PICARD GROUPS OF CERTAIN COMPACT COMPLEX PARALLELIZABLE MANIFOLDS AND RELATED SPACES

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ABSTRACT. Let $G$ be a complex simply connected semisimple Lie group and let $\Gamma$ be a torsionless uniform irreducible lattice in $G$. Then $\Gamma \setminus G$ is a compact complex non-Kähler manifold whose tangent bundle is holomorphically trivial. In this note we compute the Picard group of $\Gamma \setminus G$ when $\text{rank}(G) \geq 3$. When $\text{rank}(G) \leq 3$, we determine the group $\text{Pic}^0(\Gamma \setminus G) \subset \text{Pic}(\Gamma \setminus G)$ of topologically trivial holomorphic line bundles.

When $\text{rank}(G) \geq 2$, we also show that $\text{Pic}^0(P_{\Gamma})$ is isomorphic to $\text{Pic}^0(Y)$ where $P_{\Gamma}$ is a $\Gamma \setminus G$-bundle associated to a principal $G$-bundle over a compact connected complex manifold $Y$, and, when $\text{rank}(G) \geq 3$, we show that $\text{Pic}(Y) \to \text{Pic}(P_{\Gamma})$ is injective with finite cokernel.

1. Introduction

Let $G$ be a simply connected complex linear Lie group and $\Gamma \subset G$ be a uniform lattice in $G$ and let $M_{\Gamma} = \Gamma \setminus G$. Then $M_{\Gamma}$ is a compact homogeneous manifold which has the property that its tangent bundle is holomorphically trivial. Special cases of such manifolds include complex tori. A result of H.-C. Wang [W, Corollary 2] shows that when $G$ is non-abelian, $M_{\Gamma}$ is not a Kähler manifold. See also [BR].

Denote by $\text{Pic}(X)$ the group of isomorphism classes of holomorphic line bundles over a connected complex manifold $X$ and by $\text{Pic}^0(X)$ the subgroup of topologically trivial line bundles. Thus $\text{Pic}(X) \cong H^1(X, O_X^*)$ and $\text{Pic}^0(X)$ is the kernel of the Chern class map $c_1 : \text{Pic}(X) \to H^2(X; \mathbb{Z})$. When $X$ is compact and Kähler, $\text{Pic}^0(X)$ is a complex torus (diffeomorphic to a product of the circle group). When $\Gamma \subset V$ is a lattice in an $n$-dimensional complex vector space, $X := \Gamma \setminus V$ is a complex torus, $\text{Pic}^0(X)$ is again an $n$-dimensional complex torus, namely the dual torus $X^\vee = V^\vee / \Gamma^\vee$ where $V^\vee$ is the space of conjugate linear functions $V \to \mathbb{C}$ and $\Gamma^\vee = \{ f \in V^\vee \mid \text{Im} f(\Gamma) \subset \mathbb{Z} \}$. See [Mu].

In this note, we will be concerned with the determination of the Picard group when $M_{\Gamma} = \Gamma \setminus G$ where $G$ is a simply connected complex semisimple Lie group and $\Gamma$ is a torsionless irreducible uniform lattice in $G$. We shall also consider compact connected complex manifolds which are holomorphically fibred by $M_{\Gamma}$. Such holomorphic bundles arise naturally. Indeed, if $\pi : E \to B$ is a holomorphic principal $G$-bundle over $B$, where $G$ acts on the left of $E$, then $\Gamma \setminus E \to B$ is such an $M_{\Gamma}$-bundle over $B$.
Recall that the real rank of a linear connected real semisimple Lie group $G$, denoted $\text{rank}_R(G)$ is the maximum dimension $r$ of a diagonalizable subgroup of $G$ isomorphic to $(\mathbb{R}_{>0})^r$. This definition is applicable even if $G$ is a complex semisimple Lie group, by regarding it as a real Lie group, ignoring the complex structure. In this case, the real rank of $G$ equals the rank of $G$ viewed as a complex algebraic group, namely, the dimension of a maximal (algebraic) torus (isomorphic to $(\mathbb{C}^*)^r$) contained in $G$. For example, when $G = SL(n, \mathbb{C})$, we have $\text{rank}_R G = n - 1$. In view of this, we shall henceforth write \( \text{rank}(G) \) to mean $\text{rank}_R(G)$. Any simply connected semisimple complex Lie group is a direct product of simple Lie groups. Here, as is customary in the theory of Lie groups, a connected Lie group is simple if its only proper normal subgroups are discrete, which are necessarily contained in the centre of the Lie group. The centre of any complex semisimple Lie group is finite. We refer the reader to [V] for these and other basic facts about complex semisimple Lie groups, including their classification.

A discrete subgroup $\Gamma$ in a semisimple Lie group $G$ with finitely many connected components is called a lattice if $M_\Gamma = \Gamma \backslash G$ has a $G$-invariant measure (for the right action of $G$ on $M_\Gamma$) with respect to which it has finite volume.\(^1\) If $M_\Gamma$ is compact, then $\Gamma$ is said to be uniform (or cocompact). By a well-known result of Borel and Harish-Chandra, any (real) semisimple Lie group with finitely many connected components has both uniform as well as non-uniform lattices. A lattice $\Gamma$ in $G$ is said to be reducible if there exists a finite cover $\pi : \tilde{G} \to G$ such that $\tilde{G} = \tilde{G}_0 \times \tilde{G}_1$ where $\tilde{G}_i$ are non-compact subgroups of $\tilde{G}$ and a subgroup $\Gamma_0 \subset \Gamma$ having finite index in $\Gamma$ such that $\pi^{-1}(\Gamma_0)$ is a product $\Lambda_0 \times \Lambda_1$ where $\Lambda_j \subset \tilde{G}_j$ is a lattice for $j = 0, 1$. If $\Gamma$ is not reducible, then it is said to be irreducible. For example, any lattice in a simple Lie group is irreducible. The reader is referred to the book by Raghunathan [R] for the definition of lattices in a general Lie group and standard facts concerning them.

**Theorem 1.** Let $G$ be a simply connected complex semisimple Lie group and let $\Gamma$ be a torsionless irreducible uniform lattice in $G$. Let $M_\Gamma := \Gamma \backslash G$. Then:

(i) If $\text{rank}(G) \geq 2$, then $\text{Pic}^0(M_\Gamma) = 0$. If $\text{rank}(G) \geq 3$, then $\text{Pic}(M_\Gamma) \cong H^2(M_\Gamma; \mathbb{Z})$ is a finite group.

(ii) If $\text{rank}(G) = 1$, then $\text{Pic}^0(M_\Gamma) \cong \mathbb{C}^r / \mathbb{Z}_r$, where $r = \text{rank}(\Gamma / [\Gamma, \Gamma])$.

The value of $r$ in the rank 1 case, namely when $G = SL(2, \mathbb{C})$ can be arbitrarily large. See Remark 6(ii) below.

We have not been able to determine the image of the Chern class map $\text{Pic}(M_\Gamma) \to H^2(M_\Gamma; \mathbb{Z})$ in the case when $\text{rank}(G) \leq 2$. By the well-known classification theorem, the simply connected complex semisimple Lie groups of rank 2 are $SL(3, \mathbb{C}), SL(2, \mathbb{C}) \times SL(2, \mathbb{C}), Spin(5)$ and the exceptional Lie group $G_2$. When $\text{rank}(G) = 1$, $G$ is isomorphic to $SL(2, \mathbb{C})$.

\(^1\)Some authors assume that $G$ has no compact factors (i.e. $G$ has no proper normal compact subgroups of positive dimension) while defining a lattice.
Suppose that \( Y \) is a connected compact complex manifold and \( \pi: \mathcal{P} \to Y \) is a principal holomorphic fibre bundle with fibre and structure group \( G \) where \( G \) is a simply connected complex semisimple Lie group. We regard \( G \) as acting on the left of \( \mathcal{P} \). Let \( \mathcal{P}_\Gamma = \Gamma\backslash G\)-bundle where \( \Gamma \) is an irreducible torsionless lattice in \( G \). Thus \( \mathcal{P}_\Gamma = \Gamma\backslash \mathcal{P} \). Since \( \mathcal{M}_\Gamma \) is not a Kähler manifold, neither is \( \mathcal{P}_\Gamma \).

**Theorem 2.** Let \( \Gamma, G \) be as in Theorem 1 and suppose that \( \text{rank}(G) \geq 2 \). With the above notations, \( \pi: \mathcal{P}_\Gamma \to Y \) induces an isomorphism \( \pi^*: \text{Pic}^0(\mathcal{P}_\Gamma) \cong \text{Pic}^0(Y) \). Moreover, if \( \text{rank}(G) \geq 3 \), then \( \pi^*: \text{Pic}(Y) \to \text{Pic}(\mathcal{P}_\Gamma) \) is injective and has finite cokernel.

Elliptic curve bundles with non-Kähler total spaces first appeared in the work of H. Hopf [Ho] who showed that \( S^1 \times S^{2n-1} \) admits a complex structure. Since the work of Calabi and Eckmann [CE] who obtained elliptic bundle structures on product of two odd dimensional spheres, many researchers have constructed new classes of compact non-Kähler complex manifolds and studied their geometry. We point out but only a few: A. Blanchard, [Bl], C. Borcea [B], T. Höfer [Hö], V. Brînzănescu [Br], V. Ramani and P. Sankaran [RS], S. López de Madrano and A. Verjovsky [LV], J. J. Loeb and M. Nicolau [LN], L. Meersseman [Me], Meersseman and Verjovsky [MV], Sankaran and A. S. Thakur [ST], and M. Poddar and Thakur [PT]. The reader is referred to the memoir by J. Winkelmann [Wi] for a study of the geometry of complex parallelizable manifolds.

Proofs of both Theorems 1 and 2 uses the exponential exact sequence and the description of the \( \bar{\partial} \)-cohomology of \( \Gamma\backslash G \) as a \( G \)-module due to Akhiezer [A], and a criterion for the vanishing of \( H^q(\Gamma; \mathbb{C}) \) ([BW, §4, Ch. VII]) when \( q \leq 2 \). (These results will be recalled in the sequel.) In addition, proof of Theorem 2 uses the Borel spectral sequence [Hi, Appendix II], which we shall recall in §3.1. Theorem 1 will be proved in §2 and Theorem 2, in §3.

### 2. Cohomology of \( \Gamma\backslash G \)

Let \( G \) be a connected real semisimple linear Lie group with finite centre and with no compact factors. Let \( r = \text{rank}(G) \).

Let \( K \subseteq G \) be a maximal compact subgroup of \( G \). Then \( X = G/K \) is a globally symmetric space diffeomorphic to a cell. Let \( \Gamma \) be a uniform lattice in \( G \) which is torsionless. Then \( \Gamma \) acts on \( G/K \) freely and properly discontinuously and the locally symmetric space \( X_\Gamma := \Gamma\backslash G/K \) is a smooth connected compact manifold which is an Eilenberg-MacLane complex \( K(\Gamma, 1) \). Denote by \( \mathfrak{g} \) (resp. \( \mathfrak{k} \)) the Lie algebra of \( G \) (resp. \( K \)). One has the Cartan decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) where \( \mathfrak{p} = \mathfrak{k}^\perp \), the orthogonal complement with respect to the Killing form on \( \mathfrak{g} \). Denote by \( \mathfrak{p}_\mathbb{C} \) the complexification \( \mathfrak{p} \otimes \mathbb{C} \) of the vector space \( \mathfrak{p} \). Then \( \mathfrak{p}_\mathbb{C} \) is a complex representation space for \( K \) (given by the adjoint action).

Let \( C^\infty(\Gamma\backslash G) \) denote the complex vector space of smooth complex valued functions on \( \Gamma\backslash G \) and let \( L^2(\Gamma\backslash G) \) denote the complex Hilbert space of square integrable functions on \( \Gamma\backslash G \) with respect to the \( G \)-invariant measure obtained from the Haar measure on \( G \). The translation action (on the right) of \( G \) on \( \Gamma\backslash G \) induces an action of \( G \) on \( L^2(\Gamma\backslash G) \).
making the latter a representation space. Since the measure on \( \Gamma \backslash G \) is \( G \)-invariant, the \( G \) representation on \( L^2(\Gamma \backslash G) \) is unitary.

The cohomology \( H^*(\Gamma; \mathbb{C}) := H^*(X_\Gamma; \mathbb{C}) \) has been described by Matsushima [Ma] as the relative Lie algebra cohomology \( H^*(\mathfrak{g}, K; C^\infty(\Gamma \backslash G)(K)) \) when \( \Gamma \) is a uniform lattice. See also [MM], [BW, Chapter VII]. Here \( C^\infty(\Gamma \backslash G)(K) \subset C^\infty(\Gamma \backslash G) \subset L^2(\Gamma \backslash G) \) denotes the \((\mathfrak{g}, K)\)-module of smooth \( K \)-finite vectors of \( L^2(\Gamma \backslash G) \). More precisely, by a theorem of Gelfand and Piatetsky-Shapiro [GGP, p. 23], the Hilbert space \( L^2(\Gamma \backslash G) \) decomposes as a Hilbert direct sum \( \oplus m(\Gamma, \pi)H_\pi \) of certain irreducible unitary representations \((\pi, H_\pi)\) of \( G \) occurring with finite multiplicity \( m(\Gamma, \pi) \). We denote by \( V_{\sigma, (K)} \) the space of all smooth \( K \)-finite vectors of a unitary representation \((\sigma, V_\sigma)\) of \( G \) on a Hilbert space. Then \( V_{\sigma, (K)} \) is a \((\mathfrak{g}, K)\)-module. We are ready to state the result of Matsushima.

**Theorem 3.** ([Ma], [MM]) With the above notations, the cohomology of the compact locally symmetric space \( X_\Gamma = \Gamma \backslash G/K \) has the following description.

\[
H^*(X_\Gamma; \mathbb{C}) \cong H^*(\Gamma; \mathbb{C}) \cong H^*(\mathfrak{g}, K; L^2(\Gamma \backslash G)(K)) \cong \bigoplus m(\Gamma, \pi)H^*(\mathfrak{g}, K; H_\pi(K))
\]  

(1)

where the sum is over all isomorphism classes of irreducible unitary representations \( \pi \) of \( G \) that occur in \( L^2(\Gamma \backslash G) \) with positive multiplicity \( m(\Gamma, \pi) \).

Note that, since the left most side of (1) is a finite dimensional vector space, there are only finitely many representations \( \pi \) with \( m(\Gamma, \pi) > 0 \) with non-vanishing \((\mathfrak{g}, K)\)-cohomology. Since \( \Gamma \backslash G \) has finite volume, the space \( L^2(\Gamma \backslash G) \) contains the trivial representation \( \mathbb{C} \) arising as the space of all locally constant functions on \( \Gamma \backslash G \). Since \( G \) is connected, so is \( \Gamma \backslash G \) and hence \( m(\Gamma, \mathbb{C}) = 1 \). The homomorphism \( j^* : H^*(\mathfrak{g}, K; \mathbb{C}) \to H^*(\Gamma; \mathbb{C}) \) is referred to as the **Matsushima homomorphism**.

One has the **compact dual** \( X_u \) of the globally symmetric space \( X = G/K \) (where \( G \) is any non-compact real semisimple Lie group). The space \( X_u \) is the compact Riemannian symmetric space \( U/K \), where \( U \) is the maximal compact subgroup of the complexification of \( G \) that contains \( K \). Its Lie algebra \( u \) equals \( \mathfrak{k} \oplus \mathfrak{i} \mathfrak{p} \subset \mathfrak{g}_\mathbb{C} = \mathfrak{g} \oplus \mathfrak{i} \mathfrak{g} \) where \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) is the Cartan decomposition corresponding to \( K \). It is well-known that \( H^*(\mathfrak{g}, K; \mathbb{C}) \cong \text{Hom}_K(\Lambda^*p, \mathbb{C}) \cong H^*(X_u; \mathbb{C}) \) (See [BW, §3, Chapter II]).

We shall now describe the compact dual \( X_u \) when \( G \) is a simply connected **complex** semisimple Lie group.

Since \( G \) is a complex Lie group regarded as a real Lie group, we have \( p = i \mathfrak{k} \) and so \( u := \mathfrak{k} \oplus i \mathfrak{p} \cong \mathfrak{k} \times \mathfrak{k} \). Since \( u = \mathfrak{k} \times \mathfrak{k} \), it follows that \( U \cong K \times K \).

Moreover, \( K \) is imbedded diagonally in \( U \cong K \times K \) and we have \( U/K = K \times K/K \cong K \). See [He] for further details. Therefore \( X_u \cong K \) and we have \( H^*(\mathfrak{g}, K; \mathbb{C}) = H^*(X_u; \mathbb{C}) \cong H^*(K; \mathbb{C}) \). The following result is an immediate consequence.

**Proposition 4.** Let \( G \) be a connected complex semisimple Lie group and \( \Gamma \) be a torsionless uniform lattice in \( G \). Then the Matsushima homomorphism defines an injective homomorphism \( H^*(K; \mathbb{C}) \to H^*(\Gamma; \mathbb{C}) \).

\[\square\]
We have the following vanishing theorem.

**Theorem 5.** (Cf. [BW, Corollary 4.4(b), Chapter VII]) Let $\Gamma$ be an irreducible uniform lattice in a simply connected complex semisimple Lie group $G$. Then the Matsushima homomorphism $j^* : H^q(\mathfrak{g}, K; \mathbb{C}) \to H^q(\Gamma; \mathbb{C})$ is an isomorphism for $1 \leq q < \text{rank}(G)$. □

A more refined version, applicable for connected real semisimple Lie groups with finite centre and without compact factors, is proved in [BW, Chapter VII, §4, Corollary 4.4(b)]. As an immediate consequence we obtain the following.

**Corollary 6.** If $\text{rank}(G) \geq 2$, then $H^1(\Gamma; \mathbb{C}) = 0$. If $\text{rank}(G) \geq 3$, then $H^2(\Gamma; \mathbb{C}) = 0$.

**Proof.** We merely note that, under our restrictive hypotheses, $K$ is semisimple and simply connected. Moreover, since the compact dual $X_u$ is diffeomorphic to $K$, it follows that $H^q(\mathfrak{g}, K; \mathbb{C}) \cong H^q(X_u; \mathbb{C}) = 0$ if $q = 1, 2$ as $K$ is semisimple. So our assertion follows from Theorem 5. □

**Remark 7.** (i) The vanishing of $H^1(\Gamma; \mathbb{R})$ when $\Gamma$ is an irreducible lattice in any connected real semisimple Lie group and having rank at least 3 was first proved by Kazhdan [K]. The result was extended to rank 2 groups by S. P. Wang and by B. Kostant.

(ii) In the case when $\text{rank}(G) = 1$, $G = \text{SL}(2, \mathbb{C})$ and $X_\Gamma$ is a compact hyperbolic 3-manifold with fundamental group $\Gamma$, it is known that there are lattices $\Gamma$ such that $X_\Gamma$ has positive first Betti number. In fact, it was first shown by Millson [Mi] that for any $n \geq 3$, there are uniform arithmetic lattices in $SO_0(1, n)$ whose first Betti numbers are arbitrarily large.

Let $M$ be a compact complex manifold. As usual $H^q(M; \Omega^p_M)$ where $\Omega^p_M$ denotes the sheaf of holomorphic $p$-forms on $M$ will be denoted by $H^{p,q}(M)$. Of course, $\Omega^0_M$ is the structure sheaf $\mathcal{O}_M$ of $M$.

For the rest of this section, we assume that $\Gamma$ is a torsionless irreducible uniform lattice in a complex semisimple simply connected Lie group $G$.

Since $G$ operates on (the right of) $M_\Gamma = \Gamma \backslash G$, it acts linearly on the finite dimensional complex vector space $H^{p,q}(M_\Gamma)$. The following theorem, which is in fact true for any connected reductive linear algebraic group $G$ over $\mathbb{C}$, which is due to D. N. Akhiezer, describes this $G$-representation.

**Theorem 8.** (Akhiezer [A])

Let $H^* (\Gamma; \mathbb{C})$ be given the trivial $G$-module structure. With the $G$-module structure on $\Lambda^* (\mathfrak{g})$ arising from the adjoint action of $G$, we have an isomorphism

$$H^{p,q}(M_\Gamma) \cong H^q(\Gamma; \mathbb{C}) \otimes_{\mathbb{C}} \Lambda^p(\mathfrak{g})$$

as $G$-modules for all $p, q \geq 0$. □
Denote by $\pi : M_G \to X_G$ the projection of the principal $K$-bundle. We need to compute the integral cohomology groups in degrees $q = 1, 2$ of $M_G$. Since $K$ is semisimple and simply connected, we have $H^q(K; \mathbb{Z}) = 0$ for $q = 1, 2$. Also $H^1(X_G; \mathbb{Z}) = \text{Hom}(\Gamma, \mathbb{Z}) = 0$ when the rank of $G$ is at least 2 and moreover, $H^2(X_G; \mathbb{Z})$ is a finite group when rank of $G$ is at least 3 by Theorem 5. Therefore $H^2(X_G; \mathbb{Z}) = \text{Ext}(H_1(X_G; \mathbb{Z}), \mathbb{Z}) \cong \text{Ext}(\Gamma/\Gamma, \mathbb{Z}) \cong \Gamma/\Gamma$. Applying the Serre spectral sequence to $\pi : M_G \to X_G$, we have $E_2^{p,q} = H^p(X_G; \mathcal{H}^q(K; \mathbb{Z})) = 0$ when $q = 1, 2, p \geq 0$, and $E_2^{1,0} = H^1(X_G; \mathbb{Z}) \cong \mathbb{Z}^r$ where $r = 0$ when rank($G$) $\geq 2$. It follows that $H^q(M_G; \mathbb{Z}) \cong H^q(X_G; \mathbb{Z})$ when $q = 1, 2$. The same conclusion holds when the coefficient group is $\mathbb{C}$. Summarising, we have proved

**Proposition 9.** If rank($G$) $\geq 2$, then $H^1(M_G; \mathbb{Z}) = 0$ and $H^2(M_G; \mathbb{C}) \cong H^2(\Gamma; \mathbb{C})$. If rank($G$) $\geq 3$, then $H^2(M_G; \mathbb{Z}) \cong \text{Ext}(H_1(M_G; \mathbb{Z}); \mathbb{Z})$ which is isomorphic to the finite group $\Gamma/\Gamma$.

**Remark 10.** Let $G$ be a connected semisimple Lie group, not necessarily complex, and $\Gamma \subset G$ any torsionless lattice. Consider the principal $K$-bundle with projection $M_G \to X_G$.

**Claim:** The local coefficient system $\mathcal{H}^q(K; \mathbb{Z})$ over $X_G = \Gamma \backslash G$ is simple, i.e., $\Gamma$ acts trivially on the cohomology of the fibre.

This is true for the $K$-bundle $M_G \to X_G$ for any $G$ connected. Indeed let $\gamma \in \Gamma = \pi_1(X_G)$. The element $\bar{e} := \Gamma. K \in X_G$ is understood to be the base point. We identify $k \in K$ with $\Gamma.k \in M_G$, the fibre over $\bar{e}$. Choose a path $I \to G$, $t \mapsto \gamma_t$, that joins the identity element $e \in G$ to $\gamma \in \Gamma$.

$$
\begin{array}{ccc}
0 \times K & \xrightarrow{\phi_0} & M_G \\
\downarrow & & \downarrow \pi \\
I \times K & \xrightarrow{\phi} & X_G
\end{array}
$$

Then $t \mapsto \Gamma \gamma_t K$ is a loop that represents $\gamma \in \pi_1(X_G)$. The action of $\gamma \in \Gamma$ on $H^*(K; \mathbb{C})$ is induced by $\Phi_1 : K \cong 1 \times K \to \pi^{-1}(\bar{e}) \cong K$ where $\Phi : I \times K \to M_G$ is a lift of $\phi : I \times K \to X_G$ defined as $(t, k) \mapsto \Gamma \gamma_t K$ such that $\Phi(0, k) = \Gamma k \forall k \in K$. Evidently $(t, k) \mapsto \Gamma \gamma_t k \in M_G$. We note that $\Phi_1(k) = \Gamma \gamma k = \Gamma k$ corresponds to the identity map of $K$. This proves our claim.

We shall now prove Theorem 1.

**Proof of Theorem 1:** (i) Assume that the rank of $G$ is at least 2. Consider the long exact sequence induced by the exponential sequence of sheaves, where we have replaced $H^1(M_G; \mathcal{O}_{M_G}^*)$ by $\text{Pic}(M_G)$:

$$
\to H^1(M_G; \mathbb{Z}) \xrightarrow{i} H^1(M_G; \mathcal{O}_{M_G}) \xrightarrow{i} \text{Pic}(M_G) \xrightarrow{i} H^2(M_G; \mathbb{Z}) \to H^2(M_G; \mathcal{O}_{M_G}) \to \cdots \quad (2)
$$

where $c_1$ is the Chern class map.

We have $H^1(M_G; \mathbb{Z}) = 0 = H^1(M_G; \mathcal{O}_{M_G})$ and so it follows that $\text{Pic}^0(M_G) = 0$.

When rank($G$) $\geq 3$, we have $H^2(M_G; \mathcal{O}_{M_G}) \cong H^2(\Gamma; \mathbb{C}) \cong 0$ where the first isomorphism is by Theorem 8 and the second by Theorem 5. Proposition 9 implies $H^2(M_G; \mathbb{Z}) \cong \Gamma/\Gamma$ is a finite group and so our assertion follows.
In the rank 1 case, namely when $G = SL(2, \mathbb{C})$, we have $K = SU(2) \cong \mathbb{S}^3$ and $X$ is the three-dimensional real hyperbolic space. Let $H^1(\Gamma; \mathbb{Z}) \cong \mathbb{Z}^r$. By the Poincaré duality, we have $H^2(\Gamma; \mathbb{Z}) \cong \mathbb{Z}^r \oplus A$ where $A$ is isomorphic to the torsion subgroup of $H_1(\Gamma; \mathbb{Z}) = \Gamma/[\Gamma, \Gamma]$. Then $H^1(M\Gamma; \mathcal{O}_{M\Gamma}) \cong \mathbb{C}^r$, $H^2(M\Gamma; \mathcal{O}_{M\Gamma}) \cong H^2(\Gamma; \mathbb{C}) \cong \mathbb{C}^r$, by Theorem 8. Using the Serre spectral sequence we get that $H^q(M\Gamma; \mathbb{Z}) \cong H^q(\Gamma; \mathbb{Z})$, $q = 1, 2$. It follows from (2) that $Pic^0(M\Gamma) \cong \mathbb{C}^r/A$ where $A = H^1(M\Gamma; \mathbb{Z}) \cong \mathbb{Z}^r$. □

When rank$(G) \leq 2$, in order to determine $Pic(M\Gamma)$ it remains to compute the image of the Chern class map $H^1(M\Gamma; \mathcal{O}_{M\Gamma}^*) \to H^2(M\Gamma; \mathbb{Z})$. Equivalently, we need only to determine the kernel of $H^2(M\Gamma; \mathbb{Z}) \to H^2(M\Gamma; \mathcal{O}_{M\Gamma})$. We have the following diagram

$$
\begin{array}{ccc}
H^2(\Gamma; \mathbb{C}) & \xrightarrow{\psi} & H^2(M\Gamma; \mathcal{O}_{M\Gamma}) \\
\pi^* \downarrow & & \downarrow id \\
H^2(M\Gamma; \mathbb{C}) & \xrightarrow{\iota} & H^2(M\Gamma; \mathcal{O}_{M\Gamma})
\end{array}
$$

in which $\psi$ is the Akhiezer isomorphism of Theorem 8, $\iota: \mathbb{C} \to \mathcal{O}_{M\Gamma}$ is the inclusion where $\mathbb{C}$ denotes the constant sheaf over $X$. The vertical map $\pi^*$ is induced by the projection of the $SU(2)$-principal bundle, which is an isomorphism (in degree 2). It is plausible that the above diagram commutes and so $\iota_*$ is an isomorphism, but it appears to be hard to establish.

When rank$(G) = 2$, we have $H^{1,1}(M\Gamma) = 0$ by Theorem 8.

A result of Dolbeault [D, Théorème 2.3] establishes the Lefschetz theorem on (1,1)-classes for non-Kähler manifolds. Accordingly, an element $c \in H^2(M\Gamma; \mathbb{Z})$ is the first Chern class of a holomorphic line bundle over $M\Gamma$ if and only if it is represented by a form of type $(1,1)$, say $\omega$. However, as $M\Gamma$ is not Kähler, it does not seem to follow that $c$, regarded as an element of $H^2(M\Gamma; \mathbb{R})$, vanishes although $[\omega] \in H^{1,1}(M\Gamma) = 0$.

Remark 11. (i) When $\Gamma \subset G$ is as in Theorem 1 and rank of $G$ is at least two, the abelianisation $\Gamma/[\Gamma, \Gamma] \cong H_1(M\Gamma; \mathbb{Z})$ is finite. The torsion subgroup $A$ of $H^2(M\Gamma; \mathbb{Z})$ is isomorphic to $Ext(H_1(M\Gamma; \mathbb{Z}); \mathbb{Z}) \cong Ext(\Gamma/[\Gamma, \Gamma], \mathbb{Z}) \cong \Gamma/[\Gamma, \Gamma]$ since the abelianisation of $\Gamma$ is finite. In particular, $A$ is zero when $\Gamma$ is perfect. Moreover if rank$(G) \geq 3$ and $\Gamma$ is perfect, then $Pic^0(M\Gamma) = Pic(M\Gamma) = 0$.

(ii) If $\chi: \Gamma \to \mathbb{C}^*$ is any homomorphism, we obtain a holomorphic line bundle $L_\chi$ with total space $G \times_\Gamma \mathbb{C}$. When rank$(G) \geq 2$, the image of $\chi$ is necessarily a finite cyclic subgroup $\mu_m \subset \mathbb{C}^*$. The homomorphism $\chi_m: \Gamma/[\Gamma, \Gamma] \to \mu_m$ defined by $\chi$ determines an element $\alpha$ of $H^2(\Gamma; \mathbb{Z}) = Ext(\Gamma, \mathbb{Z})$ corresponding to the image of the generator $Ext(\mu_m; \mathbb{Z}) \cong \mu_m$ under $\chi_m^*: H^2(\mu_m; \mathbb{Z}) = Ext(\mu_m; \mathbb{Z}) \to Ext(\Gamma/[\Gamma, \Gamma], \mathbb{Z}) \hookrightarrow H^2(\Gamma; \mathbb{Z})$. Then, under the isomorphism $\pi^*: H^2(\Gamma; \mathbb{Z}) \cong H^2(M\Gamma; \mathbb{Z})$, (via the Serre spectral sequence) $c_1(L_\chi) = \pi^*(\alpha)$.

(iii) The computation of Betti numbers $b_j$ of $M\Gamma = \Gamma/SL(2, \mathbb{C})$ for a torsionless uniform lattice can be completed using the Serre spectral sequence $E_1^{p,q} = H^p(\Gamma; H^q(X\Gamma))$ of the principal $SU(2)$-bundle over the hyperbolic manifold $X\Gamma$. As we saw in the above proof, $b_1 = b_2 = \text{rank}(\Gamma/[\Gamma, \Gamma])$. Since the group local coefficient system $H^3(SU(2); \mathbb{C})$ over $X\Gamma$ is trivial (by Remark 10), we have $E_2^{0,3} = E_\infty^{0,3} \cong H^3(SU(2); \mathbb{C}) \cong \mathbb{C}$. Moreover,
$E_{\infty}^{3,0} = E_{2}^{3,0} = H^{3}(\Gamma; \mathbb{C}) = \mathbb{C}$ as $X_{\Gamma}$ is a compact connected 3-manifold. So $b_{3} = 2$. By Poincaré duality (applied to $M_{\Gamma}$) we have $b_{4} = b_{2} = r = b_{1} = b_{5}$. Of course $b_{0} = b_{6} = 1, b_{j} = 0$ for $j > 6$. This was already observed by Akhiezer [A].

3. Picard groups of $M_{\Gamma}$-bundles

In this section we prove the main result of this paper, namely, Theorem 2.

Suppose that $Y$ is a compact connected complex manifold. Let $P \to Y$ be a holomorphic principal $G$-bundle where $G$ is a simply connected complex linear semisimple Lie group. We assume that $G$ acts on the left of $P$. Let $\Gamma$ be a torsionless irreducible and uniform lattice in $G$. Let $P_{\Gamma} = \Gamma \backslash P$. Then we have the natural projection $P_{\Gamma} \to Y$ of a holomorphic $M_{\Gamma} = \Gamma \backslash G$-bundle over $Y$ with structure group $G$. The proof will involve computing $H^{q}(P_{\Gamma}; \mathcal{O}_{P_{\Gamma}})$ for $q = 1, 2$ using the Borel spectral sequence [Hi, Appendix II, §2].

3.1. Borel spectral sequence. The hypotheses of the Borel spectral sequence include the requirement that the connected components of the structure group of the holomorphic bundle act trivially on the $\bar{\partial}$-cohomology of the fibre. This is (in general) not true for $M_{\Gamma}$-bundles. However, one can still use the spectral sequence for the computation of $H^{*}(P_{\Gamma}; \mathcal{O}_{P_{\Gamma}})$, as we shall now explain.

More generally, let $\xi = (E, B, F, \pi)$ be a holomorphic fibre bundle with structure group $G$, which is a complex Lie group. Suppose that the fibre space $F$ is compact and that $E, B, F$ are connected. Assume that the connected components of $G$ act trivially on $H^{*}(F; \mathcal{O}_{F})$. Then one has a holomorphic vector bundle $\mathcal{H}^{q}(F; \mathcal{O}_{F})$ over $B$ whose fibre over $b \in B$ is the complex vector space $H^{q}(F_{b}; \mathcal{O}_{F_{b}}) \cong H^{q}(F; \mathcal{O}_{F})$ where $F_{b} = \pi^{-1}(b)$. Let $W$ be a holomorphic vector bundle over $B$ and let $\hat{W} := \pi^{*}(W)$. Then one has a spectral sequence $0^{q}E_{r}^{s,t} = {q}E_{r}^{s,t}$ in which the differential $d_{r} : {q}E_{r}^{s,t} \to {q+1}E_{r}^{s+r,t-r+1}$ has bidegree $(r, 1 - r)$. We have $qE_{r}^{s,t} = 0$ unless $t = q - s$; $s$ is the ‘base degree’, $t$ is the ‘fibre degree’. The $E_{2}$-page is given as $qE_{2}^{s,t} = H^{s}(B; W \otimes \mathcal{H}^{t}(F; \mathcal{O}_{F}))$. The spectral sequence converges to $H^{0,q}(E; \hat{W}) = H^{q}(E; \mathcal{O}(\hat{W}))$. That is, for each $q \geq 0$, there exists a filtration of $H^{q}(E; \hat{W})$ such that the associated graded space is:

$$GrH^{q}(E; \mathcal{O}(\hat{W})) = \bigoplus_{0 \leq s \leq q} qE_{\infty}^{s,q-s}.$$ 

The proof is exactly as given in [Hi, Appendix II], where we need only replace the filtration $L_{k}$ (in §4 therein) by $L_{k} \cap A_{E_{\infty}}^{0,*}(\hat{W})$. For the sake of completeness we indicate below the crucial place where the change is necessitated.

Define $L_{k}(U) \subset A_{E_{\infty}}(\hat{W}[U])$, for a small open set $U \subset E$, to be the span of monomials $dz_{J} \wedge dy_{J'}$ in which $|J| + |J'| \geq k$. Thus $L_{k}(U) \subset \bigoplus_{q \geq k} A_{E_{\infty}}^{0,q}(U)$. Here, the smallness refers to both $\xi$ and $W$ being trivial over $\pi(U) =: V$ and $\hat{U} \cong V \times V'$ (via a local analytic chart of $\xi$), where $(V, z_{i}), (V', y_{j})$ are holomorphic coordinate charts in $B, F$ respectively.
Define
\[ L_k := \{ \omega \in A_E(\tilde{W}) \mid \omega|_U \in L_k(U) \forall \text{ small open subset } U \subset E \}. \]

Then
\[ L_0 = A_E(\tilde{W}), L_k = 0 \text{ if } k > \dim_B B; L_k \supset L_{k+1}; \partial L_k \subset L_k \forall k \geq 0. \]

Also \( L_k = \sum_{q \geq 0} 0^q L_k \) where \( 0^q L_k = L_k \cap A_E^{0,q}(\tilde{W}) \). Note that \( H^*(E; \mathcal{O}_E(\tilde{W})) \) is the cohomology of the cochain complex \( (A_E^{0,q}(\tilde{W}), \partial) \). The required spectral sequence is associated to the filtration \( \{(L_k, \bar{\partial})\} \) of the differential graded complex \( (A_E^{0,q}(\tilde{W}), \partial) \). The proof is exactly as given by Borel in [Hi].

3.2. Proof of Theorem 2. Reverting back to the \( M_\Gamma \)-bundle with projection \( P_\Gamma \rightarrow Y \), first, observe that the structure group \( G \) of the \( M_\Gamma \)-bundle with projection \( P_\Gamma \rightarrow Y \) acts trivially on \( H^q(M_\Gamma; \mathcal{O}_{M_\Gamma}) \cong H^q(X_\Gamma; \mathbb{C}) \) by Theorem 8. So the Borel spectral sequence is applicable for the \( M_\Gamma \)-bundle for computing \( H^q(P_\Gamma; \mathcal{O}_{P_\Gamma}) \). The \( E_2 \)-page of the spectral sequence (with \( W \) being the trivial bundle) is given by:
\[ 0^q E_2^{s,t} = H^s(Y; \mathcal{H}^{0,q-s}(M_\Gamma)). \]

Here \( \mathcal{H}^{0,q-s}(M_\Gamma) \) denotes the holomorphic vector bundle over \( Y \) with discrete structure group. Since \( G \) is connected, the structure group of the vector bundle \( \mathcal{H}^{0,t}(M_\Gamma) \) reduces to the trivial group and so
\[ 0^q E_2^{s,t} = H^s(Y; \mathcal{O}_Y) \otimes H^t(M_\Gamma; \mathcal{O}_{M_\Gamma}) \]  

We shall suppress the type \((0, q)\) in the notation \( 0^q E_2^{s,t} \). Our main interest is in computing \( H^q(P_\Gamma; \mathcal{O}_{P_\Gamma}) \) when \( q = 1, 2 \).

We will assume that \( \text{rank}(G) \geq 2 \), so that \( H^{0,1}(M_\Gamma) = 0 \) by Theorems 5 and 8. Substituting in (3), we obtain \( E_2^{s,1} = 0 \forall s \geq 0 \). We have \( E_2^{0,q} \cong H^q(M_\Gamma; \mathcal{O}_{M_\Gamma}), q \geq 2; E_2^{s,0} \cong H^s(Y; \mathcal{O}_Y) \forall s \geq 0 \).

It follows that \( H^1(P_\Gamma; \mathcal{O}_{P_\Gamma}) \cong H^1(Y; \mathcal{O}_Y) \), the isomorphism being induced by \( \pi \). Also, \( H^2(P_\Gamma; \mathcal{O}_{P_\Gamma}) \cong H^2(Y; \mathcal{O}_Y) \oplus H^0_\infty = H^2(Y; \mathcal{O}_Y) \oplus E_4^{0,2} \). We note that \( E_4^{0,2} = E_2^{0,2} = H^2(M_\Gamma; \mathcal{O}_{M_\Gamma}) \) if \( \text{rank}(G) > 2 \). Summarising we have

**Lemma 12.** We keep the above notations. Suppose that \( \text{rank}(G) \geq 2 \). Then \( H^1(P_\Gamma; \mathcal{O}_{P_\Gamma}) \cong H^1(Y; \mathcal{O}_Y) \) and \( H^2(P_\Gamma; \mathcal{O}_{P_\Gamma}) \cong H^2(Y; \mathcal{O}_Y) \oplus V \) for some suitable vector subspace \( V \subset H^2(M_\Gamma; \mathcal{O}_{M_\Gamma}) \). If \( \text{rank}(G) \geq 3 \), then \( V = 0 \).

We now turn to the computation of \( H^q(P_\Gamma; \mathbb{Z}) \) for \( q = 1, 2 \). Using the homotopy exact sequence of the \( M_\Gamma \)-bundle with projection \( \pi : P_\Gamma \rightarrow Y \), we see that \( \pi_* : \pi_1(P_\Gamma) \rightarrow \pi_1(Y) \) is surjective. Therefore \( H^1(Y; \mathbb{Z}) \rightarrow H^1(P_\Gamma; \mathbb{Z}) \) is injective. We claim that it is an isomorphism. Using the Serre spectral sequence and the fact \( H^1(M_\Gamma; \mathbb{Z}) = 0 \) (as \( H_1(M_\Gamma; \mathbb{Z}) = \Gamma/[\Gamma, \Gamma] \) is finite), we see that \( \pi^* : H^1(Y; \mathbb{Z}) \rightarrow H^1(P_\Gamma; \mathbb{Z}) \) is an isomorphism.

Next we turn to computation of \( H^2(P_\Gamma; \mathbb{Z}) \). Assume that \( \text{rank}(G) \geq 3 \) so that \( H^2(\Gamma; \mathbb{Z}) \) is finite by Theorem 5. Since \( G \) is a complex semisimple simply connected Lie group, \( K \) is also semisimple and simply connected. Hence \( H^q(K; \mathbb{Z}) = 0 \) for \( q = 1, 2 \). Using the Serre spectral sequence for the principal \( K \)-bundle with projection \( M_\Gamma \rightarrow X_\Gamma \), we obtain that \( H^2(M_\Gamma; \mathbb{Z}) \cong H^2(\Gamma; \mathbb{Z}) \) is finite. Hence, in the Serre spectral sequence for the
$M_r$-bundle, we have $E^{0,2}_2 = H^0(Y; H^2(M_r; \mathbb{Z}))$ is finite. This, together with the vanishing of $H^1(M_r; \mathbb{Z})$, implies that that $\pi^* : H^2(Y; \mathbb{Z}) \to H^2(P_r; \mathbb{Z})$ is injective and has finite cokernel.

We shall now prove Theorem 2.

**Proof of Theorem 2.** Suppose that rank($G$) $\geq 2$. We have a commuting diagram induced by the exponential exact sequence, where $\pi : P_r \to Y$ is the projection of the $M_r$-bundle.

$$
\begin{array}{cccc}
H^1(Y; \mathbb{Z}) & \to & H^1(Y; \mathcal{O}_Y) & \to & \text{Pic}(Y) \\
\pi^* \downarrow & & \pi^* \downarrow & & \pi^* \downarrow \\
H^1(P_r; \mathbb{Z}) & \to & H^1(P_r; \mathcal{O}_{P_r}) & \to & \text{Pic}(P_r)
\end{array}
$$

As noted above, $\pi^* : H^1(Y; \mathbb{Z}) \to H^1(P_r; \mathbb{Z})$ is an isomorphism. By Lemma 12, $\pi^* : H^q(Y; \mathcal{O}_Y) \to H^q(P_r; \mathcal{O}_{P_r})$ is an isomorphism when $q = 1$ as rank($G$) $\geq 2$. Denoting the inclusion of sheaves $\mathbb{Z} \to \mathcal{O}_Y$ by $i_Y$ and a similar notation for $P_r$, it follows that the natural homomorphism $\text{Pic}^0(Y) = H^1(Y; \mathcal{O}_Y)/\text{Im}(i_Y)$ $\to$ $H^1(P_r; \mathcal{O}_{P_r})/\text{Im}(i_{P_r}) \cong \text{Pic}^0(P_r)$ is surjective. The commutativity of the left-most square in (4) and the fact that the $\pi^*$ in that square are isomorphisms implies that $\text{Pic}^0(Y) \to \text{Pic}^0(P_r)$ is an isomorphism.

Now let rank($G$) $\geq 3$. It remains to show that $\pi^*(\text{Pic}(Y)) \subset \text{Pic}(P_r)$ has finite index. Suppose that $L$ is a holomorphic line bundle over $P_r$. If $c_1(L) = 0$, then $L \in \text{Pic}^0(P_r)$ and so $L = \pi^*(L')$ for some line bundle $L'$ over $Y$ as $\pi^* : \text{Pic}^0(Y) \to \text{Pic}^0(P_r)$ is an isomorphism. Now suppose that $c_1(L) = a \neq 0$. Choose $m \geq 1$ to be the cardinality of the cokernel of $\pi^* : H^2(Y; \mathbb{Z}) \to H^2(P_r; \mathbb{Z})$. It suffices to show that $L^m$ is in the image of $\pi^* : \text{Pic}(Y) \to \text{Pic}(P_r)$. We have $ma = \pi^*(b)$ for some $b \in H^2(Y; \mathbb{Z})$. By the commutativity of the right-most square in (4) and the observation that $\pi^* : H^2(Y; \mathcal{O}_Y) \to H^2(P_r; \mathcal{O}_{P_r})$ is an isomorphism (by Lemma 12), we see that $b$ is in the kernel of $H^2(Y; \mathbb{Z}) \to H^2(Y; \mathcal{O}_Y)$. Therefore $b = c_1(L_1)$ for some line bundle $L_1$ over $Y$. Hence $L^{-m}\pi^*(L_1) \in \text{Pic}^0(P_r)$. Choose $L_0 \in \text{Pic}^0(Y)$ such that $L^{-m}\pi^*(L_1) = \pi^*(L_0)$. Then $L^m = \pi^*(L_0^{-1}L_1)$.

**Example 13.** Let $\omega$ be any vector bundle of rank $k$ over a compact connected complex manifold $Y$. Let $\xi = \Lambda^k(\omega)^*$. Then we claim that $\eta := \omega \otimes \xi$ admits a reduction of the structure group to $SL(k+1, \mathbb{C})$. This is because $\Lambda^{k+1}(\eta) = \Lambda^{k+1}(\omega \otimes \xi) = \Lambda^k(\omega) \otimes \xi = \varepsilon$, the trivial line bundle. We take $P \to Y$ to be the associated principal $SL(k+1, \mathbb{C})$-bundle. For any uniform irreducible torsionless lattice we obtain a holomorphic bundle $P_r \to Y$ with fibre $SL(k+1, \mathbb{C})/\Gamma$.

**Remark 14.** Let rank($G$) $\geq 3$. Suppose that $Y$ (as in Theorem 2) is simply connected. Then $\pi_1(M_r) \cong \Gamma \to \pi_1(P_r)$ is a surjection. Also $\pi^* : H^2(Y; \mathbb{Z}) \to H^2(P_r; \mathbb{Z})$ is a monomorphism as can be seen using the Serre spectral sequence. Suppose that $H^2(Y; \mathbb{Z}) \to H^2(Y; \mathcal{O}_Y)$ is the trivial homomorphism so that $c_1 : \text{Pic}(Y) \to H^2(Y; \mathbb{Z})$ is surjective. (For example, take $Y$ such that $H^2(Y; \mathcal{O}_Y) = 0$.) Let $C$ denote the subgroup of $\text{Pic}(M_r)$ defined as the image of the restriction to a fibre $\text{Pic}(P_r) \to \text{Pic}(M_r)$. Since rank($G$) $\geq 3$, we have $\text{Pic}(M_r) \cong \Gamma/[\Gamma, \Gamma] = A$, which is a finite abelian group. Finally, let $C_1$ denote the image of $c_1 : \text{Pic}(P_r) \to H^2(P_r; \mathbb{Z})$. 


Then we have the following commuting diagram

\[
\begin{array}{cccccc}
\text{Pic}^0(Y) & \cong & \text{Pic}^0(P_\Gamma) \\
\downarrow & & \downarrow \\
\text{Pic}(Y) & \to & \text{Pic}(P_\Gamma) & \to & \text{Pic}(M_\Gamma) \\
\downarrow & & \downarrow & \cong & \downarrow \\
H^2(Y; \mathbb{Z}) & \to & C_1 & \to & A
\end{array}
\]

An easy diagram chase reveals that we have an exact sequence

\[1 \to \text{Pic}(Y) \to \text{Pic}(P_\Gamma) \to C \to 1.\]

If \(\Gamma\) is perfect, then \(\text{Pic}(M_\Gamma) = 0\) and we have \(\text{Pic}(Y) \cong \text{Pic}(P_\Gamma)\).

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