Self-dual Varieties and networks in the lattice of varieties of Completely Regular Semigroups

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Abstract

The kernel relation $K$ on the lattice $\mathcal{L}(\mathcal{CR})$ of varieties of completely regular semigroups has been a central component in many investigations into the structure of $\mathcal{L}(\mathcal{CR})$. However, apart from the $K$-class of the trivial variety, which is just the lattice of varieties of bands, the detailed structure of kernel classes has remained a mystery until recently. Kad’ourek [RK2] has shown that for two large classes of subvarieties of $\mathcal{CR}$ their kernel classes are singletons. Elsewhere (see [RK1], [RK2], [RK3]) we have provided a detailed analysis of the kernel classes of varieties of abelian groups. Here we study more general kernel classes. We begin with a careful development of the concept of duality in the lattice of varieties of completely regular semigroups and then show that the kernel classes of many varieties, including many self-dual varieties, of completely regular semigroups contain multiple copies of the lattice of varieties of bands as sublattices.

Key Words and Phrases Semigroups, lattices, varieties, completely regular.
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1. Introduction

Following the presentation of necessary background in Section 2, we introduce in Section 3 the related concepts of the mirror image of a word in the free unary semigroup, the dual of a semigroup or variety of completely regular semigroups and self-dual varieties. The concept of duality in semigroups appears implicitly in [CP], while the concept of the dual semigroup is explicitly defined in [L]. This leads to the concepts of the mirror image or reverse of a word, dual identities and dual semigroup varieties. Petrich [Pe2007], [Pe2015a] extended these concepts to completely regular semigroups. Here we take a slightly different approach from Petrich and take as our starting point a completely unambiguous definition of the mirror image of a word in $Y^+$, the free semigroup on a set $X$ together with the symbols ( and )$^{-1}$. This incorporates the definitions introduced by Clifford and Preston, and Lallement and restricts to exactly what we want in the free unary semigroup $U_X$ and is mapped on to exactly the relation that we want in the free completely regular semigroup $CR_X = U_X/\zeta$ under the natural mapping $u \rightarrow u\zeta$. We develop some basic results specific to completely regular semigroup varieties.
Of particular interest to us are the relations $K_\ell = K \cap T_\ell$ and its dual $K_r = K \cap T_r$. These give us a finer dissection of any $K$-class into $K_\ell$ and $K_r$-classes. As these are complete congruences, all the classes are intervals of the form $[\mathcal{V}_K, \mathcal{V}_K^K]$ and $[\mathcal{V}_K, \mathcal{V}_K^K]$, respectively, which leads us to the operators $\mathcal{V} \mapsto \mathcal{V}^{K_\ell}$ and $\mathcal{V} \mapsto \mathcal{V}^{K_r}$. We use the repeated application of these operators applied to varieties that are self-dual together with their intersections to generate a family of subvarieties of $\mathcal{V}K$. In contrast with the similar situation using the operators $\mathcal{V} \mapsto \mathcal{V}^{T_\ell}$ and $\mathcal{V} \mapsto \mathcal{V}^{T_r}$, these varieties always constitute a sublattice of $\mathcal{V}K$. In certain circumstances, we are able to characterize the largest variety in $\mathcal{L}(\mathcal{C}R)$ whose intersection with $\mathcal{V}^K$ is a particular element of this sublattice. This mimics the result due to Reilly and Zhang [RZ] for the particular case where $\mathcal{V} = \mathcal{T}$ and $\mathcal{V}^K = \mathcal{B}$, the variety of bands.

In Sections 5 and 6 we show that there exist multiple copies of the lattice $\mathcal{L}(\mathcal{B})$ within certain $K$-classes in $\mathcal{L}(\mathcal{C}R)$. In Section 5 we provide a canonical method for generating such sublattices by applying the operators $\mathcal{V} \mapsto \mathcal{V}^{K_\ell}$ and $\mathcal{V} \mapsto \mathcal{V}^{K_r}$ repeatedly to three subvarieties $\mathcal{V}, \mathcal{V}_\ell, \mathcal{V}_r$ related as follows: $\mathcal{V} \subset \mathcal{V}_\ell \subset \mathcal{V}^{K_\ell}, \mathcal{V} \subset \mathcal{V}_r \subset \mathcal{V}^{K_r}$. The varieties $\mathcal{V}_\ell$ and $\mathcal{V}_r$ can be chosen freely subject to these constraints.

This abundance of sublattices isomorphic to $\mathcal{L}(\mathcal{B})$ in certain kernel classes is in sharp contrast to the situation in Kad’ourek [RK2], in which it is shown that, for two large classes of subvarieties of $\mathcal{C}R$, their kernel classes are singletons. In Section 6 we show how it is not always necessary to employ the operators $\mathcal{V} \mapsto \mathcal{V}^{K_\ell}$ and $\mathcal{V} \mapsto \mathcal{V}^{K_r}$ at each level.

2. Background

We refer the reader to the book [PR99] for a general background on completely regular semigroups and for all undefined notation and terminology. For an equivalence relation $\rho$ on a nonempty set $X$ and $x \in X$, $x \rho$ denotes the $\rho$-class of $x$. In any lattice $L$, for $a, b \in L$ such that $a \leq b$ define the interval $[a, b]$ to be $\{c \in L \mid a \leq c \leq b\}$.

Let $S$ be a completely regular semigroup. Then $E(S)$ denotes its set of idempotents and $\mathcal{C}(S)$ the lattice of congruences on $S$. For $\rho \in \mathcal{C}(S)$, the kernel and the trace of $\rho$ are given by $\ker \rho = \{a \in S \mid a \rho e \text{ for some } e \in E(S)\}$, $\tr \rho = \rho|_{E(S)}$, and the left and the right traces of $\rho$ are $\ltr \rho = \tr (\rho \lor \mathcal{L})^0$ and $\rtr \rho = \tr (\rho \lor \mathcal{R})^0$, where the join is taken within the lattice of equivalence relations on $S$. On $\mathcal{C}(S)$ we have several important relations defined by

$$\lambda K \rho \iff \ker \lambda = \ker \rho, \quad \lambda T_\ell \rho \iff \ltr \lambda = \ltr \rho,$$

$$\lambda T_\rho \iff \tr \lambda = \tr \rho, \quad \lambda T_r \rho \iff \rtr \lambda = \rtr \rho,$$

$K_\ell = K \cap T_\ell, \quad K_r = K \cap T_r, \quad T = T_\ell \cap T_r.$

Two fundamental facts concerning these relations are as follows (where $\epsilon$ denotes the identity relation):

$$K \cap T = K_\ell \cap K_r = \epsilon.$$
See [PR99] for details. The classes of these relations are intervals which we write as 
\[ \rho P = [\rho_p, \rho^P], \quad \text{for } P \in \{K, T, T, T, K, K\}. \]

This enables us to consider the associated operators \( \rho \to \rho P, \rho \to \rho^P \).

Let \( X \) be a countably infinite set. If \( x_1, \ldots, x_n \in X \), then \( w = x_1 \cdots x_n \) is a word over \( X \), \( h(w) = x_1 \) is the head of \( w \), \( t(w) = x_n \) is the tail of \( w \), and \( c(w) = \{x_1, \ldots, x_n\} \) is the content of \( w \). We denote by \( X^+ \) the free semigroup on \( X \) consisting of all words over \( X \). Let \( Y = X \cup \{\langle, \rangle^{-1}\} \) where “\( \langle \)” and “\( \rangle^{-1} \)” are two distinct elements not in \( X \).

C(i) \( X \subseteq U_X \).
C(ii) \( u, v \in U_X \Rightarrow uv \in U_X \).
C(iii) \( u \in U_X \Rightarrow (u)^{-1} \in U_X \).

We will often omit the subscript \( X \) in \( U_X \) when there is no danger of ambiguity, and sometimes write \( u^{-1} \) instead of \((u)^{-1}\). We also write \( u^0 = u(u)^{-1} \) and write \( |u|_Y \) for the length of \( u \) over the alphabet \( Y \).

By C(ii), \( U_X \) is a semigroup and by C(iii), the mapping \( u \to (u)^{-1} \) (\( u \in U_X \)) is a unary operation on \( U_X \). An alternative description of \( U_X \) due to A.H.Clifford [C] follows.

**Lemma 2.1.** ([PR99] Lemma I.10.1). The set \( U_X \) consists precisely of those elements \( w \) of \( Y^+ \) that satisfy the following conditions:
(i) the number of occurrences of \( \langle \) in \( w \) equals the number of occurrences of \( \rangle^{-1} \),
(ii) in each initial segment of \( w \), the number of occurrences of \( \langle \) is at least as great as the number of occurrences of \( \rangle^{-1} \),
(iii) The symbol \( \langle \) is never followed immediately in \( w \) by the symbol \( \rangle^{-1} \).

**Lemma 2.2.** ([PR99] Lemmas I.10.3, I.10.4, I.10.5). \( U_X \) is a free unary semigroup on \( X \).
(i) Let \( t_X : x \to x \) be the embedding of \( X \) in \( U_X \). Then \( (U_X, t_X) \) is a free unary semigroup on \( X \).
(ii) A word \( w \in U \) is irreducible if and only if \( w \in X \) or \( w = (u)^{-1} \) for some \( u \in U_X \).
(iii) Each element \( w \) of \( U \) can be expressed uniquely as a product of irreducible elements.

Let \( S \) be a completely regular semigroup. As \( S \) is a union of its (maximal) subgroups, we have a unary operation \( a \to a^{-1} \) on \( S \) where \( a^{-1} \) is the inverse of \( a \) in the maximal subgroup of \( S \) containing \( a \). Hence for the purpose of studying varieties of completely regular semigroups, they are considered with the binary operation of
multiplication and the unary operation of inversion. We write \( a^0 = aa^{-1} (= a^{-1}a) \) for any element \( a \) of \( S \).

The class \( CR \) of all completely regular semigroups constitutes a variety. It is defined, within the class of unary semigroups by the identities
\[
a(bc) = (ab)c, \quad a = aa^{-1}a, \quad (a^{-1})^{-1} = a, \quad aa^{-1} = a^{-1}a.
\]

Let \( x \in X \) and \( \zeta \) denote the least fully invariant congruence on \( U_X \) containing the pairs
\[
(x, xx^{-1}x), \quad (x, (x^{-1})^{-1}), \quad (xx^{-1}, x^{-1}x).
\]

**Lemma 2.3.** ([C] Theorem 3.1) \( \zeta \) is the least unary fully invariant congruence on \( U_X \) such that \( CR_X = U_X/\zeta \) is completely regular. Moreover, \( CR_X \) is a free completely regular semigroup on \( X \).

For any \( \mathcal{V} \in L(CR) \) we denote by \( \zeta_{\mathcal{V}} \) the fully invariant congruence on \( CR_X \) corresponding to \( \mathcal{V} \). We denote by \( CR \) the variety of all completely regular semigroups and by \( L(CR) \) the lattice of its subvarieties. Via the usual antitone isomorphism of the lattice of fully invariant congruences on a free completely regular semigroup of infinite rank and \( L(CR) \), the relations \( K, T, T_\ell \) and \( T_r \) defined above transfer to the lattice of fully invariant congruences on \( L(CR) \) in an obvious way. In this context, the relations \( K, K_\ell, K_r, T_\ell \) and \( T_r \) have the important property of being complete congruences. We use the same notation for these relations on \( L(CR) \) (and their intervals) as for the corresponding relations on semigroups. For these operators, we write for example, \( \mathcal{V}_K, \mathcal{V}^{K T_r} = (\mathcal{V}^K)^{T_r} \), and so on. For \( \mathcal{V} \in L(CR) \), \( L(\mathcal{V}) \) denotes the lattice of all subvarieties of \( \mathcal{V} \).

The following varieties occur frequently: \( T, L\mathcal{Z}, R\mathcal{Z}, RB, G, ReG, CS \) - the varieties of trivial semigroups, left zero semigroups, right zero semigroups, rectangular bands, groups, rectangular groups and completely simple semigroups, respectively, all of which are subvarieties of \( CS \), and \( S, LN\mathcal{B}, RN\mathcal{B}, LRB, RRB, ReB, SG, LRO, RRO, B, O \) - the varieties of semilattices, left normal bands, right normal bands, left regular bands, right regular bands, regular bands, semilattices of groups, left regular orthogroups, right regular orthogroups, bands and orthogroups, respectively, all of which are varieties of completely regular semigroups containing \( S \).

**Lemma 2.4.** \( CR_P = CR \) \( (P \in \{K, T, T_\ell, T_r, T, K_\ell, K_r\}) \).

**Proof.** See [PR88] Theorem 4.4 and [PT] Lemma 5.11.

If \( \mathcal{V} \in L(CR) \) has a basis \( \{u_\alpha = v_\alpha\}_{\alpha \in A} \), we write \( \mathcal{V} = [u_\alpha = v_\alpha]_{\alpha \in A} \), or simply \( \mathcal{V} = [u = v] \) if \( A \) is a singleton. We shall sometimes write an identity \( u^2 = u \) as \( u \in E \).
Lemma 2.5. Let $\mathcal{V} = \{u_a = v_a\}_{a \in A} \in [\mathcal{S}, \mathcal{CR}]$. Then

\[
\mathcal{V}^K = \{S \in \mathcal{CR} \mid S/\tau \in \mathcal{V}\} = \{xu_a y(xv_a y)^{-1} \in E\}_{a \in A}
\]

\[
\mathcal{V}^T = \mathcal{V}^{T_1} \cap \mathcal{V}^{T_r} = \{S \in \mathcal{CR} \mid S/\mathcal{H}^0 \in \mathcal{V}\} = \{u_0^a = v_0^a, (xu_a y)^0 = (xv_a y)^0\}_{a \in A}
\]

\[
\mathcal{V}^{T_i} = \{S \in \mathcal{CR} \mid S/\mathcal{L}^0 \in \mathcal{V}\} = \mathcal{LG} \circ \mathcal{V} = \{xu_a = xv_a(xv_a)^0\}_{a \in A}
\]

\[
\mathcal{V}^{K_i} = \mathcal{V}^K \cap \mathcal{V}^{T_i} = \{S \in \mathcal{CR} \mid S/(\tau \cap \mathcal{L})^0 \in \mathcal{V}\} = \{xu_a y(xv_a y)^{-1} \in E, xu_a = xv_a(xv_a)^0, xv_a = xv_a(xv_a)^0\}_{a \in A}.
\]

Proof. For the claims concerning $\mathcal{V}^K$ see [J], for those concerning $\mathcal{V}^T$ see [J] and [R], for those concerning $T_r$ see [Pa], [PR90] and [Po2], for those concerning $\mathcal{V}^{K_i}$ see [R2].

Let $\Theta$ be the set of all (nonempty) words over the alphabet $\{T_1, T_r\}$ of the form $P_1 \cdots P_n$, where $P_1 \in \{T_1, T_r\}$ and $P_i \neq P_{i+1}$ for $i = 1, \ldots, n-1$ with multiplication

\[
(P_1 \cdots P_m)(Q_1 \cdots Q_n) = \begin{cases} P_1 \cdots P_m Q_1 \cdots Q_n & \text{if } P_m \neq Q_1, \\ P_1 \cdots P_m Q_2 \cdots Q_n & \text{otherwise}. \end{cases}
\]

Clearly $\Theta$ is a semigroup. We adjoin the empty word $\emptyset$ to $\Theta$ to form $\Theta^1$.

We can consider $\Theta$ as being the free semigroup $\langle T_1, T_r \rangle^+$ on the set $\{T_1, T_r\}$ modulo the relations $T_1^2 = T_1$, $T_r^2 = T_r$. To emphasize that $\tau \in \Theta$ is a word in $T_1, T_r$ we might write $\tau = \tau(T_1, T_r)$. If we replace every occurrence of $T_1$ (respectively, $T_r$) in $\tau$ by $K_1$ (respectively, $K_r$) then we obtain a word over $\{K_1, K_r\}$ which we will denote by $\tau(K_1, K_r)$. Note that for each positive integer $n$ there are exactly two distinct words in $\Theta$ of length $n$. They are obtained, each from the other, by replacing each occurrence of $T_1$ by $T_r$ and vice versa.

It is easily seen that the following relation $\leq$ is a partial order on $\Theta^1$:

\[
\sigma \leq \tau \text{ if } |\sigma| > |\tau| \text{ or } \sigma = \tau \quad (\sigma, \tau \in \Theta^1).
\]

Lemma 2.6. (i) $(\mathcal{V}^K)_{T_1} = \mathcal{V}^K$, $(\mathcal{V}^T)_{T_1} = \mathcal{V}^{T_1}$ for $\mathcal{V} \in [\mathcal{S}, \mathcal{CR}]$. (ii) $(\mathcal{V}^T)_{K} = \mathcal{V}^T$ for $\mathcal{V} \in [\mathcal{R} \circ \mathcal{B}, \mathcal{CR}]$. (iii) $\mathcal{V}^\sigma \cap \mathcal{V}^K = \mathcal{V}^{\sigma(K_1, K_r)}$ for $\mathcal{V} \in [\mathcal{S}, \mathcal{CR}], \sigma \in \Theta^1$.

Proof. (i) See [Po2], Theorem 2.4(4) and [Po2] Theorem 1.7(3). (ii) See [K] Proposition 8.2. (iii) See [R2], Proposition 3.4.
Lemma 2.7.  (i) The mapping

\[ V \rightarrow V_K \quad (V \in \mathcal{L}(CR)) \]

is a complete \( \lor \)-endomorphism of \( \mathcal{L}(CR) \).

(ii) The mapping

\[ V \rightarrow V^K \quad (V \in \mathcal{L}(CR)) \]

is a complete endomorphism of \( \mathcal{L}(CR) \).

(iii) The mapping

\[ V \rightarrow V_{T_\ell} \quad (V \in \mathcal{L}(CR)) \]

is a complete endomorphism of \( \mathcal{L}(CR) \).

(iv) The mapping

\[ V \rightarrow V^{T_\ell} \quad (V \in \mathcal{L}(CR)) \]

is a complete \( \land \)-endomorphism of \( \mathcal{L}(CR) \).

(v) The mapping

\[ V \rightarrow V^{K_\ell} \quad (V \in \mathcal{L}(CR)) \]

is a complete \( \land \)-endomorphism of \( \mathcal{L}(CR) \).

(vi) The mapping

\[ V \rightarrow V_{K_\ell} \quad (V \in \mathcal{L}(CR)) \]

is a complete \( \lor \)-endomorphism of \( \mathcal{L}(CR) \).

Proof.  (i) See Petrich/Reilly [PR88].  (ii) See Polák [Po].  (iii) See Pastijn [Pa].  (iv) See Petrich and Reilly [PR90].  (v) and (vi) These follow immediately from the fact that \( K_\ell \) is a complete congruence on \( \mathcal{L}(CR) \).  \( \blacksquare \)

We gather here a few useful observations concerning the operators related to the \( K_\ell \) and \( K_r \) relations.

Lemma 2.8.  Let \( V \in [S, CR] \).

(i) \( (V^{K_\ell})_{T_r} = V^{K_\ell} \) and \( (V^{K_\ell})_{K_r} = V^{K_\ell} \).

(ii) \( V^K = \bigvee_{\sigma \in \Theta^1} V^{\sigma(K_\ell, K_r)} \).

(iii) \( V \subset V^K \iff \) either \( V \neq V^{K_\ell} \) or \( V \neq V^{K_r} \).
Proof. (i) By Lemma 2.6(ii) with \( \sigma = K_\ell \), we have \( V^K_{K_\ell} = V^K \cap V^{T_\ell} \). Then

\[
\begin{align*}
(V^K_{K_\ell})_{T_\ell} &= (V^K \cap V^{T_\ell})_{T_\ell} \\
&= (V^K)_{T_\ell} \cap (V^{T_\ell})_{T_\ell} \quad \text{by the dual of Lemma 2.7(iii)} \\
&= V^K \cap V^{T_\ell} \quad \text{by Lemma 2.6(i)} \\
&= V^{K_\ell},
\end{align*}
\]

which establishes the first claim. In addition,

\[
V^{K_\ell} = (V^{K_\ell})_{T_\ell} \subseteq (V^{K_\ell})_{K_\ell} \subseteq V^{K_\ell},
\]

Therefore \( (V^{K_\ell})_{K_\ell} = V^{K_\ell} \) as claimed.

(ii) See [R2], Corollary 3.8.

(iii) If \( V = V^{K_\ell} = V^{K_\ell} \), then \( V^{\sigma(K_\ell,K_\ell)} = V \) for all \( \sigma \in \Theta_1 \) so that, by (ii), \( V^K = V \). Thus the direct claim holds. Since \( V^{K_\ell} \cup V^{K_\ell} \subseteq V^K \), the converse is obvious.

In [PT], Pastijn and Trotter show that if \( V \) is a proper subvariety of \( CR \), then so also are \( V^K \) and \( V^{T_\ell} \). An interesting source of varieties for which \( V \subseteq V^K \) is provided below. But first we need a simple technical lemma.

Lemma 2.9. Let \( S \in CR \) be such that \( S/D \) has a least element \( D \). For \( P \in \{H, L, R\} \), let \( \rho_P \) be defined on \( S \) by

\[
a \rho_P b \iff \text{either } a = b \text{ or } a, b \in D \text{ and } a \mathrel{P} b.
\]

Then \( \rho_H \) (respectively, \( \rho_L \) and \( \rho_R \)) is a congruence on \( S \) such that \( \rho_H \subseteq H \), \( D/\rho_H \in RB \) (respectively, \( \rho_L \subseteq L \), \( D/\rho_L \in RZ \) and \( \rho_R \subseteq R \), \( D/\rho_R \in LZ \)).

Proof. The proof is a simple exercise.

Lemma 2.10. Let \( V = V^T \in [S, CR] \). Let \( S \in CR \setminus V \) and \( \langle S \rangle \) denote the variety generated by \( S \).

(i) Either \( \langle S \rangle \cap V^{K_\ell} \not\subseteq V \) or \( \langle S \rangle \cap V^{K_\ell} \not\subseteq V \).

(ii) Either \( V \neq V^{K_\ell} \) or \( V \neq V^{K_\ell} \).

(iii) \( V \neq V^K \).

Proof. Since \( S \in CR \setminus V \), there must exist \( u = u(x_1, \ldots, x_n), v = v(x_1, \ldots, x_n) \in U_X \) such that \( V \) satisfies the identity \( u = v \) but \( S \) does not. Consequently there exist \( a_1, \ldots, a_n \in S \) such that \( u(a_1, \ldots, a_n) \neq v(a_1, \ldots, a_n) \). The completely regular subsemigroup of \( S \) generated by \( a_1, \ldots, a_n \) must also lie in \( CR \setminus V \). Hence, we may assume that \( S = \langle a_1, \ldots, a_n \rangle \). It then follows that \( S \) has only a finite number of
\(D\)-classes. Thus there must exist a semigroup, which we again take to be \(S\), which has the smallest possible number of \(D\)-classes with \(S \not\in V\). Also \(S\) must have a least \(D\)-class, \(D\) say, in the semilattice \(S/D\).

Now \(V\) contains the variety \(S\) of semilattices and \(V\) satisfies the identity \(u = v\). Therefore \(u\) and \(v\) must have the same content. Hence \(u(a_1, \ldots, a_n) \ D \ v(a_1, \ldots, a_n)\). Let \(C = D_{u(a_1, \ldots, a_n)}\). If \(C \neq D\), then \(T = \bigcup \{D_a \mid C \leq D_a\}\) will be a subsemigroup of \(S\) with fewer \(D\)-classes and \(T \not\in V\). This contradicts the choice of \(S\). Hence \(C = D\).

\textbf{Case:} \(u(a_1, \ldots, a_n) \ H \ v(a_1, \ldots, a_n)\). Let \(\rho_H\) be defined as in Lemma 2.9. If \(S/\rho_H \in V\) then since \(\rho_H \subseteq H\), we have \(S \in V^T = V\), a contradiction. Hence \(S/\rho_H \not\in V\) and there must be another identity \(p = q\), where \(p = p(x_1, \ldots, x_m), q = q(x_1, \ldots, x_m) \in U_X\), that is satisfied by \(V\) but not by \(S/\rho_H\). Let \(b_1, \ldots, b_m \in S/\rho_H\) be such that \(p(b_1, \ldots, b_m) \neq q(b_1, \ldots, b_m)\). As for \(u(a_1, \ldots, a_n), v(a_1, \ldots, a_n)\), we must have \(p(b_1, \ldots, b_m), q(b_1, \ldots, b_m) \in D/\rho_H\). Since the restriction of \(H\) to \(D/\rho_H\) is the identity relation, we must have that either \(p(b_1, \ldots, b_m)\), \(q(b_1, \ldots, b_m)\) are not \(L\)-related or they are not \(R\)-related. This situation is covered by the next case.

\textbf{Case:} Either \((u(a_1, \ldots, a_n), v(a_1, \ldots, a_n)) \not\in L\) or \((u(a_1, \ldots, a_n), v(a_1, \ldots, a_n)) \not\in R\). It suffices to consider the case \((u(a_1, \ldots, a_n), v(a_1, \ldots, a_n)) \not\in L\). Then \((u(a_1, \ldots, a_n), v(a_1, \ldots, a_n)) \not\in \rho_L\). Hence \(S/\rho_L \not\in V\). However, \(D/\rho_L \in RZ\). If the Rees quotient \(S/D \in V\) then we have \(S/\rho_L \in (RZ \circ V) \backslash V\), that is, \(V^K \not\in V\). It remains to show that \(S/D \in V\). By way of contradiction, suppose that \(S/D \not\in V\). Then there exist \(p = p(x_1, \ldots, x_m), q = q(x_1, \ldots, x_m) \in U_X\) such that \(V\) satisfies the identity \(p = q\), but \(S/D\) does not. Let \(c_1, \ldots, c_m \in S/D\) be such that \(p(c_1, \ldots, c_m) \neq q(c_1, \ldots, c_m)\). As previously, \(p(c_1, \ldots, c_m)\) and \(q(c_1, \ldots, c_m)\) must lie in the same \(D\)-class, \(C\) say. Moreover \(C \neq D\) since \(D\) is the zero element in \(S/D\). But then \(T = \bigcup \{D_a \mid C \leq D\}\) has fewer \(D\)-classes than \(S\), contradicting the choice of \(S\). Hence \(S/D \in V\) and \(\langle S \rangle \cap V^K \not\in V\). In particular \(V^K \not\in V\).

By duality, this proves parts (i) and (ii). Part (iii) then follows trivially.

\section{Dual varieties}

The concept of duality in semigroups is introduced in [CP] where the dual semigroup is implicit, while the concept of the dual semigroup is explicitly defined in [L]. This leads to the concepts of the mirror image of a word, dual identities and dual varieties. It has played an important role in some recent papers by Petrich [Pe2007], [Pe2015a] on varieties of completely regular semigroups. In these papers, Petrich defines the mirror image of a “completely regular” word directly in the free completely regular semigroup thereby raising the awkward question as to whether the concept is well-defined. Here we take a slightly different approach and take as our starting point a completely unambiguous definition of the mirror image of a word in \(Y^+\). This incorporates the definitions introduced by Clifford and Preston [CP] and Lallement
and restricts to exactly what we want in $U_X$ and is mapped onto exactly the relation that we want in $CR_X = U_X/\zeta$ under the natural mapping.

**Definition 3.1.** Let $w = x_1x_2\cdots x_n \in Y^+$ where $x_1, x_2, \ldots, x_n \in Y$. Let $\varphi : Y \to Y$ be defined by

$$x\varphi = \begin{cases} x & \text{if } x \in X \\ \)^{-1} & \text{if } x = ) \\ ( & \text{if } x = (^{-1}. \end{cases}$$

We define the **mirror image** of $w$ to be $\overline{w} = \varphi(x_n)\varphi(x_{n-1})\cdots \varphi(x_1)$.

For example, with $w = p(q(rs)^{-1}t)^{-1}u$ where $p, q, r, s, t, u \in X$ we have $\overline{w} = u(t(sr)^{-1}q)^{-1}p$.

**Lemma 3.2.** Let $u, v, w \in Y^+$.

(i) $\overline{uv} = \overline{v} \overline{u}$.

(ii) $\overline{w} = w$, $(\overline{w})^{-1} = (\overline{w})^{-1}$, $\overline{w^0} \zeta \overline{w^0}$.

(iii) $w \in U_X \implies \overline{w} \in U_x$.

(iv) $u, v \in U_X$, $u \zeta v \implies \overline{u} \zeta \overline{v}$.

**Proof.** (i),(ii) These are self-evident.

(iii) It is straightforward to see that since $w$ satisfies the conditions (i), (ii), (iii) of Lemma 2.1, so also does $\overline{w}$. The only point worth noting is that condition (ii) of Lemma 2.1 for $w$ is equivalent to the number of occurrences of $)$ in any final segment of $w$ being at least as great as the number of occurrences of $($.

(iv) Now $\zeta$ is generated as a semigroup congruence from the identities $x = x(x)^{-1}x$, $x = ((x)^{-1})^{-1}$, $x(x)^{-1} = (x)^{-1}x$. Since $u \zeta v$, the identity $u = v$ can be formally derived from the above listed identities. Thus there exists a sequence $u = w_0, w_1, \ldots, w_{n+1} = v \in U_X$ such that, for each $i = 0, 1, \ldots, n$ there exists an endomorphism $\varphi_i$ of $U_X$, an identity $p = q$ (or $q = p$) in $I$ and $a_i, b_i \in P^+$ such that

$$w_i = a_i\varphi_i(p)b_i, \quad w_{i+1} = a_i\varphi_i(q)b_i.$$ 

Now consider the case where $p = x$, $q = x(x)^{-1}x$. Then

$$w_i = a_i\varphi_i(x)b_i, \quad w_{i+1} = a_i\varphi_i(x)(\varphi_i(x))^{-1}\varphi_i(x)b_i$$

and we have

$$\overline{w_i} = \overline{b_i\varphi_i(x)a_i}, \quad \overline{w_{i+1}} = \overline{b_i\varphi_i(x)}(\overline{\varphi_i(x)})^{-1}\overline{\varphi_i(x)a_i}.$$ 

The other possibilities for the identity $p = q$ can be dealt with similarly. Thus $\overline{w_i} \zeta \overline{w_{i+1}}$ for $i = 0, 1, \ldots, n-1$ and therefore $\overline{u} = \overline{w_0} \zeta \overline{w_{n+1}} = \overline{v}$. 

$\blacksquare$
Definition 3.3. For any semigroup \((S, \cdot)\) we define its dual \(\overline{S}\) to be \((S, \ast)\) where \(a \ast b = ba\) for all \(a, b \in S\). For any variety \(V \in \mathcal{L}(\mathcal{CR})\) we define the dual of \(V\) to be \(\overline{V} = \{S \mid S \in V\}\) and we say that \(V\) is self-dual if \(\overline{V} = V\). We denote the class of all self-dual subvarieties of \(\mathcal{CR}\) by \(\mathcal{SD}\). For any equivalence relation \(\rho\) on \(U \times X\), we define the dual \(\overline{\rho}\) of \(\rho\) by \(u \rho v \iff u \overline{\rho} v\).

When we wish to emphasize that an expression or formula is to be calculated in \(\overline{S}\) as opposed to \(S\), we will write the elements as \(a, b, c, \ldots\). For instance, if \(u(x, y) = xy \in U \times X\) and \(a, b \in S\), then \(u(a, b) = ab\) while \(u(\overline{a}, \overline{b}) = a \ast b = ba\).

The dual operator \(\overline{\cdot}\) reflects Green’s relations in the sense that, for \(a, b \in S\), \(a L S b \iff a R S b\); \(a L S 0 \iff a R S 0\).

Some elementary examples of \(\overline{V}\) are \(\mathcal{RZ} = \overline{\mathcal{LZ}}, \mathcal{RN B} = \overline{\mathcal{LN B}}, \mathcal{RG} = \overline{\mathcal{LG}}\) and \(\mathcal{RRO} = \overline{\mathcal{LRO}}\). The class of self-dual varieties clearly includes \(\mathcal{T}, \mathcal{RB}, \mathcal{G}, \mathcal{CS}, \mathcal{S}, \mathcal{NB}, \mathcal{B}, \mathcal{CR}\).

Lemma 3.4. Let \(S \in \mathcal{CR}, U, V \in \mathcal{L}(\mathcal{CR})\), \(u, v, u_\alpha, v_\alpha (\alpha \in A) \in U \times X\) and \(V = [u_\alpha = v_\alpha]_{\alpha \in A}\).

(i) \(\overline{\overline{S}} \in \mathcal{CR}\).

(ii) \(\overline{V} \in \mathcal{L}(\mathcal{CR})\).

(iii) \(\overline{\overline{V}} = V\).

(iv) If \(V \subseteq \overline{V}\) then \(V = \overline{V}\).

(v) The varieties \(V \cap \overline{V}\) and \(V \vee \overline{V}\) are self-dual.

(vi) \(\overline{\overline{V}} = [\overline{u}_\alpha = \overline{v}_\alpha]_{\alpha \in A}\).

(vii) \(u \zeta V v \implies \overline{u} \zeta \overline{V} \overline{v}\)

(viii) \(U \vee \overline{V} = U \vee \overline{V}\) and \(U \cap \overline{V} = U \cap \overline{V}\).

Proof. (i) It is clear that the dual of a group is also a group and therefore that the dual of a semigroup that is a union of groups is again a semigroup that is a union of groups. Hence \(\overline{\overline{S}} \in \mathcal{CR}\).

(ii) Clearly \(\overline{V}\) is closed under the formation of homomorphisms, subsemigroups and direct products. Therefore \(\overline{V}\) is a variety and, by part (i), \(\overline{V} \in \mathcal{L}(\mathcal{CR})\).

(iii) That \(\overline{\overline{V}} = V\) follows from the evident fact that the dual of the dual of a semigroup \((S, \cdot)\) is just \((S, \cdot)\) itself.

(iv) If \(V \subseteq \overline{V}\), then we must have \(\overline{V} \subseteq \overline{\overline{V}} = V\) and therefore \(V = \overline{V}\) as claimed.

(v) It is trivial that \(V \cap \overline{V}\) is self-dual. Let \(S \in \mathcal{V} \cap \overline{\mathcal{V}}\). Then there exist \(V_\alpha \in \mathcal{V} \cup \overline{\mathcal{V}}, \alpha \in A\), a subsemigroup \(R\) of \(\prod_{\alpha \in A} S_\alpha\) and an epimorphism \(\varphi : R \rightarrow S\). It follows
that \( \overline{S} \) is a homomorphic image of \( \overline{R} \) which, in turn, is a subsemigroup of \( \prod_{\alpha \in A} \overline{S}_\alpha \)

where \( \overline{S}_\alpha \in \mathcal{V} \cup \overline{\mathcal{V}} \). Hence \( \overline{S} \in \mathcal{V} \vee \overline{\mathcal{V}} \). That implies that \( \overline{\mathcal{V}} \vee \overline{\mathcal{V}} \subseteq \mathcal{V} \vee \overline{\mathcal{V}} \) and therefore that \( \mathcal{V} \vee \overline{\mathcal{V}} \) is self-dual. \( 0.2 \text{cm} \)

(vi) Let \( a_1, a_2, \ldots, a_n \in S \in \mathcal{V} \) and \( u = u(x_1, \ldots, x_n) \in U_X \). \( -0.2 \text{cm} \)

Claim: \( \overline{v}(\overline{u}_1, \ldots, \overline{u}_n) = u(a_1, \ldots, a_n) \). \( -0.2 \text{cm} \)

We argue by induction on \( |u|_Y \).

Case: \( |u|_Y = 1 \). Then \( u = x_1 \in X \). Hence \( \overline{u}(\overline{a}_1) = \overline{a}_1 = a_1 = u(a_1) \).

Now assume that \( |u|_Y > 1 \) and that the claim is true for all \( v \in U_X \) with \( |v|_Y < |u|_Y \).

Let \( u = u_1 \cdots u_m \) as a product of irreducibles (see [PR99], Lemma I.10.4).

Case: \( m = 1 \). Then \( u = u_1 \) is irreducible and \( |u|_Y > 1 \). The only way that that can occur is if \( u = (w)^{-1} \) for some \( w \in U_X \). By Lemma 3.2(ii), \( \overline{(w)^{-1}} = \overline{(w)}^{-1} \) so that \( \overline{u} = (w)^{-1} = \overline{(w)}^{-1} \) and

\[
\overline{u}(\overline{a}_1, \ldots, \overline{a}_n) = (w(\overline{a}_1, \ldots, \overline{a}_n))^{-1} = (\overline{w}(\overline{a}_1, \ldots, \overline{a}_n))^{-1} = (w(a_1, \ldots, a_n))^{-1} \quad \text{by induction hypothesis}
\]

\[
= u(a_1, \ldots, a_n).
\]

Case: \( m > 1 \). Then \( \overline{u} = \overline{u}_1 \overline{u}_2 \cdots \overline{u}_m = \overline{u}_m \cdots \overline{u}_2 \overline{u}_1 \) so that from the substitution \( x_i \to \overline{a}_i \) \( (1 \leq i \leq n) \) into \( \overline{S} \) we obtain (using \( * \) to denote the multiplication in \( \overline{S} \))

\[
\overline{u}(\overline{u}_1, \ldots, \overline{u}_n) = \overline{u}_m(\overline{u}_1, \ldots, \overline{u}_n) * * \overline{u}_1(\overline{u}_1, \ldots, \overline{u}_n)
\]

\[
= \overline{u}_m(\overline{u}_1, \ldots, \overline{u}_n) \cdots \overline{u}_1(\overline{u}_1, \ldots, \overline{u}_n)
\]

\[
= u_1(a_1, \ldots, a_n) \cdots u_m(a_1, \ldots, a_n) \quad \text{by the induction hypothesis}
\]

\[
= u(a_1, \ldots, a_n).
\]

Thus the claim holds.

Consequently, for all \( \alpha \in A \),

\[
\overline{u}_\alpha(\overline{a}_1, \ldots, \overline{a}_n) = u_\alpha(a_1, \ldots, a_n) = v_\alpha(a_1, \ldots, a_n) = \overline{v}_\alpha(\overline{a}_1, \ldots, \overline{a}_n).
\]

Thus \( \overline{S} \in [\overline{u}_\alpha = \overline{v}_\alpha]_{\alpha \in A} \) so that \( \overline{V} \subseteq [\overline{u}_\alpha = \overline{v}_\alpha]_{\alpha \in A} \). Similarly, \( S \in [u_\alpha = v_\alpha]_{\alpha \in A} \) implies that \( \overline{S} \in [\overline{u}_\alpha = \overline{v}_\alpha]_{\alpha \in A} = [u_\alpha = v_\alpha]_{\alpha \in A} = \mathcal{V} \). Thus \( \overline{S} = \overline{S} \in \overline{\mathcal{V}} \) whence \( [\overline{u}_\alpha = \overline{v}_\alpha]_{\alpha \in A} \subseteq \overline{\mathcal{V}} \) and equality prevails.

(vii) This follows immediately from (vi).

(viii) Let \( u, v \in U_X \). It follows from (vii) that

\[
u \in \mathcal{U} \cap \mathcal{V} \iff \overline{u} \overline{\mathcal{U} \cup \mathcal{V}} \iff \overline{u} (\zeta_\mathcal{U} \cap \zeta_\mathcal{V}) \overline{v} \iff \overline{u} \zeta_\mathcal{U} \overline{v} \text{ and } \overline{u} \mathcal{U} \overline{v} \iff \overline{u} \zeta_\mathcal{U} \overline{v} \iff u (\zeta_\mathcal{U} \cap \zeta_\mathcal{V}) v \iff u \zeta_\mathcal{U} \overline{v} \overline{v}.
\]


Therefore, \( U \lor V = \overline{U \lor V} \), establishing the first claim.

Similarly,
\[
\begin{align*}
    u \zeta_{\overline{U \lor V}} v & \iff \overline{u} \zeta_{U \lor V} \overline{v} \\
    & \iff \overline{u} (\zeta_u \lor \zeta_v) \overline{v} \\
    & \iff \text{there exist } u_i \in U \land V, \rho_i \in \{\zeta_u, \zeta_v\} \text{ with} \\
    & \quad \overline{u} = \rho_0 \rho_1 u_1 \rho_2 u_2 \ldots \rho_n u_n = \overline{v} \\
    & \iff u \rho_1 \overline{u} \rho_2 \ldots \rho_n \overline{u} \rho_n = v \\
    & \iff u (\zeta_{\overline{U}} \lor \zeta_{\overline{V}}) v \\
    & \iff u \zeta_{\overline{U} \land \overline{V}} v.
\end{align*}
\]

Therefore \( \overline{U \lor V} = \overline{U} \land \overline{V} \).

\[\text{Theorem 3.5.} \quad (i) \ \mathcal{SD} \ \text{is a complete sublattice of } \mathcal{L}(CR).\]

For the remaining parts, let \( \mathcal{V} = [u_{\alpha} = v_{\alpha}]_{\alpha \in A} \in [\mathcal{S}, CR] \) be self-dual.

(ii) \( \overline{V^T} = V^T, \ \overline{V^K} = V^K \).

(iii) \( \mathcal{V}^K \) and \( \mathcal{V}^T \) are self-dual.

Proof.

(i) Clearly \( \mathcal{SD} \) is closed under arbitrary intersections. Now let \( S \in \mathcal{V} = \bigvee_{\alpha \in A} \mathcal{V}_{\alpha} \) where \( \mathcal{V}_{\alpha} \in \mathcal{SD} \) for all \( \alpha \in A \). Then there exist \( B \subseteq A, \ S_{\alpha} \in \mathcal{V}_{\alpha} \) for \( \alpha \in B \), a subsemigroup \( R \) of \( \prod_{\alpha \in A} S_{\alpha} \) and an epimorphism \( \phi : R \to S \). Since each \( \mathcal{V}_{\alpha} \) is self-dual, we have \( \overline{S}_{\alpha} \in \overline{\mathcal{V}}_{\alpha} = \mathcal{V}_{\alpha} \) for all \( \alpha \in B \). Now \( \phi \) is also a homomorphism of \( R \) onto \( \overline{S} \) and \( \overline{R} \) is a subsemigroup of \( \prod_{\alpha \in B} \overline{S}_{\alpha} \subseteq \bigvee_{\alpha \in A} \mathcal{V}_{\alpha} = \mathcal{V} \). Thus \( \overline{S} \in \mathcal{V} \) so that \( \overline{\mathcal{V}} \subseteq \mathcal{V} \) and \( \mathcal{V} \) is self-dual.

(ii) We have
\[
S \in \mathcal{V}^{T_t} \iff S/L^0 \in \mathcal{V} \iff \overline{S}/R^0 \in \overline{\mathcal{V}} = \mathcal{V} \iff \overline{S} \in \mathcal{V}^{T_t}.
\]

Hence \( \mathcal{V}^{T_t} = \overline{\mathcal{V}^{T_t}} \). A similar argument yields \( \mathcal{V}^{K_t} = \overline{\mathcal{V}^{K_t}} \).

(iii) Case: \( \mathcal{V}^K \). By Lemma 2.5, \( \mathcal{V}^K = [x_{u_{\alpha}}y(x_{v_{\alpha}}y)^{-1} \in E]_{\alpha \in A} \). Since \( \mathcal{V} \) is self-dual, \( \mathcal{V}^K \) also satisfies \( \overline{u}_{\alpha} = \overline{v}_{\alpha} \), for all \( \alpha \in A \). Hence, by Lemma 2.5, \( \mathcal{V}^K \) satisfies the identity
\[
y_{\overline{u}_{\alpha}} x(y_{\overline{v}_{\alpha}} x)^{-1} \in E \quad (\alpha \in A).
\]

By [PR99] Lemma II.2.2(iii), \( \mathcal{V}^K \) also satisfies
\[
(y_{\overline{u}_{\alpha}} x)^{-1} y_{\overline{u}_{\alpha}} x \in E,
\]
that is,
\[
{xu_{\alpha}}(x_{v_{\alpha}}y)^{-1} \in E.
\]
Thus, by Lemma 3.4\((\nu)\),
\[
\mathcal{V}^K \subseteq \left[ xu_\alpha y (xv_\alpha y)^{-1} \in E \right]_{\alpha \in A} = \overline{\mathcal{V}^K}
\]
which implies, by Lemma 3.4\((\nu)\), that \(\mathcal{V}^K = \overline{\mathcal{V}^K}\) and \(\mathcal{V}^K\) is self-dual. -0.05cm]

Case: \(\mathcal{V}^T\). By Lemma 2.5, \(\mathcal{V}^T = [u_\alpha^0 = v_\alpha^0, (xu_\alpha y)^0 = (xv_\alpha y)^0]_{\alpha \in A}\). Since \(\mathcal{V}\) is self-dual, \(\mathcal{V}\) also satisfies \(\pi_\alpha = \pi_\alpha\) so that \(\mathcal{V}^T\) satisfies \((y\pi_\alpha x)^0 = (y\pi_\alpha x)^0\) and, equivalently, \((xu_\alpha y)^0 = (xv_\alpha y)^0\), for all \(\alpha \in A\). In a similar fashion, it follows that \(\mathcal{V}^T\) satisfies \(\pi_\alpha^0 = \pi_\alpha^0\). Thus \(\mathcal{V}^T \subseteq \overline{\mathcal{V}^T}\) and equality prevails.0.4cm

We may now augment our list of self-dual varieties with \(\mathcal{O} = \mathcal{G}^K\), \(L\mathcal{O} = \mathcal{CS}^K\), \(\mathcal{B}\mathcal{G} = \mathcal{B}^T\), \(\mathcal{B}^{T_t} \lor \mathcal{B}^{T_r}\), etc.

4. Networks

Before focussing on \(K\)-classes, we first make some general observations regarding sublattices of \(\mathcal{L}(\mathcal{CR})\) generated by the repeated application of operators of the form \(\mathcal{V} \rightarrow \mathcal{V}^\sigma, \mathcal{V} \rightarrow \mathcal{V}^{\sigma(\ell_t, K_r)}\) for \(\sigma \in \Theta^1\).

**Theorem 4.1.** Let \(\mathcal{V} = [u_\alpha = v_\alpha]_{\alpha \in A} \in [\mathcal{S}, \mathcal{CR}]\) be self-dual. In addition let \(\sigma, \tau \in \Theta\) be such that \(|\sigma| = |\tau|\) and \(h(\sigma) \neq h(\tau)\), \(t(\sigma) = T_\ell, t(\tau) = T_r\).

(i) \(\mathcal{V}^{\sigma(\ell_t, K_r)} \cap \mathcal{V}^{\tau(\ell_t, T_r)}\) and \(\mathcal{V}^{\sigma(T_t, T_r)} \cap \mathcal{V}^{\tau(\ell_t, T_r)}\) are self-dual.

(ii) \(\mathcal{V}^{\sigma(\ell_t, K_r)} \lor \mathcal{V}^{\tau(K_r, K_t)} = \mathcal{V}^{\sigma(\ell_t, K_r)K_r} \lor \mathcal{V}^{\tau(K_r, K_t)K_t}\) is self-dual.

(iii) \((\mathcal{V}^\sigma \lor \mathcal{V}^\tau) \cap \mathcal{V}^K = (\mathcal{V}^{\sigma T_\ell} \lor \mathcal{V}^{\tau T_\ell}) \cap \mathcal{V}^K = \mathcal{V}^{\sigma(\ell_t, K_r)}K_r \lor \mathcal{V}^{\tau(\ell_t, K_r)K_t} K_t\).

(iv) \((\mathcal{V}^\sigma \lor \mathcal{V}^\tau)^T = \mathcal{V}^{\sigma T_\ell} \lor \mathcal{V}^{\tau T_\ell}\).

(v) Let \(\mathcal{V} \subset \mathcal{V}^K\). The varieties of the form \(\mathcal{V}^{\rho(\ell_t, K_r)}\) \((\rho \in \Theta^1)\) and \(\mathcal{V}^{\sigma(\ell_t, K_r)} \cap \mathcal{V}^{\tau(K_r, K_t)}\) (with \(\sigma, \tau\) as above) constitute a sublattice of \([\mathcal{V}, \mathcal{V}^K]\) with distinct elements as illustrated in Diagram 4.2. Solid lines indicate \(\ell_t\)-related varieties and broken lines indicate \(K_r\)-related varieties.

(vi) Let \(\mathcal{V} \subset \mathcal{V}^K\). The varieties of the forms \(\mathcal{V}^\rho\) \((\rho \in \Theta^1)\), \(\mathcal{V}^\sigma \lor \mathcal{V}^\tau\) and \(\mathcal{V}^\sigma \lor \mathcal{V}^\tau\) (with \(\sigma, \tau\) as above) constitute a sublattice of \([\mathcal{V}, \mathcal{CR}]\) with distinct elements as illustrated in Diagram 4.3 (with the possible exception of the equality of \(\mathcal{V}^\sigma \lor \mathcal{V}^\tau\) and \(\mathcal{V}^{\sigma T_\ell} \lor \mathcal{V}^{\tau T_\ell}\) for some or all of the \(\sigma, \tau\)). Solid lines indicate \(T_\ell\)-related varieties and broken lines indicate \(T_r\)-related varieties.

(vii) \(\bigvee_{\sigma \in \Theta} \mathcal{V}^{\sigma(\ell_t, T_r)} = \mathcal{CR}\).
Proof. (i) Claim: $V^\sigma \cap V^\tau$ is self-dual. We argue by induction on $|\sigma|$. Let $|\sigma| = |\tau| = 1$ so that $\sigma = t(\sigma) = T_\ell$, $\tau = t(\tau) = T_r$. Then

$$V^T_\ell \cap V^T_r = V^T_\ell \cap \overline{V^T_r}$$

by Theorem 3.5 (ii)

which is self-dual by Lemma 3.4(v).

So now assume that $|\sigma| = |\tau| = m > 1$ and that the claim holds for $|\sigma| = |\tau| < m$.

We may assume that $\sigma = \sigma_1 T_\ell$ and $\tau = \tau_1 T_r$ for some $\sigma_1, \tau_1 \in \Theta$ with $t(\sigma_1) = T_r, t(\tau_1) = T_\ell$. Then, by Lemma 2.7(iv), we have

$$(V^{\sigma_1} \cap V^{\tau_1})^{T_\ell} = V^\sigma \cap V^{\tau_1} = V^{\tau_1}$$

since $|\tau_1| < |\sigma|$

and similarly

$$(V^{\sigma_1} \cap V^{\tau_1})^{T_r} = V^{\sigma_1} \cap V^r = V^{\sigma_1}$$

since $|\sigma_1| < |\tau|$.

By Theorem 3.5(ii), since $V^{\sigma_1} \cap V^{\tau_1}$ is self-dual by the induction hypothesis, we have

$$V^{\sigma_1} = (V^{\sigma_1} \cap V^{\tau_1})^{T_r} = (V^{\sigma_1} \cap V^{\tau_1})^{T_\ell} = \overline{V^{\tau_1}}$$

so that

$$V^{\sigma_1} \lor V^{\tau_1} = \overline{V^{\tau_1}} \lor V^{\tau_1}.$$  

By Lemma 3.4(v), $V^{\sigma_1} \lor V^{\tau_1}$ is self-dual.

Furthermore, by Lemma 2.7(iii) and with $\tau_1 = \tau_2 T_\ell$, where $\tau_2 \in \Theta^1$, we have

$$(V^{\sigma_1} \lor V^{\tau_1})^{T_\ell} = (V^{\sigma_1})^{T_\ell} \lor (V^{\tau_1})^{T_\ell} = V^{\sigma_1} \lor (V^{\tau_2})^{T_\ell}.$$  

Since $|\tau_2| < |\sigma_1|$, it follows that $V^{\tau_2} \subseteq V^{\sigma_1}$. Consequently,

$$(V^{\sigma_1} \lor V^{\tau_1})^{T_\ell} = V^{\sigma_1}$$

which implies that $(V^{\sigma_1} \lor V^{\tau_1})^{T_\ell} = V^{\sigma_1} T_\ell = V^\sigma$. Similarly, $(V^{\sigma_1} \lor V^{\tau_1})^{T_r} = V^\tau$. But $V^{\sigma_1} \lor V^{\tau_1}$ is self-dual and so, by Theorem 3.5(ii),

$$V^\tau = (V^{\sigma_1} \lor V^{\tau_1})^{T_r} = \overline{(V^{\sigma_1} \lor V^{\tau_1})^{T_\ell}} = \overline{V^\sigma}.$$  

Consequently $V^\sigma \lor V^\tau$ is self-dual.

Claim: $V^{\sigma(K_\ell, K_r)} \lor V^{\tau(K_\ell, K_r)}$ is self-dual. By Lemma 2.6(ii),

$$V^{\sigma(K_\ell, K_r)} \lor V^{\tau(K_\ell, K_r)} = V^{\sigma(T_\ell, T_r)} \lor V^K \lor V^{\tau(T_\ell, T_r)} \lor V^K$$

$$= (V^{\sigma(T_\ell, T_r)} \lor V^{\tau(T_\ell, T_r)}) \lor V^K$$

where, by the first case $V^{\sigma(T_\ell, T_r)} \lor V^{\tau(T_\ell, T_r)}$ is self-dual and, by Theorem 3.5(iii), $V^K$ is self-dual. Hence $V^{\sigma(K_\ell, K_r)} \lor V^{\tau(K_\ell, K_r)}$ is self-dual.
(ii) We have
\[ \forall^\sigma(K_\ell,K_r) \subseteq \forall^\sigma(K_\ell,K_r) \lor \forall^\tau(K_\ell,K_r) \subseteq \forall^\sigma(K_\ell,K_r)K_r \]
so that
\[ \forall^\sigma(K_\ell,K_r) \lor \forall^\tau(K_\ell,K_r) \]
similarly
\[ \forall^\tau(K_\ell,K_r) \lor \forall^\sigma(K_\ell,K_r) . \]
On the other hand
\[ \forall^\sigma(K_\ell,K_r) \subseteq \forall^\sigma(K_\ell,K_r)K_r \cap \forall^\tau(K_\ell,K_r)K_\ell \subseteq \forall^\sigma(K_\ell,K_r)K_r \]
so that
\[ \forall^\sigma(K_\ell,K_r) \lor \forall^\tau(K_\ell,K_r)K_r \cap \forall^\tau(K_\ell,K_r)K_\ell \]
and similarly
\[ \forall^\tau(K_\ell,K_r) \lor \forall^\sigma(K_\ell,K_r)K_r \cap \forall^\tau(K_\ell,K_r)K_\ell . \]
Thus we must have
\[ \forall^\sigma(K_\ell,K_r) \lor \forall^\tau(K_\ell,K_r)K_r \cap K_r \lor \forall^\sigma(K_\ell,K_r)K_r \cap \forall^\tau(K_\ell,K_r)K_\ell . \]
Since \( K_\ell \cap K_r = \varepsilon \), we have
\[ \forall^\sigma(K_\ell,K_r) \lor \forall^\tau(K_\ell,K_r) = \forall^\sigma(K_\ell,K_r)K_r \cap \forall^\tau(K_\ell,K_r)K_\ell . \]
That this variety is self-dual follows from part (i) with \( \sigma(K_\ell,K_r)K_r \) and \( \tau(K_\ell,K_r)K_\ell \) in place of \( \sigma(K_\ell,K_r) \) and \( \tau(K_\ell,K_r) \), respectively.

(iii) Denote the three expressions by \( A, B \) and \( C \), respectively. \( A \subseteq B \). We have \( \forall^\sigma, \forall^\tau \subseteq \forall^{\sigma T_r} \lor \forall^{\tau T_\ell} \) so that
\[ (\forall^\sigma \lor \forall^\tau) \cap \forall^K \subseteq (\forall^{\sigma T_r} \lor \forall^{\tau T_\ell}) \cap \forall^K . \]
\( B = C \). We have
\[ (\forall^{\sigma T_r} \lor \forall^{\tau T_\ell}) \cap \forall^K = (\forall^{\sigma T_r} \lor \forall^K) \cap (\forall^{\tau T_\ell} \lor \forall^K) \]
\[ = \forall^{\sigma(K_\ell,K_r)K_r} \cap \forall^{\tau(K_\ell,K_r)K_\ell} \]
by Lemma 2.6
\[ \forall^{\sigma(K_\ell,K_r)K_r} \cap \forall^{\tau(K_\ell,K_r)K_\ell} \subseteq (\forall^\sigma \lor \forall^K) \lor (\forall^\tau \lor \forall^K) \]
\[ = (\forall^\sigma \lor \forall^\tau) \cap \forall^K . \]
(iv) Replacing $K_t$ and $K_r$ by $T_t$ and $T_r$, respectively, in the argument in part (ii) we obtain

$$\mathcal{V}^{\sigma} \lor \mathcal{V}^{\tau} \ P \ \mathcal{V}^{\sigma T_r} \lor \mathcal{V}^{\tau T_t}$$

for $P = T_t$ and $P = T_r$ and therefore also for $T_t \cap T_r = T$. Thus $\mathcal{V}^{\sigma} \lor \mathcal{V}^{\tau} \ T \ \mathcal{V}^{\sigma T_r} \lor \mathcal{V}^{\tau T_t}$. Now $T$ is a complete congruence on $\mathcal{L}(\mathcal{CR})$ while $\mathcal{V}^{\sigma T_r} = \mathcal{V}^{\sigma T_T}$ and $\mathcal{V}^{\tau T_t} = \mathcal{V}^{\tau T_T}$ are both maximal in their $T$-class. Hence so also is $\mathcal{V}^{\sigma T_r} \lor \mathcal{V}^{\tau T_t}$, that is,

$$\mathcal{V}^{\sigma T_r} \lor \mathcal{V}^{\tau T_t} = (\mathcal{V}^{\sigma} \lor \mathcal{V}^{\tau})^T.$$

(v) Since $\mathcal{V} \subset \mathcal{V}^K$ and, by Lemma 2.8(ii), $\mathcal{V}^K = \bigvee_{\sigma \in \Theta^1} \mathcal{V}^{\sigma(K_t,K_r)}$, it follows that either $\mathcal{V} \neq \mathcal{V}^{K_t}$ or $\mathcal{V} \neq \mathcal{V}^{K_r}$. By hypothesis, $\mathcal{V}$ is self-dual. Hence $\mathcal{V} \neq \mathcal{V}^{K_t}, \mathcal{V}^{K_r}$. We also have $\mathcal{V} = \mathcal{V}^{K_t} \lor \mathcal{V}^{K_r}$ so that $\mathcal{V}^{K_t}$ and $\mathcal{V}^{K_r}$ must be incomparable. Thus we have the base $\mathcal{V}, \mathcal{V}^{K_t}, \mathcal{V}^{K_r}$ for Diagram 4.2. This, in turn, implies that $\mathcal{V}^{K_t} \lor \mathcal{V}^{K_r} \neq \mathcal{V}^{K_t}, \mathcal{V}^{K_r}$. In addition $\mathcal{V}^{K_t} = \mathcal{V}^{K_t}$ so that $\mathcal{V}^{K_t} \lor \mathcal{V}^{K_r}$ is self-dual.

Suppose that $\mathcal{V}^{K_t} \lor \mathcal{V}^{K_r} = \mathcal{V}^K$. Then

$$\mathcal{V}^K = \mathcal{V}^{K_t} \lor \mathcal{V}^{K_r} = (\mathcal{V}^{K_t} \lor \mathcal{V}^{K_r})^{K_t} \subseteq \mathcal{V}^{K_t} \lor \mathcal{V}^{K_r} \subseteq \mathcal{V}^K$$

so that $\mathcal{V}^K = \mathcal{V}^{K_t} \lor \mathcal{V}^{K_r}$. Hence, either $\mathcal{V} \subset \mathcal{V}^K$ and $\mathcal{V}^{K_t} = \mathcal{V}^K$, or $\mathcal{V}^{K_r} \subset \mathcal{V}^K$ and $\mathcal{V}^{K_t} \lor \mathcal{V}^{K_r} = \mathcal{V}^K$. This implies that either

$$(\mathcal{V}^K)^{T_r} = (\mathcal{V}^{K_t})^{T_r} = \mathcal{V}^{T_r} \subseteq \mathcal{V} \subset \mathcal{V}^K$$

or

$$(\mathcal{V}^K)^{T_t} = (\mathcal{V}^{K_t} \lor \mathcal{V}^{K_r})^{T_t} = (\mathcal{V}^{K_t})^{T_t} \subseteq \mathcal{V}^{K_t} \subset \mathcal{V}^K$$

contradicting either Lemma 2.6(i) or its dual. Hence $\mathcal{V}^{K_t} \lor \mathcal{V}^{K_r}$ is a self-dual variety that is a proper subvariety of $(\mathcal{V}^{K_t} \lor \mathcal{V}^{K_r})^K = \mathcal{V}^K$. In addition, by part (ii), with $\sigma = T_t, \tau = T_r$ and Lemma 2.7 (v),

$$(\mathcal{V}^{K_t} \lor \mathcal{V}^{K_r})^{K_t} = (\mathcal{V}^{K_t} \lor \mathcal{V}^{K_r})^{K_t} \subseteq \mathcal{V}^{K_t} \lor \mathcal{V}^{K_r} = \mathcal{V}^{K_t} \lor \mathcal{V}^{K_r}$$

and similarly $(\mathcal{V}^{K_t} \lor \mathcal{V}^{K_r})^{K_r} = \mathcal{V}^{K_t} \lor \mathcal{V}^{K_r}$. Thus we can build the next step of the diagram starting with $\mathcal{V}^{K_t} \lor \mathcal{V}^{K_r}$ and so on.

(vi) Note that $\mathcal{V}^\rho \subseteq \mathcal{V}^\sigma$ for all $\rho, \sigma \in \Theta$ with $|\rho| < |\sigma|$ so that with the help of part (iv) we see that the partially ordered set of the varieties listed in this part is as depicted in Diagram 4.3. As in part (v), we deduce that $\mathcal{V} \neq \mathcal{V}^{K_t}$ and $\mathcal{V} \neq \mathcal{V}^{K_r}$. Again, $\mathcal{V} = \mathcal{V}^{K_t} \lor \mathcal{V}^{K_r}$ so that the three varieties $\mathcal{V}, \mathcal{V}^{K_t}, \mathcal{V}^{K_r}$ are all distinct.
We have $\mathcal{V} \subseteq \mathcal{V}^K \cap \mathcal{V}^T \subseteq \mathcal{V}^K \cap \mathcal{V}^T = \mathcal{V}$. Therefore $\mathcal{V}^K$ and $\mathcal{V}^T$ are incomparable. 
Hence $\mathcal{V}^T \neq \mathcal{V}^T$ and, dually, $\mathcal{V}^T \neq \mathcal{V}^T$. But $\mathcal{V}^T = \mathcal{V}^T \cap \mathcal{V}^T$. It follows that $\mathcal{V}^T$ and $\mathcal{V}^T$ are incomparable and therefore that $\mathcal{V}^T \vee \mathcal{V}^T$ is distinct from $\mathcal{V}^T$ and $\mathcal{V}^T$. 
It is possible that $\mathcal{V}^T \vee \mathcal{V}^T = \mathcal{V}^T \cap \mathcal{V}^T$ but definitely this is not always the case, see example below. However, by Theorem 3.5(ii) and Lemma 3.4(v), $\mathcal{V}^T \vee \mathcal{V}^T$ is self-dual and therefore, by Theorem 3.5(iii), $\mathcal{V}^T \cap \mathcal{V}^T = (\mathcal{V}^T \vee \mathcal{V}^T)^T$ is self-dual.

Suppose that $\mathcal{V}^T \cap \mathcal{V}^T = \mathcal{C} \mathcal{R}$. Since $\mathcal{V}^T \subseteq \mathcal{C} \mathcal{R}$, we must have $\mathcal{V}^T = \mathcal{C} \mathcal{R}$. This means that either $\mathcal{V}^T < \mathcal{C} \mathcal{R}$ and $(\mathcal{V}^T)^T = \mathcal{C} \mathcal{R}$ or $\mathcal{V} < \mathcal{C} \mathcal{R}$ and $\mathcal{V}^T = \mathcal{C} \mathcal{R}$. 
In the former case $\mathcal{C} \mathcal{R} \subseteq \mathcal{V}^T < \mathcal{C} \mathcal{R}$ and in the latter case $\mathcal{C} \mathcal{R} \subseteq \mathcal{V} < \mathcal{C} \mathcal{R}$. 
Since $\mathcal{C} \mathcal{R} = \mathcal{C} \mathcal{R}^K$, this contradicts Lemma 2.6(i) or its dual. Thus $\mathcal{V}^T \cap \mathcal{V}^T = (\mathcal{V}^T \vee \mathcal{V}^T)^T$ is self-dual and a proper subvariety of $\mathcal{C} \mathcal{R}$.

Furthermore,

$$((\mathcal{V}^T \vee \mathcal{V}^T)^T)^T = (\mathcal{V}^T \cap \mathcal{V}^T)^T$$

by part (iv)

$$= \mathcal{V}^T \cap \mathcal{V}^T$$

by Theorem 2.7(iv)

$$= (\mathcal{V}^T)^T$$

and dually $((\mathcal{V}^T \vee \mathcal{V}^T)^T)^T = (\mathcal{V}^T)^T$.

We may now repeat the above argument starting with $(\mathcal{V}^T \vee \mathcal{V}^T)^T$ to extend Diagram 4.3 to the next level, and so on. 0.4cm

(vii) See [PR88] Theorem 4.6(iii).
Diagram 4.2

Diagram 4.3
Theorem 4.4. Let $V = V^T \in \mathcal{L}(CR), V \neq CR$ and $\sigma \in \Theta^1$. Then $V^\sigma$ is the largest variety $W \in \mathcal{L}(CR)$ such that $W \cap V^K = V^{\eta(K_\ell,K_r)}$. If $\sigma, \tau \in \Theta$ are such that $|\sigma| = |\tau|$, $h(\sigma) \neq h(\tau)$, then $V^\sigma \cap V^\tau$ is the largest variety $W \in \mathcal{L}(CR)$ such that $W \cap V^K = V^{\eta(K_\ell,K_r)} \cap V^{\eta(K_\ell,K_r)}$.

Proof. Recall from Lemma 2.6(ii) that, for any $\sigma \in \Theta^1$, we have

$$V^\sigma \cap V^K = V^{\eta(K_\ell,K_r)}.$$  

Now let $W \in \mathcal{L}(CR)$ be such that $W \cap V^K \subseteq V^{\eta(K_\ell,K_r)}$. For $\sigma = \emptyset$, this means that $W \cap V^K \subseteq V$. Suppose that $W \not\subseteq V$. By Lemma 2.10, there exists $S \in W \setminus V$ such that either $S \in V^K \setminus V$, or $S \in V^K \setminus V$ so that $S \in (W \cap V^K) \setminus V$ which is a contradiction. Hence $W \subseteq V$.

Now consider $\sigma \in \Theta$ and assume that the claim holds for shorter words than $\sigma$. By duality, if suffices to consider the case where $\sigma = \sigma_1 T_\ell$, $\sigma_1 \in \Theta^1$. Let $W \in \mathcal{L}(CR)$ be such that $W \cap V^K \subseteq V^{\eta(K_\ell,K_r)}$. Then, by Lemma 2.7(iii) and the induction hypothesis,

$$W_{T_\ell} \cap V^K = (W \cap V^K)_{T_\ell} \subseteq (V^{\eta(K_\ell,K_r)})_{T_\ell} = (V^{\eta_1(K_\ell,K_r)K_\ell})_{T_\ell} \subseteq V^{\eta_1(K_\ell,K_r)}.$$  

By the induction hypothesis $W_{T_\ell} \subseteq V^{\eta_1}$. Hence

$$W \subseteq (W_{T_\ell})^{T_\ell} \subseteq V^{\eta_1 T_\ell} = V^\sigma$$

and the induction step holds.

Therefore $V^\sigma$ must be the largest variety whose intersection with $V^K$ is $V^{\eta(K_\ell,K_r)}$.

For $\sigma, \tau$ as in the statement and $W \in \mathcal{L}(CR)$, we have

$$W \cap V^K \subseteq V^{\eta(K_\ell,K_r)} \cap V^{\eta(K_\ell,K_r)} \implies W \cap V^K \subseteq V^{\eta(K_\ell,K_r)} \implies W \subseteq V^\sigma$$

by argument above.

Similarly $W \cap V^K \subseteq V^{\eta(K_\ell,K_r)} \cap V^{\eta(K_\ell,K_r)} \implies W \subseteq V^\tau$ so that $W \subseteq V^\sigma \cap V^\tau$. Conversely

$$(V^\sigma \cap V^\tau) \cap V^K = (V^\sigma \cap V^K) \cap (V^\tau \cap V^K) = V^{\eta(K_\ell,K_r)} \cap V^{\eta(K_\ell,K_r)}$$

and the final claim holds. 

Theorem 4.4 is modelled on the result of Reilly and Zhang [RZ], Lemma 4.11, characterizing the largest subvariety of $\mathcal{L}(CR)$ whose intersection with $B$ is a specific variety of bands.

We note that, for any variety $V = [u_\alpha = v_\alpha]_{\alpha \in A} \in \mathcal{L}(CR)$, we can derive a basis of identities for each of the varieties appearing in Diagrams 4.2 and 4.3 (except for
those in 4.3 expressed as the join of two varieties) by a sequence of applications of the bases provided in Lemma 2.5 for varieties of each of the forms $U^K$, $U^{τ_l}$, $U^{τ_r}$, $U^{K_l}$, $U^{K_r}$ ($U \in L(CR)$) and taking the union of two bases for the intersection of varieties. Diagram 4.5 displays the relationship between the network based on $\mathcal{V}$ using the upper operators associated with $K_l, K_r$ and the network based on $\mathcal{V}$ using the upper operators associated with $T_l, T_r$.

\hspace{0.5cm}  

Diagram 4.5
Example 4.6. In general the vertical lines in Diagram 4.5 can represent non-trivial intervals. For example, if we take \( V = SG \), the variety of all semilattices of groups, then

\[
\begin{align*}
V^T_L &= LRO \quad \text{— the variety of all left regular orthogroups} \\
V^T_R &= RRO \quad \text{— the variety of all right regular orthogroups} \\
V^T_L \lor V^T_R &= RO \quad \text{— the variety of all regular orthogroups} \\
V^T_L \land V^T_R &= LRO \lor RRO \quad \text{— the variety of all regular orthogroups on which } R \text{ is a congruence} \\
V^T_L \land V^T_R &= \text{the dual of } R^* \\
V^T_L \land V^T_R &= RB \quad \text{— the variety of all regular bands of groups.}
\end{align*}
\]

For the equality \( LRO \lor RRO = RO \) see [PR99], Theorem V.3.3.

Let \( U \in L(CS) \), where \( CS \) is the variety of all completely simple semigroups, be a non-orthodox variety. Then we have \( RO \lor U \in (RO, RBG) \). Thus

\[
SG^T_L \lor SG^T_R = LRO \lor RRO \neq RBG = SG^{T_L} \cap SG^{T_R}.
\]

this is illustrated in Diagram 4.7.

\[\text{Diagram 4.7}\]

We discuss briefly some ways in which the diagrams 4.5 and 4.7 can be found in \( L(CR) \).

Definition 4.8. We refer to the variety \( V \) on which the Diagrams 4.2 and 4.3 are built as the base point of the diagrams.

For instance, we can start with \( SG, NBG \). Alternatively, we can start with any self-dual proper subvariety \( Z \) of \( CR \), pass to \( Z^K \) which is also self-dual, and then pass to \( Z^{KT} \) which is self-dual and a proper subvariety of \( Z^{KTK} \), and which also has
the properties required for a base point. For example, if we take $Z = T, G$ or $CS$ we obtain base points $T^{KT} = BG, G^{KT} = OT, CS^{KT} = LO^T$. Each of these is self-dual and so we can repeat the process to obtain

$$BG^{KT}, BG^{(KT)^2}, \ldots, BG^{(KT)^n}, BS^{(KT)^n}, (OT)^{(KT)^n}, (LO^T)^{(KT)^n}.$$  

Another approach is to start with any variety $V \in [S, CR]$. Then the varieties $V \cap V$ and $V \lor V$ are both self-dual and so $(V \cap V)^T$ and $(V \lor V)^T$ are both base points for diagrams and so on.

5. Multiple copies of the lattice $[S, B]$

Here we will be interested in the underlying abstract lattice of the lattice $[S, B]$ of varieties of bands containing the variety $S$ of semilattices. The sublattice $[S, B]$ is extremely well-known and has a very special role in the theory of semigroups. In this section, we will show that copies of this lattice appear multiple times as a sublattice of various kernel classes of $L(CR)$.

**Theorem 5.1.** Let $V, \ell, V_r \in [S, CR]$ be such that $V \subset V_\ell \subset V^{K_\ell}$ and $V \subset V_r \subset V^{K_r}$. Let $\rho, \sigma, \tau \in \Theta^1$ be such that $h(\sigma) = T_\ell, h(\tau) = T_r$. Then the varieties $V_\rho^{(K_\ell, K_r)}, V_\ell^{(K_\ell, K_r)}, V_r^{(K_\ell, K_r)}$, generate a sublattice $L$ of $VK$ isomorphic to $[S, B]$.

**Note:** Varieties satisfying the conditions in the hypothesis of Theorem 5.1 are not difficult to find. For example, one could take $S, LN B, LRB$ and their duals (in which case we obtain the interval $[S, B]$ itself) or $SG, LN O, LRO$ and their duals. Also, the varieties $V_\ell$ and $V_r$ may or may not be chosen independently of each other. For instance, if $V$ is self-dual, then one natural choice, after selecting $V_\ell$, would be to take $V_r = \overline{V_\ell}$. But other choices may be available, see Theorem 6.1.

**Proof.** The foundation of the lattice $L$ is shown in Diagram 5.2.
This is simply the sublattice of $L(CR)$ generated by the elements $V$, $V_{\ell}$, $V_{r}$, $V_{K\ell}$, $V_{Kr}$. However, we need to establish that all the elements are distinct and that the intersections are as indicated in the diagram. Clearly, the joins are correctly shown. We break the argument into convenient pieces.

(1) $V_{\ell}$ (respectively, $V_{r}$) is not comparable with either $V_{r}$ or $V_{Kr}$ (respectively, $V_{\ell}$ or $V_{K\ell}$). We have

$$V_{\ell} \subseteq V_{Kr} \implies V_{\ell} K_{r} \implies V_{\ell} \cap K_{r} \implies V = V$$

which is a contradiction. A similar argument applies if we assume that $V_{Kr} \subseteq V_{\ell}$. Thus $V_{\ell}$ and $V_{Kr}$ are incomparable. If $V_{\ell}$ and $V_{r}$ are comparable then, without loss of generality, we may assume that $V_{\ell} \subseteq V_{r}$. This implies that $V_{\ell} K_{\ell} \cap K_{r} \implies V$ which is again a contradiction. By duality, the claim holds.

(2) $V_{\ell} \lor V_{r}$ is not comparable with $V_{K\ell}$ (respectively, $V_{Kr}$) and is therefore distinct from $V_{\ell}$ and $V_{Kr}$ (respectively $V_{\ell}$ and $V_{K\ell}$).

That $V_{\ell} \lor V_{r}$ is not contained in $V_{K\ell}$ follows from (1). On the other hand, by Lemmas 2.8(i) and 2.7(vi),

$$V_{K\ell} \subseteq V_{\ell} \lor V_{r} \implies V_{K\ell} = (V_{K\ell})_{Kr} \subseteq (V_{\ell} \lor V_{r})_{Kr} = (V_{\ell})_{Kr} \lor V \subseteq V_{\ell}$$

which is a contradiction. This establishes the first claim and the remaining claims then follow.

(3) $(V_{\ell} \lor V_{r}) \lor V_{K\ell}$ is distinct from $V_{\ell} \lor V_{r}$ and $V_{K\ell}$. The equality is clear and the remaining claims follow from the fact that $V_{\ell} \lor V_{r}$ and $V_{K\ell}$ are incomparable.
(4) \( \mathcal{V}^{K_\ell} \) and \( \mathcal{V}^{K_r} \) are incomparable and are distinct from \( \mathcal{V}^{K_\ell} \vee \mathcal{V}^{K_r} \). If, for instance, \( \mathcal{V}^{K_\ell} \subseteq \mathcal{V}^{K_r} \), then \( \mathcal{V}_\ell \vee \mathcal{V}_r \subseteq \mathcal{V}^{K_r} \) which would contradict (2). Reversing the roles of \( \mathcal{V}^{K_\ell} \) and \( \mathcal{V}^{K_r} \) we see that these varieties are incomparable and therefore distinct from \( \mathcal{V}^{K_\ell} \vee \mathcal{V}^{K_r} \).

(5) \( \mathcal{V}^{K_\ell} \vee \mathcal{V}_r \) and \( \mathcal{V}_\ell \vee \mathcal{V}^{K_r} \) are incomparable and both are distinct from \( \mathcal{V}^{K_\ell} \vee \mathcal{V}^{K_r} \). Suppose that \( \mathcal{V}^{K_\ell} \vee \mathcal{V}_r \subseteq \mathcal{V}_\ell \vee \mathcal{V}^{K_r} \). Then by Lemmas 2.8(i) and 2.7(vi),

\[
\mathcal{V}^{K_\ell} = (\mathcal{V}^{K_\ell})_{K_r} \subseteq (\mathcal{V}^{K_\ell} \vee \mathcal{V}_r)_{K_r} \subseteq (\mathcal{V}_\ell \vee \mathcal{V}^{K_r})_{K_r} = (\mathcal{V}^{K_r})_{K_\ell} \vee (\mathcal{V}_\ell)_{K_r} \subseteq \mathcal{V} \vee \mathcal{V}_\ell = \mathcal{V}
\]

which is a contradiction. The assumption of the reverse containment also leads to a contradiction. Accordingly, the first claim holds. The second claim follows immediately from the first.

(6) The following are sets of \( K_\ell \)-related varieties

\[
\{ \mathcal{V}_r, \mathcal{V}_\ell \vee \mathcal{V}_r, \mathcal{V}^{K_\ell} \vee \mathcal{V}_r, \mathcal{V}^{K_r}_r \}, \{ \mathcal{V}^{K_r}, \mathcal{V}_\ell \vee \mathcal{V}^{K_r}, \mathcal{V}^{K_\ell} \vee \mathcal{V}^{K_r} \}
\]

and the following are sets of \( K_r \)-related varieties

\[
\{ \mathcal{V}_\ell, \mathcal{V}_\ell \vee \mathcal{V}_r, \mathcal{V}_\ell \vee \mathcal{V}^{K_r}, \mathcal{V}^{K_r}_r \}, \{ \mathcal{V}^{K_\ell}, \mathcal{V}^{K_\ell} \vee \mathcal{V}_r, \mathcal{V}^{K_\ell} \vee \mathcal{V}^{K_r} \}.
\]

These claims follow immediately from the choices of \( \mathcal{V}_\ell \) and \( \mathcal{V}_r \) together with the fact that \( K_\ell \) and \( K_r \) are complete congruences on \( \mathcal{L}(\mathcal{C} \mathcal{R}) \).

(7) \( \mathcal{V}_\ell \vee \mathcal{V}^{K_r} \subseteq \mathcal{V}^{K_r}_r \) and \( \mathcal{V}^{K_\ell} \vee \mathcal{V}_r \subseteq \mathcal{V}^{K_r}_r \). Clearly it suffices to establish just one of these containments. It is also clear that \( \mathcal{V}_\ell \vee \mathcal{V}^{K_r} \subseteq \mathcal{V}^{K_r}_r \) so that the goal is to establish that the containment is proper. Let \( F \) denote the free object in \( \mathcal{V}^{K_r}_r \) on a countably infinite set \( X \). Let \( R = F^1 \) as a set and endowed with the right zero multiplication. Let \( S \) denote the disjoint union \( F \cup R \) of these sets and define a product \( * \) on \( S \) as follows. The operation \( * \) agrees with the already defined multiplication in \( F \) and in \( R \) while, for \( a \in F, b \in R \), we have

\[
a * b = b, \quad \text{and} \quad b * a = ba
\]

where \( ba \) denotes the product within \( F^1 \), as the semigroup \( F \) with an identity adjoined. It is routine to check that this multiplication is associative and that \( S \) is a union of groups. Therefore \( S \) is a completely regular semigroup.

The Rees quotient \( S/R \) is isomorphic to \( F^0 \) and so belongs to \( \mathcal{V}^{K_r}_r \). Hence \( S \in (\mathcal{V}^{K_r}_r)_{K_r} = \mathcal{V}^{K_r}_r \). Now assume, by way of contradiction, that \( \mathcal{V}^{K_r}_r \subseteq \mathcal{V}_\ell \vee \mathcal{V}^{K_r} \). Since \( \mathcal{V}_\ell \vee \mathcal{V}^{K_r} \) is \( K_\ell \)-related to \( \mathcal{V}^{K_r} \), we must have that \( S \in (\mathcal{V}^{K_r}_r)_{K_\ell} \) which, by Lemma 2.5, means that \( S / (\tau \cap \mathcal{L}^0) \in \mathcal{V}^{K_r} \), where \( \tau \) denotes the largest idempotent pure congruence on \( S \). It is helpful to note that \( \tau \cap \mathcal{L}^0 = (\tau \cap \mathcal{L})^0 \).
Let $a, b \in S$ be such that $a \mathcal{L}^0 b$. Since $R$ is a right zero semigroup and $a \mathcal{L} b$ either $a = b \in R$ or $a, b \in F$. If $a, b \in R$ then trivially $a = b$. If $a, b \in F$ then $1 * a \mathcal{L}^0 1 * b$ so that, again since $R$ is a right zero semigroup, $1 * a = 1 * b$ and $a = b$. Consequently, $\mathcal{L}^0 = \epsilon$, the identity congruence on $S$, and also $\tau \cap \mathcal{L}^0 = \epsilon$. This implies that $S \in \mathcal{V}_{\mathcal{K}_r}$. But $F$ is a subsemigroup of $S$. Hence $F \in \mathcal{V}_{\mathcal{K}_r}$ which implies that $\mathcal{V}_{\mathcal{K}_r} \subseteq \mathcal{V}_{\mathcal{K}_r}$ and therefore that $\mathcal{V}_\ell \subseteq \mathcal{V}_{\mathcal{K}_r}$, contradicting (1). Therefore the claim holds.

(8) $\mathcal{V}_{\mathcal{K}_r}^\ell$, $\mathcal{V}_{\mathcal{K}_r}^r$ and $\mathcal{V}_{\mathcal{K}_r}^\ell \lor \mathcal{V}_{\mathcal{K}_r}^r$ are pairwise incomparable. If $\mathcal{V}_{\mathcal{K}_r}^\ell \subseteq \mathcal{V}_{\mathcal{K}_r}^r$ then

$$\mathcal{V}_\ell \subseteq \mathcal{V}_{\mathcal{K}_r}^\ell = (\mathcal{V}_{\mathcal{K}_r}^\ell)_{K_r} \subseteq (\mathcal{V}_{\mathcal{K}_r}^r)_{K_r} = (\mathcal{V}_r)_{K_r} \subseteq \mathcal{V}_r$$

contradicting (1). Reversing the roles of $\mathcal{V}_{\mathcal{K}_r}^\ell$ and $\mathcal{V}_{\mathcal{K}_r}^r$, we see that these varieties are incomparable. Next

$$\mathcal{V}_{\mathcal{K}_r}^\ell \lor \mathcal{V}_{\mathcal{K}_r}^r \subseteq \mathcal{V}_{\mathcal{K}_r}^\ell \quad \Rightarrow \quad (\mathcal{V}_{\mathcal{K}_r}^\ell \lor \mathcal{V}_{\mathcal{K}_r}^r)_{K_r} \subseteq (\mathcal{V}_{\mathcal{K}_r}^\ell)_{K_r} = (\mathcal{V}_r)_{K_r} \subseteq \mathcal{V}_r$$

$$\quad \Rightarrow \quad (\mathcal{V}_{\mathcal{K}_r}^\ell)_{K_r} \lor (\mathcal{V}_{\mathcal{K}_r}^r)_{K_r} \subseteq \mathcal{V}_r$$

$$\quad \Rightarrow \quad \mathcal{V}_{\mathcal{K}_r}^\ell \lor \mathcal{V}_{\mathcal{K}_r}^r \subseteq \mathcal{V}_r$$

$$\quad \Rightarrow \quad \mathcal{V}_{\mathcal{K}_r}^r \subseteq \mathcal{V}_r$$

contradicting the choice of $\mathcal{V}_r$. On the other hand, if $\mathcal{V}_{\mathcal{K}_r}^r \subseteq \mathcal{V}_{\mathcal{K}_r}^\ell \lor \mathcal{V}_{\mathcal{K}_r}^r$, then

$$\mathcal{V}_{\mathcal{K}_r}^r = (\mathcal{V}_{\mathcal{K}_r}^\ell)_{K_r} \subseteq (\mathcal{V}_{\mathcal{K}_r}^\ell \lor \mathcal{V}_{\mathcal{K}_r}^r)_{K_r} = \mathcal{V}_{\mathcal{K}_r}^\ell \lor \mathcal{V}_{\mathcal{K}_r}^r = \mathcal{V}_{\mathcal{K}_r}^r$$

which contradicts (7). Hence $\mathcal{V}_{\mathcal{K}_r}^r$ and $\mathcal{V}_{\mathcal{K}_r}^\ell \lor \mathcal{V}_{\mathcal{K}_r}^r$ are incomparable. The dual argument applies to $\mathcal{V}_{\mathcal{K}_r}^r$ and $\mathcal{V}_{\mathcal{K}_r}^\ell \lor \mathcal{V}_{\mathcal{K}_r}^r$.

From (6), we know that the varieties $\mathcal{V}_\ell \lor \mathcal{V}_r$, $\mathcal{V}_{\mathcal{K}_r}^\ell \lor \mathcal{V}_r$ and $\mathcal{V}_{\mathcal{K}_r}^r$ are $K_\ell$-related and by (3), (7), that

$$\mathcal{V}_\ell \lor \mathcal{V}_r \subseteq \mathcal{V}_{\mathcal{K}_r}^\ell \lor \mathcal{V}_r \subset \mathcal{V}_{\mathcal{K}_r}^r.$$

Similarly, the varieties $\mathcal{V}_\ell \lor \mathcal{V}_r$, $\mathcal{V}_{\mathcal{K}_r}^\ell \lor \mathcal{V}_r$ and $\mathcal{V}_{\mathcal{K}_r}^r$ are $K_r$-related and

$$\mathcal{V}_\ell \lor \mathcal{V}_r \subseteq \mathcal{V}_\ell \lor \mathcal{V}_{\mathcal{K}_r}^r \subset \mathcal{V}_{\mathcal{K}_r}^r.$$

Thus we may repeat the above discussion using $\mathcal{V}_\ell \lor \mathcal{V}_r$ as our starting point with $\mathcal{V}_\ell \lor \mathcal{V}_r$, $\mathcal{V}_\ell \lor \mathcal{V}_{\mathcal{K}_r}^\ell$, $\mathcal{V}_\ell \lor \mathcal{V}_{\mathcal{K}_r}^r$, replacing $\mathcal{V}_\ell$, $\mathcal{V}_r$, respectively, to obtain another nine-element sublattice which overlaps with the lattice in Diagram 5.2 to obtain the following larger sublattice of $\mathcal{V}_K$. 


Diagram 5.3

Regarding the varieties top left and top right in the main body of the diagram, since \( V_\ell \lor V_{Kr} \) is \( K_\ell \)-related to \( V_{Kr} \) we must have \((V_\ell \lor V_{Kr})^{K_\ell} = V_{K\ell,K_\ell}\) and dually \((V_{K\ell} \lor V_r)^{Kr} = V_{K\ell,Kr}\).

Each new level provides the foundation for the next level and so on, thereby generating a lattice of subvarieties of \( V^K \) which is isomorphic to the interval \([S,B]\). Then by Lemma 2.8(ii), the supremum of all the elements in this lattice is

\[
\bigvee_{u \in \Theta} V^{u(K_\ell,K_r)} = V^K.
\]

Therefore we may legitimately adjoin \( V^K \) to the top of the Diagram 5.3 to obtain a complete sublattice of \( V^K \) that is isomorphic to the interval \([S,B]\). Note that we have also proved that the lines of positive slope connect \( K_\ell \)-related varieties and the lines of negative slope connect varieties that are \( K_r \)-related.

The most important illustration of Theorem 5.1 and the only previously known example of the behaviour described there has been the lattice \([S,B]\) itself which can be viewed as illustrating Theorem 5.1 by taking \( V = S, V_\ell = LN B, V_r = RN B \).
6. More copies of the lattice \([S, B]\)

In this section we indicate ways in which the conditions in Theorem 5.1 can be relaxed.

**Theorem 6.1.** Let \(V, V_r, V, V, V_r, V_r \in [S, CR]\) be such that \(V \subset V_r \subset V \subset V_r \subset V_r \subset K_r\) and \(V \subset V \subset V_r \subset V_r \subset V_r \subset V_r\). In addition, assume that \((V^\ell)_K = V^\ell\) and \((V^r)_K = V^r\). Then we have the following:

(i) \(V K \ell V \ell V^\ell \ell V^\ell r V^r r V^r \).

(ii) The varieties

\(V, V_r, V, V_r, V_r \subset V, V \vee V^\ell, V^\ell \vee V, V^\ell \vee V, V^r \vee V^\ell \vee V^r\)

constitute a sublattice of nine distinct elements in \(L(CR)\) as depicted in Diagram 6.2. (iii) \(V^\ell \setminus V_r \subset V^\ell \ell V_r \subset V^\ell \).

(iv) By selecting varieties \((V^\ell)^r\) and \((V_r)^r\) such that

\[V^\ell \vee V^r \subset (V^\ell)^r \subset (V^\ell)^K \text{ and } ((V^\ell)^r)_K = (V^\ell)^r\]

and

\[V^\ell \vee V_r \subset (V_r)^r \subset (V_r)^K \text{ and } ((V_r)^r)_K = (V_r)^r,\]

the procedure in parts (i) - (iii) may now be repeated starting from the base consisting of the varieties

\(V^\ell \vee V_r, V^\ell \vee V_r, (V^\ell)^r, V^\ell \vee V^r, (V_r)^r,\)

Repeating the process and including \(V^K\) yields a sublattice (but not necessarily a complete sublattice) of \(VK\) isomorphic to \([S, B]\).

Note also that the varieties \(V^\ell\) and \(V_r\) may or may not be chosen independently. Once again, if \(V\) is self-dual, then one natural choice, after selecting \(V^\ell\), would be to take \(V_r = \overline{V}^\ell\). Similarly, one might choose \(V^\ell = V^K\ell\), but other choices may be available. The same applies to \(V^r\). For instance, one could choose \(SG, LNO, LRO, RNO, LRB \vee G\).

**Proof.** The proof follows the lines of the proof of Theorem 5.1.

It is natural to wonder the extent to which different lattices constructed as above starting from the same base variety \(V\) might overlap. The following simple observation sheds some light on that, especially for what might be called the default option after the choice of the starting varieties \(V, V^\ell, V^r, V_r, V^r\), namely where we always choose the largest element in each \(K_\ell\)-class and each \(K_r\)-class. This leads to the lattice consisting of all the elements of the form \(V, (V^\ell)^u, (V^r)^u, (V_r)^v, (V^r)^v\), where \(u = u(K_\ell, K_r), v = v(K_\ell, K_r), h(u) = K_r, h(v) = K_\ell\) and their intersections.

**Lemma 6.2.** Let \(U, V \in [S, CR], U K_\ell V, U \neq V\). Let \(u = u(K_\ell, K_r)\) be such that \(h(u) = T_r\). Then \(U^u \neq V^u\). 
We argue by induction on $|u|$. Since we know that $U K_\ell V$, $U \neq V$ and that $K_\ell \cap K_r = \epsilon$, it follows that $U$ and $V$ are not $K_r$-related. Hence $U K_r \neq V K_r$ and the claim holds for $|u| = 1$. Now assume that the claim is true for all words of shorter length than $u$ and that $|u| > 1$. Without loss of generality, we may assume that $t(u) = K_\ell$ and that $u = v K_\ell$ for suitable $v$ with $t(v) = K_r$. Suppose that $U u = V u$. We must have $t(v) = K_r$ so that $U v = (U v) K_\ell = (U v) K_\ell = (V v) K_\ell = (V v) K_\ell = V v$.

Since $|v| < |u|$, this contradicts the induction hypothesis and therefore $U u \neq V u$ as required.

We know that $|V K_\ell| = 3, 4$ or 5 for all $V \in [S, B]$ and Theorem 6.1 applies only in the context of $K_\ell$ and $K_r$ classes containing at least three elements. On the other hand, the cardinality of $SG K_\ell$ is $2^{\aleph_0}$. Recall the definition and basic properties of the variety $LRO$ of left regular orthogroups from [PR99]. We conclude with an analysis of the $K_\ell$-class of $SG$. This has some interesting features.

**Lemma 6.3.** Let $V \in SG K_\ell$.

(i) $SG K_\ell = [SG, LRO]$.

(ii) $V \cap B \in \{S, LN B, LRB\}$.

(iii) $V \cap B = S \iff V = SG$.

(iv) $V \cap B = LN B \iff V = LN O$.

(v) $V \cap B = LRB \iff V \in [LRB \lor G, LRO]$.

(vi) $[LRB \lor G, LRO] \cong L(G)$.

**Proof.** (i) See [RK1] Theorem 6.3(iv).(ii) We have $S = SG \cap B \subseteq V \cap B \subseteq LRO \cap B = LRB$

where $[S, LRB] = \{S, LN B, LRB\}$. Therefore the claim holds.(iii) By [PR99] Theorem IV.2.4, if $V \cap B = S$ then $V \subseteq SG$. By the hypothesis and part (i), $SG \subseteq V$ so that equality prevails. The converse implication is trivial.(iv) By [PR99] Corollary IV.2.12, if $V \cap B = LN B$ then $V \subseteq LN O$ so that $LN B \subseteq V \subseteq LN O$. On the other hand, by [PR99] Corollary IV.2.12, $LN O = S \lor LZ \lor G \subseteq V$. Hence $V = LN O$. The converse is clear.(v) We have $V \in SG K_\ell, V \cap B = LRB \implies LRB, SG \subseteq V \subseteq SG K_\ell \implies V \in [LRB \lor G, LRO]$.
Conversely, let $V \in [\mathcal{LRB} \lor \mathcal{G}, \mathcal{LRO}]$. Then

$$SG \subseteq V \subseteq \mathcal{LRO} = SG^K$$

so that $V \in SG^K$. In addition,

$$\mathcal{LRB} \subseteq V \cap B \subseteq \mathcal{LRO} \cap B = \mathcal{LRB}.$$ 

Thus $V \cap B = \mathcal{LRB}$ and the claim holds. (vi) See Reilly [R1], Theorem 6.3(iv). 

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