Spin structures and codimension-two homeomorphism extensions

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Abstract

Let \( \imath : M \to \mathbb{R}^{p+2} \) be a smooth embedding from a connected, oriented, closed \( p \)-dimensional smooth manifold to \( \mathbb{R}^{p+2} \), then there is a spin structure \( \imath^! (\varsigma^{p+2}) \) on \( M \) canonically induced from the embedding. If an orientation-preserving diffeomorphism \( \tau \) of \( M \) extends over \( \imath \) as an orientation-preserving topological homeomorphism of \( \mathbb{R}^{p+2} \), then \( \tau \) preserves the induced spin structure.

Let \( \mathcal{E}_C(\imath) \) be the subgroup of the \( C \)-mapping class group \( \text{MCG}_C(M) \) consisting of elements whose representatives extend over \( \mathbb{R}^{p+2} \) as orientation-preserving \( C \)-homeomorphisms, where \( C = \text{Top}, \text{PL} \) or \( \text{Diff} \). The invariance of \( \imath^! (\varsigma^{p+2}) \) gives nontrivial lower bounds to \( [\text{MCG}_C(M) : \mathcal{E}_C(\imath)] \) in various special cases. We apply this to embedded surfaces in \( \mathbb{R}^4 \) and embedded \( p \)-dimensional tori in \( \mathbb{R}^{p+2} \). In particular, in these cases the index lower bounds for \( \mathcal{E}_{\text{Top}}(\imath) \) are achieved for unknotted embeddings.

1 Introduction

Let \( M \) be a connected, oriented, closed \( p \)-dimensional smooth manifold, and \( \imath : M \to \mathbb{R}^{p+2} \) be a smooth embedding. We are concerned with the question: ‘how many mapping classes of \( M \) extend over \( \mathbb{R}^{p+2} \)?’ Regarding to different possible flavors of this question, we shall write \( \text{Top} \) (resp. \( \text{PL} \), or \( \text{Diff} \)) for the category of topological (resp. PL, or smooth) manifolds with continuous (resp. PL, or smooth) maps as morphisms, and generally write \( C \) for any of these categories. We speak of \( C \)-manifolds, \( C \)-homeomorphisms, \( C \)-isotopies, etc. in the usual sense.

With notations above, denote \( \text{MCG}_C(M) = \pi_0 \text{Homeo}^+_C(M) \) for the \( C \)-mapping-class-group of \( M \), i.e. the group of \( C \)-isotopy classes of orientation-preserving \( C \)-self-homeomorphisms on \( M \). A class \( [\tau] \in \text{MCG}_C(M) \) is called \( C \)-extendable over \( \imath \) if for some (hence any, cf. Lemma 5.4) representative \( \tau \), there is an orientation-preserving \( C \)-self-homeomorphism \( \tilde{\tau} \) of \( \mathbb{R}^{p+2} \) such that \( \imath \circ \tau = \tilde{\tau} \circ \imath \). We define the \( C \)-extendable subgroup with respect to \( \imath \) as:

\[
\mathcal{E}_C(\imath) = \{ [\tau] \in \text{MCG}_C(M) \mid \tau \text{ is } C\text{-extendable over } \imath \}\.
\]

Now the question makes sense by asking what is the index of \( \mathcal{E}_C(\imath) \leq \text{MCG}_C(M) \). In this paper, we prove the following criterion.
Let $\varsigma^{p+2}$ by the canonical spin structure on $\mathbb{R}^{p+2}$. For any smooth embedding $i : M \to \mathbb{R}^{p+2}$, there is a canonically induced spin structure $i^*(\varsigma^{p+2})$ on $M$ (Definition 3.2).

**Proposition 1.1.** For any smooth embedding $i : M \to \mathbb{R}^{p+2}$, the induced spin structure $i^*(\varsigma^{p+2})$ is null spin-cobordant, and is invariant under any orientation-preserving self-diffeomorphism of $M$ which extends over $i$ as an orientation-preserving topological self-homeomorphism of $\mathbb{R}^{p+2}$.

In fact, $i^*(\varsigma^{p+2})$ is naturally induced as the boundary of a spin structure on a smooth Seifert hypersurface $\Sigma$ of $i(M)$. Proposition 1.1 allows us to find nontrivial lower bounds of $[\mathrm{MCG}_C(M) : \mathcal{E}_C(i)]$ in certain cases. We may even compute the index for some specific embeddings. In this paper, we apply the criterion to smoothly embedded surfaces in $\mathbb{R}^4$ and certain smoothly embedded $p$-dimensional torus in $\mathbb{R}^{p+2}$.

**Theorem 1.2.** For any smooth embedding $i : F_g \hookrightarrow \mathbb{R}^4$ of the closed oriented surface of genus $g$ into $\mathbb{R}^4$,

$$[\mathrm{MCG}_{\mathrm{Top}}(F_g) : \mathcal{E}_{\mathrm{Top}}(i)] \geq 2^{2g-1} + g^2 - 1.$$  

**Remark 1.3.** In principle, one should be able to derive the smooth version that $[\mathrm{MCG}_{\mathrm{Diff}}(F_g) : \mathcal{E}_{\mathrm{Diff}}(i)] \geq 2^{2g-1} + g^2 - 1$ from the invariance of the Rokhlin quadratic form (\cite{Ro}). However, this, also the PL version, is immediately implied by Theorem 1.2 as the $\mathrm{MCG}_C(F_g)$’s are canonically isomorphic for $C$ being $\mathrm{Diff}$, $\mathrm{PL}$, and $\mathrm{Top}$. The lower bound in Theorem 1.2 is sharp for any unknotted embedding of $F_g$ in $\mathbb{R}^4$, namely which bounds a smoothly embedded handlebody of genus $g$ in $\mathbb{R}^4$, following from an intensive construction of Susumu Hirose (\cite{Hi}, cf. also (\cite{Mo}) for $g = 1$). See Section 4 for details.

Another interesting fact follows from the proof of Theorem 1.2

**Corollary 1.4.** For any $g \geq 1$, there exists $[\tau] \in \mathrm{MCG}_{\mathrm{Top}}(F_g)$ which is not homeomorphically extendable over any smooth embedding $i : F_g \hookrightarrow \mathbb{R}^4$.

Denote the standard $p$-dimensional smooth torus $S^1 \times \cdots \times S^1 (p \text{ copies})$ as $T^p$. The structure of $\mathrm{MCG}_C(T^p)$ is fairly well-understood except for $p = 4$, thank to the work of Allen Hatcher et al, (see Section 5 for details), which makes the following theorem attainable.

**Theorem 1.5.** For $p \geq 1$, suppose $i : T^p \hookrightarrow \mathbb{R}^{p+2}$ is a smooth embedding whose induced spin structure $i^*(\varsigma^{p+2})$ on $T^p$ is not the Lie-group spin structure, then:

$$[\mathrm{MCG}_{\mathrm{Top}}(T^p) : \mathcal{E}_{\mathrm{Top}}(i)] \geq 2^p - 1.$$  

Moreover, the lower bound is realized by unknotted embeddings (Definition 5.4).

**Remark 1.6.** A parallel proof shows $[\mathrm{MCG}_C(T^p) : \mathcal{E}_C(i)] \geq 2^p - 1$, cf. Lemma 5.1. For $p \neq 4$, this also follows from the fact that the natural forgetting functor $\mathrm{Diff} \to \mathrm{PL} \to \mathrm{Top}$ yields epimorphisms on $\mathrm{MCG}_C(T^p)$ and homomorphisms on $\mathcal{E}_C(T^p)$, and hence $[\mathrm{MCG}_{\mathrm{Diff}}(T^p) : \mathcal{E}_{\mathrm{Diff}}(i)] \geq [\mathrm{MCG}_{\mathrm{PL}}(T^p) : \mathcal{E}_{\mathrm{PL}}(i)] \geq [\mathrm{MCG}_{\mathrm{Top}}(T^p) : \mathcal{E}_{\mathrm{Top}}(i)]$.

**Corollary 1.7.** If a smoothly embedded 3-torus $T^3$ in $\mathbb{R}^5$ has a (hence any) smooth Seifert hypersurface $\Sigma^4$ of signature 0 modulo 16, then $[\mathrm{MCG}_{\mathrm{Top}}(T^3) : \mathcal{E}_{\mathrm{Top}}(i)] \geq 7$.

Note any smooth Seifert hypersurface in this case must have signature 0 modulo 8, (cf. \cite{SST} Proposition 6.1).
Corollary 1.8. If \( \iota : T^p \hookrightarrow \mathbb{R}^{p+2} \) is an unknotted embedding, then \( \text{MCG}_{\text{Diff}}(T^p) : \mathcal{E}_{\text{Diff}}(\iota) \) and \( \text{MCG}_{\text{PL}}(T^p) : \mathcal{E}_{\text{PL}}(\iota) \) are finite but at least \( 2^p - 1 \).

Remark 1.9. While these indices are indeed \( 2^p - 1 \) when \( p \leq 3 \), it is still an interesting question figuring out the indices when \( p > 3 \).

It is pretty easy to find examples where the assumption of Theorem 1.5 holds, but with some effort one can still find smooth embeddings \( \iota : T^p \hookrightarrow \mathbb{R}^{p+2} \) which induce the Lie-group spin structure on \( T^p \) when \( p \geq 3 \), (cf. [SST] for such an example). Unfortunately, Theorem 1.5 says nothing about that case. We make the following conjecture:

Conjecture 1.10. For any smooth embedding \( \iota : T^p \hookrightarrow \mathbb{R}^{p+2} \), \( \mathcal{E}_{\text{Top}}(\iota) \) is a proper subgroup of \( \text{MCG}_{\text{Top}}(T^p) \).

In Section 2 we recall some preliminary material about spin structures in terms of trivializations. In Section 3 we introduce \( \mathcal{J}(T^{p+2}) \) (Definition 3.2 using Seifert hypersurfaces, and prove Proposition 3.1. In Section 4 we consider embedded surfaces in \( \mathbb{R}^4 \) and prove Theorem 1.2 and Corollary 1.4. In Section 5, we consider embedded \( T^p \) in \( \mathbb{R}^{p+2} \). We first review some results of Hatcher about the structure of \( \text{MCG}_C(T^p) \) for \( p \neq 4 \), then prove Theorem 1.5 and its corollaries by studying the action of \( \text{MCG}_C(T^p) \) on \( S(T^p) \).

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2 Spin structure preliminaries

In this section, we recall some basic facts about spin structures, cf. [Ki] Chapter IV, [Mi].

Spin structures of a rank \( n \) vector bundle \( \xi \) over a CW complex \( X \) can be phrased with trivialization, i.e. framing. \( \xi \) can be endowed with a spin structure if \( E \oplus \epsilon^k \) has a trivialization over the 1-skeleton which may extend over the 2-skeleton, (if \( n \geq 3 \), \( k = 0 \); if \( n = 2 \), \( k = 1 \); if \( n = 1 \), \( k = 2 \)), and a spin structure is a homotopy class of such trivializations over the 1-skeleton. For a CW complex \( X \), we use \( X^{(i)} \) to denote its \( i \)-skeleton.

Lemma 2.1. Suppose \( \xi \) is a rank \( n \) vector bundle over a topological space \( M \) that admits CW structures. Then there is a natural bijection between the sets of spin structures corresponding to different CW structures.

Proof. Suppose \( X_0, X_1 \) are two CW structures on \( M \), and let \( \sigma_i \) be a spin structure of \( \xi \) with respect to \( X_i \) \( (i = 0, 1) \). There is a natural difference homomorphism:

\[
\sigma_1 - \sigma_0 : \pi_1(M) \to \mathbb{Z}_2,
\]

defined as follows. For any \( [\alpha] \in \pi_1(M) \), let \( \gamma_0, \gamma_1 : S^1 \to M \) be two closed paths in \( X_0^{(1)}, X_1^{(1)} \) respectively, both basepoint-freely homotopic to \( \alpha \) in \( M \). Let \( \gamma_t : S^1 \to M \ (t \in I) \) be any homotopy
between $\gamma_0$ and $\gamma_1$, and pick a trivialization $\varsigma$ of $e^{n+k}|S^1 \times I$. If $\gamma : S^1 \times I \to M$ extends to a bundle map $\tilde{\gamma} : e^{n+k}|S^1 \times I \to \xi \oplus e^h|_M$, such that $\tilde{\gamma}_*|_{S^1 \times 0} \cong \sigma_0$, $\tilde{\gamma}_*|_{S^1 \times 1} \cong \sigma_1$, then we define $(\sigma_1 - \sigma_0)([\alpha]) = 0$, otherwise $(\sigma_1 - \sigma_0)([\alpha]) \neq 0$. It is easy to see that $\sigma_1 - \sigma_0$ is a well-defined homomorphism, and $(\sigma_2 - \sigma_1) - (\sigma_1 - \sigma_0) = \sigma_2 - \sigma_0$.

Thus, we define two spin structures with respect to possibly different spin structures to be equivalent if their difference is zero. If $X_0$ and $X_1$ happen to be the same, it is clear that $\sigma_0$ and $\sigma_1$ are equivalent if and only if they are equal by definition. Moreover, for any CW complex structure, the space of spin structures forms an affine $H^1(M; \mathbb{Z}_2) \cong \text{Hom}(\pi_1(M), \mathbb{Z}_2)$, precisely as described by the difference homomorphism. Thus, if $\sigma_1 - \sigma_0$ is not zero, we find exactly one $\sigma'_1$ with respect to $X_1$, such that $\sigma'_1 - \sigma_0$ is zero. This implies that the spaces of spin structures corresponding to different CW structures are in natural bijection to each other, namely according to the equivalence.

For a rank $n$ vector bundle $\xi$ over a CW space $M$, a spin structure on $\xi$ is known as with respect to any CW structure on $M$, up to the natural equivalence. A spin structure of a smooth manifold $M$ is a spin structure of its tangent bundle. For any closed path $\alpha$ on $M$, we may also restrict a spin structure $\sigma$ of $M$ to $\sigma|_\alpha$, namely picking a trivialization of $\xi|_\alpha$ which extends to a trivialization on (some) 1-skeleton equivalent to $\sigma$. It is clear that two spin structures $\sigma_1, \sigma_2$ of $\xi$ are equivalent if and only if the trivialization $\sigma_1|_\alpha \cong \sigma_2|_\alpha$ for any closed path $\alpha$ on $M$. For an oriented smooth manifold $M$, $M$ has a spin structure if and only if $w_2(M) = 0$, and when $M$ has spin structures, the space $S(M)$ of spin structures on $M$ is an affine $H^1(M; \mathbb{Z}_2)$.

If $\xi = \xi' \oplus \xi''$ are bundles over a CW space $X$, then spin structures on any two determine a spin structure of the third, so that $\sigma \simeq \sigma' \oplus \sigma''|_{\chi(i)}$. The less trivial direction that $\sigma, \sigma'$ determine $\sigma''$ follows from the general fact: if $\xi = \eta \oplus \xi'$ is a trivial $(n + k)$-vector bundle where $\eta, \xi'$ has rank $n, k$, respectively, and if $\xi, \xi'$ are both trivialized over $X^{(n-1)}$, then there is a complementary trivialization of $\eta$ over $X^{(n-1)}$, which is unique over $X^{(n-2)}$ up to homotopy, (cf. [K] p. 33)).

For a spin manifold $M$ with boundary $\partial M$, $\partial M$ has a natural spin structure induced from the spin structure of $M$ and the (inward) normal vector of $\partial M$ in $M$. A manifold is called a spin boundary if there is a spin manifold bounding it, inducing its spin structure. For example, the circle $S^1$ has two spin structures: one spin-bounds the spin $D^2$, and the other is the Lie-group spin structure which is not a spin-boundary.

### 3 Invariant induced spin structure

In this section, we introduce the induced spin structures for closed oriented codimension-2 smooth submanifolds of $\mathbb{R}^{p+2}$, and prove Proposition 1.1.

Suppose $\iota : M \hookrightarrow \mathbb{R}^{p+2}$ is a connected, closed, oriented $p$-dimensional smooth submanifold of $\mathbb{R}^{p+2}$, $p \geq 1$. Since any closed oriented smooth submanifold of $\mathbb{R}^{p+2}$ has trivial Euler class ([MS, Corollary 11.4]), the normal bundle of $M$ in $\mathbb{R}^{p+2}$ is trivial as $M$ is codimension 2. On the other hand, it is well-known that there exists a Seifert hypersurface $\Sigma \subset \mathbb{R}^{p+2}$ of $\iota(M)$, namely, a compact connected oriented $(p + 1)$-dimensional smooth submanifold such that $\partial \Sigma = \iota(M)$, (cf. for example, [F] Lemma 2.2).  

Let $W$ be an inward normal vector field of $\iota(M)$ in $\Sigma$ (say, w.r.t some compatible Riemannian
metric on a collar), and $H$ be a normal vector field of $\Sigma$ in $\mathbb{R}^{p+2}$ over $i(M)$, such that the orientation $(W, H)$ of the normal bundle $N_{\mathbb{R}^{p+2}}(i(M))$ and the orientation of $M$ match up to that the canonical orientation of $\mathbb{R}^{p+2}$. The trivialization $(W, H)$ of $N_{\mathbb{R}^{p+2}}(i(M))$ defines a spin structure $\sigma$ of $N_{\mathbb{R}^{p+2}}(i(M))$, and the canonical spin structure $\varsigma^{p+2}$ of $\mathbb{R}^{p+2}$ restricts to a spin structure on $T\mathbb{R}^{p+2}|_{i(M)}$. As $T\mathbb{R}^{p+2}|_{i(M)} = i_*(TM) \oplus N_{\mathbb{R}^{p+2}}(i(M))$, we conclude there is a complementary spin structure $\sigma^\perp$ of $i_*(TM)$ such that:

$$\sigma^\perp \oplus \sigma = \varsigma^{p+2},$$

(i.e. as trivializations $\sigma^\perp \oplus \sigma \simeq \varsigma$ over $M$).

**Lemma 3.1.** The spin structure $\sigma^\perp$ on $i(M)$ is independent of the choice of $\Sigma$ and $(W, H)$.

**Proof.** It suffices to show $\sigma$ is independent of the choice of $\Sigma$ and $(W, H)$. In fact, we show for any two choices $\Sigma$, $(W, H)$ and $\Sigma'$, $(W', H')$, the trivializations $(W, H) \simeq (W', H')$ over $i(M)$.

First observe that any loop $\alpha$ on $i(M)$, when pushed into $\hat{\Sigma}$ along $W$, becomes null-homologous in $\mathbb{R}^{p+2} \setminus i(M)$. To see this, consider the map $f : \mathbb{R}^{p+2} \setminus i(M) \to S^1$ defined as follows: take a tubular neighborhood $N(\hat{\Sigma})$, where $\hat{\Sigma}$ is the interior of $\Sigma$, and let $f | : N(\hat{\Sigma}) \to S^1$ to be the composition: $N(\hat{\Sigma}) \cong \hat{\Sigma} \times I \xrightarrow{p} I \xrightarrow{q} I/\partial I \cong S^1$, where $p$ is the second-factor projection and $q$ is the quotient map; then extend $f$ to $f : \mathbb{R}^{p+2} \setminus i(M) \to S^1$ by the constant map. Then $f_* : H_1(\mathbb{R}^{p+2} \setminus i(M)) \to H_1(S^1)$ is isomorphic, but the push-off of $\alpha$ along $W$ is mapped to $0 \in H_1(S^1)$, so it is null-homologous in $\mathbb{R}^{p+2} \setminus i(M)$.

Now $(W, H)$ and $(W', H')$ differ pointwisely by an element of $GL^+(2, \mathbb{R})$, namely for any $x \in M$, $(W', H')|_{x} = (W, H)|_{x} : R(x)$ for some $R(x) \in GL^+(2, \mathbb{R})$. This gives a map $R : M \to GL^+(2, \mathbb{R})$. If for some loop $\alpha : S^1 \to M$, $R \circ \alpha$ were not null-homotopic in $GL^+(2, \mathbb{R})$, then the push-offs of $\alpha$ along $W$ and $W'$ would differ by a non-zero multiple of the meridian $\mu$, namely the loop which bounds a normal disk of $i(M)$. Since $\mu$ is the generator of $H_1(\mathbb{R}^{p+2} \setminus i(M)) \cong \mathbb{Z}$ by the Alexander duality, the two push-offs would not be both null-homologous in $\mathbb{R}^{p+2} \setminus i(M)$, which is a contradiction. Thus $R_* : \pi_1(M) \to \pi_1(GL^+(2, \mathbb{R}))$ is trivial. We conclude that $R$ is homotopic to the constant identity map as $\pi_1(GL^+(2, \mathbb{R})) \cong \pi_1(S^1) = 0$ for $i \geq 2$. This implies $(W, H) \simeq (W', H')$ over $M$.

Lemma 3.1 allows us to make the following definition.

**Definition 3.2.** For a smooth embedding $i : M \hookrightarrow \mathbb{R}^{p+2}$ of a connected, closed, oriented $p$-dimensional smooth manifold $M$ into $\mathbb{R}^{p+2}$, we define the induced spin structure as:

$$i^*(\varsigma^{p+2}) = i^*(\sigma^\perp),$$

where $\sigma^\perp$ is as described above.

**Proof of Proposition 1.1.** We first show $i^*(\varsigma^{p+2})$ is null-spin-cobordant, or equivalently that $\sigma^\perp$ is a spin boundary. In fact, for a Seifert hypersurface $\Sigma$ of $i(M)$, the normal vector field $H$ of $\Sigma$ in $\mathbb{R}^{p+2}$ defines a spin structure $\sigma_H$ on the normal bundle $N_{\mathbb{R}^{p+2}}(\Sigma)$, so there is a spin structure $\sigma_H^\perp$ on $T\Sigma$ such that $\sigma_H^\perp \oplus \sigma_H = \varsigma^{p+2}$ on $T\Sigma \oplus N_{\mathbb{R}^{p+2}}(\Sigma) = T\mathbb{R}^{p+2}|_{\Sigma}$. The spin boundary of $(\Sigma, \sigma_H^\perp)$ is clearly $(M, \sigma^\perp)$ by the construction.

We next prove the invariance of $i^*(\varsigma^{p+2})$ under homeomorphically extendable self-diffeomorphisms. Specifically, for an orientation-preserving self-diffeomorphism $\tau : M \to M$ which extends over $i$ as
an orientation-preserving topological self-homeomorphism \( \tilde{\tau} \) of \( \mathbb{R}^{p+2} \), we must show \( \tilde{\tau}(\zeta^{p+2}) \) equals \( \tau^*(\tilde{\tau}(\zeta^{p+2})) = (\iota \circ \tau)^*\tilde{\tau}(\zeta^{p+2}) \). Without loss of generality, we may assume \( p > 1 \) as there is nothing to prove for \( p = 1 \). We shall omit writing \( \iota \) identifying \( M \) as a submanifold of \( \mathbb{R}^{p+2} \), and identify \( D^2 \) as the unit disk in \( \mathbb{C} \).

Let \( \mathcal{N} \) be a closed tubular neighborhood of \( M \) in \( \mathbb{R}^{p+2} \), identified with \( M \times D^2 \) such that \( M \) is identified with \( M \times \{0\} \) and \( M \times \{1\} \) is the push-off of \( M \) along \( W \). By the uniqueness of normal bundle for codimension 2 locally flat embedding (see [KS] for the ambient dimension \( \geq 5 \), and [FQ] Section 9.3 for that = 4), we may assume \( \tilde{\tau} \) preserves \( \mathcal{N} \), namely restricted to this neighborhood, \( \tilde{\tau} \) is a bundle map \( M \times D^2 \to M \times D^2 \), \( \tilde{\tau}(x,v) \to (\tau(x), R(x)v) \), where \( R(x) \in \text{SO}(2) \).

Because \( \tilde{\tau}(\Sigma) \) is still a (topological) Seifert hypersurface, by the same argument of Lemma 3.1, \( R : M \to \text{SO}(2) \) is homotopic to the constant identity map. This implies that \( \tilde{\tau}|_{\mathcal{N}} \) may be assumed to be \( \tau \times \text{id}_{D^2} \) under the identification \( \mathcal{N} \cong M \times D^2 \). Let \( X = S^{p+2} \setminus \mathcal{N} \), where \( S^{p+2} = \mathbb{R}^{p+2} \cup \{\infty\} \). Extend \( \tilde{\tau} \) to a homeomorphism of \( S^{p+2} \), still denoted as \( \tilde{\tau} \), by defining \( \tilde{\tau}(\infty) = \infty \). We glue two (opposite) copies \( X, -X \) along the boundary via \( \tilde{\tau}|_{\partial X} : \partial X \to \partial X \), and the resulting smooth manifold is called \( Y_\tau = X \cup_\tau (-X) \). On the other hand, we may glue via \( \text{id}|_{\partial X} \) to obtain the double of \( X \), called \( Y_{\text{id}} = X \cup_{\text{id}} (-X) \). Thus \( Y_\tau \) is homeomorphic to \( Y_{\text{id}} \) via \( \tilde{\tau} \cup \text{id} \).

Observe that \( TY_{Y_{\text{id}}} \) is stably trivial. In fact, \( X \subset S^{p+2} = \partial D^{p+3} \), and we may push the interior of \( X \) into the interior of \( D^{p+3} \) so that \( (X, \partial X) \subset (D^{p+3}, S^{p+2}) \) is a proper embedding of pairs. We may further assume that on the collar neighborhood of \( \partial D^{p+3} \), diffeomorphically \( S^{p+2} \times I \), \( X \) is identified as \( \partial X \times I \). Then doubling \( D^{p+3} \) along boundary gives a codimension 1 smooth embedding \( Y_{Y_{\text{id}}} \subset S^{p+3} \). Hence clearly \( TY_{Y_{\text{id}}} \cup e^1 \) is trivial, so \( w_2(Y_{\text{id}}) = 0 \in H^2(Y_{\text{id}}; \mathbb{Z}_2) \). Because the Stiefel-Whitney class depends only on the homotopy type of the smooth manifold (cf. [Wu], also [MS]), it suffices to show that if \( \tau \) does not preserve \( \tilde{\tau}(\zeta^{p+2}) \), then \( w_2(Y_\tau) \in H^2(Y_\tau; \mathbb{Z}_2) \) does not vanish.

Suppose on the contrary that \( \tau \) does not preserve \( \tilde{\tau}(\zeta^{p+2}) \), then there is some smoothly embedded loop \( \alpha \subset M \) such that \( \tilde{\tau}(\zeta^{p+2})|_\alpha \not\cong \tau^*(\tilde{\tau}(\zeta^{p+2}))|_\alpha \). Since we assumed that \( \tilde{\tau}|_{\mathcal{N}} = \tau \times \text{id}_{D^2} \) under the identification \( \mathcal{N} \cong M \times D^2 \), by the construction of \( \tilde{\tau}(\zeta^{p+2}) \), we see that the difference \( \tau_\alpha(\zeta^{p+2}|_\alpha) - \zeta^{p+2}|_\alpha \) between the spin structures on \( T\mathbb{R}^{p+2}|_{\mathcal{N}} \) as an element in \( H^1(\mathcal{N}; \mathbb{Z}_2) \) (cf. Section 9) equals \( \tau_\alpha(\tilde{\tau}(\zeta^{p+2})) - (\tilde{\tau}(\zeta^{p+2})) \in H^1(M; \mathbb{Z}_2) \), under the natural inclusion isomorphism \( H^1(\mathcal{N}; \mathbb{Z}_2) \to H^1(M; \mathbb{Z}_2) \). Hence \( \tau_\alpha(\zeta^{p+2}|_{\alpha \times \{1\}}) \not\cong \zeta^{p+2}|_{\alpha \times \{1\}} \).

As \( \alpha \times \{1\} \) is null-homologous in \( X \) by the construction (cf. the proof of Lemma 3.1), it bounds a smoothly immersed oriented surface \( j : X \ni X \) such that \( j(F) \subset X \), and \( j \) is a smooth embedding in a collar neighborhood of \( \partial F \). This can be seen by writing \( \alpha \) as a product of commutators, so there is a continuous map \( F \to X \), which can be perturbed to be an immersion by the Whitney’s trick. Thus, there is a smoothly immersed closed oriented surface \( \widehat{j} : K = F \cup (-F) \ni Y_\tau \) defined by \( j \cup (-\tilde{\tau} \circ j) \).

However, \( \tilde{\tau}(TY_{Y_\tau})|_F = j^*(T\mathbb{R}^{p+2})|_F \) has a spin structure \( \zeta^{p+2}|_F \), and \( \tilde{\tau}(TY_{Y_\tau})|_{-F} = (-\tilde{\tau} \circ j)^*(T\mathbb{R}^{p+2})|_{-F} \) has a spin structure \(-\zeta^{p+2}|_{-F} \). They disagree along \( \alpha \times \{1\} \subset \partial X \) (corresponding to \( \tau(\alpha) \times \{1\} \subset \partial (-X) \)). This implies that \( w_2(\tilde{\tau}(TY_{Y_\tau})) \neq 0 \in H^2(K; \mathbb{Z}_2) \), and hence \( w_2(Y_\tau) \neq 0 \in H^2(Y_\tau; \mathbb{Z}_2) \) as it pulls back giving a nontrivial element of \( H^2(K; \mathbb{Z}_2) \). This contradicts that \( Y_{\text{id}} \) is homeomorphic to \( Y_\tau \), so \( \tau \) has to preserve \( \tilde{\tau}(\zeta^{p+2}) \).

\begin{remark}
\textbf{Remark 3.3.} We are aware that the induced spin structure \( \tilde{\tau}(\zeta^{p+2}) \) can also be derived from a general construction of characteristic pairs ([KT], cf. also [GAL], [E]). Recall that a pair of oriented
compact smooth manifolds \((W, M)\) is called characteristic if \(M \subset W\) is a proper codimension-2 submanifold dual to \(w_2(M)\). The space \(\text{Char}(W, M)\) of characterizations of \((W, M)\) consists of spin structures on \(W \setminus M\) which does not extend across any component of \(M\), admitting a natural free transitive action \(H^1(W; \mathbb{Z}_2)\). There is a function \(h : \text{Char}(W, M) \to S(M)\) equivariant under the natural actions of \(H^1(W; \mathbb{Z}_2)\) on \(\text{Char}(W, M)\) and \(H^1(M; \mathbb{Z}_2)\) on \(S(M)\) via the homomorphism \(H^1(W; \mathbb{Z}_2) \to H^1(M; \mathbb{Z}_2)\), where \(S(M)\) is the space of spin structures on \(M\), ([KT, Definition 6.1, Theorem 2.4, Lemma 6.2]). When \(W = S^{p+2}\) and \(M\) is connected, \(\text{Char}(W, M)\) is a single element group whose image under \(h\) coincides with \(\iota^*(\mathbb{Z}_2)\). This gives an alternative proof of Proposition [14] if one assumes \(\tau\) extends over \(S^{p+2}\) diffeomorphically rather than just homeomorphically. When \(p = 2\), one can also phrase \(\iota^*(\mathbb{Z}_2)\) in terms of the Rohlin quadratic form, see Lemma [43].

Before going to the applications, we mention the following lemma which justifies the well-definedness of \(\delta_C(\iota)\).

**Lemma 3.4.** Let \(\iota : M \hookrightarrow \mathbb{R}^{p+2}\) be a smooth embedding of an orientable closed \(p\)-dimensional manifold. Let \(\tau, \tau' : M \to M\) be two \(C\)-isotopic orientation-preserving \(C\)-homeomorphisms, then \(\tau\) is \(C\)-extendable if and only if \(\tau'\) is \(C\)-extendable over \(\iota\).

**Proof.** First assume \(\tau'\) is the identity. Take a tubular neighborhood \(N\) of \(\iota(M)\) in \(\mathbb{R}^{p+2}\), we have seen that \(N\) is diffeomorphic to \(M \times D^2\). As \(\tau\) is \(C\)-isotopic to the identity, say \(f_t : M \to M\) where \(t \in [0, 1]\), we define \(\hat{\tau} : M \times D^2 \to M \times D^2\) by \(\hat{\tau}(x, re^{i\theta}) = (f_t(x), re^{i\theta})\), where \(D^2\) is identified as the unit disk of \(\mathbb{C}\). Then \(\hat{\tau}\) is the identity restricted to \(\partial N \cong M \times \partial D^2\). We may further extend \(\hat{\tau}\) outside \(N\) over \(\mathbb{R}^{p+2}\) by the identity. This implies \(\tau\) is \(C\)-extendable.

In the general case, let \(\tau, \tau'\) be two \(C\)-isotopic orientation-preserving \(C\)-homeomorphisms, then \(\tau^{-1} \circ \tau'\) is \(C\)-isotopic to the identity and hence \(C\)-extendable. Let \(\phi : \mathbb{R}^{p+2} \to \mathbb{R}^{p+2}\) be an orientation-preserving \(C\)-homeomorphic extension of \(\tau^{-1} \circ \tau'\). If \(\tau\) is \(C\)-extendable, say as \(\hat{\tau} : \mathbb{R}^{p+2} \to \mathbb{R}^{p+2}\), then \(\tau'\) may be extended as \(\hat{\tau} \circ \phi\), and vice versa. \(\square\)

### 4 Embedded surfaces in \(\mathbb{R}^4\)

In this section, we prove Theorem 1.2. Note \(\text{MCG}_\mathcal{C}(F_g)\) for \(\mathcal{C} = \text{Diff}, \text{PL}, \text{Top}\) are all canonically isomorphic to \(\text{Out}(\pi_1(F_g))\) due to the Dehn-Nielsen-Baer theorem (cf. [KV]).

Let \(\mathcal{S}(F_g)\) be the space of spin structures on a closed connected oriented surface \(F_g\) of genus \(g\). There is a surjective map:

\[
\mathcal{S}(F_g) \to \Omega_2^{\text{Spin}} \xrightarrow{\text{Arf}} \mathbb{Z}_2,
\]

where \(\Omega_2^{\text{Spin}}\) is the second spin cobordism group and \(\text{Arf}\) is the Arf isomorphism. More precisely, for any \(\sigma \in \mathcal{S}(F_g)\), there is an associated nonsingular quadratic function \(q_\sigma : H_1(F_g; \mathbb{Z}_2) \to \mathbb{Z}_2\), such that \(q_\sigma(\alpha) = 0\) (resp. 1) if the spin structure on \(F_g\) restricted to the bounding (resp. Lie-group) spin structure on \(\alpha\). Note \(q_\sigma(\alpha + \beta) = q_\sigma(\alpha) + q_\sigma(\beta) + \alpha \cdot \beta\) where \(\alpha \cdot \beta\) is the \(\mathbb{Z}_2\)-intersection number, and \(\sigma = \sigma'\) if and only if \(q_\sigma = q_{\sigma'}\). Thus \(\text{Arf}([\sigma])\) is defined as the Arf invariant of the nonsingular quadratic form \(q_\sigma\). Recall that for a nonsingular quadratic form \(q\) on \(V \cong \mathbb{Z}_2^{2g}\), \(\text{Arf}(q) = 0\) (resp. 1) if and only if \(q\) vanishes on exactly \(2^{2g-1} + 2g-1\) (resp. \(2^{2g-1} - 2g-1\)) elements, (cf. [K1 Appendix]).
Correspondingly, $S(F_g)$ is a disjoint union:

$$S(F_g) = B_g \sqcup U_g,$$

of bounding and unbounding spin structures. We denote the cardinal numbers of $B_g, U_g$ as $b_g, u_g$, respectively.

**Lemma 4.1.** For $g \geq 1$, $b_g = 2^{2g-1} + 2^{2g-1}$ and $u_g = 2^{2g-1} - 2^{2g-1}$.

**Proof.** For $g = 1$, it is well-known that the only unbounding spin structure on $F_1 = T^2$ is the Lie-group spin structure, so $b_1 = 3$, $u_1 = 1$. In general, any pair of two spin structures $\sigma, \tau \in S(F_g)$, $\delta \in S(T^2)$ determines a bounding (resp. unbounding) spin structure on $F_{g+1} \cong F_g \# T^2$ if and only if $\text{Arf}([\sigma_\delta])$ and $\text{Arf}([\delta_\tau])$ have the same (resp. distinct) parity. This implies $b_{g+1} = b_1 b_g + u_1 u_g = 3b_g + u_g$, and $u_{g+1} = b_1 u_g + u_1 b_g = 3u_g + b_g$, so $b_{g+1} - u_{g+1} = 2(b_g - u_g) = \cdots = 2^g (b_1 - u_1) = 2^{2g+1}$. Using $b_g - u_g = 2^g$ and $b_g + u_g = 2^{2g}$, we see $b_g = 2^{2g-1} + 2^{2g-1}$, $u_g = 2^{2g-1} - 2^{2g-1}$. \(\square\)

There is a natural action of $\text{MCG}_C(F_g)$ on $S(F_g)$, where any $[\tau] \in \text{MCG}_C(F_g)$ acts as the pull-back $\tau^* : S(F_g) \to S(F_g)$.

**Lemma 4.2.** For $g \geq 1$, $\text{MCG}_C(F_g)$ acts invariantly and transitively on $B_g$ and $U_g$.

**Proof.** The invariance of the $	ext{MCG}_C(F_g)$-action on $B_g$ and $U_g$ follows immediately from, for example, counting vanishing elements of the associated quadratic forms $q_\sigma, q_{\tau^*} \sigma$ for $\sigma \in S(F_g)$ and $[\tau] \in \text{MCG}_C(F_g)$. It suffices to prove the transitivity of the action. We argue by induction on $g \geq 1$.

When $g = 1$, $F_1$ is $T^2 \cong S^1_1 \times S^1_1$, and $\text{MCG}_C(T^2) \cong \text{SL}(2, \mathbb{Z})$ is generated by the Dehn twists $D_1, D_2$ along the first and second factors. It is straightforward to check that $\text{MCG}_C(T^2)$ acts transitively on $B_1$ and $U_1$.

Suppose for some $g \geq 1$, $\text{MCG}_C(F_g)$ acts transitively on $B_g$ and $U_g$ for some $g \geq 1$. To see $\text{MCG}_C(F_{g+1})$ acts transitively on $B_{g+1}$, let $\sigma, \sigma' \in B_{g+1}$. Pick a connect sum decomposition $F_{g+1} \cong F_g \# T^2$, which induces a decomposition $H_1(F_{g+1}; \mathbb{Z}_2) \cong H_1(F_g; \mathbb{Z}_2) \oplus H_1(T^2; \mathbb{Z}_2)$. Then $\sigma$ determines spin structures $\sigma_g \in S(F_g)$ and $\delta \in S(T^2)$ so that $[\sigma] = [\sigma_g] + [\delta]$ in $\Omega^{\text{spin}}_2$, and similarly $\sigma'$ determines spin structures $\sigma'_g, \delta'$ so that $[\sigma'] = [\sigma'_g] + [\delta']$. If $[\sigma] = [\sigma'_g]$, and hence $[\delta] = [\delta']$, then by the induction assumption there are $[\tau_\sigma] \in \text{MCG}_C(F_g)$ and $[\tau_\delta] \in \text{MCG}_C(T^2)$ such that $\tau_\sigma^*(\sigma_g) = \sigma'_g, \phi^*(\delta) = \delta'$. Then one finds an element $[\tau] \in \text{MCG}_C(F_{g+1})$, where $\tau = \tau_\sigma \# \phi$, such that $\tau^*([\sigma]) = [\sigma']$.

Now we consider the case if $[\sigma_g] \neq [\sigma'_g]$, and hence $[\delta] \neq [\delta']$. Thus one of $\delta, \delta' \in S(T^2)$ is the Lie-group spin structure, and the other is a spin-boundary, but there always exists some nontrivial $[\alpha] \in H_1(T^2; \mathbb{Z}_2)$ such that $[\delta_\alpha] = [\delta']$. For any $[\beta] \in H_1(T^2; \mathbb{Z}_2)$ with $\alpha \cdot \beta = 1$, that $[\delta] \neq [\delta']$ implies $[\delta] \neq [\delta']$. On the other hand, there is some nontrivial $[\gamma] \in H_1(F_g; \mathbb{Z}_2)$, such that $[\sigma_g] \gamma \neq [\sigma'_g] \gamma$. Let $[\tilde{\beta}] = [\beta] + [\gamma] \in H_1(F_{g+1}; \mathbb{Z}_2)$, we have $\alpha \cdot \tilde{\beta} = 1$, and the difference $\alpha - \alpha' \in H^1(F_g; \mathbb{Z}_2)$ vanishes on $[\alpha]$ and $[\tilde{\beta}]$. We may take two simple closed curve representatives $\alpha, \tilde{\beta} \subset F_{g+1}$ such that $\alpha \cap \tilde{\beta}$ is a single point. A regular neighborhood of $\alpha \cup \tilde{\beta}$ on $F_{g+1}$ is a punctured torus $\hat{T}$, which gives another connected sum decomposition of $F_{g+1} = F_g \# \hat{T}$. It is clear that with respect to this decomposition, the induced spin structures $\tilde{\sigma}_g, \tilde{\sigma}'_g \in S(F_g)$, $\tilde{\delta}, \tilde{\delta}' \in S(F_g)$ satisfy $[\tilde{\sigma}_g] = [\tilde{\sigma}'_g]$, $[\tilde{\delta}] = [\tilde{\delta'}]$, so we apply the previous case to obtain some $[\tilde{\tau}] \in \text{MCG}_C(F_{g+1})$, such that $\tilde{\tau}^*([\sigma]) = [\sigma']$. This means $\text{MCG}_C(F_{g+1})$ acts transitively on $B_{g+1}$.
The proof for the transitivity of the MCG$_C(F_{g+1})$-action on $\mathcal{U}_{g+1}$ is similar, so we complete the induction. □

**Proof of Theorem 1.2.** Because MCG$_{Top}(F_g)$ are represented by self-diffeomorphisms, by Proposition 1.1, any element in $\partial_\text{Top}(i)$ preserves $\iota^*(\varsigma^4) \in B_g \subset S(F_g)$. On the other hand, MCG$_{Top}(F_g)$ acts transitively on $B_g$ (Lemma 4.2). Therefore, $|\text{MCG}_{Top}(F_g) : \partial_\text{Top}(i)| \geq |B_g| = 2^{2g-1} + 2g^{-1}$, (Lemma 4.2). □

For a smoothly embedded oriented closed surface $i : F_g \hookrightarrow \mathbb{R}^4$, the Rokhlin quadratic form $q_1 : H_1(F_g ; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ is defined so that for any smoothly embedded subsurface $P \subset \mathbb{R}^4$ with $\partial P \subset i(F_g)$, $\tilde{P} \subset \mathbb{R}^4 \setminus i(F_g)$ and transverse to $i(F_g)$ along $\partial P$, $q_1([\partial P])$ is the mod 2 number of points in $P \cap \tilde{P}$, where $P'$ is a smooth perturbed copy of $P$ so that $\partial P' \subset i(F_g)$ is disjoint parallel to $\partial P$, and that $\tilde{P}'$ is transverse to $P$, (Ro). In dimension 4, the induced spin structure $\iota^*(\varsigma^4)$ is related to it as follows.

**Lemma 4.3.** The quadratic form $q_{1,\iota^*(\varsigma^4)}$ coincides with the Rokhlin form $q_1$.

**Proof.** To see this, consider a smoothly embedded surface $P \subset \mathbb{R}^4$ as in the definition of $q_1$. Note that the normal vector field of $\partial P$ in $P$ is parallel to the vector field $W$ as in Section 3. There is a trivialization defined by a frame field $(U,V,W,H)|_{\partial P}$ such that for any $x \in \partial P$, $U_x \in T(\partial P)|_x$, $V_x \in N_i(F_g)(\partial P)|_x$, $W_x \in N_P(\partial P)|_x$, and $H_x$ is a complementary vector orthogonal to $U_x, V_x, W_x$. Now $(U,V,W,H) \simeq \varsigma^4|_{\partial P}$ if and only if it extends over $T\mathbb{R}^4|_P$. Since $(U,W)|_{\partial P}$ does not (stably) extend over $TP$, $(U,V,W,H) \simeq \varsigma^4|_{\partial P}$ if and only if $(V,H)|_{\partial P}$ fails to extend (stably) over $N_{4 \mathbb{R}^4}(P)$, i.e. $|P \cap P'|$ is odd. In this case, $\iota^*(\varsigma^4)|_{\partial P}$ is by definition given by $(U,V)|_{\partial P}$ which is the Lie-group spin structure. We conclude $q_1([\partial P]) = 1$ if and only if $q_{\iota^*(\varsigma^4)}([\partial P]) = 1$. □

It is clear from its definition that the Rokhlin form is invariant under the action of $[\tau] \in \text{MCG}_C(F_g)$ if $\tau$ extends diffeomorphically. This would imply $|\text{MCG}_{\text{Diff}}(F_g) : \partial_{\text{Diff}}(i)| \geq 2^{2g-1} + 2g^{-1}$ following Lemmas 1.3, 1.1, 1.2. On the other hand, Hirose showed in [H] that for an unknotted smooth embedding $i : F_g \hookrightarrow \mathbb{R}^4$, i.e. which bounds a smoothly embedded handlebody, $[\tau] \in \partial_{\text{Diff}}(i)$ if and only if $i$ preserves the Rokhlin quadratic form, (cf. also Mq for $g = 1$). Noting that $\partial_{\text{Diff}}(i) \leq \partial_{\text{PL}}(i) \leq \partial_\text{Top}(i)$ under the natural isomorphism $\text{MCG}_{\text{Diff}}(F_g) \cong \text{MCG}_{\text{PL}}(F_g) \cong \text{MCG}_{\text{Top}}(F_g)$, we have $|\text{MCG}_C(F_g) : \partial_\text{C}(i)| = 2^{2g-1} + 2g^{-1}$ for unknotted embeddings.

Corollary 1.4 is an easy consequence of Lemma 4.2.

**Proof of Corollary 1.4.** Observe that the action of MCG$_C(F_g)$ on $S(F_g)$ descends to an action of a group $\Gamma < \text{Aut}^+(H_1(F_g ; \mathbb{Z}_2))$: indeed, if $\tau$ projects to $\text{id} \in \text{Aut}^+(H_1(F_g ; \mathbb{Z}_2))$, $q_{\tau^\ast ([\alpha])} = q_{\tau^\ast ([\tau][\alpha])} = q_{\tau^\ast ([\alpha])}$, for any $[\alpha] \in H_1(F_g ; \mathbb{Z}_2)$, so $\tau^\ast \sigma = \sigma$ for any $\sigma \in S(F_g)$. $\Gamma$ is a finite group isomorphic to $\text{Sp}(2g, \mathbb{Z}_2)$ as it preserves the $\mathbb{Z}_2$-intersection form. Then Lemma 4.2 implies $\Gamma$ acts transitively on $B_g$, so for any $\sigma \in B_g$, $\text{Stab}_\Gamma(\sigma) \subset \Gamma$ has index $b_g = 2^{2g-1} + 2g^{-1}$. Since id $\in \text{Stab}_\Gamma(\sigma)$ for all $\sigma \in B_g$, the subset:

$$W = \bigcup_{\sigma \in B_g} \text{Stab}_\Gamma(\sigma) \subset \Gamma,$$

9
has at most \( b_g \frac{\|j\|}{b_g} - 1 \) + 1 < |\( \Gamma \)| elements. Thus for any \([\tau] \in \Gamma \mathcal{W}, \tau \) does not fix any \( \sigma \in B_g \). In particular, for any smooth embedding \( i : F_g \hookrightarrow \mathbb{R}^4, \partial^i(\gamma) \in B_g \) will not be invariant under \( \tau \). By Proposition 4.1, \([\tau] \notin \mathcal{E}_{\text{top}}(\gamma) \).

\[\square\]

## 5 Embedded \( p \)-tori in \( \mathbb{R}^{p+2} \)

In this section, we prove Theorem 1.3 and its corollaries.

Let \( T^p \) be the standard \( p \)-dimensional torus. Fix a parametrization \( T^p = S^1 \times \cdots \times S^1 \), where each \( S^1 \) is a copy of the unit circle \( S^1 \subset \mathbb{C} \). We start by some general facts about \( \text{MCG}_C(T^p) \) and its action on the space \( \mathcal{S}(T^p) \) of spin structures on \( T^p \).

For any \( p \geq 2 \), \( \text{Hom}_{\mathbb{C}}(T^p) \) has a modular subgroup \( \text{Mod}(T^p) \cong \text{SL}(p, \mathbb{Z}) \) generated by elements represented by the Dehn twists \( \tau_{i,j} (1 \leq i, j \leq p, i \neq j) \) along the \( i \)-th factor in the \( S^1 \times S^1 \) direction, namely, \( \tau_i(u_1, \cdots, u_p) = (u_1, \cdots, u_{i-1}, u_{i+1}, u_i, u_{i+1}, \cdots, u_p) \). \( \text{Mod}(T^p) \) may identically be regarded as a subgroup of \( \text{MCG}_C(T^p) \) under the natural quotient \( \pi_0 : \text{Hom}_{\mathbb{C}}(T^p) \to \text{MCG}_C(T^p) \). Thus the action of \( \text{MCG}_C(T^p) \) on \( H_1(T^p; \mathbb{Z}) \) induces a splitting sequence of groups:

\[
1 \to \mathcal{J}(T^p) \to \text{MCG}_C(T^p) \to \text{SL}(p, \mathbb{Z}) \to 1,
\]
as \( \text{Aut}^+(H_1(T^p; \mathbb{Z})) \cong \text{SL}(p, \mathbb{Z}), \) (which holds trivially for \( p = 1 \) as well). In other words, \( \text{MCG}_C(T^p) = \mathcal{J}(T^p) \times \text{Mod}(T^p) \). It is well-known that \( \text{MCG}_C(T^2) = \text{Mod}(T^2) \) (cf. [HK]), and \( \text{MCG}_C(T^3) = \text{Mod}(T^3) \) follows from general results of Hatcher for Haken 3-manifolds ([H1], [H2]). While the case \( p = 4 \) remains mysterious, for \( p \geq 5 \), the splitting is known to be nontrivial and \( \text{MCG}_C(T^p) \) are different for various \( C \)'s. Specifically, a theorem of Hatcher ([H2, Theorem 4.1], cf. also [HS]) implies \( \mathcal{J}(T^p) \) (\( p \geq 5 \)) is an infinitely generated abelian group, which can be regarded as a \( \text{SL}(p, \mathbb{Z}) \)-module with the following decomposition:

\[
\mathcal{J}_{\text{diff}}(T^p) \cong \mathcal{W}_p \oplus H^2(T^p; \mathbb{Z}_2) \oplus \bigoplus_{i=1}^{p} H^i(T^p; \Gamma_{i+1}),
\]
and \( \mathcal{J}_{\text{pl}}(T^p) \cong \mathcal{W}_p \oplus H^2(T^p; \mathbb{Z}_2) \), \( \mathcal{J}_{\text{top}}(T^p) \cong \mathcal{W}_p \) as induced by the forgetting quotients. Here \( \mathcal{W}_p \cong \mathbb{Z}_2[t_1, t_1^{-1}, \cdots, t_p, t_p^{-1}] / \mathbb{Z}_2[t_1 + t_1^{-1}, \cdots, t_p + t_p^{-1}] \cong \mathbb{Z}_2^{\oplus \infty} \) has the natural action induced by that of \( \text{SL}(p, \mathbb{Z}) \) on the monomials, and \( \Gamma_i \) is the \( i \)-th Kervaire-Milnor group of homotopy spheres which is finite abelian, \( i \geq 0 \), and the \( \text{SL}(p, \mathbb{Z}) \) acts on \( H^2(T^p; \mathbb{Z}_2), H^i(T^p; \Gamma_{i+1}) \) naturally as usual.

As the space of spin structures \( \mathcal{S}(T^p) \) is an affine \( H^1(T^p; \mathbb{Z}_2) \), there is a Lie-group spin structure and \( 2^p - 1 \) non-Lie-group spin structures. Denote the subset of non-Lie-group spin structures as \( \mathcal{S}^*(T^p) \).

The lower bound in Theorem 1.3 follows from the lemma below.

**Lemma 5.1.** \( \text{MCG}_C(T^p) \) fixes the Lie-group spin structure of \( T^p \), and acts transitively on \( \mathcal{S}^*(T^p) \). Hence \( [\text{MCG}_C(T^p) : \mathcal{E}_{\mathcal{C}}(i)] \geq 2^p - 1 \) if \( i : T^p \to \mathbb{R}^{p+2} \) induces a non-Lie-group spin structure \( \partial^i(S^{p+2}) \) on \( T^p \).

**Proof.** For the standard parametrization \( u = (u_1, \cdots, u_p) \) of \( T^p = S^1 \times \cdots \times S^1 \), the Lie group spin structure \( \sigma_0 \in \mathcal{S}(T^p) \) is represented by the standard framing \( (\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_p}) \) over \( T^p \), so for any
τ ∈ Mod(Tp), τ∗(∂u1,..., ∂un)|u = (∂u1,..., ∂un)|τ(u). A for the matrix A ∈ SL(p, Z) defining τ for any u ∈ Tp. This means pulling-back by τ fixes the framing over Tp up to homotopy, so τ∗(σ0) = σ0. On the other hand, FC(Tp) fixes σ since the action of MCGc(Tp) descends to Aut+(H1(Tp)) ∼= SL(p, Z). Thus MCGc(Tp) fixes σ0.

Let α′, α′′ ∈ S∗(Tp), the differences α′ − σ0, α′′ − σ0 ∈ H1(Tp; Z2) \ {0}. As MCGc(Tp) acts transitively on H1(Tp; Z2) \ {0} and fixes σ0, there is some [τ] ∈ MCGc(Tp) such that τ∗(α′) = α′′. Thus MCGc(Tp) acts transitively on S∗(Tp).

Finally, by Proposition 1.1, τ ∈ EC(i) only if τ fixes i(2εp + 2), so the transitivity implies |MCGc(Tp) : EC(i)| ≥ |S∗(Tp)| = 2p − 1 if i(2εp + 2) ∈ S∗(Tp).

A little more can be said about EC(i) for general smooth embeddings of Tp into Rp+2.

Lemma 5.2. For p ≥ 1, and for any smooth embedding i : Tp ↪ Rp+2, FC(Tp) ∩ EC(i) has finite index in FC(Tp). Moreover, FCtop(Tp) ≤ ECtop(i).

Proof. Without loss of generality, we may assume p ≥ 4 as FC(Tp) is trivial when p ≤ 3.

First suppose p ≥ 5. In this case, it suffices to show WP ≤ EDiff(i). Let [τ] ∈ WP where τ is a diffeomorphic representative. By Remark (4) of [Ha2 Theorem 4.1], τ is smoothly concordant to id, namely, there is a diffeomorphism f : Tp × [0, 1) → Tp × [0, 1], such that f|Tp×{0} = τ, f|Tp×{1} = idTp. Let Jr, Jr be the first and the second component of f, respectively, i.e. such that f(u, r) = (fR(u, r), fI(u, r)). Pick be a tubular neighborhood N ⊃ Tp × D2 of i(Tp) in Rp+2. Identify D2 as the unit disk of C, and define ˜τ : Tp × D2 → Tp × D2 by ˜τ(u, r eιθ) = (fR(u, r), fI(u, r) r eιθ). It is clear that ˜τ is an orientation-preserving diffeomorphism which restrict to Tp × ∂D2 as identity. We may define an orientation-preserving diffeomorphism ˆτ : R4 → R4 by extending ˜τ as identity outside N, which extends τ. This shows [τ] ∈ EDiff(i).

For p = 4, let [τ] ∈ FC(T4) where τ is a C-homeomorphic representative. Pick be a tubular neighborhood N ⊃ T4 × D2 of i(T4) in R6. We first define ˜τ : T4 × D2(4) → T4 × D2(4) as τ × idD2(4), where D2(4) is the disk of radius one half. ˜τ restricted to T4 × ∂D2(4) may be regarded as an element of FC(T5). If it lies in WP, then there is a C-concordance f : T5 × [0, 1] → T5 × [0, 1] between ˜τ|T5×∂D2(4) and the identity obtain by joining a C-isotopy between ˜τ|T5×∂D5(4) and a diffeomorphic representative φ ∈ [τ] with a smooth concordance between φ and the identity. As T5 × [0, 1] ∼= T4 × (D2 \ D2(4)), we may extend ˜τ using the C-concordance f over N such that ˜τ|∂N is the identity. Further extend ˜τ outside N by the identity, we see [τ] ∈ EC(i). This means the preimage of WP under:

FC(T4) → FC(T5),

defined by [τ] → [τ × idS1], is contained in EC(i). Since WP has finite index in FC(T5), we conclude FC(T4) ∩ EC(i) has finite index in FC(T4) as well. Moreover, FCtop(T4) ≤ ECtop(i) since WP = FCtop(T5).

We proceed to consider unknotted embeddings of Tp into Rp+2. These have been defined and studied in [DLWY]. We recall the notion and properties enough for our use here. Regard S1 and D2 as the unit circle and the unit disk of C, respectively. The standard basis of Rn is (ε1,..., εn), and the m-subspace spanned by (εi1,..., εim) will be written as Rm i1,...,im, and hence Rn = R1,...,n ⊂ Rn+1.
Example 5.3 (The standard model). Let \( t_0 : pt = T^0 \to \mathbb{R}^2 \) be \( t_0(pt) = 0 \) by convention. Inductively suppose \( t_{p-1} \) has been constructed for some \( p \geq 1 \) such that \( t_{p-1}(T^{p-1}) \subset D^p \subset \mathbb{R}^2 \) for \( p \geq 1 \). Denote the rotation of \( \mathbb{R}^{p+2} \) on the subspace \( \mathbb{R}^2_1 \subset \mathbb{R}^2_{2, \ldots, p+1} \). Therefore, \( [\text{MCG} \mathbb{R}^{p+2}] \) is the group of automorphisms of \( \mathbb{R}^{p+2} \) denoted as Aut\( (\mathbb{R}^{p+2}) \). The standard model \( \mathbb{R}^{p+2} \) is defined as follows:

\[
\begin{align*}
\mathbb{R}^{p+2} & \cong \mathbb{R}^2_1 \times \mathbb{R}^2_{2, \ldots, p+1} \\
\ & \cong \mathbb{R}^{p+2}.
\end{align*}
\]

For any unknotted embedding \( \mathcal{E}_1 \) of \( T^p \), the induced spin structure \( \mathcal{E}_1(T^p) \) is the spin boundary of a spin structure on \( T^p \). However, the spin structure \( \mathcal{E}_1 \) is not the Lie-group spin structure on \( T^p \).

**Proof.** One can easily see that the standard embedding \( t_p : T^p \to \mathbb{R}^{p+2} \) is a diffeomorphism \( t : \mathbb{R}^{p+2} \to \mathbb{R}^{p+2} \) such that \( t \) and \( g \circ t_p \) have the same image, i.e. \( t(T^p) = g \circ t_p(T) \).

**Lemma 5.5.** For any unknotted embedding \( \mathcal{E}_1 : T^p \to \mathbb{R}^{p+2} \), the induced spin structure \( \mathcal{E}_1(t(T^p)) \) is not the Lie-group spin structure on \( T^p \).

**Proof.** By the proof of Proposition \[ \ref{prop:spin-boundary} \] (\( T^p, \mathcal{E}_1(t(T^p)) \)) is the spin boundary of a spin structure on \( \Sigma \). However, the spin structures on \( \Sigma \cong D^2 \times T^{p-1} \) are \( \zeta^2 \oplus \sigma \), where \( \sigma \in S(T^{p-1}) \), and these induce \( \partial \zeta^2 \oplus \sigma \) on \( \partial \Sigma \cong S^1 \times T^{p-1} \), which disagree with the Lie-group spin structure along the loop \( S^1 \times \ast \).

**Proof of Theorem \[ \ref{thm:main} \].** Lemma \[ \ref{lem:standard-model} \] proves \( |\text{MCG} \mathbb{R}^{p+2}(T^p) : \mathcal{E}_1(T^p) | \geq 2^p - 1 \). To see that any unknotted embedding \( \mathcal{E}_1 : T^p \to \mathbb{R}^{p+2} \) realizes the lower bound, note \( \text{MCG} \mathbb{R}^{p+2}(T^p) = \mathcal{A}(T^p) \times \text{Mod}(T^p) \). By Lemma \[ \ref{lem:inv-MCG} \] \( \mathcal{A}(T^p) \leq \mathcal{E}_1(t(T^p)) \). On the other hand, [DLWY14] Theorem 1.4 showed \( \text{Mod}(T^p) \) (denoted as \( \text{Aut}(T^p) \) there) has a subgroup of index \( 2^p - 1 \) which is diffeomorphically extendable. Therefore, \( |\text{MCG} \mathbb{R}^{p+2}(T^p) : \mathcal{E}_1(t(T^p)) | \leq 2^p - 1 \), and hence the index is exactly \( 2^p - 1 \).

**Proof of Corollary \[ \ref{cor:main} \].** This follows from Rokhlin’s theorem that any closed spin 4-manifold \( X \) has signature 0 modulo 16, (cf. [K101 Theorem III.1.1]). In fact, if \( \Sigma \) is a Seifert hypersurface of \( \mathcal{E}_1 \), the
proof of Proposition 5.1 implies $\Sigma$ has a spin structure inducing $\varphi^*(\varsigma^3)$ on the boundary $T^3$. If it is the Lie-group spin structure, one can find a compact spin 4-manifold $N$ of signature $8 \mod 16$ (cf. [Ki Chapter V], also [SST Proposition 6.1]) with $\partial N = T^3$, such that one can glue $\Sigma$ and $N$ along boundary to obtain a closed spin 4-manifold $X$. Then $\text{sig}(\Sigma) + 8 \equiv \text{sig}(\Sigma) + \text{sig}(N) = \text{sig}(X) \equiv 0 \mod 16$ would imply $\Sigma$ has signature $8 \mod 16$, which violates the assumption. Thus from $\varphi^*(\varsigma^3)$ is not the Lie-group spin structure on $T^3$, and Theorem 1.3 holds in this case. Note also that in this case, any Seifert hypersurface of $i$ has signature $0 \mod 16$, (cf. for example, [SST Proposition 6.1]).

To prove Corollary 1.8, we need an elementary lemma in group theory.

**Lemma 5.6.** If $G$ is a subgroup of a semi-direct product of groups $N \rtimes H$, then $[N \rtimes H : G] \leq [N : N \cap G] \cdot [H : H \cap G]$.

**Proof.** Let $N' = N \cap G$, $H' = H \cap G$. Clearly $H'$ preserves $N'$ under the conjugation, so the subgroup $N'H'$ is also a semi-direct product. Note $[NH : N'H'] = [NH : NH'] \cdot [NH' : N'H']$. As $N$ is normal in both $NH$ and $NH'$, quotienting out $N$ yields $[NH : NH'] = [H : H']$. Because $N \cap N'H' = N'$ as $N'H'$ is a semi-direct product, the map $N \to NH'/N'H'$ descends to a bijection $N/N' \to NH'/N'H'$ between the cosets, so $[NH' : N'H'] = [N : N']$. Thus $[NH : G] \leq [NH : N'H'] = [N : N'] \cdot [H : H']$.

**Proof of Corollary 1.8.** By Lemmas 5.1, 5.5, By Lemma 5.2, $[\mathcal{F}(T^p) : \mathcal{F}(T^p) \cap \mathcal{E}(i)]$ is finite. By [DLWY] Theorem 1.4, $[\text{Mod}(T^p) : \text{Mod}(T^p) \cap \mathcal{E}(i)]$ is finite. Therefore, as $\text{MCG}_C(T^p) = \mathcal{F}(T^p) \rtimes \text{Mod}(T^p)$, $[\text{MCG}_C(T^p) : \mathcal{E}(i)]$ is also finite by Lemma 5.6. Note clearly $[\text{MCG}_C(T^p) : \mathcal{E}(i)] = 2^p - 1$ when $p \leq 3$.

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