Separability and harmony in ecumenical systems

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Abstract

The quest of smoothly combining logics so that connectives from classical and intuitionistic logics can co-exist in peace has been a fascinating topic of research for decades now. In 2015, Dag Prawitz proposed a natural deduction system for an ecumenical first-order logic. We start this work by proposing a pure sequent calculus version for it, in the sense that connectives are introduced without the use of other connectives. For doing this, we extend sequents with an extra context, the stoup, and define the ecumenical notion of polarities. Finally, we smoothly extend these ideas for handling modalities, presenting pure labeled and nested systems for ecumenical modal logics.

Keywords  Ecumenical systems; modalities; nested systems; labeled systems; cut-elimination; polarities.

1 Introduction

LC [13] is a sequent system for classical logic that separates the rules for positive and negative formulas, being a precursor of the notion of focusing in sequent systems [11]. The idea is that right rules for positive formulas are applied in the stoup, which is a differentiated context, where formulas are focused on. Negative formulas, on the other hand, are stored in a classical context, where they can be eagerly decomposed.

Sequents with one stoup have the form $\Gamma \Rightarrow \Delta; \Pi\{\}^1$ where $\Gamma, \Delta$ are sets and $\Pi$, the stoup, is a multiset containing at most one formula. In LC, the meaning of these contexts is the following.

- $\Gamma$ is the usual classical left context in well known sequent systems for classical and intuitionistic first order logics, like LK and LJ [35].

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¹Pereira, Pimentel and Sales have been partially supported by CAPES and CNPq.
²It should be mentioned that, in LC, sequents have the one-sided presentation – the left context is not present. Also, in systems like Girard’s LU [14], sequents have two stoups and linear contexts. Here we adopted the simpler possible version for sequents with stoup supporting the intuitionistic setting and avoiding structural rules in classical contexts.
- The stoup $\Pi$ is a differentiated context, where positive formulas are “worked on”. In a bottom-up read, a positive formula can be chosen from the classical right context to populate the stoup using the dereliction rule

$$ \Gamma \Rightarrow \Delta, P; P \quad \Rightarrow \Delta, P; \cdot \quad \text{D} $$

- The (classical) right context $\Delta$ carries the information of subformulas of negative formulas $N$, in the sense that $N = \neg N'$. This means that $N \in \Delta$ can be interpreted as $N' \in \Gamma$. Negative formulas are added to the classical context via the store rule

$$ \Gamma \Rightarrow \Delta, N; \cdot \quad \Rightarrow \Delta; N \quad \text{store} $$

It is interesting to note that, while the sequent $\Gamma \Rightarrow \Delta; \Pi$ is intuitionistically interpreted as $\Gamma, \neg \Delta \Rightarrow \Pi$, it only has a classical interpretation in LC if the stoup $\Pi$ is empty. Moreover, the stoup in LC is persistent, in the sense that, after applying dereliction $D$ over a positive formula $P$ in the bottom-up reading, the stoup is emptied only when either $P$ is totally consumed, or a negative subformula is reached – in which case it is stored in the classical right context.

The ecumenical systems we will study in this paper have a quite different behavior, since they are intuitionistic in nature. Hence the use of stoups will mix some of the characteristics of LC with intuitionistic systems featuring stoup such as, for example, Herbelin’s LJT and LJQ [16, 10]. The base difference is that, in the ecumenical formulation, stoups cannot be persistent since, otherwise, the logic would not be complete.

Several approaches have been proposed for combining intuitionistic and classical logics (see e.g. [7, 19, 9]), many of them inspired by Girard’s polarised system $LU$ [14]. More recently, Prawitz chose a completely different approach by proposing a natural deduction ecumenical system [31]. While it also took into account meaning-theoretical considerations, it is more focused on investigating the philosophical significance of the fact that classical logic can be translated into intuitionistic logic.

In this paper, we will proceed with a careful study of Prawitz’ ecumenical system under the view of Girard’s original idea of stoup, for separating the intuitionistic from the classical behaviors. This will also allow for a a first-order ecumenical system that avoids the use of negations in the formulation of rules. Such systems are called pure or separable [25], in the sense that connectives are introduced without the use of other connectives, hence giving a clearer notion of the meaning for that connective. This goes straight into the direction first pointed by Prawitz, and adopted by the Proof-theoretic semantics’ school [17]. Finally, we will extend this notion to modalities.

In this work, we bring new basis for ecumenical systems, where systems and results presented in [22, 23] fit smoothly. More specifically, this work improves the op.cit. in the following ways:

1. Instead of building the modal system over the sequent presentation [28] of Prawitz’ ecumenical system [31], we propose a new pure first-order ecumenical system. This not only allows for a better proof theoretic view of Prawitz’ original proposal, but it also serves as a solid ground for smoothly accommodating modalities.

2. A new pure labeled system for modalities comes naturally in this approach, and the nested system in [23] is easily proven correct and complete w.r.t. it.
3. The proof of completeness of the nested system is new, and it does not refer to the axiomatic system.

Under this new perspective, we can start new lively discussions about the nature of formulas and systems.

The rest of the paper is organized as follows: Section 2 introduces the notion of ecumenical systems with stoup (system LCE), and in in Section 3 we prove it complete and correct w.r.t. Prawitz’ ecumenical system (LE). This involves a non-trivial use of polarities, as well as a non-standard proof of cut-elimination. We show that, is one is not careful, the quest for purity ends up in collapsing; Section 4 extends the propositional fragment of LCE with modalities, resulting in a new pure labeled ecumenical modal system (labEK); Section 5 brings the nested ecumenical system nEK from [23], which is naturally seen as the label-free counterpart of labEK; Section 6 briefly discusses fragments, axioms and extensions and Section 7 discusses related and future work, and concludes the paper.

2 The system LCE

In [31] Dag Prawitz proposed a natural deduction system where classical and intuitionistic logic could coexist in peace. In this system, the classical logician and the intuitionistic logician would share the universal quantifier, conjunction, negation and the constant for the absurd, but they would each have their own existential quantifier, disjunction and implication, with different meanings. Prawitz’ main idea is that these different meanings are given by a semantical framework that can be accepted by both parties.

The sequent system LE (depicted in Fig. 1) was presented in [28] as the sequent counterpart of Prawitz’ natural deduction system.

The language L used for ecumenical systems is described as follows. We will use a subscript c for the classical meaning and i for the intuitionistic, dropping such subscripts when formulas/connectives can have either meaning.

Classical and intuitionistic n-ary predicate symbols (p_c, p_i, ...) co-exist in L but have different meanings. The neutral logical connectives {⊥, ¬, ∧, ∀} are common for classical and intuitionistic fragments, while {→_i, ∨_i, ∃_i} and {→_c, ∨_c, ∃_c} are restricted to intuitionistic and classical interpretations, respectively.

LE has very interesting proof theoretical properties including cut-elimination, together with a Kripke semantical interpretation, that allowed the proposal of a variety of ecumenical proof systems, such as multi-conclusion and nested sequent systems, as well as several fragments of such systems [28].

2.1 Ecumenical consequence and stoup

Denoting by ⊢_S A the fact that the formula A is a theorem in the proof system S, the following theorems are easily provable in LE:

1. ⊢_LE (A ∨_c B) ↔_i ¬(¬A ∧ ¬B);
2. ⊢_LE (A →_c B) ↔_i ¬(A ∧ ¬B);
3. ⊢_LE (∃_c x A) ↔_i ¬(∀x, ¬A).
Intuitionistic and neutral Rules

\[
\frac{A, B, \Gamma \Rightarrow C}{A \land B, \Gamma \Rightarrow C} \quad \land L
\]
\[
\frac{\Gamma \Rightarrow A}{A \Rightarrow A \land B} \quad \land R
\]
\[
\frac{A, B, \Gamma \Rightarrow C}{A \lor B, \Gamma \Rightarrow C} \quad \lor L
\]
\[
\frac{\Gamma \Rightarrow \exists y \, A}{\forall x, A, \Gamma \Rightarrow C} \quad \forall L
\]
\[
\frac{\exists y, [y/x], \forall x, A, \Gamma \Rightarrow C}{\forall y, [y/x], \Gamma \Rightarrow [y/x]} \quad \forall R
\]
\[
\frac{\exists y, [y/x], \Gamma \Rightarrow A}{\exists y, [y/x], \Gamma \Rightarrow C} \quad \exists L
\]

Classical rules

\[
\frac{A, \Gamma \Rightarrow \bot}{A \lor \epsilon, B, \Gamma \Rightarrow \bot} \quad \lor L
\]
\[
\frac{\Gamma \Rightarrow \neg A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \lor \epsilon, B} \quad \lor R
\]
\[
\frac{A \Rightarrow \epsilon, B, \Gamma \Rightarrow A \land \epsilon, B, \Gamma \Rightarrow \bot}{A \Rightarrow \epsilon, B, \Gamma \Rightarrow \epsilon} \quad \epsilon, L
\]
\[
\frac{\Gamma \Rightarrow \neg p_i, \Gamma \Rightarrow \bot}{p_i, \Gamma \Rightarrow \bot} \quad \epsilon, R
\]
\[
\frac{\Gamma \Rightarrow \epsilon, p_i, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow p_i, \Gamma \Rightarrow \bot} \quad \epsilon, L
\]
\[
\frac{\forall y, [y/x], \Gamma \Rightarrow \bot}{\exists y, x, A, \Gamma \Rightarrow \bot} \quad \exists L
\]
\[
\frac{\Gamma \Rightarrow \forall y, [y/x], \Gamma \Rightarrow \bot}{\Gamma \Rightarrow \exists y, x, A} \quad \exists R
\]

Initial, cut and Structural Rules

\[
\frac{p_i, \Gamma \Rightarrow p_i}{\text{init}}
\]
\[
\frac{\Gamma \Rightarrow A}{A, \Gamma \Rightarrow C} \quad \text{cut}
\]
\[
\frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow A} \quad \text{W}
\]

Figure 1: Ecumenical sequent system LE. In rules \(\forall R, \exists L, \exists L\), the eigenvariable \(y\) is fresh; \(p\) is atomic.

There equivalences are of interest since they relate the classical and the neutral operators: the classical connectives can be defined using negation, conjunction, and the universal quantifier. On the other hand,

4. \(\text{inLE} \quad \neg (\neg A) \Rightarrow A \) but \(\text{nuLE} \quad (\neg A) \Rightarrow A \) in general;
5. \(\text{nuLE} \quad (A \land (A \Rightarrow B)) \Rightarrow B \) but \(\text{inLE} \quad (A \land (A \Rightarrow \epsilon)) \Rightarrow B \) in general;
6. \(\text{nuLE} \quad \forall x, A \Rightarrow \neg \exists x, A \neg A \) but \(\text{inLE} \quad \neg \exists x, A \neg A \Rightarrow \forall x, A \) in general

Observe that (4) and (6) reveal the asymmetry between definability of quantifiers: while the classical existential can be defined from the universal quantification, the other way around is not true, in general. This is closely related with the fact that, proving \(\forall x, A\) from \(\neg \exists x, A \neg A\) depends on \(A\) being a classical formula. We will come back to this in Section 4.

On its turn, the following result states that logical consequence in LE is intrinsically intuitionistic.

Proposition 2.1 (28). \(\Gamma \vdash B\) is provable in LE iff \(\text{nuLE} \quad \epsilon \Rightarrow \bot \Rightarrow B\).

To preserve the "classical behavior", i.e., to satisfy all the principles of classical logic e.g. \textit{modus ponens} and the \textit{classical reductio}, it is sufficient that the main operator of the formula will be eventually classical (28). Thus, "hybrid" formulas, i.e.,
formulas that contain classical and intuitionistic operators may have a classical behavior. Formally,

**Definition 2.2.** Eventually externally classical (eec for short) formulas are given by the following grammar

\[ A^{ec} := A^c | A^{ec} \land A^{ec} | \forall x.A^{ec} | A \rightarrow_i A^{ec} | \neg A \]

where \( A \) is any formula and \( A^c \) is an externally classical formula given by

\[ A^c := p_c | \bot | A \lor_c A | A \rightarrow_c A | \exists_c x.A \]

For eec formulas we can prove the following theorems

7. \( \vdash_{LE} (A \land (A \rightarrow_i B^{ec})) \rightarrow_i B^{ec} \).
8. \( \vdash_{LE} \neg B^{ec} \rightarrow_i B^{ec} \).
9. \( \vdash_{LE} \neg \exists_c x.\neg B^{ec} \rightarrow_i \forall x.B^{ec} \).

More generally, notice that all classical right rules as well as the right rules for the neutral connectives in LE are invertible. Since invertible rules can be applied eagerly when proving a sequent, this entails that eec formulas can be eagerly decomposed. As a consequence, the ecumenical entailment, when restricted to eec succedents (antecedents having an unrestricted form), is classical.

**Theorem 2.3** (extended). Let \( A^{ec} \) be an eventually externally classical formula and \( \Gamma \) be a multiset of ecumenical formulas. Then

\[ \vdash_{LE} \bigwedge \Gamma \rightarrow_i A^{ec} \iff \vdash_{LE} \bigwedge \Gamma \rightarrow_i A^{ec} \]

This sums up well, proof theoretically, the ecumenism of Prawitz’ original proposal: consequence relations are intrinsically intuitionistic, but have a classical behavior when proving a formula that eventually will behave classically.

Moreover, observe that, from a proof \( \pi \) of \( \Gamma \Rightarrow A \) in LE, we can derive \( \Gamma, \neg A \Rightarrow \bot \):

\[ \Gamma, \neg A \Rightarrow A \]

\[ \Gamma, \neg A \Rightarrow \bot \neg L \]

where \( \pi^w \) is the weakened version of \( \pi \). The other direction does not hold since \( \vdash_{LE} \neg \neg A \nRightarrow A \), in general. However, for eec formulas the converse also holds.

**Proposition 2.4.** If \( \Gamma, \neg A^{ec} \Rightarrow \bot \) is provable in LE so it is \( \Gamma \Rightarrow A^{ec} \).

**Proof.** Since \( \vdash_{LE} \neg \neg A^{ec} \Rightarrow A^{ec} \) (see [8]), then

\[ \frac{\Gamma, \neg A^{ec} \Rightarrow \bot \quad \Gamma \Rightarrow \neg A^{ec} \quad \neg \neg A^{ec}, \Gamma \Rightarrow A^{ec} \quad \text{cut}}{\Gamma \Rightarrow A^{ec}} \]

This corroborates the idea that, in an ecumenical system with stoup, formulas in the classical context should hold classical subformulas of eec formulas. The stoup, on the other hand, would carry the intuitionistic or neutral information.

We are now ready to describe the ecumenical system with stoup, where the connections between the “primitive” sequent calculus in LE and the “pure” sequent calculus in LCE is established as follows:
- A sequent of the form $\Gamma, \neg \Delta \Rightarrow \Pi$ will be translated as $\Gamma \Rightarrow \Delta ; \Pi$ for some set $\Delta$ of negated formulas.

- A sequent of the form $\Gamma \Rightarrow \Delta ; \Pi$ will be translated as $\Gamma, \neg \Delta \Rightarrow \Pi$.

- The empty stoup will be translated as $\bot$.

As already mentioned, formulas will move over contexts depending on the polarity.

**Definition 2.5.** An ecumenical formula is called negative if its main connective is classical or the negation, and positive otherwise (we will use $N$ for negative and $P$ for positive formulas).

Figure 2 brings the rules for the ecumenical pure systems with stoup (LCE). Observe that rules in LCE determine positive/negative phases in derivations, and the dynamic of the rules for classical connectives in LCE is as follows: Negative formulas in the classical contexts are eagerly decomposed; if a positive formula in the right context is chosen to be worked on, it is placed in the stoup $\Pi$, and treated intuitionistically.

The analogy with focusing [1] stops there, though: explicit weakening in the stoup is needed for completeness, as the next example shows.

**Example 2.6.** The sequent

$$\neg B, A \rightarrow c B, A \vdash C$$

is provable in LE, where the succedent $C$ is weakened. This means that, in LCE, the following sequents should be provable

$$\neg B, A \rightarrow c B, A \Rightarrow \cdot ; C \quad \Rightarrow B, A \land \neg B, \neg A ; C$$

But the stoup is necessarily erased in the process.

Observe that, in LC, sequents with non-empty stoup do not have a classical interpretation. In fact, none of the sequents above are provable in LC, if $C$ is a positive formula.

### 3 Correctness of the systems

We start by stating standard proof theoretic results.

**Lemma 3.1.** In LCE:

i. The rules $\vee L, \vee R, \rightarrow c L, \rightarrow c R, \neg L, \neg R, L, \exists L, \exists R$ and $D$ are invertible, that is, in any application of such rules, if the conclusion is a provable nested sequent so are the premises.

ii. The rules $\land L, \land R, \lor L, \rightarrow i L, \exists L, \forall L, \forall R$ and $\text{store}$ are totally invertible, that is, they are invertible and can be applied in any contexts.

iii. Classical weakening and contraction are admissible

\[
\frac{\Gamma \Rightarrow \Delta; \Pi}{\Gamma, \Gamma' \Rightarrow \Delta; \Delta'; \Pi} \quad \text{Wc} \quad \frac{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'; \Pi}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'; \Pi} \quad \text{Cc}
\]

iv. The general form of initial axioms are admissible

\[
\frac{\Gamma, A \Rightarrow \Delta; A}{\Gamma, A \Rightarrow \Delta; A} \quad \text{initi} \quad \frac{\Gamma, A \Rightarrow \Delta, A; \Pi}{\Gamma, A \Rightarrow \Delta, A; \Pi} \quad \text{initc}
\]
Theorem 3.2. The sequent $\Gamma \Rightarrow \Delta; \Pi$ is provable in LCE iff $\Gamma, \neg \Delta \Rightarrow \Pi$ is provable in LE.

Proof. The proofs are by standard induction on the height of derivations. The proof of admissibility of $W_L$ does not depend on any other result, while the admissibility of $C_c$ depends on the invertibility results and the admissibility of weakening.

The proof of admissibility of the general initial axioms is by mutual induction. Below we show the cases for quantifiers where, by induction hypothesis, instances of the axioms hold for the premises.

\[
\frac{\Delta; A[y/x], \Gamma \Rightarrow \exists x.A, A[y/x], \Delta; \xi \Rightarrow \exists x.A, A[y/x], \Delta; \Pi \Rightarrow \exists x.A, A[y/x]}{\exists x.A, A[y/x], \Gamma \Rightarrow \Delta; A[y/x], \Pi \Rightarrow \exists x.A, A[y/x]} \quad \text{init}_L
\]

\[
\frac{\Delta; A[y/x], \Gamma \Rightarrow \exists x.A, A[y/x], \Delta; \xi \Rightarrow \exists x.A, A[y/x], \Pi \Rightarrow \exists x.A, A[y/x]}{\exists x.A, A[y/x], \Gamma \Rightarrow \Delta; A[y/x], \Pi \Rightarrow \exists x.A, A[y/x]} \quad \text{init}_R
\]

\[
\frac{\Delta; A[y/x], \Gamma \Rightarrow \exists x.A, A[y/x], \Delta; \xi \Rightarrow \exists x.A, A[y/x], \Pi \Rightarrow \exists x.A, A[y/x]}{\exists x.A, A[y/x], \Gamma \Rightarrow \Delta; A[y/x], \Pi \Rightarrow \exists x.A, A[y/x]} \quad \text{init}_L
\]

\[
\frac{\Delta; A[y/x], \Gamma \Rightarrow \exists x.A, A[y/x], \Delta; \xi \Rightarrow \exists x.A, A[y/x], \Pi \Rightarrow \exists x.A, A[y/x]}{\exists x.A, A[y/x], \Gamma \Rightarrow \Delta; A[y/x], \Pi \Rightarrow \exists x.A, A[y/x]} \quad \text{init}_R
\]

The following shows that LCE is correct and complete w.r.t. LE.
Proof. The only interesting cases are the ones involving classical connectives.

- Case $\lor, R$. Suppose that $\Gamma \Rightarrow \Delta, A \lor, B; \cdot$ is provable in LCE with proof

$$\Gamma \Rightarrow A, B, \Delta; \cdot \lor, R$$

By inductive hypothesis, $\Gamma, \neg A, \neg B, \neg \Delta \Rightarrow \bot$ is provable in LE. Hence

$$\Gamma, \neg A, \neg B, \neg \Delta \Rightarrow A \lor, B \lor, R$$

On the other hand, suppose that $\Gamma \Rightarrow A \lor, B$ is provable in LE with proof

$$\Gamma \Rightarrow A \lor, B \lor, R$$

By inductive hypothesis, $\Gamma \Rightarrow A, B; \cdot$ is provable in LCE. Thus

$$\Gamma \Rightarrow A \lor, B; \cdot \lor, R$$

- Case $\rightarrow, R$. Suppose that $\Gamma \Rightarrow \Delta, A \rightarrow, B; \cdot$ is provable in LCE with proof

$$\Gamma \Rightarrow A, B; \cdot \rightarrow, R$$

By inductive hypothesis, $\Gamma, A, \neg B, \neg \Delta \Rightarrow \bot$ is provable in LE. Hence

$$\Gamma, A, \neg B, \neg \Delta \Rightarrow A \rightarrow, B \rightarrow, R \bot$$

On the other hand, suppose that $\Gamma \Rightarrow A \rightarrow, B$ is provable in LE with proof

$$\Gamma \Rightarrow A \rightarrow, B \rightarrow, R$$

By inductive hypothesis, $\Gamma, A \Rightarrow B; \cdot$ is provable in LCE. Hence

$$\Gamma, A \Rightarrow B; \cdot \rightarrow, R$$

- Case $\exists, R$. Suppose that $\Gamma \Rightarrow \exists x. A, \Delta; \cdot$ is provable in LCE with proof

$$\Gamma \Rightarrow A[i/x], \exists x. A, \Delta; \cdot \rightarrow, R$$

By inductive hypothesis, $\Gamma, \neg A[y/x], \neg \exists x. A, \neg \Delta \Rightarrow \bot$ is provable in LE. Hence

$$\Gamma, \neg A[y/x], \forall x, \neg A, \neg \Delta \Rightarrow \cdot \forall L$$

\[ \text{\Letter} \]
On the other hand, suppose that $\Gamma \vdash \exists x. A$ is provable in LE with proof

$\pi$

$\frac{\pi}{\Gamma, \forall x. \neg A \Rightarrow \bot} \quad \exists x. R$

By inductive hypothesis, $\Gamma, \forall x. \neg A \Rightarrow \cdot \cdot$ is provable in LCE. Hence

$\frac{\pi_w}{\Gamma, \forall x. \neg A \Rightarrow \exists x. A \cdot} \quad \neg R \quad \frac{\Gamma, \forall x. \neg A \Rightarrow \exists x. A \cdot}{\Gamma \Rightarrow \exists x. A \cdot \cdot \cdot}$

store

$N$ - cut

where $\pi_w$ represents the translation of the weakened version of $\pi$ and the double bars in the right branch indicates an adapted proof of $E$

- Case $\rightarrow_{\epsilon} L$. Suppose that $\Gamma, A \rightarrow_{\epsilon} B \Rightarrow \Delta \cdot$ is provable in LCE with proof

$\frac{\Gamma, A \rightarrow_{\epsilon} B \Rightarrow \Delta; A \cdot}{\Gamma, A \rightarrow_{\epsilon} B \Rightarrow \Delta; :} \quad \rightarrow_{\epsilon} L$

By inductive hypothesis, $\Gamma, A \rightarrow_{\epsilon} B, \neg \Delta \Rightarrow \bot$ are provable in LE. Hence

$\frac{\Gamma, A \rightarrow_{\epsilon} B, \neg \Delta \Rightarrow \bot}{\Gamma, A \rightarrow_{\epsilon} B, \neg \Delta \Rightarrow \bot} \quad \rightarrow_{\epsilon} L$

and vice-versa. The other left-rule cases are similar.

- Case D. Suppose that $\Gamma \Rightarrow \Delta; P; \cdot$ is provable in LCE with proof

$\frac{\Gamma \Rightarrow \Delta; P; P}{\Gamma \Rightarrow \Delta; P; \cdot} \quad D$

By inductive hypothesis, $\Gamma, \neg P, \neg \Delta \Rightarrow P$ is provable in LE. Hence

$\frac{\Gamma, \neg P, \neg \Delta \Rightarrow P}{\Gamma, \neg P, \neg \Delta \Rightarrow P} \quad \neg L$

- Case store. Suppose that $\Gamma \Rightarrow \Delta; N$ is provable in LCE with proof

$\frac{\Gamma \Rightarrow \Delta, N; \cdot}{\Gamma \Rightarrow \Delta; N \cdot \cdot \cdot}$

store

By inductive hypothesis, $\Gamma, \neg N, \neg \Delta \Rightarrow \bot$ is provable in LE. Hence

$\frac{\Gamma, \neg \Delta, \neg N \Rightarrow \bot}{\Gamma, \neg \Delta \Rightarrow \neg \neg N \cdot} \quad \neg R \quad \frac{\Gamma, \neg \Delta \Rightarrow \neg \neg N \cdot \cdot \cdot}{\Gamma, \neg \Delta \Rightarrow N \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \
- Cases cut. The $P - \text{cut}$ rule in LCE trivially corresponds to cut in LE. Suppose that $\Gamma \Rightarrow \Delta; \Pi$ is provable in LCE with proof

\[
\frac{\Gamma \Rightarrow \Delta, N; \Pi^* \quad N, \Gamma \Rightarrow \Delta; \Pi}{\Gamma \Rightarrow \Delta; \Pi} \quad N - \text{cut}
\]

By inductive hypothesis, $\Gamma, \neg N, \neg \Delta \Rightarrow \Pi^*$ and $\Gamma, N, \neg \Delta \Rightarrow \Pi$ are provable in LE with proofs $\pi_1$ and $\pi_2$ respectively.

If $\Pi^* = \cdot$, then

\[
\frac{\pi_1}{\Gamma, \neg \Delta, \neg N \Rightarrow \bot} \quad \frac{\neg \neg N, \Gamma, \neg \Delta \Rightarrow N \quad \neg \neg N, \Gamma, \neg \Delta \Rightarrow \Pi}{\neg \neg N, \Gamma, \neg \Delta \Rightarrow \Pi} \quad \text{cut}
\]

where $\pi_n^\nu$ is the weakened version of $\pi_2$, and the double bar corresponds to the proof of $[8]$.

If $\Pi^* = P \in \Delta$, then the left premise derivation is substituted by

\[
\frac{\pi_1}{\Gamma, \neg \Delta, \neg N \Rightarrow P} \quad \frac{\neg \neg N, \Gamma, \neg \Delta \Rightarrow \bot \quad \neg \neg N, \Gamma, \neg \Delta \Rightarrow \Pi}{\neg \neg N, \Gamma, \neg \Delta \Rightarrow \Pi} \quad \text{cut}
\]

Observe that, other than the cut rules, $\text{store}$ introduces cut in the translation between proofs from LCE to LE, while $\exists R$ introduces cuts in the other way around. Since LE has the cut-elimination property [28], the translation from LCE to LE is not problematic. However, for proving cut-completeness from LE to LCE, we need to prove that the later has the cut-elimination property.

### 3.1 Cut-elimination

A logical connective is called harmonious in a certain proof system if there exists a certain balance between the rules defining it. For example, in natural deduction based systems, harmony is ensured when introduction/elimination rules do not contain insufficient/excessive amounts of information [9]. In sequent calculus, this property is often guaranteed by the admissibility of a general initial axiom (identity-expansion) and of the cut rule (cut-elimination) [24].

In Lemma 3.1 we proved identity-expansion for LCE. In the following, we will complete the proof of harmony for LCE, proving that it enjoys the cut-elimination property. This will also guarantee cut-completeness from LE to LCE, as mentioned above.

Proving admissibility of cut rules in sequent based systems with multiple contexts is often tricky, since the cut formulas can change contexts during cut reductions. This is the case for LCE. The proof is by mutual induction, with inductive measure $(n,m)$ where $m$ is the cut-height, the cumulative height of derivations above the cut, and $n$ is the ecumenical weight of the cut-formula, defined as:

- $\text{ew}(p_i) = \text{ew}(\bot) = 0$
- $\text{ew}(A \star B) = \text{ew}(A) + \text{ew}(B) + 1$ if $\star \in \{\wedge, \to, \vee\}$
- $\text{ew}(p_i) = 4$
- $\text{ew}(!A) = \text{ew}(A) + 1$ if $! \in \{\neg, \exists x, \forall x\}$
- $\text{ew}(!x.$A$) = \text{ew}(A) + 4$
- $\text{ew}(A \circ B) = \text{ew}(A) + \text{ew}(B) + 4$ if $\circ \in \{\neg, \lor\}$

\[\text{ew}(\exists x.$A$) = \text{ew}(A) + 4\]
Intuitively, the ecumenical weight measures the amount of extra information needed (the negations added) to define classical connectives from intuitionistic and neutral ones.

**Theorem 3.3.** The rules $N - \text{cut}$ and $P - \text{cut}$ are admissible in LCE.

**Proof.** The dynamic of the proof is the following: cut applications either move up in the proof, *i.e.* the cut-height is reduced, or are substituted by simpler cuts of the same kind, *i.e.* the ecumenical weight is reduced, as in usual cut-elimination reductions. The cut instances alternate between positive and negative (and vice-versa) in the principal cases, where the polarity of the subformulas flip. We will detail the main cut-reductions.

- **Base cases.** Consider the derivation

\[
\frac{\Gamma \Rightarrow \Delta; p_i, \Gamma, p_i \Rightarrow \Delta; \Pi}{\Gamma \Rightarrow \Delta; \Pi} \quad \text{init}
\]

If $p_i$ is principal, then $\Pi = p_i$ and the derivation reduces to $\pi$.

If $p_i$ is not principal, then there is an atom $q_i \in \Gamma \cap \Pi$ and the reduction is a trivial one. Similar analyses hold for $N - \text{cut}$, when the left premise is an instance of init, as well as for the other axioms.

- **Non-principal cases.** In all the cases where the cut-formula is not principal in one of the premises, the cut moves upwards. The only exceptions are when:

  - dereliction is applied in the left premise

\[
\frac{\Gamma \Rightarrow \Delta, P, N; P}{\Gamma \Rightarrow \Delta, P; \Pi} \quad \text{D}
\]

In this case, we substitute the version of $N - \text{cut}$ for absorbing the dereliction

\[
\frac{\pi_1 \quad \pi_2}{\Gamma \Rightarrow \Delta, P; \Pi} \quad N - \text{cut}
\]

- weakening is applied in the left premise

\[
\frac{\pi_1 \quad \pi_2}{\Gamma \Rightarrow \Delta, P; \Pi} \quad N - \text{cut}
\]

In this case, we substitute the version of $N - \text{cut}$ for absorbing the weakening

\[
\frac{\pi_1 \quad \pi_2}{\Gamma \Rightarrow \Delta, P; \Pi} \quad N - \text{cut}
\]

- **Principal cases.** If the cut formula is principal in both premises, then we need to be extra-careful with the polarities. We show two most representative cases.
- $N = P \rightarrow, Q$, with $P, Q$ positive.

\[
\begin{align*}
\pi_1 & : \Gamma, P \Rightarrow \Delta, Q ; \Rightarrow R \quad \pi_2 & : \Gamma, P \Rightarrow \Delta; \Gamma, Q \Rightarrow \Delta; \Rightarrow L \\
\Gamma \Rightarrow \Delta; \Gamma \Rightarrow \Delta; & \\pi_3 & : N - \text{cut}_0
\end{align*}
\]

reduces to

\[
\begin{align*}
\pi_1 & : \Gamma, \neg Q, P \Rightarrow \Delta, Q ; \Rightarrow R \\
\Gamma & \Rightarrow \Delta; \neg Q, P \Rightarrow \Delta; \Rightarrow L \\
\pi_2 & : \Gamma, \neg Q \Rightarrow \Delta; N - \text{cut}_1
\end{align*}
\]

where $\pi^*_1$ is the same as $\pi_1$ where every application of the rule $\mathcal{D}$ over $Q$ in the above derivation is substituted by an application of $\neg$ over $\neg Q$. Observe that the cut-formula of $N - \text{cut}_1$ has lower ecumenical weight than $N - \text{cut}_0$, while the cut-height of $N - \text{cut}_2$ is smaller than $N - \text{cut}_0$. Finally, observe that this is a non-trivial cut-reduction: usually, the cut over the implication is replaced by a cut over $Q$ first. Due to polarities, if $Q$ is positive, then $\neg Q$ is negative and cutting over it will add to the left context the classical information $Q$, hence mimicking the behavior of formulas in the right input context.

- $N = \exists x. P$, with $P$ positive.

\[
\begin{align*}
\pi_1 & : \Gamma \Rightarrow \Delta; \exists x. P, P[y/x]; \Rightarrow \exists R \\
\Gamma & \Rightarrow \Delta; \exists x. P \Rightarrow \Delta; \Rightarrow L \\
\pi_2 & : N - \text{cut}_0
\end{align*}
\]

reduces to

\[
\begin{align*}
\pi_1 & : \Gamma, \neg P[y/x] \Rightarrow \Delta; \Rightarrow \exists R \\
\Gamma & \Rightarrow \Delta; \neg P[y/x] \Rightarrow \exists x. P; \Rightarrow \exists L \\
\pi_2 & : \Gamma, \neg P[y/x], P[z/x] \Rightarrow \Delta; \Rightarrow \exists L \\
\Gamma & \Rightarrow \Delta; \neg P[y/x], \exists x. P \Rightarrow \Delta; \Rightarrow N - \text{cut}_1
\end{align*}
\]

where the same observations for the above case hold, and $\pi_2[z/y]$ indicates the renaming of fresh variables in the derivation $\pi_2$. 

\[\square\]

We finish this section noting that polarities play an important role in the cut-elimination process. In fact, without them, adding a general cut rule would collapse the system to classical logic.

**Example 3.4.** If the cut rule

\[
\begin{align*}
\Gamma & \Rightarrow \Delta; A; \Pi' \Rightarrow \Delta; \Pi \\
\Gamma & \Rightarrow \Delta; \Pi
\end{align*}
\]

was admissible in LCE for an arbitrary formula $A$, then $\therefore \therefore A \lor \neg A$ would have the
proof

\[
\begin{array}{c}
A \Rightarrow A \lor \neg A; \quad \text{init}_i \\
A \Rightarrow A \lor \neg A; A \lor \neg A \\
\vdots \Rightarrow A \lor \neg A; \neg A \\
\vdots \Rightarrow A \lor \neg A; \neg A \\
\vdots \Rightarrow A \lor \neg A; A \lor \neg A \\
\vdots \Rightarrow A \lor \neg A; A \lor \neg A \\
\vdots \Rightarrow A \lor \neg A; \neg A \\
\end{array}
\]

\[\vdots \Rightarrow \neg A \quad \text{init}_i \\
\vdots \Rightarrow A \lor \neg A \\
\]

4 Ecumenical modalities

We will now extend the propositional fragment of LCE with modalities.

The language of (propositional, normal) modal formulas consists of a denumerable set \( P \) of propositional symbols and a set of propositional connectives enhanced with the unary modal operators \( \Box \) and \( \Diamond \) concerning necessity and possibility, respectively [2].

The semantics of modal logics is often determined by means of Kripke models. Here, we will follow the approach in [33], where a modal logic is characterized by the respective interpretation of the modal model in the meta-theory (called meta-logical characterization).

Formally, given a variable \( x \), we recall the standard translation \([\cdot]\) from modal formulas into first-order formulas with at most one free variable, \( x \), as follows: if \( p \) is atomic, then \([p]_x = p(x); \bot_x = \bot\); for any binary connective \( \star \), \([A \star B]_x = [A]_x \star [B]_x\); for the modal connectives

\[
\begin{align*}
[\Box A]_x &= \forall y(R(x, y) \rightarrow [A]_y) \\
[\Diamond A]_x &= \exists y(R(x, y) \wedge [A]_y)
\end{align*}
\]

where \( R(x, y) \) is a binary predicate.

Opening a parenthesis: such a translation has, as underlying justification, the interpretation of alethic modalities in a Kripke model \( M = (W, R, V)\):

\[
\begin{align*}
M, w \models \Box A & \iff \text{for all } v \text{ such that } wRv, M, v \models A. \\
M, w \models \Diamond A & \iff \text{there exists } v \text{ such that } wRv \text{ and } M, v \models A. \\
\end{align*}
\]

(1)

\( R(x, y) \) then represents the accessibility relation \( R \) in a Kripke frame. This intuition can be made formal based on the one-to-one correspondence between classical/intuitionistic translations and Kripke modal models [33]. We close this parenthesis by noting that this justification is only motivational, aiming at introducing modalities.

The object-modal logic OL is then characterized in the first-order meta-logic ML as

\[\vdash_{OL} A \quad \text{iff} \quad \vdash_{ML} \forall x,[A]_x\]

Hence, if ML is classical logic (CL), the former definition characterizes the classical modal logic \( K \) [2], while if it is intuitionistic logic (IL), then it characterizes the intuitionistic modal logic \( IK \) [33].

In this work, we will adopt ecumenical logic as the meta-theory (given by the system LCE), hence characterizing what we will define as the ecumenical modal logic EK.
4.1 An ecumenical view of modalities

The language of ecumenical modal formulas consists of a denumerable set \( P \) of (ecumenical) propositional symbols and the set of ecumenical connectives enhanced with unary ecumenical modal operators. Unlike for the classical case, there is not a canonical definition of constructive or intuitionistic modal logics. Here we will mostly follow the approach in [33] for justifying our choices for the ecumenical interpretation for possibility and necessity.

The ecumenical translation \([\cdot]_{e}^{K}\) from propositional ecumenical formulas into \(LCE\) is defined in the same way as the modal translation \([\cdot]_{e}^{K}\) in the last section. For the case of modal connectives, observe that, due to Proposition 2.1, the interpretation of ecumenical consequence should be essentially intuitionistic. This implies that the box modality is a neutral connective. The diamond, on the other hand, has two possible interpretations: classical and intuitionistic, since its leading connective is an existential quantifier. Hence we should have the ecumenical modalities: \(\Box, \Diamond_{i}, \Diamond_{c}\), determined by the translations

\[
[\Box A]_{e}^{K} = \forall y(R(x, y) \rightarrow [A]_{e}^{K})
\]

\[
[\Diamond_{i} A]_{e}^{K} = \exists y(R(x, y) \land [A]_{e}^{K})
\]

\[
[\Diamond_{c} A]_{e}^{K} = \exists y(R(x, y) \land [A]_{e}^{K})
\]

We will denote by \(EK\) the ecumenical modal logic meta-logically characterized by \(LCE\) via \([\cdot]_{e}^{K}\). Polarities will be extended to the modal case smoothly, that is, formulas with outermost connective classical or negation are negative, all the others are positive. Relational atoms are not polarizable.

In Figure 3 we present the pure, labeled ecumenical system \(labEK\). Observe that

\[
\vdash_{labEK} x : \Diamond_{c} A \leftrightarrow [A]_{e}^{K} 
\]

\[
\vdash_{labEK} x : \Diamond_{i} A \leftrightarrow [A]_{e}^{K} 
\]

On the other hand, \(\Box\) and \(\Diamond_{i}\) are not inter-definable. Finally, if \(A^{ec}\) is eventually externally classical, then

\[
\vdash_{labEK} x : \Box A^{ec} \leftrightarrow [A^{ec}]_{e}^{K} 
\]

This means that, when restricted to the classical fragment, \(\Box\) and \(\Diamond_{c}\) are duals. This reflects well the ecumenical nature of the defined modalities.

4.2 Ecumenical birelational models

The ecumenical birelational Kripke semantics, which is an extension of the proposal in [27] to modalities, was presented in [22].

**Definition 4.1.** A birelational Kripke model is a quadruple \(M = (W, \preceq, R, V)\) where \((W, R, V)\) is a Kripke model such that \(W\) is partially ordered with order \(\preceq\), \(R \subseteq W \times W\) is a binary relation, the satisfaction function \(V : (W, \preceq) \rightarrow (2^P, \subseteq)\) is monotone and:

- **F1.** For all worlds \(w, v, v'\), if \(wRv\) and \(v \preceq v'\), there is a \(w'\) such that \(w \preceq w'\) and \(w'Rv'\);
- **F2.** For all worlds \(w', w, v, v'\), if \(w \preceq w'\) and \(wRv\), there is a \(v'\) such that \(w'Rv'\) and \(v \preceq v'\).

An ecumenical modal Kripke model is a birelational Kripke model such that truth of an ecumenical formula at a point \(w\) is the smallest relation \(\models_{E}\) satisfying
Intuitionistic and Neutral Rules

\[
\begin{align*}
\Gamma, x : A, x : B & \Rightarrow \Delta; \Pi & \Gamma \Rightarrow \Delta; x : A & \quad \Gamma \Rightarrow \Delta; x : B & \quad \land R \\
\Gamma, x : A \land B & \Rightarrow \Delta; \Pi & \land L
\end{align*}
\]

\[
\begin{align*}
\Gamma, x : A & \Rightarrow \Delta; \Pi & \Gamma, x : B & \Rightarrow \Delta; \Pi & \lor L & \quad \lor R \\
\Gamma, x : A \lor B & \Rightarrow \Delta; \Pi
\end{align*}
\]

Classical Rules

\[
\begin{align*}
\Gamma, x : A & \Rightarrow \Delta; \Pi & \Gamma, x : A & \Rightarrow \Delta; \Pi & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow \\
\Gamma, x : \neg A & \Rightarrow \Delta; x : A & \Gamma, x : \neg A & \Rightarrow \Delta; x : A & \neg L & \quad \neg R
\end{align*}
\]

Modal Rules

\[
\begin{align*}
xR_y, y : A, x : \Box A, \Gamma & \Rightarrow \Delta; \Pi & xR_y, x : \Box A, \Gamma & \Rightarrow \Delta; x : A & \Box L & \quad \Box R \\
xR_y, \Gamma & \Rightarrow \Delta; y : A & xR_y, y : A, \Gamma & \Rightarrow \Delta; A & \Box L & \quad \Box R
\end{align*}
\]

Initial, Decision and Structural Rules

\[
\begin{align*}
\Gamma, x : A & \Rightarrow \Delta; x : A & \text{init} \\
\Gamma, x : A & \Rightarrow \Delta; x : A & \text{init}
\end{align*}
\]

Cut Rules

\[
\begin{align*}
\Gamma & \Rightarrow \Delta; x : P & x : P, \Gamma & \Rightarrow \Delta; \Pi & P - \text{cut} & \Gamma & \Rightarrow \Delta; x : N & \Pi & \Rightarrow N & \Delta, x : A, \Gamma & \Rightarrow \Delta; \Pi & N - \text{cut}
\end{align*}
\]

Figure 3: Ecumenical pure modal system labEK. In rules \(\Box R, \Box L, \Box L, \) the eigenvariable \(y\) does not occur free in any formula of the conclusion; \(N\) is negative and \(P\) is positive; \(p\) is atomic; \(\Pi^*\) is either empty or some \(z : P \in \Delta.\)
Theorem 4.2 (22). The system \textsc{labEK} is sound and complete w.r.t. the ecumenical modal Kripke semantics, that is, \( \vdash_{\textsc{labEK}} x : A \) iff \( \models_{E} A \).

We end this section with a small note on the relationship between the semantics and the dynamics of proofs. On a bottom-up reading of proofs, the \textsc{store} rule is a delay on applying rules over classical connectives. It corresponds to moving the formula up w.r.t. \( \leq \) in the birelational semantics. The rule \( \triangleright R \), on the other hand, slides the formula to a fresh new world, related to the former one through the relation \( R \). Finally, rule \( \neg R \) moves up the formula w.r.t. \( \leq \).

5 A nested system for ecumenical modal logic

The criticism regarding system \textsc{labEK} is that it includes labels in the technical machinery, hence allowing one to write sequents that cannot always be interpreted within the ecumenical modal language.

This section is devoted to present the pure label free calculus for ecumenical modalities introduced in [23], where every basic object of the calculus can be translated as a formula in the language of the logic.

The inspiration comes from Straßburger’s nested system for \textsc{IK} [34]. The main idea is to add nested layers to sequents, which intuitively corresponds to worlds in a relational structure [12, 4, 30].

The structure of a nested sequent for ecumenical modal logics is a tree whose nodes are multisets of formulas, just like in [24], with the relationship between parent and child in the tree represented by bracketing [1]. The difference however is that the ecumenical formulas can be left inputs (in the left contexts – marked with a full circle *), right inputs (in the classical right contexts – marked with a triangle ▽) or a single right output (the stoup – marked with a white circle ◊).
Definition 5.1. Ecumenical nested sequents are defined in terms of a grammar of input sequents (written \(\Lambda\)) and full sequents (written \(\Gamma\)) where the left/right input formulas are denoted by \(A^*\) and \(A^\triangleright\), respectively, and \(A^\diamond\) denote the output formula. When the distinction between input and full sequents is not essential or cannot be made explicit, we will use \(\Delta\) to stand for either case.

\[
\Lambda := \emptyset | A^* | A^\triangleright | \Lambda | \Lambda \triangleright \Lambda |
\Gamma := A^\diamond, \Lambda | [\Gamma], \Lambda
\Delta := \Lambda | \Gamma
\]

As usual, we allow sequents to be empty, and we consider sequents to be equal modulo associativity and commutativity of the comma.

We write \(\Gamma \ominus\diamond\) for the result of replacing an output formula from \(\Gamma\) by \(\bot\diamond\), while \(\Lambda \ominus\diamond\) represents the result of adding anywhere of the input context \(\Lambda\) the output formula \(\bot\diamond\).

Finally, \(\Delta *\) is the result of erasing an output formula (if any).

Observe that full sequents \(\Gamma\) necessarily contain exactly one output-like formula, having the form \(\Lambda_1, [\Lambda_2, [\ldots, [\Lambda_n, A^\diamond]\ldots]]\).

Example 5.2. The nested sequent \(\blacksquare c A \triangleright, [\neg B^\diamond], [C \land D^\bullet]\) represents the following tree of sequents

\[
\begin{array}{c}
\cdot \Rightarrow \blacksquare c A \\
\cdot \Rightarrow \cdot
\end{array}
\]

\[
\begin{array}{c}
\cdot \Rightarrow \bot B \\
C \land D \Rightarrow \\
\cdot
\end{array}
\]

The next definition (of contexts) allows for identifying subtrees within nested sequents, which is necessary for introducing inference rules in this setting.

Definition 5.3. An \(n\)-ary context \(\Delta_1^1 \cdots \Delta_n^1\) is like a sequent but contains \(n\) pairwise distinct numbered holes \(\{\}\) wherever a formula may otherwise occur. It is a full or an input context when \(\Delta = \Gamma\) or \(\Lambda\) respectively.

Given \(n\) sequents \(\Delta_1, \ldots, \Delta_n\), we write \(\Delta|\Delta_1| \cdots |\Delta_n\) for the sequent where the \(i\)-th hole in \(\Delta|\{\}\cdots|\{\}\) has been replaced by \(\Delta_i\) (for \(1 \leq i \leq n\), assuming that the result is well-formed, i.e., there is at most one output formula. If \(\Delta_i = \emptyset\) the hole is removed.

Given two nested contexts \(\Gamma(i) = \Delta_{i}^1, [\Delta_{i}^2, \ldots, [\Delta_{i}^n, \{\}]]\ldots, i \in \{1, 2\}\), their merge is

\[
\Gamma_1 \otimes \Gamma_2(i) = \Delta_{1}^1, \Delta_{2}^1, \ldots, [\Delta_{n}^1, \Delta_{n}^2, \ldots, [\Delta_{n}^1, \Delta_{n}^2, \ldots, [\emptyset, \{\}]]]
\]

Figure 4 presents the nested sequent system \(n\text{EK}\) for ecumenical modal logic \(\text{EK}\).

Example 5.4. Below right is the proof that \(\blacksquare A\) is a consequence of \(\neg \Box \neg A\) for any formula \(A\). Below left the proof that, if \(N\) is negative, then \(\Box N\) is a consequence of \(\neg \blacksquare \neg N\). In fact, this holds for and only for eventually externally classical formulas (see

---

\[2\] As observed in [30][13], the merge is a “zipping” of the two nested sequents along the path from the root to the hole.
**Intuitionistic and Neutral Rules**

\[
\begin{align*}
&\frac{\Gamma[A^*, B^*]}{\Gamma[A \land B^*]} \quad \land^* \\
&\frac{\Lambda[A^d]}{\Lambda[A \land B^*]} \quad \land^o \\
&\frac{\Gamma[A^*]}{\Gamma[A \lor B^*]} \quad \lor^* \\
&\frac{\Lambda[A^i]}{\Lambda[A \lor B^*]} \quad \lor^i \\
&\frac{\Gamma}{\Gamma[\bot^*]} \quad \bot^* \\
\end{align*}
\]

**Classical Rules**

\[
\begin{align*}
&\frac{\Gamma[A \rightarrow B^*, A^*]}{\Gamma[A \rightarrow B^*]} \quad \rightarrow^* \\
&\frac{\Lambda[A^*] \lor B^*}{\Lambda[A \rightarrow B^*]} \quad \rightarrow^o \\
&\frac{\Gamma[B^*]}{\Gamma[\neg \neg A^*]} \quad \neg^* \\
&\frac{\Gamma[A^*]}{\Gamma[\neg \neg A^*]} \quad \neg^i \\
&\frac{\Gamma[A^*]}{\Gamma[\neg \neg A^*]} \quad \neg^\gamma \\
\end{align*}
\]

**Modal Rules**

\[
\begin{align*}
&\frac{\Delta_i(\Box A^d, [A^*, A^v]) \quad \Box^*}{\Delta_i(\Box A^d, [A^*, A^v])} \\
&\frac{\Lambda([A^*]) \quad \Box^o}{\Lambda([A^*])} \\
&\frac{\Gamma([A^*]) \quad \Box^i}{\Gamma([A^*])} \\
&\frac{\Delta_i([A^*, A^v]) \quad \Box^c}{\Delta_i([A^*, A^v])} \\
&\frac{\Gamma^i([A^*]) \quad \Box^*}{\Gamma^i([A^*, A^v])} \\
&\frac{\Lambda([A^*, A^v]) \quad \Box^c}{\Lambda([A^*, A^v])} \\
\end{align*}
\]

**Initial, Decision and Structural Rules**

\[
\begin{align*}
&\frac{\Lambda[A^*, A^v]}{\Gamma[A^*, A^v]} \quad \text{init}_c \\
&\frac{\Gamma[A^*, A^v]}{\Gamma[A^*, A^v]} \quad \text{init}_c \\
&\frac{\Gamma[P^y, P^y]}{\Gamma[P^y, P^y]} \quad D \\
&\frac{\Lambda[N^y, N^y]}{\Lambda[N^y]} \quad \text{store} \\
&\frac{\Gamma[N^y]}{\Gamma[N^y]} \quad \text{cut}^\gamma \\
&\frac{\Gamma[P^y]}{\Gamma[P^y]} \quad \text{cut}^\gamma \\
&\frac{\Gamma[N^y]}{\Gamma[N^y]} \quad \text{cut}^\gamma \\
\end{align*}
\]

**Cut Rules**

\[
\begin{align*}
&\frac{\Gamma[P^y]}{\Gamma[P^y]} \quad \text{cut}^o \\
&\frac{\Gamma[N^y]}{\Gamma[N^y]} \quad \text{cut}^o \\
&\frac{\Gamma[N^y]}{\Gamma[N^y]} \quad \text{cut}^o \\
&\frac{\Gamma[N^y]}{\Gamma[N^y]} \quad \text{cut}^o \\
\end{align*}
\]

Figure 4: Nested ecumenical modal system nEK. \( P \) is a positive formula, \( N \) is a negative formula. \( p \) is atomic. \( \Gamma^\phi \) denotes either \( \Gamma^\land \) or \( \Gamma^\land [P^y, P^y] \) for some \( P^y \in \Gamma \).

**Definition 2.2**

\[
\begin{align*}
&\frac{[A^*, A^v, \bot^*]}{[A^*, A^v, \bot^*]} \quad \text{init}_c \\
&\frac{[\neg A^d]}{[\neg A^d]} \quad \neg^c \\
&\frac{\neg \neg \neg A^d, [\neg A^d]}{\neg \neg \neg A^d, [\neg A^d]} \quad \neg^c \\
&\frac{\neg \neg \neg A^*, [\neg A^d]}{\neg \neg \neg A^*, [\neg A^d]} \quad \neg^c \\
&\frac{\neg \neg \neg \neg \neg A^*, [\neg A^d]}{\neg \neg \neg \neg \neg A^*, [\neg A^d]} \quad \neg^c \\
\end{align*}
\]
5.1 Proof theoretic properties

As for labEK, the properties of nEK are inherited by the ones in LCE (see Lemma 3.1). We will list them explicitly since the notation is quite different.

Theorem 5.5. In nEK:

1. The rules $\vee^\circ, \vee^\circ, \rightarrow^\circ, \rightarrow^\circ, \neg^\circ, p^\circ, p^\circ, \neg^\circ$ and $\Delta$ are invertible.
2. The rules $\wedge^\circ, \wedge^\circ, \rightarrow^\circ, \neg^\circ, \neg^\circ, \neg^\circ$ and $\text{store}$ are totally invertible.
3. The following structural rules are admissible

\[
\begin{array}{c}
\frac{}{\Gamma} \quad \frac{\Lambda \otimes \Gamma}{\Lambda \otimes \Gamma} \quad W_c \\
\end{array}
\]

4. The rules $\text{cut}^\circ$ and $\text{cut}^\circ$ are admissible. The ecumenical weight is the following extension of the measure presented in Section 3.1 for propositional connectives

\[
\text{ew}(\diamond \cap A) = \text{ew}(A) + 1 \text{ if } \diamond \in \{\neg, \cap\} \quad \text{ew}(\diamond, A) = \text{ew}(A) + 4
\]

The invertible but not totally invertible rules in nEK concern negative formulas, hence they can only be applied in the presence of empty stoups ($\perp^\circ$). Note also that the rules $W_c, \vee^\circ$, and $\neg^\circ$ are not invertible, while $\rightarrow^\circ$ is invertible only w.r.t. the right premise.

5.2 Soundness and completeness

In this section we will show that all rules presented in Figure 4 are sound and complete w.r.t. the ecumenical birelational model. The idea is to prove that the rules of the system nEK preserve validity, in the sense that if the interpretation of the premises is valid, so is the interpretation of the conclusion.

The first step is to determine the interpretation of ecumenical nested sequents. In this section, we will present the translation of nestings to labeled sequents, hence establishing, at the same time, soundness and completeness of nEK and the relation between this system with labEK.

Definition 5.6. Let $\Sigma^\circ, \Sigma^\circ, \Pi^\circ$ represent that all formulas in the each set/multiset are respectively input left, right, or output formulas. The underlying set/multiset will represent in all cases the corresponding multiset of unmarked formulas. The translation $[\cdot]_i$ from nested into labeled sequents is defined recursively by

\[
[\Sigma^\circ_1, \Sigma^\circ_2, \Pi^\circ_i, [\Delta_1], \ldots, [\Delta_n]]_i \iff ([xR_i]_i, x : \Sigma_1 \Rightarrow \Sigma_2 ; x : \Pi_3) \otimes ([\Delta_i]_i)_i
\]

where $1 \leq i \leq n, x_i$ are fresh, $\perp$ is translated to the empty set, and the merge operation on labeled sequents is defined as

\[
(\Gamma_1 \Rightarrow \Delta; \Pi_1) \otimes (\Gamma_2 \Rightarrow \Delta; \Pi_2) \iff \Gamma_1, \Gamma_2 \Rightarrow \Delta; \Pi_1, \Pi_2
\]

Since full nested sequents have exactly one output formula (which can be $\perp$), the stoup in the labeled setting will have at most one formula, and the merge above is well defined. Given $R$ a set of relational formulas, we will denote by $xR^z$ the fact that there is a path from $x$ to $z$ in $R$, i.e., there are $y_j \in R$ for $0 \leq j \leq k$ such that $x = y_0, y_{j+1}Ry_j$ and $y_k = z$. 19
Theorem 5.7. Let $\Gamma$ be a nested sequent and $x$ be any label. The following are equivalent.

1. $\Gamma$ is provable in $nEK$;
2. $[\Gamma]_x$ is provable in $labEK$.

Proof. Let $xR^*z \in R$. Observe that:

- $[\Gamma]_x$ is provable in $nEK$.
- $[\Gamma]_x$ is provable in $labEK$.

Given this transformation, (1) $\iff$ (2) is easily proved by induction on a proof of $\Gamma/[\Gamma]_x$ in $nEK/labEK$. □

Theorems 4.2 and 5.7 immediately imply the following.

Corollary 5.8. Nested system $nEK$ is sound w.r.t. ecumenical birelational semantics.

We observe that, often, passing from labeled to nested sequents is not a simple task, sometimes even impossible. In fact, although the relational atoms of a sequent appearing in $labEK$ proofs can be arranged so as to correspond to nestings, as shown here, if the relational context is not tree-like [15], the existence of such a translation is not clear. For instance, how should the sequent $xRy, yRx, x : A \Rightarrow y : B$ be interpreted in modal systems with symmetrical relations?

Also thanks to their tree shape, it is possible to interpret nested sequents as ecumenical modal formulas, and hence prove soundness in the same way as in [34]. This direct interpretation of nested sequents as ecumenical formulas means that $nEK$ is a so-called internal proof system.

We end this section by briefly showing an alternative way of proving of soundness of $nEK$ w.r.t. the ecumenical birelational semantics. Please refer to [23] for a more detailed presentation.

Definition 5.9. The formula translation $et(\cdot)$ for ecumenical nested sequents is given by

$\text{Theorem 5.7.}$ Let $\Gamma$ be a nested sequent and $x$ be any label. The following are equivalent.

1. $\Gamma$ is provable in $nEK$;
2. $[\Gamma]_x$ is provable in $labEK$.

Proof. Let $xR^*z \in R$. Observe that:

- $[\Gamma]_x$ is provable in $nEK$.
- $[\Gamma]_x$ is provable in $labEK$.

Given this transformation, (1) $\iff$ (2) is easily proved by induction on a proof of $\Gamma/[\Gamma]_x$ in $nEK/labEK$. □

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We end this section by briefly showing an alternative way of proving of soundness of $nEK$ w.r.t. the ecumenical birelational semantics. Please refer to [23] for a more detailed presentation.

Definition 5.9. The formula translation $et(\cdot)$ for ecumenical nested sequents is given by
\[
\begin{align*}
\text{et}(\emptyset) & \equiv \top \\
\text{et}(A \land \emptyset) & \equiv A \land \text{et}(\emptyset) \\
\text{et}(\emptyset \lor A) & \equiv \neg A \land \text{et}(\emptyset) \\
\text{et}(\emptyset \land A) & \equiv \text{et}(\emptyset) \\
\text{et}(\emptyset \cdot \emptyset) & \equiv A \land \text{et}(\emptyset) \\
\text{et}(\emptyset \circ \emptyset) & \equiv \text{et}(\emptyset) \\
\text{et}(\emptyset \circ \Gamma) & \equiv \text{et}(\emptyset) \rightarrow i \text{et}(\Gamma)
\end{align*}
\]

where all occurrences of \( A \land \top \) and \( \top \rightarrow i A \) are simplified to \( A \). We say a sequent is valid if its corresponding formula is valid.

The next theorem shows that the rules of \( nEK \) preserve validity in ecumenical modal frames w.r.t. the formula interpretation \( \text{et}(\cdot) \).

**Theorem 5.10.** Let \( \Gamma_1, \ldots, \Gamma_n \in \{0, 1, 2\} \) be an instance of the rule \( r \) in the system \( nEK \). Then \( \text{et}(\Gamma_1) \land \ldots \land \text{et}(\Gamma_n) \rightarrow_i \text{et}(\Gamma) \) is valid in the birelational ecumenical semantics.

## 6 Fragments, axioms and extensions

In this section, we discuss fragments, axioms and extensions of \( nEK \).

### 6.1 Extracting fragments

For the sake of simplicity, in this sub-section negation will not be considered a primitive connective, it will rather take its respective intuitionistic or classical form.

**Definition 6.1.** An ecumenical modal formula \( C \) is classical (resp. intuitionistic) if it is built from classical (resp. intuitionistic) atomic propositions using only neutral and classical (resp. intuitionistic) connectives but negation, which will be replaced by \( A \rightarrow c \bot \) (resp. \( A \rightarrow i \bot \)).

The first thing to observe is that, when only pure fragments are concerned, weakening is admissible (remember that this is not the case for the whole system \( nEK \) – see Example 2.6). Also, only positive (resp. eventually externally classical) formulas are present in the intuitionistic (resp. classical) fragment.

Let \( nEK_c \) (resp. \( nEK_i \)) be the system obtained from \( nEK \) – \( W \) by restricting the rules to the intuitionistic (resp. classical) case – see Figures 5 and 6.

The intuitionistic fragment does not have classical input formulas and it coincides with the system \( NIK \) in [34].

Regarding \( nEK_c \), since all the classical/neutral rules are invertible, the following proof strategy is complete:

i. Apply the rules \( \land^*, \land^\circ, \square^*, \square^\circ \) and store eagerly, obtaining leaves of the form \( \Lambda(\bot^\circ) \).

ii. Apply any other rule of \( nEK_c \) eagerly, until either finishing the proof with an axiom application or obtaining leaves of the form \( \Lambda(P^*) \), where \( P \) is a positive formula in \( nEK_c \), that is, having as main connective \( \land \) or \( \square \). Start again from step (i).

This discipline corresponds to the focused strategy for a fragment of the two-sided version of the polarized system defined in [6], exchanging the polarities of diamond and box (which, as observed in the op.cit., is a matter of choice since all rules are invertible).
6.2 About axioms and extensions

Classical modal logic $K$ is defined as propositional classical logic, extended with the necessity rule (presented in Hilbert style) $A/□A$ and the distributivity axiom $k : □(A → B) → (□A → □B)$.

There are, however, many variants of axiom $k$ that induce logics that are classically, but not intuitionistically, equivalent (see [29, 33]). In fact, the following axioms follow from $k$ via the De Morgan laws, but are intuitionistically independent

$$k_1 : □(A → B) → (φA → φB) \quad k_2 : φ(A \lor B) → (φA \lor φB)$$

Combining axiom $k$ with axioms $k_1 – k_4$ defines intuitionistic modal logic IK [29].

In the ecumenical setting, this discussion is even more interesting, since there are many more variants of $k$, depending on the classical or intuitionistic interpretation of implications and diamonds.

It is an easy exercise to show that the intuitionistic versions of $k_1 – k_4$ are provable in $nEK$. One could then ask: what happens if we exchange the intuitionistic versions of the connectives with classical ones?

Consider $k^{αγ}_β : □(A → B) → γ(□A → Λγ □B)$ with $α, β, γ \in \{i, c\}$. First of all, note that $k^{αγ}_β$ is not provable, for any $β, γ$. This is a consequence of the fact that $A → □B, A \not\models □B$ in $EK$ in general (see Equation [5]). Moreover, since $C → □D \models C → D$ in $EK$, $k^{αγ}_β \models k^{αγ}_β$ for any value of $β, γ$. The same reasoning can be extended to all the
The rules $b$, $4$, and $5$ match the rules ($\ast$) and ($\ast\ast$) presented in [33], and are depicted in Figure 7.

For completing the ecumenical view, the classical ($\ast$) rules for extensions are justified via translations from labeled systems to $\eta\mathbf{EK}$. We first translate the labeled rules for extensions appearing in [33] to $\mathbf{EK}$ then use the translation on derivations defined in Section 5.2 to justify the rule scheme.

For example, starting with the rule $T$ below left, which is the labeled rule corresponding to the axiom $T$ in [33], the labeled derivation on the middle justifies the classical nested rule in the right.

\[
\frac{xR, \Gamma \vdash z : C}{\Gamma \vdash z : C} \quad \frac{xR, \Sigma, \Delta \vdash A, x : \phi A ; z : \perp}{\Gamma, \Sigma, \Delta \vdash A, x : \phi A ; z : \perp} \quad \frac{\phi, R}{\Gamma, \Sigma, \Delta \vdash A'}{\Delta^\perp \{ A' \}}
\]

The rules $b^\ast$, $4^\ast$ and $5^\ast$, shown in Figure 7, are obtained in the same manner.

Restricted to the fragments described in the last section, by mixing and matching...
these rules, we obtain ecumenical modal systems for the logics in the S5 modal cube [2] not defined with axiom d.

7 Related and future work

The main idea behind Prawitz’ ecumenical system [31] is to build a proof framework in which classical and intuitionistic logics may co-exist in peace. Although one could argue that this is easily done using the well known double-negation translations by Kolmogorov, Gödel, Gentzen and others [11], Prawitz’ view matches the idea presented by Liang and Miller in their PIL system presented in [19]: not seeing classical logic as a fragment of intuitionistic logic but rather to determine parts of reasoning which are classical or intuitionistic in nature. While double negation acts on formulas, the approach in [19] and also followed here concerns proofs. For example, we do not want to interpret \( A \lor \neg A \) as “it is not the case that A does not hold and it is not the case that it is not the case that A holds”.

Rather, we aim at identifying the points in proofs where the excluded middle is valid and/or necessary.

The similarities between our work and the system presented in [19] ends there, though. Indeed, in the op.cit. there are two versions of the constant for absurdum and universal quantifier, and all connectives have a dual version. For example, the intuitionistic implication \( \supset \) comes with the intuitionistic dual \( \propto \), a form of (non-commutative) conjunction, which has no correspondent in usual classical or intuitionistic systems. Also, these dualized versions have opposite polarities (red and green), that do not match Girard’s original idea of polarities: They are, instead, defined model theoretically. In this work, we opt for smoothly extending well known systems and features (like stoup or polarities), which turns LCE and PIL incomparable. It would be interesting to investigate, for example, if PIL could be smoothly extended to the modal case, as done in this work.

There are other proposals for ecumenical systems in the literature. For, in [3] the authors present a (type) theory in \( \lambda \Pi \)-calculus modulo theory, where proofs of several logical systems can be expressed. We are planning to propose type systems related to the systems/fragments described in this paper, and it would be interesting to see the intersection that may appear from the two approaches. It would be also interesting to implement ecumenical provers, as well as to automate the cut-elimination proof in the L-Framework [26].

A complete different approach comes from the school of combining logics [7, 20, 5], where Hilbert like systems are built from a combination of axiomatic systems. As we trail the exact opposite path, it would be interesting to see if (the propositional fragment of) Prawitz’ natural deduction system is axiomatizable.

Finally, the presence of polarization and stoup paves the way for proposing focused ecumenical systems. For getting a complete focused discipline, though, it would be necessary to add polarized versions of conjunction and disjunction, as done e.g. in [19, 6]. This would give a unified focused framework, which could be used, among other things, to automatically extracting rules from axioms, as done in [21].
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