Some minimal degree Vassiliev invariants
not realizable by the HOMFLY and Kauffman polynomial

This is a preprint. I would be grateful for any comments and corrections!

A. Stoimenow

Ludwig-Maximilians University Munich, Mathematics
Institute, Theresienstraße 39, 80333 München, Germany,
e-mail: stoimeno@informatik.hu-berlin.de,

WWW: http://www.informatik.hu-berlin.de/~stoimeno

Current version: March 30, 2022 First version: June 29, 1998

Abstract. We collect some examples showing that some Vassiliev invariants are not obtainable from the HOMFLY and Kauffman polynomials in the real sense, namely, that they distinguish knots not distinguishable by the HOMFLY and/or Kauffman polynomial.

Keywords: Vassiliev invariants, HOMFLY polynomial, Kauffman polynomial, chirality.

1. Introduction

Briefly after the introduction of Vassiliev knot invariants [Va], it was discovered [BL], that a lot of such invariants can be obtained from knot polynomials [H, J, Ka], basically by taking some coefficient in a version of the polynomial where some variable was replaced by an exponential power series. In a subsequent series of papers, several authors (see [MR] and loc. cit. and [Da]) independently computed the number of Vassiliev invariants of given degree arising in this way from the HOMFLY and Kauffman polynomials. This, however, is a priori, only a lower bound for the number of Vassiliev invariants of given degree obtainable from the knot polynomials as it is not clear that no more can be generated in some other way. For example, one could take some long polynomial expression of higher degree Vassiliev invariants coming from the polynomial and little seems to be known about how to control from below the degree of the resulting Vassiliev invariant. Of course, one of the intrinsics of Vassiliev theory (see [BN]) is that Vassiliev invariants form a symmetric (i.e., commutative polynomial) algebra over a (primitive) subspace of them, but trying to prove general degree bounds by looking for the expression of a given Vassiliev invariant as a polynomial in some fixed primitive basis is practically impossible, because already the question of the number of primitive Vassiliev invariants of given degree is far from being solved and one of the most intriguing in the whole area (see [CD, Da2, Kn, Za]). See, however, [St, §3].

Given this subtlety, in this note we collect some examples showing that some Vassiliev invariants are not obtainable from the HOMFLY and Kauffman polynomials in the real sense, namely, that they distinguish knots not distinguishable by the HOMFLY and/or Kauffman polynomial. We hope these examples to be interesting at least because (to the best of my knowledge) they never appeared explicitly before in this context. All these examples are in a way related to problems of detecting chirality with the knot polynomials.

Acknowledgement. I would wish to thank to W. B. R. Lickorish, H. Morton, D. Zagier, L. Kauffman, T. Kanenobu and P. Johnson for their helpful remarks.

*Supported by a DFG postdoc grant.
2. **A degree 5 Vassiliev invariant not obtainable from the HOMFLY polynomial**

It is known that the HOMFLY polynomial can be used to obtain all Vassiliev invariants of degree at most 4 and two of the three primitive Vassiliev invariants of degree 5.

There are 4 knots with 10 crossings with self-conjugate HOMFLY polynomial but non-self-conjugate Kauffman polynomial, see figure 1. Each one of them together with its obverse provide an example of two knots not distinguishable by the HOMFLY polynomial but distinguishable by a degree 5 Vassiliev invariant, showing that the third primitive Vassiliev invariant of degree 5 is not obtainable from the HOMFLY polynomial.

![Figure 1](image1)

**Figure 1:** The 4 knots with 10 crossings with self-conjugate HOMFLY polynomial but non-self-conjugate Kauffman polynomial.

3. **A degree 6 Vassiliev invariant not obtainable from the Kauffman polynomial**

P. Johnson [Jo] used the tables [St2] to obtain all Vassiliev invariants of degree at most 5 from the Kauffman polynomial and also 4 of the 5 primitive Vassiliev invariants of degree 6, but not the fifth.

The example that the Kauffman polynomial does not contain a degree 6 Vassiliev invariant is given by the two 11 crossing alternating knots with equal Kauffman polynomials but different Conway polynomials, see figure 2. In Thistlethwaite’s notation [HT] the knots are \(11_{30}\) and \(11_{189}\). They Conway polynomials are \(\nabla(11_{30}) = -2x^6 + x^4 - x^2 + 1\) and \(\nabla(11_{189}) = x^8 + 2x^6 + x^4 - x^2 + 1\), which differ at the coefficient of \(x^6\), which is a degree 6 Vassiliev invariant [BN]. This shows that the Kauffman polynomial misses a degree 6 Vassiliev invariant, which is contained in the Conway polynomial and hence also in the HOMFLY polynomial. Thus, the HOMFLY and Kauffman polynomials together exhaust all Vassiliev invariants up to degree 6.

![Figure 2](image2)

**Figure 2:** The pair of 11 crossing knots with equal Kauffman polynomials but different Conway polynomials.
Remark 3.1 The pair of knots $11_{30}$ and $11_{189}$ was already known to Lickorish at the 1986 Santa Cruz conference [K, remark on p. 472 top], probably from the 11 crossing knot (polynomial) tabulation, but has been reconstructed in a more elucidative way by Kanenobu [K, theorem 5]. Kanenobu observed that these knots can be used to provide a counterexample to Lou Kauffman’s hope [Ka, p. 428] that the Kauffman polynomial always detects chirality when the HOMFLY polynomial does so. Consider the knot $11_{30}\#11_{189}$. Then it has self-conjugate Kauffman polynomial but non-self-conjugate HOMFLY polynomial. Note also, that whenever the signature of a knot is not divisible by 4, it has negative determinant $\Delta(-1)$, which is clearly exhibited by the HOMFLY polynomial. So the HOMFLY polynomial detects chirality in all such cases even if it is self-conjugate, as for $9_{42}$ (where the Kauffman polynomial is also self-conjugate) and $10_{125}$, although not being able to distinguish between the knot and its obverse. Therefore, care should be taken that the formulations “K has self-conjugate polynomial” and “the polynomial does not detect chirality of K” are not quite equivalent!

4. A degree 7 Vassiliev invariant not obtainable from the Kauffman and HOMFLY polynomial

Soon after the discovery of the three knot polynomials capable of detecting chirality of knots, the knot $9_{42}$ became prominent by hiding its chirality to all of them. A less noted relative of $9_{42}$ is $10_{71}$, which has a crossing more but is more interesting because it has additionally zero signature (contrarily to $9_{42}$).

Morton and Short [MS] developed a program allowing to exhibit $9_{42}$’s chirality by showing non-symmetry of the HOMFLY polynomial of its untwisted 2-cable. An attempt to apply the same procedure on $10_{71}$ meets some (resource requirement) difficulties, as $10_{71}$ has braid index 5 (see [J, appendix]). Hence, a priori, one has only a 10-braid representation of its untwisted 2-cable. This was revealed to de indeed minimal by the Morton-Williams-Franks inequality (see [J]) and the result of the calculation shown on figure 3. For its interpretation, if not self-explaining, we refer to [MS].

Now, non-symmetry of the HOMFLY polynomial can be shown in terms of Vassiliev invariants by evaluating at $i = \sqrt{-1}$ derivations of the polynomial given by the rows of the output, i.e. the coefficients of the $z$ powers (in the convention of [MS]). Then the $a$-th $v$-derivation at $v = i$ of the coefficients of $z^b$ is a Vassiliev invariant of degree at most $a + b$ (by arguments analogous to [BL]). The difference between the polynomial and its conjugate comes out as a Vassiliev invariant of degree 7 (e.g., for $b = 1$ and $a = 6$), which is therefore not contained neither in the HOMFLY nor in the Kauffman polynomial, but in the HOMFLY polynomial of the untwisted 2-cable.

Remark 4.1 Przytycki conjectures that all Vassiliev invariants of degree at most 10 are contained in the HOMFLY and Kauffman polynomials and their untwisted 2-cables. A counterexample to this conjecture would require a pair of knots not distinguishable by all of them. Such pairs are mutants [Co, LL, P]. However, there is no Vassiliev invariant known distinguishing mutants below degree 11 (see [MC]). Are there more such pairs?
Figure 4: The 2 chiral knots with at most 10 crossings with self-conjugate HOMFLY and Kauffman polynomials.

References

[BN] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology 34 (1995) 423–472.

[BL] J. S. Birman and X-S. Lin, *Knot polynomials and Vassiliev’s invariants*, Invent. Math. 111 (1993) 225–270.

[CD] S. V. Chmutov and S. V. Duzhin, *A lower bound for the number of Vassiliev knot invariants*, preprint.

[Co] J. H. Conway, *On enumeration of knots and links*, in “Computational Problems in abstract algebra” (J. Leech, ed.), 329–358. Pergamon Press, 1969.

[Da] O. T. Dasbach, *A remark on the HOMFLY-Vassiliev invariants*, Düsseldorf University preprint, October 1996.

[Da2] O. T. Dasbach, *On the combinatorial structure of primitive Vassiliev invariants III — a lower bound*, Düsseldorf University preprint, September 1997. See also math/9806086.

[HT] J. Hoste and M. B. Thistlethwaite, *KnotScape*, a knot polynomial calculation program. Available at http://www.math.utk.edu/~morwen.

[H] P. Freyd, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu and D. Yetter, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. 12 (1985), 239–246.

[Jo] P. Johnson, to appear.

[J] V. F. R. Jones, *A polynomial invariant of knots and links via von Neumann algebras*, Bull. Amer. Math. Soc. 12 (1985), 103–111.

[K] T. Kanenobu, *Examples of polynomial invariants for knots and links II*, Osaka J. Math. 26(3) (1989), 465–482.

[Ka] L. H. Kauffman, *An invariant of regular isotopy*, Trans. Amer. Math. Soc. 318 (1990), 417–471.

[Kn] J. A. Kneissler, *The number of primitive Vassiliev invariants up to degree twelve*, University of Bonn preprint, June 1997. See also q-alg/9706022.

[LL] W. B. R. Lickorish and A. S. Lipson, *Polynomials of 2-cable-like links*, Proc. Amer. Math. Soc. 100 (1987), 355–361.

[MR] M. McDaniel and Y. Rong, *On the dimensions of Vassiliev invariants coming from link polynomials*, George Washington University preprint, March 1997.

[MS] H. R. Morton and H. B. Short, *The 2-variable polynomial of cable knots*, Math. Proc. Camb. Philos. Soc. 101 (1987), 267–278.

[MC] H. R. Morton and P. R. Cromwell, *Distinguishing mutants by knot polynomials*, Jour. of Knot Theory and its Ramifications 5(2) (1996), 225–238.

[P] J. Przytycki, *Equivalence of cables of mutants of knots*, Canad. J. Math., 41 3(2), 1989, 250–273.

[St] A. Stoimenow, *On some restrictions to the values of the Jones polynomial*, Humboldt University Berlin preprint, October 1997.

[St2] A. Stoimenow, *Polynomials of knots with up to 10 crossings*. Available on my webpage.

[Va] V. A. Vassiliev, *Cohomology of knot spaces*, Theory of Singularities and its Applications (Providence) (V. I. Arnold, ed.), Amer. Math. Soc., Providence, 1990.

[Za] D. Zagier, *Vassiliev invariants and a strange identity related to the Dedekind eta-function*, Max-Planck-Institut für Mathematik (Bonn) preprint, May 1998.