The Casimir effect for parallel plates at finite temperature in the presence of one fractal extra compactified dimension

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Abstract

We discuss the Casimir effect for massless scalar fields subject to the Dirichlet boundary conditions on the parallel plates at finite temperature in the presence of one fractal extra compactified dimension. We obtain the Casimir energy density with the help of the regularization of multiple zeta function with one arbitrary exponent and further the renormalized Casimir energy density involving the thermal corrections. It is found that when the temperature is sufficiently high, the sign of the Casimir energy remains negative no matter how great the scale dimension $\delta$ is within its allowed region. We derive and calculate the Casimir force between the parallel plates affected by the fractal additional compactified dimension and surrounding temperature. The stronger thermal influence leads the force to be stronger. The nature of the Casimir force keeps attractive.

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I. Introduction

The model of higher-dimensional spacetime as a powerful ingredient is employed to unify the interactions in nature. In the Kaluza-Klein theory, one extra dimension in our Universe was introduced to be compactified in order to unify gravity and classical electrodynamics [1, 2]. The characteristic size of the additional dimension is of the order of Planck length. It may be better to describe the background whose scale is on the Planck order with the help of fractal geometry involving some non-integer dimensions in the process of the investigation of quantum gravity [3]. More attentions of the physical community are attracted to this topic. It is found that the spectral dimension of the spacetime where Quantum Einstein Gravity lives in is equal to 2 microscopically while to 4 on macroscopic scales [4]. It is also discovered that the background with a quantum symmetry has a scale-dependent fractal dimension at short scales to describe a phenomenon appeared in the quantum gravity [5]. A kind of field theory which is Lorentz invariant, power-counting renormalizable, ultraviolet finite and causal is proposed to explore a consistent theory of quantum gravity needing the fractal geometry [6, 7]. Within the frame of modern Kaluza-Klein issue it seems to be reasonable to describe the extra compactified space in virtue of fractal geometry because the quantum fluctuations is extremely tiny close to the Planck scale. I. Smolyaninov originated a fractal extra compactified dimension to modify the Kaluza-Klein model [8].

More than 60 years ago Casimir put forward an effect of boundaries [9]. The so-called Casimir effect is essentially a direct consequence of quantum field theory due to a change in the spectrum of vacuum oscillations when the quantization volume is bounded, or some background fields are inserted. More than 40 years ago Boyer found that the Casimir force for a conducting spherical shell is repulsive, meaning that the circumstance will determine the nature of the force [10]. Afterwards, a lot of effort has been contributed to the related topics and more and more results and methods have been put forward [11-18]. The precision of the measurements has been greatly improved experimentally [19-21]. The Casimir effect has something to do with various factors, so the sign of the Casimir energy and the nature of the Casimir force can become principles to be applied in many subjects. The Casimir effect can be used to explore high-dimensional spacetimes. We can study the Casimir effect for the simple device such as parallel plates in the spacetimes with extra compactified dimensions to show that the extra-dimension influence was manifest and distinct, then we open a window to probe the Kaluza-Klein model [22-37]. Recently more attention of the physical community is also paid to the Casimir effect for parallel plates or piston in the braneworld, such as Randall-Sundrum models, etc. [38-51]. During the investigation above we can estimate the properties of the warped world which may be utilized to resolve the hierarchy problem. The Casimir effect has also been discussed in the context of string theory [52-54]. In addition the Casimir effect for fermionic field within the parallel plates with various types of boundary conditions was evaluated [55, 56].

It is fundamental to research on the Casimir effect for parallel plates in the presence of fractal extra compactified dimensions. As mentioned above the size of additional space is extremely tiny
close to the Planck order, so the properties of the compactified space should be described in virtue of fractal geometry. We have studied the Casimir effect in the parallel-plate system in the background involving one fractal extra compactified dimension [57]. The number of the additional spatial dimension is fractal instead of being exactly a positive integer, so the dimensionality varies within a smaller region. Here the dimensionality of additional space is denoted as $D = D_T + \delta$, where $D_T$ is topological dimension and $\delta$ is the scale dimension. We show that the negative sign of the renormalized Casimir energy which is the difference between the regularized energy for two parallel plates and the one with no plates narrows the region of the fractal dimensionality of additional space, the scale dimension $\delta \in \left(\frac{1}{2}, 1\right)$ instead of $\delta \in [0, 1)$. In addition, it is also found that the larger scale dimension will lead to the greater revision on the original Casimir force. The Casimir force between two parallel plates can show whether the dimensionality of additional space is integer or fraction because the shapes of two kinds of Casimir force are not exactly identical.

The quantum field theory at finite temperature shares a lot of effects. The thermal influence on the Casimir effect can not be neglected, and its influence certainly modifies the effect. The Casimir energy for a rectangular cavity in a background with nonzero temperature was evaluated, and the temperature controls the energy sign [58]. The Casimir effect for parallel plates, including thermal corrections in the world with additional compactified dimensions, was discussed, and the magnitude of Casimir force as well as the sign of Casimir energy relates to the temperature [59-63]. The Casimir effect for a scalar field within two parallel plates under thermal influence in the bulk region of Randall-Sundrum models was studied [64]. We have researched on the Casimir effect for parallel plates involving massless Majorana fermions obeying the bag boundary conditions at finite temperature [65]. The Casimir force and Casimir free energy instead of Casimir energy for massless Majorana fermions with thermal modifications in a magnetic field is also investigated [66].

Here we plan to discuss the thermal corrections to the Casimir effect for parallel plates in the spacetime with a fractal extra compactified dimension in detail to generalize the results of Ref. [57]. We wonder whether the thermal influence enlarges or narrows the range belonging to the scale dimension of additional compactified space and how the influence revises the description of the Casimir effect for parallel plates under this kind of environment. At first we derive the frequency of massless scalar fields referring to Dirichlet boundary conditions at plates containing thermal influence by means of the fractal-extra-dimension Kaluza-Klein model put forward by Smolyaninov [8] and finite-temperature field theory. We regularize the total vacuum energy density to obtain the Casimir energy density by means of the regularization technique of multiple zeta function with arbitrary exponents. During this investigation we obtain the finite part of this kind of zeta function according to the procedure in Ref. [13, 14]. Further we obtain the renormalized Casimir energy density which is the difference between the regularized energy density for two parallel plates and the one with no plates at finite temperature on purpose to subtract the divergent part. The Casimir force between the parallel plates can also be gained from the renormalized Casimir energy density. Our discussions for thermal influence on this kind of Casimir effect are given at the end of this
II. The Casimir energy for parallel plates at finite temperature in the spacetime with a fractal extra compactified dimension

Within the Kaluza-Klein issue with a fractal extra compactified dimension [8], the scalar field is periodic in the fifth coordinate $x^5$, leading to the appearance of an infinite tower of solutions with a quantized $x^5$-component of the momentum like $q_n = \frac{2\pi n}{L}$. The scale dimension is defined as,

$$\delta = D - D_T$$

where $L$ is the main fractal variable denoting a length of a fractal curve, an area of a fractal surface, etc.. The coefficients $l$ and $\lambda$ are measurement scale. Here $D$ is the fractal dimensionality. $D_T$ is the topological dimension and $D_T = 1$ for a curve, $D_T = 2$ for a surface. The spectrum of momentum is,

$$q_n = 2\pi \left( \frac{n}{L_0 \delta} \right)^{\frac{1}{2+\delta}}$$

where $n$ is a nonnegative integer. $L_0$ is the length measured when $\lambda = l$. This tower of solutions will recover to be the tower in the regular five-dimensional Kaluza-Klein theory if we choose $\delta = 0$.

In finite-temperature field theories the imaginary time formalism can be employed to describe the scalar fields in thermal equilibrium. We introduce a partition function for a system,

$$Z = N \int_{\text{periodic}} \prod_k D\phi_k \exp\left[ \int_0^\beta d\tau \int d^3x \mathcal{L}(\phi_\parallel, \partial_\mathcal{E} \phi_\parallel) \right]$$

where $\mathcal{L}$ is the Lagrangian density for the system under consideration. $N$ is a constant and the "periodic" means $\phi_k(0, x) = \phi_k(\beta, x)$, $k = 0, 1, 2, \cdots$, and $\beta = \frac{1}{T}$ is the inverse of the temperature and $\tau = it$. The scalar fields $\phi_k$ satisfy the Klein-Gordon equations $(\partial_\mu \partial^\mu - \frac{k^2}{L^2})\phi_k(x) = 0$. The fields confined between the two parallel plates obey the Dirichlet boundary conditions $\phi_k(x)|_{\partial\Omega} = 0$, and $\partial\Omega$ stands for positions of the plates. According to the solutions to the Klein-Gordon equation and the boundary conditions, the generalized zeta function can be written as,

$$\zeta(s; -\partial_E) = Tr(-\partial_E)^{-s}$$

$$= \int \frac{d^2k}{(2\pi)^2} \sum_{n=1}^\infty \sum_{n_1=0}^\infty \sum_{l=-\infty}^\infty [k^2 + \frac{n^2\pi^2}{R^2} + \frac{(2\pi)^2}{(L_0 \delta)^{2+\delta}}n_1^{\frac{2}{\delta}} + (\frac{2\pi}{\beta})^2]^{-s}$$

where

$$\partial_E = \frac{\partial^2}{\partial\tau^2} + \nabla^2$$
and \( \kappa^2 = k_1^2 + k_2^2 \) denote the transverse components of the momentum. \( R \) is the separation of plates. The total energy density of the system with thermal corrections is,

\[
\varepsilon = -\frac{\partial}{\partial \beta} \left( \frac{\partial \zeta(s; -\partial E)}{\partial s} \right)_{s=0}
\]

\[
= -\frac{1}{2\pi} \Gamma(-1) \frac{\partial}{\partial \beta} E_2(-1; \frac{\pi^2}{R^2}, \frac{4\pi^2}{\beta^2})
\]

\[
-\frac{1}{2\pi} \Gamma(-1) \frac{\partial}{\partial \beta} M_3(-1; \frac{\pi^2}{R^2}, \frac{4\pi^2}{\beta^2}) (\frac{(2\pi)^2}{\pi R^2}, 2, 2, 1 + \delta)
\]

\[(6)\]

in terms of Epstein zeta function \( E_2(s; a_1, a_2) = \sum_{n_1,n_2=1}^{\infty} (a_1 n_1^2 + a_2 n_2^2)^{-s} \) and the multiple zeta function with arbitrary exponents \( M_3(s; a_1, a_2, a_3; a_1, a_2, a_3) \) defined as,

\[
M_3(s; a_1, a_2, a_3; a_1, a_2, a_3) = \sum_{n_1,n_2,n_3=1}^{\infty} (a_1 n_1^{a_1} + a_2 n_2^{a_2} + a_3 n_3^{a_3})^{-s}
\]

\[(7)\]

Here we make use of the standard method to regularize the multiple zeta function with one arbitrary exponent \( M_3(s; a_1, a_2, a_3; 2, 2, a_3) \) which will be used in this work as follow,

\[
\begin{align*}
M_3(s; a_1, a_2, a_3; 2, 2, a_3) \\
= \sum_{n_1,n_2,n_3=1}^{\infty} (a_1 n_1^2 + a_2 n_2^2 + a_3 n_3^a)^{-s} \\
= \frac{1}{2} M_2(s; a_2, a_3; 2, a_3) + \frac{1}{2} \sum_{n_1,n_2,n_3=1}^{\infty} \left( \frac{n_1 \pi}{a_1 (a_2 n_2^2 + a_3 n_3^a)} \right)^{s-\frac{1}{2}} K_{-s+\frac{1}{2}} (2\pi n_1 \frac{a_2 n_2^2 + a_3 n_3^a}{a_1})
\end{align*}
\]

\[(8)\]

which contains the tiny contribution denoted as the modified Bessel function term. The multiple zeta function \( M_2(s; a_1, a_2; 2, \alpha_2) \) has also been regularized \[57\] and is expressed as,

\[
\begin{align*}
M_2(s; a_1, a_2; 2, \alpha_2) \\
= \sum_{n_1,n_2=1}^{\infty} (a_1 n_1^2 + a_2 n_2^{a_2})^{-s} \\
= -\frac{1}{2} a_2^{-s} \zeta(a_2 s) + \frac{1}{2} a_2^{-s} \sqrt{\frac{\pi a_2}{a_1}} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \zeta(a_2 (s - \frac{1}{2})) \\
+ 2\pi a_1^{-\frac{1}{2}} a_2^{-\frac{1}{2}} \sum_{n_1,n_2}^{\infty} \frac{n_1}{n_2} \frac{1}{n_1 n_2} (2\pi n_1 \frac{a_2 n_2^{a_2}}{a_1})
\end{align*}
\]

\[(9)\]

where \( K_\nu(z) \) is the modified Bessel function of the second kind and drops exponentially with \( z \).

According to the regularization of the multiple zeta functions with one arbitrary exponent in Eq. \(8\) and \(9\), we rewrite the total energy density of the two parallel plates at finite temperature as,
\[ 
\varepsilon = -\frac{\pi^2}{720 R^3} + \frac{1}{\sqrt{2} \beta R^2} \sum_{n_1, n_2=1}^{\infty} \frac{n_2}{n_1} \frac{1}{2} K_{\frac{3}{2}} \left( \pi \frac{\beta}{R} n_1 n_2 \right) + \frac{\pi}{\sqrt{2} \beta R^2} \sum_{n_1, n_2=1}^{\infty} \frac{n_2}{n_1} \left[ K_{\frac{1}{2}} \left( \pi \frac{\beta}{R} n_1 n_2 \right) + K_{\frac{3}{2}} \left( \pi \frac{\beta}{R} n_1 n_2 \right) \right] \\
+ \frac{\pi^{\frac{3}{2}}}{2} \Gamma \left( \frac{3}{2} \right) \zeta \left( -\frac{1}{1+\delta} \right) \frac{1}{(L_0^\delta)^{1+\delta}} \\
- \frac{1}{\beta^2 \left( L_0^\delta \right)^{1+\delta}} \sum_{n_1, n_2=1}^{\infty} \frac{n_2}{n_1} \frac{1}{2} \frac{1}{2} K_{\frac{3}{2}} \left( 2\pi \frac{\beta}{(L_0^\delta)^{1+\delta}} n_1 n_2 \right) \\
- \frac{2\pi}{\beta^2 \left( L_0^\delta \right)^{1+\delta}} \sum_{n_1, n_2=1}^{\infty} \frac{n_2}{n_1} \left[ K_{\frac{1}{2}} \left( 2\pi \frac{\beta}{(L_0^\delta)^{1+\delta}} n_1 n_2 \right) + K_{\frac{3}{2}} \left( 2\pi \frac{\beta}{(L_0^\delta)^{1+\delta}} n_1 n_2 \right) \right] \\
+ K_{\frac{1}{2}} \left( 2\pi \frac{\beta}{(L_0^\delta)^{1+\delta}} n_1 n_2 \right) \\
- 4\pi^{-\frac{3}{2}} \Gamma \left( -\frac{3}{2} \right) R^2 \frac{\partial}{\partial \beta} M_2 \left( -\frac{3}{2}; \frac{4\pi^2}{\beta^2}, \frac{(2\pi)^2}{(L_0^\delta)^{1+\delta}}; 2, \frac{2}{1+\delta} \right) \\
+ 6\pi^{\frac{1}{2}} R^2 \sum_{n_1, n_2, n_3=1}^{\infty} \frac{n_2}{n_1} \left( \frac{4\pi^2}{\beta^2} n_2^2 + \frac{(2\pi)^2}{(L_0^\delta)^{1+\delta}} n_3^{\frac{2}{3}} \right)^{-\frac{1}{2}} \\
\times K_{\frac{1}{2}} \left( 2n_1 R \sqrt{\frac{4\pi^2}{\beta^2} n_2^2 + \frac{(2\pi)^2}{(L_0^\delta)^{1+\delta}} n_3^{\frac{2}{3}}} \right) \\
- 4\pi^{\frac{1}{2}} R^2 \frac{1}{\beta^2} \sum_{n_1, n_2, n_3=1}^{\infty} \frac{n_2}{n_1} \left( \frac{4\pi^2}{\beta^2} n_2^2 + \frac{(2\pi)^2}{(L_0^\delta)^{1+\delta}} n_3^{\frac{2}{3}} \right)^{-\frac{1}{2}} \\
\times K_{\frac{1}{2}} \left( \frac{4\pi^2}{\beta^2} n_2^2 + \frac{(2\pi)^2}{(L_0^\delta)^{1+\delta}} n_3^{\frac{2}{3}} \right) + K_{\frac{1}{2}} \left( \frac{4\pi^2}{\beta^2} n_2^2 + \frac{(2\pi)^2}{(L_0^\delta)^{1+\delta}} n_3^{\frac{2}{3}} \right) \right) \\
(10) 
\]

In the absence of plates the vacuum energy density at finite temperature is,

\[ 
\varepsilon_0 = -\frac{\partial}{\partial \beta} \left( \frac{\partial}{\partial \beta} \frac{1}{(2\pi)^3} \sum_{n_1, n_2=1}^{\infty} \sum_{s=0}^{\infty} \left[ k^2 + \frac{(2\pi)^2}{(L_0^\delta)^{1+\delta}} n_1 n_2 s \right] \right) \bigg|_{s=0} \\
= \frac{\pi^2}{15} \beta^4 - \frac{1}{4\pi^2} \Gamma \left( -\frac{3}{2} \right) \frac{\partial}{\partial \beta} M_2 \left( -\frac{3}{2}; \frac{4\pi^2}{\beta^2}, \frac{(2\pi)^2}{(L_0^\delta)^{1+\delta}}; 2, \frac{2}{1+\delta} \right) \\
(11) 
\]

We can subtract the part of energy density for no plates to renormalize the two-parallel-plate system energy density denoted in Eq. (10). The terms like \(-\frac{1}{4\pi^2} \Gamma \left( -\frac{3}{2} \right) \frac{\partial}{\partial \beta} M_2 \left( -\frac{3}{2}; \frac{4\pi^2}{\beta^2}, \frac{(2\pi)^2}{(L_0^\delta)^{1+\delta}}; 2, \frac{2}{1+\delta} \right)\) in Eq. (10) and Eq. (11) respectively are just compensated. We regularize the difference between the two energy densities with or without plates respectively to obtain the renormalized Casimir energy density,

\[ 
\varepsilon_{\text{ren}}^{\varepsilon} = \varepsilon - R\varepsilon_0 
\]
If we take $T$ to be, the renormalized Casimir energy density reduces for two parallel plates in the presence of one fractal extra compactified dimension. It is the constraint on the scale dimension $\delta > \frac{1}{2}$, our results in Eq.(13) will recover to be those of our previous work [57] leading to

$$\lim_{R \to \infty} \epsilon_{ren}^C = \frac{3}{\pi} \beta^3 \Gamma\left(\frac{3}{2(1+\delta)}\right) \Gamma\left(\frac{4 + \delta}{2(1+\delta)}\right) \zeta\left(\frac{4 + \delta}{1 + \delta}\right) \sin \frac{3\pi}{2(1+\delta)}$$

$$- \frac{1}{\beta^3} \sum_{n_1,n_2=1}^{\infty} \frac{n_2^{3/2}}{n_1^{1/2}} K_2^3 \left(\frac{\beta}{R} n_1 n_2\right)$$

$$- \frac{2\pi}{\beta^{3/2} L_{0}^{\delta}} \sum_{n_1,n_2=1}^{\infty} \frac{n_2^{3/2}}{n_1^{1/2}} \left[K_2^3 \left(\frac{\beta}{R} n_1 n_2\right) + K_2^3 \left(\frac{\beta}{R} n_1 n_2\right)\right]$$

If one of the plates is moved to the remote place, the renormalized Casimir energy density reduces to be,

$$\lim_{R \to \infty} \epsilon_{ren}^C = \frac{3}{\pi} \beta^3 \Gamma\left(\frac{3}{2(1+\delta)}\right) \Gamma\left(\frac{4 + \delta}{2(1+\delta)}\right) \zeta\left(\frac{4 + \delta}{1 + \delta}\right) \sin \frac{3\pi}{2(1+\delta)}$$

$$- \frac{1}{\beta^3} \sum_{n_1,n_2=1}^{\infty} \frac{n_2^{3/2}}{n_1^{1/2}} K_2^3 \left(\frac{\beta}{R} n_1 n_2\right)$$

$$- \frac{2\pi}{\beta^{3/2} L_{0}^{\delta}} \sum_{n_1,n_2=1}^{\infty} \frac{n_2^{3/2}}{n_1^{1/2}} \left[K_2^3 \left(\frac{\beta}{R} n_1 n_2\right) + K_2^3 \left(\frac{\beta}{R} n_1 n_2\right)\right]$$

If we take $T = 0$, our results in Eq.(13) will recover to be those of our previous work [57] leading the constraint on the scale dimension $\delta > \frac{1}{2}$ corresponding to the negative nature of the Casimir energy for two parallel plates in the presence of one fractal extra compactified dimension. It is
interesting that the sign of terms due to the temperature is minus, so it is easier to keep the negative nature of the Casimir energy. After numerical calculation we find that when we take ratio \( \xi = (L_0l^4)_{\frac{1}{1+\gamma}} \beta > 0.54 \), the sign of the Casimir energy will remain negative no matter how large the scale dimension is. We show the relation between \( \xi \) and \( \delta_0 \) graphically in Fig. 1. For a definite temperature or equivalently that the ratio \( \xi \) has a definite value belonging to \([0, 0.54]\), the sign of the Casimir energy for two parallel plates in the world involving one fractal additional compactified dimension keeps negative when the scale dimension for extra space \( \delta > \delta_0 \). Fig. 1 demonstrates that the parameter \( \delta_0 \) will decreases from \( \frac{1}{\beta} \) to zero when the ratio \( \xi \) increases from zero to 0.54. We can emphasize that there is no limit on the scale dimension range like \( \delta \in (0, 1) \) when the surrounding temperature is sufficiently high.

III. The Casimir force between parallel plates at finite temperature in the spacetime with a fractal extra compactified dimension

It is important to continue studying the Casimir force within the two-parallel-plate device involving the thermal influence when the dimensionality of extra space governed by Kaluza-Klein theory is not an integer. The Casimir force on the plates is given by the derivative of the Casimir energy with respect to the plate distance. Here the Casimir energy is thought as the renormalized one. The Casimir force per unit area on the plates at finite temperature in the presence of one fractal extra compactified dimension can be written as,

\[
f_C = -\frac{\partial \varepsilon_{\text{ren}}}{\partial R} = -\frac{\pi^2}{240} \frac{1}{R^4} + \frac{\pi^2}{15} + \frac{3}{2\sqrt{2}} R^2 \sum_{n_1,n_2=1}^{\infty} \left( \frac{n_2}{n_1} \right)^2 K_{\frac{\pi}{2}} \left( \frac{\beta}{R} n_1 n_2 \right)
\]

\[
+ \frac{\sqrt{2\pi}}{\beta^4 R^2} \sum_{n_1,n_2=1}^{\infty} \left( \frac{n_2}{n_1} \right)^2 \left[ K_{\frac{\pi}{2}} \left( \frac{\beta}{R} n_1 n_2 \right) + K_{\frac{\pi}{2}} \left( \frac{\beta}{R} n_1 n_2 \right) \right]
\]

\[
- \frac{\pi^2}{2\sqrt{2}} R^2 \sum_{n_1,n_2=1}^{\infty} \left( n_1 \right)^{\frac{3}{2}} \left[ K_{\frac{3}{2}} \left( \frac{\beta}{R} n_1 n_2 \right) + 2K_{\frac{3}{2}} \left( \frac{\beta}{R} n_1 n_2 \right) + K_{\frac{3}{2}} \left( \frac{\beta}{R} n_1 n_2 \right) \right]
\]

\[
+ 3\sqrt{\pi} \frac{R^2}{\beta^3} \sum_{n_1,n_2,n_3=1}^{\infty} \left( \frac{n_2}{n_1} \right)^{-\frac{3}{2}} \left( \frac{4\pi^2}{\beta^2} n_2^2 + \frac{2\pi^2}{(L_0 l^4)_{\frac{1}{1+\gamma}}} n_3^2 \right)^{-\frac{1}{2}}
\]

\[
\times K_{\frac{1}{2}} \left( 2n_1 R \left( \frac{4\pi^2}{\beta^2} n_2^2 + \frac{(2\pi)^2}{(L_0 l^4)_{\frac{1}{1+\gamma}}} n_3^2 \right) \right)
\]

\[
+ 8\sqrt{\pi} \frac{R^2}{\beta^3} \sum_{n_1,n_2,n_3=1}^{\infty} \left( \frac{n_2}{n_1} \right)^{-\frac{3}{2}} \left( \frac{4\pi^2}{\beta^2} n_2^2 + \frac{(2\pi)^2}{(L_0 l^4)_{\frac{1}{1+\gamma}}} n_3^2 \right)^{-\frac{1}{2}}
\]

\[
\times K_{\frac{1}{2}} \left( 2n_1 R \left( \frac{4\pi^2}{\beta^2} n_2^2 + \frac{(2\pi)^2}{(L_0 l^4)_{\frac{1}{1+\gamma}}} n_3^2 \right) \right)
\]
separation $R$. If we employ the piston model [30], this term will be compensated because the $R$-independent forces acting on the plates have the same magnitude and their sign are opposite each other. We rewrite the Casimir force per unit area of plates as,

$$f_C = -\frac{\pi^2}{240} \frac{1}{R^4} + C(R, \beta, \delta)$$

where the correction function $C(R, \beta, \delta)$ is shown as,

$$C(R, \beta, \delta)$$

\[
= \frac{3}{2\sqrt{2} \beta^3 R^2} \sum_{n_1, n_2, n_3=1}^{\infty} \frac{n_2}{n_1} \frac{\beta}{R} K_{\frac{3}{4}} \left( \frac{\beta}{R} n_1 n_2 \right) \\
+ \frac{\sqrt{2} \pi}{\beta^2 R^2} \sum_{n_1, n_2=1}^{\infty} \frac{n_2}{n_1} \left[ K_{\frac{3}{4}} \left( \frac{\beta}{R} n_1 n_2 \right) + K_{\frac{3}{4}} \left( \frac{\beta}{R} n_1 n_2 \right) \right] \\
- \frac{\pi^2}{2\sqrt{2} \beta^3 R^2} \sum_{n_1, n_2=1}^{\infty} \frac{n_1 n_2}{n_1} \left[ K_{\frac{3}{4}} \left( \frac{\beta}{R} n_1 n_2 \right) + 2K_{\frac{3}{4}} \left( \frac{\beta}{R} n_1 n_2 \right) + K_{\frac{3}{4}} \left( \frac{\beta}{R} n_1 n_2 \right) \right] \\
+ \frac{3\sqrt{\pi}}{R^2 \beta^3} \sum_{n_1, n_2, n_3=1}^{\infty} \frac{n_2}{n_1} \frac{4\pi^2}{\beta^2} n_2 + \frac{(2\pi)^2}{(L_0^2)^{\frac{1}{1+\frac{1}{3}}}} n_3^{-\frac{2}{3+\frac{1}{3}}}} \\
\times K_{\frac{3}{4}} \left( 2n_1 R \sqrt{\frac{4\pi^2}{\beta^2} n_2 + \frac{(2\pi)^2}{(L_0^2)^{\frac{1}{1+\frac{1}{3}}}} n_3^{-\frac{2}{3+\frac{1}{3}}}} \right) \\
+ 8\frac{\sqrt{\pi}}{R^2 \beta^3} \sum_{n_1, n_2, n_3=1}^{\infty} \frac{n_2}{n_1} \frac{4\pi^2}{\beta^2} n_2 + \frac{(2\pi)^2}{(L_0^2)^{\frac{1}{1+\frac{1}{3}}}} n_3^{-\frac{2}{3+\frac{1}{3}}}} \\
\times K_{\frac{3}{4}} \left( 2n_1 R \sqrt{\frac{4\pi^2}{\beta^2} n_2 + \frac{(2\pi)^2}{(L_0^2)^{\frac{1}{1+\frac{1}{3}}}} n_3^{-\frac{2}{3+\frac{1}{3}}}} \right) \\
+ K_{\frac{3}{4}} \left( 2n_1 R \sqrt{\frac{4\pi^2}{\beta^2} n_2 + \frac{(2\pi)^2}{(L_0^2)^{\frac{1}{1+\frac{1}{3}}}} n_3^{-\frac{2}{3+\frac{1}{3}}}} \right) \\
\left( 15 \right)
\]
$$-4\sqrt{\frac{\pi}{\beta}} \frac{R^{\frac{1}{3}}}{\beta^3} \sum_{n_1,n_2,n_3=1}^{\infty} n_1^2 n_2^2 (\frac{4\pi^2}{\beta^2} n_2^2 + \frac{(2\pi)^2}{(L_0 l^\delta)_{1+\delta}} n_3^2)^{\frac{1}{2}} \times [K_{\frac{1}{2}} (2n_1 R \sqrt{\frac{4\pi^2}{\beta^2} n_2^2 + \frac{(2\pi)^2}{(L_0 l^\delta)_{1+\delta}} n_3^2})]
 + 2K_{\frac{1}{2}} (2n_1 R \sqrt{\frac{4\pi^2}{\beta^2} n_2^2 + \frac{(2\pi)^2}{(L_0 l^\delta)_{1+\delta}} n_3^2})
 + K_{\frac{1}{2}} (2n_1 R \sqrt{\frac{4\pi^2}{\beta^2} n_2^2 + \frac{(2\pi)^2}{(L_0 l^\delta)_{1+\delta}} n_3^2})]$$

(16)

It is obvious that the first term expressed as $f_0 = -\frac{\pi^2}{240} \frac{1}{R^4}$ in Eq.(15) is the same as the original Casimir pressure on the parallel plates involving massless scalar fields obeying the Dirichlet conditions without thermal influence in the four-dimensional spacetime. It should be pointed out that the correction function represents the deviation from the fractal additional dimension while the involves thermal corrections. When the plates are moved far enough away from each other, the correction function approaches to vanish,

$$\lim_{R \to \infty} C(R, \beta, \delta) = 0$$

(17)

no matter how large or how high the scale dimension or the temperature is. We have shown that the larger scale dimension will lead the greater Casimir force [57]. For a definite value of scale dimension $\delta$ the behaviour of the Casimir pressure on the variable $\mu = \frac{R}{(L_0 l^\delta)_{1+\delta}}$ with various temperature is plotted in Fig.2. We discover that the shapes of Casimir force due to different temperature are similar. The higher temperature lets the curve of the attractive force to be lower, meaning that the stronger thermal influence will lead the greater Casimir force between two parallel plates, and the nature of the Casimir force remains negative. It is interesting that the Casimir force between the two parallel plates is revised by both the scale dimension and the surrounding temperature.

IV. Conclusion

In this work the Casimir effect for parallel plates at finite temperature in the spacetime with one fractal extra compactified dimension is studied. We obtain the total vacuum energy density according to the Kaluza-Klein model with one additional non-integer dimension [8]. We regularize the energy density by means of the regularization of multiple zeta function with one arbitrary exponent to obtain the Casimir energy density and further the renormalized Casimir energy density which is the difference between the Casimir energy densities for two parallel plates and without plates respectively. It is declared that when the temperature is high enough the sign of the renormalized Casimir energy of the system consisting of two parallel plates will keep negative no matter what the value of scale dimension $\delta$ is equal to within its own region. In addition to the fractal extra spatial dimension, the Casimir force has something to do with the thermal influence. The
hotter environment leads the Casimir force greater. The shapes of Casimir force between two parallel plates due to different temperature are similar. The two plates keep attracting each other no matter how high the temperature is. Our research can be generalized to the case that the world has more than one multiple fractal additional compactified spatial dimensions.

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Figure 1: The relation between temperature-dependent ratio $\xi = \frac{(Lat^k)^{1/2}}{\beta}$ and parameter $\delta_0$ and when the scale dimension for extra space satisfies $\delta > \delta_0$, the sign of Casimir energy for parallel plates keeps negative.
Figure 2: The solid, dot, dashed curves of Casimir force per unit area between two parallel plates as functions of ratio $\mu = \frac{R}{(L_0t^\delta)^{1+\delta}}$ at finite temperature with $\frac{(L_0t^\delta)^{1+\delta}}{\beta} = 1, 1.02, 1.04$ respectively in the presence of one fractal extra compactified dimension for $\delta = 0.6$. 