AN EFFICIENT APPROACH FOR NONCONVEX SEMIDEFINITE OPTIMIZATION VIA CUSTOMIZED ALTERNATING DIRECTION METHOD OF MULTIPLIERS

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Abstract. We investigate a class of general combinatorial graph problems, including MAX-CUT and community detection, reformulated as quadratic objectives over nonconvex constraints and solved via the alternating direction method of multipliers (ADMM). We propose two reformulations: one using vector variables and a binary constraint, and the other further reformulating the Burer-Monteiro form for simpler subproblems. Despite the nonconvex constraint, we prove the ADMM iterates converge to a stationary point in both formulations, under mild assumptions. Additionally, recent work suggests that in this latter form, when the matrix factors are wide enough, local optimum with high probability is also the global optimum. To demonstrate the scalability of our algorithm, we include results for MAX-CUT, community detection, and image segmentation benchmark and simulated examples.

Key words. ADMM, semidefinite optimization, symmetric matrix factorization, optimization with nonconvex constraints, large-scale graph problems

AMS subject classifications. 90C22, 90C26

1. Introduction. We consider rank-constrained semidefinite optimization problems (SDPs) of the type

\[
\min_{Z,X} f(Z), \quad \text{s.t.} \quad A(Z) = b, Z = XX^T, X \in \mathcal{C}
\]

where the matrix variable \( Z \in S^n_+ \) is a \( n \times n \) symmetric semidefinite matrix, and \( X \in \mathbb{R}^{n \times r} \) a low rank symmetric factor. The linear constraints \( A(Z) = b \) constrain either the diagonal or trace of \( Z \), and the set \( \mathcal{C} \) controls desirable features of the factor—e.g. nonnegativity, integer, norm-1, etc. (\( \mathcal{C} \) may be nonconvex.) The objective function \( f(x) \) is convex, differentiable everywhere, with \( L_f \)-Lipschitz gradient, but the overall problem (1.1) is nonconvex.

This problem is equivalent to many important nonconvex SDPs, such as the MAX-CUT problem and its related applications [3, 24, 41], rank-constrained nonnegative matrix factorization problem [29, 20], and constrained eigenvalue problems [17, 28, 43]. It is known that exactly solving (1.1) globally is in general a very difficult problem, as it includes many NP-Hard problems. Methods for heuristically solving (1.1) fall in three categories: i) solving the convexified SDP, where (1.1) does not have the rank-\( r \) or \( X \in \mathcal{C} \) constraint, using any convex optimization method [32, 36, 26], ii) approximately solving (1.1) using an alternating minimization method [10, 8] and relying on statistical arguments suggesting that the acquired local optimal = the global optimal [8], or iii) using other application-specific approaches [50, 24]. The methods investigated in this paper fall in the second category. Specifically, we investigate solving (1.1) using ADMM and linearized ADMM on two reformulations. We find that these flexible reformulations allow easy incorporation of low-rank and sparse structure, making the resulting algorithm extremely scalable, in both memory and computation, which we demonstrate on a number of popular applications.

However, often nonconvex formulations of SDPs are not favored because the convergence behavior of standard algorithms are not well understood. Specifically, an iterative procedure can do one of four things: diverge, oscillate within a bounded
interval, converge to an arbitrary point, or converge to a useful point. We show that linearized ADMM on a nonsymmetric reformulation of (1.1) can either converge to a stationary point, or diverge to $\pm\infty$; it cannot oscillate or converge to a non-stationary point. Additionally, for the case without linear constraints, vanilla ADMM is guaranteed to converge to a stationary point with a monotonically decreasing augmented Lagrangian term, and at a linear rate if the objective is strongly convex.

2. Applications. It is well-known that many convex optimization problems can be reformulated as SDPs (e.g. [71]). In nonconvex optimization, SDPs are studied in several key areas, as tight convex relaxations of otherwise NP-hard problems.

2.1. Combinatorial problems. A simple reparametrization of the constraint $x \in \mathbb{R}^n, x_i \in \{-1, 1\}$ is as $X = xx^T, \text{diag}(X) = 1$. This property has been heavily exploited for finding lower bounds in combinatorial optimization [48, 62, 32], and generalized further to polynomial optimization [7, 2]. Of high interest is the MAX-CUT problem

\begin{equation}
\min_{x \in \mathbb{R}^n} x^T C x, \quad \text{s.t.} \quad x_i \in \{-1, 1\}, \quad i = 1, \ldots, n
\end{equation}

where $C = (A - \text{diag}(A)) / 4$ and $A \in \mathbb{S}^n$ is the symmetric adjacency matrix of an undirected graph. Written in this way, the solution to (1.1) is exactly the maximum cut of an undirected graph with nonnegative weights $A_{ij}$.

This seemingly simple framework appears in many other applications, such as community detection [1] and image segmentation [66], and is equivalent to the non-convex SDP

\begin{equation}
\min_{Z} \text{Tr}(CZ), \quad \text{s.t.} \quad Z_{kk} = 1, Z \succeq 0, \text{rank}(Z) = 1.
\end{equation}

Lifting $x \in \mathbb{R}^n$ to a skinny matrix $X \in \mathbb{R}^{n \times k}$ generalizes this technique to partitioning [45] and graph coloring problems [44].

Related works on MAX-CUT. More generally, combinatorial methods can be solved using branch-and-bound schemes, using a linear relaxation of (1.1) as a bound [5, 18], where the binary constraint $x \in \{-1, 1\}$ is relaxed to $0 \leq (x + 1)/2 \leq 1$. Historically, these “polyhedral methods” were the main approach to find exact solutions of the MAX-CUT problem. Though this is an NP-Hard problem, if the graph is sparse enough, branch-and-bound converges quickly even for very large graphs [18]. However, when the graph is not very sparse, the linear relaxation is loose, and finding efficient branching mechanisms is challenging, causing the algorithm to run slowly. The MAX-CUT problem can also be approximated by one pass of the linear relaxation (with bound $f_{\text{max}} / f_{\text{exact}} \geq 2 \times \#\text{edges}$) [59].

A tighter approximation can be found with the semidefinite relaxation, which is also used for better bounding in branch-and-bound techniques [35, 63, 12, 4, 47]. In particular, the rounding algorithm of [32] returns a feasible $\hat{x}$ given optimal $Z$, and is shown in expectation to satisfy $\frac{\text{Tr}(C\hat{x})}{\text{Tr}(C_x)} \geq 0.878$. For this reason, the semidefinite relaxation for problems of type (1.1) are heavily studied (e.g.[58, 34, 26]).

Specialization to community detection. A small modification of the matrix $C$ generalizes problems of form (2.1) and (2.2) to community detection in machine learning. Here the problem is to identify node clusters in undirected graphs that are more likely to be connected with each other than with nodes outside the cluster. This prediction is useful in many graphical settings, such as interpreting online communities through social network or linking behavior [56], interpreting biological ecosystems [30], finding
disease sources in epidemiology, and many more. There are many varieties and methodologies in this field, and it would be impossible to list them all, though many comprehensive overviews exist (e.g. [24]).

The stochastic binary model [37] is one of the simplest generative models for this application. Given a graph with \( n \) nodes and parameters \( 0 < q < p < 1 \), the model partitions the nodes into two communities, and generates an edge between nodes in a community with probability \( p \) and nodes in two different communities with probability \( q \). Following the analysis in [1], we can define

\[
C = \frac{p+q}{2} 11^T - A,
\]

where \( A \) is the graph adjacency matrix, and the solution to (1.1) gives a solution to the community detection problem with sharp recovery guarantees.

2.2. Nonnegative factorization. For a symmetric matrix \( C \), the maximum eigenvalue / eigenvector pair of \( C \) is the solution to the nonconvex optimization problem

\[
\max_{x \in \mathbb{R}^n} x^T C x, \quad \text{s.t.} \quad ||x||_2 = 1.
\]

By inverting the sign of \( C \), we can transform this into a minimization problem, or equivalently acquire the minimum eigenvalue/eigenvector pair. Interestingly, despite the nonconvex nature of (2.3), we have many efficient globally optimal methods for finding \( x \), e.g. Lanczos, Arnoldi, etc. However, adding any additional constraints, such as nonnegativity of \( x \) [60], and simple methods generally do not work without heavy data assumptions [19]. This is of interest in problems such as phase retrieval, recommender systems with positive-only observations, clustering and topic models, etc. Here we discuss three variations of the nonnegative factorization problem appearing in literature, all of which are special instances of (1.1).

Optimization over spectrahedron. We can frame (2.3) as a linear objective over the spectrahedron

\[
\min_{Z \in \mathbb{S}^n} \text{Tr}(CZ), \quad \text{s.t.} \quad \text{Tr}(Z) = 1, Z \succeq 0.
\]

If additionally the maximum eigenvalue of \( C \) is isolated (corresponding only to one leading eigenvector) then \( Z = xx^T \) and \( Cx = \lambda_{\max}(C)x \). To see this, by definition,

\[
\lambda_{\max}(C) = \max_{x:||x||_2 = 1} x^T C x = \max_{Z=xx^T, ||x||_2 = 1} \text{Tr}(CZ) = \max_{Z: \text{Tr}(Z) = 1, Z \succeq 0} \text{Tr}(CZ).
\]

As a consequence, note that that though (2.4) is convex, the solution \( Z^* \) will always have rank 1 when \( \lambda_{\max}(C) \) has multiplicity 1. A simple extension of (2.4) often used in nonnegative PCA [75] is

\[
\min_{Z \in \mathbb{S}^n, x \in \mathbb{R}^n} \text{Tr}(CZ), \quad \text{s.t.} \quad \text{Tr}(Z) = 1, Z \succeq 0, Z = xx^T, \quad x \geq 0,
\]

which is an instance of (1.1) with \( C \) the nonnegative orthant.

Factorization with partial observations. An equivalent way of formulating the top-k nonnegative-eigenvector problem is as the nonnegative minimizer \( X \) to \( \|XX^T - C\|_2 \) where \( X \) is \( \mathbb{R}^{n \times k} \). However, in many applications, we may not have full view of the matrix \( C \), (e.g. \( C \) is a rating matrix). Suppose that an index set \( \Omega \) defines the observed entries, e.g. \( \{i, j\} \in \Omega \) implies \( C_{ij} \) is known. Then the nonnegative factorization problem can be written as

\[
\min_{Z \in \mathbb{S}^n, x \in \mathbb{R}^n} \sum_{i,j \in \Omega} (Z_{ij} - C_{ij})^2, \quad \text{s.t.} \quad Z = xx^T, \quad x \geq 0
\]

This formulation exists in [49].
**Projective Nonnegative Matrix Factorization.** A third method toward this goal is to optimize over the low rank projection matrix itself [74], a variant of nonnegative matrix factorization, solving

\[
\min_{Z \in \mathbb{S}^n, X \in \mathbb{R}^{n \times k}} \|B - ZB\|_2, \quad \text{s.t.} \quad Z = XX^T, \quad X \geq 0
\]

Here, the data matrix may not even be symmetric, but \( \frac{1}{\text{Tr}(Z)} ZB \) will approximate the projection of \( B \) to its top-\( k \) singular vectors.

3. Related work.

**Convex relaxations.** If \( r = n \) and \( C = \mathbb{S}^n \) then (1.1) is a convex problem, and can be solved using many conventional methods with strong convergence guarantees. However, even in this case, if \( n \) is large, traditional semidefinite solvers are computationally limiting. In the most general case, an interior point method solves at each iteration a KKT system of at least order \( n^6 \), and most first-order methods for general SDPs require eigenvalue decompositions, which are of order \( O(n^3) \) per iteration.

**Low-rank convex cases.** In fact, assuming low-rank solutions often allows for the construction of faster SDP methods. In [25] it is noted that the rank of primal PSD matrix variable is equal to the multiplicity of the matrix variable arising from the gauge dual formulation, and finding only those \( r \) corresponding eigenvectors can recover the primal solution. In [36], a similar observation is made of the Lagrange dual variable and thus the dual problem can be solved via a modified bundle method. More generally, the recently popularized conditional gradient algorithm (also called the Frank-Wolfe algorithm) efficiently solves norm-constrained problems for nonsymmetric matrices [40], exploiting the fact that the dual norm minimizer can be computed efficiently; see also [14, 61, 69].

**Nonconvex cases.** In close connection with these observations, [10, 11] proposed simply reformulating semidefinite matrix variables \( Z = XX^T \), solving the “standard” nonconvex SDP

\[
\min_{X \in \mathbb{R}^{n \times r}} \langle C, XX^T \rangle, \quad \text{s.t.} \quad \mathcal{A}(XX^T) = b
\]

by sequentially optimizing the Lagrangian. However, solving (1.1) is still numerically burdensome; in the augmented Lagrangian term, the objective is quartic in \( R \), and is usually solved using an iterative numerical method, such as L-BFGS.

**Global optimality of a nonconvex problem with linear objective.** A main motivation behind solving rank-constrained problems using convex optimization methods comes from key results in [57, 6] which show that for a linear SDP, when \( X^* \) is the optimum and \( r = \text{rank}(X^*) \), then \( \frac{r(r+1)}{2} \geq m \) where \( m \) is the number of linear constraints. Furthermore, a recent work [8] shows that almost all local optima of FSDP are also global optima, suggesting that any stationary point of the FSDP is also a reasonable approximation of (1.1), if the constraint space of (3.1) is compact and sufficiently smooth, e.g. \( A_i Y \) linearly independent whenever \( \langle A_i, YY^T \rangle = b_i \) for all \( i = 1, \ldots, m \). The MAX-CUT problem satisfies this constraint; an example of a linear SDP without this condition is the phase retrieval problem [13], when \( m > n \).

**Nonconvex constraint \( C \).** Although there are many cases where the linear constraint in (1.1) serves a distinct purpose, largely it is introduced to tighten the convex relaxation. When working in the nonconvex formulation, for many applications, the linear constraint becomes superfluous, and a more useful reformulation may be

\[
\min_{x, y} g(x), \quad \text{s.t.} \quad x = y, \quad y \in C,
\]
for some nonconvex set \( \mathcal{C} \) (e.g. \( \mathcal{C} = \{-1,1\}^n \)). Note that the projection on \( \mathcal{C} \) is extremely easy, despite its nonconvexity. Although less explored, this idea is not new; see [9] chapter 9.

### 3.1. ADMM for nonconvex problems.

The alternating direction method of multipliers (ADMM) [31, 27] is a now popular method [9] for convex large-scale distributed optimization problems, with understood convergence rates [23] and variations [68, 73, 33]. It is closely related to dual decomposition methods, but alternates its subproblems, and makes use of augmented Lagrangians, which smooths the subproblems and reduces the influence of the dual ascent step size. Although there are extensions to many variable blocks, most ADMM implementations use two variable block decompositions, solving

\[
\min_x g(x) + h(y), \quad \text{s.t.} \quad Ax = By
\]

by alternatingly minimizing over each variable in the augmented Lagrangian

\[
\mathcal{L}_\rho(x, y; u) = g(x) + h(y) + u^T(Ax - By) + \frac{\rho}{2} \|Ax - By\|^2_2
\]

and then incrementally updating the dual variable:

\[
x^+ = \arg\min_x \mathcal{L}_\rho(x, y; u), \quad y^+ = \arg\min_y \mathcal{L}_\rho(x^+, y; u), \quad u^+ = u + \rho(Ax^+ - By^+).
\]

Here, any \( \rho > 0 \) will achieve convergence.

In general there is a lack of theoretical justification for ADMM on nonconvex problems despite its good numerical performance. Almost all works concerning ADMM on nonconvex problems investigate when nonconvexity is in the objective functions ([38, 70, 51, 55, 53], and also [54, 72] for matrix factorization) Under a variety of assumptions (e.g. convergence or boundedness of dual objectives) they are shown to converge to a KKT stationary point.

In comparison, relatively fewer works deal with nonconvex constraints. [42] tackles polynomial optimization problems by minimizing a general objective over a spherical constraint \( \|x\|_2 = 1 \), [39] solves general QCQPs, and [65] solves the low-rank-plus-sparse matrix separation problem. In all cases, they show that all limit points are also KKT stationary points, but do not show that their algorithms will actually converge to the limit points. In this work, we investigate a class of nonconvex constrained problems, and show with much milder assumptions that the sequence always converges to a KKT stationary point.

### 4. Linearized ADMM on full SDP.

We first investigate a reformulation of (1.1) as

\[
(4.1) \quad \min_{Z, X, Y} f(Z) + \delta_{\{0\}}(A(Z) - b) + \delta_{\mathcal{C}}(Y), \quad \text{s.t.} \quad Z = (XY^T)_{\Omega}, X = Y
\]

with variables \( Z \in \mathbb{S}^{n \times n} \), \( X \in \mathbb{R}^{n \times r} \), and \( Y \in \mathbb{R}^{n \times r} \). The affine and \( \mathcal{C} \) constraints are lifted to the objective via an indicator function

\[
\delta_{\mathcal{C}}(x) = \begin{cases} 
0 & \text{if } x \in \mathcal{C}, \\
\infty & \text{else}. 
\end{cases}
\]

The notation \( A_{\Omega} \) for a symmetric matrix \( A \) is the projection of \( A \) on the sparsity pattern \( \Omega \):

\[
(A_{\Omega})_{ij} = \begin{cases} 
A_{ij} & \text{if } \{i, j\} \in \Omega \\
0 & \text{else}, 
\end{cases}
\]
and we write \( A \in S_n^{\Omega} \) if \( A_\Omega = A \). Specifically, \( \Omega \) captures the effective sparsity of the problem; that is, \( f(Z) = f(Z_\Omega) \) and \( A(Z) = A(Z_\Omega) \). We assume \( i, i \in \Omega \) for all \( i \), so the second is trivially true.

**Duality.** As shown in [64], a notion of a dual problem can be established via the augmented Lagrangian of (4.1)

\[
L_\rho(Z, X, Y; S, U) = f(Z) + \delta_C(Y) + \langle U, X - Y \rangle + \langle S, Z - XY^T \rangle
\]

\[
+ \frac{\rho}{2} \|X - Y\|_F^2 + \frac{\rho}{2} \|Z - XY^T\|_F^2
\]

(4.2)

where the dual problem is \( \max \min_{S, U} L_\rho(Z, X, Y; S, U) \). The minimization of \( L_\rho \) over \( Z \) and \( X \) is the solution to

\[
\nabla f(Z) - A^*(\nu) + S + \rho(Z - XY^T) = 0
\]

\( U - SY + \rho(XY^T - ZY) + \rho(X - Y) = 0 \)

\( A(Z) = b \)

(4.3)

where \( \nu > 0 \) is a Lagrange dual variable for the local constraint \( A(Z) = b \). The minimization of \( L_\rho \) over \( Y \) is the solution to the generalized projection problem

\[
\min_{Y \in C} \langle Y - \hat{Y}, Y - \hat{Y} \rangle_H = \text{Tr}(H(Y - \hat{Y})^T)
\]

(4.4)

where \( \hat{Y} = U + SX + \rho(X + Z^TX) \), \( H = \rho(I + X^TX) \). For general nonconvex problems, it is difficult to guarantee global minimality. Here we introduce two sought-after properties that are more reasonably attainable.

**Definition 4.1.** [15] The tangent cone of a nonconvex set \( C \) at \( x \) is given by

\[
T_C(x) = \{ d : \text{ for all } t \to 0, \dot{x} \to x, \dot{x} \in C, \text{ there exists } \dot{d} \to d, \dot{x} + t\dot{d} \in C \}.
\]

The normal cone of \( C \) at \( x \) (:= \( N_C(x) \)) is the polar of the tangent cone.

**Definition 4.2.** For a minimization of a smooth constrained function \( \min_{x \in C} f(x) \)

we say that \( x^* \) is a KKT-stationary point if \( -\nabla f(x^*) \in N_C(x^*) \).

**Definition 4.3.** For a function defined over \( M \) variables \( \mathcal{L}(X_1, \ldots, X_m) \), we say that \( X_1^*, \ldots, X_m^* \) are (block) coordinatewise minimum points if for each \( k = 1, \ldots, m \),

\( X_k^* = \text{argmin}_X \mathcal{L}(X_1^*, \ldots, X_{k-1}^*, X, X_{k+1}^*, \ldots, X_m^*) \).

Note that it is not always the case that stationarity is stronger than coordinatewise minimum. A simple example is \( C = \{-1, 1\}^n \). Then for all points \( x \in C \), the tangent cone is \( \{0\} \) and the normal cone is \( \mathbb{R}^n \). Then every point in \( C \) is stationary, no matter what the objective function.

**Proposition 4.4.** If Alg. 4.1 converges to coordinatewise minimum points \( ((X, Z)^*, (Y, S, U)^*) \), then the primal points i) satisfy (4.3) for some choice of \( \nu \geq 0 \), ii) minimize (4.4), iii) and are primal-feasible, e.g. \( X^* = Y^* \) and \( (X^*(Y^T)^*)_\Omega = Z^* \). Furthermore, \( (X^*, Y^*, Z^*, S^*, U^*) \) are stationary points of (4.2)

Proof. It is clear that the convergent points of Alg. 4.1 exactly satisfy the three conditions. To show that these points are stationary, note that the augmented Lagrangian is convex with respect to \( X, Z \) jointly, and is a projection on a compact set \( C \) with respect to \( Y \). Therefore

\[
\nabla_{X, Z, S, U} L_\rho(Z^*, X^*, Y^*; S^*, U^*) = 0,
\]

\( -\nabla_Y L_\rho(Z^*, X^*, Y^*; S^*, U^*) \in N_C(Y^*) \)

where \( L_\rho(Z, X, Y; S, U) = -\langle U, Y \rangle - \langle S, XY^T \rangle + \frac{\rho}{2} \|X - Y\|_F^2 + \frac{\rho}{2} \|Z - XY^T\|_F^2 \) with all the differentiable terms of \( L_\rho \) involving \( Y \). \(\square\)
4.1. Linearized ADMM. We propose to solve (4.1) via the linearized ADMM, e.g. where at each iteration, the objective is replaced by its current linearization

\[ f(Z) \approx \hat{f}^k(Z) := f(Z^{k-1}) + \langle \nabla f(Z^{k-1}), Z - Z^{k-1} \rangle. \]

We then build the linearized augmented Lagrangian function as

\[ \hat{L}^k(Z,X,Y;S,U) = g_k(X,Z) + h(Y) + \langle U,X - Y \rangle + \langle S,Z - XY^T \rangle + \]

\[ \frac{\rho}{2} \|X - Y\|_F^2 + \frac{\rho}{2} \|Z - XY^T\|_F^2, \]

where \( g_k(X,Z) = \hat{f}^k(Z) + \delta_{\Omega}(A(Z) - b) \), \( h(Y) = \delta_C(Y) \) and \( S \in \mathbb{R}^{n \times n} \) and \( U \in \mathbb{R}^{n \times r} \) are the dual variables corresponding to the two coupling constraints. The full algorithm is given in Alg. 4.1.

Algorithm 4.1 ADMM for solving (4.1)

1. \textbf{Inputs:} \( \rho_0 > 0, \alpha > 1, \text{tol} \in (0) \)
2. \textbf{Initialize:} \( Z^0, X^0, S^0, U^0 \) as random matrices
3. \textbf{Outputs:} \( Z, X \)
4. \textbf{for} \( k = 1 \ldots \) \textbf{do}
5. \textbf{Update} \( Y^{k+1} \) the solution of

\[ \min_{Y \in \mathbb{R}^{n \times n}} \|Z^k - X^k Y^T\|_F^2 + \frac{S^k}{\rho^k} \|Z^k - X^k Y^T\|_F^2 + \frac{U^k}{\rho^k} \|Z^k - X^k Y^T\|_F^2, \quad \text{s.t.} \quad Y \in \mathcal{C} \]

6. \textbf{Update} \( (Z,X)^{k+1} \) as the solutions of

\[ \min_{Z,X \in \mathbb{R}^{n \times n}} \hat{L}_{k+1}(Z,X,Y^{k+1};S^k,U^k;\rho^k), \quad \text{s.t.} \quad A(Z) = b \]

where \( \mathcal{C} \) is the linearized augmented Lagrangian as defined in (4.5).

7. \textbf{Update} \( S, U \) and \( \rho \) via

\[ S^{k+1} = S^k + \rho^k (Z^{k+1} - X^{k+1} (Y^{k+1} Y^{k+1}^T) \Omega \]

\[ U^{k+1} = U^k + \rho^k (X^{k+1} - Y^{k+1}) \]

\[ \rho^{k+1} = \alpha \rho^k \]

8. \textbf{if} \( \max\{\|X^k - Y^k\|, \|(Z^k - X^k (Y^k Y^k^T))_\Omega\|\} \leq \text{tol} \) \textbf{then}
9. \textbf{break}
10. \textbf{end if}
11. \textbf{end for}

Minimizing over \( Y \). The generalized projection (4.4) can be solved a number of ways. Note if \( r = 1 \) then \( H \) is a positive scalar, and the problem reduces to \( Y^+ = \text{proj}_{Y \in \mathcal{C}} \left( \frac{1}{\| Y \|} \hat{Y} \right) \). When \( \mathcal{C} = \{-1,1\}^n \), this process reduces to recovering the signs of \( \hat{Y} \) i.e., \( Y_i = \text{sign} \circ (\hat{Y}_i) \), and when \( \mathcal{C} = \{u : \|u\|_2 = 1\} \) the set of unit-norm vectors, \( Y \) is just a properly scaled version of \( \hat{Y} \). However, in general, it is difficult to compute the generalized projection over a nonconvex set. When \( \mathcal{C} \) is convex, the generalized projection problem (4.4) can be computed using projected gradient descent. Note that the objective of (4.4) is 1-strongly convex; thus we expect fast convergence in this subproblem. In practice, we find that if \( r \) is not too large, often a few tens of iterations is enough.
Minimizing over $X$ and $Z$. Using standard linear algebra techniques, the linear system (4.3) can be reduced to a few simple instructions. First, we solve for the Lagrange dual variable $\nu$ associated with the linear constraints (and localized to the minimization of $X$ and $Z$):

\[
(4.9) \quad A(\nabla f) \left(YY^T + I\right) = \rho \left(b - A(YY^T + YY^T)\right) + A((G + S)(I + YY^T)),
\]

where $D = \frac{1}{\rho}(SY - U) + Y$ and $G = \nabla f(Z^{k-1})$ the local gradient estimate. When $A = \text{diag}(\rho)$ reduces to $n$ scalar element-wise computations $\nu_i = \frac{\rho(b - \text{Tr}(DXY^T))}{\text{Tr}(YY^T) + 1}$. When $A = \text{Tr}, \nu = \frac{\rho(b - \text{Tr}(DXY^T)) + \text{Tr}((G + S)(I + YY^T))}{\text{Tr}(YY^T) + 1}$. Note that in both cases, no $n \times n$ matrix need ever be formed, so the memory requirement remains $O(nr)$. (See appendix for elaboration.) Then the primal variables are recovered via $X = BY + D$, and $Z = (XY^T)\Omega + B$, with $B = -\frac{1}{\rho}(C - A^*(\nu) + S)$. In these cases, the complexity is dominated by multiplications between $n \times n$ and $n \times r$ matrices. Thus, the method is especially efficient when $r \ll n$.

4.2. Convergence analysis.

**Theorem 4.5.** Assume that $f(Z)$ is $L_f$-smooth. Assume the dual variables are bounded, e.g. $\max\{\|S^k\|_F, \|U^k\|_F, \|Y^k\|_F\} \leq B_F < +\infty$, and $\frac{L_f}{\sigma_{\max}}$ is bounded above, where $\sigma_{\max} = 1 - \sqrt{\frac{\sigma_z^2 + \sigma_y^2 - \sigma_x^2}{2}}, \sigma_y = \|Y^{k+1}\|_2$. Then by running Alg. 4.1 with $\rho^k = \alpha \rho^{k-1} = \alpha^k \rho_0$, if $L_k$ is bounded below, then the sequence $\{P^k, D^k\}$ converges to a stationary point of (4.2).

**Proof.** See section B in the appendix.

**Corollary 4.6.** If $r \geq \left[\sqrt{2n}\right]$ and the stationary point of Algorithm 4.1 converges to a second order critical point of (1.1), then it is globally optimal for the convex relaxation of (3.1) [8].

Unfortunately, the extension of KKT stationary points to global minima is not yet known when $\frac{r(r+1)}{2} < n$ (i.e., $r = 1$). However, our empirical results suggest that even when $r = 1$, often a local solution to (3.1) well-approximates the global solution to (1.1).

5. ADMM on simplified nonconvex SDP. When the linear constraints are not present, (1.1) can be reformulated without $Z$, into

\[
(5.1) \quad \min_{X,Y} \ g(X) + \delta_C(Y), \quad \text{s.t.} \quad X = Y
\]

with matrix variables $X \in \mathbb{R}^{n \times r}, Y \in \mathbb{R}^{n \times r}$, and where $g(X) = f(XX^T)$ is smooth. We can also define an augmented Lagrangian of (5.1) as $L_{\rho}(X,Y;U) = g(X) + \delta_C(Y) + \langle U, X - Y \rangle + \frac{\rho}{2} \|X - Y\|^2_F$.

**Theorem 5.1.** The coordinatewise minimum points $X^* = Y^*$ satisfying

\[
0 = \nabla g(X^*) + U + \rho(X - Y)
\]

\[
Y = \text{proj}_C(X + \frac{1}{\rho}U)
\]

\[
X = Y
\]

are the stationary points of the problem

\[
(5.3) \quad \min_X \ g(X), \quad \text{s.t.} \quad X \in C.
\]
Proof. The KKT stationary points of (5.3) can be characterized in terms of the normal cone of \( \mathcal{C} \) at \( X^* \); specifically, \( X^* \) is stationary iff
\[
\langle \nabla g(X^*), X - X^* \rangle \leq 0, \quad \forall X \in \mathcal{C} \cap \mathcal{N}_e(X^*)
\]
where \( \mathcal{N}_e(X^*) \) is some small neighborhood containing \( X^* \). (This is an equivalent definition of the Clarke stationary point\[15\], since in a close enough neighbourhood to \( X^* \), the subdifferential of \( \delta_{\mathcal{C}}(x) \) is \( \mathcal{N}_C(x) \).)

Combining terms in (5.2) gives \( X^* = Y^* \) satisfying
\[
X^* = \text{proj}_{\mathcal{C}} \left( X^* - \frac{1}{\rho} \nabla g(X^*) \right).
\]
The optimality condition of the projection is
\[
\langle X - (X - \frac{1}{\rho} \nabla g(X^*)), X - X^* \rangle \leq 0, \quad \forall X \in \mathcal{C} \cap \mathcal{N}_e(X^*)
\]
which reduces to the desired condition.

Algorithm 5.1 ADMM for solving (5.3)

1: Inputs: \( \rho_0 > 0, \alpha > 1, \text{tol} \epsilon > 0 \)
2: Initialize: \( Z^0, X^0, S^0, U^0 \) as random matrices
3: Outputs: \( Z, X = Y \)
4: for \( k = 1 \ldots \) do
5: \( \text{Update } Y^{k+1} \) the solution of
\[
\min_{Y \in \mathbb{R}^{n \times k}} \|X^k - Y + \frac{U^k}{\rho} \|_F^2, \quad \text{s.t. } Y \in \mathcal{C}.
\]
6: \( \text{Update } X^{k+1} \) as the solution of
\[
0 = \nabla g(X) + U + \rho(X - Y).
\]
7: \( \text{Update } U \) and \( \rho \) via
\[
U^{k+1} = U^k + \rho^k(X^{k+1} - Y^{k+1}), \quad \rho^{k+1} = \alpha \rho^k.
\]
8: if \( \|X^k - Y^k\|_F \leq \epsilon \) then
9: \( \text{break} \)
10: end if
11: end for

5.1. ADMM. The alternating steps in minimizing the augmented Lagrangian over the primal variables are extremely simple, compared with the previous matrix formulation. In general we are considering \( f(X) \) linear (in which case the update of \( X \) involves only addition) or quadratic with strictly positive diagonal Hessian (which adds a small scaling step). \( \mathcal{C} = \{-1, 1\}^n \), \( \mathcal{C} = \{x : \|x\|_2 = 1\} \), even when \( r > 1 \).

5.2. Convergence analysis.

Definition 5.2. A differentiable convex function \( g(X) \) is \( L_g \)-smooth and \( H_g \)-strongly convex over \( \mathbb{R}^n \) if for any \( X, Y \),
\[
g(X) - g(Y) \geq \langle \nabla f(X), X - Y \rangle - \frac{L_g}{2} \|X - Y\|^2_F.
\]
and
\[
g(X) - g(Y) \leq \langle \nabla f(X), X - Y \rangle - \frac{H_g}{2} \|X - Y\|^2_F.
\]

Theorem 5.3. Assume \( g(X) \) is lower bounded over \( \mathcal{C} \), and is \( L_g \)-smooth. Given a sequence \( \{\rho^k\} \) such that
\[
\frac{\rho^k - 3L_g}{2} - L_g \rho^{k+1} + 9 \rho^k > 0, \quad \rho^k > L_g
\]
for all $k$, then under Algorithm 5.1 the augmented Lagrangian $\mathcal{L}(X^k, Y^k; U^k)$ is lower bounded and convergent, with $\{X^k, Y^k, U^k\} \to \{X^*, Y^*, U^*\}$ a stationary and feasible solution of (5.3).

Proof. See section C in the appendix.

Remark. Convergence is guaranteed under a constant penalty coefficient $\rho_k \equiv \rho^0 \geq \frac{3+\sqrt{17}}{2}L_g$, $\alpha = 1$. However, in implementation, we find empirically that increasing $\{\rho^k\}$ from a relatively small $\rho^0$ can encourage convergence to more useful global minima.

Theorem 5.4. If $g(X)$ is $H_g$-strongly convex and $\rho^k = \rho$ constant, with $\frac{\rho + H_g}{2} \geq \frac{L^2}{\alpha}$, $\rho > L_g$ then under Algorithm 5.1 the augmented Lagrangian $\mathcal{L}(X^k, Y^k; U^k)$ converges to $\mathcal{L}(X^*, Y^*, U^*)$ at a linear rate.

Proof. See section C.1 in the appendix.

6. Numerical experiments. In this section, we give numerical results on the proposed methods for community detection, MAX-CUT, image segmentation, and symmetric matrix factorization. In each application, we evaluate and compare these four methods. i) SD: the solution to a semidefinite relaxation of (1.1) (SDR), where $\mathcal{C} = \mathbb{R}^{n,r}$. The binary vector factor $x$ where $xx^T = Z$ is recovered using a Goemans-Williamson style rounding. [32] technique. This is our baseline method, and is described in more detail below. ii) MR1: Algorithm 4.1 with $r = 1$. iii) MRR: Algorithm 4.1 with $r = \lceil \sqrt{2n} \rceil$, then rounded to a binary vector using a nonsymmetric version of the Goemans-Williamson style rounding [32] technique. Both MR1 and MRR have the following stopping criterion $\max\{P^{(k)}, D^{(k)}\} \leq \epsilon$ for some tolerance parameter $\epsilon > 0$, where: $P^{(k)} := \left\{ \frac{\|Z^k - Z^{k-1}\|_2}{\|Z^k\|_2}, \frac{\|X^k - X^{k-1}\|_2}{\|X^k\|_2}, \frac{\|Y^k - Y^{k-1}\|_2}{\|Y^k\|_2} \right\}$, $D^{(k)} := \max \left\{ \frac{\|Z^k - X^k Y^k\|_2}{\|Z^k\|_2}, \frac{\|X^k Y^k - X^{k-1}\|_2}{\|X^{k-1}\|_2} \right\}$. (Here $D^{(k)}$ is also proportional to the difference in dual iterates, and thus $P^{(k)}$ and $D^{(k)}$ can be interpreted as primal and dual residuals, respectively.)

iv) V: Algorithm 5.1, with stopping criterion $\max\{P^{(k)}, D^{(k)}\} \leq \epsilon$ where $P^{(k)} := \left\{ \frac{\|x^k - x^{k-1}\|_2}{\|x^k\|_2}, \frac{\|y^k - y^{k-1}\|_2}{\|y^k\|_2} \right\}$, $D^{(k)} := \frac{\|x^k - y^k\|_2}{\|x^k\|_2}$. The same primal and dual residual interpretation can be used here as well. In all cases, we use the following scheme for $\rho$: $\rho^k = \min\{\rho_{\text{max}}, \rho^{k-1} + \alpha\}$, where $\rho_{\text{max}} \approx 10,000$ and $\alpha \approx 1.05$ (slightly larger than 1).

Solving the baseline (SDR). As a baseline, we compare against the solution of the semidefinite relaxed problem without factor variables $X$ (e.g. $\mathcal{C} = \mathbb{R}^{n,n}$):

$$\min_Z f(Z), \text{ s.t. } A(Z) = b, Z \succeq 0.$$  \hspace{1cm} (6.1)

For a fair comparison, we use a first-order splitting method very similar to ADMM, which is the Dougals-Rachford Splitting (DRS) method ([52, 21], see also [67, 22]). We introduce dummy variables and solve the reformulation of (6.1)

$$\min_{Z_1, Z_2, Z_3} g_1(Z_1) + g_2(Z_2) + g_3(Z_3), \text{ s.t. } Z_1 + Z_2 + Z_3$$

where $g_1(Z_1) = \text{Tr}(C Z_1)$, $g_2(Z_2) = \begin{cases} 0, & A(Z_2) = b, \\ +\infty, & \text{else} \end{cases}$, and $g_3(Z_3) = \begin{cases} 0, & Z_3 \succeq 0, \\ +\infty, & \text{else} \end{cases}$ An application of the DRS on this reformulation (see also Alg. 3.1 in [16]) is then the
following iteration scheme: for $i = 1, 2, 3$,
\[
X_i^{(k+1)} = \text{prox}_{t_f}(Z_i), Y_i = 2X_i^{(k+1)} - Z_i^{(k)},
\]
\[
Y^{(k+1)} = \frac{1}{3}(X_1^{(k+1)} + X_2^{(k+1)} + X_3^{(k+1)}),
\]
\[
Z_i^{(k+1)} = Z_i^{(k)} + \rho(Y^{(k+1)} - X_i^{(k+1)})
\]
and for a convex function $f = \text{prox}_{\ell_f}(u) \iff \arg\min_z f(z) + \frac{1}{2
\epsilon} \|z - u\|^2$.

**Rounding.** Following the technique in [32], we can estimate $x$ from a rank $r$ matrix $X \approx xx^T$ by randomly projecting the main eigenspaces on the unit sphere. The exact procedure is as follows. i) For the symmetric SDP solution $X$, we first do an eigenvalue decomposition $X = QAQ^T$ and form a factor $F = QA^{1/2}$ where the diagonal elements of $\Lambda$ are in decreasing magnitude order. Then we scan $k = 1, \ldots, n$ and find $x_{k,t} = \text{sign}(F_kz_t)$ for trials $t = 1, \ldots, 10$. Here, $F_k$ contain the first $k$ columns of $F$, and each element of $z_t \in \mathbb{R}^k$ is drawn i.i.d from a normal Gaussian distribution. We report the values for $x_r = \arg\min \{x_r^T C x_r\}$. ii) For the MRR method, we repeat the procedure using a factor $F = US\Sigma^{1/2} X = USV^T$ is the SVD of $X$. iii) For MR1 and V, we simply take $x_r = \text{sign}(x)$ as the binary solution.

**Computer information.** The following simulations are performed on a standard desktop computer with an Intel Xeon processor (3.6 GHz), and 32 GB of RAM. It is running with Matlab R2017a.

**MAX-CUT.** Table 1 gives the best MAX-CUT values using best-of-random-guesses and our approaches over four examples from the 7th DIMACS Implementation Challenge in 2002.\footnote{See \url{http://dimacs.rutgers.edu/Workshops/7thchallenge/}. Problems downloaded from \url{http://www.optiscom.es/maxcut/}} Often, we find the quality of our recovered solutions close to the best-known solutions, and often achieve similar suboptimality as the rounded SDR solutions. However, the runtime comparison (Fig. 1) suggests the ADMM methods (especially MR1 and SDR) are much more computationally efficient and scalable. All experiments are performed with $\epsilon = 1 \times 10^{-3}$.

**Image segmentation.** Both community detection and MAX-CUT can be used in image segmentation, where each pixel is a node and the similarity between pixels form the weight of the edges. Generally, solving (1.1) for this application is not preferred, since the number of pixels in even a moderately sized image is extremely large. However, because of our fast methods, we successfully performed image segmentation on several thumbnail-sized images, in figure 2.

The $C$ matrix is composed as follows. For each pixel, we compose two feature vectors: $f_i^{(c)}$ containing the RGB values and $f_i^{(p)}$ containing the pixel location. Scaling $f_i^{(c)}$ by some weight $c$, we form the concatenated feature vector $f_i^{(j)} = [f_i^{(c)}, cf_i^{(p)}]$, and form the weighted adjacency matrix as the squared distance matrix between each feature vector $A_{(ij), (kl)} = \|f_i^{(j)} - f_k^{(l)}\|^2$. For MAX-CUT, we again form $C = A - \text{Diag}(A1)$ as before. For community detection, since we do not have exact $p$ and $q$ values, we use an approximation as $C = \alpha 11^T - A$ where $\alpha = \frac{1}{n} 1^T A 1$ the mean value of $A$. Sweeping $C$ and $\rho_0$, we give the best qualitative result in figure 2.

**Symmetric factorization with partial observations.** Recall the factorization with partial observations formulation as follows
\[
(6.2) \quad \min_{Z \in \mathbb{S}_n, X \in \mathbb{R}^{n \times r}} \sum_{i,j \in \Omega} (Z_{ij} - C_{ij})^2, \quad \text{s.t.} \quad Z = XX^T, \quad X \geq 0.
\]
| Database | m | sparsity | BK | V | MR1 | MRR | SDR |
|--------|---|----------|----|---|-----|-----|-----|
| g3-8   | 512| 0.012    | 4168414 | 34105231 | 36780180 | 35943350 | 33442495 |
| g3-15  | 3375| 0.018   | 281029888 | 255681256 | 241740931 | 212669181 |
| pm3-8-50 | 512| 0.012   | 454 | 394 | 346 | 378 | 416 |
| pm3-15-50 | 3375| 0.018   | 2964 | 2594 | 2140 | 2616 |

**Table 1**

MAX-CUT values for graphs from the 7th DIMACS Challenge. MRR = matrix formulation, \( r = \lceil \sqrt{2n} \rceil \). SDR = SDP relaxation + rounding technique.
Fig. 1. Time comparisons for DIMACS problems. Left: average runtime per iteration. Right: total runtime. We observe that both V and MRR converge in relatively few number of iterations, with MR1 taking slightly longer. However, as previously observed with splitting methods, the convergence rate is sensitive to the parameter choices $\rho(t)$. For best performance, we start with a relatively small initial penalty coefficient and increase it with the iteration until the upper bound is achieved.

| n        | 1,000 | 3,000 | 5,000 | 8,000 |
|----------|-------|-------|-------|-------|
| $||/n^4$ | 0.1   | 0.5   | 0.8   | 0.8   |
| CPU time/s | 9.74  | 13.53 | 13.97 | 61.15 |
| $(Z^* - C)/(||C||)$ | 0.86  | 0.85  | 0.86  | 0.89  |
| STD      | 0.043 | 0.020 | 0.021 | 0.010 |

Table 2
Result for nonnegative factorization with partial observations from linearized ADMM (5 trials).

Note that here we generalize the aforementioned formulation with $r = 5$. In this setting, while the strongly convex $Y-$update in the proposed algorithm can no longer be solved in closed form, projected gradient descent is applied to deal with it. The relative error defined as $||(Z^* - C)||/||C||$ and CPU time with varying problem size and sparsity are demonstrated in Table 2.

7. Conclusion. We present two methods for solving quadratic combinatorial problems using ADMM on two reformulations. Though the problem has a nonconvex constraint, we give convergence results to KKT solutions under mild conditions. From this, we give empirical solutions to several graph-based combinatorial problems, specifically MAX-CUT and community detection; both can be used in additional
downstream applications, like image segmentation.

**Appendix A. Derivation of $X$, $Z$ update.** In linearized case, consider $G = \nabla f(Z^{k-1}) = G_{\Omega}$. Then the optimality conditions of $X$, $Z$ are

\[
G - A^*(\nu) + S + \rho(Z - (XY^T)_{\Omega}) = 0 \\
U - SY + \rho((XY^T)_{\Omega}Y - ZY) + \rho(X - Y) = 0 \\
A(Z) = b.
\]

Using $D = \rho^{-1}(SY - U) + Y, B = -\rho^{-1}(G - A^*(\nu) + S)$, we get

\[
-B + Z - (XY^T)_{\Omega} = 0 \\
-D + (XY^T)_{\Omega}Y - ZY + X = 0 \\
A(Z) = b.
\]

Substitute for $Z$: $Z = (XY^T)_{\Omega} + B \Rightarrow D + ((XY^T)_{\Omega} + B)Y = (XY^T)_{\Omega}Y + X \Rightarrow D + BY = X$. Since we assume the diagonal is in $\Omega, A(X_{\Omega}) = A(X)$, so to solve for $\nu$:

\[
A((XY^T)_{\Omega} + B) = A(XY^T + B) = A((D + BY)Y^T + B) = b
\]

and therefore $A(B(YY^T + I)) = b - A(DY^T)$. Insert $B$ and simplify

\[
b - A(DY^T) = A((-\rho^{-1}(G - A^*(\nu) + S))(YY^T + I)) \\
= -\rho^{-1}A((G - A^*(\nu) + S)(YY^T + I))
\]
and thus
\[ b - A(DY^T) + \rho^{-1} A((G + S)(YY^T + I)) = \rho^{-1} A(A^*(\nu)(YY^T + I)) = \rho^{-1} H \nu \]
where \( H \) is an \( m \times m \) matrix with \( H_{ij} = \langle A_i, A_j(YY^T + I) \rangle \). Thus this system reduces to \( \nu = H^{-1} (b - A(DY^T) + \rho^{-1} A((G + S)(YY^T + I))) \).

**Implicit inverse of \( H \).** When \( A = \text{diag} \), (4.9) reduces to \( n \) scalar element-wise computations \( \nu_i = \rho(b_{(b-(DY^T))_{ii}} + ((G+S)(I+YY^T))_{ii}) \). When \( A = \text{Tr} \),
\[ \nu = \frac{\rho(b - \text{Tr}(DY^T)) + \text{Tr}((G+S)(I+YY^T))}{\text{Tr}(YY^T)+1} \]. Note that in both cases, the computation for \( \nu \) can be done without ever forming an \( n \times n \) matrix. For example, for \( A = \text{diag} \), \( DY^T_{ii} = \rho^{-1} (SY^T)_{ii} - \rho^{-1} (UY^T)_{ii} + (YY^T)_{ii} \) Recall that for any two matrices \( A, B \in \mathbb{R}^{n \times n} \), \( (AB^T)_{ii} = A^2_i B_i \) where \( A_i, B_i \) are the \( i \)th rows of \( A \) and \( B \); thus an efficient way of computing \( \nu \) is i) Compute more skinny matrices \( F_1 = SY, F_2 = GY \)

ii) Compute the element-wise products \( G_1 = F_1 \circ Y, G_2 = U \circ Y, G_3 = F_2 \circ Y \), and \( G_4 = Y \circ Y \), where \( (A \circ B)_{ij} = A_{ij} B_{ij} \) (element-wise multiplication). iii) Compute the row sums \( g_i = G_i 1 \), \( i = 1, \ldots, 4 \). iv) Compute the “numerator vector” \( h_1 = \rho(b - \rho^{-1} (g_1 - g_2 + g_4) + \text{diag}(G) + \text{diag}(S) + g_3 + g_1 \) and “denominator vector” \( h_2 = g_4 + 1 \). v) Then \( \nu_i = \frac{h_i}{h_1} \).

A similar procedure can be done for \( A = \text{Tr} \), to keep memory requirements low.

**Appendix B. Convergence analysis for matrix form.**

To simplify notation, we first collect the primal and dual variables \( p^k = (Z, X, Y)^k \) and \( D^k = (\Lambda_1, \Lambda_2)^k \). We define the augmented Lagrangian at iteration \( k \) as

\[ \mathcal{L}^k = \mathcal{L}(P^k; D^k; \rho^k) = f(Z^k) + \delta_C(Y) + \langle U, X - Y \rangle + \langle S, Z - XY^T \rangle + \rho \|X - Y\|^2_F + \rho \|Z - XY^T\|^2_F \]
and its linearization at iteration \( k \) as

\[ \mathcal{L}^k(x^k, z^k, \nu^k) = f(z^{k-1}) + \langle C^{k-1}, Z - z^{k-1} \rangle \] such that \( f^k \) is the linearization of \( f \) at \( z^{k-1} \).

**Lemma B.1.** \( \nabla^2 \mathcal{L}(Z; X, Y) \succeq \rho^k \lambda^k \)

**Proof.** Given the definition of \( \mathcal{L} \), we can see that the Hessian \( \nabla^2 \mathcal{L}(X, Z) = \rho^k (M + I) \succeq \rho^k I \) where \( M = \text{blkdiag}(X^T X, X^T X, \ldots) \). \( \square \)

**Lemma B.2.** \( \nabla^2 \mathcal{L}_{(X, Z)}(Z, X) \succeq \rho^k 
\left( 1 - \frac{\sqrt{\lambda^2 + 4 \lambda - \lambda N}}{2} \right) I \)

**Proof.** For \( (X, Z) \), we have \( \nabla^2 \mathcal{L}_{(X, Z)}(Z, X) = \rho^k \left[ I + N N^T - N^T \right] \) where \( N = \text{blkdiag}(Y^T, \ldots, Y^T) \in \mathbb{R}^{n \times n} \). Note that for block diagonal matrices, \( \|N\|_2 = \|Y\|_2 \). Note also that the determinant of \( \frac{1}{\rho^k} \nabla^2 \mathcal{L}_{(X, Z)}(Z, X) \) is \( \det((I + N N^T) - N N^T) = 1 \geq 0 \), so \( \nabla^2 \mathcal{L}_{(X, Z)}(Z, X) \succeq 0 \) and equivalently \( \lambda_{\min}(\nabla^2 \mathcal{L}_{(X, Z)}(Z, X)) > 0 \).

To find the smallest eigenvalue \( \lambda_{\min}(\nabla^2 \mathcal{L}_{(X, Z)}(Z, X)) \), it suffices to find the largest \( \sigma > 0 \)
with \( c \) the largest feasible non-ascent step in the linearized augmented Lagrangian, and at least one update step if \( f - \) smaller root cannot satisfy 1. We can see that (1.4, 2) \( \rho \geq 0 \) such that the spectral norm of \( Y \) to the spectral norm of \( H \) is \( k \). The proof outline of Lemma B.3 is to show that each update step is a complement of \( H \) at 1. Positive semidefinite matrix \( A \) we have \( \min \Delta(\min(H)) = (1 - \sigma) + (\sigma Y)^2(1 - (1 - \sigma)^{-1}) \). We can see that \( (1 - \sigma)\min(H) \) is a convex function in \( (1 - \sigma) \), with two zeros at \( 1 - \sigma = \pm \sqrt{\sigma^2 + 4\lambda^2 - (\sigma Y)^2} \). In between the two roots, \( \min(H) < 0 \). Since the smaller root cannot satisfy \( 1 - \sigma > 0 \), we choose \( \sigma > 1 - \frac{\sqrt{\sigma^2 + 4\lambda^2 - (\sigma Y)^2}}{2} > 0 \) as the largest feasible \( \sigma \) that maintains \( \min(H) \geq 0 \). As a result, \( \min(\nabla^2_{(X,Z)}^2) \sigma) = \rho^k\sigma_{\min} = \rho^k \left( 1 - \frac{\sqrt{\sigma^2 + 4\lambda^2 - (\sigma Y)^2}}{2} \right) \). Figure 3 shows how this term behaves according to the spectral norm of \( Y \).

We now prove the main theorem.

**Lemma B.3.** Consider the sequence

\[
\mathcal{L}^k := \mathcal{L}(P^k; D^k) = f(Z^{k-1}) + \langle \nabla f(Z^{k-1}), Z - Z^{k-1} \rangle + \delta_c(Z) + \langle U, X - Y \rangle + \langle S, Z - XY^T \rangle + \frac{\rho}{2}\|X - Y\|_F^2 + \frac{\rho}{2}\|Z - XY^T\|_F^2.
\]

If \( f(Z) \) is \( L_f \)-Lipschitz smooth, then sequence \( \mathcal{L}^k \) generated from Alg. 4.1 satisfies

\[
\mathcal{L}^{k+1} - \mathcal{L}^k \leq -c_1^k\sqrt{X^{k+1} - X^k}\|X^{k+1} - X^k\|_F^2 - c_2^k\|Z^{k+1} - Z^k\|_F^2 - c_3^k\|Y^{k+1} - Y^k\|_F^2 - \frac{\rho}{2}\|S^{k+1} - S^k\|_F^2 + \|U^{k+1} - U^k\|_F^2.
\]

(2.4)

with \( c_1^k = \frac{\rho^k}{2} \left( 1 - \frac{\sqrt{\sigma^2 + 4\lambda^2 - (\sigma Y)^2}}{2} \right) \), \( c_2^k = \frac{\rho^k}{2} - \frac{L_f}{2} \), and \( c_3^k = \frac{\rho^k}{2} > 0 \).

**Proof.** The proof outline of Lemma B.3 is to show that each update step is a non-ascent step in the linearized augmented Lagrangian, and at least one update step
is descent. We can describe the linearized ADMM in terms of four groups of updates: the primal variable $Y$, the primal variables $X$ and $Z$, the dual variables $U$, $S$, and coefficient $\rho$.

In other words, at iteration $k$, taking i) $\mathcal{L}^k = \mathcal{L}(Z^k, X^k, Y^k; D^k, \rho^k; G^k)$, ii) $\mathcal{L}^Y = \mathcal{L}(Z^k, X^k, Y^{k+1}; D^k, \rho^k; G^k)$, iii) $\mathcal{L}^{XZ} = \mathcal{L}(P^{k+1}; D^k, \rho^k; G^k)$, and iv) $\mathcal{L}^{k+1} = \mathcal{L}(P^{k+1}; D^k, \rho^k; G^k)$ and $\mathcal{L}^{k+1} - \mathcal{L}^k = (\mathcal{L}^Y - \mathcal{L}^k) + (\mathcal{L}^{XZ} - \mathcal{L}^Y) + (\mathcal{L}^{k+1} - \mathcal{L}^{XZ})$. We now lower bound each term.

1. Update $Y$. For the update of $Y$ in (4.6), taking

$$\mathcal{L}^Y - \mathcal{L}^k \leq (\nabla_Y \mathcal{L}^Y, Y^{k+1} - Y^k) - \frac{\lambda_{\min}(\nabla^2_{r Y} \mathcal{L}^Y)}{2} \|Y^{k+1} - Y^k\|^2_F,$$

(2.5) $\leq -\frac{\rho^k}{2} \|Y^{k+1} - Y^k\|^2_F$

where (a) follows from the definition of strong convexity, and (b) the optimality of $Y^{k+1}$.

2. Update $X$, $Z$. Similarly, the update of $(Z, X)$ in (4.6), denoting

$$\mathcal{L}^{XZ} = \mathcal{L}(P^{k+1}; D^k, \rho^k; G^k),$$

we have

$$\mathcal{L}^{XZ} - \mathcal{L}^Y \leq (\nabla_Z \mathcal{L}^{XZ}, Z^{k+1} - Z^k) + (\nabla_X \mathcal{L}^{XZ}, X^{k+1} - X^k) - \frac{\lambda_{\min}(\nabla^2_{r Z} \mathcal{L}^{XZ})}{2} (\|Z^{k+1} - Z^k\|^2_F + \|X^{k+1} - X^k\|^2_F),$$

(2.6) $\leq -\frac{\lambda_{\min}(\nabla^2_{r Z} \mathcal{L}^{XZ})}{2} (\|Z^{k+1} - Z^k\|^2_F + \|X^{k+1} - X^k\|^2_F),$

where (a) follows from the definition of strong convexity, and (b) the optimality of $X^{k+1}$ and $Z^{k+1}$. To further bound $\mathcal{L}^{XZ} - \mathcal{L}^{XZ}$, we use the linearization definitions

$$\mathcal{L}^{XZ} - \mathcal{L}^{XZ} = f(Z^{k+1}) - f(Z^k) - \langle \nabla f(Z^k), Z^{k+1} - Z^k \rangle$$

(2.7) $\leq \frac{L_f}{2} \|Z^{k+1} - Z^k\|_F$

where (a) comes from the $L_f$ Lipschitz smooth property of $f$.

3. Update $S$, $U$, and $\rho$. For the update of the dual variables and the penalty coefficient, with $\mathcal{L}^k = \mathcal{L}(P^k; D^k, \rho^k)$, we have

$$\mathcal{L}^D - \mathcal{L}^{XZ} \equiv \langle S^{k+1} - S^k, Z^{k+1} - X^{k+1}(Y^{k+1})^T \rangle + \langle U^{k+1} - U^k, X^{k+1} - Y^{k+1} \rangle$$

(2.8) $+ \frac{\rho^{k+1}}{2} \|Z^{k+1} - X^{k+1}(Y^{k+1})^T\|^2_F + \frac{\rho^k}{2} \|X^{k+1} - Y^{k+1}\|^2_F + \frac{\rho^{k+1}}{2} \|S^{k+1} - S^k\|^2_F + \frac{\rho^k}{2} \|U^{k+1} - U^k\|^2_F$

where the (a) follows the definition of $\mathcal{L}$ and (b) from the dual update procedure.

The lemma statement results by incorporating (2.6), (2.5), (2.8), and (2.7).

**Lemma B.4.** If $\mathcal{L}_k$ is unbounded below, then either problem (1.1) is unbounded below, or the sequence $L_f \|Z_k - Z_{k-1}\|_F$ diverges.
Proof. First, consider the case that $\mathcal{L}_k$ is unbounded below. First rewrite $\mathcal{L}^k$ equivalently as

$$
\mathcal{L}^k = f(Z^{k-1}) + \langle \nabla f(Z^{k-1}), Z^k - Z^{k-1} \rangle + \delta_c(Y^k) + \frac{\rho}{2} \|X^k - Y^k + \frac{1}{\rho} U^k\|_F^2 + \frac{\rho}{2} Z^k - X^k(Y^k) + \frac{1}{\rho^2} S^k \|_F^2 - \frac{1}{2 \rho^2} \|U^k\|_F^2 - \frac{1}{2 \rho^2} \|S^k\|_F^2.
$$

Since $\|U^k\|_F$ and $\|S^k\|_F$ are bounded above, this implies that the linearization $g^k := f(Z^{k-1}) + \langle \nabla f(Z^{k-1}), Z^k - Z^{k-1} \rangle$ is unbounded below. Note that

$$
g^k - f(Z^k) = f(Z^{k-1}) - f(Z^k) - \nabla f(Z^{k-1}) \|Z^k - Z^{k-1}\|_F^2 \geq -\frac{L_f}{2} \|Z^k - Z^{k-1}\|_F^2
$$

which implies either $f(Z^k) \to -\infty$ or $L_f \|Z^k - Z^{k-1}\|_F^2 \to +\infty.$

**Corollary B.5.** If $\mathcal{L}_k$ is unbounded below and the objective $f(Z) = \text{Tr}(CZ)$ then it must be that (1.1) is unbounded below. This follows immediately since $L_f = 0.$

**Theorem B.6.** Assume the dual variables are bounded, e.g.

$$
\max \{\|S^k\|_F, \|U^k\|_F, \|Y^k\|_F\} \leq B_p < +\infty, \text{ and } \frac{L_f}{\sigma_{\max}} \text{ is bounded above, where } \sigma_{\max} = 1 - \sqrt{\frac{\sigma_Y^2 + \rho^2}{\sigma_Y^2}}, \quad \sigma_Y = \|Y^{k+1}\|_2. \text{ Then by running Alg. 4.1 with } \rho^k = \alpha \rho^{k-1}, \text{ if } \mathcal{L}_k \text{ is bounded below, then the sequence } \{P^k, D^k\} \text{ converges to a stationary point of (4.2).}
$$

Proof. If $f(Z)$ is linear, take $K_0 = 0.$ If $f(Z)$ is $L_f$ smooth, take $\bar{K}$ large enough such that for all $k > K_0, \alpha \rho \geq L_f \sigma_{\max}.$ By assumption, $K_0$ is always finite.

Taking $\Delta^k_{X^k Y^k} = (\|Z^{k+1} - Z^k\|_F^2 + \|X^{k+1} - X^k\|_F^2 + \|Y^{k+1} - Y^k\|_F^2)$ and $c^k = \min\{c_1, c_2, c_3\},$ the summation of (2.4) leads to

$$
\mathcal{L}^K - \mathcal{L}^{K_0} = \sum_{k=K_0}^{K-1} \mathcal{L}^k - \mathcal{L}^k \leq \sum_{k=K_0}^{K-1} \frac{\rho^{k+1} + \rho^k}{2(\rho^k)^2} (\|S^{k+1} - S^k\|_F^2 + \|U^{k+1} - U^k\|_F^2) - \sum_{k=K_0}^{K-1} c^k \Delta^k_{X^k Y^k} \leq 4B_p \sum_{k=K_0}^{K-1} \frac{\rho^{k+1} + \rho^k}{2(\rho^k)^2} - \sum_{k=K_0}^{K-1} \Delta^k_{X^k Y^k} \leq 4B_p \sum_{k=K_0}^{K-1} \frac{\rho^{k+1} + \rho^k}{2(\rho^k)^2}
$$

where (a) follows from the boundedness assumption of the dual variables, and and (b) follows from Lemma B.2, B.1, and careful construction of $\rho$ with respect to $L_f$ and $\|Y^{k+1}\|_2.$ Further simplifying, we see that $L^K$ is thus bounded above, since

$$
\mathcal{L}^K - \mathcal{L}^{K_0} \leq \lim_{K \to \infty} 4B_p \sum_{k=K_0}^{K-1} \frac{\rho^{k+1} + \rho^k}{2(\rho^k)^2} = 4B_p \frac{1 + \alpha + \frac{1}{\alpha^2} + \cdots}{2\alpha K_0 \rho} = 4B_p \frac{1}{2\alpha K_0 \rho} < +\infty.
$$

If $\mathcal{L}^k$ is not unbounded below, then

$$
(2.9) \quad 0 \leq \sum_{k=K_0}^{K-1} (c_1 \|X^{k+1} - X^k\|_F^2 + c_2 \|Z^{k+1} - Z^k\|_F^2 + c_3 \|Y^{k+1} - Y^k\|_F^2) \leq +\infty.
$$
Recall \( c^k = \frac{\rho_k}{2} \), and by boundedness assumption on \( \|Y^{k+1}\|_2 \), for \( k > K_0 \), \( c^k \), \( c^\infty \) \( \propto \rho^k \). Since additionally \( \sum_k \rho_k = +\infty \), then this immediately yields \( Z^{k+1} - Z^k \to 0 \), \( X^{k+1} - X^k \to 0 \), \( Y^{k+1} - Y^k \to 0 \).

Therefore, since the primal variables are convergent, this implies that
\[
Z^{k+1} - (X^{k+1}(Y^{k+1})^T)_{\Omega} = \frac{1}{\rho^k} (S^{k+1} - S^k), \quad X^{k+1} - Y^{k+1} = \frac{1}{\rho^k} (U^{k+1} - U^k)
\]
converges to a constant. But since \( \rho^k \to \infty \) and the dual variables are all bounded, then it must be that \( Z^{k+1} - (X^{k+1}(Y^{k+1})^T)_{\Omega} \to 0 \), \( X^{k+1} - Y^{k+1} \to 0 \). Therefore the limit points \( X^*, Y^* \), and \( Z^* \) are all feasible, and simply checking the first optimality condition will verify that this accumulation point is a stationary point of (4.2).

**Appendix C. Convergence analysis for vector form.**

**Lemma C.1.** For two adjacent iterations of Algorithm 5.1 we have
\[
(3.1) \quad \|U^{k+1} - U^k\|_2^2 \leq L_2^2\|X^{k+1} - X^k\|_2^2.
\]

**Proof.** From the first order optimality conditions for the update of \( X \)
\[
(3.2) \quad \nabla g(X^{k+1}) + U^k + \rho^k (X^{k+1} - Y^{k+1}) = 0.
\]
Combining with the dual update, we get \( \nabla g(X^{k+1}) + U^{k+1} = 0 \). Then result follows from the definition of \( L_2 \).

Next we will show that the augmented Lagrangian is monotonically decreasing and lower bounded.

**Lemma C.2.** Each step in the augmented Lagrangian update is decreasing, e.g. for
\[
(3.3) \quad \mathcal{L}(X,Y;U;\rho) := g(X) + \delta_C(Y) + \langle U,X - Y \rangle + \frac{\rho}{2}\|X - Y\|_F^2,
\]
we have
\[
\mathcal{L}(Y^{k+1},X^{k+1};U^{k+1};\rho^{k+1}) \leq \mathcal{L}(Y^{k+1},X^{k+1};U^{k};\rho^{k})
\]
\[
\leq \mathcal{L}(Y^{k+1},X^{k};U^{k};\rho^{k}) \leq \mathcal{L}(Y^{k},X^{k};U^{k};\rho^{k}).
\]

Furthermore, the amount of decrease is
\[
(3.5) \quad \mathcal{L}(Y^{k+1},X^{k+1};U^{k+1};\rho^{k+1}) - \mathcal{L}(Y^{k},X^{k};U^{k};\rho^{k})
\]
\[
\leq -\rho^k\|X^{k+1} - Y^k\|_F^2 - c^k\|X^{k+1} - X^k\|_F^2.
\]

Here,
- if \( g(X) \) is \( H_g \)-strongly convex (where \( H_g = 0 \) if \( g \) is convex but not strongly convex) then \( c^k = \frac{\rho^k + H_g}{2} - \frac{L_2^2\rho^{k+1} + \rho^k}{2\rho^k} \), and
- if \( g(X) \) is nonconvex but \( L_g \)-smooth, then \( c^k = \frac{\rho^k - 3L_g}{2} - \frac{L_2^2\rho^{k+1} + \rho^k}{2\rho^k} \).

**Proof.** Both the updates of \( Y \) and \( X \) globally minimize \( \mathcal{L} \) with respect to those variables. To minimize \( Y \) at \( (X,U) = (X^k,U^k) \):
\[
(3.6) \quad \mathcal{L}(Y^{k+1},X;U;\rho) - \mathcal{L}(Y^{k},X;U;\rho) \leq (\nabla_Y \mathcal{L}(Y^{k+1},X;U;\rho),Y^{k+1} - Y^k) - \frac{\rho^k}{2}\|Y^{k+1} - Y^k\|_2^2
\]
\[
\leq -\frac{\rho^k}{2}\|Y^{k+1} - Y^k\|_2^2.
\]
To minimize $X$ at $(Y, U) = (Y^{k+1}, U^k)$, we consider two cases. If $g$ is $H_g$-strongly convex, then

$$
\mathcal{L}(Y, X^{k+1}; U; \rho) - \mathcal{L}(Y, X^k; U; \rho) \leq \langle \nabla_X \mathcal{L}(Y, X^{k+1}; U; \rho), X^{k+1} - X^k \rangle - \frac{\rho^k + H_g}{2} \|X^{k+1} - X^k\|^2_F
$$

(3.7)

where (a) follows from the strong convexity of $\mathcal{L}$, and (b) follows from the Lipschitz gradient condition on $g$. Therefore

$$
g(X^{k+1}) - g(X^k) \leq \langle \nabla g(X^k), X^{k+1} - X^k \rangle + \frac{L_g}{2} \|X^{k+1} - X^k\|^2_F
$$

where (a) follows from adding and subtracting a term, (b) from Cauchy-Schwartz, and (c) from the Lipschitz gradient condition on $g$. Therefore

$$
\mathcal{L}(Y, X^{k+1}; U; \rho) - \mathcal{L}(Y, X^k; U; \rho) \leq \langle \nabla_X \mathcal{L}(Y, X^{k+1}; U; \rho), X^{k+1} - X^k \rangle - \frac{\rho^k + 3L_g}{2} \|X^{k+1} - X^k\|^2_F
$$

In the dual variables, using $\{X, Y\} = \{X^{k+1}, Y^{k+1}\}$ we have

$$
\mathcal{L}(Y, X; U^{k+1}; \rho^{k+1}) - \mathcal{L}(Y, X; U^k; \rho^k) \leq \langle U^{k+1} - U^k, X - Y \rangle + \frac{\rho^{k+1} + \rho^k}{2} \|X - Y\|^2_F
$$

(3.8)

where (a) follows from the definition of $\mathcal{L}$, (b) follows from the update of $U$, and (c) follows from Lemma (C.1) since $\rho^k > 0$ for all $k$. Incorporating these observations completes the proof.

**Lemma C.3.** If $\rho^k \geq L_g$ and the objective $g(X)$ is lower-bounded over $\mathcal{C}$, then the augmented Lagrangian (3.3) is lower bounded.
Proof. From the $L_g$-Lipschitz continuity of $\nabla g(X)$, it follows that

$$g(X) \geq g(Y) + \langle \nabla g(X), X - Y \rangle - \frac{L_g}{2} \| X - Y \|^2_F$$

for any $X$ and $Y$. By definition

$$\mathcal{L}(Y^k, X^k; U^k; \rho^k) = g(X^k) + \langle U^k, X^k - Y^k \rangle + \frac{\rho^k}{2} \| X^k - Y^k \|^2_F.$$

We have

(a) $\mathcal{L}(Y^k, X^k; U^k; \rho^k) = (a) g(X^k) - \langle \nabla g(X^k), X^k - Y^k \rangle + \frac{\rho^k}{2} \| X^k - Y^k \|^2_F$, 

(b) $\mathcal{L}(Y^k, X^k; U^k; \rho^k) \geq g(Y^k) + \frac{\rho^k - L_g}{2} \| X^k - Y^k \|^2_F,$

where (a) follows from the optimality in updating $X$ and (b) follows from (3.8). Since $\mathcal{L}^k$ is unbounded below, then $g(Y^k)$ is unbounded below. Since $Y^k \in C$ for all $k$, this implies that $g$ is unbounded below over $C$. \hfill $\Box$

Thus, if $g(X)$ is lower-bounded over $C$, then since the sequence $\{ \mathcal{L}(X^k, Y^k; U^k) \}$ is monotonically decreasing and lower bounded, then the sequence $\{ \mathcal{L}(X^k, Y^k; U^k) \}$ converges. Given the monotonic descent of each subproblem (Lemma C.2) and strong convexity of $\mathcal{L}^k$ with respect to $X$ and $Y$, it is clear that $X^k \to X^*$, $Y^k \to Y^*$ fixed points. Combining with Lemma C.1 gives also $U^k \to U^*$.

The proof of Theorem 5.3 easily follows from Lemma C.3.

C.1. Linear rate of convergence when $g$ is strongly convex.

Lemma C.4. Consider Alg. 5.1 with $\rho^k$ constant. Then collecting the variables all vectorized $x = (X, Y, Y)$,

$$\mathcal{L}^{k+1} - \mathcal{L}^k \leq -c_3 \| x^{k+1} - x^k \|^2,$$

where $g$ is $H_g$ strongly convex and

$$c_3 = \max_{\theta \in (0, 1)} \min \left\{ \theta \left( \frac{\rho + H_g}{2} - \frac{L^2_g}{\rho} \right), (1 - \theta) \left( \frac{\rho + H_g}{2H_g} - \frac{L^2_g}{\rho H_g} \right), -\rho \right\}.$$

Proof. From Lemma C.2 we already have that

$$\mathcal{L}^{k+1} - \mathcal{L}^k \leq -\rho \| Y^{k+1} - Y^k \|^2 - c \| X^{k+1} - X^k \|^2$$

where for constant $\rho$, $c = \frac{\rho + H_g}{2} - \frac{L^2_g}{\rho}$. Moreover, when $g(X)$ is $H_g$-strongly convex, 

$$\| U^{k+1} - U^k \|_2 = \| \nabla g(X^{k+1}) - \nabla g(X^k) \|_2 \geq H_g \| X^{k+1} - X^k \|_2.$$ 

Therefore

$$\mathcal{L}^{k+1} - \mathcal{L}^k \leq -\theta \frac{c}{H_g} \| U^{k+1} - U^k \|^2_2 - (1 - \theta)c \| X^{k+1} - X^k \|^2.$$ \hfill $\Box$

for any $\theta \in (0, 1)$. We thus have

$$\mathcal{L}^{k+1} - \mathcal{L}^k \leq -\theta c \| X^{k+1} - X^k \|^2_F - (1 - \theta) \frac{c}{H_g} \| U^{k+1} - U^k \|^2_2 - \rho \| Y^{k+1} - Y^k \|^2_F$$

$$\leq -\min \left\{ \theta c, (1 - \theta) \frac{c}{H_g}, -\rho \right\} \left[ \begin{array}{c} X^{k+1} - X^k \\ Y^{k+1} - Y^k \\ U^{k+1} - U^k \end{array} \right].$$
Note that this does not mean \( L \) is strong convex with respect to the \textit{collected} variables \( x = (X,Y,Z) \) (\( L \) is not even convex). But with respect to each variable \( X, Y, \) and \( Z \), it is strongly convex.

\textbf{Lemma C.5.} Again with \( \rho^k > 1 \) constant and collecting \( x = (X,Y,Z) \), we have

\[
L^{k+1} - L^* \leq c_4 \|x^{k+1} - x^*\|^2, \quad c_4 = \min\{L_g + \rho + 2, 2\rho, 1\}
\]

whenever \( Y^{k+1} \) and \( Y^* \) are both in \( C \).

\textit{Proof.} Over the domain \( C \), the augmented Lagrangian can be written as

\[
L(x) = g(X) + \langle U, X - Y \rangle + \frac{\rho}{2} \|X - Y\|_F^2
\]

with gradient \( \nabla L(x) = \begin{bmatrix} \nabla g(X) + U + \rho(X - Y) \\ -Y + \rho(Y - X) \\ X - Y \end{bmatrix} \) and thus

\[
\|\nabla L(x_1) - \nabla L(x_2)\|_F^2 = \|\nabla_X L(x_1) - \nabla_X L(x_2)\|_F^2 + \|\nabla_Y L(x_1) - \nabla_Y L(x_2)\|_F^2 + \|\nabla_U L(x_1) - \nabla_U L(x_2)\|_F^2
\]

\[
\leq (L_g + \rho + 2)\|X_1 - X_2\|_F^2 + (2\rho)\|Y_1 - Y_2\|_F^2 + \|U_1 - U_2\|_F^2
\]

\[
\leq \min\{L_g + \rho + 2, 2\rho, 1\} \|x_2 - x_1\|_2^2
\]

which reveals the Lipschitz smoothness constant for \( L \) as \( c_4 = \min\{L_g + \rho + 2, 2\rho, 1\} \).

Then using first-order optimality conditions,

\[
L^{k+1} \leq L^* + (\nabla L(x^*), x^{k+1} - x^*) + c_4 \|x^{k+1} - x^*\|_2^2
\]

\[
\leq L^* + c_4 \|x^{k+1} - x^*\|_2^2
\]

where (a) follows from the optimality of \( L^* \).

\textbf{Lemma C.6.} Consider \( g(x) \) \( H_g \)-strongly convex in \( x \), and \( \rho \) large enough so that \( c_3 > 0 \). Then the number of steps for \( |L^k - L^0| \leq \epsilon \) is \( O(\log(1/\epsilon)) \).

This proof is standard in the linear convergence of block coordinate descent when the objective is strongly convex. Note that \( L \) is not strongly convex or even convex, but still all the steps hold.

\textit{Proof.} Take \( x^k = \{X^k, Y^k, U^k\} \) and \( x^* = \{X^*, Y^*, U^*\} \). Then

\[
L(x^k) - L(x^*) = L(x^k) - L(x^{k+1}) + L(x^{k+1}) - L(x^*)
\]

\[
\geq c_3 \|x^{k+1} - x^k\|^2 + L(x^{k+1}) - L(x^*)
\]

\[
\geq \left( \frac{c_3}{c_4} + 1 \right) (\mathcal{L}(x^{k+1}) - \mathcal{L}(x^*))
\]

Therefore

\[
\frac{L(x^k) - L(x^*)}{L(x^0) - L(x^*)} \leq \left( \frac{c_4}{c_4 + c_3} \right)^k
\]

and so

\[
\mathcal{L}(x^k) - \mathcal{L}(x^*) \leq \epsilon
\]

if

\[
k \geq D_1 \log(1/\epsilon) + D_2
\]
where

\[ D_1 = \log^{-1}\left(\frac{c_4 + c_3}{c_4}\right) \quad \text{and} \quad D_2 = \frac{\log(L(x^0) - L(x^*))}{\log\left(\frac{c_4 + c_3}{c_4}\right)}. \]

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