Fractional smoothness of functionals of diffusion processes under a change of measure

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Abstract

Let \( v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R} \) be the solution of the parabolic backward equation \( \partial_t v + (1/2) \sum_{i,j} [\sigma \sigma^\top]_{ij} \partial_{x_i} \partial_{x_j} v + \sum_i b_i \partial_{x_i} v + kv = 0 \) with terminal condition \( g \), where the coefficients are time- and state-dependent, and satisfy certain regularity assumptions. Let \( X = (X_t)_{t \in [0,T]} \) be the

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associated $\mathbb{R}^d$-valued diffusion process on some appropriate $(\Omega, \mathcal{F}, \mathbb{Q})$. For $p \in [2, \infty)$ and a measure $d\mathbb{P} = \lambda_T d\mathbb{Q}$, where $\lambda_T$ satisfies the Muckenhoupt condition $A_\alpha$ for $\alpha \in (1, p)$, we relate the behavior of $\|g(X_T) - \mathbb{E}_\mathbb{P}^T g(X_T)\|_{L_p(\mathbb{P})}$, $\|\nabla v(t, X_t)\|_{L_p(\mathbb{P})}$ and $\|D^2 v(t, X_t)\|_{L_p(\mathbb{P})}$ to each other, where $D^2 v := (\partial_{x_i} \partial_{x_l} v)_{i,l}$ is the Hessian matrix.

1 Introduction

For a fixed time-horizon $T > 0$ let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{Q})$ be a filtered probability space where $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]})$ is complete, $\mathcal{F} = \mathcal{F}_T$, the filtration $(\mathcal{F}_t)_{t \in [0,T]}$ is right-continuous, $\mathcal{F}_0$ is generated by the null sets of $\mathcal{F}$ and where all local martingales are continuous (see Section 2). Assume for some $d \geq 1$ that the process $B = (B_t)_{t \in [0,T]}$ is a $d$-dimensional $(\mathcal{F}_t)_{t \in [0,T]}$-standard Brownian motion starting in zero. We consider an $\mathbb{R}^d$-valued diffusion process $X = (X_t)_{t \in [0,T]}$, solution to the stochastic differential equation

$$X_t = x_0 + \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds$$

for some smooth bounded coefficients $b$ and $\sigma$, and we focus on the rate of convergence of $R_p^X(t) := \|g(X_T) - \mathbb{E}(g(X_T)|\mathcal{F}_t)\|_p$ for $p \in [2, \infty)$ as $t \to T$, where $g$ satisfies a suitable growth condition ensuring $g(X_T) \in L_p$. The behavior of $R_p^X(t)$ as $t \to T$ is a measure of the fractional smoothness of $g$, see [4] for an overview. Actually it is now well-known [3, 6, 10, 5] that there is a precise correspondence between the irregularity of the terminal function $g$ and the time-singularity of the $L_p$-norms of $\nabla v(t, X_t)$ as $t \uparrow T$ where

$$v(t, x) = \mathbb{E}(g(X_T)|X_t = x).$$

The aim of this paper is to extend these quantitative equivalence results to situations where the $L_p$-norms are computed under different measures. The theory of probabilistic Muckenhoupt weights, developed as a counterpart to the deterministic ones from [14] and other papers, gives a natural way to extend various martingale inequalities to equivalent measures, see exemplary [12, 11, 13] and the references therein. A typical situation is a change of measure initiated by a Girsanov transformation, i.e. a change of the drift of $X$. Applying the results of this paper in this particular case, gives -without
going into full details- the following: if the process $Y$ differs from $X$ by another bounded drift and if $\theta \in (0, 1)$, then we have

$$
\sup_{t \in [0,T]} (T-t)^{-\theta/2} R^Y_p(t) < \infty \iff \sup_{t \in [0,T]} (T-t)^{(1-\theta)/2} \| \nabla v(t, Y_t) \|_p < +\infty \quad (1)
$$

which follows from Theorem 1 below for $q = \infty$ as explained in Remark 2(7). The parameter $\theta$ is the degree of fractional smoothness.

Regarding the references in the literature related to (1), a 1-dimensional diffusion case with $X = Y$ is considered in [3], the extension to multidimensional processes is performed in [6] in the case $X = Y$ being a Brownian motion and in [10] for diffusion processes. In [5] path-dependent functionals are considered. For an overview the reader is referred to [4]. Actually our main result (Theorem 1) takes a more general form than (1):

- we consider an $L_q([0, T), \frac{dt}{T})$-norm with $q \in [2, \infty]$ instead of the above $L_\infty$-norm with respect to $t \in [0, T]$;

- we consider an additional potential factor $k$ in our parabolic problem to define $v$;

- the change of measure, described in (1) by the change from $X$ to $Y$, is described by Muckenhoupt weights;

- we also state results regarding the second derivatives.

Applications. The tight control of the behavior of the norms $\| \nabla v(t, X_t) \|_2$ as $t \to T$ is an issue that has been raised in [3], where the purpose was to analyze discrete approximations of stochastic integrals coming from the representation

$$
g(X_T) = v(0, x_0) + \int_0^T \nabla v(t, X_t) \sigma(t, X_t) dB_t. \quad (2)
$$

Discretizing the above stochastic integral and analyzing the resulting approximation error in $L_2$, requires a better understanding how strongly the irregularity of the terminal function $g$ transfers to the blow-up of the function $t \mapsto \| \nabla v(t, X_t) \|_{L_2}$ and higher derivatives of $v$ as well. Major consequences of this analysis are the derivation of tight convergence rates for uniform time grids and the design of non-equidistant time grids to obtain optimal convergence rates.
Recently, similar results have been established in the context of Backward Stochastic Differential Equations [10, 5] to pave the way for the development of more efficient numerical schemes.

Finally, similar issues arise in the analysis of the Delta-Gamma hedging strategies in Finance, which typically result in a higher order approximation of the stochastic integral (2), see [11].

Within the applications in Stochastic Finance intrinsically two measures are involved: the historical measure for evaluating the risk, for example as $L_p$-mean, and the risk-neutral measure, under which the price and the hedging strategy are computed and which is related to the above function $v$. For this setting, the current results are particularly of interest. Moreover, the potential $k$ may be interpreted as an interest rate.

2 Setting

Notation. We denote by $| \cdot |$ the Euclidean norm of a vector. Given a matrix $C$ considered as operator $C : \ell_2^d \to \ell_2^N$, the expression $|C|$ stands for the Hilbert-Schmidt norm and $C^\top$ for the transposed of $C$. The $L_p$-norm ($p \in [1, \infty]$) of a random vector $Z : \Omega \to \mathbb{R}^n$ or a random matrix $Z : \Omega \to \mathbb{R}^{n \times m}$ is denoted by $\|Z\|_p = \|\|Z\|_p$. As usual, $\partial^\alpha \varphi$ is the partial derivative of the order of an multi-index $\alpha$ (with length $|\alpha|$) with respect to $x \in \mathbb{R}^d$. The Hessian matrix of a function $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$ is abbreviated by $D^2 \varphi$ and the gradient (as row vector) by $\nabla \varphi$. In particular, this means that $D^2$ and $\nabla$ always refer to the state variable $x \in \mathbb{R}^d$. If we mention that a constant depends on $b, \sigma$ or $k$, then we implicitly indicate a possible dependence on $T$ and $d$ as well. Finally, letting $h : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{n \times m}$ we use the notation $\|h\|_\infty := \sup_{t, x} |h(t, x)|$.

The parabolic PDE. We fix $T > 0$ and consider the Cauchy problem

$$
\mathcal{L}v = 0 \quad \text{on} \quad [0, T) \times \mathbb{R}^d,
$$

$$
v(T, x) = g(x)
$$

with

$$
\mathcal{L} := \partial_t + \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(t, x) \partial^2_{x_i, x_j} + \sum_{i=1}^d b_i(t, x) \partial_{x_i} + k(t, x),
$$
where $A := (a_{ij})_{ij} = \sigma \sigma^T$. The assumptions on the coefficients and $g$ are as follows:

(C1) The functions $\sigma_{i,j}, b_i, k$ are bounded and belong to $C^{0,2}_b([0, T] \times \mathbb{R}^d)$ and there is some $\gamma \in (0, 1]$ such that the functions and their state-derivatives are $\gamma$-Hölder continuous with respect to the parabolic metric on each compactum of $[0, T] \times \mathbb{R}^d$. Moreover, $\sigma$ is $1/2$-Hölder continuous in $t$ uniformly in $x$.

(C2) $\sigma(t, x)$ is an invertible $d \times d$-matrix with $\sup_{t,x} |\sigma^{-1}(t,x)| < +\infty$;

(C3) the terminal function $g: \mathbb{R}^d \to \mathbb{R}$ is measurable and exponentially bounded: for some $K_g \geq 0$ and $\kappa_g \in [0, 2)$ we have $|g(x)| \leq K_g \exp(K_g|x|^\kappa_g)$ for all $x \in \mathbb{R}^d$.

The condition (C2) implies that there exists a $\delta > 0$ with $\langle Ax, x \rangle \geq \delta|x|^2$ for all $x \in \mathbb{R}^d$, i.e. the operator $L$ is uniformly parabolic. Under the above assumptions there exists a fundamental solution:

**Proposition 1** ([2, Theorem 7, p. 260; Theorem 10, pp. 72-74]). Under the assumptions (C1) and (C2) there exists a fundamental solution $\Gamma(t, x; \tau, \xi) : \{0 \leq t < \tau \leq T\} \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ for $L$ and a constant $c_3 > 0$ such that for $0 \leq |a| + 2b \leq 3$ the derivatives $D_x^a D_t^b \Gamma$ exist in any order, are continuous, and satisfy

$$|D_x^a D_t^b \Gamma(t, x; \tau, \xi)| \leq c_3 (\tau - t)^{\frac{|a| + 2b}{2}} \gamma_{d}^{d/2} \frac{|x - \xi|}{c_3}$$

where $\gamma_d := e^{-\frac{|x|^2}{2}}/(\sqrt{2\pi})^d$.

For

$$v(t, x) := \int_{\mathbb{R}^d} \Gamma(t, x; T, \xi) g(\xi) d\xi,$$

$$v(T, x) := g(x),$$

and $0 \leq |a| + 2b \leq 3$ Proposition implies that the derivatives $D_x^a D_t^b v$ exist in any order, are continuous on $[0, T) \times \mathbb{R}^d$ and satisfy

$$L v = 0 \quad \text{on} \quad [0, T) \times \mathbb{R}^d,$$

$$|D_x^a D_t^b v(t, x)| \leq c(T - t)^{\frac{|a| + 2b}{2}} \exp(c|x|^\kappa_g)$$

for $x \in \mathbb{R}^d$ and $t \in [0, T)$, where $c > 0$ depends at most on $(\kappa_g, K_g, c_3, T)$. 5
The stochastic differential equation. Let \((B_t)_{t \in [0,T]}\) be a \(d\)-dimensional standard Brownian motion defined on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, Q)\), where \((\Omega, \mathcal{F}, Q)\) is complete, \((\mathcal{F}_t)_{t \in [0,T]}\) is right-continuous, \(\mathcal{F} = \mathcal{F}_T\), \(\mathcal{F}_0\) is generated by the null sets of \(\mathcal{F}\) and where all local martingales are continuous.

As we work on a closed time-interval we have to explain our understanding of a local martingale: we require that the localizing sequence of stopping times \(0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq T\) satisfies \(\lim_{n \to \infty} \mathbb{P}(\tau_n = T) = 1\). The reason for this is that we think about the extension of the filtration constantly by \(\mathcal{F}_T\) to \((T, \infty)\) and that all local martingales \((N_t)_{t \in [0,T]}\) (in our setting) are extended by \(N_T\) to \((T, \infty)\). This yields the standard notion of a local martingale. However this is not needed explicitly in our paper, we only need this implicitly whenever we refer to results about the Muckenhoupt weights \(A_\alpha(Q)\) from [13].

To shorten the notation, we denote sometimes the conditional expectation \(\mathbb{E}(\cdot|\mathcal{F}_t)\) by \(\mathbb{E}^{\mathcal{F}_t}(\cdot)\). The process \(X = (X_t)_{t \in [0,T]}\) is given as strong unique solution of

\[
X_t = x_0 + \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds.
\]

Introducing the standing notation

\[
K^X_t := e^{\int_0^t k(r, X_s) dr} \quad \text{and} \quad M_t := K^X_t v(t, X_t),
\]

Itô’s formula implies, for \(t \in [0, T]\), that

\[
M_t = v(0, x_0) + \int_0^t K^X_s \nabla v(s, X_s) \sigma(s, X_s) dB_s.
\] (4)

Moreover,

\[
\lim_{t \to T} M_t = M_T \quad \text{and} \quad \lim_{t \to T} v(t, X_t) = g(X_T)
\] (5)

almost surely and in any \(L_r(Q)\) with \(r \in [1, \infty)\). Using Proposition [4] for \(k = 0\) we also have

\[
\mathbb{P}(|X_t - x_0| > \lambda) \leq c \exp \left(-\frac{\lambda^2}{c}\right)
\]

for all \(\lambda \geq 0\) and \(t \in [0, T]\), where \(c > 0\) depends at most on \((\sigma, b)\) and is, in particular, independent from the starting value \(x_0 \in \mathbb{R}^d\). It directly implies that

\[
g(X_T) \in \bigcap_{r \in [1, \infty)} L_r(Q)
\]
so that Remark 1 applies as well. We will also use the following

**Lemma 1** ([9], [10, Proof of Lemma 1.1], [5, Remark 3 in Appendix B]). Let \( t \in (0, T] \), \( h : \mathbb{R}^d \to \mathbb{R} \) be a Borel function satisfying (C3) and \( \Gamma_X \) be the transition density of \( X \), i.e. the function \( \Gamma \) from Proposition 1 in the case \( k = 0 \). Define

\[
H(s, x) := \int_{\mathbb{R}^d} \Gamma_X(s, x; t, \xi) h(\xi) d\xi \quad \text{for} \quad (s, x) \in [0, t) \times \mathbb{R}^d.
\]

For \( r \in [0, t) \) and \( x \in \mathbb{R}^d \) let \( (Z_u)_{u \in [r, t]} \) be the diffusion based on \((\sigma, b)\) starting in \( x \) defined on some \((M, \mathcal{G}, (\mathcal{G}_u)_{u \in [r, t]}, \mu)\) equipped with a standard \((\mathcal{G}_u)_{u \in [r, t]}\)-Brownian motion, where \((M, \mathcal{G}, \mu)\) is complete, \((\mathcal{G}_u)_{u \in [r, t]}\) is right-continuous and \( \mathcal{G}_r \) is generated by the null sets of \( \mathcal{G} \). Then, for \( q \in (1, \infty) \) and \( s \in [r, t) \), one has a.s.

\[
|\nabla H(s, Z_s)| \leq \kappa_q \left[ \mathbb{E}\left( |h(Z_t) - \mathbb{E}(h(Z_t)|\mathcal{G}_s)|^q |\mathcal{G}_s\right)\right]^{1/q} (t-s)^{\frac{1}{2}},
\]

\[
|D^2 H(s, Z_s)| \leq \kappa_q \left[ \mathbb{E}\left( |h(Z_t) - \mathbb{E}(h(Z_t)|\mathcal{G}_s)|^q |\mathcal{G}_s\right)\right]^{1/q} t-s,
\]

where \( \kappa_q > 0 \) depends at most on \((\sigma, b, q)\).

**Conditions on the equivalent measure.** In addition to the given measure \( \mathbb{Q} \) we will use an equivalent measure \( \mathbb{P} \sim \mathbb{Q} \) and agree about the following standing assumption:

(P) There exists a martingale \( Y = (Y_t)_{t \in [0, T]} \) with \( Y_0 \equiv 0 \) such that

\[
\lambda_t := \mathcal{E}(Y)_t = e^{Y_t - \frac{1}{2}\langle Y \rangle_t} \quad \text{for} \quad t \in [0, T]
\]

is a martingale and

\[
d\mathbb{P} = \lambda_T d\mathbb{Q}.
\]

**Definition 1.** Assume that condition (P) is satisfied.

(i) For \( \alpha \in (1, \infty) \) we say that \( \lambda_T \in A_{\alpha}(\mathbb{Q}) \) provided that there is a constant \( c > 0 \) such that for all stopping times \( \tau : \Omega \to [0, T] \) one has that

\[
\mathbb{E}_\mathbb{Q}\left( \left| \frac{\lambda_\tau}{\lambda_T} \right|^{\frac{1}{\alpha-1}} |\mathcal{F}_\tau| \right) \leq c \quad \text{a.s.}
\]
(ii) For $\beta \in (1, \infty)$ we let $\lambda_T \in RH_\beta(Q)$ provided that there is a constant $c > 0$ such that for all stopping times $\tau : \Omega \to [0, T]$ one has that
\[
\mathbb{E}_Q(|\lambda_T|^\beta|F_\tau|^\frac{\beta}{2}) \leq c\lambda_T \quad \text{a.s.}
\]

The class $A_\alpha(Q)$ is the probabilistic variant of the Muckenhoupt condition and $RH$ stands for reverse Hölder inequality. Next we need

**Definition 2.** A martingale $Z = (Z_t)_{t \in [0, T]}$ is called BMO-martingale provided that $Z_0 \equiv 0$ and there is a $c > 0$ such that for all stopping times $\tau : \Omega \to [0, T]$ one has that
\[
\mathbb{E}_Q(|Z_T - Z_\tau|^2|F_\tau) \leq c^2 \quad \text{a.s.}
\]

It is known [13, Theorems 2.3] that $(e^{Z_{t-} - \frac{1}{2}(Z_t)})_{t \in [0, T]}$ is a martingale provided that $Z$ is a BMO-martingale.

**Proposition 2** ([13, Theorems 2.4 and 3.4]). Under condition (P) the following assertions are equivalent:

(i) $Y$ is a BMO-martingale.

(ii) $\mathcal{E}(Y) \in A_\alpha(Q)$ for some $\alpha \in (1, \infty)$.

(iii) $\mathcal{E}(Y) \in RH_\beta(Q)$ for some $\beta \in (1, \infty)$.

**Remark 1.** Under the assertions of Proposition 2 we have that $\lambda_T \in L_\beta(Q)$ and $1/\lambda_T \in L_{\alpha'}(\mathbb{P})$ with $1 = (1/\alpha) + (1/\alpha')$ so that
\[
\bigcap_{r \in (1, \infty)} L_r(Q) = \bigcap_{r \in [1, \infty)} L_r(\mathbb{P}).
\]

**Proposition 3** ([13, Theorems 2.3 and 3.19]). Let $Y$ be a BMO-martingale so that (P) is satisfied. For all $p \in (0, \infty)$ there is a $b_p(\mathbb{P}) > 0$ such that for all $Q$-martingales $N$ with $N_0 \equiv 0$ one has that
\[
\frac{1}{b_p(\mathbb{P})}||N_T^*||_{L_p(\mathbb{P})} \leq ||\sqrt{\langle N \rangle_T}||_{L_p(\mathbb{P})} \leq b_p(\mathbb{P})||N_T^*||_{L_p(\mathbb{P})}
\]
where $N_t^* := \sup_{s \in [0, t]} |N_s|$. 

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An inequality. Given a probability space \((M, \Sigma, \mu)\) with a sub-\(\sigma\) algebra \(\mathcal{G} \subseteq \Sigma\) and \(Z \in L_p(M, \Sigma, \mu)\) with \(p \in [1, \infty]\) we have that

\[
\frac{1}{2} \|Z - \mathbb{E}(Z|\mathcal{G})\|_p \leq \inf_{Z' \in L_p(M, \mathcal{G}, \mu)} \|Z - Z'\|_p \leq \|Z - \mathbb{E}(Z|\mathcal{G})\|_p.
\] (6)

3 The result

In the following \(\theta \in (0, 1]\) will be the main parameter of the fractional smoothness. Additionally, we introduce a fine-tuning parameter \(q \in [2, \infty]\) and

\[
\Phi_q(h) := \|h\|_{L_q([0,T], \frac{dt}{T-t})}
\]

for a measurable function \(h : [0, T] \to \mathbb{R}\). The aim of this paper is to prove the following result:

**Theorem 1.** Let \(p \in [2, \infty)\), \(\alpha \in (1, p)\) and \(\lambda_T \in A_\alpha(Q)\), and assume that (C1), (C2) and (P) are satisfied. Then, for \(\theta \in (0, 1)\), \(q \in [2, \infty]\), a measurable function \(g : \mathbb{R}^d \to \mathbb{R}\) satisfying (C3) and \(d\mathbb{P} = \lambda_T d\mathbb{Q}\) the following assertions are equivalent:

(i) \(\Phi_q\left((T-t)^{-\frac{\theta}{2}}\|g(X_T) - \mathbb{E}^\mathbb{P}_t g(X_T)\|_{L_p(\mathbb{P})}\right) < +\infty\).

(ii) \(\Phi_q\left((T-t)^{-\frac{1-\theta}{2}}\|\nabla v(t, X_t)\|_{L_p(\mathbb{P})}\right) < +\infty\).

(iii) \(\Phi_q\left((T-t)^{-\frac{\theta}{2}}\|D^2 v(t, X_t)\|_{L_p(\mathbb{P})}\right) < +\infty\).

**Remark 2.** (1) Using [13, Corollary 3.3] it is sufficient to require that \(\lambda_T \in A_\alpha(Q)\) as in this case there is an \(\varepsilon \in (0, p - 1)\) such that \(\lambda_T \in A_{p-\varepsilon}(Q)\). One the other hand, it would be of interest to investigate the case when \(\lambda_T \in A_\alpha(Q)\) with \(\alpha > p\). This is not done here.

(2) Examples of functions \(g\) such that (i) is satisfied are given for example in [3, 6, 7, 5].

(3) In the case \(X = B, \mathbb{P} = Q, T = 1\) and \(k = 0\) the conditions of Theorem [4] (neglecting the boundedness condition (C3)) are equivalent to that \(g\) belongs to the Malliavin Besov space \(B^\theta_{p,q}\) on \(\mathbb{R}^d\) weighted by the standard Gaussian measure (see [8]).
(4) The case \( \theta = 1 \) and \( q \in [2, \infty) \) is not considered in Theorem \( \square \) because it yields to pathologies. Let \( X = B, \mathbb{P} = \mathbb{Q}, T = 1 \) and \( k = 0 \). Condition (i\( _1 \)) implies (ii\( _1 \)) by Lemma \( \mathbb{L} \) below. Moreover, condition (ii\( _1 \)) and the monotonicity of \( \| \nabla v(t, B_t) \|_{L_\varphi(\mathbb{P})} \) ((\( \nabla v(t, B_t) \))\( _{t \in [0,1]} \)) is a martingale in this case) imply that that \( \nabla v(t, B_t) = 0 \) a.s. so that \( g(B_1) \) is almost surely constant (for example, one can use \( g(B_1) = \mathbb{E}(g(B_1)) + \int_0^1 \nabla v(t, B_t) dB_t \).

(5) As the process \( M = (M_t)_{t \in [0,T]} \) with \( M_t = K_t^X(t, X_t) \) is a martingale under \( \mathbb{Q} \) it is natural to consider condition (i\( _q \)) for the corresponding martingale under \( \mathbb{P} \) as well:

\[
(i_q') \quad \Phi_g \left( (T-t)^{-q} \| M_T - \mathbb{E}_\mathbb{P}^{F_T} M_T \|_{L_\varphi(\mathbb{P})} \right) < +\infty.
\]

One can easily check that \( (i_q) \iff (i_q') \) for \( \theta \in (0,1] \) and \( q \in [1, \infty) \): Indeed, for any random variables \( U \) and \( V \), respectively bounded and in \( L_\varphi(\mathbb{P}) \), observe that

\[
\begin{align*}
\| UV - \mathbb{E}_{\mathbb{P}}^{F_T} (UV) \|_{L_\varphi(\mathbb{P})} &\leq \|[U - \mathbb{E}_\mathbb{P}^{F_T} U]V\|_{L_\varphi(\mathbb{P})} + \|\mathbb{E}_\mathbb{P}^{F_T} (U)[V - \mathbb{E}_\mathbb{P}^{F_T} V]\|_{L_\varphi(\mathbb{P})} \\
&\quad + \|\mathbb{E}_\mathbb{P}^{F_T} (U)[\mathbb{E}_\mathbb{P}^{F_T} (V) - V]\|_{L_\varphi(\mathbb{P})} \\
&\leq \|[U - \mathbb{E}_\mathbb{P}^{F_T} U]V\|_{L_\varphi(\mathbb{P})} + 2\|U\|_\infty \|V - \mathbb{E}_\mathbb{P}^{F_T} V\|_{L_\varphi(\mathbb{P})}.
\end{align*}
\]

For \( U := e^{\int_0^T k(r, X_r) dr} \) and \( V := g(X_T) \) we have

\[
|U - \mathbb{E}_\mathbb{P}^{F_T} U| \leq 2\|k\|_\infty (T-t) e^{\|k\|_\infty T}
\]

and obtain

\[
\begin{align*}
\| e^{\int_0^T k(r, X_r) dr} g(X_T) - \mathbb{E}_\mathbb{P}^{F_T} (e^{\int_0^T k(r, X_r) dr} g(X_T)) \|_{L_\varphi(\mathbb{P})} &\leq 2e^{\|k\|_\infty T} \left[ \| k\|_\infty (T-t) \| g(X_T) \|_{L_\varphi(\mathbb{P})} + \| g(X_T) - \mathbb{E}_\mathbb{P}^{F_T} g(X_T) \|_{L_\varphi(\mathbb{P})} \right].
\end{align*}
\]

This proves \( (i_q) \implies (i_q') \). The converse is proved similarly by letting \( U := e^{-\int_0^T k(r, X_r) dr} \) and \( V := e^{\int_0^T k(r, X_r) dr} g(X_T) \).

(6) The case \( \theta = 1 \) and \( q = \infty \).
(a) One has \((i_1') \iff (ii_1) \implies (iii_1)\): First we observe that

\[
\Phi_\infty \left( (T-t)^{-\frac{1}{2}} \left( \int_t^T h(s)^2 ds \right)^{\frac{1}{2}} \right) \leq \Phi_\infty (h). \tag{7}
\]

Then \((ii_1) \implies (i_1')\) follows from \([7]\) with \(h(t) = \|\nabla v(t, X_t)\|_{L_p}(\mathbb{P})\) and Lemma \([\mathbf{7}]\). The implications \((i_1') \implies (ii_1)\) and \((i_1) \implies (iii_1)\) follow by Lemmas \([\mathbf{3}]\) and \([\mathbf{6}]\).

(b) The implication \((iii_1) \implies (ii_1)\) is not true in general. Take \(p = 2, q = \infty, X = B, \mathbb{P} = \mathbb{Q}, T = 1, k = 0\) and \(d = 1\), then the counterexample \(g(x) = \sqrt{x \vee 0}\) is discussed in \([5]\).

(7) Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) be a stochastic basis satisfying the usual conditions, i.e. \((\Omega, \mathcal{F}, \mathbb{P})\) is complete, \((\mathcal{F}_t)_{t \in [0,T]}\) is right-continuous, \(\mathcal{F}_0\) is generated by the null-sets of \(\mathcal{F}\) and where we can assume w.l.o.g. that \(\mathcal{F} = \mathcal{F}_T\). Assume further that the filtration is obtained as augmentation of the natural filtration of a standard \(d\)-dimensional Brownian motion \(W = (W_t)_{t \in [0,T]}\) starting in zero. It is known \([15, \text{Corollary 1 on p. 187}]\) that on this stochastic basis all local martingales are continuous. Assume a progressively measurable \(d\)-dimensional process \(\beta = (\beta_t)_{t \in [0,T]}\) with \(\sup_{t,\omega} |\beta_t(\omega)| < \infty\) and consider the unique strong solution of the SDE

\[
X_t = x_0 + \int_0^t \sigma(s, X_s) dW_s + \int_0^t b(s, X_s) ds - \int_0^t \beta_s ds.
\]

Letting,

\[
\gamma_s := \sigma^{-1}(s, X_s) \beta_s,
\]

\[
B_t := W_t - \int_0^t \gamma_s ds,
\]

\[
1/\lambda_t := e^{\int_0^t \gamma_s dW_s - \frac{1}{2} \int_0^t |\gamma_s|^2 ds} = e^{\int_0^t \gamma_s dW_s + \frac{1}{2} \int_0^t |\gamma_s|^2 ds},
\]

\[
d\mathbb{Q} := (1/\lambda_T)d\mathbb{P},
\]

we obtain by the Girsanov Theorem that \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{Q}), (B_t)_{t \in [0,T]}\) and \((X_t)_{t \in [0,T]}\) satisfy the assumptions of our paper (i.e. all martingales are continuous - which can be checked by expressing the conditional expectation under \(\mathbb{Q}\) by the conditional expectation under \(\mathbb{P}\), so that
local martingales are continuous as well) and that $\lambda_T \in A_\alpha$ for all $\alpha \in (1, \infty)$. Hence the passage from $\mathbb{Q}$ to $\mathbb{P}$ corresponds to adding a drift to the diffusion $X$.

(8) In the case the drift term in item (7) is Markovian, i.e. $\beta_t = \beta(t, X_t)$ for an appropriate $\beta : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, and if we let $Y_t := v(t, X_t)$ and $Z_t := \nabla v(t, X_t) \sigma(t, X_t)$, then we get the BSDE

$$-dY_t = [k(t, X_t)Y_t + Z_t \sigma^{-1}(t, X_t)\beta_t]dt - Z_t dW_t,$$

$$Y_T = g(X_T).$$

Then it is proved in [5] under certain conditions the equivalence between the following assertions for $p \in [2, \infty)$, $\theta \in (0, 1]$ and polynomially bounded $g$:

(a) $\sup_{t \in [0, T]} (T - t)^{-\frac{\theta}{2}} \|g(X_T) - \mathbb{E}^{\mathbb{F}_t}(g(X_T))\|_{L^p(\mathbb{P})} < +\infty$.

(b) $\sup_{t \in [0, T]} (T - t)^{-\frac{1}{2}} \|Z_t\|_{L^{p}(\mathbb{P})} < +\infty$.

These are the analogues of (i$\theta$) and (ii$\theta$) for $q = \infty$.

4 Proof of Theorem 1

Through the whole section we assume that the condition (P) is satisfied.

4.1 Preliminaries

To estimate $L_p$ norms under different measures, the following lemma is useful.

**Lemma 2.** For any $1 < \alpha < p < \infty$, $\lambda_T \in A_\alpha(\mathbb{Q})$, $r := \frac{\alpha}{p - \alpha}$, $U \in L_p(\Omega, \mathcal{F}, \mathbb{P})$, $V \in L_r(\Omega, \mathcal{F}, \mathbb{Q})$ and $c_8 > 0$ such that $[\mathbb{E}^{\mathbb{F}_t}(\|\frac{1}{\lambda_T}|V|^r\|)^\frac{1}{r} \leq c_8$ a.s. we have that

$$\mathbb{E}^{\mathbb{F}_t} |UV| \leq c_8 \left[ \mathbb{E}^{\mathbb{F}_t} |U|^p \right]^\frac{1}{p} \left[ \mathbb{E}^{\mathbb{F}_t} |V|^r \right]^\frac{1}{r} \ a.s. \quad (8)$$

**Proof.** Letting $1 = \frac{1}{p} + \frac{1}{p'} = \frac{1}{\alpha} + \frac{1}{\alpha'}$ one has a.s. that

$$\mathbb{E}^{\mathbb{F}_t} |UV| = \lambda_t \mathbb{E}^{\mathbb{F}_t} (|UV|/\lambda_T)$$

$$\leq \lambda_t [\mathbb{E}^{\mathbb{F}_t} |U|^p]^{\frac{1}{p}} [\mathbb{E}^{\mathbb{F}_t} (|V|^r \lambda_T^{-\frac{p'}{r}} \lambda_T^{-\frac{p'}{r}})]^{\frac{1}{r'}}$$
\[ \begin{align*}
&\leq \lambda_{1} \left[ \mathbb{E}^{\mathcal{F}_{t}}_{\mathbb{P}} |U|^{p} \right]^{\frac{1}{p}} \left[ \mathbb{E}^{\mathcal{F}_{t}}_{\mathbb{P}} \left( |V|^{r}/\lambda_{T} \right) \right]^{\frac{1}{r}} \left[ \mathbb{E}^{\mathcal{F}_{t}}_{\mathbb{P}} \lambda_{T}^{-\alpha'} \right]^{\frac{\alpha'-\tau}{\alpha'-\tau}} \\
&\leq c_{\mathcal{F}} \left[ \mathbb{E}^{\mathcal{F}_{t}}_{\mathbb{P}} |U|^{p} \right]^{\frac{1}{p}} \left[ \mathbb{E}^{\mathcal{F}_{t}}_{\mathbb{Q}} |V|^{r} \right]^{\frac{1}{r}}.
\end{align*} \]

As simple consequences of this lemma for \( V \equiv 1 \), observe that
\[ \| \mathbb{E}^{\mathcal{F}_{t}}_{\mathbb{Q}} U \|_{L_{p}(\mathbb{P})} \leq c_{\mathcal{F}} \| U \|_{L_{p}(\mathbb{P})} \quad \text{for} \quad U \in L_{p}(\mathbb{P}). \quad (9) \]

In the next step we will estimate \( \nabla v(t, X_{t}) \) and \( D^{2} v(t, X_{t}) \) in Lemmas 3 and 6 from above by conditional moments of \( M_{T} = K^{X}_{T} g(X_{T}) \) and \( g(X_{T}) \), and extend therefore Lemma 1 to the case \( k = 0 \) and allow a change of measure by Muckenhoupt weights.

**Lemma 3.** For any \( p \in (1, \infty) \), we have a.s. that
\[ |\nabla v(t, X_{t})| \leq c_{10} \left[ \left( \mathbb{E}^{\mathcal{F}_{t}}_{\mathbb{Q}} |M_{T} - \mathbb{E}^{\mathcal{F}_{t}}_{\mathbb{Q}} M_{T}|^{p} \right)^{\frac{1}{p}} + (T - t) \left( \mathbb{E}^{\mathcal{F}_{t}}_{\mathbb{Q}} |M_{T}|^{p} \right)^{\frac{1}{p}} \right], \]
where \( c_{10} > 0 \) depends at most on \( (\sigma, b, k, p) \). The same estimate holds true if the measure \( \mathbb{Q} \) is replaced by the measure \( \mathbb{P} \) with \( \lambda \in A_{\alpha}(\mathbb{Q}) \) and \( \alpha \in (1, p) \), where the constant \( c_{10} > 0 \) might additionally depend on \( \mathbb{Q} \) (and therefore implicitly on \( \alpha \)).

**Proof.** The statement for \( \mathbb{P} \) for \( p \in (1, \infty) \) can be deduced from the statement for \( \mathbb{Q} \) for \( q \in (1, p) \). Let us fix \( 1 < q < p < \infty \), define \( p_{0} := p/q \in (1, \infty) \), take \( r \in (p_{0}', \infty) \) and let \( \beta := \frac{p_{0}' - p_{0}}{r - p_{0}} \). For \( \lambda \in A_{\alpha}(\mathbb{Q}) \) with \( 1 = (1/\alpha) + (1/\beta) \) we apply Lemma 2 with \( p \) replaced by \( p_{0} \) and get
\[ \left( \mathbb{E}^{\mathcal{F}_{t}}_{\mathbb{Q}} |Z|^{q} \right)^{\frac{1}{q}} \leq c_{\mathcal{F}} \left( \mathbb{E}^{\mathcal{F}_{t}}_{\mathbb{P}} |Z|^{p} \right)^{\frac{1}{p}} \]
and, by (6),
\[ \left( \mathbb{E}^{\mathcal{F}_{t}}_{\mathbb{Q}} |Z - \mathbb{E}^{\mathcal{F}_{t}}_{\mathbb{Q}} Z|^{q} \right)^{\frac{1}{q}} \leq 2 \left( \mathbb{E}^{\mathcal{F}_{t}}_{\mathbb{Q}} |Z - \mathbb{E}^{\mathcal{F}_{t}}_{\mathbb{P}} Z|^{q} \right)^{\frac{1}{q}} \leq 2c_{\mathcal{F}} \left( \mathbb{E}^{\mathcal{F}_{t}}_{\mathbb{P}} |Z - \mathbb{E}^{\mathcal{F}_{t}}_{\mathbb{P}} Z|^{p} \right)^{\frac{1}{p}} \]
whenever \( Z \in \bigcap_{r \in [1, \infty)} L_{r}(\mathbb{Q}) \) (cf. Remark 1). Because \( \lim_{r \to \infty} \frac{p_{0}' - p_{0}}{r - p_{0}} = p_{0}' = \frac{p}{p - q} \) and the convergence is from above, we can take \( \beta \) to be in \( \left( \frac{p}{p - q}, \infty \right) \).

Sending \( q \) to 1 gives that \( \beta \in \left( \frac{p}{p - 1}, \infty \right) \) or \( \alpha \in (1, p) \).
Now we follow a martingale approach (see, for example, [9]) and prove the statement for the measure $\mathcal{Q}$.

(a) We define $(\nabla X_t)_{t \in [0,T]}$ to be the solution of a linear SDE (see [15, Chapter 5]):

$$\nabla X_t = I_d + \sum_{j=1}^{d} \int_0^t \nabla \sigma_j(s, X_s) \nabla X_s dB^j_s + \int_0^t \nabla b(s, X_s) \nabla X_s ds$$

and $\sigma(.) = (\sigma_1(.), \ldots, \sigma_d(.))$. This matrix-valued process is a.s. invertible and its inverse satisfies

$$[\nabla X_t]^{-1} = I_d - \sum_{j=1}^{d} \int_0^t [\nabla X_s]^{-1} \nabla \sigma_j(s, X_s) dB^j_s - \int_0^t [\nabla X_s]^{-1} (\nabla b(s, X_s) - \sum_{j=1}^{d} (\nabla \sigma_j(s, X_s))^2) ds.$$ 

(b) Next we show that $(N_t)_{t \in [0,T]}$ with

$$N_t := K_t^X \nabla v(t, X_t) \nabla X_t + \left( \int_0^t \nabla k(s, X_s) \nabla X_s ds \right) M_t$$

is a $\mathcal{Q}$-martingale. One way consists in using Itô’s formula to verify that $N$ is a martingale. In fact, the bounded variation term in the Itô-process decomposition of $N$ is

$$\int_0^t [K_t^X k(s, X_s) \nabla v(s, X_s) \nabla X_s + K_t^X C_s] ds + \int_0^t [\nabla k(s, X_s) \nabla X_s M_s] ds,$$

where $\int_0^t C_s ds$ is the bounded variation term of $\nabla v(t, X_t) \nabla X_t$. Hence it is sufficient to show that

$$C_s = -\nabla [v(s, X_s) k(s, X_s)] \nabla X_s.$$

The PDE for $w = \nabla v$ on $[0, T) \times \mathbb{R}^d$ reads as

$$\frac{\partial}{\partial t} w_i + \frac{1}{2} \{A, D^2 w_i\} + \langle b, (\nabla w_i)^T \rangle = -\frac{1}{2} \langle \partial_{x_i} A, D^2 v \rangle - \langle \partial_{x_i} b, w_i^T \rangle - \partial_{x_i} (vk).$$

(11)
By a simple computation this gives that the bounded variation term of 
\( \sum_{i=1}^{d} \frac{\partial u}{\partial x_i}(t, X_t)(\nabla X_t)_{i\ell} \) computes as
\(- \sum_{i=1}^{d} \frac{\partial(u_k)}{\partial x_i}(t, X_t)(\nabla X_s)_{i\ell} dt \) and step (b) is complete.

(c) Exploiting the martingale property of \( N \) between \( t \) and some deterministic \( S \in (t, T) \), we have
\[
(S - t) \left[ K_t^X \nabla v(t, X_t) + \left( \int_0^t \nabla k(s, X_s) \nabla X_s ds \right) M_t \right] = \mathbb{E}_{Q}^{F_t} \left( \int_t^S \left[ K_r^X \nabla v(r, X_r) + \left( \int_0^r \nabla k(s, X_s) \nabla X_s ds \right) M_r \right] dr \right) 
\]
\[
= \mathbb{E}_{Q}^{F_t} \left( \left[ \int_t^S K_r^X \nabla v(r, X_r) \sigma(r, X_r) dB_r \right] \left[ \int_t^r (\sigma(r, X_r)^{-1} \nabla X_r)^\top dB_r \right]^\top \right) 
\]
\[
+ (S - t) M_t \left[ \int_0^t \nabla k(s, X_s) \nabla X_s ds \right] 
+ \mathbb{E}_{Q}^{F_t} \left( M_S \int_t^S \left[ \int_t^r \nabla k(s, X_s) \nabla X_s ds \right] dr \right). \quad (12)
\]

At the last equality, we have used the \( Q \)-martingale property of \( (M_t)_{t \in [0,T]} \)
and the conditional Itô isometry
\[
\mathbb{E}_{Q}^{F_t} \left( \left[ \int_t^S A_{1,r} dB_r \right] \left[ \int_t^r A_{2,r} dB_r \right]^\top \right) = \mathbb{E}_{Q}^{F_t} \left( \int_t^S A_{1,r} A_{2,r}^\top dr \right)
\]
(available for any square integrable and progressively measurable matrix-valued processes \( (A_{1,r})_r \) and \( (A_{2,r})_r \), having \( d \) columns and an arbitrary number of rows). After simplifications, (12) writes
\[
(S - t) K_t^X \nabla v(t, X_t) \nabla X_t 
\]
\[
= \mathbb{E}_{Q}^{F_t} \left( \left[ M_S - M_t \right] \left[ \int_t^S (\sigma(r, X_r)^{-1} \nabla X_r)^\top dB_r \right]^\top \right) 
\]
\[
+ \mathbb{E}_{Q}^{F_t} \left( M_S \left[ \int_t^S (S - s) \nabla k(s, X_s) \nabla X_s ds \right] \right). 
\]

Using that \( M_S \to M_T \) in \( L_2(Q) \) we derive
\[
(T - t) K_t^X \nabla v(t, X_t)
\]
\[ E_{Q}^{F_{r}} \left( [M_{T} - M_{1}] \left[ \int_{t}^{T} (\sigma(r, X_{r})^{-1} \nabla X_{r}[\nabla X_{t}]^{-1})^{\top} dB_{r} \right]^{\top} \right) + E_{Q}^{F_{r}} \left( M_{T} \left[ \int_{t}^{T} (T - s) \nabla k(s, X_{s}) \nabla X_{t}[\nabla X_{t}]^{-1} ds \right] \right). \]

Finally, observe that \( \sup_{t \in [0, T]} \sup_{r \in [t, T]} E_{Q}^{F_{r}} (|\nabla X_{r}[\nabla X_{t}]^{-1}|^{q}) \) is a bounded random variable for any \( q \geq 1 \); therefore, standard computations complete our assertion. \( \square \)

For the following we let \( m(t, x) := v(t, x)k(t, x) \).

**Lemma 4.** For \( 0 \leq r < t \leq T \) and \( 1 < p_{0} < p < \infty \) one has a.s. that

\[ \left( E_{Q}^{F_{r}} |m(t, X_{t}) - m(t, X_{t})|^{p_{0}} \right)^{\frac{1}{p_{0}}} \leq c_{13} \left[ \sqrt{t - r} \left( E_{Q}^{F_{r}} |M^{r}|^{p} \right)^{\frac{1}{p}} + \left( E_{Q}^{F_{r}} |M_{t} - M_{r}|^{p_{0}} \right)^{\frac{1}{p_{0}}} \right] \quad (13) \]

where \( M^{r} := \sup_{s \in [0, T]} |M_{s}| \) and \( c_{13} > 0 \) depends at most on \( (p_{0}, p, \sigma, b, k) \).

**Proof.** (a) For \( \frac{1}{p_{0}} = \frac{1}{q_{k}} + \frac{1}{r_{k}} = \frac{1}{s_{k}} + \frac{1}{r_{k}} \) with \( r_{k}, s_{k}, t_{k} \in [p_{0}, \infty] \), a sub-\( \sigma \)-algebra \( G \subseteq \mathcal{F}, U_{k} := U_{1} \cdots U_{k} \) and \( U_{k} := U_{k} \cdots U_{N} \) with \( U_{j} := 1 \) and \( U_{N+1} := 1 \), and for \( U_{k-1} \in L_{k}(\mathbb{Q}), U_{k} \in L_{s_{k}}(\mathbb{Q}), U_{k+1} \in L_{r_{k}}(\mathbb{Q}) \), where \( k = 1, \ldots, N \), we get by a telescoping sum argument and the conditional Hölder inequality that

\[ \left( E_{Q}^{G} |U_{1} \cdots U_{N} - E_{Q}^{G} (U_{1} \cdots U_{N})|^{p_{0}} \right)^{\frac{1}{p_{0}}} \leq \sum_{k=1}^{N} \left( E_{Q}^{G} \left[ E_{Q}^{G} (U_{k-1}) \right] U_{k} - E_{Q}^{G} (U_{k}) \right)^{\frac{1}{q_{k}}} \left( E_{Q}^{G} |U_{k+1}|^{r_{k}} \right)^{\frac{1}{r_{k}}} \]

\[ \leq \sum_{k=1}^{N} \left( E_{Q}^{G} \left[ E_{Q}^{G} (U_{k-1}) \right] U_{k} - \left[ E_{Q}^{G} (U_{k}) \right] \right)^{\frac{1}{q_{k}}} \left( E_{Q}^{G} |U_{k+1}|^{r_{k}} \right)^{\frac{1}{r_{k}}} \]

\[ + \sum_{k=1}^{N} \left( E_{Q}^{G} \left[ E_{Q}^{G} (U_{k-1}) \right] E_{Q}^{G} U_{k} - \left[ E_{Q}^{G} (U_{k}) \right] \right)^{\frac{1}{q_{k}}} \left( E_{Q}^{G} |U_{k+1}|^{r_{k}} \right)^{\frac{1}{r_{k}}} \]

\[ \leq 2 \sum_{k=1}^{N} \left( E_{Q}^{G} |U_{k-1}|^{r_{k}} \right)^{\frac{1}{r_{k}}} \left( E_{Q}^{G} |U_{k} - E_{Q}^{G} U_{k}|^{s_{k}} \right)^{\frac{1}{s_{k}}} \left( E_{Q}^{G} |U_{k+1}|^{r_{k}} \right)^{\frac{1}{r_{k}}}. \]
Lemma 5. For $0 \leq r < t < T$ and $p \in (1, \infty)$ one has a.s. that

$$
(\mathbb{E}_{Q}^{F_t} |M_t - M_r|^p)^{\frac{1}{p}} \leq c_{[4]} \left[ \left( \frac{t-r}{T-t} \right)^{\frac{1}{2}} (\mathbb{E}_{Q}^{F_t} |M_T - M_r|^p)^{\frac{1}{p}} + (t-r)^{\frac{1}{2}} |M_r| \right]
$$

where $c_{[4]} \geq 1$ depends at most on $(p, \sigma, b, k)$.

Proof. Let $p_0 := \frac{1+p}{2}$, $\lambda_u := K_u X \nabla v(u, X_u) \sigma(u, X_u)$ and $0 \leq r \leq u \leq t$. Then Lemma 3 implies that

$$
|\lambda_u| e^{-T\|k\|_{\infty}}
\leq \|\sigma\|_{\infty} c_{[3]} (T-u)^{-\frac{1}{2}} \left( \mathbb{E}_{Q}^{F_u} |M_T - M_u|^p \right)^{\frac{1}{p_0}} + (T-u) \left( \mathbb{E}_{Q}^{F_u} |M_T|^p \right)^{\frac{1}{p_0}}
$$

and

$$
(\mathbb{E}_{Q}^{F_t} |K_t^{-1} - F_t^{-1}|^p)^{\frac{1}{p}} \leq \|\sigma\|_{\infty} c_{[4]} (T-r)^{\frac{1}{2}} |M_r|.
$$

(b) We apply (a) to $N = 3$ and $m(s, X_s) = k(s, X_s)(K_s^{-1} - M_s)$ to derive

$$
(\mathbb{E}_{Q}^{F_t} |m(t, X_t)|^p)^{\frac{1}{p}} \leq 2\|k\|_{\infty} e^{T\|k\|_{\infty}} (\mathbb{E}_{Q}^{F_t} |M_T - M_r|^p)^{\frac{1}{p_0}}
$$

and

$$
0 \leq (\mathbb{E}_{Q}^{F_t} |K_t^{-1} - F_t^{-1}|^p)^{\frac{1}{p}} \leq 2\|k\|_{\infty} (\mathbb{E}_{Q}^{F_t} |K_t^{-1}|^p)^{\frac{1}{p}}.
$$

We conclude by

$$
(\mathbb{E}_{Q}^{F_t} |k(t, X_t)|^p)^{\frac{1}{p}} \leq 2 (\mathbb{E}_{Q}^{F_t} |k(t, X_r)|^p)^{\frac{1}{p}}
$$

and

$$
(\mathbb{E}_{Q}^{F_t} |(K_t^{-1} - F_t^{-1})|^p)^{\frac{1}{p}} \leq 2\|k\|_{\infty} (t-r)e^{T\|k\|_{\infty}}.
$$

$\Box$
\[ \leq \|\sigma\|_{\infty}^{c} \|T - u\|^{\frac{1}{2}} \left( \mathbb{E}_{\mathbb{Q}}^{F} |M_{T} - M_{r}|^{p_{0}} \right)^{\frac{1}{p_{0}}} + (T - u)|M_{r}| \]
\[ \leq \|\sigma\|_{\infty}^{c} (2 + T^{\frac{3}{2}} + T) \left[ (T - t)^{-\frac{1}{2}} \left( \mathbb{E}_{\mathbb{Q}}^{F} |M_{T} - M_{r}|^{p_{0}} \right)^{\frac{1}{p_{0}}} + |M_{r}| \right]. \]

Letting \( c := e^{T\|k\|_{\infty}} \|\sigma\|_{\infty}^{c} (2 + T^{\frac{3}{2}} + T) \) we conclude the proof by using the Burkholder-Davis-Gundy inequalities in order to get

\[ \frac{1}{a_{p}} \left( \mathbb{E}_{\mathbb{Q}}^{F} |M_{t} - M_{r}|^{p} \right)^{\frac{1}{p}} \]
\[ \leq \left( \mathbb{E}_{\mathbb{Q}}^{F} \left( \int_{r}^{t} |\lambda_{u}|^{2} du \right) \right)^{\frac{1}{p}} \]
\[ \leq c \left[ (T - t)^{-\frac{1}{2}} \left( \mathbb{E}_{\mathbb{Q}}^{F} \left( \int_{r}^{t} \left( \mathbb{E}_{\mathbb{Q}}^{F} |M_{T} - M_{r}|^{p_{0}} \right)^{\frac{2}{p_{0}}} du \right) \right)^{\frac{2}{p_{0}}} \right]^{\frac{1}{p}} \]
\[ + \sqrt{T - r}|M_{r}| \]
\[ \leq c \left[ \left( \mathbb{E}_{\mathbb{Q}}^{F} \left( \sup_{u \in [r,t]} \mathbb{E}_{\mathbb{Q}}^{F} |M_{T} - M_{r}|^{p_{0}} \right)^{\frac{p}{p_{0}}} \right)^{\frac{1}{p}} \right]^{\frac{1}{p}} \]
\[ \leq c \left[ \left( \mathbb{E}_{\mathbb{Q}}^{F} \left( \sup_{u \in [r,t]} \mathbb{E}_{\mathbb{Q}}^{F} |M_{T} - M_{r}|^{p_{0}} \right)^{\frac{p}{p_{0}}} \right)^{\frac{1}{p}} \right]^{\frac{1}{p}} \]
\[ + \sqrt{T - r}|M_{r}| \]
\[ \leq c \left[ \left( \mathbb{E}_{\mathbb{Q}}^{F} \left( \frac{p}{p_{0}} \left( \sup_{u \in [r,t]} \mathbb{E}_{\mathbb{Q}}^{F} |M_{T} - M_{r}|^{p_{0}} \right)^{\frac{p}{p_{0}}} \right)^{\frac{1}{p}} \right]^{\frac{1}{p}} \]
\[ + \sqrt{T - r}|M_{r}| \]
\[ \leq c \left[ \left( \mathbb{E}_{\mathbb{Q}}^{F} \left( \sup_{u \in [r,t]} \mathbb{E}_{\mathbb{Q}}^{F} |M_{T} - M_{r}|^{p_{0}} \right)^{\frac{p}{p_{0}}} \right)^{\frac{1}{p}} \right]^{\frac{1}{p}} \]
\[ + \sqrt{T - r}|M_{r}| \] .

\[ \square \]

**Lemma 6.** For \( p \in (1, \infty) \) there is a constant \( c_{15} = c(\sigma, b, k, p) > 0 \) such that one has a.s. that

\[ |D^{2}v(r, X_{r})| \leq c_{15} \left[ \left( \mathbb{E}_{\mathbb{Q}}^{F} \left| g(X_{T}) - \mathbb{E}_{\mathbb{Q}}^{F} g(X_{T}) \right|^{p} \right)^{\frac{1}{p}} \right] + \sqrt{T - r} \left( \mathbb{E}_{\mathbb{Q}}^{F} |M^{*}|^{p} \right)^{\frac{1}{p}} . \]
The same estimate holds true if the measure $Q$ is replaced by the measure $\mathbb{P}$ with $\lambda \in A_{\alpha}(Q)$ and $\alpha \in (1,p)$, where the constant $c_{(15)} > 0$ might additionally depend on $Q$ (and therefore implicitly on $\alpha$).

**Proof.** (a) The statement for $\mathbb{P}$ for $p \in (1,\infty)$ can be deduced from the statement for $Q$ for $q \in (1,p)$ as in the first step of the proof of Lemma 3.

(b) Now we show the estimate for the measure $Q$. For $0 \leq s \leq t \leq T$, a fixed $T_0 \in (0,T)$ and $r \in [0,T_0]$ we let

$$v^t(s,x) := \mathbb{E}_Q(m(t,X_t)|X_s = x) \quad \text{and} \quad v_h(r,x) := \mathbb{E}_Q(v(T_0,X_{T_0})|X_r = x).$$

Itô’s formula applied to $v$ gives for $r \in [0,T_0]$ that

$$v(r,x) = \mathbb{E}_Q(v(T_0,X_{T_0}) + \int_r^{T_0} (kv)(t,X_t)dt|X_r = x)$$

and therefore

$$v(r,x) = v_h(r,x) + \int_r^{T_0} v^t(r,x)dt.$$  

Using Lemma [1] and the arguments from Remark 2(5) one can show for $0 \leq r < t \leq T_0 < T$ that

$$|\nabla v^t(r,x)| \leq \gamma e^{\gamma |x|^k} \quad \text{and} \quad |D^2 v^t(r,x)| \leq \frac{\gamma}{\sqrt{t-r}} e^{\gamma |x|^k}, \quad (16)$$

where $\gamma > 0$ depends at most on $(\sigma, b, k, K_g, k_g, T_0)$. From this we deduce that

$$D^2 v(r,x) = D^2 v_h(r,x) + \int_r^{T_0} D^2 v^t(r,x)dt$$

where (16) are used to interchange the integral and $D^2$. For $p_0 := \frac{1+p}{2}$, $0 \leq r < t \leq T$ and $s \in [0,T_0)$ we again use Lemma [1] to get

$$|D^2 v^t(r,X_r)| \leq \frac{k_{p_0}}{(t-r)} \left( \mathbb{E}_Q^F |m(t,X_t) - \mathbb{E}_Q^F m(t,X_t)|^{p_0} \right)^{\frac{1}{p_0}} \quad \text{a.s.},$$

$$|D^2 v_h(s,X_s)| \leq \frac{k_p}{(T_0-s)} \left( \mathbb{E}_Q^F |v(T_0,X_{T_0}) - \mathbb{E}_Q^F v(T_0,X_{T_0})|^p \right)^{\frac{1}{p}} \quad \text{a.s.}$$

From the first estimate we derive by Lemmas [1] and [5] (with $p$ replaced by $p_0$) a.s. that

$$|D^2 v^t(r,X_r)|$$
Proof. Owing to inequality (6) and applying Proposition 3, we get

\[
\begin{align*}
&\leq \frac{\kappa_{p_0}}{(t-r)} \left( \mathbb{E}_Q^{F_r} |m(t, X_t) - \mathbb{E}_Q^{F_r} m(t, X_t)|^{p_0} \right)^{\frac{1}{p_0}} \\
&\leq \frac{\kappa_{p_0} c_{[13]}(t-r)}{(t-r)} \left[ \sqrt{t-r} \left( \mathbb{E}_Q^{F_r} |M^*|^p \right)^{\frac{1}{p}} + \left( \mathbb{E}_Q^{F_r} |M_t - M_r|^{p_0} \right)^{\frac{1}{p_0}} \right] \\
&\leq \kappa_{p_0} c_{[13]} [1 + c_{[14]}] \frac{1}{\sqrt{t-r}} \left( \mathbb{E}_Q^{F_r} |M^*|^p \right)^{\frac{1}{p}} \\
&\quad + \kappa_{p_0} c_{[13]} c_{[14]} \frac{1}{\sqrt{t-r}} \left( \mathbb{E}_Q^{F_r} |M_T - M_r|^{p_0} \right)^{\frac{1}{p_0}}
\end{align*}
\]

and

\[
\begin{align*}
\int_T^r |D^2 v^r(r, X_r)| \, dt \leq c \left[ \sqrt{T-r} \left( \mathbb{E}_Q^{F_r} |M^*|^p \right)^{\frac{1}{p}} + \left( \mathbb{E}_Q^{F_r} |M_T - M_r|^{p_0} \right)^{\frac{1}{p_0}} \right]
\end{align*}
\]

with \( c := \kappa_{p_0} c_{[13]} \max \{ 2 + 2c_{[14]}, c_{[14]} \} \). The second estimate yields by \( T_0 \uparrow T \) and (5) that

\[
|D^2 v^r(r, X_r)| \leq \frac{\kappa_{p}}{(T-r)} \left( \mathbb{E}_Q^{F_r} |g(X_T) - \mathbb{E}_Q^{F_r} g(X_T)|^{p} \right)^{\frac{1}{p}}
\]

and the upper bound is independent of \( T_0 \). Combining the estimates with

\[
\left( \mathbb{E}_Q^{F_r} |M_T - M_r|^{p_0} \right)^{\frac{1}{p_0}} \leq 2 e^{\|k\|\infty T} \left[ \|k\|\infty (T-r) e^{\|k\|\infty T} \left( \mathbb{E}_Q^{F_r} |M^*|^p \right)^{\frac{1}{p}} + \left( \mathbb{E}_Q^{F_r} |g(X_T) - \mathbb{E}_Q^{F_r} g(X_T)|^{p} \right)^{\frac{1}{p}} \right]
\]

using the arguments from Remark [25] the proof is complete. \( \square \)

Lemma 7. Let \( \lambda = \mathcal{E}(Y) \), where \( Y \) is a BMO-martingale with \( Y_0 = 0 \). Then, for \( p \in (1, \infty) \), \( t \in [0, T] \) and \( c_{[14]} := 2b_p(\mathbb{P}) e^{T\|k\|\infty} \|\sigma\|\infty \) we have that

\[
\| M_T - \mathbb{E}_P^{F_r} M_T \|_{L_p(\mathbb{P})} \leq c_{[14]} \left( \left( \int_t^T |\nabla v(s, X_s)|^2 \, ds \right)^{\frac{1}{2}} \right) \|_{L_p(\mathbb{P})}.
\]

Proof. Owing to inequality (4) and applying Proposition [3] we get

\[
\begin{align*}
\| M_T - \mathbb{E}_P^{F_r} M_T \|_{L_p(\mathbb{P})} &\leq 2 \| M_T - M_t \|_{L_p(\mathbb{P})} \\
&\leq 2b_p(\mathbb{P}) \| \sqrt{\langle M \rangle_T - \langle M \rangle_t} \|_{L_p(\mathbb{P})}
\end{align*}
\]

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\[ = 2b_p(\mathbb{P}) \left\| \sqrt{\int_t^T |K^X_s \nabla v(s, X_s)\sigma(s, X_s)|^2 ds} \right\|_{L_p(\mathbb{P})}. \]

Lemma 8. For \( p \in [2, \infty) \), \( \lambda_T \in A_\alpha(\mathbb{Q}) \) with \( \alpha \in (1, p) \), \( 0 \leq s < t < T \) and \( l = 1, \ldots, d \) we have that

\[
\left\| K^X_t \partial_{x_l} v(t, X_t) - K^X_s \partial_{x_l} v(s, X_s) \right\|_{L_p(\mathbb{P})} \\
\leq c_{(18)} \left[ \left\| M_T \right\|_{L_p(\mathbb{P})} \int_s^t \frac{dr}{\sqrt{T-r}} + \left( \int_s^t \left\| D^2 v(r, X_r) \right\|_{L_p(\mathbb{P})}^2 dr \right)^{\frac{1}{2}} \right] \tag{18}
\]

with \( c_{(18)} > 0 \) depending at most on \((\sigma, b, k, p, \mathbb{P})\) (and therefore implicitly on \( \alpha \)).

Proof. Exploiting (11) and Propositions 2 and 3 we get that

\[
\left\| K^X_t \partial_{x_l} v(t, X_t) - K^X_s \partial_{x_l} v(s, X_s) \right\|_{L_p(\mathbb{P})} \\
\leq \left\| \int_s^t K^X_r (\nabla \partial_{x_l} v)(r, X_r)\sigma(r, X_r) dB_r \right\|_{L_p(\mathbb{P})} \\
+ \left\| \int_s^t K^X_r \left[ \frac{1}{2} \left\langle \partial_{x_l} A(r, X_r), D^2(r, X_r) \right\rangle + \left\langle \partial_{x_l} b(r, X_r), \nabla v(r, X_r)^\top \right\rangle \right] \right\|_{L_p(\mathbb{P})} \\
\leq b_p(\mathbb{P}) \left\| \left( \int_s^t \left| K^X_r (\nabla \partial_{x_l} v)(r, X_r)\sigma(r, X_r) \right|^2 dr \right) \right\|_{L_p(\mathbb{P})} \\
+ \frac{1}{2} \left\| \partial_{x_l} A \right\|_\infty \left\| \int_s^t \left| K^X_r D^2 v(r, X_r) \right| dr \right\|_{L_p(\mathbb{P})} \\
+ \left\| \partial_{x_l} b \right\|_\infty \left\| \int_s^t \left| K^X_r \nabla v(r, X_r) \right| dr \right\|_{L_p(\mathbb{P})} \\
+ \left\| \partial_{x_l} k \right\|_\infty \left\| \int_s^t \left| K^X_r v(r, X_r) \right| dr \right\|_{L_p(\mathbb{P})}. 
\]

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Inequality (9) directly yields
\[
\sup_{r \in [0,T]} \| K^X_t v(r, X_r) \|_{L^p(\mathbb{P})} = \sup_{r \in [0,T]} \| \mathbb{E}_{\mathbb{P}}^F_{r} M_T \|_{L^p(\mathbb{P})} \leq c |C8| \| M_T \|_{L^p(\mathbb{P})}.
\]
Moreover, by Lemma 3,
\[
\| \nabla v(r, X_r) \|_{L^p(\mathbb{P})} \leq c |C8| (T - r)^{-\frac{1}{2}} \left( 2 + T^{3/2} \right) \| M_T \|_{L^p(\mathbb{P})}.
\]
Inserting these estimates in the above upper bound for
\[
\| K^X_t \partial_{x_l} v(t, X_t) - K^X_s \partial_{x_l} v(s, X_s) \|_{L^p(\mathbb{P})}
\]
gives the announced result.

**Lemma 9** ([8, Proposition A.4]). Let \(0 < \theta < 1, 2 \leq q \leq \infty\) and \(d^k : [0,T] \to [0,\infty), k = 0, 1, 2,\) be measurable functions. Assume that there are \(A \geq 0\) and \(D \geq 1\) such that
\[
\frac{1}{D} (T - t)^{\frac{k}{2}} d^k(t) \leq d^0(t) \leq D \left( \left( \int_t^T [d^1(s)]^2 ds \right)^{\frac{1}{2}} \right),
\]
\[
d^1(t) \leq A + D \left( \left( \int_0^t [d^2(u)]^2 du \right)^{\frac{1}{2}} \right)
\]
for \(k = 1, 2\) and \(t \in [0,T)\). Then there is a constant \(c |19| > 0\), depending at most on \((D, \theta, q, T)\), such that, for \(k, l \in \{0, 1, 2\}\),
\[
A + \Phi_q \left( (T - t)^{-\frac{\theta}{2}} d^k(t) \right) \sim_{c |19|} A + \Phi_q \left( (T - t)^{-\frac{\theta}{2}} d^l(t) \right). \tag{19}
\]

**4.2 Proof of Theorem 1**

We let
\[
d^0(t) := \sqrt{T - t} + \| M_T - \mathbb{E}^F_{\mathbb{P}} M_T \|_{L^p(\mathbb{P})},
\]
\[
d^1(t) := 1 + \| \nabla v(t, X_t) \|_{L^p(\mathbb{P})},
\]
\[
d^2(t) := 1 + \| D^2 v(t, X_t) \|_{L^p(\mathbb{P})}.
\]
From Lemma 3 we get that
\[
d^1(t) = 1 + \| \nabla v(t, X_t) \|_{L^p(\mathbb{P})}
\]

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\[ \leq 1 + c_{10} (T - t)^{-\frac{1}{2}} \| MT - \mathbb{E}_P^T M_T \|_{L_p(P)} + c_{10} (T - t) \| M_T \|_{L_p(P)} \]
\[ \leq (T - t)^{-\frac{1}{2}} [1 + c_{10} + c_{10} T \| M_T \|_{L_p(P)}] \]
\[ = \left( \sqrt{T - t} + \frac{1}{T - t} \right) [1 + c_{10} T \| M_T \|_{L_p(P)}] d^0(t). \]

From Lemma 6 we get that
\[ d^2(t) = 1 + \| D^2 v(t, X_t) \|_{L_p(P)} \]
\[ \leq 1 + c_{10} \left[ \frac{\| g(X_T) - \mathbb{E}_P^T g(X_T) \|_{L_p(P)}}{T - t} + \sqrt{T - t} \| M^* \|_{L_p(P)} \right]. \]

Using Remark 2(5) we have that
\[ \| g(X_T) - \mathbb{E}_P^T g(X_T) \|_{L_p(P)} \leq 2e^{\| k \|_{\infty} T} \left( \| k \|_{\infty} (T - t) \| M_T \|_{L_p(P)} + \| M_T - \mathbb{E}_P^T M_T \|_{L_p(P)} \right). \]

Together with the previous estimate we obtain a constant \( c > 0 \) depending at most on \( c_{10}, k, T, \| M^* \|_{L_p(P)} \) such that
\[ d^2(t) \leq c(T - t)^{-1} d^0(t). \]

From Lemma 7 we get that
\[ d^0(t) = \sqrt{T - t} + \| M_T - \mathbb{E}_P^T M_T \|_{L_p(P)} \]
\[ \leq \sqrt{T - t} + c_{10} \left( \int_t^T \| \nabla v(s, X_s) \|_{L_p(P)}^2 ds \right)^{\frac{1}{2}} \]
\[ \leq [1 + c_{10}] \left( \int_t^T [1 + \| \nabla v(s, X_s) \|_{L_p(P)}^2] ds \right)^{\frac{1}{2}} \]
\[ = [1 + c_{10}] \left( \int_t^T [d^1(s)]^2 ds \right)^{\frac{1}{2}}. \]

Finally, from Lemma 8 for \( s = 0 \) we deduce that
\[ d^1(t) = 1 + \| \nabla v(t, X_t) \|_{L_p(P)} \]
\[ \leq 1 + e^{\| k \|_{\infty} T} \| K^X_t \nabla v(t, X_t) \|_{L_p(P)} \]
\[ \leq 1 + e^{\| k \|_{\infty} T} \| K^X_0 \nabla v(0, X_0) \|_{L_p(P)}. \]
\begin{equation}
+ e^{\|k\|_{\infty}} T c_{\text{LS}} \sqrt{d} \left[ \|M_T\|_{L_p(\mathbb{P})} 2 \sqrt{T} + \left( \int_0^t \|D^2 v(r, X_r)\|_{L_p(\mathbb{P})}^2 dr \right)^{\frac{1}{2}} \right]
\end{equation}

\begin{align*}
\leq & \ d_1 + d_2 \left( \int_0^t \|D^2 v(r, X_r)\|_{L_p(\mathbb{P})}^2 \right)^{\frac{1}{2}} \\
\leq & \ d_1 + d_2 \left( \int_0^t [d^2(r)]^2 \right)^{\frac{1}{2}}
\end{align*}

with
\begin{align*}
d_1 := & \ 1 + e^{\|k\|_{\infty}} T \left[ \|K_0 X_0 \nabla v(0, X_0)\|_{L_p(\mathbb{P})} + 2 c_{\text{LS}} \sqrt{dT} \|M_T\|_{L_p(\mathbb{P})} \right], \\
d_2 := & \ e^{\|k\|_{\infty}} T c_{\text{LS}} \sqrt{d}.
\end{align*}

Lemma \[9\] combined with Remark \[2\(25\)] yields the statement. \hfill \Box

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