HADAMARD MATRICES FROM BASE SEQUENCES: 
AN EXAMPLE

DRAGOMIR Ž. DOKOVIĆ

Abstract. There are several well-known methods that one can use to construct Hadamard matrices from base sequences $BS(m, n)$. In view of the recent classification of base sequences $BS(n + 1, n)$ for $n \leq 30$, it may be of interest to show on an example how prolific these methods are. For that purpose we have selected the Hadamard matrices of order 60. By using these methods and the transposition map we have constructed 1759 nonequivalent Hadamard matrices of order 60.

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1. Introduction

Recall that a Hadamard matrix of order $m$ is a $\{\pm 1\}$-matrix $H$ of size $m \times m$ such that $HH^T = mI_m$, where $T$ denotes the transpose and $I_m$ the identity matrix. Let us denote by $H(m)$ the set of Hadamard matrices of order $m$. By abuse of language, we say that $H(m)$ exist if $H(m) \neq \emptyset$. If $m > 2$ and $H(m)$ exist, then $m$ is divisible by 4. Two Hadamard matrices $A, B \in H(m)$ are said to be equivalent if $B = P AQ$ for some signed permutation matrices $P$ and $Q$.

By binary respectively ternary sequence we mean a sequence $A = a_1, a_2, \ldots, a_m$ whose terms belong to $\{\pm 1\}$ respectively $\{0, \pm 1\}$. To such a sequence we associate the polynomial $A(z) = a_1 + a_2 z + \cdots + a_m z^{m-1}$. We refer to the Laurent polynomial $N(A) = A(z)A(z^{-1})$ as the norm of $A$. Base sequences $(A; B; C; D)$ are quadruples of binary sequences, with $A$ and $B$ of length $m$ and $C$ and $D$ of length $n$, and such that $N(A) + N(B) + N(C) + N(D) = 2(m + n)$. We denote the set of such base sequences by $BS(m, n)$. We shall demonstrate that the base sequences and, their special cases, normal and near-normal sequences play an important role in the construction of Hadamard matrices [2] [12]. The recent discovery of a Hadamard matrix of order 428 [8] used a $BS(71, 36)$, constructed specially for that purpose.

As explained in [3], we can view the normal sequences $NS(n)$ and near-normal sequences $NN(n)$ as subsets of $BS(n + 1, n)$. For normal...
sequences $2n$ must be a sum of three squares, and for near-normal sequences $n$ must be even or 1. The base sequences $(A; B; C; D) \in BS(n+1, n)$ are normal respectively near-normal if $b_i = a_i$ respectively $b_i = (-1)^{i-1}a_i$ for all $i \leq n$.

There are several known methods that one can use to construct Hadamard matrices from base sequences $BS(m, n)$. In view of the recent classification [6] of base sequences $BS(n+1, n)$ for $n \leq 30$, it may be of interest to show on an example how prolific these methods are. For that purpose we have selected the Hadamard matrices of order 60. By using these methods we have constructed 1086 Hadamard matrices of order 60. Exactly 1012 of them are pairwise nonequivalent. By taking transposes of these 1012 matrices, we obtain additional 747 equivalence classes. Thus in total we have constructed 1759 equivalence classes of $H(60)$ by using base sequences and the transposition. This is probably a very tiny portion of the totality of equivalence classes of $H(60)$. In that regard we mention that the incomplete classification, carried out very recently in [9], shows that the number of classes of $H(32)$ is well over 13 millions.

In Section 3 we describe the construction of $H(8n+4)$ from $BS(n+1, n)$ and we summarize the results of our computation in Table 1. The 558 pairwise nonequivalent Hadamard matrices constructed in this section are listed in Table 2.

Yang’s paper [13] contains four powerful “multiplication theorems”. The proofs of these theorems in loc. cit. are based on Yang’s generalization of Lagrange’s theorem on the sum of four squares to the ring of Laurent polynomials $\mathbb{Z}[x, x^{-1}]$ with integer coefficients. This beautiful generalization deserves to be wider known. In a recent paper with K. Zhao [7] we have shown that Yang’s generalization is essentially unique.

As 15 is a composite number, each of the four multiplication theorems can be used to construct base sequences $BS(15, 15)$, and then construct Hadamard matrices of order 60. The results of these computations are described in Sections 4-7. For the convenience of the reader, in each of these four sections we state explicitly the multiplication theorem that we use. Three misprints in the statement of Yang theorems have been observed in [5]. The number of equivalence classes of $H(60)$ constructed in these four sections are 192, 208, 64 and 64 respectively. Their representative matrices are listed in the Appendix in Tables 3-6 respectively.

In Section 8 we describe the encoding of base sequences $BS(n+1, n)$ that we use in Table 1, and have used in several of our previous papers. The appendix contains the Tables 2-6. We also explain there how to interpret the entries of the tables.
2. Preliminaries

All computations were carried out in Magma [1] modulo the tables of base sequences constructed in [3]. In fact we only make use of Table 2 of that paper. The main reason for using Magma is that it provides a (small) database of Hadamard matrices, in particular 256 matrices of order 60, and a useful collection of functions for working with these matrices. The 1759 classes mentioned above are all different from these 256. The most valuable functions for us were “IsHadamard” for testing whether a matrix is a Hadamard matrix, “HadamardCanonicalForm” which provides a test for equivalence of Hadamard matrices, and “HadamardMatrixToInteger” and its inverse “HadamardMatrixFromInteger” for compact representation of Hadamard matrices.

All Hadamard matrices in this note are constructed by using the Goethals–Seidel array:

$$
\begin{bmatrix}
Z_0 & Z_1 R & Z_2 R & Z_3 R \\
-Z_1 R & Z_0 & -RZ_3 & RZ_2 \\
-Z_2 R & RZ_3 & Z_0 & -RZ_1 \\
-Z_3 R & -RZ_2 & RZ_1 & Z_0 \\
\end{bmatrix}
$$

where $Z_0, Z_1, Z_2, Z_3$ are suitable circulant matrices, and $R$ denotes the matrix having ones on the back-diagonal and all other entries zero.

Let us make a remark about this array. It is not hard to verify that if we permute the circulants $Z_0, Z_1, Z_2, Z_3$ then the new Hadamard matrix obtained from the above array will be equivalent to the original one provided that the permutation is even. If it is odd then the two Hadamard matrices may be nonequivalent. This is used in Section 6.

We now list additional notation and definitions that we need.

We separate the sequences by a semicolon, and use the comma as the concatenation operator. The symbol $0_s$ denotes the sequence of $s$ zeros. For a sequence $A = a_1, a_2, \ldots, a_m$ we denote by $A'$ the reversed sequence, i.e., $A' = a_m, a_{m-1}, \ldots, a_1$. Thus we have $a_k' = a_{m+1-k}$ for $k = 1, 2, \ldots, m$. If $f \in \{+1, -1\}$ then $fA$ is the sequence $fa_1, fa_2, \ldots, fa_m$. For sequences $A$ and $B$ of length $n$, $A \pm B$ denotes the sequence with terms $a_i \pm b_i$, $i = 1, 2, \ldots, n$. For sequences $A = a_1, a_2, \ldots, a_{m+1}$ and $C = c_1, c_2, \ldots, c_m$ we denote by $A/C$ the interlaced sequence

$$A/C = a_1, c_1, a_2, c_2, \ldots, a_{m}, c_{m}, a_{m+1}.$$

We say that two ternary sequences $G$ and $H$ of length $n$ are disjoint if at most one of $g_i$ and $h_i$ is nonzero for each index $i$. We recall that $T$-sequences are quadruples $(A; B; C; D)$ of pairwise disjoint ternary sequences of length $n$ such that $N(A) + N(B) + N(C) + N(D) = n$. We denote by $TS(n)$ the set of $T$-sequences of length $n$. It has been
conjectured that $TS(n) \neq \emptyset$ for all integers $n \geq 1$, and it is known that this is true for $n \leq 100$ different from 79 and 97. There is a map

$$\tag{2.1} TS(n) \rightarrow BS(n, n)$$

sending $(A; B; C; D) \rightarrow (Q, R, S, T)$ where

- $Q = A + B + C + D$;
- $R = A + B - C - D$;
- $S = A - B + C - D$;
- $T = A - B - C + D$.

3. Construction of $H(60)$ from $BS(8, 7)$

For any finite binary sequence $X$ let $Z_X$ denote the circulant matrix having $X$ as its first row. It is well known (see e.g. [12]) that there is a map

$$\tag{3.1} BS(d, d) \rightarrow H(4d).$$

sending $(M; U; V; W)$ to the Hadamard matrix $H$ constructed by plugging in the circulants $Z_M, Z_U, Z_V, Z_W$ for $Z_0, Z_1, Z_2, Z_3$ in the Goethals–Seidel array.

One can also use base sequences $BS(m, n)$ with $m$ and $n$ arbitrary to construct Hadamard matrices of order $4(m + n)$. For that purpose we just compose the above map with the map

$$BS(m, n) \rightarrow BS(m + n, m + n)$$

which sends $(A; B; C; D) \rightarrow (A, C; A, -C; B, D; B, -D)$. In particular, for $m = n + 1$, we obtain the map

$$\tag{3.2} BS(n + 1, n) \rightarrow H(8n + 4).$$

In our recent paper [6] we have introduced a new equivalence relation in $BS(n + 1, n)$ which the reader should consult for further details. As this relation is not easy to describe, we shall just say that there is a group $G_{BS}$ of order $2^{12}$, whose definition depends on the parity of $n$, which acts on $BS(n+1, n)$ so that its orbits are exactly the equivalence classes. By using this group, it is easy to generate in Magma the whole equivalence class from its representative. We point out that the map (3.2) may produce many nonequivalent Hadamard matrices from a single equivalence class of base sequences (see Table 1).

Let $E \subseteq BS(n + 1, n)$ be an equivalence class. From above it follows that the cardinality of $E$ must be a power of two, $2^k$ with $k \leq 12$. We are interested in the case $n = 7$ in which case $2n + 1 = 15$ and $8n + 4 = 60$. The set $BS(8, 7)$ splits into 17 equivalence classes.
The results of our computation in this case are summarized in Table 1. The listing of the equivalence classes of $BS(8,7)$ is taken from Table 2. The second column of Table 1 lists the representatives $(A; B; C; D)$ of the equivalence classes $E$ of $BS(8,7)$. These representatives are written in encoded form (see Section 8 for the definition and our conventions for this encoding). For each representative we record the cardinality $\#E$ of $E$ and the number of equivalence classes of $H(60)$ constructed from $E$ via (3.2).

It turns out that any two nonequivalent base sequences in $BS(8,7)$ produce two nonequivalent Hadamard matrices. We do not know whether this is true in general.

**Question**: Is it true that two nonequivalent base sequences in $BS(n+1, n)$ always produce via (3.2) two nonequivalent Hadamard matrices of order $8n + 4$?

The sum of the numbers in the last column of Table 1 is 558. Consequently, we have constructed 558 equivalence classes of $H(60)$.

| $AB; CD$ | $\#E$ | #Had. |
|----------|-------|-------|
| 1 0165; 6123 | 2048 | 32 |
| 2 0165; 6141 | 4096 | 64 |
| 3 0166; 6122 | 2048 | 32 |
| 4 0173; 6161 | 2048 | 32 |
| 5 0173; 6411 | 4096 | 64 |
| 6 0183; 6121 | 2048 | 32 |
| 7 0613; 1623 | 2048 | 32 |
| 8 0614; 1641 | 4096 | 64 |
| 9 0615; 1263 | 2048 | 32 |
| 10 0615; 1272 | 512 | 8 |
| 11 0616; 1262 | 2048 | 32 |
| 12 0618; 1261 | 2048 | 32 |
| 13 0635; 1621 | 1024 | 16 |
| 14 0638; 1620 | 2048 | 32 |
| 15 0641; 1622 | 2048 | 32 |
| 16 0646; 1222 | 256 | 6 |
| 17 0646; 1260 | 1024 | 16 |

4. **H(60) from Yang’s Theorem 1**

The theorem [13 Theorem 1] gives a map

$$(4.1) \quad NS(n) \times BS(s, t) \rightarrow TS(d), \quad d = (2n + 1)(s + t).$$
Normal sequences $NS(n)$ can be written as $(F, +; F, -; G + H; G - H)$ or $(F, -; F, +; G + H; G - H)$, where $F$, $G$ and $H$ are uniquely determined sequences of length $n$ such that $F$ is binary while $G$ and $H$ are ternary and disjoint. The map (4.1) is defined in terms of the sequences $F$, $G$ and $H$. This map sends the ordered pair, having these normal sequences as the first component and $(A; B; C; D) \in BS(s, t)$ as the second, to the $T$-sequences $(Q, R, S, T)$ where

\[
\begin{bmatrix}
Q \\
R \\
S \\
T
\end{bmatrix} = [X_1, X_2, \ldots, X_n, X_{n+1}]
\]

and the blocks $X_k$, $k = 1, 2, \ldots, n$ and $X_{n+1}$ are given by

\[
X_k = \begin{bmatrix} f'_k A, & g'_k C + h'_k D, & 0_{s+t} \\
- h'_k C + g'_k D, & 0_{s+t} \\
0_{s+t}, & g'_k A - h'_k B, & - f'_k C \\
0_{s+t}, & h'_k A + g_k B, & - f_k D
\end{bmatrix},
\]

\[
X_{n+1} = \begin{bmatrix} - B', & 0_t \\
A', & 0_t \\
0_s, & - D' \\
0_s, & C'
\end{bmatrix}.
\]

Two misprints in the expression for $\tau_k$ in [13, p. 770] have been corrected. Instead of our sequences $g'_k A - h'_k B$ and $h'_k A + g'_k B$, inside the block $X_k$, Yang has $g'_k A - h'_k B$ and $h'_k A + g'_k B$, respectively. At a first glance our sequences appear to be in error since these have to be ternary. In fact they are binary since Theorem 8.1 guarantees that $g_k = 0$ iff $g'_k = 0$ and $h_k = 0$ iff $h'_k = 0$. Hence, exactly one of $g'_k$ and $h_k$ is zero and the same is true for $h'_k$ and $g_k$.

There exists only one equivalence class of base sequences $BS(2, 1)$ and the same is true for $BS(3, 2)$. Their representatives are 03; 1 and 0; 0 in encoded form, or explicitly

\[
0; 0 = +, +; +, -; +; + \\
03; 1 = +, -, +; +, -; +, +; +, +.
\]

These are also representatives of the equivalence classes of normal sequences $NS(1)$ and $NS(2)$ respectively.

By applying the above theorem, with $s = 3$, $t = 2$ and $n = 1$, we compute the image of the whole set $NS(1) \times BS(3, 2)$ and then apply the map (5.2) to this image. We obtain 128 equivalence classes of $H(60)$. Another 64 equivalence classes are obtained by taking $s = 2$, $t = 1$ and $n = 2$, i.e., by using the set $NS(2) \times BS(2, 1)$. These $128 + 64 = 192$ equivalence classes turn out to be distinct.
5. **H(60) from Yang’s Theorem 2**

The theorem [13, Theorem 2] gives a map

\[(5.1) \quad NS(n) \times BS(s, t) \rightarrow BS(d, d), \quad d = n(s + t).\]

We write normal sequences \(NS(n)\) as in the previous section, i.e., as \((F, +; F, -; G + H; G - H)\) or \((F, -; F, +; G + H; G - H)\). The map \((5.1)\) sends the ordered pair consisting of these normal sequences and the base sequences \((A; B; C; D) \in BS(s, t)\) to the base sequences \((Q, R, S, T)\) defined by

\[
\begin{bmatrix}
Q \\
R \\
S \\
T
\end{bmatrix} = [X_1, X_2, \ldots, X_n], \quad X_k = \begin{bmatrix}
f_k A, & g_k C + h_k D \\
f_k B, & -h_k C + g_k D \\
g_k A - h_k B, & -f_k C \\
h_k A + g_k B, & -f_k D
\end{bmatrix}.
\]

There are two possibilities. First we take \(s = 3, t = 2\) and \(n = 3\). There exists only one equivalence class of normal sequences \(NS(3)\). As its representative we can take

\[
06; 11 = +, +, -; +, +, -; +, +, +; +, -; +, +, +.
\]

By computing the image of \(NS(3) \times BS(3, 2)\) under the map \((5.1)\) and applying the map \((3.2)\), we obtain 80 equivalence classes of \(H(60)\).

The second possibility is to take \(s = 2, t = 1\) and \(n = 5\). Again there exists only one equivalence class of normal sequences \(NS(5)\). As its representative we can take 016; 640, i.e.,

\[
+, +, +, +, +; +, +, +, +, -; +, +, +, +, -; +, +, +, -; +, +, +, +, -.
\]

In this case we obtain 128 equivalence classes of \(H(60)\).

These 80 + 128 = 208 equivalence classes turn out to be distinct.

6. **H(60) from Yang’s Theorem 3**

The theorem [13, Theorem 3] gives a map

\[(6.1) \quad NN(n) \times BS(s, t) \rightarrow TS(d), \quad d = (2n + 1)(s + t),\]

where \(n = 2m\) is even. To describe this map, we shall write near-normal sequences in \(NN(n)\) in the form

\[((Y, +)/X; (Y, -)/(-X); G + H; G - H),\]

where \(X\) and \(Y\) are binary sequences and \(G\) and \(H\) are disjoint ternary sequences, all of length \(n\). This map sends the ordered pair, having
these near-normal sequences as the first component and \((A; B; C; D) \in BS(s, t)\) as the second, to the \(T\)-sequences \((Q; R; S; T)\) where
\[
\begin{bmatrix}
Q \\
R \\
S \\
T
\end{bmatrix} = \begin{bmatrix}
U_1, U_2, \ldots, U_m, U_{m+1} \\
V_{m+1}, V_m, \ldots, V_1
\end{bmatrix},
\]
the blocks \(U_k\) and \(V_k\), \(k \leq m\), are given by
\[
U_k = \begin{bmatrix}
g'_{2k-1}A - h_{2k-1}B, & -y_kC, & g'_{2k}A - h_{2k}B, & -x_kD' \\
\end{bmatrix},
\]
\[
V_k = \begin{bmatrix}
x_kB, & g_{2k}C' + h_{2k}D', & y_k'A', & g_{2k-1}C' + h_{2k-1}D' \\
x_kA, & g'_{2k}D' - h'_{2k}C', & y_k'B', & g'_{2k-1}D' - h'_{2k-1}C'
\end{bmatrix},
\]
and
\[
U_{m+1} = \begin{bmatrix}
0_s, & -D', & 0_{n(s+t)} \\
0_s, & C', & 0_{n(s+t)}
\end{bmatrix}, \quad V_{m+1} = \begin{bmatrix}
0_{n(s+t)}, & -B, & 0_t \\
0_{n(s+t)}, & A, & 0_t
\end{bmatrix}.
\]
There exists only one equivalence class in \(NN(2)\). As its representative we can take
\[02; 1 = +, -, +; +, +, -; +, +, +; +, +, +, +, +, +, +, +.
\]
After computing the image of the map (6.1) with \(s = 2\), \(t = 1\) and \(n = 2\) and applying the maps (2.1) and (3.2) in succession, we obtain 32 equivalence classes of \(H(60)\). Another 32 equivalence classes are obtained by swapping the first two components of the base sequences produced by the map (2.1) (see the remark made in Section 2). These 32 + 32 = 64 equivalence classes turn out to be distinct.

7. \(H(60)\) from Yang’s Theorem 4

The theorem [13, Theorem 4] gives a map
\[
(7.1) \quad BS(m+1, m) \times BS(n+1, n) \to BS(d, d), \quad d = (2m+1)(2n+1),
\]
which sends the ordered pair \(((A; B; C; D), (F; G; H; E))\) to \((Q; R; S; T)\) defined again by the formula (1.2) but the blocks \(X_k\), \(k \leq n\), and \(X_{n+1}\) are now given by
\[
X_k = \begin{bmatrix}
f'_{k}A/g'_{k}C, & (-e_kB')/h_kD \\
f'_{k}B/g'_{k}D, & e_kA'/(-h_kC) \\
g'_{k}A/(-f_kC), & (-h_kB)/(-e_kD') \\
g_kB/(-f_kD), & h_kA/e_kC'
\end{bmatrix}, \quad X_{n+1} = \begin{bmatrix}
f_{1}A/g_{1}C \\
f_{1}B/g_{1}D \\
g_{1}A/(-f_{1}'C) \\
g_{1}'B/(-f_{1}'D)
\end{bmatrix}
\]
(a misprint in the expression for \(\beta_k\) in [13, p. 773] has been corrected).

We apply this theorem, with \(m = 1\) and \(n = 2\). We compute the image of \(BS(2, 1) \times BS(3, 2)\) and then apply the map (3.2). We obtain
32 equivalence classes of $H(60)$. Another 32 equivalence classes are obtained by using the set $BS(2,1) \times BS(3,2)$. These $32 + 32 = 64$ equivalence classes turn out to be distinct.

8. The encoding scheme

Let $(A;B;C;D) \in BS(n+1,n)$. For $n$ even (odd) set $n = 2m$ ($n = 2m+1$). Decompose $(A;B)$ into quads

$$\begin{bmatrix} a_i & a_{n+2-i} \\ b_i & b_{n+2-i} \end{bmatrix}, \quad i = 1, 2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor,$$

and, if $n$ is even, the central column $\begin{bmatrix} a_{m+1} \\ b_{m+1} \end{bmatrix}$. Similar decomposition is valid for $(C;D)$. The quad encoding is based on [11, Theorem 1].

**Theorem 8.1.** The sum of the four quad entries is $2 \pmod{4}$ for the first quad of $(A;B)$ and is $0 \pmod{4}$ for all other quads of $(A;B)$ and for all quads of $(C;D)$.

There are 8 possibilities for the first quad of $(A;B)$:

$$1' = \begin{bmatrix} - & + \\ + & + \end{bmatrix}, \quad 2' = \begin{bmatrix} + & - \\ + & + \end{bmatrix}, \quad 3' = \begin{bmatrix} + & + \\ + & - \end{bmatrix}, \quad 4' = \begin{bmatrix} + & + \\ - & + \end{bmatrix},$$

$$5' = \begin{bmatrix} + & - \\ - & - \end{bmatrix}, \quad 6' = \begin{bmatrix} - & + \\ - & - \end{bmatrix}, \quad 7' = \begin{bmatrix} - & - \\ - & + \end{bmatrix}, \quad 8' = \begin{bmatrix} - & - \\ + & - \end{bmatrix}.$$

The possibilities for the remaining quads of $(A;B)$ and the quads of $(C;D)$ are:

$$1 = \begin{bmatrix} + & + \\ + & + \end{bmatrix}, \quad 2 = \begin{bmatrix} + & + \\ - & - \end{bmatrix}, \quad 3 = \begin{bmatrix} + & + \\ - & + \end{bmatrix}, \quad 4 = \begin{bmatrix} + & + \\ - & + \end{bmatrix},$$

$$5 = \begin{bmatrix} + & - \\ + & - \end{bmatrix}, \quad 6 = \begin{bmatrix} + & - \\ + & - \end{bmatrix}, \quad 7 = \begin{bmatrix} - & - \\ + & + \end{bmatrix}, \quad 8 = \begin{bmatrix} - & - \\ - & - \end{bmatrix}.$$ 

Finally, there are 4 possibilities for the central column:

$$0 = \begin{bmatrix} + \\ + \end{bmatrix}, \quad 1 = \begin{bmatrix} + \\ - \end{bmatrix}, \quad 2 = \begin{bmatrix} - \\ + \end{bmatrix}, \quad 3 = \begin{bmatrix} - \\ - \end{bmatrix}.$$

We encode $(A;B)$ by the symbol sequence $p_1p_2\ldots p_mp_{m+1}$, where $p_i$ is the label of the $i$th quad except that for $n$ even $p_{m+1}$ is the label of the central column. Similarly, we encode $(C;D)$ by $q_1q_2\ldots q_m$ respectively $q_1q_2\ldots q_mq_{m+1}$ when $n$ is even respectively odd. Here $q_i$, $i \leq m$, is the label of the $i$th quad and, for $n$ odd, $q_{m+1}$ is the label of the central column.
As an example, the base sequences

\begin{align*}
A &= +, +, +, +, -, -, +, +, +; \\
B &= +, +, +, - +, +, +, +, -; \\
C &= +, +, -, - +, -, - , +; \\
D &= +, +, +, +, -, +, +, -;
\end{align*}

are encoded as $3'6142;1675$. In Table 1 and elsewhere in the text we write 0 instead of $3'$.

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9. Appendix

The following tables contain the list of Hadamard matrices constructed by the methods explained in Sections 3-7. Since these matrices are constructed by using the Goethals–Seidel array, i.e., via the map \([32]\), they can be stored very efficiently. Indeed it suffices to list only the base sequences \((A; B; C; D) \in BS(15, 15)\) used to construct
the Hadamard matrix. We first concatenate these four constituent binary sequences to obtain the binary sequence $A, B, C, D$ of length 60. Next we replace each $-1$ term in this sequence with 0 to obtain a $\{0,1\}$-sequence, say $S$, of length 60. Next we split $S$ into 15 pieces of length 4 each. Finally, we interpret each piece as the binary representation of a hexadecimal digit $0, 1, \ldots, 9, a, b, c, d, e, f$. Thus we obtain a sequence, $X$, of exactly 15 hexadecimal digits. Each entry in our tables is such a sequence. Clearly, one can easily reconstruct the base sequences $(A; B; C; D)$ from $X$.

Here is an example. We take $X = 0dc41a77adbf5e8$, the first hexadecimal sequence in Table 2. Each hexadecimal digit has to be replaced by its binary representation by using exactly four binary digits. Thus the hexadecimal digit 0 gets replaced by the sequence $0, 0, 0, 0$. Next the hexadecimal digit “d” is replaced by the sequence $1, 1, 0, 1$ etc. We thus obtain the sequence $S$ as

$$00011011100100001101001111110101111011011001000.$$  

Finally, we replace each 0 with $-1$ and read off the four binary sequences:

$$A = -,-,-,-,+,-,+,-,+,-,-,-,$$
$$B = -,-,-,-,+,-,+,-,+,-,+,-;$$
$$C = +,+,+,-,+,-,+,-,+,-,+,-;$$
$$D = +,+,+,-,+,-,+,-,+,-,-,-.$$  

To obtain the Hadamard matrix $H \in H(60)$, it remains to form the circulants $Z_A, Z_B, Z_C, Z_D$ and plug them (in that order) into the Goethals–Seidel array.

The Hadamard matrices within each table are pairwise nonequivalent. The same is true for the list of all Hadamard matrices made up from Tables 3, 5 and 6. However there is an overlap (32 matrices in total) between these tables and Table 4. Namely, 24 matrices in Table 4 are equivalent to some matrices in Table 3 and another 8 matrices in Table 4 are equivalent to some matrices in Table 5. Similarly, there is an overlap (42 matrices in total) between Table 2 and the other four tables. Thus these five tables together contain representatives of 1012 equivalence classes of $H(60)$.  


Table 2: \( H(60) \) from \( BS(8, 7) \)

|      |     |      |     |     |     |     |
|------|-----|------|-----|-----|-----|-----|
| 0dc41a77adbf5c8 | a73bf489fb643e57 | f2de4275f16b9d | b03bf618bace4f56 |
| e547cb71f44b6e6 | d7edae2434e86e2 | d7ceae753e427b7 | ebc5d676c245837 |
| 411283db96e72a3 | a74f7f105b0a0c8 | be6f7d2397172d | 7db8f8a897b7289 |
| eb3bd7893ca7f6 | 82ed04269e3ed2e | b03bf618bace4f57 | 0d6c1b235ebee2 |
| ebedd6253d1a7dc | e56fcb2106e2a3 | 826f0523971729d | b09160d5ee6ba3 |
| d7c9ae77ca47937 | 824f705396472b7 | 413a88a9d6bd3c8 | ebedd6253f1279d |
| f23be58a9f75b7 | 7edaa2436e06a3 | e5c5ca75f56bce8 | 7db8f8a89f5b389 |
| e5edca2506e2a0a | b0ed0275d1ebedc | 7d3a8b897b7289 | e591cad06e2a03 |
| a7984ed7f93e8d | 4176a313967e2a7 | a7994edf7d3e85 | ebd1b77ca08d586 |
| a7cd4e66f82df7a | eb0bd77eae0d866 | 416633396872af | e7d4de4686d72 |
| be4f7d639d5d2c4 | be5f7d43959ef2cc | e5f4cb610772091 | e55cf4d10732099 |
| f2335e9b5fd6b85 | f2235e5b5f96b8d | e5e5ca35f773e91 | 4f29fbb5f96b8d |
| e5f5ca15f733e99 | 4f329fb95d6bd85 | 82f504169f35399 | 82e04369f75391 |
| 4f6699f55eb8af | 4f769f135e6bba7 | bebe7c9e253bb | bea17cb9e653b3 |
| 82a104be9f5f399 | 82b1049e9f75391 | e55f6c410a6e6b | d723a8bac13d585 |
| d733af9ac17d850 | e5f4cb61048a0ee | d71b7af93c8a7e8 | d70baf93c8a7e6 |
| 824f0549777291 | f24e56d53edebe4 | 4f9889ef5e8baf | 4f888eef5e6ba7 |
| 82f50439737299 | f25fe5435d9ebcc | e5f5ca15f4cbee6 | e5e5ca35f48b5ee |
| b05f614359d9ebcc | b04f61635ddec4 | e50bcbe9f636eb3 | e51bcb9f623ebb |
| a75f4f2099c14c | a74f4f6209d144 | e53d9283d0685 | eb23db8379068d |
| ebad16bfc8cf966 | ebb1d69f8c8f99e | 0d4e1b63addf5c4 | 0da01ab5c8cebe6 |
| be57c169d9d3cc | bee57c369d3d3c4 | d777af13ace7927 | f25f6435e66b3 |
| eb67d73037d0685 | eb77d710379068d | be7771d96c72a7 | d7b1a9c935d84c |
| be677d396972a7 | d7a1aebc35968c6 | d7dace435b278d | d7dca863f2785 |
| 5946b3729ff25301 | 4f569ff33fbb678 | a77f4f0194eb2e2 | f313e77ac15d854 |
| 59c4b27796072bf | cf6f20253586e4 | e5b9ca8e6a0d43 | cfe9d25f3257f |
| a7034f9651aadcc | a74f7479f5203 | ca99f9ac3606b7 | 597eb3096e72a3 |
| cf59f53cb79b909 | b0d5065cd7b9c8 | a6b94c8f94f2fe | 4f569f53c8f9f6 |
| e57fcb01963e32a | 0d6e1b233cae7e | b0261a36d6b2e7c8 | 4f4e9273e07ea |
| f2ed2473ea67ab | cf595e6c1b8d48 | ca99f9ac3586b8 | a7b94e865faacc0 |
| f3916d34a86ea | b09160df356795 | a6814cf9f1729d | e53bcb9a9b5301 |
| a77f4f0298ed365 | f547bb27abe4d3 | e5a47c72bba63 | a63b4d89e053bf |
| d3b1a6e920e47a | d3a1a6ebe06c47 | bbe57636782c7fa | bbf57616786c72 |
| bbb176e197c350 | 61be0c29ae85dabf | bba176bec935e8 | 61a02b00ee5d5a7 |
| dda1babd879308d | ddb1ba9d87d3085 | 9e4f3d637686avf | 9e5f3d43766ca7 |
| dde5ba358683a0f | ddf5ba1586c30a7 | 11a022bf49698d | 11b0229af69685 |
| 88a110bf26c5e5f | 88b1109f2e20e5f | 88a110bf3d69d | 88a1109f2d3e9d |
| cba196bc46888f2 | cbb1969c44288fa | cbb1969c47d0885 | cba196bc479088d |
| d30ba7e9df93b8d | d3b1a7ed9df3b85 | ee4fdd632fd6585 | ee5fdd4329658d |
| Table 2 (continued) |
|---------------------|
| dda1babd853b0d8    | ddb1ba0d575b0d0 | ddf5ba15e63ca7 | dde5ba35e683caf |
| dd17bb2b190c35e    | dd07bf2194c356 | cb079f045a88ca | cb196f3fbb7729  |
| cb1797d045e88c2     | 887b110b45e9f4 | cdb1965fba7f72 | cb859f6f20b468  |
| 886b112bf4e9f      | cb9596d20fc460 | dd3fbb21ab432 | dd2fbb21af4321 |
| bb6b772a1b0431f    | bb7b770a1b44317 | bbe9762e79efc42 | 9e533d5b7f4e0ef |
| bbf9760e79af4a     | dd7bbba7b44f17 | 9e433d7b74beee8 | bb3f772974c5f6  |
| dd6bb2a7b04f1f     | bb7b770a7b44f17 | bb2f77a2790cf5e | bb6b7729e703cf9 |
| bb7b7709743c97     | 881711d34dee9c2 | bb6b772ab043f1f | 880711f34daec9a |
| 88dd01872b9f5e8    | 88ad10a726f5e90 | 77bcee872b6e5e8 | 77aceea7f0e5e0  |
| 776ae62b2c5f5c     | 777ae0b2c5e5f4 | 778ee2f0e679a1 | 778ee0f0e679a9  |
| eed1dc5f4d6ee9c2   | 116a232b4c8e90 | cfb9960fbb7729 | eec1dc7f4dae9c9a|
| 117a230bf4ce9e8    | cbe9962fba7f72 | 6116c3d2ee5f5a1 | 6106c3f2eb5da9  |
| 61e8c22f75eeec2    | 111623d32d6e5de | 9ead3c7aa7feef0 | 61f8c20f75aeeca |
| 110623f2d4ec56d    | dd07bfb2794cf56 | 9ebd3c7aa7feef0 | dd17bfb2790c5e  |
| dd07bb21ab4329     | cb7b970847a08b8 | dd17bb21af4321 | cb6b972847e0883 |
| d385a6ff7b85f774   | d395a65d7b81f77c | dd6bb2a87f0f60 | dd7bbba78e8f68  |
| 9e2f3da2ed6ed2c    | 9e3f3d82eaddca | ee3f3d843d4e9d | ee2f3d3a6d9e0de |
| 61d4c57f64d7eb7    | caf94074f0e9e2 | 9e3133daa7f54f95 | ace558774e069bf |
| d347a772b3f5601    | ca814f4f0f69f9d | 53b8a8f4dfe8e0 | ca779770d601abf |
| ca479753f469e98    | 9fd53e5e55fbcc8 | 9f153f3da75f15 | 534a7e67740e9f  |
| 9f573f1e643cb7     | 537ea07032e5e2 | 8f397f5ab0ed662 | d3b9a86f0e29bf  |
| 69d53e5585bb0c8    | f92bb3a987b3089 | cb7f900d711a9d | 9f2f3a8a1bb4039 |
| 9e2f3db3dababece  | 792fa3abeb5f6c8 | 86130d5b7579c5 | cbdf9606b2e5623 |
| ac475973c46e9f9    | 79e43e2586ab30e | ac035f6b4d16d9 | cb996f8208b57e  |
| 9f573f527a4f437    | 792fa3aa7e44f5f | 79e25f7e44f6e | 61ac3abeb7f7c89 |
| ed87daf0dd41bd7    | 7beb6e28f7be1e | 133c2786edab8a | ed15d7b1e7643  |
| 47688f2ee415fdl    | 9d33a58778e68e2 | 9d053fb6e5b0c6 | ec51d95bf476e1 |
| b88bf072847b4ee9   | b99772d2e33e0c2 | b92d3a78a1e0120 | ed79db0ec8b9a8 |
| 9dfb3a091da23cb    | b92d3a6e105c5f | 1dd323a576fca0 | c8eb902abdf09e |
| b7796ff214657      | 131427fb616fbc | edadaa3b2b163 | b786ebf0e8b9a8 |
| b7516f5e19b0c2c    | e2f3bc4a765ce4b | b7516f5c0f9b9e | edadaa3b0f6761 |
| b9fb79ae0b5c9b     | b94173f89a174b | b9bf7238b48f116 | 7e2ca8676fca0  |
| b7516f5e4f0899e    | ed3ca7a8de9b9a8 | 47048f6e4b5e0e9 | b8fb700ae5a65c |
| 13b6626b2c357f1    | 13a626b2c357f1 | b94f721872b9f7 | b9e37238b76292 |
| af4953f6b0f767e1   | af59f4f0b37679 | 9d8f3ae2836d012 | a05941e0558e0e |
| a049146e55c0c7     | f3cbea68b5e89b | 9d9f3ae2832d01a | 1360273ebcd57e |
| f5bea48e389b8     | 1370271ec957ed | a0f3401ec69cac | a0e3403ae9dd1c |
| 05a60ab2e435cf9    | 05b60a92e475cf1 | faf3f1a7564ed3 | fac3f40a75624ed |
| c8b79093d63ab8     | c8a790b3d67f4b | a0db404a7474ef1 | a0cb406a7434ef9 |
Table 2 (continued)

| 37246fb6be3d7b8 | 37346f96be7d7b0 | 9de33a3a836d012 | 9df33a1a832d01a |
|------------------|-----------------|-----------------|------------------|
| 9df33a1814d02e5  | 9de33a3814902ed | 05e20a3a776ce92 | 05f20a1a772ce9a |
| 9db73a93e9c7d47  | 9da73ab9e87d4f  | 1348276bec757f1| 1358274ebc357f9 |
| 9df33ac014d02e5  | c87b909d378fa8e| c87b90b3d7cfa86| 9df33ae014902ed|
| afeb5e684fc8986 | 9d353b9417ec8286| ee8fd8e3d6d4a4 | 9d253bb4178828e |
| afeb5e484f8898e  | ec9fd8c3d69faac| 9d353b9416782b0| 9d253bb4163828b |
| 9d593b4c16782b0  | c87b90b3d437af9| 9d493b6613628b2| 9d353bb35d3059d |
| 1370271ebedd7a4  | 1360273ebe7d7ac| af1d5fe44f8999a| af0d5fe44f8999a |
| eca7d8b3d7cfa66 | ecb7d893d7f8fa8e| 3786ee2bd257db  | 3796ee2bd257d3 |
| 9d593b4c17c8286  | 9d493b6617c8286| f5b7ea904c709f1 | f5a7eab04c309f9 |
| b7b66eb433e818  | edcfd602d705d1 | 1d203be85950cd  | e2f7c41285350d9 |
| ed8bdaeb4066873 | 85f70a12e135c59 | ed19dbc2d705d1  | 0b8a6eac65bf3 |
| 2f5c5f46d5cc5be7 | c24dc56684850fe| e209c5ee16526b4 | b7b6e6302e85ba |
| d07a116d3d9b89  | f119e9cdd5bc5  | eda3dab032c0e7 | 85d0b66ed3dcd4 |
| 85090bed1e6a3b2  | ed8bdaeb8f39798 | a1b342991fda38 | 474c8f6687dd04 |
| b75d6f442cc05e7  | e2b655621d8284 | a1b942c9128530 | 1db23a9a14842ef|
| b8b3709a84250fb  | edcfd60a8858ae | 7088f6e1626c2b | 2b3c49a84250fb |
| 1d083bec16cc2a6  | ed19dbc2d2fa3a | b7756f142cc05e7 | b7d6f47d39fa0c |
| edbbda8b0d6e7a63 | e293c4da84450f7| 47ee8e2215b42c | a1f422e2b05c49 |
| 0b38178edeedba2  | b0441777b71f69c| 1de3a2285b205c9 | ed39db8cb97a2 |
| 2f9a5ebd4c7a7e | f5b3ea982cc05e7 | f54deb642cc05e7 | d04da167d73a98 |
| a0ef4b62866d0b2  | fab3f49a179c38 | af53ebd06c7a67 | 53b0b9e1666c2b |
| 85310b9fe937d59  | 2f5e6e266d87b2 | f531eb9f4136859| f519ebcf246832 |
| 503b89e85350d9  | 54f4eb6683d084| 0b4c176d73a98 | 0b1817e00d37d9 |
| do4ca063d597acd  | 2f18f5ce5d357d9| 2b5f8abf2d798 | 0b641736be57e7 |
| f5b3e9a4b53e818  | 05300b9e85350d9| 85e70a30195092cd| a01941ce8602a |
| fab3f49a146423f | 5f5b3e9a173c298| 0b3549a173c298 | a3f159fa19684d |
| a16543341738298  | a01941ce85950cd | 56f4b36173c298 | a1654337e66737 |
| 0d201bbee3686ca6 | b0756116c4888ef | b18b62e834206fb | b17563153f7a90 |
| e512bb1bc9e893ae | 1b6f36129e83d3ae| 27204f9ed95d3c5 | 27f64e192c39d5c5 |
| e5f7ca1392b22e | 8ddf1a413dd27c5 | e48bce86942722fb| 4fa29eb34246fb |
| b15d6346b37d810 | 8d8b1ae834006ef | e509bec97d13 | 8d571b153f7a90 |
| b0b86b9342d46fb | d5b1847977f290 | 1b5c374798728ef | b0a360bac45b8f |
| 1ba236ba9f7d390 | b10963ed3e8a7ae | a75df449df3984 | 8d751b17c7910 |
| a7a34eb989f7390 | 0d5c1b46377e690 | 4ff97e125746d1 | a75df449df789d0 |
| 27b0e4d29d53c5 | 4f5c9f6e4858ef | a775f4146f789d0 | e5dfca4392f2e |
| 192c33a607ca08 | 2b054766c199b9c | 2bfa560b361f6ecb | d43ca87b36fe6a0 |
| 199632d207ac08a | 6768af2e06b0a | 81150d46658cb4 | 195033e05a40cb |
| d405a9f735e66c3 | 81d302599eb3a8 | e6afcca2065c0b4 | d487a8f334166fd |
|   |   |   |   |
|---|---|---|---|
| d43da98737ee682 | 812d03a59ebb3a8 | b30567f751e6a43 | d441a97e5cde59c3 |
| 2368472e8efda10 | 898712f027a848a | f679ed0e454f85f | f6487e9f454f85f |
| 2368472e8c151fd | a30547f4ee5b9d4 | c5d38a5b12fe220 | 918722f1bedb7a8 |
| f6ebe42a470c99e | 6f78df0e454f8d7 | 6f9ede2a44f4d8e1 | c569b27f3eee02 |
| a3d3465b73eee02 | a39746d3153ee202 | 6f50df5e270c49e | dc69b2f15062df |
| 09c2127a46be8a8 | a369472f122e220 | 898712f025404d7 | 89c3127826b84a8 |
| 89af12a1bdc3e03c | 6f2d89a68051df | 23d2465a8c151fd | a341476c8f49196 |
| a3bf46808f49196 | a3d3465b13e202 | 918722f3dabf6b8 | 6f2d467a2454f5 |
| 89eb122a40f861 | a369472f0e114fd | 6f6c3e7a26be4a8 | dc6b8828f4d196 |
| 5340a77e270c49e | dc69b927e7ebc8a | 3bea762be5a7cbb | 5304af625e4e3 |
| 918722f14c129fd | 35fa6a0a24f4e4e1 | 35be6a825e48c3 | 91c32279d029df |
| ac3d598641414fd | 9f6e3e2b734ee16 | cac3947a46fc84a0 | 91bf2282b05661 |
| 9f2d3afa773ae0a | c41589d67e5cfb4 | 89bf1280d709a9e | 35866af244148fd |
| 8979130cd7e9a82 | acf85b0a470c89e | c469892fe547cd7 | 9f6e3e288d414ac3 |
| 91bf22814de29c3 | f92d3a4eeb9da8 | 9f153fd7725e34 | c451895fe65fb4 |
| f9e6f22b10b6269 | dc51b95c7eb4e9 | 9f693f2ced41dd7 | f9ebf226cb11e9 |
| 9fa3ea084f49196 | 9f693f2ced41d7 | 5378a7e44148fd | 9141237d3e1ac3 |
| 5bc667205140dd | 25ba44a0a4e40e3 | 8f111df524ea36 | f193c2d8af9b588 |
| 814503756cee2de3 | 8139038e911525d | 81a302b8979928c | 8f8f8e435196a4d |
| 81e702316d72d1 | 81a302b96f9ad8c | f14de3675860a6f | f14de36753e10a |
| 8fb3f1e98af79509 | 8131039f69d6d45 | 81750314979928c | f165e33753dea04 |
| 2bb2509c4878ef | 81e7023291d5245 | 5be6b63205740d1 | da5db546073c098 |
| d44da967c7df884 | 8f091fecd315d9 |   |   |
Table 3: $H(60)$ from Yang’s Theorem 1

| d9d7bc22eb65e8f | f165ed4722f67bd | e963dd4b21f67dd | 9443270ba1377c5 |
|-----------------|-----------------|-----------------|-----------------|
| 851105aeec3de64 | f15bed38c2e9bbe | de17b3a288652ef | f763e14b25f675d |
| 82170ba28c25267 | de2fb3d1146216f | ef9dd0bd4ae98be | 9383288bc237ba5 |
| 9d2935dd0f22007 | c2d78a22e47d6f6 | 9ad13a2ef3de04 | c1178da2eb7d8ec |
| 831109ae883d2e4 | 8f451107cd2fa46 | eb73d96a21f47dd | 9be9385d1422167 |
| ee9bd2b8c1e9bde | c128d1d777ad0c | f65de334a2f17bd | 9be93851772d07 |
| 9a113ba8ef25207 | 84d70622883d2e4 | 831709a2eb3de84 | 9a2f3bd16f3a04 |
| 832f09d1773ad04 | c6e9825d777ad0c | e85df34b9e94de | 888318bced2fa46 |
| 82e9a05d0c222677 | c5ef8451647a6f6 | f69de2b4c2e9bbe | 94452707c237ba5 |
| 85ef01456c3e64 | f66e3e4b42eebde | 9c1137aeeb25e87 | 9aef3a150f22007 |
| 8b43190bca27ba6 | 8c83168ba127c6 | d9e9bc5d16d216f | 93852887a137c5 |
| e863df4b46eeb3e | 82290bdd6c3ae64 | eea5d2c721f67dd | f375e966224f7bd |
| f4b3e6ea22f47bd | e865df4725ee7e5 | f75de134daf18bd | c2e98a5d046236 |
| ecb5d6e621f47dd | dad1ba2ee77d0f0 | 832909dd143a164 | de11b3aeeb65e8 |
| c1298dd1471a6c | f775e1664ef4a3d | e875df662dec65e | efa3d0cb25ee7e |
| f0a5ee7c25f675d | ebb3d8ea41ebd | 9d73d46f25207 | 9d7375ae2f3e04 |
| c6ef8251147a16c | d9d1c2e88652ef | 9783028be37625 | f0a3eebcb4f6b3d |
| c2178ba2846536f | 8f851087ad37645 | daefba51076230f | f19becba2af17bd |
| 9045207ae37625 | c51186ae47d6f6c | f75be138b9f14d | c6d7822887d2ec |
| c61822eeb7de8c | e9a3dcb41eebde | 84ef051143a164 | f09beebd8a18bd |
| c5d1842e84636f | dde9b45d677af0c | 9c1737a28252e7 | e85bd3f8dae98be |
| dd29b5dd076230f | e95dd34c1e9bde | eb3d0ea2dec65e | d9efbe517762d0f |
| c2298bd647af6c | e873df6a4eeaca3e | dd17b5a2e77d0fc | 9bd7382eb25e87 |
| f0b3eeea4f4a3d | 84e9065d773ad04 | f765e14766f3b3 | f3b5e642ecbde |
| da2fbbd1677af0c | ec75d76641ebd | eb5d0e64eeaca3e | f773e16a2df45d |
| dd77b422876530f | f6a3e2cb22f67bd | 9e2937d7722d07 | 8b451907a127c6 |
| ee05d43741eebde | e9fbd0b89e94de | f05b3ee62df65d | 9de9345d6f3ae04 |
| c1118da887d2ec | 9743210bce2fa26 | c5f285d1046236f | 82d70a22ec3de64 |
| 852f05d10c22267 | 85d1024ce25267 | f09d3eb49f14d | 9c2f37d11422167 |
| 90852e87ce2fa6 | f473e6a42ecbde | 8c851687c2f1af0 | fa15ec742eebde |
| 88431f0bad37645 | de29b3dd7762d0f | efa5d0c746ebeb3e | ee5bd338a1f17dd |
| e99dcd4a1f17dd | 9bd138288252e7 | 84d17062eb3de84 | da11bbae876530f |
| 82ed3c33350853 | 4a32ac809e19dfa | ba034ec32f16dbb | be1344c3ad1f9a |
| 7afccd1109682b | 66805f4e999aea | aa416676e13fa3 | 72dded5b61d6203 |
| 9a8f0db8b159812 | 668ef5f309682b | a6f7f418a1191a1a | 4212bc02c0d0a3 |
| ba04d4ceca191a1a | b64549e91d9c02 | aa4d6c7e039e84a | 56494f739f04 |
| 86fd3503569d93 | 8edf25591d1c03 | 7af2cd00be999eaa | 92ad1dcb91d1c0 |
| a26f7c382310a5b | 92a31da36e163fb | 86fd351c159812 | 92af1db80390e4b |
Table 3 (continued)

| be1d44dc2310a5b | 9e9f05d83350853 | 9a810de73f569d3 | 564094649e19dfa | 9e9105c7bd5f992 | 4a30ac840c58fb2 | a2617c27ad1fb9a | 421cbcdfa2dfa62 | 5a7e8c1b20d6a23 | 4602b4e0aed9be2 | b6315487fc5f1b2 | 564294600c58fb2 | 92a11da7fe571b3 | 8ed125476e163fb | b63354836e1e3fa | 6290fde43c909ab | 5a708c04aed9be2 | 6eace5bf61d6203 | 72dced5ff39704b | 4a3eac9bf39f04a | 6ea2e5a09e11dfe | 460cb4ff20d6a23 | 82e33d23bd5f992 | 72d0dd449e11dfb | 629efddb29f86a | b63f54980398e4a | aa4f6c7891d9c02 | 8edd255e0390e4b | 4a3cac9f61de202 | 72d2dd400c50fb3 | 5e6084242c0ba3 | 7eccc53fb29f86a | 564e947b61de202 | 6eae5bbf39704b | a67174072f16db | 7ee2c5203c909ab | 6ea0e5a40c50fb3 | 5e6e843ba2dfa62 | aa436c63fc5f1b2 | 8ed32543fc571b3 |
Table 4: $H(60)$ from Yang's Theorem 2

| Sample Data |
|-------------|
| 317431f20d9ee45a | 10bd747b2f4831 | 63e084527485d94 | 36b41f77c6ee7ba |
| 29b2217bbaf6839 | 7b26b5de179d1f7 | 73a420fbbf4452 | 63217a2e68821eb |
| 73a411763e979bd | 6ba210f65d8f9bd | c94a2e6bbf91fbc5 | 08ba5e7d9ee839 |
| 087a2e67d9e83a | a5df7e0b9ad88 | 17bc6e7ba5f67ba | 64208bd626b85e74 |
| 10bce6ca3e975de | 107cd6c65d8f5de | 4228c7c27485d94 | 9a4e9c3b6b85217 |
| 7b7e4ba208ad59 | 08bad6ce1dce831 | 4528f84e089de74 | 7c47242149ae68 |
| 642145a27782e68 | 6ba2217bbf4831 | 087a62ed9d9ee45a | 42e8f7ce179d94 |
| 7b6e85d2179dd94 | a2d0c482779d868 | 6e62f7634828f5d | 592e413e975de |
| d1a2c77bbf917a6 | bd16c6428821eb | 10bc5f7baabf839 | 177d92759f1bc5 |
| 1076c74bb2f4452 | 6a175ae787220b | a210f84e149ae68 | 087ae6bd1c452 |
| 087bec9bc5e9446 | 17bcb946428f63e | 5a2ece6427485f17 | 31b420fba86f54a |
| 0fba8c6428f54d | 0fba6d6b6ce67ba | 7ce7742e149a20b | 45e8c8426b85e74 |
| d68c20fba6f1446 | 29721177d9ee839 | 7c6ba5e089d217 | 0fba3b1d9e79a6 |
| 7c6e28a2089d7e4 | 107dad1ba6f4466 | 64c10f7c5e9446 | bad6c9c2778220b |
| 7b277bae09a1eb | 6320b45e7851f7 | 6e0bbde68821eb | 08bae74a3e979bd |
| 6b621177d1ee831 | 0f77b9317da9bc5 | 17bda29bb9f17a6 | 181c1f77da97a6 |
| 1076c3baa6f45a | cea41177c5e9825 | cea8217d9f18f52 | a5107c26882d88 |
| 08bb9c97c5e9825 | 5aee641e79d1f7 | 6b6220fa5d8f5de | bdd6f6e09a1eb |
| 6ca22f7a428f63e | 5d2e9c9889d217 | 36742f7ba5f67ba | 10bd9d17a6f1825 |
| 736410f71dce45 | ba16f9e149a20b | 63e14a26882d88 | c98ae7f4e19bc5 |
| 1c835c902575f3f3 | df0f257566ea10 | 49df638353e04 | db9f64717c2a90 |
| 0073e647115a0b96 | 713586fd97ae428 | aec989f2d45ec3 | 6dc5ff1ca37bd4 |
| ff865d7055854c4b | fb1d247450b4e82 | 923bf63850b4e82 | 55a7c905e7661 |
| b2b0ff18555c4b | 96abff19e6b9859 | 381115b4b7c1ba | 8a5b8ed01d6898 |
| f00cd3a8133d4d | e7ecab4d05e7661 | fb9d6d1571278d9 | aad9cf6dbb70de43 |
| b23800e515a0b96 | aed87800b3d2f3 | 8e4a30245741ba | c37f1e972544c0a |
| 8ada7020b73c1ba | 3a15257b7b00c0a | b2b9fe1850b5c5b | 3c8155b3b70de43 |
| 75a5c7f993af761 | aec8390421752f3 | fb1d257458b4e02 | c7e5d932d45ec3 |
| 381114b4b33c0bba | e3fcea4997af61 | c7e5c932545e43 | b6a9f7397c278d9 |
| 8a5a7020b33c0ba | e755b5b270de43 | b62841e115a19df | 3f3ce5b493a7f61 |
| fb9d6c517427859 | 962bb63ccafe02 | c3f54b2939e898 | 246125583e98df |
| aad87900b73d3f3 | 923bbae6de6ea10 | ae49c7fa9b9fa51 | c76ea26d01e6528 |
| db9e9a8ca37af04 | 962bb73cc2f8e82 | ff8c3a83533c4d | 69d5f183133d4 |
| df86f5d2f5db6 | 92ba08c5878e9896 | aees9df2545e43 | 18131c9427541ba |
| c7e6a36d05e6428 | 923a08c583e8a96 | 69d5f183533c4d | 96ba49c187e99df |
| 713587fd93ae528 | 8ecbceda017d0a1 | c37ee26997ae848 | 4d4b7a3e7aaf04 |
| ff8dd5c66e8f59 | f0d657050b5c5b | df8f65cafe4b | 24e12d55879e9df |
| dff02475ee6e90a | 4d47b63ca77ae04 | 51378edd01e6528 | 96aa48e183e98df |
| b629bf1ccaf0c4b | 0463257587e8b96 | db1f254cafe02 | 20f16c5115a19df |
### Table 4 (continued)

| 8e4b86ff2544c0a | 3c0155b033d2f3 | b239b63d7426a10 | aa5986fe9b9e818 |
| 923bbe1dec6ea90 | 8e4b87ff2d44c8a | db1f64717426a10 | b6a9b1f1cc2fddec8 |
| df8ed88a37bd4d | 7525c7f997af661 | 51b78edd50e6428 | 96abfe19ee6f8d9 |
| e3fd592939fad1 | c76d1d9601d7ad1 | db1e9a8ca77ae04 | e7d655f3b0fdec3 |
| 6d45bf1ca77bc4d | b629f3979427859 | b6a84e111a18df | e3fd5c929b9fa51 |
| fb1c92ac3132f04 | e76caeb401e7761 | e37d14b7bf0cc8a | f0d2d55ee6f8d9 |
| 8ecb9fda09d7a51 | aec9c7fa939fad1 | c3f8e26993ae528 | 8e4a312421740ba |
| 4957f683132f04 | 1c835d9021752f3 | 04632475839e896 | 20f16d5111a18df |
| b23801e511a8a96 | aad9cdcb8f0dec3 | e7f1ed19609d7a51 | ae48390245753f3 |
| 8adb66fb70cc0a | 8a5a66fbf0cc8a | aa5987fe939e898 | 92b6f3b458b4e02 |
| 8adb88ed09d818 | 55a7ced901e7761 | 18931c921740ba | 00f3647111a096 |
| c3ff1c972d44c8a | fb1c93ace352e04 | c3f54b2b9e818 | c76f15b069d618 |
| df8edaa8a77bc4d | c76f14b601d6898 | b239b73d7c26a90 | db9f2c542f2ce82 |

### Table 5: $H(60)$ from Yang’s Theorem 3

| 3c987fcd5b0aadde | 94772e1098f92a0 | 1b743015c943897 | 8b6b102b893f098 |
| fc65f03588b26a9 | 939721d37886aef | 8801b6e8909e9e | 49692e9e5c8a6 |
| ff1b8c9388b2a6a | 1c983fcd4942897 | a1956ccbf09ede | 88b571014e909c9e |
| b39b61c8a0b0e9 | d8f07b711aaf4b0e | 3b747015db0bade | e7f9c04b53a9d8 |
| 239041d11b8836af | b3856f176a86ce6 | c0158d5c973891 | 4878f14972891 |
| 070408f52ac4e6 | 939821e18b02a9 | 039601d1aadab4e6 | 88156d7893f098 |
| b47660beaaf6ce6 | a675030fb09de6 | df1bb8c92a4e0 | c36b8029c73891 |
| e015c6d5db3bad8 | dc65be352afa4e0 | aff590f1b762d1 | 94e9272f886af |
| a80756f38b3f2d8 | b4656e438a90e9 | 8ff519146940e9 | ac87f03b762d1 |
| c7f9890d4972891 | 047a0e092aca4e6 | 3f679315b0aade | 971728d3f8cfe6 |
| 1fe639314942897 | 970928e98f92a0 | 180a36e9c943897 | afe75930b4ed7 |
| 270448538826af | f87f711b8b36a9 | 247a4938826af | 8feb192b0976091 |
| fb89f1edb8b36a9 | b6e567340a00e9 | 8c951fd70976091 | ab7950f9b3f2d8 |
| db89b1eadab3b0 | 90f7271018b30a | 380a76e9dbbode | 36b0c929db3bad8 |
| 8e8b1fe58940e9 | b7166888af90e0 | b0f867b0ab686aef | 20e84732d8836af |
| e487cf15b3aad8 | 00e8072daac4e6 | b705687eacfe6 | ac995f7c7b40ed7 |
Table 6: $H(60)$ from Yang’s Theorem 4

| DFA   | Sequence | DFA   | Sequence | DFA   | Sequence |
|-------|----------|-------|----------|-------|----------|
| dfa308e8885c365 | 7c74ef6e25dc37a | 2c604f44857176f | d461ba477071490 |
| 86f51f6dc5dc225 | 27a2fdeeb658c23a | d6e11a6e30215d0 | 27a258c0857176f |
| 8fb70de9c5dc225 | 25225dc090715cf | 753658c1d02148f | 7c744a47708c09a |
| 8af70769cd3c65  | 22e252408f7162f | 77b68ea25dc37a | 2ee04a4490715cf |
| 2ee0ef6dc52162f | 83b515edcfde365 | 8475bf44858c365 | 7536fde30d17da |
| 7ef4ea6e30dc1da | 2c60ea6dd02148f | dd23a8c30171490 | 77b65dc1c52162f |
| 8d37adc230215d0 | 7ef44f47658c23a | 292045c48f7162f | 2522f8eb708c09a |
| 72f65741cf2176f | 2ba06edcf2176f | dae302688f8c225 | 8135b5c48f8c225 |
| b96fabb29e41fbc | 74dec0de947cefe | 21881a743ed1bab | ec3b011834eace9 |
| 35d832d41e51fbb | 58ee9b9a947cefe | 504ef9fac53c4d6 | 608c987eb4faee |
| 375837d41a151f3b | 68ae883ee5bc0c6 | e41b11586ac0c1 | 22081f7f431d1b2b |
| b04fb9f2c5014d4 | 4c3ec11ab4fcaee | 0faa46103ad1bab | 0d3a43103ed1bab |
| ddf9629ec01594 | 1bea6eb01a51f3b | f86b29b8146ce9f | d5972de9e41fbc |
| dcf9609ec5014d4 | 441ed15ae5bc0c6 | 808dd87434eace9 | 7cfca09e53c4d6 |
| 196abc01e51fbb | 88adcb3465ac0c1 | f04b39f8452e4d1 | 9cfde094452e4d1 |
| 94ddfd41d15eef9 | b14ffbf2cf05194 | 114a7bf04f11593 | 3df82294ff11593 |