On the Geometric Diversity of Wavefronts for the Scalar Kolmogorov Ecological Equation

Karel Hasík · Jana Kopfová · Petra Nábělková · Sergei Trofimchuk

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Abstract
We answer three fundamental questions concerning monostable traveling fronts for the scalar Kolmogorov ecological equation with diffusion and spatiotemporal interaction: These are the questions about their existence, uniqueness and geometric shape. In the particular case of the food-limited model, we give a rigorous proof of the existence of a peculiar, yet substantive and nonlinearly determined class of non-monotone and non-oscillating wavefronts.

Keywords Non-local equation · Nonlinear determinacy · Delay · Wavefront · Existence · Uniqueness

Mathematics Subject Classification 34K12 · 35K57 · 92D25
1 Introduction and Main Results

1.1 Traveling Waves in the Scalar Kolmogorov Ecological Equation

In mathematical modeling of single-species populations, one of the key assumptions having clear biological meaning is that the specific growth rate, \( u^{-1} \Delta u / \Delta t \), of the population density \( u \) declines as \( u \) increases (at least, after \( u \) has reached certain level \( u_0 \)) (Smith 1963). In continuous models, this leads to the ordinary differential equation

\[
\frac{d u}{d t} = uG(u), \quad u \geq 0, \tag{1}
\]

where continuous function \( G : \mathbb{R}_+ \to \mathbb{R} \) satisfies the monostability condition

\[
G(u)(1 - u) > 0 \quad \text{for all} \quad u \geq 0, \quad u \neq 1. \tag{2}
\]

Even if Eq. (1) provides an efficient analytical tool for the ecologists, it does not take into account a series of relevant factors. Among them, let us mention generally non-homogeneous spatial and age distribution of the population, the possibility of the long-distance interaction of individuals, the migration effects. There are various ways to incorporate these factors into the basic Eq. (1); perhaps, the most popular ones are represented by the Mackey–Glass-type diffusive equation (Aguerrea et al. 2008; Bani-Yaghoub et al. 2015; Huang et al. 2018; Li et al. 2007; Liang and Wu 2003; Lin et al. 2014)

\[
\frac{d u}{d t}(t, x) = u_{xx}(t, x) - u(t, x) + F((K * u)(t, x)), \tag{3}
\]

and the scalar Kolmogorov ecological equation

\[
\frac{d u}{d t}(t, x) = u_{xx}(t, x) + u(t, x)G((K * u)(t, x)), \quad u \geq 0, \quad (t, x) \in \mathbb{R}^2, \tag{4}
\]

cf. Gourley and Chaplain (2002, formula (4.1)). Here, the non-local spatiotemporal interaction of individuals is expressed in terms of the convolution of their density \( u(t, x) \geq 0, \) considered at some time \( t \) and location \( x \), with an appropriate nonnegative normalized kernel \( K(s, y) \),

\[
(K * u)(t, x) := \int_{\mathbb{R}} \int_{\mathbb{R}_+} K(s, y)u(t-s, x-y)dyds, \quad \int_{\mathbb{R}} \int_{\mathbb{R}_+} K(s, y)dyds = 1. \tag{5}
\]

Among others, Eq. (4) includes the delayed (Benguria and Solar 2019a; Bocharov et al. 2016; Ducrot and Nadin 2014; Faria et al. 2006; Gomez and Trofimchuk 2011, 2014; Hasík and Trofimchuk 2014, 2015; Solar and Trofimchuk 2019; Wu and Zou 2001) and non-local (Alfaro and Coville 2012; Ashwin et al. 2002; Benguria and Solar 2019b; Berestycki et al. 2009; Fang and Zhao 2011; Gourley 2000; Hasík et al. 2016; Solar and Trofimchuk 2019) variants of the KPP–Fisher (i.e., Kolmogorov–Petrovsky–Piskunov–Fisher) equation, diffusive version of Smith’s (1963) food-limited model (Gourley 2001; Gourley and Chaplain 2002; Gourley and So 2002; Ou and Wu 2007; Trofimchuk et al. 2019; Wang and Li 2007; Wei et al. 2017) as well as the single-species model with the Allee effect analyzed in Gopalsamy and Ladas (1990), Han...
et al. (2016), Ruan (2006) and Song et al. (2004). Recall that (4) possesses the Allee effect if the maximal per capita growth $G^* := \max_{u \geq 0} G(u)$ is reached at some positive point, i.e., $G(0) < G^*$, cf. Kuang (2003).

From both mathematical and modeling points of view, the main difference between Eqs. (3) and (4) is that the term $u(t, x)$ enters (3) additively while (4) multiplicatively (Smith 2011, Section 1.1). The multiplicative coupling of the density $u(t, x)$ with its transform $(K \ast u)(t, x)$ can be considered as a complication in the studies of Eq. (4).

Clearly, Eq. (2) implies that $u = 0$ and $u = 1$ are the only nonnegative equilibria of Eq. (4). Besides them, Eq. (4) has many other bounded solutions. This work is dedicated to the studies of the key transitory regimens, wavefronts and semi-wavefronts, connecting the trivial equilibrium and some positive (possibly inhomogeneous) steady state of Eq. (4).

We recall that the classical solution $u(t, x) = \phi(x + ct)$ is a wavefront (or a traveling front) for Eq. (4) propagating with the velocity $c \geq 0$, if the profile $\phi$ is a $C^2$-smooth nonnegative function satisfying the boundary conditions $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. By replacing the condition $\phi(+\infty) = 1$ with the less restrictive requirement

$$0 < \liminf_{s \to +\infty} \phi(s) \leq \limsup_{s \to +\infty} \phi(s) < \infty,$$

we obtain the definition of a semi-wavefront. Clearly, each wave profile $\phi$ to Eq. (4) satisfies

$$\phi''(t) - c\phi'(t) + \phi(t) G((N_c \ast \phi)(t)) = 0, \quad t \in \mathbb{R},$$

(6)

where $N_c(s) := \int_{\mathbb{R}^+} K(v, s - cv)dv \geq 0$, $(N_c \ast \phi)(t) = \int_{\mathbb{R}} N_c(s)\phi(t - s)ds$, $\int_{\mathbb{R}} N_c(s)ds = 1$.

In this work, we are going to answer three fundamental questions concerning the traveling fronts for Eq. (4): These are the questions about their existence, uniqueness and their geometric shape. At the present moment, these aspects are relatively well understood in the case of the Mackey–Glass-type diffusive equation (3) and the KPP–Fisher non-local equation (i.e., when $G(u) = 1 - u$). Quite contrarily, they seem to be only sporadically investigated in the case of the general ecological equation (4). In the particular case of the food-limited model, we also give a rigorous proof of the existence of a peculiar, yet substantive class of non-monotone and non-oscillating wavefronts.

1.2 On the Existence of Semi-wavefronts

Everywhere in the sequel, we assume that $K(s, y)$ is such that the measurable function $N_c(s)$ of two arguments is well defined and depends continuously on $c$ for each fixed $s$. These assumptions are rather weak and can be easily checked in each particular case. For example, if $K(s, y) = K_1(y)\delta(s)$ (as in the KPP–Fisher model with the non-local spatial interaction) then $N_c(s) = K_1(s)$ so that it is enough to assume that $K_1 \in L^1(\mathbb{R})$. 

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By invoking the approaches developed in Hasík et al. (2016) and Hasík and Trofimchuk (2014), in Sects. 2 and 3 we establish the following general existence result:

**Theorem 1** Assume that the continuous function \( G : \mathbb{R}_+ \to \mathbb{R} \) satisfies (2) and the kernel \( K \geq 0 \) satisfies (5). In addition, there exist finite lower one-sided derivatives

\[
DG(0^+) = \liminf_{u \to 0^+} u^{-1} (G(u) - G(0)) \in \mathbb{R}, \quad DG(1^-) = \liminf_{u \to 1^-} (u - 1)^{-1} G(u) \in \mathbb{R}.
\]

Then for each speed \( c \geq 2 \sqrt{G^*} \) Eq. (4) has at least one semi-wavefront \( u(t, x) = \phi_c(x + ct) \). Moreover, Eq. (4) does not possess any semi-wavefront propagating with the speed \( c < 2 \sqrt{G(0)} \).

Theorem 1 applies to the above-mentioned population models. In particular, in the case of the food-limited model with the spatiotemporal interaction, we have that

\[
G(u) = \frac{1 - u}{1 + \gamma u}, \quad \gamma > 0, \text{ so that } G(0) = G^* = 1.
\]

Clearly, the Allee effect is not present here so that the model has at least one semi-wavefront propagating with the speed \( c \) if and only if \( c \geq 2 \).

It is worth mentioning that the papers (Gourley 2001; Gourley and Chaplain 2002; Ou and Wu 2007; Trofimchuk et al. 2019; Wang and Li 2007; Wei et al. 2017) provide a series of conditions sufficient for the presence of monotone wavefronts in the food-limited equation. On the other hand, as Trofimchuk et al. (2019) indicates and we will also discuss it later, it is rather unrealistic to expect derivation of a sharp coefficient criterion for the existence of monotone wavefronts to this equation. This explains the importance of the above simple criterion, \( c \geq 2 \), for the existence of semi-wavefronts. The papers Gourley (2001), Gourley and Chaplain (2002) and Trofimchuk et al. (2019) present computer simulations which confirm numerically the validity of this criterion. Taking \( \gamma = 0 \) in the food-limited equation, we obtain the KPP–Fisher model with the spatiotemporal interaction. In such a case, Theorem 1 slightly extends the existence criterion of Berestycki et al. (2009) and Hasík et al. (2016) proved for the KPP–Fisher model with the non-local spatial interaction (i.e., when \( K(s, y) = K_1(y)\delta(s) \)).

Next, by considering

\[
G(u) = a + bu - cu^2, \quad \text{where } a, c > 0 \text{ and } b \in \mathbb{R},
\]

we obtain a single-species model analyzed in Gourley (2000), Han et al. (2016), Ruan (2006) and Song et al. (2004). Since

\[
G(0) = G^* = a \text{ if and only if } b \leq 0 \quad \text{and } G^* = a + b^2/4c > G(0) \text{ if } b > 0,
\]

this model possesses the Allee effect for \( b > 0 \). Clearly, the birth rate per capita \( G(u) \) satisfies all the assumptions of Theorem 1 and therefore Eq. (4) with such \( G(u) \) has at least one semi-wavefront propagating with the speed \( c \geq 2 \sqrt{G^*} \) and does not have such a solution if \( c < 2 \sqrt{a} \).
For some special kernels, without admitting the Allee effect, the wavefronts to Eq. (4) with $G(u)$ given by Eq. (8) were investigated in Han et al. (2016) and Song et al. (2004). More precisely, for special spatiotemporal averaging kernels (5) (allowing the use of the so-called linear chain technique and containing some small parameter $\tau$, a delay) and $b < 0$, the existence of wavefronts for (4) and (8) was proved by Song et al. (2004) with the help of Fenichel’s invariant manifold theory. Recently, Han et al. considered (4), (8) with the singular kernels
\[ K(s, y) = K_1(y)\delta(s), \quad K_1(0) > 0, \]
and also with $b < 0$, see Han et al. (2016). Using the Leray–Schauder degree argument, they proved the existence of semi-wavefronts for (8) for each propagation speed $c \geq 2\sqrt{a}$. The above-mentioned conclusions of Han et al. (2016) and Song et al. (2004) follow from our more general existence result.

1.3 On the Semi-wavefront Uniqueness

The stability of semi-wavefronts implies their uniqueness up to translation (Lin et al. 2014) so that the uniqueness property of wave can be considered as a natural indicator of its stability. This simple observation becomes important if we take into account significant technical difficulties in proving the wave stability (Benguria and Solar 2019a; Huang et al. 2018; Lin et al. 2014). Now, we can observe a striking difference between non-local equations (3) and (4) in what concerns the uniqueness property of their wave solutions. Indeed, if the nonlinearity $-u + F(v)$ in (3) is sub-tangential at 0 (i.e., $-u + F(v) \leq -u + F'(0)v$, $u, v \geq 0$), the Diekmann–Kaper theory assures the uniqueness of each wave (including the critical one) (Aguerre et al. 2012). However, if the nonlinearity $u G(v)$ has the similar property (i.e., $u G(v) \leq u G(0)$, $u, v \geq 0$), Eq. (4) for certain kernels $K$ can possess multiple wavefronts and semi-wavefronts propagating with the same speed (Hasík et al. 2016). Even so, a remarkable fact is that, in some special cases, monotone wavefront can still be unique in the class of all monotone wavefronts (Fang and Zhao 2011; Trofimchuk et al. 2016). In particular, this is true when $G(v)$ is a linear function, $G(v) = 1 - v$. This observation is due to Fang and Zhao (2011) and it can be generalized for general ecological equation (4) as follows:

**Theorem 2** Assume that $G(u)$ is a strictly decreasing, Lipschitz continuous function which is differentiable at 1 with $G'(1) < 0$ and such that, for some $\alpha > 0$,
\[ G(u)/(u - 1) - G'(1) = O((1 - u)^\alpha), \quad u \to 1^- . \]

Furthermore, assume that for each $c > 0$ there are \( \lambda_0(c), \lambda_1(c) \in (0, +\infty) \) such that
\[ \mathcal{I}_c(z) := \int_{\mathbb{R}_+ \times \mathbb{R}} K(s, y)e^{-z(cs+y)}dsdy \in \mathbb{R}, \quad \mathcal{I}_c(0) = 1, \quad \lim_{z \to -\lambda_0(c)^+} \mathcal{I}_c(z) = +\infty, \]
for all $z \in (-\lambda_0(c), \lambda_1(c))$ and $\mathcal{I}_c(z)$ is a scalar continuous function of variables $c, z$. Suppose that $\phi_c(t), \psi_c(t)$ are two monotone wavefronts to Eq. (6) propagating with the same speed $c > 0$. Then, there exists $t' \in \mathbb{R}$, such that $\phi_c(t) \equiv \psi_c(t + t')$. 
We note that Theorem 2 is a non-trivial extension of the aforementioned uniqueness result from Fang and Zhao (2011). Indeed, the proof in Fang and Zhao (2011) uses in essential way the sub-tangency property of the function \( u(1 - v) \) at the equilibrium 1 (i.e., the inequality \( u(1 - v) \leq (1 - v) \) for \( u, v \in [0, 1] \)). This property, however, is not generally satisfied by the function \( u G(v) \) (as in the case of the food-limited model with \( \gamma > 0 \)). Therefore, in order to prove Theorem 2, it was necessary to find a completely different method. Our approach here is inspired by the recent studies in Solar and Trofimchuk (2019) and Hernández and Trofimchuk (2020). The proof of Theorem 2 is given in Sect. 4.

1.4 On the Existence of Non-monotone and Non-oscillating Wavefronts

It is well known that the classical (i.e., non-delayed and without non-local interaction) scalar monostable diffusive equation cannot admit waves other than wavefronts. Moreover, these wavefronts must have monotone profiles. This simple panorama changes drastically if the equation incorporates either delayed or non-local interaction effects. In such a model, non-monotone waves with an unusually high leading edge can appear; clearly, this type of waves might produce a major impact in the underlying biological system (Bani-Yaghoub and Amundsen 2015; Gourley 2000; Han et al. 2016). Therefore, as it was mentioned in So et al. (2001), ‘it is important and challenging, both theoretically and numerically, to find this critical value [when the wave monotonicity is lost] and to understand the mechanism behind this loss of monotonicity of wavefronts.’

Ashwin et al. (2002) were the first authors who provided numerical evidence suggesting a clear relation between the shape of wave profile and the position on the complex plane of eigenvalues to Eq. (6) linearized around the positive steady state. The heuristic paradigm suggested by Ashwin et al. (2002) can be considered as a particular case of the so-called linear determinacy principle (Lewis et al. 2002) and reads as follows: If the linearization around the positive equilibrium \( \kappa \) has a negative eigenvalue, the wave profile is monotone; next, if this linearization does not have negative eigenvalues and also does not have complex eigenvalues with the positive real part, then the profile oscillatory converges to \( \kappa \); finally, if this linearization does not have a negative eigenvalue but does have complex eigenvalues with the positive real part, then the profile develops non-decaying oscillations around \( \kappa \). In Ashwin et al. (2002) and Gourley (2000), the authors tested this principle on the non-local and delayed KPP–Fisher equations; numerical simulations realized in subsequent works also supported the above informal principle for the food-limited model (Gourley 2001; Gourley and Chaplain 2002; Ou and Wu 2007), the model with the quadratic function \( G(u) \) given by (8) (Han et al. 2016) and the Mackey–Glass-type diffusive equations (Bani-Yaghoub and Amundsen 2015; Liang and Wu 2003; Lin et al. 2014). Hence, a preliminary answer to the above concern in So et al. (2001) is based on abundant numerical evidence and might be formulated as follows: The wavefront profile loses its monotonicity and starts to oscillate at \( +\infty \) around the positive equilibrium \( \kappa \) at the moment when the negative eigenvalues of the linearization of the profile equation at \( \kappa \) coalesce and then disappear.
The above answer is very significant from the practical point of view. Indeed, it allows to indicate linearly determined ‘safe’ zone of the model parameters where we cannot expect appearance of an invasion traveling wave with dramatically high concentration of acting agents in its leading edge (what we can observe in Fig. 1, where $\kappa = 1$). The circumstance that this non-monotone wave is additionally developing oscillations at its rear part seems to be less important; actually, sometimes the oscillatory component is decaying so fast that the oscillating (at $+\infty$) wave can visually be interpreted as (or approximated by) eventually monotone wave, e.g., see Gourley (2000).

Importantly, the above-mentioned monotonicity criterion can be analytically justified for some subclasses of Eqs. (3) and (4) including the KPP–Fisher non-local (Fang and Zhao 2011) and the delayed (Ducrot and Nadin 2014; Gomez and Trofimchuk 2011, 2014; Hasik and Trofimchuk 2014, 2015; Solar and Trofimchuk 2019) equations, particular cases of the Mackey-Glass-type delayed (Gomez and Trofimchuk 2014) and non-local (Trofimchuk et al. 2016) equations. Analyzing related proofs, we can see in each of them that the reaction term is necessarily dominated by its linear part at the positive steady state. The key discovery of the present subsection is that without assuming this sub-tangency condition on the nonlinearity $u G(v)$ at the equilibrium 1, i.e., without requiring the inequality $G(v) \leq G'(1)(v - 1)$, $v \in [0, 1]$, the ecological equation (4) might not satisfy the above heuristic principle. The mechanism behind the unexpected loss of monotonicity of wavefronts in such a case is precisely the same one which causes the ‘linear determinacy principle’ (Lewis et al. 2002) to fail for the model exhibiting the Allee effect (which finally results in the appearance of pushed or nonlinearly determined waves). Specifically, we will show that the food-limited model with spatiotemporal interaction admits unexpectedly high wavefronts...
for a broad domain of parameters \( \tau, \gamma \) from an apparently ‘safe’ zone provided by the linear analysis of the model at the positive steady state, as shown in Fig. 1 and the next two Theorems.

**Theorem 3** For each fixed \( \tau > 0, \gamma > 0 \) and \( c \geq 2 \), the food-limited equation
\[
\partial_t u(t, x) = \partial_{xx} u(t, x) + u(t, x) \left( \frac{1 - (K * u)(t, x)}{1 + \gamma (K * u)(t, x)} \right), \quad x \in \mathbb{R},
\]  
with the so-called weak generic delay kernel
\[
K(s, y, \tau) = \frac{e^{-y^2/4s}}{4\pi s} \frac{1}{\tau} e^{-s/\tau}
\]  
has at least one positive wavefront \( u(t, x) = \phi_c(x + ct) \). The profile \( \phi_c(t) \) either tends to 1 as \( t \to +\infty \) or is asymptotically periodic at \( +\infty \). If, in addition, \( \tau > (1 + \gamma)/4 \), then \( \phi_c(t) \) is oscillating around 1 on some interval \([A, +\infty)\). Furthermore, if \( \tau \leq (1 + \gamma)/4 \), then there exists \( \hat{c}(\tau, \gamma) \geq 2 \) such that \( \phi_c(t) \) is eventually monotone at \(+\infty\) whenever \( c \geq \hat{c}(\tau, \gamma) \). Finally, for each \( \gamma > 7.29 \ldots \) there are positive \( c_0(\gamma), \tau_#(\gamma) < (1 + \gamma)/4 \) such that for each \( \tau \in (\tau_#(\gamma), (1 + \gamma)/4) \) and \( c \geq c_0(\gamma) \) there exists a wavefront whose profile \( \phi_c(t) \) is neither monotone nor oscillating.

Hence, the aforementioned heuristic monotonicity criterion fails for \( \tau \in (\tau_#(\gamma), (1 + \gamma)/4) \) if the propagation speed is sufficiently large. Figure 2 presents the corresponding region of parameters (lying between the graphs of \( \tau = (1 + \gamma)/4 \) and \( \tau = \tau_#(\gamma), \gamma \in [10,40])\).

It should be noted that the existence of monotone wavefronts for (9) was recently proved in Trofimchuk et al. (2019) under the condition
\[
\tau \leq \begin{cases} 
(1 + \gamma)/4, & \gamma \in (0, 1), \\
\gamma/(1 + \gamma), & \gamma \geq 1.
\end{cases}
\]

Importantly, for values of \( \gamma \in (0, 1], \) the inequality \( \tau \leq (1 + \gamma)/4 \) gives a sharp criterion for the existence of monotone wavefronts. Consequently, in such a case, the above-mentioned heuristic criterion holds. A question left unanswered in Trofimchuk et al. (2019) concerns the presence of non-monotone and non-oscillating wavefronts (and, more generally, semi-wavefronts) for the values \( \gamma > 1 \) and \( \tau > \gamma/(1 + \gamma) \). In this context, Theorem 3 explains phenomenon numerically observed in Trofimchuk (2019, Figure 1) for the values \( \gamma = 40, \tau = 9 \).

Theorem 3 (as well as Theorem 5) will be proved in Section 5 with the help of (a) Mallet-Paret and Smith theory of monotone cyclic feedback systems (Elkhader 1992; Mallet-Paret and Smith 1990; Mallet-Paret and Sell 2003) and (b) the singular perturbation theory developed by Faria et al. (2006) and Faria and Trofimchuk (2010). The latter theory provides a rigorous justification of the Canosa method (Canosa 1973; Gourley 2000; Murray 1989; Ou and Wu 2007) for the case of equations incorporating spatiotemporal effects. In Canosa (1973), Canosa constructed an analytic approximation of the monotone wavefront for the classical KPP–Fisher equation, which is
For each pair \((\gamma, \tau)\) of parameters lying in the domain between the graphs of \(\tau = (1 + \gamma)/4\) and \(\tau = \tau_\#(\gamma)\), Eq. (9) possesses fast non-monotone and non-oscillating wavefronts highly accurate for all values of \(c \geq 2\), although theoretically valid only for small \(\epsilon := c^{-2} \ll 1\). This allowed Murray to observe in Murray (1989) that ‘It is an encouraging fact that asymptotic solutions with ‘small’ parameters ... frequently give remarkably accurate solutions.’ Now, it is also known that in models with spatiotemporal effects, the wavefronts propagating with smaller speeds generally have better monotonicity and convergence properties than the wavefronts propagating with bigger speeds, cf. Hasík and Trofimchuk (2014, 2015). So, taking into account all these arguments and numerical simulations in Gourley and Chaplain (2002), we conjecture that the food-limited model (9) with the kernel (10) cannot have proper semi-wavefronts (i.e., we conjecture that always \(\phi_\epsilon(+\infty) = 1\)). It is also clear from Theorem 3 that, in difference with some Mackey–Glass-type equations (Lin et al. 2014), models (9) and (10) cannot have waves whose profiles oscillate ‘chaotically’ around 1 at +\(\infty\).

Similarly, we can establish the existence of non-monotone non-oscillating wavefronts in the linearly determined domain \(0 < \tau < (1 + \gamma)/e\) of parameters \((\gamma, \tau)\) for the food-limited model with single discrete delay

\[
\partial_t u(t, x) = \partial_{xx} u(t, x) + u(t, x) \left( \frac{1 - u(t - \tau, x)}{1 + \gamma u(t - \tau, x)} \right), \quad x \in \mathbb{R}. \tag{11}
\]

To give more complete description of the possible shapes of wavefronts, we recall the definition of sine-like slowly oscillating profile (Hasík and Trofimchuk 2014; Mallet-Paret and Sell 1996a,b):
Definition 4 Set $h := cτ$, $I = [−h, 0] ∪ {1}$. For each $v ∈ C(I) \setminus \{0\}$, we define the number of sign changes by

$$\text{sc}(v) = \sup\{k ≥ 1 : \text{there are } t_0 < \cdots < t_k \text{ such that } v(t_{i−1})v(t_i) < 0 \text{ for } i ≥ 1\}.$$ 

We set $\text{sc}(v) = 0$ if $v(s) ≥ 0$ or $v(s) ≤ 0$ for $s ∈ I$. If $φ$ is a non-monotone semi-wavefront profile to (11), we set $(φ_i)(s) = φ(t + s) − 1$ if $s ∈ [−h, 0]$, and $(φ_i)(1) = φ′(t)$. We will say that $φ(t)$ is sine-like slowly oscillating on a connected interval $J$ if the following conditions are satisfied: (d1) $φ$ oscillates around 1 and has exactly one critical point between each two consecutive intersections with level 1; (d2) for each $t ∈ J$, it holds that either $\text{sc}(φ_t) = 1$ or $\text{sc}(φ_t) = 2$.

Note that if $φ$ sine-like slowly oscillates on some interval $J$ and if $\{Q_j\}_{j≥1}$, $Q_j ∈ J$ denotes the increasing sequence of all moments $Q_j$ where $φ(Q_j) = 1$, then $Q_{j+2} − Q_j ≥ h$ for all $j ≥ 1$. Thus, every open time interval of length $h$ can contain at most two points at which the graph of $φ = φ(t)$ crosses level 1.

Observe also that the uniqueness conclusion in the next theorem is much stronger than in Theorem 2.

Theorem 5 For each fixed triple of parameters $c ≥ 2$, $τ ≥ 0, γ ≥ 0$, Eq. (11) has a unique (up to translation) positive semi-wavefront $u(t, x) = φ_c(x + ct)$. The profile $φ_c(t)$ is either eventually monotone or is sine-like slowly oscillating around 1 at $+∞$. Next, for each $0 < τ < (1 + γ)/e$ such that

$$ζ := \max_{a∈[0,1]} a \exp\left(τ + \frac{(1−a)τ}{1 + γa}\right) \left(\frac{1 + aγ}{1 + aγe^τ}\right)^{1+1/γ} > 1,$$

there exists $\hat{c}(τ, γ) ≥ 2$ such that for each $c ≥ \hat{c}(τ, γ)$ Eq. (11) has a positive wavefront propagating with the speed $c$ and whose profile $φ_c(t)$ is eventually monotone at $±∞$ and is non-monotone on $R$. In fact, $|φ_c(·)|_∞ ≥ ζ > 1$.

In Fig. 3, we present the subset of parameters $(γ, τ)$ lying in the rectangle $[1, 10] × [0, 4]$ and satisfying requirement (12) of Theorem 5.

In view of Ducrot and Nadin work (2014) on (11) with $γ = 0$ and Mallet-Paret and Sell theory (1996b), we conjecture that, in full analogy with the statement of Theorem 3, the profiles $φ_c(t)$ provided by Theorem 5 cannot oscillate ‘chaotically’ around 1 at $+∞$ and should either converge to 1 or approach a non-trivial periodic regime as $t → +∞$.

As far as we know, the food-limited Eqs. (9) and (11) are the first scalar models coming from applications where-untypical behavior due to the presence of non-monotone non-oscillating wavefronts is established analytically. Between previous studies, we would like to mention numerical simulations in Trofimchuk et al. (2019) and the theory developed in Ivanov et al. (2014) for the ‘toy’ example of the Mackey–Glass-type diffusive equation with a single delay. In view of the argumentation exposed in Gomez and Trofimchuk (2014, Subsection 2.3) and also in this work, it would be interesting to investigate whether the celebrated Nicholson’s diffusive equation with a discrete delay [i.e., if $F((K * u)(t, x)) := pu(t − τ, x)e^{−u(t−τ,x)}$ in (3)] could possess non-monotone non-oscillating wavefronts when $p > c^2$. 
2 Existence of Semi-wavefronts for $c \geq 2\sqrt{G^{*}}$: Proof of Theorem 1

In this section, we prove Theorem 1 by invoking an approach proposed in Hasík et al. (2016) and Hasík and Trofimchuk (2014). It suggests consideration of Eq. (6) together with its modification

$$
\phi''(t) - c\phi'(t) + g_\beta(\phi(t))G((N_c * \phi)(t)) = 0,
$$

where the continuous piece-wise linear function $g_\beta$ is given by

$$
g_\beta(u) = \begin{cases} 
  u, & u \in [0, \beta], \\
  \max\{0, 2\beta - u\}, & u > \beta,
\end{cases}
$$

for some appropriate large $\beta > 1$. In Hasík et al. (2016) and Hasík and Trofimchuk (2014), we analyzed the simple situation when $G(u) = 1 - u$, and in this section, we are going to show that the method developed in Hasík et al. (2016) and Hasík and Trofimchuk (2014) can also be applied to general $G(u)$ satisfying certain natural restrictions. Hence, in our subsequent exposition we are using the techniques from Hasík et al. (2016); in order to avoid the repetition, those assertions which can be proved analogously to Hasík et al. (2016) are given without proofs but referred to similar results in Hasík et al. (2016). In any event, for the reader’s convenience, we decided to include all omitted proofs into the extended arXiv version (Hasík et al. 2019) of this paper.
2.1 Some Auxiliary Results

As it is usual for the monostable systems, solutions to Eqs. (6) and (13) exhibit the following separation dichotomy at \( \pm \infty \):

Lemma 6 Assume that \( \phi \) is a nonnegative, bounded and non-constant solution of Eqs. (13) or (6). Then, \( \phi(t) > 0, \ t \in \mathbb{R} \). If, in addition, \( \phi(t_0) \to 0 \) along some sequence \( t_n \to -\infty \), then \( \phi(t) \leq 2\beta, \ t \in \mathbb{R} \), and there exists \( \rho \) such that \( \phi(t) \) is increasing on some interval \((-\infty, \rho]\), \( \phi(-\infty) = 0 \), \( \lim_{t \to +\infty} \phi(t) > 0 \) and \( c \geq 2\sqrt{\rho} \).

Proof Since Eq. (13) with \( \beta = +\infty \) coincides with (6), it suffices to consider Eq. (13) allowing \( \beta = +\infty \).

First, notice that \( y = \phi(t) \) is the solution of the following initial value problem for a linear second order ordinary differential equation

\[
y''(t) - cy'(t) + a(t)y(t) = 0, \quad (15)
\]

where

\[
a(t) := G((N_c \ast \phi)(t)) \begin{cases} 1, & 0 \leq \phi(t) \leq \beta, \\ \frac{g_{\beta}(\phi(t))}{\phi(t)}, & \phi(t) > \beta, \end{cases}
\]

is a continuous bounded function. Suppose for a moment that \( \phi(s) = 0 \). Then, also \( \phi'(s) = 0 \) since \( \phi(t) \) is nonnegative on \( \mathbb{R} \). But then, \( y(t) \equiv 0 \) due to the uniqueness theorem, a contradiction. Therefore, \( \phi(t) > 0 \) for all \( t \in \mathbb{R} \).

Next, we suppose that \( \phi(t_0) \to 0 \) along some sequence \( t_n \to -\infty \). We claim that then \( \phi(t) \leq 2\beta, \ t \in \mathbb{R} \). Indeed, on the contrary, suppose that there exists a maximal interval \((t_0, t_1)\), such that \( \phi(t) > 2\beta = \phi(t_0) \) for all \( t \in (t_0, t_1) \). Then, \( \phi'(t_s) > 0, \phi(t_s) > 2\beta \) for some \( t_s \in (t_0, t_1) \). It follows from (13) and the definition of \( g_{\beta} \) that \( \phi''(t) = c\phi'(t) \) for all \( t \in (t_0, t_1) \). Hence, \( \phi'(t) = \phi'(t_0)e^{c(t-t_0)} > 0 \), \( t \in (t_0, t_1) \) and therefore \( t_1 = +\infty, \phi(+\infty) = +\infty \), contradicting the boundedness of \( \phi \).

In what follows, the use of the Harnack inequality is suggested by Berestycki et al. (2009, Lemmas 3.7 and 3.9). Let assume that the third conclusion of the lemma is false. As \( \phi(t_0) \to 0 \) for \( t_0 \to -\infty \) and \( \phi(t) \) is not eventually monotone at \( -\infty \), there exists another sequence \( s_j \to -\infty \) such that \( \phi(t) \) attains a local minimum at \( s_j \) and \( \phi(s_j) \to 0 \). Since \( a(t) \) is a continuous bounded function and \( \phi(t) \) is bounded in \( C^2(\mathbb{R}) \), we can apply the Harnack inequality, see Gilbarg and Trudinger (2001, Theorem 8.20), to Eq. (15). We can conclude that for any \( R > 0 \) and any \( \delta > 0 \) there exists \( n_0 \in \mathbb{N} \) such that \( 0 < \phi(t) \leq \delta \) for all \( t \in (s_j - R, s_j + R) \) and \( j \geq n_0 \). In particular, \( g_{\beta}(\phi(s_j)) = \phi(s_j) > 0, j \geq n_0 \), so that it follows from (2) and (13) that \( G((N_c \ast \phi)(s_j)) \leq 0 \), implying \( (N_c \ast \phi)(s_j) \geq 1 \). On the other hand, if we take \( \delta < 1/2 \) and \( R \) sufficiently large to have \( \int_{-R}^{R} N_c(s)ds > 1 - 1/(4\beta) \) and recalling
that \( \phi(t) \leq 2\beta, \ t \in \mathbb{R} \), we obtain the following contradiction:

\[
(N_c \ast \phi)(s_j) \leq \int_{-R}^{R} \phi(s_j - s)N_c(s)ds + 2\beta \int_{|s| \geq R} N_c(s)ds < 1/2 + 1/2 = 1.
\]

To prove the fourth conclusion of the lemma, let us assume that \( \lim \inf_{t \to +\infty} \phi(t) = 0 \).
Then, there exists a sequence \( t_k \to +\infty \) such that \( \phi(t_k) \to 0 \) so that, following the above reasoning, we may conclude that \( \phi(+\infty) = 0 \) and \( \phi'(t) < 0 \) on some interval \([\varphi, +\infty)\). Consequently, \( a(t) = G(0) + o(1) \) as \( t \to +\infty \). This shows that \( c \neq 0 \), since otherwise \( \phi''(t) = -a(t)\phi(t) < 0, \ \phi'(t) < 0 \) for all large positive \( t \), implying that \( \phi(+\infty) = -\infty \). On the other hand, if \( c > 0 \) [respectively, \( c < 0 \)], then Eq. (15) is exponentially unstable [respectively, stable] at both \( +\infty \) and \( -\infty \). This means that \( \phi(t) \) can vanish only at one end of the real line, being separated from zero at the opposite end of \( \mathbb{R} \). Actually, since by our assumption \( \phi(-\infty) = 0 \), this implies that Eq. (15) is unstable and therefore \( c > 0 \) and \( \lim \inf_{t \to +\infty} \phi(t) > 0 \).

Finally, due to a classical oscillation theorem by Sturm [e.g., see Wong (1969)], the solution \( \phi(t) \) oscillates around \( 0 \) at \( -\infty \) once \( a(-\infty) = G(0) > c^2/4 \). \( \square \)

In the sequel, we assume that \( c \geq 2\sqrt{G(0)} \), then \( \lambda(c) \leq \mu(c) \) will denote positive roots of the characteristic equation \( \varepsilon^2 - \varepsilon G(0) = 0 \) at the zero equilibrium. The next three results of this subsection require the second differentiability assumption of Eq. (7).

**Lemma 7** Let a nonnegative bounded \( \phi \neq 0 \) solve either (13) or (6) and \( c \geq 2\sqrt{G(0)} \). Assume also that \( G(s) < G(0) = G^* \) for all \( s > 0 \). Then, \( \phi'(t)/\phi(t) < \lambda(c) \). If, in addition, \( \phi(-\infty) = 0, \ \phi(t) \leq 1, \ t \in \mathbb{R} \), then \( \phi'(t) > 0 \) for all \( t \in \mathbb{R} \) and \( \phi(+\infty) = 1 \).

**Proof** See (Hasík et al. 2019, Lemma 10) or (Hasík et al. 2016, Lemma 4). \( \square \)

**Lemma 8** Assume that \( G(s) < G(0) = G^* \) for all \( s > 0 \). Then, for each \( c \geq 2\sqrt{G(0)} \) and \( N \in \mathcal{S} := \{A \in L^1(\mathbb{R}, \mathbb{R}_+) : |A|_1 = 1\} \) there exists \( U(c, N) \geq 1 \) depending only on \( c \) and \( N \) such that the following holds: If \( \phi(t), \phi(-\infty) = 0, \) is a positive bounded solution of Eq. (13) with \( \beta > U(c, N) \), then

\[
0 < \phi(t) \leq U(c, N), \ t \in \mathbb{R} \tag{16}
\]

(i.e., the set of all semi-wavefronts to Eq. (13) is uniformly bounded by a constant which does not depend on a particular semi-wavefront). Moreover, given a fixed pair \((c_0, N_0) \in [2\sqrt{G(0)}, +\infty) \times \mathcal{S}, \) we can assume that the map \( U : [2\sqrt{G(0)}, +\infty) \times \mathcal{S} \to (0, +\infty) \) is locally continuous at \((c_0, N_0)\).

**Proof** See Hasík et al. (2019, Lemma 11) or Hasík et al. (2016, Lemma 5). \( \square \)

**Corollary 9** Assume that \( G(s) < G(0) = G^* \) for all \( s > 0 \) and let some \( c \geq 2\sqrt{G(0)} \) be fixed. Then, for each sufficiently large \( \beta > 1 \) Eqs. (13) and (6) share the same set of semi-wavefronts propagating at the speed \( c \).

**Proof** Due to Lemma 8 and the definition of \( g_\beta(u) \), it suffices to take \( \beta > U(c, N_c) \). \( \square \)
2.2 Proof of Theorem 1 in the Non-critical Case and Without the Allee Effect

In this section, we are going to prove Theorem 1 in the case when $G^* = G(0) > G(u)$ for all $u > 0$. By the first assumption of (7), there exists some $p \geq G(0)$ such that

$$G(u) \geq G_0(u) := G(0) - pu, \quad u \in [0, 2\beta].$$

From Lemma 6, we know that the condition $c \geq 2\sqrt{G(0)}$ is necessary for the existence of semi-wavefronts. In this subsection, we will prove that somewhat weaker inequality $c > 2\sqrt{G(0)}$ is also sufficient for the existence of semi-wavefronts, the critical case $c = 2\sqrt{G(0)}$ being left to the next subsection. So consider

$$r(\phi)(t) := b\phi(t) + g_\beta(\phi(t))G((N_c \ast \phi)(t)),$$

where $g_\beta(u)$ is defined by (14), $\beta$ is as in Corollary 9, and $b > G(0) - \min_{[0,2\beta]} G(u)$. In view of Corollary 9, it suffices to establish that the equation

$$\phi''(t) - c\phi'(t) - b\phi(t) + r(\phi)(t) = 0 \quad (17)$$

has a semi-wavefront. Observe that if a continuous function $\psi(t), \ 0 \leq \psi(t) \leq 2\beta$, satisfies $\psi(s) \leq \beta$ at some point $s \in \mathbb{R}$, then

$$r(\psi)(s) = \psi(s) [b + G((N_c \ast \psi)(s))] \geq \psi(s)(b + \min_{[0,2\beta]} G(u)) \geq \psi(s)G(0) \geq 0. \quad (18)$$

Now, if $\beta \leq \psi(s) \leq 2\beta$, then

$$r(\psi)(s) = b\psi(s) + (2\beta - \psi(s))G((N_c \ast \psi)(s)) \geq b\psi(s) + (2\beta - \psi(s)) \min_{[0,2\beta]} G(u) \\
\geq \beta(2 \min_{[0,2\beta]} G(u) + b - \min_{[0,2\beta]} G(u)) > G(0)\beta > 0. \quad (19)$$

Furthermore, the inequality $0 \leq \psi(s) \leq 2\beta, \ s \in \mathbb{R}$, implies that

$$r(\psi)(s) = b\phi(s) + g_\beta(\phi(s))G((N_c \ast \phi)(s)) \leq 2\beta(b + G(0)). \quad (20)$$

Next, we consider the non-delayed KPP–Fisher equation $u_t = u_{xx} + G(0)g_\beta(u)$. The profiles $\phi$ of the traveling fronts $u(x,t) = \phi(x + ct)$ for this equation satisfy

$$\phi''(t) - c\phi'(t) + G(0)g_\beta(\phi(t)) = 0, \quad c \geq 2\sqrt{G(0)}. \quad (21)$$

In the sequel, $\phi_+(t)$ will denote the unique monotone front to (21) normalized [cf. (Gomez and Trofimchuk 2011, Theorem 6)] by the condition

$$\phi_+(t) = (-t)^j e^{\lambda(c)t}(1 + o(1)), \quad t \to -\infty, \ j \in \{0, 1\}. $$
Let us note here that $\phi^+(t)$ satisfies the linear differential equation
\[
\phi''(t) - c\phi'(t) + G(0)\phi(t) = 0
\]
for all $t$ such that $\phi^+(t) < \beta$. In particular, if $c > 2\sqrt{G(0)}$, then there exists [see, e.g., (Gomez and Trofimchuk 2011, Theorem 6)] $C \geq 0$ such that
\[
\phi^+(t) = e^{\lambda(c)t} - Ce^{\mu(c)t}, \quad t \leq \phi^{-1}_+(\beta).
\]
Let $z_1 < 0 < z_2$ be the roots of the equation $z^2 - cz - b = 0$. Set $z_{12} = z_2 - z_1 > 0$ and consider the integral operator $A$ depending on $b$ and defined by
\[
(A\phi)(t) = \frac{1}{z_{12}} \left\{ \int_{-\infty}^{t} e^{z_1(t-s)} r(\phi)(s)\,ds + \int_{t}^{+\infty} e^{z_2(t-s)} r(\phi)(s)\,ds \right\}.
\]

**Lemma 10** Assume that $b > G(0) - \min_{[0,2\beta]} G(u)$ and let $0 \leq \phi(t) \leq \phi^+(t)$, then
\[
0 \leq (A\phi)(t) \leq \phi^+_+(t).
\]

**Proof** The lower estimate is obvious since $0 \leq \phi(t) \leq \phi^+(t) \leq 2\beta$ and therefore $r(\phi)(t) \geq 0$ in view of (18) and (19). Now, since $\phi(t) \leq \phi^+_+(t)$ and $bu + G(0)g_\beta(u)$ is an increasing function, we find that
\[
r(\phi)(t) \leq b\phi(t) + G(0)g_\beta(\phi(t)) \leq b\phi^+_+(t) + G(0)g_\beta(\phi^+_+(t)) =: R(\phi^+_+(t)).
\]
Thus,
\[
(A\phi)(t) \leq \frac{1}{z_{12}} \left\{ \int_{-\infty}^{t} e^{z_1(t-s)} R(\phi^+_+(s))\,ds + \int_{t}^{+\infty} e^{z_2(t-s)} R(\phi^+_+(s))\,ds \right\} = \phi^+_+(t),
\]
and the lemma is proved.

By Lemma 10, $\phi^+_+(t)$ is an upper solution for (17), cf. Wu and Zou (2001). Still, we need to construct a lower solution. Here, assuming that $c > 2\sqrt{G(0)}$ and that $N_c$ has a compact support, we will use the following well-known function (Wu and Zou 2001)
\[
\phi^-(t) = \max\{0, e^{\lambda t}(1 - Me^{\epsilon t})\},
\]
where $\epsilon \in (0, \lambda), \lambda := \lambda(c), \mu := \mu(c)$, and $M \gg 1$ satisfy
\[
-\chi(\lambda + \epsilon) > (Lp/M) \int_{-\infty}^{\infty} e^{-\epsilon s} N_c(s)\,ds.
\]
Here, $L := \sup_{t \in \mathbb{R}} \phi^+(t)e^{-\epsilon t}$, $\lambda + \epsilon < \mu$ and
\[
0 < \phi^-(t) < \phi^+_+(t) < e^{\epsilon t} < 1, \quad t < T_c, \text{ where } \phi^-(T_c) = 0.
\]
The above inequality $\phi_-(t) < \phi_+(t)$ holds true due to the representation (22). We also note that $(N_c * \phi_+)(t) \leq L e^{\epsilon t} \int_{\mathbb{R}} e^{-\epsilon s} N_c(s) \, ds$.

**Lemma 11** Assume that $c > 2 \sqrt{G(0)}$, $N_c$ has a compact support and $b > 2 p \beta$. Then, the inequality $\phi_-(t) \leq \phi(t) \leq \phi_+(t)$, $t \in \mathbb{R}$, implies that

$$\phi_-(t) \leq (A \phi)(t) \leq \phi_+(t), \quad t \in \mathbb{R}. \quad (23)$$

**Proof** Due to Lemma 10, it suffices to prove the first inequality in (23) for $t \leq T_c$. Since $0 < \phi(t) < 1 < \beta$, $t \leq T_c$, we have, for $t \leq T_c$, that

$$(A \phi)(t) \geq \frac{1}{z_{12}} \left\{ \int_{-\infty}^{t} e^{z_{11} (t-s)} r(\phi)(s) \, ds + \int_{t}^{T_c} e^{z_{11} (t-s)} \Gamma(s) \, ds \right\}$$

$$\geq \frac{1}{z_{12}} \left\{ \int_{-\infty}^{t} e^{z_{11} (t-s)} \Gamma(s) \, ds + \int_{t}^{T_c} e^{z_{11} (t-s)} \Gamma(s) \, ds \right\} =: Q(t),$$

where $\Gamma(s) := \phi_-(b + G_#((N_c * \phi_+)(t)))$. To estimate $Q(t)$, we find, for $t \leq T_c$, that

$$\phi''_-(t) - c \phi'_-(t) - b \phi_-(t) + b \phi_-(t) + \phi_-(t) G_#((N_c * \phi_+)(t))$$

$$:= - \chi(\lambda + \epsilon) M e^{(\lambda+\epsilon) t} - [G(0) - G_#((N_c * \phi_+)(t))] e^{\lambda t} (1 - M e^{\epsilon t})$$

$$\geq - \chi(\lambda + \epsilon) M e^{(\lambda+\epsilon) t} - e^{\epsilon t} e^{\lambda t} L \rho \int_{-\infty}^{\infty} e^{-\epsilon s} N_c(s) \, ds$$

$$= M e^{(\lambda+\epsilon) t} \left( - \chi(\lambda + \epsilon) - \frac{L \rho}{M} \int_{-\infty}^{\infty} e^{-\epsilon s} N_c(s) \, ds \right) > 0.$$ But then, rewriting the latter differential inequality in the equivalent integral form [see, e.g., (Trofimchuk et al. 2019, Lemma 18)] and using the fact that

$$\phi'_+(T_c^+) - \phi'_-(T_c^-) = -\phi'_-(T_c^-) > 0,$$

we can conclude that $Q(t) \geq \phi_-(t), \quad t \in \mathbb{R}$. Hence, $(A \phi)(t) \geq \phi_-(t), \quad t \in \mathbb{R}$. \hfill \square

Next, with some $\rho > 0$, we will consider the Banach space

$$C_m := \{ y \in C(\mathbb{R}, \mathbb{R}) : |y|_m := \sup_{s \leq 0} e^{-0.5 \lambda s} |y(s)| + \sup_{s \geq 0} e^{-\rho s} |y(s)| < + \infty \},$$

$$C_{m1} := \{ y \in C_m : y' \in C_m, \quad |y|_{1,m} := |y|_m + |y'|_m < + \infty \}.$$ In order to establish the existence of semi-wavefronts to Eq. (17), it suffices to prove that the equation $A \phi = \phi$ has at least one solution from the set

$$\mathcal{R} = \{ x \in C_m : \phi_-(t) \leq x(t) \leq \phi_+(t), \quad t \in \mathbb{R} \}.$$
Note that \( \phi_+(t)e^{-\rho t} = O(e^{-\rho t}) \) at \(+\infty\) and \( \phi_+(t)e^{-\lambda t/2} = O(|t|e^{\lambda t/2}) \) at \(-\infty\), so the norm \(|\phi_+|_m\) is finite. Since \( 0 \leq x(t) \leq \phi_+(t) \) implies \(|x|_m \leq |\phi_+|_m\), the set \( \mathcal{R} \) is bounded and non-empty. Observe also that the convergence \( x_n \to x \) in \( \mathcal{R} \) is equivalent to the uniform convergence on compact subsets of \( \mathbb{R} \).

**Lemma 12** Let \( c > 2\sqrt{G(0)} \), \( N_c \) has a compact support, \( b > 2p\beta \). Then, \( \mathcal{R} \) is a non-empty, closed, bounded and convex subset of \( C_m \) and \( A : \mathcal{R} \to \mathcal{R} \) is completely continuous. As a consequence, the equation \( A\phi = \phi \) has at least one positive bounded solution in \( \mathcal{R} \).

**Proof** The proof of the compactness of the operator \( A : \mathcal{R} \to \mathcal{R} \) is rather straightforward (cf. Hasík et al. 2016, Lemma 11) and is omitted here. The existence of at least one solution \( \phi \in \mathcal{R} \) to the equation \( A\phi = \phi \) is an immediate consequence of the Schauder fixed point theorem.

\[ \Box \]

**2.3 Proof of Theorem 1 in the General Case**

In what follows, \( C_b := C_b(\mathbb{R}, \mathbb{R}^N) \) will denote the space of all continuous and bounded functions from \( \mathbb{R} \) to \( \mathbb{R}^N \), with the supremum norm \( |y|_\infty = \sup_{x \in \mathbb{R}} |y(s)| \).

**Theorem 13** Assume that \( c \geq 2\sqrt{G^*} \). Then, the integral equation \( A\phi = \phi \) has at least one positive bounded solution in \( \mathcal{R} \).

**Proof** Assume first that \( K \) (hence, \( N_c(s) \) for each \( c > 0 \)) has a compact support and \( G(0) = G^* > G(u) \) for \( u > 0 \). If \( c > 2\sqrt{G(0)} \), then the assertion of the theorem follows from Lemma 12.

It remains to analyze the case when \( c = 2\sqrt{G(0)} \). Consider the sequence \( c_j := c + 1/j \). Since \( c_j > 2\sqrt{G(0)} \), there exists a semi-wavefront \( \phi_j \) of Eq. (17) for each \( j \), which we can normalize by the condition \( \phi_j(0) = 1/2 = \max_{s \leq 0} \phi_j(s) \). It is easy to see that the set \( \{ \phi_j, j \geq 0 \} \) is precompact in the compact-open topology of \( C_b(\mathbb{R}, \mathbb{R}) \) and therefore we can also assume that \( \phi_j \to \phi_0 \) uniformly on compact subsets of \( \mathbb{R} \), where \( \phi_0 \in C_b(\mathbb{R}, \mathbb{R}) \) and \( \phi_0(0) = 1/2 = \max_{s < 0} \phi_0(s) \). In addition, \( R_j(s) := r(\phi_j(s)) \to R_0(s) := r(\phi_0)(s) \) for each fixed \( s \in \mathbb{R} \). The sequence \( \{ R_j(t) \} \) is also uniformly bounded on \( \mathbb{R} \), see (20). All this allows us to apply Lebesgue’s dominated convergence theorem in

\[ \frac{1}{\epsilon_j} \int_{-\infty}^t e^{z_{1,j}(t-s)} R_j(s) ds + \int_t^{+\infty} e^{z_{2,j}(t-s)} R_j(s) ds = \phi_j(t), \tag{24} \]

where \( z_{1,j} < 0 < z_{2,j} \) satisfy \( z^2 - cz - b = 0 \) and \( \epsilon_j := z_{2,j} - z_{1,j} \). Taking the limit in Eq. (24), we obtain that \( A\phi_0 = \phi_0 \) with \( c = 2\sqrt{G(0)} \) and therefore \( \phi_0 \) is a nonnegative solution of Eq. (6) satisfying the condition \( \phi_0(0) = 1/2 = \max_{s \leq 0} \phi_0(s) \). Lemma 7 shows that actually \( \phi_0(t) > 0 \) for all \( t \in \mathbb{R} \). We claim, in addition, that \( \inf_{s \leq 0} \phi_0(s) = 0 \) and therefore \( \phi_0(-\infty) = 0 \) in view of Lemma 6. Indeed, otherwise there exists a positive \( k_0 \) such that \( k_0 \leq \phi_0(t) \leq 1/2 \) for all \( t \leq 0 \). This implies that

\[ 0 < 0.5k_0 \min_{u \in [k_0,1/2]} G(u) \leq \phi_0(t) G((N_c*\phi_0)(t)) \leq \max_{u \in [k_0,1/2]} G(u) \]
for all sufficiently large negative \( t \) (say, for \( t \leq t_0 \)). But then,

\[
\phi_0'(t) = \phi_0'(t_0) + c(\phi_0(t) - \phi_0(t_0)) + \int_{t_0}^t \phi_0(s)G((N_c \ast \phi_0)(s))ds \to +\infty
\]

as \( t \to -\infty \),

contradicting the positivity of \( \phi_0(t) \). Therefore, \( \phi_0 \) is a semi-wavefront for \( c = 2\sqrt{G(0)} \).

Next, consider the case when \( K \) has a compact support with \( K(0, 0) > 0 \) (hence, \( N_c(s) \) has a compact support with \( N_c(0) > 0 \) for each \( c > 0 \)) and when \( G(0) < G^* \), \( c \geq 2\sqrt{G^*} \). For each \( j \geq 2 \), we define a continuous function \( G_j : \mathbb{R}_+ \to \mathbb{R} \) with \( G_j(0) = G^* + 1/j \) which coincides with \( G(u) \) on the interval \([1/j, +\infty)\) and is linear on \([0, 1/j]\). Clearly, each \( G_j \) satisfies all conditions of the first part of this proof and for every positive \( A \) there exists integer \( j_0 \) such that \( G_j(u) = G(u) \) for all \( u \geq A \). \( j \geq j_0 \). Again, we know that for each large \( j \) there exists a semi-wavefront \( \phi_j \) of the equation

\[
\phi''(t) - c\phi'(t) - b\phi(t) + r_j(\phi(t)) = 0, \tag{25}
\]

where

\[
r_j(\phi)(t) := b\phi(t) + g_\beta(\phi(t))G_j((N_c \ast \phi)(t)).
\]

We will normalize \( \phi_j \) by the condition \( \phi_j(0) = 1/2 = \max_{s \leq 0} \phi_j(s) \). It is easy to see that the set \( \{ \phi_j, j \geq 0 \} \) is precompact in the compact-open topology of \( C_b(\mathbb{R}, \mathbb{R}) \) and therefore we can assume that \( \phi_j \to \phi_* \) uniformly on compact subsets of \( \mathbb{R} \), where \( \phi_*(0) = 1/2 = \max_{s \leq 0} \phi_0(s) \). In addition, \( R_j(s) := r_j(\phi_j)(s) \to R_*(s) := r(\phi_*)(s) \) for each fixed \( s \in \mathbb{R} \). Indeed, suppose that \( (N_c \ast \phi_*)(s) > 0 \) for some \( s \in \mathbb{R} \), then \( (N_c \ast \phi_j)(s) > 0 \) for all large \( j \) so that \( G_j((N_c \ast \phi_j)(s)) = G((N_c \ast \phi_j)(s)) \) if \( j \) is sufficiently large. In consequence, \( \lim_{j \to +\infty} R_j(s) = R_*(s) \). On the other hand, if \( (N_c \ast \phi_*)(s) = 0 \), then necessarily \( \phi_*(s) = 0 \) (recall that \( N_c(0) > 0 \)) and therefore \( \phi_j(s) \to 0 \) as \( j \to +\infty \). Thus,

\[
\lim_{j \to +\infty} \left[ b\phi_j(s) + \phi_j(s)G_j((N_c \ast \phi_j)(s)) \right] = 0 = b\phi_*(s) + \phi_*(s)G_*(((N_c \ast \phi_*)(s)).
\]

The sequence \( \{ R_j(t) \} \) is also uniformly bounded on \( \mathbb{R} \). All this allows to apply Lebesgue’s dominated convergence theorem in \( \text{(24)} \) and conclude that \( A\phi_* = \phi_* \). Thus \( \phi_* \) is a nonnegative solution of Eq. \( \text{(6)} \) satisfying the condition \( \phi_*(0) = 1/2 = \max_{s \leq 0} \phi_*(s) \). Arguing as above, we conclude that \( \phi_* \) is a semi-wavefront propagating with the speed \( c \geq 2\sqrt{G^*} \).

Finally, to prove the theorem for general kernels, we can use a similar limiting argument by constructing a sequence of compactly supported normalized kernels \( K_j \geq 0 \) converging to \( K \). Indeed, set \( K_j(s, y) = (1 - 1/j)K(s, y) + \delta_j \) for \( (s, y) \in \Pi_j := [0, j] \times [-j, j] \),

\[
\delta_j := \frac{1}{2j^2} \left( 1 - \left( 1 - \frac{1}{j} \right) \int \int_{\Pi_j} K(s, y)dsdy \right) > 0,
\]
and let $K_j(s, y) = 0$ if $(s, y) \in (\mathbb{R}_+ \times \mathbb{R}) \setminus \Pi_j$. Clearly, $K_j(0, 0) \geq \delta_j > 0$. Therefore, as we have already proved, for each fixed $c \geq 2\sqrt{G}$ and $j$ there exists a semi-wavefront $\phi_j$ propagating with the velocity $c$ and satisfying the condition $\phi_j(0) = 1/2 = \max_{s \leq 0} \phi_j(s)$. Set $N_j(s) := \int_{\mathbb{R}_+} K_j(v, s - cv)dv$, then by direct computation $\int_{\mathbb{R}_+} |N_j(s) - N_c(s)|ds \to 0$ as $j \to +\infty$. Hence, in view of Lemma 8, $0 < \phi_j(t) \leq U(c, N_c) + 1$ for all $t \in \mathbb{R}$ and large $j$. The sequence $\{\phi_j'(t)\}$ is uniformly bounded on $\mathbb{R}$ as well, so we can assume that $\phi_j \to \phi^* \in C_b(\mathbb{R}, \mathbb{R})$ uniformly on compact subsets of $\mathbb{R}$. But then, also $\phi^*(0) = 1/2 = \max_{s \leq 0} \phi^*(s)$, $(N_j * \phi_j)'(t) \to (N_c * \phi^*)(t)$, so that, arguing as in the first part of our proof, we conclude that $\phi^*(x + ct)$ must be a semi-wavefront for Eq. (6) with a general kernel. □

3 Monotone Wavefronts: The Uniqueness

We will assume in the whole section that $c \geq 2\sqrt{G(0)}$, all considered wavefronts are positive and monotone and that there exists a finite derivative $G'(1) < 0$. Then, the function $H(u) := G(u)/(u - 1), u \neq 1$, $H(1) = G'(1)$, is well defined and continuous. Set $H_* = \min_{u \in [0, 1]} H(u)$, $H^* = \max_{u \in [0, 1]} H(u)$, clearly, $H_* \leq H^* < 0$. Furthermore, the kernel $K : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ will satisfy

$$J_c(z) := \int_{\mathbb{R}_+ \times \mathbb{R}} K(s, y)e^{-z(cy + y)}dxdy \in \mathbb{R}, \quad J_c(0) = 1, \quad \lim_{z \to -\lambda_0(c)^+} J_c(z) = +\infty,$$

for all $z \in (-\lambda_0(c), \lambda_1(c)), c > 0$, and some $\lambda_0(c), \lambda_1(c) \in (0, +\infty]$.

Again, we will assume that $J_c(z)$ is a scalar continuous function of variables $c, z$.

Next, consider the characteristic function of Eq. (6) linearized along $\phi(t) \equiv 1$:

$$\chi_+(z, c) = z^2 - cz + G'(1) \int \text{e}^{-zs} N_c(s)ds = z^2 - cz + G'(1)J_c(z).$$

Observe that $\chi_+(0, c) < 0, \chi_+(-\lambda_0(c)^+, c) = -\infty$, and $\chi_+^{(4)}(z, c) < 0, z \in (-\lambda_0(c), \lambda_1(c))$), so that the function $\chi_+(z, c)$ has at most four negative zeros.

Our subsequent analysis is inspired by the argumentation in Fang and Zhao (2011), Hernández and Trofimchuk (2020) and Trofimchuk et al. (2019). Again, those assertions which can be proved analogously to Hernández and Trofimchuk (2020) and Trofimchuk et al. (2019) are given without proofs but rather referred to similar results in Hernández and Trofimchuk (2020) and Trofimchuk et al. (2019). For the reader convenience, all omitted proofs are included into the extended arXiv version (Hasík et al. 2019) of this paper.

3.1 Five Auxiliary Results

In particular, the next lemma shows the relation between the monotonicity property of wavefront and the existence of negative zeros for $\chi_+(z, c)$:
Lemma 14 For a fixed $c > 0$, function $\chi_+(z, c)$ has at least one negative zero if $\text{supp } N_c \subset (-\infty, 0)$. Next, suppose that $\text{supp } N_c \cap (0, +\infty) \neq \emptyset$ and let $\phi(t)$ be a monotone wavefront. Set $y(t) = 1 - \phi(t)$ and consider the Perron exponent $\sigma_*$ of $y(t)$, where $\sigma_* = \liminf_{t \to +\infty} t^{-1} \ln y(t)$. Then, $\chi_+(\sigma_*, c)$ is a finite and nonnegative number. In particular, $-\lambda_0(c) < \sigma_* < 0$ and $\chi_+(z, c)$ has at least one zero on the interval $[\sigma_*, 0]$. Finally, for some $C > 0$ and $\sigma < 0$ it holds that

$$y(t) \geq Ce^{\sigma t}, \quad t \geq 0. \quad (26)$$

Proof See Hasík et al. (2019, Lemma 17), or Hasík et al. (2019, Lemma 9), or Hernández and Trofimchuk (2020, Lemma 12).

Remark 15 Let $\{z : \Re z > \alpha(y)\} \subset \mathbb{C}$ be the maximal open strip where the Laplace transform $\tilde{y}(z)$ of $y(t)$ is defined. Since $y(t)$ is bounded and positive on $\mathbb{R}$, we have that $\alpha(y) \leq 0$ is a singular point of $\tilde{y}(z)$. Now, by the definition of $\sigma_*$, we obtain that $\lim_{t \to +\infty} y(t)e^{-zt} = +\infty$ for every $z < \sigma_*$. In this way, $\alpha(y) \geq \sigma_* > -\lambda_0(c)$.

Lemma 16 Suppose that $\text{supp } N_c \cap (0, +\infty) = \emptyset$ and let $\phi(t)$ be a monotone wavefront to Eq. (6). Then, $y(t) = 1 - \phi(t)$ satisfies

$$y(t) \geq y(s)e^{(s-t)H_*/c} \quad \text{for all } t \geq s, \quad t, s \in \mathbb{R}. \quad (27)$$

Proof We have that

$$y''(t) - cy'(t) + [\phi(t)H((N_c * \phi)(t))] (N_c * y)(t) = 0, \quad t \in \mathbb{R}. \quad (28)$$

Clearly, $(N_c * \phi)(t) = \int_{-\infty}^{0} \phi(t-s)N_c(s)ds \geq \int_{-\infty}^{0} \phi(t)N_c(s)ds = \phi(t), \quad t \in \mathbb{R}$, so that

$$y(t) = 1 - \phi(t) \geq 1 - (N_c * \phi)(t) = (N_c * y)(t), \quad -\phi(t)H((N_c * \phi)(t)) \leq -H_*, \quad t \in \mathbb{R}.$$ 

Using the notation

$$y'(t) = z(t), \quad r(t) = [-\phi(t)H((N_c * \phi)(t)) + H_*(N_c * y)(t) \leq 0, \quad t \in \mathbb{R}, \quad (29)$$

we find that

$$z'(t) = cz(t) - H_*(N_c * y)(t) + r(t), \quad t \in \mathbb{R}.$$ 

Since $z(\pm \infty) = 0$, we also have that

$$y'(t) = z(t) = \int_{t}^{+\infty} e^{c(t-s)} (H_*(N_c * y)(s) - r(s)) ds \quad (30)$$

$$\geq H_* \int_{t}^{+\infty} e^{c(t-s)} (N_c * y)(s)ds.$$
\[ \geq H_* \int_t^{+\infty} e^{c(t-s)} y(s)ds \geq H_* y(t)/c, \quad t \in \mathbb{R}. \]

Thus,
\[ (y(t)e^{-tH_*/c})' \geq 0, \quad t \in \mathbb{R}, \]
which implies (27). \(\square\)

We will also need the next property:

**Lemma 17** There exist \(\rho > 0\) such that
\[ (N_c \ast y)(t) \geq \rho y(t), \quad t \in \mathbb{R}. \] (31)

**Proof** We will distinguish between two situations.

**Case 1:** \(\text{supp } N_c \cap (0, +\infty) \neq \emptyset\). Then, there exists \(m > 0\) such that \(\rho_1 := \int_0^m N_c(s)ds > 0\) and
\[ \int_{\mathbb{R}} y(t-s)N_c(s)ds \geq \int_0^m y(t-s)N_c(s)ds \geq \rho_1 y(t), \quad t \in \mathbb{R}. \]

**Case 2:** \(\text{supp } N_c \cap (0, +\infty) = \emptyset\). Then, by Lemma 16, we have, for \(t \in \mathbb{R},\)
\[ (N_c \ast y)(t) = \int_{-\infty}^0 y(t-s)N_c(s)ds \geq y(t) \int_{-\infty}^0 e^{s[H_*]/c} N_c(s)ds =: \rho_2 y(t). \]

In any event, (31) holds with \(\rho \in \{\rho_1, \rho_2\}\).

\(\square\)

**Lemma 18** Suppose that \(G(0) = \max_{u \geq 0} G(u)\) and, for some \(\beta \in (0, 1],\)
\[ H(u) - G'(1) = O((1 - u)^\beta), \quad u \to 1^- . \] (32)

Let \(\phi(t)\) be a monotone wavefront to Eq. (6). Then, there exist \(P_\phi, t_1, t_2 \in \mathbb{R}, \epsilon > 0\) such that
\[ \phi(t + t_1) = e^{\lambda(c)t}((-t)^j + jP_\phi + O(e^{\epsilon t})), \quad t \to -\infty, \]
\[ \phi(t + t_2) = 1 - t^k \hat{\epsilon}(1 + o(1)), \quad t \to +\infty. \] (33)

where \(j = 0\) if \(c > 2\sqrt{G(0)}\) and \(j = 1\) when \(c = 2\sqrt{G(0)}; k \in \{0, 1, 2, 3\}\) and \(\hat{\epsilon} = \hat{\epsilon}(\phi)\) is a negative zero of the characteristic function \(\chi_+(z, c).\)
**Proof** Asymptotic representation of $\phi$ at $+\infty$. Our first step is to establish that $y(t) = 1 - \phi(t)$ has an exponential rate of convergence to 0 at $+\infty$.

Since $\phi(+\infty) = 1$, we can indicate $T_0$ sufficiently large to satisfy

$$-\phi(t)H((N_c * \phi)(t)) > -0.5 \ G'(1), \quad t \geq T_0.$$  

With the positive number $\kappa_* = -0.5 \rho \ G'(1)$, we can rewrite Eq. (28) as

$$y''(t) - cy'(t) - \kappa_* y(t) = h(t), \quad \text{where} \quad h(t) := -\phi(t)H((N_c * \phi)(t))(N_c * y)(t) - \kappa_* y(t), \ t \in \mathbb{R}.$$  

Importantly, for $t \geq T_0$,

$$h(t) > (N_c * y)(t) \left(-0.5 \ G'(1) - \frac{\kappa_*}{\rho}\right) = 0.$$  

Next, since $y''(t) - cy'(t) - \kappa_* y(t) = (D - m)(D - l)y$ where $D := d/dt$ and $l < 0 < m$ are the roots of the characteristic equation $\zeta^2 - c \zeta - \kappa_* = 0$, similarly to (30) we get

$$y'(t) - ly(t) = -\int_t^{+\infty} e^{m(t-s)}h(s)ds < 0, \quad t \geq T_0.$$  

Thus,

$$y(t) \leq y(s)e^{(t-s)}, \quad t \geq s \geq T_0, \quad \text{where} \ \alpha(y) \leq l = 0.5 \left(c - \sqrt{c^2 + 4\kappa_*}\right) < 0.$$  

Hence, by Remark 15, $\mathcal{J}_c(l)$ is a finite number. Combining the latter exponential estimate with the results of Lemma 16 (if $\text{supp } N_c \cap (0, +\infty) = \emptyset$) or inequality (26) (if $\text{supp } N_c \cap (0, +\infty) \neq \emptyset$), we conclude that $y(t)$ has an exponential rate of convergence at $+\infty$. Moreover, the same is true for $y'(t)$ because of the following estimates

$$R(t) := -[\phi(t)H((N_c * \phi)(t))](N_c * y)(t) \leq |H_*|(N_c * y)(t)$$

$$= |H_*| \int_{t-T_0}^{t} N_c(s)y(t-s)ds + |H_*| \int_{t-T_0}^{+\infty} N_c(s)y(t-s)ds \leq |H_*|$$

$$\int_{-\infty}^{t-T_0} N_c(s)y(T_0)e^{l(t-s-T_0)}ds + |H_*| \int_{t-T_0}^{+\infty} N_c(s)e^{-ls}e^{ls}ds$$

$$\leq e^{l(t-T_0)}|H_*| \int_{-\infty}^{t-T_0} N_c(s)e^{-ls}ds + e^{l(t-T_0)}|H_*| \int_{t-T_0}^{+\infty} N_c(s)e^{-ls}ds$$

$$= e^{l(t-T_0)}|H_*|\mathcal{J}_c(l).$$
and
\[
y'(t) = - \int_t^{+ \infty} e^{c(t-s)} R(s) ds \geq -3c(l) e^{-lt_0} |H_s| \int_t^{+ \infty} e^{c(t-s)} e^{ls} ds
= \frac{3c(l) e^{-lt_0} |H_s|}{l - c} e^{lt}.
\]

Similarly to (30), the latter representation of \(y'(t)\) is deduced from (28), which also implies that
\[
y''(t) - cy'(t) + (G'(1) + \epsilon(t)) (N_c * y)(t) = 0, \quad t \in \mathbb{R},
\]
where \(\epsilon(t) := [\phi(t) H((N_c * \phi)(t))] - G'(1)\). By (32),
\[
\epsilon(t) = (H((N_c * \phi)(t)) - G'(1)) - H((N_c * \phi)(t)) y(t) = O(e^{\beta t}), \quad t \to + \infty.
\]

Then, in view of Remark 15, an application of Trofimchuk et al. (2009, Lemma 22) shows that \(y(t) = w_0(t)(1 + o(1))\), \(t \to + \infty\), where \(w_0(t)\) is a nonzero eigensolution of the equation \(w''(t) - cw'(t) + G'(1)(N_c * w)(t) = 0\) corresponding to one of its negative eigenvalue \(\hat{\lambda}\). As we have already mentioned, the multiplicity of \(\hat{\lambda}\) is less or equal to 4. This proves the second representation in Eq. (33).

Asymptotic representation of \(\phi\) at \(-\infty\). Since the linear equation \(y'' - cy' + G(0)y = 0\) with \(c \geq 2\sqrt{G(0)}\) is exponentially unstable, so is the equation
\[
\phi''(t) - c\phi'(t) + (G(0) + \hat{R}(t)) \phi(t) = 0, \quad t \in \mathbb{R},
\]
where \(\hat{R}(t) = G((N_c * \phi)(t)) - G(0)\) and \(\hat{R}(- \infty) = 0\).

This assures at least the exponential rate of convergence of \(\phi(t), \phi'(t)\) to 0 at \(-\infty\). On the other hand, \(\phi(t), \phi'(t)\) has no more than exponential rate of decay at \(-\infty\), cf. Solar and Trofimchuk (2019, Lemma 6). Again, an application of Trofimchuk et al. (2009, Lemma 22) shows that \(y(t) = v_0(t)(1 + o(1)), t \to + \infty\), where \(v_0(t)\) is the nonzero eigensolution of the equation \(v''(t) - cv'(t) + G(0)v(t) = 0\) corresponding to one of the positive eigenvalues \(\lambda(c), \mu(c)\). Finally, since the function \(F(u, v) = u G(v)\) satisfies the sub-tangency condition at zero equilibrium (this assures that \(\hat{R}(t) \leq 0, t \in \mathbb{R}\)), we conclude that the correct eigenvalue in our case is precisely \(\lambda(c)\); see Gomez and Trofimchuk (2011, Section 7) for the related computations and further details. \(\Box\)

**Corollary 19** Suppose that \(G\) is a strictly decreasing function satisfying (32). Let \(\psi(t), \phi(t)\) be different monotone wavefronts to Eq. (6). Then, there exist \(t_3, t_4, P_{\phi}, P_{\psi} \in \mathbb{R}, \epsilon > 0\) such that \(\psi(t + t_3) \neq \phi(t + t_4)\) for all \(t \in \mathbb{R}\), meanwhile \(\phi(t + t_3), \psi(t + t_4)\) have equal principal asymptotic terms at \(-\infty\):
\[
\phi(t + t_3) = e^{\lambda(c)t}((-t)^j + jP_{\phi} + O(e^{et})), \quad \psi(t + t_4) = e^{\lambda(c)t}((-t)^j + jP_{\psi} + O(e^{et})), \quad t \to - \infty.
\]
\textbf{Proof} Since the operator \((Ff)(t) = [G(0) - G((N_c * f)(t))]f(t)\) is non-decreasing in \(f\), this corollary can be proved analogously to Trofimchuk et al. (2019, Corollary 14). See Hasík et al. (2019, Corollary 22) for details.

\[\square\]

### 3.1.1 Proof of Theorem 2

Suppose that there are two different wavefronts, \(\phi(t)\) and \(\psi(t)\) to Eq. (6). By Corollary 19, without restricting the generality, we can assume that, for each \(z \in \mathbb{R}\),

\[w(t) := (\psi(t) - \phi(t))e^{-zt} > 0, \quad t \in \mathbb{R}.\]  \hfill (35)

Take now some \(z \in (\lambda(c), \mu(c)) \cap (\lambda(c), \lambda(c) + \epsilon)\) if \(c > 2\sqrt{G(0)}\) and \(z = \lambda(c) = \mu(c) = \sqrt{G(0)}\) if \(c = 2\sqrt{G(0)}\). Then, \(w(t)\) is bounded on \(\mathbb{R}\) and satisfies the following equation

\[w''(t) - (c - 2z)w'(t) + (z^2 - cz + G(0))w(t) = e^{-zt} ((F\psi)(t) - (F\phi)(t)), \quad t \in \mathbb{R}.\]  \hfill (36)

Next, if \(c > 2\sqrt{G(0)}\), then \(\lambda(c) + \epsilon - z > 0\) and therefore \(w(-\infty) = w(+\infty) = 0\). This means that, for some \(t^*\),

\[w(t^*) = \max_{s \in \mathbb{R}} w(s) > 0, \quad w''(t^*) \leq 0, \quad w'(t^*) = 0.\]

Then, evaluating (36) at \(t^*\) and noting that \(z^2 - cz + G(0) < 0\), \((F\psi)(t) > (F\phi)(t)\), \(t \in \mathbb{R}\), we get a contradiction in signs. This proves the uniqueness of all non-critical wavefronts.

Supposing now that \(c = 2\sqrt{G(0)}\), Eq. (36) takes the form

\[w''(t) = e^{-zt} ((F\psi)(t) - (F\phi)(t)) > 0, \quad t \in \mathbb{R}.\]

Clearly, this inequality is not compatible with (35) and the existence of finite limits \(w(-\infty) \geq 0, \quad w(+\infty) = 0\). Hence, the uniqueness of the minimal wavefront is also proved. \(\square\)

### 4 On the Existence of Non-monotone and Non-oscillating Wavefronts

The main working tool in this section is the singular perturbation theory developed by Faria et al. (2006) and Faria and Trofimchuk (2010). More specifically, we will invoke several results from Faria and Trofimchuk (2010). For the reader’s convenience, they are resumed as Theorem 29 in “Appendix.”

#### 4.1 Non-local Food-Limited Model with a Weak Generic Delay Kernel: proof of Theorem 3

Here, following Gourley (2001), Gourley and Chaplain (2002), Ou and Wu (2007), Trofimchuk et al. (2019) and Wang and Li (2007), we study the non-local food-
limited model (9) with the so-called weak generic delay kernel (10). As in Gourley and Chaplain (2002), after introducing the function \( v(t, x) = (K * u)(t, x) \), we rewrite the model (9), (10) as the system of two coupled reaction–diffusion equations

\[
\begin{align*}
    u_t &= u_{xx} + u \left( \frac{1 - v}{1 + \gamma v} \right), \\
    v_t &= v_{xx} + \frac{1}{\tau} (\mu - v).
\end{align*}
\]  

Then, the task of determining semi-wavefronts \( u(t, x) = \varphi(x + ct) \) to (9) and (10) is equivalent to the problem of finding wave solutions

\[
\begin{align*}
    u(t, x) &= \phi(\sqrt{\epsilon} x + t), \\
    v(t, x) &= \psi(\sqrt{\epsilon} x + t), \\
    \epsilon &= c^{-2}.
\end{align*}
\]

for system (37). The profiles \( \phi, \psi \) satisfy the equations

\[
\begin{align*}
    \epsilon \phi'' - \phi' + \phi \frac{1 - \psi}{1 + \gamma \psi} &= 0, \\
    \epsilon \psi'' - \psi' + \frac{1}{\tau} (\phi - \psi) &= 0.
\end{align*}
\]  

(38)

Note that the characteristic equation for (38) at the positive equilibrium \( \phi = 1, \psi = 1 \) is

\[
(\epsilon z^2 - z)^2 - \frac{1}{\tau} (\epsilon z^2 - z) + \frac{1}{\tau (1 + \gamma)} = 0.
\]  

(39)

If \( \tau < (1 + \gamma)/4 \), it has exactly two positive and two negative simple roots. On the other hand, if \( \tau > (1 + \gamma)/4 \), then it has exactly two complex roots with positive real parts and two complex roots with negative real parts. This circumstance explains the necessity of the assumption \( \tau \leq (1 + \gamma)/4 \) for the existence of monotone wavefronts.

Since we are interested in the positive solutions \( (\phi, \psi) \) of (38), we can introduce new variable \( \eta \) by \( \phi = e^{-\eta} \). Then, (38) can be written as

\[
\begin{align*}
    \epsilon \psi' &= \xi, \\
    \xi' &= \xi/\epsilon + \frac{1}{\tau} (\psi - e^{-\eta}), \\
    \epsilon \eta' &= \zeta, \\
    \zeta' &= (\zeta + \xi^2)/\epsilon + \frac{1 - \psi}{1 + \gamma \psi}.
\end{align*}
\]  

(40)

This system belongs to the class of monotone cyclic negative feedback systems (i.e., inequalities (1.10) in Mallet-Paret and Sell (2003) are satisfied for (40) with \( \delta^* = -1 \)). If \( (\phi(t), \psi(t)) \) is the wave profile, the corresponding solution \( \Gamma(t) := (\psi(t), \xi(t), \eta(t), \zeta(t)) \) of (40) is clearly bounded on \( \mathbb{R}_+ \). Then, in view of studies realized in Elkhader (1992), we can apply the main theorem in Mallet-Paret and Smith (1990) to conclude that the omega limit set \( \omega(\Gamma) \) for \( \Gamma(t) \) is either the equilibrium \( \epsilon := (1, 0, 0, 0) \) or a non-trivial periodic orbit. (By Mallet-Paret and Smith (1990), \( \omega(\Gamma) \) cannot contain any orbit homoclinic to \( \epsilon \) since \( \Delta \text{det}(-Df(\epsilon)) = -1/(\tau (1 + \gamma)) \) is negative.) This proves the statement of Theorem 3 concerning the asymptotic shape of the profile \( \phi_c(t) \).
Lemma 20  The positive equilibrium $(1, 1)$ of the system

$$\begin{align*}
\phi'(t) &= \phi \frac{1 - \psi}{1 + \gamma \psi}; \\
\psi'(t) &= \frac{1}{\tau} (\phi - \psi),
\end{align*}$$

(41)

is locally exponentially stable and it is also globally stable in the set $Q = \{ \phi > 0, \psi \geq 0 \}$. The zero equilibrium is a saddle point: The tangent direction at the origin of the unstable [respectively, stable] manifold is $(1 + \tau, 1)$ [respectively, $(0, 1)$]. Hence, for each fixed pair of parameters $\tau, \gamma$ there exists a unique orbit connecting equilibria $(0, 0)$ and $(1, 1)$. Furthermore, if $\tau < (1 + \gamma)/4$, then the positive equilibrium is a stable node, and all positive semi-orbits, with the only exception of two trajectories, enter these equilibria in the directions

$$\pm n_1 := \pm \left(1, \frac{1 + \gamma}{2\tau} - \sqrt{\frac{(1 + \gamma)^2}{4\tau^2} - \frac{1 + \gamma}{\tau}}\right).$$

The two above-mentioned exceptional trajectories enter $(1, 1)$ in the directions

$$\pm n_2 := \pm \left(1, \frac{1 + \gamma}{2\tau} + \sqrt{\frac{(1 + \gamma)^2}{4\tau^2} - \frac{1 + \gamma}{\tau}}\right).$$

Furthermore, if $\gamma > 1$, $\tau < (1 + \gamma)/4$, then the trajectories of Eq. (41) cannot cross the half-line

$$\psi := \psi_r(\phi) = [(1 + \gamma)/(2\tau)](\phi - 1) + 1, \quad \phi > 1,$$

(42)

from right to the left. If $\tau > (1 + \gamma)/4$, then the positive equilibrium is a stable focus; in particular, the heteroclinic solution spirals into $(1, 1)$.

Observe that the exceptional direction $n_2$ is ‘steeper’ than $n_1$ and both of them are ‘steeper’ than the diagonal direction $(1, 1)$. The half-line (42) is located in between the half-lines passing through the point $(1, 1)$ in the directions $n_1$ and $n_2$.

Proof  We begin by noting that the right-hand side of system (41) is $C^\infty$-smooth on $\mathbb{R}_+^2$, where it has at most linear growth with respect to $(\phi, \psi)$. In addition, $\mathbb{R}_+^2$ is positively invariant with respect to Eq. (41). Indeed, the semi-axis $\phi = 0, \psi \geq 0$ is a union of the positive half of the stable manifold of the equilibrium $(0, 0)$ with $(0, 0)$. On the other hand, the vector field on the horizontal semi-axis has upward orientation. Therefore, Eq. (41) defines a smooth semi-flow on $\mathbb{R}_+^2$.

The characteristic polynomials at the equilibria $(0, 0)$ and $(1, 1)$ are, respectively, $z^2 - (1 - \tau^{-1})z - \tau^{-1}$ and $z^2 + z\tau^{-1} + (\tau(1 + \gamma))^{-1}$, from which we obtain the above-mentioned stability properties of both equilibria. The statement concerning the directions of the integral curves for (41) at the equilibrium $(1, 1)$ follows from a variant of the Hartman $C^1$–linearization theorem for smooth autonomous systems in a

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neighborhood of a hyperbolic attractive point, see Perko (2001, p.127). The computation of the indicated directions of tangencies is straightforward, and it is omitted here. Similarly, the above-mentioned property of $\psi_r$ amounts to the inequality

$$\frac{1 + \gamma}{2\tau} < \frac{(\phi - \psi_r(\phi))(1 + \gamma \psi_r(\phi))}{\tau \phi (1 - \psi_r(\phi))}, \quad \phi > 1,$$

which can be easily checked.

Next, consider the following Lyapunov function

$$V(\phi, \psi) = \int_{\phi}^{\psi} \frac{x - 1}{x} \, dx + \tau \int_{1}^{\psi} \frac{y - 1}{1 + \gamma y} \, dy, \quad \phi, \psi > 0.$$

It is easy to see that $V$ vanishes at the positive equilibrium only. Calculating the derivative $\dot{V}$ of $V$ along the trajectories of (41), we get

$$\dot{V} = -\frac{(\psi - 1)^2}{1 + \gamma \psi} \leq 0.$$

Since the set $\{ (\phi, \psi) : \dot{V} = 0 \} \setminus (1, 1) = \{ \psi = 1, \phi \geq 0 \} \setminus (1, 1)$ does not contain an entire orbit of Eq. (41), the positive equilibrium is globally asymptotically stable, see, e.g., Hirsch and Smale (1974, Theorem 2, p. 196).

**Lemma 21** Assume that $\tau \leq (\gamma + 1)/4$ and $\gamma > 1$. Suppose that there exists a smooth monotone function $\psi = \psi(\phi), \phi \in [0, 1]$ such that $\psi(0) = 0, \psi(1) = 1$ and

$$\psi'(0) > \frac{1}{1 + \tau}; \quad \psi'(1) > \frac{1 + \gamma}{2\tau} + \sqrt{\frac{(1 + \gamma)^2}{4\tau^2} - \frac{1 + \gamma}{\tau}}, \quad (43)$$

$$\psi'(\phi) > \frac{(\phi - \psi(\phi))(1 + \gamma \psi(\phi))}{\tau \phi (1 - \psi(\phi))}, \quad \phi \in (0, 1). \quad (44)$$

Then, each component of the heteroclinic solution $(\phi(t), \psi(t))$ to system (41) is a non-monotone and non-oscillating function with exactly one critical point, where the global maximum (bigger than 1) is reached, as shown in Fig. 4.

**Proof** Indeed, inequality (44) implies that the positive semi-orbits of system (41) starting below the arc $\psi = \psi(\phi), \phi \in [0, 1]$, cannot cross it in the direction from the right to the left. The first inequality in Eq. (43) also shows that the unstable manifold of the zero equilibrium in the first quarter lies below the graph of $\psi(\phi)$. Then, the second inequality in Eq. (43) as well as the properties of the half-line $\psi_r(\phi)$ in Lemma 20 oblige the heteroclinic trajectory to approach the equilibrium $(1, 1)$ in the direction $-n_1$. The existence of exactly one critical point for each component of the heteroclinic connection follows immediately from the elementary analysis of the vector field near $(1, 1)$. \qed
The simplest candidate for the test function $\psi(\phi)$ in Lemma 21 is the polynomial $\psi = a\phi + b\phi^n$, where $b = 1 - a$ and $n \in \mathbb{N}$.

Taking $n = 3$, we obtain the following.

**Corollary 22** The conclusion of Lemma 21 holds true whenever we can find positive $a, b = 1 - a$ such that

$$\frac{1}{\tau + 1} < a < 1.5 - \frac{1 + \gamma}{4\tau} - \sqrt{\frac{(1 + \gamma)^2}{16\tau^2} - \frac{1 + \gamma}{4\tau}},$$

$$\tau(a + 3bx^2)(1 + bx + bx^2) > b(1 + x)(1 + \gamma(ax + bx^3)), \quad x \in (0, 1).$$

Note that the latter inequality can be rewritten as $P_4(x) > 0$, where $P_4$ is a real polynomial of order 4. Since all zeros and critical points of $P_4$ can be calculated explicitly, inequality (46) admits a rigorous verification for each fixed set of parameters $a, \tau, \gamma$.

**Example 23** For $\tau = 10$ and $\gamma = 40$, numerical simulations in Trofimchuk et al. (2019) suggest the existence of non-monotone and non-oscillating wavefront propagating with speed $c = 2$. This numerical result is in good agreement with Theorem 3. Indeed, if we take $\tau = 10, \gamma = 40$, then Corollary 22 applies with $a = 0.12$, as shown in Fig. 4. In fact, for $\gamma = 40$, the numerical $\tau-$interval for the existence of non-monotone non-oscillating heteroclinics for Eq. (41) is $(\tau_{\phi}(40), 10.25) := (8.7 \ldots, 10.25)$, while a shorter interval $(\tau_{\phi}(40), 10.25) := (9.4 \ldots, 10.25)$ is provided by Corollary 22.
Corollary 24 For each $\gamma > 7.3$, there exists $1 < \tau_0(\gamma) < (\gamma + 1)/4 =: \tau_1(\gamma)$ such that the unique heteroclinic connection for system (41) is non-monotone and non-oscillating for every $\tau \in (\tau_0(\gamma), \tau_1(\gamma))$.

Proof For a fixed $\gamma > 3$ and positive parameter $\epsilon$, we take

$$a = \frac{1}{2} - \epsilon, \quad \frac{\tau}{1 + \gamma} = \frac{1}{4} - \epsilon^3.$$

Suppose that $\epsilon > 0$ is small enough to assure the inequalities

$$\tau = (1 + \gamma) \left( \frac{1}{4} - \epsilon^3 \right) > \frac{1 + 2\epsilon}{1 - 2\epsilon}, \quad \frac{1}{4} - \epsilon^3 > \max \left( \frac{1 + 2\epsilon^{3/2}}{4(1 + \epsilon)}, \frac{1 + 2\epsilon}{4(1 + \epsilon)^2} \right)$$

$$= \frac{1}{4} - \epsilon^2 + \ldots.$$

Then, it is easy to see that (45) is satisfied for all small $\epsilon > 0$ as well as inequality (46) at the end points $x = 0, 1$. In order to analyze (46) for $x \in (0, 1)$, it is convenient to rewrite it in the finite form $Q_\epsilon(x) = \sum_j \epsilon^j Q_j(x) \geq 0$, where $Q_j(x)$ are certain polynomials, which can be easily calculated. In particular,

$$Q_0(x) = (x - 1) \left( (3 - \gamma)(x^3 + 2x^2 + 2) + (\gamma + 13)x \right) > 0, \quad x \in (0, 1),$$

whenever

$$\frac{\gamma + 13}{\gamma - 3} < \min_{x \in (0, 1)} \left\{ x^2 + 2x + \frac{2}{x} \right\} = 4.729 \ldots \quad \text{or, equivalently, } \gamma > 7.2907 \ldots$$

Observe that $Q_0(1) = 0$ and therefore an additional analysis is required to prove the positivity of $Q_\epsilon(x)$ on $(0, 1)$. We have that $Q_\epsilon(1) > 0$ for all sufficiently small $\epsilon > 0$ and $Q_0'(1) = 4(7 - \gamma) < 0$. The latter implies that $Q_\epsilon'(x) < 0$ for all $x$ from some small interval $[x_0, 1], x_0 < 1$. Therefore, $Q_\epsilon'(x) < 0$ for all sufficiently small $\epsilon > 0$ (say, for $\epsilon \in (0, \epsilon_0]$) and $x \in [x_0, 1]$. Hence, $Q_\epsilon(x) = Q_\epsilon(1) - \int_x^1 Q_\epsilon'(s)ds > 0$ for all $\epsilon \in (0, \epsilon_0], x \in [x_0, 1]$. Finally, since $Q_0(x) > 0$ for $x \in [0, x_0]$, we conclude that there exists $\epsilon_1 \in (0, \epsilon_0) \text{ such that } Q_\epsilon(x) > 0 \text{ for all } \epsilon \in (0, \epsilon_1], x \in [0, 1]$. This proves (46) and therefore Corollary 24 follows from Corollary 22.

Remark 25 It is easy to see that if the assumptions of Corollary 22 are satisfied for some triple of parameters $\gamma', \tau', a'$, with $\tau' < (1 + \gamma')/4$, then they will be satisfied for all triples $\gamma', \tau, a'$, with the same $\gamma', a'$ and $\tau \in (\tau', (1 + \gamma')/4)$. Therefore, for a fixed $\gamma$, the set of all $\tau$ satisfying the assumptions of Corollary 22 with some adequate $a$, is a connected interval, say $(\tau_*(\gamma), (1 + \gamma)/4)$. Similarly, these assumptions will be satisfied for all $\gamma, \tau', a'$ such that $4\tau' - 1 < \gamma < \gamma'$. In view of Corollary 24, all this means that $\tau_*(\gamma)$ is a non-decreasing function defined on the maximal interval $(7.29 \ldots, + \infty)$.  

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Lemma 26  For each $\gamma > 1$, there exists $\tau_\#(\gamma) \leq (1 + \gamma)/4$ such that each component of the heteroclinic solution $(\phi(t, \tau, \gamma), \psi(t, \tau, \gamma))$ to system (41) is a non-monotone and non-oscillating function if and only if $\tau \in (\tau_\#(\gamma), (1 + \gamma)/4]$. The maximal value of the profile $\phi(t, \tau, \gamma)$ increases as $\tau$ increases (for a fixed $\gamma$) or $\gamma$ decreases (for a fixed $\tau$). Furthermore, $\tau_\#(\gamma)$ is a non-decreasing right-continuous function, and $\tau_\#(\gamma) \leq \tau_\#(\gamma) < (1 + \gamma)/4$ for all $\gamma > 7.29\ldots$. See Fig. 2 where the graph of $\tau = \tau_\#(\gamma)$, $\gamma \in (7.29\ldots, 40]$ is calculated numerically.

Proof  Suppose that (41) has a non-monotone and non-oscillating heteroclinic for some $\tau', \gamma'$. Let $\psi = \psi(\phi)$ be the representation for this heteroclinic on the maximal open interval for $\phi \in (0, \phi_0)$, $\phi_0 > 1$. Here, $\phi_0 := \max\{\phi(t), t \in \mathbb{R}\}$. Then, clearly (44) and the first inequality in (43) hold true on $(0, \phi_0)$ for each $\gamma < \gamma'$, $\tau = \tau'$ or $\tau > \tau'$, $\gamma = \gamma'$. Observe that the second inequality in (43) does not matter since $\psi(1) < 1$. This implies the existence and monotonicity of $\tau_\#(\gamma)$ as well as the above-mentioned properties of $\phi(t, \tau, \gamma)$. Finally, suppose for a moment that $\tau_\#$ is not right-continuous at some point $\gamma_0$. Then, $\tau_\#(\gamma_0^+) > \tau_\#(\gamma_0)$ so that for each fixed $\tau \in (\tau_\#(\gamma_0), \tau_\#(\gamma_0^+))$ and $\gamma_n = \gamma_0 + 1/n$, (41) has a monotone heteroclinic $\psi_n(\phi)$. But then, the limit function $\lim_{n \to +\infty} \psi_n(\phi)$ gives a monotone heteroclinic connection for the parameters $\gamma_0, \tau > \tau_\#(\gamma_0)$, a contradiction. □

Now we can complete the proof of Theorem 3. To establish the existence of wavefronts for (38), it suffices to check that the right-hand side of system (41) meets the hypotheses (H1)–(H4) of Theorem 29 from “Appendix.” (H1) and (H2) are obviously verified with $\kappa = (1, 1)$. (H3) and (H4) follow from Lemma 20. Now, the general oscillatory and eventual monotonicity properties of wavefront profiles follow from the related properties of roots to the characteristic equation (39), see the above discussion. Finally, take some $\tau \in (\tau_\#(\gamma), (1 + \gamma)/4]$ for $\gamma > 7.29\ldots$. From Corollary 24 and Lemma 26, we know that such a pair of $\tau, \gamma$ exists and the associated heteroclinic connection $(\phi_0(t), \psi_0(t))$ of (41) has non-monotone components. Then, Theorem 29 implies the existence of wavefronts with profiles $\phi_0(t, c)$ for all sufficiently large propagation speeds $c$. Since $\phi_0(t, c) \to \phi_0(t)$ uniformly on $\mathbb{R}$ as $c \to +\infty$, these profiles $\phi_0(t, c)$ are non-monotone. However, since $\tau < (1 + \gamma)/4$, they also are not oscillating around the level 1. □

4.2 Food-Limited Model with a Discrete Delay: Proof of Theorem 5

In this subsection, following Gourley (2001), Ou and Wu (2007), Trofimchuk et al. (2019) and Wang and Li (2007), we consider the diffusive version of the food-limited model with a discrete delay (11). Again, looking for wavefronts in the form

$$u(t, x) = \phi(\sqrt{\epsilon} x + t), \quad \epsilon = c^{-2},$$

we obtain the profile equation

$$\epsilon \phi''(t) - \phi'(t) + \phi(t) \left( \frac{1 - \phi(t - \tau)}{1 + \gamma \phi(t - \tau)} \right) = 0. \quad (47)$$
Equation (47) with $\gamma = 0$ was analyzed in Benguria and Solar (2019a), Bocharov et al. (2016), Ducrot and Nadin (2014), Faria et al. (2006), Gomez and Trofimchuk (2011, 2014), Hasík and Trofimchuk (2014, 2015), Solar and Trofimchuk (2019) and Wu and Zou (2001). The uniqueness of each positive semi-wavefront to Eq. (47) with $\gamma = 0$ was proved in Solar and Trofimchuk (2019). Remarkably, the approach of Solar and Trofimchuk (2019) can be also applied for $\gamma > 0$ since the functional $F : C([-h, 0]) \rightarrow \mathbb{R}$, $h := c\tau$, defined as $F(\phi) := \phi(0)(1 - \phi(-h))/(1 + \gamma\phi(-h))$ has the following monotonicity property

$$F(\psi) - F(\phi) \leq F'(0)(\psi - \phi),$$

for $\phi, \psi \in C([-h, 0])$ such that $0 \leq \phi(s) \leq \psi(s)$, $s \in [-h, 0]$.

Thus, the uniqueness (up to translation) of each semi-wavefront follows from Solar and Trofimchuk (2019, Theorem 1; see also Solar and Trofimchuk (2019, Corollary 2) for computation details when $\gamma = 0$).

Similarly, when $\gamma = 0$, it was proved in Hasík and Trofimchuk (2014, Section 3) that each positive solution $\phi(t)$ of (47) satisfying $v(\infty) = 0$ is either eventually monotone or it is sine-like slowly oscillating around 1 at $\infty$. It is easy to see that, due to the strict monotonicity of the function $G(u) = (1 - u)/(1 + \gamma u)$ on $\mathbb{R}_+$, the proof given in Hasík and Trofimchuk (2014, Section 3) is also valid for the case $\gamma > 0$ if we consider an additional possibility when the solution $\phi(t)$ slowly oscillates around 1 on a finite interval and then converges monotonically to 1.

Next, the limit form of Eq. (47) with $\epsilon = 0$ is

$$\phi'(t) = \phi(t) \left(1 - \frac{\phi(t - \tau)}{1 + \gamma\phi(t - \tau)}\right). \quad (48)$$

It is immediate to see that hypotheses (H1), (H2) and (H4) of Theorem 29 from “Appendix” are satisfied with $\kappa = 1$ for Eq. (48). For $\tau < 1.5(1 + \gamma)$, the assumption (H3) is also satisfied in view of Liz et al. (2003, Example 1.4). Hence, by Theorem 29, Eq. (48) has a positive heteroclinic connection $\phi = u^*(t)$ and Eq. (47) has a positive heteroclinic connection $\phi(t, \epsilon)$ for all small $\epsilon > 0$. Moreover, when $\epsilon \rightarrow 0^+$ it holds that $\phi(t, \epsilon) \rightarrow u^*(t)$ uniformly on $\mathbb{R}$.

A simple analysis shows that only the following two possibilities can happen for $u^*(t)$: Either it is strictly monotone on $\mathbb{R}$ taking values in the interval $(0, 1)$ or $u^*(t)$ crosses transversally the level $\phi = 1$ at some sequence of points $t_1 < \cdots < t_n$ where $t_{j+1} > t_j + \tau$ (this sequence can be finite, $t_n \in \mathbb{R} \cup \{\infty\}$). Linearizing (48) at the steady state $\phi(t) = 1$, we find the associated characteristic equation

$$z = -\frac{e^{-\epsilon\tau}}{1 + \gamma}.$$ 

After a straightforward computation, we can see that this equation has exactly two different real roots $z_2 < z_1 < 0$ if and only if $\tau < (1 + \gamma)/e$. Moreover, under the latter condition, each other root $z_j = \alpha_j + i\beta_j$, $\beta_j \geq 0$, $j > 2$, satisfies the inequalities $\alpha_j < z_2$ and $\beta_j > 2\pi/\tau$, see Mallet-Paret (1988, Theorem 6.1). We claim...
that $u^*(t)$ is eventually monotone at $+\infty$ if $\tau < (1 + \gamma)/e$. Indeed, otherwise, as we have mentioned above, $u^*(t)$ exponentially converges to the equilibrium 1 and slowly oscillates around it. Then, by the Cao theory of super-exponential solutions, see Cao (1990, Theorem 3.4), there exists a root $z_j = \alpha_j + i\beta_j$, $j \in \mathbb{N}$, of the characteristic equation and $C \neq 0$, $\delta > 0$, $\theta \in \mathbb{R}$, such that $u^*(t)$ admits the asymptotic representation

$$u^*(t) - 1 = Ce^{\alpha_j t} \cos(\beta_j t + \theta) + O(e^{(\alpha_j - \delta)t}), \quad t \to +\infty.$$  

However, if $z_j$ is not a real root, i.e., if $j > 2$, the above representation implies that the distance between large adjacent zeros of $u^*(t) - 1$ is less than $\tau/2$, i.e., $u^*(t)$ oscillates rapidly around 1, a contradiction. Hence,

$$u^*(t) = 1 + C_*e^{z_j t} + O(e^{(z_j - \delta)t}), \quad t \to +\infty,$$

where $j \in \{1, 2\}$, $\alpha_j = z_j < 0$, $\beta_j = 0$, and our claim is proved.

After some substantial technical work, the eventual monotonicity property of $u^*(t)$ can be extended for $\phi(t, \epsilon)$ for all small $\epsilon \geq 0$ (note that the Cao theory cannot be applied to the second order delay differential equations):

**Lemma 27** Assume that $0 < \tau < (1 + \gamma)/e$. Then, there exists $\epsilon_0 > 0$ such that the solution $\phi(t, \epsilon)$ is eventually monotone at $+\infty$ for each $\epsilon \in [0, \epsilon_0]$.

**Proof** After linearizing (47) at the equilibrium $\phi(t) = 1$, we find the related characteristic equation

$$\chi(z, \epsilon) := \epsilon z^2 - z - \frac{e^{-z\tau}}{1 + \gamma} = 0.$$  

If $0 < \tau < (1 + \gamma)/e$, it follows from Gomez and Trofimchuk (2014, Lemma 1.1) that there are $\delta > 0$ and $\epsilon_1 > 0$ such that (50) has in the half-plane $\Re z > z_2 - 2\delta$ for each $\epsilon \in (0, \epsilon_1]$ exactly three roots $z_j(\epsilon)$, $j = 0, 1, 2$. Moreover, these roots are real and $z_j(\epsilon) \to z_j$, $j = 1, 2$, and $z_0(\epsilon) \to +\infty$ as $\epsilon \to 0^-$. Therefore, the constant solution $\phi(t) = 1$ to (47) is hyperbolic and the orbit associated with the heteroclinic $\phi(t, \epsilon)$ belongs to the stable manifold of $\phi(t) = 1$. This implies that

$$\phi(t, \epsilon) = 1 + C_j(\epsilon)e^{z_j(\epsilon)t} + O(e^{(z_j - \delta)t}), \quad t \to +\infty,$$

$$\epsilon \geq 0, \quad \text{for some } j = j(\epsilon) \in \{1, 2\},$$

where, similarly to Gomez and Trofimchuk (2011, Section 7), $C_j(\epsilon)$ can be calculated as

$$C_j(\epsilon) = \text{Res}_{z=z_j(\epsilon)} \frac{\tilde{R}(z, \epsilon)}{\chi(z, \epsilon)} = \frac{\tilde{R}(z_j(\epsilon), \epsilon)}{\chi'(z_j(\epsilon), \epsilon)}, \quad \tilde{R}(z, \epsilon) := \int_{\mathbb{R}} R(s, \epsilon)e^{-zs}ds,$$

$$R(t, \epsilon) := \frac{\phi(t - \tau, \epsilon) - 1}{1 + \gamma}. \left(\phi(t, \epsilon) - 1 + \gamma(\phi(t, \epsilon) - \phi(t - \tau, \epsilon))\right), \quad \epsilon \geq 0.$$
In view of Remark 30 and Lemma 4.1 in Aguerrea et al. (2008) (it is easy to see that we can take \( h, p < 0 \) as well as the negative sign before \( y'(t) \) in the referenced lemma), \( \hat{R}(\epsilon) \) is a continuous function so that \( C_j(\epsilon) \) depends continuously on \( \epsilon \) from some non-degenerate interval \([0, \epsilon_2] \subset [0, \epsilon_1] \). Hence, if \( j = 1 \) in (49), then \( C_1(\epsilon) \neq 0 \) for all small \( \epsilon \) (say, for \( \epsilon \in [0, \epsilon_0] \)) and the conclusion of Lemma 27 follows. Suppose now that \( C_1(0) = 0 \) and that the closed set \( \mathcal{E} = \{ \epsilon \in [0, \epsilon_0] : C_1(0) = 0 \} \) is infinite and has 0 as its accumulation point. Then, \( C_2(0) \neq 0 \) so that, without loss of generality, \( C_2(\epsilon) \neq 0 \) for all \( \epsilon \in \mathcal{E} \). Since

\[
\phi(t, \epsilon) = 1 + C_2(\epsilon) e^{\varepsilon_2(\epsilon)t} + O(e^{(\varepsilon_2-\delta)t}), \quad t \to +\infty, \quad \epsilon \in \mathcal{E},
\]

\( \phi(t, \epsilon) \) is eventually monotone at \(+\infty\) for each \( \epsilon \in [0, \epsilon_0] \).

Somewhat surprising fact is that even if \( 0 < \tau < (1 + \gamma)/e \), then \( u^*(t) \) can be non-monotone for certain parameters \( \tau, \gamma \):

**Lemma 28** Assume that \( 0 < \tau < (1 + \gamma)/e \) and

\[
\zeta := \max_{a \in [0,1]} a \exp \left( \tau + \frac{(1-a)\tau}{1+\gamma a} \right) \left( 1 + a\gamma \right)^{1+1/\gamma} > 1. \tag{51}
\]

Then, solution \( u^*(t) \) is eventually monotone at \(+\infty\) and it is non-monotone on \( \mathbb{R} \).

**Proof** For a fixed \( a \in (0, 1) \), there exists a unique point \( t_0 \) such that \( u^*(t) \) is strictly increasing on \((-\infty, t_0)\) and \( u^*(t_0) = a \). Without loss of generality, we can suppose that \( t_0 = -\tau \). Then, after integrating (48) on \([-\tau, 0]\) and using the monotonicity of \( G(u) \), we find that

\[
a \exp \left( \frac{(1-a)(t+\tau)}{1+\gamma a} \right) \leq u^*(t)
\]

\[
= a \exp \int_{-\tau}^{t} \left( \frac{1-u^*(s-\tau)}{1+\gamma u^*(s-\tau)} \right) ds \leq a e^{(t+\tau)}, \quad t \in [-\tau, 0].
\]

Similarly, by integrating (48) on \([0, \tau]\), we obtain that, for \( t \in [0, \tau] \),

\[
u^*(t) \geq qa \exp \left( \frac{(1-a)\tau}{1+\gamma a} \right) \exp \int_{0}^{t} \left( \frac{1-u^*(s-\tau)}{1+\gamma u^*(s-\tau)} \right) ds
\]

\[
\geq a \exp \left( \frac{(1-a)\tau}{1+\gamma a} \right) \exp \int_{0}^{t} \left( \frac{1-a e^s}{1+\gamma a e^s} \right) ds
\]

\[
= a e^{t} \left( \frac{1 + a\gamma}{1 + a\gamma e^t} \right)^{1+1/\gamma} \exp \left( \frac{(1-a)\tau}{1+\gamma a} \right).
\]

Clearly, the last inequality evaluated at the point \( t = \tau \) together with (51) imply the conclusion of the lemma.

This completes the proof of Theorem 5. The graph of \( u^*(t) \) for the parameters \( \gamma = 9, \tau = 3 \) is shown in Fig. 1 in “Introduction.”
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Appendix

Theorem 29 follows from Faria and Trofimchuk (2010, Theorems 2.1, 3.8, 4.3; Corollaries 3.9, 3.11). To state this result, we first introduce some notation. For $d = (d_1, \ldots, d_N) \in \mathbb{R}^N$, we say that $d > 0$ (respectively, $d \geq 0$) if $d_i > 0$ (respectively $d_i \geq 0$) for $i = 1, \ldots, N$. In the Banach space $\mathcal{C} = C([-\tau, 0]; \mathbb{R}^N)$, we consider the partial orders $\geq$, resp. $>$, defined as follows: $\phi \geq \psi$ if and only if $\phi(\theta) - \psi(\theta) \geq 0$ for $\theta \in [-\tau, 0]$; in an analogous way, $\phi > \psi$ if $\phi(\theta) - \psi(\theta) > 0$ for $\theta \in [-\tau, 0]$. Next, $\mathcal{C}_+$ denotes the positive cone $C([-\tau, 0]; [0, \infty)^N)$.

We also will need the following Banach spaces:

$C_0 = \{ y \in C_b : \lim_{s \to \pm \infty} y(s) = 0 \}$ is considered as a subspace of $C_b$;

$C_{\mu} = \{ y \in C_b : \sup_{s \leq 0} e^{-\mu s} |y(s)| < \infty \}$ (for $\mu > 0$) with the norm

$$
\|y\|_{\mu} = \max\{\|y\|_{\infty}, \|y\|_{\mu}^{-}\} \quad \text{where} \quad \|y\|_{\mu}^{-} = \sup_{s \leq 0} e^{-\mu s} |y(s)|.
$$

The space $C_{\mu, 0} = C_{\mu} \cap C_0$ will be considered as a subspace of $C_{\mu}$.

We will analyze singular perturbations of the heteroclinic connection in the system of functional differential equations

$$
u'(t) = f(u), \quad t \in \mathbb{R},
$$

where $f$ is such that

(H1) $f(0) = f(\kappa) = 0$, where $\kappa$ is some positive vector;

(H2) (i) $f$ is $C^2$-smooth; furthermore, (ii) for every $M > 0$, there is $\beta > 0$ such that $f_i(\varphi) + \beta \varphi_i(0) \geq 0$, $i = 1, \ldots, N$, for all $\varphi \in \mathcal{C}$ with $0 \leq \varphi \leq M$;

(H3) for Eq. (52), the equilibrium $u = \kappa$ is locally asymptotically stable and globally attractive in the set of solutions of (1.2) with the initial conditions $\varphi \in \mathcal{C}_+$, $\varphi(0) > 0$;

(H4) for Eq. (52), its linearized equation about the equilibrium 0 has a real characteristic root $\lambda_0 > 0$, which is simple and dominant (i.e., $\Re z < \lambda_0$ for all other characteristic roots $z$); moreover, there is a characteristic eigenvector $v > 0$ associated with $\lambda_0$.

Then, the following holds.

**Theorem 29** Faria and Trofimchuk (2010) Assuming (H1)–(H4), then Eq. (52) has a positive heteroclinic solution $u^*(t)$: $u^*(-\infty) = 0$, $u^*(+\infty) = \kappa$. Next, denote by
σ(A) the set of characteristic values for
\[ u'(t) = Lu_t, \quad \text{where } L = Df(0). \]

Let positive \( \mu < \lambda_0 \) be such that the strip \( \{ \lambda \in \mathbb{C} : \Re \lambda \in (\mu, \lambda_0) \} \) does not intersect \( \sigma(A) \). Then, there exists a direct sum representation
\[ C_{\mu,0} = X_\mu \oplus Y_\mu, \quad \text{where } X_\mu, Y_\mu \text{ are subspaces of } C_{\mu,0}, \quad \dim X_\mu = 1, \]
and \( \varepsilon^* > 0, \sigma > 0 \), such that for \( 0 < \varepsilon \leq \varepsilon^* \), the following holds: For each unit vector \( w \in \mathbb{R}^P \), in a neighbourhood \( B_\sigma(0) \) of \( u^*(t) \) in \( C_\mu \), the set of all wavefronts \( u(t,x) = \psi(ct + w \cdot x) \) of
\[ \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + f(u(\cdot,x)), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^P. \]

with speed \( c = 1/\varepsilon \) and connecting 0 to \( \kappa \) forms a one-dimensional manifold (which does not depend on the choice of \( \mu \)), with the profiles
\[ \psi(\varepsilon, \xi) = u^* + \xi + \phi(\varepsilon, \xi), \quad \xi \in X_\mu \cap B_\sigma(0), \]
where \( \phi(\varepsilon, \xi) \in Y_\mu \cap B_\sigma(0) \) is continuous in \( (\varepsilon, \xi) \). Next, the profile \( \psi(\varepsilon, 0) \) is positive and satisfies \( \psi(\varepsilon, 0) \to u^* \) in \( C_\mu \) as \( \varepsilon \to 0^+ \). Moreover, the components of the profile \( \psi(\varepsilon, 0) \) are increasing in the vicinity of \(-\infty \) and \( \psi(\varepsilon, 0)(t) = O(e^{\lambda(\varepsilon)t}) \), \( \psi'(\varepsilon,0)(t) = O(e^{\lambda(\varepsilon)t}) \) at \(-\infty \), where \( \varepsilon = 1/c \) and \( \lambda(\varepsilon) \) is the real solution of
\[ \det \Delta_\varepsilon(z) = 0, \quad \Delta_\varepsilon(z) := \varepsilon^2 z^2 I - zI + L(e^z I), \]
where \( L = Df(0) \), with \( \lambda(\varepsilon) \to \lambda_0 \) as \( \varepsilon \to 0^+ \).

**Remark 30** In fact, after a slight modification of the proof of Theorem 3.8 in Faria and Trofimchuk (2010), one can note that the conclusions of Theorem 29 remain valid if we replace the space \( C_{\mu,0} \) with \( C_{\mu,\delta} = \{ y \in C_B : \| y \|_{\mu,\delta} < \infty \} \) for small \( \delta \in (0, \mu) \) and with the norm
\[ \| y \|_{\mu,\delta} = \max\{ \| y \|_{\delta}^+, \| y \|_{\mu}^- \} \quad \text{where } \| y \|_{\mu}^- = \sup_{s \leq 0} e^{-\mu s} |y(s)|, \quad \| y \|_{\delta}^+ = \sup_{s \geq 0} e^{\delta s} |y(s)|. \]

To see this, it suffices to use the change of variables \( \phi(t) = u^*(t) + e^{-\delta t} w(t) \) instead of \( \phi(t) = u^*(t) + w(t) \) before formula (3.3) in Faria and Trofimchuk (2010). As a consequence of this observation, there exist small \( \varepsilon_0 > 0 \) and some constant \( C > 0 \) which does not depend on \( \varepsilon \) such that \( |\psi(\varepsilon, 0)(t) - \kappa| \leq Ce^{-\delta t}, t \geq 0, \varepsilon \in [0, \varepsilon_0] \).

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