Clarifying spatial distance measurement

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We examine length measurement in curved spacetime, based on the 1+3-splitting of a local observer frame. This situates extended objects within spacetime, in terms of a given coordinate which serves as an external reference. The radar metric is shown to coincide with the spatial projector, but these only give meaningful results on the observer’s 3-space, where they reduce to the metric. Examples from Schwarzschild spacetime are given.

1. Introduction and motivation

Recall the textbook “radial proper distance” in Schwarzschild spacetime:

\[ ds = \left(1 - \frac{2M}{r}\right)^{-1/2} dr, \]  

(1)

which follows from setting \( dt = d\theta = d\phi = 0 \) in the line element in Schwarzschild coordinates. But what is the physical motivation for choosing a slice Schwarzschild \( t = \text{const} \), rather than some other time coordinate? Special relativity stresses length is relative to the observer, so which observers measure Equation 1, and what do others measure? The claim the \( dt = 0 \) slice is “measurement by an observer at infinity” is problematic, because there are many ways to extend a local frame, or to choose a simultaneity convention between distant frames.

Consider the same procedure repeated for Gullstrand-Painlevé coordinates:

\[ ds = dr. \]  

(2)

Painlevé hence concluded relativity is self-contradictory. Instead, as we shall see, these correspond to measurements by different observers. (Mathematically, note the differing expressions called “\( ds \)” are really restrictions to different subspaces.) We apply four complementary theoretical tools: suitably chosen coordinates, the spatial projector, the radar metric, and adapted frames.

2. Well-suited coordinates

An intuitive and pedagogical approach to length measurement is to provide coordinates suited to a given congruence of observers, if possible. Consider for example radial geodesic motion in Schwarzschild spacetime, with 4-velocity field \( u \) parametrised by the Killing energy per mass \( e := -g(u, \partial_t) \). A generalisation of Gullstrand-Painlevé coordinates has metric:

\[ ds^2 = -\frac{1}{e^2} \left(1 - \frac{2M}{r}\right) dT^2 + \frac{2}{e^2} \sqrt{e^2 - 1 + \frac{2M}{r}} dT \, dr + \frac{1}{e^2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \]  

(3)
where $T \equiv T_e$ is the Einstein-synchronised proper time of the observers, which all share the same $e \in \mathbb{R} \setminus \{0\}$. For details see Ref. 6. Setting $dT = d\theta = d\phi = 0$ gives:

$$dL = \frac{1}{e} dr,$$

(4)

where we write $dL$ in place of $ds$ for the length element, and the sign choice is mere convention. The physical justification behind $dT = 0$ is that the $T = \text{const}$ hypersurface is orthogonal to the 4-velocities, so coincides with their local 3-spaces.

Note a static observer at $r = r_0$ is identical to an observer falling from rest at $r_0$, in the sense their velocities and hence local 3-spaces coincide instantaneously. Both have $e = \sqrt{1 - 2M/r_0}$, so Equation 4 reduces to the usual quantity (Equation 1).

Some textbooks set up a false dichotomy that $dr$ is not the distance but $(1 - 2M/r)^{-1/2} dr$ is. Instead, for radial observers with $e = \pm 1$ the $r$-coordinate is precisely proper distance.

Equation 4 is remarkably little known. Gautreau & Hoffmann derived it, using a different parameter corresponding to the $0 < e < 1$ case. Taylor & Wheeler present the $e = 1$ case clearly, which is the only textbook coverage apparently. Finch showed the 3-volume inside the horizon is $1/e$ times its Euclidean value $4\pi(2M)^3$, for the $e > 0$ case. (Precedents include Lemaître, who pointed out 3-space is Euclidean for $e = 1$, and mentioned measurement. Painlevé made the same observations, but mistakenly saw contradiction.)

In general, consider a 4-velocity field $u$. Define a new coordinate $T$ by:

$$dT := -N^{-1}u^\flat,$$

(5)

where $u^\flat$ is the 1-form dual to the 4-velocity, and $N$ is a lapse. $T$ exists locally iff the velocity gradient is vorticity-free, a consequence of Frobenius’ theorem. Then the $T = \text{const}$ hypersurfaces are orthogonal to the congruence, since $dT(\xi) = 0$ for any vector $\xi$ orthogonal to $u$. Now express the metric in coordinates including $T$, and set $dT = 0$.

### 3. Spatial projector

Given a 4-velocity $u$, the metric splits into parts parallel and orthogonal to $u$ as $g_{\mu\nu} \equiv -u_\mu u_\nu + (g_{\mu\nu} + u_\mu u_\nu)$, assuming metric signature $-+++$. The latter term is the spatial projection tensor $P$, which extracts the spatial part of tensors via contraction. In particular, $P^\mu_\nu u^\nu = 0$, and $P^\mu_\nu \xi^\nu = \xi^\nu$ for any vector $\xi$ in the 3-space orthogonal to $u$. Furthermore $P_{\mu\nu} \xi^\mu \xi^\nu = g_{\mu\nu} \xi^\mu \xi^\nu$ for such a $\xi$, so $P$ is also called the spatial metric.

One may wonder if $dL^2 := P_{\mu\nu} \xi^\mu \xi^\nu$ is meaningful as a length measurement for any $\xi$, not necessarily orthogonal to $u$. For radial motion in Schwarzschild

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*If in addition the congruence is geodesic, we can set $N \equiv 1$, then $dT/d\tau = -u^\flat(u) = 1$, so $T$ measures proper time. This trick to derive $T$ was applied to relativity by Synge, and Lagrange’s 3-velocity potential in Newtonian mechanics is an antecedent.*
spacetime, the projector in Schwarzschild coordinates is

\[ P_{\mu\nu} = \begin{pmatrix}
    e^2 - 1 + \frac{2M}{r} & e\left(1 - \frac{2M}{r}\right)^{-1} \sqrt{e^2 - 1 + \frac{2M}{r}} \\
    e\left(1 - \frac{2M}{r}\right)^{-1} \sqrt{e^2 - 1 + \frac{2M}{r}} & e^2\left(1 - \frac{2M}{r}\right)^{-2}
\end{pmatrix} \tag{6} \]

in the \( t-r \) block, plus the usual 2-sphere metric components \( \text{diag}(r^2, r^2 \sin^2 \theta) \) in the \( \theta-\phi \) block. One might expect the radial direction to be the coordinate basis vector \( \partial_r \), at least for \( r > 2M \). Contracting with \( \xi = \partial_r \) picks out the \( P_{rr} \) component (boxed), for a candidate spatial measurement:

\[ dL = e\left(1 - \frac{2M}{r}\right)^{-1} dr. \tag{7} \]

But consider the “same” contraction of tensors expressed in the generalised Gullstrand-Painlevé coordinates. The projector is:

\[ P_{\mu\nu} = \begin{pmatrix}
    \frac{1}{2}e^2 - 1 + \frac{2M}{r} & \frac{1}{2\gamma} \sqrt{e^2 - 1 + \frac{2M}{r}} \\
    \frac{1}{2\gamma} \sqrt{e^2 - 1 + \frac{2M}{r}} & \frac{1}{e^2}
\end{pmatrix}, \tag{8} \]

in the \( T-r \) block, so the contraction with \( \partial_r \) yields \( dL = \frac{1}{2} dr \) as in Equation 4. But why the discrepancy with Equation 7? While Equations 6 and 8 represent the same tensor, it turns out the coordinate vectors \( \partial_r \) are distinct. By definition of coordinate basis, \( (\text{Schw}) \partial_r \) is orthogonal to \( dt \), whereas \( (\text{GP}) \partial_r \) is orthogonal to \( dT \). (In contrast, the vector \( (d\tau)^2 \) depends only on \( r \).) This potential confusion about coordinate vectors is rarely discussed explicitly. In the present context, it shows a potential pitfall for measurement, and the superficial contradiction motivates deeper study.

In fact Equation 7 does have physical meaning: it is the measurement of a falling ruler as determined in the local static frame. By this, we mean the comparison of the ruler’s length-contracted tick marks with the \( r \)-coordinate. The two results are related by the Lorentz factor \( \gamma = |e|(1 - 2M/r)^{-1/2} \), since the frames are in standard configuration. This will be examined in future work. For now we conclude \( \sqrt{P_{\xi\xi}(\xi, \xi)} \) is not a measurement in \( u \)'s frame, if \( g(\xi, u) \neq 0 \).

4. Radar metric

The sonar / radar method of distance measurement involves bouncing a signal off a distant object, and timing the return journey. In relativity a null signal is used, along with the proper time \( \Delta \tau \) of the emitter, hence the one-way distance is \( \Delta L := \Delta \tau/2 \) (assuming an isotropic speed of light \( c = 1 \)). While radar was promoted by Poincaré, Einstein, Milne, Bondi, and others, the following formula was derived by Landau & Lifshitz:

\[ \gamma_{ij} := g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}, \tag{9} \]
for \( i, j = 1, 2, 3 \), with infinitesimal length element \( dL^2 = \gamma_{ij} dx^i dx^j \). This assumes a particular coordinate system is provided, and that the radar instrument is comoving in those coordinates.

To apply this to radial motion in Schwarzschild, we need comoving coordinates. One choice is a case of Lemaître-Tolman-Bondi coordinates, using \( \rho e \) given in Refs. 2 and 6, together with \( T e \) from above. The radar metric is:

\[
\gamma_{ij} = \begin{pmatrix}
1 & e^{2 \left( e^2 - 1 + \frac{2M}{r} \right)} & 0 & 0 \\
0 & r^2 & 0 & 0 \\
0 & 0 & r^2 \sin^2 \theta & 0 \\
\end{pmatrix},
\]

with radial distance \( dL = \frac{1}{e} \sqrt{e^2 - 1 + \frac{2M}{r}} \, dp \). The interval “\( dp \)” is unfamiliar, hence reinterpret the radar metric as 4-dimensional (which simply adds terms of 0), and transform into other coordinates. It turns out Equation 10 is identical to the spatial projector (Equations 6 and 8), and the coordinate vector \( \partial_\rho \) is parallel to both \( (GP) \partial_r \) and the radial ruler seen later (Equation 11).

In fact the spatial projector is the covariant generalisation of the radar metric. In any comoving coordinates, the observer has 4-velocity \( u^\mu = (-g_{00})^{-1/2}, 0, 0, 0 \), assuming \( \partial_0 \) is future-pointing. Then \( P_{\mu\nu} \) reduces to \( \gamma_{\mu\nu} \), taking the latter as 4-dimensional. Landau & Lifshitz interpret 3-space as spanned by the coordinate vectors \( \partial_i \). However these are not necessarily orthogonal to \( \partial_0 \) and the 4-velocity. Equation 9 gives incorrect results for directions not orthogonal to \( u \), as discussed for the spatial projector. When restricted appropriately, both radar and the projector are simply \( g_{\mu\nu} \).

5. Adapted frames

Our final technical tool is local reference frames. The observer 4-velocity splits the local tangent space into “time” \( u \), and 3-dimensional “space” orthogonal to \( u \). For our purposes a single spatial vector \( \xi \) is often sufficient. While it is well known that measurements relate to expressing tensors in an observer’s frame, strangely this is not applied to spatial distance — with rare exceptions. Likewise Rindler writes, “rigid scales [rulers] are of ill repute in relativity”, but there is nothing wrong with “resilient scales”, within limits on acceleration and tidal forces.

For radial motion in Schwarzschild, the obvious choice of radial vector is that orthogonal to \( u \), \( d\theta \) and \( d\phi \):

\[
\xi^\mu := \left( -\left( 1 - \frac{2M}{r} \right)^{-1} \sqrt{e^2 - 1 + \frac{2M}{r}}, e, 0, 0 \right).
\]

Since the \( r \)-component is \( e \) and the vector has unit length, this means a coordinate interval \( \Delta r = e \) corresponds to a proper length \( \Delta L = 1 \). Hence \( dL = \frac{1}{e} \, dr \) as before.
In general, suppose a unit spatial vector $\xi$, and a scalar $\Phi$ (for instance a coordinate) are provided. The change in $\Phi$ over the extent of the ruler is $\Delta \Phi = \frac{d\Phi}{d\xi}$, hence:

$$dL = \frac{1}{d\Phi(\xi)} d\Phi = \frac{1}{(\xi^\Phi)} d\Phi,$$

where the latter expression applies if $\Phi$ is taken as a coordinate. We require $d\Phi(\xi) \neq 0$, meaning $\Phi$ is not constant along the ruler direction, so can demarcate length. In any spacetime, given a coordinate expression of a tetrad, one can simply read off a coordinate length interval by inverting the relevant component.

The “best” ruler direction to choose is typically not obvious. Given $\Phi$ and $u$, one candidate is a certain maximal direction as follows. Recall the gradient vector $(d\Phi)^\#$ is the direction of steepest increase of $\Phi$ per unit length. However this is generally not purely spatial, according to $u$. Hence, restrict $d\Phi$ to $u$’s 3-space, then take the vector dual. This vector shows the fastest increase of $\Phi$ along any possible ruler of $u$. The measurement turns out to be $(g^{-1}(d\Phi, d\Phi) + (d\Phi(u))^2)^{-1/2} d\Phi$, or:

$$dL_{\max-\Phi} = \frac{1}{\sqrt{g^{\Phi\Phi} + (u^\Phi)^2}} d\Phi,$$

if $\Phi$ is a coordinate. Alternatively, given $\Phi$, we can ask which observers $u$ make extremal measurements. If a $\Phi = \text{const}$ slice is spatial ($g^{\phi\phi} > 0$), then observers with $u^\Phi \equiv d\Phi/d\tau = 0$ are possible, these measure $(g^{\Phi\Phi})^{-1/2} d\Phi$ in the direction $(d\Phi)^\#$. No ruler orientation for observers with $u^\phi \neq 0$ can achieve this.

Consider a Schwarzschild observer parametrised by both $e$ and the Killing angular momentum per mass $\ell$. Its maximum possible $r$ and $\phi$ measurements are:

$$dL_{r,\max} = \left(e^2 - \left(1 - \frac{2M}{r}\right)\frac{\ell^2}{r^2}\right)^{-1/2} dr, \quad dL_{\phi,\max} = \frac{r d\phi}{\sqrt{1 + \ell^2/r^2}},$$

where motion is in the plane $\theta = \pi/2$. Incidentally, the corresponding ruler vectors are not orthogonal to one-another. Note that while $r$ is known as the reduced circumference, the Euclidean 2-sphere measurement $r d\phi$ only holds for zero angular momentum $\ell = 0$, under this natural ruler orientation.

Our orthonormal frame approach is trivially the zero-distance limit of Fermi coordinates, however very few exact Fermi coordinate expressions are known.

6. Discussion

There are many potential questions and objections. It is non-trivial to start from textbook material such as $\int ds$ or first-principles radar. Despite some excellent books\cite{1,2,3,4}, our results fill an independent niche.

Given two events, why not extremise the length of spatial geodesics between them? This is a 4-dimensional approach, whereas ours uses a 1+3-dimensional splitting, measuring within the rest space of a given observer.

Is the result coordinate-dependent? The coordinate $\Phi$ serves as an extrinsic standard, but the length element is simply the ruler $\xi^\phi$, which is also the metric interval restricted to a 1-dimensional subspace.
In Schwarzschild, $r$ is timelike inside the horizon, so how can it describe spatial measurement? Indeed the normal vector $(dr)^2$ is timelike, but the measurement direction $\xi$ is spatial.

References

1. P. Painlevé, La mécanique classique et la théorie de la relativité, *Comptes rendus de l’Académie des Sciences* **173**, 677 (1921).
2. R. Gautreau and B. Hoffmann, The Schwarzschild radial coordinate as a measure of proper distance, *Physical Review D* **17**, 2552 (May 1978).
3. K. Martel and E. Poisson, Regular coordinate systems for Schwarzschild and other spherical spacetimes, *American Journal of Physics* **69**, 476 (April 2001).
4. T. Finch, Coordinate families for the Schwarzschild geometry based on radial timelike geodesics, *General Relativity and Gravitation* **47**, p. 56 (May 2015).
5. D. Bini, A. Geralico and R. Jantzen, Separable geodesic action slicing in stationary spacetimes, *General Relativity and Gravitation* **44**, 603 (March 2012).
6. C. MacLaurin, Schwarzschild spacetime under generalised Gullstrand-Painlevé slicing, in *Einstein equations: Physical and mathematical aspects of general relativity*, eds. S. Cacciatori, B. Güneysu and S. Pigola (Springer, 2019).
7. E. Taylor and J. Wheeler, *Exploring black holes: Introduction to general relativity* (Addison-Wesley, 2000).
8. G. Lemaître, L’Univers en expansion, *Publication du Laboratoire d’Astronomie et de Géodésie de l’Université de Louvain* **9**, 171 (1932), translation in *General Relativity and Gravitation*, 1997.
9. L. Landau and E. Lifshitz, *Field theory* (GITTL, 1941).
10. R. Klauber, Toward a consistent theory of relativistic rotation, in *Relativity in rotating frames*, eds. G. Rizzi and M. Ruggiero (Springer, 2004) pp. 103–137.
11. W. Rindler, *Essential relativity: Special, general, and cosmological* (Springer-Verlag, 1977).
12. J. Droste, The field of a single centre in Einstein’s theory of gravitation, and the motion of a particle in that field, *Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series B Physical Sciences* **19**, 197 (Jan 1917), republished in *General Relativity and Gravitation*, 2002.
13. D. Bini, L. Lusanna and B. Mashhoon, Limitations of radar coordinates, *International Journal of Modern Physics D* **14**, 1413 (2005).
14. F. de Felice and D. Bini, *Classical measurements in curved space-times* (Cambridge, 2010).
15. R. Jantzen, P. Carini and D. Bini, *GEM [Gravitoelectromagnetism] the user manual: Understanding spacetime splittings and their relationships* (online draft, 2013).