Paschke Categories, K-homology and the Riemann-Roch Transformation

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Abstract

For a separable $C^*$-algebra $A$, we introduce an exact $C^*$-category called the Paschke Category of $A$, which is completely functorial in $A$, and show that its K-theory groups are isomorphic to the topological K-homology groups of the $C^*$-algebra $A$. Then we use the Dolbeault complex and ideas from the classical methods in Kasparov K-theory to construct an acyclic chain complex in this category, which in turn, induces a Riemann-Roch transformation in the homotopy category of spectra, from the algebraic K-theory spectrum of a complex manifold $X$, to its topological K-homology spectrum.

Introduction

The main purpose of this paper is to define a Riemann-Roch transformation from the algebraic K-theory spectrum of a complex manifold to its topological K-homology spectrum. The topological K-homology spectrum of a manifold can be defined in various ways, but in this paper, we concern ourselves with definitions that use the language of $C^*$-algebras, as they provide a natural framework for the Dolbeault complex. For a separable $C^*$-algebra $A$, the K-homology spectrum of $A$ can be defined through the K-theory spectrum of the $C^*$-algebra $\mathcal{Q}(A)$ called the Paschke dual of $A$ [Pas81]. However the definition of the Paschke dual depends on the choice of a representation of $A$ and is only functorial up to homotopy. Here for any separable $C^*$-algebra $A$ we introduce the Paschke category $(\mathcal{D}/\mathcal{C})_A$ of $A$ whose objects are representations of $A$ and morphisms are the quotient of pseudo-local modulo locally compact operators. Since we are considering all the representations, this category is completely functorial in $A$. We define structure of an exact $C^*$-category on the Paschke category which in particular, makes it a topological exact category, so that by applying Waldhausen’s $S^*$-construction on the Paschke category and considering the fat geometric realization, we obtain a functor from $C^*$-algebras to the category of spectra. The K-theory groups of the Paschke category are defined to be the (stable) homotopy groups of this spectrum.

We observe that the ample representations of $A$ form a strictly cofinal subcategory of the Paschke category and through a standard argument, show that the K-theory spectrum of the Paschke category is homotopy equivalent to the K-theory spectrum of the Paschke dual of the $C^*$-algebra $A$, which gives the K-homology spectrum of $A$. We also check that the pull-back maps of the Paschke category agree with the classically defined ones up to homotopy. This makes Paschke categories a convenient place to study K-homology of $C^*$-algebras.

By translating the arguments into the language of categories, we can replicate the constructions in bivariant K-theory [Kas80] to show that the Dolbeault complex of a complex manifold $X$ with coefficients in a holomorphic vector bundle $E$ induces an exact sequence in the Paschke category, obtained by considering the $L^2$-completions and applying functional calculus with respect to a

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normalizing function to the Dolbeault operator. Since the $L^2$-completion of sections of a bundle depends on the choice of the metric, then so does the exact sequence we obtain out of the Dolbeault complex, even though there are natural isomorphisms on the relatively compact open subsets. Therefore this process only makes sense on a certain category of vector bundles with a choice of metric. We show that this process induces an exact functor from that category, to the category of acyclic chain complexes in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C_0(X)}$.

To obtain the Riemann-Roch transformation, we need to land in the loop space of the K-theory spectrum of Paschke category. To achieve this, we first note that there is a natural construction of Grayson [Gra12] in the homotopy category of spectra, from the K-theory spectrum of the category of bounded acyclic double chain complexes to the loop space of the K-theory spectrum of the original category. Then we generalize a construction of Higson given in [Hig95] to obtain a natural functor from the category of acyclic chain complexes in the Paschke category to the category of acyclic double chain complexes in the Paschke category. The composition of these maps give us a Riemann-Roch transform from the K-theory spectrum of a certain category of vector bundles with metrics, to the loop space of the K-theory spectrum of the Paschke category. When we restrict this transformation to a relatively compact open subset, then this map factors through the K-theory spectrum of the original category of vector bundles. Then we can use the descent properties of the topological K-homology to glue all the maps on the relatively compact open subsets and obtain the Riemann-Roch transformation.

A functorial Riemann-Roch transformation of complex analytic spaces was defined by Levy in [Lev87, Lev08]. We leave to a future paper the functoriality of the Riemann-Roch transformation, with respect to proper maps of complex manifolds.

This paper is organized as follows. In section 1, for a $C^*$-algebra $A$, we define the Paschke category (and also a variant called the Calkin-Paschke category) of $A$, as an exact $C^*$-category, and investigate its basic properties, including properties of certain subcategories of the Paschke category. In section 2, we replicate Waldhausen’s arguments to prove cofinality for topological categories, then repeat a construction of Grayson to obtain a map (in the stable homotopy category) between certain K-theory spectra, and investigate if the same holds for topological categories. Then we generalize a construction of Higson to obtain an exact functor between certain categories of chain complexes in the Paschke category. In section 3, we use the Dolbeault complex of a complex manifold, together with methods commonly used in the bivariant K-theory, to define an exact sequence in the corresponding Paschke category. This procedure depends on the choice of metric, and we go through a careful argument to show that all the choices induce homotopic maps of spectra. Finally in section 4, we show that the positive K-theory groups of the Paschke category of $A$ are equal to the shifted K-homology groups of the $C^*$-algebra $A$. We also show that the natural pull-back maps agree with the classically defined ones, and use the ingredients from the previous sections to define the Riemann-Roch transformation. Finally, for a unital $C^*$-algebra $A$, we will define a natural pairing of the category of normed right projective $A$-modules, with the Paschke category of $A$, and show that it has the expected properties.

0.1 Notation and Terminology

In this paper, we are only considering separable Hilbert spaces and $C^*$-algebras. We will use the letters $\mathfrak{A}, \mathfrak{A}', \mathfrak{B}, \mathfrak{C}, \ldots$ to refer to categories "somewhat related" to categories of $C^*$-algebras, and letters $A, B, P, \ldots$ to other categories. Also, if $A, B$ are two objects in the category $\mathfrak{A}$, then we will use the notation $\mathfrak{A}(A, B)$ to denote the set (or space) of morphisms from $A$ to $B$ in the category $\mathfrak{A}$. Also, if no confusion arises, we will write $\mathfrak{A}(A)$ instead of $\mathfrak{A}(A, A)$. We will use $\mathfrak{D}, \mathfrak{A}', \mathfrak{C}, \ldots$ to refer to certain sheaves.
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1 \(C^*\)-Categories and the Paschke Category

1.1 Definitions and Basic Properties

Let us start with giving a brief history and basic definitions of \(C^*\) \textit{categories}. Karoubi first defined Banach Categories in [Kar68]. A good source for this material is [Kar08]. Later \(C^*\)-categories were defined in [GLR85]. Another good source for \(C^*\)-categories is [Mit02].

Definition 1.1. The category \(\mathfrak{A}\) is called a \textit{complex \(\ast\)-category} if:

A1 For each two objects \(A, B\) of \(\mathfrak{A}\), \(\mathfrak{A}(A, B)\) is a complex vector space and composition of arrows is bilinear.

A2 There is an involution antilinear contravariant endofunctor \(\ast\) of \(\mathfrak{A}\) which preserves objects. The image of \(x\) under \(\ast\) will be denoted by \(x^\ast\). It follows that each \(\mathfrak{A}(A, A)\) is a \(\ast\)-algebra with identity.

A3 For each \(x \in \mathfrak{A}(A, B)\), \(x^\ast x\) is a positive element of the \(\ast\)-algebra \(\mathfrak{A}(A, A)\), i.e. \(x^\ast x = y^\ast y\) for some \(y \in \mathfrak{A}(A, A)\). Furthermore, \(x^\ast x = 0\) implies \(x = 0\).

It follows that the mapping \(\mathfrak{A}(A, B) \times \mathfrak{A}(A, B) \to \mathfrak{A}(A, A)\) defined by \((x, y) \mapsto x^\ast y\) is a \(\mathfrak{A}(A, A)\)-valued inner product on the right \(\mathfrak{A}(A, A)\)-module \(\mathfrak{A}(A, B)\), where \(\mathfrak{A}(A, A)\) acts on \(\mathfrak{A}(A, B)\) by composition of arrows.

A \(\ast\)-category \(\mathfrak{A}\) is called a \textit{normed \(\ast\)-category} if:

A4 Each \(\mathfrak{A}(A, B)\) is a normed space and \(\|xy\| \leq \|x\|\|y\|\).

A normed \(\ast\)-category is called a \textit{Banach \(\ast\)-category} if:

A5 Each \(\mathfrak{A}(A, B)\) is a Banach space.

A Banach \(\ast\)-category is called a \textit{\(C^*\)-category} if:

A6 For each arrow \(x\) of \(\mathfrak{A}\), \(\|x\|^2 = \|x^\ast x\|\).

It follows that each \(\mathfrak{A}(A, A)\) is a \(C^*\)-algebra with identity. A6 shows that the norm on a \(C^*\)-category is uniquely determined by the norms on the \(C^*\)-algebra \(\mathfrak{A}(A, A)\). In fact, we can say more: Let \(\mathfrak{A}\) be a \(\ast\)-category where each \(\mathfrak{A}(A, A)\) is a \(C^*\)-algebra, then \(\mathfrak{A}\) can be made into a normed \(\ast\)-category satisfying A6 (but not A5 in general) in a unique way by setting \(\|x\| = \|x^\ast x\|^{1/2}\).

Of course any \(C^*\)-algebra with identity can be considered as a \(C^*\)-category with a single object.

Definition 1.2. Let \(\mathfrak{A}, \mathfrak{A}'\) be \(C^*\)-categories. Then a functor \(F : \mathfrak{A} \to \mathfrak{A}'\) is called a \textit{\(\ast\)-functor} if it is a linear functor (i.e. \(F : \mathfrak{A}(A, B) \to \mathfrak{A}'(F(A), F(B))\) is linear for all objects \(A, B\) of \(\mathfrak{A}\)) and also \(F(x)^\ast = F(x^\ast)\) for all morphisms \(x\) in \(\mathfrak{A}\).

Definition 1.3. [Mit02, 3.1] A \textit{non-unital category}, is a category of objects and morphisms similar to a category, except that there need not exist an identity morphism \(1 \in \text{Hom}(A, A)\) for each object \(A\). A \textit{non-unital functor} \(F : \mathfrak{A} \to \mathfrak{B}\) between (possibly non-unital) categories \(\mathfrak{A}, \mathfrak{B}\) is a
transformation similar to a functor, except that there is no condition on the identity morphisms of
the category $\mathfrak{A}$. Similarly, we can define non-unital $C^*$-categories, and $*$-functors between them.

**Definition 1.4.** [Mit02, 4.2.] Let $\mathfrak{A}$ be a $C^*$-category, then a $C^*$-ideal $\mathfrak{I}$ in the category $\mathfrak{A}$ is (a probably non-unital) subcategory of $\mathfrak{A}$ so that:

- The subcategory $\mathfrak{I}$ has the same objects as the category $\mathfrak{A}$.
- Each morphism set $\mathfrak{I}(A, B)$ is a norm closed subspace of the space $\mathfrak{A}(A, B)$.
- The composition of an arrow in the category $\mathfrak{A}$ with an arrow in the subcategory $\mathfrak{I}$ is an arrow in the subcategory $\mathfrak{I}$.

As a result of the definition above we have:

**Proposition 1.5.** [Mit02, 4.7.] Let $j \in \mathfrak{I}(A, B)$ be a morphism in the $C^*$-ideal $\mathfrak{I}$ of the $C^*$-category $\mathfrak{A}$. Then the adjoint morphism $j^*$ is also a morphism in the ideal $\mathfrak{I}$.

Also, we can define the quotient $\mathfrak{A}/\mathfrak{I}$ to be the category with the same objects as $\mathfrak{A}$ and with morphism sets the quotient Banach space $$(\mathfrak{A}/\mathfrak{I})(A, B) = \frac{\mathfrak{A}(A, B)}{\mathfrak{I}(A, B)}.$$ This is also a $C^*$-category.

**Example 1.6.** We will use $\mathfrak{B}$ to denote the category of Hilbert spaces with bounded operators between them, which is an additive $C^*$-category as products and coproducts of a finite number of Hilbert spaces is just their direct summand, and also the set of bounded operators between two Hilbert spaces forms an abelian group, as we can add the operators with each other and composition on both sides is linear.

We will denote the $C^*$-ideal of compact operators by $\mathfrak{K}$.

**Example 1.7.** Let $A$ be a $C^*$-algebra, and let $\mathfrak{A}$ denote a $C^*$-category. Let $\mathcal{R}ep\mathfrak{A}(A)$ denote the category of representations of $A$, i.e. a category whose objects are representations $\rho : A \to \mathfrak{A}(H)$, where $H$ is an object in $\mathfrak{A}$, and whose morphisms between two representations $\rho_1 : A \to \mathfrak{A}(H_1)$ and $\rho_2 : A \to \mathfrak{A}(H_2)$ is the Banach space $\mathfrak{A}(H_1, H_2)$. Notice that we are not restricting our attention to unital representations, i.e. we also consider the zero representation, and other non-unital representations.

If $\mathfrak{A}$ is additive, then it is easy to check that $\mathcal{R}ep\mathfrak{A}(A)$ is an additive $C^*$-category as well.

**Definition 1.8.** Let $A$ be a $C^*$-algebra, and let $\rho_i : A \to \mathfrak{B}(H_i)$, be representations of $A$ for $i = 1, 2$. A bounded operator $T : H_1 \to H_2$ is called pseudo-local, if $\rho_2(a)T - T\rho_1(a) \in \mathfrak{K}(H_1, H_2)$, $\forall a \in A$, and $T$ is locally compact, if both $\rho_2(a)T, T\rho_1(a)$ are in $\mathfrak{K}(H_1, H_2)$ for all $a \in A$.

**Definition 1.9.** Let $A$ be a $C^*$-algebra. Then we define the Paschke category of $A$ to be the quotient category $(\mathfrak{D}/\mathfrak{C})_A := \mathfrak{D}_A/\mathfrak{C}_A$, where $\mathfrak{D}_A$ is the category of representations $\rho : A \to \mathfrak{B}(H)$ of $A$, where the morphisms between two representations are the pseudo-local operators between them, and the $C^*$-ideal $\mathfrak{C}_A$ has the same objects, but the morphisms are locally compact operators.

We define the Calkin-Paschke category of $A$ to be the category where the objects are representations $\rho' : A \to (\mathfrak{B}/\mathfrak{K})(H)$, and morphisms are again the quotient of pseudo-local operators modulo locally compact operators. We denote the Calkin-Paschke category by $(\mathfrak{D}/\mathfrak{C})'_A$.

Note that there is a natural functor $(\mathfrak{D}/\mathfrak{C})_A \to (\mathfrak{D}/\mathfrak{C})'_A$, which sends a representation $\rho : A \to \mathfrak{B}(H)$ to $\rho' : A \to \mathfrak{B}(H) \to (\mathfrak{B}/\mathfrak{K})(H)$.
Therefore we have a natural functor \((\mathcal{D} \rightarrow \mathcal{C})\) in the Calkin-Paschke category. The functor is also full, because any morphism \((\varphi, \psi)\) that satisfies \(\varphi = \psi\) is locally compact, hence \((\mathcal{D} \rightarrow \mathcal{C})\) is equivalent to the category where objects are pairs \((A, B)\) and every morphism \(p : A \to B\) so that \(p^2 = p\), the kernel of \(p\) exists.

In case no confusion should arise, instead of writing \(T\rho(a)\) is compact for all \(a \in A\), we will simply say \(T\rho\) is compact, and similarly for \(\rho'\).

**Example 1.11** (See [HR00 5.3.2.]). One can generalize the definition above, by introducing a "relative" version. Let \(A\) be a \(C^*\)-algebra and \(I \subset A\) a \(C^*\)-ideal. Then for representations \(\rho_i : A \to \mathfrak{B}(H_i)\) for \(i = 1, 2\), define \(\mathcal{D}_A(\rho_1, \rho_2)\) to be the same as the above example, and let

\[
\mathcal{C}_{I,A}(\rho_1, \rho_2) = \{ T \in \mathcal{D}_A(\rho_1, \rho_2) | Tp(a)\,p(a)T \in \mathfrak{R}(H_1, H_2), \forall a \in I \}.
\]

Note that when \(I \subset J\) then \(\mathcal{C}_A \subset \mathcal{C}_{I,A} \subset \mathcal{C}_{J,A}\), and if \(I = A\), then we recover the definition above. All of the results on the Paschke category also holds for this relative version, however, by theorem \([\text{I+} 5.4.5.\)], this does not provide any new information.

**Definition 1.12.** [Kar08 1.1.6.7.] Let \(\mathfrak{A}\) be an additive category. Then \(\mathfrak{A}\) is called **pseudo-abelian** if for each object \(H\) of \(\mathfrak{A}\) and every morphism \(p : H \to H\) so that \(p^2 = p\), the kernel of \(p\) exists.

In the case when \(\mathfrak{A}\) is an additive \(C^*\)-category, and each self-adjoint projection has a kernel, then we say \(\mathfrak{A}\) is **weakly pseudo-abelian**.

**Proposition 1.13.** [Kar08 1.1.6.9.] Let \(\mathfrak{A}\) be a (weakly) pseudo-abelian category, let \(H\) be an object of \(\mathfrak{A}\) and let \(p : H \to H\) be such that \(p^2 = p\) (and also \(p = p^*\)). Then the object \(H\) splits into the direct sum \(H = \ker(p) \oplus \ker(1-p)\).

**Proposition 1.14.** [Kar08 1.1.6.10.] Let \(\mathfrak{A}\) be an additive category. Then there exists a pseudo-abelian category \(\hat{\mathfrak{A}}\), and an additive functor \(\phi : \mathfrak{A} \to \hat{\mathfrak{A}}\) which is fully faithful and is universal among additive functors from \(\mathfrak{A}\) to a pseudo-abelian category. The pair \((\phi, \mathfrak{A})\) is unique up to equivalence of categories.

\(\mathfrak{A}\) is equivalent to the category where objects are pairs \((H, p)\) where \(H\) is an object in \(\mathfrak{A}\) and \(p : H \to H\) is a projector \(i.e.\) \(p^2 = p\), and morphisms between \((H_1, p_1)\) and \((H_2, p_2)\) are morphisms \(f : H_1 \to H_2\) in \(\mathfrak{A}\) such that \(fp_1 = p_2f = f\) in \(\mathfrak{A}\). This category is called the pseudo-abelianization of \(\mathfrak{A}\).

The same statement is true for a \(C^*\)-category and its weakly pseudo-abelian counterpart.

**Proposition 1.15.** The weak pseudo-abelianization of the \(C^*\)-category \((\mathfrak{B}/\mathfrak{R})\) is naturally isomorphic to the Calkin-Paschke category \((\mathcal{D}/\mathcal{C})'_{\mathcal{C}}\).

**Proof.** The objects in the Calkin-Paschke category \((\mathcal{D}/\mathcal{C})'_{\mathcal{C}}\) can be considered as pairs \((H, \rho'(1))\) of a Hilbert space \(H\) and a self-adjoint projection \(p = \rho'(1) \in (\mathfrak{B}/\mathfrak{R})(H)\), and morphisms \(\rho'_1 \to \rho''_2\) in \((\mathcal{D}/\mathcal{C})'_{\mathcal{C}}\) are the pseudo-local operators modulo locally compact ones, i.e. the operators \(F \in (\mathfrak{B}/\mathfrak{R})(H_1, H_2)\) so that \(Fp_1 = p_2F\) modulo the ones that \(Fp_1 = 0 = p_2F\). In other words since \(F(1-p_1)\) is locally compact, hence \(F = Fp_1 = p_2F\) in the Calkin-Paschke category \((\mathcal{D}/\mathcal{C})'_{\mathcal{C}}\). Therefore we have a natural functor \((\mathcal{D}/\mathcal{C})'_{\mathcal{C}} \to (\mathfrak{B}/\mathfrak{R})\).

This functor is faithful, because \(F = Fp_1 = p_2F\) are all zero in the category \((\mathfrak{B}/\mathfrak{R})\) if \(F\) is locally compact in the Calkin-Paschke category. The functor is also full, because any \(F \in (\mathfrak{B}/\mathfrak{R})(H_1, H_2)\) that satisfies \(Fp_1 = p_2F\) is pseudo-local.

**Proposition 1.16.** The Calkin-Paschke category \((\mathcal{D}/\mathcal{C})'_{\mathcal{A}}\) is a weakly pseudo-abelian category.
We need the following two lemmas to prove the proposition above.

**Lemma 1.17.** Each self-adjoint projection in \((\mathcal{D}/\mathcal{C})'_{A}\) has a representative in \(\mathcal{D}_{A}\) which is a self-adjoint projection.

**Proof.** Let \(P \in \mathcal{D}_{A}(\rho')\) be a representative for a self-adjoint projection in \((\mathcal{D}/\mathcal{C})'_{A}\). Hence \(\rho'(P-P^*)\) and \((P-P^*)\rho'\) are compact operators. Set \(P' = (P+P^*)/2\). Then \(P'\) is a self-adjoint operator, hence by Weyl-Von Neumann Theorem [HR00, 2.2.5], there is a diagonal compact perturbation of \(P'\), i.e., there exists an operator \(P_1\) with an orthonormal basis \(\{e_i\}_{i=1}^{\infty}\) of eigenvectors of \(P_1\) for \(H\) with complex numbers \(\lambda_i\) as eigenvalues so that and \(P_1 - P'\) is a compact operator. Therefore \(P - P_1\) is in \(\mathcal{C}_{A}(\rho')\).

Let \(I \subset \mathbb{N}\) be the set of indices \(i\) such that \(|\lambda_i| < 1/2\). Now define the bounded operator \(Q\) by \(Q(e_i) = e_i\) if \(i \notin I\), and set \(Q(e_i) = 0\) otherwise. Evidently, \(Q\) is a self-adjoint projection in the category \(\mathfrak{B}\). We want to show that \(P_1 - Q\) is in \(\mathcal{C}_{A}(\rho')\).

Define \(D(e_i) = \frac{1}{1-\lambda_i}e_i\) if \(i \in I\), and \(D(e_i) = \frac{1}{1}\lambda_i e_i\) otherwise. Notice that \(D\) is a bounded diagonal operator (of norm at most 2). Also \((P_1 - Q)e_i = \lambda_i e_i\) when \(i \in I\), and \((P_1 - Q)e_i = (\lambda_i - 1)e_i\) otherwise. Furthermore \((P_1 - P^2_1)e_i = (\lambda_i - \lambda_i^2)e_i\) for all \(i \in \mathbb{N}\). Therefore \((P_1 - Q)(e_i) = D(P_1 - P^2_1)(e_i) = (P_1 - P^2_1)D(e_i)\) for all \(i\), hence \(P_1 - Q = D(P_1 - P^2_1)\). But since \(P_1 - P^2_1 \in \mathcal{C}_{A}(\rho')\), then \((P_1 - P^2_1)\rho', \rho'(P_1 - P^2_1)\) are compact operators. Therefore \((P_1 - Q)\rho', \rho'(P_1 - Q)\) are also compact, which proves that \(P_1 - Q \in \mathcal{C}_{A}\) and hence \(Q \in \mathcal{D}_{A}(\rho')\). \(\square\)

**Lemma 1.18.** Let \(T \in \mathcal{D}_{A}(\rho_1', \rho_2')\) be a pseudo-local operator with closed image \(V_2 \subset H_2\). Let \(V_1 \subset H_1\) be the orthogonal complement of \(\text{ker}(T)\). Then for \(i = 1, 2\), the projections \(\pi_i : H_2 \to V_i\) and the inclusions \(\iota_i : V_i \to H_i\) are pseudo-local operators.

**Proof.** Since \(T\) has a closed image, then it induces an isomorphism of Hilbert spaces from \(V_1\) to \(V_2\).

To simplify the notation, denote \(T' = \pi_2\tau_{T} : V_1 \to V_2\). Let \(S' \in \mathfrak{B}(V_2, V_1)\) be the inverse to \(T'\) and \(S = S' \oplus 0 : V_2 \oplus V_2^\perp = H_2 \to V_1 \oplus V_1^\perp = H_1\). We have \(ST = \iota_1\pi_1\) and \(TS = \iota_2\pi_2\). First we show that \(S\) is also pseudo-local. This would show that \(\iota_i, \pi_i\) are pseudo-local.

Let \(\rho_i' = \begin{pmatrix} \rho_i'^{11} & \rho_i'^{12} \\ \rho_i'^{21} & \rho_i'^{22} \end{pmatrix} \in (\mathfrak{B}/\mathfrak{R})(V_i^\perp \oplus V_i^\perp)\) for \(i = 1, 2\). Since \(T\) is pseudo-local, we have \(T'\rho_1'^{11} - \rho_2'^{11}T'\) and \(T'\rho_2'^{12}\) and \(\rho_2'^{12}T'\) are compact \(\Box\). Therefore \(\rho_1'^{12} = S'T'\rho_1'^{12} = 0\) and \(\rho_2'^{21} = \rho_2'^{21}T'S' = 0\). Also, since \(\rho_1'^{21}(a^*) = \rho_2'^{21}(a^*)\), then we can say that \(\rho_1'^{11}, \rho_2'^{21}\) are zero, for \(i = 1, 2\). Therefore \(\rho_1'S - S\rho_2 = (\rho_1'^{11}S' - S'\rho_2'^{11})\). But \(\rho_1'^{11}S' - S'\rho_2'^{11} = S'T(\rho_1'^{11}S' - S'\rho_2'^{11}) = S'T\rho_1'^{11}S' - S'T\rho_2'^{11} = 0\). Hence \(S\) is pseudo-local.

We have \(\iota_i\rho_i'^{11} - \rho_i'\iota_i = (\rho_i'^{11}, 0) - (\rho_i'^{11}, \rho_i'^{12}) = 0\). This proves that \(\iota_i\) is pseudo-local, for \(i = 1, 2\).

It only remains to show that \(\rho_i'^{11} = \pi_i\rho_i' : A \to (\mathfrak{B}/\mathfrak{R})(V_i)\) is an object of the Calkin-Paschke category \((\mathcal{D}/\mathcal{C})'_{A}\). This follows from adjointness of \(\iota_i\) and \(\pi_i\) and \(\Box\)

\[\rho_i'^{11}(ab^*) = \pi_i\rho_i'(a)\rho_i'(b^*)\iota_i = \pi_i\iota_i\pi_i\rho_i'(a)\rho_i'(b^*)\iota_i = \pi_i\rho_i'(a)\iota_i\pi_i\rho_i'(b)\iota_i.\]

\(\square\)

**Proof of Proposition.** Let \(P \in \mathcal{D}_{A}(\rho')\) be a representative for a self-adjoint projection in \((\mathcal{D}/\mathcal{C})'_{A}\), and let \(Q\) be the projection as in proof of lemma [Lemma 1.17] (we will only use the fact that \(Q\) has a closed image). Let \(\iota\) be the inclusion of \(\text{ker}(Q)\) in \(H\), and let \(\pi\) be the projection onto the kernel. Then we want to show that \(\iota\) is a kernel for \(Q\). We clearly have \(Q\iota = 0\). Also since \(Q\) is bounded from below below
on the orthogonal complement of its kernel, then it has closed image. It follows from lemma 1.18 that \( \iota \) is pseudo-local. Let \( Q' \) denote "inverse" of \( Q \) restricted to orthogonal complement of \( \ker(Q) \), i.e. \( Q' \) sends image of the projection \( Q \) isometrically to the orthogonal complement of \( \ker(Q) \).

Now let \( F \in \mathcal{D}_A(\rho_0, \rho') \) be an operator so that \( QF \in \mathcal{C}_A(\rho_0, \rho') \). Then we want to show that \( F \) factors through \( \iota \) up to locally compact operators. We have \( i\pi F = F \) modulo compact operators because \( (Id_H - i\pi)F = Q'QF = 0 \). Also, if we have \( \rho'(iG - F) = 0 \), then \( \rho'i(G - \pi F) = 0 \). Therefore \( \rho'(G - \pi F) = 0 \). This completes the proof. \( \square \)

**Definition 1.19.** Let \( \mathfrak{A} \) be an additive category. Then we say that a chain complex

\[
\cdots \xrightarrow{T_{i-1}} \rho_i \xrightarrow{T_i} \rho_{i+1} \xrightarrow{T_{i+1}} \rho_{i+2} \xrightarrow{T_{i+2}} \cdots
\]

is exact if there is a contracting homotopy, i.e. if there are morphisms \( S_i \) in \( \mathfrak{A} \) from \( \rho_{i+1} \) to \( \rho_i \) so that \( T_{i-1}S_i + S_iT_i = Id_{\rho_i} \) in \( \mathfrak{A} \).

As a result of this definition, every short exact sequence in \( \mathfrak{A} \) is split, hence \( \mathfrak{A} \) is an exact category in the sense of Quillen [Qui73, Sec 2.]. Note that this does not mean all exact sequences are split.

In particular, (using the definition above) the Paschke category \((\mathcal{D}/\mathcal{C})_A\), the Calkin-Paschke category \((\mathcal{D}/\mathcal{C})'_A\), and also \( \mathcal{D}_A, \mathfrak{B}, (\mathfrak{B}/\mathfrak{R}) \) are all exact \( C^* \)-categories.

Notice that a map \( f : A \to B \) of \( C^* \)-algebras, induces pull-back maps \( f^* : (\mathcal{D}/\mathcal{C})_B \to (\mathcal{D}/\mathcal{C})_A \) and also \( f^* : (\mathcal{D}/\mathcal{C})'_B \to (\mathcal{D}/\mathcal{C})'_A \) of categories, by precomposing with the representation. This map preserves exact sequences, hence the pull-back functor is exact, and this process is functorial.

### 1.2 Subcategories of the Paschke Category

We start this subsection by giving a definition similar to [Wal85].

**Definition 1.20.** Let \( \mathcal{B} \) be an additive category. Then a full additive subcategory \( \mathcal{A} \) is called cofinal if for every object \( B \) of the category \( \mathcal{B} \), there is an object \( B' \) in \( \mathcal{B} \) so that \( B \oplus B' \) is isomorphic to an object in \( \mathcal{A} \). If we can always take \( B' \) to be an object in \( \mathcal{A} \), then \( \mathcal{A} \) is called strictly cofinal.

In case the category \( \mathcal{B} \) is exact, we require the subcategory \( \mathcal{A} \) to be exact as well.

Let us recall some definitions and useful properties of representations.

**Definition 1.21.** A representation \( \rho : A \to \mathfrak{B}(H) \) of a \( C^* \)-algebra is called non-degenerate if \( \rho(A)H \) is a dense subset of \( H \) (or equivalently, it is the whole \( H \), cf. [HR00, 1.9.17.]). Another equivalent definition is that for each \( h \in H, \ h \neq 0 \), there exists an \( a \in A \) so that \( \rho(a)h \neq 0 \).

A representation \( \rho : A \to \mathfrak{B}(H) \) is called ample if it is non-degenerate, and also for each \( a \in A, a \neq 0, \rho(a) \) is not a compact operator.

**Proposition 1.22.** Let \( \mathfrak{A}_A \) denote the full subcategory of \((\mathcal{D}/\mathcal{C})_A\) whose objects are ample representations, together with the zero representation \( A \to 0 \). This is an exact strictly cofinal subcategory of \((\mathcal{D}/\mathcal{C})_A\).

Let \( A \) be a unital \( C^* \)-algebra and let \( \mathfrak{A}'_A \) denote the full subcategory of \((\mathcal{D}/\mathcal{C})'_A\) whose objects are unital injective representations, together with the zero representation \( A \to 0 \). This is an exact strictly cofinal subcategory of \((\mathcal{D}/\mathcal{C})'_A\).

---

\(^3\)This was originally defined for Waldhausen categories by considering coproduct instead of the direct sum. In this paper, we will only apply the definition to certain Waldhausen categories.
Proof. Note that direct sum of two non-degenerate representations is non-degenerate, and direct sum of a non-degenerate and an ample representation is ample. Given some representation $\rho : A \to \mathcal{B}(H)$, let $H_1 = \overline{\rho(A)H} \subset H$, $\pi : H \to H_1$ be the orthogonal projection onto the closed subspace, and let $\iota : H_1 \to H$ be the inclusion. Then we can define $\rho_1 : A \to \mathcal{B}(H_1)$ by $\rho_1 = \pi \rho \iota$. Since $\pi, \iota$ are adjoints to each other, and $\pi \rho = \rho = \rho \pi$ then $\rho_1$ is indeed a representation. Also these two representations are isomorphic as objects of $(\mathfrak{O}/\mathfrak{C})_A$, as $\pi, \iota$ are pseudo-local and induce the isomorphism. But $\rho_1$ is a non-degenerate representation. Hence for any object $\rho$ of $(\mathfrak{O}/\mathfrak{C})_A$, and any ample representation $\rho_0$ of $A$, $\rho \oplus \rho_0$ is isomorphic to an ample representation in $(\mathfrak{O}/\mathfrak{C})_A$.

If the $C^*$-algebra $A$ is unital and $\rho' : A \to (\mathcal{B}/\mathcal{R})(H)$ is an object of $(\mathfrak{O}/\mathfrak{C})'_A$, then by lemma 1.17 $\rho'(1)$ has a representative $\pi \in \mathcal{B}(H)$ which is a self-adjoint projection. By repeating the argument above, $\rho'$ is isomorphic to the unital representation $\rho'_1 : A \to (\mathcal{B}/\mathcal{R})(H_1)$, where $H_1 \subset H$ is image of $\pi$. Also, direct sum of an injective representation $\rho'_0$ with any representation $\rho'$ is injective. \hfill \square

Another important property of ample representations, is the following corollary of Voiculescu’s theorem \cite{Vo}, which we mention similar to as stated in \cite[3.4.2.]{HR}.

**Theorem 1.23** (Voiculescu). Let $A$ be a unital $C^*$-algebra, let $\rho : A \to \mathcal{B}(H)$ be a non-degenerate representation, and let $\nu' : A \to (\mathcal{B}/\mathcal{R})(H')$ be an object in $(\mathfrak{O}/\mathfrak{C})'_A$. Assume that for each $a \in A$ with $\rho(a) \in \mathcal{R}(H)$, we have $\nu'(a) = 0$. Then there exists an isometry $V : H' \to H$ so that $V^* \rho(a)V - \nu'(a) = 0$ for all $a \in A$.

An important corollary is the following:

**Corollary 1.24.** Let $\rho_1, \rho_2$ be two ample representations of the $C^*$-algebra $A$. Then there is a unitary operator $U : H_1 \to H_2$ so that $U\rho_1(a)U^* - \rho_2(a)$ is compact for all $a$.

In other words, any two ample representations in the Paschke category are isomorphic. Hence we can denote the isomorphism class of automorphisms of an ample representation $\rho$ of $A$ by $\Omega(A)$, which is also known as the Paschke dual.

**Remark 1.25.** The natural map $(\mathfrak{O}/\mathfrak{C})_A \to (\mathfrak{O}/\mathfrak{C})'_A$ is fully faithful, and by Voiculescu’s theorem, each object $\nu'$ of $(\mathfrak{O}/\mathfrak{C})'_A$ has an admissible monomorphism to an object which lifts to a non-degenerate representation of $A$. Since $(\mathfrak{O}/\mathfrak{C})'_A$ is weakly pseudo-abelian, therefore by Voiculescu’s theorem, the full subcategory of the Calkin-Paschke category $(\mathfrak{O}/\mathfrak{C})'_A$ consisting of objects which lift to the Paschke category $(\mathfrak{O}/\mathfrak{C})_A$ is cofinal.

2 K-Theory

2.1 Waldhausen’s Cofinality

In this subsection, we recall some standard facts about K-theory and fix our notation for the rest of the section. Aside from Waldhausen’s original paper \cite{Wal}, a good source for more information is \cite{Weg}.

**Definition 2.1.** \cite{Seg} A.] Let $X$ be a simplicial space. Then define the topological space $\|X\|$ called the *fat geometric realization* of $X$, as the quotient

$$\coprod_{n} X_n \times \Delta^n_{top} / \sim_+$$

\textsuperscript{4}We are abusing the notation for the class of $V^* \rho(a)V$ in $(\mathcal{B}/\mathcal{R})(H')$.  

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where the relation \( \sim_+ \) is generated by \((x, f \ast p) \sim_+ (f \ast x, p)\) for \(x \in X_n, p \in \Delta_{\text{top}}^m\), whenever \(f : [m] \to [n]\) in the simplex category \(\Delta\) is a face (injective) map.

Let \(A\) be a simplicial set (or a discrete simplicial space). Then define the topological space \(|X|\) called the geometric realization of \(X\), as the quotient

\[
\coprod_n X_n \times \Delta_{\text{top}}^n/ \sim
\]

where the relation \(\sim\) is generated by \((x, f \ast p) \sim (f \ast x, p)\) for \(x \in X_n, p \in \Delta_{\text{top}}^m\), for any morphism \(f : [m] \to [n]\) in the simplex category \(\Delta\).

Remark 2.2. The two definitions of the geometric realization above are equivalent for discrete simplicial spaces.

Also, for a simplicial space \(X\), there is a natural quotient map \(\|X\| \to |X|\).

The notion of fat geometric realization is better suited for simplicial topological spaces than the usual notion, as it takes the topological structure into account. In particular we have the proposition below.

Proposition 2.3. ([Seg74] A.1.) Let \(X, Y\) be simplicial topological spaces.

1. If each \(X_n\) has the homotopy type of a CW-complex, then so does \(\|X\|\).
2. If \(X \to Y\) is a simplicial map such that \(X_n \to Y_n\) is a weak homotopy equivalence for each \(n\), then \(\|X\| \to \|Y\|\) is also a weak homotopy equivalence.
3. \(\|X \times Y\|\) is weakly homotopy equivalent to \(\|X\| \times \|Y\|\).

Let us recall the general process of defining the algebraic K-theory spectrum of a small Waldhausen category \((\mathcal{A}, w)\) (for more details, see [Wal85]).

Definition 2.4. Let \(\mathcal{A}\) be a Waldhausen category. Define the simplicial category \(S\mathcal{A}\) as follows. First, consider the category of ordered pairs of integers \((j, k)\) with \(0 \leq j \leq k \leq n\) that has a unique morphism from \((j, k)\) to \((j', k')\) if \(j \leq j'\) and \(k \leq k'\). Then the objects in \(S_n\mathcal{A}\) are the functors \(A\) from this category of pairs to the category \(\mathcal{A}\), so that \(A(j, j) = 0\) and \(A(j, k) \to A(j, l) \to A(k, l)\) is a cofibration sequence in \(\mathcal{A}\) whenever \(0 \leq j \leq k \leq l \leq n\). The morphisms in \(S_n\mathcal{A}\) are the natural transformations \(A \to A'\), and the weak equivalences are the morphisms that \(A(j, k) \to A'(j, k)\) are all weak equivalences in \(\mathcal{A}\). The cofibrations are the morphisms that \(A(j, k) \to A'(j, k)\) are all cofibrations, and \(A(j, l) \coprod_{A(j, k)} A'(j, k) \to A'(j, l)\) are also cofibrations in \(\mathcal{A}\) whenever \(0 \leq j \leq k \leq l \leq n\). Note that a morphism \(f : [n] \to [m]\) in the opposite simplex category \(\Delta^{op}\) induces a functor \(S_n\mathcal{A} \to S_m\mathcal{A}\), which sends the object \((j, k)\) to \(A(j, k)\) of \(S_n\mathcal{A}\) to the object \((r, s)\) of \(S_m\mathcal{A}\). This defines a simplicial structure on \(S\mathcal{A}\).

Let \(wS\mathcal{A}\) be the simplicial category obtained by only considering the weak equivalences in \(S\mathcal{A}\), and form the nerves in each degree, which yields a bisimplicial set \(NwS\mathcal{A}\). Define the algebraic K-theory spectrum \(K_{\text{alg}}(\mathcal{A})\) of the discrete Waldhausen category \(\mathcal{A}\) as the spectrum whose \(n\)th space is the geometric realization \(|NwS\mathcal{A}|\). (Or we could have defined the algebraic K-theory space to be the loop space \(\Omega|NwS\mathcal{A}|\). In fact, they have the same (stable) homotopy groups, hence we may sometimes use the space instead of the spectrum.)

By [Mit01], we can define the topological K-theory spectrum \(K^{\text{top}}(\mathcal{A})\) of the topological Waldhausen category \(\mathcal{A}\) similar as above, i.e. as a spectrum whose \(n\)th space is the fat geometric realization \(\|NwS\mathcal{A}\|\).

\footnote{This description is taken from [TT90] 1.5.1.}
**Definition 2.5.** Let $\mathcal{A}$ be a topological Waldhausen category. Let $\mathcal{A}^\delta$ denote the discrete Waldhausen category obtained by forgetting the topological structure. There is a natural exact functor $\mathcal{A}^\delta \to \mathcal{A}$ which induces a natural map $K^{\text{top}}(\mathcal{A}^\delta) \to K^{\text{top}}(\mathcal{A})$. By remark 2.2, there is a natural equivalence of K-theory spectra $K^{\text{alg}}(\mathcal{A}^\delta) \cong K^{\text{top}}(\mathcal{A}^\delta)$.

Hence there is a natural comparison map $c : K^{\text{alg}}(\mathcal{A}^\delta) \to K^{\text{top}}(\mathcal{A})$.

**Notation 2.6.** If $\mathcal{A}$ is a topological Waldhausen category, we will simply write $K^{\text{alg}}(\mathcal{A})$ instead of $K^{\text{alg}}(\mathcal{A}^\delta)$.

Recall from [Wal85, 1.3.2.]:

**Definition 2.7.** Let $\mathcal{A}, \mathcal{B}$ be Waldhausen categories. Then we say that a sequence $F_0 \to F_1 \to F_2$ of exact functors from $\mathcal{A}$ to $\mathcal{B}$ and natural transformations between them is a short exact sequence of functors, if for each object $A$ of $\mathcal{A}$ we have a cofibration sequence $F_0(A) \to F_1(A) \to F_2(A)$ in $\mathcal{B}$.

**Theorem 2.8 (Additivity Theorem).** [Qui73, Sec 3.] [Wal85, 1.3.2.] Let $\mathcal{A}, \mathcal{B}$ be Waldhausen categories, and let $F_0 \to F_1 \to F_2$ be a short exact sequence of functors from $\mathcal{A}$ to $\mathcal{B}$. Then $F_1^*, F_0^* + F_2^* : K(\mathcal{A}) \to K(\mathcal{B})$ are homotopic to each other. By [Mit01, 4.2.], the same holds for the topological Waldhausen categories.

**Definition 2.9 (Relative K-theory Space).** [Wal85, 1.5.] Let $\mathcal{A}, \mathcal{B}$ be Waldhausen categories, and let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Then define the category $[\mathcal{A} \xrightarrow{F} \mathcal{B}]$ by $[\mathcal{A} \xrightarrow{F} \mathcal{B}]_n = S_n\mathcal{A} \times S_n\mathcal{B} S_{n+1}\mathcal{B}$. There is a natural simplicial structure on $[\mathcal{A} \xrightarrow{F} \mathcal{B}]$, and the Waldhausen category structures of $\mathcal{A}, \mathcal{B}$ induce one on $[\mathcal{A} \xrightarrow{F} \mathcal{B}]$ in a natural way.

**Proposition 2.10.** [Wal85, 1.5.5.] There are natural functors of Waldhausen categories $\mathcal{B} \to [\mathcal{A} \xrightarrow{F} \mathcal{B}] : S\mathcal{A}$ which in turn induce the homotopy fibration sequence

$$wS\mathcal{B} \to wS[\mathcal{A} \xrightarrow{F} \mathcal{B}] \to wS.S\mathcal{A}.$$ 

By [Mit01, 4.4.], the same holds for topological Waldhausen categories.

**Definition 2.11.** [Wal85, 1.5.3.] Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be Waldhausen categories, then a functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is biexact if for each object $A$ of $\mathcal{A}$ and $B$ of $\mathcal{B}$, the functors $F(A, -)$ and $F(-, B)$ are exact, and also for each cofibration $A \to A'$ in $\mathcal{A}$ and $B \to B'$ in $\mathcal{B}$, the map below is a cofibration in $\mathcal{C}$.

$$F(A, B') \cup_{F(A,B)} F(A', B) \to F(A', B').$$

**Proposition 2.12.** [Wal85, 1.5.3.] A biexact functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ of Waldhausen categories, induces a map of bisimplicial categories $wS\mathcal{A} \wedge wS\mathcal{B} \to wS.S\mathcal{C}$ which in turn induces a map of K-theory spectra $K(\mathcal{A}) \wedge K(\mathcal{B}) \to K(\mathcal{C})$.

The same holds for the topological categories [Mit01, 2.8.].

**Definition 2.13.** We say that a (topological) Waldhausen subcategory $\mathcal{A}$ of $\mathcal{B}$ is closed under extensions if for each cofibration sequence in $\mathcal{B}$ where the source, and the quotient are in $\mathcal{A}$, then the target is isomorphic to an object in $\mathcal{A}$.
Proposition 2.14. \[Wal85, 1.5.9.\] If \(A\) is a strictly cofinal (topological) Waldhausen subcategory of \(B\), then the natural map \(K(A) \to K(B)\) is a homotopy equivalence. If \(A\) is only a cofinal (topological) Waldhausen subcategory of \(B\) which is also closed under extensions, then the natural map of \(K\)-theory spectra \(K(A) \to K(B)\) induces an isomorphism on the \(i\)’th homotopy group when \(i \geq 1\).

The above statement was originally proved for discrete categories, however, in here we will need to apply it to a certain cofinal subcategory of the Paschke category. The proof goes through for topological categories with no change, but for the sake of completeness, we repeat the argument here.

Notation 2.15. Let \(A, B\) denote topological Waldhausen categories and let \(F : A \to B\) be an exact functor. Then we denote the space of objects in \(S_A\) by \(sA\), and denote the space of objects in \([A \xrightarrow{F} B]\) by \([s(A \xrightarrow{F} B)]\). Beware that the second notation is not standard.

Lemma 2.16. \[Wal85, 1.4.1.\] Let \(F : A \to B\) be an exact functor of topological Waldhausen categories. Then there is an induced map \(sF : sA \to sB\). An isomorphism between two such functors \(F_0, F_1\) induces a homotopy between \(sF_0, sF_1\).

Proof. The first statement is clear (cf. \[Mit01\] Page 6.). To prove the second part, we will explicitly write down a simplicial homotopy. Simplicial objects in a category \(C\) can be considered as functors \(X : \Delta^{op} \to C\), and maps of simplicial objects are natural transformations of such functors. Simplicial homotopies can be described similarly; namely let \(\Delta/\{1\}\) be denote the category of objects over \(\{1\}\) in the simplex category, i.e. objects are maps \([n] \to \{1\}\). For any \(X : \Delta^{op} \to C\), let \(X^*\) denote the composited functor

\[
(\Delta/\{1\})^{op} \to \Delta^{op} \xrightarrow{X} C \\
([n] \to \{1\}) \mapsto \{n\} \mapsto X[n]
\]

Then a simplicial homotopy of maps may be identified with a natural transformation \(X^* \to Y^*\).

Now, suppose there is a functor isomorphism from \(F_0\) to \(F_1\) given by \(F : A \times \{1\} \to B\). The required simplicial homotopy then is a map from \([n] \to \{1\}\) to \((\{n\} \to \{1\}) \mapsto s_nA\) given by

\[
(a : [n] \to \{1\}) \mapsto (\{A : Ar[n] \to A\}) \mapsto (B : Ar[n] \to B))
\]

where \(B\) is defined as the composition

\[
Ar[n] \xrightarrow{(A,a*)} A \times Ar[1] \xrightarrow{Id \times p} A \times [1] \xrightarrow{F} B
\]

and \(p : Ar[1] \to [1]\) is given by \((0,0) \mapsto 0, (0,1) \mapsto 1, (1,1) \mapsto 1\).

Corollary 2.17. \[Wal85\] An equivalence of Waldhausen topological categories \(A \to B\) induces a homotopy equivalence \(sA \to sB\). Therefore if weak equivalences of \(A\) are the isomorphisms, denoted by \(i\), then \(sA \to iS_A\) is a homotopy equivalence.

The first part of this corollary is clear consequence of the lemma. The second part is a result of considering the simplicial object \([n] \to i_0 sA\), the nerve of \(iS_A\) in the \(i\)-direction, and noting that \(i_0 sA = sA\) and that face and degeneracy maps are homotopy equivalences by the first part of the corollary.
Proof of Proposition 2.14. To prove that a strictly cofinal topological Waldhausen subcategory $\mathcal{A}$ of $\mathcal{B}$ and $\mathcal{B}$ have homotopy equivalent K-theory spaces (and similarly spectra), it suffices to show that the relative K-theory category $wS[\mathcal{A} \hookrightarrow \mathcal{B}]$ is contractible. By property 2 of fat geometric realization, it suffices to show that each $wS_n[\mathcal{A} \hookrightarrow \mathcal{B}]$ is contractible. Consider the inclusions $S_n\mathcal{A} \hookrightarrow S_n\mathcal{B}$. Then $wS_n[\mathcal{A} \hookrightarrow \mathcal{B}]$ is equivalent to $w[S_n\mathcal{A} \hookrightarrow S_n\mathcal{B}]$. But it is easy to show that $S_n\mathcal{A}$ is a strictly cofinal subcategory of $S_n\mathcal{B}$: take an object $\{B_{jk}\}_{0 \leq j < k \leq n}$ in $S_n\mathcal{B}$. Then for each $B_{jk}$ in $\mathcal{B}$, there exists an object $A_{jk}$ in $\mathcal{A}$ so that $B_{jk} \amalg A_{jk}$ is isomorphic to an object in $\mathcal{A}$. Hence if $\mathcal{A} = \amalg_{j,k} A_{jk}$, then for all $j, k$, $B_{jk} \amalg (\amalg_{i=1}^{k-j} A)$ is isomorphic to an object in $\mathcal{A}$, as $\mathcal{A}$ is closed under finite coproducts. Therefore $\{B_{jk} \amalg (\amalg_{i=1}^{k-j} A)\}_{0 \leq j < k \leq n}$ is isomorphic to an object in $S_n\mathcal{A}$.

To show $w[\mathcal{A} \hookrightarrow \mathcal{B}]$ is contractible, again, by property 2 of fat geometric realization, it suffices to show that $w_m[\mathcal{A} \hookrightarrow \mathcal{B}]$, (the $m$-degree part in the $w$-direction) is contractible for all $m$. Let $\mathcal{A}(m, w)$ denote a sequence $A_0 \rightarrow A_1 \rightarrow \ldots \rightarrow A_m$ of $m$-weak equivalences in $\mathcal{A}$, and similarly define $\mathcal{B}(m, w)$, and consider the inclusion $\mathcal{A}(m, w) \hookrightarrow \mathcal{B}(m, w)$. Similar to before, $\mathcal{A}(m, w)$ is strictly cofinal in $\mathcal{B}(m, w)$. It is easy to see that $w_m[\mathcal{A} \hookrightarrow \mathcal{B}] \simeq [s(\mathcal{A}(m, w) \hookrightarrow \mathcal{B}(m, w))]$. Hence it suffices to prove that when $\mathcal{A} \hookrightarrow \mathcal{B}$ is an inclusion of a strictly cofinal topological Waldhausen subcategory, then $[s(\mathcal{A} \hookrightarrow \mathcal{B})]$ is contractible (cf. 2.15).

First note that the simplicial space $[s(\mathcal{A} \xrightarrow{Id} \mathcal{A})]$, is nerve of the (topological) category of cofibrations in $\mathcal{A}$ which has an initial object, and hence is contractible. Now we want to show that the inclusion $[s(\mathcal{A} \xrightarrow{Id} \mathcal{A})] \hookrightarrow [s(\mathcal{A} \hookrightarrow \mathcal{B})]$ is a homotopy equivalence. Consider the category of cofibrations in $\mathcal{B}$. Then $[s(\mathcal{A} \hookrightarrow \mathcal{B})]$ is homotopic to a simplicial subset of the nerve of this category (through forgetting the choices of quotients $B_{jk} \simeq B_{0k}/B_{0j}$, and taking a pushout with a fixed object is a natural transformation of the identity functor on this category to the pushout functor). In other words, there is a homotopy from the identity functor on $[s(\mathcal{A} \hookrightarrow \mathcal{B})]$ to the functor $\kappa_\mathcal{A}$, where $\kappa_\mathcal{A}(A_{jk}, B_{jk}) = (A_{jk}, B_{jk})$, where $B_{jk}' = B_{jk} \amalg A$ when $j = 0$ and $B_{jk}' = B_{jk}$ otherwise. But for any simplicial set $L$ in $[s(\mathcal{A} \hookrightarrow \mathcal{B})]$, with finitely many non-degenerate simplicies, as we argued before there exists an object $\mathcal{A}$ in $\mathcal{A}$ so that $\kappa_\mathcal{A}$ applied to $L$, would send each of the non-degenerate simplicies to simplicies (weakly equivalent to simplicies) in $[s(\mathcal{A} \xrightarrow{Id} \mathcal{A})]$. But then $\kappa_\mathcal{A}$ sends all of $L$ to simplicies (weakly equivalent to simplicies) in $[s(\mathcal{A} \xrightarrow{Id} \mathcal{A})]$. Therefore there is a homotopy from the inclusion of $L$ in $[s(\mathcal{A} \hookrightarrow \mathcal{B})]$ to a map from $L$ to $[s(\mathcal{A} \xrightarrow{Id} \mathcal{A})]$.

The proof for when $\mathcal{A}$ in $\mathcal{B}$ is only cofinal, goes through similarly. To show that the connected component of the zero object in $[s(\mathcal{A} \hookrightarrow \mathcal{B})]$, is contractible, one needs to use the assumption that $\mathcal{A}$ is closed under extension, which in turn shows that the object $A$ (that was obtained by using the cofinality assumption applied to the objects $B_{jk}$) used in the paragraph above, is in fact isomorphic to an object of $\mathcal{A}$.

Remark 2.18. We will only use cofinality in the case when there exists an object $A_0$ of $\mathcal{A}$ so that for each object $B$ of $\mathcal{B}$, $A_0 \oplus B$ is isomorphic to an object in $\mathcal{A}$. The proof of lemma above for this special case is slightly simpler and easier to understand.

\footnote{The point and the 1-simplex [1] are both good simplicial spaces \cite[4.A.4.]{Segel}, hence their fat geometric realizations are homotopy equivalent to their (usual) geometric realizations \cite[5.A.5.]{Segel} which are both contractible. Hence, by property 3 of the fat geometric realization, the argument in \cite[2.1.]{Segel8} goes through to show that a natural transformation between two functors between topological categories induces a homotopy between the induced maps on the fat geometric realizations. If the topological category $\mathcal{C}$ has an initial object, then there is an induced homotopy between the fat geometric realization of the nerve of the category $\mathcal{C}$, and the fat geometric realization of a point, which is contractible.}
2.2 Grayson’s Map

We start this subsection by going through an unfortunately rather long list of notations and definitions, and then we will use a construction of Grayson to give a natural map in the homotopy category of spectra\(^7\) from the K-theory spectrum of the category of acyclic binary chain complexes in an exact category \(\mathcal{A}\), to the loop space of the K-theory spectrum of \(\mathcal{A}\). We will closely follow the construction to see the extent of which it can be applied to topological exact categories.

**Notation 2.19.** Let \(X\) be a spectrum. For \(n \in \mathbb{Z}_{\geq 0}\), let \(V^nX\) denote the \(n\)'th stage of the postnikov filtration of \(X\), obtained by killing the stable homotopy groups \(\pi_m(X)\) for \(m < n\). In particular, \(V^0X\) is the connective part of \(X\).

Following \[Wal85\] Sec 1.6., let \((\mathcal{A}, w)\) be a (topological) Waldhausen category with the subcategory \(w\mathcal{A}\) (sometimes abbreviated to just \(w\)) of weak equivalences. If \((\mathcal{A}, v)\) is also a (topological) Waldhausen category with weak equivalences so that \(w\) is a subcategory of \(v\), then let \(A^w\) denote the full (topological) Waldhausen subcategory of \((\mathcal{A}, w)\) whose objects \(A\) are the ones with the property that \(* \to A\) is in \(v\). Recall that for a (topological) category \(\mathcal{A}\) with cofibrations, if \(i\mathcal{A}\) denotes the subcategory of isomorphisms, then \((\mathcal{A}, i)\) is a (topological) Waldhausen category.

For a (topological) exact category \(\mathcal{A}\), let \(\mathcal{C}\mathcal{A}\) denote the category of **chain complexes** in \(\mathcal{A}\) and let \(\mathcal{Ch}\mathcal{A}\) be the category of **acyclic chain complexes** in \(\mathcal{A}\), both of which have chain maps as morphisms. The categories \(\mathcal{C}\mathcal{A}, \mathcal{Ch}\mathcal{A}\) have a natural (topological) exact structure; a sequence is called exact iff it is exact degreewise. This means the cofibrations are the morphisms which are degree-wise cofibrations (admissible monomorphisms) and the weak equivalences \(i\) are the degree-wise **isomorphisms**\(^8\). We introduce a different structure of a (topological) Waldhausen category on \(\mathcal{C}\mathcal{A}\) by defining the cofibrations to be the degree-wise cofibrations again, and define the weak equivalences to be the **quasi-isomorphisms**, which we denote by \(q\). Note that quasi-isomorphisms are considered with respect to embedding the exact category \(\mathcal{A}\) into an abelian category. This definition does not depend on the choice of the embedding if \(\mathcal{A}\) either **supports long exact sequences** \[Gra12\] 1.4., or if \(\mathcal{A}\) satisfies the condition in \[TT90\] 1.11.3.. These conditions are both satisfied if \(\mathcal{A}\) is a (topological) pseudo-abelian category, cf \[TT90\] 1.11.5. and \[Gra12\] 4.. Evidently, \((\mathcal{C}\mathcal{A}, q)\) is a (topological) Waldhausen category, and \(\mathcal{Ch}(\mathcal{A})\) is equal to \((\mathcal{C}\mathcal{A})^q\). Furthermore, denote the full subcategories of **bounded chain complexes** and **bounded acyclic chain complexes** by \(\mathcal{C}^b\mathcal{A}\) and \(\mathcal{Ch}^b\mathcal{A}\) respectively. Again we have \((\mathcal{C}^b\mathcal{A})^q = \mathcal{Ch}^b\mathcal{A}\).

**Definition 2.20.** \[Gra12\] 3.1. Let \(\mathcal{A}\) be a (topological) exact category. We define a **binary chain complex** in \(\mathcal{A}\) to be a chain complex in \(\mathcal{A}\) with two differentials instead of one, i.e. a pair of chain complexes with the same objects but possibly different differentials, called the **top differential** of the top chain complex, and the **bottom differential** of the bottom chain complex. A binary chain complex is **acyclic** if both the top and the bottom chain complexes are acyclic. If we denote a binary chain complex by \((A^1, d_1, d_2)\), then \(A^1\) are the objects of the complex, \(d_1\) are the top differentials, and \(d_2\) are the bottom differentials. Let \(BA\) and \(B^iA\) be the (topological) category of binary chain complexes in \(\mathcal{A}\) and acyclic binary chain complexes in \(\mathcal{A}\). Also denote the (topological) category of **bounded binary chain complexes** and the category of **bounded acyclic binary chain complexes** in \(\mathcal{A}\) by \(B^b\mathcal{A}\) and \(B^b\mathcal{A}\), respectively. A morphism between two (respectively, acyclic) binary chain complexes is a map between the underlying objects which is a chain map with respect to both chain

\(^7\)The **stable homotopy category** (cf. \[AA95\]) can be considered as the localization of the category of spectra at the weak homotopy equivalences. In particular, all homotopic maps are equivalent to each other in the homotopy category, and homotopy equivalences are invertible.

\(^8\)We will abuse notation and denote the class of isomorphisms of different categories by \(i\).
complexes; in other words, a chain map when we consider only the top chain complexes, and also a chain map when we consider only the bottom chain complexes.

Similar to before, the categories $BA, BiA, B^hA, B^bA$ have a natural (topological) exact structure given by exactness at each degree. This means the cofibrations are degree-wise cofibrations, and weak equivalences $i$ are the degree-wise isomorphisms. We can define another structure of a (topological) Waldhausen category on $BA, B^hA$ with the cofibrations being the degree-wise cofibrations and the weak equivalences being the quasi-isomorphisms which we again denote by $q$. This again may depend on the choice of embedding $A$ in an abelian category, but does not depend on that choice if $A$ is a (topological) pseudo-abelian category. Hence we have a (topological) Waldhausen category $(BA, q)$, and again $(BA)^q, (B^hA)^q$ are the categories $BiA, B^bA$, respectively.

Let us denote the morphism that sends a chain complex $(A, d')$ to the binary chain complex $(A, d, d)$ by $\Delta : CA \rightarrow BA$, and denote the morphisms that send a binary chain complex to respectively the top and the bottom chain complex by $\top, \bot : BA \rightarrow CA$. These are exact functors, and we use the same notation for their restriction to $C^bA \rightarrow B^bA, ChA \rightarrow BiA, B^bA \rightarrow C^bA$ and $B^bA \rightarrow Ch^bA$. Let $\tau, \tau^b$ denote the category of maps $f$ in $BA$ and $B^bA$ respectively, such that $\top f$ is in $qCA, qC^bA$, respectively, and let $\beta, \beta^b$ denote the category of maps $f$ in $BA$ and $B^bA$, such that $\bot f$ is in $qCA, qC^bA$, respectively. Define $F : (CA, q) \rightarrow (BA, \tau)$ by $F(A, d') = (A, d', 0)$. Then the composition $\top \circ F$ is the identity functor on $CA$, and $F \circ \top$ is an exact endofunctor of $(BA, \tau)$.

Recall from definition[2.9] that for an exact functor $F : A \rightarrow B$ between (topological) Waldhausen categories, we have the relative K-theory space denoted by $[A \xrightarrow{F} B]$. We have the following proposition from Grayson [Gra12 Sec 7.]

**Proposition 2.21.** Let $A$ be a discrete exact category. Then there is a natural homotopy equivalence of spectra

$$K[(Ch^bA, i) \xrightarrow{\Delta} (B^bA, i)] \simeq V^b\Omega K(A)$$

In particular, there is a natural isomorphism of K-theory groups $K_{n-1}[(Ch^bA, i) \xrightarrow{\Delta} (B^bA, i)] \cong K_n(A)$ when $n \geq 1$, and there is a natural map in the homotopy category of spectra

$$\tau^b_A : K(B^bA, i) \rightarrow \Omega K(A).$$  \hspace{1cm} (1)

The proposition above uses ingredients such as Waldhausen’s fibration and approximation theorems [Wal85 1.6.4, 1.6.7.], the Gillet-Waldhausen theorem [Gil81 6.2.], and Thomason’s cofinality theorem [TT90 1.10.1], which we will check for topological categories in a future paper.

Let $A$ be (a topological) an exact category. Recall that by definition of the relative K-theory space, there is an exact functor $B^b(A) \rightarrow [Ch^bA \xrightarrow{\Delta} B^bA]$ for a (topological) Waldhausen category $A$. Now, assuming that the (topological) category $A$ ”supports long exact sequences”, we give a series of maps in the homotopy category of spectra as follows. Following the proof of [Gra12 4.3.], we first give a map in the homotopy category of spectra

$$G_1 : K[(Ch^bA, i) \xrightarrow{\Delta} (B^bA, i)] \rightarrow \Omega K[(CA, i) \xrightarrow{\Delta} (BA, i)].$$

We have the following commutative diagrams:

$$
\begin{array}{ccc}
K(Ch^bA, i) & \Rightarrow & K(C^bA, i) \\
\downarrow & & \downarrow \\
K(Ch^bA, q) & \Rightarrow & K(C^bA, q)
\end{array}
\hspace{1cm}
\begin{array}{ccc}
K(B^bA, i) & \Rightarrow & K(B^bA, i) \\
\downarrow & & \downarrow \\
K(B^bA, q) & \Rightarrow & K(B^bA, q)
\end{array}
$$
By Waldhausen’s fibration theorem \cite[1.6.4.]{Wal85}, the squares above are cartesian when the category $\mathcal{A}$ is discrete. Therefore we get the cartesian square below.

$$
\begin{align*}
K[(Ch^b \mathcal{A}, i) \xrightarrow{\Delta} (Bi^b \mathcal{A}, i)] & \longrightarrow K[(Ch^b \mathcal{A}, i) \xrightarrow{\Delta} (Bb^b \mathcal{A}, i)] \\
\downarrow & \\
K[(Ch^b \mathcal{A}, q) \xrightarrow{\Delta} (Bi^b \mathcal{A}, q)] & \longrightarrow K[(Ch^b \mathcal{A}, q) \xrightarrow{\Delta} (Bb^b \mathcal{A}, q)]
\end{align*}
$$

In the case of topological exact categories, the square above is still commutative (but not necessarily cartesian). Also the argument below works for topological exact categories as well.

The lower left hand corner of the diagram is contractible as each of the two categories in the relative $K$-theory space are contractible. The map $K(Ch^b \mathcal{A}, i) \xrightarrow{K\Delta} K(Bi^b \mathcal{A}, i)$ is a homotopy equivalence, because the functor $P^j : (Ch \mathcal{A}, i) \rightarrow \mathcal{A}$ which sends a chain complex to the term in degree $j$ is exact, and by induction and the additivity theorem, induces an isomorphism $K(Ch^b \mathcal{A}, i) \cong K(\prod_{\mathbb{Z}} \mathcal{A})$ (cf. \cite[6.2.]{Gil81}). Similarly we can say the same for $K(Bb^b \mathcal{A}, i)$, and note that $\Delta$ commutes with these isomorphisms and the identity map on $K(\prod_{\mathbb{Z}} \mathcal{A})$. Thus the upper right hand corner of the diagram above is also contractible. Therefore we have the following sequence of natural maps:

$$
\Omega K \left[ ([Ch^b \mathcal{A}, i) \xrightarrow{\Delta} (Bi^b \mathcal{A}, i)] \cong K \left[ ([Ch^b \mathcal{A}, i) \xrightarrow{\Delta} (Bb^b \mathcal{A}, i)] \rightarrow 0 \right] \right] \cong
\Omega K \left[ ((Ch^b \mathcal{A}, q) \xrightarrow{\Delta} (Bi^b \mathcal{A}, q)] \cong \Omega K \left[ ((Ch^b \mathcal{A}, q) \xrightarrow{\Delta} (Bb^b \mathcal{A}, q)] \right.
$$

When $\mathcal{A}$ is a discrete category, all of the maps above are homotopy equivalences, hence $G_1$ is a homotopy equivalence. Note that if the fibration theorem holds for topological exact categories, then * (and therefore $G_1$) is a homotopy equivalence for topological categories as well.

The next step is to define the homotopy equivalence

$$
G_2 : \Omega K[(Bb^b \mathcal{A}, q) \xrightarrow{\top} (Ch^b \mathcal{A}, q)] \rightarrow K[(Ch^b \mathcal{A}, q) \xrightarrow{\Delta} (Bb^b \mathcal{A}, q)]
$$

This map is induced by the commutative diagram of \cite[4.5.]{Gra12}. To be more precise, $G_2$ is the composition of the following sequence of maps:

$$
\Omega K \left[ (Ch^b \mathcal{A}, q) \xrightarrow{\top} (Ch^b \mathcal{A}, q) \right] \cong \Omega K \left[ \begin{array}{c} 0 \\ \downarrow \end{array} \right] \rightarrow \Omega K \left[ \begin{array}{c} \top \\ \downarrow \end{array} \right] \rightarrow \Omega K \left[ \begin{array}{c} 0 \\ \downarrow \end{array} \right] \cong K \left[ (Ch^b \mathcal{A}, q) \xrightarrow{\Delta} (Ch^b \mathcal{A}, q) \right].
$$

Where we used the fact that $\top \circ \Delta = 1$ and $K$-theory of squares is a generalization of relative $K$-theory which was defined in \cite[Sec 4.9]{Gra92}.

\footnote{The proofs only rely on the additivity theorem, which holds for topological categories.}
For the next step [Gra12 4.9], Grayson defines a map \( K((B^bA)\tau,q) \to \Omega K((B^b\mathcal{A},q) \to (C^b\mathcal{A},q)) \), which is a homotopy equivalence in the case of discrete categories. Instead we define the homotopy equivalence below for the (topological) exact category \( \mathcal{A} \).

\[
G_3 : K((B^b\mathcal{A},q) \to (C^b\mathcal{A},q)) \to K((B^b\mathcal{A},q) \to (B^b\mathcal{A},\tau)).
\]

Notice that we have the following commutative diagram, where each row is a cofiber sequence \(^{10}\), and \( F : (C^b\mathcal{A},q) \to (B^b\mathcal{A},\tau) \) is defined by \( F(A',d') = (A',d',0) \).

\[
\begin{array}{c}
K(B^b\mathcal{A},q) \xrightarrow{K_T} K(C^b\mathcal{A},q) \xrightarrow{K_F} K((B^b\mathcal{A},q) \to (C^b\mathcal{A},q)) \\
\downarrow 1 \quad \downarrow K_F \quad \downarrow \exists G_3 \\
K(B^b\mathcal{A},q) \xrightarrow{} K(B^b\mathcal{A},\tau) \xrightarrow{} K((B^b\mathcal{A},q) \to (B^b\mathcal{A},\tau))
\end{array}
\]

This induces the desired map \( G_3 \). Similar to [Gra12 4.8.], we can argue that for the (topological) exact category \( \mathcal{A} \), the maps \( KF, K_T \) are inverses to each other up to homotopy. (The argument relies on the fact that weak equivalence between two functors induces a homotopy between the corresponding maps of K-theory [Wal85 1.3.1.], which also holds for topological categories; see Seg68 2.1.1.) Hence \( KF \) is a homotopy equivalence for the (topological) exact category \( \mathcal{A} \), which means that \( G_3 \) is also a homotopy equivalence.

Let \( (C^b\mathcal{A})^x \) be the subcategory of chain complexes \( (A',d') \) in \( C^b\mathcal{A} \), whose euler characteristic \( \chi(A) = \sum_n (-1)^n A^n \) is equal to zero. When \( \mathcal{A} \) is a discrete category, according to [Gra12 5.8.], as a corollary of Thomason’s cofinality theorem [TT90 1.10.1.], we have a homotopy equivalence \( K((C^b\mathcal{A})^x,q) \to V^1K(C^b\mathcal{A},q) \). Then for the discrete exact category \( \mathcal{A} \), one has the following sequence of homotopy equivalences:

\[
\Omega K((B^b\mathcal{A})\tau,q) \cong \Omega K((B^b\mathcal{A})^\beta,q) = \Omega K((B^b\mathcal{A})^\beta,\tau) \cong \Omega K((C^b\mathcal{A})^x,q) \\
\cong \Omega V^1K(C^b\mathcal{A},q) \cong \Omega V^1K(\mathcal{A}) \cong V^0\Omega K(\mathcal{A})
\]

(2)

where the first homotopy equivalence is given by interchanging the top and the bottom differentials; the second is done by observing that \( ((B^b\mathcal{A})^\beta,q) = ((B^b\mathcal{A})^\beta,\tau) \); the third map is induced by the functor \( \top : ((B^b\mathcal{A})^\beta,\tau) \to ((C^b\mathcal{A})^x,q) \) (note that this is well-defined since \( (A',d_1,d_2) \) in \( ((B^b\mathcal{A})^\beta,\tau) \) is sent to \( (A',d_1) \), but acyclicity of \( (A',d_2) \) shows that the euler characteristic is zero.), which by theorem [Gra12 5.9.1] is a homotopy equivalence for discrete categories \(^{11}\); the fourth one by [Gra12 5.8.], is a corollary of Thomason’s cofinality theorem [TT90 1.10.1.]; the last one is true for any spectrum; and finally, the fifth map is induced by the inclusion \( \mathcal{A} \to C^b\mathcal{A} \) as the chain complex concentrated in degree zero, which is a homotopy equivalence is by [Gil81 6.2.]. (Also see TT90 1.11.7.] \(^{12}\) However, since the map is going in the opposite direction, the homotopy equivalence does not necessarily induce a map for topological exact categories.

The proof of [Gil81 6.2.] relies on Waldhausen’s fibration theorem as well. The author will check whether this holds for topological exact categories in a future work.

**Lemma 2.22.** Let \( \mathcal{A} \) be a topological exact category, that "supports long exact sequences", and assume that \( K(\mathcal{A}) \to K(C^b\mathcal{A},q) \) is a homotopy equivalence, where the map is induced by inclusion as the chain complex concentrated in degree zero. Then the sequence of maps in \( \mathcal{B} \) composed with the natural map \( V^0\Omega K(\mathcal{A}) \to \Omega K(\mathcal{A}) \) (given by definition of Postnikov tower) factors through \( K((B^b\mathcal{A},q) \to (B^b\mathcal{A},\tau)) \).

\(^{10}\)Recall that fibration and cofiber sequences are the same in the category of spectra.

\(^{11}\)The proof relies on Waldhausen’s approximation theorem [Wal85 1.6.7.] whose proof is quite long!

\(^{12}\)We need the extra assumption [TT90 1.11.3.] for this theorem to hold, however we are assuming that \( \mathcal{A} \) "supports long exact sequences", which ensures that there is no problem.
Before proving the lemma above, let us summarize [Gra12, Sec 6.] on what happens when the (topological) exact category \( \mathcal{A} \) does not ”support long exact sequences”.

The pseudo-abelianization (cf. proposition 14) \( \tilde{\mathcal{A}} \) of the (topological) exact category \( \mathcal{A} \) inherits (both the topological and) the exact structure of \( \mathcal{A} \). Also \( \mathcal{A} \) embeds in \( \tilde{\mathcal{A}} \) as a cofinal subcategory. Hence \( K_n(\mathcal{A}) \to K_n(\tilde{\mathcal{A}}) \) is an isomorphism when \( n > 0 \) and is injective for \( n = 0 \). Similar to [Gra12, 6.3.], the induced inclusions \( \text{Ch}^b(\mathcal{A}) \to \text{Ch}^b(\tilde{\mathcal{A}}) \) and \( \text{B}_i^b(\mathcal{A}) \to \text{B}_i^b(\tilde{\mathcal{A}}) \) are also cofinal, and by repeating the argument in [Gra12, 6.2.], the natural map from the cofiber of \( K(\text{Ch}^b(\mathcal{A})) \to K(\text{B}_i^b(\mathcal{A})) \) to the cofiber of \( K(\text{Ch}^b(\tilde{\mathcal{A}})) \to K(\text{B}_i^b(\tilde{\mathcal{A}})) \) is a homotopy equivalence.[13] Again by cofinality, \( \text{V}^0\Omega K(\mathcal{A}) \to \text{V}^0\Omega K(\tilde{\mathcal{A}}) \) is a homotopy equivalence.

Since \( \tilde{\mathcal{A}} \) is pseudo-abelian hence as explained before, \( \tilde{\mathcal{A}} ”\) supports exact sequences”, and when \( \mathcal{A} \) is a discrete category, then there is an induced natural map in the homotopy category of spectra

\[
\tau^G_\mathcal{A} : \quad K(\text{B}_i^b(\mathcal{A})) \to K(\text{B}_i^b(\tilde{\mathcal{A}})) \xrightarrow{\tau^G} \text{V}^0\Omega K(\mathcal{A}) \xleftarrow{\sim} \text{V}^0\Omega K(\tilde{\mathcal{A}}).
\]

**Proof of Lemma.** Let \( (\mathcal{A}^1, d_1, d_2) \) be an object of \( (\text{B}_i^b \mathcal{A}, q)^\tau \). By definition, the top chain complex \( (\mathcal{A}^1, d_1) \) is acyclic. This goes to \( (\mathcal{A}^1, d_2, d_1) \) through the first map in the sequence 2, and the second map is the identity. The third map sends it to the top chain complex \( (\mathcal{A}^1, d_2) \) in \( ((\text{B}_i^b \mathcal{A}), q) \). The composition

\[
\text{V}^0\Omega K \left( ((\text{B}_i^b \mathcal{A})^x, q) \right) \sim \text{V}^1\Omega K((\text{B}_i^b \mathcal{A}, q) \sim \text{V}^1\Omega K(\mathcal{A}) \sim \text{V}^0\Omega K(\mathcal{A}) \to \text{K}(\mathcal{A})
\]

is equal to the composition

\[
G_5 : \quad \text{V}^0\Omega K \left( ((\text{B}_i^b \mathcal{A})^x, q) \right) \to \text{K}(\mathcal{A}),
\]

where the first map is induced by inclusion of categories, and the second is given by the hypothesis of the lemma.

The natural map \( \text{V}^0\Omega K((\text{B}_i^b \mathcal{A})^\tau, q) \to \text{K}((\text{B}_i^b \mathcal{A}, q) \to (\text{B}_i^b \mathcal{A}, \tau)) \) is induced by inclusion. This sends the object \( (A^1, d_1, d_2, d_{2,j})_{0 \leq j \leq k \leq n} \) of \( S_n(\text{B}_i^b \mathcal{A})^\tau \) to the pair \( (A^1, d_1, d_{2,jk})_{0 \leq j \leq k \leq n}, (0)_{0 \leq j \leq k \leq n+1} \). Now, define the map \( G_4 : \text{K}((\text{B}_i^b \mathcal{A}, q) \to (\text{B}_i^b \mathcal{A}, \tau)) \to \text{V}^0\Omega K(\mathcal{B}_i^b q) \) by

\[
G_4 ((A^1, d_1, d_{2,jk})_{0 \leq j \leq k \leq n}, (A^1, d_1, d_{2,j})_{0 \leq j \leq k \leq n}) = (A^1, d_2, d_{2,j})_{0 \leq j \leq k \leq n},
\]

where \( (A^1, d_1, d_{2,j})_{0 \leq j \leq k \leq n+1} \) is an object of \( S_{n+1}(\text{B}_i^b \mathcal{A}, \tau) \), the first term is an object of \( ((\text{B}_i^b \mathcal{A}, q) \to (\text{B}_i^b \mathcal{A}, \tau)) \) and the second term is an object of \( S_n(\text{B}_i^b \mathcal{A}, q) \). Then use the natural homotopy equivalence \( \|w. S, E\| \simeq \Omega \|w. S, E\| \) for the topological Waldhausen category \( \mathcal{E} = (\text{B}_i^b \mathcal{A}, q) \).

\[\square\]

### 2.3 Higson’s Functor

Generalizing a construction given by Higson in [Hig95, Page 6.], for a \( C^*\)-algebra \( A \) we define the functor \( \tau^H_A : \text{C}(\mathcal{O}/\mathcal{C})_A \to B(\mathcal{O}/\mathcal{C})_A \) below.

**Definition 2.23.** Let \( (\rho, T) \) be a chain complex in \( C(\mathcal{O}/\mathcal{C})_A \). Define \( \tau^H_A(\rho, T) \) to be the binary chain complex whose \( n \)’th term is the graded object \( \nu^n = (\oplus_{i=-\infty}^{n-1}(\rho^{n-1} \odot \rho^i)) \odot \rho^n \) in \( (\mathcal{O}/\mathcal{C})_A \), where the last piece is of degree \( n \). The top differential (temporarily denoted by) \( \nabla^n \) from \( \nu^n \) to

---

[13] The reason why the \( n \)’th homotopy groups are isomorphic follows from cofinality when \( n > 0 \). But for \( n = 0 \) an extra argument is needed.
\( \nu^{n+1} \) is a degree 1 map, where its \( i \)'th degree piece from \( \rho^{n-1} \oplus \rho^n \) (of degree \( i \)) to \( \rho^n \oplus \rho^{n+1} \) (of degree \( i + 1 \)) is the trivial one (i.e. is identity on \( \rho^n \) and zero on \( \rho^{n-1} \)) for \( i \leq n - 1 \), and its \( n \)'th degree piece is equal to \( \rho^n \xrightarrow{T_n} \rho^{n+1} \). The bottom differential (temporarily denoted by) \( \perp^n \) from \( \nu^n \) to \( \nu^{n+1} \) is a degree 0 map, where its \( i \)'th degree piece from \( \rho^{n-1} \oplus \rho^n \) to \( \rho^n \oplus \rho^{n+1} \) is again the trivial one (i.e. is identity on \( \rho^n \) and zero on \( \rho^{n-1} \)) for \( i \leq n - 1 \), and its \( n \)'th degree piece is the trivial inclusion \( \rho^n \xrightarrow{(Id,0)} \rho^n \oplus \rho^{n+1} \).

\[
\begin{array}{cccccccccc}
\vdots & \oplus (\rho^{n-1}) & \oplus (\rho^n) & \oplus (\rho^n) & \oplus (\rho^n) & \oplus (\rho^n) & \oplus (\rho^n) & \oplus (\rho^n) & \oplus (\rho^n) & \oplus (\rho^n) & \oplus (\rho^n) & \oplus (\rho^n) & \oplus (\rho^n) & \oplus (\rho^n) \\
& & & & & & & & & & & & & & & & \\
& & & & & & & & & & & & & & & & \\
& & & & & & & & & & & & & & & & \\
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& & & & & & & & & & & & & & & & \\
& & & & \cdots & & & & \cdots & & & & \cdots & & & & \\
\end{array}
\]

It is easy to see that the bottom chain complex is split exact, and the top chain complex is exact iff the original chain complex \( (\rho,T) \) is exact. This process is functorial with respect to chain maps in a trivial way. Finally, note that if we start with a chain complex of length \( n \), then we will get a binary chain complex of length \( n + 1 \). Hence we also have the natural functor \( \tau^H_A : Ch^b(\mathcal{D}/\mathcal{C})_A \rightarrow Bi^b(\mathcal{D}/\mathcal{C})_A \). This functor is not exact; however we can tweak the structures of the categories to obtain an exact functor.

**Definition 2.24.** Let \( A \) be a \( C^\ast \)-algebra and let \( Ch^i(\mathcal{D}/\mathcal{C})_A, Bi^i(\mathcal{D}/\mathcal{C})_A \) denote the categories with the same objects as \( Ch^b(\mathcal{D}/\mathcal{C})_A, Bi^b(\mathcal{D}/\mathcal{C})_A \) respectively, but with morphisms and exact structure coming from the category \( \mathcal{D}_A \). To be precise, a morphism in \( Ch^i(\mathcal{D}/\mathcal{C})_A \) from the chain complex \( (\rho,T) \) to \( (\nu,S) \) is given by a chain map \( f^n : \rho^n \rightarrow \nu^n \) in the category \( \mathcal{D}_A \), and morphisms in \( Bi^i(\mathcal{D}/\mathcal{C})_A \) are defined similarly as a chain map in \( \mathcal{D}_A \) with respect to both the top and the bottom chain complex. We say a sequence of chain complexes in \( Ch^i(\mathcal{D}/\mathcal{C})_A \) is exact, iff the sequence is exact at each degree in \( \mathcal{D}_A \), and similarly define the exact structure on \( Bi^i(\mathcal{D}/\mathcal{C})_A \).

There are natural functors \( Ch^i(\mathcal{D}/\mathcal{C})_A \rightarrow Ch^b(\mathcal{D}/\mathcal{C})_A \) and \( Bi^i(\mathcal{D}/\mathcal{C})_A \rightarrow Bi^b(\mathcal{D}/\mathcal{C})_A \). These functors are exact, since exactness in \( \mathcal{D}_A \) guarantees exactness in \( (\mathcal{D}/\mathcal{C})_A \).

**Lemma 2.25.** The functor \( \tau^H_A \) defined in 2.24 induces an exact functor

\[
\tau^H_A : Ch^i(\mathcal{D}/\mathcal{C})_A \rightarrow Bi^i(\mathcal{D}/\mathcal{C})_A.
\]

Therefore, we have a natural map of \( K \)-theory spectra

\[
\tau^H_A : K(Ch^i(\mathcal{D}/\mathcal{C})_A) \rightarrow K(Bi^i(\mathcal{D}/\mathcal{C})_A).
\]

This is proved by observing that infinite direct sum of identities is equal to identity in \( \mathcal{D}_A \). Note that this is not true in the paschke category \( (\mathcal{D}/\mathcal{C})_A \).

**Proof.** Let \( (\rho_i,T_i) \) denote objects in \( Ch^i(\mathcal{D}/\mathcal{C})_A \) for \( i \in \mathbb{Z} \), and let \( f_i : (\rho_i,T_i) \rightarrow (\rho_{i+1},T_{i+1}) \) be morphisms that give an exact sequence in \( Ch^i(\mathcal{D}/\mathcal{C})_A \), with the degree-wise contracting homotopy given by \( g_i : (\rho_{i+1},T_{i+1}) \rightarrow (\rho_i,T_i) \). Then we need to show that \( \tau^H_A(g_i)\tau^H_A(f_i) + \tau^H_A(f_{i-1})\tau^H_A(g_{i-1}) \) is equal to identity at each degree of \( \tau^H_A(\rho_i,T_i) \). This is true since at degree \( n \) this is given by infinite direct sum \( g^n_i f^n_i + f_{n-1}^n g_{n-1}^n \) and \( g_{n-1}^n f_{n-1}^n + f_{n-1}^n g_{n-1}^n \). But by assumption, each term is equal to identity in \( \mathcal{D}_A \), hence their infinite direct sum is also equal to identity. \( \square \)
None of this process works in the Calkin-Paschke category $(\mathcal{D}/\mathcal{E})'_A$, as infinite direct sums of $\rho : A \to (\mathcal{B}(\mathcal{H}))(H)$ is not necessarily defined, since infinite direct sum of compact operators does not have to be compact. In fact if the infinite direct sum of $\rho'$ is well-defined, then by an Eilenberg swindle argument we can show that the class corresponding to $\rho'$ in $K^\text{top}_1(A) = \text{Ext}(A)$ defined in [BDF77] is zero.

3 Complex Manifolds and the Dolbeault Complex

This section will contain a great deal of computations, and to ease the readability, we will fix some of our notations.

**Notation 3.1.** Fix $\chi(t) = \frac{t}{\sqrt{1+t^2}}$. The functions $\chi, \phi$ will be used for functional calculus. The letter $X$ denotes manifolds, $U, V$ are used for open subsets of the manifold, $\lambda$ for a partition of unity, and $\gamma$ for cutoff functions. The letters $D, d$ will be used for differential operators, and $\bar{\partial}$ will denote the Dolbeault operator.

We will use $E$ for vector bundles, $g, h$ will be reserved for a metric on the manifold, and on the bundle respectively. The letters $\alpha, \beta$ will be isomorphisms of vector bundles, $\varphi, \sigma, \psi$ will be maps of vector bundles.

The letter $I$ will be used as a map of Hilbert spaces induced by identity map on a bundle (with different choices of metrics), $\pi$ will refer to projection onto the $L^2$-integrable functions on an open subset, and $\iota$ will denote extension by zero of $L^2$ sections on an open subset to the whole space.

3.1 The Dolbeault Functor

For the definition and basic properties of functional calculus, see the appendix B. One could apply functional calculus to an essentially self-adjoint operator, and in certain cases we get interesting properties.

**Lemma 3.2.** Let $X$ be a differentiable manifold, $E$ a differentiable vector bundle over $X$, and let $D \in \text{Diff}_1(E,E)$ be a differential operator of order $1$. Consider the representation $\rho : C_0(X) \to \mathcal{B}(L^2(X,E))$.

1. Let $\phi$ be a bounded Borel function on $\mathbb{R}$, whose Fourier transform is compactly supported. Then $\phi(D)$ is a well-defined bounded operator acting on $L^2(X,E)$, which is in fact, pseudo-local. [HR00, 10.3.5, 10.6.3.]

2. Assume in addition that $D$ is an elliptic operator. Let $\phi \in C_0(\mathbb{R})$, then $\phi(D) : L^2(X,E) \to L^2(X,E)$ is a locally compact operator. [HR00, 10.5.2.]

Now we are ready to define the functor $\hat{\tau}^D_X$.

**Definition 3.3.** Let $X$ be a complex manifold of dimension $n$, and let $E$ be a holomorphic vector bundle on $X$. We will use the Dolbeault complex to define an exact sequence in the Paschke category of $C_0(X)$.

Fix some hermitian metric $g$ on $X$ and a Hermitian metric $h$ on $E$ and let $H^i$ be the space of $L^2$-integrable sections of the bundle $\wedge^{0,i}T^*X \otimes E$ over $X$. There are natural representations $\rho^i : C_0(X) \to \mathcal{B}(H^i)$ given by point-wise multiplication of a function on $X$ with the $L^2$-section. Let $\tilde{\partial}_E^*$ be the formal adjoint of the Dolbeault operator $\bar{\partial}_E$ (with respect to the metrics $g, h$), and consider the essentially self-adjoint differential operator $D_E = \tilde{\partial}_E + \bar{\partial}_E^*$ of order $1$ [HR00, 11.8.1.]. Therefore we can apply functional calculus to $D_E$ with respect to the function $\chi(t) = \frac{t}{\sqrt{1+t^2}}$, to...
obtain a bounded operator \( \frac{D_E}{\sqrt{1+D_E^2}} = \chi(D_E) \in \mathfrak{B}(\oplus_i H^i) \). By lemma 3.2 this is a pseudo-local operator with respect to the \( \rho \)'s, so if \( \chi_i(D_E) = \frac{\partial}{\sqrt{1+(D_E)^2}} \) denotes the restriction of \( \chi(D_E) \) to \( \mathfrak{B}(H^i, H^{i+1}) \), then we have the following chain complex in the Paschke category \((\mathcal{D}/\mathcal{E})_{C_0(X)}\).

\[
\hat{\tau}^D_{X,F}(E, h) : 0 \to \rho^0 \chi_0(D_E) \to \rho^1 \chi_1(D_E) \to \ldots \to \rho^{n-1}(D_E) \to \rho^n \to 0. \tag{4}
\]

To show that this is in fact an exact sequence in the Paschke category, we need to find pseudo-local operators \( P_i : H^{i+1} \to H^i \) which give a contracting homotopy \( \frac{\partial}{\sqrt{1+D_E^2}} \), i.e. \( P_i \chi_i(D_E) + \chi_{i-1}(D_E) P_{i-1} - \text{Id}_{H^i} \) is a locally compact operator. It is easy to see if \( P_i = \frac{\partial}{\sqrt{1+D_E^2}} : H^{i+1} \to H^i \), then \( P_i \chi_i(D_E) + \chi_{i-1}(D_E) P_{i-1} - \text{Id}_{H^i} = \frac{1}{1+D_E^2} \), which is locally compact by lemma 3.2. This shows that 4 is an exact sequence.

**Proposition 3.4.** Let \( X \) be a compact complex manifold and \( E \) a holomorphic vector bundle. Then the chain complex 4 considered as a complex of Hilbert spaces and bounded operators, is quasi-isomorphic to the Dolbeault complex of coefficients in \( E \).

**Proof.** It is easy to see that the diagram below commutes:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{A}^{0,0}_X(E) & \xrightarrow{\partial_0} & \mathcal{A}^{0,1}_X(E) & \xrightarrow{\partial_1} & \ldots & \xrightarrow{\partial_{n-1}} & \mathcal{A}^{0,n}_X(E) & \longrightarrow & 0 \\
& & \downarrow{(1+D_E^2)^{-1/2}} & & \downarrow{(1+D_E^2)^{-n/2}} & & & & \\
0 & \longrightarrow & H^0 & \xrightarrow{\chi_0(D_E)} & H^1 & \xrightarrow{\chi_1(D_E)} & \ldots & \xrightarrow{\chi_{n-1}(D_E)} & H^n & \longrightarrow & 0
\end{array}
\]

Since the image of the vertical maps are dense, then by the Hodge-decomposition, we can see this sends Harmonic forms isomorphically to the cohomology of the complex below.

The definition 3.3 is not very easy to work with when we restrict to open subsets, because restriction of an essentially self-adjoint operator to an open subset is not necessarily essentially self-adjoint. We will give an equivalent definition in 3.7 for any symmetric elliptic operator.

**Lemma 3.5.** [HR00, 10.8.4.] Let \( X \) be a differentiable manifold, \( E \) a differentiable vector bundle on it, \( U \subseteq X \) an open subset, and \( D_1, D_2 \in \text{Diff}_1(E, E) \) are order one essentially self-adjoint differential operators, so that \( D_1|_U = D_2|_U \). Then if \( f \in C_0(U) \), we have \( \rho(f) \chi(D_1) - \rho(f) \chi(D_2) \) is a compact operator, where \( \rho : C_0(U) \to \mathfrak{B}(L^2(X, E)) \) is given by pointwise multiplication, and \( \chi(t) = \frac{t}{\sqrt{1+t^2}} \).

**Definition 3.6.** Let \( X \) be a locally compact, and Hausdorff topological space. We say that the open cover \( \{U_j\}_j \) is a *good cover*, if it is countable, locally finite, and each open set \( U_j \) is relatively compact.

**Definition 3.7.** [HR00, 10.8.] Let \( X \) be a (non-compact) differentiable manifold, \( E \) a differentiable vector bundle on \( X \), and let \( D \in \text{Diff}_1(E, E) \) be a symmetric elliptic differential operator of order 1. Let \( \{U_j\}_j \) be a good cover (definition 3.6), and let \( \{\gamma_j\}_j \) be a partition of unity subordinate to the cover, and let \( \{\lambda_j\}_j \) be compactly supported non-negative continuous functions, so that \( \gamma_j|_{U_j} \) is equal to (the constant function) one. Then the symmetric differential operator \( D_j = \gamma_j D \gamma_j \) is supported on a compact set, hence by lemma 3.5 is essentially self-adjoint. Therefore if the

\[14\] The operators \( P_i \) are also called the parametrices.
representation \( \rho : C_0(X) \to \mathfrak{B}(L^2(X,E)) \) (where the \( L^2 \)-completion is of course defined with respect to a choice of a metric on \( X \) and one on \( E \)) is given by pointwise multiplication, then we can define

\[
\chi_D := \sum_j \rho(\lambda_j^{1/2}) \chi(D_j) \rho(\lambda_j^{1/2}),
\]

(5)
as the partial sums are bounded in norm and the series converges in the strong operator topology.

One can see that \( \chi_D \) is self-adjoint. \( \chi_D \) depends on the choice of the open cover, the partition of unity, and the cut-off functions \( \gamma_j \)'s, but if \( f \in C_0(X) \) is compactly supported, then \( \rho(f) \chi_D \) has only finitely many terms, hence by lemma 3.5 if \( D_1 \in \text{Diff}_1(E,E) \) is any essentially self-adjoint differential operator which agrees with \( D \) on support of \( f \), then \( \rho(f) \chi_D - \rho(f) \chi(D_1) \) is compact, and in particular if \( D \) is itself essentially self-adjoint, then \( \chi_D - \chi(D) \) is locally compact. Hence the choices do not matter up to locally compact operators. Therefore, we have a well-defined operator \( \chi_D \) in the Paschke category \( (\mathcal{D}/\mathcal{E})_{C_0(X)} \).

**Definition 3.8.** Let \( X \) be a complex manifold, then denote the category of holomorphic vector bundles on \( X \) by \( \mathcal{P}(X) \). This is an exact category.

It is straightforward to show that \( \mathcal{P}(X) \) has a small skeletal subcategory. For each vector bundle on \( X \), there is a set of metrics, hence if we denote the category of holomorphic vector bundles with a choice of metric by \( \mathcal{P}_m(X) \), (i.e. objects are pairs \( (E,h) \) of a holomorphic vector bundle with a hermitian metric, and morphisms are bundle maps) then without loss of generality, we can assume that this is a small category. This inherits an exact structure from the category \( \mathcal{P}(X) \).

Let \( g \) be a hermitian metric on \( X \), then denote the **bounded** category of holomorphic vector bundles with a choice of metric by \( \mathcal{P}_{m,b}(X,g) \), where objects again are pairs \( (E,h) \) of a holomorphic vector bundle with a choice of a hermitian metric, and a morphism from \( (E_1,h_1) \) to \( (E_2,h_2) \) is a map of bundles \( E_1 \to E_2 \), so that the induced map \( L^2(X,\wedge^0 T^*X \otimes E_1) \to L^2(X,\wedge^0 T^*X \otimes E_2) \) is a bounded map of Hilbert spaces. We say a sequence \( \ldots \to (E_{i-1},h_{i-1}) \to (E_i,h_i) \to (E_{i+1},h_{i+1}) \to \ldots \) is exact in \( \mathcal{P}_{m,b}(X) \), if there are smooth maps of bundles \( \sigma_i : E_{i+1} \to E_i \) which are contracting homotopy, and also the induced maps \( L^2(X,\wedge^0 T^*X \otimes E_{i+1}) \to L^2(X,\wedge^0 T^*X \otimes E_i) \) are bounded.\(^{15}\) This is again a small category.

Note that there \( \mathcal{P}_{m,b}(X) \) is a subcategory of \( \mathcal{P}_m(X) \), and there is a forgetful map \( \mathcal{P}_m(X) \to \mathcal{P}(X) \). Both of these functors are exact.

**Proposition 3.9.** Let \( X \) be a complex manifold, and let \( g \) be a hermitian metric of bounded geometry on \( X \). Recall from definition 2.4 that \( \text{Ch}^!(\mathcal{D}/\mathcal{E})_{C_0(X)} \) denotes the category of bounded acyclic chain complexes in \( (\mathcal{D}/\mathcal{E})_{C_0(X)} \) where the exact structure is induced by that of \( \mathcal{D}_{C_0(X)} \).

The map \( \tilde{\tau}^D_{X,g} \) defined in [3.3] induces an exact functor from \( \mathcal{P}_{m,b}(X) \) to \( \text{Ch}^!(\mathcal{D}/\mathcal{E})_{C_0(X)} \).

The proof of proposition above, will take the entirety of the next subsection, as we will need to prove a series of technical lemmas (which are probably known to the experts). We include them with great details for readers who may not have a background in the topic.

Throughout this section, we had fixed a single metric on the manifold \( X \) and do the rest of our computations. Let us investigate effect of the choice of the metric \( g \) on \( \tilde{\tau}^D_{X,g} \).

**Lemma 3.10.** Let \( X \) be a differentiable manifold, and let \( E \) be a differentiable vector bundle on \( X \). Let \( d \in \text{Diff}_1(E,E) \) be a differential operator, so that for each metric \( g \) on \( X \) and \( h \) on \( E \), \( D = d + d^* \) is an essentially self-adjoint elliptic differential operator. Let \( g_0, g_1 \) be two metrics on \( X \), and let \( h_0, h_1 \) be metrics on \( E \). Denote \( d + d^* \) with respect to \( (g_0, h_0), (g_1, h_1) \) by \( D_0, D_1 \) respectively.

\(^{15}\)Notice that a smooth contracting homotopy always exists [AA67, 1.4.11.]. The only condition here is boundedness of \( \sigma_i \)'s.
Then there is a unitary isomorphism $L^2(X, E; g_0, h_0) \rightarrow L^2(X, E; g_1, h_1)$ that commutes with $\chi(D)$ up to locally compact operators.

**Proof.** Let $g_t = (1-t)g_0 + tg_1, h_t = (1-t)h_0 + th_1$ for $0 \leq t \leq 1$. Then both $g_t, h_t$ are metrics (and in case both $g_0, g_1$ are hermitian, then so are $g_t$ and etc.). Denote the Hilbert space of $L^2$-sections of $E$ with respect to the metric $g_t, h_t$ by $H_t$. Let $\nu_t : X \rightarrow \mathbb{R}_+^+$ be the "square root of the measure" given by the Radon Nikodym theorem so that $d\mu_t(Z) = \int_Z \nu_t^2 d\mu_0$ for each measurable subset $Z$ of $X$ and each $t$. Let $S_t : E \rightarrow E$ be the square root of the positive definite map $E \xrightarrow{h_0} E^* \xrightarrow{h_1^*} E$. Then $T_t(x) = \eta(x)S_t(x)$ acts fiberwise, hence it is pseudo-local. Also for $L^2$ sections $\eta, \zeta$ in $H_0$,

$$\langle T_t\eta, T_t\zeta \rangle = \int_X \langle (h_t)(T_t\eta)(T_t\zeta) \rangle d\mu_t = \int_X \nu_t^2 h_t(S_t\eta)(S_t\zeta)d\mu_t = \int_X h_t(S_t^*S_t\eta)(\zeta)d\mu_0 = \int_X h_t^* h_t^* h_0(\eta)(\zeta)d\mu_0 = \langle \eta, \zeta \rangle_0.$$ 

Therefore we have unitary maps $T_t : L^2(X, E) \rightarrow L^2(X, E)$ (where the $L^2$-completions are with respect to $(g_0, h_0), (g_t, h_t)$, respectively). Consider the the path $t \mapsto T_t^*\chi(D_1)T_t$ from $\chi(D_0)$ to $T_1^*\chi(D_1)T_1$. Since $\chi^2 - 1 \in C_0(\mathbb{R})$, therefore $T_t \frac{(\chi(D_0) + 1)}{2}T_t \in (\mathcal{D}/\mathcal{E})(\rho_0)$ is a self-adjoint projection up to locally compact operators. Hence by lemma 1.17 without loss of generality we can assume $T_t^* \frac{(\chi(D_0) + 1)}{2}T_t \in \mathcal{B}(L^2(X, E))$ (where the $L^2$-completion is with respect to $(g_0, h_0)$) is a self-adjoint projection, and by [HR00, 4.1.8.] this path of projections induces a unitary operator $W_1 : L^2(X, E; g_0, h_0) \rightarrow L^2(X, E; g_t, h_t)$ such that $W_1(T_1^* \frac{(\chi(D_1) + 1)}{2}T_1)W_1 = \frac{(\chi(D_0) + 1)}{2}$. Therefore, $W_1T_1$ is the unitary isomorphism that commutes with $\chi(D)$ up to locally compact operators. 

#### 3.2 Proof of Proposition 3.9

**Notation 3.11.** Let $X$ be a differentiable manifold, and let $E_1, E_2$ be differentiable vector bundles on $X$. Choose metrics $g$ on $X$ and $h_1, h_2$ on $E_1, E_2$. To shorten the notation, we say $(X; g; E_1, h_1; E_2, h_2)$ is a metric pair.

Let $X$ be a complex manifold, and let $E_1, E_2$ be holomorphic vector bundles on $X$. Choose hermitian metrics $g$ on $X$ and $h_1, h_2$ on $E_1, E_2$, and let set $D_{E_1} = \partial E_1 + \bar{\partial}E_1$ and $D_{E_2} = \partial E_2 + \bar{\partial}E_2$ be the corresponding Dolbeault operators. To shorten the notation, we say $(X, g; E_1, h_1, D_{E_1}; E_2, h_2, D_{E_2})$ is a hermitian pair.

**Definition 3.12.** Let $(X; g; E_1, h_1; E_2, h_2)$ be a metric pair. We say an operator $T$ (or a family of operators) is locally bounded with respect to $(X; g; E_1, h_1; E_2, h_2)$, if for each relatively compact open subset $U$ of $X$, there exists an induced operator $T_U : L^2(U, E_1|_U) \rightarrow L^2(U, E_2|_U)$ so that $T_U$ is a bounded linear operator, and if for each pair of relatively compact open subsets $U_1, U_2$, we have

$$\pi_{U_1 \cap U_2}^U T_{U_1} T_U T_{U_2} \pi_{U_1 \cap U_2}^U = \pi_{U_1 \cap U_2}^U T_{U_2} T_{U_1} \pi_{U_1 \cap U_2}^U$$

where in here $\pi_U^V : L^2(U, E|_U) \rightarrow L^2(V, E|_V)$ is the projection defined by multiplication by the characteristic function of $U \cap V$, and $\nu_U^V : L^2(V, E|_V) \rightarrow L^2(U, E|_U)$ is extension by zero.

Beware that this definition is not exactly the same as more well-known definitions of local boundedness. Also note that there does not need to be a uniform bound on $\|T_U\|$. However, in case there is a uniform bound on $T_U$ (say $M$), then we can "glue" them to obtain $T : L^2(X, E_1) \rightarrow L^2(X, E_2)$, by simply choosing a relatively compact open neighborhood $U$ of $x$, and setting $T(\zeta)(x) = T_U(\pi_U^V \zeta_U(x))$. This is independent of choice of $U$ and $\|T(\zeta)\| \leq M\|\zeta\|_1$, as this holds for almost every point $x$.

---

16 Notice that by [La02, Page 150] this exists and varies continuously.

17 Recall that evaluating $\zeta \in L^2(X, E_1)$ at a point $x \in X$ only makes sense up to subsets of measure zero in $X$. 

---
Example 3.13. Let \( (X, g; E_1, h_1; E_2, h_2) \) be a metric pair (definition 3.11), and let \( \varphi : E_1 \to E_2 \) be a continuous bundle map. Then \( \varphi \) is locally bounded (definition 3.12).

Example 3.14. Let \( (X, g; E_1, h_1, E_2, h_2) \) be a metric pair (definition 3.11), and let \( L^2(X, E; h_i) = L^2(X, E; g, h_i) \) denote the space of \( L^2 \)-sections of \( E \) on \( X \) with respect to the metric \( h_i \) on \( E \) (and \( g \) on \( X \)). Then the identity map \( \text{Id} : E \to E \) induces a locally bounded map (definition 3.12) from \( L^2(X, E; h_1) \) to \( L^2(X, E; h_2) \) which we denote by \( I(h_2, h_1) \) throughout this section.

Lemma 3.15. Let \( (X, g; E_1, h_1, D_1, E_2, h_2, D_2) \) be a hermitian pair (definition 3.11). Then \( D_{E_1} - D_{E_2} \) is locally bounded (definition 3.12).

Proof. Recall from definition 3.6 that the metric \( h_i \) can be considered as a linear map of bundles from \( E \) to the dual bundle \( E^* \), which by abuse of notation we denote with \( h_i \) again. Let \( h_i^* : E^* \to E \)
denote the dual maps induced by \( h_i \), let \( \theta \) denote the composition \( E \xrightarrow{h_1} E^* \xrightarrow{h_2^*} E \), and let \( \vartheta^* \) denote the composition \( E^* \xrightarrow{h_2^*} E \xrightarrow{h_1} E^* \).

Consider \( f \otimes e \in \mathcal{C}^{\infty}(X, \wedge^0, T^*X \otimes E) \), we have:

\[
D_{E_1}(f \otimes e) - D_{E_2}(f \otimes e) = (\partial_E + (\star \otimes h_1^*)\tilde{\partial}(\star \otimes h_1)(f \otimes e) - (\partial_E + (\star \otimes h_2^*)\tilde{\partial}(\star \otimes h_2)(f \otimes e))
= (\star \otimes h_1^*)\tilde{\partial}(f \otimes h_1(e)) - (\star \otimes h_2^*)\tilde{\partial}(f \otimes h_2(e))
= (\star \otimes h_1^*)\tilde{\partial}(f \otimes h_1(e)) - (\star \otimes h_2^*)\tilde{\partial}(f \otimes h_2(e))
= (\star \otimes h_1^*)\tilde{\partial}(f \otimes e) - (\star \otimes h_2^*)\tilde{\partial}(f \otimes e)
\]

where in here, \( e^* = h_1(e) \). The term above does not have any differentials of \( f \otimes e \); recall \( \star \) is \( \pm 1 \), and \( ||\theta||, ||\vartheta^*|| \), vary continuously with respect to \( x \in X \), and \( i = 1, 2 \), hence the term \( \theta(e)\tilde{\partial}(\vartheta^*) \) is bounded with respect to both norms on the relatively compact set \( U \).

Lemma 3.16. Let \( (X, g; E_1, h_1; E_2, h_2) \) be a metric pair (definition 3.11), and let \( D_i \in \text{Diff}_i(E, E) \) be an essentially self-adjoint differential operators with respect to the metric \( h_i \) for \( i = 1, 2 \), so that \( D_1 - D_2 \) is a locally bounded operator (definition 3.12). Let \( I(h_2, h_1) \) denote the locally bounded map induced by the identity map of \( E \) (example 3.14). Then for each relatively compact open subset \( U \) of \( X \), we have

\[
\pi_1 \chi(D_1)\iota_1 I(h_1, h_2)|_U = I(h_1, h_2)|_U \pi_2 \chi(D_2)|_U
\]
in the Paschke category \( \mathcal{D}/\mathcal{C} \mathcal{C}_0(U) \), where in here, \( \pi_i : L^2(X, E; h_i) \to L^2(U, E|_U; h_i) \) is the projection and \( \iota_i \) is extension by zero, for \( i = 1, 2 \).

This proof closely follows that of [HR00] 10.9.5.\(^\text{18}\)

Proof. Similar to [HR00] 10.3.5.] we argue that if \( u, v \) be compactly supported smooth sections of \( \wedge^0, T^*X \otimes E \), and \( \phi \) is a Schwartz function, then since \( \phi(x) = \frac{1}{2\pi} \int e^{-tsx} \hat{\phi}(s) ds \), then we can pair \( \hat{\phi} \) with the smooth function \( s \mapsto \langle (I(h_2, h_1) e^{-tsD_1} I(h_1, h_2))(u, v) \rangle_2 = \langle e^{-tsD_1} I(h_1, h_2) u, v \rangle_2 \) to obtain

\[
\langle \phi(D_1)u, v \rangle_2 = \langle I(h_2, h_1)\phi(D_1) I(h_1, h_2) u, v \rangle_2 = \frac{1}{2\pi} \int \langle I(h_2, h_1) e^{-tsD_1} I(h_1, h_2) u, v \rangle_2 \hat{\phi}(s) ds,
\]

\(^{18}\text{We can not directly apply this result here, even though they look similar; the problem is that in [HR00] 10.9.5. it is required for both } D_1, D_2 \text{ to be essentially self-adjoint with respect to the same given inner product, which is not the case here.} \)
and then use the rest of the argument in [HR00, 10.3.5., to generalize this for any bounded Borel function whose Fourier transform is compactly supported.

Let $\phi, u, v$ be as above (with the extra assumption that $s\hat{\phi}(s)$ is a smooth function, also note that $D_1, D_2$ both share the invariant domain of smooth compactly supported functions.), then we have

$$\langle (I(h_2, h_1)\phi(D_1)I(h_1, h_2)-\phi(D_2))u, v\rangle_2 = \frac{1}{2\pi}\int \langle (I(h_2, h_1)e^{\sqrt{-1}sD_1}I(h_1, h_2)-e^{\sqrt{-1}sD_2})u, v\rangle_2 \hat{\phi}(s)ds.$$  

By fundamental theorem of calculus we know that

$$\langle (I(h_2, h_1)e^{\sqrt{-1}tD_1}I(h_1, h_2) - e^{\sqrt{-1}tD_2})u, v\rangle_2 = \langle (e^{\sqrt{-1}tD_1} - e^{\sqrt{-1}tD_2})u, v\rangle_2 = \sqrt{-1}\int_0^\delta \langle (I(h_2, h_1)e^{\sqrt{-1}tD_1}I(h_1, h_2)(D_1 - D_2)e^{\sqrt{-1}(s-t)D_2})u, v\rangle_2,$$

and by repeating the argument in [HR00, 10.3.6, 10.3.7.] we obtain that there exists a constant $C_\phi < \infty$ (which only depends on $\phi$) so that $\|I(h_2, h_1)\phi(D_1)I(h_1, h_2) - \phi(D_2)\|_2 \leq C_\phi \|D_1 - D_2\|_2$.

Now, let $\phi$ be a normalizing function (i.e. $\phi - \chi \in C_0(\mathbb{R})$) that satisfies the conditions above, and let $\hat{\phi}(x) = \hat{\phi}(\epsilon x)$. Then $\hat{\phi}$ is also a normalizing function, and hence $\phi_\epsilon(D_1) - \chi(D_1)$ is a locally compact operator for any $\epsilon > 0$. But as $\epsilon \to 0$, we get

$$\|I(h_2, h_1)\phi_\epsilon(D_1)I(h_1, h_2) - \phi_\epsilon(D_2)\|_2 = \|I(h_2, h_1)\phi(D_1)I(h_1, h_2) - \phi(D_2)\|_2 \leq C_\phi \|D_1 - D_2\|_2 \to 0.$$  

In other words, there are elements of equivalency class of locally compact operators equivalent to $I(h_2, h_1)\chi(D_1)I(h_1, h_2)$ and $\chi(D_2)$ respectively, which get arbitrarily close. But these are linear subspaces of pseudo-local operators, hence these subspaces have to be the same, i.e. $I(h_2, h_1)\chi(D_1)I(h_1, h_2) - \chi(D_2)$ is locally compact. This finishes the proof.

$\square$

**Corollary 3.17.** Let $(X, g; E, h_1, D_{E,1}; E, h_2, D_{E,2})$ be a hermitian pair (definition 3.11), let $I(h_2, h_1)$ be the locally bounded map induced by the identity map of $E$ (example 3.13). Let $U$ be a relatively compact open subset of $X$, and let $\pi_i : L^2(X, \wedge^0, T^*X \otimes E; h_i) \to L^2(U, \wedge^0, T^*X \otimes E|U; h_i)$ be the projection and let $i_\nu$ be its adjoint. Then $\pi_1\chi_{D_{E,1}}I(h_1, h_2)U = I(h_1, h_2)U \pi_2\chi_{D_{E,2}}U$ in the Paschke category $(\mathcal{D} \mathcal{E})_{C_0(U)}$.

**Definition 3.18.** Let $X$ be a differentiable manifold, and let $E_1, E_2$ be differentiable vector bundles on $X$. Let $\alpha : E_1 \to E_2$ be a smooth bundle map. Choose metrics $g, h_1, h_2$ on $X, E_1, E_2$ respectively. We say $\alpha$ preserves the metrics, if the dual map of bundles $\beta : E_2^* \to E_1^*$ on the dual vector bundles (defined by $\beta(e_2^*)(e_1) = e_2^*(\alpha(e_1))$) makes the diagram below commute.

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\alpha} & E_2 \\
\downarrow{h_1} & & \downarrow{h_2} \\
E_1^* & \leftarrow & E_2^*.
\end{array}
$$

**Lemma 3.19.** Let $(X, g; E_1, h_1; E_2, h_2)$ be a metric pair (definition 3.11), and let $\alpha : E_1 \to E_2$ be a smooth isomorphism of vector bundles that preserves the metrics (definition 3.13). Let $D \in \text{Diff}_1(E_1, E_1)$ be an essentially self-adjoint differential operator of order one. Then $\chi(D) = \alpha^{-1}\chi(\alpha D\alpha^{-1})\alpha$.

**Proof.** Since $\alpha$ preserves the metric on each fiber, then the induced map $\alpha : L^2(X, E_1) \to L^2(X, E_2)$ is a unitary map, i.e. $\alpha^{-1} = \alpha^*$. Since $D$ is symmetric, then $\alpha D\alpha^{-1} = \alpha D\alpha^*$ is also symmetric.
Also if \( x \in \text{Domain}(\alpha D^* \alpha^{-1}) \subset L^2(X, E_2) \), then there exists a constant \( M \) so that for each \( y \in \text{Domain}(\alpha D a^{-1}) \subset L^2(X, E_2) \), we have \( |\langle x, \alpha D^* \alpha^{-1} y \rangle| \leq M \| y \| \). But if \( x' = \alpha^{-1} x, y' = \alpha^{-1} y \in L^2(X, E_1) \), then this is equivalent to saying that \( |\langle x', D^* y' \rangle| \leq M \| y' \| \), i.e. \( x' \in \text{Domain}(D^*) \). Since \( D \) is essentially self-adjoint, then \( x' \in \text{Domain}(D) \), hence \( x \in \text{Domain}(\alpha D a^{-1}) \), i.e. \( \alpha D a^{-1} \) is also essentially self-adjoint. Hence \( \chi(\alpha D a^{-1}) \in \mathfrak{B}(L^2(X, E_2)) \) is well-defined.

Assume that the Fourier of the bounded Borel function \( \phi \) is compactly supported, then for small values of \( s > 0 \), we have \( e^{\sqrt{-is} \alpha D a^{-1}} = \alpha e^{\sqrt{-is}D^* a^{-1}} \). Hence by [HR00] 10.3.5 it is easy to argue that \( \phi(\alpha D a^{-1}) = \alpha \phi(D) \alpha^{-1} \). Now if \( \phi \) is a normalizing function, then \( \phi(D) - \chi(D), \phi(D) \alpha^{-1} - \chi(\alpha D a^{-1}) \) are locally compact.

\[ \square \]

**Lemma 3.20.** Let \( (X, g; E_1, h_1, D_{E_1}; E_2, h_2, D_{E_2}) \) be a hermitian pair (definition 3.11), and let \( \alpha : E_1 \to E_2 \) be a smooth isomorphism of vector bundles that preserves the metrics. Then \( \alpha D_{E_1} \alpha^{-1} - D_{E_2} \) is locally bounded (definition 3.12).

**Proof.** Recall from definition 3.6 that the metrics induce conjugate linear smooth bundle isomorphisms \( h_i : E_i \to E_i^* \) to the dual bundle, for \( i = 1, 2 \), and \( h_i^* : E_i^* \to E_i \) is the inverse. Let \( \beta : E_2 \to E_1^* \) be the bundle dual to \( \alpha \). Since \( \alpha \) is a smooth isomorphism of vector bundles, then \( \alpha D_{E_1} - \partial_{E_2} \alpha \) is locally bounded. Therefore by [18] to prove the lemma it suffices to show that the term below is locally compact, where \( f \otimes e_2 \) is a smooth section of \( \wedge^0 T^*X \otimes E_2 \), and \( e_1 = \alpha^{-1} e_2, e_2 = h_2(e_2) \)

\[
(\alpha^* E_1 \tilde{\partial}^* E_1 \alpha^{-1} - \tilde{\partial}^* E_2)(f \otimes e_2) = \alpha^* E_1 \tilde{\partial}(\tilde{\partial}(h_1(e_1)) \otimes h_2(e_2)) - \tilde{\partial}^* E_2 \tilde{\partial}(h_2(e_2))
\]

But \( \beta \) is also a smooth isomorphism of vector bundles hence \( \tilde{\partial} \beta - \beta \tilde{\partial} \) is locally bounded. \( \square \)

**Corollary 3.21.** Let \( 0 \to E_1 \overset{\varphi_1}{\to} E_2 \overset{\varphi_2}{\to} E_3 \to 0 \) be a short exact sequence of holomorphic vector bundles on the complex manifold \( X \). Choose a hermitian metric \( g \) on \( X \), and \( h_1 \) on \( E_1 \). Then we get an induced hermitian metric on the subbundle \( \varphi_1(E_1) \) of \( E_2 \). Extend this metric to hermitian metric \( h_2 \) on all of \( E_2 \). Then there exists a smooth map of bundles \( \sigma_2 : E_3 \to E_2 \) which is an isomorphism from \( E_3 \) to the orthogonal complement of \( \varphi_1(E_1) \) in \( E_2 \). Let \( \sigma_3 \) be the hermitian metric induced by this isomorphism.

Hence \( (X, g; E_1 \oplus E_3, h_1 \oplus h_3, D_{E_1} \oplus D_{E_3}; E_2, h_2, D_{E_2}) \) is a hermitian pair (definition 3.11), and we have a smooth isomorphism \( \varphi_2 \): \( E_1 \oplus E_3 \to E_2 \). By definition of the metrics, it is easy to check that this isomorphism preserves metrics. Therefore as a corollary of 3.20

\[
D_{E_1} \oplus D_{E_3} - (\varphi_1, \sigma_2)^{-1} D_{E_2}(\varphi_1, \sigma_2)
\]

is locally bounded.

**Corollary 3.22.** Let \( (X, g; E_1, h_1, D_{E_1}; E_2, h_2, D_{E_2}) \) be a hermitian pair (definition 3.11), and let \( \alpha : E_1 \to E_2 \) be a smooth isomorphism of vector bundles on \( X \). Let \( U \) be a relatively compact open subset of \( X \), let \( \pi_i : L^2(X, \wedge^0 T^*X \otimes E_i) \to L^2(U, \wedge^0 T^*X \otimes E_i|U) \) be the projection and let \( \iota_i \) be its adjoint. Then

\[
\alpha_U \pi_1 \chi(D_{E_i}) \iota_1 = \pi_2 \chi(D_{E_2}) \iota_2 \alpha_U
\]

in the Paschke category \( \mathfrak{D}(\mathcal{E}_{C_0(U)}) \), where by abuse of notation, we are denoting the map induced by \( \alpha_U \) from \( L^2(U, \wedge^0 T^*X \otimes E_1|U) \to L^2(U, \wedge^0 T^*X \otimes E_2|U) \) by \( \alpha_U \) as well.
Proof. Consider the hermitian metric $h'_2$ on $E_2$ (defined through the diagram in definition 3.18) so that the bundle isomorphism $\alpha : E_1 \rightarrow E_2$ preserves the metrics. Let $D'_{E_2} = \partial_{E_2} + \bar{\partial}'_{E_2}$ be the Dolbeault operator with respect to $h'_2$, let $\pi'_2 : L^2(X, \wedge^{0,*} T^* X \otimes E_2; h'_2) \rightarrow L^2(U, \wedge^{0,*} T^* X \otimes E_2|U; h'_2)$ be the projection, and let $\iota'_2$ be its adjoint. Denote the map induced by $\alpha$ from $L^2(U, \wedge^{0,*} T^* X \otimes E_1|U)$ to $L^2(U, \wedge^{0,*} T^* X \otimes E_2|U; h'_2)$ by $\alpha'_2$, and let $I(h_2, h'_2)$ denote the locally bounded map induced by the identity of $E_2$ (example 3.14). Therefore $\alpha_U = I(h_2, h'_2)_{U} \alpha'_U$ and:

$$
\alpha_U \pi_1 \chi(D_{E_1})_{i_1} = I(h_2, h'_2)_{U} \alpha'_U \pi_1 \chi(D_{E_1})_{i_1} \\
= I(h_2, h'_2)_{U} \pi'_2 \chi(D_{E_1})_{i_1} \\
= I(h_2, h'_2)_{U} \pi'_2 \chi(D_{E_1})_{i_1} \\
= I(h_2, h'_2)_{U} \pi'_2 \chi(D_{E_1})_{i_1} \\
= \pi_2 \chi(D_{E_2})_{i_2} I(h_2, h'_2)_{U} \alpha_U \\
= \pi_2 \chi(D_{E_2})_{i_2} \alpha_U.
$$

By lemma 3.19 and by the identity of $\iota_2$.

Remark 3.23. One may wonder if we can change the metric $g$ on $X$ in the corollary above as well. Consider the case where $E_1 = E_2$ is the trivial bundle of rank one, $\alpha$ is the identity map, and $h_1 = h_2$. When $g_1, g_2$ are two different hermitian metrics on $X$, the symbols of $\partial + \bar{\partial} g_1, \partial + \bar{\partial} g_2$ are not equal to each other, and there is no indication on why after applying functional calculus, we should get the same operator in the Paschke category. However for a relatively compact open subset $U$, the operator induced by identity $I(g_2, g_1)_U : L^2(U, E_1|U; g_1, h_1) \rightarrow L^2(U, E_2|U; g_2, h_2)$ is the identity on the underlying vector spaces (although these Hilbert spaces are different as they have different inner products.), hence $I(g_2, g_1)$ should not induce a map between the chain complexes $\hat{\tau}^D_{X,g_1}(E_1,h_1), \hat{\tau}^D_{X,g_2}(E_2,h_2)$.

Proof of proposition 3.30. We have already defined $\hat{\tau}^D_{X,g}$ on the objects of the category, and showed that $\hat{\tau}^D_{X,g}(E,h)$ is an exact sequence in the Paschke category $(\mathcal{O}/\mathcal{C})_{G_0}(X)$. We need to show functoriality and exactness. Before going further, let us fix some notation.

Let $\varphi : (E_1, h_1) \rightarrow (E_2, h_2)$ be a morphism in $\mathcal{P}_{m, b}(X)$. Choose a good cover (definition 3.6) $\{U_j\}_j$ so that for $i = 1, 2$ and for each $j$, there exists an open subset $V_j$ of $X$ that contains closure of $U_j$, and that $E_i|V_j$ and $\wedge^{0,*} T^* X|V_j$ is isomorphic to the trivial bundle on $V_j$. In other words, there exists holomorphic isomorphisms of bundles $\alpha_j : \wedge^{0,*} T^* X \otimes E_i|V_j \rightarrow V_j \times \mathbb{C}^k$ and $\beta_j : \wedge^{0,*} T^* X \otimes E_2|V_j \rightarrow V_j \times \mathbb{C}^m$ where $k, m$ are ranks of the corresponding bundles. Then $\psi_j = \beta_j \varphi|V_j \alpha_j^{-1} : V_j \rightarrow M_{m,k}(\mathbb{C})$ is a holomorphic matrix valued function. Let $D_{i,j} = \gamma_j D_{E_i} \gamma_j$ for $i = 1, 2$. Let $\{\lambda_j\}_j$ be a partition of unity subordinate to the cover $\{U_j\}_j$, and let $\gamma_j$ be smooth cutoff functions which are equal to one on $U_j$. Also, let $\pi_{i,j} : L^2(X, \wedge^{0,*} T^* X \otimes E_i) \rightarrow L^2(U_j, \wedge^{0,*} T^* X \otimes E_i|U_j)$ be the projection and let $\iota_{i,j}$ be its adjoint. For $n \in \mathbb{Z}_{>0}$, let $D^n$ denote the Dolbeault operator corresponding to the trivial rank $n$ bundle, let $\pi^n_j : L^2(X, X \times \mathbb{C}^n) \rightarrow L^2(U_j, U_j \times \mathbb{C}^n)$ be the projection, and let $\iota^n_j$ be its adjoint. Then by corollary 3.22 for relatively compact subset $U_j$ of $V_j$, we get that

$$
\alpha_{j,U} \pi_{1,j} \chi(D_{E_1})_{i_{1,j}} = \pi^{k}_j \chi(D^k_j \iota^k_j \alpha_{j,U}) \\
\beta_{j,U} \pi_{2,j} \chi(D_{E_2})_{i_{2,j}} = \pi^{m}_j \chi(D^m_j \iota^m_j \beta_{j,U})
$$

26
in the Paschke category \((\mathfrak{D}/\mathfrak{C})_{C_0(U_j)}\). Now, let \(f \in C_0(X)\) be compactly supported. Then there are only finitely many of the \(V_j\)'s that intersect support of \(f\), i.e. the sums below are all finite.

\[
(\varphi \chi_{D_{E_1}} - \chi_{D_{E_2}} \varphi) \rho(f) = \left( \sum_j \chi_j^{1/2} \varphi \chi(D_{E_1}) \chi_j^{1/2} \right) - \sum_j \chi_j^{1/2} \chi(D_{E_2}) \varphi \chi_j^{1/2} \rho(f) = \left( \sum_j \chi_j^{1/2} \varphi \chi(D_{E_1}) \chi_j^{1/2} \right) - \sum_j \chi_j^{1/2} \chi(D_{E_2}) \varphi \chi_j^{1/2} \rho(f) = \left( \sum_j \chi_j^{1/2} \varphi \chi(D_{E_1}) \chi_j^{1/2} \right) - \sum_j \chi_j^{1/2} \chi(D_{E_2}) \varphi \chi_j^{1/2} \rho(f) = \left( \sum_j \chi_j^{1/2} \varphi \chi(D_{E_1}) \chi_j^{1/2} \right) - \sum_j \chi_j^{1/2} \chi(D_{E_2}) \varphi \chi_j^{1/2} \rho(f) = \left( \sum_j \chi_j^{1/2} \varphi \chi(D_{E_1}) \chi_j^{1/2} \right) - \sum_j \chi_j^{1/2} \chi(D_{E_2}) \varphi \chi_j^{1/2} \rho(f)
\]

Where the last equality holds because, in the first sum \(\chi_j^{1/2}\) is pseudo-local, and hence up to compact operators, commutes with multiplication by the matrix valued continuous function \(\chi_j^{1/2}\) that vanishes at infinity, therefore \((\lambda_j^{1/2} \psi_j) \chi_{D_{E_1}} - \chi_{D_{E_2}} (\lambda_j^{1/2} \psi_j)\) is compact for each \(j\); and in the second sum \(\chi_j^{1/2} \chi_{D_{E_2}} - \chi_{D_{E_2}} \lambda_j^{1/2}\) is also compact for each \(j\), and both sums are finite.

We conclude the first part of the proof by noting that

\[
\psi_j^{\alpha_j} \chi_{D_{E_1}} \lambda_j = \psi_j^{\alpha_j} \chi(D_{E_1}) \lambda_j,
\]

hence each term in the sum above is zero, and therefore \(\varphi\) induces a map from \(\hat{\tau}_X^{D}(E_1, h_1)\) to \(\hat{\tau}_X^{D}(E_2, h_2)\) in the category \(\mathcal{CH}'(\mathfrak{D}/\mathfrak{C})_{C_0(X)}\).

It is straightforward to check that \(\hat{\tau}_X^{D}(\varphi_1 \circ \varphi_2) = \hat{\tau}_X^{D}(\varphi_1) \circ \hat{\tau}_X^{D}(\varphi_2)\). This shows that \(\hat{\tau}_X^{D}\) is a functor.

**Remark 3.24.** Note that the condition on \(\varphi: L^2(X, \wedge^0, T^*X \otimes E_1) \rightarrow L^2(X, \wedge^0, T^*X \otimes E_2)\) being bounded is not used in the proof of why \(\hat{\tau}_X^{D}\) is functorial. Also holomorphicity of \(\varphi\) was not needed in the argument above, we only needed continuity to show that multiplication by \(\lambda_j \psi_j\) commutes with \(\chi(D)\) modulo compact operators.

Now, to prove that \(\hat{\tau}_X^{D}\) is an exact functor, let

\[
0 \rightarrow (E_1, h_1) \xrightarrow{\varphi_1} (E_2, h_2) \xrightarrow{\varphi_2} (E_3, h_3) \rightarrow 0
\]

be an exact sequence in \(\mathcal{P}_{m,b}(X)\). Then by definition of exactness in this category, there exists smooth sections \(\sigma_2: E_3 \rightarrow E_2, \sigma_1: E_2 \rightarrow E_1\) so that \(\sigma_1 \varphi_1 = Id_{E_1}, \varphi_2 \sigma_2 = Id_{E_3}\), and similar to \(\varphi_i\), \(\sigma_i\) also induce a bounded map of Hilbert spaces \(L^2(X, \wedge^0, T^*X \otimes E_{i+1}) \rightarrow L^2(X, \wedge^0, T^*X \otimes E_i)\), for \(i = 1, 2\).

Let \(h_2^*\) be the hermitian metric on \(E_2\) induced by \(h_1, h_3\), i.e.

\[
h_2^* = \sigma_1^* h_1 \sigma_1 + \varphi_2^* h_3 \varphi_2: E_2 \rightarrow E_2^*
\]

where in here, \(\sigma_1^*: E_1^* \rightarrow E_2^*\) and \(\varphi_2^*: E_3^* \rightarrow E_2^*\) are the dual maps to \(\sigma_1, \varphi_2\) respectively. Then the subbundles \(\varphi_1(E_1), \sigma_2(E_3)\) of \(E_2\) are orthogonal with respect to \(h_2^*\), and the induced metrics on these subbundles match with the metrics \(h_1, h_3\) respectively, i.e. the isomorphism between \(E_1 \oplus E_3\) and \(E_2\) preserves the metric (definition 3.18). Hence by corollary 3.21 \((\sigma_1, \varphi_2) D_{E_2}^*(\varphi_1, \sigma_2) - D_{E_1} \oplus D_{E_3}\)
is locally bounded, where $D'_E$ is the Dolbeault operator on $E_2$ with respect to the metric $h'_2$. Therefore by lemma 3.19 we get that

$$\chi_{D_{E_1}} \oplus \chi_{D_{E_3}} = \chi_{D_{E_1} \oplus D_{E_3}} = (\varphi_1, \sigma_2)\chi_{(\sigma_1, \varphi_2)D_{E_2}'(\varphi_1, \varphi_2)}(\sigma_1, \varphi_2).$$

By corollary 3.17 for any relatively compact open subset $U$ of $X$, we have $I(h'_2, h_2)_U \pi_2 \chi_{D_{E_2}'} t_2 = \pi'_2 \chi_{D_{E_2}'} t'_2 I(h'_2, h_2)_U$, where $\pi_2, \pi'_2$ are the projections $L^2(X, \Lambda^0, * T^* X \otimes E_2) \to L^2(U, \Lambda^0, * T^* X \otimes E_2|_U)$ with respect to the metrics $h_2, h'_2$ and $t_2, t'_2$ are their adjoints, respectively. Also $I(h'_2, h_2)_U : L^2(U, \Lambda^0, * T^* X \otimes E_2|_U; h'_2) \to L^2(U, \Lambda^0, * T^* X \otimes E_2|_U; h_2)$ is the map induced by $Id_{E_2}$ (example 3.14). This factors through

$$L^2(U, \Lambda^0, * T^* X \otimes E_2|_U; \overline{h_2}(\varphi_1, \varphi_2)) \overset{(\varphi_1, \varphi_2)}{\longrightarrow} L^2(U, \Lambda^0, * T^* X \otimes E_1|_U) \oplus L^2(U, \Lambda^0, * T^* X \otimes E_3|_U) \overset{(\varphi_1, \varphi_2)}{\longrightarrow} L^2(U, \Lambda^0, * T^* X \otimes E_2|_U; \overline{h_2})$$

Because $(\phi_1, \sigma_1)$ has norm one, and $(\sigma_1, \varphi_2)$ has a bounded norm (independent of $U$), then norm of $I(h'_2, h_2)_U$ is also independent of $U$, therefore we can glue all the data to obtain $I(h'_2, h_2)_U \chi_{D_{E_2}'} = \chi_{D_{E_2}'} I(h'_2, h_2)$. This proves that $\hat{\tau}_X^D$ is exact.

\[ \square \]

### 3.3 Restriction to Open Subsets

**Lemma 3.25.** Let $X$ be a complex manifold, and let $U$ be an open subset. Then the diagram below commutes up to homotopy.

\[
\begin{array}{ccc}
K(P_{b,d}(X, g)) & \overset{X \hat{\delta}^D_{U,g}}{\longrightarrow} & K(Ch'(\mathcal{D}/\mathcal{C})_{C_0(X)}) \\
\downarrow res & & \downarrow res \\
K(P_{b,d}(U, g)) & \overset{X \hat{\delta}^D_{U,g}}{\longrightarrow} & K(Ch'(\mathcal{D}/\mathcal{C})_{C_0(U)})
\end{array}
\]

**Proof.** Let $(E, h)$ be an object of $P_{b,d}(X)$. It suffices to show that in the diagram below (which is not commutative on the nose), $res^X_{U,g}\hat{\tau}_{X,g}^D(E, h)$ is naturally isomorphic to $\hat{\tau}_{U,g}^D res^X_U(E, h)$.

\[
\begin{array}{ccc}
P_{b,d}(X, g) & \overset{X \hat{\delta}^D_{U,g}}{\longrightarrow} & Ch'(\mathcal{D}/\mathcal{C})_{C_0(X)} \\
\downarrow res^X & & \downarrow res^X \\
P_{b,d}(U, g) & \overset{X \hat{\delta}^D_{U,g}}{\longrightarrow} & Ch'(\mathcal{D}/\mathcal{C})_{C_0(U)}
\end{array}
\]

Denote the restriction map $L^2(X, \Lambda^0, * T^* X \otimes E) \to L^2(U, \Lambda^0, * T^* X \otimes E|_U)$ given by multiplying with the characteristic function of $U$ by $\pi$.

Let $u$ be compactly supported section of $L^2(U, \Lambda^0, * T^* X \otimes E|_U)$. Then by [HHR00, 10.3.1.] there exists $\epsilon > 0$ so that for $|s| < \epsilon$, $e^{-\epsilon \tau_{D,E}} u = e^{-\epsilon \tau_{D,E}} \pi^* u$ are supported on $U$. Let $\phi$ be a normalizing function so that its Fourier transform is supported in the interval $[-\epsilon, \epsilon]$. Then by [HHR00, 10.3.5.] we get that $\phi(D_E^i) u = \pi \phi(D_E) \pi^* u$. Since $\phi - \chi \in C_0(\mathbb{R})$, then by lemma 3.2 $\chi(D_E^i) = \pi \chi(D_E) \pi^*$ in the Paschke category $(\mathcal{D}/\mathcal{C})_{C_0(U)}$. Therefore $\pi$ is a chain map from $res^X_{U,g}\hat{\tau}_{X,g}^D(E, h)$ to $\hat{\tau}_{U,g}^D res^X_U(E, h)$.

Since $\pi \pi^* = Id$ and $\pi^* \pi - Id$ is characteristic function of $X \setminus U$ which is locally compact in $(\mathcal{D}/\mathcal{C})_{C_0(U)}$, then $\pi$ induces an isomorphism.

Therefore there is a natural transformation from $res^X_{U,g}\hat{\tau}_{X,g}^D$ to $\hat{\tau}_{U,g}^D res^X_U$, meaning these two functors induce homotopic maps of K-theory spectra. \[ \square \]
Proposition 3.26. Let $X$ be a complex manifold. Then for each relatively compact open subset $V$ of $X$ there exists an exact functor $\tau^D_V$ that makes the square below commute up to homotopy. Furthermore, these functors are compatible with further restriction to open subsets, i.e. for an open subset $W$ of $V$, the triangle on the bottom of the diagram commutes up to homotopy as well.

$$
\begin{array}{ccc}
K(\mathcal{P}_b,d(X,g)) & \xrightarrow{\tau^D_{V,g}} & K(Ch^b(\mathfrak{D}/\mathfrak{C})_{C_0(X)}) \\
\downarrow & & \downarrow res^X_U \\
K(\mathcal{P}(X)) & \xrightarrow{\tau^D_{W,g}} & K(Ch^b(\mathfrak{D}/\mathfrak{C})_{C_0(V)}) \\
\downarrow & & \downarrow res^V_U \\
K(Ch^b(\mathfrak{D}/\mathfrak{C})_{C_0(W)}) & & \\
\end{array}
$$

(7)

Proof. For each object $E$ of $\mathcal{P}(X)$, choose a hermitian metric $h(E)$ [19]. Then define $\tau^D_{V,h}(E) = res^X_V \tau^D_{X,g}(E,h(E))$. Also, for a morphism of bundles $\varphi : E_1 \to E_2$, define $\tau^D_{V,h}(\varphi)$ through the composition below, where the first map is given by projection, and the last one is given by extension by zero.

$$
L^2(X, \wedge^{0,*} T^* X \otimes E_1) \to L^2(V, \wedge^{0,*} T^* X \otimes E_1|_V) \xrightarrow{\tau^D_{V,h}(\varphi|_V)} L^2(V, \wedge^{0,*} T^* X \otimes E_2|_V) \to L^2(X, \wedge^{0,*} T^* X \otimes E_2)
$$

Note that $\tau^D_{X,g}(\varphi)$ is not necessarily defined, as $\varphi$ could induce an unbounded map of Hilbert spaces, however by restricting to the relatively compact open subset $V$, the composition above is indeed a well-defined map.

Since $\tau^D_{X,g}$ is a functor, then $\tau^D_{V,h}$ is also a functor, i.e. for composable maps of bundles $\varphi_1, \varphi_2$, we have $\tau^D_{V,h}(\varphi_2 \circ \varphi_1) = \tau^D_{V,h}(\varphi_2) \circ \tau^D_{V,h}(\varphi_1)$. Exactness of $\tau^D_{V,h}$ also follows from that of $\tau^D_{X,g}$. Hence we have an induced map of spectra

$$
\tau^D_{V,h} : K(\mathcal{P}(X)) \to K(Ch^b(\mathfrak{D}/\mathfrak{C})_{C_0(V)}).
$$

(8)

The square in the diagram [7] commutes (up to homotopy) because for any object $(E_1, h_1)$ of $\mathcal{P}_b,d(X,g)$, by corollary 3.17 the identity map of $E$ induces an isomorphism from $res^X_V \tau^D_{X,g}(E_1, h_1)$ to $res^X_V \tau^D_{X,g}(E_1, h(E_1)) = \tau^D_{V,h}(E)$. Also for a morphism $\varphi : (E_1, h_2) \to (E_2, h_2)$ in $\mathcal{P}_b,d(X,g)$, the difference $res^X_V \tau^D_{X,g}(\varphi) - \tau^D_{V,h}(\varphi)$ is locally compact, because multiplying by characteristic function of $X \setminus V$ is locally compact in $(\mathfrak{D}/\mathfrak{C})_{C_0(V)}$.

The functor defined $\tau^D_{V,h}$ commutes (up to homotopy) with restriction to further open subset $W \subset V$ because multiplying by characteristic function of $V \setminus W$ is locally compact in $(\mathfrak{D}/\mathfrak{C})_{C_0(W)}$. Therefore the triangle in the diagram [7] commutes as well.

Note that the choices of metrics $h(E)$ on $E$ do not affect the map [8] up to homotopy, because again by corollary 3.17, for any two choices $h_1, h_2$, the objects $\tau^D_{V,h_1}(E), \tau^D_{V,h_2}(E)$ are naturally isomorphic, hence all the different functors are homotopic.

\[\square\]

Corollary 3.27. Let $X$ be a complex manifold. Then the functor $\tau^D$ defined in proposition 3.26 commutes with restriction to open subsets, i.e. for open subset $U$ of $X$ and relatively compact open subset $V$ of $X$ and open subset $W$ of $U \cap V$ which is relatively compact as an open subset of $U$, the diagram below commutes up to homotopy.

\[\text{Note that we are assuming the axiom of choice. Also, we are only working over a small skeletal subcategory of } \mathcal{P}(X).\]
\[
K(\mathcal{P}(X)) \xrightarrow{\tau_0^D} K(\text{Ch}'(\mathfrak{D}/\mathfrak{E})_{C_0(V)}) \\
\downarrow \text{res}_Y \quad \downarrow \text{res}_W \\
K(\mathcal{P}(U)) \xrightarrow{\tau_0^W} K(\text{Ch}'(\mathfrak{D}/\mathfrak{E})_{C_0(W)})
\]

Proof. Consider the diagram below, where all the arrows with no labels are the natural ones.

The squares on the left and the one on the right commute because restriction maps (and the forgetful functors \( \mathcal{P}_{m,b} \to \mathcal{P} \)) are natural. By lemma 3.25 the square on the top commutes up to homotopy. By proposition 3.26 the squares in the back and on the front commute up to homotopy as well. This proves that the square on the bottom commutes up to homotopy.

Remark 3.28. All the results in this subsection hold whether we use \( K_{\text{alg}} \) or \( K_{\text{top}} \).

4 Main Results

4.1 K-theory of the Paschke category

In this subsection, we will compute the K-theory groups of the Paschke category and the Calkin-Paschke category.

Let \( A \) be a \( \mathcal{C}^* \)-algebra. Let \( K_{\text{top}}(A) \) denote the topological K-homology groups of \( A \), which are contravariant functors of the \( \mathcal{C}^* \)-algebra, and let \( K_{\text{top}}'(A) \) denote the topological K-theory groups of \( A \), which are covariant functors. The reason for the unusual naming is that we are primarily interested in the case when \( A \) is the \( \mathcal{C}^* \)-algebra \( C_0(X) \) of continuous complex valued functions on the (locally compact and Hausdorff) topological space \( X \) which vanish at infinity, and in this case functoriality matches the expectations.

Let \( A \) be a topological exact category, and recall that \( K_{\text{top}}(A) \) denotes the K-theory spectrum of \( A \) with respect to the fat geometric realization. Since the additivity theorem holds for K-theory of topological categories, then this is a connective spectrum, i.e. there are no negative K-theory groups.

Here is the main result of this subsection.

Theorem 4.1. The \((1-i)\)’th topological K-homology group \( K_{\text{top}}^{1-i}(A) \) of a \( \mathcal{C}^* \)-algebra \( A \), is isomorphic to the \( i \)’th topological K-theory groups of the exact \( \mathcal{C}^* \)-categories \( (\mathfrak{D}/\mathfrak{E})_A \) and \( (\mathfrak{D}/\mathfrak{E})'_A \) for \( i \geq 1 \). If \( A \) is unital and nuclear, then \( K_{\text{top}}^{1-i}(A) = K_{\text{top}}^{1-i}(\mathfrak{D}/\mathfrak{E})'_A \) as well. For a \( * \)-morphism \( f : A \to B \), this isomorphism commutes with respect to the pull-back maps \( f^* \).
In particular if $A = C_0(X)$ for a locally compact Hausdorff topological space $X$, then the topological $K$-homology groups $K^\text{top}_{i-1}(X)$ are isomorphic to $K_i((\mathcal{D}/\mathcal{C})_0)$ for $i \geq 0$. This isomorphism also commutes with the restriction maps to open subsets.

Proof. Recall that the $K_0$ group of a Waldhausen category is the free abelian group generated by the weak equivalence classes of objects of the category, modulo the relations induced by the cofibration sequences. The same is true for topological Waldhausen categories (cf. [Wei 4.8.4.]). In particular since all the non-zero objects of the category $\mathcal{Q}_A$ are isomorphic to each other by 1.24 hence $K_0(\mathcal{Q}_A) = 0$.

In the case when $A$ is a unital $C^*$-algebra, recall that $Ext(A)$ [BDF77] is defined as the semi-group of unitary equivalence classes of unital injective representations of $A$ to the Calkin algebra $(\mathcal{B}/\mathcal{K})(H)$ (cf. [HR00 2.7.1]). When $A$ is nuclear, then as a corollary of Voiculescu’s theorem [20] we know that $Ext(A)$ is in fact a group, and isomorphic to the first $K$-homology group $K^\text{top}_1(A)$.

Lemma 4.2. Let $A$ be a unital and nuclear $C^*$-algebra. Then $K_0(\mathcal{Q}_A) = K^\text{top}_1(A)$.

Proof. Let $\rho_i^1 : A \rightarrow (\mathcal{B}/\mathcal{K})(H_i)$ be non-zero objects in $\mathcal{Q}_A$ for $i = 1, 2$, which are isomorphic, i.e. there exists isomorphisms $T : \rho_1^1 \rightarrow \rho_2^2$ and $S : \rho_2^2 \rightarrow \rho_1^1$ which are inverses to each other. By definition of a $C^*$-category, for a positive operator $T^*T \in \mathcal{Q}_A(\rho_1^1)$, there exists an operator $F \in \mathcal{Q}_A(\rho_1^1)$ so that $F^*F = T^*T$. Since $S,T$ are invertible, then so are $F$ and $FS : \rho_2^2 \rightarrow \rho_1^1$. We have $(FS)^*(FS) = S^*F^*FS = S^*T^*TS = Id$ and $((FS)(FS)^*)F = (FSS^*F^*)F = FSS^*T^*T = FST = F$ in the category $\mathcal{Q}_A$. Hence $FS$ is a unitary isomorphism in this category. Choose representatives for $S,F$ in the category $\mathcal{Q}_A$. Because $\rho_1^1,\rho_2^2$ are unital, then it means $FS$ is also a Fredholm operator, and in particular has closed image and finite dimensional kernel and cokernel. Hence there exists closed subspaces $H'_i \subset H_i$ of finite codimension so that $\pi_1 H_i = H_i'$ is a isomorphism of Hilbert spaces, where $\pi_i : H_i \rightarrow H_i'$ is the inclusion and $\pi_1 : H_i \rightarrow H_i'$ is the projection for $i = 1, 2$. Since $\ker(FS) = \ker(S^*F^*)$, $\ker(S^*F^*) = \ker(FS)$ then $\pi_1 H_i$ is a unitary map of Hilbert spaces. Let $\nu_i^1 = \pi_i \rho_i : A \rightarrow (\mathcal{B}/\mathcal{K})(H_i)$ for $i = 1, 2$. Then we just showed that $\nu_i^1,\nu_i^2$ are unitarily equivalent. Since the difference between $\rho_i^1,\nu_i^1$ is a finite dimensional Hilbert space, then they represent the same class in $Ext(A)$ [21]. This shows that $Ext(A) \rightarrow K_0(\mathcal{Q}_A)$ is injective. Surjectivity follows from the definition.

By propositions 2.14 and 1.22 the maps of spectra induced by inclusion of subcategories $K(\mathcal{Q}_A) \rightarrow K((\mathcal{D}/\mathcal{C})_A)$ and $K(\mathcal{Q}_A') \rightarrow K((\mathcal{D}/\mathcal{C})'_A)$ are both homotopy equivalences. By remark 1.25 and proposition 2.14 the map $K((\mathcal{D}/\mathcal{C})_A) \rightarrow K((\mathcal{D}/\mathcal{C})'_A)$ induces an isomorphism on the $i$'th K-groups for $i \geq 1$.

Let $A$ be a $C^*$-algebra, let $\rho$ be an ample representation of $A$, and let $\mathcal{R}$ be a full subcategory of $\mathcal{Q}_A$ with two objects: the zero representation $A \rightarrow 0$ and $\rho$. Then this is a $C^*$-category and also a skeleton for the category $\mathcal{Q}_A$, as all ample representaions are isomorphic to each other. But since every short exact sequence in $\mathcal{Q}_A$ splits, then by result of Mitchell [Mit01], $\Omega||wS\mathcal{Q}_A||$ is homotopy equivalent to $BGL_\infty(Sk(\mathcal{Q}_A))$, where the latter is defined in [Mit01 6.1] and $Sk(\mathcal{Q}_A)$ denotes the skeleton of the additive category $\mathcal{Q}_A$. Hence the K-theory space of $\mathcal{Q}_A$ is homotopy equivalent to $BGL(\mathcal{R})$, which by definition is homotopy equivalent to $BGL_\infty(\Omega(A))$.

20 Note that for a nuclear $C^*$-algebra $A$ and $C^*$-algebra $B$ with a $C^*$-ideal $K$, a $*$-morphism $A \rightarrow B/K$ lifts to a completely positive map $A \rightarrow B$ (cf. [HR00 3.3.6.]), and Stinespring’s theorem (cf. [HR00 3.1.3.]) shows that each completely positive map to $\mathcal{B}(H)$, can be written as $V^*\rho V$, where $V : H \rightarrow H'$ is an isometry and $\rho$ is a representation to $\mathcal{B}(H')$. Then the restriction of $\rho$ to the orthogonal complement of image of $V$ induces a representation of $A$ to the Calkin algebra (cf. [HR00 3.1.6.]).

21 Let $H'$ denote the orthogonal complement of $H_1'$ in $H_1$, which is finite dimensional. Then the direct sum of the zero representation from $A$ to $(\mathcal{B}/\mathcal{K})(H'_1)$ and $\nu_1$ is equal to $\rho_1$. 

31
Therefore when \( i \geq 1 \) we have the following sequence of isomorphisms of abelian groups, where the first isomorphism is given by Paschke duality \cite{Pas81}, the second isomorphism is one of the equivalent definitions of topological K-theory groups, the third and the fourth one were explained above, and the last one follows from propositions 2.14 and 1.22.

\[
K^i_{\text{top}}(\mathcal{A}) \cong K^i_{\text{top}}(\mathcal{G}(\mathcal{A})) = \pi_i(BGL(\mathcal{G}(\mathcal{A}))) \cong \pi_i(BGL_\infty(\mathcal{R})) \\
\cong K^i_{\text{top}}(\mathcal{O}) \cong K^i_{\text{top}}((\mathcal{D}/\mathcal{E})_\mathcal{A}).
\]

(9)

This means we have proved the first part of the theorem.

Let \( A, B \) be unital \( C^* \)-algebras with ample representations \( \rho_A : A \rightarrow \mathcal{B}(H_A) \) and \( \rho_B : B \rightarrow \mathcal{B}(H_B) \). Let \( \alpha : A \rightarrow B \) be a unital map of \( C^* \)-algebras, then by Voiculescu’s theorem there exists an isometry \( V : H_B \rightarrow H_A \) so that \( V^*\rho_A(a)V - \rho_B(\alpha(a)) \) is compact for all \( a \in A \). Note that \( VV^* \in \mathcal{B}(H_A) \) is a projection which commutes with the representation \( \rho_A \) \cite{HR00, 3.1.6.} modulo compact operators. Also note that \( V : H_B \rightarrow VV^*H_A \) is an isomorphism of Hilbert spaces. Now the function \( Ad_V(T) = VTV^* \) gives a map

\[
Ad_V(\cdot) : (\mathcal{D}/\mathcal{E})_\mathcal{B}(\rho_B) \cong \mathcal{O}(B) \rightarrow \mathcal{O}(A) \cong (\mathcal{D}/\mathcal{E})_\mathcal{A}(\rho_A),
\]

and hence induces a map on the K-homology groups which only depends on \( \alpha \), i.e. does not depend on the choices of ample representations \( \rho_A, \rho_B \) and the isometry \( V \).

Let \( T \in (\mathcal{D}/\mathcal{E})_\mathcal{B}(\rho_B) \) be a unitary element. The pull-back map of K-homology groups sends \( T \) to the unitary

\[
V(T - Id_{H_B})V^* + Id_{H_A} = VTV^* \oplus (Id - VV^*) \in (\mathcal{D}/\mathcal{E})_\mathcal{A}(\rho_A) \cong \mathcal{O}(A).
\]

On the other hand, pulling back in the Paschke category is given by precomposing with the representation. Hence \( T \in (\mathcal{D}/\mathcal{E})_\mathcal{B}(\rho_B) \) is sent to \( T \in (\mathcal{D}/\mathcal{E})_\mathcal{A}(\rho_B \circ \alpha) \). These two procedures give two different maps from the unitaries (or invertible elements) in \( (\mathcal{D}/\mathcal{E})_\mathcal{B}(\rho_B) \) to the topological space \( \Omega^2\|S(\mathcal{D}/\mathcal{E})_\mathcal{A}\| \).

Let \( S = Id_{H_B} \oplus (VTV^* \oplus (Id_{H_A} - VV^*)) \in \mathcal{B}(H_B \oplus H_A) \). Consider the following two "prisms" in \( wS(\mathcal{D}/\mathcal{E})_\mathcal{A} \) where the cofibration and the quotient maps on the left diagram are the trivial ones, but the cofibrations on the right diagram are given by \( (0, V) : H_B \rightarrow H_B \oplus H_A \) and the quotient maps are given by \( V + (Id_{H_A} - VV^*) : H_B \oplus H_A \rightarrow H_A \). By \cite{HR00, 3.1.6.} these maps are pseudo-local. It is also easy to check that both diagrams below commute.

Since fat geometric realization of a point is the infinite dimensional ball which is contractible, then in the fat geometric realization of \( wS(\mathcal{D}/\mathcal{E})_\mathcal{A} \), the side of the prism \( \Delta^1_{\text{top}} \times \Delta^2_{\text{top}} \) corresponding to the identity map is contractible. Hence we get a homotopy from \( S \) to \( VTV^* \oplus (Id - VV^*) \) induced by the left diagram and also from \( S \) to \( T \) induced by the right diagram, by "sliding" one side of the prism towards the other along the contractible side (corresponding to identity map). \footnote{To be more precise, by the additivity theorem we know \( t,s+q : \mathcal{E}(\mathcal{D}/\mathcal{E})_\mathcal{A} \rightarrow (\mathcal{D}/\mathcal{E})_\mathcal{A} \) are homotopic, where \( \mathcal{E} \) is temporarily denoting the category of cofibration sequences, and \( s,t,q \) refer to the first (source), second (target), and the third (quotient) object in the cofibration sequence. By applying this to the prism on the right we get that}
the two different pushforward maps are homotopic to each other and they induce the same map on the level of K-homology groups. \(^{23}\)

In the context of Paschke categories restriction maps are defined similar to pull-back maps, i.e. by precomposing with the representation. To be more precise, let \(X\) be a locally compact and Hausdorff topological space, and let \(U\) be an open subset. The inclusion \(j : U \to X\) induces an inclusion \(j_* : C_0(U) \hookrightarrow C_0(X)\) of \(C^*\)-algebras, given by extending functions by zero. Then the restriction map sends the object \(\rho : C_0(X) \to \mathfrak{B}(H)\) to \(j^* (\rho) := \rho \circ j_* : C_0(U) \to \mathfrak{B}(H)\).

We follow [RS13] to recall the process of defining the (wrong-way) restriction maps on the classical topological K-homology. Let \(X, U, j, \rho : C_0(X) \to \mathfrak{B}(H)\) be as before. If we extend \(\rho\) to the Borel functions on \(X\), then \(\rho(\mathbb{1}_U)\) is a self-adjoint projection, where \(\mathbb{1}_U\) is the characteristic function of the open subset \(U\) of \(X\). Let \(H_U\) be the image of this projection, and define the representation \(\rho_U : C_0(U) \to \mathfrak{B}(H_U)\) by \(\rho_U(f) = \pi_U \rho(j_*(f)) \mathbb{1}_U\) where \(\mathbb{1}_U : H_U \to H\) is the inclusion, \(\pi_U : H \to H_U\) is the projection, and \(j_* : C_0(U) \to C_0(X)\) is extension by zero. The linear map \(\mathfrak{B}(H) \to \mathfrak{B}(H_U)\) defined by \(T \mapsto \pi_U T \mathbb{1}_U\) maps \(\mathfrak{D}_C(X)(\rho)\) to \(\mathfrak{D}_C(U)(\rho_U)\), and \(\mathfrak{C}_C(X)(\rho)\) to \(\mathfrak{C}_C(U)(\rho_U)\). Hence there is an induced map \(\mathfrak{D}_\rho(C_0(X)) \to \mathfrak{D}_\rho(C_0(U))\), which induces the restriction map from the K-homology groups of \(X\) to the K-homology groups of \(U\).

But the representations \(j^* \rho, \rho_U\) are naturally isomorphic; in fact the maps \(\pi_U, \mathbb{1}_U\) induce the isomorphisms. To show this, first note that \(\pi_U, \mathbb{1}_U\) commute with these representations since for \(f \in C_0(U)\), we have \(\rho_U(f) \pi_U = \rho_U(f) \rho(\mathbb{1}_U) = \rho(j_* f) = \pi_U j^* \rho(f)\). Also \(\rho_U(\mathbb{1}_U \mathbb{1}_U - \Id_{H_U})\) and \(j^* \rho(\Id_U \mathbb{1}_U - \Id_{H_U})\) are both zero. Therefore these two restriction functors are homotopic to each other as a maps to the \(\Omega \| wS. (\mathfrak{D}/\mathfrak{C})_{C_0(U)} \|\), which means the two induced restriction maps on K-homology groups are equal to each other.

\[\square\]

### 4.2 Descent and The Riemann-Roch Transformation

**Definition 4.3.** Let \(\mathcal{V}_C\) denote the topological category where the objects are finite dimensional complex vector spaces, and morphisms are invertible linear maps. Let \(A\) be a nuclear \(C^*\)-algebra. Then there is a biexact functor \(\mathcal{V}_C \times (\mathfrak{D}/\mathfrak{C})'_A \to (\mathfrak{D}/\mathfrak{C})'_A\) induced by taking the tensor product of the corresponding Hilbert space with the finite dimensional vector space. This induces a map of spectra

\[\text{ku} \wedge K^{top}((\mathfrak{D}/\mathfrak{C})'_A) \to K^{top}((\mathfrak{D}/\mathfrak{C})'_A)\]  

(10)

where \(\text{ku}\) is the K-theory spectrum \(K^{top}(\mathcal{V}_C)\), also known as the connective complex K-theory spectrum.

Let \(KU\) denote the (non-connective) complex K-theory spectrum.

Since we have only defined the functor \(\tau_{D,g}^{\mathcal{D}}\) on relatively compact open subsets, we will need a descent argument to glue them together. So far we have defined a connective K-homology spectrum for \(C^*\)-algebras. We need a non-connective K-homology spectrum to make the descent work. It is

\[T + \Id\text{ is homotopic to }S, \text{ and by applying it on the prism on the left we get that } (VTV^* \oplus (\Id - VV^*)) + \Id\text{ is homotopic to }S, \text{ but since }\Id\text{ is contractible in the fat geometric realization, then we get that }T\text{ is homotopic to }VTV^* \oplus (\Id - VV^*).\]

\(^{23}\)One may wonder why we did not simply say that \(VTV^* \oplus (\Id - VV^*)\) is direct sum of \(VTV^*\) and \(\Id - VV^*\), and argue similar to above that the identity on the Hilbert space \((\Id - VV^*)H_A\) corresponds to a contractible side of a prism, and then "slide" \(VTV^* \oplus (\Id_{H_A} - VV^*)\) directly onto \(VTV^*\) which is isomorphic to \(T\) in the category \((\mathfrak{D}/\mathfrak{C})_A\), to obtain a homotopy. The reason is that the restriction of \(\rho_A\) to \((\Id - VV^*)H_A\) is only a representation up to compact operators, and the Hilbert space \((1 - VV^*)H_A\) does not come with a representation, which means we can not simply consider \((1 - VV^*)H_A\) as an object in \((\mathfrak{D}/\mathfrak{C})_A\). Note however, we can consider the Hilbert space \(VV^*H_A\) together with the representation \(V(\rho_B\alpha)V^*\).
well known that the process below will give us the non-connective spectrum we need. We will only provide a sketch proof for this lemma.

**Lemma 4.4.** By definition above, we can consider the smash product of spectra $K^\text{top} ((\mathcal{D}/\mathcal{C})'_A) \wedge_{\text{ku}} KU$. This has the same homotopy groups as the non-connective topological K-homology of $A$.

**Sketch Proof:** Recall that for a ring spectrum $X$ and $b \in X$, multiplication by $b$ induces a map $X \to \Sigma^{-n} X$. Then we define $X_\beta [b^{-1}]$ to be the homotopy colimit of the telescope

$$X \xrightarrow{b} \Sigma^{-n} X \xrightarrow{\Sigma^{-n} b} \Sigma^{-2n} X \xrightarrow{\Sigma^{-2n} b} \Sigma^{-3n} X \to \ldots .$$

Also, (stable) homotopy groups commute with this mapping telescope (cf. [EKM07, 5.1.14.]). Let $\beta \in \text{ku}_2$ denote the bottle element. Then it is well-known that $KU$ is naturally homotopy equivalent to $\text{ku}[\beta^{-1}]$ (cf. [Sna81]).

Since $K^\text{top} ((\mathcal{D}/\mathcal{C})'_A)$ is a $\text{ku}$-module, then there is a natural weak equivalence $K^\text{top} ((\mathcal{D}/\mathcal{C})'_A) \wedge_{\text{ku}} \text{ku}[\beta^{-1}] \to K^\text{top} ((\mathcal{D}/\mathcal{C})'_A)[\beta^{-1}]$ (cf. [EKM07, 5.1.15.]). This is easy to see that the homotopy groups of the latter is $2$-periodic, as one could disregard the first $n$-term in the mapping telescope, and that positive homotopy groups of $K^\text{top} ((\mathcal{D}/\mathcal{C})'_A)$ are $2$-periodic. The fact that positive homotopy groups of $K^\text{top} ((\mathcal{D}/\mathcal{C})'_A) \wedge_{\text{ku}} KU$ are isomorphic to the positive homotopy groups of $K^\text{top} ((\mathcal{D}/\mathcal{C})'_A)$ follows from the (strongly converging) Atiyah-Hirzebruch spectral sequence (cf. [EKM07, 4.3.7.]) and the fact that positive homotopy groups of $\text{ku}$ and $KU$ agree with each other. This finishes the proof.

**Proposition 4.5.** Let $A$ be a $C^*$-algebra, and let $I \subset A$ be an ideal, so that the projection $\pi : A \to A/I$ has a completely positive section. Then

$$K^\text{top} ((\mathcal{D}/\mathcal{C})'_A/I) \wedge_{\text{ku}} KU \to K^\text{top} ((\mathcal{D}/\mathcal{C})'_A) \wedge_{\text{ku}} KU \to K^\text{top} ((\mathcal{D}/\mathcal{C})'_I) \wedge_{\text{ku}} KU$$

is a homotopy fiber sequence.

**Proof.** It is easy to observe that the composition of the two maps above is null-homotopic. Hence it suffices to show that the homotopy groups of the sequence above induce a long exact sequence of homotopy groups. This is a direct consequence of the six-term exact sequence of K-homology groups [HHR00 5.3.10.], lemma 4.4 which says that the homotopy groups agree with the K-homology groups and also theorem 9, which says that the pull-back maps of the Paschke category agree with the classical pull-backs.

We will give definition of descent with respect to hypercovers [DH104 4.2.] below. The definition essentially states when a presheaf of spectra is in fact a sheaf (up to homotopy). A hypercover over a site $X$ is a simplicial presheaf $U$. [Jar87 1.] with an augmentation $U \to X$, which satisfies certain conditions. This can be thought of as a generalization of Čech covers, which have the form

$$\ldots \longrightarrow \prod_{j_0,j_1,j_2} (U_{j_0} \cap U_{j_1} \cap U_{j_2}) \longrightarrow \prod_{j_0,j_1} (U_{j_0} \cap U_{j_1}) \longrightarrow \prod_{j_0} U_{j_0} \longrightarrow X$$

In fact, Čech covers are hypercovers of height zero. Also, for our purposes, homotopy limits refer to limits in the homotopy category of spectra.

**Definition 4.6.** [DH104 4.3.] Let $X$ be an object in the site $\mathcal{C}$ (which can be thought of as a topological space). An object-wise fibrant simplicial presheaf $\mathcal{F}$ satisfies descent for a hypercover $U \to X$ if the natural map from $\mathcal{F}(X)$ to the homotopy limit of the diagram

$$\prod_i \mathcal{F}(U^i_0) \longrightarrow \prod_i \mathcal{F}(U^i_1) \longrightarrow \prod_i \mathcal{F}(U^i_2) \ldots$$

(11)
is a weak equivalence. Here the products range over the representable summands of each $U_n$. If $F$ is not object-wise fibrant, we say it satisfies descent if some object-wise fibrant replacement for $F$ does.

The fact that topological K-homology satisfies descent is essentially a result of the Atiyah-Hirzebruch spectral sequence (cf. [Bro73] [AH61]). In fact, by [DI04, 4.3.], for a hypercover $U \rightarrow X$, we have weak equivalences $\text{holim} U \rightarrow |U| \rightarrow X$, where the second map is induced by taking geometric realization. Since taking smash product in the homotopy category of spectra preserves colimits, then by applying the smash product with an $\Omega$-spectrum $E$, the colimit of the diagram \[11\] is weakly equivalent to $E \wedge X$. When the $\Omega$-spectrum $E = KU$ is the (non-connective) topological K-theory spectrum, this proves descent. (Also see [AW14, 2.2.] for the case of twisted topological K-theory of CW-complexes.)

Now we are ready to define the Riemann-Roch transformation over the relatively compact open subsets of a complex manifold, which in turn induce the Riemann-Roch transformation over the manifold itself.

**Definition 4.7.** Let $X$ be a complex manifold, and let $V$ be a relatively compact open subset of $X$. Define the functor $\tau_{X,V}$ in the homotopy category of spectra as the composition below

\[
\begin{align*}
K^{alg}(\mathcal{P}(X)) & \xrightarrow{\tau^{D}_{V}} \nu^{alg}(\mathcal{D}/\mathcal{E}_{C_{0}(V)}) \\
K^{alg}(Ch'(\mathcal{D}/\mathcal{E}_{C_{0}(V)}) & \xrightarrow{\tau^{H}} K^{alg}((\mathcal{D}/\mathcal{E}_{C_{0}(V)}) \\
K^{alg}(Bi^{b}(\mathcal{D}/\mathcal{E}_{C_{0}(V)}) & \xrightarrow{\tau^{G}_{(\mathcal{D}/\mathcal{E}_{C_{0}(V)})}} \\
\Omega K^{alg}((\mathcal{D}/\mathcal{E}_{C_{0}(V)}) & \xrightarrow{c} \\
\Omega K^{top}((\mathcal{D}/\mathcal{E}_{C_{0}(V)}) & \xrightarrow{\Omega K^{top}((\mathcal{D}/\mathcal{E}_{C_{0}(V)})} \wedge ku KU \\
K^{top}(V) & \xrightarrow{\cong} \\
\end{align*}
\]

Note that except for the first one, all the maps above are functorially defined. Also, we showed in proposition \[11\] that in the homotopy category of spectra, $\tau^{D}$ is compatible with restriction to further open subsets. Therefore for a hypercover $V$, so that all the open sets in $V_n$ are relatively compact subsets of the open sets in $V_{n-1}$ (and in particular, all are relatively compact in $X$), there is an induced map $\tau : K^{alg}(\mathcal{P}(X)) \rightarrow \text{holim} K^{top}(V)$ in the homotopy category of spectra, where the latter is referring to the homotopy colimit of the diagram \[11\] for the hypercover $V \rightarrow X$. Since topological K-homology satisfies descent, then there is an induced map in the homotopy category
of spectra
\[ \tau_X : K^{alg}(\mathcal{P}(X)) \to K^{top}(X). \] (12)

By taking a finer cover if necessary, one can see that the map above is independent of the choice of the hypercover \( V \).

**Proposition 4.8.** The Riemann-Roch transformation defined above commutes with restriction to open subsets. In other words, for a complex manifold \( X \) and an open subset \( U \) of \( X \), the diagram below commutes in the homotopy category of spectra.

\[
\begin{array}{c}
K^{alg}(\mathcal{P}(X)) \xrightarrow{\tau_X} K^{alg}(\mathcal{P}(U)) \\
\downarrow \quad \downarrow \tau_U \\
K^{top}(X) \xrightarrow{} K^{top}(U)
\end{array}
\]

**Proof.** Let \( V \to X \) be a hypercover so that all the open sets in \( V_n \) are relatively compact in \( V_{n-1} \). Choose a hypercover \( W \to U \) with the same condition as \( V \) so that \( W_n \) is finer than \( V_n \cap U \), i.e. each open set in \( W_n \) (which is a relatively compact open subset of \( U \)) is contained in (the intersection of \( U \) with) some open set in \( V_n \). Hence for any relatively compact open set \( W^j_n \) in the hypercover \( W \), there exists relatively compact open set \( V^j_n \) of the hypercover \( V \) so that \( W^j_n \subset V^j_n \cap U \) is relatively compact. Hence by corollary 3.27 and by naturality of all the other maps in definition of \( \tau \), the diagram below commutes in the homotopy category of spectra.

\[
\begin{array}{c}
K^{alg}(\mathcal{P}(X)) \xrightarrow{\tau_X} K^{alg}(\mathcal{P}(U)) \\
\downarrow \rho^D_{V^j_n} \quad \downarrow \rho^D_{W^j_n} \\
K^{alg}(Ch'(\mathcal{D}/\mathcal{C})^C_0(V^j_n)) \xrightarrow{} K^{alg}(Ch'(\mathcal{D}/\mathcal{C})^C_0(W^j_n)) \\
\downarrow \quad \downarrow \\
K^{top}(X) \xrightarrow{} K^{top}(V^j_n) \xrightarrow{\tau^D} K^{top}(W^j_n)
\end{array}
\]

Hence there is a unique map from \( K^{top}(X) \) to the holim \( K^{top}(W) \) that makes the diagram above commute (in the homotopy category of spectra), where the homotopy limit is taken on the diagram 11 for the hypercover \( W \to U \). But this homotopy limit is weakly equivalent to \( K^{top}(U) \) because topological K-homology satisfies descent. This finishes the proof.

We will investigate functoriality of \( \tau \) with respect to proper morphisms of complex manifolds in a future work.

### 4.3 Cap Product

In the last subsection, let us emphasize on the case when the \( C^* \)-algebra \( A \) is unital, and how one could define a pairing between the K-theory and K-homology of \( A \) using the Paschke category.

When \( A \) is unital, we will define the *Euler characteristic* of an exact sequence in the Paschke category \( (\mathcal{D}/\mathcal{C})_A \). Recall that if

\[ \ldots \to \rho_i \xrightarrow{T_i} \rho_{i+1} \xrightarrow{T_{i+1}} \rho_{i+2} \to \ldots \]
is a chain complex in \((\mathcal{D}/\mathcal{C})_A\), and \(T_i\) are representatives for \(T'_i\) in \(\mathcal{D}_A\), then the composition \(T_{i+1} \circ T_i\) may not be zero; it only has to be locally compact. If the representations are all unital, then \(T_{i+1} \circ T_i\) has to be compact. This is still not enough for us to be able to take the quotient of \(\ker(T_{i+1})\) by the image of \(T_i\), as the image may not even be a subset of the kernel. However, we can get around this issue. Recall the following definition from [Seg70, Sec 1.] and the main result of [Tar07].

**Definition 4.9.** Let \(\ldots \rightarrow V_i \xrightarrow{T_i} V_{i+1} \xrightarrow{T_{i+1}} V_{i+2} \rightarrow \ldots\) be a complex of Hilbert spaces and bounded linear operators. Then it is called a Fredholm complex if all the \(T_i\)'s have closed images and the cohomology is finite dimensional at every step.

Equivalently, we may define a complex of bounded operators between Hilbert spaces as before to be a Fredholm complex if there exists bounded operators \(S_i : V_{i+1} \rightarrow V_i\) so that \(T_{i-1}S_{i-1} + S_iT_i - Id_{V_i}\) is a compact operator for all \(i\).

If the complex is bounded, i.e. \(V_i = 0\) for all but finitely many values of \(i\), then define its Euler characteristic by

\[
\chi(V_i) = \sum_i (-1)^i H^i(V_i)
\]

We can consider the Euler characteristic as a formal difference of two finite dimensional subspaces of \(\oplus_i V_i\).

**Proposition 4.10.** Let \(\ldots \rightarrow V_i \xrightarrow{T_i} V_{i+1} \xrightarrow{T_{i+1}} \ldots\) be a bounded above exact sequence in \((\mathcal{B}/\mathcal{R})\). Then there are morphisms \(T_i \in \mathcal{B}(V_i, V_{i+1})\) so that \(T_i\) is a representative for \(T'_i\), and also \(T_{i+1} \circ T_i = 0\). Hence the new complex has a well-defined cohomology. Also the sequence \(\ldots \rightarrow V_i \xrightarrow{T_i} V_{i+1} \xrightarrow{T_{i+1}} \ldots\) is a Fredholm complex, and if the complex is both bounded above and below, then the Euler characteristic of the complex is independent of the choices of \(T_i\)'s. (In the sense that for the finite dimensional subspaces \(V^+, V^-, W\) of the Hilbert space \(H\), we consider the formal differences \(V^+ - V^-\) and \(V^+ \oplus W - V^- \oplus W\) to be equivalent.)

Let \(A\) be a unital \(C^*\)-algebra, and let

\[
\ldots \rightarrow \rho_i \xrightarrow{T'_i} \rho_{i+1} \xrightarrow{T'_{i+1}} \rho_{i+2} \rightarrow \ldots
\]

be a bounded exact sequence (i.e. there are only finitely many non-zero objects) in the Paschke category \((\mathcal{D}/\mathcal{C})_A\). Then by the argument proving proposition 4.10 we know there exists natural choices of unital representations \(\hat{\rho}_i\) which are isomorphic to \(\rho_i\). This induces a new exact sequence

\[
\ldots \rightarrow \hat{\rho}_i \xrightarrow{T_i} \hat{\rho}_{i+1} \xrightarrow{T_{i+1}} \hat{\rho}_{i+2} \rightarrow \ldots
\]

where all the representations are unital and hence \(T_{i+1} \hat{T}_i\) is compact for all \(i\), and this induces an exact sequence in \((\mathcal{B}/\mathcal{R})\). Therefore by proposition 4.10 the exact sequence above has a well-defined euler characteristic. Note that this process can not be replicated for the Calkin-Paschke category, as the choice of the projection \(\pi \in \mathcal{B}(H)\) corresponding to \(\rho'(1)\) may affect the index. To sum it all up:

**Corollary 4.11.** Let \(A\) be a unital \(C^*\)-algebra, and let \((\rho, T)\) be an exact sequence in the Paschke category \((\mathcal{D}/\mathcal{C})_A\) with finitely many non-zero objects. Then the procedure above defines the Euler characteristic of this complex. The Euler characteristic defined is additive with respect to exact sequences in the category \(Ch'(\mathcal{D}/\mathcal{C})_A\). (This is not true for the category \(ChB(\mathcal{D}/\mathcal{C})_A\).)
Remark 4.12. When $A$ is not unital, the argument in [Tar07] does not work anymore; as it relies on the fact that if $1d_H - \Delta \in \mathfrak{H}(H)$, then $\Delta$ has a closed image. This is no longer the case if we replace $\mathfrak{H}(H)$ by locally compact operators.

In a slightly different direction, let us define a natural pairing between projective modules and representations of a $C^*$-algebra.

Definition 4.13. Let $R$ be a ring. Denote the exact category of finitely generated projective right modules on $R$ by $\mathcal{P}^r(R)$. When $R$ is commutative, we drop the superscript $r$. Note that for any (right) projective $R$-module, there exists an integer $n$, and an inclusion $\iota : P \to R^n$ of (right) $R$-modules.

We are interested in the particular case when $R = A$ is a unital $C^*$-algebra. Let $\mathcal{P}^r_m(A)$ denote the category of finitely generated projective $A$-modules with an inner-product structure. One can consider the inclusion $\iota : P \to A^n$ of right $A$-modules to be norm preserving. Morphisms in $\mathcal{P}^r_m(A)$ are the (not necessarily norm-preserving) morphisms between the projective modules.

Let $X$ be a compact Hausdorff space, in particular a compact manifold. Then denote the exact category of topological (complex) vector bundles on $X$ by $\mathcal{P}^b(X)$. Recall the category $\mathcal{P}^b_m(X)$ from definition 3.8. Note that by the Serre-Swan theorem [Swa62, Thm 2.], $\mathcal{P}^b(X) = \mathcal{P}(C(X))$, hence $\mathcal{P}^b_m(X) = \mathcal{P}_m(C(X))$.

Definition 4.14. Let $A$ be a unital $C^*$-algebra and let $P \in \mathcal{P}^r_m(A)$. Let $\rho$ be an object in $(\mathfrak{O}/\mathfrak{C})_A$. Then we define the representation $P \otimes \rho : A \to \mathfrak{B}(P \otimes_A H)$, where we are considering $H$ as a left $A$-module through the representation $\rho$.

We follow [Al70] to show that $P \otimes_A H$ is in fact a Hilbert space, and hence the definition above makes sense.

Since $P$ is a finitely generated projective (right) module, there exists a norm preserving (right) $A$-module surjection $\pi : A^n \to P$, with a norm preserving (right) $A$-module section $\iota : P \to A^n$. Without loss of generality, we can assume that $\iota \pi$ is a self-adjoint projection on $A^{\oplus n}$. Now let $\iota \pi(e_i) = \sum_j a^j_i$ for $1 \leq i \leq n$, where $e_1, \ldots, e_n$ are the standard basis for $A^n$, and $a^j_i \in A$. Define the linear operator $\hat{\pi}$ on $H^{\oplus n} \cong A^n \otimes_A H$ by

$$\hat{\pi}(e_i \otimes_A h) = \sum_j e_j \otimes_A \rho(a^j_i)h.$$  

It is easy to check that this is in fact a self-adjoint projection, and since $\iota \pi$ is $A$-linear, then $\hat{\pi}$ also commutes with $\rho^{\oplus n}$, because

$$\hat{\pi}\rho^{\oplus n}(a)(h_1, \ldots, h_n) = \sum_i \hat{\pi}(e_i \otimes_A \rho(a)h_i) = \sum_i (e_j \otimes_A \rho(a^j_i)\rho(a)h_i) = \sum_i (e_j \otimes_A \rho(a^j_i)h_i) = \rho^{\oplus n}(a) \sum_i \pi(e_i \otimes_A h_i) = \rho^{\oplus n}(a)\hat{\pi}(h_1, \ldots, h_n).$$

Now let $V \subset H^{\oplus n}$ be the image of $\hat{\pi}$, and let $\hat{\iota} : V \to H^{\oplus n}$ be the inclusion, then consider the composition $A \xrightarrow{\rho^{\oplus n}} \mathfrak{B}(H^{\oplus n}) \to \mathfrak{B}(V)$, where the last map sends $T$ to $\hat{\pi}T\hat{\iota}$ (we are abusing the notation and denoting the composition of $\hat{\pi}$ with the orthogonal projection $H^{\oplus n} \to V$ by $\hat{\pi}$ as well.). It is easy to check that the compositions below are inverses to each other.

$$V \xrightarrow{\hat{\iota}} H^{\oplus n} \cong A^n \otimes_A H \xrightarrow{\pi \otimes AId} P \otimes_A H$$

$$V \leftarrow \hat{\pi} H^{\oplus n} \cong A^n \otimes_A H \xleftarrow{\iota \otimes AId} P \otimes_A H$$
Since all the maps above commute with multiplication by \( A \) in \( \mathcal{D}_A \), this induces structure of a Hilbert space on \( P \otimes A H \).

**Proposition 4.15.** Let \( A \) be a unital \( C^* \)-algebra. The tensor product introduced in definition 4.14 induces a biequivalent functor

\[
\cap_A : \mathcal{P}_n^r(A) \times (\mathcal{D}/\mathcal{C})_A \to (\mathcal{D}/\mathcal{C})_A
\]

which we will call the cap product. This induces a pairing on the level of \( K \)-theory spectra

\[
\cap_A : K^{alg}(\mathcal{P}_n^r(A)) \wedge K((\mathcal{D}/\mathcal{C})_A) \to K((\mathcal{D}/\mathcal{C})_A).
\]

**Proof.** First we need to check functoriality. Let \( F : P_1 \to P_2 \) be a morphism in \( \mathcal{P}_n^r(A) \). Then we can consider the morphism \( F \otimes_A Id : P_1 \otimes_A \rho \to P_2 \otimes_A \rho \) in the category \( \mathcal{D}_A \) where \( p \otimes_A h \mapsto F(p) \otimes_A h \). Note that for \( a \in A \), \( F(p) \otimes_A \rho(a) = F(p) \otimes_A h = F(p.a) \otimes_A h \). Hence this is well defined and commutes with multiplication by \( A \). It is clear that this process is functorial, i.e.

\[
(F_2 \otimes_A Id) \circ (F_1 \otimes_A Id) = F_2F_1 \otimes_A Id \text{ in } \mathcal{D}_A.
\]

Let \( T : \rho_1 \to \rho_2 \) be a morphism in the Paschke category \( (\mathcal{D}/\mathcal{C})_A \). Then we define \( Id \otimes_A T : P \otimes_A \rho_1 \to P \otimes_A \rho_2 \) as follows. Let \( \pi : A^m \to P \) be a norm preserving surjective map of right \( A \)-modules, let \( \iota : P \to A^m \) be the corresponding inclusion of right \( A \)-modules, and let \( \hat{\pi}_i \in \mathcal{B}(H_i^{\otimes m}) \) be the projection corresponding to \( P \otimes_A H_i \) for \( i = 1,2 \), and \( \hat{i}_i \) be the inclusion of its image \( V_i \) in the corresponding Hilbert space. Consider \( T^\otimes m : A^m \otimes_A \rho_1 \cong \hat{\pi}_1 \otimes \rho_2 \cong A^m \otimes_A \rho_2 \), and define \( Id \otimes_A T = \hat{\pi}_2T^\otimes m\hat{i}_1 \). Note that this is a pseudo-local operator, as both \( \hat{\pi}_2, \hat{i}_1 \) commute with the representations and \( T \) is pseudo-local. Also, since \( T \) commutes with multiplication by \( a_j^1 \) in \( A \) modulo compact operators, then \( \hat{\pi}_2T^\otimes m - T^\otimes m\hat{\pi}_1 \) is compact. This shows that the definition for \( Id \otimes_A T \) is independent of the choice of the projection \( \pi \) up to locally compact operators. This also shows that for morphisms \( T_1 : \rho_1 \to \rho_2 \) and \( T_2 : \rho_2 \to \rho_3 \) in the Paschke category, the compositions

\[
(Id \otimes_A T_2) \circ (Id \otimes_A T_1) = \hat{\pi}_3\hat{\pi}_2T^\otimes m_t_1\hat{\pi}_1 = \hat{\pi}_3\hat{\pi}_2T^\otimes m_t_1\hat{\pi}_1
\]

are equal to each other modulo locally compact operators. Hence this process is functorial.

**Remark 4.16.** The map \( Id \otimes_A T : P \otimes_A \rho_1 \to P \otimes_A \rho_2 \) is not well-defined in the category \( \mathcal{D}_A \).

Let \( \rho_1, \rho_2 \) be two objects of \( (\mathcal{D}/\mathcal{C})_A \), let \( T : \rho_1 \to \rho_2 \) be a morphism, let \( P_1, P_2 \) be two objects of \( \mathcal{P}_n^r(A) \), and let \( F : P_1 \to P_2 \) be a morphism of right \( A \)-modules. Choose the norm preserving inclusions \( \iota_i : P_i \to A^{n_i} \) of right \( A \)-modules, so that there exist a map of right \( A \)-modules \( F' : A^{n_1} \to A^{n_2} \) that makes the corresponding diagram commute. Then the square on the left and the one on the right commute in the category \( \mathcal{D}_A \). Consider \( e_i \otimes_A h \in A^{n_1} \otimes_A H_i \), we have \( (F' \otimes_A Id)T^\otimes n_1(e_i \otimes h) = (b_{1,1}T(h), \ldots, b_{n_2,i}T(h)) \), where \( b_{j,i} \) is the \( j \)th term in \( F'(e_i) \in A^{n_2} \), and \( T^\otimes n_2(F' \otimes_A Id)(e_i) = (T(b_{1,1}h), \ldots, T(b_{n_2,i}h)) \). Since \( T \) is pseudo-local, then the square in the center also commutes in the Paschke category \( (\mathcal{D}/\mathcal{C})_A \). Hence functoriality in two directions are compatible with each other.
It is easy to check that if $T$ is invertible, then so is $Id \otimes_A T$, and if $F$ is an isomorphism of $A$-modules, then $F \otimes_A Id$ is a pseudo-local isomorphism of Hilbert spaces. This procedure is exact in both variables, because if $T_2 \circ T_1 = 0$ then $(Id \otimes_A T_2) \circ (Id \otimes_A T_1) = 0$ and similarly for $F$. Also, an exact sequence $(\rho, T)$ in $(\mathcal{D}/\mathcal{C})_A$ has a contracting homotopy $S$, which translates into a contracting homotopy $Id \otimes_A S$ for the sequence $(P \otimes_A \rho, Id \otimes_A T)$. Also a short exact sequence $(P, F)$ of projective modules splits, i.e. has a contracting homotopy which again gives a contracting homotopy for the sequence $(P \otimes_A \rho, F \otimes_A Id)$.

Let $F_1 : P_1 \to P_2$ be an admissible monomorphism in $\mathcal{P}^*_m(A)$ and consider an exact sequence

$$0 \to \rho_1 \xrightarrow{T_1} \rho_2 \xrightarrow{T_2} \rho_3 \to 0$$

with a choice of contracting homotopy in the Paschke category $(\mathcal{D}/\mathcal{C})_A$. Then there is a map

$$P_1 \otimes_A \rho_3 \oplus P_2 \otimes_A \rho_1 \xrightarrow{(Id \otimes_A S_2) \oplus Id} P_1 \otimes_A \rho_2 \oplus P_2 \otimes A \rho_1 \to (P_1 \otimes_A \rho_2) \cup (P_1 \otimes_A \rho_1) (P_2 \otimes A \rho_1)$$

which induces an isomorphism between the first object and the last object in the Paschke category $(\mathcal{D}/\mathcal{C})_A$. The map $P_1 \otimes_A \rho_3 \oplus P_2 \otimes_A \rho_1 \xrightarrow{(F_1 \otimes_A S_2, Id \otimes_A T_3)} P_2 \otimes A \rho_2$ is an admissible monomorphism, whose cokernel is $P_2 \otimes A \rho_2 \xrightarrow{F_2 \otimes A T_2} P_3 \otimes A \rho_3$, where $F_2 : P_2 \to P_3$ is cokernel of $F_1$, and the contracting homotopies are the trivial ones induced by contracting homotopies of $\rho$ and $F$. This proves that $\cap_A$ is biexact, hence by proposition 2.12 induces a map of K-theory spectra. \hfill $\square$

Let $f : A \to B$ be a unital map between unital $C^*$-algebras. Recall there is an exact push-forward functor $f_* : \mathcal{P}^*_m(A) \to \mathcal{P}^*_m(B)$ and $f_* : \mathcal{P}^*_m(A) \to \mathcal{P}^*_m(B)$ defined by $f_*(P) = P \otimes_A B$. There is also a pull-back map $f^* : (\mathcal{D}/\mathcal{C})_B \to (\mathcal{D}/\mathcal{C})_A$. One could ask about the relation between the pairing defined above and these structures.

**Proposition 4.17.** The pairing defined in proposition 4.15 is natural in the sense that for a unital map $f : A \to B$ of unital $C^*$-algebras, the diagram below commutes up to homotopy

$$
\begin{array}{ccc}
K(\mathcal{P}^*_m(A)) \wedge K((\mathcal{D}/\mathcal{C})_B) & \xrightarrow{f_* \times Id} & K(\mathcal{P}^*_m(B)) \wedge K((\mathcal{D}/\mathcal{C})_B) \\
\downarrow & & \downarrow \\
K(\mathcal{P}^*_m(A)) \wedge K((\mathcal{D}/\mathcal{C})_A) & \xrightarrow{\cap_A} & K((\mathcal{D}/\mathcal{C})_B) \\
\downarrow \scriptsize{\cap_B} & & \downarrow \scriptsize{f^*} \\
K((\mathcal{D}/\mathcal{C})_B) & \xrightarrow{f^*} & K((\mathcal{D}/\mathcal{C})_A).
\end{array}
$$

**Proof.** Consider the diagram below
Let $\rho : B \to \mathcal{B}(H)$ be a representation, and let $P$ be an object in $P_{r}(A)$. We can consider $H$ as a left $A$-module through the representation $f^{*}\rho : A \to B \to \mathcal{B}(H)$. It is straightforward to check that the natural map of Hilbert spaces $P \otimes_{A} H \to (P \otimes_{A} B) \otimes_{B} H$ defined by $p \otimes_{A} h \mapsto (p \otimes_{A} 1) \otimes_{B} h$ is well-defined, and has a two-sided inverse given by $(p \otimes_{A} b) \otimes_{B} h \mapsto p \otimes_{A} \rho(f(b))h$. This isomorphism is pseudo-local, hence induces a natural isomorphism between $f^{*}((P \otimes_{A} B) \otimes_{B} \rho)$ and $P \otimes_{A} f^{*}\rho$ in the category $\mathcal{D}_{A}$. Hence the diagram above commutes up to natural isomorphisms.

Remark 4.18. One can replace the Paschke category $(\mathcal{D}/\mathcal{E})_{A}$ with the category $Ch^b(\mathcal{D}/\mathcal{E})_{A}$ (or $C(\mathcal{D}/\mathcal{E})_{A}, C^{b}(\mathcal{D}/\mathcal{E})_{A}, Ch(\mathcal{D}/\mathcal{E})_{A}$) in propositions 4.15 and 4.17 and the same result would still hold. However, we can not necessarily replace $Ch^{l}(\mathcal{D}/\mathcal{E})_{A}$, as morphisms in $Ch^{l}(\mathcal{D}/\mathcal{E})_{A}$ come from $\mathcal{D}_{A}$, but pairing a morphism with the identity on a projective module is only well-defined up to compact operators.

Fix an object $(\rho, T)$ of $Ch^{l}(\mathcal{D}/\mathcal{E})_{A}$, since for a morphism $F : P_{1} \to P_{2}$ in $P_{m}^{r}(A)$, the morphism $F \otimes_{A} Id : P_{1} \otimes_{A} \rho \to P_{2} \otimes_{A} \rho$ is well-defined in $\mathcal{D}_{A}$, hence we obtain a functor

$$P_{m}^{r}(A) \to Ch^{l}(\mathcal{D}/\mathcal{E})_{A}$$

which maps $P$ to $P \cap_{A} (\rho, T) = (P \otimes_{A} \rho, Id \otimes_{A} T)$.

Definition 4.19. Let $X$ be a compact complex manifold, let $g$ be a hermitian metric on $X$, let $X \times \mathbb{C}$ denote the trivial rank one bundle on $X$, and let $E$ be a topological vector bundle on $X$. Then denote the map $[13]$ obtained through pairing with $\tau^{D}_{X,g}(X \times \mathbb{C}) \in Ch^{l}(\mathcal{D}/\mathcal{E})_{C(X)}$ defined in [4] by $- \cap \tau^{D}[X]$.

We have to emphasize that for a non-holomorphic vector bundle, the Dolbeault complex is not well-defined. Let $X$ be a compact complex manifold, let $E$ be a holomorphic vector bundle on $X$, and let $g, h$ be hermitian metrics on $X, E$, respectively. Recall from [5] that we have an exact sequence $\tau^{D}_{X,g}(E, h)$ in the Paschke category $(\mathcal{D}/\mathcal{E})_{C(X)}$ corresponding to the Dolbeault complex.

Proposition 4.20. Let $X$ be a compact complex manifold, and let $E$ be a holomorphic vector bundle on $X$. Choose hermitian metrics $g, h$ on $X, E$ respectively. Then the chain complexes $\tau^{D}_{X,g}(E, h)$ and $E \cap \tau^{D}[X]$ are isomorphic to each other in the category $Ch^{l}(\mathcal{D}/\mathcal{E})_{C(X)}$.

Proof. Let $\pi : X \times \mathbb{C}^{m} \to E$ be a smooth projection onto the bundle $E$, and let $\zeta : E \to X \times \mathbb{C}^{m}$ be the inclusion. Denote the Dolbeault operator on the trivial bundle $X \times \mathbb{C}^{k}$ of rank $k$ by $D^{k} = \bar{\partial} + \partial^{*}$, and let $D_{E} = \partial_{E} + \bar{\partial}_{E}$ denote the Dolbeault operator on $E$. Note that by definition, $Id \otimes \chi(D^{1}) \in \mathfrak{A}(E \otimes L^{2}(X, \wedge^{0, *} T^{*}X))$ is defined as $\pi \chi(D^{1}) \otimes_{m} = \pi \chi(D^{m})$. By remark 3.24, $\pi \chi(D^{m}) - \pi \chi(D_{E})$ is locally compact in the Paschke category $(\mathcal{D}/\mathcal{E})_{C(X)}$. Since $\pi_{*} = Id_{E}$, this proves the assertion.

\[\square\]
Corollary 4.21. Let $X$ be a compact complex manifold, and let $E$ be a topological vector bundle. Then $E \cap \tau_X^{0,q}$ is an exact sequence in the Paschke category $(\mathcal{D}/\mathcal{E})_{C(X)}$, and by corollary 4.11 has a well-defined Euler characteristic. By propositions 4.20 and 5.4, this concept of Euler characteristic is equal to the classical concept of the Euler characteristic of the Dolbeault complex when $E$ is a holomorphic vector bundle.

Appendix A  Complex Manifolds

Let us give the basics and the notation used for complex manifolds in here. A good source for reading more on the topic is [We07].

Let $X$ be a complex manifold and let $E$ be a holomorphic vector bundle, then denote the sheaf of holomorphic, real analytic, differentiable, and continuous sections of $E$ by $\mathcal{O}(E)$, $\mathcal{C}^\infty(E)$, $\mathcal{C}(E)$, respectively. Notice that each of the four sheaves just mentioned, is a subsheaf of the next ones. Also if $X$ is only real analytic, then we can still consider the sheaves $\mathcal{C}^\infty(E)$, $\mathcal{C}^\infty(E)$, $\mathcal{C}(E)$, and similar statements can be repeated for differentiable or topological manifolds. Let $\mathcal{F}$ be one of the four sheaves above, then for an open subset $U$ of $X$, denote the space of sections of $E$ on $U$ by $\mathcal{F}(U, E)$. In the case when $E$ is the trivial line bundle $X \times \mathbb{C}$, then we will just denote $\mathcal{F}(U)$ instead of $\mathcal{F}(U, E)$ and also denote the structure sheaf by $\mathcal{F}_X$.

Let $T^*X$ denote the cotangent bundle of the complex manifold $X$. Then the (almost) complex structure of $X$ induces the decomposition $T^* X \otimes \mathbb{R} \mathbb{C} = T^* (X)^{1,0} \oplus T^* (X)^{0,1}$, which in turn induces the Dolbeault operator $\bar{\partial} : \mathcal{C}^\infty(\wedge^{p,q} T^* X) \rightarrow \mathcal{C}^\infty(\wedge^{p,q+1} T^* X)$, that vanishes on the holomorphic sections. Hence for a holomorphic vector bundle $E$, we get an induced differential operator

$$\bar{\partial} \otimes 1 : \mathcal{C}^\infty(\wedge^{p,q} T^* X) \otimes \mathcal{O}(E) \rightarrow \mathcal{C}^\infty(\wedge^{p,q+1} T^* X) \otimes \mathcal{O}(E),$$

which is also known as the Dolbeault operator. But we have $\mathcal{C}^\infty(\wedge^{p,q} T^* X) \otimes \mathcal{O}(E) \cong \mathcal{O}(\wedge^{p,q} T^* X \otimes \mathbb{C} E)$. From now on, we will abbreviate the latter to $\mathcal{A}_{X}^{p,q}(E)$ (or just $\mathcal{A}_{p,q}(E)$, if $X$ is clear from the context.), and we denote the Dolbeault operator by $\bar{\partial}_E : \mathcal{A}_{p,q}(E) \rightarrow \mathcal{A}_{p,q+1}(E)$. We will also call the following as the Dolbeault complex with coefficients in $E$:

$$0 \rightarrow \mathcal{A}_{X}^{0,0}(E) \xrightarrow{\partial_E} \mathcal{A}_{X}^{0,1}(E) \xrightarrow{\partial_E} \mathcal{A}_{X}^{0,2}(E) \xrightarrow{\partial_E} \mathcal{A}_{X}^{0,n}(E) \rightarrow 0 \quad (16)$$

where in here, $n = \text{dim}_\mathbb{C}(X)$.

Definition A.1. We follow [We07, 4.2] to recall the definition of symbol of a differential operator. First let $X$ be a differentiable manifold, and consider differentiable vector bundles $E,F$ on $X$. A linear operator $D : \mathcal{C}^\infty(X,E) \rightarrow \mathcal{C}^\infty(X,F)$ is a differential operator of order $k$, if no derivations of order $\geq k+1$ appear in its local representation. We denote the vector space of all such operators with $\text{Diff}_k(E,F)$.

Let $T'X$ denote the cotangent bundle $T^* X$ of $X$ with the zero section deleted, and let $\pi : T'X \rightarrow X$ denote the projection. For $k \in \mathbb{Z}$ set

$$\text{Smbl}_k(E,F) := \{ \sigma \in \text{Hom}(\pi^*E,\pi^*F) : \sigma(x,\rho v) = \rho^k \sigma(x,v), \text{ where } (x,v) \in T'X, \rho > 0 \}.$$  

We now define the $k$-symbol of a differential operator as a linear map $\sigma_k : \text{Diff}_k(E,F) \rightarrow \text{Smbl}_k(E,F)$ by

$$\sigma_k(D)(x,v) = D(\frac{d^k}{k!} (g-g(x))^k f)(x) \in F_x$$

\textsuperscript{24}All the vector bundles and vector spaces we are considering in this section are over the complex numbers. Some of the arguments still hold over the real numbers as well.
where in here \((x, v) \in T^* X, e \in E_x\) are given and \(g \in \mathcal{C}^\infty(X), f \in \mathcal{C}^\infty(X, E)\) are chosen so that \(f(x) = e, dg_x = v\). We can see that we have a linear mapping \(\sigma_k(D)(x, v) : E_x \to F_x\), and that the symbol does not depend on the choices made.

One can also define pseudo-differential operator of order \(k\) for \(k \in \mathbb{Z}\) (which we will denote by \(\text{PDiff}_k\)), and their symbol, but since definitions are somewhat more technical, and will not be used here, we refer the interested reader to \([\text{Wel07}, 4.3.\])

Symbols of (pseudo-) differential operators have the following important properties:

\[
\sigma_{k+m}(D_2 D_1) = \sigma_m(D_2) \sigma_k(D_1) \quad \text{when} \quad D_1 \in \text{PDiff}_k(E_1, E_2), D_2 \in \text{PDiff}_m(E_2, E_3)
\]

\[
\sigma_k(D^*) = (−1)^k \sigma_k(D)^* \quad \text{if} \quad D \in \text{PDiff}_k(F, E)
\]

where in here \(D^* \in \text{PDiff}_k(F, E)\) is the formal adjoint of \(D\) \([\text{Wel07}, 4.1.5.\]), and \(\sigma_k(D)^*\) is the adjoint of the linear map \(\sigma_k(D)(x, v) : E_x \to F_x\). Note that both \(D^*\) and \(\sigma(D^*) = \sigma(D)^*\) depend on the choice of metric on \(X\) and the bundles \(E, F\).

**Definition A.2.** \([\text{Wel07} 4.4.]\) Let \(E, F\) be differentiable vector bundles on the differentiable manifold \(X\) and let \(D \in \text{Diff}_k(E, F)\). Then we say that \(D\) is an **elliptic differential operator** if for all \((x, v) \in T^* X\), the linear map \(\sigma_k(D)(x, v) : E_x \to F_x\) is an isomorphism. In particular both \(E, F\) have the same fiber dimension. The same can be defined for pseudo-differential operators.

Let \(E_0, E_1, \ldots, E_m\) be a sequence of differentiable vector bundles on \(X\) and for some fixed \(k\), let \(D_i \in \text{Diff}_k(E_i, E_{i+1})\) for all \(i = 0, 1, \ldots, m - 1\). We say this is an elliptic complex if \(D_i \circ D_{i+1} = 0\) for all \(i\), and also if the associated symbol sequence

\[
0 \to \pi^*E_0 \xrightarrow{\sigma_k(D_0)} \pi^*E_1 \xrightarrow{\sigma_k(D_1)} \pi^*E_2 \xrightarrow{\sigma_k(D_2)} \cdots \xrightarrow{\sigma_k(D_{m-1})} \pi^*E_m \to 0
\]

is exact, where \(\pi : T^* X \to X\) is the projection.

**Remark A.3.** In the literature, elliptic complexes are usually defined for compact differentiable manifolds, since Sobelov spaces over non-compact spaces don’t behave as well as they do on compact spaces (e.g. Rellich’s lemma works for Sobelov spaces over a fixed compact subset of the manifold.), which makes elliptic complexes over non-compact manifolds not as easy to work with. For example, the Hodge decomposition theorem (mentioned later in this section) is no longer true for non-compact complex manifolds.

**Example A.4.** Let \(E\) be a holomorphic vector bundle on the complex manifold \(X\). The Dolbeault operator \(\bar{\partial}_E : \mathcal{A}^{p,q}_X(E) \to \mathcal{A}^{p,q+1}_X(E)\) is a differential operator of order 1, and for \((x, v) \in T^* X, f \in \mathcal{C}^\infty(X, E)\), \(f \otimes e \in \wedge^p \bar{\partial}_x \otimes E\),

\[
\sigma_1(\bar{\partial}_E)(x, v)f \otimes e = (iv^{0,1} \wedge f) \otimes e
\]

where in here \(v = v^{1,0} + v^{0,1} \in T^*_x(X)^{1,0} + T^*_x(X)^{0,1}\). It is easy to check that the symbol sequence is exact, and hence the Dolbeault complex \([\text{Wel07} 4.6]\) is an elliptic complex.

**Theorem A.5** (The Hodge decomposition). Let \(X\) be a compact complex manifold, and let \(E\) be a holomorphic vector bundle on \(X\). Choose hermitian metrics on \(X\) and on \(E\) and let \(\bar{\partial}^*_i : \mathcal{A}^{0,i}_X(E) \to \mathcal{A}^{0,i+1}_X(E)\) be the formal adjoint of \(\bar{\partial}_i : \mathcal{A}^{0,i}_X(E) \to \mathcal{A}^{0,i+1}_X(E)\) (with respect to the metrics chosen). Let \(\Delta_i = \bar{\partial}_i \bar{\partial}_{i-1} + \bar{\partial}_{i-1} \bar{\partial}_i\) and let \(\mathcal{H}^{0,i}(X, E) = \ker \Delta_i \subset \mathcal{A}^{0,i}_X(E)\) denote the harmonic \((0, i)\)-forms. Then we have the orthogonal decomposition

\[
\mathcal{A}^{0,i}_X(E) \cong \mathcal{H}^{0,i}(X, E) \oplus \text{im}(\bar{\partial}_{i-1}) \oplus \text{im}(\bar{\partial}_i^*),
\]

and also there is an isomorphism \(\mathcal{H}^{0,i}(X, E) \cong H^i(X, E)\), where the latter, is the cohomology of \(X\) with coefficients in \(E\)
**Definition A.6** (Hodge Star operator). [Wel07, 5.1.] Let \( V \) be a (complex) vector space of dimension \( n \). Choose an inner product on \( V \) and then choose an orthonormal basis \( e_1, \ldots, e_n \) for \( V \). Then define the Hodge \( \ast \)-operator
\[
\ast : \wedge^k V \to \wedge^{n-k} V
\]
defined by \( \ast(e_{i_1} \wedge \ldots \wedge e_{i_k}) = \pm(e_{j_1} \wedge \ldots \wedge e_{j_{n-k}}) \), where \( \{j_1, \ldots, j_{n-k}\} \) is complement of \( \{i_1, \ldots, i_k\} \) in \( \{1, \ldots, n\} \), and we assign the plus sign if \( \{i_1, \ldots, i_k, j_1, \ldots, j_{n-k}\} \) is an even permutation of \( \{1, \ldots, n\} \), and assign the minus sign if it is an odd permutation.

It is easy to extend \( \ast \) by linearity, and also to observe that \( \ast \) does not depend on the choice of the orthonormal basis, and depends only on the inner-product structure.

Let \( X \) be a complex manifold of dimension \( n \), and choose a hermitian metric \( g \) on \( X \). Then similar to above, we can define the Hodge \( \ast \)-operator
\[
\ast : \wedge^k T^* X \to \wedge^{n-k} T^* X
\]
and it is easy to see that there is an induced \( \ast \)-operator
\[
\ast : \wedge^{p,q} T^* X \to \wedge^{n-p,n-q} T^* X.
\]

Let \( E \) be a holomorphic vector bundle on \( X \), and choose a hermitian metric \( h \) on \( E \). We can consider the metric as a linear map \( h : E \to E^* \), where \( E^* \) is the dual vector bundle to \( E \), and we also have the dual linear map \( h^* : E^* \to E \), and these satisfy \( h^* h = \Id_E, hh^* = \Id_{E^*} \). Let \( \bar{s}(f) := \ast(\bar{f}) \) for a section \( f \) of \( \wedge^{\ast} T^* X \). Define
\[
\bar{s}_E = \bar{s} \otimes h : \wedge^{p,q} T^* X \otimes E \to \wedge^{n-q, n-p} T^* X \otimes E^*.
\]
Then one can show [Wel07, 5.2.4.a.] the following relation between the adjoint \( \bar{\partial}^\ast \) of the Dolbeault operator \( \bar{\partial} \) and the Hodge \( \ast \)-operator.

\[
\bar{\partial}^\ast = -\bar{s}_E \cdot \bar{\partial} \bar{s}_E : \mathcal{A}^{p,q}_X(E) \to \mathcal{A}^{p,q-1}_X(E). \tag{18}
\]

**Appendix B  Functional Calculus**

Let us follow [HR00] to give a quick introduction to functional calculus.

**Definition B.1.** Let \( T \in \mathfrak{B}(H) \). Then let \( C^\ast(T) \) temporarily denote the Banach subalgebra of \( \mathfrak{B}(H) \), generated by \( T \), its adjoint \( T^\ast \), and the identity operator.

We say that the operator \( T \) is normal if \( TT^\ast = T^\ast T \). If \( T \) is normal then \( C^\ast(T) \) is a commutative Banach algebra.

Let \( A \) be a unital Banach algebra. Then for \( a \in A \), we define
\[
\text{Spectrum}_A(a) = \{ \lambda \in \mathbb{C} : a - \lambda.1 \text{ is not invertible in } A \}.
\]

**Proposition B.2** (Spectral Theorem). [HR00, 1.1.11.] Let \( T \in \mathfrak{B}(H) \) be a bounded normal operator acting on the Hilbert space \( H \), then the map \( \alpha \mapsto \alpha(T) \) is a homomorphism from dual of \( C^\ast(T) \) onto \( \text{Spectrum}_\mathfrak{B}(H)(T) \), and the induced Gelfand transform \( C^\ast(T) \to \mathcal{C}(\text{Spectrum}_\mathfrak{B}(H)(T)) \) is an isometric \( \ast \)-isomorphism.

**Definition B.3.** Let \( T \in \mathfrak{B}(H) \) be a bounded normal operator acting on the Hilbert space \( H \), and let \( f \in \mathcal{C}(\text{Spectrum}_\mathfrak{B}(H)(T)) \). Denote the corresponding element in \( C^\ast(T) \) by \( f(T) \). The \( \ast \)-homomorphism (inverse of the Gelfand transform) \( \mathcal{C}(\text{Spectrum}_\mathfrak{B}(H)(T)) \to \mathfrak{B}(H) \) defined by \( f \mapsto f(T) \) is called functional calculus for \( T \).
**Definition B.4.** [HR00, 1.8.] Let $T$ be an unbounded operator, defined over a dense subset of the Hilbert space $H$. Then we say $T$ is **symmetric** if for each $x, y \in H$ which are in domain of $T$, we have $\langle Tx, y \rangle = \langle x, Ty \rangle$.

We say $T$ is **essentially self-adjoint** if domain of $T$ is a subset of domain of $T^*$, and for any $x$ in domain of $T$, $Tx = T^*x$, and also $x$ is in domain of $T^*$ if there is a sequence of points $\{x_i\}_{i=1}^\infty$ in domain of $T$ so that $x_i$’s converge to $x$ and $\|T(x_i)\|$ remains bounded.

Note that the first two conditions are equivalent to $T$ being symmetric. In other words, every essentially self-adjoint unbounded operator is symmetric.

**Lemma B.5.** [HR00, 10.2.6.] Every symmetric differential operator on a compact manifold is essentially self-adjoint. More generally, every compactly supported symmetric differential operator on a (non-compact) manifold is essentially self-adjoint.

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