FINITE GROUPS WITH ABELIAN AUTOMORPHISM GROUPS: A SURVEY

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Abstract. In this article we present an extensive survey on the developments in the theory of non-abelian finite groups with abelian automorphism groups, and pose some problems and further research directions.

1. Introduction

All the groups considered here are finite groups.

Any cyclic group has abelian automorphism group. By the structure theorem for finite abelian groups, it is easy to see that among abelian groups, only the cyclic ones have abelian automorphism groups. A natural question, posed by H. Hilton [22] Appendix, Question 7] in 1908, is:

Can a non-abelian group have abelian group of automorphisms?

An affirmative answer to this question was given by G. A. Miller [16] in 1913. He constructed a non-abelian group of order $2^6$ whose automorphism group is elementary abelian of order $2^7$. Observe that a non-abelian group with abelian automorphism group must be a nilpotent group of nilpotency class 2. Hence it suffices to study the groups of prime power orders. Investigation on the structure of groups with abelian automorphism groups was initiated by C. Hopkins [23] in 1927. He, among other things, proved that such a group can not have a non-trivial abelian direct factor, and if such a group is a $p$-group, then so is its automorphism group, where $p$ is a prime integer.

Unfortunately, the topic was not investigated for about half a century after the work of Hopkins. But days came when examples of such odd prime power order groups were constructed by D. Jonah and M. Konvisser [29] in 1975 and a thesis was written on the topic by B. E. Earnley [12] during the same year, in which, attributing to G. A. Miller, a group with abelian automorphism group

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was named as 'Miller group'. Following Earnley, we call a group to be Miller if it is non-abelian and its automorphism group is abelian. Earnley pointed out that the former statement of Hopkins is not true as such for $2$-groups. He constructed Miller $2$-groups admitting a non-trivial abelian direct factor. However, the statement is correct for odd order groups. He also presented a generalization of examples of Jonah-Konvisser, which implies that the number of elements in a minimal generating set for a Miller group can be arbitrarily large. But he obtained a lower bound on the number of generators, which was later improved to an optimal bound by M. Morigi \[38\]. He also gave a lower bound on the order of Miller $p$-groups, which was again improved to an optimal bound by M. Morigi \[37\].

Motivated by the work done on the topic, various examples of Miller groups were constructed by several mathematicians via different approaches during the next 20 years, which were mostly special $p$-groups. A $p$-group $G$ is said to be special if $Z(G) = G'$ is elementary abelian, where $Z(G)$ and $G'$ denote the center and the commutator subgroup of $G$ respectively. These include the works by R. Faudree \[14\], H. Heineken and M. Liebeck \[20\], D. Jonah and M. Konvisser \[29\], B. E. Earnley \[12\], H. Heineken \[19\], A. Hughes \[24\], R. R. Struik \[39\], S. P. Glasby \[16\], M. J. Curran \[8\], and M. Morigi \[37\]. The existence of non-special Miller groups follows from \[29\] Remark 2.

The abelian $p$-groups of minimum order, which can occur as automorphism groups of some $p$-groups were studied by P. V. Hegarty \[18\] in 1995 and G. Ban and S. Yu \[2\] in 1998. More examples of Miller groups were constructed by A. Jamali \[26\] and Curran \[10\].

Neglecting Remark 2 in \[29\], A. Mahalanobis \[33\], while studying Miller groups in the context of MOR cryptosystems, conjectured that Miller $p$-groups are all special for odd $p$. Again neglecting \[29\] Remark 2, non-special Miller $p$-groups were constructed by V. K. Jain and the second author \[27\], V. K. Jain, P. K. Rai and the second author \[28\], A. Caranti \[6\] and the authors \[31\].

The motivation for many of the examples comes from some natural questions or observations on previously known examples. The construction of examples of Miller groups of varied nature have greatly contributed to complexify the structure of a Miller group. The structure of a general Miller group has not yet been well understood.

We record here the known information about structure of Miller $p$-groups and their automorphism groups. Let $G$ be a Miller $p$-group, and assume that it has no abelian direct factor. Then the following hold:
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(1) $|G| \geq p^6$ if $p = 2$ and $|G| \geq p^7$ if $p > 2$. 
(2) Minimal number of generators of $G$ is 3 for $p$ even, and 4 for $p$ odd. 
(3) The exponent of $G$ is at least $p^2$. 
(4) If $|G'| > 2$, then $G'$ has at least two cyclic factors of the maximum order in its cyclic decomposition. 
(5) If Aut($G$) is elementary abelian, then $\Phi(G)$ is elementary abelian, $G' \leq \Phi(G) \leq Z(G)$ and at least one equality holds. There are Miller $p$-groups in which only one equality holds. 
(6) For $p > 2$ (resp. $p = 2$), the abelian $p$-groups of order $< p^{11}$ (resp. $< 2^7$) are not automorphism groups of any $p$-group.

The aim of this article is to present an extensive survey on the developments in the theory of Miller groups since 1908, and pose some problems and further research directions. There are, at least, three survey articles on automorphisms of $p$-groups and related topics (see [21], [34], [35]); the present one does not overlap with any of them.

We conclude this section with setting some notations for a multiplicatively written group $G$. We denote by $\Phi(G)$, the Frattini subgroup of $G$. For an element $x$ of $G$, $\langle x \rangle$ denotes the cyclic subgroup generated by $x$, and $o(x)$ denotes its order. For subgroups $H, K$ of a group $G$, $H < K$ (or $K > H$) denotes that $H$ is proper subgroup of $K$. The exponent of $G$ is denoted by $\text{exp}(G)$. By $C_n$, we denote the cyclic group of order $n$. For a $p$-group $G$ and integer $i \geq 1$, $\Omega_i(G)$ denotes the subgroup of $G$ generated by those $x$ in $G$ with $x^{p^i} = 1$, and $\Phi_i(G)$ denotes the subgroup generated by $x^{p^i}$ for all $x$ in $G$. By Inn($G$), Autcent($G$) and Aut($G$) we denote, respectively, the group of inner automorphisms, central automorphism, and all automorphisms of $G$. Let a group $K$ acts on a group $H$ by automorphisms, then $H \rtimes K$ denotes the semidirect product of $H$ by $K$. All other notations are standard.

2. Reductions

Starting from the fundamental observations of Hopkins, in this section, we summarize all the known results (to the best of our knowledge) describing the structure of Miller groups. Although we preserve the meaning, no efforts are made to preserve the original statements from the source.

As commented in the introduction, a non-abelian Miller group must be nilpotent (of class 2), and therefore it is sufficient to study Miller $p$-groups. The following result is an easy exercise.

**Proposition 1** (Hopkins, [23]). Every automorphism of a Miller group centralizes $G'$.
An automorphism of a group $G$ is said to be central if it induces the identity automorphism on the central quotient $G / Z(G)$. Note that $\text{Aut}_{\text{cent}}(G)$, the group of all central automorphisms of $G$, is the centralizer of $\text{Inn}(G)$ in $\text{Aut}(G)$.

A group $G$ is said to be purely non-abelian if it has no non-trivial abelian direct factor. Hopkins made the following important observation:

**Theorem 2** (Hopkins, [23]). A Miller $p$-group is purely non-abelian.

Unfortunately, the statement, as such, is not true for $p = 2$ as shown by Earnley [12] (see Theorem 5 below for the correct statement).

In a finite abelian group $G$, there exists a minimal generating set $\{x_1, \ldots, x_n\}$ such that $\langle x_i \rangle \cap \langle x_j \rangle = 1$ for $i \neq j$. Hopkins observed that Miller groups possess a set of generators with a property similar to the preceding one. More precisely, he proved the following result.

**Theorem 3** (Hopkins, [23]). If $G$ is a Miller $p$-group, then there exists a set $\{x_1, \ldots, x_n\}$ of generators of $G$ such that

1. for $p > 2$, $\langle x_i \rangle \cap \langle x_j \rangle = 1$ for all $i \neq j$;
2. for $p = 2$, $\langle x_i \rangle \cap \langle x_j \rangle$ is of order at most 2, for all $i \neq j$.

In fact the theorem is true for any $p$-group of class 2, as shown below. Let $G$ be a $p$-group of class 2. Choose a set $\{y_1, \ldots, y_k\}$ of generators of $G$ with the property

$$\prod \ o(y_i) \text{ is minimum.}$$

Let $p > 2$ and $o(y_i) = p^{n_i}$, $1 \leq i \leq k$. If $\langle y_1 \rangle \cap \langle y_2 \rangle \neq 1$ then $y_2^p \in \langle y_1 \rangle$. Assuming that $o(y_1) \geq o(y_2)$, we can write $y_2^p = (y_1^p)^a$ for some integer $a$. Take $y_2' = y_2y_1^{-a}$. Then $\{y_1, y_2', y_3, \ldots, y_k\}$ is a generating set for $G$ and $o(y_2') \leq p^{n_a} < o(y_2)$, which contradicts \(^(*)\). The case $p = 2$ can be handled in a similar way.

**Theorem 4** (Hopkins, [23]). If $G$ is a Miller $p$-group, then $\text{Aut}(G)$ is a $p$-group.

This can be proved easily in the following way. In the case when $G$ is purely non-abelian, by a result of Adney-Yen [11] Theorem 1, $\text{Aut}_{\text{cent}}(G)$ (= $\text{Aut}(G)$) has order equal to $|\text{Hom}(G, Z(G))|$, which is clearly a power of $p$. Now consider the case when $G$ has an abelian direct factor, which occurs only when $p = 2$ and it must be cyclic of order at least $2^2$ (see Theorem 5). Let $G = H \times C_{2^n}$, where $H$ is purely non-abelian and $n \geq 2$. By the main theorem in [4],

$$|\text{Aut}(G)| = |\text{Aut}(H)| \cdot |\text{Aut}(C_{2^n})| \cdot |\text{Hom}(H, C_{2^n})| \cdot |\text{Hom}(C_{2^n}, Z(H))|.$$
and each factor on the right side has order a power of 2.

The investigation of structure of Miller groups remained unattained for about half a century until it was revisited by Earnley [12] in 1974. He pointed out that a Miller 2-group can have an abelian direct factor. So the correct form of Theorem 2 is

**Theorem 5** (Earnley, [12]). Let \( G \) be a finite p-group such that \( G = A \times N \) with \( A \neq 1 \) an abelian group and \( N \) a purely non-abelian group. Then \( G \) is a Miller group if and only if \( p = 2 \) and \( A, N \) satisfy the following conditions:

1. \( A \) is cyclic of order \( 2^n > 2 \);
2. \( N \) is a special Miller 2-group.

**Theorem 6** (Earnley, [12]). If \( G \) is a Miller p-group, then the following hold.

1. The exponent of \( G \) is greater than \( p \);
2. If \( p > 2 \), then \( Z(G) \cap \Phi(G) \) is non-cyclic.

For the first statement of the preceding theorem, we can assume that \( p > 2 \) and the proof now follows by noting that if \( \exp(G) = p \), then the map \( x \mapsto x^{-1} \) is a non-central automorphism of \( G \). The second statement follows from the following result of Adney-Yen.

**Theorem 7** (Adney-Yen, [11]). If \( G \) is a p-group of class 2 such that \( G' \) has only one cyclic factor of maximum order in the direct product decomposition, then \( G \) possesses a non-central automorphism.

Since, in p-group of class 2, \( G' \leq Z(G) \cap \Phi(G) \), the preceding theorem restricts the structure of the commutator subgroup of a Miller p-group.

**Theorem 8.** If \( G \) is a Miller p-group, \( p \) odd, then \( G' \) possesses at least two cyclic factors of maximum order in the direct product decomposition.

This raises a natural question for \( p = 2 \). The analogue of the preceding theorem for \( p = 2 \) holds except when \( |G'| = 2 \). This can be obtained from the following generalization of Theorem 7.

**Theorem 9** (Faudree, [13]). Let \( G \) be a p-group of class 2 with the following conditions:

1. \( G' = \langle u \rangle \times U \), where \( o(u) = p^{m_1} > p^m = \exp(U) \).
2. \([g, h] = u \) and \( h^{p^{m_1}+m'} = 1 \).
3. \( m'' = m' \) if \( p \) is odd, and \( m'' = \max(1, m') \) if \( p = 2 \).

Let \( H = \langle g, h \rangle \) and \( L = \{ x \in G : [g, x], [h, x] \in U \} \). Then \( G = HL \) and the map

\[
\begin{align*}
g &\mapsto gh^{p^{m''}}, \\
h &\mapsto h, \\
x &\mapsto x \quad (x \in L)
\end{align*}
\]
defines an automorphism of \( G \) which centralizes \( Z(G) \).

Notice that if \( p = 2 \) and the exponent of \( G' \) is at least 4, then the automorphism defined in the preceding theorem is non-central, and therefore \( G \) is not Miller. But if \( |G'| = 2 \), then the theorem is no-longer applicable to produce a non-central automorphism of \( G \). So the following question remains open.

**Question 1.** Can a finite 2-group \( G \) with \( G' \) cyclic of order 2 be Miller?

Earnley obtained lower bounds for the order and the minimum number of generator of a Miller group, which were later sharpened to the optimal level by Morigi \[37, 38\]

**Theorem 10.** Let \( G \) be a Miller \( p \)-group.

1. For any prime \( p \), \( G \) is generated by at least 3 elements. (Earnley, \[12\])
2. If \( p \) is odd, then \( G \) is generated by at least 4 elements. (Morigi, \[37\])

The example of a Miller 2-group constructed by Miller is minimally generated by 3 elements (see Section 3 (3.1)). For \( p \) odd, Jonah-Konvisser constructed Miller \( p \)-groups which are minimally generated by 4 elements (see Section 4 (4.3) for more details).

**Theorem 11.** Let \( G \) be a Miller \( p \)-group.

1. For any prime \( p \), \( |G| \geq p^6 \). (Earnley, \[12\])
2. If \( p \) is odd, then \( |G| \geq p^7 \) and there exists a Miller group of order \( p^7 \). (Morigi, \[38\])

Again the Miller 2-group constructed by Miller is of order \( 2^6 \) having automorphism group of order \( 2^7 \). Morigi constructed a special Miller \( p \)-group \( G \) of order \( p^7 \) with \( \text{Aut}(G) \) elementary abelian of order \( p^{12} \). In fact, it is one among an infinite family of Miller groups constructed by Morigi (see Section 4 (4.6) for more details).

The question whether an abelian \( p \)-group of order smaller than \( p^{12} \) for an odd prime \( p \) can occur as the automorphism group of a \( p \)-group, was addressed by Hegarty \[18\] and Ban and Yu \[2\] independently. They proved

**Theorem 12.** For \( p \) odd, there is no abelian \( p \)-group of order smaller than \( p^{12} \) which can occur as the automorphism group of a \( p \)-group.

Finally we state some results on the structure of \( \text{Aut}(G) \) for a Miller group \( G \). Many known examples of Miller groups are special \( p \)-groups. Certainly for such
groups, the automorphism group is elementary abelian. The following problem appears as an old problem in [3] problem 722.

**Problem 2.** Study the \( p \)-groups with elementary abelian automorphism groups.

Let \( G \) be a \( p \)-group. If \( \text{Aut}(G) \) is elementary abelian, then so is \( G/Z(G) \); hence \( G' \leq \Phi(G) \leq Z(G) \). In fact, it is interesting to see that at least one equality always holds.

**Theorem 13** (Jain-Rai-Yadav, [28]). Let \( G \) be a \( p \)-group, \( p \) odd, such that \( \text{Aut}(G) \) is elementary abelian. Then \( \Phi(G) \) is elementary abelian, and one of the following holds:

1. \( Z(G) = \Phi(G) \).
2. \( G' = \Phi(G) \).

Jain-Rai-Yadav [28] constructed \( p \)-groups with elementary abelian automorphism group, in which exactly one of the above two conditions holds.

For \( 2 \)-groups with elementary abelian automorphism groups, there are analogous necessary conditions, stated below. Since a Miller \( 2 \)-group can have abelian (cyclic) direct factor, we consider two cases.

An abelian \( p \)-group of type \( (p^n, p, \ldots, p) \) with \( n > 1 \) is called a ce-group. If \( G = A \times B \) with \( A \cong C_{p^n} \) (\( n > 1 \)) and \( B \cong C_p \times \cdots \times C_p \), then call \( A \) cyclic part and \( B \) elementary part of \( G \).

**Theorem 14** (Jafari, [25]). Let \( G \) be a purely non-abelian \( 2 \)-group. Then \( \text{Autcent}(G) \) is elementary abelian if and only if one of the following holds.

1. \( G/G' \) is of exponent 2.
2. \( Z(G) \) is of exponent 2.
3. \( \gcd(\exp(G/G'), \exp(Z(G))) = 4 \) and \( G/G', Z(G) \) are ce-groups such that elementary part of \( Z(G) \) is contained in \( G' \) and there is an element \( z \) of order 4 in cyclic part of \( Z(G) \) with \( zG' \) lying in cyclic part of \( G/G' \) satisfying \( o(zG') = \exp(G/G')/2 \).

In particular, if \( G \) is a purely non-abelian \( 2 \)-group with \( \text{Aut}(G) \) elementary abelian, then one of the above three conditions holds. The example constructed by Miller shows that it satisfies only condition (3). Jain-Rai-Yadav [28] constructed purely non-abelian Miller \( 2 \)-groups which satisfy only condition (1) or only condition (2). Finally, for \( 2 \)-groups \( G \) with abelian direct factor, the following theorem gives necessary conditions for \( \text{Aut}(G) \) to be elementary abelian.
Theorem 15 (Karimi-Farimani, \cite{30}). Let $G = A \times N$ with $A$ cyclic 2-group and $N$ purely non-abelian 2-group of class 2. Then $\text{Autcent}(G)$ is elementary abelian if and only if

(1) $|A|$ is 4 or 8.

(2) $N$ is special 2-group.

With the notations of the preceding theorem, if $G$ is Miller then (1) and (2) hold.

Theorem\cite{8} tells us that a Miller $p$-group, $p$ odd, can not admit a non-abelian direct factor. It is natural to ask whether a Miller $p$-group can occur as a direct product of non-abelian groups. This situation has been considered by Curran \cite{10}. He determined necessary and sufficient conditions on a direct product $H \times K$ to have abelian automorphism group. Note that for any non-trivial group $H$, $\text{Aut}(H \times H)$ is non-abelian. Before we state the result of Curran, we set some notations.

Let $H$ be a $p$-group of class 2.

(1) Let $a, b, c, d$ denote the exponents of $H/H'$, $H/Z(H)$, $Z(H)$ and $H'$ respectively.

(2) If $H'$ and $Z(H)$ have the same rank, then define $d_s$ to be the largest integer
\[
\leq \exp(Z(H)) \text{ such that } \Omega_{d_s}(H') = \Omega_{d_s}(Z(H)).
\]

(3) If $H/Z(H)$ and $H/H'$ have the same rank, then define $b_t$ to be the largest integer
\[
\leq \exp(H/H') \text{ such that } \Omega_{b_t}(H/Z(H)) = \Omega_{b_t}(H/H').
\]

(4) Let $\Gamma^i(H)$ denote the subgroup of $H$ with $\Gamma^i(H)/H' = \Omega^i(H/H')$.

Replacing $H$ by $K$ in (1)-(4), the corresponding terms $a', b', c', d'$, $d'_s$, $b'_t$ and $\Gamma^i(K)$ are similarly defined. For simplicity, we denote by $r(A)$, the rank of an abelian group $A$. With this setting, we have

Theorem 16 (Curran, \cite{10}). Let $G = H \times K$, where $H, K$ are $p$-groups of class 2 with no common direct factor. Then $\text{Aut}(G)$ is abelian if and only if $\text{Aut}(H)$ and $\text{Aut}(K)$ are abelian and one of the following holds:

(1) $Z(H) = H'$ and $Z(K) = K'$.

(2) $Z(H) > H'$ and $Z(K) = K'$, where $r(Z(H)) = r(H')$, $a' \leq d_s \leq a \leq c$ and $\Omega_a(Z(H)) \leq \Gamma^c(H)$.

(3) $Z(H) = H'$ and $Z(K) > K'$, where $r(Z(K)) = r(K')$, $a \leq d'_s \leq a' \leq c'$ and $\Omega_{a'}(Z(K)) \leq \Gamma^c(K)$.

(4) $Z(H) > H'$ and $Z(K) > K'$, where $r(Z(H)) = r(H')$, $r(Z(K)) = r(K')$ and $a = d_s = d'_s = a'$. 
(5) $Z(H) > H'$ and $Z(K) = K'$, where $r(H/Z(H)) = r(H/H')$, $a' \leq b_i \leq c \leq a$ and $\Omega_a(Z(H)) \leq \Gamma_a(H)$.

(6) $Z(H) = H'$ and $Z(K) > K'$, where $r(K/Z(K)) = r(K/K')$, $a \leq b_i' \leq c' \leq a'$ and $\Omega_a(Z(K)) \leq \Gamma_a'(K)$.

(7) $Z(H) > H'$ and $Z(K) > K'$, where $r(H/Z(H)) = r(H/H')$, $r(K/Z(K)) = r(K/K')$ and $c = b_1 = b_i' = c'$.

**Remark 17.** Observe that in the above theorem, if $\text{Aut}(H \times K)$ is abelian then either $H'$ and $Z(H)$ have the same rank or $H/H'$ and $H/Z(H)$ have the same rank; the same is true for the other component $K$.

An analogous problem for central product may be stated as follows.

**Problem 3.** Find necessary and/or sufficient condition such that central product of two Miller groups is again a Miller group.

### 3. Examples of Miller 2-groups

In this section, we discuss examples of Miller 2-groups in chronological order.

**3.1** As mentioned above several times, the first example of a Miller 2-group was constructed by Miller himself, which comes as a semi-direct product of the cyclic group of order 8 by the dihedral group of order 8, and presented by

$$G_1 = C_8 \rtimes D_8 = \langle x, y, z \mid x^8, y^4, z^2, xyz^{-1} = y^{-1}, yxy^{-1} = x^5, zxz^{-1} = x \rangle.$$

Miller proved that each coset of $Z(G_1)$ is invariant under every automorphism of $G_1$ and that every automorphism has order dividing 2. Thus $\text{Aut}(G_1)$ is elementary abelian, and it can be shown that its order is $2^7$.

By Theorem 3, the order of a Miller 2-group is at least 64. Having an example of order 64, a natural idea which peeps in ones mind is to explore groups of order 64 to find more Miller groups. This was done by Earnley who proved that there are (exactly) two more Miller groups of the minimal order, which are presented as follows.

$$G_2 = (C_4 \times C_4) \rtimes C_4 = \langle x, y, z \mid yxy^{-1} = x^{-1}, zxz^{-1} = xy^2, zyx^{-1} = y \rangle,$$

$$G_3 = (C_4 \times C_4 \times C_2) \rtimes C_2 = \langle x, y, z, t \mid x^4, y^4, z^2, t^2, xy = yx, xz = zx, yz = yz, txy^{-1} = xy^2, txy^{-1} = tz, t^{-2}zt = z \rangle.$$

Note that in all the groups $G_1, G_2$ and $G_3$, the center and Frattini subgroups coincide and are elementary abelian of order 8, whereas the commutator subgroup
is elementary abelian of order 4. Further, \( \text{Aut}(G_2) \) and \( \text{Aut}(G_3) \) are elementary abelian of order \( 2^9 \).

A generalization of \( G_1 \) appears, as an exercise, in the book [32] Exercise 46, Page 237 by Macdonald, written in 1970. For \( n \geq 3 \), the group is presented as follows.

\[
G_{1,n} = \langle a, b, c \mid a^{2^n} = b^4 = c^2 = 1, b^{-1}ab = a^{1+2^{n-1}}, c^{-1}bc = b^{-1}, [c, a] = 1 \rangle.
\]

The group \( G_{1,n} \) is of order \( 2^{n+3} \) and its automorphism group is an abelian 2-group of type \((2^{n-2}, 2, 2, 2, 2, 2, 2)\). In 1982, Struik [39], independently, obtained the same example with a different presentation.

A generalization of \( G_2 \) and \( G_3 \) has also been obtained by Glasby [16], which is described as follows. For \( n \geq 3 \), let \( G_{2,n} \) denotes the 2-group of class 2 with generators \( x_1, x_2, \ldots, x_n \) with following additional relations:

\[
x_i^4 = 1, \ (1 \leq i \leq n), \ [x_i, x_n] = x_{i+1}^2, \ (1 \leq i \leq n-1),
\]

and set \([x_k, x_j] = 1\) in the remaining cases. Here \( \text{Aut}(G_{2,n}) \) is elementary abelian group of order \( 2^{2n} \). If \( n = 3 \) then \( G_{2,n} \) is isomorphic to \( G_2 \).

Again for \( n \geq 3 \), let \( G_{3,n} \) denotes the 2-group of class 2 with generators \( y_0, y_1, \ldots, y_n \) with following additional relations:

\[
y_0^2 = y_i^4 = y_n^2 = 1, \ (1 \leq i \leq n-1), \ [y_i, y_n] = y_{i+1}^2, \ (1 \leq i \leq n-2)
\]

and set \([y_k, y_j] = 1\) in the remaining cases. Here also \( \text{Aut}(G_{3,n}) \) is elementary abelian group of order \( 2^{2n} \). If \( n = 3 \) then \( G_{3,n} \) is isomorphic to \( G_3 \).

(3.2) It should be noted that, in 1974, Jonah and Konvisser constructed special Miller groups of order \( p^8 \), which were generalized to an infinite family of Miller \( p \)-groups by Earnley, and those groups include the case \( p = 2 \) too (see Section 4 (4.3) and (4.4)).

(3.3) Heineken and Liebeck (see Section 4 (4.2) for details) proved that \textit{given a finite group} \( K \), \textit{there exists a special} \( p \)-\textit{group} \( G \), \( p \) odd, \textit{with} \( \text{Aut}(G) / \text{Autcent}(G) \cong K \). In particular, if \( K = 1 \), then the corresponding group \( G \) is Miller \( p \)-group.

In 1980, A. Hughes [24] proved that one can construct a special 2-group as well with the above property. The method of Hughes is a modification of the graph theoretic method of Heineken-Liebeck. It should be noted that the method of Heineken-Liebeck uses \textit{digraphs}, whereas that of Hughes considers \textit{graphs}, and is described below.

Let \( K \) be any finite group and associate to \( K \) a connected graph \( D(K) \) which satisfies following conditions:
(1) Each vertex of the graph has degree at least 2.
(2) Every cycle in the graph contains at least 4 vertices.
(3) \( \text{Aut}(D(K)) \cong K \).

Associate a special 2-group \( G \) to \( D(K) \) as follows. If the graph has \( n \) vertices \( v_1, \ldots, v_n \), then consider the free group \( F_n \) on \( x_1, \ldots, x_n \). Let \( R \) be the normal subgroup of \( F_n \) generated by \( x_2, [x_i, x_j, x_k] \) (for all \( i, j, k \)) and \([x_r, x_s]\) whenever the vertices \( v_r \) and \( v_s \) are adjacent. Define \( G \) to be the group \( F_n / R \); it is a special 2-group of order \( 2^n + \binom{n}{2} - e \), where \( n \) is the number of generators of \( G \) (so \( |G / G'| = 2^n \)) and \( e \) is the number of edges of the graph. It turns out that \( \text{Aut}(G) / \text{Autcent}(G) \cong K \) (see [24] for proof).

(3.4) In 1987, Curran [8] studied automorphisms of semi-direct product, and suggested a method to construct many more examples of Miller 2-groups similar to \( G_1 \). We describe the method briefly and see that the above \( G(n) \) can be constructed by this method. Let \( A = \langle a \rangle \) be a cyclic group of order \( 2^n \), \( n \geq 3 \) and \( N \) a special 2-group acting on \( A \) in the following way: a maximal subgroup \( J \leq N \) acts trivially, and any \( x \in N \setminus J \) acts by \( xax^{-1} = a^{1+2^n-1} \).

Let \( G = A \rtimes N \) be the semi-direct product of \( A \) and \( N \) with this action. Then we get

**Theorem 18** (Curran, [8]). Let \( G = A \rtimes N \) be as above along with the conditions:

1. \( A \times J \) is characteristic in \( A \rtimes N \).
2. Any automorphism of \( N \) leaving \( J \) invariant is central automorphism of \( N \).

Then \( A \rtimes N \) is a Miller group.

To elaborate, consider \( A = \langle a \rangle \), the cyclic group of order \( 2^n \), \( n \geq 3 \) and \( N = \langle b, c \mid b^4, c^2, [b, c] = b^2 \rangle \), the dihedral group of order 8. Consider the action of \( N \) on \( A \) by

\[ b^{-1}ab = a^{1+2^n-1}, \quad c^{-1}ac = a. \]

Define \( G = A \rtimes N \), the semi-direct product with this action. Note that \( Z(G) = \langle a^2, b^2 \rangle = \Phi(G) \), and \( G / \Phi(G) \) is elementary abelian of order 8.

Take \( J = \langle b^2, c \rangle \), the largest subgroup of \( N \) acting trivially on \( A \). Then \( A \times J \) is characteristic in \( A \rtimes N \), since it is the unique abelian subgroup of index 2 (if there were more than one abelian subgroups of index 2, then center would have index 4). Also, in \( N \), the subgroup \( \langle b \rangle \) is characteristic. Hence if \( \phi \in \text{Aut}(N) \) leaves \( J \) invariant, then it leaves invariant the subgroup \( J \cap \langle b \rangle = \langle b^2 \rangle \) (the center of \( N \)), and one can see that \( \phi \) is central automorphism of \( N \). The group \( A \rtimes N \)
is therefore a Miller group by Theorem 18. Note that this group is isomorphic to $G(n)$ described above.

(3.5) After a considerable time gap, Jamali [26] in 2002 constructed the following infinite family of Miller 2-groups in which, the number of generators and the exponent of the group can be arbitrarily large. For integers $m \geq 2$ and $n \geq 3$, let $G_n(m)$ be the group generated by $a_1, \ldots, a_n, b$, subject to following relations:

\begin{align*}
a_1^2 &= a_2^{2m} = a_i^4 = 1 \quad (3 \leq i \leq n) \\
a_n^{2} &= b^2, \\
[a_1, b] &= [a_i, a_j] = 1 \quad (1 \leq i < j \leq n), \\
[a_n, b] &= a_1, [a_{i-1}, b] = a_i^2 \quad (3 \leq i \leq n).
\end{align*}

The group $G_n(m)$ has order $2^{2n+m-2}$ and its automorphism group is abelian of type $(2, 2, \ldots, 2^{m-2})$. For $n = 3$ and $m = 2$, the group $G_3(2)$ is isomorphic to $G_3$ (see Section 3(3.1)).

4. Examples of Miller $p$-groups, $p$-odd

This section, a lifeline for Miller groups in a sense, presents evolution of the topic. We’ll see the influence of examples of varied nature on Miller groups towards understanding the structure. Some examples of Miller groups occurred in other related contexts without any pointer to the topic. The deriving force behind many of the examples comes from natural questions on previously known Miller groups or certain natural optimistic expectations on the structure of such groups. The evolution process certainly helped, although minimally, in studying the structure of Miller groups, especially, in turning down the natural optimistic expectations.

During 1971 to 1979, there were three occasions, in which certain $p$-groups were constructed with a specific property, and these groups turned out to be Miller groups. The motive of the construction of these groups had no obvious connection with Miller groups.

(4.1) We now start the discussion of the (possibly) first example of Miller group of odd order. It was conjectured that a finite group in which every element commutes with its epimorphic image is abelian. It was disproved by R. Faudree [14] in 1971. He constructed a non-abelian group $G$ in which every element commutes with its epimorphic image. The group $G$ is a special $p$-group, and therefore one can easily deduce that $\text{Aut}(G)$ is abelian. The group $G$ is described as follows.
Let $G = \langle a_1, a_2, a_3, a_4 \rangle$ be the $p$-group of class 2 with following additional relations:

\[
[a_1, a_2] = a_1^p, \quad [a_1, a_3] = a_3^p, \quad [a_1, a_4] = a_4^p,
\]
\[
[a_2, a_3] = a_2^p, \quad [a_2, a_4] = 1, \quad [a_3, a_4] = a_3^p.
\]

The group $G$ is a special $p$-group of order $p^8$ and $Z(G) = G'$ is elementary abelian of order $p^4$. If $p$ is odd, then $\text{Aut}(G)$ is elementary abelian of order $p^{16}$.

(4.2) In 1974, Heineken-Liebeck [20], constructed a $p$-group $G$ of class 2, $p$ odd, for a given finite group $K$ such that $\text{Aut}(G)/\text{Aut}_{\text{cent}}(G) \cong K$. Associate to the finite group $K$, a connected digraph (directed graph) $X$ as follows. If $K$ is cyclic of order $> 2$, then take $X$ to be the cyclic digraph with $|K|$ vertices. In the remaining cases, we associate a digraph $X$ with following conditions:

1. $X$ is strongly connected,
2. Any two non-simple vertices are not adjacent,
3. Every vertex belongs to a (directed) cycle of length at least 5,
4. Every non-simple vertex has at least two outgoing edge,
5. $\text{Aut}(X) \cong K$,

where by a simple vertex we mean a vertex with exactly one incoming and exactly one outgoing edge. Associate to such $X$, a $p$-group $G_X$ of class 2 as follows. If $X$ has $n$ vertices, we take $n$ generators for $G_X$. If the vertex $i$ has outgoing edges to $j_1, \ldots, j_r$ precisely, then we put the relation $x_i^p = [x_{i_1}, x_{i_2}, \ldots, x_{i_r}]$. Finally we put $[[x_i, x_j], x_k] = 1$ for all $i, j, k$, making $G_X$ of class 2. It is easy to show that $G_X$ is a special $p$-group of order $p^{n+\binom{n}{2}}$.

The conditions (1)-(4) imply that $\text{Aut}(G)/\text{Aut}_{\text{cent}}(G)$ is isomorphic to the automorphism group of $X$, which is isomorphic to $K$ by (5)).

Before proceeding for the example of Miller $p$-group by this method, we make some comments. If $|K| \geq 5$, then there is a systematic procedure to construct graph $X$ satisfying conditions (1) to (5); it is obtained by a specific subdivision of a Cayley digraph of $K$. If $|K| < 5$, then one constructs $X$ satisfying (1)-(5) by some ad-hoc method.

For $K = 1$, the following digraph satisfies conditions (1)-(5).
The group $G_X$ associated to this digraph is a special Miller $p$-group of order $p^{9+\binom{2}{2}} = p^{45}$, $p$ odd.

(4.3) The existence of Miller 2-groups of order $2^6$ served as a base to the belief that $p^{45}$ is too big to be a minimal order for a Miller $p$-group for odd $p$. This motivated Jonah and Konvisser [29] to construct Miller $p$-groups of smaller orders. In 1975, they constructed, for each prime $p$, $p + 1$ non-isomorphic Miller $p$-groups of order $p^8$ as described below. Let $\lambda = (\lambda_1, \lambda_2)$ be a non-zero vector with entries in field $\mathbb{F}_p$ and $G_\lambda = \langle a_1, a_2, b_1, b_2 \rangle$ be the $p$-group of class 2 with following additional relations:

$$
\begin{align*}
    a_1^p &= [a_1, b_1], \\
    a_2^p &= [a_1, b_1^2b_2], \\
    b_1^p &= [a_2, b_1b_2], \\
    b_2^p &= [a_2, b_2], \\
    [a_1, a_2] &= [b_1, b_2] = 1.
\end{align*}
$$

The proof of the fact that $G_\lambda$ is Miller group is very elegant. We briefly describe the idea here as it will be useful in later discussions.

Fix a non-zero vector $\lambda$ and write $G = G_\lambda$. Note that $G$ is special $p$-group of order $p^8$ and $G'$ is of rank 4 with generators $[a_i, b_j], i, j = 1, 2$.

The subgroups $A = \langle a_1, a_2, G' \rangle$ and $B = \langle b_1, b_2, G' \rangle$ are the only abelian normal subgroups such that $[A : G'] = p^2 = [B : G']$; hence they are permuted by every automorphism of $G$. There exists $x \in A$ such that $A^p \leq [x, G]$, but there is no $y \in B$ with $B^p \leq [y, G]$; hence $A, B$ are characteristic in $G$. The only $x \in A$ with $A^p \leq [x, G]$ are $a_1^iz$ with $p \nmid i$ and $z \in G'$; they generate characteristic subgroup $\langle a_1, G' \rangle$. Similarly, those $x \in A$ with $B^p \leq [x, G]$ generate a characteristic subgroup, namely $\langle a_2, G' \rangle$ and those $x \in B$ with $x^p \in [x, G]$ generate the characteristic subgroup $\langle b_2, G' \rangle$. Thus, if $\varphi \in \text{Aut}(G)$, then

$$
\varphi(a_1) \equiv a_1^i, \quad \varphi(a_2) \equiv a_2^i, \quad \varphi(b_1) \equiv b_1^{k_1}b_2^{k_2}, \quad \varphi(b_2) \equiv b_2^l \pmod{G'},
$$

where $i, j, l$ are integers not divisible by $p$, and $(k_1, k_2) \neq (0, 0)$. Using relations in $G$, it is now easy to deduce that $i = j = k_1 = l = 1 \pmod{p}$ and $k_2 = 0$. Therefore
every automorphism of $G$ is central. Since $G' = Z(G)$, it follows that Aut$(G)$ is abelian.

The groups $G_{\lambda}$ and $G_{\mu}$, for non-zero vectors $\lambda, \mu \in \mathbb{F}_p \times \mathbb{F}_p$, are isomorphic only if these vectors are dependent.

Remark 19. Note that in the group $G_{\lambda}$, replacing, in the power relations, all the $p$-th power by $p^k$-th power, it can be shown by the same arguments that the resulting groups are still Miller. If $k > 1$, then $G'_{\lambda} = Z(G_{\lambda})$ and has exponent $p^k > p$; hence $G_{\lambda}$ is a non-special Miller group.

We must emphasize the negligence of the preceding remark [29], Remark 2] by the authors of many recent papers claiming the existence of non-special Miller groups was not known in the literature until recently.

(4.4) Examples of Jonah and Konvisser were generalized by Earnley [12], during the same year, by increasing number of generators, in the following way. Fix $n \geq 2$ and a non-zero $n$-tuple $(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in \mathbb{F}_p$. Let $G_{\lambda} = \langle x_1, x_2, y_1, y_2, \ldots, y_n \rangle$ be the $p$-group of class 2 with the following additional relations:

\[
x^p_1 = [x_1, y_1], \quad x^p_2 = [x_1, y_1^{\lambda_1} \ldots y_n^{\lambda_n}],
\]

\[
y^p_i = [x_2, y_i y_{i+1}], \quad i = 1, 2, \ldots, n - 1,
\]

\[
y^p_n = [x_2, y_n], \quad [x_1, x_2] = [y_i, y_j] = 1 \quad \text{for all } i, j.
\]

It can be shown easily that $G'_{\lambda} = Z(G_{\lambda})$ is elementary abelian and generated by the $2n$ elements $[x_i, y_j]$ for $i = 1, 2$ and $j = 1, 2, \ldots, n$; hence order of $G_{\lambda}$ is $p^{(n+2)2n} = p^{2+3n}$.

For Aut$(G_{\lambda})$ to be abelian, the obvious necessary condition is that every automorphism should be central, but since $G_{\lambda}$ is special, this is sufficient too. We briefly describe the beautiful linear algebra techniques evolved by Earnley to prove that every automorphism of $G_{\lambda}$ is central as, with a little variation, these techniques have been used in different contexts by Morigi [37], Hegarty [18] and Earnley himself.

For simplicity, fix non-zero vector $\lambda$ and write $G = G_{\lambda}$. Let $f : G / G' \to G'$ denote the map $xG' \mapsto x^p$, which is a homomorphism for odd $p$. An automorphism $\alpha$ of $G / G'$ determines its action on $G'$ by

\[
\hat{\alpha} : G' \to G', \quad \hat{\alpha}([x, y]) = [\alpha(xG'), \alpha(yG')].
\]

The automorphism $\alpha$ of $G / G'$ is induced by an automorphism $\varphi$ of $G$ if and only if $\hat{\alpha} \circ f = f \circ \alpha$, i.e. the following diagram commutes:
Consequently, every automorphism of $G$ is central if and only if identity is the only automorphism of $G/G'$ which fits in the above commutative diagram.

Note that $G/G'$ and $G'$ can be considered as vector spaces over $\mathbb{F}_p$; hence the automorphisms $\hat{\alpha}$ and $\alpha$ in the above diagram can be considered as (invertible) linear maps. However, if $p = 2$, the map $f$ may not be linear. But, in the group $G$ under consideration, consider the abelian subgroups $A = \langle x_1, x_2, G' \rangle$ and $B = \langle y_1, \ldots, y_n, G' \rangle$; they generate $G$ and the restriction of $f$ to $A/G'$ and $B/G'$ are homomorphisms, and therefore linear.

Next, $A$ and $B$ are the only abelian subgroups containing $G'$ such that modulo $G'$ each of them have order at least $p^2$. For $n > 2$, both $A$ and $B$ are characteristic as their orders are different. But even for $n = 2$, they are characteristic, since, in this case, $A$ contains an element $t$ such that $A^p \leq [t, G]$, but $B$ has no element with similar property.

Thus, consider an automorphism $\alpha = (\alpha_1, \alpha_2)$ of $G/G' = A/G' \oplus B/G'$, where $\alpha_1 \in \text{Aut}(A/G')$ and $\alpha_2 \in \text{Aut}(B/G')$. As a vector space, $G' = [A, B]$ is isomorphic to $A/G' \otimes B/G'$, the tensor product of $A/G'$ and $B/G'$. Hence the automorphism induced by $\alpha$ on $G'$ is nothing but $\hat{\alpha}_1 \otimes \hat{\alpha}_2$. Then $\alpha$ is induced by an automorphism of $G$ if and only if the following diagrams commute:

For the first diagram, consider the following ordered bases:

$$\{x_1 G', x_2 G'\}$$ for $A/G'$ and $$\{[x_1, y_1], [x_2, y_1], \ldots, [x_1, y_n], [x_2, y_n]\}$$ for $G'$.

Also for the second diagram, consider the following bases:

$$\{y_1 G', \ldots, y_n G'\}$$ for $B/G'$ and $$\{[x_1, y_1], \ldots, [x_1, y_n], [x_2, y_1], \ldots, [x_2, y_n]\}$$ for $G'$. 

\[
\begin{array}{ccc}
G/G' & \xrightarrow{f} & G' \\
\downarrow \alpha & & \downarrow \hat{\alpha} \\
G/G' & \xrightarrow{f} & G'.
\end{array}
\]
The matrix of $f$ with respect to these bases can be easily written from the power-commutator relations in $G$. Writing the matrices of $\alpha_1, \alpha_2, \alpha_1 \otimes \alpha_2$ with respect to these bases, one expresses the commutativity of above diagram in terms of two matrix equations, and a simple matrix computation shows that $\alpha_1 = 1$ and $\alpha_2 = 1$ are the only solutions. This implies that every automorphism of $G$ is central. Note that this also covers the case $p = 2$ as the restrictions of $f$ to $A/G'$ and $B/G'$ are linear.

For more detailed module-theoretic formulation of the above arguments for arbitrary special $p$-groups, one may refer to [6].

(4.5) The third instance of examples of Miller groups, not in the context of the topic, is by Heineken [19] in 1979. He constructed a family of $p$-groups in which every normal subgroup is characteristic. The groups, actually, possess more interesting properties, which force the groups actually become Miller. The construction is as follows. Let $\mathbb{F}_q$ ($q = p^n$) be the field of order $p^n$ and $U(3, \mathbb{F}_q)$ denote the group of $3 \times 3$ unitriangular matrices over $\mathbb{F}_q$. Identify the elements of $U(3, \mathbb{F}_q)$ as triples over $\mathbb{F}_q$ by

$$(x, y, z) \sim \begin{bmatrix} 1 & x & z \\ 1 & y \\ 1 \end{bmatrix}, \quad x, y, z \in \mathbb{F}_q.$$  

The sets $A = \{(x, 0, z) : x, z \in \mathbb{F}_q\}$ and $B = \{(0, y, z) : y, z \in \mathbb{F}_q\}$ constitute abelian subgroups of order $q^2 = p^{2n}$ and exponent $p$; they generate $U(3, \mathbb{F}_q)$ and their intersection is the center of $U(3, \mathbb{F}_q)$. Also the commutator of an element of $A$ with that of $B$ is given by

$$[(x, 0, z_1), (0, y, z_2)] = (0, 0, xy).$$

Heineken constructed a group $G$ by a little modification in the commutator and power relations of $U(3, \mathbb{F}_q)$ as follows. The group $G$ consists of triples $(x, y, z)$ over $\mathbb{F}_q$, in which

$$A^* = \{(x, 0, z) \mid x, z \in \mathbb{F}_q\} \quad \text{and} \quad B^* = \{(0, y, z) \mid y, z \in \mathbb{F}_q\}$$

constitute abelian subgroups of order $q^2 = p^{2n}$. The power relations are

$$(x, 0, z)^p = (0, 0, x^p - x),$$

$$(0, y, z)^p = (0, 0, y + y^p + \cdots + y^{p^{n-1}}).$$
For a fixed generator $t$ of the multiplicative group of $\mathbb{F}_q$, the commutator relations between $A^*$ and $B^*$ are defined as

$$[(x, 0, z_1), (0, y, z_2)] = (0, 0, xy - tx^py^2).$$

Then $G = \{(x, y, z) \mid x, y, z \in \mathbb{F}_q\}$ is a group, which is the product of abelian subgroups $A^*$ and $B^*$, and $Z(G) = \{(0, 0, z) \mid z \in \mathbb{F}_q\} = A^* \cap B^*$. Fix $x \neq 0$ and vary $y \in \mathbb{F}_q$, then the elements $xy - tx^py^2$ exhaust whole $\mathbb{F}_q$. It follows that $G' = Z(G)$, and therefore $G$ is a special $p$-group of order $p^{3n}$. With the above setup, we have

**Theorem 20.** Let $n$ be odd positive integer. If $n \geq 5$, $p > 2$, or $n = 3$, $p \geq 5$, then every automorphism of $G$ is identity on $Z(G)$ as well as on $G/Z(G)$; hence $\text{Aut}(G)$ is abelian.

The groups $G$ in the preceding theorem have the following remarkable property: for every $x \in G - G'$, the conjugacy class of $x$ in $G$ is $xG'$. The groups satisfying this property are called Camina groups. The readers interested in Camina groups are referred to [11] and the references therein.

An automorphism $\alpha$ of a group is said to be class-preserving if it takes each element of the group to its conjugate. It is an easy exercise to show that each automorphism of the group $G$ is class-preserving. This is not only true for the groups $G$ considered above, but also for any Camina $p$-group which is Miller as well.

**Theorem 21.** Let $G$ be a $p$-group which is Miller as well as a Camina group. Then automorphisms of $G$ are all class-preserving.

As a simple consequence, we have

**Corollary 22.** Let $G$ be a $p$-group which is Miller as well as a Camina group. Then normal subgroups of $G$ are all characteristic.

We remark that the examples of Jonah-Konvisser are not Camina groups and extraspecial $p$-groups are Camina groups but not Miller. As the structure of Camina $p$-groups of class 2 in general is not well understood, it will be interesting to study $p$-groups which are both Camina as well as Miller.

**Problem 4.** Determine the structure of $p$-groups which are Camina as well as Miller.

This structural information will also, on the one hand, help in understanding $p$-groups of class 2 whose automorphisms are all class-preserving and, on the
other hand, shed some light on the study of $p$-groups of class 2 in which all normal subgroups are characteristic.

(4.6) Note that the Miller groups constructed by Heineken in the preceding discussion are of order at least $p^9$. But, as we already mentioned, the lower bound on the order of Miller groups is $p^7$ for odd $p$. This was Morigi [37], who, in 1994, constructed examples of Miller groups of minimal order as a part of a general construction of an infinite family of such groups. The construction is briefly described as follows. For any natural number $n$, let $G^n$ denote the $p$-group of class 2 generated by $a_1, a_2, b_1, \ldots, b_{2n}$ with the following additional commutator and power relations:

\[ [a_1, b_{2i+1}] = [a_2, b_{2i+2}] = 1, \quad i = 0, 1, \ldots, n - 1, \]
\[ [b_1, b_2] = [b_3, b_4] = \cdots = [b_{2n-1}, b_{2n}] = 1, \]
\[ [b_i, b_j] = 1 \text{ if } i \equiv j \pmod{2}. \]

Further $a_1^p = a_2^p = 1$ and $b_1^p$ is the product of following $n^2 + n + 1$ commutators:

\[ (**): [a_1, a_2], [a_1, b_{2i+2}], [a_2, b_{2i+1}], [b_{2i+1}, b_{2j+2}], \quad i, j = 0, 1, \ldots, n - 1, i \neq j. \]

Finally powers of $b_i$’s are related by

\[ b_2^p = b_1^p [a_1, b_2]^{-1}, \]
\[ b_{2i+1}^p = b_{2i}^p [a_2, b_{2i+1}]^{-1}, \quad b_{2i+2}^p = b_{2i+1}^p [a_1, b_{2i+2}]^{-1}, \quad i = 1, \ldots, n - 1. \]

Then $G^n$ is a special $p$-group, with $|G^n/(G^n)'| = p^{2n+2}$ and $(G^n)'$ is elementary abelian $p$-group generated by $n^2 + n + 1$ commutators in (**); so $|G^n| = p^{n^2 + 3n + 5}$. Further, $\text{Aut}(G^n)$ is (elementary) abelian, and

\[ |\text{Aut}(G^n)| = |\text{Autcent}(G^n)| = p^{(2n+2)(n^2+n+1)}. \]

For $n = 1$, $G^1$ is a special $p$-group of order $p^7$. This is an example of a Miller group of the smallest order for $p$ odd. The following problem is interesting:

**Problem 5.** Describe all Miller $p$-groups of order $p^7$ for $p > 2$. 

(4.7) Upto this point, we have only considered the examples of special Miller $p$-groups (modulo Remark 19). Now we’ll consider non-special groups. Before we proceed further with more examples, we record a very useful result of Adney and Yen [11].

Let $G$ be a purely non-abelian $p$-group of class 2. Let $G/G' = \prod_{i=1}^{r}(x_iG')$, with $o(x_iG') \geq o(x_{i+1}G')$. Let $p^a, p^b, p^c$ denote the exponents of $Z(G)$, $G'$ and $G/G'$ respectively. Finally, let $R$ be the subgroup of $Z(G)$ generated by all the
homomorphic images of $G$ in $Z(G)$, and $K$ denote the intersection of the kernels of all the homomorphisms $G \to G'$.

**Theorem 23** (Adney-Yen, [11]). *With the above notations, Autcent$(G)$ is abelian if and only if the following hold:

1. $R = K$.
2. either $\text{min}(a, c) = b$ or $\text{min}(a, c) > b$ and $R/G' = \langle x^b \rangle$. 

Motivated by the conjecture of Mahalanobis as stated in the introduction, Jain and the second author [27], in 2012, constructed the following infinite family of non-special Miller $p$-groups. For $n \geq 2$, and $p$ odd, let $G_n = \langle x_1, x_2, x_3, x_4 \rangle$ be the $p$-group of class 2 with the following additional relations:

$$x_1^{p^n} = x_2^{p^2} = x_3^{p^2} = x_4^{p^2} = 1,$$

$$[x_1, x_2] = x_2^{p^2}, \ [x_1, x_3] = [x_1, x_4] = x_3^{p^2},$$

$$[x_2, x_3] = x_1^{p^{n-1}}, \ [x_2, x_4] = x_2^{p^2}, \ [x_3, x_4] = 1.$$

It is then easy to see that

1. $Z(G_n) = \Phi(G_n) = \langle x_1^{p}, x_2^{p^2}, x_3^{p^2} \rangle$.
2. $G'_n = \langle x_1^{p^{n-1}}, x_2^{p^2}, x_3^{p^2} \rangle$ is elementary abelian of order $p^3$.
3. $G_n$ is special only when $n = 2$ (follows by (1) and (2)).

The proof that Aut$(G_n) = \text{Autcent}(G_n)$ is constructive and rely on detailed careful calculations. An application of Theorem 23 now shows that $G_n$ is Miller.

In the preceding examples of non-special Miller $p$-groups $G$, one can easily notice that

$$G' < Z(G) = \Phi(G).$$

One might desire that for a Miller $p$-group $G$, one of the following always holds true: (i) $G' = Z(G)$; (ii) $Z(G) = \Phi(G)$.

In 2013, Jain, Rai and the second author [28] constructed the following infinite family of Miller $p$-groups $G$ such that $G' < Z(G) < \Phi(G)$, which we again denote by $G_n$. For $n \geq 4$, let $G_n = \langle x_1, x_2, x_3, x_4 \rangle$ be a $p$-group of class 2 with the following additional relations:

$$x_1^{p^n} = x_2^{p^4} = x_3^{p^4} = x_4^{p^2} = 1,$$

$$[x_1, x_2] = [x_1, x_3] = x_2^{p^2}, \ [x_1, x_4] = x_3^{p^2},$$

$$[x_2, x_3] = x_1^{p^{n-2}}, \ [x_2, x_4] = x_3^{p^2}, \ [x_3, x_4] = x_2^{p^2}. $$
Then $G_n$ is a $p$-group of order $p^{n+10}$ with

$$Z(G_n) = \langle x_1^p, x_2^p, x_3^p \rangle, \quad \Phi(G_n) = \langle x_1^p, x_2^p, x_3^p, x_4^p \rangle, \quad G'_n = \langle x_1^{p^{n-2}}, x_2^p, x_3^p \rangle.$$

It follows that $G'_n \leq Z(G_n) < \Phi(G_n)$ and $G'_n = Z(G_n)$ only when $n = 4$. As in the previous examples, the proof of the fact that $G_n$ is Miller is constructive.

As mentioned in Theorem\[3\] if Aut($G$) is elementary abelian, then $\Phi(G)$ is elementary abelian and one of the following holds: (1) $Z(G) = \Phi(G)$; (2) $G' = \Phi(G)$. Jain, Rai and the second author constructed the following Miller $p$-groups in which exactly one of these conditions hold.

For any prime $p$, let $G_4 = \langle x_1, x_2, x_3, x_4 \rangle$ denote the $p$-group of class 2 with the following additional relations:

$$x_1^{p^2} = x_2^{p^2} = x_3^{p^2} = x_4^{p^2} = 1,$$

$$[x_1, x_2] = 1, \quad [x_1, x_3] = x_1^p, \quad [x_1, x_4] = x_4^p,$$

$$[x_2, x_3] = x_1^p, \quad [x_2, x_4] = x_2^p, \quad [x_3, x_4] = x_4^p.$$

Then $G_4$ is a $p$-group of order $p^8$ and the following holds.

1. $G'_4 = \langle x_1^p, x_2^p, x_3^p \rangle$ is elementary abelian of order $p^3$.
2. $\Phi(G_4) = Z(G_4) = \langle x_1^p, x_2^p, x_3^p, x_4^p \rangle$ is elementary abelian of order $p^4$.
3. Aut($G_4$) is elementary abelian of order $p^{16}$.

Again for any prime $p$, consider the $p$-group $G_5 = \langle x_1, x_2, x_3, x_4, x_5 \rangle$ of class 2 with the following additional relations:

$$x_1^{p^2} = x_2^{p^2} = x_3^{p^2} = x_4^{p^2} = x_5^{p^2} = 1,$$

$$[x_1, x_2] = x_1^p, \quad [x_1, x_3] = x_5^p, \quad [x_1, x_4] = 1, \quad [x_1, x_5] = x_1^p, \quad [x_2, x_3] = x_2^p,$$

$$[x_2, x_4] = 1, \quad [x_2, x_5] = x_4^p, \quad [x_3, x_4] = 1, \quad [x_3, x_5] = x_3^p, \quad [x_4, x_5] = 1.$$

Then $G_5$ is a $p$-group of order $p^9$ and the following holds.

1. $G'_5 = \Phi(G_5) = \langle x_1^p, x_2^p, x_3^p, x_4^p \rangle$ is elementary abelian of order $p^4$.
2. $Z(G_5) = \langle x_4, G'_5 \rangle$.
3. Aut($G_5$) is elementary abelian of order $p^{20}$.

It is natural to ask

**Question 6.** Does there exist a Miller $p$-group $G$ in which $Z(G) \not\subseteq \Phi(G)$ and $\Phi(G) \not\subseteq Z(G)$?

\[4.8\] The proofs of the results in subsection \[4.7\] involve heavy computations. To remedy the problem, in 2015, Caranti [6] [7] suggested a simple module theoretic approach to construct non-special Miller $p$-groups from special ones. The
arguments given in [6] are not sufficient to prove the results as stated. The authors of the present survey proved that the results are valid under an additional hypothesis. The construction is briefly described as follows.

For an odd prime \( p \), let \( H \) be a special Miller \( p \)-group satisfying the following hypotheses:

(i) \( \mathcal{U}_1(H) \) is a proper subgroup of \( H' \).

(ii) The map \( H/H' \to H' \) defined by \( h \mapsto h^p \) is injective.

Let \( K = \langle z \rangle \) be the cyclic group of order \( p^2 \), and \( M \) be a subgroup of order \( p \) in \( H' \) but not in \( \mathcal{U}_1(H) \). Let \( G \) be a central product of \( H \) and \( K \) amalgamated at \( M \).

Note that \( G' = \Phi(G) < Z(G) \); hence \( G \) is non-special. With this setting, we have

**Theorem 24.** If \( H/M \) is a Miller group, then so is \( G \).

Before we proceed, we make a comment on the preceding theorem. Caranti claimed that \( G \) is Miller without the condition ‘\( H/M \) is Miller’. Unfortunately, this is not always true, as shown in the following example.

Let \( H = \langle a, b, c, d \rangle \) be the \( p \)-group of class 2 with the following additional relations:

\[
\begin{align*}
a^p &= [a, c], & b^p &= [a, bcd], & c^p &= [b, cd], & d^p &= [b, d].
\end{align*}
\]

Then \( H \) is a special Miller \( p \)-group of order \( p^{10} \) and satisfies conditions (i)-(ii). It can be proved that if \( M = \langle [a, b] \rangle \), then \( G \) is Miller, and if \( M = \langle [a, d] \rangle \), then \( G \) is not a Miller group. That \( H \) and \( H/\langle [a, b] \rangle \) are Miller can be proved following the arguments similar to those in [29] (for details see [31]).

As noted above, the Miller groups \( G \) are such that \( G' = \Phi(G) < Z(G) \). Now with a little variation in the preceding construction, we obtain Miller \( p \)-groups \( G \) with \( G' < \Phi(G) = Z(G) \). Assume that the special Miller group \( H \) also satisfies the following hypothesis in addition to (i)-(ii) above:

(iii) If \( H \) is minimally generated by \( x_1, \ldots, x_n \), then \( H' \) is minimally generated by \( [x_i, x_j] \) for \( 1 \leq i < j \leq n \).

Let \( L = \langle z \rangle \) be a cyclic group of order \( p^n \), \( n \geq 3 \) and let \( z \) act on \( H \) via a non-inner central automorphism \( \sigma \) of \( H \) (which always exists in a Miller \( p \)-group). Let \( N \) be a subgroup of order \( p \) in \( H' \) but not in \( \mathcal{U}_1(H) \). Now define \( G \) to be the partial semi-direct product of \( H \) by \( L \) amalgamated at \( N \) (cf. [17] or [31]). With this setting, we finally have

**Theorem 25.** If \( H/N \) is a Miller group, then so is \( G \).
Again, we remark that the condition \( H / N \) is Miller, in the preceding theorem, can not be dropped, as shown by the following example. Let \( H = \langle a, b, c, d \rangle \) be the \( p \)-group of class 2 described above. Then \( \sigma \) defined by
\[
\begin{align*}
    a &\mapsto ad^p, \\
    b &\mapsto b, \\
    c &\mapsto c, \\
    d &\mapsto d
\end{align*}
\]
extends to a non-inner central automorphism of \( H \). Let \( L \) be of order \( p^3 \) acting on \( H \) via \( \sigma \). Then \( G \) is Miller if \( N = \langle [a, b] \rangle \), and \( G \) is not if \( N = \langle [a, d] \rangle \).

Remark 26. The above construction of non-special Miller groups \( G \) from special Miller groups \( H \) is valid even without hypotheses (i)-(iii) on \( H \). That \( \text{Aut}(G) = \text{Aut}_{\text{cent}}(G) \) can be proved using the same arguments as in [31], which do not rely on hypotheses (i)-(iii). Then one can apply Theorem 23 to show that \( \text{Aut}(G) \) is abelian.

We conclude with the following remarks. In all known examples of Miller \( p \)-groups \( G \), \( G / Z(G) \) is homocyclic. It will be interesting to know whether this happens in all Millers \( p \)-groups. If true, one might expect if \( \gamma_2(G) \) is always homocyclic. Again in the known Millers \( p \)-groups one can observe that either \( G' \) and \( Z(G) \) have same ranks or \( G / Z(G) \) and \( G / G' \) have same ranks. We wonder whether this is true for all Miller \( p \)-groups.

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