We examine the set of objects which can be built in type IIA string theory by matrix methods using an infinite number of D0-branes. In addition to stacks of ordinary Dp-branes and branes in background fields, we find exotic states which cannot be constructed by other means. These states exhibit strongly noncommutative geometry, (e.g., partial derivatives on them do not commute) and some are conjectured to have $\mathbb{Z}_N$-valued charges similar to those of the type I D-instanton. Real-valued charges are forbidden by Dirac quantization, leading to a nontrivial relationship between noncommutative topological invariants.

It is already widely known how several p-brane type objects may be built out of zero-branes by matrix methods; flat membranes, compact membranes of various geometries, and 4-branes. [1] The purpose of this paper is to determine what the most general state is that can be built by matrix methods in IIA string theory with an infinite number of 0-branes. The guiding idea is that, with a finite number of branes, one sees what looks like a higher D-brane wrapping a fuzzy space. (Like the usual fuzzy sphere construction) However, infinite matrices allow a lot of qualitatively more exciting things to happen; what one sees is a D-brane wrapping a generic noncommutative space.[29] All of the ordinary D-branes of type IIA can be constructed out of (infinitely many) zero-branes in this manner, as well as D-branes in background fields; but more interestingly, there are a large number of exotic states which appear as bound states of infinitely many zero-branes which cannot be constructed by other means. These states do not necessarily have good finite-N approximations; they exist only in the strict large-N limit. Finally, I can conjecture based on these results (but not, alas, prove) that some of these exotic D-branes have charges taking values in $\mathbb{Z}_N$ rather than $\mathbb{Z}$. (This is like the D-instanton in type I, which has a $\mathbb{Z}_2$-valued charge) The exclusion of real-valued charges by Dirac quantization leads to an interesting relationship between two noncommutative topological invariants.

All of these results hold at the level of the corrected non-Abelian brane action; in fact, I was recently pleasantly surprised when a paper appeared [5] computing the corrections to the Born-Infeld action to order $\alpha'^4 F^8$, giving a result consistent with those given here. However, calculating beyond the Yang-Mills level requires a great deal of mathematical machinery (pullbacks of forms onto noncommutative spaces and so on) which is not particularly physically illuminating, so the body of this paper will work at the Yang-Mills level, and the demonstration that this continues to work at all orders is left to the appendix. Also, nontrivial backgrounds (a metric with cycles or nonzero background fields, for example) add significant complications, so here I will deal only with trivial backgrounds, with occasional notes on the generalizations which would be needed for other situations.

This paper is therefore divided as follows: I will begin by writing down the action for an infinite number of D0-branes, which is parametrized by an algebra $A$ and a collection $X^\mu$ of ten covariant derivatives. (Plus the fermions $\theta^\alpha$, of course, but these are suppressed for notational clarity) Using this, I demonstrate that certain choices of $(A, X)$ correspond to stacks of coincident D-branes wrapping any given cycle, to D-branes in a background B-field, and to states which correspond to neither of these. The natural question to ask about these states is which of them are stable; the result is that there are two topological invariants, both of which must be nonzero for there to be a stable bound state. The first of these is an element of noncommutative K-theory which takes the role of the ordinary conserved charge; it obstructs decay of the brane to closed string states by nucleation of brane-antibrane pairs. The second is something called Hochschild cohomology (explained below) which obstructs processes such as the classical collapse of the fuzzy sphere down to a point under its own gravity.[30] Applied to ordinary D-branes, this leads to the usual K-theoretic conditions for stability; for noncommutative branes, more interesting things may happen, such as the above-mentioned exotic states.

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I. SETTING UP THE ACTION AND BUILDING ORDINARY BRANES

The Yang-Mills action for a finite collection of D0-branes is [31]

\[ S = \mu_0 \int dt \text{Tr} \left( \frac{1}{4} F^2 + \text{fermions} + RR \text{ fields} \right), \]

where \( F_{\mu\nu} = [X_\mu, X_\nu] \) is the field strength and \( \mu_0 = (2\pi/g)(2\pi l_s)^{-1} \). Certain aspects of its generalization to infinite \( N \) are obvious; the fields now take values in an algebra \( A \) more general than \( M_N \), and the trace becomes a trace on the algebra. What happens to \( X \) is a bit more subtle. \( X \) is not an ordinary (\( A \)-valued) field, but rather a derivation on \( A \). (i.e., a map \( A \to A \) that satisfies the product rule) When we say that \( X \) is some matrix \( \hat{X} \), we really mean that as a derivation, \( X(f) = [\hat{X}, f] \); this satisfies the product rule thanks to the Jacobi identity. One obvious way to see that \( X \) is a derivation rather than a field is that in the \( p \)-brane action, the components of \( X \) parallel to the brane become ordinary covariant derivatives.

It is worth recalling a few of the properties of derivations on algebras. (This is summarized from [6], a good general textbook on the subject) The set \( \Delta(A) \) of all derivations on \( A \) forms a Lie algebra, since the commutator of two derivations is also a derivation. A theorem states that every continuous automorphism of \( A \) can be written in the form \( a \to e^{i\theta} a \), where \( \theta \) is a real parameter and \( \partial \) is some derivation; thus this Lie algebra is the algebra of infinitesimal automorphisms of \( A \). (The proof is very similar to the derivation of angular momentum from rotational symmetry in quantum mechanics) For example, when \( A = C(\mathbb{R}^n) \), the ordinary partial derivatives form a basis for \( \Delta(A) \), and the automorphisms are general coordinate transformations. When \( A = C(\mathbb{R}^n) \otimes M_k \), (so the elements of the algebra are matrix-valued functions) a basis for the derivations is \( \partial_i \otimes 1 \) and \([a, \cdot] \), where \( a \) is any matrix-valued function; the automorphisms generated by this new generator are \( U(k) \) gauge transformations. (Thus \( \Delta(A) \) is truly the symmetry algebra of the system)

Note that, since \( X \in \Delta(A) \), it transforms under this symmetry algebra by commutation. (i.e., the ordinary action of a Lie algebra) Thus \( X \) is manifestly a gauge-covariant quantity, as is \( F = [X, X] \). In terms of this quantity, the action (1) is also valid at infinite \( N \). Alternatively, one may absorb the integral (and thus the time coordinate) into an additional \( C(\mathbb{R}) \) factor in \( A \); then using the fact that the trace on \( C(\mathbb{R}) \) is \((2\pi l_s)^{-1} \int dt \), (the coefficient being for later convenience and dimensional consistency) we may write the action at infinite \( N \) in the simpler form

\[ S = \frac{\pi}{2g} \text{Tr}_A F^2 . \]

When \( A = C(\mathbb{R}) \otimes M_N \), the possible derivations are \( \partial_i \) and \([a, \cdot] \). We set \( X_0 = \partial_t + A_0 \) and \( X_i = \langle X_i \rangle + A_i \); we can always choose our basis so that none of the \( \langle X_i \rangle \) contain \( \partial_t \). This action then immediately reduces to the ordinary finite-\( N \) action. When \( A = C(\mathbb{R}^{p+1}) \otimes M_N \), \( p + 1 \) of the \( \langle X \rangle \)'s can be partial derivatives, and the trace is \((2\pi l_s)^{-p-1} \int d^{p+1} \xi \text{Tr}_N \), so the action (2) becomes

\[ S_{C(\mathbb{R}^{p+1}) \otimes M_N} = \frac{2\pi}{g(2\pi l_s)^{p+1}} \int d^{p+1} \xi \text{Tr}_N \frac{1}{4} F^2 , \]

the standard action for a stack of Dp-branes. Similarly, when \( A = C(\mathbb{R}) \otimes A_0 \), the action looks structurally like the D2-brane action, but the multiplication rule for functions is now the Moyal star product; the resulting action is that of a 2-brane in a constant background B-field.[32] Note that these are all the same action, with different choices for \( A \).

Three related points have been subtly glossed over here. First, how do we determine which derivatives are available? Second, how is it that \( A \) is ordinarily thought of as an \( A \)-valued field, one which cannot take the value (e.g.) \( \partial_t \), rather than a derivation, while \( X \) is a generic derivation? Third, the trace is normally defined in terms of ordinary elements of \( A \), but equation (2) includes a trace of a derivation. In the simple cases considered above, there seemed to be a natural interpretation, but how should this be generalized?

Note that, for any algebra \( A \) and any \( a \in A \), \([a, \cdot] \) is a derivation. The set \( \Delta_I(A) \) of all such derivations is called the set of inner derivations, and it is a Lie subalgebra of \( \Delta(A) \). The first Hochschild cohomology class \( H^1(A) \) is defined to be the coset \( \Delta(A)/\Delta_I(A) \), forming a group under addition; it is a known topological invariant of algebras. [8] For \( A = C(\mathbb{R}^n) \), all inner derivations are clearly zero, so \( H^1 \) is generated by the ordinary partial derivatives. For \( A = M_N \), one can show that \( H^1 \) is zero, so all derivations are inner. Unfortunately, \( H^1 \) is in general very difficult to calculate; the known results are that

1. When \( \mathcal{M} \) is a manifold, \( H^1(C(\mathcal{M})) \) is generated by the ordinary derivatives on \( \mathcal{M} \), and its dimension is equal to the dimension of \( \mathcal{M} \).
2. $H^1(\mathcal{A}_0)$ is two-dimensional, generated by the ordinary partial derivatives with respect to $x_1$ and $x_2$. [9]

3. When $\mathcal{A}$ is a von Neumann algebra (a class of algebras that includes $M_N$, $\mathbb{K}$ and $\mathbb{B}[33]$) $H^1(\mathcal{A})$ is trivial.

4. Higher Hochschild cohomology classes (see the references for details) obey the Künneth formula

$$H^n(\mathcal{A} \otimes \mathcal{B}) = \bigoplus_{p+q=n} H^p(\mathcal{A}) \otimes H^q(\mathcal{B}) ,$$

and $H^0(\mathcal{A})$ is trivial for every $\mathcal{A}$; thus

$$H^1(\mathcal{A} \otimes \mathcal{B}) = H^1(\mathcal{A}) \oplus H^1(\mathcal{B}) .$$

5. Hochschild cohomology is continuous over inductive limits, i.e. $H^n(\lim_{\to} \mathcal{A}_i) = \lim_{\to} H^n(\mathcal{A}_i)$. This follows from theorem 21.3.1 of [13]. In particular, $H^1(\mathcal{A}) = 0$ whenever $\mathcal{A}$ is an AF-algebra. (An AF-algebra is a C*-algebra that is the limit of a sequence of finite-dimensional algebras)

$H^1$ is a vector space over $\mathcal{A'}$, the set of elements of $\mathcal{A}$ which commute with all of $\mathcal{A}$.

This answers the first question, of how the set of allowable derivations was determined. The answer to the second question is that the fluctuation $A$ of $X$ is an “ordinary field in $\mathcal{A}$,” i.e. an inner derivation in $\Delta(\mathcal{A})$. The physical intuition of this is that $A$ should correspond to a bounded fluctuation, which a partial derivative such as $\partial$ is not. (Its eigenvalues spread over all of $\mathbb{R}$). Another way to show this is that $A$ is an inner derivation for the same reason that on commutative space it is a function: if one begins from the definition of a bundle in terms of transition functions, the connection is the Poincaré dual of the derivative of the log of the transition function, and therefore an ordinary 1-form. (With function-valued coefficients) This argument continues unchanged in the noncommutative case. [17]

The third question can now be answered by saying that the trace acting on an inner derivation is the ordinary algebraic trace, and acting on an outer derivation

$$\text{Tr} \, \partial = 0 .$$

This is simply the fundamental theorem of calculus, generalized to noncommutative space.

Note that adding a B-field (even a nonconstant one) to an ordinary brane will not create all of these states, since a B-field can deform the algebra by changing $[x_i, x_j]$ but it leaves $[\partial_i, \partial_j] = 0$. Since $\Delta(\mathcal{A})$ is a Lie algebra, in general partial derivatives on these noncommutative spaces do not commute!

II. BRANE STABILITY AND SOME EXTRAORDINARY BRANES

We now wish to determine the criteria for a configuration specified by some $\mathcal{A}$ and $\langle X \rangle$ to be stable. We cannot, in general, rely on supersymmetry alone since an arbitrary brane configuration preserves no supercharges. Even in the commutative case, the BPS condition alone is not sufficient to describe stability, as the stability constraints often involve nontrivial anomaly cancellations. These are summarized in the commutative case by K-theory, which is known to reproduce the correct phases in the M-theory partition function [14] and to summarize all BPS and anomaly conditions. [15] We would like to generalize these considerations to noncommutative branes.

In our case there are two stability issues to consider: quantum-mechanical decay to closed string states and semiclassical collapse similar to that of the unsupported fuzzy sphere. The first of these two concerns whether or not the brane can decay to closed string states by nucleating brane-antibrane pairs. (Since there are only branes in this theory, this is the only type of quantum process which can occur) In order to do this, we need to consider stacks of noncommutative branes and antibranes.

The most convenient language for this is the language of modules.[34] Modules over $\mathcal{A}$ are the noncommutative version of bundles over a space, in the following way. On a commutative space $\mathcal{M}$, to every bundle $\mathcal{E}$ there corresponds the $C(\mathcal{M})$-module $\Gamma(\mathcal{E})$ of its sections. The Serre-Swan theorem [16] states that this is actually a duality between the set of all bundles on $\mathcal{M}$ and the set of all finitely generated, projective modules over $C(\mathcal{M})$.[35] For noncommutative spaces, we simply define a “bundle” to be such a module over $\mathcal{A}$. Now on a commutative space, there is an obvious relationship (again, a pairing) between a covariant derivative $X$ and its associated bundle; this relationship extends to the noncommutative case [17], so we can associate to the covariant derivative $X$ an $\mathcal{A}$-module which (by abuse of notation) we also call $X$. Physically, this corresponds to the boundary state description of a brane; the elements of $\mathcal{A}$ are composed of position and momentum operators, and are thus open string operators, and the module $X$ is a collection of kets on which they act, i.e. boundary states. The module notation is simply the boundary state description in the noncommutative case.
The reason we want to use this formalism is that it makes it easy to consider systems of multiple branes. Consider two noncommutative branes corresponding to modules \( X \) and \( Y \). Then placing both branes together corresponds to the module \( X \oplus Y \), with the operators acting on them being \( 2 \times 2 \) matrices in \( \mathcal{A} \). (These correspond to \( XX \) and \( YY \) strings on the diagonal, and \( XY \) strings on the off-diagonal) Similarly a stack of \( N \) branes would be described by a direct sum of modules, acted upon by \( M_N(\mathcal{A}) \). These clearly form an additive semigroup under \( \oplus \).

Now imagine that we take a brane \( X \) and an antibrane \( Y \). Again the module is \( X \oplus Y \), acted upon by a \( 2 \times 2 \) matrix of elements of \( \mathcal{A} \), but now the off-diagonal elements correspond to \( DD \) strings and thus have the opposite GSO projection. [18] In operator language, this corresponds to a graded algebra where the diagonal elements of \( M_2(\mathcal{A}) \) have positive sign and the off-diagonal elements have negative sign; the total GSO projection is the product of the ordinary GSO projection and this sign factor. For notational clarity, we write \( X \) as \( (X,0) \) and \( Y \) as \( (0,Y) \), emphasizing that there are modules and “antimodules;” their sum is the graded module \( (X,Y) \).

We can now describe the quantum stability condition. We want to allow the brane (initially described by some \( (X,Y) \)) to undergo nucleation of an arbitrary brane-antibrane pair, described by \( (Z,Z) \) for some \( Z \). Thus the brane is defined only up to the equivalence relation

\[
(X,Y) \sim (X \oplus Z, Y \oplus Z).
\]

Using the fact that \( X \) and \( Y \) are finitely generated projective modules, we know that there is some \( Z \) for which \( Y \oplus Z = 0 \); thus without loss of generality, we can continue to denote our brane by \( X \) alone. The set of graded modules under addition, modulo this equivalence relation, is precisely the definition of the group \( K_0(\mathcal{A}) \), the noncommutative generalization of K-theory:[36] modulo brane nucleation, \( X \) is defined only as an element of this group.[37]

Two things can help clarify this: comparing it to the analogous commutative calculation and examining known cases. In the commutative case, one begins with a stack of space-filling 9-branes described by a bundle \( E \). (This bundle is a collection of ten covariant derivatives, analogous to our \( \mathcal{A} \).) In the commutative case, one begins with a stack of space-filling 9-branes described by a bundle \( E \). (This bundle is a collection of ten covariant derivatives, analogous to our \( \mathcal{A} \).) In the commutative-case, one begins with a stack of space-filling 9-branes described by a bundle \( E \). (This bundle is a collection of ten covariant derivatives, analogous to our \( \mathcal{A} \).) In the commutative case, one begins with a stack of space-filling 9-branes described by a bundle \( E \). (This bundle is a collection of ten covariant derivatives, analogous to our \( \mathcal{A} \).)

In our formalism, we used zero-branes but their infinite number allows all components to be turned on as well.) Stacks of branes and antibranes can be described by pairs of bundles \( (E,F) \), again with opposite GSO projections for brane-antibrane strings. By creating a brane-antibrane pair, we transform this to \( (E \oplus H,F \oplus H) \), and so we say that the brane state is only defined modulo transformations of this sort. The set of bundles on the spacetime \( \mathcal{M} \) modulo this relationship defines the commutative K-theory class \( K^0(\mathcal{M}) \sim K_0(C(\mathcal{M})) \). Then using the Sen construction [20] to build lower branes out of higher branes by repeated anihilation, this result can be extended to \( p < 9 \), giving the same result where now \( \mathcal{M} \) is the manifold wrapped by the brane. [18]

Several things should be noted about this. First, the commutative argument has a remarkably similar structure to our argument above, in that it involves pairs of branes and antibranes forming a group under addition modulo anihilation. They differ in that the noncommutative construction begins from 0-branes and “builds upwards” to form higher branes by noncommutative methods, while the commutative argument begins from 9-branes and “builds downwards” via the Sen conjecture to form lower branes. Also, the commutative argument began from stable 9-branes, and is therefore appropriate to type IIB; there is an analogous argument for type IIA string theory [22] which gives the result that \( X \in K^{-1}(\mathcal{M}) \), where \( \mathcal{M} \) is the spatial part of the manifold wrapped by the brane, and \( K^{-1} \) is a certain higher K-class. (Its details are discussed in the references; we do not need them here.)

We can (and should) compare our results to these in the cases where they both coincide, namely \( \mathcal{A} = C(\mathbb{R}^{p+1}) \oplus M_N \). In order to do this, we must first be a bit more careful about what we mean by \( C(\mathbb{R}) \). Normally this is used to denote \( C_0(\mathbb{R}) \), the set of complex-valued functions of \( \mathbb{R} \) vanishing at infinity. (This is the algebra dual to \( \mathbb{R} \) by the standard pairing of algebras and spaces) This is clearly correct for the spacelike components of \( \mathcal{A} \), since we want to consider gauge fluctuations which become trivial at spacelike infinity. However, it is not correct for the time component; there is no boundary condition that forces \( X \) to go to zero at timelike infinity! Instead, the time component ought to be \( C(\mathbb{R}_\infty) \), the set of continuous functions on \( \mathbb{R} \cup \{-\infty, \infty\} \). (So that a nontrivial boundary condition may be set at \( t = \pm \infty \)) This interval is homeomorphic to the closed unit interval \([0,1]\), which is homeomorphic to a point, so

\[
C(\mathbb{R}_\infty) \otimes C_0(\mathbb{R}^p) \sim C(\mathbb{R}_\infty) \otimes C(\mathbb{R}^p) \sim C_0(\mathbb{R}^p).
\]

Since \( K_0 \) is invariant under homeomorphisms of algebras, we can perform this substitution inside \( K_0 \). Also, by the stability theorem for K-theory \( K_0(\mathcal{A} \otimes M_N) = K_0(A) \) for any \( \mathcal{A} \). Thus \( K_0(C(\mathbb{R}_\infty) \otimes C_0(\mathbb{R}^p) \otimes M_N) = K_0(C_0(\mathbb{R}^p)) \), which (by a standard result) is \( \mathbb{Z} \) when \( p \) is even and the trivial group when \( p \) is odd.[23]

Physically, this means that for \( p \) even, the set of modules \( X \) up to brane nucleation is isomorphic to the integers, and so there is an integer-valued charge associated with \( X \) which is not changed by nucleation. If one examines the calculation of \( K_0(C_0(\mathbb{R}^p)) \) in detail, (see references) one finds that this charge is simply the number of branes. When \( p \) is odd, however, every \( X \) is equivalent (under brane nucleation) to the trivial module, so this \( \mathcal{A} \) leads to unstable branes for all \( X \). This is clearly the right answer for type IIA string theory.

This concludes the discussion of quantum stability and K-theory. The second stability issue occurs once we have ‘fixed’ this, i.e. once \( \mathcal{A} \) is given and \( X \) is a module (i.e. a covariant derivative) which represents a nonzero class...
in $K_0(A)$. We are now left with possible classical fluctuations of $X$, i.e. continuous variations of $A$. The potential stability issue is one that was alluded to before, namely the possibility of gravitational collapse down to a point – a situation which, while it does include an infinite number of 0-branes, is certainly not particularly interesting. At a simple level, this happens because (if we fix timelike gauge $A_0 = 0$) the potential is

$$V = \text{Tr}_A \frac{1}{4}(X_i, X_j)^2$$

(8)

where the indices go over space coordinates $1 \ldots 9$. The minimum of this clearly happens when the $X$’s commute, but in general even partial derivatives on noncommutative spaces don’t commute ($\Delta(A)$ is, after all, a Lie algebra) and so the minimum ends up at $X = 0$ rather than $A = 0$. (This is exactly what happens for the fuzzy sphere, where the $\langle X \rangle$ are a set of matrices describing a sphere at finite $N$ [24, 25])

The obstruction to such a collapse in a trivial background is the fact (mentioned above) that $X$ is a general derivation in $\Delta(A)$ but $A$ is restricted to $\Delta_I(A)$. Thus classical fluctuations can only move $X$ within a fixed element of $H^1(A)$; if a given component of $X$ has an outer VEV, then no classical fluctuation can ever cancel it, and so $X = 0$ is topologically excluded from the space of possible solutions. Clearly the number of linearly independent components of $X$ which can be stabilized in this manner is no greater than the dimension of $H^1(A)$, so we can refer to this latter number as the effective dimension of the brane.

This statement is true only in the case of a trivial background. Supersymmetry adds extra terms to the potential which stabilize the system when $X$ wraps supersymmetric cycles. In a flat background, the only such cycles are infinite flat hyperplanes, and this issue does not apply; however, even a finite-$N$ fuzzy torus can wrap a $T^2$ of the background space and be stable. [26] Similarly nontrivial Ramond-Ramond backgrounds can stabilize configurations, as a background $c^{(3)}_{\mu\nu\lambda}$ does a fuzzy sphere. Both of these can cause additional states (beyond those given here) to have nontrivial spatial extent; the considerations discussed here are those which determine whether a brane can be stable even in the absence of such additional forces.

The two topological conserved charges $H^1$ and $K_0$ are compatible in the sense that fluctuations of $A$ within $\Delta_I(A)$ lead to continuous deformations of $X$, which leave $K_0$ invariant. Thus classical fluctuations never change the quantum charge.

If we examine this charge for ordinary branes, we see a very unsurprising result. For any manifold $M$ wrapped by a brane, $H^1(C(M))$ is generated by the ordinary derivatives on $M$ (covariant derivatives in the GR sense) and the number of stabilized dimensions is exactly the ordinary dimension of $M$. (Note that this happens even if the brane is initially curved non-supersymmetrically, so that there is a nonzero potential; the dimension of the brane nonetheless remains fixed) Similarly for the Moyal plane, only two components can be stabilized. For a finite number of D0-branes, however, $H^1$ is generated by $\partial_t$ alone and there is no stabilization; all finite-$N$ systems collapse.

Thus we find that in order for a noncommutative brane $(A, X)$ to be stable, $X$ must represent a nonzero class in $K_0(A)$ and have more than one component with a nonzero projection onto $H^1(A)$. This, of course, requires that both $K_0(A)$ and $H^1(A)$ be nontrivial. $K_0(A)$ then forms the “group of conserved charges,” in the sense that when multiple branes are adjoined to one another, the charges add using the addition rule of $K_0$.

This leads to some interesting conjectures about the types of branes which may be built by this method. In order to build a brane other than a commutative brane or a brane in a background $B$-field, one must find an algebra $A$ such that $K_0(A)$ is nontrivial, $H^1(A)$ is at least two-dimensional, and at least two derivations in $H^1(A)$ do not commute. (cf. the note at end of section 1)

The main obstacle to explicitly constructing such an algebra is that it is hard to calculate $H^1$, and the list of algebras known to have $H^1 \neq 0$ is unfortunately short.[38] Getting a nontrivial $H^1$ by means of direct products alone, i.e. by taking an algebra $A = C(\mathbb{R}^{p+1}) \otimes A_0$ where $H^1(A_0) = 0$, doesn’t stabilize the brane in an interesting manner; it simply builds the unstable algebra $A_0$ out of $p$-branes rather than 0-branes. However, all three of the properties specified above are believed to be generic properties of $C^*$-algebras; therefore we can strongly conjecture that such states exist and should be easily accessible given better means of computing $H^1$.

An interesting feature of these exotic branes is that not all of them have integer-valued conserved charges. A classical theorem [27] states that every Abelian group is $K_0$ of some algebra; this therefore raises the possibilities of $\mathbb{Z}_N$ and $\mathbb{R}$. The former possibility is not surprising, being a generalization of the type I D-instanton which has a $\mathbb{Z}_2$ charge, also for K-theoretic reasons. [18] Real-valued charges, on the other hand, are excluded by Dirac quantization. The brane has a monopole coupling to the $C^{(p+1)}$ Ramond-Ramond field, where $p + 1$ is the effective (Hochschild) dimension; this is simply the continuum limit of the Myers dielectric coupling. (See appendix) Similarly it couples magnetically to a $(d - p - 3)$-form potential, so the dual monopole (if such exists; we conjecture that this is the case but the argument may be subtle) is a $(d - p - 3)$-dimensional noncommutative object. If both objects share a time direction but are otherwise transverse to one another, there remain three spatial dimensions transverse to both branes.

The coupling terms are therefore $C^{(p+1)}_{0,\ldots,p}$ for the electric brane and $C^{(d-p-3)}_{0,p+1,\ldots,d-3}$, with the electric charge valued in
the Abelian group $K_0(A)$ and the magnetic charge in its dual group. If we reduce to the transverse dimensions and
time, these become point particles coupled electrically and magnetically to a 1-form gauge field in 3+1 dimensions.
We may therefore apply the ordinary Dirac quantization argument in this case, showing that the product of the two
charges must be $2\pi$ times an integer. This can only happen if both the electric and magnetic charges are integral,
and thus if $K_0$ is either $Z_N$ or $Z$.

We therefore conclude that a continuous charge is inconsistent with Dirac quantization for noncommutative branes
in the same way as it is for commutative ones, since the branes carry a monopole coupling to the Ramond-Ramond
fields. This leads to a conjecture (no more since the proof above is not rigorous) that so long as $A$ is a simple $C^*$-algebra and $K_0(A)$ is continuous, then $H^1(A) = 0$. This result is consistent with known results about $C^*$-algebras.

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III. APPENDIX: TECHNICAL DETAILS AT THE DBI LEVEL

In order to make these arguments rigorous at the DBI level, we must give a detailed prescription for the action
with infinite matrices, and demonstrate that the identification of a stack of $p$-branes with the algebra $C(\mathbb{R}^{p+1}) \otimes M_N$
continues to hold. At the Yang-Mills level, this happened because the action depended only on the full field strength $F$
and not independently on the components parallel and transverse to the brane. This meant that whether a given
component of $X_\mu$ was a brane direction or not could be determined simply by whether or not its VEV contained a
partial derivative term.

For the Neveu-Schwarz term in the Born-Infeld action, a similar argument holds, since the action may be written
(following equation (26) of [28]) as

$$S = \mu_0 \int dt \text{Tr} \ e^{-\phi} \sqrt{P[G + B] + F} , \quad (9)$$

where $G = \eta$ is the (flat) background metric, $B$ is the background tensor field (zero in our case) and $\phi$ is the dilaton.
Pullbacks are defined as in the finite-matrix case.

For the Ramond-Ramond term, however, there are subtleties due to the Myers effect. The action for a set of
D0-branes is

$$S = \mu_0 \int P[e^{i X_1 X} C e^{B}] , \quad (10)$$

where $C$ is the formal sum of background Ramond-Ramond fields, $B$ is the Neveu-Schwarz tensor field, (here zero)
and $\iota_X$ is the inner product of forms with derivatives. For finite matrices, this satisfies $\iota_X (C_{\mu} dx^\mu) = X^\mu C_{\mu}$, and the commutator terms coming from multiple insertions of $X$ leads to the well-known Myers coupling of the brane to higher Ramond-Ramond charges. (Since each insertion of an $X$ lowers the degree of the form by one)

To define $\iota_X (C)$ for an algebra with outer $X$, consider the case for an ordinary manifold $A = C(M)$, where $X$ is a covariant derivative. Let $d\xi^a$ be a basis for world-volume forms and $dx^\mu$ be a basis for spacetime forms. By linearity of the inner product of vector fields and forms, $\iota_X (C) = \langle d\xi^a X_a , C_{\mu} dx^\mu \rangle = d\xi^a \langle X_a , dx^\mu \rangle C_{\mu}$, so we only need the latter product. This is defined since 1-forms are defined to be dual objects to vectors, i.e. maps $\Delta(A) \to A$. For the exterior derivative of a function $f$ in $A$, we can use the definition

$$df : df(\partial) = \partial(f) \quad (11)$$

for all $\partial \, \in \, \Delta(A)$. Thus

$$\langle X_a , dx^\mu \rangle = dx^\mu (X_a) = X_a (x^\mu) , \quad (12)$$

where the latter is the action of $X_a$ as a derivative on the embedding function $x^\mu$. Since $X_a = D_a$ is a covariant
derivative, we can put this together to give

$$\iota_X (C) = d\xi^a D_a (x^\mu) C_{\mu} \quad (13)$$

which gives the inner product when $X_a$ is outer. When some $X$’s are outer and some are inner, these combine to give

$$\iota_X (C) = d\xi^a D_a (x^\mu) C_{\mu} + X^i C_i \quad (14)$$
where $a$ indexes the outer $X$'s and $i$ the inner.

For $\mathcal{A} = C(\mathbb{R}^{p+1}) \otimes M_N$, with $\langle X_a \rangle = \partial_a$ for $a = 0 \ldots p$, then, the Myers coupling becomes

$$
\mu_0 \int_{\mathbb{R}^{p+1}} \text{Tr}_{C(\mathbb{R}^{p+1}) \otimes M_N} P[X^2 C^{(3)} + \cdots] = \frac{\mu_0}{(2\pi)^p} \int_{\mathbb{R}^{p+1}} \text{Tr}_N P[\xi^a D_a X^\mu d\xi^b D_b X^\nu C^{(3)}_{\mu\nu\lambda} dx^\lambda + \cdots] = \mu_p \int_{\mathbb{R}^{p+1}} \text{Tr}_N \xi^a d\xi^b d\xi^c D_a X^\mu D_b X^\nu D_c X^\rho C^{(3)}_{\mu\nu\rho} + \cdots = \mu_p \int_{\mathbb{R}^{p+1}} \text{Tr}_N P_{p+1} \left[ C^{(3)}_{\mu\nu\lambda} dx^\mu dx^\nu dx^\lambda \right] + \cdots (15)
$$

where $P_{p+1}$ is the pullback onto a $(p+1)$-dimensional worldvolume and $P$ is the pullback onto the original D0-brane worldline, and the ordinary formula for this was used in the second step. The ellipses represent similar terms for each higher $C^{(k)}$. The final integral keeps only terms with $p+1$ $d\xi$'s, so the term involving $C^{(p+1)}$ contracted entirely with outer indices gives the ordinary monopole Ramond-Ramond coupling of a $p$-brane. For higher forms, $p$ indices may be contracted with outer components of $X$, and the remaining components must be contracted with inner components, which gives the usual Myers dielectric coupling of a brane to higher-rank fields.

This construction continues to hold for algebras other than $C(\mathbb{R}^{p+1}) \otimes M_N$. Let $\Delta^*(\mathcal{A})$ be the dual space of $\Delta(\mathcal{A})$; to every outer derivation $\partial_a$ of $\Delta(\mathcal{A})$ there exists a dual basis 1-form $d\xi^a$ satisfying

$$
d\xi^a(\partial_b) = \eta^a_b \quad (16)
$$

for some metric function $\eta^a_b$. The metric cannot be set equal to one unless $\mathcal{A}$ is unital, but the $d\xi$'s are nonetheless paired 1-1 with the partials since $\Delta^*(\mathcal{A}) \simeq \Delta(\mathcal{A})$ whenever $\Delta(\mathcal{A})$ is reflexive as a Banach space. (This is true whenever $\mathcal{A}$ is a $C^*$-algebra)

Finally, the fermion terms of the action are straightforward. The basis forms $d\xi^a$ form an $\mathcal{A}$-module which is a natural $O(n)$-structure on $\mathcal{A}$, where $n = \dim H^1(\mathcal{A})$. The condition for this to lift to a spin structure can be derived by exactly the same computation as for a manifold, with ordinary cohomology now replaced by Hochschild cohomology. (i.e., the integral cohomology classes $H^{(1,2)}(\mathcal{A},\mathbb{Z}_2)$ must both vanish to guarantee orientability and spin, respectively) This allows the definition of spin bundles, and $\Gamma$-matrices and a Dirac operator may be constructed out of the metric in the usual manner. The fermions themselves are $\mathcal{A}$-valued fields (not derivations) and so their action may be written down in the same manner as for non-Abelian fermions on a commutative space. Supersymmetry is manifestly maintained by checking it in terms of components. (Of course, any particular configuration $X_\mu$ will usually violate supersymmetry)

Corrections to the Born-Infeld action due to nontrivial commutators have been computed up to order $\alpha'^4 F^8$. [5] These actions affect only the Neveu-Schwarz term and are purely in terms of the full $F$, so the previous argument continues through without change. Thus the Born-Infeld action continues to be defined for the case of infinite matrices, and for the particular case $\mathcal{A} = C(\mathbb{R}^{p+1}) \otimes M_N$ the action reproduces that of a stack of $N$ $p$-branes.

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A finitely generated projective module is a module \( A \)

The idea here is that there is a pairing between topological spaces and commutative algebras that maps

This is actually the definition of

See, e.g., M. Rørdam, F. Larsen and N. J. Laustsen, K-theory and C*-algebras: A friendly approach (Cambridge University Press, New York, 2000) theorem 13.4.5.

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The idea here is that there is a pairing between topological spaces and commutative algebras that maps \( M \) onto \( \text{Cu}(M) \), the algebra of continuous functions on \( M \) vanishing at infinity with pointwise addition and multiplication. When we do physical calculations, typically everything is in terms of functions anyway, so we are already implicitly working in terms of \( \text{Cu}(M) \). A general noncommutative algebra \( A \) is then defined to be dual to a noncommutative space. Most ideas about commutative space have a fairly natural generalization; e.g., integrals over all of space become traces on the algebra, and compactness of the space corresponds to the algebra having a multiplicative unit. A “fuzzy space” is the space corresponding to a finite-dimensional (matrix) algebra \( M_N \) for some \( N \). (All fuzzy spaces have the same algebra; in fact, this is the only possible finite-dimensional algebra, part of the reason that something qualitatively new ever happens at finite \( N \).) Another algebra that comes up often is the irrational rotation algebra \( A_\theta \), generated by elements \( u \) and \( v \) satisfying \( uv = e^{2\pi i \theta} vu \); these are the functions on the Moyal plane. For any algebra \( A \), \( M_N(A) \) will denote \( A \otimes M_N \), the algebra of \( N \times N \) matrices with elements in \( A \).

This is the fact that the fuzzy sphere has \( X = (X) + A \), where \( (X) \) encodes the spherical geometry and \( A \) is a fluctuation, but the minimum of the potential is at \( X = 0 \) rather than \( A = 0 \). The Hochschild cohomology class \( H^1 \) essentially measures the extent to which \( A \) is incapable of cancelling \( (X) \).

There are two standard actions for a membrane in string theory. The Born-Infeld action (a Yang-Mills action with corrections in \( \alpha'^2 F \)) is known to describe D-branes by analysis of the S-matrix for strings scattering from a brane. \[ 2 \] The Nambu-Goto action, on the other hand, is the “obvious” action for a geometric membrane, and looks (for a 2-brane) like a gauge theory of world-volume Poisson brackets and for a higher brane like a gauge theory of higher-order bracket-like objects. The former is appropriate to D-branes (which we consider here) and the latter to membranes. Both of these are gauge theories (with gauge group elements taking values in \( \text{Cu}(M) \otimes M_N \) and the Poisson algebra \( P(M) \) respectively, for \( N \) branes and world-volume \( M \) and thus have matrix approximations which correspond to approximating the gauge algebra with some finite-dimensional algebra. For the Born-Infeld case, we can approximate \( C(\mathbb{R}) \otimes C(\Sigma) \otimes M_N \) (factoring out the time component of \( M \)) by \( C(\mathbb{R}) \otimes M_\ell \otimes M_N \); this is the approximation of the surface \( \Sigma \) by a fuzzy space, and generalizes the usual fuzzy sphere construction. For the Nambu-Goto action, the algebra being approximated is the Poisson algebra; (e.g. the Heisenberg algebra \( [x, p] = 1 \) when the membrane is a flat plane) this leads to the usual matrix membrane \[ 3 \] and matrix 4-brane \[ 4 \] constructions. The relationship between the two matrix constructions is the relationship between the Born-Infeld and Nambu-Goto actions for a brane, which is not well understood. In the remainder of this work we will consider only D-branes and thus the Born-Infeld action; the infinite matrices we consider are a generalization of the fuzzy (sphere, torus, etc.) construction.

That is, the action of a single (commutative) D2-brane in a background B-field is equivalent under a gauge transformation of the 2-form to the action of the given noncommutative brane without a background field. \[ 7 \]

\( \mathbb{K} \) is the algebra of compact operators, the set of infinite matrices whose eigenvalues, listed in descending order, go to zero; it is the algebra of infinite matrices that appears e.g. in the matrix membrane construction. \[ 10–12 \] \( \mathbb{B} \) is the algebra of countably infinite matrices with bounded eigenvalues.

A module over an algebra is like a vector space, but with \( A \)-valued rather than real-valued coefficients.

A finitely generated projective module is a module \( E \) for which there is a second module \( F \) such that \( E \oplus F = \text{Cu} \). This “cancellation condition” for the modules is the module analogue of Swan’s Theorem, which states that for every vector bundle \( E \) there is a bundle \( F \) such that their direct sum \( E \oplus F \) is a trivial bundle. This is actually the definition of \( K_0(A) \), a precursor of K-theory which agrees with \( K_0 \) when the noncommutative space is “compact,” i.e. when \( A \) is unital. (The distinction is similar to the requirement of compact support in the commutative case) When \( A \) does not contain the identity, the transition from covariant derivatives to modules does not work properly. Instead one can define modules on the unitization \( A^\oplus = A \oplus 1 \otimes \mathbb{C} \) of \( A \), and then mod out the group of graded modules both by the equivalence relation \( (7) \) and by the condition that \( X \) be trivial on the extra \( \mathbb{C} \) factor. (In the commutative case, adding the extra \( C(pt.) \simeq \mathbb{C} \) factor is like adjoining a point at infinity, and the extra condition is that the bundles be
trivial at that point) The group defined by this pair of equivalence relations is $K_0(A)$.

[37] A technical aside about the K-theory argument and why I (tentatively) believe it: Arguments for the quantum stability of branes based on brane-antibrane nucleation are the standard way of arriving at K-theoretic charges, since they are both physically intuitive and quickly lead to the correct result. (The physical correctness of K-theory as the conserved charge, at least in the commutative case, is strongly indicated by the nontrivial matching of phases in the IIA partition function obtained using K-theory to those obtained from M-theory. [14]) However, one should take these arguments with a grain of salt. First, they are only strictly valid at weak coupling, since at strong coupling the perturbation expansion implicit in talking about intermediate virtual states is not guaranteed to exist. Second, in the commutative case the brane-antibrane argument does not make clear the origin of certain anomaly cancellation conditions which it implies; in technical language, one can describe commutative K-theory as a series of refinements of the ordinary cohomological charges by means of the Atiyah-Hirzebruch Spectral Sequence. (AHSS) The brane-antibrane annihilation condition, strictly interpreted, only explains the first term in this sequence; higher terms must be thought of in terms of anomaly influxes to various instanton configurations. (The careful commutative argument based on the AHSS was given in [15]) Nonetheless, these brane annihilation arguments repeatedly seem to give the “right answer,” namely K-theory, in the commutative case, naturally including all higher terms in the AHSS, and (based on the non-perturbative test of [14]) correctly even in strong coupling.

I am therefore continuing to use this argument in the noncommutative case. We are implicitly assuming weak coupling here, but one may expect it to continue into the strong-coupling limit as it does in the commutative case, for similar reasons. The second argument is harder to analyze in this case since there is no good noncommutative analogue of the AHSS approximation to K-theory, but we should note that in the case of vanishing torsion (which is implicit in our assumption of no background fields) the terms beyond the first in the commutative AHSS are automatically zero. In the presence of torsion, moreover, the algorithm for going from a covariant derivative to a module will be torsion-dependent, so one could reasonably expect the correct torsion dependence to arise in such a manner. This belief is further reinforced by the result from category theory that K-theory is essentially the only possible topological invariant of an algebra which satisfies reasonable covariance properties. [21]

[38] The earlier list of results represents the present state of the art. I attempted to extend this somewhat by computing $H^1$ for algebras freely generated on $N$ operators with a generic normal-ordering prescription. The results (after much computation) were that for two generators, the only algebras with nonzero $H^1$ are $C(R^2)$, $C(T^2)$ and $A_0$, either with or without a unit adjoined. (These correspond geometrically to the noncommutative torus and plane, respectively) For all other algebras — which includes the Heisenberg algebra $[x, p] = 1$, as well as various algebras with rules like $uv = \alpha u^p v^q + \gamma = H^1$ is zero. For three generators, the problem rapidly became intractable even with the aid of a computer; it is an interesting open problem.

[39] Simplicity is required to avoid the trivial solution $A = A_0 \odot A_1$, where the nonzero $H^1$ comes from $A_0$ and the large $K_0$ from $A_1$ with $H^1(A_1) = 0$, as when an unstable noncommutative brane $A_1$ is built out of $p$-branes rather than 0-branes. In this case, we may simply factor out $A_0$ by considering only dimensions transverse to it, and repeat the previous argument for $A_1$. This argument may also fail if $H^1(A)$ is large enough that there are fewer than three transverse dimensions; in these cases, however, back-reaction can no longer be ignored and so the construction of branes becomes subtle for other reasons. As a mathematical statement about algebras (rather than a physical one about branes), this relationship between invariants should continue to hold because this argument could be done in any sufficiently high dimension.