Realization of POVMs using measurement-assisted programmable quantum processors

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We study possible realizations of generalized quantum measurements on measurement-assisted programmable quantum processors. We focus our attention on the realization of von Neumann measurements and informationally complete POVMs. It is known that two unitary transformations implementable by the same programmable processor require mutually orthogonal states. It turns out that the situation with von Neumann measurements is different. Specifically, in order to realize two such measurements one does not have to use orthogonal program states. On the other hand, the number of the implementable von Neumann measurements is still limited. As an example of a programmable processor we use the so-called quantum information distributor.

I. INTRODUCTION

General quantum measurements are formalized as \textit{positive operator valued measures} (POVM), i.e. sets of positive operators \( \{ F_j \} \) summing up to identity, \( \sum_k F_k = I \) (see, for instance, Refs. [1–4]). From the quantum theory it follows that each collection of such operators corresponds to a specific quantum measurement. However, the theory does not tell us anything about particular physical realization of a specific POVM. The aim of this paper is to exploit the so-called \textit{measurement-assisted quantum processors} to perform POVMs.

The \textit{Stinespring-Kraus theorem} [5] relates quantum operations (\textit{linear completely positive trace-preserving maps}) with unitary transformations. In particular, any quantum operation \( \mathcal{E} \) realized on the system \( A \) corresponds to a unitary transformation \( U \) performed on a larger system \( A + B \), i.e.

\[
\mathcal{E}[\varrho] = \text{Tr}_B [G_\varrho \otimes \xi G^\dagger],
\]

where \( \xi \) is a suitably chosen state of the system \( B \) and \( \text{Tr}_B \) denotes a \textit{partial trace} over the ancillary system \( B \). The assignment \( \mathcal{E} \mapsto (G, \xi) \) is one-to-many, because the dilation of the Hilbert space of the system \( A \) can be performed in many different ways. However, if we fix the transformation \( G \), states \( \xi \) of the ancillary system \( B \) control and determine quantum operations that are going to be performed on the system \( A \). In this way one obtains a concept of a \textit{programmable quantum processor}, i.e. a fixed piece of hardware taking as an input a data register (system \( A \)) and a program register (system \( B \)). Here the state of the program register \( \xi \) encodes the operation \( \varrho \to \varrho' = \mathcal{E}_\xi[\varrho] \) that is going to be performed on the data register.

In a similar way, any quantum generalized measurement (POVM), that is represented by a set of positive operators \( \{ F_j \} \), can be understood as a \textit{von Neumann measurement} performed on the larger system [4]. von Neumann measurements are those for which \( F_j \equiv E_j \) are mutually orthogonal projectors, i.e. \( E_j E_k = \delta_{jk} E_k \). The \textit{Neumark theorem} (see, e.g. Ref. [6]) states that for each POVM \( \{ F_j \} \) there exists a von Neumann measurement \( \{ E_j \} \) on a larger Hilbert space \( \mathcal{H}_{AB} \) and \( \text{Tr}_B F_j = \text{Tr}[(\varrho \otimes \xi) E_j] \) for all \( \varrho \), where \( \xi \) is some state of the system \( B \). Moreover, it is always possible to choose a von Neumann measurement such that \( E_j = G^\dagger (I \otimes Q_j) G \) are \( G \) is a unitary transformation and \( Q_j \) are projectors defined on the system \( B \). Using the cyclic property of a trace operation, i.e. \( \text{Tr}[(\varrho \otimes \xi) G^\dagger (I \otimes Q_j) G] = \text{Tr}[G(\varrho \otimes \xi) G^\dagger (I \otimes Q_j)] \), we see that the von Neumann measurement can be understood as a unitary transformation \( G \) followed by a von Neumann measurement \( M \leftrightarrow \{ Q_j \} \) performed on the ancillary system only.

As a result we obtain the couple \( (G, M) \) that determines a programmable quantum processor assisted by a measurement of the program register, i.e. \textit{measurement-assisted programmable quantum processor}. Such device can be used to perform both generalized measurements as well as quantum operations.

Programmable quantum processors (gate arrays of a finite extent) has been studied first by Nielsen and Chuang [7]. They have shown that no programmable quantum processor can perform all unitary transformations of a data register. To be specific, in order to encode \( N \) unitaries into a program register one needs \( N \) mutually orthogonal program states. Consequently, the required program register has to be described by an inseparable Hilbert space, because the number of unitaries is uncountable. However, if we work with a measurement-assisted programmable quantum processor, then with a certain probability of success we can realize all unitary transformations [8–11]. The probability of success can be increased arbitrarily close to unity utilizing conditioned loops with a specific set of error correcting program states [8,12–14].
“quantum multimeters” for discrimination of quantum state has been introduced in Ref. [16] and subsequently studied in Refs. [17–19]. An analogous setting of a unitary transformation followed by a measurement has been used in Ref. [20] to evaluate/measure the expectation value of any operator. The quantum network based on a controlled-SWAP gate can be used to estimate nonlinear functionals of quantum states [21] without any recourse to quantum tomography. Recently D’Ariano and co-workers [22–24] have studied how programmable quantum measurement devices. In particular, we will show how von Neumann measurements and informationally complete POVMs can be realized via programmable quantum measurement devices. In particular, we will show that this goal can be achieved using the so-called quantum information distributor [25,26].

II. GENERAL CONSIDERATION

Let us start our investigation with an assumption that the program register is always prepared in a pure state, i.e. \( \xi = |\Xi\rangle\langle \Xi| \). In this case the action of the processor can be written in the following form

\[
G|\psi\rangle \otimes |\Xi\rangle = \sum_k A_k(\Xi)|\psi\rangle \otimes |k\rangle ,
\]

where \(|k\rangle\) is some basis in the Hilbert space of the program register and \(A_k(\Xi) = \langle k|G(\Xi)\rangle\). In particular, we can use the basis in which the measurement \(M\) is performed, i.e. \(Q_a = \sum_{k \in J_a} |k\rangle\langle k|\), where \(J_a\) is a subset of indices \(\{k\}\). Note that \(J_a \cap J_{a'} = \emptyset\), because \(\sum_a Q_a = I\).

Measuring the outcome \(a\) the data evolve according to the following rule (the projection postulate)

\[
\varrho \rightarrow \varrho' = \frac{1}{p_a} \Tr_p[(I \otimes Q_a)G(\varrho \otimes |\Xi\rangle\langle \Xi|)G]\]

\[
= \frac{1}{p_a} \sum_{k \in J_a} A_k(\Xi)\varrho A_k^\dagger(\Xi) ,
\]

with the probability \(p_a = \Tr[(I \otimes Q_a)G(\varrho \otimes |\Xi\rangle\langle \Xi|)G]\) = \(\Tr[\varrho \sum_{k \in J_a} A_k^\dagger(\Xi)A_k(\Xi)] = \Tr[\varrho G]\). Consequently for the elements of the POVM we obtain

\[
F_a = \sum_{k \in J_a} A_k^\dagger(\Xi)A_k(\Xi) .
\]

If we consider a general program state with its spectral decomposition in the form \(\xi = \sum_n \pi_n |\Xi_n\rangle\langle \Xi_n|\), then the transformation reads

\[
\varrho \rightarrow \varrho' = \frac{1}{p_a} \sum_{n,k \in J_a} \pi_n A_{kn}\varrho A_{kn}^\dagger ,
\]

with \(A_{kn} = \langle k|G(\Xi_n)\rangle\) and \(p_a = \sum_{n,k \in J_a} \pi_n \Tr[\varrho A_{kn}^\dagger A_{kn}]\). Therefore the operators

\[
F_a = \sum_{n,k \in J_a} \pi_n A_{kn}^\dagger A_{kn}
\]

constitute the realized POVM.

Given a processor \(G\) and some measurement \(M\) one can easily determine which POVM can be performed. Note that the same POVM can be realized in many physically different ways. Two generalized measurements \(M_1, M_2\) are equivalent, if the resulting functionals \(f_k(x) = \Tr G(x)\) (\(x = 1, 2\)) coincide for all \(k\), i.e. they result in the same probability distributions. For the purpose of the realization of POVMs, the state transformation during the process is irrelevant. However, two equivalent realizations of POVM can be distinguished by the induced state transformations (for more on quantum measurement see Ref. [4]).

Let us consider, for instance, the trivial POVM, which consists of operators \(F_k = c_k I\) (\(c_k \geq 0, \sum c_k = 1\)). In this case the observed probability distribution is data-independent and some quantum operation is realized. In all other cases, the state transformation depends on the initial state of the data register, and is not linear [12]. In these cases the resulting distribution is nontrivial and contains some information about the state \(\varrho\). In the specific case when the state \(\varrho\) can be determined (reconstructed) perfectly, the measurement is informationally complete. In this case we can perform the complete state reconstruction. Any collection of \(d^2\) linearly independent positive operators \(F_k\) determine such informationally complete POVM. In particular, they form an operator basis, i.e. any state \(\varrho\) can be written as a linear combination \(\varrho = \sum_j \varrho_j F_j\). Using this expression the probabilities read

\[
p_j = \Tr[\varrho F_k] = \sum_k \varrho_k \Tr[F_j F_k] = \sum_k \varrho_k L_{jk} ,
\]

where the coefficients \(L_{jk} = \Tr[F_j F_k]\) define a matrix \(L\). In this setting the (inverse) problem of the state reconstruction reduces to a solution of a system of linear equations \(p_j = \sum_k L_{jk} \varrho_k\), where \(\varrho_k\) are unknown. The solution exists only if the matrix \(L\) is invertible and then \(\varrho_k = \sum_j L^{-1}_{jk} p_j\).

The purpose of any measurement is to provide us information about the state of the physical system based on the results of measurement. The presented scheme of measurement-assisted quantum processor represents quite general picture of the physical realization of any POVM.

III. QUANTUM INFORMATION DISTRIBUTOR

In this section we will present a specific example of a quantum processor the so-called quantum information distributor (QID) [25]. This device uses as an input a two-qubit program register and a single-qubit data register.
The processor consists of four CNOT gates. Its name reflects the property [25] that in special cases of program states the QID acts as an optimal cloner and the optimal universal CNOT gate, i.e. it optimally distributes quantum information according to a specific prescription. Moreover, it can be used to perform an arbitrary qubit rotation with the probability $p = 1/4$ [10]. The action of the QID can be written in the form [12]

$$\psi \rightarrow QID\psi = \sum_{k} \sigma_k A(\Xi) \sigma_k \psi \otimes \left| k \right>,$$  

(3.1)

where $\sigma_k$ are sigma matrices, $A(\Xi) = \left< k | G_{\text{QID}} | \Xi \right>$ and $\left| k \right> \in \{\left| + \right>, \left| 1+ \right>, \left| - \right>, \left| 1- \right>\}$ is a two-qubit program-register basis in which the measurement $M$ is performed ($\left< \pm \right> = \frac{1}{\sqrt{2}}(\left| 0 \right> \pm \left| 1 \right>)$).

In what follows we shall extend the list of applications of the QID processor and show how to realize a complete POVM, i.e. a complete state reconstruction. For a general program state $|\Xi\rangle = \sum_k \alpha_k |\Xi_k\rangle$ with $|\Xi_k\rangle = f(\sigma_k \otimes I) |\Xi_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$ the POVM consists of the following four operators

$$F_k = \sigma_k F_0 + \sigma_k = \sigma_k A(\Xi)^\dagger A(\Xi) \sigma_k,$$  

(3.2)

with $F_0^+ = \frac{1}{4} I + \frac{1}{\sqrt{2}} |00\rangle \langle 00 | + \frac{1}{\sqrt{2}} |11\rangle \langle 11 |$ and $\sigma^+ = \sigma_1^+ \sigma_2^+ \sigma_3^+$ and $\sigma = \sigma_1 \sigma_2 \sigma_3$.

Note that for the initial program state $|\Xi\rangle$ with $\alpha_0 = \cos \mu$, $\bar{\alpha} = \frac{\sin \mu}{\mu}$ ($\mu = ||\vec{m}||$) the probabilities $p_{0+} = \text{Tr} F_0^+ = 1/4$ are $\phi$-independent, and a unitary operation $U_{\phi} = \exp(i\vec{m} \cdot \vec{\sigma})$ is realized [10]. The question of interest is whether an informationally complete POVM can be encoded into a program state. In fact, the problem reduces to the question of the linear independency of operators $F_k$ for some $|\Xi\rangle$. Using the vector representation of operators, $F_k = 1/4(I + \vec{r}_k \cdot \vec{\sigma})$, one can show that the operators $F_k$ are linearly independent only if none of the coefficients of $\vec{r}_k$ are $\alpha_0 \sigma^+ + \alpha_1 \sigma^+ + \alpha_2 \sigma^+ + \alpha_3 \sigma^+$.

The elements of a POVM can be represented in the Bloch-sphere picture. This is due to the fact that operators $F_k = \frac{1}{2} g_k$, and $g_k$ represent quantum states. Choosing the program state

$$|\Xi_{\text{POVM}}\rangle = \frac{1}{\sqrt{2}} |\Xi_0\rangle + \frac{1}{\sqrt{6}} (|\Xi_1\rangle + |\Xi_2\rangle + |\Xi_3\rangle),$$  

(3.3)

we obtain the informationally complete POVM with a very symmetric structure. In particular, the operators $F_k$ are proportional to pure states associated with vertexes of a tetrahedron drawn inside the Bloch sphere (see Fig. 1). These operators read

$$F_{0+} = \frac{1}{4} (I + \frac{1}{\sqrt{3}} [\sigma_x + \sigma_y + \sigma_z]);$$  

(3.4)

$$F_{0-} = \frac{1}{4} (I + \frac{1}{\sqrt{3}} [-\sigma_x - \sigma_y + \sigma_z]);$$  

(3.5)

$$F_{1+} = \frac{1}{4} (I + \frac{1}{\sqrt{3}} [\sigma_x - \sigma_y - \sigma_z]);$$  

(3.6)

$$F_{1-} = \frac{1}{4} (I + \frac{1}{\sqrt{3}} [-\sigma_x - \sigma_y + \sigma_z]).$$  

(3.7)

It is obvious that these operators are not mutually orthogonal, but $\text{Tr} F_j^\dagger F_k = \frac{1}{4} \delta_{jk} + \frac{1}{4} (1 - \delta_{jk})$. Using this identity one can easily compute the relation (2.6) between the observed probability distribution and the data state $\varrho$

$$\varrho = \sum_k (-\frac{21}{5} p_k + \frac{9}{5} \sum_{j \neq k} p_j) |Q_k\rangle \langle Q_k|,$$  

(3.8)

where we used the notation $F_k = \frac{1}{2} |Q_k\rangle \langle Q_k|$. The last equation completes the task of the state reconstruction task.

**FIG. 1.** The Bloch sphere can be used to illustrate any POVM realized on the QID processor. Each POVM is given by four operators that determine four points in the Bloch sphere. Using this picture one can see the structure and some properties of the realized POVM. The depicted points forming a tetrahedron correspond to POVM elements of the symmetric informationally complete POVM associated with the program state $|\Xi\rangle = \frac{1}{\sqrt{2}} |\Xi_0\rangle + \frac{1}{\sqrt{6}} (|\Xi_1\rangle + |\Xi_2\rangle + |\Xi_3\rangle)$.

Because of the identity $\text{Tr} F_j F_k = \text{const}$ for $j \neq k$ the realized POVM $\{F_k\}$ is of a special form. It belongs to a family of the so-called symmetric informationally complete measurements (SIC POVM) [27]. These measurements are of interest in several tasks of quantum information processing and possess many interesting properties. It is known (see, e.g. Ref. [27]) that for qubits there essentially exist only two (up to unitaries) such measurements. Above we have shown how one of them can be performed using the QID processor.
IV. VON NEUMANN MEASUREMENTS

An important class of measurements is described by the projector valued measures (PVM), which under specific circumstances enable us to distinguish between orthogonal states in a single shot, i.e. no measurement statistics is required. A set of operators \{E_k\} form a PVM, if \(E_j = E_j^\dagger\) and \(E_j E_k = E_j \delta_{jk}\), i.e. it contains mutually orthogonal projectors. The total number of (nonzero) operators \{E_k\} cannot be larger than the dimension of the Hilbert space \(d\).

Usually the von Neumann measurements are understood as those that are compatible with the projection postulate, i.e. the result \(j\) associated with the operator \(E_j = |e_j\rangle\langle e_j|\) induces the state transformation

\[
\rho \rightarrow \rho'_j = \frac{E_j \rho E_j}{\text{Tr} E_j} = \frac{|e_j\rangle\langle e_j|}{\langle e_j|\langle e_j|} = |e_j\rangle\langle e_j| = E_j \, . \tag{4.1}
\]

That is, the state after the measurement is described by the corresponding projector \(E_j\).

However, each PVM can be realized in many different ways and a particular von Neumann measurement is only a specific case. In our settings the realized PVM \(\{F_k\}\) is related to the state transformation via the identity

\[
F_k = A_k^\dagger A_k, \quad \text{where} \quad \rho \rightarrow \rho'_k = A_k \rho A_k^\dagger.
\]

The set of operators \(A_k = U_k E_k\), with \(E_k\) projectors and \(U_k\) unitary transformations, define the same PVM given by \(\{E_k\}\). In particular, \(A_k^\dagger A_k = E_k U_k^\dagger U_k E_k = E_k E_k = E_k\), but the state transformation results in

\[
\rho \rightarrow \rho'_k = U_k E_k U_k^\dagger \neq E_k \, . \tag{4.2}
\]

Thus the final state is described by a projector, but not in accordance with the projection postulate. We refer to the PVMs that are compatible with the projection postulate as the von Neumann measurements. Moreover, for a simplicity we shall assume that the projectors are always one-dimensional, i.e. the PVM is associated with non-degenerate hermitian operators.

The action of the processor \(G\) implementing two von Neumann measurements \(\{E_j\}\) and \(\{G_j\}\) can be written as

\[
G|\psi\rangle \otimes |\Xi_E\rangle = \sum_j E_j |\psi\rangle \otimes |j\rangle \, ; \tag{4.3}
\]

\[
G|\psi\rangle \otimes |\Xi_G\rangle = \sum_j G_j |\psi\rangle \otimes |j\rangle \, . \tag{4.4}
\]

It is well known [11] that when two sets of Kraus operators are realizable by the same processor \(G\), then the following necessary relation holds \(\sum_j E_j G_j = \langle \Xi_E | \Xi_G \rangle I\).

Using this relation for the projections \(E_j = |e_j\rangle\langle e_j|,\ G_j = |g_j\rangle\langle g_j|\) we obtain the identity

\[
\sum_j u_{jj} |e_j\rangle\langle g_j| = k I \, , \tag{4.5}
\]

where \(u_{jj} = \langle g_j|e_j\rangle\). For general measurements, the operator on the left-hand side of the previous equation contains off-diagonal elements. In this case the corresponding program states must be orthogonal, i.e. \(k = 0\).

This result is similar to the one obtained by Nielsen and Chuang [7] who have studied the possibility of the realization of unitary transformations via programmable gate arrays. They have shown that in order to perform (with certainty) two unitary transformations on a fixed quantum processor one needs two orthogonal program states. However, in our case we still cannot be sure that measurement-assisted processor realizing two von Neumann measurements exists. Moreover, there are possibilities, when the condition holds also for non-orthogonal program states (see the case study below).

In order to realize a projective measurement on a \(d\)-dimensional data register the program space must be at least \(d\) dimensional. Let us start with the assumption that the Hilbert space of the program register is \(d\) dimensional. In this case the program states have to be orthogonal (this is due to the fact the expression (4.5) contain off-diagonal elements). Let us consider \(d\) different (non-degenerate) von Neumann measurements \(M_\alpha\) that are determined by a set of operators \(E_k^\alpha = |\alpha_k\rangle\langle \alpha_k|\) \((E_k^\alpha E_j^\beta = \delta_{kj} E_k^\alpha\) and \(\sum_k E_k^\alpha = I\) for all \(\alpha\)). Let \(|\alpha\rangle\) denote the associated program states and \(|\alpha\langle \beta| = \delta_{\alpha\beta}\). It is easy to see that for general measurements the resulting operator

\[
G = \sum_{k,\alpha} E_k^\alpha \otimes |k\rangle\langle \alpha| \tag{4.6}
\]

is not unitary. In particular, \(G^* G = \sum E_k^\alpha E_k^\beta \otimes |\alpha\langle \beta| \neq I\). The equality would require that the identity \(\sum_k E_k^\alpha \otimes |k\rangle\langle \alpha| \neq I\). Therefore, we conclude that no two orthogonal states do guarantee the existence of a programmable processor that performs desired set of measurements. This result makes the case of programming the unitaries and von Neumann measurements different.

| measurement | \(M_1\) | \(M_2\) | \ldots | \(M_d\) |
|-------------|--------|--------|----------|--------|
| result 1    | \(|\alpha_1\rangle\) | \(|\beta_1\rangle\) | \ldots | \(|\omega_1\rangle\) |
| result 2    | \(|\alpha_2\rangle\) | \(|\beta_2\rangle\) | \ldots | \(|\omega_2\rangle\) |
| \vdots      | \vdots | \vdots | \ddots | \vdots |
| result \(d\)| \(|\alpha_d\rangle\) | \(|\beta_d\rangle\) | \ldots | \(|\omega_d\rangle\) |

TABLE I. The measurements \(M_1, M_2, \ldots, M_d\) are realizable by a \(d\) dimensional program register only if all vectors in the rows are mutually orthogonal. Moreover, no two columns can be related by a permutation. The orthogonality of the vectors in columns is ensured by the fact that they form a PVM. It turns out that the number of realizable measurements equals at most to \(d - 2\), i.e. neither for qutrit one can encode more than a single von Neumann measurement. Moreover, the measurements that can be performed are not arbitrary.
For instance, let us consider a two-dimensional program register and let us denote $E_{01}^0 = E_{01}$ and $E_{10}^1 = G_{01}$. Then the above condition reads $E_0 G_0 = E_1 G_1 = 0$. Using the definition $E_k = |e_k\rangle\langle e_k|$ and $G_k = |g_k\rangle\langle g_k|$ we obtain the orthogonality conditions $\langle e_0 | g_0 \rangle = \langle e_1 | g_1 \rangle = 0$. Consequently, because in the two-dimensional case the orthogonal state is unique, we obtain $|g_0\rangle = |f_1\rangle$ and $|g_1\rangle = |e_0\rangle$, i.e. the measurements are the same. For larger $d$ the situation is different. The realizable measurements must possess the derived property which can be summarized with the help of Tab. I.

In order to realize more von Neumann measurements on a qudit one has to work with a larger-dimensional program space, i.e. dim $\mathcal{H}_p = d_p > d$. In general, in this case we work with $d_p$ outcomes and $d_p$ projective operators $Q_k$ that define the realized measurement of the program register. However, each PVM consists of maximally $d$ projectors. Therefore, $d_p - d$ of the induced operators $E_k$ should represent the zero operator. It means that when we are realizing the von Neumann measurement such that some of the outcomes do not occur, i.e. probability of them vanishes for all data states. However, there is one more option that the set of operators $\{E_k\}$ ($k = 1,\ldots,d_p$) contains exactly only $d$ different operators (projectors). This means that more results can specify the same projection and define a single result of the realized von Neumann measurement.

The idea of additional, the so called, “zero” operators can be used to formulate a general statement about the implementation of any collection of arbitrary von Neumann measurements. Let us consider $N$ von Neumann measurements $M_\alpha$ given by non-zero operators $\{E_\alpha^p\}$ (number of $k$ equals to $d$). We can define the sets of $d_p$ operators $\{\tilde{E}_k^\alpha\}$ by adding to these sets zero operators so that the condition $\sum_k E_\alpha^p \tilde{E}_k^\beta = \delta_{\alpha\beta} I$ holds. Using this approach we find out that any collection of $N$ von Neumann measurements can be realized on a single quantum processor given by Eq.(4.6) with (maximally) $Nd$ dimensional program space.

**A. Case study: Projective measurements on a qubit.**

Let us consider two von Neumann measurements $M = \{E_0, E_1\}$ and $N = \{G_0, G_1\}$ on a qubit. Further, let us assume a three-dimensional program space and define measurements $M_1 = \{E_0, E_1, 0\}$ and $M_2 = \{0, G_1, G_2\}$, respectively. It is easy to see that neither of these two sets of operators do satisfy the condition $0 = \sum_k E_k G_k = 0 E_0 + E_1 G_1 + 0 G_2 = E_1 G_1$. The equality holds only if $E_1 G_1 = 0$, i.e. $E_1 = |\psi\rangle\langle \psi|$ and $G_2 = |\psi_\perp\rangle\langle \psi_\perp|$, but this implies that both measurements are the same. Consequently, the dimension of the program space has to increase by one. Then we have $M_1 = \{E_0, E_1, 0, 0\}$, $M_2 = \{0, 0, G_0, G_1\}$ and the condition holds for all possible measurements $M_1, M_2$. It follows that the implementation of $N$ von Neumann measurements on a qubit requires $N$-qubit program space.

The program space of the QID processor given by Eq.(3.1) consists of two qubits. Using the conclusion of the previous paragraph it follows that two von Neumann measurements could be performed with the help of this processor. It is easy to see that the operators $A_k = A(E)\sigma_k$ with $A(E) = \frac{1}{2} \sum_j \alpha_j \sigma_j$ are not projectors. Moreover, they do not vanish for certain $j$. Consequently, the projective measurement cannot be realized in the same way as described above. However, the QID-processer can still be exploited to perform a von Neumann measurement.

Using the program state $|\Xi\rangle = \frac{1}{\sqrt{2}}(|\Xi_0\rangle + |\Xi_1\rangle)$ the operator $A = \frac{1}{\sqrt{2}}[I + \sigma_x]$ (i.e., $F_0 = A^1 A = \frac{1}{2} P_+$, where $P_+ = \frac{1}{2}[I + \sigma_x]$) is a projection onto the vector $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + |1\rangle$. It is obvious that $F_1 = \sigma_x F_0 \sigma_x = F_0$ and $F_2 = F_3 = \frac{1}{\sqrt{2}} P_-$, where $P_- = \frac{1}{2}[I - \sigma_x]$. It turns out that we have realized PVM described by $P_{\pm}$, i.e. the eigenvectors of the $\sigma_x$ measurement. The state transformation reads $\varrho \rightarrow \tilde{\varrho} = P_{\pm}$ (if $p_k \neq 0$), respectively. It follows that the realization of the measurement of $\sigma_x$ is in accordance with the projection postulate. In the same way we can realize $\sigma_y$ and $\sigma_z$ measurement (in these cases different results must be paired). Basically, this corresponds to a choice of different two-valued measurements, but in reality we perform only a single four-valued measurement. As a result we find that on QID we can realize three different von Neumann measurements. Note that we have used only two qubits as the program register. Moreover, the associated program states $|\tilde{\Xi}_{\sigma_j}\rangle = \frac{1}{\sqrt{2}}(|\Xi_0\rangle + |\Xi_j\rangle)$ are not mutually orthogonal, but $|\tilde{\Xi}_{\sigma_j}\rangle|\tilde{\Xi}_{\sigma_k}\rangle = \frac{1}{2}$ (for $j \neq k$) and Eq. (4.5) holds. Namely, for the measurements of $\sigma_x \leftrightarrow \{P_\pm\}$ and $\sigma_z \leftrightarrow \{P_0 = |0\rangle\langle 0|, P_1 = |1\rangle\langle 1|\}$ the condition (4.5) reads $\frac{1}{2}[P_0 + P_1 + P_+ + P_-] = \frac{1}{2} I$.

Till now we have always assumed that states that encode two von Neumann measurements have to be orthogonal. The last paragraph describes an counterexample. As we already mentioned in some specific cases the condition of the orthogonality can be relaxed.

**B. Note. Projection valued measures**

If we relax the compatibility with the projection postulate more PVMs can be realized on a single processor. Let us consider that the dimension of the program space equals $d$ and $|\alpha\rangle$ is the state that encodes the PVM given by a set $\{E_k^\alpha\}$. The action of $G$ can be written as

$$G|\psi\rangle \otimes |\alpha\rangle = \sum_k U_k^\alpha E_k^\alpha |\psi\rangle \otimes |k\rangle \quad (4.7)$$

and the condition $\sum_k E_k^\alpha U_k^\beta U_k^\beta E_k^\alpha = \delta_{\alpha\beta} I$ must hold. Let us consider two PVMs on a qubit $\{E_0 = |0\rangle\langle 0|, E_1 = |1\rangle\langle 1|\}$ and $\{E_0 = |1\rangle\langle 1|, E_1 = |0\rangle\langle 0|\}$.
where \( \tilde{G} \) is given by projectors \( |\phi\rangle \langle \phi| \) and \( \langle \Xi | \tilde{G} \rangle = 0 \). Direct calculation shows that \( E_0 \tilde{G}_0 + E_1 \tilde{G}_1 = |0\rangle \langle 0| |\phi\rangle \langle \phi| + |1\rangle \langle 1| |\phi\rangle \langle \phi| = 0 \), i.e. \( G \) is unitary. From here it follows that if one does not require the validity of the projection postulate, then any two PVMs can be performed on a processor with two-dimensional program space.

This result holds in general. Let us consider a set of \( d \) PVMs \( \{E_k^\alpha\} \) on a qudit. There always exist unitary transformations \( U^\alpha \) such that operators \( E_k^\alpha = U^\alpha E_k \) satisfy the condition \( \sum_k E_k^{\alpha^*} E_k^\alpha = \delta_{\alpha,\beta} I \). Without the loss of generality we can consider that measurement \( M_0 \) is given by projectors \( |0\rangle \langle 0|, \ldots, |d-1\rangle \langle d-1| \) and \( M_d \) by \( |\phi_1^d\rangle \langle \phi_1^d|, \ldots, |\phi_2^d\rangle \langle \phi_2^d| \) (see Tab. II).

V. CONCLUSION

In this paper we have studied how POVMs can be physically realized using the so-called measurement-assisted quantum processors. In particular, we have analyzed how to perform complete state reconstruction and von Neumann measurements. As a result we have found that an arbitrary collection of von Neumann measurements cannot be realized on a single programmable quantum processor. We have shown how to use the QID processor to perform the state reconstruction.

The number of implementable von Neumann measurements is limited by the dimensionality of the program register. Our main result is that with a program register composed of \( N \) qudits one can surely define a processor which performs arbitrary \( N \) von Neumann measurements. In fact, in general one can do much better. We have shown that the usage of non-orthogonal program states can be helpful. In particular, the QID processor can be exploited to perform three von Neumann measurements by using only two qubits as a program register and nonorthogonal states. Using only \( d \) dimensional program space one can realize maximally \( N = d - 2 \) von Neumann measurements on a qudit (for a qubit we have \( N = 1 \)).

Relaxing the condition on compatibility with the projection postulate the processor allows us to realize any collection of \( N \) PVMs just by using \( d_p = N \) dimensional program space. An open question is whether we can perform more POVMs, or not. The two tasks can be performed by programmable processors: the realization of von Neumann measurements and the application unitary transformations on the data register, are different. According to Nielsen and Chuang [7], any collection of \( N \) unitary transformations requires \( N \) dimensional program space. For \( N \) von Neumann measurements the upper bound reads \( d_p = Nd \) and any improvement strongly depends on the specific set of these measurements. The characterization of these classes of measurements is an interesting topic that will be studied elsewhere.

ACKNOWLEDGMENTS

This work was supported in part by the European Union projects QUPRODIS, QGATES and CONQUEST. We thank Peter Stelmachovič for valuable discussions. We would also like to thank Mariá n Roško for reading the manuscript and having no comments.

| measurement | \( M_1 \) | \( M_2 \) | \ldots | \( M_d \) |
|-------------|--------|--------|------|--------|
| result 1    | \( |\alpha_1\rangle \) | \( |\beta_1\rangle \) | \ldots | \( |\omega_1\rangle \) |
| result 2    | \( |\alpha_2\rangle \) | \( |\beta_2\rangle \) | \ldots | \( |\omega_2\rangle \) |
| \vdots      | \vdots | \vdots | \vdots | \vdots |
| result \( d \) | \( |\alpha_d\rangle \) | \( |\beta_d\rangle \) | \ldots | \( |\omega_d\rangle \) |

TABLE II. The measurements \( M_1, M_2, \ldots, M_d \) are realizable by a \( d \) dimensional program register only if all vectors in the rows are mutually orthogonal. Moreover, no two columns can be related by a permutation. The orthogonality of the vectors in columns is ensured by the fact that they form a PVM. It turns out that the number of realizable measurements equals at most to \( d - 2 \), i.e. neither for qutrit one can encode more than a single von Neumann measurement. Moreover, the measurements that can be performed are not arbitrary.
[1] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1993).
[2] J. Preskill, *Quantum theory of Information and Computation*, www.theory.caltech.edu/people/preskill
[3] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum information* (Cambridge University Press, Cambridge, 2000).
[4] P. Busch, P. Lahti, and P. Mittalstead, *Quantum Theory of Measurement* (Lecture Notes in Physics m2, Springer Verlag, 1996).
[5] D. E. Evans and J. T. Lewis, *Dilations of Irreversible Evolutions in Algebraic Quantum Theory*, Communications of Dublin Institute of Advanced Studies, Series A (Theoretical Physics), No. 24, Dublin, DIAS (1977).
[6] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[7] M. A. Nielsen and I. L. Chuang, *Programmable quantum gate arrays*, Phys. Rev. Lett. 79, 321 (1997).
[8] G. Vidal, L. Masanes, and J. I. Cirac, *Storing quantum dynamics in quantum states: A stochastic programmable gate*, Phys. Rev. Lett. 88, 047905 (2002).
[9] M. Hillery, V. Bužek, and M. Ziman, *Programmable quantum gate arrays*, Fortschritte der Physik 49, 987 (2001).
[10] M. Hillery, V. Bužek, and M. Ziman, *Probabilistic implementation of quantum processors*, Phys. Rev. A 65, 022301 (2002).
[11] M. Hillery, M. Ziman, and V. Bužek, *Implementation of quantum maps by programmable quantum processors*, Phys. Rev. A 66, 042302 (2002).
[12] M. Ziman and V. Bužek, *Realization of unitary maps via programmable quantum processors*, Int. Journal of Quantum Inf. 1, 527 (2003).
[13] M. Hillery, M. Ziman, and V. Bužek, *Improving performance of probabilistic programmable quantum processors*, Phys. Rev. A 69 (2004).
[14] A. Brazier, V. Bužek, and P. L. Knight, *Probabilistic programmable quantum processors with multiple copies of program state*, submitted to Phys. Rev. A.
[15] A. Yu. Vlasov, *Aleph-QP: Universal Hybrid quantum processors*, quant-ph/0205074.
[16] M. Dušek and V. Bužek, *Quantum-controlled measurement device for quantum-state discrimination*, Phys. Rev. A 66, 022112 (2002).
[17] J. Fiurášek, M. Dušek, and R. Filip, *Universal measurement apparatus controlled by quantum software*, Phys. Rev. Lett. 89, 190401 (2002).
[18] J. Fiurášek and M. Dušek, *Probabilistic quantum meters*, Phys. Rev. A 69, 032302 (2004).
[19] J. A. Bergou, M. Hillery, and V. Bužek, *Programmable quantum state discriminator with simple programs*, unpublished.
[20] J. P. Paz and A. Roncaglia, *A quantum gate array can be programmed to evaluate the expectation value of any operator*, Phys. Rev. A 68, 052316 (2003).
[21] A. K. Ekert, C. M. Alves, D. K. L. Oi, M. Horodecki, P. Horodecki, and L. C. Kwek, *Direct estimation of linear and non-linear functionals of a quantum state*, Phys. Rev. Lett. 88, 217901 (2002).
[22] G. M. D’Ariano, P. Perinotti, and M. F. Sacchi, *Quantum universal detectors*, Europhys. Lett. 65, 165 (2004).
[23] G. M. D’Ariano, P. Perinotti, and M. F. Sacchi, *Optimization of quantum universal detectors*, in “Proceedings of the 8th Int. Conf. on Squeezed States and Uncertainty Relations”, ed. by H. Moya-Cessa et al., (Rinton, Princeton, 2003) p. 86.
[24] G. M. D’Ariano and P. Perinotti, *Efficient universal programmable quantum measurements*, quant-ph/0410169.
[25] S. Braunstein, V. Bužek, and M. Hillery, *Quantum Information Distributor: Quantum network for symmetric and anti-symmetric cloning in arbitrary dimension and continuous limit*, Phys. Rev. A 63, 052313 (2001).
[26] M. Rosko, V. Bužek, P. R. Chouha, and M. Hillery, *Generalized measurements via programmable quantum processor*, Phys. Rev. A 68, 062302 (2003).
[27] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, *Symmetric informationally complete quantum measurements*, J. Math. Phys. 45, 2171 (2004).