q-Painlevé equations on cluster Poisson varieties via toric geometry

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Abstract

We provide a relation between the geometric framework for q-Painlevé equations and cluster Poisson varieties by using toric models of rational surfaces associated with q-Painlevé equations. We introduce the notion of seeds of q-Painlevé type by the negative semi-definiteness of symmetric bilinear forms associated with seeds, and classify the mutation equivalence classes of these seeds. This classification coincides with the classification of q-Painlevé equations given by Sakai. We realize q-Painlevé systems as automorphisms on cluster Poisson varieties associated with seeds of q-Painlevé type.

1 Introduction

Sakai introduced a geometric framework for discrete Painlevé equations [Sak01]. In his framework, discrete Painlevé systems are realized as birational automorphisms on families of rational surfaces. On the level of Picard groups of rational surfaces, these automorphisms are called Cremona isometries, and related to the action of affine Weyl groups on root lattices. Moreover, he classified discrete Painlevé equations based on the classification of rational surfaces. According to Sakai’s classification, there are three types of discrete Painlevé equations: elliptic type, multiplicative type, and additive type. A discrete Painlevé equation of multiplicative type is often called a q-Painlevé equation, which is the main topic of this paper.

In the recent paper [BGM18], Bershtein, Gavrylenko, and Marshakov showed that all the q-Painlevé systems in Sakai’s classification can be realized on cluster varieties. Cluster varieties are introduced by Fock and Goncharov [FG09] as geometric counterparts of cluster algebras, which are introduced by Fomin and Zelevinsky [FZ02]. Fock and Goncharov also define cluster modular groups, which are groups of automorphisms on cluster varieties. Cluster modular groups have a nice combinatorial description since an element of these groups can be expressed as a composition of mutations, which are operations among quivers described by a combinatorial way. Bershtein, Gavrylenko, and Marshakov proved that groups of Cremona isometries for q-Painlevé systems can be embedded into cluster modular groups associated with appropriate quivers. Note that the relation between q-Painlevé equations and the cluster theory was already noticed by Okubo [Oku13] in several cases.

The purpose of this paper is to provide a more precise relation between Sakai’s theory and the cluster theory. The basic idea is that rational surfaces together with their anti-canonical divisors associated with q-Painlevé equations appearing in Sakai’s theory have toric models in the sense of [GHK15a, GHK15b]. From combinatorial data of these toric models, we can define cluster varieties by using a construction given by Gross, Hacking, and Keel [GHK15a]. It turns out that the resulting cluster varieties are essentially the families of the interiors of rational surfaces on which Cremona isometries act in Sakai’s theory. Moreover, the null space of a skew-symmetric bilinear form associated with such a cluster variety is identified with a root lattice of affine type as a sublattice of the Picard group of a rational surface. The relation between the cluster theory and Sakai’s theory for q-Painlevé equations is roughly summarized in Table 1.

Our main results are the following.

Theorem 1.1 (See Theorem 3.31 and 3.32). The seeds of q-Painlevé type (Definition 3.23) modulo mutation equivalence are classified into the ten seeds in Figure 1.

Theorem 1.2 (See Theorem 3.35). Let s be a seed of q-Painlevé type, and i be a free cover of s (Definition 3.3). Then the action of the cluster modular group \( \Gamma_i \) on the cluster Poisson variety \( X_i \) gives q-Painlevé systems in the sense of [Sak01].
Realizing $q$-Painlevé systems using the cluster theory has the following advantages:

- Cluster varieties have positivity in the sense that they can be defined over the semiring $\mathbb{N}$. So $q$-Painlevé systems can be defined over $\mathbb{N}$. In particular, we obtain $q$-Painlevé systems over any semifield. It is worth mentioning that $\mathbb{Z}^{\text{Trop}}$-valued points of a cluster variety play an important role in the construction of theta bases of its Langlands dual cluster algebra [GHKK18], where $\mathbb{Z}^{\text{Trop}} = (\mathbb{Z} \sqcup \{-\infty\}, \max, +)$ is the max-plus semifield of integers.

- Cluster Poisson varieties have quantization [FG09]. Using this quantization, we obtain quantum $q$-Painlevé systems. Bershtein, Gavrylenko, and Marshakov present a formal solution of a quantum $q$-Painlevé equation of type $E_{1}^{(1)'}$ using $q$-deformed conformal blocks or 5-dimensional Nekrasov functions [BGM18].

Remark 1.3. There is a relationship between cluster varieties and dimer models, called cluster integrable systems [GK13]. In [BGM18], they find quivers associated with $q$-Painlevé systems based on cluster integrable systems, except for types $E_{7}^{(1)}$ and $E_{8}^{(1)}$. This method seems to be different from the method in this paper based on toric geometry. One advantage of our method is that we can deal with all the ten types in a unified way. It would be interesting to study the relation between these two methods.

This paper is organized as follows. In Section 2, we review basic definitions in the cluster theory. We define cluster Poisson varieties and the action of cluster modular groups on these varieties. In Section 3.1, we study seeds in lattices of rank 2. We define a symmetric bilinear form associated with a free cover of a full dimensional primitive seed in lattices of rank 2. We show that this symmetric bilinear form is invariant under seed cluster transformations. In Section 3.2, we introduce seeds of $q$-Painlevé type. We say that a seed is of $q$-Painlevé type if the symmetric bilinear form defined in Section 3.1 is negative semi-definite but not negative definite. By the results in Section 3.1, we see that this notion is invariant under cluster transformations. For a seed of $q$-Painlevé type, we define its symmetry type $R^{\perp}$. In Section 3.3, we classify the mutation equivalence classes of seeds of $q$-Painlevé type. This classification is based on a classification for Fano polygons given by Kasprzyk, Nill, and Prince [KNP17]. In fact, we associate Fano polygons with seeds of $q$-Painlevé type by using null root for these seeds. It turns out that there are exactly ten mutation equivalence classes, and this classification agree with the classification of the $q$-Painlevé equations given by Sakai [Sak01]. In Section 3.4, we show that (the opposite of) the groups of Cremona isometries are embedded in cluster modular groups. This induces the (right) action of the groups of Cremona isometries on cluster Poisson varieties, and we get $q$-Painlevé systems on cluster Poisson varieties. As an example, we realize the sixth $q$-Painlevé system as an alternating actions of two involutive cluster transformations on a cluster Poisson variety. In Appendix A, we provide basic data for seeds associated with $q$-Painlevé equations. Some proofs in the body of this paper depend on the computations in this appendix.
2 Preliminaries on cluster varieties

We review the definition of cluster varieties and the action of cluster modular groups on cluster varieties. We basically follow the formulation in [FG09] and [GHK15a], except for the following minor modifications:

- We only define $\mathcal{X}$-varieties, and not define $\mathcal{A}$-varieties.
- We only deal with cluster varieties of skew-symmetric type, not skew-symmetrizable type.
- Our ambient lattices are torsion-free abelian groups, not just finitely generated free abelian groups (see Remark 2.6).
- We allow seed isomorphisms to reverse a sign of a skew-symmetric form as in [ASS12].
- Mutations are equipped with signs as in [Kel11].

Seeds and mutations In this paper, a lattice means a torsion-free abelian group, written additively, equipped with a skew-symmetric bilinear form. For a lattice $N$, we denote its skew-symmetric bilinear form by $\{\cdot, \cdot\} : N \times N \to \mathbb{Z}$.

Definition 2.1. Given a lattice $N$, a finite multiset on $N$ is called a seed in $N$.

A seed is sometimes called an unlabeled seed. On the other hand, a finite tuple of elements of $N$ is called a labeled seed in $N$. For a labeled seed $(e_i)_{i \in I}$ in $N$, we have a seed in $N$ obtained by forgetting label, which we denote by $[(e_i)_{i \in I}]$. We have a bijection

$$\{\text{seed in } N\} \cong \{\text{labeled seed in } N\}/\langle \text{change of index sets} \rangle,$$

where the change of index sets we mean the equivalence relation such that $(e_i)_{i \in I}$ and $(f_i)_{i \in I'}$ are equivalent if there is a bijection $\sigma : I \to I'$ such that $e_i = f_{\sigma(i)}$ for any $i \in I$. For a seed $i$, a labeled seed $(e_i)_{i \in I}$ such that $i = [(e_i)_{i \in I}]$ is called a labeling of $i$.

Given a lattice $N$, an element $v \in N$, and a sign $\varepsilon = +$ or $-$, we define a piecewise linear map $\mu^\varepsilon : N \to N$ by

$$\mu^\varepsilon(n) := n + [\varepsilon\{n, v\}]_+v,$$

where $[x]_+ := \max(x, 0)$.

Given a multiset $A$ on a set $X$ and an element $a \in X$, we denote by $A \cup \{a\}$ the multiset on $X$ obtained from $A$ by adding 1 to the multiplicity of $a$ in $A$, and denote by $A \setminus \{a\}$ the multiset on $X$ obtained from $A$ by subtracting 1 from the multiplicity of $a$ in $A$. Given a multiset $A$ on $X$ and a function $f : X \to Y$, we denote by $f(A)$ the multiset on $Y$ obtained by applying $f$ to each element of $A$. 

Table 1: The dictionary between the cluster theory and $q$-Painlevé equations. This dictionary is based on toric geometry.

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Lemma 2.4. Suppose that seeds $i$ and $i'$ are related by the following formula:

$$i' = (\mu^\varepsilon_i(i) \setminus \{v\}) \cup \{-v\}.$$  

We denote by $\mu^\varepsilon_i : i \rightarrow i'$ the seed mutation $(i, i', v, \varepsilon)$.

Definition 2.3. A seed isomorphism is a tuple $(i, i', \sigma, \varepsilon)$ such that

- $i$ and $i'$ are seeds in lattices $N$ and $N'$, respectively;
- $\sigma : N \rightarrow N'$ is an isomorphism of abelian groups;
- $i' = \sigma(i)$;
- $\sigma$ preserves the skew-symmetric forms after multiplying $\varepsilon$, that is, the following diagram commutes:

$$
\begin{array}{ccc}
N \times N & \xrightarrow{\sigma \times \sigma} & N' \times N' \\
\{\cdot, \cdot\} \downarrow & & \downarrow \{\cdot, \cdot\}' \\
\mathbb{Z} & \xrightarrow{\varepsilon} & \mathbb{Z}
\end{array}
$$

We denote by $\varepsilon\sigma : i \rightarrow i'$ the seed isomorphism $(i, i', \sigma, \varepsilon)$. The seed isomorphism $+\sigma : i \rightarrow i'$ will be denoted by $\sigma : i \rightarrow i'$.

We denote by Seed the free groupoid whose objects are the seeds, and morphisms are generated by the seed mutations and the seed isomorphisms. A morphism in Seed is called a seed cluster transformation. If $c$ is a seed cluster transformation from $i$ to $i'$, we denote it by $c : i \rightarrow i'$. We say that seeds $i$ and $i'$ are mutation equivalent if there is a seed cluster transformation from $i$ to $i'$.

The following lemma is obvious.

Lemma 2.4. Suppose that $i$ is a seed in $N$.

1. For any $v \in i$ and sign $\varepsilon$, there is a unique seed $i'$ in $N$ such that $\mu^\varepsilon_i : i \rightarrow i'$ is a seed mutation.
2. For any isomorphism $\sigma : N \rightarrow N'$ and sign $\varepsilon$, there is a unique seed $i'$ in $N'$ such that $\varepsilon\sigma : i \rightarrow i'$ is a seed isomorphism.
3. For any $v \in i$ and sign $\varepsilon$, there is a unique seed $i'$ in $N$ such that $\mu^-v : i \rightarrow i'$ is a seed mutation.
4. For any isomorphism $\sigma : N' \rightarrow N$ and sign $\varepsilon$, there is a unique seed $i'$ in $N'$ such that $\varepsilon\sigma : i' \rightarrow i$ is a seed isomorphism.

Remark 2.5. Similarly to Definition 2.2, we say that a tuple $((e_i)_{i \in I}, (e'_i)_{i \in I}, k, \varepsilon)$ is a labeled seed mutation if $(e_i)_{i \in I}$ and $(e'_i)_{i \in I}$ are labeled seeds in the same lattice with the same index set, $k \in I$, $\varepsilon$ is a sign, and the following relation holds:

$$e'_i = \begin{cases} 
e e_i + [\varepsilon e_k] e_k & \text{if } i \ne k, \\ -e_k & \text{if } i = k. \end{cases}$$

In this case, it is easy to see that $((e_i)_{i \in I}, (e'_i)_{i \in I}, e_k, \varepsilon)$ is a seed mutation. Moreover, Lemma 2.4 also holds for labeled seed mutations. We will use labeled seed mutations to define seed mutations specifically in Appendix A.

Remark 2.6. In usual literature of cluster algebras, a lattice $N$ is assumed to be a finitely generated free abelian group, and seeds in $N$ are assumed to be bases in $N$. We do not impose these assumptions in order to treat seeds for $q$-Painlevé systems in a unified way. See Section 3. In particular, we remark that a seed for the $q$-Painlevé system of type $E_{1}^{(1)'}$ or $E_{0}^{(1)}$ does not span the underlying lattice $N$, although it spans $N \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$. See Figure 1.
Cluster modular groupoid  For any torsion-free abelian group $N$, we denote by $D(N)$ the affine scheme over $\mathbb{C}$ associated with the group algebra of $N$:

$$D(N) := \text{Spec} \mathbb{C}[N].$$

Note that $\mathbb{C}[N]$ is an integral domain since $N$ is a torsion-free abelian group. We denote by $\text{Scheme}_{\text{rat}}/\mathbb{C}$ the category whose objects are the integral schemes that are separated over $\mathbb{C}$, and morphisms are the dominant rational maps over $\mathbb{C}$.

**Definition 2.7.** We define a functor

$$\mathcal{X} : \text{Seed} \to \text{Scheme}_{\text{rat}}/\mathbb{C}$$

as follows.

- We define $\mathcal{X}(i) := D(N)$ for a seed $i$ in a lattice $N$.
- For a seed mutation $\mu^+_i : i \to i'$, we define a birational map

$$\mathcal{X}(\mu^+_i) : D(N) \dashrightarrow D(N),$$

via the following isomorphism of rings:

$$\mathcal{X}(\mu^+_i)^* : \mathbb{C}[N]_{1+z^v} \to \mathbb{C}[N]_{1+z^v}, \quad \mathcal{X}(\mu^+_i)^* z^n := z^n(1 + z^v)^{-\{n,v\}},$$

where $\mathbb{C}[N]_f$ is the localization of $\mathbb{C}[N]$ by $f \in \mathbb{C}[N]$. Explicitly, we have an isomorphism between dense open subsets $\mathcal{X}(\mu^+_i) : U \to U'$, where $U = D(N) \setminus V(1 + z^v) \cong \text{Spec} \mathbb{C}[N]_{1+z^v}$.
- For a seed isomorphism $\varepsilon\sigma : i \to i'$, we define an isomorphism

$$\mathcal{X}(\varepsilon\sigma) : D(N) \to D(N')$$

via the following isomorphism of rings:

$$\mathcal{X}(\varepsilon\sigma)^* : \mathbb{C}[N'] \to \mathbb{C}[N], \quad \mathcal{X}(\varepsilon\sigma)^* z^n := z^{\varepsilon\sigma^{-1}(n)}.$$

For a seed cluster transformation $c : i \to i'$, we define its source and target by $\text{src}(c) := i$ and $\text{tar}(c) := i'$. We say that seed cluster transformations $c_1$ and $c_2$ are equivalent if $\text{src}(c_1) = \text{src}(c_2)$, $\text{tar}(c_1) = \text{tar}(c_2)$, and $\mathcal{X}(c_1) = \mathcal{X}(c_2)$.

**Definition 2.8.** The groupoid $\text{CMG} := \text{Seed}/(\text{equivalence})$ is called the cluster modular groupoid. In other words, $\text{CMG}$ is the groupoid whose objects are the seeds, and morphisms are the seed cluster transformations modulo equivalence. The fundamental group $\Gamma_i := \text{Aut}_{\text{CMG}}(i)$ of the cluster modular groupoid based at a seed $i$ is called the cluster modular group at $i$.

A seed cluster transformation $c : i \to i'$ gives an isomorphism $\Gamma_i \to \Gamma_{i'}$ by $c' \mapsto c \circ c' \circ c^{-1}$.

**Example 2.9** ([FZ02, Section 6],[FG09, Section 2.5]). Let $N = \mathbb{Z}^2$. Let $i = \{e_i \mid i = 1, 2\}$ be the standard basis of $N$. We define a skew-symmetric form on $N$ by $\{e_1, e_2\} = -\{e_2, e_1\} = 1$. Then the cluster modular group at $i$ is given by

$$\Gamma_i = \langle \sigma \circ \mu^{+}_{e_1} \rangle \times \langle -\iota \rangle \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

where $\sigma : (e_1, e_2) \mapsto (-e_2, e_1)$ and $\iota : (e_1, e_2) \mapsto (e_2, e_1)$.

We now see some relations in the cluster modular groupoid. Given a seed $i$ in $N$, an element $v \in N$, and a sign $\varepsilon$, we define a seed isomorphism $t^\varepsilon_v : i \to i'$ by the isomorphism $N \to N$ given by $t^\varepsilon_v(n) = n + \varepsilon\{n, v\}v$. 

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Proposition 2.10. Seed mutations and seed isomorphisms satisfy the following relations in the cluster modular groupoid:

\[
\mu_{\nu}^+ \circ \mu_{\nu}^- = \mu_{\nu}^- \circ \mu_{\nu}^+ = \text{id},
\]

\[
\mu_{\nu}^\varepsilon \circ \mu_{\nu}^\varepsilon = t_{\nu}^\varepsilon,\]

\[
\varepsilon \sigma \circ \varepsilon' \sigma' = \varepsilon \varepsilon' \sigma \sigma',
\]

\[
\varepsilon \sigma \circ \mu_{\nu}^\varepsilon = \mu_{\sigma(v)}^\varepsilon \circ \varepsilon \sigma.
\]

Proof. These relations are proved by direct calculations. We only prove the fourth relation:

\[
\begin{align*}
\mathcal{X}(\varepsilon \sigma \circ \mu_{\nu}^\varepsilon)^* z^{\sigma(n)} &= \mathcal{X}(\mu_{\nu}^\varepsilon)^* z^{\varepsilon n} \\
&= (z^n(1 + z^{\varepsilon \sigma(v)})^{-(n,v)})^\varepsilon \\
&= \mathcal{X}(\varepsilon \sigma)^*(z^{\sigma(n)}(1 + z^{\varepsilon \sigma(v)}))^{-\{\sigma(n),\sigma(v)\}} \\
&= \mathcal{X}(\mu_{\sigma(v)}^\varepsilon \circ \varepsilon \sigma)^* z^{\sigma(n)}.
\end{align*}
\]

Corollary 2.11. In the cluster modular groupoid, any seed cluster transformation \( c \) can be expressed as

\[
c = \varepsilon \sigma \circ \mu_{\nu_1}^+ \circ \cdots \circ \mu_{\nu_m}^+
\]

for some seed mutations \( \mu_{\nu_1}, \ldots, \mu_{\nu_m} \) and seed isomorphism \( \varepsilon \sigma \).

Cluster varieties and the action of cluster modular groups We will now define cluster varieties. The main tool needed for the definition is the gluing construction of schemes using birational map. We first generalize the gluing construction given by [GHK15a, Proposition 2.4] so that it can be applied to our setting. Give a scheme \( S \), we denote by \( \text{Scheme}_{\text{rat}}/S \) the category whose objects are the integral schemes that are separated over \( S \) and morphisms are the dominant rational maps over \( S \). Given a pair \((i,j)\) of objects in a groupoid, we write \( i \sim j \) if there is a morphism from \( i \) to \( j \). We say that a groupoid is thin (resp. connected) if there is at most (resp. at least) one morphism for each pair of objects.

Lemma 2.12. Suppose that \( \mathcal{C} \) is a thin groupoid, and \( \alpha : \mathcal{C} \to \text{Scheme}_{\text{rat}}/S \) is a functor. For any pair of objects \((i,j)\) in \( \mathcal{C} \) such that \( i \sim j \), we denote by \( f_{ij} : \alpha(i) \longrightarrow \alpha(j) \) the dominant rational map obtained by applying \( \alpha \) to the unique morphism from \( i \) to \( j \). For any pair of objects \((i,j)\) in \( \mathcal{C} \) such that \( i \sim j \), we define \( U_{ij} \) to be the union of open sets \( U \subseteq \alpha(i) \) such that there is an open immersion \( \varphi : U \to \alpha(j) \) over \( S \) such that \((\varphi,U)\) is a representative of \( f_{ij} \), and we define \( \varphi_{ij} : U_{ij} \to \alpha(j) \) to be the morphism over \( S \) obtained by gluing all open immersions \( \varphi : U \to \alpha(j) \) over \( S \) such that \((\varphi,U)\) is a representative of \( f_{ij} \). We also define \( U_{ij} \) and \( \varphi_{ij} \) to be empty if \( i \not\sim j \). Then:

1. \( \varphi_{ij} \) is an open immersion whose image is \( U_{ji} \) for any \( i,j \).
2. \( U_{ii} = \alpha(i) \) and \( \varphi_{ii} = \text{id}_{\alpha(i)} \) for any \( i \).
3. \( \varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk} \) for any \( i,j,k \).
4. \( \varphi_{jk} \circ \varphi_{ij} = \varphi_{ik} \) on \( U_{ij} \cap U_{ik} \) for any \( i,j,k \).

Consequently, we have a scheme \( \mathcal{X} \) over \( S \) obtained by gluing schemes \( \alpha(i) \) by open immersions \( \varphi_{ij} \). Moreover, if \( \mathcal{C} \) is connected and non-empty, then \( \mathcal{X} \) is integral.

Proof. By [Gro60, 8.2.8], \( \varphi_{ij} \) is an open immersion. It is easy to see that \( \varphi_{ij}(U_{ij}) = U_{ji} \). Thus (1) is proved. (2) is obvious. To prove (3) and (4), it suffices to show that \( \varphi_{ij}(U_{ij} \cap U_{ik}) \subseteq U_{ji} \cap U_{jk} \). Let \( V = \varphi_{ij}(U_{ij} \cap U_{ik}) \). Suppose that \( \varphi_{ij}(x) \in V \). Then \( \varphi_{ij}(x) \in U_{ji} \) is obvious. Since \( (\varphi_{ik} \mid \varphi_{ji}) \circ \varphi_{ji} \mid V \) is a representative of \( f_{ji} \), and \( \varphi_{ik} \mid \varphi_{ji} \mid V \) is an open immersion over \( S \), we have \( \varphi_{ij}(x) \in U_{ji} \). The remaining statements follow from [Har77, Exercises II 2.12].
We now use this construction to define cluster varieties. Let \( i \) be a seed. Let \( i \downarrow \text{Seed} \) be the groupoid whose objects are the morphisms in \( \text{Seed} \) with source \( i \), and morphisms from \( c_1 : i \to i_1 \) to \( c_2 : i \to i_2 \) are the morphisms \( c : i_1 \to i_2 \) in \( \text{Seed} \) such that \( c_2 = c \circ c_1 \). The groupoid \( i \downarrow \text{Seed} \) is thin and connected since \( \text{Hom}_{i \downarrow \text{Seed}}(c_1, c_2) \) is a singleton whose element is \( c_2 \circ c_1^{-1} : i_1 \to i_2 \). We have a functor \( (i \downarrow \text{Seed}) \to \text{Seed} \) that sends seed cluster transformations to their target seeds, and morphisms to themselves. By composing this functor with \( \mathcal{X} : \text{Seed} \to \text{Scheme}_{\text{rat}}/\mathbb{C} \), we get the functor \( (i \downarrow \text{Seed}) \to \text{Scheme}_{\text{rat}}/\mathbb{C} \). Applying Lemma 2.12 to this functor, we obtain an integral scheme over \( \mathbb{C} \), which we denote by \( \mathcal{X}_i \):

\[
\mathcal{X}_i := \bigcup_{c \in (i \downarrow \text{Seed})} \mathcal{X}(\text{tar}(c)).
\]

The scheme \( \mathcal{X}_i \) is called the \textit{cluster variety} associated with \( i \). This is also called the \textit{cluster} \( \mathcal{X} \)-\textit{variety} or the \textit{cluster Poisson variety} associated with \( i \). The cluster variety \( \mathcal{X}_i \) is not separated over \( \mathbb{C} \) in general.

For any seed cluster transformation \( c : i \to i' \), we have an isomorphism \( (i \downarrow \text{Seed}) \to (i' \downarrow \text{Seed}) \) given by \( c' \mapsto c' \circ c^{-1} \). This induces an isomorphism between the corresponding cluster varieties:

\[
\mathcal{X}_i \to \mathcal{X}_{i'}.
\]

In particular, any element \( c \in \Gamma_i \) acts on \( \mathcal{X}_i \) as an automorphism, and we obtain the action of the cluster modular group on the cluster variety:

\[
\Gamma_i \to \text{Aut}(\mathcal{X}_i).
\]

The open immersion \( D(N) \to \mathcal{X}_i \) corresponding to the object \( \text{id} : i \to i \) in \( i \downarrow \text{Seed} \) is called the \textit{initial affine chart} of \( \mathcal{X}_i \). On this chart, the action of \( c \in \Gamma_i \) coincides with the birational map \( \mathcal{X}(c) : D(N) \to D(N) \). On the other hand, we emphasize that the action on the whole \( \mathcal{X}_i \) is given by isomorphisms, not just birational maps.

Given a lattice \( N \), we denote by \( K \) the orthogonal complement of itself:

\[
K = \{ n \in N \mid \forall n' \in N, \{ n, n' \} = 0 \}. \tag{2.1}
\]

We define a functor \( (\cdot)_0 : \text{Seed} \to \text{Seed} \) as follows. Given a seed \( i \) in \( N \), we define a seed \( i_0 \) in \( K \) to be the image of \( i \) under the zero morphism \( N \to K \). Given a seed mutation \( \mu' : i \to i' \), we define \( \mu'_0 : i_0 \to i'_0 \) to be the seed mutation \( \mu'_0 = \mu' \circ i_0 \to i'_0 \). Given a seed isomorphism \( \varepsilon \sigma : i \to i \), we define \( \varepsilon \sigma_0 : i_0 \to i'_0 \) by \( \varepsilon_0 := \varepsilon \) and \( \sigma_0 := \sigma|_K \).

**Lemma 2.13.** Let \( i \) be a seed in \( N \). Then \( D(K) \cong \mathcal{X}_{i_0} \), where the isomorphism is given by the initial affine chart. Under this identification, we have a map \( \lambda : \mathcal{X}_i \to D(K) \) induced by the inclusion \( K \subseteq N \).

**Proof.** Noting that \( \mathcal{X}((\mu'_0)_0^*)^* \) is the identity map, the assertions follow from the following commutative diagrams:

\[
\begin{array}{ccc}
\mathbb{C}[K] & \to & \mathbb{C}[N]_{1+z^\varepsilon}^* \\
\text{id} \downarrow & & \downarrow \mathcal{X}(\mu'_0)^* \\
\mathbb{C}[K] & \to & \mathbb{C}[N]_{1+z^\varepsilon}^* \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathbb{C}[K'] & \to & \mathbb{C}[N]' \\
\mathcal{X}((\varepsilon_0^*)^*) \downarrow & & \downarrow \mathcal{X}(\varepsilon_0)^* \\
\mathbb{C}[K] & \to & \mathbb{C}[N] \\
\end{array}
\]

In particular, the cluster modular group \( \Gamma_i \) acts on \( D(K) \), and the map \( \lambda : \mathcal{X}_i \to D(K) \) is equivariant with respect to the actions of \( \Gamma_i \) on \( \mathcal{X}_i \) and \( D(K) \).

**Remark 2.14.** Let \( i \) be a seed in a lattice \( N \). Then the cluster modular group \( \Gamma_i \) and the cluster variety \( \mathcal{X}_i \) are large in the sense that they are proper classes. However, we can identify them to small ones in the following way. Let \( \text{Seed}_N \) be the subgroupoid of \( \text{Seed} \) generated by the seed mutations and the seed isomorphisms between seeds in \( N \). Let \( \Gamma_i^{\text{small}} \) (resp. \( \mathcal{X}_i^{\text{small}} \)) be the cluster modular group at \( i \) (resp. the cluster variety associated with \( i \)) defined by using \( \text{Seed}_N \) instead of \( \text{Seed} \). Then Corollary 2.11 implies that the natural maps \( \Gamma_i^{\text{small}} \to \Gamma_i \) and \( \mathcal{X}_i^{\text{small}} \to \mathcal{X}_i \) are isomorphisms. The correspondence \( i \mapsto \Gamma_i^{\text{small}} \) (resp. \( i \mapsto \mathcal{X}_i^{\text{small}} \)) induces a functor \( \text{CMG} \to \text{Group} \) (resp. \( \text{CMG} \to \text{Scheme} \)).
Remark 2.15. In the definition of cluster variety in [GHK15a], their gluing data involve only seed mutations, not seed isomorphisms. However, the resulting cluster variety is the same as that defined in this paper thanks to Corollary 2.11. We use seed isomorphisms because we are interested in the action of the cluster modular group.

3 \( q \)-Painlevé systems on cluster Poisson varieties

In this section, we realize \( q \)-Painlevé systems on cluster Poisson varieties via toric geometry. See [CLS11, Ful93] for the basics on toric geometry.

3.1 Seeds in lattices of rank 2

Definition 3.1. We say that a lattice \( \tilde{N} \) is of rank 2 if \( \tilde{N} \) is a free abelian group of rank 2, and the skew-symmetric bilinear form on \( \tilde{N} \) comes from an isomorphism \( \Lambda^2 \tilde{N} \cong \mathbb{Z} \).

For a lattice \( \tilde{N} \) of rank 2, we can identify \( \tilde{N} \) and its dual \( \text{Hom}_\mathbb{Z}(\tilde{N}, \mathbb{Z}) \) by \( n \mapsto \cdot \wedge n \).

Definition 3.2. Let \( s \) be a seed in a lattice \( \tilde{N} \).

1. We say that \( s \) is primitive if \( w \) is primitive in \( \tilde{N} \) for any \( w \in s \).
2. We say that \( s \) is full dimensional if \( \tilde{N} \otimes_{\mathbb{Z}} \mathbb{Q} \) is generated by \( s \) as a vector space over \( \mathbb{Q} \).

It is easy to see that these two properties are invariant under seed cluster transformations.

Definition 3.3. Let \( s \) be a full dimensional primitive seed in a lattice \( \tilde{N} \). A free cover \( \gamma \) of \( s \) is a tuple of the following data:

- \( N \): a lattice;
- \( i \): a seed in \( N \) such that \( i \) is a basis of \( N \otimes_{\mathbb{Z}} \mathbb{Q} \) as a vector space over \( \mathbb{Q} \);
- \( \psi \): a \( \mathbb{Q} \)-linear map from \( N \otimes_{\mathbb{Z}} \mathbb{Q} \) to \( \tilde{N} \otimes_{\mathbb{Z}} \mathbb{Q} \) such that \( N = \psi^{-1}(\tilde{N}) \), \( \psi(i) = s \), and \( \{n, n'\} = \{\psi(n), \psi(n')\} \) for any \( n, n' \in N \).

Definition 3.4. A blowup data \( \beta \) is a tuple of the following data:

- \( \tilde{N} \): a lattice of rank 2;
- \( s \): a full dimensional primitive seed in \( \tilde{N} \);
- \( \gamma = (N, i, \psi) \): a free cover of \( s \);
- \( (e_i)_{i \in I} \): a labeling of \( i \), where \( I = \{1, 2, \ldots, |i|\} \);
- \( \Sigma \): a smooth complete fan in \( N \otimes_{\mathbb{Z}} \mathbb{R} \) such that \( \mathbb{R}_{\geq 0} w \) is a ray of \( \Sigma \) for any \( w \in s \);
- \( \phi \in D(K)(\mathbb{C}) = \text{Hom}_\mathbb{Z}(K, \mathbb{C}^\times) \), where \( K = \ker \psi \).

Lemma 3.5. Let \( s \) be a full dimensional primitive seed in a lattice \( \tilde{N} \) of rank 2.

1. There is a free cover \( \gamma \) of \( s \).
2. For any free cover \( \gamma \) of \( s \), there is a blowup data \( \beta \) whose free cover of \( s \) is \( \gamma \).
3. There is a blowup data whose full dimensional primitive seed in \( \tilde{N} \) is \( s \).

Proof. We define an index set \( J \) by

\[ J = \{(w, i) \mid w \in s, \ 1 \leq i \leq m_w\}, \]

where \( m_w \) is the multiplicity of \( w \) in \( s \). Let \( V = \mathbb{Q}^J \), and let \( \mathfrak{i} = \{e_{w,i} \in N \mid (w, i) \in J\} \) be the standard basis of \( V \). We define a \( \mathbb{Q} \)-linear map \( \psi : V \to \tilde{N} \otimes_{\mathbb{Z}} \mathbb{Q} \) by \( e_{w,i} \mapsto w \). We define a torsion-free abelian group \( N \) by \( N := \psi^{-1}(\tilde{N}) \), and a skew-symmetric bilinear form on \( N \) by \( \{n, n'\} := \psi(n) \wedge \psi(n') \). Then \( \gamma = (N, i, \psi) \) is a free cover of \( s \), which proves (1). (2) follows from [CLS11, Theorem 10.1.10], which says that any fan has a smooth refinement. (3) follows from (1) and (2). \( \square \)
Lemma 3.6. Let \( \beta \) be a blowup data. Then the lattice \( K \) in (2.1) for \( N \) is equal to \( \ker \psi \). In particular, \( K \) is a vector subspace of \( N \otimes \mathbb{Q} \), and \( \dim K = |s| - 2 \).

Proof. First we prove the inclusion \( K \subseteq \ker \psi \). Let \( n \in K \). Then \( \psi(n) \wedge \psi(n') = 0 \) for any \( n' \in N \).

Since \( \bar{N} \otimes \mathbb{Q} \) is generated by \( (\psi(e_i))_{i \in I} \) and the skew-symmetric bilinear form on \( \bar{N} \) is non-degenerate, we have \( \bar{\psi}(n) = 0 \). Next we prove the inclusion \( \ker \psi \subseteq K \). Let \( n \in \ker \psi \). Then \( n \in N \) since \( 0 \in \bar{N} \). We have \( \{n, n'\} = \psi(n) \wedge \psi(n') = 0 \) for any \( n' \in N \). Thus \( n \in K \).

Corollary 3.7. We have an isomorphism \( N/K \cong \bar{N} \) induced by \( \psi \).

We will now see that a blowup data determines a blowup of the toric surface associated with its fan. Let \( \beta \) be a blowup data. For each cone \( \sigma \in \Sigma \), we define an affine scheme \( U_\sigma := \text{Spec} \mathbb{C}[S_\sigma] \), where

\[
S_\sigma := \{ n \in N \mid \psi(n) \in \sigma^\vee \}, \quad \sigma^\vee := \{ w \in \bar{N} \otimes \mathbb{R} \mid w \wedge w' \geq 0 \text{ for any } w' \in \sigma \}.
\]

Let \( X_\Sigma \) be a scheme obtained by gluing \( U_\sigma \) for \( \sigma \in \Sigma \) in the same way as the standard construction of toric varieties. We have a map

\[
\bar{\lambda} : X_\Sigma \to D(K)
\]

induced by the inclusion \( K \subseteq N \). The fiber \( \tilde{Y} := \bar{\lambda}^{-1}(\phi) \) is isomorphic to a smooth complete toric surface associated with the fan \( \Sigma \). Let \( \bar{D} \) be the toric boundary of \( \tilde{Y} \).

For each \( i \in I \), we denote by \( D_i \) the toric divisor of \( X_\Sigma \) corresponding to the ray in \( \Sigma \) generated by \( \psi(e_i) \). We define a subscheme \( Z_i \subseteq D_i \) by

\[
Z_i := D_i \cap \tilde{V}(1 + z^{e_i}),
\]

where \( \tilde{V}(f) \) is the closure of \( V(f) \subseteq \text{Spec} \mathbb{C}[N] \) in \( X_\Sigma \). Since \( \psi(e_i) \) is primitive, there is a section \( q_i : D(K) \to X_\Sigma \) of \( \bar{\lambda} \) such that the underlying set of \( Z_i \) is the image of \( q_i \) (see [GHK15a, Lemma 5.1]). Thus \( Z_i \cap \bar{\lambda}^{-1}(\phi) \) consists of a single point, which we denote by \( p_i \). We define \( \pi : (Y, D) \to (\tilde{Y}, \bar{D}) \) to be the composition of the blowups at the points \( p_1, \ldots, p_{|I|} \) (with infinitely near points allowed).

Let \( \Sigma(1) \) be the set of the rays in \( \Sigma \). For each \( \rho \in \Sigma(1) \), let \( u_\rho \) be the primitive generator of \( \rho \). Let \( \bar{D}_\rho \) be the toric divisor of \( \tilde{Y} \) corresponding to \( \rho \). For each \( i \in I \), let \( w_i := \psi(e_i) \). We define a map \( \xi : I \to \Sigma(1) \) by \( \xi(i) = \rho \), where \( \rho \) is the ray generated by \( w_i \). Note that the map \( \xi \) is not injective or surjective in general. Let \( D_\rho \) be the proper transform of \( \bar{D}_\rho \), and \( E_i \) be the exceptional divisor associated with the blowup at \( p_i \). Let \( D^\perp \) be the kernel of the linear map

\[
\text{Pic}(Y) \to \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}D_\rho, \quad C \mapsto \sum_{\rho \in \Sigma(1)} (C \cdot D_\rho)D_\rho,
\]

where \( \text{Pic}(Y) \) is the Picard group of \( Y \), which is equipped with the intersection form.

Proposition 3.8. Let \( \beta \) be a blow up data. Let \( K_\Sigma := K \cap \bigoplus_{i \in I} \mathbb{Z}e_i \subseteq N \). Then we have an isomorphism

\[
K_\Sigma \to D^\perp, \quad \sum_{i \in I} a_i e_i \mapsto \pi^* C - \sum_{i \in I} a_i E_i
\]

where \( C \) is the unique divisor class of \( \tilde{Y} \) such that

\[
C \cdot D_\rho = \sum_{\xi(i) = \rho} a_i.
\]

Proof. This is proved by the same way as the proof of [GHK15a, Theorem 5.5].

Remark 3.9. In [GHK15a, Theorem 5.5], they also show that \( \phi \in \text{Hom}_\mathbb{Z}(K, \mathbb{C}^\times) \) coincides with the period point of \((Y, D)\) under the isomorphism in Proposition 3.8. Moreover, they show that \( \mathcal{Y} \setminus \mathcal{D} \) is an approximation of the cluster variety \( X_\Sigma \), where \( \mathcal{Y} \) is the scheme obtained from \( X_\Sigma \) by the composition of the blowups at \( Z_1, \ldots, Z_{|I|} \), and \( \mathcal{D} \) is the proper transform of the toric boundary of \( X_\Sigma \). See [GHK15a, Lemma 5.2] for a precise statement.
By Proposition 3.8, we have a symmetric bilinear form $K \times K \to \mathbb{Q}$ $((\alpha, \beta) \mapsto \alpha \cdot \beta)$ induced by the intersection form on Pic$(Y)$. We call it the intersection form associated with $\beta$. We will now see that this intersection form only depends on the seed $i$ in $N$ (Proposition 3.16), and only depends on up to isomorphisms the mutation equivalence class of the seed $s$ in $\hat{N}$ (Theorem 3.20). The proof is divided into several lemmas.

**Lemma 3.10.** Let $\beta$ and $\beta'$ be blowup data. Suppose that $\beta$ and $\beta'$ are identical except for $(e_i)_{i \in I}$ and $(e'_i)_{i \in I}$. Then the intersection forms associated with $\beta$ and $\beta'$ are identical.

**Proof.** Let $\tau : I \to I$ be the bijection defined by $e'_i = e_{\tau(i)}$. Let $\alpha \in K$. Then $\alpha$ has the following two expressions: $\alpha = \sum_{i \in I} a_i e_i = \sum_{i \in I} a_{\tau(i)} e'_i$. Let $C$ and $C'$ be the divisor class of $Y$ in Proposition 3.8 corresponding to these two expression of $\alpha$. Then $C = C'$ because

$$C' \cdot \hat{D}_\rho = \sum_{\xi'(i) = \rho} a_{\tau(i)} = \sum_{\xi(i) = \rho} a_{\tau(i)} = \sum_{i \in I} a_i = C \cdot \hat{D}_\rho.$$ 

Now the lemma follows from the following calculation:

$$\left(\sum_{i \in I} a_{\tau(i)} E_i\right) \cdot \left(\sum_{i \in I} b_{\tau(i)} E_i\right) = -\sum_{i \in I} a_{\tau(i)} b_{\tau(i)} = -\sum_{i \in I} a_i b_i = \left(\sum_{i \in I} a_i E_i\right) \cdot \left(\sum_{i \in I} b_i E_i\right).$$

$\square$

**Lemma 3.11.** Let $\beta$ and $\beta'$ be blowup data. Suppose that $\beta$ and $\beta'$ are identical except for $\Sigma$ and $\Sigma'$. Then the intersection form associated with $\beta$ and $\beta'$ are identical.

**Proof.** Choose a smooth complete fan $\Sigma''$ that refines both $\Sigma$ and $\Sigma'$. Then $\Sigma''$ is obtained from both $\Sigma$ and $\Sigma'$ by sequences of star subdivisions [CLS11, Lemma 10.4.2]. Thus we can assume that $\Sigma$ and $\Sigma'$ are related by a star subdivision. Let $\tau : (Y', D') \to (Y, D)$ be the blowup at a node of $D$ corresponding to the star subdivision. Then the pullback of divisors induced by $\tau$ gives an isomorphism from $D^\perp$ to $(D')^\perp$. This isomorphism commutes with the maps (3.2) for $D^\perp$ and $(D')^\perp$ by definition. $\square$

**Lemma 3.12.** Let $\beta$ and $\beta'$ be blowup data. Suppose that $\beta$ and $\beta'$ are identical except for $\phi$ and $\phi'$. Then the intersection forms associated with $\beta$ and $\beta'$ are identical.

**Proof.** This is obvious from (3.2). $\square$

**Lemma 3.13.** The intersection form associated with a blowup data $\beta$ only depends on $\gamma$.

**Proof.** Noting that $\hat{N}$ and $s$ are determined by $\gamma$, the assertion follows from Lemma 3.10, 3.11, and 3.12. $\square$

In Lemma 3.14, 3.15, 3.17, 3.18, and Proposition 3.16 below, we impose the following common assumptions:

- $\hat{N}$ and $\hat{N'}$ are lattices of rank 2.
- $s$ and $s'$ are full dimensional primitive seeds in $\hat{N}$ and $\hat{N'}$, respectively.
- $\gamma$ and $\gamma'$ are free covers of $s$ and $s'$, respectively.

Under these assumptions, we have the intersection forms associated with $\gamma$ and $\gamma'$ by Lemma 3.13.

**Lemma 3.14.** Suppose that we have a seed mutation $\mu_i^n : i \to i'$. Then the intersection forms associated with $\gamma$ and $\gamma'$ are identical.
Proof. This is proved by the same way as the proof of [GHK15a, Theorem 5.6].

Lemma 3.15. Suppose that we have a seed isomorphism $\varepsilon \sigma : i \to i'$. Then the isomorphism $\sigma|_K : K \to K'$ preserves the intersection forms.

Proof. By Lemma 3.5, choose a blowup data $\beta$ associated with $\gamma$. We define a blowup data $\beta'$ associated with $\gamma'$ by setting $c'_i = \sigma(c_i)$, $\Sigma' = \sigma(\Sigma)$, and $\phi' = \sigma(\phi)$, where $\sigma : \hat{N} \to \hat{N}'$ and $\sigma : D(K) \to D(K')$ are the isomorphisms induced by $\sigma$. Then we have isomorphisms $f : \hat{Y} \to \hat{Y}'$ and $\beta : Y \to Y'$. Since $f^* : \text{Pic}(\hat{Y}') \to \text{Pic}(\hat{Y})$ preserves the intersection forms, it suffices to prove that the following diagram commutes:

$$
\begin{array}{ccc}
K & \to & K' \\
\downarrow & & \downarrow \\
D^\perp & \to & (D')^\perp
\end{array}
$$

where the map in the first row is $\sigma|_K$, the map in the second row is $(f^*|_{(D')^\perp})^{-1}$, and the vertical maps are the isomorphisms in Proposition 3.8 for $\beta$ and $\beta'$. The commutativity of the diagram is equivalent to the following equality:

$$
\pi^* C - \sum_{i \in I} a_i E_i = f^* \left( (\pi')^* C' - \sum_{i \in I} a_i E'_i \right),
\tag{3.3}
$$

where $f^* C' = C$ and $f^* E'_i = E_i$. The above equality follows from $\pi' \circ f = f \circ \pi$. $\square$

Proposition 3.16. The intersection form associated with $\gamma$ only depends on $i$.

Proof. This is proved by applying Lemma 3.15 to the seed isomorphism $id : i \to i$. $\square$

Lemma 3.17. Suppose that we have a seed mutation $\mu_{i\omega} : s \to s'$. Then there exists a free cover $\gamma''$ of $s'$ and $v \in i$ such that $\mu^v_{i\omega} : i \to \hat{i}'$ is a seed mutation.

Proof. Choose $v \in i$ such that $w = \psi(v)$. By Lemma 2.4, we define a seed $\hat{i}'$ by the seed mutation $\mu^v_{i\omega} : i \to \hat{i}'$. Then $\gamma'' = (N, \hat{i}', \psi)$ satisfies the desired property. $\square$

Lemma 3.18. Suppose that we have a seed isomorphism $\varepsilon \sigma : s \to s'$. Then there exists an isomorphism $\overline{\sigma} : N \to N'$ such that $\varepsilon \overline{\sigma} : i \to i'$ is a seed isomorphism.

Proof. Since $s' = \sigma(s)$, there is a bijection $\tau$ from the set $i$ to the set $i'$ such that $\sigma(\psi(i)) = \psi'(\tau(i))$ for any $i \in i$. Then the isomorphism $N \otimes_\mathbb{Z} \mathbb{Q} \to N' \otimes_\mathbb{Z} \mathbb{Q}$ given by $i \mapsto \tau(i)$ restricts to an isomorphism $\tau : N \to N'$. This gives a seed isomorphism $\varepsilon \tau : i \to i'$. $\square$

We now prove the invariance of the intersection forms under seed cluster transformations. For any seed $i$, let $P(i)$ be the following property: there is a lattice $\hat{N}$ of rank 2, a full dimensional primitive seed $s$ in $\hat{N}$, and a free cover $\gamma$ of $s$, such that $i$ is the seed in $\gamma$. It is easy to see that $P(i)$ is invariant under seed cluster transformations. By Proposition 3.16, for any seed $i$ satisfying $P(i)$, the intersection form associated with $i$ is well-defined.

Theorem 3.19. Let $i$ and $i'$ be seeds. Suppose that $P(i)$ and $P(i')$ hold. Suppose that we have a seed cluster transformation $\epsilon c : i \to i'$. Then the isomorphism $K \to K'$ induced by $\epsilon$ preserves the intersection forms. In particular, the cluster modular group $\Gamma_1$ acts on $K$ as isometries.

Proof. This follows from Lemma 3.14 and 3.15. $\square$

Theorem 3.20. Let $\hat{N}$ and $\hat{N}'$ be lattices of rank 2. Let $s$ and $s'$ be full dimensional primitive seeds in $\hat{N}$ and $\hat{N}'$, respectively. Let $\gamma$ and $\gamma'$ be free covers of $s$ and $s'$, respectively. Suppose that we have a seed cluster transformation $\epsilon c : s \to s'$. Then there is an isomorphism $K \to K'$ that preserves the intersection forms.

Proof. This follows from Lemmas 3.14, 3.15, 3.17, and 3.18. $\square$
We will use the following construction of seed isomorphisms.

**Definition 3.21.** Let $N$ and $N'$ be lattices of rank 2. Let $s$ and $s'$ be full dimensional primitive seeds in $N$ and $N'$, respectively. Let $\gamma$ and $\gamma'$ be free covers of $s$ and $s'$, respectively. Let $(e_i)_{i \in I}$ and $(e'_i)_{i \in I}$ be labelings of $i$ and $i'$, respectively. Let $\sigma : I \to I$ be a bijection. Suppose that there is a seed isomorphism $\varepsilon \sigma : s \to s'$ such that $\sigma(\psi(e_i)) = \psi'(\varepsilon'(e''(i)))$ for any $i \in I$. Then the isomorphism $\sigma : N \to N'$ given by $e_i \mapsto e''(i)$ gives a seed isomorphism $\varepsilon \sigma : i \to i'$. By abuse of notation, we will denote it by $\varepsilon \sigma : i \to i'$. We say that $\varepsilon \sigma : i \to i'$ is the seed isomorphism induced by the permutation $\sigma$.

### 3.2 Seeds of q-Painlevé type

A pair $(Y, D)$ is called a **log Calabi-Yau surface with maximal boundary** if $Y$ is a smooth rational projective surface over $\mathbb{C}$, and $D \in |{-K_Y}|$ is a reduced nodal curve with at least one singular point. Such a pair is called a **Looijenga pair** in [GHK15b]. We will call a log Calabi-Yau surface with maximal boundary simply a **log CY surface**. For any blowup data $\beta$, the pairs $(\tilde{Y}, \tilde{D})$ and $(Y, D)$ defined in Section 3.1 are log CY surfaces.

Let $(Y, D)$ be a log CY surface. We define a **simple toric blowup** $(\tilde{Y}, \tilde{D}) \to (Y, D)$ to be the blowup at a node of $D$ such that $\tilde{D}$ is the reduced inverse image of $D$. Then $(\tilde{Y}, \tilde{D})$ is again a log CY surface, and $\tilde{Y} \setminus \tilde{D} = Y \setminus D$. A **toric blowup** is a composition of simple toric blowups.

For any log CY surface $(Y, D)$, we define $D^\perp \subseteq \text{Pic}(Y)$ by

$$D^\perp = \{ \alpha \in \text{Pic}(Y) \mid \alpha \cdot C = 0 \text{ for any irreducible component } C \text{ of } D \}. \quad (3.4)$$

If $(\tilde{Y}, \tilde{D}) \to (Y, D)$ is a toric blowup, then it induces an isomorphism $D^\perp \cong \tilde{D}^\perp$ that preserves the intersection forms. When $(Y, D)$ is a log CY surface associated with a blowup data $\beta$, the kernel of the map (3.1) coincides with $D^\perp$ in (3.4).

Let $\beta$ be a blowup data. We have a unique cyclic order on $\Sigma(1)$ that is compatible with the orientation on $N \otimes_\mathbb{Z} \mathbb{R}$ given by the fixed isomorphism $\wedge^2 N \cong \mathbb{Z}$. For any $\rho \in \Sigma(1)$, we denote by $\rho + 1$ and $\rho - 1$ the rays next to $\rho$ in this cyclic order. Let $n_\rho$ be the self-intersection number $\tilde{D}_\rho \cdot \tilde{D}_\rho$, and $m_\rho$ be the number of blowups on $D_\rho$. Then the intersection matrix $(D_i \cdot D_j)_{i,j \in \Sigma(1)}$ of the irreducible components of $D$ is given by

$$D_i \cdot D_j = \begin{cases} n_i - m_i & \text{if } i = j, \\ 1 & \text{if } i = j \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 3.22.** Let $\beta$ be a blowup data. Then the followings are equivalent:

1. The intersection form on $K$ is negative semi-definite but not negative definite.
2. The intersection matrix $(D_i \cdot D_j)_{i,j \in \Sigma(1)}$ is negative semi-definite but not negative definite.
3. There exists a unique primitive vector $(c_\rho) \in (\mathbb{Z}_{>0})^{\Sigma(1)}$ such that $(\sum_{\rho \in \Sigma(1)} c_\rho D_\rho)^2 = 0$.
4. For any toric blowup $(Y, D) \to (Y^\rho, D^\rho)$ such that $D^\rho$ does not have $(-1)$-components, $D^\rho \in (D^\rho)^\perp$.

**Proof.** The equivalence of (1), (2), and (4) is proved by [Man19, Theorem 4.2]. The equivalence of (2) and (3) follows from [Kac90, Theorem 4.3 and Lemma 4.5].

**Definition 3.23.** We say that a blowup data $\beta$ is of **q-Painlevé type** if it satisfies the conditions in Proposition 3.22.

This definition is motivated by [Sak01, Definition 2]. By (3) of Theorem 3.20 and the fact that the condition (1) of Proposition 3.22 is preserved under isomorphisms of vector spaces that preserve intersection forms, the property that a blowup data $\beta$ is of q-Painlevé type only depends on the mutation equivalence class of $s$. In other words, the property that a seed $s$ in a lattice of rank 2 is of q-Painlevé type is well-defined, and this property is invariant under seed cluster transformations.
Suppose that $\beta$ is a blowup data of $q$-Painlevé type. Let $(c_\rho)_{\rho \in \Sigma(1)}$ be the vector in (3) of Proposition 3.22. We define an element $\delta \in N$ by the following formula:

$$\delta := \sum_{i \in I} c_{\xi(i)} e_i.$$  \hfill (3.5)

**Lemma 3.24.** $\delta$ only depends on the free cover $\gamma$.

**Proof.** Clearly, $\delta$ does not depend on the choice of the labeling and the choice of $\phi \in D(K)(\mathbb{C})$. We now prove that $\delta$ does not depend on the choice of the fan. Suppose that we have two choices $\Sigma$ and $\Sigma'$. As in the proof of Lemma 3.11, we can assume that $\Sigma$ and $\Sigma'$ are related by a star subdivision at a cone $\sigma = \text{Cone}(u_\pi, u_{\pi+1})$. Then we have

$$\Sigma' = (\Sigma \cup \sigma) \cup \{\text{Cone}(u_\pi + u_{\pi+1}), \text{Cone}(u_\pi, u_\pi + u_{\pi+1}), \text{Cone}(u_{\pi+1}, u_\pi + u_{\pi+1})\}.$$  

Under the star subdivision, the numbers $(n_\rho)_{\rho \in \Sigma(1)}$ and $(m_\rho)_{\rho \in \Sigma(1)}$ change into $(n'_\rho)_{\rho \in \Sigma'(1)}$ and $(m'_\rho)_{\rho \in \Sigma'(1)}$ where

$$n'_\rho = \begin{cases} -1 & \text{if } \rho = \text{Cone}(u_\pi + u_{\pi+1}), \\ n_\rho - 1 & \text{if } \rho = \pi \pm 1, \\ n_\rho & \text{otherwise,} \end{cases} \quad m'_\rho = \begin{cases} 0 & \text{if } \rho = \text{Cone}(u_\pi + u_{\pi+1}), \\ m_\rho & \text{otherwise.} \end{cases}$$

Thus the numbers $(c_\rho)_{\rho \in \Sigma(1)}$ change into $(c'_\rho)_{\rho \in \Sigma'(1)}$ where

$$c'_\rho = \begin{cases} c_\rho + c_{\rho+1} & \text{if } \rho = \text{Cone}(u_\pi + u_{\pi+1}), \\ c_\rho & \text{otherwise.} \end{cases}$$

Therefore, $c_{\xi(i)} = c'_{\xi(i)}$ for any $i \in I$. 

**Proposition 3.25.** $\delta \in K$, and $\delta \mapsto \sum_{\rho \in \Sigma(1)} c_\rho D_\rho$ under the isomorphism in Proposition 3.8. In particular, we have $\delta \cdot \delta = 0$.

**Proof.** We have

$$\psi(\delta) = \sum_{i \in I} c_{\xi(i)} w_i = \sum_{\rho \in \Sigma(1)} m_\rho c_\rho u_\rho = \sum_{\rho \in \Sigma(1)} (n_\rho c_\rho + c_{\rho-1} + c_{\rho+1}) u_\rho = \sum_{\rho \in \Sigma(1)} (-c_\rho u_{\rho-1} - c_\rho u_{\rho+1} + c_{\rho-1} u_\rho + c_{\rho+1} u_\rho) = 0,$$

where we use the well-known relation $n_\rho u_\rho + u_{\rho-1} + u_{\rho+1} = 0$ (see [CLS11, Theorem 10.4.4]) at the fourth equality. Thus $\delta \in K$ by Lemma 3.6. Moreover, $\delta \in K_\Sigma$ by definition. We have

$$\delta \mapsto \pi^* \left( \sum_{\rho \in \Sigma(1)} c_\rho \tilde{D}_\rho \right) - \sum_{i \in I} c_{\xi(i)} E_i = \sum_{\rho \in \Sigma(1)} c_\rho D_\rho \quad \hfill (3.6)$$

since

$$\left( \sum_{\rho \in \Sigma(1)} c_\rho \tilde{D}_\rho \right) \cdot \tilde{D}_\rho = n_\rho c_\rho + c_{\rho-1} + c_{\rho+1} = c_\rho m_\rho = \sum_{i \in I, \xi(i)=\rho} c_{\xi(i)}.$$

**Definition 3.26.** We say that $\delta \in K$ defined by (3.5) is the null root associated with $\gamma$. We denote by $q$ the function $z^\delta \in \mathbb{C}[K]$, which gives a globally defined function on the cluster variety $\mathcal{X}_1$ by the map $\lambda : \mathcal{X}_1 \to D(K)$.

**Lemma 3.27.** Let $\beta$ be a blowup data of $q$-Painlevé type. Suppose that $\rho, -\rho \in \Sigma(1)$. Let $\varepsilon$ be a sign. Then

$$c_\rho + c_{-\rho} = \sum_{\rho' \in \Sigma(1)} \left[ \varepsilon u_{\rho'} \wedge u_\rho \right] + m_{\rho'} c_{\rho'}.$$

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Suppose that $\delta$ and $\delta'$ are the null root associated with $\gamma$ and $\gamma'$, respectively. Suppose that we have a seed cluster transformation $c : i \to i'$. Then $\delta$ maps to $\delta'$ by the isomorphism $K \to K'$ induced by $c$. In particular, $\delta$ only depends on $i$. 

**Proof.** If $c$ is a seed isomorphism, the claim follows from (3.3) and (3.6). Applying this claim for $\text{id} : i \to i$, we see that $\delta$ only depends on $i$. Now we consider the case where $c$ is a seed mutation $\mu_c$. Since $\delta$ only depends on $i$, we can assume that $N = N'$ and $\psi = \psi'$. Let $w = \psi(v)$. Choose a blowup data $\beta$ and $\beta'$ associated with $\gamma$ and $\gamma'$, respectively, such that $\Sigma'$ is obtained from $\Sigma$ by the piecewise linear map $u \mapsto u + [\varepsilon(u \wedge w) + w] w$. Let $\sigma : \Sigma(1) \to \Sigma'(1)$ be the bijection induced by this piecewise linear map. It is easy to see that $\rho'_{\sigma(\rho)} - m'_{\sigma(\rho)} = \rho - m_\rho$ for any $\rho \in \Sigma(1)$, which implies that $\rho'_{\sigma(\rho)} = \rho$ for any $\rho \in \Sigma(1)$. Now the claim follows from Definition 2.2 and Lemma 3.27. □

Suppose that $\beta$ is a blowup data of $q$-Painlevé type. We construct an affine root system associated with $\beta$ as follows. First we perform a toric blowdown $(Y,D) \to (Y^0,D^0)$ such that $D^0$ does not have $(-1)$-curves. By (4) and [Sak01, Proposition 2], there is a birational map $Y^0 \to \mathbb{P}^2$ that is a composition of nine successive blowups of points, which are possibly infinitely near points. This implies that $\{\alpha \in \text{Pic}(Y^0) \mid \alpha \cdot D^0 = 0\}$ is isomorphic to a root lattice of type $E_8^{(1)}$, and the irreducible components of $D^0$ form simple roots for a root system of type $A_7^{(1)}$, where $r + 1$ is the number of the irreducible components of $D^0$. By Theorem 3.8, $K_Z$ is isomorphic to $Q(R)^{1}$, where $Q(R)$ is the root lattice of type $R = A_7^{(1)}$, and the complement is taken in the root lattice $Q(E_8^{(1)})$.

There are ten embeddings (up to the action of the Weyl group of type $E_8^{(1)}$) of the root system $A_7^{(1)}$ into the root system $E_8^{(1)}$, as shown in Figure 2 (see [Cox34]). There is exactly one embedding for $r = 0, 1, 2, 3, 4, 5, 6, 8$, and there are two embeddings for $r = 7$. We use the symbol $A_7^{(1)}$ for the one which has no orthogonal real roots of $E_8^{(1)}$ in its complement, and $A_7^{(1)\prime}$ for the other one. The arrows

![Diagram](image-url)
$R \to R'$ in the figure mean inclusions $Q(R) \subset Q(R')$. For any symbol $R = A_s^{(1)}$, it is convenient to introduce the symbol $R^\perp = E_{8-r}^{(1)}$ and define $Q(R^\perp) := Q(R)\perp$, as in Figure 3. Using Sakai’s symbols in [Sak01], $R^\perp$ is expressed as follows:

$$
\begin{array}{cccccccc}
E_{6,7,8}^{(1)} & E_{5}^{(1)} & E_{4}^{(1)} & E_{3}^{(1)} & E_{2}^{(1)} & E_{1}^{(1)} & E_{1}^{(1)'} & E_{0}^{(1)} \\
E_{6,7,8}^{(1)} & D_{5}^{(1)} & A_{4}^{(1)} & (A_{2} + A_{1})^{(1)} & (A_{1} + A_{1})^{(1)} & A_{1}^{(1)} & A_{1}^{(1)} & A_{0}^{(1)} \\
\end{array}
$$

In summary, there is an isomorphism $K_Z \cong Q(R^\perp)$ preserving the intersection forms, where $R^\perp$ is a symbol in Figure 3. By (3) of Theorem 3.20, the symbols $R$ and $R^\perp$ only depends on the mutation equivalence class of $s$. Conversely, we will prove in the next section that any two seeds that are of $q$-Painlevé type and have the same $R^\perp$ (or $R$) are mutation equivalent (Corollary 3.33).

### 3.3 Fano polygons and the classification of seeds of $q$-Painlevé type

In this section, we will classify the seeds of $q$-Painlevé type up to mutation equivalence by using a classification result for Fano polygons in [KNP17]. Let $\bar{N}$ be a lattice of rank 2. We say that a convex polytope $P$ in $\bar{N} \otimes \mathbb{R}$ is a Fano polygon if the origin $0 \in \bar{N}$ lies in the interior of $P$, and each vertex of $P$ is primitive element in $\bar{N}$. Let $H_{w,c}$ and $H_{w,c}^\perp$ be the affine hyperplane and the closed half-space given by

$$
H_{w,c} = \{ v \in \bar{N} \otimes \mathbb{R} \mid \{ v, w \} = -c \}, \quad H_{w,c}^\perp = \{ v \in \bar{N} \otimes \mathbb{R} \mid \{ v, w \} \geq -c \},
$$

where $w \in N$ and $c \in \mathbb{Z}$. Then each facet $f$ of a Fano polygon $P$ is uniquely expressed as $f = P \cap H_{w_f,c_f}$ by a primitive vector $w_f \in N$ and a positive integer $c_f$, and we have

$$
P = \bigcap_{f : \text{facet}} H_{w_f,c_f}^\perp.
$$

For each facet $f$, we define a positive integer $l_f$ as the greatest degree of divisibility of $v - v'$ in $\bar{N}$, where $v$ and $v'$ are the endpoints of $f$. We also set $l_v = 0$ for each vertex $v$ of $P$. We say that a Fano polygon $P$ has no remainders if $l_f$ is a multiple of $c_f$ for any facet $f$ of $P$.

Suppose that $\beta$ is a blowup data of $q$-Painlevé type. We define $P_\beta \subseteq \bar{N} \otimes \mathbb{R}$ by

$$
P_\beta = \bigcap_{i \in I} H_{w_i,c_{\xi(i)}}^+(1)
$$

where $c_\rho$ for $\rho \in \Sigma(1)$ is given in (3) of Proposition 3.22. Since the seed $s$ in $\beta$ is full dimensional, $P_\beta$ is a convex polytope.

**Proposition 3.29.** $P_\beta$ only depends on the seed $s$.

**Proof.** By Lemma 3.24, $P_\beta$ only depends on $\gamma$. Suppose that $\gamma = (N, i, \psi)$ and $\gamma' = (N', i', \psi')$ are free covers of $s$. Let $P$ and $P'$ be the convex polytopes associated with $\gamma$ and $\gamma'$, respectively. Choose a bijection $\tau : i \to i'$ as in the proof of Lemma 3.18. Let $\eta$ be the map $\eta : i \to \Sigma(1)$ given by $i \mapsto \mathbb{R}_{\geq 0} \psi(i)$. Let $\eta' : i' \to \Sigma(1)$ be the same map for $i'$. Then we have

$$
P = \bigcap_{i \in I} H_{\psi(i),c_{\eta(i)}}^+(1) = \bigcap_{i \in I} H_{\psi'(\tau(i)),c'_{\eta'(\tau(i))}}^+(1) = \bigcap_{i' \in I'} H_{\psi'(i'),c'_{\eta'(i)}}^+(1) = P'.
$$

Suppose that $s$ is a seeds of $q$-Painlevé type. By Proposition 3.29, we have the polygon associated with $s$, which we denote by $P_s$.

**Proposition 3.30.** The correspondence $s \mapsto P_s$ is a bijection from the set of the seeds of $q$-Painlevé type to the set of the Fano polygons that have no remainders.
Proof. We first show that $P_s$ is a Fano polygon that has no remainders. Fix a blowup data $\beta$ associated with $s$. We define a polygon $Q_\beta$ by

$$Q_\beta = \bigcap_{\rho \in \Sigma(1)} H_{u_{\rho},c_{\rho}}^+.$$  

Let $f_{\rho} := Q_\beta \cap H_{u_{\rho},c_{\rho}}$ and $l_{\rho} := f_{\rho}$ for each $\rho \in \Sigma(1)$. If $v$ and $v'$ are the vertex of $P$ in $f_{\rho} \cap f_{\rho - 1}$ and $f_{\rho} \cap f_{\rho + 1}$, respectively, we have

$$\{v - v', u_{\rho}\} = 0,$$

and

$$\{v - v', u_{\rho + 1}\} = \{v, u_{\rho + 1}\} + c_{\rho + 1} = \{v, -n_{\rho}u_{\rho} - u_{\rho - 1}\} + c_{\rho + 1} = n_{\rho}c_{\rho} + c_{\rho - 1} + c_{\rho + 1} = m_{\rho}c_{\rho}.$$ 

Since $u_{\rho}$ and $u_{\rho + 1}$ form a basis of $N$ for any $\rho \in \Sigma(1)$, we have $l_{\rho} = m_{\rho}c_{\rho}$ for any $\rho \in \Sigma(1)$. This relation in particular implies that $P_s = Q_\beta$. Since $c_{\rho}$ is positive for any $\rho \in \Sigma(1)$, the origin lies in the interior of $Q_\beta$. Suppose that $v$ is a vertex of $P$. By using the relation $(n_{\rho} - m_{\rho})c_{\rho} + c_{\rho - 1} + c_{\rho + 1} = 0$ for $\rho \in \Sigma(1)$ repeatedly, we see that for any $\rho \in \Sigma(1)$ there is an element $u \in N$ such that $\{v, u\} = c_{\rho}$. Since $(c_{\rho})_{\rho \in \Sigma(1)}$ is primitive, $v$ is also primitive. Thus $Q_\beta$ is a Fano polygon. Since $l_{\rho} = m_{\rho}c_{\rho}$ for any $\rho \in \Sigma(1)$, we see that $Q_\beta$ has no remainders.

We next construct the inverse map. Suppose that $P$ is Fano polygon that has no remainders. Let $F$ be the set of the facets in $P$. We define a finite index set $J$ by

$$J = \{(f, i) \mid f \in F \text{ and } 1 \leq i \leq l_{f}/c_{f}\}.$$ 

We also define a collection of primitive elements $(w_{f,i})(f,i) \in J$ in $\tilde{N}$ by $w_{f,a} := w_{f}$. Let $s$ be the seed for $\tilde{N}$ defined by $s = \{w_{f,i} \in \tilde{N} \mid (f, i) \in J\}$. Now we show that $s$ is of q-Painlevé type. It is obvious that $s$ is full dimensional and primitive. Choose a blowup data $\beta$ associated with $s$. To show the condition (3) in Proposition 3.22, we define an integer $c_{\rho}$ for each $\rho \in \Sigma(1)$ as follows. We set $c_{\rho} := c_{f}$ if $u_{\rho} = w_{f,i}$ for some $(f, i) \in J$. If this is not the case, there are two faces $f$ and $f'$ such that $u_{\rho}$ lies in the interior of the cone generated by $w_{f}$ and $w_{f'}$. In this case, we set $c_{\rho} := -\{v, u_{\rho}\}$ where $\{v\} = f \cap f'$. We have $n_{\rho}c_{\rho} + c_{\rho - 1} + c_{\rho + 1} = 0$ in this second case since $n_{\rho}u_{\rho} + w_{\rho - 1} + w_{\rho + 1} = 0$. We now suppose that $c_{\rho} = c_{f}$ for some face $f$. Let $f \pm 1 := P \cap H_{u_{\rho},c_{\rho \pm 1}}$ be the neighboring facets or vertices of $f$. Let $v$ and $v'$ be the vertices of $P$ given by $v \in (f - 1) \cap f$ and $v' \in (f + 1) \cap f$. Then we have

$$\{v - v', u_{\rho}\} = 0, \quad \{v - v', u_{\rho + 1}\} = n_{\rho}c_{\rho} + c_{\rho - 1} + c_{\rho + 1}$$

as before. Since $u_{\rho}$ and $u_{\rho + 1}$ form a basis of $N$, we have $n_{\rho}c_{\rho} + c_{\rho - 1} + c_{\rho + 1} = l_{f} = m_{\rho}c_{\rho}$. Therefore, the seed $s$ is of q-Painlevé type. It follows by definition that this gives the inverse map of $s \mapsto P_s$.  

Theorem 3.31. All the seeds in Figure 1 are of q-Painlevé type, and the symbol at the left of each seed is $R^+$ for this seed. Moreover, Fano polygons associated with these seeds are given in Figure 4.

Proof. This is proved by case-by-case analysis given in Appendix A. We explain the detail in the case of $E_7^{(1)}$. Let $\tilde{N} = \mathbb{Z}^2$, which is equipped with the standard exterior product given by $(1,0) \wedge (0,1) = 1$. Let $s$ be a seed in $\tilde{N}$ given by

$$s = \{(1,0), (1,0), (1,0), (0,1), (0,1), (0,1), (0,1), (-1,0), (0,-1), (0,-1)\}.$$ 

Let $u_1 = (0,1), u_2 = (-1,0), u_3 = (0,-1)$, and $u_4 = (1,0)$. Let $\rho_i = \mathbb{R}_{\geq 0}u_i$. Let $\Sigma$ be the smooth complete fan in $\mathbb{R}^2$ whose rays are $\rho_1, \ldots, \rho_4$. Choose a free cover $\gamma = (N, i, \psi)$ of $s$ and a labeling $(e_1, \ldots, e_{10})$ of $i$ such that

$$w_i = \begin{cases} 
(0,1) & \text{if } i = 1, \ldots, 6, \\
(-1,0) & \text{if } i = 7, \\
(0,-1) & \text{if } i = 8, 9, 10, \\
(1,0) & \text{if } i = 11, 
\end{cases}$$
where \( w_i = \psi(e_i) \). Choose an element \( \phi \in D(K)(\mathbb{C}) \). These data define a blowup data \( \beta \) associated with \( s \). We simply write \( \bar{D}_i \rho \) and \( D_i \rho \) as \( \bar{D}_i \) and \( D_i \), respectively. We have \( (D_1 + 3D_2 + 2D_3 + D_4)^2 = 0 \), which shows that \( s \) is of \( q \)-Painlevé type by Proposition 3.22. We define the following basis of \( K \):

\[
\alpha_0 = e_4 - e_3, \quad \alpha_1 = e_3 - e_2, \quad \alpha_2 = e_2 - e_1, \quad \alpha_3 = e_1 + e_6, \\
\alpha_4 = e_5 + e_8, \quad \alpha_5 = e_9 - e_8, \quad \alpha_6 = e_{10} - e_9, \quad \alpha_7 = e_7 - e_6.
\]

By the isomorphism in Proposition 3.8, these elements correspond to the following divisor classes:

\[
\begin{align*}
\alpha_0 &\mapsto E_3 - E_4, & \alpha_1 &\mapsto E_2 - E_3, & \alpha_2 &\mapsto E_1 - E_2, & \alpha_3 &\mapsto \pi^* \bar{D}_2 - E_1 - E_6, \\
\alpha_4 &\mapsto \pi^* \bar{D}_1 - E_5 - E_8, & \alpha_5 &\mapsto E_8 - E_9, & \alpha_6 &\mapsto E_9 - E_{10}, & \alpha_7 &\mapsto E_6 - E_7.
\end{align*}
\]

Thus the intersection matrix for these elements is given by

\[
\alpha_i \cdot \alpha_j = \begin{cases} 
-2 & \text{if } i = j \\
1 & \text{if there is an edge between } \alpha_i \text{ and } \alpha_j \text{ in the graph below} \\
0 & \text{otherwise,}
\end{cases}
\]

which shows that \( s \) is of type \( E_7^{(1)} \). By (3.8), the Fano polygon \( P_\beta \) is given by

\[
P_\beta = H_{u_{1,1}}^{+} \cap H_{u_{2,3}}^{+} \cap H_{u_{3,2}}^{+} \cap H_{u_{4,1}}^{+} = 
\]

\[\begin{array}{ccccccc}
\end{array}\]

**Theorem 3.32.** Suppose that \( s \) is a seed of \( q \)-Painlevé type. Then \( s \) is mutation equivalent to one and only one of the seeds in Figure 1.

**Proof.** This follows from Proposition 3.30, Theorem 3.31, and the classification of Fano polygons up to mutation equivalence given by [KNP17, Theorem 6].

**Corollary 3.33.** Suppose that \( s \) and \( s' \) are seeds of \( q \)-Painlevé type. Then \( s \) and \( s' \) are mutation equivalent if and only if they have the same \( R^\perp \).
3.4 Actions of Cremona isometries on cluster Poisson varieties

Let $\beta$ be a blowup data of $q$-Painlevé type. Suppose that $\beta$ is of type $R^\perp$. Let $(Y, D) \to (Y^\flat, D^\flat)$ be a toric blowup such that $D^\flat$ does not have $(-1)$-components as in Section 3.2. Let $W(R^\perp)$ be the Weyl group associated with $R^\perp$:

$$W(R^\perp) = \langle s_\alpha \mid \alpha \text{ is a real root in } Q(E_8^{(1)}) \text{ and } \alpha \in Q(R^\perp) \rangle \subseteq W(E_8^{(1)}),$$

where $s_\alpha$ is a reflection on $Q(E_8^{(1)})$ associated with $\alpha$. The Weyl group $W(R^\perp)$ can be embedded in $	ext{Aut}(\text{Pic}(Y^\flat))$ by setting $s_\alpha(\lambda) = \lambda + (\alpha \cdot \lambda)\alpha$ for any $\lambda \in \text{Pic}(Y^\flat)$. We denote by $	ext{Aut}(R^\perp)$ the group of automorphism of the Dynkin diagram of type $R^\perp$, where the Dynkin diagram is the graph given in Appendix A for each type $R^\perp$.

An automorphism $w \in \text{Aut}(\text{Pic}(Y^\flat))$ is called a Cremona isometry if $w$ preserves the intersection form on $\text{Pic}(Y^\flat)$, the canonical class of $Y^\flat$, and the monoid generated by the effective classes. We denote by $\text{Cr}(Y^\flat)$ the group of the Cremona isometries. This group is computed by Sakai:

**Theorem 3.34** ([Sak01, Theorem 26]). Suppose that $(Y^\flat, D^\flat)$ is generic, that is, there is no smooth rational curve with self-intersection $-2$ that is disjoint from the irreducible components of $D^\flat$. Then

1. $\text{Cr}(Y^\flat) \cong W(R^\perp) \times \text{Aut}(R^\perp)$ for $R^\perp \neq E_2^{(1)}, E_1^{(1)}, E_1^{(1)'},$ and $E_0^{(1)}$,
2. $\text{Cr}(Y^\flat) \cong W(A_1^{(1)}) \times \mathbb{Z} \times S_2$ for $R^\perp = E_2^{(1)},$
3. $\text{Cr}(Y^\flat) \cong \mathbb{Z} \times S_2$ for $R^\perp = E_1^{(1)},$
4. $\text{Cr}(Y^\flat) \cong W(A_1^{(1)}) \times D_8$ for $R^\perp = E_1^{(1)'},$
5. $\text{Cr}(Y^\flat) \cong D_3$ for $R^\perp = E_0^{(1)}$,

where $S_n$ is the symmetric group of order $n!$ and $D_n$ is the dihedral group of order $2n$. In particular, the group $\text{Cr}(Y^\flat)$ for $R^\perp = E_r^{(1)}$ contains a free abelian group of rank $r$.

We denote by $\text{Cr}(R^\perp)$ the group of Cremona isometries for type $R^\perp$ in Theorem 3.34. Let $\text{Cr}(R^\perp)^{\text{op}} = \{w^* \mid w \in \text{Cr}(R^\perp)\}$ be its opposite group.

The main result of this section is the following theorem.

**Theorem 3.35.** Let $s$ be a seed of $q$-Painlevé type. Suppose that $s$ is of type $R^\perp$. Let $\gamma = (N, i, \psi)$ be a free cover of $s$. Then there is an injective morphism

$$\text{Cr}(R^\perp)^{\text{op}} \to \Gamma_1$$

such that, under this embedding, the action of $\text{Cr}(R^\perp)^{\text{op}}$ on $D(K)$ is given by the pull-back of functions

$$w(z^\alpha) = z^{\text{sgn}(w)w(\alpha)}$$

for any $\alpha \in K \cong D^\perp \otimes \mathbb{Z} \mathbb{Q}$ and $w \in \text{Cr}(R^\perp)$, where $\text{sgn}(w) \in \{\pm 1\}$ is determined by $w(q) = q^{\text{sgn}(w)}$.

**Proof.** Suppose that $s$ and $s'$ are seeds of $q$-Painlevé type, and both are of type $R^\perp$. Then $s$ and $s'$ are mutation equivalent by Corollary 3.33. Therefore, by the results in Section 3.1 and Proposition 3.28, it is enough to prove the theorem for a free cover associated with each seed in Figure 1. The proof is given by the explicit case-by-case computations in Appendix A. The construction of the embedding $\text{Cr}(R^\perp)^{\text{op}} \to \Gamma_1$ in Appendix A is based on [BGM18]. We explain the detail in the case of $R^\perp = E_7^{(1)}$.

Let $s$ be the seed of type $E_7^{(1)}$ in Figure 1. Let $\beta$ be the blowup data associated with $s$ given in Appendix A. See also the proof of Theorem 3.31. By Theorem 3.34, the group of Cremona isometries is given by

$$\text{Cr}(E_7^{(1)}) = \langle s_0, \ldots, s_7 \rangle \cong W(E_7^{(1)}) \times \text{Aut}(E_7^{(1)})$$,
where \( s_0, \ldots, s_7 \) are reflections associated with \( \alpha_0, \ldots, \alpha_7 \), and \( \iota \) is the automorphism of the Dynkin diagram of type \( E_7^{(1)} \) given by \( \iota(\alpha) = \alpha_{10} \), where the notation means \( \alpha_0 \mapsto \alpha_6 \) and so on. We define a map \( \text{Cr}(E_7^{(1)})^{\text{op}} \to \Gamma_1 \) by

\[
\begin{align*}
\text{s}^*_1 &\mapsto (3,4), \quad \iota \mapsto (2,3), \quad \text{s}^*_2 \mapsto (1,2), \quad \text{s}^*_3 \mapsto \mu_1 \circ (1,6) \circ \mu_1^+, \\
\text{s}^*_4 &\mapsto \mu_2 \circ (5,8) \circ \mu_5^+, \quad \text{s}^*_5 \mapsto (8,9), \quad \text{s}^*_6 \mapsto (9,10), \quad \text{s}^*_7 \mapsto (6,7), \\
\iota^* &\mapsto \mu_3 \circ -(1,5) (2,8) (3,9) (4,10) \circ \mu_7^+.
\end{align*}
\]

We explain the notation in the right hand sides by using the image of \( s^*_3 \) as an example. By Remark 2.5, there are unique labeled seeds \( (\epsilon'_i)_{i \in I} \) and \( (\epsilon''_i)_{i \in I} \) such that \( \mu_1^+ : (\epsilon_i)_{i \in I} \mapsto (\epsilon'_i)_{i \in I} \) and \( \mu_1^- : (\epsilon''_i)_{i \in I} \mapsto (\epsilon_i)_{i \in I} \) are labeled seed mutations. We can check that the permutation \((1,6)\) on \( I \) satisfies the assumption of Lemma 2.31. Consequently, we have the actions \( \mu_1^+ \circ (1,6) \circ \mu_1^- : i \mapsto i \), where \( \mu_1^+ \) and \( \mu_1^- \) in this equation are understood as seed mutations \( \mu^+_5 \) and \( \mu^-_6 \), respectively.

We can verify that this map is well-defined by directly checking that the following defining relations of the Weyl group hold in \( \Gamma_1 \):

\[
\begin{align*}
\text{s}_i^2 &= \text{id}, \quad \text{s}_i \text{s}_j = \text{s}_j \text{s}_i \quad (\text{if } \alpha_i \cdot \alpha_j = 0), \quad \text{s}_i \text{s}_j \text{s}_i = \text{s}_j \text{s}_i \text{s}_j \quad (\text{if } \alpha_i \cdot \alpha_j = 1),
\end{align*}
\]

\[
\begin{align*}
\iota^2 &= \text{id}, \quad \iota \cdot s^0_{0,1,2,3,4,5,6,7} = s^0_{0,1,2,3,4,5,6,7} \iota.
\end{align*}
\]

We can also verify that the action of \( \text{Cr}(E_7^{(1)})^{\text{op}} \) on \( D(K) \) induced by this map is given by the pull-back of functions

\[
\text{s}_i(z^\alpha_i) = z^{s_i(\alpha_i)}, \quad \iota(z^\alpha) = z^{-\iota(\alpha)}.
\]

This in particular implies that the group homomorphism \( \text{Cr}(E_7^{(1)})^{\text{op}} \to \Gamma_1 \) is injective. The remaining statement follows from \( s_i(z^\delta) = z^\delta \) and \( \iota(z^\delta) = z^{\delta} \).

Consequently, we have the actions

\[
\begin{align*}
\text{Cr}(R^\perp)^{\text{op}} &\to \text{Aut}(X_1), \quad \text{Cr}(R^\perp)^{\text{op}} \to \text{Aut}(D(K))
\end{align*}
\]

such that the natural map \( \lambda : X_1 \to D(K) \) is equivariant with respect to this action. The map \( \lambda : X_1 \to D(K) \) can be essentially thought of as a family of surfaces obtained by blowup a collection of points on the boundary of a toric variety, and deleting the proper transform of the boundary (Remark 3.9). The action of the group of Cremona isometries on the cluster Poisson variety yields \( q \)-Painlevé systems as in [Sak01].

**Example 3.36.** Let \( \beta \) be the blowup data of type \( E_6^{(1)} \) in Appendix A. Using the embedding \( \text{Cr}(E_6^{(1)})^{\text{op}} \to \Gamma_1 \) in Appendix A, we define two seed cluster transformations \( \textbf{c}_1, \textbf{c}_2 \in \Gamma_1 \) by

\[
\begin{align*}
\textbf{c}_1 &= (\iota_3 \circ s_0 \circ s_1 \circ (s_4 \circ s_5 \circ s_3)^2)^*, \\
\textbf{c}_2 &= (\iota_1 \circ s_0 \circ s_5 \circ (s_0 \circ s_1 \circ s_2)^2)^*,
\end{align*}
\]

where \( \iota_3^* := (\iota_2 \circ \iota_1 \circ \iota_3)^* = -(3,7)(4,8) \). It is easy to verify that these are involutions. The discrete dynamical system on the cluster variety \( X_1 \) obtained by alternately iterating \( \textbf{c}_1 \) and \( \textbf{c}_2 \) is the sixth \( q \)-Painlevé system \( (q-P_{VI}) \) [JS96, Sak01]. We see this by writing explicit expressions for these two actions on an appropriate coordinate system. Let \( \alpha_i = z^{\alpha_i}, \quad q = z^3 = a_0 a_1 a_2 a_3 a_4 a_5 f = z^{(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5)/4}, \) and

\[
\begin{align*}
g &= z^{(\alpha_3 + \alpha_4 - \alpha_5 + \alpha_6 - \alpha_7 + \alpha_8)/4}.
\end{align*}
\]

Note that \( f, g \in \mathbb{C}[N] \), so these are functions on the initial affine chart of \( X_1 \). On the other hand, \( a_0, \ldots, a_5 \) and \( q \) are globally defined functions on \( X_1 \). In terms of these coordinates, the actions of \( \textbf{c}_1 \) and \( \textbf{c}_2 \) on the initial affine chart of \( X_1 \) are given by

\[
\begin{align*}
\mathcal{X}(\textbf{c}_1)^*: (a_0, a_1, a_2, a_3, a_4, a_5; f, g; q) &\mapsto (a_0, a_1, q^{-1} a_2, a_3, a_4, a_5; f, g; q^{-1}), \\
\mathcal{X}(\textbf{c}_2)^*: (a_0, a_1, a_2, a_3, a_4, a_5; f, g; q) &\mapsto (a_0, a_1, a_2, q^{-1} a_3, a_4, a_5; f, g; q^{-1}),
\end{align*}
\]
where \( \overline{f} := \mathcal{X}(c_1)^* f \) and \( \underline{g} := \mathcal{X}(c_2)^* g \), and these satisfy the following relations:

\[
\begin{align*}
\overline{f} \overline{f} &= b_7 b_8 \frac{(g + b_3)(g + b_4)}{(g + b_7)(g + b_8)}, \\
\underline{g} \underline{g} &= b_1 b_2 \frac{(f + b_5)(f + b_6)}{(f + b_1)(f + b_2)},
\end{align*}
\]

where

\[
\begin{align*}
b_1 &:= z^{-e_1} f = (a_0 a_1^{-1} a_2^{-2})^{1/4}, \\
b_2 &:= z^{-e_2} f = (a_0^{-3} a_1^{-1} a_2^{-2})^{1/4}, \\
b_3 &:= z^{-e_3} g = (a_3^{-2} a_4 a_5^{-1})^{1/4}, \\
b_4 &:= z^{-e_4} g = (a_3^{-2} a_4^{-3} a_5^{-1})^{1/4}, \\
b_5 &:= z^{e_5} f = (a_0 a_1 a_2^2)^{1/4}, \\
b_6 &:= z^{e_6} f = (a_0 a_1^3 a_2^3)^{1/4}, \\
b_7 &:= z^{e_7} g = (a_3 a_4 a_5^{-1})^{1/4}, \\
b_8 &:= z^{e_8} g = (a_3^2 a_4 a_5^3)^{1/4}.
\end{align*}
\]

The system of relations (3.9) and (3.10) is called the \textit{sixth q-Painlevé equation} \cite{js96}.

\section{Cluster data for \(q\)-Painlevé equations}

In this Appendix, we provide blowup data associated with \(q\)-Painlevé systems. The meaning of these data is explained especially in the proof of Theorem 3.31 and 3.35. We also give a quiver for each type \(R^\perp\), which is the directed graph whose signed adjacency matrix is \((w_i \wedge w_j)_{i,j \in \mathcal{J}}\). When we draw a quiver, we enclose vertices that have the same rows in the signed adjacency matrix. Note that there are infinitely many seeds for each type \(R^\perp\), which are related by seed cluster transformations, and it seems that there is no canonical choice of a representative. The seeds given in this appendix are chosen so that the associated quivers are relatively simple.

\textbf{Type} \(E_8^{(1)}\)

\textbf{Fan, seed, and quiver}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{quiver.png}
\caption{Quiver for \(E_8^{(1)}\) type.}
\end{figure}

\textbf{Root data}

\[
\begin{align*}
\alpha_0 &= e_6 - e_5, & \alpha_1 &= e_5 - e_4, & \alpha_2 &= e_4 - e_3, & \alpha_3 &= e_3 - e_2, & \alpha_4 &= e_2 - e_1, \\
\alpha_5 &= e_1 + e_8, & \alpha_6 &= e_9 - e_8, & \alpha_7 &= e_9 - e_9, & \alpha_8 &= e_7 + e_11, \\
\delta &= \alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 6\alpha_6 + 2\alpha_7 + 3\alpha_8 \\
&= e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + 3e_7 + 2e_8 + 2e_9 + 2e_{10} + 3e_{11}.
\end{align*}
\]

\[
\begin{align*}
\alpha_0 \mapsto E_5 - E_6, & \quad \alpha_1 \mapsto E_4 - E_5, & \quad \alpha_2 \mapsto E_3 - E_4, & \quad \alpha_3 \mapsto E_2 - E_3, & \quad \alpha_4 \mapsto E_1 - E_2, \\
\alpha_5 \mapsto \pi^* D_2 - E_1 - E_8, & \quad \alpha_6 \mapsto E_8 - E_9, & \quad \alpha_7 \mapsto E_9 - E_{10}, & \quad \alpha_8 \mapsto \pi^* D_1 - E_7 - E_{11}, \\
\delta \mapsto D_1 + 3D_2 + 2D_3 + 3D_4.
\end{align*}
\]

\[
\begin{align*}
\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6 - \alpha_7 - \alpha_8 &= 0, \\
\alpha_i^2 &= -2 & \text{for any } i.
\end{align*}
\]
Embedding \( \text{Cr}(R^\perp)^{\text{op}} \to \Gamma_1 \)

\[
\text{Cr}(E_8^{(1)}) = \langle s_0, \ldots, s_8 \rangle \cong W(E_8^{(1)}),
\]

\[
s_0^* = (5, 6), \quad s_1^* = (4, 5), \quad s_2^* = (3, 4), \quad s_3^* = (2, 3), \quad s_4^* = (1, 2), \quad s_5^* = \mu_1^- \circ (1, 8) \circ \mu_1^+, \quad s_6^* = (8, 9), \quad s_7^* = (9, 10), \quad s_8^* = \mu_7^- \circ (7, 11) \circ \mu_7^+.
\]

Action on \( D(K) \)

\[
s_i(z^\alpha) = z^{s_i(\alpha)}.
\]

Type \( E_7^{(1)} \)

Fan, seed, and quiver

Root data

\[
\begin{align*}
\alpha_0 &= e_4 - e_3, & \alpha_1 &= e_3 - e_2, & \alpha_2 &= e_2 - e_1, & \alpha_3 &= e_1 + e_6, \\
\alpha_4 &= e_5 + e_8, & \alpha_5 &= e_9 - e_8, & \alpha_6 &= e_{10} - e_9, & \alpha_7 &= e_7 - e_6, \\
\delta &= \alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7 \\
&= e_1 + e_2 + e_3 + e_4 + 3e_5 + 2e_6 + 2e_7 + e_8 + e_9 + e_{10}.
\end{align*}
\]

\[
\begin{align*}
\alpha_0 &\mapsto E_3 - E_4, & \alpha_1 &\mapsto E_2 - E_3, & \alpha_2 &\mapsto E_1 - E_2, & \alpha_3 &\mapsto \pi^*D_2 - (E_1 - E_6), \\
\alpha_4 &\mapsto \pi^*D_1 - E_5 - E_8, & \alpha_5 &\mapsto E_8 - E_9, & \alpha_6 &\mapsto E_9 - E_{10}, & \alpha_7 &\mapsto E_6 - E_7, \\
\delta &\mapsto D_1 + 3D_2 + 2D_3 + D_4.
\end{align*}
\]

\[
\begin{align*}
\alpha_7 \\
\alpha_0 &- \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6 \\
\alpha_i^2 &= -2 \quad \text{for any } i.
\end{align*}
\]

Embedding \( \text{Cr}(R^\perp)^{\text{op}} \to \Gamma_1 \)

\[
\text{Cr}(E_7^{(1)}) = \langle s_0, \ldots, s_7 \rangle \rtimes \langle \iota \rangle \cong W(E_7^{(1)}) \rtimes \text{Aut}(E_7^{(1)}),
\]

\[
s_0^* = (3, 4), \quad s_1^* = (2, 3), \quad s_2^* = (1, 2), \quad s_3^* = \mu_1^- \circ (1, 6) \circ \mu_1^+, \quad s_4^* = \mu_5^- \circ (5, 8) \circ \mu_5^+, \quad s_5^* = (8, 9), \quad s_6^* = (9, 10), \quad s_7^* = (6, 7), \quad \iota^* = \mu_5^- \circ -(1, 5)(2, 8)(3, 9)(4, 10) \circ \mu_5^+.
\]

Action on \( D(K) \)

\[
s_i(z^\alpha) = z^{s_i(\alpha)}, \quad \iota(z^\alpha(0,1,2,3,4,5,6,7)) = z^{-\alpha(6,5,4,3,2,1,0,7)}.
\]
Type $E_6^{(1)}$

Fan, seed, and quiver

Root data

$$\alpha_0 = e_9 - e_8, \quad \alpha_1 = e_3 - e_2, \quad \alpha_2 = e_2 - e_1, \quad \alpha_3 = e_1 + e_5,$$
$$\alpha_4 = e_6 - e_5, \quad \alpha_5 = e_7 - e_6, \quad \alpha_6 = e_4 + e_8,$$
$$\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$$
$$= e_1 + e_2 + e_3 + 2e_4 + e_5 + e_6 + e_7 + e_8 + e_9.$$

$$\alpha_0 \mapsto E_8 - E_9, \quad \alpha_1 \mapsto E_2 - E_3, \quad \alpha_2 \mapsto E_1 - E_2, \quad \alpha_3 \mapsto E_t^* D_2 - E_1 - E_5,$$
$$\alpha_4 \mapsto E_5 - E_6, \quad \alpha_5 \mapsto E_6 - E_7, \quad \alpha_6 \mapsto E_t^* D_1 - E_4 - E_8,$$
$$\delta \mapsto D_1 + 2D_2 + D_3 + D_4.$$

$$\alpha_0$$
$$\quad \alpha_6$$
$$\quad \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$$

$$\alpha_i^2 = -2 \quad \text{for any } i.$$

Embedding $Cr(R^\perp)^{\text{op}} \to \Gamma_1$

$$Cr(E_6^{(1)}) = \{s_0, \ldots, s_6\} \times \langle \nu_1, \nu_2 \rangle \cong W(E_6^{(1)}) \rtimes \text{Aut}(E_6^{(1)}),$$

$$s_0^* = (8,9), \quad s_1^* = (2,3), \quad s_2^* = (1,2), \quad s_3^* = \mu_1^* \circ (1,5) \circ \mu_4^+, \quad s_4^* = (5,6), \quad s_5^* = (6,7), \quad s_6^* = \mu_4^* \circ (4,8) \circ \mu_4^+, \quad s_7^* = -(1,5)(2,6)(3,7), \quad s_8^* = \mu_4^* \circ -(1,4)(2,8)(3,9) \circ \mu_4^+.$$

Action on $D(K)$

$$s_i(z^{\alpha_j}) = z^{s_i(\alpha_j)}, \quad \nu_1(z^{\alpha_{(0,1,2,3,4,5,6)}}) = z^{-\alpha_{(0,5,4,3,2,1,6)}}, \quad \nu_2(z^{\alpha_{(0,1,2,3,4,5,6)}}) = z^{-\alpha_{(1,0,6,3,4,5,2)}}.$$

Type $E_5^{(1)}$

Fan, seed, and quiver
Root data

\[
\begin{align*}
\alpha_0 &= e_2 - e_1, \quad \alpha_1 = e_6 - e_5, \quad \alpha_2 = e_4 + e_3, \quad \alpha_3 = e_4 + e_7, \quad \alpha_4 = e_4 - e_3, \quad \alpha_5 = e_8 - e_7, \\
\delta &= \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 = e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8.
\end{align*}
\]

\[
\begin{align*}
\alpha_0 &\mapsto E_1 - E_2, \quad \alpha_1 &\mapsto E_5 - E_6, \quad \alpha_2 &\mapsto \pi^* D_2 - E_1 - E_5, \\
\alpha_3 &\mapsto \pi^* D_1 - E_3 - E_7, \quad \alpha_4 &\mapsto E_3 - E_4, \quad \alpha_5 &\mapsto E_7 - E_8, \\
\delta &\mapsto D_1 + D_2 + D_3 + D_4.
\end{align*}
\]

\begin{align*}
\alpha_0 + \alpha_5 &
\begin{array}{|c|c|}
\hline
\alpha_1 & \alpha_2 \\
\hline
\alpha_3 & \alpha_4 \\
\hline
\end{array}
\end{align*}

\[
\alpha_i^2 = -2 \quad \text{for any } i.
\]

Embedding \( \text{Cr}(R^\perp)^{\text{op}} \to \Gamma_1 \)

\[
\text{Cr}(E_5^{(1)}) = \langle s_0, \ldots, s_5 \rangle \rtimes \langle \iota_1, \iota_2 \rangle \cong W(D_5^{(1)}) \rtimes \text{Aut}(D_5^{(1)}),
\]

\[
\begin{align*}
 s_0 &= (1, 2), \quad s_1 = (5, 6), \quad s_2 = \mu_1 \circ (1, 5) \circ \mu_3^+, \\
 s_3 &= \mu_2 \circ (3, 7) \circ \mu_3^+, \quad s_4 = (3, 4), \quad s_5 = (7, 8), \\
 \iota_1 &= -(1, 5)(2, 6), \quad \iota_2 = -(1, 3)(2, 4)(5, 7)(6, 8).
\end{align*}
\]

Action on \( D(K) \)

\[
 s_i(z^{\alpha_{s}}) = z^{s_i(\alpha_{s})}, \quad \iota_1(z^{\alpha_{(0, 1, 2, 3, 4, 5)}}) = z^{-\alpha_{(0, 1, 2, 3, 4, 5)}}, \quad \iota_2(z^{\alpha_{(0, 1, 2, 3, 4, 5)}}) = z^{-\alpha_{(4, 5, 3, 2, 0, 1)}}.
\]

Type \( E_6^{(1)} \)

Fan, seed, and quiver

Root data

\[
\begin{align*}
\alpha_0 &= e_2 + e_4 + e_6, \quad \alpha_1 = e_5 - e_4, \quad \alpha_2 = e_1 + e_4, \quad \alpha_3 = e_3 + e_6, \quad \alpha_4 = e_7 - e_6, \\
\delta &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7.
\end{align*}
\]

\[
\begin{align*}
\alpha_0 &\mapsto \pi^*(D_1 + \tilde{D}_2 + \tilde{D}_3) - E_2 - E_4 - E_6, \quad \alpha_1 &\mapsto E_4 - E_5, \\
\alpha_2 &\mapsto \pi^* D_5 - E_1 - E_4, \quad \alpha_3 &\mapsto \pi^* D_4 - E_3 - E_6, \quad \alpha_4 &\mapsto E_6 - E_7, \\
\delta &\mapsto D_1 + D_2 + D_3 + D_4 + D_5.
\end{align*}
\]

\begin{align*}
\alpha_0 &
\begin{array}{|c|c|}
\hline
\alpha_1 & \alpha_2 \\
\hline
\alpha_3 & \alpha_4 \\
\hline
\end{array}
\end{align*}

\[
\alpha_i^2 = -2 \quad \text{for any } i.
\]
Embedding $\text{Cr}(R^{\perp})^{\text{op}} \to \Gamma_1$

$$\text{Cr}(E^{(1)}_4) = \langle s_0, \ldots, s_4 \rangle \rtimes \langle \iota_1, \iota_2 \rangle \cong W(A^{(1)}_4) \rtimes \text{Aut}(A^{(1)}_4),$$

$$s_0^* = \mu_2^+ \circ \mu_4^+ \circ (4, 6) \circ \mu_2^+ \circ \mu_4^+, \quad s_1^* = (4, 5),$$

$$s_2^* = \mu_2^+ \circ (1, 4) \circ \mu_4^+, \quad s_3^* = \mu_3^+ \circ (3, 6) \circ \mu_3^+, \quad s_4^* = (6, 7),$$

$$\iota_1 = (1, 7, 5, 3, 2)(4, 6) \circ \mu_4^+, \quad \iota_2 = -(1, 3)(4, 6)(5, 7).$$

Action on $D(K)$

$$s_i(z^{\alpha_j}) = z^{s_i(\alpha_j)}, \quad \iota_1(z^{\alpha_{(0, 1, 2, 3, 4)}}) = \varepsilon^{\alpha_{(3, 4, 0, 1, 2)}}, \quad \iota_2(z^{\alpha_{(0, 1, 2, 3, 4)}}) = \varepsilon^{-\alpha_{(0, 4, 3, 2, 1)}}.$$

Type $E^{(1)}_3$

Fan, seed, and quiver

Root data

$$\alpha_0 = e_1 + e_4, \quad \alpha_1 = e_2 + e_5, \quad \alpha_2 = e_3 + e_6,$$

$$\alpha_3 = e_1 + e_3 + e_5, \quad \alpha_4 = e_2 + e_4 + e_6,$$

$$\delta = \alpha_0 + \alpha_1 + \alpha_2 = \alpha_3 + \alpha_4 = e_1 + e_2 + e_3 + e_4 + e_5 + e_6.$$

$$\alpha_0 \mapsto \varepsilon^*(\tilde{D}_5 + \tilde{D}_6) - E_1 - E_4, \quad \alpha_1 \mapsto \varepsilon^*(\tilde{D}_3 + \tilde{D}_4) - E_2 - E_5, \quad \alpha_2 \mapsto \varepsilon^*(\tilde{D}_1 + \tilde{D}_2) - E_3 - E_6,$$

$$\alpha_3 \mapsto \varepsilon^*(\tilde{D}_4 + \tilde{D}_5 + \tilde{D}_6) - E_1 - E_3 - E_5, \quad \alpha_4 \mapsto \varepsilon^*(\tilde{D}_1 + \tilde{D}_2 + \tilde{D}_3) - E_2 - E_4 - E_6,$$

$$\delta \mapsto D_1 + D_2 + D_3 + D_4 + D_5 + D_6.$$

$$\begin{array}{ccc}
\alpha_0 & \\
\alpha_1 & \alpha_2 & \alpha_3 \quad 2 \quad \alpha_4 \\
\alpha_i^2 & = & -2 \quad \text{for any } i.
\end{array}$$

Embedding $\text{Cr}(R^{\perp})^{\text{op}} \to \Gamma_1$

$$\text{Cr}(E^{(1)}_3) = \langle s_0, \ldots, s_4 \rangle \rtimes \langle \iota_1, \iota_2 \rangle \cong W((A_2 + A_1)^{(1)}) \rtimes \text{Aut}((A_2 + A_1)^{(1)})$$

$$s_0^* = \mu_2^+ \circ (1, 4) \circ \mu_4^+, \quad s_1^* = \mu_2^+ \circ (2, 5) \circ \mu_2^+, \quad s_2^* = \mu_3^+ \circ (3, 6) \circ \mu_3^+, \quad s_3^* = \mu_3^+ \circ (3, 5) \circ \mu_3^+) \circ \mu_4^+, \quad s_4^* = \mu_2^+ \circ (4, 6) \circ \mu_4^+ \circ \mu_2^+, \quad s_5^* = (1, 2, 3, 4, 5, 6), \quad \iota_2 = -(1, 4)(2, 3)(5, 6).$$

Action on $D(K)$

$$s_i(z^{\alpha_j}) = z^{s_i(\alpha_j)}, \quad \iota_1(z^{\alpha_{(0, 1, 2, 3, 4)}}) = \varepsilon^{\alpha_{(2, 0, 1, 4, 3)}}, \quad \iota_2(z^{\alpha_{(0, 1, 2, 3, 4)}}) = \varepsilon^{-\alpha_{(0, 2, 1, 4, 3)}}.$$
Type $E_2^{(1)}$

Fan, seed, and quiver

Root data

\[
\begin{align*}
\alpha_0 &= e_1 + e_3 + e_4, \quad \alpha_1 = e_2 + e_5, \\
\alpha_2 &= e_1 + 3e_3 - e_4 + 2e_5, \quad \alpha_3 = e_2 - 2e_3 + 2e_4 - e_5, \\
\delta &= \alpha_0 + \alpha_1 = \alpha_2 + \alpha_3 = e_1 + e_2 + e_3 + e_4 + e_5
\end{align*}
\]

\[
\begin{align*}
\alpha_0 &\mapsto \pi^*(\bar{D}_2 + 2\bar{D}_3 + \bar{D}_4) - E_1 - E_3 - E_4, \\
\alpha_1 &\mapsto \pi^*(\bar{D}_4 + \bar{D}_5) - E_2 - E_5, \\
\alpha_2 &\mapsto \pi^*(\bar{D}_2 + 2\bar{D}_3 + \bar{D}_4 + 2\bar{D}_5) - E_1 - 3E_3 + E_4 - 2E_5, \\
\alpha_3 &\mapsto \pi^*(\bar{D}_4 - \bar{D}_5) - E_2 + 2E_3 - 2E_4 + E_5, \\
\delta &\mapsto D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7.
\end{align*}
\]

\[
\begin{align*}
\alpha_0 &\mapsto 2\alpha_1 \\
\alpha_2 &\mapsto 14\alpha_3
\end{align*}
\]

\[
\begin{align*}
\alpha_4^2 &= \alpha_2^2 = -2, \quad \alpha_3^2 = \alpha_4^2 = -14.
\end{align*}
\]

Embedding $\text{Cr}(R^\perp)^{op} \to \Gamma_1$

\[
\text{Cr}(E_2^{(1)}) = \langle s_0, s_1 \rangle \times \langle \tau \rangle \times \langle \iota \rangle \cong W(A_1^{(1)}) \times \mathbb{Z} \times S_2,
\]

\[
s_0 = \mu_1^- \circ \mu_3^- \circ (3, 4) \circ \mu_2^+ \circ \mu_1^+, \quad s_1 = \mu_2^- \circ (2, 5) \circ \mu_3^+,
\]

\[
\tau^* = (1, 5, 3, 4, 2) \circ \mu_4^+,
\]

\[
\iota^* = (2, 5)(3, 4) \circ \tau^*.
\]

Action on $D(K)$

\[
\begin{align*}
s_i(z^{\alpha_j}) &= z^{s_i(\alpha_j)},
\tau(z^{\alpha(0, 1)}) &= z^{\alpha(1, 0)},
\tau(z^{\alpha(2)}) &= z^{\alpha + \delta},
\tau(z^{\alpha(3)}) &= z^{\alpha - \delta},
\iota(z^{\alpha(0, 1)}) &= z^{-\alpha(1, 0)},
\iota(z^{\alpha(2, 3)}) &= z^{-\alpha(3, 2)}.
\end{align*}
\]

Type $E_3^{(1)}$

Fan, seed, and quiver
Root data

\[ \alpha_0 = e_1 + 2e_3 - e_4, \quad \alpha_1 = e_2 - e_3 + 2e_4, \]
\[ \delta = \alpha_0 + \alpha_1 = e_1 + e_2 + e_3 + e_4. \]
\[ \alpha_0 \mapsto \pi^*(2\tilde{D}_7 + \tilde{D}_8) - E_1 - 2E_3 + E_4, \quad \alpha_1 \mapsto \pi^*(\tilde{D}_5 + 2\tilde{D}_6) - E_2 + E_3 - 2E_4, \]
\[ \delta \mapsto D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8. \]

\[ \alpha_0 \overset{8}{\mapsto} \alpha_1 \]
\[ \alpha_0^2 = \alpha_1^2 = -8. \]

Embedding \( \text{Cr}(R^\perp)^{\text{op}} \to \Gamma_1 \)

\[ \text{Cr}(E_1^{(1)}) = (\tau) \times (\iota) \cong \mathbb{Z} \times S_2, \]
\[ \tau^* = (1, 3, 4, 2) \circ \mu^+_3; \quad \iota^* = -(1, 2)(3, 4). \]

Action on \( D(K) \)

\[ \tau(z^{\alpha_0}) = z^{\alpha_0 + \delta}, \quad \tau(z^{\alpha_1}) = z^{\alpha_1 - \delta}, \quad \iota(z^{\alpha_{(0,1)}}) = z^{-\alpha_{(1,0)}}. \]

Type \( E_1^{(1)'} \)

Fan, seed, and quiver

\[ \begin{align*}
\alpha_0 &= e_1 + e_3, \quad \alpha_1 = e_2 + e_4, \\
\delta &= \alpha_0 + \alpha_1 = e_1 + e_2 + e_3 + e_4.
\end{align*} \]
\[ \begin{align*}
\alpha_0 \mapsto \pi^*(\tilde{D}_2 + 2\tilde{D}_3 + \tilde{D}_4) - E_1 - E_3, \quad \alpha_1 \mapsto \pi^*(\tilde{D}_4 + 2\tilde{D}_5 + \tilde{D}_6) - E_2 - E_4, \\
\delta \mapsto D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8.
\end{align*} \]
\[ \begin{align*}
\alpha_0 \overset{2}{\mapsto} \alpha_1 \\
\alpha_0^2 = \alpha_1^2 = -2.
\end{align*} \]

Embedding \( \text{Cr}(R^\perp)^{\text{op}} \to \Gamma_1 \)

\[ \text{Cr}(E_1^{(1)'}) = \langle s_0, s_1 \rangle \times \langle \iota_1, \iota_2 \rangle \cong W(A_1^{(1)}) \times \mathcal{D}_4, \]
\[ s_0^* = \mu_1^+ \circ (1, 3) \circ \mu_1^+, \quad s_1^* = \mu_2^+ \circ (2, 4) \circ \mu_2^+, \quad \iota_1^* = (1, 2, 3, 4), \quad \iota_2^* = -(1, 3). \]
Action on $D(K)$

$$s_i(z^{a_j}) = z^{s_i(a_j)}, \quad \iota_1(z^{a(0,1)}) = z^{a(1,0)}, \quad \iota_2(z^{a(0,1)}) = z^{-a(0,1)}.$$ 

Type $E_6^{(1)}$

Fan, seed, and quiver

Root data

$$\alpha_0 = \delta = e_1 + e_2 + e_3.$$ 

$$\alpha_0, \delta \mapsto D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8 + D_9.$$ 

$$\alpha_0^2 = 0.$$ 

Embedding $\text{Cr}(R^\perp)^{\text{op}} \to \Gamma_1$

$$\text{Cr}(E_6^{(1)}) = \langle \iota_1, \iota_2 \rangle \cong D_3,$$ 

$$\iota_1^* = (1, 2, 3), \quad \iota_2^* = -(1, 2).$$ 

Action on $D(K)$

$$\iota_1(z^{a_0}) = z^{a_0}, \quad \iota_2(z^{a_0}) = z^{-a_0}.$$ 

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