A Böcherer-Type Conjecture for Paramodular Forms

Nathan C. Ryan, Gonzalo Tornaría *
Department of Mathematics, Bucknell University
nathan.ryan@bucknell.edu
Centro de Matemática, Universidad de la República
tornaria@cmat.edu.uy

In the 1980s Böcherer formulated a conjecture relating the central value of the quadratic twists of the spinor $L$-function attached to a Siegel modular form $F$ to the coefficients of $F$. He proved the conjecture when $F$ is a Saito-Kurokawa lift. Later Kohnen and Kuss gave numerical evidence for the conjecture in the case when $F$ is a rational eigenform that is not a Saito-Kurokawa lift. In this paper we develop a conjecture relating the central value of the quadratic twists of the spinor $L$-function attached to a paramodular form and the coefficients of the form. We prove the conjecture in the case when the form is a Gritsenko lift and provide numerical evidence when it is not a lift.

1. Introduction

Conjectures about central values of $L$-functions abound; for example the Conjecture of Birch and Swinnerton-Dyer predicts that the order of vanishing of the central critical values of an elliptic curve $L$-functions is equal to the rank of the elliptic curve’s Mordell-Weil group. Finding an asymptotic formula for $V_E(x)$, the number of quadratic twists by discriminant $d \leq x$ of an elliptic curve $E$ with vanishing central value, might provide some insight into how often the twists of $E$ have rank at least 2.

Sarnak has predicted the asymptotic size of this count based on formulas of Waldspurger [19] and Kohnen-Zagier [10] which relate the central value of the quadratic twists of an elliptic curve $L$-function to squares of Fourier coefficients of half-integral weight modular forms. These formulas have been used (e.g., in [5]) to give more refined conjectures as to the asymptotic size of $V_E(x)$. They can also be used to compute a large number of central critical values of twisted $L$-functions as, generally, computing coefficients of a form is easier than computing the central critical values of an $L$-function. Moreover, since the Fourier coefficients are integral, the computation allows one to determine exactly whether or not there is a vanishing as there will be no error introduced by rounding.

In the 1980s and 1990s similar formulas were conjectured for Siegel modular forms. In [3] a conjecture was formulated relating central critical values of quadratic

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twists of spinor $L$-functions associated to a Siegel modular form $F$ of full level and sums of coefficients of $F$. The conjecture when $F$ is a Maass lift was proved in [3]. Numerical evidence for the conjecture when $F$ is not a Maass lift was provided in [11] for rational eigenforms of level 1 and weight up to 26, and only recently for higher weights and non-rational eigenforms in [13]. For modular forms of (squarefree) level $N > 1$ the conjecture has been proved in [2] in the case when the form is a Yoshida lift.

In what follows we investigate a version of Böcherer’s Conjecture in the setting of paramodular forms. Paramodular forms are similar to Siegel modular forms on $Sp(4, \mathbb{Z})$ in that they are multivariate modular forms on a group of rank 4. Recently they have been studied in the context of the Paramodular Conjecture which identifies the spinor $L$-functions of certain paramodular forms with the Hasse-Weil $L$-functions of certain abelian surfaces [12], [4].

We give evidence for our generalization of Böcherer’s Conjecture in two ways: first, we prove the conjecture when the form is a Gritsenko lift and, second, we verify the conjecture computationally in a number of cases. For a fundamental discriminant $D < 0$ coprime to the level, our conjecture takes the form

$$L(F, 1/2, \chi_D) = C_F |D|^{1-k} A(D)^2$$

where $F$ is a paramodular form, $C_F > 0$ is a constant that only depends on $F$, and $A(D)$ is an average of the coefficients of $F$. It turns out that computing the Fourier coefficients of such an $F$ is computationally very expensive: we use data from [12] to compute the right-hand side of the formula.

For the left-hand side, however, the data of [12] yield at most the Euler factors at the primes 2, 3, 5, 7 and, in some cases, 11. In particular, in all cases the first $L$-series coefficient that is unknown is less than 17. In order to compute central critical values to any reasonable precision, we do not have enough coefficients. Instead, we assume the Paramodular Conjecture and compute central critical values $L(F, 1/2, \chi_D)$ for many $D$ by computing the central critical value of the corresponding Hasse-Weil $L$-functions and showing that, numerically, $L(F, 1/2, \chi_D)/(C_F |D|^{1-k})$ is the square of $A(D)$ for the $D$ for which we have data. The constant $C_F$ is computed from the formula applied to the smallest possible $|D|$.

The paper is organized as follows. In the rest of this section we define all the terminology and notation needed to understand the conjecture. In the following section we prove the conjecture for paramodular forms that are lifts. In the third section we describe our experimental results. The fourth section deals with the case when the form $F$ is in the minus space with respect to the Atkin-Lehner operator—the conjecture holds in the case but some interesting computational phenomena arise. We conclude with tables reporting the data we generated.

1.1. Notation

The main objects of study in this paper are paramodular forms of prime level $p$ and their $L$-functions.
Let $R$ be a commutative ring with identity. The symplectic group is $Sp(4, R) := \{ x \in GL(4, R) : x^t J_2 x = J_2 \}$, where the transpose of matrix $x$ is denoted $x^t$ and for the $n \times n$ identity matrix $I_n$ we set $J_n = \left( \begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix} \right)$. When $R \subseteq \mathbb{R}$, the group of symplectic similitudes is $GSp(4, \mathbb{R}) := \{ x \in GL(4, R) : \exists \mu \in \mathbb{R}_{>0} : x^t J_2 x = \mu J_2 \}$.

The paramodular group of level $p$ is

$$\Gamma_{\text{para}}[p] := Sp(4, \mathbb{Q}) \cap \left\{ \begin{pmatrix} * & * & * / p & * \\ p* & * & * & * \\ p* & p* & * & p* \\ p* & * & * & * \end{pmatrix} \right\}, \text{ where } * \in \mathbb{Z}. $$

1.2. Modular Form Notation

Let $\mathcal{H}_n := \{ Z = X + iY \in M_{n \times n}(\mathbb{C}) : Z' = Z, Y > 0 \}$ be the Siegel upper half space. The group $GSp^+(4, \mathbb{R})$ acts on $\mathcal{H}_2$ by $\gamma(Z) = (AZ + B)(CZ + D)^{-1}$ where $\gamma = (A \ B \ C \ D \gamma)$. The complex vector space of paramodular forms of degree 2, level $p$ and weight $k$ is the set of holomorphic $F : \mathcal{H}_2 \to \mathbb{C}$ such that

$$(F|\gamma)(Z) := \det(CZ + D)^{-k}F(\gamma(Z)) = F(Z)$$

for all $\gamma \in \Gamma_{\text{para}}[p]$ and such that for all positive definite $Y_0$ and for all $\gamma \in Sp(4, \mathbb{Z})$, $F|\gamma$ is bounded on $\{ Z \in \mathcal{H}_2 : \exists Z > Y_0 \}$. We denote the space of paramodular forms by $M^k(\Gamma_{\text{para}}[p])$. For $F \in M^k(\Gamma_{\text{para}}[p])$ we define the Siegel $\Phi$-operator as $\Phi(F)(Z) := \lim_{\lambda \to \infty} F \left( \left( \begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^2 \end{smallmatrix} \right) Z \right)$ and the space of cusp forms $S^k(\Gamma_{\text{para}}[p])$ as the space of all paramodular forms so that $F | \gamma \in \ker \Phi$ for all cusps $\gamma$.

By the Köcher principle, every $F \in M^k(\Gamma_{\text{para}}[p])$ has a Fourier expansion of the form

$$F(Z) = \sum_{T \in \mathcal{X}_2} a(T; F) q^m \zeta^r q^n$$

where $q := e^{2\pi i z}$, $q' := e^{2\pi i z'}$ ($z, z' \in \mathcal{H}_1$) $\zeta := e^{2\pi i \tau}$ (where $\tau \in \mathbb{C}$) and

$$p\mathcal{X}_2 := \left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathbb{Z}^2 : a, b, c \in \mathbb{Z} \right\}.$$

For $F \in S^k(\Gamma_{\text{para}}[p])$, we have $a(T[U]; f) = \det(U)^k a(T; f)$ for every $U \in \Gamma_0(p)$ where $\Gamma_0(p) := \left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) : \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathbb{Z}^2, \text{det} \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \equiv 1 \text{ mod } p \right\}$ (here $\Gamma_0(p)$ is the congruence subgroup of $\text{SL}(2, \mathbb{Z})$ with lower lefthand entry congruent to 0 mod $p$.) Moreover, cusp forms are supported on the positive definite matrices in $p\mathcal{X}_2$.

The space $S^k(\Gamma_{\text{para}}[p])$ can be split into a plus space and a minus space. Define an operator

$$\mu_p = \frac{1}{\sqrt{p}} \left( \begin{smallmatrix} 0 & 1 & 0 & 0 \\ -p & 0 & 0 & 0 \\ 0 & 0 & 0 & p \\ 0 & 0 & -1 & 0 \end{smallmatrix} \right),$$

an involution. Then, we define $S^k(\Gamma_{\text{para}}[p])^\pm = \{ f \in S^k(\Gamma_{\text{para}}[p]) : f | \mu = \pm f \}$. 

1.3. L-function Notation

Following Andrianov [11], one can define operators $T(n)$ in terms of double cosets. Specifically, using [9], one can define the action of the operator $T(q^\delta)$ for $S^k(\Gamma_{\text{para}}[p])$ for primes $(p, q) = 1$.

Suppose we are given a paramodular form $F \in S^k(\Gamma_{\text{para}}[p])$ so that for all $n \in \mathbb{Z}$, $F|T(n) = \lambda_{F,n}F = \lambda_nF$. Then we can define the spinor $L$-series by the Euler product

$$L(F, s) := \prod_{q \text{ prime}} L_q(q^{-s-k+3/2})^{-1}, \quad (1.1)$$

where the local Euler factors are given by

$$L_q(X) := 1 - \lambda_qX + (\lambda_q^2 - \lambda_q^2 - q^{2k-4})X^2 - \lambda_q^2q^{2k-3}X^3 + q^{4k-6}X^4$$

for $q \neq p$, and $L_p(X)$ has a similar formula but of degree 3 (this will be investigated further by the first author in future work).

As our computations are in weight 2, we will compute the $L$-series of a paramodular form of prime level by assuming the following conjecture.

**Conjecture 1.1 (Paramodular Conjecture).** Let $p$ be a prime. There is a bijection between lines of Hecke eigenforms $F \in S^2(\Gamma_{\text{para}}[p])$ that have rational eigenvalues and are not Gritsenko lifts and isogeny classes of rational abelian surfaces $A$ of conductor $p$. In this correspondence we have that

$$L(A, s, \text{Hasse-Weil}) = L(F, s).$$

We remark that it is a conjecture that the two $L$-series mentioned above have an analytic continuation and satisfy a functional equation.

In order to compute central values we need the Selberg data for the $L$-function: specifically for an $L$-series $L(s) = \sum_{n \geq 0} a(n)n^{-s}$

- we complete $L(s)$ by multiplying it by some $\Gamma$-factors of the form
  $$\gamma(s) := \Gamma\left(\frac{s + \lambda_1}{2}\right) \cdots \Gamma\left(\frac{s + \lambda_d}{2}\right)$$
  and an exponential factor $A^s$; i.e., $L^*(s) := A^s\gamma(s)L(s)$ and we need $\lambda_1, \ldots, \lambda_d$ and $A$; and
- we require that $L^*(s)$ satisfies a functional equation of the form
  $$L^*(s) = \pm L^*(1 - s).$$

We note that we use the analytic normalization $s \mapsto 1 - s$ and that the factor $\gamma(s)$ is not unique as it can be rewritten using the duplication formula.

A table in [6] summarizes the data that we use: in particular, as we have a degree 4 $L$-function attached to a paramodular form of weight 2 and level $p$ (that corresponds to an abelian surface isogenous to a curve of genus 2) we have

$$L^*(F, s) = \left(\frac{\sqrt{p}}{4\pi^2}\right)^s \Gamma(s + 1/2)\Gamma(s + 1/2)L(F, s). \quad (1.2)$$
so that conjecturally
\[ L^*(F, s) = \epsilon L^*(F, 1 - s), \]
when \( F \in S^2(\Gamma_{\text{para}}[p])^r \).

Let \( D \) be a fundamental discriminant, and denote by \( \chi_D \) the unique quadratic character of conductor \( D \). For the spinor \( L \)-series \( L(F, s) = \sum_{n \geq 1} a(n) n^{-s} \) of a paramodular form \( F \), we define the quadratic twist
\[ L(F, s, \chi_D) := \sum_{n \geq 1} \chi_D(n) a(n) n^{-s}. \]
In our case, most of the Selberg data for the \( L \)-function is expected to be the same as the data for the non-twisted \( L \)-function, except for the exponential factor and the sign of the functional equation. For instance, assuming \( p \nmid D \), the exponential factor increases by a factor of \( D^2 \) and the sign of the functional equation is changed by a factor of \( \left( \frac{D}{p} \right) \).

The computation of the central values of the \( L \)-functions was done using Mike Rubinstein’s \texttt{lcalc} package [14].

### 1.4. Gritsenko Lifts

A Gritsenko lift [8] is a paramodular form that comes from a Jacobi form. The standard reference for Jacobi forms is [7]. We summarize the relevant terminology here.

**Definition 1.2.** A Jacobi form of level 1, weight \( k \) and index \( m \) is a function \( \phi(z, \tau) \) for \( z \in \mathbb{H}_1 \) and \( \tau \in \mathbb{C} \) such that:

1. \( \phi(a\tau + b, c\tau + d) = (cz + d)^k e^{2\pi imc^{-2}} \phi(\tau, z) \) for \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \);
2. \( \phi(z, \tau + \lambda z + \mu) = e^{-2\pi im(\lambda^2 z + 2\lambda \tau)} \phi(z, \tau) \) for all integers \( \lambda, \mu \); and
3. \( \phi \) has a Fourier expansion
   \[ \phi(z, \tau) = \sum_{n \geq 0} \sum_{r \leq 4m} c(n, r) q^n \zeta^r. \]

Our first main theorem is the proof of Conjecture [14] for Gritsenko lifts and so we make the following definition:

**Definition 1.3 (Gritsenko Lift).** Let \( \phi \in S_{k, p} \) and suppose \( \phi(\tau, z) = \sum_{n>0, r \in \mathbb{Z}} c(n, r) q^n \zeta^r \) is its Fourier expansion. Then the Gritsenko lift of \( \phi \) is \( \text{Grit}(\phi) \in S^k(\Gamma_{\text{para}}[p])^+ \) given by

\[ \text{Grit}(\phi) \left( \frac{z}{w} \right) = \sum_{n, r, m} \left( \sum_{\delta | (n, r, m)} \delta^{k-1} c \left( \frac{mn}{\delta^2}, \frac{r}{\delta} \right) \right) q^{mp} \zeta^r q^n. \tag{1.3} \]

We remark that it also makes sense to talk about \( \text{Grit}(f) \) where \( f \) is a modular cuspform of level \( p \), in the minus space and weight \( 2k - 2 \) that corresponds to a
\( \phi \in S_{k,p} \) (this can be done via the inverse of the map constructed in Theorem 5 of [10]).

1.5. Summary of Main Results

We define

\[
A_F(D) := \sum_{\{T > 0 : \text{disc } T = D\} \mod \tilde{\Gamma}_0(p)} \frac{a(T; F)}{\varepsilon(T)}
\]

where \( \varepsilon(T) := \# \{U \in \tilde{\Gamma}_0(p) : T[U] = T\} \). We often write \( A(D) \) when \( F \) is obvious from context. Our main goal is to give evidence for the following conjecture

**Conjecture 1.4 (Paramodular Böcherer’s Conjecture).** Suppose \( F \in S^k(\Gamma_{\text{para}}[p])^+ \). Then, for fundamental discriminants \( D < 0 \) we have

\[
L(F, 1/2, \chi_D) = \ast C_F |D|^{1-k} A(D)^2
\]

where \( C_F \) is a positive constant that depends only on \( F \), and \( \ast = 1 \) when \( p \nmid D \), and \( \ast = 2 \) when \( p \mid D \).

The evidence we provide is in two forms. First, we prove the conjecture in the case that the form is a Gritsenko lift:

**Theorem 1.5.** Let \( F = \text{Grit}(f) \in S^k(\Gamma_{\text{para}}[p])^+ \) where \( p \) is prime and \( f \) is a Hecke eigenform of degree 1, level \( p \) and weight \( 2k-2 \). Then there exists a constant \( C_F > 0 \) so that

\[
L(F, 1/2, \chi_D) = \ast C_F |D|^{1-k} A(D)^2
\]

for \( D < 0 \) a fundamental discriminant, and \( \ast = 1 \) when \( p \nmid D \), and \( \ast = 2 \) when \( p \mid D \).

The idea of the proof is to combine four ingredients: (i) the factorization of the \( L \)-function of the Gritsenko lift as in [15], (ii) Dirichlet’s class number formula, (iii) an explicit description of the Fourier coefficients of the Gritsenko lift and (iv) Waldspurger’s theorem relating the central values of quadratic twists to sums of coefficients of modular forms of half-integer weight [19].

Second, we verify the conjecture computationally in a number of cases. The computations we do are based on [12] and [18]. On the one hand, [12] provides Fourier coefficients for all paramodular forms of prime level up to 600 that are not Gritsenko lifts and on the other [18] provides most of the curves that correspond to the paramodular forms. Armand Brumer kindly provided a curve that was not in [18] but corresponds to one of the paramodular forms in [12]. By matching levels of modular forms and discriminants of hyperelliptic curves we show the following complement to Theorem 1.5. Suppose \( F \in S^2(\Gamma_{\text{para}}[p])^+ \) for \( p \) a prime less than 600 is not a Gritsenko lift. Then, numerically, there exists a coefficient \( C_F > 0 \) so that

\[
L(F, 1/2, \chi_D) = \ast C_F |D|^{1-k} A(D)^2
\]
for $D < 0$ a fundamental discriminant listed in Tables 2–8.

Note that in case $(\frac{D}{p}) = −1$ the twisted central value is expected to be zero due to the sign of the functional equation being $−1$, and on the right hand side the average $A(D)$ is an empty sum. For this reason, we exclude these discriminants from our computation.

2. The Case of Lifts

Assume $p$ is an odd prime. Recall ([7, Theorem 2.2, p. 23]) that $c(u, r)$ depends only on $D = r^2 − 4np$; call this number $c^*(D)$, i.e.

$$c^*(D) := c\left(\frac{r^2 − D}{4p}, r\right),$$

for any $r \in \mathbb{Z}$ such that $r^2 \equiv D \pmod{4p}$. We let $c^*(D) := 0$ otherwise.

**Lemma 2.1.** Let $D$ be a fundamental discriminant. Then

$$\sum_{T \in \mathbb{X}_2/\Gamma_0(p) \atop \text{disc } T = D} \frac{1}{\varepsilon(T)} = \frac{h(D)}{w_D},$$

where $h(D)$ and $w_D$ are the class number and the number of units of the quadratic order of discriminant $D$, respectively.

**Proof.** We start by noting that $\#\{U \in \Gamma_0(p) : T[U] = T\} = w_D$ for any $T$ with $\text{disc } T = D$. Also, $\#\mathbb{X}_2/\Gamma_0(p) = 2h(D)$.

The lemma then follows from the fact that

$$\sum_{T \in \mathbb{X}_2/\Gamma_0(p) \atop \text{disc } T = D} \frac{1}{\varepsilon(T)} = \frac{1}{[\Gamma_0(p) : \Gamma_0(p)]} \sum_{T \in \mathbb{X}_2/\Gamma_0(p) \atop \text{disc } T = D} \frac{1}{w_D} = \frac{1}{2w_D} 2h(D) \quad \Box$$

**Proposition 2.2.** Let $F = \text{Grit}(\phi)$. For $D < 0$ a fundamental discriminant we have:

$$A(D; F) = \frac{h(D)}{w_D} c^*(D)$$

**Proof.** By the definition of the Gritsenko lift, we know that

$$A(T; F) = c^*(\text{disc } T)$$

provided $T$ is primitive; this is always the case when $\text{disc } T = D$ is a fundamental discriminant.

Thus

$$A(D; F) = \sum_{T \in \mathbb{X}_2/\Gamma_0(p) \atop \text{disc } T = D} \frac{c^*(D)}{\varepsilon(T)}$$

The result follows from the lemma. \qed
Proposition 2.3. Let $F = \text{Grit}(f) \in S^k(\Gamma_{\text{para}}[p])^+$. 

(1) $L(F, s, \chi_D)$ has an analytic continuation to an entire function.

(2) $L(F, 1/2, \chi_D) = \frac{4\pi^2}{w_D} \cdot \frac{b(D)^2}{\sqrt{|D|}} \cdot L(f, 1/2, \chi_D)$ where $D < 0$ is a fundamental discriminant.

Proof. It is a standard fact that $L(F, s) = \zeta(s + 1/2) \zeta(s - 1/2) L(f, s)$ (using the analytic normalization, so that the center is at $s = 1/2$). Twisting by $\chi_D$ we obtain

$$L(F, s, \chi_D) = L(s + 1/2, \chi_D) L(s - 1/2, \chi_D) L(f, s, \chi_D)$$

valid on the region of convergence. Since the right hand side has an analytic continuation, (1) follows.

To prove (2), we evaluate the above equation at $s = 1/2$, and use the Dirichlet class number formula for $L(0, \chi_D)$ and $L(1, \chi_D)$.

Proof of Theorem 1.5. By Waldspurger’s formula [19], we have

$$L(f, 1/2, \chi_D) = \star k_f \cdot \frac{c^r(D)^2}{|D|^{k-3/2}},$$

with $k_f > 0$. The theorem thus follows directly from Proposition 2.2 and part (2) of Proposition 2.3.

3. The Case of Nonlifts

In this section we describe numerical experiments that support Conjecture 1.4 in the case when the form is not a Gritsenko lift. We discuss how to compute the $L$-series that correspond to paramodular forms of weight 2 and prime conductor $p < 600$. For the rest of this section, let $F$ be such a form. In order to compute the central values $L(F, 1/2, \chi_D)$ for several twists $\chi_D$ we would need a large number of coefficients of $F$ since the exponential factor grows as $D^2$. As already mentioned in the introduction, the data in [12] are not enough for this purpose.

To remedy this we do the following. In the same paper [12], Poor and Yuen describe and verify Conjecture 1.3. For our purposes, this conjecture asserts that to compute the $L$-series of $F$, we can compute the Hasse-Weil $L$-series of a related Abelian surface $A$. In particular, Table 1 associates each $F$ to a hyperelliptic curve $C$ isogenous to $A$. There are two forms of level 587 that are nonlifts; one is in the plus space the other is in the minus space (see Section 4).

By the Paramodular Conjecture, then, the $L$-function of the curve $C$ of conductor $p$ is equal to the $L$-function corresponding to the paramodular form $F$ of level $p$. Thus, we compute the $L$-function by counting points on the curve. In computing the $L$-function of $F$ in this way, we provide evidence for the Paramodular Conjecture as well. Since the Paramodular Böcherer’s Conjecture is verified for the $L$-function of $F$ computed via this correspondence, it strongly suggests that the Hasse-Weil and spinor $L$-functions agree.
Table 1. Hyperelliptic curves $C$ used to compute $L$-series associated to paramodular forms of level $p$ that are not lifts.

| $p$ | $\epsilon$ | $\lambda$ | $C$ |
|-----|-------------|-----------|-----|
| 277 | +           | 8         | $y^2 + y = x^6 - 2x^4 + 2x^2 - x$ |
| 349 | +           | 12        | $y^2 + y = -x^3 - 2x^3 - x^3 + x^2 + x$ |
| 353 | +           | -9        | $y^2 + (x^3 + x + 1) y = x^2$ |
| 389 | +           | -10       | $y^2 + xy = -x^6 - 3x^4 - 4x^3 - 3x^2 - x$ |
| 461 | +           | 0         | $y^2 + y = -2x^6 + 3x^5 - 3x^3 + x$ |
| 523 | +           | 24        | $y^2 + xy = -x^3 + 4x^4 - 5x^3 + x^2 + x$ |
| 587 | +           | -6        | $y^2 = -3x^6 + 18x^3 + 9x^2 - 54x + 57$ |
| 587 | -           | -36       | $y^2 + (x^3 + x + 1) y = -x^3 - x^2$ |

Assuming Conjecture [1,4], we compute the $L$-series for $F$ by counting points on its corresponding $C$. The reciprocal of the $q$-th Euler factor of $L$-series in [1,1] for $q \neq p$, specialized to $k = 2$ is of the form

$$L_q(X) = 1 - \lambda_q X + (\lambda_q^2 - \lambda_q^2 - 1) X^2 - \lambda_q q X^3 + q^2 X^4.$$  

Writing the Euler factor as $L_q(X)^{-1} = \sum_{i=0}^\infty a(q^i) X^i$ and matching it up with the Hasse-Weil $L$-series allows us to conclude

$$\lambda_q = a(q) = 1 + q - N_1$$

$$\lambda_{q^2} + 1 = a(q^2) = 1 + q + q^2 - (1 + q) N_1 + (N_1^2 - N_2)/2$$

where $N_1$ is the number of points on $C$ over $\mathbb{F}_q$ and $N_2$ is the number of points on $C$ over $\mathbb{F}_{q^2}$. We determined $N_1$ and $N_2$ by counting points on $C$ using Sage [17]. The $p$-th Euler factor of $C$ is of degree 3 and given by

$$L_p(X) = (1 - \epsilon X) (1 - \lambda X + pX^2),$$

where $\epsilon$ and $\lambda$ for each $C$ are given in Table [1,1].

Having all the local Euler factors, we computed the central value of the $L$-function and its quadratic twists using Mike Rubinstein’s lcalc [14]. We recall Conjecture [1,4]

$$\frac{L(F,s,\chi_0)}{C_F} \mid D \mid = \sum_{\{T > 0 : \text{disc } T = D \}} \frac{a(T; F)}{\varepsilon(T)}^2$$

where the constant at $C_F$ is positive. In Tables [2,5] the Conjecture is verified in the case of forms that are not Gritsenko lifts and have been computed in [12]. We first determine $C_F$ by solving for it in the case of the first discriminant $D$ in the table. The second column is the average $A(D)$ of the coefficients of discriminant $D$ of the form $F$. The third column is the quantity $\frac{L(F,1/2,\chi_0)}{C_F} \mid D \mid$ which, numerically, for the discriminants $\mid D \mid < 200$ are $A(D)^2$. For the paramodular form of level 277,
we also include data for \( D = -3 \cdot 277 \) and \( D = -4 \cdot 277 \); in this case, since \( p \mid D \), the quantity in the third column is expected to agree with \( 2A(D)^2 \).

All the computations described above were done using an eight core Xeon E5520 system. Computing the first \( 10^6 \) coefficients of the Hasse-Weil \( L \)-series for the eight curves took a total of about 60 cpu-days using a combination of Sage and custom code written in Python and Cython. Computing the central values of the \( L \)-functions and their quadratic twists for the discriminants with \( |D| < 200 \) took less than 1 cpu-hour using \lcalc.

4. The minus space

Suppose \( F \in S^k(\Gamma_{\text{para}}[p])^- \), and let \( D < 0 \) be a fundamental discriminant. In case \( \left( \frac{D}{p} \right) = +1 \), the formula of Conjecture 1.4 holds trivially. Indeed, note that for such \( F \) the sign of the functional equation is \(-1\) and so the central critical value \( L(F, s, \chi_D) \) is zero. On the other hand, \( A(D) \) can be shown to be zero in the following way: Poor and Yuen \cite{poor_yuen} Definition 3.9] define an involution \( \text{Twin} \) over the set \( X_2^p \) with discriminant \( D \) for which \( a(\text{Twin}(T); F) = a(T; F | \mu) \). Since we are in the minus space

\[
A(D) = \sum_{\{T > 0 : \text{disc } T = D \}/\hat{\Gamma}_0(p)} \frac{a(T; F)}{\varepsilon(T)}
= \sum_{\{T > 0 : \text{disc } T = D \}/\hat{\Gamma}_0(p)} \frac{a(\text{Twin}(T); F)}{\varepsilon(T)}
= \sum_{\{T > 0 : \text{disc } T = D \}/\hat{\Gamma}_0(p)} \frac{a(T; F | \mu)}{\varepsilon(T)}
= -\sum_{\{T > 0 : \text{disc } T = D \}/\hat{\Gamma}_0(p)} \frac{a(T; F)}{\varepsilon(T)}
= -A(D).
\]

On the other hand, the formula of Conjecture 1.4 fails to hold in case \( \left( \frac{D}{p} \right) = -1 \). Since \( A(D) \) is an empty sum for this type of discriminants, the right hand side of the formula vanishes trivially. However, the left hand side is still an interesting central value, not necessarily vanishing. For example, as can be seen in Table 9 we have that \( L(F_{587,1/2,\chi_D}, s) \) \( |D| \) seems to always be the square of an integer, and frequently nonzero, in spite of \( A(D) \) being zero. In a future paper we will investigate this phenomenon.

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Table 2. Data for the paramodular form of level 277, based on the Hasse-Weil $L$-series for the curve $y^2 + y = x^5 - 2x^3 + 2x^2 - x$. The constant $C_{277} = 6.49630674438$.

| $D$ | $A(D; F_{277})$ | $\frac{L(F_{277}, 1/2, \chi_D)}{C_{277}} |D|$ | $D$ | $A(D; F_{277})$ | $\frac{L(F_{277}, 1/2, \chi_D)}{C_{277}} |D|$ |
|-----|----------------|-----------------|-----|----------------|----------------|
| -3  | -1             | 1.000000        | -83 | 6              | 36.000000     |
| -4  | -1             | 1.000000        | -84 | 1              | 1.000000      |
| -7  | -1             | 1.000000        | -87 | -3             | 9.000000      |
| -19 | -2             | 4.000000        | -88 | -2             | 4.000000      |
| -23 | 0              | -0.000000       | -91 | -1             | 1.000000      |
| -39 | 1              | 1.000000        | -116 | 3            | 9.000000      |
| -40 | -6             | 36.000000       | -120 | -2            | 4.000000      |
| -47 | 0              | 0.000000        | -123 | -1            | 1.000000      |
| -52 | 5              | 25.000000       | -131 | -10            | 100.000000    |
| -55 | -2             | 4.000000        | -136 | -6             | 36.000000     |
| -59 | 3              | 9.000000        | -155 | -10            | 100.000000    |
| -67 | -8             | 64.000000       | -164 | -5             | 25.000000     |
| -71 | 2              | 4.000000        | -187 | 8             | 64.000001     |
| -79 | 0              | 0.000000        | -191 | 2             | 3.999999      |

Table 3. Data for the paramodular form of level 349, based on the Hasse-Weil $L$-series for the curve $y^2 + y = -x^5 - 2x^4 - x^3 + x^2 + x$. The constant $C_{349} = 7.91921340249$.

| $D$ | $A(D; F_{349})$ | $\frac{L(F_{349}, 1/2, \chi_D)}{C_{349}} |D|$ | $D$ | $A(D; F_{349})$ | $\frac{L(F_{349}, 1/2, \chi_D)}{C_{349}} |D|$ |
|-----|----------------|-----------------|-----|----------------|----------------|
| -3  | 1              | 1.000000        | -95 | -4             | 15.999431     |
| -4  | 1              | 1.000000        | -104 | 2             | 4.001714      |
| -15 | -1             | 1.000000        | -111 | -1             | 0.986454      |
| -19 | 4              | 16.000000       | -115 | 14            | 196.009480    |
| -20 | 1              | 1.000000        | -116 | 6             | 36.007307     |
| -23 | 0              | -0.000000       | -123 | 18            | 323.991266    |
| -31 | -1             | 1.000000        | -139 | -11           | 120.948205    |
| -51 | 5              | 25.000000       | -143 | -2             | 4.045737      |
| -56 | 2              | 4.000001        | -148 | -1             | 0.949639      |
| -67 | 13             | 168.999985      | -151 | -1             | 0.948102      |
| -68 | 3              | 8.999976        | -155 | -9             | 81.114938     |
| -83 | 2              | 3.999214        | -164 | 0             | 0.144191      |
| -87 | -4             | 16.000248       | -168 | 6             | 36.150448     |
| -88 | 2              | 4.000085        | -191 | 0             | 0.177733      |
| -91 | 4              | 15.999774       |
Table 4. Data for the paramodular form of level 353, based on the Hasse-Weil $L$-series for the curve $y^2 + (x^3 + x + 1) y = x^2$. The constant $C_{353} = 9.48552733703$.

| $D$ | $A(D; F_{353})$ | $\frac{L(F_{353}, 1/2; \chi_D)}{C_{353}} |D|$ | $D$ | $A(D; F_{353})$ | $\frac{L(F_{353}, 1/2; \chi_D)}{C_{353}} |D|$ |
|-----|----------------|--------------------------------|-----|----------------|--------------------------------|
| -4  | 1              | 1.000000                       | -11 | -6             | 36.005797                     |
| -8  | 1              | 1.000000                       | -16 | 2              | 3.989115                      |
| -11 | 1              | 1.000000                       | -120| 8              | 63.996789                     |
| -15 | 0              | 0.000000                       | -127| 3              | 9.018957                      |
| -19 | -3             | 9.000000                       | -131| 5              | 24.986828                     |
| -23 | 1              | 1.000000                       | -136| -3             | 9.020983                      |
| -35 | 2              | 4.000000                       | -152| 3              | 9.036669                      |
| -39 | 2              | 4.000000                       | -155| 2              | 3.982909                      |
| -43 | -5             | 25.000000                      | -159| -4             | 16.059848                     |
| -47 | 1              | 1.000000                       | -164| 2              | 3.986694                      |
| -68 | -1             | 1.000011                       | -167| 0              | 0.018414                      |
| -83 | -3             | 8.999872                       | -168| -2             | 4.150487                      |
| -84 | -6             | 36.000088                      | -184| 9              | 81.067576                     |
| -88 | 1              | 1.000097                       | -187| -1             | 0.910705                      |
| -91 | 0              | 0.000490                       | -191| 2              | 3.754734                      |

Table 5. Data for the paramodular form of level 389, based on the Hasse-Weil $L$-series for the curve $y^2 + xy = -x^5 - 3x^4 - 4x^3 - 3x^2 - x$. The constant $C_{389} = 10.7918126629$.

| $D$ | $A(D; F_{389})$ | $\frac{L(F_{389}, 1/2; \chi_D)}{C_{389}} |D|$ | $D$ | $A(D; F_{389})$ | $\frac{L(F_{389}, 1/2; \chi_D)}{C_{389}} |D|$ |
|-----|----------------|--------------------------------|-----|----------------|--------------------------------|
| -4  | -1             | 1.000000                       | -91 | -2             | 3.999647                      |
| -7  | 1              | 1.000000                       | -95 | 2              | 4.000340                      |
| -11 | -1             | 1.000000                       | -111| 0              | 0.006107                      |
| -19 | -3             | 9.000000                       | -119| 0              | 0.006107                      |
| -20 | 0              | 0.000000                       | -120| -8             | 63.995136                     |
| -24 | -2             | 4.000000                       | -127| 5              | 24.993422                     |
| -35 | -2             | 4.000000                       | -143| 0              | -0.012195                     |
| -52 | -4             | 16.000000                      | -159| 0              | -0.0012195                    |
| -55 | 2              | 4.000000                       | -164| 2              | 3.933087                      |
| -59 | -3             | 8.999999                       | -168| 0              | 0.076723                      |
| -67 | -2             | 4.000022                       | -179| 0              | -0.062437                     |
| -68 | 2              | 4.000001                       | -183| 4              | 16.008922                     |
| -79 | 5              | 25.000105                      | -184| -12            | 144.143109                    |
| -87 | 2              | 4.000124                       | -187| -2             | 3.453849                      |
Table 6. Data for the paramodular form of level 461, based on the Hasse-Weil $L$-series for the curve $y^2 + y = -2x^6 + 3x^5 - 3x^3 + x$. The constant $C_{461} = 12.0599439822$.

| $D$  | $A(D; F_{461})$ | $\frac{L(F_{461}; 1/2, \chi_D)}{C_{461}} |D|$ | $D$  | $A(D; F_{461})$ | $\frac{L(F_{461}; 1/2, \chi_D)}{C_{461}} |D|$ |
|------|----------------|------------------|------|----------------|------------------|
| -4   | 1              | 1.000000         | -103 | -6             | 35.997894        |
| -19  | -1             | 1.000000         | -104 | 2              | 4.000584         |
| -20  | 1              | 1.000000         | -107 | 0              | -0.006352        |
| -23  | 1              | 1.000000         | -111 | -4             | 16.005971        |
| -24  | -2             | 4.000000         | -115 | 7              | 48.982236        |
| -39  | 2              | 4.000000         | -120 | 2              | 4.003979         |
| -43  | 4              | 16.000000        | -132 | -6             | 36.033483        |
| -56  | 0              | 0.0000001        | -139 | 10             | 99.938609        |
| -59  | -2             | 3.999996         | -143 | 0              | 0.022373         |
| -67  | 2              | 4.000039         | -151 | -7             | 48.911564        |
| -68  | -1             | 0.999934         | -163 | -3             | 9.016937         |
| -84  | 6              | 36.000642        | -164 | -4             | 15.940089        |
| -87  | -2             | 4.000088         | -167 | 1              | 0.940123         |
| -88  | -2             | 4.002180         | -191 | -1             | 1.102735         |
| -91  | -4             | 15.999578        | -195 | -2             | 3.518855         |
| -95  | 3              | 9.000945         | -199 | 5              | 24.577953        |
Table 7. Data for the paramodular form of level 523, based on the Hasse-Weil $L$-series for the curve $y^2 + xy = -x^5 + 4x^4 - 5x^3 + x^2 + x$. The constant $C_{523} = 6.8275178004$.

| $D$ | $A(D; F_{523})$ | $\frac{L(F_{523}, 1/2, \chi_D)}{C_{523}} | D|$ | $D$ | $A(D; F_{523})$ | $\frac{L(F_{523}, 1/2, \chi_D)}{C_{523}} | D|$ |
|-----|-----------------|-------------------------------|-----|-----------------|-------------------------------|
| -3  | -1 1.000000     | -103 24.997989                | -5  |                |                                |
| -8  | -2 4.000000     | -104 16.010823                | -4  |                |                                |
| -20 | 2   4.000000     | -115 35.990265                | 6   |                |                                |
| -35 | -4 16.000000    | -120 16.007819                | -2  |                |                                |
| -39 | 2   4.000000     | -123 4.017347                 | -2  |                |                                |
| -47 | -5 25.000000    | -127 0.041002                 | 0   |                |                                |
| -51 | 0   -0.000000   | -132 16.040526                | -4  |                |                                |
| -55 | -4 15.999997    | -136 15.950713                | 4   |                |                                |
| -56 | 0   -0.000003   | -139 63.909162                | -8  |                |                                |
| -59 | -2 4.000011     | -148 0.064277                 | 0   |                |                                |
| -67 | 5   24.999910   | -152 64.090096                | -8  |                |                                |
| -79 | 0   -0.001005   | -155 0.137795                 | 0   |                |                                |
| -83 | 3   8.995907    | -159 0.313322                 | 9   |                |                                |
| -84 | 4   15.999928   | -163 80.912682                | -1  |                |                                |
| -87 | -8 64.001544    | -167 4.081331                 | -2  |                |                                |
| -88 | 4   16.001550   | -184 16.041808                | 4   |                |                                |
| -95 | 0   0.000162    | -199 2.233478                 |     |                |                                |

Table 8. Data for the paramodular form of level 587 (in the plus space), based on the Hasse-Weil $L$-series for the curve $y^2 = -3x^6 + 18x^4 + 6x^3 + 9x^2 - 54x + 57$. The constant $C_{587} = 15.8250549126$.

| $D$ | $A(D; F_{587}^+)$ | $\frac{L(F_{587}^+, 1/2, \chi_D)}{C_{587}} | D|$ | $D$ | $A(D; F_{587}^+)$ | $\frac{L(F_{587}^+, 1/2, \chi_D)}{C_{587}} | D|$ |
|-----|-------------------|-------------------------------|-----|-------------------|-------------------------------|
| -8  | 1                 | 1.000000                      | -107| 3                 | 8.992246                      |
| -11 | 1                 | 1.000000                      | -111| 0                 | 0.001034                      |
| -15 | -1                | 1.000000                      | -123| 2                 | 3.967416                      |
| -19 | 1                 | 1.000000                      | -127| -1                | 1.021384                      |
| -20 | 1                 | 1.000000                      | -131| 3                 | 8.952729                      |
| -23 | 1                 | 1.000000                      | -132| 5                 | 24.978160                     |
| -24 | 1                 | 1.000000                      | -136| -2                | 4.004657                      |
| -35 | 1                 | 1.000000                      | -139| 1                 | 0.975103                      |
| -39 | -1                | 1.000000                      | -148| -8                | 63.992830                     |
| -52 | 1                 | 1.000000                      | -155| -8                | 64.026128                     |
| -56 | 3                 | 9.000000                      | -164| 4                 | 15.945677                     |
| -71 | 2                 | 4.000060                      | -168| -1                | 1.251997                      |
| -91 | -5                | 25.000542                     | -183| -5                | 25.081099                     |
| -103| -3                | 8.997703                      | -187| -2                | 4.172874                      |
Table 9. Data for the paramodular form of level 587 (in the minus space), based on the Hasse-Weil $L$-series for the curve $y^2 + (x^3 + x + 1) y = -x^3 - x^2$. The constant $C_{587} = 12.6406580054$.

| $D$ | $\frac{L(F_{587}, 1/2; \chi_D)}{C_{587}}$ | $|D|$ | $D$ | $\frac{L(F_{587}, 1/2; \chi_D)}{C_{587}}$ | $|D|$ |
|-----|----------------------------------|------|-----|----------------------------------|------|
| -3  | 1.000000                         | -95  | 0.988953 |
| -4  | 1.000000                         | -104 | 0.998569 |
| -7  | 1.000000                         | -115 | 528.955763 |
| -31 | 4.000000                         | -116 | 0.987458 |
| -40 | 9.000000                         | -119 | 0.022887 |
| -43 | 144.000000                       | -120 | 25.008747 |
| -47 | 1.000001                         | -143 | 0.856318 |
| -51 | 4.000000                         | -151 | 0.151989 |
| -55 | 9.000004                         | -152 | 0.975679 |
| -59 | 16.000007                        | -159 | 1.086667 |
| -67 | 143.999732                       | -163 | 898.880486 |
| -68 | 4.000069                         | -167 | 4.308248 |
| -79 | 0.000216                         | -179 | 10.246464 |
| -83 | 4.000474                         | -184 | 25.006321 |
| -84 | 0.999879                         | -191 | 0.537613 |
| -87 | 9.000104                         | -195 | 8.602606 |
| -88 | 0.999560                         | -199 | 0.158427 |