Nonextensive statistical mechanics and central limit theorems II - Convolution of $q$-independent random variables

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Abstract.
In this article we review recent generalisations of the central limit theorem for the sum of specially correlated (or $q$-independent) variables, focusing on $q \geq 1$. Specifically, this kind of correlation turns the probability density function $G_q(X) = A_q \left[1 + (q-1)\beta_q (X - \bar{\mu}_q)^2\right]^{\frac{1}{1-q}}$, which emerges upon maximisation of the entropy $S_q = k \left(1 - \int [p(X)]^q \, dX\right) / (1 - q)$, into an attractor in probability space. Moreover, we also discuss a $q$-generalisation of $\alpha$-stable Lévy distributions which can as well be stable for this special kind of correlation. Within this context, we verify the emergence of a triplet of entropic indices which relate the form of the attractor, the correlation, and the scaling rate, similar to the $q$-triplet that connects the entropic indices characterising the sensitivity to initial conditions, the stationary state, and relaxation to the stationary state in anomalous systems.

Keywords: generalised central limit theorem, $q$-independence, nonextensive statistical mechanics
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INTRODUCTION

In our previous article [1], we have verified the standard central limit theorem and its Lévy-Gnedenko extension for the case of independent and identically distributed random variables associated with a $q$-Gaussian distribution,

$$G_q(x) \equiv A_q e^{-\beta_q (x-\bar{\mu}_q)^2},$$

with $e_q^+ \equiv [1 + (1 - q)x]^{\frac{1}{1-q}}$ (if $1 + (1 - q)x \geq 0$, and zero otherwise) ($e_1^+ \equiv e^+$). Distribution (1) optimises the continuous version of the nonadditive entropy [2] $S_q \equiv k \left(1 - \sum_{i=1}^{W} p_i^q\right) / (1 - q)$, where $q \in \mathbb{R}$. In this article we review recent generalisations of the central limit theorem which have been formulated within nonextensive statistical mechanical concepts. After numerical indications suggesting the existence of a generalisation, within nonextensive statistical mechanics, of the central limit theorem [3, 4] $^1$, such

$^1$ The models introduced in references [3] and [4] have recently been analytically solved in Ref. [5]. It was verified that, for these two cases, the limiting distributions are not in fact $q$-Gaussians, but are instead other distributions. These distributions are, however, so close to $q$-Gaussians “that it becomes extremely
generalisation has indeed been proved for variables with finite or infinite $q$-generalized second-order moment [6, 7].

CENTRAL LIMIT THEOREMS FOR $q$-INDEPENDENT VARIABLES

The $q$-Gaussian case

As we have reviewed and illustrated, in the case of convolution of independent random variables (including random variables associated with $G_q(X)$ distributions), there are only two stable forms in probability space, namely, the Gaussian and the $\alpha$-stable Lévy probability density functions. We have also referred in Part I [1] to other versions of the central limit theorem that are available in the literature. In what follows we discuss the addition of random variables that are correlated in such a special way that a new algebra, the $q$-algebra [8, 9], is necessary for a suitable analysis. In this context, it has been introduced in Ref. [6] a non-linear integral transform, the $q$-Fourier Transform,

$$
\mathcal{F}_q[f](k) \equiv \int_{-\infty}^{\infty} e^{ikX} \otimes_q f(X) \, dX.
$$

(2)

Applying the definition of $q$-product, $x \otimes_q y \equiv \left[x^{1-q} + y^{1-q} - 1\right]^{\frac{1}{1-q}}$, in Eq. (2), it is possible to write $\mathcal{F}_q[f](k)$ without the explicit use of the $q$-product, $\mathcal{F}_q[f](k) = \int_{-\infty}^{\infty} f(X) \exp_q \left[\frac{ikX}{\{f(X)\}^{1-q}}\right] \, dX$. As an application of $\mathcal{F}_q[f](k)$, it is provable that,

$$
\mathcal{F}_q[G_q(X)](k) = \left\{ \exp_q \left[-\frac{k^2}{4\beta^2 - qC_q^{2(q-1)}} \right] \right\}^{\frac{3-q}{2}},
$$

(3)

where $C_q = \sqrt{\beta}/A_q$, and $\beta = B$ with $A_q$ and $B$ as defined in Eq. (9) of Part I [1].

Another distribution for which it is simple to obtain its $q$-Fourier Transform, is the uniform distribution, $U(X)$, $U(X) = \frac{1}{2a} (-a \leq X \leq a, a > 0)$. Its $q$-Fourier Transform is, $\mathcal{F}_q[U(X)](k) = \frac{q^{(2a)\left[q(1-q)/q\right]}}{ak} \sin_q \left[\frac{ak}{q(2a)^{q(1-q)/q}}\right]$, where $\sin_q(x)$ represents the $q$-generalisation of $\sin(x)$ [10], and $q = 2 - \frac{1}{\beta}$.

Introducing a function,

$$
v(u) = \frac{1 + u}{3 - u}, \quad (u < 3),
$$

(4)

difficult to distinguish the true curve from its $q$-Gaussian approximant” [5]. These interesting results show that the probabilistic correlations included in these two specific models do not correspond exactly to $q$-independence, but are only very close to it instead.

2 In this case, as well as for all distributions with compact support, integration must be done over that support. Otherwise the integral does not converge.
whose inverse is,
\[ v^{-1}(u) = \frac{3u - 1}{1 + u}, \quad (u > -1), \] (5)

and assuming \( q_1 = v(q) \) and \( q_{-1} = v^{-1}(q) \), it is possible to rewrite Eq. (3) as follows:
\[ \mathcal{F}_q[G_q(X)](k) = \exp_{q_1}[-\beta'_q k^2], \] (6)

and \( \mathcal{F}_{q-1}[G_{q-1}(X)](k) = \exp_{q_{-1}}[-\beta'_{q-1} k^2] \), where
\[ \beta'_q = \frac{3 - q}{8 \beta^{2-q} C_q^2(q-1)}. \] (7)

Equation (7) can be rewritten as
\[ \left[ \beta'_q \right]^{\frac{1}{\sqrt{2-q}}} \beta^{\frac{1}{\sqrt{2-q}}} = \left( \frac{3 - q}{8 C_q^{2(q-1)}} \right)^{\frac{1}{\sqrt{2-q}}} \equiv K(q), \quad q < 2. \] (8)

We might consider the case \( q = 1 \) in Eq. (8). In this situation, the Fourier Transform of \( G_1(X) \) has the same functional form, \( G_1(k) = \exp[-\beta' k^2] \). For Gaussian functions like these two, the width (and the inflexion point in linear-linear scale) are related to \( \beta \) (actually \( \frac{1}{\sqrt{2\beta}} \)). Hence, Eq. (8) reflects a relation between uncertainties in real and reciprocal spaces. In the general case, relation (8) is a sort of \( q \)-analogue of the quantum mechanical uncertainty principle by WERNER HEISENBERG [11]. In Fig. 1 we represent \( K(q) \) for values of \( q \) between \(-5\) and \( 2\).

![Graph](image.png)

FIGURE 1. Representation of Eq. (8) for values of \( q \) between \(-5\) and \( 2\). For \( q = 1 \) \( K(q) = 1/4\); \( \lim_{q \to -\infty} K(q) = 0 \). We are presently focusing on \( q \geq 1 \).

Equation (6) has permitted to verify the mapping, through \( \mathcal{F}_q \),
\[ G_q \xrightarrow{\mathcal{F}_q} G_{q_1}, \quad q_1 = v(q), \quad 1 \leq q < 3, \] (9)

\[ G_{q-1} \xrightarrow{\mathcal{F}_{q-1}} G_q, \quad q_{-1} = v^{-1}(q), \quad 1 < q, \]
and to prove the existence of the inverse q-Fourier Transform, \( \mathcal{F}_q^{-1}, \mathcal{G}_q \rightarrow \mathcal{G}_q \) \( (q_1 = v(q), \ 1 < q < 3) \), and \( \mathcal{G}_q \rightarrow \mathcal{G}_{q^{-1}} (q_{-1} = v^{-1}(q), \ 1 < q) \). It is worth to mention that \( q_1 \) and \( q_{-1} \) fulfil the dual relation \( q_{-1} + \frac{1}{q_1} = 2 \), that has also appeared in the context of phase space self-invariant occupancy \([12]\), i.e., such as that all marginal probabilities of the system composed by \( N \) equal and distinguishable subsystems coincide or asymptotically approach (for \( N \rightarrow \infty \)) the joint probabilities of the \((N-1)\)-system. If we consider a sequence of applications of \( v(q) \), \( q_n = v_n(q) \), it can be seen that
\[
\frac{2}{1 - q} = \frac{2}{1 - q} + n, \quad n = 0, \pm 1, \pm 2, \ldots \quad \text{and} \quad q = v_0(q_0).
\]
If \( q = 1 \), then \( q_n = 1 \) for all \( n \). Otherwise, i.e. if \( q \neq 1 \), in the limit \( n \rightarrow \pm \infty \), \( q_n \rightarrow 1 \). This result can be interpreted in the following way: the simple successive application of the q-Fourier Transform,
\[
\mathcal{F}_q^m \equiv \mathcal{F}_q \circ \mathcal{F}_q \circ \ldots \mathcal{F}_q, \quad m = 1, 2, \ldots \quad \text{and} \quad n = 0, \pm 1, \pm 2, \ldots
\]
on a distribution \( \mathcal{G}_q \) leads to the Gaussian form, \( \lim_{m \rightarrow \pm \infty} \mathcal{F}_q^m[\mathcal{G}_q] = \mathcal{G} \).

Let us now present the scheme within which the functional form Eq. (1), i.e., \( \mathcal{G}_q(X) \), turns out to be stable.

Two random variables, \( X \) and \( Z \), are said q-independent if
\[
\mathcal{F}_q, [\mathcal{P}(X + Z)](k) = \mathcal{F}_q, [p(X)](k) \otimes_q \mathcal{F}_q, [p(Z)](k),
\]
where \( q_\ast = q_{-1} \), \( \mathcal{P}(X + Z) \), \( p(X) \), and \( p(Z) \) are the probability density functions for \( X + Z \), \( X \) and \( Z \), respectively. As an example of q-independency we refer two variables, \( X \) and \( Z \), both associated with a \( q_\ast \)-Gaussian probability density function, \( \mathcal{G}_{q_\ast}(X) \) and \( \mathcal{G}_{q_\ast}(Z) \), with \( \beta_X \) and \( \beta_Z \), respectively. If the variables are \( q \)-independent then the sum \( X + Z \) is also associated with a \( q_\ast \)-Gaussian whose width is \( \delta = \left( \frac{3 - q}{8(\beta_X + \beta_Z)c_{q_\ast}^{2(q_\ast - 1)}} \right)^{1/2} \),
where \( c_{q_\ast} = \frac{3 - q}{8\beta_X^2c_{q}^{2(q_\ast - 1)}} \).

Consider now the variable, \( Y \equiv \frac{X_1 + X_2 + \ldots + X_N - N\mu_{q_\ast}}{D_{N/q_\ast}} \), where \( X_i, i \in (1, N) \), are identically distributed random variables.

Let \( X_1, \ldots, X_N \) be a sequence of \( q \)-independent identically distributed random variables with finite \( q_\ast \)-mean, \( \mu_{q_\ast} \), and finite second \((2q_\ast - 1)\)-order moment, \( \sigma_{2q_\ast - 1}^2 \). Under these conditions, with
\[
D_{N/q_\ast}(q_\ast) \equiv \left( \sqrt{N \mathcal{F}_q^{-1}[\mathcal{G}_{q_{-1}}]} \right)^{1/2}\! q_\ast,
\]
\( \mathcal{P}(Y) \) is said to be \( q \)-convergent to a \( q_{-1} \)-normal distribution as \( N \rightarrow \infty \), with \( \mathcal{L}_u = \int_{-\infty}^{\infty} [f(X)]^{\mu} dX \). Analytically, this can be written as,
\[
\mathcal{F}_q, [\mathcal{P}(Y)](k) = \mathcal{G}_q(k) = \mathcal{F}_q^{-1}[\mathcal{G}_{q_{-1}}(X)](k).
\]
The proof of Eq. (14) \cite{6} follows along the lines of the standard central limit theorem where, instead of $\mathcal{F} [f] (k)$ and the usual product, we use $\mathcal{F}_q [f] (k)$ with

$$\mathcal{F}_q [f] (k) = 1 + i \mu_q, \mathcal{Z}_q, k - \frac{q}{2} \sigma^2_{q-1} \mathcal{Z}_q, k^2 + O(k^2), \quad (k \to 0), \quad (15)$$

and the $q$-product. Thus, the convolution of $q$-independent random variables $Y$ is written as,

$$\mathcal{F}_q [\mathcal{P} (Y)] (k) = \mathcal{F}_q [p (X)] (k) \otimes q \ldots \otimes q \mathcal{F}_q [p (X)] (k), \quad \text{(N factors),} \quad (16)$$

where $1 \leq q \leq 2$ and consequently $1 \leq q_s \leq \frac{2}{3}$.

Using Eq. (16) in Eq. (15), together with properties of $q$-logarithm when $k \to 0$ ,

$$\ln_q \mathcal{F}_{q-1} [\mathcal{P} (Y)] (k) = N \ln_q \left( 1 - \frac{q}{2} \frac{k^2}{N} + O \left( \frac{k^2}{N} \right) \right), \quad \text{i.e.,}$$

$$\lim_{N \to \infty} \mathcal{F}_{q-1} [\mathcal{P} (Y)] (k) = \exp_q \left[ -\frac{q-1}{2} k^2 \right]. \quad (17)$$

In other words, $Y$ is $q_{-1}$-convergent to the random variable $Z$ whose $q_{-1}$-Fourier Transform is given by Eq. (17). Hence, according to the mapping relations, the explicit form of the corresponding $q_{-1}$-Gaussian, $\mathcal{G}_{q_{-1}} (X)$, yields

$$\mathcal{G}_{q_{-1}} (X) = \frac{1}{\mathcal{Z}_{q_{-1}} \left( 2 \sqrt{1 + q-1} \right)^{1/(2-q_{-1})}} \exp_{q_{-1}} \left[ -\beta_{q_{-1}} X^2 \right], \quad (18)$$

where $\beta_{q_{-1}} \equiv \left[ (3 - q_{-1}) / \left( 4 q_{-1} \mathcal{Z}_{q_{-1}}^{2(q_{-1}-1)} \right) \right]^{1/2-q_{-1}}$. Obviously, when we $q$-convolute $q_s$-Gaussians, the resulting probability density function is a $q_s$-Gaussian, i.e., $\mathcal{G}_{q_s} (X)$ is a stable attractor for $q$-independent random variables upon the condition stated above, in the same way $\mathcal{G} (X)$ and $L_\alpha (X)$ are the stable attractors for the sum of independent random variables.

Concerning the exponent of Eq. (13), it is easy to verify that $(2 - q_s)^{-1} = v (v(q_{-1})) \equiv q_1$. Defining $\delta \equiv \left( 2 - q_{-1} \right)^{-1}$, as the scaling rate exponent, we have $\delta = q_{-1}$. We are then allowed to define, for generic $n$ (following Eq. (10)), a $q$-triplet, $\{ q_{att}, q_{corr}, q_{scal} \}$, which relates entropic indices for the attractor, $q_{att} = q_{n-1}$, the correlation, $q_{corr} = q_n$, and the scaling, $q_{scal} = q_{n+1}$.

Consider now the variable $Y' = Y D_N (q_s)$, that is composed by the sum of $N$ $q$-independent random variables all following a $q_s$-Gaussian with the same $\beta_{q_s}$. Using the associative property of the $q$-product in Eq. (16), we are able to obtain the scaling relation,

$$\beta'_{q_s} (Y) = N \beta'_{q_s}, \quad (19)$$

in Fourier space (see Eq. (7)). For the standard ($q_s = 1$) and generalised ($2q_s - 1$)-variances, we have obtained the relations

$$\sigma^2 (Y) = N^{\frac{1}{2 - q_s}} \sigma^2, \quad \text{and} \quad \sigma^2_{2q_s-1} (Y) = N^{\frac{1}{2 - q_s}} \sigma^2_{2q_s-1}, \quad (20)$$
with \( \sigma^2 = [\beta_{q_*} (5 - 3 q_*)]^{-1} \) and \( \sigma^2_{2q_*-1} = [\beta_{q_*} (1 + q_*)]^{-1} \). When \( q = 1 \), Eq. (20) turns out into the well known relation for the sum of independent variables, \( \sigma^2(Y) = N \sigma^2 \). The way on which variance scales upon addition is a standard tool to evaluate the character of a time series whose elements \( X_i \) have a variance \( \sigma^2 \). It is well known that the nature of a signal is characterised by its Hurst exponent, \( H \), \( \sigma^2_N = N^{2H} \sigma^2 \) where \( \sigma^2_N \) represents the variance of a new variable obtained from the sum of \( N \) elements of the time series with variance \( \sigma^2 \). The series is considered as anti-persistent if \( 0 < H < 1/2 \), Brownian if \( H = 1/2 \), and persistent if \( 1/2 < H < 1 \) \([13]\). By comparing the Hurst exponent definition with Eq. (20) we verify that a connection can be established. Expressly, and from Eq. (20), we conjecture that \( q \)-independent stochastic signals should respect the following relation \( H = \left[ 2 \left( 2 - q_* \right) \right]^{-1} \).

**Verification of the \( q \)-generalised CLT**

We verify here that correlations of the form of Eq. (12) yield a \( q \)-Gaussian attractor.

Let us start with the case of \( q_* \)-Gaussians with \( q_* = 3/2 \) and \( \beta = 1 \) which are \( q \)-independent. By direct evaluation of its \( q_* \)-Fourier Transform, \( \mathcal{F}_{q_*} [G_{q_*}(X)](k) \), and taking into account the Cauchy principal value, we obtain

\[
\mathcal{F}_{q_*} [G_{q_*}(X)](k) = \left[ 1 + \frac{1}{2\sqrt{2\pi}k^2} \right]^{-\frac{1}{2}},
\]

which corresponds exactly to a \( S_{\frac{3}{2}}(k) \) function with the same \( q \) and \( \beta \) as indicated by Eq. (7). Using Eq. (21) in Eq. (12), and the mapping relations (4) and (5), we have obtained the convolution of two \( S_{\frac{3}{2}}(X) \) distributions which is also a \( \frac{3}{2} \)-Gaussian. From the latter, and applying the associative property of the \( q \)-product, we have calculated the convolution of \( N = 2, 4, 8, 16 \) \( \left( q = \frac{3}{2} \right) \)-Gaussian distributions. Contrarily to what happens for \( (q = 1) \)-independent variables, the resulting distribution is always a \( (q = \frac{3}{2}) \)-Gaussian that will never converge to a Gaussian, even when \( N \to \infty \), and despite the finiteness of the \( S_{\frac{3}{2}}(X) \) standard deviation. On the panels of Fig. 2 we depict the behaviour we have just described.

Another illustration is presented in Fig. 3 for the case of the sum of \( \left( q = \frac{7}{3} \right) \)-independent random variables associated with a \( S_{\frac{7}{3}}(X) \) distribution with \( \beta = 1 \). As we have verified when the random variables have the same probability density function but are \( (q = 1) \)-independent instead, the convolution leads to a \( \alpha \)-stable distribution, \( L_{\alpha}(Y_{N \to \infty}) \), with \( \alpha = \frac{3}{2} \). In the case of \( (q = \frac{7}{3}) \)-independence, the limiting (stable) distribution is the \( \frac{7}{3} \)-Gaussian in accordance with the \( q \)-generalised central limit theorem. Consistently, \( \beta^{-1}_{q_*}(N) = N^5 \) and \( \beta'_{q_*}(N) = N \beta'_{q_*}(1) \).
Within nonextensive statistical mechanics, the $\alpha$-stable Lévy distribution has also been generalised [7]. A distribution $f(x)$ has been considered whose asymptotic form, $|X| \to \infty$, corresponds to $f(x) \sim C |X|^{-1+\alpha(q-1)}$ (for $q = 1$, $f(x)$ recovers the already mentioned $\alpha$-stable Lévy distribution). Within this new class, we can define, e.g., a $q$-Cauchy distribution when $\alpha = 1$, which leads to the classical Cauchy distribution for $q = 1$. Just as the usual Lévy distribution, this $q$-generalisation of the $\alpha$-stable distribution, $\mathcal{L}_{q,\alpha}(X)$, is defined by its $q$-Fourier Transform

$$\mathcal{F}_q [\mathcal{L}_{q,\alpha}(X)](k) = C' \exp_{q_1} \left[ -\beta' |k|^\alpha \right],$$

\[ (22) \]
\[ Y_N \equiv \sum_{i=1}^{N} X_i \], \( X_i \) being \((q = \frac{7}{3})\)-independent random variables associated with a \(G_{\frac{5}{9}}(X)\) distribution with \(\beta = 1\) (left), and the respective \((q = \frac{9}{5})\)-Fourier Transform, \(\tilde{P}(k)\), vs. \(k\) (right). Middle panels: Same as above, in \(\ln \frac{7}{3}\)-squared scale (left), and \(\ln \frac{9}{5}\)-squared scale (right). The straight lines indicate that \(P(Y_N)\) and \(\tilde{P}(k)\) are \(q\)-Gaussians with \(q = \frac{9}{5}\) and \(q = \frac{7}{3}\) respectively. Their slopes are \(\beta - 1\frac{q}{q^*} = \frac{9}{5}\) for left panel curves and \(\beta' q^* = \frac{9}{5}\) for right panel curves. Lower panels: \(\beta^{-1} q^* = \frac{9}{5}\) vs. \(N\), which is a straight line with slope 1 (left); \(\beta' q^* = \frac{9}{5}\) vs. \(N\), which is also a straight line, but with slope \(\frac{3-\alpha-1}{8c_{q^*+1}} = 0.030995\ldots\) (right).

where, as stated previously, \(q_1 = \frac{1+q}{3-q} = \nu(q)\). Distribution \(L_{q,\alpha}(X)\) presents an infinite \((2q-1)\)-variance, when \(1 \leq q < 2\), \(0 < \alpha < 2\), and \(\alpha < \frac{1}{1-q}\). From this point on we denote, \(G_{q,\alpha}(z) \equiv A \exp_q [-c|z|^\alpha]\). Parameter \(A\) is related to the normalisation of the distribution and \(c\) basically controls its width.

Along the lines of the previous generalisation, it has been shown that the sum of \(q_+\)-independent random variables, all having the same distribution \(f(x)\), leads to Eq. (22). Since \(f(x)\) is stable, i.e., after an appropriate scaling, the sum of \(X\) variables has the same form for the probability distribution. In other words, \(f(x)\) is an attractor in the probability space, being a \(L_{q,\alpha}(X)\). In addition, distribution \(f(X)\) is asymptotically equivalent to \(G_{q_L}(X) \equiv G_{q_L,\alpha}(X)\), where \(q_L = (2q\alpha - \alpha + 3) / (\alpha + 1)\). In this way, we can say that the following mapping has been asymptotically introduced, \(G_{q_L,\alpha} \rightarrow \nu(q)\).
In the second paper by the same authors [7], an extension for all $1 \leq q < 2, 0 < \alpha < 2$ has been introduced. This extension is based on the asymptotic equivalence between $f (X)$ and the $(q, \alpha)$-distribution, $\nu_q (X) \sim |X|^{-\frac{q}{\alpha}} (|X| \to \infty)$, and the fact that, for $0 < \alpha \leq 2$, and arbitrary $q_1$, there exists an index $q_2$, together with a one-to-one mapping, such that $\nu_{q_1, \alpha} (X) \mapsto \nu_{q_2, \alpha} (X)$. Using this mapping, it has been obtained the extension of Eq. (9) for $\nu_{q, \alpha} (X)$, which is written in the following way

$$\nu_{q, \alpha} \xrightarrow{(a)} \nu_{q^*, \alpha}, \quad 1 < q < 2, \ 0 < \alpha \leq 2,$$

(23)

where

$$\frac{\alpha}{1-q'} = 1 + \frac{\alpha}{1-q} \quad \text{and} \quad q^* = \frac{\alpha - 2 (1-q)}{\alpha},$$

(24)

(a) standing for asymptotic behaviour. Moreover, it has been shown that this inverse $q$-Fourier transform exists, i.e.,

$$\nu_{q', \alpha} \xrightarrow{(a)} \nu_{q, \alpha}. \quad (25)$$

The consecutive application of $n$ $q$-Fourier transforms, $\mathcal{F}_{q^*_n}$, each one following Eq. (24), leads to,

$$\frac{\alpha}{1-q_n} = \frac{\alpha}{1-q} + n, \quad n = 0, \pm 1, \pm 2, \ldots$$

(26)

Hence, following the previous scheme, it has also been proved [7] that, if we consider the sum $Y = \frac{X_1 + X_2 + \ldots + X_N}{D_N (q)}$ of symmetric variables $X_i$ mutually $q_1$-independent, and having a probability density function $f (X) \sim |X|^{-\frac{q}{1-\alpha(q-1)}} (|X| \to \infty, D_N (q) \propto N^{\frac{1}{1-\alpha(q-1)}})$, then, when $N \to \infty$, $\mathcal{F}_q (\frac{\nu}{\nu} (Y)) (k) = \exp_{q_1} [-|k|^{\alpha}], a (q_1, \alpha)$-stable distribution. Bearing in mind Eq. (24) and Eq. (25), we have that $Y$ is convergent to a $\nu_{\hat{q}, \alpha} (X)$ distribution with

$$\hat{q} = \frac{2 (1-q) - \alpha (1+q)}{2 (1-q) - \alpha (3-q)}, \quad (27)$$

Regarding scaling, $D_N (q)$, assumed for variable $Y \equiv \frac{X_1 + X_2 + \ldots + X_N}{D_N (q)}$, we can define the scaling rate exponent for $\nu_{q, \alpha} (X)$ (asymptotically equal to $f (X)$) as, $\delta = [\alpha (2-q)]^{-1}$.

Summarising, we have two different approaches to the attractor of the sum of random variables following a $(q, \alpha)$-stable distribution. These two approaches can be understood as the existence of a crossover between two regimes in the attractor for $(q, \alpha)$-stable distributions. The first regime corresponds to the intermediate region of the attractor in which it is asymptotically equal to a $\nu_{\hat{q}, \alpha} (X)$, whose tail exponent is

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3 It should be emphasised that $\nu_{q, \alpha} (X)$ is not a $(q, \alpha)$-stable distribution since it does not respect Eq. (22).
When $\alpha = 2$, this exponent coincides with the exponent of the attractor for the convolution of $q$-Gaussians, $2/(q - 1)$. The second regime, which tends to the $\alpha$-stable Lévy distribution when $q = 1$, represents the behaviour for large $|X|$ where the attractor, $\mathcal{L}_{q,\alpha}(X)$, is asymptotically equivalent to a $\mathcal{G}_{q,2}(X)$ distribution which has a tail exponent equal to $(\alpha + 1)/(1 + \alpha q - \alpha)$. When $q = 1$ this slope coincides with the exponent for a $\alpha$-stable Lévy distribution, $\alpha + 1$. Yet in this regime, and $q = 1$, we verify $q_L = \frac{\alpha + 1}{\alpha - 1}$, which coincides with the relation obtained in reference [14]. For $\alpha \to 0$, we have $q_L \to 3$, upper limit for normalisation of $\mathcal{G}_q(X)$, and when $\alpha \to 2$, $q_L \to \frac{5}{3}$, i.e., the upper limit for finiteness of the second-order moment, $\sigma_2^1$, of $\mathcal{G}_q(X)$.

The former analysis is evocative of Fig. 5 of Ref. [1]. In other words, as $\alpha \to 2$, $\mathcal{L}_{q,\alpha}(X)$ approaches the $q$-Gaussian, $\mathcal{G}_q(X)$ and, at some critical value $X_c$, it changes its behaviour to a power-law decay with exponent $(\alpha + 1)/(1 + \alpha q - \alpha)$. With this picture in mind, we sketch in Fig. 4 the attractor for the case $q_1 = 2 = \nu(q = \frac{5}{3})$ and values of $\alpha$ approaching $\alpha = 2$. We can verify that, for all $1 < q < 2$, the inequalities

$$
\frac{2}{q - 1} \geq \frac{2(1-q) - \alpha(3-q)}{2(1-q)} > \frac{\alpha + 1}{1 + \alpha(q-1)}
$$

hold. Our sketch might be corroborated in the near future, as soon as the form of the inverse $q$-Fourier Transform becomes analytically available. A summary of the whole situation is presented in Table 1.

**FIGURE 4.** Outline of $(q, \alpha)$-stable distributions for the case in which the correlation is given by $q_1 = 2$. As $\alpha$ of Lévy distributions approaches 2, the distributions $(q, \alpha)$-stable becomes more and more similar to a $G_2(X)$ with an exponent $[2(1-q) - \alpha(3-q)]/[2(1-q)]$. However, since $\alpha \neq 2$, for some critical value $X^*$, the distribution changes to another regime with a tail exponent $(\alpha + 1)/(1 + \alpha q - \alpha)$.

**FINAL REMARKS**

In this article we have reviewed the generalised central limit theorems presented in [6, 7]. These theorems are based on nonextensive statistical mechanics and they address the sum of $q_1$-correlated random variables (with $q_1 = \frac{1+q}{3-q}$) whose attractor scales as $N^{-1/\alpha(2-q)]}$. Introducing an appropriate nonlinear generalisation for the Fourier Transform, it has been possible to verify the existence of a new attractive subspace in probability space, namely $\mathcal{G}_{q,\alpha}(X)$, which contains the Gaussian, $\mathcal{G}(X) \equiv \mathcal{G}_{1,2}(X)$, and (in an asymptotic way) $\alpha$-stable Lévy distributions, $\mathcal{L}_{1,\alpha}(X) \equiv \mathcal{F}^{-1} [\mathcal{G}_{1,\alpha}(k)](X)$. These special attractors are both related to the sum of independent variables. Within $\mathcal{G}_{q,\alpha}(X)$,
**Résumé** of the main results presented in the article: Central limit theorems which present a $N^{1/(\alpha(2-q))}$-scaled attractor $F(X)$ for the sum of $N \to \infty$ $q$-independent identical random variables with symmetric distribution $f(X)$; $q_1 \equiv \frac{q+1}{3-q}$; we focus on $q \geq 1$. The term *intermediate* must be read as an infinity, however not so large as the infinity associated with the *distant* regime. For $q \neq 1$ and $\alpha = 2$, when the random variables are associated with a $q$-Gaussian, we verify the scaling relation, $\beta'(N) = N^{\beta'}$, where $\beta'$ is the (inverse) width for the $q$-Fourier Transform.

| $q = 1$ [independent] | $q \neq 1$ [globally correlated] |
|-----------------------|----------------------------------|
| $\sigma_{2q-1} < \infty$ ($\alpha = 2$) | $F(X) = \mathcal{G}_q(X) = \mathcal{G}_{3q-1}^{\infty/q}$ (with same $\sigma_1$ of $f(X)$) stable distribution [with same $\sigma_{2q-1}$ of $f(X)$] |
| $\propto X_c(q, 2)$ & $\mathcal{G}_q(X) \sim \begin{cases} \mathcal{G}(X) & \text{if } |X| \ll X_c(q, 2); \\ C_2/|X|^{2/(q-1)} & \text{if } |X| \gg X_c(q, 2) \end{cases}$ for $q > 1$, with $\lim_{q \to 1} X_c(q, 2) = \propto$ |
| $L_\alpha(X) \sim \begin{cases} \mathcal{G}(X) & \text{if } |X| \ll X_c(1, \alpha); \\ C_1,\alpha/|X|^{\alpha+1} & \text{if } |X| \gg X_c(q, 2) \end{cases}$ with $\lim_{\alpha \to 2} X_c(1, \alpha) = \propto$ Lévy – Gnedenko CLT | $F(X) = \mathcal{L}_{q,\alpha}(X)$ stable distribution [with same $|X| \to \infty$ behaviour of $f(X)$] |
| $|X| \ll X_c(1, \alpha)$ | $\mathcal{L}_{q,\alpha}(X) \sim \begin{cases} \mathcal{G}(X) \sim C_{q,\alpha}^1/|X|^{2(1-q)/(3-q)} & \text{intermediate regime} \\ X_c(1, \alpha) \ll |X| \ll X_c(2)(q, \alpha) \end{cases}$ |
| $|X| \gg X_c(1, \alpha)$ | $\mathcal{L}_{q,\alpha}(X) \sim \begin{cases} \mathcal{G}(X) \sim C_{q,\alpha}^2/|X|^{1+(\alpha-1)/(3-q)} & \text{distant regime} \\ |X| \gg X_c(2)(q, \alpha) \end{cases}$ |
considering \( q_+ \)-correlated variables (Eq. (12)) sharing the same distribution, and presenting a finite \((2q - 1)\)-variance, \( \sigma_{2q-1}^2 \), it is possible to observe the existence of a line of attraction, \( \mathcal{L}_{q,\alpha}(X) \), \((\alpha = 2)\) in \( q - \alpha \) space. We have also dealt with a \( q_+ \)-generalisation of the \( \alpha \)-stable Lévy distribution, \( \mathcal{L}_{q,\alpha}(X) \equiv \mathcal{F}_{q}^{-1}[\mathcal{G}_{q,\alpha}(k)](X) \). Since \( \mathcal{L}_{q,\alpha}(X) \) is stable, the convolution of such distributions, assuming they have an infinite \((2q - 1)\)-variance, converges towards a \( \mathcal{L}_{q,\alpha}(X) \) distribution, which for large \(|X|\), is equivalent to a \( \mathcal{G}_{qL,2}(X) \) distribution with \( q_L = \frac{2q(\alpha - 1)}{\alpha + 1} \). Removing the restriction of infinite \((2q - 1)\)-variance, the convolution of \( \mathcal{L}_{q,\alpha}(X) \) distributions, asymptotically equivalent to \( \mathcal{G}_{qL,2}(X) \) distribution with \( q_L = \frac{2q_0 - \alpha + 1}{\alpha_0} \), leads to a \( \mathcal{G}_{q,\alpha}(X) \) distribution for which \( q_0 = \frac{2(1-q) - \alpha}{2(1-q) - \alpha(q-1)} \). These two asymptotic laws for \( \mathcal{L}_{q,\alpha}(X) \) correspond to the existence of a crossover that we have depicted in Fig. 4.

In both cases referred above, i.e., addition of random variables with finite and incommensurable standard deviation, it is possible to define a triplet of parameters which characterises the attractor, the correlation, and the scaling rate, \( \{P_{at}, P_{cor}, P_{scal}\} \) [7] reminiscent of the \( q \)-triplet \( \{q_{sen}, q_{rel}, q_{stat}\} \) conjectured in Ref. [15]. In that article, it was proposed that the values of \( \{q_{sen}, q_{rel}, q_{stat}\} \) for say systems like long-range Hamiltonian systems characterised by the interaction-decay exponent \( \alpha \) and the dimension \( d \) would respect inequalities such as \( q_{rel}, q_{sta} > 1 \) and \( q_{sen} < 1 \). Considering the convolution of \( q \)-independent random variables associated with \( \mathcal{G}_{q-1}(X) \), it has been shown that the triplet of parameters \( \{P_{at}, P_{cor}, P_{scal}\} \) corresponds in fact to \( \{\frac{3q - 1}{1+q}, q, \frac{1+q}{3-q}\} \), or simply \( \{q_{k-1}, q_k, q_{k+1}\} \) following mapping relations. In this way, we can replace \( P \), that stands for parameter, by \( q \) used to represent entropic indices. Hence, the triplet of parameters in actually leading to the \( q \)-triplet \( \{q_{at}, q_{corr}, q_{scal}\} \). Establishing a bridge between the triplet porposed in Ref. [15] and the triplet we have discussed, we argue that \( q_{k-1} \), the entropic index for the attractor, should equal \( q_{stat} \). Taking into account Eq. (10), we can write

\[
q_{k-1} = 2 - \frac{1}{q_{k+1}} \tag{28}
\]

It is noteworthy that the \( q \)-triplet conjecture [15] was first observed by NASA using data from Voyager 1 related to the solar wind at the distant heliosphere [16], and also in a paradigmatic complex system such as a financial market [17]. For the solar wind observations, it has been inferred the relations \( q_{stat} + 1/q_{rel} = 2 \), and \( q_{rel} + 1/q_{sen} = 2 \), which are consistent, within experimental error, with the results obtained from the NASA data set. Again, for reasons presented above, \( q_{stat} = q_{k-1} \). Using Eq. (28) we have \( q_{rel} = q_{k+1} \), and \( q_{sen} = q_{k+3} \). These two cases of correspondence between \( \{q_{stat}, q_{rel}, q_{sen}\} \) and \( \{q_k, q_{k+1}, q_{k+3}\} \) represent strong candidates for the description of the \( q \)-triplets for conservative and dissipative systems.

In addition, let us also mention two transformations that appear quite often in problems discussed within nonextensive concepts, namely \( q_{a}(q) = 2 - q \) and \( q_{m}(q) = 1/q \). In fact, these transformations, \( additive \) and \( multiplicative \) dualities respectively, have shown to be at the basis of the relations between entropic indices. Explicitly, if we apply both transformations \( n \) times in a row, we obtain \( [q_a q_m]^n(q) = \frac{q_0^n + n(1-q)}{1+m(1-q)} \). Looking to Eq. (26), we notice that sequences with \( \alpha = 2 \) and \( n = 0, \pm 2, \pm 4, \ldots \), or \( \alpha = 1 \) and
\( n = 0, \pm 1, \pm 2, \ldots \) coincide with \([q_a q_m]^n\). This fact is quite remarkable since it reveals a connection between the sequences that emerge from the \( q \)-generalised central limit theorems and the dualities presented here above. The physical interpretation of these as well as other relations between entropic indices constitutes an interesting open challenge.

Finally, let us mention the fact that the Central Limit Theorem results appear to also be applicable to the sum of deterministic variables \([18]\). It has been proved that, when the maximum Lyapunov exponent is positive, the sum of deterministic variables obtained from some dynamical process leads to a Gaussian distribution. Recently, studies on the sum of deterministic variables obtained from dissipative and conservative systems have been made. For the logistic map \([19]\) (dissipative system) at the edge of the chaos (vanishing Lyapunov exponent) it has been verified that the intermediate part and the tail of the distribution are consistent with a \( q \)-Gaussian distribution. Concerning conservative systems, studies on the Hamiltonian Mean Field model \([20]\) at its metastable state have shown the emergence of non-Gaussian attracting probability density functions when the sum of velocities is performed. These resulting distributions have been numerically quite well approached by \( q \)-Gaussians over its whole range of values. Further analysis of both conservative and dissipative systems might clarify the emergence of a new generalisation of the central limit theorem, but for deterministic variables.

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