Existence and classification of maximally non-integrable distributions of derived length one

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Abstract

The $h$-principles for maximally non-integrable tangent distributions of derived length one on manifolds are studied in this paper. Such distributions of odd rank are dealt with. The formal structures for such distributions are introduced. From the viewpoint of the $h$-principles, we discuss the existence and classification of such structures.

1 Introduction

Existence and Classification of geometric structures on manifolds are fundamental problems in differential topology. In this paper, we discuss such problems for maximally non-integrable tangent distributions of derived length one of odd rank. We obtain the necessary and sufficient condition for the existence of such distributions on manifolds possibly closed. In addition, we discuss classification of such structures up to isotopy.

First, we introduce the structures that we deal with in this paper. Let $V$ be a manifold of dimension $n$. A tangent distribution on $V$ of rank $r$ is a subbundle $\mathcal{D} \subset TV$ of the tangent bundle of $V$ whose fiber is of dimension $r$. At each point $x \in V$, a subspace $\mathcal{D}^2_x = [\mathcal{D}, \mathcal{D}]_x + \mathcal{D}_x \subset T_x V$ is derived from $\mathcal{D}$ by the Lie bracket (see Section 2.1 for precise definition). If $\mathcal{D}^2 = \mathcal{D}$ at any $x \in V$, $\mathcal{D}$ is said to be integrable. When a distribution is integrable, it generates a foliation on $V$. We deal with non-integrable distributions on this paper. We impose further conditions on tangent distributions. A tangent distribution $\mathcal{D} \subset TV$ is said to be completely non-holonomic with derived length 1 if $\mathcal{D}^2 = TV$. We deal with such tangent distributions of odd rank $r = 2k + 1$. A distribution $\mathcal{D} \subset TV$ on $V$ of rank $r$ is locally described as a kernel of an $(n-r)$-tuple of 1-forms $\alpha_1, \alpha_2, \ldots, \alpha_{n-r}$:

$$\mathcal{D} = \{ \alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_{n-r} = 0 \}.$$

We say a distribution $\mathcal{D}$ is maximally non-integrable if $(d\alpha_1)^k, \ldots, (d\alpha_{n-r})^k$ are linearly independent $(2k)$-forms at each point (see Section 2.1 for precise definition). We remark that, from this condition, $\mathcal{D}$ is of derived length one (see Section 2.1). In this paper we deal with such tangent distributions, maximally non-integrable tangent distributions of derived
length one and of odd rank. Note that, if the corank $n - r = n - (2k + 1)$ of such a tangent distribution is 1, it is the so-called even-contact structure or quasi-contact structure. In that sense, maximally non-integrable distribution of derived length one is regarded as a generalization of even-contact structure.

We introduce, in this paper, the formal structure for maximally non-integrable tangent distribution of derived length one. Let $V$ be a manifold of dimension $n$, and $\mathcal{D} \subset TV$ a tangent distribution of rank $r = 2k + 1$. We say the distribution $\mathcal{D}$ is almost maximally non-integrable of derived length one if there exist an $(n - r)$-tuple of 2-forms $\omega_1, \omega_2, \ldots, \omega_{n-r}$ for that $(\omega_1)^k, (\omega_2)^k, \ldots, (\omega_{n-r})^k$ are linearly independent on $\mathcal{D}$ at any point. Clearly, maximally non-integrable distribution of derived length one is almost maximally non-integrable distribution of derived length one for $\omega_i = d\alpha_i$.

Now, the existence theorem for maximally non-integrable distributions of derived length one is described as follows.

**Theorem A.** Let $V$ be a manifold of dimension $n$ possibly closed. The manifold $V$ admits a maximally non-integrable distribution of derived length one and of rank $2k + 1$ if and only if it admits an almost maximally non-integrable distribution of derived length one and of rank $2k + 1$.

Recall that, if the corank $n - r = n - (2k + 1)$ of the distribution $\mathcal{D} \subset TV$ is 1, the tangent distribution $\mathcal{D}$ is an even-contact structure. We should remark that, in this case, this theorem is equivalent to the theorem on McDuff’s $h$-principle for even-contact structures (see [Mc]). In this sense, Theorem A in this paper can be regarded as a generalization of McDuff’s $h$-principle.

The first explicit generalization may be a distribution of type $(3, 5)$. It is, by definition, a tangent distribution $\mathcal{D}$ of rank 3 on a manifold $V$ of dimension 5 that satisfies $\mathcal{D}^2 = TV$. When we deal with orientable distributions, we obtain the following as a corollary of Theorem A.

**Corollary B.** Let $V$ be a manifold of dimension 5 possibly closed. The manifold $V$ admits an orientable distribution of type $(3, 5)$ if and only if it admits a trivial subbundle $E \subset TV$ of rank 3 of the tangent bundle.

In Section 5, we discuss a more general corollary of Theorem A for orientable distributions than Corollary B. Then the result concerning the classification is described as follows. It is also considered as a generalization of McDuff’s $h$-principle (see [Mc]).

**Theorem C.** Let $\mathcal{D}_1, \mathcal{D}_2 \subset TV$ be maximally non-integrable distributions of derived length one and of odd rank on a manifold $V$. If they are homotopic through almost maximally non-integrable distributions then they are isotopic.

This implies that the isotopic classification of maximally non-integrable distributions of derived length one follows a “topological” classification. In other words, all such structures are “flexible”, and there is no “rigid” class. The explicit classification is given by algebraic topology on each base manifold $V$.

These theorems are proved from the view point of the $h$-principles. Gromov’s convex integration method is applied to show the $h$-principles. As we mentioned above, these results are generalizations of McDuff’s $h$-principle for even-contact structures. It is originally
proved by using Gromov’s convex integration method (see [Mc]), although an alternative proof is given in [EM]. The proof in this paper is influenced by the original one.

Let us observe some related results around. A completely non-integrable distribution of derived length one is a contact structure if it is of corank 1 and even rank. The existence of a contact structure on a closed orientable 3-manifold is proved by Martinet [Ma] by a constructive method. For higher-dimensional manifolds, it is proved by Borman, Eliashberg, and Murphy [BEM] from the view point of the h-principle that, if a manifold admits an almost contact structure, it admits a genuine contact structure. One of the properties that make contact topology important is the existence of the rigid (tight) class. It is proved by Eliashberg [E] that contact structures on 3-manifolds are divided into two classes, tight (rigid) and overtwisted (flexible). For higher-dimensional manifolds, the overtwistedness is introduced in [BEM]. On the other hand, McDuff’s h-principle [Mc] implies that there is no rigid class for even-contact structures. In this sense, the result in this paper implies that there is no rigid class for maximally non-integrable distributions of derived length one of odd rank.

This paper is organized as follows. In the following section, we introduce some basic notions for this paper. First, we review something on tangent distributions, and then we introduce the h-principles and convex integration method. In Section 3 we set the problem to be solved in terms of the h-principles. Then we show the h-principles by Gromov’s convex integration method in Section 4. The proof is concluded in Section 5.

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2 preliminaries

In this section, we discuss basic things which are needed in the proof of Theorems. In Subsection 2.1, we introduce tangent distributions on manifolds. In Subsection 2.2 the notion of the h-principles and the method of convex integration are introduced.

2.1 Tangent distribution

First of all, we introduce the notion of tangent distributions on manifolds and something concerning the notion. Let $M$ be a smooth manifold of dimension $n$. A subbundle $\mathcal{D} \subset TM$ of the tangent bundle $TM$ with $r$-dimensional fibers is called a tangent distribution (or distribution for short) of rank $r$ on $M$. Let $\text{Sec}(\mathcal{D})$ be the set of all cross-sections (vector fields) of the subbundle $\mathcal{D} \to M$, and $\mathcal{D} \subset TM$ the sheaf of the vector fields. Then $\mathcal{D}$ can be considered as a distribution of the $r$-dimensional tangent subspace $\mathcal{D}_p \subset T_p M$ at each point $p \in M$, or a $r$-dimensional plane field on $M$.

In this paper, we deal with completely non-holonomic distributions, which are defined as follows. Let $\mathcal{D}$ be a distribution of rank $r$ on an $n$-dimensional manifold $M$. A distribution $\mathcal{D} \subset TM$ is said to be completely non-holonomic if, for any local frame $\{X_1, \ldots, X_k\}$ of $\mathcal{D}$, its iterated Lie brackets $X_i, [X_i, X_j], [X_i, [X_j, X_l]], \ldots$ spans the tangent bundle $TM$. This condition is observed from the view point of distributions derived by the Lie brackets as follows. For a given local vector field $X \in \mathcal{D}$ defined on a neighborhood of a point $x \in M$, we have the germ $X_x \in \mathcal{D}(x)$ as an element of the stalk $\mathcal{D}(x)$ at $x \in M$. Then by
inductively setting
\[
\mathcal{D}(x)^2 = ([\mathcal{D}, \mathcal{D}] + \mathcal{D})_x \\
:= \text{Span} \{ [X, Y] + Z_x \in TM(x) \mid X, Y, Z \in \mathcal{D}(x) \},
\]
\[
\mathcal{D}(x)^{l+1} = ([\mathcal{D}, \mathcal{D}^l] + \mathcal{D})_x \\
:= \text{Span} \{ [X, Y] + Z_x \in TM(x) \mid X \in \mathcal{D}(x), Y, Z \in \mathcal{D}(x)^{l} \}, \quad (l = 2, 3, \ldots),
\]
we have a flag \( \mathcal{D} \subset \mathcal{D}^2 \subset \cdots \subset \mathcal{D}^l \subset \cdots \subset TM \) of subsheaves. The condition that the distribution \( \mathcal{D} \subset TM \) is completely non-holonomic implies that \( \mathcal{D}^l = TM \) holds for some \( l \in \mathbb{N} \). We have the subspace \( \mathcal{D}^l_x \subset T_x M \) as \( \mathcal{D}^l := \{ X \in T_x M \mid X \in \mathcal{D}(x)^l \} \). If \( \dim(\mathcal{D}^l) \) is constant for any \( x \in M \), there corresponds a distribution \( \mathcal{D}^l \subset TM \) on \( M \). It is called the derived distribution of \( \mathcal{D} \). In this paper, we deal with completely non-holonomic distributions of derived length one. In other words, completely non-holonomic distributions that satisfy \( \mathcal{D}^2 = TM \). When the rank of such a tangent distribution \( \mathcal{D} \) on \( M \) is \( r \) and the dimension of the manifold \( M \) is \( n \), then \( \mathcal{D} \) is said to be of type \( (r, n) \).

Now, we introduce basic property concerning completely non-holonomic distributions of derived length one. Let \( \mathcal{D} \subset TM \) be a distribution of rank \( r \) on an \( n \)-dimensional manifold \( M \). Let \( \mathcal{F}(\mathcal{D}) \subset T^*M \) be the bundle of covectors that annihilate \( \mathcal{D} \). Note that \( \mathcal{F}(\mathcal{D}) \) is a vector bundle of fiber dimension \( n - r \). The distribution \( \mathcal{D} \) can be locally described by 1-forms \( \omega_1, \omega_2, \ldots, \omega_{n-r} \in \text{Sec}(T^*M) \) as \( \mathcal{D} = \{ \omega_1 = 0, \omega_2 = 0, \ldots, \omega_{n-r} = 0 \} \). We observe the choice of the basis \( \mathcal{F}(\mathcal{D}) \) annihilating the distribution \( \mathcal{D} \). If \( \mathcal{D} \) is completely non-holonomic with derived length one, then we can take a basis \( \{ \omega_1, \ldots, \omega_{n-r} \} \) as follows (see [A3] for example).

**Proposition 2.1.** Let \( \mathcal{D} \subset TM \) be a distribution of rank \( r \) on an \( n \)-dimensional manifold \( M \). The distribution \( \mathcal{D} \) is completely non-holonomic with derived length one if and only if \( \mathcal{F}(\mathcal{D}) \) has a local basis \( \{ \omega_1, \omega_2, \ldots, \omega_{n-r} \} \) which satisfies the condition that \( (n - r + 2) \) forms \( \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_{n-r} \wedge d\omega_i, i = 1, 2, \ldots, n - r \), are vanishing nowhere and pointwise linearly independent.

In addition, we impose further conditions to distributions that we are dealing with in this paper. Let \( \mathcal{D} \subset TM \) be a distribution of rank \( r \) on an \( n \)-dimensional manifold \( M \). Let \( \{ \omega_1, \omega_2, \ldots, \omega_{n-r} \} \) be the defining 1-forms of \( \mathcal{D} \). When \( \mathcal{D} \) is of odd rank \( r = 2k + 1 \), we say \( \mathcal{D} \) is maximally non-integrable if it satisfies the following condition:

\[ \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_{n-2k-1} \wedge (d\omega_i)^k, \quad i = 1, 2, \ldots, n - 2k - 1, \] are pointwise linearly independent.

Note that such distributions are completely non-holonomic with derived length one from Proposition 2.1. When the rank of \( \mathcal{D} \) is \( r = 3 \), such notions are equivalent to completely non-holonomic with derived length one. The dimension \( n \) of manifolds for such distributions of rank \( 2k + 1 \) should be \( 2k + 2 \leq n \leq 2(2k + 1) \).

**Examples.** (1) A contact structure \( \xi \) on a \((2n + 1)\)-dimensional manifold \( M \) is completely non-holonomic with derived length one. Actually, \( \xi \subset TM \) is a distribution of corank 1 on \( M \) which is completely non-integrable by definition. By the Darboux theorem, it is locally contactomorphic to the standard contact structure \( \xi_0 = \{ \alpha_0 = 0 \} \), \( \alpha_0 = dz - \sum_{i=1}^{n} y_idx_i \) on \( \mathbb{R}^{2n+1} \) with coordinates \( (z, x_1, y_1, \ldots, x_n, y_n) \). This \( \xi_0 \) is spanned
by $2n$ vector fields $(\partial/\partial x_1) + y_1(\partial/\partial z_1), (\partial/\partial y_1), \ldots, (\partial/\partial x_n) + y_n(\partial/\partial z), (\partial/\partial y_n)$. Since $[(\partial/\partial x_1) + y_1(\partial/\partial z_1), (\partial/\partial y_1)] = (\partial/\partial z)$, we have $\xi^2 = TM$. We should remark that, from the same discussion, an even-contact structure is also completely non-holonomic with derived length one and maximally non-integrable.

(2) The distribution $\mathcal{D} = \{dy - z_1 dx_1 = 0\}$ on $\mathbb{R}^5$ with coordinates $(x_1, x_2, y, z, z_2)$ is completely non-holonomic with derived length one. However, it is not completely non-integrable.

(3) The canonical distribution on the jet space $J^1(\mathbb{R}, \mathbb{R}^k) \cong \mathbb{R}^{2k+1}$ is completely non-holonomic with derived length one. Let $(x, y_1, \ldots, y_k, z_1, \ldots, z_k) \in J^1(\mathbb{R}, \mathbb{R}^k)$ be the coordinates, where $z_i$ corresponds to $dy_i/dx$. The canonical distribution $\mathcal{D}_0$ is given as

$$\mathcal{D}_0 := \{\alpha_1 = 0, \ldots, \alpha_k = 0\}, \quad \text{where } \alpha_i := dy_i - z_i dx.$$

It is a distribution of rank $k + 1$, or corank $k$, spanned by the following vector fields:

$$\frac{\partial}{\partial x} + \sum_{j=1}^k z_j \frac{\partial}{\partial y_j}, \quad \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_k}.$$

Since $\left[ (\partial/\partial x) + \sum_{j=1}^k z_j (\partial/\partial y_j), (\partial/\partial z) \right] = (\partial/\partial y_i), \ i = 1, 2, \ldots, k$, we have $\mathcal{D}_0^2 = TR^{2k+1}$. Note that, if $k = 1$, it is the standard contact structure on $\mathbb{R}^3$. If $k = 2$, it is completely non-integrable. However, for $k > 2$ it is not completely non-integrable.

In this paper we deal with maximally non-integrable distributions of odd rank.

## 2.2 Homotopy principle and Convex integration

We introduce the notion of the $h$-principles in Section 2.2.1. It is a key tool to show Theorems A and C. In order to show that our objects satisfy the $h$-principles, we use the convex integration method due to Gromov. In Section 2.2.2 we introduce that method. In order to make this paper self-contained, some basic things that are needed in this paper are defined there. Readers should refer to the literature for further study on the $h$-principles (see [G2], [EM], [Am], [S]).

### 2.2.1 Differential relations and homotopy principle

We introduce a philosophy to “solve” some “problems” related to differentiation. In other words, we introduce the notion of the $h$-principle for partial differential relations.

First, we review basic notions concerning fibrations and jets. Let $p: X \to V$ be a fibration over a manifold $V$. The $r$-jet extension $p^r: X^{(r)} \to V$ of the fibration $p: X \to V$ is defined as the manifold $X^{(r)}$ consisting of all $r$-jets of cross-sections $V \to X$ of the fibration at all $v \in V$ and the projection $p^r: X^{(r)} \to V$. $X^{(r)}$ is called the $r$-jet space. The cross-section $J^r_f: V \to X^{(r)}$ of $p^r: X^{(r)} \to V$ is the $r$-jet extension of a cross-section $f: V \to X$.

Then, we introduce a description method for certain problems. Let $p: X \to V$ be a fibration. A differential relation of order $r$ for sections of $p: X \to V$ is defined as a subset $\mathcal{R}$ of the $r$-jet space $X^{(r)}$. For example, a system of differential equations is regarded as a subset of the jet space $J^r(\mathbb{R}^n, \mathbb{R}^m)$, that is, a differential relation. There are two kinds of solutions to a differential relation. A formal solution to the differential relation $\mathcal{R} \subset X^{(r)}$ is
defined to be a section $F: V \to X^{(r)}$ of the fibration $p^{r}: X^{(r)} \to V$ that satisfies $F(V) \subset \mathcal{R}$. Let Sec$\mathcal{R}$ denote the space of formal solution to $\mathcal{R}$. On the other hand, a genuine solution to $\mathcal{R} \subset X^{(r)}$ is defined to be a section $f: V \to X$ of $p: X \to V$ whose $r$-jet satisfies $(J_{f})^{r}(V) \subset \mathcal{R}$. Let Sol$\mathcal{R}$ denote the space of genuine solutions to $\mathcal{R}$.

Some examples of differential relations come from singularity theory. Roughly speaking, in some cases, a singular point is a point where certain expressions related to derivatives vanish. Then, to singular points, there corresponds the set $\Sigma \subset X^{(r)}$ called the singularity. Setting $\mathcal{R} := X^{(r)} \setminus \Sigma$, we have an open differential relation. In many cases, it is important for singularity theory to solve such differential relations.

Now, we introduce the notion of the homotopy principle. In general, the formal solvability of a differential relation is just a necessary condition for the genuine solvability of the differential relation. In some cases, the formal one is sufficient for the genuine solvability.

The notion, the homotopy principle, is introduced by Gromov and Eliashberg to formalize such properties (see [GE]). Let $p: X \to V$ be a fibration, and $\mathcal{R} \subset X^{(r)}$ a differential relation. $\mathcal{R}$ is said to satisfy the homotopy principle (or $h$-principle for short) if every formal solution to $\mathcal{R}$ is homotopic in Sec$\mathcal{R}$ to a genuine solution to $\mathcal{R}$. In this paper, we use the notion to show the existence of a genuine solution. In addition to that, there are some flavors of the $h$-principle (see [EM]). In this paper we introduce one of them. The differential relation $\mathcal{R} \subset X^{(1)}$ is said to satisfy the one-parametric $h$-principle if every family $\{f_{t}\}_{t \in [0,1]}$ of formal solutions to $\mathcal{R}$ between genuine solutions $f_{0}$, $f_{1}$ can be deformed inside Sec$\mathcal{R}$ to a family $\{f_{t}\}_{t \in [0,1]}$ of genuine solutions to $\mathcal{R}$ keeping both the ends $f_{0}$, $f_{1}$. The notion is used to classifications.

In general, it is difficult to show if a differential relation satisfies the $h$-principles. For each relation, we should apply each suitable method. We introduce one of such methods in the next section.

### 2.2.2 Convex integration method

In this section, we introduce a method due to Gromov to show the $h$-principle for certain differential relations, which is called the convex integration theory.

First, we recall ampleness of subsets. Let $P$ be an affine space, and $\Omega \subset P$ a subset. The minimum convex set that include $\Omega$ is called the convex hull of $\Omega$. The subset $\Omega \subset P$ is said to be ample if the convex hull of each path-connected component of $\Omega$ is $P$. The empty set is also defined to be ample.

In order to define the ampleness of differential relations, we first introduce the principal directions for fibrations. Let $p: X \to V$ be a fibration over an $n$-dimensional manifold $V$ with the fiber dimension $q$. For the 1-jet $p^{1}: X^{(1)} \to V$, there exists the natural projection $p^{1}_{0}: X^{(1)} \to X^{(0)} = X$, which is an affine bundle. Let

$$E_{x} := (p^{1}_{0})^{-1}(x), \quad x \in X$$

denote the fiber of $p^{1}_{0}: X^{(1)} \to X$ over $x \in X$, and Vert$_{x} \subset T_{x}X$ the $q$-dimensional tangent space at $x \in X$ of the fiber $p^{-1}(p(x)) \subset X$ of $p: X \to V$ over $p(x) \in V$. Then the fiber $E_{x} \subset X$ can be identified with

$$\text{Hom}(T_{p(x)}V, \text{Vert}_{x}) \cong \text{Hom}(\mathbb{R}^{n}, \mathbb{R}^{q}) \cong \{q \times n \text{ matrices}\}$$

as follows. As $X^{(1)}$ is the 1-jet space, a point of $X^{(1)}$ is regarded as a non-vertical $n$-dimensional subspace $P_{x} \subset T_{x}X$, where “non-vertical” implies that $P_{x}$ is transverse to
Vert_x \subset T_x X. Then, regarding P_x \subset T_x X \cong T_{p(x)} V \oplus \text{Vert}_x as a graph, there corresponds a linear mapping. Then, for a fixed point x \in X, the fiber E_x \subset X^{(1)} is identified with Hom(T_{p(x)} V, \text{Vert}_x). Now we define the principal directions for fibrations. Suppose that a hyperplane \tau \subset T_{p(x)} V and a linear mapping l: \tau \to \text{Vert}_x are given. For these \tau, l, let P^l_\tau \subset E_x denote the affine subspace of E_x defined as
\[ P^l_\tau := \{ L \in \text{Hom}(T_{p(x)} V, \text{Vert}_x) \mid L|_\tau = l \} \subset E_x. \]

Such affine subspaces of E_x are said to be \textit{principal}. Note that the principal subspaces are parallel q-dimensional affine spaces if the hyperplane \tau \subset T_{p(x)} V is fixed. The direction of the principal subspaces P^l_\tau determined by \tau is called the \textit{principal direction}.

Now we define the ampleness of differential relations. Let p: X \to V a fibration and \mathcal{R} \subset X^{(1)} a differential relation. A differential relation \mathcal{R} is said to be \textit{ample} if the intersection of \mathcal{R} with any principal subspaces is ample in the affine fiber.

Then the key tool in this paper is the following theorem due to Gromov (see [G1], [G2], [EM], [Am], [S]).

\textbf{Theorem 2.2.} Let p: X \to V be a fibration and \mathcal{R} \subset X^{(1)} an open differential relation. If \mathcal{R} \subset X^{(1)} is ample then \mathcal{R} satisfies the \textit{h-principle}. In addition, under this condition, \mathcal{R} also satisfies the one-parametric \textit{h-principle}.

\textbf{Remark.} Gromov showed by this method that other flavors of the \textit{h-principles} also holds from the ampleness of open differential relations. (see [G1], [EM])

As we see in Section 2.2.1, singularity theory is a rich source of open differential relations. Let p: X \to V a fibration. A singularity \Sigma \subset X^{(1)} is said to be \textit{thin} if, at any point a \in \Sigma, the intersection P \cap \Sigma with any principal subspace P through a \in \Sigma is a stratified subset of dimension greater than or equal to 2 in P. When the singularity \Sigma \subset X^{(1)} is thin, the open differential relation \mathcal{R} := X^{(1)} \setminus \Sigma is ample. Then, as a corollary of Theorem 2.2 we obtain the following.

\textbf{Corollary 2.3.} Let p: X \to V a fibration. If a singularity \Sigma \subset X^{(1)} is thin, then the differential relation \mathcal{R} := X^{(1)} \setminus \Sigma satisfies the \textit{h-principle}. In addition, \mathcal{R} also satisfies the one-parametric \textit{h-principle} under the same condition.

Next, we introduce the homotopy principle for differential sections for a linear differential operator \mathcal{F}, following [EM]. Let p_x: X \to V and p_z: Z \to V be vector bundles over a manifold V. We first introduce an operator \mathcal{F}: \text{Sec}(X^{(1)}) \to \text{Sec}(Z) called a first order linear differential operator as follows. Let F: X^{(1)} \to Z be a fiberwise homomorphism between vector bundles over V. Then there corresponds a mapping \tilde{F}: \text{Sec}(X^{(1)}) \to \text{Sec}(Z) as \sigma \mapsto F \circ \sigma for a section \sigma of the jet extension (p_x)^1: X^{(1)} \to V. By the composition with the differential operator J^1: \text{Sec}(X) \to \text{Sec}(X^{(1)}), we have the linear operator
\[ \mathcal{F} := \tilde{F} \circ J^1: \text{Sec}(X) \to \text{Sec}(Z). \]

The operators of this type are called the \textit{first order linear differential operators}. The bundle homomorphism F for a first order linear differential operator \mathcal{F} is called the \textit{symbol} of \mathcal{F}. Let \text{Symb} \mathcal{F} = F denote it.

For example, the exterior derivative of differential p-forms
\[ d: \text{Sec} \left( \bigwedge^p T^* V \right) \to \text{Sec} \left( \bigwedge^{p+1} T^* V \right) \]
is a first order linear differential operator. Its symbol $D := \text{Symb} d$ is a fiberwise epimorphism $D: (\wedge^p T^*V)^{(1)} \to \wedge^{p+1} T^*V$.

We now introduce the homotopy principle for differential sections. Let $p_x: X \to V$ and $p_z: Z \to V$ be vector bundles over a manifold $V$, and $\mathcal{F}: \text{Sec}(X) \to \text{Sec}(Z)$ a first order linear differential operator. An $\mathcal{F}$-section is, by definition, a section $s_z: V \to Z$ of the bundle $p_z: Z \to V$ for which there exists a section $s_x: V \to X$ of $p_x: X \to V$ that satisfies $s_z = \mathcal{F}(s_x)$. For example, when $\mathcal{F} = d$ is the exterior derivative $d: \text{Sec}(\wedge^p T^*V) \to \text{Sec}(\wedge^{p+1} T^*V)$ of differential $(p-1)$-forms, the $\mathcal{F}$-sections are exact differential $p$-forms. In order to state a property of differential sections, we introduce the following notation. For the given subset $S \subset Z$, let $\text{Sec}_\mathcal{F}(Z \setminus S)$ denote the space of $\mathcal{F}$-sections $s: V \to Z$ of $p_z: Z \to V$ that satisfies $s(V) \subset Z \setminus S$. Then the following theorem concerning the $h$-principle for differential sections is introduced in [EM].

**Theorem 2.4.** Let $\mathcal{F}: \text{Sec}(X) \to \text{Sec}(Z)$ be a first order linear differential operator for vector bundles $p_x: X \to V$ and $p_z: Z \to V$ over a manifold $V$. Assume that the symbol $F = \text{Symb} \mathcal{F}: X^{(1)} \to Z$ is fiberwise epimorphic. For the given subset $S \subset Z$, set

$$\Sigma := F^{-1}(S) \subset X^{(1)}.$$ 

If the differential relation $\mathcal{R} := X^{(1)} \setminus \Sigma$ satisfies the $h$-principle, then the inclusion $\text{Sec}_\mathcal{F}(Z \setminus S) \to \text{Sec}(Z \setminus S)$ also satisfies the $h$-principle. In other words, any section $s_0 \in \text{Sec}(Z \setminus S)$ is homotopic in $\text{Sec}(Z \setminus S)$ to a section $s_1 \in \text{Sec}_\mathcal{F}(Z \setminus S)$. In addition, the same claim also holds for the one-parametric $h$-principle.

### 3 Key theorem and corresponding differential relation

We introduce the key theorem to be proved for the proof of Theorems [A] and [C] in Subsection 3.1. In order to show the theorem by the $h$-principles, we define the differential relation for Theorem 3.1 in Subsection 3.2. Then we show the ampleness of the differential relations in Subsection 4.

#### 3.1 Key theorem for Theorems [A] and [C]

We introduce the key theorem in this paper. The method to prove the theorem is the $h$-principles. We need to translate the essence of Theorems [A] and [C] to the philosophy of the $h$-principle. First, we introduce the formal structure for the structures we are dealing with. Then we formulate the theorem to be proved.

In terms of the $h$-principle, roughly speaking, the formal things are obtained from genuine things by forgetting differentiations. The structure dealt in Theorems [A] and [C] is maximally non-integrable distributions of derived length one and of odd-rank (see Section 2.1 for definition). For such structures, the formal structures are introduced as follows. Let $V$ be a manifold of dimension $n$, and $\mathcal{D}$ a distribution of odd rank $2k+1$. We say $\mathcal{D}$ to be almost maximally non-integrable if there exists an $(n-2k-1)$-tuple $\{\omega_1, \omega_2, \ldots, \omega_{n-2k-1}\}$ of 2-forms that satisfies that $(\omega_1)^k, \ldots, (\omega_{n-2k-1})^k$ are linearly independent on $\mathcal{D}$ at each
point of $V$. Let

$$\text{AMNI}(V) := \left\{ (\mathscr{D}, \{\omega_1, \ldots, \omega_{n-2k-1}\}) \mid \begin{array}{c} \mathscr{D} \text{ is maximally non-integrable} \\ \text{with respect to 2-forms } \omega_1, \ldots, \omega_{n-2k-1} \end{array} \right\}$$

denote the set of pairs of an almost maximally non-integrable distribution $\mathscr{D}$ and its defining 2-forms on $V$.

We should remark that if $\mathscr{D}$ is maximally non-integrable defined by $(n-2k-1)$-tuple $\{\alpha_1, \ldots, \alpha_{n-2k-1}\}$ of 1-forms, then $\{d\alpha_1, \ldots, d\alpha_{n-2k-1}\}$ is a tuple of 2-forms that satisfies the condition of the definition above. Then $\mathscr{D}$ is almost maximally non-integrable. If the corank of $\mathscr{D}$ is $n-2k-1 = 1$, an almost maximally non-integrable distribution is an almost even-contact structure.

By using the notion, almost maximally non-integrable distribution, the key theorem is formulated as follows.

**Theorem 3.1.** Let $V$ be a manifold of dimension $n$, and $\mathscr{D}$ a distribution of odd rank $2k+1$, $k \in \mathbb{N}$, on $V$. Assume that $\mathscr{D}$ is almost maximally non-integrable with respect to $(n-2k-1)$-tuple $\{\omega_1, \omega_2, \ldots, \omega_{n-2k-1}\}$ of 2-forms. Then there exists a distribution $\mathcal{E}$ of rank $2k+1$ on $V$ and defining $(n-2k-1)$-tuple $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-2k-1}\}$ of 1-forms for which $(\mathscr{D}, \{\omega_i\})$ is homotopic to $(\mathcal{E}, \{d\alpha_i\})$ in $\text{AMNI}(V)$. In addition, the one-parametric version also holds. In other words, any path $(\mathscr{D}_t, \{\omega^t_i\}) \in \text{AMNI}(V)$, $t \in [0,1]$, of almost maximally non-integrable distributions of derived length one between two genuine maximally non-integrable distributions $\mathscr{D}_0$, $\mathscr{D}_1$ of derived length one can be deformed in $\text{AMNI}(V)$ to a path $(\tilde{\mathscr{D}}_t, \{d\alpha^t_i\})$ of genuine maximally non-integrable distributions of derived length one keeping both $\mathscr{D}_0 = \tilde{\mathscr{D}}_0$ and $\mathscr{D}_1 = \tilde{\mathscr{D}}_1$.

This theorem is proved in the next section by using the $h$-principles introduced in Section 2.2. Theorem A and Theorem C follows this theorem (see Section 5).

### 3.2 Differential relations for the problems

In order to show Theorem 3.1, we apply Gromov’s convex integration method, Theorem 2.2, and the $h$-principle for differential sections, Theorem 2.4. Then, in this subsection, we define the first order linear differential operator and the differential relations concerning the problems dealt in Theorem 3.1 in order to apply the theorems above.

First, we define a first order linear differential operator so that the target is concerning Theorem 3.1 in order to apply Theorem 2.4. Let $V$ be a manifold of dimension $n$. Let $X \to V$ and $Z \to V$ be vector bundles over $V$ defined as

$$X := \bigoplus_{n-2k-1} T^*V \to V,$$

$$Z := \left( \bigoplus_{n-2k-1} T^*V \right) \oplus \left( \bigwedge^2 T^*V \right) \to V,$$

where $k \in \mathbb{N}$ is the same integer as in the statement of Theorem 3.1. Then we define the first order linear differential operator

$$\mathcal{F} : \text{Sec}(X) \to \text{Sec}\left( \bigoplus_{n-2k-1} Z \right)$$
as follows. Let $\mathcal{F}_i$ : $\text{Sec}(X) \rightarrow \text{Sec}(Z)$, $i = 1, 2, \ldots, n - 2k - 1$, be a first order linear differential operator defined as $\mathcal{F}_i$ : $(\alpha_1, \alpha_2, \ldots, \alpha_{n-2k-1}) \mapsto (\alpha_1, \alpha_2, \ldots, \alpha_{n-2k-1}, da_i)$. Then we define the operator $\mathcal{F}$ as

$$
\mathcal{F} := (\mathcal{F}_1, \ldots, \mathcal{F}_{n-2k-1}) : (\alpha_1, \ldots, \alpha_{n-2k-1}) \mapsto
((\alpha_1, \ldots, \alpha_{n-k-1}, da_1), \ldots, (\alpha_1, \ldots, \alpha_{n-k-1}, da_{n-k-1})). \quad (3.2)
$$

We should remark that the symbol $F : X^{(1)} \rightarrow \bigoplus_{2}^{n-2k-1} Z$ of $\mathcal{F}$ is fiberwise epimorphic.

Next, we define a differential relation in the source side of $\mathcal{F}$ that corresponds to a critical condition for Theorem 3.1 in the target side. Let $S \subset \bigoplus_{2}^{n-2k-1} Z$ be the subset defined as

$$
S := \{(\alpha_1, \ldots, \alpha_{n-2k-1}, \omega_1) \in \bigoplus_{2}^{n-2k-1} T^*V \oplus \bigwedge^{2} T^*V \cap \bigoplus_{2}^{n-2k-1} Z |
(\alpha_1 \land \cdots \land \alpha_{n-2k-1} \land (\omega_1)^k)_v, \ldots, (\alpha_1 \land \cdots \land \alpha_{n-2k-1} \land (\omega_1)^k)_v
\text{ are linearly dependent}\}.\quad (3.3)
$$

In other words, in the fiber $\bigoplus_{2}^{n-2k-1} Z_v$, $S$ is defined by the condition that $(\alpha_1 \land \cdots \land \alpha_{n-2k-1} \land (\omega_1)^k)_v, \ldots, (\alpha_1 \land \cdots \land \alpha_{n-2k-1} \land (\omega_1)^k)_v$ are linearly dependent. Then we take the inverse image of $S \subset \bigoplus_{2}^{n-2k-1} Z$ by the symbol $F : X^{(1)} \rightarrow \bigoplus_{2}^{n-2k-1} Z$ of $\mathcal{F} : \text{Sec}(X) \rightarrow \text{Sec}(\bigoplus_{2}^{n-2k-1} Z)$ (see Section 2.2 for definition). Set $\Sigma := F^{-1}(S) \subset X^{(1)}$.

We remark that, from the construction of $\mathcal{F}$, the subset $\Sigma \subset X^{(1)}$ is the same as the inverse image $F^{-1}(\tilde{S}) \subset X^{(1)}$ of the set

$$
\tilde{S} := \{(\alpha_1, \ldots, \alpha_{n-2k-2}, \omega_1) \in \bigoplus_{2}^{n-2k-1} T^*V \oplus \bigwedge^{2} T^*V \cap \bigoplus_{2}^{n-2k-1} Z |
(\alpha_1 \land \cdots \land \alpha_{n-2k-2} \land (\omega_1)^k)_v, \ldots, (\alpha_1 \land \cdots \land \alpha_{n-2k-2} \land (\omega_1)^k)_v
\text{ are linearly dependent}\}. \quad (3.3)
$$

Then as the subset $\tilde{S} \subset \bigoplus_{2}^{n-2k-1}$ corresponds to where the maximally non-integrability collapses, the subset $\Sigma = F^{-1}(\tilde{S}) \subset X^{(1)}$ is regarded as the singularity to be considered. Set

$$
\mathscr{R} := X^{(1)} \setminus \Sigma \subset X^{(1)}. \quad (3.4)
$$

It is the open differential relation to be considered in the next section to show Theorem 3.1.

4 Proof of Theorem 3.1

In this section, we show Theorem 3.1. As mentioned above, we apply Gromov’s convex integration method, and the $h$-principle for differential sections. Then the key of the proof is the following proposition. The notations introduced in the previous section are used. Let $V$ be an $n$-dimensional manifold, $X = \bigoplus_{2}^{n-2k-1} T^*V$ and $Z = \bigoplus_{2}^{n-2k-1} T^*V$
\( \left( \bigoplus^k (\wedge^2 T^*V) \right) \) the vector bundles over \( V \), \( \mathcal{F} : \text{Sec} \, (X) \to \text{Sec} \left( \bigoplus^{n-2k-1} Z \right) \) the linear differential operator, and \( \mathcal{R} = X^{(1)} \setminus \Sigma \subset X^{(1)} \) the differential relation.

**Proposition 4.1.** *The open differential relation \( \mathcal{R} \subset X^{(1)} \) is ample.*

**Proof.** In order to show that the differential relation \( \mathcal{R} = X^{(1)} \setminus \Sigma \subset X^{(1)} \) is ample, we will show that the singularity \( \Sigma \subset X^{(1)} \) is thin.

It is enough that we discuss by using local coordinates of the base manifold \( V \) of the bundles. Let \((x_1, x_2, \ldots, x_n)\) be local coordinates of \( V \). Then \( \{(dx_1)_v, \ldots, (dx_n)_v\} \) is a basis of the fiber \( T^*_v V \). Let \((a_1, \ldots, a_n)\) be coordinates on the fiber of \( T^*V \). Similarly, \( \{(dx_i \wedge dx_j)_v\}_{1 \leq i < j \leq n} \) is a basis of the fiber \( \left( \wedge^2 T^*V \right)_v \). Let \((z_{12}, \ldots, z_{(n-1)n})\) be coordinates on the fiber of \( \wedge^2 T^*V \). Further, let \((a_1, \ldots, a_n, y_{11}, y_{12}, \ldots, y_{nn})\) be coordinates of the fiber \( (T^*V)_v \), where \( y_{ij} \) corresponds to \( \partial a_i / \partial x_j \).

By using such local coordinates, we write down the singularity \( \Sigma \subset \left( \bigoplus^{n-2k-1} T^*V \right)^{(1)} = X^{(1)} \) as follows. Recall that \( \Sigma \) is defined as the inverse image \( \Sigma = F^{-1}(\tilde{S}) \) by the symbol \( F : X^{(1)} \to \text{Sec} \left( \bigoplus^{n-2k-1} Z \right) \) of the linear differential operator \( \mathcal{F} : \text{Sec} \, (X) \to \text{Sec} \left( \bigoplus^{n-2k-1} Z \right) \) defined by Equation (3.1) concerning exterior derivative of differential forms. Note that, by the exterior derivative, the coordinates \( y_{ij} - y_{ji} \) correspond to \( z_{ij} \).

Recall that the set \( \tilde{S} \subset \text{Sec} \left( \bigoplus^{n-2k-1} Z \right) \) is defined by the following condition (see Equation (3.3)): \((n - 2k - 1)\) pieces of \((n - 1)\)-forms

\[
\alpha_1 \wedge \cdots \wedge \alpha_{n-2k-1} \wedge (\omega_1)^k, \ldots, \alpha_1 \wedge \cdots \wedge \alpha_{n-2k-1} \wedge (\omega_{n-2k-1})^k
\]  

(4.1)

are linearly independent on each fiber, where \( \alpha_i \in \text{Sec} \,(T^*V) \) are \( 1 \)-forms and \( \omega_i \in \text{Sec} \,(\wedge^2 T^*V) \) are \( 2 \)-forms on \( V \). Then, in order to write down the singularity \( \Sigma = F^{-1}(\tilde{S}) \subset X^{(1)} \) by using local coordinates, we represent \( \alpha_i \) and \( \omega_i \) on a fiber over \( v \in V \) by such coordinates as follows:

\[
\alpha_i = \sum_{j=1}^{n} a^i_j dx_j = a^i_1 dx_1 + a^i_2 dx_2 + \cdots + a^i_n dx_n, \quad (i = 1, 2, \ldots, n - 2k - 1),
\]

\[
\omega_i = \sum_{1 \leq j \leq k \leq n} z^i_{jk} dx_j \wedge dx_k = z^i_{12} dx_1 \wedge dx_2 + \cdots + z^i_{(n-1)n} dx_{n-1} \wedge dx_n, \quad (i = 1, 2, \ldots, n - 2k - 1).
\]

Following this representation, the \((n - 1)\)-forms in Equation (4.1) are written down as follows. First, we have

\[
(\omega_i)^k = \sum_{1 \leq j_1 < \cdots < j_{2k} \leq n} \left( A^i_{j_1 \ldots j_{2k}} \right) dx_{j_1} \wedge \cdots \wedge dx_{j_{2k}}, \quad i = 1, 2, \ldots, n - 2k - 1,
\]

where the coefficients are

\[
A^i_{j_1 \ldots j_{2k}} = \sum_{\{l_1, \ldots, l_{2k}\} = \{j_1, \ldots, j_{2k}\}} \sigma(l_1, \ldots, l_{2k}) \cdot z^i_{l_1 l_2} \cdots z^i_{l_{2k} l_{2k}}, \quad i = 1, 2, \ldots, n - 2k - 1, \quad 1 \leq j_1 < j_2 < \cdots < j_{2k} \leq n,
\]  

(4.2)
for the sign $\sigma(l_1, \ldots, l_{2k}) = \text{sgn} \left( j_1 \cdots j_{2k} \right)$ of permutations. Note that $z_j^i$ is valid for $j < k$. Then the $(n - 1)$-forms in Equation (4.1) are

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_{n-2k-1} \wedge (\omega_i)^k = \sum_{i=1}^{n} (B^i_r) dx_1 \wedge \cdots \wedge \widehat{dx_r} \wedge \cdots \wedge dx_n,$$  

where “$\widehat{dx_r}$” implies “without $dx_r$.” The coefficients are

$$B^i_r = \sum_{\{p_1, \ldots, p_{n-1}\} = \{1, \ldots, r, \ldots, n\}} \sigma(p_1, \ldots, p_{n-1}) \cdot a^1_{p_1} \cdots a^{n-2k-1}_{p_{n-2k-1}} \cdot \left( A^i_{p_{n-2k} \cdots p_{n-1}} \right),$$  

where $A^i_{p_{n-2k} \cdots p_{n-1}}$ is valid for $p_{n-2k} < p_{n-2k+1} < \cdots < p_{n-1}$. At last, we obtain a representation of the singularity $\Sigma \subset X^{(1)}$ by the local coordinates. From the expression in Equation (4.3), the defining condition of $\Sigma \subset X^{(1)}$ is that the following $n - 2k - 1$ vectors

$$\beta_i := (B^i_1, B^i_2, \ldots, B^i_n), \quad i = 1, 2, \ldots, n - 2k - 1,$$

are linearly independent. In other words, there exists $(c_1, \ldots, c_{n-2k-1})$ vanishing nowhere that satisfies

$$c_1 \beta_1 + c_2 \beta_2 + \cdots + c_{n-2k-1} \beta_{n-2k-1} = 0. \tag{4.5}$$

In order to show that the singularity $\Sigma \subset X^{(1)}$ is thin, we observe the intersections of $\Sigma$ with principal subspaces, using local coordinates. To make the discussion simple, we take a principal direction of $P$ that concerns $x_1$ in the local coordinates of the base manifold $V$. In other words, in the defining condition of $\Sigma \subset X^{(1)}$, only $z_{11}^i$ are variables. Other $z_{ml}^i$ and $a_j^i$ are constant on the intersection in a fiber over $v \in V$. This implies, from Equations (4.2) and (4.3), that $B^i_1, i = 1, 2, \ldots, n - 2k - 1$, are constant, and

$$B^i_r = \overline{C}^i_r + \sum_{\{p_1, \ldots, p_{n-2k} = 1, \ldots, p_{n-1}\} = \{1, \ldots, r, \ldots, n\}} \sigma(p_1, \ldots, p_{n-1}) \cdot \left( A^i_{1(p_{n-2k+1} \cdots p_{n-1})} \right),$$

$$= \overline{C}^i_r + \sum_{m=2}^{n} (C^i_r(m)) \cdot z_{1m}^i,$$

$$= \overline{C}^i_r + (C^i_r(2)) \cdot z_{12}^i + (C^i_r(3)) \cdot z_{13}^i + \cdots + (C^i_r(r)) \cdot z_{1r}^i + \cdots + (C^i_r(n)) \cdot z_{1n}^i,$$  

$$i = 1, 2, \ldots, n - 2k - 1, \quad r = 2, 3, \ldots, n,$$

where $\overline{C}^i_r$ and

$$C^i_r(2) = 0,$$

$$C^i_r(m) = k \sum_{\{p_1, \ldots, p_{n-3}\} = \{1, \ldots, n\} \setminus \{1, r, m\}} a^i_{p_1} \cdots a^i_{p_{n-2k-1}} A^i_{p_{n-2k+1} \cdots p_{n-3} p_{n-1}},$$

are constant. We should remark here that

$$C^i_r(m) = \pm C^i_m(r). \tag{4.7}$$
Now, we observe the intersection of the singularity $\Sigma \subset X^{(i)}$ with the principal subspace $P$ with respect to $x_1$ direction, by using local coordinates. First, we confirm the system of linear equations that represents $\Sigma \cap P$. Recall that $\Sigma \subset X^{(i)}$ is defined by the condition (4.5). In other words, the system of linear equations that represents $\Sigma$ is defined by the following system of linear equations with respect to $(1)$:

$$\begin{align*}
&c_1 B_1^1 + c_2 B_2^1 + \cdots + c_{n-2k-1} B_{n-2k-1}^1 = 0, \\
&c_1 B_1^2 + c_2 B_2^2 + \cdots + c_{n-2k-1} B_{n-2k-1}^2 = 0, \\
&\vdots \\
&c_1 B_1^n + c_2 B_2^n + \cdots + c_{n-2k-1} B_{n-2k-1}^n = 0.
\end{align*}$$

Note that, since $B_i^j, i = 1, 2, \ldots, n-2k-1$, are constant, $c_1, c_2, \ldots, c_{n-2k-1}$ should satisfy the first equation. If there are no such $c_1, c_2, \ldots, c_{n-2k-1}$, there is no intersection between $\Sigma$ and $P$ on the fiber $X^{(j)}_v$ over $v$. Then, from the second to $n$-th equations, and by the Equation (1.0), the intersection of $\Sigma$ with $P$ is defined by the following system of linear equations with respect to $(n-1) \times (n-2k-1)$ variables $z_{12}, z_{13}, \ldots, z_{1n}, z_{21}, z_{22}, \ldots, z_{2n}$:

$$\begin{align*}
&c_1 C_1^1(3) z_{13}^1 + c_1 C_1^2(4) z_{14}^1 + \cdots + c_1 C_1^n(n) z_{1n}^1 \\
&+ c_2 C_2^1(3) z_{13}^2 + \cdots + c_{n-2k-1} C_{n-2k-1}^n(n) z_{1n}^2 = -c_1 \bar{C}_1^1 - c_2 \bar{C}_2^1 - \cdots - c_n \bar{C}_n^n \\
&c_1 C_1^1(2) z_{12}^1 + c_1 C_1^2(3) z_{13}^1 + \cdots + c_1 C_1^n(n) z_{1n}^1 \\
&+ c_2 C_2^1(2) z_{12}^2 + \cdots + c_{n-2k-1} C_{n-2k-1}^n(n) z_{1n}^2 = -c_1 \bar{C}_1^1 - c_2 \bar{C}_2^1 - \cdots - c_n \bar{C}_n^n \\
&\vdots \\
&c_1 C_1^1(2) z_{12}^1 + c_1 C_1^2(3) z_{13}^1 + \cdots + c_1 C_1^n(n-1) z_{1(n-1)}^1 \\
&+ c_2 C_2^1(2) z_{12}^2 + \cdots + c_{n-2k-1} C_{n-2k-1}^n(n-1) z_{1(n-1)}^2 = -c_1 \bar{C}_1^1 - c_2 \bar{C}_2^1 - \cdots - c_n \bar{C}_n^n.
\end{align*}$$

Then we observe the codimension of $\Sigma \cap P \subset P$ to show $\Sigma \subset X^{(i)}$ is thin. The rank of the coefficient matrix of the system of equations above is the codimension of $\Sigma \cap P \subset P$. The coefficient matrix is the $(n-1) \times (n-1)(n-2k-1)$-matrix written down as

$${\begin{pmatrix} 0 & \tilde{C}_1^1(3) & \tilde{C}_1^2(4) & \cdots & \tilde{C}_1^n(n) & 0 & \tilde{C}_2^2(3) & \cdots & \tilde{C}_{n-2k-1}^n(n) \\
\tilde{C}_2^1(3) & 0 & \tilde{C}_3^1(4) & \cdots & \tilde{C}_3^n(n) & \tilde{C}_2^2(2) & \cdots & \tilde{C}_{n-2k-1}^n(n) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\tilde{C}_n^1(3) & \tilde{C}_n^1(4) & \cdots & 0 & \tilde{C}_2^2(2) & \tilde{C}_2^2(3) & \cdots & 0 
\end{pmatrix}},$$

where $\tilde{C}_i^j(m) = c_i C_i^j(m)$. When the rank of this matrix is 0, the differential relation $R = X^{(i)} \setminus \Sigma$ is ample by definition. We claim that the rank of this matrix never be 1. Suppose the $i$-th column $a_i \in \mathbb{R}^{n-1}$ is non-zero. Assume $i \equiv \bar{i} \pmod{n-1}$ for $1 \leq \bar{i} \leq n-1$, and $i = \bar{i} + j(n-1)$. Then the $\tilde{i}$-th element of $a_i$ is $\tilde{C}_{i+1}^{j+1}(\bar{i}+1) = c_{i+1} C_{i+1}^{j+1}(\bar{i}+1) = 0$. As $a_i \in \mathbb{R}^{n-1}$ is not zero, there should be a non-zero element, say $\tilde{C}_{\bar{i}+1}^{j+1}(\bar{i}+1) \neq 0$ for $r \neq \bar{i} + 1$. From Equation (1.7), we have $\tilde{C}_{\bar{i}+1}^{j+1}(r) = \tilde{C}_{\bar{i}+1}^{j+1}(\bar{i}+1) \neq 0$. This implies that the $(r-1 + j(n-1))$-st column $a_{r-1+j(n-1)} \in \mathbb{R}^{n-1}$ of the coefficient matrix is not zero, and the $\tilde{i}$-th element of $a_{r-1+j(n-1)}$ is $\tilde{C}_{i+1}^{j+1}(r) \neq 0$. Therefore, $a_i, a_{r-1+j(n-1)} \in \mathbb{R}^{n-1}$ are linearly independent. Thus we conclude that the rank of the coefficient matrix is greater than 1.

Thus, we have proved that the singularity $\Sigma$ is thin and the differential relation $R = X^{(i)} \setminus \Sigma$ is ample. □
In the rest of this section, we prove Theorem 3.1.

**Proof of Theorem 3.1.** Let $V$ be a manifold of dimension $n$, and $\mathcal{D}$ a distribution of rank $2k + 1$ on $V$. Suppose that the 1-forms $\alpha_i$ and 2-forms $\omega_i$ satisfy the assumptions in Theorem 3.1 for $\mathcal{D}$. We prove in the following two steps.

First, we apply Proposition 4.1 and Gromov’s $h$-principle for ample differential relations (Theorem 2.2). Let the vector bundle $X := \bigoplus (n-2k-1)^r T^*V$ over $V$ be as in Equations (3.1). Now, we take the differential relation $\mathcal{R} \subset X^{(1)}$ to be considered as in Equation (3.4). Then, from Proposition 4.1, $R \subset X^{(1)}$ is ample. This implies, according to Theorem 2.2, that the differential relation $\mathcal{R} \subset X^{(1)}$ satisfies the $h$-principle and the one-parametric $h$-principle.

Next, we discuss the $h$-principles for differential operators. For the vector bundle $X$ above, let the vector bundle $Z := (\bigoplus (n-2k-1)^r T^*V) \oplus (\bigwedge^2 T^*V)$ over $V$ be as in Equations (3.1), and the linear differential operator $\mathcal{F} : \text{Sec}(X) \rightarrow \text{Sec}\left(\bigoplus (n-2k-1)^r Z\right)$ be as in Equation (3.2). Note that the symbol $F : X^{(1)} \rightarrow \bigoplus (n-2k-1)^r Z$ of $\mathcal{F}$ is fiberwise epimorphic. Recall that the differential relation $\mathcal{R} \subset X^{(1)}$ is defined as $\mathcal{R} = X^{(1)} \setminus \Sigma = X^{(1)} \setminus F^{-1}(\tilde{S})$ for the set $\tilde{S} \subset \bigoplus (n-2k-1)^r Z$ defined in Equation (3.3). Now, we have proved that the differential relation $\mathcal{R} \subset X^{(1)}$ satisfies the $h$-principle and the one-parametric $h$-principle. Then, from Theorem 2.4, we obtain that the $h$-principle and the one-parametric $h$-principle hold for the inclusion

$$\text{Sec}_{\mathcal{F}}\left(\left(\bigoplus (n-2k-1)^r Z\right) \setminus \tilde{S}\right) \rightarrow \text{Sec}\left(\left(\bigoplus (n-2k-1)^r Z\right) \setminus \tilde{S}\right),$$

where

$$\text{Sec}_{\mathcal{F}}\left(\left(\bigoplus (n-2k-1)^r Z\right) \setminus \tilde{S}\right) := \left\{s : V \rightarrow \left(\bigoplus (n-2k-1)^r Z\right) \setminus \tilde{S}, \left(\bigoplus (n-2k-1)^r Z\right) \setminus \tilde{S} \right\}$$

is the space of $\mathcal{F}$-sections. Since $\text{Sec}_{\mathcal{F}}\left(\left(\bigoplus (n-2k-1)^r Z\right) \setminus \tilde{S}\right)$ corresponds to the set of genuine maximally non-integrable distributions and $\text{Sec}\left(\left(\bigoplus (n-2k-1)^r Z\right) \setminus \tilde{S}\right)$ corresponds to the set of almost maximally non-integrable distributions, this implies Theorem 3.1.

5 Proof of Theorems $\mathbb{A}$ and $\mathbb{C}$ and further observations

Following Theorem 3.1, we prove Theorems $\mathbb{A}$ and $\mathbb{C}$ in Subsection 5.1. Further, we discuss some miscellaneous things on the case when the tangent distributions are orientable. Corollary $\mathbb{B}$ is proved in Subsection 5.2.
5.1 Proof of theorems

First, we show Theorem A. Since a genuine maximally non-integrable distribution of derived length one of odd rank is an almost maximally non-integrable distribution of derived length one, the necessity of Theorem A is clear. Then we show the sufficiency of the theorem. Let \( V \) be a manifold of dimension \( n \), and \( \mathcal{D} \subset TV \) a distribution of rank \( r = 2k + 1 \). Suppose that it is almost maximally non-integrable of derived length one. In other words, there is an \( (n - 2k - 1) \)-tuple of 2-forms \( \{\omega_1, \omega_2, \ldots, \omega_{n-2k-1}\} \) for which \((\omega_1)^k, \ldots, (\omega_{n-2k-1})^k \) are pointwise linearly independent on \( \mathcal{D} \). Then, from Theorem 3.1 on the manifold \( V \) there exists an almost maximally non-integrable distribution \( \mathcal{E} = \{\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_{n-2k-1} = 0\} \subset TV \) of rank \( 2k + 1 \) of derived length one with respect to \( (n - 2k - 1) \)-tuple \( \{\alpha_1, \alpha_2, \ldots, \alpha_{n-2k-1}\} \) of 2-forms. This implies that \( \mathcal{E} \subset TV \) is a genuine maximally non-integrable distribution of derived length one on \( V \).

This concludes the proof of Theorem A.

Then, we show Theorem C. We apply the one-parametric \( h \)-principle in Theorem 3.1. Let \( \mathcal{D}_0, \mathcal{D}_1 \subset TV \) be maximally non-integrable distributions of derived length one of odd rank on a manifold \( V \). Suppose that they are homotopic through almost maximally non-integrable distributions of derived length one. In other words, there is a path \( (\mathcal{D}_t, \{\omega_t^i\}) \) of almost maximally non-integrable distributions of derived length one between them. Then, from Theorem 3.1, the path \( (\mathcal{D}_t, \{\omega_t^i\}) \) can be deformed to a path \( (\mathcal{D}_t, \{\hat{\omega}_t^i\}) \) of genuine maximally non-integrable distributions of derived length one keeping both \( \mathcal{D}_0 = \mathcal{D}_0, \mathcal{D}_1 = \mathcal{D}_1 \). This implies that \( \mathcal{D}_0 \) and \( \mathcal{D}_1 \) are isotopic. This concludes the proof of Theorem C. □

5.2 Orientable cases

In this subsection, we only deal with orientable tangent distributions. In some cases, the topological condition on manifolds in the existence theorem, Theorem A, can be described simpler. First, we introduce more general description. Then we obtain Corollary B as a special case.

For orientable distributions, fixing dimension of manifolds and rank of tangent distributions, we discuss the existence of maximally non-integrable distributions of derived length one. Recall that, for such a distribution of rank \( 2k + 1 \), the dimension \( n \) of manifolds should be \( 2k + 2 \leq n \leq 2(2k + 1) = 4k + 2 \) (see Section 2.1). When corank of distribution is large, we obtain the following proposition as a corollary of Theorem A. In other words, we obtain the condition on manifolds to have a maximally non-integrable orientable distribution of type \((2k + 1, 4k + 1)\) or \((2k + 1, 4k + 2)\).

**Proposition 5.1.** Let \( V \) be a manifold possibly closed. \( V \) admits a maximally non-integrable orientable distribution of type \((2k + 1, 4k + 1)\) or \((2k + 1, 4k + 2)\) if and only if there exists a trivial subbundle \( \mathcal{E} \subset TV \) of rank \( 2k + 1 \).

**Proof.** First, we discuss the sufficiency. Suppose that there exists a trivial subbundle \( \mathcal{E} \subset TV \) of rank \( 2k + 1 \) on a manifold \( V \) of dimension \( n = 4k + 1 \) or \( 4k + 2 \). Then there exists a coframe \( X_1^*, \ldots, X_{2k+1}^* \) of \( \mathcal{E}^* \subset T^*V \). Setting for each \( i = 1, 2, \ldots, 2k + 1 \),

\[
\omega_i := X_{p_1}^* \wedge X_{p_2}^* \wedge \cdots \wedge X_{p_{2k-1}}^* \wedge X_{p_{2k}}^*, \quad \{p_1, \ldots, p_{2k}\} = \{1, \ldots, i, \ldots, 2k + 1\},
\]

we have \((2k + 1)\)-tuple \( \{\omega_1, \ldots, \omega_{2k+1}\} \) of 2-forms. This tuple of 2-forms satisfies that \((\omega_1)^k, \ldots, (\omega_{2k+1})^k \) are linearly independent \( 2k \)-forms on the distribution \( \mathcal{E} \) of rank \( 2k + 1 \).
Since
\[(\omega_i)^k = \pm (k!) X_i^* \wedge \cdots \wedge X_{i+k}^*, \quad i = 1, 2, \ldots , 2k+1.\]
When the dimension of \(V\) is \(n = 4k + 1\) (resp. \(n = 4k + 2\)), the corank of \(\mathcal{E} \subset TV\) is \(2k\) (resp. \(2k + 1\)). Then \(\mathcal{E}\) with \(\{\omega_1, \ldots , \omega_{2k}\}\) (resp. \(\{\omega_1, \ldots , \omega_{2k+1}\}\)) is almost maximally non-integrable of derived length one. Then, from Theorem A the manifold \(V\) admits a maximally non-integrable distribution \(\mathcal{D} \subset TV\) of derived length one of rank \(2k + 1\). In other words, \(\mathcal{D} \subset TV\) is a maximally non-integrable distribution of type \((2k + 1, 4k + 1)\) or \((2k + 1, 4k + 2)\).

Next, we discuss the necessity. Suppose that a manifold \(V\) admits a maximally non-integrable orientable distribution of type \((2k+1, 4k+1)\) or \((2k+1, 4k+2)\). From Theorem A, we may suppose that there exists an almost maximally non-integrable orientable distribution \(\mathcal{D} \subset TV\) of derived length one of rank \(2k + 1\). When the dimension of the manifold \(V\) is \(n = 4k + 1\) (resp. \(n = 4k + 2\)), the codimension of \(\mathcal{D} \subset TV\) is \(2k\) (resp. \(2k + 1\)). Then there exists a \((2k)\)-tuple of \(2\)-forms \(\{\omega_1, \ldots , \omega_{2k}\}\) (resp. \((2k + 1)\)-tuple of \(2\)-forms \(\{\omega_1, \ldots , \omega_{2k+1}\}\)) for the almost maximal non-integrability (see Section 2.1). This implies that \(\{\omega_1)^k, \ldots , \omega_{2k+1}\}\) is a \((2k+1)\)-tuple (resp. \(\{\omega_1\)^k, \ldots , \omega_{2k+1}\}\) is a \((2k+1)\)-tuple) of linearly independent \((2k)\)-forms on the distribution \(\mathcal{D}\) of rank \(2k + 1\).

Now we observe the tangent distribution \(\mathcal{D} \subset TV\). Since \(\mathcal{D}\) is orientable, \(\bigwedge^{2k+1} \mathcal{D}\) is trivial. Then \(\bigwedge^{2k+1} \mathcal{D}^*\) is also trivial. Taking a non-zero section, or \((2k+1)\)-form, \(\gamma\) of \(\bigwedge^{2k+1} \mathcal{D}^*\), we have an equivalence \(\mathcal{D} \cong \bigwedge^{2k} \mathcal{D}^*\) by the contraction \(X \mapsto \iota_X \gamma\). Then \(\bigwedge^{2k} \mathcal{D}^*\) is also orientable.

When the dimension of \(V\) is \(n = 4k + 1\), we have linearly independent \(2k\) non-zero sections \((\omega_1)^k, \ldots , (\omega_{2k})^k\) of the vector bundle \(\bigwedge^{2k} \mathcal{D}^* \cong \mathcal{D}\) of rank \(2k + 1\). From the orientation of \(\mathcal{D} \cong \bigwedge^{2k} \mathcal{D}^*\), we have another independent non-zero section. Then, \(\mathcal{D}\) is trivial. When the dimension of \(V\) is \(n = 4k + 2\), we already have linearly independent \(2k + 1\) non-zero sections \((\omega_1)^k, \ldots , (\omega_{2k+1})^k\) of the vector bundle \(\bigwedge^{2k} \mathcal{D}^* \cong \mathcal{D}\) of rank \(2k + 1\). Then \(\mathcal{D}\) is trivial.

This concludes the proof of Proposition 5.1.\[\square\]

From Proposition 5.1, in the case when \(k = 1\) and the dimension of the manifold \(V\) is \(n = 4k + 1 = 5\), we obtain Corollary [B] as follows.

Proof of Corollary [B]. For \(k = 1\), we obtain from Proposition 5.1 that a manifold \(V\) of dimension 5 admits a maximally non-integrable orientable distribution of type (3, 5) if and only if there exists a trivial subbundle \(\mathcal{E} \subset TV\) of rank 3. As we mentioned in Section 2.1 a tangent distribution of type (3, 5) is automatically maximally non-integrable. Thus we have proved Corollary [B].\[\square\]

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