MAPPINGS BETWEEN MODULE LATTICES

Patrick F. Smith

Received: 7 October 2013
Communicated by Edmund Puczylowski

Dedicated to the memory of Professor Efraim P. Armendariz

Abstract. We examine the properties of certain mappings between the lattice of ideals of a commutative ring $R$ and the lattice of submodules of an $R$-module $M$, in particular considering when these mappings are lattice homomorphisms. We prove that the mapping $\lambda$ from the lattice of ideals of $R$ to the lattice of submodules of $M$ defined by $\lambda(B) = BM$ for every ideal $B$ of $R$ is a (lattice) isomorphism if and only if $M$ is a finitely generated faithful multiplication module. Moreover, for certain but not all rings $R$, there is an isomorphism from the lattice of ideals of $R$ to the lattice of submodules of an $R$-module $M$ if and only if the mapping $\lambda$ is an isomorphism.

Mathematics Subject Classification (2010): 06B99, 13F05, 13E05, 13C99

Keywords: Lattice homomorphism, commutative ring, Prufer domain, multiplication module, Noetherian ring.

1. Introduction

This paper is concerned with mappings, in particular homomorphisms, between lattices. Let $L$ and $L'$ be lattices. As usual, given $a$ and $b$ in $L$, the least upper bound and greatest lower bound of $a$ and $b$ are denoted by $a \lor b$ and $a \land b$, respectively. Given mappings $\varphi$ and $\theta$ from the lattice $L$ to the lattice $L'$ we define $\varphi \leq \theta$ provided $\varphi(a) \leq \theta(a)$ for all $a \in L$. Clearly, $\varphi \leq \theta$ and $\theta \leq \varphi$ together imply $\varphi = \theta$.

We begin with a very simple result.

Lemma 1.1. Let $L$, $L_1$ and $L_2$ be lattices, let $\varphi$, $\varphi_1$ and $\varphi_2$ be mappings from $L$ to $L_1$ and let $\theta$, $\theta_1$ and $\theta_2$ be mappings from $L_1$ to $L_2$ such that $\varphi_1 \leq \varphi_2$, $\theta_1 \leq \theta_2$ and $\theta(a) \leq \theta(b)$ for all $a, b \in L_1$ with $a \leq b$. Then $\theta \varphi_1 \leq \theta \varphi_2$ and $\theta_1 \varphi \leq \theta_2 \varphi$.

Proof. Let $a \in L$. Then $\varphi_1(a) \leq \varphi_2(a)$ and hence

$$\theta \varphi_1(a) = \theta(\varphi_1(a)) \leq \theta(\varphi_2(a)) = \theta \varphi_2(a).$$

On the other hand,

$$\theta_1 \varphi(a) = \theta_1(\varphi(a)) \leq \theta_2(\varphi(a)) = \theta_2 \varphi(a).$$

The result follows. □
A mapping \( \varphi \) from a lattice \( L \) to a lattice \( L' \) is a homomorphism provided
\[
\varphi(a \vee b) = \varphi(a) \vee \varphi(b) \quad \text{and} \quad \varphi(a \wedge b) = \varphi(a) \wedge \varphi(b),
\]
for all \( a, b \in L \). A bijective (respectively, injective, surjective) homomorphism is called an isomorphism (respectively, monomorphism, epimorphism).

The next result is absolutely standard and easy to prove.

**Lemma 1.2.** The following statements are equivalent for a bijection \( \varphi \) from a lattice \( L \) to a lattice \( L' \).

1. \( \varphi \) is an isomorphism.
2. \( \varphi(a \vee b) = \varphi(a) \vee \varphi(b) \) for all \( a, b \in L \).
3. \( \varphi(a \wedge b) = \varphi(a) \wedge \varphi(b) \) for all \( a, b \in L \).

Moreover, in this case the inverse mapping \( \varphi^{-1} : L' \to L \) is also an isomorphism.

Throughout this note all rings will be commutative with identity and all modules will be unital. Let \( R \) be a ring and \( M \) be any \( R \)-module. The collection of submodules of \( M \) form a lattice which we shall denote by \( \mathcal{L}(R \cdot M) \) with respect to the following definitions:
\[
L \vee N = L + N \quad \text{and} \quad L \wedge N = L \cap N,
\]
for all submodules \( L \) and \( N \) of \( M \). Note that \( \mathcal{L}(R \cdot M) \) is a lattice with least element the zero submodule, greatest element \( M \) and, for any given submodules \( L \) and \( N \) of \( M \),
\[
L \leq N \text{ in } \mathcal{L}(R \cdot M) \iff L \subseteq N \text{ in } M.
\]
In particular, we shall denote the lattice \( \mathcal{L}(R) \) of ideals of \( R \) by \( \mathcal{L}(R) \). We shall be interested in mappings between \( \mathcal{L}(R) \) and \( \mathcal{L}(R \cdot M) \).

Let \( R \) be a ring and \( M \) an \( R \)-module. Let \( L \) and \( N \) be submodules of \( M \). Then \( (L : R \cdot N) \) will denote the set of elements \( r \in R \) such that \( rN \subseteq L \). Note that \( (L : R \cdot N) \) is an ideal of \( R \). In particular, \( (0 : R \cdot M) \) is the annihilator of \( M \) in \( R \) and we shall denote it simply by \( \text{ann}_R(M) \). As usual, \( M \) is called faithful in case \( \text{ann}_R(M) = 0 \).

Define a mapping \( \lambda : \mathcal{L}(R) \to \mathcal{L}(R \cdot M) \) by
\[
\lambda(B) = BM,
\]
for all ideals \( B \) of \( R \) and define a mapping \( \mu : \mathcal{L}(R \cdot M) \to \mathcal{L}(R) \) by
\[
\mu(N) = (N : R \cdot M),
\]
for every submodule \( N \) of \( M \). Note that \( \lambda(B) \leq \lambda(C) \) for all ideals \( B, C \) of \( R \) with \( B \leq C \) and \( \mu(L) \leq \mu(N) \) for all submodules \( L, N \) of \( M \) with \( L \leq N \). Note further that for each ideal \( B \) of \( R \),
\[
B \subseteq (BM : R \cdot M) = \mu(\lambda(B)),
\]
and thus
\[ 1 \leq \mu \lambda. \]

On the other hand, for each submodule \( N \) of \( M \),
\[ \lambda \mu (N) = (N :_{R} M)M \subseteq N, \]
gives that
\[ \lambda \mu \leq 1. \]
From Lemma 1.1 it follows that \( \lambda = \lambda 1 \leq \lambda (\mu \lambda) = (\lambda \mu) \lambda \leq 1 \lambda = \lambda \) and thus
\[ \lambda = \lambda \mu \lambda. \]
Similarly, Lemma 1.1 gives that \( \mu = 1 \mu \leq (\mu \lambda) \mu = \mu (\lambda \mu) \leq 1 \mu = \mu \), so that
\[ \mu = \mu \lambda \mu. \]

**Lemma 1.3.** With the above notation, the following statements are equivalent.

(i) \( \lambda \) is a surjection.

(ii) \( \lambda \mu = 1 \).

(iii) \( N = (N :_{R} M)M \) for every submodule \( N \) of \( M \).

(iv) \( \mu \) is an injection.

**Proof.** (i) \( \Rightarrow \) (iii) Because \( \lambda \mu \lambda = \lambda \). (ii) \( \Leftrightarrow \) (iii) Clear. (ii) \( \Rightarrow \) (iv) Clear. (iv) \( \Rightarrow \) (ii) Because \( \mu \lambda \mu = \mu \). (ii) \( \Rightarrow \) (i) Clear. \( \square \)
The proof of the next result is similar to the proof of Lemma 1.3

**Lemma 1.4.** With the above notation, the following statements are equivalent.

(i) \( \lambda \) is an injection.

(ii) \( \mu \lambda = 1 \).

(iii) \( B = (BM :_{R} M) \) for every ideal \( B \) of \( R \).

(iv) \( \mu \) is a surjection.

Moreover, in this case \( M \) is faithful.

**Corollary 1.5.** With the above notation, the mapping \( \lambda \) is a bijection if and only if \( \mu \) is a bijection. In this case \( \lambda \) and \( \mu \) are inverses of each other.

**Proof.** By Lemmas 1.3 and 1.4. \( \square \)

Again let \( R \) be a ring and let \( M \) be an \( R \)-module. Let \( A = \text{ann}_{R}(M) \). By defining
\[ (r + A)m = rm \quad (r \in R, m \in M), \]
\( M \) becomes a faithful \((R/A)\)-module with the property that a subset \( X \) of \( M \) is an \( R \)-submodule of \( M \) if and only if \( X \) is an \((R/A)\)-submodule of \( M \). Thus the lattice
\( \mathcal{L}(R M) \) is identical to the lattice \( \mathcal{L}(R/A M) \). We define a mapping \( \overline{\lambda} : \mathcal{L}(R/A) \to \mathcal{L}(R M) \) by

\[
\overline{\lambda}(B/A) = BM,
\]

for every ideal \( B \) of \( R \) containing \( A \). In addition, we define a mapping \( \overline{\mu} : \mathcal{L}(R M) \to \mathcal{L}(R/A) \) by

\[
\overline{\mu}(N) = (N : R M)/A,
\]

for every submodule \( N \) of \( M \), noting that, of course, \( A \subseteq (N : R M) \) for every submodule \( N \) of \( M \). Lemmas 1.3 and 1.4 have analogues for \( \overline{\lambda} \) and \( \overline{\mu} \). Now we define \( \pi : \mathcal{L}(R) \to \mathcal{L}(R/A) \) by

\[
\pi(C) = (C + A)/A,
\]

for every ideal \( C \) of \( R \). It is clear that

\[
\lambda = \overline{\lambda} \pi \text{ and } \overline{\mu} = \pi \mu.
\]

We shall prove that a domain \( R \) is Prüfer if and only if the above mapping \( \lambda : \mathcal{L}(R) \to \mathcal{L}(R M) \) is a (lattice) homomorphism for every (cyclic) \( R \)-module \( M \) (Theorem 2.3). Given a general ring \( R \), if \( M \) is a faithful multiplication module then the mapping \( \lambda \) is a homomorphism (Theorem 2.12). Moreover, \( \lambda \) is an isomorphism if and only if \( M \) is a finitely generated faithful multiplication module and in this case the inverse of \( \lambda \) is \( \mu \) which is also an isomorphism (Theorem 4.3). Furthermore, in case \( R \) is a semilocal Noetherian domain or a Dedekind domain then the lattices \( \mathcal{L}(R) \) and \( \mathcal{L}(R M) \) are isomorphic if and only if the above mapping \( \lambda \) is an isomorphism (Corollary 5.4 and Theorem 5.5).

2. The Mapping \( \lambda \)

Let \( R \) be a ring and let \( M \) be an \( R \)-module. Let the mapping \( \lambda : \mathcal{L}(R) \to \mathcal{L}(R M) \) be as before, so that \( \lambda(B) = BM \) for every ideal \( B \) of \( R \). It is clear that

\[
\lambda(B \lor C) = \lambda(B) \lor \lambda(C),
\]

for all ideals \( B \) and \( C \) of \( R \). Thus \( \lambda \) is a homomorphism if and only if

\[
\lambda(B \land C) = \lambda(B) \land \lambda(C),
\]

for all ideals \( B \) and \( C \) of \( R \). It will be convenient for us to call the module \( M \) a \( \lambda \)-module in case the above mapping \( \lambda \) is a homomorphism.

**Lemma 2.1.** The following statements are equivalent for an \( R \)-module \( M \).

(i) \( M \) is a \( \lambda \)-module.

(ii) \( (B_1 \cap \cdots \cap B_n)M = (B_1 M) \cap \cdots \cap (B_n M) \) for every positive integer \( n \) and ideals \( B_i (1 \leq i \leq n) \) of \( R \).

(iii) \( (B \cap C)M = BM \cap CM \) for all finitely generated ideals \( B \) and \( C \) of \( R \).
Proof. (i) $\Rightarrow$ (ii) By induction on $n$,

$$(B_1 \cap \cdots \cap B_n)M = \lambda(B_1 \wedge \cdots \wedge B_n) = \lambda(B_1) \wedge \cdots \wedge \lambda(B_n) = (B_1M) \cap \cdots \cap (B_nM).$$

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (i) Suppose that (iii) holds. Let $G$ and $H$ be any ideals of $R$. Then

$$\lambda(G \wedge H) = (G \cap H)M \subseteq GM \cap HM = \lambda(G) \wedge \lambda(H),$$

so that $\lambda(G \wedge H) \leq \lambda(G) \wedge \lambda(H)$. Let $m \in GM \cap HM$. There exist a positive integer $n$ and elements $g_i \in G, h_i \in H (1 \leq i \leq n)$ with

$$m \in (g_1M + \cdots + g_nM) \cap (h_1M + \cdots + h_nM) = G'M \cap H'M,$$

where $G' = Rg_1 + \cdots + Rg_n \subseteq G$ and $H' = Rh_1 + \cdots + Rh_n \subseteq H$. Now by hypothesis, $G'M \cap H'M = (G' \cap H')M \subseteq (G \cap H)M$. Thus $m \in (G \cap H)M$. We have proved that $GM \cap HM \subseteq (G \cap H)M$ so that $\lambda(G \wedge H) \leq \lambda(G) \wedge \lambda(H)$. It follows that $\lambda(G \wedge H) = \lambda(G) \wedge \lambda(H)$ for all ideals $G$ and $H$ of $R$. Thus $\lambda$ is a homomorphism and $M$ is a $\lambda$-module.

Let $R$ be a domain with field of fractions $F$. Let $I$ be any non-zero ideal of $R$. Then $I^*$ is the $R$-submodule of $F$ consisting of all elements $f \in F$ such that $fI \subseteq R$. Note that $I^*I$ is an ideal of $R$. The ideal $I$ is called invertible provided $I^*I = R$ (see, for example, [5, p. 67]). It is well known that $I$ is an invertible ideal of $R$ if and only if there exists an $R$-submodule $X$ of $F$ such that $IX = R$. The domain $R$ is called Prüfer in case every non-zero finitely generated ideal of $R$ is invertible. For a very comprehensive account of the properties of Prüfer domains see [5].

Lemma 2.2. Let $R$ be a domain, let $B$ and $C$ be invertible ideals of $R$ and let $M$ be the $R$-module $B + C$. Then $M$ is a $\lambda$-module only if $M$ is also an invertible ideal of $R$.

Proof. We are given that

$$(B \cap C)(B + C) = \lambda(B \wedge C) = \lambda(B) \wedge \lambda(C) = B(B + C) \cap C(B + C).$$

Note that

$$(B \cap C)(B + C) \subseteq BC,$$

and

$$BC \subseteq B(B + C) \cap C(B + C).$$

Thus $BC = (B \cap C)(B + C)$. It follows that the ideal $B + C$ is invertible. \hfill \Box

The next result is presumably known but we do not have a reference for it.
Theorem 2.3. The following statements are equivalent for a (commutative) domain $R$.

(i) $R$ is Pr"ufer.

(ii) Every $R$-module is a $\lambda$-module.

(iii) Every homomorphic image of a $\lambda$-module is a $\lambda$-module.

(iv) Every cyclic $R$-module is a $\lambda$-module.

(v) Every ideal of $R$ is a $\lambda$-module.

(vi) Every 2-generated ideal of $R$ is a $\lambda$-module.

Proof. (i) $\Rightarrow$ (ii) Let $M$ be any $R$-module. Let $B$ and $C$ be non-zero finitely generated ideals of $R$. Let $D = B + C$. Then $D$ is a finitely generated ideal of $R$ and hence $D^*D = R$. Let $G = D^*B$ and $H = D^*C$ so that $G$ and $H$ are ideals of $R$ such that $G + H = D^*B + D^*C = D^*(B + C) = R$. Note that

$$BH = BD^*C = C(D^*B) = CG.$$ 

Now

$$B \cap C = (B \cap C)(G + H) \subseteq BH + CG = BH = CG \subseteq B \cap C,$$

so that

$$BM \cap CM = (G + H)(BM \cap CM) \subseteq GMC + HBM = (B \cap C)M \subseteq BM \cap CM,$$

and hence

$$BM \cap CM = (B \cap C)M.$$ 

By Lemma 2.1, $M$ is a $\lambda$-module.

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (iv) Because the $R$-module $R$ is clearly a $\lambda$-module.

(ii) $\Rightarrow$ (v) $\Rightarrow$ (vi) Clear.

(iv) $\Rightarrow$ (i) Let $B$, $C$ and $E$ be ideals of $R$. By (iv) and Lemma 2.1 we know that

$$B(R/E) \cap C(R/E) = (B \cap C)(R/E)$$

and hence that

$$B + E \cap (C + E) = (B \cap C) + E.$$

It follows that $B \cap (C + E) \subseteq (B \cap C) + E$ and hence that $B \cap (C + E) \subseteq (B \cap C) + (B \cap E)$. Therefore

$$B \cap (C + E) = (B \cap C) + (B \cap E)$$

for all ideals $B$, $C$ and $E$ of $R$. By [5, Theorem 25.2], $R$ is a Pr"ufer domain.

(vi) $\Rightarrow$ (i) By Lemma 2.2, every non-zero 2-generated ideal of $R$ is invertible because every non-zero principal ideal is clearly invertible. By [5, Theorem 22.1], $R$ is a Pr"ufer domain.

Note that if $R$ is a domain which is not Pr"ufer then Theorem 2.3 shows that, despite the fact that the $R$-module $R$ is a $\lambda$-module, there exists an ideal of $R$ which is not a $\lambda$-module and there exists a homomorphic image of $R$ which is not a $\lambda$-module. Thus in general the class of $\lambda$-modules is not closed under taking submodules and homomorphic images.

There is another class of rings for which every module is a $\lambda$-module. If $R$ is a ring then an $R$-module $M$ will be called a chain module provided the lattice $\mathcal{L}(rM)$
is a chain, that is for any submodules $N$ and $L$ of $M$ either $N \subseteq L$ or $L \subseteq N$. The ring $R$ will be called a chain ring if the $R$-module $R$ is a chain module.

**Proposition 2.4.** Let $R$ be any chain ring. Then every $R$-module is a $\lambda$-module.

**Proof.** Let $M$ be any $R$-module. Let $B$ and $C$ be ideals of $R$. Without loss of generality we can suppose that $B \subseteq C$ because $R$ is a chain ring. Then $BM \subseteq CM$ and we have:

$$(B \cap C)M = BM = BM \cap CM.$$  

By Lemma 2.1 $M$ is a $\lambda$-module. $\square$

To find other examples of $\lambda$-modules we first prove a simple lemma.

**Lemma 2.5.** Let $R$ be any ring. Then

(a) Every direct summand of a $\lambda$-module is a $\lambda$-module.

(b) Every direct sum of $\lambda$-modules is also a $\lambda$-module.

**Proof.** (a) Let $K$ be a direct summand of a $\lambda$-module $M$. Let $B$ and $C$ be any ideals of $R$. Then

$$BK \cap CK = (K \cap BM) \cap (K \cap CM) = K \cap (BM \cap CM)$$

$$= K \cap (B \cap C)M = (B \cap C)K.$$  

By Lemma 2.1 $K$ is a $\lambda$-module.

(b) Let $L_i (i \in I)$ be any collection of $\lambda$-modules and let $L = \oplus_{i \in I} L_i$. Given any ideals $B$ and $C$ of $R$ we have:

$$BL \cap CL = (\oplus_{i \in I} BL_i) \cap (\oplus_{i \in I} CL_i) = \oplus_{i \in I} (BL_i \cap CL_i)$$

$$= \oplus_{i \in I} (B \cap C)L_i = (B \cap C)L.$$  

By Lemma 2.1 $L$ is a $\lambda$-module. $\square$

**Corollary 2.6.** Let $R$ be any ring and let an $R$-module $M = M_1 \oplus M_2$ be the direct sum of a projective submodule $M_1$ and a semisimple submodule $M_2$. Then the module $M$ is a $\lambda$-module.

**Proof.** Clearly every simple module and the $R$-module $R$ are $\lambda$-modules. Apply Lemma 2.5. $\square$

For any ring $R$, every semisimple $R$-module is a $\lambda$-module but the mapping $\lambda$ is not a monomorphism (if $R$ is not von Neumann regular) and seldom an epimorphism, as the following result shows.

**Proposition 2.7.** Let $R$ be any ring and let $M$ be a semisimple $R$-module. Then the above mapping $\lambda$ is a homomorphism. Moreover,

(a) $\lambda$ is a monomorphism only if $R$ is von Neumann regular, and

(b) $\lambda$ is an epimorphism only if $M$ is cyclic.
Proof. The first part follows by Corollary 2.6.

Let $M = \oplus_{i \in I} U_i$ for some non-empty collection of simple $R$-modules $U_i (i \in I)$. For each $i \in I$ let $P_i = \text{ann}_R (U_i)$. First suppose that $\lambda$ is a monomorphism. Let $0 \neq a \in R$. Then $\lambda(Ra) = \oplus_{j \in J} U_j$, where $J$ is the set of elements $i \in I$ such that $a \notin P_i$. Clearly we have

$$\lambda(Ra^2) = \oplus_{j \in J} U_j = \lambda(Ra),$$

and hence $Ra = Ra^2$. This proves (a).

To prove (b), we suppose now that $\lambda$ is an epimorphism. Suppose that $I$ contains at least two elements. For each $i \in I$ let $B_i = \cap_{j \neq i} P_j$. By [4, Theorem 2.2] $R = P_i + B_i$ for all $i \in I$. It follows that $P_i \neq P_j$ for all distinct $i, j$ in $I$. Moreover if $A = \cap_{i \in I} P_i$ then $R/A$ is semisimple Artinian. Thus $I$ is finite and $M$ is a finite direct sum of non-isomorphic simple modules. In this case it is well known that $M$ is cyclic. $\square$

Let $\mathbb{Z}$ denote the ring of rational integers. Let $M$ denote the $\mathbb{Z}$-module $\oplus_{p \in S} (\mathbb{Z}/p)$, for any infinite set $S$ of primes in $\mathbb{Z}$. Then Proposition 2.7 shows that the mapping $\lambda : \mathcal{L}(\mathbb{Z}) \to \mathcal{L}(\mathbb{Z}M)$ is a homomorphism that is neither a monomorphism nor an epimorphism. The question remains, for a given ring $R$, which $R$-modules $M$ are $\lambda$-modules. This is certainly the case if $\lambda : \mathcal{L}(R) \to \mathcal{L}(RF)$ is a bijection (see Lemma 1.2) but note the following simple fact.

Example 2.8. Let $R$ be any ring and let $F$ be a free $R$-module of rank $\geq 2$. Then the mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}(RF)$ is a monomorphism but not an epimorphism.

Proof. Given ideals $B$ and $C$ of $R$, $\lambda(B) = \lambda(C)$ implies that $BF = CF$ and hence $B = C$. It follows that $\lambda$ is an injection. By Corollary 2.6 $\lambda$ is a monomorphism. However, if $F$ has a basis $f_i (i \in I)$ then, for each $j \in I$, $RF_j \neq \lambda(B)$ for any ideal $B$ of $R$. Thus $\lambda$ is not an epimorphism. $\square$

Next we consider when the mapping $\lambda$ is a surjection for a given module $M$. The module $M$ is called a multiplication module in case $\lambda$ is a surjection, in which case $M$ satisfies the equivalent conditions of Lemma 1.3. In other words, $M$ is a multiplication module if and only if for each submodule $N$ of $M$ there exists an ideal $B$ of $R$ such that $N = BM$. Multiplication modules have been extensively studied (see, for example, [1] - [4], [8] - [11]). Note the following simple fact about multiplication modules that is included for completeness.

Lemma 2.9. Let $R$ be a ring and let $M$ be an $R$-module with annihilator $A$ in $R$. Then the $R$-module $M$ is a multiplication module if and only if the $(R/A)$-module $M$ is a multiplication module.
Proof. Let \( N \) be any submodule of \( M \) and let \( B \) be any ideal of \( R \). Then \( N = BM \) if and only if \( N = ((B + A)/A)M \). The result follows.

It is quite possible for the mapping \( \lambda \) to be a surjection but not an injection, as we shall see shortly. Note the following fact about multiplication modules which we shall require later.

**Lemma 2.10.** (See [4, Theorem 1.2].) Let \( R \) be any ring. Then an \( R \)-module \( M \) is a multiplication module if and only if for each maximal ideal \( P \) of \( R \) either

(a) for each \( m \) in \( M \) there exists \( p \) in \( P \) such that \((1 - p)m = 0\), or

(b) there exist \( x \in M \) and \( q \in P \) such that \((1 - q)M \subseteq Rx\).

**Corollary 2.11.** Let \( R \) be any ring. Then an \( R \)-module \( M \) is a finitely generated multiplication module if and only if for each maximal ideal \( P \) of \( R \) there exist \( m \in M \), \( p \in P \) such that \((1 - p)M \subseteq Rm\).

Proof. Suppose that \( M \) is a finitely generated multiplication module. There exist a positive integer \( n \) and elements \( m_i \in M \) \((1 \leq i \leq n)\) such that \( M = Rm_1 + \cdots + Rm_n \). Let \( P \) be any maximal ideal of \( R \). By the lemma, either (a) or (b) in the lemma holds. Suppose that (a) holds. Then for each \( 1 \leq i \leq n \) there exists \( p_i \in P \) such that \((1 - p_i)m_i = 0\). Let \( p = 1 - [(1 - p_1)\ldots(1 - p_n)] \in P \) and note that \((1 - p)m_i = 0\) for all \( 1 \leq i \leq n \). Thus \((1 - p)M = 0\). This proves the necessity.

Conversely, suppose that the module \( M \) has the stated property. Suppose that the ideal \( \sum_{x \in M} (Rx :_R M) \) is proper and let \( Q \) be a maximal ideal of \( R \) such that \( \sum_{x \in M} (Rx :_R M) \subseteq Q \).

By hypothesis, there exist \( u \in M \), \( q \in Q \) such that \((1 - q)M \subseteq Ru \) and hence \( 1 - q \in (Ru :_R M) \subseteq Q \), a contradiction. Thus \( R = \sum_{x \in M} (Rx :_R M) \) and there exist a positive integer \( k \) and elements \( u_i \in M \) \((1 \leq i \leq k)\) such that \( R = (Ru_1 :_R M) + \cdots + (Ru_k :_R M) \). It follows that

\[
M = RM = (Ru_1 :_R M)M + \cdots + (Ru_k :_R M)M \subseteq Ru_1 + \cdots + Ru_k \subseteq M.
\]

Hence \( M = Ru_1 + \cdots + Ru_k \) and \( M \) is finitely generated. By [4, Corollary 1.5] \( M \) is a multiplication module.

The next result shows that multiplication modules are \( \lambda \)-modules in certain cases.

**Theorem 2.12.** Let \( R \) be any ring. Then every faithful multiplication \( R \)-module is a \( \lambda \)-module.

Proof. By Lemma 2.1 and [4, Theorem 1.6].

**Corollary 2.13.** Let \( R \) be a ring and let \( M \) be any multiplication module with \( A = \text{ann}_R(M) \). Then the mapping \( \overline{\lambda} : \mathcal{L}(R/A) \to \mathcal{L}(R_M) \) is a homomorphism.
Proof. By Lemma 2.9 and Theorem 2.12. □

Corollary 2.14. Let $R$ be any ring and let an $R$-module $M$ be a direct summand of a direct sum of faithful multiplication $R$-modules. Then $M$ is a $\lambda$-module.

Proof. By Lemma 2.5 and Theorem 2.12. □

For any ring $R$, the $R$-module $R \oplus R$ is a faithful $\lambda$-module which is not a multiplication module and therefore the converse of Theorem 2.12 is false. Note further that in general if $R$ is a domain which is not Prüfer then there exists a cyclic $R$-module which is not a $\lambda$-module (see Theorem 2.3). Clearly cyclic modules are multiplication modules. This shows that the modules in Theorem 2.12 need to be faithful. Moreover in Theorem 2.12, although $\lambda$ is an epimorphism it need not be an isomorphism as the following result shows.

Proposition 2.15. Let $R$ be a ring and let $I$ be a proper ideal of $R$ which is generated by idempotent elements such that $\text{ann}_R(I) = 0$. Then the $R$-module $I$ is a faithful multiplication module and the mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}(RI)$ is an epimorphism but not a monomorphism.

Proof. Let $J$ be any ideal of $R$ with $J \subseteq I$. Let $a \in J$. Then $a \in Re_1 + \cdots + Re_n$ for some positive integer $n$ and idempotent elements $e_i (1 \leq i \leq n)$ of $I$. It is well known that there exists an idempotent $e \in I$ such that $Re_1 + \cdots + Re_n = Re$. Then $a = be$ for some $b \in R$ and hence $a = ae \in JI$. It follows that $J = JI$. Hence the $R$-module $I$ is a faithful multiplication module. By Theorem 2.12 $\lambda$ is an epimorphism. However, $\lambda(R) = RI = I^2 = \lambda(I)$, so that $\lambda$ is not an injection. □

To illustrate Proposition 2.15 we have the following example.

Example 2.16. Let a ring $R = \prod_{i \in I} F_i$ be the direct product of any infinite collection of fields $F_i (i \in I)$. Then $R$ is a commutative von Neumann regular ring whose socle $S = \oplus_{i \in I} F_i$. Moreover for any proper ideal $B$ of $R$ with $S \subseteq B$ the $R$-module $B$ is a faithful multiplication module such that the mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}(RB)$ is an epimorphism but not a monomorphism.

Proof. It is clear that $R$ is a commutative von Neumann regular ring. Let $f = \{f_i\}$ be any non-zero element of $R$ where the $i$th component $f_i$ is an element of $F_i$ for each $i \in I$. For each $j \in I$ let $e_{(j)}$ denote the element in $R$ whose $j$th component is 1 and all of whose other components are 0. There exists $k \in I$ such that $f_k \neq 0$. Then $e_{(k)}f$ is a non-zero element of $U_k$ where $U_k$ is the ideal of $R$ consisting of all elements $\{u_i\}$ in $R$ such that $u_i = 0$ for all $i \neq k$. Clearly $U_i$ is a simple submodule of $R$ for each $i \in I$ and $\oplus_{i \in I} U_i$ is an essential submodule of $R$ so that $S = \oplus_{i \in I} U_i = \oplus_{i \in I} F_i$ is the socle of $R$. Let $B$ be any proper ideal of $R$ with $S \subseteq B$. Then $B$ is generated by idempotent elements and $\text{ann}_R(B) = 0$. Apply Proposition 2.15. □
Note that in Proposition 2.15 the ideal $I$ is not finitely generated.

3. The Mapping $\mu$

Given a ring $R$, in this section we shall consider the mapping $\mu$ from the lattice $\mathcal{L}(RM)$ of submodules of an $R$-module $M$ to the lattice $\mathcal{L}(R)$ of ideals of $R$ defined by $\mu(N) = (N:_RM) = \text{ann}_R(M/N)$. We shall say that $M$ is a $\mu$-module in case the mapping $\mu$ is a homomorphism. First note the following simple fact.

**Lemma 3.1.** Let $R$ be a ring and let $M$ be an $R$-module. Then $M$ is a $\mu$-module if and only if $(N + L:_RM) = (N:_RM) + (L:_RM)$ for all submodules $N$ and $L$ of $M$.

**Proof.** Let $N$ and $L$ be any submodules of $M$. Clearly

$$\mu(N \cap L) = (N \cap L:_RM) = (N:_RM) \cap (L:_RM) = \mu(N) \wedge \mu(L).$$

Next note that

$$\mu(N \vee L) = \mu(N) \vee \mu(L) \iff (N + L:_RM) = (N:_RM) + (L:_RM).$$

The result follows. \qed

**Corollary 3.2.** Let $M$ be a $\mu$-module for a ring $R$ such that $M = N + L$ for some submodules $N$ and $L$. Then there exists $a \in R$ such that $aM \subseteq N$ and $(1 - a)M \subseteq L$.

**Proof.** Note that Lemma 3.1 gives:

$$R = (M:_RM) = (N + L:_RM) = (N:_RM) + (L:_RM).$$

Clearly the result follows. \qed

In contrast to Theorem 2.3, no (non-zero) ring $R$ has the property that the mapping $\mu : \mathcal{L}(RM) \rightarrow \mathcal{L}(R)$ is a homomorphism for every $R$-module $M$, as the next result shows.

**Corollary 3.3.** Let $R$ be any (non-zero) ring and let $M$ be any non-zero $R$-module. Then the $R$-module $M \oplus M$ is not a $\mu$-module.

**Proof.** Suppose that $M \oplus M$ is a $\mu$-module. By Corollary 3.2, there exists $a \in R$ such that $a(M \oplus M) \subseteq M \oplus 0$ and $(1 - a)(M \oplus M) \subseteq 0 \oplus M$, so that $aM = 0$ and $(1 - a)M = 0$ giving $M = 0$, a contradiction. \qed

Compare the next result with Lemma 2.10

**Corollary 3.4.** Let $M$ be a $\mu$-module for a ring $R$. Then for each maximal ideal $P$ of $R$ either $M = PM$ or there exist $m \in M$ and $p \in P$ such that $(1 - p)M \subseteq Rm$. 

Proof. Let $P$ be a maximal ideal of $R$ such that $M \neq PM$. Note that $M/PM$ is a non-zero semisimple module and hence contains a maximal submodule. Let $L$ be a maximal submodule of $M$ such that $PM \subseteq L$. Let $m$ be any element of $M$ with $m \notin L$. Clearly $M = L + Rm$. By Corollary 3.2 there exists an element $p \in R$ such that $pM \subseteq L$ and $(1 - p)M \subseteq Rm$. If $p \notin P$ then $R = P + Rp$ and hence $M = PM + pM \subseteq L$, a contradiction. Thus $p \in P$, as required. □

Corollary 3.5. Let $R$ be a ring and let $M$ be a $\mu$-module over $R$ such that $M \neq PM$ for every maximal ideal $P$ of $R$. Then the $R$-module $M$ is a multiplication module. Moreover the mapping $\mu$ is a monomorphism.

Proof. By Lemmas 1.3 and 2.10 and Corollary 3.4. □

We now aim to prove a partial converse of Corollary 3.5. It is proved in Theorem 2.3 that a domain is Prüfer if and only if every homomorphic image of a $\lambda$-module is a $\lambda$-module. Contrast this fact with the following result.

Proposition 3.6. Every homomorphic image of a $\mu$-module is a $\mu$-module.

Proof. Let $K$ be any submodule of $M$ and let $\overline{M}$ be the module $M/K$. Let $\overline{N}$ be any submodule of $\overline{M}$. There exists a submodule $N$ of $M$ containing $K$ such that $N = N/K$. Then

$$(\overline{N} : \overline{M}) = \text{ann}_R(\overline{M}/\overline{N}) = \text{ann}_R(M/N) = (N :_R M).$$

Apply Lemma 3.1. □

Corollary 3.7. For any ring $R$, every cyclic $R$-module is a $\mu$-module.

Proof. Consider the $R$-module $R$. Let $B$ and $C$ be any ideals of $R$. Then

$$((B + C :_R R) = B + C = (B :_R R) + (C :_R R).$$

By Lemma 3.1 $R$ is a $\mu$-module. Now apply Proposition 3.6. □

We now characterize which finitely generated modules are $\mu$-modules.

Theorem 3.8. Given any ring $R$, a finitely generated $R$-module is a $\mu$-module if and only if $M$ is a multiplication module.

Proof. The necessity is proved in Corollary 3.5. Conversely suppose that $M$ is a multiplication module. Let $N, L$ be any submodules of $M$. Clearly $(N :_R M) + (L :_R M) \subseteq (N + L :_R M)$. Suppose that $(N :_R M) + (L :_R M) \neq (N + L :_R M)$. Let $a \in (N + L :_R M)$ such that $a \notin (N :_R M) + (L :_R M)$. Let $B = \{r \in R : ra \in (N :_R M) + (L :_R M)\}$. Then $B$ is a proper ideal of $R$ and hence $B \subseteq P$ for some maximal ideal $P$ of $R$. Because $M$ is a finitely generated multiplication module, Corollary 2.11 shows that there exist $m \in M$ and $p \in P$ such that $(1 - p)M \subseteq Rm$. 


In particular, note that \((1 - p)N\) and \((1 - p)L\) are both submodules of the cyclic \(R\)-module \(Rm\). Next,
\[(1 - p)aM \subseteq (1 - p)(N + L) = (1 - p)N + (1 - p)L.\]
Because \(Rm\) is a \(\mu\)-module by Corollary 3.7, we now see that
\[(1 - p)a \in ((1 - p)N + (1 - p)L :_R Rm) = ((1 - p)N :_R Rm) + ((1 - p)L :_R Rm).\]
Therefore \((1 - p)a = b + c\) for some \(b \in ((1 - p)N :_R Rm)\) and some \(c \in ((1 - p)L :_R Rm)\). Note that
\[(1 - p)bM \subseteq bRm \subseteq (1 - p)N \subseteq N,\]
and hence that \((1 - p)b \in (N :_R M)\). Similarly \((1 - p)c \in (L :_R M)\). But this implies that
\[(1 - p)^2a = (1 - p)b + (1 - p)c \in (N :_R M) + (L :_R M),\]
which in turn implies that \((1 - p)^2 \in B \subseteq P\), a contradiction. It follows that
\[(N + L :_R M) = (N :_R M) + (L :_R M)\]
for all submodules \(N\) and \(L\) of \(M\). By Lemma 3.1 \(M\) is a \(\mu\)-module.

**Corollary 3.9.** The following statements are equivalent for a module \(M\) over a ring \(R\).

(i) Every finitely generated submodule of \(M\) is a \(\mu\)-module.
(ii) Every 2-generated submodule of \(M\) is a \(\mu\)-module.
(iii) \(R = (Rx :_R Ry) + (Ry :_R Rx)\) for all elements \(x, y \in M\).
(iv) \(R + (N :_R L) + (L :_R N)\) for all finitely generated submodules \(N\) and \(L\) of \(M\).
(v) Every finitely generated submodule of \(M\) is a multiplication module.

**Proof.** (i) \(\Leftrightarrow\) (v) By Theorem 3.8.

(ii) \(\Rightarrow\) (iii) Let \(x, y \in M\). By Lemma 3.1,
\[R = (Rx + Ry :_R Rx + Ry) = (Rx :_R Rx + Ry) + (Ry :_R Rx + Ry)\]
\[= (Rx :_R Ry) + (Ry :_R Rx).\]

(iii) \(\Rightarrow\) (v) Let \(K\) be any finitely generated submodule of \(M\). There exist a positive integer \(n\) and elements \(u_i \in K\) \((1 \leq i \leq n)\) such that \(K = Ru_1 + \cdots + Ru_n\). If \(n = 1\) then \(K\) is a multiplication module. Suppose that \(n \geq 2\). By induction on \(n\) the submodule \(L = Ru_1 + \cdots + Ru_{n-1}\) is a multiplication module. Let \(P\) be any maximal ideal of \(R\). By Corollary 2.11, there exist \(v \in L, p \in P\) such that \((1 - p)L \subseteq Rv\). Let \(u = u_n\) and note that \(K = L + Ru\). By hypothesis, \(R = (Ru :_R Rv) + (Rv :_R Ru)\) so that either \((Ru :_R Rv) \not\subseteq P\) or \((Rv :_R Ru) \not\subseteq P\).
Suppose first that \((Ru :_R Rv) \not\subseteq P\). Then there exists \(p_1 \in P\) such that \((1 - p_1)v \in Ru\). This implies that
\[
(1 - p_1)(1 - p)K \subseteq (1 - p_1)(1 - p)L + Ru \subseteq Ru.
\]
Now suppose that \((Rv :_R Ru) \not\subseteq P\). Then there exists \(p_2 \in P\) such that \((1 - p_2)u \in Rv\). In this case,
\[
(1 - p_2)(1 - p)K \subseteq (1 - p_2)(1 - p)L + (1 - p)(1 - p_2)Ru \subseteq Rv.
\]
In case \(n \geq 2\) we have proved that for each maximal ideal \(P\) of \(R\) there exist an element \(w\) in \(K\) and an element \(q\) in \(P\) such that \((1 - q)K \subseteq Rw\). By Lemma 2.10 \(K\) is a multiplication module.

(iv) \(\Rightarrow\) (iii) Clear.

(i) \(\Rightarrow\) (iv) Given any finitely generated submodules \(N, L\) of \(M\), the submodule \(N + L\) is also finitely generated and hence a \(\mu\)-module. By Lemma 3.1,
\[
R = (N + L :_R N + L) = (N :_R N + L) + (L :_R N + L)
\]
\[
= (N :_R L) + (L :_R N).
\]
This completes the proof.

Next we give examples to show that the condition that the module \(M\) be finitely generated is necessary for both implications in Theorem 3.8.

**Example 3.10.** Let \(F\) be any field and let \(R\) denote the collection of all sequences \(\{f_1, f_2, f_3, \ldots\}\) of elements of \(F\) with the property that there exists a positive integer \(n\) (depending on the particular sequence) such that \(f_n = f_{n+1} = f_{n+2} = \ldots\). Then \(R\) is a commutative von Neumann regular ring whose socle \(S\) is a multiplication module but not a \(\mu\)-module.

**Proof.** It is not difficult to see that the commutative ring \(R\) is von Neumann regular. Next it is clear that \(S\) consists of all sequences \(\{f_1, f_2, f_3, \ldots\}\) of elements of \(F\) such that \(0 = f_n = f_{n+1} = f_{n+2} = \ldots\) for some positive integer \(n\). By Proposition 2.15, the module \(S\) is a faithful multiplication module. Let \(S_1\) denote the subset of \(S\) consisting of all sequences \(\{f_1, f_2, f_3, \ldots\}\) such that \(f_{2k} = 0 (k \geq 1)\) and let \(S_2\) denote the subset of \(S\) consisting of all elements \(\{f_1, f_2, f_3, \ldots\}\) with \(f_{2k-1} = 0 (k \geq 1)\). Clearly \(S_1\) and \(S_2\) are submodules of the \(R\)-module \(S\) such that \(S = S_1 \oplus S_2\), \((S_1 :_R S) = S_1\) and \((S_2 :_R S) = S_2\). By Corollary 3.2 the \(R\)-module \(S\) is not a \(\mu\)-module.

**Example 3.11.** Again let \(\mathbb{Z}\) denote the ring of rational integers and let \(M\) denote the Prüfer \(p\)-group for any prime \(p\) in \(\mathbb{Z}\). Then the \(\mathbb{Z}\)-module \(M\) is a \(\mu\)-module but not a multiplication module.
Proof. If \( N \) and \( L \) are proper submodules of \( M \) then \( (N:_R M) = (L:_R M) = 0 \) by [7, Proposition 2.6]. Now Lemma 3.1 gives that \( M \) is a \( \mu \)-module. On the other hand, if \( K \) is any proper non-zero submodule of \( M \) then \( K \neq BM \) for any ideal \( B \) of \( \mathbb{Z} \) and hence \( M \) is not a multiplication module. \( \square \)

Recall that a module \( M \) over a general ring \( R \) is called hollow in case \( M \) is not the sum of two proper submodules. Next we generalise Example 3.11

**Proposition 3.12.** Let \( R \) be a domain which is not a field and let \( M \) be a non-zero injective \( R \)-module. Then

(a) The above mapping \( \lambda \) is a homomorphism but is neither a monomorphism nor an epimorphism.

(b) The mapping \( \mu \) is a homomorphism if and only if \( M \) is a hollow module. However \( \mu \) is neither a monomorphism nor an epimorphism.

**Proof.** Note that \( M = BM = \lambda(B) \) for every non-zero ideal \( B \) of \( R \) and \( 0 = (N:_R M) = \mu(N) \) for every proper submodule \( N \) of \( M \) by [7, Proposition 2.6].

(a) For all ideals \( B \) and \( C \) of \( R \) we have \( (B \cap C)M = BM \cap CM \). By Lemma 2.1 \( \lambda \) is a homomorphism. Let \( a \) be any non-zero element of \( R \) such that \( a \) is not a unit in \( R \). Then \( \lambda(Ra) = M = \lambda(R) \) so that \( \lambda \) is not a monomorphism. For any maximal ideal \( P \) of \( R \), \( P \neq 0 \) and hence \( M = PM \). Thus \( M \) is not simple. If \( L \) is a non-zero proper submodule of \( M \) then \( L \neq \lambda(E) \) for any ideal \( E \) of \( R \). Thus \( \lambda \) is not an epimorphism.

(b) It is clear that \( \mu \) is a homomorphism if and only if whenever \( N \) and \( L \) are proper submodules of \( M \) then the submodule \( N + L \) is also proper, that \( M \) is hollow. We have seen above that \( M \) is not simple and hence \( \mu(L) = 0 = \mu(0) \) for every non-zero proper submodule \( L \) of \( M \). Thus \( \mu \) is not a monomorphism. Moreover if \( a \) is as before we have \( Ra \neq \mu(K) \) for every submodule \( K \) of \( M \), so that \( \mu \) is not an epimorphism. \( \square \)

The next result is an analogue of Theorem 2.3. It is an immediate consequence of Corollary 3.9 because domains with the property that every finitely generated ideal is a multiplication module are precisely Prüfer domains.

**Theorem 3.13.** The following statements are equivalent for a domain \( R \).

(i) \( R \) is Prüfer.

(ii) Every finitely generated ideal of \( R \) is a \( \mu \)-module.

(iii) Every 2-generated ideal of \( R \) is a \( \mu \)-module.

**Proof.** By Corollary 3.9. \( \square \)

Note that Theorem 3.13 shows that the class of \( \mu \)-modules is not closed under submodules in general.
Lemma 3.14. Given any ring $R$, the following statements are equivalent for an $R$-module $M$.

(i) $R = (N :_RL) + (L :_RN)$ for all submodules $N$ and $L$ of $M$.

(ii) Every submodule of $M$ is a $\mu$-module.

Proof. (i) $\Rightarrow$ (ii) Let $K$ be any submodule of $M$ and let $N$ and $L$ be submodules of $K$. Then

$$(N + L :_RK) = R(N + L :_RK) = [(N :_RL) + (L :_RN)](N + L :_RK)$$

$$= (N :_RN + L)(N + L :_RK) + (L :_RN + L)(N + L :_RK)$$

$$\subseteq (N :_RK) + (L :_RK).$$

It is clear that $(N :_RK) + (L :_RK) \subseteq (N + L :_RK)$. By Lemma 3.1 $K$ is a $\mu$-module.

(ii) $\Rightarrow$ (i) Let $N$ and $L$ be submodules of $M$. Then the submodule $N + L$ is a $\mu$-module. Lemma 3.1 then gives

$$R = (N + L :_RN + L) = (N :_RN + L) + (L :_RN + L) = (N :_RL) + (L :_RN).$$

The next result should be compared with Proposition 2.4. Recall that a (non-zero) ring $R$ is called local in case it contains precisely one maximal ideal.

Proposition 3.15. Given any local ring $R$, the following statements are equivalent for an $R$-module $M$.

(i) $M$ is a chain module.

(ii) Every submodule of $M$ is a $\mu$-module.

(iii) Every finitely generated submodule of $M$ is a $\mu$-module.

Proof. (i) $\Rightarrow$ (ii) Given any submodules $N$ and $L$ of $M$ either $N \subseteq L$ or $L \subseteq N$ and hence $R = (N :_RL) + (L :_RN)$. By Lemma 3.14 every submodule of $M$ is a $\mu$-module.

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (i) Let $U$ and $V$ be any submodules of $M$ such that $U \not\subseteq V$. Let $u \in U \setminus V$. Let $v \in V$. Then $Ru + Rv$ is a $\mu$-module and Lemma 3.1 gives that

$$R = (Ru + Rv :_R(Ru + Rv)) = (Ru :_RRu + Rv) + (Rv :_RRu + Rv)$$

$$= (Ru :_RRv) + (Rv :_RRu).$$

Because $R$ is local either $1 \in (Ru :_RRv)$ and $v \in Ru$ or else $1 \in (Rv :_RRu)$ and $u \in Rv \subseteq V$. By the choice of $u$ it follows that $v \in Ru$ for all $v \in V$ and hence $V \subseteq Ru \subseteq U$. We have proved that $U \subseteq V$ or $V \subseteq U$ for any submodules $U$ and $V$ of $M$. Thus $M$ is a chain module.

We next consider semisimple modules.
Proposition 3.16. Let $R$ be any ring. Then the following statements are equivalent for a semisimple $R$-module $M$.

(i) $M$ is a $\mu$-module.

(ii) $(N + L :_R M) = (N :_R M) + (L :_R M)$ for all submodules $N$ and $L$ of $M$ with $N \cap L = 0$.

(iii) $R = \text{ann}_R(N) + \text{ann}_R(L)$ for all submodules $N$ and $L$ of $M$ with $N \cap L = 0$.

Proof. (i) $\Rightarrow$ (ii) Clear by Lemma 3.1.

(ii) $\Rightarrow$ (iii) Let $N$ and $L$ be any submodules of $M$ with $N \cap L = 0$. There exists a submodule $K$ of $M$ such that $M = N \oplus L \oplus K$. By (ii),

$$R = (M :_R M) = (L \oplus K :_R M) + (N :_R M) = \text{ann}_R(N) + \text{ann}_R(L \oplus K)$$

$$\subseteq \text{ann}_R(N) + \text{ann}_R(L) \subseteq R,$$

so that $R = \text{ann}_R(N) + \text{ann}_R(L)$.

(iii) $\Rightarrow$ (i) Let $N$ and $L$ be submodules of $M$. Suppose first that $N \cap L = 0$. Let $\text{ann}_R(N) = B$ and $\text{ann}_R(L) = C$. By hypothesis, $R = B + C$. If $M = N \oplus L$ then $(N + L :_R M) = R = C + B = (N :_R M) + (L :_R M)$. Now suppose that $M \neq N \oplus L$. There exists a non-zero submodule $K$ of $M$ such that $M = N \oplus L \oplus K$. Note that $(N + L :_R M) = \text{ann}_R(K) = D$ (say), $(N :_R M) = \text{ann}_R(L \oplus K) = C \cap D$ and $(L :_R M) = \text{ann}_R(N \oplus K) = B \cap D$. By hypothesis $R = B + D$ implies that $B \cap D = (B \cap D)(B + D) \subseteq BD \subseteq B \cap D$. Thus $B \cap D = BD$. Similarly $C \cap D = CD$. Combining this information gives us the following:

$$(N + L :_R M) = D = D(B + C) = CD + BD = (N :_R M) + (L :_R M).$$

Now suppose that $N \cap L$ need not be zero. There exists a submodule $H$ of $M$ such that $L = (N \cap L) \oplus H$. This implies that $N + L = N \oplus H$. By the above proof we have

$$(N + L :_R M) = (N \oplus H :_R M) = (N :_R M) + (H :_R M)$$

$$\subseteq (N :_R M) + (L :_R M) \subseteq (N + L :_R M).$$

Thus in any case $(N + L :_R M) = (N :_R M) + (L :_R M)$ for all submodules $N$ and $L$ of $M$. By Lemma 3.1 $M$ is a $\mu$-module.

Example 3.17. Let $R$ be the ring in Example 2.16. Then the socle $S$ of $R$ is a semisimple $\mu$-module over $R$.

Proof. We saw in Example 2.16 that $S = \oplus_{i \in I} U_i$. For each $i \in I$ let $\pi_i : S \to U_i$ denote the canonical projection. Let $N$ and $L$ be submodules of the $R$-module $S$ such that $N \cap L = 0$. It is rather easy to see that $N \cap U_i = \pi_i(N)$ for all $i \in I$. Let $I_1 = \{i \in I : \pi_i(N) \neq 0\}$. Then $N = \oplus_{i \in I_1} U_i$. Similarly, if $I_2 = \{i \in I : \pi_i(L) \neq 0\}$ then $L = \oplus_{i \in I_2} U_i$. Because $N \cap L = 0$, the set $I_1 \cap I_2$ must be empty or, in
other words, \( I_2 \subseteq I \setminus I_1 \). Let \( f = \{f_i\} \in R \). Let \( g \) denote the element \( \{g_i\} \) such that \( g_i = f_i \) (\( i \in I_2 \)) and otherwise \( g_i = 0 \). Then
\[
f = g + (f - g) \in \text{ann}_R(N) + \text{ann}_R(L).
\]
This proves that \( R = \text{ann}_R(N) + \text{ann}_R(L) \). By Proposition 3.16 it follows that the \( R \)-module \( S \) is a \( \mu \)-module. \( \square \)

We saw in Corollary 2.6 that every semisimple module is a \( \lambda \)-module. However Proposition 3.16 shows that if \( M \) is the semisimple \( \mathbb{Z} \)-module \( \bigoplus_{p \in \Pi} (\mathbb{Z}/\mathbb{Z}p) \), where \( \Pi \) is the collection of all primes \( p \) in \( \mathbb{Z} \), then \( M \) is not a \( \mu \)-module.

4. Finitely Generated Modules

Again \( R \) is a commutative ring and \( M \) a unital \( R \)-module. We begin this section with the following elementary result.

**Lemma 4.1.** Let \( P \) be a maximal ideal of a ring \( R \) and let \( M \) be an \( R \)-module. Then \( M \neq PM \) if and only if \( P = (PM :_R M) \).

**Proof.** Clear. \( \square \)

The next result can be found essentially in [4, Theorem 3.1].

**Lemma 4.2.** Let \( R \) be any ring and let \( M \) be an \( R \)-module with annihilator \( A \) in \( R \). If \( M \) is finitely generated then \( M \neq PM \) for every maximal ideal \( P \) of \( R \) with \( A \subseteq P \). Moreover, the converse holds in case \( M \) is a multiplication module.

**Proof.** Suppose first that \( M \) is finitely generated. Suppose further that \( M = PM \) for some maximal ideal \( P \) of \( R \). Then the usual determinant argument gives that \((1 - p)M = 0 \) for some \( p \in P \). Thus \( 1 - p \in A \) and hence \( A \not\subseteq P \). Conversely, now suppose that \( M \) is a multiplication module such that \( M \neq PM \) for every maximal ideal \( P \) of \( R \) with \( A \subseteq P \). We have to prove that \( M \) is finitely generated. Suppose that \( R \neq \sum_{m \in M} (Rm :_R M) \). Then there exists a maximal ideal \( Q \) of \( R \) such that \( \sum_{m \in M} (Rm :_R M) \subseteq Q \). Note that \( A = (R0 :_R M) \subseteq Q \). By hypothesis, there exists an element \( x \) in \( M \) such that \( x \notin QM \). Because \( M \) is a multiplication module, \( Rx = BM \) for some ideal \( B \) of \( R \). Clearly \( B \not\subseteq Q \). But \( B \subseteq (Rx :_R M) \subseteq Q \), a contradiction. Thus \( R = \sum_{m \in M} (Rm :_R M) \). There exist a positive integer \( n \) and elements \( m_i \in M \) (\( 1 \leq i \leq n \)) such that \( 1 \in (Rm_1 :_R M) + \cdots + (Rm_n :_R M) \). In this case, we have
\[
M = 1M \subseteq (Rm_1 :_R M)M + \cdots + (Rm_n :_R M)M \subseteq Rm_1 + \cdots + Rm_n \subseteq M,
\]
and hence \( M = Rm_1 + \cdots + Rm_n \). \( \square \)
Recall that we are interested in the mapping $\lambda$ from the lattice $\mathcal{L}(R)$ of ideals of a ring $R$ to the lattice $\mathcal{L}(R M)$ of submodules of an $R$-module $M$ defined by $\lambda(B) = BM$ for every ideal $B$ of $R$ and the mapping $\mu : \mathcal{L}(R M) \to \mathcal{L}(R)$ defined by $\mu(N) = (N :_R M)$ for every submodule $N$ of $M$. Now we come to our main result.

**Theorem 4.3.** Let $R$ be a ring and let $M$ be an $R$-module. Then the following statements are equivalent.

(i) The mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}(R M)$ is an isomorphism.

(ii) The mapping $\mu : \mathcal{L}(R M) \to \mathcal{L}(R)$ is an isomorphism.

(iii) $M$ is a multiplication module such that $B = (BM :_R M)$ for every ideal $B$ of $R$.

(iv) $M$ is a finitely generated faithful multiplication module.

**Proof.** (i) $\Leftrightarrow$ (ii) By Corollary 1.5.

(i) $\Rightarrow$ (iii) Because $\lambda$ is an isomorphism, $\lambda$ is a bijection. In particular, $\lambda$ is a surjection and hence $M$ is a multiplication module. Next Lemma 1.4 shows that $\mu \lambda = 1$ because $\lambda$ is an injection. Therefore, for each ideal $B$ of $R$,

$$B = \mu \lambda(B) = \mu(BM) = (BM :_R M).$$

This proves (iii).

(iii) $\Rightarrow$ (iv) Note that $\text{ann}_R(M) = (0 :_R M) = 0$ by (iii) and hence $M$ is faithful. Moreover, for any maximal ideal $Q$ of $R$, $QM = M$ implies that $R = (M :_R M) = (QM :_R M) = Q$, a contradiction. Thus $M \neq QM$ for every maximal ideal $M$ of $R$. By Lemma 4.2 $M$ is finitely generated.

(iv) $\Rightarrow$ (i) By [4, Theorem 3.1], if $C$ and $D$ are ideals of $R$ such that $\lambda(C) = \lambda(D)$ then $CM = DM$ and hence $C = D$. Thus $\lambda$ is an injection. But $M$ being a multiplication module gives that $\lambda$ is a surjection. Being a bijection, $\lambda$ is an isomorphism by Lemma 1.2.

**Corollary 4.4.** Let $R$ be a ring and let $M$ be an $R$-module with annihilator $A$ in $R$. Then $M$ is a finitely generated multiplication module if and only if the mapping $\overline{\lambda} : \mathcal{L}(R/A) \to \mathcal{L}(R M)$ is an isomorphism.

**Proof.** Suppose that $M$ is a finitely generated multiplication module. Then the $(R/A)$-module $M$ is a finitely generated faithful multiplication module (Lemma 2.9) and the mapping $\overline{\lambda}$ is an isomorphism by Theorem 4.3. Conversely, suppose that $\overline{\lambda}$ is an isomorphism. By Theorem 4.3, the $(R/A)$-module $M$ is a finitely generated multiplication module. It follows that the $R$-module $M$ is a finitely generated multiplication module by Lemma 2.9.
Corollary 4.5. Let $R$ be a domain and let $M$ be an $R$-module. Then the mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}(RM)$ is an isomorphism if and only if $M$ is a faithful multiplication module.

Proof. Suppose that $M$ is a (non-zero) faithful multiplication module. By [4, Lemma 4.3 and Theorem 4.4] $M$ is finitely generated. Now apply Theorem 4.3. □

5. Isomorphisms of Lattices

Let $R$ be a ring and $M$ an $R$-module. Suppose that $\rho : \mathcal{L}(R) \to \mathcal{L}(R)$ is a lattice isomorphism (in particular, $\rho$ could be the lattice isomorphism induced from an automorphism of the ring $R$) and suppose that $\sigma : \mathcal{L}(RM) \to \mathcal{L}(R)$ is a lattice isomorphism (in particular, $\sigma$ could be the lattice isomorphism induced from an $R$-isomorphism from $M$ to $M$). Then $\sigma \lambda \rho : \mathcal{L}(R) \to \mathcal{L}(RM)$ is a mapping which is an isomorphism if and only if $\lambda$ is an isomorphism. Thus in general there can be many mappings from $\mathcal{L}(R)$ to $\mathcal{L}(RM)$ each of which may or may not be an isomorphism.

Now let $R$ be a ring which is not Hopfian, that is there exists a ring epimorphism $\nu : R \to R$ which is not a monomorphism. Let $A$ denote the non-zero kernel of $\nu$. Then $R \cong R/A$ so that $\mathcal{L}(R)$ is isomorphic to $\mathcal{L}(R/A)$ which in turn is isomorphic to $\mathcal{L}(R(R/A))$. However in this case $\lambda : \mathcal{L}(R) \to \mathcal{L}(R(R/A))$ is not an isomorphism because $R/A$ is not a faithful $R$-module. For a specific example, let $R$ denote the polynomial ring $F[x_1, x_2, x_3, \ldots]$ in indeterminates $x_i (i \geq 1)$ over a field $F$. Let $n$ be any positive integer. The linear mapping $\nu$ from the $F$-vector space $R$ to $R$ defined by $\nu(x_i) = 0$ if $1 \leq i \leq n$ and $\nu(x_i) = i - n$ if $i \geq n + 1$ induces a ring epimorphism with non-zero kernel $A_n = Rx_1 + \cdots + Rx_n$.

This leads us to ask the following question.

Question 5.1. Let $R$ be a ring and let $M$ be an $R$-module such that the lattices $\mathcal{L}(R)$ and $\mathcal{L}(RM)$ are isomorphic. Then what can one say about the module $M$? For example, need $M$ be a multiplication module?

As a contribution towards the answer to this question we offer some modest results. We look at Noetherian rings and modules. Any Noetherian module has Krull dimension (see, for example, [6, Lemma 6.2.3]). For the definition and properties of Krull dimension see [6, Chapter 6]. We shall denote the Krull dimension of the ring $R$ by $\text{kdim}(R)$ and of an $R$-module $M$ by $\text{kdim}(RM)$, if either exists.

Lemma 5.2. Let $R$ be a Noetherian domain and let $M$ be an $R$-module such that the lattices $\mathcal{L}(R)$ and $\mathcal{L}(RM)$ are isomorphic. Then $M$ is a finitely generated faithful $R$-module.

Proof. Let $A = \text{ann}_R(M)$. Clearly the module $M$ is Noetherian and $\text{kdim}(R) = \text{kdim}(RM)$. But $M$ is a finitely generated $(R/A)$-module. By [6, Lemma 6.2.5],
kdim(M) \leq kdim(R/A). Thus kdim(R) \leq kdim(R/A). Finally, by [6, Proposition 6.3.11], A = 0 and hence M is faithful. \qed

Recall that a ring R is called \textit{semilocal} provided it contains only a finite number of maximal ideals.

**Theorem 5.3.** Let R be a ring and let M be a finitely generated R-module such that the lattices \( L(R) \) and \( L(RM) \) are isomorphic. Suppose further that either (a) R is a local ring or (b) R is a semilocal ring and M is faithful. Then M is cyclic.

**Proof.** (a) Suppose that R has unique maximal ideal P. Then the lattice \( L(RM) \) must have only one maximal submodule which must be the submodule PM. Let \( m \in M \setminus PM \). Then \( M = Rm + PM \). It follows that \( M/Rm = P(M/Rm) \). But M is finitely generated so that Nakayama's Lemma gives that \( M = Rm \).

(b) Let \( Q_1, \ldots, Q_n \) denote the maximal ideals of R, for some positive integer n. If \( M = Q_iM \), for some \( 1 \leq i \leq n \), then by the usual determinant argument we have \( (1 - q)M = 0 \) for some \( q \in Q_i \), contradicting the fact that M is faithful. Thus \( M \neq Q_iM \) \( (1 \leq i \leq n) \). Because the lattices \( L(R) \) and \( L(RM) \) are isomorphic it follows that the maximal submodules of M are precisely the submodules \( Q_iM \) \( (1 \leq i \leq n) \). Let \( Q = Q_1 \cap \cdots \cap Q_n \), the Jacobson radical of R. Then QM is the Jacobson radical of M and hence \( QM = Q_1M \cap \cdots \cap Q_nM \). The lattice isomorphism then gives \( M/QM \cong R/Q \), which is a cyclic R-module. The proof of (a) then gives that M is cyclic. \qed

**Corollary 5.4.** Let R be a semilocal Noetherian domain. Then the following statements are equivalent for a non-zero R-module M.

(i) The above mapping \( \lambda : L(R) \to L(RM) \) is an isomorphism.

(ii) The lattices \( L(R) \) and \( L(RM) \) are isomorphic.

(iii) \( M \cong R \).

**Proof.** (i) \( \Rightarrow \) (ii) Clear.

(ii) \( \Rightarrow \) (iii) By Lemma 5.2 M is faithful and by Theorem 5.3 M is cyclic. Therefore \( M \cong R \).

(iii) \( \Rightarrow \) (i) By Theorem 4.3. \qed

Next we give another situation where we can settle when there is an isomorphism from the lattice of ideals of a ring R to the lattice of submodules of an R-module M.

**Theorem 5.5.** Let R be a Dedekind domain (which is not a field). Then the following statements are equivalent for a non-zero R-module M.

(i) The above mapping \( \lambda : L(R) \to L(RM) \) is an isomorphism.

(ii) The lattices \( L(R) \) and \( L(RM) \) are isomorphic.
(iii) $M \cong B$ for some non-zero ideal $B$ of $R$.

Proof. (i) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (iii) Suppose that $\tau : \mathcal{L}(RM) \to \mathcal{L}(R)$ is an isomorphism. Because $R$ is a Noetherian ring, the $R$-module $M$ must be Noetherian. Next note that if $N$ and $L$ are submodules of $M$ such that $N \cap L = 0$ then $\tau(N) \cap \tau(L) = 0$. This implies that $\tau(N)\tau(L) = 0$ and hence $\tau(N) = 0$ or $\tau(L) = 0$. Thus $N = 0$ or $L = 0$. It follows that the $R$-module $M$ is uniform. Since $R$ has zero socle it follows that the $R$-module $M$ has zero socle and thus $M$ is a torsion-free $R$-module. Putting this information together gives us that $M$ is isomorphic to a non-zero ideal of $R$.

(iii) $\Rightarrow$ (i) Every non-zero ideal of $R$ is invertible and hence is a multiplication module. By Corollary 4.5, (i) follows. \hfill $\Box$

Combining Corollaries 4.5 and 5.4 and Theorem 5.5 we see that if $R$ is a semilocal Noetherian domain or a Dedekind domain then the lattices $\mathcal{L}(R)$ and $\mathcal{L}(RM)$ are isomorphic if and only if $M$ is a faithful multiplication module, and in this case the $R$-module $M$ is finitely generated and the mappings $\lambda$ and $\mu$ are both isomorphisms.

Acknowledgement. Some of this work was done during visits to Akdeniz University, Antalya, Turkey, and Adnan Menderes University, Aydin, Turkey in May 2013 and the author would like to thank both universities for their hospitality and their financial support. He would also like to thank TUBITAK once again for its financial support.

References

[1] D. D. Anderson, Some remarks on multiplication ideals, Math. Japonica, 4(1980), 463-469.
[2] D. D. Anderson and Y. Al-Shaniafi, Multiplication modules and the ideal $\theta(M)$, Comm. Algebra, 30(7)(2002), 3383-3390.
[3] A. Barnard, Multiplication modules, J. Algebra, 71(1981), 174-178.
[4] Z. A. El-Bast and P. F. Smith, Multiplication modules, Comm. Algebra, 16(4)(1988), 755-779.
[5] R. Gilmer, Multiplicative Ideal Theory, Marcel Dekker, New York, 1972.
[6] J. C. McConnell and J. C. Robson, Noncommutative Noetherian Rings, Wiley, Chichester, 1987.
[7] D. W Sharpe and P. Vamos, Injective Modules, Cambridge Tracts in Mathematics and Mathematical Physics 62, Cambridge Univ. Press, Cambridge, 1972.
[8] S. Singh and F. Mehdi, Multiplication modules, Canad. Math. Bull., 22 (1979), 93-98.
[9] S. Singh and Y. Al-Shaniafi, *Multiplication modules*, Comm. Algebra, 29(6)(2001), 2597-2609.

[10] S. Singh and Y. Al-Shaniafi, *A companion ideal of a multiplication module*, Period. Math. Hungar., 46(1)(2003), 1-8.

[11] P. F. Smith, *Some remarks on multiplication modules*, Arch. Math., 50(1988), 405-422.

**Patrick F. Smith**
Department of Mathematics
University of Glasgow
Glasgow G12 8QW, Scotland UK
e-mail: Patrick.Smith@glasgow.ac.uk