Non-parametric threshold estimation for classical risk process perturbed by diffusion

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Abstract
In this paper we consider a macro approximation of the flow of a risk reserve, which is first introduced by [16]. The process is observed at discrete time points. Because we cannot directly observe each jump time and size then we will make use of a technique for identifying the times when jumps larger than a suitably defined threshold occurred. We estimate the jump size and survival probability of our risk process from discrete observations.

Keywords: survival estimation, Integrated Squared Error, goodness-of-fit test

1. Introduction
In the field of financial and insurance mathematics, the following model is the most commonly used:

\[ X_t = x + ct + \sigma W_t - \sum_{i=1}^{N_t} \gamma_i, \]

where \( x \) is the initial value and \( c > 0 \) is a premium rate, \( \sigma > 0 \) represents the diffusion volatility, \( N_t \) is a Poisson process with rate \( \lambda \) counting the number of jumps up to time \( t > 0 \), \( \gamma_1, \gamma_2, ... \) are independent and identically distributed (i.i.d.) positive random variables with distribution \( F \), \( \sum_{i=1}^{N_t} \gamma_i \) represents the aggregate jumps up to time \( t \), \( W_t \) is a Wiener process independent of \( N_t \) and \( \gamma_i \). In order to avoid the almost sure ruin of (1), we assume that the premium rate can be written as \( c = (1 + \theta_0)\lambda \mu \) where the premium loading factor \( \theta_0 \) is positive and \( \mu = \int_0^\infty uF(du) \).

This model was first introduced by [8] and further studied by many authors during the last few years ([6, 7, 12, 9]). Many results on ruin probability and other ruin problems have been obtained by the works mentioned above. If we define the survival probability of (1) as \( \Phi(x) \) then the problem is that, apart from some special cases, a general expression for \( \Phi(x) \) does not exist. So in this paper we will construct an estimator of \( \Phi(x) \) without specifying any parametric model on \( F \), the distribution function of \( \gamma_i \).

In our model we will assume that \( \sigma \) and \( \lambda \) are unknown parameters. In the classical Poisson risk model, the estimation of ruin probability has been considered by many authors ([2, 11, 14, 15, 10, 13]). Not the same observation rules as in the work of [18], we only consider a discrete record of \( n + 1 \) observations \( \{X_{t_0}^n, X_{t_1}^n, ..., X_{t_{n-1}}^n, X_{t_n}^n\} \) where \( t_i^n = ih_n, T_n = t_n^n \) and \( h_n > 0 \). This observation is also important since the real data is always obtained at discrete time points.
When we want to estimate the distribution of $\gamma_i$, we have to separate the contributions of the diffusion part with respect to the jump of the Poisson process so that the threshold estimation which has been introduced in [3, 16, 17, 20, 19] can be used. We only accept there is a jump for the Poisson process between the interval $(t_{n_i-1}, t_{n_i}]$ if and only if the increment $\Delta_i X = X_{t_{n_i}} - X_{t_{n_i-1}}$ has a too big absolute value. We define this threshold function $\vartheta(h_n)$ which goes to zero when $h_n \to 0$.

Recall that the estimator of the empirical distribution function of $\gamma_i$ can be written by

$$\hat{F}_{N_t}(u) = \frac{1}{N_t} \sum_{i=1}^{N_t} I_{\{\gamma_i \leq u\}},$$

then we can try to write the estimator as

$$\hat{F}_n(u) = \frac{1}{\sum_{i=1}^{n} I_{\{\Delta_i X > \vartheta(h_n)\}}} \sum_{i=1}^{n} I_{\{\Delta_i X \leq u\}}. \quad (2)$$

In this paper we will try to study the properties of the estimator defined in (2), such as the strong consistency, asymptotic normality. We also obtain the weak consistency in a sense of the integrated squared error ($ISE$) of the estimator of survival probability such as in [13].

The rest of the paper is organized as follows, in section 2 we will give some notations and construct the estimators for all the unknown parameters and the distribution of $\gamma_i$. In section 3, we will give the asymptotic properties of the estimators given in section 2. In section 4, we obtain the weak consistency in a sense of $ISE$ of the estimator of survival probability of our risk process.

2. Preliminaries

2.1. General Notation

Throughout the paper, we use the primary notations and assumptions.

- Symbols $P$ and $D$ stand for the convergence in probability, and in law, respectively;
- $L_F$ (respectively, $l_F$) is the Laplace (respectively, Stieltjes) transform for a function $F$: for $s > 0$
  $$L_F(s) = \int_0^{\infty} e^{-su} F(u) du; \quad l_F(s) = \int_0^{\infty} e^{-su} F(du)$$
  where in $L_F$, $F$ is for every function and in $l_F$, $F$ is the distribution function.
- $\|f\|_K := (\int_0^K |f(t)|^2 dt)^{1/2}$ for a function $f$. In particular, $\|f\| = \|f\|_\infty$. We say that $f \in L^2(0, K)$ if $\|f\|_K < \infty$ and $f \in L^2(0, \infty)$ if $\|f\| < \infty$, respectively.
- For a stochastic sequence $X_n$, we denote by $X_n = O_P(R_n)$ if $X_n / R_n \to 0$ in probability and $X_n = o_P(R_n)$ if $X_n / R_n \to 0$.
- $E$ is a compact subset of $(0, \infty)$.
- $k_n > 0$ is a real-value sequence.

We make the following assumptions.

A. Let $\sigma, \gamma$ satisfy: $\sigma < Q$, $0 < \Gamma \leq \gamma < Q$, with $Q > 0$. 


B. For $\delta \in (0, \frac{1}{2})$, $p, q > 1$ with $p^{-1} + q^{-1} = 1$ and $s \in \mathbb{E}$,

1. $\alpha_n^\delta(k_n^2) \to 0$ as $n \to \infty$

2. $\sqrt{T_n}(\alpha_n^\delta(k_n) + \omega_n(s, p, q) + \Gamma(s, \varphi_n)) \to 0$ as $n \to \infty$,

where

$$\alpha_n^\delta(k_n) = k_n h_n^{\frac{1-\delta}{2}}$$

and

$$\omega_n(s, p, \rho) = \frac{1}{h_n^{\frac{\delta}{2}}} \left( \lambda \int_0^\infty |M_s(x) - \varphi_n \circ M_s(x)| F(dx) \right)^{\frac{1}{\beta}}.$$

2.2. Estimator of unknown parameters and $l_F$

Now we will try to construct the estimator of $\sigma^2$, $\lambda$, $\rho = \frac{h \varphi}{c}$ and $l_F$. Because we only consider the case where observations are discrete, that is to say the jumps are not observable. To overcome the problem, [3], [20] and [17] have used a jump-discriminant filter of the form

$$C_n^\vartheta(\vartheta(h_n)) = \{ \omega \in \Omega; |\Delta_i X| > \vartheta(h_n) \}$$

to discriminate between jumps and large Brownian shocks in an interval $(t_{n-1}, t_n]$ in the cases of jump-diffusions. They judged that no jump had occurred if $|\Delta_i X| \leq \vartheta(h_n)$ and that a single jump had occurred if $|\Delta_i X| > \vartheta(h_n)$ by choosing the threshold $\vartheta(h_n)$ suitably.

According to the work of [20], we can define the following estimators of $\sigma^2$, $\lambda$, $\rho$ and $l_F$ :

$$\tilde{\sigma}_n^2 = \frac{\sum_{i=1}^{n} |\Delta_i X - c h_n|^2 I(|\Delta_i X| \leq \vartheta(h_n))}{h_n \sum_{i=1}^{n} I(|\Delta_i X| \leq \vartheta(h_n))}, \quad \tilde{\lambda}_n = \frac{\sum_{i=1}^{n} I(|\Delta_i X| > \vartheta(h_n))}{T_n},$$

$$\tilde{\rho}_n = \frac{1}{c} \frac{\sum_{i=1}^{n} |\Delta_i X| I(|\Delta_i X| > \vartheta(h_n))}{T_n}, \quad \tilde{l}_F(s) = \frac{\sum_{i=1}^{n} (e^{-s|\Delta_i X|} I(|\Delta_i X| > \vartheta(h_n)))}{\sum_{i=1}^{n} I(|\Delta_i X| > \vartheta(h_n))}$$

where $\vartheta(h_n) = L h_n^p$, $L > 0$, $\omega \in (0, \frac{1}{2})$ and $s \in \mathbb{E}$.

Remark 2.1. Here, we choose $\vartheta(h_n) = L h_n^p$ for a constant $L > 0$ and $\omega \in (0, \frac{1}{2})$. In [3], the author proposed the jump-discriminant threshold function such as $\vartheta(h_n) = L h_n \log \frac{1}{h_n}$ for a constant $L > 0$. In [20], the author proposed $\vartheta(h_n) = L h_n^p$ for a constant $L > 0$ and $\omega \in (0, \frac{1}{2})$. First of all, it is obvious that their threshold functions satisfy the conditions: $\vartheta(h_n) \to 0$ and $\sqrt{h_n} \vartheta(h_n)^{-1} \to 0$. If $\vartheta(h_n)$ does not converge to zero then one cannot detect small jumps even when $n \to \infty$. Moreover, if $\vartheta(h_n)$ converges to zero faster than the order of $\sqrt{h_n}$ then the filter $C_n^\vartheta(\vartheta(h_n))$ would possibly misjudge the Brownian noise as a small jump since the variation of the Brownian motion is of order $\sqrt{h_n} \log \frac{1}{h_n}$. Actually, the threshold function $\vartheta(h_n)$ in [20] and [3] are all suited for our aim. For the convenience of proof on our results, we choose $\vartheta(h_n)$ in [20]. For a detailed account on the threshold functions we refer to [17].
3. Properties of the Estimators

3.1. Asymptotic Normality of $F_n(u)$

First we will try to get the asymptotic normality of $\hat{F}_n(u)$. Before that we will define $\hat{F} = 1 - F$ where $F$ is the distribution of $\gamma_i$ and $\hat{F}_n(u) = 1 - \hat{F}_n(u)$. Then

$$\hat{F}_n(u) = \frac{1}{\sum_{i=1}^{n} I(\{\Delta_iX_i > \phi(h_n)\})} \sum_{i=1}^{n} I(\Delta_iX_i \{\Delta_iX_i > \phi(h_n)\} > u),$$

then we have the following result:

**Theorem 3.1.** Suppose that $nh_n \to \infty$, $h_n \to 0$ as $n \to \infty$ and the condition $A$ in section 2 is satisfied, then

$$\sqrt{T_n} \left( \hat{F}_n(u) - F(u) \right) \xrightarrow{D} \mathbb{N}(0, \frac{F(u)(1-F(u))}{\lambda}).$$

then we change to $\bar{F}_n(u)$ and $F(u)$ that is

$$\sqrt{T_n} \left( \hat{F}_n(u) - F(u) \right) \xrightarrow{D} \mathbb{N}(0, \frac{F(u)(1-F(u))}{\lambda}).$$

To prove this theorem, we need the following lemma:

**Proposition 3.2.** Following from the condition of Theorem 3.1, for any $\epsilon > 0$,

$$\lim_{n \to \infty} P \left( |\Delta_iX I(\{\Delta_iX > \phi(h_n)\}) - \gamma_1 I(\Delta_i \geq 1)| > \epsilon \right) = 0,$$

where $\gamma_1$ is the eventual jump in time interval $(t_{i-1}^n, t_i^n]$.

*Proof.*

\[
P \left( |\Delta_iX I(\{\Delta_iX > \phi(h_n)\}) - \gamma_1 I(\Delta_i \geq 1)| > \epsilon \right) \\
\leq P \left( |\Delta_iX I(\{\Delta_iX > \phi(h_n), \Delta_i = 0\})| > \frac{\epsilon}{3} \right) \\
+ P \left( |\Delta_iX I(\{\Delta_iX > \phi(h_n)\}) - \gamma_1 I(\Delta_i = 1)| > \frac{\epsilon}{3} \right) \\
+ P \left( |\Delta_iX I(\{\Delta_iX > \phi(h_n)\}) - \gamma_1 I(\Delta_i \geq 2)| > \frac{\epsilon}{3} \right).
\]

Due to $P\{|\sigma \int_{t_{i-1}^n}^{t_i^n} dW_s| \geq c\} \leq 2e^{-c^2/2\sigma^2h_n}$, the first term in (6) is dominated by

$$P \left( \sigma \int_{t_{i-1}^n}^{t_i^n} dW_s \right) > \frac{\epsilon}{6}$$

$$\leq P \left( \sigma \int_{t_{i-1}^n}^{t_i^n} dW_s \right) > \frac{\epsilon}{6} + 2e^{-c^2/2\sigma^2h_n}.$$ 

The term

$$|\Delta_iX I(\{\Delta_iX > \phi(h_n)\}) - \gamma_1 I(\Delta_i \geq 1)|$$
is equal to
\[
|ch_n + \sigma \int_{t_{n-1}^n} dw_s + \gamma_{\tau(t)} | I_{\{\Delta_t \geq \vartheta(h_n)\}} - \gamma_{\tau(t)} | I_{\{\Delta_t, N=1\}}
\]
\[
\leq |ch_n + \sigma \int_{t_{n-1}^n} dw_s| + |\gamma_{\tau(t)} | I_{\{\Delta_t \leq \vartheta(h_n), \Delta_t, N=1\}}.
\]

Thus, (6) is dominated by
\[
2P(ch_n > \epsilon) + 4e^{-\epsilon^2/2} Q(h_n) + P(\Delta_t, N = 1) + P(\Delta_t, N \geq 2).
\]

Therefor, (6) tends to zero as \( n \to \infty \).

Now we will prove the Theorem 3.1. Because
\[
\sqrt{T_n} \left[ \frac{1}{\sum_{i=1}^{n} I_{\{\Delta_t \geq \vartheta(h_n)\}} \sum_{i=1}^{n} I_{\{\Delta_t, X_{\{\Delta_t \geq \vartheta(h_n)\}} > u\}} - F(u) \right]
\]
\[
= \sqrt{T_n} \left[ \frac{\sum_{i=1}^{n} I_{\{\Delta_t, X_{\{\Delta_t \geq \vartheta(h_n)\}} > u\}} - F(u) I_{\{\Delta_t, X > \vartheta(h_n)\}}}{\sum_{i=1}^{n} I_{\{\Delta_t \geq \vartheta(h_n)\}}} \right]
\]
\[
= \frac{J}{G}.
\]

where
\[
J = \sum_{i=1}^{n} I_{\{\Delta_t, X_{\{\Delta_t \geq \vartheta(h_n)\}} > u\}} - F(u) I_{\{\Delta_t, X > \vartheta(h_n)\}}
\]
and
\[
G = \sum_{i=1}^{n} I_{\{\Delta_t \geq \vartheta(h_n)\}}
\]

First, we will calculate the limit of the expectation of \( J \):
\[
\lim_{n \to \infty} E[J]
\]
\[
= \lim_{n \to \infty} \sqrt{T_n} \left[ E(I_{\{\Delta_t, X_{\{\Delta_t \geq \vartheta(h_n)\}} > u\}} - F(u) I_{\{\Delta_t, X > \vartheta(h_n)\}}) \right]
\]
\[
= \lim_{n \to \infty} \sqrt{T_n} E(H)
\]
\[
= \lim_{n \to \infty} \sqrt{T_n} \left[ P(\Delta_t, X I_{\{\Delta_t \geq \vartheta(h_n)\}} > u) - F(u) P(|\Delta_t, X > \vartheta(h_n)\}) \right],
\]
where, \( H = I_{\{\Delta_t, X_{\{\Delta_t \geq \vartheta(h_n)\}} > u\}} - F(u) I_{\{\Delta_t, X > \vartheta(h_n)\}} \).
By Proposition 3.2 and 3, we have

\[
\lim_{n \to \infty} P(\Delta_i X \{ |\Delta_i X| > \vartheta(h_n) \} > u) = P(\gamma_{\tau(i)} I(\Delta_i N \geq 1) > u) = P(\gamma_{\tau(i)} > u | \Delta_i N \geq 1) P(\Delta_i N \geq 1) = P(\gamma_{\tau(i)} > u) P(\Delta_i N \geq 1) = \mathcal{F}(u)(\lambda h_n + o(h_n))
\]  

and

\[
\lim_{n \to \infty} P(\|\Delta_i X\| > \vartheta(h_n)) = P(\Delta_i N \geq 1) = \lambda h_n + o(h_n).
\]  

Therefore,

\[
\lim_{n \to \infty} E[J] = 0.
\]

Now, we will calculate the limit of the variation of \(J\):

\[
\lim_{n \to \infty} Var[J] = \lim_{n \to \infty} \frac{n}{T_n} \left[ Var(I_{\{\Delta_i X \| > \vartheta(h_n)\}}) - \mathcal{F}(u)I_{\{\|\Delta_i X\| > \vartheta(h_n)\}} \right]
\]

\[
= \lim_{n \to \infty} \frac{n}{T_n} Var(H)
\]

\[
= \lim_{n \to \infty} \frac{n}{T_n} [E(H)^2 - E^2(H)]
\]

\[
= \lim_{n \to \infty} \frac{n}{T_n} [E(H)^2]
\]

The term \(E(H)^2\) is equal to

\[
E(I_{\{\Delta_i X \| > \vartheta(h_n)\}}^2 I_{\{\|\Delta_i X\| > \vartheta(h_n)\}}) + \mathcal{F}(u)^2 I_{\{\|\Delta_i X\| > \vartheta(h_n)\}} - 2\mathcal{F}(u)I_{\{\Delta_i X \| > \vartheta(h_n)\}} I_{\{\|\Delta_i X\| > \vartheta(h_n)\}}.
\]

Due to \(I^2 = I\) and

\[
I_{\{\Delta_i X \| > \vartheta(h_n)\}} I_{\{\|\Delta_i X\| > \vartheta(h_n)\}} = I_{\{\Delta_i X \| > \vartheta(h_n)\}} I_{\{\|\Delta_i X\| > \vartheta(h_n)\}}
\]

\(E(H)^2\) is equal to

\[
E(I_{\{\Delta_i X \| > \vartheta(h_n)\}}^2 I_{\{\|\Delta_i X\| > \vartheta(h_n)\}}) + \mathcal{F}(u)^2 I_{\{\|\Delta_i X\| > \vartheta(h_n)\}} - 2\mathcal{F}(u)I_{\{\Delta_i X \| > \vartheta(h_n)\}} I_{\{\|\Delta_i X\| > \vartheta(h_n)\}}.
\]

By (11) and (10),

\[
\lim_{n \to \infty} Var[J] = \lim_{n \to \infty} \frac{n}{T_n} \left[ \lambda h_n \mathcal{F}(u)(1 - \mathcal{F}(u)) + o(h_n) \right] = \lambda \mathcal{F}(u)(1 - \mathcal{F}(u)).
\]
Applying the central limit theorem, we have

\[ J \xrightarrow{D} N(0, \lambda F(u)(1 - F(u))) \]

as \( n \to \infty \).

From the Lemma 3.3 we have \( G \xrightarrow{P} \lambda \) and then by the Slutsky’s theorem, we have

\[ J \xrightarrow{G} N(0, \lambda F(u)(1 - F(u))) \]

3.2. Study of the estimators in section 2.2

As is well known in the inference for discretely observed diffusions that:

\[ \tilde{\sigma}^2_n \xrightarrow{P} \sigma^2, \quad n \to \infty. \]

In addition, if \( nh_n^2 \to 0 \),

\[ \sqrt{n}(\tilde{\sigma}^2_n - \sigma^2) \xrightarrow{D} N(0, 2\sigma^4) \]

as \( n \to \infty \). The proof is from Theorem 3.1 in [16].

In order to find the asymptotic normality of \( \tilde{\lambda}_n, \tilde{\rho}_n \) and \( \tilde{l}_F \), We will first define a truncation function \( \varphi_n(x) \) satisfying the following conditions:

1. \( |\varphi_n(x)| \leq \beta_n \) a.e. for a real-valued sequence \( \beta_n \) such that \( \beta_n \uparrow \infty \) as \( n \to \infty \).
2. \( \varphi_n(x) \to x \) a.e. as \( n \to \infty \).

Through this \( \varphi_n \) we will redefine the estimator of \( \lambda, \rho \) and \( l_F \). Moreover, we consider the estimator of the product of \( \lambda l_F \). They are defined as

\[ \tilde{\rho}_n^* = \frac{1}{c} \sum_{i=1}^{n}(\varphi_n \circ M_1^1)(|\Delta_i X|)I_{|\Delta_i X| > \theta(h_n)} \]

\[ \tilde{\lambda}_n l_F^*(s) = \frac{\sum_{i=1}^{n}(\varphi_n \circ M_2^2)(|\Delta_i X|)I_{|\Delta_i X| > \theta(h_n)} \}}{T_n} \]

where \( M_1^1(x) = x, M_2^2(x) = e^{-sx} \),

\[ \varphi_n \circ M_1^1(x) = \begin{cases} M_1^1(x) & \text{if } (M_1^1(x) \lor \sup_{s \in \mathbb{E}}(M_1^1(x)))' \lor \sup_{s \in \mathbb{E}}(M_1^1(x)')' \leq \kappa_n, \\ 0 & \text{Otherwise} \end{cases} \]

and \( \kappa_n > 0 \) is a real-value sequence, we have the following results:

Lemma 3.3. Suppose that \( nh_n \to \infty, h_n \to 0 \) as \( n \to \infty \), then

\[ \tilde{\lambda}_n \xrightarrow{P} \lambda, \quad \tilde{\rho}_n \xrightarrow{P} \rho, \]

and

\[ \sup_{\{s \in \mathbb{E}\}} |\tilde{\lambda}_n l_F^*(s) - \lambda l_F(s)| \xrightarrow{P} 0. \]

In addition, if the Condition B in section 2.1 is satisfied, \( \int_0^\infty x^2 F(dx) < \infty \) and \( nh_n^{1+\beta} \to 0 \) for some \( \beta \in (0, 1) \), we have

\[ \sqrt{T_n}(\tilde{\lambda}_n - \lambda) \xrightarrow{D} N(0, \lambda), \]
\[
\sqrt{T_n} (\rho_n^* - \rho) \xrightarrow{D} N(0, \frac{\lambda}{c^2} \int_0^\infty x^2 F(dx)) \tag{14}
\]

and
\[
\sqrt{T_n} \left( \lambda F_n(s) - \lambda F(s) \right) \xrightarrow{D} N(0, \lambda \int_0^\infty e^{-2sx} F(dx)), \tag{15}
\]
as \(n \to \infty\).

Proof. First from the theorem 3.2 and 3.4 in [16], it is easy to get (11) and (12). So we only need to check (13), (14) and (15). We define
\[
\nu_n(s) = \frac{\sum_{i=1}^{n} \varphi_n \circ M_s(|\Delta_i|X)I_{|\Delta_i| > \vartheta(h_n)}}{T_n}
\]
and
\[
\nu(s) = \lambda \int_0^\infty M_s(x) F(dx)
\]
Notice that
\[
\sqrt{T_n} (\nu_n(s) - \nu(s)) = \sum_{i=1}^{n} Y_{n,i}
\]
where
\[
Y_{n,i} = \frac{1}{\sqrt{T_n}} \left( \varphi_n \circ M_s(|\Delta_i|X)I_{|\Delta_i| > \vartheta(h_n)} - h_n \lambda \int_0^\infty M_s(x) F(dx) \right)
\]
we can apply the central limit theorem for \(\{Y_{n,i}\}_{1 \leq i \leq n}\). If the following conditions (17)-(19) are satisfied, we can obtain (13)-(15) by [1].

\[
\sum_{i=1}^{n} |E[Y_{n,i}]| \xrightarrow{P} 0, \tag{17}
\]
\[
\sum_{i=1}^{n} E[Y_{n,i}]^2 \xrightarrow{P} \lambda \int_0^\infty M_s^2 F(dx), \tag{18}
\]
\[
\sum_{i=1}^{n} E[Y_{n,i}]^4 \xrightarrow{P} 0 \tag{19}
\]
The details of the proofs of (17)-(19) may refer to Theorem [17].

4. Integrated Squared Error (ISE) of the Estimator

4.1. Weak consistency in the ISE sense

We will present our important result that states a convergence in probability of the ISE of the estimator.

First let us recall that let \(\tau(x)\) be the time of ruin with the initial reserve \(x\): \(\tau(x) = \inf\{t > 0, X_t \leq 0\}\). The survival probability \(\Phi(x)\) is defined as follows:
\[
\Phi(x) = P\{\tau(x) = \infty\}.
\]
As we know, a general expression for $\Phi(x)$ does not exist, but the corresponding Laplace transform of $\Phi(x)$ can be obtained by [18], there we take $\omega \equiv 1$ and $\delta = 0$ which is the Laplace transform of our model:

$$L_\Phi(s) = \int_0^\infty e^{-sx} \Phi(x)dx = \frac{1 - \lambda \mu}{s + \frac{\sigma^2}{2} s^2 - \frac{\lambda}{c}(1 - l_F(s))}$$ \hspace{1cm} (20)

with the estimators of the parameters in (20), we define $\tilde{L}_\Phi(s)$ as the estimator of $L_\Phi(s)$:

$$\tilde{L}_\Phi(s) = \frac{1 - \tilde{\rho}^*_n}{s + \frac{\sigma^2}{2} s^2 - \frac{\lambda_n}{c}(1 - \tilde{l}_{F_n}(s))}.$$ \hspace{1cm} (21)

In order to estimate the original functions $\Phi(x)$, we will apply a regularized inversion of the Laplace transform proposed by [4] to (21). The regularized inversion is defined as follows, which is available for any $L^2$ functions.

**Definition 4.1.** Let $m > 0$ be a constant. The regularized Laplace inversion $L^{-1}_m : L^2 \to L^2$ is given by

$$L^{-1}_m g(t) = \frac{1}{\pi^2} \int_0^\infty \int_0^\infty \Psi_m(y) y^{-\frac{1}{2}} e^{-tvy} g(v) dv dy$$

for a function $g \in L^2$ and $t \in (0, \infty)$, where

$$\Psi_m(y) = \int_{a_m}^{\infty} \cosh(\pi x) \cos(x \log y) dx$$

and $a_m = \pi^{-1} \cosh^{-1}(\pi m) > 0$.

**Remark 4.2.** It is well known that the norm $L^{-1}$ is generally unbounded, which causes the ill-posedness of the Laplace inversion. However, $L^{-1}_m$ is bounded for each $m > 0$, in particular,

$$\|L^{-1}_m\| \leq m\|L\| = \sqrt{\pi m}$$

see [4], equation (3.7).

By (20) and (21), it is obvious that $L_\Phi(s) \notin L^2(0, \infty)$ and $\tilde{L}_\Phi(s) \notin L^2(0, \infty)$. The regularized laplace transform inversion $L^{-1}_m$ of Definition 4.1 does not apply at once. In order to ensure they are in $L^2(0, \infty)$, these functions $L_\Phi(s)$ and $\tilde{L}_\Phi(s)$ will be slightly modified.

For arbitrary fixed $\theta > 0$, we define $\Phi_\theta(x) = e^{-\theta x} \Phi(x)$. It is obvious that

$$L_{\Phi_\theta}(s) = L_\Phi(s + \theta), \quad \tilde{L}_{\Phi_\theta}(s) = \tilde{L}_\Phi(s + \theta).$$

Let us define an estimator of $\Phi(x)$: for given numbers $\theta > 0$ and $m(n)$, we denote by

$$\tilde{\Phi}_n(x) = e^{\theta x} \tilde{\Phi}_{\theta,m(n)}(x),$$ \hspace{1cm} (22)

where $\Phi_{\theta,m(n)}(x) = L^{-1}_m \Phi_{\theta}(s)$.

Now we shall present our important result which states a convergence in probability of the $ISE$ on compacts.
**Theorem 4.3.** Suppose the same assumption as in Lemma 3.3 and the net profit condition: $c > \lambda \mu$. Moreover, suppose that there exist a constant $K > 0$ such that $0 \leq \Phi'(x) = g(x) \leq K < \infty$. Then, for numbers $m(n)$ such that $m(n) = \sqrt{\frac{T_n}{\log T_n}}$ as $n \to \infty$ and for any constant $B > 0$, we have

$$\|\tilde{\Phi}_{m(n)} - \Phi\|_B^2 = O_P((\log T_n)^{-1}) \quad (n \to \infty).$$

### 4.2. The proof of Theorem 4.3

To prove Theorem 4.3, we need the following lemma:

**Lemma 4.4.** Suppose that, for a function $f \in L^2$ with the derivative $f'$, $\int_0^\infty [t(t^2 f(t))']^2 t^{-1} dt < \infty$, then

$$\|L^{-1}_n L f - f\| = O \left((\log n)^{-\frac{1}{2}}\right) \quad (n \to \infty).$$

Now we will prove the Theorem 4.3.

**Proof.** By (22), we have

$$\|\tilde{\Phi}_{m(n)} - \Phi\|_B^2 \leq e^{2B} \|	ilde{\Phi}_{\theta,m(n)} - \Phi_\theta\|_B^2 \leq 2e^{2B} \{\|L_m^{-1}(L_{\Phi_\theta} - L_{\Phi_\theta}^{-1})L_{\Phi_\theta}\|^2 + \|\Phi_{\theta,m(n)} - \Phi_\theta\|^2\} = 2e^{2B} \|[I_1 + I_2]\|.$$

In order to deal with $I_2$, let us write $\Phi'_\theta = g_\theta$ and note that

$$\int_0^\infty [x(\sqrt{T} \Phi_\theta(x))]^2 \frac{1}{x} dx = \int_0^\infty [x(\frac{1}{2}\sqrt{x} \Phi_\theta(x) + x \sqrt{x} g_\theta(x))]^2 \frac{1}{x} dx \leq \int_0^\infty 2 \frac{1}{x} [x(\frac{1}{2}\sqrt{x} \Phi_\theta(x))]^2 + \int_0^\infty \frac{2}{x} [x \sqrt{x} g_\theta(x)]^2 dx = \int_0^\infty \frac{1}{2} \Phi_\theta^2(x) dx + 2 \int_0^\infty x^2 g_\theta^2(x) dx \leq \int_0^\infty \frac{1}{2} e^{-2\theta x} dx + 2 \int_0^\infty x^2 [g(x)e^{-\theta x} - \theta \Phi(x)e^{-\theta x}]^2 dx \leq \frac{1}{4\theta} + 4 \int_0^\infty x^2 g_\theta^2(x) e^{-2\theta x} dx + 4\theta^2 \int_0^\infty \Phi_\theta^2(x) x^2 e^{-2\theta x} dx \leq \frac{1}{4\theta} + 4(K^2 + \theta^2) \int_0^\infty x^2 e^{-2\theta x} dx < +\infty.$$  

Therefore, by the Lemma 4.4, we may conclude that

$$\|\Phi_{\theta,m(n)} - \Phi_\theta\|^2 = O\left(\frac{1}{\log m(n)}\right). \quad (23)$$
Next, we consider the formula $I_1$. By (20) and (21), we have

$$
\frac{||L_{\theta} - L_{\theta}||^2}{2} \leq \int_0^\infty \frac{1 - \tilde{\rho}^*}{D(s + \theta)} \frac{1 - \rho}{D(s + \theta)}^2 ds
$$

$$
= \int_0^\infty \frac{1 - \tilde{\rho}^*}{D(s + \theta)} \frac{1 - \rho}{D(s + \theta)} + \frac{1 - \rho}{D(s + \theta)} - \frac{1 - \rho}{D(s + \theta)}^2 ds
$$

$$
\leq \int_0^\infty \frac{2}{(s + \theta)^4(1 - \tilde{\rho}^*)^2} ds + 2 \int_0^\infty \frac{\tilde{\rho}^* - \rho}{1 - \tilde{\rho}^*} \frac{1}{(s + \theta)^2} ds
$$

$$
= \frac{(s + \theta)^2}{2c}(\tilde{\sigma}^2 - \sigma^2) + \frac{1}{c} \left( \lambda - \tilde{\lambda} + \frac{\lambda}{\tilde{\lambda} F_n(s + \theta) - \lambda F(s + \theta)} \right)
$$

$$
= O_P(T_n^{-\frac{1}{2}}) + \frac{1}{c} \left( \tilde{\lambda} F_n(s + \theta) - \lambda F(s + \theta) \right).
$$

Note that we used here the fact that $\tilde{\sigma}^2 - \sigma^2 = o_P(T_n^{-\frac{1}{2}})$. Now we consider the second term in (24). By (25), we have

$$
\int_{T_n^{-1}} \frac{\tilde{D}(s + \theta)}{(s + \theta)^4(1 - \tilde{\rho}^*)^2} ds
$$

$$
\leq O_P(T_n^{-1}) \int_{T_n^{-1}} \frac{1}{(s + \theta)^4} ds + \frac{2}{c^2} \int_{T_n} \frac{\tilde{\lambda} F_n(s + \theta) - \lambda F(s + \theta)}{(s + \theta)^4(1 - \tilde{\rho}^*)^2} ds
$$

$$
\leq O_P(T_n^{-1}) \int_{T_n^{-1}} \frac{1}{(s + \theta)^4} ds + \frac{4}{c^2} \int_{T_n} \frac{\tilde{\lambda}^2 + \lambda^2}{(s + \theta)^4(1 - \tilde{\rho}^*)^2} ds
$$

$$
= O_P(T_n^{-1}) \int_{T_n^{-1}} \frac{1}{(s + \theta)^4} ds + O_P(1) \int_{T_n} \frac{1}{(s + \theta)^4} ds
$$

$$
= O_P(1) \int_{T_n^{-1}} \frac{1}{(s + \theta)^4} ds
$$

$$
= O_P(\frac{1}{T_n}).
$$
By Lemma 3.3, the first term of (24) is
\[ 2 \int_0^T \frac{(\bar{D}(s+\theta) - D(s+\theta))^2}{(s+\theta)^2(1-\rho_n^\ast)^2} ds = O_P(T_n^{-1}) \]
and the last term is
\[ 2 \int_0^\infty \frac{(\bar{\rho}_n^\ast - \rho_1^\ast)^2}{(s+\theta)^2} ds = O_P(T_n^{-1}). \]
Therefore,
\[ \|L^{-1}_m n\|_2 \|\bar{\Phi}_\theta - L\Phi_\theta\|_2 = O_p\left(\frac{m^2(n)}{T_n}\right) \]
Combining (23) and (26), we have
\[ \|\bar{\Phi}_m(n) - \Phi_B\|^2 = O_p\left(\frac{m^2(n)}{T_n}\right) + O_p\left(\frac{1}{\log m(n)}\right). \]
With an optimal \(m(n) = \sqrt{T_n \log T_n}\), balancing the the right hand two terms in (27), the order becomes \(O_p((\log T_n)^{-1})\).

**Remark 4.5.** The explicit integral expression
\[ \bar{\Phi}_m(n)(u) = \frac{\pi \theta}{\pi^2} \int_0^\infty \int_0^\infty e^{-usy} \bar{L}_\Phi(s) \Psi_m(n)(y) y^{-\frac{3}{2}} dsdy \]
where \(\Psi_m(n)(y) = \int_0^{a_m(n)} \cosh(\pi x) \cos(x \log(y)) dx\) and \(a_m(n) = \pi^{-1} \cosh^{-1}(\pi m(n)) > 0\) and \(m(n) = \sqrt{T_n \log T_n}\).

**5. Goodness-of-fit Test for \(\gamma_i\)**

In theorem 3.1, we have proved the asymptotic properties of \(\hat{F}(u)\), but as we can see the variance depends on the unknown parameter \(\lambda\), when we want to do the problem of goodness-of-fit test for the distribution of \(\gamma_i\), we want to find the estimator with the distribution free just with the null hypothesis: \(H_0 : F(x) = F_0(x)\). To achieve this goal, we need the following convergence:

\[ \sqrt{n} \sum_{i=1}^n I_{\{\Delta_i X_i > \theta(n)\}} \left( \hat{F}_n(u) - F(u) \right) \overset{D}{\to} N(0, F(u)(1 - F(u))) \]

The proof is the same as in theorem 3.1. Now we construct the Cramér-von Mises \(W_n^2\) and Kolmogorov-Smirnov \(D_n\) statistics are
\[ W_n^2 = n \int_{-\infty}^\infty \left( \hat{F}_n(x) - F_0(x) \right)^2 dF_0(x), \quad D_n = \sup_x |\hat{F}_n(x) - F_0(x)| \]
From the equation (28), we can easily get the limit behavior of these statistics
\[ W_n^2 \overset{d}{\to} \int_0^1 W_0(s)^2 ds, \quad \sqrt{n}D_n \overset{d}{\to} \sup_{0 \leq s \leq 1} |W_0(s)| \]
where \(\{W_0(s), 0 \leq s \leq 1\}\) is a Brownian bridge, the same as the i.i.d case.
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