Weighted almost convergence and related infinite matrices

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Abstract
The purpose of this paper is to introduce the notion of weighted almost convergence of a sequence and prove that this sequence endowed with the sup-norm \( \| \cdot \|_{\infty} \) is a BK-space. We also define the notions of weighted almost conservative and regular matrices and obtain necessary and sufficient conditions for these matrix classes. Moreover, we define a weighted almost \( A \)-summable sequence and prove the related interesting result.

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1 Introduction and preliminaries
Let \( \omega \) denote the space of all complex sequences \( s = (s_j)_{j=0}^{\infty} \) (or simply write \( s = (s_j) \)). Any vector subspace of \( \omega \) is called a sequence space. By \( \mathbb{N} \) we denote the set of natural numbers, and by \( \mathbb{R} \) the set of real numbers. We use the standard notation \( \ell_\infty \), \( c \) and \( c_0 \) to denote the sets of all bounded, convergent and null sequences of real numbers, respectively, where each of the sets is a Banach space with the sup-norm \( \| \cdot \|_\infty \) defined by \( \| s \|_\infty = \sup_{j \in \mathbb{N}} |s_j| \).

We write the space \( \ell_p \) of all absolutely \( p \)-summable series by

\[
\ell_p = \left\{ s \in \omega : \sum_{j=0}^{\infty} |s_j|^p < \infty \ (1 \leq p < \infty) \right\}.
\]

Clearly, \( \ell_p \) is a Banach space with the following norm:

\[
\| s \|_p = \left( \sum_{j=0}^{\infty} |s_j|^p \right)^{1/p}.
\]

For \( p = 1 \), we obtain the set \( l_1 \) of all absolutely summable sequences. For any sequence \( s = (s_j) \), let \( s^{(n)} = \sum_{j=0}^{n} s_j e_j \) be its \( n \)-section, where \( e_j \) is the sequence with 1 in place \( j \) and 0 elsewhere and \( e = (1, 1, 1, \ldots) \).

A sequence space \( X \) is called a BK-space if it is a Banach space with continuous coordinates \( p_j : X \to \mathbb{C} \), the set of complex fields, and \( p_j(s) = s_j \) for all \( s = (s_j) \in X \) and every \( j \in \mathbb{N} \). A BK-space \( X \supseteq \psi \), the set of all finite sequences that terminate in zeros, is said to have AK if every sequence \( s = (s_j) \in X \) has a unique representation \( s = \sum_{j=0}^{\infty} s_j e_j \).
Let $X$ and $Y$ be two sequence spaces, and let $A = (a_{n,k})$ be an infinite matrix. If, for each $s = (s_k)$ in $X$, the series

$$A_n s = \sum_k a_{n,k} s_k = \sum_{k=0}^{\infty} a_{n,k} s_k$$

(1)

converges for each $n \in \mathbb{N}$ and the sequence $A s = (A_n s)$ belongs to $Y$, then we say that matrix $A$ maps $X$ into $Y$. By the symbol $(X, Y)$ we denote the set of all such matrices which map $X$ into $Y$. The series in (1) is called $A$-transform of $s$ whenever the series converges for $n = 0, 1, \ldots$. We say that $s = (s_k)$ is $A$-summable to the limit $\lambda$ if $A_n s$ converges to $\lambda$ $(n \to \infty)$.

The sequence $s = (s_k)$ of $\ell_\infty$ is said to be almost convergent, denoted by $f$, if all of its Banach limits [1] are equal. We denote such a class by the symbol $f$, and one writes $f$-lim $s = \lambda$ if $\lambda$ is the common value of all Banach limits of the sequence $s = (s_k)$. For a bounded sequence $s = (s_k)$, Lorentz [2] proved that $f$-lim $s = \lambda$ if and only if

$$\lim_{k \to \infty} \frac{s_m + s_{m+1} + \cdots + s_{m+k}}{k+1} = \lambda$$

uniformly in $m$. This notion was later used to (i) define and study conservative and regular matrices [3]; (ii) introduce related sequence spaces derived by the domain of matrices [4–6]; (iii) study some related matrix transformations [7–9]; (iv) define related sequence spaces derived as the domain of the generalized weighted mean and determine duals of these spaces [10, 11]. As an extension of the notion of almost convergence, Kayaduman and Şengönül [12, 13] defined Cesàro and Riesz almost convergence and established related core theorems. The almost strongly regular matrices for single sequences were introduced and characterized [14], and for double sequences they were studied by Mursaleen [15] (also refer to [16–19]). As an application of almost convergence, Mohiuddine [20] proved a Korovkin-type approximation theorem for a sequence of linear positive operators and also obtained some of its generalizations. Başar and Kirişçi [21] determined the duals of the sequence space $f$ and other related spaces/series and investigated some useful characterizations.

We now recall the following result.

**Lemma 1.1** ([22]) Let $X$ and $Y$ be BK-spaces. (i) Then $(X, Y) \subset B(X, Y)$, that is, every $A \in (X, Y)$ defines an operator $L_A \in B(X, Y)$ by $L_A(x) = Ax$ for all $x \in X$, where $B(X, Y)$ denotes the set of all bounded linear operators from $X$ into $Y$. (ii) Then $A \in (X, \ell_\infty)$ if and only if $\|A\|_{(X, \ell_\infty)} = \sup_n \|A_n\|_X < \infty$. Moreover, if $A \in (X, \ell_\infty)$, then $\|L_A\| = \|A\|_{(X, \ell_\infty)}$.

**2 Weighted almost convergence**

**Definition 2.1** Let $t = (t_k)$ be a given sequence of nonnegative numbers such that $\liminf t_k > 0$ and $T_m = \sum_{k=0}^{m-1} t_k \neq 0$ for all $m \geq 1$. Then the bounded sequence $s = (s_k)$ of real or complex numbers is said to be weighted almost convergent, shortly $f(\bar{N})$-convergent, to $\lambda$ if and only if

$$\lim_{m \to \infty} \frac{1}{T_m} \sum_{k=r}^{r+m-1} t_k s_k = \lambda \quad \text{uniformly in } r.$$
We shall use the notation \( f(\bar{N}) \) for the space of all sequences which are \( f(\bar{N}) \)-convergent, that is,

\[
f(\bar{N}) = \left\{ s \in l_\infty : \exists \lambda \in \mathbb{C} \ni \lim_{m \to \infty} \frac{1}{T_m} \sum_{k=r}^{r+m-1} t_k s_k = \lambda \text{ uniformly in } r; \lambda = f(\bar{N})-\lim s \right\}. \tag{2}
\]

We remark that if we take \( t_k = 1 \) for all \( k \), then (2) is reduced to the notion of almost convergence introduced by Lorentz \[2\]. Clearly, a convergent sequence is \( f(\bar{N}) \)-convergent to the same limit, but its converse is not always true.

**Example 2.2** Consider a sequence \( s = (s_k) \) defined by \( s_k = 1 \) if \( k \) is odd and 0 for even \( k \). Also, let \( t_k = 1 \) for all \( k \). Then we see that \( s = (s_k) \) is \( f(\bar{N}) \)-convergent to \( 1/2 \) but not convergent.

**Definition 2.3** The matrix \( A \) (or a matrix map \( A \)) is said to be weighted almost conservative if \( As \in f(\bar{N}) \) for all \( s = (s_k) \in c \). One denotes this by \( A \in (c,f(\bar{N})) \). If \( A \in (c,f(\bar{N})) \) with \( f(\bar{N})-\lim As = \lim s \), then we say that \( A \) is weighted almost regular matrix; one denotes such matrices by \( A \in (c,f(\bar{N}))_R \).

**Theorem 2.4** The space \( f(\bar{N}) \) of weighted almost convergence endowed with the norm \( \| \cdot \|_\infty \) is a BK-space.

**Proof** To prove our results, first we have to prove that \( f(\bar{N}) \) is a Banach space normed by

\[
\| s \|_{f(\bar{N})} = \sup_{m,r} |\Psi_{m,r}(s)|, \tag{3}
\]

where

\[
\Psi_{m,r}(s) = \frac{1}{T_m} \sum_{k=r}^{r+m-1} t_k s_k.
\]

It is easy to verify that (3) defines a norm on \( f(\bar{N}) \). We have to show that \( f(\bar{N}) \) is complete. For this, we need to show that every Cauchy sequence in \( f(\bar{N}) \) converges to some number in \( f(\bar{N}) \). Let \( (s^j) \) be a Cauchy sequence in \( f(\bar{N}) \). Then \( (s^j_k) \) is a Cauchy sequence in \( \mathbb{R} \) (for each \( j = 1, 2, \ldots \)). By using the notion of the norm of \( f(\bar{N}) \), it is easy to see that \( (s^j) \to s \). We have only to show that \( s \in f(\bar{N}) \).

Let \( \epsilon > 0 \) be given. Since \( (s^j) \) is a Cauchy sequence in \( f(\bar{N}) \), there exists \( M \in \mathbb{N} \) (depending on \( \epsilon \)) such that

\[
\| s^j_k - s^i_k \| < \epsilon/3 \quad \text{for all } k, i > M,
\]

which yields

\[
\sup_{m,r} |\Psi(s^j_k - s^i_k)| < \epsilon/3.
\]

Therefore we have \( |\Psi(s^j_k - s^i_k)| < \epsilon/3 \). Taking the limit as \( m \to \infty \) gives that \( |\lambda^j - \lambda^i| < \epsilon/3 \) for each \( m, r \) and \( k, i > M \), where \( \lambda^j = f(\bar{N})-\lim_m s^j \) and \( \lambda^i = f(\bar{N})-\lim_m s^i \). Let \( \lambda = \lim_{r \to \infty} \lambda^j \).
Letting $i \to \infty$, one obtains

$$\left| \Psi_{mr}(s^k - s^i) \right| < \epsilon/3 \quad \text{and} \quad |\lambda^k - \lambda| < \epsilon/3 \quad (4)$$

for each $m$, $r$ and $k > M$. Now, for fixed $k$, the above inequality holds. Since $s^k \in f(\bar{N})$, for fixed $k$, we get

$$\lim_{m \to \infty} \Psi_{mr}(s^k) = \lambda^k \quad \text{uniformly in } r.$$ 

For given $\epsilon > 0$, there exists positive integers $M_0$ (independent of $r$, but dependent upon $\epsilon$) such that

$$\left| \Psi_{mr}(s^k) - \lambda^k \right| < \epsilon/3 \quad (5)$$

for $m > M_0$ and for all $r$. It follows from (4) and (5) that

$$\left| \Psi_{mr}(s) - \lambda \right| = \left| \Psi_{mr}(s) - \Psi_{mr}(s^k) + \Psi_{mr}(s^k) - \lambda^k + \lambda^k - L \right|$$

$$\leq \left| \Psi_{mr}(s) - \Psi_{mr}(s^k) \right| + \left| \Psi_{mr}(s^k) - \lambda^k \right| + |\lambda^k - L| < \epsilon.$$

This proves that $f(\bar{N})$ is a Banach space normed by (3).

Since $c \subset f(\bar{N}) \subset l_\infty$, there exist positive real numbers $\alpha$ and $\beta$ with $\alpha < \beta$ such that $\alpha \parallel s \parallel_\infty \leq \parallel s \parallel_{f(\bar{N})} \leq \beta \parallel s \parallel_\infty$. That is to say, two norms $\parallel \cdot \parallel_\infty$ and $\parallel \cdot \parallel_{f(\bar{N})}$ are equivalent. It is well known that the spaces $c$ and $l_\infty$ endowed with the norm $\parallel \cdot \parallel_\infty$ are BK-spaces, and hence the space $f(\bar{N})$ endowed with the norm $\parallel \cdot \parallel_\infty$ is also a BK-space.

We prove the following characterization of weighted almost conservative matrices.

**Theorem 2.5** The matrix $A = (a_{n,k})$ is weighted almost conservative, that is, $A \in (c,f(\bar{N}))$ if and only if

$$\sup \left\{ \sum_{k=0}^{\infty} \frac{1}{T_m^r} \sum_{n=r}^{r+m-1} t_n a_{n,k} \mid m \in \mathbb{Z}^+ \right\} < \infty; \quad (6)$$

$$\lim_{m \to \infty} \frac{1}{T_m^r} \sum_{n=r}^{r+m-1} t_n a_{n,k} = \lambda_k \quad \text{exists (} k = 0, 1, 2, \ldots \text{) uniformly in } r; \quad (7)$$

$$\lim_{m \to \infty} \frac{1}{T_m^r} \sum_{n=r}^{r+m-1} \sum_{k=0}^{\infty} t_n a_{n,k} = \lambda \quad \text{exists uniformly in } r. \quad (8)$$

**Proof** Necessity. Let $A \in (c,f(\bar{N}))$. Since the sequences $e$ and $e_k$ both are convergent, so $A$-transforms of the sequences $e_k$ and $e$ belong to $f(\bar{N})$ and exist uniformly in $r$. It follows that (7) and (8) are valid. Let $r$ be any nonnegative integer. One writes

$$\Phi_{mr}(s) = \frac{1}{T_m^r} \sum_{n=r}^{r+m-1} t_n a_n(s),$$
where
\[ \alpha_n(s) = \sum_{k=0}^{\infty} a_{n,k} s_k. \]

It follows that \( \alpha_n \in c' \) for all \( n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) and this yields \( \Phi_{mr} \in c' \) \( (m \geq 1) \). Since \( A \in (c, f(\mathbb{N})) \),
\[ \lim_{m \to \infty} \Phi_{mr}(s) = \Phi(s) \quad \text{exists uniformly in } r. \]

It is clear that \( (\Phi_{mr}(s)) \) is bounded for \( s = (s_k) \in c \) and fixed \( r \). Hence, by the uniform boundedness principle, \( (\| \Phi_{mr} \|) \) is bounded. For each \( p \in \mathbb{Z}^+ \) (the positive integers), the sequence \( x = (x_k) \) is defined by
\[ x_k = \begin{cases} 
\text{sgn} \sum_{n=r}^{r+m-1} t_n a_{n,k} & \text{if } 0 \leq k \leq p, \\
0 & \text{if } k > p.
\end{cases} \]

Then a sequence \( x \in c, \| x \| = 1 \) and
\[ |\Phi_{mr}(x)| = \frac{1}{T_m} \sum_{k=0}^{p} \sum_{n=r}^{r+m-1} t_n a_{n,k} \leq \| \Phi_{mr} \| \| x \| = \| \Phi_{mr} \|. \quad (9) \]

Therefore, we obtain
\[ |\Phi_{mr}(x)| \leq \| \Phi_{mr} \| \| x \| = \| \Phi_{mr} \|. \quad (10) \]

Equations (9) and (10) give that
\[ \frac{1}{T_m} \sum_{k=0}^{p} \sum_{n=r}^{r+m-1} t_n a_{n,k} \leq \| \Phi_{mr} \| < \infty, \]

it follows that (6) is valid.

Sufficiency. Let conditions (6)-(8) hold. Let \( r \) be any nonnegative integer, and let \( s_k \in c \). Then
\[ \Phi_{mr}(s) = \frac{1}{T_m} \sum_{n=r}^{r+m-1} \sum_{k=0}^{\infty} t_n a_{n,k} s_k \]
\[ = \frac{1}{T_m} \sum_{k=0}^{\infty} \sum_{n=r}^{r+m-1} t_n a_{n,k} s_k, \]
which gives
\[ |\Phi_{mr}(s)| \leq \frac{1}{T_m} \sum_{k=0}^{\infty} \sum_{n=r}^{r+m-1} t_n a_{n,k} \| s \|. \]

It follows from hypothesis (6) that \( |\Phi_{mr}(s)| \leq B_r \| s \| \), where \( B_r \) is a constant independent of \( r \). Thus we have \( \Phi_{mr} \in c' \) for each \( m \geq 1 \), which gives that a sequence \( (\| \Phi_{mr} \|) \) is bounded.
for each nonnegative integer \( r \). Hypotheses (7) and (8) imply that the limit of \( \Phi_{mr}(e_k) \) and \( \Phi_{mr}(e) \) must exist for all nonnegative integers \( k \) and \( r \). Since \( \{e, e_0, e_1, \ldots\} \) is a fundamental set in \( c \), it follows from [23, p. 252] that \( \lim_{m} \Phi_{mr}(s) = \Phi(s) \) exists and \( \Phi_r \in c' \). Therefore \( \Phi_r \) has the following form (see [23, p. 205]):

\[
\Phi_r(s) = \xi \left( \Phi_r(e) - \sum_{k=0}^{\infty} \Phi_r(e_k) \right) + \sum_{k=0}^{\infty} s_k \Phi_r(e_k),
\]

where \( \xi = \lim s_k \). From (7) and (8), we see that \( \Phi_r(e_k) = \lambda_k \) for a nonnegative integer \( k \) and \( \Phi_r(e) = \lambda \). Therefore, for each \( s \in c \) and a nonnegative integer \( r \), we have

\[
\lim_{m \to \infty} \Phi_{mr}(s) = \Phi(s)
\]

with the following expression:

\[
\Phi(s) = \xi \left( \lambda - \sum_{k=0}^{\infty} \lambda_k \right) + \sum_{k=0}^{\infty} s_k \lambda_k. \quad (11)
\]

Since \( \Phi_{mr} \in c' \), so it has the representation

\[
\Phi_{mr}(s) = \xi \left( \Phi_{mr}(e) - \sum_{k=0}^{\infty} \Phi_{mr}(e_k) \right) + \sum_{k=0}^{\infty} s_k \Phi_{mr}(e_k). \quad (12)
\]

We observe from (11) and (12) that the convergence of \( \Phi_{mr}(s) \) to \( \Phi(s) \) is uniform since \( \lim_{m \to \infty} \Phi_{mr}(e_k) = \lambda_k \) and \( \lim_{m \to \infty} \Phi_{mr}(e) = \lambda \) uniformly in \( r \). Hence, \( A \) is a weighted almost conservative matrix. \( \square \)

In the following theorem, we obtain the characterization of weighted almost regular matrices.

**Theorem 2.6** The matrix \( A \in (c,f(\bar{N}))_R \) if and only if

\[
\sup \left\{ \sum_{k=0}^{\infty} \frac{1}{T_m} \left| \sum_{n=r}^{r+m-1} t_n a_{n,k} \right| : m \in \mathbb{Z}^+ \right\} < \infty; \quad (13)
\]

\[
\lim_{m \to \infty} \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} = 0 \quad \text{uniformly in } r (k \in \mathbb{N}_0); \quad (14)
\]

\[
\lim_{m \to \infty} \frac{1}{T_m} \sum_{n=r}^{r+m-1} \sum_{k=0}^{\infty} t_n a_{n,k} = 1 \quad \text{uniformly in } r. \quad (15)
\]

**Proof** Necessity. Let \( A \in (c,f(\bar{N}))_R \). We see that condition (13) holds by using the fact that \( A \) is also weighted almost conservative. Take \( e_k, e \in c \). Then \( A \)-transforms of the sequences \( e_k \) and \( e \) are weighted almost convergent to 0 and 1, respectively, since \( e_k \to 0 \) and \( e \to 1 \). Hence \( e_k \in c \) gives condition (14) and \( e \in c \) proves the validity of (15).

Sufficiency. Let conditions (13)-(15) hold. It is easy to see that \( A \) is weighted almost conservative. So, for each \( (s_k) \in c \), \( \lim_{m \to \infty} \Phi_{mr}(s) = \Phi(s) \) uniformly in \( r \). Thus we obtain
from (11) and our hypotheses (13)-(15) that $\Phi(s) = \xi = \lim s_k$. This yields $A$ is weighted almost regular.

We now obtain necessary and sufficient conditions for the matrix $A$ which transform the absolutely convergent series into the space of weighted almost convergence.

**Theorem 2.7** The matrix $A \in (l_1, f(\tilde{N}))$ if and only if

\[
\sup_{k,m,r} \left| \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} \right| < \infty,
\]

\[
\lim_{m \to \infty} \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} = \lambda_k \quad \text{exists for each } k \in \mathbb{N}_0 \text{ uniformly in } r.
\]

**Proof** Necessity. Let $A \in (l_1, f(\tilde{N}))$. Condition (17) follows since $e_k \in l_1$. Let $\Phi_{mr}$ be a continuous linear functional on $l_1$ defined by

\[
\Phi_{mr}(s) = \frac{1}{T_m} \sum_{k=0}^{\infty} \sum_{n=r}^{r+m-1} t_n a_{n,k} s_k.
\]

Then we have

\[
|\Phi_{mr}(s)| \leq \sup_k \left| \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} s_k \right| ||s||_1,
\]

which yields

\[
\|\Phi_{mr}\| \leq \sup_k \left| \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} s_k \right|.
\]

For any fixed $k \in \mathbb{N}_0$, we define a sequence $s = (s_j)$ by

\[
s_j = \begin{cases} 
\text{sgn} \sum_{n=r}^{r+m-1} t_n a_{n,k} & \text{if } j = k, \\
0 & \text{if } j \neq k.
\end{cases}
\]

Then we have $||s||_1 = 1$ and

\[
|\Phi_{mr}(s)| = \left| \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} s_k \right| = \left| \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} s_k \right| ||s||_1,
\]

so

\[
\|\Phi_{mr}\| \geq \sup_k \left| \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} s_k \right|.
\]

We obtain from (18) and (19) that

\[
\|\Phi_{mr}\| = \sup_k \left| \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} s_k \right|.
\]
Since \( A \in (l_1, f(\bar{N})) \), for any \( s \in l_1 \), we have

\[
\sup_{m,r} |\Phi_{mr}(s)| = \sup_{m,r} \left| \frac{1}{T_m} \sum_{k=0}^{\infty} \sum_{n=r}^{r+m-1} t_n a_{n,k} s_k \right| < \infty. \tag{20}
\]

By using the uniform boundedness theorem, Equation (20) becomes

\[
\sup_{m,r} \|\Phi_{mr}\| = \sup_{k,m,r} \left| \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} \right| < \infty.
\]

This proves the validity of (16).

**Sufficiency.** Let conditions (16) and (17) hold, and let \( s = (s_k) \in l_1 \). In virtue of these conditions, we see that

\[
\lim_{m \to \infty} \frac{1}{T_m} \sum_{k=0}^{\infty} \sum_{n=r}^{r+m-1} t_n a_{n,k} s_k = \sum_{k=0}^{\infty} \lambda_k s_k \quad \text{uniformly in } r, \tag{21}
\]

it also converges absolutely. Furthermore, \( \frac{1}{T_m} \sum_{k=0}^{\infty} \sum_{n=r}^{r+m-1} t_n a_{n,k} s_k \) converges absolutely for each \( m \) and \( r \).

Let \( \epsilon > 0 \) be given. Then there exists \( k_0 \in \mathbb{N} \) such that

\[
\sum_{k > k_0} |s_k| < \epsilon. \tag{22}
\]

By condition (17), we can find some \( m_0 \in \mathbb{N} \) such that

\[
\left| \sum_{k \leq k_0} \left[ \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} - \lambda_k \right] s_k \right| < \epsilon \tag{23}
\]

for all \( m > m_0 \) uniformly in \( r \). Now

\[
\left| \sum_{k=0}^{\infty} \left[ \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} - \lambda_k \right] s_k \right| \leq \sum_{k \geq k_0} \left| \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} - \lambda_k \right| s_k + \sum_{k < k_0} \left| \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} - \lambda_k \right| |s_k| \tag{24}
\]

for all \( m > m_0 \) uniformly in \( r \). By using Equations (22) and (23) and our hypotheses in the above inequality, we see that (21) holds, and hence the sufficiency part.

**Theorem 2.8** If the matrix \( A \) in \((l_1, f(\bar{N})))\), then \( \|L_A\| = \|A\| \).

**Proof** Let \( A \in (l_1, f(\bar{N})) \). Then we have

\[
\|L_A(s)\| = \sup_{m,r} \left| \frac{1}{T_m} \sum_{k=0}^{\infty} \sum_{n=r}^{r+m-1} t_n a_{n,k} s_k \right| \leq \sup_{m,r} \left| \frac{1}{T_m} \sum_{k=0}^{\infty} \sum_{n=r}^{r+m-1} t_n a_{n,k} \right| |s_k|,
\]
which gives \( \| \mathcal{L}_A(s) \| \leq \| A \| \| s \|_1 \). This implies that \( \| \mathcal{L}_A \| \leq \| A \| \). Also, \( \mathcal{L}_A \in B(l_1, f(\bar{N})) \) gives

\[
\| \mathcal{L}_A \| = \| A \| \leq \| \mathcal{L}_A \| \| s \|_1.
\]

Taking \( s = (e_k) \) and using the fact that \( \| e_k \|_1 = 1 \) \( \forall k \), one obtains \( \| A \| \leq \| \mathcal{L}_A \| \). Hence we conclude that \( \| \mathcal{L}_A \| = \| A \| \).

**Definition 2.9** Let \( t = (t_k)_{k \in \mathbb{N}} \) be a given sequence of nonnegative numbers such that \( \liminf_k t_k > 0 \) and \( T_m = \sum_{k=0}^{m-1} t_k \neq 0 \) for all \( m \geq 1 \). A sequence \( s = (s_k) \) is said to be weighted almost \( A \)-summable to \( \lambda \in \mathbb{C} \) if the \( A \)-transform of sequence \( s = (s_k) \) is weighted almost convergent to \( \lambda \); equivalently, we can write

\[
\lim_m \sigma_{mr}(s) = \lambda \quad \text{uniformly in } r,
\]

where

\[
\sigma_{mr}(s) = \frac{1}{T_m} \sum_{n=r}^{r+m-1} \sum_{k=0}^{\infty} t_n a_{n,k} s_k.
\]

In the applications of summability theory to function theory, it is important to know the region in which \( S = (S_k(z)) \), the sequence of partial sums of the geometric series is \( A \)-summable to \( \frac{1}{1-z} \) for a given matrix \( A \). In the following theorem, we find the region in which \( S \) is weighted almost \( A \)-summable to \( \frac{1}{1-z} \).

**Theorem 2.10** Let \( A = (a_{n,k}) \) be a matrix such that (15) holds. The sequence \( (S_k(z)) \) is weighted almost \( A \)-summable to \( \frac{1}{1-z} \) if and only if \( z \in R \), where

\[
R = \left\{ z = (z^k) : \lim_m \sigma_{mr}(z) = 0 \text{ uniformly in } r \right\}.
\]

**Proof** One writes

\[
\sigma_{mr} = \frac{1}{T_m} \sum_{n=r}^{r+m-1} \sum_{k=0}^{\infty} t_n a_{n,k} S_k(z)
= \frac{1}{T_m} \sum_{n=r}^{r+m-1} \sum_{k=0}^{\infty} t_n a_{n,k} \frac{1 - z^{k+1}}{1-z}
= \frac{1}{(1-z)T_m} \sum_{n=r}^{r+m-1} \sum_{k=0}^{\infty} t_n a_{n,k} \frac{z^k}{1-z}.
\]

Taking the limit as \( m \to \infty \) in the above equality and using condition (15), one obtains

\[
\lim_{m \to \infty} \sigma_{mr} = \frac{1}{1-z} \quad \text{uniformly in } r
\]

if and only if \( z \in R \). This completes the proof.
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Competing interests
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Authors’ contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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