INVARINATS OF 3D TRANSFORMATION FOR POINT ROTATION COORDINATE FRAMES

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In Nature there is no so small thing which, by intend look turned to it, would not grow up to infinity
L. N. Tolstoi

I. INTRODUCTION

Recently the general linear transformation for point rotation coordinate frames was considered [1]. A distinguishing feature of the frame, in contrast to the Cartesian one, is the existence of the rotation axis at every point. The frame coordinates are an angle and time, the frequency of rotation is a parameter. The concept of the frame originated from the optical indicatrix (index ellipsoid) [2]. Rotation of the optical indicatrix arises in three-fold electrooptical crystals under the action of the rotating electric field applied perpendicular to the optical axis [3]. Such a rotation is possible as in the Pockels as Kerr crystals and also in the isotropic Kerr medium. The rotation is used in single-sideband modulators [4], [5]. The single-sideband modulation has very interesting features from the theoretical viewpoint. In applications it may be used for the frequency modulation and frequency shifting. In contrast to usual modulation such a shifting is ”100% transformation” of the initial into output frequency. However at present the modulation practically is not in use. It is connected with the high controlling voltage of bulk modulators [5]; creating waveguide single-sideband modulators calls for considerable technological efforts [6].

Two point are essential by the consideration of a plane circularly polarized wave propagating through a medium with the rotating optical indicatrix. That is the necessity to use the non-Cartesian point rotation frame and the necessity to know what is the frequency superposition law by the transition from one rotating frame to another. The usual description in the Cartesian frame tacitly assumes that this law is Galilean or linear one. In this law the frequency of any field may be infinitely large.

In the general case considered in [1] the reverse frequency, i.e., the frequency of the second frame relative to the first one, is a function of the direct frequency. However both the frequencies are assumed to be symmetric, i.e., the direct frequency is the same function of the reverse frequency. Using symmetry of the transformation under interchanging coordinates and assuming that this function is kept by such a interchange, it was shown that three different types of the transformation are possible. The first type is a generalization of the Lorentz transformation. The second and third types are principally different and possess unusual properties, in particular, an uncertainty of time determination and solutions with lower and upper frequency boundaries.

The point rotation frames have not transverse coordinates, however a coordinate along the axis of rotation can be used as the space coordinate. The approach developed in [1] for two-dimensional (2D) case is unacceptable for the 3D case. In this paper we somewhat modify the approach using again only symmetry. The main idea of the modification is a ”velocity invariant” which remains unchanged by replacing any coordinate by another and the use of the invariant for the construction of an additional condition. The condition together with the condition of the speed of light constancy defines the transformation parameters.

II. TWO-DIMENSIONAL TRANSFORMATION

In this section we demonstrate the approach on an example of the two-dimensional transformation and show as the Lorentz transformation may be deduced in such an approach.

The general form of the linear transformation for the transition from one frame to another can be written as follows

$$\tilde{z} = q(z - ut), \quad \tilde{t} = \frac{q^2 q - 1}{q^2 u} z - q z t,$$

where $z$ and $t$ is an space coordinate and time, $u$ is the velocity of the second frame relative to first one, the tilde corresponds to reverse values. Eq. (1) turns out into the reverse transformation if variables with the tilde change to variables without the tilde and vice versa. Moreover the transformation is invariant by interchanging the coordinates $z$ and $t$ correspondingly normalized.
In the general case $\tilde{u}$ and $u$ are connected by some dependence. This dependence is assumed to be symmetric about $\tilde{u}$ and $u$: if $\tilde{u}$ is a function of $u$, then $u$ is the same function of $\tilde{u}$. Parameters $I_1 = (\tilde{u} + u)/2$, $I_2 = (\tilde{u} - u)^2/4$ are invariant under the reverse transformation therefore the dependence may be rewritten in the form $I_2$ as some function of $I_1$ or vice versa. Obviously the same is valid for another pair of parameters $I_1$ and $\tilde{u}u \equiv (I_1^2 - I_2^2)$ used early in [1]. The function is assumed to be differentiable infinite number of times and may be presented as power series:

$$I_2 = \sum_{n=1}^{\infty} \lambda_n I_1^n, \quad \text{or} \quad I_1 = \sum_{n=1}^{\infty} \lambda_n I_2^n$$  \hspace{1cm} (2)

In [1] it was assumed that if to substitute into \((2)\) corresponding velocity obtained by interchanging coordinates then the equality in \((2)\) is kept and this equality may be used as a condition for the determination of $q$. However this approach cannot be used for 3D case with the space and angle coordinate since interchange these coordinates corresponds to replacing velocity by frequency. Below we modify this approach. We form a "velocity invariant" keeping by such an interchange and construct from this invariant an additional condition for $q$.

We start from the normalization of \((1)\) as that the reverse velocity is equal direct one with the opposite sign. An example of the normalization is $z_n = z\sqrt{-u/u}$, $t_n = t$, $\tilde{z}_n = \tilde{z}\sqrt{-u/u}$, $q_n = q(-\tilde{u}/u)$, $u = \sqrt{-u} = -\tilde{u}$, where the subscript "n" corresponds to a normalized value.

Consider symmetry of the normalized transformation \((1)\) about the change of coordinates $z \rightarrow h\zeta$, $t \rightarrow b\zeta$, where $h, b$ are dimensional coefficients. In the new variables the transformation has the same form \((1)\) if the new and old parameters are connected by relations

$$Q = q, \quad U = \frac{h}{b} \frac{\tilde{q}q - 1}{\tilde{qq}u},$$  \hspace{1cm} (3)

where $Q, U$ are new $q, u$ respectively.

The expression ("velocity invariant")

$$I = bu + h \frac{\tilde{q}q - 1}{\tilde{qq}u}$$  \hspace{1cm} (4)

is kept under the above change. We may form two even invariants from the velocity invariant: $I_1 = \tilde{I} + I$, and $I_2 = (\tilde{I} - I)^2$. The invariants are kept under the sign change of $u$. In the given two-dimensional case $\tilde{I} + I = 0$ and, substituting $I_1, I_2$ into Eq. \((2)\) we find that the equation has a "fundamental solution" $I = 0$. $h/b$ has the dimension of velocity squared. Let $h/b = c^2$, where $c$ is the speed of light, then the equality $I = 0$ is non other than the condition of the speed of light constancy. From the equality one follows

$$\tilde{q}q = \frac{1}{1 - u^2/c^2}.$$  \hspace{1cm} (5)

We assume the equivalence of $\tilde{q}$ and $q$ that $q$ must be an even function of $u$: $\tilde{q}(u) \equiv q(-u) = q(u)$.

### III. THREE-DIMENSIONAL TRANSFORMATION

The transformation in the 3-dimensional case may be written in the following general form

$$\tilde{\varphi} = q_1 \varphi + p_1 (z - ut) - \nu t,$$
$$\tilde{z} = q_2 [p_2 (\varphi - \nu t) + z - ut],$$
$$\tilde{t} = q_3 [\varphi + p_3 z + q_3 t].$$  \hspace{1cm} (6)

We consider this transformation in application to electrooptics but assume that such a transformation has more general character. In \((6)\) $\varphi$ is the angle between the electric vector of a plane circularly polarized light wave and an direction in the first frame, $t$ is time, $\nu$ is the frequency of the second frame relative to first one, $u$ is the velocity of the second frame relative to the first one, all parameters are functions of $\nu, u$. The terms $p_1(z - ut)$ and $p_2(\varphi - \nu t)$ describe the velocity and frequency mismatch between the modulating and modulated wave. It may be shown that parameters of the transformation are connected by five independent equations:

$$q_{31} = \frac{1}{u} (\tilde{p}_2 q_1 + q_2 p_2), \quad q_{32} = \frac{1}{\nu} (q_1 p_1 + \tilde{p}_1 q_2),$$
$$q_{33} = \frac{\nu'}{\nu} q_1 = \frac{u'}{u} q_2, \quad (\tilde{q}_1 - \frac{\nu'}{u} \tilde{p}_2 q_2)(q_1 - \frac{\nu}{u} p_2 q_2) = 1.$$  \hspace{1cm} (7, 8)
where \( \nu' = \nu + p_1 u, \ u' = u + p_2 \nu \). Except \( \nu, u \), the transformation (1) has 7 parameters therefore 2 from them are indeterminate and two extra conditions are necessary. The first condition, as in the Lorentz case, is the speed of light constancy. The second one follows from symmetry considerations.

Now we make normalization \( \varphi_n = \zeta \varphi, \ z_n = \zeta z, \ q_{1n} = \zeta^2 q_1, \ q_{2n} = \zeta^2 q_2, \ p_{1n} = \zeta p_1, \ p_{2n} = \zeta p_2, \ \nu_n = \sqrt{-\nu \nu}, \ \nu_n = -\sqrt{-\nu \nu}, \ u_n = \sqrt{-uu}, \ \bar{u}_n = -\sqrt{-uu}, \)

\[
\zeta = \sqrt{\frac{\upsilon}{\upsilon'}}, \ \zeta = \sqrt{-\frac{\nu}{\nu'}}, \ \xi = \sqrt{\frac{-u}{u}}, \ \bar{\xi} = \sqrt{-\frac{u}{u'}}.
\]  

(9)

Analogously to the Lorentz case we assume equivalence not only the direct and reverse transformation but also the direct and opposite motion. It means that in the normalized units \( q_k(u, \nu) = q_k(-u, -\nu) = q_k(-u, \nu) \), where \( k = 1, 2 \).

**A. The speed of light constancy condition**

We use the speed of light constancy as one from conditions for the definition of the transformation parameters.

Consider circularly polarized light propagating through a medium with rotating optical indicatrix (a single-sideband modulator \([4, 5]\)). Let \( \omega = \varphi/t, V = z/t \) to be the frequency and velocity of the light wave in the first frame. If after passing the modulator the polarization is kept then frequency and velocity at the output remains unchanged. If the polarization is reversed \( \tilde{\omega} \rightarrow -\tilde{\omega} \) (as it is in the single-sideband modulator at the half-wave condition \([5]\)) then, using the direct and reverse transformation (6), we can find the output velocity in the initial frame

\[
V' = \frac{V - 2\bar{p}_2 \bar{q}_2 q_1 (\omega + p_1 V - \nu')}{1 - 2q_3 q_1 (\omega + p_1 V - \nu')}.
\]

(10)

If \( V \) equals the speed of light \( c \) then \( V' = c \) and we obtain from (10) the first condition

\[
q_{31} c = q_2 p_2.
\]

(11)

Due to this condition the velocity in the second frame

\[
\tilde{V} = \frac{q_2 (p_2 \omega + V - u' \bar{\nu})}{q_{31} \omega + q_{32} V + q_{33}}
\]

(12)

equals \( c \) as well. Moreover we can express all parameters in terms of \( \nu, u \) and \( \gamma \equiv q_2/q_1 \). For the determination of \( \gamma \) an additional condition is necessary.

**B. The second condition**

The transformation (6) is kept under the change \( (\varphi, z, t) \rightarrow (a_\varphi \varphi, h_\varphi z, b_\varphi t), \ (\varphi, z, t) \rightarrow (h_t z, b_\varphi z, a_\varphi \varphi), \ (\varphi, z, t) \rightarrow (b_t z, a_\varphi \varphi, h_t t) \), where \( a, b, h \) with corresponding indices are dimensional coefficients. Starting from (6) and using the change, we can construct the form (“velocity invariant”)

\[
\rho \frac{q_1 p_1}{q_2} + \tau \bar{\nu} + \frac{1}{\rho} \frac{q_2 p_2}{q_1} + \frac{1}{\beta} \bar{u} + \frac{1}{\tau} \frac{q_3}{q_1} + \beta \frac{q_{32}}{q_2},
\]

(13)

where \( \beta^2 \equiv h_\varphi/b_\varphi, \ \tau^2 \equiv a_\varphi/h_\varphi, \ \rho^2 \equiv a_1/b_1, \ h_\varphi^2 = a_1 b_1, \ b_\varphi^2 = a_\varphi h_\varphi, \ a_\varphi^2 = b_\varphi h_\varphi \) and

\[
\rho = \beta \tau.
\]

(14)

The constants \( \rho, \tau, \beta \) have the dimension of length, time and velocity respectively. Analogously to the two-dimensional case we equate

\[
\beta^2 = c^2.
\]

(15)

Exclude the constant \( \rho \) with help of (14), then in the normalized units \( \tau \nu \rightarrow \nu, \ u/\beta \rightarrow u \) the invariant may be rewritten as

\[
I = \frac{\nu^2 (1 - u^2) - 1}{u^2 \nu (1 + u) + 1} [\nu - u (1 + u)] + \frac{\nu (\gamma - 1)}{u \gamma} - \nu - 2u + 2 \frac{\nu^2 (1 - u^2) - 1}{\nu (1 + u) + 1}.
\]  

(15)
where the positive value of \( \beta \) is used. For negative \( \beta \) the last term in (15) is excluded, this case is connected with negative \( \rho \) or \( \tau \), i.e., with the reflection of \( z \) or \( t \) at interchange coordinates. In contrast to the two-dimensional case the invariant (15) cannot be equal zero for arbitrary \( \nu, u \). Apparently the most probable value of the characteristic constants \( \tau \) is of the order of "nuclear" time \( \sim 10^{-23} \) sec. With this values normalized frequencies of the electromagnetic field in the optical range are of the order of \( 10^{-8} + 10^{-9} \).

Because of assumed \( q_1(u, \nu) \) symmetry, \( \gamma \equiv q_2/q_1 \) also must keep under the reverse transformation \( \tilde{\gamma}(u, \nu) \equiv \gamma(-u, -\nu) = \gamma(u, \nu) \). With this in mind we first symmetrize the invariant (15) with respect to the reverse transformation

\[
I_s = \frac{1}{2}[\tilde{I} + I] = \frac{d'}{u\gamma D\nu} - \frac{d}{\gamma D} - \frac{2\ d\gamma u}{\nu\ D},
\]

where

\[
d = \gamma^2(1 - u^2) - 1, \quad d' = \gamma^2(1 + u^2) - 1, \quad D = d + 2\gamma + 2.
\]

Moreover we assume that \( \gamma \) must be a even function under the sign change of the velocity \( \gamma(u, \nu) = \gamma(-u, \nu) \). The second condition also must possess this property. Therefore we construct two even invariants.

\[
I_1 \equiv \frac{1}{2}[I_s + I_s(-u)] = \frac{d}{\gamma D}, \quad I_2 \equiv \frac{1}{4}[I_s - I_s(-u)]^2 = \left( \frac{d\nu}{u\gamma D} - \frac{2\ d\gamma u}{\nu\ D} \right)^2
\]

The invariants play the same role as in the two-dimensional case. Accordingly to our approach we must postulate some dependence between \( I_1, I_2 \) and consider this dependence as the second condition. Analogously to the two-dimensional case we assume that this dependence is differentiable infinite number of times and can be presented in the form of a power series

\[
\left( \frac{d\nu}{u\gamma D} - \frac{2\ d\gamma u}{\nu\ D} \right)^2 = \sum_{n=1}^{\infty} \lambda_n \left( \frac{d}{\gamma D} \right)^n.
\]

The dependence \( I_1 \) equals power series in \( I_2 \) give the same results and the choice of Eq. (19) is dictated by simplicity of the expression.

Since coefficients \( \lambda_n \) are unknown we restrict ourselves to the expansion \( \gamma \) in the vicinity of \( u = 0 \) or \( \nu = 0 \). Consider parameters \( p_1, p_2 \). Using the condition (11) we find from (7), (8)

\[
p_1 = \frac{\nu}{c\ u^2[\gamma(1 + u) + 1]} - \frac{\gamma^2(1 - u^2) - 1}{\nu\gamma(1 + u) + 1}, \quad p_2 = \frac{\nu\gamma(1 + u) + 1}{c\ u^2[\gamma(1 + u) + 1]}.
\]

We require that \( p_1, p_2 \) must be bounded at \( u \to 0 \). Therefore the expansion of \( \gamma \) in power series in \( u^2 \) is \( \gamma = 1 + \vartheta_2 u^2 + \vartheta_4 u^4 + \ldots \), where \( \vartheta_n \) is a function of \( \nu \). For \( \vartheta_2 \) we obtain from (19)

\[
\vartheta_2 = \frac{3 - \sqrt{1 - 2\nu^2/\lambda_1}}{2[1 + \sqrt{1 - 2\nu^2/\lambda_1}]}.
\]

With this value the transformation (6) in the limit \( u \to 0 \) has the form

\[
\hat{\varphi} = \varphi + \nu \cdot \frac{1 - \sqrt{1 - 2\nu^2/\lambda_1}}{1 + \frac{\sqrt{1 - 2\nu^2/\lambda_1}}{\lambda_1}} z - \nu t, \quad \hat{z} = z, \quad \hat{t} = t,
\]

where the normalized values \( z \to z/\rho, \ t \to t/\tau \) are used. For the positive \( \lambda_1 \) values of \( \nu^2 \) have the upper boundary \( \nu^2 \leq \lambda_1/4 \).

From the other hand if \( \nu = 0 \) then the solution of Eq. (19) \( d = 0 \) with \( \gamma = 1/\sqrt{1 - u^2} \) exist. This value may be used as the first term of the expansion \( \gamma = 1/\sqrt{1 - u^2} + \gamma_2 u^2 + \gamma_4 u^4 + \ldots \), where \( \gamma_n \) is a function of \( u \). In this case

\[
\gamma_2 = \frac{1}{2\sqrt{1 - u^2}} - 1 + \frac{8u^2}{\lambda_1}, \quad \gamma_4 = \frac{1}{2\sqrt{1 - u^2}} - 1 + \frac{8u^4}{\lambda_1}.
\]

where \( \lambda = \lambda_1(1 - u^2)(\sqrt{1 - u^2} + 1) \). If \( u \to 0 \) and \( u \to 1 \), then \( \gamma_2 \to u^2/2\lambda_1 \), and \( \gamma_2 \to 1/(2\lambda_1\sqrt{1 - u^2}) \). With the solution (23) the transformation (6) exactly corresponds to the Lorentz transformation in the limit \( \nu \to 0 \)

\[
\hat{\varphi} = \varphi, \quad \hat{z} = \frac{z - ut}{\sqrt{1 - u^2}}, \quad \hat{t} = \frac{-uz + t}{\sqrt{1 - u^2}}.
\]
The solution in the form of an expansion of $\gamma$ in a power series in $\nu^2$ represents a generalization of the Lorentz transformation. It may be straightforwardly shown that if the characteristic time $\tau$ is of the order of "nuclear time" $\sim 10^{-23}\text{sec}$ then the generalized Lorentz transformation as well as the transformation in the form (22) cannot give an extra frequency shift by the single-sideband modulation discussed in [1].

Numerical calculations of $\gamma$ for different dependences $I_2 = I_2(I_1)$ as in the form of powers series as other forms demonstrate a whimsical variety of solutions.

However the question on the explicit form of this dependence remains open.

IV. CONCLUSION

We have considered 3D transformation for the point rotation coordinate frames with coordinates angle, length and time. Main problem of such a transformation is two additional conditions for the definition of the transformation parameters and the expression of parameters as functions of velocity and frequency. The speed of light constancy can be used as one condition. However there is no a principle like relativity principle for the second condition. Instead we use a "velocity invariant". The invariant, constructed from parameters corresponding to velocity at every interchanging the coordinates, is kept by such an interchange as well as the transformation itself. In approach considered in the paper two even invariants are extracted from this invariant. A dependence between these invariants defines the second condition. This dependence is not completely definite therefore the transformation may be investigated only in vicinity of zeroth velocity or frequency. If the frequency tends to zero then in this limit the transformation coincides with the Lorentz transformation.

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