when \( \mu \) admits moments of order \( r \).

- the reciprocal Cauchy transform of \( \mu \), \( F_{\mu} : \mathbb{C}^+ \to \mathbb{C}^+ \) with \( F_{\mu}(z) = \frac{1}{m_{\mu}(z)} \). If \( \mu \) admits moments of order two, we have the following important formula [MS17, Lemma 3.20], which will be used throughout the paper,
  \[ F_{\mu}(z) = z - \mu(1) + \text{Var}(\mu)m_{\mu}(z), \]
  for some probability measure \( \rho \). In particular,
  \[ \Im[F_{\mu}(z)] \geq \Im(z). \]

When \( \mu \) admits a moment of order three, then \( \rho \) has a moment of order one which is given by the formula
\[ \rho(1) = \mu(3) - 2\mu(1)\mu(2) + \mu(3)^2. \]

- the \( h \)-transform of \( \mu \), \( h_{\mu} : \mathbb{C}^+ \to \mathbb{C}^+ \) and \( h_{\mu}(z) = \text{Var}(\mu)m_{\mu}(z) - \mu(1) \) for \( z \in \mathbb{C}^+ \).

**Probability measure with positive support.** Suppose that \( \mu \) has a positive support; up to a rescaling, we can assume that \( \mu(1) = 1 \). Then several new analytic transformation will be useful in the sequel. Note first that
\[ \tilde{m}_{\mu}(z) := 1 + zm_{\mu}(z) = \int_{\mathbb{R}^+} \frac{t}{1+t-z} d\mu(t) = \int_{\mathbb{R}^+} \frac{1}{1+t-z} d\tilde{\mu}(t), \]
with \( \tilde{\mu} \) being the probability measure absolutely continuous with respect to \( \mu \) with density \( d\tilde{\mu}(t) = t d\mu(t) \). Moments of \( \tilde{\mu} \) are directly related to moments of \( \mu \) by the relation \( \tilde{\mu}(k) = \mu(k+1) \). In particular,
\[ \text{Var}(\tilde{\mu}) = \tilde{\mu}(2) - \tilde{\mu}(1) = \tilde{\mu}(3) - \tilde{\mu}(2)^2. \]

We denote by \( \tilde{F}_{\mu} \) the reciprocal Cauchy transform of \( \tilde{\mu} \), and set
\[ \tilde{F}_{\mu} := 1 + \tilde{F}_{\mu}. \]

Remark that \( \tilde{F}_{\mu} \) is again the reciprocal Cauchy transform of a measure \( \tilde{\mu} \) with support on \( \mathbb{R}^+ \). Indeed, \( -\frac{1}{\tilde{F}_{\mu}} \) takes values in \( \mathbb{C}^+ \) and \( -\frac{1}{\tilde{F}_{\mu}} \sim \frac{1}{z} \) as \( z \) goes to infinity, so by Theorem 3.1, there exists a measure \( \hat{\mu} \) such that \( \frac{-1}{\tilde{F}_{\mu}} = m_{\hat{\mu}} \). Moreover, at \( t_0 < 0 \),
\[ \tilde{F}_{\mu}(t_0) = \frac{-1}{m_{\mu}(t_0)} = \frac{-1}{\int_{\mathbb{R}^+} \frac{1}{1+t_0} d\tilde{\mu}(t)} > \frac{-1}{\int_{\mathbb{R}^+} \frac{1}{1+t} d\tilde{\mu}(t)}. \]
By Jensen’s inequality, \( \frac{1}{\int_{\mathbb{R}^+} \frac{1}{1+t} d\tilde{\mu}(t)} < 1 \), thus \( \tilde{F}_{\mu}(t_0) > -1 \) and \( \tilde{F}_{\mu}(t_0) > 0 \). Hence, \( \frac{-1}{\tilde{F}_{\mu}} \) extends continuously on \( \mathbb{R}_{<0} \) with values in \( \mathbb{R} \), which by Stieltjes inversion formula implies that \( \tilde{\mu}(\mathbb{R}_{<0}) = 0 \). Actually, \( \tilde{F}_{\mu} \) is related to \( h_{\mu} \) by the relation
\[ \frac{z}{\tilde{F}_{\mu}(z)} = 1 - \frac{1}{1+zm_{\mu}(z)} = \frac{1}{zm_{\mu}(z)} = z - F_{\mu}(z) = -h_{\mu}(z). \]
We finally introduce a last transformation which is useful in the multiplicative case. When $\mu$ is a probability measure on $\mathbb{R}^+$ with $\mu(1) = 1$, introduce the $h$-transform $L_\mu$ on $\mathbb{C}^+$ as

$$L_\mu(z) = -\log(-h_\mu(z)).$$

Since $h_\mu$ takes values in $\mathbb{C}^+$, $L_\mu(\mathbb{C}^+) \subset \mathbb{C}^+$. By [10], as $z$ goes to infinity, $h_\mu(z) = -\mu(1) - \frac{\mu(2) - \mu(1)^2}{z} + \frac{\mu(3) - 2\mu(1)\mu(2) + \mu(1)^3}{z^2} + o(z^{-2})$, so that using $\mu(1) = 1$, as $z$ goes to infinity

$$\log -h_\mu = \frac{\mu(2) - \mu(1)^2}{z} + \frac{\mu(3) - 2\mu(1)\mu(2) + \mu(1)^3 - (\mu(2) - \mu(1)^2)^2/2}{z^2} + o(z^2)$$

$$= \frac{\text{Var}(\mu)}{z} + \frac{\text{Var}(\tilde{\mu}) + \text{Var}(\mu)^2/2}{z^2} + o(z^2).$$

Thus, by Theorem 3.1 there exists a probability measure $\rho_L$ with mean $\frac{\text{Var}(\tilde{\mu}) + \text{Var}(\mu)^2/2}{\text{Var}(\mu)}$ such that

$$L_\mu(z) = \text{Var}(\mu)m_{\rho_L}(z).$$

3.2. Free convolution of measures. From the seminal work of Voiculescu [Vo91], it is known that as $n$ goes to infinity while the support of $\mu_A$ and $\mu_X$ remain bounded, the spectral distribution of $M$ converges in probability to a deterministic measure called the free additive (resp. multiplicative) convolution of $\mu_A$ and $\mu_X$ and denoted by $\mu_A \boxplus \mu_X$ (resp. $\mu_A \boxtimes \mu_X$). For more background on free convolutions and their relation with random matrices, see [MS17]. In this manuscript, we will only use the following characterization of free additive/multiplicative convolution, called the subordination method. This method has been fully developed by [BB07, Bel05], after having been introduced by [Bia98] and [Voi00].

- Suppose that $\mu_1 \boxplus \mu_2 = \mu_3$. Then, for $z \in \mathbb{C}^+$, we have $m_{\mu_3}(z) = m_{\mu_2}(\omega_2(z)) = m_{\mu_1}(\omega_1(z))$, where $\omega_2(z)$ is the unique fixed point of the function $K_z : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ given by

$$K_z(w) = h_{\mu_1}(h_{\mu_2}(w) + z) + z,$$

and $\omega_1$ and $\omega_2$ satisfy the relation

$$\omega_1(z) + \omega_2(z) = z - \frac{1}{m_{\mu_1}(z)}. \tag{11}$$

Moreover, we have

$$\omega_2(z) = \lim_{n \to \infty} K_z^{\otimes n}(w)$$

for all $w \in \mathbb{C}^+$.

- Suppose that $\mu_1 \boxtimes \mu_2 = \mu_3$. Then, for $z \in \mathbb{C}^+$, we have $\hat{m}_{\mu_3}(z) = \hat{m}_{\mu_2}(\omega_2(z)) = \hat{m}_{\mu_1}(\omega_1(z))$, where $\omega_2(z)$ is the unique fixed point of the function $H_z : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ given by

$$H_z(w) = -\frac{z}{h_{\mu_1}(\hat{m}_{\mu_2}(w))},$$

and $\omega_1(z)$ and $\omega_2(z)$ satisfy the relation

$$\omega_1(z)\omega_2(z) = z \frac{zm_{\mu_3}(z)}{1 + zm_{\mu_3}(z)} = z\hat{F}_{\mu_3}(z). \tag{12}$$

Moreover, we have

$$\omega_2(z) = \lim_{n \to \infty} H_z^{\otimes n}(w)$$

for all $w \in \mathbb{C}^+$.

These two iterative procedures should be understand as the main implementation scheme for concrete applications, whereas the fixed point equations give the precise definition of both convolutions. Two fundamental results relating free probability to random matrices are the convergence of the spectral distribution of sums or products of random matrices conjugated by Haar unitaries towards free additive or multiplicative convolutions.
Theorem 3.2. \([\text{Vol91, Spe93, PV00, Vas01}]\) Suppose that \((A_N, B_N)_{N \geq 0}\) are two sequences of matrices, with \(A_N, B_N \in M_N(\mathbb{C})\) self-adjoint, and let \(U_N\) be a random unitary matrix distributed according to the Haar state. Then, if \(\mu_{A_N} \rightarrow \mu_1\) and \(\mu_{B_N} \rightarrow \mu_2\) with \(\sup_N (\max(\mu_{A_N}(2), \mu_{B_N}(2))) < +\infty\), then

\[\mu_{A_N+UB_NU^*} \quad \text{a.s.

in the } W^*\text{- topology} \quad \mu_1 \boxplus \mu_2,\]

and, assuming \(A \geq 0, \)

\[\mu_{A^{1/2}UB_NU^*A^{1/2}} \quad \text{a.s.

in the } W^*\text{- topology} \quad \mu_1 \boxtimes \mu_2.\]

Since those original results, several progresses have been made towards a better comprehension of the above convergences. Let us mention particularly results of \([\text{BES17, Kar15}]\) in the additive case, which prove the convergence of the spectral norm in a much stronger topology, establishing the local laws of the spectral distribution up to an optimal scale (see also \([\text{EKN20}]\) for polynomial of matrices). Let us mention also the recent results of \([\text{BGH20}]\), which establish a large deviations principle for the convergence of the spectral distribution in the additive case.

3.3. Matrix subordination. In \([PV00]\), Pastur and Vasilchuk noticed that, since the asymptotic spectral behavior of the addition/multiplication of matrices is close to a free additive/multiplicative convolution, and the latter are described by subordination functions, there may exist subordination functions directly at the level of random matrices. They actually found such subordination functions and use them to study the convergence of the spectral distribution of the matrix models towards the free convolution. This approach is in particular fundamental to remove the boundedness assumption of \(\mu_1\) and \(\mu_2\) in Theorem 3.2. In \([\text{Kar12, Kar15}]\), Kargin greatly improved the method to provide concentration bounds for the additive convolution when the support of \(\mu_A\) and \(\mu_B\) remain bounded.

The goal of Section 4 is to improve Kargin’s result in the additive case by removing the boundedness assumption and computing explicit bounds, and to provide similar results in the multiplicative case. We review here the matricial subordinations functions in the additive and multiplicative case. Note that we modify the subordination function of \([\text{Vas01}]\) in the multiplicative case to get better control on the convergence towards the free multiplicative convolution.

**Additive case.** For \(w \in \mathbb{C}^+\), set \(f_A(w) = \text{Tr}(A G_H(w))\) and \(f_B(w) = \text{Tr}(B G_H(w))\). Then, set

\[\omega_A(w) = z - \frac{\mathbb{E}(f_B(w))}{\mathbb{E}(m_H(w))}, \quad \omega_B(w) = z - \frac{\mathbb{E}(f_A(z))}{\mathbb{E}(m_H(z))}.\]

A crucial point is that

\[\omega_A(z) + \omega_B(z) = z - \frac{1}{\mathbb{E}m_H(z)},\]

which is the same relation as the one satisfied by the subordination functions for the free additive convolution in \([11]\). After a small modification of Kargin’s formulation \([\text{Kar15}]\), we get the following approximate subordination relation.

**Lemma 3.3.** For \(z \in \mathbb{C}^+\),

\[\mathbb{E}G_H(z) = G_A(\omega_A(z)) + R_A(z),\]

with \(R_A(z) := \frac{1}{\mathbb{E}m_H}G_A(\omega_A(z))\mathbb{E}A,\) and

\[\Delta_A(z) = \mathbb{E}((m_H - \mathbb{E}m_H)(U B U^* G_H - \mathbb{E}(B G_H)) - (f_B - \mathbb{E}(f_B))(G_H - \mathbb{E}(G_H))).\]

Moreover, \(\mathbb{E}\Delta_A\) is diagonal and \(\text{Tr}\mathbb{E}\Delta_A = 0.\)

**Proof.** By \([\text{Kar15}]\) Eqs. (12), (13),

\[\mathbb{E}U G_H(z) = G_A(\omega_A(z)) + R_A(z),\]

with \(R_A(z) := \frac{1}{\mathbb{E}m_H}G_A(\omega_A(z))(A - z)\mathbb{E}U \tilde{\Delta}_A,\) and

\[\tilde{\Delta}_A = -(m_H - \mathbb{E}m_H)G_H - (f_B - \mathbb{E}(f_B))G_A G_H.\]
Since \((A - z)\) is deterministic, \((A - z)\mathbb{E}\hat{\Delta}_A = \mathbb{E}[(A - z)\Delta_A]\), and we have
\[
(A - z)\mathbb{E}\hat{\Delta}_A = \mathbb{E}(-(m_H - \mathbb{E}m_H)(A - z)G_H - (f_B - \mathbb{E}(f_B))G_H)
\]
\[
= \mathbb{E}(-m_H - \mathbb{E}m_H)(1 - UBU^*G_H) - (f_B - \mathbb{E}(f_B))G_H)
\]
\[
= \mathbb{E}((m_H - \mathbb{E}m_H)(UBU^*G_H - \mathbb{E}(UBU^*G_H)) - (f_B - \mathbb{E}(f_B))(G_H - \mathbb{E}(G_H))
\]
\[
:= \mathbb{E}\Delta_A.
\]

where we have used on the penultimate step that \(\mathbb{E}(X - \mathbb{E}(X)) = 0\) for any random variable \(X\). This proves the first part of the lemma. For the second part, note that if \(V\) is any diagonal unitary matrix, setting \(U = VU\) and using the invariance of the Haar measure and the fact that \(VAV^* = A\) yields that
\[
V\mathbb{E}((m_H - \mathbb{E}m_H)G_H) = V\mathbb{E}((\text{Tr}((A + UBU^* - z)^{-1}) - \mathbb{E}m_H)(A + UBU^* - z)^{-1})
\]
\[
= V\mathbb{E}((\text{Tr}(V^*(VAV^* + VUBU^*V^* - z)^{-1}V) - \mathbb{E}m_H)
\]
\[
(VAV^* + VUBU^*V^* - z)^{-1}V)
\]
\[
= \mathbb{E}((\text{Tr}((A + UBU^*V^* - z)^{-1}) - \mathbb{E}m_H)V^*(A + VBU^*V^* - z)^{-1})V)
\]
\[
= \mathbb{E}((\text{Tr}((A + UBU^* - z)^{-1}) - \mathbb{E}m_H)(A + UBU^* - z)^{-1})V,
\]
where we used the trace property on the third equality. Likewise,
\[
V\mathbb{E}((f_B - \mathbb{E}(f_B))G_AG_H) = \mathbb{E}((f_B - \mathbb{E}(f_B))G_AG_H)V,
\]
and thus \(V\) commutes with \(\mathbb{E}\hat{\Delta}_A\). Since \(\mathbb{E}\hat{\Delta}_A\) commutes with any diagonal unitary matrix, it is also diagonal, and so is \(\mathbb{E}\Delta_A = (A - z)\mathbb{E}\hat{\Delta}_A\). Finally,
\[
\text{Tr}(\Delta_A) = \mathbb{E}((m_H - \mathbb{E}m_H) \text{Tr}(UBU^*G_H - \mathbb{E}(UBU^*G_H)) - (f_B - \mathbb{E}(f_B)) \text{Tr}(G_H - \mathbb{E}(G_H)))
\]
\[
= \mathbb{E}((m_H - \mathbb{E}m_H)(f_B - \mathbb{E}(f_B)) - (f_B - \mathbb{E}(f_B))(m_H - \mathbb{E}m_H)) = 0.
\]

Moreover, an algebraic manipulation yields that
\[
\omega_A = A - (\mathbb{E}G_H)^{-1}(-\mathbb{E}G_H)^{-1}1 - \mathbb{E}U\Delta_A,
\]
Following [Kar15, Lemma 2.1] (see also Lemma 4.3), we have also
\[
-(\mathbb{E}G_H)^{-1} + A - z \in \mathbb{H}(M_n(\mathbb{C})),
\]
where \(\mathbb{H}(M_n(\mathbb{C}))\) denotes the half-space \(\{M \in M_n(\mathbb{C}), \frac{1}{z}(M - M^*) \geq 0\}\).

**Multiplicative case.** This section adapts Kargin’s approach to the multiplicative case. Matricial subordination functions already appeared in the multiplicative case in [Vas01], but we chose to create new matricial subordination functions which are closer to the ones encoding the free multiplicative convolution in Section 3.

Recall here that \(M = A^{1/2}UBU^*A^{1/2}\), \(m_M(z) = \text{Tr}((M - z)^{-1})\) and \(\tilde{m}_M(z) = \text{Tr}(M(M - z)^{-1}) = 1 + zm_M(z)\). Like in the additive case, we define \(f_A(z) = \text{Tr}(A(M - z)^{-1})\) and introduce for \(z \in \mathbb{C}^+\) the subordination functions
\[
\omega_A = \frac{z\mathbb{E}f_A(z)}{\mathbb{E}\tilde{m}_M(z)}, \quad \omega_B = \frac{z\mathbb{E}m_M(z)}{\mathbb{E}f_A(z)}.
\]

Remark that there is an asymmetry between \(\omega_A\) and \(\omega_B\), which reflects the different role played by \(A\), and \(B\) in \(M\). This symmetry can be restored by studying \(AUBU^*\) instead of \(A^{1/2}UBU^*A^{1/2}\) at the cost of loosing the self-adjointness. The two subordination functions however still satisfy the relation
\[
\omega_A(z)\omega_B(z) = \frac{z\mathbb{E}m_M(z)}{1 + z\mathbb{E}m_M(z)},
\]
which is similar to [12].
Lemma 3.4. For $z \in \mathbb{C}^+$,

$$\mathbb{E}(MG_M(z)) = AG_A(\omega_A(z)) + R_A(z),$$

with $R_A(z) = \omega_A(z)G_A(\omega_A(z))\Delta_A$, where

$$\Delta_A(z) = \frac{z}{\mathbb{E}(f_A(z))} \mathbb{E}((f_A(z) - \mathbb{E}(f_A(z)))G_M - (m_M(z) - \mathbb{E}(m_M(z)))AG_M).$$

Similarly, setting $M' = B^{1/2}UABU^*B^{1/2}$,

$$\mathbb{E}(M'G_{M'}) = BG_B(\omega_B(z)) + R_B(z),$$

with $R_B(z) = BG_B(\omega_B(z))\Delta_B$, where

$$\Delta_B = \frac{z}{\mathbb{E}(f_A(z))}(-(f_A - \mathbb{E}(f_A))G_{M'} + (m_M - \mathbb{E}(m_M))UA^{1/2}G_MA^{1/2}U^*).$$

Moreover, $\mathbb{E}\Delta_A$ and $\mathbb{E}\Delta_B$ are diagonal and $\mathbb{E}\text{Tr}\Delta_A = \mathbb{E}\text{Tr}\Delta_B = 0$.

Proof. This lemma is deduced from Lemma 3.3. Suppose first that $A$ is invertible. Then, we have

$$G_M(z) = (A^{1/2}UBU^*A^{1/2} - z^{-1}) = (A^{1/2}(-zA^{-1} + UBU^*)A^{1/2})^{-1} = A^{1/2}(UBU^* - zA^{-1})^{-1}A^{-1/2}.$$

Set $\tilde{A} = -zA^{-1}$, $\tilde{B} = B$. Set $\sigma = \sup(\Delta(\text{Spec}(-zA^{-1}))$, and $\tilde{M} = \tilde{A} + UBU^*$. Applying (3.3) to $\tilde{A}, \tilde{B}, w \in \mathbb{C}_\sigma$ instead of $A, B$ yields that

$$\mathbb{E}G_{\tilde{M}}(w) = G_{\tilde{A}}(\omega_{\tilde{A}}(w)) + R_{\tilde{A}}(\omega_{\tilde{A}}(w)),$$

where $\omega_{\tilde{A}}$ and $R_{\tilde{A}}$ are respectively defined by

$$\omega_{\tilde{A}}(w) = w - \frac{\mathbb{E}(f_{\tilde{B}}(w))}{\mathbb{E}(m_{\tilde{M}}(w))}$$

with

$$f_{\tilde{B}}(w) = \text{Tr}(U\tilde{B}U^*G_{\tilde{M}}(z)).$$

and

$$R_{\tilde{A}} = \frac{1}{\mathbb{E}(m_{\tilde{M}})}G_{\tilde{A}}(\omega_{\tilde{A}}(\tilde{A} - w)\mathbb{E}_{U}\Delta_{\tilde{A}},$$

where

$$\Delta_{\tilde{A}} = -(m_{\tilde{M}} - \mathbb{E}(m_{\tilde{M}}))G_{\tilde{M}} - (f_{\tilde{B}} - \mathbb{E}(f_{\tilde{B}}))G_{\tilde{A}}G_{\tilde{M}}.$$

First, since $z \in \mathbb{C}^+$ and $A > 0$, $\sup(\Delta(\text{Spec}(-zA^{-1})) < 0$. Hence $0 \in \mathbb{C}_\sigma$, and

$$f_{\tilde{B}}(0) = 1 + \text{Tr}(zA^{-1}(UBU^* - zA^{-1}))^{-1}$$

$$= 1 + z\text{Tr}(A^{-1/2}(UBU^* - zA^{-1})^{-1}A^{-1/2})$$

$$= 1 + zm_M(z),$$

where we have used (21) in the last equality. Similarly,

$$m_{\tilde{M}}(0) = \text{Tr}((UBU^* - zA^{-1})^{-1}) = \text{Tr}(A^{1/2}G_M(z)A^{1/2}) = f_A(z).$$

Hence,

$$\omega_{\tilde{A}}(0) = -\frac{\mathbb{E}(f_{\tilde{B}}(0))}{\mathbb{E}(m_{\tilde{M}}(0))} = -\frac{\mathbb{E}(m_{\tilde{M}}(z))}{\mathbb{E}(\text{Tr}(A^{1/2}G_M(z)A^{1/2}))} = -z\omega_A(z)^{-1},$$

and, using again (21),

$$\Delta_{\tilde{A}} = -(f_A(z) - \mathbb{E}(f_A(z)))(UBU^* - zA^{-1})^{-1} - (zm_M(z) - \mathbb{E}(zm_M(z)))(z^{-1}A)(UBU^* - zA^{-1})^{-1}$$

$$= -(f_A(z) - \mathbb{E}(f_A(z))A^{1/2}(A^{1/2}UBU^*A^{1/2} - z^{-1})A^{1/2} + z^{-1}(zm_M(z) - \mathbb{E}(zm_M(z)))AA^{1/2}G_MA^{1/2}$$

$$= A^{1/2}(m_M(z) - \mathbb{E}(m_M(z)))AG_M - (f_A(z) - \mathbb{E}(f_A(z))(A^{1/2}UBU^*A^{1/2} - z^{-1})A^{1/2}).$$
Putting the latter expression in (22) and using (21) gives then
\[ R_A = \frac{1}{\mathbb{E}(f_A(\omega))} (-zA^{-1} + z\omega(z)^{-1} - zA^{-1}) \]
\[ A^{1/2} \mathbb{E} \left( (m_M(z) - \mathbb{E}(m_M(z)))AG_M - (f_A(z) - \mathbb{E}(f_A(z)))G_M \right) A^{1/2} \]
\[ = \frac{1}{\mathbb{E}(f_A(z))} (A^{-1} - \omega(z)^{-1}) A^{1/2} \]
\[ \mathbb{E} \left( (m_M(z) - \mathbb{E}(m_M(z)))AG_M - (f_A(z) - \mathbb{E}(f_A(z)))G_M \right) A^{1/2}. \]

Putting the latter expression in (22) and using (21) gives then
\[ \mathbb{E}(G_M) = A^{-1/2}G_M(z)A^{-1/2} \]
\[ = A^{-1/2}(-zA^{-1} + z\omega(z)^{-1}) A^{-1/2} + A^{-1/2}R_A(z)A^{-1/2} \]
\[ = z^{-1}\omega(z)(A - \omega(z))^{-1} \frac{\omega(z)}{\mathbb{E}(f_A(z))} (A - \omega(z))^{-1} \mathbb{E}((f_A(z) - \mathbb{E}(f_A(z)))G_M \]
\[ - (m_M(z) - \mathbb{E}(m_M(z)))AG_M). \]

Hence, we get
\[ (23) \quad z\mathbb{E}(G_M) = \omega(z)G_M(\omega(z)) + \omega(z)G_A(\omega(z))\Delta_A = \omega(z)G_A(\omega(z))(1 + \Delta_A), \]
with
\[ \Delta_A = \frac{z}{\mathbb{E}(f_A(z))} \mathbb{E}((f_A(z) - \mathbb{E}(f_A(z)))G_M - (m_M(z) - \mathbb{E}(m_M(z)))AG_M). \]

Finally, we have
\[ \mathbb{E}(M(G_M(z))) = 1 + z\mathbb{E}(G_M(z)) = 1 + \omega_A G_A(\omega_A) + R_A(z) = AG_{A\omega}(\omega_A) + R_A(z). \]

Let us do the same computation for the subordination involving \( \omega_B \). Using the subordination on \( B \) for \( B = zU_A^{-1}U^* \) together with (21) yields
\[ \mathbb{E}(U_{A^{1/2}}G_M A^{1/2}U^*) = (B - \omega_B)^{-1} + R_B(z), \]
with \( \omega_B(z) = \frac{-f_A(0)}{m_M(0)} = \frac{z\mathbb{E}(m_M(z))}{\mathbb{E}(f_A(z))} \) and \( R_B(z) = G_B(\omega_B)B\mathbb{E}\Delta_B \)
\[ \tilde{\Delta}_B = \frac{1}{\mathbb{E}(f_A(z))} \left( -(f_A - \mathbb{E}f_A)A^{1/2}G_MA^{1/2}U^* + z(m_M - \mathbb{E}m_M)B^{-1}U_{A^{1/2}}G_MA^{1/2}U^* \right). \]

Since \( B^{1/2}U_{A^{1/2}}G_M A^{1/2}U^* B^{1/2} = B^{1/2}\tilde{A}B^{1/2}G_M = 1 + zG_M, \) we get
\[ B^{1/2}\mathbb{E}\tilde{\Delta}_BB^{1/2} = \frac{z}{\mathbb{E}(f_A(z))} \left( -(f_A - \mathbb{E}f_A)G_M + (m_M - \mathbb{E}m_M)U_{A^{1/2}}G_MA^{1/2}U^* \right). \]

Hence,
\[ B^{1/2}\mathbb{E}(U_{A^{1/2}}G_M A^{1/2}U^*)B^{1/2} = \mathbb{E}(M'G_M') = BG_B(\omega_B) + R_B(z), \]
with \( R_B(z) = BG_B(\omega_B)B\mathbb{E}\tilde{\Delta}_B, \) where
\[ \Delta_B = \frac{z}{\mathbb{E}(f_A(z))} \left( -(f_A - \mathbb{E}f_A)G_M + (m_M - \mathbb{E}m_M)U_{A^{1/2}}G_MA^{1/2}U^* \right). \]

The proof that \( \text{Tr} (\Delta_A) = \text{Tr} (\Delta_B) = 0 \) and that \( \Delta_A, \Delta_B \) are diagonal is then the same as in Lemma 3.3

In order to end the proof, it remains to deal with the case where \( A \) is non invertible. Let \( z \in \mathbb{C}^+ \) be fixed. Then,
\[ \Phi(A) = z\mathbb{E}(G_M(z)) - w_A G_A(w_A) - R_A(z) \]
is a map from \( M_n(\mathbb{C}) \approx \mathbb{R}^{2n^2} \) to \( \mathbb{R}^{2n^2} \). By [Vas01, Proposition 3.1], \( A \mapsto G_A(z) \) is Lipschitz with Lipschitz constant \( \frac{1}{\mathfrak{m}(z)^2} \). Hence, \( \Phi(A) \) is rational expression of continuous functions of \( A \). In order to prove continuity of \( \Phi \), it suffices therefore to prove that no denominator vanishes when

A is non zero. When checking each term in \( \Phi \), the only non trivial ones are \( 1 + z\mathbb{E}m_M(z) \) and \( \mathbb{E}f_M(z) \). First, expanding \( m_M \) yields
\[
m_M(z) = -\frac{1}{z} - \frac{\mathbb{E}(\text{Tr}(A^{1/2}UBU^*A^{1/2}))}{z^2} + \frac{\text{Tr}(A^{1/2}UBU^*A^{1/2})}{z^3} + o(z^{-3})
\]
at infinity. Moreover, set \( v \in \mathbb{C}^n \) be such that \( A^{1/2}v = w \not= 0 \). Then, \( U^*v \) is uniformly distributed on the sphere of radius \( |w| \), and thus \( \langle BU^*w, U^*w \rangle \) is almost surely non-zero (provided \( B \) is non-zero). Hence, \( A^{1/2}UBU^*A^{1/2} \) is almost surely non-zero, which implies that
\[
\text{Tr}(UBU^*A^{1/2}) > 0
\]
which implies that \( m_M \) is almost-surely not equal to the function \( z \mapsto z^{-1} \). Hence, for all \( z \in \mathbb{C}^+ \), and thus \( \Im(1 + zm_M(z)) > 0 \) almost surely, and after averaging \( 1 + zm_M(z) \) never vanishes on \( \mathbb{C}^+ \). The function \( f_M \) is analytic from \( \mathbb{C}^+ \) to \( \mathbb{C}^+ \), and \( f_M = \frac{\text{Tr}(A)}{z} + o(z) \) at infinity, thus there exists a finite positive measure \( \rho \) on \( \mathbb{R} \) such that
\[
f_M(z) = \int_{\mathbb{R}} \frac{1}{t-z} d\rho(z).
\]
Therefore, \( \Im(f_M(z)) > 0 \) almost surely for \( z \in \mathbb{C}^+ \), which yields that \( \mathbb{E}(f_M(z)) \) never vanishes.

Remark that rearranging terms in (19) yields
\[
\omega_A A = A^2 - (A + \omega_A \Delta_A) (A \mathbb{E}[MG_M]^{-1}),
\]
where \( A \mathbb{E}[MG_M]^{-1} = \mathbb{E}[UB^{1/2}G_M B^{1/2}U^*]^{-1} \) is always defined (see Lemma 4.8). Likewise, rearranging terms in (20) yields
\[
\omega_B(z) = B - B(\mathbb{E}M'G_M')^{-1} + B\Delta_B(\mathbb{E}M'G_M')^{-1}.
\]

4. Bound on the subordination method

We have seen in the previous section that matricial subordination function already satisfy exactly the same relations as for the subordinations functions for the free convolutions. In this section we estimate the subordination property of these function. Namely we show that \( \mathbb{E}m_H \) (resp. \( \mathbb{E}m_M \)) and \( m_A(\omega_A) \) or \( m_B(\omega_B) \) (resp. \( \tilde{m}_A \) or \( \tilde{m}_B \)) are approximately the same. In the additive case, this has been already done in [Kar13]; hence the goal of the study of the additive case is just giving precise estimate in the approach of Kargin. Up to our knowledge, the multiplicative case has not been done with the subordination approach of Kargin (see however [EKN20] for similar result for general polynomial in Wigner matrices).

4.1. Subordination in the additive case. The goal of this subsection is to prove the following convergence result.

**Proposition 4.1.** For
\[
N \geq \sqrt{\max(C_{\text{thres},A}(\eta), C_{\text{thres},B}(\eta))},
\]
with \( C_{\text{thres},A}(\eta) \) given in Proposition 4.5 then \( \Im \omega_A \geq 2\eta/3, \Im \omega_B \geq 2\eta/3 \) and
\[
|m_H(z) - m_{\mu_1}(\omega_A(z))| \leq \frac{C_{\text{bound},A}(\eta)}{|z|N^2},
\]
and
\[
|m_H(z) - m_{\mu_2}(\omega_B(z))| \leq \frac{C_{\text{bound},B}(\eta)}{|z|N^2},
\]
with
\[
C_{\text{bound},A}(\eta) = 36 \frac{\sigma_2^2 \sigma_1}{\eta^3} \left(1 + \frac{\sigma_1^2}{\eta^2} + \frac{\sigma_2^2}{\eta^2}\right) \left(1 + \frac{\sigma_1^2}{\eta^2} + \frac{b_4}{\eta^2 \sigma_2^2}\right) \left(1 + \sqrt{a_4 \mu_1(\mathbb{E}[B^{1/2}]^2, 12^2) + b_6^{2/3} \alpha_6^{1/3}}\right),
\]
and \( C_{\text{bound},B} \) obtained from \( C_{\text{bound},A} \) by switching \( A \) and \( B \).
The latter is a direct consequence of the following proposition with $\alpha = 1/2$, adding the error term $\delta_A$ and $\delta_H$.

**Proposition 4.2.** For $z \in \mathbb{C}^+$, and suppose that $N \geq \sqrt{\frac{1}{\eta^2}}$, with $C_{\text{thres}, A}(\eta)$ given in Proposition 4.3, then $3 \omega_A \geq 2\eta/3$ and

$$|E(m_H) - m_A(\omega_A(z))| \leq \frac{C_{\text{bound}, A}(\eta)}{|z|N^2},$$

with

$$C_{\text{bound}, A}(\eta) = 36 \frac{\sigma_2^2 \sigma_1}{\eta^3} \left(1 + \frac{\sigma_1^2 + \sigma_2^2}{\eta^2} + \frac{b_4}{\eta^2 \sigma_2^2} \right) \left(1 + \frac{\sqrt{a_4 m_{\mu(A^2)\|B^2\|}(1,1^2)} + b_6/\eta^{1/3}}{\sigma_2^2 \sigma_1^2 \eta^2} \right).$$

We postpone the proof of Proposition 4.2 to the end of the section, proving some intermediary steps. First, using the fact that $E \text{Tr}((A + U B U^*)^2) = \text{Tr}(A^2) + \text{Tr}(B^2) + 2 \text{Tr}(A) \text{Tr}(B) = \text{Tr}(A^2) + \text{Tr}(B^2)$, by (6),

$$|E(m_H(z)) + z| \leq \frac{\text{Tr}((A + U B U^*)^2)}{3(z)} \leq \frac{\mu_2(A) + \mu_2(B)}{3(z)},$$

for all $z \in \mathbb{C}^+$. We can obtain a similar bound for $(E(G_H))^{-1}$, as next lemma shows.

**Lemma 4.3.** The matrix $E(G_H)^{-1}$ is diagonal and satisfies the bound

$$|E(G_H)^{-1}_{ii} - A + z| \leq \frac{\mu_2(B)}{\eta}.$$

**Proof.** We know by Lemma 3.3 that $E(G_H)$ commutes with $A$, and thus is diagonal. Define the map $I : \mathbb{C}^+ \rightarrow \mathbb{C}$ as $I(z) = -E(G_H)^{-1}_{ii} = -E(G_H)_{ii}^{-1}$. By (6), $I$ maps $\mathbb{C}^+$ to $\mathbb{C}^+$. Moreover, as $z$ goes to infinity, $G_H = -z^{-1} + E(A + U B U^*)z^{-2} - E(A + U B U^*)^2 z^{-3} + o(z^{-3}).$

A quick computation yields that $E(U B U^*) = \text{Tr}(B) = 0$ and

$$E((A + U B U^*)^2) = A^2 + E(U B U^*)A + A E(U B U^*) + E(U B^2 U^*) = A^2 + \mu_2(B).$$

Hence,

$$E(G_H)_{ii} = -z^{-1} + \lambda_i^A z^{-2} - ((\lambda_i^A)^2 + \mu_2(B))z^{-3} + o(z^{-3}).$$

Applying Theorem 3.3 to the map $I$ yields the existence of a probability measure $\rho$ on $\mathbb{R}$ such that

$$(-E(G_H)_{ii}^{-1}) = z - \lambda_i^A + \mu_2(B) \int_{\mathbb{R}} \frac{1}{z - t} d\rho(t).$$

In particular,

$$E(G_H)_{ii}^{-1} + z - \lambda_i^A \leq \frac{\mu_2(B)}{\eta}.$$

We now provide a bound on $T \Delta_A$ for $T \in M_\mu(\mathbb{C})$.

**Lemma 4.4.** For $T \in M_\mu(\mathbb{C})$,

$$|E \text{Tr}(A \Delta_A)| \leq \frac{8 \text{Tr}(B^2)\|TA\|_\infty}{\eta^4 N^2},$$

$$|E \text{Tr}(T \Delta_A)| \leq \frac{8 \text{Tr}(B^2)\|TA\|_\infty}{\eta^4 N^2},$$

and

$$|E \text{Tr}(T \Delta_B)| \leq \frac{4 \|A\|_\infty}{\eta^4 N^2} (\sqrt{\text{Tr}(A^2)\text{Tr}(B^2)} + \|A\|_\infty \sqrt{\text{Tr}(B^2)\text{Tr}(T^2)}),$$

and in case (II),

$$|E \text{Tr}(T \Delta_A)| \leq \frac{4 \text{Tr}(B^2)^{1/4}}{\eta^4 N^2} \left[\sqrt{\text{Tr}(B^2)} \text{Tr}(T A^4)^{1/4} + \text{Tr}(B^4)^{1/4} \text{Tr}(T^4)^{1/4} \text{Tr}(A^4)^{1/4}\right],$$

and in case (III),

$$|E \text{Tr}(T \Delta_A)| \leq \frac{8 \text{Tr}(B^2)^{1/4}}{\eta^4 N^2} \left[\sqrt{\text{Tr}(B^2)} \text{Tr}(T A^4)^{1/4} + \text{Tr}(B^4)^{1/4} \text{Tr}(T^4)^{1/4} \text{Tr}(A^4)^{1/4}\right].$$
and
\[ |\mathbb{E} \text{Tr}(T\Delta_B)| \leq \frac{4 \text{Tr}(A^4)^{1/4}}{\eta^4 N^2} \left( \sqrt{\text{Tr}(A^2)} \text{Tr}((TB)^4)^{1/4} + (\text{Tr}(A^4) \text{Tr}(B^4) \text{Tr}(T^4))^{1/4} \right). \]

**Proof.** Using the definition of \( \Delta_A \), we get
\[
- \text{Tr}(T\Delta_A) = (m_H - \mathbb{E} m_H)T(A - z)G_H + (f_B - \mathbb{E} f_B)TG_H \\
= (m_H - \mathbb{E} m_H)f_{T_A} - z(m_H - \mathbb{E} m_H)f_T + (f_B - \mathbb{E} f_B)f_T.
\]

Using the fact that \( zm_H = f_A + f_B - 1 \) yields finally
\[
- \text{Tr}(T\Delta_A) = (m_H - \mathbb{E} m_H)f_{T_A} - f_T(f_A - \mathbb{E} f_A) - f_T(f_B - \mathbb{E} f_B) + (f_B - \mathbb{E} f_B)f_T \\
= (m_H - \mathbb{E} m_H)f_{T_A} - f_T(f_A - \mathbb{E} f_A).
\]

Then, Lemma C.3 and Cauchy-Schwartz inequality give
\[
\mathbb{E} |\text{Tr}(T\Delta_A)| \leq \sqrt{\text{Var} f_{T_A} \text{Var} m_H} + \sqrt{\text{Var} f_A \text{Var} f_T} \\
= \frac{4}{\eta^4 N^2} \left[ \sqrt{\text{Tr}(B^2)} \min((\text{Tr}(A^4) \text{Tr}((TA)^4))^{1/4}, \sqrt{\text{Tr}(B^2)} \| T \|_\infty) \\
+ \min(\sqrt{\text{Tr}(A^4)}(\text{Tr}(T^4) \text{Tr}(A^4))^{1/4}, \sqrt{\text{Tr}(B^2)}(\text{Tr}(T^4))^{1/4}, \sqrt{\text{Tr}(B^2) \text{Tr}(T^2)} \| A \|_\infty^2) \right].
\]

Doing the same for \( \Delta_B \) yields
\[
\mathbb{E} |\text{Tr}(T\Delta_B)| \leq \sqrt{\text{Var} f_{T_B} \text{Var} m_H} + \sqrt{\text{Var} f_B \text{Var} f_T} \\
= \frac{4}{\eta^4 N^2} \left[ \sqrt{\text{Tr}(A^2)} \min((\text{Tr}(A^4) \text{Tr}((TB)^4))^{1/4}, \sqrt{\text{Tr}(B^2)} \| A \|_\infty) \\
+ \min(\sqrt{\text{Tr}(A^4)}(\text{Tr}(T^4) \text{Tr}(A^4))^{1/4}, \sqrt{\text{Tr}(B^2)}(\text{Tr}(T^4))^{1/4}, \sqrt{\text{Tr}(B^2) \text{Tr}(T^2)} \| B \|_\infty^2) \right].
\]

We deduce the following bound on the subordination functions \( \omega_A, \omega_B \).

**Proposition 4.5.** Let \( z \in \mathbb{C} \) with \( \Im(z) := \eta \). Then,
\[
|\omega_A - z| \leq \frac{\text{Tr}(B^2)}{\eta} + \frac{C_{\mathrm{thres}, A}}{3N^2 \eta^2},
\]
and
\[
\Im \omega_A \geq \eta - \frac{C_{\mathrm{thres}, A}}{3N^2 \eta^2},
\]

with
\[
C_{\mathrm{thres}, A}(\eta) = 12 \sqrt{3} \sigma_1^2 \left( 1 + \frac{\sigma_1^2 + \sigma_2^2}{\eta^2} \right) \left( \sqrt{1 + \frac{\sigma_1^2}{\eta^2} + \frac{b_1}{\sigma_2^2 \eta^2}} + \frac{\theta_B^2 \theta_A^2}{\eta^2 \sigma_1^2} \right) + \frac{\sigma_2^2}{\eta^2} \left( \frac{\theta_B^2 a_4^{1/4}}{4} \right) + \frac{\sigma_2^2}{\eta^2} \left( \frac{\theta_B^2 a_4^{1/4}}{4} \right) \left( \sqrt{1 + \frac{m_{\cal A}^2 \sigma_2^2 (1^2, 1^2)^{1/2} a_4^{1/4} + b_6^{2/3} a_6^{1/3}}{\eta^2 \sigma_1^2 \sigma_2^2}} \right).
\]

**Proof.** We modify the original proof of Kargin to best explicit bound as possible. Indeed, we already know that \( \mathbb{E}(G_H) \) commutes with \( A \), and by Lemma 3.3,
\[
\mathbb{E} (\Delta_A) = \mathbb{E} (m_H)(A - \omega_A(z))(\mathbb{E} G_H - G_A(\omega_A(z))),
\]
Then, using Lemma C.4 and switching with \( \epsilon_1 \in \mathbb{H}_2 \) by [16] and \( |(\epsilon_1)| \leq \frac{\mu_2(B)}{3N} \). Hence, taking the trace yields

\[
\omega_A = z + \text{Tr}(\epsilon_1) + \delta,
\]

with \( \delta = \text{Tr}((-E G_H)^{-1}\frac{1}{\text{Em}_H}\text{E}_U \Delta_A) \) and \( \text{Tr}(\epsilon_1) \in \mathbb{C}^- \). By [26], \( \frac{1}{\text{Em}_H} = z + \epsilon_2 \) with \( |\epsilon_2| \leq \frac{\mu_2(A) + \mu_2(B)}{3N} \). Therefore,

\[
\delta = \text{Tr}((A - z + \epsilon_1)(z + \epsilon_2)\text{E}_U(\Delta_A)) \]
\[
= \text{Tr}((z + \epsilon_2)(A + \epsilon_1)\text{E}_U(\Delta_A) - z(z + \epsilon_2)\text{E}_U(\Delta_A))
\]
\[
=(1 + \epsilon_2/z)\text{E}_U[\text{Tr}((A + \epsilon_1)(z\Delta_A))],
\]

where we used that \( \text{Tr}(E \Delta_A) = 0 \). By Lemma [4.4] we have

\[
|z \text{Tr}((A + \epsilon_1)(\Delta_A))| \leq \sqrt{\text{Var}(f_A)\text{Var}(zm_H)} + \sqrt{\text{Var}(f_B)\text{Var}(zf_A)} + \sqrt{\text{Var}(f_{e_1}A)\text{Var}(zm_H)} + \sqrt{\text{Var}(f_{e_1})\text{Var}(zf_A)} \leq \sqrt{\text{Var}(zm_H)}(\sqrt{\text{Var}(f_A)} + \sqrt{\text{Var}(f_{e_1}A)}) + \sqrt{\text{Var}(zf_A)}(\sqrt{\text{Var}(f_B)} + \sqrt{\text{Var}(f_{e_1})}).
\]

Then, using Lemma C.4 and switching \( A \) and \( B \) yields

\[
\text{Var}(zm_H) \leq \frac{12}{N^2\eta^2} \left( \text{Tr}(A^2) + \frac{\text{Tr}(A^2)\text{Tr}(B^2) + \text{Tr}(A^4)}{\eta^2} \right),
\]

and with \( \alpha_1, \beta_1 = 4 \) and \( \alpha_2, \beta_2 = 6 \),

\[
\text{Var}(zf_A) \leq \frac{12}{N^2\eta^2} \left( \text{Tr}(A^2)\text{Tr}(B^2) + \frac{\text{E}(\text{Tr}((A B^2 A)^2))^1/2\text{Tr}(A^4)^1/2 + \text{Tr}(B^6)^2/3\text{Tr}(A^6)^1/3)}{\eta^2} \right)
\]
\[
\leq \frac{12}{N^2\eta^2} \left( \text{Tr}(A^2)\text{Tr}(B^2) + \frac{(m_0^2 B^2)(12)^1/2\text{Tr}(A^4)^1/2 + \text{Tr}(B^6)^2/3\text{Tr}(A^6)^1/3)}{\eta^2} \right).
\]

Then, by Lemma C.3

\[
\text{Var}(f_{e_1}) \leq \frac{4||\epsilon_1||_N^2\text{Tr}(B^2)}{\eta^4 N^2} \leq \frac{4\text{Tr}(B^2)^3}{N^2\eta^6},
\]

\[
\text{Var}(f_{e_1}A) \leq \frac{4\text{Tr}(B^2)^2}{N^2\eta^6} \sqrt{\text{Tr}(B^4)}\text{Tr}(A^4),
\]

and

\[
\text{Var}(f_B) \leq \frac{4\sqrt{\text{Tr}(B^4)}\text{Tr}(A^4)}{\eta^4 N^2}.
\]

Finally, by Lemma C.5

\[
\text{Var}(f_A) \leq \frac{4}{N^2\eta^4} \left( \eta^2 \left( \text{Tr}(A^2)\text{Tr}(B^2) + \sqrt{\text{Tr}(A^4)\text{Tr}(B^4)} \right) + \sqrt{m_0^2 B^2(12)^1/2}\text{Tr}(B^4)^{1/4} \right).
\]
Hence, applying Lemma 4.4 yields

\[
|z \text{ Tr}((A + \epsilon_1)(\Delta_A))| \\
\leq \frac{4\sqrt{3} \text{ Tr}(B^2)}{N^2\eta^4} \sqrt{\eta^2 + \text{Tr}(A^2) + \text{Tr}(B^4)/\text{Tr}(B^2)} \\
+ \frac{4\sqrt{3} \text{ Tr}(B^2)}{N^2\eta^4} \left( \frac{\text{Tr}(B^2)}{\eta} + \frac{(\text{Tr}(B^4) \text{Tr}(A^4)^{1/4})}{\sqrt{\text{Tr}(B^2)}} \right) \\
+ \frac{4\sqrt{3} \text{ Tr}(B^2)}{N^2\eta^4} \sqrt{\eta^2 \text{Tr}(A^2) + \frac{(m_A^2B^2(1^2, 1^2)^{1/2} \text{Tr}(A^4)^{1/2} + \text{Tr}(B^6)^{2/3} \text{Tr}(A^6)^{1/3}}{\text{Tr}(B^2)}} \\
\leq \frac{4\sqrt{3} \sigma_2^2}{N^2\eta^4} \left( \sqrt{\eta^2 + \alpha^2_1 + \frac{\text{Tr}(B^4)}{\sigma_2^2} \eta^2 \sqrt{\eta^2 \alpha_2^4 + \theta_B^2 \text{Tr}(A^4)^{1/2} + \theta_B^4 \sqrt{m_A^2\sigma_2^4(1^2, 1^2)} + \sigma_2^4 \theta_B^2 \text{Tr}(A^4)^{1/2}} \\
+ \alpha_2^2 \sqrt{\eta^2 \alpha_2^4 + \frac{(m_A^2\sigma_2^4(1^2, 1^2)^{1/2} \text{Tr}(A^4)^{1/2} + \text{Tr}(B^6)^{2/3} \text{Tr}(A^6)^{1/3}}{\text{Tr}(B^2)}} \\
\leq \frac{4\sqrt{3} \sigma_2^2 \alpha_1 \sigma_1 C}{N^2\eta^2},
\]

with

\[
C = \left(1 + \frac{\alpha_2^2}{\eta^2} \right) \left( \frac{\sigma_1^2 + \alpha_2^2}{\eta^2} + \frac{b_4}{\sigma_2^4 \eta^2} \right) \left(1 + \frac{\theta_B^2 \alpha_2^4 \sigma_1^2 \alpha_1 C}{\eta^2 \sigma_2^4} + \frac{\alpha_2^4}{\eta^2 \sigma_1^4 \sigma_2^4} \theta_B^2 \alpha_2^4 \right) \\
+ \left(1 + \frac{\theta_B^2 \alpha_2^4 \sigma_1^2 \alpha_1 C}{\eta^2 \sigma_2^4} \right) \left(1 + \frac{(m_A^2\sigma_2^4(1^2, 1^2)^{1/2} \text{Tr}(A^4)^{1/2} + \text{Tr}(B^6)^{2/3} \text{Tr}(A^6)^{1/3}}{\eta^2 \sigma_2^4 \sigma_1^2} \right).
\]

The two bounds of the statement are deduced from the latter expressions and [27].

**Proof.** By Lemma 3.3, we have to estimate \( \text{Tr}(R_A(z)) = \frac{1}{\text{Em}_H} \text{Tr}(G_A(\omega_A)E_U \Delta_A) \). By Proposition 4.5, for \( N \geq \sqrt{12\sqrt{3}\sigma_2^4 \alpha_1 \sigma_1 C \text{hres}_A \alpha_1 C / \alpha_4} \), \( \Im(\omega_A) \geq 2\eta/3 \), which yields

\[
\|G_A(\omega_A)\|_\infty \leq \frac{3}{2\eta}.
\]

Hence, applying Lemma 4.4 yields

\[
|\text{Tr}(R_A(z))| = \left| \frac{1}{\text{Em}_H} \text{Tr}(G_A(\omega_A)E_U \Delta_A) \right| \\
\leq \frac{1}{|z^2 \text{Em}_H(z)|} \left( \text{Var}(zm_H) \text{Var}(zf_{AGA}(\omega_A)) + \sqrt{(\text{Var}(zf_A) \text{Var}(zf_{GA}(\omega_A))) \right} \\
\leq \frac{3}{|z| \cdot |zm_H(z)|} \text{Var}(zf_A) \text{Var}(zm_H).
\]

By Lemma C.4, we get

\[
\text{Var}(zm_H) \leq \frac{12}{N^2\eta^4} (\text{Tr}(B^2) \eta^2 + \text{Tr}(B^2) \text{Tr}(A^2) + \text{Tr}(B^4)),
\]
and
\[ \text{Var}(zf_A) \leq \frac{12}{N^2 \eta^4} \left( \text{Tr}(A^2) \text{Tr}(B^2) \eta^2 + (m_{A \otimes B^2}(1^2, 1^2)^{1/2} \text{Tr}(A^4)^{1/2} + \text{Tr}(B^6)^{2/3} \text{Tr}(A^6)^{1/3}) \right). \]

Hence,
\[ \sqrt{\text{Var}(zf_A) \text{Var}(zm_H)} \leq \frac{12 \text{Tr}(B^2)}{N^2 \eta^4} \sqrt{\eta^2 + \text{Tr}(A^2) + \text{Tr}(B^4)/\text{Tr}(B^2)} \]
\[ \frac{1}{\eta^2} \sqrt{\text{Tr}(A^2) \eta^2 + \left( m_{A \otimes B^2}(1^2, 1^2)^{1/2} \text{Tr}(A^4)^{1/2} + \text{Tr}(B^6)^{2/3} \text{Tr}(A^6)^{1/3} \right) \text{Tr}(B^2) \text{Tr}(A^2) \eta^2}. \]

Hence, using (26) yields
\[ |\text{Tr}(R_A(z))| \leq \frac{C_{\text{bound},A}}{|z| N^2}, \]
with
\[ C_{\text{bound},A} = \frac{36 \text{Tr}(B^2)}{\eta^3} \sqrt{\text{Tr}(A^2)} \left( 1 + \frac{\sigma_1^2 + \sigma_2^2}{\eta^2} \right) \sqrt{1 + \frac{\text{Tr}(A^2) + \text{Tr}(B^4)/\text{Tr}(B^2)}{\eta^2}} \]
\[ \frac{1}{\eta^2} \sqrt{1 + \left( m_{A \otimes B^2}(1^2, 1^2)^{1/2} \text{Tr}(A^4)^{1/2} + \text{Tr}(B^6)^{2/3} \text{Tr}(A^6)^{1/3} \right) \text{Tr}(B^2) \text{Tr}(A^2) \eta^2}. \]

\[ \square \]

4.2. Subordination in the multiplicative case. Building in the latter results, we prove an analogue of Proposition 4.1 in the multiplicative case, which gives the following.

**Proposition 4.6.** If
\[ N^2 \geq \frac{|z|}{\eta^2} \max \left( C_{\text{thres},A}(\eta), C_{\text{thres},B}(\eta) \right), \]
then
\[ |\tilde{m}_{\mu_1}(\omega_A) - \tilde{m}_M(z)| \leq \frac{C_{\text{bound},A}(\eta)}{N^2}, \]
and
\[ |\tilde{m}_{\mu_2}(\omega_B) - \tilde{m}_M(z)| \leq \frac{C_{\text{bound},B}(\eta)}{N^2}, \]
with \( C_{\text{bound},A}(\eta) \) given in Proposition 4.1 and \( C_{\text{bound},B}(\eta) \) given in Proposition 4.1.

**Proposition 4.7.** Suppose that
\[ N^2 \geq \frac{|z|}{\eta^2} C_{\text{thres},A}(\eta), \]
with \( C_{\text{thres},A} \) given in Proposition 4.1. Then \( \exists \omega_A \geq 2\eta/3 \) and
\[ |zm_M - \omega_A m_A(\omega_A)| \leq \frac{C_{\text{bound},A}(\eta)}{N^2}, \]
with
\[ C_{\text{bound},A}(\eta) = 24 \frac{a_3^2 b_2}{\eta^3} \left( 1 + \frac{m_{A,B}(1^2, 21^2)}{\eta^2 b_2} \right) \cdot \left( 1 + \frac{a_2}{\eta} + \frac{a_3 \sigma_2^2 + \sigma_1^2}{(1 - N^{-2}) \eta^2} \right). \]

As in the additive case, we first need to control the behavior of \( \omega_A \) and \( \omega_B \). Let us first apply Nevanlinna’s theory to the various analytic functions involved in the subordination. Let us first compute second moment of \( M \) using Weingarten calculus.

**Lemma 4.8.** There exist a probability measure \( \rho \) and \( n \) probability measures \( \rho_i, 1 \leq i \leq n \) on \( \mathbb{R} \) such that
\[ \frac{1}{E(f_A)} = -z + \mu_2(\omega_A) + \gamma G_\rho(z), \]
with \( \gamma \leq \frac{1}{1-m^2} \left( \text{Tr}((A^*)^2) \text{Var}(\mu_B) + \text{Tr}((A^*)^3) - \text{Tr}((A^*)^2)^2 \right), \]
\[ \text{E}(G_M(z))_{ii}^{-1} = -z + A_{ii} + \gamma G_{\rho_i}(z), \]
with \( \gamma_i \leq \frac{1}{1-n^2} A_{ii} \text{Var}(\mu_B) \), and
\[
\mathbb{E}(UA^{1/2}U^*G_M(z)UA^{1/2}U^*)^{-1} = -z + \tilde{B}_{ii} + \frac{\text{Var}(\mu_A)}{1-1/n^2} + \gamma_i G_{\rho_i}(z),
\]
with \( 0 \leq \tilde{B} \leq B \) and \( \gamma_i' \leq \gamma_i \) where
\[
\gamma_i' \leq \frac{1}{(1-1/N^2)(1-4/N^2)} (\text{Tr}(A^3) - 3 \text{Tr}(A^2) + 2 (\text{Tr}(B^2) - \text{Var}(\mu_A)) \text{Var}(\mu_A) + \epsilon_N),
\]
where \( \epsilon_N \leq \frac{1}{N} (\text{Tr}(B^2)(\text{Tr}(A^3) + 2 \text{Tr}(A^2)^2) + 2 \text{Tr}(A^2)^2 \text{Var}(\mu_A)) \).

Proof. First, note that for \( \gamma = \frac{1}{1-n^2} (\mu_2(\mu_A)\mu_2(B) + \mu_3(\mu_A) - \mu_2(\mu_A)^2 - \mu_2(\mu_A)^2) \).

On the other hand, using Lemma B.2 and the fact that \( \mathbb{E}(MAA^*M) \) is diagonal yields
\[
\mathbb{E}(\text{Tr}((AA^*M)^2)) = \text{Tr}((A^*A)^2 \mathbb{E}(BA^*\tilde{B}))
\]
\[
= \frac{1}{1-1/n^2} \sum_{i=1}^{n} (A^*A)_{ii}^2 (\text{Tr}((A^*A)\text{Tr}(B^2) - \text{Tr}(A^*A)\text{Tr}(B^2))^2
\]
\[
+ (A^*A)_{ii} (\text{Tr}(B^2) - \frac{1}{n^2} \text{Tr}(B^2))
\]
\[
= \frac{1}{1-1/n^2} \left( \text{Tr}((A^*A)^2) \text{Tr}(B^2) - \text{Tr}((A^*A)^2) + \text{Tr}((A^*A)^3)(1 - \frac{1}{n^2} \text{Tr}(B^2)) \right).
\]

Hence, by Nevanlinna’s theory, there exists a probability measure \( \rho \) such that
\[
\frac{1}{\mathbb{E}(f_A)} = -z + \mu_2(\mu_A^*) + \gamma \rho_G(z),
\]
with
\[
\gamma = \mathbb{E}(\text{Tr}((AA^*M)^2)) - \mathbb{E}(\text{Tr}(AA^*M))^2 \leq \frac{1}{1-1/n^2} (\mu_2(\mu_A)^2 - \mu_2(\mu_A)^2 + \mu_2(\mu_A)^2 - \mu_2(\mu_A)^2).
\]

Likewise, as \( n \) goes to infinity,
\[
\mathbb{E}(G_M) = -z^{-1} - \mathbb{E}(M)z^{-2} - \mathbb{E}(M^2)z^{-3} + o(z^{-3}).
\]

A quick computation yields that \( \mathbb{E}(M) = \text{Tr}(B)A = A \) and Lemma B.2 yields that for \( 1 \leq i \leq n \)
\[
\mathbb{E}(M^2)_{ii} = (A^{1/2} \mathbb{E}(UBU^*AU^*)A^{1/2})_{ii} = \frac{1}{1-1/n^2} A_{ii} \left( \text{Tr}(A) \text{Tr}(B^2) - \text{Tr}(A) \text{Tr}(B^2) + A_{ii} \text{Tr}(B^2) - \frac{1}{n^2} \text{Tr}(B^2) \right)
\]
\[
= \frac{1}{1-1/n^2} \left( A_{ii}^2 (1 - \frac{1}{n^2} \text{Tr}(B^2)) + A_{ii} \text{Tr}(B^2) - 1 \right).
\]

Hence, by Nevanlinna’s theory, there exists a probability measure \( \rho_i \) such that
\[
\mathbb{E}(G_M)_{ii}^{-1} = -z + A_{ii} + \gamma_i G_{\rho_i}(z),
\]
where
\[
\gamma_i = \mathbb{E}(M^2)_{ii} - (\mathbb{E}(M))_{ii}^2 = \frac{1}{1-1/n^2} A_{ii} \text{Var}(\mu_B) - A_{ii}^2 \text{Var}(\mu_B)/(n^2 - 1) \leq \frac{1}{1-n^2} A_{ii} \text{Var}(\mu_B).
\]

Similarly, \( \mathbb{E}(UA^{1/2}U^*G_M(z)UA^{1/2}U^*) \) maps \( \mathbb{C}^+ \) to \( \mathbb{C}^+ \), and as \( n \) goes to infinity
\[
\mathbb{E}(UA^*)z^{-1} - \mathbb{E}(UA^*BUA^*)z^{-2} - \mathbb{E}(UA^*BUAU^*)z^{-3} + o(z^{-3}).
\]
Since $E(UA^*) = \text{Tr}(A) \text{Id} = \text{Id}$, by Nevanlinna’s theory there exists $\rho'_i$ such that

$$E(UA^{1/2}U^*G_M(z)UA^{1/2}U^*)^{-1} = -z + E(UA^*BUA^*)_{ii} + (E(UA^*BUA^*)_{ii})^2 - E(UA^*BUA^*)^2 G_{\rho'_i}(z).$$

By Lemma B.2,

$$E(UA^*BUA^*)_{ii} = \frac{1}{1 - 1/N^2} B(1 - \frac{1}{N^2} \text{Tr}(A^2)) + \frac{1}{1 - 1/n^2} \text{Var}(\mu_A).$$

Since $\text{Tr}(A^2) \geq 1$, $1 - \frac{1}{1/N^2} \text{Tr}(A^2) \leq 1$, which shows that $E(UA^*BUA^*) - \frac{1}{1-1/n^2} \text{Var}(\mu_A) \leq B$. Likewise, we have by Lemma B.2

$$(E(UA^*BUA^*BUA^*)_{ii}) \leq E(UA^*BUA^*BUA^*)_{ii} - (B(1 - 1/N^2 \text{Tr}(A^2)) + \text{Var}(\mu_A))^2 \leq E(UA^*BUA^*BUA^*)_{ii} - 2B \text{Var}(\mu_A) - \text{Var}(\mu_A)^2 + \delta_n,$$

with $\delta_n \leq \frac{1}{2} \text{Tr}(A^2)(B^2 + \text{Var}(\mu_A))$. Using then the expression of $E(UA^*BUA^*BUA^*)$ from Lemma B.2, simplifying, removing negative terms and using the fact that $\frac{B^2}{N} \leq \text{Tr}(B^2)$ gives the result. \hfill \Box

**Proposition 4.9.** Let $z \in \mathbb{C}$ with $\Im(z) := \eta$. Then, whenever

$$N^2 \geq \frac{|z|}{\eta^3} C_{\text{thres},A}(\eta),$$

with

$$C_{\text{thres},A}(\eta) = 48b_2a_\infty^3 \left( 1 + \frac{m_B(13, 212)}{\eta^2 a_\infty^2} \right) \cdot \left( 1 + \frac{\text{Tr}(M^2)}{\eta^2} + \frac{\sigma_B^2}{\eta^2} \right) \cdot \left( 1 + \frac{k_3(\mu_B) + \sigma_B^2(a_2 - \sigma_B^2)}{1 - 4N^{-2}a_\infty^2} \right),$$

then,

$$\Im(\omega_A) \geq \frac{2}{3} \eta \quad \text{and} \quad \|G_A(\omega_A)\| \leq \frac{3}{2\eta},$$

**Proof.** Write $A(EMG_M)^{-1} = -z + \tilde{A} + Y$ with $\tilde{A} \leq A$ and $|Y_{ii}| \leq \frac{\eta'}{\eta}$, with $\gamma'$ as in Lemma 4.8 with $A$ and $B$ switched. By \[(24),\]

$$\omega_A A = A^2 - (A + \omega_A \Delta_A)(AE[MG_M]^{-1}) = A^2 + zA - A\tilde{A} - AY + z\omega_A \Delta_A - \omega_A \Delta_A(\tilde{A} + Y).$$

Hence, using the fact that $\text{Tr}(A) = 1$ and $\text{Tr}(\Delta_A) = 0$, we get by taking the trace above \[(28),\]

$$\omega_A = z + \text{Tr}(A(\tilde{A} - \tilde{A}) - \text{Tr}(AY) - \omega_A \text{Tr}(\Delta_A(\tilde{A} + Y)).$$

Remark that $\text{Tr}(A(A - \tilde{A})) = \text{Tr}(A^{1/2}(A - \tilde{A})A^{1/2})$ and $\text{Tr}(AY) = \text{Tr}(A^{1/2}YA^{1/2})$. Hence, since $\tilde{A}$ is self-adjoint and $Y \in \mathbb{H}_n^+$, $\text{Tr}(A(A - \tilde{A}) - \text{Tr}(AY) \in \mathbb{C}^+$. Therefore,

$$\Im(\omega_A) \geq \Im(z) - \|\omega_A \text{Tr}(\Delta_A(\tilde{A} + Y))\|.$$

On the other hand, by the definition of $\omega_A$ we have

$$\omega_A \text{Tr}(\Delta_A(\tilde{A} + Y)) = \frac{zE_{f_A}}{E_M(z)} \cdot E \left( (z f_A - zE f_A)(\tilde{A} + Y)G_M - (m_M - E M_M)(\tilde{A} + Y)AG_M \right)$$

$$= \frac{1}{m_M(z)} E \left( (z f_A - zE f_A) \cdot (z f_{\tilde{A} + Y} - zE f_{\tilde{A} + Y}) \right)$$

$$+ (z_m - zE m_M) \cdot (z f_{\tilde{A} + Y} - zE f_{\tilde{A} + Y}).$$
Hence, by Cauchy-Schwartz inequality and Lemma \[C.7\]
\[
\left| \omega_A \text{Tr}(\Delta_A(\tilde{A} + \Upsilon)) \right| \leq \frac{1}{|\tilde{m}_M(z)|} \left( \sqrt{\text{Var}(zf_A) \text{Var}(zf_{\tilde{A}+\Upsilon})} + \sqrt{\text{Var}(zm_M) \text{Var}(zf_{A+\tilde{\Upsilon}})} \right)
\leq \frac{1}{|\tilde{m}_M(z)|} \frac{16||A||_\infty^3}{N^2\eta^2} \left( \text{Tr}(B^2) + \frac{m_{\text{ASB}}(1^3, 21^2)}{\eta^2} \right)
\leq \frac{1}{|\tilde{m}_M(z)|} \frac{16||A||_\infty^3}{N^2\eta^2} \left( 1 + \frac{\gamma'}{||A||_\infty \eta} \right) \left( \text{Tr}(B^2) + \frac{m_{\text{ASB}}(1^3, 21^2)}{\eta^2} \right).
\]
Therefore, whenever
\[
N^2 \geq \frac{1}{|\tilde{m}_M(z)|} \frac{48||A||_\infty^3}{\alpha \eta^3} \left( 1 + \frac{\gamma'}{||A||_\infty \eta} \right) \left( \text{Tr}(B^2) + \frac{m_{\text{ASB}}(1^3, 21^2)}{\eta^2} \right) \leq \frac{|z|}{\eta^3} C_{\text{thres,}A}(\eta),
\]
for some $\alpha < 1$,
\[
\exists \omega_A \geq 2\eta/3 \text{ and } ||G_A(\omega_A)|| \leq \frac{3}{2\eta}.
\]
Since $\frac{1}{|\tilde{m}_M(z)|} = z - \text{Tr}(M^2) + \tilde{\sigma}_M^2 m_{\rho M}(z)$,
\[
\frac{1}{|\tilde{m}_M(z)|} \frac{48||A||_\infty^3}{\alpha \eta^3} \left( 1 + \frac{\gamma'}{||A||_\infty \eta} \right) \left( \text{Tr}(B^2) + \frac{m_{\text{ASB}}(1^3, 21^2)}{\eta^2} \right) \leq \frac{|z|}{\eta^3} C_{\text{thres,}A}(\eta),
\]
with
\[
C_{\text{thres,}A}(\eta) = 48\sigma_2^2 ||A||_\infty^3 \left( 1 + \frac{m_{\text{ASB}}(1^3, 21^2)}{\eta^2 \sigma_2^2} \right) \cdot \left( 1 + \frac{\gamma'}{||A||_\infty \eta} \right) \cdot \left( 1 + \frac{\text{Tr}(M^2)}{\eta} + \frac{\tilde{\sigma}_M^2}{\eta^2} \right).
\]

Proof of Proposition \[4.7\]. Suppose that $N$ satisfies the lower bound of Proposition \[4.9\] with $\alpha > 1$. Then, by this proposition,
\[
||G_A(\omega_A)||_\infty \leq \frac{||A||_\infty}{(1-\alpha)\eta}.
\]
Hence, by Lemma \[C.6\]
\[
||\text{Tr}(G_A(\omega_A)\mathbb{E}\Delta_A)|| \leq \frac{1}{z \text{E}f_A} \sqrt{\text{Var}(zf_A) \text{Var}(zf_{\text{AG}_A(\omega_A)})} + \sqrt{\text{Var}(zf_{\text{AG}_A(\omega_A)}) \text{Var}(zm_M)}
\leq \frac{16||A||_\infty^3}{(1-\alpha)N^2\eta^3 |\omega_A \text{E}m_M(z)|} \left( \text{Tr}(B^2) + \frac{m_{\text{ASB}}(1^3, 21^2)}{\eta^2} \right),
\]
where we have used the definition of $\omega_A$. Hence, by \[19\] and the fact that $\mathbb{E} \text{Tr} \Delta_A = 0$,
\[
|zm_M - \omega_A m_A(\omega_A)| = |\omega_A \text{Tr}(G_A(\omega_A)\mathbb{E}\Delta_A)|
= |\text{Tr}(G_A(\omega_A)\mathbb{E}\Delta_A)|
\leq \frac{16||A||_\infty^3}{(1-\alpha)N^2\eta |\omega_A \text{E}m_M(z)|} \left( \frac{\text{Tr}(B^2)}{\eta^2} + \frac{m_{\text{ASB}}(1^3, 21^2)}{\eta^4} \right).
\]
Using that $\omega_A \text{E}m_M = z \text{E}f_A$, we get
\[
\frac{1}{|\omega_A| \text{E}m_M} \leq (Ef_A)^{-1}/z \leq 1 + \frac{\text{Tr}(A^2)}{\eta} + \frac{\gamma}{\eta^2},
\]
which yields the final result. □

We next turn to the subordination for $\omega_B$. 

Lemma 4.10. For $N^2 \geq \frac{|C_{\text{thres}, B}(\eta)|}{\eta^4}$, then $\Im(\omega_B) \geq 2\eta/3$, and
\[
\|G_B(\omega_B)\| \leq \frac{3}{2\eta},
\]
with
\[
C_{\text{thres}, B}(\eta) = 24a\sqrt{b_2} \left( 1 + \frac{m_{A,B}^N(1^3, 21^2)}{b_2\eta^2} \right) \left( 1 + \frac{\sigma_1^2}{\eta} + \frac{\sigma_2^2}{(1 - N^2)\eta^2} \right)
\]
\[
\left( \sqrt{1 + \frac{m_{A,B}^N(1^3, 21^2)}{b_2\eta^2}} + \frac{\sigma_0^2\sqrt{\theta_2^2}}{\sqrt{\eta}} + \frac{\sigma_0^2}{\sqrt{\eta}} + \frac{\sigma_0^2}{\eta} \right) \left( \frac{k_3(\mu_A) + \sigma_1^2(b_2 - \sigma_1^2) + \frac{b_3(3\sigma_2^2 + 2\sigma_0^2)}{(1 - 4N^{-2})\eta^2}}{1 + \frac{m_{A,B}^N(1^3, 21^2)}{b_2\eta^2}} \right).
\]

Proof. Taking the trace in (25) yields
\[
\omega_B = \text{Tr}(B) - \text{Tr}(BE(MG_M)^{-1}) + E\text{Tr}(\Delta_B BE(MG_M)^{-1}).
\]
Writing $BE(MG_M)^{-1} = z - B + \frac{\text{Var}(\mu_A)}{1/N^2} + \Upsilon$ with $\|\Upsilon\| \leq \frac{\gamma'}{\eta}$ and using a similar reasoning as in Proposition 4.9 gives
\[
\Im(\omega_B) \geq z - \delta,
\]
with $|\delta| \leq |\text{Tr}((B + \Upsilon)\Delta_B)|$. Expanding $\Delta$ yields
\[
|\delta| \leq \frac{1}{|E_{fA}(z)|} \left( \sqrt{\text{Var}(zf_A)\text{Var}(f_B + \Upsilon)} + \sqrt{\text{Var}(\tilde{m}_M)\text{Var}((B + \Upsilon)U A^{1/2}G_M A^{1/2}U^*)} \right)
\]
\[
\leq \frac{1}{|E_{fA}(z)|} \left( \sqrt{\text{Var}(zf_A)(\sqrt{\text{Var}(f_B)} + \sqrt{\text{Var}(f_\Upsilon)})} + \sqrt{\text{Var}(\tilde{m}_M)}(\sqrt{\text{Var}(\tilde{m}_M)}) \right)
\]
\[
+ \sqrt{\text{Var}(\text{Tr}(U^*\Upsilon UA^{1/2}G_M A^{1/2}))}.
\]
By Lemma [C.7]
\[
\text{Var}(zf_A) \leq \frac{8\|A\|_\infty^3}{\eta^2 N^2} \left( \frac{\text{Tr}(B^2) + m_{A,B}^N(1^3, 21^2)}{\eta^2} \right),
\]
and
\[
\text{Var}(\tilde{m}_M) = \text{Var}(zm) \leq \frac{8\|A\|_\infty^3}{\eta^2 N^2} \left( \frac{\text{Tr}(B^2) + \frac{m_{A,B}^N(1^3, 21^2)}{\eta^2}}{\eta^2} \right).
\]
By the second part of Lemma [C.6] (switching $A$ and $B$) with $\alpha = 4$ and $\beta = 4$,
\[
\text{Var}(f_B) \leq \frac{4\text{Tr}(B^4)\|A\|_\infty^2}{N^2 \eta^6},
\]
and by the same lemma,
\[
\text{Var}(f_\Upsilon) \leq \frac{4\gamma'^2 \text{Tr}(B^2)\|A\|_\infty^2}{N^2 \eta^6}.
\]
Finally, by the second part of Lemma [C.9] with $\alpha = \infty$ and $\beta = 2$,
\[
\text{Var}(\text{Tr}(U^*\Upsilon UA^{1/2}G_M A^{1/2})) \leq \frac{8\gamma'^2\|A\|_\infty^3 \text{Tr}(B^2)}{\eta^6 N^2}.
\]
Hence putting all the previous bounds together yields
\[
|\delta| \leq \frac{4\sqrt{2}\|A\|_\infty \sqrt{\text{Tr}(B^2) + \frac{m_{A,B}^N(1^3, 21^2)}{\eta^2}}}{|E_{fA}(z)| \eta^3 N^2} \left( \|A\|_\infty^{3/2} \left( \sqrt{\text{Tr}(B^4) + \gamma' \sqrt{\text{Tr}(B^2)}/\eta} \right) + \sqrt{2} \left( \eta \sqrt{\text{Tr}(B^2) + \frac{m_{A,B}^N(1^3, 21^2)}{\eta^2}} + \gamma' \|A\|_\infty \sqrt{\text{Tr}(B^2)} / \eta \right) \right).
\]
By Lemma 4.8 \( \frac{1}{|E|} \leq (|z| + \text{Tr}(A^2) + \frac{2}{3}) \). Hence,

\[ |\delta| \leq \frac{|z|C_{\text{thres}, B}(\eta)}{3\eta^2 N^2}, \]

with

\[ C_{\text{thres}, B}(\eta) = 24a_{\infty}b_2\sqrt{1 + \frac{m_{A,B}^N(1^3, 21^2)}{b_2\eta^2}} \cdot \left( 1 + \frac{a_2}{\eta} \right) + \frac{\sigma_1^2 + a_2^2}{(1 - N^{-2})\eta^2} \left( 1 + \frac{m_{A,B}^N(1^3, 21^2)}{b_2\eta^2} \right). \]

Then, when

\[ N^2 \geq \frac{C_{\text{thres}, B}(\eta)|z|}{\eta^3}, \]

we have \( |\delta| \leq 2\eta/3 \), which yields,

\[ \|G_B\omega_B\|_{\infty} \leq \frac{3}{2\eta}. \]

\[ \square \]

**Proposition 4.11.** For \( N^2 \geq \frac{C_{\text{thres}, B}(\eta)|z|}{\eta^3} \),

\[ |\tilde{m}_M(z) - \tilde{m}_B(\omega_B)| \leq \frac{C_{\text{bound}, B}(\eta)}{N^2}, \]

with

\[ C_{\text{bound}, B} = \frac{4\sqrt{2}a_{\infty}b_2^2}{\eta^2} \left( \sqrt{1 + \frac{9}{4\eta^2}} + \frac{9b_6}{b_2^2\eta^2} \right) + \sqrt{2} \left( 1 + \frac{m_{A,B}^N(1^3, 21^2)}{b_2\eta^2} \right)^{3/2} \left( \frac{b_4 + m_{A,B}^N(1^3, 21^2)}{b_2\eta^2}/(b_2\eta^2)^{3/2} \right). \]

**Proof.** By (20),

\[ \tilde{m}_M(z) = \tilde{m}_B(\omega_B) + \text{Tr}(BG_B(\omega_B)\Delta_B). \]

Let us bound the error term by first rewriting it as

\[ \text{Tr}(BG_B(\omega_B)\Delta_B) = \omega_B^{-1}\text{Tr}((B + B^2G_B(\omega_B))(\Delta_B)) \]

\[ = \frac{1}{\text{E}z_m(M)} \left( \mathbb{E}((zf_A - zEf_A)(f_B + B^2G_B(\omega_B))) - \mathbb{E}f_B + B^2G_B(\omega_B) \right) \]

\[ + \mathbb{E}((zm - zEm)(\text{Tr}((B + B^2G_B(\omega_B)U^*)A^{1/2}G_MA^{1/2}U^*))) \]

\[ - \mathbb{E}\text{Tr}((B + B^2G_B(\omega_B)U^*)A^{1/2}G_MA^{1/2}U^*)), \]

where we used the definition of \( \omega_B \). Thus,

\[ |\tilde{m}_M(z) - \tilde{m}_B(\omega_B)| \leq \frac{1}{\text{E}z_m(M)} \left( \sqrt{\text{Var}(zf_A)^2 \text{Var}(f_A)} + \sqrt{\text{Var}(zf_A)^2 \text{Var}(f_B^2G_B(\omega_B))} \right) \]

\[ + \sqrt{\text{Var}(zm_m(z))^2 \text{Var}(\tilde{m}_M(z)) + \sqrt{\text{Var}(zm_m(z)) \text{Var}(\text{Tr}(B^2G_B(\omega_B)U^*)A^{1/2}G_MA^{1/2}U^*))} \]

By Lemma C.6

\[ \text{Var}(zf_A) \leq \frac{8\|A\|^3_{\infty}}{\eta^2 N^2} \left( \text{Tr}(B^2) + \frac{m_{A,B}^N(1^3, 21^2)}{\eta^2} \right), \]
and

\[ \Var(\tilde{m}_M(z)) = \Var(zm_z) \leq \frac{8\|A\|_\infty}{\eta^2 N^2} \left( \Tr(B^2) + \frac{m_{A,B}(1^3, 2^1)}{\eta^2} \right). \]

By the first part of Lemma \([\text{C.9}]\) with \(\alpha = \beta = 4\),

\[ \Var(f_B) \leq \frac{4 \Tr(B^4)\|A\|_\infty^2}{N^2 \eta^4}, \]

and by the first part of Lemma \([\text{C.9}]\) with \(\alpha = 3, \beta = 6\)

\[ \Var(f_B^2 G_B(\omega_B)) \leq \frac{4 \Tr((B^2 G_B(\omega_B))^3) 3/2}{N^2 \eta^4} \Tr(B^6)^{1/3} \|A\|_\infty^2 \leq \frac{9 \Tr(B^6)\|A\|_\infty^2}{N^2 \eta^6}, \]

where we used the hypothesis on \(N\) and Lemma \([\text{4.10}]\) to get \(\|G_B(\omega_B)\|_\infty \leq \frac{3}{\eta^2}\). Finally, by the second part of Lemma \([\text{C.9}]\) with \(\alpha = 1/3\) and \(\beta = 1/6\), and using the fact that \(\|G_B(\omega_B)\|_\infty \leq \frac{3}{\eta^6}\),

\[ \Var(\Tr(A^{1/2}U(B^2 G_B(\omega_B))U^* A^{1/2}G_M) \leq \frac{18\|A\|_\infty}{N^2 \eta^4} \left( \E \Tr(A^{1/2}UB^2U^* A^{1/2}) + \frac{\E \Tr((A^{1/2}UB^2U^* A^{1/2})^3) 2/3}{\eta^2} \right) \]

\[ \leq \frac{18\|A\|_\infty}{N^2 \eta^4} \left( \E \Tr(A^{1/2}UB^2U^* A^{1/2}) + \frac{\E \Tr((A^{1/2}UB^2U^* A^{1/2})^3)}{\eta^2} \right) \]

\[ \leq \frac{18\|A\|_\infty}{N^2 \eta^4} \left( \Tr(B^4) + \frac{m_{A,B}(1^3, 2^1)}{\eta^2} \right). \]

Then, putting all latter bounds together and recalling that \(C_1(\eta) = \Tr(B^2) + \frac{m_{A,B}(1^3, 21^2)}{\eta^2}\) yield

\[ |\tilde{m}_M(z) - \tilde{m}_B(\omega_B)| \leq \frac{1}{|z|E_m(z)} \left( \frac{4 \sqrt{2}\|A\|_\infty^{1/2}}{\eta^3 N^2} \left( \sqrt{\Tr(B^4)} + \frac{3 \sqrt{\Tr(B^2)}}{2 \eta} \right) \sqrt{\Tr(B^2)} + \frac{m_{A,B}(1^3, 21^2)}{\eta^2} \right) \]

\[ + \frac{8\|A\|_\infty}{\eta^3 N^2} \left( \eta(\Tr(B^2) + \frac{m_{A,B}(1^3, 21^2)}{\eta^2}) + 2 \sqrt{\Tr(B^2) + \frac{m_{A,B}(1^3, 21^2)}{\eta^2}} \left( \sqrt{\Tr(B^4)} + \frac{9 \Tr(B^6)}{4 \eta^2} \right) \right) \]

\[ \leq \frac{4 \sqrt{2}\|A\|_\infty^{1/2}}{|z|E_m(z)|\eta^3 N^2} \left( \left( \|A\|_\infty^{1/2} \right) \left( \sqrt{\Tr(B^4)} + \frac{9 \Tr(B^6)}{4 \eta^2} \right) \right) \]

\[ + \sqrt{2}\sqrt{\Tr(B^2) + \frac{m_{A,B}(1^3, 21^2)}{\eta^2}} + \frac{3}{2 \eta} \sqrt{\Tr(B^4) + \frac{m_{A,B}(1^3, 2^1)}{\eta^2}} \right). \]

Since \(\frac{1}{|z|E_m|} = z - \E(\Tr(M)) + (\E(\Tr(M^2)) - \E(\Tr(M)^2)) m_p(z) = z - 1 + (\sigma_1^2 + \sigma_2^2) m_p(z),\)

\[ \frac{1}{|z|E_m|} \leq (1 + (1 + \frac{\sigma_1^2 + \sigma_2^2}{\eta})/|z|) \leq 1 + \frac{\sigma_1^2 + \sigma_2^2}{\eta^2}. \]

Hence,

\[ |\tilde{m}_M(z) - \tilde{m}_B(\omega_B)| \leq \frac{C_{\text{bound}, B}}{N^2}, \]

with

\[ C_{\text{bound}, B} = \frac{4 \sqrt{2} a_{b_1} b_{b_2}^2}{\eta^2} \left( 1 + \frac{\sigma_1^2 + \sigma_2^2}{\eta^2} \right) \cdot \sqrt{1 + \frac{m_{A,B}(1^3, 21^2)}{b_2 \eta^2}} \cdot \left( \frac{a_{b_1}^{3/2}}{\eta} \left( \frac{b_1}{b_2} + \sqrt{\frac{9 b_6}{4 b_2 \eta^2}} \right) + \sqrt{2} \sqrt{1 + \frac{m_{A,B}(1^3, 21^2)}{b_2 \eta^2}} + \frac{3}{2} \sqrt{\frac{b_1 + m_{A,B}(1^3, 2^1)}{b_2 \eta^2}} \right). \]

\(\square\)
5. Stability result for the deconvolution

In this section, we need to take into account the error term from the fluctuations of $m_H$ or $m_M$ around their average and fluctuations from $A$ around $\mu_1$. To this end, introduce in the additive case

$$\delta_H(z) = m_H(z) - \mathbb{E}m_H(z), \quad \delta_A(z) = m_A(\omega_A(z)) - m_{\mu_1}(\omega_A(z)),$$

and in the multiplicative case

$$\bar{\delta}_M(z) = \bar{m}_M(z) - \mathbb{E}\bar{m}_M(z), \quad \bar{\delta}_A(z) = \bar{m}_A(\omega_A(z)) - \bar{m}_{\mu_1}(\omega_A(z)).$$

Both additive and multiplicative cases are dealt with using the coercive property of the reciprocal Cauchy transform, which is summarized in the next lemma.

**Lemma 5.1.** Let $\mu$ be a probability measure with variance $\sigma^2$. For all $z, z' \in \mathbb{C}^+$,

$$F_\mu(z) - F_\mu(z') = (z - z')(1 + \tau_\mu(z, z')),$$

with $\delta(\Im(z), \Im(z')) \leq \frac{\sigma^2}{3z \bar{z}'}$.

**Proof.** By (6),

$$F_\mu(z) = z - \mu(1) + \sigma^2 m_\rho(z),$$

with $\rho$ a probability measure on $\mathbb{R}$. Then, for $z, z' \in \mathbb{C}_\sigma$,

$$F_\mu(z) - F_\mu(z') = z - z' + \sigma^2(m_\rho(z) - m_\rho(z')).$$

Moreover,

$$m_\rho(z) - m_\rho(z') = \int_{\mathbb{R}} \frac{1}{t-z} d\mu(t) - \int_{\mathbb{R}} \frac{1}{t-z'} d\mu(t) = (z - z') \int_{\mathbb{R}} \frac{1}{(t-z)(t-z')} d\mu(t),$$

which implies the first statement of the lemma. The second statement is given by the inequality

$$\int_{\mathbb{R}} \frac{1}{(t-z)(t-z')} d\mu(t) \leq \frac{1}{3z \bar{z}'}.$$

5.1. **Additive case.** For $z \in \mathbb{C}_{2\sqrt{2} \sigma_1}$, let $(\omega_1, \omega_3)$ be the solution of the system

$$(29) \quad \begin{cases} \omega_1 + z = \omega_3 + F_{\mu_1}(\omega_3), \\ \omega_1 + z = \omega_3 + F_{\mu_1}(\omega_1). \end{cases}$$

Remark in particular that by Theorem 2.4,

$$\xi \geq \frac{\Im(\omega_1)}{4\sigma_1} \geq \frac{3}{\sqrt{2}} \geq 2.$$ 

Moreover, since $F_{\mu_1}(\omega_1) = F_{\mu_3}(\omega_3)$, (6) yields

$$\Im(\omega_1) \geq \Im(F_{\mu_1}(\omega_1)) - \frac{\sigma_1^2}{3\omega_1} \geq \Im(F_{\mu_3}(\omega_3)) - \frac{\sigma_1^2}{3\omega_1} \geq \Im(\omega_3) - \frac{\sigma_1^2}{3\omega_1},$$

which implies (using the fact that $\Im(\omega_3) > 2\sigma_1$) and $\Im(\omega_1) > \sigma_1$ thanks to Theorem 2.4 that

$$\Im(\omega_1)/\sigma_1 \geq \frac{\xi + \sqrt{\xi^2 - 4}}{2} := k(\xi).$$

Moreover, we suppose that

$$\sigma_A \leq \sigma_1/\sqrt{k(\xi)}.$$

Let $\omega_A, \omega_B$ be the subordination functions introduced in [13] for $\omega_3$.

**Lemma 5.2.** For $z \in \mathbb{C}^+$ such that $\Im(z) \geq 2\sqrt{2} \sigma_1$,

$$(m_B(z) - m_H(\omega_3)) = \frac{A m_B(z)}{m_{\mu_1}(\omega_A)} (m_A(\omega_A) - \mathbb{E}m_H(\omega_3)) + m_B(\omega_B) - \mathbb{E}m_H(\omega_3)$$

$$+ \frac{A m_B(z)}{m_{\mu_1}(\omega_A)} \delta_A + \left( \frac{m_B(z)}{m_H(\omega_3)} A \tau_{\mu_1}(\omega_B, \omega_A) - 1 \right) \delta_H,$$

with

$$A = \left( 1 + \frac{m_B(\omega_B) - \mathbb{E}m_H(\omega_3)}{\mathbb{E}m_H(\omega_3)} \right) \left( 1 + \frac{\tau_B(\omega_B, z)}{1 + \tau_{\mu_1}(\omega_1, \omega_A)} \right).$$
Proof. Note that
\[ m_B(z) - m_H(\omega_3) = m_B(z) - m_B(\omega_B) + m_B(\omega_B) - \mathbb{E}m_H(\omega_3) + \mathbb{E}m_H(\omega_3) - m_H(\omega_3). \]
First,
\[ m_B(z) - m_B(\omega_B) = -\frac{1}{F_B(z)} + \frac{1}{F_B(\omega_B)} = (F_B(z) - F_B(\omega_B))m_B(z)m_B(\omega_B) \]
\[ = (z - \omega_B)(1 + \tau_B(\omega_B, z))m_B(z)m_B(\omega_B), \]
where we used Lemma 5.1 in the last inequality. Then, using the relation satisfied by \( \omega_B \) and \( z \) yields
\[ \omega_B - z = \omega_3 + F_{H}(\omega_3) - \omega_3 - F_{H}(\omega_3) + \omega_1 \]
\[ = \omega_1 - \omega_A + F_{H}(\omega_3) - F_{H}(\omega_3), \]
where \( F_{H} = \frac{-1}{\mathbb{E}m_H}. \) Then, by Lemma 5.1 and the relation \( F_{\mu_1}(\omega_1) = F_{H}(\omega_3) \), with \( \tau_1 = \mu_1(\omega_1, \omega_A) \),
\[ \omega_B - z = \frac{F_{\mu_1}(\omega_1) - F_{\mu_A}(\omega_A)}{1 + \tau_1(\omega_1, \omega_A)} + F_{H}(\omega_3) - F_{H}(\omega_3) \]
\[ = \frac{F_{H}(\omega_3) - F_{\mu_A}(\omega_A)}{1 + \tau_1} + F_{H}(\omega_3) - F_{H}(\omega_3) \]
\[ = \frac{F_{H}(\omega_3) - F_{H}(\omega_3)}{1 + \tau_1} \frac{\tau_1}{1 + \tau_1} + \frac{F_{H}(\omega_3) - F_{\mu_1}(\omega_A)}{1 + \tau_1} \]
Write temporarily \( \epsilon_B = \frac{m_B(\omega_B) - \mathbb{E}m_H(\omega_3)}{\mathbb{E}m_H(\omega_3)} \), \( \epsilon_A = \frac{m_{\mu_A}(\omega_A) - \mathbb{E}m_H(\omega_3)}{m_{\mu_A}(\omega_A)} \) and \( \epsilon_H = \frac{m_H(\omega_3) - \mathbb{E}m_H(\omega_3)}{m_H(\omega_3)}. \) Hence, putting the latter relation in (30) yields
\[ m_B(z) - m_B(\omega_B) = m_B(z)(\tau_1 \epsilon_H + A \epsilon_A), \]
with
\[ A = \frac{m_B(\omega_B)}{\mathbb{E}m_H(\omega_3)} \frac{(1 + \tau_2)}{1 + \tau_1} = (1 + \epsilon_B) \frac{(1 + \tau_2)}{1 + \tau_1}, \]
with \( \tau_2 = \tau_B(\omega_B, z) \). Hence, using the first relation of the proof yields
\[ (m_B(z) - m_H(\omega_3)) = \frac{Am_B(\omega_A) - \mathbb{E}m_H(\omega_3)}{m_{\mu_A}(\omega_A) \delta_A + \left( \frac{m_B(z)}{m_H(\omega_3)} A\tau_1 - 1 \right) \delta_H}. \]

From the latter lemma we deduce the distance between \( m_{\mu_3}(z) \) and \( m_{\mu_H}(\omega_3) \).

**Proposition 5.3.** Suppose that \( N^2 \geq \max(C_{\text{thres},A}(\xi \sigma_1), C_{\text{thres},B}(\xi \sigma_1)) \xi \sigma_1 \). Then
\[ |m_B(z) - m_H(\omega_3)| \leq C_1(N) \frac{|z|^{N^2}}{|z|} + C_2(N) \frac{|\omega_A \delta_A|}{|z|} + C_3(N) \frac{|\omega_3 \delta_H|}{|z|}, \]
with \( C_1(N), C_2(N) \) and \( C_3(N) \) are respectively given in (31), (32) and (33).

**Proof.** By Proposition 4.1 for \( N \geq \sqrt{\max(C_{\text{thres},A}(\xi \sigma_1), C_{\text{thres},B}(\xi \sigma_1)) \xi \sigma_1} \), with \( C_{\text{thres},A}(\xi \sigma_1) \) given in Proposition 4.5 then \( \Im A, \Im B \geq 2 \xi \sigma_1/3 \) and
\[ |\Im m_H(\omega_3) - m_{\mu_1}(\omega_A(z))| \leq \frac{C_{\text{bound},A}(\xi \sigma_1)}{|\omega_3| N^2}, \]
and
\[ |\text{Em}_H(\omega_3) - m_B(\omega_B(z))| \leq \frac{C_{\text{bound},B}(\xi\sigma_1)}{|\omega_3|^2 N^2}, \]
with \(C_{\text{bound},A}(\xi\sigma_1)\) and \(C_{\text{bound},B}(\xi\sigma_1)\) given in Proposition 4.2. Hence, in particular,
\[ |A| \leq \left( 1 + \frac{F_H(\omega_3)C_{\text{bound},B}(\xi\sigma_1)}{|\omega_3|^2 \xi^2 \sigma_1^2 N^2} \right) \left( 1 + \tau_B(\omega_B, z) \right), \]
Moreover, since \(3\omega_A, 3\omega_B \geq 2\ell \omega_3/3 \geq \eta/2\), and \(3\omega_1 \geq \eta/2\), with \(\eta = 3\zeta\), which yields
\[ |\tau_{\mu_1}(\omega),\omega_A)\rangle = \int R_1 \frac{\sigma^2 dt}{(\omega_1 - t)(\omega_A - t)} \leq \frac{4\sigma^2}{\eta^2}, \quad |\tau_B(\omega_B, z)| = \int R_1 \frac{\sigma^2 dt}{(\omega_B - t)(z - t)} \leq \frac{2\sigma^2}{\eta^2}. \]
Hence, using that
\[ \left| \frac{F_H(\omega_3)}{\omega_3} \right| \leq 1 + \frac{\text{Tr}(M^2)}{3}\omega_3^2 \text{ yields} \]
\[ A \leq \left( 1 + \frac{C_{\text{bound},B}(\xi\sigma_1)(1 + \frac{\text{Tr}(M^2)}{\xi^2 \sigma_1^2})}{N^2} \right) \left( 1 + \frac{2\sigma^2}{\eta^2} \right) \leq \frac{1}{1 - 4\sigma^2/\eta^2} = K(N). \]
Therefore, by Lemma 4.2
\[ \left| m_B(z) - m_H(\omega_3) \right| \leq C_1(N) + \frac{C_2(N)}{|z|} \left| \omega_A\delta_A \right| + C_3(N) \left| \omega_3 \delta_H \right|, \]
with, using that \(3\omega_A \geq \eta/2\) and \(\omega_A - \omega_3 \leq \frac{\sigma^2}{(\xi\sigma_1^3)} + \xi\sigma_1/3\) (cf Lemma 4.5),
\[ C_1(N) = K(N) \frac{|F_H(\omega_3)|}{|\omega_A|} \frac{|m_B(z)|}{|z|} \leq C_{\text{bound},B}(\xi\sigma_1) \left( 1 + \frac{\sigma^2}{\eta} \right) \]
(31)
\[ = \left( 1 + \frac{C_{\text{bound},B}(\xi\sigma_1)(1 + \frac{\text{Tr}(M^2)}{\xi^2 \sigma_1^2})}{N^2} \right) \left( 1 + \frac{2\sigma^2}{\eta^2} \right) \left( 1 + \frac{4\sigma^2}{\eta^2} \right) \left( 1 + \frac{\sigma^2}{(\xi\sigma_1^3)} \right) \frac{C_{\text{bound},A}(\xi\sigma_1)}{N^2}, \]
\[ C_2(N) = K(N) \frac{|F_H(\omega_3)|}{|\omega_A|} \frac{|m_B(z)|}{|z|} \leq C_{\text{bound},B}(\xi\sigma_1) \left( 1 + \frac{\sigma^2}{\eta} \right) \]
(32)
\[ = \left( 1 + \frac{C_{\text{bound},B}(\xi\sigma_1)(1 + \frac{\text{Tr}(M^2)}{\xi^2 \sigma_1^2})}{N^2} \right), \]
and
\[ C_3(N) = \left| \tau_{\mu_1}(\omega_1,\omega_A) \frac{z m_B(z)}{\omega_3 m_H(\omega_3)} A_{\tau_{\mu_1}(\omega_1,\omega_1)} \right| \]
Expanding the last term gives with \(z = \omega_3 = h_{\mu_1}(\omega_1)\)
\[ = \tau_{\mu_1}(\omega_1,\omega_A) \left( 1 + \frac{\sigma^2}{\omega_3} m_B(\omega_3) \right) \left( 1 + \tau_B(\omega_B, z) \right) - \frac{z}{\omega_3} \]
\[ = \tau_B(\omega_B, z) \tau_{\mu_1}(\omega_1,\omega_A) - \frac{z}{\omega_3} \]
\[ \left[ \epsilon_B \tau_{\mu_1}(\omega_1,\omega_A) \left( 1 + \frac{\sigma^2}{\omega_3} m_B(\omega_3) \right) + \frac{\sigma^2}{\omega_3} m_B(\omega_3) \right], \]
which yields
\[ C_3(N) \leq \frac{1 + 2\frac{\sigma_1^2\sigma_2^2}{\eta^4}}{1 - \frac{4\sigma_1^2}{\eta^2}} + \frac{8\sigma_1^2}{\eta^2} + \frac{4\sigma_1^2}{\eta^2} - 4\frac{\sigma_1^2}{\eta^2}. \]

(33) \[ \left( \frac{\sigma_2}{\eta}(1 + \frac{16\sigma_2^2}{9\eta^2}) + \frac{16\sigma_2^2}{9\eta^2} + \frac{C_{\text{bound},B}(\xi\sigma_1)(1 + \frac{16\eta(M^2)}{9\eta^2})}{N^2}\right) \left(1 + \frac{\sigma_2}{\eta}(1 + \frac{16\sigma_2^2}{9\eta^2})\right). \]

5.2. Multiplicative case. We now turn to the multiplicative case, which follows a similar pattern. We first express the difference between \( \tilde{m}_B(z) \) and \( \tilde{m}_M(\omega_3) \).

**Lemma 5.4.** Set \( \epsilon_A = \tilde{m}_A(\omega_A) - \mathbb{E}\tilde{m}_M(\omega_3) \) and \( \epsilon_B = \tilde{m}_B(\omega_B) - \mathbb{E}\tilde{m}_M(\omega_3) \). Then
\[ \tilde{m}_B(z) - \tilde{m}_M(\omega_3) = Le_A + \left[(1 + \tau_B(\omega_B, z))\tilde{m}_B(z)(z - \omega_B) + 1\right] \epsilon_B + L\delta_A + [L' - 1] \delta_M, \]
with \( L = \frac{z\tilde{m}_B(z)\tilde{F}_M(\omega_M(\omega_3))}{\omega_M(1 + \tau_M(\omega_1, \omega_A))} \) and \( L' = \frac{z\tilde{m}_B(z)(1 + \tau_B(\omega_B, z))\tilde{F}_M(\omega_M(\omega_3))}{\omega_M(1 + \tau_M(\omega_1, \omega_A))} \).

**Proof.** We have
\[ \tilde{m}_B(z) - \tilde{m}_M(\omega_3) = \tilde{m}_B(z) - \tilde{m}_B(\omega_B) + \tilde{m}_B(\omega_B) - \mathbb{E}\tilde{m}_M(\omega_3) - \delta_M, \]
and setting \( \epsilon_B = \tilde{m}_B(\omega_B) - \mathbb{E}\tilde{m}_M(\omega_3) \),
\[ \tilde{m}_B(z) - \tilde{m}_B(\omega_B) = (1 + \tau_B(\omega_B, z))\tilde{m}_B(\omega_B)\tilde{m}_B(z)(z - \omega_B) \]
(34) \[ = (1 + \tau_B(\omega_B, z))\mathbb{E}\tilde{m}_M(\omega_3)\tilde{m}_B(z)(z - \omega_B) + (1 + \tau_B(\omega_B, z))\epsilon_B\tilde{m}_B(z)(z - \omega_B). \]

By Theorem 2.6, \( \omega_1 = \omega_3 \tilde{F}_M(\omega_3) \), and by (18), \( \omega_1 \omega_B = \omega_3 \tilde{F}_M(\omega_3) \), with \( \tilde{F}_M \) denoting \( 1 + \tilde{F}_M \) and \( \tilde{F}_M = \frac{\tilde{m}_B}{\tilde{m}_M} \). Hence,
\[ \omega_B - z = \omega_3 \left( \frac{\tilde{F}_M(\omega_M(\omega_3))}{\omega_A} - \frac{F_M(\omega_M(\omega_3))}{\omega_1} \right) = \omega_3 \left( \frac{\tilde{F}_M(\omega_M(\omega_3)) - F_M(\omega_M(\omega_3))}{\omega_A} \right). \]

Then, since
\[ \omega_1 - \omega_A = \frac{1}{1 + \tau_M(\omega_1, \omega_A)}(\tilde{F}_{\mu_1}(\omega_1) - \tilde{F}_{\mu_1}(\omega_A)) = \frac{1}{1 + \tau_M(\omega_1, \omega_A)}(\tilde{F}_{\mu_1}(\omega_1) - \tilde{F}_{\mu_1}(\omega_M(\omega_3))) \]
we get, using again the relation \( \omega_B = \omega_3 \tilde{F}_M(\omega_3) \) and \( \tilde{F}_{\mu_1}(\omega_1) = \tilde{F}_M(\omega_3) \),
\[ \omega_B - z = \omega_3 \tilde{F}_M(\omega_3) - \tilde{F}_M(\omega_3) + \tilde{F}_{\mu_1}(\omega_1) - \tilde{F}_{\mu_1}(\omega_A) \]
\[ = - \omega_3 \tilde{F}_M(\omega_3) + \tilde{F}_M(\omega_3) + \tilde{F}_{\mu_1}(\omega_1) - \tilde{F}_{\mu_1}(\omega_A) \]
\[ = \tilde{F}_M(\omega_3)(\omega_A - \omega_3) \delta_M(\omega_A) + z \tilde{F}_M(\omega_3) \tilde{F}_M(\omega_3) \frac{\tilde{m}_M(\omega_3) - \tilde{m}_M(\omega_A)}{\omega_A(1 + \tau_M(\omega_1, \omega_A))} \]
\[ + z \frac{\tilde{m}_M(\omega_3)}{\omega_A(1 + \tau_M(\omega_1, \omega_A))} \delta_M(z) - \frac{\tilde{m}_M(\omega_3)}{\omega_A(1 + \tau_M(\omega_1, \omega_A))} \delta_A(\epsilon_A, \omega_3), \]
with \( \epsilon_A = \tilde{m}_A(\omega_A) - \mathbb{E}\tilde{m}_M(\omega_3) \). Putting the latter equality in (34) yields then
\[ \tilde{m}_B(z) - \tilde{m}_B(\omega_B) = (1 + \tau_B(\omega_B, z))\tilde{m}_B(z) \left[ \tilde{F}_M(\omega_3) \frac{\omega_3 - z}{\omega_A(1 + \tau_M(\omega_1, \omega_A))} \delta_M(z) \right] \]
\[ + \frac{\tilde{F}_{\mu_1}(\omega_A)}{\omega_A(1 + \tau_M(\omega_1, \omega_A))} (\delta_A + \epsilon_A) + (1 + \tau_B(\omega_B, z))\epsilon_B\tilde{m}_B(z)(z - \omega_B). \]
Since $\omega_3 = -zh_{\mu_1}(\omega_1)$ (see Theorem 2.6) and $\hat{F}_M(\omega_3) = \hat{F}_{\mu_1}(\omega_1)$, we can further simplify the above expression as

\[
(\omega_3 - z)\hat{F}_M(\omega_3) = z(-h_{\mu_1}(\omega_1) - 1)\hat{F}_{\mu_1}(\omega_1) = z\left[\frac{-1}{m_{\mu_1}(\omega_1)} - \frac{1}{1 + \omega_1m_{\mu_1}(\omega_1)} - \hat{F}_{\mu_1}(\omega_1)\right]
\]

\[
= z\left[\frac{-1}{m_{\mu_1}(\omega_1)} - \frac{1}{1 + \omega_1m_{\mu_1}(\omega_1)} - \hat{F}_{\mu_1}(\omega_1)\right] = z(F_{\mu_1}(\omega_1) - \hat{F}_{\mu_1}(\omega_1)),
\]

yielding

\[
\tilde{m}_B(z) - \tilde{m}_B(\omega_B) = \frac{z\tilde{m}_B(z)(1 + \tau_B(\omega_B, z))}{\omega_A(1 + \tau_{\mu_1}(\omega_1, \omega_A))} \left[ (F_{\mu_1}(\omega_1) - \hat{F}_{\mu_1}(\omega_1))\delta_M(z) + \hat{F}_{\mu_1}(\omega_A)(\delta_A + \epsilon_A) \right]
+ (1 + \tau_B(\omega_B, z))\epsilon_B\tilde{m}_B(z)(z - \omega_B).
\]

Hence,

\[
\tilde{m}_B(z) - \tilde{m}_M(\omega_3) = \frac{z\tilde{m}_B(z)(1 + \tau_B(\omega_B, z))}{\omega_A(1 + \tau_{\mu_1}(\omega_1, \omega_A))} \epsilon_A + \left[ (1 + \tau_B(\omega_B, z))\tilde{m}_B(z)(z - \omega_B) + 1 \right] \epsilon_B
+ \frac{z\tilde{m}_B(z)(1 + \tau_B(\omega_B, z))(F_{\mu_1}(\omega_1) - \hat{F}_{\mu_1}(\omega_1))}{\omega_A(1 + \tau_{\mu_1}(\omega_1, \omega_A))} - 1 \right] \delta_M
\]

Estimating the different contributions from latter lemma yields the following control on the deconvolution procedure in the multiplicative case.

**Proposition 5.5.** Let $z \in \mathbb{C}^+$ satisfy $\Im(z) > g^{-1}(\xi_0)\bar{\sigma}_1$, and consider the system of equations

\[
\begin{align*}
\omega_1z &= \omega_3\hat{F}_{\mu_1}(\omega_3) \\
\omega_1z &= \omega_3\hat{F}_{\mu_1}(\omega_1).
\end{align*}
\]

Then this system admits a solution, and for

\[
N^2 \geq \frac{\omega_3}{\xi_3\bar{\sigma}_1^3} \max \left( C_{\text{thres}, A}(\xi\bar{\sigma}_1), C_{\text{thres}, B}(\xi\bar{\sigma}_1) \right),
\]

we have

\[
|\tilde{m}_M(\omega_3)/z - m_B(z)| \leq \frac{C_1(\eta)}{N^2} + C_2(\eta)\delta_A + C_3(\eta)\delta_M,
\]

with $C_1(\eta), C_2(\eta), C_3(\eta)$, respectively given in (36), (37) and (38).

**Proof.** We have to bound the different contribution from Lemma 5.4. Suppose that

\[
N^2 \geq \frac{|\omega_3|}{\xi_3\bar{\sigma}_1^3} \max \left( C_{\text{thres}, A}(\xi\bar{\sigma}_1), C_{\text{thres}, B}(\xi\bar{\sigma}_1) \right).
\]

Then, $\Im\omega_A \geq 2\Im\omega_3/3$. Hence,

\[
\left| \frac{\hat{F}_{\mu_1}(\omega_A)}{\omega_A} \right| \leq 1 + \frac{\text{Tr}(A^2)}{3\omega_A} + \frac{\bar{\sigma}_1^2}{(3\omega_A)^2} \leq 1 + 3\text{Tr}(A^2) + \frac{9}{4\xi^2}.
\]

Moreover, $\Im\omega_1 \geq k(\bar{\xi})\bar{\sigma}_1$, thus $|\tau_{\mu_1}(\omega_1, \omega_A)| \leq \frac{\bar{\sigma}_1^2}{3\omega_1\Im\omega_A} \leq \frac{3}{2k(\bar{\xi})}. \xi_3\bar{\sigma}_1$. Similarly, $\Im\omega_B \geq \Im\omega_3/3$, thus $\tau_B(z, \omega_B) \leq \frac{3\bar{\sigma}_1^2}{2k(\bar{\xi})}. \xi_3\bar{\sigma}_1$. Hence, since $z\tilde{m}_B(z) = 1 + \int_\mathbb{R} \frac{t^2}{1-t^2}d\mu_B(t)$,

\[
L \leq \left(1 + \frac{\text{Tr}(B^2)}{\eta}\right) \left(1 + 3\text{Tr}(A^2) + \frac{9}{4\xi^2}\right) \cdot \frac{1 + 3\bar{\sigma}_1^2}{1 - 2k(\bar{\xi})}. \xi_3\bar{\sigma}_1.
\]
and, using the fact that $F_{\mu_1}(\omega_1) - \tilde{F}_{\mu_1}(\omega_1) = \sigma_1^2 + \sigma_1^2 m_{\mu}(\omega_1) - \tilde{\sigma}_1^2 m_{\mu'}(\omega_1)$,

$$L' \leq \frac{3}{2\xi \tilde{\sigma}_1} \cdot \left(1 + \frac{\text{Tr}(B^2)}{\eta}\right) \cdot \left(\sigma_1^2 + \frac{\sigma_1^2}{k(\xi) \tilde{\sigma}_1} + \frac{\tilde{\sigma}_1^2}{k(\xi) \tilde{\sigma}_1}\right) \cdot \frac{1 + \frac{3\tilde{\sigma}_1^2}{2n(\xi)\tilde{\sigma}_1}}{1 - \frac{3}{2k(\xi)}}.$$

Then, we have by (17)

$$\frac{\omega_B}{z} = \frac{\omega_B \omega_3}{\omega_3 z} = -h_{\mu_1}(\omega_1) \frac{\mathbb{E} \tilde{m}_M(\omega_3)}{\mathbb{E} f_A}.$$

Since $\omega_3 \mathbb{E} \tilde{m}_M(\omega_3) = -1 + \mathbb{E} \tilde{m}_M(\omega_3)$,

$$\left|\mathbb{E} \tilde{m}_M(\omega_3) / \mathbb{E} f_A\right| \leq \left(1 + \frac{\sigma_1^2}{k(\xi) \tilde{\sigma}_1}\right) \left(1 + \frac{\text{Tr}(A^2)}{\xi \tilde{\sigma}_1} + \frac{\text{Tr}(A^2) \text{Var}(\mu_B) + \tilde{\sigma}_1^2}{(1 - N^{-2}) \xi^2 \tilde{\sigma}_1^2}\right),$$

which yields

$$\left|\frac{\omega_B}{z}\right| \leq \left(1 + \frac{\sigma_1^2}{k(\xi) \tilde{\sigma}_1}\right) \left(1 + \frac{\text{Tr}(A^2)}{\xi \tilde{\sigma}_1} + \frac{\text{Tr}(A^2) \text{Var}(\mu_B) + \tilde{\sigma}_1^2}{(1 - N^{-2}) \xi^2 \tilde{\sigma}_1^2}\right).$$

Hence,

$$|G\tau_B(\omega_B, z)) \tilde{m}_B(z)(z - \omega_B) + 1| \leq \left(1 + \frac{3 \text{Tr}(B^2)}{2n(\xi)\tilde{\sigma}_1}\right) \left(1 + \frac{\text{Tr}(B^2)}{\eta}\right) \left(1 + \frac{\sigma_1^2}{k(\xi) \tilde{\sigma}_1}\right) \left(1 + \frac{\text{Tr}(A^2)}{\xi \tilde{\sigma}_1} + \frac{\text{Tr}(A^2) \text{Var}(\mu_B) + \tilde{\sigma}_1^2}{(1 - N^{-2}) \xi^2 \tilde{\sigma}_1^2}\right).$$

Putting all the above bounds together, and using that $\epsilon_A \leq \frac{C_{\text{bound,A}}}{N^2}$ and $\epsilon_B \leq \frac{C_{\text{bound,B}}}{N^2}$, we obtain

$$|\tilde{\nu}_2(z) - \tilde{m}_M(z)| \leq \frac{C_1(\eta)}{N^2} + \frac{C_2(\eta)}{N^2} + \frac{C_3(\eta)}{N^2},$$

with

$$C_1(\eta) = \left(1 + \frac{b_1}{\eta}\right) \left(1 + \frac{3a_2}{2\xi \tilde{\sigma}_1} + \frac{9}{4\xi^2}\right) \cdot \frac{1 + \frac{3\tilde{\sigma}_1^2}{2n(\xi)\tilde{\sigma}_1}}{1 - \frac{3}{2k(\xi)}},$$

$$C_2(\eta) = \left(1 + \frac{b_2}{\eta}\right) \left(1 + \frac{3a_2}{2\xi \tilde{\sigma}_1} + \frac{9}{4\xi^2}\right) \cdot \frac{1 + \frac{3\tilde{\sigma}_1^2}{2n(\xi)\tilde{\sigma}_1}}{1 - \frac{3}{2k(\xi)}},$$

and

$$C_3(\eta) = \left(1 + \frac{3}{2\xi \tilde{\sigma}_1}\right) \left(1 + \frac{\text{Tr}(B^2)}{\eta}\right) \left(1 + \frac{\sigma_1^2}{k(\xi) \tilde{\sigma}_1} + \frac{\tilde{\sigma}_1^2}{k(\xi) \tilde{\sigma}_1}\right) \cdot \frac{1 + \frac{3\tilde{\sigma}_1^2}{2n(\xi)\tilde{\sigma}_1}}{1 - \frac{3}{2k(\xi)}}.$$

5.3. $L^2$-estimates. Building on the previous stability results, we deduce the following concentration inequalities. In this section, we fix a parameter $\eta > 0$ which denotes the imaginary part of the line on which the first part of the deconvolution process is achieved (see Section 2.3 for an explanation of the method). Then, for each $t \in \mathbb{R}$, the deconvolution process associates to each sample of $H$ or $M$ an estimator $\tilde{m}_{\mu_2}(t) := \tilde{m}_{\mu_2}(t + i\eta)$ of $m_{\mu_2}(t) := m_{\mu_2}(t + i\eta)$. Using the hypothesis previous results and Condition 2.1, we deduce the following estimates.

**Proposition 5.6.** Suppose that $N^2 \geq \frac{\max(C_{\text{thr susceptibility}}, C_{\text{thr susceptibility,B}})}{\xi^4 \tilde{\sigma}_1^2}$. Then,

$$\mathbb{E}(\tilde{m}_{\mu_2,\eta} - m_{\mu_2,\eta})^2 \leq \frac{K_1(\eta, N)}{N^2} + \frac{K_2(\eta, N)}{N^3} + \frac{K_3(\eta, N)}{N^4},$$
Hence, with Proof. By Proposition 5.3, for \( t \in \mathbb{R} \),
\[
K_1(\eta, N) = \frac{2^{6} \pi C_A C_2(N)^2}{\eta^3} + \frac{2^{6} \pi C_3(N)^2}{3^{2} N^2 \eta^3} \left( \sigma_1^2 + 4^{2} \sigma_1^2 \sigma_2^2 + a_4 \right),
\]

\[
K_2(\eta, N) = \frac{2 \pi C_1(N)}{\eta} \left( \frac{4 \sqrt{C_A C_2(N)}}{\eta^2} + \frac{4 \sqrt{2 C_3(N)}}{3 \eta} \sqrt{\sigma_1^2 + 4^{2} \sigma_1^2 \sigma_2^2 + a_4} \right),
\]

and
\[
K_3(\eta, N) = \frac{\pi C_1(N)^2}{\eta}.
\]

Proof. By Proposition 5.3, for \( z = t + i \eta \) with \( \eta > 2 \sqrt{2} \sigma_1 \),
\[
| m_B(z) - m_H(\omega_3) | \leq \frac{C_1(N)}{|z| N^2} + \frac{C_2(N)}{|z|} |\omega_A \delta_A| + \frac{C_3(N)}{|z|} |\omega_A \varepsilon_H|,
\]

with \( C_1(N), C_2(N), C_3(N) \) given in Proposition 5.3 as long as \( N^2 \geq \frac{4^{3} \max(C_{\text{threshold}}(3 \eta/4), C_{\text{threshold}}(3 \eta/4))}{\sigma_1^2 \eta^4} \).

Hence,
\[
\mathbb{E}(|m_B(z) - m_H(\omega_3)|^2) \leq \frac{1}{|z|^2} \left[ \frac{C_1(N)^2}{N^4} + 2 \frac{C_1(N)}{N^2} (C_2(N) \mathbb{E} |\omega_A \delta_A|) + C_3(N) \mathbb{E} |\omega_3 \delta_H| \right]
\]

\[
+ \frac{2}{N^2} \left( C_2(N)^2 \mathbb{E} |\omega_A \delta_A|^2 + C_3(N)^2 \mathbb{E} |\omega_3 \delta_H|^2 \right).
\]

First, since \( z \mapsto z \mu_1(z) \) is Lipschitz with constant \( \frac{\sigma_1}{(3 \omega_3)^2} \), by Condition 2.1 we have \( \mathbb{E}(|\omega_A \delta_A(\omega_A)|^2) \leq \frac{C_1 \sigma_1^2 (3 \omega_3)^4}{N^4} \leq \frac{\pi C_A}{\xi \eta^4 N^2} \), and the lower bound on \( \xi \) from Theorem 2.4 we thus have
\[
\mathbb{E}(|\omega_A \delta_A|^2) \leq \frac{24 C_A}{\eta^4 N^2}.
\]

Finally, by Lemma C.4
\[
\mathbb{E} \left| \omega_3 \delta_H(\omega_3) \right| \leq \frac{8}{N^2 (3 \omega_3)^2} \left( \mathbb{E} (\omega_A \delta_A(\omega_A)) + \mathbb{E} |\omega_3 \delta_H(\omega_3)| \right)
\]

\[
\leq \frac{25}{3^2 N^2 \eta^2} \left( \mathbb{E} (\omega_A \delta_A(\omega_A)) + \mathbb{E} |\omega_3 \delta_H(\omega_3)| \right).
\]

Hence,
\[
\left[ \frac{C_1(N)^2}{N^4} + 2 \frac{C_1(N)}{N^2} (\mathbb{E} |\omega_A \delta_A(\omega_A)| + \mathbb{E} |\omega_3 \delta_H(\omega_3)|) \right]
\]

\[
+ \frac{2}{N^2} \left( C_2(N)^2 \mathbb{E} |\omega_A \delta_A|^2 + C_3(N)^2 \mathbb{E} |\omega_3 \delta_H|^2 \right)
\]

\[
\leq \frac{C_1(N)^2}{N^4} + 2 \frac{C_1(N)}{N^2} \left( \frac{4 \sqrt{C_A C_2(N)}}{\eta^2} + \frac{4 \sqrt{2 C_3(N)}}{3 \eta} \sqrt{\mathbb{E} (\omega_A \delta_A(\omega_A)) + \mathbb{E} |\omega_3 \delta_H(\omega_3)|} \right)
\]

\[
+ \frac{25}{3^2 N^2 \eta^2} \left( \mathbb{E} (\omega_A \delta_A(\omega_A)) + \mathbb{E} |\omega_3 \delta_H(\omega_3)| \right).
\]

Since, \( \int_{\mathbb{R}} \frac{dt}{|t+i\eta|^2} = \frac{\pi}{\eta} \), the latter inequality yields
\[
\mathbb{E} \left( |\hat{m}_{\mu_2,\eta} - m_{\mu_2,\eta}|^2 \right)
\]

\[
\leq \frac{\pi C_1(N)}{\eta N^4} + 2 \frac{\pi C_1(N)}{\eta N^3} \left( \frac{4 \sqrt{C_A C_2(N)}}{\eta^2} + \frac{4 \sqrt{2 C_3(N)}}{3 \eta} \sqrt{\mathbb{E} (\omega_A \delta_A(\omega_A)) + \mathbb{E} |\omega_3 \delta_H(\omega_3)|} \right)
\]

\[
+ \frac{25}{3^2 N^2 \eta^2} \left( \mathbb{E} (\omega_A \delta_A(\omega_A)) + \mathbb{E} |\omega_3 \delta_H(\omega_3)| \right)
\]

\[
\leq \frac{K_1(\eta, N)}{N^2} + \frac{K_2(\eta, N)}{N^3} + \frac{K_3(\eta, N)}{N^4},
\]
with
\[ K_1(\eta, N) = \frac{2^5\pi C_A C_2(N)^2}{\eta^5} + \frac{2^6\pi C_3(N)^2}{3^2N^2\eta^4}\left(\text{Tr}(A^2) + \frac{4^2\text{Tr}(A^2)\text{Tr}(B^2) + \text{Tr}(A^4)}{3^2\eta^2}\right), \]
\[ K_2(\eta, N) = \frac{2\pi C_1(N)}{\eta}\left(\frac{4\sqrt{C_A^2 C_2(N)}}{\eta^2} + \frac{4\sqrt{2}C_3(N)}{3\eta}\sqrt{\text{Tr}(A^2) + \frac{4^2\text{Tr}(A^2)\text{Tr}(B^2) + \text{Tr}(A^4)}}\right), \]

and
\[ K_3(\eta, N) = \frac{\pi C_1(N)^2}{\eta}. \]

\[ \Box \]

Specifying the latter proposition for \( \eta = 2\sqrt{2}\sigma_1 \) yields then the proof of Theorem 2.8.

We get a similar result for the multiplicative case.

**Proposition 5.7.** Suppose that \( N^2 \geq \max(C_{\text{thres}, A}, C_{\text{thres}, B})(1 - \frac{1}{\log^{-1}(\kappa)})\frac{3\alpha}{\xi^3\sigma_1^4} \), and set
\[ t_N = N^2 \frac{\sqrt{3\xi^3\sigma_1^4}}{2\max(C_{\text{thres}, A}, C_{\text{thres}, B}) \left(1 - \frac{1}{\log^{-1}(\kappa)}\right)}. \]

Then,
\[ \mathbb{E}(|\hat{m}_{\mu_2, \eta} - m_{\mu_2, \eta}|^2_{L^2([-t_n, t_n])}) \leq \frac{K_1}{N^2} + \frac{K_2}{N^3} + \frac{K_3}{N^4}, \]
with
\[ K_1(\eta) = \frac{2\pi}{\kappa \sigma_1} \left(\frac{3^4C_2(\eta)^2C_A}{2^4g^{-1}(\kappa)^4\sigma_1^4} + \frac{\Delta(\kappa)C_3(\eta)^2}{g^{-1}(\kappa)^2\sigma_1^2}\right), \]
with \( \Delta(\kappa) \) is given in (39),
\[ K_2(\eta) = \frac{2\pi C_1(\eta)}{\eta} \left(\frac{9\sqrt{C_A}C_2(\eta)}{4g^{-1}(\kappa)^2\sigma_1^2} + \frac{\sqrt{\Delta(\kappa)}C_3(\eta)}{g^{-1}(\kappa)^2\sigma_1}\right), \]
and
\[ K_3(\eta) = \frac{\pi C_1(\eta)^2}{\eta}. \]

**Proof.** The proof is similar to the additive case, but we have to take into account the fact that the bound we got in Proposition 5.5 only holds on a sub-interval of \( \mathbb{R} \). Indeed, by this Proposition, for \( z = t + ik\sigma_1 \) with \( \kappa > g(\xi_0) \) and when
\[ N^2 \geq \frac{|\omega_3|}{\xi^3\sigma_1^4} \max\left(C_{\text{thres}, A}(\xi_0\sigma_1), C_{\text{thres}, B}(\xi_0\sigma_1)\right), \]
we have
\[ |\hat{m}_M(\omega_3)/z - m_B(z)| \leq \frac{C_1(\eta)}{N^2} + C_2(\eta)\delta_A + C_3(\eta)\delta_M \]
with \( C_1(\eta), C_2(\eta), C_3(\eta) \) given in Proposition 5.5. Hence, Since \( \omega_3(z) = h_{\mu_1}(\omega_1)z \) and \( \omega_1(z) \geq k(3\omega_3)\sigma_1 \geq k \circ g^{-1}(\kappa)\sigma_1 \), the condition on \( N \) is fulfilled when
\[ |z| \leq \frac{\xi^3\sigma_1^4N^2}{\max(C_{\text{thres}, A}, C_{\text{thres}, B})|h_{\mu_1}(\omega_1)|} \leq \frac{\xi^3\sigma_1^4N^2}{\max(C_{\text{thres}, A}, C_{\text{thres}, B})(1 + \frac{\sigma_1^2}{\kappa \log^{-1}(\kappa)\sigma_1})} \leq \frac{\xi^3\sigma_1^4N^2}{\max(C_{\text{thres}, A}, C_{\text{thres}, B})(1 + \frac{1}{\kappa \log^{-1}(\kappa)})}. \]

and, using the hypothesis on \( N \), this is satisfied when
\[ t \leq N^2 \frac{\sqrt{3\xi^3\sigma_1^4}}{2\max(C_{\text{thres}, A}, C_{\text{thres}, B}) \left(1 - \frac{1}{\log^{-1}(\kappa)}\right)}. \]
Set $t_N = N^2\frac{\sqrt{3}}{2\max(C_{\text{thres}, A}, C_{\text{thres}, B})\left(1 + \frac{1}{\log^{-1}(\xi)}\right)}$. Then,

$$
E(\hat{m}_{\mu_2, \eta} - m_{\mu_2, \eta})^2_{L^2([-t_N, t_N])} \leq \int_{\mathbb{R}} \frac{1}{|z|^2} \left( \frac{C_1(\eta)}{N^2} + 2C_2(\eta)E|\delta_A| + 2C_3(\eta)E|\delta_M| \right)
$$

$$
+ 2C_2(\eta)^2E|\delta_A|^2 + 2C_3(\eta)^2E|\delta_M|^2.
$$

By Lemma C.8

$$
E(|\delta_M|)^2 \leq \frac{\Delta(\kappa)}{g^{-1}(\kappa)^2\sigma_1^2 N^2},
$$

with

$$
\Delta(\kappa) = 8\left( \sqrt{\alpha_0(b^2_0 + \sigma_1^2\sigma_2^2)} + \frac{\alpha_\infty}{g^{-1}(\kappa)^2\sigma_1^2}(m_{A\overline{A}B}(1^3, 21^2) - 2m_{A\overline{A}B}(1^3, 1^3) + m_{A\overline{A}B}(21, 1^2) \right)
$$

(39) $\Delta(\kappa) = 8\left( \sqrt{\alpha_0(b^2_0 + \sigma_1^2\sigma_2^2)} + \frac{\alpha_\infty}{g^{-1}(\kappa)^2\sigma_1^2}(m_{A\overline{A}B}(1^3, 21^2) - 2m_{A\overline{A}B}(1^3, 1^3) + m_{A\overline{A}B}(21, 1^2) \right)$

Since $E(|\delta_A(z)|^2) \leq \frac{C_A(\frac{1}{N^2})}{N^2}$ by Condition 2.1 and since $\Im \omega_A \geq 3\Im \omega_3/3 \geq 2g^{-1}(\kappa)/3\sigma_1$ by Proposition 4.9

$$
E(|\delta_A(\omega_A)|^2) \leq \frac{3^4C_A}{2^4g^{-1}(\kappa)^4\sigma_1^4 N^2}
$$

Putting all the above bound together and using that $\int_{\mathbb{R}} \frac{dt}{|t|^2} = \frac{\pi}{\sigma_1}$ yields

$$
E(\hat{m}_{\mu_2, \eta} - m_{\mu_2, \eta})^2_{L^2([-t_N, t_N])} \leq \frac{K_1}{N^2} + \frac{K_2}{N^3} + \frac{K_3}{N^4},
$$

with

$$
K_1 = \frac{2\pi}{\kappa \sigma_1} \left( \frac{3^4C_2(\eta)^2C_A}{2^4g^{-1}(\kappa)^4\sigma_1^2 N^2} + \frac{C_3(\eta)^2\Delta(\eta)}{g^{-1}(\kappa)^2\sigma_1^2} \right),
$$

$$
K_2 = \frac{2\pi C_1(\eta)}{\eta} \left( \frac{9C_2(\eta)\sqrt{C_A}}{4g^{-1}(\kappa)^2\sigma_1^2} + \frac{2C_3(\eta)\sqrt{\Delta(\eta)}}{g^{-1}(\kappa)\sigma_1} \right),
$$

and

$$
K_3 = \frac{\pi C_1(\eta)^2}{\eta}.
$$

It remains to estimates the contribution of $m_{\mu_2, \eta}$ on $\mathbb{R} \setminus [-t_N, t_N]$ to the $L^2$ norm of $m_{\mu_2, \eta}$. Remark that we are only interested in the imaginary part of this function. Hence, we have the following estimates.

**Lemma 5.8.** Suppose that $N^2 \geq \max(C_{\text{thres}, A}, C_{\text{thres}, B})(1 - \frac{1}{\log^{-1}(\xi)}\frac{3e}{\xi\sigma_1})$. Then,

$$
\|\Im m_{\mu_2, \eta}\|_{L^2, \mathbb{R} \setminus [-t_N, t_N]} \leq \|\Im m_{\mu_2, \eta}\|_{L^2(\mathbb{R} \setminus [-t_N, t_N])} \leq \frac{\max(C_{\text{thres}, A}, C_{\text{thres}, B})^3}{N^6\Im^{3/2}(\xi\sigma_1)}^\frac{3}{2}.
$$

**Proof.** Note first that for a $\mu$ probability measure with second moment,

$$
m_{\mu}(z) = -\frac{1}{z} + \frac{1}{z^2}(-\mu(1) + \int_{\mathbb{R}} \frac{t^2}{t - z} dt).
$$

Hence, for $z$ such that $z = t + \imath \eta$,

$$
|\Im m_{\mu}(z)| \leq |\Im(z^{-1})| + \frac{\mu(1) + \mu(2)}{|\eta|^2} \leq \frac{1}{|z|^2} \left( \eta + \mu(1) + \frac{\mu(2)}{\eta} \right).
$$
Thus,
\[
\int_{t_N}^{+\infty} |\Im(m_B(t + i\eta))|^2 dt \leq \left( \eta + 1 + \frac{\text{Tr}(B^2)}{\eta} \right)^2 \int_{t_N}^{+\infty} \frac{dt}{(t^2 + \eta^2)^2} \leq \left( \eta + 1 + \frac{\text{Tr}(B^2)}{\eta} \right)^2,
\]
and using the definition of \( t_N \) yields
\[
\|\Im m_{\mu,\eta}\|_{L^2(\mathbb{R}[-t_N,t_N])} \leq \frac{2^3 \max(C_{\text{thres}}, A, C_{\text{thres}}, B) 3 (1 - \frac{1}{k_{\text{log}}^{-1}(\kappa)})^3 \left( \eta + 1 + \frac{\text{Tr}(B^2)}{\eta} \right)^2}{N^6 3^{3/2} (\xi \sigma_1)^9}.
\]

We can now prove Theorem 2.9

**Proof.** Set \( \eta = \kappa \sigma_1 \) with \( \kappa > g(\xi_0) \). Then,
\[
\mathbb{E}((\hat{C}_B(\eta) - C_B(\eta))^2) = \frac{1}{\pi^2} \int_{\mathbb{R}[-t_N,t_N]} \mathbb{E}(|\Im m_{B,\eta}(t + i\eta)|^2 dt) + \frac{1}{\pi^2} \int_{-t_N}^{t_N} \mathbb{E}(|\Im m_{B,\eta}(t + i\eta) - \Im m_{\mu,\eta}(t + i\eta)|^2 dt
\leq \int_{\mathbb{R}[-t_N,t_N]} |\Im m_{\mu,\eta}(t + i\eta)|^2 dt + \int_{-t_N}^{t_N} \mathbb{E}|\hat{m}_{\mu,\eta}(t + i\eta) - m_{\mu,\eta}(t + i\eta)|^2 dt.
\]
On the one hand, Lemma 5.8 yields
\[
\int_{\mathbb{R}[-t_N,t_N]} |\Im m_{\mu,\eta}(t + i\eta)|^2 dt \leq \frac{C_{\text{thres}}}{N^6},
\]
with
\[
K_4 = \max(C_{\text{thres}}, A(g^{-1}(\kappa), C_{\text{thres}}, B(g^{-1}(\kappa)))^3 (1 - \frac{1}{k_{\text{log}}^{-1}(\kappa)})^3.
\]
On the other hand, by Proposition 5.7,
\[
\int_{-t_N}^{t_N} \mathbb{E}|\hat{m}_{\mu,\eta}(t + i\eta) - m_{\mu,\eta}(t + i\eta)|^2 dt \leq \frac{K_1}{N^2} + \frac{K_2}{N^3} + \frac{K_3}{N^4},
\]
with \( K_1, K_2, K_3 \) given in Proposition 5.7. Hence,
\[
\mathbb{E}((\hat{C}_B(\eta) - C_B(\eta))^2_{L^2}) \leq \frac{K_1}{N^2} + \frac{K_2}{N^3} + \frac{K_3}{N^4} + \frac{K_4}{N^6}.
\]

**APPENDIX A. SUBORDINATION IN THE MULTIPLICATIVE CASE**

We prove here Theorem 2.6 which we recall here.

**Theorem A.1.** There exist two analytic functions \( \omega_1, \omega_3 : C_{g(\xi_0)} \sigma_1 \to \mathbb{C}^+ \) such that
\[
\omega_1 z = \omega_3 F_{\mu_3}(\omega_3) = \omega_3 F_{\mu_3}(\omega_1)
\]
for all \( z \in C_{g(\xi_0)} \sigma_1 \). Moreover, setting \( K_2(w) = -h_{\mu_3}(w) F_{\mu_3}(w) / z' \) for \( z \in \mathbb{C}^+ \) and \( w \in \mathbb{C}^+ \).

1. If \( \Re z < -K \) with \( K \) given in Lemma A.7, then
\[
\omega_3(z) = \lim_{n \to \infty} K_{\omega_3}^{\text{con}}(z),
\]
2. if \( z \in C_{g(\xi_0)} \sigma_1 \), then for all \( z' \in C_K \cap B(z, R(\xi_0)) \), with \( R(\xi_0) > 0 \) given in (45),
\[
\omega_3(z') = \lim_{n \to \infty} K_{\omega_3}^{\text{con}}(\omega_3(z)).
\]

Define the function \( k \) on \([2, +\infty[\) by \( k(t) = \frac{t + \sqrt{t^2 - 4}}{2} \).

**Lemma A.2.** Let \( \mu \) be a probability measure with finite variance \( \sigma^2 \). If \( w \in \mathbb{C}^+ \) is such that \( \Im \omega > 2\sigma \), then there exists \( z \in \mathbb{C}^+ \) with \( \Im z > k(\Im \omega / \sigma) \sigma \) such that \( F_{\mu}(z) = \omega \).
Proof. By Lemma 24, \( F_{\mu}^{< -1} \) is well-defined on \( \mathbb{C}_{2\sigma} \) and takes values in \( \mathbb{C}_\sigma \). Hence, if \( w \in \mathbb{C}^+ \) is such that \( \Im w > 2\sigma \), there exists \( z \in \mathbb{C}_\sigma \) such that \( F_{\mu}(z) = w \). By (6), \( |F_{\mu}(z) - z| \leq \frac{\sigma^2}{3|z|} \), which yields

\[
\Im w - \Im z \leq \frac{\sigma^2}{\Im(z)}.
\]

Hence, dividing the latter inequality by \( \sigma \) and setting \( t = \Im w/\sigma, \xi = \Im(z)/\sigma \), we have

\[
t - \xi \leq \frac{1}{\xi},
\]

or \( \xi^2 - t\xi + 1 \geq 0 \). Since \( t > 2 \) and \( \xi > 1 \), this implies that \( \xi \geq k(t) \), or equivalently

\[
\Im z > k(\Im w/\sigma \sigma),
\]

with \( k(t) = \frac{t + \sqrt{t^2 - 4}}{2} \).

For \( z \in \mathbb{C} \), set

\[
\Phi_z(\omega_1, \omega_3) = \left( \omega_1 z - \omega_3 \hat{F}_{\mu_1}(\omega_1) \right) / \left( \omega_1 z - \omega_3 \hat{F}_{\mu_3}(\omega_3) \right),
\]

where \( \hat{F}_{\mu}(w) = 1 + \frac{1}{w - \mu(t)} \) is defined in Section 3.1. Recall that we assume that \( \mu_1(1) = \mu_3(1) = 1 \), and we have \( \sigma_i = \mu_i(3) - \mu_i(2)^2 \) for \( i = 1, 3 \).

We first have the following relations between \( z, \omega_1 \) and \( \omega_3 \) when \( \Phi_z(\omega_1, \omega_3) = 0 \).

Lemma A.3. If \( \Im \omega_3 > 2\sigma_1 \) there exist \( z \in \mathbb{C}, \omega_1 \in \mathbb{C}^+ \) such that \( \Phi_z(\omega_1, \omega_3) = 0 \). Moreover, if we write \( \Im z = k_3 \sigma_1 + \Im \omega_3 = k_3 \sigma_3 \), we have

\[
k_3 \leq k_3 + 1 \left( k_3 \sigma_1 + \frac{\omega_3}{k_3 \sigma_1} \right)
\]

where \( k(t) = \frac{t + \sqrt{t^2 - 4}}{2} \in \mathbb{R} \) for \( t > 2 \).

Proof. Suppose that \( \Im \omega_3 > 2\sigma_1 \). Then, \( \Im \hat{F}_{\mu_3}(\omega_3) \geq \Im \omega_3 > 2\sigma_1 \), and thus by Lemma A.2 there exists \( \omega_1 \) such that \( \Im \omega_1 \geq k(\Im \hat{F}_{\mu_3}(\omega_3) / \sigma_1) \) such that \( \hat{F}_{\mu_1}(\omega_1) = \hat{F}_{\mu_3}(\omega_3) \). Since the function \( k \) is increasing, we have in particular \( \Im \omega_1 \geq k(\Im \omega_3) \). Since \( \hat{F}_{\mu_1}(\omega_1) = \hat{F}_{\mu_3}(\omega_3) \), we have by using (6)

\[
|\omega_1 - \omega_3| \leq |\omega_1 - \hat{F}_{\mu_1}(\omega_1) - \omega_3 + \hat{F}_{\mu_3}(\omega_3)|
\]

\[
\leq |\sigma_1^2 - \sigma_3^2 + \sigma_3^2 m_3(\omega_3) - \sigma_1^2 m_1(\omega_1)|
\]

\[
\leq \left( \frac{1}{k(\omega_3)} + \frac{\sigma_3^2}{k_3 \sigma_1} \right) \sigma_1 + |\sigma_3^2 - \sigma_1^2|,
\]

Setting \( z = \frac{\omega_3}{\omega_1} \hat{F}_{\mu_3}(\omega_3) \) yields then

\[
\Phi_z(\omega_1, \omega_3) = 0.
\]

Writing \( \hat{F}_{\mu_1}(\omega_1) = \omega_1 - \Var(\mu_1) - \sigma_1^2 G_\rho(\omega_1) \) gives also

\[
z = \omega_3 \frac{\hat{F}_{\mu_1}(\omega_1)}{\omega_1}
\]

\[
= \omega_3 - \frac{\omega_3}{\omega_1} \left( \Var(\mu_1) + \sigma_1^2 G_\rho(\omega_1) \right)
\]

\[
= \omega_3 - \left( 1 + \frac{\omega_3 - \omega_1}{\omega_1} \right) \left( \Var(\mu_1) + \sigma_1^2 G_\rho(\omega_1) \right).
\]

Hence, since \( \Var(\mu_1) \) is real,

\[
\Im z \leq \Im \omega_3 + \frac{\sigma_1^2}{3|\Im(\omega_1)|} + \frac{1}{3|\Im(\omega_1)|} \left( \frac{\sigma_1}{k(\omega_3)} + \frac{\sigma_3^2}{k_3 \sigma_1} + |\sigma_3^2 - \sigma_1^2| \right) \left( \Var(\mu_1) + \frac{\sigma_1^2}{3|\Im(\omega_1)|} \right).
\]

Using that \( \Im \omega_1 \geq k(\Im \omega_3) \) implies then

\[
\Im z \leq k_3 \sigma_1 + \frac{\sigma_1}{k(\omega_3)} + \frac{1}{k(\omega_3)} \left( \frac{1}{k(\omega_3)} + |\sigma_3^2 - \sigma_1^2| + \frac{\sigma_3^2}{k_3 \sigma_1} + \frac{\sigma_3^2}{k_3 \sigma_1} \right) \left( \frac{\sigma_1^2}{\sigma_1} + \frac{1}{k(\omega_3)} \right) \sigma_1.
\]
The inequality of the statement is then obtained by dividing by \( \tilde{\sigma}_1 \).

In the sequel, set \( H_3(w) = w \tilde{F}_{\mu_3}(w) \) and \( K_z(w) = -h_{\mu_1}(H_3(w)/z)z \).

**Lemma A.4.** Suppose that \( \Phi_z(\omega_1, \omega_3) = 0 \) with \( k_3 := \Im\omega_3/\tilde{\sigma}_1 > 2 \). Then, \( K_z(\omega_3) = \omega_3 \) and if \( |w - \omega_3| \leq r k_3 \tilde{\sigma}_1 \) with \( r < \min(\theta(k_3), 1/4) \), then \( K_z(w) \) is well-defined and

\[
|K_z'(w)| \leq (1 + rs(k_3))t((1 - r) k_3),
\]

where

\[
\theta(k_3) = \frac{k_3}{4} \left( \frac{5}{4} + \frac{\sigma_1^2}{k_3 \tilde{\sigma}_1} \right)^{-1} \left( 1 + \frac{2 \sigma_3^2}{3 k_3 \tilde{\sigma}_1} + \frac{16 \sigma_3^2}{9 k_3^2 \tilde{\sigma}_1^2} \right)^{-1},
\]

and

\[
s(k_3) = \frac{8k_3}{3k_3} \left( \frac{\sigma_1^2}{k_3 \tilde{\sigma}_1} + \frac{2 \sigma_3^2}{3 k_3 \tilde{\sigma}_1} + \frac{16 \sigma_3^2}{9 k_3^2 \tilde{\sigma}_1^2} \right) \left( 1 + \frac{\sigma_1^2}{k_3 \tilde{\sigma}_1} + \frac{1}{k_3} \right) \left( 1 + \frac{2 \sigma_1^2}{k_3 \tilde{\sigma}_1} \right).
\]

**Proof.** Note first that since \( \Phi_z(\omega_1, \omega_3) = 0, \omega_1 = \frac{3 \Im F_{\mu_3}(\omega_3)}{\omega_3} = H_3(\omega_3)/z \). Hence, using again the relation \( \Phi_z(\omega_1, \omega_3) = 0 \) yields \( K_z(\omega_3) = \omega_3 \). Moreover, for \( w \) such that \( H_3(w)z \in \mathbb{C}^+ \)

\[
K_z'(w) = -h_{\mu_1}'(H_3(w)/z)H_3'(w) = -\frac{zh_{\mu_1}(H_3(w))h_{\mu_3}'(H_3(w))}{wh_{\mu_1}(H_3(w))} \frac{H_3'(w)}{F_{\mu_3}(w)}.
\]

Since \( H_3'(w) = w \tilde{F}_{\mu_3}'(\omega_3) + F_{\mu_3}(\omega_3) \),

(41) \[ K_z'(w) = \frac{zh_{\mu_1}(H_3(w))h_{\mu_1}'(H_3(w))}{wh_{\mu_1}(H_3(w))} \left( 1 + \frac{\tilde{F}_{\mu_3}'(w)}{F_{\mu_3}(w)} \right) = \frac{zh_{\mu_1}(H_3(w))}{w} u_1 (2 - u_3), \]

with \( u_1 = \frac{\omega_1 h_{\mu_1}'(\omega_1)}{h_{\mu_1}(\omega_1)} \) and \( u_3 = 1 - \frac{\omega_3}{F_{\mu_3}(\omega_3)} \tilde{F}_{\mu_3}'(\omega_3) \). Remark then that [9] implies

\[
1 - \frac{z}{F_{\mu_3}(z)} \tilde{F}_{\mu_3}'(z) = 1 - h_{\mu_3}(z) \left( -\frac{1}{h_{\mu_3}(z)} + \frac{zh_{\mu_3}(z)}{h_{\mu_3}(z)^2} \right) = \frac{zh_{\mu_3}'(z)}{h_{\mu_3}(z)}.
\]

Moreover, by [10], for \( \mu \) a probability measure,

\[
\frac{zh_{\mu_3}'(z)}{h_{\mu_3}(z)} = -z L_{\mu}''(z) = -\text{Var}(\mu)(zm_{\mu_3}'(z)) = \text{Var}(\mu) \left( m_{\mu_3}(z) + \int_{\mathbb{R}} \frac{t}{(z-t)^2} d\rho_L(t) \right),
\]

which implies, when \( \mu \) is supported on \( \mathbb{R}^+ \) and using the formula \( \rho_L(1) = \frac{\text{Var}(\mu) + \text{Var}(\mu)^2/2}{\text{Var}(\mu)} \) before [10] yields

\[
\left| \frac{zh_{\mu_3}'(z)}{h_{\mu_3}(z)} \right| \leq \frac{\text{Var}(\mu)}{3z} + \text{Var}(\mu) \int_{\mathbb{R}} \left| \frac{t}{(z-t)^2} \right| d\rho_L(t) \leq \text{Var}(\mu) \left( \frac{1}{3z} + \frac{1}{3z^2} \int_{\mathbb{R}} |t| d\rho_L(t) \right) \leq \frac{\text{Var}(\mu)}{3z} + \frac{\text{Var}(\mu) + \text{Var}(\mu)^2/2}{3z^2}.
\]

Hence, putting this bound in (41) gives

\[
|K_z'(w)| \leq \left| \frac{zh_{\mu_1}(H_3(w))}{w} \right| \left( \frac{\sigma_1^2}{3\omega_1} + \frac{\sigma_1^2}{3(\omega_1)^2} \right) \left( 2 + \left( \frac{\sigma_3^2}{3\omega_3} + \frac{\sigma_3^2}{3(\omega_3)^2} \right) \right) \leq \left| \frac{zh_{\mu_1}(H_3(w))}{w} \right| \left( \frac{\sigma_1^2}{k_3 \tilde{\sigma}_1} + \frac{\sigma_1^2}{(k_3 \tilde{\sigma}_1)^2} \right) \left( 2 + \left( \frac{\sigma_3^2}{k_3 \tilde{\sigma}_1} + \frac{\sigma_3^2}{k_3^2 \tilde{\sigma}_1^2} \right) \right) = \left| \frac{zh_{\mu_1}(H_3(w))}{w} \right| t(\Im \omega_3/\tilde{\sigma}_1) \leq \left| \frac{zh_{\mu_1}(H_3(w))}{w} \right| t((1 - r) k_3)\]
when \(|w - \omega_3| \leq r k_3 \tilde{\sigma}_1\), where we used that \(t\) is decreasing. Temporarily write \(L(w) = \frac{z h_{\mu_1}(H_3(w))}{w}\), and remark that \(L(\omega_3) = \left| \frac{z h_{\mu_1}(\omega_3)}{\omega_3} \right| = 1\). First, by (6),

\[
H'(w) = \frac{1}{z} \left( \dot{F}_{\nu_3}(w) + w \dot{F}'_{\nu_3}(\omega) \right)
\]

(42)

\[
= w (2 + \frac{-\sigma_3^2 + \sigma_3^2 m_\rho(w)}{w} + \frac{\sigma_3^2}{w} \int_0^1 \frac{1}{(t - w)^2} d\rho(t)).
\]

Hence, since by (2) we have \(g(\xi) \geq \xi\),

\[
\frac{w}{z} = \frac{w - \omega_3}{z} + \frac{\omega_3}{z} \leq \frac{r k_3 \tilde{\sigma}_1}{g(k_3) \tilde{\sigma}_1} + |h_{\mu_1}(\omega_1)| \leq r + 1 + \frac{\sigma_3^2}{k_3 \tilde{\sigma}_1},
\]

which implies

\[
\left| \frac{1}{z} H'(w) \right| \leq 2 \left( 1 + r + \frac{\sigma_3^2}{k_3 \tilde{\sigma}_1} \right) \cdot \left( 1 + \frac{\sigma_3^2}{2 (1 - r) k_3 \tilde{\sigma}_1} + \frac{\sigma_3^2}{(1 - r)^2 k_3^2 \tilde{\sigma}_1^2} \right)
\]

\[
\leq 2 \left( \frac{5}{4} + \frac{\sigma_3^2}{k_3 \tilde{\sigma}_1} \right) \cdot \left( 1 + \frac{2 \sigma_3^2}{3 k_3 \tilde{\sigma}_1} + \frac{16 \sigma_3^2}{9 k_3^2 \tilde{\sigma}_1^2} \right),
\]

when we assume \(r \leq \frac{1}{2}\). Hence, first, for

\[
r \leq \frac{k_3 \tilde{\sigma}_1}{4} \left( \frac{5}{4} + \frac{\sigma_3^2}{k_3 \tilde{\sigma}_1} \right)^{-1} \cdot \left( 1 + \frac{2 \sigma_3^2}{3 k_3 \tilde{\sigma}_1} + \frac{16 \sigma_3^2}{9 k_3^2 \tilde{\sigma}_1^2} \right)^{-1} = \theta(k_3),
\]

we have

\[
\exists \frac{H_3(w)}{z} \geq \frac{H_3(\omega_3)}{z} - \frac{k_3 \tilde{\sigma}_1}{2} \geq \frac{k_3 \tilde{\sigma}_1}{2},
\]

so that \(h_{\mu_1}(H_3(w)/z)\) and \(L(w)\) are well-defined. Then,

\[
L'(w) = \frac{H'_3(w)}{w} h_{\mu_1}(H_3(w)/z) - \frac{z}{w^2} h_{\mu_1}(H_3(w)/z).
\]

By (42) and (44) for \(r \leq 1/4\)

\[
\left| \frac{H'_3(w)}{w} h_{\mu_1}(H_3(w)/z) \right| \leq \frac{8 \sigma_3^2}{k_3 \tilde{\sigma}_1^2} \left( 1 + \frac{2 \sigma_3^2}{3 k_3 \tilde{\sigma}_1} + \frac{16 \sigma_3^2}{9 k_3^2 \tilde{\sigma}_1^2} \right),
\]

and since \(|z/\omega| \leq 2 |z/\omega_3| \leq 2 \left| \frac{\dot{F}_{\nu_3}(\omega_1)}{\omega_1} \right|\), using again (44) yields for \(t \leq 1/4\)

\[
\left| \frac{z}{w^2} h_{\mu_1}(H_3(w)/z) \right| \leq \frac{8 \sigma_3^2}{3 k_3 \tilde{\sigma}_1} \left( 1 + \frac{\sigma_3^2}{k_3 \tilde{\sigma}_1} + \frac{1}{k_3 \tilde{\sigma}_1^2} \right) \left( 1 + \frac{2 \sigma_3^2}{k_3 \tilde{\sigma}_1} \right).
\]

Hence, with \(k_3 \geq k_3\),

\[
|L'(w)| \leq \frac{8 \sigma_3^2}{3 k_3 \tilde{\sigma}_1} \left( \frac{\sigma_3^2}{k_3 \tilde{\sigma}_1} \left( 3 + \frac{2 \sigma_3^2}{k_3 \tilde{\sigma}_1} + \frac{16 \sigma_3^2}{3 k_3^2 \tilde{\sigma}_1^2} \right) + \left( 1 + \frac{\sigma_3^2}{k_3 \tilde{\sigma}_1} + \frac{1}{k_3 \tilde{\sigma}_1^2} \right) \left( 1 + \frac{2 \sigma_3^2}{k_3 \tilde{\sigma}_1} \right) \right).
\]

Therefore, since \(L(\omega_3) = 1\), when \(|w - \omega_3| \leq t k_3 \tilde{\sigma}_1\) with \(r \leq \min(1/4, \theta(k_3))\),

\[
L(w) \leq 1 + r \frac{8 r k_3}{3 k_3} \left( \frac{\sigma_3^2}{k_3 \tilde{\sigma}_1} \left( 3 + \frac{2 \sigma_3^2}{k_3 \tilde{\sigma}_1} + \frac{16 \sigma_3^2}{3 k_3^2 \tilde{\sigma}_1^2} \right) + \left( 1 + \frac{\sigma_3^2}{k_3 \tilde{\sigma}_1} + \frac{1}{k_3 \tilde{\sigma}_1^2} \right) \left( 1 + \frac{2 \sigma_3^2}{k_3 \tilde{\sigma}_1} \right) \right) \cdot r k_3 \tilde{\sigma}_1 (1 - t((1 - r) k_3))(1 + r s(k_3)),
\]

\[
:= 1 + r s(k_3).
\]

Let us show that \((\omega_1(z), \omega_3(z))\) can be extended as long as \(\Im(z) \geq g(\xi_0)\) (recall the definition of \(\xi_0\) from Section 2.3). For \(k_3 > \xi_0\), set

\[
d(k_3) = \max_{0 \leq r \leq \max(\theta(k_3), 1/4)} \left( r k_3 \tilde{\sigma}_1 (1 - t((1 - r) k_3))(1 + r s(k_3)) \right),
\]
with $\theta_3(k_3)$ and $s(k_3)$ given in Lemma A.4. Note that $d(k_3)$ is well defined and positive, since $k_3 \mapsto (rk_3\tilde{\sigma}_1(1-t((1-r)k_3)(1+rs(k_3))))$ is continuous, equal to 0 at 0 and positive in the neighborhood of 0.

**Lemma A.5.** Suppose $z \in \mathbb{C}^+$ is such that there exists $\omega_1, \omega_3 \in \mathbb{C}^+$ with $\Im(\omega_3)/\tilde{\sigma}_1 := k_3 > \xi_0$, such that $\Phi_z(\omega_1, \omega_3) = 0$. Then, for all $z' \in B(z, R(\xi))$ with

$$R(k_3) = \frac{d(k_3)k(3)^2\tilde{\sigma}_1^2}{k(3)^2\tilde{\sigma}_1^2 + 4(\mu_1(3) - 2\mu_1(2)\mu_1(1) + \mu_1(1)^2)},$$

with $d(k_3)$ defined in (45), there exist $\omega_1(z'), \omega_3(z')$ such that $\Phi_{z'}(\omega_1(z'), \omega_3(z')) = 0$. Moreover, the function $\lambda' \mapsto (\omega_1(z'), \omega_3(z'))$ is analytic, and for $z' \in B(0, R(k_3))$,

$$K_{z'}^{\circ n}(\omega_3) \underset{n \to \infty}{\longrightarrow} \omega_3(z').$$

**Proof.** Set

$$r_0 = \arg\max_{0 \leq r \leq \max(\theta(k_3), 1/4)}(rk_3\tilde{\sigma}_1(1-t((1-r)k_3)(1+rs(k_3))).$$

From the bound $\left|K_{z}(\omega_3)\right|$ given in Lemma A.4 and the condition $r_0 \leq \max(\theta(k_3), 1/4)$, we deduce that for $w \in B(\omega_3, r_0k_3\tilde{\sigma}_1)$, we have

$$\left|K_{z}'(w)\right| \leq t((1-r_0)k_3)(1+r_0s(k_3)).$$

Hence, since $d(k_3) > 0$, $\left|K_{z}'(w)\right| < 1$ on $B(\omega_3, r_0k_3\tilde{\sigma}_1)$ and moreover

$$\left|d(K_{z}'(B(\omega_3, r_0k_3\tilde{\sigma}_1)), \partial B(\omega_3, r_0k_3\tilde{\sigma}_1))\right| \leq r_0k_3\tilde{\sigma}_1(1-r_0k_3\tilde{\sigma}_1)t((1-r_0)k_3)(1+r_0s(k_3)) \leq d(k_3).$$

We will now make $z$ vary. Since $h_{\mu_1}(w) = -1 - \sigma_1^2 \int_{R} \frac{1}{w-z}d\rho(t)$,

$$\frac{\partial}{\partial z}K_{z}(w) = -h_{\mu_1}(H_3(w)/z) - \frac{H_3(w)}{z}h'_{\mu_1}(H_3(w)/z) + \frac{1}{\frac{\sigma_1^2}{z}}\int_{R} \frac{1}{H_3(w)/z-t}d\rho(t)$$

$$= 1 + \sigma_1^2 \int_{R} \frac{1}{H_3(w)/z-t}d\rho(t) - \frac{\sigma_1^2}{z} \int_{R} \frac{1}{H_3(w)/t-z}d\rho(t)$$

$$= 1 - \int_{R} \frac{\sigma_1^2}{H_3(w)/z-t}d\rho(t).$$

Hence, by (44) on $B(\omega_3, \theta(k_3))$ we have $H_3(w)/z \in \mathbb{C}_{k(3)/2\tilde{\sigma}_1}$,

$$\left|\frac{\partial}{\partial z}K_{z}(w)\right| \leq 1 + \int_{R} \frac{\sigma_1^2}{k(3)/2\tilde{\sigma}_1}dt \leq 1 + \frac{4(\mu_1(3) - 2\mu_1(2) + 1)}{k(3)^2\tilde{\sigma}_1^2},$$

where we used (8) in the last inequality. Hence, if

$$|z' - z| < \frac{d(k_3)k(3)^2\tilde{\sigma}_1^2}{k(3)^2\tilde{\sigma}_1^2 + 4(\mu_1(3) - 2\mu_1(2)\mu_1(1) + \mu_1(1)^2)} := R(k_3),$$

then by (47),

$$K_{z'}(B(\omega_3, s(k_3))) \subset B(\omega_3, s(k_3)),$$

and by Denjoy-Wolf theorem, there exists $\omega_3(z') \in B(\omega_3, s(k_3))$ such that $K_{z'}(\omega_3(z')) = \omega_3(z')$, and

$$K_{z'}^{\circ n}(\omega_3) \underset{n \to \infty}{\longrightarrow} \omega_3(z').$$

The analyticity of the function $z' \mapsto \omega_3(z')$ is deduced by the implicit function theorem and the above bound on $K_{z}'$. 

An important property of the radius $R(\xi)$ is to be increasing in $\xi$, which reflects the fact that the subordination equation is more stable as the imaginary part of $z$ grows.

**Lemma A.6.** The function $\xi \mapsto R(\xi)$ is increasing from $[\xi_0, +\infty]$ to $[0, \infty]$. 
Proof. First, since $\xi \mapsto k(\xi)$ is increasing and $\xi \mapsto \frac{\xi}{k(\xi)}$ is decreasing, a careful look at the expression of $\theta(\xi)$ and $s(\xi)$ from Lemma A.4 shows that the former is increasing while the latter is decreasing. Hence, since $\xi \mapsto t(\xi)$ is decreasing in $\xi$ (see [3]), for all $r > 0$ fixed, $\xi \mapsto \xi (1 - (1 + rs(\xi))t((1 - r)\xi))$ is increasing in $\xi$. Hence, for $\xi \leq \xi'$,
\[
d(\xi) = \max_{0 \leq r \min(\theta(\xi),1/4)} (r\xi \sigma_1 (1 - t((1 - r)\xi))(1 + rs(\xi)))
\leq \max_{0 \leq r \min(\theta(\xi'),1/4)} (r\xi \sigma_1 (1 - t((1 - r)\xi))(1 + rs(\xi)))
\leq \max_{0 \leq r \min(\theta(\xi'),1/4)} (r\xi \sigma_1 (1 - t((1 - r)\xi'))(1 + rs(\xi'))) = d(\xi').
\]
Therefore, since $\frac{k(\xi)^2 \sigma_1^2}{k(\xi)^2 \sigma_1^2 + 4(m(3) - 2m(2))m(1) + m(1)^3}$ is increasing in $\xi$, $\xi \mapsto R(\xi)$ is increasing. \qed

We establish now a result similar to the one of [ATV17] Proposition 3.4, with slightly different hypothesis.

**Lemma A.7.** Suppose that $z \in \mathbb{C}^+$ is such that $d(z, [0, +\infty[) > K$, with $K$ being the positive root of
\[
K^2/3 - \sigma_1^2 \frac{27\sigma_3^3 + 2K/3}{4} \left( 1 + \frac{27(a_3 - 2a_2a_1 + a_1^2)(\sigma_3^2 + 2K/3)}{4K^2} \right) = 0.
\]
Then, $K_{zn}^\alpha(z)$ converges to $\omega_3(z)$ as $n$ goes to infinity.

**Proof.** The proof of this lemma is similar to the one of [ATV17] Proposition 3.4. \qed

We can now prove Theorem 2.6. Recall from Section 2.3 that $\xi_0$ is the unique positive root in $[\xi_g, +\infty[ of the the equation
\[
\xi_0 = \inf \left( \xi \geq \xi_g, \left( \frac{\sigma_1^2}{\xi k(\xi)} + \frac{1 + \sigma_1^2}{k(\xi)^2} \right) \left( 2 + \frac{\sigma_3^2}{\xi \sigma_1} + \frac{\sigma_2^3 + \sigma_1^3/2}{\xi^2 \sigma_1^2} \right) < 1 \right),
\]
and set $K = g(\xi_0)$, where $g$ is defined in (2). Note that the above equation implies (with $\sigma_3 > \sigma_1$), that $k(\xi_0)^2 \geq (2 + 1/\xi_0^2) \geq \frac{9}{4}$ which yields $\xi_0 \geq \frac{3}{2} + \frac{3}{2} = \frac{15}{4}$.

**Proof.** Let us fix $\eta > K$, and write $z_t = t + i\eta \tilde{\sigma}_1$ for $t \in \mathbb{R}$. Since $K = g(\xi_0)$ with $\xi_0 > \xi_g$, $\xi = g^{-1}(\eta)$ is well-defined. We write
\[
I = \left\{ t \in \mathbb{R}, 3\omega_3(z_t), K_z(z_t, \omega_3(z_t)) = \omega_3(z_t), 3\omega_3(z_t) \geq g^{-1}(\eta) \tilde{\sigma}_1 \right\}.
\]
We will show that $I = \mathbb{R}$. By Lemma A.7 if $t < -K$, then $K_{zn}^\alpha(z_t)$ converges to a fixed point $\omega_3(z_t)$ of $K_{zn}$ as $n$ goes to infinity and $\omega_3(z_t) \geq \xi_0 \tilde{\sigma}_1$. Hence, writing $3\omega_3(z_t) = k \tilde{\sigma}_1$, by Lemma A.3, $\eta \leq g(k)$ and since $g$ is increasing on $[\xi_g, +\infty[, k \geq g^{-1}(\eta)$. Hence, there exist $K'$, such that $-\infty, K'] \subset I$, and $I$ is non void.

If $t \in I$, then there exists $\omega_3(z_t)$ such that $K_z(\omega_3(z_t)) = \omega_3(z_t)$ and $3(\omega_3(z_t)) \geq g^{-1}(\eta) \tilde{\sigma}_1 \geq \xi_0 \tilde{\sigma}_1$. Hence, by Lemma A.6, for all $z' \in B(z_t, R(3(\omega_3(z_t))/\tilde{\sigma}_1))$, where $R$ is defined in (45), there exists $\omega_3(z')$ such that $K_z(\omega_3(z')) = \omega_3(z')$. By Lemma A.6 $R(\xi)$ is increasing in $\xi$, and $3\omega_3(z_t) \geq g^{-1}(\eta) \tilde{\sigma}_1$, thus $B(z_t, R(g^{-1}(\eta))) \subset B(z_t, R(3(\omega_3(z_t))/\tilde{\sigma}_1))$. Hence, considering $B(z_t, R(g^{-1}(\eta))) \cap \mathbb{R} + i\eta$ yields an open interval $I_t \subset \mathbb{R}$ such that for all $t' \in I_t$, there exists $\omega_3(z_{t'}) \in B(\omega_3(z_t), \theta(k))$ fixed point of $K_{zn}_z$ and
\[
\omega_3(z_{t'}) = \lim_{n \to +\infty} K_{zn}^\alpha(z_{t'}).
\]
Remark that assuming $\sigma_3 > \sigma_1$ yields $\theta(k) \leq \frac{k(k(k_3)^2)}{6} \sigma_1$ (see Lemma A.4), implying that $\exists \omega_3(z_{t'}) \geq \exists (\omega_3(z_{t'})) - \frac{1}{6} \exists (\omega_1(z_{t'})) \geq (k_3 - \frac{k(k_3)}{6} \exists (\omega_3(z_{t'})))$. Since $\exists (\omega_3(z_{t'})) > \xi_0 \tilde{\sigma}_1 > \frac{12\xi_0}{5} \tilde{\sigma}_1$, this implies that $\exists (\omega_3(z_{t'})) > \left( \frac{12\xi_0}{5} - \frac{1}{5} \right) \tilde{\sigma}_1 > 2 \tilde{\sigma}_1$. Hence, by Lemma A.3 $\exists (\omega_3(z_{t'})/\tilde{\sigma}_1 \geq g^{-1}(\exists z_t/\tilde{\sigma}_1) \geq g^{-1}(\eta)$. Hence, $I_t \subset I$, and thus $[t, t + R(g^{-1}(\eta))] \subset I$. The interval $I$ contains some interval $]-\infty, K[ and for all $t \in I$, $[t, t + R(g^{-1}(\eta))] \subset I$, thus $I = \mathbb{R}$.

By the previous argument, $\omega_3(z)$ is defined on $\mathbb{C}_{K \tilde{\sigma}_1}$. Using then Lemma A.5 yields the local analyticity and the convergence result of the lemma. \qed
Appendix B. Integration on the unitary group and Weingarten calculus

We prove here the integration formulas which are used in the paper. The main ingredient in the proofs is the Weingarten calculus as developed by Collins and Sniady [Col03, CS06], which allows to integrate polynomials of entries of a random unitary matrix with respect to the Haar measure. We only state the result for polynomials up to order six, which are the ones we use in this paper.

**Theorem B.1** (Weingarten calculus, [Col03].) Let $i, i', j, j' \in \mathbb{N}^r$ with $r \geq 1$. Then,

$$
\int_{U_N} u_{i_1 j_1} \cdots u_{i_r j_r} \bar{u}_{i'_1 j'_1} \cdots \bar{u}_{i'_r j'_r} = \sum_{\sigma, \tau \in S_r \atop \iota \sigma = \iota' \tau = \tau'} W_{N,r}(\sigma \tau^{-1}),
$$

where $S_r$ denotes the symmetric group of size $r$ and $W_{N,r} : S_r \to \mathbb{Q}$ is the Weingarten function whose values at $\sigma$ only depends on the cycle structure of the permutation. Moreover,

$$
W_{N,1}(\text{Id}) = \frac{1}{N},
$$

$$
W_{N,2}(1^2) = \frac{1}{N^2(1 - N^{-2})}, \quad W_{N,3}(2) = \frac{-1}{N^3(1 - N^{-2})},
$$

$$
W_{N,3}(1^3) = \frac{1 - 2N^{-2}}{N^3(1 - N^{-2})(1 - 4N^{-2})}, \quad W_{N,3}(21) = \frac{-1}{N^4(1 - N^{-2})(1 - 4N^{-2})},
$$

$$
W_{N,3}(3) = \frac{2}{N^5(1 - N^{-2})(1 - 4N^{-2})},
$$

where $(3^a2^b1^c)$ denotes a permutation with $a$ cycles of length 3, $b$ cycles of length 2 and $c$ cycles of length 1.

Using the latter theorem, we prove the following asymptotic formula for product of matrices $A$ and $UBU^*$.

**Lemma B.2.** Let $A, B \in M_n(\mathbb{C})$ and $U \in U_n$ Haar unitary, and suppose that $A, B$ are diagonal. Then, $\mathbb{E}[UBU^* A] = \text{Tr}(B)A$,

$$
(1 - 1/N^2)\mathbb{E}[UBU^* A] = \left( \text{Tr}(A) \text{Tr}(B^2) - \text{Tr}(A) \text{Tr}(B)^2 + A(\text{Tr}(B)^2 - \frac{1}{N^2} \text{Tr}(B^2)) \right),
$$

and when $\text{Tr}(B) = 1$,

$$
(1 - 1/N^2)(1 - 4/N^2)\mathbb{E}[UBU^* A] = A^2 \left( 1 + (1 + 4/N^2) \text{Tr}(B^3)/N^2 - 6/2 \text{Tr}(B^3) \right)
$$

$$
+ A \left( 2(\text{Tr}(B^2) - 1) + 4/N^2(\text{Tr}(B^2) - \text{Tr}(B^3)) \right)
$$

$$
+ \left( \text{Tr}(B^3) + \text{Tr}(B^2) \text{Tr}(A^2) + 2 - \text{Tr}(A^2) - 3 \text{Tr}(B^2) \right).
$$

**Proof.** We only explain the proof of the first equality, since the proof of the second uses similar pattern. Note first that $\mathbb{E}[UBU^* A]$ commutes with $A$, and thus is diagonal when $A$ has distinct diagonal entries. By a continuity argument, $\mathbb{E}[UBU^* A]$ is thus diagonal. Write $U = (u_{ij})_{1 \leq i, j \leq n}$ and expand $\mathbb{E}[UBU^* A]_{ij}$ as

$$
\mathbb{E}[UBU^* A]_{ij} = \sum_{k,j,s=1}^n \mathbb{E}(u_{ik} B_{kk} u_{jk} A_{jj} u_{js} B_{ss} \bar{u}_{is})
$$

$$
= \sum_{k,j,s=1}^n B_{kk} A_{jj} B_{ss} \mathbb{E}(u_{ik} u_{jk} u_{js} \bar{u}_{is}).
$$
Let $1 \leq i, j \leq n$ and $1 \leq k, s \leq n$. Then, by Theorem B.1 and summing on permutations of $S_2$,

$$\mathbb{E}(u_{ik}u_{is}u_{js}u_{jk}) = \begin{cases} -\frac{1}{N(N^2-1)} & \text{if } i \neq j, k \neq s \\ \frac{1}{N(N+1)} & \text{if } i = j, k \neq s \text{ or } i \neq j, k = s \\ \frac{2}{N(N+1)} & \text{if } i = j, k = s \end{cases}$$

Hence, using the latter formula yields

$$\mathbb{E}(UBU^*AUBU^*)_{ij} = \sum_{j \neq i} A_{jj} \left[ -\frac{1}{N(N^2-1)} B_{kk} B_{ss} + \sum_{k \neq s} \frac{1}{N(N+1)} B_{kk}^2 \right]$$

$$+ A_{ii} \left[ \sum_{k \neq s} \frac{2}{N(N+1)} B_{kk} B_{ss} + \sum_{k = 1}^n \frac{2}{N(N+1)} B_{kk}^2 \right]$$

$$= (\text{Tr}(A) - A_{ii}/N) \left[ -\frac{1}{1 - 1/N^2} \text{Tr}(B)^2 + \text{Tr}(B^2)(\frac{1}{N+1} + \frac{1}{N^2 - 1}) \right]$$

$$+ A_{ii} \left[ \frac{1}{1 + 1/N} \text{Tr}(B^2) - \frac{1}{1 + N} \text{Tr}(B^2) \right]$$

$$= \frac{1}{1 - 1/N^2} \left[ \text{Tr}(A) \text{Tr}(B^2) - \text{Tr}(A) \text{Tr}(B^2) + A_{ii}(\text{Tr}(B)^2 - \frac{1}{N^2} \text{Tr}(B^2)) \right]$$

A similar computation yields the third equality. We used [PKN19] to achieve the computation.

Using Lemma B.2 yields directly formulas for expectation of trace of products. For two finite sequences $\pi, \pi'$ of $N$ of length $r \geq 1$, set

$$m_{A\pi B}(\pi, \pi') = \mathbb{E} \text{Tr}(A^{\pi_1}U B^{\pi_2}U^* \ldots A^{\pi_r}U B^{\pi'_r}U^*)$$

**Lemma B.3.** Suppose that $A, B \in \mathcal{M}_N(\mathbb{C})$. Then,

$$m_{A\pi B}(1, 1) = \text{Tr}(A) \text{Tr}(B),$$

$$m_{A\pi B}(1^2, 1^2) = \frac{1}{1 - N^{-2}} \left[ \text{Tr}(A^2) \text{Tr}(B^2) + \text{Tr}(A)^2 \text{Tr}(B^2) - \text{Tr}(A)^2 \text{Tr}(B^2) - \frac{1}{N^2} \text{Tr}(A^2) \text{Tr}(B^2) \right]$$

and when $\text{Tr}(B) = 1$,

$$m_{A\pi B}(1^3, 1^3) = \frac{1}{(1 - 1/N^2)(1 - 4/N^2)} \left( \text{Tr}(B^3) + 3 \text{Tr}(B^2)^2 \text{Var}(\mu_A) \right.$$ 

$$+ (\text{Tr}(A^3) - 2 \text{Tr}(A)^2 + 2 \text{Tr}(A)^2) + \tilde{\epsilon}_N),$$

with

$$\tilde{\epsilon}_N = \frac{6}{N^2} (\text{Tr}(A^2) \text{Tr}(B^2) - \text{Tr}(B^2) \text{Tr}(A^3) - \text{Tr}(A)^2 \text{Tr}(B^4)) + \frac{4}{N^4} \text{Tr}(A^3) \text{Tr}(B^4),$$

and

$$m_{A\pi B}(1^3, 21^2) = \frac{1}{(1 - 1/N^2)(1 - 4/N^2)} \left( \text{Tr}(B^4) + (\text{Tr}(B^3)^2 + 2 \text{Tr}(B^3)) \text{Var}(\mu_A) \right.$$ 

$$+ \text{Tr}(B^2)(\text{Tr}(A^3) - 2 \text{Tr}(A)^2 + 2 \text{Tr}(A)^3) + \tilde{\epsilon}_N),$$

with

$$\tilde{\epsilon}_N = \frac{1}{N^2} (\text{Tr}(A^3)(\text{Tr}(B^4) - 2 \text{Tr}(B^2)^2 - 4 \text{Tr}(B^3)) + \text{Tr}(A^3)(2 \text{Tr}(B^2)^2$$

$$- 6 \text{Tr}(B^4) + 4 \text{Tr}(B^2)) + \frac{1}{N^4} \text{Tr}(A^3) \text{Tr}(B^4).$$

**Appendix C. Analysis on the unitary group**

We provide here concentration inequalities on the unitary group which imply all our concentration results concerning the Stieltjes transform. Proofs are adapted from Kargin’s approach to get bounds only depending on first moments of the matrix involved.
C.1. Poincaré inequality and concentration results. Several concentration inequalities exist on the unitary group [AGZ10 BE85]. In this paper, we only use Poincaré’s inequality, which has the fundamental property of having an error term which is averaged on the unitary group. Poincaré’s inequality exists on every compact Riemannian manifolds without boundary, for which the Laplacian has a discrete spectrum.

**Theorem C.1** (Poincaré’s inequality). Suppose that $M$ is a compact manifold without boundary and with volume form $\mu$, and let $\lambda_1 > 0$ be the first non-zero eigenvalue of the Laplacian on $M$. Then, for all $f \in C^2(M)$ such that $\int_M f d\mu = 0$,

$$\int_M f^2 d\mu \leq \frac{1}{\lambda_1} \int_M \|\nabla f\|^2 d\mu.$$ 

Proof of this theorem is a direct consequence of the integration by part formula on $M$. In the case of the unitary group $U_N$ the spectrum of the Laplacian can be explicitly computed using the representation theory of the group (see [Hum72]), and the first eigenvalue of the Laplacian is simply equal to $N$. Hence, we deduce from Poincaré’s inequality the following concentration inequality for the unitary group.

**Corollary C.2** (Poincaré’s inequality). For all $f \in C^2(U_N)$ such that $\int_{U_N} f d\mu = 0$, where $\mu$ denotes the Haar measure on $U_N$,

$$\int_{U_N} f^2 d\mu \leq \frac{1}{N} \int_{U_N} \|\nabla f\|^2 d\mu.$$ 

In the sequel the functions $f$ we will studied are traces of matrices involving the various resolvent. We will use several times the generalized matrix Hölder inequality for Schatten $p$-norm. Recall that the Schatten $p$-norm of a matrix $X \in M_N(\mathbb{C})$ is defined by

$$\|X\|_p = [N \text{Tr}((X^*X)^{p/2})]^{1/p}.$$ 

Then, if $X_1, \ldots X_k \in M_N(\mathbb{C})$ and $\alpha_1, \ldots, \alpha_k \in [1, +\infty]$, then

$$(49) \quad \|X_1 \ldots X_k\|_r \leq \prod_{i=1}^k \|X_i\|_{\alpha_i},$$

where $\frac{1}{r} = \sum_{i=1}^k \frac{1}{\alpha_i}$. Remark that the matrix Hölder is not a trivial consequence of the usual Holder inequality, and its proof is quite involve (see [Ser10] 7.3).

C.2. Application to the additive convolution.

**Lemma C.3.** For $H = A + UBU^*$ and $z \in \mathbb{C}^+$ with $\eta = \Im(z)$ and for $T \in M_n(\mathbb{C})$,

$$\mathbb{E} \left( |f_T(z) - \mathbb{E}(f_T(z))|^2 \right) \leq \frac{4 \min(\text{Tr}(B^2)^{2/\alpha} \text{Tr}(T^2)^{2/\beta}, \text{Tr}(B^2)\|T\|_\infty^2, \text{Tr}(T^2)\|B\|_\infty)}{\eta^4 N^2},$$

where $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{2}$.

**Proof.** We prove only the second part, since the first part is a straightforward adaption of the proof of [Kar15] using $L^2$-norm. By (C.2), for all function $f$ which is $C^2$ on $U_N$ of mean zero, $\mathbb{E}(|f|^2) \leq \frac{1}{2} \mathbb{E}(\|\nabla f\|^2)$. Let us apply this to the map $f_T$. Since $d\chi(X - z)^{-1} = (X - z)^{-1}X(X - z)^{-1}$, applying the chain rule for $f_T$ at $U \in U_n$ yields for $X$ anti-hermitian

$$\nabla_U f_T(X) = \text{Tr}(TG[X, \hat{B}]G) = \text{Tr}([\hat{B}, GTG]X),$$

where $\hat{B} = UBU^*$. Hence,

$$\|\nabla_U f_T\|_2 = \frac{1}{\sqrt{N}} \|\hat{B}, GTG\|_2 \leq \frac{2}{\sqrt{N} \eta^2} \min(\|B\|_{\alpha} \|T\|_{\beta}, \|B\|_2 \|T\|_\infty)$$

with $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{2}$, where we applied matrix Hölder inequality in the last inequality. Therefore,

$$\text{Var}(f_T) \leq \frac{4}{N^2 \eta^4} \min(\|B\|_{\alpha}^2 \|T\|_{\beta}^2, \|B\|_2^2 \|T\|_\infty^2) \leq \frac{4 \min(\text{Tr}(B^2)^{2/\alpha} \text{Tr}(T^2)^{2/\beta}, \text{Tr}(B^2)\|T\|_\infty^2, \text{Tr}(T^2)\|B\|_\infty)}{N^2 \eta^4}.$$ 

$\square$
Lemma C.4. For $H = A + UBU^*$ and $z \in \mathbb{C}^+$ with $\eta = \Im(z)$,
\[
\text{Var}(zm_H) \leq \frac{8}{N^2\eta^2} \left( \text{Tr}(A^2) + \frac{\text{Tr}(A^2)\text{Tr}(B^2) + \text{Tr}(A^4))}{\eta^2} \right),
\]
and for $T \in M_n(\mathbb{C})$,
\[
\text{Var}(zf_T) \leq \frac{12}{N^2\eta^2} \left( \text{Tr}(|T|^2) \text{Tr}(B^2) + \min \left( \frac{||T||_\infty^2 \text{Tr}(A^2) \text{Tr}(B^2) + \text{Tr}(B^4))}{\eta^2}, \frac{\mathbb{E}(\text{Tr}((A\tilde{B}^2A)^{\alpha_1/2})^{2}||T||_{\beta_1}^2 + \text{Tr}(B^{2\alpha_2})^{2/\alpha_2}||T||_{\beta_2}^2)^2}{\eta^2} \right) \right)
\]
for any $\alpha_1, \beta_1, \alpha_2, \beta_2$ satisfying
\[
\frac{1}{\alpha_1} + \frac{1}{\beta_1} = \frac{1}{\alpha_2} + \frac{1}{\beta_2} = \frac{1}{2}.
\]

Proof. We only prove the second statement, since the first one follows a similar pattern. As in the latter lemma, taking the derivative of $zf_T$ at $U \in U_N$ yields for $X$ anti-hermitian
\[
\nabla_U(zf_T)(X) = \text{Tr}(TG[X, \tilde{B}]G) = \text{Tr}([\tilde{B}, zGTG]X) = \text{Tr}(\tilde{B}TG + GT\tilde{B} + \tilde{B}(A + \tilde{B})GTG - GTG(A + B)\tilde{B}, X),
\]
where $\tilde{B} = UBU^*$. Hence,
\[
\|\nabla_Uzf_T\|^2 = \frac{1}{N} \left( 2||T\tilde{B}||_2 + 2||\tilde{B}AG\tilde{T}G||_2 + 2||\tilde{B}^2GTG||_2 \right)^2 \leq \frac{12}{N^2} \left( ||T\tilde{B}||_2^2 + ||\tilde{B}AG\tilde{T}G||_2^2 + ||\tilde{B}^2GTG||_2^2 \right).
\]
First, $\mathbb{E}(||T\tilde{B}||_2^2) = \mathbb{E}(\text{Tr}(TT^*\tilde{B}^2) = \text{Tr}(TT^*)\mu_2(B)$. To get the bound with $||T||_\infty$, we simply bound the last two terms by
\[
\mathbb{E}(||\tilde{B}AG\tilde{T}G||_2) \leq \frac{||T||_\infty^2 \text{Tr}(A^2) \text{Tr}(B^2)}{\eta^4},
\]
and by
\[
\mathbb{E}(||\tilde{B}^2GTG||_2^2) \leq \frac{||T||_\infty^2 \text{Tr}(B^4)}{\eta^4}.
\]
To get the bound with finite moments of $T$, we apply matrix Hölder inequality to get
\[
\mathbb{E}(||\tilde{B}AG\tilde{T}G||_2^2) \leq \frac{1}{\eta^2} \mathbb{E}(||\tilde{B}A||_\alpha ||T||_\beta^2) \leq \frac{1}{\eta^2} \mathbb{E}(||\tilde{B}A||_\alpha^{2}||T||_\beta^2)
\]
and
\[
\mathbb{E}(||\tilde{B}^2GTG||_2^2) \leq \frac{1}{\eta^2} \text{Tr}(B^{2\alpha})^{2/\alpha}||T||_{\beta}^2
\]
for any $\alpha, \beta$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{2}$. Hence, using Poincaré inequality yields
\[
\text{Var}(zf_T) \leq \frac{12}{N^2\eta^2} \left( \text{Tr}(|T|^2) \text{Tr}(B^2) + \min \left( \frac{||T||_\infty^2 \text{Tr}(A^2) \text{Tr}(B^2) + \text{Tr}(B^4))}{\eta^2}, \frac{\mathbb{E}(||\tilde{B}A||_\alpha^{2}||T||_{\beta_1}^2 + \text{Tr}(B^{2\alpha_2})^{2/\alpha_2}||T||_{\beta_2}^2)^2}{\eta^2} \right) \right)
\]
for any $\alpha_1, \beta_1, \alpha_2, \beta_2$ satisfying
\[
\frac{1}{\alpha_1} + \frac{1}{\beta_1} = \frac{1}{\alpha_2} + \frac{1}{\beta_2} = \frac{1}{2}.
\]
Since $||\tilde{B}A||_\alpha = \text{Tr}((A\tilde{B}^2A)^{\alpha/2}$, the result follows.

We give a similar result when the matrix $T$ of the latter lemma also depends on $UBU^*$.  \qed
Lemma C.5. Let $z \in \mathbb{C}^+$ and for $T \in M_n(\mathbb{C})$ set $\tilde{f}_T = \text{Tr}(TUBU^*G_H)$. Then,
\[
\mathbb{E}\left( \left| \tilde{f}_T(z) - \mathbb{E}(\tilde{f}_T(z)) \right|^2 \right) \leq \frac{4}{N^2\eta^4} \left( \eta^2 \text{Tr}(|T|^2) \text{Tr}(B^2) + \sqrt{\text{Tr}(|T|^4) \text{Tr}(B^4)} \right) + \sqrt{\text{Tr}(B^4)m_{\mathbb{R}^2B^2}(1^2, 1^2)}.\]

Proof. Consider the map $\tilde{f}_T : U \mapsto \text{Tr}(TUBU^*G_H)$. Then,
\[
d_U \tilde{f}_T(X) = \text{Tr}(T[\tilde{B}, X]G_H + T\tilde{B}G_H[\tilde{B}, X]G_H) = \text{Tr}([G_HT\tilde{B}]X) + \text{Tr}([G_H\tilde{T}\tilde{B}G_H, \tilde{B}]X).
\]
Hence, by Cauchy-Schwarz inequality,
\[
\|d_U \tilde{f}_T\|^2 \leq 4\frac{\text{Tr}(|T|^2) \text{Tr}(B^2) + \sqrt{\text{Tr}(|T|^4) \text{Tr}(B^4)}}{\eta^2} + \frac{4\mathbb{E}(\text{Tr}([TB^4]^{1/2} \text{Tr}(B^4)^{1/2})}{\eta^4}.
\]
Since by Lemma B.2,
\[
\mathbb{E}(\text{Tr}(|TB^4|^{1/2})) = \mathbb{E}(\text{Tr}((B^2TB)^{1/2})) = \text{Tr}(B^4) \text{Tr}(|T|^{1/2} \text{Tr}(B^2)^{1/2} + \text{Tr}(|T|^4) \text{Tr}(B^2)^2 - \text{Tr}((B^2)^2 \text{Tr}(|T|^2)^2),
\]
the results follows by Poincaré’s inequality. \qed

C.3. Application to the multiplicative convolution. We now state the concentration results for the multiplicative case.

Lemma C.6. For $M = A^{1/2}UBU^*A^{1/2}$ and $z \in \mathbb{C}^+$ with $\eta = \mathbb{E}(z)$ and for $T \in M_n(\mathbb{C})$,
\[
\mathbb{E}\left( \left| f_T(z) - \mathbb{E}(f_T(z)) \right|^2 \right) \leq \frac{4}{N^2\eta^4} \min \left( K\|T\|_\infty^2, \|B\|_\infty^2, \|T\|_2^2, \|A\|_2^2 \right).
\]
with $K = \min \left( \text{Tr}(B^4)\|A\|_\infty \sqrt{\text{Tr}(A^2)m_{\mathbb{R}^2B^2}(1^2, 1^2)} \right)$, and $\alpha, \beta > 0$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{2}$.

Proof. Like in the previous lemmas, the aim is to compute bound the derivative of the map $f_T : U \mapsto \text{Tr}(T(z - A^{1/2}UBU^*A^{1/2}))$. Using the chain rule, we get
\[
d_U f_T(X) = \text{Tr}(TGA^{1/2}X, \tilde{B}A^{1/2}G) = \text{Tr}([\tilde{B}, A^{1/2}GTGA^{1/2}]X),
\]
and with $\tilde{B} = UBU^*$. Hence, for all $U \in U_n$,
\[
\|d_U f_T\|_2 \leq \frac{1}{\sqrt{N}} \|[\tilde{B}, A^{1/2}GTGA^{1/2}]\|_2 \leq \frac{2}{\sqrt{N}} \|\tilde{B}A^{1/2}GTGA^{1/2}\|_2.
\]
We deduce that
\[
\mathbb{E}\left( \|d_U f_T\|_2^2 \right) \leq \frac{4}{N^2} \mathbb{E}(\|\tilde{B}A^{1/2}GTGA^{1/2}\|_2^2) \leq \frac{4}{N^2} \|\tilde{B}A^{1/2}GTGA^{1/2}\|_2^2.
\]
Then, either
\[
\mathbb{E}\left( \|d_U f_T\|_2^2 \right) \leq 4\frac{\|A\|_\infty\|T\|_\infty^2}{N\eta^4} \mathbb{E}(\text{Tr}(A\tilde{B}^2)) \leq 4\frac{\|B\|_\infty^2\|T\|_2^2 \|A\|_2^2}{N\eta^4},
\]
where we have used that $\mathbb{E}(\text{Tr}(F_1UF_2U^*)) = \text{Tr}(F_1) \text{Tr}(F_2)$ for $F_1, F_2 \in M_N(\mathbb{C})$, or by applying the matrix Hölder’s inequality,
\[
\mathbb{E}\left( \|d_U f_T\|_2^2 \right) \leq 4\frac{\|B\|_\infty^2\|T\|_2^2 \|A\|_2^2}{N\eta^4},
\]
for $\alpha, \beta > 0$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{2}$. To get a bound in terms of moments of $A$, we used Cauchy-Schwarz inequality to get
\[
\mathbb{E}\left( \|d_U f_T\|_2^2 \right) \leq \frac{4}{N^2\eta^4} \mathbb{E}(\sqrt{\text{Tr}(AB^2AB^2)}/\sqrt{\text{Tr}(A^2)}) \leq \frac{4\|T\|_2^2}{N\eta^4} \sqrt{\mathbb{E}(\text{Tr}((AB^2AB^2)/\sqrt{\text{Tr}(A^2))} \leq \frac{4\|T\|_2^2}{N\eta^4} \sqrt{\text{Tr}(A^2)} \sqrt{m_{\mathbb{R}^2B^2}(1^2, 1^2)}. \]
Using Poincaré inequality on the unitary group concludes the proof.

Lemma C.7. For $M = A^{1/2}UBU^*A^{1/2}$ and $z \in \mathbb{C}^+$ with $\eta = \mathcal{O}(z)$, for $T \in M_{n}(\mathbb{C})$,

$$\mathbb{E}\left(\left|z^{f_T}(z) - \mathbb{E}(z^{f_T}(z))\right|^2\right) \leq \frac{8\|T\|_{\infty}^2 \|A\|_{\infty}^2}{\eta^2 N^2} (\text{Tr}(B^2) + m_{A \otimes B}(1^3, 21^2)/\eta^2).$$

Proof. As in the previous lemma, we have

$$d_{U}f_{T}(X) = \text{Tr}(TGA^{1/2}[X, \tilde{B}]A^{1/2}G) = \text{Tr}([\tilde{B}, A^{1/2}GTA^{1/2}]X),$$

and with $\tilde{B} = UBU^*$. Moreover, for all $U \in U_n$,

$$z\tilde{B}A^{1/2}GTA^{1/2} = \tilde{B}A^{1/2}(-1 + A^{1/2}\tilde{B}A^{1/2}G)TA^{1/2} = -\tilde{B}A^{1/2}GTA^{1/2} + \tilde{B}A^{1/2}GTA^{1/2},$$

and likewise

$$zA^{1/2}GTA^{1/2}\tilde{B} = -A^{1/2}GTA^{1/2}\tilde{B} + A^{1/2}GTA^{1/2}\tilde{B}A\tilde{B}.$$

Hence,

$$\mathbb{E}(\|zd_{U}f_{T}(X)\|^2) \leq \frac{8\|T\|_{\infty}^2 \|A\|_{\infty}^2}{\eta^2 N} \mathbb{E}(\text{Tr}(A\tilde{B})^2) + \frac{\|A\|_{\infty}^2 \|T\|_{\infty}^2 \mathbb{E}(\text{Tr}(A\tilde{B}A\tilde{B}A\tilde{B}))}{\eta^4}.$$  

A computation gives that $\mathbb{E} \text{Tr}(A\tilde{B}^2) = \text{Tr}(A)\text{Tr}(B^2) = \text{Tr}(B^2)$ and by Lemma B.3 we have $\mathbb{E} \text{Tr}(A\tilde{B}A\tilde{B}A\tilde{B}) = m_{A \otimes B}(1^3, 21^2)$. Forgetting the negative terms in the error term yields then

$$\mathbb{E}(\|zd_{U}f_{T}(X)\|^2) \leq \frac{8\|T\|_{\infty}^2 \|A\|_{\infty}^2}{\eta^2 N} \left(\frac{\text{Tr}(B^2)}{\eta^2} + \frac{\text{Tr}(B^4) + (\text{Tr}(B^2)^2 + 2\text{Tr}(B^3))\text{Var}(\mu_A) + \text{Tr}(B^2)k_3(\mu_A)}{(1 - 1/N^2)(1 - 4/N^2)\eta^2} + \frac{1}{\eta^2} \epsilon_N \right),$$

with $\epsilon_N \leq \frac{1}{\text{Tr}(\text{Tr}(A^3)\text{Tr}(B^4))(1 + 4/N^2) + \text{Tr}(A^2)(2\text{Tr}(B^2)^2 + 4\text{Tr}(B^2))}$. Using Poincaré inequality on the unitary group concludes the proof of the lemma.

In the case where $T = \text{Id}$ we can get a better bound. This improvement is important, since this gives the main contribution of our concentration bounds.

Lemma C.8. For $M = A^{1/2}UBU^*A^{1/2}$ and $z \in \mathbb{C}^+$ with $\eta = \mathcal{O}(z)$,

$$\mathbb{E}\left(\left|z\hat{m}_M(z) - \mathbb{E}(z\hat{m}_M(z))\right|^2\right) \leq \frac{8}{N} \left(\frac{\sqrt{\text{Tr}(A^2)(\text{Tr}(B_0^2) + \text{Var}(A))\eta^2}}{\eta^2} + \frac{\|A\|_{\infty}^2 (m_{A \otimes B}(1^3, 21^2) - 2m_{A \otimes B}(1^3, 1^3) + \text{Tr}(B^2)m_{A \otimes B}(21, 1^2))}{\eta^2} \right).$$

Proof. We have

$$d_{U}m_{M}(X) = \text{Tr}(GA^{1/2}[X, \tilde{B}]A^{1/2}G) = \text{Tr}([\tilde{B}, A^{1/2}G^2A^{1/2}]X),$$

and with $\tilde{B} = UBU^*$. Since $\text{Id}$ commutes with $A^{1/2}G^2A^{1/2}$, we can replace $B$ by $B_0 = B - \text{Tr}(B)$ in the latter equality. Moreover, for all $U \in U_n$,

$$z\tilde{B}_0A^{1/2}G^2A^{1/2} = \tilde{B}_0A^{1/2}(-1 + A^{1/2}\tilde{B}A^{1/2}G)GA^{1/2} = -\tilde{B}_0A^{1/2}GA^{1/2} + \tilde{B}_0A\tilde{B}A^{1/2}G^2A^{1/2},$$

and likewise

$$zA^{1/2}G^2A^{1/2}\tilde{B} = -A^{1/2}GA^{1/2}\tilde{B}_0 + A^{1/2}G^2A^{1/2}\tilde{B}A\tilde{B}_0.$$

Hence,

$$\|zd_{U}m_{M}(X)\|_2^2 \leq \frac{8}{N} \left(\|A^{1/2}GA^{1/2}\tilde{B}_0\|_2^2 + \|\tilde{B}_0A\tilde{B}A^{1/2}G^2A^{1/2}\|_2^2\right).$$
By the matrix Holder inequality \[ \|A^{1/2}GA^{1/2}B_0\|_2 \leq \frac{1}{\eta^2} \sqrt{\text{Tr}(A^2) \text{Tr}(AB_0 A B_0)}, \]
and, using \(\text{Tr}(B) = 1\),
\[ \|B_0 A B A^{1/2} G A^{1/2}\|_2^2 \leq \frac{\|A\|_\infty}{\eta^4} \text{Tr}(AB A B_0 A B_0). \]
Hence, after integration on the unitary group, and using the classical Holder inequality,
\[
\mathbb{E}\|d_U m_M\|^2 \leq \frac{8}{N} \left( \sqrt{\text{Tr}(A^2)(\text{Tr}(B_0^2) + \text{Var}(A)\sigma_2^2)} \right. \\
+ \left. \frac{\|A\|_\infty}{\eta^2} (m_{\text{AEB}}(1^3, 2^1) - 2 m_{\text{AEB}}(1^3, 1^3) + \text{Tr}(B^2) m_{\text{AEB}}(21, 1^2)) \right)
\]
Using Poincaré inequality on the unitary group concludes then the proof of the lemma. \(\square\)

**Lemma C.9.** For \( M = A^{1/2} U B U^* A^{1/2} \) and \( z \in \mathbb{C}^+ \) with \( \eta = \Im(z) \) and for \( T \in \mathbb{M}_n(\mathbb{C}) \),
\[
\mathbb{E}\left( |f_T(z) - \mathbb{E}(f_T(z))|^2 \right) \leq 4 \frac{\text{Tr}(|T|^{2\alpha})^2}{N^2 \eta^4} \mathbb{E}\left( (A^{1/2} U B U^* A^{1/2})^\beta \right)^2/\beta \|A\|_\infty,
\]
and writing \( \tilde{T} = U T U^* \), then if \( T \) is normal,
\[
\mathbb{E}(\|T^{1/2} T A^{1/2} G_M - \text{Tr}(A^{1/2} U T U^* A^{1/2} G_M)\|) \leq 8 \frac{\|A\|_\infty}{N^2 \eta^2} \left( \mathbb{E}\left( (A^{1/2} \tilde{T} A^{1/2} G_M)\right)^2/\alpha (\mathbb{E}\left( (A^{1/2} U B U^* A^{1/2})^\beta \right)^2/\beta \right) \]
for all \( \alpha, \beta > 1 \) satisfying \( \frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{2} \).

**Proof.** The first part of the lemma is a direct adaptation of the proof of Lemma C.7 with the Hölder inequality
\[
\|A^{1/2} G_M T G_M A^{1/2} B\|_2 \leq \|T\|_\alpha \|A^{1/2} B\|_\beta
\]
for all \( \alpha, \beta > 1 \) satisfying \( \frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{2} \). In view of applying the same method for the second part,
we compute the derivative of the map \( f : U \mapsto \text{Tr}(A^{1/2} U T U^* A^{1/2} G_M) \), yielding to
\[
d_U f(X) = \text{Tr}([\tilde{T}, X] A^{1/2} G_M A^{1/2}) + \text{Tr}(A^{1/2} \tilde{T} A^{1/2} G_M A^{1/2} [B, X] A^{1/2} G_M)
\]
\[
= \text{Tr} \left( ([A^{1/2} G_M A^{1/2}, \tilde{T}] + [A^{1/2} G_M A^{1/2} \tilde{T} A^{1/2} G_M A^{1/2}, \tilde{B}]) X \right).
\]
Hence,
\[
\sqrt{N} \|d_U f\|_2 \leq 2 \|A^{1/2} G_M A^{1/2} \tilde{T}\|_2 + 2 \|A^{1/2} G_M A^{1/2} \tilde{T} A^{1/2} G_M A^{1/2} B\|_2.
\]
Using Holder inequality yields then
\[
\sqrt{N} \|d_U f\|_2 \leq \frac{2 \|A^{1/2}\|_\infty \|A^{1/2} \tilde{T}\|_2}{\eta} + \frac{2 \|A^{1/2}\|_\infty \|A^{1/2} \tilde{T} A^{1/2}\|_\alpha \|A^{1/2} B\|_\beta}{\eta^2},
\]
for any \( \alpha, \beta > 1 \) such that \( \frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{2} \). Hence,
\[
N \|d_U f\|_2^2 \leq \frac{8 \|A\|_\infty}{\eta^2} \left( \|A^{1/2} \tilde{T}\|_2^2 + \frac{\|A^{1/2} \tilde{T} A^{1/2}\|_\alpha^2 \|A^{1/2} B\|_\beta^2}{\eta^2} \right)
\]
\[
\leq \frac{8 \|A\|_\infty}{\eta^2} \left( \text{Tr}(A^{1/2} \tilde{T}^2 A^{1/2}) + \frac{\text{Tr}((A^{1/2} \tilde{T} A^{1/2})^\alpha) \text{Tr}((A^{1/2} B^2 A^{1/2})^{\beta/2})}{\eta^2} \right),
\]
where we used that \( \|A T A\|_\alpha \leq \|A T\|_\alpha \|A\|_\alpha \) when \( T \) is normal. Integrating on \( U_n \) and applying Holder inequality on the last term of the latter sum yields the result. \(\square\)
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