Practical Frank-Wolfe algorithms

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Abstract

In the last decade there has been a resurgence of interest in Frank-Wolfe (FW) style methods for optimizing a smooth convex function over a polytope. Examples of recently developed techniques include Decomposition-invariant Conditional Gradient (DiCG), Blended Condition Gradient (BCG), and Frank-Wolfe with in-face directions (IF-FW) methods. We introduce two extensions of these techniques. First, we augment DiCG with the working set strategy, and show how to optimize over the working set using shadow simplex steps. Second, we generalize in-face Frank-Wolfe directions to polytopes in which faces cannot be efficiently computed, and also describe a generic recursive procedure that can be used in conjunction with several FW-style techniques. Experimental results indicate that these extensions are capable of speeding up original algorithms by orders of magnitude for certain applications.

1 Introduction

We consider the problem of optimizing a convex smooth function $f$ over a bounded polytope $P \subset \mathbb{R}^n$: 

$$
\min_{x \in P} f(x)
$$

We assume that the access to $P$ is provided only through a min-oracle (or a linear minimization oracle) that returns vector $s \in \arg \min_{s \in P} \langle g, s \rangle$ (called an atom) for a given vector $g \in \mathbb{R}^n$. The classical algorithm for solving this problem is the Frank-Wolfe method [7, 11], also known as Conditional Gradient method. This method and its variants have enjoyed renewed popularity in recent years due to their ability to solve efficiently certain large scale optimization problems in data analysis, e.g. over combinatorial polytopes. Frank-Wolfe techniques have been applied to traffic assignment problems [17], low-rank matrix approximation [23], structural SVMs [16], video co-localization [13], infinite RBMs [21], MAP inference in graph models [24].

While the original FW algorithm has a sublinear convergence rate, many linearly convergent modifications have been proposed; see e.g. [10, 15, 9, 1, 3, 5], to name just a few. In this paper we will consider two such techniques: Decomposition-invariant Conditional Gradient (DiCG) [10, 9, 1] and Blended Conditional Gradient (BCG) [3].

Our first contribution is the CacheDiCG algorithm that augments DiCG with the working set strategy: we cache atoms returned by the min-oracle in a working set $W$, and interleave DiCG steps applied to the current iterate $x$ with optimization over $\text{conv}(\{x\} \cup W)$. Note that caching planes a standard idea in the context of FW algorithms, and has been used in [12] for standard FW and in [22, 20] for Block-Coordinate FW. Our technical contribution here is a procedure for optimizing over $\text{conv}(\{x\} \cup W)$ (we use shadow steps from [19] for this purpose), and also a criterion for deciding which step to perform that preserves linear convergence guarantees of DiCG.

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This criterion is a slight modification of that in [22] (which was formulated without theoretical guarantees), and is based on measuring runtimes of different steps.

The main focus of this paper, however, is on the method by Freund et al. [8] called Frank-Wolfe with in-face directions. It computes the minimal face \( P_x \subseteq P \) containing current iterate \( x \), and performs a FW step within the face \( P_x \) if a certain condition is satisfied. Such step can be much faster than a FW step in the original polytope, since the dimension of \( P_x \) is typically much smaller than the dimension of \( P \). We explore further this idea, and make the following contributions.

- We observe that FW-style steps can be performed in lower-dimensional polytopes that are not necessarily faces of \( P \). For example, such polytope can be obtained by fixing variables \( i \) with \( x_i = 0 \) to 0 (in the case when \( P \subseteq [0,1]^n \)). Another possibility is to rank nodes \( i \) with \( x_i = 0 \) according to how likely they belong to an optimal solution (using e.g. values of dual variables for this purpose), and then fix \( k \leq n \) lowest-ranked variables to 0.

- Freund et al. used in-face directions only with FW steps, which have sublinear convergence. A natural idea is to replace them with linearly convergent variants of FW. The question that arises is how to formulate the criterion for selecting which step to perform (low- or full-dimensional). We design a generic procedure that can be used in conjunction with several FW variants, in particular with DiCG, BCG and CacheDiCG. Our procedure is recursive, and allows nested contractions. We again use a criterion based on measuring runtimes of different computations. We prove that this criterion guarantees linear convergence, and also that the bound on the total time of good steps is roughly the same as the respective bound for the original algorithms.

Note that the criterion used in [8] needs an upper bound on the Lipschitz constant \( L \), which may not always be easy to obtain. Furthermore, their criterion does not use runtimes. Since the main advantage of low-dimensional steps is that they are faster, we argue that it is important to take into account actual runtimes.

- We apply these ideas to several combinatorial polytopes, in particular to the polytope of \( s-t \) paths in a DAG and the polytope of perfect matchings. To our knowledge, so far in-face directions have only been used for the low-rank matrix completion problem [8].

The rest of the paper is organized as follows. In Section 2 we describe our notation and assumptions, and summarize FW, DiCG and BCG methods. Section 3 presents the CacheDiCG method, while Section 4 describes a recursive procedure for FW-style methods. Section 5 gives experimental results, and Section 6 contains conclusions and discussion of future work.

### 2 Background and preliminaries

For simplicity, we assume that the min-oracle always returns a vertex of \( P \), and denote the set of vertices as \( \text{Atoms}(P) \). The simplex over \( \text{Atoms}(P) \) will be denoted as \( \Delta_P = \{ \alpha \in [0,1]^\text{Atoms}(P) : ||\alpha||_1 = 1 \} \). We use letter \( \alpha \) to denote the current state of the algorithm, and \( \overline{P} \) to be the set of valid states \( \alpha \). Every state \( \alpha \in \overline{P} \) is associated with vector \( x[\alpha] \in P \), where \( x[\cdot] \) is a mapping \( \overline{P} \rightarrow P \). Two primary examples are as follows:

1. \( \overline{P} = P \), with \( x[\alpha] = \alpha \). In that case we will sometimes use letter \( x \) instead of \( \alpha \). Some algorithms also maintain a working set of atoms \( \mathcal{W} \subseteq \text{Atoms}(P) \); in that case we can define \( \overline{P} = (P, 2^\text{Atoms}(P)), \alpha = (x, \mathcal{W}) \) and \( x[(x, \mathcal{W})] = x \).

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1For certain polytopes (e.g. if \( P \) is a simplex) this would actually give the face \( P_x \). However, this is not always the case. If, for example, \( P = \text{conv}(A) \) where \( A \subseteq [0,1]^E \) is the set of perfect matchings of a non-bipartite graph \((V,E)\) then the result is not necessarily a face of \( P \). Note, computing the minimal face \( P_x \) in the latter case is a rather expensive operation.
II. \( \overline{P} = \Delta_P \), with \( x[\alpha] = \sum_{s \in \text{Atoms}(P)} \alpha_s \cdot s \in P \). Of course, in practice it is sufficient to store only atoms in the active set

\[
\text{supp}(\alpha) = \{ s \in \text{Atoms}(P) \mid \alpha_s > 0 \}
\]

We will sometimes refer to these two cases as type-I and type-II methods. With a slight abuse of notation, we will often write \( f(\alpha) \) to mean \( f(x[\alpha]) \), and for type-I methods define

\[
\text{supp}(\alpha) = \bigcup \{ \text{supp}(\beta) \mid \beta \in \Delta_P, x[\alpha] = \sum_{s \in \text{Atoms}(P)} \beta_s \cdot s \}
\]

For \( \alpha \in \overline{P} \) let us define the following quantities:

\[
w(\alpha) = \max_{s \in P, v \in \text{supp}(\alpha)} \langle \nabla f(x[\alpha]), v - s \rangle
\]

\[
w^+(\alpha) = \max_{s \in P} \langle \nabla f(x[\alpha]), x[\alpha] - s \rangle
\]

\[
w^-(\alpha) = \max_{u, v \in \text{supp}(\alpha)} \langle \nabla f(x[\alpha]), v - u \rangle
\]

We also define \( h(\alpha) \) to be the suboptimality gap:

\[
h(\alpha) = f(\alpha) - f(\alpha^*) , \quad \alpha^* \in \arg \min_{\alpha \in \overline{P}} f(\alpha)
\]

Note that \( w^+(\alpha) \) and \( h(\alpha) \) depend only on \( x[\alpha] \). The following facts are well-known:

(i) Quantities \( w(\alpha), w^+(\alpha), w^-(\alpha) \) are non-negative and satisfy \( \max \{ w^+(\alpha), w^-(\alpha) \} \leq w(\alpha) \leq w^+(\alpha) + w^-(\alpha) \).

(ii) \( h(\alpha) \leq w^+(\alpha) \).

(iii) \( w^+(\alpha) = w^-(\alpha) = 0 \) if and only if \( \alpha \) is optimal, i.e. \( h(\alpha) = 0 \).

Unless stated otherwise, we assume throughout the paper that the following assumption hold.

(Geometric Strong Convexity) There exists constant \( \mu > 0 \) such that

\[
h(\alpha) \leq \frac{(w(\alpha))^2}{2\mu} \quad \forall \alpha \in \overline{P}
\]

This holds, in particular, if \( f \) is a strongly convex function (see [15] and [1] \(^2\)). By the results of [2] this can also be extended to the case when \( f(x) = g(Ex) + \langle b, x \rangle \) where \( E \) is some matrix, \( b \) is some vector and \( g \) is a strongly convex and continuous differentiable function.

The second assumption that we make is

(L-smoothness) There exists value \( L > 0 \) such that

\[
f(y) \leq f(x) - \langle \nabla f(x), x - y \rangle + \frac{L}{2} ||y - x||^2 \quad \forall x, y \in P
\]

It implies, in particular, the following property (with \( C \leq \frac{1}{2} LD^2 \) where \( D \) is the diameter of \( P \)).

(Bounded Curvature) There exists value \( C > 0 \) such that

\[
f(\gamma s + (1 - \gamma)x) \leq f(x) - \gamma \langle \nabla f(x), x - s \rangle + C\gamma^2 / 2 \quad \forall x, s \in P, \gamma \in [0, 1]
\]

\(^2\)In [1] this inequality follows from their eq. (4) combined with the fact that \( h(x) \geq \langle \nabla f(x), x - x^* \rangle \).
2.1 Algorithms

A classical algorithm for solving problem (1) is the Frank-Wolfe (FW) method. It starts with an arbitrary initial vector \( x \in P \), and then repeatedly applies the update given in Algorithm 1.

**Algorithm 1:** Procedure \( \text{FWstep} : x \mapsto x' \)

1. compute \( g = \nabla f(x) \)
2. compute \( s \in \arg\min_{s \in P} \langle g, s \rangle \)
3. compute \( y \leftarrow \arg\min_{y \in [x, s]} f(y) \) where \( [x, s] \) is the line segment between \( x \) and \( s \)
4. return \( y \)

The Bounded Curvature assumption ensures that each step decreases the objective by the amount

\[
f(x) - f(y) \geq \max_{\gamma \in [0, 1]} \left[ \gamma \cdot w^+(x) - \gamma^2 \cdot C/2 \right] \geq \min \left\{ \frac{(w^+(\alpha))^2}{2C}, \frac{w^+(\alpha)}{2} \right\}
\]

which is \( \Omega((w^+(\alpha))^2) \) if \( w^+(\alpha) \) is sufficiently small. In general, FW has a sublinear convergence rate: \( h(x^{(t)}) \leq O(1/t) \) where \( x^{(t)} \) is the iterate at the \( t \)-th step (see e.g. [11]). Several alternative methods have been proposed in the literature that achieve a linear convergence rate. In general, each step in these methods is either good or bad. Good steps are guaranteed to decrease the objective by \( \Omega((w^+(\alpha))^2) \) (which is \( \Omega(h(\alpha)) \)) by the Geometric Strong Convexity assumption), while bad steps make the current iterate sparser and thus cannot happen too often. In later sections we will review two such techniques: Decomposition-invariant Conditional Gradient (DiCG) [9, 1] and Blended Condition Gradient (BCG) [3].

Both of them employ a max-oracle in addition to the min-oracle. Given vectors \( \alpha \in \overline{P} \) and \( g = \nabla f(x[\alpha]) \), it computes \( v \in \arg\max_{v \in \text{supp}(\alpha)} \langle g, v \rangle \). For a type-II method (i.e. when \( \overline{P} = \Delta_P \)) this optimization problem can be solved by explicitly going through atoms \( v \in \text{supp}(\alpha) \) and computing inner products \( \langle g, v \rangle \). Implementing the max-oracle for a type-I method requires additional assumptions on the polytope \( P \); they are discussed in Section 2.2.

Note that an access to the min- and max-oracles allows to compute quantities \( w(\alpha) \) and \( w^+(\alpha) \): we can compute \( x = x[\alpha], g = \nabla f(x) \) and let

\[
s \leftarrow \arg\min_{s \in P} \langle g, s \rangle \quad v \leftarrow \arg\min_{v \in \text{supp}(\alpha)} \langle g, v \rangle \quad w(\alpha) = \langle g, v - s \rangle \quad w^+(\alpha) = \langle g, x - s \rangle \tag{6}
\]

In type-II methods we can also compute \( w^-(\alpha) \), since vector \( u \in \arg\min_{u \in \text{supp}(\alpha)} \langle g, u \rangle \) can be computed by explicitly going through atoms in \( \text{supp}(\alpha) \).

2.2 Decomposition-invariant Conditional Gradient (DiCG) [10, 9, 1]

A version of this method was introduced by Guélat and Marcotte [10] for polytopes of the form \( P = \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \} \). Garber and Meshi [9] presented several other versions of DiCG and strengthened the analysis in the case when the vertices satisfy \( A \subseteq \{0, 1\}^n \). Bashiri and Zhang [1] further extended the method and analysis to more general polytopes of the form \( P = \{ x \in \mathbb{R}^n : Ax \leq b, Cx = d \} \); the only assumption stated in [1] is that the system \( Ax \leq b \) does not have redundant constraints. Note that works [9, 1] analyzed several line search strategies; for simplicity, below we review only DiCG with the exact line search.

DiCG is a type-I method: it works directly with vector \( x \in \overline{P} = P \). It selects an arbitrary \( x = x^{(0)} \in P \) and then repeatedly updates \( x \) using Algorithm 2. Note that there are two versions of DiCG: with away steps (AFW) and with pairwise steps (PFW). These are standard steps in the context of type-II FW algorithms [11].
the AFW case every step is either a drop step or a good step assuming that

where ρ claim in [1, Sec. 3.5] that “clamping never happens for FW-steps”.

P in [1], if oracle has the following specification:

consequently

have 0 = ρ

A

where x

The next method that we consider is a type-II method, i.e. it maintains vector

2.3 Blended Conditional Gradient (BCG) [3]

is larger for PFW).

Algorithm 2: Procedure DiCGstep(x) ↦ x′

1 compute s ← arg minx∈P ⟨g, s⟩
2 compute v ← arg maxv∈supp(x) ⟨g, v⟩
3 select direction d as follows:
4 [option 1: AFW] let d ← arg maxx∈{dFW, dA} ⟨g, −d⟩ where dFW def = s − x and dA def = x − v
5 [option 2: PFW] let d = dFW def = s − v = dPW + dh
6 compute η = arg max{η ≥ 0 | x + ηd ∈ P} and let u = x + ηd
7 compute y ← arg miny∈[x, u] f(y)
8 return y

Besides calling the min-oracle, the algorithm also calls the max-oracle in line 2. As observed in [1], if P = {x ∈ Rn : Ax ≤ b, Cx = d} then this is equivalent to maximizing ⟨g, v⟩ over the set

P x = {z ∈ P | ⟨a j, x⟩ = b j ⇒ ⟨a j, z⟩ = b j}

where a j is the j-th row of matrix A. (As with the min-oracle, we assume that the max-oracle returns a vertex of P x). It can be easily checked that P x is the minimal face of P containing x.

With a slight abuse of notation, we write dim(x) to denote the dimension of P x. Clearly, we have 0 ≤ dim(x) ≤ n. Let Δdim be the maximum increase in dim(x) after applying one step of Algorithm 2. One obvious bound is Δdim ≤ n; for some polytopes a better bound can be derived.

The call DiCG(x) will be called a drop step if dim(y) < dim(x). It is a good step if dim(y) ≥ dim(x) and y ∈ {u, x} ∪ {s}. The theorem below summarizes the analysis from [1] (with minor variations 3). For completeness, we give a proof in Appendix A.

Theorem 1. (a) If DiCGstep(x) makes a good step then h(x) − h(x′) ≥ min{ρ · (w(x))^2, 1 2 w(x)}

where ρ = 1 8LD2 for AFW and ρ = 1 12LD2 for PFW.

Consider the run of the algorithm until encountering vector x with h(x) ≤ ε, and let h0 = h(x(0)). (b) The number of good steps satisfies Ngood ≤ ⌈1 2 ln h0 θ⌉ where θ = min{2ρµ, 1 2}. (c) In the AFW case every step is either a drop step or a good step assuming that h(x) > h(x′), and consequently Ndrop ≤ 2Δdim Ngood + dim(x(0)).

Note that in the PFW case there can be “abnormal” steps which are neither good steps nor drop steps; this makes the analysis of PFW challenging. While there are no good bounds on the number of abnormal steps, in practice they seem to be rather rare, and PFW often outperforms AFW (which could potentially be attributed to the fact that the constant ρ in the theorem above is larger for PFW).

2.3 Blended Conditional Gradient (BCG) [3]

The next method that we consider is a type-II method, i.e. it maintains vector x as a convex combination x = x[α] for α ∈ P = ΔP. It assumes the existence of a “Simplex Descent Oracle” (SiDO) oracle that optimizes f(x[α]) over α′ ∈ conv(supp(α)) for the current vector α ∈ ΔP. This oracle has the following specification:

SiDO(α)

Input: vector α ∈ ΔP with |supp(α)| ≤ n + 1

Output: vector α′ ∈ conv(supp(α)) satisfying one of the following:

(i) f(α) − f(α′) ≥ 0 and |supp(α′)| < |supp(α)|; {“drop step”}

3Note that instead of dim(x), [1] used an alternative (but related) expression. Also, we did not understand the claim in [1, Sec. 3.5] that “clamping never happens for FW-steps”.

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(ii) \( f(\alpha) - f(\alpha') \geq \frac{(w^-(\alpha))^2}{2\nu} \) for some constant \( \nu > 0 \).

In [3] this oracle is implemented as one step of the projected gradient descent method over \( \text{conv}(\text{supp}(\alpha)) \) with an exact line search. As shown in [3], such step does satisfy the specification above assuming that function \( f \) is strongly convex.

At each step the BCG algorithm calls either \( \text{SiDO} \) oracle or applies the FW step (Algorithm 1). The former would decrease the objective by \( \Omega((w^-)(\alpha))^2 \) (unless it is a drop step), while the latter would decrease the objective by \( \Omega((w^+)(\alpha))^2 \). Since \( w^+(\alpha) + w^-(\alpha) \geq w(\alpha) \), at least one of those steps would decrease the objective by \( \Omega((w(\alpha))^2) \), which is \( \Omega(h(\alpha)) \) by the Geometric Strong Convexity assumption. Thus, with the appropriate choice each step would either decrease the objective by \( \Omega(h(\alpha)) \), or be a drop step. The number of drop steps cannot exceed the number of FW steps (plus the initial number of atoms minus 1), since the FW step adds at most one atom to \( \text{supp}(\alpha) \) while a drop step removes at least one atom from \( \text{supp}(\alpha) \). Thus, such scheme would lead to a linear rate of convergence.

To select the right subroutine, one could potentially compute quantities \( w^-(\alpha) \) and \( w^+(\alpha) \) and check which one is larger. Computing \( w^-(\alpha) \) requires evaluating inner products \( \langle g, v \rangle \) for all \( v \in \text{supp}(\alpha) \) whereas computing \( w^+(\alpha) \) requires calling the min-oracle for vector \( g = \nabla f(x[\alpha]) \). The latter computation is often much more expensive (assuming that \( |\text{supp}(\alpha)| \) is small, which is typically the case). To avoid calling the min-oracle, the BCG method maintains an estimate \( \Phi \) on \( w(\alpha) \), and calls the \( \text{SiDO} \) oracle if \( w^-(\alpha) \geq \Phi/2 \). Formally, BCG selects initial vector \( \alpha \in \Delta_P \), computes value \( \Phi = w(\alpha) \), and then repeatedly updates pair \( (\alpha, \Phi) \) using Algorithm 3.

**Algorithm 3:** Procedure BCGstep : \( (\alpha, \Phi) \mapsto (\alpha', \Phi') \)

1. compute \( x = x[\alpha] \) and \( g = \nabla f(x) \)
2. compute \( u \leftarrow \arg \min_{u \in \text{supp}(\alpha)} \langle g, u \rangle \)
3. compute \( v \leftarrow \arg \max_{v \in \text{supp}(\alpha)} \langle g, v \rangle \)
4. if \( w^-(\alpha) = \langle g, v - u \rangle \geq \Phi/2 \) then
   5. update \( \alpha \leftarrow \text{SiDO}(\alpha) \)
5. else
   6. try to find \( s \in P \) with \( \langle g, x - s \rangle \geq \Phi/(2K) \) \{optional: “lazy FW”. \( K \geq 1 \) is a constant.\}
   8. if failure then
      9. compute \( s \in \arg \min_{s \in P} \langle g, s \rangle \)
   10. if \( w^+(\alpha) = \langle g, x - s \rangle < \Phi/2 \) then
      11. return \( (\alpha, \Phi') \) where \( \Phi' = \min\{w(\alpha), \Phi/2\} \) and \( w(\alpha) = \langle g, v - s \rangle \)
   12. compute \( y \leftarrow \arg \min_{y \in [x, s]} f(y) \)
   13. update vector \( \alpha \) accordingly to restore equality \( y = x[\alpha] \)
14. return \( (\alpha, \Phi) \)

**Remark 1.** The idea of line 7 comes from the “lazy FW” method [4]: it is not necessary to call the expensive min-oracle if we can find a sufficiently good vector \( s \in P \) with a cheaper computation. [4] gives two examples of such computations: (1) check vectors \( s \in \text{supp}(\alpha) \); (2) use a min-oracle that supports an “early termination” option, e.g. an LP solver that solves the primal problem. Note, while option (1) gave a substantial speed-up in the setting of [4], it is arguably much less effective when used inside BCG, since optimization over \( \text{conv}(\text{supp}(\alpha)) \) is already performed by the \( \text{SiDO} \) oracle.

**Remark 2.** Theoretical guarantees for BCG are established in [3] under the assumption that \( |\text{supp}(\alpha)| \leq n + 1 \). This could be achieved by choosing vector \( \alpha \) in line 13 that satisfies \( x' = x[\alpha] \) and has a minimal support; by Caratheodory’s theorem, such choice does satisfy \( |\text{supp}(\alpha)| \leq n + 1 \).
However, this would be a very expensive computation. As remarked in [3], in practice this is not needed since $|\text{supp}(\alpha)|$ remains invariably small.

Remark 3. The original BCG algorithm [3] returns $\Phi' = \Phi/2$ in line 11 instead of $\Phi' = \min\{\omega(\alpha), \Phi/2\}$. Our change does not affect the theoretical analysis (see below), and ensures that there are no two failed steps in a row (i.e. steps that terminate in line 11).

Next, we summarize the analysis of Algorithm 3 given in [3] (slightly rephrased and with different constants). We say that a pair $(\alpha, \Phi)$ is valid if $h(\alpha) \leq \min\{2\Phi^2/\mu, 2\Phi\}$ where $\mu$ is the Geometric Strong Convexity constant. A call to Algorithm 3 is a good step if $|\text{supp}(\alpha')| \geq |\text{supp}(\alpha)|$ and $\Phi' = \Phi$, a drop step if $|\text{supp}(\alpha')| < |\text{supp}(\alpha)|$ and $\Phi' = \Phi$, and a halve step if $\alpha' = \alpha$ and $\Phi' \leq \Phi/2$. The following theorem is proven in Appendix B.

Theorem 2. (a) Iterates $(\alpha, \Phi)$ remain valid during the algorithm. (b) If $\text{BCGstep}(\alpha, \Phi)$ makes a good step then $h(\alpha) - h(\alpha') \geq \min\{\rho \Phi^2, \Phi/(2K)\}$ where $\rho = \min\{\frac{1}{2\epsilon}, \frac{1}{2\epsilon K^2}\}$. (c) Consider the run of the algorithm until encountering vector $\alpha$ with $h(\alpha) \leq \epsilon$. Denote $h_0 = h(\alpha(0))$ where $\alpha(0)$ is the initial vector. The numbers of good, drop and halve steps satisfy

$$N_{\text{good}} \leq \left[\frac{1}{2} \ln \frac{\|x\|}{\epsilon}\right] \quad N_{\text{drop}} \leq N_{\text{good}} + |\text{supp}(\alpha(0))| - 1 \quad N_{\text{halve}} \leq \min\{N_{\text{good}} + N_{\text{drop}}, \left\lfloor \log_2 \frac{w(\alpha(0))}{\epsilon} \right\rfloor\}$$

where $\theta = \min\{\frac{\omega(\theta)}{2}, \frac{1}{4\epsilon}\}$.

3 DiCG with a working set

As our first contribution, we extend the DiCG method by maintaining a “working set” of atoms $W$. We will interleave DiCG steps with steps that optimize over the set $\text{conv}(W \cup \{x\})$. In previous works such optimization was performed either by calling a quadratic programming solver [12] or by iteratively calling the same Frank-Wolfe technique (namely, BCFW) used to optimize the main objective. Note that the former method is restricted to quadratic objectives, and also requires computing inner products for every pair of elements in $W \cup \{x\}$.

We instead adopt “shadow steps” from [19]. Assume that $W = \{v_1, \ldots, v_m\}$, and denote $v_0 = x$. Let $\Delta^m = \{\lambda \in \mathbb{R}^{m+1}_0 : \sum_{i=0}^m \lambda_i = 1\}$ be the $m$-dimensional simplex. Define function $\hat{f} : \Delta^m \to \mathbb{R}$ via $\hat{f}(\lambda) = f(\sum_{i=0}^m \lambda_i v_i)$, so that vector $\lambda(0) = (1, 0, \ldots, 0)$ corresponds to vector $x$. Our goal is to minimize function $\hat{f}$ over $\Delta^m$. It is argued in [19] that the best direction to move from $\lambda(0)$ is given by

$$d = \lim_{\epsilon \to 0} \frac{\Pi(x - \epsilon w) - x}{\epsilon}, \quad w = \nabla \hat{f}(\lambda(0))$$

(7)

where $\Pi$ is the projection operator onto the set $\Delta^m$. Vector $d$ is called the shadow of the gradient $w = \nabla \hat{f}(\lambda(0))$. In [19] this shadow was computed by solving a quadratic program using Gurobi (for arbitrary polytopes). In the case of the simplex polytope expression (7) can be computed more explicitly, as described in Algorithm 4. Note that value $\delta$ at line 3 is indeed unique since function $h(\delta) = \sum_{i=0}^m c_i^\delta$ is strictly decreasing with $h(-\infty) = +\infty$ and $h(+\infty) = -\infty$. In Appendix C.1 we discuss how to find this $\delta$ efficiently, and also prove the following fact.

Proposition 3. Vector $c^\delta = (c_0^\delta, c_1^\delta, \ldots, c_m^\delta)$ at line 4 equals $-d$ where $d$ is as defined in (7).

Algorithm 4: Procedure ShadowSimplexStep($x, \{v_1, \ldots, v_m\}$) $\rightarrow y$

1. Compute $g = \nabla f(x)$
2. Compute $c_i = (g, v_i)$ for $i = 0, 1, \ldots, m$ where $v_0 = x$
3. Find unique $\delta \in \mathbb{R}$ with $\sum_{i=0}^m c_i^\delta = 0$ where $c_0^\delta = c_0 - \delta$ and $c_i^\delta = \min\{c_i - \delta, 0\}$ for $i \neq 0$
4. If $c_0^\delta = 0$ then return $x$, otherwise compute normalized vector $\nu = -\frac{(1/c_0^\delta) \cdot c^\delta}{\langle -1, \ldots \rangle}$
5. Compute $u = \sum_{i=0}^m (\lambda(0) + \nu) i \cdot v_i = \sum_{i=1}^m \nu_i \cdot v_i$ and $y \leftarrow \arg \min_{y \in [x, u]} \hat{f}(y)$
6. Return $y$
Next, we describe procedure \texttt{CacheDiCGstep} (Algorithm 5). It performs one DiCG step followed by one or more simplex steps. After each simplex step we need to decide whether or not to perform another simplex step. We use a criterion similar to the one in [22]: we measure the “progress” of each step (defined as the improvement in the objective function divided by the time of the step), and terminate the current iteration if the progress of the simplex step is smaller than the progress of the initial DICG step. We introduce two modifications compared to [22]:

(i) When measuring the progress of the DiCG step, we define the improvement as $\Delta = f(x) - \min_{\xi \geq 0} f(x + \xi d)$ instead of the “actual” improvement $f(x) - \min_{\eta \geq 0; x + \eta d \in \mathcal{P}} f(x + \eta d)$. This ensures that the progress made by simplex steps is sufficiently large (see Theorem 5 below). Note, if the objective function is quadratic then $\Delta_0$ can be computed “for free”.

(ii) We use an additional stopping criterion controlled by parameter $\kappa \in (0, +\infty)$. If $\kappa < +\infty$ then the runtime of \texttt{CacheDiCGstep} is within a constant factor from the runtime of \texttt{DiCGstep}, which will be important in the next section.

**Algorithm 5:** Procedure \texttt{CacheDiCGstep}(x, W) $\mapsto (x', W')$

1. call $x \leftarrow \text{DiCGstep}(x)$ (Algorithm 2) with the following additional steps:
   2. update $\mathcal{W} \leftarrow \mathcal{W} \cup \{s\}$ where $s$ is the atom at line 1
   3. [optional] remove some atoms from $\mathcal{W}$
   4. at line 6 compute $\Delta_0 = f(x) - \min_{\xi \geq 0} f(x + \xi d)$
   5. measure time $T_0$ of this call

   for $t = 1, 2, \ldots$ do
   
   \begin{enumerate}
   \item call $x \leftarrow \text{ShadowSimplexStep}(x, W)$ (Algorithm 4) with the following additional steps:
   \item at line 7 compute $\Delta_t = f(x) - f(y)$
   \item measure time $T_t$ of this call
   \item if $\Delta_t/T_t \leq \Delta_0/T_0$ or $\sum_{t=1}^{\infty} T_t \geq \kappa \cdot T_0$ then return $(x, W)$
   \end{enumerate}

Let us call the $t$-th simplex step at line 7 successful if $\Delta_t/T_t > \Delta_0/T_0$, and unsuccessful otherwise. Note that there can be at most one unsuccessful step (namely, the last one).

**Theorem 4.** Let $\Delta$ be the improvement in the objective during successful steps, and $T$ be their total time. Then $\Delta_0 \geq \rho \cdot (w(x))^2$ and $\Delta \geq \Delta_0 \cdot T/T_0$ where $\rho$ is the constant from Theorem 1.

Now let us consider algorithm \texttt{CacheDiCG} that starts with an arbitrary $\alpha = (x^{(0)}, \emptyset)$ and repeatedly updates $\alpha$ by calling \texttt{CacheDiCGstep}. To control the size of the working set, we use a common strategy: if $|\mathcal{W}|$ exceeds a certain constant $M$ (10 in our experiments) then we remove an atom with the oldest timestamp. A timestamp for a given atom is set to the current time if it is returned by the \texttt{DiCGstep} call, or if it has non-zero weight $\alpha_i$ in the simplex step.

The call to \texttt{CacheDiCGstep} is called good if the call to \texttt{DiCGstep} at line 1 is good. A drop step is defined in a similar way. We assume that if $y = x + \eta(v - x)$ where $x, y \in \mathcal{P}, v \in \text{Atoms}(P)$ and $\eta \geq 0$ then $\dim(y) \leq \dim(x) + \Delta_{\dim}$. Then calling \texttt{CacheDiCGstep}(x, W) may increase $\dim(x)$ by at most $(M + 1)\Delta_{\dim}$, since updates in lines 7-10 can be written of a sequence of at most $M$ updates of the form $x \leftarrow x + \eta(v - x)$ for $v_i \in \mathcal{W}$.

**Theorem 5.** Consider the run of the algorithm until encountering vector $\alpha$ with $h(\alpha) \leq \varepsilon$. Denote $h_0 = h(x^{(0)})$. (a) The number of good steps satisfies $N_{\text{good}} \leq \left\lceil \frac{1}{\varepsilon} \ln \frac{h_0}{\theta} \right\rceil$ where $\theta = \min\{2\rho\mu, \frac{1}{\varepsilon}\}$. (b) The total time spent in successful simplex steps (excluding the last one) is at most $\frac{T_{\max}}{\rho^2} \ln \frac{h_0}{\varepsilon}$ where $T_{\max}$ is the maximum value of $T_0$ from line 6. (b) In the AFW case every step is either a drop step or a good step, and consequently $N_{\text{drop}} \leq (M + 1)\Delta_{\dim}N_{\text{good}} + \dim(x^{(0)})$. 

4 FW algorithms with self-reducible oracles

Our second contribution is to extend in-face FW directions [8] to more general scenarios. We assume that polytope $P \subset \mathbb{R}^n$ comes with a procedure \texttt{Reduce}_P(\alpha, s)$ (called a self-reducible oracle) that takes vector $\alpha \in \mathcal{P}$ and atom $s \in \text{Atoms}(P)$ and produces polytope $Q \subset \mathbb{R}^{\tilde{n}}$, $\tilde{n} \leq n$ together with a linear injective mapping $\mathcal{L}^{Q \rightarrow P} : Q \rightarrow P$. We require the following properties:

[P1] Polytope $\hat{P} = \mathcal{L}^{Q \rightarrow P}(Q)$ satisfies $\text{supp}(\alpha) \cup \{s\} \subseteq \hat{P} \subseteq P$.

[P2] Mappings $\mathcal{L}^{Q \rightarrow P} : Q \rightarrow \hat{P}$ and $(\mathcal{L}^{Q \rightarrow P})^{-1} : \hat{P} \rightarrow Q$ can be efficiently evaluated for any inputs $\hat{y} \in Q$ and $y \in \hat{P}$, respectively.

[P3] Polytope $Q$ admits an efficient min-oracle (and an efficient max-oracle, in the case of type-II methods).

[P4] Function $\hat{f} : Q \rightarrow \mathbb{R}$ defined via $\hat{f}(\hat{y}) = f(\mathcal{L}^{Q \rightarrow P}(\hat{y}))$ can be efficiently evaluated at any $\hat{y} \in Q$ together with its gradient.

Clearly, mapping $\mathcal{L}^{Q \rightarrow P} : Q \rightarrow P$ can be extended in a natural way to the mapping $\mathcal{L}^{Q \rightarrow P} : \overline{Q} \rightarrow \mathcal{P}$ for the examples considered in this paper. The inverse mapping can also be efficiently computed (for states $\beta \in \mathcal{P}$ for which it exists). For brevity, we will denote $\beta^{Q \rightarrow P} = \mathcal{L}^{Q \rightarrow P}(\beta)$ and $\beta^{P \rightarrow Q} = \mathcal{L}^{P \rightarrow Q}(\beta)$.

\textbf{Example 1.} Suppose that $P = \text{conv}(A)$ for some set of atoms $A \subseteq \{0, 1\}^n$. One valid choice for procedure \texttt{Reduce} is as follows: given vectors $x = x[\alpha] \in P$ and $s \in P$, for each $\ell \in \{0, 1\}$ compute $V_\ell = \{i \in [n] | x_i = s_i = \ell\}$ and let $Q$ be the $\tilde{n}$-dimensional polytope for $\tilde{n} = n - |V_0| - |V_1|$ obtained from $P$ by fixing variables in $V_\ell$ to $\ell$. Clearly, with a proper data structure mapping $\mathcal{L}^{Q \rightarrow P}$ and its inverse can be evaluated in $O(n)$ time. We refer to Sec. 4.1 for further examples.

The following important fact can be deduced from properties [P1]-[P4] and from the definition of $\text{supp}(\alpha)$.

[P5] If $\texttt{Reduce}_P(\alpha, s)$ returns $Q$ then $\hat{w}(\alpha^{P \rightarrow Q}) = w(\alpha)$ where $\hat{w}(\cdot)$ is the quantity in (4a) for $Q$.

Let us assume that we have a procedure $\text{CGstep}(\alpha, \Phi) \mapsto (\tilde{\alpha}, \tilde{\Phi}, \tilde{T})$ that returns updated vector $\tilde{\alpha}$, estimate $\tilde{\Phi}$ of $w(\alpha)$, and time $\tilde{T}$ taken by the call. In Algorithm 6 we describe how to wrap it inside a recursive procedure using the self-reducible oracle. It should be called via $\text{CG}'_{\text{recursive}}(0, P, \alpha(0), *, *)$ where $P$ is the original polytope and $* \ast$ are arbitrary values (they will not be used). Lines 2-8 reduce the problem and make a recursive call for a smaller subproblem; if we are already at the smallest possible problem, then we exit at line 11 and go to the second stage (lines 10-12) that calls $\text{CGstep}(\ldots)$. We expect that calling $\text{CGstep}(\ldots)$ decreases the objective by $\Omega(\Phi^2)$. We want the relative progress at a given level to be at least as large as the progress that $\text{CGstep}(\ldots)$ would have made at the parent level. Accordingly, the algorithm backtracks if $\Phi$ is too small compared to $\Phi_{\text{parent}}$ normalized by the runtimes at the respective levels.

\footnote{If $\alpha = (x, \mathcal{W})$ (as in the CacheDiCG method) then it may happen that some vectors $v \in \mathcal{W}$ do not belong to $\hat{P}$. In that case we simply remove such $v$ from the working set.}
Algorithm 6: \( \text{CG}_{\text{recursive}}(\text{depth}, P, \alpha, \Phi_{\text{parent}}, T_{\text{parent}}) \mapsto \alpha' \): recursive FW

1. for \( t = 0, 1, \ldots \) do
   2. compute \( x = x[\alpha] \) and \( g = \nabla f(x) \)
   3. compute \( s \leftarrow \arg \min_{s \in \mathcal{B}} (g, s) \)
   4. compute \( w(\alpha) \), set \( \Phi \leftarrow w(\alpha) \)
   5. let \( T = \min(T', T_{\text{parent}}) \) where \( T' \) is the time spent in lines 2-4
   6. if \( \text{depth} > 0 \) and \( t > 0 \) and \( \Phi^2/T < \Phi_{\text{parent}}^2/T_{\text{parent}} \) then return \( \alpha \)
   7. call \( Q \leftarrow \text{Reduce}_P(\alpha, s) \); if \( Q = P \) then break
   8. call \( \tilde{\alpha} \leftarrow \text{CG}_{\text{recursive}}(\text{depth} + 1, Q, \alpha^{P \rightarrow Q}, \Phi, T) \), set \( \alpha \leftarrow \tilde{\alpha}^{Q \rightarrow P} \)

9. for \( t = 0, 1, \ldots \) do
   10. call \( (\alpha', \Phi', T) \leftarrow \text{CGstep}(\alpha, \Phi) \)
   11. if \( \text{depth} > 0 \) and \( t > 0 \) and \( \Phi^2/T < \Phi_{\text{parent}}^2/T_{\text{parent}} \) then return \( \alpha \)
   12. update \( \alpha \leftarrow \alpha' \)

We will consider three possible implementations for \( \text{CGstep}(\alpha, \Phi) \):

1. Call \( x' \leftarrow \text{DiCGstep}(x) \) for \( x = \alpha \), compute \( \Phi' = w(x) = (g, v - s) \) in this call, measure time \( T' \) of this call and return \( (\alpha', \Phi', T') \).
2. Call \( (\alpha', \Phi') \leftarrow \text{BCGstep}(\alpha, \Phi) \), measure time \( T' \) of this call and return \( (\alpha', \Phi', T') \).
3. Call \( (x', W') \leftarrow \text{CacheDiCGstep}(x, W) \) for \( (x, W) = \alpha \) with \( \kappa < \infty \), compute \( \Phi' = w(x) = (g, v - s) \) in the call to \( \text{DiCGstep} \), measure time \( T' \) of the call to \( \text{DiCGstep} \) and return \( ((x', W'), \Phi', T') \).

Next, we analyze Algorithm 6 with these choices. We assume for simplicity that at lines 10-12 we always have \( \text{depth} > 0 \). Note, once this assumption is violated then the algorithm will stay forever in the loop 10-12, and so it reduces to one of the three algorithms analyzed previously.

We say that the call at line 10 is a **inner step**, while computations at lines 2-4 are an **outer step**. An inner step is **failed** if the test at line 11 fails. Similarly, an outer step is **failed** if the test at line 6 fails. By definition, steps with \( t = 0 \) are non-failed. This means that the number of failed steps does not exceed the number of non-failed steps (since the loops 2-8 and 10-12 terminate after a failed step). Accordingly, we will be interested in the number and runtime of non-failed steps.

An inner step is **good** (drop, halve, ...) if it is non-failed and the corresponding call to \( \text{CGstep} \) is good (resp. drop, halve, ...), as defined in previous sections. As shown previously, in each case there exist constants \( \rho, K \) with the following properties.

**Proposition 6.** Consider inner step \( (\alpha', \Phi', T) \leftarrow \text{CGstep}(\alpha, \Phi) \). (a) Its runtime is \( O(T) \). (b) If \( \Phi = w(\alpha) \) then \( \Phi' = \Phi \). (c) If this step is good then \( f(\alpha) - f(\alpha') \geq \min\{\rho(\Phi')^2, \Phi'/(2K)\} \).

For an outer step \( O \) we define \( \Phi_O \) and \( T_O \) to be the values \( \Phi \) and \( T \) computed at lines 4-5. Similarly, for an inner step \( I \) of the form \( (\alpha', \Phi', T) \leftarrow \text{CGstep}(\alpha, \Phi) \) define \( \Phi_I = \Phi' \) and \( T_I = T \). Value \( T_S \) will be called the effective runtime of step \( S \). We expect computations at larger depths to be faster than that at lower depths, and so for outer step \( O \) value \( T_O \) will usually be equal to the actual runtime of \( O \). By Proposition 6(a), \( T_I \) for inner step \( I \) is within a constant factor of the actual runtime of \( I \). We will thus focus on bounding effective rather than actual runtimes.

An outer step contains all inner and outer steps that happen inside a call to \( \text{CG}_{\text{recursive}}(\ldots) \) at line 8, assuming that line 8 is reached. If instead the algorithm exits at line 7 then this outer step contains all inner steps at lines 10-12 (an no outer steps). Note, a failed outer step does not contain any inner or outer steps. By construction, a non-failed outer step \( O \) contains at least one inner step \( I \). The first such step will be denoted as \( I_{\text{first}}(O) \). Note that \( I = I_{\text{first}}(O) \) is non-failed since it has \( t = 0 \). Furthermore, we will have \( \Phi_I = \Phi_O \) by property [P5] and Proposition 6(b).
Let $T_{\text{max}}, T_{\text{min}}$ be respectively the maximum and the minimum effective runtimes (of either outer or inner steps), and $\tau_{\text{max}}$ be the maximum ratio $T_{\text{inner}}/T_{\text{outer}}$ where $T_{\text{inner}}$ is the effective runtime value of an inner step $I$ with $t = 0$, and $T_{\text{outer}}$ is the effective runtime of an outer call that immediately precedes $I$. A non-failed step $S$ will be called large if $\Phi_S \geq \frac{1}{2K\rho}$, and small otherwise. Note, good large steps $S$ improve the objective by at least $\Phi_S/(2K) \geq \frac{1}{4K\rho}$, by Proposition 6.

At each moment current state $\alpha$ corresponds to some state $\alpha_{\text{root}}$ at the root level. We will mark quantities related to $\alpha_{\text{root}}$ with the index “root”, e.g. $h_{\alpha_{\text{root}}} = h(\alpha_{\text{root}}) = f(\alpha_{\text{root}}) - f^*$.

**Theorem 7.** Consider a run of Algorithm 6 until encountering vector $\alpha$ with $h_{\alpha_{\text{root}}} \leq \varepsilon$. Let $h_0 = h_{\alpha_{\text{root}}} (\alpha^{(0)})$. (a) Good inner steps satisfy

$$N_{\text{good}}^{\text{small}} \leq \left[ \frac{T_{\text{max}}}{T_{\text{min}}} \cdot \frac{\tau_{\text{max}}}{2\rho \mu} \cdot \ln \frac{h_0}{\varepsilon} \right]$$

$$N_{\text{good}}^{\text{large}} \leq K \sqrt{\frac{T_{\text{max}}}{T_{\text{min}}} \cdot \frac{\tau_{\text{max}}h_0}{\mu}}$$

$$T_{\text{good}}^{\text{small}} \leq T_{\text{max}} \cdot \frac{\tau_{\text{max}}}{2\rho \mu} \ln \frac{h_0}{\varepsilon}$$

(8)

where “$N_{\text{good}}^{\text{small}}$” stands for the number and “$T_{\text{good}}^{\text{small}}$” stands for the effective time of good steps of the appropriate type. (b) Outer steps at a fixed depth $d$ satisfy

$$N_{\text{good}}^{\text{small}} \leq \left[ \frac{T_{\text{max}}}{T_{\text{min}}} \cdot \frac{1}{2\rho \mu} \cdot \ln \frac{h_0}{\varepsilon} \right]$$

$$N_{\text{good}}^{\text{large}} \leq K \sqrt{\frac{T_{\text{max}}}{T_{\text{min}}} \cdot \frac{h_0}{\mu}}$$

$$T_{\text{good}}^{\text{small}} \leq T_{\text{max}} \cdot \frac{1}{2\rho \mu} \ln \frac{h_0}{\varepsilon}$$

(9)

(c) The number of non-failed outer steps at a fixed depth $d$ does not exceed the number of non-failed inner steps. (d) In the case of (Cache-)DiCGstep with AFW every good step is either good or drop, and $N_{\text{drop}} \leq (M + 1) \Delta \text{dim} N_{\text{good}}^{\text{small}} + \text{dim}(x[\alpha^{(0)}])$, where $M = 0$ in the case of DiCGstep. (e) In the case of BCGstep we have $N_{\text{drop}} \leq N_{\text{good}}^{\text{small}} + |\text{supp}(\alpha^{(0)})| - 1$ and $N_{\text{halve}} \leq N_{\text{good}} + N_{\text{drop}}$.

The main conclusion from Theorem 7 is that good steps have roughly the same upper bound on the runtime as that of the corresponding non-recursive algorithm (up to a multiplicative constant and up to an initial burn-in period that does not depend on $\varepsilon$). The bound on steps of other types is in general worse in the recursive version compared to the non-recursive version. However, in practice good steps often dominate other steps, and so we believe it makes sense to try to optimize the runtime of good steps.

4.1 Implementation of the Reduce procedure

In this section we describe an implementation of procedure Reduce for the following combinatorial polytopes of the form $P = \text{conv}(A)$ for $A \subseteq \{0, 1\}^n$:

(a) $A = \{x \in \{0, 1\}^n : ||x||_1 = 1\}$ (then $P$ is the simplex with $n$ coordinates).

(b) Given a directed acyclic graph $G = (V, A)$ with a single source $s$ and single sink $t$, $A \subseteq \{0, 1\}^{V \cup A}$ is the set of $s$-$t$ paths in $G$ expressed via indicator variables.

(c) Given an undirected graph $G = (V, E)$, $A \subseteq \{0, 1\}^E$ is the set of perfect matchings in $G$ expressed via indicator variables.

In each case we can write $A \subseteq \{0, 1\}^{A \cup B}$ where $A, B$ are disjoint finite sets such that $x_A$ uniquely determines $x_B$ for $x \in A$. In particular, $B = \emptyset$ in (a,c), and $B = V$ in (b).

Let us fix parameter $d_{\text{max}} \geq 1$ (maximum depth). Procedure Reduce($\alpha, s$) will depend on the current depth and on the vector $g = \nabla f(x)$, $x = x[\alpha]$. (For brevity of notation, this dependence is not reflected in Algorithm 6). If $\text{depth} = d_{\text{max}}$ then we return the current polytope $P$. Otherwise we compute set $A_0 = \{e \in A \mid x_e = s_e = 0\}$ of size $N_0 = |A_0|$, then compute subset $\tilde{A}_0 \subseteq A_0$ of fixed size $|\tilde{A}_0| = \tilde{N}_0 \in [0, N_0]$, and construct polytope $Q$ by fixing variables in $\tilde{A}_0$ to 0. These steps are elaborated below.

We set $\tilde{N}_0 = \left\lceil N_0 \cdot (N_0/N)^{d_{\text{max}}-\text{depth}-1} \right\rceil$, then each reduction shrinks the size of vector $x$ by approximately the same factor. We assign weight $z_e$ to each $e \in A_0$, and then select $\tilde{N}_0$ elements
in $A_0$ with the smallest values of $z_e$ (by using a linear-time QuickSelect algorithm). In the cases (a,b) we set $z_e = \min_{v \in A} g_{v,e}$. Note, in the case (a) this simply means $z_e = g_e$, while in (b) all values $z_e$ can be computed in linear time by forward and backward passes through the DAG. In the case (c) computing values above would be too expensive. Instead, we set $z_e$ to be the dual slack variable for edge $e$ computed when solving $\min_{x \in A} \langle g, v \rangle$ (see e.g. [14]).

After fixing variables in $A_0$ to 0 we obtain a smaller problem. In the cases (b,c) we try to reduce the size of the problem further by exploiting the combinatorial structure. Namely, in the case (b) we take all directed edges $e = (i,j)$ such that $e$ is the only outgoing edge of $i$ and the only incoming edge of $j$, and contract such edges (i.e. merge nodes $i,j$ into a single node). Furthermore, we remove all edges that do not lie on any $s-t$ path. In the case (c) we find vertices $i$ with degree 1, fix the incident edge $(i,j)$ to 1 and edges $(j,k)$ for $k \neq i$ to 0, remove these edges from the graph, and repeat the procedure until the graph stops changing. (These computations can easily be implemented in linear time).

5 Experimental results

In all examples below (except for video co-localization) the objective function is defined as follows: $f(x) = \|Ax - b\|^2, b = Ax^* + w$ where $A \in \mathbb{R}^{m \times n}$ is a matrix with i.i.d. components generated uniformly at random from $[0,1]$, $w \sim \mathcal{N}(0,\sigma^2I_m)$ is a noise, and $x^* \in \mathcal{P}$ is a fixed vector. Unless noted otherwise, $\sigma$ was computed as follows: we took the smallest integer $\sigma$ for which we could determine that $\min_{x \in P} f(x) > 0$, and added 1 to it. Experimentally, most problems appear to exhibit a sharp transition: $\min_{x \in P} f(x)$ suddenly drops to 0 once $\sigma$ is below a certain threshold. One exception was the sparse signal recovery problem; we took $\sigma = 0.1$ in that case. We always used $m = 200$. Next, we discuss polytopes $P = \text{conv}(A)$ that we used.

**Sparse signal recovery** In our first example we consider problem $\min_{x \in \mathbb{R}^n: \|x\|_1 \leq \|x^*\|_1} \|Ax - b\|_2^2$. This corresponds to the lasso formulation of the sparse signal recovery problem [25]. We set $n = 500$ and $x^* = (1, \ldots, 1, 0, \ldots, 0)$ with $\|x\|_1 = 20$. Following [6], we use an equivalent reformulation of the problem:

$$\min_{z \in \tau \Delta_{2n}} \|y - A(z_{1:n} - z_{n+1:2n})\|_2^2$$

where $\Delta_{2n}$ is the standard simplex with $2n$ variables and $\tau = \|x^*\|$. The polytope $P$ is now the simplex $\Delta_{2n}$, and $\Delta \mathcal{CG}$ can be easily applied.

**Paths in a DAG** Given a directed acyclic graph $G = (V,A)$ with a single source $s$ and a single sink $t$, let $A$ be the set of $s-t$ paths in $G$, where we represent each path via an indicator vector in $\{0,1\}^{V \cup A}$. Min- and max-oracles reduce to respectively computing min- and max-weight $s-t$ paths in a DAG, which can be done in linear time. We use the following graph $G$: the set of nodes is $V = \{s,t\} \cup \{ia \mid i \in [N], a \in [K]\}$ for positive integers $N,K$, and the edges are $A = \{(s,1b), (ia,(i+1)b), (na,t) \mid a \in [K], i \in [N-1]\}$. Note, $s-t$ paths in $G$ correspond to an MRF optimization problem on a chain, which is frequently used for sequence labeling problems.

**Video co-localization** We consider the task of video co-localization on the aeroplane class of the YouTube-Objects dataset, using the problem formulation of [13]. The set of atoms $A$ corresponds to 5 independent $s-t$ path problems in a DAG, with a total of 660 nodes (excluding terminals). The objective function is $f(x) = \frac{1}{2} x^\top Ax + b^\top x$, where matrix $A \in \mathbb{R}^{660 \times 660}$ and vector $b \in \mathbb{R}^{660}$ are given explicitly.

**Perfect matchings** Given an undirected graph $G = (V,E)$ with $|V| = n$, let $A$ be the set of perfect matchings in $G$, where we represent a perfect matching via an indicator vector in $\{0,1\}^E$. We use three types graphs:

(a) Random regular bipartite graph with $2n$ nodes and $kn$ edges (“pm-bipartite”); if $k = n$ then
we instead use the complete bipartite graph \( K_{n,n} \), so that \( P \) is the Birkhoff polytope.

(b) Random graph with \( n \) vertices and \( kn \) edges ("pm-random").

(c) \( k \)-nearest-neighbor graph for a random graph whose nodes have 2D-components ("pm-euclid").

Note, in the case (a) the polytope is defined by \( P = \{ x \in \mathbb{R}_E^E : \sum_{e=\{i,j\} \in E} x_e = 1 \forall i \in V \} \), and so DiCG can be easily applied. We used the Blossom V algorithm [14] to compute minimum-cost perfect matchings.

**Algorithms**  We compared the following algorithms: BCG, DiCG, CacheDiCG (that do not use self-reducible oracle) and BCGd, DiCGd, CacheDiCGd where \( d = d_{\text{max}} \in \{1, 2\} \) is the maximum depth of recursion (see Section 4.1)\(^5\). For sparse signal recovery, paths in a DAG and video co-localization problems we used (Cache-)DiCG with pairwise steps (PFW), while for \( \text{pm-random} \) we used away steps (AFW). (PFW in the latter case performed poorly, perhaps because the majority of steps were not good steps). Note that DiCG is not applicable for \( \text{pm-random} \) and \( \text{pm-euclid} \).

All algorithms were implemented in C++. We used a compact representation of atoms (depending on the problem at hand), and implemented fast operations with these atoms (such as computing an inner product with a gradient). The tests were run on an Intel(R) Core(TM) i5-5200U CPU @ 2.20GHz processor.

Results are given in Fig. 1. Each plot shows simultaneously function value (upper curve) and lower bound (lower curve). Note, the former is always monotonic; we also made the latter curve monotonic by taking the maximum of all lower bounds computed so far. For each instance we took the best values of function values and lower bounds that we could compute (\( f_{\text{best}} \) and \( \ell_{\text{best}} \)), and subtracted \( \frac{1}{2}(f_{\text{best}} + \ell_{\text{best}}) \) from all values. We then used symmetric log scaling for the Y-axis ("symlog" in Python matplotlib, with parameter "linthresh" set manually).

From the plots we can make the following conclusions:

- Self-reducible oracles reduce the runtime of respective algorithms by one order of magnitude or more.

- CacheDiCG outperforms DiCG in the cases when the min-/max-oracles are slow compared to other computations, as is the case for perfect matchings.

- Recursive algorithms with \( d_{\text{max}} = 2 \) can outperform corresponding algorithms with \( d_{\text{max}} = 1 \), but this is not always the case. We conjecture then the advantage of larger \( d_{\text{max}} \) is more pronounced when the runtime of the oracles grows superlinearly with the instance size.

6 **Conclusions and discussion**

We believe the two main messages of this paper are as follows: (i) any practical implementation of a FW-style algorithm should use self-reducible oracles; (ii) if min-/max-oracles in DiCG are expensive, then one should combine DiCG with a caching strategy.

In our work we left out one important class of FW algorithms, namely Block-Coordinate FW algorithms [16, 22, 20]. We expect that self-reducible oracles should also speed up BCFW. However, the criterion of when to terminate computations at a given depth may need to be modified, since current BCFW algorithms do not compute an analogue of quantity \( \Phi = w(\alpha) \). We envisage two possibilities: (i) use the improvement in the objective function during one iteration instead of \( \Phi^2 \); (ii) terminate the call at depth \( d \) when the time of this call exceeds some constant times the runtime at depth \( d - 1 \). This is left as a future work.

\(^5\)Note that procedures \( \text{BCGstep} \) and \( \text{CGstep}^{\text{recursive}} \) reject certain steps; this is needed to get a bound on non-good steps in some pathological cases. We believe such pathological cases do not occur in practice; to simplify the implementation, we modify the procedures so that all steps are accepted. Also, we set \( \kappa = +\infty \) in \( \text{CacheDiCGstep} \) to reduce the number of parameters that need to be set.
Figure 1: Plots of function values and lower bounds (Y-axis) vs. time (X-axis, in seconds).
A Analysis of DiCG: proof of Theorem 1

(a) Let $\gamma \geq 0$ be the value with $x' = y = x + \gamma d$. The $L$-smoothness assumption gives

$$ f(x) - f(y) \geq -\langle g, \gamma d \rangle - \frac{L}{2}\gamma^2 \geq -\gamma \langle g, d \rangle - \frac{L}{2}\gamma^2 D^2 $$

If the call is good then two cases are possible:

- $y \neq u$, then $\min \{ f(x + \gamma d) \mid \gamma \geq 0 : x + \gamma d \in P \} = \min \{ f(x + \gamma d) \mid \gamma \geq 0 \}$ and so

  $$ f(x) - f(y) \geq \min_{\gamma \geq 0} \{ -\gamma \langle g, d \rangle - \frac{L}{2}\gamma^2 D^2 \} = \frac{(g, d)^2}{2LD^2} $$

assuming that $\langle g, d \rangle \leq 0$. Denote $d_F = s - v = d^F + d^A$, then $\langle g, -d_F \rangle = w(x)$. In the PFW case we have $\langle g, -d \rangle = \langle g, -d_F \rangle = w(x)$, while in the AFW case $\langle g, -d \rangle \geq \frac{1}{2} \left( \langle g, -d_F \rangle + \langle g, -d^A \rangle \right) \geq w(x)/2$. Thus, in both cases $h(x) - h(x') \geq \rho \cdot (w(x))^2$.

- $y = u = s$ and $\dim(y) \geq \dim(x)$. Since $y$ is an atom, we have $P_y = \{ y \}$ and $\dim(y) = 0$, and so $\dim(x) = 0$, i.e. $P_x = \{ x \}$. This means that $v = x$, and so $d = s - v$ and $\langle g, d \rangle = -w(x)$. Therefore,

  $$ f(x) - f(y) \geq \max_{\gamma \in [0,1]} \{ \gamma \cdot w(x) - \frac{L}{2}\gamma^2 D^2 \} \geq \min \left\{ \frac{(w(x))^2}{2LD^2}, \frac{w(x)}{2} \right\} $$

(b) Two cases are possible:

- $w(x) \leq \frac{1}{2\rho}$. Then $h(x) - h(x') \geq \rho \cdot (w(x))^2 \geq \rho \cdot (2\mu h(x))$ by part (a) and the Geometric Strong Convexity.

- $w(x) > \frac{1}{2\rho}$. Then $h(x) - h(x') \geq \frac{1}{2} w(x) \geq \frac{1}{2} w^+(x) \geq \frac{1}{2} h(x)$.

In each case $h(x) - h(x') \geq \theta h(x)$, or equivalently $h(x') \leq (1 - \theta) h(x)$. This implies that $N_{\text{good}} \leq \left\lceil \frac{1}{\theta} \ln \frac{\rho}{\mu} \right\rceil$.

(c) Suppose that the step is neither a drop step nor a good step, then $\dim(y) \geq \dim(x)$ and $y = u \neq s$. Suppose that $d = d_F$, then $u = x + \eta(s - x)$. Since $x \in P$, $s$ is a vertex of $P$ and $\eta$ is chosen as the maximum value with $x + \eta(s - x) \in P$, we must have either $x = s$ or $\eta = 1$. But then $u = s$, a contradiction. We showed that $d \neq d_F$, and so $d = d_A = x - v$ (recall that we assumed the AFW case). Note that $v \neq x$ since $h(y) \neq h(x)$ by assumption. Since $x, v \in P_x$, we must have $y = u = x + \eta(x - v) \in P_x$ and so $P_y \subseteq P_x$. Vector $y$ cannot lie in the relative interior of $P_x$ (since $\eta$ is chosen as the maximum value with $x + \eta(x - v) \in P_x$). Thus, face $P_y$ is a strict subset of face $P_x$, which implies that $\dim(y) < \dim(x)$ - a contradiction.

B Analysis of BCG: proof of Theorem 2

(a) If $\Phi \geq w(\alpha)/2$ then $(\alpha, \Phi)$ is valid by the Geometric Strong Convexity assumption and by the bound $h(\alpha) \leq \alpha(\alpha) \leq w(\alpha)$. Thus, the initial pair $(\alpha(0), w(\alpha(0)))$ is valid, and the pair $(\alpha, \Phi')$ returned in line 11 is also valid (since we have $w(\alpha) \leq w^- + \alpha(\alpha) \leq \Phi/2 + \Phi/2 = \Phi$, and so $\Phi' = \min \{ w(\alpha), \Phi/2 \} \geq w(\alpha)/2$). If the call BCGStep$(\alpha, \Phi)$ for a valid $(\alpha, \Phi)$ does not terminate at line 11 then we output $(\alpha', \Phi')$ satisfies $h(\alpha') \leq h(\alpha)$, and thus is valid.

(b) Suppose that BCGStep$(\alpha, \Phi)$ makes a good step. If the S1DO oracle is called in line 5, then we must have $\Phi \geq w^- + (\alpha)$, and so the specification of the oracle gives $f(\alpha) - f(\alpha') \geq \frac{(w^- (\alpha))^2}{2\rho} \geq \frac{\Phi^2}{2\rho}$. 

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Now suppose that the algorithm reached line 12, either by calling the “lazy oracle” in line 7 or the exact oracle in line 9. In each case we have \((g, x - s) \geq \Phi / K\). We have \(f(\alpha') = \min_{\gamma \in [0,1]} f(\gamma s + (1 - \gamma)x)\), so by the Bounded Curvature assumption

\[
f(\alpha) - f(\alpha') \geq \max_{\gamma \in [0,1]} [\gamma(g, x - s) - \gamma^2 \cdot C / 2] \\
\geq \max_{\gamma \in [0,1]} [\gamma \cdot \Phi / K - \gamma^2 \cdot C / 2] = \min \left\{ \frac{(\Phi / K)^2}{2C}, \frac{\Phi / K}{2} \right\}
\]

(c) Consider a good step \((\alpha, \Phi) \mapsto (\alpha', \Phi)\) for valid \((\alpha, \Phi)\). Denote \(h = h(\alpha)\) and \(h' = h(\alpha')\). Two cases are possible:

- \(\Phi \leq \frac{1}{2RK}\), then \(\min\{\rho \Phi^2, \Phi / (2K)\} = \rho \Phi^2\) and so \(h - h' \geq \rho \Phi^2 \geq \frac{\rho \Phi}{4} h\).
- \(\Phi > \frac{1}{2RK}\), then \(\min\{\rho \Phi^2, \Phi / (2K)\} = \Phi / (2K)\) and so \(h - h' \geq \frac{1}{2K} \Phi \geq \frac{1}{4K} h\).

In each case \(h - h' \geq \theta h\), or equivalently \(h' \leq (1 - \theta)h\). This implies that

\[N_{\text{good}} \leq \left\lceil \frac{\ln \frac{h_0}{\epsilon}}{\ln \frac{1}{1 - \theta}} \right\rceil \leq \left\lceil \frac{1}{\theta} \ln \frac{h_0}{\epsilon} \right\rceil\]

A drop step decreases \(|\text{supp}(\alpha)|\) by at least one, while a good step increases \(|\text{supp}(\alpha)|\) by at most one. This implies the bound for \(N_{\text{drop}}\).

Finally, we cannot have two consecutive halve steps (since for the second step we would have \(\Phi = w(\alpha) / 2\), and so at least one of the quantities \(w^+(\alpha), w^-(-\alpha)\) will be at least \(\Phi\)). Thus, \(N_{\text{halve}} \leq N_{\text{good}} + N_{\text{drop}}\). Also, we have \(N_{\text{halve}} \leq \left\lceil \log_2 \frac{w(\alpha(0)^+)/2}{\epsilon/2} \right\rceil\) since each halve step decreases \(\Phi\) by a factor of 2 or more, and for the penultimate pair \((\alpha, \Phi)\) we have \(\Phi \geq h(\alpha)/2 > \epsilon/2\).

C DiCG with a working set

C.1 Simplex Shadow step: proof of Proposition 3

[19, Section 2] describes an alternative characterization of vector \(d\) in eq. (7), due to [18]. Applying this characterization to our problem gives that \(d\) is the projection of vector \(-\nabla \tilde{f}(\lambda_0)\) onto the set

\[\Phi = \{d \in \mathbb{R}^{m+1} : d_1 \geq 0, \ldots, d_m \geq 0, d_0 + d_1 + \ldots + d_m = 0\}\]

Note that \(\nabla \tilde{f}(\lambda_0) = c\), where \(c\) is the vector computed at line 2 of Algorithm 4. Thus, we need to show vector \(-c^\delta\) is the projection of \(-c\) onto \(\Phi\), or equivalently that \(c^\delta\) is a projection of \(c\) onto \(\Phi = -\Phi\).

Let \(c^* = \arg \min_{c^* \in \Phi} ||c^* - c||\) be the projection of \(c\) onto \(\Phi\), and let \(I = \{i \in [m] : c^* < 0\}\). Define

\[
\Phi_I = \{d \in \mathbb{R}^{m+1} : d_i \leq 0 \ \forall i \in [m] - I, d_0 + d_1 + \ldots + d_m = 0\} \\
\Psi_I = \{d \in \mathbb{R}^{m+1} : d_i = 0 \ \forall i \in [m] - I, d_0 + d_1 + \ldots + d_m = 0\}
\]

\(\Phi_I\) is obtained from \(\Phi\) by removing constraints that are not tight at \(c^*\). Thus, \(c^*\) is also a projection of \(c\) onto \(\Phi_I\). We have \(c^* \in \Psi_I \subseteq \Phi_I\), thus \(c^*\) is also a projection of \(c\) onto \(\Psi_I\). The latter is a linear subspace, and projection onto it can be trivially computed. We obtain that \(c_I^* = c_i - \delta^*\) for all \(i \in \{0\} \cup I\), where \(\delta^*\) is some constant.

We claim that \(c_k - \delta^* \geq 0\) for all \(k \in [m] - I\). Indeed, it suffices to prove this claim in the case when \(I = [k - 1]\) (since renaming indexes for elements in \([m]\) does not affect the claim). Let \(\epsilon > 0\) be a sufficiently small number (to be specified later). Define vector \(\hat{c}\) by \(\hat{c}_k = -k\epsilon, \hat{c}_i = c_i^* + \epsilon\) for
\begin{itemize}
  \item Note that \[ c - c^* = (\delta^*, \delta^*, \ldots, \delta^*, c_k, c_{k+1}, \ldots, c_m) \]
  \end{itemize}

We can write
\[ \|c - \hat{c}\|^2 - \|c - c^*\|^2 = k \left( (\delta^* - \varepsilon)^2 - (\delta^*)^2 \right) + \left( (c_k + k\varepsilon)^2 - c_k^2 \right) = k\varepsilon(2(c_k - \delta^*) + (k + 1)\varepsilon) \]

If \( c_k - \delta^* < 0 \) then the last expression will be negative for a sufficiently small \( \varepsilon \), contradicting the fact that \( c^* \) is a projection of \( c \) onto \( \hat{\Phi} \). This proves the claim.

We can now conclude that \( c^* = \hat{c}^\delta \). Since \( \sum_{i=0}^m c_i^\delta = 0 \), we have \( \delta = \delta^* \) and \( \hat{c}^\delta = c^* \).

### C.2 Implementation of the Simplex Shadow step

Given vector \( c = (c_0, c_1, \ldots, c_m) \), our goal is to compute \( \delta \in \mathbb{R} \) such that \( \sum_{i=0}^m c_i^\delta = 0 \) where \( c_0^\delta = c_0 - \delta \) and \( c_i^\delta = \min\{c_i - \delta, 0\} \) for \( i \neq 0 \). Function \( h(\delta) = \sum_{i=0}^m c_i^\delta \) is strictly decreasing with \( h(-\infty) = +\infty \) and \( h(+\infty) = -\infty \), therefore such \( \delta \) exists and is unique. To find it, we first sort values \( c_1, \ldots, c_m \) in the non-decreasing order and then find the smallest \( k \in [m + 1] \) such that \( h(c_k) = c_0 + c_1 + \ldots + c_{k-1} - k c_k \leq 0 \) (where we assume that \( c_{m+1} = +\infty \)). Clearly, we have \( \delta \in [c_{k-1}, c_k] \) if \( k > 1 \), or \( \delta < c_1 \) if \( k = 1 \). \( \delta \) can now be trivially computed. The formal procedure is given in Algorithm 7.

**Algorithm 7**: Implementation of line 3 of Algorithm 4.

1. sort \((c_1, \ldots, c_m)\) so that \( c_1 \leq \ldots \leq c_m \)
2. set \( s \leftarrow c_0 \)
3. for \( k = 1 \) to \( m \) do
   4. if \( s - k c_k \leq 0 \) then break
   5. update \( s \leftarrow s + c_k \)
4. return \( \delta = s/k \)

### C.3 Proof of Theorem 4

We have \( \Delta_0 = f(x) - \min_{q \geq 0} f(x + q\bar{d}) \geq \frac{(q, \bar{d})^2}{2\mu \bar{d}^2} \geq \rho \cdot (w(x))^2 \) by the same argument as in Section A. We also \( \Delta = \sum_t \Delta_t \geq \sum_t \bar{T}_t \cdot (\Delta_0/T_0) = \bar{T} \cdot (\Delta_0/T_0) \) where the sum is taken over successful steps \( t \).

### C.4 Proof of Theorem 5

Parts (a) and (c) can be shown in exactly the same way as in Section A. Let us prove (b). Let \( \Delta_i \) be the improvement of objective during \( i \)-th successful simplex step, \( T_i \) be its runtime, \( x_i \) be the vector at the beginning of this step with suboptimality \( h_i = f(x_i) - f(x^*) \), and let \( n \) be the maximum index with \( h_i - \Delta_i \geq \varepsilon \). Denote \( h_{n+1} = h_n - \Delta_n \geq \varepsilon \). For each \( i \in [n] \) we have \( h_i - h_{i+1} \geq \Delta_i \geq \frac{T_i}{T_{\text{max}}} \rho \cdot (w(x_i))^2 \geq \frac{T_i}{T_{\text{max}}} \rho \cdot (2\mu h_i) \). Therefore,
\[
\sum_{i=1}^n T_i \leq \frac{T_{\text{max}}}{2\mu \rho} \sum_{i=1}^n \frac{h_i - h_{i+1}}{h_i} \leq \frac{T_{\text{max}}}{2\mu \rho} \sum_{i=1}^n \ln \frac{h_i}{h_{i+1}} = \frac{T_{\text{max}}}{2\mu \rho} \ln \frac{h_0}{h_{n+1}} \leq \frac{T_{\text{max}}}{2\mu \rho} \ln \frac{h_0}{\varepsilon}
\]
where we used the fact that \( \ln x \geq \frac{x-1}{x} \) for \( x > 1 \).
D FW algorithm with recursive oracles: proof of Theorem 7

It can be checked that $I_{\text{first}}(O_1)$ and $I_{\text{first}}(O_2)$ are distinct for distinct outer steps $O_1, O_2$ at the same level. This implies, in particular, part (c) of Theorem 7.

An outer step $O$ is called an ancestor of inner or outer step $S$ if $O$ contains $S$ and the depth of $O$ is strictly smaller than the depth of $S$.

**Proposition 8.** If outer step $O$ is an ancestor of non-failed step $S$ then $\Phi_S^2/T_S \geq \Phi_O^2/T_O$ if step $S$ is outer, and $\Phi_S^2/T_S \geq \frac{1}{\tau_{\max}}\Phi_O^2/T_O$ if $S$ is inner.

**Proof.** It suffices to prove the claim in the case when $O$ is an immediate ancestor on $S$. Note that the depth of $S$ cannot be zero. Four cases are possible.

- $S$ is an outer step with $t = 0$. Then $\Phi_S = \Phi_O = w(\alpha)$ (by [P5]) and $T_S \leq T_{\text{parent}} = T_O$.
- $S$ is an outer step with $t > 0$. Since $S$ is non-failed, we have $\Phi_S^2/T_S \geq \Phi_{\text{parent}}^2/T_{\text{parent}} = \Phi_O^2/T_O$.
- $S$ is an inner step with $t = 0$. Let $O'$ be the ancestor outer step at the same level, then $\Phi_S = \Phi_O'$ (by Proposition 6(b)) and $T_S \leq \tau_{\max}T_{O'}$. The claim then follows from one of the two cases above.
- $S$ is an inner step with $t > 0$. Since $S$ is non-failed, we have $\Phi_S^2/T_S \geq \Phi_{\text{parent}}^2/T_{\text{parent}} = \Phi_O^2/T_O$.

Consider good step $S$, and denote $I = I_{\text{first}}(S)$ (if step $S$ is outer) or $I = S$ (if $S$ is inner). Recall that we have $\Phi_S = \Phi_I$ in both cases. Suppose that step $I$ transforms vector $\alpha$ to $\alpha'$. Denote $h = h_{\text{root}}(\alpha)$ and $h' = h_{\text{root}}(\alpha')$. Let $O_{\text{root}}$ be the root ancestor of $S$ and $\alpha_{\text{root}}$ be the vector at the beginning of $O_{\text{root}}$. By Proposition 8 we have

$$\frac{\Phi_S^2}{T_S} \geq \gamma \frac{\Phi_{O_{\text{root}}}^2}{T_{O_{\text{root}}}} = \frac{(w(\alpha_{\text{root}}))^2}{T_{O_{\text{root}}}} \geq \frac{2\mu h_{\text{root}}(\alpha_{\text{root}})}{T_{O_{\text{root}}}} \geq \frac{2\mu h}{T_{O_{\text{root}}}} \geq \frac{2\mu h}{T_{\max}}$$

where $\gamma = 1$ if $S$ is outer and $\gamma = \frac{1}{\tau_{\max}}$ if $S$ is inner. Proposition 6(c) gives

$$h - h' \geq \min\{\rho \Phi_S^2, \Phi_S/(2K)\}$$

Recall that step $S$ is called small if $\Phi_S \leq \frac{1}{2K\rho}$ (in which case $h - h' \geq \rho \Phi_S^2$), and large if $\Phi_S > \frac{1}{2K\rho}$ (in which case $h - h' \geq \frac{\Phi_S}{2K} \geq \frac{1}{4K^2\rho}$). We can now prove parts (a,b) of Theorem 7. Below $N_{\text{good}}$ and $T_{\text{good}}$ refer to the number and effective time of steps of the appropriate type (either inner steps for part (a), or outer steps at a fixed depth for part (b)).

- By the results above, for a small good step $S$ we have $h - h' \geq \rho \Phi_S^2 \geq \rho \gamma \frac{2\mu h}{T_{\max}}$. This implies that the number of small good steps is at most $N_{\text{small}} \leq \left\lfloor \frac{1}{\theta_{\text{small}}} \ln \frac{h_0}{\varepsilon} \right\rfloor$.

For large steps we have $h - h' \geq \Phi_S^2 \geq \frac{2}{K} \sqrt{\gamma \mu T_{\max} \sqrt{h}}$. Denote $c = \frac{2}{K} \sqrt{\gamma \mu T_{\max} \sqrt{h}}$, then

$$\sqrt{h} - \sqrt{h'} \geq \frac{(\sqrt{h} - \sqrt{h})(\sqrt{h} + \sqrt{h'})}{2\sqrt{h}} = \frac{h - h'}{2\sqrt{h}} \geq \frac{c}{2}.$$ Summing this inequality over good large steps gives $\sqrt{h_0} \geq N_{\text{large}} \cdot \frac{c}{2}$, and so $N_{\text{large}} \leq K \sqrt{\frac{h_0}{\gamma \mu T_{\max}}}$. 

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• For a small good step $S$ we have $h - h' \geq \rho \Phi^2 S \geq \rho \gamma \frac{2\rho \nu}{\tau_{max}} T_S$, and so $\ln \frac{h}{h'} \geq \ln \frac{1}{1 - \frac{2\rho \nu}{\tau_{max}} T_S} \geq \frac{2\rho \nu}{\tau_{max}} T_S$. Summing these inequalities gives $\ln \frac{h_0}{\varepsilon} \geq \frac{2\gamma\rho\mu T_{small}}{M + 1} \Delta_{dim}$.

It remains to prove parts (d,e) of Theorem 7. It follows from definitions that transforming vector $\alpha$ during the reduction in lines 7-8 (and transforming it back) does not affect $\dim(x[\alpha])$ (in the case of (Cache)-DiCGstep) or $|\text{supp}(\alpha)|$ (in the case of BCGstep). The claim about (Cache)-DiCGstep with AFW follows by the same arguments as in Section B. (Recall that a good step increases $\dim(x[\alpha])$ be at most $(M + 1)\Delta_{dim}$, while a drop step decreases it by at least 1).

Let us now consider the case of BCGstep. We have $N_{\text{drop}} \leq N_{\text{good}} + |\text{supp}(\alpha(0))| - 1$ since a good step increases $|\text{supp}(\alpha)|$ by at most one while a drop step decreases it by at least 1. Consider loop 10-12. The first inner step (with $t = 0$) cannot be a halve step since in this step we have $\Phi = w(\alpha)$. Also, we cannot have two consecutive halve steps by the same argument as in Section B. Thus, the number of non-failed halve steps in the loop 10-13 cannot exceed the number of non-failed non-halve steps. This implies that $N_{\text{halve}} \leq N_{\text{good}} + N_{\text{drop}}$.

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