ABSTRACT

A series of sigma models with torsion are analysed which generate their mass dynamically but whose ultra-violet fixed points are non-trivial conformal field theories – in fact SU(2) WZW models at level $k$. In contrast to the more familiar situation of asymptotically free theories in which the fixed points are trivial, the sigma models considered here may be termed “asymptotically CFT”. These theories have previously been conjectured to be quantum integrable; this is confirmed by postulating a factorizable S-matrix to describe their infra-red behaviour and then carrying out a stringent test of this proposal. The test involves coupling the theory to a conserved charge and evaluating the response of the free-energy both in perturbation theory to one loop and directly from the S-matrix via the Thermodynamic Bethe Ansatz with a chemical potential at zero temperature. Comparison of these results provides convincing evidence in favour of the proposed S-matrix; it also yields the universal coefficients of the beta-function and allows for an evaluation of the mass gap (the ratio of the physical mass to the $\Lambda$-parameter) to leading order in $1/k$. 

* Supported by a fellowship from the EU Human Capital and Mobility programme
** On leave from: Department of Physics, University of Wales, Swansea, SA2 8PP, U.K.
1. Introduction

A theory like QCD with massless quarks in four dimensions has no explicit mass parameters in its classical Lagrangian; instead a mass scale $\Lambda$ is generated dynamically at the quantum level. The quantity $\Lambda$ sets the scale of low-energy physics so that the masses of all states in the theory, glue-balls, protons, etc., are simply numbers times $\Lambda$. These numbers are notoriously difficult to extract in QCD, either on the lattice or analytically. At energies much greater than $\Lambda$, on the other hand, the theory is asymptotically free and perturbation theory can be used reliably. In the language of the renormalization group (RG), QCD is described by a trajectory emanating from a fixed point which corresponds to a free theory of gluons and quarks, the direction of the trajectory being determined by an operator which is marginally relevant, by which we mean that it is marginal but not truly marginal. The fact that the operator is marginal means that no explicit mass scale is introduced at the fixed point itself, whilst the fact that it is not truly marginal means that conformal invariance is broken by the dynamical generation of the scale $\Lambda$ as one moves away from the fixed point. The RG trajectory is specified by a running coupling constant $e(h)$ which depends upon the mass scale scale $h$ being probed and which in the ultra-violet regime (large $h$) behaves like

$$\frac{1}{e(h)} = \beta_0 \ln(h/\Lambda) + \frac{\beta_1}{\beta_0} \ln \ln(h/\Lambda) + O\left(\frac{\ln \ln(h/\Lambda)}{\ln(h/\Lambda)}\right),$$

(1.1)

which in fact serves to define $\Lambda$ precisely. In the above $\beta_0$ and $\beta_1$ are universal numbers which appear as the first two coefficients of the beta-function in perturbation theory.

A more general situation can be envisaged for a theory with dynamical mass generation, namely the ultra-violet fixed point of the theory, while necessarily conformally invariant, need not be free. The purpose of this paper is to analyse such a situation in two dimensions in which the ultra-violet fixed point is a non-trivial Conformal Field Theory (CFT) – in fact a WZW model. The direction of the RG trajectory is again determined by some marginally relevant operator and we say that the theory is “asymptotically CFT”.

In the case of QCD the main difficulty is the absence of non-perturbative calculational techniques which can be applied in the low-energy regime. In two dimensions, however, there is a rich class of asymptotically-free theories which are integrable: the $O(N)$ sigma models; the principal chiral models; and the Gross-Neveu models. In these theories the existence of higher spin conserved charges means that the S-matrix factorizes, a property which allows in some cases for its complete determination (see [1], [2] and [3] respectively) yielding an exact description of the low-energy physics. These integrable theories are
therefore particularly interesting from a theoretical point of view since they provide an arena in which one can attempt to understand the connection between the infra-red and ultra-violet regimes. Such a connection is also important in order to confirm the S-matrices written down for these models. This is because the S-matrices must, in the first instance, be regarded as conjectures which should be tested; in particular the question of CDD ambiguities must be resolved.\footnote{In the case of the SU(N) chiral Gross-Neveu model there is a derivation of the S-matrix from first principles via the Bethe Ansatz \cite{4}.}

In a series of papers (\cite{5,6} for the O(N) sigma model, \cite{7} for the SU(N) principal chiral model, \cite{8} for the SO(N) and Sp(N) principal chiral models and \cite{9} for the O(N) Gross-Neveu model) various authors have used a technique relying on integrability to relate the infra-red and ultra-violet physics of families of integrable models, building on the original work of \cite{10,11}. The idea is to compute a particular physical quantity – the free-energy in the presence of a coupling to a conserved charge – in two ways: firstly from the S-matrix using a technique known as the Thermodynamic Bethe Ansatz (TBA) and secondly from the lagrangian via perturbation theory. For the cases mentioned above the results of the two calculations are found to be in perfect agreement in the ultra-violet regime, thus resolving the problem of CDD ambiguities and, as a bonus, yielding an exact expression for the mass gap (the ratio of the physical mass to the $\Lambda$-parameter). As well as providing a very stringent test of the form the S-matrix, knowing the mass-gap ratios is interesting in its own right as they can be compared directly with the results of lattice simulations.

In this paper we analyse a class of theories which are asymptotically WZW models based on the group SU(2) at level $k$ (see e.g. \cite{12,13}). In accordance with the general situation described above, the model will correspond to an RG trajectory defined by the marginally relevant operator $\text{Tr}(J_L J_R)$ in the WZW theory, where $J_L$ and $J_R$ are the usual left/right conserved Kac-Moody currents. This family of examples fits into the general scheme of “massive current algebras” set out in \cite{14}. It is crucial that the theories we consider can also be described explicitly at the lagrangian level: they are in fact sigma models with a Wess-Zumino (WZ) term defined on the group manifold SU(2). This family of lagrangians was first written down by Balog et al in \cite{15} who argued further that the resulting theories should be quantum integrable. What was not so clear in their work was whether the models would lie in the class of massive current algebras at the quantum level. Our strategy for showing that these models do lie in that class, and in particular that they are quantum integrable, is to use the exact S-matrices that has been proposed by Ahn et al \cite{16} to describe perturbations of WZW models. We shall then use the ideas of \cite{5–9} to carry out a highly non-trivial consistency check between the lagrangian formulation of \cite{15} and the S-matrix written down in \cite{16} in the manner we have already outlined above. As a by-product we will extract an expression for the mass gap valid to leading order in $1/k$.\footnote{In the case of the SU(N) chiral Gross-Neveu model there is a derivation of the S-matrix from first principles via the Bethe Ansatz \cite{4}.}
The paper is organized as follows. In section 2 we discuss the lagrangian for the model, its current algebra, and its renormalization to one-loop. In section 3 we write down the S-matrix conjectured to describe the quantum scattering and in section 4 this is used in conjunction with TBA techniques to calculate the response of the free-energy to an external field. Section 5 contains a calculation of this same quantity in perturbation theory, after which we compare the expressions to confirm the choice of S-matrix and extract the mass gap of the model. We conclude with some further remarks in section 6.

2. The lagrangian, current algebra and one-loop renormalization

The integrable field theories that we shall investigate are described in two-dimensional Minkowski space-time (with coordinates $\xi^\mu = (\tau, \sigma)$) by the lagrangian density [15]

$$L_0 = \frac{1}{2e^2} \left\{ \frac{1}{x^2 - 1} (\partial_\mu w)^2 + \frac{\beta(w)}{x + 1} (\partial_\mu n_a)^2 + \frac{1}{x + 1} \left[ \frac{1}{\sqrt{x^2 - 1}} \left( \frac{\pi}{2} - w \right) - \alpha(w) \right] \epsilon_{abc} \epsilon^{\mu\nu} n_a \partial_\mu n_b \partial_\nu n_c \right\},$$

(2.1)

with

$$\beta(w) = \frac{\cos^2 w}{x + \cos 2w}, \quad \alpha(w) = \sqrt{\frac{x - 1}{x + 1}} \sin w \cos w.$$

(2.2)

The fields $(w, n_a)$ parameterize the SU(2) group manifold in such a way that a general group element can be written $g = \cos w + in_a \sigma_a \sin w$ where the $\sigma_a$’s are the Pauli matrices and the fields $n_a$ are constrained via $n_1^2 + n_2^2 + n_3^2 = 1$. $e$ and $x$ are coupling constants with $x > 1$.

The complicated form of $L_0$ requires some explanation. The most important point is that it ensures that the resulting theory is classically integrable – in fact it ensures the existence of a canonical structure consisting of two commuting current algebras [15] – precisely the structure studied in [16]. We shall elaborate on this point below.

The theory has an SU(2) global symmetry generated by transformations $n_a \mapsto n_a + \epsilon_{abc} q_b n_c$ for parameters $q_b$. Finite symmetry transformations are given by the adjoint action $g \mapsto hgh^{-1}$, using some $h \in$ SU(2). This is to be contrasted with the principal sigma-model and WZW model which both have chiral SU(2) × SU(2) global symmetries. Our models are invariant under just the diagonal subgroup.

The antisymmetric term in (2.1) is an example of a Wess-Zumino (WZ) term and as usual its presence leads to a quantization condition on coupling constants which is essential in order to obtain a consistent quantum theory. In the present case this condition is

$$\frac{2\pi}{e^2(x + 1)\sqrt{x^2 - 1}} = k \in \mathbb{N}.$$  

(2.3)
One way to derive this is to consider the integral of the curl or exterior derivative of the WZ term over an arbitrary three-sphere, as in [12], and to demand that this always be a multiple of $2\pi$. Alternatively one can require that the WZ term itself, although not globally well-defined, is ambiguous only up to multiples of $2\pi$. In our case we can choose the ranges of our coordinates to be, for example, $0 \leq w < \pi$ with $n_a$ labelling any point on a two-sphere, which covers SU(2) except for one point. Then we demand that the integral of the WZ term should be changed by $2\pi$ on sending $w \rightarrow w + \pi$ which gives exactly the condition above. Yet a third possibility is to appeal to the general representation theory of Kac-Moody algebras because, as we shall see below, the combination in (2.3) appears as a central term in the current algebras which are responsible for the integrability of this model.

All the information concerning the model (2.1) that we shall need in the remainder of this paper has now been set down. However, in view of the brevity of the presentation in [15] (and because there appear to be a number of numerical misprints in the relevant equations which can only be detected after long calculations) we shall, before proceeding, elaborate on the current algebra structure which is responsible for the particular form of the Lagrangian (2.1). We shall also supply some details of the one-loop renormalizability of the model which were left implicit in [15].

The theory (2.1) is of the general form

$$L_0 = \frac{1}{2e^2} \left\{ G_{ij} (\phi) \partial^\mu \phi^i \partial^\mu \phi^j + \epsilon^{\mu\nu} B_{ij}(\phi) \partial_\mu \phi^i \partial_\nu \phi^j \right\} .$$

(2.4)

where the fields $\phi^i (\xi^\mu)$ describe a map from two-dimensional Minkowski space-time to some target manifold. Motivated by the example of WZW models, one can ask when such a general sigma-model exhibits a classical current algebra. We restrict attention to the case in which the target manifold is the group SU(2) and it is convenient for this part of our discussion to choose antihermitean generators normalized so that

$$\lambda_a = -\frac{i}{2} \sigma_a , \quad [\lambda_a, \lambda_b] = \epsilon_{abc} \lambda_c ,$$

(2.5)

which corresponds to choosing the single simple root of SU(2) to have length one. We shall make no distinction between upper and lower SU(2) indices. A natural Ansatz for the light-cone components $I^a_{\pm} = I^a_0 \pm I^a_1$ of a current in the SU(2) Lie algebra is

$$I^a_+ = -\frac{1}{e^2} I^a_i \partial_+ \phi^i , \quad I^a_- = -\frac{1}{e^2} R^a_i \partial_- \phi^i ,$$

(2.6)

\footnote{The precise relationship between this criterion and the previous one is quite subtle in the general case; see e.g. [17].}
where $L^a_i$ and $R^a_i$ are vielbeins for the sigma-model metric:

$$L^a_i L^a_j = R^a_i R^a_j = G_{ij},$$  \hspace{1cm} (2.7)$$

The equations of motion following from (2.4) ensure that these currents are conserved $\partial_\mu I^{a\mu} = 0$. It can also be shown by tedious calculation that, with the canonical structure defined by (2.4), these currents obey a classical (equal-$\tau$) Poisson bracket algebra

$$\{I^a_\pm(\sigma), I^b_\pm(\sigma')\} = \epsilon^{abc} (aI^c_\pm(\sigma) + bI^c_\mp(\sigma)) \delta(\sigma - \sigma') \pm \frac{2}{e^2} \delta'(\sigma - \sigma'),$$  \hspace{1cm} (2.8)$$

with $a$ and $b$ constants, provided that the quantities $L^a_i$ and $R^a_i$ satisfy certain conditions.

To express these conditions compactly it is convenient to introduce differential forms on the group manifold:

$$L = \lambda_a L^a_i d\phi^i, \quad R = \lambda_a R^a_i d\phi^i.$$  \hspace{1cm} (2.9)$$

Then the current algebra above will hold provided

$$hL + Rh = 0$$
$$dL + aL^2 - bh^{-1}L^2h = 0$$
$$dR + aR^2 - bhR^2h^{-1} = 0$$
$$3H = -a \text{Tr}L^3 - 3b \text{Tr}RL^2 = a \text{Tr}R^3 + 3b \text{Tr}LR^2,$$  \hspace{1cm} (2.10)$$

where $h$ is some group-valued function on SU(2) and $2H = dB$ is the field-strength three-form corresponding to $B$. We use the conventions $B = \frac{1}{2i} B_{ij} d\phi^i \wedge d\phi^j$ and $H = \frac{1}{3!} H_{ijk} d\phi^i \wedge d\phi^j \wedge d\phi^k$ for the components of two-forms and three-forms respectively; as a result $2H_{ijk} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}$.

The SU(2) WZW model corresponds to a special solution of the current algebra conditions above in which

$$L = g^{-1} dg, \quad R = -dgg^{-1}, \quad h = g, \quad a = 1, \quad b = 0,$$  \hspace{1cm} (2.11)$$

and in this case (2.8) clearly collapses to two commuting Kac-Moody algebras. The action written in (2.1) corresponds to a slightly more complicated solution of (2.10) which can be motivated as follows. First consider the Ansatz

$$L = c(h^{-1} dh) + \lambda_a n_a f(\rho) d\rho, \quad R = c(-dh h^{-1}) - \lambda_a n_a f(\rho) d\rho,$$  \hspace{1cm} (2.12)$$

where $h = \exp(-\lambda_a n_a \rho)$, $c$ is some constant, $f(\rho)$ is a function to be determined, and the variable $\rho(w)$ is itself some function of our SU(2) coordinate $w$ which we will fix in
a convenient way at the end, after finding a solution. This seems on the face of it to be a rather redundant procedure, but it turns out to simplify some technical aspects of the discussion. The Ansatz above is clearly a straightforward modification of the WZW case, and it is chosen so as to satisfy the first equation in (2.10) automatically. It is not difficult to check that the remaining conditions in (2.10) hold if

\[ a = 2x + 1, \quad b = -1, \quad c = \frac{1}{2(x + 1)}, \quad f(\rho) = \frac{1}{x + 1} \frac{\cos \rho - 1}{x + \cos \rho}. \]  

(2.13)

This solution can also be expressed in the form

\[ L = \lambda_a (\alpha \delta n_a - \beta \epsilon_{abc} \delta n_b \delta n_c + \gamma \delta n_a \delta \rho) \]

\[ R = -\lambda_a (\alpha \delta n_a + \beta \epsilon_{abc} \delta n_b \delta n_c + \gamma \delta n_a \delta \rho), \]

(2.14)

where the functions \( \alpha, \beta \) and \( \gamma \) are given by

\[ \alpha = -\frac{\sin \rho}{2(x + 1)}, \quad \beta = \frac{1 - \cos \rho}{2(x + 1)}, \quad \gamma = -\frac{1}{2(x + \cos \rho)}. \]  

(2.15)

The sigma-model metric and WZ term are now determined as functions of \( \rho \) and \( n_a \) by the equations (2.7) and (2.10) respectively.

The final step is to relate \( \rho \) to \( w \), which can be done in such a way that the expression for the WZ term can be written in closed form. This is achieved by choosing

\[ \frac{\alpha}{\beta} = -\cot \frac{\rho}{2} = \sqrt{\frac{x - 1}{x + 1}} \tan w. \]  

(2.16)

Using this one can deduce the expressions \( \alpha(w) \) and \( \beta(w) \) given in (2.2) and, after some effort, one then recovers (2.1).

The current algebra corresponding to the solution (2.13) above was first considered by Rajeev [18], who showed that it could be decomposed into two commuting Kac-Moody algebras. The combinations which achieve this are

\[ J^a_\pm = \frac{1}{4} \left\{ \left( \frac{1}{x + 1} + \frac{1}{\sqrt{x^2 - 1}} \right) I^a_\pm + \left( \frac{1}{x + 1} - \frac{1}{\sqrt{x^2 - 1}} \right) I^a_\mp \right\}, \]  

(2.17)

obeying

\[ \{ J^a_\pm(\sigma), J^b_\pm(\sigma') \} = e^{abc} J^c_\pm(\sigma) \delta(\sigma - \sigma') \pm \frac{1}{2e^2(x + 1) \sqrt{x^2 - 1}} \delta'(\sigma - \sigma') \]

\[ \{ J^a_\pm(\sigma), J^b_\mp(\sigma') \} = 0. \]  

(2.18)

The \( \pm \) signs occur in the central terms because these are classical Kac-Moody algebras, and the quantization condition (2.3) can now be recovered by comparison with some standard reference (e.g. equation (2.3.14) of [13]). Unlike the WZW case, the components of
these Kac-Moody currents are not chirally conserved (although the original current $I^a_\mu$ is conserved by construction). Notice, however, that the WZW case can be recovered by taking the limit $x \to \infty$, $k$ fixed, provided we rescale the fields appropriately.

Since the theory (2.1) is a generalized sigma model (a sigma model with a WZ term) its renormalization group flow can be analysed using the background field method (see for example [19]). We can simply quote the well-known results for the way that the metric and WZ term in (2.4) run with the renormalization scale to one-loop, but we must then ensure that these equations are indeed consistent with the specific Ansatz of (2.1). In our discussion of the current algebra, it was convenient to keep the coupling constant dependence explicit, but to apply the general renormalization results of sigma-models it is better to absorb the coupling constant $e$ into our definitions of the metric and WZ term by defining $g_{ij} = G_{ij}/e^2$, $b_{ij} = B_{ij}/e^2$ and $h_{ijk} = H_{ijk}/e^2$. The coefficients of the beta-function are calculated in terms of the generalized curvature $\hat{R}_{ijkl}$ corresponding to the connection

$$\hat{\Gamma}^i_{jk} = \Gamma^i_{jk} + h^i_{jk}$$

which involves the usual Christoffel connection $\Gamma^i_{jk}$ (constructed from the metric $g_{ij}$) modified by a torsion term. To one loop one finds [19] that under the renormalization group transformation of the subtraction point $\mu$ the metric and anti-symmetric field satisfy

$$\mu \frac{\partial g_{ij}}{\partial \mu} = -\frac{1}{2\pi} \hat{R}_{(ij)}, \quad \mu \frac{\partial b_{ij}}{\partial \mu} = -\frac{1}{2\pi} \hat{R}_{[ij]},$$

where $\hat{R}_{ij} = \hat{R}^k_{ij}k$ is the generalized Ricci tensor.

We now apply these formulae to the theory (2.1). First of all, we define the coordinates $\theta$ and $\psi$ via $n_a = (\cos \theta, \sin \theta \cos \psi, \sin \theta \sin \psi)$. In these coordinates the metric has non-zero components

$$g_{ww} = \frac{1}{e^2(x^2 - 1)}, \quad g_{\theta\theta} = \frac{\beta}{e^2(x + 1)}, \quad g_{\psi\psi} = \frac{\beta}{e^2(x + 1)} \sin^2 \theta,$$  

and the anti-symmetric field has non-zero components

$$b_{\theta\psi} = -b_{\psi\theta} = \frac{1}{e^2(x + 1)} \left[ \frac{1}{\sqrt{x^2 - 1}} \left( \frac{\pi}{2} - w \right) - \alpha \right] \sin \theta.$$  

From these we find that the non-zero components of the generalized Ricci tensor are

$$\hat{R}_{ww} = \frac{2}{x^2 - 1} + 4\alpha' \sqrt{\frac{x + 1}{x - 1}},$$

$$\hat{R}_{\theta\theta} = -\frac{2x}{x + 1} \beta + 2\beta \alpha' \sqrt{x^2 - 1},$$

$$\hat{R}_{\psi\psi} = \hat{R}_{\theta\theta} \sin^2 \theta,$$

$$\hat{R}_{\theta\psi} = -\hat{R}_{\psi\theta} = 2\beta \beta' \sqrt{x^2 - 1} \sin \theta.$$
where the prime denotes a derivative with respect to \( w \) at constant \( x \). Using the expression for the Ricci tensor in (2.23) in the equations (2.20) shows that under renormalization group flow the form of the lagrangian is preserved up to a renormalization of the coupling constants \( e \) and \( x \):

\[
\mu \frac{\partial e}{\partial \mu} = \frac{1}{2\pi} (1-2x)e^3 + \mathcal{O}(e^5), \quad \mu \frac{\partial x}{\partial \mu} = \frac{1}{\pi} (x^2 - 1)e^2 + \mathcal{O}(e^4),
\]

(2.24)

and a diffeomorphism of the field \( w \) given by

\[
\mu \frac{\partial w}{\partial \mu} = -\frac{1}{\pi} (x^2 - 1) \frac{\cos w \sin w}{x + \cos 2w} e^2 + \mathcal{O}(e^4).
\]

(2.25)

These one-loop results (2.24) agree with the analysis of [15]. Notice that to this order \( k \) defined in (2.3) is constant under renormalization group flow as we expect. In the ultraviolet, \( \mu \to \infty \), \( e \) runs to zero and \( x \) runs to infinity. In this limit one can easily show that

\[
\mathcal{L}_0 = \mathcal{L}_{\text{WZW}} + \frac{k}{8\pi x} \text{Tr} (J_L J_R) + \mathcal{O}(1/x^2),
\]

(2.26)

where \( \mathcal{L}_{\text{WZW}} \) is the usual SU(2) WZW lagrangian at level \( k \), and \( J_L = g^{-1} \partial_+ g \) and \( J_R = -(\partial_- g) g^{-1} \) are its left and right conserved Kac-Moody currents. This expression justifies our earlier statement that the theories (2.1) are SU(2) WZW models perturbed by the operator \( \text{Tr}(J_L J_R) \). It is also easy to see that this perturbation breaks the chiral SU(2) \( \times \) SU(2) symmetry of \( \mathcal{L}_{\text{WZW}} \) to the diagonal, or adjoint, SU(2) subgroup mentioned above.

Assuming that \( k \) is indeed constant we can eliminate \( x \) from (2.24) to get the flow equation just involving \( e \). Later we shall be interested in this flow equation for large but finite values of \( k \). In this regime we deduce from (2.24) that

\[
\mu \frac{\partial e}{\partial \mu} = -\beta_0 e^2 - \beta_1 e^3 - \mathcal{O}(e^4),
\]

(2.27)

where

\[
\beta_0 = \sqrt{\frac{2}{\pi k}} (1 + \mathcal{O}(1/k)), \quad \beta_1 = -\frac{1}{\pi} (1 + \mathcal{O}(1/k)).
\]

(2.28)

Notice that terms coming from higher loops can produce corrections of lower order in \( 1/k \) assuming that the coefficient of \( e^p \) in \( \mu(\partial e/\partial \mu) \) is polynomial in \( x \). It is important to remember that the expressions (2.28) are universal.

We have now described in some detail the lagrangian field theory we wish to study, and in the next section we conjecture an S-matrix to describe the scattering of states in this theory. We will subsequently undertake a non-trivial check of the form of the S-matrix by using the ideas of [5-9]. To do this we need to couple the theory to a conserved charge.
The idea is to modify the Hamiltonian $H \rightarrow H - hQ$, where $Q$ is a conserved charge corresponding to some generator of the SU(2) symmetry of the model. In the Minkowski space Lagrangian picture this is achieved simply by replacing the derivative of $n_a$ in the time-direction by the “covariant derivative”:

$$\partial_0 n_a \rightarrow \partial_0 n_a + 2h \epsilon_{abc} q_b n_c,$$

where the $q_a$ are a set of parameters which later we take to be $q = (1, 0, 0)$ without loss of generality (due to the SU(2) symmetry). We will then compute the response of the free-energy per unit volume $\delta f(h) = f(h) - f(0)$ in the ultra-violet regime in two ways: using the S-matrix along with thermodynamic Bethe Ansatz techniques and using conventional perturbation theory.

3. The S-matrix

Consider, for a moment, the most general way of associating S-matrices to the Lie algebra SU($N$). The particles form multiplets associated to the fundamental, or completely anti-symmetric, representations of the algebra, and each particle carries in general say $m$ copies of the quantum numbers of that representation. The general two-body S-matrix element – from which all the others may be deduced by factorization – has the block form [20,21]

$$S_{ab}^{(k_1, k_2, \ldots, k_m)}(\theta) = X_{ab}^{(\theta)} S_{(k_1)}^{ab}(\theta) \otimes S_{(k_2)}^{ab}(\theta) \otimes \cdots \otimes S_{(k_m)}^{ab}(\theta),$$

(3.1)

where factor $S_{(k_j)}^{ab}(\theta)$ acts between the $j^{th}$ copies of the fundamental representations $a$ and $b$ ($a, b = 1, 2, \ldots, N - 1$) and the $k_j$’s are parameters or coupling constants. The part $X_{ab}^{(\theta)}$ is a scalar factor which ensures that the overall S-matrix has the right analytic structure. Each block $S_{(k_j)}^{ab}(\theta)$ is invariant under the action of the quantum loop-group $U_q(SU(N) \otimes \mathbb{C}[\theta, \theta^{-1}])$ where the deformation parameter is $q = -\exp(-i\pi/(N + k))$. In the limit $k = \infty$ the quantum loop-group reduces to the ordinary loop-group and the block $S_{(\infty)}^{ab}(\theta)$ is invariant under the action of the group SU($N$) itself. When $k$ is a natural number the blocks are proportional to RSOS solutions of the Yang-Baxter equation and $S_{(1)}^{ab}(\theta) = 1$.

The S-matrix that was proposed in [16] to describe the perturbation of the WZW model of level $k$ is $S_{(k, \infty)}^{ab}(\theta)$. For the case of SU(2) there is only one particle and $X_{11}^{(\theta)} = -1$. It is worth pointing out that this general form subsumes the well-known S-matrices of the SU($N$) chiral Gross-Neveu model, given by $S_{(\infty)}^{ab}(\theta) \equiv S_{(\infty, 1)}^{ab}(\theta)$, and the SU($N$) principal chiral model, given by $S_{(\infty, \infty)}^{ab}(\theta)$. Remarkably, this implies that the model (2.1) is equivalent at the quantum level to the SU(2) chiral Gross-Neveu and principal chiral
models, for \( k = 1 \) and \( k = \infty \), respectively. We shall make a comment about these equivalences at the end of the paper.

We now write down the S-matrix that is proposed to describe the theory (2.1). As is conventional, we take the kinematic variable to be the rapidity difference \( \theta \) of the incoming particles.\(^3\) The S-matrix describes one massive particle with internal quantum numbers and for the two-body process it has the product form mentioned above [16]:

\[
S(\theta) = S_{SU(2)}(\theta) \otimes S_{(k)}^{\text{kink}}(\theta),
\]

with \( k \in \mathbb{N} \) being identified with (2.3). The product form means that the particle carries two sets of quantum numbers and each factor acts on one of the sets only. The first factor \( S_{SU(2)}(\theta) \) is the S-matrix of the SU(2) chiral Gross-Neveu model; hence the particle transforms in the two-dimensional representation of SU(2) and the two-body S-matrix elements may be written [3]

\[
S_{SU(2)}(\theta) = \frac{\Gamma \left( 1 - \frac{\theta}{2 \pi i} \right) \Gamma \left( \frac{1}{2} + \frac{\theta}{2 \pi i} \right)}{\Gamma \left( 1 + \frac{\theta}{2 \pi i} \right) \Gamma \left( \frac{1}{2} - \frac{\theta}{2 \pi i} \right)} \left[ \mathbb{P}_t + \left( \frac{\theta + 2 \pi i}{\theta - 2 \pi i} \right) \mathbb{P}_s \right],
\]

where \( \mathbb{P}_t, s \) indicate the triplet and singlet channels. This part is equal to \( -S_{(\infty)}^{11}(\theta) \) as written above and is invariant under SU(2).

The other factor in (3.2) describes the scattering of kink degrees-of-freedom carried by the particle. The particle can either be a kink or an anti-kink which interpolates between a set \( k + 1 \) vacua \( \{1, 2, \ldots, k + 1\} \) with the following selection rule: a kink can connect vacuum \( a \) with \( a + 1 \) and an anti-kink \( a \) with \( a - 1 \). The S-matrix is that of soliton scattering in the restricted sine-Gordon model [22] so the S-matrix element for the process \( K_{da}(\theta_1) + K_{ab}(\theta_2) \rightarrow K_{dc}(\theta_2) + K_{cb}(\theta_1) \) is

\[
S_{(k)}^{\text{kink}} \left( \begin{array}{ccc}
  a & b \\
  d & c \\
  
\end{array} \right)(\theta) = u(\theta) \left( \frac{\sinh(\pi a/p) \sinh(\pi c/p)}{\sinh(\pi d/p) \sinh(\pi b/p)} \right)^{-\theta/2 \pi i}
\]

\[
\times \left\{ \sinh \left( \frac{\theta}{p} \right) \delta_{db} \left( \frac{\sinh(\pi a/p) \sinh(\pi c/p)}{\sinh(\pi d/p) \sinh(\pi b/p)} \right)^{1/2} + \sinh \left( \frac{i \pi - \theta}{p} \right) \delta_{ac} \right\},
\]

where \( p = k + 2 \) and

\[
u(\theta) = \Gamma \left( \frac{1}{p} \right) \Gamma \left( 1 + \frac{i \theta}{p} \right) \prod_{n=1}^{\infty} \frac{R_n(\theta) R_n(i \pi - \theta)}{R_n(0) R_n(i \pi) }
\]

\[
R_n(\theta) = \frac{\Gamma \left( \frac{2n}{p} + \frac{i \theta}{p} \right) \Gamma \left( 1 + \frac{2n}{p} + \frac{i \theta}{p} \right) \Gamma \left( 1 + \frac{2n-1}{p} + \frac{i \theta}{p} \right)}{\Gamma \left( \frac{2n+1}{p} + \frac{i \theta}{p} \right) \Gamma \left( 1 + \frac{2n+1}{p} + \frac{i \theta}{p} \right)}.
\]

We remark that the form of the S-matrix reflects the form of the lagrangian: the SU(2) part manifests the SU(2) symmetry of the model and the kink part describes degrees-of-freedom associated to the periodic field \( w \).

\(^3\) The velocity and rapidity of a particle are related by \( v = \tanh \theta \).
4. The free-energy from the S-matrix

In this section we evaluate the response of the free-energy $\delta f(h)$ to the coupling with the charge directly from the S-matrix. The technique we use is known as the thermodynamic Bethe Ansatz [23]. In its most general form it leads to an expression for the free-energy in a cylindrical geometry coupled to a conserved charge which plays the rôle of a chemical potential. In our case we wish to evaluate the free-energy in the plane, i.e. at zero temperature.

Consider the (one-dimensional) statistical mechanics of a gas of particles described by the S-matrix (3.2). Since this theory is integrable, the number of particles is conserved under interaction and it is meaningful to consider single particle energy levels. In a free-theory the energy of these levels would simply be $\epsilon(\theta) = mc \cosh \theta - h$, where $h$ is the chemical potential, and the free-energy (per unit volume) at zero temperature would be that of a free one-dimensional relativistic fermion gas:\footnote{The fact that particles should be treated as fermions in the thermodynamic Bethe Ansatz results from that fact that the S-matrix satisfies $S(0) = -1$.}

$$f(h) = \frac{m}{2\pi} \int_{-\theta_F}^{\theta_F} d\theta \epsilon(\theta) \cosh \theta,$$

where $\theta_F$ the Fermi rapidity is determined by the condition that $\epsilon(\pm \theta_F) = 0$. In our case, the complications are two-fold: the theory is, after all, not free and furthermore the particles carry internal quantum numbers. As a result of the former $\epsilon(\theta)$ now satisfies an integral equation involving kernels related to the S-matrix of the theory. The effect of the internal quantum numbers is to couple the energy $\epsilon(\theta)$ to the “magnon” energy levels $\xi_p(\theta)$, $p = 1, 2, \ldots, k - 1$, and $\zeta_q(\theta)$, $q = 1, 2, \ldots, \infty$; where the former result from the kink part of the S-matrix and the latter from the SU(2) part. The free-energy is then still given by (4.1). The equations are known as the TBA equations and they have been derived at finite temperature and zero chemical potential for our S-matrix in [24].\footnote{The TBA equations for the more general S-matrices (3.1) was considered in [21].} At $T = 0$ and in the presence of a chemical potential coupling to the charge of the SU(2) symmetry, the
TBA equations adopt the form

$$
\epsilon^+(\theta) + R \ast \epsilon^-(\theta) + \sum_{p=1}^{k-1} a_p^{(k)} \ast \xi_p^+(\theta) + \sum_{q=1}^{\infty} a_q^{(\infty)} \ast \zeta_q^+(\theta) = m \cosh \theta - h,
$$

$$
\xi_p^-(\theta) + \sum_{q=1}^{k-1} A_{pq}^{(k)} \ast \xi_q^+(\theta) = a_p^{(k)} \ast \epsilon^-(\theta),
$$

\( (4.2) \)

$$
\zeta_p^- + \sum_{q=1}^{\infty} A_{pq}^{(\infty)} \ast \zeta_q^+(\theta) = a_p^{(\infty)} \ast \epsilon^-(\theta) - 2hp,
$$

where star denotes the convolution \( f \ast g(\theta) = \int d\theta' f(\theta - \theta')g(\theta') \) and \( f^\pm(\theta) \) denote the positive/negative decomposition of \( f(\theta) = f^+(\theta) + f^-(\theta) \), i.e. \( f^\pm(\theta) = f(\theta) \) if \( f(\theta) > 0 \) or \( f(\theta) < 0 \), respectively, being otherwise zero. The kernels in (4.2) are given by

$$
R(\theta) = \int_0^{\infty} dx \frac{\cos(\theta x)}{\pi} \frac{\sinh^2(\pi x/2)}{\sinh(\pi k x/2) \sinh(\pi x)} \exp(k\pi x/2),
$$

$$
A_{pq}^{(k)}(\theta) = \int_0^{\infty} dx \frac{\cos(\theta x)}{\pi} \frac{2 \sinh(\pi px/2) \sinh(\pi(k-q)x/2) \cosh(\pi x/2)}{\sinh(\pi x) \sinh(\pi x/2)},
$$

\( (4.3) \)

$$
a_p^{(k)}(\theta) = \frac{1}{\pi k} \cdot \frac{\sin(\pi p/k)}{\cosh(2\theta/k) - \cos(\pi p/k)},
$$

for \( q \geq p \) (\( A_{pq}^{(k)}(\theta) = A_{qp}^{(k)}(\theta) \)). The dependence of the free-energy (per unit volume) on the chemical potential is then given by

$$
\delta f(h) = f(h) - f(0) = \frac{m}{2\pi} \int_{-\infty}^{\infty} d\theta \epsilon^-(\theta) \cosh \theta.
$$

\( (4.4) \)

The problem before us is to solve the coupled integral equations (4.2). Our strategy will implicitly assume that the solution of the equations (4.2) is unique. Given this the crucial observation is that \( a_p^{(k)}(\theta) \) is a positive kernel; hence the solution of the TBA equations is \( \xi_p^+(\theta) = \zeta_q^+(\theta) = 0 \) with \( \xi_p^-(\theta) = a_p^{(k)} \ast \epsilon^-(\theta), \zeta_p^-(\theta) = a_p^{(\infty)} \ast \epsilon^-(\theta) - 2hp \) and

$$
\epsilon^+(\theta) + R \ast \epsilon^-(\theta) = m \cosh \theta - h.
$$

\( (4.5) \)

The solution \( \epsilon(\theta) \) to (4.5) is some symmetric convex function with zeros at the Fermi rapidity \( \pm \theta_F(h) \). When \( h < m \) the system is below threshold; the external field is too weak to excite any particle states and hence \( \delta f(h) \) is zero. Beyond the threshold \( h = m \) the chemical potential forces the system into a state where the particles line up their spins with the external field. From the form of the TBA equations we see that the external field does not couple to the kink number and it turns out that the ground-state has total kink number zero.
Notice that the reasoning which led to identifying the solution of the TBA equations in terms of a particular configuration of the quantum numbers of the particles was arrived at from studying the full TBA equations. This is to be contrasted with the more heuristic arguments used in [5-9] leading to the hypothesis that only one particle-state contributed to the ground-state.

We have arrived at an expression for the free-energy in terms of a single function \( \epsilon(\theta) \) which satisfies the single integral equation (4.5). This equations are of the same form, but with a different kernel, as those of the O(\( N \)) sigma model [5,6], principal chiral models [7,8] and Gross-Neveu models [9].

It is not possible to solve the equation (4.5) in closed form; however, we will be interested in the solution only in the deep ultra-violet \( h \gg m \) for which one can develop a series solution using generalized Wiener-Hopf techniques [5,25] (see the appendix of [9] for a clear summary). Rather than explain these techniques we simply follow the manipulations of [9] required to extract the series solution.

The method starts by decomposing the Fourier transform of the kernel \( R(\theta) \) in the following way:

\[
\frac{\sinh^2(\pi x/2)}{\sinh(\pi kx/2)\sinh(\pi x)} \exp(k\pi x/2) = \frac{1}{G_+(x)G_-(x)},
\]

where \( G_-(x) = G_+(\text{−}x) \) and \( G_+(x) \) are analytic in the upper/lower half-planes, respectively. So

\[
G_+(x) = \sqrt{2k} \frac{\Gamma(1 \text{−} i\theta/2)}{\Gamma(1 \text{−} ikx/2)\Gamma(1 \text{−} i\theta)} \exp\left(\text{i}x b - \text{i} \frac{kx}{2} \ln(\text{i}x)\right),
\]

where

\[
b = \frac{k}{2} - \ln 2 - \frac{k}{2} \ln \frac{k}{2}.
\]

Following [9] we now define the function \( \alpha(x) = \exp(2\text{i}x\theta_F)G_-(x)/G_+(x) \), where \( \epsilon(\pm\theta_F) = 0 \). \( \alpha(x) \) has a cut along the positive imaginary axis and we define \( \gamma(\xi) \) in terms of the discontinuity:

\[
\alpha(i\xi + 0) - \alpha(i\xi - 0) = -2\text{i}e^{-2\xi\theta_F} \gamma(\xi),
\]

giving in this case

\[
\gamma(\xi) = \exp(-k\xi \ln \xi + 2\xi b) \frac{\Gamma^2(1 \text{−} \xi/2)\Gamma(1 + k\xi/2)\Gamma(1 + \xi)}{\Gamma^2(1 + \xi/2)\Gamma(1 + \xi/2)\Gamma(1 - \xi)} \sin(\pi k\xi/2).
\]

If one consults [9] then it soon becomes apparent that \( \gamma(\xi) \) for our model has the same functional form as a fermion model, rather than a sigma model, namely

\[
\gamma(\xi) = \pi e^{-k\xi \ln \xi} \sum_{n=1}^{\infty} d_n \xi^n.
\]
The expansion of the free-energy is given in terms of the quantities \( d_j \). It turns out that \( \delta f(h)/h^2 \) is power series in the effective coupling \( u = u(h) \) defined through

\[
\frac{1}{u} - k \ln u = \frac{1}{z},
\]

where

\[
\frac{1}{z} = \ln \left[ \frac{h^2}{m^2} \left( \frac{2G_+(0)}{G_+}(i) \right)^2 \right].
\]

Putting these expression together with the results of [9] allows us to extract the first few terms in the expansion of the free-energy as a function of \( h/m \)

\[
\delta f(h) = \frac{h^2}{2\pi} G_+(0)^2 \left\{ 1 - 2d_1 z + 2kd_1 z^2 \ln z - 2 \left[ 2d_1 - \Gamma'(2)kd_1 - d_1^2 + d_2 \right] z^2 \\
- 2k^2 d_1 z^3 \ln^2 z + 2k \left[ 4d_1 - \Gamma'(3)kd_1 - 2d_1^2 + 2d_2 \right] z^3 \ln z + \mathcal{O}(z^3) \right\}.
\]

In the above, \( \delta f(h) \) is an expansion in terms of the form \( z^m \ln^n z \) where \( m > n \). From (4.10) we find

\[
d_1 = \frac{k}{2}, \quad d_2 = -k \ln 2 - \frac{k^2}{2} \ln \left( \frac{k}{2} \right) + \frac{k^2}{2} \Gamma'(2).
\]

So putting everything together for the models that we are considering we find that the free-energy extracted from the S-matrix yields the following expansion in the ultra-violet:

\[
\delta f(h) = -\frac{h^2 k}{\pi} \left\{ 1 - \frac{k}{2} s + \frac{k^2}{4} s^2 \ln s - \frac{k}{2} \left[ 1 - \frac{1}{2} \ln \frac{64k}{\pi} - \frac{k}{2} \ln \frac{k}{4} \right] s^2 \\
- \frac{k^3}{8} s^3 \ln^2 s + \frac{k^2}{2} \left[ 1 - \frac{1}{2} \ln \frac{64k}{\pi} - \frac{k}{2} \ln \frac{k}{4} \right] s^3 \ln s + \mathcal{O}(s^3) \right\},
\]

where \( s^{-1} = \ln(h/m) \).

5. The free-energy from perturbation theory

In this section, we develop the expansion of the free-energy \( \delta f(h) \) in perturbation theory. We assume, following the discussion to one-loop in section 2 and [15], that under renormalization group flow \( k \) is constant. Therefore, we may express \( 1/x \) in terms of \( e \):

\[
\frac{1}{x} = \sqrt{\frac{k}{2\pi}} e + \frac{k}{4\pi} e^2 + \mathcal{O}(e^3),
\]

\[ (5.1) \]
and we shall find that in the ultra-violet \( e \) runs to zero and hence \( x \) runs to infinity. The loop expansion parameter is \( e^2(x + 1)^2 \) which is expressed in terms of \( e \) as
\[
e^2(x + 1)^2 = \frac{2\pi}{k} \left( 1 + \sqrt{\frac{k}{2\pi}} e + \frac{k}{2\pi} e^2 + O(e^3) \right).
\] (5.2)

This means that the contributions from higher loops can lead to terms of the same order in \( e \), but their coefficients will be suppressed by higher powers of \( 1/k \). Our result to one-loop will therefore be valid in the large but finite \( k \) limit.

The Minkowski space lagrangian in the presence of the chemical potential is given by substituting (2.29) in (2.1) which on Wick rotating to Euclidean space becomes
\[
\mathcal{L} = \mathcal{L}_0 - \frac{2h^2 \beta(w)}{e^2(x + 1)} (1 - n_1^2) + \frac{2h}{e(x + 1)} \left\{ \beta(w)(n_2 \partial_0 n_3 - n_3 \partial_0 n_2) + \left[ \frac{1}{\sqrt{x^2 - 1}} \left( \frac{\pi}{2} - w \right) - \alpha(w) \right] \partial_1 n_1 \right\}.
\]

The ground-state of the system is given by \( n_1 = w = 0 \) (modulo some discrete ambiguity depending on the precise way we parametrize SU(2)). We wish to calculate the change in the free-energy per-unit-volume \( \delta f(h) \) to one-loop, so it suffices to expand the lagrangian to quadratic order around the ground state with \( (n_2, n_3) = \sqrt{1 - n_1^2} (\cos \psi, \sin \psi) \). On suitably re-scaling the field \( w \) we find
\[
\mathcal{L} = \frac{1}{2e^2(x + 1)^2} \left\{ (\partial_\mu w)^2 + (\partial_\mu n_1)^2 + (\partial_\mu \psi)^2 - \frac{8hx}{x + 1} w \partial_1 n_1 - 4h^2 + 4h^2 n_1^2 + 4h^2 w^2 \left( \frac{x - 1}{x + 1} \right)^2 + \ldots \right\},
\]

where the ellipsis represent the interaction.

We can simply read off the tree-level contribution to \( \delta f(h) \):
\[
\delta f(h)_0 = -\frac{2h^2}{e^2(x + 1)^2} = -\frac{h^2 k}{\pi} \left\{ 1 - \sqrt{\frac{k}{2\pi}} e + O(e^3) \right\}.
\]

The one-loop contribution is, from the quadratic Euclidean lagrangian written above,
\[
\delta f(h)_1 = \frac{1}{2} \ln \text{Det} \left\{ \frac{M}{e^2(x + 1)^2} \right\},
\]

where the operator \( M \) acts on \( (n_1, w) \) according to
\[
M = \frac{1}{\mu^2} \begin{pmatrix}
-\partial^2 + 4h^2 & \frac{4hx}{x + 1} \partial_1 \\
-\frac{4hx}{x + 1} \partial_1 & -\partial^2 + 4h^2 \left( \frac{x - 1}{x + 1} \right)^2
\end{pmatrix}.
\]

Here $\mu$ is a mass-scale introduced to make the eigenvalues of $M$ dimensionless. Note also that the field $\psi$ is completely decoupled to this order in the loop expansion.

The operator above is rather unconventional in nature and it proves convenient to evaluate its determinant using zeta-function techniques. Some basic facts concerning these methods, together with their application to operators of the above type, are summarized in the appendix. The important point for our purposes is that

$$\ln \text{Det } M = -\zeta'_M(0), \quad (5.8)$$

where the zeta-function $\zeta_M(s)$ corresponding to the operator $M$ can be represented by

$$\zeta_M(s) = \frac{2V\mu^{2s}}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \int \frac{d^2p}{(2\pi)^2} e^{-t(p^2 + \lambda)} \cosh \left( t(\eta^2 p_1^2 + \rho^2)^{1/2} \right), \quad (5.9)$$

with

$$\lambda = \frac{4h^2(x^2 + 1)}{(x + 1)^2}, \quad \rho = \frac{8h^2x}{(x + 1)^2}, \quad \eta = \frac{4hx}{x + 1}. \quad (5.10)$$

The factor $V$ denotes formally the volume of two-dimensional spacetime (strictly speaking this should be dealt with using some explicit infra-red regularization but these details are irrelevant for our purposes). Unfortunately the integral above cannot be evaluated in closed form, but we shall sketch below how it can be successfully expanded in powers of $1/x$ to the order which we need. (Recall that the ultra-violet limit of our models is $x \to \infty$ with $k$ fixed.)

The strategy is to expand the cosh factor in (5.9) as a series and collect terms of a given power in $p_1$ so that the momentum integrals can be evaluated. The $t$ integrals can then be expressed in terms of $\Gamma$-functions, and one obtains the result

$$\zeta_M(s) = \frac{V\mu^{2s}}{2\pi \Gamma(s)} \sum_{m=-\infty}^{\infty} C_m \lambda^{-(s+m)} \Gamma(s + m), \quad (5.11)$$

where the coefficients are given by the rather complicated expressions

$$C_m = \sum_{\substack{m = 2n - r - 1 \leq r \leq n \leq \infty \\text{with} \quad 0 \leq r \leq n \leq \infty}} \frac{(2r)!n!}{(2n)!r!} \frac{\eta^{2r} \rho^{2(n-r)}}{(n-r)!2^{2r}}. \quad (5.12)$$

It is not obvious how to sum the series above completely, but we note that $\rho$ in these expressions is $O(1/x)$ which will enable us to simplify things presently.

On differentiating and setting $s = 0$ we find

$$\zeta'_M(0) = \frac{V}{2\pi} \left\{ -\lambda + \left( \lambda - \frac{\eta^2}{4} \right) \ln \frac{\lambda}{\mu^2} + \sum_{m=1}^{\infty} C_m \lambda^{-m} \Gamma(m) \right\}. \quad (5.13)$$
At this stage our result is still exact, but we must now consider its behaviour as a power series in \( \rho \) (and hence \( 1/x \)) to make further progress. The leading contribution is \( \mathcal{O}(\rho^0) \); extracting all such terms from the expressions (5.12) for \( C_m \) with \( m \geq 1 \) gives us an infinite series which can be summed\(^6\) to yield

\[
\frac{V}{2\pi} \lambda \left\{ \left( 1 - \frac{\eta^2}{4\lambda} \right) \left( \ln \left( 1 - \frac{\eta^2}{4\lambda} \right) - 1 \right) + 1 \right\}.
\]  

(5.14)

The next contributions are \( \mathcal{O}(\rho^2) \) and again the series resulting from the coefficients \( C_m \) with \( m \geq 1 \) can be summed\(^7\) to give

\[
\frac{V}{2\pi} \frac{\rho^2}{2\eta\sqrt{\lambda}} \left\{ \ln \left( 1 + \frac{\eta}{2\sqrt{\lambda}} \right) - \ln \left( 1 - \frac{\eta}{2\sqrt{\lambda}} \right) \right\}.
\]  

(5.15)

The other contributions to (5.13) are \( \mathcal{O}(\rho^4) = \mathcal{O}(1/x^4) \) and we neglect them. Now we substitute (5.14) and (5.15) in (5.13) and expand each of \( \lambda, \rho \) and \( \eta \) in powers of \( 1/x \) using (5.10). It turns out that although (5.14) is \( \mathcal{O}(\rho^0) \) and \( \rho = \mathcal{O}(1/x) \) the particular form of the expressions for \( \lambda \) and \( \eta \) imply that (5.14) is, more precisely, \( \mathcal{O}(1/x^2) \). The final result is

\[
\zeta_M'(0) = \frac{V}{2\pi} \frac{4h^2}{x^2} \left[ \ln \frac{16h^2}{\mu^2} - 1 \right] + \mathcal{O}(1/x^3).
\]  

(5.16)

Now to obtain the required determinant in (5.6) all we need do is restore the factor \( e^2(x+1)^2 \) (which we dropped in (5.8) for simplicity) by rescaling \( \mu \) and use the consequence (5.1) of the quantization condition (2.3) to eliminate \( x \) in favour of \( e \) and \( k \). This gives a final result for the change in the one-loop contribution to the free-energy per unit volume of

\[
\delta f(h)_1 = \frac{h^2k}{\pi} \left\{ \frac{e^2}{2\pi} \left( 1 - \ln \frac{8kh^2}{\pi\mu^2} \right) + \mathcal{O}(e^3) \right\}.
\]  

(5.17)

The change in the free-energy \( \delta f_0 + \delta f_1 \) which we have just calculated is a renormalization group invariant quantity, \( i.e. \) it is independent of \( \mu \) when the coupling constant runs with \( \mu \). This fact allows us to determine the running of the coupling, as expressed by the beta-function

\[
\frac{\partial e}{\partial \mu} = -\sqrt{\frac{2}{\pi k}} e^2 - \beta_1 e^3 + \mathcal{O}(e^4),
\]  

(5.18)

although the coefficient \( \beta_1 \) is not determined to the order that we are working. Since the first two coefficients of the beta-function are universal, it is comforting that (5.18) is consistent with the calculation of the beta-function via the background field method (2.28).

---

\(^6\) \( \sum_{n=2}^{\infty} \frac{1}{n(n-1)} y^n = (1 - y)(\ln(1 - y) - 1) + 1 \) for suitable \( y \).

\(^7\) \( \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1} = \frac{1}{2}(\ln(1 + z) - \ln(1 - z)) \) for suitable \( z \).
It is convenient to use the fact that the free-energy is independent of $\mu$ to set $\mu = h$ and then, integrating (5.18), one obtains

$$e(h) = \sqrt{\frac{\pi k}{2}} \xi + \beta_1 \left( \frac{\pi k}{2} \right)^{3/2} \frac{1}{4} \xi^2 \ln \xi + \beta_1^2 \left( \frac{\pi k}{2} \right)^{5/2} \xi^3 \left( \ln^2 \xi + \ln \xi \right) + O(\xi^3),$$

(5.19)

where $\xi^{-1} = \ln(h/\Lambda_\zeta)$ is defined in terms of the $\Lambda$-parameter of the zeta-function regularization scheme and the expansion is in terms of the form $\xi^m \ln^n \xi$ with $m > n$.

Hence to one-loop we deduce

$$\delta f(h) = -\frac{h^2 k}{\pi} \left\{ 1 - \frac{k}{2} \xi - \frac{\pi k^2 \beta_1}{4} \xi^2 \ln \xi - \frac{k}{4} \left( 1 - \ln \frac{8k}{\pi} \right) \xi^2 - 
\frac{\pi^2 k^3 \beta_1^2}{8} \xi^3 \ln^2 \xi - \frac{\pi k^2 \beta_1}{4} \left( \frac{k \beta_1}{2} + 1 - \ln \frac{8k}{\pi} \right) \xi^3 \ln \xi + O(\xi^3) \right\}$$

(5.20)

The effect of higher loops would be to introduce corrections at the same order in $\xi$ but with coefficients which are suppressed by powers of $1/k$.

We now reach our main result: at the order to which we are working the two expansions (4.16) and (5.20) are consistent, a fact which provides a highly non-trivial check of the conjectured S-matrix. Furthermore, by comparing the two expressions we can extract the mass-gap ratio for large $k$:

$$\ln \frac{m}{\Lambda_\zeta} = -\frac{1}{2} + \frac{3}{2} \ln 2 + \frac{k}{4} + \frac{k}{2} \ln \frac{k}{4} + O(1/k).$$

(5.21)

We also deduce that the second coefficient of the beta-function is, for large $k$, simply $\beta_1 = -1/\pi + O(1/k)$, a result which is in perfect agreement with the second coefficient of the beta-function computed directly using the background field method (2.28).

6. Conclusions

We have investigated a series of theories that generate their mass dynamically but which are asymptotically (in the ultra-violet limit) non-trivial CFTs. The theories are in addition integrable, a property implying that their S-matrices factorize, allowing us to conjecture a form for these S-matrices. The calculations of the free energy which we carried out provide a highly non-trivial check on the form of the S-matrices since, as pointed out in [7], the addition of any CDD factors would drastically alter the thermodynamics of the system and consequently destroy the remarkable consistency between the TBA calculation and the perturbative result. We can conclude with some confidence therefore that the proposed S-matrices are correct and that the classical integrability of these models...
extends to the quantum regime. Notice that the S-matrix has a quantum group symmetry, an invariance which does not seem to be manifested at the lagrangian level in any simple fashion.

It is worth pointing out that the leading order behaviour \(-h^2k/\pi\) of the free-energy near the ultra-violet fixed-point can easily be deduced from the knowledge that in this limit the theory is an SU(2) WZW model of level \(k\). At the fixed point the chemical potential couples to the combination \(\tilde{J} = \tilde{J}_L + \tilde{J}_R\), where \(\tilde{J}_L\) and \(\tilde{J}_R\) are the left and right currents of a U(1) subalgebra of the SU(2) current algebra. Following [26], the response of the free-energy in a finite volume \(V\) is given by

\[
\delta f(h) = -\frac{h^2}{2V} \int \frac{d^2z}{2\pi} \int \frac{d^2w}{2\pi} \langle \tilde{J}(z)\tilde{J}(w) \rangle, \tag{6.1}
\]

where the expectation value is evaluated in the WZW model. The final result is proportional to the anomaly in the U(1) currents, however one must take careful account of the form of the operator products in a finite volume [26]:

\[
\langle \tilde{J}_L(z)\tilde{J}_L(w) \rangle = \frac{k}{(z-w)^2} + 2\pi kV, \quad \langle \tilde{J}_R(z)\tilde{J}_R(w) \rangle = \frac{k}{(z-w)^2} + 2\pi kV, \tag{6.2}
\]

\[
\langle \tilde{J}_L(z)\tilde{J}_R(w) \rangle = 2\pi k\delta^{(2)}(z_w) - \frac{2\pi k}{V}.
\]

It is then straightforward to extract [26]

\[
\delta f(h) = -\frac{h^2k}{\pi}, \tag{6.3}
\]

in agreement with the leading order term in (4.16) and (5.20). Obviously one could extend this calculation away from the fixed-point by perturbation theory and hope to reproduce the series (5.20); a calculation which would be interesting since it would be valid not just in the large \(k\) regime.

We should also emphasize a remarkable consequence of the equivalence of the lagrangian and S-matrix descriptions we have established in this paper: namely that the field theory (2.1) for \(k = 1\) and \(k = \infty\) is quantum equivalent to the SU(2) chiral Gross-Neveu model and principal chiral model, respectively (since our Ansatz for the S-matrix then reduces to these well-known cases). Let us consider the latter equivalence in more detail. In taking the limit \(k \to \infty\), at fixed \(\epsilon\), it is necessary to introduce the field \(r = ((\pi/2) - w)/\sqrt{x^2 - 1}\). The lagrangian then has a well-defined limit

\[
\mathcal{L}_0 = \frac{1}{2\epsilon^2} \left\{ (\partial_{\mu}r)^2 + \frac{x^2}{1 + 4r^2} (\partial_{\mu}n_a)^2 + \frac{2r^3}{1 + 4r^2} \epsilon_{abc} \epsilon^{\mu\nu} n_a \partial_{\mu}n_b \partial_{\nu}n_c \right\}. \tag{6.4}
\]
With a simple change of variables $\phi_a = r n_a$, this may be written

$$\mathcal{L}_0 = \frac{1}{2e^2} \left( \frac{1}{1 + 4\phi^2} \right) \left\{ \left( \delta_{ab} + 4\phi_a \phi_b \right) \partial_\mu \phi_a \partial^\mu \phi_b + 2\epsilon_{abc} \epsilon^{\mu\nu} \phi_a \partial_\mu \phi_b \partial_\nu \phi_c \right\}.$$  (6.5)

This lagrangian is actually the non-abelian dual of the SU(2) principal chiral model [27] and hence it is indeed known to be quantum equivalent to it. It would be interesting to show, in a similar spirit, that (2.1) with $k = 1$ was a bosonized form of the SU(2) chiral Gross-Neveu model.

Finally, it would clearly be interesting to extend the various results above to larger groups and also to S-matrices of the more general form (3.1).

TJH would like to thank Michel Bauer for many interesting discussions.

Appendix : Some basic facts about zeta-functions

The zeta-function $\zeta_M$ associated with an operator $M$ with eigenvalues $\lambda_i$ can be thought of formally as

$$\zeta_M(s) = \sum_i \lambda_i^{-s}. \quad \text{(A.1)}$$

It is not immediately obvious when this formula makes sense beyond the simplest circumstances in which $M$ is finite-dimensional with $\lambda_i > 0$ and $\text{Re}(s)$ sufficiently large. By comparison with this simplest case, however, it should seem reasonable that when $\zeta_M$ exists it can be used to calculate the dimensions and determinant of $M$ via

$$\dim M = \zeta_M(0), \quad \text{Det} M = -\zeta_M'(0). \quad \text{(A.2)}$$

To make use of this we need some means of calculating $\zeta_M$ (without first finding all its eigenvalues and using (A.1)!) and this is provided by the heat-kernel representation. The key is the observation that (A.1) is formally equivalent to

$$\zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} \text{Tr} \mathcal{G}(t), \quad \text{(A.3)}$$

where $\mathcal{G}(t) = \exp(-tM)$. Then we note that $\mathcal{G}(t) = \exp(-tM)$ can also be characterized by the ‘heat’ equation and boundary condition

$$\frac{\partial \mathcal{G}}{\partial t} = -M \mathcal{G}, \quad \mathcal{G}(0) = 1, \quad \text{(A.4)}$$

(where in the boundary condition 1 is the identity operator on the relevant space). Now we can dispense with (A.1) entirely by adopting (A.3) and (A.4) as a definition of $\zeta_M(s)$.

20
for any given (suitably well-behaved) operator $M$ and we can take (A.2) as providing definitions of the dimension and determinant of $M$.

The case of interest for us is an operator in two Euclidean space dimensions of the form

$$M = \frac{1}{\mu^2} \begin{pmatrix} -\partial^2 + a^2 & c\partial_1 \\ -c\partial_1 & -\partial^2 + b^2 \end{pmatrix}. \quad (A.5)$$

A solution of (A.4) can be found by Fourier transforming to momentum space:

$$G(\xi, t) = \int \frac{d^2p}{(2\pi)^2} e^{ip\cdot\xi} e^{-tM(p)}, \quad M(p) = \frac{1}{\mu^2} \begin{pmatrix} p^2 + a^2 & icp_1 \\ -icp_1 & p^2 + b^2 \end{pmatrix}. \quad (A.6)$$

After substituting in (A.3) we can evaluate the functional part of the trace immediately to obtain a factor $V$ which is the two-dimensional space-time-volume. This leaves us still with a matrix trace

$$\text{Tr} e^{-tM(p)} = 2e^{-(t/\mu^2)(p^2 + \lambda)} \cosh \left( (t/\mu^2)(c^2p_1^2 + \rho^2)^{1/2} \right), \quad (A.7)$$

where

$$\lambda = \frac{1}{2}(a^2 + b^2), \quad \rho = \frac{1}{2}(a^2 - b^2). \quad (A.8)$$

On combining this with (A.3) and (A.6) and re-scaling $t$ we obtain the expression (5.9) in the text with $\eta = c$.

References

[1] A.B. Zamolodchikov and Al. B. Zamolodchikov, Ann. Phys. 120 (1979) 253
[2] E. Ogievetsky, N. Reshetikhin and P. Wiegmann, Nucl. Phys. B280 (1987) 45
[3] B. Berg and P. Weisz, Nucl. Phys. B146 (1979) 205
  V. Kurak and J.A. Swieca, Phys. Lett. B82 (1979) 289
  M. Karowski and H.J. Thun, Nucl. Phys. B190 (1981) 61
[4] N. Andrei and J.H. Lowenstein, Phys. Lett. B90 (1980) 106
[5] P. Hasenfratz, M. Maggiore and F. Niedermayer, Phys. Lett. B245 (1990) 522
[6] P. Hasenfratz and F. Niedermayer, Phys. Lett. B245 (1990) 529
[7] J. Balog, S. Naik, F. Niedermayer and P. Weisz, Phys. Rev. Lett. 69 (1992) 873;
  S. Naik, Nucl. Phys. B (Proc. Suppl.) 30 (1993) 232
[8] T.J. Hollowood, Phys. Lett. B329 (1994) 450
[9] P. Forgács, F. Niedermayer and P. Weisz, Nucl. Phys. B367 (1991) 123
[10] A. Polyakov and P.B. Wiegmann, Phys. Lett. B131 (1983) 121
[11] P.B. Wiegmann, Phys. Lett. B141 (1984) 217
[12] E. Witten, Commun. Math. Phys. 92 (1984) 455
[13] P. Goddard and D. Olive, Int. J. Mod. Phys. A1 (1986) 303
[14] D. Bernard, Commun. Math. Phys. 137 (1991) 191
[15] J. Balog, P. Forgács, Z. Horváth and L. Palla, Phys. Lett. B324 (1994) 403
[16] C. Ahn, D. Bernard and A. LeClair, Nucl. Phys. B346 (1990) 409
[17] O. Alvarez, Commun. Math. Phys. 100 (1985) 279; R. Rohm and E. Witten, Ann. Phys. 170 (1986) 454
[18] S. Rajeev, Phys. Lett. B217 (1989) 123
[19] D. Friedan, Ann. Phys. 163 (1985) 318
    T.L. Curtright and C.K. Zachos, Phys. Rev. Lett. 53 (1984) 1799
    E. Braaten, T.L. Curtright and C.K. Zachos, Nucl. Phys. B260 (1985) 630
    S. Mukhi, Nucl. Phys. B264 (1986) 640
    C.M. Hull and P.K. Townsend, Phys. Lett. B191 (1987) 115
    D. Zanon, Phys. Lett. B191 (1987) 363
    D.R.T. Jones, Phys. Lett. B192 (1987) 391
    H. Osborn, Ann. of Phys. 200 (1990) 1
[20] T.J. Hollowood, Nucl. Phys. B414 (1994) 379
[21] T.J. Hollowood, Phys. Lett. B230 (1994) 43
[22] A. LeClair, Phys. Lett. B227 (1989) 417
    D. Bernard and A. LeClair, Nucl. Phys. B340 (1990) 721
[23] E.H. Lieb and W. Liniger, Phys. Rev. 130 (1963) 1605
    A.B. Zamolodchikov, Nucl. Phys. B342 (1990) 695
[24] A.B. Zamolodchikov, Nucl. Phys. B366 (1991) 122
[25] G. Japaridze, A. Nersesyan and P. Wiegmann, Nucl. Phys. B230 (1984) 511
[26] P. Fendley and K. Intriligator, Phys. Lett. B319 (1993) 132
[27] B. Fridling and A. Jevicki, Phys. Lett. B134 (1984) 70
    E. Fradkin and A. Tseytlin, Ann. Phys. 162 (1985) 31
    C.K.Zachos and T.L. Curtright, Phys. Rev. D49 (1994) 5408