Turbulent magnetized gas is ubiquitous in astrophysics. In a single system (e.g., stellar and planetary atmospheres, interstellar medium, etc.), the physical quantities, like density, gas pressure, entropy and temperature can vary by orders of magnitude. Let’s take our closest star, the Sun, as an example. It has several distinct regions. In the core, energy is generated by nuclear fusion and diffused outward by radiation (gamma and X-rays). Soon the temperature drops sufficiently enough for ions to recombine, which in turn increase the opacity of the plasma and the radiative diffusion can no longer efficiently transport the energy outwards and the plasma starts to ‘boil’. Here, the convective zone begins. The transition between the radiative core and the convective zone is rather narrow, on the order of 10 Mm (Elliott & Gough 1999) and is called the tachocline. The convective zone spans roughly 200 Mm and makes up the outer 30% of the solar interior. The energy flux causes the plasma to swirl in turbulent eddies and spirals called convective flows. The convective motions are seen at the solar surface (the photosphere) as granules and supergranules. The convective turnover timescale varies by 4 orders of magnitude - near the tachocline is of the order of 25 – 30 days; and near the photosphere it is approximately 300 seconds. Above the photosphere, in the chromosphere, the thermal energy of the gas no longer dominates the total energy budget, plasma β drops below unity, indicating, that magnetic pressure becomes higher than the gas pressure, and the gas flows must turn the magnetic field configuration. Even higher up, the gas temperature suddenly goes back up to millions degrees Kelvin. The physical quantities, like density, gas pressure, temperature, varies by many orders of magnitude from the deep interior all the way up to the outermost parts of the Sun. However, in the convective zone, radially averaged entropy per unit mass is nearly constant.

With the advent of exascale computing we are able to run numerical simulations of increasing complexity - either covering larger range of physical and temporal scales, or including more physics, like radiative heat transfer, conductivity, chemistry, non-local thermal equilibrium, generalized Ohm’s law, etc. It requires a very flexible solver for the fluid equations of conservation to deal with a large range of physical conditions. The DISPATCH framework (Norland et al. 2018) can use several different solvers for magnetohydrodynamics (MHD), we currently use STAGGER-based and Riemann solver groups:

- an internal energy based STAGGER solver taken from the Bifrost (Gudiksen et al. 2011) code, which then was adapted to work well with local timesteps;
- an entropy based STAGGER solver, used in (Popovas et al. 2018, 2019);
- several Godunov-type Riemann solvers:
  - HLL, HLLC and HLLD solvers, based on ones from the public domain RAMSES code (Teyssier 2002; Teyssier et al. 2006; Fromang et al. 2006)
  - Roe solver based on works by Winters & Gassner (2016); Derigs et al. (2016)

Both of these solver groups have their own benefits and limitations. The first, STAGGER group, is good under conditions where magnetic fields are dominating the energy budget, as they were developed with upper Solar atmosphere and corona in mind. On the other hand, they are very dissipative and perform poorly in low Mach number conditions. The Riemann group of
solv...
The system in \[ \text{[3]} \] can be written in vectorial form, similar to e.g. \[ \text{[4]} \].

\[
\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = \Psi,
\]

where

\[
U = (\rho, \rho u_x, \rho u_y, \rho u_z, \rho S)^T,
\]

\[
\Psi = (0, -\Phi_x, -\Phi_y, -\Phi_z, -Q/T_{\text{gas}} - S_{\text{gen}})^T,
\]

and

\[
F = \begin{pmatrix}
\rho u_x \\
\rho u_x^2 + \rho P + B_x^2 \\
\rho u_x u_y - B_x B_y \\
\rho u_x u_z - B_x B_z \\
\rho u_y S
\end{pmatrix}
\]

is the flux function. The expressions for the terms \( G \) and \( H \) are completely analogous. These equations have seven eigenvalues, corresponding to four magneto-acoustic (two slow and two fast), two Alfvén waves and an entropy wave:

\[
\lambda_{1,7} = u \mp c_f, \quad \lambda_{3,5} = u \mp c_s, \quad \lambda_{2,6} = u \mp a, \quad \lambda_4 = u,
\]

where

\[
c_f = \pm \sqrt{d^2 - \alpha B^2/\rho}, \quad c_s = \frac{|B_1|}{\sqrt{\rho}}
\]

with

\[
d = \frac{a^2 + |B|^2}{2},
\]

and \( a \) is the speed of sound. For convenience, similarly to \[ \text{[5]} \] and \[ \text{[6]} \] we define \( \alpha_f \) and \( \alpha_s \) parameters, which measure how closely the fast/slow waves approximate the behavior of acoustic waves \[ \text{[7]} \].

\[
\alpha_f^2 = \frac{a^2 - c_s^2}{c_f^2 - c_s^2}, \quad \alpha_s^2 = \frac{c_f^2 - a^2}{c_f^2 - c_s^2},
\]

and they have useful properties, e.g. \( \alpha_f^2 + \alpha_s^2 = 1 \), \( \alpha_f^2 c_f^2 + \alpha_s^2 c_s^2 = a^2 \). These \( \alpha \) values will become relevant when we will get to the \( S_{\text{gen}} \).

More often than not some eigenvalues in \[ \text{[8]} \] coincide,

\[
\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \lambda_5 \leq \lambda_6,
\]

depending on the direction and strength of the magnetic field. A solution to the Riemann problem then may be composed not only of ordinary shock and rarefaction waves, but also compound waves \[ \text{[9]} \] and \[ \text{[10]} \].

### 3.1. The numerical method

In Godunov-type schemes the volume-averaged conservative physical quantities at a next time-step are given by integrating an approximate solution to the Riemann problem with left and right states, \( U_L \) and \( U_R \) at the cell interface. An HLL Riemann solver \[ \text{[11]} \] is constructed by assuming an average intermediate state between the fastest and slowest waves. The information is lost, as slow magneto-acoustic waves are merged together with Alfvén and entropy waves. An HLLC \[ \text{[12]} \] solver expands and estimates the middle wave of the contact surface. Lastly, HLLD \[ \text{[13]} \] expands into four intermediate states. We start with computing the HLL wave speed. Usually the wave speeds are notated by \( S_L \) and \( S_R \), but to avoid the confusion with entropy, we will note them with \( \Sigma_L \) and \( \Sigma_R \):

\[
\Sigma_L = \min(u_l, u_r) - \max(c_{f,l}, c_{f,r}) \quad \Sigma_R = \max(u_l, u_r) + \max(c_{f,l}, c_{f,r})
\]

we define the Lagrangian sound speed,

\[
u_L = u_L - \Sigma_L \quad \nu_R = \Sigma_R - u_r,
\]

and then compute the acoustic star:

\[
\nu^*_L = \frac{\rho L u_L + \rho L u_L + (P_{\text{tot}, L} - P_{\text{tot}, R})}{\rho L u_L + \rho L u_L}
\]

\[
\nu^*_R = \frac{\rho R u_R + \rho R u_R + (P_{\text{tot}, L} - P_{\text{tot}, R})}{\rho R u_R + \rho R u_R}
\]

The left star region variables are

\[
\rho^*_L = -\frac{\rho L u_L}{\Sigma_L - \nu^*_L},
\]

\[
\nu^*_L = \frac{\nu_L - A B L (u^*_L - u_L)}{e^*_L},
\]

\[
\nu^*_L = \frac{B L e^*_L}{e^*_L},
\]

\[
C^*_L = \frac{C L e^*_L}{e^*_L},
\]

where \( \nu_L \) and \( \nu_R \) are left tangential velocities, \( A = A_L = A_L \) is the normal component of the magnetic field, \( B_L \) and \( C_L \) are the left states of the tangential components of the magnetic field,

\[
e^*_L = -\frac{\rho L u_L}{\Sigma_L - \nu^*_L} - A^2,
\]

and

\[
e^*_L = -\frac{\rho L u_L}{\Sigma_L - \nu^*_L} - A^2.
\]
Entropy is updated as a passive scalar:

\[ S_0^* = -\frac{\mu_L(S_l - S_0)}{\Sigma_R - u^*} + S_0. \]

Left state Alfvén wave speed is

\[ \Sigma_{a,l} = u^* + \frac{|A|}{\sqrt{\rho^*_l}}. \]

Correspondingly, the right region variables are:

\[ \rho^*_r = -\frac{\rho_r v_r}{\Sigma_R - u^*}, \]
\[ v^*_r = \frac{v_r - AB_r(u^* - u_l)}{e_r^*}, \]
\[ w^*_r = \frac{w_r - AC_r(u^* - u_l)}{e_r^*}, \]
\[ B^*_r = B_r e_r, \]
\[ C^*_r = C_r e_r, \]

where \( v_r \) and \( w_r \) are right tangential velocities, \( A \) is the normal component of the magnetic field, \( B_r \) and \( C_r \) are the tangential components of the magnetic field,

\[ e_r^* = -\frac{\rho_r v_r}{\Sigma_R - u^*} - A^2, \]

and

\[ e_r = -\frac{\rho_r v_r}{\Sigma_R - u_l} - A^2. \]

Right state Alfvén wave speed is

\[ \Sigma_{a,r} = u^* + \frac{|A|}{\sqrt{\rho^*_r}}. \]

Lastly, the double star region variables are:

\[ v^{**} = \frac{v^*_l \sqrt{\rho^*_l} + v^*_r \sqrt{\rho^*_r} + \Xi(B^*_l - B^*_r)}{\sqrt{\rho^*_l} + \sqrt{\rho^*_r}} , \]
\[ w^{**} = \frac{w^*_l \sqrt{\rho^*_l} + w^*_r \sqrt{\rho^*_r} + \Xi(C^*_l - C^*_r)}{\sqrt{\rho^*_l} + \sqrt{\rho^*_r}} , \]
\[ B^{**} = \frac{B^*_l \sqrt{\rho^*_l} + B^*_r \sqrt{\rho^*_r} + \Xi \rho^*_l \sqrt{\rho^*_r} (v^*_l - v^*_r)}{\sqrt{\rho^*_l} + \sqrt{\rho^*_r}} , \]
\[ C^{**} = \frac{C^*_l \sqrt{\rho^*_l} + C^*_r \sqrt{\rho^*_r} + \Xi \rho^*_l \sqrt{\rho^*_r} (w^*_l - w^*_r)}{\sqrt{\rho^*_l} + \sqrt{\rho^*_r}} , \]

where \( \Xi \) is a sign function, \( \Xi = \text{sign}(A) \). Note, that there is no double star variable for entropy.

### 3.2. Godunov fluxes

The fluxes are given by

\[ F_{L*} = \begin{cases} F_L, & \text{if } \Sigma_l > 0 \\ F_L^*, & \text{if } \Sigma_l \leq 0 \leq \Sigma_R \\ F_L^{**}, & \text{if } \Sigma_R \leq 0 \leq \Sigma_l \\ F_L^{***}, & \text{if } \Sigma_l \leq 0 \text{ and } u^* > 0 \\ F_L^R, & \text{if } \Sigma_R < 0 \end{cases} \]

\[ F_{R*} = \begin{cases} F_R, & \text{if } \Sigma_R > 0 \\ F_R^*, & \text{if } \Sigma_R \leq 0 \leq \Sigma_l \text{ and } u^* > 0 \\ F_R^{**}, & \text{if } \Sigma_l \leq 0 \text{ and } u^* > 0 \\ F_R^{***}, & \text{if } \Sigma_R \leq 0 \text{ and } u^* > 0 \end{cases} \]

### 3.3. Entropy generation

The entropy production in shocks from dissipation of kinetic energy has long been a topic of discussion, as nicely summarized by Salas & Tolle ([1995](#salas1995)). Such historical figures as Stokes, Kelvin and Rayleigh questioned the validity of shock discontinuity as it violated the conservation of entropy. A very curious result was put forward by Morduchow & Libby ([1949](#morduchow1949)). The equilibrium entropy has a maximum inside a Navier-Stokes shock profile, indicating that entropy was decreasing after passing the shock, strengthening the doubts by Stokes, Kelvin and Rayleigh.

Salas & Tolle ([1995](#salas1995)) explored this curiosity and concluded, that because of this phenomenon, the entropy propagation equation...
cannot be used as a conservation law. Using a single jump condition for propagation of entropy is not adequate, as going from equations with two jumps \([P]\) and \([\mu]\) to an equation with a single jump \([S]\) information is lost. See section 5 in [Salas & Tollo (1995)] for details. Shocks in the infinitesimally small area cannot be considered equilibrium, as locally the adiabatic index changes because of the rapid compression. Rankine-Hugoniot jump conditions are only applicable when elements are far enough from the discontinuity to assume the local equilibrium is present on each of the both sides, and it is not resolved, what happens in the discontinuity. [Margolin (2017)] studies non-equilibrium entropy in a shock and shows, that it increases monotonically inside the shock and a certain modification can be done to the equilibrium formulation that it would follow the non-equilibrium formulation better.

[Thorber et al. (2008)] derives analytical formulae for the rate of increase of entropy at arbitrary jumps in primitive variables for Godunov methods. It is then later used for total energy corrections. We take it a step further and use entropy as the main ‘energy’ variable. For the main part, the entropy is advected through the Riemann fan as a passive scalar. In addition to that, we compute the increase in entropy when we move along the states of the Riemann fan. We may subdivide the latter part into two closely interconnected components - hydrodynamic and one, influenced by magnetic fields. If no magnetic fields are present, entropy generation between two, L and R states can be written

\[
S_{\text{gen}} = (u_L - u_R) \left[ \ln \frac{P_L}{P_R} + \gamma \ln \frac{\rho_L}{\rho_R} \right].
\]

(49)

Shocks and MHD waves interact with each other and readily convert kinetic and magnetic energy into thermal energy. A perfect example of all the wave interactions can be seen in the Solar corona, see e.g. [Nakariakov & Verwichte (2005)]. This must be taken into consideration, thus we expand equation (49) for all the states of the Riemann fan:

\[
S_{\text{gen}} = \begin{cases} 
\alpha f_j(u_t - u_r) \left[ \ln \left( \frac{P_{\text{tot},L}}{P_{\text{tot},R}} \right) + \gamma \ln \left( \frac{\rho_{\text{tot},L}}{\rho_{\text{tot},R}} \right) \right] \\
(\alpha^* - u_r) \left[ \ln \left( \frac{P_{\text{tot},L}}{P_{\text{tot},R}} \right) + \gamma \ln \left( \frac{\rho_{\text{tot},L}}{\rho_{\text{tot},R}} \right) \right] - \xi L \\
(u_t - u*) \left[ \ln \left( \frac{P_{\text{tot},L}}{P_{\text{tot},R}} \right) + \gamma \ln \left( \frac{\rho_{\text{tot},L}}{\rho_{\text{tot},R}} \right) \right] + \xi R \\
\alpha f_j(u_t - u_r) \left[ \ln \left( \frac{P_{\text{tot},L}}{P_{\text{tot},R}} \right) + \gamma \ln \left( \frac{\rho_{\text{tot},L}}{\rho_{\text{tot},R}} \right) \right]
\end{cases}
\]

(50)

where

\[
\xi L = \text{sgn}(u_t - u^*) \frac{C_{\ell} f_{\ell} x y}{2} \left[ \frac{P L}{P_{\text{tot},L}} \right]
\]

(51)

\[
\xi R = \text{sgn}(u_t - u^*) \frac{C_{\ell} f_{\ell} x y}{2} \left[ \frac{P R}{P_{\text{tot},R}} \right]
\]

(52)

and \(\text{sgn}()\) is a sign function.

It is clear, that in this formulation \(S_{\text{gen}}\) can be both positive and negative. But in a closed system entropy can only increase. We note, that \(S_{\text{gen}}\) is calculated for an interface between two cells and both positive and negative values make perfect sense – the sign indicates, which side of the interface the \(S_{\text{gen}}\) should be applied to. Positive value indicates positive flow generation, thus the \(S_{\text{gen}}\) is applied to the right side of the interface, and to the left if the value is negative:

\[
s_{\text{gen}}^{n+1} = s_n^{n+1} + \left[ \max(0, S_{\text{gen},n-1}^{n+1}) - \min(0, S_{\text{gen},n+1}^{n+1}) \right] + \frac{Q'}{T_{\text{gas}}} \Delta t,
\]

(53)

where \(s = \rho S\) is entropy per unit volume, \(t\) and \(\Delta t\) is time and time-step respectively, \(Q\) is heating per unit volume and \(T_{\text{gas}}\) is the gas temperature.

### 3.4. Summary

To reformulate a total-energy-based HLLD Riemann solver, we need to replace the total energy with entropy in both predictor and corrector steps, if Godunov method is used. Here are the steps:

1. Replace thermal energy-based EOS \(E\) with entropy-based EOS \(S\).
2. For the predictor step, convert \(S\) to primitive variable, \(s = \rho S\).
3. In the predictor step, it is sufficient to advect \(S\) as a passive scalar to get the source term \(\sigma\), e.g. for the right state:

\[
\sigma_S = \Delta t \left[ -\frac{\partial \xi}{\partial t} - \frac{\Delta x}{2} \frac{\partial \xi}{\partial x} \right],
\]

(54)

where \(\xi = [x, y, z]\) is the spatial dimension;
4. Compute the Godunov flux for entropy instead of total energy;
5. Compute the \(S_{\text{gen}}\);
6. Apply Godonov flux, heating and \(S_{\text{gen}}\) to get \(S^{n+1}\).

In the next section we show a number of tests we have put this new solver through.

### 4. The numerical tests

To check the validity of the HLLS solver, we conduct a series of experiments. They are in 1D, 2D and 3D, both hydrodynamic and MHD. We run 1D tests in all principal directions, 2D tests in all 3 planes, to make sure the solver is well balanced in all dimensions.

#### 4.1. Entropy wave

Riemann solvers are generally very good at preserving shocks. Very slowly moving waves are, however, sometimes more difficult. To test the diffusion and dispersion error, we launch a
very slowly moving discontinuity (entropy wave) in one dimension. Assuming the experiment is carried out in \( x \) direction, the generic quantities are \((\rho, p) = [0.9, 1.0] \) when \( x < 0 \) and \([1.1, 1.0] \) when \( x \geq 0 \). All the other quantities are set to 0, with \( \gamma = \frac{4}{3} \) and \( S = \log(\frac{p}{\rho}) \) in the whole experiment. The velocity \( u_x \) of the whole box is set constant to \([1.0, 0.1, 0.01, 0.001] \) and the runtime of the experiment is set to respectively \( t_{\text{run}} = [10, 10, 100, 1000] \) time units. This means the number of updates is increasing tenfold with each run. If the solver is diffusive, the wave form should smear out. Figure 2 shows the test results at the end time, when the wave crosses one period. Naturally the corners are slightly diffused, but we can clearly see, that with lower velocities the increase in diffusion is negligible. The diffusion is resolution-sensitive, and depends on the slope limiters. The cyan curve is at time=0, red - \( u_x = 0.1 \), green - \( u_x = 0.01 \), blue dotted - at \( u_x = 0.001 \). For the latter velocity we do already see distortion, stemming from gas turning to incompressible gas at such small Mach numbers and Riemann solvers are normally not good with \( M \leq 0.2 \).

4.2. Shu & Osher shocktube

This test is a 1D Mach=3 shock interacting with sine waves in density (Shu & Osher 1989). It tests the solver’s ability to capture both shocks and small-scale smooth flow. The computational domain is 9 length units long and is split into two regions with different conditions in each of them. Assuming the experiment is carried out in \( x \) direction, the quantities are \((\rho, u_x, p) = [3.857143, 2.629369, 10.33333] \) when \( x < -4 \) and \([1 + 0.2 \sin(5x), 0, 0] \) when \( x \geq -4 \). All the other quantities are set to 0, with \( \gamma = 1.4 \) and \( S = \log(\frac{p}{\rho}) \) in the whole experiment. Figure 2 shows the test results at time = 1.8 time units (red curve). The reference run is done with the HLLD solver (black curve, nearly completely covered by the red curve). Both the test and the reference runs had 1500 cells. We additionally show a run with 500 cells (dots).

4.3. Brio & Wu shocktube

This is a classical test of an MHD shocktube, described by Brio & Wu (1988, section V), where the right and left states are initialized to different values. The left state is initialized as \((\rho, u_x, u_y, B_x, B_y, p) = [1, 0, 0, 1, 0, 1] \), and the right state \([0.125, 0, 0, -1, 0, 0.1] \). \( B_x = 0.75 \) and \( \gamma = 2 \). This test shows whether the solver can accurately represent the shocks, rarefaction waves, compound structures and contact discontinues. Figure 3 shows the test results at time=1.0. Black curve is for reference HLLD run with 1200 cells, red - HLLS run; blue dots - HLLS run with 400 cells. Both high and low resolution runs overlap each other nearly perfectly and follow the profiles in the literature very accurately. From left to right we can identify fast rarefaction fan, compound wave, contact discontinuity, slow shock and a fast rarefaction wave again.

4.4. Kelvin-Helmholtz instability

Kelvin-Helmholtz instability occurs when velocity shear is present within a continuous fluid or across fluid boundaries. We conduct the test as described in McNally et al. (2012). For convenience, here we summarize the setup. The domain is 1 unit by 1 unit in \( x \)– and \( y \)– directions, with resolutions of 256 x 256

Fig. 2. Shu & Osher shocktube test. Density profile at time=1.8: black - reference HLLD solver with 1500 cells, red - HLLS solver with 1500 cells, blue dots - HLLS solver with 500 cells.

Fig. 3. Brio & Wu shocktube test. Density profile (left) and y component of magnetic field (right) at time=1.0; black - reference HLLD solver with 1200 cells, red - HLLS solver with 1200 cells, blue dots - HLLS solver with 400 cells. Black and red overlap each other nearly perfectly.

Fig. 4. 1D Kelvin-Helmholtz instability. Density profile (left) and time evolution of kinetic energy (right). Density evolution movie available online.

Fig. 5. MHD Kelvin-Helmholtz instability. Density profile (left) and time evolution of magnetic energy (right). Density evolution movie available online.
the density at enough to suppress it, as can be seen in figure 5, where we show

\[ \rho \begin{cases} \rho_1 - \rho_0 e^{\frac{-y^2}{2}} & \text{if } 1/4 > y \geq 0 \\ \rho_2 + \rho_0 e^{\frac{-y^2}{2}} & \text{if } 1/2 > y \geq 1/4 \\ \rho_2 + \rho_0 e^{\frac{-y^2}{2}} & \text{if } 3/4 > y \geq 1/2 \\ \rho_1 - \rho_0 e^{\frac{-y^2}{2}} & \text{if } 1 > y \geq 3/4 \end{cases} \]  

(55)

where \( \rho_{in} = (\rho_1 + \rho_2)/2, \rho_1 = 1.0, \rho_2 = 2.0 \) and \( L = 0.025 \). The \( x \)-direction velocity is given by

\[ u_x = \begin{cases} u_1 - u_{in} e^{\frac{-y^2}{2}} & \text{if } 1/4 > y \geq 0 \\ u_2 + u_{in} e^{\frac{-y^2}{2}} & \text{if } 1/2 > y \geq 1/4 \\ u_2 + u_{in} e^{\frac{-y^2}{2}} & \text{if } 3/4 > y \geq 1/2 \\ u_1 - u_{in} e^{\frac{-y^2}{2}} & \text{if } 1 > y \geq 3/4 \end{cases} \]  

(56)

where \( u_{in} = (u_1 - u_2)/2, u_1 = 0.5 \) and \( u_2 = -0.5 \). The background shear is perturbed by adding velocity in \( y \)-direction,

\[ u_y = 0.01 \sin (4\pi x) \]  

(57)

An ideal EOS with \( \gamma = 5/3 \) is used. Initial gas pressure \( P_{gas} = 2.5 \). We run the simulation until time \( t = 10 \). To do an exact comparison to codes in McNally et al. (2012), we show the gas density at \( t = 1.5 \) and the maximal value of vertical kinetic energy evolution in time in figure 4. Note, that we used 256x256 resolution, but our result is directly comparable to 512x512 and 4096x4096 runs in McNally et al. (2012). On the right panel of the figure we show the maximum value of vertical kinetic energy in the simulation for three resolutions. From this figure it can be seen, that primary instability sets on at exactly the same time, secondary instabilities occur around the same time as well, indicating that we get a converged solution even at the low resolution of 256x256.

### 4.4.1. MHD

Additionally, we have added an ambient magnetic field \( B_z = 0.2 \). We chose this value for the magnetic field as initially it is weak enough to allow the instability to occur, but later it strengthens enough to suppress it, as can be seen in figure 5, where we show the density at \( t = 2.8 \), when the instability is starting to be actively suppressed. Magnetic energy peaks at \( t = 4.2 \) and simulation starts to relax into equilibrium later on.

### 4.5. Rayleigh–Taylor instability

Another classic test of a code’s ability to handle subsonic perturbations is the Rayleigh–Taylor instability and has been described in a number of studies, see e.g. Stone & Gardiner (2007), Abel (2011), Hopkins (2015). In this test a layer of heavier fluid is placed on top of a layer of lighter fluid. With gravitational source term added to the forces we test two things: whether the explicit addition of force is correct and the solver’s ability to preserve instabilities while keeping things symmetric where it should be, during the linear phase. During the non-linear phase, in a sufficiently high resolution the symmetry is expected to break by construction. The initial setup we use is similar to Abel (2011), Hopkins (2015). Here we recap the setup for convenience. In two dimensional domain with \( 0 \leq x \leq 0.5 \) (128 cells for low and 768 for high resolution runs) and \( 0 \leq y \leq 1 \) (256 and 1536 cells for low/high resolution runs respectively); we use periodic boundary condition in \( x \) direction and constant boundary conditions in \( y \). In this test we use \( \gamma = 1.4 \) and density profile is initialized as

\[ \rho(y) = \rho_1 + (\rho_2 - \rho_1)/(1 + e^{-y/(\sqrt{2}/\Delta)}) \]  

where \( \rho_1 = 1 \) and \( \rho_2 = 2 \) are the density below and above the contact discontinuity \( (y_\gamma) \) respectively, with \( \Delta = 0.025 \) being its width. The pressure gradient is in hydrostatic equilibrium with a uniform gravitational acceleration \( g = -0.5 \) in the \( y \) direction, \( P = \rho_2/y + g * \rho(y) * (y - y_\gamma) \), then entropy \( S = ln(P) - ln(\rho)^2 \). An initial \( y \)-velocity perturbation \( \delta v_y = \delta v_y(1 + cos(8\pi x (x + 0.25)))(1 + cos(5\pi(y - 0.5))) \) with \( \delta v_y = 0.025 \) is applied in the range \( 0.3 \leq y \leq 0.7 \) (Hopkins 2015).

Figure 6 shows the evolution of the instability at different times. The initial velocity grows and the heavier fluid starts to sink. Note the single-cell resolution of contact discontinuities and mixing. Both blobs are nearly perfectly symmetric during the linear phase. Soon enough Kelvin–Helmholtz instabilities develop at the shear surface between the fluids and the symmetry is harder to keep, but the low resolution run maintains perfect symmetry throughout the whole simulation. Figure 7 shows the high resolution run. Secondary instabilities are more pronounced and symmetry is harder to maintain. The figure can be directly compared to figure 22 in Hopkins (2015). Our high resolution (768x1536) run shows very similar features as in Hopkins (2015), although we maintain much better symmetries until the blobs reach the bottom.

### 4.6. MHD blast

This is a very popular test and various papers have presented results with slightly different problem setups. It is a very good test of the code ability to handle the evolution of strong MHD waves and look for directional biases. We choose to follow the setup by Ramsey et al. (2012), Clarke (2010), Stone et al. (2008). We use a rectangular domain with \( -0.5 \leq x, y \leq 0.5, 512 \times 512 \) cells resolution. All boundaries are periodic. \( (\rho, u, B_z, B_y, B_x) = \) \( (1, 0.5 \sqrt{2}, 5 \sqrt{2}, 0) \). The ambient pressure \( P_{gas} = 1 \) with \( P_{gas} = 100 \) within radius \( r = 0.125 \). Figure 8 shows the density and magnetic pressure in the experiment at time \( t = 0.021 \) and can be directly compared to Ramsey et al. (2012), Clarke (2010), Stone et al. (2008). The shock front is sharp, fast magneto-acoustic wave is traveling at the correct speed, the blast is symmetric.

### 4.7. Orszag–Tang vortex

This test has become a staple for MHD codes. The initial setup is identical to Stone et al. (2008), Ramsey et al. (2012). We use a rectangular domain with \( -0.5 \leq x, y \leq 0.5, 1024 \times 1024 \) cells for high resolution run. All boundaries are periodic. Initially the pressure and density are constant, \( P_{gas} = 5/12\pi \) and \( \rho = 25/36\pi \). The ratio of specific heats \( \gamma = 5/3 \). Initial velocity \( (u_x, u_y, u_z) = (-\sin(2\pi y), \sin 2\pi x, 0 \) and the magnetic field is set through the vector potential

\[ A_z = \frac{B_0}{4\pi} \cos(4\pi x) + \frac{B_0}{2\pi} \cos(4\pi y) \]  

where \( B_0 = 1/\sqrt{4\pi} \). In figure 9 we show density, entropy per unit mass and magnetic pressure at two times, \( t = 0.5 \) and \( t = 0.75 \). The first time is the typical time this test is shown. When compared to Stone et al. (2008), Ramsey et al. (2012), we can recognize similar features. Notice the very sharp features and perfect symmetry between sides. When advancing the simulation further, the vortex starts producing plasmoids. Here it is very difficult to maintain symmetry, but the lower panels show, that even plasmoids are released symmetrically.

This experiment is great to test the solver stability. In high resolution discontinuities and rarefactions are more severe and
Fig. 6. Rayleigh-Taylor instability in low (128x256) resolution. We plot density at different times.

Fig. 7. Same as 6 but in 768x1536 resolution. Movie available online.

Fig. 8. MHD blast experiment. Density (left) and magnetic pressure (right) at time=0.021. Note the sharp shock-fronts. Movie available online.

Riemann solvers can crash from negative pressure or thermal energy values, if they cannot deal with them. Since we have a HLLD solver available as well, we can see, that the HLLS solver preserves the thermal energy better.

4.8. Current sheet

The current sheet test problem is designed to see what an algorithm will do with a perturbed current sheet. Although an analytic solution for this test problem is not available, it is still a great test to check the robustness of the algorithm. The experimental setup is similar to Hawley & Stone (1995). The test is run in two dimensions using a periodic, square grid with \(-0.5 \leq [x, y] \leq 0.5, 256 \times 256 \) cells resolution, upon which there is a uniform magnetic field that discontinuously reverses direction at some point. In the whole box \((\rho, P_{gas}) = (1, \beta/2)\), where \(\beta\) is an input parameter. We set \(B_y\):

\[
B_y = \begin{cases} 
1/\sqrt{(4\pi)} & \text{if } |x| > 0.25 \\
-1/\sqrt{(4\pi)} & \text{if } |x| \leq 0.25
\end{cases}
\]  

(58)

For \(|x| > 0.25\), velocities \((u_x, u_y) = (A \sin (2\pi y), 0)\), where \(A\) is an amplitude. We ran the test with a range of \(\beta\) and \(A\) values, but here we will show only with the "standard" values, \(\beta = A = 0.1\). In figure 10, we show the density and magnetic pressure at different times. Initially linearly polarized Alfvén waves propagate along the field in the y-direction and quickly start generating magneto-acoustic waves since the magnetic pressure does not remain constant. Since there are two current sheets in the setup (at \(x = \pm0.25\), reconnection inevitably occurs. If \(\beta < 1\), this reconnection drives strong over-pressurized regions that launch magneto-acoustic waves transverse to the field. Moreover, as reconnection changes the topology of the field lines, magnetic islands form, grow, and merge. The point of the test is to make sure the algorithm can follow this evolution for as long as possible without crashing. We ran tests for \(0.1 \leq (\beta, A) \leq 10\) and the HLLS solver deals with it with ease.

4.9. Magnetic field loop advection

This is a very powerful test to check whether the scheme preserves \(\nabla \cdot B = 0\). The experiment is similar to Tóth & Odstrčil (1996); Stone et al. (2008) with two dimensional domain, \(0 \leq x \leq 2\) (128 cells) and \(0 \leq y \leq 1\) (64 cells). We use periodic boundary condition in the whole domain. In this test we use...
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\[ \gamma = \frac{5}{3} \]

and both density and gas pressure are constant throughout the box, \( \rho = \rho_{\text{gas}} = 1.0 \). The magnetic field is initialized using a vector potential \( A_z = \max[A \ast (r_0 - r), 0] \), with \( A = 10^{-3} \), \( r_0 = 0.3 \) and \( r \) is the radial distance from the domain centre. The flow velocity \( (u_x, u_y) = (2, 1) \), thus the problem is essentially an advection test for the vector potential.

Although this test doesn’t test the HLLS solver directly, but nonetheless, the CT scheme is an integral part of the solver. Figure 11 shows the initial magnetic pressure and after the magnetic loop has been advected twice through the domain. The shape of the loop is very well preserved.

4.10. Gresho vortex

The Gresho vortex is a time-independent rotation pattern. Angular velocity depends only on the radius and centrifugal force is balanced by the pressure gradient. We use the slightly modified initial condition, which permits the variation of the Mach number, it can be found in Happenhofer et al. (2013), Grimm-Strele et al. (2014). For convenience, we summarize the setup here.

The simulation is in two dimensions, \( 0 \leq x \leq 1 \) (48 cells) and \( 0 \leq y \leq 1 \) (48 cells) with periodic boundary conditions everywhere. The low resolution is deliberate - from our tests we see, that with higher resolution (e.g. [Grimm-Strele et al. 2014] the experiment becomes much easier. In the whole domain \( \rho = 1 \),

\[ u_\phi = \begin{cases} 5r & \text{if } 0 \leq r < 0.2 \\ 2 - 5r & \text{if } 0.2 \leq r \leq 0.4 \\ 0 & \text{if } 0.4 < r \end{cases} \]

(59)

\[ P_{\text{gas}} = \begin{cases} P_0 + \frac{32}{5} r^2 & \text{if } 0 \leq r < 0.2 \\ P_0 + \frac{32}{5} r^2 + 4(1 - 5r - \ln(0.2) + \ln(r)) & \text{if } 0.2 \leq r \leq 0.4 \\ P_0 - 2 + 4 \ln(2) & \text{if } 0.4 < r \end{cases} \]

(60)

where \( r = \sqrt{x^2 + y^2} \) is the radial distance from the centre of the domain, \( u_\phi \) is the angular velocity in terms of the polar angle \( \phi = \text{atan2}(y, x) \) and

\[ P_0 = \frac{\rho}{\gamma \text{Ma}_{\text{ref}}^2} \]

(61)

with \( \text{Ma}_{\text{ref}} \) being a reference Mach number, which is the highest Mach number in the resulting flow. We execute 5 runs, with \( \text{Ma}_{\text{ref}} = [0.1, 0.01, 0.001] \). The last \( \text{Ma}_{\text{ref}} \) is repeated in two runs - the nominal resolution and a higher resolution, 128x128 cells. Figure 12 shows the results. We deliberately did not run the experiment with larger Mach numbers, as those are just too easy to maintain. In the figure we can see, that the vortex is maintained very well down to \( \text{Ma} = 0.01 \), and with \( \text{Ma} = 0.001 \) we get

Fig. 9. Orszag-Tang vortex test at two separate times. Density (left), entropy per unit mass (middle) and magnetic pressure (right). Notice the plasmoids forming. Movie available online.
4.11. Magnetic rotor

The magnetic rotor tests the propagation of strong torsional Alfvén waves. A dense disc of fluid rotates within a static fluid background, with a gradual velocity tapering layer between the disc edge and the ambient fluid. An initially uniform magnetic field is present, which is twisted with the disc rotation. The magnetic field is strong enough that as it wraps around the rotor diminishing the rotor’s angular momentum. The increased magnetic pressure around the rotor compresses the fluid in the rotor, giving it an oblong shape. The experimental setup is introduced by Balsara & Spicer (1999) and we use a more stringent variant of it from Guillet et al. (2019), which is summarized below.

The simulation is in two dimensional square, $0 \leq x, y \leq 1$ (512x512 cells) with periodic boundary conditions everywhere. The gas pressure and magnetic fields are uniform in the whole domain, with $P_{gas} = 1$, $\gamma = 1.4$ and $B = (5/4\pi, 0, 0)$. The gas density is

$$
\rho = \begin{cases} 
10 & \text{if } r < r_0 \\
1 + 9f & \text{if } r_0 \leq r \leq r_1 \\
1 & \text{if } r_1 < r 
\end{cases} 
$$

where $r_0 = 0.1$, $r_1 = 0.115$, $r$ is the radial distance from the centre of the box $c$ and $f = (r_1 - r)/(r_1 - r_0)$ is the tapering function for the taper region between the disc and the background. Velocities are

$$
(u_x, u_y) = \begin{cases} 
\left( \frac{v_0(c-y)}{r_0}, \frac{v_0(x-c)}{r_0} \right) & \text{if } r < r_0 \\
(0, 0) & \text{if } r_1 < r 
\end{cases} 
$$

with $v_0 = 2$. The experiment was run until time $t = 0.15$, by which the torsional Alfvén waves have almost reached the boundary. Figure 14 shows the density $\rho$, magnetic pressure $P_B$, Mach number and the normalized magnetic field divergence, $\nabla \cdot \mathbf{B} / |\mathbf{B}|$, which, unlike in Guillet et al. (2019), we do not rescale to cell size $\Delta x$. We note the very sharp details with no distortions outside the now almond-shaped disc. In the compressed areas the Mach number becomes very high, but it does not pose any issues. The magnetic field is divergence-free within the noise level.

5. Discussion and conclusions

In this work we presented a new approximate entropy-based HLLD Riemann solver. It works well in both sub- and supersonic regimes, preserves positive temperature and gas pressure. The numerical tests are very encouraging and indicate that the HLLS solver can be readily used in a wide range of physical conditions and experimental setups.

5.1. Very low Mach number regimes

Our tests show, that HLLS solver can easily handle Mach numbers to as low as 0.01 and with lower values it becomes rather diffusive. Of course, the perturbations at such Mach numbers are on the order of numerical precision (see Gresho vortex in subsection 4.10 for details) and can be absolutely indistinguishable.
from the background. This is very encouraging, as normally Godunov and Roe type Riemann solvers with single precision struggle with Mach numbers below 0.1. There are different good attempts to modify solvers to go to very low Mach numbers, either adding correction terms into star states, e.g. Shima & Kitamura (2011); Dellacherie et al. (2016); Minoshima & Miyoshi (2021); sheng Chen et al. (2022), or by using well-balanced schemes. In the latter a hydrostatic equilibrium is imposed directly in the set of dynamic equations, separating primitive variables into equilibrium (stationary) states and dynamical perturbations, as it is done in e.g. (Greenberg & Leroux [1996]; Hotta et al. (2022); Canivete Cuissa & Teyssier [2022]). This approach has less success and not as flexible, as the first option, as although helps with hydrostatic equilibrium in deep atmospheres, numerical precision is better when only perturbations are considered, there is nothing really preventing gas in incompressible state to act as compressible gas and ignore the rotational flows. Our initial attempts to use a simple ad-hoc modification to  and  to imitate the transition to incompressible gas at around  = 0.2, as in Minoshima & Miyoshi (2021) did not result in any significant improvement and more work has to be done to balance out the equations.

We will continue our work on very low Mach number regimes as this is rather crucial to the applications we intend to use the solver for.

5.2. Applications

The solver was developed with primary intent to use it in the context of Solar, stellar and planetary atmospheres. The from the outermost regions, towards the cores, density, temperature and pressure varies by orders of magnitude, but entropy per unit mass stays almost constant. Additionally, HLLS is the preferred solver in situations where magnetic energy and kinetic energy strongly dominate over thermal energy, e.g. Solar chromosphere and corona.

We are already employing this solver to simulate the whole Solar convective region from 0.655 to 0.995 R⊙ [Popovas et al. (2022, in prep.)] and we can see, that the HLLS solver can maintain the hydrostatic equilibrium much better than a HLLD solver without introducing a well-balanced scheme, as in e.g. Canivete Cuissa & Teyssier (2022).

5.3. Future work

In future papers we will discuss the divergence cleaning procedures and the addition of tabular EOS. The latter is needed to do more realistic simulations in astrophysical applications. The HLLS solver requires a slight more care when adding the more general, non-ideal gas EOS. We will keep developing the solver so it would be able to cover a larger range of Mach numbers and in general perform better:

- we intend to implement better slope limiters, e.g. Sekora & Colella (2009);
- solver modifications for very low Mach number regimes;
- solver modifications for relativistic regimes;
- heavy optimization for better memory alignment and vectorization

Acknowledgements. This research was supported by the Research Council of Norway through its Centres of Excellence scheme, project number 262622, and through grants of computing time from the Programme for Supercomputing, as well as through the Synergy Grant number 810218 (ERC-2018-SyG) of the European Research Council.

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Fig. 14. Magnetic rotor test. From left to right: density, magnetic pressure, Mach number and $\nabla \cdot B$. See text for more details.

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