Rapid mixing from spectral independence beyond the Boolean domain

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Glauber dynamics

Sampling from joint distribution

Set of variables $V$

Finite domain $[q] = \{1,2, \ldots, q\}$ for $q \geq 2$

Joint distribution $\mu$ over $\Omega = \text{supp}(\mu) \subseteq [q]^V$

Problem draw random samples from $\mu$

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**Fundamental MCMC: Glauber dynamics**

Start from an arbitrary feasible configuration $X \in \Omega$;

**For** each $t$ from 1 to $T$ **do**

- pick a variable $v \in V$ uniformly at random;
- resample $X_v \sim \mu_v(\cdot|X_{V\backslash\{v\}})$;

Return $X$;
Example: proper $q$-coloring

**Uniform proper $q$-coloring**

**Undirected graph** $G = (V, E)$

**Finite set of colors** $[q] = \{1, 2, \ldots, q\}$

**Gibbs distribution** $\mu$ uniform distribution over $\Omega$

$$\Omega = \{ X \in [q]^V \mid X \text{ is a proper coloring} \}$$

Graph $G = (V, E)$

Colors $[q] = \{ \text{red, green, yellow} \}$

$\Omega$: set of all proper colors
Example: proper $q$-coloring

Uniform proper $q$-coloring

Undirected graph $G = (V, E)$
Finite set of colors $[q] = \{1, 2, \ldots, q\}$
Gibbs distribution $\mu$ uniform distribution over $\Omega = \{X \in [q]^V \mid X$ is a proper coloring$\}$
Problem sample proper coloring u.a.r.

Glauber dynamics for proper $q$-coloring

Start from an arbitrary proper coloring $X \in \Omega$;
For each $t$ from 1 to $T$ do
  • pick a vertex $v \in V$ uniformly at random;
  • resample $X_v$ from $[q] \setminus \{X_u \mid u \in \Gamma(v)\}$ uniformly at random;
Return $X$;
Convergence

Glauber dynamics: Markov chain over $\Omega$

Transition Matrix $P \in \mathbb{R}^{\Omega \times \Omega}$

Glauber dynamics is *reversible*

detailed balance equation with respect to $\mu$

$$\forall X, Y \in \Omega, \mu(X)P(X, Y) = \mu(Y)P(Y, X)$$

*Stationary distribution* $\mu P = \mu$

move among any states with positive probability

**Proposition (convergence)**

If Glauber dynamics is **connected**, it converges to **unique** stationary distribution $\mu$.

If $q \geq \Delta + 2$, Glauber dynamics converges to uniform distribution over $q$-colorings.
Mixing time

*How fast* does the Glauber dynamics converge to stationary distribution $\mu$?

Glauber dynamics $X_0, X_1, X_2, \ldots$ where each $X_t \in \Omega \subseteq [q]^V$

Mixing time

\[
T_{\text{mix}} = \max_{X_0 \in \Omega} \min \left\{ t \left| d_{TV}(X_t, \mu) \leq \frac{1}{4e} \right. \right\},
\]

$d_{TV}(X_t, \mu)$: the **total variation distance** between $X_t$ and $\mu$.

The Glauber dynamics is **rapid mixing** if

\[
T_{\text{mix}} = \text{Poly}(n) \quad n = |V| = \#\text{variables}
\]

✓ Sample from an exponential space $|\Omega| = \exp(O(n))$ within polynomial steps $T_{\text{mix}} = \text{poly}(n)$. 

Open problems

Under what **condition** of the distribution $\mu$ the Glauber dynamics for $\mu$ rapid mixing?

Under what **relation** between $q$ and max degree $\Delta$ the Glauber dynamics for coloring rapid mixing?
## Previous works

### Glauber dynamics for graph coloring

**General graphs** [Jer95, Vig00, SS97, CDMPP19]

Current best result: \( q \geq \left( \frac{11}{6} - \epsilon_0 \right) \Delta \) [CDMPP19]

**Special graphs** [DF01, Hay03, HV03, GMP05, HV06, Mol04, Hay13, DFHV13]

### High-dimensional expansion (HDX)

**Strongly log-concave distribution** [ALOV19, CGM19]

**Spectral independence** with **Boolean domain** \( \{0, 1\}^V \) [ALO20]

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**Spectral Independence**

Mixing up to uniqueness for **hardcore model** [ALO20]  
**anti-ferro 2-spin systems** [CLV20]
Our results

- A spectral independence condition for general distribution.
- Rapid mixing of Glauber dynamics from spectral independence.
  - combinatorial proof: coupling;
  - algebraic proof for Boolean variables [ALO20].
- Application: a new rapid mixing regime for graph coloring.
  - relate spectral independence with correlation decay;
  - a refined recursive coupling [GMP05] argument.
Our results

Result (I). A spectral independence condition beyond the Boolean domain.

\( \mu \): a distribution over \( \Omega \subseteq [q]^V \)

\(|V| \times |V| \) influence matrix \( \Psi \in \mathbb{R}^{V \times V} \) with \( \Psi(u, u) = 0 \) and

\[ \Psi(u, v) = \max_{i,j \in [q]} d_{TV}(\mu_v(\cdot | u \leftarrow i), \mu_v(\cdot | u \leftarrow j)) \]

maximum influence on \( v \) caused by a disagreement on \( u \)

For any subset \( S \subseteq V \), any feasible \( \sigma \in [q]^{V \setminus S} \)

\( \mu^\sigma_S \) distribution on \( S \) conditional on \( \sigma \)

influence matrix \( \Psi^\sigma_s \in \mathbb{R}^{S \times S} \) for conditional distribution

\[ \Psi^\sigma_s(u, v) = \max_{i,j \in [q]} d_{TV}(\mu^\sigma_v(\cdot | u \leftarrow i), \mu^\sigma_v(\cdot | u \leftarrow j)) \]
Our results

Result (I). A spectral independence condition beyond the Boolean domain

Spectral independence [This work]

There is a constant $C > 0$ such that

for all conditional distributions $\mu^g_S$,

spectral radius of influence matrices $\rho(\Psi^g_S) \leq C$.

Spectral independence for Boolean variables [Anari, Liu, Oveis Gharan 20]

Distribution over Boolean domain $\{0,1\}^V$

signed influence matrix: $I^g_S(u,v) = \mu^g_v(1 | u \leftarrow 1) - \mu^g_v(1 | u \leftarrow 0)$.

Relation: $\Psi^g_S(u,v) = |I^g_S(u,v)|$.

Spectral independence: for all influence matrices, max eigenvalue $\lambda_{\text{max}}(I^g_S) \leq C$. 
Our results

Result (II). **Rapid mixing** of Glauber dynamics from **spectral independence**

Theorem [This work]

\[ \mu \text{ is spectrally independent with constant } C \]

\[ T_{\text{mix}} = O \left( n^{1+2C} \log \left( \frac{1}{\mu_{\text{min}}} \right) \right), \]

where \( \mu_{\text{min}} = \min_{X \in \Omega} \mu(X) \).

**Bounded one-to-all influence**

\[ \text{spectral radius } \leq \sum_{v \in S} \psi_S(u, v) \leq C \]

**Bounded all-to-one influence**

\[ \text{spectral radius } \leq \sum_{u \in S} \psi_S(u, v) \leq C \]
Our results

**Result (III).** Rapid mixing for $q$-coloring on triangle-free graph with $q > 1.763\Delta$

Theorem [This work]
Triangle free graph and $q \geq (\alpha + \delta)\Delta$ where $\alpha \approx 1.763$ s.t. $\alpha = \exp(1/\alpha)$

\[ T_{\text{mix}} \leq n^{2+O(1/\delta)} \log q. \]

| Work     | Regime       | Girth | Addition condition                      | Mixing time         |
|----------|--------------|-------|----------------------------------------|---------------------|
| [GMP05]  | $q > \alpha \Delta$ | $\geq 4$ | $\Delta = O(1) + \text{amenable graph}$ | $O(n^2)$           |
| [HV06]   | $q \geq (\alpha + \delta)\Delta$ | $\geq 4$ | $\Delta = \Omega(\log n)$           | $O(n \log n)$      |
| [DFHV13] | $q \geq (\alpha + \delta)\Delta$ | $\geq 5$ | $\Delta \geq \Delta_0(\delta)$      | $O(n \log n)$      |
| This work| $q \geq (\alpha + \delta)\Delta$ | $\geq 4$ | --                                    | $n^{2+O(1/\delta)} \log q$ |
Proof outline

Spectral Independence

Rapid mixing of Glauber dynamics

Bridge: HDX

Rapid mixing of global walk

local-to-global [AL20]

Rapid mixing of local walks

Boolean domain [ALO20]

General domain [this work]

Graph Coloring

Decay analysis [this work]

Based on recursive coupling [GMP05]

Spectral Independence

Rapid mixing of Glauber dynamics
Lazy local random walk

State space \( U = \{ (v, i) \mid v \in V, i \in [q] \} \)

Current state \((v, i) \in U\). Transition \((v, i) \rightarrow (u, j)\)

- pick a vertex \( u \in V \) uniformly at random;
- sample \( j \sim \mu_u (\cdot \mid v \gets i) \).

graph \( G = (V, E) \)

colors \([q] = \{ \red, \green, \yellow \} \)
Our technique: coupling

**Coupling** \((X_t, Y_t)_{t \geq 0}\) of local walk

- start from two states \(X_0, Y_0 \in U\)
- two chains \((X_t)_{t \geq 0}\) and \((Y_t)_{t \geq 0}\) follow local walk

\[
\begin{align*}
\text{Current state } & \quad X_t = (u, i) \text{ and } Y_t = (v, j) \\
\text{Next state } & \quad X_{t+1} = (u', i') \text{ and } Y_{t+1} = (v', j')
\end{align*}
\]

- Pick the same \(u' = v' \in V\) uniformly at random;
- Sample \((i', j')\) from the **optimal coupling** between \(\mu_{u'}(\cdot | u \leftarrow i)\) and \(\mu_{v'}(\cdot | v \leftarrow j)\).
Observation: for any $t \geq 1$, $X_t$ and $Y_t$ must be on the same vertex.

$X_t = (v, i)$ and $Y_t = (v, j)$ (same vertex, different color)

- Pick the same vertex $u \in V$ uniformly at random.
- Couple the colors on $u$ optimally, the coupling fails with probability

$$d_{TV}(\mu_u(\cdot | v \leftarrow i), \mu_u(\cdot | v \leftarrow j)) \leq \Psi(v, u).$$

(Influence $v \to u$)

Remark (conditional distributions)

- [AL20] requires local mixing on all conditional distributions.
- Our coupling also works for all conditional distributions.
Spectral independence for coloring

List coloring instance
• graph \( G = (V, E) \) with max degree \( \Delta \);
• each vertex \( v \in V \) has a color list \( L_v \).

Proper list coloring \( X \)
• \( X_v \in L_v \) for all \( v \in V \);
• \( X_u \neq X_v \) for all \( \{u, v\} \in E \).

Gibbs distribution \( \mu \)
• uniform distribution over all proper list colorings.

Theorem [this work]
In triangle-free graph, if for all \( v \in V \),
\[ |L(v)| \geq (\alpha + \delta)\Delta \approx (1.763 + \delta)\Delta, \]
then under any pinning,
\[ \text{one-to-all influence} = O\left(\frac{1}{\delta}\right), \]
\( \mu \) is spectrally independent with \( C = O\left(\frac{1}{\delta}\right) \).
Recursive coupling

Influence from $u$ to $v$

$$\text{Inf}(u \to v) = \max_{i,j \in L(u)} d_{TV}(\mu_v(\cdot | u \leftarrow i), \mu_v(\cdot | u \leftarrow j))$$

One-to-all influence

$$\sum_{v \in V \setminus \{u\}} \text{Inf}(u \to v)$$

Proof sketch

by recursive coupling [Goldberg, Martin, Paterson 05]

Construct a coupling $(c_v, c'_v)$ between $\mu_v(\cdot | u \leftarrow i)$ and $\mu_v(\cdot | u \leftarrow j)$

$$d_{TV}(\mu_v(\cdot | u \leftarrow i), \mu_v(\cdot | u \leftarrow j)) \leq \Pr[c_v \neq c'_v]$$

Bound one-to-all influence by coupling inequality

$$\sum_{v \in V \setminus \{u\}} \text{Inf}(u \to v) \leq \sum_{v \in V \setminus \{u\}} \Pr[c_v \neq c'_v]$$
Recursive coupling

- Staring from the “disagreement vertex” $u$.
- Coupling vertex one by one in a “DFS-manner”.
- If the coupling on $v$ fails (i.e. $c_v \neq c'_v$) then **there is a path** $P$ from $u$ to $v$  
  **ALL vertices** in $P$ **FAIL** in coupling.

Bound **one-to-all influence** by enumerating all the paths from $u$

$$\sum_{u \neq v} \inf(u \to v) \leq \sum_{\text{all paths } P \text{ from } u} \text{Influence alone the path } P \leq O\left(\frac{1}{\delta}\right)$$

Triangle-free  Many colors  Coupling succeeds with high prob.  Bounded total influence
Theorem [Chen, Galanis, Štefankovič, Vigoda 20]

The Glauber dynamics for coloring is rapid mixing if \( q \geq \alpha \Delta + 1 \)

- **different definition** of spectral independence
- **different method** to prove spectral independence for coloring
Summary

- A definition of spectral independence for general distribution (generalize def. in [ALO20])
- Rapid mixing of Glauber dynamics from spectral independence
- Application: sampling uniform $q$-coloring on triangle-free graph when $q \geq (\alpha + \delta)\Delta \approx (1.763 + \delta)\Delta$.

Future work

- Improve the $n^{O_C}$ mixing time for general distribution.
- Improve the $n^{O(1/\delta)}$ mixing time for spin systems (including coloring)
  - $O(n \log n)$ optimal mixing for spin systems with spectral independence and $\Delta = O(1)$ [CLV20, arXiv:2011.02075].
- Better condition for spectral independence
  - example: prove spectral independence for graph coloring with fewer colors.

Thank you