The Poincaré constant of the Koch snowflake

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Abstract

We compute sharp upper and lower bounds for the Poincaré constant of the Koch snowflake. Our method comprises four crucial steps: domain monotonicity, conformal mapping of the region, a grad-div system formulation of the eigenvalue problem and an application of the quadratic method. We describe each one of these steps in detail and report on the results of a numerical implementation involving the finite element method.

1 Introduction

We establish a numerical procedure for computing sharp upper and lower bounds for the Poincaré constant of a Koch snowflake $\Sigma$. This is the optimal constant $k_\Sigma > 0$ such that

$$\int_\Sigma |u|^2 \leq k_\Sigma \int_\Sigma |\nabla u|^2 \quad \forall u \in H^1_0(\Sigma).$$

Our best estimate with high confidence that all the digits are correct is

$$13.11601 \leq \frac{1}{k_\Sigma} \leq 13.11623 \quad (1)$$

for $\partial \Sigma$ inscribed in a circle of radius 1; see Table 1. Remarkably, neither the computation nor the validation of the bounds we report require any asymptotic argumentation.

Eigenvalue problems on fractal domains and in particular on the Koch snowflake have fascinated mathematicians and physicists for many years. Various numerical results have been produced in the past [18, 15, 21, 1]. Equally, much work has been conducted on determining the analytic behaviour of eigenvalues
and eigenmodes [19, 23]. Beautiful sculptures of fractal eigenmodes have been built by Helaman Ferguson\textsuperscript{1} based on images produced by Lapidus et al. [18].

The present paper is a natural continuation of the research initiated in [1]. As there, we make use of conformal mapping techniques to achieve sharp estimates. However, a main difference is that now we are able to enclose the ground eigenvalue of $\Sigma$ inside a small interval. For this, we implement the quadratic method [20], which is based on the computation of second order relative spectra [24, 11]. Our approach comprises four main steps summarised as follows.

**Workflow of the method** (Upper and lower bounds for $k_g$).

**W1 Embedding of the region and domain monotonicity.** Consider embeddings $T_j \subset \Sigma \subset H_j$ for suitable open polygons $T_j$ and $H_j$. By domain monotonicity, upper (and lower) bounds for the Poincaré constant in $H_j$ (and $T_j$) give upper (and lower) bounds for $k_g$. Here and elsewhere $\Sigma_j$ denotes either $H_j$ or $T_j$ and is referred to as the level $j$.

**W2 Conformal transplantation.** Determine conformal maps $\Sigma_0 \rightarrow \Sigma_j$. The eigenvalue problem associated to the Laplacian with Dirichlet boundary conditions on $\Sigma_j$ transforms into a pencil eigenvalue problem on $\Sigma_0$ with singular $j$-dependent right hand side. The latter is the problem to be solved.

**W3 Formulation of the eigenvalue problem as a system.** Write the pencil eigenvalue problem for $\Sigma_0$ at level $j$ as a first order system involving the gradient operator, the divergence operators and the singular term.

**W4 Computation of the upper and lower bounds.** Compute enclosures for the smallest positive eigenvalue of the singular first order system at level $j$ by means of the quadratic method.

Some comments about this procedure are in place.

To obtain sharp estimates for the ground eigenvalue of $\Sigma$, the polygons $\Sigma_j$ in W1 should fit tightly. We have chosen the natural construction of the Koch snowflake from an equilateral triangle for $T_j$ and from a hexagon for $H_j$; see Figure 1.

In W2, $T_0$ is an equilateral triangle and $H_0$ a hexagon. The modified eigenvalue problem on these domains is sufficiently regular to allow accurate computation by the finite element method. For details see Remark 3.6.

The step W3 is not standard in the context of eigenvalue computation for the Laplacian. It is also counterintuitive, as one is left with an indefinite eigenvalue problem which is prone to spectral pollution due to variational collapse. However, an order reduction of the differential operator is crucial for accuracy in W4. As the quadratic method avoids spectral pollution, the eigenvalue bounds are guaranteed and we do not have to worry about variational collapse.

We elaborate more on each one of the steps W1-W4 in §2-5. In §6 we describe details of the implementation and report on the concrete calculations leading towards the estimate (1). The appendix A is devoted to various technical details which justify the validity of step W3.

\textsuperscript{1}Images of these can currently be seen at the author’s website http://helasculpt.com/
Embedding of the region and domain monotonicity

In this section we expand on step W1. Consider the eigenvalue problem associated with the Dirichlet Laplacian,

\[
\begin{aligned}
-\Delta u &= \omega^2 u & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega
\end{aligned}
\]  

(2) on an open region \( \Omega \). By virtue of the min-max principle, \( k_{\omega}^{g} = \frac{1}{\omega_1^2} \) where \( \omega_1 \equiv \omega_1(\Sigma) \) is the square root of the first eigenvalue of (2) for \( \Omega = \Sigma \). Domain monotonicity ensures that

\[
\Omega \subset \tilde{\Omega} \Rightarrow \omega_1^2(\tilde{\Omega}) \leq \omega_1^2(\Omega).
\]  

(3)

We will, in particular, repeatedly use this property for \( k = 1 \) and \( k = 2 \).

Let \( T_0 \) be an equilateral triangle of side length \( \sqrt{3} \) and \( H_0 \) a hexagon of side length 1, both centred at 0, such that \( T_0 \) is inscribed in \( H_0 \). The polygon \( T_j \) is constructed by attaching to the central third of each side of \( T_{j-1} \) an equilateral triangle, whereas the polygon \( H_j \) is constructed from \( H_{j-1} \) by subtracting an equilateral triangle. The two procedures are described in the following image, with \( T_j \) on the top and \( H_j \) on the bottom.

The resulting polygons for levels \( j = 0, 1, 2 \) are shown in Figure 1. Thereby defined, the polygons satisfy

\[ T_0 \subset \ldots \subset T_{j-1} \subset T_j \subset \ldots \subset \Sigma \subset \ldots \subset H_j \subset H_{j-1} \subset \ldots \subset H_0. \]

See Lemma 2.1. Hence

\[ \omega_1^2(H_j) \leq \omega_1^2(\Sigma) \leq \omega_1^2(T_j). \]

In [4] we compute upper bounds for \( \omega_1^2(T_j) \) and lower bounds for \( \omega_1^2(H_j) \) in the range \( j = 0 : 10 \). In turn, these lead to appropriate upper and lower bounds for \( \omega_1^2(\Sigma) \).

Let \( D \) be the unit disk. Then \( \omega_1^2(D) = j_{0,1} \approx 5.784 \) and

\[ 14.68 < j_{1,1} = \omega_2^2(D) \leq \begin{cases} \omega_2^2(T_j) & j \geq 1 \\ \omega_2^2(H_j) & j \geq 0. \end{cases} \]  

(4)

These estimates will be required in the step W4 as described in [3]
Figure 1: The two types of polygons, $T_j$ and $H_j$, for three different levels.

2.1 Eigenvalues from inner and outer domains

Let

$$\Sigma_T = \bigcup_{j=0}^{\infty} T_j \quad \text{and} \quad \Sigma_H = \text{int} \left( \bigcap_{j=0}^{\infty} H_j \right).$$

As we shall see next, these two open sets coincide. We define $\Sigma$ as either of them.

**Lemma 2.1.** We have $T_{j-1} \subset T_j \subset H_j \subset H_{j-1}$ for all $j \in \mathbb{N}$ and $\Sigma_T = \Sigma_H$.

**Proof.** We claim that polygon $H_j$ is obtained by attaching to each edge of $T_j$ an isosceles triangle whose base is this edge and whose height is $\frac{1}{2\sqrt{3}}$ times the length of the edge. An obvious consequence of this claim is that $T_j \subset H_j$. As $T_j$ has $3 \times 4^j$ edges each of length $3^{-j}\sqrt{3}$, the area of $S_j = H_j \setminus T_j$ is

$$|S_j| = \frac{3\sqrt{3}}{4} \left( \frac{2}{3} \right)^{2j} \to 0 \text{ as } j \to \infty.$$ 

Since $\Sigma_T$, $\Sigma_H$ and $S_j$ are all open sets it follows that indeed $\Sigma_T = \Sigma_H$.

It remains to prove the claim by induction. It is not difficult to check that the claim holds for $j = 0$. Next, assume that it holds for some $j = k > 0$. After rotation and translation let $AB$ with $A = (0,0)$, $B = (\ell,0)$ and $\ell = 3^{-k}\sqrt{3}$ be an edge of $T_k$; see Figure 2. Then by assumption $BC$ and $CA$ are edges
Figure 2: A diagram in aid of Lemma 2.1 explaining the connection between $T_j$ and $H_j$.

of $H_k$ where $C = (\frac{\ell}{2}, \frac{\ell}{2\sqrt{3}})$. By the definition of polygons $T_j$, $AD$ and $DC$ are edges of $T_{k+1}$ where $D = (\frac{\ell}{3}, 0)$. Further, by the definition of polygons $H_j$, $AE$, $ED$, $DF$ and $FC$ are edges of $H_{k+1}$ where $E = (\frac{\ell}{6}, \frac{\ell}{6\sqrt{3}})$ and $F = (\frac{\ell}{3}, \frac{\ell}{3\sqrt{3}})$. As triangles $ADE$ and $DCF$ are of the required shape, we have proved the claim for $j = k + 1$ and hence by induction for all $j \geq 0$.

Here and everywhere below $\Sigma = \Sigma_T = \Sigma_H$. As we shall see next,

$$\omega_T^2(T_j) \downarrow \omega_T^2(\Sigma) \quad \text{and} \quad \omega_H^2(H_j) \uparrow \omega_T^2(\Sigma), \quad (5)$$

in the regime $j \to \infty$.

**Lemma 2.2.** For all $j \geq 1$,

$$\omega_T^2(T_j) - \omega_T^2(H_j) \leq j^4 \frac{3^{3/4}}{2^{5/4} \pi^{9/4}} \left( \frac{1}{\sqrt{3}} \right)^j.$$

**Proof.** Apply directly Pang’s Theorem [23, Theorem 1.1] taking $\varepsilon = \frac{1}{3j+1}$, $\gamma = j_{0,1}$, $R = 1$ and $\Omega = H_0$. Note that

$$T_j \subset \{ x \in H_j : \text{dist}(x, \partial H_j) \geq \frac{1}{3j+1} \}.$$
3 Conformal transplantation

We now describe in detail step W2. Let \( f = f_j : \Sigma_0 \to \Sigma_j \) be the conformal map taking \( \Sigma_0 \) to the corresponding level \( \Sigma_j \). At times it will be useful to consider \( \Sigma_0 \) as a subset of the complex \( z \)-plane and \( \Sigma_j \) subset of the complex \( w \)-plane. When instead the polygons are viewed as lying in the two-dimensional real plane \( \mathbb{R}^2 \), the coordinates are denoted by \( y = (y_1, y_2) \in \Sigma_0 \) and \( x = (x_1, x_2) \in \Sigma_j \).

The following standard manipulations involving the composition map associated with \( f \) will be useful in later sections. We also denote by \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) the map given in real coordinates,

\[ y \mapsto x = f(y) = (f^1(y), f^2(y)). \]

As \( f \) is analytic, it satisfies the Cauchy-Riemann equations:

\[ \frac{\partial}{\partial x} f^1 = \frac{\partial}{\partial y} f^2 \quad \text{and} \quad \frac{\partial}{\partial x} f^1 = -\frac{\partial}{\partial y} f^2. \]

Hence

\[ \nabla_y f = \begin{bmatrix} \frac{\partial}{\partial x} f^1 & -\frac{\partial}{\partial y} f^1 \\ \frac{\partial}{\partial x} f^2 & \frac{\partial}{\partial y} f^2 \end{bmatrix}. \]

Thus

\[ \det(\nabla_y f) = |f'|^2 \quad \text{and} \quad (\nabla_y f)(\nabla_y f)^T = |f'|^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \]

Here

\[ |f'|^2 = (\frac{\partial}{\partial x} f^1)^2 + (\frac{\partial}{\partial y} f^1)^2 = (\frac{\partial}{\partial x} f^2)^2 + (\frac{\partial}{\partial y} f^2)^2. \]

Let \( u \in C^\infty(\Sigma_j) \) and \( v = u \circ f \). Then

\[ \nabla_y v = \nabla_y (u \circ f) = (\nabla_y f)^T \nabla_x u \circ f \]

and

\[ \Delta_y v = \text{div}_y \nabla_y v = \nabla_y \cdot [(\nabla_y f)^T \nabla_x u \circ f] \]

\[ = [\nabla_x u \circ f] \cdot \Delta f + \text{Tr} [(\nabla_y f)^T (D_x^2 u \circ f)(\nabla_y f)] \]

\[ = 0 + \text{Tr} [(\nabla_y f)^T (D_x^2 u \circ f)(\nabla_y f)] \]

\[ = |f'|^2 \Delta_x u \circ f. \]

Here \( D_x^2 u \) denotes the Hessian

\[ D_x^2 u = \begin{bmatrix} \frac{\partial^2}{\partial x_1^2} u & \frac{\partial^2}{\partial x_1 \partial x_2} u \\ \frac{\partial^2}{\partial x_2 \partial x_1} u & \frac{\partial^2}{\partial x_2^2} u \end{bmatrix} \]

and \( \Delta f \) the vector Laplacian

\[ \Delta f = \begin{bmatrix} \Delta f^1 \\ \Delta f^2 \end{bmatrix}. \]
Also
\[ \int_{\Sigma_0} |v(y)|^2 |f'(y)|^2 dy = \int_{\Sigma_j} |u(x)|^2 dx < \infty \] (6)
and
\[ \int_{\Sigma_0} |\nabla_y v(y)|^2 dy = \int_{\Sigma_j} |\nabla_x u(x)|^2 dy < \infty, \] (7)
whenever \( u \in H^1(\Sigma_j) \).

The above calculations indicate that if \( u \) is an eigenfunction of (2) for \( \Omega = \Sigma_j \), then \( v = u \circ f \) solves the transplanted eigenvalue problem

\[ \begin{cases} -\Delta v = \omega^2 |f'|^2 v & \text{in } \Sigma_0 \\ v = 0 & \text{on } \partial \Sigma_0. \end{cases} \] (8)

Moreover, if \( v \) is an eigenfunction of (8), then \( u = v \circ f^{-1} \) is an eigenfunction of (2) associated with the same eigenvalue. As we shall see later in Theorem 4.2 there is a one-to-one correspondence between the eigenfunctions of the Dirichlet Laplacian on \( \Sigma_j \) and those of a selfadjoint operator associated to (8). This is neither obvious nor an immediate consequence of classical principles, as \( |f'| \) has zeros and poles on the boundary of the domain.

There are two reasons for preferring (8) over (2). One is that even though both \( u \) and \( v \) have singularities, \( v \) is more regular than \( u \). The other reason is related to the fact that our polygons will have thousands of vertices. With techniques developed in [2, 3] we are able to efficiently and accurately compute the conformal map \( f \), even in these extreme situations. Solving the eigenvalue problem on \( \Sigma_0 \), especially for the ground eigenvalue, requires much simpler and smaller meshes than we would have needed on domains \( \Sigma_j \) with a highly complex boundary. This approach was used in [1] to compute eigenvalues of fractal regions. A possible alternative approach that avoids conformal maps would be to use a triangular mesh overlapping the boundary and enforcing the boundary condition in a weak sense, e.g., by means of the composite finite element method [10].

### 3.1 The Schwarz-Christoffel maps

We denote by \( w_k \) the corners of the polygon \( \Sigma_j \) and by \( z_k \) their pre-images under the map \( f \), so that \( f(z_k) = w_k \). In the case of the polygons \( T_0 \) and \( T_j \), we order the vertices so that \( w_k = z_k \) for \( k = 1, 2, 3 \). That is, the first three vertices of \( T_j \) are the vertices of the original triangle \( T_0 \). Similarly, for polygons \( H_0 \) and \( H_j \) we require that \( w_k = z_k \) for \( k = 1, \ldots, 6 \). The ordering of the remaining vertices is not important.

**Remark 3.1.** A conformal map between two domains is not unique, but can be made so by fixing three boundary points. For \( T_j \) this immediately ensures uniqueness. For \( H_j \), due to symmetries we are able to fix 6 vertices.

We denote the interior angles of \( \Sigma_j \) by \( \pi \alpha_k \).
\( \Sigma_j = T_j \), \( \alpha_k = 1/3 \) for \( k = 1, 2, 3 \) and \( \alpha_k = 1/3 \) or 4/3 for \( k > 3 \). The total number of corners of \( T_j \) is

\[
n(j) = 4^j 3 = 3 + (4^j - 1) + 2(4^j - 1) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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Proposition 3.2. Let \( f(z) : \Sigma_0 \rightarrow \Sigma_j \) be the conformal map defined by (11). Then, \( f \) is analytic in a neighbourhood of \( z_1, \ldots, z_m \). Moreover

\[
f(z) = w_k + (z - z_k)^{\alpha_k} \tilde{f}_k(z) \quad \forall k > m
\]

where \( \tilde{f}_k(z) \) is analytic in a neighbourhood of \( z_k \) and \( \tilde{f}_k(z_k) \neq 0 \).

The Schwarz-Christoffel formula (11) is semi-explicit as the position of the pre-vertices \( z_k \) for \( j > 0 \) is not a-priori known and needs to be computed as the solution of a non-linear parameter problem. Using a simple iteration due to Davis [12] and accelerating the computation using the fast multipole method [14], we solve this problem for hundreds of thousands of pre-vertices. The details and required modifications to the original algorithms are described in [2, 3].

3.2 Singularities of the eigenmodes

Let \( u : \Sigma_j \rightarrow \mathbb{R} \) be an eigenfunction of (2) associated with an eigenvalue \( \omega^2 \) for \( \Omega = \Sigma_j \).

Proposition 3.3. Let \( (r, \theta) \) be the local polar coordinates of \( x \in \Sigma_j \) with origin at the vertex \( w_k \). Let \( R > 0 \) be such that \( R < \min_{i \neq k} \text{dist}(w_k, w_i) \) and \( R < \frac{\pi}{2|\omega|} \).

Then for \( r \in (0, R) \)

\[
u(x) = \sum_{n=1}^{\infty} a_n J_{\frac{n\alpha_k}{\alpha_k}}(|\omega|r) \sin \left( \frac{n\theta}{\alpha_k} \right),
\]

where

\[
a_n = \frac{2}{\alpha_k \pi J_{\frac{n\alpha_k}{\alpha_k}}(|\omega|R)} \int_{0}^{\frac{\alpha_k \pi}{2}} u(R, \theta) \sin \left( \frac{n\theta}{\alpha_k} \right) \, d\theta, \quad n \in \mathbb{N}.
\]

Proof. The proof is obtained by separation of variables. \( \square \)

Remark 3.4. The condition \( R < \frac{\pi}{2|\omega|} \) ensures that \( J_{\frac{n\alpha_k}{\alpha_k}}(|\omega|R) \) is non-zero. Further, we have

\[
\left| a_n J_{\frac{n\alpha_k}{\alpha_k}}(|\omega|r) \right| \leq C \left( \frac{r}{R} \right)^{\frac{n}{\alpha_k}},
\]

giving absolute convergence of the series; see [8]. The following expansion of Bessel functions [22, 10.2.2] will be useful below

\[
J_{\nu}(z) = \left( \frac{1}{2} \right)^{\nu} \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{\left( \frac{1}{2} z^2 \right)^\ell}{\ell! \Gamma(\nu + \ell + 1)}.
\]

Now, let \( v = u \circ f : \Sigma_0 \rightarrow \mathbb{R} \) be the transplanted eigenfunction for the eigenvalue \( \omega^2 \) of [8], where \( f : \Sigma_0 \rightarrow \Sigma_j \) is the Schwarz-Christoffel map (11).

Proposition 3.5. Let \( (\rho, \varphi) \) be the local polar coordinates of \( y \in \Sigma_0 \) with origin at the pre-vertex \( z_k \). Then
1. For $\Sigma_j = T_j$

$$v(y) = b_k(\varphi)\varrho^3 + O(\varrho^5), \quad \text{for} \ k = 1, 2, 3,$$

$$v(y) = b_k(\varphi)\varrho + O(\varrho^{5/3}), \quad \text{for} \ k > 3 \text{ and } \alpha_k = 1/3,$$

$$v(y) = b_k(\varphi)\varrho + O(\varrho^2), \quad \text{for} \ k > 3 \text{ and } \alpha_k = 4/3.$$ 

2. and for $\Sigma_j = H_j$

$$v(y) = b_k(\varphi)\varrho^{3/2} + O(\varrho^{7/2}), \quad \text{for} \ k = 1, \ldots, 6,$$

$$v(y) = b_k(\varphi)\varrho + O(\varrho^2), \quad \text{for} \ k > 6 \text{ and } \alpha_k = 2/3,$$

$$v(y) = b_k(\varphi)\varrho + O(\varrho^2), \quad \text{for} \ k > 6 \text{ and } \alpha_k = 5/3.$$ 

In the above, $b_k(\varphi)$ are analytic functions of $\varphi$ for $\varrho e^{i\varphi} \in \Sigma_0$. These functions are different for each corner.

**Proof.** Combining Proposition 3.2 and Proposition 3.3 it follows that

$$v(y) = v(z) = \sum_{n=1}^{\infty} a_n J_{\frac{n}{\alpha_k}}(\varrho \rho) \sin\left(\frac{n\theta}{\alpha_k}\right)$$

near $z_k$, where

$$r e^{i\theta} = \varrho^{\alpha_k} e^{i\alpha_k \varphi} \tilde{f}_k(z)$$

and the analytic function $\tilde{f}_k(z)$ is as in Proposition 3.2. In order to make use of the expansion (12), consider the terms of the form

$$r^{n/\alpha_k+2\ell} = \varrho^{n+2\alpha_k} |\tilde{f}_k(z)|^{n/\alpha_k+2\ell} \quad n = 1, 2, \ldots \text{ and } \ell = 0, 1, \ldots.$$ 

Note that $|\tilde{f}_k(z)|^{n/\alpha_k+2\ell}$ is an analytic function of $\varrho$ and $\varphi$ in the vicinity of $z_k$ since $\tilde{f}_k(z_k) \neq 0$. Further

$$\theta = \alpha_k \varphi + \text{Arg} \tilde{f}_k(z).$$

The claimed conclusion is obtained by isolating the leading term in each one of the cases. The details for $k > m$ are omitted. For $k \leq m$, note that the singularity at $z_k$ of $v$ is the same as that of $u$, since $f$ is analytic near $z_k$ and $f(z_k) = z_k$. \qed

**Remark 3.6.** As a consequence of this proposition, it follows that the strongest singularity for $T_j$ is near the angles $\alpha_k = 1/3$ and for $H_j$ near the 6 original corners. Therefore, overall, the transplantation has reduced the strongest singularity from $r^{3/4}$ to $\varrho^{5/3}$ for $T_j$ and from $r^{3/5}$ to $\varrho^{3/2}$ for $H_j$. This implies, for example, that the first derivative of $v$ is bounded but the first derivative of $u$ is unbounded.
4 Formulation of the eigenvalue problem as a system

In this section we set the theoretical framework of the step W3. For this purpose we define a selfadjoint operator $\mathcal{T}_y$ of order 1 associated with the eigenvalue problem (8), whose squared non-zero spectrum coincides with the eigenvalues of (2). In §5 we will formulate a procedure for computing lower and upper bounds for $\text{spec}(\mathcal{T}_y)$, which involves the square of this operator. For this, the trial functions are required to lie in the operator domain of $\mathcal{T}_y$ (the form domain of $\mathcal{T}_y^2$). In §4.2 we describe explicitly an operator core $D$ in terms of the derivative of the conformal map $|f'|$.

4.1 The div-grad operator

Let

$$
\begin{pmatrix}
0 & i \text{div}_x \\
\text{grad}_x & 0
\end{pmatrix}
\begin{pmatrix}
\mathcal{G}_x \\
\mathcal{D}(\mathcal{G}_x)
\end{pmatrix}
\begin{pmatrix}
\mathcal{H}^1_0(\Sigma_j) \\
\mathcal{H}(\text{div}, \Sigma_j)
\end{pmatrix}
\rightarrow
\begin{pmatrix}
L^2(\Sigma_j)^3 \\
\times \\
\times \\
L^2(\Sigma_j)^2 \\
\rightarrow
\end{pmatrix}
\begin{pmatrix}
L^2(\Sigma_j)
\times
L^2(\Sigma_j)
\end{pmatrix}.
$$

The densely defined operator $\mathcal{G}_x : D(\mathcal{G}_x) \rightarrow L^2(\Sigma_j)^3$ is selfadjoint, because the adjoint of the minimal operator $i \text{grad}_x : H^1_0(\Sigma_j) \rightarrow L^2(\Sigma_j)^2$ is the maximal operator $i \text{div}_x : H(\text{div}, \Sigma_j) \rightarrow L^2(\Sigma_j)$ and vice versa.

Denote the selfadjoint operator associated to (2) on $\Omega = \Sigma_j$ by

$$
-\Delta_x : D(\Delta_x) \rightarrow L^2(\Sigma_j).
$$

Here the domain of the Dirichlet Laplacian is defined via von Neumann’s Theorem [17, p.275], as

$$
D(\Delta_x) = \{u \in H^1_0(\Sigma_j) : \text{grad} u \in H(\text{div}, \Sigma_j) \} \subset L^2(\Sigma_j).
$$

See Appendix A.

Lemma 4.1. The vector $\begin{pmatrix} u \\ s \end{pmatrix} \in D(\mathcal{G}_x)$ is an eigenfunction of $\mathcal{G}_x$ if and only if,

1. either $u \in D(\Delta_x)$, $-\Delta_x u = \omega^2 u$ and $s = \frac{\pm i}{|\omega|} \text{grad}_x u$

2. or $u = 0$ and $\text{div}_x s = 0$.

Moreover, $\begin{pmatrix} u \\ s \end{pmatrix}$ is associated to the eigenvalue $\pm \omega$ in the case 1 and to the eigenvalue 0 in the case 2.

Proof. See Lemma A.1 and the proof of Lemma A.2. □
Denote by \( \{u_k\}_{k \in \mathbb{N}} \subset D(\Delta_x) \) an orthonormal basis of eigenfunctions such that \( -\Delta_x u_k = \omega_k^2 u_k \). As a consequence of Lemma 4.1, the family
\[
\mathcal{E} = \left\{ \begin{bmatrix} u_k \\
 \pm s_k \end{bmatrix} \right\}_{k \in \mathbb{N}} \cup \left\{ \begin{bmatrix} 0 \\
 \sigma_n \end{bmatrix} \right\}_{n \in \mathbb{N}}
\]
where \( s_k = \frac{\pm i}{|\omega_k|} \text{grad}_x u_k \) and we pick \( \{\sigma_n\}_{n=1}^\infty \subset H(\text{div}, \Sigma_j) \) an orthonormal basis of \( \ker(\text{div}) \), is a complete family of eigenfunctions of \( G_x \) and
\[
\text{spec}(G_x) = \{ \pm \omega_k(\Sigma_j), 0 \}.
\]
In fact \( \mathcal{E} \) is an orthonormal basis of \( L^2(\Sigma_j)^3 \). Each non-zero eigenvalue is discrete and the eigenvalue zero is degenerate (infinite multiplicity).

4.2 The transplanted selfadjoint operator

Let
\[
D := \left\{ \begin{bmatrix} \tilde{v} \\
 t \end{bmatrix} \in L^2(\Sigma_0)^3 : |f'|^{-1} \tilde{v} \in H^1_0(\Sigma_0), |f'|^{-1} \text{div}_y t \in L^2(\Sigma_0) \right\}
\]
and define
\[
\mathcal{T}_y = \begin{bmatrix} 0 & i|f'|^{-1} \text{div}_y t \\
 i \text{grad}_y |f'|^{-1} & 0 \end{bmatrix} : D \rightarrow L^2(\Sigma_0)^3.
\]
Then \( \mathcal{T}_y \) is a densely defined symmetric operator.

**Theorem 4.2.** The operator \( (\mathcal{T}_y, D) \) on \( L^2(\Sigma_0)^3 \) has an orthonormal basis of eigenfunctions in its domain. The closure
\[
\overline{\mathcal{T}_y} : D(\mathcal{T}_y) \rightarrow L^2(\Sigma_0)^3
\]
is selfadjoint. Moreover,
\[
\text{spec}(\overline{\mathcal{T}_y}) = \text{spec}(G_x) = \{ \pm \omega_k(\Sigma_j), 0 \}.
\]

The remainder of this section is devoted to the proof of this theorem. Our first task will be to verify that the transplanted eigenfunctions are in the domain of \( \mathcal{T}_y \). Let
\[
v_k = u_k \circ f, \quad \tilde{v}_k = |f'| v_k, \quad t_k = (\nabla_y f)^T s_k \circ f \quad \text{and} \quad \tau_n = (\nabla_y f)^T \sigma_n \circ f.
\]

**Lemma 4.3.**
\[
\mathcal{\tilde{E}} = \left\{ \begin{bmatrix} \tilde{v}_k \\
 \pm t_k \end{bmatrix} \right\}_{k,n \in \mathbb{N}} \subset D.
\]

**Proof.** Let us first show that
\[
\begin{bmatrix} \tilde{v}_k \\
 \pm t_k \end{bmatrix} \in D.
\]
From (7) with \( u = u_k \) and \( v = v_k \), it follows that
\[
\int_{\Sigma_0} |\nabla_y v_k|^2 dy < \infty.
\]
Since \( \Sigma_0 \) is compact, by Sobolev embedding it then follows that also
\[
\int_{\Sigma_0} |v_k|^2 dy < \infty
\]
and \( |f'|^{-1} \tilde{v}_k = v_k \in H_0^1(\Sigma_0) \). This is the first condition in the definition of \( D \).

Now the second condition. Since
\[
\int_{\Sigma_0} |t_k|^2 dy = \int_{\Sigma_0} \left[ \left( \nabla_y f \right) \left( \nabla_y f \right)^T \left( \sigma_k \circ f \right) \right] \cdot \left( \sigma_k \circ f \right) dy = \int_{\Sigma_j} |\sigma_k|^2 dx
\]
we gather that \( t_k \in L^2(\Sigma_0)^2 \). Then
\[
\text{div}_y t_k = (\sigma_k \circ f) \cdot \Delta f + \text{Tr} \left( (\nabla_y f)^T \nabla_y (\sigma_k \circ f) \right)
\]
\[
= 0 + \text{Tr} \left( (\nabla_y f)^T \left( \nabla_y \sigma_k \right)^T \circ f \right) (\nabla_y f)
\]
\[
= |f'|^2 \text{div}_x \sigma_k \circ f.
\]
Hence
\[
|f'|^{-1} \text{div}_y t_k = |f'| \text{div}_x \sigma_k \circ f,
\]
so
\[
\int_{\Sigma_0} |f'|^{-1} \text{div}_y t_k|^2 dy = \int_{\Sigma_0} |f'|^2 |\text{div}_x \sigma_k \circ f|^2 dy = \int_{\Sigma_j} |\text{div}_x \sigma_k|^2 dx < \infty.
\]
This is the second condition in the definition of \( D \).

It is only left to show that
\[
\begin{bmatrix}
0 \\
\pm \Sigma_n
\end{bmatrix} \in D.
\]

On the one hand,
\[
\int_{\Sigma_0} |\sigma_n|^2 dy = \int_{\Sigma_0} \left[ \left( \nabla_y f \right) \left( \nabla_y f \right)^T \left( \varpi_n \circ f \right) \right] \cdot \left( \varpi_n \circ f \right) dy = \int_{\Sigma_j} |\varpi_n|^2 dx.
\]

On the other hand,
\[
\text{div}_y \varpi_n = |f'|^2 \text{div}_x \varpi_n \circ f = 0 \in L^2(\Sigma_0).
\]
\[\square\]
The family $E$ in this lemma is a family of eigenfunctions of $T_y$. Indeed

$$
\begin{align*}
T_y \begin{pmatrix} \tilde{E}_k^+ \\ \pm E_k^- \end{pmatrix} &= \begin{pmatrix} \pm i |f'|^{-1} \text{div}_y \xi_k \\ i \text{grad}_y \psi_k \end{pmatrix} = \begin{pmatrix} \pm i |f'| (\text{div}_y \xi_k) \circ f \\ i (\nabla_y f)^T (\text{grad}_x u_k) \circ f \end{pmatrix} \\
&= \begin{pmatrix} |f'| & 0 \\ 0 & (\nabla_y f)^T \end{pmatrix} G_x \begin{pmatrix} u_k \\ \pm s_k \end{pmatrix} \circ f \\
&= \pm \omega_k \begin{pmatrix} |f'| u_k \circ f \\ \pm (\nabla_y f)^T s_k \circ f \end{pmatrix} = \pm \omega_k \begin{pmatrix} \tilde{E}_k^+ \\ \pm E_k^- \end{pmatrix}
\end{align*}
$$

and

$$T_y \begin{pmatrix} 0 \\ \tau_n \end{pmatrix} = \begin{pmatrix} i |f'| (\text{div}_x g_n \circ f) \\ 0 \end{pmatrix} = 0.$$

In fact it is a complete family of eigenfunctions as we shall see next.

**Lemma 4.4.**

$$\text{Span } \tilde{E} = L^2(\Sigma_0)^3.$$

**Proof.** We verify that $\tilde{E}^\perp = \{0\}$. Suppose that

$$\int_{\Sigma_0} \tilde{E}_k^+ \cdot \nu \, dy = 0 = \int_{\Sigma_0} \tau_n \cdot \nu \, dy \quad \forall k, n \in \mathbb{N}. \quad (13)$$

Let $g = f^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ be the inverse map to $f$. Then $u = v \circ g$ and $s = t \circ g$ and

$$0 = \int_{\Sigma_0} |f'| u_k \circ f \\ \pm (\nabla_y f)^T s_k \circ f \, dy = \int_{\Sigma_0} u_k \circ f \\ \pm s_k \circ f \, dy = \int_{\Sigma_0} |f'| u_k \circ f \\ (\nabla_y f)^T s_k \circ f \, dy$$

for all $k \in \mathbb{N}$. Further

$$0 = \int_{\Sigma_j} |g'|^2 g_n : (\nabla_y f \circ g) \xi \, dx$$

for all $n \in \mathbb{N}$. Since $E$ is an orthonormal basis of $L^2(\Sigma_j)^3$, then

$$|g'| u = 0 \quad \text{and} \quad |g'|^2 (\nabla_y f \circ g) \xi = 0.$$

Hence, since $|g'| \neq 0$ a.e. and $\nabla_y f \circ g = |g'|^{-2} \neq 0$ a.e., $u = 0$ and $s = 0$. Thus (13) implies $\nu \tau = 0$. \hfill \square

In order to generate an orthonormal family of eigenfunctions apply Gram-Schmidt to $\tilde{E}$ which might not be orthonormal a priori. Note that in fact $T_y$ is essentially selfadjoint, [10, Lemma 1.2.2]. This completes the proof of Theorem 4.2.

**Remark 4.5.** Since $|f'|$ has singularities on $\partial \Sigma_0$, it is not a priori clear whether $(T_y, D)$ is already closed. This is a rather subtle point. We are unaware of any investigation in this respect.
5 Computation of the upper and lower bounds

We now describe how to determine bounds for the eigenvalues of the operator $T_y$ for the step $W4$. We have chosen the quadratic method \cite{20} which fully avoids spectral pollution \cite{24} and is reliable for computing eigenvalues. For a full list of references see \cite{4, Section 6.1}. For alternative approaches see \cite{26, 7}.

5.1 The quadratic method

Given a subspace $L \subset D(T_y)$ of dimension $d < \infty$, the second order spectrum \cite{11} of $T_y$ relative to $L$ is the spectrum of the following quadratic matrix polynomial eigenvalue problem which we write in weak form.

**Problem 5.1.** Find $\lambda \in \mathbb{C}$ and $0 \neq \begin{bmatrix} v \\ t \end{bmatrix} \in L$ such that

$$\left\langle (T_y - \lambda) \begin{bmatrix} v \\ t \end{bmatrix}, (T_y - \lambda^*) \begin{bmatrix} v \\ t \end{bmatrix} \right\rangle = 0 \quad \forall \begin{bmatrix} v \\ t \end{bmatrix} \in L.$$

Given a basis for the subspace $L$,

$$L = \text{span}\{b_j\}_{j=1}^d,$$

and writing

$$\begin{bmatrix} v \\ t \end{bmatrix} = \sum_{j=1}^d \alpha_j b_j \quad \text{for} \quad \alpha = (\alpha_j)_{j=1}^d \in \mathbb{C}^d,$$

this problem becomes equivalent to

$$Q(\lambda) \alpha = 0 \quad \text{for} \quad Q(z) = K - 2zL + z^2M,$$

where

$$K = [(T_y b_j, T_y b_k)]_{j,k=1}^d \quad L = [(T_y b_j, b_k)]_{j,k=1}^d \quad M = [(b_j, b_k)]_{j,k=1}^d.$$

The $\lambda \in \mathbb{C}$ solutions to Problem 5.1 are therefore the spectrum of the quadratic matrix polynomial $Q(z)$. Since $Q(z)$ is selfadjoint, this set is symmetric with respect to the real line. Since $\det M \neq 0$, it consists of at most $2d$ distinct isolated points.

The following relation between the second order spectra and the spectrum of $T_y$ is crucial below. Let

$$\mathbb{D}(a, b) = \left\{ z \in \mathbb{C} : \left| z - \frac{a + b}{2} \right| < \frac{b - a}{2} \right\}.$$

Then,

$$\left\{ \begin{array}{l} (a, b) \cap \text{spec } T_y = \{\omega\} \\ \det Q(\lambda) = 0 \\ \lambda \in \mathbb{D}(a, b) \end{array} \right\} \Rightarrow \text{Re } \lambda - \frac{|\text{Im } \lambda|^2}{b - \text{Re } \lambda} < \omega < \text{Re } \lambda + \frac{|\text{Im } \lambda|^2}{\text{Re } \lambda - a} \quad (14)$$
See [25, Remark 2.3] and [6, Corollary 2.6].

To get lower bounds for $\omega_1(\Sigma_j)$, we consider (14) fixing $a = 0$ and $b \leq \omega_2(D)$ known. In practice we choose $b < \sqrt{J_{1,1}},$

recall (4).

5.2 Finite element approximation of the eigenvalue bounds

Let $\Xi_h$ be a uniform triangulation of $\Sigma_0$, define the corresponding space of piecewise polynomials to be

$$\hat{L} = \left\{ \begin{bmatrix} v \\ t \end{bmatrix} \in C^0(\Sigma_0)^3 : \begin{bmatrix} v|_K \\ t|_K \end{bmatrix} \in P_p(K)^3 \forall K \in \Xi_h, v|_{\partial\Sigma_0} = 0 \right\},$$

and let

$$F = \begin{bmatrix} |f'| & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The problem we solve numerically is the following.

**Problem 5.2.** Find $\lambda$ and $0 \neq \begin{bmatrix} v \\ t \end{bmatrix} \in \hat{L}$ such that for all $\begin{bmatrix} \tilde{v} \\ \tilde{t} \end{bmatrix} \in \hat{L}$

$$\left\langle F^{-1}\hat{G}_y\begin{bmatrix} v \\ t \end{bmatrix}, F^{-1}\hat{G}_y\begin{bmatrix} \tilde{v} \\ \tilde{t} \end{bmatrix} \right\rangle - 2\lambda \left\langle \hat{G}_y\begin{bmatrix} v \\ t \end{bmatrix}, \begin{bmatrix} \tilde{v} \\ \tilde{t} \end{bmatrix} \right\rangle + \lambda^2 \left\langle F\begin{bmatrix} v \\ t \end{bmatrix}, F\begin{bmatrix} \tilde{v} \\ \tilde{t} \end{bmatrix} \right\rangle = 0.$$

The substitution $\begin{bmatrix} v \\ t \end{bmatrix} = F^{-1}\begin{bmatrix} w \\ \ell \end{bmatrix}$ and $\begin{bmatrix} \tilde{v} \\ \tilde{t} \end{bmatrix} = F^{-1}\begin{bmatrix} \tilde{w} \\ \tilde{\ell} \end{bmatrix}$ yields an equivalence between Problem 5.2 and Problem 5.1 where the subspaces are deformed by the action of $|f'|$,

$$\mathcal{L} = F\hat{L}.$$

Indeed Problem 5.2 is equivalent to finding $\lambda$ and $0 \neq \begin{bmatrix} w \\ \ell \end{bmatrix} \in \mathcal{L}$ such that for all $\begin{bmatrix} \tilde{w} \\ \tilde{\ell} \end{bmatrix} \in \mathcal{L}$

$$\left\langle \mathcal{T}_y\begin{bmatrix} w \\ \ell \end{bmatrix}, \mathcal{T}_y\begin{bmatrix} \tilde{w} \\ \tilde{\ell} \end{bmatrix} \right\rangle - 2\lambda \left\langle \mathcal{T}_y\begin{bmatrix} w \\ \ell \end{bmatrix}, \begin{bmatrix} \tilde{w} \\ \tilde{\ell} \end{bmatrix} \right\rangle + \lambda^2 \left\langle \begin{bmatrix} w \\ \ell \end{bmatrix}, \begin{bmatrix} \tilde{w} \\ \tilde{\ell} \end{bmatrix} \right\rangle = 0.$$

The latter is exactly Problem 5.1 for $\mathcal{L}$ given by (16). For the quadratic method to be free from spectral pollution we require $\mathcal{L} \subset D(\mathcal{T}_y)$. As we shall see next, this is indeed the case.

**Lemma 5.3.** Let $\mathcal{L}$ be given by (15) and $\mathcal{L}$ be given by (16). Then $\mathcal{L} \subset D$.
Proof. Let \( \begin{bmatrix} \tilde{v} \\ \tilde{t} \end{bmatrix} \in \hat{\mathcal{L}} \). Since

\[
\hat{\mathcal{L}} \subset \frac{H^1_0(\Sigma_0)}{H^1(\Sigma_0)^2},
\]

then \( \tilde{v} \in H^1_0(\Sigma_0) \) and \( \tilde{t} \in H^1(\Sigma_0)^2 \subset H(\text{div}, \Sigma_0) \). As the first entry of \( F \begin{bmatrix} \tilde{v} \\ \tilde{t} \end{bmatrix} \) is \( |f'|\tilde{v} \), it indeed satisfies the first condition in the definition of \( \mathcal{D} \).

Now, the second entry of \( F \begin{bmatrix} \tilde{v} \\ \tilde{t} \end{bmatrix} \) is \( \tilde{t} \). Since \( \tilde{t} \) is continuous and piecewise polynomial its divergence is bounded on \( \Sigma_0 \). Hence

\[
\int_{\Sigma_0} |f'|^{-1} \text{div}_y \tilde{t} |^2 \, dy \leq \| \text{div}_y \tilde{t} \|_\infty^2 \int_{\Sigma_0} |f'|^{-2} \, dy.
\]

The function \( |f'|^{-1} \) has singularities only on \( \partial \Sigma_0 \). From (11) it follows that the exponent of the strongest singularity is \( -\frac{2}{3} \) in the radial coordinate for \( \Sigma_0 = H_0 \) and \( -\frac{1}{3} \) for \( \Sigma_0 = T_0 \). As both are square integrable in \( \mathbb{R}^2 \),

\[
\int_{\Sigma_0} |f'|^{-2} \, dy < \infty
\]

so that indeed \( |f'|^{-1} \text{div}_y \tilde{t} \in L^2(\Sigma_0) \), ensuring the second condition in the definition of \( \mathcal{D} \).

The steps W2-W4 are valid for any non-degenerate polygons, provided a good lower bound for the second eigenvalue is at hand.

**Remark 5.4.** Let

\[
\begin{bmatrix} \tilde{v}_1 \\ \tilde{t}_1 \end{bmatrix} \in \hat{\mathcal{E}}
\]

be the normalised eigenfunction associated to the first positive eigenvalue

\( \omega_1(\Sigma_j) \in \text{spec}(\mathcal{T}_y) \).

A convergence analysis of the finite element method at individual \( \Sigma_j \) can be carried out in the context of [5, Theorem 3.2]. If there exists \( \left[ \begin{array}{c} w_h \\ \mathcal{L}_h \end{array} \right] \in \mathcal{L} \) such that

\[
\left\| \begin{bmatrix} \tilde{v}_1 \\ \tilde{t}_1 \end{bmatrix} - \left[ \begin{array}{c} w_h \\ \mathcal{L}_h \end{array} \right] \right\|_{L^2(\Sigma_0)^3} + \left\| T_y \left( \begin{bmatrix} \tilde{v}_1 \\ \tilde{t}_1 \end{bmatrix} - \left[ \begin{array}{c} w_h \\ \mathcal{L}_h \end{array} \right] \right) \right\|_{L^2(\Sigma_0)^3} < c_1 h^p,
\]

then there exists \( \lambda_h \) such that \( \det Q(\lambda_h) = 0 \) and

\[
|\lambda_h - \omega_1| < c_2 h^{p/2}.
\]

The hypothesis translates into the subspace \( \hat{\mathcal{L}} \) as follows,

\[
\left\| \begin{bmatrix} |f'|\tilde{v}_1 \\ \tilde{t}_1 \end{bmatrix} - \left[ \begin{array}{c} |f'|w_h \\ \mathcal{L}_h \end{array} \right] \right\|_{L^2(\Sigma_0)^3} + \left\| F^{-1} G_y \left( \begin{bmatrix} |f'|\tilde{v}_1 \\ \tilde{t}_1 \end{bmatrix} - \left[ \begin{array}{c} |f'|w_h \\ \mathcal{L}_h \end{array} \right] \right) \right\|_{L^2(\Sigma_0)^3} < c_1 h^p
\]

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for \( \begin{bmatrix} v_h \\ \tau_h \end{bmatrix} \in \hat{L} \). Here \( v_1 = u_1 \circ f \). This convergence analysis will be investigated in more detail elsewhere.

6 Numerical estimation of \( \omega_1^2(\Sigma) \)

In this final section we report on the specific numerical estimation of upper and lower bounds for \( \omega_1^2(\Sigma) \) from a concrete computer implementation of the steps W1-W4. We recall that in [9, Table 2] an approximation of the ground eigenvalue for the hexagon was reported as \( \omega_1^2(H_0) \leq 7.155339146 \). Later in [1, Table 2] it was found that \( \omega_1^2(\Sigma) = 13.1161843 \) with doubt over the last digit. Note that the latter estimate is within the estimate (1).

Set \( j = 1 : 10 \) in step W1. We compute the coefficients of the Schwarz-Christoffel map in step W2 by means of the highly accurate procedure described in [3, 2] coded in C++. We pick standard finite element spaces for the construction of (15) for step W4. We consider uniform meshes on \( \Sigma_0 \) made of uniformly distributed equilateral triangles of identical area. We always pick Lagrange elements of order 5 on these meshes to guarantee high accuracy. The code for the solution of Problem 5.2 was written in Comsol Multiphysics and was ran in Comsol Livelink for Matlab.

6.1 Our best estimate

Table 1 shows our best approximation for \( \omega_1^2(\Sigma) \), achieved from \( H_{10} \) for the lower bound and from \( T_{10} \) for the upper bound. The appropriate numerical values come from the last rows and they render the estimate (1) claimed in [1].

In all the calculations shown, the computer finds the eigenvalue bounds on meshes refined several times starting from an initial coarse mesh. This coarse mesh has 4 equilateral triangles for \( T_0 \) and 6 equilateral triangles for \( H_0 \). At each refinement the number of equilateral triangles is multiplied by 4 (the number of degrees of freedom is reported in the second row of Table 2).

For \( T_{10} \) accuracy stalls from the 6th to the 7th refinement, then it jumps by a considerable margin. We believe that this phenomenon is linked to the structure of the eigenfunction for \( T_{10} \) near the boundary, but we can say no more at present. Similarly a stall in accuracy occurs between the 3rd and 5th refinement for \( H_{10} \).

6.2 The optimal rate of interior-exterior domain approximation

In order to test optimality of the decreasing rate of

\[ \omega_1^2(T_j) - \omega_1^2(H_j) \]
Table 1: For level $j = 10$. Upper and lower bounds for the ground eigenvalue on $H_{10}$ and $T_{10}$. The mesh is made of equilateral triangles. At each refinement we increase the number of triangles by a factor of 4.

| $j$ (level) | $\omega_{j}^{2}(H_{j})_{\text{upper}}$ | DOF (ref) | $\omega_{j}^{2}(T_{j})_{\text{lower}}$ | DOF (ref) |
|------------|------------------|-----------|-----------------|-----------|
| 0          | 7.15533433       | 5823 (4)  | 17.5459633       | 2583 (4)  |
| 1          | 11.78144197      | 231843 (5)| 13.4027432       | 39123 (5) |
| 2          | 12.51986577      | 924483 (6)| 13.2685769       | 39123 (6) |
| 3          | 13.89178892      | 3692163 (7)| 13.1704575       | 155063 (7) |
| 4          | 13.03710567      |           | 13.1353754       | 617283 (8) |
| 5          | 13.08768534      |           | 13.1232294       |           |
| 6          | 13.10593527      |           | 13.1185291       |           |
| 7          | 13.1129516       |           | 13.1179332       |           |
| 8          | 13.11486239      |           | 13.1169426       |           |
| 9          | 13.11570298      |           | 13.1164312       | 2463363 (9) |

Table 2: For level $j = 0 : 9$. Upper and lower bounds for the ground eigenvalue on $H_{j}$ and $T_{j}$. The mesh is made of equilateral triangles. The degrees of freedom and corresponding mesh refinement are as shown.

Established in Lemma 2.2, consider Table 2 where the computer estimation of the eigenvalues at different levels ($j = 0 : 9$) is shown. Note that

$$\omega^{2}(T_{0}) = \frac{16\pi^{2}}{9} \leq 17.54597.$$ 

Therefore the lower bound for level $j = 0$ is not given, because the $b$ chosen in (14) is not below $\omega_{1}(T_{0})$.

Let

$$\tilde{\omega}^{2}(\Sigma_{j}) = \frac{\omega^{2}(\Sigma_{j})_{\text{upper}} + \omega^{2}(\Sigma_{j})_{\text{lower}}}{2}$$

be the mean of the computed bounds at corresponding level for region $\Sigma_{j}$. In Figure 3 we show a semilog (vertical axis) plot of

$$r(j) = \tilde{\omega}^{2}(T_{j}) - \tilde{\omega}^{2}(H_{j})$$

versus $j = 2 : 10$. Remarkably, the picture shows a near straight line, suggesting that, to high accuracy, the law

$$r(j) \approx C \rho^{j}$$
Figure 3: Semilog plot of $r(j)$ for $j = 2 : 10$. Remarkably the curve is very near a straight line.

is satisfied. Our computed values give $\rho \approx 0.35958 < 0.36 = \frac{9}{25}$ and

$$\frac{29}{5} = 5.8 < C \approx 5.8688 < 6.$$  

On these grounds we formulated the Remark 2.3. Note that this clean behaviour was used in [1] to accelerate by extrapolation the convergence to the eigenvalues of the fractal.

A Spectrum of matrix operators

The results presented in this appendix might be well known to specialists. However, as we could not find a suitable reference to the specific statement we required in §3, we include full details of proofs.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two possibly different separable Hilbert spaces. Let $T : D(T) \rightarrow \mathcal{H}_2$ be a densely defined closed operator from $D(T) \subset \mathcal{H}_1$ and let

$$\mathcal{E} = \begin{bmatrix} 0 & T^* \\ T & 0 \end{bmatrix}.$$  

The operator

$$\mathcal{E} : D(T) \oplus D(T^*) \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2$$

is selfadjoint, indeed note that $(T^*)^* = T^* = T$. Moreover, by von Neumann’s Theorem [17, p.275], we know that both $TT^* : D(TT^*) \rightarrow \mathcal{H}_1$ and $T^*T : D(T^*T) \rightarrow \mathcal{H}_2$ are selfadjoint in the corresponding domains of operator multiplication (of closed operators). Also $D(T^*T) \subset \mathcal{H}_1$ is a core for $T$ and $D(TT^*) \subset \mathcal{H}_1$ is a core for $T^*$. As we shall see next, the spectrum of $\mathcal{E}$ is fully characterised by the spectra of $TT^*$ and $T^*T$. Below, the point spectrum is denoted by $\text{spec}_p$.  

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Lemma A.1. \( 0 \in \text{spec}_p(\mathcal{E}) \) if and only if \( 0 \in \text{spec}_p(T^*T) \cup \text{spec}_p(T^*T) \). Moreover
\[
\text{Tr} \, \mathbb{1}_0(\mathcal{E}) = \text{Tr} \, \mathbb{1}_0(T^*T) + \text{Tr} \, \mathbb{1}_0(TT^*).
\]

Proof. Since
\[
\mathcal{E} \begin{bmatrix} u \\ v \end{bmatrix} = 0 \iff T^*v = 0 \quad \text{and} \quad Tu = 0 \iff TT^*v = 0 \quad \text{and} \quad T^*Tu = 0,
\]
the first claim follows.

For the second claim note that there is a one-to-one correspondence between a set of eigenvectors \( \left\{ \begin{bmatrix} u_n \\ \pm v_n \end{bmatrix} \right\} \) associated to \( 0 \in \text{spec}_p(\mathcal{E}) \) and \( \left\{ \begin{bmatrix} u_n \\ 0 \\ v_n \end{bmatrix} \right\} \), which possibly has some zero vectors.

In the above statement zero can be in the point spectrum of one of the operators \( TT^* \) or \( T^*T \) but not necessarily the other. This is for example the case for \( T \) the standard shift in \( \ell^2(\mathbb{N}) \).

Lemma A.2. Let \( \lambda \neq 0 \). The following are equivalent
- \( \lambda \in \text{spec}_p(\mathcal{E}) \)
- \( -\lambda \in \text{spec}_p(\mathcal{E}) \)
- \( \lambda^2 \in \text{spec}_p(TT^*) \)
- \( \lambda^2 \in \text{spec}_p(T^*T) \).

Moreover
\[
\text{Tr} \, \mathbb{1}_\lambda(\mathcal{E}) = \text{Tr} \, \mathbb{1}_{-\lambda}(\mathcal{E}) = \text{Tr} \, \mathbb{1}_{\lambda^2}(T^*T) = \text{Tr} \, \mathbb{1}_{\lambda^2}(TT^*).
\]

Proof. Let \( \lambda \in \text{spec}_p(\mathcal{E}) \) and \( \text{Tr} \, \mathbb{1}_\lambda(\mathcal{E}) = m \). Then there exists a linearly independent set
\[
\left\{ \begin{bmatrix} u_n \\ v_n \end{bmatrix} \right\}_{n=1}^m \subset \text{D}(\mathcal{E}) \quad \text{such that} \quad (\mathcal{E} - \lambda) \begin{bmatrix} u_n \\ v_n \end{bmatrix} = 0.
\]

Then \( T^*v_n = \lambda u_n \) and \( Tu_n = \lambda v_n \) and, necessarily, \( u_n \neq 0 \) and \( v_n \neq 0 \) for all \( n \in \{1, \ldots, m\} \). Thus also the set \( \left\{ \begin{bmatrix} u_n \\ -v_n \end{bmatrix} \right\}_{n=1}^m \) is linearly independent and
\[
(\mathcal{E} + \lambda) \begin{bmatrix} u_n \\ -v_n \end{bmatrix} = 0. \text{Therefore} \quad -\lambda \in \text{spec}_p(\mathcal{E}) \text{and} \quad \text{Tr} \, \mathbb{1}_{-\lambda}(\mathcal{E}) = m.
\]

Now, as
\[
\left\{ \begin{bmatrix} T^*v_n \\ v_n \end{bmatrix} \right\}_{n=1}^m = \left\{ \begin{bmatrix} u_n \\ v_n \end{bmatrix} \right\}_{n=1}^m,
\]
the former is a linearly independent set of eigenvectors with \( v_n \in \text{D}(T^*) \) and \( T^*v_n \in \text{D}(T) \). Then \( TT^*v_n = \lambda^2 v_n \) for the set of non-zero vectors \( \{v_n\}_{n=1}^m \subset \text{D}(TT^*) \). Assume that \( \text{Tr} \, \mathbb{1}_{\lambda^2}(TT^*) = l < m \). Then
\[
\{v_n\}_{n=1}^m \subset \text{span}\{\hat{v}_j\}_{j=1}^l
\]
for a linearly independent set \{\tilde{v}_j\}_{j=1}^l. Hence
\[
v_k = \sum_{j=1}^l a_j \tilde{v}_j \quad \text{and} \quad T^* v_k = \sum_{j=1}^l a_j T^* \tilde{v}_j
\]
for some \(k \in \{1, \ldots, m\}\). Thus
\[
\left[ \frac{1}{\lambda} T^* v_k \right] = \sum_{j=1}^l a_j \left[ \frac{1}{\lambda} T^* \tilde{v}_j \right]
\]
which is a contradiction. Therefore \(l \geq m\). But let \(\hat{u}_j = \frac{1}{\lambda} T^* \tilde{v}_j\) and consider the set \(\left\{ \hat{u}_j \right\}_{j=1}^l \subset D(\mathcal{E})\). This is a linearly independent set of eigenvectors of \(\mathcal{E}\) for \(\lambda\). This shows that necessarily \(l = m\).

All the above, and a symmetric argument involving \(u_n\) instead of \(v_n\) and \(T^* T\) instead of \(T T^*\), are enough to prove the claim.

\begin{proof}
\end{proof}

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