Characters of the Unitarizable Highest Weight Modules over the N=2 Superconformal Algebras

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The N=2 superconformal (or super-Virasoro) algebras in two dimensions are three complex Lie superalgebras: the Neveu-Schwarz superalgebra \[^1\], the Ramond superalgebra \[^1\], the twisted superalgebra \[^2\], which are denoted as \(\mathcal{A}\), \(\mathcal{P}\), \(\mathcal{T}\), resp., or \(\mathcal{G}\) when a statement holds for all three superalgebras. They have the following nontrivial super-Lie brackets:

\[
\begin{align*}
[L_m, L_n] &= (m-n) L_{m+n} + \frac{1}{4} z (m^3 - m) \delta_{m,-n} \quad (1a) \\
[L_m, G^j_n] &= \left(\frac{1}{2} m - n\right) G^j_{m+n} , \quad j = 1, 2 \quad (1b) \\
[L_m, Y_n] &= - n Y_{m+n} , \quad [Y_m, Y_n] = z m \delta_{m,-n} \quad (1c) \\
[Y_m, G^j_n] &= i \epsilon^{jk} G^k_{m+n} , \quad \epsilon^{jk} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (1d) \\
[G^j_m, G^k_n]^{+} &= 2 \delta^{jk} L_{m+n} + i \epsilon^{jk} (m-n) Y_{m+n} + z \left(m^2 - \frac{1}{4}\right) \delta^{jk} \delta_{m,-n} \quad (1e)
\end{align*}
\]

where \(m \in \mathbb{Z}\) in \(L_m\) for all superalgebras; \(m \in \mathbb{Z}\) in \(Y_m\) and \(m \in \frac{1}{2} + \mathbb{Z}\) in \(G^j_m\) for \(\mathcal{A}\); \(m \in \mathbb{Z}\) in \(Y_m\) and \(G^j_m\) for \(\mathcal{P}\); \(m \in \mathbb{Z}\) in \(G^1_m\) and \(m \in \frac{1}{2} + \mathbb{Z}\) in \(Y_m\) and \(G^2_m\) for \(\mathcal{T}\).

The standard triangular decomposition of \(\mathcal{G}\) is:

\[
\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{H} \oplus \mathcal{G}_- \quad (2)
\]

\[
\mathcal{H} = \text{l.s.}\{z, L_0, Y_0\} \quad \text{for } \mathcal{A}, \mathcal{P} \quad (3a)
\]

\[
= \text{l.s.}\{z, L_0, G^1_0\} \quad \text{for } \mathcal{T}, \quad (G^1_0)^2 = L_0 - z/8 \quad (3b)
\]

\[
\mathcal{G}_+ = \text{l.s.}\{L_m, m > 0, Y_n, n > 0, G^j_p, p > 0\} \oplus \text{l.s.}\{\bar{G}_0\}_\mathcal{P} \quad (4a)
\]

\[
\mathcal{G}_- = \text{l.s.}\{L_m, m < 0, Y_n, n < 0, G^j_p, p < 0\} \oplus \text{l.s.}\{G_0\}_\mathcal{P} \quad (4b)
\]

where the generators \(G_0, \bar{G}_0\) which appear for \(\mathcal{P}\) in \(\mathcal{G}\) are the zero modes of:

\[
G_n = \frac{1}{2} (G^1_n + iG^2_n) , \quad \bar{G}_n = \frac{1}{2} (G^1_n - iG^2_n) \quad (5)
\]

\[^1\] This is a slightly extended version of an Encyclopedia entry.
A highest weight module (HWM) over $\mathcal{G}$ is characterized by its highest weight $\lambda \in \mathcal{H}^*$ and highest weight vector $v_0$ so that $Xv_0 = 0$, for $X \in \mathcal{G}_+$, $Hv_0 = \lambda(H)v_0$ for $H \in \mathcal{H}$. Denote $\lambda(L_0) = h$, $\lambda(z) = c$, $\lambda(Y_0) = q$. [Note that interchanging $G_0$ and $\bar{G}_0$ in (4) means to pass from $P^+$ to $P^-$ modules in the terminology of [4].] The largest HWM with these properties is the Verma module $V^\lambda = V^{h,c,q}$ ($= V^{h,c}$ for $T$), which is isomorphic to $U(\mathcal{G}_-)v_0$, where $U(\mathcal{G}_-)$ denotes the universal enveloping algebra of $\mathcal{G}_-$. Denote by $L^\lambda$ (resp. $L^{h,c,q}$, $L^{h,c}$) the factor-module $V^\lambda/I^\lambda$, where $I^\lambda$ is the maximal proper submodule of $V^\lambda$. Then every irreducible HWM over $\mathcal{G}$ is isomorphic to some $L^\lambda$.

A Verma module $V^{h,c,q}$ ($V^{h,c}$) over $\mathcal{G}$ is reducible if and only if [2]:

$$f_{r,s}^A = 2h(c - 1) - q^2 - \frac{1}{4}(c - 1)^2 + \frac{1}{4}((c - 1)r + s)^2 = 0, \quad \text{for some } r \in \mathbb{N}, s \in 2\mathbb{N}, \quad (6a)$$

or

$$g_n^A \equiv 2h - 2nq + (c - 1)(n^2 - \frac{1}{4}) = 0, \quad \text{for some } n \in \frac{1}{2} + \mathbb{Z}, \quad \text{for } A; \quad (6b)$$

$$f_{r,s}^P = 2(c - 1)(h - \frac{1}{8}) - q^2 + \frac{1}{4}((c - 1)r + s)^2 = 0, \quad \text{for some } r \in \mathbb{N}, s \in 2\mathbb{N}, \quad (7a)$$

or

$$g_n^P \equiv 2h - 2nq + (c - 1)(n^2 - \frac{1}{4}) - \frac{1}{4} = 0, \quad \text{for some } n \in \mathbb{Z}, \quad \text{for } P; \quad (7b)$$

$$f_{r,s}^T = 2(c - 1)(h - \frac{1}{8}) + \frac{1}{4}((c - 1)r + s)^2 = 0, \quad \text{for some } r \in \mathbb{N}, s \in 2\mathbb{N} - 1, \quad \text{for } T \quad (8)$$

The necessary conditions for the unitarity of $L^{h,c,q}$ ($L^{h,c}$) are [2]:

**case** $A_3$ :  
$c \geq 1$, $g_n^A \geq 0$, \quad for all $n \in \frac{1}{2} + \mathbb{Z}$ ; \quad (9a)

**case** $A_2$ :  
$c \geq 1$, $f_{1,2}^A \geq 0$, $g_n^A = 0$, $g_{n + \text{sign}(n)} \leq 0$, \quad for some $n \in \frac{1}{2} + \mathbb{Z}$ ; \quad (9b)

**case** $A_0$ :  
$c < 1$, $c = 1 - \frac{2}{m}$, $h = \frac{1}{m}(jk - \frac{1}{4})$, $q = \frac{1}{m}(j - k)$,

for $m \in 1 + \mathbb{N}$, $j, k \in \frac{1}{2} + \mathbb{Z}$, \quad $0 < j, k, j + k \leq m - 1$ ; \quad (9c)

**case** $P_3$ :  
$c \geq 1$, $g_n^P \geq 0$, \quad for all $n \in \mathbb{Z}$ ; \quad (10a)

**case** $P_2$ :  
$c \geq 1$, $f_{1,2}^P \geq 0$, $g_n^P = 0$, $g_{n + \text{sign}(n)} < 0$, \quad for some $n \in \mathbb{Z}$ , \quad (10b)

$\text{sign}(0) = \pm 1$ for $P^{\pm}$ ;

**case** $P_0$ :  
$c < 1$, $c = 1 - \frac{2}{m}$, $h = \frac{1}{8}c + \frac{1k}{m}$, $q = \pm \frac{1}{m}(j - k)$,

for $m \in 1 + \mathbb{N}$, $j, k \in \mathbb{Z}$, \quad $0 \leq j - 1, k, j + k \leq m - 1$ ; \quad (10c)

**case** $T_2$ :  
$c \geq 1$, \quad $h \geq \frac{1}{8}c$ ; \quad (11a)

**case** $T_0$ :  
$c < 1$, $c = 1 - \frac{2}{m}$, $h = \frac{1}{8}c + \frac{1}{16m}(m - 2r)^2$,

for $m \in 1 + \mathbb{N}$, $r \in \mathbb{N}$, \quad $1 \leq r \leq \frac{1}{2}m$ ; \quad (11b)

Further write $V^{h,c,q}, L^{h,c,q}$ in the cases when a statement holds for $V^{h,c,q}, L^{h,c,q}$ over $A, P$ as written and for $V^{h,c}, L^{h,c}$ over $T$ after deleting $q$ and all related quantities.
The weight decomposition of $V^{h,c}(q)$ is:

$$V^{h,c}(q) = \bigoplus_{n,m} V_{n,m}^{h,c}(q)$$

(12a)

$$V_{n,m}^{h,c}(q) = \{ v \in V^{h,c}(q) \mid L_0 v = (h+n)v, \text{ for } \mathcal{G}, \quad Y_0 v = (q+m)v, \text{ for } \mathcal{A}, \mathcal{P} \}$$

(12b)

where the ranges of $n, m$ in (12) are:

$$n \in \mathbb{Z}^+, \quad m \in 2n + 2 \mathbb{Z}, \quad |m| \leq \sqrt{2n}, \text{ for } \mathcal{A}$$

(13a)

$$n \in \mathbb{Z}^+, \quad m \in \mathbb{Z}, \quad \frac{1}{2}(1 - \sqrt{8n + 1}) \leq m \leq \frac{1}{2}(1 - \sqrt{8n + 1}), \quad \text{for } \mathcal{P}$$

(13b)

$$n \in \frac{1}{2} \mathbb{Z}^+, \quad \text{for } \mathcal{T}$$

(13c)

$n$ is called the level of $V^{h,c}(q)$, $m$ - its relative charge.

Then the character of $V_{n,m}^{h,c}(q)$ may be defined as follows [2]:

$$\text{ch } V^{h,c,q} = \sum_{n,m} (\dim V_{n,m}^{h,c}(q)) x^{h+n} y^{q+m} = \sum_{n,m} P(n,m) x^{h+n} y^{q+m} = x^h y^q \psi(x,y)$$

(14a)

$$\text{ch } V^{h,c} = \sum_n (\dim V_n^{h,c}(q)) x^{h+n} = \sum_n P_T(n) x^{h+n} = x^h \psi_T(x),$$

(14b)

$$\psi_A(x,y) \equiv \sum_{n,m} P_A(n,m) x^n y^m = \prod_{k \in \mathbb{N}} \frac{(1+x^{k-1/2}y)(1+x^{k-1/2}y^{-1})}{(1-x^k)^2}$$

(15a)

$$\psi_P(x,y) \equiv \sum_{n,m} P_P(n,m) x^n y^{m-1/2} = (y^{1/2} + y^{-1/2}) \prod_{k \in \mathbb{N}} \frac{(1+x^k y)(1+x^k y^{-1})}{(1-x^k)^2}$$

(15b)

$$\psi_T(x) \equiv \prod_n P_T(n) x^n = \prod_{k \in \mathbb{N}} \frac{(1+x^k)(1+x^k^{-1/2})}{(1-x^k)(1-x^k^{-1/2})}$$

(15c)

(for $P$ - representations one should write $y^{m+1/2}$ instead of $y^{m-1/2}$ [2]).

**Proposition 1:** [2],[3] The character formulae for the unitary cases $A_3$, $(P_3)$, with either $c > 1$ and $g_n > 0, \forall n \in \frac{1}{2} + \mathbb{Z}$, ($\forall n \in \mathbb{Z}$), or $c = 1$, and cases $T_2$ are given by:

$$\text{ch } L^{h,c,q} = \text{ch } V^{h,c,q}$$

(16a)

$$\text{ch } L^{h,c} = \text{ch } V^{h,c}, \quad h \neq \frac{c}{8}, \quad \text{ch } L^{\frac{c}{8},c} = \frac{1}{2} \text{ ch } V^{\frac{c}{8},c}$$

(16b)

Note that the Verma modules involved are irreducible except in the last case, where $V^{\frac{c}{8},c} = I^{\frac{c}{8},c} \oplus V^{\frac{c}{8},c}/I^{\frac{c}{8},c}$, $I^{\frac{c}{8},c} \cong V^{\frac{c}{8},c}/I^{\frac{c}{8},c}$.
Proposition 2: The character formulae for the unitary cases $A_3$, $(P_3)$, with $c > 1$, $q/(c-1) = n_0 \in \frac{1}{2} + \mathbb{Z}$, $(n_0 \in \mathbb{Z})$, and $g_{n_0} = 0$, and for the cases $A_2$, $(P_2)$, with $f_{1,2} > 0$, are given by:

$$
\text{ch} L^{h,c,q} = \tilde{\text{ch}}_n V^{h,c,q} \equiv \frac{1}{(1 + x|n| y^{\text{sign}(n)})} \text{ch} V^{h,c,q}
$$

where for $A_3$, $P_3$, $n = n_0$, and for $A_2$, $P_2$, $n$ is such that $g_n = 0$, $g_{n+\text{sign}(n)} < 0$.

Proof: Actually, the Proposition holds in a more general situation beyond the unitary cases, namely, when for a fixed $V^{h,c,q}$ (6b), ((7b)) holds for some $n$, possibly also for some $n'$ such that $\text{sign}(n) = \text{sign}(n')$ and $|n'| > |n|$, and (6a), ((7a)) does not hold for any $r, s$. [In the statement of Proposition 2 the additional reducibility appears in the cases $A_2$, $(P_2)$ when $2q(c-1) \in \mathbb{Z}$, then $n' = M - n$, $M = 2q(c-1)$ and $g_{M-n} = 0$.] In this situation there is a singular vector $v^s_n$ and possibly a singular vector $v^s_{n'}$, however, the latter (when existing) is a descendant of $v^s_n$. Thus, there is the following embedding diagram:

$$
V^{h,c,q} \rightarrow V^{h+|n|,c,q+\text{sign}(n)}
$$

where is used the convention that the arrow points to the embedded module. This embedding has a kernel, since there is an infinite chain of embeddings of Verma modules:

$$
\cdots \rightarrow V_t \rightarrow V_{t+1} \rightarrow \cdots
$$

where $V_t \equiv V^{h+t|n|,c,q+t\text{sign}(n)}$, $t \in \mathbb{Z}$. Using the Grassmannian properties of the odd generators one can show that this chain of embedding maps is exact. Due to the kernel one has:

$$
\text{ch} L^{h,c,q} = \text{ch} V^{h,c,q} - \tilde{\text{ch}}_n V^{h+|n|,c,q+\text{sign}(n)} = \tilde{\text{ch}}_n V^{h,c,q}
$$

Proposition 3: The character formulae for the unitary cases $A_2$, $P_2$, with $f_{1,2} = 0$ is given by:

$$
\text{ch} L^{h,c,q} = \frac{(1 - x)}{(1 + x|n| y^{\text{sign}(n)}) (1 + x|n|+1 y^{\text{sign}(n)})} \text{ch} V^{h,c,q}
$$

Proof: The character relevant structure of $V^{h,c,q}$ is given by the embedding diagram:

$$
\begin{align*}
V_0 &= V^{h,c,q} \\
V_0' &= V^{h+1,c,q} \\
V_1 &= V^{h+|n|,c,q+\text{sign}(n)} \\
V_1' &= V^{h+|n|+2,c,q+\text{sign}(n)}
\end{align*}
$$
where the dashed arrows denote even embeddings, \( V_0 \) is reducible v.r.t. \( g_n = 0 = f_{1,2} \) from the statement; then (with \( \mu = 0 \) for \( A \), \( \mu = 1 \) for \( P \)):

\[
h = \frac{1}{8} (c-1)(2n + \epsilon)^2 + n\epsilon + \frac{1}{8} \mu, \quad q = \frac{1}{2} (c-1)(2n + \epsilon), \quad \epsilon \equiv \text{sign}(n)
\]  

(23)

The other reducibilities relevant for the structure are: \( V_1 \) w.r.t. \( g_n = 0 = f_{1,2} \), \( V_0' \) and \( V_1' \) w.r.t. \( g_{n+\text{sign}(n)} = 0 \). Thus for the character formula follows:

\[
\chi L^{h,c,q} = \chi V^{h+1,c,q} - \chi V^{h+|n|,c,q+\text{sign}(n)} + \widetilde{c}_n V^{h+|n|+2,c,q+\text{sign}(n)}
\]

which after substituting the definitions gives (21). ♦

**Proposition 4:** [9], [4], [3] The character formulae for the unitary cases \( A_0, P_0^\pm \), is given by:

\[
\chi L_{m,j,k}(x,y) = \sum_{n \in \mathbb{Z}_+} x^{mn^2+(j+k)n} \left\{ 1 - x^{(m-j-k)(2n+1)} + \right.
\]

\[
+ x^{mn+k} y \left[ \frac{x^{2(m-j-k)(n+1)}}{1 + x^{mn+m-j}y} - \frac{1}{1 + x^{mn+k}y} \right] +
\]

\[
+ x^{mn+k} y^{-1} \left[ \frac{x^{2(m-j-k)(n+1)}}{1 + x^{mn+m-k}y^{-1}} - \frac{1}{1 + x^{mn+j}y^{-1}} \right] \chi V_{m,j,k}(x,y)
\]

(25)

where \( L_{m,j,k} = L^{h,c,q} \), \( V_{m,j,k} = V^{h,c,q} \), when \( h, c, q \) are expressed through \( m, j, k \) as in (9c), (10c).

**Proof:** The structure of \( V_0 \equiv V_{m,j,k} \) is given by the following embedding diagram:

\[
\begin{align*}
V_0 &= V^{h+mn^2+(j+k)n,c,q}, \quad V_0' = V^{h+mn^2+m(2n+1)-(k+j)(n+1),c,q}, \\
V_n &= V^{h+mn^2+(m+k+j)n+k,c,q+1}, \quad V_n' = V^{h+mn^2+m(3n+2)-(k+j)(n+2)+k,c,q+1}, \\
V_n^- &= V^{h+mn^2+(m+k+j)n+j,c,q-1}, \quad V_n' = V^{h+mn^2+m(3n+2)-(k+j)(n+2)+j,c,q-1}
\end{align*}
\]  

(27)
From this follows:

$$
\text{ch } L_{m,j,k}(x,y) = \sum_{n \in \mathbb{Z}^+} \left[ \text{ch } V_n - \text{ch } V'_n - \tilde{c}_{mn+k} V^+_n - \tilde{c}_{mn+j} V^-_n + \tilde{c}_{mn+m-j} V'^+_n + \tilde{c}_{mn+m-k} V'_{n-} \right]
$$

(28)

which after substituting the definitions gives (25). 

**Remark:** It should be stressed that diagram (26) is used only as representing the structure of the Verma module $V_{m,j,k}$. In particular, later it was shown that each even embedding between the Verma modules $V_n$ and $V'_n$, $n = 1, 2, \ldots$, and between the Verma modules $V'_n$ and $V_{n+1}$, $n = 1, 2, \ldots$, is generated by two uncharged fermionic singular vectors [6]. However, this has no relevance for the character formulae.

**Proposition 5:** [3],[4],[5] Let $V_{r,s}$, $r \in \mathbb{N}$, $s \in \mathbb{N} - 1/2$, be the Verma module $V^{h,c}$ with $h = h^{T}_{r,s} = [(tr - ms)^2 - t^2] / 4mt + 1/8 = h^{T}_{m-r,t-s}$, $c = 1 - 2t/m$, $t, m \in \mathbb{N}$, $tr \leq ms$, $s < t < m$, $t, m$ have no common divisor. Then the character formula for the corresponding irreducible quotient $L_{r,s}$ is given by:

$$
\text{ch } L_{r,s}(x) = \text{ch } V_{r,s}(x) \sum_{j \in \mathbb{Z}} x^{j(tmj+tr-ms)}(1 - x^{s(2mj+r)})
$$

(29)

In particular, the character formula for the $T_0$ unitary cases $r \leq m/2$ is obtained from (29) by setting $t = 1$, $s = 1/2$.

The Proof relies on the realization that the Verma modules $V_{r,s}$ has exactly the structure of certain Virasoro and $N = 1$ super-Virasoro (Neveu-Schwarz and Ramond) Verma modules for which the character formulae were known (see also the corresponding encyclopedia entry). 

♦
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