Phase Space Isometries and Equivariant Localization of Path Integrals in Two Dimensions

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Abstract

By considering the most general metric which can occur on a contractable two dimensional symplectic manifold, we find the most general Hamiltonians on a two dimensional phase space to which equivariant localization formulas for the associated path integrals can be applied. We show that in the case of a maximally symmetric phase space the only applicable Hamiltonians are essentially harmonic oscillators, while for non-homogeneous phase spaces the possibilities are more numerous but ambiguities in the path integrals occur. In the latter case we give general formulas for the Darboux Hamiltonians, as well as the Hamiltonians which result naturally from a generalized coherent state formulation of the quantum theory which shows that again the Hamiltonians so obtained are just generalized versions of harmonic oscillators. Our analysis and results describe the quantum geometry of some two dimensional systems.
1. Introduction

There are a number of known examples of quantum systems for which the Feynman path integral is given exactly by the WKB approximation [1]. This fact has been studied in detail in some recent literature where conditions under which path integrals could be WKB exact are outlined and general localization formulas are given [2–9]. This development is particularly interesting in two respects: It has the possibility of expanding the number of known examples of quantum systems where the Feynman path integral can be evaluated exactly and it promises deeper insights into the geometrical structure of quantum systems. It also forms a convenient approach to topological quantum field theories [3,4] and is the basis of a conceptual geometric description of Poincaré supersymmetric quantum field theories [5].

In this Paper, we shall explore the applicability of these recently derived equivariant localization formulas for quantum mechanical path integrals by considering the case where the phase space of a quantum system is a two dimensional contractable symplectic manifold. Equivariant localization formulas are based on the existence of a metric on the phase space and also on the requirement that the Hamiltonian must generate an isometry of this metric. In two dimensions, the Lie algebra of isometries is either zero, one or three dimensional [10]. We examine the case of a three dimensional isometry group in detail and show that there are only a small, few-parameter families of Hamiltonians which fit the localization framework. We also make some observations of the case of a one dimensional isometry group wherein the applications of the localization formulas are the most non-trivial.

The localization formulas are sometimes viewed as infinite dimensional generalizations of the Duistermaat-Heckman theorem [11,12]. Consider the finite dimensional integral

\[ \tilde{Z}(\beta_0) = \int_{\Gamma^{2n}} \frac{\omega^n}{n!} e^{-\beta_0 H} \]  \hspace{1cm} (1.1)

over a compact 2n dimensional symplectic manifold \((\Gamma^{2n}, \omega)\) (i.e. a partition function of classical statistical mechanics). If the critical points of the Hamiltonian function \(H\) on \(\Gamma^{2n}\) are isolated and the Hamiltonian vector field \(V\) generates a symplectic U(1) group action on \(\Gamma^{2n}\), then the Duistermaat-Heckman integration formula asserts that (1.1) is given exactly by a sum over the critical points of \(H\):

\[ \tilde{Z}(\beta_0) = \sum_{x \in I(H)} e^{-\beta_0 H(x)} \frac{e^{-\beta_0 H(x)}}{\beta_0^m W(x)}, \]  \hspace{1cm} (1.2)
where $I(H)$ is the critical point set of $H$ and

$$W(x) = \frac{\det^{1/2}_{(\mu\nu)} \left[ \frac{\partial^2 H}{\partial x^\mu \partial x^\nu} \right]}{\det^{1/2}_{(\mu\nu)}(\omega_{\mu\nu}(x))}$$

is the determinant arising from a Gaussian integral near $x$. Here $x^\mu$ are local coordinates on the phase space $\Gamma^{2n}$ in which the symplectic 2-form is locally $\omega = \frac{1}{2} \omega_{\mu\nu}(x) dx^\mu \wedge dx^\nu$, and the Hamiltonian vector field $V = V^\mu \frac{\partial}{\partial x^\mu}$ is defined by the equation

$$dH = -i_V (\omega),$$

where $i_V : \Lambda^p(\Gamma^{2n}) \to \Lambda^{p-1}(\Gamma^{2n})$ is the nilpotent interior multiplication acting on the DeRham complex $\Lambda^*(\Gamma^{2n})$ by contracting forms with the vector field $V$.

The problem for a path integral which describes the dynamics of a quantum mechanical system is to evaluate the bosonic phase space functional integral

$$Z(T) = \int_{L\Gamma^{2n}} Dx x^{\mu}(t) \det^{1/2}\|\omega_{\mu\nu}\| \exp \left[ i \int_0^T dt \left( \theta_{\mu} \dot{x}^\mu - H(x) \right) \right],$$

where $\theta = \theta_\mu(x) dx^{\mu}$ is the symplectic 1-form which locally generates $\omega$ as $\omega = d\theta$. The path integral (1.4) is taken over the loop space $L\Gamma^{2n}$ of the phase space, which is the space of trajectories $x(t) : [0, T] \to \Gamma^{2n}$ satisfying periodic boundary conditions $x^{\mu}(0) = x^{\mu}(T)$, and it naturally inherits an infinite dimensional symplectic structure $\Omega$ from $\Gamma^{2n}$ defined by lifting of the symplectic 2-form $\omega$, $\Omega_{\mu\nu}(x; t, t') = \omega_{\mu\nu}(x(t)) \delta(t - t')$.

It has been asserted [2–5,7] that a set of sufficient criteria for the WKB approximation to the path integral (1.4) to be exact are

(a) The classical action functional should have only isolated critical points.

(b) The Hamiltonian vector field should generate a symplectic U(1) action on the phase space.

(c) The phase space should admit a metric with respect to which the Hamiltonian vector field is a Killing vector.

There has recently been a generalization of these localization formulas which avoids the necessity of using action functionals which have only isolated critical points [6,7], and also for Hamiltonians which themselves do not satisfy (1.3), but are generic functionals $F(H)$ of some observable $H$ generating the U(1) action on $\Gamma^{2n}$ as in (1.3) [7]. These more
general integration formulas localize the path integral onto finite dimensional integrals over submanifolds of the original phase space $T^{2n}$ and can be applied to quantum systems for which the WKB approximation is unsuitable. Moreover, the same principles can be applied to Hamiltonians which are constructed from generators of some non-Abelian Lie algebra acting on $T^{2n}$, yielding applications such as localization formulas for two dimensional Yang-Mills theory [9,13] and representations of the infinitesimal Lefschetz number of Dirac operators [8].

It has recently been suggested that for two dimensional phase spaces, the set of Hamiltonians for which these criteria apply is rather small [14]. In the following we shall consider some two dimensional symplectic manifolds which obey the criterion that there exists at least one Killing vector field. We first examine the case of two dimensional maximally symmetric spaces which have three Killing vector fields [10]: the plane, sphere and Lobascevsky plane (these are essentially the contractable spaces of constant curvature). We show that, if the phase space has the topology of the plane $\mathbb{R}^2$, the only “admissible” Hamiltonian is that of the displaced harmonic oscillator. In this case, we find that we could replace the criterion that the Hamiltonian generates a circle action with the requirement that it is semi-bounded.

On the sphere we obtain a similar result that the only Hamiltonian is essentially the height function on the sphere, a Hamiltonian which is familiar from the coadjoint orbit quantization of SU(2) and which has been discussed in connection with the quantization of spin systems [15,16]. It is known that the path integral in this case is given exactly by the WKB approximation [3,4,17]. The Lobascevsky plane turns out to be the same as the sphere except that it is the coadjoint orbit quantization of the group SU(1,1) [15–18]. Here there are two inequivalent Hamiltonians, corresponding to a choice of “spacelike” and “timelike” Killing vectors. For these latter two cases of the sphere and Lobascevsky plane, the admissible Hamiltonian in Darboux coordinates is a displaced harmonic oscillator where, unlike the case of the plane, the Darboux phase space is not the plane but the unit disc $D^2 = \{ z \in \mathbb{C}^1 : z \bar{z} \leq 1 \}$ for the sphere and the complement of the unit disc for the Lobascevsky plane.

The two dimensional geometries which have a single Killing vector are more numerous and we discuss only some specific examples. We do, however, give a general prescription for

\footnote{Note that homogeneous symplectic manifolds are essentially coadjoint orbits of Lie groups $G$ [10], and so they can be expressed group theoretically as coset spaces $G/H$ whose points can be used to construct appropriate coherent state representations of $G$ [15,16].}
computing the alloted Hamiltonians in Darboux coordinates and discuss some of the ambiguities associated with the metric dependence of the localization formulas in these cases which was first pointed out in [14]. We also consider a general coherent state formalism [19] corresponding to the one-parameter isometry group action on the manifold and show that in this formulation of the quantum theory the Hamiltonians are again just “generalized” harmonic oscillators. This final result includes the three cases mentioned above as special examples, and is therefore a general expression for the set of two dimensional Hamiltonian systems whose phase space path integrals may be equivariantly localized. Furthermore, we show how these and the more general geometric structures of the phase space are explicitly realized in the relevant quantized Hamiltonian systems, which gives a further probe into the geometrical nature of quantum integrability.

2. The Equivariant Localization Principle

The derivations of the standard localization formulas are based on equivariant cohomology [12,20] and a supersymmetry of the underlying Hamiltonian system. In this Section we begin by briefly sketching how these ideas lead to the principle of Abelian equivariant localization for path integrals and the constraints it imposes on the form of the Hamiltonian system \((\Gamma^{2n}, \omega, \mathcal{F}(H))\). We assume henceforth that the observable \(H\) generates, through the relation (1.3), a global U(1) action on \(\Gamma^{2n}\).

Let \(\text{Fun}(S^1)\) be the algebra of polynomial functions on the Lie algebra of U(1) graded so that an \(n\)-th order homogeneous polynomial is considered to be of degree \(2n\), and consider the complex \(\Lambda_{\text{inv}}^{*}(\Gamma^{2n}) \subset \Lambda^{*}(\Gamma^{2n}) \otimes \text{Fun}(S^1)\) of differential forms on \(\Gamma^{2n}\) which are invariant under the U(1) action on the phase space, i.e. the equivariant differential forms. The U(1)-equivariant cohomology of \(\Gamma^{2n}\) is defined by endowing \(\Lambda_{\text{inv}}^{*}(\Gamma^{2n})\) with the derivative \(D_{V} = d + i_{V}\). Then \(D_{V}^{2} = \mathcal{L}_{V}\), where

\[\mathcal{L}_{V} = di_{V} + i_{V}d\]

is the Lie derivative along the Hamiltonian vector field \(V\). Thus \(D_{V}^{2} = 0\) precisely on \(\Lambda_{\text{inv}}^{*}(\Gamma^{2n})\), and the cohomology of the operator \(D_{V}\) on \(\Lambda_{\text{inv}}^{*}(\Gamma^{2n})\) is called the U(1)-equivariant cohomology of \(\Gamma^{2n}\), \(H_{\text{inv}}^{*}(\Gamma^{2n})\). By lifting of the phase space coordinates we also obtain the loop space equivariant cohomology \(H_{\text{inv}}^{*}(L\Gamma^{2n})\) associated with the loop space Hamiltonian vector field \(V_{L}^{\mu}(x; t) = \dot{x}^{\mu}(t) - V^{\mu}(x(t))\) corresponding to the action functional in (1.4).
The action functional

\[ S[x] = \int_0^T dt \left( \theta_\mu x^\mu - \mathcal{F}(H) \right) \]  

(2.1)
generates a series \( \alpha \in H^*_\text{inv}(L\Gamma^{2n}) \) of equivariantly closed differential forms integrated over the loop space in the partition function (1.4) [2–7]: \( Z(T) = \int_{L\Gamma^{2n}} \alpha \). It follows that for any \( s \in \mathbb{R}^+ \) and for any equivariant form \( \beta \in \Lambda^*_\text{inv}(L\Gamma^{2n}) \), the integral

\[ \int_{L\Gamma^{2n}} \alpha = \int_{L\Gamma^{2n}} \alpha e^{-sD_V\beta} \]  

(2.2)
is formally independent of the parameter \( s \) [9], and so the right-hand side of (2.2) can be evaluated in the limit \( s \to +\infty \) (if this limit in fact exists) giving a localization of the path integral (1.4) onto the zeroes of the loop space form \( \beta \).

We now come to the main assumption invoked in the equivariant localization principle. We assume that the phase space \( \Gamma^{2n} \) admits a globally defined \( U(1) \)-invariant Riemannian structure with metric tensor \( g = \frac{1}{2} g_{\mu\nu}(x) dx^\mu \otimes dx^\nu \) satisfying

\[ \mathcal{L}_V g = 0 \]  

(2.3)
In other words, \( V \) is a Killing vector of the metric \( g \). If \( \Gamma^{2n} \) is compact and the observable \( H \) generates a smooth global \( U(1) \) action on \( \Gamma^{2n} \), then such a metric can always be obtained from any smooth metric \( g' \) on \( \Gamma^{2n} \) by averaging \( g' \) over the group \( U(1) \). In the following we shall always suppose that (2.3) generally holds here, and the metric \( g \) naturally induces a loop space metric \( G_{\mu\nu}(x; t, t') = g_{\mu\nu}(x(t)) \delta(t - t') \).

Various localization formulas can now be obtained by taking \( \beta \) to be the dual 1-form \( \beta = G(W, \cdot) \) of some loop space vector field \( W \). Then the formula (2.2) becomes

\[ \int_{L\Gamma^{2n}} \alpha = \int_{L\Gamma^{2n}} \alpha e^{-sD_Vg(W,V_L)} \]

and yields a localization onto the zeroes of \( G(W, V_L) \), which for suitably chosen \( W \) will be the same as the zeroes of \( W \). If the Hessian of \( G(W, V_L) \) is non-degenerate then the large-\( s \) limit can be evaluated by Gaussian integration. For example, if \( \mathcal{F}(H) = H \) and \( W = V_L \) is the loop space Hamiltonian vector field corresponding to the action functional (2.1), then we can formally obtain the path integral generalization of the Duistermaat-Heckman formula (1.2) [2–5]

\[ Z(T) = \sum_{x \in I(S)} \frac{\det^{1/2} |\omega_{\mu\nu}|}{\det^{1/2} |\delta_{\mu} - \partial_\nu(\omega_{\mu\lambda} \partial_{\lambda} H)|} e^{iS[x]} \]  

(2.4)
where $\omega^{\mu\nu}$ is the matrix inverse of $\omega_{\mu\nu}$. The formula (2.4) supposes that the determinant of the Jacobi fields is non-trivial. Thus under these circumstances the path integral (1.4) localizes onto classical trajectories and we obtain a WKB localization of (1.4).

More generally, however, we can set $W^\mu = \frac{1}{2}\dot{x}^\mu(t)$ and formally obtain a localization of (1.4) onto the time-independent modes $x^\mu_0$ of $x^\mu(t)$ [6,7]:

$$Z(T) = \int_{\Gamma_0} d\phi_0 \, dx^\mu_0 \det^{1/2}_{(\mu\nu)}[\omega_{\mu\nu}] \, e^{-iT(F(\phi_0) + \phi_0 H)} \det^{1/2}_{(\mu\nu)} \left[ \frac{1}{2}(\phi_0 \tilde{\Omega}^\mu_\nu + R^\mu_\nu) \right] \sinh \left[ \frac{T}{2}(\phi_0 \tilde{\Omega}^\mu_\nu + R^\mu_\nu) \right].$$

(2.5)

In (2.5), which is now an ordinary finite dimensional integral over some submanifold $\Gamma_0$ of the phase space $\Gamma^{2n}$, $F(\phi)$ determines a functional Fourier transform of the Hamiltonian,

$$\exp \left( -i \int_0^T dt \, F(H) \right) = \int_{L\Gamma^{2n}} D\phi \exp \left( -i \int_0^T dt \, F(\phi) \right) \exp \left( -i \int_0^T dt \phi(x)H(x) \right),$$

(2.6)

and $\phi_0$ are the zero modes of the auxilliary field $\phi(x)$. $R^\mu_\nu = R^\mu_{\nu\lambda\rho}(x)dx^\lambda \wedge dx^\rho$ is the Riemann curvature 2-form of $g$, while $\tilde{\Omega}^\mu_\nu(x) = 2\nabla_\nu V^\mu(x)$ is the Riemann moment map associated with the U(1) action on the Riemannian manifold $(\Gamma^{2n}, g)$ [20] (Notice that, by the Killing equation (2.3), the matrix $\tilde{\Omega}$ is antisymmetric and so has only $n(2n-1)$ independent components).

The exponential factor in (2.5) is an equivariant generalization of the Chern character, while the second determinant is an equivariant $\hat{A}$-genus [6,7]. Thus in this case we obtain, when the above formal steps actually carry through, a relatively simple localization onto equivariant characteristic classes of the manifold $\Gamma^{2n}$ (i.e. an equivariant generalization of the Atiyah-Singer index density for a Dirac operator) [20]. However, although the functional $F(H)$ is a priori arbitrary, there is no reason to believe that the functional Fourier transform (2.6) can exist arbitrarily. If $F(H)$ is an unbounded functional, then a Wick rotation off the real time axis may produce an analytically continued propagator which is not a tempered distribution. Thus we expect that the general localization formula (2.5) is valid only for Hamiltonians $F(H)$ which are bounded functionals of the observable $H$.

We remark that there are also weaker localization formulas for Hamiltonian sytems which are not necessarily completely integrable (i.e. for which the equivariant localization procedure above does not carry through). In particular, consider a Hamiltonian $F(H) = H$
with $m < n$ independent integrals of motion $I_k(x)$ in involution,

$$\{H, I_k\} = \{I_k, I_\ell\} = 0, \quad (2.7)$$

where the Poisson bracket is defined by

$$\{f, g\} = -\omega^{-1}(df, dg) \quad ; \quad f, g \in C^\infty(\Gamma^{2n}). \quad (2.8)$$

In this case we can take $\beta = \int_0^T dt \dot{I}_k dI_k$ and formally obtain the integration formula [4]

$$Z(T) = \int_{L\Gamma^{2n}} Dx^\mu(t) \prod_{k=1}^m \delta(\dot{I}_k) e^{iS[x]}, \quad (2.9)$$

a weaker version of the above formulas which localizes the path integral (1.4) onto the reduced symplectic subspace of the loop space determined by the constant values of the conserved charges $I_k$, and it can be viewed as a quantum generalization of the classical reduction theorem [21].

The above results indicate that equivariant cohomology might therefore provide a natural geometric framework for understanding quantum integrability (in the sense that the path integral can be reduced to a finite dimensional expression). The derivative $D_{\mathcal{V}_k}$ above has also been shown to be a supersymmetry operator on $\Lambda^*_\text{inv}(L\Gamma^{2n})$ for which the effective action functional in (1.4) is supersymmetric, with respect to the Poisson algebra (2.8) given by the underlying symplectic structure [2–7]. In the next Section we shall use the global constraint (2.3) on the phase space geometry to construct the possible Hamiltonians to which the above localization formulas apply in two dimensions, and show how the geometry of the phase space is thus explicitly realized in the underlying Hamiltonian system.

### 3. Equivariant Hamiltonian Systems in Two Dimensions

We now concentrate on the case of a two dimensional symplectic manifold $(\Gamma^2, \omega)$, and take $g$ to be of Euclidean signature. We assume that $\Gamma^2$ is a (compact or non-compact) Riemann surface which is simply connected, $\pi_1(\Gamma^2) = H^1(\Gamma^2; \mathbb{Z}) = 0$. Since the fundamental group of $\Gamma^2$ is trivial, it follows from the Riemann uniformization theorem that via a diffeomorphism and a Weyl transformation of the coordinates the metric can be put globally into the form

$$g_{\mu\nu}(x) = e^{\phi(x)} \delta_{\mu\nu}. \quad (3.1)$$
Here $\varphi(x)$ is a globally-defined real-valued function on $\Gamma^2$ which we call the conformal factor of the metric. We put the standard complex structure on $\Gamma^2$ and define the complex coordinates $z, \bar{z} = x^1 \pm ix^2$, and set $V^z, \bar{V} = V^1 \pm iV^2$ and $\partial, \bar{\partial} = \frac{1}{2}(\partial_1 \mp i\partial_2)$.

In local coordinates on $\Gamma^2$ the Killing equation (2.3) has the form
\[ g_{\mu\lambda} \partial_\nu V^\lambda + g_{\nu\lambda} \partial_\mu V^\lambda + V^\lambda \partial_\lambda g_{\mu\nu} = 0 \] (3.2)
which, in complex coordinates for the metric (3.1), becomes
\[ \bar{\partial} V^z = 0, \quad \partial \bar{V}^z = 0 \] (3.3)
\[ \partial V^z + \bar{\partial} \bar{V}^z + V^z \partial \varphi + \bar{V}^z \bar{\partial} \varphi = 0. \] (3.4)
The equations (3.3) are the Cauchy-Riemann equations and they say that, in the coordinates specified by (3.1), the Killing vector field $V^z$ is a holomorphic function on $\Gamma^2$, while (3.4) is a source equation for $V^z$ and $\bar{V}^z$. Rather than attempting to solve the general equation (3.4) directly in the coordinates involved, we turn to a few facts from the theory of isometry groups of Riemannian manifolds [10]. For simply connected spaces, the generators of the isometry group $\mathcal{I}(\Gamma^2, g)$ of the phase space form a locally compact Lie algebra $\mathcal{K}(\Gamma^2, g) = \{ U \in T\Gamma^2 : \mathcal{L}_U g = 0 \}$ whose dimension is either 1 or 3. We begin by considering the case $\dim \mathcal{K}(\Gamma^2, g) = 3$ when the phase space $(\Gamma^2, g)$ is maximally symmetric (i.e. homogeneous and isotropic about each of its points).

In the case of maximal symmetry it can be shown [10] that the Gaussian curvature $K$ of $(\Gamma^2, g)$, which is defined through the Riemann curvature tensor by
\[ R_{\lambda\mu\nu\rho} = -K(g_{\lambda\nu}g_{\mu\rho} - g_{\lambda\rho}g_{\mu\nu}), \] (3.5)
is constant. Moreover, the converse is also true, and the constant Gaussian curvature of the space uniquely determines $(\Gamma^2, g)$, in that any two maximally symmetric metrics on $\Gamma^2$ with the same $K$ and the same signature are equivalent up to a local diffeomorphism of $\Gamma^2$. In terms of the conformal factor defined in (3.1) the Gaussian curvature is
\[ K = -\frac{1}{2} e^{-\varphi} \Delta \varphi, \]
where $\Delta = \partial \bar{\partial}$ is the two dimensional scalar Laplacian on $\Gamma^2$, and we now consider separately the cases $K = 0, K > 0$ and $K < 0$. Notice that if $\Gamma^2$ were a compact

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2 In general, for a Hamiltonian system with $n$ degrees of freedom we have that $\dim \mathcal{K}(\Gamma^{2n}, g) \leq n(2n + 1)$. 

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Riemann surface of genus \( h \), then an application of the Gauss-Bonnet theorem would give 
\[ \int_{\Gamma^2} d\text{vol}(g) K = 4\pi(1 - h). \]
Thus a maximally symmetric compact space \( \Gamma^2 \) of constant negative curvature must have genus \( h \geq 2 \). Therefore, under our topological assumptions, in the cases \( K = 0 \) and \( K > 0 \) the phase space \( \Gamma^2 \) may be either compact or non-compact, while for \( K < 0 \) it is necessarily non-compact.

### 3.1. Planar Geometries

When \( K = 0 \), the space \((\Gamma^2, g)\) is locally flat, and the conformal factor satisfies the two dimensional Laplace equation \( \Delta \varphi = 0 \) whose general solutions are 
\[ \varphi(z, \bar{z}) = f_0(z) + \bar{f}_0(\bar{z}), \]
where \( f_0(z) \) is an arbitrary holomorphic function on \( \Gamma^2 \). Since two dimensional conformal transformations are precisely the analytic coordinate transformations \( z \to w(z) \), we can conformally map \((\Gamma^2, g)\) onto the plane \( \mathbb{R}^2 \) with its standard flat Euclidean metric \( g_{E^2} \). The most general solution to the Killing equations in these new coordinates (\( \varphi = 0 \) in (3.4)) is then
\[ V_0 = (-i\Omega_0 w + \alpha_0) \frac{\partial}{\partial w} + (i\Omega_0 \bar{w} + \alpha_0) \frac{\partial}{\partial \bar{w}}, \quad (3.6) \]
where \( \Omega_0 \in \mathbb{R}^1 \) and \( \alpha_0 \in \mathbb{C}^1 \) are arbitrary parameters. Notice that the \( \Omega_0 \)-term in (3.6) generates the single possible rotation of the plane while the \( \alpha_0 \)-term generates the two possible translations in Euclidean 2-space \( \mathbb{R}^2 \), and they together generate \( \mathcal{I}(\mathbb{R}^2, g_{E^2}) = E^2 \), the Euclidean group of the plane.

The local form of the Hamiltonian equations (1.3) is
\[ \partial H = \frac{i}{2} \omega(w, \bar{w}) V^\bar{w}, \quad \bar{\partial} H = -\frac{i}{2} \omega(w, \bar{w}) V^w \quad (3.7) \]
where \( \omega = \frac{i}{2} \omega(w, \bar{w}) dw \wedge d\bar{w} \). The symplectic 2-form can be explicitly determined by recalling that the U(1) action on the phase space is required to be symplectic. This means that \( \omega \in \Lambda^2_{\text{inv}}(\Gamma^2) \) so that
\[ 0 = \mathcal{L}_V \omega = \partial_{\mu} \left( V^\lambda \omega_{\mu\lambda}(x) \right) dx^\mu \wedge dx^\nu. \quad (3.8) \]
For the planar Killing vectors (3.6) the condition (3.8) implies that the function \( \omega(w, \bar{w}) \) is constant on \( \mathbb{R}^2 \); i.e. \( \omega \) is the volume (or, in this case, the Darboux) 2-form on flat \( \mathbb{R}^2 \). The equations (3.7) can now be solved on \((\mathbb{R}^2, g_{E^2})\) by substituting in (3.6) and \( \omega(w, \bar{w}) = 1 \),
which gives the observable $H$ on $(\mathbb{R}^2, g_{E^2})$, and on the original manifold $(\Gamma^2, g)$ in terms of the metric (3.1), the most general solution of (1.3) is therefore given by

$$H_0(z, \bar{z}) = \Omega_0 w_f(z) \overline{w}_f(z) + \overline{\alpha}_0 w_f(z) + \alpha_0 \overline{w}_f(z) + C_0,$$

(3.9)

where

$$w_f(z) = \int_{C_z} d\xi \ e^{f_0(\xi)}$$

(3.10)

is the conformal transformation mapping $(\Gamma^2, g)$ onto flat $\mathbb{R}^2$ with $C_z \subset \Gamma^2$ a simple curve from some fixed base point $z_0$ to $z$, and $C_0 \in \mathbb{R}^1$ is a constant of integration.

3.2. Spherical Geometries

Now let us consider the case of a positive Gaussian curvature $K > 0$. In this case the conformal factor solves the Liouville field equation $\Delta \varphi(z, \bar{z}) = -2K e^{\varphi(z, \bar{z})}$ [22], a completely integrable system whose general solutions are

$$\varphi(z, \bar{z}) = \log \left[ \frac{\partial f_1(z) \overline{\partial f_1(z)}}{(K + f_1(z) \overline{f_1(z)})^2} \right],$$

(3.11)

where $f_1(z)$ is any holomorphic function on $\Gamma^2$. By the essential uniqueness of maximally symmetric spaces, there exists a local coordinate transformation $(z, \bar{z}) \rightarrow (w(z, \bar{z}), \overline{w}(z, \bar{z}))$ mapping $(\Gamma^2, g)$ onto the sphere $S^2$ of radius $K^{-1/2}$ with its standard round metric $g_{S^2}$ [10]. That $(S^2, g_{S^2})$ is maximally symmetric follows from the fact that it is the one-point compactification of the maximally symmetric space $(\mathbb{R}^2, g_{E^2})$. The Hamiltonian equations (1.3) on $\Gamma^2$ in this case can therefore be solved by considering the corresponding problem for a spherical geometry.

The induced metric on $S^2$ from its embedding in Euclidean 3-space $\mathbb{R}^3$ is

$$g_{S^2} = \frac{1}{4K} \left[ \frac{\overline{w}^2}{1 - w\overline{w}} dw \otimes dw + \frac{w^2}{1 - w\overline{w}} d\overline{w} \otimes d\overline{w} + 2 \left( 2 + \frac{w\overline{w}}{1 - w\overline{w}} \right) dw \otimes d\overline{w} \right]$$

(3.12)

where $w\overline{w} \leq 1$. By considering the manifest invariances of the embedding condition $\overline{x}^2 = K^{-1}, \overline{x} \in \mathbb{R}^3$, simultaneously with those of the standard Euclidean metric on $\mathbb{R}^3$, it is straightforward to show that the Killing vectors of the metric (3.12) have the general form

$$V_{S^2} = \left(-i\Omega_0 w + \alpha_0(1 - w\overline{w})^{1/2}\right) \frac{\partial}{\partial w} + \left(i\Omega_0 \overline{w} + \overline{\alpha}_0(1 - w\overline{w})^{1/2}\right) \frac{\partial}{\partial \overline{w}}.$$

(3.13)
The $\Omega_0$-term in (3.13) generates rigid rotations of the sphere while the $\alpha_0$-term generates the two independent quasi-translations on $S^2$, and the isometry group of the sphere is just $\mathcal{I}(S^2, g_{S^2}) = \text{SO}(3)$.

The U(1)-invariance condition (3.8) in the case at hand then implies again that the symplectic 2-form $\omega$ is just the volume form on $(S^2, g_{S^2})$, i.e. $\omega(w, \overline{w}) = K^{-1}(1 - w\overline{w})^{-1/2}$, which is a non-trivial element of $H^2(S^2; \mathbb{Z}) = \mathbb{Z}$. Substituting this and the Killing vectors (3.13) into (3.7) gives the function $H$ on $(S^2, g_{S^2})$. We can now use a generalized stereographic projection from the south pole of $S^2$,

$$(w(z, \overline{z}), \overline{w}(z, \overline{z})) = \frac{4K^{-1/2}}{1 + 4K^{-1}f_1(z)\overline{f}_1(\overline{z})}(f_1(z), \overline{f}_1(\overline{z})),$$

(3.14)

to map (3.12) back onto the original Riemannian manifold $(\Gamma^2, g)$ defined by (3.1) and (3.11), and we find that the most general observable $H$ on $(\Gamma^2, g)$ for $K > 0$ is

$$H_1(z, \overline{z}) = \Omega_0 \left( \frac{K^4}{4} - f_1(z)\overline{f}_1(\overline{z}) \right) + \frac{\alpha_0 f_1(z) + \overline{\alpha}_0 f_1(\overline{z})}{K^4 + f_1(z)\overline{f}_1(\overline{z})} + C_0.$$  

(3.15)

The case of the sphere $S^2$ corresponds of course to the choice of $f_1(z) = (K^{1/2}/2)z$ in (3.15).

3.3. Hyperbolic Geometries

The construction for the case $K < 0$ is identical to that in Section 3.2 above, except that now we map onto the maximally symmetric space $H^2$, the Lobashevsky plane of constant negative curvature, with its standard curved metric $g_{H^2}$ [10]. This space is defined by the embedding of the surface $\eta_{ij}x^ix^j = K^{-1}$, $\vec{x} = \{x^i\} \in \mathbb{R}^3$, in $\mathbb{R}^3$ with flat Minkowskian metric $\eta_{ij} = \text{diag}(1, 1, -1)$, from which it can be shown that the Killing vectors of the corresponding induced metric $g_{H^2}$ on $H^2$ have the general form

$$V_{H^2} = \left(-i\Omega_0 w + \alpha_0(1 + w\overline{w})^{1/2}\right) \frac{\partial}{\partial w} + \left( i\Omega_0 \overline{w} + \overline{\alpha}_0(1 + w\overline{w})^{1/2}\right) \frac{\partial}{\partial \overline{w}}$$

(3.16)

and generate the isometry group $\mathcal{I}(H^2, g_{H^2}) = \text{SO}(2, 1)$. The rest of the above steps now carry through analogously for the case at hand, and the final result for the observable $H$ on $(\Gamma^2, g)$ for $K < 0$ is identical to (3.15). Notice that now, however, the functions $f_1(z)$ map $(\Gamma^2, g)$ onto the Poincaré disc of radius $|K|^{1/2}/2$ (whereas in the spherical case the mapping is onto the entire complex plane with the usual Kähler geometry of $S^2$).
3.4. Geometries with Non-maximal Symmetry

The above solutions do not, of course, constitute the complete solution set of (1.3) for the metric (3.1). We still need to consider the cases where the Gaussian curvature of \((\Gamma^2, g)\) is a non-constant function of the coordinates; i.e. \(\dim \mathcal{K}(\Gamma^2, g) = 1\). The case where \((\Gamma^2, g)\) admits only a one-parameter group of isometries, or equivalently \((\Gamma^2, g)\) contains a one-dimensional maximally symmetric subspace, is the richest in its applications to non-trivial problems.

In this case we can change coordinates \(x \rightarrow x'\) on \(\Gamma^2\) so that the single isometry generator \(V\) has components \(V'^1 = a_0\), where \(a_0 \in \mathbb{R}^1\) is an arbitrary constant, and \(V'^2 = 0\) (i.e. the Killing vector of the flat line \(\mathbb{R}^1\)), and so that \(g'_{12}(x') = 0\). This can be explicitly accomplished by introducing two differentiable functions \(\chi^1\) and \(\chi^2\) on \(\Gamma^2\) such that the Jacobian \(\det_{(\mu\nu)}[\partial_{\mu}\chi^\nu]\) is non-trivial and such that \(\chi^2(x^1, x^2)\) is the unique solution of the homogeneous first order linear partial differential equation

\[
V(\chi^2) = V^\mu \partial_\mu \chi^2(x^1, x^2) = 0. \tag{3.17}
\]

These yields a set of coordinates \(x'_0 = \chi\) in which the Killing vector has components \(V'^2_0 = 0\) and \(V'^1_0 \neq 0\). The desired coordinates are now obtained by defining \(x'^1 = a_0 \int dx'^1_0/V'^1_0\) and \(x'^2 = x^2 = \chi^2\), and then choosing the characteristic curves of the coordinate \(x'^2 = \chi^2\) as found from (3.17) orthogonal to the paths defined by the isometry generator \(V\) (i.e. choosing the initial data for the solutions of (3.17) to lie on a non-characteristic surface) to give \(g'_{12} = 0\). In the new \(x'\)-coordinates, the Killing equation (3.2) implies that \(g'^{11}\) and \(g'^{22}\) are functions only of \(x'^2\), and so the phase space describes a surface of revolution (e.g. a cylinder or the “cigar-shaped” geometries used in black hole theories).

In these new coordinates \(\omega\) can be taken to be the Darboux 2-form, and the Hamiltonian equations (1.3) can be solved as above to give \(H\) in terms of the original coordinates defined by (3.1) as

\[
H(x^1, x^2) = a_0 \chi^2(x^1, x^2) \tag{3.18}
\]

where the coordinate transformation function \(\chi^2\), determined from (3.17), is constant along the integral curves of the Killing vector field \(V\). Therefore, if the general metric (3.1) admits a sole isometry, the Hamiltonians which result from the equivariant localization constraints are given by the transformation \(x \rightarrow x'\) in (3.18) to coordinates in which the corresponding Killing vector generates translations in the coordinate \(x'^1\) (which is an explicit \(U(1)\) action on \(\Gamma^2\)). We will discuss what Hamiltonian systems can explicitly arise
under these circumstances in the next Section. Notice that in the case of a one-parameter isometry group the equivariant condition (3.8) does not uniquely determine the symplectic 2-form $\omega$, and in most cases $\omega$ can even be taken to be the Darboux 2-form in the original coordinates (3.1).

4. Integrable Quantum Systems and Coherent State Formulations

In the previous Section we have obtained the explicit geometric dependence of the Hamiltonian systems which result from the equivariant localization constraints. We now consider explicitly what Hamiltonians result from the above analysis as pertaining to the quantization of some known systems, and examine the exactness of the localization formulas (2.4) and (2.5) for the corresponding partition functions $Z(T|\Omega_0, \alpha_0; \Gamma^2, \mathcal{F}(H))$ in the maximally symmetric cases and $Z(T|\alpha_0; \chi, \mathcal{F}(H))$ in the non-homogeneous cases. We also show that these Hamiltonians can be written as coherent state matrix elements associated with the canonical Poisson bracket action (2.8) of the corresponding isometry groups on the phase space [15,16,19].

In the case where the phase space admits only a one-parameter isometry group the calculations can be carried out straightforwardly, but in the maximally symmetric cases where the phase space isometry group is three dimensional some care must be taken in applying the localization formulas. This is because the equivariant localization principle discussed in Section 2 was based on the assumption that the function $H$ generates a global U(1) action on the phase space. However, as written above the observables $H$ in general are linear combinations of functions $H_a$ with $dH_a = -iV_a(\omega)$, where $V_a$ is the vector field on $\Gamma^2$ which is a generator of the non-Abelian Lie algebra $\mathcal{K}(\Gamma^2, g)$. While each of the $H_a$ individually generate U(1) actions on $\Gamma^2$, the collection of them generate a Hamiltonian action of the isometry group $\mathcal{I}(\Gamma^2, g)$ represented in $C^{\infty}(\Gamma^2)$ with Lie bracket given by the Poisson bracket (2.8). To apply the above formalism we must therefore consider Hamiltonians which are functionals of only one of the $H_a$, corresponding to an Abelian group action. This is also what one expects from standard integrability theory [22], which tells us that an integrable Hamiltonian arises by taking functionals of action variables $I_k$ which are in involution, as in (2.7). In the present context, this means that an integrable Hamiltonian is obtained by considering functionals of Cartan elements of $\mathcal{K}(\Gamma^2, g)$ only, which is equivalent to the requirement that $H$ generate a U(1) action on the phase space.
acting on $T^2$ can also be considered with modifications of the equivariant localization principle discussed in Section 2 [9], but we will consider only the Abelian localization formulas (2.4) and (2.5) in the following.

4.1. Maximally Symmetric Phase Spaces

To determine what (flat space) quantum systems (3.9) describes, we map onto Darboux coordinates, defined by $\omega = \frac{i}{2} dz \wedge d\bar{z}$, to see what possible one dimensional quantum mechanical models can arise from this observable. This is accomplished by the conformal transformation $z \rightarrow w_f(z)$ defined in (3.10) mapping $(T^2, g)$ onto $(\mathbb{R}^2, g_{E^2})$, which gives (3.9) in the Darboux form

$$H_0^D(z, \bar{z}) = \Omega_0 z\bar{z} + \alpha_0 z + \alpha_0 \bar{z} ; \quad z \in \mathbb{C}^1.$$  \hspace{1cm} (4.1)

Equation (4.1) shows that we can obtain only the harmonic oscillator ($\alpha_0 = 0$ in (3.9) and $\mathcal{F}(H) = H$ in (2.1)), and the free particle ($\alpha_0 \in \mathbb{R}^1$, $\Omega_0 = 0$ and $\mathcal{F}(H) = H^2$).

The localization formula (2.5) was considered in [14] for the harmonic oscillator (which is already known to be WKB exact) and shown to give the exact result for its path integral (as found from the WKB formula (2.4) or the Schrödinger equation)

$$Z \left( T \bigg| \frac{1}{2}, 0; \mathbb{R}^2, H \right) = \frac{1}{2 \sin \frac{T}{2}}.$$

For the free particle, we find from (2.6) with $\mathcal{F}(H) = H^2$ that $F(\phi) = \phi^2$, and $R_\nu^\mu = \tilde{\Omega}_\nu^\mu = 0$ in (2.5). The $\phi_0$-integral in (2.5) is thus a trivial Gaussian one, and we end up with the standard free particle partition function$^3$

$$Z \left( T \bigg| 0, \frac{1}{2}; \mathbb{R}^2, H^2 \right) = \int_{-\infty}^{+\infty} dp \ dq \ e^{-iTp^2}.$$

The remaining Hamiltonian systems, defined by (3.9), are merely holomorphic copies of the displaced harmonic oscillators (4.1) (i.e. $z \rightarrow z + \alpha_0$) defined by the conformal transformation (3.10) of the phase space.

---

$^3$ Dykstra et al. [14] verified the free particle case as well, when $\mathcal{F}(H) = H$. However, as we have shown in Section 3 above, the Hamiltonian vector field corresponding to $p^2$ cannot generate isometries of any metric of the form (3.1), which is what leads to the singular metrics used by Dykstra et al. It is possible to go through a similar analysis as in Section 3 using such singular geometries, and obtain the same results as in [14].
Notice that the three independent terms in (4.1) can also be written as the matrix elements $L(z, \bar{z}) = (z|L|z)/(z|z)$ of the usual Heisenberg-Weyl group generators $L = a^\dagger a, a^\dagger, a$, respectively, in the canonical coherent states [15]

$$|z\rangle = e^{za^\dagger} |0\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle ; \quad z \in \mathbb{C}^1.$$ 

The Heisenberg-Weyl algebra is explicitly realized by the corresponding planar Poisson algebra of the functions $L(z, \bar{z})$ on the phase space $\mathbb{C}^1$, and as discussed at the beginning of this Section an integrable Hamiltonian is obtained by considering functionals only of Cartan elements of the Heisenberg-Weyl algebra (e.g. the harmonic oscillator $a^\dagger a$ or the free particle $a + a^\dagger$).

In the case of a spherical or hyperbolic geometry, the transformation to Darboux coordinates on $(\Gamma^2, \omega)$ is given by the diffeomorphism $(z, \bar{z}) \to (v(z, \bar{z}), \bar{v}(z, \bar{z}))$, where

$$v(z, \bar{z}) = f_1(z) \left(1 + \frac{|K|}{4} + \text{sign}(K)f_1(z)f_1(\bar{z})\right)^{1/2}$$

maps $\Gamma^2$ onto $D^2_K = \{z \in \mathbb{C}^1 : \text{sign}(K)z\bar{z} \leq \text{sign}(K)\}$, the unit disk $D^2$ in $\mathbb{R}^2$ for $K > 0$ and its complement $\mathbb{R}^2 \setminus \text{int}(D^2)$ for $K < 0$. The observable (3.15) in Darboux coordinates is therefore

$$H^D_1(z, \bar{z}) = \Omega_0 z\bar{z} + (\alpha_0 z + \alpha_0 \bar{z}) (1 - \text{sign}(K)z\bar{z})^{1/2} ; \quad z \in D^2_K. \quad (4.2)$$

Equation (4.2) shows that the other general Hamiltonian systems defined by (3.15) above are just holomorphic copies of each other, as they all map onto the same Darboux system defined by (4.2), namely the quasi-displaced harmonic oscillator; i.e.

$$z \to z + \alpha_0 (1 - \text{sign}(K)z\bar{z})^{1/2}.$$ 

We therefore consider the quantum problems defined on the phase spaces $S^2$ and $H^2$ only, and normalize the local coordinates so that now $|K| = 1$.

The three independent observables in (3.15) on the sphere are just the realization of the SU(2) Lie algebra on $S^2$ by its standard Poisson structure. Indeed, if we write

$$J_3(z, \bar{z}) = -j \frac{1 - z\bar{z}}{1 + z\bar{z}} , \quad J_+(z, \bar{z}) = 2j \frac{\bar{z}}{1 + z\bar{z}} , \quad J_-(z, \bar{z}) = 2j \frac{z}{1 + z\bar{z}} , \quad (4.3)$$

then the Poisson algebra of these functions

$$\{J_3, J_\pm\} = \pm J_\pm , \quad \{J_+, J_-\} = 2J_3$$
is just SU(2). The general Hamiltonians $\mathcal{F}(H)$ defined by (3.15) are then functions on the coadjoint orbit $SU(2)/U(1) = S^2$ of SU(2), and the corresponding path integral gives the quantization of spin [16]. The generators (4.3) are in fact the matrix elements $J_i(z,\bar{z}) = (z|J_i|z)/(z|z)$ in the SU(2) coherent states [15]

$$|z\rangle = e^{zJ_3}|j, -j\rangle = \sum_{m=-j}^j \left( \frac{2j}{j+m} \right)^{1/2} z^{j+m}|j, m\rangle \quad ; \quad z \in \mathbb{C}$$

for the irreducible spin-$j$ representation of SU(2), $j = \frac{1}{2}, 1, \ldots$.

As before, an integrable Hamiltonian is obtained by considering functionals only of Cartan elements, which for SU(2) means taking $H = J_3$ above, the height function on $S^2$. The WKB localization formula (2.4) was computed in [3,4,17] and the general formula (2.5) for $\mathcal{F}(H) = H$ in [7], and shown to give the exact Weyl character formula for SU(2) [16]

$$Z(T|j, 0; S^2, H) = \frac{\sin(2j+1)T}{\sin T}.$$

The formula (2.5) was considered in this case for $\mathcal{F}(H) = H^2$ as well in [7] and shown to give the exact result

$$Z(T|j, 0; S^2, H^2) = \sum_{m=-j}^j e^{-iTm^2} = \text{Tr}_{(j)} e^{-iTJ_3^2}.$$

The path integrals here yield the quantization of the harmonic oscillator on the sphere (see (4.2)), and therefore the only integrable quantum system, up to holomorphic equivalence, we can consider within the equivariant localization framework on a general spherical geometry is the harmonic oscillator (on the reduced phase space $D^2$).

The situation is analogous for the function (3.15) on the Lobashevsky plane $H^2$. If we define

$$S_3(z, \bar{z}) = k \frac{1 + z\bar{z}}{1 - z\bar{z}}, \quad S_+(z, \bar{z}) = 2k \frac{\bar{z}}{1 - z\bar{z}}, \quad S_-(z, \bar{z}) = 2k \frac{z}{1 - z\bar{z}} \quad (4.4)$$

then the Poisson algebra of these observables is just the SU(1,1) algebra

$$\{S_3, S_{\pm}\} = \pm S_{\pm}, \quad \{S_+, S_-\} = -2S_3.$$

The Hamiltonians obtained from (3.15) are therefore functions on the coadjoint orbit $SU(1,1)/U(1) = H^2$ of the non-compact Lie group SU(1,1) with the generators (4.4)
being matrix elements in the SU(1,1) coherent states

$$|z\rangle = e^{zS_+}|k; 0\rangle = \sum_{n=0}^{\infty} \left( \frac{2k + n + 1}{n} \right)^{1/2} z^n |k; n\rangle; n \in \text{int}(D^2)$$

for the discrete representation of SU(1,1) characterized by $k = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$ [15,16,18]. Again integrable Hamiltonian systems are obtained by taking $H = S_3$, which is the height function on $H^2$, and the corresponding path integral gives the quantization of the harmonic oscillator on the open infinite space $H^2$ (and up to holomorphic equivalence these are the only integrable systems possible on a general hyperbolic geometry). The WKB approximation for this coadjoint orbit path integral has been shown to be exact and gives the usual Weyl character formula for SU(1,1) [17]

$$Z(T|k, 0; H^2, H) = 2^i e^{-iT(k - \frac{1}{2})} \frac{\sin \frac{T}{2}}{\sin \frac{T}{2}}.$$  

4.2. Phase Spaces with One-parameter Isometry Groups

In the case where $(T^2, g)$ isn’t maximally symmetric, we showed in Section 3.4 that it can have local coordinates defined on it such that its single isometry can be taken to be translations in a single coordinate, and then the Hamiltonian, in the coordinates (3.1), is given by the corresponding transformation function of the other coordinate, which is constant along the integral curves of the isometry generator. For example, if the metric (3.1) is initially radially symmetric (i.e. $\varphi$ depends only on the product $z\overline{z}$) then the desired coordinate transformation is to the usual polar coordinates

$$\theta(x^1, x^2) = \arctan \left( \frac{x^2}{x^1} \right)$$

$$r^2(x^1, x^2) = (x^1)^2 + (x^2)^2$$

in which the Killing vector $V = \partial_\theta$ generates translations in $\theta$. In the original coordinates on $T^2$ the corresponding Hamiltonian function is nothing but the harmonic oscillator $H = \frac{1}{2}r^2$, as expected from (4.5) and the general arguments of Section 3.4 above.

Thus one would initially expect that the localization formulas (2.4) and (2.5) are exact for the harmonic oscillator with any radially symmetric geometry on the underlying phase space. This is true of the WKB formula (2.4), wherein the phase space metric does not enter directly into its evaluation, but the more general formula (2.5) is explicitly metric dependent through the equivariant $\hat{A}$-genus. In fact, it was shown by Dykstra et al. [14]
that (2.5) (with $F(H) = H$) gives the correct result for the harmonic oscillator path integral only if the phase space metric satisfies the additional constraint

$$\lim_{r \to 0} \frac{\partial_r g_{\theta\theta}(r)}{\sqrt{g(r)}} = -1$$

(4.6)

where $g(r) = \det(\mu\nu)[g_{\mu\nu}(r)]$. For the metric (3.1) with radial symmetry, this implies that

$$\lim_{r \to 0} r \frac{d}{dr} \varphi(r) = 0$$

(4.7)

so that the conformal factor $\varphi(r)$ must be an analytic function of $r$ about $r = 0$. This puts a somewhat strong restriction on the general form of the metric (3.1) (i.e. on the functional properties of the conformal factor $\varphi$) and it ensures that $Z(T)$ is independent of the phase space metric, as it should be. As discussed in [14], this appears to be a general problem with the more general localization formulas (2.5), in that one must essentially know the quantum theory ab initio in order to resolve the ambiguities associated with the arbitrariness of the metric (3.1). Nevertheless, the above simple example shows that quite general geometries can still be used to equivariantly localize the path integral (1.4) using the formulas (2.4) and (2.5).

This problem can also be seen to be the case for quantum systems which are not WKB exact. In [14] a similar metric ambiguity was shown to occur for the one dimensional hydrogen atom Hamiltonian $H(x^1, x^2) = \frac{1}{2}(x^1)^2 - 1/|x^2|$, a quantum system whose classical bound state orbits coalesce and so cannot be WKB localized [23]. They argued that the localization formula (2.5) with $F(H) = H$ is however exact for this quantum mechanical example. In our context this Hamiltonian would occur, following the analysis of Section 3.4, if the phase space metric $g_{\mu\nu}(x^1, x^2)$ in (3.1) is invariant under translations $\tilde{\chi}^1 \to \tilde{\chi}^1 + a_0$ of the variable

$$\tilde{\chi}^1(x^1, x^2) = -\frac{1}{\sqrt[3]{\frac{2}{|x^2|} - (x^1)^2}} \left[ x^1|x^2| \sqrt{\frac{2}{|x^2|} - (x^1)^2} + 2 \arctan \left( \frac{x^1}{\sqrt{\frac{2}{|x^2|} - (x^1)^2}} \right) \right].$$

(4.8)

The coordinate transformation $(x^1, x^2) \to (\tilde{\chi}^1(x^1, x^2), \tilde{\chi}^2(x^1, x^2))$, where $\tilde{\chi}^1$ is given by (4.8) and $(\tilde{\chi}^2(x^1, x^2))^2 = 2/|x^2| - (x^1)^2$, then produces new coordinates $x'$ in which the metric is independent of $x'^1$ and the Killing vector has the single non-vanishing component $V'^1 = 1$. From (3.18) we see that under these circumstances we do indeed obtain the hydrogen atom Hamiltonian $H = -\frac{1}{2}(\tilde{\chi}^2)^2$. The problem, however, is the existence of a
well-defined metric which is translation invariant in the variable (4.8), but the results of [14] show that such a metric can be globally defined on the phase space, and then imposing a further constraint similar to (4.6) the localization formula (2.5) gives the usual hydrogen atom partition function [14,23]

$$Z\left( T \left| -\frac{1}{2}; \tilde{\chi}, H \right. \right) = \sum_{n=0}^{\infty} e^{i T/2n^2}. $$

This example shows that non-trivial quantum systems can arise from the equivariant localization constraints, but only for phase space geometries which have complicated and unusual symmetries (such as translations in (4.8) above). Thus aside from the above noted problem of resolving the metric ambiguity in the localization formulas, there is further the problem as to whether or not a two dimensional geometry can in fact possess the required symmetry. Of course we do not expect that any two dimensional quantum mechanical Hamiltonian will have an exactly solvable path integral, and in the present point of view the cases where the path integral fails to be effectively computable will be the cases where a required symmetry of the phase space geometry does not lead to a globally well-defined metric tensor. For example, in [14] it was argued that, with the exception of the harmonic oscillator, equivariant localization fails for all one dimensional quantum mechanical Hamiltonians with static potentials which are bounded below, due to the non-existence of a single-valued metric satisfying the Killing equation (2.3) in this case.

We can alternatively formulate the quantum theory of systems with non-homogeneous phase spaces within the framework of Section 4.1 by using a formalism for coherent states associated with non-transitive group actions [19]. Let us consider the metric (3.1) in the coordinates $x'$ described in Section 3.4, which we label by $z = x' e^{i x'/1}$. Let $f(z \overline{z})$ be an analytic solution of the ordinary differential equation

$$\frac{d}{d(z \overline{z})} \frac{d}{d(z \overline{z})} \log f(z \overline{z}) = \frac{1}{2} e^{\theta(z \overline{z})}. \tag{4.9}$$

The symplectic 2-form can be chosen to be the volume form of $(\Gamma^2, g)$,

$$\omega = i \frac{d}{d(z \overline{z})} \frac{d}{d(z \overline{z})} \log f(z \overline{z}) dz \wedge d \overline{z}, \tag{4.10}$$

so that the canonical 1-form is

$$\theta = \frac{i}{2} \frac{d}{d(z \overline{z})} \log f(z \overline{z}) (\overline{z} dz - z d \overline{z}). \tag{4.11}$$
Let $N_\varphi$, $0 < N_\varphi \leq \infty$, be the integer such that the function $f(z\bar{z})$ has the Taylor expansion

$$f(z\bar{z}) = \sum_{n=0}^{N_\varphi} (z\bar{z})^n f_n,$$

and let $\rho(z\bar{z})$ be an integrable function whose moments are

$$\int_0^P d(z\bar{z}) (z\bar{z})^n \rho(z\bar{z}) = \frac{1}{f_n} , \quad 0 \leq n \leq N_\varphi,$$

where $P$ is a real number with $0 < P \leq \infty$. Let $a^\dagger$ and $a$ be bosonic creation and annihilation operators on some representation space of the isometry group, and let $|n>$ be the complete system of eigenstates of the corresponding number operator, $a^\dagger a|n> = n|n>$. The desired coherent states are defined as

$$|z> = \sum_{n=0}^{N_\varphi} \sqrt{f_n} z^n |n> . \quad (4.12)$$

These states have the normalization $(z|z) = f(z\bar{z})$ and satisfy the completeness relation

$$\int_{\Gamma^2} d\mu(z,\bar{z}) \frac{|z>\langle z|}{(z|z)} = 1 \quad (4.13)$$

in the U(1)-invariant measure

$$d\mu(z,\bar{z}) = \frac{i}{2\pi} f(z\bar{z}) \rho(z\bar{z}) \Theta(P - z\bar{z}) dz \wedge d\bar{z}.$$

Notice that for $f(z\bar{z}) = e^{z\bar{z}}$, $(1 + z\bar{z})^{2j}$ and $(1 - z\bar{z})^{-2k}$, (4.12) reduces to, respectively, the Heisenberg-Weyl group, spin-$j$ SU(2) and level-$k$ SU(1,1) coherent states in Section 4.1 above, as anticipated from (4.9). The isometry group acts on the states (4.12) as $V_\tau|z> = |e^{i\tau}z>$; $V_\tau \in I(\Gamma^2, g)$, $\tau \in \mathbb{R}^1$, and the metric tensor (3.1) and symplectic 1-form (4.11) can be expressed in the standard coherent state forms [15]

$$g = 4 \left[ \frac{||d|z||}{\sqrt{(z|z)}} \otimes \frac{||d|z||}{\sqrt{(z|z)}} - \frac{(z|d|z)}{(z|z)} \otimes \frac{(z|d|z)}{(z|z)} \right]$$

$$\theta = \frac{i(z|d|z)}{(z|z)} .$$

As before, we consider the matrix elements $H(z,\bar{z}) = (z|H|z)/(z|z)$ of some operator $\mathcal{H}$ on the underlying Hilbert space, and then using (4.11) and (4.13) the standard coherent
state path integral may be constructed as [15]

\[ Z(T|a_0; f, \mathcal{F}(H)) = \int_{\mathcal{L}^2} \prod_t d\mu(z(t), \overline{z}(t)) \]

\[ \times \exp \left[ i \int_0^T dt \left( \frac{1}{2} \frac{d}{dz} \log f(z \overline{z}) \left( \dot{z} \overline{z} - \overline{z} \dot{z} \right) - \mathcal{F}(H) \right) \right]. \]

(4.14)

The observable \( H(z, \overline{z}) \) appearing in (4.14) can now be found as before from the equivariant localization constraints and is given in terms of the phase space metric as

\[ H(z, \overline{z}) = a_0 z \overline{z} \frac{d}{dz} \log f(z \overline{z}) + C_0 = a_0 \frac{\langle z|a^\dagger a|z \rangle}{\langle z|z \rangle} + C_0 \]

(4.15)

where the function \( f(z \overline{z}) \) is related to the metric (3.1) by equation (4.9). For the maximally symmetric cases of Section 4.1 above (4.15) reduces to the observables \( H \) given there (as does the coherent state path integral (4.14) for these cases) \(^4\). Thus the formula (4.15) is a general formula valid for any geometry on the underlying phase space, be it maximally symmetric or otherwise. This is not surprising, since as remarked in Section 4.1 the Hamiltonian functions obtained for the case of maximal symmetry are just displaced harmonic oscillators, and the oscillator Hamiltonians arise from the rotation generators of the isometry groups, i.e. translations in \( \text{arg}(z) = x' \), and so the coordinate system used in Section 4.1 coincides with that defined in Section 3.4 and used in (4.15) (this also agrees with what one expects from integrability arguments). In fact, the expression (4.15) shows explicitly that the function \( H \) is really just a harmonic oscillator Hamiltonian on some general geometry.

The main difference in the present context between the maximally symmetric and non-homogeneous cases lies in the path integral (4.14) itself: In the former case the measure \( d\mu(z, \overline{z}) \) which must be used in the Feynman measure in (4.14) coincides with the volume form (4.10), because if the isometry group acts transitively on \((\mathcal{L}^2, g)\) then there is a unique left-invariant measure (i.e. a unique solution to equation (3.8)) and so \( d\mu = \omega \) yields the standard Liouville measure on the loop space. In the latter case \( d\mu \neq \omega \) in general (i.e. (4.14) is not necessarily in the form (1.4), but the standard localization formulas still hold with the obvious replacements corresponding to the change of integration measure). Of course the standard Liouville path integral measure can be used if instead one follows the prescription of Section 3.4. It is essentially the non-uniqueness of a U(1)-invariant symplectic 2-form in the case of non-transitive isometry group actions which leads

\(^4\) For the cases of the Heisenberg-Weyl, SU(2) and SU(1,1) group actions on \( \mathcal{L}^2 \), the weight functions are \( \rho(z \overline{z}) = e^{-z \overline{z}} (P = \infty) \), \((2j + 1)(1 + z \overline{z})^{-2(j+1)} (P = \infty)\) and \((2k - 1)(1 - z \overline{z})^{2(k-1)} (P = 1)\), respectively.
to numerous possibilities for the Hamiltonians on such geometries. If one consistently makes the “natural” choice of the volume form (4.10), then indeed the only admissible Hamiltonian functions $H$ are generalized harmonic oscillators.

5. Summary and Discussion

In this Paper we have discussed what Hamiltonian systems with two dimensional simply connected phase spaces can have equivariantly localized Feynman path integrals. We derived general formulas for the Hamiltonian functions in terms of the underlying phase space geometry, which shows how equivariant localization of path integrals is explicitly metric dependent and provides a picture of how the quantum geometry affects the exact solvability of some two dimensional quantum mechanical systems. This is interesting in that the underlying quantum theory is always ab initio metric-independent so that an analysis such as the one presented here probes into how the classical phase space geometry is modified by quantum effects and the role geometry plays towards the understanding of quantum integrability. For instance, we showed that the classical trajectories of a harmonic oscillator must be embedded, at the quantum level, into a rotationally invariant geometry. For more complicated systems the quantum geometries are less familiar and endow the phase space with unusual Riemannian structures.

We showed that all Hamiltonians so obtained are essentially harmonic oscillators, with modification terms to take into account the possible non-trivial geometry (3.1) of the phase space (see (4.15)). These observables arise naturally in coherent state formalisms corresponding to the Poisson Lie group actions of the appropriate isometry groups on the phase space, and the fact that they always correspond rather trivially to harmonic oscillator Hamiltonians is equivalent to the original constraint that they generate a circle action on the phase space. Non-trivial systems (such as the one dimensional hydrogen atom for which the semi-classical approximation is unsuitable) arise only under somewhat ad-hoc restrictions on the symmetries the given geometry can possess (see (4.8)) and there seems to be no general way to obtain the set of all Hamiltonians which arise from a general non-homogeneous geometry. Although we have obtained a formal prescription in Section 3.4 which in principle allows one to obtain such systems, in practice introducing such a definite geometry into the problem is quite non-trivial. For these latter non-homogeneous cases there is also the problem that the quantum theory must be essentially known beforehand in order to resolve the ambiguities associated with the single degree of freedom of
the metric (3.1). Thus the localization formulas, although being nice formal insights into quantum integrability and its geometric nature, are only practically applicable to rather trivial systems. Our results here also impose restrictions the class of topological quantum field theories and supersymmetric models which arise from these geometric localization principles [3–5].

Of course our analysis and comments above are all made under the rather severe topological restriction $\pi_1(\Gamma^2) = 0$ of simple connectivity of the phase space. We expect that more non-trivial systems can arise for more complicated topologies, such as the 2-torus $T^2 = S^1 \times S^1$, which has fundamental group $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$. For example, it is known that the Duistermaat-Heckman formula (1.2) fails for the Hamiltonian system whose phase space is the torus and whose Hamiltonian function is the height function on $T^2$ [4].

Notice further that the analysis presented in this Paper carries through rather nicely only in two dimensions. For higher dimensional symplectic manifolds $(\Gamma^{2n}, \omega)$ ($n > 1$), the possibilities are far more numerous. First of all, the phase space metric will have more than one degree of freedom, and a general expression of the form (3.1) is not possible. Thus a complete geometric classification as above is not possible. Secondly, the isometry group $\mathcal{I}(\Gamma^{2n}, g)$ can now have up to $n(2n + 1)$ generators [10], and the total number of possible dimensions of the isometry group will in general be more than 2 (the case of isometries of two dimensional Riemannian manifolds is quite unique in its properties as compared to higher dimensions [10]). There are also the additional cases where the manifold itself isn’t maximally symmetric but contains a smaller, maximally symmetric subspace. Thirdly, even for the standard maximally symmetric spaces in $2n$ dimensions, whose Killing vectors can be readily constructed [10], the Hamiltonian equations (1.3) are more difficult to solve because the symplectic 2-form is no longer a top-form on the manifold. We therefore expect that our analysis here is rather difficult to generalize to arbitrary dimensional symplectic manifolds. Nonetheless, the two dimensional analysis presented here gives an indication of the range of applicability of the localization formulas in general.

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