Structure of Matrix Elements in Quantum Toda Chain.

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Abstract. We consider the quantum Toda chain using the method of separation of variables. We show that the matrix elements of operators in the model are written in terms of finite number of “deformed Abelian integrals”. The properties of these integrals are discussed. We explain that these properties are necessary in order to provide the correct number of independent operators. The comparison with the classical theory is done.
1 Introduction

As it became clear recently [1, 2, 3] there is a close connection between the formulae for the matrix elements in integrable field theory (form factors) [4] and the method of separation of variables developed by Sklyanin [5].

The form factors are typically given by certain integrals. These kind of formulae can be interpreted as follows. Consider an integrable model which allows the separation of variables. The separated variables naturally split into two equal parts: one of them can be considered as “coordinates” and another as “momenta” (of course they have nothing to do with original canonical variables in which the model is formulated). The formulae for the form factors are understood as matrix elements written in “coordinate” representation, i.e. in terms of integrals with respect to the “coordinates”.

Another observation made in [6], and used intensively in [1, 2] is that the integrals in the formulae for the form factors in models with $\hat{sl}(2)$ Lie-Poisson symmetry (Sine-Gordon, for example) can be considered as deformations of hyper-elliptic integrals. This fact must be also related to the method of separation of variables because the “coordinates” describe classically a divisor on the spectral hyper-elliptic curve. The important conclusion made in [2] is that these deformed hyper-elliptic integrals must have similar properties to the usual hyper-elliptic integral in order that the correct number of equations of motion exists in the quantum case.

In paper [3] we performed the quasi-classical analysis of the matrix elements in CFT in finite volume. This is a much more complicated case than the case of infinite volume. The method of separation of variables seems to give the only possible approach to the calculation of form factors. The main difficulty of the problem in finite volume is due to the fact that the separation of variables leads to the Baxter equations whose solutions describe the wave-functions in “coordinate” representation. So, one must consider integrals of solutions over Baxter equations.

In the present paper we consider a much simpler model which nevertheless exhibits difficulties similar to those of integrable field theory in finite volume. This is the periodical Toda chain. Historically this is the first model to which the method of separation of variables was applied [5]. In this case the problem of describing the spectrum leads to Baxter equations with non-trivial solutions in entire functions. The matrix elements are given by integrals over these solutions. We show that these integrals can be considered as deformed hyper-elliptic integrals allowing deformation of all the important properties of hyper-elliptic integrals. Similarly to [2] these properties are needed for the correct counting of operators, they are actually equivalent to the equations of motion.

2 Classical Toda chain

The periodical Toda chain is described by the Hamiltonian:

$$H = \sum_{j=1}^{n} \frac{p_j^2}{2} + e^{q_j+1-q_j}$$

where $p_j, q_j$ are canonical variables, $q_{n+1} \equiv q_1$.

The exact solution is due to existence of Lax representation. Consider the L-operator

$$L_j(\lambda) = \left( \begin{array}{cc} \lambda - p_j & e^{q_j} \\ e^{-q_j} & 0 \end{array} \right)$$

and the monodromy matrix

$$M(\lambda) = L_n(\lambda) \cdots L_1(\lambda) = \left( \begin{array}{cc} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{array} \right)$$

Obviously, det $M(\lambda) = 1$. The monodromy matrix satisfies Sklyanin’s Poisson brackets:

$$\{ M(\lambda) \otimes M(\mu) \} = [r(\lambda - \mu), M(\lambda) \otimes M(\mu)]$$

where

$$r(\lambda) = -\frac{P}{\lambda}$$
P is the permutation. The coefficients of $T(\lambda) \equiv \text{tr} M(\lambda)$ are in involution:

$$\{T(\lambda), T(\mu)\} = 0$$

Moreover,

$$T(\lambda) = \lambda^n - P\lambda^{n-1} + \left(\frac{1}{2} P^2 - H\right)\lambda^{n-2} + \ldots$$

where $P = \sum p_j$ is the total momentum and $H$ is the Hamiltonian \(\mathbf{[1]}\). Thus $T(\lambda)$ generates $n$ integrals of motion in involution providing complete integrability of the system.

From here on we can forget about the Toda chain saying that we consider an orbit of Lie-Poisson group \(\mathbf{[2]}\) i.e. the polynomial matrix $M(\lambda)$ with $\det M(\lambda) = 1$ satisfying the Poisson brackets \(\mathbf{[2]}\) (determinant is in the center of these Poisson brackets) and characterized by certain reality conditions which we shall discuss later.

Let us consider the elements of $M(\lambda)$ in some more details. We introduce the notations

$A(\lambda) = \lambda^n + \lambda^{n-1}a_1 + \cdots + a_n$

$B(\lambda) = b\left(\lambda^{n-1} + \lambda^{n-2}b_1 + \cdots + b_{n-1}\right)$

$C(\lambda) = \lambda^{n-1}c_2 + \cdots + c_{n+1}$

$D(\lambda) = \lambda^{n-2}d_2 + \cdots + d_n$

The variables $b$ and $a_1$ have the Poisson brackets

$$\{a_1, b\} = b$$

and Poisson commute with the rest of variables. In terms of Toda chain $a_1 = P$ and $b = e^{\gamma_0}$ describe the motion of the center of mass. Our nearest concern is the algebra of observables $A$. We define this algebra as the one generated by all the monomials of finite degree of the variables $a_j, b_j, c_j, d_j$ and $b$. It is important that the polynomial structure of $M(\lambda)$ introduces grading of $A$. Namely, we can prescribe the degree $i$ to every of elements $a_i, b_i, c_i, d_i$ and degree 0 to $b$. The degrees of the leading coefficients of $C(\lambda)$ and $D(\lambda)$ are chosen in order that the coefficients of the determinant

$$\det M(\lambda) = \lambda^{2n-2}f_2 + \cdots + f_{2n}$$

are homogeneous. The variable $b$ is a kind of zero-mode, it is of minor dynamical value. The algebra $A$ contains a subalgebra $A_0$ of polynomial functions of $a_i, b_i, d_i$ and $c_i = bc_i$. So, this subalgebra does not have $b$ as a separate generator, the change in definition of $c_i$ is needed in order that the Poisson brackets are closed for $A(\lambda), D(\lambda), B(\lambda) = \prod (\lambda - \gamma_j)$ and $C(\lambda) = bC(\lambda)$. We shall deal only with this subalgebra.

The algebra $A_0$ as a vector space splits into direct sum of subspaces of different degrees. Let us denote by $\delta(n)$ the dimension of the subspace of the degree of $n$. The generating function of $\delta(n)$ (character) is given by

$$\chi(q) \equiv \sum_{n=0}^{\infty} \delta(n)q^n = \frac{1}{[n]!} \frac{1}{[n-1]!} \left[ \frac{1}{[n+1]!} \frac{[1]}{[n]!} \frac{1}{[1]} \frac{[2n]!}{[1]} \right]$$

where $[n] = 1 - q^n$, $[n]! = [1][2] \cdots [n]$. The first four multipliers come from monomials of of $a_j, b_j, c_j, d_j$ respectively, the last multiplier comes from factorization by the condition $\det M(\lambda) = 1$.

Notice that

$$\chi(q) = \frac{1}{[n]! [n-1]!} \left( \left[ \frac{2n-1}{n-1} \right] - q \left[ \frac{2n-1}{n-2} \right] \right)$$

where we introduced the q-binomial coefficients

$$\binom{n}{m} = \frac{[n]!}{[m]! [n-m]!}$$

Later we shall provide an interesting interpretation of this formula.
Let us return to more traditional consideration of classical Toda chain. We do not give a complete list of references which can be found in [5], the important for us fact concerning the classical system is that it allows the separation of variables [8, 9]. Consider zeros of the polynomial $B(\lambda)$:

$$B(\lambda) = b \prod_{j=1}^{n-1} (\lambda - \gamma_j)$$

and the variables $\Lambda_j \equiv D(\gamma_j)$. Notice that $\Lambda_j = \Lambda(\gamma_j)$ where $\Lambda(\lambda)$ is the eigenvalue of $M(\lambda)$. The variables $\gamma_j$, $\log \Lambda_j$ are canonically conjugated which can be shown following [5] using (2):

$$\{\gamma_i, \log \Lambda_j\} = \delta_{i,j}$$

From $\det M(\lambda) = 1$ it follows that $A(\gamma_j) = \Lambda_j^{-1}$. One can reconstruct the matrix $M(\lambda)$ from $\gamma_1, \cdots, \gamma_{n-1}, \Lambda_1, \cdots, \Lambda_{n-1}, a_1$ and $b$. The symplectic form is written as

$$\omega = \sum_{j=1}^{n-1} d\log \Lambda_j \wedge d\gamma_j + d\log b \wedge da_1$$

The 1-form $\alpha (\omega = d\alpha)$ is

$$\alpha = \sum_{j=1}^{n-1} \log \Lambda_j d\gamma_j + \log b \, da_1$$

Let us take other coordinates on the phase space, namely, $\gamma_1, \cdots, \gamma_{n-1}, t_2, \cdots, t_n$ (defined by $T(\lambda) = \lambda^n + \lambda^{n-1} t_1 + \lambda^{n-2} t_2 + \cdots + t_n), t_1 \equiv a_1$ and $b$. From

$$\Lambda_j = \Lambda(\gamma_j) = \frac{1}{2} \left( T(\gamma_j) + \sqrt{T(\gamma_j)^2 - 4} \right)$$

equivalently, one easily finds the expression for the symplectic form in these variables

$$\omega = \sum_{j=1}^{n-1} \sum_{k=2}^{n} \frac{\gamma_j^{n-k}}{P(\gamma_j)} \, dt_k \wedge d\gamma_j + d\log b \wedge da_1$$

where $P(\lambda) = T^2(\lambda) - 4$. Thus the equations of motion take the form

$$\{T(\lambda), \gamma_j\} = \sqrt{P(\gamma_j)} \prod_{k \neq j} \frac{\lambda - \gamma_k}{\gamma_j - \gamma_k}$$

$$\{T(\lambda), b\} = \lambda^{n-1} b$$

(certainly, only the first $n-1$ equations are really interesting. They are linearized by the Abel transformation:

$$\left\{ T(\lambda), \sum_{k=1}^{n-1} \gamma_k \int \sigma_j \right\} = \lambda^{j-1}$$

where $\sigma_j$ are the first kind abelian differentials on the spectral curve $\mu^2 = P(\lambda)$:

$$\sigma_j = \frac{\lambda^{j-1}}{\sqrt{P(\lambda)}} \, d\lambda$$

We associate the “times” $\tau_1, \cdots, \tau_{n-1}$ with $t_2, \cdots, t_n$:

$$\frac{\partial}{\partial \tau_j} F = \partial_j F = \{t_{j+1}, F\}$$

The evolution of $\sum_{k=1}^{n-1} \int \gamma_k \sigma_j$ with respect to times is linear.
The above considerations apply to any orbit of the Lie-poisson group. We want now to consider specific reality conditions which correspond to Toda chain. It can be shown [3, 9] that the conditions in question are:

1. The polynomial $T(\lambda)$ of degree $n$ has $n$ real zeros. Moreover its local maxima are not below $2$ and its local minima are not above $-2$. So, all the zeros of the polynomial $P(\lambda)$ are also real, they are denoted by $\lambda_1 < \lambda_2 < \cdots < \lambda_{2n}$. 

2. The polynomial $B(\lambda)$ has real zeros $\gamma_1, \cdots, \gamma_{n-1}$ which belong to the “forbidden zones”: $\lambda_{2k} < \gamma_k < \lambda_{2k+1}$.

The equations of motion [3] preserve these conditions. The hyper-elliptic Riemann surface $\mu^2 = P(\lambda)$ has $2n$ branch points $(\lambda_j)$. Its genus equals $n - 1$. We present the surface as two complex planes with the cuts along $(-\infty, \lambda_1]$, $[\lambda_2, \lambda_3]$, $\cdots$, $[\lambda_{2n}, \infty)$ identifying the banks of the cuts on two sheets in usual way. The canonocal $a$-cycles $a_j$ are taken as ones encircling the cuts $[\lambda_{2j}, \lambda_{2j+1}]$ for $j = 1, \cdots, n - 1$. Topologically the points $\gamma_j$ move along the cycles $a_j$.

Define the normalized holomorphic differentials

$$\omega_j = A_{jk} \sigma_k$$

such that

$$\frac{1}{2\pi} \int_{a_j} \omega_k = \delta_{j,k}$$

Then

$$\theta_j = \sum_{k=1}^{n-1} \int \gamma_k \omega_j$$

are real angles on the Jacobi variety, and the dynamics describes linear motion along this real torus. One can invert the Abel transformation expressing the symmetric functions of $\gamma_1, \cdots, \gamma_{n-1}$ (recall that they coincide with $b_1, \cdots, b_{n-1}$) as functions on the Jacobi variety (functions of $\theta$'s) using the Riemann theta-function but we shall not need explicit formulae. The angles $\theta$ and the times $\tau$ are related linearly:

$$\theta_j = \sum_{l=1}^{n-1} A_{jl} \tau_{n-l}$$

so, the using the theta-function formulae mentioned above one can resolve the equations of motion expressing $b_j$ as $b_j = b_j(\tau_1, \cdots, \tau_{n-1})$.

From the point of view of algebraic geometry the monodromy matrix $M(\lambda)$ gives an affine model of hyper-elliptic Jacobian, and the functions $b_j(\tau)$ are generalized Weierstrass functions [10]. In the case of genus one ($n = 2$) the function $\gamma(\tau_1) = b_1(\tau_1)$ is the usual Weierstrass function which satisfies the second order differential equation

$$\partial_1^2 \gamma = \frac{1}{2} \frac{d}{d\gamma} P(\gamma)$$

(5)

One of results of our further analysis will be in finding certain second order partial differential equations for generalized Weierstrass functions which can be thought about as generalizations of (4).

Let us consider the ring of generalized Weierstrass functions with coefficients in $t_1, \cdots, t_{n-1}$, i.e. the ring of polynomials

$$F(t_1, \cdots, t_{n-1}, b_1, \cdots, b_{n-1})$$

Consider further all possible derivatives of these polynomials with respect to $\tau_i$:

$$\partial_1^{k_1} \cdots \partial_{n-1}^{k_{n-1}} F(t_1, \cdots, t_{n-1}, b_1, \cdots, b_{n-1})$$

(6)

The equations we are looking for correspond to all possible linear combinations of the functions (5) which vanish due to equations of motion. To understand the origin of these equations we have to return to our mechanical considerations.

Mechanically one understands the derivatives $\partial_i$ as action of hamiltonian vector-fields. Using the Poisson brackets (3) one can express (5) as a function of $a_1, \cdots, a_n, b_1, \cdots, b_{n-1}, c_2, \cdots, c_{n+1}, d_2, \cdots, d_n$ i.e. as an element of the algebra $A_0$. We put forward the following

**Conjecture 1.** Every element of $A_0$ can be presented as linear combination of the expressions (5).
We were not able to find a complete proof of this statement, however, the consideration of examples supports it. Further indirect support of this conjecture will be provided by the calculation of characters given below.

Assuming that the conjecture is true one realizes that the way of presenting an element of \( A_0 \) as a linear combination of the expressions (7) may be not unique. Indeed, let us calculate the character of the space span by (8). We prescribe the degree \( i \) to \( \partial_i \) which is consistent with the Poisson brackets (9). Obviously, the character is

\[
\frac{1}{[n-1]! [m]! [n-1]!} \chi(q) = \chi(q)
\]

where \( \chi(q) \) is the character (8). So, there must be linear dependence between the functions (8) which is responsible for differential equations on the generalized Weierstrass functions. Moreover, there is a criterium which allows to judge whether the set of equations is complete. Indeed, to show the completeness of the equations one has, obviously, to prove that taking them into account leads to correct character (8).

Let us find the equations in question. To this end we shall use the Fourier transform. Consider a function \( F(t_1, \cdots t_n, b_1, \cdots, b_{n-1}) \). The variables \( t_j \) are the integrals of motion (and the moduli of the Riemann surface) and the variables \( b_j \) are the functions on the Jacobi variety due to the equations of motion. Hence

\[
F(t_1, \cdots t_n, b_1(\tau), \cdots, b_{n-1}(\tau)) = \sum_{k_1, \cdots, k_{n-1}} e^{-i\Sigma k_j \theta_j} \int_0^{2\pi} d\theta_1' \cdots \int_0^{2\pi} d\theta_{n-1}' F(t_1, \cdots t_n, b_1(\theta'), \cdots, b_{n-1}(\theta')) e^{i\Sigma k_j \theta_j}
\]

where \( \theta_j = \sum_{i} A_{ij} \tau_{n-i} \). Let us undo the Abel transformation inside the integrals:

\[
F(t_1, \cdots t_n, b_1(\tau), \cdots, b_{n-1}(\tau)) = \frac{1}{\det(A)} \sum_{k_1, \cdots, k_{n-1}} e^{-i\Sigma k_j \theta_j} \int_{\alpha_1}^{2\pi} \frac{d\gamma_1}{\sqrt{P(\gamma_1)}} \cdots \int_{\alpha_{n-1}}^{2\pi} \frac{d\gamma_1}{\sqrt{P(\gamma_{n-1})}} \prod_{i<j}(\gamma_i - \gamma_j) \tilde{F}(t_1, \cdots t_n, \gamma_1, \cdots, \gamma_{n-1}) \prod_j e^{ik_j \gamma_j}
\]

(7)

where \( \tilde{F}(t, \gamma) = F(t, b(\gamma)) \) (recall that \( b_j \) are elementary symmetric functions of \( \gamma \)'s), for any \( k = \{k_1, \cdots, k_{n-1}\} \) we define

\[
\Phi_k(\gamma) = \int k_j \omega_j
\]

Deriving (7) we needed to take into account the Jacobian of the Abel transformation. Later we shall relate the integrals in (7) to the quasi-classical limit of the matrix elements in quantum Toda chain. The equations of motion correspond to vanishing of all the integrals in (7). Let us explain the possible reasons for these integrals to vanish.

Consider first the term in (7) with \( k = 0 \) which is nothing but the average of \( F \) over the Jacobi variety (the trajectory):

\[
\langle F \rangle = \frac{1}{\det(A)} \int_{\alpha_1}^{2\pi} \frac{d\gamma_1}{\sqrt{P(\gamma_1)}} \cdots \int_{\alpha_{n-1}}^{2\pi} \frac{d\gamma_1}{\sqrt{P(\gamma_{n-1})}} \prod_{i<j}(\gamma_i - \gamma_j) \tilde{F}(t, \gamma)
\]

(8)

There are two reasons for this integral to vanish. First one is obvious, it is due to existence of exact forms. With arbitrary polynomial \( L(\gamma) \) associate the polynomial

\[
D_t(L)(\gamma) = P(\gamma) \frac{dL(\gamma)}{d\gamma} + \frac{1}{2} \frac{dP(\gamma)}{d\gamma} L(\gamma)
\]

We marked explicitly the dependence on the moduli (integrals of motion) \( t = \{t_1, \cdots, t_n\} \) which enters through \( P(\gamma) \).

There is an obvious

**Proposition 1.** The following integral vanishes

\[
\int_{c} \frac{1}{\sqrt{P(\gamma)}} D_t(L)(\gamma) = 0
\]

(9)
for any polynomial \( L \) and any closed cycle \( c \).

Hence the integral \( \tilde{F}(t, \gamma) \) vanishes if

\[
\tilde{F}(t, \gamma) = \sum_{i=1}^{n-1} \frac{1}{\prod_{j \neq i} (\gamma_i - \gamma_j)} D_t(L)(\gamma_i) G(\gamma_i, \gamma_{i+1}, \ldots, \gamma_{n-1})
\]

for any polynomial \( L \) and any symmetric polynomial of \( n-2 \) variables \( G \) (both of them can be also polynomials of parameters \( t \)). This property means in particular that by adding exact forms one can reduce the degree of the polynomial \( \tilde{F} \) in every \( \gamma_j \) up to \( n \).

The second reason for the integral \( \tilde{F} \) to vanish is due to Riemann bilinear identity. Consider the anti-symmetric polynomial of two variables

\[
C_t(\gamma_1, \gamma_2) = R_t(\gamma_1, \gamma_2) - R_t(\gamma_2, \gamma_1)
\]

where

\[
R_t(\gamma_1, \gamma_2) = \sqrt{P(\gamma_1)} \frac{d}{d\gamma_1} \left( \frac{1}{\gamma_1 - \gamma_2} \sqrt{P(\gamma_1)} \right)
\]

For any two cycles on the Riemann surface one has

\[
\int_c \int_c C_t(\gamma_1, \gamma_2) = c_1 \circ c_2
\]

where \( \circ \) means the intersection number. Since the cycles \( a_j \) do not intersect one has the following Proposition 2.

For any two \( a \)-cycles \( a_j \) and \( a_k \) the following integral vanishes:

\[
\int \int \frac{1}{\sqrt{P(\gamma_1)}} \frac{1}{\sqrt{P(\gamma_1)}} C_t(\gamma_1, \gamma_2) = 0
\]

Hence the integral \( \tilde{F} \) vanishes if

\[
\tilde{F}(t, \gamma) = \sum_{i<j} \frac{1}{(\gamma_i - \gamma_j) \prod_{l \neq i,j} (\gamma_i - \gamma_l)(\gamma_j - \gamma_l)} C_t(\gamma_i, \gamma_j) G(\gamma_1, \gamma_{i+1}, \gamma_{j+1}, \gamma_{n-1})
\]

Let us consider now the case of arbitrary \( k = \{k_1, \ldots, k_{n-1}\} \). Introduce the polynomials \( S_{t,k} \):

\[
i \sum_{j=i}^{n-1} k_j \omega_j(\gamma) = \frac{S_{t,k}(\gamma)}{\sqrt{P(\gamma)}} d\gamma
\]

Integrating by parts one gets the following three simple propositions

Proposition 1’. For any polynomial \( L \) define the polynomial

\[
D_{t,k}(L)(\gamma) = D_t(L)(\gamma) - S_{t,k}(\gamma) \int_0^{\gamma} L(\gamma') S_{t,k}(\gamma') d\gamma'
\]

then the following integral vanishes for any \( a \)-cycle:

\[
\int_{a_j} \frac{d\gamma}{\sqrt{P(\gamma)}} e^{i \Phi_k(\gamma)} D_{t,k}(L)(\gamma) = 0
\]

Proposition 2’. Define the polynomial

\[
C_{t,k}(\gamma_1, \gamma_2) = C_t(\gamma_1, \gamma_2) - S_k(\gamma_1) \int_0^{\gamma_2} \frac{S_{t,k}(\gamma) - S_{t,k}(\gamma_2)}{\gamma - \gamma_2} d\gamma + S_k(\gamma_2) \int_0^{\gamma_1} \frac{S_{t,k}(\gamma) - S_{t,k}(\gamma_1)}{\gamma - \gamma_1} d\gamma
\]
then for any two $a$-cycles $a_j$ and $a_k$ the following integral vanishes:

\[ \int_{a_j} \frac{d\gamma_1}{\sqrt{P(\gamma_1)}} \int_{a_k} \frac{d\gamma_2}{\sqrt{P(\gamma_2)}} e^{i\Phi_k(\gamma_2)} e^{i\Phi_k(\gamma_1)} C_{t,k}(\gamma_1, \gamma_2) = 0 \]

There is one more identity which is trivial in the case $k = 0$:

**Proposition 3'.** For any $a$-cycle the following integral vanishes:

\[ \int_{a_j} \frac{d\gamma}{\sqrt{P(\gamma)}} e^{i\Phi_k(\gamma_2)} S_{t,k}(\gamma) = 0 \]

From these three Propositions one finds the following partial differential equations on the symmetric functions of $\gamma$:

1. For any polynomial one variable $L$ and any symmetric polynomial of $n - 2$ variables $G$ the equation holds:

\[
\sum_{i=1}^{n-1} \frac{1}{\prod_{j \neq i} (\gamma_i - \gamma_j)} D_t(L)(\gamma_i) G(\gamma_1, \ldots, \hat{\gamma}_i, \ldots, \gamma_{n-1}) - \\
- \sum_{l,m=1}^{n-1} \partial_l \partial_m \left[ \sum_{i<j} \frac{1}{\prod_{j \neq i} (\gamma_i - \gamma_j)} \gamma_i^{n-1-l} \int_0^\gamma L(\gamma) \gamma^{n-2-l} d\gamma G(\gamma_1, \ldots, \hat{\gamma}_i, \ldots, \gamma_{n-1}) \right] = 0 \quad (11)
\]

2. For any symmetric polynomial of $n - 3$ variables $G$ the equation holds:

\[
C(G) \equiv \sum_{i<j} \frac{1}{\prod_{l \neq i,j} (\gamma_i - \gamma_l)(\gamma_j - \gamma_l)} C_t(\gamma_i, \gamma_j) G(\gamma_1, \ldots, \hat{\gamma}_i, \hat{\gamma}_j, \ldots, \gamma_{n-1}) - \\
- \sum_{l,m=1}^{n-1} \partial_l \partial_m \left[ \sum_{i<j} \frac{1}{\prod_{j \neq i} (\gamma_i - \gamma_j)} \gamma_i^{n-1-l} \int_0^\gamma L(\gamma) \gamma^{n-2-l} d\gamma G(\gamma_1, \ldots, \hat{\gamma}_i, \hat{\gamma}_j, \ldots, \gamma_{n-1}) \right] \times \\
\left( \gamma_i^{n-1-l} \int_0^\gamma \gamma^{n-2-l} d\gamma - \gamma_i^{n-2-l} \int_0^\gamma \gamma^{n-3-l} d\gamma \right) = 0 \quad (12)
\]

3. For any symmetric polynomial of $n - 2$ variables $G$ the equation holds:

\[
Q(G) \equiv \sum_{l=1}^{n-1} \partial_l \left[ \frac{1}{\prod_{j \neq i} (\gamma_i - \gamma_j)} \gamma_i^{n-2-l} G(\gamma_1, \ldots, \hat{\gamma}_i, \ldots, \gamma_{n-1}) \right] = 0 \quad (13)
\]

One must add to the equations (11), (12), (13) their trivial consequences i.e. the equations obtained from them by applying arbitrary number of $\partial_l$. We claim that in this way all the equations of motion can be described.

Let us illustrate this point considering the simple example $n = 2$. In that case genus is equal to 1 and we have only one variable $\gamma$. The equations (12), (13) are trivial, so, we are left with (11). Let us take $L(\gamma) = \gamma^p$, $p = 0, 1, \ldots$. Then (11) turns into

\[ p\gamma^{p-1} P(\gamma) + \frac{1}{2} \gamma^p \frac{d}{d\gamma} P(\gamma) = \partial_t^2 \left( \frac{1}{p+1} \gamma^p \right), \quad (14) \]

we have only one time $\tau_1$ in that case. As it has been said earlier the function $\gamma$ is the Weierstrass $P$-function. The equation (14) coincides with the usual equation (5) when $p = 0$. For other $p$ we get the equations on degrees of $\gamma$ which can be verified for the Weierstrass function. Recall that we were considering the space of functions of the kind

\[ \partial_t^2 F(t_1, t_2, \gamma) \quad (15) \]
Let us calculate the character of this space modulo the equations (14) and their trivial consequences (those obtained by application of \( \partial_1 \)). Obviously, using (14) we can express any function of the kind (15) as linear combination of

\[
F_0(t_1, t_2), ~ F_1(t_1, t_2) \partial_1^n, ~ F_2(t_1, t_2) \partial_1^n \gamma^2
\]

So, the character is

\[
\frac{1}{[2]!} \left( 1 + \frac{q}{[1]} + \frac{q^2}{[1]} \right) = \frac{1}{[2]!}(1 + q^2)
\]

which coincides with (3).

Let us consider the general case. We have the space of symmetric functions of \( \gamma_1, \cdots, \gamma_{n-1} \) with coefficients in \( t_1, \cdots, t_n \) on which the derivatives \( \partial_1, \cdots, \partial_{n-1} \) act. We can consider the derivatives as coefficients identifying this space with \( H_{n-1} \) which is the space of symmetric polynomials of \( \gamma_1, \cdots, \gamma_{n-1} \) with coefficients in \( t_1, \cdots, t_n \) and \( \partial_1, \cdots, \partial_{n-1} \) (recall that \( \partial_i \) and \( t_j \) commute). Substracting the exact forms (11) we finish with the space \( \hat{H}_{n-1} \) which is the subspace of \( H_{n-1} \) defined by the condition that the degree of the polynomials in every \( \gamma_j \) does not exceed \( n \). We define the spaces \( \hat{H}_j \) \((j \leq n - 1)\) of symmetric polynomials of \( j \) variables \( \gamma_i \) whose degree in every variable does not exceed \( 2n - 1 - j \) with coefficients in \( t_1, \cdots, t_n \) and \( \partial_1, \cdots, \partial_{n-1} \). The action of the operators \( \mathcal{C} \) and \( \mathcal{Q} \) defined by (12) (13) can be obviously extended to the action from \( H_{n-1} \) to \( \hat{H}_n \) and from \( H_{n-2} \) to \( \hat{H}_{n-1} \) respectively. It is also clear that the images of respectively \( \hat{H}_{n-3} \) and \( \hat{H}_{n-2} \) belong to \( \hat{H}_{n-1} \). One can easily generalize the definitions of \( \mathcal{C} \) and \( \mathcal{Q} \) allowing them to act respectively from \( \hat{H}_{j-2} \) to \( \hat{H}_j \) and from \( \hat{H}_{j-1} \) to \( \hat{H}_j \). For these operators one has

\[
[\mathcal{C}, \mathcal{Q}] = 0, \quad \mathcal{Q}^2 = 0, \quad \text{Ker}[\hat{H}_{j-2} \rightarrow \hat{H}_j](\mathcal{C}) = 0
\]

Now noticing that \( \deg(\mathcal{C}) = 2 \) and \( \deg(\mathcal{Q}) = 1 \) one evaluates the character:

\[
\frac{1}{[n-1]!}[n!] \left\{ \left( \begin{array}{c} 2n-1 \\ n-1 \end{array} \right) - q \left( \begin{array}{c} 2n-1 \\ n-2 \end{array} \right) + q^2 \left( \begin{array}{c} 2n-1 \\ n-3 \end{array} \right) - \cdots + (-q)^{n-1} \left( \begin{array}{c} 2n-1 \\ 0 \end{array} \right) \right\} - \\
- q^2 \left( \begin{array}{c} 2n-1 \\ n-3 \end{array} \right) - q \left( \begin{array}{c} 2n-1 \\ n-4 \end{array} \right) + q^2 \left( \begin{array}{c} 2n-1 \\ n-5 \end{array} \right) - \cdots + (-q)^{n-3} \left( \begin{array}{c} 2n-1 \\ 0 \end{array} \right) \right\} = \\
= \frac{1}{[n!] [n-1]!} \left( \begin{array}{c} 2n-1 \\ n-1 \end{array} \right) - q \left( \begin{array}{c} 2n-1 \\ n-2 \end{array} \right)
\]

which coincides with the character (3). There is obvious similarity of what we have done with the paper [2].

3 Quantum Toda chain

Following Sklyanin [3] we shall use the same notations for the quantum analogues of classical objects that have been used in the classical case. The Hamiltonian of the quantum periodical Toda chain is given by (3) with \( p_j, q_j \) being the canonical operators

\[
[p_i, q_j] = i\hbar \delta_{i,j}
\]

Consider the same definition of L-operator and monodromy matrix as in classics. The monodromy matrix satisfies the relations

\[
R(\lambda - \mu)(M(\lambda) \otimes M(\mu)) = (M(\mu) \otimes I)(M(\lambda) \otimes I)R(\lambda - \mu)
\]

with \( R(\lambda) \) being the quantum R-matrix:

\[
R(\lambda) = \lambda - i\hbar P
\]

The trace of the monodromy matrix provides \( n \) commutative integrals of motion. The center of the algebra is created by the quantum determinant

\[
A(\lambda)D(\lambda + i\hbar) - B(\lambda)C(\lambda + i\hbar) = 1
\]

The idea of using the separated variables in quantum case goes back to [1]. It was developed as universal method by Sklyanin. Let us briefly review the method of separation of variables following [3]. From the relations (17) one finds, in particular, that

\[
[B(\lambda), B(\mu)] = 0
\]
So, presenting the operator $B(\lambda)$ in the form

$$B(\lambda) = b \prod_{j=1}^{n-1} (\lambda - \gamma_j)$$

one gets a family of commuting operators:

$$[b, \gamma_j] = 0, \quad [\gamma_i, \gamma_j] = 0$$

As it was in the classical case the operators $b$ and $a_1 = P$ commute with everything except between themselves:

$$[a_1, b] = i\hbar b$$

The idea of the method of separation of variables is in considering the model in $b, \gamma$-representation. The canonically conjugated operator to $b$ exists already: this is $a_1$. The canonically conjugated operators for $\gamma_j$ are constructed as follows. Consider the operators

$$A(\lambda) = \lambda^n + \lambda^{n-1} a_1 + \cdots + a_n$$
$$D(\lambda) = \lambda^{n-2} d_2 + \cdots + d_n$$

then it can be shown that the operators

$$\tilde{\Lambda}_j = \gamma_j^n + \gamma_j^{n-1} a_1 + \cdots + a_n$$
$$\Lambda_j = \gamma_j^{n-2} d_2 + \cdots + d_n$$

satisfy

$$\tilde{\Lambda}_j \Lambda_j = 1, \quad [\Lambda_j, \gamma_g] = i\hbar \delta_{j,k} \Lambda_j$$

The order of multipliers in (19) is important.

It is possible to reconstruct the operators $A(\lambda), B(\lambda), C(\lambda), D(\lambda)$ using $a_1, b, \gamma_j, \Lambda_j$. In particular,

$$T(\lambda) = \sum_{k=1}^{n-1} \prod_{j \neq k} \left( \frac{\lambda - \gamma_j}{\gamma_k - \gamma_j} \right) (\Lambda_k + \Lambda_k^{-1}) + \prod_{j \neq 1}^{n-1} (\lambda - \gamma_j) (\lambda + a_1 + \sum \gamma_j)$$

(21)

The Hilbert space splits into a direct sum of orthogonal subspaces $H_p$ corresponding to different eigenvalues $p$ of the zero-mode $a_1$. Let us consider the space $H_0$, the solution in other subspaces $H_p$ are obtained from the one for $H_0$ by simple transformation. In $H_0$ the eigenfunctions of $T(\lambda)$ with the eigenvalue $t(\lambda)$ in $\gamma$-representation can be looked for in the form

$$\langle t | \gamma \rangle = \prod_{j=1}^{n-1} Q(\gamma_j)$$

Applying (21) one finds with the following equation for $Q(\gamma)$:

$$t(\gamma)Q(\gamma) = Q(\gamma + i\hbar) + Q(\gamma - i\hbar)$$

(22)

where $t(\lambda)$ is the eigenvalue of $T(\lambda)$ on $| t \rangle$. In the subspace $H_0$ this eigenvalue must be a polynomial of the kind

$$t(\lambda) = \lambda^n + O(\lambda^{n-2})$$

The equation (22) is one equation for two unknowns: $t$ and $Q$, so, at the first glance it is rather useless. However, assuming certain analytical properties of $Q(\gamma)$ this single equation actually defines the spectrum. Namely, require that the function $Q(\gamma)$ is an entire function of $\gamma$ with infinitely many real zeros. Moreover, impose the following asymptotic:

$$Q(\lambda) \sim \cos \left( \frac{\lambda n}{\hbar} \log \left( \frac{\lambda}{e} \right) + \frac{\pi}{4} \right), \quad \lambda \sim \infty,$$
$$Q(\lambda) \sim e^{\pi \lambda n} \cos \left( \frac{\lambda n}{\hbar} \log \left( -\frac{\lambda}{e} \right) + \frac{\pi}{4} \right), \quad \lambda \sim -\infty$$

(23)
The normalization of $Q$ is not the same as in [5]. For $n = 4k$ the function $Q$ differs from the function $\varphi$ from [5] by the multiplier $\exp(\pi \lambda n/2h)$ which is in this case a quasi-constant: does not disturb the equations (22). If $n \neq 4k$ there is essential discrepancy with [5], however, we claim that these are (23) which describe correct asymptotical behaviour. Our normalization basically coincides with the one accepted in [12]. According to [12] it provides the only way to have correct quasi-classical limit. We are going to discuss this limit in the next section.

The main conjecture of the paper [5] is that the spectrum of the model is defined by all the solutions to the equations [22] with the analytical properties of $t$ and $Q$ described above and the asymptotic behaviour of $Q$ given by the formulae (23). This conjecture was proven by Gaudin and Pasquier [12].

Now we want to discuss the properties of the matrix elements of the operators. To consider the matrix elements we need to know the scalar product in the space of functions of $\gamma_j$. This scalar product was found by Sklyanin [5], let us repeat the essential steps because again there will be a minor difference if $n \neq 4k$. Consider an operator $O$ which is given by a symmetric function $F(\gamma_1, \ldots, \gamma_{n-1})$. The wave-functions are real, so, the matrix element is given by

$$\langle t | O | t' \rangle = \int_{-\infty}^{\infty} d\gamma_1 \cdots \int_{-\infty}^{\infty} d\gamma_{n-1} \prod_{j=1}^{n-1} Q(\gamma_j)Q'(\gamma_j)F(\gamma_1, \ldots, \gamma_{n-1})w(\gamma_1, \ldots, \gamma_{n-1})$$

where $w$ is certain weight. Requiring that the operator $T(\lambda)$ for real $\lambda$ is self-adjoint and using the formula (24) one concludes that

$$w(\gamma_1, \ldots, \gamma_{n-1}) = \prod_{i<j}(\gamma_i - \gamma_j)\bar{w}(\gamma_1, \ldots, \gamma_{n-1})$$

where $\bar{w}(\gamma_1, \ldots, \gamma_{n-1})$ is $\bar{h}$-periodic entire function of its arguments. There are two formal reasons for choosing particular $\bar{w}$. First, everything under the integral is symmetric with respect to $\gamma_1, \ldots, \gamma_{n-1}$ except for the multiplier $\prod_{i<j}(\gamma_i - \gamma_j)$ in $w$, so it does not make sense to put in $\bar{w}$ anything that can be killed by anti-symmetrization. Second, requiring convergence of the integrals one finds from the asymptotic (23) that $\bar{w}$ can contain $\exp(-2\pi m/h)$ with $1 \leq m \leq n-1$. These two requirements fix $\bar{w}$ up to anti-symmetrization:

$$\bar{w}(\gamma_1, \ldots, \gamma_{n-1}) = \prod_{j=1}^{n-1} e^{\frac{\pi h}{4} \gamma_j(j-n)}$$

Again, if $n = 4k$ this essentially coincides with [5], otherwise there is a minor discrepancy.

Informal reason for taking $\bar{w}$ in the form (24) refers to the quasi-classics which will be discussed in the next section.

Similarly to the classical case the algebra of observables $A_0$ is defined as the algebra generated by all the coefficients of $A(\lambda), \bar{B}(\lambda) = b^{-1}B(\lambda), \bar{C}(\lambda) = b C(\lambda), D(\lambda)$. The commutation relations (17) and the quantum determinant (18) provide sufficiently many relations to show that the quantum algebra of observables has the same size as the classical one. To make this statement mathematically rigorous one says that their characters coincide.

Now we formulate an analogue of the Conjecture 1 of the previous section.

**Conjecture 2.** Every quantum observable $O$ can be presented in the form

$$O = G_L(t_1, \ldots, t_n)F(b_1, \ldots, b_{n-1})G_R(t_1, \ldots, t_n)$$

where $G_L$, $F$, $G_R$ are polynomials.

This conjecture looks more natural than its classical counterpart and explains the mystery of the latter. The point is that there is no closed formula for the commutation relations of $T(\lambda)$ and $\bar{B}(\mu)$ which would allow ordering of the operators $t_j$ and $b_j$.

Taking the matrix element between two eigenstates of the Hamiltonians one can essentially neglect the polynomials $G_L$ and $G_R$ because acting on the eigenstates they produce the eigenvalues. Hence we are mostly interested in the matrix elements of the operators $O_0 = F(b_1, \ldots, b_{n-1})$ which are given by the integrals of the kind

$$\langle t | O_0 | t' \rangle = \int_{-\infty}^{\infty} d\gamma_1 \cdots \int_{-\infty}^{\infty} d\gamma_{n-1} \prod_{j=1}^{n-1} Q(\gamma_j)Q'(\gamma_j)\prod_{i<j}(\gamma_i - \gamma_j)\prod_{j=1}^{n-1} F(\gamma_1, \ldots, \gamma_{n-1})\prod_{j=1}^{n-1} e^{\frac{\pi h}{4} \gamma_j(j-n)}$$

(25)
where
\[
\tilde{F}(\gamma_1, \cdots, \gamma_{n-1}) = F(b_1(\gamma), \cdots, b_{n-1}(\gamma))
\] (26)

The function
\[
\prod_{i<j}(\gamma_i - \gamma_j) \prod_{j=1}^{n-1} \tilde{f}(\gamma_1, \cdots, \gamma_{n-1})
\]
is an anti-symmetric polynomial of \(\gamma_1, \cdots, \gamma_{n-1}\). Every anti-symmetric polynomial can be presented as a linear combination of Schur functions, so, without loss of generality we can replace (26) by
\[
\det(F_i(\gamma_j))
\]
for some polynomials of one variable \(F_1, \cdots, F_{n-1}\). Then the matrix element can be presented as a determinant of \((n-1) \times (n-1)\)-matrix composed of one-fold integrals:
\[
\det\left(\int_{-\infty}^{\infty} Q(\gamma)Q'(\gamma)F_i(\gamma)e^{\frac{2\pi i}{n} \gamma(j-n)} d\gamma\right)
\] (27)

We consider the integral
\[
\int_{-\infty}^{\infty} Q(\gamma)Q'(\gamma)F(\gamma)e^{\frac{2\pi i}{n} - \frac{2\pi i}{n} k} d\gamma
\] (28)
as a deformation of hyper-elliptic integral. The study of the properties of these integrals is very important for understanding of matrix elements of operators in the model.

Since the quantum algebra of observables has the same character as the classical one the equations (11), (12), (13) must have their quantum counterparts. Hence there must be identities for the integrals (28) from which these quantum counterparts follow. These identities can indeed be found, we are going to describe them in the same order as it has been done in classical case.

Let us introduce the operation \(\Delta\) which attaches to every function \(F(\gamma)\) the function
\[
\Delta(F)(\gamma) = F(\gamma + i\hbar) - F(\gamma - i\hbar)
\]
The operation \(\Delta^{-1}\) is not always defined, but on polynomials it is. For any polynomial \(L\) one can define a polynomial \(F\) such that \(F(\gamma) = \Delta^{-1}(L)(\gamma)\). For uniqueness we require also that \(\Delta^{-1}(L)(0) = 0\). Consider the integral (28) with \(Q(\gamma), Q'(\gamma)\) satisfying the equations (22) with the eigenvalues \(t(\gamma)\) and \(t'(\gamma)\). For any given polynomial \(L(\lambda)\) construct the polynomial
\[
D(L)_{t,v}(\gamma) = t(\gamma)\Delta^{-1}(Lt)(\gamma) + t'(\gamma)\Delta^{-1}(Lt')(\gamma) - t(\gamma)\Delta^{-1}(Lt')(\gamma - i\hbar) - t'(\gamma)\Delta^{-1}(Lt)(\gamma - i\hbar) - L(\gamma)t(\gamma)t'(\gamma) + L(\gamma + i\hbar) - L(\gamma - i\hbar)
\] (29)

Then we have the following analogue of the Propositions 1 and 1’:

**Proposition 1**. For any \(1 \leq k \leq n - 1\) the following integral vanishes:
\[
\int_{-\infty}^{\infty} Q(\gamma)Q'(\gamma)D(L)_{t,v}(\gamma)e^{-\frac{2\pi i}{n} \gamma k} d\gamma = 0
\] (30)

Using these q-exact forms (24) we can always reduce the degree of the polynomial \(F(\gamma)\) under the integral (28) in order that it does not exceed \(2n - 2\). So, we are left with finite number of different integrals (28) with \(F(\gamma) = \gamma^j, j = 0, 1, \cdots, 2n - 2\). These integrals are subject to further relations.

In order to define a quantum analogue of the polynomial \(C_t(\gamma_1, \gamma_2)\) (14) we shall need some preparations. Consider the function
\[
U(\gamma, \delta) = \frac{t(\gamma) - t(\delta)}{\gamma - \delta}
\]
The notation $\Delta^{-1}(U(\cdot, \delta))(\gamma)$ means that $\Delta^{-1}$ is applied to the first argument, i.e.

$$\Delta(\Delta^{-1}(U(\cdot, \delta))(\gamma)) = U(\gamma, \delta)$$

The function $U'$ is defined in the same way replacing $t$ by $t'$.

Consider the anti-symmetric polynomial of two variables

$$C_{t,t'}(\gamma_1, \gamma_2) = R_{t,t'}(\gamma_1, \gamma_2) - R_{t,t'}(\gamma_2, \gamma_1)$$

where $R_{t,t'}(\gamma_1, \gamma_2)$ is defined as follows:

$$R_{t,t'}(\gamma_1, \gamma_2) = t(\gamma_1)\Delta^{-1}(U(\cdot, \gamma_2))(\gamma_1) + t'(\gamma_1)\Delta^{-1}(U'(\cdot, \gamma_2))(\gamma_1) - t(\gamma_1)\Delta^{-1}(U'(\cdot, \gamma_2))(\gamma_1 - i\hbar) - \frac{1}{2}\frac{1}{\gamma_1 - \gamma_2}(t(\gamma_1) - t(\gamma_2))$$

We have the

**Proposition 2**. For any $1 < k, l < n - 1$ the following integral vanishes:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(\gamma_1)Q'(\gamma_1)Q(\gamma_2)Q'(\gamma_2)C_{t,t'}(\gamma_1, \gamma_2)e^{-\frac{2\pi}{\hbar}\gamma_1k}e^{-\frac{2\pi}{\hbar}\gamma_2l}d\gamma_1d\gamma_2 = 0$$  \hspace{1cm} (31)

Finally, let us define

$$S_{t,t'}(\gamma) = t(\gamma) - t'(\gamma)$$

then we have

**Proposition 3**. For any $1 < k < n - 1$ the following integral vanishes:

$$\int_{-\infty}^{\infty} Q(\gamma)Q'(\gamma)S_{t,t'}(\gamma)e^{-\frac{2\pi}{\hbar}\gamma k}d\gamma = 0$$  \hspace{1cm} (32)

This is in fact the simplest relation, we consider it last for historical reasons. One obvious consequence of this relation is orthogonality of the wave-function.

Let us say a few words about proof of all these relations. Consider the simplest one \((32)\) we have

$$Q(\gamma)Q'(\gamma)S_{t,t'}(\gamma) = Q(\gamma)Q'(\gamma)(t(\gamma) - t'(\gamma)) = Q(\gamma + i\hbar)Q'(\gamma) + Q(\gamma - i\hbar)Q'(\gamma) - Q(\gamma)Q'(\gamma + i\hbar) - Q(\gamma)Q'(\gamma - i\hbar)$$

So,

$$\int_{-\infty}^{\infty} Q(\gamma)Q'(\gamma)S_{t,t'}(\gamma)e^{-\frac{2\pi}{\hbar}\gamma k}d\gamma =$$

$$= \int_{-\infty}^{\infty} Q(\gamma + i\hbar)Q'(\gamma)e^{-\frac{2\pi}{\hbar}\gamma k}d\gamma + \int_{-\infty}^{\infty} Q(\gamma - i\hbar)Q'(\gamma)e^{-\frac{2\pi}{\hbar}\gamma k}d\gamma -$$

$$- \int_{-\infty}^{\infty} Q(\gamma)Q'(\gamma + i\hbar)e^{-\frac{2\pi}{\hbar}\gamma k}d\gamma - \int_{-\infty}^{\infty} Q(\gamma)Q'(\gamma - i\hbar)e^{-\frac{2\pi}{\hbar}\gamma k}d\gamma$$

by shift of contour the first integral cancels the forth one and the second cancels the third one. The shift of contour is possible because the function $Q$ behaves asymptotically on the line $\text{Im}(\gamma) = \text{const}$ essentially in the same way as it behaves at the real axis. The relations \((30)\) and \((31)\) are proven similarly, but require more sophisticated calculations.

Using the equations \((30), (31), (32)\) we can write down equations similar to \((11), (12), (13)\) which would guarantee that the number of operators (character of the algebra of observables) is correct. The corresponding calculation does not differ from the one presented earlier for the classical case.
Quasi-classical case

We had a number of identities described by Propositions 1, 1', 1''; 2, 2', 2''; 3', 3". They look as reflection of the same structure. The goal of this section is to explain the relation between different levels of deformation.

Consider the quasi-classical quantization of the Toda lattice. We had the following formulae for the symplectic form and corresponding 1-form in the coordinates $t$ and $\gamma$:

$$\alpha = \sum_{j=1}^{n-1} \log \Lambda(\gamma_j) d\gamma_j$$

$$\omega = d\alpha = \sum_{j=1}^{n} \sum_{k=2}^{n} \frac{\gamma^{n-k}}{P(\gamma_j)} dt_k \wedge d\gamma_j,$$

it this section we shall ignore the contribution from the center of mass variables $a_1$ and $b$ which is easy to find if needed. Using these formulae one immediately writes a quasi-classical proposal for the wave-function of the states with eigenvalues $I$:

$$\Psi_t(\gamma) = \mu^\frac{i}{\hbar} \exp \left( \frac{1}{\hbar} \int_\gamma^\gamma \alpha \right) = \prod_{i<j} (\gamma_i - \gamma_j) \prod_{i=1}^{n-1} \left( \frac{1}{P(\gamma_j)} \right)^{\frac{i}{\hbar}} \exp \left( \frac{1}{\hbar} \int_\gamma^\gamma \log \Lambda(\gamma) d\gamma \right)$$

where $\mu$ is defined as follows

$$\wedge^{n-1}(\omega) = \mu \, dt_1 \wedge \cdots \wedge dt_n$$

The relation of the formula (34) to the exact quantum formulae that we had before is clear.

1. The multiplier $\prod_{i<j} (\gamma_i - \gamma_j)\frac{i}{\hbar}$ is a piece of the weight of integration $\omega$ (23). Another one will come from the second wave-function in the matrix elements. The wave function without this multiplier will be denoted by $\tilde{\Psi}_t(\gamma)$.

2. The function $\tilde{\Psi}_t(\gamma)$ splits into product of the expressions

$$\left( \frac{1}{P(\gamma_j)} \right)^{\frac{i}{\hbar}} \exp \left( \frac{1}{\hbar} \int_\gamma^\gamma \log \Lambda(\gamma) d\gamma \right)$$

which has to be related to the quasi-classical limit of function $Q$. One should be careful here because the function $\log \Lambda(\gamma)$ is multi-valued, so, the branches must be defined. Moreover, we could probably need linear combination of the wave-functions corresponding to different branches.

Recall that

$$\Lambda(\gamma) = \frac{1}{2} \left( T(\gamma) + \sqrt{T^2(\gamma) - 4} \right)$$

where $T(\lambda)$ is a polynomial of degree $n$. The polynomial $P(\gamma) = T^2(\gamma) - 4$ has $2n$ real simple roots $\lambda_1 < \cdots < \lambda_{2n}$. We define the function $\Lambda(\gamma)$ on the plane with the cuts along the intervals; $I_0 = (-\infty, \lambda_1]$, $I_1 = [\lambda_2, \lambda_3], \ldots, I_n = [\lambda_{2n}, \infty)$. Require that $\log \Lambda(\gamma \pm i0)$ are real for $\gamma > \lambda_{2n}$ then, obviously,

$$\log \Lambda(\gamma + i0) + \log \Lambda(\gamma - i0) = 0, \quad \gamma \in I_n$$

Continuing analytically $\log \Lambda(\gamma)$ into the plane with the cuts one finds that

$$\log \Lambda(\gamma) = \log \Lambda(\gamma), \quad \log \Lambda(\gamma + i0) + \log \Lambda(\gamma - i0) = 2\pi i(n - j), \quad \gamma \in I_j$$

The variable $\gamma_j$ belongs classically to $I_j$. So, we need the wave function real on entire real axis and not containing the factors $\exp(-\frac{1}{\hbar} s)$ with real $s$ when all the variables are in the classically permited places. This requires, first, taking different branches of $\log \Lambda$ for different $\gamma_j$ and, second, taking a sum of two wave functions with $\gamma_j + i0$ and $\gamma_j - i0$ for every $\gamma_j$. The result is

$$\tilde{\Psi}_t(\gamma) = \prod e^{\frac{\pi}{\hbar} (j-n)\gamma_j} Q_{qc}(\gamma_j)$$
where
\[ Q_{qc}(\gamma) = V(\gamma + i0) + V(\gamma - i0), \quad V(\gamma) = \left( -\frac{1}{P(\gamma)} \right)^+ \exp \left( \frac{1}{\hbar} \int_0^\gamma \log \Lambda(\gamma)d\gamma \right) \]
and the branch of \( \log \Lambda \) is defined above, \( \left( -\frac{1}{P(\gamma)} \right)^+ \) is real positive for \( \lambda_{2j-1} < \gamma < \lambda_{2j} \).

To ensure that \( Q_{qc}(\gamma) \) is a single-valued function in the plane with the cuts the Bohr-Sommerfeld quantization condition must hold
\[ J_j = \int_{a_j} \log \Lambda(\gamma)d\gamma = \pi \hbar(2n_j + 1) \tag{35} \]
the integrals \( J_j \) are the classical actions. The cuts of \( Q_{qc} \) come from condensation of zeros of the quantum \( Q \) in the quasi-classical limit.

Let us consider now the quasi-classical limit \( \hbar \to 0 \) of the matrix elements. This limit makes sense literally if two condition are satisfied:
1. The quantum numbers are large. In our case it means that the zones \([\lambda_{2j}, \lambda_{2j+1}]\) do not collapse.
2. The states \(|t\rangle \) and \(|t'\rangle \) are close, i.e. the eigenvalues of Hamiltonians are close: \( t'_j - t_j = O(\hbar) \).

Consider the matrix element \([23]\) for such close states. It consists of the integrals
\[ \int_{-\infty}^{\infty} Q(\gamma)Q'(\gamma)F_i(\gamma)e^{\frac{i}{h}(2-n)\gamma}d\gamma \]
From the quasi-classical estimation of \( Q, Q' \) one concludes that
\[ \int_{-\infty}^{\infty} Q(\gamma)Q'(\gamma)F_i(\gamma)e^{\frac{i}{h}(2-n)\gamma}d\gamma \overset{\hbar \to 0}{\to} \]
\[ 4 \int \frac{1}{\sqrt{P(\gamma)}} \cos \left( \frac{1}{h} \int_0^\gamma \text{Re}(\log \Lambda(\gamma' + i0))d\gamma' + \frac{\pi}{4} \right) \cos \left( \frac{1}{h} \int_0^\gamma \text{Re}(\log \Lambda'(\gamma' + i0))d\gamma' + \frac{\pi}{4} \right) F_i(\gamma)d\gamma \]
We have
\[ 2 \cos \left( \frac{1}{h} \int_0^\gamma \text{Re}(\log \Lambda(\gamma' + i0))d\gamma' + \frac{\pi}{4} \right) \cos \left( \frac{1}{h} \int_0^\gamma \text{Re}(\log \Lambda'(\gamma' + i0))d\gamma' + \frac{\pi}{4} \right) = \]
\[ = \cos \left( \frac{1}{h} \int_0^\gamma \text{Re}(\log \Lambda'(\gamma' + i0) - \log \Lambda(\gamma' + i0))d\gamma' \right) - \]
\[ - \sin \left( \frac{1}{h} \int_0^\gamma \text{Re}(\log \Lambda'(\gamma' + i0) + \log \Lambda(\gamma' + i0))d\gamma' \right) \]
The second term in the RHS can be thrown away because it is rapidly oscillating in the classical limit. The variation \( \log \Lambda(\gamma) - \log \Lambda'(\gamma) \) is estimated as follows. Consider the classical solution with \( T_{cl}(\lambda) = t(\lambda) \) where \( t(\lambda) \) is one of the eigenvalues. This is the place where using the same notations for classical and quantum observables can be misleading, so, we mark explicitly the classical ones. The eigenvalue \( t'(\lambda) = T_{cl}(\lambda) + \delta T(\lambda) \). One has (in further calculations we neglect contributions of order \( o(\hbar) \)):
\[ \delta \log \Lambda(\gamma) \equiv \log \Lambda'(\gamma) - \log \Lambda(\gamma) = \frac{\delta T(\gamma)}{\sqrt{P(\gamma)}} \]
How to find \( \delta T(\gamma) \)? The quasi-classical states are subject to the Bohr-Sommerfeld quantization conditions \([23]\). The quantum numbers \( n_j \) are quasi-classically large: \( n_j = O(\hbar^{-1}) \), but their differences for the close states are finite \( k_j \equiv n'_j - n_j = O(1) \). Hence
\[ \delta \int_{a_j} \log \Lambda(\gamma)d\gamma = \int_{a_j} \frac{\delta T(\gamma)}{\sqrt{P(\gamma)}}d\gamma = \sum_{l=1}^{n-1} \delta t_{n-l+1} \int_{a_j} \frac{\zeta_{l-1}}{\sqrt{P(\gamma)}}d\gamma = \hbar k_j \tag{36} \]
The matrix
\[ A_{ij}^{-1} = \int_{a_j}^{b_j} \frac{\gamma^{j-1}}{\sqrt{P(\gamma)}} d\gamma \]
is the inverse for the matrix used in definition of the normalized Abelian differentials \( \omega_j \). Hence
\[ \delta t_{n-t+1} = \hbar k_j A_{jt} \]
which means that
\[ \delta \log \Lambda(\gamma) d\gamma = \hbar \sum k_j \omega_j \]
Thus the quasi-classical matrix element for the close states is
\[ \langle t \mid O \mid t' \rangle = \int_{I_1}^{I_2} \frac{d\gamma_1}{\sqrt{P(\gamma_1)}} \cdots \int_{I_{n-1}}^{I_n} \frac{d\gamma_n}{\sqrt{P(\gamma_n)}} \prod_{i<j} (\gamma_i - \gamma_j) F(\gamma_1, \cdots, \gamma_{n-1}) \prod_j 2 \cos(\Phi_k(\gamma_j)) \] (37)
recall the notation \( \Phi_k(\gamma) = \int \gamma k_j \omega_j \). This is the same expression as for the Fourier coefficient in (38). At the first glance there are two disagreements: in (38) we integrate over \( a_j \), and we have \( \exp(i \Phi_k(\gamma_j)) \) under the integral. Actually, these two disagreements compensate each other because \( a_j = (I_j + i0) - (I_j - i0) \) and \( \Phi_k(\gamma + i0) + \Phi_k(\gamma - i0) = 0 \) when \( \gamma \in I_j \). Notice that the states \( \langle t \rangle \) are not normalized.

Certainly it is not a wonder that we have found the Fourier coefficient as the quasi-classical limit of the matrix element. We have performed all the calculations in order to have complete mathematical picture. On the other hand from the point of view of physics one can argue as follows.

Consider the action-angle variables \( J_1, \cdots, J_{n-1}, \theta_1, \cdots, \theta_j \). The Bohr-Sommerfeld quantization in this variables does not give correct quantum result, but still it is correct quasi-classically. Consider the eigenstate of Hamiltonians \( \langle t \rangle \) and the eigenstates of the angles \( \langle \theta \rangle \). Quasi-classically one has for the wave-function:
\[ \langle t \mid \theta \rangle = \frac{1}{\sqrt{\prod J_k}} \exp \left( \frac{i}{\hbar} \sum J_k \theta_k \right) \] (38)
Consider an operator \( O \). This operator can be, at least quasi-classically, ordered in such a way that \( J \)'s are to the left of \( \theta \)'s. Then the classical shape of the corresponding observable on the solution with given values of integral \( (t) \) is
\[ O_{cl}(\theta_1, \cdots, \theta_{n-1}) = \lim_{h \to 0} \frac{\langle t \mid O \mid \theta \rangle}{\langle t \mid \theta \rangle} \]
Insert the complete set of eigenstates into this formula
\[ O_{cl}(\theta_1, \cdots, \theta_{n-1}) = \lim_{h \to 0} \frac{\langle t \mid O \mid \theta \rangle}{\langle t \mid \theta \rangle} = \sum_{t'} \frac{\langle t \mid O \mid t' \rangle}{\langle t \mid t' \rangle} \frac{\langle t' \mid \theta \rangle}{\langle t' \mid \theta \rangle} \] (39)
Quasi-classically only close states are important for which we have from (38):
\[ \frac{\langle t' \mid \theta \rangle}{\langle t \mid \theta \rangle} = e^{-i \sum J_i \theta_j} \]
Now it is obvious that (38) gives the Fourier transformation of \( O_{cl}(\theta_1, \cdots, \theta_{n-1}) \). It is clear from the equation (37) that quasi-classically
\[ \langle t' \mid t \rangle = \langle t \mid t \rangle + O(h) \]
so, there is complete agreement with the formula (37).

Let us consider two close states introducing the notation: \( S_{t,k}(\gamma) = t'(\gamma) - t(\gamma) \). The quantum matrix element goes to the classical Fourier coefficient when \( \hbar \to 0 \). If we consider the classical Fourier coefficient with \( k = 0 \) it describes the classical limit of the quantum expectation value \( \langle t \mid O \mid t \rangle \). So, there is no wonder that we have the following sequences:
\[ D_{t,\gamma}(\gamma) \xrightarrow{\hbar \to 0} D_{t,k}(\gamma) \xrightarrow{k=0} D_{t}(\gamma) \]
\[ C_{t,\gamma}(\gamma_1, \gamma_2) \xrightarrow{\hbar \to 0} C_{t,k}(\gamma_1, \gamma_2) \xrightarrow{k=0} C_{t}(\gamma_1, \gamma_2) \]
\[ S_{t,\gamma}(\gamma) \xrightarrow{\hbar \to 0} S_{t,k}(\gamma) \xrightarrow{k=0} 0 \] (40)
So, we have two levels of deformation of hyper-elliptic differentials. The impression is that the quantum deformation is very natural, and that the classical mechanics appears as a strange intermediate case.
5 Conclusions

The identities (30), (31), (32) present the main result of this paper. They show that the matrix elements of arbitrary operator in quantum Toda chain can be expressed with help of finitely many integrals which possess remarkable properties.

There is a difference with what we have in Sine-Gordon theory in infinite volume. Indeed, the matrix elements for Toda chain are given by integrals of arbitrary deformed differentials with respect to fixed half-basis of deformed cycles. In Sine-Gordon case we took arbitrary half-basis. The reason for that is the difference in the type of reality conditions.

In this connection it would be very important to consider the deformation of Toda chain with trigonometric R-matrix and more complicated reality conditions which is more close to Sine-Gordon case. We hope that in this situation there will be a complete duality between deformed differentials and deformed cycles.

Acknowledgments. I am grateful to O. Babelon, D. Bernard, L. Faddeev, E. Frenkel and N. Reshetikhin for discussions. I would like to thank RIMS at Kyoto University where the work was finished for hospitality.

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