VECTOR BUNDLES ON FANO VARIETIES OF GENUS TEN

MICHAL KAPUSTKA AND KRISTIAN RANESTAD

Abstract. In this note we describe a unique linear embedding of a prime Fano 4-fold $F$ of genus 10 into the Grassmannian $G(3, 6)$. We use this to construct some moduli spaces of bundles on sections of $F$. In particular the moduli space of bundles with Mukai vector $(3, L, 3)$ on a generic polarized K3 surface $(S, L)$ is constructed as a double cover of $\mathbb{P}^2$ branched over a smooth sextic.

1. Introduction

Mukai showed that a polarized Fano 4-fold $(F, L)$ of genus 10 and index 2 has a unique linear embedding as a hyperplane section of the homogeneous variety $G_2 \subset \mathbb{P}^{13}$, the closed orbit of the adjoint representation of the simple Lie Group $G_2$ [1, ch 22]. This is part of his famous linear section theorem [11, thm 2] and [12], on smooth varieties whose general linear curve section is a canonical curve of genus at most 10. Kuznetsov [8, sec 6.4, App. B] showed that every hyperplane section of $G_2$ admits a pair of possibly isomorphic vector bundles of rank 3 with 6 independent sections. In this paper we show

Theorem 1.1. Any smooth hyperplane section of $G_2$ admits a unique linear embedding as a linear section in the Grassmannian $G(3, 6)$.

Together with Mukai’s theorem this means that any smooth Fano fourfold of genus 10 and index 2 has the above property. We also note that the variety $G_2$ does not admit a linear embedding into $G(3, 6)$ (cf. Corollary 3.11). As a consequence, we get explicit models for moduli spaces of vector bundles on linear sections of $F$. In particular, we construct moduli spaces of vector bundles on generic K3 surfaces and Fano 3-folds of genus 10. More precisely, let $(S, L)$ be a general K3 surface and let $(X, L)$ be a general prime Fano 3-fold of genus 10. By Mukai’s linear section theorem [14] the surface $S$ and the 3-fold $X$ are complete codimension 3 (resp. 2) linear section of $G_2$. For a linear section $Y$ of $G_2$, we let $\Pi_Y \subset (\mathbb{P}^{13})^*$ be the linear space orthogonal to the linear span of $Y$. A point in $\Pi_Y$ is therefore represented by a hyperplane section $F$ of $G_2$ that contains $Y$. Now a generic point $[F]$ on $\Pi_Y$ corresponds to a linear embedding of $Y$ into $G(3, 6)$ coming from the embedding of $F$. When $Y$ is a general complete linear section of dimension at least 1, we show that the pullbacks of the two universal bundles on $G(3, 6)$ by this embedding are stable vector bundles.

Let $M_{S}(3, L, 3)$ denote the moduli space of stable sheaves $E$ on $S$ with Mukai vector

$$(\text{rk} E, c_1(E), \frac{1}{2}c_1(E)^2 - c_2(E) + \text{rk} E) = (\text{rk} E, c_1(E), \chi(E) - \text{rk} E) = (3, L, 3).$$

We show
Theorem 1.2. The space $M_S(3, L, 3)$ is a double cover of the plane $\Pi_S$ branched over the intersection of $\Pi_S$ with the sextic dual variety $\hat{G}_2$. In particular, $M_S(3, L, 3)$ is a K3 surface with a genus 2 polarization.

Thus $S$ and $M_S(3, L, 3)$ is an explicit geometric example of a Mukai dual pair of K3 surfaces (cf. [13]). Furthermore, let $M_X(3, L, \sigma, 2)$ be the moduli space of stable rank 3 vector bundles $E$ on $X$ with Chern classes $c_1(E) = L$, $c_2(E) = \sigma$ where $\sigma$ is the class of a curve of degree 9 and genus 2 on $X$ and $\deg(c_3(E)) = 2$.

Theorem 1.3. The moduli space $M_X(3, L, \sigma, 2)$ has an irreducible component $M_X$ that is a double cover of the line $\Pi_X$ branched over the intersection of $\Pi_X$ with the sextic dual variety $\hat{G}_2$. In particular, $M_X$ is a genus 2 curve.

In the proof of Theorem 1.1 we study the Hilbert scheme of conic sections and cubic surface scrolls in $F$ and $G_2$. In particular we observe that if the line through two points on $G_2$ is not contained in $G_2$, then there is a unique conic section in $G_2$ through the two points. Combining this with J. Sawons and D. Markushevich results on Lagrangian fibrations on the Hilbert scheme of points on a K3 surface, see [22] and [10], we prove

Corollary 1.4. The Hilbert scheme $H(X)$ of conic sections on a generic Fano 3-fold $X$ of genus 10 is isomorphic to the Jacobian of the genus 2 curve $M_X$.

A similar example is worked out in [6].

In section 2 we formulate the main theorem and show an explicit embedding of $F$ into $G(3, 6)$. In section 3 we prove the uniqueness of the embedding by analyzing conic sections on $F$ which appear to be the zero loci of the generic sections of the considered bundles. The key is to prove the following result on the Hilbert scheme of conic sections and of cubic surface scrolls on $F$.

Proposition 1.5. The Hilbert scheme of conic sections on $F$ is isomorphic to the graph of the Cremona transformation on $\mathbb{P}^5$ defined by the linear system of quadrics in the ideal of a Veronese surface.

The Hilbert scheme of cubic surface scrolls on $F$ is isomorphic to the union of two disjoint projective planes.

We extend this analysis to get the following result suggested to us by Frederic Han.

Proposition 1.6. The Hilbert scheme of lines on $F$ is isomorphic to a smooth divisor of bidegree $(1, 1)$ in $\mathbb{F}^2 \times \mathbb{F}^2$.

In section 3 we prove stability of the constructed bundles and injectivity of the map to the moduli space. In section 4 we find a geometric way to describe all linear sections of $G(3, 6)$ which are isomorphic to $F$. More precisely $F$ is obtained as a component of the intersection of $G(3, 6)$ with a maximal dimensional linear space in a quadric of rank 12 containing $G(3, 6)$. The second component being a $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. An analogous construction is valid for the variety $G_2$ in $G(2, 7)$, but there the second component is $\mathbb{P}^2 \times \mathbb{P}^2$. In the last section we deduce the results concerning moduli spaces of bundles on sections of $F$.

2. The embedding

This section is devoted to proving the existence part of Theorem 1.1 and analyzing the constructed example. First we need a result on the orbits of the adjoint action of the simple Lie group $G_2$. 
Lemma 2.1. The projectivized adjoint representation of $G_2$ admits a pencil of invariant sextic hypersurfaces $D = \{D_\mu\}_{\mu \in \mathbb{P}^1}$. Let $B$ be the base locus of this pencil.

1. There is a distinguished element $D_l \in D$ which is the discriminant variety of the adjoint variety
2. For each sextic $D_\mu \in D$ different from $D_l$ the set $D_\mu \setminus B$ is an orbit of the representation.

Proof. By [21, Ex.30], the Lie algebra $g_2$ is a subalgebra of $\mathfrak{gl}(7, \mathbb{C})$, given by matrices of the following form:

$$A = \begin{pmatrix}
g & h & i & 0 & f & -e & a \\
j & k & l & -f & 0 & d & b \\
m & n & -g - k & e & -d & 0 & c \\
0 & -c & b & -g & -j & m & d \\
c & 0 & -a & -h & -k & n & e \\
-b & a & 0 & -i & -l & g + k & f \\
2d & 2e & 2f & 2a & 2b & 2c & 0
d
guideline
$$

It is a simple Lie algebra of rank 2 and dimension 14. Let us denote by $t$ its Cartan subalgebra spanned by $g, k$. Let $Q_2 = \det(t, \text{Id} - ad(x)) = \sum_{i=1}^{14} \delta_i(x)t^i$ be the characteristic polynomial of the adjoint operator (in our case it is the restriction of the adjoint operator on $\mathfrak{g}(7, \mathbb{C})$ to the subalgebra $g_2$, which is given by a $14 \times 14$ matrix with linear entries). By [23, thm 8.25] the polynomial $\Delta(x) := \delta_2(x)$ is a polynomial of degree 12 which can be decomposed into $\Delta(x) = \Delta_l(x).\Delta_s(x)$ in such a way that $\Delta_l$ is the discriminant of the adjoint variety (the projectivization of the orbit of the highest weight vector) and $\Delta_s$ is the discriminant of the variety $\mathbb{P}(\mathcal{O}_s)$, where $\mathcal{O}_s$ is the orbit of any short root vector. The polynomials $\Delta_s$ and $\Delta_l$ restricted to $t$ are just the products of short and long roots respectively. It follows that they are irreducible of degrees equal to the numbers of long and short root vectors respectively, which is 6 in both cases. They define hypersurfaces $D_s$ and $D_l$, respectively, that generate a pencil of invariant sextics, which proves the first sentence of the lemma together with (1). Moreover, the ring of invariants of $G_2$ in the coordinate ring of the adjoint representation is $\mathbb{C}[Q, \Delta_s]$, where $Q = 48(ad + be + cf) + 16(g^2 + k^2 + (g + k)^2 + jh + im + nl))$ is the quadric defined by the Killing form. It follows that we have a $\mathbb{C}^2$ of closed orbits. Using in addition the fact that all orbits of the representation which are not contained in the discriminant are closed, this proves that the codimension 1 orbits fill the complement of the discriminant.

Remark 2.2. The sextic $D_l$ is identified with the dual variety of $G_2$ via the Killing form and by [5] it’s singular locus is of codimension two.

Proof of existence in Theorem 1.1. Consider first the whole variety $G_2$. It is a linear section of the Grassmannian $G(5, V)$ in it’s Plücker embedding parameterizing 5-spaces in a 7-dimensional vector space $V$ isotropic with respect to a non-degenerate four-form $\omega$. The span of $G_2$ is the projectivization of the adjoint representation of the simple Lie group $G_2$. From lemma 2.1 this projectivized representation has a one-dimensional family of codimension 1 orbits.
In the above coordinates \((a, \ldots, n)\) for the Lie-algebra \(P^{13} = P(g_2)\), the variety \(G_2\) is defined by the \(4 \times 4\) Pfaffians of the matrix

\[
\begin{pmatrix}
0 & -f & e & g & h & i & a \\
f & 0 & -d & j & k & l & b \\
-e & d & 0 & m & n & -g - k & c \\
-g & -j & -m & 0 & c & -b & d \\
-h & -k & -n & -c & 0 & a & e \\
-i & -l & g + k & b & -a & 0 & f \\
-a & -b & -c & -d & -e & -f & 0
\end{pmatrix}.
\]

Consider now a pencil of hyperplane sections given by the hyperplanes \(g = -\lambda(g + k)\). Observe that this pencil corresponds by the Killing form to the projectivization of the Cartan subalgebra \(t\) from the proof of Lemma 2.1. The latter meets \(D_l\) in the set of long roots while \(D_s\) in the set of short roots. As these are disjoint, the corresponding line do not meet \(B = D_l \cap D_s\). It follows now from Lemma 2.1 that this pencil has a representative in each orbit of codimension 1 which is not the one contained in the discriminant \(D_l\).

For the above pencil of hyperplane sections we explicitly construct a family of embeddings as linear sections of the Grassmannian \(G(3, 6)\). To do this it is enough to observe that for each \(\lambda \neq 0, -1\) the ideal generating the section of \(G_2\) is also generated by the Grassmann quadrics corresponding to the following data:

\[
\begin{pmatrix}
c & b & -\lambda e \\
-t & l & a \\
m & j & \lambda c
\end{pmatrix}, \quad \begin{pmatrix}
-(1 + \lambda)f & -e & b \\
-\lambda(1 + \lambda)t & \lambda n & d \\
-(1 + \lambda)i & -h & -f
\end{pmatrix}, (1 + \lambda)a,
\]

where \(t = g + k = \frac{2}{\lambda}\).

\[\square\]

**Remark 2.3.** We can also find a different family of embeddings of sections of \(G_2\) given by the hyperplanes \(j = e + \lambda a\) for \(\lambda \neq 0\). The embeddings are then described by the data

\[
f, \left(\begin{array}{ccc}
\lambda b - d & e + \lambda a & b + \lambda t \\
e + \lambda a & b + \lambda i & a + \lambda i \\
h + \lambda b & -k & a + \lambda i \\
-k & b & a + \lambda i \\
-d & \lambda b & m - \lambda d
\end{array}\right), \quad n - \lambda e.
\]

For \(\lambda = 0\) the above map is no more an embedding but a projection from the only singular point \((0, \ldots, 1, \ldots, 0)\), where \(i = 1\), which is a node. The image of this map is then a proper codimension 2 section of \(LG(3, 6) \subset G(3, 6)\) (for more details see \([3]\)).

The proof of uniqueness is more delicate and will be postponed to section 3 after we have given some results on the geometry of \(G_2\) and its smooth hyperplane sections. We start with the above pencil of examples.

2.1. **The example.** Let us choose a coordinate system \(e_1, \ldots, e_6\) on a 6-dimensional vector space \(U\) and denote by \(e_{ijk}\), with \(1 \leq i < j < k \leq 6\), the corresponding Plücker coordinates on \(P(\wedge^3 U)\). Let \(x_{ijk}\) denote the respective dual linear forms. The considered family of hyperplane sections of \(G_2\) is then described as a family of sections of \(G(3, U)\) by the linear spaces \(H_{12}^i \subset P(\wedge^3 U)\) given by equations:
We can explicitly compute that for each $\lambda$

Consider moreover the set of reducible 3-forms

that $\Pi \lambda$

details see [4], that for each $\lambda$

scroll over a Veronese surface in two ways. Indeed, $\Omega$

projections onto 5-dimensional projective spaces $P$

projection, we need only to solve the system of equations

$v$

this system of equations has a nontrivial solution and the set of solution is then a

line. Analogously we consider the second projection. The following follows:

Lemma 2.4. Let $Y$ be a variety isomorphic to $G(2, 5)$ linearly embedded in $G(3, U)$, then the linear span of $Y$ intersects each linear space $H_{12}^\lambda$ in a plane or a 4-dimensional projective space.

Proof. Observe that $Y$ is equal to one of the following

• the Grassmannian $G(3, W)$ for some 5-dimensional vector subspace $W \subset U$,

• the partial Flag variety $F(w, 3, U)$ of 3-dimensional subspaces of $U$ that contains a given vector $w \in U$.

In both cases $< Y >^\perp \subset \Omega$ is the closure of a fiber of one of the two projections of $\Omega \setminus G(3, U^*)$. But we checked above that the intersection of $\Pi_6^\lambda = (H_{12}^\lambda)^\perp$ is either empty or it is a line. The assertion follows by duality. □

Corollary 2.5. The set of $W$ (resp. $w$) for which the intersection $< G(3, W) > \cap H_{12}^\lambda$ (resp. $F(w, 3, U) \cap H_{12}^\lambda$) is of dimension 4 is a Veronese surface in $P(U^*)$ (resp. $P(U)$).

Proof. The proof follows directly from the proof of the lemma and the discussion above it. □

3. Conic sections and cubic surface scrolls on $G_2$

In this section we study the Hilbert scheme of conic sections and cubic surface scrolls on $G_2$. Let $V$ be a 7-dimensional vector space, let $\omega \in \Lambda^4 V$ be a 4-form defining $G_2 \subset P(\Lambda^2 V)$, and let $\omega^*$ be the corresponding dual 3-form. Then, as in [2] fig.1 and [21] Ex. 30, we may choose coordinates $V^* =< x_0, ..., x_6 >$ such that $\omega^* \in \Lambda^3 V^*$ is defined as

$$
\omega^* = x_0 \wedge x_1 \wedge x_2 + x_3 \wedge x_4 \wedge x_5 + (x_0 \wedge x_3 + x_1 \wedge x_4 + x_2 \wedge x_5) \wedge x_6.
$$
The variety $G_2 \subset G(2, V)$ is defined by the Pfaffians of the skew symmetric matrix $(x_{ij})_{0 \leq i, j \leq 6, x_{ij} + x_{ji} = 0}$, restricted to the 2-vectors that are killed by $\omega^*$. With coordinates $x_{ij} = x_i \wedge x_j$ the variety $G_2$ is then given by the Pfaffians of

$$\begin{pmatrix}
0 & -x_{56} & x_{46} & x_{03} & x_{04} & x_{05} & x_{06} \\
-x_{56} & 0 & -x_{36} & x_{13} & x_{14} & x_{15} & x_{16} \\
-x_{46} & x_{36} & 0 & x_{23} & x_{24} & x_{25} & x_{26} \\
-x_{03} & -x_{13} & -x_{23} & 0 & x_{06} & x_{06} & x_{36} \\
-x_{04} & -x_{14} & -x_{24} & -x_{26} & 0 & x_{06} & x_{46} \\
-x_{05} & -x_{15} & -x_{25} & x_{16} & -x_{06} & 0 & x_{56} \\
-x_{06} & -x_{16} & -x_{26} & -x_{36} & -x_{46} & -x_{56} & 0
\end{pmatrix}$$

with $x_{03} + x_{14} + x_{25} = 0$ as in the previous section.

**Lemma 3.1.** There are no planes on $G_2$.

**Proof.** We compute directly in any chosen point $p \in G_2$ that the intersection of $G_2$ with the tangent space to $G_2$ in $p$ is a cone over a twisted cubic curve, hence do not contain a plane. Indeed, let $p = (0, \ldots, 0, 1, 0, \ldots, 0)$ where $x_{05} = 1$. Then the tangent space at $p$ is given by $x_{13} = x_{14} = x_{16} = x_{23} = x_{24} = x_{26} = x_{36} = x_{46} = 0$ and its intersection with $G_2$ is additionally given by the Pfaffians of

$$\begin{pmatrix}
0 & -x_{56} & 0 & x_{03} & x_{04} & x_{05} & x_{06} \\
-x_{56} & 0 & 0 & 0 & 0 & x_{15} & 0 \\
0 & 0 & 0 & 0 & 0 & x_{25} & 0 \\
-x_{03} & 0 & 0 & 0 & 0 & 0 & 0 \\
-x_{04} & 0 & 0 & 0 & 0 & x_{06} & 0 \\
-x_{05} & -x_{15} & -x_{25} & 0 & -x_{06} & 0 & x_{56} \\
-x_{06} & 0 & 0 & 0 & 0 & -x_{56} & 0
\end{pmatrix}$$

with $x_{03} + x_{25} = 0$ i.e. by the $2 \times 2$ minors of

$$\begin{pmatrix}
x_{56} & x_{06} & x_{04} \\
x_{15} & x_{56} & x_{06}
\end{pmatrix}$$

and $x_{03} + x_{25} = x_{03}^2 = 0$. Therefore the tangent cone is a cone over a twisted cubic curve. We conclude by homogeneity. □

**Lemma 3.2.** Let $Q \in \mathbb{P}(V^*)$ be the $G_2$ invariant smooth quadric hypersurface, and let $U \subset V$, be a 6-dimensional subspace. Then the intersection $G_2 \cap G(2, U)$ is a fourfold of degree 6 that spans a $\mathbb{P}^7$. Furthermore

- If $[U] \notin Q$, then $G_2 \cap G(2, U)$ is a smooth hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$.
- If $[U] \in Q$, then $G_2 \cap G(2, U)$ is a hyperplane section of $\mathbb{P}(O_{\mathbb{P}^2}(2) \oplus T_{\mathbb{P}^2}(-1)))$.

**Proof.** The group $G_2$ has two orbits on $\mathbb{P}(V^*)$, so for the intersection $G_2 \cap G(2, U)$ there are two isomorphism classes. We compute these directly. If $U \subset V$ is the 6-dimensional subspace $x_6 = 0$, then $[U] \notin Q$ and the intersection $G_2 \cap G(2, U)$ is defined by $\{ \alpha \in G(2, U) : (x_0 \wedge x_1 \wedge x_2 + x_3 \wedge x_4 \wedge x_5)(\alpha) = (x_0 \wedge x_3 + x_1 \wedge x_4 + x_2 \wedge x_5)(\alpha) = 0 \}$ so it is defined by the $2 \times 2$ minors of

$$\begin{pmatrix}
x_{03} & x_{04} & x_{05} \\
x_{13} & x_{14} & x_{15} \\
x_{23} & x_{24} & x_{25}
\end{pmatrix}$$

with $x_{03} + x_{14} + x_{25} = 0$, i.e. a smooth fourfold Fano hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$. On the other hand if $U \subset V$ is the 6-dimensional subspace $x_6 = 0$, then $[U] \in Q$ and the intersection $G_2 \cap G(2, U)$ is defined by $\{ \alpha \in G(2, U) : (x_3 \wedge x_4 \wedge x_5 + x_1 \wedge$
\[ x_4 + x_2 \wedge x_3 \wedge x_6(\alpha) = (x_1 \wedge x_2 + x_3 \wedge x_6)(\alpha) = 0, \] so it is defined by the Pfaffians of
\[
\begin{pmatrix}
0 & -x_{36} & x_{13} & -x_{25} & x_{15} & x_{16} \\
x_{36} & 0 & x_{23} & x_{24} & x_{25} & x_{26} \\
-x_{13} & -x_{23} & 0 & x_{26} & -x_{16} & x_{36} \\
x_{25} & -x_{24} & -x_{26} & 0 & 0 & 0 \\
-x_{15} & -x_{25} & x_{16} & 0 & 0 & 0 \\
x_{16} & -x_{26} & -x_{36} & 0 & 0 & 0 \\
\end{pmatrix}
\]
i.e. by the 2 \times 2 minors of the symmetric matrix
\[
\begin{pmatrix}
x_{24} & x_{25} & x_{26} \\
x_{25} & -x_{15} & -x_{16} \\
x_{26} & -x_{16} & x_{36} \\
\end{pmatrix}
\]
and the three quadratic entries in the product matrix
\[
(x_{13}, x_{23}, x_{12}) \cdot \begin{pmatrix} x_{24} & x_{25} & x_{26} \\ x_{25} & -x_{15} & -x_{16} \\ x_{26} & -x_{16} & x_{36} \end{pmatrix}
\]
together with the linear relation \( x_{12} + x_{36} = 0 \). But these quadrics define the linear embedding of the \( \mathbb{P}^2 \)-bundle \( \mathbb{P}(O_{\mathbb{P}^2}(2) \oplus T_{\mathbb{P}^2}(-1)) \), so the lemma follows.

\[ \square \]

**Lemma 3.3.** Let \( W \subset V \), be a 5-dimensional subspace. One of the following holds

- the intersection \( G_2 \cap G(2, W) \) in \( G(2, V) \) is proper. It is a possibly singular conic section.
- the intersection \( G_2 \cap G(2, W) \) in \( G(2, V) \) is not proper. It is a possibly singular cubic surface scroll.

There is a 7-fold subvariety in \( G(5, V) \) of 5-dimensional subspaces \( W \subset V \) such that the intersection \( G_2 \cap G(2, W) \) is a cubic scroll.

**Proof.** Let \( W \subset V \), be a 5-dimensional subspace, and let \( U \) be a general 6-dimensional subspace of \( V \) that contains \( W \). We first assume that the intersection \( G_2 \cap G(2, U) \) is isomorphic to a hyperplane section of \( \mathbb{P}^2 \times \mathbb{P}^2 \). From the above proof, the variety \( \mathbb{P}^2 \times \mathbb{P}^2 \subset G(2, U) \) may be identified with 2-dimensional subspaces that has a 1-dimensional intersection with each of 3-dimensional subspaces \( U_1 \) and \( U_2 \) that together span \( U \). The further intersection with \( G(2, W) \) therefore depends on whether \( W \) contains \( U_1 \) or \( U_2 \). If neither is the case, the intersection \( G_2 \cap G(2, W) \) is a linear section of the set of 2-dimensional subspaces that intersect \( U_1 \cap W \) and \( U_2 \cap W \) so it is a conic section. Similarly, if \( W \supset U_1 \), then \( G_2 \cap G(2, W) \) is a smooth cubic surface scroll.

If \( G_2 \cap G(2, U) \) is a hyperplane section of \( \mathbb{P}(O_{\mathbb{P}^2}(2) \oplus T_{\mathbb{P}^2}(-1)) \), then we may use the above equations to identify \( G_2 \cap G(2, W) \). In fact if \( U \) is defined by \( x_0 = 0 \), then \( G_2 \cap G(2, U) \) spans a \( \mathbb{P}^7 \) on which we may choose coordinates
\[
x_{12}, x_{13}, x_{23}, x_{15}, x_{24}, x_{25}, x_{26},
\]
as above, in which the equations defining the threefold \( X_U = G_2 \cap G(2, U) \) are the 2 \times 2-minors of the symmetric matrix
\[
\begin{pmatrix}
x_{24} & x_{25} & x_{26} \\
x_{25} & -x_{15} & -x_{16} \\
x_{26} & -x_{16} & -x_{12} \\
\end{pmatrix}
\]
and the quadrics
\[
(x_{13} x_{23} x_{12}) \cdot \begin{pmatrix} x_{24} & x_{25} & x_{26} \\ x_{25} & -x_{15} & -x_{16} \\ x_{26} & -x_{16} & -x_{12} \end{pmatrix}.
\]

So \(X_U\) is a threefold contained in the fourfold cone over a Veronese surface with vertex a line \(L_U\). Since the Veronese surface contains no lines, the only lines in \(X_U\) are lines that intersect \(L_U\) and lie over a unique point on the Veronese surface. On the other hand, for each point on the Veronese surface the symmetric matrix has rank 1, so the three remaining quadrics restricted to the corresponding plane in the cone have a common linear factor, so they define a line, except in one plane: In the plane \(P_U : x_{15} = x_{16} = x_{24} = x_{25} = x_{26} = 0\) the intersection with \(X_U\) is the double line defined by \(x_{12}^2 = 0\). Let \(W \subset U\) be a 5-dimensional subspace. The intersection \(G_2 \cap G(2, W)\) is a complete linear section of both \(G(2, W)\) and of \(X_U\). Any complete linear surface section of \(X_U\) is either a Veronese surface, a cone over a linear section a Veronese surface, or it contains the vertex line \(L_U\). Any complete surface linear section of \(G(2, 5)\) that contains a Veronese surface, contains also a plane. A surface cone over a linear section of the Veronese surface is either a quadric or a quartic surface, but neither are contained in linear sections of \(G(2, 5)\) that do not also contain a plane, so only linear sections of \(X_U\) that contain \(L_U\) can be linear sections of \(G(2, 5)\). These are easily seen to correspond to subspaces \(W\) defined by \(x_0 = \ell = 0\), where \(\ell \in \langle x_4, x_5, x_6 \rangle\), and for each such \(\ell\) the corresponding intersection \(G(2, W) \cap G_2\) is a cubic scroll. When the intersection \(G(2, W) \cap G_2\) is proper in \(G(2, V)\) it is clearly a conic as above.

Setting up an incidence variety we find,
\[
I = \{(W, U) : \dim G(2, W) \cap G_2 > 1\} \subset F = \{(W, U) : W \subset U\}
\]
We have seen that the fiber over each \(U\) is 2-dimensional, so \(\dim I = 6 + 2 = 8\). The fiber over each \(W\) is clearly 1-dimensional, so there is a 7-dimensional variety of 5-dimensional subspace \(W \subset V\) such that \(G(2, W) \cap G_2\) is a cubic surface scroll. \(\Box\)

**Corollary 3.4.** Let \(Z\) be a scheme of length 2 in \(G_2\) contained in a line \(\ell\). Then one of the following holds:

1. The line \(\ell\) is contained in \(G_2\).
2. There is a unique conic section in \(G_2\) that contains \(Z\).

**Proof.** Assume that \(\ell\) does not lie in \(G_2\). Then any conic section in \(G_2\) through \(Z\) spans a plane contained in the linear span of any \(G(2, 5)\) in \(G(2, 7)\) that contains the \(Z\). But any \(G(2, 5)\) intersects \(G_2\) in either a conic section or a cubic scroll. In either case there is a unique conic section in the intersections that contains \(Z\). \(\Box\)

**Remark 3.5.** Jarek Buczyński, private communication, showed that this property is common to the closed orbit of the adjoint representation of any semisimple Lie group.

**Corollary 3.6.** If three points \(p, q, r \in G_2\) do not lie on a conic section, there is at most one 5-dimensional subspace \(W \subset V\) such that \(p, q, r \in G(2, W)\).

**Proof.** If \(W\) and \(W'\) are distinct 5-dimensional subspace then \(G(2, W) \cap G(2, W') \cap G_2\) is either a line or a conic section. \(\Box\)

**Corollary 3.7.** Let \(F\) be a smooth hyperplane section of \(G_2\) and \(W \subset V\) a 5-dimensional subspace. Then \(F \cap G(2, W)\) is a conic section if and only if \(F \cap G(2, W) = G_2 \cap G(2, W)\) and \(G_2 \cap G(2, W)\) is a conic section. \(F \cap G(2, W)\) is a
cubic scroll if and only if \(F \cap G(2, W) = G_2 \cap G(2, W)\) and \(G_2 \cap G(2, W)\) is a cubic scroll.

**Proof.** Since \(F\) defines a hyperplane section, it either contains \(G_2 \cap G(2, W)\) or it intersects it in a hyperplane section. But \(G_2 \cap G(2, W)\) is either a conic section or a cubic scroll, so the corollary follows.

**Corollary 3.8.** Let \(L \subset \mathbb{P}^{13}\) be a linear subspace of codimension 2 or 3, such that \(Y = G_2 \cap L\) is a smooth 3-fold (resp. surface) with Picard group generated by \(H\). If \(F\) and \(F'\) are distinct hyperplane sections of \(G_2\) that contain \(Y\), and \(S_W\) and \(S_W'\) are cubic scrolls on \(F\) and \(F'\) respectively, then \(S_W \cap Y\) and \(S_W' \cap Y\) are either empty or distinct subschemes on \(Y\).

**Proposition 3.9.** Let \(F\) be any smooth hyperplane section of \(G_2\). Let \(U\) be a 6-dimensional vector space and let \(u\) and \(W\) be a nontrivial vector and a 5-dimensional subspace of \(U\) respectively. Let \(\phi : F \to G(3, U)\) be a linear embedding. Then the intersections \(\phi(F) \cap G(3, W), \phi(F) \cap F(u, 3, U)\) are linearly isomorphic to one of the following:

1. a conic section
2. a cubic surface scroll in \(\mathbb{P}^4\)

Moreover every conic section in \(\phi(F)\) lies in the intersection of a unique \(F(u, 3, U)\) and a unique \(G(3, W)\), while every cubic scroll in \(\phi(F)\) lies in a unique \(F(u, 3, U)\) or a unique \(G(3, W)\).

**Proof.** Observe first that \(G(3, W)\) and the partial Flag variety \(F(u, 3, U)\) are 6-dimensional Schubert cycles on \(G(3, U)\) that intersect the 4-dimensional subvariety \(\phi(F)\) in dimension at least one. Furthermore, both \(G(3, W)\) and \(F(u, 3, U)\) are isomorphic to the Grassmannian \(G(2, 5)\), so the considered intersections are linearly isomorphic to some linear sections of the Grassmannian \(G(2, 5) \subset \mathbb{P}^9\). Therefore, let us consider a linear subspace \(L \subset \mathbb{P}^9 = \langle G(2, 5) \rangle\), and see when \(L \cap G(2, 5)\) is spanned by a variety that is linearly isomorphic to a subvariety of \(F\) and hence of \(G_2\). We have the following possibilities:

- \(T = L \cap G(2, 5)\) is a variety of codimension 3 in \(L\). Then this intersection is proper, hence \(\deg(T) = 5\) and a generic linear section of appropriate codimension is a set of five isolated points spanning a \(\mathbb{P}^1\). Such five points in \(F \subset G(2, 7)\) lie in a unique \(G(2, 5) \subset G(2, 7)\), so by Lemma 3,3, the intersection \(G(2, 5) \cap G_2\) must be a cubic scroll. Therefore \(F \cap G(2, 5)\) has codimension 2 in its linear span, against the assumption.

- \(T = L \cap G(2, 5)\) is a variety of codimension 2 in \(L\). Then the generic intersection with a plane in \(L\) is a set of isolated points. By [3] prop 2.5, the number of these points cannot be greater than 3. It follows that \(T\) has degree 3. To be contained in \(G_2\) it cannot contain planes nor quadric surfaces, hence it has to be a smooth or singular cubic surface scroll.

- \(T = L \cap G(2, 5)\) is a variety of codimension 1 in \(L\). Then as above \(T\) has to be of degree 2 and hence to be contained in \(G_2\) it needs to be a conic section.

- \(L \subset G(2, 5)\). Then to be contained in \(G_2\) it has to be a line. But by adjunction a line cannot be the intersection of \(G(2, 5)\) with \(\phi(F)\). Indeed, if \(C\) is a smooth curve of intersection of \(G(2, 5)\) with \(\phi(F)\) then \(C\) is linearly isomorphic to the zero locus of a rank 3 bundle on \(F\) with determinant \(H\). By adjunction \(K_C = (H + K_F)|_C = -H|_C\) which implies \(C\) is a conic section.

Hence the first part of the Proposition follows. To prove the second part we consider each case separately. Each conic section in \(G(3, 6)\) is contained in a unique partial
Flag variety $F(A, 3, B)$ where $A$ and $B$ are fixed vector spaces of dimension 1 and 5 respectively. So the conic section lies in both $G(3, B)$ and in $F(A, 3, U)$. For a cubic surface scroll $S$ we need the following:

**Lemma 3.10.** For any cubic surface scroll $S \subset G(3, U)$ such that $S =< S > \cap G(3, U)$ there is a subspace $A$ of dimension 1 or a subspace $B$ of dimension 5, such that $S \subset F(A, 3, U)$ or $S \subset G(3, B)$.

**Proof.** The union of planes in $\mathbb{P}(U)$ parameterized by $S \subset G(3, U)$ form a variety $X_S \subset \mathbb{P}(U)$. The lines in $S$ defines pencil of planes in $\mathbb{P}(U)$ that each have a common line and fill a $\mathbb{P}^3$ in $X_S$. The common lines form a scroll $Y_S \subset X_S$. The degree of $S$ in the Plücker embedding equals the sum of degrees of $X_S$ and $Y_S$. In fact the degree of $S$ is, by Schubert calculus, the sum of the degree of the intersection of $S$ with the cycle of planes that meet a general line, and the general cycle of planes that meet a general $\mathbb{P}^3$ in a line. The former is clearly the degree of $X_S$, while the latter is the degree of $Y_S$, since a general $\mathbb{P}^3$ meet a plane in $S$ in a line if and only if it meets the common line of the pencil that the planes belong to in a point. If $X_S$ is not a fourfold, all its $\mathbb{P}^i$’s coincide and $< S > \subset G(3, 6)$, contrary to the assumption. If $X_S$ is a fourfold of degree 1 it spans a hyperplane $W \subset \mathbb{P}(U)$, and $S \subset G(3, W)$. If $X_S$ is a fourfold of degree 2, then $Y_S$ has degree 1, which means that $Y_S$ is a plane and the lines in $Y_S$ have a common point $u$. So in this case $S \subset F(u, 3, U)$. If $X_S$ is a fourfold of degree 3, then $Y_S$ is a line, a common line for all the planes in $S$, in which case $< S > \subset G(3, 6)$, contrary to the assumption. □

The intersection $G(3, B) \cap F(A, 3, U) = F(A, 3, B)$ so the intersection with $\phi(F)$ cannot contain $S$, so any cubic surface scroll $S \subset \phi(F)$ is contained in precisely one of them. □

**Corollary 3.11.** The variety $G_2$ does not admit a linear embedding into the Grassmannian $G(3, 6)$.

**Proof.** Let $U$ be a 6-dimensional vector space. An embedding $G_2 \subset G(3, U)$ induces an embedding of any hyperplane section $F$ of $G_2$. By Proposition 3.9 the intersection $F \cap G(3, B)$ is a conic section for a general 5-dimensional subspace $B \subset U$. But then $G_2 \cap G(3, B)$ must be a quadric surface, against the fact that there is either a line or a unique conic section through two points on $G_2$ (cf. Corollary 3.4). □

**Remark 3.12.** On any cubic surface scroll $\Sigma$, the Hilbert scheme $H_\Sigma$ of conic sections is a $\mathbb{P}^2$. In a cubic cone, the conic sections are all singular while the double lines form themselves a conic section $C_\Sigma$ in $H_\Sigma$. Any pencil of singular conics on $\Sigma$ with a fixed line form a tangent line to $C_\Sigma$. On a smooth cubic scroll the general conic section is smooth, while the singular ones form a line in $H_\Sigma$.

The Proposition 3.13 is an immediate corollary of the following

**Proposition 3.13.** The Hilbert scheme $H_F$ of conic sections lying on a smooth hyperplane section $F$ of $G_2$ is isomorphic to the graph in $\mathbb{P}^5_1 \times \mathbb{P}^5_2$ of the Cremona transformation $\gamma : \mathbb{P}^1_1 \cdots > \mathbb{P}^1_2$ defined by the linear system of quadrics that contain a Veronese surface $V_1 \subset \mathbb{P}^1_1$. The inverse Cremona transformation is defined by the quadrics that contain a Veronese surface $V_2 \subset \mathbb{P}^5_2$.

The Hilbert scheme $V_F$ of cubic surface scrolls in $F$ has two components each of which is isomorphic to $\mathbb{P}^2$. Each component has a natural embedding as the Veronese surface, $V_i \subset \mathbb{P}^5_2$, $i = 1, 2$, and the Hilbert scheme of conics $H_F$ restricts to a $\mathbb{P}^2$-bundle over these two Veronese surfaces.

**Proof.** Let $U$ be a 6-dimensional vector space and let $F \rightarrow G(3, U)$ be a linear embedding as in example 2.1. By Proposition 3.9 each conic $C$ in $F$ is the intersection $F(u_C, 3, U) \cap G(3, V_C) \cap F$ for a unique one dimensional subspace $< u_C > \in U$
and a unique codimension one subspace $V_G \subset U$. Therefore the Hilbert scheme $H_F$ has a natural embedding in $\mathbb{P}(U) \times \mathbb{P}(U^*)$ taking a conic $C$ to the unique pair $([u_C], [V_C])$. Furthermore, by Lemma 2.4 and Corollary 2.5, for $u \in U$, the intersection $F(u, 3, U) \cap F$ either a conic section or a cubic surface scroll and similarly for $G(3, V) \cap F$ when $V \subset U$ is a codimension one subspace. The Hilbert scheme $V_F$ of cubic surfaces in $F$ is isomorphic to two Veronese surfaces $V_1 \subset \mathbb{P}(U)$ and $\mathbb{P}(U^*)$ via the maps $[u] \mapsto \mathbb{P}(U)$ and $[V] \mapsto \mathbb{P}(U^*)$ respectively.

We can then define a rational map $\gamma : \mathbb{P}(U) \ni [u] \mapsto [V] \in \mathbb{P}(U^*)$ such that $F(u, 3, U) \cap G(3, V) \cap F \in H_F$. By the above, the map is birational and defined outside the Veronese surface $V_1$, so it is a Cremona transformation. The fiber in $H_F$ over $[u] \in V_1$ is the Hilbert scheme $H_{\Sigma}$ of conic sections on the cubic scroll $\Sigma = F(u, 3, U) \cap F$. This fiber is therefore isomorphic to $\mathbb{P}^2$, which is also the fiber of the projectivized normal bundle of $V_1$ in $\mathbb{P}(U)$. Each conic in $H_{\Sigma}$ is mapped to a unique $[V] \in \mathbb{P}(U^*)$ by the second projection, so $H_F \subset \mathbb{P}(U) \times \mathbb{P}(U^*)$ is the graph of the Cremona transformation $\gamma$. It remains to show that $\gamma$ is the Cremona transformation defined by the linear system of quadrics that contain $V_1$.

By symmetry, the inverse is mapped by the linear system of quadrics that contain $V_2$. Observe first that the projections $p_1 : H_F \to \mathbb{P}(U)$ and $p_2 : H_F \to \mathbb{P}(U^*)$ map the plane of conics $H_{\Sigma}$ lying on a cubic surface scroll to a point by one projection and to a plane by the other. Indeed, let $\Sigma \subset F(u, 3, U)$ be a cubic surface scroll for some one dimensional subspace $< u > \subset U$. Consider the 5-dimensional quotient $W = U/ < u >$ and the cubic scroll $\Sigma$ as a subvariety of the Grassmannian $G(2, W)$. The lines in $\mathbb{P}(W)$ corresponding to points in $\Sigma$ all meet a line $E_{\Sigma}$ and are contained in a nodal quadric hypersurface. Now any $G(2, 4) \subset G(2, W)$ intersects the scroll in conic section if and only if it is defined by a 4-space that contains $E_{\Sigma}$. Therefore this family of Grassmannians $G(2, 4)$ is defined by a linear family of codimension one subspaces in $W$, which means that the corresponding family of codimension one subspaces $V \subset U$ is a plane in $\mathbb{P}(U^*)$. These planes are the image by the second projection of the fibers $H_{\Sigma} \cong \mathbb{P}^2$ in $H_F$ over the Veronese surface $V_1$. Any two of these planes intersect only on the Veronese surface. In particular the union of these planes form a hypersurface. The Cremona transformation $\gamma^{-1}$ must contract the planes, while the Veronese surface is the base locus, so it must be given by the quadrics defining the Veronese surface.

We now pass to the proof of uniqueness.

*Proof of uniqueness in Theorem 1.1.* Let $\psi : F \to G(3, U)$ be any linear embedding. By Proposition 3.9 this embedding gives rise to two projections from the Hilbert scheme of conic sections on $F$ onto $\mathbb{P}^5$. Again by Proposition 3.9 and Proposition 3.13 these projections must coincide with the projections in the constructed examples. Let $Q$ be the universal quotient bundle and $E$ the universal bundle on $G(3, U)$. It follows that there is a linear isomorphism $\pi$ between $H^0(F, \phi^*(Q))$ and one of the spaces $H^0(F, \phi^*(Q))$ or $H^0(F, \phi^*(E^*))$ such that the sections $s$ and $\pi(s)$ have the same zero locus. Therefore $\psi$ is the embedding defined by one of the bundles $\phi^*(Q)$, $\phi^*(E^*)$. We conclude that there exists a linear automorphism of $G(3, 6)$ mapping $\phi(F)$ to $\psi(F)$. 

We end the section by applying our analysis of conics and cubic scrolls on $F$ to the Hilbert scheme of lines on $F$. The Proposition 1.6 was suggested to us by Frederic Han.

*Proof of Proposition 1.6.* We keep the notation from the proof of Proposition 3.13. We shall use the word cubic scroll for both a smooth cubic scroll and a cone over a cubic curve. In the latter case every lines in the scroll is both exceptional and a
ruling. Let \( l \subset F \subset G(3, 6) \) be a line. There exist spaces \( U_1 \) and \( V_i \) of dimension 2 and 4 such that \( l = F(U_1, 3, V_i) \). Consider the set \( H_l \subset H_F \) of conics contained in \( F \) and containing \( l \). This set is mapped by both \( p_1 : H_F \to \mathbb{P}(U) \) and \( p_2 : H_F \to \mathbb{P}(U^*) \) surjectively onto the lines \( l_1 \subset \mathbb{P}(U) \) and \( l_2 \subset \mathbb{P}(U^*) \) defined by \( \langle \langle u, \subset U_1 \rangle \rangle \) and \( \{ V \supset V_1 \} \). Observe that \( H_l \) has at most two components each one isomorphic to \( \mathbb{P}^1 \). Indeed, each component of \( H_l \) is either contracted or maps surjectively onto \( l_1 \) or \( l_2 \). But both \( p_1 \) and \( p_2 \) are one to one over the complements of the Veronese surfaces. It follows that \( H_l \) has either two components each contracted by one of the projections \( p_1 \) and \( p_2 \) or \( H_l \) has one component which is not contracted in any direction. Notice, that in the first case, \( l_1 \) and \( l_2 \) are contracted by the Cremona transformation \( \gamma \) and \( \gamma^{-1} \) respectively. Therefore \( l_1 \) and \( l_2 \) must intersect \( V_1 \), respectively \( V_2 \), in a scheme of length two. In the latter case the line \( l_1 \) is mapped to the line \( l_2 \) by the Cremona transformation \( \gamma : \mathbb{P}(U) \to \mathbb{P}(U^*) \), so \( l_1 \) and \( l_2 \), intersect \( V_1 \) and \( V_2 \), respectively, transversely in one point.

Let \( l \cup l' \subset F \) be a conic section. Let \( v \in H_l \) be its parameter point. Assume that \( H_l \) has a component through \( v \) that is contracted by the projection \( p_1 \). Then \( l \cup l' \) lies in a cubic scroll \( \Sigma_{v_1} \), where \( v_1 = p_1(v) \in V_1 \), and \( l \) must be contained in a pencil of conic sections on this scroll. In particular \( l \) must be an exceptional line on the scroll. If \( H_l \) does not have a component through \( v \) that is contracted by the projection \( p_1 \) and \( v_1 = p_1(v) \in V_1 \), then \( l \cup l' \) belongs to a pencil of conics on the scroll \( \Sigma_{v_1} \) with \( l' \) as a fixed component, i.e. \( l' \) is exceptional on the scroll.

When \( H_l \) has two components, one component is contracted to a point \( v_1(l) \in V_1 \). In this case \( v_1(l) \in l_1 \cap V_1 \), and since \( l_1 \cap V_1 \) has length two, there is a unique point \( w_1(l) \) residual to \( v_1(l) \) in \( l_1 \cap V_1 \). Similarly there is a unique point \( w_2(l) \) residual to \( v_1(l) \) in \( l_2 \cap V_2 \). When \( H_l \) has a unique component, we set \( w_i(l) = l_i \cap V_i \) for \( i = 1, 2 \). For each \( i \) there is a unique singular conic section \( l \cup l(i) \in H_l \) which is mapped to \( w_i(l) \) by the projection \( p_i \). Clearly \( l(i) \) is a line in the cubic scroll \( \Sigma_{w_i(l)} \), and \( l \cup l(i) \) belongs to a pencil of conics with \( l(i) \) as a fixed component.

Summing up, we may define a morphism from the Hilbert scheme \( \text{Hilb}_l(F) \) of lines to \( \mathbb{P}^2 \times \mathbb{P}^2 \):

\[
\rho : \text{Hilb}_l(F) \to V_1 \times V_2
\]

by

\[
l \mapsto (w_1(l), w_2(l)).
\]

The scroll \( \Sigma_{w_i(l)} \) contains \( l \) as a ruling, and every line \( l' \) in this ruling has \( w_1(l') = w_1(l) \). Indeed, this \( \mathbb{P}^1 \) of lines is a fiber of \( \rho \) composed with the projection to the first factor. Clearly, this fiber is mapped to a line in the second factor: Indeed, the pencil of conic sections in \( \Sigma_{w_i(l)} \) with fixed component \( l(1) \) form a line in \( V_2 \). Likewise the fiber of the composition of \( \rho \) with the second factor is mapped isomorphically to a line in the first factor.

In this construction clearly \( \rho \) is an embedding, and all fibers of the projections to the two factors are mapped to isomorphically to lines in the other factor, so \( \text{Hilb}_l(F) \) is smooth and the proposition follows.

\begin{remark}
Observe that the intersection of two scrolls from the same family may contain a line that is exceptional on one of the scrolls, and that the intersection of two scrolls from different families can be a smooth conic.
\end{remark}

\begin{remark}
Observe that in the case where \( H_l \) has two components if \( w_1(l) = v_1(l) \) the scroll \( \Sigma_{w_1(l)} \) has to be a cone. And analogously for \( w_2(l) = v_2(l) \). Moreover, when both \( w_1(l) = v_1(l) \) and \( w_2(l) = v_2(l) \) the conic corresponding to the intersection of the two components is a double line. Note that \( w_1(l) = v_1(l) \) does not imply \( w_2(l) = v_2(l) \) and vice versa.
\end{remark}
4. The bundles

Recall that $(S, L)$ is a generic $K3$ surface of genus 10 which is a general linear section of $G_2$. Consider a smooth hyperplane sections $F$ of $G_2$, containing $S$. By Theorem 1.1 $F$ correspond to a pair $(E, E')$ of bundles of rank 3 each, the pullbacks of the universal quotient bundle and the dual to the universal subbundle on $G(3, 6)$. We shall prove that all such bundles are stable and different. Let us first prove stability.

**Proposition 4.1.** The bundles $E$ and $E'$ are $L$-stable.

**Proof.** Consider a generic curve $C$ in $|L|$. It is enough to prove that the bundles are stable on $C$. Observe first that by construction $E$ is a rank 3 bundle on $C$ such that $c_1(E) = K_C$, $h^0(E) = 6$. We have two possibilities to destabilize $E$. Either there is a subbundle of $E$ of rank 1 and degree $\geq 6$ or a rank 2 subbundle of degree $\geq 12$. We consider the two cases separately.

Assume first that there is a line bundle $M$ on $X$ of degree $d \geq 6$ which is a subbundle of $E$. We get the exact sequence:

$$0 \longrightarrow M \longrightarrow E \longrightarrow N \longrightarrow 0,$$

with $c_1(N) = K_C \otimes M^*$. It follows that $h^0(M) + h^0(N) \geq h^0(E) = 6$. On the other hand by the Riemann-Roch formula we have $h^0(M) - h^0(K_C \otimes M^*) = d - 9 \geq -3$. We claim that $h^0(M) \geq 1$. Indeed, if $M$ had no sections then the map $H^0(E) \rightarrow H^0(N)$ would be an embedding. Hence the bundle $N$ would be globally generated by 6 sections. We could then consider a nonvanishing section of $N$. Its wedge products with the remaining generators would give 5 linearly independent sections of $c_1(N)$. The latter contradicts the Riemann-Roch formula. Hence $h^0(M) \geq 1$ and each of its sections defines a hyperplane in $\mathbb{P}(H^0(E))$ which contains $d$ fibers of the natural image of $\mathbb{P}(E)$ in $\mathbb{P}(H^0(E))$. It follows that $C$ intersects a $G(3, W)$ (for some hyperplane $W \subset V$) in $d \geq 6$ points. By Proposition 3.6 this implies that $C$ is either a conic section or is contained in a cubic scroll. This contradicts genericity.

Assume now that there is a rank 2 bundle $N$ on $X$ of degree $d \geq 12$ which is a subbundle of $E$. We get the exact sequence:

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0.$$

Then $M = K_C \otimes c_1(N)^*$ has at most one section by assumption on $C$. Hence, $h^0(N) \geq 5$. But we have proved above that $E$ does not admit any line subbundle with 2 sections (as by assumption on $C$ it would have degree $> 6$), hence neither does $N$. It follows that the projectivization of the kernel of the map $\bigwedge^2 H^0(N) \rightarrow H^0(\bigwedge^2 N) = H^0(c_1(N))$ does not meet the Grassmannian $G(2, H^0(N))$. Finally $h^0(c_1(N)) \geq 7$ which contradicts the Riemann Roch for $M$. \[\square\]

Let us now prove that all these bundles are different.

**Proposition 4.2.** Consider two smooth hyperplane sections $F_1$ and $F_2$ of $G_2$ containing $S$. Denote by $E_1, E_1', E_2, E_2'$ their corresponding bundles. Then no two of these bundles are isomorphic.

Before we pass to the proof of the proposition let us prove the following lemmas.

**Lemma 4.3.** For every cubic scroll $S$ contained in $G(2, V)$ such that $S = < S \cap G(2, V), there is a 5-dimensional subspace $W \subset V$ such that $S \subset G(2, W)$.

**Proof.** The union of lines in $\mathbb{P}(V)$ parameterized by $S \subset G(2, V)$ form a variety $X_S \subset \mathbb{P}(V)$. The lines in $S$ define pencils of lines in $\mathbb{P}(V)$ that each have a common point and fill a $\mathbb{P}^2$ in $X_S$. The common points form a curve $Y_S \subset X_S$. The degree
of $S$ in the Pl"{u}cker embedding equals the sum of degrees of $X_S$ and $Y_S$. In fact the degree of $S$ is, by Schubert calculus, the sum of the degree of the intersection of $S$ with the cycle of lines that meet a general $\mathbb{P}^3$, and the general cycle of lines that is contained in a general $\mathbb{P}^3$. The former is clearly the degree of $X_S$, while the latter is the degree of $Y_S$, since a general $\mathbb{P}^3$ contains a line in $S$ if and only if it meets the common point of the pencil that the line belongs to in a point. If $X_S$ is not a threefold, all its $\mathbb{P}^2$s coincide and $< S > \subset G(2, V)$, contrary to the assumption. If $X_S$ is a threefold of degree 1 it is a $\mathbb{P}^3 \subset \mathbb{P}(U)$, and $< S > \cap G(2, V)$ is a quadric hypersurface, contrary to the assumption. If $X_S$ is a threefold of degree 2, then $X_S$ is a quadric hypersurface in a hyperplane $\mathbb{P}(W) \subset \mathbb{P}(U)$ and $S \subset G(2, W)$.

If $X_S$ is a threefold of degree 3, then $Y_S$ is a point, i.e. all the lines parameterized by $S$ have a common point, so $< S > \subset G(2, V)$, contrary to the assumption. □

Lemma 4.4. If $s$ is a section of one of the bundles $\mathcal{E}_1, \mathcal{E}_1', \mathcal{E}_2$, and $\mathcal{E}_2'$ then its zero locus on $S$ is the intersection of $S$ with $G(2, W)$ for some $W$ of dimension 5. In particular it is either empty or is a 0-dimensional scheme of length 2 or 3 spanning a line or a plane respectively. Moreover in the case of length 3 the subspace $W$ is uniquely determined.

Proof. Observe that all considered bundles admit exactly six global sections. Hence, these global sections are restrictions of global sections of appropriate universal bundles on $G(3, 6)$. It follows that the zero loci of these sections are intersection of the image of $S$ by the embedding considered in Theorem 1.1 with some $G(2, 5) \subset G(3, 6)$ (there are two types of such as in Proposition 3.9). By Proposition 3.9 these are linear sections of conic sections or cubic scrolls by space of codimension 2. As both a conic section and a cubic scroll are contained in $G(2, W)$ for some $W$ of dimension 5, and $S$ do not contain neither a conic section nor a twisted cubic curve we get the assertion together with the classification of examples. For the last statement we use Corollary 3.6. □

Proof of Proposition 4.2. Observe first that $S$ does not contain any twisted cubic curve. It follows that $F_1$ and $F_2$ do not contain any common cubic scroll. We prove that $\mathcal{E}_1$ is different from the other bundles, the rest being analogous. Let $D$ be a cubic scroll in $F_1$ from the chosen family corresponding to $\mathcal{E}_1$. Then $D \cap S$ is the zero locus of a section of $\mathcal{E}_1$. By Lemma 4.4 it is a 0-dimensional scheme of length 3 which is contained in $G(2, W)$ for a unique $W \subset V$ of dimension 5. We claim that it is not the zero locus of any section of the remaining bundles. This is a consequence of the fact that $D$ is contained neither in $F_2$ nor in the other family of scrolls on $F_1$ and Corollary 3.7 and proves that the bundles are not isomorphic. □

5. Geometry and general construction

In this section we shall give a geometric construction of $F \subset G(3, 6)$. Let us start by analyzing possible constructions coming from geometry of the variety $G_2$. We shall describe the space spanned by $G_2$ in terms of quadrics containing $G(2, V)$, for a 7-dimensional vector space $V$. By [15, prop. 1.3.] the linear system of quadrics vanishing on $G(2, V)$ is naturally isomorphic to $P^*(\wedge^4 V)$. The orbits on this space under the natural $SL(7)$ action are described in [2].

Lemma 5.1. There is a 7-dimensional vector space of quadrics in the ideal of the Grassmannian $G(2, V)$ vanishing on the linear span of $G_2$, and all of them have rank 12. Moreover every variety which is a linear section of $G(2, V)$ with this property is isomorphic to $G_2$.

Proof. Let $\omega \in \wedge^4 V$ be a 4-form defining $G_2 = \{ \alpha \in G(2, W) : \alpha \wedge \omega = 0 \} \subset \mathbb{P}(\wedge^4 V)$, and let $\omega^*$ be the corresponding dual 3-form. The forms $\beta \in \wedge^4 V^*$ define...
quadric forms $q_3$ that vanish on $G(2, V)$ in the following sense $q_3(\alpha) = \beta(\alpha \wedge \alpha)$. We easily check that the quadrics vanishing on the span of $G$ of rank 12 correspond to 4-forms $\omega$ with linear forms. Dually, this wedge product corresponds to reductions of $\omega$ be the linear coordinates of $V$.

To prove the second assertion, let us consider a 7-dimensional linear space $L$ of quadrics of rank 12. We shall prove that there exists a form $\omega$ such that its reductions modulo the coordinates generate $L$. Observe that the seven quadrics of rank 12 correspond to 4-forms $x_1 \wedge \omega_1, \ldots, x_7 \wedge \omega_7$ for some $x_1, \ldots, x_7 \in V, \omega_1, \ldots, \omega_7 \in \bigwedge^3 V$. From the assumption we have that for any $\lambda_1, \ldots, \lambda_7 \in \mathbb{C}$ there exist $x_8 \in V, \omega_8 \in \bigwedge^3 V$ such that:

$$\lambda_1 x_1 \wedge \omega_1 + \cdots + \lambda_7 x_7 \wedge \omega_7 = x_8 \wedge \omega_8.$$ 

Observe moreover that no two forms corresponding to quadrics in $L$ have the same linear entry in the above description. Indeed, the line joining two forms $v \wedge \omega_1$ wedge $v \wedge \omega_2$ cuts the set corresponding to quadrics of smaller rank. Hence $x_1, \ldots, x_7$ form a basis of $V$. We can then easily prove by induction that the above conditions imply that there exits a form $\omega$ such that $x_i \wedge \omega_i = x_i \wedge \omega$ for all $i \in \{1, \ldots, 7\}$. □

It follows that $G_2$ is contained in a linear subspace in each one of the above quadrics. Note that a rank 12-quadric in $\mathbb{P}^{20}$ has two families of maximal dimensional linear spaces of dimension 14.

We may now formulate the following statement.

**Proposition 5.2.** Let $Q$ be a generic quadric of rank 12 containing the Grassmanian $G(2, V)$. Then there are two isomorphic families of 14-dimensional projective spaces contained in $Q$, say $\mathcal{F}$ and $\mathcal{G}$, such that the following holds:

- If $R \in \mathcal{F}$, then $R \cap G(2, W)$ is linearly isomorphic to $G_2 \cup D$, where $D$ is the intersection of the singular locus of $Q$ with $G(3, W)$ and is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$.
- If $R \in \mathcal{G}$, then $R \cap G(2, W)$ is linearly isomorphic to a Fano fivefold of degree 24 containing $D$.

**Proof.** By [13] prop. 1.3.] the quadrics containing $G(2, V)$ correspond to 3-forms on $V$. By [2] fig. 1] the generic quadric of rank 12 (i.e. $Q$) corresponds in a suitable coordinate system $v_1, \ldots, v_6$ to the 3 form $v_1 \wedge v_2 \wedge v_3 + v_4 \wedge v_5 \wedge v_6$. We can then recover the equation of $Q$ and check directly that the singular locus meets $G(2, V)$ in a fourfold $Z$ linearly isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$. Now, let $R$ be a generic (from one of the two families) 14-dimensional projective space contained in $Q$ and let $T$ be a generic 15-dimensional space containing it that intersects $G(3, V)$ properly. Then $T$ intersects $Q$ in $R$ and another linear space $R'$, and the intersection of $T$ with $G(3, V)$ has two components $X_1 \subset R$ and $X_2 \subset R'$, both of dimension 5. The union $X_1 \cup X_2$ has degree 42. Observe that $Z$ is contained in exactly one of the two varieties $X_1$ and $X_2$ say $X_2$ and in both $R$ and $R'$. Now $X_1 \cap Z = X_1 \cap X_2 \cap Z$ is contained in a linear section of $Z$, hence $X_1$ does not span the whole space $R$, i.e. it is contained in a 13-dimensional linear subspace. Moreover $R \cap R' = (X_1 \cap X_2) \cup Z$ is a hyperplane section of $X_2$ and $X_1 \cap X_2$ is a hyperplane section of $X_1$. It follows that $\deg(X_1) = 18$ and $\deg(X_2) = 24$. Using the standard example we have $X_1$ is smooth. Finally we get by adjunction that $K_{X_1} = -3H$. It is therefore a Fano 5-fold of index 3, degree 18 and Picard number 1. By the theorem of Mukai (see [7] thm. 5.2.3]) it is a section of $G_2$ hence is isomorphic to $G_2$. □

**Remark 5.3.** An alternative way of proving the above proposition is to perform a dimension count. Indeed, we can set up an incidence relation containing pairs each consisting of a quadric and a $P^{13}$ contained in it meeting $G(2, 7)$ in a variety.
linearly isomorphic to $G_2$. The family of quadrics of rank 12 is 25 dimensional (see [2] fig. 1) each contains a 15-dimensional family of 14-dimensional subspaces, hence the incidence has dimension 40. From the other side let us start with $G_2$. In $G(2,W)$ there is a 34-dimensional family of varieties linearly isomorphic to $G_2$ (parameterized by 3-forms on $W$). Now each of them is contained in a 6-dimensional family of quadrics which is in the incidence. It follows that each variety obtained in the construction is linearly isomorphic to $G_2$.

Let us now pass to the description of $F$ in $G(3, U)$. We shall use the following.

**Lemma 5.4.** All nontrivial quadrics containing $< F > \cup G(3, U)$ are of rank 12.

**Proof.** It is easy to see that there is a surjective map from $U \otimes U^*$ to the space of quadrics generating $G(3, U)$. The map is given by: $U \otimes U^* \ni u \otimes v \mapsto \langle \Lambda^3 U \ni \alpha \mapsto (\alpha \wedge u)(v) \wedge \alpha \in \Lambda^6 U \rangle \in S^2(\Lambda^3(U))$. We check directly in the introduced basis for the constructed family of examples that for each of the space of matrices corresponding to the space of quadrics containing $F^3 = H^3 \cap G(3, U)$ is as follows:

\[
\begin{pmatrix}
(\lambda + 1)B & 0 & -\lambda(\lambda + 1)F & (\lambda + 1)E & -\lambda D & -C \\
0 & (\lambda + 1)B & 0 & A & 0 & -G \\
D & 0 & 0 & 0 & 0 & -E \\
-(\lambda + 1)G & (\lambda + 1)C & 0 & 0 & \lambda A & -\lambda F \\
-F & 0 & 0 & -C & \lambda B & 0 \\
-A & (\lambda + 1)E & \lambda(\lambda + 1)G & \lambda D & 0 & \lambda B
\end{pmatrix}
\]

Then we check possible eigenforms to see that they all correspond to rank 12 quadrics. 

It follows that we have a 7-dimensional space of quadrics of rank 12 containing $< F > \cup G(3, U)$. We also observe that the generic rank 12 quadric corresponds to a matrix with diagonal eigenform and three (i.e. three pairs of) distinct eigenvalues. It follows that the singular set of such a quadric meets the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Now we can copy the proof of Proposition 5.2 in this context using our constructed example instead of the standard description of $G_2$. This proves the following.

**Proposition 5.5.** Let $Q$ be a generic quadric of rank 12 containing the Grassmannian $G(3, U)$. Then there are two isomorphic families of 13-dimensional projective spaces contained in $Q$, say $F$ and $G$ such that the following holds:

- If $R \in F$, then $R \cap G(3, U)$ is linearly isomorphic to $F \cup D$, where $D$ is the intersection of the singular locus of $Q$ with $G(3, W)$ and is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.
- If $R \in G$, then $R \cap G(2, W)$ is linearly isomorphic to a Fano threefold of degree 24 containing $D$.

**Remark 5.6.** Performing a dimension count in this context we get a two parameter subgroup of $SL(6)$ acting on $F$. Indeed, let us first compute the dimension of the family of quadrics of rank 12 containing $G(3, W)$. The generic such quadric has a singular $\mathbb{P}^8$ meeting $G(3, 6)$ in a $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ corresponding to three skew lines in $W$. Triples of such lines are parameterized by a 24-dimensional space. We easily check (by Macaulay 2) that the projection of $G(3, 6)$ from such a $\mathbb{P}^8$ is a complete intersection of two quadrics. It follows that we have a 25-dimensional family of quadrics. Each considered quadric has a 15-dimensional (two components) family of 13-dimensional projective spaces. Now each smooth Fano fourfold of genus 10 and index 2 contained in $G(3, W)$ is contained in a 6-dimensional space of quadrics. As there is a one-parameter family of such Fano varieties the group $SL(6)$ acting on $G(3, 6)$ makes a 33-dimensional family of varieties linearly isomorphic to $F$. 
Corollary 5.7. The projections from the span of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset G(3, W)$ and from the span of $\mathbb{P}^2 \times \mathbb{P}^2 \subset G(2, V)$ define birational maps from $F$ to quadrics in $\mathbb{P}^5$.

Proof. We check directly, with Macaulay 2 [9], that the projection from $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ maps $G(3, W)$ birationally to a complete intersection of a pencil of quadrics in $\mathbb{P}^{11}$, and likewise that the projection of $G(2, V)$ from $\mathbb{P}^2 \times \mathbb{P}^2$ defines a rational fibration of $G(2, V)$ over a complete intersection of a pencil of quadrics in $\mathbb{P}^{11}$. From the constructions above it follows that the corresponding $F$ and $G_2$ map through these maps to quadrics in $\mathbb{P}^5$. The assertion follows. \hfill $\Box$

Remark 5.8. The two pencils of quadrics appearing in the above proof are distinct pencils of quadrics in $\mathbb{P}^{11}$. The image of the projection of $G(3, 6)$ is defined by a pencil of quadrics of rank 12 with three degenerate elements of rank 8, whereas the pencil of quadrics defining the image of the projection of $G(2, V)$ has two degenerate elements of rank 6.

6. The moduli space

In this section we construct the announced moduli spaces. We start with a generic $K3$ surface $(S, L)$ of genus 10. Then $S$ is a proper section of $G_2$ by a 10-dimensional linear space. The variety $G_2$ spans a 13-dimensional space $R$, hence there is a plane $\Pi_S \subset R^*$ of hyperplanes of $R$ containing $S$. By Theorem 5.4 a generic hyperplane $H \in \Pi_S$, outside a smooth sextic curve, corresponds each to a unique embedding of $S$ in $G(3, U)$. Such an embedding gives rise to two stable and distinct bundles of rank 3 with a 6-dimensional vector space of sections and determinant $L$ by Proposition 5.5 and Proposition 5.2 respectively. Since $L$ is nondivisible every semi-stable bundle in $M_S(3, L, 3)$ is stable, therefore, by [13], the moduli space $M_S(3, L, 3)$ is a smooth $K3$ surface. By our Proposition 3.5 and Proposition 4.2 the family of bundles we obtain is a 2-dimensional algebraic family of pairwise non-isomorphic bundles. It follows that this family of bundles is an open subset of the $K3$ surface $M_S(3, L, 3)$. We constructed a rational map from this surface to $\mathbb{P}^2$ which is a two-to-one morphism over the complement of a smooth sextic. It follows that the $K3$ surface is a double cover of $\Pi_S \cap G_2$, where $G_2$ denotes the dual hypersurface to $G_2$. Indeed, by [3] the generic point on the sextic dual to $G_2$ corresponds to a nodal hyperplane section which admits a unique projection from its node to $LG(3, 6)$. It follows that for a chosen $S$ the sextic $\Pi_S \cap G_2$ parameterizes bundles which are pullbacks by these projections of the universal quotient bundle on $LG(3, 6)$. Notice that on $LG(3, 6)$ the universal quotient bundle is isomorphic to the dual of the universal sub-bundle.

For a general Fano 3-fold $(X, L)$ of genus 10, we do not know a general structure theorem for the moduli space of stable bundles, so we do not aim for a complete description in our case. Still, in analogy with the surface case we get an irreducible component of the moduli space. We start with $X$ as a proper section of $G_2$ by a 11-dimensional linear space. As above, the pencil of hyperplane sections $F$ of $G_2$ that contains $X$ gives rise to a pencil of pairs of rank 3 vector bundles in $M_X(3, L, \sigma, 2)$ that are pairwise distinct. For the six singular hyperplane sections in the pencil, the two bundles coincide, so we get a complete family of vector bundles. To complete the proof of [13] we need only show that this family is dense in its closure. But for each bundle $E$ in the family the natural map $\phi_E : \wedge^3 H^0(X, E) \rightarrow H^0(X, L)$ is surjective, and for the general $E$ in the family, $\phi_E$ even induces an embedding of $X$ into $G(3, 6)$. These two properties clearly define open conditions on the moduli space $M_X(3, L, \sigma, 2)$. So it suffices to show that if $E$ is a bundle in $M_X(3, L, \sigma, 2)$, such that $\phi_E$ is surjective and defines an embedding of $X$ into $G(3, 6)$, then $E$ is the restriction of a bundle on a Fano 4-fold $F$ that contains $X$. By abuse of notation.
we denote again by $X$ the image of the embedding into $G(3, 6)$. Then $E$ is the restriction to $X$ of the universal quotient bundle on $G(3, 6)$. Let $S$ be a generic hyperplane section of $X$. By the above, the surface $S$ is the codimension 2 linear section of a unique Fano 4-fold $F$ of genus 10 and index 2 in $G(3, 6)$. Furthermore, this 4-fold $F$ is a complete linear section of $G(3, 6)$ with a $\mathbb{P}^{12}$. We claim that $X$ is a hyperplane section of $F$. Assume it is not. Then the linear span $H_{13} = \langle F \cup X \rangle$ is a $\mathbb{P}^{13}$. Consider a general quadric hypersurface $Q$ in $\mathbb{P}^{13} = \langle G(3, 6) \rangle$ that contains $H_{13} \cup G(3, 6)$. We observed above that all quadrics containing $\langle F \cup G(3, 6) \rangle$ are of rank 12. Since $H_{13} = \langle F \cup X \rangle$ contains $\langle F \rangle$ we conclude that $Q$ has rank 12. Let $H_{14}$ be a general $\mathbb{P}^{14}$ that contains $H_{13}$. Then $H_{14} \cap Q$ splits into the union of $H_{13}$ and another projective space $H_{13}$ of dimension 13. By genericity $H_{13}$ does not contain $X$: The singular locus of $Q$ is the base locus of the system of $H'_{13} \cap Q$ obtained as $H_{14}$ varies. This singular locus $\text{Sing}(Q)$ is a linear space of dimension 7, and hence does not contain $X$. Now, every component of $H_{14} \cap G(3, 6)$ has dimension at least 4, so every component of $H_{13} \cap G(3, 6)$ which is not contained in $\text{Sing}(Q)$ must also have dimension at least 4. On the other hand $F$ is a complete hyperplane section of $H_{13} \cap G(3, 6)$, so the latter have dimension at most 5. This leaves only two possibilities:

- $H_{13} \cap G(3, 6)$ is a 5-fold $F_5$. Since $F$ is smooth, it can only have isolated singularities. Then a generic hyperplane section of $F_5$ containing $X$ is also a smooth Fano 4-fold $F'$ of degree 18 and index 2. But $S$ is contained in $X$ and therefore also in $F'$, contrary to the unicity of $F$.

- $H_{13} \cap G(3, 6)$ is a 4-fold with at least two components, each of degree 18 and none containing $\text{Sing}(Q) \cap G(3, 6)$, which is a threefold in a $\mathbb{P}^{7}$. It follows that the generic element of the system $H'_{13}$ intersects $G(3, 6)$ in a 4-fold of degree $\leq 42 - 36 = 6$, so the generic element of the system $H_{13}$ in $Q$ intersects $G(3, 6)$ in a 4-fold of degree $\geq 36$. Consider the intersection $H_{13} \cap H'_{13} \cap G(3, 6)$. It is a hyperplane section of both $H_{13} \cap G(3, 6)$ and $H'_{13} \cap G(3, 6)$. Then $H'_{13} \cap G(3, 6)$ must also have degree at least 36 which gives a contradiction.

This concludes the proof of Theorem 1.3

We end this section with the proof of Corollary 1.4, a corollary of our study of the Hilbert scheme of conic sections in a different direction. Consider first $S^{[2]} = \text{Hilb}^2 S$, the Hilbert scheme of pairs of points on a general K3 surface section of $G_2$. We may assume that $S$ contains no lines, so by Lemma 5.4 there is a unique conic section through any pair of, possibly infinitesimally close, points. We may also assume that $S$ contains no conic sections, so each conic section lies in a unique Fano 3-fold section $X$ of $G_2$ that contains $S$. Thus we obtain a morphism $S^{[2]} \to \mathbb{P}^2$. The fiber $F_X$ over $X$ is precisely the Hilbert schemes of conic sections contained in $X$. Sawon proved, [22] Theorem 2], that $S^{[2]}$ has a Lagrangian fibration to $\mathbb{P}^2$. Markushevich identified the target $\mathbb{P}^2$ with the linear system of the ample generator of the Picard group on $M_S(3, L, 3)$. He proved, [10] Theorem 4.3], that the general fiber is the Jacobian of the corresponding curve in the linear system. In our case the linear system is $|M_X|$, so we conclude that the Hilbert scheme of conics sections on $X$ coincides with the Jacobian of the genus 2 curve $M_X$.

**Acknowledgements**

The work was done during the first author’s stay at the University of Oslo between March 2009 and March 2010. We would like to thank G. Kapustka, J. Weyman, A. Kuznetsov, F. Han, S. Mukai, J. Buczyński, D. Anderson, and L. Manivel for discussions and remarks.
References

[1] William Fulton, Joe Harris, *Representation theory, a first course*, GTM 129, Springer-Verlag, New York 1991.
[2] Hirotachi Abo, Giorgio Ottaviani, Chris Peterson, *Non-Defectivity of Grassmannians of planes*, preprint 2009, [arXiv:0901.2601v1 [math.AG]]
[3] Michał Kapustka, *Relation between equations of Mukai varieties*, preprint 2009, [arXiv:1005.5557v2 [math.AG]]
[4] Michał Kapustka, Macaulay 2 scripts for “Vector bundles on Fano varieties of genus 10”, available at [http://folk.uio.no/ranestad/papers.html](http://folk.uio.no/ranestad/papers.html)
[5] F. Holweck, *Lieu singulier des variétés duales: approche géométrique et applications aux variétés homogènes*. Ph. D thesis, Université Paul Sabatier, Toulouse 2004.
[6] Atanas Iliev, Kristian Ranestad, *The abelian fibration on the Hilbert cube of a K3 surface of genus 9*. International Journal of Mathematics, 18, 1, (2007), 1-26
[7] Vasili Alekseevich Iskovskikh, Yuri Prokhorov, *Fano varieties*. Algebraic geometry, V, 1-247, Encyclopaedia Math. Sci., 47, Springer, Berlin, 1999.
[8] A. G. Kuznetsov, *Hyperplane sections and derived categories*. Izvestiya: Mathematics 70:3 (2006), 447-547.
[9] Dan R. Grayson and Mike E. Stillman: *Macaulay 2, a software system for research in algebraic geometry*. Available at [http://www.math.uiuc.edu/Macaulay2/](http://www.math.uiuc.edu/Macaulay2/)
[10] D. Markushevich: *Rational Lagrangian fibration on punctual Hilbert schemes of K3 surfaces*, Manuscripta Math. 120, 2, (2006) 131–150
[11] Shigeru Mukai, *Biregular classification of Fano 3-folds and Fano manifolds of coindex 3*. Proc. Matl. Acad. Sci. USA, 86, (1989) 3000-3002.
[12] Shigeru Mukai, *Fano 3-folds*. London Math. Soc. Lect. Note Ser. 179, (1992) 255-263
[13] Shigeru Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, Invent. math. 77 (1984), 101-116.
[14] Shigeru Mukai, *Curves, K3 surfaces and Fano 3-folds of genus ≤ 10*, in Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata, pp. 357-377, Kinokuniya, Tokyo, 1988.
[15] Shigeru Mukai, *Curves and Grassmannians*, Inchoen, Korea (Algebraic Geometry and related Topics, pp.19-40, International Press, 1993, Cambridge, MA
[16] Shigeru Mukai, *Non-abelian Brill-Noether theory and Fano 3-folds* [translation of Sugaku 49 (1997), no. 1, 1–24]. Sugaku Expositions 14 (2001), no. 2, 125–153.
[17] Kristian Ranestad, *Non-abelian Brill-Noether Loci and the Lagrangian Grassmannian LG(3,6)*
[18] Kristian Ranestad, *Geometry of the Lagrangian Grassmannian LG(3,6) and Applications to Brill-Noether Loci*, Michigan Math. J. 53 (2005), 383–417
[19] Kristian Ranestad, Frank-Olaf Schreyer, *Varieties of sums of powers*, J. reine angew. Math. 525 (2000), 147–181.
[20] Miles Reid, Stavros Papadakis, *Kustin-Miller unprojection without complexes*, J. Algebraic Geometry 13 (2004) 563-577.
[21] M.Sato, T. Kimura, *A classification of irreducible prehomogeneous spaces and their relative invariants*, Nagoya Math. J. 65 (1977) 1-155.
[22] Justin Sawon, *Lagrangian fibrations on Hilbert schemes of points on K3 surfaces*, Journal of Algebraic Geometry 16 (2007) 477-497.
[23] Evgenii A. Tevelev, *Projective duality and Homogeneous Spaces*, Encyclopaedia of Mathematical Sciences, 133. Invariant Theory and Algebraic Transformation Groups, IV. Springer-Verlag, Berlin, 2005.

Institute of Mathematics, Jagiellonian University of Kraków, ul.Łojasiewicza 6, 30-348 Kraków, Poland
E-mail address: Michał.Kapustka@im.uj.edu.pl

Department of Mathematics, University of Oslo, PB 1053 Blindern, 0316 Oslo, Norway
E-mail address: ranestad@math.uio.no