An extension of Milman’s reverse Brunn-Minkowski inequality

by

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0. Introduction

The classical Brunn-Minkowski inequality states that for $A_1, A_2 \subset \mathbb{R}^n$ compact,

$$|A_1 + A_2|^{1/n} \geq |A_1|^{1/n} + |A_2|^{1/n}$$

where $| \cdot |$ denotes the Lebesgue measure on $\mathbb{R}^n$. Brunn [Br] gave the first proof of this inequality for $A_1, A_2$ compact convex sets, followed by an analytical proof by Minkowski [Min]. The inequality (1) for compact sets, not necessarily convex, was first proved by Lusternik [Lu]. A very simple proof of it can be found in [Pi 1], Ch. 1.

It is easy to see that one cannot expect the reverse inequality to hold at all, even if it is perturbed by a fixed constant and we restrict ourselves to balls (i.e. convex symmetric compact sets with the origin as an interior point). Take for instance by a fixed constant and we restrict ourselves to balls (i.e. convex symmetric compact sets with the origin as an interior point). Take for instance $A_1 = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x_1| \leq \varepsilon, |x_i| \leq 1, 2 \leq i \leq n\}$ and $A_2 = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x_n| \leq \varepsilon, |x_i| \leq 1, 1 \leq i \leq n-1\}$.

In 1986 V. Milman [Mil 1] discovered that if $B_1$ and $B_2$ are balls there is always a relative position of $B_1$ and $B_2$ for which a perturbed inverse of (1) holds. More precisely: “There exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and any balls $B_1, B_2 \subset \mathbb{R}^n$ we can find a linear transformation $u : \mathbb{R}^n \to \mathbb{R}^n$ with $|\det(u)| = 1$ and

$$|u(B_1) + B_2|^{1/n} \leq C(|B_1|^{1/n} + |B_2|^{1/n})$$

The nature of this reverse Brunn-Minkowski inequality is absolutely different from others (say reverse Blaschke-Santaló inequality, etc.). Brunn-Minkowski inequality is an isoperimetric inequality, (in $\mathbb{R}^n$ it is its first and most important consequence till now) and there is no inverse to isoperimetric inequalities. So, it was a new idea that in the class of affine images of convex bodies there is some kind of inverse.

The result proved by Milman used hard technical tools (see [Mil 1]). Pisier in [Pi 2] gave a new proof by using interpolation and entropy estimates. Milman in [Mil 2] gave another proof by using the “convex surgery” and achieving also some entropy estimates.

The aim of this paper is to extend this Milman’s result to a larger class of sets. Note that simple examples show that some conditions on a class of sets are clearly necessary.

For $B \subset \mathbb{R}^n$ body (i.e. compact, with non empty interior), consider $B_1 = B - x_0$, where $x_0$ is an interior point. If we denote by $N(B_1) = \cap_{|\alpha| \geq 1} aB_1$ the balanced kernel of $B_1$, it is clear that $N(B_1)$ is a balanced compact neighbourhood of the origin, so there exists $c > 0$ such that $B_1 + B_1 \subset cN(B_1)$. The Aoki-Rölewicz theorem (see [Ro], [K-P-R]) implies that there is $0 < p \leq 1$, namely $p = \log_2^{-1}(c)$, such that $B_1 \subset B \subset 2^{1/p}B_1$, where $B$ is the unit ball of some $p$-norm. This observation will allow us to work in a $p$-convex environment.

The above construction allows us to define the following parameter. For $B$ a body let $p(B), 0 < p(B) \leq 1$, be the supremum of the $p$ for which there exist a measure preserving affine transformation

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of $B$, $T(B)$, and a $p$-norm with unit ball $	ilde{B}$ verifying $T(B) \subseteq \tilde{B}$ and $|\tilde{B}| \leq |B|^{1/p}$, (by suitably adapting the results appearing in [Mil 2], it is clear that $p(B) \geq p$ for any $p$-convex body $B$).

Our main theorem is,

**Theorem 1.** Let $0 < p \leq 1$. There exists $C = C(p) \geq 1$ such that for all $n \in \mathbb{N}$ and all $A_1, A_2 \subset \mathbb{R}^n$ bodies such that $p(A_1), p(A_2) \geq p$, there exists an affine transformation $T(x) = u(x) + x_0$ with $x_0 \in \mathbb{R}^n$, $u: \mathbb{R}^n \rightarrow \mathbb{R}^n$ linear and $|\det(u)| = 1$ such that

$$|T(A_1) + A_2|^{1/n} \leq C(|A_1|^{1/n} + |A_2|^{1/n})$$

In particular, for the class of $p$-balls the constant $C$ is universal (depending only on $p$).

We prove this theorem in section 2. The key is to estimate certain entropy numbers. We will use the convexity of quasi-normed spaces of Rademacher type $r > 1$, as well as interpolation results and iteration procedures.

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1. Notation and background

Throughout the paper $X, Y, Z$ will denote finite dimensional real vector spaces. A quasi-norm on a real vector space $X$ is a map $\| \cdot \|: X \rightarrow \mathbb{R}^+$ such that

i) $\|x\| > 0 \forall x \neq 0$.

ii) $\|tx\| = |t| \|x\| \forall t \in \mathbb{R}, x \in X$.

iii) $\exists C \geq 1$ such that $\|x + y\| \leq C(\|x\| + \|y\|) \forall x, y \in X$

If iii) is substituted by

iii’) $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for $x, y \in X$ and some $0 < p \leq 1$,

$\| \cdot \|$ is called a $p$-norm on $X$. Denote by $B_X$ the unit ball of a quasi-normed or a $p$-normed space.

The above observations concerning the $p$-convexification of our problem can be restated using $p$-norm and quasi-norm notation. Recall that any compact balanced set with 0 in its interior is the unit ball of a quasi-norm.

By the concavity of the function $t^p$, any $p$-norm is a quasi-norm with $C = 2^{1/p-1}$. Conversely, by the Aoki-Rolewicz theorem, for any quasi-norm with constant $C$ there exists $p$, namely $p = \log_2(2C)$, and a $p$-norm $\| \cdot \|$ such that $|x| \leq \|x\| \leq 4^{1/p} |x|$, $\forall x \in X$.

A set $K \subset X$ is called $p$-convex if $\lambda x + \mu y$, whenever $x, y \in K$, $\lambda, \mu \geq 0$, $\lambda^p + \mu^p = 1$. Given $K \subset X$, the $p$-convex hull (or $p$-convex envelope) of $K$ is the intersection of all $p$-convex sets that contain $K$. It is denoted by $p$-conv $(K)$. The closed unit ball of a $p$-normed space $(X, \| \cdot \|)$ will simply be called a $p$-ball. Any symmetric compact $p$-convex set in $X$ with the origin as an interior point is the $p$-ball associated to some $p$-norm.

We say that a quasi-normed space $(X, \| \cdot \|)$ is of (Rademacher) type $q, 0 < q \leq 2$ if for some constant $T_q(X) > 0$ we have

$$\frac{1}{2^n} \sum_{\varepsilon_i = \pm 1} \| \sum_{i=1}^n \varepsilon_i x_i \| \leq T_q(X) \left( \sum_{i=1}^n \|x_i\|^q \right)^{1/q}, \forall x_i \in X, 1 \leq i \leq n, \forall n \in \mathbb{N}$$

Kalten, [Ka], proved that any quasi-normed space $(X, \| \cdot \|)$ of type $q > 1$ is convex. That is, the quasi-norm $\| \cdot \|$ is equivalent to a norm and moreover, the equivalence constant depends only on $T_q(X)$, (for a more precise statement and proof of this fact see [K-S]).

Given $f, g: \mathbb{N} \rightarrow \mathbb{R}^+$ we write $f \sim g$ if there exists a constant $C > 0$ such that $C^{-1}f(n) \leq g(n) \leq Cf(n), \forall n \in \mathbb{N}$. Numerical constants will always be denoted by $C$ (or $C_p$ if it depends only on $p$) although their value may change from line to line.
Let $u: X \to Y$ be a linear map between two quasi-normed spaces and $k \geq 1$. Recall the definition of the following numbers:

**Kolmogorov numbers:** $d_k(u) = \inf \{\|Q_S \circ u\| : S \subset Y \text{ subspace and } \dim(S) < k\}$ where $Q_S: Y \to Y/S$ is the quotient map.

**Covering numbers:** For $A_1, A_2 \subset X$, $N(A_1, A_2) = \inf \{N \in \mathbb{N} : \exists x_1 \ldots x_N \in X \text{ such that } A_1 \subset \bigcup_{1 \leq i \leq N} (x_i + A_2)\}$.

**Entropy numbers:** $e_k(u) = \inf \{\varepsilon > 0 : N(u(B_X), \varepsilon B_Y) \leq 2^{k-1}\}$

The following two lemmas contain useful information about these numbers. The first one extends to the $p$-convex case its convex analogue due to Carl ([Ca]). Its proof mimics the ones of Theorem 5.1 and 5.2 in [Pi 1] (see also [T]) with minor changes. In particular we identify $X$ as a quotient of $\ell_p(I)$, for some $I$, and apply the metric lifting property of $\ell_p(I)$ in the class of $p$-normed spaces (see Proposition C.3.6 in [Pie]). The second one contains easy facts about $N(A, B)$ and its proof is similar to the one of Lemma 7.5. in [Pi 1].

**Lemma 1.** For all $\alpha > 0$ and $0 < p < 1$ there exists a constant $C_{\alpha, p} > 0$ such that for all linear map $u: X \to Y$, $X, Y$ $p$-normed spaces and for all $n \in \mathbb{N}$ we have

$$\sup_{k \leq n} k^\alpha e_k(u) \leq C_{\alpha, p} \sup_{k \leq n} k^\alpha d_k(u)$$

**Lemma 2.**

i) For all $A_1, A_2, A_3 \subset X$, $N(A_1, A_3) \leq N(A_1, A_2)N(A_2, A_3)$

ii) For all $t > 0$ and $0 < p < 1$ there is $C_{p,t} > 0$ such that for all $X$ $p$-normed space of dimension $n$, $N(B_X, tB_X) \leq C_{p,t}^n$.

iii) For any $A_1, A_2, K \subset \mathbb{R}^n$, $|A_1 + K| \leq N(A_1, A_2)|A_2 + K|$

iv) Let $B_1, B_2$ be $p$-balls in $\mathbb{R}^n$ for some $p$ and $B_2 \subset B_1$; then $\frac{|B_1|}{|B_2|} \sim N(B_1, B_2)$.

For any $B \subset \mathbb{R}^n$ $p$-ball the polar set of $B$ is defined as

$$B^\circ := \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \forall y \in B\}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^n$. Given $B, D$ $p$-balls in $\mathbb{R}^n$ we define the following two numbers:

$$s(B) := (|B| \cdot |B^\circ|)^{1/n}$$

and

$$M(B, D) := \left(\frac{|B + D|}{|B \cap D|} \cdot \frac{|B^\circ + D^\circ|}{|B^\circ \cap D^\circ|}\right)^{1/n}$$

Observe that for any linear isomorphism $u: \mathbb{R}^n \to \mathbb{R}^n$ we have $s(u(B)) = s(B)$ and

$$M(u(B), u(D)) = M(B, D).$$

Recall that $s(B_{\ell_p^n}) \sim n^{-1/p} \sim s(B_{\ell_2^n}^{1/p})$, $0 < p \leq 1$ ([Pi 1] pg. 11).

The following estimates on these numbers are known:
a) [Sa]. For every symmetric convex body $B \subset \mathbb{R}^n$, $s(B) \leq s(B_{\ell_2^n})$ with equality only if $B$ is an ellipsoid. (Blaschke-Santaló’s inequality).

b) [B-M]. There exists a numerical constant $C > 0$ such that for any $n \in \mathbb{N}$ and any symmetric convex body $B \subset \mathbb{R}^n$, $s(B) \geq Cs(B_{\ell_2^n})$.

c) [Mil 1]. There exists a numerical constant $C > 0$ such that for any $n \in \mathbb{N}$ and any symmetric convex body $B \subset \mathbb{R}^n$, there is an ellipsoid (called Milman ellipsoid) $D \subset \mathbb{R}^n$ such that $M(B, D) \leq C$. (Milman ellipsoid theorem).

2. Entropy estimates and reverse Brunn-Minkowski inequality

We first introduce some useful notation: Let $B_1, B_2 \subset \mathbb{R}^n$ be two $p$-balls and $u: \mathbb{R}^n \to \mathbb{R}^n$ a linear map. We denote $u: B_1 \to B_2$ the operator between $p$-normed spaces $u: (\mathbb{R}^n, \| \cdot \|_{B_1}) \to (\mathbb{R}^n, \| \cdot \|_{B_2})$ where $\| \cdot \|_{B_i}$ is the $p$-norm on $\mathbb{R}^n$ whose unit ball is $B_i$.

**Proof of Theorem 1:**

Let $A_1, A_2$ be two bodies in $\mathbb{R}^n$ such that $p(A_1), p(A_2) \geq p$. It’s clear from the definition that there exist two $\bar{p}$-balls, $B_1, B_2$, (for instance, $\bar{p} = p/2$) and two measure preserving affine transformations $T_1, T_2$, verifying

$$|T_2^{-1}T_1(A_1) + A_2| \leq |B_1 + B_2|$$

and

$$|B_1|^{1/n} + |B_2|^{1/n} \leq C_p \left(|A_1|^{1/n} + |A_2|^{1/n}\right).$$

So, we only have to prove the theorem for $p$-balls.

In the convex case a way to obtain the reverse Brunn-Minkowski inequality is to prove that, for any symmetric convex body $B$, there exists an ellipsoid $D$ verifying $|B| = |D|$ and

$$|B + \Delta|^{1/n} \leq C|D| + \Delta|^{1/n}$$

for any, say compact, subset $\Delta \subseteq \mathbb{R}^n$ ($C$ is an universal constant independent of $B$ and $n$).

Indeed, let $B_1, B_2$ be two balls in $\mathbb{R}^n$. Suppose w.l.o.g. that $D_i$, the ellipsoids associated to $B_i$ satisfy $u_2D_i = \alpha_iB_{\ell_2^n}$, where $u_i$ are linear mappings with $|\det u_i| = 1$ and $|B_i|^{1/n} = \alpha_i|B_{\ell_2^n}|^{1/n}$. Then

$$|u_1B_1 + u_2B_2|^{1/n} \leq C^2|u_1D_1 + u_2D_2|^{1/n} = C^2(\alpha_1 + \alpha_2)|B_{\ell_2^n}|^{1/n} = C^2(|B_1|^{1/n} + |B_2|^{1/n})$$

In view of the preceding comments and of straightforward computations deduced from Lemma 2, in order to obtain (2) for $p$-balls it is sufficient to associate an ellipsoid $D$ to each $p$-ball $B \subset \mathbb{R}^n$ in such a way that the corresponding covering numbers verify $N(B, D), N(D, B) \leq C^n$ for some constant $C$ depending only on $p$.

It is important to remark now the fact that, what we deduce from covering numbers estimate is that the ellipsoid $D$ associated to $B$ actually verifies the stronger assertion

$$C^{-1}|B + \Delta|^{1/n} \leq |D + \Delta|^{1/n} \leq C|B + \Delta|^{1/n}$$

for any compact set $\Delta$ in $\mathbb{R}^n$, with constant depending only on $p$. Furthermore, the role of the ellipsoid can be played by any fixed $p$-ball in a “spetial position”.

Denote by $\hat{B}$ the convex hull of $B$.

By definition of $e_n$, if $e_n(id: B \to D) \leq \lambda$ then $N(B, 2\lambda D) \leq 2n-1$ and by Lemma 2-ii), $N(B, D) \leq \lambda^n$. (Of course, the same can be done with $N(D, B)$). Therefore our problem reduces to estimating entropy numbers. What we are going to prove is really a stronger result than we need, in the line of Theorem 7.13 of [Pi 2].
Lemma 3. Given $\alpha > 1/p - 1/2$, there exists a constant $C = C(\alpha, p)$ such that, for any $n \in \mathbb{N}$ and for any $p$-ball $B \in \mathbb{R}^n$ we can find an ellipsoid $D \in \mathbb{R}^n$ such that

$$d_k(D \to B) + e_k(B \to D) \leq C\left(\frac{n}{k}\right)^\alpha$$

for every $1 \leq k \leq n$.

Proof of the Lemma. From Theorem 7.13 of [Pi 2] we can easily deduce the following fact: There exists a constant $C(\alpha) > 0$ such that for any $1 \leq k \leq n$, $n \in \mathbb{N}$ and any ball $\hat{B} \subseteq \mathbb{R}^n$, there is ellipsoid $D_0 \subseteq \mathbb{R}^n$ such that the identity operator $id: \mathbb{R}^n \to \mathbb{R}^n$ verifies

$$d_k(id: D_0 \to \hat{B}) \leq C(\alpha)\left(\frac{n}{k}\right)^\alpha \quad \text{and} \quad e_k(id: \hat{B} \to D_0) \leq C(\alpha)\left(\frac{n}{k}\right)^\alpha$$

(3)

This let us to introduce the constant $C_n$ as the infimum of the constants $C > 0$ for which the conclusion of lemma 3 is true for all $p$-ball in $\mathbb{R}^n$. Trivially $C_n \leq C(\alpha)\left(1 + n^{1/p-1}\right)$. Let $D_1$ be an almost optimal ellipsoid such that

$$d_k(D_1 \to B) \leq 2C_n\left(\frac{n}{k}\right)^\alpha \quad \text{and} \quad e_k(B \to D_1) \leq 2C_n\left(\frac{n}{k}\right)^\alpha$$

(4)

for every $1 \leq k \leq n$.

Use the real interpolation method with parameters $\theta, 2$ to interpolate the couple $id: B \to B$ and $id: D_1 \to B$. It is straightforward from its definition that for $B_\theta := (B, D_1)_{\theta, 2}$, we have

$$d_k(B_\theta \to B) \leq \|B \to B\|^{1-\theta}(d_k(D_1 \to B))^\theta \quad \forall k \leq n$$

and therefore,

$$d_k(B_\theta \to B) \leq \left(2C_n\left(\frac{n}{k}\right)^\alpha\right)^\theta \quad \forall k \leq n$$

Write $\lambda = 4C_n\left(\frac{n}{k}\right)^\alpha$. By definition of the entropy numbers, there exist $x_i \in \mathbb{R}^n$ such that $B \subseteq \bigcup_{i=1}^{2^k-1} x_i + 2\lambda D_1$. But by perturbing $\lambda$ with an absolute constant we can suppose w.l.o.g. that $x_i \in B$. For all $z \in B$, there exists $x_i \in B$ such that $\|z - x_i\|_{D_\theta} \leq 2\lambda$. Also by $p$-convexity, $\|z - x_i\|_B \leq 2^{1/p}$.

A general result (see [B-L] Ch. 3.) assures the existence of a constant $C_p > 0$ such that

$$\|x\|_{B_\theta} \leq C_p\|x\|_B^{1-\theta}\|x\|^\theta_{D_1}.$$ 

Therefore, for all $z \in B$, there exists $x_i \in B$ such that $\|z - x_i\|_{B_\theta} \leq C_p\lambda^\theta$ which means

$$e_k(B \to B_\theta) \leq C_p\left(2C_n\left(\frac{n}{k}\right)^\alpha\right)^\theta.$$

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Since $\alpha > 1/p - 1/2$, then we can pick $\theta \in (0,1)$ such that \( \frac{2(1-p)}{2-p} < \theta < \min\{1, 1-1/2\alpha\} \). Then $B_{\theta}$ has Rademacher type strictly bigger than 1 because \( \frac{1-\theta}{p} + \frac{\theta}{2} < 1 \).

By Kalton’s result quoted before, we can suppose that $B_{\theta}$ is a ball and therefore we can apply to it (3) for $\gamma = \alpha(1-\theta) > 1/2$ and assure the existence of another ellipsoid $D_{2}$ such that

\[
d_{k}(D_{2} \to B_{\theta}) \leq C(\gamma) \left( \frac{n}{k} \right)^{\gamma} \quad \text{and} \quad e_{k}(B_{\theta} \to D_{2}) \leq C(\gamma) \left( \frac{n}{k} \right)^{\gamma} \quad \text{and} \quad \forall \, k \leq n
\]

Recall that $d_{2k-1}(D_{2} \to B) \leq d_{k}(D_{2} \to B_{\theta})d_{k}(B_{\theta} \to B)$ and the same for the $e_{k}$’s. Thanks to the monotonicity of the numbers $s_{k}$ we can use the what is known about $s_{2k-1}$ for all $s_{k}$. Using the estimates obtained above we get $\forall \, k \leq n$, 

\[
d_{k}(D_{2} \to B) \leq C(p, \alpha)^{2\theta} C_{n}^{\alpha} \left( \frac{n}{k} \right)^{\gamma + \alpha \theta} \quad \text{and} \quad e_{k}(B \to D_{2}) \leq C(p, \alpha)^{2\theta} C_{n}^{\alpha} \left( \frac{n}{k} \right)^{\gamma + \alpha \theta}.
\]

Hence by the election of $\gamma$ and by minimality we obtain $C_{n}^{\gamma - \theta} \leq C(p, \alpha)^{2\theta}$, and the conclusion of the lemma holds.

The theorem follows now from the estimate we achieved in Lemma 3 and by Lemma 1. Indeed, given any $\alpha > 1/p - 1/2$, if $D$ is the ellipsoid associated to $B$ by Lemma 3, we have

\[
n^{\alpha} e_{n}(D \to B) \leq \sup_{k \leq n} k^{\alpha} e_{k}(D \to B) \leq C(\alpha, p) \sup_{k \leq n} k^{\alpha} d_{k}(D \to B) \leq C(\alpha, p) \sup_{k \leq n} k^{\alpha} \frac{n^{\alpha}}{k^{\alpha}} = C(\alpha, p)n^{\alpha}
\]

and so, $e_{n}(D \to B) \leq C(\alpha, p)$. On the other hand just take $k = n$ in Lemma 3 and so, $e_{n}(B \to D) \leq C(\alpha, p)$.

Finally observe that since the constant $C(\alpha, p)$ depends only on $p$ and $\alpha$ and we can take any $\alpha > 1/p - 1/2$ the thesis of the theorem as stated immediately follows.
3. Concluding remarks

We conclude this note by stating the corresponding versions of a) Blaschke-Santaló, b) reverse Blaschke-Santaló and c) Milman ellipsoid theorem, cited in section 1, in the context of $p$-normed spaces.

**Proposition 1.** Let $0 < p \leq 1$. There exists a numerical constant $C_p > 0$ such that for every $p$-ball $B \subseteq \mathbb{R}^n$,

$$C_p \left( s(B_{B^p}) \right)^{1/p} \leq s(B) \leq s(B_{B^p})$$

and in the second inequality, equality holds if only if $B$ is an ellipsoid.

**Proof:** Denote by $\hat{B}$ the convex envelope of $B$. Since $\hat{B}^o = B^o$ we have $s(B) \leq s(\hat{B}) \leq s(B_{B^p})$. If $s(B) = s(B_{B^p})$, then $B$ is an ellipsoid. We will show that $B = \hat{B}$. Every $x$ in the boundary of $B$ can be written as $x = \sum \lambda_i x_i$, $x_i \in B$, $\sum \lambda_i = 1$; but since $\hat{B}$ is an ellipsoid, $x$ is an extreme point of $\hat{B}$ and so $x = x_i$ for some $i$ that is $x \in B$. This shows $B = \hat{B}$ and we are done.

For the first inequality, $B \subseteq \hat{B} \subseteq B n^{1/p - 1}$ easily implies $\left( \frac{|\hat{B}|}{|B|} \right)^{1/n} \leq n^{1/p - 1}$ and so,

$$s(B) = (|B| \cdot |B^o|)^{1/n} = (|B| \cdot |\hat{B}^o|)^{1/n} = \left( \frac{|B|}{|\hat{B}|} \right)^{1/n} (|\hat{B}| \cdot |B^o|)^{1/n} \geq \frac{s(\hat{B})}{\left( \frac{|B|}{|\hat{B}|} \right)^{1/n}} = C n^{-1/p} = C_p \left( s(B_{B^p}) \right)^{1/p}$$

///

The left inequality above is sharp since $s(B_{B^p}) = C_p \left( s(B_{B^p}) \right)^{1/p}$. The right inequality is also sharp since every ball is a $p$-ball for every $0 < p < 1$. And it is sharp even if we restrict ourselves to the class of $p$-balls which are not $q$-convex for any $q > p$, as it is showed by the following example: Let $\varepsilon > 0$ and $C_\varepsilon$ be a relatively open cap in $S^{n-1}$ centered in $x = (0, \ldots, 0, 1)$ of radius $\varepsilon$. Write $K = S^{n-1} \setminus \{C_\varepsilon \cup -C_\varepsilon\}$. The $p$-ball $p$-conv $(K)$ is not $q$-convex for any $q > p$ and we can pick $\varepsilon$ such that $s(p$-conv $(K)) / s(B_{B^p}) \sim 1$.

Observe that the left inequality is actually equivalent to the existence of a constant $C_p > 0$ such that for every $p$-ball $B$, $\left( \frac{|\hat{B}|}{|B|} \right)^{1/n} \leq C_p n^{1/p - 1}$ and by Lemma 2 iv), this is also equivalent to the inequality $N(\hat{B}, B) \leq C_p n^{1/p - 1}$.

With respect to the Milman ellipsoid theorem we obtain

**Proposition 2.** Let $0 < p < 1$. There exists a numerical constant $C_p > 0$ such that for every $p$-ball $B$ there is an ellipsoid $D$ such that $M(B, D) \leq C_p n^{1/p - 1}$.

**Proof:** Given a $p$-ball $B$ let $D$ be the Milman ellipsoid of $\hat{B}$. Then,

$$M(B, D) = \left( \frac{|B + D|}{|B \cap D|}, \frac{|B^o + D|}{|B^o \cap D|} \right)^{1/n} \leq M(\hat{B}, D) \left( \frac{|\hat{B} \cap D|}{|\hat{B} \cap D|} \right)^{1/n} \leq C_p n^{1/p - 1}$$
The bound for $M(B, D)$ is sharp. Indeed, if there was a function $f(n) << n^{1/p-1}$ such that for every a $p$-ball $B$ there was an ellipsoid $D$ with $M(B, D) \leq f(n)$, then

$$\frac{s(B_2^n)}{f(n)} = \frac{s(D)}{f(n)} \leq (|B \cap D| \cdot |B^o \cap D|)^{1/n} \leq s(B)$$

and we would have, $s(B) \geq \frac{s(B_2^n)}{f(n)} >> n^{-1/p}$ which is not possible.

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References

[B-L] Bergh, J. Löfström, J.: Interpolation Spaces. An Introduction. Springer-Verlag, 223. Berlin Heidelberg (1976).

[B-M] Bourgain, J., Milman, V.D.: On Mahler’s conjecture on the volume of a convex symmetric body and its polar. Preprint I.H.E.S., March 1985.

[Br] Brunn, H.: Über Ovale und Eiflächen. Inaugural dissertation. München 1887.

[Ca] Carl B.: Entropy numbers, $s$-numbers, and eigenvalue problems. J. Funct. Anal. 41, 290-306 (1981).

[G-K] Gordon, Y., Kalton, N.J.: Local structure for quasi-normed spaces. Preprint.

[Ja] Jarchow, H.: Locally convex spaces. B.G. Teubner Stuttgart, (1981).

[Ka] Kalton, N.: Convexity, Type and the Three Space Problem. Studia Math. 69, 247-287 (1980-81).

[K-S] Kalton, N., Sik-Chung Tam: Factorization theorems for quasi-normed spaces. Houston J. Math. 19 (1993), 301-317.

[K-P-R] Kalton, N., Peck, N.T., Roberts, J.W.: An F-space sampler. London Math. Soc. Lecture Notes 89. Cambridge Univ. Press. Cambridge (1985).

[Lu] Lusternik, L.A.: Die Brunn-Minkowskische Ungleichung für beliebige messbare Mengen. C.R. Acad.Sci.URSS 8, 55-58 (1935).

[Mil 1] Milman, V.D.: Inégalité de Brunn-Minkovsky inverse et applications à la théorie locale des espaces normés. C.R. Acad.Sci.Paris 302, Sér 1, 25-28 (1986).

[Mil 2] Milman, V.D.: Isomorphic symmetrizations and geometric inequalities. In “Geometric Aspects of Functional Analysis” Israel Sem. GAFA. Lecture Notes in Mathematics, Springer, 1317, (1988), 107-131.

[Min] Minkowski, H.:Geometrie der Zahlen. Teubner, Leipzig (1910).

[Pe] Peck, T.: Banach-Mazur distances and projections on $p$-convex spaces. Math. Zeits. 177, 132-141 (1981).

[Pie] Pietsch, A.: Operator ideals. North-Holland, Berlin 1979.

[Pi 1] Pisier, G.: The volume of convex bodies and Banach Space Geometry. Cambridge University Press, Cambridge (1989).

[Pi 2] Pisier, G.: A new approach to several results of V. Milman. J. reine angew. Math. 393, 115-131 (1989).
[Ro] Rolewicz, S.: *Metric linear spaces*. MM 56. PWN, Warsaw (1972).

[Sa] Santaló, L.A.: *Un invariante afín para los cuerpos convexos de n dimensiones*. Portugal Math. 8, 155-161 (1949).

[T] Triebel, H.: *Relations between approximation numbers and entropy numbers*. Journal of Approx. Theory 78, (1994), 112-116.