Minimal Gaussian Curvature Surface

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March 23, 2022

Abstract

This paper deals with finding surfaces in $\mathbb{R}^3$ which are as close as possible to being flat and span a given contour such that the contour is a geodesic on the sought surface. We look for a surface which minimizes the total Gaussian curvature squared. We show that by a change of coordinates the curvature of the optimal surface is controlled by a PDE which can be reduced to the biharmonic equation with an easy-to-define Dirichlet boundary condition and Neumann boundary condition zero. We then state a system of PDEs for the function whose graph is the optimal surface.

1 Introduction

This paper deals with finding surfaces in $\mathbb{R}^3$ which are as close as possible to being flat and span a given contour such that the contour is a geodesic on the sought surface. Explicitly:

Given a smooth closed simple curve in $\mathbb{R}^3$, we will give a system of PDEs for which its solution describes the surface with minimum total Gaussian curvature squared, that spans the given contour. The boundary condition is that the given curve is a geodesic on the target surface (we give definitions in the following section).

I wish to elaborate on the specific choice of energy used and the geodesic boundary condition as this is a new concept, which relates to challenges in computer graphics and computer and human vision which appeared in previous publications (see [1] and references therein). I wanted to be able to recover "nice" surfaces from networks of geodesic curves in 3d space. I was looking for an energy on a surface that will allow me to translate the geodesic requirement for the contour to the boundary conditions for an Euler-Lagrange equation. I could then consider each "cell" in the network of curves separately. I had in mind a piece of leather that is stretched on a shoe last or on a baseball - one starts with a flat piece of leather so it seemed reasonable to check the square of the Gaussian curvature (the seam on a baseball is not a geodesic but may

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have low geodesic curvature). The material, stretchable to some extent but intrinsically flat, would incline to stay flat. I was able to show that the geodesic boundary condition is equivalent to taking Neumann boundary condition zero for the domain of the PDE for the conformal factor for the optimal metric. I prove this in this text.

We use a novel concept where we consider the projection of an ambient open surface containing the unknown optimal surface on an arbitrary affine plane to be the local coordinates, and we then find two coordinate transformation maps which relate the "projection coordinates" to isothermal coordinates. The isothermal coordinates are adjusted so they identify with the projection coordinates when restricted to the contour. The affine plane can be arbitrary as long as the projection is one-to-one. The novelty is in having the two charts, related by transformations: the projection one which allows to ask for a function whose graph is the surface, and the isothermal chart where the Euler-Lagrange with its boundary conditions are computed and proved. (We assume that the local coordinates on the surface can be expressed by one isothermal coordinates patch. In addition, we assume that the optimal surface can be projected one-to-one on at least one affine plane.)

In numerical applications, we do not use the full Euler-Lagrange equations, but we take its linear terms which are a biharmonic operator applied to the conformal factor of the metric.

A related question is the question of prescribing the Gaussian curvature of a surface. This question has been considered in the past. For an open set \( V \subset \mathbb{R}^2 \), and for a function \( u : V \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) and a prescribed Gaussian curvature \( K : V \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) the equation for \( u \) having curvature \( K \) is the following:

\[
\frac{u_{xx}u_{yy} - u_{xy}^2}{(1 + u_x^2 + u_y^2)^2} = K(x, y)
\]  

(1)

Considering a general smooth \( K(x, y) \) defined in \( V \), this equation poses difficulties for even proving the existence of a solution. See Kazdan [4] for more information. We remark that this is a Monge-Ampère equation as it involves the determinant of the Hessian matrix.

This work was done with applications in computer graphics in mind. I wanted to be able to recover a "flat" surface from a network of geodesic curves. However, I believe that this work and its generalization to scalar curvature and in a pseudo-Riemannian setting can be useful in physics, for example in general relativity. In [1] I show how to numerically compute approximations of the minimal Gaussian curvature surfaces. I am currently exploring applications of these surfaces in the area of magnetic field computation and fluid mechanics, where the magnetic field or flow is normal relative to these surfaces.

2 Preliminaries

We give the following definitions.
Definition 1. A surface is a dimension 2 regular submanifold of $\mathbb{R}^3$. A surface inherits a Riemannian metric from the Euclidean metric on $\mathbb{R}^3$. We will regard a surface as a Riemannian manifold with this metric. (See [7], [8].)

Definition 2. We say that a surface $S$ has the graphicality property with respect to an affine plane $H$, if the projection of $S$ on $H$ is one-to-one.

Definition 3. A surface $S$ which can be covered with one local coordinates patch is homeomorphic to $\mathbb{R}^2$. Denote this homeomorphism by $\phi$. If $\Gamma$ is a simple closed curve on $S$ then we can use the Jordan curve theorem to define the interior of $S$ with respect to $\Gamma$ as the preimage under $\phi$ of the interior component in $\mathbb{R}^2$ divided by $\phi(\Gamma)$. We denote it by $\tilde{S}_\Gamma$. We let $S_\Gamma = \tilde{S}_\Gamma \cup \Gamma$.

Let $\Gamma$ be a smooth simple, closed curve in $\mathbb{R}^3$. Throughout the text $\Gamma$ will denote such a curve. We sometimes regard $\Gamma$ as a subset of $\mathbb{R}^3$. We now make reasonable assumptions about $\Gamma$:

1. There exists an open surface $\tilde{S}$ which can be covered with one isothermal coordinates patch with minimum total Gaussian curvature squared on the interior component of the surface with respect to $\Gamma$. The minimum is taken over all the two-dimensional Riemannian manifolds which can be covered with one isothermal coordinates patch, such that the image of the new chart contains, as a subset, the image under the chart of $\tilde{S}$ of the interior component of $\tilde{S}$ with respect to $\Gamma$. In addition, we require that the preimage of the boundary of the original "interior component" under the new chart is a geodesic, and that the preimage (under this new chart) of the boundary of the "interior component" agrees with the metric induced on $\Gamma \subset \tilde{S}$ by $\mathbb{R}^3$ under the natural correspondence induced by the two charts. The total Gaussian curvature is computed on the "interior component" of the two-dimensional manifold under consideration.

2. There exists a plane $\tilde{H}$ such that $\tilde{S}$ has the graphicality property with respect to $\tilde{S}$.

From now on we assume that $\Gamma$ satisfies Properties (1) and (2).

Definition 4. $E_\Gamma(S) = \int_{S_\Gamma} K^2 \text{dvol}_g$ where $g$ is the Riemann metric of $S$ (inherited from $\mathbb{R}^3$). $K$ is the Gaussian curvature at each point of $S$.

If Property (1) holds for $\Gamma$ then $E_\Gamma(\tilde{S}) \leq E_\Gamma(S)$ for any surface $S$ which contains $\Gamma$ as a geodesic and can be covered by one isothermal coordinates patch (see proof of Theorem 2).

3 Main Theorems

We will state the two main theorems shortly and prove them, but we first give an exposition of the strategy that we use.
We assume that we are given $\Gamma$ for which Properties (1) and (2) hold. Let $\tilde{H}$ be the affine plane as in the description of Property (2). The strategy is to use the expression for the Gaussian curvature squared in isothermal coordinates.

Assume for a minute that we know the optimal surface $\tilde{S}$, and we look at the local coordinates which are given by the projection of $\tilde{S}$ to $\tilde{H}$ (identified with $\mathbb{R}^2$). By a change of coordinates we can then assume that in local coordinates the Riemannian metric of the unknown surface is given by:

$$g = e^{2f(x,y)}(dx^2 + dy^2),$$

where $f$ is a smooth real-valued function. These coordinates are called isothermal coordinates. In these coordinates the Gaussian curvature is given by

$$K(x, y) = -e^{-2f(x,y)}\Delta f(x,y),$$

$\Delta$ being the Laplacian operator (computed by The Theorema Egregium, see \[8, p. 90\], appears in Prof. Xianfeng David Gu’s lectures notes found online and also in a more general form in \[2, Section 3.5: Yamabe equation\]). For the existence of isothermal coordinates, see \[5, p. 135–138\] or \[6, p. 376–378\].

We then compute the Euler-Lagrange equation for the Gaussian curvature squared given in the isothermal coordinates, as the energy. We are then able to reduce the Euler-Lagrange equation to be the biharmonic equation for the function $f$ in the exponent, which relates the original metric to the flat metric.

The following PDE is the result of the Euler-Lagrange computation:

$$
\begin{pmatrix}
    f_{xx}^2 \\
    f_{xx}f_{yy} \\
    f_x^2f_{xx} \\
    f_x^2f_{yy} \\
    f_xf_{xxx} \\
    f_xf_{xyy} \\
    f_{xxxx} \\
    f_{xxyy} \\
    f_y^2 \\
    f_y^2f_{xx} \\
    f_y^2f_{yy} \\
    f_yf_{yyy} \\
    f_yf_{xxy} \\
    f_{yyyy}
\end{pmatrix}
^T
\begin{pmatrix}
    \frac{-3}{2} \\
    -6 \\
    4 \\
    4 \\
    -4 \\
    -4 \\
    1 \\
    2 \\
    -3 \\
    4 \\
    4 \\
    -4 \\
    -4 \\
    1
\end{pmatrix}
= 0. \quad (2)
$$

This equation looks intimidating but heuristically in applications one can replace it with the biharmonic equation:

$$f_{xxxx} + 2f_{xxyy} + f_{yyyy} = 0, \quad (3)$$

as all other terms consist of two or more factors, and are expected to be small. The biharmonic equation is a well-studied equation in the literature.

The boundary conditions for the PDE constraining the function $f$ are deduced by simple arguments using the fact that the metric is conformal, and that boundary of the surface is a geodesic in the sense we defined it.

Recall that $\Gamma$ is the original contour in $\mathbb{R}^3$, and let $\tilde{S}$ and $\tilde{H}$ be as in Property (1) and (2). We denote by $\Gamma$ the projection of $\Gamma$ on the affine plane $\tilde{H}$. Let
\( \pi : \Gamma \to \bar{\Gamma} \) be the projection map. Let \( \gamma(t) \) be a parameterization (arc-length or not) of \( \bar{\Gamma} \) with length \( L \). In order to be able to take a derivative of \( \gamma \) at 0, we regard its domain as \( \mathbb{R}/L\mathbb{Z} \). We denote by \( \phi : \bar{\Gamma} \to \mathbb{R}^2 \), the projection of \( \bar{\Gamma} \) on \( \bar{\bar{\Gamma}} \) composed with an identification of \( \bar{\bar{\Gamma}} \) with \( \mathbb{R}^2 \). Lastly, let \( \Omega = \phi \big( \bar{\Gamma} \big) \).

We prove the following theorem,

**Theorem 1.** Given a smooth simple closed curve \( \Gamma \) for which Properties (1) and (2) hold. Let \( \bar{\Gamma} \) and \( \bar{\bar{\Gamma}} \) be as in the statement of the Property (1) and (2), with respect to \( \Gamma \). Let \( g = e^{2f(x,y)}(dx^2 + dy^2) \) be a Riemannian metric on \( \bar{\Gamma} \) for a coordinate chart which maps \( \Gamma \) to \( \bar{\Gamma} \) by projection, then \( f \) satisfies the following PDE:

\[
\begin{align*}
\text{Equation (2) in } \Omega, \\
f(\gamma(t)) &= \frac{1}{2} \cdot \log \left( t \right), \quad t \in [0, L], \quad \text{on } \partial \Omega, \\
\frac{\partial f}{\partial n} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(4)

We give concise proofs for the PDE in \( \Omega \), and the Dirichlet and Neumann boundary conditions.

**Proof (PDE for \( f \) inside the domain).** The volume form for the surface \( \bar{\Gamma} \) in the isothermal coordinates is given by \( \sqrt{\det g} \, dxdy = e^{2f(x,y)} dxdy \), therefore the functional to be minimized is:

\[
\int_{\bar{\Gamma}} K^2 d\text{vol}_g = \int_{\Omega} e^{-2f(x,y)}(f_{xx} + f_{yy})^2 dxdy.
\]

Working out the Euler-Lagrange equation yields Equation (2).

**Proof (Dirichlet Boundary Condition).** For any two points \( x, y \in \partial \Omega \), we know the length on \( \bar{\Gamma} \) of the preimage of each of the two arcs connecting \( x \) to \( y \). It is the length of the segment of \( \Gamma \) connecting \( \pi^{-1}(x) \) and \( \pi^{-1}(y) \). The Dirichlet boundary condition is set to agree with this observation.

**Proof (Neumann Boundary Condition).** Assume that there is an \( x \in \partial \Omega \) for which \( \frac{\partial f}{\partial n}(x) \neq 0 \). The function \( f \) is smooth, therefore there exists a neighborhood of \( x \) (in the image of the coordinate chart of \( \bar{\Gamma} \)) on which \( \frac{\partial f}{\partial n} \neq 0 \). The neighborhood we choose can be arbitrarily small. This shows that there is a shorter closed curve than \( \Gamma \) on \( \bar{\Gamma} \) which is equal to \( \Gamma \) except for the preimage of a curve segment in an arbitrarily small neighborhood of \( x \) under the coordinate chart. To see this: we can always make a detour around \( x \) in the image of the coordinate chart of \( \bar{\Gamma} \) which increases as little as we wish the factor \( dx^2 + dy^2 \) in the metric taken along the new curve, while \( e^{2f} \) is bounded from above by \( e^{2f(x)} \cdot \delta \) for some \( \delta > 0 \). This is a contradiction to the assumption that \( \Gamma \) is a geodesic on \( \bar{\Gamma} \). (See [3, Chapter 1] for details.)
We now prove the following theorem, which can be viewed as a corollary of Theorem 1.

**Theorem 2.** Given a smooth simple closed curve $\Gamma$ for which Properties (1) and (2) hold. Let $\tilde{S}$ and $\tilde{H}$ be as in the statement of the Property (1) and (2), with respect to $\Gamma$. Assume $\tilde{H} = \mathbb{R}^2$, and let $\Omega$ be the projection of $\tilde{S}\Gamma$ on $\mathbb{R}^2$. Let $h : \Omega \to \mathbb{R}$ be a function which is $C^\infty$ on $\Omega$ and continuous on $\overline{\Omega}$, such that the graph of $h$ is $\tilde{S}\Gamma$. (By Properties (1) and (2) such a function exists.) Let $f$ be the solution of Equation (4) in Theorem 1. Let $\tilde{K}(x, y) = -e^{-2f(x,y)}\Delta f(x, y)$ be a function on $\Omega$. Let $E(u, v) = 1 + h_u^2(u, v)$, $F(u, v) = h_u(u, v)h_v(u, v)$ and $G(u, v) = 1 + h_v^2(u, v)$. Then there is a coordinate change $x(u, v), y(u, v)$, such that $h, x, y$ satisfy the following equations (for $(u, v) \in \Omega$ in the first three equations):

\[
\begin{align*}
\frac{h_{uu}h_{vv} - h_{uv}^2}{(1 + h_u^2 + h_v^2)^2} &= \tilde{K}(x(u, v), y(u, v)), \\
\frac{\partial}{\partial v} F_{xu} - Ex_v + \frac{\partial}{\partial u} F_{xv} - Ex_u &= 0, \\
\frac{\partial}{\partial v} F_{yu} - Ey_v + \frac{\partial}{\partial u} F_{yv} - Ey_u &= 0,
\end{align*}
\]

(5)

**Proof.** If the isothermal coordinates covering $\tilde{S}$, which exist by Property (1), are given by $(\xi, \eta)$, then a coordinate change to new isothermal coordinates $(x, y)$ is given by the following equations: $x_{\xi\xi} + x_{\eta\eta} = 0$, $y_{\xi\xi} + y_{\eta\eta} = 0$ (see [1, p. 135–138]). By prescribing Dirichlet boundary values agreeing with the conditions in Theorem 1 for these Laplace equations, we can obtain isothermal coordinates $(x, y)$. This shows that isothermal coordinates, which agree with the conditions in Theorem 1 exist.

We assume that $(u, v)$ are the coordinates on $\tilde{S}$ obtained by the projection of the surface on $\mathbb{R}^2$. If $h$ is the inverse function of this projection then $E, F, G$ are given by the identities stated. Then the maps $x(u, v), y(u, v)$ are the solutions of the two Laplace-Beltrami equations in the above system with the two Dirichlet boundary conditions. See [1, p. 135–138] for details.

If $f(x, y)$ satisfies Equation (4) then $\tilde{K}(x, y) = -e^{-2f(x,y)}\Delta f(x, y)$ is the optimal Gauss curvature at $(x, y)$. The function $h$, whose graph is $\tilde{S}\Gamma$, is the solution of the Monge-Ampère equation stated.

\[\square\]

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