Direct exposure to volatility has been made easier for a wide range of underlyings by the creation of standardized instruments. The widespread use and increasing liquidity of volatility products, such as index futures and variance swaps, clearly show that investors are taking a keen interest in volatility. In addition to short-term trading ideas, some investors have been seeking structural exposure to volatility because they consider it either a well-identified asset class or, at the very least, a set of strategies with strong diversifying potential for their portfolios. Basically, standard exposure to volatility can be achieved through two complementary strategies—on the one hand, long exposure to implied volatility and, on the other hand, exposure to the volatility risk premium. Both strategies are consistent with the traditional rationales required by investors to invest in an asset class—potential for return enhancement and risk diversification.

Being long implied volatility is compelling to investors for diversification purposes (Daigler and Rossi [2006] and Dash and Moran [2005]). The remarkably strong negative correlation between implied volatility and equity prices during market downturns offers timely protection against the risk of capital loss. This relationship has been well documented in the academic literature (Turner, Starz, and Nelson [1989]), Haugen, Talmor, and Torous [1991], and Glosten, Jangannathan, and Runkle [1993]) and has led to two theoretical explanations. The first explanation is the leverage effect (Black [1976], Christie [1982], and Schwert [1989]); namely, an equity downturn increases the leverage of the firm and thus is expected to increase the risk of the stock. The alternative explanation (French, Schwert, and Stambaugh [1987], Bekker and Wu [2000], Wu [2001], and Kim, Morley, and Nelson [2004]) is the volatility feedback effect in which, assuming volatility is incorporated into stock prices, a positive volatility shock increases the future required return on equity, so that the stock’s price will be expected to simultaneously fall.

Historically, exposure to the volatility risk premium has delivered very attractive risk-adjusted returns (Egloff, Leippold, and Wu [2006] and Hafner and Wallmeier [2008]). As documented by Bakshi and Kapadia [2003], Bondarenko [2006], and Carr and Wu [2009], implied variance is higher on average than ex post realized variance and can be explained by the risk asymmetry between a short volatility position (a net seller of options faces an unlimited potential loss) and a long volatility position (where the loss is capped at the premium). Moreover, going long the variance swap contract can be seen as a hedge against the risks associated with the random arrival of discontinuous price movements. To make up for the uncertainty on the future level of realized volatility, sellers of implied volatility demand compensation in the form
of a premium over the expected realized volatility.\textsuperscript{1} Qualitatively, the variance risk premium is consistent with the capital asset pricing model (CAPM) framework and the well-documented negative correlation between stock index returns and their variance. Carr and Wu [2009] showed, however, that this negative correlation does not fully account for the negative variance risk premium. Other traditional equity risk factors, such as size, book-to-market, and momentum, likewise cannot explain the negative variance risk premium. The majority of the premium is thus explained by an independent variance risk factor, which relates to the willingness of investors to receive an excess return not only because volatility hikes are seen as signals of equity market downturns, but also because these hikes by themselves are seen as unfavorable shocks on investors’ portfolios (through the reduction of Sharpe ratios, for instance).

The scope of this article is to analyze investors’ portfolio choices when volatility is added to their investment opportunity set. Taking the perspective of a long-term U.S. equity investor, we first design standard volatility strategies and then build efficient frontiers within a mean-modified Value at Risk (VaR) framework. Our study is related to the strand of the literature that examines the asset allocation problem in the presence of derivatives. Carr and Madan [2001] studied how to choose options at different strikes to span the random jump risk in the stock price, and Liu and Pan [2003] analyzed spanning the variance risk. In the same vein, Egloff, Leippold, and Wu [2006] used variance swaps at different maturities to span the variance risk and benefit from large variance risk premia. Furthermore, volatility as an investment theme is often associated with the universe of alternative strategies, identified as a source of \textit{alternative beta} (Kuenzi [2007]), that is to say a source of returns linked to systematic exposure to a risk factor, but not directly investable through conventional asset classes. Another strand of investment research related to our article analyzes the interest of having different sources of alternative beta, such as hedge funds (Amin and Kat [2003] and Amenc, Goltz, and Martellini [2005]), in a portfolio.

We believe our research is original for two reasons. First, it offers a framework for analyzing the inclusion of volatility strategies in a portfolio, departing from most of the previous literature, which employs mean–variance optimization. Adding volatility strategies to the investment opportunity set raises the issue of how to measure a portfolio’s expected utility when returns are non-normal. This issue is crucial because the returns to volatility strategies are asymmetric and leptokurtic. Accordingly, the goal of minimizing risk through a conventional mean–variance optimization framework can be misleading, because extreme risks are not properly captured (Sornette, Andersen, and Simonetti [2000] and Goetzmann et al. [2002]). For volatility premium strategies, low volatility of returns is generally countered by higher negative skewness and higher kurtosis, which could prove costly for investors if not properly taken into account (Amin and Kat [2003]). Thus, appropriate optimization techniques must be used in order to assess risk through measures that capture higher-order moments of the return distribution. For our purposes, modified Value at Risk meets that requirement (Favre and Galeano [2002], Agarwal and Naik [2004], and Martellini and Ziemann [2007]). The practical implementation of exposing a portfolio to volatility also needs to be addressed. Because these strategies are implemented through derivative products, they require limited capital, so that leverage is the key factor. In this case, the amount of risk to be taken in the portfolio needs to be properly calibrated.

Second, our research combines two complementary sets of volatility exposures, which to date have been examined separately. Daigler and Rossi [2006] analyzed the effect of adding a long volatility strategy to an equity portfolio, and Dash and Moran [2005] studied the impact of adding a long volatility strategy to a fund of hedge funds. Egloff, Leippold, and Wu [2006] and Hafner and Wallmeier [2008] examined the contribution to an equity portfolio of volatility risk premium strategies. Because investing in the volatility premium is a similar strategy to selling an insurance premium—which exhibits very high downside risk—Egloff, Leippold, and Wu highlighted the need to hedge such an investment, at least partially. They showed that under two-factor risk dynamics two distinct variance swap contracts, of any maturity, span the variance risk. They proposed a partial hedge of the short-term volatility risk premium by assuming a short position in the stock index and a long position in the long-term volatility risk premium.

From this point of view, our research is related to the work of Egloff, Leippold, and Wu and adds to this strand of the literature by demonstrating the usefulness of approaching volatility through the combination of two standard, complementary strategies. A long implied-volatility strategy is an excellent hedge against the risks engendered by investing in the volatility risk premium.
In general, adding a long position in implied volatility leads to lower levels of risk as this strategy typically hedges downside equity risk. Meanwhile, investing in the volatility risk premium enhances returns for a given level of risk. Finally, a combination of the two strategies is very appealing because they tend to hedge each other in adverse events. This combination can deliver compelling results for the portfolio, not just in terms of enhanced returns, but also in terms of risk diversification.

**INVESTING IN VOLATILITY**

We begin by presenting two standard and complementary approaches to investing in volatility. We then address the practical issues of risk measurement and calibration in order to properly add the two designed strategies to the investment opportunity set of an equity investor.

**Long Exposure to Volatility**

One approach to volatility investing is to expose a portfolio to implied-volatility changes in an underlying asset. The primary rationale for this kind of investment is the diversification benefits that arise from the strong negative correlation between the performance and the implied volatility of the underlying, particularly during market downturns (Daigler and Rossi [2006]). Tracking implied volatility for a specific underlying requires the computation of a synthetic volatility indicator. A volatility index, expressed in annualized terms, prices a portfolio of available options across a wide range of strikes (volatility skew) and with a constant maturity (interpolation on the volatility term structure). Within the family of volatility indices, the Chicago Board Options Exchange (CBOE) Volatility Index® (VIX®) is widely used as a benchmark by investors. The VIX is the expression of the 30-day implied volatility generated from S&P 500 traded options. The details of the index calculation are given in a white paper published by the CBOE [2004]. Because the VIX reflects a consensus view of short-term volatility in the equity market, the index is widely used as a measure of market participants’ risk aversion, also known as the investor fear gauge. A time series of the VIX is plotted in Exhibit 1 for the period February 1990–August 2008.

Although the VIX is not a tradable product, the Chicago Futures Exchange launched futures contracts on VIX in March 2004. Consequently, investors now have a direct way of exposing their portfolios to variations in the short-term implied volatility of the S&P 500. VIX futures provide a better alternative to achieving such exposure than traditional approaches, which rely on the use of delta-neutral combinations of options, such as straddles (at-the-money call and put), strangles (out-of-the-money call and put), or more complex strategies (volatility-weighted combinations of calls and puts). On short maturities (less than three months), the impact of neutralizing the delta exposure of these portfolios can easily dominate the impact of implied volatility variations.

The approach we take in establishing a structurally long investment in implied volatility attempts to take advantage of the mean-reverting nature of volatility (Dash and Moran [2005]). We achieve this by calibrating the exposure according to the absolute levels of the VIX. The highest exposure occurs when implied volatility is at historically low levels, and exposure is reduced as volatility rises. Implementing the long volatility (LV) strategy consists of buying a number of VIX futures such that the impact of a one-point variation in the price of the future is equal to $\frac{1}{F_t} \times 100\%$. The profit and loss generated between $t-1$ (contract date) and $t$ (maturity date) can then be written as

$$PL_{t}^{LV} = \frac{1}{F_{t-1}} (F_{t} - F_{t-1})$$

where $F_t$ is the price of the future at time $t$.

In practice, VIX futures prices are available only since 2004. They represent the one-month forward market price for 30-day implied volatility. This forward-looking component is reflected in the existence of a term premium between the VIX future and the VIX. The premium tends to be positive when volatility is low.
Capturing the Volatility Risk Premium

The second volatility strategy involves creating an exposure to the difference between implied and realized volatility. As already stressed earlier in the article, this difference has, on average, been historically strongly positive for equity indices (Bakshi and Kapadia [2003] and Bondarenko [2006]), delivering very attractive risk-adjusted returns (Egloff, Leippold, and Wu [2006] and Carr and Wu [2009]).

The variance risk premium, as shown in Exhibit 2 for the period February 1990–August 2008, is captured by entering into a variance swap (i.e., a swap contract on the spread between the implied and realized variances). Through an over-the-counter transaction, the two parties agree to exchange a specified implied variance level for the level of variance realized over a specified period. The implied variance at inception is the level that gives the swap a zero net present value. Variance swaps on major equity indices are actively traded today.

We consider a short variance swap strategy on the S&P 500 held over a one-month period. The profit and loss of a short variance swap position between the start date, \( t - 1 \), and the end date, \( t \), can be written as

\[
PL^{VARSWAP}_t = \frac{N_{VEGA}}{2K_{t-1}} \left[ K_{t-1}^2 - RV_{t-1,t}^2 \right] 
\]

where \( K_{t-1} \) is the volatility strike of the variance swap contract entered at date \( t - 1 \), \( RV_{t-1,t} \) is the realized volatility between \( t - 1 \) and \( t \), and \( N_{VEGA} \) is the vega notional of the contract (see the appendix for further details on variance swaps).

A primary attraction of the variance swap investment is that it is a linear contract in variance risk. Thus, a variance swap does not create additional delta exposure to the underlying as would be the case with a strategy involving vanilla options. From a theoretical point of view, the strike level of the variance swap is computed from the price of the portfolio of options that is used to calculate the volatility index itself. Therefore, the theoretical strike of a one-month variance swap on the S&P 500 is the value of the VIX (Carr and Wu [2006, 2009]).

In practice, a number of difficulties arise in replicating a volatility index, and synthetic variance swap rates are exposed to measurement errors due to the no-price-jump hypothesis, bid–ask spreads, and state dependencies. Carr and Wu [2009] conducted a large panel of robustness checks and concluded that the errors generated are usually small. According to Standard & Poor’s [2008], the VIX level has to be reduced by one point to fully reflect the average replication costs borne by arbitragers. Furthermore, computing realized volatility also depends on the type of returns used (log returns versus standard returns), the data frequency (high-frequency data versus daily data), and the annualization method, as outlined by Wu [2005], Carr and Wu [2009], and Bollerslev, Gibson, and Zhou [forthcoming]. Referring to the most liquid and standardized variance swap contracts, for which realized volatility is computed with daily log returns annualized on a 252–business–day basis (JPMorgan [2006] and Standard and Poor’s [2008]), we have chosen this market standard for our computations.

Adequate Risk Measure

A key issue in implementing volatility strategies is the non-normality of returns distributions, which is discussed in the next section. When returns are not normally distributed, the mean–variance criterion of Markowitz
[1952] is no longer adequate. To compensate for this, many authors have sought to include higher-order moments of the return distribution in their analyses. Lai [1991] and Chunhachinda et al. [1997], for example, introduced the third moment of the return distribution (i.e., skewness) and showed that it leads to significant changes in the optimal portfolio construction. Extending portfolio selection to a four-moment criterion brings a further significant improvement (Jondeau and Rockinger [2006, 2007]).

Within the proposed volatility investment framework, the main danger for the investor is the risk of substantial losses in extreme market scenarios or, in other words, the left tail of the return distribution. Because returns on volatility strategies are not normally distributed, we chose modified Value at Risk as our reference measure of risk. Value at Risk is defined as the maximum potential loss over a period of time given a specified probability, \( \alpha \). Within normally distributed returns, VaR can be written as

\[
VaR(1 - \alpha) = - (\mu + z_\alpha \sigma)
\]

where \( \mu \) and \( \sigma \) are the mean and standard deviation of the return distribution, respectively, and \( z_\alpha \) is the \( \alpha \)-quantile of the standard normal distribution \( N(0,1) \).

To capture the effect of non-normal returns, we replace the quantile of the standard normal distribution with the “modified” quantile of the distribution \( w_\alpha \), approximated by the Cornish–Fisher expansion based on a Taylor series approximation of the moments (Stuart, Ord, and Arnold [1999]). This substitution enables us to correct the distribution \( N(0,1) \) by taking skewness and kurtosis into account. Modified VaR is written as

\[
ModVaR(1 - \alpha) = - (\mu + w_\alpha \sigma)
\]

where

\[
w_\alpha = z_\alpha + \frac{1}{6} (z_\alpha^3 - 1) S + \frac{1}{24} (z_\alpha^4 - 3z_\alpha^2) EK - \frac{1}{36} (2z_\alpha^5 - 5z_\alpha^3) S^2
\]

The variable \( w_\alpha \) is the modified percentile of the distribution at threshold \( \alpha \), \( S \) is the skewness, and \( EK \) is the excess kurtosis of the portfolio.

Two portfolios that offer the same expected return for a given level of volatility will have the same “normal” VaR, but will have a different modified VaR if their returns present different skewness or/and excess kurtosis. In particular, modified VaR will be greater for the portfolio that has negative skewness (left-handed return distribution) and/or higher excess kurtosis (leptokurtic return distribution). In addition to being simple to implement in the context of constructing the investor’s risk budget, modified VaR explicitly takes into account how the investor’s utility function changes in the presence of returns that are not normally distributed. A risk-averse investor will prefer a return distribution in which the odd moments (expected return and skewness) are positive and the even moments (variance and kurtosis) are low.

Calibrating Volatility Strategies

In practice, cash requirements are very limited when volatility strategies are implemented because the required exposure is achieved through listed or OTC derivatives. Capital requirements are limited to the collateral needed for entering the swap contracts and for futures margin deposits of listed products. A key step in the process of investing in the volatility asset class is the proper calibration of the strategies.

Based on raw computations of modified VaR for each asset, each volatility exposure is normalized by setting its monthly modified 99% VaR to the targeted level of the equity asset class. Once calibrated, the return of each volatility strategy is thus the risk-free rate of return plus a fixed proportion—we will refer to this as the degree of leverage—of the strategy’s profit and loss. The leverage is determined ex ante by our calibration methodology,

\[
\begin{align*}
  r_t^{LV} &= r_L + L_1 \cdot P L_{LV}^{fix} \\
  r_t^{VRP} &= r_L + L_2 \cdot P L_{VRP}^{fix}
\end{align*}
\]

where \( r_t^{LV} (r_t^{VRP}) \) is the monthly return of the LV (VRP) strategy, \( r_L \) is the cash return, and \( L_1 (L_2) \) is the degree of leverage calibrated on the LV (VRP) strategy.

BUILDING AN EFFICIENT PORTFOLIO WITH VOLATILITY

We will now discuss the investment case of adding the two volatility strategies to the opportunity set of a
long-term investor whose portfolio is fully invested in U.S. equities.

**Data**

The S&P 500 and CBOE VIX indices and the one-month U.S. LIBOR time series are used for the equity asset class, volatility strategies, and risk-free rate, respectively. The study covers the period between February 1990 and August 2008. To gauge the stability of our results, the data sample is divided into two equal subperiods covering approximately nine years of data. The first subperiod is used to calibrate the risk exposure to volatility strategies, and the second is used to measure out-of-sample performance. Moreover, we also control for the stability of our results depending on the choice of the starting date. Four different time series of non-overlapping one-month returns are compared by beginning each monthly investment at four different dates during the month. The reported results exhibit average summary statistics computed on each of the four series of returns.

**Summary Statistics**

Exhibit 3 presents the descriptive statistics of the equity, LV, and VRP strategies during the entire sample period. Based on the Sharpe ratio and annualized return, the VRP strategy appears to be the most attractive, with a global Sharpe ratio of 2.3 and an annualized return of 37.1%. The equity and LV strategies rank second and third with Sharpe ratios of 0.4 and 0.2, respectively, and annualized returns of 7.90% and 7.03%, respectively. Although the LV strategy is last in the ranking, it offers a considerable benefit during market downturns.

The VRP strategy is the most consistent winner. It exhibits relatively stable performance, except during periods of rapidly increasing realized volatility, such as the onset of crises and unexpected market shocks, when returns are strongly negative (i.e., realized volatility rises above implied volatility) and much greater in amplitude than for the traditional asset classes. These periods are usually of short duration, accounting for only 15% of the months in the study period.

Given the calibration we used in the analysis, the modified VaR for the three strategies—equity, LV, and VRP—is 10.6%, 7.8%, and 11.4%, respectively. Analysis of extreme returns (minimum and maximum monthly returns) highlights the asymmetry of the two volatility strategies. The LV strategy offers the highest monthly return (31.3%) and a minimum return of −12.5%, whereas the VRP strategy posts the worst monthly performance (−16.3%) with its highest monthly return equal to 11.7%. The higher-order moments clearly highlight that returns are not normally distributed, especially for the two volatility strategies. The skewness of the equity returns is slightly negative (−0.2), but the skewness of the VRP strategy returns is very strongly negative (−1.7). These empirical results are consistent with the evidence of a strong significant jump component in the variance rate process in addition to a significant jump component in the stock index return (Eraker, Johannes, and Polson [2003] and Wu [2005]). The LV strategy is the only strategy showing positive skewness (1.6), which suggests that a long position in implied volatility provides a partial hedge to the leftward asymmetry of the other assets. All three assets exhibit kurtosis in excess of 3.0. Kurtosis for the equity strategy is 4.4, and for the LV and VRP strategies, 8.9 and 10.2, respectively.

**Co-dependencies**

The multivariate characteristics of returns are also of great interest. Correlations are provided in Exhibit 4. As expected, the LV strategy offers strong diversification
power relative to the equities with a −44% correlation coefficient, a phenomenon already well publicized by other studies (Daigler and Rossi [2006]). The VRP strategy shows different characteristics, offering little diversification to equity exposure with a 43% correlation coefficient. The two volatility strategies are mutually diversifying, however, with a −44% correlation coefficient.

The importance of extreme risks also requires the analysis of the coskewness and cokurtosis matrices in Exhibits 5 and 6. Positive coskewness value $s_{ij}$ suggests that asset $j$ has a high return when the volatility of asset $i$ is high (i.e., $j$ is a good hedge against an increase in the volatility of $i$). This relationship is particularly true for the LV strategy, which offers a good hedge for the VRP strategy and for equities. In contrast, the VRP strategy does not efficiently hedge the other assets, because it tends to perform poorly when their volatility increases.

Positive cokurtosis value $k_{uij}$ means that the return distribution of asset $i$ is more negatively skewed when the return on asset $j$ is lower than expected, (i.e., $i$ is a poor hedge against a decrease in the value of $j$). Again, we find that the LV strategy is an excellent hedge against equities, unlike the VRP strategy; however, the two volatility strategies hedge each other very well.

Lastly, positive cokurtosis $k_{uij}$ is a sign that the covariance between $j$ and $k$ increases when the volatility of asset $i$ increases. We find that in periods of rising equity volatility, the VRP/LV correlation is negative. Thus, during periods of stress in the equity market, the VRP strategy and equities perform badly at the same time, while the LV strategy performs better.

This initial analysis already allows us to highlight different advantages of the two volatility strategies within an equity portfolio—the LV strategy delivers excellent diversification relative to equities and the VRP strategy allows for a very substantial increase in returns, but at the expense of a broadly increased risk profile due to extreme risks and co-dependencies with equities. A combination of the two volatility strategies appears particularly attractive because they tend to hedge each other, especially in extreme market scenarios.

Efficient Portfolios

As previously described, the volatility strategies are collateralized in our analytical framework. Also, although long and short positions are permitted with volatility strategies, net short exposure to the equity asset class is restricted.
Moreover, optimal portfolios are designed under an additional budget constraint, which requires that the sum of the percentage shares in the three assets must equal 100%.

To determine the potential benefit for an equity investor in adding a systematic exposure to volatility, we consider four investment cases: 1) equities only, 2) equities and the LV strategy, 3) equities and the VRP strategy, and 4) equities and both volatility strategies. In the first half of the sample, we focus on portfolio construction, minimizing modified VaR, and in the second half, we backtest the results to gauge the portfolios’ out-of-sample performance. Exhibits 7 and 8 report each portfolio’s composition and performance in the two subperiods.15

In sample, the addition of the LV strategy, with a weight of 37%, to an equity investment reduces risk and improves risk-adjusted performance. The modified VaR of the portfolio decreases from 7.5% to 3.0%, and the Sharpe ratio increases from 1.0 to 1.2. The resulting allocation delivers slightly lower performance (15.9% versus 18.9%), but overall the portfolio is less sensitive to extreme events with a monthly maximum loss more than halved from −9.9% to −4.0%, and a distribution of returns that offers much higher positive skewness of 0.8 compared to 0.2, and smaller kurtosis of 4.1 compared to 4.9. Adding the VRP strategy, with a weight of 37%, to the equity-only portfolio, makes it possible to achieve significantly higher returns of 27.4% compared to 19.0%, as well as a lower modified VaR of 6.3%. The success rate of the portfolio is improved to 83%, and the Sharpe ratio rises to 2.0. Overall, however, the portfolio’s return distribution shows a more pronounced leftward asymmetry of −0.1 versus 0.2 and higher kurtosis of 6.0 versus 4.9. This profile demonstrates a higher sensitivity to extreme events, and for the most risk-averse investors, thus reduces the appeal of adding a VRP strategy for an equity investor.

The most interesting risk–return profile is obtained by adding a combination of the two volatility strategies. Adding both the LV and the VRP strategies with respective weights of 29% and 44% makes it possible to achieve the smallest modified VaR (1.7%) and the highest Sharpe ratio (2.7) among the portfolios presented. Overall, the portfolio shows a significantly decreased sensibility to extreme risks as measured by the worst monthly loss of −2.9% compared to −9.9% for the equity-only portfolio. Also, higher-order moments of the portfolio’s return distribution are improved with higher positive skewness (0.7 compared to 0.2) and an almost equivalent kurtosis (4.7 compared to 4.9).

As already stated, the turbulence of equity markets during the second part of our study period affects the returns and Sharpe ratios of all portfolios. Nevertheless, all the results support a strong rationale for introducing a systematic exposure to volatility strategies for equity investors. We observe in the out-of-sample performances all of the positive characteristics previously outlined for the volatility strategies—the LV strategy’s capacity to reduce risks, the VRP strategy’s capacity to boost returns, and the reciprocal hedge of LV/VRP strategies. When part of the portfolio is dedicated to the LV strategy, the worst monthly loss and the modified VaR are roughly halved—from −13.4% to −7.5% and from 11.8% to 5.9%, respectively—relative to an equity-only investment. The Sharpe ratio at 0.1 turns slightly positive compared to a Sharpe ratio of −0.1 for the equity-only portfolio. The portfolio invested in both equities and the VRP strategy achieves a strong annualized return of 8.2% and a Sharpe ratio of 0.4. Furthermore, the risks of the equity/VRP strategy portfolio are smaller than those of the equity-only investment. The combined strategy portfolio has a modified VaR of 9.6% versus 11.8% for the equity-only investment and a maximum monthly loss of −12.0% versus −13.4% for the equity-only investment, even if

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**Exhibit 7**

| Portfolio Allocation: Minimum Modified VaR, U.S., February 1990–July 1999 |
|------------------|------------------|------------------|------------------|
|                    | Equity + LV       | Equity + VRP     | Equity + LV + VRP |
| Ann. Geometric Mean | 18.95%           | 15.88%           | 27.39%           | 24.54%           |
| Ann. Std. Dev.     | 12.48%           | 8.64%            | 9.68%            | 6.32%            |
| Skewness           | 0.24             | 0.81             | -0.12            | 0.73             |
| Kurtosis           | 4.87             | 4.08             | 5.96             | 4.74             |
| Max Monthly Loss   | -9.92%           | -3.95%           | -7.75%           | -2.94%           |
| Max Monthly Gain   | 14.27%           | 9.41%            | 12.40%           | 8.26%            |
| Mod. VaR (99%)     | 7.51%            | 3.04%            | 6.32%            | 1.71%            |
| Sharpe Ratio       | 1.04             | 1.15             | 2.00             | 2.68             |
| Success Rate       | 68.20%           | 67.76%           | 83.11%           | 86.62%           |

| Equity     | 100% | 63%  | 63%  | 26%  |
| LV         | -    | 37%  | -    | 29%  |
| VRP        | -    | -    | 37%  | 44%  |
skewness (−0.7 versus −0.3) and kurtosis (4.3 versus 3.5) are higher in absolute terms.

Finally, the most attractive portfolio for investors proves to be the combination of the two volatility strategies with equities, which achieves the best performance (11.6%) with the lowest volatility (6.9%) and obtains the highest Sharpe ratio (1.2). This allocation creates the only portfolio with a success rate higher than 75%. The good news for risk-adverse investors is that these results are not obtained at the expense of extreme risks; the portfolio returns distribution has a leftward asymmetry comparable to that of equities (−0.3), even though kurtosis is still high (6.1) and the maximum monthly loss is −6.7% compared to −13.4% for equities.

**CONCLUSION**

Recent literature has begun to show the merit of including a long exposure to implied volatility in a pure equity portfolio (Daigler and Rossi [2006]), in a portfolio of funds of hedge funds (Dash and Moran [2006]), and an investigation of the benefits of volatility risk premium strategies (Egloff, Leippold, and Wu [2006] and Hafner and Wallmeier [2008]). The purpose of this article was to examine a classic equity allocation with a combination of volatility strategies, since little has been written so far on the subject. Among the standardized strategies for adding volatility exposure to the investment opportunity set, we identified buying implied volatility as well as investing in the volatility risk premium. Although these strategies have attracted considerable interest on the part of some market professionals, especially hedge fund managers (and, more recently, more sophisticated managers of traditional funds), the academic literature has to date paid little attention to them.

We explored two very simple types of volatility strategy exposure added to an equity portfolio. Our results from a historical analysis of the past 20 years show a strong benefit from including these volatility strategies in such a portfolio. Taken separately, each of the strategies moves the efficient frontier significantly outward, but a combination of the two produces even better results. A long exposure to volatility is particularly valuable for diversifying a portfolio holding equities; because of its negative correlation to the asset class, its hedging function during equity market downturns clearly has potential benefits.

A volatility risk premium strategy boosts returns. Although such a strategy provides little diversification to equities because it loses significantly when equity prices fall, it provides good diversification with respect to implied volatility. Combining the two strategies—equities and long volatility—offers a major advantage of fairly effective reciprocal hedging during periods of market stress and significantly improves the portfolio return for a given level of risk. With the presence of volatility strategies, investors radically change their portfolio composition, giving less weight to equity investments and replacing them with volatility exposure. We show that doing this would have improved investment performance, both in sample and out of sample.

One of the limits of our work relates to the period analyzed. Although markets experienced several major crises over the period from 1990 to 2008, and therefore several significant volatility spikes, we can provide no assurance that, in the future, crises will not be more acute than those experienced over the testing period and that losses on variance swap positions will not be greater, thus partly erasing the high reward associated with the volatility risk premium indicated by our study. An interesting continuation of this work would be to explore the extent to which long exposure to volatility is a satisfactory hedge of the
volatility risk premium strategy during periods of stress and sharp increases in realized volatility. It would also be important to analyze the dynamics of the volatility risk premium and its determinants, as envisaged by Bollerslev, Gibson, and Zhou [forthcoming].

Like fixed income and equity, volatility as an asset class can be approached not only in terms of directional volatility strategies, but also in terms of interclass arbitrage strategies, such as relative value, correlation trades, and so forth. Tactical strategies can also be envisaged. The possibilities are numerous, and they deserve further investigation to precisely measure both the benefits and the risks to an investor who incorporates such strategies into an existing portfolio.

**Appendix**

From a theoretical standpoint, a variance swap can be viewed as a representation of the structure of implied volatility, known as the volatility “smile,” because the strike price of the swap is determined by the prices of options of the same maturity and different strikes (all available calls and puts in, at, or out of the money) that make up a static portfolio replicating the payoff at maturity. The calculation methodology for the VIX represents the theoretical strike of a variance swap on the S&P 500 Index with a maturity of one month (interpolated from the closest maturities in order to keep the maturity constant).

From a practical standpoint, the two markets are closely linked through the hedging activity of market makers; to a first approximation, a market maker that sells a variance swap will typically hedge the vega risk on its residual position by buying the 95% out-of-the-money put on the listed options market.

The profit and loss of a variance swap is expressed as follows (Demeterfi et al. [1999]):

\[ P & L = N_{\text{variance}} \times [RV_{0,T}^2 - K_T^2] \]

where \( K_T \) is the volatility strike of a variance swap of maturity \( T \), \( K_T^2 \) is the delivery price of the variance, \( RV_{0,T} \) is the realized volatility of the asset underlying the variance swap over the term of the swap, and \( N_{\text{variance}} \) is the variance notional.

Realized volatility \( RV_{0,T} \) is calculated from closing prices of the S&P 500 Index according to the following formula:

\[ RV_{0,T} = \sqrt{\frac{252}{T} \sum_{t=1}^{T} \ln \left( \frac{S&P500_t}{S&P500_{t-1}} \right)} \]

In terms of the Greek letter parameters popularized by the Black-Scholes-Merton option pricing model, the notional of a variance swap is expressed as a vega notional, which represents the mean profit and loss of a variation of 1% (one vega) in volatility. Although the variance swap is linear in variance, it is convex in volatility (a variation in volatility has an asymmetric impact). The relationship between the two notional is expressed as

\[ N_{\text{vega}} = N_{\text{variance}} \times 2K \]

where \( N_{\text{vega}} \) is the vega notional.

**Endnotes**

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1 Other components can provide partial explanations of this premium, such as the convexity of the profit and loss of the variance swap, and the fact that investors tend to be structural net buyers of volatility to hedge equity exposure or to meet risk constraint requirements (Bollen and Whaley [2004] and Carr and Wu [2009]).

2 The method of calculation of the VIX changed in September 2003. The current method (applied retroactively to the index since 1990) takes into account S&P 500 traded options at all strikes in contrast to the previous index, VXO, which was based solely on at-the-money S&P 100 options.

3 The Chicago Futures Exchange is part of the Chicago Board Options Exchange (CBOE).

4 Empirical tests have shown that an exposure, which is inversely proportional to the observed level of implied volatility, markedly increases the profitability of the strategy.

5 For instance, the one-point impact is 5% when the VIX is 20.

6 The most widely traded indices include the S&P 500 in U.S., EuroSTOXX 50 in the EMU, and Nikkei 225 in Japan.

7 The data were provided by Datastream (S&P 500 Index) and Bloomberg (CBOE VIX and one-month U.S. LIBOR).

8 The start dates are the 7th, 14th, 21st, and 28th of each month. When a date is not a business day, we use quotes from the previous business day. We thank an anonymous referee for having made this suggestion.

9 The volatility strategies have been calibrated to have the same risk as the equity strategy (7.5%) in the first sample period. This calibration can be considered adequate, given that the out-of-sample VaR of the volatility strategies (not reported here for the sake of simplicity) does not exceed that of the equity strategy.

10 For the returns on all three strategies, the null hypothesis of a normality test is significantly rejected.

11 We provide a summary presentation of these matrices. For \( n = 3 \) assets, it suffices to calculate 10 elements for the
coskewness matrix of dimension (3, 9) and 15 elements for the cokurtosis matrix of dimension (3, 27).

12 The general formula for coskewness is

\[
s_{ki} = \frac{E[(r_i - \mu_i)(r_j - \mu_j)(r_l - \mu_l)]}{\sigma_i \sigma_j \sigma_l}
\]

where \( r_j \) is the return on asset \( i \) and \( \mu_i \) is its mean.

13 The general formula for cokurtosis is

\[
k_{ijkl} = \frac{E[(r_i - \mu_i)(r_j - \mu_j)(r_k - \mu_k)(r_l - \mu_l)\sigma_{ij} \sigma_{jk} \sigma_{kl} \sigma_{il}]}{\sigma_{ij} \sigma_{jk} \sigma_{kl} \sigma_{il}}.
\]

14 The null hypothesis of a multivariate normality test (Kotz, Balakrishnan, and Johnson [2000]) is significantly rejected.

15 We ran portfolio optimizations and computed performance for each of the four starting dates considered in the month. We report the average summary statistics.

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