Anderson localisation for an interacting two-particle quantum system on $\mathbb{Z}$

Victor Chulaevsky $^1$, Yuri Suhov $^2$

$^1$ Département de Mathématiques et Informatique, Université de Reims, Moulin de la Housse, B.P. 1039, 51687 Reims Cedex 2, France
E-mail: victor.tchoulaevski@univ-reims.fr

$^2$ Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, UK
E-mail: Y.M.Suhov@statslab.cam.ac.uk
Abstract

We study spectral properties of a system of two quantum particles on an integer lattice $\mathbb{Z}$ with a bounded short-range two-body interaction, in an external random potential field $x \mapsto V(x, \omega)$ with independent, identically distributed values. The main result is that if the common probability density $f$ of random variables $V(x, \cdot)$ is analytic in a strip around the real line and the amplitude constant $g$ is large enough (i.e. the system is at high disorder), then, with probability one, the spectrum of the two-particle lattice Schrödinger operator $H(\omega)$ (bosonic or fermionic) is pure point, and all eigenfunctions decay exponentially. The proof given in this paper is based on a refinement of a multiscale analysis (MSA) scheme proposed by von Dreifus and Klein ([9]), adapted to incorporate lattice systems with interaction.

1 Introduction

1.1 Random operators

Random self-adjoint operators appear in various problems of physical origin, in particular, in solid state physics. For example, they model properties of an ideal or non-ideal crystal where immovable atoms create an external potential field for moving electrons. Typically, it is difficult to analyse spectral properties of each sample operator $H(\omega)$. However, one rarely, if ever, needs such a detailed information. A more subtle approach is to consider almost every operator and establish properties held with probability one. Perhaps the most popular model of a random operator is a lattice Schrödinger operator (LSO) $H(\omega) = H_{V,g}(\omega)$ with a random external potential. Operator $H$ has the form $H^0 + gV$ and acts on functions $f$ from $\ell_2(\mathbb{Z}^d)$:

$$H\phi(x) = H^0\phi(x) + gV\phi(x) = \sum_{y \in \mathbb{Z}^d} f(y) + gV(x,\omega)f(x), \quad x \in \mathbb{Z}^d. \quad (1.1)$$

Here $H^0$ stands for the kinetic energy operator (the lattice Laplacian) and $V$ for the potential energy operator. Further, $(y, x)$ indicates a nearest-neighbor pair of lattice sites $y, x \in \mathbb{Z}^d$. Finally, $g$ is an amplitude constant.

In particular, the Anderson model is where $V(x, \omega), x \in \mathbb{Z}$, are real-valued independent, identically distributed (IID) random variables (RVs).
This model describes the motion of a single lattice electron in a potential field generated by random ‘impurities’ present at sites $x$ of the cubic lattice $\mathbb{Z}^d$ independently for different sites. The question here is about the character of the spectrum of LSO $H$ in (1.1).

The single-particle Anderson model generated a substantial literature, and Anderson’s localisation in a single-particle system is now well understood. The initial result was suggested by Sinai in the mid-70’s and proved in [14] for one-dimensional case ($d = 1$). We refer the reader to subsequent works [12], [17], [11], [8], and particularly [9]. A multi-scale analysis (MSA) scheme proposed in [9] proved to be very general and flexible and has been applied to different models of disordered media. The scheme was re-fined in [2] and [1]. The general result of these papers is that for the Anderson model in any dimension $d \geq 1$, with a fairly general distribution of $V(\cdot, \omega)$ and a sufficiently large amplitude $|g|$, operator $H_{V,g}$ has with probability one a pure point spectrum, and all its eigen-functions (EFs) decay exponentially fast at infinity (”exponentially localised”, in physical terminology). This phenomenon is often called Anderson, or exponential, localisation.

1.2 Interacting systems

This paper considers a two-particle Anderson system on a one-dimensional lattice $\mathbb{Z}$, with interaction, in a random external potential. The Hamiltonian/LSO $H \left( = H_{U,V,g}^{(2)}(\omega) \right)$ is of the form $H^0 + U + g(V_1 + V_2)$:

$$H\phi(x) = H^0\phi(x) + \left[(U + gV_1 + gV_2)\phi\right](x)$$

$$= \phi(x_1 - 1, x_2) + \phi(x_1 + 1, x_2) + \phi(x_1, x_2 - 1) + \phi(x_1, x_2 + 1)$$

$$+ [U(x) + gV(x_1, \omega) + gV(x_2, \omega)] \phi(x), \quad x = (x_1, x_2) \in \mathbb{Z}^2_{\geq}.$$

Here, as before, $H^0$ stands for the kinetic energy operator (the lattice Laplacian), and $U + gV_1 + gV_2$ is the potential energy operator; all operators act in the two-particle Hilbert space $\ell^2(\mathbb{Z}^2_{\geq})$. Next, $\mathbb{Z}^2_{\geq}$ is the ’sub-diagonal half’ of the two-dimensional lattice $\mathbb{Z}^2$:

$$\mathbb{Z}^2_{\geq} = \{x = (x_1, x_2) : x_1, x_2 \in \mathbb{Z}, \ x_1 \geq x_2\}.$$

A boundary condition on the diagonal

$$\partial\mathbb{Z}^2_{\geq} = \{x = (x_1, x_2) : x_1 = x_2\}$$
specifies the statistics of the two-particle system: it is a reflection condition for a bosonic and zero (Dirichlet’s) condition for a fermionic system. Consequently, in the RHS of (1.2), in the bosonic case
\[ H^0 \phi(x) = 2\phi(x_1 + 1, x_2) + 2\phi(x_1, x_2 - 1), \quad x \in \partial \mathbb{Z}^2_\geq, \]
while in the fermionic case \( H \) is considered on functions \( f \) vanishing on \( \partial \mathbb{Z}^2_\geq \).

**Remark.** The method used in this paper had been specifically designed for bosonic and fermionic systems. An extension of our results to the Maxwell–Boltzmann statistics is possible but would require additional technical constructions.

The interaction potential \( U : x \in \mathbb{Z}^2_\geq \mapsto \mathbb{R} \) is a fixed real-valued function vanishing when \( x_1 - x_2 \) exceeds a given value \( d < \infty \):
\[ U(x) = 0, \quad \text{if} \quad x_1 - x_2 > d. \tag{1.4} \]

In addition, there is given a family of real IID random variables \( V(x, \cdot) \), \( x \in \mathbb{Z} \), representing the external field. Constant \( g \) (the amplitude parameter) will be assumed big, but may be of positive or negative sign.

As a working approximation for \( H \) we consider a Hermitian \( |\Lambda| \times |\Lambda| \) matrix \( H_\Lambda \left( = H^{(2)}_{\Lambda, U, V, g}(\omega) \right) \) where \( \Lambda \subset \mathbb{Z}^2_\geq \) is a finite set of cardinality \( |\Lambda| \). Matrix \( H_\Lambda \) is of the form \( H^0_\Lambda + U + g(V_1 + V_2) \) and represents an LSO in \( \Lambda \):
\[ H_\Lambda \phi(x) = H^0_\Lambda \phi(x) + \left[ (U + gV_1 + gV_2) \phi \right](x) \]
\[ = \begin{bmatrix}
\phi(x_1 - 1, x_2)1_\Lambda(x_1 - 1, x_2) + \phi(x_1 + 1, x_2)1_\Lambda(x_1 + 1, x_2) \\
+ \phi(x_1, x_2 - 1)1_\Lambda(x_1, x_2 - 1) + \phi(x_1, x_2 + 1)1_\Lambda(x_1, x_2 + 1)
\end{bmatrix} \tag{1.5} \]
\[ + [U(x) + gV(x_1, \omega) + gV(x_2, \omega)] \phi(x), \quad x = (x_1, x_2) \in \Lambda, \]
\( 1_\Lambda \) being the indicator function of \( \Lambda \). In fact, we focus on lattice squares or their intersections with \( \mathbb{Z}^2_\geq \), and use the notation
\[ \Lambda_L(u) = \left( [u_1 - L, u_1 + L] \times [u_2 - L', u_2 + L'] \right) \cap \mathbb{Z}^2_\geq. \tag{1.6} \]
Such a set is called a (lattice) sub-square.

Given a finite set \( \Lambda^{(1)} \subset \mathbb{Z} \), we can also consider a single-particle LSO \( H^{(1)}_{\Lambda^{(1)}} \left( = H^{(1)}_{\Lambda^{(1)}, V, g} \right) \) of the form
\[ H^{(1)}_{\Lambda^{(1)}} \phi(x) = \begin{bmatrix}
\phi(x - 1)1_{\Lambda^{(1)}}(x) + \phi(x + 1)1_{\Lambda^{(1)}}(x + 1)
\end{bmatrix} + gV(x, \omega)\phi(x), \quad x \in \Lambda^{(1)}. \tag{1.7} \]
Of particular interest to us are (lattice) segments:

\[ \Lambda^{(1)} = \Lambda^{(1)}_L(u) = [u - L, u + L] \cap \mathbb{Z}. \]  

(1.8)

Matrix \( H^{(1)}_{\Lambda^{(1)}_L(u)} \) gives a finite-volume approximation to a single-particle LSO \( H^{(1)} \) on \( \mathbb{Z} \):

\[
H^{(1)} \phi(x) = \left[ \phi(x - 1) + \phi(x + 1) \right] + gV(x, \omega) \phi(x), \quad x \in \mathbb{Z},
\]

(1.9)

which acts in the single-particle Hilbert space \( \ell^2(\mathbb{Z}) \).

Next, a system of two particles in a finite volume \( \Lambda \in \mathbb{Z}^2 \) with no interaction is described by the LSO \( H^{n-i}_{\Lambda} \left( = H^{(2),n-i}_{\Lambda,V_V} \right) \) of the form \( H^{0}_{\Lambda} + g(V_1 + V_2) \):

\[
H^{n-i}_{\Lambda} \phi(x) = H^{0}_{\Lambda} \phi(x) + g \left[ (V_1 + V_2) \phi \right] (x)
= \left[ \phi(x_1 - 1, x_2)1_{\Lambda}(x_1 - 1, x_2) + \phi(x_1 + 1, x_2)1_{\Lambda}(x_1 + 1, x_2)
+ \phi(x_1, x_2 - 1)1_{\Lambda}(x_1, x_2 - 1) + \phi(x_1, x_2 + 1)1_{\Lambda}(x_1, x_2 + 1) \right]
+ g \left[ V(x_1, \omega) + V(x_2, \omega) \right] \phi(x), \quad \mathbf{x} = (x_1, x_2) \in \Lambda.
\]

(1.10)

In this paper we work with matrices \( H^{n-i}_{\Lambda} \) where \( \Lambda \) is a (lattice) square \( \Lambda_L(u) = \Lambda^{(1)}_L(u_1) \times \Lambda^{(1)}_L(u_2) \) lying inside \( \mathbb{Z}^2 \), where segments \( \Lambda^{(1)}_L(u_j) \) are as in (1..). In this case we can use the straightforward representation

\[
H^{n-i}_{\Lambda_L(u)} = H^{(1)}_{\Lambda^{(1)}_L(u_1)} \otimes I_{\Lambda^{(1)}_L(u_2)} + I_{\Lambda^{(1)}_L(u_1)} \otimes H^{(1)}_{\Lambda^{(1)}_L(u_2)}.
\]

(1.11)

Of course, the spectrum of matrix \( H^{n-i}_{\Lambda_L(u)} \) will be formed by the sums of the eigen-values (EVs) of \( H^{(1)}_{\Lambda^{(1)}_L(u_1)} \) and \( H^{(1)}_{\Lambda^{(1)}_L(u_2)} \).

This brings us to the observation that the principal difference between a single-particle random LSO (1.1) on \( \mathbb{Z}^2 \) and a two-particle LSO (1.2) on \( \mathbb{Z}^2 \) is that the values of the external potential field

\[
\mathbf{x} \mapsto g \left[ V_1(x_1, \omega) + V_2(x_2, \omega) \right]
\]

(1.12)

in (1.2) are ‘strongly’ dependent. For example, for any two points \( \mathbf{x} = (x_1, x_2) \) and \( \mathbf{x}' = (x_1 + a, x_2) \) from \( \mathbb{Z}^2 \), with \( a \geq 1 \), the values

\[ gV(x_1, \omega) + gV(x_2, \omega) \quad \text{and} \quad gV(x_1 + a, \omega) + gV(x_2, \omega) \]
are coupled, as RV \( V(x_2, \omega) \) is present in both sums. On the other hand, LSOs (1.1) and (1.2) bear essential similarities, owing to the fact that the approximating matrix \( H_\Lambda \) for a square \( \Lambda_L(\omega) \) ‘deeply inside’ \( \mathbb{Z}^2 \geq \), coincides with \( H^{n-1}_{\Lambda_L(\omega)} \). This allows us to apply a number of results and techniques from the single-particle MSA scheme, while some other key points of the scheme have to be modified or extended.

1.3 The main result

Our assumptions throughout the paper are as follows.

(A) RV’s \( V(x, \cdot) \), \( x \in \mathbb{Z} \), are IID and have a probability density function (PDF) \( f \) which is bounded on \( \mathbb{R} \):

\[
\| f \|_\infty = \sup \{ f(y) : y \in \mathbb{R} \} < \infty,
\]

and is such that the characteristic function

\[
\mathbb{E} \left[ e^{itV(x, \cdot)} \right] = \int_{\mathbb{R}} dy e^{ity} f(y)
\]

admits the bound

\[
|\mathbb{E} \left[ e^{itV(x, \cdot)} \right]| \leq be^{-a|t|},
\]

(1.13)

where \( a > 0 \) and \( b \geq 1 \) are constants.

(B) \( U \) is a real bounded function on \( \mathbb{Z}^2 \geq \) satisfying (1.4).

Bound (1.13) implies that PDF \( f(y), y \in \mathbb{R} \), admits the analytic continuation into a strip \( \{ z \in \mathbb{C} : |\text{Im } z| < a \} \).

As was indicated, the statistics of the system is defined by the type of the boundary conditions on \( \partial \mathbb{Z}^2 \). In both cases, LSO \( H \) formally defined by (1.2) is initially considered on the set of functions \( f \) with compact support. Here, with probability one, it is essentially self-adjoint, and we take its self-adjoint extension which is again denoted by \( H(= H^{(2)}_{U,V,g}(\omega)) \). Theorem 1.1 below addresses both cases.

**Theorem 1.1** Assume that conditions (A) and (B) are fulfilled. Then there exists \( g_0 \in (0, \infty) \) such that if \( |g| \geq g_0 \) then LSO \( H \) in (1.2) satisfies the following property. With probability one,

(a) the spectrum of \( H \) is pure point: \( \sigma(H) = \sigma_{\text{pp}}(H) \), and
∀ eigen-value \( E \in \sigma_{pp}(H) \), every corresponding EF \( \psi(x; E) \) exhibits an exponential decay:

\[
\limsup_{||x|| \to \infty} \log \frac{||\psi(x; E)||}{||x||} = -m < 0.
\]

(1.14)

Here, \( ||x|| \) stands for the Euclidean norm \( (|x_1|^2 + |x_2|^2)^{1/2} \); the value \( m \) \((= m(\psi(\cdot; E)))\) is called the mass (of eigen-function \( \psi(\cdot; E) \)).

The threshold \( g_0 \) in Theorem 1.1 can be assessed in terms of the sup–norm \( ||f||_\infty \), the constants \( a \) and \( b \) in Assumption (A) and the radius of interaction \( d \) and the maximum \( \max \{ ||U(r)| : r \in \mathbb{Z}_+ \} \) in Assumption (B).

Throughout the paper, symbol \( \Box \) is used to mark the end of a proof.

2 Wegner-type estimates

One of the key ingredients of MSA is an estimate of the probability to find an EV of LSO \( H_\Lambda \) (see (1.5)) in an interval \((E_0 - r, E_0 + r)\). The Wegner estimate, used for IID values of the external potential, does not apply directly to our problem. So, we need an analog of the Wegner estimate of the density of states. For definiteness, we assume that \( \Lambda \) is a lattice rectangle.

Let \( d\kappa_\Lambda(\lambda) \) be the averaged spectral measure of \( H_\Lambda \) such that

\[
\mathbb{E} \left[ \langle \delta_\mathbf{u}, \varphi(H_\Lambda) \delta_\mathbf{u} \rangle \right] = \int \varphi(\lambda)d\kappa_\Lambda(\lambda), \quad \mathbf{u} \in \Lambda,
\]

(2.1)

for any bounded test function \( \varphi \). Here and below, \( \delta_\mathbf{u} \) stands for the Dirac’s delta, and \( \langle \cdot, \cdot \rangle \) and \( || \cdot || \) denote the inner product and the norm in \( \ell^2(\mathbb{Z}^2) \). It is well-known that measure \( d\kappa_\Lambda \) is independent of the choice of an element \( \phi = \sum_{\mathbf{u} \in \Lambda} \langle \phi, \delta_\mathbf{u} \rangle \delta_\mathbf{u} \) with \( ||\phi|| = 1 \): for any such \( \phi \),

\[
\mathbb{E} \left[ \langle \phi, \varphi(H_\Lambda) \phi \rangle \right] = \int \varphi(\lambda)d\kappa_\Lambda(\lambda).
\]

(2.2)

Actually, \( d\kappa_\Lambda \) is a normalised (i.e. a probability) measure on \( \mathbb{R} \). Let \( \widehat{k}_\Lambda(t) \) be its inverse Fourier transform (the characteristic function, in a probabilistic terminology),

\[
\widehat{k}_\Lambda(t) = \mathbb{E} \left[ \langle \delta_\mathbf{u}, e^{itH_\Lambda} \delta_\mathbf{u} \rangle \right], \quad \mathbf{u} \in \Lambda, \ t \in \mathbb{R}.
\]

(2.3)
then

\[ k_\Lambda(s) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ist} \hat{k}_\Lambda(t) dt. \] (2.4)

**Lemma 2.1.** The quantity \( \hat{k}_{\Lambda, u} \) defined in (2.3) obeys

\[
|\hat{k}_{\Lambda, u}(t)| \leq e^{-B|t|}, \quad u \in \Lambda, \ t \in \mathbb{R},
\] (2.5)

where

\[ B = 2(a|g| - b - 1), \] (2.6)

independently of \( \Lambda \). Therefore, \( k_\Lambda(\lambda) \) is analytic in a strip around real line, so it has a bounded derivative on any interval.

**Proof of Lemma 2.1.** For definiteness, we focus on the fermionic case. To estimate \( |\hat{k}_{\Lambda, u}(t)| \), we use Molchanov’s formula expressing matrix elements of \( e^{itH_\Lambda} \) in terms of the integral over trajectories of a Markov jump process on the time interval \([0, |t|]\). Namely,

\[
\langle \delta_{\hat{u}}, e^{itH_\Lambda} \delta_{\hat{u}} \rangle = e^{4|t|} E_{\hat{u}} \left( 1(X(t) = \hat{u}) \cdot i^{K(|t|)(\text{sign } t)} \exp \left[ i(\text{sign } t) \int_0^{|t|} W(X(s)) ds \right] \right).
\]

Here \( \{X(s), s \geq 0\} \) is the continuous-time Markov jump process on \( \Lambda \), with holding times of rate 2, equiprobable jumps to four nearest neighbour sites and Dirichlet’s boundary conditions outside \( \Lambda \). Next, \( E_{\hat{u}} \) denotes the expectation generated by the distribution of the process when the initial site is \( \hat{u} \). Further, \( K(|t|) (= K(|t|; \{X(s)\})) \) is the number of jumps of \( \{X(s)\} \) between times 0 and \( |t| \). Now,

\[
\hat{k}_{\Lambda, u}(t) = e^{4|t|} E_{\hat{u}} \left[ \mathbb{E}_{\hat{u}} \left[ 1(X(t) = \hat{u}) \cdot i^{K(|t|)(\text{sign } t)} \exp \left( i(\text{sign } t) \int_0^{|t|} W(X(s)) ds \right) \right] \right]
\]

\[
= e^{4|t|} E_{\hat{u}} \left[ \mathbb{E}_{\hat{u}} \left[ 1(X(t) = \hat{u}) \cdot i^{K(|t|)(\text{sign } t)} \exp \left( i(\text{sign } t) \int_0^{|t|} W(X(s)) ds \right) \right] \right]
\]

\[
= e^{4|t|} E_{\hat{u}} \left[ \mathbb{E}_{\hat{u}} \left[ 1(X(t) = \hat{u}) \cdot i^{K(|t|)(\text{sign } t)} \exp \left( i(\text{sign } t) \int_0^{|t|} W(X(s)) ds \right) \right] \right]
\]

8
the change of order of integration is justified by the boundedness of the integrand.

For simplicity we assume from now on that $t > 0$. In our case,

$$W(u) = V(u_1) + V(u_2) + U(u), \quad u = (u_1, u_2).$$

Given trajectory $X(s), s \geq 0$, the values $K(t)$ and $U(u), u \in \Lambda$, are non-random. Hence, the internal expectation

$$E\left(1(X(t) = u)i^{K(t)} \exp \left[i \int_0^t W(X(s))ds\right]\right)$$

$$= 1(X(t) = u)i^{K(t)} \exp \left[i \int_0^t U(X(s))ds\right] E\exp \left(ig \sum_{j=1}^2 \int_0^t V(X_j(s))ds\right),$$

where $X_1(s), X_2(s)$ are the components of $X(s)$.

Write

$$\sum_{j=1}^2 \int_0^t V(X_j(s))ds = \sum_{z \in \mathbb{Z}} V(z) \sum_{j=1}^2 \tau_j(z),$$

where $\tau_j(z)$ is the time spent at $z$ by process $\{X_j(s)\}$ between 0 and $t$. This yields

$$E\left[1(X(t) = u)i^{K(t)} \exp \left(i \int_0^t W(X(s))ds\right)\right]$$

$$= E_u\left[1(X(t) = u)i^{K(t)}\exp \left(ig \sum_{z \in \mathbb{Z}} V(z) \sum_{j=1}^2 \tau_j(z)\right)\right].$$

Then

$$E\left[|E_u\left(1(X(t) = u)i^{K(t)} \exp \left[i \int_0^t W(X(s))ds\right]\right)|\right]$$

$$\leq E_u\left[1(X(t) = u)\left|\exp \left(ig \sum_{z \in \mathbb{Z}} V(z) \sum_{j=1}^2 \tau_j(z)\right)\right|\right] E\left[i \int_0^t W(X(s))ds\right]$$

$$= E_u\left[1(X(t) = u) \prod_{z \in \mathbb{Z}} \left|E e^{igV(z)(\tau^1(z) + \tau^2(z))}\right|\right];$$

the last equality holds as RVs $V(z)$ are independent for different $z$. 

9
By (1.13), the last expression is
\[\leq E_u \left[ 1(X(t) = u)b^{M(t)} \exp \left( -a|g| \sum_{z \in \mathbb{Z}} (\tau^1(z) + \tau^2(z)) \right) \right]\]
which equals \(e^{-a|g|2t}E_u b^{M(t)}\), as the sum \(\sum_{z \in \mathbb{Z}} (\tau^1(z) + \tau^2(z)) = 2t\). Here \(M(t) = M(t; \{X(s)\})\) is the total number of sites in \(\mathbb{Z}\) visited by processes \(\{X_j(s)\}, j = 1, 2\), between times 0 and \(t\). Since \(M(t) \leq K(t)\), we have that
\[e^{-2at}E_u b^{M(t)} \leq e^{-2a|g|t}E_u b^{K(t)} = e^{-2t(a|g| - b + 1)}.
\]

For the matrix elements \(\langle \delta_u, e^{itH} \delta_u \rangle\) we get the bound
\[\left| \langle \delta_u, e^{itH} \delta_u \rangle \right| \leq e^{-2t(a|g| - b - 1)}.
\]
This completes the proof of Lemma 2.1. \(\square\)

**Remark.** Molchanov’s formula has been used in [7], Proposition VI.3.1, to prove analyticity of the integrated density of states in the single-particle Anderson model with an IID random potential of the same type as in the present paper. As we will see, path integration techniques can be adapted to multi-particle lattice systems in any dimension.

We see that \(d\kappa_\Lambda(\lambda)\) admits a density: \(d\kappa_\Lambda(\lambda) = k_\Lambda(\lambda)d\lambda\).

**Theorem 2.1. (A Wegner-type estimate)** Consider LSO \(H_\Lambda\), as in (1.5), with \(\Lambda = \Lambda_{L_1, L_2}(x) = \Lambda_{L_1}(x_1) \times \Lambda_{L_2}(x_2)\). Under conditions (A) and (B), \(\forall E \in \mathbb{R}, L_1, L_2 \geq 1, r > 0\), and \(\forall x = (x_1, x_2) \in \mathbb{Z}^2\), probability
\[P \left\{ \text{dist} [E, \sigma(H_\Lambda)] < r \right\} \]
\[\leq \frac{2}{\pi B} (2L_1 + 1)(2L_2 + 1)r,\]
where \(B\) is the same as in Equation (2.6). In particular, for \(r = e^{-(L_1 \wedge L_2)\beta/2}\),
\[P \left\{ \text{dist} [E, \sigma(H_\Lambda)] < e^{-(L_1 L_2)^{\beta/2}} \right\} \leq \frac{2}{\pi B} (2L_1 + 1)(2L_2 + 1)e^{-(L_1 \wedge L_2)^{\beta/2}}. \quad (2.7)
\]
Here and below, \(L_1 \wedge L_2 = \min\{L_1, L_2\}\).
Proof of Theorem 2.1. We begin with an elementary inequality (cf. [7]). Let \( \Pi^\Lambda_{(E-r,E+r)} \) be the spectral projection on the subspace spanned by the corresponding EFs of \( H_\Lambda \). Then

\[
\mathbb{P}\left\{ \text{dist } [E, \sigma (H_\Lambda)] < r \right\} \leq \mathbb{E} \operatorname{tr} \Pi^\Lambda_{(E-r,E+r)}.
\] (2.8)

Further, in the Dirac's delta-basis:

\[
\operatorname{tr} \Pi^\Lambda_{(E-r,E+r)} = \sum_{u \in \Lambda} \langle \delta_u, \Pi^\Lambda_{(E-r,E+r)} \delta_u \rangle,
\]

and

\[
\mathbb{E} \operatorname{tr} \Pi^\Lambda_{(E-r,E+r)} = \sum_{u \in \Lambda} \int_{E-r}^{E+r} k_\Lambda(s) \, ds.
\]

The assertion of Theorem 2.1 now follows easily from Lemma 2.1 and Equations (2.3)–(2.4). \( \square \)

We will also need a variant of the Wegner-type estimate where either the horizontal or vertical projection sample of the potential is fixed. In Lemma 2.2 and Theorem 2.2 it is assumed that the lattice rectangle \( \Lambda = \Lambda_1^{(1)} \times \Lambda_2^{(1)} \) has \( \Lambda_1^{(1)} \cap \Lambda_2^{(1)} = \emptyset \). In Lemma 2.2 we consider the conditional expectation \( \tilde{k}_{\mathbb{W},\Lambda}(t|\mathcal{Y}(\Lambda_2^{(1)})) \):

\[
\tilde{k}_{\Lambda}(t|\mathcal{Y}(\Lambda_2^{(1)})) = \mathbb{E} \left[ \langle \delta_u, e^{itH_\Lambda} \delta_u \rangle \bigg| \mathcal{Y}(\Lambda_2^{(1)}) \right], \ u \in \Lambda, \ t \in \mathbb{R}, \quad (2.9)
\]

where the sigma-algebra \( \mathcal{Y}(\Lambda_2^{(1)}) = \{ V(x, \cdot), \ x \in \Lambda_2^{(1)} \} \) is generated by the values of the potential potential over segment \( \Lambda_2^{(1)} \).

Lemma 2.2. The quantity \( \tilde{k}_{\mathbb{W},\Lambda}(t|\mathcal{Y}(\Lambda_2^{(1)})) \) defined in (2.9) obeys

\[
\sup_{t} \left| \tilde{k}_{\mathbb{W},\Lambda}(t|\mathcal{Y}(\Lambda_2^{(1)})) \right| \leq e^{-Bt/2}, \ u \in \Lambda, \ t \in \mathbb{R}, \quad (2.10)
\]

independently of \( \Lambda \). Here, as in (2.6), \( B = 2(a|g| - b - 1) \). Therefore, \( k_\Lambda(\lambda) \) is analytic in a strip around real line, so it has a bounded derivative on any interval.
Lemma 2.2 is proved in the same way as Lemma 2.1. A direct corollary of Lemma 2.2 is

**Theorem 2.2. (A conditional Wegner-type estimate)** For LSO $H_\Lambda$, as in (1.5), with $\Lambda = \Lambda_{L_1,L_2}(x) = \Lambda_{L_1}(x_1) \times \Lambda_{L_2}(x_2)$ and $\Lambda_{L_1}(x_1) \cap \Lambda_{L_2}(x_2) = \emptyset$, under assumptions (A) and (B), $\forall E \in \mathbb{R}$, $L_1, L_2 \geq 1$, $r > 0$, and $\forall x = (x_1, x_2) \in \mathbb{Z}^2$, the conditional probability $\mathbb{P} \left\{ \text{dist} [E, \sigma(H_\Lambda)] < r | \mathfrak{U}(\Lambda_2^{(1)}) \right\}$ satisfies

$$
\sup \mathbb{P} \left\{ \text{dist} [E, \sigma(H_\Lambda)] < r | \mathfrak{U}(\Lambda_2^{(1)}) \right\} \leq \frac{4}{\pi B} (2L_1 + 1)(2L_2 + 1)r,
$$

where $B$ is the same as in Equation (2.6). In particular, for $r = e^{-(L_1 \wedge L_2)\beta/2}$,

$$
\sup \mathbb{P} \left\{ \text{dist} [E, \sigma(H_\Lambda)] < e^{-(L_1 \wedge L_2)\beta/2} | \mathfrak{U}(\Lambda_2^{(1)}) \right\} \leq \frac{4}{\pi B} (2L_1 + 1)(2L_2 + 1)e^{-(L_1 \wedge L_2)\beta/2}.
$$

(2.11)

**Remark.** Obviously, similar estimate holds for the conditional expectation with respect to the sigma-algebra $\mathfrak{U}(\Lambda_1^{(1)}) = \{V(x, \cdot), x \in \Lambda_1^{(1)}\}$.

We conclude this section with the statement which is a straightforward refinement of Theorem 2.2 and can be proved in a similar fashion.

**Theorem 2.3.** Consider segments $I_1 = [a_1, a_1 + L_1'] \cap \mathbb{Z}$, $I_2 = [a_2, a_2 + L_2'] \cap \mathbb{Z}$, $J_1 = [c_1, c_1 + L_2'] \cap \mathbb{Z}$ and $J_2 = [c_2, c_2 + L_2'] \cap \mathbb{Z}$. Assume that

either $I_1 \cap (J_1 \cup I_2 \cup J_2) = \emptyset$ or $J_1 \cap (I_1 \cup I_2 \cup J_2) = \emptyset$.

Set $\Lambda' = I_1 \times J_1$, $\Lambda'' = I_2 \times J_2$. Let $\mathfrak{U}(I_2 \cup J_2)$ stand for the sigma-algebra $\{V(x, x), x \in I_2 \cup J_2\}$. Consider an arbitrary function $\mathcal{E}$ measurable relative to $\mathfrak{U}(I_2 \cup J_2)$. Then

$$
\sup \mathbb{P} \left\{ \text{dist} [\mathcal{E}, \sigma(H_{\Lambda'})] < r | \mathfrak{U}(I_2 \cup J_2) \right\} \leq \frac{4}{\pi B} (2L_1' + 1)(2L_2' + 1)r.
$$

In particular, for $r = e^{-(L_1' \wedge L_2')\beta/2}$,

$$
\sup \mathbb{P} \left\{ \text{dist} [\mathcal{E}, \sigma(H_{\Lambda'})] < r | \mathfrak{U}(I_2 \cup J_2) \right\} \leq \frac{4}{\pi B} (2L_1' + 1)(2L_2' + 1)e^{-(L_1' \wedge L_2')\beta/2}.
$$

(2.12)
Lemma 2.3. Let $\Lambda' = \Lambda'_L = I_1 \times J_1$, $\Lambda'' = \Lambda''_L = I_2 \times J_2$, be two sub-squares with $I_j = [a_j, b_j]$, $J_j = [c_j, d_j]$, $j = 1, 2$ and such that

$$\Lambda' \cap D_d \neq \emptyset, \Lambda'' \cap D_d \neq \emptyset.$$ 

Assume that the max-norm distance

$$d_\infty(\Lambda', \Lambda'') > 5L$$

and that $L > d$. Then the coordinate projections of $\Lambda'$ are disjoint from those of $\Lambda''$: $(I_1 \cup J_1) \cap (I_2 \cup J_2) = \emptyset$, and so the potential samples in $\Lambda'$ and $\Lambda''$ are independent.

Proof of Lemma 2.3. Indeed, since $\Lambda' \cap D_d \neq \emptyset$, then $(a_2, d_2) \in D_d$, so that $a_2 - d_2 \leq d$. Further, $a_2 - c_2 - 2L$, so we have

$$c_2 \geq d_2 - 2L \geq a_2 - d - 2L.$$ 

On the other hand, since $\Lambda' \cap Z_2^2$, we have $(b_1, c_1) \in Z_2^2$, so that $c_1 \leq b_1$. Therefore,

$$d_1 \leq c_1 + 2L \leq b_1 + 2L.$$ 

Combining the above inequalities, we see that

$$c_2 - d_1 \geq (a_2 - d - 2L) - (b_1 - 2L) = (a_2 - b_1) - 4L - d > 5L - (4L + d) > 0,$$

so that $J_1 \cap J_2 = \emptyset$. Taking into account that $I_1 \cap I_2 = \emptyset$, we conclude that

$$(I_1 \cup J_1) \cap (I_2 \cup J_2) = \emptyset.$$ 

Definition 2.1. We call a pair of sub-squares $\Lambda'$, $\Lambda''$ $L$-distant ($L$-D, for short), if

$$d_\infty(\Lambda', \Lambda'') > 8L.$$ 

Lemma 2.4. Let $\Lambda' = (I_1 \times J_1) \cap Z_2^2$ and $\Lambda'' = (I_2 \times J_2) \cap Z_2^2$ be two sub-squares in $Z_2^2$. Assume that: (a) the lengths of four segments $I_j$ and $J_j$ is $\leq 2L$, $j = 1, 2$, and (b) $\Lambda'$, $\Lambda''$ are $L$-D, i.e. $d_\infty(\Lambda', \Lambda'') > 8L$.

Then either

(A) at least one of sub-squares $\Lambda', \Lambda''$ is off-diagonal (and hence is a square), in which case at least one of their coordinate projections is disjoint from the three others,

or

(B) the projections of $\Lambda'$ are disjoint from those of $\Lambda''$: $(I_1 \cup J_1) \cap (I_2 \cup J_2) = \emptyset$. 

13
Proof of Lemma 2.4. Denote by $J$ the union of four segments $I_1 \cup J_1 \cup I_2 \cup J_2$ and call it disconnected if (i) there exists a segment, among the four, disjoint from the three others, or (ii) there are two pairs of segments disjoint from each other, although within each pair the segments have non-empty intersections. Otherwise, $J$ is called connected.

First, note that had set $J$ been connected, its diameter would have been bounded by $8L$, since each interval has length $\leq 2L$. Then we would have had $d_\infty(I_1, I_2) \leq 4L, d_\infty(J_1, J_2) \leq 8L \Rightarrow d_\infty(\Lambda', \Lambda'') \leq 8L$, which is impossible by assumption (b).

Thus, assume that $J$ is disconnected. It is straightforward that in case (i) the assertion (A) of the Lemma 2.4 holds true. Hence we only have to show that in case (ii), both $\Lambda'$ and $\Lambda''$ are diagonal sub-squares.

In case (ii) we call the unions of segments within a given pair a connected component (of $J$). By assumption (b), either dist$(I_1, I_2) > 8L$ or dist$(J_1, J_2) > 8L$. For definiteness, suppose that dist$(I_1, I_2) > 8L$. Then $I_1$ is disjoint from $I_2$, and the connected component of $J$ containing $I_1$ should include either $J_1$ or $J_2$. Suppose first that $I_1 \cap J_1 \neq \emptyset, \text{ and } I_2 \cap J_2 \neq \emptyset$, \hspace{1cm} (2.13)

then $\Lambda' \cap D_d \neq \emptyset$ and $\Lambda'' \cap D_d \neq \emptyset$. By virtue of property (b), Lemma 2.3 applies, and assertion (B) in this case holds true.

Now suppose that $I_1 \cap J_2 \neq \emptyset$ and $I_2 \cap J_1 \neq \emptyset$.

Then $I_1 \cap J_1 = \emptyset, I_2 \cap J_2 = \emptyset$ and $J_1 \cap J_2 = \emptyset$. \hspace{1cm} (2.14)

We see that both $\Lambda'$ and $\Lambda''$ are off-diagonal squares. Write $I_j = [a_j, b_j], J_j = [c_j, d_j], j = 1, 2$. Since $I_2 \cap I_1 = \emptyset$, we can assume without loss of generality that $a_1 < b_1 < a_2 < b_2$.

Further, $(a_1, d_1) \in \Lambda' \subset \mathbb{Z}_2^2$, so that $c_1 < d_1 < a_2 < b_2$ implying that $I_2 \cap J_1 = \emptyset$.

In turn, this yields $I_2 \cap (I_1 \cup J_1) = \emptyset$. 14
But as $I_2 \cap J_2 = \emptyset$ (see (2.14)), then
\[ I_2 \cap (J_2 \cup I_1 \cup J_1) = \emptyset, \]
which is impossible: we are in case (ii), so no interval among $I_1$, $I_2$, $J_1$, $J_2$ is disjoint from the remaining three. This completes the proof of Lemma 2.4.

\[ \square \]

3 The MSA scheme: a single-particle case

Throughout this section we assume that condition (A) holds, although the scheme works for a much larger class of IID RVs $V(x, \cdot)$, $x \in \mathbb{Z}$. (In fact, the MSA scheme does not even require dimension one.)

For reader’s convenience, we reproduce here the principal points of the proof of localisation given in [9]. To simplify the future adaptation of the MSA scheme to the case of two particles, we choose particular values of parameters $p$, $q$, $\alpha$ and $\beta$ figuring in the specification of the scheme. This does not reduce the generality of the construction.

**Definition 3.1.** Fix $\beta = 1/2$. Given $E \in \mathbb{R}$, a segment $\Lambda^{(1)}_L(x) = [x - L, x + L] \cap \mathbb{Z}$, $x \in \mathbb{Z}$, is called $E$-resonant ($E$-R, for short) if the spectrum $\sigma(H^{(1)}_{\Lambda^{(1)}_L(x)})$ of $H^{(1)}_{\Lambda^{(1)}_L(x)}$, the single-particle LSO in $\Lambda^{(1)}_L(x)$ (see (1.7)), satisfies
\[ \text{dist} \left[ E, \sigma(H^{(1)}_{\Lambda^{(1)}_L(x)}) \right] < e^{-L^\beta}. \] (3.1)

Otherwise, $\Lambda^{(1)}_L(x)$ is called $E$-non-resonant ($E$-NR).

**Definition 3.2.** Given $E \in \mathbb{R}$ and $m > 0$, a segment $\Lambda^{(1)}_L(x) = [x - L, x + L] \cap \mathbb{Z}$, $x \in \mathbb{Z}$, is called $(E, m)$-non-singular ($(E, m)$-NS, for short) if
\[ \max_{u : |u - x| = L} \left| G^{(1)}_{\Lambda^{(1)}_L(x)}(x, u; E) \right| \leq e^{-mL}. \] (3.2)

Otherwise it is called $(E, m)$-singular ($(E, m)$-S). Here, $G^{(1)}_{\Lambda^{(1)}_L(x)}(y, u; E)$, $y, u \in \Lambda^{(1)}_L(x)$, stands for the Green’s function of $H^{(1)}_{\Lambda^{(1)}_L(x)}$:
\[ G^{(1)}_{\Lambda^{(1)}_L(x)}(y, u; E) = \left( H^{(1)}_{\Lambda^{(1)}_L(x)} - E \right)^{-1} \delta_y, \delta_u \), \quad y, u \in \Lambda^{(1)}_L(x). \] (3.3)
In Theorems 3.1 and 3.2 we consider intervals $I \subset \mathbb{R}$ of length $\leq 1$. However, the statements of both theorems can be easily extended to any finite interval.

**Theorem 3.1.** Let $I \subset \mathbb{R}$ be an interval of length $\leq 1$. Given $L_0 > 0$, $m_0 > 0$, $p = 6$, $q = 24$ and $\beta = 1/2$, consider the following properties (S1.0) and (S2.0) of single-particle LSOs $H^{(1)}_{\Lambda_{L_0}^{(1)}}$ in (1.7):

(S1.0) $\forall x, y \in \mathbb{Z}$ and disjoint segments $\Lambda_{L_0}^{(1)}(x)$ and $\Lambda_{L_0}^{(1)}(y)$,

$\mathbb{P} \left\{ \forall E \in I : \text{ both } \Lambda_{L_0}^{(1)}(x) \text{ and } \Lambda_{L_0}^{(1)}(y) \text{ are } (E, m_0) - S \right\} < L_0^{-2p}$. (3.4)

(S2.1) $\forall L \geq L_0$ and $\forall E$ with $\text{dist } [E, I] \leq \frac{1}{2} e^{-L^\beta}$,

$\mathbb{P} \left\{ \Lambda_{L_0}^{(1)}(x) \text{ is } E - R \right\} < L^{-q}$. (3.5)

Take $\alpha = 3/2$. Next set $L_{k+1} = L_0^\alpha$, $k = 0, 1, \ldots$. Given a number $m \in (0, m_0)$, $\exists Q^0 = Q^0(m_0, m) < \infty$ such that if properties (S1.0) and (S2.0) hold for $L_0 > Q^0$, property (S1.0) is valid for $L_k$, $k \geq 1$. That is, single-particle LSOs $H^{(1)}_{\Lambda_{L_k}^{(1)}}$ satisfy

(S1.k) $\forall x, y \in \mathbb{Z}$ and disjoint segments $\Lambda_{L_k}^{(1)}(x)$ and $\Lambda_{L_k}^{(1)}(y)$,

$\mathbb{P} \left\{ \forall E \in I : \text{ both } \Lambda_{L_k}^{(1)}(x) \text{ and } \Lambda_{L_k}^{(1)}(y) \text{ are } (E, m) - S \right\} < L_k^{-2p}$. (3.6)

**Remark.** A detailed analysis of proofs given in [9] shows that in fact, the parameters $p$ and $q$ can be chosen arbitrarily big, provided that the amplitude $|g|$ of the random external potential is large enough:

$p \geq p(g) \underset{|g| \rightarrow \infty}{\longrightarrow} + \infty$, $q \geq q(g) \underset{|g| \rightarrow \infty}{\longrightarrow} + \infty$. (3.7)

**Theorem 3.2.** Let $I \subset \mathbb{R}$ be an interval of length $\leq 1$, and fix $L_0 > 0$, $m > 0$, $p = 6$, $\alpha = 3/2$ and $m > 0$. Set $L_{k+1} = L_k^\alpha$, $k = 0, 1, \ldots$. Suppose that for any $k = 0, 1, 2, \ldots$, the single-particle LSOs $H^{(1)}_{\Lambda_{L_k}^{(1)}}$ in (1.7) obey the bound from (S1.k). That is,

$\mathbb{P} \left\{ \forall E \in I \text{ both } \Lambda_{L_k}^{(1)}(x) \text{ and } \Lambda_{L_k}^{(1)}(y) \text{ are } (E, m) - S \right\} \leq L_k^{-2p}$. (3.8)

$\forall x, y \in \mathbb{Z}$ and disjoint segments $\Lambda_{L_k}^{(1)}(x)$, $\Lambda_{L_k}^{(1)}(y)$. 

16
Then, with probability one, the spectrum of single-particle LSO \(H^{(1)}\) (cf. (1.8)) in \(I\) is pure point, and the EFs corresponding to EVs in \(I\) decay exponentially fast at infinity.

The proof of Theorem 3.2 is purely deterministic and does not rely upon probabilistic properties of the random process of potential values \(gV(x, \omega)\), \(x \in \mathbb{Z}\). The core technical statement to be adapted to our two-particle model is the above Theorem 3.1. We will see, however, that methods and results of one-particle localisation theory also play an important role in the two-particle theory.

Apart from probabilistic estimates of the Green’s functions in finite volumes, we will also need the following result on the exponential decay of EFs of one-dimensional LSOs in finite volumes. It is convenient here to introduce the definition of “tunneling”.

**Definition 3.3.** Given \(x \in \mathbb{Z}\) and an integer \(L > 0\), let \(\psi_j, j = 1, \ldots, 2L + 1\), be the EFs of matrix \(H^{(1)}_{\Lambda_L^{(1)}(x)}\), the single-particle LSO in segment \(\Lambda_L^{(1)}(x) = [x - L, x + L] \cap \mathbb{Z}\) (cf. (1.7)). We say that \(\Lambda_L^{(1)}(x)\) is \(m\)-non-tunneling (\(m\)-NT, for short), if the following inequality holds:

\[
\sum_j \sum_{y = x \pm L} |\psi_j(x)\psi_j(y)| \leq e^{-mL}.
\] (3.9)

Otherwise, \(\Lambda_L^{(1)}(x)\) is called \(m\)-tunneling (\(m\)-T).

The rest of the presentation, in Sections 3 and 4, is based on a sequence of technical lemmas related to single- and two-particle systems.

**Lemma 3.1.** Fix \(\beta = 1/2\). Given \(E \in \mathbb{R}\), \(x \in \mathbb{Z}\) and an integer \(L \geq 1\), consider segment \(\Lambda_L^{(1)}(x) = [x - L, x + L] \cap \mathbb{Z}\) and the single-particle LSO \(H_{\Lambda_L^{(1)}(x)}^{(1)}\) in (1.7). Assume that \(\Lambda_L^{(1)}(x)\) is \(E\)-NR and \(m\)-NT where \(m \geq 2\). Then \(\Lambda_L^{(1)}(x)\) is also \((E, m')\)-NS where \(m'\) satisfies

\[
m' \geq m - L^{-(1-\beta)}.
\] (3.10)

For the proof, use the formula for the Green’s functions \(G^{(1)}_{\Lambda_L^{(1)}(x)}(u, y; E)\) (cf. (3.3)):

\[
G^{(1)}_{\Lambda_L^{(1)}(x)}(u, y; E) = \sum_{j=1}^{2L+1} \frac{\psi_j(u)\bar{\psi}_j(y)}{E_j - E}.
\]
where $E_j$ is the EV of the EF $\psi_j$ of $H^{(1)}_{\Lambda^{(1)}_L(x)}$.

In the one-dimensional, single-particle Anderson model, it is well-known that the probability of tunneling in segment $\Lambda^{(1)}_L(x)$ is exponentially small with respect to $L$; see, e.g., [14], [15]). For convenience, we state here the corresponding assertion in the form used below, with a power-like bound. In this form it has been proven in higher dimensions, for large values of $|g|$; see [9], Theorem 2.3 and Lemma 3.1. We note that in [2] a stronger bound was established, by using the method of fractional moments of the resolvent.

Lemma 3.2. Consider segment $\Lambda^{(1)}_L(x) = [x - L, x + L] \cap \mathbb{Z}$ and the LSO $H^{(1)}_{\Lambda^{(1)}_L(x)}$. Then for any $m > 0$ there exist constants $g_1 = g_1(m) \in (0, +\infty)$ and $L_0 = L_0(m)$ such that for all $|g| \geq g_1$ and $L \geq L_0$ we have

$$P\{\Lambda^{(1)}_L(x) \text{ is } m-\text{NT}\} \geq 1 - L^{-q}. \quad (3.11)$$

Lemma 3.2 plays an important role in the proof of Lemma 4.8; see below. We want to note that such strong (in fact, optimal) probabilistic estimates, both for continuous and discrete one-dimensional random Schrödinger operators, go back to earlier works, viz. [13], [14], [16], [15], [5]. The reader can find a detailed account of specifically one-dimensional methods and an extensive bibliography in the monographs [7] and [18] (cf., in particular, Theorem VIII.3.7 and Section VIII.3 in [7]).

4 An MSA for a two-particle system

In this section, we propose a modification to the von Dreifus – Klein MSA scheme so as to adapt it to two-particle systems. The scheme allows any finite number of "singular" areas in a given finite volume $\Lambda \subset \mathbb{Z}^2$, provided that the "disorder" is high enough ($|g| \gg 1$). This feature is a serious advantage of the MSA scheme which makes it flexible and applicable to the random field (1.12) generated by the potential $V(x,\omega)$, $x, x_1, x_2 \in \mathbb{Z}^1$.

As was said before, we follow the general strategy of [9], but introduce some technical changes. It was noted that the MSA scheme includes values $p, q, \alpha$ and $\beta$ which are subject to certain restrictions. For us, it is convenient to set, throughout Sections 4 and 5:

$$p = 6, \quad q = 24, \quad \alpha = 3/2, \quad \text{and} \quad \beta = 1/2. \quad (4.1)$$
similarly to Section 3. However, to make the presentation consistent with that in [9], we continue referring to parameters \( p, q, \alpha \) and \( \beta \) in our constructions below. The main components of the MSA scheme are an increasing sequence of positive integer lengths \( L_0, L_1, L_2, \ldots \) and a decreasing sequence of positive masses \( m_0, m_1, m_2, \ldots \). In Sections 4 and 5 these sequences are assumed to be as follows:

(i) for \( k \geq 1 \):

\[
L_k = \text{the smallest integer } \geq L_{k-1}^\alpha
\]
and
\[
m_k = m_0 \prod_{j=1}^{k} \left(1 - 8L_0^{-j/2}\right),
\]

(ii) \( L_0 \) is positive integer and \( m_0 \) is positive such that

\[
m_0 > 2, \ L_0 \geq 256 \text{ and } e^{-m_0L_0} \leq e^{-L_0^\beta}.
\]

Observe that, owing to the bound \( L_0 \geq 256 \), the infinite product

\[
\prod_{j \geq 1} (1 - 8L_0^{-j/2}) \geq 1/2.
\]

Thus,

\[
m_\infty := \liminf_{k \to \infty} m_k \geq m_0/2.
\]

In addition, we will have to assume that \( L_0 \) is large enough; such restrictions will appear in various lemmas below. Ultimately, the lower bound on \( L_0 \) will depend on a particular choice of \( m_0 \).

In this respect, it should be noted that the choice of \( m_0 \) and \( L_0 \) dictates the choice of the value of \( |g| \) in Theorem 1.1. More precisely, if \( |g| \) large enough (roughly, \( \ln |g| \simeq O(m_0L_0) \)), then the (modified) MSA scheme will guarantee the exponential decay of the EFs of the two-particle LSO \( H \) from (1.2) with mass \( \geq m_\infty \).

As in [9], we define the notions of resonant and singular sub-squares.

**Definition 4.1.** Given \( E \in \mathbb{R} \), a sub-square \( \Lambda_L(x) \) of size \( L \) with center at \( x \) is called \( E \)-resonant (E-R, for short) if the spectrum \( \sigma(H_{\Lambda_L(x)}^{(2)}) \) of \( H_{\Lambda_L(x)}^{(2)} \), the two-particle LSO in \( \Lambda_L(x) \), satisfies

\[
dist\left[E, \sigma(H_{\Lambda_L(x)}^{(2)})\right] < e^{-L^\beta}.
\]

Otherwise, \( \Lambda_L(x) \) is called \( E \)-non-resonant (E-NR).
Definition 4.2. Given $E \in \mathbb{R}$ and $m > 0$, a sub-square $\Lambda_L(x)$ is called $(E,m)$-non-singular ($(E,m)$-NS, for short), if
\[ \max_{u \in \partial \Lambda_L(x)} \left| G^{(2)}_{\Lambda_L(x)}(x, u; E; \omega) \right| \leq e^{-mL}. \] (4.7)
Otherwise it is called $(E,m)$-singular ($(E,m)$-S). Here, $G^{(2)}_{\Lambda_L(x)}(y, u; E), y, u \in \Lambda_L(x)$, stands for the Green’s function of $H^{(2)}_{\Lambda_L(x)}$:
\[ G^{(2)}_{\Lambda_L(x)}(y, u; E) = \left( H^{(2)}_{\Lambda_L(x)} - E \right)^{-1} \delta_y \delta_u, \quad y, u \in \Lambda_L(x). \] (4.8)

Similar definitions hold for $H^{n-i}_{\Lambda_L(x_1) \times \Lambda_L(x_2)}$, the LSO of a two-particle system with no interaction, in square $\Lambda_L(x_1) \times \Lambda_L(x_2)$; cf. (1.9).

Recall, in this paper, the interaction potential $U$ has finite range $d$ (cf. (1.4)). So, there are two kinds of sub-squares: those which are disjoint from the diagonal strip
\[ D = \left\{ z = (z_1, z_2) \in \mathbb{Z}^2_\geq : z_1 - z_2 \leq d \right\}, \]
and those having common points with $D$. The former are called off-diagonal sub-squares (actually, squares), and the latter diagonal sub-squares. On an off-diagonal square $\Lambda_L(y)$, the interaction potential is identically zero, and so the two-particle LSO $H^{n-i}_{\Lambda_L(y)}$ coincides with $H^{n-i}_{\Lambda_L(y)}$ (cf. (1.9)) and is written as the sum (1.11) involving single-particle LSOs $H^{(1)}_{\Lambda_L(y)}$ and $H^{(1)}_{\Lambda_L(y)}$. The distinction between off-diagonal and diagonal sub-squares requires different techniques.

Our version of the two-particle MSA scheme can be summarised in a form similar to that in Section 3. More precisely, the following assertions hold, whose structure is similar to Theorems 3.1 and 3.2.

Theorem 4.1. Let $I \subset \mathbb{R}$ be an interval of length $\leq 1$. Given $L_0 > 0$, $m_0 > 0$, consider the following properties (T1.0) and (T2.0) of two-particle LSOs $H_{\Lambda_L}$ from (1.5):

(T1.0) \quad $\forall \bar{x}, \bar{y} \in \mathbb{Z}^2_\geq$ and $L_0$-D sub-squares $\Lambda_{L_0}(\bar{x})$ and $\Lambda_{L_0}(\bar{y})$,\[
\mathbb{P} \left\{ \forall E \in I : \text{both } \Lambda_{L_0}(\bar{x}) \text{ and } \Lambda_{L_0}(\bar{y}) \text{ are } (E, m_0)\text{-S} \right\} < L_0^{-2p}. \] (4.11)
∀ L ≥ L_0 and ∀ E with \text{dist} [E, I] ≤ \frac{1}{2} e^{-L^a}, \quad \mathbb{P} \left\{ \Lambda_L(x) \text{ is } E-R \right\} < L^{-q}. \quad (4.12)

Next, define values \( L_k \) and \( m_k \), \( k \geq 1 \), as in (4.2) and (4.3). There exists \( Q_0 = Q_0(m_0) \in (0, \infty) \) such that if properties (T1.0) and (T2.0) are valid for \( L_0 > Q_0 \) and \( m_0 \), then property (T1.0) holds for \( L_k \) and \( m_k \), \( k \geq 1 \). That is, the two-particle LSOs \( H_{\Lambda_{L_k}} \), \( k = 1, 2, \ldots \), obey

\begin{equation}
\forall \, x, y \in \mathbb{Z}^2 \geq \text{ and } L_k-D \text{-sub-squares} \, \Lambda_{L_k}(x) \text{ and } \Lambda_{L_k}(y), \\
\mathbb{P} \left\{ \forall E \in I: \text{ both } \Lambda_{L_k}(x) \text{ and } \Lambda_{L_k}(y) \text{ are } (E, m_{\infty})-S \right\} < L_k^{-2p}. \quad (4.13)
\end{equation}

**Theorem 4.2.** Let \( I \subset \mathbb{R} \) be an interval of length \( \leq 1 \), and fix \( L_0 > 0 \), \( m_0 > 0 \). Define values \( L_k \), \( k \geq 1 \), as in (4.2), and suppose that for some finite constant \( C \), for any \( k = 0, 1, 2, \ldots \), two-particle LSOs \( H_{\Lambda_{L_k}} \) from (1.5) satisfy the bound

\begin{equation}
\mathbb{P} \left\{ \forall E \in I: \text{ both } \Lambda_{L_k}(x) \text{ and } \Lambda_{L_k}(y) \text{ are } (E, m_{\infty})-S \right\} \leq C L_k^{-2p} \quad (4.14)
\end{equation}

whenever \( x, y \in \mathbb{Z}^2 \geq \text{ and } L_k-D \text{-sub-squares} \, \Lambda_{L_k}(x) \text{ and } \Lambda_{L_k}(y) \). Here, \( m_{\infty} \) is defined in (4.3) and (4.5). Then, with probability one, the spectrum of the two-particle LSO \( H \) (cf. (1.2)) in \( I \) is pure point, and the EFs corresponding to the EVs in the interval \( I \) decay exponentially fast at infinity, with mass \( \geq m_{\infty} \).

As in Section 3, the assumption that \( I \) has length \( \leq 1 \) is introduced for technical convenience and does not restrict generality.

However, the reader should note a difference between Theorem 3.1 and Theorem 4.1. Namely, in Equations (4.11) and (4.13), sub-squares \( \Lambda_{L_0}(x) \), \( \Lambda_{L_0}(y) \) and \( \Lambda_{L_k}(x) \), \( \Lambda_{L_k}(y) \), are assumed to be not simply disjoint but \( L_0-D \) and \( L_k-D \), respectively. In other words, in the two-particle MSA inductive scheme from this paper it is required less and assumed less compared with the single-particle one from [9]. Formally, the original argument developed in [9] estimates, at each inductive step, the probability that any disjoint pair of volumes (cubes) is simultaneously singular or simultaneously resonant, is sufficiently small. However, a careful analysis shows that it suffices to consider pair of volumes satisfying a stronger requirement:

\[ d_{\infty}(\Lambda_{L_k}(x), \Lambda_{L_k}(y)) \geq C L_k, \]
for any given positive constant $C$.

The initial step in the inductive scheme described in Theorems 4.1 and 4.2 is provided by Theorem 4.3 below.

**Theorem 4.3.** $\forall L_0 \geq 256 \exists g_0 \in (0, \infty)$ such that, for $g$ with $|g| \geq g_0$, $\forall$ interval $I$ of length $\leq 1$ assumptions (T1.0) and (T2.0) hold true.

Theorem 1.1 follows directly from Theorems 4.1 – 4.3. In turn, Theorem 4.3 follows from Lemmas 4.1 and 4.2 below.

**Lemma 4.1.** Given $m, L > 0$ and $E \in \mathbb{R}$, assume that sub-square $\Lambda_L(x)$ is $(E, m)$-S. Then $\Lambda_L(x)$ contains at least one site $u = (u_1, u_2)$, with

$$|U(u) + gV(u_1, \omega) + gV(u_2, \omega) - E| < e^{mL} + \|H^0\| = e^{mL} + 4.$$  \hspace{1cm} (4.15)

Therefore, $\forall \tilde{p} > 0$

$$L^\tilde{p} \cdot \mathbb{P}\left\{ \Lambda_L(x) \text{ is } (E, m) - S \right\} \xrightarrow{|g| \to \infty} 0.$$  

Here $\|H^0\|$ stands for the operator norm of $H^0$.

Therefore, for any $\tilde{p} > 0$ and all sufficiently large $|g|$, $\mathbb{P}\left\{ \Lambda_L(x) \text{ is } (E, m) - S \right\} \leq L^{-\tilde{p}}$.

**Lemma 4.2.** Given $E \in \mathbb{R}$, $\forall x \in \mathbb{Z}^2$ and $L \geq 1$,

$$\mathbb{P} \left\{ \Lambda_L(x) \text{ is } E-R \right\} \leq |\Lambda_L(x)|^2 \|f\|_\infty e^{-L^\beta}.$$ \hspace{1cm} (4.16)

Here and below, $|\Lambda_L(x)|$ stands for the number of points in sub-square $\Lambda_L(x)$ (which is $\leq (2L + 1)^2$), and $\|f\|_\infty$, as before, is the sup-norm of PDF $f$.

Therefore, for $L > 0$ large enough,

$$\mathbb{P} \left\{ \Lambda_L(x) \text{ is } E-R \right\} \leq L^{-q}.$$  \hspace{1cm} (4.17)
Lemmas 4.1 and 4.2 follow directly from our Wegner-type estimate in Theorem 2.1 (cf. Theorem A.1.3(i) in [9]). The meaning of this lemma is that if a finite (and fixed) size sub-square $\Lambda_L(x)$ is singular and the coupling constant $g$ is large enough, then $\Lambda_L(x)$ contains necessarily resonant points. The importance of such a relation between resonant and singular domains is explained by the fact that the probability of being resonant is much simpler to estimate than that of being singular.

Both Lemma 4.2 and Lemma 4.2 do not use the recursive scheme from (4.2) and (4.3).

The estimates provided by Lemma 4.1 and Lemma 4.2 will also be used in the proof of Lemma 4.8 in the same way as a similar estimate was used in [9].

The statement of Theorem 4.2 is similar to the assertion of Lemma 3.1 from [9]. We want to note that Lemma 3.1 in [9] is a general statement based only on probabilistic estimates provided by Lemma 3.2 in [9], so that the Borel-Cantelli lemma (which is the key ingredient of the proof of Lemma 3.1 in [9]), applies. In our situation, the proof of Theorem 4.2 goes along the same line and is based on probabilistic estimates from Theorem 4.1.

Therefore, to prove Theorem 1.1, it suffices to establish Theorem 4.1. This is the subject of the rest of the paper. The specification (4.1) will help with producing fairly explicit bounds. In essence, Theorem 4.1 constitutes an inductive assertion, in the value of $k$, guaranteeing a reproduction of property $(T_{1.k})$ from properties $(T_{1.k-1})$ and $(T_{2.0})$. More precisely, in our approach to inequality (4.13), we estimate (by using different methods), the probability in the LHS of (4.13) for pairs of $L_k$-D sub-squares $\Lambda_L(x)$ and $\Lambda_L(y)$ of three types:

(I) Both $\Lambda_L(x)$ and $\Lambda_L(y)$ are off-diagonal; see Lemma 4.5.

(II) Both $\Lambda_L(x)$ and $\Lambda_L(y)$ are diagonal; see Lemmas 4.7 and 4.8.

(III) One of sub-squares $\Lambda_L(x)$ and $\Lambda_L(y)$ is off-diagonal and one diagonal; see Lemma 4.9.

As was said earlier, the aim is to show that the probability figuring in $(T_{1.k})$, that $\forall \ E \in I$, both $\Lambda_L(x)$ and $\Lambda_L(y)$ are $(E, m_k)$-S, in each of cases (I) – (III) is bounded from above by $L_k^{-2p}$. Here and below, $L_k$ and $m_k$ are assumed to be as in (4.2) and (4.3).

The rest of Section 4 contains various bounds on probabilities related to $LSO \ H_{\Lambda_L(x)}$. The first such bound is
Lemma 4.3. Fix $m > 2$. There exists $Q_1 = Q_1(m)$ such that for $L \geq Q_1$ the following property holds. Let $x \in \mathbb{Z}^2$ and assume that $\Lambda_L(x) = \Lambda_L^{(1)}(x_1) \times \Lambda_L^{(1)}(x_2)$ is an off-diagonal square. Assume that segments $\Lambda_L^{(1)}(x_1)$ and $\Lambda_L^{(1)}(x_2)$ are $(E, m)$-NT, in the sense of Definition 3.3. Consider the LSO $H_{\Lambda_L(x)} = H_{\Lambda_L^{(1)}(x)}$. Then there exists positive $m'$ satisfying

$$m' \geq m - 3L^{-(1-\beta)}$$

such that if square $\Lambda_L(x)$ is $E$-NR, then it is $(E, m')$-NS.

For the proof of Lemma 4.3, see Section 5.

The next assertion, Lemma 4.4, helps to understand several parts of the two-particle MSA scheme. Consider standard coordinate projections $\Pi_j : \mathbb{Z}^2 \to \mathbb{Z}$, $j = 1, 2$, so that, for a given subset of the lattice $\Lambda \subset \mathbb{Z}^2$, its coordinate projections are given by

$$\Pi_1(\Lambda) = \{ u_1 \in \mathbb{Z} : (u_1, u_2) \in A \text{ for some } u_2 \in \mathbb{Z} \},$$

and

$$\Pi_2(\Lambda) = \{ u_2 \in \mathbb{Z} : (u_1, u_2) \in A \text{ for some } u_1 \in \mathbb{Z} \}.$$

Lemma 4.4. Fix an interval $I \subset \mathbb{R}$ of length $\leq 1$. Suppose that property (T2.0) holds, that is

$$\mathbb{P}\{ \Lambda_L(x) \text{ is } E\text{-R} \} < L^{-q}, \forall L \geq L_0 \text{ and } E \text{ with } \text{dist} \ [E, I] \leq \frac{1}{2}e^{-L^\beta}.$$ 

Next, let $\Lambda' = \Lambda_L'(u')$ and $\Lambda'' = \Lambda_L''(u'')$ be two sub-squares such that both their horizontal projections are disjoint and their vertical projections are disjoint:

$$\Pi_1(\Lambda') \cap \Pi_1(\Lambda'') = \emptyset, \quad \Pi_2(\Lambda') \cap \Pi_2(\Lambda'') = \emptyset.$$

Set $L = \min\{L', L''\}$. If $L \geq L_0$, then

$$\mathbb{P}\left\{ \exists E \in I : \text{ both } \Lambda' \text{ and } \Lambda'' \text{ are } E-\text{R} \right\} \leq L^{-q}.$$  

The proof of Lemma 4.4 is straightforward; see Section 5. Observe that Lemmas 4.1 – 4.4 are ”non-recursive” statements: they do not refer the recursive scheme introduced in (4.2) and (4.3).
We now pass to Lemma 4.5 which covers the probability in the LHS of (4.13) for two off-diagonal squares \( \Lambda_{L_k}(x) \) and \( \Lambda_{L_k}(y) \). The estimate provided in this lemma is similar to that in Lemma 4.1 from [9]. However, the difference is that in Lemma 4.5 the assumption is made for all pairs of disjoint sub-squares and reproduced for pairs of disjoint off-diagonal squares.

**Lemma 4.5.** Let \( I \subset \mathbb{R} \) be an interval of length \( \leq 1 \). Suppose that, \( \forall \ L \geq L_0 \), property (3.12) is fulfilled. That is:

\[
P\left\{ \Lambda^{(1)}(x) \text{ is } m-\text{NT} \right\} \geq 1 - L^{-q}.
\]

Then, \( \forall \) pair of \( L_k \)-D, off-diagonal squares \( \Lambda_{L_k+1}(x) \) and \( \Lambda_{L_k+1}(y) \),

\[
P\left\{ \exists \ E \in I : \text{ both } \Lambda_{L_k+1}(x) \text{ and } \Lambda_{L_k+1}(y) \text{ are } (E, m_{k+1})-S \right\} \leq L_k^{-2p}.
\]  

(4.20)

For the proof of Lemma 4.5 see Section 5.

The assertion of Lemma 4.6 below is close to Lemma 4.2 in [9] and can be proved in essentially the same way, for it only relies upon singularity/non-singularity properties of the sub-squares residing in a larger sub-square.

**Lemma 4.6.** Fix \( E \in \mathbb{R} \) and an integer \( K > 0 \). There exists a constant \( Q_2 = Q_2(K) \in (0, +\infty) \) with the following property. Assume that \( L_0 \geq Q_2 \). Next, given \( k \geq 0 \), assume that a sub-square \( \Lambda_{L_k}(x) \) is \( E\text{-NR} \) and does not contain more than \( K \) disjoint sub-squares \( \Lambda_{L_k}(u_j) \subset \Lambda_{L_k+1}(x) \) that are \( (E, m_k)\text{-S} \). Then sub-square \( \Lambda_{L_k+1}(x) \) is \( (E, m_{k+1})\text{-NS} \).

Next, consider an assertion

\[
\textbf{DS}(k, I) : \forall \ \text{pair of diagonal } L_k\text{-D sub-squares } \Lambda_{L_k}(u) \text{ and } \Lambda_{L_k}(v)
\]

\[
P\left\{ \exists \ E \in I : \text{ both } \Lambda_{L_k}(u) \text{ and } \Lambda_{L_k}(v) \text{ are } (E, m_k)\text{-S} \right\} \leq L_k^{-2p}.
\]  

(4.21)

The following lemma is used in the proof of Lemma 4.8.

**Lemma 4.7.** Given \( k \geq 0 \), assume that property \( \textbf{DS}(k, I) \) in (4.21) holds true. Consider a sub-square \( \Lambda := \Lambda_{L_k+1}(x) \) and let \( N(\Lambda; E) \) be the maximal number of \( (E, m_k)\text{-S} \), pair-wise \( L_k\text{-D} \) diagonal sub-squares \( \Lambda_{L_k}(u^{(j)}) \subset \Lambda \). Then \( \forall \ n \geq 1 \),

\[
P\left\{ \exists \ E \in I : N(\Lambda; E) \geq 2n \right\} \leq L_k^{n(1+\alpha)} \cdot L_k^{-np/2}.
\]  

(4.22)
Now comes a statement which extends Lemma 4.1 from [9] to pairs of diagonal sub-squares.

**Lemma 4.8.** There exists a constant $Q_3 \in (0, +\infty)$ such that if $L_0 \geq Q_3$, then, $\forall k \geq 0$, the property $DS(k, I)$ in (4.21) implies $DS(k + 1, I)$.

Finally, the case of a pair with one diagonal and one off-diagonal sub-square is covered by

**Lemma 4.9.** There exists a constant $Q_4 \in (0, +\infty)$ with the following property. Assume that $L_0 \geq Q_4$ and that, given $k \geq 0$, property $DS(k, I)$ in (4.21) holds. Let $\Lambda' = \Lambda_{L_k+1}(x')$ be a diagonal sub-square and $\Lambda'' = \Lambda_{L_k+1}(x'')$ an off-diagonal square, and let $\Lambda'$ and $\Lambda''$ be $L_{k+1}$-D. Then

$$P\{\exists E \in I : \text{both } \Lambda' \text{ and } \Lambda'' \text{ are } (E, m_{k+1})-S\} \leq L_{k+1}^{-2p}.$$  \hspace{1cm} (4.23)

From Lemmas 4.5, 4.8, and 4.9, Theorem 4.1 is deduced by following the remaining parts of the MSA scheme [9].

## 5 Proof of MSA Lemmas

**Proof of Lemma 4.3** Let $\{\psi'_j\}$ be normalised EFs of single-particle LSO $H_{\Lambda_{L_k}(x_1)}^{(1)}$ with EVs $E'_j$ and $\{\psi''_k\}$ be normalised eigen-functions of $H_{\Lambda_{L_k}(x_2)}^{(1)}$ with EVs $E''_k$. As $\|\psi'_j\|_2 = \|\psi''_k\|_2 = 1$, we have that

$$\max_{u \in \Lambda_{L_k}(x_1)} |\psi'_j(u)| \leq 1, \quad \max_{v \in \Lambda_{L_k}(x_2)} |\psi''_k(v)| \leq 1.$$  \hspace{1cm}

Next, for $\underline{u} = (u, u'), \underline{v} = (v, v') \in \Lambda_{L_k}(x)$, the two-particle Green’s functions have the form

$$G_{\Lambda_k}(\underline{u}, \underline{v}; E) = \left( (H_{\Lambda_{L_k}(x)}^{(n-i)} - E)^{-1} \delta_{\underline{u}, \underline{v}} \right)_{j, k} = \sum_{j, k} \frac{\psi'_j(u')\overline{\psi'_j(v')}\psi''_k(u'')\overline{\psi''_k(v'')}}{E - (E'_j + E''_k)}. \hspace{1cm} (5.1)$$

26
Further, assuming that $\Lambda_L(\mathbf{x})$ is $E$-NR, we get
\[
|E - (E_j + E_k')|^{-1} \leq e^{L\beta}.
\]
Finally, for $y \in \partial\Lambda_{L,L'}(\mathbf{x}),$
\[
|G_{\Lambda_{L,L'}(\mathbf{x}, y; E)}(x, y; E)| \leq (2L + 1)^2 \min_{j,k} \left| E - (E_j' + E_k') \right| e^{-mL} \leq (2L + 1)^2 e^{-mL} e^{L\beta} \leq e^{-m'L},
\]
with
\[
m' = m - L^{-1}(2 \ln(2L + 1) + L\beta) \geq m - C''L^{-1(1-\beta)}. \quad (5.3)
\]

**Proof of Lemma 4.4:** Since $\Lambda'$ and $\Lambda''$ have both coordinate projections disjoint, the respective samples of potential in these two sub-squares are independent, as in the single-particle theory with IID potential. So, we can use exactly the same argument (conditioning on the potential in $\Lambda'$, combined with the Wegner-type estimate for a single-particle model) as in the proof of Lemma 4.1 in [9]. \qed

**Proof of Lemma 4.5:** If $\Lambda'$ and $\Lambda''$ are off-diagonal and $L_k$-D, then, by virtue of Lemma 2.4, at least one of their horizontal and vertical projections among
\[
I_1 = \Pi_1 \Lambda', \quad J_1 = \Pi_2 \Lambda', \quad I_2 = \Pi_1 \Lambda'', \quad J_2 = \Pi_2 \Lambda''
\]
is disjoint with the three others. Without loss of generality, suppose that
\[
I_1 \cap (J_1 \cup I_2 \cup J_2) = \emptyset;
\]
three other possible cases are similar.

Consider the following events:
\[
B = \{ \text{ both } \Lambda' \text{ and } \Lambda'' \text{ are } (E, m_{k+1})-S \},
\]
\[
C = \{ \text{ both } \Lambda' \text{ and } \Lambda'' \text{ are } E-R \}. \quad (5.4)
\]
Then we can write
\[
\mathbb{P} \{ C \} = \mathbb{E} \left[ \mathbb{P} \{ C \mid \mathfrak{M}(\Pi_1 \Lambda' \cup \Pi_2 \Lambda'') \} \right].
\]
where the sigma-algebra $\mathcal{V}(\Pi_1 \Lambda' \cup \Pi_2 \Lambda'')$ is generated by potential values \(\{ V(x, \cdot), x \in \Pi_1 \Lambda' \cup \Pi_2 \Lambda'' \}\).

By Theorem 2.3, the conditional probability $\mathbb{P} \{ C | \mathcal{V}(\Pi_1 \Lambda' \cup \Pi_2 \Lambda'') \}$ is a.s. bounded by $L_k^{-q}$, and so is its expectation.

Now let
$$\tilde{J} = \Pi_1 \Lambda' \cup \Pi_1 \Lambda'' \cup \Pi_2 \Lambda' \cup \Pi_2 \Lambda''$$
and consider the event
$$T = \left\{ \exists \text{ an } (E, m_0 + 1) \cdot T \text{ interval } \tilde{J} \subset J \right\}. \quad (5.5)$$

By Lemma 3.2 and Corollary 3.1, $\mathbb{P} \{ T \} \leq L_k^{-q}$. On the other hand, if the potential sample belongs to $\bar{T}$ and both $\Lambda'$ and $\Lambda''$ are $(E, m_{k+1})$-S, then both sub-squares must be $E$-R.

Now we can write that
$$\mathbb{P} \{ B \} \leq \mathbb{P} \{ B \cap \bar{C} \cap \bar{T} \} + \mathbb{P} \{ B \cap C \} + \mathbb{P} \{ B \cap T \}.$$

By Lemma 4.3, if $\Lambda'$ (resp., $\Lambda''$) is both $(E, m_{k+1})$-NT (property $\bar{T}$) and $E$-NR (property $\bar{C}$), then it cannot be $E$-R, so that $B \cap \bar{C} \cap \bar{T} = \emptyset$. Finally,
$$\mathbb{P} \{ B \} \leq \mathbb{P} \{ B \cap C \} + \mathbb{P} \{ B \cap T \} < L_k^{-q} + L_k^{-q} = 2L_k^{-q} < L_k^{-2p}. \quad \Box \quad (5.6)$$

**Proof of Lemma 4.7** Suppose we have diagonal sub-squares $\Lambda_{L_k}(u^{(1)})$, \ldots, $\Lambda_{L_k}(u^{(2n)})$, such that
a) any two of them are $L_k$-D, i.e., are at the distance $\geq 6L_k + 2d$,
b) all the sub-squares $\Lambda_{L_k}(u^{(1)})$, \ldots, $\Lambda_{L_k}(u^{(2n)})$ lie in $\Lambda = \Lambda_{L_k+1}(\underline{x})$.

Without loss of generality, one can assume that points $u^{(i)} = (u_1^{(i)}, u_2^{(i)})$ have
$$u_1^{(1)} < u_1^{(2)} < \cdots < u_1^{(2n)}.$$

Indeed, one can always sort entries $u_1^{(i)}$ in the non-decreasing order, and if two of them coincide, say $u_1^{(j)} = u_1^{(j+1)}$, then sub-squares $\Lambda_{L_k}(u^{(j)})$ and $\Lambda_{L_k}(u^{(j+1)})$ cannot be disjoint, which is impossible by our hypothesis. Then it is readily seen that:

(i) By virtue of Lemma 2.3, $\forall$ pair $\Lambda_{L_k}(u^{(j)})$, $\Lambda_{L_k}(u^{(j+1)})$, the respective (random) LSOs $H_{\Lambda_{L_k}(u^{(j)})}$ and $H_{\Lambda_{L_k}(u^{(j+1)})}$ are independent, and so are their spectra and Green’s functions.
(ii) Moreover, the pairs of LSOs,

\[
\left( H_{\Lambda L_k(u^{(2j+1)})}, H_{\Lambda L_k(u^{(2j+2)})} \right), \quad j = 0, \ldots, n - 1,
\]

form an independent family. Thus, any collection of events \( A_0, \ldots, A_{n-1} \) related to the corresponding pairs \( \left( H_{\Lambda L_k(u^{(2j+1)})}, H_{\Lambda L_k(u^{(2j+2)})} \right), \quad j = 0, \ldots, n - 1, \) also form an independent family.

Indeed, LSO \( H_{\Lambda L_k(u^{(j)})} \) is measurable with respect to the sigma-algebra \( \mathfrak{V}(I_j \cup J_j) \) generated by random variables \( V(u, \cdot), u \in I_j \cup J_j \), where

\[
I_j = \Lambda_{L_k}^{(1)}(u_{1}^{(j)}) = \Pi_1(\Lambda_{L_k}(u^{(j)})), \quad J_j = \Lambda_{L_k}^{(1)}(u_{2}^{(j)}) = \Pi_2(\Lambda_{L_k}(u^{(j)})).
\]

By virtue of Lemma 2.3, sigma-algebras \( \mathfrak{V}(I_j \cup J_j), \quad j = 1, \ldots, 2n, \) are independent. Then the sigma-algebras

\[
\mathfrak{V}(I_{2j+1} \cup J_{2j+1}) \vee \mathfrak{V}(I_{2j+2} \cup J_{2j+2}), \quad j = 0, \ldots, n - 1,
\]

generated by subsequent pairs are also independent.

Now, for \( j = 0, \ldots, n - 1 \), set

\[
A_j = \{ \exists \, E \in I : \Lambda_{L_k}(u^{(2j+1)}) \text{ and } \Lambda_{L_k}(u^{(2j+2)}) \text{ are (}E, m_k\text{-}S\}) \}. \quad (5.7)
\]

Then, by the hypothesis \( DS(k, I) \),

\[
P \left\{ A_j \right\} \leq L_k^{-p}, \quad 0 \leq j \leq n - 1,
\]

and by virtue of independence of events \( A_0, \ldots, A_{n-1} \), we obtain

\[
P \left\{ \bigcap_{j=0}^{n-1} A_j \right\} = \prod_{j=0}^{n-1} P \left\{ A_j \right\} \leq (L_k^{-q})^n. \quad (5.9)
\]

To complete the proof, it suffices to notice that the total number of different families of \( 2n \) sub-squares with required properties is bounded by \( 2 \cdot L_k \cdot L_{k+1} \), since their centres must belong to a strip \( \{ (x_1, x_2) \in \mathbb{Z}^2 : x_1 \leq x_2 + 2L_k \} \cap \Lambda_{L_{k+1}} \) of width \( 2L_k \) adjacent to the diagonal \( \partial \mathbb{Z}^2 \geq \). \( \square \)

**Proof of Lemma 4.8** The strategy of this proof is very close in spirit to that of Lemma 4.1 in [9]. This similarity is due to a simple geometrical fact: samples of potential corresponding to two \( L_{k+1}\text{-}D \) diagonal sub-squares
\( \Lambda', \Lambda'' \) of size \( L_{k+1} \) are independent, for both their horizontal projections are disjoint and their vertical projections are disjoint. This makes the situation quite similar to that of lemma 4.1 in [9]. The difference, though, is that inside each of the sub-squares, smaller scale sub-squares \( \Lambda_{L_k}(u) \) are not pairwise independent, so we need to use a more involved proof based on our conditional Wegner-type estimates.

Let \( k \in \mathbb{N} \) and \( \Lambda_{L_k}(u), \Lambda_{L_k}(v) \) be two diagonal \( L_k \)-D sub-squares. Consider the following event:

\[
B_k(u,v) = \left\{ \exists E \in I : \text{both } \Lambda_{L_k}(u) \text{ and } \Lambda_{L_k}(v) \text{ are } (E,m_k)\text{-S} \right\}. \tag{5.10}
\]

Assuming that the estimate

\[
P \{ B_k(u,v) \} \leq L^{-2p}_k \tag{5.11}
\]

holds for all diagonal \( L_k \)-D sub-squares \( \Lambda_{L_k}(u), \Lambda_{L_k}(v) \), we have to obtain the similar estimate at scale \( L_{k+1} \). Namely, fix two diagonal and \( L_{k+1} \)-D sub-squares, \( \Lambda_{L_{k+1}}(u) \) and \( \Lambda_{L_{k+1}}(v) \). Then we have to prove that

\[
P \{ B_k(u,v) \} \leq L^{-2p}_{k+1}. \tag{5.12}
\]

We will do so by covering the event \( B_k(u,v) \) by a union of several events the probability of which will be estimated separately. We shall use shortened notations \( B_k = B_k(u,v), \Lambda' = \Lambda_{k+1}(u), \Lambda'' = \Lambda_{k+1}(v) \).

It is convenient to introduce three events:

\[
C = \{ \text{either } \Lambda' \text{ or } \Lambda'' \text{ contains at least } 2n, (E,m_k)\text{-S,}
\]
\[
\text{pair-wise } L_k \text{-D sub-squares } \Lambda_{L_k}(u_i), i = 1, \ldots, 2n \}, \tag{5.13}
\]

\[
D = \{ \text{either } \Lambda' \text{ or } \Lambda'' \text{ contains at least two disjoint,}
\]
\[
\text{off-diagonal, } (E,m_k)\text{-S sub-squares } \Lambda_{L_k}(\xi'), \Lambda_{L_k}(\xi'') \},
\]

\[
E = \{ \text{both } \Lambda' \text{ and } \Lambda'' \text{ are } E\text{-R} \}.
\]

Then \( P \{ B_k \} \) is bounded by

\[
P \{ B_k \cap C \} + P \{ B_k \cap D \} + P \{ B_k \cap E \} + P \{ B_k \cap \bar{C} \cap \bar{D} \cap \bar{E} \} \leq P \{ C \} + P \{ D \} + P \{ E \} + P \{ B_k \cap \bar{C} \cap \bar{D} \cap \bar{E} \}. \tag{5.14}
\]

So, it suffices to estimate probabilities \( P \{ C \}, P \{ D \}, P \{ E \} \) and \( P \{ B_k \cap \bar{C} \cap \bar{D} \cap \bar{E} \} \).

First of all, note that

\[
B_k \cap \bar{C} \cap \bar{D} \cap \bar{E} = \emptyset.
\]
Indeed, if the potential sample belongs to $\bar{C} \cap \bar{D}$, then either of the sub-squares $\Lambda', \Lambda''$ contain less than $2n$ $(E, m_k)$-S sub-squares which are diagonal and $L_k$-D, and at most one which is off-diagonal. So, the total number of $(E, m_k)$-S sub-squares of size $L_k$ inside each of the sub-squares $\Lambda', \Lambda''$ is bounded by $2n - 1 + 1 = 2n$. In addition, property $\bar{E}$ implies that either $\Lambda'$ or $\Lambda''$ must be $E$-NR. 

Therefore, 

$$P\{B_k\} \leq P\{C\} + P\{D\} + P\{E\}. \quad (5.15)$$ 

The probability $P\{C\}$ can be estimated with the help of Lemma 4.7. Indeed, set 

$$C' = \{ \Lambda' \text{ contains at least } 2n \text{ pair-wise } L_k \text{-D}, \quad (E, m_k) \text{-S sub-squares } \Lambda_k(u_i), \ i = 1, \ldots, 2n \},$$ 

$$C'' = \{ \Lambda'' \text{ contains at least } 2n \text{ pair-wise } L_k \text{-D}, \quad (E, m_k) \text{-S sub-squares } \Lambda_k(u_i), \ i = 1, \ldots, 2n \}. \quad (5.16)$$ 

By virtue of Lemma 4.7, 

$$P\{C'\} \leq L_{k+1}^{-n(p-1-\alpha)/\alpha}, \quad P\{C''\} \leq L_{k+1}^{-n(p-1-\alpha)/\alpha}. \quad (5.17)$$ 

With our choice (4.1) and with $n \geq 6$, we get that 

$$\frac{n(p-1-\alpha)}{\alpha} > \frac{6(6-1-3/2)}{3/2} = 14 = 2p + 2 > 2p.$$ 

Since $C \subset C' \cup C''$, we obtain that 

$$P\{C\} < 2L_{k+1}^{-2p-2}. \quad (5.18)$$ 

Next, consider the events 

$$D' = \{ \Lambda' \text{ contains at least two disjoint, off-diagonal, } \quad (E, m_k) \text{-S sub-squares } \Lambda_{L_k}(x') \text{ and } \Lambda_{L_k}(x'') \},$$ 

$$D'' = \{ \Lambda'' \text{ contains at least two disjoint, off-diagonal, } \quad (E, m_k) \text{-S sub-squares } \Lambda_{L_k}(x') \text{ and } \Lambda_{L_k}(x'') \}. \quad (5.19)$$
Notice that $D \subset D' \cup D''$, so that $\mathbb{P} \{ D \} \leq \mathbb{P} \{ D' \} + \mathbb{P} \{ D'' \}$. Probabilities $\mathbb{P} \{ D' \}$ and $\mathbb{P} \{ D'' \}$ are estimated in a similar way, so consider, say, the event $D'$. Obviously, $D'$ is a union of events $D'((x, y))$ of the form

$$D'((x, y)) = \{ N \text{ contains two disjoint, off-diagonal,}$$

$$(E, m_k)-S \text{ sub-squares,} \Lambda_{L_k}(x) \text{ and } \Lambda_{L_k}(y) \};$$

and the number of such pairs $(x, y)$ is bounded by $L_{k+1}^2$. Hence,

$$\mathbb{P} \{ D' \} \leq L_{k+1}^2 \max_{(x, y)} \mathbb{P} \{ D'((x, y)) \}. \quad (5.21)$$

Now fix a pair of disjoint, off-diagonal sub-squares $\Lambda_{L_k}(x), \Lambda_{L_k}(y)$ and consider either of them, e.g., $\Lambda_{L_k}(x)$. By virtue of Lemma 4.3, if both coordinate projections $J_1 = \Pi_1 \Lambda_{L_k}(x)$ and $J_2 = \Pi_2 \Lambda_{L_k}(x)$ are $(E, m_k)$-NT, then either $\Lambda_{L_k}(x)$ is $E$-R or $\Lambda_{L_k}(x)$ is $(E, m_k)$-NS; the latter is impossible by our hypothesis. Set

$$\tilde{J} = \Pi_1 \Lambda'_{L_{k+1}} \cup \Pi_1 \Lambda''_{L_{k+1}} \cup \Pi_2 \Lambda'_{L_{k+1}} \cup \Pi_2 \Lambda''_{L_{k+1}}$$

and consider the event

$$T = \{ \exists \text{ an } (E, m_0 + 1)-T \text{ interval } J_{L_k} \subset \tilde{J} \}. \quad (5.22)$$

Then

$$\mathbb{P} \{ D'((x, y)) \} = \mathbb{P} \{ D'((x, y)) \cap T \} + \mathbb{P} \{ D'((x, y)) \cap \tilde{T} \} \leq \mathbb{P} \{ T \} + \mathbb{P} \{ D'((x, y)) \cap \tilde{T} \}.$$  \quad (5.23)

Recall that, in the same way as in Lemma 4.5, the value $m_0$ is chosen so as to guarantee that properties $(E, m_0 + 1)$-NT and $E$-NR imply property $(E, m_k)$-NS for any $k \geq 0$.

By Lemma 3.2 and Corollary 3.1, $\mathbb{P} \{ T \} \leq L_k^{-q}$. On the other hand, if the potential sample belongs to $D'((x, y)) \cap \tilde{T}$, then both $\Lambda_{L_k}(x)$ and $\Lambda_{L_k}(y)$ must be $E$-R. Therefore,

$$\mathbb{P} \{ D'((x, y)) \cap \tilde{T} \} \leq \mathbb{P} \{ \text{ both } \Lambda_{L_k}(x) \text{ and } \Lambda_{L_k}(y) \text{ are } E$-R \};$$

Recall that both sub-squares are off-diagonal. Then, by Lemma 4.4, the above probability is not greater than $L_k^{-q}$. Finally,

$$\mathbb{P} \{ D'((x, y)) \} \leq L_k^{-q} + L_k^{-q} \quad (5.24)$$
and
\[ \mathbb{P} \{ D' \} \leq L_{k+1}^2 (L_k^{-q} + L_k^{-q}) \] (5.25)
yielding
\[ \mathbb{P} \{ D \} \leq 2L_{k+1}^2 (L_k^{-q} + L_k^{-q}). \] (5.26)

Finally, probability \( \mathbb{P} \{ E \} \) is estimated again with the help of Lemma 4.4. In fact, the sub-squares \( \Lambda' \) and \( \Lambda'' \), being diagonal and \( L_k \)-D, have both their horizontal projections disjoint and their vertical projections disjoint. But just one of these properties would suffice for Lemma 4.4 to be applied:
\[ \mathbb{P} \{ E \} = \mathbb{P} \{ \text{both } \Lambda' \text{ and } \Lambda'' \text{ are } E-R \} \leq L_{k+1}^{-q}. \] (5.27)

Combining bounds (5.15)–(5.27), we see that
\[ \mathbb{P} \{ B_k \} \leq \mathbb{P} \{ C \} + \mathbb{P} \{ D \} + \mathbb{P} \{ E \} \]
\[ \leq 2L_{k+1}^{-2p} + 2L_{k+1}^2 (L_k^{-q} + L_k^{-q}) + L_{k+1}^{-q} \] (5.28)
This proves \( \text{DS}(k+1,I) \). \( \square \)

**Proof of Lemma 4.9:** **Step 1.** Consider sub-squares \( \Lambda' = \Lambda_{L_{k+1}}(x'), \Lambda'' = \Lambda_{L_{k+1}}(x'') \) and set
\[ \tilde{J} = \Pi_1 \Lambda' \cup \Pi_2 \Lambda' \cup \Pi_1 \Lambda'' \cup \Pi_2 \Lambda'' \]
Let \( C \) stand for the following event:
\[ C = \left\{ \exists \text{ an } (m_k)\text{-T segment } I_{L_k} \subset \tilde{J} \right\}. \] (5.29)
As before, the tunneling property (i.e. delocalisation, or insufficient localisation) for segments is related to single-particle spectra. Thus, we can use results of the single-particle localisation theory (cf. [9]). By Lemma 3.2, \( \mathbb{P} \{ C \} \leq L_k^{-q} \). Next, for
\[ B_k = \{ \text{ both } \Lambda' \text{ and } \Lambda'' \text{ are } (E,m_{k+1})\text{-S} \} \] (5.30)
we obtain that
\[ \mathbb{P} \{ B_k \} \leq \mathbb{P} \{ C \} + \mathbb{P} \{ B_k \cap C \} \leq L_k^{-q} + \mathbb{P} \{ B_k \cap C \}. \] (5.31)
It now remains to bound probability $\mathbb{P}\{B_k \cap \bar{C}\}$.

**Step 2.** By Lemma 4.3, if one-dimensional projections of the off-diagonal sub-square (actually, a square) $\Lambda''$ are non-tunneling, then either it is $E$-$R$, or it is $(E, m_{k+1})$-$NS$. The latter is impossible for potential samples in $B_k$ (for both $\Lambda'$ and $\Lambda''$ must be resonant), so $\Lambda''$ has to be $E$-$R$. Introduce the following event:

$$D = \{ \text{ both } \Lambda' \text{ and } \Lambda'' \text{ are } E$-$R \}. \quad (5.32)$$

Since $\Lambda' \cap \Lambda'' = \emptyset$ and $\Lambda''$ is off-diagonal, we can apply Lemma 2.4 and Lemma 4.4 and write

$$\mathbb{P}\{D\} \leq L_k^{-q}, \quad (5.33)$$

so that

$$\mathbb{P}\{B_k \cap \bar{C}\} \leq \mathbb{P}\{D\} + \mathbb{P}\{B_k \cap \bar{C} \cap \bar{D}\} \leq L_k^{-q} + \mathbb{P}\{B_k \cap \bar{C} \cap \bar{D}\}. \quad (5.34)$$

**Step 3.** Assuming now the non-resonance of $\Lambda''$ (due to $\bar{D}$ and the resonance of $\Lambda'$), we see that, due to Lemma 4.6, in order to be resonant, square $\Lambda''$ must contain at least $K = 2n$ $(E, m_k)$-$S$ sub-squares $\Lambda_{L_k}(u_i)$ of size $L_k$. There are two types of them: diagonal and off-diagonal. By Lemma 4.7, the probability to have $\geq 2n$ diagonal $L_k$-$D$, $E$-$R$ sub-squares $\Lambda_{L_k}(u_i)$, $i = 1, \ldots, 2n$, is not greater than $L_k^{n(1+\alpha)}L_k^{-\alpha p/2}$. On the other hand, the probability to have both an off-diagonal (sub-)square $\Lambda'$ and an off-diagonal (sub-)square $\Lambda_{L_k}(u)$ $E$-$R$ is bounded by $L_k^{-q}$. Combining these two bounds, we conclude that

$$\mathbb{P}\{B_k \cap \bar{C} \cap \bar{D}\} \leq L_k^{-q} + L_k^{-q} = 2L_k^{-q}. \quad (5.35)$$

**Step 4.** With estimates of Steps 1–3 (see Equations (5.31)–(5.35)), we have that

$$\mathbb{P}\{B_k\} \leq L_k^{-q} + L_k^{-q} + 2L_k^{-q} = 4L_k^{-q} < L_{k+1}^{-2p}, \quad (5.36)$$

with our choice of exponents $p = 6$, $q = 24$, $\alpha = 3/2$. $\square$

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