KORTEWEG–DE VRIES LIMIT FOR THE FERMI–PASTA–ULAM SYSTEM

YOUNGHUN HONG, CHULKWANG KWAK, AND CHANGHUN YANG

ABSTRACT. In this paper, we develop dispersive PDE techniques for the Fermi–Pasta–Ulam (FPU) system with infinitely many oscillators, and we show that general solutions to the infinite FPU system can be approximated by counter-propagating waves governed by the Korteweg–de Vries (KdV) equation as the lattice spacing approaches zero. Our result not only simplifies the hypotheses but also reduces the regularity requirement in the previous study [45].

1. INTRODUCTION

The Fermi–Pasta–Ulam (FPU) system is a simple nonlinear dynamical lattice model describing a long one-dimensional chain of vibrating strings with nearest neighbor interactions. This model was first introduced by Fermi, Pasta, and Ulam in the original Los Alamos report [12] in 1955 with regard to their numerical studies on nonlinear dynamics. At that time, it was anticipated that the energy initially given to only the lowest frequency mode would be shared by chaotic nonlinear interactions and it would be eventually thermalized to equilibrium. However, numerical simulations showed the opposite behavior. The energy is shared among only a few low-frequency modes and it exhibits quasi-periodic behavior. This phenomenon is known as the FPU paradox. Since then, the FPU paradox has emerged as one of the central topics in various fields, and it has stimulated extensive studies on nonlinear chaos.

Among the various important studies in this regard, the most remarkable one is the fundamental work of Zabusky and Kruskal [49], in which the problem was solved for the first time by discovering a connection to the Korteweg–de Vries (KdV) equation. The authors showed that the FPU system is formally approximated by the KdV equation and its quasi-periodic dynamics is thus explained in connection with solitary waves for the KdV equation. From an analysis perspective, Friesecke and Wattis proved that the FPU system has solitary waves [17], confirming the numerical observation [11], whereas Friesecke and Pego established their convergence to the soliton solutions to the KdV equation [13]. Moreover, various qualitative properties have been proved for the FPU solitary waves [14, 15, 16, 38].

The KdV approximation problem has also been investigated for general states without restriction to solitary waves. For an infinite chain, Schneider and Wayne showed that the FPU flow can be approximated by counter-propagating KdV flows (see [19] below) via the multi-scale method [45]. This approach has been applied to a periodic setting [40] as well as
to generalized discrete models \[37, 4, 19\]. Furthermore, with a different scaling, the cubic nonlinear Schrödinger equation is derived from the FPU system \[44\] (see also \[21, 22\]).

In contrast, the FPU paradox can be explained in a completely different manner, i.e., by the approach of Izrailev and Chirikov \[27\], which involves the Kolmogorov–Arnold–Moser (KAM) theory: quasi-periodicity occurs because the FPU system can be approximated by a finite-dimensional integrable system (see \[39, 41, 42\] for this direction). We also note that the quasi-periodic dynamics vanishes after a sufficiently long time-scale as predicted originally \[18\]. This phenomenon is called *metastability*, and it has been investigated rigorously (e.g., \[2, 1\]). Overall, the dynamics problem for the FPU system has garnered considerable research attention and it has been explored from various perspectives. We refer to the surveys \[48, 20\] and the references therein for a more detailed history and an overview of the problem.

In this article, we follow the approach of Zabusky and Kruskal \[49\]. Our objective is to provide a rigorous justification of the KdV approximation for general solutions, including solitary waves, to infinite FPU chains. Let us begin with introducing the setup of the problem. Consider the FPU Hamiltonian

\[
H(q, p) := \sum_{x \in \mathbb{Z}} \frac{p(x)^2}{2} + V(q(x + 1) - q(x))
\]

(1.1)

for a function \((q, p) = (q(x), p(x)) : \mathbb{Z} \to \mathbb{R} \times \mathbb{R}\). Here, \((q(x), p(x))\) denotes for the position and momentum of the \(x\)-th string, and the potential function \(V : \mathbb{R} \to \mathbb{R}\) determines the potential energy from nearest-neighbor interactions. We assume that

\[
V \in C^5, \quad V(0) = V'(0) = 0, \quad V''(0) =: a > 0 \quad \text{and} \quad V'''(0) =: b \neq 0.
\]

(1.2)

Such potentials include the cubic FPU potential \(\frac{1}{2}ar^2 + \frac{1}{6}br^3\), a more general polynomial potential \(\sum_{k=2}^{N} c_k r^k\), the Lennard-Jones potential \(e[(1 + \frac{r}{\sigma})^{-12} - 2(1 + \frac{r}{\sigma})^{-6} + 1]\) and the Toda potential \(\alpha(e^{\beta r} - \beta r - 1)\).

The above-mentioned Hamiltonian generates the FPU system

\[
\begin{align*}
\partial_t q(t, x) &= p(t, x), \\
\partial_t p(t, x) &= V'(q(t, x + 1) - q(t, x)) - V'(q(t, x) - q(t, x - 1)),
\end{align*}
\]

(1.3)

where \((q, p) = (q(t, x), p(t, x)) : \mathbb{R} \times \mathbb{Z} \to \mathbb{R} \times \mathbb{R}\). By combining the two equations in the system and then rewriting them for the relative displacement between two adjacent points, \(r(t, x) = q(t, x + 1) - q(t, x)\), we can simplify the system as

\[
\partial_t^2 r = \Delta_1(V'(r))
\]

(1.4)

where \(\Delta_1 u = u(\cdot + 1) + u(\cdot - 1) - 2u\). Next, by rescaling with

\[
\tilde{r}_h(t, x) := \frac{1}{h^2}r\left(\frac{t}{h^3}, \frac{x}{h}\right) : \mathbb{R} \times h\mathbb{Z} \to \mathbb{R}
\]

(1.5)

for small \(h > 0\), we obtain

\[
h^6 \partial_t^2 \tilde{r}_h = \Delta_h(V'(h^2 \tilde{r}_h))
\]

(1.6)
where $\Delta_h$ is a discrete Laplacian on $h\mathbb{Z}$, i.e.,

$$\Delta_h u = \frac{u(\cdot + h) + u(\cdot - h) - 2u}{h^2}.$$ 

Finally, by extracting the linear term from the right-hand side of (1.6), we derive a discrete nonlinear wave equation, which we refer to hereafter as the FPU system

$$\begin{align*}
\partial_t^2 \tilde{r}_h - \frac{a}{h^4} \Delta_h \tilde{r}_h &= \frac{1}{h^6} \Delta_h \left\{ V'(h^2 \tilde{r}_h) - ah^2 \tilde{r}_h \right\}, \\
\tilde{r}_h(0) &= \tilde{r}_{h,0}, \\
\partial_t \tilde{r}_h(0) &= \tilde{r}_{h,1},
\end{align*}$$

(1.7)

where $\tilde{r}_h = \tilde{r}_h(t, x) : \mathbb{R} \times h\mathbb{Z} \to \mathbb{R}$. This reformulated equation is still a Hamiltonian equation with the Hamiltonian

$$H_h(\tilde{r}_h) = h \sum_{x \in h\mathbb{Z}} \left\{ \frac{1}{2} \left( \frac{h^2}{\sqrt{-\Delta_h}} \partial_t \tilde{r}_h \right)^2 + \frac{1}{h^4} V(h^2 \tilde{r}_h) \right\}.$$ 

(1.8)

Through the formal analysis described in Section 2, one would expect that the solutions to FPU (1.7) are approximated by counter-propagating waves

$$\tilde{r}_h(t, x) \approx w^+_h(t, x - \frac{t}{h^2}) + w^-_h(t, x + \frac{t}{h^2}),$$

(1.9)

where each $w^+_h = w^+_h(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a solution to the KdV equation

$$\begin{align*}
\partial_t w_{\pm} \pm \frac{\sqrt{a}}{24} \partial_x^3 w_{\pm} + \frac{b}{4\sqrt{a}} \partial_x(w_{\pm}^2) &= 0, \\
w_{\pm}(0) &= w_{\pm,0}.
\end{align*}$$

(1.10)

This method of deriving the two KdV flows can be regarded as an infinite-lattice version of the method of Zabusky and Kruskal [49].

In this study, we revisit the KdV limit problem for general solutions, albeit through a rather different approach. Indeed, in a broad sense, a dynamical system approach was adopted in all the aforementioned studies [45, 21, 40, 37, 22, 41, 6, 19]. By regarding the FPU system (1.7) as a nonlinear dispersive equation, we exploit its dispersive and smoothing properties, and we then employ them to justify the KdV approximation. This approach enables us to not only simplify the assumptions on the initial data in the previous study but also reduce the regularity requirement.

For the statement of the main theorem, we introduce the basic definitions of function spaces, the Fourier transform and differentials on a lattice domain, and the linear interpolation operator. For $1 \leq p \leq \infty$, the Lebesgue space $L^p(h\mathbb{Z})$ is defined by the collection of

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1It is derived from (1.1).
real-valued functions on a lattice domain \( h\mathbb{Z} \) equipped with the \( L^p \)-norm

\[
\|f_h\|_{L^p(h\mathbb{Z})} := \left\{ \begin{array}{ll}
\left( \frac{1}{h} \sum_{x \in h\mathbb{Z}} |f_h(x)|^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\
\sup_{x \in h\mathbb{Z}} |f_h(x)| & \text{if } p = \infty.
\end{array} \right.
\]

For \( f_h \in L^1(h\mathbb{Z}) \), we define its (discrete) Fourier transform by

\[
(F_h f_h)(\xi) := h \sum_{x \in h\mathbb{Z}} f_h(x) e^{-ix\xi}, \quad \forall \xi \in \mathbb{R}/(\frac{2\pi}{h}\mathbb{Z}) = [-\frac{\pi}{h}, \frac{\pi}{h}).
\]

Meanwhile, for a periodic function \( f \in L^1([-\frac{\pi}{h}, \frac{\pi}{h})) \), its inverse Fourier transform is given by

\[
(F_h^{-1} f)(x) := \frac{1}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} f(\xi) e^{ix\xi} d\xi, \quad \forall x \in h\mathbb{Z}.
\]

Then, Parseval’s identity,

\[
h \sum_{x \in h\mathbb{Z}} f(x)\overline{g(x)} = \frac{1}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} (F_h f)(\xi)\overline{(F_h g)(\xi)} d\xi,
\]

extends the discrete Fourier transform (resp., its inversion) to \( L^2(h\mathbb{Z}) \) (resp., \( L^2([-\frac{\pi}{h}, \frac{\pi}{h})) \)).

There are several ways to define differentials on a lattice domain \( h\mathbb{Z} \). Throughout the paper, we use the following different types of differentials, all of which are consistent with differentiation on the real line as the Fourier multiplier of the symbol \( i\xi \) as \( h \to 0 \).

**Definition 1.1 (Differentials on \( h\mathbb{Z} \)).** (i) \( \nabla_h \) (resp., \( |\nabla_h| \), \( \langle \nabla_h \rangle \)) denotes the discrete Fourier multiplier of the symbol \( \frac{2i}{h} \sin(\frac{k}{2}) \) (resp., \( \frac{2}{h} \sin(\frac{kb}{2}) \), \( \frac{2}{h} \sin(\frac{kb}{2}) \)), where \( \langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}} \).

(ii) \( \partial_h \) (resp., \( |\partial_h| \), \( \langle \partial_h \rangle \)) denotes the discrete Fourier multiplier of the symbol \( i\xi \) (resp., \( |\xi| \), \( \langle \xi \rangle \)).

(iii) \( \partial_h^+ \) denotes the discrete right-hand side derivative naturally defined by

\[
(\partial_h^+ f_h)(x) := \frac{f_h(x + h) - f_h(x)}{h}, \quad \forall x \in h\mathbb{Z}.
\]

For \( s \in \mathbb{R} \), we define the Sobolev space \( W^{s,p}(h\mathbb{Z}) \) (resp., \( \dot{W}^{s,p}(h\mathbb{Z}) \)) by the Banach space equipped with the norm

\[
\|f_h\|_{W^{s,p}(h\mathbb{Z})} := \|\langle \partial_h \rangle^s f_h\|_{L^p(h\mathbb{Z})} \quad \text{(resp., } \|f_h\|_{\dot{W}^{s,p}(h\mathbb{Z})} := \|\partial_h^s f_h\|_{L^p(h\mathbb{Z})}).\]

In particular, when \( p = 2 \), we denote

\[
H^s(h\mathbb{Z}) := W^{s,2}(h\mathbb{Z}) \quad \text{(resp., } \dot{H}^s(h\mathbb{Z}) := \dot{W}^{s,2}(h\mathbb{Z})).\]

To compare functions on different domains, we introduce the linear interpolation

\[
(l_h f_h)(x) := f_h(hm) + (\partial_h^+ f_h)(hm) \cdot (x - hm)
\]

\[
= f_h(hm) + \frac{f_h(hm + h) - f_h(hm)}{h}(x - hm)
\]

\[
(1.13)
\]

These definitions are consistent with the discrete Laplacian \( \Delta_h \), because \( (-\Delta_h) \) is the Fourier multiplier of the symbol \( \frac{2i}{h} \sin^2(\frac{k}{2}) \); thus, \( |\nabla_h| = \sqrt{-\Delta_h} \) and \( \langle \nabla_h \rangle = \sqrt{1 - \Delta_h} \).
for all $x \in [hm, hm + h]$ with $m \in \mathbb{Z}$. Note that the linear interpolation converts a function $f_h : h\mathbb{Z} \to \mathbb{R}$ on a lattice into a continuous function on the real line.

Now, we are ready to state our main result.

**Theorem 1.2 (KdV limit for FPU).** If $V$ satisfies (1.2), then for any $R > 0$, there exists $T(R) > 0$ such that the following holds. Suppose that for some $s \in (\frac{3}{4}, 1]$,

$$
\sup_{h \in [0, 1]} \left\| \left(\tilde{r}_{h, 0}, h^2\nabla_h^{-1}\tilde{r}_{h, 1}\right) \right\|_{H^s(h\mathbb{Z}) \times H^s(h\mathbb{Z})} \leq R. \quad (1.14)
$$

Let $\tilde{r}_h(t) \in C_t([-T, T]; H^s_x(h\mathbb{Z}))$ be the solution to FPU (1.1) with initial data $(\tilde{r}_{h, 0}, \tilde{r}_{h, 1})$, and let $w^+_h(t) \in C_t([-T, T]; H^s_x(\mathbb{R}))$ be the solution to KdV (1.10) with interpolated initial data $\tilde{r}_{h, 0} = h^2\nabla_h^{-1}\tilde{r}_{h, 1}$.

(i) (Continuum limit)

$$
\sup_{t \in [-T, T]} \left\| (l_h \tilde{r}_h)(t, x) - w^+_h(t, x - \frac{t}{h^2}) - w^-_h(t, x + \frac{t}{h^2}) \right\|_{L^2_2(\mathbb{R})} \lesssim h^{\frac{2}{4}}. \quad (1.15)
$$

(ii) (Small amplitude limit) Scaling back,

$$
r(t, x) = h^2 \tilde{r}_h(h^3 t, hx) : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}
$$

is a solution to FPU (1.1). Moreover, it satisfies

$$
\sup_{t \in [-\frac{T}{h^2}, \frac{T}{h^2}]} \left\| (l_1 r)(t, x) - h^2 w^+_h(h^3 t, h(x - t)) - h^2 w^-_h(h^3 t, h(x + t)) \right\|_{L^2_2(\mathbb{R})} \lesssim h^{\frac{3}{4} + \frac{2}{5}}. \quad (1.16)
$$

We remark that the assumption on the initial data is simplified compared to the previous work [15]. We assume only a uniform bound on the size of the initial data (see (1.14)) in a natural Sobolev norm (without any weight), and the mean-zero momentum condition $\sum_{x \in h\mathbb{Z}} \tilde{r}_{h, 1}(x) = 0$ is not imposed. Furthermore, the regularity requirement is reduced to $s > \frac{3}{4}$.

As for the regularity issue, we emphasize that reducing the regularity is not only a matter of mathematical curiosity but it may also lead to a significant improvement in the continuum limit (1.15). As stated in our main theorems, the KdV approximation is stated in the form of either a continuum limit or a small amplitude limit. Mathematically, they are equivalent; however, the continuum limit (1.15) seems rather weaker because it holds only in a short time interval $[-T, T]$, whereas the small amplitude limit (1.16) is valid almost globally in time $[-\frac{T}{h^2}, \frac{T}{h^2}] \to (-\infty, \infty)$ as $h \to 0$. Thus, it would be desirable to extend the time interval $[-T, T]$ arbitrarily for the continuum limit. For comparison, we state that for discrete nonlinear Schrödinger equations (DNLS), the continuum limit is established in a compact time interval of any size [24], and an exponential-in-time bound is obtained. In the proof, conservation laws obviously play a crucial role. However, unlike DNLS, the FPU system does not have a conservation law controlling a higher regularity norm, say the $H^1$ norm. Only an $L^2$-type quantity is controlled by its Hamiltonian (1.8). Therefore, it would be desirable to establish the continuum limit for $L^2$-data. If such a low regularity convergence is achieved, then one may try to employ the conservation law to extend the size of the interval to be arbitrarily large. Although the regularity is significantly reduced in this study, our assumption that $s > \frac{3}{4}$ is still far from the desired case of $s = 0$. At the end
of this section, we mention the technical obstacle that prevents us from going below $s = \frac{3}{4}$. Instead of sharpening the estimates, a new idea seems necessary to reduce the regularity.

The main contribution of this article is to present a new approach to the KdV limit problem from the perspective of the theory of nonlinear dispersive PDEs. In spite of the dispersive nature of the FPU system, which is clear from its connection to the KdV equation, to the best of authors’ knowledge, there has been no attempt to tackle the problem using dispersive PDE techniques thus far.

Our approach is achieved on the basis of the following two observations. First, as outlined in Section 2, we reformulate the FPU system (1.7) by separating its Duhamel formula into two coupled equations (2.3), which we refer to as the coupled FPU. Indeed, this is a standard method to deal with inhomogeneous wave equations; however after implementing it, we realized that it is much easier to understand the limit procedure by analyzing the symbols of the linear propagators and their asymptotics (see Remark 2.1). By this reformulation, we introduce a different convergence scheme to the KdV equation via the decoupled FPU (2.12). It makes the problem more suitable and clearer for analysis by dispersive PDE techniques.

Second, we discover that the linear propagators $S_h^\pm(t) = e^{\mp \frac{ith^2}{2} (\nabla h - \partial h)}$ for the coupled and decoupled FPU flows exhibit properties similar to those of the Airy flows $S(t) = e^{\mp \frac{ith^2}{2} \partial^3_x}$ in many aspects. A technical but crucial feature of our analysis is that the phase functions of the linear FPU propagators are comparable with those of the Airy propagators at different derivative levels. Indeed, direct calculations show that

\[
\frac{1}{\pi^2} (\xi - \frac{2}{\pi} \sin(h\xi)) \sim \xi^3,
\]

\[
\{ \frac{1}{\pi^2} (\xi - \frac{2}{\pi} \sin(h\xi)) \}' = \frac{1}{\pi^2} (1 - \cos(h\xi)) \sim \xi^2,
\]

\[
\{ \frac{1}{\pi^2} (\xi - \frac{2}{\pi} \sin(h\xi)) \}'' = \frac{1}{2h} \sin(\frac{h\xi}{2}) \sim \xi,
\]

on the frequency domain $[-\frac{\pi}{h}, \frac{\pi}{h}]$ for the discrete Fourier transform (see Figure 1.2). This allows us to recover the Strichartz estimates, the local smoothing and maximal function estimates (Proposition 5.1), and the bilinear estimates (Lemma 6.1) for the linear FPU flows owing to the “magical” property of Zabusky and Kruskal’s transformation of the FPU system in their original study [49]. Indeed, dispersive equations on a lattice domain do not enjoy smoothing in general. For instance, the phase function for the linear Schrödinger flow $e^{ith^2} \Delta_h$ is comparable with that for the linear Schrödinger flow on $\mathbb{R}$, i.e., $-\frac{2}{\pi} (1 - \cos(h\xi)) \sim -t\xi^2$ on $[-\frac{\pi}{h}, \frac{\pi}{h}]$; however, its derivative $-\frac{2t}{\pi} \sin(h\xi)$ is far from $-t\xi$ near the high frequency edge $\xi = \pm \frac{\pi}{h}$ (see Figure 1.1). Therefore, the discrete linear Schrödinger flow does not enjoy local smoothing at all (see [26]). With various dispersive and smoothing estimates for the linear FPU flows, we follow a general strategy (see [24] for instance) to prove the convergence from the coupled to the decoupled FPU and the convergence from the decoupled FPU to the KdV equation. First, we employ the linear and bilinear estimates to obtain $h$-uniform bounds for solutions to the coupled and decoupled FPUs. Then, using the uniform bounds, we directly measure the differences to prove the convergences.

We conclude the introduction with a comment on the obstacle to reducing the regularity below $s = \frac{3}{4}$. As mentioned above, our strategy heavily employs uniform bounds for the coupled and decoupled FPUs, and their proofs resemble those of the local well-posedness of the KdV equation. The $X^{s,b}$-norm (see Section 3.3) is well known as a powerful tool
Figure 1.1. Comparison between the discrete and the continuous Schrödinger flows: The symbol $\frac{2}{h^2}(1 - \cos(h\xi))$ is comparable with the symbol $\xi^2$ on the frequency domain $[-\frac{\pi}{h}, \frac{\pi}{h}]$. However, their derivatives are not comparable particularly near the endpoints $\xi = \pm \frac{\pi}{h}$.

Figure 1.2. Comparison between the linear FPU flow and the Airy flow: The derivatives of the symbol $\frac{1}{h^2}(\xi - \frac{2}{h} \sin(\frac{h\xi}{2}))$ are comparable with those of the symbol $\xi^3$ on the frequency domain $[-\frac{\pi}{h}, \frac{\pi}{h}]$.

for the low regularity theory. Indeed, Kenig, Ponce, and Vega established the local well-posedness of KdV in $H^s(\mathbb{R})$ for $s > -\frac{3}{4}$ using the bilinear estimates in $X^{s,b}$. Thus, one may attempt to employ the $X^{s,b}$-norm for the KdV limit problem. However, this norm is too sensitive to the linear propagator, because a certain weight is imposed away from the characteristic curve on the space-time Fourier side. Hence, it is not suitable to measure two different linear flows having different characteristic curves at the same time. Indeed, Proposition A.1 shows that linear FPU flows are not uniformly bounded in the $X^{s,b}$-norm associated with the Airy flows (the other direction can be proved similarly). Therefore, we do not use the $X^{s,b}$-norm. We employ the Strichartz estimates, the local smoothing and maximal function estimates, and their corresponding norms (see Proposition 5.1 and 5.4), because they are not sensitive to the propagators. These norms have been employed in the previous work of Kenig, Ponce, and Vega for the local well-posedness of KdV in $H^s$ for
\[ s > \frac{3}{4}. \] However, it is known that the maximal function estimate holds only when \( s > \frac{3}{4}. \) Therefore, we are currently unable to go below \( s = \frac{3}{4}. \)

1.1. **Organization of the paper.** The remainder of this paper is organized as follows. In Section 2 the outline of the proof of Theorem 1.2 is presented. In particular, FPU systems are reformulated and Theorem 1.2 is reduced to two propositions. In Section 3 some definitions and estimates, in particular, well-known estimates and the Littlewood–Paley theory on a lattice and \( X^{s,b} \) space are introduced. In Section 4 the local well-posedness of FPU is established. In Section 5 Strichartz, local smoothing, and maximal function estimates of linear FPU flows are discussed in comparison with linear KdV flows. In Section 6 \( X^{s,b} \) bilinear estimates are discussed, respectively. In Section 7 the remainder of this paper is organized as follows. In Section 8 the main proposition is proven by combining the proofs of two propositions. Finally, in Appendices A and B justification of the non-triviality of the approximation via \( X^{s,b} \) analysis and the estimate of the higher-order term are discussed, respectively.

1.2. **Notations and basic definitions.** In this article, we deal with two different types of functions, i.e., functions on the real line \( \mathbb{R} \) and functions on the lattice domain \( h\mathbb{Z}. \) To avoid possible confusion, we use the subscript \( h \) for functions on \( h\mathbb{Z} \) with no exception. For instance, \( u_h, v_h, \) and \( w_h \) are defined on \( h\mathbb{Z}, \) while \( u, v, \) and \( w \) are defined on \( \mathbb{R}. \)

If there is no confusion, we assign lower-case letters \( x, y, z, ... \) to spatial variables regardless of whether they are on the lattice or on the real line; for instance, \( u_h(x) : h\mathbb{Z} \to \mathbb{R} \) and \( u(x) : \mathbb{R} \to \mathbb{R}. \) Note that the subscript \( h \) determines the space of the spatial variable.

For notational convenience, we may abbreviate the domain and codomain of a function in the norm. For example, for \( f_h = f_h(x) : h\mathbb{Z} \to \mathbb{R} \) (resp., \( f = f(x) : \mathbb{R} \to \mathbb{R} \)),

\[
\|f_h\|_{L^p} = \|f_h\|_{L^p(h\mathbb{Z})} \quad (\text{resp., } \|f\|_{L^p} = \|f\|_{L^p(\mathbb{R})}),
\]

\[
\|f_h\|_{W^{s,p}} = \|f_h\|_{W^{s,p}(h\mathbb{Z})} \quad (\text{resp., } \|f\|_{W^{s,p}} = \|f\|_{W^{s,p}(\mathbb{R})}),
\]

and for \( F_h = F_h(t,x) : h\mathbb{Z} \to \mathbb{R} \) (resp., \( F = F(t,x) : \mathbb{R} \to \mathbb{R} \)),

\[
\|F_h\|_{L_t^q L_x^r} = \|F_h\|_{L_t^q([-T,T];L_x^r(h\mathbb{Z}))} \quad (\text{resp., } \|F\|_{L_t^q L_x^r} = \|F\|_{L_t^q([-T,T];L_x^r(\mathbb{R}))}),
\]

\[
\|F_h\|_{L_t^q W_x^{s,r}} = \|F_h\|_{L_t^q([-T,T];W_x^{s,r}(h\mathbb{Z}))} \quad (\text{resp., } \|F\|_{L_t^q W_x^{s,r}} = \|F\|_{L_t^q([-T,T];W_x^{s,r}(\mathbb{R}))}).
\]

Similarly, for a vector-valued function \( (f^+_h, f^-_h) : h\mathbb{Z} \to \mathbb{R} \times \mathbb{R}, \) we have

\[
\| (f^+_h, f^-_h)\|_{L^p} = \left\| (f^+_h, f^-_h)\right\|_{L^p(h\mathbb{Z})} = \left\{ \left( (f^+_h) \right)^2 + \left( f^-_h \right)^2 \right\}^{1/2}_{L^p(h\mathbb{Z})}.
\]

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2. OUTLINE OF THE PROOF

The proof of the main theorem (Theorem 1.2) is outlined in this section. Although our proof is strongly inspired by the original idea of Zabusky and Kruskal [49], it is reorganized to fit into the theory of dispersive PDEs.

First, we note that the constants $a$ and $b$ in the assumption (1.2) can be normalized by constant multiplication and scaling $\tilde{r}_h(t, x) \mapsto \frac{a}{b} \tilde{r}_h(\sqrt{ab}, x)$. Thus, we assume that $a = b = 1$. By Taylor’s theorem, the nonlinear term in FPU (1.7) can be expressed as

$$
\frac{1}{h^6} \Delta_h \left\{ V'(h^2 \tilde{r}_h) - h^2 \tilde{r}_h \right\} = \frac{1}{2h^2} \Delta_h \left\{ (\tilde{r}_h)^2 + h^2 \mathcal{R} \right\},
$$

with the higher-order remainder

$$
\mathcal{R} := \frac{V^{(4)}(h^2 \tilde{r}_h)}{3} (\tilde{r}_h)^3,
$$

where $V^{(4)}$ denotes the fourth-order derivative of $V$ and $\tilde{r}_h$ is some number between 0 and $\tilde{r}_h$. To avoid non-essential complexity, we suggest that readers consider the FPU with simple quadratic nonlinearity, i.e., $\mathcal{R} = 0$, which corresponds to the normalized standard FPU potential $V(r) = \frac{r^2}{2} + \frac{r^4}{6}$.

2.1. Reformulation of FPU as a coupled system. By Duhamel’s formula, the initial data problem for FPU (1.7) is written as

$$
\tilde{r}_h(t) = \cos \left( \frac{t \sqrt{-\Delta_h}}{h^2} \right) \tilde{r}_{h,0} + \sin \left( \frac{t \sqrt{-\Delta_h}}{h^2} \right) \frac{h^2}{\sqrt{-\Delta_h}} \tilde{r}_{h,1}
- \frac{1}{2} \int_0^t \sin \left( \frac{(t - t_1) \sqrt{-\Delta_h}}{h^2} \right) \sqrt{-\Delta_h} \left\{ (\tilde{r}_h(t_1))^2 + h^2 \mathcal{R}(t_1) \right\} dt_1.
$$

Observe that

$$
\cos \left( \frac{t \sqrt{-\Delta_h}}{h^2} \right) = \frac{1}{2} (e^{\frac{t}{h^2} \nabla_h} + e^{-\frac{t}{h^2} \nabla_h}), \quad \sin \left( \frac{t \sqrt{-\Delta_h}}{h^2} \right) = \frac{1}{2h} (e^{\frac{t}{h^2} \nabla_h} - e^{-\frac{t}{h^2} \nabla_h}) \mathcal{H},
$$

where $\nabla_h$ is the discrete Fourier multiplier of the symbol $\frac{2i}{h} \sin(\frac{h \xi}{2})$ (see Definition 1.1) and $\mathcal{H}$ is the Hilbert transform, i.e., the Fourier multiplier of the symbol $-i \text{sign}(\xi)$. Indeed, $\cos(\frac{t}{h^2} \sin(\frac{h \xi}{2})) = \cos(\frac{2i}{h} \sin(\frac{h \xi}{2})) = \frac{1}{2} (e^{\frac{2i}{h^2} \sin(\frac{h \xi}{2})} + e^{-\frac{2i}{h^2} \sin(\frac{h \xi}{2})})$, and the other identity can be shown similarly. Thus, by inserting these into the Duhamel formula (2.2) and separating the operators $e^{\pm \frac{i}{h^2} \nabla h}$, we deduce that if

$$(\tilde{r}_h^+, \tilde{r}_h^-) : \mathbb{R} \times h \mathbb{Z} \to \mathbb{R} \times \mathbb{R}
$$
solves the system of coupled equations

$$
\tilde{r}_h^+(t) = e^{\pm \frac{i}{h^2} \nabla h} \tilde{r}_{h,0} \mp \frac{1}{4} \int_0^t e^{\pm \frac{(t - t_1)}{h^2} \nabla h} \nabla_h \left\{ (\tilde{r}_h(t_1))^2 + h^2 \mathcal{R}(t_1) \right\} dt_1
$$

with initial data

$$
\tilde{r}_h^+,_{t=0} = \frac{1}{2} \left\{ \tilde{r}_{h,0} \mp h^2 \nabla_h^{-1} \tilde{r}_{h,1} \right\},
$$

then

$$
\tilde{r}_h(t, x) = \tilde{r}_h^+(t, x) + \tilde{r}_h^-(t, x)
$$
is the solution to FPU (2.2).

Next, we introduce
\[ u_h^\pm(t) := e^{\pm \frac{4}{h^2} \partial_h} \tilde{r}_h^\pm(t), \]  
where \( \partial_h \) is given in Definition 1.1. Then, they solve the coupled integral equation
\[ u_h^\pm(t) = e^{\pm \frac{4}{h^2} (\nabla_h - \partial_h)} r_{h,0}^\pm \mp \frac{1}{4} \int_0^t e^{\mp \frac{4}{h^2} (\nabla_h - \partial_h)} \nabla_h e^{\pm \frac{4}{h^2} \partial_h} \left\{ \tilde{r}_h(t_1) + h^2 \mathcal{R}(t_1) \right\} dt_1. \]

Note that the main nonlinear term in the integral can be written as
\[ e^{\pm \frac{4}{h^2} \partial_h} \{ \tilde{r}_h(t_1)^2 \} = e^{\pm \frac{4}{h^2} \partial_h} \{ \tilde{r}_h(t_1) + \tilde{r}_h(t_2) \} = \left\{ u_h^\pm(t_1) + e^{\pm \frac{4}{h^2} \partial_h} u_h^\mp(t_1) \right\}^2, \]

because \( e^{a \partial_h} (u_h^2) = (e^{a \partial_h} u_h)^2 \) holds. Therefore, the equation (2.3) is reformulated as a coupled system of integral equations, which we refer to as the coupled FPU,
\[ u_h^\pm(t) = S_h^\pm(t) u_{h,0}^\pm \mp \frac{1}{4} \int_0^t S_h^{-\pm}(t - t_1) \nabla_h \left[ \left\{ u_h^\pm(t_1) + e^{\pm \frac{4}{h^2} \partial_h} u_h^\mp(t_1) \right\}^2 \right. \]
\[ \left. + h^2 e^{\pm \frac{4}{h^2} \partial_h} \mathcal{R}(t_1) \right\} dt_1, \]

with initial data
\[ u_{h,0}^\pm = \frac{1}{2} \left\{ \tilde{r}_{h,0} \mp h^2 \nabla_h^{-1} \tilde{r}_{h,1} \right\}, \]

where the linear FPU propagator is denoted by
\[ S_h^\pm(t) = e^{\mp \frac{4}{h^2} (\nabla_h - \partial_h)}. \]

**Remark 2.1.** (i) By construction, FPU (1.7) can be recovered from the equation (2.8) via
\[ \tilde{r}_h(t, x) = e^{-\frac{4}{h^2} \partial_h} u_h^+(t, x) + e^{\frac{4}{h^2} \partial_h} u_h^-(t, x). \]

(ii) \( e^{\pm \frac{4}{h^2} \partial_h} \) is an almost translation in that at each discrete time \( t = n/h \) with \( n \in \mathbb{N} \),
\[ (e^{\pm \frac{4}{h^2} \partial_h} u_h^\pm(t, x)) = (e^{\pm (h \partial_h) u_h^\pm}(t, x) = u_h^\pm(t, x \mp h) = u_h^\pm(t, x \mp \frac{1}{h^2}). \]

Thus, at least formally, if \( u_h^\pm(t, x) \approx w^\pm(t, x) \), then by (2.11), the solution \( \tilde{r}_h(t, x) \) to the FPU becomes asymptotically decoupled into the counter-propagating flows \( w^+ (t, x - \frac{1}{h^2}) \) (moving to the right) and \( w^- (t, x + \frac{1}{h^2}) \) (moving to the left).

(iii) If the nonlinear solution \( u_h^\pm(t) \) behaves almost linearly in a short time interval, the coupled term \( e^{\pm \frac{2}{h^2} \partial_h} u_h^\pm(t_1, x) \) in (2.8) is approximated by
\[ e^{\pm \frac{4}{h^2} \partial_h} u_h^\pm(0, x) = \frac{1}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\pm \frac{4}{h^2} \partial_h} \left( \frac{\xi + \frac{2}{h} \sin(\frac{\xi}{h}) + \imath \xi }{\mathcal{F}_h u_{h,0}^\pm}(\xi) d\xi. \]

\[ \mathcal{F}_h \left( e^{a \partial_h} u_h^\pm \right)(\xi) = \frac{e^{i\alpha h}}{2\pi} \int_{-\pi/h}^{\pi/h} (\mathcal{F}_h u_h)(\xi - \eta) (\mathcal{F}_h u_h)(\eta) d\eta \]
\[ = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{ia(\xi - \eta)} (\mathcal{F}_h u_h)(\xi - \eta) e^{ia\eta} (\mathcal{F}_h u_h)(\eta) d\eta = \mathcal{F}_h \left( e^{a \partial_h} u_h^2 \right)(\xi). \]

This computation can be extended to any polynomial of finite degree.
This term is expected to vanish as $h \to 0$ owing to fast dispersion. Note that its group velocity $\mp \frac{1}{h^2} (1 + \cos(\frac{k^2}{2})) = \mp \frac{1}{h^2} (2 - \frac{h^2 \xi^2}{8} + \cdots)$ diverges to $\mp \infty$ for $|\xi| \leq \frac{2}{h}$. The higher-order remainder $e^{\pm \frac{2 h}{h} \partial_t} R(t_1)$ is also expected to vanish owing to the spare $h$ of order 2.

(iv) The linear propagator $S^\pm_h(t)$ formally converges to the Airy flow, because by Taylor’s theorem, $\mp \frac{1}{h^2} (\frac{2}{h^2} \sin(\frac{k^2}{2}) - \xi) = (\frac{\xi^3}{h^3} - \frac{h^2 \xi^5}{10!} + \cdots) \to \pm \frac{\xi^3}{h^3}$ as $h \to 0$.

2.2. From coupled to decoupled FPU. As mentioned in Remark 2.1 (iii) and (v), one would expect that the coupled terms $e^{\pm \frac{2 h}{h} \partial_t} u^\pm_h(t_1, x)$ in (2.8), as well as the $O(h^2)$-order remainder term, vanish as $h \to 0$. Thus, dropping them in (2.8), we derive the following decoupled system, which we refer to as the decoupled FPU:

$$v^\pm_h(t) = S^\pm_h(t) u^\pm_h(0, 0) \mp \frac{1}{4} \int_0^t S^\pm_h(t - t_1) \nabla_h \{ v^\pm_h(t_1) \}^2 dt_1, \quad (2.12)$$

where

$$v^\pm_h = v^\pm_h(t, x) : \mathbb{R} \times h\mathbb{Z} \to \mathbb{R}.$$  

It is easy to show that both the coupled and the decoupled FPUs are well-posed (see Proposition 1.1). However, their well-posedness is not sufficient for rigorous reduction to the decoupled equation. Indeed, the time interval of existence, given by the well-posedness, may shrink to zero as $h \to 0$; however, some regularity is also required to measure the difference between two solutions. Thus, we exploit the dispersive and smoothing properties of the linear FPU flows in Section 5 and 6, and we obtain finer uniform-in-$h$ bounds for nonlinear solutions (Proposition 7.3). In Section 8.1 by using these uniform bounds, we justify the convergence from the coupled to the decoupled FPU.

**Proposition 2.2** (Convergence to the decoupled FPU). If $V$ satisfies (1.2), then for any $R > 0$, there exists $T(R) > 0$ such that the following holds. Let $s \in (0, 1]$. Suppose that

$$\sup_{h \in (0, 1]} \| (u^+_h, 0, u^-_h, 0) \|_{H^s(h\mathbb{Z}) \times H^s(h\mathbb{Z})} \leq R.$$  

Let $(u^+_h(t), u^-_h(t))$ (resp., $(v^+_h(t), v^-_h(t))$) be the solution to the coupled FPU (2.8) (resp., the decoupled FPU (2.12)) with initial data $(u^+_h, 0, u^-_h, 0)$. Then,

$$\| u^+_h(t) - v^+_h(t) \|_{C_t([-T, T], L^2_h(\mathbb{Z}))} \lesssim h^s \| u^+_0 \|_{H^s(\mathbb{R})}.$$  

2.3. From decoupled FPU to KdV. As mentioned in Remark 2.1 (iv), by convergence of symbols, each equation in the decoupled FPU (2.12) is expected to converge to the Korteweg–de Vries equation (KdV)

$$w^\pm(t) = S^\pm(t) u^\pm_0 \mp \frac{1}{4} \int_0^t S^\pm(t - t_1) \partial_x \{ w^\pm(t_1) \}^2 dt_1, \quad (2.13)$$

where

$$w^\pm = w^\pm(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

and

$$S^\pm(t) = e^{\pm \frac{1}{2t} \partial_x^3} \quad (2.14)$$
denotes the Airy flow.

In Section 8.2, we establish the convergence from the decoupled FPU (2.12) to KdVs (2.13). Its proof uses uniform bounds for nonlinear solutions (see Section 7).

**Proposition 2.3** (Convergence to KdVs). If \( V \) satisfies (1.2), then for any \( R > 0 \), there exists \( T(R) > 0 \) such that the following holds. Let \( s \in (\frac{1}{4}, 1] \). Suppose that

\[
\sup_{h \in (0, 1]} \| (u_{h,0}^+, u_{h,0}^-) \|_{H^s(h\mathbb{Z}) \times H^s(h\mathbb{Z})} \leq R.
\]

Let \( (v_h^+(t), v_h^-(t)) \) (resp., \( (w_h^+(t), w_h^-(t)) \)) be the solution to the decoupled FPU (2.12) (resp., the KdVs (2.13)) with initial data \( (u_{h,0}^+, u_{h,0}^-) \) (resp., with initial data \( (l_h u_{h,0}^+, l_h u_{h,0}^-) \)), where \( l_h \) is the linear interpolation operator to be defined in (3.9). Then,

\[
\| l_h v_h^+(t) - w_h^+(t) \|_{C_r([-T,T]; L^2_{\mathbb{R}})} \lesssim h^{\frac{2s}{5}}.
\]

**Remark 2.4.** To avoid confusion, we explain here that the solutions \( w_h^\pm \) to KdVs (2.13) with initial data \( l_h v_{h,0}^\pm \) are real-valued functions posed not on \( h\mathbb{Z} \) but on \( \mathbb{R} \) because, as seen, their initial data depend on \( h \). Therefore, \( w_h^\pm \) involve subscript \( h \).

Finally, by combining Proposition 2.2 (with Lemma 3.8) and Proposition 2.3, we complete the proof of our main theorem. Figure 2.1 shows the convergence scheme outlined in this section.

**Figure 2.1.** Convergence scheme from FPU to KdV.

### 3. Preliminaries

In this section, we summarize the basic analysis tools for functions on a lattice.

**3.1. Basic inequalities and Littlewood-Paley theory on a lattice.** By definition, \( L^p(h\mathbb{Z}) = \ell^p(\mathbb{N}) \) but \( \| f_h \|_{L^p(h\mathbb{Z})} = h^{1/p} \| f_h \|_{\ell^p(\mathbb{N})} \). Thus, we have Hölder’s inequality

\[
\| f_h g_h \|_{L^p(h\mathbb{Z})} \leq \| f_h \|_{L^{p_1}(h\mathbb{Z})} \| g_h \|_{L^{p_2}(h\mathbb{Z})}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},
\]

the standard duality relation

\[
\| f_h \|_{L^p(h\mathbb{Z})} = \sup_{\| g \|_{L^{p'}(h\mathbb{Z})} \leq 1} h \left| \sum_{x \in h\mathbb{Z}^d} f_h(x) \overline{g_h(x)} \right|, \quad \frac{1}{p} + \frac{1}{p'} = 1,
\]
Lemma 3.2

and the following types of Leibniz rule:

Lemma 3.1 (Norm equivalence, Proposition 1.2 in [25]). For any $1 < p < \infty$, we have

$$
\|f_h\|_{W^{s,p}(\mathbb{Z})} \sim \|\nabla_h^s f_h\|_{L^p(\mathbb{Z})} \quad \forall s \in \mathbb{R}
$$

and

$$
\|f_h\|_{\dot{W}^{1,p}(\mathbb{Z})} \sim \|\nabla_h f_h\|_{L^p(\mathbb{Z})} \sim \|\partial_h^+ f_h\|_{L^p(\mathbb{Z})}.
$$

Lemma 3.2 (Sobolev embedding, Proposition 2.5 in [25]). Let $h \in (0,1]$. If $1 \leq p < q \leq \infty$ and $\frac{1}{q} > \frac{1}{p} - s$, then

$$
\|f_h\|_{L^q(\mathbb{Z})} \lesssim \|f_h\|_{W^{s,p}(\mathbb{Z})}.
$$

In contrast to the continuous domain case, differential operators are bounded on a lattice; however, the bound blows up as $h \to 0$.

Lemma 3.3 (Boundedness of differential operators, Lemma 2.2 in [24]). For $h > 0$ and $0 \leq s_1 \leq s_2$, we have

$$
\|f_h\|_{\dot{H}^{s_2}(\mathbb{Z})} \lesssim \frac{1}{h^{s_2-s_1}} \|f_h\|_{\dot{H}^{s_1}(\mathbb{Z})}.
$$

Lemma 3.4 (Leibniz rule for discrete differentials). Differential operators $\partial_h^+$ and $\nabla_h$ allow the following types of Leibniz rule:

$$
\partial_h^+(f_h g_h) = \partial_h^+ f_h \cdot g_h + f_h(\cdot + h) \cdot \partial_h^+ g_h,
$$

(3.4)

$$
\nabla_h(f_h g_h) = \nabla_h f_h \cdot \cos\left(\frac{-ih\partial_h}{2}\right)g_h + \cos\left(\frac{-ih\partial_h}{2}\right)f_h \cdot \nabla_h g_h,
$$

(3.5)

where $\cos\left(\frac{-ih\partial_h}{2}\right)$ denotes the Fourier multiplier of the symbol $\cos(h\xi/2)$.

**Proof.** Here, (3.4) follows from the definition. For (3.5), we take the Fourier transform of the left-hand side:

$$
\mathcal{F}_h(\nabla_h(f_h g_h))(\xi) = \frac{2i}{h} \sin\left(\frac{h\xi}{2}\right) \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} (\mathcal{F}_h f_h)(\eta)(\mathcal{F}_h g_h)(\xi - \eta) d\eta.
$$

We apply the identity $\frac{2i}{h} \sin\left(\frac{h\xi}{2}\right) = \frac{2i}{h} \sin\left(\frac{hn}{2}\right) \cos\left(\frac{h(\xi-\eta)}{2}\right) + \cos\left(\frac{h\eta}{2}\right) \frac{2i}{h} \sin\left(\frac{h(\xi-\eta)}{2}\right)$ to the integral and then take the inversion.

Let $\phi : \mathbb{R} \to [0,1]$ be an even smooth bump function such that $\phi(\xi) = 1$ for $|\xi| \leq 1$ and $\phi(\xi) = 0$ for $|\xi| \geq 2$. For a dyadic number $N \in 2^\mathbb{Z}$ with $N \leq 1$, set $\psi_N$ by

$$
\psi_N(\xi) = \phi\left(\frac{h\xi}{\pi N}\right) - \phi\left(\frac{2h\xi}{\pi N}\right).
$$
Note that $\text{supp} \psi_N \subset \{ \xi : \frac{\pi N}{2} \leq |\xi| \leq \frac{2\pi N}{2} \}$, and $\{ \psi_N \}$ is a partition of unity on $\mathbb{T}_h$, i.e., $\sum_{N \leq 1} \psi_N \equiv 1$. Now we define the Littlewood-Paley projection operator $P_N = P_{N;h}$ as the Fourier multiplier operator given by

$$F_h(P_N f)(\xi) = \psi_N(\xi)(F_h f)(\xi).$$

Moreover, we define $P_{\leq N}$ by $F_h(P_{\leq N} f) = \sum_{M \leq N} \psi_M F_h f$.

**Remark 3.5.** For each $h \in (0,1]$, there exists $N_0 = N_{0,h} \in 2\mathbb{Z}$ such that $\pi N \leq h$ holds for all $N \leq N_0$. The projection $P_{\leq N_0}$ on the lattice corresponds to $P_{\leq 1}$ on $\mathbb{R}$ (referred to as a low frequency piece). Indeed,

$$\sum_{N \leq 1} \psi_N(\xi) \sim \phi(\xi) + \sum_{N_0 < N \leq 1} \psi_N(\xi) \equiv 1, \ \forall \xi \in \mathbb{T}_h.$$

**Proposition 3.6** (Littlewood-Paley inequality, Theorem 4.2 in [25]). For $1 < p < \infty$, we have

$$\|f_h\|_{L^p(h\mathbb{Z})} \lesssim \left\| \left( \sum_{N \leq 1} |P_N f_h|^2 \right)^{1/2} \right\|_{L^p(h\mathbb{Z})} \lesssim \|f_h\|_{L^p(h\mathbb{Z})}.$$

**Lemma 3.7** (Bernstein’s inequality, Lemma 2.3 in [25]). Let $h \in (0,1]$. If $1 \leq p \leq q \leq \infty$, then we have

$$\|P_N f_h\|_{L^q(h\mathbb{Z})} \lesssim \left( \frac{N}{h} \right)^{\frac{q}{p} - \frac{q}{q}} \|P_N f_h\|_{L^p(h\mathbb{Z})}.$$

### 3.2. Properties of Linear interpolation

A function $f_h : h\mathbb{Z} \to \mathbb{R}$ on a lattice domain becomes continuous by linear interpolation

$$(l_h f_h)(x) := f_h(x_m) + (\partial^+_h f_h)(x_m) \cdot (x - x_m), \ \forall x \in [x_m, x_m + h).$$

This operator is bounded in Sobolev spaces.

**Lemma 3.8** (Boundedness of linear interpolation, Lemma 5.2 in [24]). Let $0 \leq s \leq 1$. Then, for $f_h \in H^s(h\mathbb{Z})$, we have

$$\|l_h f_h\|_{H^s(h\mathbb{Z})} \lesssim \|f_h\|_{H^s(h\mathbb{Z})}.$$  

**Proof.** See Lemma 5.2 in [24] for the proofs.

The linear interpolation operator and the differential (in some sense) are exchangeable at the cost of one additional derivative.

**Proposition 3.9.** If $f_h \in \dot{H}^2(h\mathbb{Z})$, then

$$\|l_h \nabla_h f_h - \partial_x l_h f_h\|_{L^2(\mathbb{R})} \lesssim h \|f_h\|_{\dot{H}^2(h\mathbb{Z})}.$$  

**Proof.** By definition, we have

$$l_h \nabla_h f_h(x) - \partial_x l_h f_h(x) = \nabla_h f_h(x_m) + \partial^+_h (\nabla_h f_h)(x_m) \cdot (x - x_m) - \partial^+_h f_h(x_m)$$

for $x \in [x_m, x_m + h)$; thus,

$$\|l_h \nabla_h f_h - \partial_x l_h f_h\|_{L^2(\mathbb{R})} \leq \|\nabla_h f_h - \partial^+_h f_h\|_{L^2(h\mathbb{Z})} + h \|\partial^+_h \nabla_h f_h\|_{L^2(\mathbb{R})}.$$
Plancherel’s theorem and the norm equivalence \( \|u\|_{L^2(\mathbb{R})} \) yield
\[
\|\nabla_h f_h - \partial_h^+ f_h\|_{L^2(h\mathbb{Z})} = \left\| \left\{ \frac{i}{h} \xi \frac{\sin(h \xi/2)}{2} \right\} (\mathcal{F}_h f_h)(\xi) \right\|_{L^2_h([-\pi/h, \pi/h])}
= \left\| \left\{ \frac{\cos(h \xi/2)}{h} - \frac{\sin(h \xi/2)}{2} \right\} (\mathcal{F}_h f_h)(\xi) \right\|_{L^2_h([-\pi/h, \pi/h])}
\leq h\|f\|_{H^2(h\mathbb{Z})} + h^2\|f_h\|_{\dot{H}^3(h\mathbb{Z})} \lesssim h\|f\|_{H^2(h\mathbb{Z})},
\]
where, in the last step, we use Lemma 3.3.

\[\square\]

**Proposition 3.10** (Almost distribution).
\[
\|\partial_x l_h(f_h^2) - \partial_x(l_h f_h)^2\|_{L^2(\mathbb{R})} \lesssim h\|\partial_h^+ f_h\|_{L^2(h\mathbb{Z})}^2.
\]

**Proof.** From the definition of \( l_h \) and (3.4), we write for \( x \in [x_m, x_m + h) \) that
\[
\partial_x l_h(f_h^2)(x) - \partial_x(l_h f_h)^2(x) = \partial_h^+(f_h^2)(x_m) - 2l_h f_h(x)\partial_h^+ f_h(x_m)
\]
\[
= \partial_h^+ f_h(x_m) f_h(x_m + h) + \partial_h^+ f_h(x_m) f_h(x_m) - 2f_h(x_m + h) \partial_h^+ f_h(x_m)
\]
\[
= h(\partial_h^+ f_h(x_m))^2 - 2(\partial_h^+ f_h(x_m))^2(x - x_m).
\]

Taking the \( L^2(\mathbb{R}) \) norm, we complete the proof. \[\square\]

### 3.3. \( X^{s,b} \) spaces.
In this subsection, we introduce the \( X^{s,b} \) spaces\(^4\), introduced by Bourgain \([4]\) and further developed by Kenig, Ponce, and Vega \([33]\) and Tao \([46]\).

First, we define the function space in a general setting. In subsequent applications, the spatial domain \( \Lambda \) will be either the real line \( \mathbb{R} \) or the lattice \( h\mathbb{Z} \), and the associated symbol \( P \) is chosen according to the model considered. Since the following are stated in a general setting, they can be applied in a unified way. We refer to \([47]\) for the details and proofs.

**Definition 3.11** (\( X^{s,b} \) spaces). Let \( \Lambda \) be either \( \mathbb{R} \) or \( h\mathbb{Z} \). Let \( P \) be a real-valued continuous function. For \( s, b \in \mathbb{R} \), we define the \( X^{s,b}_P(\mathbb{R} \times \Lambda) \) spaces \((X^{s,b} \text{ in short})\) as the completion of \( \mathcal{S}(\mathbb{R} \times \Lambda) \) with respect to the norm
\[
\|u\|_{X^{s,b}} := \left\{ \int_{\mathbb{R} \times \Lambda} |\xi|^{2s} (|\tau - P(\xi)|)^{2b} |\hat{u}(\tau, \xi)|^2 \, d\xi d\tau \right\}^{1/2},
\]
where \( \hat{u} \) denotes the space-time Fourier transform of \( u \) defined by\(^5\)
\[
\hat{u}(\tau, \xi) = \int_{\mathbb{R} \times \Lambda} e^{-i\tau \cdot x} e^{-ix \xi} u(t, x) \, dt dx
\]
and \( \hat{\Lambda} \) is the Pontryagin dual space of \( \Lambda \), i.e., \( \hat{\mathbb{R}} = \mathbb{R} \) and \( \hat{h\mathbb{Z}} = T_h \).

The following are well-known properties of \( X^{s,b} \) spaces (see, for instance, \([47]\) for the proofs).

---

\(^4\)They are sometimes called the Bourgain spaces or dispersive Sobolev spaces.
\(^5\)In particular, when \( \lambda = h\mathbb{Z}, \) \( \tilde{u} \) (as in Definition 3.14) is defined by
\[
\tilde{u}_h(\tau, \xi) = h \sum_{x \in h\mathbb{Z}} \int_{\mathbb{R}} e^{-i\tau \cdot x} e^{-ix \xi} u_h(t, x) \, dt.
\]
Lemma 3.12. Let $s, b \in \mathbb{R}$ and $X^{s, b}$ spaces be defined as in Definition 3.11. Let $\theta \in \mathcal{S}(\mathbb{R})$ be a (compactly supported) cut-off function. Then, the following properties hold:

1. (Nesting) $X^{s', b'} \subset X^{s, b}$ whenever $s \leq s', \ b \leq b'$.
2. (Well-defined for linear solutions) For any $f \in H^s$, we have

$$
\left\| \theta(t) e^{it\mathcal{P}(-i\nabla)} f \right\|_{X^{s, b}} \lesssim_{s, b} \| f \|_{H^s}.
$$

3. (Transference principle) Let $Y$ be a Banach space such that the inequality

$$
\| e^{it\tau_0} e^{it\mathcal{P}(-i\nabla)} f \|_Y \lesssim_b \| f \|_{H^s}
$$

holds for all $f \in H^s$ and $\tau_0 \in \mathbb{R}$. If, additionally, $b > \frac{1}{2}$, then we have the embedding

$$
\| u \|_Y \lesssim \| u \|_{X^{s, b}}.
$$

In particular, we have

$$
\| u \|_{C_1 H^s} \lesssim \| u \|_{X^{s, b}}. \tag{3.11}
$$

4. (Stability with respect to time localization) Let $0 < T < 1$, $b > \frac{1}{2}$ and $f \in H^s$. We have

$$
\left\| \theta\left(\frac{t}{T}\right) e^{it\mathcal{P}(-i\nabla)} f \right\|_{X^{s, b}} \lesssim_{s, b} T^{\frac{1}{2} - b} \| f \|_{H^s}.
$$

If $-\frac{1}{2} < b' \leq b < \frac{1}{2}$, then we have

$$
\left\| \theta\left(\frac{t}{T}\right) u \right\|_{X^{s, b'}} \lesssim_{s, b, b'} T^{b - b'} \| u \|_{X^{s, b}}.
$$

5. (Inhomogeneous estimate) Let $b > \frac{1}{2}$. Then, we have

$$
\left\| \theta(t) \int_0^t e^{i(t-t_1)\mathcal{P}(-i\nabla)} F(t_1) dt_1 \right\|_{X^{s, b}} \lesssim \| F \|_{X^{s, b-1}}. \tag{3.12}
$$

Remark 3.13. The proof of the above-mentioned lemma under the discrete setting is analogous to the one under the continuous setting, since the proof is based on the temporal Fourier analysis.

Now, we fix the symbols associated with the discrete linear FPU flows, and we focus on the corresponding $X^{s, b}$ spaces, because they are our main function spaces.

Definition 3.14 ($X^{s, b}_{h, \pm}$ spaces). For $s, b \in \mathbb{R}$, we define the discrete Bourgain spaces $X^{s, b}_{h, \pm} = X^{s, b}_{h, \pm}(\mathbb{R} \times h\mathbb{Z})$ as the completion of $\mathcal{S}(\mathbb{R} \times h\mathbb{Z})$ with respect to the norm

$$
\| u_h \|_{X^{s, b}_{h, \pm}} := \left\{ \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} (\xi)^{2s} (\tau + s_h(\xi))^{2b} |\tilde{u}_h(\tau, \xi)|^2 d\xi d\tau \right\}^{1/2},
$$

where $\tilde{u}_h$ denotes the (discrete) space-time Fourier transform of $u_h$, and

$$
s_h(\xi) := \frac{1}{h^2} \left( \xi - \frac{2}{h} \sin \left( \frac{\pi}{2} \right) \right). \tag{13.13}
$$

Remark 3.15. The Littlewood-Paley theory ensures

$$
\| f_h \|_{X^{s, b}_{h, \pm}}^2 \sim \sum_{N_0 \leq N \leq 1} \left( \frac{N}{h} \right)^{2s} \| P_N f_h \|_{X^{0, b}_{h, \pm}}^2.
$$

This facilitates a type of fractional Leibniz rule; see Lemma 5.5.
We end this section with the following temporal Sobolev embedding property.

**Lemma 3.16** (Temporal Sobolev embedding). Let \( 2 \leq p < \infty \) and \( u_h^\pm \) be a smooth function on \( \mathbb{R} \times h\mathbb{Z} \). Then for \( b \geq \frac{1}{2} - \frac{1}{p} \), we have

\[
\|f\|_{L^p_t(L^s_x(\mathbb{H}^p)))} \lesssim \|f\|_{X^{s,b}_{h,\pm}}.
\]

When \( p = \infty \), the usual Sobolev embedding \( (b > \frac{1}{2}) \) holds.

**Proof.** The proof directly follows from the Sobolev embedding with respect to the temporal variable \( t \). For \( S_h^\pm(t)u_h^\pm(t, x) = \mathcal{F}_h^{-1}[e^{\mp iht\hbar}\xi]F(u_h^\pm)(t, \xi) \), we know that \( \|S_h^\pm(-t)u_h^\pm\|_{H^s_h} = \|u_h^\pm\|_{H^s_h} \). Thus,

\[
\|u_h^\pm\|_{L^p_t(L^s_x(\mathbb{H}^p)))} = \|S_h^\pm(-t)u_h^\pm\|_{H^s_h} \lesssim \|S_h^\pm(-t)u_h^\pm\|_{H^s_h} = \|u_h^\pm\|_{X^{s,b}_{h,\pm}},
\]

which completes the proof. \( \square \)

4. **Well-posedness of coupled and decoupled FPUs**

The well-posedness of a nonlinear difference (or discrete differential) equation is obvious in most cases owing to the boundedness of discrete differential operators. Nevertheless, the proof of the local well-posedness of the coupled and decoupled FPUs is included for the readers' convenience.

**Proposition 4.1** (Local well-posedness of coupled and decoupled FPUs). Let \( h \in (0, 1] \) be fixed. For any \( R > 0 \), there exists \( T(R, h) > 0 \) such that the following holds. Suppose that

\[
\|(u_h^+, 0, u_h^-)\|_{L^2_t(L^2_x(\mathbb{H}^2)))} \leq R.
\]

Then, there exists a unique solution \((u_h^+, u_h^-)\) \( \in C_t([-T, T]; L^2_x(\mathbb{H}^2)) \) (resp., \((v_h^+, v_h^-)\) \( \in C_t([-T, T]; L^2_x(\mathbb{H}^2)) \)) to the coupled FPU (2.8) (resp., the decoupled FPU (2.12)) with initial data \((u_h^+, 0, u_h^-)\). Moreover, \((u_h^+, u_h^-)\) preserves the Hamiltonian \( H_h(\hat{r}_h) \) (see (1.8)), where \( \hat{r}_h \) is given by (2.11).

**Remark 4.2.** In the proof below, we do not estimate the higher-order remainder term in (2.8), since the higher-order term contains a spare \( \hbar \) of order 2 and it is thus small and nonessential in our analysis. For readers’ convenience, we refer to Lemma 3.16 for the proof of the estimate of the higher-order remainder term.

**Proof of Proposition 4.1.** We drop the time interval \([-T, T]\) in the notation \( C_t([-T, T]) \). We consider only the coupled FPU, because the decoupled FPU can be dealt with in the same way.

We define a nonlinear map \( \Phi = (\Phi^+, \Phi^-) \) by

\[
\Phi^\pm(u_h^+, u_h^-) := S_h^\pm(t)u_h^\pm + \frac{1}{4} \int_0^t S_h^\pm(t-t_1)\nabla_h \left\{ u_h^\pm(t_1) + e^{\pm 2\hbar\partial_h u_h^\pm(t_1)} \right\}^2 dt_1.
\]

Let \( T > 0 \) be a small number to be chosen later. Then, by unitarity, it follows that

\[
\|\Phi^\pm(u_h^+, u_h^-)\|_{C_t L^2_x} \leq \|u_h^+, 0\|_{L^2_x} + \frac{T}{2} \|\nabla_h \left\{ u_h^\pm + e^{\pm 2\hbar\partial_h u_h^\pm} \right\}^2 \|_{C_t L^2_x}.
\]
For the nonlinear term, we observe that by the boundedness of the discrete differential operator $\nabla_h$ (see Definition 1.11) and the trivial inequality $\|f_h\|_{L^2} \leq h^{-\frac{1}{2}} \|f_h\|_{L^2}$, we have
\[
\|\nabla_h \left\{ u_h^\pm + e^{\pm \frac{2\pi i}{h} \partial_h u_h^\pm} \right\} \|_{L^2_h}^2 \leq \frac{2}{h} \left\| u_h^\pm + e^{\pm \frac{2\pi i}{h} \partial_h u_h^\pm} \right\|_{L^2_h}^2 = \frac{2}{h} \left\| u_h^\pm + e^{\pm \frac{2\pi i}{h} \partial_h u_h^\pm} \right\|_{L^2_h}^2 
\]
\[
\leq \frac{2}{h^{3/2}} \left\| u_h^\pm \|_{L^2_h} + \left\| e^{\pm \frac{2\pi i}{h} \partial_h u_h^\pm} \right\|_{L^2_h} \right\|_{L^2_h}^2 \leq \frac{4}{h^{3/2}} \|(u_h^+, u_h^-)\|_{L^2_h}^2.
\]
Thus, we obtain
\[
\|\Phi(u_h^+, u_h^-)\|_{C^1L^2} \leq R + \frac{2T}{h^{3/2}} \|(u_h^+, u_h^-)\|_{C^1L^2}^2.
\]

Similarly, one can show that
\[
\|\Phi(u_h^+, u_h^-) - \Phi(\tilde{u}_h^+, \tilde{u}_h^-)\|_{C^1L^2} \leq \frac{2T}{h^{3/2}} \left\{ \|(u_h^+, u_h^-)\|_{C^1L^2} + \|(\tilde{u}_h^+, \tilde{u}_h^-)\|_{C^1L^2} \right\} \|(u_h^-, \tilde{u}_h^-)\|_{C^1L^2}.
\]

Taking $T = \frac{h^{3/2}}{2\pi}$, we prove that $\Phi$ is contractive on the ball in $C^1L^2$ of radius $2R$ centered at zero. Therefore, local well-posedness follows from the contraction mapping principle.

By a straightforward computation, we prove the conservation law,
\[
\frac{d}{dt} H_h(\tilde{r}_h) = h \sum_{x \in hZ} h^4 \frac{1}{\sqrt{-\Delta_h}} \partial_t \tilde{r}_h \cdot \frac{1}{\sqrt{-\Delta_h}} \partial_t^2 \tilde{r}_h + \frac{1}{h^2} V'(h^2 \tilde{r}_h) \partial_t \tilde{r}_h
\]
\[
= h^5 \sum_{x \in hZ} \left(-\Delta_h\right)^{-1} \partial_t \tilde{r}_h \cdot \left\{ \partial_t^2 \tilde{r}_h - \frac{1}{h^6} \Delta_h V'(h^2 \tilde{r}_h) \right\} = 0,
\]
where in the last step, we use $\tilde{r}_h$ to solve (1.7).

\section{5. Linear FPU flows}

We investigate various dispersive and smoothing properties for the linear FPU flows (Proposition 5.1), and we then show how these discrete flows can be approximated by the Airy flows as $h \to 0$ (Proposition 5.10). Later, in Section 8.2, the main results of this section will be employed to prove the convergence from the decoupled FPU to KdVs.

\subsection{5.1. Estimates for the linear FPU flows.}

We establish dispersive and smoothing inequalities for the linear FPU propagator $S_h^\pm(t)$, i.e., the discrete Fourier multipliers of the symbol $e^{\mp \frac{2\pi i}{h} \left( \frac{2\pi i}{h} \sin(h^2) - \xi \right) \cdot \xi}$.

**Proposition 5.1** (Estimates for the linear FPU flows). Let $s > \frac{3}{4}$. Suppose that $2 \leq q, r \leq \infty$ and $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$ with $(q, r) \neq (4, \infty)$.

(i) (Strichartz estimates)
\[
\|\partial_t^{\frac{1}{2}} S_h^\pm(t) u_{h,0}\|_{L_t^q(R;L_x^r(hZ))} \lesssim \|u_{h,0}\|_{L^2(hZ)}.
\]

(ii) (Local smoothing estimate)
\[
\|\partial_t S_h^\pm(t) u_{h,0}\|_{L_t^q(hZ;L_x^r(hZ))} \lesssim \|u_{h,0}\|_{L^2(hZ)}.
\]

(iii) (Maximal function estimate)
\[
\|S_h^\pm(t) u_{h,0}\|_{L_t^q(hZ;L^\infty([-1,1]))} \lesssim \|u_{h,0}\|_{H^s(hZ)}.
\]
In all the three above-mentioned inequalities, the implicit constants are independent of $h \in (0,1]$. Moreover, the differential operator $\partial_h$ in (i) and (ii) can be replaced by $\partial_h^+$ or $\nabla_h$ (see Definition 1.1).

Remark 5.2. Proposition 5.1 (i) may hold at $(q,r) = (4,\infty)$; however, it is excluded here to simplify the proof. Indeed, this endpoint case is not necessary in this article.

As a direct consequence of Proposition 5.1 and the transference principle (Lemma 3.12 (3)), we obtain the bounds in the associated Bourgain spaces.

**Corollary 5.3.** Let $s > 3/4$ and $b > 1/2$. Suppose that $2 \leq q,r \leq \infty$ and $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$ with $(q,r) \neq (4,\infty)$. Then, we have

$$
\| \partial_h^{\frac{1}{q}} u_h \|_{L^q_t(L^r_x(h\mathbb{Z}))} + \| \partial_h u_h \|_{L^\infty(h\mathbb{Z};L^2_x(\mathbb{R}))} \lesssim \| u_h \|_{X^{s,b}_{h,\pm}}
$$

and

$$
\| u_h \|_{L^2(h\mathbb{Z};L^\infty([-1,1]))} \lesssim \| u_h \|_{X^{s,b}_{h,\pm}},
$$

where the implicit constants are independent of $h \in (0,1]$ and $\partial_h$ can be replaced by $\partial_h^+$ or $\nabla_h$.

Before presenting the proof of Proposition 5.1, let us recall and compare with the linear estimates for the Airy propagator $S^\pm(t) = e^{\pm \frac{1}{2} t \partial_x^2}$ from Kenig, Ponce and Vega [30].

**Proposition 5.4** (Linear estimates for the Airy flows). Let $s > 3/4$. Suppose that $2 \leq q,r \leq \infty$ and $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$.

(i) (Strichartz estimates)

$$
\| \partial_x^{\frac{1}{q}} S^\pm(t) u_0 \|_{L^q_t(L^r_x(\mathbb{R}))} \lesssim \| u_0 \|_{L^2(\mathbb{R})}.
$$

(ii) (Local smoothing estimate)

$$
\| \partial_x S^\pm(t) u_0 \|_{L^\infty(\mathbb{R};L^2_x(\mathbb{R}))} \lesssim \| u_0 \|_{L^2(\mathbb{R})}.
$$

(iii) (Maximal function estimate)

$$
\| S^\pm(t) u_0 \|_{L^2(\mathbb{R};L^\infty([-1,1]))} \lesssim \| u_0 \|_{H^s(\mathbb{R})}.
$$

Let $X^{s,b}_{\pm}$ denote the Bourgain space with the norm

$$
\| u \|_{X^{s,b}_{\pm}} := \| \langle \xi \rangle^s \tau^{\frac{s}{4} + \frac{b}{2}} \hat{u}(\tau,\xi) \|_{L^2_\tau L^2_\xi(\mathbb{R} \times \mathbb{R})}.
$$

(5.1)

By the transference principle, we have the following corollary.

**Corollary 5.5.** Let $s > 3/4$ and $b > 1/2$. Suppose that $2 \leq q,r \leq \infty$ and $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$. Then, we have

$$
\| \partial_x^{\frac{1}{q}} w \|_{L^q_t(L^r_x(\mathbb{R}))} + \| \partial_x w \|_{L^\infty(\mathbb{R};L^2_x(\mathbb{R}))} \lesssim \| w \|_{X^{s,b}_{\pm}}
$$

and

$$
\| w \|_{L^2(\mathbb{R};L^\infty([-1,1]))} \lesssim \| w \|_{X^{s,b}_{\pm}}.
$$
Proposition 5.1 will be proved below by adapting the argument in [30] and the references therein. For the proof, we decompose the linear propagator into dyadic pieces,

\[ S_h^\pm(t) = \sum_{N \leq 1} S_h^\pm(t)P_N = S_h^\pm(t)P_{\leq N_0} + \sum_{N_0 < N \leq 1} S_h^\pm(t)P_N, \]

where \( N_0 \) is the dyadic number introduced in Remark 3.5. We denote the kernel of a dyadic piece of the linear propagator \( S_h^\pm(t)P_N \) (resp., \( S_h^\pm(t)P_{\leq N_0} \)) by \( K_N^\pm(t, x) \) (resp., \( K_{\leq N_0}^\pm(t, x) \)). By the discrete Fourier transform in addition to (3.7), their kernels can be expressed as oscillatory integrals

\[ K_N^\pm(t, x) = \frac{1}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\pm \frac{ih}{2} \xi} \xi \psi_N(\xi) d\xi \]

and

\[ K_{\leq N_0}^\pm(t, x) = \frac{1}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\pm \frac{ih}{2} \xi} \xi \phi(\xi) d\xi. \]

These kernels obey the following decay properties.

**Lemma 5.6.** (i) For every \( N = 2^k \leq 1 \), we have

\[ |K_N^\pm(t, x)| \lesssim \left( \frac{h}{N|t|} \right)^{1/2}. \]

(ii) Suppose that \( |t| \leq 1 \). Then, we have

\[ |K_{\leq N_0}^\pm(t, x)| \lesssim \frac{1}{1 + x^2}. \]  \hspace{1cm} (5.2)

For every \( N_0 < N \leq 1 \), we have

\[ |K_N^\pm(t, x)| \lesssim \begin{cases} \frac{N}{h} & \text{if } |x| \leq \frac{h}{N}, \\ \left( \frac{N}{h|x|} \right)^{1/2} & \text{if } \frac{h}{N} \leq |x| \leq \frac{N^2}{h^2}, \\ \frac{h}{Nx^2} & \text{if } |x| \geq \frac{N^2}{h^2}. \end{cases} \]  \hspace{1cm} (5.3)

**Remark 5.7.** It is easy to see that \( s_h(\xi) = \frac{1}{h} (\xi - \frac{2}{h} \sin(\frac{h}{2} \xi)) \) is comparable with \( \xi^3 \) on the frequency domain \([-\frac{\pi}{h}, \frac{\pi}{h}]\) in the sense that

\[ s_h''(\xi) = \frac{1}{h^2} \left( 1 - \cos \left( \frac{h}{2} \xi \right) \right) \sim |\xi|^2, \quad |s_h''(\xi)| \sim |\xi|, \quad |s_h'''(\xi)| \sim 1. \]  \hspace{1cm} (5.4)

**Proof.** (i). The proof follows from the van der Corput lemma with \( |(\pm ts_h(\xi) + x\xi)''| = |ts_h''(\xi)| \sim |t||\xi| \sim \frac{N}{h} |t| \) on the support of \( \psi_N \).

(ii). For (5.2), by integration by parts twice, we write

\[ K_{\leq N_0}^\pm(t, x) = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \frac{1}{(ix)^2} (e^{ix\xi}'' e^{\pm it s_h(\xi)} \phi(\xi)) d\xi = -\frac{1}{x^2} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{ix\xi} (e^{\pm it s_h(\xi)} \phi(\xi))'' d\xi. \]
A straightforward computation shows that
\[
|e^{\pm its_h(\xi)}\phi(\xi)| = |(t^2 s_h(\xi))^2 \mp its_h''(\xi)\phi(\xi) + 2its_h(\xi)\phi'(\xi) - \phi''(\xi)|
\leq |(\xi^4 + |\xi|)(\phi(\xi) + \xi^2\phi'(\xi) + |\phi''(\xi)|) < \infty.
\]

Thus, it follows that $|K_{\pm}^{\pm}(0, t, x)| \lesssim \frac{1}{2\pi}$. Together with the trivial bound $|K_{\pm}^{\pm}(t, x)| \lesssim 1$, we obtain (5.2).

For (5.3), it is obvious that $|K_{\pm}^{\pm}(t, x)| \lesssim \frac{N}{h^2}$. Suppose that $|x| \geq \frac{N^2}{h^2} \geq 1$. By integration by parts twice, we have
\[
|K_{\pm}^{\pm}(t, x)| \leq \int_{-\pi}^{\pi} \left| \frac{1}{x \mp ts_0'(\xi)} \left( \frac{\psi_N(\xi)}{x \pm ts_0'(\xi)} \right)' \right| d\xi.
\]
A direct computation gives
\[
\left( \frac{1}{x \pm ts_0'(\xi)} \left( \frac{\psi_N(\xi)}{x \pm ts_0'(\xi)} \right)' \right)' = \frac{\psi_N''(\xi)}{(x \pm ts_0'(\xi))^2} + \frac{3t^2\psi_N(\xi)(s''_N(\xi))^2}{(x \pm ts_0'(\xi))^4}
\approx \frac{ts''_N(\xi)\psi_N(\xi) + 3ts''_N(\xi)\psi_N''(\xi)}{(x \pm ts_0'(\xi))^3}.
\]
Note that in this case, $|x \pm ts_0'(\xi)| = |x| - \frac{2|t|}{h^2} \sin^2(\frac{h\xi}{4}) \gtrsim |x|$ on supp$\psi_N$. Thus, by (5.4), we obtain that
\[
|K_{\pm}^{\pm}(t, x)| \lesssim \int_{-\pi}^{\pi} \frac{|\psi_N''(\xi)|}{|x|^2} + \frac{\xi^2|\psi_N(\xi)|}{|x|^4} + \frac{|\psi_N(\xi)| + |\xi||\psi_N''(\xi)|}{|x|^3} d\xi
\lesssim \frac{h}{N|x|^2} + \left( \frac{N}{h} \right)^3 \frac{1}{|x|^4} + \frac{N}{h^3} \lesssim \frac{h}{N|x|^2}.
\]

It remains to consider the case $\frac{h}{N} \leq |x| \leq \frac{N^2}{h^2}$ for (5.3). When $t = 0$, by integration by parts, we obtain
\[
|K_{\pm}^{\pm}(0, x)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix\xi}\psi_N(\xi)d\xi \right| \lesssim \frac{1}{|x|} \leq \left( \frac{N}{h|x|} \right)^{1/2}.
\]

When $t \neq 0$, by simple change of variables, we may assume that $t, x > 0$, since $s_h(\xi) = -s_h(-\xi)$ and $\psi_N$ is an even function. For $K_{\pm}^{\pm}(t, x)$, by integration by parts with $(x\xi + ts_h(\xi))' = x + \frac{2t}{h^2} \sin^2(\frac{h\xi}{4}) \geq x$, we obtain
\[
|K_{\pm}^{\pm}(t, x)| \lesssim \int_{-\pi}^{\pi} \left| \frac{\psi_N''(\xi)}{x + ts_0'(\xi)} \right| d\xi = \int_{-\pi}^{\pi} \frac{|\psi_N''(\xi)|}{x + ts_0'(\xi)} d\xi
\lesssim \frac{1}{|x|} + \frac{N^2}{h^2|x|^2} \lesssim \left( \frac{N}{h|x|} \right)^{1/2}.
\]

For $K_{\mp}^{\pm}(t, x)$, we note that its phase function may have stationary points. Thus, by splitting the frequency domain $[-\frac{\pi}{h}, \frac{\pi}{h}] = \{ \xi : |s_h'(\xi)| \leq \frac{2}{h^2} \} \cup \{ \xi : |s_h'(\xi)| > \frac{2}{h^2} \} =: \Omega_1 \cup \Omega_2$, we decompose
\[
K_{\mp}^{\pm}(t, x) = \frac{1}{2\pi} \left( \int_{\Omega_1} + \int_{\Omega_2} \right) e^{ix\xi - ts_h(\xi)} \psi_N(\xi)d\xi =: K_{\mp, 1}^{\pm}(t, x) + K_{\mp, 2}^{\pm}(t, x).
\]
Note that the phase function does not have a stationary point in $\Omega_1$. More precisely, we have a lower bound $(x^2 - ts_\xi(t))^2 = x^2 - ts_\xi(t) \geq \frac{\pi^2}{4}$. Hence, by repeating (5.5), we obtain $|K_{N,1}^N(t,x)| \leq \left( \frac{N}{|t|} \right)^{1/2}$. For $K_{N,2}^N(t,x)$, we observe from (5.3) that $|\xi^2| = s_\xi(t) \geq \frac{\pi^2}{4}$ on $\Omega_2$; consequently, $|x^2 - ts_\xi(t)| = |t|s_\xi(t) \sim |t||\xi| \geq \left( \frac{N}{|t|} \right)^{1/2}$. Thus, by the van der Corput lemma, we obtain $|K_{N,2}^N(t,x)| \leq \left( \frac{N}{|t|} \right)^{1/2}$. By combining all the above-mentioned results, we complete the proof.

\[ \square \]

**Proof of Proposition 5.1**. (i) We use Lemma 5.6 (i) to obtain

$$
\|S_h^\pm(t)P_Nu_{h,0}\|_{L^\infty} = \left\| h \sum_{y \in h\mathbb{Z}} K_N(t,x - y)u_{h,0}(y) \right\|_{L^\infty} \lesssim \left( \frac{h}{N|t|} \right)^{1/2} \|u_{h,0}\|_{L^1}.
$$

By interpolating with the trivial inequality $\|S_h^\pm(t)P_Nu_{h,0}\|_{L^r} \leq \|u_{h,0}\|_{L^2}$, it follows that

$$
\|S_h^\pm(t)P_Nu_{h,0}\|_{L^r} \lesssim \left( \frac{h}{N|t|} \right)^{1/2-r'/2} \|u_{h,0}\|_{L^2},
$$

for $2 \leq r \leq \infty$, where $r'$ is the Hölder conjugate of $r$. Hence, a standard $TT^*$ argument yields

$$
\|S_h^\pm(t)P_Nu_{h,0}\|_{L^r} \lesssim \left( \frac{N}{h} \right)^{1/2} \|u_{h,0}\|_{L^2}.
$$

The Littlewood-Paley theory (Lemma 3.9) and the norm equivalence (Lemma 3.1) enable us to show (i) for all $\partial_h$, $\partial_h^+$ and $\nabla_h$.

(ii). We follow the argument in [10]. First, by changing the variable $\tau = \pm \frac{1}{\pi^2}(\xi - \frac{2}{\pi} \sin(\frac{h\xi}{2}))$ such that $d\tau = \pm \frac{1}{\pi^2}(1 - \cos(\frac{h\xi}{2}))d\xi$, we write

$$
\partial_h S_h^\pm(t)u_{h,0}(x) = \frac{1}{2\pi} \int_{-\pi/\xi}^{\pi/\xi} i\xi e^{\pm i\xi x}(\xi - \frac{2}{\pi} \sin(\frac{h\xi}{2}))e^{i\xi(x,F_hu_{h,0})}(\xi)d\xi
$$

$$
= \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{i\xi \tau} 1_{[-\frac{\pi}{\xi}, \frac{\pi}{\xi}]}(\tau)e^{i\xi(x,F_hu_{h,0})}(\xi)\frac{\pm h^2\xi}{1 - \cos(\frac{h\xi}{2})}d\tau,
$$

where, in the last integral, $\xi = \xi(\tau)$ is a function of $\tau$. Thus, Plancherel’s theorem yields

$$
\|\partial_h S_h^\pm(t)u_{h,0}\|_{L^2} \sim \left\| 1_{[-\frac{\pi}{\xi}, \frac{\pi}{\xi}]}(\tau)e^{i\xi(x,F_hu_{h,0})}(\xi)\frac{\pm h^2\xi}{1 - \cos(\frac{h\xi}{2})} \right\|_{L^2}.
$$

By changing the variable back to $\tau = \pm \frac{1}{\pi^2}(\xi - \frac{2}{\pi} \sin(\frac{h\xi}{2})) \mapsto \xi$ and using $1 - \cos(\frac{h\xi}{2}) = 2\sin^2(\frac{h\xi}{4}) \sim h^2\xi^2$ on $[-\frac{\pi}{\xi}, \frac{\pi}{\xi}]$, we prove that

$$
\|\partial_h S_h^\pm(t)u_{h,0}\|_{L^2}^2 \sim \int_{-\pi}^{\pi} |(F_hu_{h,0})(\xi)|^2 \frac{h^2\xi^2}{1 - \cos(\frac{h\xi}{2})}d\xi \sim \|u_{h,0}\|_{L^2}^2.
$$

Similarly, one can show the inequality with $\partial_h^+$ (resp., $\nabla_h$) by replacing the symbol $i\xi$ for $\partial_h$ by $\frac{e^{ih\xi} - 1}{h}$ (resp., $\frac{2i}{h} \sin(\frac{h\xi}{2})$).

(iii). We claim that if $N_0 < N \leq 1$, then

$$
\|S_h^\pm(t)P_Nu_{h,0}\|_{L^2} \lesssim \left( \frac{N}{h} \right)^{\frac{3}{4}} \|u_{h,0}\|_{L^2},
$$

where the time interval $[-1,1]$ is omitted in the norm. Indeed, by a $TT^*$ argument, the claim is equivalent to
\[
\left\| \int_{-1}^{1} S_h^+(t-t_1) P_N g(t_1) dt_1 \right\|_{L_x^2 L_t^\infty} \leq \left( \frac{N}{h} \right)^{\frac{3}{2}} \| g \|_{L_x^2 L_t^1}.
\]
We observe that
\[
\left| \int_{-1}^{1} S_h^+(t-t_1) P_N g(t_1) dt_1 \right| = h \sum_{x_1 \in h \mathbb{Z}} \left| \int_{-1}^{1} K_N(t-t_1, x_1) g(t, \cdot - x_1) dt_1 \right|
\]
\[
\leq h \sum_{x_1 \in h \mathbb{Z}} \| K_N(t, x_1) \|_{L_t^\infty} \| g(t, \cdot - x_1) \|_{L_t^1}.
\]
Using (5.3) and (5.3) (in particular $\| K_N(t, x) \|_{L_t^1 L_x^\infty} \leq \left( \frac{N}{h} \right)^{3/2}$), we conclude that
\[
\left\| \int_{-1}^{1} S_h^+(t-t_1) P_N g(t_1) dt_1 \right\|_{L_x^2 L_t^\infty} \leq \| K_N \|_{L_h^1 L_t^\infty} \| g \|_{L_x^2 L_t^1} \leq \left( \frac{N}{h} \right)^{\frac{3}{2}} \| g \|_{L_x^2 L_t^1}.
\]
Similarly but by using (5.2), one has
\[
\| S_h^+(t) P_{\leq N_0} u_{h,0} \|_{L_x^2 L_t^\infty} \lesssim \| u_{h,0} \|_{L^2}.
\]
Summing them up for $s > \frac{3}{4}$, we obtain
\[
\| S_h^+(t) u_{h,0} \|_{L_x^2 L_t^\infty} \leq \| S_h^+(t) P_{\leq N_0} u_{h,0} \|_{L_x^2 L_t^\infty} + \sum_{N_0 < N \leq 1} \| S_h^+(t) P_N \tilde{P}_N u_{h,0} \|_{L_x^2 L_t^\infty}
\]
\[
\lesssim \| u_{h,0} \|_{L^2} + \sum_{N_0 < N \leq 1} \left( \frac{N}{h} \right)^{\frac{3}{4}} \| \tilde{P}_N u_{h,0} \|_{L^2}
\]
\[
\lesssim \left\{ 1 + \sum_{N_0 < N \leq 1} \left( \frac{N}{h} \right)^{-(s-\frac{3}{4})} \right\} \| u_{h,0} \|_{H^s} \lesssim \| u_{h,0} \|_{H^s}.
\]
Therefore, we complete the proof.

As mentioned in Remark 5.2, the time-averaged $L^\infty(h \mathbb{Z})$ bound corresponding to the endpoint $(q,r) = (4,\infty)$ is excluded here. Nevertheless, together with the Sobolev inequality, we still have the following bound.

**Corollary 5.8.** If $\frac{3}{4} < s < \frac{3}{2}$ and $4 < q < \frac{6 - s}{3 - 2s}$, then
\[
\| \partial_h S_h^+(t) u_{h,0} \|_{L_t^q L_x^\infty(h \mathbb{Z})} \lesssim \| u_{h,0} \|_{H^s(h \mathbb{Z})}.
\]
(5.7)

Here, $\partial_h$ can be replaced by $\partial_h^+$ and $\nabla_h$.

**Proof.** Let $r$ satisfy $\frac{3}{q} + \frac{1}{r} = \frac{1}{2}$. Then, it follows from the Sobolev inequality (3.3) and Strichartz estimates (Proposition 5.1 (i)) that
\[
\| \partial_h S_h^+(t) u_{h,0} \|_{L_x^2 L_t^\infty} \lesssim \| (\partial_h)^{\frac{1}{2} + \delta} \partial_h S_h^+(t) u_{h,0} \|_{L_t^2 L_x^2}
\]
\[
\lesssim \| (\partial_h)^{\frac{1}{2} + \delta} \partial_h |\partial_h|^{-\frac{1}{2}} u_{h,0} \|_{L^2} \lesssim \| (\partial_h)^{s} u_{h,0} \|_{L^2}
\]
with \( \delta = s + \frac{2}{n} - \frac{2}{3} > 0 \). Thus, (5.7) follows from the norm equivalence (Lemma 5.1). Similarly, one can show the desired inequalities for different operators. \( \square \)

**Remark 5.9.** Let \( \tilde{S}_h^\pm(t) \) denote another linear propagator with the kernel

\[
\tilde{K}_N^\pm(t, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\pm i \frac{n}{2} (\xi + \frac{2}{h} \sin(h \frac{\xi}{2}))} e^{ix\psi_N(\xi)} d\xi.
\]

By repeating the proof of Lemma 5.12 (i), one can show the same kernel estimate,

\[
|K_N^\pm(t, x)| \lesssim \left( \frac{h}{N|t|} \right)^{1/2}.
\]

Consequently, the Strichartz estimates for the propagator \( \tilde{S}_h^\pm(t) \) of the form in Proposition 5.11 (i) follow (see the proof of Proposition 5.1 (i) above). This will enable us to control the cubic term for the analysis on a general nonlinearity (see Appendix B).

### 5.2. Approximation of linear FPU flows by Airy flows

We now compare the linear FPU flows with the Airy flows with respect to the space-time norm

\[
\|u\|_{S([-T, T])} := \|u\|_{C_t([-T, T], L^2_x(R))} + \|\partial_x u\|_{L^2_t(R, \dot{L}^2_x([-T, T])))}
\]

using the linear interpolation \( l_h \) defined by (3.9).

**Proposition 5.10** (Comparison between linear FPU and Airy flows). For \( s \in (0, 1] \), we have

\[
\|l_h(S_h^\pm(t)u_{h, 0}) - S_h^\pm(t)(l_hu_{h, 0})\|_{S([-1, 1])} \lesssim h^{2s} \|u_{h, 0}\|_{H^{s}(\mathbb{R}^n)}.
\]

**Remark 5.11.** We note that the linear interpolation can be regarded as a Fourier multiplier (see [21] Lemma 5.5).

**Lemma 5.12** (Symbol of linear interpolation operator). The interpolation operator \( l_h \) is a Fourier multiplier operator in the sense that

\[
\mathcal{F}(l_h f_h)(\xi) = \mathcal{L}_h(\xi)(\mathcal{F}_h f_h)(\xi), \quad \forall \xi \in \mathbb{R},
\]

where

\[
\mathcal{L}_h(\xi) = \frac{1}{h} \int_0^h e^{-i\xi x} dx + e^{ih\frac{\xi}{2}} - \frac{1}{h^2} \int_0^h x e^{-i\xi x} dx = \frac{4 \sin^2(h \xi/2)}{h^2 \xi^2}
\]

and \( \mathcal{F}_h \) denotes the \([-\pi/h, \pi/h]\)-periodic extension of the discrete Fourier transform \( \mathcal{F}_h \), precisely,

\[
(\mathcal{F}_h f_h)(\xi) = (\mathcal{F}_h f_h)(\xi'),
\]

where \( \xi' = \xi - \frac{2\pi m}{h} \in [-\pi/h, \pi/h] \) for some \( m \in \mathbb{Z} \).

A straightforward computation of the Fourier transform and Lemma 5.12 give

\[
l_h(S_h^\pm(t)u_{h, 0})(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{L}_h(\xi)(\mathcal{F}_h S_h^\pm(t)u_{h, 0})(\xi) e^{i\xi x} d\xi
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{L}_h(\xi)e^{\pm i \frac{2}{nh}(\xi' + \frac{2}{h} \sin(h \xi'/2))}(\mathcal{F}^\pm_h u_{h, 0})(\xi') e^{i\xi x} d\xi
\]

\[
= \tilde{S}_h^\pm(t)(l_hu_{h, 0})(x),
\]
where the new linear propagator $\tilde{S}_h^\pm(t) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is given by

$$\mathcal{F}(\tilde{S}_h^\pm(t)u_0)(\xi) = e^{\pm \frac{ht}{h^2}(\xi - \frac{2}{h^2}\sin(\frac{\xi}{2}))}(\mathcal{F}u_0)(\xi)$$

with $\xi' = \xi - \frac{2\pi m}{h} \in [-\frac{\pi}{h}, \frac{\pi}{h})$ for some $m \in \mathbb{Z}$.

**Proof of Proposition 5.10.** First, by (5.10), we write

$$l_h(S_h^\pm(t)u_{h,0}) - S^\pm(t)(l_hu_{h,0}) = (S_h^\pm(t) - S^\pm(t))P_{low}(l_hu_{h,0}) + \tilde{S}_h^\pm(t)P_{high}(l_hu_{h,0})$$

$$=: I + II + III,$$

where $P_{low}$ (resp., $P_{high}$) is the Fourier multiplier of the symbol $1_{|\xi| \leq h^{-2/5}}$ (resp., $1_{|\xi| \geq h^{-2/5}}$).

For $III$, we use Proposition 5.4 and Lemma 3.8 to obtain

$$\|III\|_{S([-1,1])} \lesssim \|P_{high}(l_hu_{h,0})\|_{L^2} \lesssim h^{2\alpha} \|l_hu_{h,0}\|_{H^s} \lesssim h^{2\alpha} \|u_{h,0}\|_{H^s}.$$  

For $I$, we observe from (5.10) that

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm \frac{ht}{h^2}(\xi - \frac{2}{h^2}\sin(\frac{\xi}{2}))} - e^{\pm \frac{ht}{h^2}\xi^3} = \frac{1}{2\pi} \int_{0}^{1} e^{\pm it\frac{ht}{h^2}(\xi - \frac{2}{h^2}\sin(\frac{\xi}{2})) + \frac{1-\alpha}{24}\xi^3} d\alpha.$$  

The fundamental theorem of calculus yields

$$e^{\pm \frac{ht}{h^2}(\xi - \frac{2}{h^2}\sin(\frac{\xi}{2}))} - e^{\pm \frac{ht}{h^2}\xi^3} = \pm it\left(\xi - \frac{2}{h^2}\sin(\frac{\xi}{2}) - \frac{\xi^3}{24}\right) \int_{0}^{1} e^{\pm it\frac{ht}{h^2}(\xi - \frac{2}{h^2}\sin(\frac{\xi}{2})) + \frac{1-\alpha}{24}\xi^3} d\alpha.$$  

Thus, introducing the linear propagator $S_{h,\alpha}^\pm(t)$ given by

$$\mathcal{F}(S_{h,\alpha}^\pm(t)u_0)(\xi) = e^{\pm it\frac{ht}{h^2}(\xi - \frac{2}{h^2}\sin(\frac{\xi}{2})) + \frac{1-\alpha}{24}\xi^3}(\mathcal{F}u_0)(\xi),$$

we write

$$I = \pm it \int_{0}^{1} S_{h,\alpha}^\pm(t)\mathcal{F}^{-1}\left(1_{|\xi| \leq h^{-\frac{1}{2}}} \left(\xi - \frac{2}{h^2}\sin(\frac{\xi}{2}) - \frac{\xi^3}{24}\right)\mathcal{F}_h(l_hu_{h,0})\right)d\alpha.$$  

Analogously, one can show for the new propagator $S_{h,\alpha}^\pm(t)$ that

$$\|S_{h,\alpha}^\pm(t)P_{low}u_0\|_{C_t,L^2_x} \lesssim \|u_0\|_{L^2}$$

and

$$\|\partial_x S_{h,\alpha}^\pm(t)P_{low}u_0\|_{L^\infty_tL^2_x} \lesssim \|u_0\|_{L^2}.$$  

These results, together with Plancherel’s theorem, the Taylor series expansion $\frac{\xi - \frac{2}{h^2}\sin(\frac{\xi}{2})}{h^2} = \frac{\xi^3}{24} = O(h^2\xi^5)$, and Lemma 3.8 yield

$$\|I\|_{S([-1,1])} \lesssim \|\mathcal{F}^{-1}\left(1_{|\xi| \leq h^{-\frac{1}{2}}} \left(\xi - \frac{2}{h^2}\sin(\frac{\xi}{2}) - \frac{\xi^3}{24}\right)\mathcal{F}_h(l_hu_{h,0})\right)\|_{L^2}$$

$$\lesssim h^2\|\xi^5 1_{|\xi| \leq h^{-\frac{1}{2}}}\mathcal{F}_h(l_hu_{h,0})\|_{L^2_x} \lesssim h^2h^{-\frac{2}{5}(5-s)}\|\xi^s \mathcal{F}(l_hu_{h,0})(\xi)\|_{L^2_x}$$

$$\lesssim h^{2\alpha} \|l_hu_{h,0}\|_{H^s} \lesssim h^{2\alpha} \|u_{h,0}\|_{H^s}.$$
For $II$, by the unitarity of $\hat{S}_{h}^{\pm}(t)$ and Lemma 3.8, we obtain

$$\|II\|_{L_{2}^{\infty}} \lesssim \|P_{h\infty}(l_{h}u_{h,0})\|_{L^{2}} \lesssim h^{2}\|l_{h}u_{h,0}\|_{H^{s}} \lesssim h^{2}\|u_{h,0}\|_{H^{s}}.$$  

It remains to estimate $\|\partial_{x}II\|_{L_{2}^{\infty}L_{2}^{\infty}}$. By (5.10), we write

$$II = \frac{1}{2\pi} \int_{|\xi| \geq \frac{1}{h^{2/3}}} \mathcal{L}_{h}(\xi) e^{\pm \frac{i}{\pi}(\xi - \frac{2}{\pi}\sin(\frac{h\xi}{2}))}(\mathcal{F}_{h}u_{h,0})(\xi') e^{ix\xi} d\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{h^{2/3}} e^{\pm \frac{i}{\pi}(\xi - \frac{2}{\pi}\sin(\frac{h\xi}{2}))}(\mathcal{F}_{h}u_{h,0})(\xi') e^{ix\xi} d\xi$$

$$+ \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{1}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \mathcal{L}_{h}(\xi + \frac{2m\pi}{h}) e^{\pm \frac{i}{\pi}(\xi - \frac{2}{\pi}\sin(\frac{h\xi}{2}))}(\mathcal{F}_{h}u_{h,0})(\xi') e^{ix\xi} d\xi$$

$$=: II_{0} + II_{\neq 0}.$$ 

For $II_{0}$, we apply the local smoothing estimate (Proposition 5.1 (ii)) to obtain

$$\|\partial_{x}II_{0}\|_{L_{2}^{\infty}L_{2}^{\infty}} = \left\|\partial_{x}\hat{S}_{h}^{\pm}(t) \mathcal{F}_{x}^{-1} \left( \frac{1}{h^{2/3}} \mathcal{L}_{h}(\xi)(\mathcal{F}_{h}u_{h,0})(\xi) \right) \right\|_{L_{2}^{\infty}L_{2}^{\infty}}$$

$$\lesssim \left\|\mathcal{F}_{x}^{-1} \left( \frac{1}{h^{2/3}} \mathcal{L}_{h}(\xi)(\mathcal{F}_{h}u_{h,0})(\xi) \right) \right\|_{L_{2}^{\infty}} \lesssim h^{s}\|u_{h,0}\|_{H^{s}}.$$  

(5.11)

Now, we consider $II_{\neq 0}$. A direct computation with (5.9) gives

$$\partial_{x}II_{\neq 0}(t, x) = \frac{1}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{it(\pm \frac{2m\pi}{h}\sin(\frac{h\xi}{2}))} e^{ix\xi} \sum_{m \neq 0} \frac{4\sin^{2}(\frac{h\xi}{2})}{h(2m\pi + h\xi)} e^{i\frac{2m\pi x}{h}} (\mathcal{F}_{h}u_{h,0})(\xi) d\xi.$$ 

Analogously to the proof of Proposition 5.1 (ii), one can show that

$$\|\partial_{x}II_{\neq 0}\|_{L_{2}^{\infty}} \lesssim \left\|\frac{h}{\sqrt{1 - \cos(\frac{h\xi}{2})}} \sum_{m \neq 0} \frac{4\sin^{2}(\frac{h\xi}{2})}{h(2m\pi + h\xi)} e^{i\frac{2m\pi x}{h}} (\mathcal{F}_{h}u_{h,0})(\xi) \right\|_{L_{2}^{\infty}}$$

$$\lesssim \left\|\left( \sum_{m \neq 0} \frac{e^{i\frac{2m\pi x}{h}}}{2m\pi + h\xi} \right) \sin \left( \frac{h\xi}{2} \right) (\mathcal{F}_{h}u_{h,0})(\xi) \right\|_{L_{2}^{\infty}}.$$ 

Note that

$$\left| \sum_{m \neq 0} \frac{e^{imx}}{m} \right| = 2 \left| \sum_{m=1}^{\infty} \frac{\sin(mx)}{m} \right| \lesssim \int_{0}^{\infty} \frac{\sin(xt)}{t} dt < \infty,$$

uniformly in $x$, which implies that

$$\left| \sum_{m \neq 0} \frac{e^{i\frac{2m\pi x}{h}}}{2m\pi + h\xi} \right| = \sum_{m \neq 0} \frac{e^{i\frac{2m\pi x}{h}}}{2m\pi} + \sum_{m \neq 0} \left( \frac{e^{i\frac{2m\pi x}{h}}}{2m\pi + h\xi} - \frac{e^{i\frac{2m\pi x}{h}}}{2m\pi} \right) \lesssim 1 + \sum_{m \neq 0} \frac{1}{m^{2}} < \infty,$$

uniformly in $x \in \mathbb{R}$. Thus, we prove that

$$\|\partial_{x}II_{\neq 0}\|_{L_{2}^{\infty}} \lesssim \|\sin(\frac{h\xi}{2})(\mathcal{F}_{h}u_{h,0})(\xi)\|_{L_{2}^{\infty}([-\frac{\pi}{h}, \frac{\pi}{h}])} \lesssim h^{s}\|u_{h,0}\|_{H^{s}}.$$ 

which, in addition to (5.11), implies that

$$\|\partial_{x}II\|_{L_{2}^{\infty}L_{2}^{\infty}} \lesssim h^{\frac{2s}{3}}\|u_{h,0}\|_{H^{s}}.$$
By combining all the results, we complete the proof of the proposition.

\[ \square \]

6. Bilinear estimates

In this section, we prove a series of $X^{s,b}$ bilinear estimates, which are the key estimates in our analysis.

**Lemma 6.1 (Bilinear estimate I).** For $s \geq 0$, there exist $b = b(s) > \frac{1}{2}$ and $\delta = \delta(b) > 0$ such that

\[
\| \nabla_h (u^+_h \cdot v^+_h) \|_{X^{s,-(1-b-\delta)}} \lesssim \| u^+_h \|_{X^{s,b}_h} \| v^+_h \|_{X^{s,b}_h},
\]

(6.1)

for any $u^+_h, v^+_h \in X^{s,b}_h$.

The following elementary integral estimates will be employed.

**Lemma 6.2 (Lemma 2.3 in [33]).** Let $\alpha, \beta \in \mathbb{R}$. For $b > \frac{1}{2}$, we have

\[
\int_{-\infty}^{\infty} \frac{dx}{(x-\alpha)^{2b}(x-\beta)^{2b}} \lesssim \frac{1}{(\alpha-\beta)^{2b}}
\]

(6.2)

and

\[
\int_{-\infty}^{\infty} \frac{1}{(x)^{2b}\sqrt{|x-\beta|}} \lesssim \frac{1}{(\beta)^{\frac{1}{2}}}.
\]

(6.3)

**Proof of Lemma 6.1.** We prove Lemma 6.1 only for the $\| \nabla_h (u^+_h, v^+_h) \|_{X^{s,-(1-b-\delta)}}$ case, otherwise, an analogous argument is applicable.

By Parseval’s identity, we write

\[
\int_{-\infty}^{\infty} \sum_{x \in h \mathbb{Z}} \nabla_h (u^+_h, v^+_h) (t, x) w(t, x)
\]

\[
\sim \int \int \int \int \frac{2i}{h} \sin \left( \frac{h^2}{2} \right) \hat{u}^+_h (\tau_1, \xi_1) \hat{v}^+_h (\tau - \tau_1, \xi - \xi_1) \hat{w}_h (\tau, \xi) d\xi d\tau d\tau_1 d\tau,
\]

where $\hat{u}$ is the space-time Fourier transform, and the intervals of integration are omitted for notational convenience. By the symmetry and duality, it suffices to show that

\[
\int \int \int \int \frac{2i}{h} \sin \left( \frac{h^2}{2} \right) \hat{u}^+_h (\tau_1, \xi_1) \hat{v}^+_h (\tau - \tau_1, \xi - \xi_1) \hat{w}_h (\tau, \xi) d\xi d\tau d\tau_1 d\tau
\]

\[
\lesssim \| u^+_h \|_{X^{s,b}_h} \| v^+_h \|_{X^{s,b}_h} \| w_h \|_{X^{-s,1-b-\delta}_h},
\]

which is equivalent to

\[
\int \int \int \frac{2i}{h} \sin \left( \frac{h^2}{2} \right) \hat{u}^+_h (\tau_1, \xi_1) \hat{v}^+_h (\tau - \tau_1, \xi - \xi_1) \hat{w}_h (\tau, \xi) d\xi d\tau d\tau_1 d\tau
\]

\[
\lesssim \| F \|_{L^2_{\tau,\xi}} \| G \|_{L^2_{\tau,\xi}} \| W \|_{L^2_{\tau,\xi}},
\]

where

\[
s_h (\xi) := \frac{1}{h^2} (\xi - \frac{2}{h} \sin \left( \frac{h^2}{2} \right))
\]
and

\[
\begin{aligned}
F(\tau, \xi) &= (\xi)^s (\tau - s_h(\xi))^b \tilde{u}_h^+(\tau, \xi), \\
G(\tau, \xi) &= (\tau - s_h(\xi))^b \tilde{v}_h^+(\tau, \xi), \\
W(\tau, \xi) &= (\xi)^{-s} (\tau - s_h(\xi))^{1-b-\delta} \tilde{w}_h(\tau, \xi).
\end{aligned}
\]

Hence, by the trivial inequality

\[
\frac{(\xi)^s}{(\xi_1)^s (\xi - \xi_1)^s} \lesssim 1
\]

the Cauchy-Schwarz inequality for the \(\tau_1\)- and \(\xi_1\)-variables and (6.2), we have

\[
\begin{aligned}
&\left| \iiint \frac{2}{\pi} \sin\left(\frac{h\xi}{2}\right) (\xi)^s F(\tau_1, \xi_1) G(\tau - \tau_1, \xi - \xi_1) W(\tau, \xi) \, d\xi_1 \, d\tau_1 \, d\tau \right| \\
&\lesssim \left( \sup_{\tau \in \mathbb{R}} \sup_{|\xi| \leq \frac{\pi}{2}} \int \frac{1}{\pi} \sin^2\left(\frac{h\xi}{2}\right) \left( \frac{\tau_1 - s_h(\xi_1)}{\tau - s_h(\xi_1)} \right)^{2b} \left( \tau - s_h(\xi_1) - s_h(\xi - \xi_1) \right)^{2b} \, d\tau_1 \right)^{\frac{1}{2}} \\
&\quad \cdot \|F\|_{L^2_{\tau_1,\xi}} \|G\|_{L^2_{\tau_1,\xi}} \|W\|_{L^2_{\tau_1,\xi}}.
\end{aligned}
\]

Therefore, it suffices to show that

\[
\sup_{\tau \in \mathbb{R}} \sup_{|\xi| \leq \frac{\pi}{2}} \frac{1}{\pi} \sin^2\left(\frac{h\xi}{2}\right) \int \frac{d\xi_1}{\pi} \left( \frac{\tau_1 - s_h(\xi_1)}{\tau - s_h(\xi_1) - s_h(\xi - \xi_1)} \right)^{2b} \lesssim 1.
\]

Note that the left-hand side of (6.5) vanishes when \(\xi = 0\). In what follows, we assume that \(\xi \neq 0\). By the symmetry \(s_h(-\xi) = -s_h(\xi)\), we may assume that \(\xi > 0\). To show (6.5), by the sum-to-product rule for sine functions, we write

\[
I_{\tau, \xi} := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\xi_1}{\pi} \left( \frac{\tau - s_h(\xi_1)}{\tau - s_h(\xi_1) - s_h(\xi - \xi_1)} \right)^{2b} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\xi_1}{\pi} \left( \frac{\tau - \xi_1}{\tau - \xi_1^+ + \frac{h}{\pi^2} \sin\left(\frac{h\xi_1}{2}\right) + \sin\left(\frac{h(\xi - \xi_1)}{2}\right)} \right)^{2b}
\]

\[
= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\xi_1}{\pi} \left( \frac{\tau - \xi_1}{\tau - \xi_1^+ + \frac{h}{\pi^2} \sin\left(\frac{h(\xi - \xi_1)}{2}\right)} \right)^{2b}.
\]
Then, by changing the variables $\frac{h(\xi - 2\xi)}{4} \mapsto \xi_1$ and since $\cos \xi_1$ is an even function, it follows that

$$I_{\tau, \xi} = \frac{2}{h} \int_{\frac{\pi}{2} + \frac{h\xi}{4}}^{\frac{\pi}{2} - \frac{h\xi}{4}} \frac{d\xi_1}{\langle \tau - \frac{\xi}{4\pi} + \frac{4}{h^4} \sin(\frac{h\xi}{4}) \cos \xi_1 \rangle^{2b}}$$

$$= \frac{2}{h} \left( \int_{0}^{\frac{\pi}{2} - \frac{h\xi}{4}} + \int_{0}^{\frac{\pi}{2} + \frac{h\xi}{4}} \right) \frac{d\xi_1}{\langle \tau - \frac{\xi}{4\pi} + \frac{4}{h^4} \sin(\frac{h\xi}{4}) \cos \xi_1 \rangle^{2b}}$$

$$\leq \frac{4}{h} \int_{0}^{\frac{\pi}{2} - \frac{h\xi}{4}} \frac{d\xi_1}{\langle \tau - \frac{\xi}{4\pi} + \frac{4}{h^4} \sin(\frac{h\xi}{4}) \rangle^{2b}}$$

$$I_{\tau, \xi} \leq \frac{4}{h^4} \sin^2(\frac{h\xi}{4}) \int_{-\infty}^{\infty} \frac{d\mu}{\langle \tau - \frac{\xi}{4\pi} + \frac{4}{h^4} \sin(\frac{h\xi}{4}) \rangle^{2b}}$$

$$\leq \frac{4}{h^4} \sin^2(\frac{h\xi}{4}) \int_{-\infty}^{\infty} \frac{d\mu}{\langle \mu - (\tau - \frac{\xi}{4\pi} + \frac{4}{h^4} \sin(\frac{h\xi}{4})) \rangle^{2b}}.$$
whenever $0 < \delta < \frac{1}{4}$ and $\frac{1}{2} < b < \frac{3}{2} - \delta$. Thus, taking $\zeta = \tau - \frac{\xi}{\eta^{2}}$, $\alpha = -\frac{\eta}{h^{4}} \sin\left(\frac{h\xi}{4}\right)$ and $\beta = -\frac{4}{h^{4}} \sin\left(\frac{h\xi}{4}\right)$ in (6.7), and using trigonometric identities, we conclude that

$$\frac{\frac{4}{h^{4}} \sin^{2}\left(\frac{h\xi}{4}\right)}{\langle \tau - sh(\xi) \rangle^{2(1-b-\delta)}} I_{\tau,\xi} \lesssim \frac{\frac{1}{h^{4}} \sin^{2}\left(\frac{h\xi}{4}\right)}{\sqrt{\frac{4}{h^{4}} \sin\left(\frac{h\xi}{4}\right) \sin\left(\frac{h\xi}{4}\right) - \frac{2}{h^{4}} \sin\left(\frac{h\xi}{2}\right)}} = \frac{\frac{4}{h^{4}} \sin^{2}\left(\frac{h\xi}{4}\right)}{\sqrt{\frac{4}{h^{4}} \sin\left(\frac{h\xi}{4}\right) \sin\left(\frac{h\xi}{4}\right) - \frac{1}{h^{4}} \sin\left(\frac{h\xi}{2}\right) \cos\left(\frac{h\xi}{2}\right)}} \frac{4}{h} \frac{\sin\left(\frac{h\xi}{4}\right) \cos\left(\frac{h\xi}{4}\right)}{\sqrt{1 - \cos\left(\frac{h\xi}{2}\right)}} = 4\sqrt{2} \cos\left(\frac{h\xi}{8}\right) \cos^{2}\left(\frac{h\xi}{4}\right) \lesssim 1,$$

which proves the desired bound (6.5). □

**Lemma 6.3** (Bilinear estimate II). For $s \geq 0$, there exist $b = b(s) > 1/2$ and $\delta = \delta(b) > 0$ such that if $s \leq s' \leq s + 1$, then

$$\left\| \nabla_{h} (e^{\frac{2i}{h} \partial_{h} u_{h}^{+}} \cdot e^{\frac{2i}{h} \partial_{h} v_{h}^{+}}) \right\|_{X_{h,\pm}^{s,b-1+\delta}} \lesssim h^{s'-s} \left\| u_{h}^{+} \right\|_{X_{h,\pm}^{s',b}} \left\| v_{h}^{+} \right\|_{X_{h,\pm}^{s',b}}. \tag{6.9}$$

*Proof.* We consider the case of $\left\| \nabla_{h} (e^{-\frac{2i}{h} \partial_{h} u_{h}^{+}} \cdot e^{-\frac{2i}{h} \partial_{h} v_{h}^{+}}) \right\|_{X_{h,\pm}^{s-1,b-\delta}}$ only. The proof closely follows from that of Lemma [6.1]. By Parseval’s identity, we write

$$\int_{-\infty}^{\infty} \sum_{x \in hZ} \nabla_{h} (e^{-\frac{2i}{h} \partial_{h} u_{h}^{+}} \cdot e^{-\frac{2i}{h} \partial_{h} v_{h}^{+}})(t, x)w_{h}(t, x)$$

$$\sim \int \int \int \int \frac{2i}{h} \sin\left(\frac{h\xi}{4}\right) \langle \xi \rangle^{s} F(\tau, \xi) G(\tau - \xi^{1}, \xi^{2} - s_{h}(\xi)) W(\tau, \xi) d\xi_{1} d\xi_{2} d\tau_{1} d\tau$$

where

$$\begin{align*}
F(\tau, \xi) &= \langle \xi \rangle^{s} \langle \tau + \frac{2\xi^{2}}{h^{4}} - s_{h}(\xi) \rangle^{b} \bar{u}_{h}^{+}(\tau + \frac{2\xi^{2}}{h^{4}}, \xi), \\
G(\tau, \xi) &= \langle \tau + \frac{2\xi^{2}}{h^{4}} - s_{h}(\xi) \rangle^{b} \bar{v}_{h}^{+}(\tau + \frac{2\xi^{2}}{h^{4}}, \xi), \\
W(\tau, \xi) &= \langle \xi \rangle^{-s} \langle \tau + s_{h}(\xi) \rangle^{1-b-\delta} \bar{w}_{h}(\tau, \xi)
\end{align*}$$

and the second integral has the same structure. Hence, by repeating the reduction to (6.5) but using $\frac{\langle \xi \rangle^{s'}}{\langle \xi \rangle^{s}} \lesssim 1$ instead of (6.4), one can reduce the proof of Lemma 6.3 to get a uniform bound for

$$\frac{4}{h^{4}} \sin^{2}\left(\frac{h\xi}{2}\right) \int_{-\infty}^{\infty} \int \frac{d\xi_{1}}{\langle \tau + \frac{\xi^{2}}{h^{4}} + \frac{2\xi^{2}}{h^{4}} (\sin\left(\frac{h\xi_{1}}{4}\right) + \sin\left(\frac{h(\xi_{1} + \epsilon)}{4}\right))\rangle} \lesssim h^{s'-s}. \tag{6.10}$$
for all $|\xi| \leq \frac{\pi}{2}$ and $\tau \in \mathbb{R}$. We may assume that $\xi > 0$. We denote the integral in (6.10) by $I_{\tau,\xi}$. Then, by following the argument in the proof of Lemma 6.1, we write

\[
I_{\tau,\xi} = \int_{\tau}^{\tau + \frac{\pi}{\sqrt{b}}} \int_{-\frac{\pi}{\sqrt{b}}}^{\frac{\pi}{\sqrt{b}}} \frac{d\xi_1}{\left(\tau + \frac{\xi}{\sqrt{b}} + \mu \sin(\frac{\pi}{4})\right)^{2(1-b-\delta)}}
\]

Next, changing the variable $\mu = \frac{4}{\pi^2} \sin(\frac{\pi}{4}) \cos \xi_1$ yields

\[
I_{\tau,\xi} \leq \frac{4}{h} \int_{\tau}^{\tau + \frac{\pi}{\sqrt{b}}} \int_{-\frac{\pi}{\sqrt{b}}}^{\frac{\pi}{\sqrt{b}}} \frac{d\mu}{\left(\tau + \frac{\xi}{\sqrt{b}} + \mu \sin(\frac{\pi}{4})\right)^{2(1-b-\delta)}}
\]

Thus, by inserting this bound in (6.10), we prove that

\[
\langle \xi \rangle^{s' - s} (\tau + s h(\xi))^{2(1-b-\delta)} I_{\tau,\xi}
\]

\[
\leq \frac{4}{h^2 \sin^2(\frac{\pi}{4})} \int_{\tau}^{\tau + \frac{\pi}{\sqrt{b}}} \int_{-\frac{\pi}{\sqrt{b}}}^{\frac{\pi}{\sqrt{b}}} \frac{d\xi_1}{\left(\tau + \frac{\xi}{\sqrt{b}} + \mu \sin(\frac{\pi}{4})\right)^{2(1-b-\delta)}}
\]

\[
\leq \frac{16}{h^2 \sin^2(\frac{\pi}{4})} \int_{\tau}^{\tau + \frac{\pi}{\sqrt{b}}} \int_{-\frac{\pi}{\sqrt{b}}}^{\frac{\pi}{\sqrt{b}}} \frac{d\xi_1}{\left(\tau + \frac{\xi}{\sqrt{b}} + \mu \sin(\frac{\pi}{4})\right)^{2(1-b-\delta)}(1 + \cos(\frac{\pi}{4}))}
\]

By (6.7), we prove (6.10).

Lemma 6.4 (Bilinear estimate III). For $s \geq 0$, there exist $b = b(s) > 1/2$ and $\delta = \delta(b) > 0$ such that if $s \leq s' \leq s + 1$, then

\[
\left\| \nabla_h (u_h^+ \cdot e^{\pm \frac{2\pi}{h^2 \tau} \partial_\theta} v_0^\mp) \right\|_{X_{h,\tau,1}^{b-1+\delta}} \lesssim h^{s' - s} \| u_h^\pm \|_{X_{h,h_{0}^\pm}^{b'}} \| v_0^\mp \|_{X_{h,h_{0}^\pm}^{b'}}
\]
Proof. Again, we consider the case of \( \| \nabla_h(u_h e^{-\frac{\theta}{\mu}} v_h^+)}\|_{X_{h,-a,b}} \) only, and we write

\[
\int_{-\infty}^{\infty} \sum_{x \in h\mathbb{Z}} \nabla_h(u_h e^{-\frac{\theta}{\mu}} v_h^+)(t, x) w_h(t, x) \sim \int \int \int \int \frac{4\pi}{\sin(\frac{h}{2})} \langle \xi \rangle^s F(\tau, \xi_1) G(\tau - \tau_1, \xi - \xi_1) W(\tau, \xi) d\xi_1 d\xi d\tau d\tau_1 d\xi_1,
\]

where

\[
F(\tau, \xi) = \langle \xi \rangle^s (\tau + s_1(\xi)) b_1(\tau, \xi),
\]
\[G(\tau, \xi) = (\tau + 2\frac{\xi}{h^2} - s_1(\xi)) b_1(\tau + 2\frac{\xi}{h^2}, \xi),
\]
\[W(\tau, \xi) = \langle \xi \rangle^{-s} (\tau + s_1(\xi))^{1-b-\delta} w_1(\tau, \xi).
\]

Thus, similarly to the proof of the previous two lemmas, one can reduce to the bound

\[
\int \frac{4\pi}{\sin(\frac{h}{2})} \langle \xi \rangle^{s-s} (\tau + s_1(\xi))^{2(1-b-\delta)} \left( \int \frac{d\xi_1}{(\tau + \frac{\xi}{h^2} - 2\frac{\xi^2}{h^2}\sin(\frac{h\xi}{2}) - \sin(\frac{h(\xi-\xi_1)}{2}))^{2b}} \right)
\]

We may assume that \( \xi > 0 \). Let \( I_{\tau, \xi} \) denote the integral in (6.13). Changing the variable \( \xi - 2\xi_1 \to \xi_1 \) and using the sum-to-product formula yields

\[
I_{\tau, \xi} = \int \frac{d\xi_1}{(\tau + \frac{\xi}{h^2} + \frac{4\xi^2}{h^2} \cos(\frac{h\xi}{4}) \sin(\frac{h(\xi-\xi_1)}{2}))^{2b}} = \int \frac{d\xi_1}{(\tau + \frac{\xi}{h^2} + \frac{4\xi^2}{h^2} \cos(\frac{h\xi}{4}) \sin(\xi_1)^{2b}}.
\]

By the trivial identity \( \sin(\theta) = \sin(\pi - \theta) \),

\[
\int \frac{d\xi_1}{(\tau + \frac{\xi}{h^2} + \frac{4\xi^2}{h^2} \cos(\frac{h\xi}{4}) \sin(\xi_1)^{2b}} = \int \frac{d\xi_1}{(\tau + \frac{\xi}{h^2} + \frac{4\xi^2}{h^2} \cos(\frac{h\xi}{4}) \sin(\xi_1)^{2b}}.
\]

Thus,

\[
I_{\tau, \xi} \leq \frac{4h}{h^2} \int \frac{d\xi_1}{(\tau + \frac{\xi}{h^2} + \frac{4\xi^2}{h^2} \cos(\frac{h\xi}{4}) \sin(\xi_1)^{2b}}.
\]

By performing change of variables \( \mu = \frac{4h}{h^2} \cos(\frac{h\xi}{4}) \sin(\xi_1) \), we write

\[
I_{\tau, \xi} \leq \frac{4h}{h^2} \int \frac{d\mu}{(\tau + \frac{\xi}{h^2} + \mu) + \sqrt{\frac{4h}{h^2} \cos(\frac{h\xi}{4}) \sin(\xi_1)^{2b}} - \mu}
\]

\[
\leq \frac{1}{h \sqrt{\frac{4h}{h^2} \cos(\frac{h\xi}{4})(1 - \cos(\frac{h\xi}{4}))}} \int \frac{d\mu}{(\tau + \frac{\xi}{h^2} + \mu) + \sqrt{\frac{4h}{h^2} \cos(\frac{h\xi}{4}) - \mu}}
\]

\[
\leq \frac{1}{\sqrt{\frac{4h}{h^2} \cos(\frac{h\xi}{4})(1 - \cos(\frac{h\xi}{4}))}} \int \frac{d\mu}{(\tau + \frac{\xi}{h^2} + \mu) + \sqrt{\frac{4h}{h^2} \cos(\frac{h\xi}{4}) - \mu}}.
\]
which, in addition to the half-angle formula and (6.3), implies that

\[ I_{\tau, \xi} \lesssim \frac{\sqrt{h}}{\sin(\frac{h \xi}{8})} \left( \langle \tau + \frac{\xi}{h^2} + \frac{4}{h^4} \cos\left(\frac{h \xi}{4}\right) \right)^{\frac{1}{2}}. \]

Finally, by applying (6.7) and the half-angle formula, we prove that

\[
\frac{4}{h^2} \sin^2\left(\frac{h \xi}{2}\right) \langle \xi \rangle^{s''-s} \langle \tau + s_h(\xi) \rangle^{2(1-b-\delta)} I_{\tau, \xi} \leq \frac{\sqrt{h}}{\frac{4}{h^2} \sin^2\left(\frac{h \xi}{2}\right)} \langle \xi \rangle^{s''-s} \sin\left(\frac{h \xi}{8}\right) \sim \langle \xi \rangle^{s''-s} \sin\left(\frac{h \xi}{8}\right) \lesssim \langle \xi \rangle^{s''-s} \lesssim \langle \xi \rangle^{s''.}
\]

Therefore, we complete the proof. \( \square \)

Finally, we show the regularity gain for the bilinear estimates in higher regularity norms.

**Lemma 6.5.** Let \( 0 < T \leq 1 \) and \( s > \frac{3}{4} \). Then, we have

\[ \| u_h \cdot v_h \|_{L^2([-T; T]; H^s_\theta(h \mathbb{Z}))} \lesssim \| \nabla_h^{-1} u_h \|_{X^{s,b}_h} \| v_h \|_{X^{s,b}_h}, \tag{6.14} \]

for \( \nabla_h^{-1} u_h, v_h \in X^{s,b}_h \).

**Proof.** The Littlewood-Paley theory yields

\[
\| u_h \cdot v_h \|_{L^2_t H^s_x} \lesssim \| P_{\leq N_0}(u_h v_h) \|_{L^2_t H^s_x} + \left( \sum_{N_0 \leq N \leq 1} \left( \frac{N}{h} \right)^2 \| P_N(u_h v_h) \|_{L^2_t L^2_x}^2 \right)^{\frac{1}{2}}, \tag{6.15}
\]

where \( N_0 < 1 \) is the maximum dyadic number satisfying \( N \leq h \). Here, the time interval \([T, T] \) is omitted in the norms.

The first term on the right-hand side of (6.15) is easily treated compared to the second one. Indeed, from the fact that \( \langle \xi \rangle \sim 1 \) on the support of \( P_{N_0} \) and Corollary 5.3, we show that

\[
\| P_{\leq N_0}(u_h v_h) \|_{L^2_t H^s_x} \lesssim \| u_h v_h \|_{L^2_t L^2_x} \lesssim \| u_h \|_{L^\infty_t L^2_x} \| v_h \|_{L^2_t L^\infty_x} \lesssim \| \nabla_h^{-1} u_h \|_{X^{0,b}_h} \| v_h \|_{X^{s,b}_h}. \tag{6.16}
\]

For the second term, we further decompose

\[
\| P_N(u_h v_h) \|_{L^2_t L^2_x} \lesssim \| P_N((P_{\leq \frac{N}{h}} u_h) v_h) \|_{L^2_t L^2_x} + \| P_N((P_{> \frac{N}{h}} u_h) v_h) \|_{L^2_t L^2_x} =: I + II.
\]
For $I$, we observe that $P_N((P_{\leq \frac{N}{4}} u_h) v_h) = P_N((P_{\leq \frac{N}{4}} u_h)(P_{\frac{N}{16} < \cdot < \frac{16N}{16}} v_h))$ owing to the support property\footnote{Roughly speaking, in a $(k)$-multilinear form, one has a frequency relation $\xi_1 + \cdots + \xi_k = \xi$; thus, the multilinear form vanishes unless the maximum two frequencies are comparable.}. Thus, by the Hölder and Bernstein inequalities\footnote{Roughly speaking, in a $(k)$-multilinear form, one has a frequency relation $\xi_1 + \cdots + \xi_k = \xi$; thus, the multilinear form vanishes unless the maximum two frequencies are comparable.} and Corollary\footnote{Roughly speaking, in a $(k)$-multilinear form, one has a frequency relation $\xi_1 + \cdots + \xi_k = \xi$; thus, the multilinear form vanishes unless the maximum two frequencies are comparable.} we obtain
\[
\|P_N((P_{\leq \frac{N}{4}} u_h) v_h)\|_{L^2_t L^2_x} \lesssim \|P_N((P_{\leq \frac{N}{4}} u_h)(P_{\frac{N}{16} < \cdot < \frac{16N}{16}} v_h))\|_{L^2_t L^2_x} \\
\lesssim \|P_{\leq \frac{N}{4}} u_h\|_{L^4_t L^\infty_x}\|P_{\frac{N}{16} < \cdot < \frac{16N}{16}} v_h\|_{L^\infty_t L^2_x} \\
\lesssim T^{\frac{1}{4} - \frac{1}{4}} \|\partial_t u_h\|_{L^4_t L^\infty_x}\|P_{\frac{N}{16} < \cdot < \frac{16N}{16}} v_h\|_{L^\infty_t L^2_x} \\
\lesssim T^{\frac{1}{4} - \frac{1}{4}} \|\partial_t \|_{L^4_t L^\infty_x} \|P_{\frac{N}{16} < \cdot < \frac{16N}{16}} v_h\|_{X_{h, \pm}^{0,b}} \|P_{\frac{N}{16} < \cdot < \frac{16N}{16}} v_h\|_{X_{h, \pm}^{0,b}} .
\]
Thus, it follows that
\[
\sum_{N_0 \leq N \leq 1} \left( \frac{N}{h} \right)^{2s} \|P_N((P_{\leq \frac{N}{4}} u_h) v_h)\|^2_{L^2_t L^2_x} \\
\lesssim T^{1 - \frac{1}{2}} \|\partial_t u_h\|_{L^\infty_x} \|P_{N_0 \leq N \leq 1} \left( \frac{N}{h} \right)^{2s} \|P_{\frac{N}{16} < \cdot < \frac{16N}{16}} v_h\|^2_{X_{h, \pm}^{0,b}} .
\]  \hspace{1cm} (6.17)

For $II$, by repeating the estimates in (6.16), we obtain
\[
\|P_N((P_{\leq \frac{N}{4}} u_h) v_h)\|_{L^2_t L^2_x} \lesssim \|P_{\leq \frac{N}{4}} u_h\|_{L^4_t L^\infty_x}\|v_h\|_{L^2_t L^\infty_x} \lesssim \|P_{\leq \frac{N}{4}} \nabla_h^{-1} u_h\|_{X_{h, \pm}^{0,b}} \|v_h\|_{X_{h, \pm}^{s,b}} .
\]
Inserting this result and by Fubini’s theorem for the sum, we obtain
\[
\sum_{N_0 \leq N \leq 1} \left( \frac{N}{h} \right)^{2s} \|P_N((P_{\geq \frac{N}{4}} u_h) v_h)\|^2_{L^2_t L^2_x} \\
\lesssim \|v_h\|^2_{X_{h, \pm}^{s,b}} \sum_{N_0 \leq N \leq 1} \left( \frac{N}{h} \right)^{2s} \sum_{M \geq \frac{N}{4}} \|P_M \nabla_h^{-1} u_h\|^2_{X_{h, \pm}^{0,b}} \\
\sim \|v_h\|^2_{X_{h, \pm}^{s,b}} \sum_{\frac{N}{4} \leq M \leq 1} \left( \frac{M}{h} \right)^{2s} \|P_M \nabla_h^{-1} u_h\|^2_{X_{h, \pm}^{0,b}} \\
\lesssim \|\nabla_h^{-1} u_h\|_{X_{h, \pm}^{s,b}} \|v_h\|^2_{X_{h, \pm}^{s,b}} .
\]  \hspace{1cm} (6.18)

By combining (6.16), (6.17) and (6.18) for the right-hand side of (6.15), we complete the proof. □

7. Uniform bounds for nonlinear solutions

This section is devoted to bounds for solutions to the three equations in consideration. In Section 7.1 we briefly review the well-posedness of the KdV equation and state several mixed norm bounds for nonlinear solutions. In Section 7.2 we obtain analogous uniform
bounds for the coupled and decoupled FPU s (Proposition 7.3 and 7.3). The main results in this section play a crucial role in our analysis.

7.1. Bounds for solutions to KdVs. We consider the KdV equations

\[ \partial_t w_{\pm} + \frac{1}{24} \partial_x^3 w_{\pm} + \frac{1}{4} \partial_x (w_{\pm}^2) = 0, \]  

(7.1)

where \( w_{\pm} = w_{\pm}(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), i.e., the differential form of (24.13). This equation is nothing but the standard formulation of the KdV equation \( \partial_t w + \partial_x^3 w - \partial_x (w^2) = 0 \), because by simple changes of variables, \( w_{\pm}(t,x) = w(\sqrt{3}t, \sqrt{6}x) \) and \( w_{\pm}(t,x) = w_{\pm}(t,-x) \).

KdV has been a central research topic in various fields of mathematics, especially because of its complete integrability. From an analysis perspective, its well-posedness has been investigated by many authors. We refer to, for instance, [23, 29, 30, 32, 31, 33, 8, 9, 23, 36, 35], and the references therein. It should be noted that among the various important results, in [33], Kenig, Ponce, and Vega established local well-posedness in the negative Sobolev space \( H^s \) for \( s > -\frac{3}{4} \); later, in [9], Colliander, Keel, Staffilani, Takaoka, and Tao extended the previous result to global well-posedness at the same low regularity level. It was further improved by Guo [23] and Kishimoto [36] independently at the end-point \( s = -\frac{3}{4} \).

These low regularity well-posedness results are known to be the best possible ones via the contraction mapping argument, because uniform continuity of the data-to-solution map fails when \( s < -\frac{3}{4} \) (see [34], [7]). Remarkably, very recently in [35], Killip and Visan established global well-posedness in \( H^{-1} \) by exploiting integrability of the equation.

Coming back to our discussion, we restrict ourselves to non-negative Sobolev spaces, and state the following well-posedness theorem of Bourgain [5, Appendix 2].

**Theorem 7.1** (Well-posedness of KdVs). Let \( s \geq 0 \).

(i) (Local well-posedness) There exist \( b \in (\frac{1}{4}, 1) \) such that for the given initial data \( w_0^\pm \in H^s(\mathbb{R}) \), there exist \( T > 0 \), depending on \( \|w_0^\pm\|_{H^s} \), and a unique solution \( w^\pm(t) \) to KdV (7.1) in the interval \([-T,T]\) satisfying \( w^\pm \in C_t([-T,T];H^s(\mathbb{R})) \) and \( w^\pm \in X_{s,b}^\pm \), where \( X_{s,b}^\pm \) is the Bourgain space equipped with the norm as in (5.1). Moreover, the solution \( w^\pm(t) \) conserves the momentum

\[ P^\pm(t) = \int_{-\infty}^{\infty} w^\pm(t,x)^2 dx = P^\pm(0). \]

(ii) (Global well-posedness) The solution \( w^\pm(t) \) exists globally in time.

Combining the previous theorem and Proposition 5.4 with the transference principle, we deduce the following mixed norm bounds.

**Corollary 7.2** (Mixed norm bounds for the KdVs). For \( s > \frac{3}{4} \) and \( b \in (\frac{1}{4}, 1) \), let \( w^\pm(t) \in C_t([-T,T];H^s_{X_{s,b}^+}(\mathbb{R})) \) be the solution to KdV (7.1) with initial data \( w_0^\pm \in H^s(\mathbb{R}) \), constructed in Theorem 7.1. Then, we have

\[
\begin{align*}
\| \partial_x^\frac{1}{2} (\partial_x^b)^s w^\pm \|_{L_t^4([-T,T];L_x^{\infty}(\mathbb{R}))} & \lesssim \|w_0^\pm\|_{H^s(\mathbb{R})}, \\
\| \partial_x (\partial_x^b)^s w^\pm \|_{L_t^\infty(\mathbb{R};L_x^2([-T,T])))} & \lesssim \|w_0^\pm\|_{H^s(\mathbb{R})}, \\
\| w^\pm \|_{L_t^2(\mathbb{R};L_x^\infty([-T,T])))} & \lesssim \|w_0^\pm\|_{H^s(\mathbb{R})}.
\end{align*}
\]
7.2. Uniform bounds for coupled and decoupled FPUs. Next, we state the main results of this section. They assert uniform bounds for the coupled and decoupled FPUs, analogous to those for the KdV equations.

**Proposition 7.3** (Uniform bound for coupled and decoupled FPU). Let $s \geq 0$. Suppose that

$$\sup_{h \in [0,1]} \sum_{\pm} \|u_{h,0}^\pm\|_{H^s(h\mathbb{Z})} \leq R.$$ 

Then, there exists $T > 0$, depending on $R > 0$ but not on $h \in (0,1]$, such that the solution $(u_h^+(t), u_h^-(t))$ (resp., $(v_h^+(t), v_h^-(t))$) to the coupled FPU (2.8) (resp., decoupled FPU (2.12)) with initial data $(u_{h,0}^+, u_{h,0}^-)$ such that

$$\|\theta(t T)u_h^\pm(t)\|_{X^{s,b}_{h,\pm}} \lesssim \|u_h^\pm\|_{H^s(h\mathbb{Z})}\quad \text{(resp.,}\quad \|\theta(t T)v_h^\pm(t)\|_{X^{s,b}_{h,\pm}} \lesssim \|u_{h,0}^\pm\|_{H^s(h\mathbb{Z})}),$$

where $\theta \in C_c^\infty$ is a non-negative cut-off, and the Bourgain space $X^{s,b}_{h,\pm}$ is given in Definition 3.14.

**Remark 7.4.** For the same reason mentioned in Remark 4.2, the proof below does not include the estimate of the higher-order remainder term in (2.8). See Lemma 3.1 for the proof of the estimate of the higher-order remainder term.

**Remark 7.5.** As a consequence of the bound (7.2), in addition to (3.11), we have (also for $v_h^\pm$)

$$\|(u_h^+(t), u_h^-(t))\|_{C_h([-T,T];H^s_{h}(h\mathbb{Z}))} \lesssim \|(u_{h,0}^+, u_{h,0}^-)\|_{H^s(h\mathbb{Z})}.$$ 

Analogous to Corollary 7.2, we have the following result from Propositions 5.1 and 7.3.

**Corollary 7.6** (Mixed norm bounds for the decoupled FPU). Let $s \geq 0, \frac{3}{4} < s' < \frac{3}{2}$ and $b > \frac{1}{2}$. Let $v_h^\pm(t) \in C([T,T];L^2_{h}(h\mathbb{Z}))$ be the solution to the decoupled FPU (2.12) with initial data $u_{h,0}^\pm$ given in Proposition 7.3. Let $m(D_h)$ be the Fourier multiplier, i.e., $\mathcal{F}_h[m(D_h)f_h](-\xi) = m(\xi)\hat{f}_h(\xi)$, with a uniform bound such that

$$|m(\xi)| \leq C|\xi|^s, \quad \text{for all} \quad \xi \in \mathbb{T}_h,$$

where $C$ is independent of $h$. Then, we have for $q$ as in Corollary 5.18

$$\|m(D_h)\partial_h v_h^\pm\|_{L^q([-T,T];L^\infty_{h}(h\mathbb{Z}))} \lesssim \|u_{h,0}^\pm\|_{H^{s+s'(h\mathbb{Z})}},$$

$$\|m(D_h)\partial_h v_h^\pm\|_{L^\infty_{h}(h\mathbb{Z};L^2([-T,T]))} \lesssim \|u_{h,0}^\pm\|_{H^s(h\mathbb{Z})},$$

$$\|m(D_h)v_h^\pm\|_{L^2_{h}(h\mathbb{Z};L^\infty([-T,T]))} \lesssim \|u_{h,0}^\pm\|_{H^{s+s'(h\mathbb{Z})}}.$$

The discrete differential operator $\partial_h$ can be replaced by $\nabla_h$ or $\partial_h^+$. 

Our proof of Proposition 7.3 uses the standard iteration scheme (also known as the “Picard iteration method”) via the Fourier restriction norm method, and the main ingredients are the bilinear estimates established in Section 6 (in particular, Lemma 6.1, 6.3 and 6.4). Here, we present the proof of only the coupled FPU, because the proof of the decoupled one closely follows. Indeed, the latter is simpler owing to the absence of the coupled terms.
Proof of Proposition 7.3. Let $\theta \in C^\infty_c(\mathbb{R})$ be a time cut-off function satisfying $\theta(t) \equiv 1$ on $[-1, 1]$ and $\theta(t) = 0$ for $|t| > 2$. For sufficiently small $T \in (0, 1]$ to be chosen later, we define
\[
\Phi(u^+_{h}, u^-_{h}) = (\Phi^+(u^+_{h}), \Phi^-(u^-_{h}))
\]
by
\[
\Phi^\pm(u^\pm_{h}) := \theta(\frac{t}{T})S^\pm_h(t)u^\pm_{h} + \frac{\theta(t)}{4} \int_0^t S^\pm_h(t - t_1)\theta(\frac{t_1}{2T})\nabla_h\left\{ u^\pm_{h}(t_1) + e^{\pm \frac{2\pi}{h} \partial_t} u^\pm_{h}(t_1) \right\}^2 dt_1.
\]
Let $X^{s, b}_h$ denote the solution space for $(u^+_{h}, u^-_{h})$ equipped with the norm
\[
\|(u^+_{h}, u^-_{h})\|_{X^{s, b}_h} := \sum_{\pm} \|u^\pm_{h}\|_{X^{s, b}_h},
\]
where the exponent $b$ will be chosen later.

Lemma 3.12 (4) and (5) yield
\[
\|\Phi^\pm(u^\pm_{h})\|_{X^{s, b}_h} \leq CT^{\frac{1}{2} - b}R + CT^\delta \left\| \nabla_h\left\{ u^\pm_{h} + e^{\pm \frac{2\pi}{h} \partial_t} u^\pm_{h} \right\}^2 \right\|_{X^{s, b} - 1 + \delta} \tag{7.3}
\]
for some $C \geq 1$ and $\delta > 0$ (to be chosen later). Then, by applying Lemma 6.1, 6.3 and 6.4 (with $s' = s$) to the second term on the right-hand side of (7.3), we obtain
\[
\|\Phi^\pm(u^\pm_{h})\|_{X^{s, b}_h} \leq CT^{\frac{1}{2} - b}R + \tilde{C}T^\delta \left\| (u^+_{h}, u^-_{h})\right\|_{X^{s, b}_h}^2.
\]
Thus, by summing in $\pm$, we have
\[
\|\Phi(u^+_{h}, u^-_{h})\|_{X^{s, b}_h} \leq 2CT^{\frac{1}{2} - b}R + 2\tilde{C}T^\delta \left\| (u^+_{h}, u^-_{h})\right\|_{X^{s, b}_h}^2. \tag{7.4}
\]
Now, we choose $b$ and $\delta$ satisfying
\[
\frac{1}{2} < b < \frac{3}{4} - \delta \quad \text{and} \quad b - \frac{1}{2} < \delta < \frac{1}{4},
\]
and we take $T > 0$ such that
\[
32\tilde{C}T^{\frac{1}{2} + \delta - b}R \leq 1. \tag{7.5}
\]
Then, $\Phi$ maps from the set
\[
\mathcal{X} := \left\{ (u^+_{h}, u^-_{h}) \in X^{s, b}_h : \left\| (u^+_{h}, u^-_{h})\right\|_{X^{s, b}_h} \leq 4CT^{\frac{1}{2} - b}R \right\}
\]
to itself. Indeed, it follows from (7.4) that
\[
\|\Phi(u^+_{h}, u^-_{h})\|_{X^{s, b}_h} \leq 2CT^{\frac{1}{2} - b}R + 2\tilde{C}T^\delta \cdot (4CT^{\frac{1}{2} - b}R)^2
\]

\[
= 2CT^{\frac{1}{2} - b}R + 32\tilde{C}T^{\frac{1}{2} + \delta - b}R \cdot CT^{\frac{1}{2} - b}R
\]
\[
\leq 4CT^{\frac{1}{2} - b}R.
\]
We repeat the procedure for the difference. By Lemma 3.12 (4) and (5), it follows that
\[
\|\Phi^\pm(u^\pm_{h}) - \Phi^\pm(\tilde{u}^\pm_{h})\|_{X^{s, b}_h}
\]
\[
\leq CT^\delta \left\| \nabla_h\left( u^\pm_{h} + e^{\pm \frac{2\pi}{h} \partial_t} u^\pm_{h} \right)^2 - \nabla_h\left( \tilde{u}^\pm_{h} + e^{\pm \frac{2\pi}{h} \partial_t} \tilde{u}^\pm_{h} \right)^2 \right\|_{X^{s, b} - 1 + \delta}
\]
\[
= CT^\delta \left\| \nabla_h\left( (u^\pm_{h} + \tilde{u}^\pm_{h}) + e^{\pm \frac{2\pi}{h} \partial_t} (u^\pm_{h} + \tilde{u}^\pm_{h}) \right) \left( (u^\pm_{h} - \tilde{u}^\pm_{h}) + e^{\pm \frac{2\pi}{h} \partial_t} (u^\pm_{h} - \tilde{u}^\pm_{h}) \right) \right\|_{X^{s, b} - 1 + \delta}.
\]
Next, by applying Lemmas 6.1, 6.3, and 6.4 (with $s' = s$), we obtain
\[ \| \Phi^{\pm}(u_h^\pm) - \Phi^{\pm}(\tilde{u}_h^\pm) \|_{X^{s,b}} \leq CT^\delta \left( \sum_{\pm} \| u_h^\pm \|_{X^{s,b}} + \| \tilde{u}_h^\pm \|_{X^{s,b}} \right) \sum_{\pm} \| u_h^\pm - \tilde{u}_h^\pm \|_{X^{s,b}}. \]

Hence, if $(u_h^+, u_h^-), (\tilde{u}_h^+, \tilde{u}_h^-) \in \mathcal{X}$, then
\[ \| \Phi(u_h^+, u_h^-) - \Phi(\tilde{u}_h^+, \tilde{u}_h^-) \|_{X^{s,b}} = \sum_{\pm} \| \Phi^{\pm}(u_h^\pm) - \Phi^{\pm}(\tilde{u}_h^\pm) \|_{X^{s,b}} \leq 2 \cdot CT^\delta \cdot SCT^{\frac{1}{2}-b} R \cdot \| (u_h^+ - \tilde{u}_h^+, u_h^- - \tilde{u}_h^-) \|_{X^{s,b}} \quad (7.6) \]
\[ \leq \frac{1}{2} \| (u_h^+ - \tilde{u}_h^+, u_h^- - \tilde{u}_h^-) \|_{X^{s,b}}. \]

Therefore, we conclude that $\Phi$ is contractive on $\mathcal{X}$. Consequently, $(u_h^+, u_h^-)$ is a solution to the coupled FPU (2.8), which by uniqueness coincides with the solution in $C_t([-T, T]; L^2_\mathbb{Z}(h \mathbb{Z}))$, constructed in Proposition 4.1. Moreover, it satisfies the desired bound $\| u_h^\pm \|_{X^{s,b}} \leq 4CT^{\frac{1}{2}-b}R. \]

8. CONVERGENCE FROM FPU TO COUNTER-PROPAGATING KdVS

8.1. FROM COUPLED TO DECOPLED FPUS: PROOF OF PROPOSITION 2.2

This section is devoted to showing that solutions to the coupled system (2.8) approximate to those to the decoupled system (2.12) in $L^2_\mathbb{Z}(h \mathbb{Z})$ as $h \to \infty$.

The remainder of this section will be (roughly) presented as follows (see also Fig. 8.1). For given (slightly) regular initial data $u_{h,0}^\pm$ (in $H^s(h \mathbb{Z})$, $0 < s \leq 1$), we have $H^s(h \mathbb{Z})$ local solutions $u_h^\pm$ and $v_h^\pm$ (see Proposition 4.1). Then, a suitable choice of $0 < T \ll 1$ and uniform bounds of $u_h^\pm$ and $v_h^\pm$ (see Propositions 7.3 and 7.4) ensure that
\[ \| u_h^\pm(t) - v_h^\pm(t) \|_{L^2_\mathbb{Z}(h \mathbb{Z})} \lesssim O(h^s), \quad \text{for all} \quad |t| \leq T. \]

\[ (u_h^+, u_h^-) \text{ in } C_t([-T, T]; H^s(h \mathbb{Z})) \]

\[ (u_{h,0}^+, u_{h,0}^-) \text{ in } H^s(h \mathbb{Z}) \]

\[ \text{Proposition 4.1} \]

\[ \text{Proposition 2.2 with } h \to \infty \]

\[ (v_h^+, v_h^-) \text{ in } C_t([-T, T]; H^s(h \mathbb{Z})) \]

\[ (u_h^+, u_h^-) \approx (v_h^+, v_h^-) \text{ in } C_t([-T, T]; L^2_\mathbb{Z}(h \mathbb{Z})) \]

**Figure 8.1.** Schematic representation for decoupling (2.12) from (2.8).

**Remark 8.1.** As $h^2$ is involved in the higher-order remainder term in (2.8) (see Lemma B.1), the estimate of the higher-order term is not essential in the proof below; thus, we omit it.
Proof of Proposition 2.3. For given initial data $u^\pm_{h,0}$, let $u^\pm_h(t)$ (resp., $v^\pm_h(t)$) be the solution to the coupled FPU (resp., the decoupled FPU) constructed in Proposition 4.4 and let $T = T(R) > 0$ be the minimum of the existence times for two solutions. Moreover, Proposition 7.3 implies that the $X_{h,\pm}^{s,b}$ norms of the solutions are uniformly bounded by the size of the initial data, i.e.,

$$\|u^\pm_h\|_{X_{h,\pm}^{s,b}}, \|v^\pm_h\|_{X_{h,\pm}^{s,b}} \leq M = M(R). \tag{8.1}$$

Note that $M(R) = 4CT^{\frac{1}{2} - b}R$.

First, by subtracting (2.12) from (2.8), we write

$$u^\pm_h(t) - v^\pm_h(t) = \pm \frac{\theta(t)}{4} \int_0^t S^\pm_h(t - s)\partial_t \left\{ (u^\pm_h(t_1) + \varphi_h(t_1))\left( u^\pm_h(t_1) - v^\pm_h(t_1) \right) \right\} ds.$$

We take the $X_{h,\pm}^{0,b}$ norm\footnote{One can fix $b > \frac{1}{2}$ here, and $\delta > 0$ below such that the argument of the local well-posedness is valid.} on both sides. Similarly to the proof of Proposition 7.3, we apply Lemma 6.1 to the first integral and Lemma 6.3 and 6.4 to the second integral to obtain

$$\|u^\pm_h(t) - v^\pm_h(t)\|_{X_{h,\pm}^{0,b}} \leq \tilde{C}T^{\delta} \left( \|u^\pm_h\|_{X_{h,\pm}^{s,b}} + \|v^\pm_h\|_{X_{h,\pm}^{s,b}} \right) \|u^\pm_h - v^\pm_h\|_{X_{h,\pm}^{0,b}}$$

$$+ \tilde{C}h^sT^{\delta} \left( \|u^\pm_h\|_{X_{h,\pm}^{s,b}} \|u^\pm_h\|_{X_{h,\pm}^{s,b}} + \|u^\pm_h\|_{X_{h,\pm}^{s,b}}^2 \right),$$

for some uniform constant $\tilde{C} > 0$ as in (7.6). With the bounds (8.1), we extend the time interval $[-T, T]$ to be as long as $2CT^{\delta}M < \frac{1}{2}$. The local existence time $T > 0$ satisfying (7.5) enables us to conclude that

$$4CM^2h^s \geq \|u^\pm_h(t) - v^\pm_h(t)\|_{X_{h,\pm}^{0,b}} \gtrsim \|u^\pm_h(t) - v^\pm_h(t)\|_{C_tL^2_x},$$

where the embedding $X_{h,\pm}^{0,b} \hookrightarrow C_t([-T, T]; L^2_x(h\mathbb{Z}))$ is used in the last step (see Lemma 3.12 (3)).

8.2. From decoupled FPU to KdV: Proof of Proposition 2.3. We now prove that solutions to the decoupled FPU (2.12) can be approximated by those to KdV (2.13) as $h \to 0$ (Proposition 2.3). To compare the two solutions, we employ the linear interpolation $l_h$ defined as in (3.9) and the spacetime norm $S([-T,T])$ given by (5.8).
Proof of Proposition 7.3. Let $T > 0$ be a sufficiently small number to be chosen later independently of $h \in (0,1]$. Using (2.12) and (2.13), we write\footnote{In what follows, as mentioned in Remark 2.4, we denote the solutions to KdVs (2.13) by $w_h^\pm$, even if they are posed on $\mathbb{R}$.}

\[
\begin{align*}
l_h v_h^\pm(t) - w_h^\pm(t) &= \left\{ l_h S_h^\pm(t) u_{h,0}^\pm - S_h^\pm(t) l_h u_{h,0}^\pm \right\} \\
&= \frac{1}{4} \int_0^T (l_h S_h^\pm(t - t_1) - S_h^\pm(t - t_1) l_h) \nabla_h (v_h^\pm(t_1)^2) dt_1 \\
&= \frac{1}{4} \int_0^T S_h^\pm(t - t_1) \left\{ l_h \nabla_h (v_h^\pm(t_1)^2) - \partial_x ((l_h v_h^\pm(t_1)^2)) \right\} dt_1 \\
&= \frac{1}{4} \int_0^T S_h^\pm(t - t_1) \partial_x \left\{ (l_h v_h^\pm(t_1)^2) - w_h^\pm(t_1)^2 \right\} dt_1.
\end{align*}
\]

Then, Propositions 5.10 and 5.4 enable us to obtain

\[
\begin{align*}
\| l_h v_h^\pm - w_h^\pm \|_{S([-T, T])} &\lesssim h^{\frac{5}{2}} \| u_{h,0} \|_{H^s_h} + h^{\frac{5}{2}} \| \nabla_h (v_h^\pm) \|_{L^1_t H^s_x} \\
&\quad + \left\| l_h \nabla_h (v_h^\pm)^2 - \partial_x ((l_h v_h^\pm)^2) \right\|_{L^1_t L^2_x} \\
&\quad + \left\| \partial_x \left\{ (l_h v_h^\pm)^2 - (w_h^\pm)^2 \right\} \right\|_{L^1_t L^2_x} \\
&=: I_1 + I_2 + I_3 + I_4.
\end{align*}
\]

For $I_2$, we apply the Leibniz rule (3.5) and the Hölder inequality,

\[
I_2 \leq 2h^{\frac{5}{2}} \left\| \nabla_h v_h^\pm(t) \cos \left( \frac{-i \partial_h}{2h} \right) \right\|_{L^1_t H^s_x} \\
\lesssim h^{\frac{5}{2}} T^{\frac{3}{2}} \left\| \partial_h \left\{ \nabla_h v_h^\pm(t) \cos \left( \frac{-i \partial_h}{2h} \right) \right\} \right\|_{L^1_t L^2_x}.
\]

Since the operator $\cos \left( \frac{-i \partial_h}{2h} \right)$ is bounded in $L^2_h$ (independent of $h$), from Lemma 6.5 and Proposition 7.3, we obtain

\[
I_2 \lesssim h^{\frac{5}{2}} T^{\frac{3}{2}} \left\| v_h^\pm(t) \right\|_{X_h^{s,b}} \left\| v_h^\pm(t) \right\|_{X_h^{s,b}} \lesssim T^{\frac{3}{2}} h^{\frac{5}{2}} \| u_{h,0} \|_{H^s}.
\]

For $I_3$, Propositions 3.9 and 3.10 immediately yield

\[
I_3 \leq \left\| l_h \nabla_h (v_h^\pm)^2 - \partial_x l_h (v_h^\pm)^2 \right\|_{L^1_t L^2_x} + \left\| \partial_x \left\{ l_h (v_h^\pm)^2 - (l_h v_h^\pm)^2 \right\} \right\|_{L^1_t L^2_x} \\
\lesssim h \left\{ \| (v_h^\pm)^2 \|_{L^1_t H^s_x} + \| (l_h v_h^\pm)^2 \|_{L^1_t L^2_x} \right\}. \tag{8.4}
\]

For the first term in (8.4), we apply (3.5) twice to get

\[
\nabla_h^2 \left\{ (v_h^\pm)^2 \right\} = 2 \nabla_h v_h^\pm \cdot \cos^2 \left( \frac{-i \partial_h}{2h} \right) v_h^\pm + 2 \left\{ \cos \left( \frac{-i \partial_h}{2h} \right) \nabla_h v_h^\pm \right\}^2.
\]
Then, by (3.3), the Hölder inequality and Corollary 7.6, we estimate
\[
\| (v_k^±)^2 \|_{L_t^1 H_x^2} \sim \| \nabla_h^2 \{ (v_k^±)^2 \} \|_{L_t^1 L_x^2}
\]
\[
\lesssim T^{-1/2} \| \nabla_h^2 v_k^± \|_{L_t^2 L_x^\infty} \| \cos(\frac{-ih\partial_h}{2})v_k^± \|_{L_t^\infty L_x^2}
+ T^{-1/2} \| \nabla_h \cos(\frac{-ih\partial_h}{2})v_k^± \|_{L_t^1 L_x^\infty} \| \nabla_h \cos(\frac{-ih\partial_h}{2})v_k^± \|_{L_t^\infty L_x^2}
\]
\[
\lesssim T^{-1/2} \| u_{h,0}^± \|_{H^1} \| u_{h,0}^± \|_{H^s}.
\]
Similarly, we estimate the second term on the right-hand side of (8.4) as
\[
\| (\partial_h^+ v_k^±)^2 \|_{L_t^1 L_x^2} \lesssim T^{-1/2} \| \partial_h^+ v_k^± \|_{L_t^1 L_x^\infty} \| \partial_h^+ v_k^± \|_{L_t^\infty L_x^2}
\]
\[
\lesssim T^{-1/2} \| u_{h,0}^± \|_{H^s} \| u_{h,0}^± \|_{H^1}.
\]
Thus, by Lemma 3.3, we conclude that
\[
I_3 \lesssim h^s T^\frac{3}{2} \| u_{h,0}^± \|_{H^s}^2. \tag{8.5}
\]
Before dealing with $I_4$, we first observe that a direct computation gives
\[
\sup_{x \in R} |\partial_x l_h f_h(x)| = \sup_{x_m \in h\mathbb{Z}} \| (\partial_h^+ f_h)(x_m)\|
\]
and
\[
\int_{\mathbb{R}} (\sup_t |l_h f_h(t, x)|)^2 \, dx = \sum_{x_m \in h\mathbb{Z}} \int_0^h (\sup_t |f_h(x_m) + (\partial_h^+ f_h)(x_m) \cdot x|)^2 \, dx
\]
\[
\lesssim h \sum_{x_m \in h\mathbb{Z}} (\sup_t |f_h(t, x_m)|)^2 + h^3 \sum_{x_m \in h\mathbb{Z}} (\sup_t |\partial_h^+ f_h|(t, x_m)),
\]
which implies that
\[
\| \partial_x l_h f_h \|_{L_t^1 L_x^\infty} = \| \partial_h^+ f_h \|_{L_t^1 L_x^\infty}
\]
and
\[
\| l_h f_h \|_{L_t^2 L_x^\infty} \lesssim \| f_h \|_{L_t^2 L_x^\infty} + h \| \partial_h^+ f_h \|_{L_t^2 L_x^\infty},
\]
respectively. With these observations, by the Hölder inequality and Corollary 5.3 and 5.5, we obtain
\[
I_4 \leq T^\frac{3}{4} \left( \| \partial_x l_h v_k^± \|_{L_t^1 L_x^\infty} + \| \partial_x w_k^± \|_{L_t^1 L_x^\infty} \right) \| l_h v_k^± - w_h^± \|_{L_t^2 L_x^2}
\]
\[
+ T^\frac{3}{4} \left( \| l_h v_k^± \|_{L_t^2 L_x^\infty} + \| w_k^± \|_{L_t^2 L_x^\infty} \right) \| \partial_x (l_h v_k^± - w_h^±) \|_{L_t^2 L_x^2}
\]
\[
\lesssim T^\frac{3}{4} \left( \| v_k^± \|_{X^h \pm} + h \| \partial_h^± v_k^± \|_{X^h \pm} + \| w_k^± \|_{X^h \pm} \right) \| l_h v_k^± - w_h^± \|_{S([-T, T])}.
\]
Owing to Theorem 7.1 and Proposition 7.3 in addition to Lemma 3.3, by choosing sufficiently small $0 < T \ll 1$, we have
\[
I_4 \leq \frac{1}{2} \| l_h v_k - w_h^± \|_{S([-T, T])}. \tag{8.6}
\]
Finally, going back to (8.2), we employ (8.3), (8.5) and (8.6), as well as Lemma 3.3, to complete the proof.
8.3. Proof of continuum limit. Propositions 2.2 and 2.3 do not immediately guarantee Theorem 1.2 owing to a lack of commutativity between the linear interpolation and translation operators in the following sense

$$\ell_h e^{\pm \frac{i}{\kappa^2} \partial_x} u_h^\pm \neq e^{\pm \frac{i}{\kappa^2} \partial_x} \ell_h u_h^\pm.$$ 

However, for every $t = h^3 k \in [-T, T]$, $k \in \mathbb{Z}$,

$$\ell_h (e^{\pm i \kappa h \partial_x} u_h^\pm) = e^{\pm i \kappa h \partial_x} \ell_h (u_h^\pm)$$ 

in $L^2$ holds, owing to Lemma 5.1, thus, Propositions 2.2 and 2.3 ensure Theorem 1.2. To complete the proof of Theorem 1.2, it is necessary to extend our result at $t = h^3 k$ for all $t \in [-T, T]$. For given $t \in [-T, T]$, there exists $k \in \mathbb{Z}$ such that $t \in [h^3 k, h^3 (k + 1))$. Since

$$\| (\ell_h \tilde{r}_h) (t, x) - w^+_h (t, x - \frac{t}{h^3}) - w^+_h (t, x + \frac{t}{h^3}) \|_{L^2_x (\mathbb{R})} \leq \| (\ell_h \tilde{r}_h) (t, x) - w^+_h (t, x - \frac{t}{h^3}) \|_{L^2_x (\mathbb{R})} + \| (\ell_h \tilde{r}_h) (t, x) - w^+_h (t, x + \frac{t}{h^3}) \|_{L^2_x (\mathbb{R})},$$

we only deal with the “+” term, since the other part follows analogously. A straightforward computation yields

$$\| (\ell_h \tilde{r}_h) (t, x) - w^+_h (t, x - \frac{t}{h^3}) \|_{L^2_x (\mathbb{R})} = \| \ell_h e^{\pm \frac{i}{\kappa^2} \partial_x} u_h^\pm (t, x) - e^{\pm \frac{i}{\kappa^2} \partial_x} w^+_h (t, x) \|_{L^2_x (\mathbb{R})} \leq \| \ell_h e^{\pm \frac{i}{\kappa^2} \partial_x} u_h^\pm (t, x) - (\ell_h e^{- h k \partial_x} u_h^+ (h^3 k, x)) \|_{L^2_x (\mathbb{R})} + \| \ell_h e^{- h k \partial_x} u_h^+ (h^3 k, x) - w^+_h (h^3 k, x) \|_{L^2_x (\mathbb{R})} + \| e^{- h k \partial_x} u_h^+ (h^3 k, x) - e^{- \frac{i}{\kappa^2} \partial_x} w^+_h (t, x) \|_{L^2_x (\mathbb{R})},$$

$$=: I + II + III.$$

Propositions 2.2 and 2.3 show that $II \lesssim h^{\frac{2s}{3}}$.

For $I$, we further split it by

$$\| e^{- \frac{i}{\kappa^2} \partial_x} u_h^+ (t) - e^{- h k \partial_x} u_h^+ (t) \|_{L^2_x (h^3)} + \| e^{- h k \partial_x} u_h^+ (t) - e^{- h k \partial_x} u_h^+ (h^3 k) \|_{L^2_x (h^3)} =: I_1 + I_2.$$

Here, we use the boundedness of the linear interpolation operator. Note that

$$| e^{- \frac{i}{\kappa^2} \xi} - e^{- i h k \xi} | \lesssim \frac{1}{h^2} | t - h^3 k | | \xi | \lesssim h | \xi |, \quad t \in [h^3 k, h^3 (k + 1)).$$

Applying Plancherel theorem, the continuity of $e^{i \theta}$ and Lemma 3.3 to $I_1$, we obtain

$$I_1 \lesssim h \| u_h^+ (t) \|_{H^1_x (h^3)} \lesssim h^s \| u_h^+ (t) \|_{H^s_x (h^3)},$$

for $\frac{3}{4} < s \leq 1$. For $I_2$, since

$$u_h^+ (t) = S^+_h (t - h^3 k) u_h^+ (h^3 k) + \frac{1}{4} \int_{h^3 k}^t S^+_h (t - t_1) \nabla_h \left\{ u_h^+ (t_1) + e^{\frac{2 i}{\kappa^2} \partial_x} u_h^- (t_1) \right\}^2 + h^2 e^{\frac{i}{\kappa^2} \partial_x} R (t_1) dt_1,$$

and it suffices to deal with

$$\| S^+_h (t - h^3 k) u_h^+ (h^3 k) - u_h^+ (h^3 k) \|_{L^2_x (h^3)} \leq (8.7)$$
and the nonlinear terms. Mixed and $u_h^-$ terms and the higher-order term in the nonlinear part can be controlled by at least $h^s$, owing to Lemmas 6.3, 6.4 and B.1. Meanwhile, the $u_h^+$ quadratic term can be roughly estimated by

$$|t - h^3 k| h^{-1} \| u_h^+(t) \|_{L_x^\infty L_t^1}^2 \lesssim h^\frac{3}{2} \| u_h^+(t) \|_{L_x^\infty L_t^1}^2,$$

which itself is sufficient. Moreover, (8.7) is bounded by $h^s \| u_h^+/h^3 k \|_{H^s(h\mathbb{Z})}$ analogously to $I_1$, owing to

$$|S_k^+(t - h^3 k) - 1| = |e^{-\frac{1}{h^3} (\xi - \frac{2}{5} \sin(\frac{\gamma^3}{h}))} - 1| \lesssim h \left( \| \xi \| + \left\| \frac{2}{h} \sin \left( \frac{h\xi}{2} \right) \right\| \right), \quad t \in [h^3 k, h^3(k + 1))$$

and Lemma 3.1. An analogous argument is available for the estimate of $III$.

**APPENDIX A. FAILURE OF THE LINEAR ESTIMATE**

In subsection 5.2 we measured the size of linear interpolation of the FPU flows in $C_1([-T, T] : H^s(h\mathbb{Z}))$ in order to approximate the FPU flows by the Airy flows. More precisely, the crucial estimates were that for $0 \leq s \leq 1$,

$$\| l_h S_h^+(t) f_h \|_{C_1([0, T]; H^s(h\mathbb{Z}))} \lesssim \| l_h f_h \|_{H^s(h\mathbb{Z})} \lesssim \| f_h \|_{H^s(h\mathbb{Z})}.$$

However, such uniform estimates fail if we consider instead the $X^{s,b}$ spaces associated to KdVs (7.1) as approximation spaces, which means that even though FPU and KdVs are shown to be well-posed in $L^2$ via $X^{s,b}$, justification of approximation from FPU to KdVs via $X^{s,b}$ is nontrivial.

**Proposition A.1.** Let $0 \leq s \leq 1$ and $b > 0$. Then,

$$\sup_{h > 0, f_h \in H^s(h\mathbb{Z})} \frac{\| \theta(t) l_h S_h^+(t) f_h \|_{X^{s,b}_{+}}}{\| f_h \|_{H^s(h\mathbb{Z})}} = \infty,$$

where $X^{s,b}_{\pm}$ is defined as in (5.1).

**Proof.** We claim that there exist a constant $C_b > 0$ independent of $h > 0$ such that

$$\| \theta(t) l_h S_h^+(t) f_h \|_{X^{s,b}_{+}}^2 \gtrsim C_b \| f_h \|_{H^s}^2.$$

for $f_h \in H^s_h$ satisfying $\text{supp} \mathcal{F}(f_h) \subset \{ \xi \in \mathbb{T}_h : |\xi| \geq \frac{\pi}{2h} \}$. Then (A.1) immediately follows.

We prove only (A.2) for the $+$ case, since the other case can be treated similarly. Using Lemma 5.12 we compute

$$\| \theta(t) l_h S_h^+(t) f_h \|_{X^{s,b}_{+}}^2 = \sum_{m \in \mathbb{Z}} \int_{\gamma_{m,h} + [-\frac{\pi}{T}, \frac{\pi}{T}]} \| \langle \xi \rangle^s \left( \tau - \frac{\xi^3}{24} \right)^b \mathcal{F}_{t,x} \theta(t) l_h S_h^+(t) f_h(\tau, \xi) \|_{L_x^2}^2 d\xi$$

$$= \sum_{m \in \mathbb{Z}} \int_{\frac{\pi}{T}, \frac{\pi}{T}} \| \tilde{\theta}(\tau) \left( \tau - \frac{1}{24} (\xi + \gamma_{m,h})^3 + s_h^+(\xi) \right) \|_{L_x^2}^2 \langle \xi + \gamma_{m,h} \rangle^{2s} \mathcal{L}_h(\xi + \gamma_{m,h})^2 |\mathcal{F}(f_h)(\xi)|^2 d\xi.$$
for \( \gamma_{m,h} := \frac{2m\pi}{h} \). First, let us compute the \( L^2 \) norm. A direct computation gives
\[
\frac{1}{24}(\xi + \gamma_{m,h})^3 - s_h^+(\xi) = \frac{\pi^3}{3} \left( \frac{m}{h} \right)^3 + \frac{\pi^2\xi}{6} \left( \frac{m}{h} \right)^2 + \frac{\pi\xi^2}{4} \left( \frac{m}{h} \right) + \frac{\xi^3}{24} + \frac{1}{h^2} \left( \xi - \frac{2}{h} \sin \left( \frac{h\xi}{2} \right) \right)
\]
and it is easy to verify that
\[
\left| \frac{\pi^2\xi}{6} \left( \frac{m}{h} \right)^2 + \frac{\pi\xi^2}{4} \left( \frac{m}{h} \right) + \frac{\xi^3}{24} \right| \lesssim \frac{m^2}{h^3}, \quad \text{for all } \xi \in \mathbb{T}_h,
\]
which indicates that \( \frac{\pi^3}{3} \left( \frac{m}{h} \right)^3 \) is the dominant part in \( \frac{1}{24}(\xi + \gamma_{m,h})^3 - s_h^+(\xi) \). In particular, there exists \( m_0 \gg 1 \), independent of \( h \), such that for \( m \leq -m_0 \),
\[
-\frac{1}{24}(\xi + \gamma_{m,h})^3 + s_h^+(\xi) \gtrsim \left( \frac{|m_0|}{h} \right)^3 \gg 1, \quad \text{for all } \xi \in \mathbb{T}_h. \quad (A.3)
\]
Using the above-mentioned observation, we have for \( m \leq m_0 \)
\[
\left\| \hat{\theta}(\tau) \left\langle \tau - \frac{1}{24}(\xi + \gamma_{m,h})^3 + s_h^+(\xi) \right\rangle \right\|^2_{L^2_{\mathbb{T}_h}} \gtrsim \int_{0}^{\infty} |\hat{\theta}(\tau)|^2 \left\langle \tau - \frac{1}{24}(\xi + \gamma_{m,h})^3 + s_h^+(\xi) \right\rangle^{2b} d\tau
\]
\[
\gtrsim \left( \frac{|m_0|}{h} \right)^{6b} \int_{0}^{\infty} |\hat{\theta}(\tau)|^2 d\tau
\]
\[
\gtrsim \left( \frac{|m_0|}{h} \right)^{6b},
\]
which implies that
\[
\left\| \theta(t) S_h^+(t) f_h \right\|_{X_{\theta,t}^+, h} \gtrsim \left( \frac{|m_0|}{h} \right)^{6b} \sum_{m \leq -m_0} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \left\langle \xi + \gamma_{m,h} \right\rangle^{2s} L_h(\xi + \gamma_{m,h})^2 |F_h(f_h)(\xi)|^2 d\xi
\]
\[
\gtrsim \left( \frac{|m_0|}{h} \right)^{6b} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \left( \frac{4\sin \left( \frac{h\xi}{2} \right)}{h^2(\xi + \gamma_{m,h})^2} \right)^2 |F_h(f_h)(\xi)|^2 d\xi.
\]
Since
\[
|\xi| \lesssim |\xi + \gamma_{m_0,h}| \lesssim |\gamma_{m_0,h}| \quad \text{and} \quad \sin^2 \left( \frac{h\xi}{2} \right) \geq \frac{1}{2}
\]
for all \( \xi \in \text{supp} F_h(f_h) \), we conclude
\[
\left\| \theta(t) S_h^+(t) f_h \right\|_{X_{\theta,t}^+, h} \gtrsim \left( \frac{|m_0|}{h} \right)^{6b} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \left( \xi + \gamma_{m_0,h} \right)^{-4} |F_h(f_h)(\xi)|^2 d\xi
\]
\[
\gtrsim \left( \frac{|m_0|}{h} \right)^{6b-4} \| f_h \|_{H^6_h}^2.
\]
APPENDIX B. ANALYSIS FOR GENERAL NONLINEARITIES

This appendix is devoted to some estimates for the higher-order remainder term introduced in Section 2 to complete our analysis established in Sections 4 and 7.2. For any real number \( \rho \in \mathbb{R} \), we write \( \rho^+ \) if there exists a small \( 0 < \epsilon \ll 1 \) such that \( \rho^+ = \rho + \epsilon \). Analogously, we use \( \rho^- \). The main estimate dealt with in this section is as follows:

**Lemma B.1.** Let \( 0 \leq s \leq 1 \) and \( 0 < h \leq 1 \) be given. Assume that

\[
\| u_h^\pm \| \left\| \mathcal{F}^s \right\|_{X^{s,h}_h} \leq M,
\]

for some constant \( M > 0 \). Then, for \( \mathcal{R} \) as in (2.1), we have

\[
\left\| \int_0^t S_h^\pm (t - t_1) \nabla_h e^{\pm \frac{eh}{h^2} \partial_h h^2 \mathcal{R}(t_1)} dt_1 \right\| \left\| \mathcal{F}^s \right\|_{X^{s,h}_h} \lesssim \frac{h^{\min\left(\frac{3}{2} - s, \frac{3}{4} + s\right)}}{h^{\frac{3}{4}}} M^3 \sup_{|r| \leq C h^{\frac{3}{2}} M} |V^{(4)}(r)|,
\]

where the constant \( C \) in supremum depends only on \( \frac{1}{2} \).

**Remark B.2.** As seen in the proofs of Propositions 4.1, \( M \) depends on the initial condition. Meanwhile, in the proofs of Propositions 7.3 and 2.2, \( M \) depends not only on the initial condition but also on the local existence time, especially, \( T_0^+ \). However, owing to \( T_0^+ \), the right-hand side of (B.1) can be sufficiently small by choosing a suitable time \( T \) independent of \( h \).

**Remark B.3.** Lemma B.1 indeed completes the proof of Proposition 7.3.

**Remark B.4.** Together with the embedding property (Lemma 3.12 (3)), Lemma B.1 completes the proofs of Propositions 4.1 and 2.2.

**Remark B.5.** Lemma B.1 ensures that the higher-order term in (2.8) is indeed the error term as \( h \to 0 \) in the proof of Proposition 2.2. More precisely, in a strong contrast to the quadratic error terms

\[
\int_0^t S_h^\pm (t - t_1) \nabla_h \left( 2 u_h^\pm (t_1) (e^{\pm \frac{2t_1}{h^2} \partial_h h^2 \mathcal{R}^\pm (t_1)}) + (e^{\pm \frac{2t_1}{h^2} \partial_h h^2 \mathcal{R}^\pm (t_1)})^2 \right) dt_1
\]

in the proof of Proposition 2.2 (see also Lemmas 6.3 and 6.4), Lemma B.1 ensures that the higher-order term itself in (2.8) can be understood as a strong error term as \( h \to 0 \) in the sense that the smoothness condition on the data is not necessary.

**Proof of Lemma B.1.** By assumption, we consequently have

\[
\| u_h^\pm \| \left\| \mathcal{F}^s \right\|_{C_1 H^s_x} \lesssim M \quad \text{and} \quad \| \tilde{r}_h \| \left\| \mathcal{F}^s \right\|_{C_1 H^s_x} \lesssim M.
\]

By (3.12), we estimate the higher-order remainder

\[
\left\| \int_0^t S_h^\pm (t - t_1) \nabla_h e^{\pm \frac{1}{h^2} \partial_h h^2 \mathcal{R}(t_1)} dt_1 \right\| \left\| \mathcal{F}^s \right\|_{X^{s,h}_h} \lesssim \| \nabla_h e^{\pm \frac{1}{h^2} \partial_h h^2 \mathcal{R}(t)} \| \left\| \mathcal{F}^s \right\|_{X^{s,h}_h}.
\]

Interpolating the dualization of the Strichartz estimates (Corollary 5.3), i.e.,

\[
\| u_h \| \left\| \mathcal{F}^s \right\|_{X^{s,h}_h} \lesssim \| \nabla_h \| \left\| \mathcal{F}^s \right\|_{L^4_x} \leq \| \mathcal{F}^s \|_{L^4_x} \frac{h^{\frac{3}{4}}}{h^{\frac{3}{4}}} M^3 \sup_{|r| \leq C h^{\frac{3}{2}} M} |V^{(4)}(r)|,
\]

we conclude the desired estimate (B.1).
with the trivial identity \( \| u_h \|_{X_{h,\pm}^{0,0}} = \| u_h \|_{L^2_t L^2_x} \), we have
\[
\| u_h \|_{X_{h,\pm}^{0,-}(\frac{1}{4}^-)} \lesssim \| \nabla_h |^{-\frac{1}{4}^-} u_h \|_{L^4_t L^4_x}.
\]
Using this bound and the Hölder inequality, we obtain
\[
\| \nabla_h e^{\pm \frac{1}{h^2} \partial_h h^2 R(t)} \|_{X_{h,\pm}^s(\frac{1}{4}^-)} \\
\lesssim h^{(\frac{1}{4}^-) - s} \left\| e^{\pm \frac{1}{h^2} \partial_h R(t)} \right\|_{L^4_t L^4_x} \\
\lesssim h^{(\frac{1}{4}^-) - s} \left\| \left( e^{\pm \frac{1}{h^2} \partial_h V(4) (h^2 \tilde{r}_h^s)} \right) \cdot \left( e^{\pm \frac{1}{h^2} \partial_h \tilde{r}_h} \right) \right\|_{L^4_t L^4_x} \\
\lesssim h^{(\frac{1}{4}^-) - s} \left\| T^4 \left\| e^{\pm \frac{1}{h^2} \partial_h V(4) (h^2 \tilde{r}_h^s)} \right\|_{L^\infty_t L^\infty_x} \right\|_{L^4_t L^4_x} h^{\frac{3}{4}} \left\| \tilde{r}_h \right\|_{L^2_t L^2_x}.
\]
By unitarity (with the algebra in footnote 1), we remove the translation operator as follows:
\[
\left\| e^{\pm \frac{1}{h^2} \partial_h V(4) (h^2 \tilde{r}_h^s)} \right\|_{L^\infty_t L^\infty_x} \lesssim \left\| V(4) (h^2 \tilde{r}_h^s) \right\|_{L^\infty_t L^\infty_x} \left\| \tilde{r}_h \right\|_{L^\infty_t L^\infty_x}.
\]
By assumption, we have
\[
\| h^2 \tilde{r}_h^s \|_{C_t L^\infty_x} \leq \| h^2 \tilde{r}_h \|_{C_t L^\infty_x} \leq h^{\frac{3}{4}} \left\| \tilde{r}_h \right\|_{C_t L^\infty_x} \leq \frac{1}{2} \left\{ \| u^+_h \|_{C_t L^\infty_x} + \| u^-_h \|_{C_t L^\infty_x} \right\} \leq C h^{3/2} M.
\]
Hence, it follows that
\[
\| V(4) (h^2 \tilde{r}_h^s) \|_{L^\infty_t L^\infty_x} \leq \sup_{|r| \leq C h^{3/2} M} \| V(4) (r) \| < \infty.
\]
For \( \| e^{\pm \frac{1}{h^2} \partial_h \tilde{r}_h} \|_{L^\infty_t L^\infty_x} \), if \( 0 \leq s \leq \frac{1}{2} \), then by the Sobolev inequality, unitarity and Lemma
\[
\| e^{\pm \frac{1}{h^2} \partial_h \tilde{r}_h} \|_{L^\infty_x} \lesssim \| e^{\pm \frac{1}{h^2} \partial_h \tilde{r}_h} \|_{H^s} \lesssim h^{\frac{1}{4} - \frac{s}{4}} \| \tilde{r}_h \|_{H^s} \lesssim h^{\frac{1}{4} - \frac{s}{4}} M.
\]
Meanwhile, if \( \frac{1}{2} < s \leq 1 \), then
\[
\| e^{\pm \frac{1}{h^2} \partial_h \tilde{r}_h} \|_{L^\infty_x} \lesssim \| e^{\pm \frac{1}{h^2} \partial_h \tilde{r}_h} \|_{H^s} = \| \tilde{r}_h \|_{H^s} \lesssim M.
\]
Therefore, by combining all these results, we complete the proof of \((B.1)\). \(\square\)

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Department of Mathematics, Chung-Ang University, Seoul 06974, Korea
E-mail address: yhhong@cau.ac.kr

Facultad de Matemáticas, Pontificia Universidad Católica de Chile and Institute of Pure and Applied Mathematics, Jeonbuk National University
E-mail address: chkwak@mat.uc.cl

Korea Institute for Advanced Study, Seoul 20455 and Institute of Pure and Applied Mathematics, Jeonbuk National University, Jeonju 54896, Korea
E-mail address: maticionych@kias.re.kr