Quantifying the performance of approximate teleportation and quantum error correction via symmetric two-PPT-extendibility

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The ideal realization of quantum teleportation relies on having access to a maximally entangled state; however, in practice, such an ideal state is typically not available and one can instead only realize an approximate teleportation. With this in mind, we present a method to quantify the performance of approximate teleportation when using an arbitrary resource state. More specifically, after framing the task of approximate teleportation as an optimization of a simulation error over one-way local operations and classical communication (LOCC) channels, we establish a semi-definite relaxation of this optimization task by instead optimizing over the larger set of two-PPT-extendible channels. The main analytical calculations in our paper consist of exploiting the unitary covariance symmetry of the identity channel to establish a significant reduction of the computational cost of this latter optimization. Next, by exploiting known connections between approximate teleportation and quantum error correction, we also apply these concepts to establish bounds on the performance of approximate quantum error correction over a given quantum channel. Finally, we evaluate our bounds for various examples of resource states and channels.

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I. INTRODUCTION

Teleportation is one of the most basic protocols in quantum information science [BBC*93]. By means of two bits of classical communication and an entangled pair of qubits (a so-called resource state), it is possible to transmit a qubit from one location to another. This protocol demonstrates the fascinating possibilities available under the distant laboratories paradigm of local operations and classical communication (LOCC), and it prompted the development of the resource theory of entanglement [BDW96]. Teleportation is so ubiquitous in quantum information science now, that nearly every subfield (fault-tolerant computing, error correction, cryptography, communication complexity, Shannon theory, etc.) employs it in some manner. A number of impressive teleportation experiments have been conducted over the past few decades [BPM*97, FSB*98, BBDM*98, RHR*04, UJA*04, SKO*06, MHS*12, RXY*17].

The teleportation protocol assumes an ideal resource state; however, if the resource state shared between the two parties is imperfect, then the teleportation protocol no longer simulates an ideal quantum channel, but rather some approximation of it [Pop94, HHH99]. This problem has been studied considerably in the literature and is related to the well-known problem of entanglement distillation [BBP*96, BDSW96]. Recently, it has been addressed in a precise and general operational way, in terms of a meaningful channel distinguishability measure [KW17, Definition 19].

In the seminal work [BDW96], a connection was forged between entanglement distillation and approximate quantum error correction. There, it was shown that certain one-way LOCC entanglement distillation protocols can be converted to approximate quantum error correction protocols, and vice versa. Thus, techniques for analyzing entanglement distillation can be used to analyze quantum error correction and vice versa.

In this paper, we obtain bounds on the performance of teleportation when using an imperfect resource state, and by exploiting the aforementioned connection, we address a related problem for approximate quantum error correction. We thus consider our paper to offer two distinct, yet related contributions. The conceptual approach that we take here is linked to that of [SW20], which was concerned with a more involved protocol called bidirectional teleportation; it is also linked to [KDWW19, KDWW21], which introduced the set of \(k\)-extendible channels as a semi-definite relaxation of the set of one-way LOCC channels. Our approach has strong links as well with that taken in [BBFS21], the latter concerned with bounding the performance of approximate quantum error correction by means of \(k\)-PPT-extendible channels; these channels were introduced in [BBFS21] as a semi-definite relaxation of the set of one-way LOCC channels that forms a tighter containment than \(k\)-extendible channels alone. In fact, our method applied to the problem of approximate quantum error correction can be understood as exploiting further symmetries available when simulating the identity channel, in order to reduce the computational complexity required to calculate the bounds given in [BBFS21].

Let us discuss our first contribution in a bit more detail. Suppose that the goal is to use a bipartite resource state \(\rho_{AB}\) along with one-way LOCC to simulate a perfect quantum channel of dimension \(d\). It is not always possible to perform this simulation exactly, and for most resource states, an error will occur. We can quantify the simulation error either in terms of the diamond distance [Kit97] or the channel infidelity [GLN05]. However, we prove here that the simulation error is the same, regardless of whether we use the channel infidelity or the diamond distance, when quantifying the deviation between the simulation and an ideal quantum channel (note that a similar
result was found previously in [SW20] and we exploit similar techniques to arrive at our conclusion here). Next, in order to obtain a lower bound on the simulation error, and due to the fact that it is computationally challenging to optimize over one-way LOCC channels, we optimize the error over the larger set of two-PPT-extendible channels (defined in Section II B 5) and show that the resulting quantity can be calculated by means of a semi-definite program. By exploiting the unitary covariance symmetry of the ideal quantum channel, we reduce the computational complexity of the semi-definite program to depend only on the dimension of the resource state $\rho_{AB}$ being considered. This constitutes our main contribution to the analysis of teleportation with an imperfect resource state. We also provide a general formulation of the simulation problem when trying to simulate an arbitrary channel using one-way LOCC and a resource state.

The second contribution of our paper employs a similar line of reasoning to obtain a lower bound on the simulation error of approximate quantum error correction. In this setting, instead of a bipartite state, two parties have at their disposal a quantum channel $\mathcal{N}_{A\rightarrow B}$, for which they can prepend an encoding and append a decoding in order to simulate a perfect quantum channel of dimension $d$. This encoding and decoding can be understood as a superchannel [CDPO8b] that transforms $\mathcal{N}_{A\rightarrow B}$ into an approximation of the perfect quantum channel. It is clear that the simulation error cannot increase by allowing for a superchannel realized by one-way local operations and common randomness (LOCR), and here, following the approach outlined above, we find a lower bound on the simulation error by optimizing instead over the larger class of two-PPT-extendible superchannels with an extra non-signaling constraint. Critically, this lower bound can be calculated by means of a semi-definite program. As indicated above, this problem was previously considered in [BBFS21], but our contribution is that the semi-definite programming lower bound reported here has a substantially reduced computational complexity, depending only on the input and output dimensions of the channel $\mathcal{N}_{A\rightarrow B}$ of interest.

A. Organization of the paper

Our paper is organized into two major parts, according to the contributions mentioned above. The first part (Sections II-IV) details our contribution to quantifying the performance of approximate teleportation. The second part (Sections V-VII) details our contribution to quantifying the performance of approximate quantum error correction.

The first part of our paper is organized as follows: Section II provides some background on quantum states and channels, with an emphasis on LOCC and LOCR bipartite channels. Section III establishes a measure for the performance of quantum channel simulation, namely, in terms of the normalized diamond distance and channel infidelity. We prove here that these two error measures are equal when the goal is to simulate the identity channel, following as a consequence of the unitary covariance symmetry of the identity channel. Section IV presents the major contribution of the first part, a semi-definite program (SDP) that gives a lower bound on the simulation error of approximate teleportation when using an arbitrary bipartite resource state and one-way LOCC channels. This SDP is further simplified by exploiting the aforementioned symmetry of the identity channel to reduce the computational cost of the optimization task significantly.

The second part of our paper is organized as follows: Section V provides background on quantum superchannels to generalize the concepts of one-way LOCC and LOCR bipartite channels to superchannels. Section VI explores the task of channel simulation, i.e., simulating a quantum channel from an arbitrary quantum channel and LOCR superchannels. The performance of channel simulation is again quantified with the normalized diamond distance and channel infidelity, and again the error measures are equal when the goal is to simulate the identity channel with the assistance of common randomness. Section VII presents the major contribution of the second part, an SDP that gives a lower bound on the error in simulating a quantum channel with an arbitrary channel and LOCR superchannels. We detail a much simplified SDP for the simulation of an identity channel, the case of interest in approximate quantum error correction, by leveraging its unitary covariance symmetry.

Section VIII presents plots that result from numerical calculations of our SDP error bounds. The first example in Section VIII A bounds the error in approximate teleportation using a certain mixed state as the resource state, demonstrating that two-PPT-extendibility constraints can achieve tighter bounds when compared to PPT constraints alone. The second example in Section VIII B considers the bounds when using a lower dimensional resource state to simulate a higher dimensional identity channel. The next example in Section VIII C considers the bounds for qubit and qutrit depolarizing channels. The penultimate example in Section VIII D bounds the error in approximate teleportation when using two-mode squeezed states as the resource state. The final example in Section VIII E bounds the error in simulating an identity channel when using the three-level amplitude damping channel [CG21], and it is thus an example of our bound applied to approximate quantum error correction.

Section IX concludes by discussing several open questions for future work. We note here that Python code for calculating the SDPs in our paper is available with its arXiv posting.

II. BACKGROUND ON STATES, CHANNELS, AND BIPARTITE CHANNELS

We recall some basic facts about quantum information theory in this section to fix our notation before proceeding; more detailed background can be found in [Hay17, Hol19, Wat18, Wil17, KW20].

A. States and channels

A quantum state or density operator, usually denoted by $\rho_A$, $\sigma_A$, etc., is a positive semi-definite, unit trace operator
acting on a Hilbert space $\mathcal{H}_A$. The Heisenberg–Weyl operators are unitary transformations of quantum states, defined for all $x, z \in \{0, 1, \ldots, d-1\}$ as

$$Z(z) := \sum_{k=0}^{d-1} e^{\frac{2\pi ikz}{d}} |k\rangle\langle k|,$$  \hspace{1cm} (1)

$$X(x) := \sum_{k=0}^{d-1} |k \oplus_d x\rangle\langle k|,$$  \hspace{1cm} (2)

$$W_{z \prec x} := Z(z)X(x),$$  \hspace{1cm} (3)

where $\oplus_d$ denotes addition modulo $d$.

A quantum channel is a completely positive (CP), trace-preserving (TP) map. Let $\mathcal{N}_{A \rightarrow B}$ denote a quantum channel that accepts as input a linear operator acting on a Hilbert space $\mathcal{H}_A$ and outputs a linear operator acting on a Hilbert space $\mathcal{H}_B$. For short, we say that the channel takes system $A$ to system $B$, where systems are identified with Hilbert spaces. Let $\Gamma_{RB}^{N}$ denote the Choi operator of a channel $\mathcal{N}_{A \rightarrow B}$:

$$\Gamma_{RB}^{N} := \mathcal{N}_{A \rightarrow B}(\Gamma_{RA}),$$  \hspace{1cm} (4)

where

$$\Gamma_{RA} := \sum_{i,j=0}^{d_A-1} |i\rangle\langle j| \otimes |i\rangle\langle j|_A$$  \hspace{1cm} (5)

is the unnormalized maximally entangled operator and $\{|i\rangle\}$ and $\{|i\rangle_\text{A}_j\}$ are orthonormal bases.

The Choi representation of a channel is isomorphic to the superoperator representation and provides a convenient means of characterizing a channel. Namely, a channel $\mathcal{M}_{A \rightarrow B}$ is completely positive if and only if its Choi operator $\Gamma_{RB}^{M}$ is positive semi-definite and a channel $\mathcal{M}_{A \rightarrow B}$ is trace preserving if and only if its Choi operator $\Gamma_{RB}^{M}$ satisfies $\text{Tr}_B[\Gamma_{RB}^{M}] = I_R$.

**B. Bipartite channels**

A bipartite channel $\mathcal{N}_{AB \rightarrow A'B'}$ maps input systems $A$ and $B$ to output systems $A'$ and $B'$, in this model, a single party Alice has access to systems $A$ and $A'$, while another party Bob has access to systems $B$ and $B'$. The Choi operator for a bipartite channel $\mathcal{N}_{AB \rightarrow A'B'}$ is as follows:

$$\Gamma_{AB'A'B'}^{N} = \mathcal{N}_{AB \rightarrow A'B'}(\Gamma_{AA} \otimes \Gamma_{BB}).$$  \hspace{1cm} (6)

### 1. One-way LOCC channels

A bipartite channel $\mathcal{L}_{AB \rightarrow A'B'}$ is a one-way LOCC (1WL) channel if it can be written as follows:

$$\mathcal{L}_{AB \rightarrow A'B'} = \sum_{x} \mathcal{E}_{A \rightarrow A'}^x \otimes \mathcal{D}_{B \rightarrow B'}^x,$$  \hspace{1cm} (7)

where $\{\mathcal{E}_{A \rightarrow A'}^x\}_x$ is a set of completely positive maps, such that the sum map $\sum_x \mathcal{E}_{A \rightarrow A'}^x$ is trace preserving, and $\{\mathcal{D}_{B \rightarrow B'}^x\}_x$ is a set of quantum channels. The idea here is that Alice acts on her system $A$ with a quantum instrument described by $\{\mathcal{E}_{A \rightarrow A'}^x\}_x$, transmits the classical outcome $x$ of the measurement over a classical communication channel to Bob, who subsequently applies the quantum channel $\mathcal{D}_{B \rightarrow B'}^x$ to his system $B$. A key example of a one-way LOCC channel is in the teleportation protocol: given that Alice and Bob share a maximally entangled state in systems $AB$ and Alice has prepared the system $A_0$ that she would like to teleport, the one-way LOCC channel consists of Alice performing a Bell measurement on systems $A_0A$ (quantum instrument), sending the measurement outcome to Bob (classical communication), who then applies a Heisenberg–Weyl correction operation on system $B$ conditioned on the classical communication from Alice. One-way LOCC channels are central in our analysis of approximate teleportation.

### 2. LOCR channels

A subset of one-way LOCC channels consists of those that can be implemented by local operations and common randomness (LOCR). These channels have the following form:

$$\mathcal{C}_{AB \rightarrow A'B'} = \sum_{y} p(y)\mathcal{E}_{A \rightarrow A'}^y \otimes \mathcal{D}_{B \rightarrow B'}^y,$$  \hspace{1cm} (8)

where $\{p(y)\}_y$ is a probability distribution and $\{\mathcal{E}_{A \rightarrow A'}^y\}_y$ and $\{\mathcal{D}_{B \rightarrow B'}^y\}_y$ are sets of quantum channels. The main difference between one-way LOCC and LOCR is that, in the latter case, the channel is simply a probabilistic mixture of local channels. In order to simulate them, classical communication is not needed, and only the weaker resource of common randomness is required. Thus, the following containment holds:

$$\text{LOCR} \subset \text{1WL}. \hspace{1cm} (9)$$

These channels play a role in our analysis of approximate quantum error correction and channel simulation.

### 3. Two-extendible channels

A bipartite channel $\mathcal{N}_{AB \rightarrow A'B'}$ is two-extendible [KDWW19, KDWW21], if there exists an extension channel $\mathcal{M}_{AB_1B_2 \rightarrow A'B'_1B'_2}$ satisfying permutation covariance:

$$\mathcal{M}_{AB_1B_2 \rightarrow A'B'_1B'_2} \circ \mathcal{F}_{B_1B_2} = \mathcal{F}_{B'_1B'_2} \circ \mathcal{M}_{AB_1B_2 \rightarrow A'B'_1B'_2} \quad (10)$$

and the following non-signaling constraint:

$$\text{Tr}_{B_2'} \circ \mathcal{M}_{AB_1B_2 \rightarrow A'B'_1B'_2} = \mathcal{N}_{AB_1 \rightarrow A'B'_1} \otimes \text{Tr}_{B_2}. \quad (11)$$

In the above, $\mathcal{F}_{B_1B_2}$ is the unitary swap channel that permutes systems $B_1$ and $B_2$, and $\mathcal{F}_{B'_1B'_2}$ is defined similarly. Also, $\text{Tr}$ denotes the partial trace channel. Note that the two conditions in (10) and (11) imply that the original channel $\mathcal{N}_{AB \rightarrow A'B'}$ is non-signaling from Bob to Alice, i.e.,

$$\text{Tr}_{B'} \circ \mathcal{N}_{AB \rightarrow A'B'} = \text{Tr}_{B'} \circ \mathcal{N}_{AB \rightarrow A'B'} \circ \mathcal{R}_{B}^\pi. \quad (12)$$
where
\[
\mathcal{R}_B^N(\cdot) := \text{Tr}[\cdot]\pi_B
\] (13)
is a replacer channel that traces out its input and replaces it with the maximally mixed state \(\pi_B := \frac{1}{d_B}\). We provide a proof of (12) in Appendix \(A\).

More generally, \(k\)-extendible channels were defined in [KDDWW19, KDWW21], and a resource theory was constructed based on them. However, we only make use of two-extendible channels in this work, and we leave the study of our problem using \(k\)-extendible channels for future work. See [BBFS21] for an alternative definition of \(k\)-extendible channels that appeared after the original proposal of [KDDWW19]. A key insight of [KDDWW19, KDWW21] is that the set of one-way LOCC channels is contained in the set of two-extendible channels, and we make use of this observation in our paper.

A bipartite channel \(N_{AB\to A'B'}\) is two-extendible if and only if its Choi operator \(\Gamma^N_{AB;B};A'B'\) satisfying [KDDWW19, KDWW21]
\[
(\mathcal{F}_{B_1}\otimes \mathcal{F}_{B_2'}')(\Gamma^M_{AB;B_1A'B'_1B'_2} = \Gamma^M_{AB;B_1A'B'_1B'_2} \otimes I_{B_2},
\]
\[
\text{Tr}_{B_2}[\Gamma^N_{AB;B};A'B'] = \frac{1}{d_B} \text{Tr}_{B_2'}[\Gamma^N_{AB;B};A'B'] \otimes I_{B_2},
\] (14)
which is equivalent to (12).

4. Completely positive-partial-transpose preserving channels

A bipartite channel \(N_{AB\to A'B'}\) is completely positive-partial-transpose preserving (C-PPT-P) [Rai99, Rai01] if the map \(T_B \circ N_{AB\to A'B'} \circ T_B\) is completely positive. Here, \(T_B\) is the partial transpose map, defined as the following superoperator:
\[
T_B(\cdot) := \sum_{i,j} |ij\rangle_B\langle j|_{B'}.
\] (19)
See also [CdVGG20]. The set of one-way LOCC channels is contained in the set of C-PPT-P channels [Rai99, Rai01], and we also make use of this observation in our paper. A bipartite channel \(N_{AB\to A'B'}\) is C-PPT-P if and only if its Choi operator \(\Gamma^N_{AB;A'B'}\) satisfies
\[
\Gamma^N_{AB;A'B'} \geq 0,
\] (20)
\[
\text{Tr}_{A'B'}[\Gamma^N_{AB;A'B'}] = \mathcal{I}_{AB},
\]
\[
T_{B'B'}[\Gamma^N_{AB;A'B'}] \geq 0,
\] (22)
where \(T_{B'B'}\) is the partial transpose acting on systems \(B\) and \(B'\). We note that the C-PPT-P constraint has been used in prior work on bounding the simulation error in bidirectional teleportation [SW20]. See also [LM15, WX18, WD16b, WD16a, BW18] for other contexts.

5. Two-PPT-extendible channels

We can combine the above constraints in a non-trivial way to define the set of two-PPT-extendible channels, and we note that this was considered recently in [BBFS21, Remark after Lemma 4.10], as a generalization of the concept employed for bipartite states [DPS02, DPS04]. Explicitly, a bipartite channel \(N_{AB\to A'B'}\) is two-PPT-extendible if there exists an extension channel \(M_{AB;B_1A'B'_1B'_2}\) satisfying the following conditions of permutation covariance, non-signaling, and being completely-PPT-preserving:
\[
M_{AB;B_1A'B'_1B'_2} \circ T_{B_1} = T_{B'_1} \circ M_{AB;B_1A'B'_1B'_2},
\]
\[
\text{Tr}_{B_1} \circ M_{AB;B_1A'B'_1B'_2} = N_{AB;A'B'} \otimes \text{Tr}_{B_2},
\]
\[
\text{Tr}_{B_2'} \circ M_{AB;B_1A'B'_1B'_2} = T_{B_2'} \circ M_{AB;B_1A'B'_1B'_2} \otimes \text{Tr}_{B_1},
\]
\[
\text{Tr}_{B'_1} \circ M_{AB;B_1A'B'_1B'_2} \otimes T_A \in \mathbb{CP}.
\] (26)
It is redundant to demand further that the following constraints hold:
\[
T_{B'_1} \circ M_{AB;B_1A'B'_1B'_2} \circ T_{B_1} \in \mathbb{CP},
\]
\[
T_{A'B'_1} \circ M_{AB;B_1A'B'_1B'_2} \circ T_{B_1} \in \mathbb{CP},
\]
\[
T_{A'B'_2} \circ M_{AB;B_1A'B'_1B'_2} \circ T_{B_2} \in \mathbb{CP},
\]
\[
T_{B'_1} \circ M_{AB;B_1A'B'_1B'_2} \circ T_{B_2} \in \mathbb{CP},
\]
\[
T_{B'_2} \circ M_{AB;B_1A'B'_1B'_2} \circ T_{B_1} \in \mathbb{CP},
\]
\[
T_{B'_1} \circ M_{AB;B_1A'B'_1B'_2} \circ T_{B_2} \in \mathbb{CP},
\] (30)
because they follow as a consequence of (25) and (23), (25), (27), and (26), respectively. A bipartite channel \(N_{AB\to A'B'}\) is two-PPT-extendible if and only if its Choi operator \(\Gamma^N_{AB;A'B'}\) is such that there exists a Hermitian operator \(\Gamma^M_{AB;B_1A'B'_1B'_2}\) satisfying
\[
(\mathcal{F}_{B_1}\otimes \mathcal{F}_{B_2'}')(\Gamma^M_{AB;B_1A'B'_1B'_2} = \Gamma^M_{AB;B_1A'B'_1B'_2} \otimes I_{B_2},
\]
\[
\text{Tr}_{B_2}[\Gamma^N_{AB;A'B'}] = \frac{1}{d_B} \text{Tr}_{B_2'}[\Gamma^N_{AB;A'B'}] \otimes I_{B_2},
\] (14)
\[
\text{Tr}_{B'_1} \circ M_{AB;B_1A'B'_1B'_2} \circ T_{B_1} \in \mathbb{CP},
\]
\[
\text{Tr}_{A'B'_1} \circ M_{AB;B_1A'B'_1B'_2} \circ T_{B_1} \in \mathbb{CP},
\]
\[
T_{A'B'_2} \circ M_{AB;B_1A'B'_1B'_2} \circ T_{B_2} \in \mathbb{CP},
\]
\[
T_{B'_1} \circ M_{AB;B_1A'B'_1B'_2} \circ T_{B_2} \in \mathbb{CP},
\] (30)
because they follow as a consequence of (25) and (23), (25), (27), and (26), respectively. A bipartite channel \(N_{AB\to A'B'}\) is two-PPT-extendible if and only if its Choi operator \(\Gamma^N_{AB;A'B'}\) is such that there exists a Hermitian operator \(\Gamma^M_{AB;B_1A'B'_1B'_2}\) satisfying
\[
(\mathcal{F}_{B_1}\otimes \mathcal{F}_{B_2'}')(\Gamma^M_{AB;B_1A'B'_1B'_2} = \Gamma^M_{AB;B_1A'B'_1B'_2} \otimes I_{B_2},
\]
\[
\text{Tr}_{B_2}[\Gamma^N_{AB;A'B'}] = \frac{1}{d_B} \text{Tr}_{B_2'}[\Gamma^N_{AB;A'B'}] \otimes I_{B_2},
\] (31)
\[
\text{Tr}_{B'_1} \circ M_{AB;B_1A'B'_1B'_2} \circ T_{B_1} \in \mathbb{CP},
\]
\[
\text{Tr}_{A'B'_1} \circ M_{AB;B_1A'B'_1B'_2} \circ T_{B_1} \in \mathbb{CP},
\]
\[
\text{Tr}_{A'B'_2} \circ M_{AB;B_1A'B'_1B'_2} \circ T_{B_2} \in \mathbb{CP},
\]
\[
\text{Tr}_{B'_1} \circ M_{AB;B_1A'B'_1B'_2} \circ T_{B_2} \in \mathbb{CP},
\] (36)
Observe that a bipartite channel \(N_{AB\to A'B'}\) is C-PPT-P if it is two-PPT-extendible. This follows from (24) and (26).
Every one-way LOCC channel of the form in (7) is two-PPTextendible by considering the following extension channel:

$$\sum_x E^{x}_{A\rightarrow A'} \otimes D^{x}_{B_1\rightarrow B'_1} \otimes D^{x}_{B_2\rightarrow B'_2},$$  \hspace{1cm} (37)

which manifestly satisfies the constraints in (23)–(26). We thus employ two-PPT-extendible channels as a semi-definite relaxation of the set of one-way LOCC channels.

6. Two-PPT-extendible non-signaling channels

We can add a further constraint to the channels discussed in the previous section, i.e., a non-signaling constraint of the following form:

$$\text{Tr}_{A'} \circ M_{AB:B_2\rightarrow A'B_1'} = \text{Tr}_{A'} \circ M_{AB:B_1\rightarrow A'B_2'} \circ R^\pi_A,$$  \hspace{1cm} (38)

which ensures that the extension channel $M_{AB:B_2\rightarrow A'B_1'}$ is also non-signaling from Alice to both Bobs. The constraint on the Choi operator $\Gamma^M_{AB:B_2A'B_1'}$ is as follows:

$$\text{Tr}_{A'}[\Gamma^M_{AB:B_2A'B_1'}] = \frac{1}{d_A} \text{Tr}_{A'}[\Gamma^M_{AB:B_2A'B_1'}] \otimes I_A.$$  \hspace{1cm} (39)

Every LOCR channel of the form in (8) is two-PPTextendible non-signaling, as is evident by choosing the following extension channel:

$$\sum_y p(y) E^{y}_{A\rightarrow A'} \otimes D^{y}_{B_1\rightarrow B'_1} \otimes D^{y}_{B_2\rightarrow B'_2}. $$  \hspace{1cm} (40)

We thus employ two-PPT-extendible non-signaling channels as a semi-definite relaxation of the set of LOCR channels, and we note here that [BBFS21] previously used this approach.

Let us state explicitly here that extensions of one-way LOCC channels of the form in (7) generally do not satisfy the non-signaling constraint in (38), due to the fact that each map $E^{x}_{A\rightarrow A'}$ in (7) is not necessarily trace preserving.

III. QUANTIFYING THE PERFORMANCE OF APPROXIMATE TELEPORTATION

In approximate teleportation, Alice and Bob are allowed to make use of a fixed bipartite state $\rho_{\hat{A}\hat{B}}$ and an arbitrary one-way LOCC channel $\mathcal{L}_{\hat{A}\hat{B}\rightarrow B'}$ with the goal of simulating an identity channel of dimension $d$. To be clear, the one-way LOCC channel $\mathcal{L}_{\hat{A}\hat{B}\rightarrow B}$ has the following form:

$$\mathcal{L}_{\hat{A}\hat{B}\rightarrow B}(\omega_{\hat{A}\hat{B}}) = \sum_x D^{x}_{\hat{B}\rightarrow B}(\text{Tr}_{A}[\Lambda^{x}_{\hat{A}\hat{A}} \omega_{\hat{A}\hat{B}}]),$$  \hspace{1cm} (41)

where $\{\Lambda^{x}_{\hat{A}\hat{A}}\}_{x}$ is a positive operator-valued measure (satisfying $\Lambda^{x}_{\hat{A}\hat{A}} \geq 0$ for all $x$ and $\sum_x \Lambda^{x}_{\hat{A}\hat{A}} = I_{\hat{A}A}$) and $\{D^{x}_{B\rightarrow B'}\}_{x}$ is a set of quantum channels. We assume that the dimension of the systems $\hat{A}\hat{B}$ is finite, and we assume that the dimension of $\hat{A}$ as $d_{\hat{A}}$ and the dimension of $\hat{B}$ as $d_{\hat{B}}$. The approximate teleportation protocol realizes the following simulation channel $\mathcal{S}_{A\rightarrow B}$ [HHH99, Eq. (11)]:

$$\mathcal{S}_{A\rightarrow B}(\omega_A) := \mathcal{L}_{\hat{A}\hat{B}\rightarrow B}(\omega_A \otimes \rho_{\hat{A}\hat{B}}),$$  \hspace{1cm} (42)

In the following subsections, we discuss two seemingly different ways of quantifying the simulation error.

A. Quantifying simulation error with normalized diamond distance

The standard metric for quantifying the distance between quantum channels is the normalized diamond distance [K1997]. See the related paper [SW20] for discussions of the operational significance of the diamond distance (see also [KW20]). For channels $\mathcal{N}_{C\rightarrow D}$ and $\mathcal{N}_{C\rightarrow D}$, the diamond distance is defined as

$$\left\| \mathcal{N}_{C\rightarrow D} - \mathcal{N}_{C\rightarrow D} \right\|_1,$$  \hspace{1cm} (43)

where the optimization is over every bipartite state $\rho_{RC}$ with the reference system $R$ arbitrarily large. The following equality is well known (see, e.g., [KW20])

$$\left\| \mathcal{N}_{C\rightarrow D} - \mathcal{N}_{C\rightarrow D} \right\|_1 = \sup_{\psi_{RC}} \left\| \mathcal{N}_{C\rightarrow D}(\psi_{RC}) - \mathcal{N}_{C\rightarrow D}(\psi_{RC}) \right\|_1, $$  \hspace{1cm} (44)

where the optimization is over every pure bipartite state $\psi_{RC}$ with the reference system $R$ isomorphic to the channel input system $C$. The normalized diamond distance is then given by

$$\frac{1}{2} \left\| \mathcal{N}_{C\rightarrow D} - \mathcal{N}_{C\rightarrow D} \right\|_1,$$  \hspace{1cm} (45)

so that the resulting error takes a value between zero and one. The reduction in (44) implies that it is a computationally tractable problem to calculate the diamond distance, and in fact, one can do so by means of the following semi-definite program [Wat09]:

$$\inf_{\lambda, Z_{RD} \geq 0} \left\{ \lambda : \lambda I_R \geq T_{RD}[Z_{RD}], \quad Z_{RD} \geq \Gamma_{RD}^N - \Gamma_{RD}^{\tilde{N}} \right\},$$  \hspace{1cm} (46)

where $\Gamma_{RD}^N$ and $\Gamma_{RD}^{\tilde{N}}$ are the Choi operators of $\mathcal{N}_{C\rightarrow D}$ and $\mathcal{N}_{C\rightarrow D}$, respectively.

The simulation error when using a bipartite state $\rho_{\hat{A}\hat{B}}$ and a one-way LOCC channel to simulate an identity channel $\mathcal{id}_{A\rightarrow B}$ of dimension $d$ is given by

$$\epsilon_{\text{WL}}(\rho_{\hat{A}\hat{B}}, \mathcal{L}_{\hat{A}\hat{B}\rightarrow B}) := \frac{1}{2} \left\| \mathcal{id}_{A\rightarrow B} - \mathcal{S}_{A\rightarrow B} \right\|_1, $$  \hspace{1cm} (47)
where the simulation channel $\tilde{S}_{A\to B}$ is defined in (42). Employing (44), we find that
\[
e_{1WL}(\rho_{AB}, \mathcal{L}_{\tilde{S}_{A\to B}}) = \sup_{\psi_{RA}} \frac{1}{2} \left\| \psi_{RA} - \mathcal{L}_{\tilde{S}_{A\to B}}(\psi_{RA} \otimes \rho_{\tilde{A}\tilde{B}}) \right\|_1, \tag{48}
\]
with $\psi_{RA}$ a pure bipartite state such that system $R$ is isomorphic to system $A$. We are interested in the minimum possible simulation error, and so we define
\[
e_{1WL}(\rho_{\tilde{A}\tilde{B}}) := \inf_{\mathcal{L} \in 1WL} e_{1WL}(\rho_{\tilde{A}\tilde{B}}, \mathcal{L}_{\tilde{S}_{A\to B}}), \tag{49}
\]
where we recall that $1WL$ denotes the set of one-way LOCC channels. The error $e_{1WL}(\rho_{\tilde{A}\tilde{B}})$ is one kind of simulation error on which we are interested in obtaining computationally efficient lower bounds. Indeed, it is a computationally difficult problem to calculate $e_{1WL}(\rho_{\tilde{A}\tilde{B}})$ directly, and so we instead resort to calculating lower bounds.

**B. Quantifying simulation error with channel infidelity**

Another measure of the simulation error is by means of the channel infidelity. Let us recall that the fidelity of states $\omega$ and $\tau$ is defined as [Uhl76]
\[
F(\omega, \tau) := \left\| \sqrt{\omega} \sqrt{\tau} \right\|_1^2, \tag{50}
\]
where $\|X\|_1 := \text{Tr}[\sqrt{X^2}]$. From this measure, we can define a channel fidelity measure for channels $\mathcal{N}_{C\to D}$ and $\mathcal{N}_{C\to D}$ as follows:
\[
F(\mathcal{N}, \mathcal{N}) := \sup_{\mathcal{N}_{C\to D}(\rho_{RC})} F(\mathcal{N}_{C\to D}(\rho_{RC}), \mathcal{N}_{C\to D}(\rho_{RC})), \tag{51}
\]
where the optimization is over every bipartite state $\rho_{RC}$ with the reference system $R$ arbitrarily large. Similar to the diamond distance, it suffices to optimize the channel fidelity over every pure bipartite state $\psi_{RC}$ with reference system $R$ isomorphic to the channel input system $C$ (see, e.g., [KW20]):
\[
F(\mathcal{N}, \mathcal{N}) := \sup_{\psi_{RC}} F(\mathcal{N}_{C\to D}(\psi_{RC}), \mathcal{N}_{C\to D}(\psi_{RC})). \tag{52}
\]
The square root of the channel fidelity can be calculated by means of the following semi-definite program [YF17, KW21]:
\[
\sqrt{F}(\mathcal{N}, \mathcal{N}) = \max_{\lambda \geq 0, \mathcal{Q}_{RD}} \lambda \tag{53}
\]
such that
\[
\mathcal{Q}_{RD} \succeq \text{Re}[\text{Tr}_{D}\mathcal{Q}_{RD}], \tag{54}
\]
\[
\begin{bmatrix} \Gamma_{RD} \mathcal{Q}_{RD}^T \\ \mathcal{Q}_{RD} \Gamma_{RD} \end{bmatrix} \succeq 0. \tag{55}
\]
An alternative method for quantifying the simulation error is to employ the channel infidelity, defined as $1 - F(\mathcal{N}, \mathcal{N})$. Indeed, we can measure the simulation error as follows, when using a bipartite state $\rho_{\tilde{A}\tilde{B}}$ and a one-way LOCC channel $\mathcal{L}_{\tilde{S}_{A\to B}}$:
\[
e_{1WL}(\rho_{\tilde{A}\tilde{B}}, \mathcal{L}_{\tilde{S}_{A\to B}}) := 1 - F(\rho_{\tilde{A}\tilde{B}}, \mathcal{L}_{\tilde{S}_{A\to B}}), \tag{56}
\]
where the simulation channel $\tilde{S}_{A\to B}$ is defined in (42). By employing (52), we find that
\[
e_{1WL}(\rho_{\tilde{A}\tilde{B}}, \mathcal{L}_{\tilde{S}_{A\to B}}) = \sup_{\psi_{RA}} \left( 1 - F(\psi_{RA}, \mathcal{L}_{\tilde{S}_{A\to B}}(\psi_{RA} \otimes \rho_{\tilde{A}\tilde{B}})) \right), \tag{57}
\]
where the optimization is over every pure bipartite state $\psi_{RA}$ with system $R$ isomorphic to the channel input system $A$. Since we are interested in the minimum possible simulation error, we define
\[
e_{1WL}(\rho_{\tilde{A}\tilde{B}}) := \inf_{\mathcal{L} \in 1WL} e_{1WL}(\rho_{\tilde{A}\tilde{B}}, \mathcal{L}_{\tilde{S}_{A\to B}}). \tag{58}
\]
This is the other kind of simulation error on which we are interested in obtaining lower bounds.

**C. One-way LOCC simulation of general point-to-point channels**

Beyond the case of simulating an ideal channel, more generally we can consider using a resource state $\rho_{\tilde{A}\tilde{B}}$ along with a one-way LOCC channel $\mathcal{L}_{\tilde{S}_{A\to B}}$ in order to simulate a general channel $\mathcal{N}_{A\to B}$. In this case, the simulation channel has the following form:
\[
\tilde{N}_{A\to B}(\omega_A) := \mathcal{L}_{\tilde{S}_{A\to B}}(\omega_A \otimes \rho_{\tilde{A}\tilde{B}}). \tag{59}
\]
The simulation error when employing a specific one-way LOCC channel $\mathcal{L}_{\tilde{S}_{A\to B}}$ is
\[
e_{1WL}(\mathcal{N}_{A\to B}, \rho_{\tilde{A}\tilde{B}}, \mathcal{L}_{\tilde{S}_{A\to B}}) := \frac{1}{2} \left\| N - \tilde{N} \right\|_1, \tag{60}
\]
and the simulation error minimized over all possible one-way LOCC channels is
\[
e_{1WL}(\mathcal{N}_{A\to B}, \rho_{\tilde{A}\tilde{B}}) := \inf_{\mathcal{L} \in 1WL} e_{1WL}(\mathcal{N}_{A\to B}, \rho_{\tilde{A}\tilde{B}}, \mathcal{L}_{\tilde{S}_{A\to B}}). \tag{61}
\]
We note here that this is a special case of the simulation problem considered in [FWTB20, Section II].
Alternatively, we can employ the infidelity to quantify the simulation error as follows:
\[
e_{1WL}^F(\mathcal{N}_{A\to B}, \rho_{\tilde{A}\tilde{B}}, \mathcal{L}_{\tilde{S}_{A\to B}}) := 1 - F(\mathcal{N}, \tilde{N}), \tag{62}
\]
\[
e_{1WL}^F(\mathcal{N}_{A\to B}, \rho_{\tilde{A}\tilde{B}}) := \inf_{\mathcal{L} \in 1WL} e_{1WL}^F(\mathcal{N}_{A\to B}, \rho_{\tilde{A}\tilde{B}}, \mathcal{L}_{\tilde{S}_{A\to B}}). \tag{63}
\]
D. Equality of simulation errors when simulating the identity channel

Proposition 1 below states that the following equality holds for every bipartite state $\rho_{AB}$:

$$e_{1WL}(\rho_{A\bar{B}}) = e_{1WL}(\rho_{A\bar{B}}).$$

(64)

We provide an explicit proof in Appendix B. This equality follows as a consequence of the unitary covariance symmetry of the identity channel being simulated and the fact that an optimal simulating channel should respect the same symmetries. Indeed, consider that the identity channel $\text{id}_{A \rightarrow B}$ possesses the following unitary covariance symmetry:

$$\text{id}_{A \rightarrow B} \circ \mathcal{U}_B = \mathcal{U}_B \circ \text{id}_{A \rightarrow B},$$

(65)

which holds for every unitary channel $\mathcal{U}(\cdot) = U(\cdot)U^\dagger$, with $U$ a unitary operator. As a consequence, the theory simplifies in the sense that we need only focus on bounding the simulation error with respect to a single measure. We note here that a similar result was found in [SW20] for the case of simulating the bipartite swap channel by means of LOCC.

**Proposition 1** The optimization problems in (49) and (58), for the error in simulating the identity channel $\text{id}_{A \rightarrow B}$, simplify as follows:

$$e_{1WL}(\rho_{A\bar{B}}) = e_{1WL}(\rho_{A\bar{B}})$$

(66)

$$= 1 - \sup_{K_{AB}, \mathcal{L}_{A\bar{B}} \geq 0} \text{Tr}[K_{AB} \rho_{A\bar{B}}],$$

(67)

subject to $K_{AB} + \mathcal{L}_{A\bar{B}} = 1_{A\bar{B}}$ and the following channel $\mathcal{L}_{A\bar{A}B \rightarrow B}$ being a one-way LOCC channel:

$$\mathcal{L}_{A\bar{A}B \rightarrow B}(\omega_{A\bar{A}B}) = \text{id}_{A \rightarrow B}(\text{Tr}_{A}[K_{AB} \omega_{A\bar{A}B}]) + \mathcal{D}_{A \rightarrow B}(\text{Tr}_{A}[\mathcal{L}_{A\bar{A}B} \omega_{A\bar{A}B}]),$$

(68)

where $\mathcal{D}_{A \rightarrow B}$ is the following channel:

$$\mathcal{D}_{A \rightarrow B}(\sigma_A) := \frac{1}{d^2 - 1} \sum_{(z,z') \neq (0,0)} W_{z,z'} \sigma(W_{z,z'})^\dagger,$$

(69)

and $W_{z,z'}$ is defined in (3). The constraint that $\mathcal{L}_{A\bar{A}B \rightarrow B}$ is a one-way LOCC channel is equivalent to the existence of a positive operator-valued measure (POVM) $\{\Lambda_{AB}^x\}_x$ and a set $\{\mathcal{D}_{A \rightarrow B}^x\}_x$ of channels such that

$$K_{AB} = \frac{1}{d^2} \sum_x \text{Tr}_B(\Lambda_{AB}^x \mathcal{D}_{A \rightarrow B}^x),$$

(70)

where $\mathcal{D}_{A \rightarrow B}^x$ is the Choi operator of the channel $\mathcal{D}_{A \rightarrow B}^x$.

**Proof.** See Appendix B. ■

IV. SDP LOWER BOUNDS ON THE PERFORMANCE OF APPROXIMATE TELEPORTATION BASED ON TWO-PPT-EXTENDIBILITY

A. SDP lower bound on the error in one-way LOCC simulation of a channel

It is difficult to compute the simulation error $e_{1WL}(\mathcal{N}_{A \rightarrow B}, \rho_{A\bar{B}})$ defined in (61) because it is challenging to optimize over the set of one-way LOCC channels [Gur04, Gha10]. Here we enlarge the set of one-way LOCC channels to the set of two-PPT-extendible bipartite channels, with the goal of simplifying the calculation of the simulation error. The result is that we provide a lower bound on the one-way LOCC simulation error in terms of a semi-definite program, which follows because the set of two-PPT-extendible channels is specified by semi-definite constraints, as indicated in (31)–(36).

In more detail, recall that a bipartite channel is two-PPT-extendible if the conditions in (23)–(26) hold. As indicated previously at the end of Section IIIB5, every one-way LOCC channel is a two-extendible channel, and the containment is strict.

Thus,

$$1WL \subset 2PE,$$

(71)

where $2PE$ denotes the set of two-PPT-extendible channels, as defined in Section IIIB5.

We can then define the simulation error under two-PPT-extendible channels, as a semi-definite relaxation of (61), as follows:

$$e_{2PE}(\mathcal{N}_{A \rightarrow B}, \rho_{A\bar{B}}) := \inf_{\tilde{\mathcal{N}}_{A \rightarrow B}} \frac{1}{2} \left\| \mathcal{N}_A - \tilde{\mathcal{N}}_A \right\|,$$

(72)

where

$$\tilde{\mathcal{N}}_{A \rightarrow B}(\omega_A) := \mathcal{K}_{A\bar{A}B \rightarrow B}(\omega_A \otimes \rho_{A\bar{B}})$$

(73)

and $\mathcal{K}_{A\bar{A}B \rightarrow B}$ is a two-PPT-extendible channel, meaning that there exists an extension channel $\mathcal{M}_{A\bar{A}B_1B_2 \rightarrow B_1B_2}$ satisfying the following conditions:

$$\text{Tr}_{B_1} \circ \mathcal{M}_{A\bar{A}B_1B_2 \rightarrow B_1B_2} = \mathcal{K}_{A\bar{A}B_1 \rightarrow B_1} \otimes \text{Tr}_{B_2},$$

(74)

$$\mathcal{M}_{A\bar{A}B_1B_2 \rightarrow B_1B_2} \circ \mathcal{T}_{B_1B_2} = \mathcal{T}_{B_1B_2} \circ \mathcal{M}_{A\bar{A}B_1B_2 \rightarrow B_1B_2},$$

(75)

$$\mathcal{T}_{B_1} \circ \mathcal{M}_{A\bar{A}B_1B_2 \rightarrow B_1B_2} \circ \mathcal{T}_{B_2} \in \mathbb{C}P,$$

(76)

$$\mathcal{M}_{A\bar{A}B_1B_2 \rightarrow B_1B_2} \circ \mathcal{T}_{A\bar{A}} \in \mathbb{C}P.$$ (77)

As a consequence of the containment in (71), the following bound holds

$$e_{2PE}(\mathcal{N}_{A \rightarrow B}, \rho_{A\bar{B}}) \leq e_{1WL}(\mathcal{N}_{A \rightarrow B}, \rho_{A\bar{B}}).$$

(78)

We now show that the simulation error in (72) can be calculated by means of a semi-definite program.

**Proposition 2** The simulation error in (72) can be calculated by means of the following semi-definite program:

$$e_{2PE}(\mathcal{N}_{A \rightarrow B}, \rho_{A\bar{B}}) = \inf_{\mathcal{M}_{A\bar{A}B_1B_2 \rightarrow B_1B_2} \geq 0} \mu,$$

(79)
subject to
\[ \mu A \geq Z_A, \]  
\[ Z_{AB} \geq 1^{N_{AB}} - \operatorname{Tr}_{\tilde{A}B_1} \left[ T_{\tilde{A}B_1}(\rho_{\tilde{A}B_1}) M_{\tilde{A} \tilde{B}_1} B_1 \frac{M_{\tilde{A} \tilde{B}_1} B_1}{d_{\tilde{B}}} \right], \]  
(80)  
(81)

The objective function in (79) and the first two constraints in (80) and (81) follow from the semi-definite program in (46) for the normalized diamond distance. The quantity \( \operatorname{Tr}_{\tilde{A}B_1} \left[ T_{\tilde{A}B_1}(\rho_{\tilde{A}B_1}) M_{\tilde{A} \tilde{B}_1} B_1 \frac{M_{\tilde{A} \tilde{B}_1} B_1}{d_{\tilde{B}}} \right] \) in (81) is the Choi operator corresponding to the composition of the appending channel and the simulation channel \( K_{A \tilde{B}_1 \rightarrow B_1} \), with Choi operator \( \frac{M_{\tilde{A} \tilde{B}_1} B_1}{d_{\tilde{B}}} \), where \( K_{A \tilde{B}_1 \rightarrow B_1} \) is the marginal channel of \( M_{\tilde{A} \tilde{B}_1} B_1 \rightarrow B_1 B_2 \), defined as
\[ K_{A \tilde{B}_1 \rightarrow B_1} (\omega_{A \tilde{B}_1}) := \operatorname{Tr}_{\tilde{B}_2} \left[ M_{A \tilde{B}_1 \tilde{B}_2} B_1 (\omega_{A \tilde{B}_1} \otimes \pi_{B_2}) \right]. \]  
(87)

The constraint in (82) forces \( M_{A \tilde{B}_1 \tilde{B}_2} \rightarrow B_1 B_2 \) to be trace preserving, that in (83) forces \( M_{A \tilde{B}_1 \tilde{B}_2} \rightarrow B_1 B_2 \) to be permutation covariant with respect to the \( B \) systems (see (75)), and that in (84) forces \( M_{A \tilde{B}_1 \tilde{B}_2} \rightarrow B_1 B_2 \) to be the extension of a marginal channel \( K_{A \tilde{B}_1 \rightarrow B_1} \). The final two PPT constraints are equivalent to the C-PPT-P constraints in (76) and (77), respectively.

### B. SDP lower bound on the simulation error of approximate teleportation

The semi-definite program in Proposition 2 can be evaluated for an important case of interest, i.e., when \( N_{A \rightarrow B} = \text{id}^{d_A}_{A \rightarrow B} \). Recall from Section III that this special case corresponds to approximate teleportation. The semi-definite program in Proposition 2 is efficiently computable with respect to the dimensions of the systems \( A \), \( \tilde{A} \), \( \tilde{B} \), and \( B \). However, it is in our interest to reduce the computational complexity of these optimization tasks even further for this important case, and we can do so by exploiting the unitary covariance symmetry of the identity channel, as stated in (65).

In this section, we provide a semi-definite program for evaluating the simulation error
\[ e_{2\text{PE}}(\rho_{\tilde{A}B}) \equiv e_{2\text{PE}}(\text{id}^{d_A}_{A \rightarrow B}, \rho_{\tilde{A}B}), \]  
(88)

with reduced complexity, i.e., only polynomial in the dimensions \( d_A \) and \( d_B \) of the resource state \( \rho_{\tilde{A}B} \). We provide a proof of Proposition 3 in Appendix C.

### Proposition 3
The semi-definite program in Proposition 2, for the special case of simulating the identity channel \( \text{id}^{d_A}_{A \rightarrow B} \), simplifies as follows for \( d \geq 3 \):

\[ e_{2\text{PE}}(\rho_{\tilde{A}B}) = e_{2\text{PE}}(\rho_{\tilde{A}B}) \]

\[ = e_{2\text{PE}}(\rho_{\tilde{A}B}) \]

\[ = 1 - \operatorname{Tr} \left[ T_{\tilde{A}B_1}(\rho_{\tilde{A}B_1}) P_{\tilde{A}B_2} \frac{P_{\tilde{A}B_2}}{d_{\tilde{B}}} \right], \]

\[ \text{subject to} \]

\[ \left[ M^0 + M^3, M^1 + i M^2, M^0 - M^3 \right] \geq 0, \]

\[ I_{\tilde{A}B_1 \rightarrow B_2} := M_{\tilde{A}B_1 \rightarrow B_2} \]

\[ M^0_{\tilde{A}B_1 \rightarrow B_2} = \left[ \begin{array}{cc}
M^0_{\tilde{A}B_1 \rightarrow B_2} & M^0_{\tilde{A}B_1 \rightarrow B_2} \\
M^0_{\tilde{A}B_1 \rightarrow B_2} & M^0_{\tilde{A}B_1 \rightarrow B_2}
\end{array} \right],
\]

\[ M^1_{\tilde{A}B_1 \rightarrow B_2} = \left[ \begin{array}{cc}
M^1_{\tilde{A}B_1 \rightarrow B_2} & M^1_{\tilde{A}B_1 \rightarrow B_2} \\
M^1_{\tilde{A}B_1 \rightarrow B_2} & M^1_{\tilde{A}B_1 \rightarrow B_2}
\end{array} \right],
\]

\[ M^2_{\tilde{A}B_1 \rightarrow B_2} = \left[ \begin{array}{cc}
M^2_{\tilde{A}B_1 \rightarrow B_2} & M^2_{\tilde{A}B_1 \rightarrow B_2} \\
M^2_{\tilde{A}B_1 \rightarrow B_2} & M^2_{\tilde{A}B_1 \rightarrow B_2}
\end{array} \right],
\]

\[ P_{\tilde{A}B_1 \rightarrow B_2} := \frac{1}{2d} \left( dM^0 + M^1 + \sqrt{d^2 - 1} M^2 \right), \]

\[ \left( \begin{array}{cc}
2M^0_{\tilde{A}B_1 \rightarrow B_2} & 2M^0_{\tilde{A}B_1 \rightarrow B_2} \\
2M^0_{\tilde{A}B_1 \rightarrow B_2} & 2M^0_{\tilde{A}B_1 \rightarrow B_2}
\end{array} \right) \geq 0,
\]

\[ \left( \begin{array}{cc}
2M^1_{\tilde{A}B_1 \rightarrow B_2} & 2M^1_{\tilde{A}B_1 \rightarrow B_2} \\
2M^1_{\tilde{A}B_1 \rightarrow B_2} & 2M^1_{\tilde{A}B_1 \rightarrow B_2}
\end{array} \right) \geq 0,
\]

\[ G^0_{\tilde{A}B_1 \rightarrow B_2} := T_A \left( M^+ + M^- + \frac{M^0 - dM^0}{2} \right),
\]

\[ G^1_{\tilde{A}B_1 \rightarrow B_2} := T_A \left( M^+ - M^- + \frac{M^1 - dM^1}{2} \right),
\]

\[ G^2_{\tilde{A}B_1 \rightarrow B_2} := \frac{3}{2} (d^2 - 1) T_A \left( M^2_{\tilde{A}B_1 \rightarrow B_2} \right),
\]

\[ G^3_{\tilde{A}B_1 \rightarrow B_2} := \frac{3}{2} (d^2 - 1) T_A \left( M^3_{\tilde{A}B_1 \rightarrow B_2} \right),
\]

\[ \left( \begin{array}{cc}
dM^+ & M^0 - M^1 - \sqrt{d^2 - 1} M^2 \\
M^0 - M^1 - \sqrt{d^2 - 1} M^2 & dM^-
\end{array} \right) \geq 0,
\]

\[ T_{\tilde{A}B_1} \left( \frac{dM^+}{d^2 + 2} + M^0 - M^1 - \sqrt{d^2 - 1} M^2 \right) \geq 0,
\]

\[ T_{\tilde{A}B_1} \left( \frac{dM^-}{d^2 - 2} - M^0 - M^1 - \sqrt{d^2 - 1} M^2 \right) \geq 0,
\]

\[ \left[ E^0 + E^3, E^1 - i E^2, E^1 + i E^2, E^0 - E^3 \right] \geq 0. \]  
(91)  
(92)  
(93)  
(94)  
(95)  
(96)  
(97)  
(98)  
(99)  
(100)  
(101)  
(102)  
(103)  
(104)  
(105)  
(106)
A. Basics of superchannels

A superchannel \( \Theta \equiv \Theta_{(A \rightarrow B) \rightarrow (C \rightarrow D)} \) is a physical transformation of a channel \( N_{A \rightarrow B} \) that accepts as input the channel \( N_{A \rightarrow q} \) and outputs a channel with input system \( C \) and output system \( D \). Mathematically, a superchannel is a linear map that preserves the set of quantum channels, even when the quantum channel is an arbitrary bipartite channel with external input and output systems that are arbitrarily large. Superchannels are thus completely CPTP preserving in this sense. A general theory of superchannels was introduced in [CDP08b] and developed further in [CDP09, CDP08a, Gou19].

In more detail, let us denote the output of a superchannel \( \Theta \) by \( K_{C \rightarrow D} \), so that

\[
\Theta_{(A \rightarrow B) \rightarrow (C \rightarrow D)}(N_{A \rightarrow B}) = K_{C \rightarrow D}. \tag{114}
\]

The superchannel \( \Theta_{(A \rightarrow B) \rightarrow (C \rightarrow D)} \) is completely CPTP preserving in the sense that the following output map

\[
(id_{R_{(R)}}) \otimes \Theta_{(A \rightarrow B) \rightarrow (C \rightarrow D)}(N_{RA \rightarrow RB}) \tag{115}
\]

is a quantum channel for every input quantum channel \( M_{RA \rightarrow RB} \), where \( id_{R_{(R)}} \) denotes the identity superchannel [CDP08b].

The fundamental theorem of superchannels from [CDP08b] is that \( \Theta_{(A \rightarrow B) \rightarrow (C \rightarrow D)} \) has a physical realization in terms of a pre-processing channel \( E_{C \rightarrow AQ} \) and a post-processing channel \( D_{BQ \rightarrow D} \) as follows:

\[
\Theta_{(A \rightarrow B) \rightarrow (C \rightarrow D)}(N_{A \rightarrow B}) = D_{BQ \rightarrow D} \circ N_{A \rightarrow B} \circ E_{C \rightarrow AQ}. \tag{116}
\]

where \( Q \) is a quantum memory system. Furthermore, every superchannel \( \Theta_{(A \rightarrow B) \rightarrow (C \rightarrow D)} \) is in one-to-one correspondence with a bipartite channel of the following form:

\[
P_{CB \rightarrow AD} := D_{BQ \rightarrow D} \circ E_{C \rightarrow AQ}. \tag{117}
\]

Note that \( P_{CB \rightarrow AD} \) is completely positive, trace preserving, and obeys the following non-signaling constraint:

\[
\text{Tr}_D \circ P_{CB \rightarrow AD} = \text{Tr}_D \circ P_{CB \rightarrow AD} \circ R^p_B, \tag{118}
\]

where the replacer channel \( R^p_B \) is defined in (13). Related to this, \( \Gamma^p_{CBAD} \) is the Choi operator of a superchannel if and only if it satisfies the following constraints:

\[
\Gamma^p_{CBAD} \succeq 0, \tag{119}
\]

\[
\text{Tr}_{AD}[\Gamma^p_{CBAD}] = I_{CB}, \tag{120}
\]

\[
\text{Tr}_D[\Gamma^p_{CBAD}] = \frac{1}{d_B} \text{Tr}_{BD}[\Gamma^p_{CBAD}] \otimes I_B. \tag{121}
\]

The first two constraints correspond to complete positivity and trace preservation, respectively, and the last constraint is a non-signaling constraint corresponding to \( P_{CB \rightarrow AD} \) having the factorization in (117), so that \( P_{CB \rightarrow AD} \) is in correspondence with a superchannel. To determine the Choi operator for the
output channel $\mathcal{K}_{C \to D}$ in (114), we can use the following propagation rule [CDP08b, Gou19]:

$$\Gamma_{CD}^N = \text{Tr}_{AB}[T_{AB}(\Gamma_{AB}^N \Gamma_{CBAD}^P)],$$

(122)

where $\Gamma_{CBAD}^P$ is the Choi operator of $\mathcal{P}_{CB \to AD}$ and $\Gamma_{AB}^N$ is the Choi operator of $\mathcal{N}_{A \to B}$.

B. One-way LOCC superchannels

A superchannel $\Lambda \equiv \Lambda_{(A \to B) \to (C \to D)}$ is implementable by one-way LOCC if it can be written in the following form:

$$\Lambda(\mathcal{N}_{A \to B}) := \sum_x \mathcal{D}^x_{B \to D} \circ \mathcal{N}_{A \to B} \circ \mathcal{E}^x_{C \to A},$$

(123)

where $\{\mathcal{E}^x_{C \to A}\}_x$ is a set of completely positive maps such that the sum map $\sum_x \mathcal{E}^x_{C \to A}$ is trace preserving and $\{\mathcal{D}^y_{B \to D}\}_y$ is a set of quantum channels. This is equivalent to the quantum memory system $Q$ in (116) being a classical system $X$, with

$$\mathcal{E}_{C \to AX}(\rho C) := \sum_x \mathcal{E}^x_{C \to A}(\rho C) \otimes |x\rangle \langle x|,$$

(124)

$$\mathcal{D}_{BX \to D}(\omega_{BX}) := \sum_y \mathcal{D}^y_{B \to D}(\langle x| \omega_{BX} |x\rangle),$$

(125)

so that

$$\Lambda(\mathcal{N}_{A \to B}) = \mathcal{D}_{BX \to D} \circ \mathcal{N}_{A \to B} \circ \mathcal{E}_{C \to AX}.$$  

(126)

In this case, the bipartite channel in (117), but corresponding to $\Lambda$ in (123), becomes the following one-way LOCC channel:

$$\mathcal{L}_{CB \to AD} := \sum_x \mathcal{E}^x_{C \to A} \otimes \mathcal{D}^x_{B \to D}.$$  

(127)

Thus, the set of one-way LOCC superchannels is in direct correspondence with the set of one-way LOCC bipartite channels.

C. LOCR superchannels

A superchannel $\mathcal{Y} \equiv \mathcal{Y}_{(A \to B) \to (C \to D)}$ is implementable by local operations and common randomness (LOCR) if it can be written in the following form:

$$\mathcal{Y}(\mathcal{N}_{A \to B}) := \sum_y p(y) \mathcal{D}^y_{B \to D} \circ \mathcal{N}_{A \to B} \circ \mathcal{E}^y_{C \to A},$$

(128)

where $\{p(y)\}_y$ is a probability distribution and $\{\mathcal{E}^y_{C \to A}\}_y$ and $\{\mathcal{D}^y_{B \to D}\}_y$ are sets of quantum channels. In more detail, the superchannel $\mathcal{Y}_{(A \to B) \to (C \to D)}$ can be realized as

$$\mathcal{Y}(\mathcal{N}_{A \to B}) = \mathcal{D}_{BY \to D} \circ \mathcal{N}_{A \to B} \circ \mathcal{E}_{CY \to A} \circ \mathcal{P}_{YA|YB},$$

(129)

where $\mathcal{P}_{YA|YB}$ is a preparation channel that prepares the common randomness state

$$\sum_y p(y) |y\rangle \langle y| \otimes |y\rangle \langle y|,$$

(130)

and the channels $\mathcal{E}_{CY \to A}$ and $\mathcal{D}_{BY \to D}$ are defined as

$$\mathcal{E}_{CY \to A}(\rho_{CYA}) := \sum_y \mathcal{E}^y_{C \to A}(\langle y| \rho_{CYA} |y\rangle),$$

(131)

$$\mathcal{D}_{BY \to D}(\omega_{BY}) := \sum_y \mathcal{D}^y_{B \to D}(\langle y| \omega_{BY} |y\rangle).$$

(132)

In this case, the bipartite channel in (117), but corresponding to $\mathcal{Y}$ in (128), becomes the following LOCR bipartite channel:

$$\mathcal{C}_{CB \to AD} := \sum_y p(y) \mathcal{E}^y_{C \to A} \otimes \mathcal{D}^y_{B \to D}.$$  

(133)

Thus, the set of LOCR superchannels is in direct correspondence with the set of LOCR bipartite channels.

D. Two-extendible superchannels

A superchannel $\Theta_{(A \to B) \to (C \to D)}$ is defined to be two-extendible if there exists an extension channel $\mathcal{M}_{CB_1B_2 \to AD_1D_2}$ of its corresponding bipartite channel $\mathcal{P}_{CB \to AD}$ that obeys the conditions in (10) and (11). Furthermore, due to the fact that (10) and (11) imply (12), there is no need to explicitly indicate that (118) holds. Two-extendible superchannels were considered in [BBFS21], but this terminology was not employed there.

The specific constraints on the Choi operator of $\mathcal{M}_{CB_1B_2 \to AD_1D_2}$ are precisely the same as those in (14)–(17), with the identifications $C \leftrightarrow A$, $B \leftrightarrow B^\prime$, $A \leftrightarrow A^\prime$, and $D \leftrightarrow B^\prime$. Explicitly, a superchannel $\Theta_{(A \to B) \to (C \to D)}$ is two-extendible if the Choi operator $\Gamma_{CBAD}^P$ of its corresponding bipartite channel $\mathcal{P}_{CB \to AD}$ satisfies the following conditions: there exists a Hermitian operator $\mathcal{M}_C^M_{CB_1B_2AD_1D_2}$ such that

$$\mathcal{F}_{B_1B_2} \otimes \mathcal{D}_{D_1D_2}(\mathcal{M}^M_{CB_1B_2AD_1D_2}) = \mathcal{M}^M_{CB_1B_2AD_1D_2},$$

(134)

$$\text{Tr}_{D_2} [\mathcal{M}^M_{CB_1B_2AD_1D_2}] = \mathcal{I}_{B_2},$$

(135)

$$\sum_{A_1A_2} \mathcal{M}^M_{CB_1B_2AD_1D_2} \geq 0,$$

(136)

$$\text{Tr}_{AD_1D_2} [\mathcal{M}^M_{CB_1B_2AD_1D_2}] = \mathcal{I}_{CB_1B_2}.$$  

(137)

Every one-way LOCC superchannel is two-extendible.

E. Completely PPT preserving superchannels

A superchannel $\Theta_{(A \to B) \to (C \to D)}$ is C-PPT-P if its corresponding bipartite channel $\mathcal{P}_{CB \to AD}$ in (117) is C-PPT-P and obeys the non-signaling constraint in (118) [LM15]. This implies the following for its Choi operator $\Gamma_{CBAD}^P$:

$$\Gamma_{CBAD}^P \geq 0,$$

(138)

$$\text{Tr}_{AD} [\Gamma_{CBAD}^P] = \mathcal{I}_{CB},$$

(139)

$$\text{Tr}_{D} [\Gamma_{CBAD}^P] = \frac{1}{d_B} \text{Tr}_{BD} [\Gamma_{CBAD}^P \otimes \mathcal{I}_B],$$

(140)

$$\text{Tr}_{BD} (\Gamma_{CBAD}^P) \geq 0.$$  

(141)
F. Two-PPT-extendible superchannels

A superchannel $\Theta_{A\rightarrow B}\rightarrow(C\rightarrow D)$ is two-PPT-extendible if its corresponding bipartite channel $\mathcal{P}_{CB\rightarrow AD}$ in (117) is two-PPT-extendible. Again, there is no need to explicitly indicate that (118) holds. The following conditions hold for the Choi operator $\Gamma^{CB}_{C\rightarrow B\rightarrow D}$ of a two-PPT-extendible superchannel: there exists a Hermitian operator $\Gamma^{M}_{C\rightarrow B\rightarrow D\rightarrow AD}$ such that (134)–(137) hold, as well as

$$T_{B:D_{2}}(\Gamma^{M}_{C\rightarrow B\rightarrow D\rightarrow AD}) \geq 0, \quad T_{C:A}(\Gamma^{M}_{C\rightarrow B\rightarrow D\rightarrow AD}) \geq 0.$$ 

Similar to what was already discussed in Section II B 5, the following constraints are redundant:

$$T_{B_{1}:D_{1}}(\Gamma^{M}_{C\rightarrow B_{2}\rightarrow AD_{2}}) \geq 0, \quad T_{C_{1}:A_{1}}(\Gamma^{M}_{C_{2}\rightarrow B_{2}\rightarrow AD_{2}}) \geq 0,$$

$$T_{C_{2}:A_{2}}(\Gamma^{M}_{C\rightarrow B_{1}\rightarrow AD_{1}}) \geq 0,$$

$$T_{B_{1}:D_{1},B_{2}:D_{2}}(\Gamma^{M}_{C\rightarrow B_{1}\rightarrow AD_{1}}) \geq 0.$$ 

Note that every one-way LOCC superchannel is two-PPT-extendible.

G. Two-PPT-extendible non-signaling superchannels

Finally, we can impose an additional non-signaling constraint on two-PPT-extendible superchannels, such that the extension of its corresponding bipartite channel is non-signaling from Alice to both Bobs. The additional constraint on the Choi operator $\Gamma^{M}_{C\rightarrow B_{2}\rightarrow AD_{2}}$ of the extension channel $\mathcal{M}_{C\rightarrow B_{2}\rightarrow AD_{2}}$ is as follows:

$$\text{Tr}_{A}[\Gamma^{M}_{C\rightarrow B_{2}\rightarrow AD_{2}}] = \frac{1}{d_{C}} \text{Tr}_{AC}[\Gamma^{M}_{C\rightarrow B_{2}\rightarrow AD_{2}}] \otimes I_{C}. \quad (146)$$

Every LOCR superchannel is non-signaling and two-PPT-extendible, which follows from definitions and the form of the corresponding bipartite channel in (133). This fact plays an important role in our analysis of approximate quantum error correction. In more detail, we obtain our tightest lower bound on the simulation error of approximate quantum error correction by relaxing the set of LOCR superchannels to the set of non-signaling and two-PPT-extendible superchannels. We note here that this approach was already considered in [BBFS21], and our main contribution here is to employ unitary covariance symmetry of the identity channel to reduce the complexity of the SDPs from that work.

VI. QUANTIFYING THE PERFORMANCE OF APPROXIMATE QUANTUM ERROR CORRECTION

A. Quantifying simulation error with normalized diamond distance and channel infidelity

In approximate quantum error correction [SW02] or quantum communication [BDSW96], the resource available is a quantum channel $N_{A\rightarrow B}$ and the goal is to use it, along with an encoding channel $\mathcal{E}_{A\rightarrow A}$ and a decoding channel $\mathcal{D}_{B\rightarrow B}$, to simulate a $d$-dimensional identity channel $id_{A\rightarrow B}$, and we can use the normalized diamond distance to quantify the error for a fixed encoding and decoding, as follows:

$$e(N_{A\rightarrow B}, (\mathcal{E}_{A\rightarrow A}, \mathcal{D}_{B\rightarrow B})) := \frac{1}{2} \left\| \mathcal{D}_{B\rightarrow B} \circ N_{A\rightarrow B} \circ \mathcal{E}_{A\rightarrow A} \right\|. \quad (147)$$

By minimizing over all encodings and decodings, we arrive at the error in using the channel $N_{A\rightarrow B}$ to simulate the identity channel:

$$e(N_{A\rightarrow B}) := \inf_{(\mathcal{E}, \mathcal{D})} e(N_{A\rightarrow B}, (\mathcal{E}_{A\rightarrow A}, \mathcal{D}_{B\rightarrow B})). \quad (148)$$

We can alternatively employ channel infidelity to quantify the error:

$$e^{F}(N_{A\rightarrow B}, (\mathcal{E}_{A\rightarrow A}, \mathcal{D}_{B\rightarrow B})) := 1 - F(id_{A\rightarrow B}, \mathcal{D}_{B\rightarrow B} \circ N_{A\rightarrow B} \circ \mathcal{E}_{A\rightarrow A}). \quad (149)$$

$$e^{F}(N_{A\rightarrow B}) := \inf_{(\mathcal{E}, \mathcal{D})} e^{F}(N_{A\rightarrow B}, (\mathcal{E}_{A\rightarrow A}, \mathcal{D}_{B\rightarrow B})). \quad (150)$$

Note that the transformation of the channel given by

$$\mathcal{D}_{B\rightarrow B} \circ N_{A\rightarrow B} \circ \mathcal{E}_{A\rightarrow A} \quad (151)$$

is a superchannel, as discussed in Section V, with corresponding bipartite channel

$$\mathcal{P}_{A\rightarrow \hat{B} \rightarrow \hat{A} : B} := \mathcal{E}_{A\rightarrow \hat{A}} \otimes \mathcal{D}_{B\rightarrow B}. \quad (152)$$

As this bipartite channel is a product channel, it is contained within the set of LOCR superchannels, which in turn is contained in the set of one-way LOCC superchannels.

By supplementing the encoding and decoding with common randomness, the resulting error correction scheme $\mathcal{Y} \equiv \mathcal{Y}_{(A\rightarrow \hat{B}): (A\rightarrow B)}$ realizes the following simulation channel:

$$\mathcal{Y}(N_{A\rightarrow B}) := \sum_{y} p(y) D^{y}_{B\rightarrow \hat{B}} \circ N_{A\rightarrow B} \circ \mathcal{E}^{y}_{A\rightarrow A}. \quad (153)$$

where $\{p(y)\}_y$ is a probability distribution and $\{\mathcal{E}^{y}_{A\rightarrow A}\}_y$ and $\{D^{y}_{B\rightarrow \hat{B}}\}_y$ are sets of quantum channels. Recall from Section V C that $\mathcal{Y}$ is an LOCR superchannel, and let LOCR denote the set of all LOCR superchannels. Then we can quantify the simulation error under LOCR in a manner similar to Section III A: we can use the normalized diamond distance to quantify the error for a fixed LOCR superchannel $\mathcal{Y}$, as follows:

$$e_{\text{LOCR}}(N_{A\rightarrow B}, \mathcal{Y}_{(A\rightarrow \hat{B}): (A\rightarrow B)}) := \frac{1}{2} \left\| \mathcal{D}_{A\rightarrow B} - \mathcal{Y}_{(A\rightarrow \hat{B}): (A\rightarrow B)}(N_{A\rightarrow B}) \right\|. \quad (154)$$
By minimizing over all such superchannels, we arrive at the
error in using the channel \(N_{\hat{A} \rightarrow \hat{B}}\) to simulate the identity channel:

\[
e_{\text{LOCR}}(N_{\hat{A} \rightarrow \hat{B}}) : = \inf_{Y \in \text{LOCR}} e(N_{\hat{A} \rightarrow \hat{B}}, Y(\hat{A} \rightarrow \hat{B} \rightarrow (A \rightarrow B))). \tag{155}
\]

As before, we can alternatively employ channel infidelity to quantify the error:

\[
e_{\text{LOCR}}^F(N_{\hat{A} \rightarrow \hat{B}}, Y(\hat{A} \rightarrow \hat{B} \rightarrow (A \rightarrow B))) := 1 - F(id_{\hat{A} \rightarrow \hat{B}}, Y(\hat{A} \rightarrow \hat{B} \rightarrow (A \rightarrow B))(N_{\hat{A} \rightarrow \hat{B}})), \tag{156}
\]

\[
e_{\text{LOCR}}^F(N_{\hat{A} \rightarrow \hat{B}}) := \inf_{Y \in \text{LOCR}} e_{\text{LOCR}}^F(N_{\hat{A} \rightarrow \hat{B}}, Y(\hat{A} \rightarrow \hat{B} \rightarrow (A \rightarrow B))). \tag{157}
\]

However, we have the following:

**Proposition 6** For a channel \(N_{\hat{A} \rightarrow \hat{B}}\), the LOCR simulation errors defined from normalized diamond distance and channel infidelity are equal to each other:

\[
e_{\text{LOCR}}(N_{\hat{A} \rightarrow \hat{B}}) = e_{\text{LOCR}}^F(N_{\hat{A} \rightarrow \hat{B}}). \tag{158}
\]

**Proof.** The proof of this equality is similar to the proof of Proposition 1, following again from the symmetry of the target channel, which is an identity channel having the symmetry in (65), and the fact that a channel twirl can be implemented by means of LOCR. Note that a channel twirl of a channel \(M_{A \rightarrow B}\) has the following form:

\[
\int dU \ U_B^\dagger \circ M_{A \rightarrow B} \circ U_A,
\]

where \(U\) is a unitary channel. ■

By exploiting the fact that a superchannel of the form in (151) is contained in the set of LOCR superchannels, the following inequality holds

\[
e_{\text{LOCR}}(N_{\hat{A} \rightarrow \hat{B}}) \leq \min \{e(N_{\hat{A} \rightarrow \hat{B}}), e^F(N_{\hat{A} \rightarrow \hat{B}})\}. \tag{160}
\]

It is unclear if \(e(N_{\hat{A} \rightarrow \hat{B}})\) is equal to \(e^F(N_{\hat{A} \rightarrow \hat{B}})\) in general: a critical aspect of the proof of Proposition 6 is the fact that LOCR superchannels are allowed for free, so that the symmetrizing twirling superchannel can be used. In the unassisted setting, we cannot use twirling because it is an LOCR superchannel and thus not allowed for free.

Recall again that the identity channel \(id_{\hat{A} \rightarrow \hat{B}}\) possesses the unitary covariance symmetry in (65). Considering this leads to the following proposition:

**Proposition 7** The optimization problems in (155) and (157), for the error in simulating the identity channel \(id_{\hat{A} \rightarrow \hat{B}}\), simplify as follows:

\[
e_{\text{LOCR}}(N_{\hat{A} \rightarrow \hat{B}}) = e_{\text{LOCR}}^F(N_{\hat{A} \rightarrow \hat{B}}) \tag{161}
\]

\[
e_{\text{LOCR}}^F(N_{\hat{A} \rightarrow \hat{B}}) := 1 - \sup_{\mathcal{P}} E_F(N_{\hat{A} \rightarrow \hat{B}}; \mathcal{P}), \tag{162}
\]

where the optimization in (162) is over every LOCR protocol \(\mathcal{P}\), defined as

\[
\mathcal{P} : = \{(p(y), E^y_{\hat{A} \rightarrow \hat{A}' \rightarrow (A \rightarrow B)}(\mathcal{D}^y_{\hat{B} \rightarrow B}))\}_y,
\]

and \(E_F(N_{\hat{A} \rightarrow \hat{B}}; \mathcal{P}) \in [0, 1]\) is the entanglement fidelity:

\[
E_F : = \sum_y p(y) \text{Tr}[\Phi^d_{A\hat{B}}(\mathcal{D}^y_{\hat{B} \rightarrow B} \circ N_{\hat{A} \rightarrow \hat{B}} \circ E^y_{\hat{A} \rightarrow \hat{A}})(\Phi^d_{A\hat{B}})]. \tag{164}
\]

An optimal LOCR simulation channel for both \(e_{\text{LOCR}}(N_{\hat{A} \rightarrow \hat{B}})\) and \(e_{\text{LOCR}}^F(N_{\hat{A} \rightarrow \hat{B}})\) has the following form:

\[
E_{\text{LOCR}}(N_{\hat{A} \rightarrow \hat{B}}) := E_F(\text{id}_{\hat{A} \rightarrow \hat{B}} + (1 - E_F) \mathcal{D}_{A \rightarrow B}), \tag{166}
\]

where \(\mathcal{D}_{A \rightarrow B}\) is the channel defined in (69). Thus, the LOCR simulation channel applies the identity channel \(id_{\hat{A} \rightarrow \hat{B}}\) with probability \(E_F\) and the randomizing channel \(\mathcal{D}_{A \rightarrow B}\) with probability \(1 - E_F\).

**Proof.** See Appendix D. ■

### B. LOCR simulation of general point-to-point channels

We can use a point-to-point channel \(N_{\hat{A} \rightarrow \hat{B}}\), along with LOCR, to simulate another general point-to-point channel \(O_{A \rightarrow B}\). In this case, the simulation channel \(\overline{O}_{A \rightarrow B}\) has the form

\[
\overline{O}_{A \rightarrow B} := Y(\hat{A} \rightarrow \hat{B} \rightarrow (A \rightarrow B))(N_{\hat{A} \rightarrow \hat{B}}), \tag{167}
\]

where \(Y(\hat{A} \rightarrow \hat{B} \rightarrow (A \rightarrow B))\) is an LOCR superchannel, as discussed in Section V C. The simulation error when employing a specific LOCR superchannel \(Y(\hat{A} \rightarrow \hat{B} \rightarrow (A \rightarrow B))\) is

\[
e_{\text{LOCR}}(O_{A \rightarrow B}, N_{\hat{A} \rightarrow \hat{B}}, Y(\hat{A} \rightarrow \hat{B} \rightarrow (A \rightarrow B)))
\]

\[
:= \frac{1}{2} \left\| O_{A \rightarrow B} - \overline{O}_{A \rightarrow B} \right\|_F, \tag{168}
\]

and the simulation error minimized over all possible LOCR superchannels is

\[
e_{\text{LOCR}}(O_{A \rightarrow B}, N_{\hat{A} \rightarrow \hat{B}}) := \inf_{Y \in \text{LOCR}} e_{\text{LOCR}}(O_{A \rightarrow B}, N_{\hat{A} \rightarrow \hat{B}}, Y(\hat{A} \rightarrow \hat{B} \rightarrow (A \rightarrow B))). \tag{169}
\]

Again we can alternatively consider quantifying simulation error in terms of the channel infidelity:

\[
e_{\text{LOCR}}^F(O_{A \rightarrow B}, N_{\hat{A} \rightarrow \hat{B}}, Y(\hat{A} \rightarrow \hat{B} \rightarrow (A \rightarrow B)))
\]

\[
:= 1 - F(O_{A \rightarrow B}, \overline{O}_{A \rightarrow B}), \tag{170}
\]

\[
e_{\text{LOCR}}^F(O_{A \rightarrow B}, N_{\hat{A} \rightarrow \hat{B}}) := \inf_{Y \in \text{LOCR}} e_{\text{LOCR}}^F(O_{A \rightarrow B}, N_{\hat{A} \rightarrow \hat{B}}, Y(\hat{A} \rightarrow \hat{B} \rightarrow (A \rightarrow B))). \tag{171}
\]
VII. SDP LOWER BOUNDS ON THE PERFORMANCE OF APPROXIMATE QUANTUM ERROR CORRECTION BASED ON TWO-PPT EXTENDIBILITY AND NON-SIGNALING CONSTRAINTS

A. SDP lower bound on the error in LOCR simulation of a channel

Using (169) to calculate the simulation error, we again encounter an intractable optimization task. Employing the same idea from Section IV A, we enlarge the set of LOCR superchannels to two-PPT-extendible, non-signaling superchannels (abbreviated henceforth as 2PENS). As noted in Section V G, the 2PENS set strictly contains the set of LOCR superchannels. Thus, we can obtain a lower bound on the simulation error by optimizing over all 2PENS superchannels. We define the simulation error under 2PENS superchannels as

\[ e_{2\text{PENS}}(O_{A \rightarrow B}, \mathcal{N}_{\hat{A} \rightarrow \hat{B}}) := \inf_{\bar{O} \in 2\text{PENS}} \frac{1}{2} \left\| O_{A \rightarrow B} - \bar{O}_{A \rightarrow B} \right\|_1, \]

(172)

where \( \bar{O}_{A \rightarrow B} \) is defined in (167).

As a result of the strict containment

\[ \text{LOCR} \subset 2\text{PENS}, \]

(173)

we have the relation

\[ e_{2\text{PENS}}(O_{A \rightarrow B}, \mathcal{N}_{\hat{A} \rightarrow \hat{B}}) \leq e_{\text{LOCR}}(O_{A \rightarrow B}, \mathcal{N}_{\hat{A} \rightarrow \hat{B}}). \]

(174)

We now state that the simulation error in (172) can be calculated by means of a semi-definite program.

Proposition 8 The simulation error in (172) can be calculated by means of the following semi-definite program:

\[ e_{2\text{PENS}}(O_{A \rightarrow B}, \mathcal{N}_{\hat{A} \rightarrow \hat{B}}) = \inf_{\mu \geq 0, Z_{AB} \geq 0, \mu} \mu, \]

(175)

subject to

\[ \mu I_A \geq Z_A, \]

(176)

\[ Z_{AB} \geq \Gamma^Q_{AB} - \text{Tr}_{\hat{A}\hat{B}} [T_{\hat{A}\hat{B}} (\Gamma^N_{\hat{A}\hat{B}}) M_{\hat{A}\hat{B}} / d_{\hat{B}}], \]

(177)

\[ \text{Tr}_{\hat{A}\hat{B}} [M_{\hat{A}\hat{B}}] = I_{\hat{A} \hat{B}}, \]

(178)

\[ (\mathcal{F}_{\hat{B}1} \otimes \mathcal{F}_{\hat{B}2}) (M_{\hat{A}\hat{B}}) = M_{\hat{A}\hat{B}} \mathcal{F}_{\hat{B}1} \mathcal{F}_{\hat{B}2} \]

(179)

\[ \text{Tr}_{\hat{B}} (M_{\hat{A}\hat{B}}) = M_{\hat{A}\hat{B}} / d_{\hat{B}}, \]

(180)

\[ T_{\hat{A}\hat{B}} (M_{\hat{A}\hat{B}}) \geq 0, \]

(181)

\[ T_{\hat{B}1\hat{B}2} (M_{\hat{A}\hat{B}}) \geq 0, \]

(182)

\[ \text{Tr}_{\hat{A}} [M_{\hat{A}\hat{B}} / d_{\hat{B}}] = I_A \otimes \frac{1}{d_A} \text{Tr}_{\hat{A}\hat{B}} [M_{\hat{A}\hat{B}}]. \]

(183)

The objective function and the first two constraints follow from the semi-definite program in (46) for the normalized diamond distance. The quantity

\[ \text{Tr}_{\hat{A}\hat{B}} [T_{\hat{A}\hat{B}} (\Gamma^N_{\hat{A}\hat{B}}) M_{\hat{A}\hat{B}} / d_{\hat{B}}] \]

(184)

in (177) is the Choi operator of the serial composition of the available channel \( \mathcal{N}_{\hat{A} \rightarrow \hat{B}} \) and the superchannel with corresponding bipartite channel \( \mathcal{K}_{\hat{A}\hat{B}1 \rightarrow \hat{A}\hat{B}1} \), with Choi operator \( M_{\hat{A}\hat{B}1\hat{B}2} / d_{\hat{B}} \), where \( \mathcal{K}_{\hat{A}\hat{B}1 \rightarrow \hat{A}\hat{B}1} \) is the marginal channel of \( M_{\hat{A}\hat{B}1\hat{B}2} \), defined as

\[ \mathcal{K}_{\hat{A}\hat{B}1 \rightarrow \hat{A}\hat{B}1} (\omega_{\hat{A}\hat{B}1}) := \text{Tr}_{\hat{B}} [M_{\hat{A}\hat{B}1\hat{B}2} \otimes \pi_{\hat{B}}]. \]

(185)

The constraint in (178) forces \( M_{\hat{A}\hat{B}1\hat{B}2} \) to be trace preserving, that in (179) forces \( M_{\hat{A}\hat{B}1\hat{B}2} \) to be permutation covariant with respect to the \( B \) systems (see (75)), and that in (180) forces \( M_{\hat{A}\hat{B}1\hat{B}2} \) to be the extension of the marginal channel \( \mathcal{K}_{\hat{A}\hat{B}1 \rightarrow \hat{A}\hat{B}1} \). The final two PPT constraints are equivalent to the C-PPT-P constraints in (76) and (77), respectively.

B. SDP lower bound on the error of approximate quantum error correction

The semi-definite program in Proposition 8 can be simplified for the special case \( \mathcal{N}_{\hat{A} \rightarrow \hat{B}} = \text{id}^d_{A \rightarrow B} \) by exploiting the unitary covariance symmetry of the identity channel, as stated in (65).

Proposition 9 The semi-definite program in Proposition 8, for the special case of simulating the identity channel \( \text{id}^d_{A \rightarrow B} \), simplifies as follows for \( d \geq 3 \):

\[ e_{2\text{PENS}}(\mathcal{N}_{\hat{A} \rightarrow \hat{B}}) = e_{2\text{PENS}}^F(\mathcal{N}_{\hat{A} \rightarrow \hat{B}}) \]

(186)

\[ = 1 - \sup_{M^+, M^- \geq 0, M^+, M^- \in \text{LinOp}} \text{Tr} \left[ T_{\hat{A}\hat{B}} (\Gamma^N_{\hat{A}\hat{B}}) \frac{P_{\hat{A}\hat{B}} / d_{\hat{B}}}{d_{\hat{A}}^2} \right], \]

(187)

subject to

\[ \begin{bmatrix} M^0 + M^3 & M^1 + iM^2 \\ M^1 - iM^2 & M^0 - M^3 \end{bmatrix} \geq 0, \]

(188)

\[ \begin{bmatrix} I_{\hat{B}1\hat{B}2} & \text{Tr}_{\hat{B}} [P_{\hat{A}\hat{B}} / d_{\hat{B}}] \otimes I_{\hat{B}1} \\ \text{Tr}_{\hat{B}} [P_{\hat{A}\hat{B}} / d_{\hat{B}}] \otimes I_{\hat{B}2} & I_{\hat{B}1\hat{B}2} \end{bmatrix} \]

(189)

\[ \begin{bmatrix} M^0_{\hat{A}\hat{B}1\hat{B}2} & M^-_{\hat{A}\hat{B}1\hat{B}2} + M^0_{\hat{A}\hat{B}1\hat{B}2} \end{bmatrix} , \]

(190)

\[ \begin{bmatrix} M^0_{\hat{A}\hat{B}1\hat{B}2} & M^-_{\hat{A}\hat{B}1\hat{B}2} \end{bmatrix} \]

(191)

\[ \begin{bmatrix} P_{\hat{A}\hat{B}1\hat{B}2} / d_{\hat{B}} & Q_{\hat{A}\hat{B}1\hat{B}2} / d_{\hat{B}} \end{bmatrix} \]

(192)

\[ \begin{bmatrix} P_{\hat{A}\hat{B}1\hat{B}2} / d_{\hat{B}} & Q_{\hat{A}\hat{B}1\hat{B}2} / d_{\hat{B}} \end{bmatrix} \]

(193)

\[ \begin{bmatrix} dM^0 + M^1 + \sqrt{d^2 - 1} M^2 \end{bmatrix}, \]

(194)
We now provide expository remarks similar to Remarks 4 and 5, as well as an additional remark about approximate quantum error correction assisted by one-way LOCC.

**Remark 10** The SDP in the statement of Proposition 9 is rather lengthy, and so we provide some explanation here. The constraint in (188) and the constraints \(M^+, M^-, M^0 \geq 0\) in (187) correspond to the constraint of complete positivity in (175) (i.e., \(M_{AB}^B_B \geq 0\)). The constraint in (189) corresponds to the constraint of trace preservation in (178). The constraints in (190)–(191) correspond to the constraint of permutation covariance in (179). The constraints in (192)–(193) correspond to the non-signaling constraint in (180). The constraints in (196)–(198) correspond to the PPT constraint in (181), and the constraints in (203)–(205) correspond to the PPT constraint in (182). Finally, the constraints in (213)–(217) correspond to the non-signaling constraint in (183).

**Remark 11** Even though the number of constraints in the SDP above appears to increase when compared with the SDP from Proposition 8, we note that the runtime of the SDP above is significantly reduced because the size of the matrices involved in each of the constraints is much smaller. This is the main advantage that we get by incorporating unitary covariance symmetry of the identity channel.

If we only optimized over the larger set of two-extendible channels instead of the set of two-PPT-extendible non-signaling channels, the SDP would be much simpler, given by (187)–(193). However, optimizing over the smaller set of two-PPT-extendible non-signaling channels gives tighter bounds at a marginal increase in computational cost, and thus we also include the PPT constraints in (196)–(198) and (203)–(205) and the non-signaling constraints in (213)–(217).

**Remark 12** By excluding the non-signaling constraints in (213)–(217), the resulting SDP gives a lower bound on the simulation error of approximate quantum error correction assisted by a one-way LOCC channel. That is, the resulting SDP gives a lower bound on

\[
e_{\text{WL}}(\mathcal{A}_{\rightarrow B}) := \inf_{\Lambda \in \mathcal{W}_L} e(\mathcal{N}_{\Lambda_{\rightarrow B} \rightarrow \mathcal{A} \rightarrow B}),
\]

where
\( e_{\text{WL}}(N_{\hat{A} \rightarrow \hat{B}}, \Lambda_{(\hat{A} \rightarrow \hat{B})}) := \frac{1}{2} \| \text{id}_{\hat{A} \rightarrow \hat{B}} - \Lambda_{(\hat{A} \rightarrow \hat{B})} \| \),

(219)

with \( \Lambda \) a one-way LOCC superchannel, as defined in (123). By the same reasoning given for Proposition 6, this error is no different if we use infidelity instead of normalized diamond distance.

VIII. EXAMPLES

In this section we present some numerical results from our semi-definite programs. To perform these numerical calculations, we employed CVXPY \([\text{DB16, AVDB18}]\) with the interior point optimizer MOSEK. All of our Python source code is available with the arXiv posting of our paper.

A. Approximate teleportation and quantum error correction using special mixed states and channels

First, we provide bounds on the performance of approximate teleportation (i.e., on the error in simulating an identity channel), when using a particular set of imperfect resource states. In the past, PPT constraints alone (i.e., without two-extendibility) have been used to obtain bounds on objective functions involving an optimization over the set of LOCC channels (see, e.g., \([\text{LM15, WXD18, WD16b, WD16a, BW18, SW20}]\)). We can also use them to obtain a lower bound on the simulation error of approximate teleportation. By following techniques similar to those in \([\text{LM15, SW20}]\), we find the following SDP gives a lower bound on the simulation error of approximate teleportation:

\[
1 - \sup_{K_{\hat{A}\hat{B}} \succeq 0} \left\{ \begin{array}{c} \text{Tr}[K_{\hat{A}\hat{B}} \rho_{\hat{A}\hat{B}}] : \\ K_{\hat{A}\hat{B}} \preceq I_{\hat{A}\hat{B}}, \\ -I_{\hat{A}\hat{B}} \preceq d T_{\hat{B}} (K_{\hat{A}\hat{B}}) \preceq I_{\hat{A}\hat{B}} \end{array} \right\}. 
\]

(220)

See Appendix F for a proof. We note here that PPT constraints are implied by the two-PPT-extendibility constraints given in Proposition 3, so that the optimal value in (220) is not smaller than the optimal value in (187). We also note that an SDP bearing some similarities to that in (220) was presented in \([\text{FWT19}]\), but that SDP calculates a bound on one-shot distillable entanglement, whereas the SDP in (220) calculates a bound on the error of approximate teleportation.

In the following example, we show that two-PPT-extendibility gives strictly better bounds than PPT constraints alone, when optimizing over one-way LOCC channels. Consider the following mixed state:

\[
p \Phi_{\hat{A}\hat{B}} + (1 - p) \pi_{\hat{A}} \otimes \sigma_{\hat{B}},
\]

where \( p \in [0, 1] \), \( \Phi_{\hat{A}\hat{B}} \) is the maximally entangled state of Schmidt rank three, \( \pi_{\hat{A}} \) is the maximally mixed state of dimension three, and \( \sigma_{\hat{B}} \) is a randomly selected \( 3 \times 3 \) density matrix. Using the state in (221) as the resource for approximate teleportation, lower bounds on the simulation error, as given by two-PPT-extendibility, are stronger than those given by PPT constraints alone, for small values of \( p \). Figure 1 compares the lower bounds obtained for different values of \( p \) and randomly generated \( \sigma_{\hat{B}} \). The state \( \sigma_{\hat{B}} \) that was used to generate data for Figure 1 is as follows:

\[
\begin{bmatrix}
0.140 & 0.043 + 0.024i & -0.143 + 0.028i \\
0.043 - 0.024i & 0.222 & -0.257 + 0.006i \\
-0.143 - 0.028i & -0.257 - 0.006i & 0.638
\end{bmatrix}
\]

(222)

We note here that the SDP calculations depend on the choice of \( \sigma_{\hat{B}} \). For certain choices of \( \sigma_{\hat{B}} \), the difference in the errors disappears for all values of \( p \), e.g., when \( \sigma_{\hat{B}} \) is a maximally mixed state. It still remains open to determine the full set of resource states for which two-PPT-extendibility gives stronger bounds on the simulation error. Regardless, this example demonstrates that including two-PPT-extendibility constraints can improve the bounds obtained using PPT constraints alone.

One can consider the same comparison for approximate quantum error correction. Using similar techniques, we derive the following SDP lower bound on the simulation error of approximate quantum error correction for a channel \( N_{\hat{A} \rightarrow \hat{B}} \), when using PPT and non-signaling constraints only:

\[
1 - \sup_{K_{\hat{A}\hat{B}}, \sigma_{\hat{A}} \succeq 0} \left\{ \begin{array}{c} \text{Tr}[K_{\hat{A}\hat{B}} F^N_{\hat{A} \rightarrow \hat{B}}] : \\ K_{\hat{A}\hat{B}} \preceq \sigma_{\hat{A}} \otimes I_{\hat{B}}, \\ d^2 \text{Tr}[K_{\hat{A}\hat{B}}] = I_{\hat{B}}, \\ \sigma_{\hat{A}} \otimes I_{\hat{B}} \preceq d T_{\hat{B}} (K_{\hat{A}\hat{B}}) \geq 0, \\ \text{Tr}[\sigma_{\hat{A}}] = 1. \end{array} \right\}.
\]

(223)

See Appendix G for a proof. We note here that essentially the same SDP was given in \([\text{LM15}]\) (up to a transpose in the objective function). The SDP in \([\text{LM15}]\) resulted from taking the error criterion to be in terms of entanglement fidelity when transmitting the maximally entangled state. Our proof here clarifies that essentially the same SDP results when using
Simulation error

0.1 0.2 0.3 0.4 0.5 0.6
Simulation error

0.45 0.50 0.55 0.60 0.65 0.70 0.75 0.80

0.10 0.15 0.20 0.25 0.30 0.35 0.40 0.45 0.50

0.10 0.20 0.30 0.40 0.50 0.60 0.70 0.80 0.90

FIG. 2. Comparison between two-PPT-extendibility and PPT constraints for bounding the simulation error in approximate quantum error correction when using the resource channel with Choi state $\rho_{\hat{A}B} = p \Phi_{\hat{A}B} + (1 - p) \pi_{\hat{A}} \otimes \sigma_B$, where $p \in [0, 1]$ and $\sigma_B$ is defined in (222). PPTNS and 2PENS are the curves obtained using the SDPs in (223) and Proposition 9, respectively, giving lower bounds on the error in approximate quantum error correction. There is no significant difference in the numerical values obtained from these two conditions. PPT and 2PE are the curves obtained using the same SDPs but without the non-signaling constraints, hence, giving lower bounds on the error in one-way LOCC-assisted approximate error correction.

FIG. 3. Comparison between bounds on the simulation error for approximate teleportation when using a two-dimensional special mixed state and a three-dimensional special mixed state as a resource. The resource state is of the form $\rho_{\hat{A}B} = p \Phi_{\hat{A}B} + (1 - p) \pi_{\hat{A}} \otimes \sigma_B$, where $p \in [0, 1]$ and $\sigma_B$ is chosen to be (225) when $d_B = 2$ and (226) when $d_B = 3$. The bounds on the simulation error are calculated using both the 2PE constraints given in Proposition 3 and the PPT constraints given in (220). There is no significant difference in the numerical values obtained from both the constraints for $d_B = 2$.

We use a similar resource state as in (221):

$$\rho_{\hat{A}B} = p \Phi_{\hat{A}B} + (1 - p) \pi_{\hat{A}} \otimes \sigma_B'.$$  \hspace{1cm} (224)

but the maximally entangled and maximally mixed states are two-dimensional. Additionally, $\sigma_B'$ was generated randomly and is taken as

$$\sigma_B' = \begin{bmatrix} 0.287 & -0.347 + 0.132i \\ -0.347 - 0.132i & 0.713 \end{bmatrix}. \quad (225)$$

In Figure 3, we plot the bounds on the simulation error versus the parameter $p$ in (224), when using the 2PE constraints given in Proposition 3 and the PPT constraints given in (220). We also compare this to the bounds on the simulation error when using a three-dimensional special mixed state instead. The resource state used is the same as the state in (221), but $\sigma_B$ is chosen as follows:

$$\begin{bmatrix} 0.287 & -0.347 + 0.132i & 0 \\ -0.347 - 0.132i & 0.713 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (226)$$

in order to provide a closer comparison with the two-dimensional case in (224).

We see from Figure 3 that a two-dimensional resource state with a small amount of imperfection can outperform a three-dimensional resource with higher amounts of imperfection for the task of three-dimensional approximate teleportation. We also notice that the 2PE constraints and the PPT constraints give the same error values when $d_B = 2$, but give different values when $d_B = 3$, as seen in Figure 1 as well.

B. Three-dimensional approximate teleportation using two-dimensional special mixed states

In this example, we investigate the simulation error in approximate teleportation when a lower dimensional imperfect resource state is used to teleport a higher dimensional state.

normalized diamond distance or channel infidelity as the error criterion. The second constraint in the SDP ($d^2 \text{Tr}_\hat{A} [K_{\hat{A}B}] = I_B$) corresponds to the non-signaling condition. Following the same reasoning as in Remark 10, removing this constraint leads to an SDP that provides a lower bound on the simulation error of approximate quantum error correction assisted by one-way LOCC.

The example state in (221) can also serve as the Choi state of a channel, due to the fact that the reduced state of system $\hat{A}$ is maximally mixed. In Figure 2, we plot the lower bound in (223) and the lower bound from Proposition 9 for the corresponding channel. Additionally, we also plot the simulation errors that result from excluding the non-signaling constraints from both SDPs. The resulting SDPs provide lower bounds on the errors in approximate quantum error correction assisted by one-way LOCC using PPT and two-PPT-extendibility, respectively. Figure 2 demonstrates that the lower bound in Proposition 9 improves upon (223) for one-way LOCC simulation but provides no advantage for LOCR simulation. The difference between all four curves becomes very small (less than $10^{-3}$) for higher values of $p$.

0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9

$\hat{d}$ = 3 with 2PE constraints
$\hat{d}$ = 3 with PPT constraints
$\hat{d}$ = 2 with 2PE constraints
$\hat{d}$ = 2 with PPT constraints

$\hat{d}$ = 3 with 2PE constraints
$\hat{d}$ = 3 with PPT constraints
$\hat{d}$ = 2 with 2PE constraints
$\hat{d}$ = 2 with PPT constraints

In this example, we investigate the simulation error in approximate teleportation when a lower dimensional imperfect resource state is used to teleport a higher dimensional state.

$\hat{d}$ = 3 with 2PE constraints
$\hat{d}$ = 3 with PPT constraints
$\hat{d}$ = 2 with 2PE constraints
$\hat{d}$ = 2 with PPT constraints

In this example, we investigate the simulation error in approximate teleportation when a lower dimensional imperfect resource state is used to teleport a higher dimensional state.

$\hat{d}$ = 3 with 2PE constraints
$\hat{d}$ = 3 with PPT constraints
$\hat{d}$ = 2 with 2PE constraints
$\hat{d}$ = 2 with PPT constraints

In this example, we investigate the simulation error in approximate teleportation when a lower dimensional imperfect resource state is used to teleport a higher dimensional state.

$\hat{d}$ = 3 with 2PE constraints
$\hat{d}$ = 3 with PPT constraints
$\hat{d}$ = 2 with 2PE constraints
$\hat{d}$ = 2 with PPT constraints

In this example, we investigate the simulation error in approximate teleportation when a lower dimensional imperfect resource state is used to teleport a higher dimensional state.

$\hat{d}$ = 3 with 2PE constraints
$\hat{d}$ = 3 with PPT constraints
$\hat{d}$ = 2 with 2PE constraints
$\hat{d}$ = 2 with PPT constraints

In this example, we investigate the simulation error in approximate teleportation when a lower dimensional imperfect resource state is used to teleport a higher dimensional state.

$\hat{d}$ = 3 with 2PE constraints
$\hat{d}$ = 3 with PPT constraints
$\hat{d}$ = 2 with 2PE constraints
$\hat{d}$ = 2 with PPT constraints

In this example, we investigate the simulation error in approximate teleportation when a lower dimensional imperfect resource state is used to teleport a higher dimensional state.

$\hat{d}$ = 3 with 2PE constraints
$\hat{d}$ = 3 with PPT constraints
$\hat{d}$ = 2 with 2PE constraints
$\hat{d}$ = 2 with PPT constraints

In this example, we investigate the simulation error in approximate teleportation when a lower dimensional imperfect resource state is used to teleport a higher dimensional state.

$\hat{d}$ = 3 with 2PE constraints
$\hat{d}$ = 3 with PPT constraints
$\hat{d}$ = 2 with 2PE constraints
$\hat{d}$ = 2 with PPT constraints

In this example, we investigate the simulation error in approximate teleportation when a lower dimensional imperfect resource state is used to teleport a higher dimensional state.

$\hat{d}$ = 3 with 2PE constraints
$\hat{d}$ = 3 with PPT constraints
$\hat{d}$ = 2 with 2PE constraints
$\hat{d}$ = 2 with PPT constraints

In this example, we investigate the simulation error in approximate teleportation when a lower dimensional imperfect resource state is used to teleport a higher dimensional state.

$\hat{d}$ = 3 with 2PE constraints
$\hat{d}$ = 3 with PPT constraints
$\hat{d}$ = 2 with 2PE constraints
$\hat{d}$ = 2 with PPT constraints

In this example, we investigate the simulation error in approximate teleportation when a lower dimensional imperfect resource state is used to teleport a higher dimensional state.

$\hat{d}$ = 3 with 2PE constraints
$\hat{d}$ = 3 with PPT constraints
$\hat{d}$ = 2 with 2PE constraints
$\hat{d}$ = 2 with PPT constraints

In this example, we investigate the simulation error in approximate teleportation when a lower dimensional imperfect resource state is used to teleport a higher dimensional state.

$\hat{d}$ = 3 with 2PE constraints
$\hat{d}$ = 3 with PPT constraints
$\hat{d}$ = 2 with 2PE constraints
$\hat{d}$ = 2 with PPT constraints
where $p$ of the depolarizing channel ($\text{SDP in (223)}$ with PPT constraints only, for different dimensions the SDP in Proposition 9 with two-PPT-extendibility constraints, and two-dimensional identity channel. The bounds are calculated using the two-PPT-extendibility conditions from Proposition 9, respectively. We note from Figure 4 that two-PPT-extendibility constraints give better bounds compared to PPT constraints when using a three-dimensional depolarizing channel to simulate a two-dimensional identity channel. However, both the constraints give the same bounds when a two-dimensional depolarizing channel is used to simulate a two-dimensional identity channel. This was also observed in the numerical calculations of [BBFS21].

We also note that a three-dimensional depolarizing channel provides little advantage over a two-dimensional depolarizing channel for simulating two-dimensional identity channel. Therefore, a two-dimensional depolarizing channel with slightly higher value of the parameter $p$ can outperform a three-dimensional depolarizing channel with a lower value of $p$, for the purpose of approximating a qubit identity channel.

D. Approximate teleportation using the two-mode squeezed vacuum state

Two-mode squeezed vacuum states are easily prepared in laboratories and have entanglement content that can be parameterized by $\lambda \geq 0$. They are defined as [Ser17]

$$\sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} \lambda^n |n\rangle |n\rangle .$$

They are used as the resource state in continuous-variable quantum teleportation [BK98] and have also been used as a resource in experiments on teleportation of photonic qubits [FSB*98, TMF*13]. Here we investigate bounds on the performance of qudit teleportation with the two-mode squeezed vacuum state as the resource state.

The parameter $\lambda$ denotes the strength of squeezing applied ($\lambda = \tanh(r)$, where $r$ is the squeezing parameter). For low squeezing strength, we can ignore higher order terms in $\lambda$ without inducing much error. We use the following state in our calculations for qudit teleportation:

$$\frac{1}{\sqrt{1 + \lambda^2 + \lambda^4}} \sum_{n=0}^{2} \lambda^n |n\rangle |n\rangle .$$

However, for higher values of the squeezing strength (i.e., $\lambda$ near to one), we do not expect this approximation to be good.

Figure 5 demonstrates that the simulation error increases with $d$ for fixed values of $\lambda$, where $d$ is the dimension of the target identity channel that the protocol is simulating. The simulation error does not go to zero for $d > 3$, even for
maximally entangled qutrit resource states. Therefore, projecting this trend further, we conclude that simulation of a higher-dimensional identity channel with a lower-dimensional resource state incurs larger errors in the simulation. We note here that we observed no difference in the values calculated by the SDPs in (223) and Proposition 9.

### E. Approximate quantum error correction for a three-level amplitude damping channel

Here we present an example of our bound for the simulation error in approximate error correction. We consider a three-level amplitude damping channel, as defined in [CG21], to demonstrate our SDP in Proposition 9.

The channel can be defined using three decay parameters, labeled by the states involved: \( \gamma_{10}, \gamma_{21}, \gamma_{20} \). See Figure 6 for a depiction. The Kraus operators for the three-level amplitude damping channel are as follows:

\[
K_0 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1 - \gamma_{10}} & 0 \\ 0 & 0 & \sqrt{1 - \gamma_{21} - \gamma_{20}} \end{bmatrix}, \tag{230}
\]

\[
K_1 := \begin{bmatrix} 0 & \sqrt{\gamma_{10}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{231}
\]

\[
K_2 := \begin{bmatrix} 0 & 0 & \sqrt{\gamma_{21}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{232}
\]

\[
K_3 := \begin{bmatrix} 0 & 0 & \sqrt{\gamma_{20}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{233}
\]

so that its action on an input state \( \rho \) is given by \( \Sigma_{i=0}^{3} K_i \rho K_i^\dagger \).

For the map to be completely positive and trace preserving, the decay parameters must obey

\[
\begin{cases}
0 \leq \gamma_i \leq 1 & \forall i \in \{10, 21, 20\} \\
\gamma_{21} + \gamma_{20} \leq 1
\end{cases}. \tag{234}
\]

Figure 7 plots the lower bound on the simulation error as a function of the decay parameter \( \gamma_{10} \), for various values of the other decay parameters. We notice in Figure 7 that the simulation error monotonically increases with the decay parameters. As all three decay parameters approach zero, the channel becomes close to an identity channel. This is reflected in the plot as the simulation error also approaches zero. We note here that we observed no difference in the values calculated by the SDPs in (223) and Proposition 9.

### F. Comparison of computational runtimes

In this section we present the average runtime to execute various SDPs listed in this work. The calculations were performed on a computer with 16 GB RAM and an Intel i7-9750H processor.

All calculations that generated the entries in Table I employed the two-dimensional maximally entangled state. For approximate teleportation, the input is the maximally entangled state of Schmidt rank two, and for approximate error correction, the input is the qubit identity channel. The simulated channel is also the qubit identity channel in all cases. The runtimes were calculated using time.time() function in Python. They are only presented for the purpose of comparison and can vary moderately.

All runtimes are listed in Table I, where We see that the unsimplified SDP for approximate teleportation with two-PPT-extendibility, given in Proposition 2, is around 25 times slower than the simplified SDP for the same in Proposition 3. The SDP for the simulation error in approximate teleportation using PPT constraints that is given in (220) is several times faster than when two-PPT-extendibility constraints are employed, but we have seen in the examples that two-PPT-extendibility constraints can give tighter lower bounds on the simulation error.

Similarly, we see that the unsimplified SDP for approximate error correction when using two-PPT-extendibility constraints
| SDP                                | Runtime (seconds) |
|-----------------------------------|------------------|
| Teleportation unsimplified 2PE    | 253.03           |
| Teleportation 2PE                 | 10.34            |
| Teleportation PPT                 | 0.19             |
| Error Correction unsimplified     | 147.75           |
| Error Correction unsimplified 2PENS| 158.22           |
| Error Correction 2PENS             | 5.65             |
| Error Correction 2PE              | 5.38             |
| Error Correction PPTNS            | 0.20             |
| Error Correction PPT              | 0.16             |

TABLE I. Comparing the runtime of different SDPs presented in this work. 2PE refers to two-PPT-extendibility constraints and NS indicates that non-signaling conditions were used. All calculations are done for two-dimensional resource and simulating two-dimensional identity channel.

(Proposition 8) is several times slower than the simplified SDP given in Proposition 9. Again, the SDP with PPT constraints given in (223) is much faster than the SDP with two-PPT-extendibility constraints, but we have demonstrated examples for which two-PPT-extendibility constraints provide a tighter lower bound on the simulation error.

IX. CONCLUSION

In this work, we developed a technique for quantifying the performance of approximate teleportation using an arbitrary resource state, by establishing a lower bound on the error in simulating a teleportation protocol that uses an imperfect resource state and one-way LOCC channels. We accomplished this by combining the notions of C-PPT-P channels and two-extendible channels to give a relaxation of the set of one-way LOCC channels, as was done previously in [BBFS21] but for approximate quantum error correction. We significantly reduced the complexity of our semi-definite program by exploiting the unitary covariance symmetry of the simulated identity channel. This symmetry is useful in semi-definite programs and can have much wider applications with respect to dynamical resource theories. As an example, we evaluated our lower bound when using a two-mode squeezed vacuum state as the resource state for approximate teleportation.

We used related techniques to quantify the performance of approximate quantum error correction. Incorporating two-PPT-extendibility constraints again led to computationally feasible semi-definite optimizations for evaluating lower bounds on the error in approximate quantum error correction. We further exploited the unitary covariance symmetry of the identity channel to give a less computationally taxing semi-definite program to calculate the error. Finally, we demonstrated some calculations for amplitude damping channels as the resource channels.

The SDPs in this work provide computational support to ongoing experimental research in quantum information by providing tools to analyse available resources and identify valuable states and channels.

Several directions for future work remain open:

1. We have only considered two-extendible channels; incorporation of \( k \)-extendible channels for \( k > 2 \) into our semi-definite optimization could offer tighter bounds on the measures we have described. The recent work of [GO22] might be helpful for addressing this problem. The notion of two-PPT-extendible channels is interesting in its own right via its connection with one-way LOCC channels.

2. It would also be interesting to find semi-definite constraints on one-way LOCC and LOCR channels, beyond those presented here, which include \( k \)-extendibility, PPT, and non-signaling.

3. One could also try to find semi-definite tightenings of one-way LOCC and LOCR, which would lead to upper bounds on the simulation errors.

4. The paper [SW20] shows that PPT constraints are sufficient to determine the exact simulation error in bidirectional teleportation for certain special states. Future work can identify a class of resource states that saturate the error bound using two-PPT-extendibility constraints, e.g., states that are PPT but two-unextendible. Such a class of states can offer insight not only in the study of teleportation protocols, but also to entanglement of states and channels.

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**Appendix A: Proof of Equation (12)**

We provide a proof of (12) here. Consider that

\[
\begin{align*}
\text{Tr}_{B_1'} &\circ \mathcal{N}_{AB_1 \rightarrow A'B_1'} \\
&= \text{Tr}_{B_1'B_2'} \circ (\mathcal{N}_{AB_1 \rightarrow A'B_1'} \otimes \mathcal{P}_{B_2'}^\pi) \\
&= \text{Tr}_{B_1'B_2'} \circ \mathcal{M}_{AB_1B_2 \rightarrow A'B_1'B_2'} \circ \mathcal{P}_{B_2'}^\pi \\
&= \text{Tr}_{B_1'B_2'} \circ \mathcal{F}_{B_1'B_2'} \circ \mathcal{M}_{AB_1B_2 \rightarrow A'B_1'B_2'} \circ \mathcal{F}_{B_2'} \circ \mathcal{P}_{B_2'}^\pi \\
&= \text{Tr}_{B_1'B_2'} \circ \mathcal{M}_{AB_1B_2 \rightarrow A'B_1'B_2'} \circ \mathcal{P}_{B_2'}^\pi \circ \text{id}_{B_1 \rightarrow B_2} \\
&= \text{Tr}_{B_1'} \circ \mathcal{N}_{AB_1 \rightarrow A'B_1'} \circ \mathcal{P}_{B_2'}^\pi \circ \text{id}_{B_1 \rightarrow B_2} \\
&= \text{Tr}_{B_1'} \circ \mathcal{N}_{AB_1 \rightarrow A'B_1'} \circ R_{B_1'}^\pi \ .
\end{align*}
\]

The first equality follows because \( \mathcal{P}_{B_2'}^\pi \) is a preparation channel that prepares the maximally mixed state \( \pi_{B_2} \) on system \( B_2 \), and then we trace it out. The second equality follows by using the non-signaling property in (11). The third equality follows from permutation covariance of the channel \( \mathcal{M}_{AB_1B_2 \rightarrow A'B_1'B_2'} \) (i.e., the assumption that (10) holds). The fourth equality follows because \( \mathcal{F}_{B_1'B_2'} \) is a unitary channel, so that

\[
\text{Tr}_{B_1'B_2'} \circ \mathcal{F}_{B_1'B_2'} = \text{Tr}_{B_1'B_2'} .
\]

Additionally, we used the fact that

\[
\mathcal{F}_{B_1B_2} \circ \mathcal{P}_{B_1}^\pi = \mathcal{P}_{B_1}^\pi \circ \text{id}_{B_1 \rightarrow B_2},
\]

where \( \text{id}_{B_1 \rightarrow B_2} \) is an identity channel that transforms system \( B_1 \) to \( B_2 \). The fifth equality again invokes the non-signaling property in (11). The last equality follows because

\[
\mathcal{P}_{B_1}^\pi \circ \text{id}_{B_2 \rightarrow B_{2'}} = R_{B_1'}^\pi.
\]

That is, \( \text{Tr}_{B_1} \circ \text{id}_{B_2 \rightarrow B_{2'}} \) is equivalent to \( \text{Tr}_{B_1'} \), so that this action combined with \( \mathcal{P}_{B_1}^\pi \) realizes a replacer channel.

**Appendix B: Proof of Proposition 1**

The proof given here is similar to that of the proof of Proposition 1 in [SW20]. We include it for completeness.
1. Exploiting unitary covariance symmetry of the identity channel

The main idea behind this proof is to simplify the optimization problems in (49) and (58) by exploiting the symmetries of the identity channel, as stated in (65). To begin, let us note that the Choi operator of the identity channel is simply $\Gamma_{AB}$, where

$$\Gamma_{AB} := \sum_{i,j} |i\rangle\langle j|_{A} \otimes |i\rangle\langle j|_{B}. \tag{B1}$$

and $\{|i\rangle_{A}\}$ and $\{|i\rangle_{B}\}$ are orthonormal bases. Defining the unitary channel $\mathcal{U}(\cdot) = U(\cdot)U^\dagger$, let us recall from (65) the following covariance property of the identity channel:

$$\text{id}_{A\rightarrow B}^{B} = \mathcal{U}_{B} \circ \text{id}_{A\rightarrow B}^{A} \circ \mathcal{U}_{A}, \tag{B2}$$

which holds for every unitary channel $\mathcal{U}$. Let $\mathcal{R}_{AB}^{p}$ denote the channel that appends the bipartite state $\rho_{AB}$ to its input:

$$\mathcal{R}_{AB}^{p}(\omega_{A}) = \omega_{A} \otimes \rho_{AB}. \tag{B3}$$

Let us begin our analysis with the diamond distance, but we note here that all of the reasoning employed in the first part of our proof applies to the channel infidelity error measure as well. Let $\mathcal{L}_{A\hat{A}B\rightarrow B}$ be an arbitrary one-way LOCC channel to consider for the optimization problem in (49). Exploiting the unitary invariance of the diamond distance with respect to input and output unitaries [Wat18, Proposition 3.44], we find that

$$\left\| \mathcal{L}_{A\hat{A}B\rightarrow B} \circ \mathcal{R}_{AB}^{p} - \text{id}_{A\rightarrow B}^{d} \right\|_{\diamond}$$

$$= \left\| \mathcal{U}_{B} \circ \mathcal{L}_{A\hat{A}B\rightarrow B} \circ \mathcal{R}_{AB}^{p} \circ \mathcal{U}_{A} - \mathcal{U}_{B} \circ \text{id}_{A\rightarrow B}^{A} \circ \mathcal{U}_{A} \right\|_{\diamond}$$

$$= \left\| \mathcal{U}_{B} \circ \mathcal{L}_{A\hat{A}B\rightarrow B} \circ \mathcal{U}_{A} \circ \mathcal{R}_{AB}^{p} - \text{id}_{A\rightarrow B}^{d} \right\|_{\diamond}. \tag{B4}$$

Thus, the channels $\mathcal{L}_{A\hat{A}B\rightarrow B}$ and $\mathcal{U}_{B} \circ \mathcal{L}_{A\hat{A}B\rightarrow B} \circ \mathcal{U}_{A}$ perform equally well for the optimization. Now let us exploit the convexity of diamond distance with respect to one of the channels [Wat18], as well as the Haar probability measure, to conclude that

$$\left\| \mathcal{L}_{A\hat{A}B\rightarrow B} \circ \mathcal{R}_{AB}^{p} - \text{id}_{A\rightarrow B}^{d} \right\|_{\diamond}$$

$$= \int dU \left\| \mathcal{U}_{B} \circ \mathcal{L}_{A\hat{A}B\rightarrow B} \circ \mathcal{U}_{A} \circ \mathcal{R}_{AB}^{p} - \text{id}_{A\rightarrow B}^{d} \right\|_{\diamond} \tag{B5}$$

$$\geq \int \tilde{\mathcal{L}}_{A\hat{A}B\rightarrow B} \circ \mathcal{R}_{AB}^{p} - \text{id}_{A\rightarrow B}^{d} \right\|_{\diamond}, \tag{B6}$$

where

$$\tilde{\mathcal{L}}_{A\hat{A}B\rightarrow B} := \int dU \mathcal{U}_{B} \circ \mathcal{L}_{A\hat{A}B\rightarrow B} \circ \mathcal{U}_{A}. \tag{B7}$$

Thus, we conclude that it suffices to optimize (49) over one-way LOCC channels that possess this symmetry. It is important to note that the channel swirl in (B7) can be realized by one-way LOCC, so that $\tilde{\mathcal{L}}_{A\hat{A}B\rightarrow B}$ is a one-way LOCC channel.

Let us determine the form of one-way LOCC channels that possess this symmetry. Let $\mathcal{L}_{A\hat{A}B\rightarrow B}$ denote the Choi operator of the channel $\mathcal{L}_{A\hat{A}B\rightarrow B}$. Then the Choi operator $\tilde{\mathcal{L}}_{A\hat{A}B\rightarrow B}$ of the twirled channel $\mathcal{L}_{A\hat{A}B\rightarrow B}$ is as follows:

$$\tilde{\mathcal{L}}_{A\hat{A}B\rightarrow B} = \int dU \left( \mathcal{U}_{A} \otimes \overline{\mathcal{U}}_{B} \right) (\mathcal{L}_{A\hat{A}B\rightarrow B}). \tag{B8}$$

where $\overline{\mathcal{U}}(\cdot) := \overline{U(\cdot)U^\dagger}$, with the overbar denoting the entrywise complex conjugate. Now let us recall the following identity from [Wer89, HH99, Wat18]:

$$\tilde{T}_{CD}(X_{CD})$$

$$:= \int dU \left( \mathcal{U}_{C} \otimes \overline{\mathcal{U}}_{D} \right) (X_{CD})$$

$$= \Phi_{CD} \text{Tr}_{CD} \left[ \Phi_{CD} X_{CD} \right]$$

$$+ \frac{I_{CD} - \Phi_{CD}}{d^2 - 1} \text{Tr}_{CD} \left[ (I_{CD} - \Phi_{CD}) X_{CD} \right]. \tag{B10}$$

We can apply this identity to find that

$$\tilde{T}_{A\hat{A}B\rightarrow B} = \tilde{T}_{AB}(\mathcal{L}_{A\hat{A}B\rightarrow B})$$

$$= \Phi_{AB} \text{Tr}_{AB} \left[ \Phi_{AB} \mathcal{L}_{A\hat{A}B\rightarrow B} \right]$$

$$+ \frac{I_{AB} - \Phi_{AB}}{d^2 - 1} \text{Tr}_{AB} \left[ (I_{AB} - \Phi_{AB}) \mathcal{L}_{A\hat{A}B\rightarrow B} \right]. \tag{B11}$$

Now defining

$$K'_{\hat{A}B} := \text{Tr}_{AB} \left[ \Phi_{AB} \mathcal{L}_{A\hat{A}B\rightarrow B} \right], \tag{B13}$$

$$L'_{\hat{A}B} := \text{Tr}_{AB} \left[ (I_{AB} - \Phi_{AB}) \mathcal{L}_{A\hat{A}B\rightarrow B} \right], \tag{B14}$$

we can write

$$\tilde{T}_{A\hat{A}B\rightarrow B} = \Phi_{AB} \otimes K'_{\hat{A}B} + \frac{I_{AB} - \Phi_{AB}}{d^2 - 1} \otimes L'_{\hat{A}B}. \tag{B15}$$

Let us determine the conditions on $K'_{\hat{A}B}$ and $L'_{\hat{A}B}$ for $\tilde{T}_{A\hat{A}B\rightarrow B}$ to be the Choi operator of a channel. Consider that $\tilde{T}_{A\hat{A}B\rightarrow B}$ is a Choi operator if and only if

$$\text{Tr}_{B}(\tilde{T}_{A\hat{A}B\rightarrow B}) = I_{\hat{A}B},$$

$$\text{Tr}_{B}(\tilde{T}_{A\hat{A}B\rightarrow B}) = I_{\hat{A}B}. \tag{B17}$$

This implies that

$$K'_{\hat{A}B}, L'_{\hat{A}B} \geq 0 \tag{B18}$$

and

$$\pi_{A} \otimes K'_{\hat{A}B} + \pi_{A} \otimes L'_{\hat{A}B} = I_{\hat{A}B}, \tag{B19}$$

which is equivalent to the following condition:

$$K'_{\hat{A}B} + L'_{\hat{A}B} = d I_{\hat{A}B}. \tag{B20}$$

Let us define

$$K_{\hat{A}B} := \frac{1}{d} K'_{\hat{A}B}, \quad L_{\hat{A}B} := \frac{1}{d} L'_{\hat{A}B}. \tag{B21}$$
Then the conditions in (B18) and (B20) are equivalent to

$$K_{\hat{A}\hat{B}}, L_{\hat{A}\hat{B}} \geq 0,$$  \hspace{1cm} \text{(B22)}

$$K_{\hat{A}\hat{B}} + L_{\hat{A}\hat{B}} = I_{\hat{A}\hat{B}},$$  \hspace{1cm} \text{(B23)}

and we can write the Choi operator \(L_{\hat{A}\hat{B}}\) as

$$L_{\hat{A}\hat{B}} = \Gamma_{\hat{A}\hat{B}} \otimes K_{\hat{A}\hat{B}} + \frac{dI_{\hat{A}} - \Gamma_{\hat{A}\hat{B}}}{d^2 - 1} \otimes L_{\hat{A}\hat{B}}.$$  \hspace{1cm} \text{(B24)}

The Choi operator of the composed channel \(\mathcal{L}_{\hat{A}\hat{B}} \rightarrow B \circ \mathcal{A}_{\hat{A}\hat{B}}\) is given by

$$\text{Tr}_{\hat{A}\hat{B}}[T_{\hat{A}\hat{B}}(\rho_{\hat{A}\hat{B}})\mathcal{L}_{\hat{A}\hat{B}} \rightarrow B] = \Gamma_{\hat{A}\hat{B}} \text{Tr}[T_{\hat{A}\hat{B}}(\rho_{\hat{A}\hat{B}})K_{\hat{A}\hat{B}}] + \frac{dI_{\hat{A}} - \Gamma_{\hat{A}\hat{B}}}{d^2 - 1} \text{Tr}[T_{\hat{A}\hat{B}}(\rho_{\hat{A}\hat{B}})L_{\hat{A}\hat{B}}].$$ \hspace{1cm} \text{(B25)}

We can make the substitutions \(T_{\hat{A}\hat{B}}(K_{\hat{A}\hat{B}}) \rightarrow K_{\hat{A}\hat{B}}\) and \(T_{\hat{A}\hat{B}}(L_{\hat{A}\hat{B}}) \rightarrow L_{\hat{A}\hat{B}}\) without changing the optimization problem because the conditions in (B22) and (B23) hold if and only if the following conditions hold

$$T_{\hat{A}\hat{B}}(K_{\hat{A}\hat{B}}), T_{\hat{A}\hat{B}}(L_{\hat{A}\hat{B}}) \geq 0,$$  \hspace{1cm} \text{(B27)}

$$T_{\hat{A}\hat{B}}(K_{\hat{A}\hat{B}}) + T_{\hat{A}\hat{B}}(L_{\hat{A}\hat{B}}) = I_{\hat{A}\hat{B}}.$$  \hspace{1cm} \text{(B28)}

Additionally, the channel \(\mathcal{L}_{\hat{A}\hat{B}} \rightarrow B\) remains a one-way LOCC channel under these substitutions. Thus, we can take the Choi operator of the composed channel \(\mathcal{L}_{\hat{A}\hat{B}} \rightarrow B \circ \mathcal{A}_{\hat{A}\hat{B}}\) to be

$$\Gamma_{\hat{A}\hat{B}} \text{Tr}[K_{\hat{A}\hat{B}} \rho_{\hat{A}\hat{B}}] + \frac{dI_{\hat{A}} - \Gamma_{\hat{A}\hat{B}}}{d^2 - 1} \text{Tr}[L_{\hat{A}\hat{B}} \rho_{\hat{A}\hat{B}}],$$  \hspace{1cm} \text{(B29)}

which corresponds to the following channel:

$$\sigma \rightarrow \text{Tr}[\rho_{\hat{A}\hat{B}} K_{\hat{A}\hat{B}}] \frac{d^2 - 1}{d^2 - 1} \sum_{(z,x) \neq (0,0)} W_{z,x} \sigma(W_{z,x})^\dagger,$$  \hspace{1cm} \text{(B30)}

which follows because

$$dI_{\hat{A}\hat{B}} - \Gamma_{\hat{A}\hat{B}} = \sum_{(z,x) \neq (0,0)} W_{z,B} \Gamma_{A}(W_{z,B})^{\dagger}.$$  \hspace{1cm} \text{(B31)}

Let us now incorporate the one-way LOCC constraint and justify the claim in (70). The general form of a one-way LOCC channel \(\mathcal{L}_{\hat{A}\hat{B}} \rightarrow B\), with forward classical communication from Alice to Bob, is as follows:

$$\mathcal{L}_{\hat{A}\hat{B}} \rightarrow B(\omega_{\hat{A}\hat{B}}) = \sum_{x} \mathcal{D}_{x} \rightarrow B(\text{Tr}_{A}[\Lambda_{x} \hat{A} \hat{B} \omega_{\hat{A}\hat{B}}]),$$  \hspace{1cm} \text{(B32)}

where \(\{\Lambda_{x}\}_{x}\) is a POVM, satisfying \(\Lambda_{x} \geq 0\) for all \(x\) and \(\sum_{x} \Lambda_{x} = I_{\hat{A} \hat{B}}\), and \(\{\mathcal{D}_{x} \rightarrow B\}_{x}\) is a set of quantum channels. The Choi operator of such a channel has the form

$$\mathcal{L}_{\hat{A}\hat{B}} \rightarrow B.$$  \hspace{1cm} \text{(B33)}

where \(\mathcal{D}_{x} \rightarrow B\) is the Choi operator of the channel \(\mathcal{D}_{x} \rightarrow B\). Since the conditions for being a POVM are invariant under a full transpose and since we are performing an optimization over all one-way LOCC channels, we can take the Choi operator to be as follows without loss of generality:

$$\Gamma_{\hat{A}\hat{B}} \rightarrow B = \sum_{x} \Lambda_{x} \hat{A} \hat{B} \otimes \mathcal{D}_{x} \rightarrow B.$$  \hspace{1cm} \text{(B34)}

After performing a bilateral twirl of the systems \(A\) and \(B\), the Choi operator becomes

$$\text{Tr}_{AB} \left[ \sum_{x} \Lambda_{x} \hat{A} \hat{B} \otimes \mathcal{D}_{x} \rightarrow B \right] = \Phi_{AB} \text{Tr}_{AB} \left[ \Phi_{AB} \sum_{x} \Lambda_{x} \hat{A} \hat{B} \otimes \mathcal{D}_{x} \rightarrow B \right].$$  \hspace{1cm} \text{(B35)}

Now consider that

$$\text{Tr}_{AB} \left[ \Phi_{AB} \sum_{x} \Lambda_{x} \hat{A} \hat{B} \otimes \mathcal{D}_{x} \rightarrow B \right] = \frac{1}{d} \sum_{x} \text{Tr}_{B} \left[ \mathcal{D}_{x} \rightarrow B \right].$$  \hspace{1cm} \text{(B36)}

Revisiting (B15), this implies that we can take

$$K_{\hat{A}\hat{B}} = \frac{1}{d} \sum_{x} \text{Tr}_{A} \left[ \mathcal{D}_{x} \rightarrow B \right].$$  \hspace{1cm} \text{(B47)}

After a rescaling as in (B21), we set

$$K_{\hat{A}\hat{B}} = \frac{d}{d} \sum_{x} \text{Tr}_{B} \left[ \mathcal{D}_{x} \rightarrow B \right].$$  \hspace{1cm} \text{(B48)}
proceed through the rest of the steps given in (B22)–(B30), and arrive at the claim in (70).

2. Evaluating normalized diamond distance

With the reasoning from the previous section, we have reduced the optimization problem in (49), for simulating the identity channel, to the following:

\[ e_{WL}(\rho_{AB}) = \inf_{\mathcal{L}_{A\hat{A}\hat{B}}\rightarrow B} \frac{1}{2} \left\| \mathcal{L}_{A\hat{A}\hat{B}} \circ \mathcal{J}_{AB}^d - \text{id}_{A-B}^d \right\|, \quad \text{(B47)} \]

subject to

\[ \mathcal{L}_{A\hat{A}\hat{B}}(\sigma_A \otimes \rho_{AB}) = \text{Tr} [\rho_{AB} K_{\hat{A}B}^d (\sigma_A)] + \text{Tr} [\rho_{AB} L_{\hat{A}B}^d (\sigma_A)] \]
\[ + \frac{1}{d^2 - 1} \sum_{(z,x) \neq (0,0)} W_{z,x} \sigma_A (W_{z,x}^*)^\dagger, \quad \text{(B48)} \]

and there existing a POVM \( \{ \Lambda_{AB}^x \} \) and a set \( \{ D_{AB}^x \} \) of channels such that

\[ K_{\hat{A}B} = \frac{1}{d^2} \sum_x \text{Tr}_B [\Lambda_{AB}^x \Gamma_{D_{AB}^x}], \quad \text{(B49)} \]

where \( \Gamma_{D_{AB}^x} \) is the Choi operator of the channel \( D_{AB}^x \). That is, there is no need to optimize over all one-way LOCC channels, but only those satisfying the constraints in (B48) and (B49).

We now exploit the form of the optimization of the normalized diamond distance from (46), in order to rewrite the optimization problem as

\[ \inf_{\mu, Z_{AB}, K_{\hat{A}B}, L_{\hat{A}B} \geq 0} \mu, \quad \text{(B50)} \]

subject to

\[ \mu I_A \geq Z_A, \quad (B51) \]
\[ Z_{AB} \geq \Gamma_{AB} (1 - \text{Tr} [K_{\hat{A}B} \rho_{AB}]), \quad \text{(B52)} \]
\[ K_{\hat{A}B} + L_{\hat{A}B} = I_{\hat{A}B}. \quad \text{(B53)} \]

with \( K_{\hat{A}B} \) further subject to having the form in (B49). Since we are minimizing \( \mu \) with respect to \( \mu \) and \( Z_{AB} \), we can choose \( Z_{AB} \) to be the smallest positive semi-definite operator such that the operator inequality in (B52) holds. This smallest positive semi-definite operator is the positive part of the operator on the right-hand side of the inequality, which is given by

\[ \Gamma_{AB} (1 - \text{Tr} [K_{\hat{A}B} \rho_{AB}]). \quad \text{(B54)} \]

This follows because \( \Gamma_{AB} \) and \( \frac{dI_{AB} - \Gamma_{AB}}{d^2 - 1} \text{Tr} [L_{\hat{A}B} \rho_{AB}] \) are each positive semi-definite and orthogonal to each other. Thus, an optimal solution is

\[ Z_{AB} = \Gamma_{AB} (1 - \text{Tr} [K_{\hat{A}B} \rho_{AB}]), \quad \text{(B55)} \]

for which the smallest \( \mu \) possible is

\[ \mu = 1 - \text{Tr} [K_{\hat{A}B} \rho_{AB}], \quad \text{(B56)} \]

because

\[ Z_A = \text{Tr}_B [Z_{AB}], \quad \text{(B57)} \]
\[ = \text{Tr}_B [\Gamma_{AB} (1 - \text{Tr} [K_{\hat{A}B} \rho_{AB}])], \quad \text{(B58)} \]
\[ = I_A (1 - \text{Tr} [K_{\hat{A}B} \rho_{AB}]). \quad \text{(B59)} \]

We then conclude that

\[ e_{WL}(\rho_{AB}) = 1 - \sup_{K_{\hat{A}B}, L_{\hat{A}B} \geq 0} \text{Tr} [K_{\hat{A}B} \rho_{AB}], \quad \text{(B60)} \]

subject to \( K_{\hat{A}B} + L_{\hat{A}B} = I_{\hat{A}B} \) and the following channel \( \mathcal{L}_{A\hat{A}\hat{B}}\rightarrow B \) being a one-way LOCC channel:

\[ \mathcal{L}_{A\hat{A}\hat{B}}(\omega_{\hat{A}B}) = \text{id}_{A-B} (\mathcal{T}_{AB} [K_{\hat{A}B} \omega_{\hat{A}B}] + D_{A-B} (\mathcal{T}_{\hat{A}B} [L_{\hat{A}B} \omega_{\hat{A}B}])). \quad \text{(B61)} \]

As indicated previously, this latter constraint is equivalent to (B49).

3. Evaluating channel infidelity

Let us recall the symmetries of the identity channel in (B2), which implies the following symmetry for its Choi operator:

\[ \Gamma_{AB} = (\mathcal{U}_A \otimes \mathcal{U}_B) (\Gamma_{AB}), \quad \text{(B62)} \]

for every unitary channel \( \mathcal{U} (\cdot) = U (\cdot) U^\dagger \). This implies that

\[ \Gamma_{AB} = \int dU (\mathcal{U}_A \otimes \mathcal{U}_B) (\Gamma_{AB}). \quad \text{(B63)} \]

Now applying the semi-definite program in (53) for channel infidelity, we find that

\[ e_{WL}(\rho_{AB}) = 1 - \left[ \sup_{\lambda \geq 0, L_{\hat{A}A\hat{B}}} \left( \lambda \left( I_{\hat{A}A\hat{B}} - \text{Tr} [\mathcal{T}_{\hat{A}B} (\mathcal{Q}_{\hat{A}B} - \lambda I_{\hat{A}B}) L_{\hat{A}A\hat{B}}] \right) \right) \right]^2, \quad \text{(B64)} \]

subject to

\[ \lambda I_A \leq \text{Re} [\text{Tr}_B [Q_{AB}]], \quad \text{(B65)} \]
\[ \text{Tr}_B [L_{\hat{A}A\hat{B}}] = I_{\hat{A}A\hat{B}}, \quad \text{(B66)} \]
\[ \begin{align*}
\Gamma_{AB} & \quad (1 - \text{Tr} [K_{\hat{A}B} \rho_{AB}]), \\
Q_{AB} & \quad \text{Tr}_{\hat{A}B} [\mathcal{T}_{\hat{A}B} (\rho_{\hat{A}B}) L_{\hat{A}A\hat{B}}]], \\
\end{align*} \quad \text{(B67)} \]

and \( L_{\hat{A}A\hat{B}} \) is the Choi operator for a one-way LOCC channel. Note that the last constraint is equivalent to

\[ |0\rangle \langle 0| \otimes \Gamma_{AB} + |0\rangle \langle 1| \otimes Q_{\hat{A}B}^\dagger + |1\rangle \langle 0| \otimes Q_{AB} + |1\rangle \langle 1| \otimes \text{Tr}_{\hat{A}B} [\mathcal{T}_{\hat{A}B} (\rho_{\hat{A}B}) L_{\hat{A}A\hat{B}}] \geq 0. \quad \text{(B68)} \]
Let $\lambda$, $L_{AABB}$, and $Q_{AB}$ be an optimal solution. Then it follows that $\lambda, \left(\overline{U}_A \otimes \overline{U}_B\right) (L_{AABB})$, and $\left(\overline{U}_A \otimes \overline{U}_B\right) (Q_{AB})$ is an optimal solution also. This follows because all of the constraints are satisfied for these choices while still obtaining the same optimal value. To see this, consider that

$$
\lambda A \leq \text{Re}[\text{Tr}_B(Q_{AB})]
$$

(B69)

$$
\Leftrightarrow \lambda A \leq \overline{U}_A (\text{Re}[\text{Tr}_B(Q_{AB})])
$$

(B70)

$$
\Leftrightarrow \lambda A \leq \text{Re}[\text{Tr}_B(\overline{U}_A Q_{AB})]
$$

(B71)

$$
\Leftrightarrow \lambda A \leq \text{Re}[\text{Tr}_B((\overline{U}_A \otimes \overline{U}_B)(Q_{AB}))]
$$

(B72)

$$
\text{Tr}_B[L_{AABB}] = I_A\hat{A}_B
$$

(B73)

$$
\text{Tr}_B[(\overline{U}_A \otimes \overline{U}_B)L_{AABB}] = I_A\hat{A}_B.
$$

(B74)

and

$$
|0\rangle\langle 0| \otimes \Gamma_{AB} + |0\rangle\langle 1| \otimes Q_{AB} + |1\rangle\langle 0| \otimes Q_{AB} + |1\rangle\langle 1| \otimes \text{Tr}_A\hat{B} (T_{\hat{A}}(\overline{U}_A \otimes \overline{U}_B)L_{AABB}) \geq 0
$$

(B75)

$$
\Leftrightarrow (\text{id} \otimes (\overline{U}_A \otimes \overline{U}_B))(|0\rangle\langle 0| \otimes \Gamma_{AB} + |0\rangle\langle 1| \otimes Q_{AB} + |1\rangle\langle 0| \otimes Q_{AB} + |1\rangle\langle 1| \otimes \text{Tr}_A\hat{B} (T_{\hat{A}}(\overline{U}_A \otimes \overline{U}_B)L_{AABB})) \geq 0
$$

(B76)

$$
\Leftrightarrow |0\rangle\langle 0| \otimes (\overline{U}_A \otimes \overline{U}_B)(\Gamma_{AB} + |0\rangle\langle 1| \otimes (\overline{U}_A \otimes \overline{U}_B)(Q_{AB}) + |1\rangle\langle 0| \otimes (\overline{U}_A \otimes \overline{U}_B)(Q_{AB}) + |1\rangle\langle 1| \otimes (\text{Tr}_A\hat{B} (T_{\hat{A}}(\overline{U}_A \otimes \overline{U}_B)L_{AABB}))) \geq 0.
$$

(B77)

Note that $(\overline{U}_A \otimes \overline{U}_B)(L_{AABB})$ is the Choi operator for a one-way LOCC channel if $L_{AABB}$ is. Also, due to the fact that the objective function is linear and the constraints are linear operator inequalities, it follows that convex combinations of solutions are solutions as well. So this implies that if $\lambda, L_{AABB}$, and $Q_{AB}$ is an optimal solution, then so is $\lambda$.

$$
\tilde{L}_{AABB} = \int dU (\overline{U}_A \otimes \overline{U}_B)(L_{AABB}),
$$

(B79)

$$
\tilde{Q}_{AB} = \int dU (\overline{U}_A \otimes \overline{U}_B)(Q_{AB}).
$$

(B80)

Additionally, $\tilde{L}_{AABB}$ is the Choi operator for a one-way LOCC channel if $L_{AABB}$ is. As argued in Appendix B 1, $\tilde{L}_{AABB}$ has a simpler form as

$$
\tilde{L}_{AABB} = \Gamma_{AB} \text{Tr}[K_{\hat{A}\hat{B}}(\overline{U}_A \otimes \overline{U}_B)] + \frac{dL_{AB} - \Gamma_{AB}}{d^2 - 1} \text{Tr}[L_{\hat{A}\hat{B}}(\overline{U}_A \otimes \overline{U}_B)],
$$

(B81)

with the constraints in (B16)–(B17) on $\tilde{L}_{AABB}$ simplifying to

$$
K_{\hat{A}\hat{B}}, L_{\hat{A}\hat{B}} \geq 0,
$$

(B82)

$$
K_{\hat{A}\hat{B}} + L_{\hat{A}\hat{B}} = I_{\hat{A}\hat{B}},
$$

(B83)

such that $K_{\hat{A}\hat{B}}$ and $L_{\hat{A}\hat{B}}$ are measurement operators for a one-way LOCC channel (i.e., satisfy (B49)). We also find that $\tilde{Q}_{AB}$ simplifies to

$$
\tilde{Q}_{AB} = q_1 \Gamma_{AB} + q_2 \frac{dL_{AB} - \Gamma_{AB}}{d^2 - 1},
$$

(B84)

where $q_1, q_2 \in \mathbb{C}$. The constraint in (B57) reduces to

$$
\lambda \leq \text{Re}[q_1 + q_2],
$$

(B85)

because

$$
\text{Tr}_B(\tilde{Q}_{AB}) = (q_1 + q_2) I_A.
$$

(B86)

The constraint in (B68) reduces to

$$
|0\rangle\langle 0| \otimes \Gamma_{AB} + |0\rangle\langle 1| \otimes \left[q_1 \Gamma_{AB} + q_2 \frac{dL_{AB} - \Gamma_{AB}}{d^2 - 1}\right] + |1\rangle\langle 0| \otimes \left[q_1 \Gamma_{AB} + q_2 \frac{dL_{AB} - \Gamma_{AB}}{d^2 - 1}\right] + |1\rangle\langle 1| \otimes \left[q_1 \Gamma_{AB} + q_2 \frac{dL_{AB} - \Gamma_{AB}}{d^2 - 1}\right] + |1\rangle\langle 1| \otimes \left[q_1 \Gamma_{AB} + q_2 \frac{dL_{AB} - \Gamma_{AB}}{d^2 - 1}\right]
$$

(B87)

Exploiting the orthogonality of the operators $\Gamma_{AB}$ and $\frac{dL_{AB} - \Gamma_{AB}}{d^2 - 1}$, we conclude that the single constraint above is equivalent to the following two constraints:

$$
|0\rangle\langle 0| + q_1^* |0\rangle\langle 1| + q_1 |1\rangle\langle 0| + \text{Tr}[K_{\hat{A}\hat{B}}(\overline{U}_A \otimes \overline{U}_B)L_{AABB}] \geq 0.
$$

(B88)

$$
q_2^* |0\rangle\langle 1| + q_2 |1\rangle\langle 0| + \text{Tr}[L_{\hat{A}\hat{B}}(\overline{U}_A \otimes \overline{U}_B)L_{AABB}] \geq 0.
$$

(B89)

We can rewrite these as the following two matrix inequalities:

$$
\begin{bmatrix}
1 & q_1^* & 0 & q_2^* \\
q_1 & \text{Tr}[K_{\hat{A}\hat{B}}(\overline{U}_A \otimes \overline{U}_B)L_{AABB}] & 0 & \text{Tr}[L_{\hat{A}\hat{B}}(\overline{U}_A \otimes \overline{U}_B)L_{AABB}] \\
0 & 0 & 1 & q_2 \\
q_2 & \text{Tr}[L_{\hat{A}\hat{B}}(\overline{U}_A \otimes \overline{U}_B)L_{AABB}] & q_2 & 0
\end{bmatrix} \geq 0.
$$

(B90)

Since $\text{Tr}[K_{\hat{A}\hat{B}}(\overline{U}_A \otimes \overline{U}_B)L_{AABB}]$ and $\text{Tr}[L_{\hat{A}\hat{B}}(\overline{U}_A \otimes \overline{U}_B)L_{AABB}]$ are non-negative, we know that the inequalities above hold if and only if

$$
\text{Tr}[K_{\hat{A}\hat{B}}(\overline{U}_A \otimes \overline{U}_B)L_{AABB}] \geq |q_1|^2, \quad q_2 = 0.
$$

(B92)

As a consequence, the optimization problem in (B64) becomes

$$
1 - \sup_{\lambda \geq 0, K_{\hat{A}\hat{B}}, L_{\hat{A}\hat{B}} \geq 0, q_1 \in \mathbb{C}} |\lambda|^2,
$$

(B93)

subject to

$$
\text{Tr}[K_{\hat{A}\hat{B}}(\overline{U}_A \otimes \overline{U}_B)L_{AABB}] \geq |q_1|^2,
$$

(B94)

$$
\lambda \leq \text{Re}[q_1],
$$

(B95)
\[ K_{\hat{A}\hat{B}} + L_{\hat{A}\hat{B}} = I_{\hat{A}\hat{B}}, \] (B96)

and the following channel being one-way LOCC:

\[ L_{\hat{A}\hat{B}\rightarrow B}(\omega_{\hat{A}\hat{B}}) = \text{id}_{A-B}(\text{Tr}_{\hat{A}\hat{B}}[K_{\hat{A}\hat{B}}(\omega_{\hat{A}\hat{B}})]) + D_{A-B}(\text{Tr}_{\hat{A}\hat{B}}[L_{\hat{A}\hat{B}}(\omega_{\hat{A}\hat{B}})]). \] (B97)

As indicated previously (see (B49)), \( L_{\hat{A}\hat{B}\rightarrow B} \) is one-way LOCC if there exists a POVM \( \{\Lambda_{B\hat{A}}^{x}\} \) and a set \( \{D_{B\rightarrow B}^{x}\} \) of channels such that

\[ K_{\hat{A}\hat{B}} = \frac{1}{d^2} \sum_{x} \text{Tr}_{B}[\Lambda_{B\hat{A}}^{x} \Gamma_{B\hat{B}}^{D_{x}}], \] (B98)

where \( \Gamma_{B\hat{B}}^{D_{x}} \) is the Choi operator of the channel \( D_{B\rightarrow B}^{x} \). Since we are trying to maximize \( \lambda \) subject to these constraints, we should then set \( \lambda = q_{1} = \sqrt{\text{Tr}[K_{\hat{A}\hat{B}}]}. \) This concludes the proof.

Appendix C: Proof of Proposition 3

Some of the reasoning here is similar conceptually to that given in Appendix B, but the details in many places are rather different. We divide our proof into the subsections given below.

1. Twirled two-extendible channel is optimal

We begin by considering two-extendible channels exclusively and then later bring in the PPT constraints. Our first goal is to prove that a two-extendible channel of the form in (C16), with extension channel in (C17), is optimal for performing the minimization in (72) if \( N_{A-B} \) is the identity channel \( \text{id}_{A-B}^{d} \). To this end, let \( \mathcal{K}_{\hat{A}\hat{B}\rightarrow B} \) be an arbitrary two-extendible channel, meaning that there exists an extension channel \( M_{\hat{A}\hat{B}_{1}\hat{B}_{2}\rightarrow B_{1}B_{2}} \) satisfying the constraints in (10)–(11), i.e.,

\[ \mathcal{F}_{B_{1}B_{2}} \circ M_{\hat{A}\hat{B}_{1}\hat{B}_{2},B_{1}B_{2}} = M_{\hat{A}\hat{B}_{1}B_{2}-B_{1}B_{2}} \circ \mathcal{F}_{B_{1}B_{2}} \circ \mathcal{K}_{\hat{A}\hat{B}_{1}\rightarrow B_{1}}, \quad \text{Tr}_{B_{1}} \circ M_{\hat{A}\hat{B}_{1},B_{1}B_{2}} = \mathcal{K}_{\hat{A}\hat{B}_{1}\rightarrow B_{1}} \circ \text{Tr}_{B_{1}}. \] (C1)

For an arbitrary unitary channel \( \mathcal{U} \), it then follows that the channel

\[ \mathcal{K}_{\hat{A}\hat{B}\rightarrow B}^{\mathcal{U}} = \mathcal{U}_{B} \circ \mathcal{K}_{\hat{A}\hat{B}\rightarrow B} \circ \mathcal{U}_{A} \] (C3)

is two-extendible with extension channel

\[ M_{\hat{A}\hat{B}_{1}\hat{B}_{2},B_{1}B_{2}} = (\mathcal{U}_{B}^{x} \circ \mathcal{U}_{B}) \circ \mathcal{M}_{\hat{A}\hat{B}_{1}\hat{B}_{2},B_{1}B_{2}} \circ \mathcal{U}_{A} \] (C4)

and achieves the same value of the normalized diamond distance that \( \mathcal{K}_{\hat{A}\hat{B}\rightarrow B} \) does. That is,

\[ \left\| \mathcal{K}_{\hat{A}\hat{B}\rightarrow B} \circ \mathcal{A}_{\hat{A}\hat{B}}^{d} \circ \text{id}_{A-B}^{d} \right\|_{\diamond} \]

\[ = \left\| \mathcal{K}_{\hat{A}\hat{B}_{1}\hat{B}_{2},B_{1}B_{2}} \circ \mathcal{U}_{B} \circ \mathcal{K}_{\hat{A}\hat{B}_{1}\rightarrow B_{1}} \circ \text{id}_{A-B}^{d} \right\|_{\diamond}. \] (C5)

This equality follows by the same reasoning used to justify (B4). The claim that \( \mathcal{K}_{\hat{A}\hat{B}\rightarrow B}^{\mathcal{U}} \) is two-extendible follows because the extension channel \( M_{\hat{A}\hat{B}_{1}\hat{B}_{2},B_{1}B_{2}} \) satisfies

\[ \mathcal{F}_{B_{1}B_{2}} \circ M_{\hat{A}\hat{B}_{1}\hat{B}_{2},B_{1}B_{2}} = \mathcal{F}_{B_{1}B_{2}} \circ (\mathcal{U}_{B_{1}}^{x} \circ \mathcal{U}_{B_{2}}^{x}) \circ \mathcal{M}_{\hat{A}\hat{B}_{1}B_{2}-B_{1}B_{2}} \circ \mathcal{U}_{A} \] (C6)

\[ = \mathcal{F}_{B_{1}B_{2}} \circ (\mathcal{U}_{B_{1}}^{x} \circ \mathcal{U}_{B_{2}}^{x}) \circ \mathcal{F}_{B_{1}B_{2}} \circ \mathcal{F}_{B_{1}B_{2}} \circ \mathcal{M}_{\hat{A}\hat{B}_{1}B_{2}-B_{1}B_{2}} \circ \mathcal{U}_{A} \] (C7)

\[ = \mathcal{F}_{B_{1}B_{2}} \circ \mathcal{M}_{\hat{A}\hat{B}_{1}B_{2}-B_{1}B_{2}} \circ \mathcal{U}_{A} \] (C8)

\[ = \mathcal{F}_{B_{1}B_{2}} \circ \mathcal{M}_{\hat{A}\hat{B}_{1}B_{2}-B_{1}B_{2}} \circ \mathcal{F}_{B_{1}B_{2}} \circ \mathcal{U}_{A} \] (C9)

\[ = \mathcal{M}_{\hat{A}\hat{B}_{1}B_{2}-B_{1}B_{2}} \circ \mathcal{F}_{B_{1}B_{2}} \circ \mathcal{U}_{A} \] (C10)

Additionally,

\[ \text{Tr}_{B_{1}} \circ M_{\hat{A}\hat{B}_{1}B_{2}-B_{1}B_{2}} = \text{Tr}_{B_{1}} \circ (\mathcal{U}_{B_{1}}^{x} \circ \mathcal{U}_{B_{2}}^{x}) \circ \mathcal{M}_{\hat{A}\hat{B}_{1}B_{2}-B_{1}B_{2}} \circ \mathcal{U}_{A} \] (C11)

\[ = \mathcal{U}_{B_{1}}^{x} \circ \text{Tr}_{B_{2}} \circ \mathcal{M}_{\hat{A}\hat{B}_{1}B_{2}-B_{1}B_{2}} \circ \mathcal{U}_{A} \] (C12)

\[ = \mathcal{U}_{B_{1}}^{x} \circ \mathcal{K}_{\hat{A}\hat{B}_{1}\rightarrow B_{1}} \circ \text{Tr}_{B_{2}} \circ \mathcal{U}_{A} \] (C13)

\[ = \mathcal{K}_{\hat{A}\hat{B}_{1}\rightarrow B_{1}} \circ \text{Tr}_{B_{2}} \circ \mathcal{U}_{A} \] (C14)

Then consider that

\[ \left\| \mathcal{K}_{\hat{A}\hat{B}\rightarrow B} \circ \mathcal{A}_{\hat{A}\hat{B}}^{d} \circ \text{id}_{A-B}^{d} \right\|_{\diamond} \]

\[ \geq \left\| \mathcal{K}_{\hat{A}\hat{B}\rightarrow B} \circ \mathcal{A}_{\hat{A}\hat{B}}^{d} \circ \text{id}_{A-B}^{d} \right\|_{\diamond}, \] (C15)

where \( \mathcal{K}_{\hat{A}\hat{B}\rightarrow B} \) is the following twirled two-extendible channel:

\[ \mathcal{K}_{\hat{A}\hat{B}\rightarrow B} = \int dU \mathcal{K}_{\hat{A}\hat{B}\rightarrow B}^{U} \] (C16)

with extension channel

\[ \mathcal{M}_{\hat{A}\hat{B}_{1}B_{2}-B_{1}B_{2}} = \int dU \mathcal{M}_{\hat{A}\hat{B}_{1}B_{2}-B_{1}B_{2}}^{U} \] (C17)

The inequality in (C15) follows from the same reasoning given for (B6). Furthermore, \( \mathcal{K}_{\hat{A}\hat{B}\rightarrow B} \) is two-extendible with extension \( \mathcal{M}_{\hat{A}\hat{B}_{1}B_{2}-B_{1}B_{2}} \) because the extension channel \( \mathcal{M}_{\hat{A}\hat{B}_{1}B_{2}-B_{1}B_{2}} \) satisfies

\[ \mathcal{F}_{B_{1}B_{2}} \circ \mathcal{M}_{\hat{A}\hat{B}_{1}B_{2}-B_{1}B_{2}} = \mathcal{F}_{B_{1}B_{2}} \circ \int dU \mathcal{M}_{\hat{A}\hat{B}_{1}B_{2}-B_{1}B_{2}}^{U} \] (C18)

\[ = \int dU \mathcal{F}_{B_{1}B_{2}} \circ \mathcal{M}_{\hat{A}\hat{B}_{1}B_{2}-B_{1}B_{2}}^{U} \] (C19)

\[ = \int dU \mathcal{M}_{\hat{A}\hat{B}_{1}B_{2}-B_{1}B_{2}}^{U} \circ \mathcal{F}_{B_{1}B_{2}} \] (C20)

\[ = \mathcal{M}_{\hat{A}\hat{B}_{1}B_{2}-B_{1}B_{2}} \circ \mathcal{F}_{B_{1}B_{2}} \] (C21)
where

\[ \text{Tr}_{B_2} \circ \tilde{M}_{AA\hat{B}_1B_2} = \text{Tr}_{B_2} \circ \int dU \mathcal{M}^U_{AA\hat{B}_1B_2} = \int dU \text{Tr}_{B_2} \circ \mathcal{M}^U_{AA\hat{B}_1B_2} = \int dU \mathcal{K}^U_{AA\hat{B}_1B_2} \circ \text{Tr}_{B_2}, \tag{C23} \]

\[ = \tilde{K}_{AA\hat{B}_1B_2} \circ \text{Tr}_{B_2}. \tag{C25} \]

In the above, we made use of (C6)–(C10) and (C11)–(C14).

As a consequence of (C15), it follows that it suffices to minimize the following objective function

\[ \| \mathcal{K}_{AA\hat{B}_1B_2} \circ \mathcal{R}^p_{AB} - id_{A\rightarrow B} \|. \tag{C26} \]

with respect to two-extendible channels obeying the symmetries in (C16) and (C17). Thus, our goal is to characterize the form of two-extendible channels possessing the symmetries in (C16) and (C17). To this end, we consider channels of the form in (C17) that obey the following constraints:

\[ \tilde{\mathcal{T}}_{B_1B_2} \circ \tilde{M}_{AA\hat{B}_1B_2} = \tilde{M}_{AA\hat{B}_1B_2} \circ \tilde{\mathcal{T}}_{B_1B_2}, \tag{C27} \]

\[ \text{Tr}_{B_2} \circ \tilde{M}_{AA\hat{B}_1B_2} = \tilde{K}_{AA\hat{B}_1B_2} \circ \text{Tr}_{B_2}. \tag{C28} \]

Equivalently, we work with the Choi operator of such channels.

First consider that a channel obeying the following symmetry:

\[ \tilde{M}_{AA\hat{B}_1B_2} = \int dU (U'_{B_1} \otimes U'_{B_2}) \circ \tilde{M}_{AA\hat{B}_1B_2} \circ U_A, \tag{C29} \]

is equivalent to its Choi operator obeying the following symmetry:

\[ \tilde{\mathcal{M}}_{AA\hat{B}_1B_2} = \tilde{\mathcal{T}}_{AB, B_2} (\tilde{M}_{AA\hat{B}_1B_2} (B_2 B_1, B_2)), \tag{C30} \]

where

\[ \tilde{\mathcal{T}}_{AB, B_2} (\cdot) := \int dU (U_A \otimes U_{B_1} \otimes U_{B_2}) (\cdot). \tag{C31} \]

It was shown in [EW01, Section VI-A] and [JV13] that the tripartite twirling channel \( \tilde{\mathcal{T}}_{AB, B_2} \) has the following effect on an arbitrary operator \( X_{AB, B_2} \):

\[ \tilde{\mathcal{T}}_{AB, B_2} (X_{AB, B_2}) = \sum_{i \in \{+, -\}} \text{Tr}_{AB, B_2} [ S^i_{AB, B_2} X_{AB, B_2} ] \frac{S^i_{AB, B_2}}{\text{Tr}[S^i_{AB, B_2}]} \], \tag{C32} \]

where

\[ g(i) := \begin{cases} i & \text{if } i \in \{+, -\} \\ 0 & \text{if } i \in \{0, 1, 2, 3\} \end{cases}. \tag{C33} \]
while \( S_{AB1B2}^1, S_{AB1B2}^2, S_{AB1B2}^3 \) are Pauli-like operators acting on the subspace onto which \( S_{AB1B2}^0 \) projects (see [EW01, Section VI-A] and [JV13]). All \( S_{AB1B2}^0 \) operators are Hermitian and satisfy the following for their Hilbert–Schmidt inner product:

\[
\langle S_{AB1B2}^i, S_{AB1B2}^j \rangle = \text{Tr}[(S_{AB1B2}^i)^\dagger S_{AB1B2}^j] = \text{Tr}[S_{AB1B2}^i S_{AB1B2}^j] = \text{Tr}[S_{AB1B2}^{(i)} \delta_{i,j}].
\]

As a consequence of the fact that \( S_{AB1B2}^0 \) is a projector with trace equal to \( \frac{d(d-2)(d+1)}{2} \), it follows that \( S_{AB1B2}^0 = 0 \) for the case of \( d = 2 \), so that this operator and its associated operators \( M_{AB1B2}^0 \) and \( M_{AB1B2}^- \) given below are not involved for the case of \( d = 2 \) (when simulating a qubit identity channel).

By applying (C32) to (C30), we find that

\[
\tilde{M}_{AA\tilde{A}B1B2B3} = \sum_{i \in \{+,-,0,1,2,3\}} M_{AB1B2}^{qi} \otimes \frac{S_{AB1B2}^i}{\text{Tr}[S_{AB1B2}^{(i)}]},
\]

where, for \( i \in \{+,-,0,1,2,3\} \),

\[
M_{AB1B2}^{qi} := \text{Tr}_{AB1B2}[S_{AB1B2}^i \tilde{M}_{AA\tilde{A}B1B2B3}].
\]

Thus, our goal from here is to determine conditions on the \( M_{AB1B2}^{qi} \) operators such that \( M_{AA\tilde{A}B1B2B3} \) corresponds to a Choi operator for a channel \( \tilde{M}_{AA\tilde{A}B1B2B3} \to AB1B2B3 \), satisfying (C28) and (C29). These conditions are the same as those specified in (79), (82), (83), and (84), and we repeat them here for convenience:

\[
\tilde{M}_{AA\tilde{A}B1B2B3} \geq 0, \quad \text{Tr}_{B1B2}[\tilde{M}_{AA\tilde{A}B1B2B3}] = I_{AA\tilde{A}B1B2B3}.
\]

2. Complete positivity condition

We begin by considering the condition in (C58), as applied to (C56). By exploiting the previously stated facts that \( S_{AB1B2}^+, S_{AB1B2}^-, S_{AB1B2}^3 \) and \( S_{AB1B2}^0 \) are orthogonal projectors and \( S_{AB1B2}^1 \) are Pauli-like operators acting on the subspace onto which \( S_{AB1B2}^0 \) projects, it follows that (C58) holds if and only if

\[
M_{AB1B2}^+ \geq 0,
\]

\[
\sum_{i \in \{0,1,2,3\}} M_{AB1B2}^i \otimes S_{AB1B2}^i \geq 0.
\]

We can now exploit the inner product relations in (C53)–(C55) to conclude that the map

\[
X \to \sum_{i \in \{0,1,2,3\}} S_{AB1B2}^i \text{Tr}[\sigma^i X],
\]

from a qubit input to the systems \( AB1B2 \) is an isometry with inverse

\[
Y \to \sum_{i \in \{0,1,2,3\}} \sigma^i \text{Tr}[\sigma^i Y],
\]

where \( \{\sigma^i\}_i \) is the set of Pauli operators:

\[
\sigma_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

Exploiting this map, we find that (C64) holds if and only if

\[
\sum_{i \in \{0,1,2,3\}} M_{AB1B2}^i \otimes \frac{\sigma^i}{\text{Tr}[S_{AB1B2}^{(i)}]} \geq 0,
\]

the latter being equivalent to

\[
\left[ M^0 + M^3, M^1 - iM^2 \right] \geq 0, \quad M^1 + iM^2, \quad M^0 - M^3 \right] \geq 0,
\]

after defining

\[
M_{AB1B2}^i := \frac{2M_{AB1B2}^{qi}}{\text{Tr}[S_{AB1B2}^{(i)}]} = \frac{M_{AB1B2}^{qi}}{d},
\]

for \( i \in \{0,1,2,3\} \) and using the shorthand \( M^i \equiv M_{AB1B2}^i \).

Defining

\[
M_{AB1B2}^i := \frac{M_{AB1B2}^i}{d}
\]

for \( i \in \{+, -, \} \), we conclude from (C62), (C63), and (C70) that (C58) holds if and only if the following operator inequalities hold

\[
M_{AB1B2}^+ \geq 0, \quad M_{AB1B2}^- \geq 0, \quad M_{AB1B2}^3 \geq 0, \quad M_{AB1B2}^0 - M_{AB1B2}^1 \geq 0,
\]

From (C75), we conclude that \( M_0 \geq 0 \) because it implies \( M^0 + M^3 \geq 0 \) and \( M^0 - M^1 \geq 0 \), which in turn implies \( M_0 \geq 0 \). We have thus justified the constraints on \( M^+, M^-, M^0, M^1, M^2, M^3 \) in (90) and (91).
3. Trace preservation condition

We now consider the condition in (C59), as applied to (C56). Consider that

$$I_{\tilde{A}\tilde{B},\tilde{B}_2} = Tr_{B_1,B_2}\left[ \tilde{M}_{\tilde{A}\tilde{B},\tilde{B}_2,B_2} \right]$$

(C76)

$$= \sum_{i \in \{+, -, 0, 1, 2, 3\}} M^q_{\tilde{A}\tilde{B},\tilde{B}_2} \otimes \frac{Tr_{B_1,B_2}\left[ S^i_{AB,B_2,B_2} \right]}{Tr\left[ S^{i(i)}_{AB,B_2} \right]}.$$  

(C77)

Thus, we need to calculate $Tr_{B_1,B_2}\left[ S^i_{AB,B_2,B_2} \right]$ for all $i \in \{+, -, 0, 1, 2, 3\}$. To do so, we first find that

$$Tr_{B_1,B_2}\left[ S^+_{AB,B_2,B_2} \right] = \frac{1}{2} \left[ \begin{array}{c} d^2 I_A + dI_A \\ - \frac{d^2 I_A - dI_A}{d+1} \end{array} \right]$$

(C84)

$$= \frac{(d+2)(d-1)}{2} I_A.$$  

(C85)

$$Tr_{B_1,B_2}\left[ S^-_{AB,B_2,B_2} \right] = \frac{1}{2} \left[ \begin{array}{c} d^2 I_A - dI_A \\ - \frac{d^2 I_A - dI_A}{d+1} \end{array} \right]$$

(C86)

$$= \frac{(d-2)(d+1)}{2} I_A.$$  

(C87)

$$Tr_{B_1,B_2}\left[ S^0_{AB,B_2,B_2} \right] = \frac{1}{d^2 - 1} \left[ \begin{array}{c} d (dI_A + dI_A) \\ - (I_A + I_A) \end{array} \right]$$

(C88)

$$= 2I_A.$$  

(C89)

$$Tr_{B_1,B_2}\left[ S^1_{AB,B_2,B_2} \right] = \frac{1}{d^2 - 1} \left[ \begin{array}{c} d (I_A + I_A) \\ - (dI_A + dI_A) \end{array} \right]$$

(C90)

$$= 0.$$  

(C91)

$$Tr_{B_1,B_2}\left[ S^2_{AB,B_2,B_2} \right] = \frac{1}{\sqrt{d^2 - 1}} \left( dI_A - dI_A \right)$$

(C92)

$$= 0.$$  

(C93)

$$Tr_{B_1,B_2}\left[ S^3_{AB,B_2,B_2} \right] = \frac{i}{\sqrt{d^2 - 1}} (I_A - I_A)$$

(C94)

$$= 0.$$  

(C95)

Combining with (C76)–(C77), we conclude that

$$I_{\tilde{A}\tilde{B},\tilde{B}_2} = M^+_{\tilde{A}\tilde{B},\tilde{B}_2} \otimes \frac{Tr_{B_1,B_2}\left[ S^+_{AB,B_2,B_2} \right]}{Tr\left[ S^{+(+)}_{AB,B_2} \right]}$$

$$+ M^-_{\tilde{A}\tilde{B},\tilde{B}_2} \otimes \frac{Tr_{B_1,B_2}\left[ S^-_{AB,B_2,B_2} \right]}{Tr\left[ S^{-(-)}_{AB,B_2} \right]}$$

(C96)

$$= M^+_{\tilde{A}\tilde{B},\tilde{B}_2} \otimes \frac{(d+2)(d-1)}{2} I_A$$

$$+ M^-_{\tilde{A}\tilde{B},\tilde{B}_2} \otimes \frac{(d-2)(d+1)}{2} I_A$$

$$+ M^0_{\tilde{A}\tilde{B},\tilde{B}_2} \otimes \frac{2I_A}{2d}.$$  

(C97)

which is equivalent to

$$I_{\tilde{A}\tilde{B},\tilde{B}_2} = \frac{1}{d} \left( M^+_{\tilde{A}\tilde{B},\tilde{B}_2} + M^-_{\tilde{A}\tilde{B},\tilde{B}_2} + M^0_{\tilde{A}\tilde{B},\tilde{B}_2} \right).$$  

(C98)

4. Non-signaling condition

We now consider the condition in (C60), as applied to (C56). Consider that

$$Tr_{B_2}\left[ \tilde{M}_{\tilde{A}\tilde{B},\tilde{B}_2,B_2,B_2} \right] = \sum_{i \in \{+, -, 0, 1, 2, 3\}} M^q_{\tilde{A}\tilde{B},\tilde{B}_2} \otimes \frac{Tr_{B_1,B_2}\left[ S^i_{AB,B_2,B_2} \right]}{Tr\left[ S^{i(i)}_{AB,B_2} \right]}$$

(C101)

Thus, we need to calculate $Tr_{B_2}\left[ S^i_{AB,B_2,B_2} \right]$ for all $i \in \{+, -, 0, 1, 2, 3\}$. To do so, we first find that

$$Tr_{B_2}\left[ I_{AB,B_2,B_2} \right] = dI_{AB}.$$  

(C102)

$$Tr_{B_2}\left[ V_{AB,B_2} \right] = I_{AB}.$$  

(C103)

$$Tr_{B_2}\left[ V^T_{AB,B_2} \right] = dI_{AB}.$$  

(C104)

$$Tr_{B_2}\left[ \tilde{M}_{\tilde{A}\tilde{B},\tilde{B}_2} \right] = I_{AB}.$$  

(C105)

$$Tr_{B_2}\left[ \tilde{M}_{\tilde{A}\tilde{B},\tilde{B}_2} \right] = \Gamma_{AB} = d\Phi_{AB}.$$  

(C106)

$$Tr_{B_2}\left[ \tilde{M}_{\tilde{A}\tilde{B},\tilde{B}_2} \right] = I_{AB}.$$  

(C107)

which implies from (C34)–(C39), that

$$Tr_{B_2}\left[ S^+_{AB,B_2,B_2} \right] = \frac{1}{2} \left[ \begin{array}{c} dI_{AB} + I_{AB} \\ - \left( d^2\Phi_{AB} + d\Phi_{AB} + d\Phi_{AB} \right) \end{array} \right]$$

(C96)
\[
\begin{align*}
\text{Tr}_B \left[ S_{AB,B_2} \right] &= \frac{1}{2} \left[ \left( d + 1 - \frac{1}{d+1} \right) I_{AB} - \frac{d^2 + 2d}{d + 1} \Phi_{AB} \right] = \frac{d(d+2)}{2(d+1)} (I_{AB} - \Phi_{AB}), \\
\text{Tr}_B \left[ S^0_{AB,B_2} \right] &= \frac{1}{d^2 - 1} \left[ d \left( d^2 \Phi_{AB} + I_{AB} \right) - d \Phi_{AB} + dI_{AB} \right] = \frac{d}{d^2 - 1} \left[ \left( d^2 - 1 \right) \Phi_{AB} - I_{AB} - \Phi_{AB} \right], \\
\text{Tr}_B \left[ S^1_{AB,B_2} \right] &= \frac{1}{d^2 - 1} \left[ d \Phi_{AB} + dI_{AB} \right] = \frac{1}{d^2 - 1} \left[ \left( d^2 - 1 \right) \Phi_{AB} - \Phi_{AB} \right]. \\
\text{Tr}_B \left[ S^2_{AB,B_2} \right] &= \frac{1}{\sqrt{d^2 - 1}} \left( 2d \Phi_{AB} - I_{AB} \right) = \frac{1}{\sqrt{d^2 - 1}} \left( \left( d^2 - 1 \right) \Phi_{AB} - \Phi_{AB} \right). \\
\text{Tr}_B \left[ S^3_{AB,B_2} \right] &= \frac{i}{\sqrt{d^2 - 1}} (d \Phi_{AB} - dI_{AB}) = 0.
\end{align*}
\]

We thus conclude from \( (C101) \) and the above that

\[
\begin{align*}
\text{Tr}_B \left[ \hat{M}_{\hat{A}\hat{B}_1 \hat{B}_2 \hat{B}_3} \right] &= \sum_{i \in \{+,-,0,1,2\}} M^i_{\hat{A}\hat{B}_1 \hat{B}_2} \otimes \frac{\text{Tr}_B \left[ S^i_{AB,B_2} \right]}{\text{Tr}[S^{(i)}_{AB,B_2}]} \quad \text{where} \\
\hat{P}'_{\hat{A}\hat{B}_1 \hat{B}_2} &:= \frac{1}{2} \left[ d \left( M^0_{\hat{A}\hat{B}_1 \hat{B}_2} + M^1_{\hat{A}\hat{B}_1 \hat{B}_2} \right) + \sqrt{d^2 - 1} M^2_{\hat{A}\hat{B}_1 \hat{B}_2} \right], \\
\hat{Q}'_{\hat{A}\hat{B}_1 \hat{B}_2} &:= \frac{1}{2} \left[ 2d \left( M^+_{\hat{A}\hat{B}_1 \hat{B}_2} + M^-_{\hat{A}\hat{B}_1 \hat{B}_2} \right) + dM^0_{\hat{A}\hat{B}_1 \hat{B}_2} \right] - \sqrt{d^2 - 1} M^2_{\hat{A}\hat{B}_1 \hat{B}_2}.
\end{align*}
\]
\( = d I_{\hat{A}B_1B_2} - P'_{\hat{A}B_1B_2} \) \tag{C120}

where the last equality follows from (C100). Now consider that

\[
\frac{1}{d_B} \text{Tr}_{\hat{B}_2B_2} [\bar{M}_{\hat{A}\hat{B}_1B_2B_1B_2} \otimes I_{\hat{B}_2} = \sum_{i \in \{+,-0,1,2,3\}} \frac{1}{d_B} \text{Tr}_{\hat{B}_2} [M'_{\hat{A}\hat{B}_1B_2} \otimes I_{\hat{B}_2} \otimes \text{Tr}_{B_1B_2} [S_{AB_1B_2}^i] / \text{Tr}[S_{AB_1B_2}^{g(i)}],
\tag{C121}
\]

which reduces to

\[
\frac{1}{d_B} \text{Tr}_{\hat{B}_2} [P'_{\hat{A}\hat{B}_1B_2} \otimes I_{\hat{B}_2} \otimes \Phi_{AB_1}
+ \frac{1}{d_B} \text{Tr}_{\hat{B}_2} [Q'_{\hat{A}\hat{B}_1B_2} \otimes I_{\hat{B}_2} \otimes \left( I_{AB_1} - \Phi_{AB_1} \right) / d^2 - 1 }).
\tag{C122}
\]

Then the constraint in (C60) reduces to the following two constraints:

\[
P'_{\hat{A}\hat{B}_1B_2} = \frac{1}{d_B} \text{Tr}_{\hat{B}_2} [P'_{\hat{A}\hat{B}_1B_2} \otimes I_{\hat{B}_2},
\tag{C123}
\]

\[
Q'_{\hat{A}\hat{B}_1B_2} = \frac{1}{d_B} \text{Tr}_{\hat{B}_2} [Q'_{\hat{A}\hat{B}_1B_2} \otimes I_{\hat{B}_2},
\tag{C124)
\]

by applying (C118) and (C122), and using the orthogonality of \( \Phi_{AB_1} \) and \( I_{AB_1} - \Phi_{AB_1} \). The second constraint in (C124) is redundant, following from the first one in (C123), because

\[
Q'_{\hat{A}\hat{B}_1B_2} = d I_{\hat{A}B_1B_2} - P'_{\hat{A}B_1B_2}.
\tag{C125}
\]

This justifies the constraint in (95), up to an inconsequential scale factor.

5. Permutation covariance condition

We now consider the condition in (C61), as applied to (C56). Consider that

\[
(\mathcal{F}_{\tilde{B}_1B_1} \otimes \mathcal{F}_{\tilde{B}_1B_1})(\bar{M}_{\hat{A}\hat{B}_1B_2B_1B_2})
= \sum_{i \in \{+,-0,1,2,3\}} \mathcal{F}_{\tilde{B}_1B_1}(M'_{\hat{A}\hat{B}_1B_2} \otimes I_{\hat{B}_2} \otimes \text{Tr}_{B_1B_2} [S_{AB_1B_2}^i] / \text{Tr}[S_{AB_1B_2}^{g(i)}],
\tag{C126}
\]

Observing from (C45)–(C48) that

\[
\mathcal{F}_{\tilde{B}_1B_1}(I_{AB_1B_2}) = I_{AB_1B_2},
\tag{C127}
\]

\[
\mathcal{F}_{\tilde{B}_1B_1}(V_{B_1B_2}) = V_{B_1B_2},
\tag{C128}
\]

\[
\mathcal{F}_{\tilde{B}_1B_1}(V^T_{AB_1}) = V^T_{AB_2},
\tag{C129}
\]

\[
\mathcal{F}_{\tilde{B}_1B_1}(V^T_{AB_2}) = V^T_{AB_1},
\tag{C130}
\]

\[
\mathcal{F}_{\tilde{B}_1B_1}(V^T_{AB_1B_1A'}) = V^T_{AB_1B_2A'},
\tag{C131}
\]

\[
\mathcal{F}_{\tilde{B}_1B_1}(V^T_{AB_2B_1A'}) = V^T_{AB_1B_2A'},
\tag{C132}
\]

we conclude from (C34)–(C39) that

\[
\mathcal{F}_{\tilde{B}_1B_1}(S_{AB_1B_2}^0) = S_{AB_1B_2}^0,
\tag{C133}
\]

\[
\mathcal{F}_{\tilde{B}_1B_1}(S_{AB_1B_2}^1) = S_{AB_1B_2}^1,
\tag{C134}
\]

\[
\mathcal{F}_{\tilde{B}_1B_1}(S_{AB_1B_2}^2) = S_{AB_1B_2}^2,
\tag{C135}
\]

\[
\mathcal{F}_{\tilde{B}_1B_1}(S_{AB_1B_2}^3) = S_{AB_1B_2}^3,
\tag{C136}
\]

\[
\mathcal{F}_{\tilde{B}_1B_1}(S_{AB_1B_2}^4) = S_{AB_1B_2}^4,
\tag{C137}
\]

\[
\mathcal{F}_{\tilde{B}_1B_1}(S_{AB_1B_2}^5) = S_{AB_1B_2}^5.
\tag{C138}
\]

This implies that the condition in (C61) is equivalent to

\[
\sum_{i \in \{+,-0,1,2,3\}} M'_{\hat{A}\hat{B}_1B_2} \otimes \frac{S_{AB_1B_2}^i} {\text{Tr}[S_{AB_1B_2}^{g(i)}]} = \sum_{i \in \{+,-0,1,2,3\}} \mathcal{F}_{\tilde{B}_1B_1}(M'_{\hat{A}\hat{B}_1B_2}) \otimes \frac{(-1)^f(i) S_{AB_1B_2}^i} {\text{Tr}[S_{AB_1B_2}^{g(i)}]},
\tag{C139}
\]

where

\[
f(i) = \begin{cases} 0 & \text{if } i \in \{+,-0,1\} \\
1 & \text{if } i \in \{2,3\} \end{cases}.
\tag{C140}
\]

Due to the fact that \( S_{AB_1B_2}^+, S_{AB_1B_2}^-, S_{AB_1B_2}^0 \) and \( S_{AB_1B_2}^3 \) are orthogonal projectors, and \( S_{AB_1B_2}^1, S_{AB_1B_2}^2, S_{AB_1B_2}^3 \) only act on the subspace onto which \( S_{AB_1B_2}^0 \) projects, the condition in (C139) is equivalent to the following three conditions:

\[
\begin{bmatrix} M^0 + M^3 & M^1 - iM^2 \\
M^1 + iM^2 & M^0 - M^3 \end{bmatrix} = \begin{bmatrix} \mathcal{F}(M^0) - \mathcal{F}(M^3) & \mathcal{F}(M^1) + i\mathcal{F}(M^2) \\
\mathcal{F}(M^1) - i\mathcal{F}(M^2) & \mathcal{F}(M^0) + \mathcal{F}(M^3) \end{bmatrix}.
\tag{C143}
\]

where we have employed the abbreviation \( \mathcal{F} = \mathcal{F}_{\tilde{B}_1B_1} \) and the isometry in (C65)–(C66). By considering that (C143) is equivalent to

\[
\begin{bmatrix} M^0 + M^3 & M^1 - iM^2 \\
M^1 + iM^2 & M^0 - M^3 \end{bmatrix} = \begin{bmatrix} \mathcal{F}(M^0) - \mathcal{F}(M^3) & \mathcal{F}(M^1) + i\mathcal{F}(M^2) \\
\mathcal{F}(M^1) - i\mathcal{F}(M^2) & \mathcal{F}(M^0) + \mathcal{F}(M^3) \end{bmatrix}.
\tag{C143}
\]

and adding and subtracting these equations, we conclude that (C143) is actually equivalent to the following four conditions:

\[
\begin{bmatrix} M^0 + M^3 & M^1 - iM^2 \\
M^1 + iM^2 & M^0 - M^3 \end{bmatrix} = \begin{bmatrix} \mathcal{F}(M^0) - \mathcal{F}(M^3) & \mathcal{F}(M^1) + i\mathcal{F}(M^2) \\
\mathcal{F}(M^1) - i\mathcal{F}(M^2) & \mathcal{F}(M^0) + \mathcal{F}(M^3) \end{bmatrix}.
\tag{C143}
\]
This justifies the constraints in (93)–(94).

At this point, let us summarize the results of Appendices C2 through C5: we have reduced the constraints in (C58)–(C61) to the following ones:

\[
\begin{bmatrix}
M^0 + M^3 & M^1 - iM^2 \\
M^1 + iM^2 & M^0 - M^3
\end{bmatrix} \geq 0,
\]

\(M^+_A B \hat{b}_1 \hat{b}_2 \geq 0,\) \hspace{1cm} (C152)

\(M^-_A B \hat{b}_1 \hat{b}_2 \geq 0,\) \hspace{1cm} (C153)

\(I^i \hat{A} \hat{b}_1 \hat{b}_2 = M^+_A B \hat{b}_1 \hat{b}_2 + M^-_A B \hat{b}_1 \hat{b}_2 + M^0_A B \hat{b}_1 \hat{b}_2 \) \hspace{1cm} (C154)

\(P'^i \hat{A} \hat{b}_1 \hat{b}_2 = \text{Tr} \hat{B}_2 (P_A \hat{B}_1 \hat{b}_2) \otimes \pi \hat{B}_2,\) \hspace{1cm} (C155)

\(M^i_A \hat{b}_1 \hat{b}_2 = \mathcal{T} \hat{B}_2 (M^i_A \hat{B}_1 \hat{b}_2) \) \hspace{1cm} \(\forall i \in \{+,-,0,1\},\) \hspace{1cm} (C156)

\(M^j_A \hat{b}_1 \hat{b}_2 = -\mathcal{F} \hat{B}_2 (M^j_A \hat{B}_1 \hat{b}_2) \) \hspace{1cm} \(\forall j \in \{2,3\},\) \hspace{1cm} (C157)

where \(P'^i \hat{A} \hat{b}_1 \hat{b}_2\) is defined in (C119).

6. PPT constraints

We now consider the PPT constraints. Suppose that \(\mathcal{K} \hat{A} \hat{A} \hat{b} \hat{b} \rightarrow \hat{b} \hat{b}\) is an arbitrary two-PPT-extendible channel, meaning that there exists an extension channel \(M_M \hat{A} \hat{A} \hat{b}_1 \hat{b}_2 \rightarrow \hat{b}_1 \hat{b}_2\) satisfying the constraints considered in Appendices C2 through C5, as well as

\[T_B \circ M_M \hat{A} \hat{A} \hat{b}_1 \hat{b}_2 \rightarrow \hat{b}_1 \hat{b}_2 \circ T_B \in \text{CP},\] \hspace{1cm} (C158)

\[M_M \hat{A} \hat{A} \hat{b}_1 \hat{b}_2 \rightarrow \hat{b}_1 \hat{b}_2 \circ T_M \hat{A} \hat{A} \circ T_M \hat{A} \hat{A} \in \text{CP}.\] \hspace{1cm} (C159)

Then it follows that \(\mathcal{K}^u \hat{A} \hat{A} \hat{b} \hat{b} \rightarrow \hat{b} \hat{b}\) and its extension \(M^u \hat{A} \hat{A} \hat{b}_1 \hat{b}_2 \rightarrow \hat{b}_1 \hat{b}_2\) as defined in (C3) and (C4), respectively, satisfy the following conditions

\[T_B \circ M^u \hat{A} \hat{A} \hat{b}_1 \hat{b}_2 \rightarrow \hat{b}_1 \hat{b}_2 \circ T_B \in \text{CP},\] \hspace{1cm} (C160)

\[M^u \hat{A} \hat{A} \hat{b}_1 \hat{b}_2 \rightarrow \hat{b}_1 \hat{b}_2 \circ T_M \hat{A} \hat{A} \circ T_M \hat{A} \hat{A} \in \text{CP}.\] \hspace{1cm} (C161)

because

\[T_B \circ M^u \hat{A} \hat{A} \hat{b}_1 \hat{b}_2 \rightarrow \hat{b}_1 \hat{b}_2 \circ T_B = T_B \circ \left( \mathcal{U}^i \hat{B}_1 \otimes \mathcal{U}^i \hat{B}_2 \right) \circ M_M \hat{A} \hat{A} \hat{b}_1 \hat{b}_2 \rightarrow \hat{b}_1 \hat{b}_2 \circ T_B,\] \hspace{1cm} (C162)

\[T_B \circ \left( \mathcal{U}^i \hat{B}_1 \otimes \mathcal{U}^i \hat{B}_2 \right) \circ T_B = T_B \circ M_M \hat{A} \hat{A} \hat{b}_1 \hat{b}_2 \rightarrow \hat{b}_1 \hat{b}_2 \circ \mathcal{U}_A \circ T_B,\] \hspace{1cm} (C163)

\[\left( \mathcal{U}^i \hat{B}_1 \otimes \left( T_B \circ \mathcal{U}^i \hat{B}_2 \circ T_B \right) \right) \circ \left[ T_B \circ M_M \hat{A} \hat{A} \hat{b}_1 \hat{b}_2 \rightarrow \hat{b}_1 \hat{b}_2 \circ T_B \right] \circ \mathcal{U}_A.\] \hspace{1cm} (C164)
\[ T_A^3(S^3_{AB,B_2}) = \frac{i}{\sqrt{d^2 - 1}} (V_{AB,B_2} - V_{B_2,B_1}). \]  
(C179)

It is helpful for us to make use of the following operators, defined in [EW01, Section II-A]:

\[ R^r_{AB,B_2} = \frac{1}{6} \begin{bmatrix} I_{AB,B_2} + V_{AB,B_2} + V_{B_2,B_1} \\ +V_{AB,B_2} + V_{B_2,B_1} \\ +V_{AB,B_2} + V_{B_2,B_1} \end{bmatrix}, \]  
(C180)
\[ R^d_{AB,B_2} = \frac{1}{6} \begin{bmatrix} I_{AB,B_2} - V_{AB,B_2} - V_{B_2,B_1} \\ -V_{AB,B_2} - V_{B_2,B_1} \end{bmatrix}, \]  
(C181)
\[ R^0_{AB,B_2} = \frac{1}{3} \begin{bmatrix} 2I_{AB,B_2} - V_{AB,B_2} - V_{B_2,B_1} \end{bmatrix}, \]  
(C182)
\[ R^i_{AB,B_2} = \frac{1}{3} \begin{bmatrix} 2V_{B_2,B_1} - V_{AB,B_2} - V_{B_2,B_1} \end{bmatrix}, \]  
(C183)
\[ R^2_{AB,B_2} = \frac{1}{\sqrt{3}} \begin{bmatrix} V_{AB,B_2} - V_{AB,B_2} \end{bmatrix}, \]  
(C184)
\[ R^3_{AB,B_2} = \frac{i}{\sqrt{3}} \begin{bmatrix} V_{AB,B_2} - V_{B_2,B_1} \end{bmatrix}. \]  
(C185)

These operators have the following properties:

\[ \text{Tr}[R^r_{AB,B_2}] = \frac{d}{6}(d + 1)(d + 2), \]  
(C186)
\[ \text{Tr}[R^d_{AB,B_2}] = \frac{d}{6}(d - 1)(d - 2), \]  
(C187)
\[ \text{Tr}[R^0_{AB,B_2}] = \frac{2d(d^2 - 1)}{3}, \]  
(C188)
\[ \text{Tr}[R^i_{AB,B_2}] = \text{Tr}[R^2_{AB,B_2}] = \text{Tr}[R^3_{AB,B_2}] = 0. \]  
(C189)

Furthermore, \( R^r_{AB,B_2}, R^d_{AB,B_2}, \) and \( R^0_{AB,B_2} \) are orthogonal projectors, satisfying

\[ R^r_{AB,B_2} + R^d_{AB,B_2} + R^0_{AB,B_2} = I_{AB,B_1}. \]  
(C190)

while \( R^i_{AB,B_2}, R^2_{AB,B_2}, \) and \( R^3_{AB,B_2} \) are Pauli-like operators acting on the subspace onto which \( R^0_{AB,B_2} \) projects. All \( R^i_{AB,B_2} \) operators are Hermitian and satisfy the following for their Hilbert–Schmidt inner product:

\[ \left\langle R^i_{AB,B_2}, R^j_{AB,B_2} \right\rangle = \text{Tr}[ (R^i_{AB,B_2})^\dagger R^j_{AB,B_2} ] \]  
(C191)
\[ = \text{Tr}[ R^i_{AB,B_2} R^j_{AB,B_2} ] \]  
(C192)
\[ = \text{Tr}[ R^{(i)}_{AB,B_2} ] \delta_{i,j}. \]  
(C193)

Making the identifications

\[ V^r_{AB,B_2} \equiv I_{AB,B_2}, \]  
(C194)
\[ V^d_{AB,B_2} \equiv V_{AB,B_2}, \]  
(C195)
\[ V^0_{AB,B_2} \equiv V_{AB,B_2}, \]  
(C196)
\[ V^i_{AB,B_2} \equiv V_{B_2,B_1}, \]  
(C197)
\[ V^2_{AB,B_2} \equiv V_{B_2,B_1}, \]  
(C198)
\[ V^3_{AB,B_2} \equiv V_{B_2,B_1,A}, \]  
(C199)
we find, for \( i \in \{+, -, 0, 1, 2, 3\} \), that

\[ S^i_{AB,B_2} = \sum_{j \in \{+, -, 0, 1, 2, 3\}} [Y]_{i,j} \text{Tr}[(V^i_{AB,B_2})], \]  
(C200)
\[ R^j_{AB,B_2} = \sum_{i \in \{+, -, 0, 1, 2, 3\}} [Z]_{i,j} \text{Tr}[(V^j_{AB,B_2})], \]  
(C201)
where the matrix \( Y \) with elements \([Y]_{i,j}\) is given by

\[ Y = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \]  
(C202)
and the matrix \( Z \) with elements \([Z]_{i,j}\) is given by

\[ Z = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ -1 & 0 & -1 & -1 \end{pmatrix}, \]  
(C203)
with inverse

\[ Z^{-1} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}, \]  
(C204)

Now using the fact that

\[ YZ^{-1} = \begin{pmatrix} \frac{d-1}{d^2-1} & 0 & \frac{d+1}{d^2-1} & \frac{d+1}{d^2-1} \\ 0 & \frac{d+1}{d^2-1} & \frac{d+1}{d^2-1} & \frac{d+1}{d^2-1} \\ \frac{d+1}{d^2-1} & 0 & -\frac{d^2}{d^2-1} & 0 \\ 0 & -\frac{d^2}{d^2-1} & \frac{d^2}{d^2-1} & 0 \end{pmatrix}, \]  
(C205)

we find, for \( i \in \{+, -, 0, 1, 2, 3\} \) that

\[ T_A(S^i_{AB,B_2}) = \sum_{j \in \{+, -, 0, 1, 2, 3\}} [Y]_{i,j} V^j_{AB,B_2}, \]  
(C206)
\[ = \sum_{j \in \{+, -, 0, 1, 2, 3\}} [Z]_{i,j} R^j_{AB,B_2}, \]  
(C207)
\[ = \sum_{k \in \{+, -, 0, 1, 2, 3\}} [Y^{-1}]_{i,k} R^k_{AB,B_2}. \]  
(C208)
and we conclude that
\[
T_{AA}(\tilde{M}_{A\tilde{A}B_1B_2B_3}) = \sum_{i \in \{+,-0,1,2,3\}} T_A(M_{A\tilde{A}B_1B_2}^i) \otimes T_A(S_{AB_1B_2}^i) \tag{C209}
\]
\[
= \sum_{i \in \{+,-0,1,2,3\}} [YZ^{-1}]_{i,k} \frac{T_A(M_{A\tilde{A}B_1B_2}^i)}{\text{Tr}[S_{AB_1B_2}^i]} \otimes R_{AB_1B_2}^k \tag{C210}
\]
\[
= \sum_{k \in \{+,-0,1,2,3\}} G_{\tilde{A}B_1B_2}^k \otimes R_{AB_1B_2}^k \tag{C211}
\]
where
\[
G_{\tilde{A}B_1B_2}^k := \sum_{i \in \{+,-0,1,2,3\}} [YZ^{-1}]_{i,k} \frac{T_A(M_{A\tilde{A}B_1B_2}^i)}{\text{Tr}[S_{AB_1B_2}^i]} \tag{C212}
\]
In detail, we find that
\[
G_{\tilde{A}B_1B_2}^+ = \frac{d - 1}{d + 1} T_A(M_{A\tilde{A}B_1B_2}^+) \frac{1}{d} \right) \frac{d + 2}{d + 1} \frac{T_A(M_{A\tilde{A}B_1B_2}^0) + T_A(M_{A\tilde{A}B_1B_2}^1)}{2d} + \frac{2}{d + 1} \frac{T_A(M_{A\tilde{A}B_1B_2}^1)}{2d} \tag{C213}
\]
\[
= \frac{d - 1}{d + 1} T_A \left( \frac{2M_{A\tilde{A}B_1B_2}^+}{d + 2} + M_{A\tilde{A}B_1B_2}^0 + M_{A\tilde{A}B_1B_2}^1 \right) \tag{C214}
\]
\[
G_{\tilde{A}B_1B_2}^- = \frac{d + 1}{d - 1} T_A(M_{A\tilde{A}B_1B_2}^-) \frac{1}{d} \right) \frac{d + 2}{d - 1} \frac{T_A(M_{A\tilde{A}B_1B_2}^0) + T_A(M_{A\tilde{A}B_1B_2}^1)}{2d} - \frac{2}{d - 1} \frac{T_A(M_{A\tilde{A}B_1B_2}^1)}{2d} \tag{C215}
\]
\[
= \frac{d + 1}{d - 1} T_A \left( \frac{2M_{A\tilde{A}B_1B_2}^-}{d + 2} + M_{A\tilde{A}B_1B_2}^0 - M_{A\tilde{A}B_1B_2}^1 \right) \tag{C216}
\]
\[
G_{\tilde{A}B_1B_2}^0 = \frac{d + 2}{d + 1} T_A(M_{A\tilde{A}B_1B_2}^0) \frac{1}{d} \right) \frac{d + 2}{d + 1} \frac{T_A(M_{A\tilde{A}B_1B_2}^+)}{2d} + \frac{d - 2}{d - 1} T_A(M_{A\tilde{A}B_1B_2}^-) \frac{1}{d} \right) \frac{d + 2}{d - 1} \frac{T_A(M_{A\tilde{A}B_1B_2}^0)}{2d} - \frac{2}{d - 1} \frac{T_A(M_{A\tilde{A}B_1B_2}^-)}{2d} \tag{C217}
\]
\[
= \frac{d + 2}{d + 1} T_A \left( \frac{M_{A\tilde{A}B_1B_2}^+}{d + 2} + M_{A\tilde{A}B_1B_2}^0 - M_{A\tilde{A}B_1B_2}^- \right) \tag{C218}
\]
\[
T_{AB_1B_2} \circ M_{A\tilde{A}B_1B_2} \rightarrow B_1B_2 \circ T_{AA} \tilde{B_1} \in \text{CP}. \tag{C229}
\]
Consider that
\[
T_{A\tilde{A}B_1B_2}(\tilde{M}_{A\tilde{A}B_1B_2B_3}) = \sum_{i \in \{+,-0,1,2,3\}} T_{A\tilde{A}B_1B_2} \frac{M_{A\tilde{A}B_1B_2}^i}{\text{Tr}[S_{AB_1B_2}^i]} \tag{C230}
\]
we conclude from (C34)–(C39) that

\[ T_{AB}(I_{AB}, B_z) = I_{AB}, B_z, \]
\[ T_{AB}(V_{AB}^A, B_z) = V_{AB}^A, \]
\[ T_{AB}(V_{AB}^B, B_z) = V_{AB}, \]
\[ T_{AB}(V_{AB, B_z}) = V_{B_z, B_z}, \]
\[ T_{AB}(V_{T_{AB, B_z}}) = V_{T_{B_z, B_z}}, \]
\[ T_{AB}(V_{T_{AB, B_z}}) = V_{T_{B_z, B_z}}, \]
\[ T_{AB}(V_{T_{AB, B_z}}) = V_{T_{B_z, B_z}}, \]
\[ T_{AB}(V_{T_{AB, B_z}}) = V_{T_{B_z, B_z}}. \]

we conclude from (C34)–(C39) that

\[ T_{AB}(S^+_{AB, B_z}) = \frac{1}{2} \left[ \begin{array}{c} I_{AB, B_z} + V_{T_{AB, B_z}} \\ \left( \frac{I_{AB, B_z} + V_{T_{AB, B_z}} + V_{T_{AB, B_z}} - V_{T_{AB, B_z}}}{d+1} \right) \end{array} \right], \]
\[ T_{AB}(S^-_{AB, B_z}) = \frac{1}{2} \left[ \begin{array}{c} I_{AB, B_z} - V_{T_{AB, B_z}} \\ \left( \frac{I_{AB, B_z} + V_{T_{AB, B_z}} - V_{T_{AB, B_z}}}{d-1} \right) \end{array} \right], \]
\[ T_{AB}(S^0_{AB, B_z}) = \frac{1}{d^2 - 1} \left[ \begin{array}{c} d \left( V_{T_{AB, B_z}} + V_{AB, B_z} \right) \\ \left( \frac{d V_{T_{AB, B_z}} + V_{AB, B_z} - V_{AB, B_z} - V_{T_{AB, B_z}}}{d-1} \right) \end{array} \right], \]
\[ T_{AB}(S^1_{AB, B_z}) = \frac{1}{d^2 - 1} \left[ \begin{array}{c} d \left( V_{T_{AB, B_z}} + V_{AB, B_z} \right) \\ \left( \frac{d V_{T_{AB, B_z}} + V_{AB, B_z} - V_{AB, B_z} - V_{T_{AB, B_z}}}{d-1} \right) \end{array} \right], \]
\[ T_{AB}(S^2_{AB, B_z}) = \frac{1}{\sqrt{d^2 - 1}} \left( V_{T_{AB, B_z}} - V_{AB, B_z} \right), \]
\[ T_{AB}(S^3_{AB, B_z}) = \frac{i}{\sqrt{d^2 - 1}} \left( V_{T_{AB, B_z}} - V_{AB, B_z} \right). \]

Let us define the following operators:

\[ C^+_{AB, B_z} := \frac{1}{2} \left[ \begin{array}{c} W^+_{AB, B_z} + W^1_{AB, B_z} \\ \left( \frac{W^+_{AB, B_z} + W^1_{AB, B_z} + W^2_{AB, B_z} + W^3_{AB, B_z}}{d+1} \right) \end{array} \right], \]
\[ C^-_{AB, B_z} := \frac{1}{2} \left[ \begin{array}{c} W^+_{AB, B_z} - W^1_{AB, B_z} \\ \left( \frac{W^+_{AB, B_z} + W^1_{AB, B_z} - W^2_{AB, B_z} - W^3_{AB, B_z}}{d-1} \right) \end{array} \right], \]
\[ C^0_{AB, B_z} := \frac{1}{d^2 - 1} \left[ \begin{array}{c} d \left( W^+_{AB, B_z} + W^0_{AB, B_z} \right) \\ \left( \frac{d W^+_{AB, B_z} + W^0_{AB, B_z} - W^2_{AB, B_z} - W^3_{AB, B_z}}{d+1} \right) \end{array} \right], \]
\[ C^1_{AB, B_z} := \frac{1}{d^2 - 1} \left[ \begin{array}{c} d \left( W^2_{AB, B_z} + W^3_{AB, B_z} \right) \\ \left( \frac{d W^2_{AB, B_z} + W^3_{AB, B_z} - W^0_{AB, B_z}}{d+1} \right) \end{array} \right], \]
\[ C^2_{AB, B_z} := \frac{1}{\sqrt{d^2 - 1}} \left( W^2_{AB, B_z} - W^0_{AB, B_z} \right), \]
\[ C^3_{AB, B_z} := \frac{i}{\sqrt{d^2 - 1}} \left( W^2_{AB, B_z} - W^3_{AB, B_z} \right). \]

where

\[ W^+_{AB, B_z} := I_{AB, B_z}, \]
\[ W^-_{AB, B_z} := V_{T_{AB, B_z}}, \]
\[ W^0_{AB, B_z} := V_{T_{AB, B_z}}, \]
\[ W^1_{AB, B_z} := V_{T_{AB, B_z}}, \]
\[ W^2_{AB, B_z} := V_{T_{AB, B_z}}, \]
\[ W^3_{AB, B_z} := V_{T_{AB, B_z}}. \]

Observe that the \( C \) operators have the same algebraic relations as the \( S \) operators, because the \( W \) operators defined above are related to the original \( V \) operators by the system permutations \( B_1 \rightarrow A, B_2 \rightarrow B_1, \) and \( A \rightarrow B_2. \) Given the above definitions, we find that, for \( i \in \{+, -, 0, 1, 2, 3\}, \)

\[ C_{AB, B_z} := \sum_{j \in \{+, -, 0, 1, 2, 3\}} [Y]_{i,j} W^j_{AB, B_z}. \]

Also, observe that, for \( j \in \{+, -, 0, 1, 2, 3\}, \)

\[ W^j_{AB, B_z} := \sum_{k \in \{+, -, 0, 1, 2, 3\}} [P]_{j,k} T_{B_1}(V^k_{AB, B_z}), \]

where \( P \) is the following permutation matrix:

\[
P = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix},
\]

and we made use of the ordering in (C194)–(C199). Thus,

\[ T_{B_1}(V^i_{AB, B_z}) = \sum_{j \in \{+, -, 0, 1, 2, 3\}} [P^{-1}Y^{-1}]_{i,j} C^j_{AB, B_z}. \]

and this means that

\[ T_{AB}(S^i_{AB, B_z}) = \sum_{j \in \{+, -, 0, 1, 2, 3\}} [Y]_{i,j} T_{AB}(V^j_{AB, B_z}) \]

\[ = \sum_{j \in \{+, -, 0, 1, 2, 3\}} [Y]_{i,j} T_{B_1}(V^j_{AB, B_z}) \]

\[ = \sum_{j \in \{+, -, 0, 1, 2, 3\}} [YP^{-1}Y^{-1}]_{i,j} C^j_{AB, B_z}. \]

Now considering that

\[
Y^{-1} = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & d & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \sqrt{d^2 - 1} \\
0 & 0 & d & \frac{1}{2} & -\frac{1}{2} \sqrt{d^2 - 1} & 0 \\
1 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \sqrt{d^2 - 1} \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \sqrt{d^2 - 1} \\
\end{bmatrix}
\]
we find that
\[ Y^{P-1}Y^{-1} = \]
\[
\begin{bmatrix}
\frac{d}{2d(d+1)} & \frac{d(d+2)}{2d(d+1)} & -\frac{d(d+2)}{2d(d+1)} \\
\frac{d}{2d(d+1)} & \frac{-d(d-2)}{2d(d+1)} & \frac{-d(d-2)}{2d(d+1)} \\
\frac{-d}{2d(d+1)} & \frac{1}{2d(d+1)} & \frac{1}{2d(d+1)} \\
\frac{1}{2d(d+1)} & \frac{1}{2d(d+1)} & \frac{1}{2d(d+1)} \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
\frac{d}{2d(d+1)} & \frac{d(d+2)}{2d(d+1)} & -\frac{d(d+2)}{2d(d+1)} \\
\frac{d}{2d(d+1)} & \frac{-d(d-2)}{2d(d+1)} & \frac{-d(d-2)}{2d(d+1)} \\
\frac{-d}{2d(d+1)} & \frac{1}{2d(d+1)} & \frac{1}{2d(d+1)} \\
\frac{1}{2d(d+1)} & \frac{1}{2d(d+1)} & \frac{1}{2d(d+1)} \\
\end{bmatrix}
\]

so that
\[
T_{\bar{A}\bar{A}B_1B_2}(\tilde{M}_{\bar{A}\bar{A}B_1B_2}) = \sum_{i \in \{\pm, 0, 1, 2, 3\}} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{i}) \otimes T_{\bar{A}B_1}(S_{\bar{A}B_1B_2}^{i}) \quad \text{(C264)}
\]
\[
= \sum_{i, j \in \{\pm, 0, 1, 2, 3\}} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{i}) \otimes \left[ Y^{P-1}Y^{-1} \right]_{i, j} C_{\bar{A}B_1B_2}^{j} \quad \text{(C265)}
\]
\[
= \sum_{j \in \{\pm, 0, 1, 2, 3\}} E_{\bar{A}B_1B_2}^{j} \otimes C_{\bar{A}B_1B_2}^{j} \quad \text{(C266)}
\]

where
\[
E_{\bar{A}B_1B_2}^{j} := \sum_{i \in \{\pm, 0, 1, 2, 3\}} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{i}) \left[ Y^{P-1}Y^{-1} \right]_{i, j} \quad \text{(C267)}
\]

In detail, we find that
\[
E_{\bar{A}B_1B_2}^{+} = \frac{d}{2(d+1)} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{+}) + \frac{d-2}{2(d-1)} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{-}) + \frac{1}{2\sqrt{d^2-1}} \frac{1}{2d} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{0}) - \frac{1}{\sqrt{d^2-1}} \frac{1}{2d} \frac{1}{2d} \frac{1}{2d} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{2}) \quad \text{(C268)}
\]
\[
E_{\bar{A}B_1B_2}^{-} = \frac{d+2}{2(d+1)} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{+}) + \frac{d}{2(d-1)} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{-}) + \frac{1}{2\sqrt{d^2-1}} \frac{1}{2d} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{0}) - \frac{1}{\sqrt{d^2-1}} \frac{1}{2d} \frac{1}{2d} \frac{1}{2d} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{2}) \quad \text{(C269)}
\]
\[
E_{\bar{A}B_1B_2}^{0} = \frac{d}{2d(d+1)} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{+}) + \frac{1}{2d(d-1)} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{-}) + \frac{1}{\sqrt{d^2-1}} \frac{1}{2d} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{0}) - \frac{1}{\sqrt{d^2-1}} \frac{1}{2d} \frac{1}{2d} \frac{1}{2d} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{2}) \quad \text{(C270)}
\]
\[
E_{\bar{A}B_1B_2}^{1} = \frac{d}{2d(d+1)} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{+}) + \frac{1}{2d(d-1)} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{-}) + \frac{1}{\sqrt{d^2-1}} \frac{1}{2d} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{0}) - \frac{1}{\sqrt{d^2-1}} \frac{1}{2d} \frac{1}{2d} \frac{1}{2d} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{2}) \quad \text{(C271)}
\]
\[
E_{\bar{A}B_1B_2}^{2} = \frac{d}{2d(d+1)} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{+}) + \frac{1}{2d(d-1)} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{-}) + \frac{1}{\sqrt{d^2-1}} \frac{1}{2d} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{0}) - \frac{1}{\sqrt{d^2-1}} \frac{1}{2d} \frac{1}{2d} \frac{1}{2d} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{2}) \quad \text{(C272)}
\]
\[
E_{\bar{A}B_1B_2}^{3} = \frac{d}{2d(d+1)} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{+}) + \frac{1}{2d(d-1)} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{-}) + \frac{1}{\sqrt{d^2-1}} \frac{1}{2d} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{0}) - \frac{1}{\sqrt{d^2-1}} \frac{1}{2d} \frac{1}{2d} \frac{1}{2d} T_{\bar{A}B_1}(M_{\bar{A}B_1B_2}^{2}) \quad \text{(C273)}
\]
The final condition is equivalent to

\[ E^3_{A\tilde{B}_1,\tilde{B}_2} = \frac{T_{A\tilde{B}_1}(M^3_{A\tilde{B}_1,\tilde{B}_2})}{2d} = \frac{T_{A\tilde{B}_1}(M^3_{A\tilde{B}_1,\tilde{B}_2})}{2} \tag{C278} \]

Then the condition that \( T_{A\tilde{B}_1,\tilde{B}_2}(\tilde{M}_{A\tilde{A},\tilde{B}_1,\tilde{B}_2}) \geq 0 \) is equivalent to

\[ E^+_{A\tilde{B}_1,\tilde{B}_2} \geq 0, \tag{C279} \]
\[ E^-_{\tilde{B}_1,\tilde{B}_2} \geq 0, \tag{C280} \]
\[ \sum_{k \in [0,1,2,3]} E^k_{A\tilde{B}_1,\tilde{B}_2} \otimes S^k_{AB,\tilde{B}_2} \geq 0. \tag{C281} \]

The final condition is equivalent to

\[ \begin{bmatrix} E^0 + E^3 & E^1 - iE^2 \\ E^1 + iE^2 & E^0 - E^3 \end{bmatrix} \geq 0, \tag{C282} \]

by reasoning similar to that given for (C70). Note that we can scale the conditions in (C279) and (C280) by \( d^2 - 1 \) without changing them. This justifies the conditions in (104)–(106).

### 7. Final evaluation of normalized diamond distance objective function

It now remains to evaluate the normalized diamond distance by means of its semi-definite programming formulation in (46):

\[ \frac{1}{2} \left\| \tilde{K}_{A\tilde{A},B} \right\|_{\text{SDP}} = \sup_{\mu \geq 0, Z_{AB} \geq 0} \left\{ \frac{\mu I_A \geq Z_A}{Z_{AB} \geq \Gamma_{AB} - \text{Tr}_{AB} [\rho^T_{\tilde{A}\tilde{B}_1,\tilde{B}_2}] \tilde{K}_{A\tilde{A},B} ] \right\}, \tag{C283} \]

Consider that the Choi operator of the channel \( \tilde{K}_{A\tilde{A},B} \) is given by

\[ \tilde{K}_{A\tilde{A},B} = \frac{1}{d_B} \text{Tr}_{\tilde{B}_1} [P'_{A\tilde{B}_1,\tilde{B}_2}] \otimes \Phi_{AB} \]
\[ + \frac{1}{d_B} \text{Tr}_{\tilde{B}_1} [Q'_{A\tilde{B}_1,\tilde{B}_2}] \otimes \left( \frac{I_{AB} - \Phi_{AB}}{d^2 - 1} \right) \tag{C284} \]
\[ = \frac{1}{d_B} \text{Tr}_{\tilde{B}_1} [P_{A\tilde{B}_1,\tilde{B}_2}] \otimes \Gamma_{AB} \]
\[ + \frac{1}{d_B} \text{Tr}_{\tilde{B}_1} [Q_{A\tilde{B}_1,\tilde{B}_2}] \otimes \left( \frac{dI_{AB} - \Gamma_{AB}}{d^2 - 1} \right). \tag{C285} \]

where

\[ P_{A\tilde{B}_1,\tilde{B}_2} := \frac{1}{d} P'_{A\tilde{B}_1,\tilde{B}_2}, \tag{C286} \]

\[ Q_{A\tilde{B}_1,\tilde{B}_2} := \frac{1}{d} Q'_{A\tilde{B}_1,\tilde{B}_2} = I_{A\tilde{B}_1,\tilde{B}_2} - P_{A\tilde{B}_1,\tilde{B}_2} \tag{C287} \]

which implies that the Choi operator of \( \tilde{K}_{A\tilde{A},\tilde{B}} \circ \mathcal{A}_{\tilde{A}}^\mu \) is

\[ \text{Tr}_{\tilde{A}B} [\rho^T_{\tilde{A}} \tilde{K}_{A\tilde{A},\tilde{B}} ] = \frac{1}{d_B} \text{Tr} [\rho^T_{\tilde{A}} \text{Tr}_{AB} P_{A\tilde{B}_1,\tilde{B}_2} ] \Gamma_{AB} \]
\[ + \frac{1}{d_B} \text{Tr} [\rho^T_{\tilde{A}} \text{Tr}_{AB} Q_{A\tilde{B}_1,\tilde{B}_2} ] \left( \frac{dI_{AB} - \Gamma_{AB}}{d^2 - 1} \right). \tag{C288} \]

Then it follows that

\[ \Gamma_{AB} - \text{Tr}_{\tilde{A}B} [\rho^T_{\tilde{A}} \tilde{K}_{A\tilde{A},\tilde{B}} ] \]
\[ = \left( 1 - \frac{1}{d_B} \text{Tr} [\rho^T_{\tilde{A}} P_{A\tilde{B}_1,\tilde{B}_2} ] \right) \Gamma_{AB} \]
\[ - \frac{1}{d_B} \text{Tr} [\rho^T_{\tilde{A}} Q_{A\tilde{B}_1,\tilde{B}_2} ] \left( \frac{dI_{AB} - \Gamma_{AB}}{d^2 - 1} \right). \tag{C289} \]

By the same reasoning used to justify (B52)–(B56), we conclude that optimal choices for \( Z_{AB} \) and \( \mu \) are

\[ Z_{AB} = \mu \Gamma_{AB}, \tag{C290} \]
\[ \mu = 1 - \frac{1}{d_B} \text{Tr} [\rho^T_{\tilde{A}} P_{A\tilde{B}_1,\tilde{B}_2} ] \tag{C291} \]

Putting everything together, we conclude that the semi-definite program in Proposition 2 reduces to the form stated in Proposition 3. This concludes the proof.

### 8. The case \( d = 2 \)

Several steps in the previous calculations involve a denominator of \( d - 2 \), making it unsuitable to get lower bounds on the simulation error of an identity channel for qubits. However, it is easy to navigate around this problem.

The trace of \( S^\prime_{AB,\tilde{B}_2} \) is \( d(d-2)(d+1)/2 \); hence, it is traceless for \( d = 2 \). Since \( S^\prime_{AB,\tilde{B}_2} \) is positive semi-definite, as well, it follows that \( S_{AB,\tilde{B}_2} = 0 \). Therefore, \( M^\prime_{A\tilde{B}_1,\tilde{B}_2} \) is not involved in (C56) and consequently \( M^\prime_{A\tilde{B}_1,\tilde{B}_2} \) is not involved in any of the constraints. This is also equivalent to setting \( M^\prime_{A\tilde{B}_1,\tilde{B}_2} = 0 \) to 0.

Similarly, \( R^\prime_{A\tilde{B}_1,\tilde{B}_2} \), defined in (C181) is positive semi-definite and traceless for \( d \geq 2 \), implying that \( R^\prime_{A\tilde{B}_1,\tilde{B}_2} = 0 \) when \( d = 2 \). This means \( G^\prime_{A\tilde{B}_1,\tilde{B}_2} \) is not involved in (C21), which in turn corresponds to removing (98) from the constraints in the SDP of Proposition 3.

Finally, \( C^\prime_{A\tilde{B}_1,\tilde{B}_2} = 0 \) because \( C^\prime_{A\tilde{B}_1,\tilde{B}_2} \) is a traceless positive semi-definite operator when \( d = 2 \). Therefore, \( E^\prime_{A\tilde{B}_1,\tilde{B}_2} \) is not involved in (C266), which in turn corresponds to removing (105) from the constraints in this SDP.

This leads us to the claim at the end of Proposition 3, that removing (97) and (105) and setting \( M^\prime_{A\tilde{B}_1,\tilde{B}_2} = 0 \) gives the SDP for \( d = 2 \).
Appendix D: Proof of Proposition 7

The proof of Proposition 7 bears similarities with the proof of Proposition 1, but we give a somewhat detailed proof here for completeness and clarity. The main idea is again to simplify the optimization problems in (155) and (156) by exploiting the symmetries of the identity channel, as stated in (65).

Let us again begin by analyzing the diamond distance. Let $Y_{(\hat{A} \rightarrow \hat{B})} : (A \rightarrow B) \rightarrow (\hat{A} \rightarrow \hat{B})$ be an arbitrary LOCR superchannel, with corresponding bipartite channel $C_{\hat{A} \hat{B} \rightarrow \hat{A} B}$ having the form in (133). That is, let $C_{\hat{A} \hat{B} \rightarrow \hat{A} B}$ denote the bipartite channel that is in direct correspondence with $Y_{(\hat{A} \rightarrow \hat{B})} : (A \rightarrow B)$, i.e.,

$$C_{\hat{A} \hat{B} \rightarrow \hat{A} B} := \sum_y p(y) E^Y_{A \rightarrow \hat{A}} \otimes D^Y_{\hat{B} \rightarrow B}.$$  \hspace{1cm} (D1)

Additionally, the notation $Y_{(\hat{A} \rightarrow \hat{B})} : (A \rightarrow B)$ indicates that $Y_{(\hat{A} \rightarrow \hat{B})} : (A \rightarrow B)$ is a physical transformation of the channel $N_{\hat{A} \rightarrow \hat{B}}$ by means of the encoding-decoding scheme $(p(y), \{E^Y_{A \rightarrow \hat{A}}\}_y, \{D^Y_{\hat{B} \rightarrow B}\}_y)$, and the transformation takes a channel with input and output systems $\hat{A}$ and $\hat{B}$, respectively, to a channel with input and output systems $A$ and $B$, respectively. Then by the same reasoning that led to (B6), we find that

$$\| Y_{(\hat{A} \rightarrow \hat{B})} : (A \rightarrow B) (N_{\hat{A} \rightarrow \hat{B}}) - \text{id}_{A \rightarrow B} \|_{\diamond} \geq \| \tilde{Y}_{(\hat{A} \rightarrow \hat{B})} : (A \rightarrow B) (N_{\hat{A} \rightarrow \hat{B}}) - \text{id}_{A \rightarrow B} \|_{\diamond},$$ \hspace{1cm} (D2)

where $\tilde{Y}_{(\hat{A} \rightarrow \hat{B})} : (A \rightarrow B)$ is the twirled version of the superchannel $Y_{(\hat{A} \rightarrow \hat{B})} : (A \rightarrow B)$ and is defined as

$$\tilde{Y}_{(\hat{A} \rightarrow \hat{B})} : (A \rightarrow B) := \int dU_{\hat{B}} U_{\hat{B}}^\dagger \circ Y_{(\hat{A} \rightarrow \hat{B})} : (A \rightarrow B) \circ U_{\hat{A}}.$$ \hspace{1cm} (D3)

Importantly, $\tilde{Y}_{(\hat{A} \rightarrow \hat{B})} : (A \rightarrow B)$ is an LOCR channel because the channel twirl above can be implemented by means of LOCR. Furthermore, observe that

$$\tilde{Y}_{(\hat{A} \rightarrow \hat{B})} : (A \rightarrow B) = U_{\hat{B}}^\dagger \circ \tilde{Y}_{(\hat{A} \rightarrow \hat{B})} : (A \rightarrow B) \circ U_{\hat{A}}.$$ \hspace{1cm} (D4)

for every unitary channel $U$. Thus, as a consequence of (D2), it suffices to minimize the error with respect to superchannels having the symmetry in (D4). The corresponding twirled bipartite channel $\tilde{C}_{\hat{A} \hat{B} \rightarrow \hat{A} B}$ then has the following form:

$$\tilde{C}_{\hat{A} \hat{B} \rightarrow \hat{A} B} = \int dU (U \otimes \bar{U}) (Y_{\hat{A} \hat{B} \rightarrow \hat{A} B}).$$ \hspace{1cm} (D5)

Let us determine the form of LOCR superchannels possessing this symmetry. Let $\tilde{Y}_{\hat{A} \hat{B} \rightarrow \hat{A} B}$ denote the Choi operator for $\tilde{Y}_{(\hat{A} \rightarrow \hat{B})} : (A \rightarrow B)$ and $C_{\hat{A} \hat{B} \rightarrow \hat{A} B}$, and let $Y_{\hat{A} \hat{B} \rightarrow \hat{A} B}$ denote the Choi operator for $Y_{(\hat{A} \rightarrow \hat{B})} : (A \rightarrow B)$ and $C_{\hat{A} \hat{B} \rightarrow \hat{A} B}$. They are related as follows:

$$\tilde{Y}_{\hat{A} \hat{B} \rightarrow \hat{A} B} = \int dU (U \otimes \bar{U}) (Y_{\hat{A} \hat{B} \rightarrow \hat{A} B}).$$ \hspace{1cm} (D6)

Recalling the form of the bipartite channel $C_{\hat{A} \hat{B} \rightarrow \hat{A} B}$, we find that its Choi operator $Y_{\hat{A} \hat{B} \rightarrow \hat{A} B}$ is given by

$$Y_{\hat{A} \hat{B} \rightarrow \hat{A} B} = \sum_y p(y) E^Y_{A \hat{A}} \otimes D^Y_{\hat{B} B}.$$ \hspace{1cm} (D7)

where, for all $y$, $E^Y_{A \hat{A}} \otimes D^Y_{\hat{B} B} \geq 0$, $\text{Tr}_{\hat{A}}[E^Y_{A \hat{A}}] = I_A$, and $\text{Tr}_{\hat{B}}[D^Y_{\hat{B} B}] = I_B$. Recall the action of the twirling channel from (B10), so that

$$\tilde{T}_{\hat{A} \hat{B}} (E^Y_{A \hat{A}} \otimes D^Y_{\hat{B} B}) = \Phi_{\hat{A} \hat{B}} \otimes \text{Tr}_{\hat{B}} [\Phi_{\hat{A} \hat{B}} (E^Y_{A \hat{A}} \otimes D^Y_{\hat{B} B})] + \frac{I_{\hat{A} \hat{B}} - \Phi_{\hat{A} \hat{B}}}{d^2 - 1} \otimes \text{Tr}_{\hat{B}} [(I_{\hat{A} \hat{B}} - \Phi_{\hat{A} \hat{B}}) (E^Y_{A \hat{A}} \otimes D^Y_{\hat{B} B})].$$ \hspace{1cm} (D8)

Consider that the Choi operator of the composition $\tilde{Y}_{(\hat{A} \rightarrow \hat{B})} : (A \rightarrow B)$ is as follows:

$$\text{Tr}_{\hat{A} \hat{B}} [\tilde{T}_{\hat{A} \hat{B}} (\Gamma_{\hat{A} \hat{B}}) \tilde{Y}_{\hat{A} \hat{B} \rightarrow \hat{A} B}]$$ \hspace{1cm} (D10)

by applying the propagation rule in (122). This means that

$$\text{Tr}_{\hat{A} \hat{B}} [\tilde{T}_{\hat{A} \hat{B}} (\Gamma_{\hat{A} \hat{B}}) \tilde{Y}_{\hat{A} \hat{B} \rightarrow \hat{A} B}] = \Gamma_{\hat{A} \hat{B}} \otimes \sum_y p(y) \frac{1}{d} \text{Tr} ([\Phi_{\hat{A} \hat{B}} \otimes \tilde{T}_{\hat{A} \hat{B}} (\Gamma_{\hat{A} \hat{B}})] (E^Y_{A \hat{A}} \otimes D^Y_{\hat{B} B})) + \frac{dI_{\hat{A} \hat{B}} - \Gamma_{\hat{A} \hat{B}}}{d^2 - 1} \otimes \sum_y p(y) \frac{1}{d} \times \text{Tr} ([(I_{\hat{A} \hat{B}} - \Phi_{\hat{A} \hat{B}}) \otimes \tilde{T}_{\hat{A} \hat{B}} (\Gamma_{\hat{A} \hat{B}})] (E^Y_{A \hat{A}} \otimes D^Y_{\hat{B} B})).$$ \hspace{1cm} (D11)

Then we find that

$$\sum_y p(y) \frac{1}{d} \text{Tr} ([\Phi_{\hat{A} \hat{B}} \otimes \tilde{T}_{\hat{A} \hat{B}} (\Gamma_{\hat{A} \hat{B}})] (E^Y_{A \hat{A}} \otimes D^Y_{\hat{B} B})) = \sum_y p(y) \frac{1}{d} \text{Tr} (\Phi_{\hat{A} \hat{B}} \Gamma_{\hat{A} \hat{B}} D^Y_{\hat{B} B} \circ N_{\hat{A} \hat{B}} \circ E^Y_{A \hat{A}} (\Phi_{\hat{A} \hat{B}}))$$ \hspace{1cm} (D12)

$$= \sum_y p(y) \frac{1}{d} \text{Tr} (\Phi_{\hat{A} \hat{B}} \Gamma_{\hat{A} \hat{B}} D^Y_{\hat{B} B} \circ N_{\hat{A} \hat{B}} \circ E^Y_{A \hat{A}} (\Phi_{\hat{A} \hat{B}}))$$ \hspace{1cm} (D13)

$$= \sum_y p(y) \frac{1}{d} \text{Tr} (\Phi_{\hat{A} \hat{B}} (D^Y_{\hat{B} B} \circ N_{\hat{A} \hat{B}} \circ E^Y_{A \hat{A}}) (\Phi_{\hat{A} \hat{B}}))$$ \hspace{1cm} (D14)

Consider that

$$\text{Tr}_{\hat{A} \hat{B}} [\Phi_{\hat{A} \hat{B}} (E^Y_{A \hat{A}} \otimes D^Y_{\hat{B} B})] = \text{Tr}_{\hat{A} \hat{B}} [\Phi_{\hat{A} \hat{B}} (T_{\hat{B}} (E^Y_{A \hat{A}}) D^Y_{\hat{B} B})]$$ \hspace{1cm} (D15)
Thus, we find that
\[
\tilde{T}_{AB}(Y_{\hat{A}B\hat{A}B}) = \Gamma_{AB} \otimes K_{\hat{A}B} + \frac{dI_{AB} - \Gamma_{AB}}{d^2 - 1} \otimes L_{\hat{A}B},
\]  
(D19)

where
\[
K_{\hat{A}B} := \frac{1}{d^2} \sum_y p(y) \text{Tr}_B[T_B(E_{AB}^x)D_{\hat{B}B}^y],
\]  
(D20)

\[
L_{\hat{A}B} := \frac{1}{d} \sum_y p(y) \text{Tr}_A[E_{\hat{A}A}^x] \otimes I_{\hat{B}} - K_{\hat{A}B}.
\]  
(D21)

Observe that the operator
\[
\tau_A := \frac{1}{d} \sum_y p(y) \text{Tr}_A[E_{\hat{A}A}^x]
\]  
(D22)

is a state, because it is positive semi-definite and has trace equal to one.

Consider that the Choi operator of the composition \(Y_{(A \rightarrow B) \rightarrow (A \rightarrow B)}(\mathcal{N}_{\hat{A} \rightarrow B})\) is as follows
\[
\text{Tr}_{\hat{A}B}[T_{\hat{A}B}(\Gamma_{\hat{A}B}^N)\tilde{Y}_{\hat{A}B\hat{A}B}]
\]  
(D23)

by applying the propagation rule in (122). This implies that the simulation channel has the following Choi operator:
\[
\text{Tr}_{\hat{A}B}[T_{\hat{A}B}(\Gamma_{\hat{A}B}^N)\tilde{Y}_{\hat{A}B\hat{A}B}] = \Gamma_{AB} \text{Tr}[\tilde{T}_{AB}(\Gamma_{AB}^N)K_{\hat{A}B}]
\]  
(D24)

\[
= \frac{dI_{AB} - \Gamma_{AB}}{d^2 - 1} \text{Tr}[T_{\hat{A}B}(\Gamma_{\hat{A}B}^N)L_{\hat{A}B}]
\]  
(D25)

Observe that the optimization does not change under the substitutions \(T_{\hat{A}B}(K_{\hat{A}B}) \rightarrow K_{\hat{A}B}\) and \(T_{\hat{A}B}(L_{\hat{A}B}) \rightarrow L_{\hat{A}B}\), we find that the Choi operator is given by
\[
\text{Tr}_{\hat{A}B}[T_{\hat{A}B}(\Gamma_{\hat{A}B}^N)\tilde{Y}_{\hat{A}B\hat{A}B}] = \frac{dI_{AB} - \Gamma_{AB}}{d^2 - 1} \text{Tr}[L_{\hat{A}B}\Gamma_{\hat{A}B}^N].
\]  
(D26)

This corresponds to a channel of the following form:
\[
\text{Tr}[K_{\hat{A}B}\Gamma_{\hat{A}B}^N] \text{id}_{A \rightarrow B}^d + \text{Tr}[L_{\hat{A}B}\Gamma_{\hat{A}B}^N]D_{A \rightarrow B}.
\]  
(D27)

where the randomizing channel \(D_{A \rightarrow B}\) is defined in (69).

The channel in (D27) has the interpretation that the identity channel \(\text{id}_{A \rightarrow B}^d\) is applied with probability \(\text{Tr}[K_{\hat{A}B}\Gamma_{\hat{A}B}^N]\) and the randomizing channel \(D_{A \rightarrow B}\) is applied with probability \(\text{Tr}[L_{\hat{A}B}\Gamma_{\hat{A}B}^N]\). The total probability is indeed equal to one because
\[
\text{Tr}[K_{\hat{A}B}\Gamma_{\hat{A}B}^N] + \text{Tr}[L_{\hat{A}B}\Gamma_{\hat{A}B}^N] = \text{Tr}[(\tau_A \otimes I_{\hat{B}})\Gamma_{\hat{A}B}^N] = 1.
\]  
(D28)

(D29)

This justifies the claim in (166).

The claim in (161), that the simulation errors are equal, follows the same reasoning given in Appendix B.

Appendix E: Proof of Proposition 9

The proof of Proposition 9 bears some similarities with the proof of Proposition 3. However, there are some key differences that we detail here. As before, symmetry plays a critical role.

Our goal is to minimize the following objective function
\[
\frac{1}{2} \left\| \Theta_{(A \rightarrow B) \rightarrow (A \rightarrow B)}(\mathcal{N}_{\hat{A} \rightarrow B}) - \text{id}_{A \rightarrow B}^d \right\|_2,
\]  
(E1)

with respect to every two-PPT-extendible non-signaling super-channel \(\Theta \equiv \Theta_{(A \rightarrow B) \rightarrow (A \rightarrow B)}\) (see Section V G), where \(\mathcal{N}_{\hat{A} \rightarrow B}\) is a given channel and \(\text{id}_{A \rightarrow B}^d\) is the d-dimensional identity channel. By exploiting the covariance symmetry of the identity channel \(\text{id}_{A \rightarrow B}^d\), as given in (65), it suffices to optimize over the set of two-PPT-extendible non-signaling superchannels that satisfy the same symmetry. Let \(\mathcal{K}_{\hat{A}B \rightarrow A\hat{B}}\) be the bipartite channel corresponding to \(\Theta\), and since \(\Theta\) is two-PPT-extendible and non-signaling, this implies that there exists an extension channel \(\mathcal{M}_{\hat{A}B\hat{A}B \rightarrow A\hat{B}B_{\hat{A}B}}\) satisfying the constraints for two-PPT-extendibility and non-signaling. Following the same arguments given in Appendix C 1, the simulation error in (E1) is minimized by a symmetrized or twirled superchannel \(\tilde{\Theta}_{(A \rightarrow B) \rightarrow (A \rightarrow B)}\), such that its corresponding bipartite channel \(\tilde{\mathcal{K}}_{\hat{A}B \rightarrow A\hat{B}}\) is twirled, as well as its extension \(\tilde{\mathcal{M}}_{\hat{A}B\hat{A}B \rightarrow A\hat{B}B_{\hat{A}B}}\), so that
\[
\tilde{\mathcal{K}}_{\hat{A}B \rightarrow A\hat{B}} := \int dU \ U_{\hat{B}}^\dagger \circ \mathcal{K}_{\hat{A}B \rightarrow A\hat{B}} \circ U_A,
\]  
(E2)

\[
\tilde{\mathcal{M}}_{\hat{A}B\hat{A}B \rightarrow A\hat{B}B_{\hat{A}B}} := \int dU (U_{B_1}^\dagger \otimes U_{B_2}^\dagger) \circ \mathcal{M}_{\hat{A}B\hat{A}B \rightarrow A\hat{B}B_{\hat{A}B}} \circ U_A.
\]  
(E3)

The corresponding Choi operators \(\tilde{\mathcal{K}}_{\hat{A}B \hat{A}B}\) and \(\tilde{\mathcal{M}}_{\hat{A}B\hat{A}B \rightarrow A\hat{B}B_{\hat{A}B}}\) satisfy
\[
\tilde{\mathcal{K}}_{\hat{A}B \hat{A}B} = \int dU (\overline{U}_A \otimes U_B) (\mathcal{K}_{\hat{A}B \hat{A}B}),
\]  
(E4)

\[
\tilde{\mathcal{M}}_{\hat{A}B\hat{A}B \rightarrow A\hat{B}B_{\hat{A}B}} = \int dU (\overline{U}_A \otimes U_{B_1} \otimes U_{B_2}) (\mathcal{M}_{\hat{A}B\hat{A}B \rightarrow A\hat{B}B_{\hat{A}B}}).
\]  
(E5)
As before, this implies that the Choi operator of $\tilde{M}_{A\hat{B}_1 B_2 \hat{A} B_1 B_2}$ has the following form:

$$
\tilde{M}_{A\hat{B}_1 B_2 \hat{A} B_1 B_2} = d \sum_{i \in \{+,-0,1,2,3\}} M_{A\hat{B}_1 B_2}^i \otimes \frac{S_{A\hat{B}_1 B_2}^{(i)}}{\text{Tr}[S_{A\hat{B}_1 B_2}^{(i)}]}, \quad (E6)
$$

where the $S_{A\hat{B}_1 B_2}^{(i)}$ operators are defined in (C34)–(C39) and the function $g(i)$ in (C33). We now consider the individual constraints on $M_{A\hat{B}_1 B_2 \hat{A} B_1 B_2}$, which are imposed on it such that it is a two-PPT-extensible non-signaling Choi operator.

1. Complete positivity condition

Complete positivity is equivalent to $\tilde{M}_{A\hat{B}_1 B_2 \hat{A} B_1 B_2}$ being positive semi-definite. By applying the development in Appendix C2, we conclude that $M_{A\hat{B}_1 B_2 \hat{A} B_1 B_2} \geq 0$ if and only if

$$
M_{A\hat{B}_1 B_2}^+ \geq 0, \quad (E7)
$$

$$
M_{A\hat{B}_1 B_2}^- \geq 0, \quad (E8)
$$

$$
\begin{bmatrix}
M^0 + M^3 & M^1 - iM^2 \\
M^1 + iM^2 & M^0 - M^3
\end{bmatrix} \geq 0. \quad (E9)
$$

As justified previously in Appendix C2, the last inequality implies that $M^0 \geq 0$. This justifies the constraints in (187)–(188).

2. Trace preservation condition

Trace preservation is equivalent to

$$
\text{Tr}_{A\hat{B}_1 B_2}[\tilde{M}_{A\hat{B}_1 B_2 \hat{A} B_1 B_2}] = I_{A\hat{B}_1 B_2}, \quad (E10)
$$

which by applying (E6), is equivalent to

$$
\sum_{i \in \{+,-0,1,2,3\}} d \text{Tr}_A[M_{A\hat{B}_1 B_2}^i] \otimes \frac{\text{Tr}_{B_1 B_2}[S_{A\hat{B}_1 B_2}^{(i)}]}{\text{Tr}[S_{A\hat{B}_1 B_2}^{(i)}]} = I_{A\hat{B}_1 B_2}. \quad (E11)
$$

Applying the analysis in Appendix C3, we find that

$$
\sum_{i \in \{+,-0,1,2,3\}} d \text{Tr}_A[M_{A\hat{B}_1 B_2}^i] \otimes \frac{\text{Tr}_{B_1 B_2}[S_{A\hat{B}_1 B_2}^{(i)}]}{\text{Tr}[S_{A\hat{B}_1 B_2}^{(i)}]} = \text{Tr}_A[M_{A\hat{B}_1 B_2}^+ + M_{A\hat{B}_1 B_2}^- + M_{A\hat{B}_1 B_2}^0] \otimes I_A, \quad (E12)
$$

so that trace preservation is equivalent to the following condition:

$$
\text{Tr}_A[M_{A\hat{B}_1 B_2}^+ + M_{A\hat{B}_1 B_2}^- + M_{A\hat{B}_1 B_2}^0] \otimes I_A = I_{A\hat{B}_1 B_2} \quad (E13)
$$

which in turn is equivalent to

$$
\text{Tr}_A[M_{A\hat{B}_1 B_2}^+ + M_{A\hat{B}_1 B_2}^- + M_{A\hat{B}_1 B_2}^0] = I_{\hat{B}_1 B_2}. \quad (E14)
$$

This justifies the constraint in (189).

3. Non-signaling conditions

We have two different non-signaling conditions:

$$
\text{Tr}_A[\tilde{M}_{A\hat{B}_1 B_2 \hat{A} B_1 B_2}] = I_A \otimes \frac{1}{d} \text{Tr}_{A\hat{A}}[\tilde{M}_{A\hat{B}_1 B_2 A\hat{A} B_1 B_2}], \quad (E15)
$$

$$
\text{Tr}_{B_1}[\tilde{M}_{A\hat{B}_1 B_2 \hat{A} B_1 B_2}] = I_{\hat{B}_1} \otimes \frac{1}{d} \text{Tr}_{B_2}[\tilde{M}_{A\hat{B}_1 B_2 A\hat{A} B_1 B_2}], \quad (E16)
$$

The second condition we have already investigated in Appendix C4, and we found that it reduces to the following:

$$
P_{\hat{A} \hat{B}_1 B_2} = \frac{1}{d} \text{Tr}_{\hat{B}_1} [P_{\hat{A} \hat{B}_1} \otimes I_{B_2}], \quad (E17)
$$

$$
Q_{\hat{A} \hat{B}_1 B_2} = \frac{1}{d} \text{Tr}_{\hat{B}_1} [Q_{\hat{A} \hat{B}_1} \otimes I_{B_2}], \quad (E18)
$$

where

$$
P_{\hat{A} \hat{B}_1 B_2} := \frac{1}{2d} \left[dM^0 + M^1 + \sqrt{d^2 - 1}M^2\right], \quad (E19)
$$

$$
Q_{\hat{A} \hat{B}_1 B_2} := \frac{1}{2d} \left[2d \left(M_{\hat{A} \hat{B}_1 B_2}^+ + M_{\hat{A} \hat{B}_1 B_2}^-\right) + dM_{\hat{A} \hat{B}_1 B_2}^0 \right], \quad (E20)
$$

Note that we need to incorporate the extra condition on $Q_{\hat{A} \hat{B}_1 B_2}$ because (92) no longer holds—we instead have (189) for the case of approximate quantum error correction and we cannot use this to eliminate the variable $Q_{\hat{A} \hat{B}_1 B_2}$.

Let us now consider the condition in (E15). Consider that

$$
\text{Tr}_A[\tilde{M}_{A\hat{B}_1 B_2 \hat{A} B_1 B_2}] = \sum_{i \in \{+,-0,1,2,3\}} \frac{d}{2} \frac{\text{Tr}_A[M_{A\hat{B}_1 B_2}^i]}{\text{Tr}[S_{A\hat{B}_1 B_2}^{(i)}]} \otimes S_{A\hat{B}_1 B_2}^i \quad (E21)
$$

Also,

$$
I_A \otimes \frac{1}{d} \text{Tr}_{A\hat{A}}[\tilde{M}_{A\hat{B}_1 B_2 A\hat{A} B_1 B_2}] = \sum_{i \in \{+,-0,1,2,3\}} \frac{d}{2} \frac{\text{Tr}_A[M_{A\hat{B}_1 B_2}^i]}{\text{Tr}[S_{A\hat{B}_1 B_2}^{(i)}]} \otimes I_A \otimes \text{Tr}_A[S_{A\hat{B}_1 B_2}^i]. \quad (E22)
$$

Consider that

$$
\text{Tr}_A[I_{A \hat{B}_1 B_2}] = dI_{B_1 B_2}, \quad (E23)
$$

$$
\text{Tr}_A[V^T_{A \hat{B}_1}] = I_{B_1 B_2}, \quad (E24)
$$

$$
\text{Tr}_A[I_{A \hat{B}_1 B_2}] = dI_{B_1 B_2}, \quad (E25)
$$
which from (C34)–(C39) implies that

\[
\text{Tr}_A [S^+_{AB;B_2}] = \frac{1}{2} \left[ -\frac{d I_{B_1;B_2} + d V_{B_1;B_2} + d V_{B_1;B_2}}{d+1} \right] 
\]

\[
= \frac{(d+2)(d+1)-(d-1)}{d+1} \Pi^S_{B_1;B_2}, 
\]

\[
\text{Tr}_A [S^-_{AB;B_2}] = \frac{1}{2} \left[ -\frac{d I_{B_1;B_2} - d V_{B_1;B_2}}{d-1} \right] 
\]

\[
= \frac{(d-2)(d+1)-(d-1)}{d-1} \Pi^\Lambda_{B_1;B_2}, 
\]

\[
\text{Tr}_A [S^0_{AB;B_2}] = \frac{1}{d^2-1} \left[ d (I_{B_1;B_2} + I_{B_1;B_2}) - (V_{B_1;B_2} + V_{B_1;B_2}) \right] 
\]

\[
= \frac{2}{d^2-1} (I_{B_1;B_2} - I_{B_1;B_2}) 
\]

\[
= \frac{2}{d+1} \Pi^S_{B_1;B_2} + \frac{2}{d-1} \Pi^\Lambda_{B_1;B_2}, 
\]

\[
\text{Tr}_A [S^1_{AB;B_2}] = \frac{1}{d^2-1} \left[ d (V_{B_1;B_2} + V_{B_1;B_2}) - (I_{B_1;B_2} + I_{B_1;B_2}) \right] 
\]

\[
= \frac{2}{d^2-1} (V_{B_1;B_2} - I_{B_1;B_2}) 
\]

\[
= \frac{2}{d+1} \Pi^S_{B_1;B_2} - \frac{2}{d-1} \Pi^\Lambda_{B_1;B_2}, 
\]

\[
\text{Tr}_A [S^2_{AB;B_2}] = \frac{1}{\sqrt{d^2-1}} (I_{B_1;B_2} - I_{B_1;B_2}) = 0, 
\]

\[
\text{Tr}_A [S^3_{AB;B_2}] = \frac{i}{\sqrt{d^2-1}} (V_{B_1;B_2} - V_{B_1;B_2}) = 0, 
\]

where we have defined the projections onto the symmetric and antisymmetric subspaces of systems \(B_1B_2\) as

\[
\Pi^S_{B_1;B_2} := \frac{I_{B_1;B_2} + V_{B_1;B_2}}{2}, 
\]

\[
\Pi^\Lambda_{B_1;B_2} := \frac{I_{B_1;B_2} - V_{B_1;B_2}}{2}. 
\]

Note that

\[
\Pi^S_{B_1;B_2} \Pi^\Lambda_{B_1;B_2} = 0, 
\]

\[
\text{Tr} [\Pi^S_{B_1;B_2}] = \frac{d(d+1)}{2}, 
\]

\[
\text{Tr} [\Pi^\Lambda_{B_1;B_2}] = \frac{d(d-1)}{2}. 
\]

So we find that

\[
I_A \otimes \frac{1}{d} \text{Tr}^{\Lambda_A} [M^+_{AB;B_1} \Lambda_{B_1B_2}] = \frac{2}{d(d+1)} \text{Tr} [M^+_{AB;B_2}] \otimes I_A \otimes \frac{d+2}{d+1} \Pi^S_{B_1B_2} 
\]

\[
+ \frac{2}{d(d+1)} \text{Tr} [M^-_{AB;B_2}] \otimes I_A \otimes \frac{d-2}{d+1} \Pi^\Lambda_{B_1B_2} 
\]

\[
+ \frac{1}{2d} \text{Tr} [M^0_{AB;B_2}] \otimes I_A \otimes \frac{2}{d^2+1} \Pi^S_{B_1B_2} - \frac{2}{d^2-1} \Pi^\Lambda_{B_1B_2}. 
\]

\[
(E41) 
\]

\[
I_A \otimes \frac{1}{d} \text{Tr}^{\Lambda_A} [M^+_{AB;B_1} \Lambda_{B_1B_2}] = \frac{2}{d+1} \text{Tr} [M^+_{AB;B_2}] \otimes I_A \otimes \Pi^S_{B_1B_2} 
\]

\[
+ \frac{2}{d+1} \text{Tr} [M^-_{AB;B_2}] \otimes I_A \otimes \Pi^\Lambda_{B_1B_2} 
\]

\[
+ \frac{1}{2d} \text{Tr} [M^0_{AB;B_2}] \otimes I_A \otimes \frac{1}{d+1} \Pi^S_{B_1B_2} + \frac{1}{d+1} \Pi^\Lambda_{B_1B_2}. 
\]

\[
(E42) 
\]

Now consider that

\[
I_A \otimes \Pi^S_{B_1B_2} = \sum_{i \in \{+,-,0,1,2,3\}} \frac{\text{Tr} [(I_A \otimes \Pi^S_{B_1B_2}) S^i_{AB;B_2}]}{\text{Tr} [S^i_{AB;B_2}]} \otimes S^i_{AB;B_2}. 
\]

\[
(E45) 
\]

\[
= \sum_{i \in \{+,-,0,1,2,3\}} \frac{\text{Tr} [\Pi^S_{B_1B_2} \text{Tr} (S^i_{AB;B_2})]}{\text{Tr} [S^i_{AB;B_2}]} \otimes S^i_{AB;B_2}. 
\]

\[
(E46) 
\]

\[
= \frac{(d+2)(d+1)}{d(d+2)(d+1)/2} \text{Tr} [S^0_{AB;B_2}] 
\]

\[
+ \frac{1}{2d} \text{Tr} [S^0_{AB;B_2}] \left( 2 \frac{I_{B_1;B_2} + V_{B_1;B_2}}{2} - \frac{d+1}{d+1} \Pi^S_{B_1B_2} \right) \otimes S^0_{AB;B_2} 
\]

\[
+ \frac{1}{2d} \text{Tr} [S^0_{AB;B_2}] \left( 2 \frac{I_{B_1;B_2} + V_{B_1;B_2}}{2} - \frac{d+1}{d+1} \Pi^S_{B_1B_2} \right) \otimes S^1_{AB;B_2}. 
\]

\[
(E47) 
\]

We also have that

\[
I_A \otimes \Pi^\Lambda_{B_1B_2} = \sum_{i \in \{+,-,0,1,2,3\}} \frac{\text{Tr} [(I_A \otimes \Pi^\Lambda_{B_1B_2}) S^i_{AB;B_2}]}{\text{Tr} [S^i_{AB;B_2}]} \otimes S^i_{AB;B_2}. 
\]

\[
(E49) 
\]

\[
= \sum_{i \in \{+,-,0,1,2,3\}} \frac{\text{Tr} [\Pi^\Lambda_{B_1B_2} \text{Tr} (S^i_{AB;B_2})]}{\text{Tr} [S^i_{AB;B_2}]} \otimes S^i_{AB;B_2}. 
\]

\[
(E50) 
\]
\(= (d - 2) (d + 1) \frac{\text{Tr} [\Pi^A_{B_1 B_2}]}{d - 1} \frac{d}{d - 2} (d + 1) / 2 \otimes S^1_{AB_1 B_2} \)
\[+ \frac{1}{2d} \text{Tr} \left [ \frac{2}{d + 1} \Pi^S_{B_1 B_2} + \frac{2}{d - 1} \Pi^A_{B_1 B_2} \right ] \otimes S^0_{AB_1 B_2} \]
\[+ \frac{1}{2d} \text{Tr} \left [ \frac{2}{d + 1} \Pi^S_{B_1 B_2} - \frac{2}{d - 1} \Pi^A_{B_1 B_2} \right ] \otimes S^1_{AB_1 B_2} \]
\[= S^0_{AB_1 B_2} + \frac{1}{2} \left (S^0_{AB_1 B_2} - S^1_{AB_1 B_2} \right ). \]

We then find that
\[I_A \otimes \frac{1}{d} \text{Tr}_{\bar{A}} \left [ \bar{M}^0_{AB_1 B_2} \right ] = \frac{\text{Tr}_{\bar{A}} [2M^* + M^0 + M^1]}{d (d + 1)} \otimes I_A \otimes \Pi^S_{B_1 B_2} \]
\[+ \frac{\text{Tr}_{\bar{A}} [2M^* + M^0 - M^1]}{d (d - 1)} \otimes I_A \otimes \Pi^A_{B_1 B_2} \]
\[= \frac{\text{Tr}_{\bar{A}} [2M^* + M^0 + M^1]}{d (d + 1)} \otimes \left (S^0_{AB_1 B_2} + \frac{1}{2} \left (S^0_{AB_1 B_2} + S^1_{AB_1 B_2} \right ) \right ) \]
\[+ \frac{\text{Tr}_{\bar{A}} [2M^* + M^0 - M^1]}{d (d - 1)} \otimes \left (S^0_{AB_1 B_2} + \frac{1}{2} \left (S^0_{AB_1 B_2} - S^1_{AB_1 B_2} \right ) \right ) \]
\[= \frac{\text{Tr}_{\bar{A}} [2M^* + M^0 + M^1]}{d (d + 1)} \otimes S^1_{AB_1 B_2} \]
\[+ \frac{\text{Tr}_{\bar{A}} [2M^* + M^0 - M^1]}{d (d - 1)} \otimes S^0_{AB_1 B_2} \]
\[+ \frac{1}{2d} \left ( \frac{\text{Tr}_{\bar{A}} [2M^* + M^0 + M^1]}{d (d + 1)} \right ) \otimes S^0_{AB_1 B_2} \]
\[+ \frac{1}{2d} \left ( \frac{\text{Tr}_{\bar{A}} [2M^* + M^0 - M^1]}{d (d - 1)} \right ) \otimes S^1_{AB_1 B_2} \]
\[= \frac{\text{Tr}_{\bar{A}} [2M^* + M^0 + M^1]}{d (d + 1)} \otimes S^1_{AB_1 B_2} \]
\[+ \frac{\text{Tr}_{\bar{A}} [2M^* + M^0 - M^1]}{d (d - 1)} \otimes S^0_{AB_1 B_2} \]
\[+ \frac{1}{2d} \left ( \frac{\text{Tr}_{\bar{A}} [2M^* + M^0 + M^1]}{d (d + 1)} \right ) \otimes S^0_{AB_1 B_2} \]
\[+ \frac{1}{2d} \left ( \frac{\text{Tr}_{\bar{A}} [2M^* + M^0 - M^1]}{d (d - 1)} \right ) \otimes S^1_{AB_1 B_2} \]
\[= \frac{\text{Tr}_{\bar{A}} [2M^* + M^0 + M^1]}{d (d + 1)} \otimes S^1_{AB_1 B_2} \]
\[+ \frac{1}{2d} \left ( \frac{\text{Tr}_{\bar{A}} [2M^* + M^0 + M^1]}{d (d + 1)} \right ) \otimes S^0_{AB_1 B_2} \]
\[+ \frac{1}{2d} \left ( \frac{\text{Tr}_{\bar{A}} [2M^* + M^0 - M^1]}{d (d - 1)} \right ) \otimes S^1_{AB_1 B_2} \]
\[+ \frac{1}{2d} \left ( \frac{\text{Tr}_{\bar{A}} [2M^* + M^0 - M^1]}{d (d - 1)} \right ) \otimes S^0_{AB_1 B_2} \]
\[+ \frac{1}{2} \text{Tr}_{\bar{A}} [dM^* - dM^- + dM^1] \otimes S^0_{AB_1 B_2}. \]

Now considering (E15) and (E22), as well as the facts stated after (C51), we find that the non-signaling condition is equivalent to the following conditions:
\[\frac{\text{Tr}_{\bar{A}} [2M^*]}{d (d + 1)} = \frac{\text{Tr}_{\bar{A}} [2M^* + M^0 + M^1]}{d (d + 1)}, \]
\[\frac{\text{Tr}_{\bar{A}} [2M^-]}{d (d - 1)} = \frac{\text{Tr}_{\bar{A}} [2M^* + M^0 - M^1]}{d (d - 1)}. \]
\[\frac{1}{2} \left [ \text{Tr}_{\bar{A}} [M^0 + M^1] \right ] \text{Tr}_{\bar{A}} [M^1 + iM^2] \text{Tr}_{\bar{A}} [M^0 - M^2] \]
\[\frac{1}{2} \left [ \text{Tr}_{\bar{A}} [M^0 + M^3] \text{Tr}_{\bar{A}} [M^1 - iM^2] \right ] = \frac{1}{d (d^2 - 1)} \left [ y^0 y^1 \right ] \]
\[Y^0 := dR_{B_1 B_2} + \text{Tr}_{\bar{A}} [M^- - M^* - M^1], \]
\[Y^1 := -dR_{B_1 B_2} + \text{Tr}_{\bar{A}} [dM^* - dM^- + dM^1]. \]

Note that the last condition in (E60) simplifies because it reduces to
\[\frac{1}{2} \text{Tr}_{\bar{A}} [M^0 + M^2] = \frac{y^0}{d (d^2 - 1)}, \]
\[\frac{1}{2} \text{Tr}_{\bar{A}} [M^0 - M^3] = \frac{y^1}{d (d^2 - 1)}, \]
which implies that \(\text{Tr}_{\bar{A}} [M^3] = 0\). Similarly,
\[\frac{1}{2} \text{Tr}_{\bar{A}} [M^1 + iM^2] = \frac{y^1}{d (d^2 - 1)}, \]
\[\frac{1}{2} \text{Tr}_{\bar{A}} [M^1 - iM^2] = \text{Tr}_{\bar{A}} [M^1 - iM^2], \]
\[\text{Tr}_{\bar{A}} [M^2] = \text{Tr}_{\bar{A}} [M^2] = 0. \]

4. Permutation covariance condition

Permutation covariance is equivalent to
\[\bar{M}_{AB_1 B_2 A B_1 B_2} = (\bar{T}_{B_1 B_2} \otimes \bar{T}_{B_1 B_2})(\bar{M}_{AB_1 B_2 A B_1 B_2}), \]
which, by applying (E6), is equivalent to
Applying the analysis in Appendix C.5, we find that permuta-
tion covariance is equivalent to the following conditions:

\[ M^i_{\tilde{A}\tilde{B},\tilde{B}_2} = \mathcal{F}_{\tilde{A}\tilde{B}_1}(M^i_{\tilde{A}\tilde{B}_1,\tilde{B}_2}), \quad \forall i \in \{+,-,0,1\}, \]  
\[ M^j_{\tilde{A}\tilde{B},\tilde{B}_2} = -\mathcal{F}_{\tilde{A}\tilde{B}_1}(M^j_{\tilde{A}\tilde{B}_1,\tilde{B}_2}), \quad \forall j \in \{2,3\}. \]  
\[ (E72) \]
\[ (E73) \]

5. PPT constraints

We finally consider the PPT constraints, which are equiva-
lent to

\[ T_{A\tilde{A}}(\tilde{M}_{\tilde{A}\tilde{B}_1,\tilde{B}_1}) \geq 0, \]  
\[ T_{A\tilde{A}\tilde{B}_1}(\tilde{M}_{\tilde{A}\tilde{B}_1,\tilde{B}_2}) \geq 0. \]  
\[ (E74) \]
\[ (E75) \]

We have already evaluated these precise conditions in Ap-
pendix C.6, and we found that they are equivalent to the con-
ditions listed in (196)–(198) and (203)–(205).

6. Final evaluation of the normalized diamond distance
objective function

We now finally evaluate the following objective function

\[ \frac{1}{2} \left\| \Theta_{(\tilde{A}\leftrightarrow B)\rightarrow (A\leftrightarrow B)}(\mathcal{N}_{\tilde{A}\rightarrow \tilde{B}}) - \text{id}_{A\rightarrow B}^B \right\|, \]  
\[ (E76) \]

subject to the constraint that the twirled superchannel
\[ \Theta_{(\tilde{A}\leftrightarrow B)\rightarrow (A\leftrightarrow B)} \] is two-PPT-extendible and non-signaling. By employing the SDP for the normalized diamond distance in (46), as well as the propagation rule in (122), we find that (E76) is equal to

\[ \inf_{\mu, Z_{AB} \geq 0} \left\{ \frac{\mu}{\mu I_A \geq Z_{AB} - \text{Tr}_{\tilde{A}\tilde{B}}[T_{\tilde{A}\tilde{B}}(\Gamma_{AB}^N_{\tilde{A}B}) \tilde{K}_{\tilde{A}B\tilde{B}}] \right\}. \]  
\[ (E77) \]

where \( \tilde{K}_{\tilde{A}B\tilde{B}} \) is the Choi operator corresponding to the twirled superchannel \( \Theta_{(\tilde{A}\leftrightarrow B)\rightarrow (A\leftrightarrow B)} \). Recalling (E16), (E17)–(E18), and (C122), this Choi operator is given by

\[ \tilde{K}_{\tilde{A}B\tilde{B}} = \frac{1}{d_{\tilde{B}}} \text{Tr}_{\tilde{B}_2}[P_{\tilde{A}\tilde{B}_1,\tilde{B}_2}] \otimes \Gamma_{AB_1}, \]  
\[ + \frac{1}{d_{\tilde{B}}} \text{Tr}_{\tilde{B}_1}[Q_{\tilde{A}\tilde{B}_2,\tilde{B}_2}] \otimes \left( \frac{d I_{AB_1} - \Gamma_{AB_1}}{d^2 - 1} \right), \]  
\[ (E78) \]

which implies that

\[ \text{Tr}_{\tilde{B}_1}[T_{\tilde{A}\tilde{B}}(\Gamma_{AB}^N_{\tilde{A}B}) \tilde{K}_{\tilde{A}B\tilde{B}}] = \frac{1}{d_{\tilde{B}}} \text{Tr}[T_{\tilde{A}\tilde{B}}(\Gamma_{AB}^N_{\tilde{A}B}) P_{\tilde{A}\tilde{B}_1,\tilde{B}_2}] \otimes \Gamma_{AB_1}, \]  
\[ + \frac{1}{d_{\tilde{B}}} \text{Tr}[T_{\tilde{A}\tilde{B}}(\Gamma_{AB}^N_{\tilde{A}B}) Q_{\tilde{A}\tilde{B}_1,\tilde{B}_2}] \otimes \left( \frac{d I_{AB_1} - \Gamma_{AB_1}}{d^2 - 1} \right). \]  
\[ (E79) \]

The rest of the analysis follows along the lines given in Ap-
pendix C.7, and so we conclude the statement of Proposition
9, after minimizing the objective function over all two-
PPT-extendible and non-signaling twirled superchannels.

7. The case \( \hat{d} = 2 \)

We follow the same reasoning from Appendix C.8. Due to
the similarities between the SDP in Proposition 3 and the SDP
in Proposition 9, it is clear that we should set \( M_{\tilde{A}\tilde{B}_1,\tilde{B}_2} = 0 \) and remove the constraints in (197) and (204) when \( \hat{d} = 2 \). In addition, we also remove the constraint in (214), as it comes from comparing the coefficients of \( S_{AB}^{\tilde{A}B\tilde{B}} \), which is equal to zero when \( \hat{d} = 2 \). This leads to the claim at the end of Proposition 9.

Appendix F: Proof of Equation (220)

In this appendix, we work out the SDP for approximate
teleportation when using PPT constraints alone. Here we do
not show as many details as previous appendices because the
methods being used are similar. The simulation error for
approximate teleportation, when using a resource state \( P_{AB} \)
and a C-PPT-P channel \( P_{A\tilde{A}\tilde{B} \rightarrow B} \) for free, is as follows:

\[ \inf_{\mathcal{P} \in \text{PPT}} \frac{1}{2} \left\| \mathcal{P}_{A\tilde{A}\tilde{B} \rightarrow B} \circ \mathcal{R}^\mathcal{P}_{AB} - \text{id}_{A\rightarrow B}^B \right\|, \]  
\[ (F1) \]

where we have abbreviated the set of C-PPT-P channels by PPT.
By the same arguments given in (B5)–(B6) (i.e., exploiting the
unitary covariance symmetry of the identity channel \( \text{id}_{A\rightarrow B}^B \)), it suffices to restrict the optimization in (F1) to twirled C-PPT-P
channels, which satisfy

\[ P_{A\tilde{A}\tilde{B} \rightarrow B} = \int dU \mathcal{U}_B \circ \mathcal{P}_{A\tilde{A}\tilde{B} \rightarrow B} \circ \mathcal{U}_A. \]  
\[ (F2) \]

The Choi operator \( P_{AB\tilde{A}\tilde{B}} \) thus satisfies

\[ P_{AB\tilde{A}\tilde{B}} = \Phi_{AB} \otimes \mathcal{R}_{\tilde{A}B}(P_{AB\tilde{A}\tilde{B}}), \]  
\[ (F3) \]

where

\[ \Phi_{AB} := \int dU (\mathcal{U}_A \otimes \mathcal{U}_B)(P_{AB\tilde{A}\tilde{B}}) \]  
\[ = \tilde{T}_{AB}(P_{AB\tilde{A}\tilde{B}}). \]  
\[ (F4) \]

By invoking the twirling identity in (B10), we find that

\[ \tilde{P}_{AB\tilde{A}\tilde{B}} = \Phi_{AB} \otimes \mathcal{R}_{\tilde{A}B}[\Phi_{AB} P_{AB\tilde{A}\tilde{B}}] \]  
\[ + \frac{1}{d^2 - 1} \otimes \mathcal{R}_{\tilde{A}B}[\{I_{AB} - \Phi_{AB}\} P_{AB\tilde{A}\tilde{B}}]. \]  
\[ (F5) \]
Now setting

\[ K_{\tilde{A}\tilde{B}} := \frac{1}{d} \text{Tr}_{AB}[\Phi_{AB} P_{AB\tilde{A}\tilde{B}}], \quad (F7) \]

\[ L_{\tilde{A}\tilde{B}} := \frac{1}{d} \text{Tr}_{AB}[(I_{AB} - \Phi_{AB}) P_{AB\tilde{A}\tilde{B}}]. \quad (F8) \]

we can write

\[ \tilde{P}_{AB\tilde{A}\tilde{B}} = \Gamma_{AB} \otimes K_{\tilde{A}\tilde{B}} + \frac{dI_{AB} - \Gamma_{AB}}{d^2 - 1} \otimes L_{\tilde{A}\tilde{B}}. \quad (F9) \]

We can then determine the conditions on \( K_{\tilde{A}\tilde{B}} \) and \( L_{\tilde{A}\tilde{B}} \) in order for \( \tilde{P}_{AB\tilde{A}\tilde{B}} \) to be a legitimate C-PPT-P channel. We know that \( P_{AB\tilde{A}\tilde{B}} \) is a Choi operator for a C-PPT-P channel (see Section II-B-4) if the following conditions hold

\[ \tilde{P}_{AB\tilde{A}\tilde{B}} \geq 0, \] (F10)
\[ \text{Tr}_{B}[\tilde{P}_{AB\tilde{A}\tilde{B}}] = I_{A\tilde{A}\tilde{B}}, \] (F11)
\[ T_{B\tilde{B}}(\tilde{P}_{AB\tilde{A}\tilde{B}}) \geq 0. \] (F12)

So we determine what each of these conditions impose on \( K_{\tilde{A}\tilde{B}} \) and \( L_{\tilde{A}\tilde{B}} \). By considering that \( \Gamma_{AB} \) is orthogonal to \( \frac{dI_{AB} - \Gamma_{AB}}{d^2 - 1} \), we conclude that \( \tilde{P}_{AB\tilde{A}\tilde{B}} \geq 0 \) if and only if

\[ K_{\tilde{A}\tilde{B}}, L_{\tilde{A}\tilde{B}} \geq 0. \] (F13)

Now consider that

\[ I_{A\tilde{A}\tilde{B}} = \text{Tr}_{B}[\tilde{P}_{AB\tilde{A}\tilde{B}}] = \text{Tr}_{B}[\Gamma_{AB}] \otimes K_{\tilde{A}\tilde{B}} + \text{Tr}_{B}\left[ \frac{dI_{AB} - \Gamma_{AB}}{d^2 - 1} \otimes L_{\tilde{A}\tilde{B}} \right] = I_A \otimes K_{\tilde{A}\tilde{B}} + I_A \otimes L_{\tilde{A}\tilde{B}} = I_A \otimes (K_{\tilde{A}\tilde{B}} + L_{\tilde{A}\tilde{B}}). \] (F14)

The equality \( I_{A\tilde{A}\tilde{B}} = I_A \otimes (K_{\tilde{A}\tilde{B}} + L_{\tilde{A}\tilde{B}}) \) is then equivalent to

\[ K_{\tilde{A}\tilde{B}} + L_{\tilde{A}\tilde{B}} = I_{\tilde{A}\tilde{B}}. \] (F15)

Finally, by making use of the identity \( T_B(\Gamma_{AB}) = V_{AB} \) and the definitions in (E36) and (E37) of the symmetric and antisymmetric subspace projectors, respectively, consider that

\[ T_{B\tilde{B}}(\tilde{P}_{AB\tilde{A}\tilde{B}}) = T_B(\Gamma_{AB}) \otimes T_B(K_{\tilde{A}\tilde{B}}) + T_B\left( \frac{dI_{AB} - \Gamma_{AB}}{d^2 - 1} \right) \otimes T_B(L_{\tilde{A}\tilde{B}}) = V_{AB} \otimes T_B(K_{\tilde{A}\tilde{B}}) + \frac{dI_{AB} - V_{AB}}{d^2 - 1} \otimes T_B(L_{\tilde{A}\tilde{B}}) = (\Pi_{AB}^S - \Pi_{AB}^A) \otimes T_B(K_{\tilde{A}\tilde{B}}) + \frac{d}{d^2 - 1} \left[ \Pi_{AB}^S + \frac{\Pi_{AB}^A}{d+1} - \frac{\Pi_{AB}^S - \Pi_{AB}^A}{d-1} \right] \otimes T_B(L_{\tilde{A}\tilde{B}}) = (\Pi_{AB}^S - \Pi_{AB}^A) \otimes T_B(K_{\tilde{A}\tilde{B}}) + \frac{\Pi_{AB}^S + \Pi_{AB}^A}{d+1} \otimes T_B(L_{\tilde{A}\tilde{B}}). \] (F16)

Since \( \Pi_{AB}^S \) and \( \Pi_{AB}^A \) are projectors and orthogonal to each other, it follows that \( T_{B\tilde{B}}(\tilde{P}_{AB\tilde{A}\tilde{B}}) \geq 0 \) if and only if

\[ T_B(K_{\tilde{A}\tilde{B}} + \frac{1}{d+1} L_{\tilde{A}\tilde{B}}) \geq 0, \] (F20)
\[ T_B\left( \frac{1}{d-1} L_{\tilde{A}\tilde{B}} - K_{\tilde{A}\tilde{B}} \right) \geq 0. \] (F21)

We can simplify these conditions by employing (F15). Substituting, we find that they reduce to

\[ 0 \leq T_B(K_{\tilde{A}\tilde{B}} + \frac{1}{d+1} L_{\tilde{A}\tilde{B}}), \] (F23)
\[ = T_B(K_{\tilde{A}\tilde{B}} + \frac{1}{d+1} (I_{\tilde{A}\tilde{B}} - K_{\tilde{A}\tilde{B}})), \] (F24)
\[ = T_B\left( \frac{I_{\tilde{A}\tilde{B}}}{d+1} + \frac{d}{d+1} K_{\tilde{A}\tilde{B}} \right), \] (F25)

which is equivalent to

\[ -I_{\tilde{A}\tilde{B}} \leq d T_B(K_{\tilde{A}\tilde{B}}). \] (F26)

Additionally,

\[ 0 \leq T_B\left( \frac{1}{d-1} L_{\tilde{A}\tilde{B}} - K_{\tilde{A}\tilde{B}} \right), \] (F27)
\[ = T_B\left( \frac{1}{d-1} (I_{\tilde{A}\tilde{B}} - K_{\tilde{A}\tilde{B}}) - K_{\tilde{A}\tilde{B}} \right), \] (F28)
\[ = T_B\left( \frac{1}{d-1} I_{\tilde{A}\tilde{B}} - \frac{d}{d-1} K_{\tilde{A}\tilde{B}} \right), \] (F29)

which is equivalent to

\[ d T_B(K_{\tilde{A}\tilde{B}}) \leq I_{\tilde{A}\tilde{B}}. \] (F30)

So we conclude that the operators \( K_{\tilde{A}\tilde{B}} \) and \( L_{\tilde{A}\tilde{B}} \) correspond to a C-PPT-P channel if

\[ K_{\tilde{A}\tilde{B}}, L_{\tilde{A}\tilde{B}} \geq 0, \] (F31)
\[ K_{\tilde{A}\tilde{B}} + L_{\tilde{A}\tilde{B}} = I_{\tilde{A}\tilde{B}}, \] (F32)
\[ -I_{\tilde{A}\tilde{B}} \leq d T_B(K_{\tilde{A}\tilde{B}}) \leq I_{\tilde{A}\tilde{B}}. \] (F33)

Now employing reasoning similar to that in Appendix C7, we conclude that

\[ \inf_{\varphi \in \text{PPT}} \frac{1}{2} \left\| P_{\tilde{A}\tilde{B}} - B \circ \mathcal{A}_{\tilde{A}\tilde{B}} - \text{id}^d \right\|_{\text{op}} = 1 - \sup_{K_{\tilde{A}\tilde{B}}, L_{\tilde{A}\tilde{B}} \geq 0} \left\{ \begin{array}{l} \text{Tr}[K_{\tilde{A}\tilde{B}} T(\rho_{\tilde{A}\tilde{B}})] : \\ K_{\tilde{A}\tilde{B}} + L_{\tilde{A}\tilde{B}} = I_{\tilde{A}\tilde{B}}, \\ -I_{\tilde{A}\tilde{B}} \leq d T_B(K_{\tilde{A}\tilde{B}}) \leq I_{\tilde{A}\tilde{B}} \end{array} \right\}. \] (F34)
\[ 1 - \sup_{K_{\hat{A}\hat{B}} \geq 0} \left\{ \text{Tr}\left[ K_{\hat{A}\hat{B}} T(\rho_{\hat{A}\hat{B}}) \right] : K_{\hat{A}\hat{B}} \preceq I_{\hat{A}\hat{B}}, I_{\hat{A}\hat{B}} \pm d T_B(K_{\hat{A}\hat{B}}) \geq 0 \right\}, \tag{F35} \]

where the last simplification follows because the operator \( L_{\hat{A}\hat{B}} \) does not appear in the objective and thus can be considered a slack variable. Finally, we can eliminate the transpose on \( K_{\hat{A}\hat{B}} \) in the objective function by making the substitution \( K_{\hat{A}\hat{B}} \rightarrow T(K_{\hat{A}\hat{B}}) \) and noticing that

\[ \text{Tr}\left[ K_{\hat{A}\hat{B}} T(\rho_{\hat{A}\hat{B}}) \right] = \text{Tr}[T(K_{\hat{A}\hat{B}}) \rho_{\hat{A}\hat{B}}], \tag{F36} \]

\[ K_{\hat{A}\hat{B}} \preceq I_{\hat{A}\hat{B}} \Leftrightarrow T(K_{\hat{A}\hat{B}}) \preceq I_{\hat{A}\hat{B}}, \tag{F37} \]

\[ K_{\hat{A}\hat{B}} \preceq I_{\hat{A}\hat{B}} \Leftrightarrow T(I_{\hat{A}\hat{B}} \pm d T_B(K_{\hat{A}\hat{B}})) \geq 0, \tag{F38} \]

\[ I_{\hat{A}\hat{B}} \pm d T_B(K_{\hat{A}\hat{B}}) \geq 0 \]

\[ \Leftrightarrow T(I_{\hat{A}\hat{B}} \pm d T_B(K_{\hat{A}\hat{B}})) \geq 0 \]

\[ \Leftrightarrow I_{\hat{A}\hat{B}} \pm d T_B(T(K_{\hat{A}\hat{B}})) \geq 0. \tag{F39} \]

\[ \text{Appendix G: Proof of Equation (223)} \]

In this appendix, we derive the SDP lower bound on the simulation error of approximate quantum error correction, when using PPT constraints alone. Ref. [LM15] already worked this out, but we provide another derivation here for completeness. The simulation error for approximate quantum error correction, when using a resource channel \( N_{\hat{A}\rightarrow\hat{B}} \) and a C-PPT-P, non-signaling superchannel \( \Theta_{(\hat{A} \rightarrow \hat{B}) \rightarrow (A \rightarrow B)} \) for free, is as follows:

\[ \inf_{\Theta \in \text{PPT} \cap \text{NS}} \frac{1}{2} \left\| \Theta_{(\hat{A} \rightarrow \hat{B}) \rightarrow (A \rightarrow B)} (N_{\hat{A}\rightarrow\hat{B}} - \text{id}_{\hat{A}\rightarrow\hat{B}}) \right\|, \tag{G1} \]

where we have abbreviated the set of C-PPT-P, non-signaling superchannels by \( \text{PPT} \cap \text{NS} \). Some of the arguments in this appendix are almost identical to those in Appendix F, and so we provide fewer details and explanations here. The Choi operator for the channel \( \Theta_{(\hat{A} \rightarrow \hat{B}) \rightarrow (A \rightarrow B)} (N_{\hat{A}\rightarrow\hat{B}}) \) is given by the propagation rule in (122):

\[ \text{Tr}_{\hat{A}\hat{B}}[\Gamma_{\hat{A}\hat{B}}(I-N_{\hat{A}\rightarrow\hat{B}}) \Theta_{\hat{A}\rightarrow\hat{B}}], \tag{G2} \]

where \( \Gamma_{\hat{A}\hat{B}} \) is the Choi operator for \( \Theta_{(\hat{A} \rightarrow \hat{B}) \rightarrow (A \rightarrow B)} \). In order to be a legitimate C-PPT-P, non-signaling superchannel, the following constraints should be satisfied:

\[ \Gamma_{\hat{A}\hat{B}} \geq 0, \tag{G3} \]

\[ \text{Tr}_{\hat{A}\hat{B}}[\Gamma_{\hat{A}\hat{B}}] = I_{\hat{A}\hat{B}}, \tag{G4} \]

\[ \text{Tr}_{\hat{A}\hat{B}}[\Gamma_{\hat{A}\hat{B}}] = \frac{1}{d_{\hat{B}}} \text{Tr}_{\hat{A}\hat{B}}[\Gamma_{\hat{A}\hat{B}}] \otimes I_{\hat{B}}. \tag{G5} \]

\[ \text{Tr}_{\hat{A}\hat{B}}[\Gamma_{\hat{A}\hat{B}}] = \frac{1}{d} \text{Tr}_{\hat{A}\hat{B}}[\Gamma_{\hat{A}\hat{B}}] \otimes I_{A}. \tag{G6} \]

\[ \text{Tr}_{\hat{A}\hat{B}}[\Gamma_{\hat{A}\hat{B}}] \geq 0. \tag{G7} \]

Due to the unitary covariance symmetry of the identity channel to be simulated, it suffices to restrict the optimization in (G1) to superchannels with Choi operators having the following form:

\[ \Gamma_{\hat{A}\hat{B}A\hat{B}B} = \Gamma_{A\hat{B}} \otimes K_{\hat{A}\hat{B}} + \frac{d I_{AB} - \Gamma_{AB}}{d^2 - 1} \otimes L_{\hat{A}\hat{B}}. \tag{G8} \]

The condition in (G3) is equivalent to \( K_{\hat{A}\hat{B}} , L_{\hat{A}\hat{B}} \geq 0 \). The condition in (G4) implies that

\[ I_{\hat{AB}} = \text{Tr}_{\hat{A}\hat{B}}[\Gamma_{\hat{A}\hat{B}A\hat{B}B}] \]

\[ = \text{Tr}_{\hat{B}}[\Gamma_{AB} \otimes \text{Tr}_{\hat{A}}[K_{\hat{A}\hat{B}}] + \text{Tr}_{\hat{B}} \left[ \frac{d I_{AB} - \Gamma_{AB}}{d^2 - 1} \otimes \text{Tr}_{\hat{A}}[L_{\hat{A}\hat{B}}] \right] \] \]

\[ = I_{A} \otimes \text{Tr}_{\hat{A}}[K_{\hat{A}\hat{B}}] + I_{A} \otimes \text{Tr}_{\hat{A}}[L_{\hat{A}\hat{B}}], \tag{G11} \]

\[ = I_{A} \otimes \text{Tr}_{\hat{A}}[K_{\hat{A}\hat{B}} + L_{\hat{A}\hat{B}}], \tag{G12} \]

which is equivalent to

\[ \text{Tr}_{\hat{A}}[K_{\hat{A}\hat{B}} + L_{\hat{A}\hat{B}}] = I_{\hat{B}}. \tag{G13} \]

Now consider that

\[ \text{Tr}_{\hat{B}}[\Gamma_{\hat{A}\hat{B}A\hat{B}B}] \]

\[ = \text{Tr}_{\hat{B}}[\Gamma_{AB} \otimes \text{Tr}_{\hat{B}}[\Gamma_{\hat{A}\hat{B}}] + \text{Tr}_{\hat{B}} \left[ \frac{d I_{AB} - \Gamma_{AB}}{d^2 - 1} \otimes \text{Tr}_{\hat{A}}[L_{\hat{A}\hat{B}}] \right] \] \]

\[ = I_{A} \otimes (K_{\hat{A}\hat{B}} + L_{\hat{A}\hat{B}}), \tag{G15} \]

and

\[ \text{Tr}_{\hat{B}}[\Gamma_{\hat{A}\hat{B}A\hat{B}B}] \otimes I_{\hat{B}} \]

\[ = \left( \text{Tr}_{\hat{B}}[\Gamma_{AB} \otimes \text{Tr}_{\hat{B}}[K_{\hat{A}\hat{B}}] + \text{Tr}_{\hat{B}} \left[ \frac{d I_{AB} - \Gamma_{AB}}{d^2 - 1} \otimes \text{Tr}_{\hat{B}}[L_{\hat{A}\hat{B}}] \right] \right) \otimes I_{\hat{B}} \]

\[ = I_{A} \otimes \left( \text{Tr}_{\hat{B}}[K_{\hat{A}\hat{B}} + L_{\hat{A}\hat{B}}] \right) \otimes I_{\hat{B}}. \tag{G17} \]

So the constraint in (G5) is equivalent to

\[ K_{\hat{A}\hat{B}} + L_{\hat{A}\hat{B}} = \text{Tr}_{\hat{B}}[K_{\hat{A}\hat{B}} + L_{\hat{A}\hat{B}}] \otimes \frac{1}{d_{\hat{B}}} I_{\hat{B}}. \tag{G18} \]

The other non-signaling constraint in (G6) is

\[ \text{Tr}_{\hat{A}}[\Gamma_{\hat{A}\hat{B}A\hat{B}B}] = \frac{1}{d} \text{Tr}_{\hat{A}}[\Gamma_{\hat{A}\hat{B}A\hat{B}B}] \otimes I_{A}. \tag{G19} \]

Consider that

\[ \text{Tr}_{\hat{A}}[\Gamma_{\hat{A}\hat{B}A\hat{B}B}] \]

\[ = \text{Tr}_{\hat{A}}[\Gamma_{AB} \otimes \text{Tr}_{\hat{A}}[K_{\hat{A}\hat{B}}] + \text{Tr}_{\hat{A}} \left[ \frac{d I_{AB} - \Gamma_{AB}}{d^2 - 1} \otimes \text{Tr}_{\hat{A}}[L_{\hat{A}\hat{B}}] \right] \] \]

while

\[ \text{Tr}_{\hat{A}}[\Gamma_{\hat{A}\hat{B}A\hat{B}B}] \]

\[ = I_{\hat{B}} \otimes \text{Tr}_{\hat{A}}[K_{\hat{A}\hat{B}}] + I_{A} \otimes \text{Tr}_{\hat{A}}[L_{\hat{A}\hat{B}}] \tag{G22} \]
Then the equality in (G6) is equivalent to
\[ \Gamma = \frac{1}{d} I_{AB} \otimes (\text{Tr}_{A}[K_{\hat{A}B} + L_{\hat{A}B}]) . \] (G23)

which simplifies to
\[ 1 - \sup_{K_{\hat{A}B}, L_{\hat{A}B} \geq 0} \begin{cases} 
\text{Tr}[T(K_{\hat{A}B})\Gamma^N_{\hat{A}B}] : \\
\text{Tr}_{A}[K_{\hat{A}B} + L_{\hat{A}B}] = I_{\hat{B}}, \\
(d^2 - 1) \text{Tr}_{A}[K_{\hat{A}B}] = \text{Tr}_{A}[L_{\hat{A}B}], \\
K_{\hat{A}B} + L_{\hat{A}B} = \\
\text{Tr}_{\hat{B}}[K_{\hat{A}B} + L_{\hat{A}B}] \otimes \frac{1}{d_{\hat{B}}} I_{\hat{B}}, \\
T_{\hat{B}} \left( K_{\hat{A}B} + \frac{1}{d^2} L_{\hat{A}B} \right) \geq 0, \\
T_{\hat{B}} \left( \frac{1}{d_{\hat{B}}} L_{\hat{A}B} - K_{\hat{A}B} \right) \geq 0 
\end{cases} \] (G34)

The transpose on $K_{\hat{A}B}$ in the objective function can be eliminated by making the substitution $K_{\hat{A}B} \rightarrow T(K_{\hat{A}B})$ and noticing that
\[ K_{\hat{A}B}, L_{\hat{A}B} \geq 0 \iff T(K_{\hat{A}B}), T(L_{\hat{A}B}) \geq 0, \] (G35)

\[ d^2 \text{Tr}_{A}[K_{\hat{A}B}] = I_{\hat{B}} \iff d^2 T_{\hat{B}}(\text{Tr}_{A}[T(K_{\hat{A}B})]) = T_{\hat{B}}(I_{\hat{B}}) \] (G36)

\[ d^2 T_{\hat{B}}(\text{Tr}_{A}[T(K_{\hat{A}B})]) = I_{\hat{B}} \iff d^2(\text{Tr}_{A}[T(K_{\hat{A}B})]) = I_{\hat{B}}. \] (G37)

\[ K_{\hat{A}B} + L_{\hat{A}B} = T_{\hat{B}}[K_{\hat{A}B} + L_{\hat{A}B}] \otimes \frac{1}{d_{\hat{B}}} I_{\hat{B}} \iff T(K_{\hat{A}B} + T(L_{\hat{A}B})) \] (G39)

so that this is equivalent to the following two constraints:
\[ \text{Tr}_{A}[K_{\hat{A}B}] = \frac{1}{d^2} \text{Tr}_{A}[K_{\hat{A}B} + L_{\hat{A}B}], \] (G28)

\[ \frac{1}{d^2 - 1} \text{Tr}_{A}[L_{\hat{A}B}] = \frac{1}{d^2} \text{Tr}_{A}[K_{\hat{A}B} + L_{\hat{A}B}], \] (G29)

which in turn are equivalent to
\[ (d^2 - 1) \text{Tr}_{A}[K_{\hat{A}B}] = \text{Tr}_{A}[L_{\hat{A}B}]. \] (G30)

By applying (F21)–(F22), the condition in (G7) is equivalent to
\[ T_{\hat{B}} \left( K_{\hat{A}B} + \frac{1}{d + 1} L_{\hat{A}B} \right) \geq 0, \] (G31)

\[ T_{\hat{B}} \left( \frac{1}{d - 1} L_{\hat{A}B} - K_{\hat{A}B} \right) \geq 0. \] (G32)

Thus, the optimization in (G1) reduces as follows:
\[ 1 - \sup_{K_{\hat{A}B}, L_{\hat{A}B} \geq 0} \begin{cases} 
\text{Tr}[T(K_{\hat{A}B})\Gamma^N_{\hat{A}B}] : \\
(d^2 - 1) \text{Tr}_{A}[K_{\hat{A}B}] = \text{Tr}_{A}[L_{\hat{A}B}], \\
\text{Tr}_{A}[K_{\hat{A}B} + L_{\hat{A}B}] = I_{\hat{B}}, \\
\text{Tr}_{\hat{B}}[K_{\hat{A}B} + L_{\hat{A}B}] \otimes \frac{1}{d_{\hat{B}}} I_{\hat{B}}, \\
T_{\hat{B}} \left( K_{\hat{A}B} + \frac{1}{d^2} L_{\hat{A}B} \right) \geq 0, \\
T_{\hat{B}} \left( \frac{1}{d_{\hat{B}}} L_{\hat{A}B} - K_{\hat{A}B} \right) \geq 0 
\end{cases} \] (G33)
\[ T_B \left( T(K_{AB}) + \frac{1}{d+1} T(L_{AB}) \right) \geq 0, \quad \text{(G47)} \]

and similarly,
\[ T_B \left( \frac{1}{d-1} L_{AB} - K_{AB} \right) \geq 0 \quad \text{(G48)} \]

\[ T_B \left( \frac{1}{d-1} T(L_{AB}) - T(K_{AB}) \right) \geq 0. \quad \text{(G49)} \]

Thus, the optimization reduces to
\[ 1 - \sup_{K_{AB}, L_{AB} \geq 0} \left\{ \begin{array}{l}
\text{Tr}[K_{AB}^N] : \\
\text{Tr}_A[K_{AB} + L_{AB}] = I_B, \\
K_{AB} + L_{AB} = \\
T_B[K_{AB} + L_{AB} \otimes \frac{1}{d_B} I_B], \\
(d^2 - 1) \text{Tr}_A[K_{AB}] = \text{Tr}_A[L_{AB}], \\
T_B \left( K_{AB} + \frac{1}{d_B} L_{AB} \right) \geq 0, \\
T_B \left( \frac{1}{d_B} L_{AB} - K_{AB} \right) \geq 0, \\
\sigma_A = \frac{1}{d_B} \text{Tr}_B[K_{AB} + L_{AB}], \\
\text{Tr}[\sigma_A] = 1.
\end{array} \right. \quad \text{(G50)} \]

To recover the form in [LM15], starting from (G50), we can introduce a new optimization variable \( \sigma_A \) with the constraint
\[ \sigma_A = \frac{1}{d_B} \text{Tr}_B[K_{AB} + L_{AB}]. \quad \text{(G51)} \]

This operator is positive semi-definite because \( K_{AB} \) and \( L_{AB} \) are, and it has trace equal to one because
\[ \text{Tr}[\sigma_A] = \frac{1}{d_B} \text{Tr}_A[K_{AB} + L_{AB}] = \frac{1}{d_B} \text{Tr}_B[I_B] = 1. \quad \text{(G52)} \]

Thus, the optimization in (G50) above is equivalent to
\[ 1 - \sup_{K_{AB}, L_{AB} \geq 0} \sigma_A \geq 0 \left\{ \begin{array}{l}
\text{Tr}[K_{AB}^N] : \\
\text{Tr}_A[K_{AB} + L_{AB}] = I_B, \\
K_{AB} + L_{AB} = \\
T_B[K_{AB} + L_{AB} \otimes \frac{1}{d_B} I_B], \\
(d^2 - 1) \text{Tr}_A[K_{AB}] = \text{Tr}_A[L_{AB}], \\
T_B \left( K_{AB} + \frac{1}{d_B} L_{AB} \right) \geq 0, \\
T_B \left( \frac{1}{d_B} L_{AB} - K_{AB} \right) \geq 0, \\
\sigma_A = \frac{1}{d_B} \text{Tr}_B[K_{AB} + L_{AB}], \\
\text{Tr}[\sigma_A] = 1.
\end{array} \right. \quad \text{(G55)} \]

However, now we can substitute to find that
\[ K_{AB} + L_{AB} = \sigma_A \otimes I_B, \quad \text{(G56)} \]

which implies that
\[ K_{AB} + \frac{1}{d+1} L_{AB} = K_{AB} + \frac{1}{d+1} (\sigma_A \otimes I_B - K_{AB}) \quad \text{(G57)} \]

\[ = \frac{1}{d+1} (dK_{AB} + \sigma_A \otimes I_B) \quad \text{(G58)} \]

\[ = \frac{1}{d+1} \left( \sigma_A \otimes I_B - K_{AB} \right) \quad \text{(G59)} \]

Thus,
\[ T_B \left( K_{AB} + \frac{1}{d+1} L_{AB} \right) \geq 0, \quad \text{(G60)} \]

is equivalent to
\[ d \text{Tr}_B(K_{AB}) + \sigma_A \otimes I_B \geq 0, \quad \text{(G61)} \]

\[ \sigma_A \otimes I_B - d \text{Tr}_B(K_{AB}) \geq 0, \quad \text{(G62)} \]

which in turn is equivalent to
\[ \sigma_A \otimes I_B \pm d \text{Tr}_B(K_{AB}) \geq 0. \quad \text{(G63)} \]

Also, the constraint \( \text{Tr}_A[K_{AB} + L_{AB}] = I_B \) becomes
\[ I_B = \text{Tr}_A[K_{AB} + L_{AB}] = \text{Tr}_A[K_{AB} + \sigma_A \otimes I_B - K_{AB}] \]
\[ = I_B. \quad \text{(G64)} \]

and is thus redundant. The constraint \( (d^2 - 1) \text{Tr}_A[K_{AB}] = \text{Tr}_A[L_{AB}] \) becomes
\[ \left( d^2 - 1 \right) \text{Tr}_A[K_{AB}] = \text{Tr}_A[L_{AB}] \]
\[ = \text{Tr}_A[\sigma_A \otimes I_B - K_{AB}] \]
\[ = I_B - \text{Tr}_A[K_{AB}], \quad \text{(G65)} \]

which is equivalent to
\[ d^2 \text{Tr}_A[K_{AB}] = I_B \quad \text{(G66)} \]

Employing these observations, the SDP above reduces to
\[ 1 - \sup_{K_{AB}, L_{AB} \geq 0} \sigma_A \geq 0 \left\{ \begin{array}{l}
\text{Tr}[K_{AB}^N] : \\
K_{AB} + L_{AB} = \sigma_A \otimes I_B, \\
d^2 \text{Tr}_A[K_{AB}] = I_B, \\
\sigma_A \otimes I_B \pm d \text{Tr}_B(K_{AB}) \geq 0, \\
\text{Tr}[\sigma_A] = 1.
\end{array} \right. \quad \text{(G73)} \]

We now notice that the constraint \( \sigma_A = \frac{1}{d_B} \text{Tr}_B[K_{AB} + L_{AB}] \) is a consequence of the constraint \( K_{AB} + L_{AB} = \sigma_A \otimes I_B \), from applying a partial trace over \( B \). It is thus redundant and can be eliminated, leading to
\[ 1 - \sup_{K_{AB}, L_{AB} \geq 0} \sigma_A \geq 0 \left\{ \begin{array}{l}
\text{Tr}[K_{AB}^N] : \\
K_{AB} + L_{AB} = \sigma_A \otimes I_B, \\
d^2 \text{Tr}_A[K_{AB}] = I_B, \\
\sigma_A \otimes I_B \pm d \text{Tr}_B(K_{AB}) \geq 0, \\
\text{Tr}[\sigma_A] = 1.
\end{array} \right. \quad \text{(G74)} \]
Now we can understand $L_{\hat{A}\hat{B}}$ as a slack variable and replace the equality constraint $K_{\hat{A}\hat{B}} + L_{\hat{A}\hat{B}} = \sigma_{\hat{A}} \otimes I_{\hat{B}}$ with an inequality constraint and eliminate $L_{\hat{A}\hat{B}}$, leading to the final form in (223):

$$1 - \sup_{K_{\hat{A}\hat{B}}, \sigma_{\hat{A}} \geq 0} \left\{ \frac{\text{Tr}[K_{\hat{A}\hat{B}} \Gamma_{N_{\hat{A}\hat{B}}}] :}{K_{\hat{A}\hat{B}} \leq \sigma_{\hat{A}} \otimes I_{\hat{B}}, \quad d^2 \text{Tr}_{\hat{A}}[K_{\hat{A}\hat{B}}] = I_{\hat{B}}, \quad \sigma_{\hat{A}} \otimes I_{\hat{B}} = d T_{\hat{B}}(K_{\hat{A}\hat{B}}) \geq 0, \quad \text{Tr}[\sigma_{\hat{A}}] = 1.} \right\}.$$