The Nonparanormal: Semiparametric Estimation of High Dimensional Undirected Graphs

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Abstract

Recent methods for estimating sparse undirected graphs for real-valued data in high dimensional problems rely heavily on the assumption of normality. We show how to use a semiparametric Gaussian copula—or “nonparanormal”—for high dimensional inference. Just as additive models extend linear models by replacing linear functions with a set of one-dimensional smooth functions, the nonparanormal extends the normal by transforming the variables by smooth functions. We derive a method for estimating the nonparanormal, study the method’s theoretical properties, and show that it works well in many examples.

Keywords: Graphical models, Gaussian copula, high dimensional inference, sparsity, $\ell_1$ regularization, graphical lasso, paranormal, occult
I. INTRODUCTION

The linear model is a mainstay of statistical inference that has been extended in several important ways. An extension to high dimensions was achieved by adding a sparsity constraint, leading to the lasso (Tibshirani, 1996). An extension to nonparametric models was achieved by replacing linear functions with smooth functions, leading to additive models (Hastie and Tibshirani, 1999). These two ideas were recently combined, leading to an extension called sparse additive models (SpAM) (Ravikumar et al., 2008b,a). In this paper we consider a similar nonparametric extension of undirected graphical models based on multivariate Gaussian distributions in the high dimensional setting. Specifically, we use a high dimensional Gaussian copula with nonparametric marginals, which we refer to as a nonparanormal distribution.

If $X$ is a $p$-dimensional random vector distributed according to a multivariate Gaussian distribution with covariance matrix $\Sigma$, the conditional independence relations between the random variables $X_1, X_2, \ldots, X_p$ are encoded in a graph formed from the precision matrix $\Omega = \Sigma^{-1}$. Specifically, missing edges in the graph correspond to zeroes of $\Omega$. To estimate the graph from a sample of size $n$, it is only necessary to estimate $\Sigma$, which is easy if $n$ is much larger than $p$. However, when $p$ is larger than $n$, the problem is more challenging. Recent work has focused on the problem of estimating the graph in this high dimensional setting, which becomes feasible if $G$ is sparse. Yuan and Lin (2007) and Banerjee et al. (2008) propose an estimator based on regularized maximum likelihood using an $\ell_1$ constraint on the entries of $\Omega$, and Friedman et al. (2007) develop an efficient algorithm for computing the estimator using a graphical version of the lasso. The resulting estimation procedure has excellent theoretical properties, as shown recently by Rothman et al. (2008) and Ravikumar et al. (2009).

While Gaussian graphical models can be useful, a reliance on exact normality is limiting. Our goal in this paper is to weaken this assumption. Our approach parallels the ideas behind sparse additive models for regression (Ravikumar et al., 2008b,a). Specifically, we replace the Gaussian with a semiparametric Gaussian copula. This means that we replace the random variable $X = (X_1, \ldots, X_p)$ by the transformed random variable $f(X) = (f_1(X_1), \ldots, f_p(X_p))$, and assume that $f(X)$ is multivariate Gaussian. This semiparametric copula results in a nonparametric extension of the normal that we call the nonparanormal distribution. The nonparanormal depends on the functions $\{f_j\}$, and a mean $\mu$ and covariance matrix $\Sigma$, all of which are to be estimated from data. While the resulting family of distributions is much richer than the standard parametric normal (the paranormal), the independence relations among the variables are still encoded in the precision matrix $\Omega = \Sigma^{-1}$. We propose a nonparametric estimator for the functions $\{f_j\}$, and show how the graphical lasso can be used to estimate the graph in the high dimensional setting. The relationship between linear
| Assumptions   | Dimension | Regression          | Graphical Models                 |
|--------------|-----------|---------------------|----------------------------------|
| parametric   | low       | linear model        | multivariate normal              |
|              | high      | lasso               | graphical lasso                  |
| nonparametric| low       | additive model      | nonparanormal                    |
|              | high      | sparse additive model | $\ell_1$-regularized nonparanormal |

Figure 1: Comparison of regression and graphical models. The nonparanormal extends additive models to the graphical model setting. Regularizing the inverse covariance leads to an extension to high dimensions, which parallels sparse additive models for regression.

Most theoretical results on semiparametric copulas focus on low or at least finite dimensional models (Tsukahara, 2005). Models with increasing dimension require a more delicate analysis; in particular, simply plugging in the usual empirical distribution of the marginals does not lead to accurate inference. Instead we use a truncated empirical distribution. We give a theoretical analysis of this estimator, proving consistency results with respect to risk, model selection, and estimation of $\Omega$ in the Frobenius norm.

In the following section we review the basic notion of the graph corresponding to a multivariate Gaussian, and formulate different criteria for evaluating estimators of the covariance or inverse covariance. In Section 3 we present the nonparanormal, and in Section 4 we discuss estimation of the model. We present a theoretical analysis of the estimation method in Section 5, with the detailed proofs collected in an appendix. In Section 6 we present experiments with both simulated data and gene microarray data, where the problem is to construct the isoprenoid biosynthetic pathway.

**II. ESTIMATING UNDIRECTED GRAPHS**

Let $X = (X_1, \ldots, X_p)$ denote a random vector with distribution $P = N(\mu, \Sigma)$. The undirected graph $G = (V, E)$ corresponding to $P$ consists of a vertex set $V$ and an edge set $E$. The set $V$ has $p$ elements, one for each component of $X$. The edge set $E$ consists of ordered pairs $(i, j)$ where $(i, j) \in E$ if there is a edge between $X_i$ and $X_j$. The edge between $(i, j)$ is excluded from $E$ if and only if $X_i$ is independent of $X_j$ given the other variables $O \setminus \{i, j\} \equiv \{X_s : 1 \leq s \leq p, \ s \neq i, j\}$, written

$$X_i \perp\!
\!
\!
\!
\!
\!
\perp X_j \mid O \setminus \{i, j\}.$$  \hspace{1cm} (2.1)

3
It is well-known that, for multivariate Gaussian distributions, (2.1) holds if and only if \( \Omega_{ij} = 0 \) where \( \Omega = \Sigma^{-1} \).

Let \( X^{(1)}, X^{(2)}, \ldots, X^{(n)} \) be a random sample from \( P \), where \( X^{(i)} \in \mathbb{R}^p \). If \( n \) is much larger than \( p \), then we can estimate \( \Sigma \) using maximum likelihood, leading to the estimate \( \hat{\Omega} = S^{-1} \), where

\[
S = \frac{1}{n} \sum_{i=1}^{n} (X^{(i)} - \bar{X}) (X^{(i)} - \bar{X})^T
\]

is the sample covariance, with \( \bar{X} \) the sample mean. The zeroes of \( \Omega \) can then be estimated by applying hypothesis testing to \( \hat{\Omega} \) (Drton and Perlman, 2007, 2008).

When \( p > n \), maximum likelihood is no longer useful; in particular, the estimate \( \hat{\Sigma} \) is not positive definite, having rank no greater than \( n \). Inspired by the success of the lasso for linear models, several authors have suggested estimating \( \Sigma \) by minimizing

\[
-\ell(\mu, \Omega) + \lambda \sum_{j \neq k} |\Omega_{jk}|
\]

where

\[
\ell(\mu, \Omega) = \frac{1}{2} \left( \log |\Omega| - \text{tr}(\Omega S) - p \log(2\pi) \right)
\]

is the average log-likelihood and \( S \) is the sample covariance matrix. The estimator \( \hat{\Omega} \) can be computed efficiently using the glasso algorithm (Friedman et al., 2007), which is a block coordinate descent algorithm that uses the standard lasso to estimate a single row and column of \( \Omega \) in each iteration. Under appropriate sparsity conditions, the resulting estimator \( \hat{\Omega} \) has been shown to have good theoretical properties (Rothman et al., 2008; Ravikumar et al., 2009).

There are several different ways to judge the quality of an estimator \( \hat{\Sigma} \) of the covariance or inverse covariance \( \hat{\Omega} \). We discuss three in this paper, persistency, norm consistency, and sparsistency. Persistency means consistency in risk, when the model is not assumed to be correct. Suppose the true distribution is \( P \) has mean \( \mu_0 \), and that we use a multivariate normal \( p(x; \mu_0, \Sigma) \) for prediction. We do not assume that \( P \) is normal. We observe a new vector \( X \sim P \) and define the prediction risk to be

\[
R(\Sigma) = -\mathbb{E} \log p(X; \mu_0, \Sigma) = -\int \log p(x; \mu_0, \Sigma) dP(x).
\]

It follows that

\[
R(\Sigma) = \frac{1}{2} \left( \text{tr}(\Sigma^{-1} \Sigma_0) + \log |\Sigma| - p \log(2\pi) \right)
\]

where \( \Sigma_0 \) is the covariance of \( X \) under \( P \). If \( S \) is a set of covariance matrices, the oracle is defined to be the covariance matrix \( \Sigma_* \) that minimizes \( R(\Sigma) \) over \( S \):

\[
\Sigma_* = \arg\min_{\Sigma \in S} R(\Sigma).
\]
Thus $p(x; \mu_0, \Sigma_*)$ is the best predictor of a new observation among all distributions in $\{p(x; \mu_0, \Sigma) : \Sigma \in S\}$. In particular, if $S$ consists of covariance matrices with sparse graphs, then $p(x; \mu_0, \Sigma_*)$ is, in some sense, the best sparse predictor. An estimator $\hat{\Sigma}_n$ is persistent if

$$R(\hat{\Sigma}_n) - R(\Sigma_*) \xrightarrow{P} 0$$

as the sample size $n$ increases to infinity. Thus, a persistent estimator approximates the best estimator over the class $S$, but we do not assume that the true distribution has a covariance matrix in $S$, or even that it is Gaussian. Moreover, we allow the dimension $p = p_n$ to increase with $n$. On the other hand, norm consistency and sparsistency require that the true distribution is Gaussian. In this case, let $\Sigma_0$ denote the true covariance matrix. An estimator is norm consistent if

$$\|\hat{\Sigma}_n - \Sigma\| \xrightarrow{P} 0$$

where $\| \cdot \|$ is a norm. If $E(\Omega)$ denotes the edge set corresponding to $\Omega$. An estimator is sparsistent if

$$\mathbb{P}(E(\Omega) \neq E(\hat{\Omega}_n)) \rightarrow 0.$$

Thus, a sparsistent estimator identifies the correct graph consistently. We summarize known results on these properties for the multivariate normal in Section 5, before presenting our theoretical analysis of the nonparanormal.

### III. The Nonparanormal

We say that a random vector $X = (X_1, \ldots, X_p)^T$ has a nonparanormal distribution if there exist functions $\{f_j\}_{j=1}^p$ such that $Z \equiv f(X) \sim N(\mu, \Sigma)$, where $f(X) = (f_1(X_1), \ldots, f_p(X_p))$. We then write

$$X \sim NPN(\mu, \Sigma, f).$$

When the $f_j$’s are monotone and differentiable, the joint probability density function of $X$ is given by

$$p_X(x) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} (f(x) - \mu)^T \Sigma^{-1} (f(x) - \mu) \right\} \prod_{j=1}^p |f_j'(x_j)|. \quad (3.1)$$

**Lemma 3.1.** The nonparanormal distribution $NPN(\mu, \Sigma, f)$ is a Gaussian copula when the $f_j$’s are monotone and differentiable.

**Proof.** By Sklar’s theorem (Sklar, 1959), any joint distribution can be written as

$$F(x_1, \ldots, x_p) = C\{F_1(x_1), \ldots, F_p(x_p)\}$$
where the function $C$ is called a copula. For the nonparanormal we have

$$F(x_1, \ldots, x_p) = \Phi_{\mu, \Sigma}(\Phi^{-1}(F_1(x_1)), \ldots, \Phi^{-1}(F_p(x_p)))$$

where $\Phi_{\mu, \Sigma}$ is the multivariate Gaussian cdf and $\Phi$ is the univariate standard Gaussian cdf. Thus, the corresponding copula is

$$C(u_1, \ldots, u_p) = \Phi_{\mu, \Sigma}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_p)).$$

This is exactly a Gaussian copula with parameters $\mu$ and $\Sigma$. If each $f_j$ is differentiable then the density of $X$ has the same form as (3.1).

Note that the density in (3.1) is not identifiable; to make the family identifiable we demand that $f_j$ preserve means and variances:

$$\mu_j = \mathbb{E}(Z_j) = \mathbb{E}(X_j) \quad \text{and} \quad \sigma^2_j \equiv \Sigma_{jj} = \text{Var}(Z_j) = \text{Var}(X_j). \quad (3.2)$$

Note that these conditions only depend on diag($\Sigma$) but not the full covariance matrix.

Let $F_j(x)$ denote the marginal distribution function of $X_j$. Then

$$F_j(x) = \mathbb{P}(X_j \leq x) = \mathbb{P}(Z_j \leq f_j(x)) = \Phi\left(\frac{f_j(x) - \mu_j}{\sigma_j}\right)$$

which implies that

$$f_j(x) = \mu_j + \sigma_j \Phi^{-1}(F_j(x)).$$

The following basic fact says that the independence graph of the nonparanormal is encoded in $\Omega = \Sigma^{-1}$, as for the parametric normal.

**Lemma 3.2.** If $X \sim NPN(\mu, \Sigma, f)$ is nonparanormal and each $f_j$ is differentiable, then $X_i \perp X_j | O_{\{i,j\}}$ if and only if $\Omega_{ij} = 0$, where $\Omega = \Sigma^{-1}$.

**Proof.** From the form of the density (3.1), it follows that the density factors with respect to the graph of $\Omega$, and therefore obeys the global Markov property of the graph.

Next we show that the above is true for any choice of identification restrictions.

**Lemma 3.3.** Define

$$h_j(x) = \Phi^{-1}(F_j(x)) \quad (3.3)$$

and let $\Lambda$ be the covariance matrix of $h(X)$. Then $X_j \perp X_k | O_{\{j,k\}}$ if and only if $\Lambda_{jk}^{-1} = 0$. 

6
Figure 2: Densities of three 2-dimensional nonparanormals. The component functions have the form \( f_j(x) = \text{sign}(x)|x|^{\alpha_j} \). Left: \( \alpha_1 = 0.9, \alpha_2 = 0.8 \); center: \( \alpha_1 = 1.2, \alpha_2 = 0.8 \); right \( \alpha_1 = 2, \alpha_2 = 3 \). In each case \( \mu = (0, 0) \) and \( \Sigma = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix} \).

**Proof.** We can rewrite the covariance matrix as

\[
\Sigma_{jk} = \text{Cov}(Z_j, Z_k) = \sigma_j \sigma_k \text{Cov}(h_j(X_j), h_k(X_k)).
\]

Hence \( \Sigma = D\Lambda D \) and

\[
\Sigma^{-1} = D^{-1}\Lambda^{-1}D^{-1}.
\]

where \( D \) is the diagonal matrix with \( \text{diag}(D) = \sigma \). The zero pattern of \( \Lambda^{-1} \) is therefore identical to the zero pattern of \( \Sigma^{-1} \).

Thus, it is not necessary to estimate \( \mu \) or \( \sigma \) to estimate the graph.

Figure 2 shows three examples of 2-dimensional nonparanormal densities. In each case, the component functions \( f_j(x) \) take the form

\[
f_j(x) = a_j \text{sign}(x)|x|^{\alpha_j} + b_j
\]
where the constants $a_j$ and $b_j$ are set to enforce the identifiability constraints (3.2). The covariance in each case is $\Sigma = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}$ and the mean is $\mu = (0, 0)$. The exponent $\alpha_j$ determines the nonlinearity. It can be seen how the concavity of the density changes with the exponent $\alpha$, and that $\alpha > 1$ can result in multiple modes.

The assumption that $f(X) = (f_1(X_1), \ldots, f_p(X_p))$ is normal leads to a semiparametric model where only one dimensional functions need to be estimated. But the monotonicity of the functions $f_j$, which map onto $\mathbb{R}$, enables computational tractability of the nonparanormal. For more general functions $f$, the normalizing constant for the density

$$p_X(x) \propto \exp \left\{ -\frac{1}{2} (f(x) - \mu)^T \Sigma^{-1} (f(x) - \mu) \right\}$$

cannot be computed in closed form.

**IV. Estimation Method**

Let $X^{(1)}, \ldots, X^{(n)}$ be a sample of size $n$ where $X^{(i)} = (X_1^{(i)}, \ldots, X_p^{(i)}) \in \mathbb{R}^p$. In light of (3.3) we define

$$\widehat{h}_j(x) = \Phi^{-1}(\widehat{F}_j(x))$$

where $\widehat{F}_j$ is an estimator of $F_j$. A natural candidate for $\widehat{F}_j$ is the marginal empirical distribution function

$$\widehat{F}_j(t) \equiv \frac{1}{n} \sum_{i=1}^{n} 1\{X_{j}^{(i)} \leq t\}.$$ 

Now, let $\theta$ denote the parameters of the copula. Tsukahara (2005) suggests taking $\widehat{\theta}$ to be the solution of

$$\sum_{i=1}^{n} \phi \left( \widehat{F}_1(X_1^{(i)}), \ldots, \widehat{F}_p(X_p^{(i)}), \theta \right) = 0$$

where $\phi$ is an estimating equation and $\widehat{F}_j(t) = n\widehat{F}_j(t)/(n + 1)$. In our case, $\theta$ corresponds to the covariance matrix. The resulting estimator $\widehat{\theta}$, called a rank approximate Z-estimator, has excellent theoretical properties. However, we are interested in the high dimensional scenario where the dimension $p$ is allowed to increase with $n$; the variance of $\widehat{F}_j(t)$ is too large in this case. Instead, we use the following truncated or Winsorized$^1$ estimator:

$$\widetilde{F}_j(x) = \begin{cases} 
\delta \quad & \text{if } \widehat{F}_j(x) < \delta \\
\widehat{F}_j(x) & \text{if } \delta \leq \widehat{F}_j(x) \leq 1 - \delta \\
1 - \delta & \text{if } \widehat{F}_j(x) > 1 - \delta,
\end{cases} \quad (4.1)$$

$^1$After Charles P. Winsor, whom John Tukey credited with converting him from topology to statistics (Mallows, 1990).
where $\delta_n$ is a truncation parameter. Clearly, there is a bias-variance tradeoff in choosing $\delta_n$. In what follows we use

$$\delta_n \equiv \frac{1}{4n^{1/4} \sqrt{\pi \log n}}. \quad (4.2)$$

This provides the right balance so that we can achieve the desired rate of convergence in our estimate of $\Omega$ and the associated undirected graph $G$.

Given this estimate of the distribution of variable $X_j$, we then estimate the transformation function $f_j$ by

$$\tilde{f}_j(x) \equiv \tilde{\mu}_j + \tilde{\sigma}_j \tilde{h}_j(x) \quad (4.3)$$

where

$$\tilde{h}_j(x) = \Phi^{-1}\left(\tilde{F}_j(x)\right)$$

and $\tilde{\mu}_j$ and $\tilde{\sigma}_j$ are the sample mean and the standard deviation:

$$\tilde{\mu}_j \equiv \frac{1}{n} \sum_{i=1}^{n} X_j^{(i)} \quad \text{and} \quad \tilde{\sigma}_j = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left(X_j^{(i)} - \tilde{\mu}_j\right)^2}.$$ 

Now, let $S_n(\tilde{f})$ be the sample covariance matrix of $\tilde{f}(X^{(1)}), \ldots, \tilde{f}(X^{(n)})$; that is,

$$S_n(\tilde{f}) \equiv \frac{1}{n} \sum_{i=1}^{n} \left(\tilde{f}(X^{(i)}) - \mu_n(\tilde{f})\right) \left(\tilde{f}(X^{(i)}) - \mu_n(\tilde{f})\right)^T \quad (4.4)$$

$$\mu_n(\tilde{f}) \equiv \frac{1}{n} \sum_{i=1}^{n} \tilde{f}(X^{(i)}).$$

We then estimate $\Omega$ using $S_n(\tilde{f})$. For instance, the maximum likelihood estimator is $\widehat{\Omega}_n = S_n(\tilde{f})^{-1}$. The $\ell_1$-regularized estimator is

$$\widehat{\Omega}_n = \arg\min_{\Omega} \left\{ \operatorname{tr} \left(\Omega S_n(\tilde{f})\right) - \log |\Omega| + \lambda \|\Omega\|_1 \right\} \quad (4.5)$$

where $\lambda$ is a regularization parameter, and $\|\Omega\|_1 = \sum_{j\neq k} |\Omega_{jk}|$. The estimated graph is then $\widehat{E}_n = \{(j, k) : \widehat{\Omega}_{jk} \neq 0\}$. In the following section we analyze the theoretical properties of this $\ell_1$-regularized estimator.

V. THEORETICAL RESULTS

In this section we present our theoretical results on risk consistency, model selection consistency, and norm consistency of the covariance $\Sigma$ and inverse covariance $\Omega$. From Lemma 3.3,
the estimate of the graph does not depend on $\sigma_j$, $j \in \{1, \ldots, p\}$ and $\mu$, so we assume that $\sigma_j = 1$ and $\mu = 0$. Our key technical result is an analysis of covariance of the Winsorized estimator defined in (4.1), (4.3), and (4.4). In particular, we show that under appropriate conditions,

$$\max_{j,k} \left| S_n(\tilde{f})_{jk} - S_n(f)_{jk} \right| = O_P \left( \sqrt{\frac{\log p \log^2 n}{n^{1/2}}} \right)$$

where $S_n(\tilde{f})_{jk}$ denotes the $(j, k)$ entry of the matrix. This result allows us to leverage the recent analysis of Rothman et al. (2008) and Ravikumar et al. (2009) in the Gaussian case to obtain consistency results for the nonparanormal. More precisely, our main theorem is the following.

**Theorem 5.1.** Suppose that $p = n^\xi$ and let $\tilde{f}$ be the Winsorized estimator defined in (4.3) with $\delta_n = \frac{1}{4n^{1/4} \sqrt{\pi \log n}}$. Define

$$C(M, \xi) \equiv \frac{48}{\sqrt{\pi \xi}} \left( \sqrt{2M - 1} \right) (M + 2)$$

for $M, \xi > 0$. Then for any $\epsilon \geq C(M, \xi) \sqrt{\frac{\log p \log^2 n}{n^{1/2}}}$ and sufficiently large $n$, we have

$$\mathbb{P} \left( \max_{jk} \left| S_n(\tilde{f})_{jk} - S_n(f)_{jk} \right| > \epsilon \right) \leq \frac{c_1p}{(n\epsilon^2)^{2\xi}} + \frac{c_2p}{n^{M\xi-1}} + c_3 \exp \left( -\frac{c_4n^{1/2}c^2}{\log p \log^2 n} \right),$$

where $c_1, c_2, c_3, c_4$ are positive constants.

The proof of the above theorem is given in Section 7. The following corollary is immediate, which specifies the scaling of the dimension in terms of sample size.

**Corollary 5.2.** Let $M > 1 + \frac{1}{\xi}$. Then

$$\mathbb{P} \left( \max_{jk} \left| S_n(\tilde{f})_{jk} - S_n(f)_{jk} \right| > C(M, \xi) \sqrt{\frac{\log p \log^2 n}{n^{1/2}}} \right) = o(1) + c_3 \exp \left( -c_4C^2(M, \xi) \right).$$

Hence,

$$\max_{j,k} \left| S_n(\tilde{f})_{jk} - S_n(f)_{jk} \right| = O_P \left( \sqrt{\frac{\log p \log^2 n}{n^{1/2}}} \right). \quad (5.1)$$
The following corollary yields estimation consistency in both the Frobenius norm and the \( \ell_2 \)-operator norm. The proof follows the same arguments as the proof of Theorem 1 and Theorem 2 from Rothman et al. (2008), replacing their Lemma 1 with our Theorem 5.1.

For a matrix \( A = (a_{ij}) \), the Frobenius norm \( \| \cdot \|_F \) is defined as \( \| A \|_F \equiv \sqrt{\sum_{i,j} a_{ij}^2} \). The \( \ell_2 \)-operator norm \( \| \cdot \|_2 \) is defined as the magnitude of the largest eigenvalue of the matrix, \( \| A \|_2 \equiv \max_{\| x \|_2 = 1} \| Ax \|_2 \). In the following, we write \( a_n \approx b_n \) if there are positive constants \( c \) and \( C \) independent of \( n \) such that \( c \leq a_n/b_n \leq C \).

**Corollary 5.3.** Suppose that the data are generated as \( X^{(i)} \sim NPN(\mu_0, \Sigma_0, f_0) \), and let \( \Omega_0 = \Sigma_0^{-1} \). If the regularization parameter \( \lambda_n \) is chosen as

\[
\lambda_n \approx \sqrt{\frac{\log p \log^2 n}{n^{1/2}}}
\]

then the nonparanormal estimator \( \hat{\Omega}_n \) of (4.5) satisfies

\[
\| \hat{\Omega}_n - \Omega_0 \|_F = O_P \left( \sqrt{\frac{(s + p) \log p \log^2 n}{n^{1/2}}} \right)
\]

and

\[
\| \hat{\Omega}_n - \Omega_0 \|_2 = O_P \left( \sqrt{\frac{s \log p \log^2 n}{n^{1/2}}} \right),
\]

where

\[
s \equiv \text{Card} (\{(i, j) \in \{1, \ldots, p\} \times \{1, \ldots, p\} \mid \Omega_0(i, j) \neq 0, \ i \neq j\})
\]

is the number of nonzero off-diagonal elements of the true precision matrix.

To prove the model selection consistency result, we need further assumptions. We follow Ravikumar (2009) and let the \( p^2 \times p^2 \) Fisher information matrix of \( \Sigma_0 \) be \( \Gamma \equiv \Sigma_0 \otimes \Sigma_0 \) where \( \otimes \) is the Kronecker matrix product, and define the support set \( S \) of \( \Omega_0 = \Sigma_0^{-1} \) as

\[
S \equiv \{(i, j) \in \{1, \ldots, p\} \times \{1, \ldots, p\} \mid \Omega_0(i, j) \neq 0\}.
\]

We use \( S^c \) to denote the complement of \( S \) in the set \( \{1, \ldots, p\} \times \{1, \ldots, p\} \), and for any two subsets \( T \) and \( T' \) of \( \{1, \ldots, p\} \times \{1, \ldots, p\} \), we use \( \Gamma_{TT'} \) to denote the sub-matrix with rows and columns of \( \Gamma \) indexed by \( T \) and \( T' \) respectively.
Assumption 5.4. There exists some $\alpha \in (0, 1]$, such that $\|\Gamma S_c S^{-1}\|_{\infty} \leq 1 - \alpha$.

As in Ravikumar et al. (2009), we define two quantities $K_{\Sigma_0} \equiv \|\Sigma_0\|_{\infty}$ and $K_{\Gamma} \equiv \|(\Gamma SS)^{-1}\|_{\infty}$. Further, we define the maximum row degree as

$$d \equiv \max_{i=1, \ldots, p} \text{Card} \left( \{ j \in 1, \ldots, p | \Omega_0(i, j) \neq 0 \} \right).$$

Assumption 5.5. The quantities $K_{\Sigma_0}$ and $K_{\Gamma}$ are bounded, and there are positive constants $C_1$ and $C_2$ such that

$$\min_{(j,k) \in S} |\Omega_0(j, k)| \geq C_1 \sqrt{\frac{\log n}{n}} \quad \text{and} \quad n \geq C_2 d^2 \log p$$

for large enough $n$.

The proof of the following uses our Theorem 5.1 in place of equation (12) in the analysis of Ravikumar et al. (2009),

Corollary 5.6. Suppose the regularization parameter is chosen as

$$\lambda_n \asymp \sqrt{\frac{\log p \log^2 n}{n^{1/2}}}.$$

Then the nonparanormal estimator $\hat{\Omega}_n$ satisfies

$$\mathbb{P} \left( \mathcal{G} \left( \hat{\Omega}_n, \Omega_0 \right) \right) \geq 1 - o(1)$$

where $\mathcal{G}(\hat{\Omega}_n, \Omega_0)$ is the event

$$\left\{ \text{sign} \left( \hat{\Omega}_n(j, k) \right) = \text{sign} \left( \Omega_0^{-1}(j, k) \right), \quad \forall j, k \in S \right\}.$$

Our persistency (risk consistency) result parallels the persistency result for additive models given in Ravikumar et al. (2008a), and allows model dimension that grows exponentially with sample size. The definition in this theorem uses the fact (from Lemma 7.1) that $\sup_x \Phi^{-1} \left( \bar{F}_j(x) \right) \leq \sqrt{2 \log n}$ when $\delta_n = 1/(4n^{1/4} \sqrt{\pi \log n})$.

In the next theorem, we do not assume the true model is nonparanormal and define the population and sample risks as

$$R(f, \Omega) = \frac{1}{2} \left\{ \text{tr} \left[ \Omega \mathbb{E}(f(X)f(X)^T) - \log |\Omega| - p \log(2\pi) \right] \right\}$$

$$\hat{R}(f, \Omega) = \frac{1}{2} \left\{ \text{tr} \left[ \Omega S_n(f) \right] - \log |\Omega| - p \log(2\pi) \right\}.$$
Theorem 5.7. Suppose that $p \leq e^{n^{\xi}}$ for some $\xi < 1$, and define the classes

$$\mathcal{M}_n = \{ f : f \text{ is monotone with } \|f\|_\infty \leq C \sqrt{\log n} \}$$

$$\mathcal{C}_n = \{ \Omega : \|\Omega^{-1}\|_1 \leq L_n \}.$$

Let $\hat{\Omega}_n$ be given by

$$\hat{\Omega}_n = \arg\min_{\Omega \in \mathcal{C}_n} \left\{ \text{tr} \left( \Omega S_n(\tilde{f}) \right) - \log |\Omega| \right\}.$$

Then

$$R(\tilde{f}_n, \hat{\Omega}_n) - \inf_{(f, \Omega) \in \mathcal{M}_n \oplus \mathcal{C}_n} R(f, \Omega) = O_P \left( L_n \sqrt{\frac{\log n}{n^{1-\xi}}} \right).$$

Hence the Winsorized estimator of $(f, \Omega)$ with $\delta_n = 1/(4n^{1/4} \sqrt{\pi \log n})$ is persistent over $\mathcal{C}_n$ when $L_n = o \left( n^{(1-\xi)/2} / \sqrt{\log n} \right)$.

The proofs of Theorems 5.1 and 5.7 are given in Section 7.

VI. Experimental Results

In this section, we report experimental results on synthetic and real datasets. We mainly compare the $\ell_1$-regularized nonparanormal and Gaussian (paranormal) models, computed using the graphical lasso algorithm (glasso) of Friedman et al. (2007). The primary conclusions are: (i) When the data are multivariate Gaussian, the performance of the two methods is comparable; (ii) when the model is correct, the nonparanormal performs much better than the graphical lasso in many cases; (iii) even for distributions that are not nonparanormal, the new method often performs better; (iv) for gene microarray data, our method behaves differently from the graphical lasso, and may support different biological conclusions.

Note that we can reuse the glasso implementation to fit a sparse nonparanormal. In particular, after computing the Winsorized sample covariance $S_n(\tilde{f})$, we pass this matrix to the glasso routine to carry out the optimization

$$\hat{\Omega}_n = \arg\min_{\Omega} \left\{ \text{tr} \left( \Omega S_n(\tilde{f}) \right) - \log |\Omega| + \lambda_n \|\Omega\|_1 \right\}.$$ 

A. Neighborhood graphs

We begin by describing a procedure to generate graphs as in (Meinshausen and Bühlmann, 2006), with respect to which several distributions can then be defined. We generate a $p$-dimensional sparse graph $G \equiv (V, E)$ as follows: Let $V = \{1, \ldots, p\}$ corresponding to
variables $X = (X_1, \ldots, X_p)$. We associate each index $j$ with a point $(Y_j^{(1)}, Y_j^{(2)}) \in [0, 1]^2$ where

$$Y_1^{(k)}, \ldots, Y_n^{(k)} \sim \text{Uniform}[0, 1]$$

for $k = 1, 2$. Each pair of nodes $(i, j)$ is included in the edge set $E$ with probability

$$P((i, j) \in E) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{\|y_i - y_j\|^2}{2s} \right)$$

where $y_i = (y_i^{(1)}, y_i^{(2)})$ is the observation of $(Y_i^{(1)}, Y_i^{(2)})$ and $\| \cdot \|_n$ represents the Euclidean distance. Here, $s = 0.125$ is a parameter that controls the sparsity level of the generated graph. We restrict the maximum degree of the graph to be four and build the inverse covariance matrix $\Omega_0$ according to

$$\Omega_0(i, j) = \begin{cases} 
1 & \text{if } i = j \\
0.245 & \text{if } (i, j) \in E \\
0 & \text{otherwise},
\end{cases}$$

(6.1)

where the value 0.245 guarantees positive definiteness of the inverse covariance matrix.

Given $\Omega_0$, $n$ data points are sampled from

$$X^{(1)}, \ldots, X^{(n)} \sim \text{NPN}(\mu_0, \Sigma_0, f_0)$$

(6.2)

where $\mu_0 = (1.5, \ldots, 1.5)$, $\Sigma_0 = \Omega_0^{-1}$. For simplicity, the transformations functions for all dimensions are the same $f_1 = \ldots = f_p = f_0$. To sample data from the nonparanormal distribution, we also need $g_0 \equiv f_0^{-1}$, two different transformations $g_0$ are employed:

Next, we define the following two transformation families:

**Definition 6.1.** (Gaussian CDF Transformation) Let $g_0$ be a one-dimensional Gaussian cumulative distribution function with mean $\mu_{g_0}$ and the standard deviation $\sigma_{g_0}$, i.e.,

$$g_0(t) = \Phi \left( \frac{t - \mu_{g_0}}{\sigma_{g_0}} \right).$$

We define the transformation function $g_j = f_j^{-1}$ for the $j$-th dimension as

$$g_j(z_j) \equiv \sigma_j \left( \frac{g_0(z_j) - \int g_0(t) \phi \left( \frac{t - \mu_j}{\sigma_j} \right) dt}{\sqrt{\int \left( g_0(y) - \int g_0(t) \phi \left( \frac{t - \mu_j}{\sigma_j} \right) dt \right)^2 \phi \left( \frac{y - \mu_j}{\sigma_j} \right) dy}} \right) + \mu_j$$

where $\sigma_j = \Sigma_0(j, j)$. 
Definition 6.2. (Symmetric Power Transformation) Let \( g_0 \) be the symmetric and odd transformation given by
\[
g_0(t) = \text{sign}(t)|t|^{\alpha}
\]
where \( \alpha > 0 \) is a parameter. We define the power transformation for the \( j \)-th dimension as
\[
g_j(z_j) \equiv \sigma_j \left( \frac{g_0(z_j - \mu_j)}{\sqrt{\int g_0^2(t - \mu_j) \phi \left( \frac{t - \mu_j}{\sigma_j} \right) dt}} \right) + \mu_j.
\]
These transformation are constructed to preserve the marginal mean and standard deviation. In the following experiments, we refer to them as the cdf transformation and the power transformation, respectively. For the cdf transformation, we set \( \mu_{g_0} = 0.05 \) and \( \sigma_{g_0} = 0.4 \). For the power transformation, we set \( \alpha = 3 \).

To visualize these two transformations, we sample 5000 data points from a one-dimensional normal distribution \( N(0.5, 1.0) \) and then apply the above two transformations; the results are shown in Figure 3. It can be seen how the cdf and power transformations map a univariate normal distribution into a highly skewed and a bi-modal distribution, respectively.

To generate synthetic data, we set \( p = 40 \), resulting in \( \binom{40}{2} + 40 = 820 \) parameters to be estimated, and vary the sample sizes from \( n = 200 \) to \( n = 1000 \). Three conditions are considered, corresponding to using the cdf transform, the power transform, or no transformation. In each case, both the glasso and the nonparanormal are applied to estimate the graph.

A.1 Comparison of regularization paths

We choose a set of regularization parameters \( \Lambda \); for each \( \lambda \in \Lambda \), we obtain an estimate \( \hat{\Omega}_n \) which is a \( 40 \times 40 \) matrix. The upper triangular matrix has 780 parameters; we can vectorize it to get a 780-dimensional parameter vector. A regularization path is trace of these parameters over all the regularization parameters within \( \Lambda \). The regularization paths for both methods are plotted in Figure 4. For the cdf transformation and the power transformation, the nonparanormal separates the relevant and the irrelevant dimensions very well. For the glasso, relevant variables are mixed with irrelevant variables. If no transformation is applied, the paths for both methods are almost the same.

A.2 Estimated transformations

For sample size \( n = 1000 \), we plot the estimated transformations for three of the variables in Figure 5. It is clear that Winsorization plays a significant role for the power transformation.
Figure 3: The power and cdf transformations. The densities are estimated using kernel density estimator with bandwidths selected by unbiased cross-validation.

This is intuitive due to the high skewness of the nonparanormal distribution resulting from the power transformations.

A.3 Quantitative comparison

To evaluate the performance for structure estimation quantitatively, we use false positive and false negative rates. Let $G = (V, E)$ be a $p$-dimensional graph (which has at most $\binom{p}{2}$ edges) in which there are $|E| = r$ edges, and let $\hat{G}^{\lambda} = (V, \hat{E}^{\lambda})$ be an estimated graph using the regularization parameter $\lambda$. The number of false positives at $\lambda$ is

$$\text{FP}(\lambda) \equiv \text{number of edges in } \hat{E}^{\lambda} \text{ not in } E$$

The number of false negatives at $\lambda$ is defined as

$$\text{FN}(\lambda) \equiv \text{number of edges in } E \text{ not in } \hat{E}^{\lambda}.$$
Figure 4: Regularization paths for the glasso and nonparanormal with $n = 500$ (top) and $n = 200$ (bottom). The paths for the relevant variables (nonzero inverse covariance entries) are plotted as solid (black) lines; the paths for the irrelevant variables are plotted as dashed (red) lines.
The oracle regularization level $\lambda^*$ is then

$$
\lambda^* = \arg \min_{\lambda \in \Lambda} \{\text{FP}(\lambda) + \text{FN}(\lambda)\}.
$$

The oracle score is $\text{FP}(\lambda^*) + \text{FN}(\lambda^*)$. Figure 6 shows boxplots of the oracle scores for the two methods, calculated using 100 simulations.

To illustrate the overall performance of these two methods over the full paths, ROC curves are shown in Figure 7, using

$$
\left(1 - \frac{\text{FN}(\lambda)}{r}, 1 - \frac{\text{FP}(\lambda)}{\left(\frac{r}{2}\right)}\right).
$$
Figure 6: Boxplots of the oracle scores for \( n = 1000, 500, 200 \) (top, center, bottom).

The curves clearly show how the performance of both methods improves with sample size, and that the nonparanormal is superior to the Gaussian model in most cases.

Let \( \text{FPE} \equiv \text{FP}(\lambda^*) \) and \( \text{FNE} \equiv \text{FN}(\lambda^*) \), Tables 1, 2, and 3 provide numerical comparisons of both methods on datasets with different transformations, where we repeat the experiments 100 times and report the average FPE and FNE values with the corresponding standard deviations. It’s clear from the tables that the nonparanormal achieves significantly smaller errors than the glasso if the true distribution of the data is not multivariate Gaussian and achieves comparable performance as the glasso when the true distribution is exactly multivariate Gaussian.
Figure 7: ROC curves for sample sizes $n = 1000, 500, 200$ (top, middle, bottom).
Table 1: Quantitative comparison on the dataset using the cdf transformation

| $n$  | Nonparanormal | glasso |
|------|---------------|--------|
|      | FPE (sd(FPE))| FNE (sd(FNE)) | FPE (sd(FPE)) | FNE (sd(FNE)) |
| 1000 | 0.10 (0.3333) | 0.05 (0.2190) | 3.73 (2.3904) | 7.24 (3.2910) |
| 900  | 0.18 (0.5389) | 0.16 (0.4197) | 3.31 (2.4358) | 8.94 (3.2808) |
| 800  | 0.16 (0.5069) | 0.23 (0.5659) | 3.80 (2.9439) | 9.91 (3.4789) |
| 700  | 0.26 (0.6295) | 0.43 (0.7420) | 3.45 (2.5519) | 12.26 (3.5862) |
| 600  | 0.33 (0.6039) | 0.41 (0.6371) | 3.31 (2.8804) | 14.25 (4.0735) |
| 500  | 0.58 (0.9658) | 1.10 (1.0396) | 3.18 (2.9211) | 17.54 (4.4368) |
| 400  | 0.71 (1.0569) | 1.52 (1.2016) | 1.58 (2.3535) | 21.18 (4.9855) |
| 300  | 1.37 (1.4470) | 2.97 (2.0123) | 0.67 (1.6940) | 23.14 (5.0232) |
| 200  | 2.03 (1.9356) | 7.13 (3.4514) | 0.01 (0.1000) | 24.03 (4.9816) |

Table 2: Quantitative comparison on the dataset using the power transformation

| $n$  | Nonparanormal | glasso |
|------|---------------|--------|
|      | FPE (sd(FPE))| FNE (sd(FNE)) | FPE (sd(FPE)) | FNE (sd(FNE)) |
| 1000 | 0.27 (0.7086) | 0.35 (0.6571) | 2.89 (1.9482) | 4.97 (2.9213) |
| 900  | 0.38 (0.6783) | 0.41 (0.6210) | 2.98 (2.3697) | 5.99 (3.0467) |
| 800  | 0.25 (0.5751) | 0.73 (0.8270) | 4.10 (2.7834) | 6.39 (3.3571) |
| 700  | 0.69 (0.9067) | 0.90 (1.0200) | 4.42 (2.8891) | 8.80 (3.9848) |
| 600  | 0.92 (1.2282) | 1.59 (1.5314) | 4.64 (3.3830) | 10.58 (4.2168) |
| 500  | 1.17 (1.3413) | 2.56 (2.3325) | 4.00 (2.9644) | 13.09 (4.4903) |
| 400  | 1.88 (1.6470) | 4.97 (2.7687) | 3.14 (3.4699) | 17.87 (4.7750) |
| 300  | 2.97 (2.4181) | 7.85 (3.5572) | 1.36 (2.3805) | 21.24 (4.7505) |
| 200  | 2.82 (2.6184) | 14.53 (4.3378) | 0.37 (0.9914) | 24.01 (5.0940) |
Table 3: Quantitative comparison on the dataset without any transformation

| n   |玻璃o | (sd(FPE)) | FNE | (sd(FNE)) |玻璃o | (sd(FPE)) | FNE | (sd(FNE)) |
|-----|------|-----------|-----|-----------|------|-----------|-----|-----------|
| 1000| 0.10 | (0.3333)  | 0.05| (0.2190)  | 0.09 | (0.3208)  | 0.06| (0.2386)  |
| 900 | 0.24 | (0.7537)  | 0.14| (0.4025)  | 0.22 | (0.6447)  | 0.15| (0.4113)  |
| 800 | 0.17 | (0.4277)  | 0.16| (0.3949)  | 0.16 | (0.4431)  | 0.19| (0.4191)  |
| 700 | 0.25 | (0.6871)  | 0.33| (0.8534)  | 0.29 | (0.8201)  | 0.27| (0.7501)  |
| 600 | 0.37 | (0.7740)  | 0.36| (0.7456)  | 0.36 | (0.7722)  | 0.37| (0.6459)  |
| 500 | 0.28 | (0.5874)  | 0.46| (0.7442)  | 0.25 | (0.5573)  | 0.45| (0.6571)  |
| 400 | 0.55 | (0.8453)  | 1.37| (1.2605)  | 0.47 | (0.7713)  | 1.35| (1.2502)  |
| 300 | 1.24 | (1.3715)  | 3.07| (1.7306)  | 0.98 | (1.2058)  | 3.04| (1.8905)  |
| 200 | 1.62 | (1.7219)  | 5.89| (2.7373)  | 1.55 | (1.6779)  | 5.62| (2.6620)  |

A.4 Visualization of typical runs

Figure 8 shows typical runs for the cdf and power transformations. It’s clear that when the glasso estimates the graph incorrectly, the mistakes include both false positives and negatives.

B. Gene microarray data

In this study, we consider a dataset based on Affymetrix GeneChip microarrays for the plant *Arabidopsis thaliana*, (Wille, 2004). The sample size is $n = 118$. The expression levels for each chip are pre-processed by log-transformation and standardization. A subset of 40 genes from the isoprenoid pathway are chosen, and we study the associations among them using both the paranormal and nonparanormal models. Even though these data are generally treated as multivariate Gaussian in the previous analysis (Wille, 2004), our study shows that the results of the nonparanormal and the glasso are very different over a wide range of regularization parameters. This suggests the nonparanormal could support different scientific conclusions.

B.1 Comparison of the regularization paths

We first compare the regularization paths of the two methods, in Figure 9. To generate
Figure 8: Typical runs for the two methods for \( n = 1000 \) using the cdf and power transformations. The dashed (black) lines in the symmetric difference plots indicate edges found by the glasso but not the nonparanormal, and vice-versa for the solid (red) lines.
the paths, we select 50 regularization parameters on an evenly spaced grid in the interval $[0.16, 1.2]$. Although the paths for the two methods look similar, there are some subtle differences. In particular, variables become nonzero in a different order, especially when the regularization parameter is in the range $\lambda \in [0.2, 0.3]$. As shown below, these subtle differences in the paths lead to different model selection behaviors.

![glasso path vs nonparanormal path](image)

**Figure 9:** The regularization paths of both methods on the microarray dataset.

**B.2 Comparison of the selected graphs**

Figure 10 compares the estimated graphs for the two methods at several values of the regularization parameter $\lambda$ in the range $[0.16, 0.37]$. For each $\lambda$, we show the estimated graph from the nonparanormal in the first column. In the second column we show the graph obtained by scanning the full regularization path of the glasso fit and finding the graph having the smallest symmetric difference with the nonparanormal graph. The symmetric difference graph is shown in the third column. The closest glasso fit is different, with edges selected by the glasso not selected by the nonparanormal, and vice-versa. Several estimated transformations are plotted in Figure 11, which are are nonlinear. Interestingly, several of the differences between the fitted graphs are related to these variables.
Figure 10: The nonparanormal estimated graph for three values of $\lambda = 0.2448, 0.2661, 0.30857$ (left column), the glasso estimated graph (middle) and the symmetric difference graph (right).

Figure 11: Estimated transformations for the microarray dataset, indicating non-Gaussian marginals. The corresponding genes are among the nodes appearing in the symmetric difference graphs of Figure 10.
VII. Proofs

We assume, without loss of generality from Lemma 3.3, that \( \mu_j = 0 \) and \( \sigma_j = 1 \) for all \( j = 1, \ldots, p \). Thus, define \( \tilde{f}_j(x) \equiv \Phi^{-1}(\tilde{F}_j(x)) \) and \( f_j(x) \equiv \Phi^{-1}(F_j(x)) \), and let \( g_j \equiv f_j^{-1} \).

A. Proof of Theorem 5.1

We start with some useful lemmas; the first is from Abramovich et al. (2006).

**Lemma 7.1.** (Gaussian Distribution function vs. Quantile function) Let \( \Phi \) and \( \phi \) denote the distribution and density functions of a standard Gaussian random variable. Then

\[
\frac{\phi(t)}{2t} \leq 1 - \Phi(t) \leq \frac{\phi(t)}{t} \quad \text{if } t \geq 1
\]  

and

\[
(\Phi^{-1})'(\eta) = \frac{1}{\phi(\Phi^{-1}(\eta))}. 
\]  

Also, for \( \eta \geq 0.99 \), we have

\[
\Phi^{-1}(\eta) = \sqrt{2 \log \left( \frac{1}{1-\eta} \right) - r(\eta)}
\]  

where \( r(\eta) \in [0, 1.5] \).

**Lemma 7.2.** (Distribution function of the transformed random variable) For any \( \alpha \in (-\infty, \infty) \)

\[
\Phi^{-1} \left( F_j \left( g_j(\alpha \sqrt{\log n}) \right) \right) = \alpha \sqrt{\log n}.
\]  

**Proof.** The statement follows from

\[
F_j(t) = P(X_j \leq t) = P(g_j(Z_j) \leq t) = P(Z_j \leq g_j^{-1}(t)) = \Phi \left( g_j^{-1}(t) \right)
\]  

which holds for any \( t \). \( \Box \)

**Lemma 7.3.** (Gaussian maximal inequality) Let \( W_1, \ldots, W_n \) be independently and identically distributed standard Gaussian random variables. Then for any \( \alpha > 0 \)

\[
P \left( \max_{1 \leq i \leq n} W_i > \sqrt{\alpha \log n} \right) \leq \frac{1}{n^{\alpha/2 - 1}}.
\]  

26
Proof. Using Mill’s inequality, we have
\[
P \left( \max_{1 \leq i \leq n} W_i > \sqrt{\alpha \log n} \right) \leq \sum_{i=1}^{n} P \left( W_i > \sqrt{\alpha \log n} \right) \leq n \frac{\phi(\sqrt{\alpha \log n})}{\sqrt{\alpha \log n}} = \frac{1}{n^{\alpha/2-1/2}} \sqrt{2\pi \alpha \log n},
\]
from which the result follows.

\[\square\]

Lemma 7.4. For any \( \alpha > 0 \) that satisfies \( 1 - \delta_n - \Phi(\sqrt{\alpha \log n}) > 0 \) for all \( n \), we have
\[
P \left[ \hat{F}_j \left( g_j \left( \sqrt{\alpha \log n} \right) \right) > 1 - \delta_n \right] \leq \exp \left\{ -2n \left( 1 - \delta_n - \Phi(\sqrt{\alpha \log n}) \right)^2 \right\}. \tag{7.5}
\]
and
\[
P \left[ \hat{F}_j \left( g_j \left( -\sqrt{\alpha \log n} \right) \right) < \delta_n \right] \leq \exp \left\{ -2n \left( 1 - \delta_n - \Phi(\sqrt{\alpha \log n}) \right)^2 \right\}. \tag{7.6}
\]

Proof. Using Hoeffding’s inequality,
\[
P \left[ \hat{F}_j \left( g_j \left( \sqrt{\alpha \log n} \right) \right) > 1 - \delta_n \right] \\
= P \left[ \hat{F}_j \left( g_j \left( \sqrt{\alpha \log n} \right) \right) - F_j \left( g_j \left( \sqrt{\alpha \log n} \right) \right) > 1 - \delta_n - F_j \left( g_j \left( \sqrt{\alpha \log n} \right) \right) \right] \\
\leq \exp \left\{ -2n \left( 1 - \delta_n - F_j \left( g_j \left( \sqrt{\alpha \log n} \right) \right) \right)^2 \right\}.
\]
Equation (7.5) then follows from equation (7.4). The proof of equation (7.6) uses the same argument.

\[\square\]

Now let \( M > 2 \) be some constant and set \( \beta = \frac{1}{2} \). We split the interval
\[
\left[ g_j(-\sqrt{M \log n}), g_j(\sqrt{M \log n}) \right]
\]
into two parts, the middle
\[
\mathcal{M}_n \equiv \left( g_j \left( -\sqrt{\beta \log n} \right), g_j \left( \sqrt{\beta \log n} \right) \right)
\]
and ends
\[
\mathcal{E}_n \equiv \left[ g_j \left( -\sqrt{M \log n} \right), g_j \left( -\sqrt{\beta \log n} \right) \right] \cup \left[ g_j \left( \sqrt{\beta \log n} \right), g_j \left( \sqrt{M \log n} \right) \right].
\]
The behaviors of the function estimates in these two regions is different, and so we first establish bounds on the probability that a sample can fall in the end region \( \mathcal{E}_n \).
Lemma 7.5. Let \( A \equiv \sqrt{\frac{2}{\pi}}(\sqrt{M} - \sqrt{\beta}) \). Then
\[
\mathbb{P}(X_{1j} \in \mathcal{E}_n) \leq A\sqrt{\frac{\log n}{n^q}}, \ \forall j \in \{1, \ldots, p\}.
\]

Proof. Using equation (7.4) and the mean value theorem, we have
\[
\mathbb{P}(X_{1j} \in \mathcal{E}_n) = \mathbb{P}(X_{1j} \in [g_j(\sqrt{\beta \log n}), g_j(\sqrt{M \log n})]) + \mathbb{P}(X_{1j} \in [g_j(-\sqrt{M \log n}), g_j(-\sqrt{\beta \log n})])
\]
\[
= F_j(g_j(\sqrt{M \log n}) - F_j(g_j(\sqrt{\beta \log n})) + F_j(g_j(-\sqrt{\beta \log n})) - F_j(g_j(-\sqrt{M \log n}))
\]
\[
= 2\left(\phi(\sqrt{M \log n}) - \phi(\sqrt{\beta \log n})\right)
\]
\[\leq 2\phi(\sqrt{\beta \log n})\left(\sqrt{M \log n} - \sqrt{\beta \log n}\right).
\]
The result of the lemma follows directly. \(\square\)

We next bound the error of the Winsorized estimate of a component function over the end region.

Lemma 7.6. For all \( n \), we have
\[
\sup_{t \in \mathcal{E}_n} \left| \Phi^{-1}(\tilde{F}_j(t)) - \Phi^{-1}(F_j(t)) \right| < \sqrt{2(M + 2)\log n}, \ \forall j \in \{1, \ldots, p\}.
\]

Proof. From Lemma 7.2 and the definition of \( \mathcal{E}_n \), we have
\[
\sup_{t \in \mathcal{E}_n} \left| \Phi^{-1}(F_j(t)) \right| \in \left[0, \sqrt{M \log n}\right].
\]
Given the fact that \( \delta_n = \frac{1}{4n^{1/4} \sqrt{\pi \log n}} \), we have \( \tilde{F}_j(t) \in \left(\frac{1}{n}, 1 - \frac{1}{n}\right) \). Therefore, from equation (7.3),
\[
\sup_{t \in \mathcal{E}_n} \left| \Phi^{-1}(\tilde{F}_j(t)) \right| \in \left[0, \sqrt{2\log n}\right].
\]
The result follows from the triangle inequality and \( \sqrt{M} + \sqrt{2} \leq \sqrt{2(M + 2)}. \) \(\square\)
Now for any $\epsilon > 0$, we have

\[
\mathbb{P}\left( \max_{j,k} \left| S_n(\tilde{f})_{jk} - S_n(f)_{jk} \right| > \epsilon \right)
\]

\[
= \mathbb{P}\left( \max_{j,k} \left| \frac{1}{n} \sum_{i=1}^{n} \left\{ \tilde{f}_j(X_{ij})\tilde{f}_k(X_{ik}) - f_j(X_{ij})f_k(X_{ik}) - \mu_n(\tilde{f}_j)\mu_n(\tilde{f}_k) + \mu_n(f_j)\mu_n(f_k) \right\} \right| > \epsilon \right)
\]

\[
\leq \mathbb{P}\left( \max_{j,k} \left| \frac{1}{n} \sum_{i=1}^{n} \left( \tilde{f}_j(X_{ij})\tilde{f}_k(X_{ik}) - f_j(X_{ij})f_k(X_{ik}) \right) \right| > \frac{\epsilon}{2} \right) + \mathbb{P}\left( \max_{j,k} \left| \mu_n(\tilde{f}_j)\mu_n(\tilde{f}_k) - \mu_n(f_j)\mu_n(f_k) \right| > \frac{\epsilon}{2} \right).
\]

We only need to analyze the rate for the first term above, since the second one is of higher order (Cai et al., 2008). Let

\[
\Delta_i(j, k) \equiv \tilde{f}_j(X_{ij})\tilde{f}_k(X_{ik}) - f_j(X_{ij})f_k(X_{ik})
\]

and

\[
\Theta_{i,s}(j, k) \equiv \tilde{f}_j(t)\tilde{f}_k(s) - f_j(t)f_k(s).
\]

We define the event $\mathcal{A}_n$ as

\[
\mathcal{A}_n \equiv \left\{ g_j \left( -\sqrt{M \log n} \right) \leq X_{ij}, \ldots, X_{nj} \leq g_j \left( \sqrt{M \log n} \right), j = 1, \ldots, p \right\}.
\]

Then

\[
\mathbb{P}(\mathcal{A}_n^c) \leq p\mathbb{P}\left( \min_i f_j(X_{ij}) < -\sqrt{M \log n} \right) + p\mathbb{P}\left( \max_i f_j(X_{ij}) > \sqrt{M \log n} \right) \leq \frac{c_1 n^{\xi}}{n^{M/2-1}}.
\]

with $c_1$ as a generic positive constant. Therefore

\[
\mathbb{P}\left( \max_{j,k} \left| \frac{1}{n} \sum_{i=1}^{n} \Delta_i(j, k) \right| > \epsilon \right) \leq \mathbb{P}\left( \max_{j,k} \left| \frac{1}{n} \sum_{i=1}^{n} \Delta_i(j, k) \right| > \epsilon, \mathcal{A}_n \right) + \mathbb{P}(\mathcal{A}_n^c)
\]

\[
\leq \mathbb{P}\left( \max_{j,k} \left| \frac{1}{n} \sum_{i=1}^{n} \Delta_i(j, k) \right| > \epsilon, \mathcal{A}_n \right) + \frac{c_1 n^{\xi}}{n^{M/2-1}}.
\]

Thus, we only need to carry out our analysis on the event $\mathcal{A}_n$. On this event, we have the following decomposition:

\[
\mathbb{P}\left( \max_{j,k} \left| \frac{1}{n} \sum_{i=1}^{n} \Delta_i(j, k) \right| > \epsilon, \mathcal{A}_n \right)
\]

\[
\leq \mathbb{P}\left( \max_{j,k} \frac{1}{n} \sum_{X_{ij} \in \mathcal{M}_n, X_{ik} \in \mathcal{M}_n} |\Delta_i(j, k)| > \frac{\epsilon}{4} \right) + \mathbb{P}\left( \max_{j,k} \frac{1}{n} \sum_{X_{ij} \in \mathcal{E}_n, X_{ik} \in \mathcal{E}_n} |\Delta_i(j, k)| > \frac{\epsilon}{4} \right)
\]

\[
+ 2\mathbb{P}\left( \max_{j,k} \frac{1}{n} \sum_{X_{ij} \in \mathcal{M}_n, X_{ik} \in \mathcal{E}_n} |\Delta_i(j, k)| > \frac{\epsilon}{4} \right).
\]
We now analyze each of these terms separately.

**Lemma 7.7.** On the event $A_n$, let $\beta = 1/2$ and $\epsilon \geq C(M, \xi) \sqrt{\frac{\log p \log^2 n}{n^{1/2}}}$, then

$$\mathbb{P} \left( \max_{j,k} \frac{1}{n} \sum_{X_{ij} \in E_n, X_{ik} \in E_n} |\Delta_i(j,k)| > \frac{\epsilon}{4} \right) = o(1).$$

**Proof.** We define

$$\theta_1 \equiv \frac{n^{3/2} \epsilon}{8A \sqrt{\log n}}$$

with the same parameter $A$ as in Lemma 7.5. Such a $\theta_1$ guarantees that

$$\frac{1}{n} \left( n \epsilon - nA \sqrt{\frac{\log n}{n^{\beta}}} \right) = nA \sqrt{\frac{\log n}{n^{\beta}}} > 0.$$

By Lemma 7.5, we have

$$\mathbb{P} \left( \frac{1}{n} \sum_{i=1}^n 1_{\{X_{ij} \in E_n, X_{ik} \in E_n\}} > \frac{\epsilon}{4\theta_1} \right) \leq \mathbb{P} \left( \sum_{i=1}^n 1_{\{X_{ij} \in E_n\}} > \frac{n \epsilon}{4\theta_1} \right)$$

$$= \mathbb{P} \left( \sum_{i=1}^n \left( 1_{\{X_{ij} \in E_n\}} - \mathbb{P}(X_{1j} \in E_n) \right) > \frac{n \epsilon}{4\theta_1} - n \mathbb{P}(X_{1j} \in E_n) \right)$$

$$\leq \mathbb{P} \left( \sum_{i=1}^n \left( 1_{\{X_{ij} \in E_n\}} - \mathbb{P}(X_{1j} \in E_n) \right) > \frac{n \epsilon}{4\theta_1} - nA \sqrt{\frac{\log n}{n^{\beta}}} \right).$$

Using the Bernstein’s inequality, for $\beta = \frac{1}{2}$,

$$\mathbb{P} \left( \frac{1}{n} \sum_{i=1}^n 1_{\{X_{ij} \in E_n, X_{ik} \in E_n\}} > \frac{\epsilon}{4\theta_1} \right) \leq \mathbb{P} \left( \sum_{i=1}^n \left( 1_{\{X_{ij} \in E_n\}} - \mathbb{P}(X_{1j} \in E_n) \right) > nA \sqrt{\frac{\log n}{n^{\beta}}} \right)$$

$$\leq \exp \left( -\frac{c_1 n^{2-\beta} \log n}{c_2 n^{1-\beta/2} \log n + c_3 n^{1-\beta/2} \log n} \right) = o(1),$$

where $c_1, c_2, c_3 > 0$ are generic constants.
Therefore,

\[
\mathbb{P}\left( \max_{j,k} \frac{1}{n} \sum_{X_{ij} \in \mathcal{E}_n, X_{ik} \in \mathcal{E}_n} |\Delta_i(j, k)| > \frac{\epsilon}{4} \right)
\]

\[
= \mathbb{P}\left( \max_{j,k} \frac{1}{n} \sum_{X_{ij} \in \mathcal{E}_n, X_{ik} \in \mathcal{E}_n} |\Delta_i(j, k)| > \frac{\epsilon}{4}, \max_{j,k} \sup_{t \in \mathcal{E}_n, s \in \mathcal{E}_n} |\Theta_{t,s}(j, k)| > \theta_1 \right)
\]

\[
+ \mathbb{P}\left( \max_{j,k} \frac{1}{n} \sum_{X_{ij} \in \mathcal{E}_n, X_{ik} \in \mathcal{E}_n} |\Delta_i(j, k)| > \frac{\epsilon}{4}, \max_{j,k} \sup_{t \in \mathcal{E}_n, s \in \mathcal{E}_n} |\Theta_{t,s}(j, k)| \leq \theta_1 \right)
\]

\[
\leq \mathbb{P}\left( \max_{j,k} \sup_{t \in \mathcal{E}_n, s \in \mathcal{E}_n} |\Theta_{t,s}(j, k)| > \theta_1 \right) + \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} 1_{\{X_{ij} \in \mathcal{E}_n, X_{ik} \in \mathcal{E}_n \}} > \frac{\epsilon}{4\theta_1} \right)
\]

\[
= \mathbb{P}\left( \max_{j,k} \sup_{t \in \mathcal{E}_n, s \in \mathcal{E}_n} |\Theta_{t,s}(j, k)| > \theta_1 \right) + o(1).
\]

Now, we analyze the first term

\[
\mathbb{P}\left( \max_{j,k} \sup_{t \in \mathcal{E}_n, s \in \mathcal{E}_n} |\Theta_{t,s}(j, k)| > \theta_1 \right) \leq p^2 \mathbb{P}\left( \sup_{t \in \mathcal{E}_n, s \in \mathcal{E}_n} |\Theta_{t,s}(j, k)| > \theta_1 \right)
\]

\[
= p^2 \mathbb{P}\left( \sup_{t \in \mathcal{E}_n, s \in \mathcal{E}_n} |\tilde{f}_j(t)\tilde{f}_k(s) - f_j(t)f_k(s)| > \theta_1 \right).
\]

By adding and subtracting terms \(f_j(t)\) and \(f_k(t)\), we have

\[
\mathbb{P}\left( \sup_{t \in \mathcal{E}_n, s \in \mathcal{E}_n} |\tilde{f}_j(t)\tilde{f}_k(s) - f_j(t)f_k(s)| > \theta_1 \right)
\]

\[
\leq \mathbb{P}\left( \sup_{t \in \mathcal{E}_n, s \in \mathcal{E}_n} |(\tilde{f}_j(t) - f_j(t))(\tilde{f}_k(s) - f_k(s))| > \frac{\theta_1}{3} \right)
\]

\[
+ \mathbb{P}\left( \sup_{t \in \mathcal{E}_n, s \in \mathcal{E}_n} |(\tilde{f}_j(t) - f_j(t))| \cdot |f_k(s)| > \frac{\theta_1}{3} \right)
\]

\[
+ \mathbb{P}\left( \sup_{t \in \mathcal{E}_n, s \in \mathcal{E}_n} |(\tilde{f}_k(s) - f_k(s))| \cdot |f_j(t)| > \frac{\theta_1}{3} \right).
\]

The first term can further be decomposed to be

\[
\mathbb{P}\left( \sup_{t \in \mathcal{E}_n, s \in \mathcal{E}_n} |(\tilde{f}_j(t) - f_j(t))(\tilde{f}_k(s) - f_k(s))| > \frac{\theta_1}{3} \right)
\]

\[
\leq \mathbb{P}\left( \sup_{t \in \mathcal{E}_n} |\tilde{f}_j(t) - f_j(t)| > \sqrt{\frac{\theta_1}{3}} \right) + \mathbb{P}\left( \sup_{s \in \mathcal{E}_n} |\tilde{f}_k(s) - f_k(s)| > \sqrt{\frac{\theta_1}{3}} \right).
\]
Also, from the definition of $E_n$, we have

$$\sup_{t \in E_n} |f_j(t)| = \sup_{t \in E_n} |g_j^{-1}(t)| \leq \sqrt{M \log n}.$$ 

Since $\epsilon \geq C(M, \xi)\sqrt{\frac{\log p \log^2 n}{n^{1/2}}}$, we have

$$\frac{\theta_1}{3} = \frac{n^{3/2}\epsilon}{24A \sqrt{\log n}} \geq \frac{C(M, \xi)\sqrt{\log p \log^2 n}}{24A \sqrt{\log n}} = 2(M + 2) \log n.$$

This implies that

$$\sqrt{\frac{\theta_1}{3}} \geq \sqrt{2(M + 2) \log n} \quad \text{and} \quad \frac{\theta_1}{3M \log n} \geq \frac{\sqrt{2(M + 2) \log n}}{3M \log n}.$$

Then, from Lemma 7.6, we get

$$\mathbb{P}\left( \sup_{t \in E_n} |(\tilde{f}_j(t) - f_j(t))| > \sqrt{\frac{\theta_1}{3}} \right) = 0$$

and

$$\mathbb{P}\left( \sup_{t \in E_n, s \in E_n} |(\tilde{f}_j(t) - f_j(t))| \cdot |f_k(s)| > \frac{\theta_1}{3} \right) = 0.$$

The claim of the lemma then follows directly. \qed

**Remark 7.8.** From the above analysis, we see that the data in the tails doesn’t affect the rate. Using exactly the same argument, we can also show that

$$\mathbb{P}\left( \max_{j,k} \frac{1}{n} \sum_{X_{ij} \in M_n, X_{ik} \in E_n} |\Delta_i(j, k)| > \frac{\epsilon}{4} \right) = o(1).$$

**Lemma 7.9.** On the event $\mathcal{A}_n$, let $\beta = 1/2$ and $\epsilon \geq C(M, \xi)\sqrt{\frac{\log p \log^2 n}{n^{1/2}}}$. There exist generic constants $c_1, c_2, c_3, c_4$, such that

$$\mathbb{P}\left( \max_{j,k} \frac{1}{n} \sum_{X_{ij} \in M_n, X_{ik} \in M_n} |\Delta_i(j, k)| > \frac{\epsilon}{4} \right) \leq c_2 \exp\left(-\frac{c_1 n^{1-\beta} \epsilon^2}{\log p \log^2 n}\right) + c_3 \exp\left(-\frac{c_4 n^{1-\beta} \epsilon^2}{\log p (\log n)}\right).$$
Proof. We have

\[
\mathbb{P} \left( \max_{j,k} \frac{1}{n} \sum_{X_{ij} \in M_n, X_{ik} \in M_n} |\Delta_i(j, k)| > \frac{\epsilon}{4} \right) \leq p^2 \mathbb{P} \left( \sup_{t \in M_n, s \in M_n} |\tilde{f}_j(t) \tilde{f}_k(s) - f_j(t)f_k(s)| > \frac{\epsilon}{4} \right)
\]

\[
\leq p^2 \mathbb{P} \left( \sup_{t \in M_n, s \in M_n} |(\tilde{f}_j(t) - f_j(t))(\tilde{f}_k(s) - f_k(s))| > \frac{\epsilon}{12} \right)
\]

\[
+ 2p^2 \mathbb{P} \left( \sup_{t \in M_n, s \in M_n} |(\tilde{f}_j(t) - f_j(t)) \cdot |f_k(s)| > \frac{\epsilon}{12} \right).
\]

Further, since

\[
\sup_{t \in M_n} |f_j(t)| = \sup_{t \in M_n} |g_j^{-1}(t)| = \sqrt{\beta \log n}
\]

and

\[
\sup_{t \in M_n, s \in M_n} |(\tilde{f}_j(t) - f_j(t))(\tilde{f}_k(s) - f_k(s))| \text{ is of higher order than } \sup_{t \in M_n, s \in M_n} |(\tilde{f}_j(t) - f_j(t)) \cdot |f_k(s)|, \]

we only need to analyze the term

\[
\mathbb{P} \left( \sup_{t \in M_n} |\tilde{f}_j(t) - f_j(t)| > \frac{\epsilon}{12 \sqrt{\beta \log n}} \right).
\]

Since \( \delta_n = \frac{1}{4n^{\beta/2} \sqrt{2 \pi \beta \log n}} \), using Mill’s inequality we have

\[
2\delta_n = \frac{\phi(\sqrt{\beta \log n})}{2\sqrt{\beta \log n}} \leq 1 - \Phi(\sqrt{\beta \log n}).
\]

This implies that

\[
1 - \delta_n - \Phi(\sqrt{\beta \log n}) \geq \delta_n > 0.
\]

Using Lemma 7.4, we have

\[
p^2 \mathbb{P} \left( \tilde{F}_j \left( g_j \left( \sqrt{\beta \log n} \right) \right) > 1 - \delta_n \right) \leq \exp \left( -\frac{n\delta_n^2}{2 \log p} \right) = \exp \left( -\frac{n^{1-\beta}}{\log p(64\pi\beta \log n)} \right) \quad (7.7)
\]

and

\[
p^2 \mathbb{P} \left( \tilde{F}_j \left( g_j \left( -\sqrt{\beta \log n} \right) \right) < \delta_n \right) \leq \exp \left( -\frac{n^{1-\beta}}{\log p(64\pi\beta \log n)} \right). \quad (7.8)
\]

Define an event \( B_n \) as

\[
B_n \equiv \left\{ \delta_n \leq \tilde{F}_j \left( g_j \left( \sqrt{\beta \log n} \right) \right) \leq 1 - \delta_n, j = 1, \ldots, p \right\}.
\]

From (7.7) and (7.8), it is easy to see that

\[
\mathbb{P} (B_n^c) \leq c_3 \exp \left( -\frac{c_4 n^{1-\beta}}{\log p(\log n)} \right)
\]

33
where $c_3$ and $c_4$ are generic positive constants.

From the definition of $\tilde{F}_j$, we have
\[
\begin{align*}
p^2 \mathbb{P} \left( \sup_{t \in M_n} |(\tilde{f}_j(t) - f_j(t))| > \frac{\epsilon}{12 \sqrt{\beta \log n}} \right) & \\
& \leq p^2 \mathbb{P} \left( \sup_{t \in M_n} \left| \Phi^{-1}(\tilde{F}_j(t)) - \Phi^{-1}(F_j(t)) \right| > \frac{\epsilon}{12 \sqrt{\beta \log n}}, B_n \right) + \mathbb{P}(B_n^c).
\end{align*}
\]
\[
\begin{align*}
& \leq p^2 \mathbb{P} \left( \sup_{t \in M_n} \left| \Phi^{-1}(\tilde{F}_j(t)) - \Phi^{-1}(F_j(t)) \right| > \frac{\epsilon}{12 \sqrt{\beta \log n}} \right) + c_3 \exp \left( - \frac{c_4 n^{1-\beta}}{\log p(\log n)} \right).
\end{align*}
\]

Define
\[
T_{1n} \equiv \max \left\{ F_j \left( g_j \left( \sqrt{\beta \log n} \right) \right), 1 - \delta_n \right\} \quad \text{and} \quad T_{2n} \equiv 1 - \min \left\{ F_j \left( g_j \left( -\sqrt{\beta \log n} \right) \right), \delta_n \right\}.
\]

From equation (7.4) and the fact that $1 - \delta_n \geq \Phi \left( \sqrt{\beta \log n} \right)$, we have that
\[
T_{1n} = T_{2n} = 1 - \delta_n.
\]

Thus, by the mean value theorem,
\[
\begin{align*}
& \mathbb{P} \left( \sup_{t \in M_n} \left| \Phi^{-1}(\tilde{F}_j(t)) - \Phi^{-1}(F_j(t)) \right| > \frac{\epsilon}{12 \sqrt{\beta \log n}} \right) \\
& \leq \mathbb{P} \left( (\Phi^{-1})' \left( \max \{T_{1n}, T_{2n}\} \right) \sup_{t \in M_n} \left| \tilde{F}_j(t) - F_j(t) \right| > \frac{\epsilon}{12 \sqrt{\beta \log n}} \right) \\
& = \mathbb{P} \left( (\Phi^{-1})' \left( 1 - \delta_n \right) \sup_{t \in M_n} \left| \tilde{F}_j(t) - F_j(t) \right| > \frac{\epsilon}{12 \sqrt{\beta \log n}} \right).
\end{align*}
\]

Finally, using the Dvoretzky-Kiefer-Wolfowitz inequality,
\[
\begin{align*}
& \mathbb{P} \left( \sup_{t \in M_n} \left| \Phi^{-1}(\tilde{F}_j(t)) - \Phi^{-1}(F_j(t)) \right| > \frac{\epsilon}{12 \sqrt{\beta \log n}} \right) \\
& \leq \mathbb{P} \left( \sup_{t \in M_n} \left| \tilde{F}_j(t) - F_j(t) \right| > \frac{\epsilon}{(\Phi^{-1})' \left( 1 - \delta_n \right) 12 \sqrt{\beta \log n}} \right) \\
& \leq \mathbb{P} \left( \sup_{t \in M_n} \left| \tilde{F}_j(t) - F_j(t) \right| > \frac{\epsilon}{(\Phi^{-1})' \left( 1 - \delta_n \right) 12 \sqrt{\beta \log n}} \right) \\
& \leq 2 \exp \left( - \frac{2 \epsilon^2}{144 \beta \log n \left[(\Phi^{-1})' \left( 1 - \delta_n \right)\right]^2} \right).
\end{align*}
\]

Furthermore, by Lemma 7.1,
\[
(\Phi^{-1})' \left( 1 - \delta_n \right) = \frac{1}{\phi(\Phi^{-1}(1 - \delta_n))} \leq \frac{1}{\phi \left( \frac{1}{\sqrt{2 \log \frac{1}{\delta_n}}} \right)} = \sqrt{2 \pi} \frac{1}{\frac{1}{\delta_n}} = 8 \pi n^{3/2} \sqrt{\beta \log n}.
\]

34
This implies that
\[ p^2 P \left( \sup_{t \in \mathcal{M} \cap \mathbb{R}^d} \left| \Phi^{-1} \left( \hat{F}_j(t) \right) - \Phi^{-1} (F_j(t)) \right| > \frac{\epsilon}{12 \sqrt{\beta \log n}} \right) \leq 2 \exp \left( -\frac{c_1 n^{1-\beta} \epsilon^2}{\log p \log^2 n} \right) \]
where \( c_1 \) is a generic constant.

In summary, we have
\[ P \left( \max_{j,k} \frac{1}{n} \sum_{X_{ij} \in \mathcal{M} \cap \mathbb{R}^d, X_{ik} \in \mathcal{E}} |\Delta_i(j, k)| > \frac{\epsilon}{4} \right) \leq c_2 \exp \left( -\frac{c_1 n^{1-\beta} \epsilon^2}{\log p \log^2 n} \right) + c_3 \exp \left( -\frac{c_4 n^{1-\beta}}{\log p (\log n)} \right) \]
where \( c_1, c_2, c_3, c_4 \) are generic constants. \( \square \)

The conclusion of Theorem 5.1 follows from Lemma 7.7 and Lemma 7.9.

**B. Proof of Theorem 5.7**

**Proof.** First note that the population and sample risks are
\[
R(f, \Omega) = \frac{1}{2} \left\{ \text{tr} \left[ \Omega \mathbb{E} (f f^T) - \log |\Omega| - p \log(2\pi) \right] \right\}
\]
\[
\hat{R}(f, \Omega) = \frac{1}{2} \left\{ \text{tr} \left[ \Omega S_n(f) \right] - \log |\Omega| - p \log(2\pi) \right\}.
\]

Therefore, for all \((f, \Omega) \in \mathcal{M}^p_p \oplus \mathcal{C}_n\), we have
\[
|\hat{R}(f, \Omega) - R(f, \Omega)| = \frac{1}{2} |\text{tr} \left[ \Omega (\mathbb{E} ff^T - S_n(f)) \right]| \\
\leq \frac{1}{2} \|\Omega\|_1 \max_{j,k \in \mathcal{M} \cap \mathbb{R}^d} \sup_{f_j, f_k \in \mathcal{M}_n} \|\mathbb{E} (f_j X_j f_k X_k) - S_n(f)_{jk}\| \\
\leq \frac{L_n}{2} \max_{j,k \in \mathcal{M} \cap \mathbb{R}^d} \sup_{f_j, f_k \in \mathcal{M}_n} \|\mathbb{E} (f_j X_j f_k X_k) - S_n(f)_{jk}\|.
\]

Now, if \( \mathcal{F} \) is a class of functions, we have
\[
\mathbb{E} \left( \sup_{g \in \mathcal{F}} |\hat{\mu}(g) - \mu(g)| \right) \leq C J_{\|\|}(\|F\|_\infty, \mathcal{F}) \sqrt{\frac{1}{n}} \tag{7.9}
\]
for some \( C > 0 \), where \( F(x) = \sup_{g \in \mathcal{F}} |g(x)|, \mu(g) = \mathbb{E}(g(X)) \) and \( \hat{\mu}(g) = n^{-1} \sum_{i=1}^n g(X_i) \) (see Corollary 19.35 of van der Vaart (1998)). Here the bracketing integral is defined to be
\[
J_{\|\|}(\delta, \mathcal{F}) = \int_0^\delta \sqrt{\log N_{\|\|}(u, \mathcal{F})} \, du
\]

35
where $\log N_{||}(\epsilon, \mathcal{F})$ is the bracketing entropy. For the class of one dimensional, bounded and monotone functions, the bracketing entropy satisfies

$$\log N_{||}(\epsilon, \mathcal{M}) \leq K \left( \frac{1}{\epsilon} \right)$$

for some $K > 0$ (van der Vaart and Wellner, 1996).

Now, let $\mathcal{P}_{n,p}$ be the class of all functions of the form $m(x) = f_j(x_j)f_k(x_k)$ for $j, k \in \{1, \ldots, p\}$, where $f_j \in \mathcal{M}_n$ for each $j$. Then the bracketing entropy satisfies

$$\log N_{||}(C\sqrt{\log n}, \mathcal{P}_{n,p}) \leq 2\log p + K \left( \frac{1}{\epsilon} \right)$$

and the bracketing integral satisfies $J_{||}(C\sqrt{\log n}, \mathcal{P}_{n,p}) = O(\sqrt{\log n \log p})$. It follows from (7.9) and Markov’s inequality that

$$\max_{jk} \sup_{f_j, f_k \in \mathcal{M}_n} |S_n(f)_{jk} - \mathbb{E}(f_j(X_j)f_k(X_k))| = O_P \left( \sqrt{\frac{\log n \log p}{n}} \right) = O_P \left( \sqrt{\frac{\log n}{n^{1-\xi}}} \right).$$

Therefore,

$$\sup_{(f, \Omega) \in \mathcal{M}_n^{p} \otimes \mathcal{C}_n} |\hat{R}(f, \Omega) - R(f, \Omega)| = O_P \left( \frac{L_n \sqrt{\log n}}{n^{(1-\xi)/2}} \right).$$

As a consequence, we have

$$R(f^*, \Omega^*) \leq R(\tilde{f}_n, \tilde{\Omega}_n)$$

$$\leq \hat{R}(\tilde{f}_n, \tilde{\Omega}_n) + O_P \left( \frac{L_n \sqrt{\log n}}{n^{(1-\xi)/2}} \right)$$

$$\leq \hat{R}(f^*, \Omega^*) + O_P \left( \frac{L_n \sqrt{\log n}}{n^{(1-\xi)/2}} \right)$$

$$\leq R(f^*, \Omega^*) + O_P \left( \frac{L_n \sqrt{\log n}}{n^{(1-\xi)/2}} \right)$$

and the conclusion follows.

\[\Box\]

VIII. Concluding Remarks

In this paper we have introduced the nonparanormal, a type of Gaussian copula with nonparametric marginals that is suitable for estimating high dimensional undirected graphs. The nonparanormal can be viewed as an extension of sparse additive models to the setting of graphical models. We proposed an estimator for the component functions that is based on
thresholding the tails of the empirical distribution function at appropriate levels. A theoretical analysis was given to bound the difference between the sample covariance with respect to these estimated functions and the true sample covariance. This analysis was leveraged with the recent work of Ravikumar et al. (2009) and Rothman et al. (2008) to obtain consistency results for the nonparanormal. Computationally, fitting a high dimensional nonparanormal is no more difficult than estimating a multivariate Gaussian, and indeed one can exploit existing software for the graphical lasso. Our experimental results indicate that the sparse nonparanormal can give very different results than a sparse Gaussian graphical model, suggesting that it may be a useful tool for relaxing the normality assumption, which is often made only for convenience.

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