1. Introduction

In this paper, we consider the following p-Laplacian equation in $\mathbb{R}^N_+$ with critical boundary nonlinearity

\[
\begin{cases}
-\Delta_p u = 0, & \text{in } \mathbb{R}^N_+,

|\nabla u|^{p-2} \frac{\partial u}{\partial n} + |u|^{p-2}u = |u|^{q-2}u, & \text{on } \mathbb{R}^{N-1} = \partial \mathbb{R}^N_+,
\end{cases}
\]

where $1 < p < N$, $\max\{p, \frac{p}{p-1}\} < q < \bar{p} = \frac{(N-1)p}{N-p}$, $\mu > 0$ and $\Delta_p$ is the p-Laplacian operator. We are looking for axial solutions of the Equation (1) that are solutions of $u(x) = u(|y|, s)$, where we denote $x \in \mathbb{R}^N_+$ by $x = (y, s) \in \mathbb{R}^{N-1} \times [0, \infty)$ and we identify $\mathbb{R}^{N-1} = \partial \mathbb{R}^N_+$, $y = (y, 0)$ for $y \in \mathbb{R}^{N-1}$ if there is no confusion.

Introduce in $C_0^\infty(\mathbb{R}^N_+)$ a norm by

\[
\|\varphi\| = \left( \int_{\mathbb{R}^N_+} |\nabla \varphi|^p \, dx + \int_{\mathbb{R}^{N-1}} |\varphi|^{q} \, dy \right)^{\frac{1}{p}}.
\]

Let $W$ be the completion of $C_0^\infty(\mathbb{R}^N_+)$ with respect to this norm and $W_r$ be the subspace of $W$ of axial functions, that is,

\[
W_r = \{ u \mid u \in W, u(x) = u(|y|, s), x = (y, s) \in \mathbb{R}^N_+ \}.
\]

The problem (1) has a variational structure given by the functional

\[
I(u) = \frac{1}{p} \left( \int_{\mathbb{R}^N_+} |\nabla u|^p \, dx + \int_{\mathbb{R}^{N-1}} |u|^p \, dy \right) - \frac{1}{p} \int_{\mathbb{R}^{N-1}} |u|^q \, dy - \frac{\mu}{q} \int_{\mathbb{R}^{N-1}} |u|^q \, dy, \; u \in W_r.
\]

Notice that $\bar{p} = \frac{(N-1)p}{N-p}$ is the critical exponent for the Sobolev imbedding from $W^{1,p}(\mathbb{R}^N_+)$ to $L^q(\mathbb{R}^{N-1})$, $p \leq q \leq \bar{p}$. Moreover, the imbedding from $W_r$ to $L^q(\mathbb{R}^{N-1})$ is continuous for $p \leq q \leq \bar{p}$ and compact only for $p < q < \bar{p}$ due to the dilations. Therefore, the Palais–Smale condition is not satisfied by the functional $I$ and the problem (1) lacks the necessary compactness property. Since the
pioneering work of Brezis and Nirenberg [1], significant progress has been made in recent decades for these kinds of problems lacking compactness. In particular, the authors of [2] dealt with the Laplacian equation with critical growth in the bounded domain

\[
\begin{aligned}
-\Delta u &= |u|^{2^* - 2}u + \lambda u, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^N, N \geq 3 \) is a regular bounded domain, and \( 2^* = \frac{2N}{N-2} \). While the authors of [3] considered the Laplacian equation with subcritical nonlinear term in the whole space \( \mathbb{R}^N \)

\[
\begin{aligned}
-\Delta u + a(x)u &= |u|^{q-2}u, \quad \text{in } \mathbb{R}^N, \\
u(x) &\to 0, \text{ as } |x| \to \infty,
\end{aligned}
\]

where \( 2 < q < 2^* \) and \( a(x) \) is the potential function. As to the \( p \)-Laplacian equation, there is a lot of significant work, whether in the field of ordinary differential equations [4–6] or partial differential equations [7–9], the authors of [7] considered

\[
\begin{aligned}
-\Delta_p u + a(x)u &= |u|^{p^* - 2}u + \lambda u, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( p^* = \frac{Np}{N-p} \). All of these authors found the solutions as limits of approximated equations with subcritical growth in bounded domains. The lack of compactness due to dilations (in the case (2) and (4)) and shifts (in the case (3)) does not allow for deducing that a sequence of approximate solutions must have a convergent subsequence, but the fact that they solve the approximated problems gives, with use of a local Pohožaev identity, some extra estimates which lead to a proof of desired compactness.

In the Existing literature, some researchers considered the existence of finite multiple solutions [10,11]. While the subcritical problems in bounded domains have infinitely many solutions. In order to show the existence of multiple solutions of the original problems, we need to check that multiple solutions of approximated problems do not converge to the same solution of the limit problems. This is hard work. In both [2,3], some estimates on the Morse index are employed, which has been used as one of the possible devices to distinguish the limit of the multiple approximate solutions by their original variational characterization. For general \( p \)-Laplacian equations, we have no information on the Morse index; therefore, the approach in this last step in [2,3] can not be extended in a straightforward way to problems involving the \( p \)-Laplacian operator. Here, we will use the truncation method, as we did in [8,9]. First, we consider some truncated problems, the solutions of which will be used as approximate solutions. By a concentration–compactness analysis, similar to that in [2,3,7], in particular with use of a local Pohožaev identity, the theorem of convergence of approximate solutions is proved. We show that, by a careful choice of the approximate nonlinear terms, the approximated problems and the original problem share more and more solutions, as the approximation parameter tends to zero. For more references, we refer the readers to [12–18].

Let us describe the truncation method in more details. Let \( \psi \in C^\infty_0(\mathbb{R}, [0,1]) \) be an even function such that \( \psi(t) = 1 \) for \( |t| \leq 1, \psi(t) = 0 \) for \( |t| \geq 2 \) and \( \psi \) is decreasing in \([1,2]\). Define the auxiliary functions for \( \Lambda \in (0,1), s \in \mathbb{R} \)

\[
\begin{aligned}
b_\Lambda(s) &= \psi(\lambda s), \\
m_\Lambda(s) &= \int_0^s b_\Lambda(\tau) \, d\tau \\
F_\Lambda(s) &= \frac{1}{p^*} |s|^p |m_\Lambda(s)|^{p-q}, \\
f_\Lambda(s) &= \frac{d}{ds} F_\Lambda(s).
\end{aligned}
\]
Instead of the original problem (1), we consider the truncated problem
\[
\begin{cases}
-\Delta_p u = 0, & \text{in } \mathbb{R}^N, \\
|\nabla u|^{p-2} \nabla u + |u|^{p-2} u = f_\lambda(u) + \mu|u|^{q-2} u, & \text{on } \mathbb{R}^{N-1}.
\end{cases}
\] (6)

In addition, the problem (6) has a variational structure given by the functional
\[
I_\lambda(u) = \frac{1}{p} \left( \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \int_{\mathbb{R}^{N-1}} |u|^p \, dy \right) - \int_{\mathbb{R}^{N-1}} F_\lambda(u) \, dy - \frac{\mu}{q} \int_{\mathbb{R}^{N-1}} |u|^q \, dy, \quad u \in W_r.
\]

Notice that the functional $I_\lambda, \lambda > 0$ is subcritical at the infinity and the imbedding from $W_r$ to $L^q(\mathbb{R}^{N-1}), p < q < \overline{p}$ is compact. Therefore, the functional $I_\lambda, \lambda > 0$ satisfies the Palais–Smale condition.

Here are our main results.

**Theorem 1.** Assume $\max\{p, \overline{p} - 1\} < q < \overline{p}$. Given $L > 0$, there exists $\nu = \nu(L)$, independent of $\lambda$, such that if $u \in W_r, D_{I_\lambda}(u) = 0$ and $I_\lambda(u) \leq L$, then it holds that
\[
|u(y)| \leq \frac{1}{\nu}, \quad \text{for } y \in \mathbb{R}^{N-1} = \partial \mathbb{R}^N.
\]

Consequently, if $\lambda < \nu$, then $u$ is a solution of the problem (1).

**Theorem 2.** Assume $\max\{p, \overline{p} - 1\} < q < \overline{p}$. Then, the problem (1) has infinitely many axial solutions.

Throughout the paper, we use the following notations: we use $\| \cdot \|$ and $| \cdot |_q$ to denote the norms of $W$ and $L^q(\mathbb{R}^{N-1})$, respectively, $\rightharpoonup$ and $\rightarrow$ to denote the weak and the strong convergence, respectively. In addition, we use the notations $B^+_\delta(x_0) = \{ x | x \in \mathbb{R}^N, |x - x_0| < \delta \}$, $D_\delta(y_0) = \{ y | y \in \mathbb{R}^{N-1}, |y - y_0| < \delta \}$, $B^+_\delta = B^+_\delta(0)$, $D_\delta = D_\delta(0)$.

The paper is organized as follows. In Section 2, we do the concentration–compactness analysis of the approximate solution sequence and prove the convergence Theorem 1. In Section 3, we construct a sequence of critical values of the truncated functionals by the symmetric mountain pass lemma. Finally, we prove the existence Theorem 2 by showing that approximated solutions are also solutions of the original problem for a sufficiently small parameter.

2. Concentration–Compactness Analysis

2.1. The Profile Decomposition

In this section, we analyze the concentration behavior for the solutions of the problem (6) as $\lambda \rightarrow 0$ and prove Theorem 1. First, we list the properties of the auxiliary functions, defined in (5) in the following lemma.

**Lemma 1.** It holds that for $s \in \mathbb{R}$

(a) $0 \leq b_\lambda(s) \leq 1$.
(b) $\text{sm}\. \lambda(s) \geq 0, \quad 0 \leq \frac{sb_\lambda(s)}{m_\lambda(s)} \leq 1$.
(c) $m_\lambda(s) = s$ for $|s| \leq \frac{1}{\lambda}$.
(d) $\min\{|s|, \frac{1}{\lambda}\} \leq |m_\lambda(s)| \leq \min\{|s|, \frac{2}{\lambda}\}$.
(e) $|f_\lambda(s)| \leq |s|^{q-1}|m_\lambda(s)|^{\overline{p} - q} \leq |s|^{\overline{p} - 1}$.
(f) $\frac{1}{\overline{p}} f_\lambda(s) - F_\lambda(s) = \left( \frac{1}{\overline{p}} - \frac{1}{p} \right) |s|^{q+1}|m_\lambda(s)|^{\overline{p} - q - 1} b_\lambda(s) \geq 0$.
(g) $F_\lambda(s) - \frac{1}{p} f_\lambda(s) = \left( \frac{1}{p} - \frac{1}{q} \right) |s|^{q}|m_\lambda(s)|^{\overline{p} - q - 1} \left( 1 - \frac{sb_\lambda(s)}{m_\lambda(s)} \right) \geq 0$. 
Proof. The proof is straightforward. We verify (e)-(g). By the definition of $F_{\lambda}$ and $f_{\lambda}$, we have
\begin{equation}
    f_{\lambda}(s) = \frac{d}{ds} F_{\lambda}(s) = \frac{q}{p} |s|^{q-2} s |m_{\lambda}(s)|^{p-2} + \frac{p-q}{p} |s|^q |m_{\lambda}(s)|^{p-q-2} m_{\lambda}(s) b_{\lambda}(s). 
\end{equation}
(7)
(f) and (g) follow from (7), and (e) follows from (7) and (a), (d) of this lemma. \qed

Lemma 2. Let $\lambda_n \geq 0$, $u_n \in W_r$ such that $DI_{\lambda_n}(u_n) = 0$, $I_{\lambda_n}(u_n) \leq L$. Then, $\{u_n\}$ is bounded in $W_r$.

Proof. By Lemma 1 (f), we have
\begin{equation}
    L \geq I_{\lambda_n}(u_n) = I_{\lambda_n}(u_n) - \frac{1}{q} (DI_{\lambda_n}(u_n), u_n)
    = \left(1 - \frac{1}{q}\right) \left( \int_{\mathbb{R}^n} |\nabla u_n|^p \, dx + \int_{\mathbb{R}^{n-1}} |u_n|^p \, dy \right) + \int_{\mathbb{R}^{n-1}} \left( \frac{1}{q} u_n f_{\lambda_n}(u_n) - F_{\lambda_n}(u_n) \right) \, dx
    \geq \left(1 - \frac{1}{q}\right) \left( \int_{\mathbb{R}^n} |\nabla u_n|^p \, dx + \int_{\mathbb{R}^{n-1}} |u_n|^p \, dy \right) = \left(1 - \frac{1}{q}\right) \|u_n\|^p.
\end{equation}
Hence, $\{u_n\}$ is bounded in $W_r$. \qed

Let $D = D^p(\mathbb{R}^N_+)$ be the completion of $C^\infty_0(\mathbb{R}^N_+)$ with respect to the norm
\begin{equation}
    \|\varphi\|_D = \left( \int_{\mathbb{R}^n} |\nabla \varphi|^p \, dx \right)^{\frac{1}{p}}
\end{equation}
and $D_r$ be the subspace of $D$ of axial functions,
\begin{equation}
    D_r = \{ u \mid u \in D, u(x) = u(|y|, s), x = (y, s) \in \overline{\mathbb{R}^N_+} = \mathbb{R}^{N-1} \times [0, \infty) \}.
\end{equation}
Let $D$ be the dilation group
\begin{equation}
    D = \{ g_\sigma | g_\sigma u(x) = \sigma^{\frac{N-p}{p}} u(\sigma x), x \in \overline{\mathbb{R}^N_+}, \sigma > 0 \}.
\end{equation}
Notice that the operator $g_\sigma$ of $D$ is an isometry in both $D$ and $L^p(\mathbb{R}^{N-1})$. The imbedding from $D_r$ to $L^p(\mathbb{R}^{N-1})$ is compact with respect to the group $D$ that is a sequence $\{u_n\}$ of $D_r$, satisfying $g_\sigma u_n \to 0 \quad$ in $D_r$ for any sequence $\{g_\sigma\}$ of $D$, denoted by $u_n \xrightarrow{D} 0 \quad$ in $D_r$, must converge to zero in $L^p(\mathbb{R}^{N-1})$.

We refer to [19,20] for general concepts of compactness and the profile decomposition and relevant results. For reader’s convenience, we consider the compactness of the imbedding from $D_r$ to $L^p(\mathbb{R}^{N-1})$ with respect to the dilation group $D$.

Lemma 3. Assume $\lambda_n > 0$, $\lambda_n \to 0$ as $n \to \infty$, $u_n \in W_r$, $DI_{\lambda_n}(u_n) = 0$ and $I_{\lambda_n}(u_n) \leq L$. Assume that the profile decomposition (9) holds. Then,
(1) \( v = |u| \) satisfies the inequality
\[
\int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \varphi \, dx + \int_{\mathbb{R}^{N-1}} \sigma^{p-1} \varphi \, dy \leq \int_{\mathbb{R}^{N-1}} \sigma^{p-1} \varphi \, dy + \mu \int_{\mathbb{R}^{N-1}} \varphi^{\sigma^{p-1}} \, dy,
\]
for \( \varphi \geq 0, \varphi \in W_r. \) Consequently, for some \( c > 1, \)
\[
\int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \varphi \, dx \leq c \int_{\mathbb{R}^{N-1}} \varphi^{\sigma^{p-1}} \, dy, \quad \text{for } \varphi \geq 0, \varphi \in W_r.
\]

(2) \( V_k = |U_k| \) satisfies the inequality
\[
\int_{\mathbb{R}^N} |\nabla V_k|^{p-2} \nabla V_k \varphi \, dx \leq \int_{\mathbb{R}^{N-1}} V_k^{p-1} \varphi \, dy \text{ for } \varphi \geq 0, \varphi \in \mathcal{D}_r.
\]

**Proof.** We prove the conclusion for the function \( V_k. \) \( u_n \) satisfies the equation in the weak form
\[
\int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \varphi \, dx + \int_{\mathbb{R}^{N-1}} |u_n|^{p-2} u_n \varphi \, dy = \int_{\mathbb{R}^{N-1}} f(u_n) \varphi \, dy + \mu \int_{\mathbb{R}^{N-1}} |u_n|^{p-2} u_n \varphi \, dy, \quad \varphi \in W_r.
\]
Denote \( \tilde{u}_n = \frac{S_{n-k}^{-1}}{\lambda_n^{p-1}} u_n. \) For \( \varphi \in W_r, \) take \( g_{\alpha_k} \varphi \) as a test function in (13). By a variable change, we obtain
\[
\int_{\mathbb{R}^N} |\nabla \tilde{u}_n|^{p-2} \nabla \tilde{u}_n \varphi \, dx + \int_{\mathbb{R}^{N-1}} |\tilde{u}_n|^{p-2} \tilde{u}_n \varphi \, dy = \int_{\mathbb{R}^{N-1}} f_{\lambda_n} (\tilde{u}_n) \varphi \, dy + \mu \int_{\mathbb{R}^{N-1}} |\tilde{u}_n|^{p-2} \tilde{u}_n \varphi \, dy, \quad \varphi \in W_r
\]
where \( \lambda_n = \lambda_n^{p-1}/\sigma_{n,k}^{-p}. \) In the above, we have used the fact that
\[
v^{-1} f_{\lambda_n} (\varphi(s)) = f_{\lambda_n} (s), \quad \lambda, \nu > 0, \quad s \in \mathbb{R}
\]
which can be proved by the very definition of the function \( f_{\lambda_n}. \)

Since \( \int_{\mathbb{R}^{N-1}} |\tilde{u}_n|^p \, dy = \int_{\mathbb{R}^{N-1}} |u_n|^p \, dy \) is bounded and \( \tilde{u}_n \) is axial, for any \( y \in \mathbb{R}^{N-1} \setminus \{0\} \)
\[
\lim_{s \to 0} \limsup_{n \to \infty} \int_{\mathcal{D}_y} |\tilde{u}_n|^p \, dy = 0.
\]

Choose \( \delta = \delta (y), \) independent of \( n, \) such that
\[
\int_{\mathcal{D}_y} |\tilde{u}_n|^p \, dy \leq \frac{1}{2} S_{n-1}^p,
\]
where \( S_{n-1} \) is the Sobolev constant of the imbedding \( \mathcal{D} \hookrightarrow L^p(\mathbb{R}^{N-1}). \) By Lemma A4, \( \tilde{u}_n \) is uniformly bounded in \( D_{2}(y). \) Consequently, by Equation (14) and the following elementary inequality (15), \( \tilde{u}_n \) converges in \( W^{1,p}(B_{\delta}(y)) \) and in \( W^{1,p}_{0}(\mathbb{R}^N). \) The following inequality (15) is useful for problems involving the \( p \)-Laplacian operator [21]. There exists a constant \( c_p \) such that, for \( \xi, \eta \in \mathbb{R}^N, \)
\[
(|\xi|^{p-2} \xi - |\eta|^{p-2} \eta, \xi - \eta) \geq c_p |\xi - \eta|^p, \quad \text{if } p \geq 2,
\]
\[
(|\xi|^{p-2} \xi - |\eta|^{p-2} \eta, \xi - \eta) \geq c_p |\xi - \eta| \cdot (|\xi|^p + |\eta|^p)^{\frac{p-2}{p}}, \quad \text{if } 1 < p < 2.
\]
Let $\tilde{v}_n = |\tilde{u}_n|$ converge to $V_k = |U_k|$ in $W^{1,p}_{loc}(\mathbb{R}^N)$ and satisfy the inequality
\[
\int_{\mathbb{R}^N} |\nabla \tilde{v}_n|^{p-2} \nabla \tilde{v}_n \nabla \varphi \, dx + \sigma_{n,k}^{-(p-1)} \int_{\mathbb{R}^N} \tilde{v}_n^{p-1} \varphi \, dy 
\leq \int_{\mathbb{R}^N} \tilde{v}_n^{p-1} \varphi \, dy + \mu n^{-(N-1)} \int_{\mathbb{R}^N} \tilde{v}_n^{p-1} \varphi \, dy
\] (16)
for $\varphi \in W_r$, $\varphi \geq 0$. Assume $\varphi \in C_0^\infty(\mathbb{R}^N \setminus \{0\}) \cap D_r$. Taking the limit $n \to \infty$ in (16), we obtain
\[
\int_{\mathbb{R}^N} |\nabla V_k|^{p-2} \nabla V_k \nabla \varphi \, dx \leq \int_{\mathbb{R}^N} V_k^{p-1} \varphi \, dy.
\] (17)
By a density argument, (17) holds for $\varphi \in D$, $\varphi \geq 0$.

Lemma 4. The index set $\Lambda$ in the profile decomposition (9) is finite.

Proof. By Lemma 3, $V_k$ satisfies the inequality (12). Choose $\varphi = V_k$ in (12). By the Sobolev imbedding theorem
\[
\int_{\mathbb{R}^N} |\nabla V_k|^p \, dx \leq \int_{\mathbb{R}^N} V_k^p \, dy \leq \left( S^{-1}_p \int_{\mathbb{R}^N} |\nabla V_k|^p \, dx \right)^{\frac{p}{p}}
\]
hence
\[
\int_{\mathbb{R}^N} |\nabla U_k|^p \, dx = \int_{\mathbb{R}^N} |\nabla V_k|^p \, dx \geq S^{-\frac{N-1}{p}}_p.
\]
By the property (3) of the decomposition (9), $\Lambda$ is a finite set. □

2.2. Safe Regions

Assume the profile decomposition (9) with a finite index set $\Lambda$. Denote
\[
\sigma_n = \min\{\sigma_{n,k} \mid k \in \Lambda\}
\]
and define the so-called safe regions [2]
\[
\begin{align*}
A^i_n &= \{x \mid x \in \mathbb{R}^N, i \sigma_n^{-\frac{1}{p}} < |x| < (7 - i) \sigma_n^{-\frac{1}{p}}\}, \\
T^i_n &= \{y \mid y \in \mathbb{R}^{N-1}, i \sigma_n^{-\frac{1}{p}} < |y| < (7 - i) \sigma_n^{-\frac{1}{p}}\}, \quad i = 1, 2, 3.
\end{align*}
\] (18)
For these regions, we have a good estimate.

Proposition 1. There exists a constant $c$, independent of $n$, such that
\[
|u_n(x)| \leq c \quad \text{for } x \in A^2_n \cup T^2_n.
\]

Corollary 1. There exists a constant $c$, independent of $n$, such that
\[
\int_{A^3_n} |\nabla u_n|^p \, dx \leq c.
\]
In order to prove these estimates, we start with the following definition.

Definition 1. Suppose $1 \leq p_2 < \overline{p} < p_1$, $\sigma > 1$ and $\alpha > 0$. Consider the following system of inequality
Assume Proposition 2. Hence, for \( V \)

By Lemma 3, profile decomposition

\[
|u|_{p_1,p_2,\sigma} = \inf\{\alpha | \text{there exist } u_1, u_2 \text{ such that } |u| \leq u_1 + u_2 \text{ and (19) holds}\}.
\]

**Proposition 2.** Assume \( \lambda_\alpha > 0, \lambda_\alpha \to 0 \) as \( N \to \infty, u_n \in W_r, D1_{\lambda_\alpha} (u_n) = 0 \) and \( I_{\lambda_\alpha} (u_n) \leq L \). Assume the profile decomposition (9) holds. Denote \( \sigma_n = \min \{\sigma_{n,k} | k \in \Lambda\} \). Then, for any \( p_1, p_2 \) satisfying

\[
(1 - \frac{1}{p}) \varphi < p_2 < \varphi < p_1,
\]

there exists a constant \( c = c(p_1, p_2) \) such that

\[
|u_n|_{p_1,p_2,\sigma_n} \leq c.
\]

**Proof.** By Lemma 3, \( v = |u| \) satisfies the inequality

\[
\int_{\mathbb{R}^N_+} |\nabla v|^{p-2} \nabla v \nabla \varphi dx \leq c \int_{\mathbb{R}^N-1} v^{p-1} \varphi dy, \ \varphi \geq 0, \varphi \in W_r.
\]

By Lemma A4, \( u \in L^\infty (\mathbb{R}^N-1) \), hence for \( p_1 > \varphi \),

\[
|u|_{p_1} \leq |u|_{\infty}^{1-\frac{p}{p_1}} |u|_{\infty}^{\frac{p}{p_1}} \leq c.
\]

By Lemma 3, \( V_k = |U_k| \) satisfies the inequality

\[
\int_{\mathbb{R}^N_+} |\nabla v|^{p-2} \nabla v \nabla \varphi dx \leq \int_{\mathbb{R}^N-1} v^{p-1} \varphi dy, \ \varphi \geq 0, \varphi \in W_r.
\]

By Theorem 2.2 of [22], there exists a constant \( c \) such that

\[
|U_k(y)| = V_k(y) \leq c \left(1 + |y|^{\frac{p}{p-1}}\right)^{-\frac{N-p}{p}}, \ y \in \mathbb{R}^{N-1}.
\]

Hence, for \( (1 - \frac{1}{p}) \varphi < p_2 < \varphi \), we have

\[
|g_{\sigma_{n,k}} U_k|_{p_2} \leq \left( \int_{\mathbb{R}^N-1} (\sigma_{n,k}^{\frac{N-p}{p}} (1 + |\sigma_{n,k} y|^{\frac{p}{p-1}})^{-\frac{N-p}{p}} p_2 dy \right)^{\frac{1}{p_2}}
\]

\[
= cc_{n,k}^{\frac{N-p}{p} - \frac{N-1}{p_2}} \left( \int_{\mathbb{R}^N-1} (1 + |y|^{\frac{p}{p-1}})^{-\frac{N-p}{p}} p_2 dy \right)^{\frac{1}{p_2}}
\]

\[
\leq cc_{n,k}^{\frac{N-1}{p} - \frac{N-1}{p_2}} \leq cc_{n,k}^{\frac{N-1}{2} - \frac{N-1}{2}}.
\]

By (20) and (21), we have

\[
|u|_{p_1,p_2,\sigma_n} + \sum_{k \in \Lambda} |g_{\sigma_{n,k}} U_k|_{p_1,p_2,\sigma_n} \leq c.
\]

Define \( w, W_k, R \in D_r \) by

\[
-\Delta_p w = 0, \ \text{in } \mathbb{R}^N, \ 
|\nabla w|^{p-2} = w^{p-1}, \ \text{on } \mathbb{R}^{N-1},
\]

\[
\left\{ \begin{array}{ll}
-\Delta_p w = 0, & \text{in } \mathbb{R}^N, \\
|\nabla w|^{p-2} = w^{p-1}, & \text{on } \mathbb{R}^{N-1},
\end{array} \right.
\]
\[
\begin{aligned}
\begin{cases}
- \Delta_p W_k = 0, & \text{in } \mathbb{R}_+^N, \\
|\nabla W_k|^{p-2} \frac{\partial W_k}{\partial n} = W_k^{\gamma N-1}, & \text{on } \mathbb{R}^{N-1}.
\end{cases}
\end{aligned}
\] (24)

By the Wolff potential estimate([2], Corollary 4.13), we have
\[
|u_n| = v_n \leq c \left( w + \sum_{k \in \Lambda} W_k + R \right).
\] (26)

By Lemma A3, for \((1 - \frac{1}{p})p < p_2 < p < p_1\), we have
\[
|\omega|_{p_1, p_2, \sigma_n} \leq c |\omega|_{p, p_1, p_2, \sigma_n}^{\frac{p-1}{p}} \cdot |\omega|_{p_1, p_2, \sigma_n}
\] (27)
\[
\leq c |u|_{p_1, p_2, \sigma_n}^{\frac{p-1}{p}} \cdot |u|_{p_1, p_2, \sigma_n} \leq c |u|_{p_1, p_2, \sigma_n} \leq 2
\]
\[
|\nabla W_k|_{p_1, p_2, \sigma_n} \leq c |g_{\epsilon_k} U_k|^\frac{p}{p-1} \cdot |g_{\epsilon_k} U_k|_{p_1, p_2, \sigma_n}
\] (28)

and
\[
|\nabla \sigma_n|_{p_1, p_2, \sigma_n} \leq c |\nabla \sigma_n|^\frac{p}{p-1} \cdot |\nabla \sigma_n|_{p_1, p_2, \sigma_n}
\]
\[
= o(1) |\nabla \sigma_n|_{p_1, p_2, \sigma_n}
\]
\[
= o(1) \left( |u_n|_{p_1, p_2, \sigma_n} + |u|_{p_1, p_2, \sigma_n} + \sum_{k \in \Lambda} |g_{\epsilon_k} U_k|_{p_1, p_2, \sigma_n} \right)
\] (29)

We have
\[
|u_n|_{p_1, p_2, \sigma_n} \leq c \left( |\omega|_{p_1, p_2, \sigma_n} + \sum_{k \in \Lambda} |W_k|_{p_1, p_2, \sigma_n} + |\nabla \sigma_n|_{p_1, p_2, \sigma_n} \right)
\]
\[
\leq c + o(1)|u_n|_{p_1, p_2, \sigma_n}
\]

and
\[
|u_n|_{p_1, p_2, \sigma_n} \leq c.
\]
\]

Lemma 5. Assume \(\lambda_n > 0\), \(u_n \in W_r, DI_{\lambda_n}(u_n) \leq L\). Assume the profile decomposition (9) holds. Then, for \(\gamma \in \left( p - 1, \frac{(p-1)p}{p-1+p} \right)\) there exists \(c = c(\gamma), \text{ independent of } n\), such that
\[
\left( \gamma^{-N} \int_{B^*_r} |u_n|^\gamma dx + \gamma^{-N+1} \int_{D_r} |u_n|^\gamma dy \right)^{\frac{1}{\gamma}} \leq c \quad \text{for } \gamma \geq \sigma_n^{\frac{1}{p}}.
\]

Proof. By Lemma A6 for \(\gamma < 1\), we have
\[
\left( \gamma^{-N} \int_{B_1^c} |u_n|^\gamma \, dx + \gamma^{-N+1} \int_{B_1} |u_n|^\gamma \, dy \right)^{\frac{1}{\gamma}} \\
\leq c \left( \int_{B_1^c} |u_n|^\gamma \, dx + \int_{D_1} |u_n|^\gamma \, dy \right)^{\frac{1}{\gamma}} + c \int_r^1 \left( \int_{D_1} |u_n|^{\gamma-1} \, dy \right)^{\frac{1}{\gamma-1}} \frac{dt}{t^{1+\frac{N-\gamma}{\gamma}}}.
\]

(30)

By Proposition 2, we have \(|u_n|_{p_1,p_2,o_n} \leq c\) for any \(p_1, p_2\) such that \((1 - \frac{1}{\gamma})p < p_2 < p < p_1\). Let \(p_2 = \frac{p}{2} - 1, p_1 = Np\). Choose \(v_1, v_2\) such that \(|u_n| \leq v_1 + v_2, \frac{|v_1|_{p_1}}{c} \leq |v_2|_{p_2} \leq c \sigma_n^{\frac{N-p}{p-1}}\). Then,

\[
\int_r^1 \left( \int_{D_1} v_1^{\gamma-1} \, dt \right)^{\frac{1}{\gamma-1}} \frac{dt}{t^{1+\frac{N-\gamma}{\gamma}}} \\
\leq \int_r^1 \left( \int_{D_1} N^{\frac{\gamma-1}{p-1}} \, dy \right)^{\frac{N-1}{p-1}} \left( \int_{D_1} v_1 \, dy \right)^{\frac{1}{p-1}} \frac{dt}{t^{1+\frac{N-\gamma}{\gamma}}}.
\]

(31)

and

\[
\int_r^1 \left( \int_{D_1} v_1^{\gamma-1} \, dt \right)^{\frac{1}{\gamma-1}} \frac{dt}{t^{1+\frac{N-\gamma}{\gamma}}} \\
\leq \int_r^1 \sigma_n^{\frac{N-p}{p(\gamma-1)}} \left( \int_{D_1} v_1 \, dy \right)^{\frac{1}{\gamma-1}} \frac{dt}{t^{1+\frac{N-\gamma}{\gamma}}}.
\]

(32)

provided \(r \geq c \sigma_n^{-\frac{1}{\gamma}}\). Hence,

\[
\left( \gamma^{-N} \int_{B_1^c} |u_n|^\gamma \, dx + \gamma^{-N+1} \int_{B_1} |u_n|^\gamma \, dy \right)^{\frac{1}{\gamma}} \\
\leq c + c \int_r^1 \left( \int_{D_1} |u_n|^{\gamma-1} \, dy \right)^{\frac{1}{\gamma-1}} \frac{dt}{t^{1+\frac{N-\gamma}{\gamma}}}.
\]

\[
\leq c + c \int_r^1 \left( \int_{D_1} |v_1|^{\gamma-1} \, dy \right)^{\frac{1}{\gamma-1}} \frac{dt}{t^{1+\frac{N-\gamma}{\gamma}}} + c \int_r^1 \left( \int_{D_1} |v_2|^{\gamma-1} \, dy \right)^{\frac{1}{\gamma-1}} \frac{dt}{t^{1+\frac{N-\gamma}{\gamma}}} \leq c \quad \text{for } r \geq c \sigma_n^{-\frac{1}{\gamma}}.
\]

\(\square\)

**Proof of Proposition 1 and Corollary 1.** Let \(w_n(x) = \sigma_n^{\frac{N-p}{p-1}} |u_n|((\sigma_n^{-\frac{1}{\gamma}} x), w_n\) satisfy

\[
\begin{cases}
-\Delta_p w_n \leq 0, & \text{in } \mathbb{R}_{+}^N, \\
|\nabla w_n|^{p-2} \frac{\partial w_n}{\partial t} \leq cw_n^{\gamma-1}, & \text{on } \mathbb{R}^{N-1}.
\end{cases}
\]

By the profile decomposition (9), we have
We complete the proof of Proposition 1. To prove Corollary 1, we choose a function \( \phi \) that satisfies Lemma 6. 

\begin{equation}
\int_{1 \leq |y| \leq 6} |w_n|^p dy = \int_{T_3^3} |u_n|^p dy 
\leq \int_{T_3^3} |u|^p dy + c \sum_{k \in A} \int_{T_3^3} |g_{n,k} U_k|^p dy + c \int_{T_3^3} |r_n|^p dy 
\leq c \sigma_n^{- \frac{N-1}{p}} + c \sum_{k \in A} \int_{|y| \geq \sigma_n} \left( \frac{N-p}{\sigma_n^p} U_k \sigma_n^p y \right)^p dy + o(1) 
\leq o(1) + c \sum_{k \in A} \int_{|y| \geq \sigma_n} \frac{1}{p} \left( 1 + |y|^p \right)^{- \frac{N-p}{p}} dy = o(1). 
\end{equation} 

By Lemma A4 and Lemma 5, for \( 2 \leq x \leq 5, x \in \mathbb{R}_+^N \cup \mathbb{R}^{N-1} \), we have 

\[ w_n(x) \leq c \left( \int_{1 \leq |x| \leq 6} w_n^p(x) dx + \int_{1 \leq |x| \leq 6} w_n^p(y) dy \right)^{\frac{1}{p}} = c \sigma_n^{- \frac{N-p}{p}} \int_{A_3^n} |u_n|^{\gamma} dx + \sigma_n^{- \frac{N-1}{p}} \int_{T_3^3} |u_n|^p dy \leq c \sigma_n^{- \frac{N-p}{p}}. \]

Hence, 

\[ |u_n(x)| \leq c \sigma_n^{- \frac{1}{p}} w_n(x) \leq c \sigma_n^{- \frac{1}{p}} \sigma_n^\frac{1}{p} x \leq c \quad \text{for } x \in A_3^n \cup T_3^3. \]

We complete the proof of Proposition 1. To prove Corollary 1, we choose a function \( \varphi \in C_0^\infty(\mathbb{R}^N) \) such that \( \varphi(x) = 1 \) for \( x \in A_3^n \cup T_3^3 \) and \( \varphi(x) = 0 \) for \( x \not\in A_3^n \cup T_3^3 \) and \( |\nabla \varphi| \leq 2c_0^{- \frac{1}{p}} \). Testing the Equation (13) by \( \varphi u_n \), we obtain 

\[ \int_{A_3^n} |\nabla u_n|^p \varphi^p dx = \int_{A_3^n} |\nabla u_n|^{p-2} \nabla u_n \varphi dx - p \int_{A_3^n} |\nabla u_n|^{p-2} \nabla u_n u_n \varphi^{p-1} \nabla \varphi dx \]

\[ \leq \int_{T_3^3} (-|u_n|^p + |u_n|^{\gamma} + \mu |u_n|^{q}) \varphi^p dx + \frac{1}{2} \int_{A_3^n} |\nabla u_n|^p \varphi^p dx + c \int_{A_3^n} |u_n|^p |\nabla \varphi|^p dx. \]

Hence, 

\[ \int_{A_3^n} |\nabla u_n|^p dx \leq \int_{A_3^n} |\nabla u_n|^p \varphi^p dx 
\leq c \int_{T_3^3} |u_n|^p dx + c \int_{A_3^n} |u_n|^p |\nabla \varphi|^p dx 
\leq c \sigma_n^{- \frac{1}{p}(N-1)} + c \sigma_n^{- \frac{N-p}{p} + \frac{1}{p} - \frac{1}{p}} \leq c \sigma_n^{- \frac{N-p}{p}}. \]

\( \square \)

2.3. Požvažovec Identity

In the remainder of this section, following the idea of [2,3], we apply the local Požvažovec identity to prove the convergence Theorem 1.

Lemma 6. (Local Požvažovec identity) Assume that \( u \in W \) satisfies the equation 

\[ \begin{cases} 
-\Delta_p u = 0, & \text{in } \mathbb{R}_+^N, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} + |u|^{p-2} u = f_\lambda(u) + \mu |u|^{q-2} u, & \text{on } \mathbb{R}^{N-1}. 
\end{cases} \]

Let \( \varphi \in C_0^\infty(\mathbb{R}^N) \), then
We estimate (38). Notice that the integrals of the right-hand side of (38) are taken over the domains $T^3$ and $T^3$. By Proposition 1 and Corollary 1, we know

$$
- \frac{p - 1}{p} \int_{R^{N - 1}} |u|^p \varphi dy + \int_{R^{N - 1}} ((N - 1)F_\lambda(u) - \frac{N - p}{p} f_\lambda(u)u) \varphi dy \\
+ \left( \frac{N - 1}{q} - \frac{N - p}{p} \right) \mu \int_{R^{N - 1}} |u|^q \varphi dy \\
= \frac{1}{p} \int_{R^N} |\nabla u|^p(x, \nabla \varphi) dx - \int_{R^N} |\nabla u|^{p - 2}(\nabla u, x)(\nabla u, \nabla \varphi) dx - \frac{N - p}{p} \int_{R^N} |\nabla u|^{p - 2} u(\nabla u, \nabla \varphi) dx \\
+ \int_{R^{N - 1}} \left( \frac{1}{p} |u|^p - F_\lambda(u) - \frac{\mu}{q} |u|^q \right) (y, \nabla y \varphi) dy.
$$

**Proof.** Multiplying (34) by $(x, \nabla u) \varphi$ and integration by parts, we obtain

$$
(N - 1) \int_{R^{N - 1}} \left( - \frac{1}{p} |u|^p + F_\lambda(u) + \frac{\mu}{q} |u|^q \right) \varphi dy \\
= \frac{N - p}{p} \int_{R^N} |\nabla u|^p \varphi dx - \int_{R^N} |\nabla u|^{p - 2}(\nabla u, x)(\nabla u, \nabla \varphi) dx + \frac{1}{p} \int_{R^N} |\nabla u|^p(x, \nabla \varphi) dx \\
+ \int_{R^{N - 1}} \left( \frac{1}{p} |u|^p - F_\lambda(u) - \frac{\mu}{q} |u|^q \right) (y, \nabla y \varphi) dy.
$$

Multiplying (34) by $u \varphi$ and integration by parts, we obtain

$$
\int_{R^{N - 1}} (|u|^p - F_\lambda(u)u - \mu |u|^q) \varphi dy \\
= - \int_{R^N} |\nabla u|^p \varphi dx - \int_{R^{N - 1}} |\nabla u|^{p - 2} u(\nabla u, \nabla \varphi) dx.
$$

Eliminating the term $\int_{R^N} |\nabla u|^p \varphi dx$, we obtain the local Pohozaev identity. 

**Proof of the convergence Theorem 1.** We apply the local Pohozaev identity to the function $u_n$. Let

$$
B_n^+ = \{ x \mid x \in R^N, |x| < 4c_n^{\frac{1}{2}} \},
$$

$$
D_n = \{ y \mid y \in R^{N - 1}, |y| < 4c_n^{\frac{1}{2}} \}.
$$

Choose $\varphi \in C_0^\infty([R^N, [0, 1])]$ such that $\varphi(x) = 1$ for $|x| \leq 3c_n^{\frac{1}{2}}, \varphi(x) = 0$ for $|x| \geq 4c_n^{\frac{1}{2}}$ and $|\nabla \varphi| \leq 2c_n^{\frac{1}{2}}$. By Lemma 6, the local Pohozaev identity, we have

$$
- \frac{p - 1}{p} \int_{D_n} |u_n|^p \varphi dy + \int_{D_n} ((N - 1)F_\lambda(u_n) - \frac{N - p}{p} f_\lambda(u_n)u_n) \varphi dy \\
+ \left( \frac{N - 1}{q} - \frac{N - p}{p} \right) \mu \int_{D_n} |u_n|^q \varphi dy \\
= \frac{1}{p} \int_{B_n^+} |\nabla u_n|^p(x, \nabla \varphi) dx - \int_{B_n^+} |\nabla u_n|^{p - 2}(\nabla u_n, x)(\nabla u_n, \nabla \varphi) dx \\
+ \int_{D_n} \left( \frac{1}{p} |u_n|^p - F_\lambda(u_n) - \frac{\mu}{q} |u_n|^q \right) (y, \nabla y \varphi) dy.
$$

We estimate (38). Notice that the integrals of the right-hand side of (38) are taken over the domains $A_n^3$ and $T_n^3$. By Proposition 1 and Corollary 1, we know
Without loss of generality, assume we have

\[ \int A_0^\lambda |\nabla u_n|^p |x| |\nabla \varphi| \, dx + c \int A_0^\lambda |\nabla u_n|^{p-1} |\nabla \varphi| \, dx + c \int_{I_0^\lambda} |y| |\nabla_y \varphi| \, dy \]

\[ \leq c\sigma_n^{-\frac{N-1}{p}} + c\sigma_n^{-\frac{N-1}{p}} \leq c\sigma_n^{-\frac{N-1}{p}}. \]  

(39)

On the other hand, by Lemma 1 (7), we have

\[ \text{LHS of (38)} \geq \left( \frac{N-1}{q} - \frac{N-p}{p} \right) \mu \int_{D_n} |u_n|^q \varphi \, dy - \frac{1}{p} \int_{D_n} |u_n|^p \varphi \, dy \]

\[ \geq \frac{1}{2} \left( \frac{N-1}{q} - \frac{N-p}{p} \right) \int_{D_n} |u_n|^q \varphi \, dy - c \int_{D_n} |u_n|^q \, dy \]

\[ \geq c \int_{D_n} |u_n|^q \varphi \, dy - c\sigma_n^{-\frac{N-1}{p}}. \]  

(40)

Without loss of generality, assume \( \sigma_{n,1} = \sigma_n = \min \{ \sigma_{n,k} \mid k \in \Lambda \} \). Choose \( L \) large enough such that

\[ \int_{D_L} |U_1|^q \, dy = \beta > 0 \]

where \( D_L = \{ y \mid y \in \mathbb{R}^{N-1}, |y| < L \} \). Since \( \tilde{u}_n = \sigma_n^{-\frac{N-p}{p}} u_n(\sigma_n^{-1}) \) weakly converges to \( U_1 \) in \( D_L \), we have

\[ \int_{D_n} |u_n|^q \varphi \, dy \geq \int_{|y| \geq L\sigma_n^{-1}} |u_n|^q \, dy \]

\[ = \sigma_n^{-\frac{N-p}{p}q-(N-1)} \int_{D_L} |\tilde{u}_n|^q \, dy \]

\[ \sim \sigma_n^{-\frac{N-p}{p}q-(N-1)} \cdot \beta \]  

(41)

we arrive at a contradiction

\[ \sigma_n^{-\frac{N-p}{p}q-(N-1)} \leq \beta \]

for \( \sigma_n \) large enough, since \( q + 1 > \frac{(N-1)p}{N-p} \). The index set \( \Lambda \) in the profile decomposition (9) must be empty, and (9) reduces to

\[ u_n = u + r_n, \quad r_n \to 0 \text{ in } L^\gamma(\mathbb{R}^{N-1}) \text{ as } n \to \infty. \]  

(42)

That is, \( u_n \to u \) in \( L^\gamma(\mathbb{R}^{N-1}) \). By Lemma A4, \( u_n \) is uniformly bounded, and there exists \( \nu = \nu(L) \) such that

\[ |u_n(y)| \leq \frac{1}{\nu} \text{ for } y \in \mathbb{R}^{N-1}. \]

\[ \square \]

3. Existence of Multiple Solutions

We define a sequence of critical values of the truncated functional \( I_\lambda \), \( \lambda > 0 \) by the symmetric mountain pass lemma due to Ambrosetti and Rabinowitz, and prove that the corresponding critical points are solutions of the original problem (1) for sufficiently small parameter \( \lambda \).

**Definition 2.** Define the critical values of \( I_\lambda \),

\[ c_k(\lambda) = \inf_{A \in \Gamma_k} \sup_{u \in A} I_\lambda(u), \quad k = 1, 2, \ldots \]

where

\[ \Gamma_k = \{ A \mid A \subset W_r, A \text{ compact}, -A = A, \gamma(A \cap \sigma^{-1}(S_\rho)) \geq k \} \]

\[ G = \{ \sigma \mid \sigma \in C(W_r, W_r), \sigma \text{ odd}, \sigma(u) = u \text{ if } I_1(u) \leq 0 \} \]
Lemma 7. The functional $I_{\lambda}$, $\lambda > 0$ satisfies the Palais–Smale condition.

Proof. Let $u_n$ be a Palais–Smale sequence of $I_{\lambda}$, and we have

$$I_{\lambda}(u_n) - \frac{1}{q} \langle DI_{\lambda}(u_n), u_n \rangle = \left( \frac{1}{p} - \frac{1}{q} \right) \left( \int_{\mathbb{R}^N} |\nabla u_n|^p dx + \int_{\mathbb{R}^{N-1}} |u_n|^p dy \right) + \int_{\mathbb{R}^{N-1}} \left( \frac{1}{q} f_\lambda(u_n) u_n - F_\lambda(u_n) \right) dy \geq \left( \frac{1}{p} - \frac{1}{q} \right) \|u_n\|_p^p$$

hence $u_n$ is bounded in $W_r$. Since the imbedding from $W_r$ to $L^q(\mathbb{R}^{N-1})$ is compact, we assume $u_n \rightharpoonup u$ in $W_r$, $u_n \to u$ in $L^q(\mathbb{R}^{N-1})$. By Lemma 1, we have

$$\int_{\mathbb{R}^N} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m, \nabla u_n - \nabla u_m) dx$$

$$+ \int_{\mathbb{R}^{N-1}} (|u_n|^{p-2} u_n - |u_m|^{p-2} u_m)(u_n - u_m) dy$$

$$= \int_{\mathbb{R}^{N-1}} (f_\lambda(u_n) - f_\lambda(u_m))(u_n - u_m) dy + \mu \int_{\mathbb{R}^{N-1}} (|u_n|^{q-2} u_n - |u_m|^{q-2} u_m)(u_n - u_m) dy$$

$$+ \langle DI_\lambda(u_n) - DI_\lambda(u_m), u_n - u_m \rangle$$

$$\leq c \left( \frac{2}{\lambda} \right)^{q-q} + 1 \int_{\mathbb{R}^{N-1}} (|u_n|^{q-1} + |u_m|^{q-1})|u_n - u_m| dy + o(1)$$

$$\leq c |u_n - u_m| + o(1) \to 0,$$

as $n, m \to \infty$.

By the elementary inequalities (15), $u_n$ is a Cauchy sequence. □

The following proposition is well known ([23,24]).

Proposition 3. (Ambrosetti–Rabinowitz) Assume $0 < \lambda \leq 1$. Then,

1. $c_k(\lambda) \geq \beta > 0$, $k = 1, 2, \ldots$ are critical values of $I_{\lambda}$.
2. If $c_k(\lambda) = c_{k+1}(\lambda) = \cdots = c_{k+m-1}(\lambda) = c$, then $\gamma(K_c(\lambda)) \geq m$, where

$$K_c(\lambda) = \{ u | u \in W_r, I_{\lambda}(u) = c, DI_{\lambda}(u) = 0 \}.$$
Proof of Theorem 2. Given an integer $k$, let $u_j(\lambda) \in W_r$, $j = 1, \ldots, k$ such that $I_k(u_j(\lambda)) = c_j(\lambda)$, $D I_k(u_j(\lambda)) = 0$. By Proposition 3 (2), we assume that $u_j(\lambda)$, $j = 1, \ldots, k$ are different from each other. Since $I_1 \leq I_k$ for $0 < \lambda \leq 1$, we have $c_1(\lambda) \leq \cdots \leq c_k(\lambda) \leq c_k(1)$. By Theorem 1, there exists $v_k$ such that
$$|u_j(\lambda)(y)| \leq \frac{1}{v_k}, \quad j = 1, \ldots, k, \quad y \in \mathbb{R}^{N-1}.$$ 

Now, for $\lambda < v_k$, we have
$$|u_j(\lambda)(y)| \leq \frac{1}{v_k} < \frac{1}{\lambda}, \quad j = 1, \ldots, k, \quad y \in \mathbb{R}^{N-1},$$

hence $f_\lambda(u_j(\lambda)) = f(u_j(\lambda))$, $u_j(\lambda)$, $j = 1, \ldots, k$ are solutions of the original problem (1). Since the integer $k$ is arbitrary, the problem (1) has infinitely many solutions. \(\square\)

For more details and background material, we refer the readers to the Appendices A–C of this paper.

4. Results

The main results of this paper are Theorem 1 and Theorem 2.

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Appendix A. Estimates on Solutions of $p$-Laplacian Equations in $\mathbb{R}^N$

Lemma A1. Let $u \in D$ satisfy the equation
$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \int_{\mathbb{R}^N} f \varphi \, dx, \quad \varphi \in D \tag{A1}$$

where $f \geq 0$, $f \in L^q(\mathbb{R}^{N-1}) \cap L^{\gamma}(\mathbb{R}^{N-1})$, $1 < q < \frac{N-1}{p-1}$. Then, there exists a constant $c = c(p,q)$ such that
$$|u|_s \leq c|f|^{\frac{1}{q}} \tag{A2}$$

with $\frac{1}{s} = \frac{1}{(p-1)q} - \frac{1}{N-1}$.

Proof. Denote
$$\gamma = \frac{q(p-1)}{qp - \frac{p}{q(q-1)}}$$

then
$$\gamma > 1 - \frac{1}{p} \left(1 + p(\gamma - 1)\right) \frac{q}{q-1} = \gamma^p = \frac{(N-1)(p-1)q}{N-1 - p(q-1)} = s. \tag{A3}$$

First, we assume $\gamma \geq 1$. Take the test function $\varphi = u \cdot u_T^{p(\gamma - 1)} \in D$ in (A1), where $u_T = \min\{u, T\}$ for $T > 0$. We have
$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \int_{u > T} |\nabla u|^{p} u_T^{p(\gamma - 1)} \, dx + (1 + p(\gamma - 1)) \int_{u \leq T} |\nabla u|^{p} u_T^{p(\gamma - 1)} \, dx$$
$$\geq c \int_{\mathbb{R}^N} |\nabla (uu_T^{\gamma - 1})|^{p} \, dx \geq c \left( \int_{\mathbb{R}^{N-1}} (uu_T^{\gamma - 1})^{p} \, dy \right)^{\frac{p}{p}} \tag{A4}$$
and
\[
\int_{\mathbb{R}^N} f \varphi \, dx = \int_{\mathbb{R}^N} f u u_T^{(\gamma - 1)} \, dy \\
\leq |f|_q \left( \int_{\mathbb{R}^N} (u u_T^{(\gamma - 1)})^\frac{q}{q-1} \, dy \right)^{\frac{q-1}{q}}.
\]  
(A5)

In the above, we have used (A3) and \( \frac{q}{q-1} \leq \gamma \) if \( \gamma \geq 1 \). By (A4) and (A5), we have
\[
\left( \int_{\mathbb{R}^N} (u u_T^{(\gamma - 1)})^\gamma \, dy \right)^{\frac{1}{\gamma}} \leq c |f|_q \left( \int_{\mathbb{R}^N} (u u_T^{(\gamma - 1)})^\gamma \, dy \right)^{\frac{\gamma-1}{\gamma}}.
\]  
(A6)

Notice that \( \frac{q}{\gamma} - \frac{q-1}{\gamma} = (p-1)s, (1 + p(\gamma - 1)) \frac{q}{q-1} = s \). Letting \( T \to \infty \) in (A6), we obtain (A2).

Next, we assume \( 1 - \frac{1}{q} < \gamma < 1 \). Let \( \varphi = (u + \theta)^{1+p(\gamma-1)} - \theta^{1+p(\gamma-1)} \), \( \theta > 0 \), then \( \varphi \in \mathcal{D} \). In fact,
\[
|\nabla \varphi| = (1 + p(\gamma - 1))(u + \theta)^{p(\gamma-1)}|\nabla u| \leq \theta^p(\gamma-1)|\nabla u|.
\]
Taking \( \varphi \) as a test function in (A1), we have
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \nabla u \nabla \varphi \, dx = c \int_{\mathbb{R}^N} (u + \theta)^{p(\gamma-1)}|\nabla u|^p \, dx \\
= c \int_{\mathbb{R}^N} |\nabla ((u + \theta)\gamma - \theta^\gamma)|^p \, dx \geq c \left( \int_{\mathbb{R}^N} ((u + \theta)\gamma - \theta) \, dy \right)^p.
\]  
(A7)

and
\[
\int_{\mathbb{R}^N} f \varphi \, dy = \int_{\mathbb{R}^N} f ((u + \theta)^{1+p(\gamma-1)} - \theta^{1+p(\gamma-1)}) \, dy \\
\leq |f|_q \left( \int_{\mathbb{R}^N} ((u + \theta)^{1+p(\gamma-1)} - \theta^{1+p(\gamma-1)}) \frac{q}{q-1} \, dy \right)^{\frac{q-1}{q}}.
\]  
(A8)

In the above, we used the following elementary inequality:
\[
((1 + s)^{1+p(\gamma-1)} - 1) \frac{q}{q-1} \leq c ((1 + s)^\gamma - 1), \quad \text{for } s \geq 0.
\]

By (A7) and (A8), we obtain
\[
\left( \int_{\mathbb{R}^N} ((u + \theta)\gamma - \theta) \, dy \right)^p \leq c |f|_q \left( \int_{\mathbb{R}^N} ((u + \theta)\gamma - \theta) \, dy \right)^{\frac{\gamma-1}{\gamma}}.
\]  
(A9)

Letting \( T \to \infty \) in (A9), we obtain (A2). \( \square \)

**Lemma A2.** Given \( f \in L^4(\mathbb{R}^{N-1}) \cap L^\frac{p}{p-1}(\mathbb{R}^{N-1}), 1 < q < \frac{N-1}{p-1} \). Then, there exists a unique function \( u \in \mathcal{D} \) satisfying the equation
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \nabla u \nabla \varphi \, dx = \int_{\mathbb{R}^N} f \varphi \, dx, \quad \varphi \in \mathcal{D}.
\]

Moreover, \( u \in L^4(\mathbb{R}^{N-1}), |u|_s \leq c |f|_q^{\frac{1}{p-1}}, \) where \( \frac{1}{s} = \frac{1}{(p-1)q} - \frac{1}{N-1} \).
Proof. Consider the functional $J$ defined on $\mathcal{D}$

$$J(u) = \frac{1}{p} \left( \int_{\mathbb{R}^N_+} |\nabla u|^p \, dx + \int_{\mathbb{R}^{N-1}} |u|^p \, dy \right) - \int_{\mathbb{R}^{N-1}} fu \, dy, \quad u \in \mathcal{D}. \tag{A10}$$

$J$ is lower semi-continuous and bounded from below. Therefore, $J$ assumes its infimum at a function $u \in \mathcal{D}$, which solves the equation. By the elementary inequalities (15), the solution is unique. □

Lemma A3. Let $u \in \mathcal{D}$ and satisfy the equation

$$\int_{\mathbb{R}^N_+} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \int_{\mathbb{R}^{N-1}} a|\varphi|^{p-2} \varphi \, dy, \quad u \in \mathcal{D} \tag{A11}$$

where $a \in L^{\frac{N-1}{p-1}}(\mathbb{R}^{N-1})$, $\varphi \in L^q(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$, $q > \frac{p}{1 - \frac{1}{p}}$). Then, there exists a constant $c = c(p, q) > 0$ such that

$$|\varphi| \leq c|a|^{\frac{1}{p-1}} |\varphi|. \tag{A12}$$

Consequently, for $p(1 - \frac{1}{p}) < p_2 < p_1$, $\sigma > 0$, we have

$$|u|_{p_1, p_2, \sigma} \leq c|a|^{\frac{1}{p-1}} |\varphi|_{p_1, p_2, \sigma}. \tag{A13}$$

Proof. Let $\frac{1}{p} = \frac{p-1}{N-1} + \frac{1}{q}$. Then, $1 < \sigma < \frac{N-1}{p-1}$ and $\frac{1}{q} = \frac{1}{p-1} - \frac{1}{N-1}$. By Lemma A1 and the Hölder inequality, we have

$$|\varphi| \leq c|a|^{\frac{1}{p-1}} |\varphi|. \tag{A14}$$

Lemma A4. Let $u \geq 0$, $u \in \mathcal{D}$ and satisfy the inequality

$$\int_{\mathbb{R}^N_+} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx \leq \int_{\mathbb{R}^{N-1}} |u|^{p-1} \varphi \, dy, \quad \varphi \geq 0, \varphi \in \mathcal{D}. \tag{A15}$$

Assume

$$\int_{\mathbb{R}^N_+} u^p \, dx \leq (S_p - \delta)^{-\frac{N-p}{q-p}}. \tag{A16}$$

Then, for any $\gamma > 0$, there exists a constant $c = c(p, \gamma, \delta)$ such that

$$|u|_{L^\infty(B_{\frac{R}{2}}^+)} + |u|_{L^\infty(B_{\frac{R}{2}}^+)} \leq c \left( R^{-\frac{N-1}{q}} |u|_{L^\gamma(D_R)} + R^{-\frac{N}{q}} |u|_{L^\gamma(B_{\frac{R}{2}}^+)} \right). \tag{A17}$$

Proof. We only need to consider the case $R = 1$. The general case can be obtained by a rescaling $u(x) \mapsto R^{\frac{N-p}{q-p}} u(Rx)$. Then, the proof is a standard Moser’s iteration and divided into three steps:

Step 1. There exists $\varepsilon > 0$, $\bar{p} = (1 + \varepsilon)p$ such that

$$\left( \int_{\mathbb{R}^N_+} u^\bar{p} \, dx \right)^\frac{1}{\bar{p}} \leq c \left( \int_{B_{\frac{R}{2}}^+} u^p \, dx \right)^{\frac{p}{\bar{p}}}, \quad 0 < R < 1. \tag{A18}$$

Let $\varepsilon > 0, \eta \in C_0^\infty(\mathbb{R}^N, [0, 1])$ such that $\eta(x) = 1, |x| \leq R; \eta(x) = 0, |x| \geq 1$ and $|\nabla \eta| \leq \frac{2}{R^2}$. Take $\varphi = u^p u_T^p \eta^p$ as test function in (A15), where $T > 0, u_T = \min\{u, T\}$. We have
By (A17) and (A18), we have
\[ \int_{\mathbb{R}^N_+} |\nabla u|^p - 2 |\nabla u \nabla \varphi dx \]
\[ = (1 + p\varepsilon) \int_{u < T} |\nabla u|^p u^p \eta^p dx + \int_{u > T} |\nabla u|^p T^p \eta^p dx + p \int_{\mathbb{R}^N_+} |\nabla u|^p - 2 |\nabla u u_t^{(p-1)} - \eta u_t \nabla \eta dx \]
\[ \geq \frac{1}{(1 + \varepsilon)^p} \int_{\mathbb{R}^N_+} |\nabla (uu_T^\varepsilon \eta)|^p dx - c \epsilon \int_{\mathbb{R}^N_+} (uu_T^\varepsilon)^p |\nabla \eta|^p dx \]
\[ \geq \frac{\tilde{S}_p}{(1 + \varepsilon)^p} \left( \int_{\mathbb{R}^N_+} (uu_T^\varepsilon \eta)^p dy \right)^p - \frac{c \epsilon}{(1 - R)^p} \int_{B_{1/2}^+} (uu_T^\varepsilon)^p dx \]
and
\[ \int_{\mathbb{R}^N_+} u^{p-1} \varphi dy = \int_{\mathbb{R}^N_+} u^{p-1} (uu_T^\varepsilon \eta)^p dx \]
\[ \leq \left( \int_{D_1} u^p dy \right)^{1/p} \left( \int_{\mathbb{R}^N_+} (uu_T^\varepsilon \eta)^p dy \right) \]
\[ \leq \left( \tilde{S}_p - \delta \right) \left( \int_{\mathbb{R}^N_+} (uu_T^\varepsilon \eta)^p dy \right) \]

Choose \( \varepsilon > 0 \) such that
\[ \frac{\tilde{S}_p}{(1 + \varepsilon)^p} \geq \frac{\tilde{S}_p}{2} - \frac{1}{2} \delta, \; (1 + \varepsilon)p < p^*. \]

By (A17) and (A18), we have
\[ \left( \int_{D_1} (uu_T^\varepsilon \eta)^p dy \right)^{1/p} \leq \left( \int_{\mathbb{R}^N_+} (uu_T^\varepsilon \eta)^p dy \right)^{1/p} \leq \frac{c \epsilon}{(1 - R)^p} \int_{B_{1/2}^+} (uu_T^\varepsilon)^p dx. \]

Letting \( T \rightarrow \infty \), we obtain
\[ \left( \int_{D_R} u^{(1+\varepsilon)p} dy \right)^{1/p} \leq \frac{c \epsilon}{(1 - R)^p} \int_{B_{1/2}^+} u^{(1+\varepsilon)p} dx \leq \frac{c \epsilon}{(1 - R)^p} \left( \int_{B_{1/2}^+} u^p dx \right)^{1/(1+\varepsilon)p} \]
and (A16) follows.

**Step 2.** Assume \( 0 < r < R \leq R_0 < 1 \). Then, there exists \( c_{R_0} > 0 \) such that
\[ |u|_{L^\infty(D_r)} + |u|_{L^\infty(B_{1/2}^+)} \leq \frac{c_{R_0}}{(R - r)} \left( |u|_{L^p(D_R)} + |u|_{L^{p\varepsilon}(B_{1/2}^+)} \right). \]

Let \( \varphi = uu_T^{(p-1)} \eta^p \), \( s > 1 \), \( \eta \in C_0^\infty(\mathbb{R}^N_+, [0, 1]) \) such that \( \eta(x) = 1, |x| \leq r \), \( \eta(x) = 0, |x| \geq R \) and \( |\nabla \eta| \leq \frac{2}{R - r} \). Taking \( \varphi \) as a test function in (a15), we have
\[ \int_{\mathbb{R}^N_+} |\nabla u|^p - 2 |\nabla u \nabla \varphi dx \]
\[ = \frac{1 + p(s - 1)}{s^p} \int_{t \leq T} |\nabla uu_T^{s-1}|^p \eta^p dx + \int_{t > T} |\nabla uu_T^{s-1}|^p \eta^p dx \]
\[ + p \int_{\mathbb{R}^N_+} |\nabla u|^p - 2 |\nabla uu_T^{(p-1)(s-1)} - \eta uu_T^{s-1} \nabla \eta dx \]
\[ \geq \frac{1}{2s^p} \int_{\mathbb{R}^N_+} |\nabla (uu_T^{s-1} \eta)|^p dx - c \int_{\mathbb{R}^N_+} (uu_T^{s-1} \eta)^p |\nabla \eta|^p dx \]
and
Letting \( d = \frac{p}{p-r+r+1} < 1 \), it follows from (A22), (A23), and the Sobolev imbedding theorem

\[
\left( \int_{D_r} (uu_T^{e-1})^p \, dy \right)^{\frac{1}{p}} + \left( \int_{B_r^+} (uu_T^{e-1})^{p^*} \, dx \right)^{\frac{1}{p^*}} \leq \left( \int_{D_r} (uu_T^{e-1})^p \, dy \right)^{\frac{1}{p}} + \left( \int_{B_r^+} (uu_T^{e-1})^{p^*} \, dx \right)^{\frac{1}{p^*}} \leq \left( c \int_{B_r^+} |\nabla (uu_T^{e-1})|^p \, dx \right)^{\frac{1}{p}}
\]

\[
\leq (cs)^\frac{1}{2} \left( \int_{D_r} (uu_T^{e-1})^p \, dy \right)^{\frac{1}{p}} + \left( \int_{B_r^+} (uu_T^{e-1})^{p^*} \, dx \right)^{\frac{1}{p^*}} \leq (cs)^\frac{1}{2} \left( \int_{D_r} (uu_T^{e-1})^p \, dy \right)^{\frac{1}{p}} + \left( \int_{B_r^+} (uu_T^{e-1})^{p^*} \, dx \right)^{\frac{1}{p^*}}
\]

In the above, we have used \( p < p^*d \). Assume

\[
\int_{D_r} u^{p^*d} \, dy < +\infty, \quad \int_{B_r^+} u^{p^*d} \, dx < +\infty.
\]

Letting \( T \to \infty \) in (A24), we obtain

\[
\left( \int_{D_r} u^{p^*d} \, dy \right)^{\frac{1}{p}} + \left( \int_{B_r^+} u^{p^*d} \, dx \right)^{\frac{1}{p^*}} \leq \left( \int_{D_r} u^{p^*d} \, dy \right)^{\frac{1}{p}} + \left( \int_{B_r^+} u^{p^*d} \, dx \right)^{\frac{1}{p^*}}
\]

Let \( \chi = \frac{1}{2}, x_j = \chi', r_j = r + \frac{1}{2}T (R - r), j = 1, 2, \ldots \). By Moser’s iteration, for some \( t > 0 \), we obtain

\[
|u|_{L^\infty(D_{r_j})} + |u|_{L^{\infty}(B_{r_j}^+)} \leq \frac{c}{(R - r)^t} \left( |u|_{L^\infty(D_{r_j})} + |u|_{L^{p^*d}(B_{r_j}^+)} \right).
\]

(A26)

**Step 3.** By (A26), there exists \( t', c' \) such that

\[
|u|_{L^\infty(D_{r_j})} + |u|_{L^{\infty}(B_{r_j}^+)} \leq \frac{1}{2} \left( |u|_{L^\infty(D_{r_j})} + |u|_{L^{\infty}(B_{r_j}^+)} \right) + \frac{c}{(R - r)^t} \left( |u|_{L^\gamma(D_{r_j})} + |u|_{L^{\gamma}(B_{r_j}^+)} \right).
\]

By iteration, we obtain

\[
|u|_{L^\infty(D_{r_j})} + |u|_{L^{\infty}(B_{r_j}^+)} \leq \frac{c'}{(R - r)^t} \left( |u|_{L^\gamma(D_{r_j})} + |u|_{L^{\gamma}(B_{r_j}^+)} \right).
\]

In particular
Then, for any $\gamma$ we have

$$|u|_{L^\gamma(D_{\frac{1}{2}})} + |u|_{L^\gamma(B^+_{\frac{1}{2}})} \leq c \left( |u|_{L^\gamma(D_1)} + |u|_{L^\gamma(B^+_1)} \right) \leq c \left( |u|_{L^\gamma(D_1)} + |u|_{L^\gamma(B^+_1)} \right).$$

$\square$

We also have inner estimate

**Lemma A5.** Let $u \geq 0$, $u \in D$ and satisfy

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx \leq 0, \varphi \geq 0, \varphi \in D.$$

Then, for any $\gamma > 0$, there exists $c = c(p, \gamma)$ such that

$$|u|_{L^\gamma(B_{\frac{1}{2}R})} \leq c R^{-\frac{N}{\gamma}} |u|_{L^\gamma(B_R)}.$$

**Appendix B. Estimate via the Wolff Potential**

**Lemma A6.** Let $f \geq 0$, $u \in D$ satisfy the equation

$$\begin{cases}
  -\Delta_p u = 0, & \text{in } \mathbb{R}^N_+,
  \\
  |\nabla u|^{p-2} \frac{\partial u}{\partial n} = f, & \text{on } \mathbb{R}^{N-1}.
\end{cases} \quad (A27)$$

Then, for $\gamma \in (p-1, \frac{(p-1)p}{(p-1)p + p+1})$, there exists a constant $c = c(p, \gamma)$ such that

$$\left( \gamma^{-N} \int_{B^+_1} |u|^{\gamma} \, dx + \gamma^{-N+1} \int_{D_1} |u|^{\gamma} \, dx \right)^{\frac{1}{\gamma}} \leq c \left( \int_{B^+_1} |u|^{\gamma} \, dx + \int_{D_1} |u|^{\gamma} \, dy \right)^{\frac{1}{\gamma}} + c \int_1^r \left( \int_{D_t} f \, dy \right)^{\frac{1}{p-1}} \frac{dt}{t^{1+\frac{N-p}{p+1}}}, \quad 0 < r < 1. \quad (A28)$$

**Proof.** Let $0 < R \leq 1$, $r_j = 2^{j-1}/R$, $j = 1, 2, \ldots$ and $a_0$. Define

$$a_{j+1} = a_j + \frac{1}{\delta} \left( r_j^{-N+1} \int_{B_{r_j+1}^+ \cap \{u > a_j\}} + r_j^{-N+1} \int_{D_{r_j+1} \cap \{u > a_j\}} (u - a_j)^\gamma \, dy \right), \quad (A29)$$

where $B^+_j = \{ x \in \mathbb{R}^N, |x| < r_j \}$, $D_j = \{ y \in \mathbb{R}^{N-1}, |y| < r_j \}$, $\delta$ is a small positive constant. By Lemma A2 of [22](and refer [25]) for $\delta$ small enough, there exists a constant $c = c(p, \gamma)$ such that

$$a_k \leq 2a_1 + c \sum_{j=1}^k \left( \frac{1}{r_j^{N-p}} \int_{D_j} f \, dy \right)^{\frac{1}{p-1}}. \quad (A30)$$

We have

$$a_1 = \frac{1}{\delta} \left( \int_{B^+_R} |u|^{\gamma} \, dx + R^{-N+1} \int_{D_R} |u|^{\gamma} \, dx \right) \quad (A31)$$

and
where

\[ \int_{D_{r_k}} f \, dy \leq c \int_{D_{r_k}} |u|^\gamma \, dx \]

and by the proof of Proposition 3 [22], we have

\[ \left( r_k^{-N} \int_{B_{r_k}^c} |u|^\gamma \, dx + r_k^{-N+1} \int_{D_{r_k}} |u|^\gamma \, dy \right)^{1/\gamma} \leq c a_k. \]  

(A33)

Now, it follows that

\[ \left( r_k^{-N} \int_{B_{r_k}^c} |u|^\gamma \, dx + r_k^{-N+1} \int_{D_{r_k}} |u|^\gamma \, dy \right)^{1/\gamma} \leq c \left( r_k^{-N} \int_{B_{r_k}^c} |u|^\gamma \, dx + r_k^{-N+1} \int_{D_{r_k}} |u|^\gamma \, dy \right)^{1/7} + c \int_{r_k}^R \left( \int_{D_{r_k}} f \, dy \right)^{1/7} \frac{dt}{t^{1+N-\frac{r}{p-1}}} \]  

(A34)

Assume \( 2^{-k} < r \leq 2^{-k+1} \). Let \( R = 2^{k-1} r \), \( \frac{1}{2} < R \leq 1 \). By (A30)–(A34), we obtain

\[ \left( r_k^{-N} \int_{B_{r_k}^c} |u|^\gamma \, dx + r_k^{-N+1} \int_{D_{r_k}} |u|^\gamma \, dy \right)^{1/7} \leq c \left( r_k^{-N} \int_{B_{r_k}^c} |u|^\gamma \, dx + r_k^{-N+1} \int_{D_{r_k}} |u|^\gamma \, dy \right)^{1/7} + c \int_{r_k}^R \left( \int_{D_{r_k}} f \, dy \right)^{1/7} \frac{dt}{t^{1+N-\frac{r}{p-1}}} \]

Appendix C. The Sobolev Imbedding Theorem

**Lemma A7.** \( W^{1,p}(\mathbb{R}_+^N) \subset W \subset W^{1,p}(\mathbb{R}^{N-1} \times (0, 1)) \).

**Proof.** (1) By the Sobolev imbedding theorem, the imbedding from \( W^{1,p}(\mathbb{R}_+^N) \) to \( L^q(\mathbb{R}^{N-1}) \), \( p \leq q \leq \frac{N}{N-1} \) is continuous, hence \( W^{1,p}(\mathbb{R}_+^N) \subset W \). On the other hand, there exist functions that belong to \( W \setminus W^{1,p}(\mathbb{R}_+^N) \). Here, we give an example. Let \( \varphi \in C_0^\infty(B_1, [0, 1]) \). Define

\[ \varphi_n(x) = \frac{n^{-\frac{N-p}{p}}}{} \varphi(n^{-1}(x - 2^{n+1}e)), x \in \mathbb{R}_+^N, \ n = 1, 2, \cdots, \]

\[ u_n = \sum_{k=1}^n \frac{1}{n} \varphi_n \]

where \( e = (0, \cdots, 0, 1) \in \mathbb{R}^N \), \( \varphi_n(n = 1, 2, \cdots) \) are supports, and we have

\[ \int_{\mathbb{R}_+^N} |\nabla \varphi_n|^p \, dx = \int_{\mathbb{R}_+^N} |\nabla \varphi|^p \, dx, \ \int_{\mathbb{R}_+^N} \varphi_n^p \, dx = n^p \int_{\mathbb{R}_+^N} \varphi^p \, dx, \ \int_{\mathbb{R}_+^N} \varphi_n \, dy = 0. \]

Hence,

\[ \|u_n - u_m\|_W^p = \sum_{k=n+1}^m \frac{1}{n^p} \int_{\mathbb{R}_+^N} |\nabla \varphi|^p \, dx \to 0 \]
\[ |u|_p^p = \sum_{k=1}^{n} \frac{1}{n^p} \cdot n^p \int_{\mathbb{R}^N} \varphi^p \, dx \to \infty. \]

Let \( u = \lim_{n \to \infty} u_n \), \( u \in W \) but \( u \not\in W^{1,p}(\mathbb{R}^N) \).

(2) Letting \( u \in W \), we have

\[
|u|^p(y, s) = |u|^p(y, 0) + \int_0^s \frac{\partial}{\partial s} |u|^p(y, \tau) \, d\tau \\
= |u|^p(y, 0) + p \int_0^s |u|^{p-2} u \cdot \frac{\partial u}{\partial s} \, d\tau \\
\leq |u|^p(y, 0) + c \int_0^1 \left| \frac{\partial u}{\partial s} \right|^p \, d\tau + \varepsilon \int_0^1 |u|^p \, d\tau.
\]

Integrating over \( x \in \mathbb{R}^{N-1} \times (0, 1) \), we obtain

\[
\int_{\mathbb{R}^{N-1} \times (0, 1)} |u|^p \, dx \leq \int_{\mathbb{R}^{N-1} \times (0, 1)} |u|^p(y, 0) \, dy + c \int_{\mathbb{R}^{N-1} \times (0, 1)} \frac{\partial u}{\partial s} |u|^p \, dx + \varepsilon \int_{\mathbb{R}^{N-1} \times (0, 1)} |u|^p \, dx,
\]

hence

\[
\int_{\mathbb{R}^{N-1} \times (0, 1)} |u|^p \, dx \leq c \left( \int_{\mathbb{R}^{N-1}} |u|^p \, dy + \int_{\mathbb{R}^N} |\nabla u|^p \, dx \right) = c||u||^p_W
\]

and

\[
||u||_{W^{1,p}(\mathbb{R}^{N-1} \times (0, 1))} \leq c||u||_W.
\]

\[ \Box \]

**Lemma A8.** The imbedding from \( W \) to \( L^q(\mathbb{R}^{N-1}) \), \( p < q < \overline{p} \) is compact.

**Proof.** Denote \( Q = (-1, 1)^{N-1} \subset \mathbb{R}^{N-1} \). For \( y \in \mathbb{R}^{N-1} \), \( |y| \geq R \), we find orthogonal transformation \( \tau_i \in O(N-1) \subset O(N), i = 1, \ldots, N(R) \) such that \( \tau_i = Id \) and \( \tau_i(Q_+) \), \( i = 1, \ldots, N(R) \) are mutually disjoint. Obviously, \( N(R) \to +\infty \) as \( R \to +\infty \).

Let \( u \in W_n, z \in \mathbb{R}^{N-1}, |z| \geq R \). We have

\[
\int_{Q^+} |u|^q \, dy = \frac{1}{N(R)} \sum_{i=1}^{N(R)} \int_{\tau_i(Q+y)} |u|^q \, dy \leq \frac{1}{N(R)} \int_{Q+y} |u|^q \, dy \quad (A35)
\]

and

\[
\int_{Q^+} |u|^q \, dy \leq c \left( \int_{Q^+} |\nabla u|^p + |u|^p \, dx \right) \left( \int_{Q^+} |u|^q \, dy \right)^{1-\frac{p}{q}} \quad (A36)
\]

Taking sum over \( z \in \mathbb{R}^{N-1}, |z| \geq R \), we obtain

\[
\int_{\mathbb{R}^{N-1}\setminus D_R} |u|^q \, dy \leq c \left( \int_{\mathbb{R}^{N-1}\setminus D_R} |\nabla u|^p \, dx + \int_{\mathbb{R}^{N-1}} |u|^p \, dy \right) \left( \frac{1}{N(R)} \int_{\mathbb{R}^{N-1}} |u|^q \, dy \right)^{1-\frac{p}{q}} \quad (A37)
\]

Now, assume \( u_n \in W_n, u_n \to 0 \) in \( W \). Then,

\[
\int_{\mathbb{R}^{N-1}} |u_n|^q \, dy = \int_{D_R} |u_n|^q \, dy + \int_{\mathbb{R}^{N-1}\setminus D_R} |u_n|^q \, dy \leq \int_{D_R} |u_n|^q \, dy + c\|u\|_{W_R}(1) \to 0 \text{ as } n \to \infty.
\]
Proposition A1. The imbedding from $D_r$ to $L^r(\mathbb{R}^{N-1})$ is compact with respect to the dilation group $D$ (defined by (8)).

Proof. (Adapted from [19]) Choose $\chi \in C_0^\infty(\mathbb{R}, [0, 1])$ such that $\chi(t) = |t|, 1 \leq |t| \leq 2^{-\frac{N-p}{p}}$; $\chi(t) = 0, |t| \leq 2^{-\frac{N-p}{p}}$ or $|t| \geq 2^{2-\frac{N-p}{p}}$. $Q = (-1, 1)^{N-1}$ and $N(R)$ as defined in Lemma A8. Assume $u_n \overset{D}{\to} 0$ in $D_r$.

Step 1. $\int_{\mathbb{R}^{N-1}} \chi^p(u_n) \, dy \to 0$ as $n \to \infty$.

For $z \in \mathbb{R}^{N-1}, |z| \geq R$, we have

$$\int_{Q_{z+2}} \chi^p(u_n) \, dy \leq c \left( \int_{(Q_{z+2}) \times (0, 1)} (|\nabla \chi(u_n)|^p + \chi^p(u_n)) \, dx \right) \left( \int_{Q_{z+2}} \chi^p(u_n) \, dy \right)^{-\frac{1}{p}}$$

$$\leq c \left( \int_{(Q_{z+2}) \times (0, 1)} |u_n|^p \, dx + \int_{Q_{z+2}} \chi^p(u_n) \, dy \right) \left( \frac{1}{N(R)} \int_{\mathbb{R}^{N-1}} \chi^p(u_n) \, dy \right)^{-\frac{1}{p}} \quad \text{(A38)}$$

Taking sum over $z \in \mathbb{R}^{N-1}$ and $|z| \geq R$,

$$\int_{\mathbb{R}^{N-1} \setminus D_R} \chi^p(u_n) \, dy \leq c \left( \int_{\mathbb{R}^{N-1}} |\nabla u_n|^p \, dx + \int_{\mathbb{R}^{N-1}} |u_n|^p \, dy \right) \left( \frac{1}{N(R)} \int_{\mathbb{R}^{N-1}} |u_n|^p \, dy \right)^{-\frac{1}{p}} \quad \text{(A39)}$$

and

$$\int_{\mathbb{R}^{N-1}} \chi^p(u_n) \, dy = \int_{D_R} \chi^p(u_n) \, dy + \int_{\mathbb{R}^{N-1} \setminus D_R} \chi^p(u_n) \, dy$$

$$\leq c \int_{D_R} |u_n|^p \, dy + o_R(1) \to 0 \quad \text{as } n \to \infty. \quad \text{(A40)}$$

Step 2. For $j \in \mathbb{Z}$, define $h_j \in D$ by $h_j u(x) = z^{\frac{N-p}{p}} u(2^j x)$. Then, for any sequence $j_n \in \mathbb{Z}, h_{j_n} u_n \overset{D}{\to} 0$ in $D_r$. By Step 1, we have

$$\int_{\mathbb{R}^{N-1}} \chi^p(h_j u_n) \, dy \to 0, \quad \text{as } n \to \infty. \quad \text{(A41)}$$

Step 3. We estimate $\int_{\mathbb{R}^{N-1}} |u_n|^p \, dy$. Since

$$\int_{2^{j-1}}^{2^j} \left( \frac{N-p}{p} \right) |u_n|^p \, dy \leq \int_{\mathbb{R}^{N-1}} \left( \frac{N-p}{p} \right) \chi(2^{j-1} \frac{N-p}{p} u_n(x)) \, dy$$

$$= \int_{\mathbb{R}^{N-1}} \left( \chi(2^{j-1} \frac{N-p}{p} u_n(x)) \right)^p \, dy$$

$$\leq c \int_{2^{j-1}}^{2^j} \frac{N-p}{p} \chi(2^{j-1} \frac{N-p}{p} u_n(x)) \, dy \cdot \left( \int_{\mathbb{R}^{N-1}} \chi(2^{j-1} \frac{N-p}{p} u_n(2^{-j} x)) \, dy \right)^{\frac{p}{p-1}}$$

$$\leq c \int_{2^{j-1}}^{2^j} \frac{N-p}{p} \chi(2^{j-1} \frac{N-p}{p} u_n(x)) \, dy \cdot \sup_{j \in \mathbb{Z}} \left( \int_{\mathbb{R}^{N-1}} \chi(h_j(u_n)) \, dy \right)^{\frac{p}{p-1}}, \quad \text{(A42)}$$
choosing $j_n \in Z$ such that
\[
\sup_{j \in Z^{N-1}} \int_{\mathbb{R}^{N-1}} \left(\chi(h_j(u_n))\right)^p dy \leq 2 \int_{\mathbb{R}^{N-1}} \left(\chi(h_j(u_n))\right)^p dy \tag{A43}
\]
Then,
\[
\int_{2^{j-1} \frac{N-p}{p} \leq |u_n| \leq 2^{j+1} \frac{N-p}{p}} |u_n|^p \, dy \\
\leq c \int_{2^{j-1} \frac{N-p}{p} \leq |u_n| \leq 2^{j+1} \frac{N-p}{p}} |\nabla u_n|^p \, dx \cdot \left(\int_{\mathbb{R}^{N-1}} \left(\chi(h_j(u_n))\right)^p dy\right)^{1 - \frac{p}{p'}} . \tag{A44}
\]
Taking sum over $j \in Z^{N-1}$ and taking into account that the sets $2^{(j-1) \frac{N-p}{p}} \leq |u_n| \leq 2^{(j+1) \frac{N-p}{p}}$ cover $\mathbb{R}$ with uniformly finite multiplicity, by Step 2, we obtain
\[
\int_{\mathbb{R}^{N-1}} |u_n|^p \, dy \leq c \int_{\mathbb{R}^{N-1}} |\nabla u_n|^p \, dx \cdot \left(\int_{\mathbb{R}^{N-1}} \left(\chi(h_j(u_n))\right)^p dy\right)^{1 - \frac{p}{p'}} \\
\leq c \left(\int_{\mathbb{R}^{N-1}} \left(\chi(h_{j_n}(u_n))\right)^p dy\right)^{1 - \frac{p}{p'}} \to 0 \quad \text{as } n \to \infty .
\]
\[\square\]

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