Approximate AF flows

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Dedicated to George A. Elliott on the occasion of his sixtieth birthday

Abstract

When $\alpha$ is a flow on a unital AF algebra $A$ such that there is an increasing sequence $(A_n)$ of finite-dimensional $\alpha$-invariant C*-subalgebras of $A$ with dense union, we call $\alpha$ an AF flow. We show that an approximate AF flow is a cocycle perturbation of an AF flow.

1 Introduction

Let $\alpha$ be a flow on a unital C*-algebra $A$, i.e., $t \mapsto \alpha_t$ is a group homomorphism of the real line $\mathbb{R}$ into the automorphism group of $A$ such that $t \mapsto \alpha_t(x)$ is continuous for each $x \in A$. If $u$ is a continuous map of $\mathbb{R}$ into the unitary group $U(A)$ of $A$ such that $u_s \alpha_s(u_t) = u_{s+t}$, $s, t \in \mathbb{R}$, then $u$ is called an $\alpha$-cocycle. In this case $t \mapsto \text{Ad} u_t \alpha_t$ is a flow again and is called a cocycle perturbation of $\alpha$.

We denote by $\delta_\alpha$ the generator of $\alpha$, which is a closed derivation defined on a dense $^*$-subalgebra $\mathcal{D}(\delta_\alpha)$ \cite{3,20}. If the $\alpha$-cocycle $u$ is differentiable and $h = -i du_t/dt|_{t=0} \in A_{sa}$, then the generator of the flow $t \mapsto \text{Ad} u_t \alpha_t$ is given by $\delta_\alpha + \text{ad} h$. If the $\alpha$-cocycle $u$ is a coboundary, i.e., $u_t = w_\alpha(w^*)$ for some $w \in U(A)$, then the generator of the flow $t \mapsto \text{Ad} u_t \alpha_t$ is given by $\text{Ad} w \circ \delta_\alpha \circ \text{Ad} w^*$ on $\text{Ad} w(\mathcal{D}(\delta_\alpha))$. In general the $\alpha$-cocycle is given as a combination of the above two types \cite{12}.

If $A$ is an AF algebra \cite{1} and has an increasing sequence $(A_n)$ of finite-dimensional C*-subalgebras of $A$ such that $\bigcup_n A_n$ is dense in $A$ and $\alpha_t(A_n) = A_n$ for all $t \in \mathbb{R}$ and all $n$, then $\alpha$ is called an AF flow. Note that in this case there is a self-adjoint element $h_n \in A_n$ such that $\alpha_t|A_n = \text{Ad} e^{ith_n}|A_n$. It also follows that $\mathcal{D}(\delta_\alpha) \supset A_n$, $\delta_\alpha|A_n = \text{ad} h_n|A_n$, and $\bigcup_n A_n$ is a core for $\delta_\alpha$. Since $[h_m, h_n] = 0$, the generator of this type is called commutative and is studied in Sakai’s book \cite{20} (see also \cite{12,2,7}). In particular $\alpha$ is approximately inner in the sense that $\lim_n \text{Ad} e^{ith_n}(x) = \alpha_t(x)$ uniformly in $t$ on every bounded subset of $\mathbb{R}$ for all $x \in A$.
AF flows (on a UHF algebra) appear as time-flows for classical lattice models in physics and look manageable for analysis (e.g., the KMS states have explicit expressions \[^20\]). There are time-flows for quantum lattice models which are not obviously cocycle perturbations of AF flows, but we still lack a rigorous proof to that effect though we know that there are flows which are not cocycle perturbations of AF flows. Our main concern is to distinguish the class of cocycle perturbations of AF flows among the flows which occur in physical models and thus to understand the flows beyond this class better. In this note we give a characterization of this class.

When \( B \) and \( C \) are \( \mathbb{C}^\ast \)-subalgebras of \( A \), we write \( B^\delta \subset C \) if for any \( x \in B \) there is \( y \in C \) such that \( \| x - y \| \leq \delta \| x \| \). We define the distance of \( B \) and \( C \) by

\[
\text{dist}(B, C) = \inf\{\delta > 0 \mid B^\delta \subset C, C^\delta \subset B\}.
\]

If \( \alpha \) is an AF flow, then a cocycle perturbation \( \alpha' \) of \( \alpha \) may not be an AF flow but an approximate AF flow in the sense that \( \sup_{t \in [0,1]} \text{dist}(\alpha'_t(A_n), A_n) \to 0 \), where the sequence \( (A_n) \) is chosen for \( \alpha \) as above. Our main purpose is to show the converse:

**Theorem 1.1** Let \( \alpha \) be a flow on a unital AF algebra \( A \). Then the following conditions are equivalent:

1. \( \alpha \) is a cocycle perturbation of an AF flow.

2. \( \alpha \) is an approximate AF flow, i.e., there is an increasing sequence \( (A_n) \) of finite-dimensional \( \mathbb{C}^\ast \)-subalgebras of \( A \) such that \( \bigcup_n A_n \) is dense in \( A \) and

\[
\sup_{t \in [0,1]} \text{dist}(\alpha_t(A_n), A_n) \to 0
\]

as \( n \to \infty \).

As noted above, the former implies the latter. In the rest of this note we shall prove that the latter implies the former. Since the latter condition is preserved under cocycle perturbations, it suffices to show:

**Lemma 1.2** For any \( \epsilon > 0 \) there exists a \( \delta > 0 \) satisfying the following condition: If \( \alpha \) is an approximate AF flow, i.e., there is an increasing sequence \( (A_n) \) of finite-dimensional \( \mathbb{C}^\ast \)-subalgebras of \( A \) such that \( \bigcup_n A_n \) is dense in \( A \) and

\[
\delta_n \equiv \sup_{t \in [0,1]} \text{dist}(\alpha_t(A_n), A_n) \to 0
\]

as \( n \to \infty \) with \( \delta_1 \leq \delta \), there is an \( \alpha \)-cocycle \( u \) such that \( \| u_t - 1 \| < \epsilon \) for \( t \in [0,1] \) and

\[
\text{Ad} u_t \alpha_t(A_1) = A_1.
\]

Moreover if \( A_0 \) is a \( \mathbb{C}^\ast \)-subalgebra of \( A_1 \) such that \( \alpha_t(A_0) = A_0 \), then \( u \) can be chosen from \( A \cap A_0' \).
Remark 1.3 For a single automorphism $\alpha$ of a unital AF algebra $A$ the following conditions are equivalent:

1. For any $\epsilon > 0$ there is a $u \in \mathcal{U}(A)$ and an increasing sequence $(A_n)$ of finite-dimensional $C^*$-subalgebras of $A$ such that $\bigcup_n A_n$ is dense in $A$ and $\text{Ad } u \alpha(A_n) = A_n$ for all $n$.

2. There is an increasing sequence $(A_n)$ of finite-dimensional $C^*$-subalgebras of $A$ such that $\bigcup_n A_n$ is dense in $A$ and

$$\text{dist}(\alpha(A_n), A_n) \to 0$$

as $n \to \infty$.

This follows from a deep result of Christensen’s (see 5.3 of [4]), which will be also used in the proof of our main result.

There is a totally different sufficient condition for $\alpha$ that implies the above condition 1, i.e., which says that $\alpha$ has the Rohlin property and $\alpha_s$ is the identity on the dimension group $K_0(A)$. See [21, 11, 8]. It is certainly desirable to find a sufficient condition like this for AF flows.

Remark 1.4 Theorem 1.1 might hold for non-unital AF algebras, where the $\alpha$-cocycle should be understood as a function into the multiplier algebra with continuity for the strict topology. But in this case it follows, by the same proof, that 2 implies 1, but it is not obvious how to prove that 1 implies 2.

The main theorem is what we should have settled sooner or later after singling out AF flows, but it is not likely to be useful to distinguish the class of cocycle perturbations of AF flows. However there are some other attempts to characterize this class. In section 2 we will briefly survey them. In section 3 we will present a key idea for proving the main theorem in the setting of matrix algebras, which is a degenerate case of Prop. 1.1 for the UHF algebra (see below). Letting $M_N$ to be the $C^*$-algebra of $N \times N$ matrices, what will be shown is as follows:

Proposition 1.5 For any $\epsilon > 0$ there exists a $\delta > 0$ satisfying the following condition: Let $A = M_N$ for some $N \in \mathbb{N}$, $B$ a unital *-subalgebra of $A$ with $B \cong M_K$ for some $K$ dividing $N$, and $H \in A_{sa}$ such that

$$e^{itH}Be^{-itH} \subset B$$

for $t \in [0, 1]$. Then there exist $u \in \mathcal{U}(A)$ and $h \in A_{sa}$ such that $\|u - 1\| < \epsilon$, $\|h\| < \epsilon$, and

$$u^*e^{it(H+h)}Bu^*e^{-it(H+h)}u = B, \quad t \in \mathbb{R}.$$
In the above situation let \( \alpha_t = Ad e^{itH} \), which defines a flow on \( A = M_N \), and let 
\[ v_t = u^* e^{it(H+h)} u e^{-itH}. \]
Then note that \( Ad v_t \alpha_t(B) = B \) and that \( v : t \mapsto v_t \) is an \( \alpha \)-cocycle such that \( \sup_{t \in [0,1]} \|v_t - 1\| < 3\epsilon. \)

In section 4 we will give a version \[4.1\] of the main theorem for UHF flows \[14\], where \( \alpha \) is a UHF flow on \( A \) if \( A \) is a UHF algebra and there is a sequence \((A_n)\) of unital matrix \( C^*\)-subalgebras of \( A \) with dense union and \( \alpha_t(A_n) = A_n, n \in \mathbb{N}, t \in \mathbb{R} \). Some of the technical results hold for more general unital \( C^*\)-algebras.

To go from UHF flows to AF flows, we will need a version of the following result, which is valid for any \( C^*\)-algebras.

**Proposition 1.6** For any \( \epsilon > 0 \) there exists a \( \delta > 0 \) satisfying the following condition:
Let \( A \) be a unital \( C^*\)-algebra and \( \alpha \) be a flow on \( A \). If \( D \) is a unital finite-dimensional \( C^*\)-subalgebra of \( A \) such that 
\[ \sup_{|t| \leq 1} \| (\alpha_t - \text{id})|D\| < \delta, \]
then there is an \( \alpha \)-cocycle \( u \) such that \( Ad u_t \alpha_t \|D\| = \text{id} \) and 
\[ \max_{|t| \leq 1} \| u_t - 1 \| < \epsilon. \]

What is important here is that \( \delta \) does not depend on \( D \) (or the size of \( D \)). In section 5 we will prove the above proposition and then derive the main result.

## 2 AF flows

For a \( C^*\)-algebra \( A \) we denote by \( \ell^\infty(A) \) the \( C^*\)-algebra of bounded sequences in \( A \) and by \( c_0(A) \) the ideal of \( \ell^\infty(A) \) consisting of \( x = (x_n) \) for which \( \lim_{n \to \infty} \|x_n\| = 0 \) and let 
\[ A^\infty = \ell^\infty(A)/c_0(A). \]
We embed \( A \) into \( A^\infty \) by regarding each \( x \in A \) as the constant sequence \( (x,x,...) \). Given a flow \( \alpha \) on \( A \) we denote by \( \ell^\infty_\alpha(A) \) the \( C^*\)-subalgebra of \( x = (x_n) \in \ell^\infty(A) \) for which \( t \mapsto \alpha_t(x) = (\alpha_t(x_n)) \) is norm-continuous and define \( A^\infty_\alpha \) as its image in \( A^\infty \). We naturally have the flow \( \overline{\alpha} \) on \( A^\infty_\alpha \) induced by \( \alpha \). We will also denote by \( \alpha \) the restriction of \( \overline{\alpha} \) to \( A' \cap A^\infty_\alpha \).

The following properties shared by AF flows and their cocycle perturbations could be used to distinguish them from other flows \[13\] \[2\]; similar properties are also considered for Rohlin flows \[15\] \[16\].

**Proposition 2.1** Let \( A \) be a unital AF algebra and let \( \alpha \) be a cocycle perturbation of an AF flow on \( A \). Then \( (A^\infty_\alpha \cap A')^\alpha \) has real rank zero and has trivial \( K_1 \). Moreover \( (A^\infty_\alpha)^\alpha \) has real rank zero and trivial \( K_1 \).

**Proof.** The latter part is shown in 3.8 of \[2\] and 3.6 and 4.1 of \[13\]. The first part also follows similarly; but we will indicate how to prove it.
Apparently we may suppose that $\alpha$ is an AF flow. Hence we suppose that there is an increasing sequence $(A_n)$ of $\alpha$-invariant finite-dimensional $C^*$-subalgebras of $A$ with dense union.

Let $b^* = b \in (A^\infty_n \cap A')^\alpha$. Then there is a bounded sequence $(b_n)$ in $A_{sa}$ such that $b \sim (b_n)$ (i.e., $b = (b_n) + c_0(A)$). We may suppose that $\|\delta_\alpha(b_n)\| \to 0$ as $n \to \infty$ and that there are increasing sequences $(k_n)$ and $(\ell_n)$ in $\mathbb{N}$ such that $k_n < \ell_n$, $k_n \to \infty$, and $b_n \in B_n \equiv A_{k_n} \cap A_k'$. (The latter follows because $\bigcup_n A_n$ is a core for $\delta_\alpha$.) Since $B_n$ is $\alpha$-invariant and finite-dimensional, there is a $h_n^* = h_n \in B_n$ such that $\delta_\alpha|B_n = \text{ad} h_n|B_n$. Since $\|[h_n, b_n]\| \to 0$ and $h_n, b_n \in (B_n)_{sa}$, we get $h_n', b_n' \in B_n$ such that $\|h_n - h_n'\| \to 0$, $\|b_n - b_n'\| \to 0$, and $[h_n', b_n'] = 0$ (see 3.1 of [2], which is an improvement of Lin’s result [17]).

Let $\epsilon > 0$ and let $F$ be a finite subset of the spectrum $\text{Spec}(b)$ of $b$ such that any $\lambda \in \text{Spec}(b)$ has $p \in F$ such that $|\lambda - p| < \epsilon$. Then we find a $b_n'' \in (B_n)_{sa}$ such that $b_n''$ is a function of $b_n'$, $\limsup_n \|b_n' - b_n''\| < \epsilon$, and $\text{Spec}(b_n'') \subseteq F$. Then $(b_n'')$ defines a self-adjoint element $c \in (A^\infty_n \cap A')^\alpha$ such that $\|c - b\| < \epsilon$ and $\text{Spec}(c) \subseteq F$, which is finite. This concludes the proof that $(A^\infty_n \cap A')^\alpha$ has real rank zero.

Let $u$ be a unitary in $(A^\infty_n \cap A')^\alpha$. Then as before we may suppose that there is a sequence $(u_n)$ in $U(A)$ and increasing sequences $(\ell_n)$ and $(k_n)$ in $\mathbb{N}$ such that $k_n < \ell_n$, $k_n \to \infty$, $u_n \in A_{k_n} \cap A_k'$, and $\|\delta_\alpha(u_n)\| \to 0$. There is an $h_n^* = h_n \in B_n \equiv A_{k_n} \cap A_k'$ such that $\delta_\alpha|B_n = \text{ad} i h_n|B_n$. Then by using the condition that $\|[u_n, h_n]\| \to 0$, we apply 4.1 of [13].

There are some examples of approximately inner flows on an AF algebra without the above types of properties (see Section 3 of [13] and 3.11 of [2], where only the properties for $(A^\infty_n)^\alpha$ are explicitly mentioned). Those examples are of the following type. Let $C$ be a maximal abelian $C^*$-subalgebra (masa) of $A$ and choose a sequence $(h_n)$ in $C_{sa}$ such that the graph limit $\delta$ of $(\text{ad} i h_n)$ is densely-defined and hence generates a flow [3] [20]. This is what we have as the examples and might be called a quasi AF flow (or a commutative flow following [20]). Note that the domain $\mathcal{D}(\delta)$ of $\delta$ contains the masa $C$ (which is actually a Cartan masa in our examples); but depending on $(h_n)$ it may contain another masa as well. (We know of no example of a generator whose domain does not contain a masa.)

Remark 2.2 For a flow $\alpha$ on a unital simple AF algebra $A$ it is shown in [13] that $\alpha$ is a cocycle perturbation of an AF flow if and only if the domain $\mathcal{D}(\delta_\alpha)$ contains a canonical AF masa of $A$, where $C$ is a canonical AF masa if there is an increasing sequence $(A_n)$ of finite-dimensional $C^*$-subalgebras of $A$ with dense union such that $C$ is the closure of $\bigcup_n C \cap A_n \cap A_{n-1}' = A_0 = 0$.

We note the following uniqueness result for canonical AF masas (cf. [19]).

Proposition 2.3 The canonical AF masas of an AF algebra are unique up to automorphism, i.e., if $A$ is an AF algebra and $C_1$ and $C_2$ are canonical AF masas of $A$, then there is an automorphism $\phi$ of $A$ such that $\phi(C_1) = C_2$. 
Proof. There are increasing sequences \((A_n)\) and \((B_n)\) of finite-dimensional \(C^*\)-subalgebras of \(A\) such that \(\bigcup_n A_n\) and \(\bigcup_n B_n\) are dense in \(A\) and \(C_1\) (resp. \(C_2\)) is the closure \(\bigcup_n C_1 \cap A_n \cap A'_n\) (resp. \(\bigcup_n C_2 \cap B_n \cap B'_n\)). By passing to subsequences, we find sequences \((u_n)\) and \((v_n)\) in \(\mathcal{U}(A)\) such that \(u_1 = 1\), \(\|u_n - 1\| < 2^{-n}\), \(\|v_n - 1\| < 2^{-n}\), \(u_{n+1} \in \text{Ad}(u_n u_{n-1} \cdots u_1)(A_n)\)’, \(v_{n+1} \in \text{Ad}(v_n \cdots v_1)(B_n)\)’, and

\[
\text{Ad}(u_n \cdots u_1)(A_n) \subset \text{Ad}(v_n \cdots v_1)(B_n) \subset \text{Ad}(u_{n+1} u_n \cdots u_1)(A_{n+1}),
\]

for all \(n\). Let \(u = \lim_n u_n u_{n-1} \cdots u_1\) and \(v = \lim_n v_n \cdots v_1\). Then \(u\) and \(v\) are unitaries in \(A\) (or \(A + C1\) if \(A \not\cong 1\)) such that \(u A_n u^* \subset v B_n v^* \subset u A_{n+1} u^*\) for all \(n\). Since \(u C_1 u^*\) and \(v C_2 v^*\) are also canonical AF masas, we may suppose that \(A_n \subset B_n \subset A_{n+1}\). We choose maximal abelian \(C^*\) subalgebras \(D_n\) of \(B_n \cap A_n'\) for \(n = 1, 2, \ldots\) and \(E_n\) of \(A_{n+1} \cap B_n\) for \(n = 0, 1, 2, \ldots\) with \(B_0 = 0\). Then the \(C^*\)-subalgebra \(D\) generated by \(D_n\) and \(E_n\) for all \(n\) is a canonical AF masa of \(A\). Since the \(C^*\)-algebra generated by \(D_n\) and \(E_n\) is isomorphic to \(C_1 \cap A_{n+1} \cap A'_n\) for \(n = 0, 1, 2, \ldots\) with \(D_0 = 0\), there is a unitary \(u_n \in A_{n+1} \cap A'_n + 1\) such that \(\text{Ad} u_n (C_1 \cap A_{n+1} \cap A'_n) = C^*(D_n, E_n)\). Since the limit of \(\text{Ad}(u_0 u_1 u_2 \cdots u_n)\) defines an automorphism of \(A\), there is an automorphism \(\phi_1\) of \(A\) such that \(\phi_1(C_1) = D\). Similarly, since the \(C^*\)-algebra generated by \(E_n\) and \(D_{n+1}\) is isomorphic to \(C_2 \cap B_{n+1} \cap B'_n\), there is an automorphism \(\phi_2\) such that \(\phi_2(C_2) = D\). Thus \(\phi_2^{-1} \phi_1(C_1) = C_2\), which concludes the proof.  

\[\square\]

3 Matrix algebras

In this section we shall prove Proposition \[15\].

Let \(\tau\) denote the unique tracial state of \(A = M_{N}\), i.e., \(\tau = (1/N)\text{Tr}\), and define an inner product on \(A\) by \(\langle x, y \rangle = \tau(y^* x) = \tau(x y^*)\), \(x, y \in A\). Equipped with this inner product, \(A\) is an \(N^2\)-dimensional Hilbert space, which we will denote by \(A_{\tau}\). We define a representation \(\rho\) on \(A_{\tau}\) of the tensor product \(A \otimes A\) by \(\rho(x \otimes y) \xi = x \xi y^*\), \(\xi \in A_{\tau}\), and a representation \(\pi\) of \(A\) by \(\pi(x) = \rho(x \otimes 1)\), \(x \in A\). Here \(y^t\) denotes the transpose of \(y \in A = M_N\). Note that \(\rho\) is irreducible and the state \(\omega\) of \(A \otimes A\) defined by \(\omega(x \otimes y) = \langle \rho(x \otimes y), 1 \rangle = \tau(xy^t)\) satisfies the condition that \(\omega|B \otimes B^t\) is pure because \(B \otimes B^t \cong M_{K^2}\) and the subspace \(\rho(B \otimes B^t)1 = B\) is \(K^2\)-dimensional.

Let \(U_t = \exp it(H \otimes 1 - 1 \otimes H^t) = e^{it(H \otimes 1)} e^{-it(1 \otimes H^t)}\) and let \(\gamma\) denote the flow \(t \mapsto \text{Ad} U_t\) on \(A \otimes A\). Then \(\gamma_t(x \otimes y^t) = \alpha_t(x) \otimes \alpha_t(y)^t\) for \(x, y \in A\), where \(\alpha\) is the flow on \(A\) defined by \(\alpha_t(x) = \text{Ad} e^{itH}(x)\). We should note that \(U\) has the following properties: \(\rho(U_t)1 = 1\) and \(\text{Ad} \rho(U_t) \pi(x) = \pi \alpha_t(x)\), \(x \in A\).

If \(\alpha_s(B) \subset B\) for \(s \in [0, 1]\) for a \(\delta > 0\) with \(\delta < 10^{-4}\), then Christensen \[6\] shows that there is a unitary \(v_s \in \mathcal{U}(A)\) such that \(\|1 - v_s\| < 120 \delta^{1/2}\) and

\[v_s \alpha_s(B)v_s^* = B.\]
Since \((v_s \otimes \overline{v}_s)(\alpha_s(B) \otimes \alpha_s(B)^t)(v_s \otimes \overline{v}_s)^* = B \otimes B^t\), we have that
\[
\gamma_t(B \otimes B^t) \overset{480s^{1/2}}{\subset} B \otimes B^t,
\]
for \(t \in [0, 1]\).

Hence the flow \(\gamma\) on \(A \otimes A\) satisfies the condition of Lemma 3.1 below for \((A, \alpha)\) with \(B \otimes B^t\) and \(\omega\) in place of \(B\) and \(\phi\) respectively if we start with the small enough \(\delta > 0\). Thus we get a \(u \in \mathcal{U}(A \otimes A)\) and \(h \in (A \otimes A)_{sa}\) such that \(\|u - 1\| \approx 0\), \(\|h\| \approx 0\), and \(\gamma_t'(B \otimes B^t) = B \otimes B^t,\ t \in \mathbb{R},\) where
\[
\gamma_t' = \text{Ad}(u^* e^{i(H \otimes 1 - 1 \otimes H^t + h)}u)
\]
is a cocycle perturbation of \(\gamma_t = \text{Ad} U_t\).

Let \(\phi\) be a pure ground state of \(A \otimes A\) with respect to \(\gamma\). Then \(\phi\) is a product state, i.e., \(\phi = (\phi|B \otimes B^t) \otimes (\phi|(A \cap B') \otimes (A \cap B')^t)\). We consider \(B\) and \(B^t\) (resp. \(A \cap B' \cong M_{N/K}\) and \((A \cap B'^t)\)) irreducibly acting on \(C^K\) (resp. \(C^{N/K}\)); and then \(B \otimes B^t\) on \(C^K \otimes C^K\) (resp. \(A \cap B' \otimes (A \cap B'^t)\) on \(C^{N/K} \otimes C^{N/K}\)). Then there are unit vectors \(\Phi_1 \in C^K \otimes C^K\) and \(\Phi_2 \in C^{N/K} \otimes C^{N/K}\) such that
\[
\phi(x \otimes y) = \langle x \Phi_1, \Phi_1 \rangle \langle y \Phi_2, \Phi_2 \rangle, \ x \in B \otimes B^t, \ y \in A \cap B' \otimes (A \cap B'^t).
\]
Let \(\Phi = \Phi_1 \otimes \Phi_2\) and note that
\[
u^*(H \otimes 1 - 1 \otimes H^t + h)u\Phi = E_0\Phi,
\]
where \(E_0\) is the minimum of the spectrum of \(H \otimes 1 - 1 \otimes H^t + h\).

We may assume that \(\min \text{Spec}(H) = 0\) and let \(E_1 = \max \text{Spec}(H), \ i.e.,\ 0, E_1 \in \text{Spec}(H) \subset [0, E_1]\). From \(H \otimes 1 - 1 \otimes H^t - \|h\| \leq H \otimes 1 - 1 \otimes H^t + h \leq H \otimes 1 - 1 \otimes H^t + \|h\|\), it follows that \(-E_1 - \|h\| \leq E_0 \leq -E_1 + \|h\|\). Hence we have that
\[
u^*(H \otimes 1 - 1 \otimes H^t + E_1)u\Phi = (E_0 + E_1)\Phi - u^* hu\Phi
\]
has norm less than \(2\|h\| \approx 0\). Hence the distance of \(u\Phi\) to the spectral subspace of \(H \otimes 1 - 1 \otimes H^t\) corresponding to \([-E_1, -E_1 + \epsilon]\) is sufficiently small for some small \(\epsilon > 0\) (depending on \(\|h\|\) and \(\|u - 1\|\)). Since \(u\Phi \approx \Phi\), we thus find a unit vector \(\Psi\) in the spectral subspace of \(H \otimes 1 - 1 \otimes H^t\) corresponding to \([-E_1, -E_1 + \epsilon]\) such that \(\Psi \approx \Phi \equiv \Phi_1 \otimes \Phi_2\). Specifically we may assume that
\[
\langle \Phi, \Psi \rangle = \text{Re}\langle \Phi, \Psi \rangle > 1 - \epsilon.
\]
We should note that \(\Psi\) belong to the spectral subspace of \(H \otimes 1\) corresponding to \([0, \epsilon]\).

Let \(P_1\) (resp. \(P_2\)) be a maximal set of mutually orthogonal one-dimensional projections in \(B^t\) (resp. \((A \cap B'^t)\)). Since \(\sum_{p \in P_1} p = 1\) and \(\sum_{p \in P_2} p = 1\), we have that
\[
\sum_{p_1 \in P_1, p_2 \in P_2} \|\langle 1 \otimes p_1 p_2 \rangle \Phi\| \|\langle 1 \otimes p_1 p_2 \rangle \Psi\| \geq \sum \text{Re}\langle (1 \otimes p_1 p_2)\Phi, \Psi \rangle = \langle \Phi, \Psi \rangle > 1 - \epsilon
\]
and
\[ \sum_{p_1 \in P_1, p_2 \in P_2} \| (1 \otimes p_1 p_2) \Phi \| (1 \otimes p_1 p_2) \Psi \| \Phi_{p_1 p_2} - \Psi_{p_1 p_2} \|^2 < 2 \epsilon, \]
where \( \Phi_{p_1 p_2} \) (resp. \( \Psi_{p_1 p_2} \)) is the unit vector \( c(1 \otimes p_1 p_2) \Phi \) (resp. \( c(1 \otimes p_1 p_2) \Psi \)) with normalization constant \( c > 0 \). (If \( (1 \otimes p_1 p_2) \Phi = 0 \) (resp. \( (1 \otimes p_1 p_2) \Psi = 0 \), we can disregard it.) Hence there must be \( p_1 \in P_1 \) and \( p_2 \in P_2 \) such that \( (1 \otimes p_1 p_2) \Phi \neq 0 \), \( (1 \otimes p_1 p_2) \Psi \neq 0 \), and
\[ \| \Phi_{p_1 p_2} - \Psi_{p_1 p_2} \|^2 < 3 \epsilon \]
(if \( \epsilon \leq 1/3 \)). We define a representation \( \pi_0 \) of \( A \) on \( \mathcal{H}_{\pi_0} = \text{Ran}(1 \otimes p_1 p_2) \) by \( x \mapsto x \otimes p_1 p_2 \).

Since \( \pi_0 \) is irreducible and \( \Phi_{p_1int p_2}, \Psi_{p_1 int p_2} \in \mathcal{H}_{\pi_0} \), there is a unitary \( v \in A \) such that \( v \approx 1 \) (depending on \( \epsilon^{1/2} \)) and \( \pi_0(v) \Phi_{p_1 p_2} = \Psi_{p_1 p_2} \). Furthermore, by the choice of \( \Psi \), there is a \( k \in A_{sa} \) such that \( k \approx 0 \) (depending on \( \epsilon \)) and \( \pi_0(H) \Psi_{p_1 p_2} = -\pi_0(k) \Psi_{p_1 p_2} \). Hence we have that
\[ \pi(v^*(H + k)v) \Phi_{p_1 p_2} = 0. \]

Note that the state \( \phi' \) of \( A \cong A \otimes 1 \) defined through \( \pi_0 \) by the unit vector \( \Phi_{p_1 p_2} = c(1 \otimes p_1) \Phi_1 \otimes (1 \otimes p_2) \Phi_2 \) with normalization constant \( c > 0 \) is a pure product state with respect to \( A = B \otimes (A \cap B') \). Since \( \text{Ad}(v^* e^{it(H+k)} v)(B) \delta' \subseteq B \) with \( \delta' = \delta + 2 \| v - 1 \| + 2 \| k \| \) for \( t \in [0, 1] \), we again reach the situation where the following lemma is applicable, which is already used once before.

**Lemma 3.1** For any \( \epsilon > 0 \) there exists a \( \delta > 0 \) satisfying the following condition: Let \( A = M_N \) for some \( N \in \mathbb{N} \), \( B \) a unital \( \ast \)-subalgebra of \( A \) with \( B \cong M_K \) for some \( K \) dividing \( N \), \( H \in A_{sa} \), \( \phi_1 \) a pure state of \( B \), and \( \phi_2 \) a pure state of \( A \cap B' \) such that
\[ e^{itH} B e^{-itH} \\delta \subseteq B \]
for \( t \in [0, 1] \) and the state \( \phi_1 \otimes \phi_2 \) of \( A = B \otimes (A \cap B') \) is left invariant under the action \( \alpha : t \mapsto \text{Ad} e^{itH} \). Then there exist a \( u \in \mathcal{U}(A) \) and \( h \in A_{sa} \) such that \( \| u - 1 \| < \epsilon \), \( \| h \| < \epsilon \), and
\[ u^* e^{it(H+h)} u B u^* e^{-it(H+h)} u = B, \quad t \in \mathbb{R}. \]

**Proof.** In the GNS representation \( (\pi_\phi, \mathcal{H}_\phi, \Omega_\phi) \) of \( A \) associated with \( \phi = \phi_1 \otimes \phi_2 \) we define a projection \( E \) on \( \mathcal{H}_\phi \) by \( E = [\pi_\phi(B) \Omega_\phi] \). Here \([S] \) means either the projection onto the closed linear span of \( S(\subseteq \mathcal{H}_\phi) \) or the closed linear span itself. Then, by Lemma 3.3 below, \( \max_{t \in [0,1]} \| e^{itH} E e^{-itH} - E \| \) is arbitrarily small depending on \( \delta \). Hence, by Lemma 3.4 below, there is a \( u \in \mathcal{U}(A) \) and \( h \in A_{sa} \) such that \( u \approx 1, h \approx 0, \pi_\phi(u) \Omega_\phi = \Omega_\phi, \pi_\phi(h) \Omega_\phi = 0 \), and
\[ \text{Ad} \pi_\phi (u^* e^{it(H+h)} u)(E) = E. \]

Thus we may just as well assume that \( \text{Ad} \pi_\phi (e^{itH})(E) = E \) retaining the assumptions of the Lemma.
Note that $E \in \pi_\phi(B)'$. Furthermore if $x \in B$, then $\pi_\phi\alpha_t(x)E = \pi_\phi(e^{itH}xe^{-itH})E = E\pi_\phi\alpha_t(x)$, which implies that $\pi_\phi\alpha_t(B)E = \pi_\phi(B)E$. Since $x \mapsto \pi(x)E$ is an isomorphism of $B$ onto $\pi_\phi(B)E$, one can find a map $\beta_t$ of $B$ into $B$ such that $\pi_\phi\alpha_t(x)E = \pi_\phi\beta_t(x)E$, $x \in B$. We can show that $t \mapsto \beta_t$ is a flow on $B$; e.g., $\pi_\phi(\beta_t(x))E = \pi_\phi(\alpha_t(\beta_t(x)))E = \pi_\phi(e^{itH})\pi_\phi(\beta_t(x))E\pi_\phi(e^{-itH}) = \pi_\phi(e^{itH})\pi_\phi(\alpha_t(x))E\pi_\phi(\beta_t(x))E$.

Let $e_1 \in B$ be a minimal projection such that $\pi_\phi(e_1)\Omega_\phi = \Omega_\phi$. Since $\pi_\phi\alpha_t(e_1)\Omega_\phi = \Omega_\phi$ (because $\phi$ is $\alpha$-invariant), it follows that $\pi_\phi(\alpha_t(e_1))E \geq \pi_\phi(e_1)E$, which implies that $\pi_\phi(\alpha_t(e_1))E = \pi_\phi(e_1)E$ and $\beta_t(e_1) = e_1$. Since $\alpha_t(B) \subset B$ for $t \in [0, 1]$, there is an $e_t \in B$ such that $\|\alpha_t(e_1) - e_t\| \leq \delta$ for $t \in [0, 1]$. This implies that $\|\pi_\phi(e_1 - e_t)E\| \leq \delta$, i.e., $\|e_1 - e_t\| \leq \delta$. Hence we get that $\|\alpha_t(e_1) - e_1\| \leq 2\delta$ for $t \in [0, 1]$.

We will then argue that there is a projection $p \in A$ and a unitary $v \in A$ such that $p \approx e_1$, $\|[H, p]\| \approx 0$, $v \approx 1$, $ve_1v^* = p$, $\pi_\phi(p)\Omega_\phi = \Omega_\phi$, and $\pi_\phi(E) = E$.

Let $f$ be a non-negative $C^\infty$ function on $\mathbb{R}$ of compact support such that $\int f(t)dt = 1$, and consider

$$
\int f(t)\alpha_t(e_1)dt,
$$

which is still close to $e_1$ for a small $\delta > 0$. By functional calculus construct a projection $p$ out of it, which satisfies that $\|[H, p]\| \approx 0$ depending on $\int |f'(t)|dt$ (see the proof of 3.4 below). Since

$$
\pi_\phi(\int f(t)\alpha_t(e_1)dt)E = \pi_\phi(e_1)E,
$$

we get that $\pi_\phi(E) = E\pi_\phi(p) = E\pi_\phi(e_1)$.

Consider

$$
z = pe_1 + (1 - p)(1 - e_1),
$$

which satisfies that $z \approx 1$, $\pi_\phi(z)E = E\pi_\phi(z) = E$, and $ze_1 = pz$, and construct a unitary $v$ by the polar decomposition of $z = |z|v$. Then it follows that $\pi_\phi(v)E = E$ and $ve_1v^* = p$.

Let $h = -[H, p]p + [H, p] = -(1 - p)Hp - pH(1 - p)$, which has small norm as we have assumed that $\|[H, p]\| \approx 0$. Then we have that $\pi_\phi(h)E = 0$ and $[H + h, p] = [H, p] - (1 - p)Hp + pH(1 - p) = 0$. Hence we have reached the following situation:

The flow $t \mapsto Ad(v^*e^{it(H+h)}v)$ leaves $e_1$ invariant and also the projection $E$ onto $\pi_\phi(B)\Omega_\phi$ invariant. Thus we may just as well assume that $\alpha_t = AdU_t$ leaves $e_1 \in B$ invariant, in addition to the conditions already assumed.

Recall that we have defined the flow $\beta$ on $B$ by $\pi_\phi(\beta_t(x))E = \pi_\phi(\alpha_t(x))E$, $x \in B$. Since $\alpha_t(B) \subset B$ for $t \in [0, 1]$, we have, for $x \in B$ with $\|x\| \leq 1$ and $t \in [0, 1]$, an $x_t \in B$ such that $\|\alpha_t(x) - x_t\| \leq \delta$, which implies that $\|\pi_\phi(\alpha_t(x)) - \pi_\phi(\alpha_t(x))E\| \leq \delta$. Hence we have that $\|\pi_\phi(x_t) - \pi_\phi(\beta_t(x))\| \leq \delta$ or $\|x_t - \beta_t(x)\| \leq \delta$. Thus we obtain that $\|\alpha_t(x) - \beta_t(x)\| \leq 2\delta$

for $x \in B$ with $\|x\| \leq 1$ and $t \in [0, 1]$. 

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Since $\beta$ is a flow on $B \cong M_K$, there is a set $(a_{ij})$ of matrix units for $B$ such that $e_{11} = e_1$ and $\beta_t(e_{ij}) = e^{i(p_1 - p_2)}e_{ij}$ for some $p_1, p_2, \ldots, p_K \in \mathbb{R}$. We then define

$$v_t = \sum_{j=1}^{K} e^{ip_j}e_{j1}\alpha_t(e_{ij}).$$

Then, since $\alpha_t(e_{11}) = e_{11}$, $v_t$ is a unitary in $A$ and $t \mapsto v_t$ is an $\alpha$-cocycle such that $v_te_{11} = e_{11}$ and $\pi_\phi(v_t)E = E$. It follows that $\Ad v_t\alpha_t | B = \beta_t | B$, since

$$\Ad v_t\alpha_t(e_{ij}) = e^{ip_t}e_{i1}\alpha_t(e_{11})\alpha_t(e_{ij})\alpha_t(e_{j1})e_{ij}e^{-ip_t} = e^{i(p_1 - p_2)}e_{ij}.$$

Since $\|\| (\alpha_t - \beta_t)B \| \leq 2\delta$ for $t \in [0, 1]$, we get that $\|\| (\Ad v^*_t - \Id)B \| = \|\| (\Ad v^*_t\beta_t - \beta_t)B \| \leq 2\delta$. Now we assert that $\max_{t \in [0, 1]} \| v_t - 1 \|$ is small depending on $\delta$ (but without depending on the sizes of $A$ and $B$).

Since $v^*_t B v_t = \alpha_t(B) \subseteq B$ for $t \in [0, 1]$, there is a $w_t \in \mathcal{U}(A)$ such that $\| w_t - 1 \| \leq 120\delta^{1/2}$ and $w^*_t B v_t w_t = B$ [13]. Since $\|\| (\Ad (w_t v^*_t)B) \| \leq 240\delta^{1/2} + 2\delta$, there exists a $u_t \in \mathcal{U}(B)$ such that $\| u_t - 1 \|$ is of the order $\|\| \Ad (w_t v^*_t) - \Id \|^{1/2}$ and $\Ad (w_t v^*_t)B = \Ad u^*_t | B$ (see 8.7.5 of [18]). That is, we have that $z_t = u_t w_t v^*_t \in A \cap B'$, or

$$v_t = z^*_tu_tw_t.$$

Since $e_{11} = e_{11}v_t = e_{11}z_t \cdot e_{11}u_tw_t$ and $e_{11}u_tw_t \approx e_{11}$ (because $u_t \approx 1 \approx w_t$), it follows that $e_{11}z_t \approx e_{11}$. Since $\| z_t - 1 \| = \| (z_t - 1)e_{11} \|$ (because $z_t \in A \cap B'$), it follows that $z_t \approx 1$. Thus we get that $v_t \approx 1$. Hence incorporating all the cocycle perturbations made we have reached the following situation: There is a cocycle $u$ with respect to $\alpha : t \mapsto \Ad e^{itH}$ on $A$ such that $\sup_{t \in [0, 1]} \| u_t - 1 \| < \epsilon$ and $\Ad u_t\alpha_t(B) = B$. Then the conclusion will follow from the following lemma.

\[\square\]

**Lemma 3.2** For any $\epsilon > 0$ there exists a $\delta > 0$ satisfying the following condition: If $\alpha$ is a flow on a unital $C^*$-algebra $A$ and $u$ is an $\alpha$-cocycle such that $\| u_t - 1 \| < \delta$, $t \in [0, 1]$, there are $v \in \mathcal{U}(A)$ and a differential $\alpha$-cocycle $w$ such that $\| v - 1 \| < \epsilon$, $\| dw_t/\| t=0 \| < \epsilon$ for $t \in [0, 1]$, and $u_t = vw_t\alpha_t(v^*)$.

**Proof.** This can be shown by using the 2 by 2 trick due to Connes (see [12]). Namely define a flow $\gamma$ on $M_2 \otimes A$ by $\gamma_t(e_{11} \otimes x) = e_{11} \otimes \alpha_t(x)$, $\gamma_t(e_{12} \otimes x) = e_{12} \otimes \alpha_t(x) u_t^*$, etc. and note that $\gamma_t(e_{21} \otimes 1) = e_{21} \otimes u_t$. We approximate $e_{21} \otimes 1$ by $e_{21} \otimes v$ with $v \in \mathcal{U}(A)$ such that $d\gamma_t(e_{21} \otimes v)/dt$ has small norm. Let $w_t = v^* u_t \alpha_t(v)$, which is an $\alpha$-cocycle. Since

$$\gamma_t(e_{21} \otimes v) = e_{21} \otimes u_t \alpha_t(v) = e_{21} \otimes vw_t,$$

this concludes the proof. \[\square\]
Lemma 3.3 For any $\epsilon > 0$ there exists a $\delta > 0$ satisfying the following condition: Let $A$ be a unital $C^*$-algebra, $A_1$ a unital $C^*$-subalgebra of $A$ with $A_1 \cong M_K$ for some $K \in \mathbb{N}$, and $\alpha$ a flow on $A$ such that

$$\sup_{t \in [0,1]} \text{dist}(A_1, \alpha_t(A)) < \delta.$$ 

Let $\phi$ be an $\alpha$-invariant pure state of $A$ such that $\phi|A_1$ is pure and let $E = [\pi_\phi(A_1)\Omega_\phi].$ 

If $U$ denotes the unitary flow on the Hilbert space $\mathcal{H}_\phi$ defined by $U_t\pi_\phi(x)\Omega_\phi = \pi_\phi(\alpha_t(x))\Omega_\phi, \ x \in A,$ then it follows that $$\sup_{t \in [0,1]} \|U_tEU_t^* - E\| < \epsilon.$$ 

Proof. Since $\phi|A_1$ is pure, there is a minimal projection $e \in A_1$ such that $\pi_\phi(e)\Omega_\phi = \Omega_\phi.$ Note that $\pi_\phi(\alpha_t(e))\Omega_\phi = U_t\pi_\phi(e)\Omega_\phi = \Omega_\phi.$ We will assert that $\|e - \alpha_t(e)\|$ is small.

Let $t \in [0,1].$ There is an $x \in \alpha_t(A_1)$ such that $\|x - e\| < \delta$ and $x^* = x.$ Since

$$\pi_\phi(\alpha_t(e)x\alpha_t(e))\Omega_\phi - \Omega_\phi = \pi_\phi(\alpha_t(e)x)\Omega_\phi - \pi_\phi(\alpha_t(e)e)\Omega_\phi = \pi_\phi(\alpha_t(e)(x - e))\Omega_\phi,$$

we have that $\|\pi_\phi(\alpha_t(e)x\alpha_t(e))\Omega_\phi - \Omega_\phi\| < \delta.$ Since $\alpha_t(e)$ is a minimal projection in $\alpha_t(A_1)$, there is a $\lambda \in \mathbb{R}$ such that $\alpha_t(e)x\alpha_t(e) = \lambda\alpha_t(e).$ The above inequality shows that $|\lambda - 1| < \delta$ or $\|\alpha_t(e)x\alpha_t(e) - \alpha_t(e)\| < \delta.$ Since $\|x - e\| < \delta,$ we then have that $$\|\alpha_t(e) - \alpha_t(e)e\alpha_t(e)\| < 2\delta.$$ 

Hence it follows that $$\|\alpha_t(e) - \alpha_t(e)e\|^2 = \|\alpha_t(e)(1-e)\alpha_t(e)\| < 2\delta.$$ 

In the same way we get that $\|e - e\alpha_t(e)\|^2 < 2\delta.$ Since $\|e - \alpha_t(e)\| \leq \|e - e\alpha_t(e)\| + \|e\alpha_t(e) - \alpha_t(e)\|,$ we obtain that $\|e - \alpha_t(e)\| < 2\sqrt{2}\delta < 3\delta^{1/2}.$

Let $w \in A_1$ be a partial isometry such that $w^*w = e.$ We will assert that there is a partial isometry $y \in \alpha_t(A_1)$ such that $y^*y = \alpha_t(e)$ and $\|y - w\|$ is small.

Let $z \in \alpha_t(A_1)$ such that $\|z - w\| < \delta.$ Note that $\|z\alpha_t(e) - w\| \leq \|(z - w)\alpha_t(e)\| + \|w(\alpha_t(e) - e)\| < \delta + 3\delta^{1/2}.$ If $\delta$ is sufficiently small, then $\lambda = \|z\alpha_t(e)\|$ is close to $1,$ as $|\lambda - 1| < \delta + 3\delta^{1/2}. Then y = \lambda^{-1}z\alpha_t(e)$ is a partial isometry in $\alpha_t(A_1)$ such that $y^*y = \alpha_t(e)$ and $\|y - w\| < 2\delta + 6\delta^{1/2}.$ The latter follows because $\|y - w\| = \|\lambda^{-1}z\alpha_t(e) - w\| \leq \|z\alpha_t(e) - w\| + \|(\lambda^{-1} - 1)z\alpha_t(e)\| < \delta + 3\delta^{1/2} + |1 - \lambda|.$
Let $\overline{\alpha}_t = \text{Ad} U_t$ as a weakly continuous flow on $B(\mathcal{H}_\phi)$ and note that $\overline{\alpha}_t(E)\mathcal{H}_\phi = [\pi_\phi(\alpha_t(A_1))\Omega_\phi]$. We will assert that

$$\inf \text{Spec}(E\overline{\alpha}_t(E)) = \inf_{\xi} \|\overline{\alpha}_t(E)E\xi\|^2$$

is close to 1, where the spectrum is taken as an operator on $E\mathcal{H}_\phi$ and the infimum is taken over all unit vectors $\xi \in E\mathcal{H}_\phi = [\pi_\phi(A_1)\Omega_\phi]$. Note that this infimum is obtained as

$$\inf_{w} \sup_{y} |\langle\pi_\phi(y)\Omega_\phi, \pi_\phi(w)\Omega_\phi\rangle|^2,$$

where $w$ runs over all $w \in A_1$ with $w^*w = e$ and $y$ runs over all $y \in \alpha_t(A_1)$ with $y^*y = \alpha_t(e)$. For each $w \in A_1$ with $w^*w = e$, we choose $y \in \alpha_t(A_1)$ with $y^*y = \alpha_t(e)$ such that $\|w - y\| < 2\delta + 6\delta^{1/2}$. Since $\|y^*w - e\| \leq \|(y^* - w^*)w\| \leq \|y - w\|$ and

$$\langle\pi_\phi(y)\Omega_\phi, \pi_\phi(w)\Omega_\phi\rangle = 1 - \langle\Omega_\phi, \pi_\phi(e - y^*w)\Omega_\phi\rangle,$$

We have that

$$|\langle\pi_\phi(y)\Omega_\phi, \pi_\phi(w)\Omega_\phi\rangle| > 1 - (2\delta + 6\delta^{1/2}).$$

Hence we get that

$$\inf \text{Spec}(E\overline{\alpha}_t(E)) > 1 - (4\delta + 12\delta^{1/2}).$$

Thus, if $t \in [0, 1]$, we have that $\|E\overline{\alpha}_t(E)E - E\| < \delta_1$, where $\delta_1 = 4\delta + 12\delta^{1/2}$. In the same way we have that $\|\pi_t(E)E\overline{\alpha}_t(E) - \pi_t(E)\| < \delta_1$. Then we get $\|E - \pi_t(E)\| < 3\delta_1^{1/2}$ as before. Since $\delta_1^{1/2} \approx 2\sqrt{3}\delta^{1/4}$, this concludes the proof. $\square$

**Lemma 3.4** For any $\epsilon > 0$ there exists a $\delta > 0$ satisfying the following condition: Let $\mathcal{H}$ be a Hilbert space, $E$ a projection on $\mathcal{H}$, and $H$ a self-adjoint operator on $\mathcal{H}$ such that

$$\|e^{itH}Ee^{-itH} - E\| < \delta$$

for $t \in [0, 1]$. Then there exist a unitary $u$ on $\mathcal{H}$ and a bounded self-adjoint operator $h$ on $\mathcal{H}$ such that $\|u - 1\| < \epsilon$, $\|h\| < \epsilon$, and

$$u^*e^{it(H+h)}uEe^{-it(H+h)}u = E.$$

Moreover if $\Omega \in \mathcal{H}$ is a unit vector such that $E\Omega = \Omega$ and $e^{itH}\Omega = \Omega$, the above $u$ and $h$ can be chosen so that $u\Omega = \Omega$ and $h\Omega = 0$.

**Proof.** Let $f$ be a non-negative $C^\infty$ function on $\mathbb{R}$ of compact support such that $\int f(t)dt = 1$. Then for any projection $E$ and a self-adjoint operator $H$ on $\mathcal{H}$, we define, for $n \in \mathbb{N}$,

$$E_n = \frac{1}{n} \int f(t/n)\text{Ad} e^{itH}(E)dt$$
and estimate
\[
\|[iH, E_n]\| \leq \frac{1}{n} \int |f'(t)| dt,
\]
which is close to zero depending on \(n\) only. If \(\delta\) is sufficiently small, i.e., \(\|E_n - E\| \approx 0\), then we can define a projection \(F_n\) by functional calculus by
\[
F_n = \frac{1}{2\pi i} \oint_C (E_n - z)^{-1} dz,
\]
where \(C\) is the path \(|z - 1| = 1/2\) (see, e.g., [20]). Here we may assume that the distance between \(C\) and \(\text{Spec}(E_n)\) is greater than \(1/4\). Since \(F_n\) is close to \(E_n\), this is also close to \(E\) depending on \(\|E - E_n\|\). Since
\[
[iH, F_n] = -\frac{1}{2\pi i} \oint_C (E_n - z)^{-1} [iH, E_n](E_n - z)^{-1} dz,
\]
the norm of \([iH, F_n]\) is smaller than \(16\|[iH, E_n]\|\). Since we may define \(u\) to be the unitary obtained from the polar decomposition of \(z = F_n E + (1 - F_n)(1 - E) = 1 - (1 - F_n)E - F_n(1 - E) \approx 1\) and \(h\) as \(-(1 - F_n)HF_n - F_n H(1 - F_n) \approx 0\), this completes the proof of the first part.

If \(\Omega\) is a unit vector such that \(E\Omega = \Omega\) and \(H\Omega = 0\), it follows that \(E_n \Omega = \Omega\), \(F_n \Omega = \Omega\), \(z \Omega = \Omega\), \(h \Omega = 0\), i.e., the last conditions follow automatically. \(\square\)

## 4 UHF algebras

The previous result 1.5 can be extended to approximate UHF flows.

**Proposition 4.1** For any \(\epsilon > 0\) there exists a \(\delta > 0\) satisfying the following condition: If \(\alpha\) is a flow on a UHF algebra \(A\) and \((A_n)\) is an increasing sequence \((A_n)\) of finite dimensional unital \(C^*\)-subalgebras of \(A\) such that \(\bigcup_n A_n = A\), \(A_n \cong M_{K_n}\) for all \(n\), and \(\delta_n \equiv \sup_{t \in [0, 1]} \text{dist}(\alpha_t(A_n), A_n) \to 0\) with \(\delta_1 \leq \delta\), then there is an \(\alpha\)-cocycle \(u\) such that \(\sup_{t \in [0, 1]} \|u_t - 1\| < \epsilon\) and \(\text{Ad } u_t \alpha_t(A_n) = A_n\) for \(n = 1\) and infinitely many \(n\).

**Proof.** Let \(\tau\) denote the tracial state of the UHF algebra \(A\). We denote by \((\pi_\tau, \mathcal{H}_\tau, \Omega_\tau)\) the GNS representation associated with \(\tau\) and recall that the canonical conjugation operator \(J\) is defined by
\[
J \pi_\tau(x) \Omega_\tau = \pi_\tau(x^*) \Omega_\tau, \quad x \in A.
\]
By using the fact that \(J \pi_\tau(A)^*J = \pi_\tau(A)'\), we define an irreducible representation \(\rho\) of \(A \otimes A^{op}\) in \(\mathcal{H}_\tau\) by
\[
\rho(x \otimes y) = \pi_\tau(x) J \pi_\tau(y^*) J, \quad x \otimes y \in A \otimes A^{op},
\]
where \(A^{op}\) is the opposite \(C^*\)-algebra of \(A\); i.e., \(A^{op} = A\) as a Banach space with the same involution and the new product \(x \circ y = yx\). We check that \(\rho\) is indeed a representation of
A \otimes A^\op by \rho(x \otimes y)\rho(a \otimes b) = \pi_\tau(xa)J\pi_\tau(y^*b^*)J = \pi_\tau(xa)J\pi_\tau((y \circ b)^*)J = \rho(xa \otimes y \circ b) and note that A \otimes A^\op is a UHF algebra with dense \bigcup_n A_n \otimes A_n. We define a state \omega of A \otimes A^\op by \omega(z) = (\rho(z)\Omega_\tau, \Omega_\tau). Since \rho(A_n \otimes A_n)\Omega_\tau = \pi_\tau(A_n)\Omega_\tau is \mathbb{K}_n^2-dimensional, we know that \omega|A_n \otimes A_n is pure for all n. (If (e_{ij}) is a family of matrix units for A_n, then p = \mathbb{K}_n^{-1} \sum_{ij} e_{ij} \otimes e_{ji} is a minimal projection in A_n \otimes A_n such that \rho(p)\Omega_\tau = \Omega_\tau.)

We define a unitary flow U in \mathcal{H}_\tau by

\[ U_t\pi_\tau(x)\Omega_\tau = \pi_\tau(\alpha_t(x))\Omega_\tau, \quad x \in A. \]

Since \[ U_t\rho(x \otimes y)\Omega_\tau = \pi_\tau(\alpha_t(x))\pi_\tau(\alpha_t(y))\Omega_\tau = \rho(\alpha_t(x) \otimes \alpha_t(y))\Omega_\tau, \]
we get that \[ \text{Ad} U_t \rho = \rho \circ (\alpha_t \otimes \alpha_t). \] Note that \[ \omega \circ (\alpha_t \otimes \alpha_t) = \omega. \]

If \delta_n is sufficiently small and \[ t \in [0,1], \]
there is a unitary v_n \in A, by [8], such that \[ \|v_n - 1\| \leq 120\delta_n^{1/2} \] and \[ v_n \alpha_t(A_n)v_n^* = A_n. \] Hence we get that

\[ \text{Ad}(v_n \otimes v_n^*)(\alpha_t \otimes \alpha_t)(A_n \otimes A_n) = A_n \otimes A_n. \]

This implies that \[ \text{dist}((\alpha_t \otimes \alpha_t)(A_n \otimes A_n), A_n \otimes A_n) \leq 480\delta_n^{1/2} \text{ for } t \in [0,1]. \]

Hence we shall first show the following weaker version of this proposition. \[ \square \]

**Lemma 4.2** For any \epsilon > 0 there exists a \delta > 0 satisfying the following condition: If \alpha is a flow on a UHF algebra A, \phi is an \alpha-invariant pure state of A, and (A_n) is an increasing sequence (A_n) of unital finite dimensional C*-subalgebras of A such that \bigcup_n A_n = A, A_n \cong M_{K_n} for all n, \phi|A_n is pure for all n, and \delta_n \equiv \sup_{t \in [0,1]} \text{dist}(\alpha_t(A_n), A_n) \to 0 as n \to \infty with \delta_1 \leq \delta, then there is an \alpha-cocycle u such that \|u_t - 1\| < \epsilon for all t \in [0,1] and \text{Ad} u_t \alpha_t(A_n) = A_n for n = 1 and infinitely many n.

We will prove the above lemma by induction. What we need is the following lemma for a unital C*-algebra A.

**Lemma 4.3** For any \epsilon > 0 there is a \delta > 0 satisfying the following condition. If \alpha is a flow on a unital C* algebra A, \phi is an \alpha-invariant pure state of A, and A_1 is a unital C*-subalgebra of A such that A_1 \cong M_K for some K, \phi|A_1 is pure, and

\[ \sup_{t \in [0,1]} \text{dist}(\alpha_t(A_1), A_1) < \delta, \]

then there is an \alpha-cocycle u such that \[ \|u_t - 1\| < \epsilon \text{ for all } t \in [0,1], \phi \text{Ad} u_t \alpha_t = \phi, \text{ and } \text{Ad} u_t \alpha_t(A_1) = A_1. \]

**Proof.** In the GNS representation (\pi_\phi, \mathcal{H}_\phi, \Omega_\phi) associated with \phi, we define a unitary flow U by \[ U_t \pi_\phi(x)\Omega_\phi = \pi_\phi(\alpha_t(x))\Omega_\phi, \quad x \in A \]
and denote by H the self-adjoint generator of U: \[ U_t = e^{itH}. \]

We define a projection E_1 in \mathcal{H}_\phi by

\[ E_1 = [\pi_\phi(A_1)\Omega_\phi]. \]
Note that $E_1$ is a projection in $\pi_\phi(A_1)'$ of rank $K$ and $A_1 \ni x \mapsto \pi_\phi(x)E_1$ is an irreducible representation of $A_1$. As in the proof of \[15\] $\|U_t E_1 U_t^* - E_1\|$ is close to zero depending on $\delta$. Then we find a projection $F$ of finite rank such that $E_1 \approx F$, $F \Omega_\phi = \Omega_\phi$, $\|H,F\| \approx 0$. We also find a unitary $Z$ on the subspace $L$ spanned by $E_1 H_\phi$ and $F H_\phi$ such that $Z \approx 1$, $Z \Omega_\phi = \Omega_\phi$, and $F = Z E_1 Z^*$. By using Kadison’s transitivity \[9, 18\], we find a $u \in \mathcal{U}(A)$ such that $\pi_\phi(u) L = Z L$ and $\|u - 1\| < 2 \|Z - 1\|$. We also find an $h \in A_{sa}$ such that $\pi_\phi(h) = -(1 - F)HF - FH(1 - F)$ on the subspace spanned by $F H_\phi$ and $(1 - F)HF H_\phi$, and $\|h\| < 2 \|1 - F\|HF\|$. Then it follows that $\pi_\phi(u) \Omega_\phi = \Omega_\phi$, $\pi_\phi(h) \Omega_\phi = 0$, $\pi_\phi(u) E_1 \pi_\phi(u^*) = F$, and $[H,F] = -[\pi_\phi(h),F]$ as well as $u \approx 1$ and $h \approx 0$. Note that all the estimates depend on $\delta$ but not on the size of $A_1$ and that $\text{Ad}(\pi_\phi(u)^* e^{itH + \pi_\phi(h)}) \pi_\phi(u^*) E_1 = 0$. If $e_1$ denotes the minimal projection of $A_1$ such that $\pi_\phi(e_1) \Omega_\phi = \Omega_\phi$, we may also suppose that $(\delta_\alpha + \text{ad}(ih)) \text{Ad} u(e_1) = 0$ (see the proof of \[15\]).

Let $\alpha'_t = \text{Ad} u e^{it(\delta_\alpha + \text{ad}(ih))} \text{Ad} u^*$, which is a small cocycle perturbation of $\alpha$. We then define a flow $\beta$ on $A_1$ by \[
\pi_\phi(\alpha'_t(x)) E_1 = \pi_\phi(\beta_t(x)) E_1, \quad x \in A_1.
\]

It follows from the commutativity of $\pi_\phi(u)^* e^{it(H + \pi_\phi(h))} \pi_\phi(u)$ and $E_1$ that $\beta$ is indeed a flow. Then we derive that $\|\langle \alpha'_t - \beta_t \rangle A_n\| \leq 2 \delta'$, where $\delta' = \sup_{t \in [0,1]} \text{dist}(\alpha'_t(A_1), A_1)$ which is small depending on the original $\delta \geq \sup_{t \in [0,1]} \text{dist}(\alpha_t(A_1), A_1)$. We choose a family ($e_{i_1}$) of matrix units for $A_1$ such that $\beta_t(e_{i_1}) = e^{ip_t} e_{i_1}$ for some $p_1, p_2, \ldots, p_K$ in $\mathbb{R}$ with $e_{11} = e_n$. Noting that $e_{11} = \alpha'_t(e_{i_1})$, we define an $\alpha'$-cocycle \[
v_t = \sum_i e^{ip_t} e_{i_1} \alpha'_t(e_{i_1}).
\]

We show that $\pi_\phi(v_t) \Omega_\phi = \Omega_\phi$, $v_t \approx 1$ (depending on $\delta'$; see the proof of \[5.1\]), and $\text{Ad} v_t \alpha'_t | A_1 = \beta_t | A_1$. Thus $\alpha''_t = \text{Ad} v_t \alpha'_t$, which is a small cocycle perturbation of $\alpha$, satisfies the required condition. Note that the estimate of how far $\alpha''$ is from $\alpha$ does not depend on the size of $A_1$, thanks to the estimate in 5.3 of \[6\]. \hfill $\square$

To prove Lemma \[1.2\] we apply the above lemma inductively. By choosing $\delta$ small enough, we guarantee the above lemma applies to $A_1$; thus we find an $\alpha$-cocycle $u^1$ such that $\max_{t \in [0,1]} \|u^1_t - 1\| < \epsilon/2$, $\alpha^1_t = \text{Ad} u^1_t \alpha_t$ fixes $A_1$, and $\phi^1_t = \phi$. To proceed to the next step, we just note that \[
\delta_n \equiv \sup_{t \in [0,1]} (\alpha^1_t(A_n \cap A'_1), A_n \cap A'_1) \to 0
\]
as $n \to \infty$. We find $n_2 > 1$ such that $\delta_{n_2}$ is sufficiently small so that we find an $\alpha^1$-cocycle $u^2$ in $A \cap A'_1$ such that $\max_{t \in [0,1]} \|u^2_t - 1\| < \epsilon/4$, $\alpha^2_t = \text{Ad} u^2_t \alpha^1_t$ fixes $A_{n_2}$, and $\phi^2_t = \phi$. We repeat this process inductively.
Lemma 4.4 For any $\epsilon > 0$ there exists a $\delta > 0$ satisfying the following condition: Let $\alpha$ be a flow on a unital $C^*$-algebra $A$ and $A_1$ a unital $C^*$-subalgebra of $A$ such that $A_1 \equiv M_K$ for some $K$. Let $u$ be an $\alpha \otimes \alpha$-cocycle, where $t \mapsto \alpha_t \otimes \alpha_t$ is a flow on $A \otimes A^{op}$, such that $\max_{t \in [0,1]} \|u_t - 1\| < \delta$ and $\text{Ad } u_t(\alpha_t \otimes \alpha_t)$ fixes $A_1 \otimes A_1$. Let $\phi$ be a pure ground state of $A \otimes A^{op}$ with respect to the flow $t \mapsto \text{Ad } u_t(\alpha_t \otimes \alpha_t)$ (such that $\phi | A_1 \otimes A_1$ is pure). Then there is an $\alpha$-cocycle $v$ such that $\max_{t \in [0,1]} \|v_t - 1\| < \epsilon$ and $\text{Ad } v_t \alpha_t$ fixes $A_1$.

Proof. We may suppose that $u$ is given as $w v_t(\alpha_t \otimes \alpha_t)(w^*)$, where $w \in U(A \otimes A^{op})$ and $v$ is a differentiable $\alpha \otimes \alpha$-cocycle with $ih =dv_t/dt|_{t=0}$ such that $\|w - 1\| < \delta$ and $\|h\| < \delta$ (see [3]).

Let $(\pi, \mathcal{H}_\phi, \Omega_\phi)$ be the GNS representation of $A \otimes A^{op}$ associated with $\phi$. Since $\phi_1 = \phi | A_1 \otimes A_1$ and $\phi_2 = \phi | (A \cap A_1') \otimes (A \cap A_1')$ are pure, $(\pi, \mathcal{H}_\phi, \Omega_\phi)$ is identical with $(\pi_{\phi_1} \otimes \pi_{\phi_2}, \mathcal{H}_{\phi_1} \otimes \mathcal{H}_{\phi_2}, \Omega_{\phi_1} \otimes \Omega_{\phi_2})$, where $(\pi_{\phi_1}, \mathcal{H}_{\phi_1}, \Omega_{\phi_1})$ is the GNS triple for $\phi_1$.

We define a unitary flow $U$ in $\mathcal{H}_\phi$ by

$$U_t \pi_\phi(x) \Omega_\phi = \pi_\phi(\alpha_t \otimes \alpha_t)(x) \Omega_\phi, \quad x \in A \otimes A^{op}.$$ 

Let $H$ be the generator of $U$. Since $\phi$ is a pure ground state, we have that $H \geq 0$ and $H \Omega_\phi = 0$. Let $H_0 = \pi_\phi(w^*)H \pi_\phi(w) - \pi_\phi(h)$, which is a self-adjoint operator in $\mathcal{H}_\phi$ with the domain $\mathcal{D}(H_0) = \pi_\phi(w^*)\mathcal{D}(H)$ and let $E_0$ be the spectral measure of $H_0$. Then, since $\text{Ad } u_t \circ (\alpha_t \otimes \alpha_t) = \text{Ad } w \circ \text{Ad } v_t \circ (\alpha_t \otimes \alpha_t) \circ \text{Ad } w^*$, we have that

$$\text{Ad } e^{ith_0} \pi_\phi(x \otimes y) = \pi_\phi(\alpha_t(x) \otimes \alpha_t(y)), \quad x, y \in A.$$ 

Since $H_0 \geq -\|h\| > -\delta$ and $\|H_0 \pi_\phi(w^*)\Omega_\phi\| = \|\pi_\phi(hw^*)\Omega_\phi\| < \delta$, we should note that $E_0[-\delta, \delta^{1/2}] \pi_\phi(w^*)\Omega_\phi$ has norm close to 1, or more concretely, $\|[E_0(\delta^{1/2}, \infty) \pi_\phi(w^*)\Omega_\phi]\| < \delta^{1/2}$.

Let $\pi_\phi(x \otimes x)$ be the representation of $A$ in $\mathcal{H}_\phi$ defined by $\pi_\phi(x \otimes x)$, $x \in A$. Since $\pi_\phi$ is irreducible, $\pi(A)^{\mu}$ is a factor. Since $H_0$ is bounded below, the Borchers’ theorem [20] tells us there is a unitary flow $V$ in $\pi(A)^{\mu}$ such that $V_t \pi_\phi(x)V_t^* = U_t \pi_\phi(x)U_t^*$, $x \in A$. Let $H_1$ be the generator of $V$ and let $E_1$ be the spectral measure of $H_1$.

Note that $H_1$ is bounded below and is unique up to constant.

Let $P' = [\pi(A)E_0[-\delta, \delta^{1/2}]H_\phi]$ and $P = [\pi(A)'E_0[-\delta, \delta^{1/2}]H_\phi]$. Note that $P' \in \pi(A)' \cap U'$, $P \in \pi(A)' \cap U'$, and $PP' \geq E_0[-\delta, \delta^{1/2}]$. Then we have:

Lemma 4.5 $PP' \leq E_0[-\delta, \delta + 2\delta^{1/2}]$.

Proof. This is proved in [11], but we give a proof here.

Let $\lambda \in \text{Spec}(UPP')$. Since $[PP'H_\phi] = [P \pi(A)PE_0[-\delta, \delta^{1/2}]H_\phi]$, for any $\epsilon > 0$ there is a $Q \in P \pi(A)P$ and $\xi \in E_0[-\delta, \delta^{1/2}]H_\phi$ such that $Q \xi \neq 0$ and $\text{Spec}_U(Q \xi) \subset (\lambda - \epsilon', \lambda + \epsilon')$, where $\text{Spec}_U(\xi)$ is the $U$-spectrum of $\xi$, meaning the least closed subset $B$ of $\mathbb{R}$ with $E_0(B) \xi = \xi$, for $\xi \in H_\phi$ (see [18]). We may assume that $\text{Spec}_U(\xi) \subset (\lambda - \epsilon' - \delta^{1/2}, \lambda + \epsilon' + \delta)$, where $\overline{\pi}$ is the (weakly continuous) flow $t \mapsto \text{Ad } U_t$ on $B(H_\phi)$ and $\text{Spec}_U(\xi)$ is the $\overline{\pi}$-spectrum of $Q \in B(H_\phi)$. Since $Q^*E_0[-\delta, \delta^{1/2}] \neq 0$, there is an $\eta \in E_0[-\delta, \delta^{1/2}]H_\phi$ such
that $Q^*\eta \neq 0$. Since $\text{Spec}_U(Q^*\eta) \subset \text{Spec}_{\overline{\pi}}(Q^*) + \text{Spec}_U(\eta) \subset (-\lambda - \epsilon' - 2\delta, -\lambda + \epsilon' + 2\delta^{1/2}) \cap \text{Spec}(UPP') \neq \emptyset$ or
\[
\lambda \in -\text{Spec}(UPP') + (-\epsilon' - 2\delta, \epsilon' + 2\delta^{1/2}).
\]

Since $\epsilon' > 0$ is arbitrary, we get that
\[
\lambda \in -\text{Spec}(UPP') + [-2\delta, 2\delta^{1/2}] \subset (-\infty, \delta + 2\delta^{1/2}].
\]

Since $\lambda \geq -\delta$, this concludes the proof. \hfill \square

Hence it follows that for a small $\delta > 0$,
\[
\text{Spec}_{\overline{\pi}}(P\pi(A)^nPP') \subset [-2\delta - 2\delta^{1/2}, 2\delta + 2\delta^{1/2}] \subset [-3\delta^{1/2}, 3\delta^{1/2}].
\]

Since $P\pi(A)^nP \ni Q \mapsto QP' \in P\pi(A)^nPP'$ is an isomorphism intertwining $\overline{\pi}$, it also follows that $\text{Spec}_{\overline{\pi}}(P\pi(A)^nP) \subset [-3\delta^{1/2}, 3\delta^{1/2}]$ and that max $\text{Spec}(H_1P) - \min \text{Spec}(H_1P) \leq 3\delta^{1/2}$. Hence adjusting a constant to $H_1$ we may suppose that
\[
P \leq E_1[0, 3\delta^{1/2}].
\]

Then it automatically follows that $H_1 \geq -\delta^{1/2} - \delta > -2\delta^{1/2}$. Because if $E_1(-\infty, -\delta^{1/2} - \delta') \neq 0$ for some $\delta' > \delta$, there is a non-zero $Q \in \pi(A)^n$ such that $Q = E_1(-\infty, -\delta^{1/2} - \delta')QP$ as $\pi(A)^n$ is a factor. Then it follows that $\text{Spec}_{\overline{\pi}}(Q) \subset (-\infty, -\delta^{1/2} - \delta')$. Since there is a $\xi \in E_0[-\delta, \delta^{1/2}]H_\phi$ such that $Q\xi \neq 0$, we reach the contradiction that $\text{Spec}_U(Q\xi) \subset (-\infty, -\delta') \subset (-\infty, -\delta)$.

Since $P \geq E_0[-\delta, \delta^{1/2}]$, we have that $E_0[-\delta, \delta^{1/2}] \leq E_1[0, 3\delta^{1/2}]$.

Let $\pi_1$ (resp. $\pi_2$) denote the representation of $A_1$ defined by $\pi_1(x) = \pi_{\phi_1}(x \otimes 1)$, $x \in A_1$ (resp. $\pi_2(x) = \pi_{\phi_2}(x \otimes 1)$, $x \in A \cap A_1^\prime$). Note that $\pi = \pi_1 \otimes \pi_2$ while $A = A_1 \otimes (A \cap A_1^\prime)$. Let $F = [\pi_1(A)\Omega_{\phi_1}] \otimes [\pi_2(A \cap A_1^\prime)\Omega_{\phi_2}] \in \pi(A)'$. We will restrict the representation $\pi$ to the cyclic subspace $FH_{\phi}$ below.

Since $\|(1 - F)\pi_{\phi}(w^*)\Omega_{\phi}\| < \delta$ and
\[
\|(1 - E_1[0, 3\delta^{1/2}])\pi_{\phi}(w^*)\Omega_{\phi}\| \leq \|E_0(\delta^{1/2}, \infty)\pi_{\phi}(w^*)\Omega_{\phi}\| < \delta^{1/2},
\]
we have that $\|FE_1[0, 3\delta^{1/2}]\pi_{\phi}(w^*)\Omega_{\phi} - \Omega_{\phi}\| < 2\delta + \delta^{1/2}$. Let $\Psi = cFE_1[0, 3\delta^{1/2}]\pi_{\phi}(w^*)\Omega_{\phi}$ have norm one with $c > 0$. Then we may suppose that $\|\Psi - \Omega_{\phi}\| < 2\delta^{1/2}$ (for a small $\delta > 0$).

Note that $F$ is given as $F_1 \otimes F_2$, where $F_1 = [\pi_1(A_1)\Omega_{\phi_1}]$ and $F_2 = [\pi_2(A \cap A_1^\prime)\Omega_{\phi_2}]$. We choose a maximal abelian $W^*$-subalgebra $C_1$ (resp. $C_2$) of $F_1\pi_1(A_1)\Omega_{\phi_1}$ (resp. $F_2\pi_2(A \cap A_1^\prime)\Omega_{\phi_2}$) so that $C = C_1 \otimes C_2$ is maximal abelian in $F\pi(A)'F$, where $C_1$ is finite-dimensional. We apply the decomposition theory to the cyclic representation $F\pi$ with the cyclic vector $\Omega_{\phi} = \Omega_{\phi_1} \otimes \Omega_{\phi_2}$ with respect to this maximal abelian $W^*$-subalgebra $C$. Let $K_i$ be the character space of $C_i$ and let $\nu_i$ be the probability measure on $K_i$ defined by
\( Q \in C \mapsto \langle Q \Omega_{\phi_i}, \Omega_{\phi_i} \rangle \). Then one can express \((F \pi, F \mathcal{H}_\phi, \Omega_{\phi})\) as a direct integral over \((K_1 \times K_2, \nu_1 \otimes \nu_2)\); e.g.,

\[
F \mathcal{H}_\phi = \int_{K_1 \times K_2}^{\oplus} \mathcal{H}_1(s_1) \otimes \mathcal{H}_2(s_2) d\nu_1(s_1) d\nu_2(s_2),
\]

where \((\pi_i(s), \mathcal{H}_i(s), \xi_i(s))\) is the representation associated with \(s \in K_i\). Note that \(\pi_i(s)\) is irreducible for almost all \(s \in K_i\).

Recall that \(\Psi \in F \mathcal{H}_\phi\) and let \(\mu\) be the probability measure on \(K_1 \times K_2\) defined by \(Q \in C \mapsto \langle Q \Psi, \Psi \rangle\). Then \(\mu\) is absolutely continuous with respect to \(\nu = \nu_1 \otimes \nu_2\); so we set \(f = d\mu/d\nu\). Then we have

\[
\Psi = \int_{K_1 \times K_2}^{\oplus} \eta(s_1, s_2) f(s_1, s_2) d\nu_1(s_1) d\nu_2(s_2).
\]

where \(\eta(s_1, s_2)\) is a unit vector in \(\mathcal{H}_1(s_1) \otimes \mathcal{H}_2(s_2)\).

Since \((H_1 + 1)^{-1} \in \pi(A)^\prime\), we get a measurable function \((s_1, s_2) \mapsto B(s_1, s_2) \in (\pi_1 \otimes \pi_2)(A)^\prime\) such that

\[
(H_1 + 1)^{-1} F = \int_{K_1 \times K_2}^{\oplus} B(s_1, s_2) d\nu_1(s_1) d\nu_2(s_2).
\]

For almost all \((s_1, s_2) \in K_1 \times K_2\), \(B(s_1, s_2)\) is given as \((H_1(s_1, s_2) + 1)^{-1}\), where \(H_1(s_1, s_2)\) is self-adjoint with \(H_1(s_1, s_2) \geq -2\delta^{-1/2}\). Setting \(V_t(s_1, s_2) = e^{itH_1(s_1, s_2)}\), we get that

\[
V_tF = \int_{K_1 \times K_2}^{\oplus} V_t(s_1, s_2) d\nu_1(s_1) d\nu_2(s_2), \quad t \in \mathbb{R}.
\]

Since \(E_1[0, 3\delta^{1/2}][\Psi = \Psi\), it also follows for almost all \((s_1, s_2)\) that

\[
E_1(s_1, s_2)[0, 3\delta^{1/2}]\eta(s_1, s_2) = \eta(s_1, s_2),
\]

where \(E_1(s_1, s_2)\) is the spectral measure of \(H_1(s_1, s_2)\), and that

\[
\text{Ad} V_t(s_1, s_2)(\pi_1(s_1) \otimes \pi_2(s_2)) = (\pi_1(s_1) \otimes \pi_2(s_2)) \alpha_t.
\]

Note that

\[
\|\Omega_{\phi} - \Psi\| = \int_{K_1 \times K_2} \|\xi_1(s_1) \otimes \xi_2(s_2) - \sqrt{\eta(s_1, s_2)}\eta(s_1, s_2)\|^2 d\nu_1(s_1) d\nu_2(s_2)
\]

\[
= \int (1 + f(s_1, s_2) - 2\sqrt{f(s_1, s_2)} \text{Re}\langle \xi_1(s_1) \otimes \xi_2(s_2), \eta(s_1, s_2) \rangle) d\nu_1(s_1) d\nu_2(s_2).
\]

Since \(\|\Omega_{\phi} - \Psi\|^2 < 4\delta\), we have \((s_1, s_2) \in K_1 \times K_2\) such that all the previous conditions are satisfied for this \((s_1, s_2)\) and \(f(s_1, s_2) \leq 1\) and

\[
1 + f(s_1, s_2) - 2\sqrt{f(s_1, s_2)} \text{Re}\langle \xi_1(s_1) \otimes \xi_2(s_2), \eta(s_1, s_2) \rangle < 4\delta.
\]
The latter two conditions imply that
\[ 1 - \Re(\xi_1(s_1) \otimes \xi_2(s_2), \eta(s_1, s_2)) \leq 1 - \sqrt{\int (s_1, s_2) \Re(\xi_1(s_1) \otimes \xi_2(s_2), \eta(s_1, s_2)) < 2\delta^{1/2}}, \]
which in turn implies that
\[ \|\xi_1(s_1) \otimes \xi_2(s_2) - \eta(s_1, s_2)\| < 4\delta^{1/2}. \]

Thus we find \( s_1 \in K_1 \) and \( s_2 \in K_2 \) such that the above norm estimate holds, \( \pi_i(s_i) \) is irreducible, \( \Ad V_i(s_1, s_2) \) implements \( \alpha_t \) in \( \pi_1(s_1) \otimes \pi_2(s_2) \), and
\[ E_i(s_1, s_2)[0, 3\delta^{1/2}]\eta(s_1, s_2) = \eta(s_1, s_2). \]
We can then find a \( z \in \mathcal{U}(A) \) and \( b \in A_{sa} \) such that \( z \approx 1, \pi_1(s_1) \otimes \pi_2(s_2)(z)\xi_1(s_1) \otimes \xi_2(s_2) = \eta(s_1, s_2), \|b\| \approx 0, \) and \( (H_1(s_1, s_2) + (\pi_1(s_1) \otimes \pi_2(s_2))(b))\eta(s_1, s_2) = 0. \) Thus we have a small \( \alpha \)-cocycle \( w \) such that the perturbed \( \Ad w \alpha_t \) has a pure invariant state \( \psi \) which is pure on \( A_1 \), i.e., \( w_t = zv_t \alpha(z^*) \), where \( v \) is the \( \alpha \)-cocycle defined by \( \frac{dv_t}{dt}|_{t=0} = ib \) and \( \psi(x) = \langle \pi_1(s_1) \otimes \pi_2(s_2)(x)\xi_1(s_1) \otimes \xi_2(s_2), \xi_1(s_1) \otimes \xi_2(s_2) \rangle, x \in A. \) Now we are in the situation where we can invoke Lemma 4.3.

\section{AF algebras}

To prove Proposition 1.6 we first show it in a special case as follows:

\textbf{Lemma 5.1} For any \( \epsilon > 0 \) there exists a \( \delta > 0 \) satisfying the following condition: Let \( A \) be a unital \( C^* \)-algebra and let \( \alpha \) be a flow on \( A. \) If \( D \) is a unital finite-dimensional abelian \( C^* \)-subalgebra of \( A \) such that
\[ \sup_{|t| \leq 1} \|\alpha_t - \id\|D\| < \delta, \]
then there is an \( \alpha \)-cocycle \( u \) such that \( \max_{|t| \leq 1} \|u_t - 1\| < \epsilon \) and \( \Ad u_t \alpha_t(x) = x, x \in D. \)

\textbf{Proof.} The main point here is that we can choose \( \delta > 0 \) independently of the dimensionality of \( D; \) otherwise the lemma would be obvious.

Suppose that we are given a unital \( C^* \)-algebra \( A \) and a flow \( \alpha \) on \( A. \)

Let \( f \) be a non-negative \( C^\infty \)-function on \( \mathbb{R} \) of compact support such that \( \int f(t)dt = 1. \) We define a unital completely positive (or CP for short) map \( \alpha_f \) on \( A \) by
\[ \alpha_f(x) = \int f(t)\alpha_t(x)dt. \]

Since \( \delta_{\alpha} \alpha_f(x) = -\int f'(t)\alpha_t(x)dt, \) we get that
\[ \|\delta_{\alpha} \alpha_f\| \leq \int |f'(t)|dt. \]
Given a small $\epsilon > 0$ we choose such an $f$ satisfying
\[\int |f'(t)| dt < \epsilon,\]
which entails $\|\delta_0 \alpha_f\| < \epsilon$. Then we choose $\delta > 0$ such that
\[\delta \int f(t)(1 + |t|) dt < \epsilon^2 / 2.\]

Suppose that we are given a unital abelian finite-dimensional C*-subalgebra $D$ of $A$ such that $\sup_{|t| \leq 1} |(\alpha_t - id)|D\| < \delta$, which entails that
\[\|(\alpha_f - id)|D\| < \epsilon^2 / 2.\]
Then we get that
\[\alpha_f(u)\alpha_f(u)^* \geq 1 - \epsilon^2\]
for any unitary $u \in D$.

What we do in the following is to find a homomorphism $\Phi$ of $D$ into $A$ in a close neighborhood of $\alpha_f$ such that the norm $\|\delta_0 \Phi\|$ is small depending on $\|\delta_0 \alpha_f\|$. To find such a $\Phi$ we will follow the strategy taken by Christensen for the proof of 3.3 of [5].

Let $(\pi, U)$ be a covariant faithful representation of $(A, \alpha)$. By Stinespring's theorem we get a representation $\rho$ of $A$ such that $\mathcal{H}_\rho \supset \mathcal{H}_\pi$, $[\rho(A)\mathcal{H}_\pi] = \mathcal{H}_\rho$, and
\[\pi(\alpha_f(x)) = P \rho(x)|\mathcal{H}_\pi, \quad x \in A,\]
where $P$ is the projection onto $\mathcal{H}_\pi$, i.e., $P = \pi(1)$. This representation $\rho$ can be obtained as follows: We define an inner product on the algebraic tensor product $A \otimes \mathcal{H}_\pi$ by
\[\langle \sum_i x_i \otimes \xi_i, \sum_j y_j \otimes \eta_j \rangle = \sum_i \sum_j \langle \alpha_f(y_j^* x_j) \xi_i, \eta_j \rangle.\]
We then define a representation $\rho$ of $A$ on $A \otimes \mathcal{H}_\phi$ by
\[\rho(a) \sum_i x_i \otimes \xi_i = \sum_i ax_i \otimes \xi_i.\]
Then $\mathcal{H}_\rho$ is defined as the completion of the quotient of $A \otimes \mathcal{H}_\pi$ by the linear subspace consisting of $\sum_i x_i \otimes \xi_i$ with $\|\sum_i x_i \otimes \xi_i\| = 0$ and $\rho$ naturally induces a representation of $A$ on $\mathcal{H}_\rho$, which we will also denote by $\rho$. We regard $\mathcal{H}_\pi$ as a subspace of $\mathcal{H}_\rho$ by mapping $\mathcal{H}_\pi$ into $A \otimes \mathcal{H}_\pi$ by $\xi \mapsto 1 \otimes \xi$, which we can easily check is isometric. For each $t \in \mathbb{R}$ we define an operator $V_t$ on $A \otimes \mathcal{H}_\pi$ by
\[V_t \sum_i x_i \otimes \xi_i = \sum_i \alpha_t(x_i) \otimes U_t \xi_i.\]
Then $V = (V_t, t \in \mathbb{R})$ induces a unitary flow on $\mathcal{H}_\rho$, which is denoted by the same symbol. Then we check that $V_tP = PV_t$, $U_t = V_tP$, and $V_t\rho(x)V_t^* = \rho\alpha_t(x)$, $x \in A$.

Let $B$ be the $C^*$-algebra generated by $\rho(A)$ and $P$; then it follows that $PBP \subseteq \pi(A)$. We define a flow $\beta$ on $B$ by $\beta_t = \text{Ad} V_t|B$, which satisfies that $\beta_t\rho(x) = \rho\alpha_t(x)$, $x \in A$, $\beta_t(P) = P$, and $\beta_t\pi(x) = \pi\alpha_t(x)$ for $x \in A$ with $\pi(x) \in PBP$.

By mimicking the proof of 3.3 of Christensen [5], we define

$$R = \int_{U(D)} \rho(u)P\rho(u)^*d\nu(u) \in \rho(D)' \cap B,$$

where $\nu$ is normalized Haar measure on the compact group $U(D)$. Since $\|\rho(u)P - P\rho(u)\| = \|(1 - P)\rho(u)P - P\rho(u)(1 - P)\|$ is equal to

$$\max_{z = u, u^*} \left\|P\rho(z)(1 - P)\rho(z)^*P\right\|^{1/2} = \max_{z = u, u^*} \left\|1 - \alpha_f(z)\alpha_f(z)^*\right\|^{1/2},$$

we get that $\|R - P\| \leq \epsilon$. Since $P$ is a projection and $0 \leq R \leq 1$, we have that $\text{Spec}(R) \subseteq [0, \epsilon] \cap [1 - \epsilon, 1]$.

If $Q$ denotes the spectral projection of $R$ corresponding to $[1 - \epsilon, 1]$, it follows that $Q \in \rho(D)' \cap B$ and

$$\|P - Q\| \leq \|P - R\| + \|R - Q\| \leq 2\epsilon.$$

Let $N$ be the dimension of $D$ and let $(\epsilon_i)_{i=1}^N$ be the family of minimal projections in $D$. Then the above $R$ is also given as $R = \sum_{i=1}^N \rho(\epsilon_i)\rho(\epsilon_i)\rho(\epsilon_i)$. We set $a_i = \pi\alpha_f(\epsilon_i)$. Since

$$PR^n = \sum_{i=1}^N a_i^{n+1}$$

for any $n \in \mathbb{N}$ and $n = 0$, we get that

$$Pe^{itP} = \sum_{i=1}^N \sum_{n=0}^{\infty} \frac{(it)^n}{n!} a_i^{n+1} = \sum_{i=1}^N a_i e^{ita_i}.$$ 

Since

$$\delta_\beta(P\rho(u)P\rho(u)^*P) = \delta_\beta(\pi(\alpha_f(u)\alpha_f(u)^*)) = \pi(\delta_\alpha\alpha_f(u)\alpha_f(u)^* + \alpha_f(u)\delta_\alpha\alpha_f(u)^*)$$

for $u \in U(D)$, we get that $\|\delta_\beta(P\rho\rho^\ast\rho)\| < 2\epsilon$. Similarly we get that

$$\|\delta_\beta(P\rho^n\rho^\ast\rho)\| < (n + 1)\epsilon.$$

This implies that $Pe^{itP}$ belongs to the domain of $\delta_\beta$ for $t \in \mathbb{R}$ and that $\|\delta_\beta(Pe^{itP})\| \leq \epsilon(e^{\epsilon t} + e^{\epsilon t}t - 1)$, which grows too rapidly in $t$. Since $\|\delta_\beta(e^{ita_i})\| \leq |t||\delta_\beta(a_i)| \leq \epsilon|t|$, we get, from the expression of $Pe^{itP}$ in terms of $a_i$’s, that

$$\|\delta_\beta(Pe^{itP})\| \leq N\epsilon(1 + |t|),$$

which depends on the dimension $N$ of $D$. We shall give another estimate of $\|\delta_\beta(Pe^{itP})\|$, which grows polynomially in $t$ and is independent of $N$:
Lemma 5.2 With $P, R$ as above and $t \in \mathbb{R}$ it follows that $\|\delta_\beta(P e^{itR})\| \leq \epsilon (2|t| + t^2/2)$.

Proof. Note that

$$\delta_\beta(P e^{itR}) = \sum_{i=1}^N \sum_{n=0}^\infty \sum_{k=0}^n \frac{(it)^n}{n!} a_i^k \delta_\beta(a_i) a_i^{n-k}.$$ 

By using

$$\frac{1}{k! \cdot (n-k)!} \int_0^1 s^k (1-s)^{n-k} ds = \frac{1}{(n+1)!},$$

the above equation equals

$$\sum_{i=1}^N \sum_{n=0}^\infty \sum_{k=0}^n \int_0^1 ds \frac{(ist)^k (i(1-s)t)^{n-k}}{k!(n-k)!} \delta_\beta(a_i)^s a_i^{n-k}.$$ 

Substituting $n + 1 = 1 + k + \ell$ in the above formula, we get that

$$\delta_\beta(P e^{itR}) = \sum_{k=0}^\infty \sum_{\ell=0}^\infty \sum_{i=1}^N \int_0^1 ds \frac{(ist)^k (i(1-s)t a_i)^\ell}{k! (k-1)!} \delta_\beta(a_i)^s a_i^{n-k}.$$ 

Note that the sum over $i$ for $k = 0$ and $\ell = 0$ of the first term is zero because $\delta_\beta(\sum_i a_i) = \delta_\beta(P) = 0$. We will evaluate the norm of the rest of the first term by splitting it into three terms $\Sigma_1 = \sum_{k \geq 1, \ell \geq 1}$, $\Sigma_2 = \sum_{k=0, \ell \geq 1}$, and $\Sigma_3 = \sum_{k \geq 1, \ell = 0}$, where $\sum_i \cdots$ is omitted. Similarly we will split the second term into two terms $\Sigma_2 = \sum_{k \geq 1, \ell \geq 1}$, and $\sum_{k \geq 1, \ell = 0}$ and the third term into two $\sum_{k \geq 1, \ell \geq 1}$ and $\sum_{k=0, \ell \geq 1}$ before evaluation. We can easily see that the unnamed terms can be expressed in terms of the named $\Sigma_1, \ldots, \Sigma_4$.

We set

$$T = \sum_{i=1}^N \rho(e_i) P \delta_\beta(a_i) P \rho(e_i).$$

Since $\|\delta_\beta(a_i)\| \leq \epsilon$, we get that $\|T\| \leq \epsilon$. If $k \geq 1$ and $\ell \geq 1$, it follows that

$$\sum_{i=1}^N a_i^k \delta_\beta(a_i) a_i^\ell = PR^{k-1} T R^{\ell-1} P.$$ 

We set

$$S = \sum_{i=1}^N P \delta_\beta(a_i) P \rho(e_i) = \sum_{i=1}^N \delta_\beta(a_i) P \rho(e_i).$$
If \( \ell \geq 1 \), it follows that
\[
\sum_{i=1}^{N} \delta_{\beta}(a_{i})a_{i}^\ell = SR^{\ell-1}P.
\]

We assert that \( \|S\| \leq \epsilon \). Since \( SS^* = \sum_{i=1}^{N} \delta_{\beta}(a_{i})\delta_{\beta}(a_{i}) \) and \( 0 \leq a_{i} \leq 1 \), we get that
\[
SS^* \leq \sum_{i=1}^{N} \delta_{\beta}(a_{i})^2 = XX^*,
\]
where \( X = (\delta_{\beta}(a_{1}), \delta_{\beta}(a_{2}), \ldots, \delta_{\beta}(a_{N})) \in M_{1N}(A) \). We naturally extend \( \alpha \) to a flow on \( M_{1N}(A) \subset M_{N}(A) \), which we will also denote by \( \alpha \). Similarly we extend \( \pi \) to a representation of \( M_{N}(A) = M_{N} \otimes A \) on \( C^{N} \otimes H_{\pi} \). Then, since \( \delta_{\beta}(a_{i}) = \pi \delta_{\alpha} \alpha f(e_{i}) \), it follows that
\[
X = \pi \delta_{\alpha} \alpha f(e),
\]
where \( e = (e_{1}, e_{2}, \ldots, e_{N}) \), which has norm 1. Since \( \|\delta_{\alpha} \alpha f\| \leq \epsilon \), which depends only on \( f \), we get that \( \|X\| \leq \epsilon \). This implies that \( \|S\| \leq \epsilon \).

Define a \( C^\infty \)-function \( h \) on \( \mathbb{R} \) by
\[
h(t) = \frac{e^{it} - 1}{it} = \sum_{k=1}^{\infty} \frac{(it)^{k-1}}{k!}
\]
and note that \( |h(t)| \leq 1 \). Since
\[
\sum_{k=1}^{\infty} \frac{(ist)^{k}}{k!} R^{k-1} = ist \cdot h(stR),
\]
we get that
\[
\Sigma_{1}(t) \equiv \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \sum_{i=1}^{N} \int_{0}^{1} ds \frac{(ist_{a_{i}})^{k}}{k!} \delta_{\beta}(a_{i}) \frac{(i(1-s)t_{a_{i}})^\ell}{\ell!}
\]
\[
= \int_{0}^{1} ds(-s(1-s)t^{2})Ph(stR)Th((1-s)tR)P,
\]
which has the estimate \( \|\Sigma_{1}(t)\| \leq \epsilon t^{2}/6 \). We also have the following:
\[
\Sigma_{2}(t) \equiv \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \sum_{i=1}^{N} \int_{0}^{1} ds \frac{(ist_{a_{i}})^{k}}{(k-1)!} \delta_{\beta}(a_{i}) \frac{(i(1-s)t_{a_{i}})^\ell}{\ell!}
\]
\[
= \int_{0}^{1} ds(-s(1-s)t^{2})Pe^{istR}Th((1-s)tR)P,
\]
which has the estimate $\|\Sigma_2(t)\| \leq \epsilon t^2/6$. Furthermore we compute:

$$
\Sigma_3(t) \equiv \sum_{\ell=1}^{\infty} \sum_{i=1}^{N} \int_{0}^{1} ds \delta_\beta(a_i) \frac{(i(1-s)t a_i)\ell}{\ell!}
= \int_{0}^{1} ds \cdot i(1-s)t S h((1-s)t R) P,
$$

which implies that $\|\Sigma_3(t)\| \leq \epsilon |t|/2$. We also compute:

$$
\Sigma_4(t) \equiv \sum_{\ell=1}^{\infty} \sum_{i=1}^{N} \int_{0}^{1} ds \delta_\beta(a_i) \frac{(i(1-s)t a_i)\ell}{(\ell-1)!}
= \int_{0}^{1} ds \cdot i(1-s)t S e^{i(1-s)t R} P,
$$

which implies that $\|\Sigma_4(t)\| \leq \epsilon |t|/2$. Since

$$
\delta_\beta(P e^{it R} P) = \Sigma_1(t) + \Sigma_2(t) + \Sigma_2(-t)^* + \Sigma_3(t) + \Sigma_3(-t)^* + \Sigma_4(t) + \Sigma_4(-t)^*,
$$

we get the desired estimate. $\square$

Let $g$ be a $C^\infty$-function on $\mathbb{R}$ of compact support such that $g(t) = 0$ for $t \in [0, 1/3]$ and $g(t) = 1$ for $t \in [2/3, 1]$. We assume that $\epsilon$ is sufficiently small; in particular, $\epsilon < 1/3$. Then $Q = g(R)$ is the spectral projection of $R$ corresponding to $[1-\epsilon, 1]$. We have already noted that $\|P - Q\| \leq 2\epsilon$, which implies that $\text{Spec}(P Q P) \subset [1 - 2\epsilon, 1]$ on $P H_\rho = H_\pi$.

Thus the support projection of $P Q P$ (resp. $Q P Q$) is $P$ (resp. $Q$).

We define a function $\hat{g}$ on $\mathbb{R}$ by

$$
\hat{g}(s) = \frac{1}{2\pi} \int e^{-ist} g(t) dt.
$$

Then we know that $\hat{g}$ is a rapidly decreasing $C^\infty$-function and that

$$
g(R) = \int \hat{g}(t) e^{it R} dt.
$$

Since $P g(R) P \in D(\delta_\beta)$ and

$$
\delta_\beta(P g(R) P) = \int \hat{g}(t) \delta_\beta(P e^{it R} P) dt,
$$

we get, by the previous lemma, the estimate

$$
\|\delta_\beta(P g(R) P)\| \leq \epsilon \int \frac{|\hat{g}(s)| (2|t| + t^2/2)}{|s|} ds.
$$
Suppose that we give such a function $g$ beforehand and let

$$C_1 = \int |\dot{g}(t)|(2|t| + t^2/2)dt.$$  

Then we choose $\epsilon > 0$ so small that $C_1\epsilon$ is sufficiently small. Since $\text{Spec}(PQP) \subset [2/3, 1]$ (on $PH_\rho = H_\pi$), we have that $(PQP)^{-1/2} \in D(\delta_\beta)$. In the same way as above we have a constant $C_2$ such that

$$\|\delta_\beta((PQP)^{-1/2})\| \leq C_2\|\delta_\beta(PQP)\| \leq C_1C_2\epsilon,$$

which we assume is sufficiently small.

We define a CP map $\Phi$ of $D$ into $PBP \subset \pi(A)$ by

$$\Phi(x) = (PQP)^{-1/2}PQP\rho(x)QP(PQP)^{-1/2}, \quad x \in D.$$  

Let $W = (PQP)^{-1/2}PQ \in B$, which is a partial isometry such that $WW^* = P$, $W^*W = Q$, and $\Phi(x) = W\rho(x)W^*$, $x \in D$. Since $[\rho(x), Q] = 0$ for $x \in D$, $\Phi$ is a unital homomorphism. Since $\|PQ - P\| \leq 2\epsilon$, we have the estimate $\|W - P\| \leq 3\epsilon$ (see 2.7 of [1]). Since $P\rho(x)P = \pi\alpha_f(x) \approx \pi(x)$ (up to $\epsilon^2/2$) we get for $x \in D$ with $\|x\| \leq 1$

$$\|\Phi(x) - \pi(x)\| \leq 3\epsilon + \epsilon^2/2.$$  

Thus $\Phi$ is an injective homomorphism from $D$ into $PBP \subset \pi(A)$ such that $\Phi$ is close to $\pi|D$.

Let $x = \sum_{i=1}^N x_i e_i \in D$ be such that $\|x\| \leq 1$, i.e., $\max_i |x_i| \leq 1$. For the same reasoning as for $Pe^{itR}P$ in the above lemma, we can conclude that $Pe^{itR}\rho(x)P \in D(\delta_\beta)$ with the same estimate

$$\|\delta_\beta(Pe^{itR}\rho(x)P)\| \leq \epsilon(2|t| + t^2/2).$$  

(In the proof of the above lemma we just have to replace $T$ and $S$ by

$$T' = \sum_{i=1}^N x_i e_i P\delta_\beta(a_i)P\rho(e_i) \quad \text{and} \quad S' = \sum_{i=1}^N x_i P\delta_\beta(a_i)P\rho(e_i),$$

respectively. Both $T'$ and $S'$ have the same estimates $\|T'\| \leq \epsilon$ and $\|S'\| \leq \epsilon$ as before since $\|x\| \leq 1$.) Thus we get that $PQP\rho(x)QP \in D(\delta_\beta)$ and hence $\Phi(x) \in D(\delta_\beta)$; moreover, since $\|\delta_\beta((PQP)^{-1/2})\| \leq C_1C_2\epsilon$, $\|(PQP)^{-1/2}\| \leq (1 - 2\epsilon)^{-1/2}$, and $\|\delta_\beta(PQP\rho(x)P)\| \leq C_1\epsilon$, we have that

$$\|\delta_\beta(\Phi(x))\| \leq 2C_1C_2\epsilon(1 - 2\epsilon)^{-1/2} + (1 - 2\epsilon)^{-1}C_1\epsilon$$

for $x \in D$ with $\|x\| \leq 1$.

Identifying $A$ with $\pi(A)$ and summing up the above arguments, we have proved the following assertion. For any $\epsilon > 0$ there is an injective homomorphism $\Phi$ of $D$ into $A$
such that \( \| \Phi - \text{id} \| < \epsilon \) and \( \Phi(D) \subset D(\delta) \) and \( \| \delta_{\alpha} \Phi(D) \| < \epsilon \). By 4.2 of [5] such a \( \Phi \) is implemented by a unitary \( w \in A \) such that \( \| w - 1 \| \leq 2\epsilon \) (and \( \Phi(x) = w x w^*, \ x \in D \)).

Let \( D_1 = \Phi(D) \). We define \( h = h^* \in A \) by

\[
ih = \int_{U(D_1)} \delta_{\alpha}(u) u^* d\nu(u),
\]

where \( \nu \) is normalized Haar measure on the compact unitary group \( U(D_1) \). Then it follows that \( \| h \| \leq \epsilon \) and \( [ih, x] = \delta_{\alpha}(x), \ x \in D_1 \). Let \( u \) denote the differentiable \( \alpha \) cocycle such that \( \frac{d}{dt} u_t|_{t=0} = -ih \). We set \( v_t = w u_t \alpha_t(w) \), which is an \( \alpha \)-cocycle such that

\[
\max_{|t| \leq 1} \| v_t - 1 \| \leq 2 \| w - 1 \| + \epsilon \leq 5\epsilon
\]

and

\[
\text{Ad} v_t \alpha_t(x) = \text{Ad} w^* \text{Ad} u_t \alpha_t \Phi(x) = \text{Ad} w^* \Phi(x) = x,
\]

for \( x \in D \). This concludes the proof of Lemma 5.1. \( \square \)

It is instructive to see what \( \Phi \) is. By computation, since \( Q = g(R) \), we have that \( PQ \rho(e_i) P = a_i g(a_i) \). Since \( \text{Spec}(a_i) \subset [0,\epsilon] \cup [1-\epsilon,1] \), if \( g_1 \) is a continuous function on \( R \) such that \( g = 0 \) on \( [0,1/3] \) and \( g(t) = t \) on \( [2/3,1] \), we get \( g_1(a_i) = a_i g(a_i) \). Thus it follows, with such a \( g_1 \), that

\[
\Phi(e_i) = b_{g_1}^{-1/2} g_1(\alpha_f(e_i)) b_{g_1}^{-1/2},
\]

where \( b_{g_1} = \sum_{j=1}^N g_1(\alpha_f(e_j)) \). (Since there is freedom in the above proof, we may as well take \( g \) instead of \( g_1 \) in the above formula (with a small but different \( \| \delta_{\alpha} \Phi(D) \| \)). Thus the spectral projections \( q_i \)'s of \( \alpha_f(e_i) \)'s corresponding to \( [1-\epsilon,1] \) may not be mutually orthogonal, but \( (\sum_j q_j)^{-1/2} q_i (\sum_j q_j)^{-1/2} \)'s are mutually orthogonal projections.)

Proof of Proposition 1.10

What we have to show is that Lemma 5.1 follows without the assumption that \( D \) is abelian. Hence suppose that \( D \) is a unital finite-dimensinal \( C^* \)-subalgebra of \( A \) such that

\[
\sup_{|t| \leq 1} \| (\alpha_t - \text{id}) D \| < \delta,
\]

where \( \delta \) is the one obtained for \( \epsilon \) in Lemma 5.1. Let \( Z \) be a maximal abelian \( C^* \)-subalgebra of \( D \). Then by Lemma 5.1 we get an \( \alpha \)-cocycle \( u \) such that \( \text{Ad} u_t \alpha_t | Z = \text{id} \) and \( \max_{|t| \leq 1} \| u_t - 1 \| < \epsilon \).

Let \( (e_{ij}^{(k)}) \) be a family of matrix units of \( D \) such that the linear span of \( e_{ii}^{(k)} \) for all \( i \) and \( k \) equals \( Z \). We set

\[
v_t = \sum_k \sum_i e_{ij}^{(k)} \text{Ad} u_t \alpha_t(e_{ij}^{(k)}),
\]

for \( t \in R \), which is an \( \text{Ad} u \alpha \)-cocycle. Then we have that

\[
\text{Ad}(v_t u_t) \alpha_t(e_{ab}^{(c)}) = e_{a1}^{(c)} \text{Ad} u_t \alpha_t(e_{1a}^{(c)} e_{ab}^{(c)} e_{b1}^{(c)}) e_{1b}^{(c)} = e_{ab}^{(c)},
\]

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Then for any $\epsilon > 0$ and $(\alpha_t)_t$ i.e., $\Ad(v_tu_t)\alpha_t D = \id$. Since $\|v_t - 1\|$ is equal to

$$\| \sum_{i,k} e_{i1}(k) (\Ad u_t \alpha_t(e_{i1}(k)) - e_{i1}(k))\| = \max_{i,k} \|\Ad u_t \alpha_t(e_{i1}(k)) - e_{i1}(k)\| \leq \|(\Ad u_t \alpha_t - \id)|D||,$$

it follows that $\max_{|t|\leq 1} \|v_tu_t - 1\| \leq 3\epsilon + \delta$. Since $t \mapsto v_tu_t$ is an $\alpha$-cocycle, this concludes the proof.

Now we turn to the proof of Lemma 1.2. We start with the following lemma.

**Lemma 5.3** Let $\alpha$ be a flow on a unital AF algebra $A$. Suppose that there is an increasing sequence $(A_n)$ of finite-dimensional $C^*$-subalgebras of $A$ such that $\bigcup_n A_n$ is dense in $A$ and

$$\sup_{t\in[0,1]} \text{dist}(\alpha_t(A_n), A_n) \to 0.$$ 

Then for any $\epsilon > 0$ there is an $\alpha$-cocycle $u$ and a subsequence $(n_i)$ in $\mathbb{N}$ such that $\Ad u_t \alpha_t|A_{n_i} \cap A'_{n_i} = \id$ for all $i$ and $\max_{t\in[0,1]} \|u_t - 1\| < \epsilon$.

**Proof.** Let $e$ be a non-zero central projection of $A_n$. Then for any projection $p \in A_n$ different from $e$, we have that $\|e - p\| = 1$.

If $\text{dist}(\alpha_t(A_n), A_n) < \delta$ for $t \in [0,1]$ for a small $\delta$, then it follows that $\|\alpha_t(e) - e\| < \delta$ for some $e_t \in A_n$. We may assume that $e_t$ is a projection by replacing $\delta$ by $2\delta$. Then from the above remark we get that $e_t = e$. Thus we have that $\|\alpha_t(e) - e\| < 2\delta$ for all projections $e \in A_n \cap A_n'$ and for $t \in [0,1]$. Since any element $x \in A_n \cap A_n'$ with $0 \leq x \leq 1$ is a convex combination of projections in $A_n \cap A_n'$, it follows that $\sup_{t\in[0,1]} \|(\alpha_t - \id)|A_n \cap A_n'\|$ converges to zero as $n \to \infty$.

By Lemma 3.1 there is an $n_1 \in \mathbb{N}$ and an $\alpha$-cocycle $u^{(1)}$ such that $\max_{t\in[0,1]} \|u^{(1)} - 1\| < \epsilon/2$ and $\Ad u^{(1)}_t \alpha_t|A_{n_1} \cap A'_{n_1} = \id$. Let $Z_1 = A_{n_1} \cap A'_{n_1}$. Since $\Ad u^{(1)} \alpha$ leaves $A \cap Z_1'$ invariant, $\bigcup_{n>n_1} A_n \cap Z_1'$ is dense in $A \cap Z_1'$, and $\sup_{t\in[0,1]} \text{dist}(\Ad u^{(1)}_t \alpha_t(A_n \cap Z_1'), A_n \cap Z_1') \to 0$ as $n \to \infty$, we can repeat this argument for $\Ad u^{(1)} \alpha|A \cap Z_1'$ and $\sup_{t\in[0,1]} \|u^{(2)} - 1\| < \epsilon/4$. It then follows that $\Ad(u^{(2)}_t u^{(1)}_t) \alpha_t|A_{n_i} \cap A'_{n_i} = \id$ for $i = 1, 2$ and that $t \mapsto u^{(2)}_t u^{(1)}_t$ is an $\alpha$-cocycle such that $\max_{t\in[0,1]} \|u^{(2)}_t u^{(1)}_t - 1\| < \epsilon/2 + \epsilon/4$. In this way we can complete the proof. \qed

Now we assume that $\alpha$ fixes each central projection of $A_n$ for all $n$. Let $C$ denote the $C^*$-subalgebra generated by $\bigcup_n (A_n \cap A_n')$, which is an AF abelian $C^*$-algebra. Let $B = A \cap C'$. Since $\alpha$ fixes each element of $C$, $\alpha$ restricts to a flow on $B$, which we will denote by $\beta$.

Let $B_n = A_n \cap C'$, which is the relative commutant of the $C^*$-subalgebra $C_n$ in $A_n$, where $C_n$ is generated by $\bigcup_{k=1}^n (A_k \cap A_k')$. Thus there is a norm one projection of $A$ onto $A \cap C'$, sending $A_n$ onto $B_n = A_n \cap C_n'$ and $B$ is an AF algebra. We identify $C$ with
the continuous functions $C(\Gamma)$, where $\Gamma$ is the compact Hausdorff space of characters of $C$. Then we can regard $B$ as the $C^*$-algebra of continuous sections over $\Gamma$; the fiber at $\gamma \in \Gamma$ will be denoted by $B^\gamma$, which is a UHF algebra (or a matrix algebra), and the canonical map of $B$ onto $B^\gamma$ will be denoted by $\Phi^\gamma : x \mapsto x(\gamma)$. To see what $B^\gamma$ is, we find a decreasing sequence $(e_n)_{n \in \mathbb{N}}$ of projections such that $e_n \in C_n$ is minimal with $\gamma(e_n) = 1$; then $B^\gamma$ is obtained as the inductive limit of the sequence $e_1 A_1 e_1 \rightarrow e_2 A_2 e_2 \rightarrow e_3 A_3 e_3 \rightarrow \cdots$, where the map of $e_n A_n e_n = B_n e_n = (A_n \cap C_n) e_n$ into $e_{n+1} A_{n+1} e_{n+1}$ is given by $x \mapsto x e_{n+1}$. Let $B_n^\gamma$ denote the image of $e_n A_n e_n$ in $B^\gamma$. We define a flow $\beta^\gamma$ on $B^\gamma$ by the requirement that $\Phi^\gamma(\beta_t(x)) = \beta_t^\gamma(\Phi^\gamma(x))$, $x \in B$.

Since $\sup_{t \in [0,1]} \text{dist}(\alpha_t(A_n), A_n) \to 0$, we obtain that

$$\sup_{t \in [0,1]} \text{dist}(\beta_t^\gamma(B_n^\gamma), B_n^\gamma) \to 0$$

for any $\gamma \in \Gamma$. Thus, by Lemma 4.2 each $\beta^\gamma$ is a cocycle perturbation of a UHF flow. To show that $\alpha$ is a cocycle perturbation of an AF flow, we would have to use the fact that the convergence in the above display is uniform in $\gamma \in \Gamma$.

We shall prove the first half of Lemma 4.2 as follows. Suppose, in particular, that $\sup_{t \in [0,1]} \text{dist}(\alpha_t(A_1), A_1)$ is sufficiently small. We have to show that there is an $\alpha$-cocycle $w$ such that $\sup_{t \in [0,1]} \|w_t - 1\|$ is arbitrarily small and $\text{Ad} w_t \alpha_t(A_1) = A_1$.

Suppose that the center of $A_1$ is $K$-dimensional, being spanned by minimal central projections $z_1, \ldots, z_K$, and choose $\gamma_1, \gamma_2, \ldots, \gamma_K \in \Gamma$ such that $\gamma_i(z_i) = 1$ for each $i = 1, 2, \ldots, K$. We apply Lemma 4.1 to each $(B^\gamma_1, \beta_t^\gamma)$ (or apply Prop. 1.5 if $B^\gamma_1$ is a matrix algebra); thus we get a $\gamma_i$-cocycle $w^i$ such that $\|w_t^i - 1\|$ is very small for $t \in [0,1]$ and $\text{Ad} w^i_t \beta^\gamma_t$ fixes $B^\gamma_1$. What is important here is there is a pure ground state $\phi_i$ for $(B^\gamma_1, \text{Ad} w^i \beta^\gamma)$, which is still pure on $B^\gamma_1$. We may regard $\phi_i$ as a pure state $\phi_i \Phi^\gamma$ on $z_i A_i z_i$, which is pure on $z_i A_i z_i (\supset B^\gamma_1)$. We consider the system $(z_i A_i z_i, \alpha^i = \alpha|z_i A_i z_i)$. By lifting $w^i$ (see below), we then find an $\alpha^i$-cocycle $w^i$ such that $\max_{t \in [0,1]} \|w^i_t - 1\|$ is sufficiently small and $\phi_i$ is $\text{Ad} w^i_t \alpha_t^i$-invariant. Since $\text{dist}(\text{Ad} w^i_t \alpha_t^i(A_1 z_i), A_1 z_i) \leq 2\|w^i_t - 1\| + \text{dist}(\alpha_t(A_1), A_1)$, which is very small, we apply Lemma 1.3 to get an $\alpha^i$-cocycle $w^i$ such that $\max_{t \in [0,1]} \|w^i_t - 1\|$ is small and $\text{Ad} w^i_t \alpha^i$ fixes $A_1 z_i (\supset z_i A_i z_i)$. By combining these $w^i$ we get an $\alpha$-cocycle $w$ such that $\max_{t \in [0,1]} \|w_t - 1\|$ is small and $\text{Ad} w_t \alpha_t$ fixes $A_1$.

**Lemma 5.4** Let $\gamma \in \Gamma$ and let $u$ be a $\beta^\gamma$-cocycle. Then there is an $\alpha$-cocycle $v$ such that $v_t \in B = A \cap C'$ and $\Phi^\gamma(v_t) = u_t$, $t \in \mathbb{R}$. If $\sup_{t \in [0,1]} \|u_t - 1\| < \delta$ holds, $\sup_{t \in [0,1]} \|v_t - 1\| < \delta$ can be imposed.

**Proof.** We find a $w \in \mathcal{U}(B^\gamma)$ and a differentiable $\beta^\gamma$-cocycle $z$ such that $\|w - 1\|$ is small and $u_t = w z_t \beta_t^\gamma(w^*)$. We can then find a $W \in B = A \cap C'$ and $H \in B_{sa}$ such that $\Phi^\gamma(W) = w$ and $\Phi^\gamma(H) = -idz_t/|t|_{t=0}$. Then we find a $\beta$-cocycle $Z$ by solving the equation $dZ_t/|t| = Z_t \beta_t(iH)$ with $Z_0 = 1$, which satisfies that $\Phi^\gamma(Z_t) = z_t$. We set $v_t = W Z_t \beta_t ^\gamma(W^*)$, which is an $\alpha$-cocycle with $\Phi^\gamma(v_t) = u_t$.

Since $\sigma \mapsto \|\Phi^\sigma(v_t) - 1\|$ is continuous, $t \mapsto \|\Phi^\sigma(v_t) - 1\|$ is equi-continuous in $\sigma \in \Gamma$, and $\Gamma$ is totally disconnected, the last condition is satisfied by replacing $v$ by $t \mapsto v_te + 1 - e$, where $e \in C$ is a projection with $\gamma(e) = 1$. □
Let \( \epsilon_1 > 0 \). For \( \epsilon = \epsilon_1/2 \) we choose a \( \delta_1 \equiv \delta > 0 \) as in Lemma 4.3 where we may assume that \( \delta_1 < \epsilon_1/2 \). For \( \epsilon = \delta_1/3 > 0 \) we choose a \( \delta_2 \equiv \delta > 0 \) as in Lemma 4.4 (or Prop. 1.5). We may assume that \( \delta_2 < \delta_1 \).

Suppose that \( \sup_{t \in [0,1]} \text{dist}(\alpha_t(A_1), A_1) < \delta_2 \). Let \( z_1, z_2, \ldots, z_K \) be the minimal central projections in \( A_1 \) and let \( \gamma_1, \ldots, \gamma_K \in \Gamma \) be such that \( \gamma_i(z_i) = 1 \). Then we apply 4.4 to \( (B^n, \beta^n) \) to get a \( \beta^n \)-cocycle \( u^i \) such that \( \sup_{t \in [0,1]} \| u^i_t - 1 \| < \delta_1/3 \) and \( \text{Ad} u^i_t \beta^n_1(B^1_1) = B^1_1 \). Since \( \beta^n \) is a UHF flow, there is a pure ground state \( \phi_i \) for \( \text{Ad} u^i \beta^n \). Note that \( \phi_i|B^1_1 \) is pure. We regard \( \phi_i \) as a state on \( z_i A z_i \) by denoting \( \phi_i \Phi^n|z_i A z_i \) again by \( \phi_i \). Note that \( \phi_i|z_i A z_i \) is pure. We lift \( u^i \) to an \( \alpha \)-cocycle \( v^i \) such that \( v^i_t \in B \) and \( \sup_{t \in [0,1]} \| v^i_t - 1 \| < \delta_1/3 \). We regard \( v^i \) as a cocycle in \( z_i A z_i \) with respect to \( \alpha|z_i A z_i \). Then it follows that

\[
\sup_{t \in [0,1]} \text{dist}(\text{Ad} v^i_t \alpha_t(z_i A_1), z_i A_1) < \delta_1.
\]

Then applying 4.3 to \( (z_i A z_i, \text{Ad} v^i_t \alpha_t|z_i A z_i) \) with a pure invariant state \( \phi_i \), we get a cocycle \( w^i \) with respect to \( \text{Ad} v^i_t \alpha_t|z_i A z_i \) such that

\[
\sup_{t \in [0,1]} \| w^i_t - 1 \| < \epsilon/2 \quad \text{and} \quad \text{Ad}(w^i_t v^i_t) \alpha_t(z_i A_1) = z_i A_1.
\]

Note that \( t \mapsto w^i_t v^i_t \) is an \( \alpha \)-cocycle in \( z_i A z_i \) such that \( \sup_{t \in [0,1]} \| w^i_t v^i_t - 1 \| < \epsilon_1 \). Then \( t \mapsto \sum_i w^i_t v^i_t \) is the desired \( \alpha \)-cocycle. This completes the proof of the first half of Lemma 1.2.

To prove the latter part, let \( A_0 \) be a C*-subalgebra of \( A_1 \) such that \( \alpha_t(A_0) = A_0 \). We have to show that the above \( \alpha \)-cocycle can be chosen from \( A_0 \cap A_0' \).

Since \( \text{dist}(\alpha_t(A_0 \cap A_0'), A_0 \cap A_0') \leq \text{dist}(\alpha_t(A_0), A_0) \), we apply the first part to \( \alpha|A_0 \cap A_0' \) to find an \( \alpha \)-cocycle \( u \) in \( A_0 \cap A_0' \) such that \( \sup_{t \in [0,1]} \| u_t - 1 \| < \epsilon \) and \( \text{Ad} u_t \alpha_t(A_0 \cap A_0') = A_0 \cap A_0' \). Since \( \text{Ad} u_t \alpha_t(A_0) = A_0 \), we get that \( \text{Ad} u_t \alpha_t(A_1 \cap Z_0') = A_1 \cap Z_0', \) where \( Z_0 = A_0 \cap A_0' \). Note that \( \sup_{t \in [0,1]} \text{dist}(\text{Ad} u_t \alpha_t(A_1), A_1) < \delta_1 + 2\epsilon \).

Let \( z_1, \ldots, z_K \) be the minimal central projections in \( A_1 \) as before. If \( (A_1 \cap Z_0') z_i \) is the direct sum of \( L_i + 1 \) factors, there are \( L_i \) partial isometries \( w_1, \ldots, w_{L_i} \) in \( A_1 z_i \) such that \( w_j^* w_j = w_i^* w_i \) is a minimal projection invariant under \( \text{Ad} u \alpha \), \( w_j w_j^* \) is a minimal projection invariant under \( \text{Ad} u \alpha \), and \( A_1 z_i \) is generated by \( (A_1 \cap Z_0') z_i \) and \( w_j, j = 1, \ldots, L_i \).

Since \( \text{Ad} u_t \alpha_t(w_j) \) is almost contained in \( A_1 \) (up to the order of \( \delta_1 + 2\epsilon \)) for \( t \in [0,1] \), it follows that \( \| \text{Ad} u_t \alpha_t(w_j) - c_t w_j \| < \delta_1 + 2\epsilon \) for some \( c_t \in \mathbb{C} \) for \( t \in [0,1] \). Note that \( t \mapsto w_j \text{Ad} u_t \alpha_t(w_j^*) \) is a cocycle with respect to \( \text{Ad} u \alpha w_j w_j^* A w_j^* \).

**Lemma 5.5** For any \( \epsilon > 0 \) there exists a \( \delta > 0 \) satisfying the following condition: If \( v \) is an \( \alpha \)-cocycle such that \( v_t \in \mathbb{C} \) for \( t \in [0,1] \), then there is a \( p \in \mathbb{R} \) such that \( \| v_t - e^{i pt} 1 \| < \epsilon \) for \( t \in [0,1] \).

Thus we find a \( \lambda_j \in \mathbb{R} \) such that \( \sup_{t \in [0,1]} \| w_j \text{Ad} u_t \alpha_t(w_j^*) - e^{i \lambda_j t} w_j w_j^* \| \approx 0 \). We extend a cocycle \( t \mapsto e^{-i \lambda_j t} w_j \text{Ad} u_t \alpha_t(w_j^*) \) in \( w_j w_j^* (A \cap (A_1 \cap Z_0')) \) to a cocycle \( v^j \) in \( p_j(A \cap (A_1 \cap Z_0')) \), where \( p_j \) is the central support projection of \( w_j w_j^* \) in \( A_1 \cap Z_0' \) (or
Letting $f$ be replaced by $f^*$, we have that $\|v^j_t - p_j\| = \|e^{-i\lambda_j t} w_j^* \text{Ad} u_t \alpha_t (w_j^*) - w_j^* w_j^*\|$. We define $v_t = \sum_j v^j_t + p_0$, where $p_0$ is the central support of $w_j^* w_j = w_j^* w_1$ in $z_i(A_1 \cap Z_0')$. Then $v$ is a cocycle in $z_iA \cap (A_1 \cap Z_0)'$ with respect to $\text{Ad} u \alpha$, sup$_{t \in [0,1]} \|v_t - z_t\| \approx 0$, and

$$\text{Ad}(v_t u_t) \alpha_t(w_j) = e^{-i\lambda_j t} w_j.$$ 

Thus, since $z_iA_1$ and all $w_j$ generate $z_iA_1$ and $A_1 \cap Z_0' \supset A_0$, we get that

$$\text{Ad}(v_t u_t) \alpha_t(z_i A_1) = z_i A_1 \quad \text{and} \quad \text{Ad}(v_t u_t) \alpha_t(z_i A_0) = z_i A_0.$$ 

We apply this argument to each $z_i$. This concludes the proof of Lemma 1.2.

**Proof of 5.3** We suppose that for a small $\delta > 0$ there is a continuous function $f : [0, 1] \rightarrow \mathbb{R}$ such that $\|u_t - e^{if(t)}1\| < \delta$ and $f(0) = 0$. For $t_1, t_2 \in [0, 1]$ with $t_1 + t_2 \leq 1$, we have that $\|u_t \alpha_t(u_{t_1}^{-1} - e^{if(t_1)}1\| < 2\delta$, which implies that $|e^{i(f(t_1 + t_2) - f(t_1) - f(t_2))} - 1| < 3\delta$.

Letting $\delta_0 = \arcsin 3\delta$, we get

$$|f(t_1 + t_2) - f(t_1) - f(t_2)| < \delta_0.$$ 

We replace $f$ by $f(t) - f(1)t$, which still satisfies the above inequality for $t_1, t_2, t_1 + t_2 \in [0, 1]$. With this replacement we have assumed that $f(0) = 0 = f(1)$.

Let $\mu = \max\{|f(t)| \mid t \in [0, 1]\}$. Suppose that $\mu > 3\delta_0$. There is an $s \in [0, 1]$ such that $|f(s)| = \mu$. If $s \leq 1/2$, then $|f(2s)| > 2|f(s)| - \delta_0 = 2\mu - \delta_0 > \mu + 2\delta_0$, which is a contradiction. Hence we have that $s > 1/2$. Then $|f(1 - s)| > |f(s)| - \delta_0$, and hence $|f(2(1 - s))| > 2|f(1 - s)| - \delta_0 > 2|f(s)| - 3\delta_0 > \mu$, which is again a contradiction. Thus we should have that $\mu \leq 3\delta_0$. This implies that

$$\|u_t - e^{if(1)t}1\| < |e^{if(t)} - e^{if(1)t}| + \delta < 3\delta_0 + \delta.$$

Since $\delta_0 \approx 3\delta$, this concludes the proof.

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