An algorithm to find the spectral radius of nonnegative tensors and its convergence analysis *

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Abstract

In this paper we propose an iterative algorithm to find out the spectral radius of nonnegative tensors. This algorithm is an extension of the smoothing method for finding the largest eigenvalue of a nonnegative matrix [12]. For nonnegative irreducible tensors, we establish the converges of the algorithm. Finally we report some numerical results and conclude this paper with some remarks.

Keywords nonnegative tensor, spectral radius, smoothing method, diagonal transformation

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1 Introduction

Eigenvalue problems of high order tensor have become an important topic of study in a new applied mathematics branch, numerical multilinear algebra, and they have a wide range of practical applications, for more references, see [4, 5, 2, 10, 9, 8]. In recent years, the largest eigenvalue problem for nonnegative tensors has attracted special attention. Chang et al [1] generalized the Perron-Frobenius theorem from nonnegative irreducible matrices to nonnegative irreducible tensors. Ng et al [6] gave a method to find the largest eigenvalue of a nonnegative irreducible tensor. Yang and Yang [11] defined the spectral radius of a tensor and gave further results for the Perron-Frobenius theorem and proved that the spectral radius is the largest eigenvalue of any nonnegative tensor and all eigenvalues with the spectral radius as their modulus distribute uniformly on the circle. In this paper, we propose a method to find the spectral radius of a class of nonnegative tensors. This method is an extension of a method in [12] for calculating the spectral radius of a nonnegative matrix. We show that for a nonnegative irreducible tensor, the sequence generated by this algorithm converges to the spectral radius.

This paper is organized as follows: In section 2 we recall some definitions and theorems; we give our algorithm in section 3 and lay down the proof of the algorithm in section 4; some numerical results are reported in section 5.

We first add a comment on the notation that is used in the sequel. Vectors are written as lowercase letters (x, y, . . .), matrices correspond to italic capitals (A, B, . . .), and tensors are written as calligraphic capitals (A, B, . . .). The entry with row index i and column index j in a matrix A, i.e. (A)ij is symbolized by aij(also (A)i1···ip,j1···jq = a(i1···ip,j1···jq). The symbol |·| used on a matrix A(or tensor A) means that (|A|)ij = |aij|(or (|A|)i1···ip,j1···jq = |ai1···ip,j1···jq|).

Rn + (Rn++ ) denotes the cone {x ∈ Rn | xi ≥ (>0, i = 1, . . . , n}. The symbol A ≥ (> ≤, <)B means that aij ≥ (> ≤, <)b(ij for every i, j and it is the same for rectangular tensors.

2 Preliminaries

First we recall the definition of tensor: a tensor is a multidimensional array, and a real order m dimension n tensor A consists of n^m real entries:

\[ A_{i_1\ldots i_m} \in R, \]

where \( i_j = 1, \ldots, n \) for \( j = 1, \ldots, m \). If a number \( \lambda \) and a nonzero vector \( x \) are solutions of the following homogeneous polynomial equations:

\[ \mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \]

then \( \lambda \) is called the eigenvalue of \( \mathcal{A} \) and \( x \) the eigenvector of \( \mathcal{A} \) associated with \( \lambda \), where \( \mathcal{A}x^{m-1} \) and \( x^{[m-1]} \) are vectors, whose ith component are

\[ (\mathcal{A}x^{m-1})_i = \sum_{i_2, \ldots, i_m=1}^{n} a_{i_2\ldots i_m} x_{i_2} \ldots x_{i_m}, \]

\[ (x^{[m-1]})_i = x_i^{m-1}. \]
respectively. This definition was introduced by Qi [7] where he supposed that \( \mathcal{A} \) is an order \( m \) dimension \( n \) symmetric tensor and \( m \) is even. Independently, Lim [4] gave such a definition but restricted \( x \) to be a real vector and \( \lambda \) to be a real number. Here we use the definition given in [1].

The Perron-Frobenius theorem for nonnegative tensors is related to measuring high-order connectivity in linked objects and hypergraphs, see [5, 2]. Let us recall the Perron-Frobenius theorem for nonnegative tensors given in [1]:

**Theorem 2.1** (see theorem 1.3 of [1]) If \( \mathcal{A} \) is a nonnegative tensor of order \( m \) dimension \( n \), then there exists \( \lambda_0 \geq 0 \) and a nonnegative vector \( x_0 \neq 0 \) such that
\[
\mathcal{A}x^{m-1} = \lambda_0 x_0^{m-1}.
\] (2.1)

**Theorem 2.2** (see theorem 1.4 of [1]) If \( \mathcal{A} \) is an irreducible nonnegative tensor of order \( m \) dimension \( n \), then the pair \( (\lambda_0, x_0) \) in equation (2.1) satisfy:

1. \( \lambda_0 > 0 \) is an eigenvalue.
2. \( x_0 > 0 \), i.e. all components of \( x_0 \) are positive.
3. If \( \lambda \) is an eigenvalue with nonnegative eigenvector, then \( \lambda = \lambda_0 \). Moreover, the nonnegative eigenvector is unique up to a multiplicative constant.
4. If \( \lambda \) is an eigenvalue of \( \mathcal{A} \), then \( |\lambda| \leq \lambda_0 \).

And the reducibility of tensor is defined as follow:

**Definition 2.1** (Reducibility, see definition 2.1 of [1]) A tensor \( C = (c_{i_1 \cdots i_m}) \) of order \( m \) dimension \( n \) is called reducible, if there exists a nonempty proper index subset \( I \subset \{1, \cdots, n\} \) such that
\[
c_{i_1 \cdots i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \cdots, i_m \notin I.
\]

If \( C \) is not reducible, then we call \( C \) irreducible. In [11], Yang and Yang prove that for any nonnegative tensor, the spectral radius is the largest eigenvalue of it, which is an enhancement of theorem 2.1:

**Theorem 2.3** (See theorem 2.3 of [11]) If \( \mathcal{A} \) is a nonnegative tensor of order \( m \) dimension \( n \), then \( \rho(\mathcal{A}) \) is an eigenvalue with a nonnegative eigenvector \( y \in \mathbb{R}^n_+ \) corresponding to it.

**Definition 2.2** The spectral radius of tensor \( \mathcal{A} \) is defined as
\[
\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\}.
\]

For positive tensors, the following theorem holds:

**Theorem 2.4** (See theorem 2.4 of [11]) Let \( \mathcal{A} \) be a positive order \( m \) dimension \( n \) tensor, if \( \lambda \) is an eigenvalue of \( \mathcal{A} \) except \( \rho(\mathcal{A}) \), then \( \rho(\mathcal{A}) > |\lambda| \).
3 Algorithm

Before presenting our algorithm, we give the definition of diagonal similar tensors, which was first used by Yang et al [11]:

**Definition 3.1** (diagonal similar tensors) Let \( A = (a_{i_1 \cdots i_m}) \), \( B = (b_{i_1 \cdots i_m}) \) be two order \( m \) dimension \( n \) tensors, if there is a nonsingular diagonal matrix \( D = (d_{ij}) \), such that

\[
A = B \cdot D^{-(m-1)} \cdot D \cdot \cdots \cdot D,
\]

where

\[
a_{i_1 i_2 \cdots i_m} = d_{i_1, i_1}^{-(m-1)} b_{i_1 i_2 \cdots i_m} d_{i_2, i_2} \cdots d_{i_m, i_m}, \quad i_1, \ldots, i_m \in \{1, \ldots, n\}.
\]

On the diagonal similar tensors we have the following proposition:

**Proposition 3.1** Let \( A, B, D \) be defined as above, if \( A, B \) have eigenvalues, then they have the same eigenvalues, i.e., if \( \lambda \) be an eigenvalue of \( A \) with corresponding eigenvector \( x \neq 0 \), then \( \lambda \) is also an eigenvalue of \( B \) with corresponding eigenvector \( Dx \); if \( \rho \) is an eigenvalue of \( B \) with corresponding eigenvector \( y \neq 0 \), then \( \rho \) is also an eigenvalue of \( A \) with corresponding eigenvector \( D^{-1}y \).

**Proof.**

\[
(B \cdot (Dx)^{m-1})_i = \sum_{i_2, \ldots, i_m=1} b_{i_1 i_2 \cdots i_m} d_{i_2, i_2} \cdots d_{i_m, i_m} x_{i_2} \cdots x_{i_m}
\]

\[
= d_{i, i}^{m-1} \sum_{i_2, \ldots, i_m=1} a_{i_1 i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}
\]

\[
= \lambda d_{i, i}^{m-1} x_i^{m-1} = \lambda(Dx)_i, \quad i = 1, \ldots, n.
\]

The proof of \( A(D^{-1}y)^{m-1} = \rho(D^{-1}y)^{m-1} \) is the same. \( \square \)

One easily gets following estimation:

**Lemma 3.1** (See lemma 5.6 of [11]) Let \( A \geq 0 \) be an order \( m \) dimension \( n \) tensor. Denote \( \rho(A) \) the spectral radius of \( A \). Then

\[
\min_{1 \leq i \leq n} \sum_{i_2, \ldots, i_m=1} a_{i, i_2 \cdots i_m} \leq \rho(A) \leq \max_{1 \leq i \leq n} \sum_{i_2, \ldots, i_m=1} a_{i, i_2 \cdots i_m}.
\]

Let \( R_i = \sum_{i_2, \ldots, i_m=1} a_{i, i_2 \cdots i_m} \). If \( R_i \equiv C \), where \( C \) is a constant, we have the following proposition:

**Lemma 3.2** (See lemma 5.5 of [11]) Suppose \( A \geq 0 \) be an order \( m \) dimension \( n \) tensor, if \( R_i = \sum_{i_2, \ldots, i_m=1} a_{i, i_2 \cdots i_m} \equiv C \) (constant) for \( i = 1, \ldots, n \), then

\[
\rho(A) = C.
\]
Given an order \( m \) dimension \( n \) nonnegative irreducible tensor \( \mathcal{B} \), we calculate its spectral radius as follow:

**Algorithm 3.1**  
**step 1.** Let \( A = \mathcal{B} + I \). Denote \( A^{(0)} = A \), set \( k := 0 \), compute

\[
R_i^{(0)} = \sum_{i_2, \ldots, i_m=1} a_{i, i_2, \ldots, i_m},
\]

\[
R^{(0)} = \max_{1 \leq i \leq n} R_i^{(0)}, \quad r^{(0)} = \min_{1 \leq i \leq n} R_i^{(0)}.
\]

if \( R^{(0)} = r^{(0)} \), let \( v(A) = e = (1, \ldots, 1)^T \), goto step 3, else goto step 2.  
**step 2.** Compute

\[
A^{(k+1)} = A^{(k)} \cdot D(k)^{(m-1)} \cdot D(k) \ldots D(k),
\]

where

\[
D(k) = \text{diag}(\{R_1^{(k)} \frac{1}{R_1}, \ldots, R_n^{(k)} \frac{1}{R_n}\})
\]

and

\[
a_{i, i_2, \ldots, i_m}^{(k+1)} = (R^{(k)})^{-1} a_{i, i_2, \ldots, i_m}^{(k)} \frac{1}{R_1^{(k)}} \ldots \frac{1}{R_n^{(k)}}.
\]

compute

\[
R_i^{(k+1)} = \sum_{i_2, \ldots, i_m=1} a_{i, i_2, \ldots, i_m}^{(k+1)}
\]

\[
R^{(k+1)} = \max_{1 \leq i \leq n} R_i^{(k+1)}, \quad r^{(k+1)} = \min_{1 \leq i \leq n} R_i^{(k+1)}
\]

\[
v^{(k+1)}(A) := \text{diag}(\{\prod_{j=0}^{k} \frac{R_1^{(j)}}{R_1^{(j)}}, \ldots, \prod_{j=0}^{k} \frac{R_n^{(j)}}{R_n^{(j)}}\}) \cdot (\prod_{j=0}^{k} \frac{R_1^{(k+1)}}{R_1^{(k+1)}}, \ldots, \prod_{j=0}^{k} \frac{R_n^{(k+1)}}{R_n^{(k+1)}})^T
\]

if \( R^{(k+1)} = r^{(k+1)} \), goto step 3, else loop step 2.  
**step 3.** Output \( \rho(\mathcal{B}) = R^{(k)} - 1, \) \( v(A) \) as the eigenvalue and eigenvector of \( \mathcal{B} \).

**Remark.** When \( m = 2 \), this algorithm reduces to the smoothing method in [12], and \( R_i^{(k)} \) is the sum of the \( i \)th row of matrix \( A^{(k)} \).

It is easy to notice that \( A \) satisfies the following condition:

**Condition .1**

\[
R_i = \sum_{i_2, \ldots, i_m} a_{i, i_2, \ldots, i_m} > 0, \quad i = 1, \ldots, n.
\]

Hence algorithm 3.1 is well-defined. Moreover, we have the following theorem to ensure that \( R^{(k)} - r^{(k)} \) is nonincreasing as \( k \to \infty \) (and in fact we can prove it decreases strictly):

**Theorem 3.1** Under Condition .1,

\[
r \leq r^{(1)} \leq \cdots \leq r^{(k)} \leq \cdots \leq \rho(A) \leq \cdots \leq R^{(k)} \leq \cdots \leq R.
\]

Under the assumption of irreducibility of \( \mathcal{B} \), we have
Theorem 3.2 If \( A = B + I \), then
\[
\lim_{k \to \infty} r^{(k)} = \lim_{k \to \infty} R^{(k)} = \rho(A).
\]

This theorem shows that the algorithm can find the spectral radius. We will prove these theorems in the next section.

4 Convergence analysis

In this section, we will prove theorems 3.1 and 3.2. First we give a lemma:

Lemma 4.1
\[
R^{(k-1)} \geq R^{(k)}, r^{(k-1)} \leq r^{(k)}, k = 1, 2, \ldots,
\]
where \( R^{(k)}, r^{(k)} \) are defined in algorithm 3.1.

Proof. Without loss of generality we suppose that \( R^{(k)} = R^{(k)}_s, r^{(k)} = R^{(k)}_t, s, t \in \{1, 2, \ldots, n\} \).
We have
\[
R^{(k)} = \sum_{i_2, \ldots, i_m = 1} a^{(k-1)}_{s, i_2 \cdots i_m} \frac{(R_{i_2})^{(k-1)}}{R_s^{(k-1)}} \cdots \frac{(R_{i_m})^{(k-1)}}{R_s^{(k-1)}} = R^{(k-1)}
\]
\[
r^{(k)} = \sum_{i_2, \ldots, i_m = 1} a^{(k-1)}_{t, i_2 \cdots i_m} \frac{(R_{i_2})^{(k-1)}}{R_t^{(k-1)}} \cdots \frac{(R_{i_m})^{(k-1)}}{R_t^{(k-1)}} \geq \sum_{i_2, \ldots, i_m = 1} a^{(k-1)}_{s, i_2 \cdots i_m} \frac{r^{(k-1)}}{R_t^{(k-1)}} = r^{(k-1)}.
\]

Proof of theorem 3.1 From lemma 3.1 and 4.1 we can easily see that theorem 3.1 holds.

The following proposition shows that the sequence \( \{R^{(k)} - r^{(k)}\} \) is decreasing strictly:

Proposition 4.1 Let \( A \) be defined in algorithm 3.1, \( i \in I = \{1, \ldots, n\} \), and without loss of generality suppose that \( R^{(k)} = R^{(k)}_s, r^{(k)} = R^{(k)}_t \); let \( J(k) = \{(i_2, \ldots, i_m) | a^{(k)}_{i_2 \cdots i_m} \frac{R^{(k)}_s}{R^{(k)}_t} \geq a^{(k)}_{s, i_2 \cdots i_m} \}, N = \{(i_2, \ldots, i_m) | i_2, \ldots, i_m \in I \} \). Then we have
\[
R^{(k)} - r^{(k)} \leq (R^{(k-1)} - r^{(k-1)}) \left[ 1 - \frac{1}{R^{(k-1)}} \sum_{(i_2, \ldots, i_m) \in N - J} a^{(k-1)}_{s, i_2 \cdots i_m} + \sum_{(i_2, \ldots, i_m) \in J} a^{(k-1)}_{t, i_2 \cdots i_m} \right] \quad (4.1)
\]
Proof. We only prove the case when \( k = 1 \), i.e.

\[
R^{(1)} - r^{(1)} \leq (R - r)[1 - \frac{1}{R} \left( \sum_{(i_2, \cdots, i_m) \in N - J} a_{s, i_2 \cdots i_m} + \sum_{(i_2, \cdots, i_m) \in J} a_t, i_2 \cdots i_m \right)]. \tag{4.2}
\]

For \( k = 2, 3, \cdots \) the proof is the same.

If \( R^{(1)} = r^{(1)} \), then (4.2) holds easily, so we suppose that \( R^{(1)} > r^{(1)} \).

\[
R^{(1)} - r^{(1)} = R_s^{(1)} - R_t^{(1)}
= \sum_{i_2, \cdots, i_m = 1}^n a_{s, i_2 \cdots i_m} \frac{R^{\frac{1}{12}} \cdots R^{\frac{1}{m-1}}}{R_s} - \sum_{i_2, \cdots, i_m = 1}^n a_{t, i_2 \cdots i_m} \frac{R^{\frac{1}{12}} \cdots R^{\frac{1}{m-1}}}{R_t}
= \sum_{i_2, \cdots, i_m = 1}^n \left( \frac{a_{s, i_2 \cdots i_m}}{R_s} - \frac{a_{t, i_2 \cdots i_m}}{R_t} \right) \frac{R^{\frac{1}{12}} \cdots R^{\frac{1}{m-1}}}{R_s} \cdots \frac{R^{\frac{1}{12}} \cdots R^{\frac{1}{m-1}}}{R_t}. \tag{4.3}
\]

By definition of \( J \) and \( R^{(1)} > r^{(1)} \) we see that \( J, N - J \neq \emptyset \), respectively, thus we have

\[
\sum_{(i_2, \cdots, i_m) \in J} \left( \frac{a_{s, i_2 \cdots i_m}}{R_s} - \frac{a_{t, i_2 \cdots i_m}}{R_t} \right) = - \sum_{(i_2, \cdots, i_m) \in N - J} \left( \frac{a_{s, i_2 \cdots i_m}}{R_s} - \frac{a_{t, i_2 \cdots i_m}}{R_t} \right). \tag{4.4}
\]

Combining (4.3) and (4.4) we have

\[
R^{(1)} - r^{(1)} = \sum_{(i_2, \cdots, i_m) \in J} \left( \frac{a_{s, i_2 \cdots i_m}}{R_s} - \frac{a_{t, i_2 \cdots i_m}}{R_t} \right) \frac{R^{\frac{1}{12}} \cdots R^{\frac{1}{m-1}}}{R_s} \cdots \frac{R^{\frac{1}{12}} \cdots R^{\frac{1}{m-1}}}{R_t}
+ \sum_{(i_2, \cdots, i_m) \in N - J} \left( \frac{a_{s, i_2 \cdots i_m}}{R_s} - \frac{a_{t, i_2 \cdots i_m}}{R_t} \right) \frac{R^{\frac{1}{12}} \cdots R^{\frac{1}{m-1}}}{R_s} \cdots \frac{R^{\frac{1}{12}} \cdots R^{\frac{1}{m-1}}}{R_t}
\leq R \sum_{(i_2, \cdots, i_m) \in J} \left( \frac{a_{s, i_2 \cdots i_m}}{R_s} - \frac{a_{t, i_2 \cdots i_m}}{R_t} \right) + r \sum_{(i_2, \cdots, i_m) \in N - J} \left( \frac{a_{s, i_2 \cdots i_m}}{R_s} - \frac{a_{t, i_2 \cdots i_m}}{R_t} \right)
= R \sum_{(i_2, \cdots, i_m) \in J} \left( \frac{a_{s, i_2 \cdots i_m}}{R_s} - \frac{a_{t, i_2 \cdots i_m}}{R_t} \right) - r \sum_{(i_2, \cdots, i_m) \in J} \left( \frac{a_{s, i_2 \cdots i_m}}{R_s} - \frac{a_{t, i_2 \cdots i_m}}{R_t} \right)
= (R - r) \left( \sum_{(i_2, \cdots, i_m) \in J} \frac{a_{s, i_2 \cdots i_m}}{R_s} - \sum_{(i_2, \cdots, i_m) \in J} \frac{a_{t, i_2 \cdots i_m}}{R_t} \right)
= (R - r) \left[ 1 - \left( \sum_{(i_2, \cdots, i_m) \in N - J} \frac{a_{s, i_2 \cdots i_m}}{R_s} + \sum_{(i_2, \cdots, i_m) \in J} \frac{a_{t, i_2 \cdots i_m}}{R_t} \right) \right]
\leq (R - r) \left[ 1 - \frac{1}{R} \left( \sum_{(i_2, \cdots, i_m) \in N - J} a_{s, i_2 \cdots i_m} + \sum_{(i_2, \cdots, i_m) \in J} a_{t, i_2 \cdots i_m} \right) \right].
\]

The proof is completed. \( \square \)

From Proposition 4.1 we see that \( \{R^{(k)} - r^{(k)}\} \) is a nonnegative and strictly monotone decreasing sequence, so it has limit, but it is not sufficient to show theorem 5.2. Before prove it, we denote \( M_i^{(k)} = (\prod_{l=0}^k \frac{R^{(l)}}{R^{(l+1)}})^{\frac{1}{m-1}}, i = 1, 2, \cdots, n \), hence \( v^{(k)}(A) \) defined in algorithm 3.1 is \( (M_1^{(k)}, \cdots, M_n^{(k)})^T \) and \( V^{(k)}(A) \) the matrix representation of \( v^{(k)}(A) \), i.e:

\[
V^{(k)}(A) = diag(M_1^{(k)}, \cdots, M_n^{(k)}).
\]

Note that the limit of \( M_i^{(k)} = (\prod_{l=0}^k \frac{R^{(l)}}{R^{(l+1)}})^{\frac{1}{m-1}} \) exists. Denote it by \( M_i, i = 1, \cdots, n \), and \( v^*(A) \), \( V^*(A) \) the limit of \( v^{(k)}(A) \), \( V^{(k)}(A) \), respectively. We introduce a notation:

\[
V(A)^{m-1} \cdot A \cdot V(A)^{-(m-1)} \cdot V(A) \cdots V(A)
\]
We will prove Proposition 4.2. By theorem 3.1, \( \mathcal{A} \cdot (V(A)^{m-1}V(A)^{-(m-1)}) \cdot V(A) \cdots V(A) \)

Where the superscript \(^{'}(k)\) is ignored. Then we have

\[
\mathcal{A}v^{(k)}(A) = \mathcal{A}(V^{(k)}(A)e)^{m-1}
\]

\[
\mathcal{A} = \mathcal{A} \cdot I \cdot (V^{(k)}(A) \cdots V^{(k)}(A))e^{m-1}
\]

\[
\geq V^{(k)}(A)_{m-1} \cdot (A \cdot V^{(k)}(A))^{-(m-1)} \cdot V^{(k)}(A) \cdots V^{(k)}(A)_{m-1}e^{m-1}
\]

Letting \( k \to \infty \) we have

\[
\mathcal{A}v^{*}(A)_{m-1} \geq r^{*}v^{*}(A)_{m-1}. \tag{4.5}
\]

By theorem 3.1 \( r^{*} \) exists, where \( r^{*} \) is the limit of \( r^{(k)} \), and it satisfies the proposition:

**Proposition 4.2** \( r^{*} \geq r^{(k)}, k = 1, 2, \cdots \). If \( \text{(4.5)} \) is an equation, by theorem 2.2 \( r^{*} \) is the spectral radius. Suppose not, denote

\[
x_{0}(v^{*}(A)) = (\mathcal{A}v^{*}(A)_{m-1})_{\frac{1}{m-1}},
\]

\[
y_{0}(v^{*}(A)) = (r^{*}v^{*}(A)_{m-1})_{\frac{1}{m-1}};
\]

\[
x_{1}(v^{*}(A)) = (\mathcal{A}x_{0}^{m-1})_{\frac{1}{m-1}},
\]

\[
y_{1}(v^{*}(A)) = (\mathcal{A}y_{0}^{m-1})_{\frac{1}{m-1}};
\]

\[
\vdots
\]

\[
x_{(n-1)}(v^{*}(A)) = (\mathcal{A}x_{(n-2)}^{m-1})_{\frac{1}{m-1}},
\]

\[
y_{(n-1)}(v^{*}(A)) = (\mathcal{A}y_{(n-2)}^{m-1})_{\frac{1}{m-1}}.
\]

We will prove

\[
x_{(n-1)}(v^{*}(A)) > y_{(n-1)}(v^{*}(A)) \tag{4.6}
\]

Recall theorem 6.6 given in [11]:

**Theorem 4.1** (see theorem 6.6 of [11]) Let \( \mathcal{B} \geq 0 \) be an order \( m \) dimension \( n \) tensor. Then \( \mathcal{B} \) is irreducible if and only if for all \( x \in R_{+}^{m} \), \( x \neq 0 \), let \( x_{0} = x \) and \( x_{k+1} = (\mathcal{B} + I)x_{k}^{m-1} \). Then \( x_{n-1} > 0 \).

We follow this theorem to prove a lemma:

**Lemma 4.2** Let \( \mathcal{B} \) be defined as above. Denote \( \mathcal{A} = \mathcal{B} + I \). Suppose \( x, y \in R_{+}^{m} \) and \( x \geq y \). let \( x_{0} = x, y_{0} = y \) and \( x_{k+1} = Ax_{k}^{m-1}, y_{k+1} = Ay_{k}^{m-1} \). Then \( x_{(n-1)} > y_{(n-1)} \).
Proof. It is easy to see that \( x_{(n-1)} \geq y_{(n-1)} \). Let \( I_k = \{i|x_i = y_i\} \) and \( M_k = |I_k| \), so does \( M_{k+1} \). All we need to do is to prove that \( M_{k+1} < M_k \). First we consider \( x_{k+1} = (\mathcal{A} x_k^{m-1} + x^m) \) where \( i \notin I_k \). We have \( x_{k+1} > y_{k+1} \), because \( (\mathcal{A} x_k^{m-1})_i \geq (\mathcal{A} y_k^{m-1})_i \) and \( x_k > y_k \). For all \( i \in I_k \), \( x_{k+1} = \sum_{i_2 \cdots i_m=1} a_{i_2 \cdots i_m} x_{k_{i_2}} \cdots x_{k_{i_m}} \). We claim that at least a \( x_{k+1} > y_{k+1}, i \in I_k \). Suppose not, then \( \sum_{i_2 \cdots i_m=1} a_{i_2 \cdots i_m} (x_{k_{i_2}} \cdots x_{k_{i_m}} - y_{k_{i_2}} \cdots y_{k_{i_m}}) = 0 \), \( i \in I_k \). It means that \( a_{i_2 \cdots i_m} = 0 \), \( i_2, \cdots, i_m \notin I_k \) and \( i \in I_k \), which contradicts to the irreducibility of \( \mathcal{A} \). Thus at least a \( x_{k+1} > y_{k+1}, i \in I_k \), which means that \( M_{k+1} < M_k \). Repeat at most \( n-1 \) times, we have \( x_{n-1} > y_{n-1} \).

The same proof can apply to \( r \) if \( k \) is not an equation, then \( (4.6) \) holds. By the continuity of \( x_{(n-1)}(\cdot) - y_{(n-1)}(\cdot) \), when \( k \) sufficiently large, one has
\[
x_{(n-1)}(v^k(\mathcal{A})) > y_{(n-1)}(v^k(\mathcal{A})),
\]
and
\[
x_0(v^k(\mathcal{A})) = (\mathcal{A} v^k(\mathcal{A}))^{m-1} = (R^{(k+1)})^{m-1} v^{(k+1)}(\mathcal{A}),
\]
\[
x_1(v^k(\mathcal{A})) = (R^{(k+1)} R^{(k+2)})^{m-1} v^{(k+2)}(\mathcal{A}),
\]
\[
\cdots
\]
\[
x_{(n-1)}(v^k(\mathcal{A})) = (\prod_{i=1}^{n} R^{(k+i)})^{m-1} v^{(k+n)}(\mathcal{A}), \text{ and}
\]
\[
y_{(n-1)}(v^k(\mathcal{A})) = r^* (\prod_{i=1}^{n-1} R^{(k+i)})^{m-1} v^{(k+n-1)}(\mathcal{A}).
\]
Then \( (4.7) \) means that
\[
R_i^{(k+n)} > r^*, \quad i = 1, \cdots, n,
\]
especially \( r^{(k+n)} > r^* \) when \( k \) sufficiently large, which contradicts with proposition 4.2. Thus
\[
(\mathcal{A} v^*(\mathcal{A}))^{m-1} = r^* v^*(\mathcal{A})^{m-1}.
\]
The same proof can apply to \( \mathcal{A} v^*(\mathcal{A})^{m-1} = R^* v^*(\mathcal{A})^{m-1} \). Hence theorem 3.2 holds and \( v^*(\mathcal{A}) \) is the positive corresponding eigenvector of \( \rho(\mathcal{A}) \).

5 Numerical results

In this section, we first give numerical result on a 3-order 3-dimension nonnegative irreducible tensor; then we generate some random tensors to test our algorithm. We use the termination condition given in [9]:

1. \( k \geq 100 \),
2. \( R^{(k)} - r^{(k)} \leq 10^{-7} \).

Example 1. Consider the 3-order 3-dimensional tensor
\[
\mathcal{B} = [B(1, :, :), B(2, :, :), B(3, :, :)],
\]
where
\[
B(1, :, :) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 3.72 & 0 \\
0 & 0 & 0 \\
\end{pmatrix},
\]

9
\[
B(2,\ldots) = \begin{pmatrix}
  9.02 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix},
\]

\[
B(3,\ldots) = \begin{pmatrix}
  9.55 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix}.
\]

**Example 2.** We use some randomly generated tensors to test algorithm 3.1. Each entry of these tensors is between 0 and 10.

Table 1 is the numerical results of algorithm 3.1 for example 1 where the tensor is \( \mathcal{A} = \mathcal{B} + I \). From the result, we get \( \rho(\mathcal{B}) = 6.79262 - 1 = 5.79262 \) and the positive corresponding eigenvector is \((0.46224, 0.57681, 0.593515)^T\). But if we directly apply the algorithm to \( \mathcal{B} \), it does not converge. Table 2 shows some numerical results on randomly generated tensors of different order and dimension. In our experiment, the algorithm converges to the spectral radius for all the tensors although there may be some reducible tensors in these randomly generated tensors.

**Table 1: numerical results of algorithm 3.1 for example 1**

| \( k \) | \( r^{(k)} \) | \( R^{(k)} \) | \( R^{(k)} - r^{(k)} \) | \( 0.5 \cdot (R^{(k)} + r^{(k)}) \) |
|---|---|---|---|---|
| 1 | 4.72 | 10.55 | 5.83 | 7.635 |
| 2 | 5.24894 | 8.89712 | 3.64818 | 7.07303 |
| 3 | 5.65898 | 8.2097 | 2.55071 | 6.93434 |
| 4 | 5.96904 | 7.7527 | 1.78366 | 6.86087 |
| 5 | 6.19911 | 7.45402 | 1.25491 | 6.82656 |
| 6 | 6.36745 | 7.25147 | 0.88402 | 6.80946 |
| ... | | | | |
| 48 | 6.79262 | 6.79262 | 3.83995e-007 | 6.79262 |
| 49 | 6.79262 | 6.79262 | 2.70932e-007 | 6.79262 |
| 50 | 6.79262 | 6.79262 | 1.9116e-007 | 6.79262 |
| 51 | 6.79262 | 6.79262 | 1.34875e-007 | 6.79262 |
| 52 | 6.79262 | 6.79262 | 9.51629e-008 | 6.79262 |

**6 Conclusion and remarks**

We give an algorithm to find the spectral radius of nonnegative tensors. When the tensor is irreducible, our algorithm can assure to find out the spectral radius and its corresponding positive eigenvector. This result is better than the algorithm proposed by Michael et al [6]. We can also apply algorithm 3.1 to \( \mathcal{A}(\alpha) = \mathcal{B} + \alpha I \) where \( \alpha \) is a positive number. The choice of \( \alpha \) will affect the convergence rate of the algorithm. This needs further research.
Table 2: numerical results of algorithm 3.1 for some randomly generated tensors

| (n, m) | k | ρ(A) | R(k) − r(k) | ∥Av(A)−1 − ρ(A)v(A)|m−1||∞ |
|--------|---|-------|-------------|---------------------------------|
| (10,3) | 7 | 444.247 | 9.56E-09    | 5.21E-09                         |
| (5,3)  | 8 | 111.111 | 2.56E-08    | 1.52E-08                         |
| (20,3) | 6 | 1791.51  | 2.15E-08    | 9.03E-09                         |
| (30,3) | 6 | 4057.86  | 1.34E-09    | 6.32E-10                         |
| (50,3) | 5 | 11225.7  | 3.60E-08    | 1.81E-08                         |
| (100,3)| 5 | 45013    | 2.57E-09    | 1.21E-09                         |
| (20,4) | 6 | 35983.8  | 2.28E-11    | 6.85E-11                         |
| (5,4)  | 7 | 556.015  | 2.44E-10    | 1.58E-10                         |
| (10,4) | 6 | 4494.69  | 1.92E-10    | 1.04E-10                         |
| (15,4) | 6 | 15144    | 1.78E-11    | 2.38E-11                         |
| (30,4) | 5 | 121554   | 1.02E-10    | 1.75E-10                         |
| (5,5)  | 5 | 2765.93  | 1.83E-08    | 9.43E-09                         |
| (10,5) | 5 | 44913.9  | 7.23E-10    | 7.02E-10                         |
| (15,5) | 4 | 227923   | 7.56E-09    | 6.42E-09                         |
| (20,5) | 4 | 720407   | 2.95E-08    | 4.01E-08                         |
| (5,6)  | 4 | 14091.6  | 2.69E-08    | 1.11E-08                         |
| (10,6) | 4 | 450133   | 1.27E-08    | 2.16E-08                         |
| (15,6) | 4 | 3.42E+06 | 5.09E-08    | 4.03E-07                         |

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