On the intersection of the shell, collective and cluster models of atomic nuclei I:
Multi-shell excitations

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(Dated: April 15, 2014)

The relation of the shell, collective and cluster models of the atomic nuclei is discussed from the viewpoint of symmetries. In the fifties the U(3) symmetry was found as their common part for a single shell problem. For multi major-shell excitations, considered here, the U(3)⊗U(3) dynamical symmetry turns out to be their intersection.

PACS numbers: 21.10.Re, 21.60.Cs, 21.60.Fw, 21.60.Gx, 27.30.+t

INTRODUCTION

The fundamental models of nuclear structure are based on different physical pictures. The shell model indicates that the atomic nucleus is something like a small atom, while the collective model says that it is a microscopic liquid drop. Therefore, in order to understand the nuclear structure we need to study (among others) the interrelation of these models, find their common intersection, etc.

The basic connections were found in the fifties. Elliott showed how the quadrupole deformation and collective rotation can be derived from the spherical shell model: the states belonging to a collective band are determined by their specific SU(3) symmetry. Wildermuth and Kanellopoulos (following Perring and Skyrme) established the relation between the shell and cluster models. They proved that the Hamiltonians of the two models can be rewritten into each other exactly in the harmonic oscillator approximation. This relation results in a close connection between the corresponding eigen-vectors, too: the wavefunction of one model is a linear combination of those of the other, which belong to the same energy. Later on this relation was interpreted by Bayman and Bohr in terms of the SU(3) symmetry. As a consequence, the cluster states are also selected from the shell model space by their specific SU(3) symmetries. (In fact only one kind of cluster states have this feature, and there are other kinds, too, as discussed later.)

We will refer to this interrelation among the three basic structure models as the SU(3) connection. It was established in 1958 for a single major shell problem, for simple symmetries and small deformations of the ground-state region. In this paper we consider its extension to multi-major-shells, and in a following one the more general symmetries and the case of large deformations are considered. The concept is illustrated by the example of the $^{20}$Ne nucleus.

The connection of the shell model and the cluster model is especially interesting due to the fact that both models have a complete set of basis states, i.e. any nucleon states can be expanded in both bases. Depending on the simple or complicated nature of these expansions we can distinguish four different cases. i) If it is simple in the shell basis, but complicated in the cluster one, then we can speak about a simple shell state, which is a poor cluster state. ii) If it is simple in the cluster basis, but complicated in the shell basis, one has a good cluster state, which is a poor shell state. (Later on we will refer to these states as rigid molecule-like cluster states.) iii) The expansion may be simple in both bases. We call this situation as a shell-like cluster state. iv) If it is complicated in both bases, then the state is not a simple one.

Of course it is not the name that matters. The important point is that there are three different kinds of simple states when we investigate them from the viewpoint of the cluster-shell competition. Sometimes only the kind ii), i.e. rigid molecule-like states are considered as cluster states. The reason for our vocabulary, i.e. speaking about two kinds of cluster states: shell-like and molecule-like is inspired by their experimentally observable characteristics. Both of them prefer a certain reaction channel, have a large cluster spectroscopic factor, etc. They represent two different quantum phases of the nucleus, as discussed in $^{[6]}$. By choosing these names we are in line with the general definition of a simple nuclear state in terms of experimental observation: a state is called simple if its wavefunction has a large overlap, or form a large matrix element with the wavefunction of a reaction channel, in which it can be observed $^{[6]}$.

The structure of this paper is as follows. First we review some aspects of the shell, collective and cluster models, which are relevant for their symmetry-based relations, then we discuss their intersection.

SHELL MODELS

Single major shell

Elliott’s U(3) model bridges the spherical shell model and the collective model, and describes rotational states in the $p$ and $sd$ shell nuclei $^{[7]}$. Here the U(3) symmetry is that of the space part, while the spin-isospin section
is characterized by Wigner’s $U^{ST}(4)$ \cite{8}. Thus the group structure of the model is $U^{ST}(4) \otimes U(3)$. In constructing the model space the spin-isospin degrees of freedom are essential, of course. The physical operators, however, often contain simply spin-isospin zero terms, in which case only the generators of the $U(3)$ group contribute.

For a single-particle problem the $U(3)$ group is generated by the oscillator quantum creation ($\pi^+_\mu, \mu = -1, 0, +1$) and annihilation ($\pi^-_\mu$) operators, which carry $\ell = 1$ angular momentum, i.e. they are vector bosons. Their 9 number-conserving bilinear products can be recast into a scalar number operator, the three components of the angular momentum, and the five components of the (algebraic, i.e. no major shell changing) quadrupole operator, as follows:

$$n = \sqrt{3}[\pi^+ \times \pi^-]_0^{(0)}, \quad L_m = \sqrt{2}[\pi^+ \times \pi^-]_m^{(1)}, \quad Q_m = \sqrt{3}/2[\pi^+ \times \pi^-]_m^{(2)}, \quad (1)$$

where $\pi^-_m = (-1)^{1-m} \pi^-_{-m}$ . Removing the number operator $n$ the remaining 8 operators generate the $SU(3)$ subgroup. For the $A$-nucleon system the operators are obtained by summing on the particle number: $O = \Sigma_{\nu=1}^A O(t)$, but the Equations (1) remain valid for the many-body operators, too.

The space part of the basis states is defined by the representation labels of the group chain:

$$U(3) \supset SU(3) \supset SO(3) \supset SO(2)$$

$$[[n_1, n_2, n_3], (\lambda, \mu), K, L, M]. \quad (2)$$

Here $n = n_1 + n_2 + n_3$, $\lambda = n_1 - n_2$, $\mu = n_2 - n_3$, $K$ distinguishes the multiple occurrence of the angular momentum $L$ in the $SU(3)$ irreducible representation ($\lambda, \mu$), and $M$ is the projection of the angular momentum.

The quadrupole-quadrupole interaction plays a major role, and it can be written as a linear combination of the quadratic invariant operators ($C^{(2)}$) of the group chain: $SU(3) \supset SO(3)$: $QQ = \frac{1}{2}C^{(2)}_{SU(3)} - \frac{3}{2}C^{(2)}_{SO(3)}$. The electromagnetic transition rates are obtained by applying effective charge. The many-nucleon states are classified according to their $SU(3)$ symmetry, and in the simplest case the interactions are expressed in terms of its generators. More sophisticated calculations allowing mixing of $SU(3)$ configurations within a single shell were performed, too, in order to study the combined effects of the quadrupole-quadrupole, pairing and spin-orbit interactions (for a recent review see e.g. \cite{3}).

The single major shell model has an algebraic structure defined by a larger group, which contains the symmetry groups of both the space part $U(3)$, and the spin-isospin part $U^{ST}(4)$. It is the unitary group of $U(4\Omega)$, where $\Omega$ denotes the orbital degeneracy: $\Omega = 1, 3, 6, ...$ for the the $s, p, sd, ...$ shells, respectively. The orbital and spin-isospin decomposition of the wavefunction is described by the algebraic reduction:

$$U(4\Omega) \supset U^{ST}(4) \otimes U(\Omega)$$

$$\left[1^M\right], [f_1, f_2, f_3, f_4], [\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4], \quad (3)$$

where $M$ is the number of nucleons, and the $\left[1^M\right]$ overall antisymmetry of the wavefunction requires conjugate symmetries for $U^{ST}(4)$ and $U(\Omega)$. The orbital wavefunction is determined by the $U(3)$ content of $U(\Omega)$:

$$U(\Omega) \supset U(3)$$

$$[f_1, f_2, f_3, f_4], [n_1, n_2, n_3]. \quad (4)$$

Thus the requirement of the antisymmetry is completely incorporated. (The operators of Eq. (1) are expressed in terms of nucleon coordinates, therefore, the bosonic realisation does not introduce simplifying approximation \cite{1}.)

$U(4\Omega)$ is a dynamical group of the single major shell model in the sense that the physical operators are obtained in terms of its generators, and the whole spectrum is provided by a single irreducible representation (irrep) of it.

### Multi major shells

The description of the electromagnetic transitions without an effective charge requires the incorporation of the major shell excitations; i.e. a vertical extension of the $SU(3)$ shell model. For this purpose the symplectic group proved to be very useful (see e.g. \cite{10}), and in the formalism developed in \cite{11} the symplectic shell model has been widely applied.

The $Sp(3,\mathbb{R})$ group is generated by the position vectors of the nucleons, and their canonically conjugate momenta. An alternative set of its generators is expressed in terms of harmonic oscillator operators, containing the $9$ generators of the $U(3)$ group, which preserve the number of oscillator quanta, and in addition $6$ creation $B_{m}^{(l)} = [\pi^+ \times \pi^-]^{(l)}_m$, $l = 0, 2$ and $6$ annihilation $\tilde{B}_{m}^{(l)} = [\pi \times \pi^-]^{(l)}_m$ operators, which ladder by 2 or -2 quanta. The creation operators are $[2, 0, 0]$ $U(3)$ tensors, therefore, their products also carry $U(3)$ labels: $[n_1^e, n_2^e, n_3^e]$ (e stands for excitation). Since these operators commute with each other, only the symmetrically coupled products are non-vanishing. (Their coupling always produces a set of unique irreps thus there is no need to introduce an additional multiplicity label.) Note that all the symplectic generators are fully symmetric one-body operators, so they conserve the permutational symmetry. Therefore, if the band-head $U(3)$ irrep is Pauli-allowed, then so are all others in the symplectic band.

The model has a rich group structure, one of its physically important subgroup chain, called shell model chain, is associated with Elliott’s $SU(3)$:
Here \([n_1^s, n_2^s, n_3^s]\) denotes the symplectic bandhead, which is a U(3) irrep, being a lowest-weight \(\text{Sp}(3,R)\) state, while \(\rho\) distinguishes multiple occurrence of \([n_1, n_2, n_3]\) in the product \([n_1^s, n_2^s, n_3^s] \otimes [n_1^s, n_2^s, n_3^s]\). Note that this basis is not orthonormal; such a basis can be constructed inductively, by diagonalizing the norm matrix (provided by the inner products of the basis states) in each major shell.

In collective terms the sympletic model includes monopole and quadrupole vibrations as well as vorticity degrees of freedom for the description of the rotational dynamics in a continuous range from the irrotational to rigid rotor flows.

The algebraic structure of the symplectic shell model is provided by the \(\text{Sp}(3,R)\) group in the sense that the physical operators are expressed in terms of its generators, and the collective bands are given by its (finite dimensional) irreducible representations (which are built on U(3) irreps).

**COLLECTIVE MODEL**

Several algebraic collective models have been developed, for their recent review we refer to \([12]\). E.g. the \(\text{Sp}(3,R)\) model can also be considered as a microscopic realization of the Bohr–Mottelson–Frankfurt model. In what follows we focus on a model with a simple algebraic structure, which is very illuminative from the viewpoint of the interrelation of the fundamental structure models. It is the large \(n\) limit the \(\text{Sp}(3,R)\) symplectic model.

The dynamical group in this limit simplifies to \(U_s(3) \otimes U_b(6)\), i.e. to a compact group, as opposed to the noncompact \(\text{Sp}(3,R)\). Technically the simplification is achieved by replacing the creation and annihilation operators of the \(\text{Sp}(3,R)\) model (in which the description is based on nucleon degrees of freedom) by the contracted boson operators: \(\xi^b_m = (1/\epsilon) B^t_m, \xi^b_m = (1/\epsilon) B^l_m\), where \(\epsilon\) denotes \([n_s^*]^2\); \(n_s^* = n_1^s + n_2^s + n_3^s\). The \(U_s(3)\) is Elliott’s shell model symmetry of the \(0\)\(\omega\) shell, and \(U_b(6)\) is the group of the six dimensional oscillator, generated by the bilinear products of the \((l = 0\) and \(2)\) boson creation and annihilation operators. It is realised in a similar way as the \(U(6)\) group of the interacting boson model (IBM) \([13]\), nevertheless physically it is different, because in the case of the contracted symplectic model the bosons are associated to internuclear excitations, not to intrashell ones. (For the relation to the other algebraic collective models see e.g. \([14]\).)

This model is called \(U(3)\) boson model \([15]\), or contracted symplectic model \([16]\). Mathematical justification for the simplifying assumptions is provided through the application of the group deformation mechanism. The contracted boson operators which generate the internuclear excitations do introduce here a simplifying approximation (in particular a large \(n\) limit). Therefore, the antisymmetry requirement is not fully incorporated in this description.

This model is more easily applicable (compared to the \(\text{Sp}(3,R)\) model), e.g. it has an orthonormal set of basis states:

\[
U_s(3) \otimes U_b(6) \supset U_s(3) \otimes U_b(3) \qquad \supset U(3) \supset SU(3) \supset SO(3) \supset SO(2)
\]

\[\left[n_1^s, n_2^s, n_3^s\right], \left[n_b^s, n_b^s, n_b^s\right], \rho, \left[n_1, n_2, n_3\right], (\lambda, \mu), K, L, M.\]

\[
(6)
\]

**CLUSTER MODELS**

There are different kinds of cluster models, which all share the general picture of dividing the relevant degrees of freedom into two categories: those belonging to the relative motion of the clusters (usually in a large variety) and those of the internal structure of them (usually in a rather limited number). For recent reviews we refer to the works \([17, 18]\) while for the treatment of the coupling to the continuum see \([19, 20]\).

In searching for the symmetry-based relations to the shell and collective models, two aspects of the models are especially important: i) how much they are micro-
scopic, and ii) to what extent they are equipped with an algebraic structure.

A model is called fully microscopic if the antisymmetrization is completely involved, and the interactions are (effective) nucleon-nucleon forces. It is semimicroscopic, if the exclusion principle is appreciated, but the interactions are (phenomenologic) cluster-cluster interactions. It is fully phenomenologic, if cluster-cluster interactions are applied in a model space, which is constructed phenomenologically, i.e. without taking into account the Pauli-principle.

The algebraic structure on the other hand reveals the symmetries of the model, and this can provide us with a connection to the shell and collective picture.

The first fully algebraic cluster model, in which not only the basis states are characterized by the irreps of some groups, but the physical operators are also expressed in terms of its generators was constructed on the phenomenologic level. It is the nuclear vibron model \[21\], in which the internal structure of the clusters is accounted for by the interacting boson model (of U(6) group structure) \[13\], and the relative motion is described by the vibron model (with U(4) dynamical algebra) \[22\]. This is an algebraic model of a two-body system which can rotate and vibrate in the three dimensional space. The nuclear vibron model has a rich structure of symmetries (starting from the U\(_C\), (6)\(\otimes\)U\(_C\), (6)\(\otimes\)U\(_R\), (4) algebra for a binary configuration). From the viewpoint of the connection to the shell and collective models, however, the microscopic and semimicroscopic descriptions (which incorporate the exclusion principle) are more easily applicable.

Microscopic Cluster Models (MCM)

For the sake of simplicity we consider here binary cluster configurations. When the SU(3) shell model \[1\] is applied for the description of the internal structure of the clusters then the spin-isospin degrees of freedom of the clusters are coupled together

\[
U^{ST}_{C_1}(4) \otimes U^{ST}_{C_2}(4) \supset U^{ST}(4) \supset U^{S}(2) \otimes U^{T}(2)
\]

where \(\eta\) is a multiplicity label. The space-part is characterized by the group-chain:

\[
U_{C_1}(3) \otimes U_{C_2}(3) \otimes U_{R}(3) \supset U_{C}(3) \otimes U_{R}(3) \supset U(3) \supset SU(3) \supset SO(3) \supset SO(2)
\]

here \(U_{C}(3)\) stands for the coupled space symmetry of the two clusters. The U(3) generators are obtained similarly to those of Eqs. \([7]\), except for the number of oscillator quanta:

\[n = n_{C} + n_{R}, \quad Q = Q_{C} + Q_{R}, \quad L = L_{C} + L_{R},\]

since in this case the relative motion of the clusters changes in steps of 1 quantum.

\[
\langle n'_{1}^{C_{1}}, n'_{2}^{C_{1}}, n'_{3}^{C_{1}}|n_{1}^{C_{1}}, n_{2}^{C_{1}}, n_{3}^{C_{1}}\rangle, \quad \rho_{C}, \quad [n_{1}^{C_{1}}, n'_{1}^{C_{2}}, n_{2}^{C_{2}}, n'_{3}^{C_{2}}], \quad \rho_{C}, \quad [n_{1}^{C_{1}}, n'_{1}^{C_{2}}, n_{2}^{C_{2}}, n'_{3}^{C_{2}}]\quad [n_{R}', 0, 0], \quad [n_{1}', n_{2}', n_{3}'], \quad (\lambda, \mu), \quad K', \quad L', \quad M\rangle
\]

This basis is especially useful for treating the exclusion principle, since the U(3) generators commute with those of the permutation group, therefore, all the basis states of an irrep are either Pauli-allowed, or forbidden \[23\].

On the fully microscopic level much work has been done in developing an algebraic description. Especially great progress was made in the works \[23, 24\], in calculating the necessary matrix elements in an U(3) basis. In \[24\] the matrix elements

\[
\langle n_{1}^{C_{1}}, n_{2}^{C_{1}}, n_{3}^{C_{1}}|n'_{1}^{C_{1}}, n'_{2}^{C_{1}}, n'_{3}^{C_{1}}\rangle, \quad \rho_{C}, \quad [n_{1}^{C_{1}}, n_{2}^{C_{1}}, n_{3}^{C_{1}}]\quad [n_{1}', n_{2}', n_{3}'], \quad (\lambda, \mu), \quad K, \quad L\rangle
\]

were investigated, (\(O\) stands here either for the Hamiltonian or for the unit operator, and \(A\) is the antisymmetrizer). It was shown that the calculation of the norm
and overlap matrix elements ($O = 1$) can be reduced to purely algebraic techniques.

(In the language of the Resonating Group Method [23], which is considered as the prototype of the microscopic cluster models, and is based on integro-differential equations, these quantities are called norm and overlap kernels.)

In the fully microscopic description usually effective nucleon-nucleon forces are applied without specific symmetry character. Therefore, this is not a fully algebraic description (the physical operators are not expressed in terms of the generators of a dynamical group), contrary to the single major shell model of U(4Ω), the symplectic model of Sp(3,R), and the contracted symplectic model of $U_s(3) \otimes U_h(6)$.

**Semimicroscopic Algebraic Cluster Model (SACM)**

The semimicroscopic algebraic cluster model [26] is a fully algebraic approach with transparent symmetry-properties. The internal structure of the clusters is described here by the Elliott model [1] too, therefore, this part of the wavefunction has a $U_C^S T(4) \otimes U_C(3)$ symmetry. The relative motion of the clusters is accounted for by the modified vibron model [22]. The U(3), i.e. harmonic oscillator basis is applied, thus the basis states of the relative motion ($R$) are characterized by the group-chain:

$$U_R(4) \supset U_R(3) \supset SU_R(3) \supset SO_R(3) \supset SO_R(2) \supset [N_R, 0, 0, 0], [n_R, 0, 0], (n_R, 0), L_R , M_R(12)$$

The exclusion of the Pauli-forbidden states amounts up to a truncation of the coupled U(3) basis from the side of the small number of oscillator quanta. Some major shells (belonging to certain $n_R$-s) are completely missing, and from some other ones parts of the single-nucleon states are excluded. This is the modification [26] with respect to the original vibron model, as it is applied e.g. in molecular physics [28]. One way to exclude the Pauli-forbidden U(3) irreps is to make an intersection with the U(3) irreps of the totally antisymmetric shell model space of the nucleus [22]. (For core-plus-alpha systems of light nuclei this is a simple procedure.)

The relation of the SACM and fully microscopic description from the viewpoint of the model space is that they contain the same $U(3)$ and $U_C^S T (4)$ irreps, but the complete antisymmetrization is carried out only in the fully microscopic descriptions. Therefore, the calculation of the cluster spectroscopic amplitude in the semimicroscopic model is being done by the introduction of phenomenologic parameters [29].

The connection between the fully microscopic and the semimicroscopic cluster models is somewhat similar to the relation between the symplectic shell model and the contracted symplectic model. The previous ones are fermionic models, accounting for the antisymmetrization to full details, while the latter ones are their simplifying (bosonic) approximations. Therefore, their wavefunctions are not the same, though the U(3) content is identical in both cases (see Table I).

The overlap of the semimicroscopic and fully microscopic cluster wavefunctions is given by Equation (11) (with $O = 1$), which is the eigenvalue of the norm kernel:}

$$D_m = [\sigma_+ \times \sigma_+ + \sigma_- \times \sigma_-]^{(1)} \cdot$$

Thus, the introduction of the $U_R(4)$ group results in: i) the truncation of the infinite basis, and ii) a simple and compact expression for the dipole operator. Another group-chain of the vibron model is:

$$U_R(4) \supset O_R(4) \supset SO_R(4) \supset SO_R(3) \supset SO_R(2). (14)$$

The coupling between the relative motion and internal cluster degrees of freedom for a binary cluster system results in a group structure:

$$G_{2C} \equiv U_C^S T (4) \otimes U_C(3) \otimes U_C^S T (4) \otimes U_C(3) \otimes U_R(4). (15)$$

The spin and isospin degrees of freedom are essential in this case, too, from the viewpoint of the construction of the model space. However, if one is interested only in a single supermultiplet [$U_C^S T (4)$] symmetry, which is typical in cluster problems, then the relevant group structure simplifies to that of the space part. In particular the U(3) (strong) coupled basis is defined by the group chain:

$$U(3) \supset SU(3) \supset SO(3) \supset SO(2) \supset [n_1, n_2, n_3], (\lambda, \mu), K \ , \ L \ , \ M \ . (16)$$
TABLE I. SU(3) irreps of the shell, collective and cluster models of $^{20}$Ne. The first column of $(\lambda, 0)$ is present in each model, the others are included in the symplectic and contracted symplectic schemes. The upper index shows the multiplicity of the representation.

| \( \hbar \omega \) | $(\lambda, \mu)$ |
|----------------|-----------------|
| 0              | (8, 0)          |
| 1              | (9, 0)          |
| 2              | (10, 0), (8, 1), (6, 2) |
| 3              | (11, 0), (9, 1), (7, 2) |
| 4              | (12, 0), (10, 1), (8, 2)$^2$, (6, 3), (4, 4), (7, 1), (6, 0) |
| ...            | ...             |

**INTERSECTION**

When major shell excitations are incorporated, then both the (symplectic) shell model, and the (contracted symplectic) collective model, as well as the (microscopic or semimicroscopic algebraic) cluster model has a set of basis states characterised by the irreps of the group chain, (see Eq. (4.1.9) of [23]). Its value depends on the U(3) representation labels, but does not depend on the labels of its subgroup, e.g. on L and M. Near the forbidden states the overlap is small, but in the fully Pauli-allowed major shells it is close to 1. For the $^{16}$O+$^4$He system its numerical values (for different \( n_{F} \)'s) are $^2$: 8: 0.229, 9: 0.344, 10: 0.510, 11: 0.620, 12: 0.719, 13: 0.790, 14: 0.846, 15: 0.887, 16: 0.918, 17: 0.940, 18: 0.957, 19: 0.969, 20: 0.978, ...

**FIG. 1.** Experimental and U(3) model spectra in the $^{20}$Ne nucleus. The experimental bands are labelled by the \( K^* \), and the model states by the $(\lambda, \mu)$ quantum numbers. (For more details see the text.)

The increasing number of shell model basis states in the expansion of the cluster wavefunction is very illuminative in this respect [30]. E.g. in case of $^{20}$Ne the $0 \hbar \omega$ (8, 0) and the $1 \hbar \omega$ (9, 0) states can be expressed only in a single shell model irrep, while for the more excited states the number of contributing shell model irreps are as follows, (10, 0): 5, (11, 0): 12, (12, 0): 34...

As for the relation of the model spaces (wavefunctions) of the three basic structure models (shell, collective and cluster models) is considered the following can be said. When multi major shells are involved then the (quadrupole) collective or (dipole) cluster bands can be picked up from the microscopic shell model basis according to their \( U_x(3) \otimes U_y(3) \subset U(3) \) symmetry. Some irreducible representations are present in each of the three models. Having the same SU(3) basis, however, does not necessarily mean 100% overlap of the wavefunction, it can be less, too. This situation is similar to what was
found for the single major shell problem in terms of the historical SU(3) connection.

Another interesting question of the intersection of the different structure models is how their spectra compare to each other, when similar interactions applied. In this respect first we should note that the Wildermuth-connection between the shell and cluster model states was found originally for harmonic oscillator Hamiltonians (with exact SU(3) symmetry). The Elliott-connection, on the other hand, between the shell and collective states is valid for more general interactions. In particular, it survives not only for the exactly SU(3) symmetric Hamiltonian, but also for the ones having a dynamical symmetry, which can be written in terms of the invariant operators of the SU(3) ⊃ SO(3) algebra-chain. (Sometimes it is called dynamically broken symmetry.) In fact, however, the Wildermuth-connection is also valid for this kind of more general interactions. In particular, the complete symmetry of the A-nucleon system is given by the U(3A) ⊃ U(3) ⊗ U(A) algebra, where U(3) refers to the space part, while U(A) acts in the pseudo-space of the particle numbers. When the system is fully symmetric with respect to this latter one, then the shell-cluster connection is valid for the dynamically broken SU(3) symmetry, too. For a recent discussion in details we refer to [33].

If the Hamiltonian is expressed in terms of the invariant operators of the group chain [17] then the eigenvalue has an analytical solution i.e. a (broken) dynamical symmetry is present. This U_x(3) ⊗ U_y(3) ⊃ U(3) dynamical symmetry is the common intersection of the shell, collective and cluster models of multi major shells.

Figure 1. shows a comparison between the experimental and the U(3) dynamically symmetric [17] spectrum of the 20Ne nucleus. The energy (in MeV) was obtained with the formula \( E = 13.19\lambda - 0.4579\lambda(\lambda + 3) + 0.8389\frac{2\theta}{L(L + 1)} \). The oscillator energy is determined according to the systematics [24], while \( \theta \) is the moment of inertia calculated classically for the rigid shape determined by the U(3) quantum numbers. The parameters of the quadratic and the rotational terms were fitted to the experimental data. (The ground state energy is taken to be zero.) The \( E2 \) transitions (in W.u.) were calculated with the operator \( T^{(E2)} = 1.6303Q \). The experimental data are from [35], and in the band-assignment the conclusions of [36] were taken into account. (The energies of the \( 6^+ \) and \( 8^+ \) states of the \( 0_2^+ \) band are obtained as the average of two and three candidate states, respectively.)

The purpose of this calculation is not to give a detailed description of the experimental data, rather to show how the common intersection of the three basic structure models compares to the experiment. The right panel in Fig. 1. can be considered as a (part of a) shell, collective or cluster spectrum, when the basis states and the operators can be characterized by the SU(3) (and subgroup) symmetries, i.e. for the case of the dynamical symmetry. These circumstances are similar again to those of the single-shell problem.

SUMMARY AND CONCLUSION

In this paper we have discussed the interrelation of the fundamental nuclear structure models, the shell, collective and cluster models from the viewpoint of symmetries. These models are based on different physical pictures, and their connection was established first in terms of the SU(3) symmetry for a single shell problem [1, 2, 4]. We have considered here the generalization of this relation along the major shell excitations.

Algebraic models have been constructed for the description of this vertical extension in each of the three approaches. The most relevant ones from the viewpoint of the symmetry-based interrelations are the symplectic shell model of Sp(3,R) algebraic structure [11], the contracted symplectic model of \( U_6(6) \otimes U_6(3) \), which is the large \( n \) limit of the multi major-shell symplectic model [15, 16], and the fully microscopic [23, 24], as well as the semimicroscopic algebraic cluster models [26], with \( U_6(3) \otimes U_6(3) \) basis. The common intersection of these models is provided by the \( U_x(3) \otimes U_y(3) \otimes U(3) \) dynamical symmetry, i.e. for the many major-shell problem this symmetry substitutes the simple SU(3). Figure 1 shows how the schematic spectrum of this dynamical symmetry, which can be considered as a shell, collective or cluster description, compares with the experimental one.

ACKNOWLEDGEMENT

This work was supported by the OTKA (Grant No K106035), as well as by the MTA-BAS (No. 7) and MTA-JSPS (SNK 6/2013) bilateral projects. Inspiring discussions with Professors J. Draayer, A. Georgieva, K. Katô and Y. Suzuki are gratefully acknowledged.

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