Classical and Quantized Tensionless Strings

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Abstract: From the ordinary tensile string we derive a geometric action for the tensionless ($T = 0$) string and discuss its symmetries and field equations. The Weyl symmetry of the usual string is shown to be replaced by a global space-time conformal symmetry in the $T \rightarrow 0$ limit. We present the explicit expressions for the generators of this group in the light-cone gauge. Using these, we quantize the theory in an operator form and require the conformal symmetry to remain a symmetry of the quantum theory. Modulo details concerning zero-modes that are discussed in the paper, this leads to the stringent restriction that the physical states should be singlets under space-time diffeomorphisms, hinting at a topological theory. We present the details of the calculation that leads to this conclusion.

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1 Introduction

The high energy limit of string theory is still quite poorly understood, despite many important and interesting results on high energy scattering, [1]-[6] and high temperature behaviour [7]-[10]. Just like the massless limit in particle theory sheds light on short distance field theory, the zero tension limit, $T \to 0$, of strings is expected to illuminate some short-distance properties of string theory. In particular we hope that the intriguing high energy symmetries discussed by Gross [5] may be studied in this limit. Thereby one would presumably be able to probe the conjectured unbroken "topological" phase of general covariance [11, 12]. Though much work has been done in topological field theory, [13, 14, 15, 16], much less is known about the relation to string theory. The present work supports and substantiates such a connection. In fact, our results point in this direction in quite an unexpected way.

We have approached the problem by first formulating the exactly tensionless case, where additional symmetries relating different mass levels, in particular space-time conformal symmetry, should appear. In a previous series of papers we have studied various aspects of the tensionless case of the bosonic string [19], the superstring [20] and the spinning string [21, 22]. We have also reported on the main result of this paper in a brief letter [17].

In addition to the questions concerning the deep symmetries of string theory that may be revealed in the tensionless limit, there are fundamental problems of string perturbation theory that can be illuminated once a tensionless theory has become available. It was found in [23] that the string tension is unchanged by one-loop corrections (for type II superstrings), but mass level shifts are larger than the level separation for high levels, no matter how small the coupling constant is. Such a behaviour is bound to cause severe problems for the tensile string perturbation theory, and it is probably one of the simplest manifestations of the rapid large order growth of the string perturbation series for massless external states discussed by Gross and Periwal [24]. In similar problems appearing in quantum mechanics, one turns to quasi-degenerate perturbation theory. Taken over to string physics, this recipe would mean that we start from the degenerate case with vanishing zeroth order level separation. But this is precisely the limit of vanishing tension. Thus a future interacting tensionless theory could become a key to non-perturbative string physics.

Other authors have also discussed tensionless strings and their quantization [23, 24, 25, 26] ever since they were first discussed by Schild [27].

One expects either a continuous or a massless spectrum when the scale given by the string tension is removed from the theory. Correspondence with classical tensionless strings would favour a continuous spectrum, but on the other hand all $T \neq 0$ string states approach zero mass as $T \to 0$. The result of the present work indicates that the massless spectrum is the correct answer, but we also find extremely restrictive constraints on the spectrum, effectively allowing only states invariant with respect to general coordinate transformations. One could envisage a
spectrum of string states characterized by topology, but we have not yet found a concrete construction of a satisfactory Hilbert space.

In this paper we will be concerned with the quantization of the tensionless closed bosonic string. In particular we will explore the (space-time!) conformal symmetry of this model and investigate under what condition this symmetry survives at the quantum level. When the two-dimensional reparametrization invariance of ordinary strings is gauge-fixed in the light-cone gauge, anomalies in the local symmetry are reflected in a breakdown of the Lorentz algebra (in non-critical dimensions). Similarly, inconsistencies in the quantum geometry of the tensionless string can be probed by checking the conformal algebra in the light-cone gauge. We are further motivated to demand space-time conformal invariance in the quantized theory because we find that this is the symmetry that replaces Weyl-invariance in the $T \to 0$ limit. Hence when we find obstructions to conformal invariance, rather than conclude that the symmetry is broken, we interpret the obstructions as conditions on the physical states of fundamental strings. In a future tensionless limit of QCD strings, it might instead be more appropriate to accept breakdown of conformal invariance.

We have chosen to make the article relatively self-contained and to include some new result on the classical theory. The article is organized as follows: In Section 2 we present the classical theory. We derive actions for a tensionless string (or tensionless $p$-branes), discuss the symmetries of the string action we choose to work with, present the equations of motion, introduce the light-cone gauge and finally give the compensating reparametrizations needed to stay in that gauge after applying a conformal transformation. Section 3 contains the quantum theory. We begin with a brief discussion of BRST-quantization. The main part of the paper is then the detailed derivation of the anomalies in the light-cone operator algebra and a discussion of their implications. Finally we end the article with our conclusions. A discussion of the relation between vacuum in the tensile and in the tensionless model is given as an appendix.

2 The classical theory

2.1 Actions

In this section we discuss the classical theory of strings in the limit that the tension $T \to 0$. At essentially no extra cost we can discuss such a limit also for higher dimensional objects commonly known as $p$-branes. Although the rest of the article deals exclusively with strings, in this section we keep the discussion general, and the formulae for strings used in other sections are obtained by setting $p = 1$.

Consider a theory given by an action of the form

$$S = T \int d^{p+1} \xi \mathcal{L},$$

(1)
e.g., a $p$-brane with space time coordinates $X^m$, world volume coordinates $\xi^\alpha$ and "tension" $T$. There are numerous ways of rewriting the action so that the $T \to 0$ limit may be taken. We will settle on a formulation that has a geometric interpretation, but first display the simplest generalization of the point-particle action, $(p = 0, T = m)$, involving an auxiliary field $\phi$:

$$ S = \frac{1}{2} \int d^{p+1} \xi \left[ \phi \mathcal{L}^2 + \phi^{-1} T^2 \right]. $$

(The equivalence is seen by integrating out $\phi$.) Here the limit $T \to 0$ can be readily taken. With $\phi \to e$ this procedure yields the reparametrization invariant action involving the einbein $e$ for the massless point-particle. With $p = 1$ and $T$ the string tension this procedure was used in, e.g., [19] to obtain an action for the tensionless string. It is only in the point-particle case that there is a connection to the world-volume geometry, however. We have found it useful to try to maintain such a relation and have therefore chosen a different route to the $T \to 0$ limit.

The starting point is the Nambu-Goto-Dirac world volume action

$$ S = T \int d^{p+1} \xi \sqrt{-\det \gamma_{\alpha\beta}} $$

where $X^m = X^m(\xi)$ and

$$ \gamma_{\alpha\beta} \equiv \partial_\alpha X^m \partial_\beta X^n \eta_{mn} $$

is the metric induced on the world volume from the Minkowski space-time metric $\eta_{mn}$. We reformulate the theory in phase space. The generalized momenta derived from the Lagrangian in (3) are

$$ P_m = T \sqrt{-\gamma_{\alpha0}} \partial_\alpha X_m. $$

where $\gamma^{\alpha\beta}$ is the inverse of $\gamma_{\alpha\beta}$. They satisfy the constraints

$$ P^2 + T^2 \gamma^{00} = 0 $$

$$ P_m \partial_\alpha X^m = 0, \quad a = 1, ..., p. $$

Here $\gamma \equiv \det \gamma_{\alpha\beta}$. As usual for a diffeomorphism invariant theory, the naive Hamiltonian vanishes and the total Hamiltonian consists of the sum of the constraints (6) multiplied by Lagrange multipliers, which we shall call $\lambda$ and $\rho^a$:

$$ \mathcal{H} = \lambda (P^2 + T^2 \gamma^{00}) + \rho^a P \cdot \partial_a X $$

The phase space action thus becomes

$$ S^{PS} = \int d^{p+1} \xi \left\{ P \cdot \dot{X} - \lambda (P^2 + T^2 \gamma^{00}) - \rho^a P \cdot \partial_a X \right\}. $$

We integrate out the momenta to find the configuration space action

$$ S^{CS} = \frac{1}{2} \int d^{p+1} \xi \frac{1}{2\lambda} \left\{ \dot{X}^2 - 2 \rho^a X^m \partial_a X_m + \rho^a \rho^b \partial_b X^m \partial_a X_m - 4 \lambda^2 T^2 \gamma^{00} \right\}. $$
For \( p = 1 \) we may identify

\[
g^{\alpha\beta} = \left( \frac{-1}{\rho} - \rho^2 + 4\lambda^2 T^2 \right) \tag{10}\]

which leads to the usual Weyl invariant tensile string action

\[
S = -\frac{1}{T} \int d^2 \xi \sqrt{-g} \left\{ g^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n \eta_{mn} \right\}. \tag{11}\]

For \( p > 1 \) it is not possible to directly identify the geometric fields in (9). We first have to rewrite it as

\[
S^{CS} = \frac{1}{2} \int d^{p+1} \xi \left\{ \frac{h^{\alpha\beta} \gamma_{\alpha\beta}}{2\lambda} - 2\lambda T^2 G(p - 1) + 2\lambda T^2 G^{ab} \gamma_{ab} \right\} \tag{12}\]

where

\[
h^{\alpha\beta} = \left( \frac{1}{-\rho^a - \rho^b} \right) \tag{13}\]

is a rank 1 auxiliary matrix and \( G_{ij} \) is a \( p \)-dimensional auxiliary metric with determinant \( G \). (Integrating out \( G_{ij} \) we recover (9).) Now the identification

\[
g^{\alpha\beta} = \frac{1}{4} T^{-2} \lambda^{-2} G^{-1} \left( \frac{-1}{\rho^a} - \rho^a \rho^b + 4\lambda^2 T^2 G^{ab} \right) \tag{14}\]

produces the usual \( p \)-brane action involving the world volume metric \( g_{\alpha\beta} \):

\[
S = -\frac{1}{2} T \int d^{p+1} \xi \sqrt{-g} \left\{ g^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n \eta_{mn} - (p - 1) \right\}. \tag{15}\]

The identification (14) tells us the transformation properties of the Lagrange multipliers. Note that for \( p = 0, 1 \) the auxiliary metric \( G_{ij} \) never appears in (12), and the configuration space action is the usual manifestly reparametrization invariant massive point-particle action and Brink-Howe-DiVecchia-Deser-Zumino reparametrization invariant tensile string action [31, 32], respectively.

It is clear from the above procedure that we may take the limit \( T \to 0 \) anywhere between (7) and (14). The identification (14) will differ in that limit, however. The metric density \( T \sqrt{-g} g^{\alpha\beta} \) becomes degenerate and gets replaced by a rank 1 matrix which can be written as \( V^\alpha V^\beta \) in terms of the vector density \( V^\alpha \)

\[
V^\alpha \leftrightarrow \frac{1}{\sqrt{2\lambda}} (1, \rho^a). \tag{16}\]

In fact, using this prescription the \( T \to 0 \) limit of the \( p \)-brane action is

\[
S = \int d^{p+1} \xi V^\alpha V^\beta \partial_\alpha X^m \partial_\beta X^n \eta_{mn}. \tag{17}\]
We will henceforth be concerned with the string case

\[ S = \int d^2 \xi V^\alpha V^\beta \partial_\alpha X^m \partial_\beta X^n \eta_{mn} \]  

(18)

where \( V^\alpha \) are world sheet vector densities of weight \( \frac{1}{4} \) and the world sheet coordinates are \( \xi^\alpha \equiv (\tau, \sigma) \). There exists many formulations of the bosonic tensionless string. The present has many advantages due to its geometric form. First, in the quantum theory in going from the phase space action (8) to the configuration space action (9) one generates a functional determinant which may be interpreted as a modification of the path integral measure. This modification is precisely what is needed to render the measure invariant under 2D diffeomorphisms, i.e., it leads to the Fujikawa variables. This is all the more important since those variables are normally defined by weighting with various powers of the determinant of the 2D metric and here we have no such determinant available. Second, the action (18) is easy to supersymmetrize [20]. In contrast, the action derived as in (4), e.g.;

\[ I_1^0 = \int d^2 \xi \phi \det \gamma_{\alpha\beta} \]  

(19)

cannot be easily extended to the superstring because of the Siegel symmetry of the superstring which transforms the Lagrangian \( \sqrt{- \det \gamma_{\alpha\beta}} \) into a 2D total derivative. In general, we see from the relation between the actions (1) and (2) that whereas \( \delta \mathcal{L} = \partial \cdot \omega \) leaves the action (1) invariant for some transformation with parameter \( \omega \), for invariance of the action (3) in the limit \( T \to 0 \), we need a transformation \( \delta \phi = -\phi \mathcal{L}^{-1} \partial \cdot \omega \). Clearly this leads to difficulties at the level of field equations where \( \mathcal{L} = 0 \). Third, when introducing spin via world sheet supersymmetry it leads naturally to a new 2D superspace geometry which in turn allows for a compact treatment of many different classes of models [22].

2.2 Symmetries

The action (18) for the tensionless string is invariant under world-sheet diffeomorphisms and space-time conformal transformations. Under the diffeomorphisms \( X^m \) transforms as a scalar field

\[ \delta_\varepsilon X^m = \varepsilon \cdot \partial X^m, \]  

(20)

and \( V^\alpha \) as a vector density:

\[ \delta_\varepsilon V^\alpha = -V \cdot \partial \varepsilon^\alpha + \varepsilon \cdot \partial V^\alpha + \frac{1}{2} (\partial \cdot \varepsilon) V^\alpha. \]  

(21)

There are of course many different gauge choices possible for fixing the reparametrization symmetry (21). We have found the following transverse gauge particularly useful:

\[ V^\alpha = (v, 0), \]  

(22)
with $v$ a constant. The transverse gauge corresponds to the conformal gauge $g_{\alpha\beta} = e^\phi \eta_{\alpha\beta}$ in the tensile theory. For classical string propagation, where the world-sheet is cylindrical, one can always reach this gauge, except in the particular case when field lines of the vector density are closed around the cylinder. In this exceptional case, which we do not consider here, one may instead choose $V^\alpha = (0, v)$. For a toroidal geometry there is a continuum of globally inequivalent vector densities. The physical consequences of this fact are currently under investigation.

Just as in the tensile case there is a residual symmetry that leaves (22) invariant;

$$\delta \xi^\alpha = \lambda^\alpha, \quad \lambda^\alpha = (f'(\sigma)\tau + g(\sigma), f(\sigma))$$

(23)

with $f$ and $g$ arbitrary functions of $\sigma$ only. If we define the generators

$$R(f) \equiv f'(\sigma)\tau \partial_\tau + f(\sigma) \partial_\sigma$$
$$T(g) \equiv g(\sigma) \partial_\tau,$$

(24)

we find the following algebra

$$[R(f_1), R(f_2)] = R(f_1 f_2' - f_2 f_1'),$$
$$[R(f), T(g)] = T(f g' - g f'),$$
$$[T(g_1), T(g_2)] = 0$$

(25)

If we furthermore fourier expand $f$

$$f(\sigma) = \frac{1}{2\pi} \sum_n a_n^f e^{2\pi in\sigma},$$

(26)

we may write

$$R(f) = \frac{1}{2\pi} \sum_n a_n^f e^{2\pi in\sigma} (2\pi in\tau \partial_\tau + \partial_\sigma) \equiv -i \sum_n a_n^f R_n$$
$$T(g) = \frac{1}{2\pi} \sum_n a_n^g e^{2\pi in\sigma} \partial_\tau \equiv -i \sum_n a_n^g T_n.$$

(27)

Despite their somewhat complicated form the generators $R^m$ then satisfy the algebra of infinitesimal diffeomorphisms of $S^1$, i.e., the Virasoro algebra:

$$[R_m, R_n] = (m - n)R_{m+n} + \frac{1}{12}(C_R - C_L)(m^3 - m)\delta_{m+n}$$

(28)

and for the mixed commutator we find

$$[R_m, T_n] = (m - n)T_{m+n} + \frac{1}{12}(C_R + C_L)(m^3 - m)\delta_{m+n}.$$

(29)

Here we have also displayed the most general central extension compatible with the Jacobi identities. A simple $T \to 0$ limit of the tensile string algebra yields zero central charges, i.e., $C_L = C_R \to 0$, (but other models are not logically excluded). Thus

Note that although we give the infinitesimal form of the transformations in (23), the finite form looks the same with $\xi^\alpha = \lambda^\alpha$ and different functions $f$ and $g$. 

\[7\]
the residual symmetry (23) is the semi-direct product of \( \sigma \)-dependent \( \tau \)-translations with a Virasoro symmetry.

The Poincaré symmetry is extended to conformal symmetry for massless particles and massless free fields. The conformal group preserves the causal structure of Minkowski space and maps light cones onto light cones. We shall see that in the transverse gauge the tensionless string can be viewed as a collection of massless particles and it is thus natural to expect that classical tensionless strings should enjoy conformal symmetry. In fact, since a conformal transformation will scale the D-dimensional line element, it will also scale the induced metric. This can be compensated by a \((X^m\text{-dependent})\) scaling of \(V^\alpha\), and the action (18) thus be left invariant (see (32) below). Note that this is not possible for the action of the tensile string,

\[
S_1 = -\frac{T}{2} \int d^2\xi \sqrt{-\det g^{\alpha\beta}} g^{\alpha\beta},
\]

since any rescaling of \(g_{\alpha\beta}\) is an invariance of \(\sqrt{-\det g^{\alpha\beta}}\) alone. In fact, in this sense \textit{world-sheet} Weyl-invariance is replaced by space-time conformal invariance in the limit \(T \to 0\).

The infinitesimal transformations form the conformal algebra:

\[
\begin{align*}
[m_{m}, m_{n}] &= \delta^r_{m} m_{n}^r - \delta^r_{n} m_{m}^r + \delta^s_{m} m_{n}^s - \delta^s_{n} m_{m}^s, \\
[m_{m}, p_{n}] &= -\delta^s_{n} pm, \\
[m_{m}, k_{l}] &= -\delta^m_{n} k^m - \delta^m_{l} k^n, \\
[p_{m}, k_{n}] &= -\delta^m_{n} s + m_{m}^n \\
[s, m_{m}] &= 0 \\
[s, p_{m}] &= p_{m} \\
[s, k_{m}] &= -k^m
\end{align*}
\]

Here \(m_{m}, p_{m}, k_{m}\) and \(s\) are the generators of Lorentz transformations, translations, conformal boosts and dilatations, respectively. Note that the whole algebra can be generated from repeated brackets of \(p_{m}\) and \(k_{m}\).

Under Poincaré transformations \(X^m\) behaves as a Lorentz vector and \(V^\alpha\) as a scalar. In contrast, conformal boosts (generator \(k^m\)) and dilations (generator \(s\)) rescale \(V^\alpha\) in order to leave the action invariant when the induced metric (11) is rescaled by the ordinary action of conformal transformations on the coordinates \(X^m\). The infinitesimal conformal boosts and dilatations act as follows:

\[
\begin{align*}
\delta_b X^m &= [b \cdot k, X^m] = (b \cdot X)X^m - \frac{1}{2}X^2b^m, \\
\delta_b V^\alpha &= -b \cdot XV^\alpha \\
\delta_s X^m &= [as, X^m] = aX^m, \\
\delta_s V^\alpha &= -aV^\alpha
\end{align*}
\]

where \(b_m\) and \(a\) are transformation parameters. The finite form of the special conformal transformation of \(V^\alpha\) reads

\[
V'^\alpha = V^\alpha \sqrt{1 + 2b \cdot X + b^2X^2},
\]

8
which shows that conformal transformations may take $V^\alpha$ to zero for some string solutions. Therefore the transverse gauge cannot be imposed globally on the world-sheet for such string states. The connection, if any, of this fact to the local problem of the infinitesimal quantum transformations described below remains to be elucidated.

Let us finally mention in passing that there exists a formulation of the zero tension string where the conformal transformations act linearly. This is achieved by describing the string in a target space with one additional spacelike and one additional timelike coordinate [19]. Recently a hamiltonian treatment of this formulation was given [35]. The symmetries (23) are then enlarged to a particular semi-direct sum between an $SU(1,1)$ affine Kač-Moody and a Virasoro algebra. Essentially, the two additional symmetry generators are needed to compensate for the extra dimensions.

2.3 Equations of motion

The field equations that follow from the action (18) are:

$$V^\beta \gamma_{\alpha\beta} = 0, \quad \partial_\alpha (V^\alpha V^\beta \partial_\beta X^m) = 0 \quad (34)$$

The first of these equations states that $\gamma_{\alpha\beta}$ has an eigenvector with eigenvalue zero which implies that it is a degenerate matrix:

$$\det \gamma_{\alpha\beta} = 0 \quad (35)$$

This means that the world sheet spanned by the tensionless string is a null surface. For this reason tensionless strings are sometimes referred to as "null strings".

The second of the field equations is most easily interpreted in the gauge (22). In this gauge the equations (34) become

$$\ddot{X}^m = 0 \quad (36)$$

$$\dot{X}^2 = \dot{X} \cdot X' = 0.$$ 

Clearly the string behaves classically as a collection of massless particles, one at each $\sigma$ position, constrained to move transversally to the direction of the string.

For open strings there are also edge conditions. With $\sigma \in [0, 1]$, we find from the derivation of (34), that we need to demand

$$\left[ V^1 V^\alpha \partial_\alpha X_m \delta X^m \right]_{\sigma=0,1} = 0 \quad (37)$$

We may implement this by requiring either

$$V^1(\tau, 0) = V^1(\tau, 1) = 0, \quad (38)$$

9
or

$$V^\alpha \partial_\alpha X^m(\tau, 0) = V^\alpha \partial_\alpha X^m(\tau, 1) = 0.$$  

(39)

The condition (37) can be satisfied simply by choosing a gauge which approaches the transverse gauge (22) at the edges of the string, thus fulfilling (38). If we want to be able to impose a "non-transverse" gauge where $V^1 \neq 0$, (39) has to be satisfied. For example, in the gauge $V^\alpha = (0, v)$ it happens to yield the usual $T \neq 0$ open string edge conditions

$$X^m(\tau, 0) = X^m(\tau, 1) = 0$$  

(40)

The first equation in (34) is the $V^\alpha$-field equation. It corresponds to the $g_{\alpha\beta}$-equation $T^\alpha_{\beta\gamma} = 0$ in the tensile theory, i.e., to the Virasoro constraints. In our case the energy-momentum tensor cannot be derived as a field-equation in the same way, but it is nevertheless related to the $V^\alpha$-field equation. The energy momentum mixed tensor(density) with one covariant and one contravariant index is derived as the translation current. It reads

$$T^\beta_{\alpha\gamma} = V^\beta V^\sigma \partial_\sigma X^m \partial_\alpha X_m - \frac{1}{2} V^\sigma V^\rho \partial_\sigma X^m \partial_\rho X_m \delta^\beta_{\alpha}.$$  

(41)

It is traceless and comparing to (34) we see that it vanishes on $V^\alpha$-shell. Furthermore, in analogy to the tensile case, we expect it to be covariantly conserved using the second equation in (34) only. To discuss covariant conservation one has to add more geometric structure to the theory than is needed in the action principle. Introducing a covariant derivative as in [21] or [22], we find that

$$\nabla_\alpha T^\alpha_{\beta} = 0$$  

(42)

provided that

$$0 = \nabla_\alpha V^\beta \equiv \partial_\alpha V^\beta + \Gamma^\beta_{\alpha\sigma} V^\sigma - \frac{1}{2} \Gamma^\sigma_{\alpha\sigma} V^\beta,$$  

(43)

the analogue of the metricity condition in the non-degenerate case. It is interesting that only the weaker condition

$$\nabla_\alpha V^\alpha = 0$$  

(44)

was needed in [21] and [22]. We shall find no more use for the connection $\Gamma^\beta_{\alpha\sigma}$ in this article.

In the transverse gauge (22) the components of the energy-momentum tensor (41) are

$$T^0_0 = -T^1_1 = \frac{1}{2} v^2 \gamma_{00}, \quad T^0_1 = v^2 \gamma_{10} \quad T^1_0 = 0.$$  

(45)

Since there is no metric to raise and lower indices, we cannot ascribe the usual symmetry properties to $T^\alpha_{\beta}$. Nevertheless it still has only two independent components.
Using Poisson brackets the components in \((45)\) is readily seen to generate the semi-direct product of \(\sigma\)-dependent \(\tau\)-translations and a Virasoro symmetry discussed in the previous section. Thus the energy momentum tensor generates a symmetry which becomes the residual symmetry \((23)\) after gauge fixing to transversal gauge, in complete analogy to the tensile case.

### 2.4 The light-cone gauge

We introduce light cone coordinates \((X^+, X^-, X^i)\), where \(X^\pm \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1})\) and \(i = 1 \ldots D - 2\). This is only a choice of coordinates, but next we use the residual symmetry \((23)\), and the equation of motion \((36)\), to fix a light cone gauge. From

\[
\tilde{\tau} = F^i(\sigma)\tau + G(\sigma), \quad \dot{X}^+ = 0
\]

we see that we may take \(\tilde{\tau} \propto X^+\), and we choose

\[
X^+ = \frac{p^+}{v^2}\tau
\]

where \(p^+\) is the conserved momentum. This completely fixes the diffeomorphism gauge, except for rigid \(\sigma\)-translations.

In light cone coordinates the \(V^\alpha\) equations of motion read

\[
V^\alpha \partial_\alpha X^i \partial_\beta X^i - V^\alpha \partial_\alpha X^- \partial_\beta X^+ - V^\alpha \partial_\alpha X^+ \partial_\beta X^- = 0,
\]

and in transverse gauge \((22)\) they give the constraints

\[
\dot{X}^i \dot{X}^i - 2\dot{X}^- \dot{X}^+ = 0
\]

\[
\dot{X}^i X'^i - \dot{X}^- X'^+ - \dot{X}^+ X'^- = 0.
\]

We now use \((49)\) in the light cone gauge \((17)\) to eliminate \(X^-\), except for a zero-mode \(x^- (\tau)\):

\[
X'^- = \frac{v^2}{p^+} \dot{X}^i X'^i
\]

\[
\dot{X}^- = \frac{v^2}{2p^+} \dot{X}^i \dot{X}^i
\]

\[
x^- \equiv \int d\sigma X^- \equiv x^-_0 + \frac{1}{v^2 p^-} \dot{\tau},
\]

where \(x^-_0\) and \(p^-\) are constants. Having eliminated \(X^\pm\) we are left with the equations of motion for the transverse components, \(X^i\), (in transverse gauge)

\[
\ddot{X}^i = 0
\]
These may be derived from the light cone action

$$S_{LC} = \frac{v^2}{2} \int d^2 \xi \dot{X}^i \dot{X}^i$$  \hfill (52)

We now wish to find a canonical formulation of the generators in order to prepare for quantization. The transverse conjugate momenta can be read off from (52), they are

$$P^i = v^2 \dot{X}^i$$  \hfill (53)

In addition $-p^\tau$ is canonically conjugate to $x^-$. We now write the first two equations in (54) using (53)

$$X'^- = \frac{1}{p^\tau} X^n P^i$$

$$\dot{X}^- = \frac{1}{2p^\tau v^2} P^i P^i \equiv \frac{1}{v^2} P^-$$  \hfill (55)

The action (52) then corresponds to a Hamiltonian

$$\mathcal{H} = \frac{1}{2v^2} \int d\sigma P^i P^i = \frac{1}{v^2} \int d\sigma p^\tau P^-,$$  \hfill (56)

which indeed generates $\tau$-translations, c.f. (47,57). Equations (47,53) and (55) give the translation operators:

$$p^m \equiv \int d\sigma P^m(\sigma).$$  \hfill (57)

Here we have introduced the convention that lower case letters denote zero modes or integrated quantities.

The generators of the conformal group additional to the Lorentz and translation generators, can now be written (at $\tau = 0$):

$$p^i = \int d\sigma P^i$$
$$p^- = \int d\sigma P^-$$
$$p^+ = p^\tau$$
$$m^{ij} = \int d\sigma \{ X^i P^j - X^j P^i \}$$
$$m^{i-} = \int d\sigma \{ X^i P^- - X^- P^i \}$$
$$m^{i+} = \int d\sigma \{ p^+ X^i \} = p^+ x^i$$
$$m^{+-} = -\int d\sigma \{ p^+ X^- \} = -p^+ x^-$$

12
\[ s = \int d\sigma \left\{ X^i P^i - p^+ X^- \right\} \]
\[ k^i = \int d\sigma \left\{ X^i X^j P^j - X^i p^+ X^- - \frac{1}{2} X^j X^j P^i \right\} \]
\[ k^- = \frac{1}{p^+} \int d\sigma \left\{ p^+ X^j X^- P^j - p^+ X^- p^+ X^- - \frac{1}{2} p^+ X^i X^i P^i \right\} \]
\[ k^+ = -\frac{1}{2} \int d\sigma \left\{ p^+ X^i X^i \right\} \]

(58)

The generators in (58) generate transformations that include precisely the compensating reparametrizations derived in the next section.

2.5 Compensating reparametrizations

In this section we wish to find the precise form of the combined special conformal transformations and diffeomorphisms that preserve both the choice of transverse gauge and light cone gauge. They may be found following the procedure used by Goddard et al. [33] for the Lorentz transformations of the ordinary string.

Under special conformal transformations the fields in the action (18) transform as (c.f. (32)):
\[ \delta_b X^m = b \cdot XX^m - \frac{1}{2} X^2 b^m \]
\[ \delta_b \partial_\alpha X^m = b \cdot (\partial_\alpha X) X^m + b \cdot X \partial_\alpha X^m - X \cdot \partial_\alpha b^m \]
\[ \delta_b V^\alpha = -b \cdot X V^\alpha \]
\[ \delta_b \gamma_{\alpha\beta} = 2b \cdot X \gamma_{\alpha\beta} \]

where \( b^m \) is a constant vector. The transformation rule for \( V^\alpha \) follows from demanding invariance of (18).

Under infinitesimal reparametrizations \( \xi^\alpha \to \xi^\alpha - \varepsilon^\alpha (\xi^\beta) \) the field transformations are (c.f. (20,21)):
\[ \delta_\varepsilon X^m = \varepsilon^\delta \partial_\delta X^m \]
\[ \delta_\varepsilon \partial_\alpha X^m = \varepsilon^\delta \partial_\delta \partial_\alpha X^m + (\partial_\delta X^m) \partial_\alpha \varepsilon^\beta \]
\[ \delta_\varepsilon V^\alpha = \varepsilon^\delta \partial_\delta V^\alpha - V^\delta \partial_\delta \varepsilon^\alpha + \frac{1}{2} (\partial_\delta \varepsilon^\delta) V^\alpha \]
\[ \delta_\varepsilon \gamma_{\alpha\beta} = \varepsilon^\delta \partial_\delta \gamma_{\alpha\beta} + (\partial_\delta \varepsilon^\gamma) \gamma_{\alpha\beta} + (\partial_\delta \varepsilon^\gamma) \gamma_{\alpha\gamma} \]

i.e. they transform as a scalar, covariant vector, contravariant vector density and contravariant tensor respectively.

Let \( \delta_b^{l.c.} \) denote the combined actions of reparametrizations and special conformal transformations i.e. \( \delta_b^{l.c.} = \delta_\varepsilon + \delta_b \). In order to stay in the transverse gauge (22) we must have
\[ 0 = \delta_b^{l.c.} V^0 = -b \cdot X v - v \varepsilon^0 + \frac{1}{2} v \varepsilon^0 + \frac{1}{2} v \varepsilon^1 \]
\[ 0 = \delta_b^{l.c.} V^1 = -v \varepsilon^1 \]

(61)
where dot (prime) refers to $\tau = \sigma^0$ ($\sigma = \xi^1$) derivatives. This implies
\[
\begin{align*}
\varepsilon^0 - \varepsilon^1 &= -2b \cdot X \\
\varepsilon^1 &= 0
\end{align*}
\] (62)
and in particular we see that $\varepsilon^1$ depends on $\sigma$ only. Note also that as long as (22) is
the only gauge condition imposed we can choose a compensating reparametrization in several ways for any given special conformal transformation. However, we also
wish to stay in the light cone gauge, which we choose to define as
\[
X^+(\tau, \sigma) = \dot{X}^+(\tau, \sigma) \tau, \quad \partial_\alpha \dot{X}^+(\tau, \sigma) = 0.
\] (63)
This formulation, using a Lagrangian language rather than a Hamiltonian one, is
useful since it makes manifest the transformation properties of all factors under
reparametrizations. To stay in this gauge requires further restrictions on $\varepsilon^\mu$ as we
will now see.

For our purposes it suffices to study transformations on shell. In the transverse
gauge (22) we have the equations of motion and constraints (36) which are solved
by
\[
X^m(\tau, \sigma) = \dot{X}^m_0(\sigma) = \dot{X}_0(\sigma) \cdot X'_0(\sigma) = 0
\] (64)
where $\dot{X}^m_0(\sigma) \equiv \dot{X}^m(0, \sigma)$ and $X'_0(\sigma) \equiv X^m(0, \sigma)$ are initial values. The require-
ment of the invariance of (63) reads
\[
0 = \delta^{l.c.}(X^+ - X^+_\tau) = \delta^{l.c.}X^+ - (\delta^{l.c.}X^+)\tau = b \cdot XX^+ - \frac{1}{2} b^+ X^2 + \varepsilon^0 \dot{X}^+ -
= (b \cdot X X^+ + b \cdot X \dot{X}^+ - b^+ X \cdot \dot{X} + \varepsilon^0 \dot{X}^+) \tau
\] (65)
where we have used (61, 62, 36). Using (64) then gives
\[
b \cdot \dot{X}_0 \dot{X}^+_0 \tau^2 + \frac{1}{2} b^+ X^2_0 = \dot{X}^+_0 (\varepsilon^0 - \varepsilon^0 \tau).
\] (66)
To solve this we make the ansatz
\[
\begin{align*}
\varepsilon^0(\tau, \sigma) &= r_1(\sigma) + r_2(\sigma) \tau + r_3(\sigma) \tau^2
\end{align*}
\] (67)
with coefficient functions $r_1, r_2$ and $r_3$, to be determined. Inserting this into (66)
immediately gives us
\[
\varepsilon^0 = \frac{1}{2\dot{X}^+_0} b^+ X^2_0 + r_2 \tau - b \cdot \dot{X}_0 \tau^2
\] (68)
where $r_2$ is still undetermined. This solution for $\varepsilon^0$, as can be easily checked, is also
consistent with $\ddot{\varepsilon}^0 = -2b \cdot \dot{X}$ which follows from (62). Inserting this solution into
(62) gives
\[
\begin{align*}
\varepsilon^1 &= \dot{\varepsilon}^0 + 2b \cdot X = -2b \cdot \dot{X}_0 \tau + r_2 + 2b \cdot X = r_2 + 2b \cdot X_0 \\
\varepsilon^1 &= 2b \cdot \int^\sigma X_0 d\sigma' + \int^\sigma r_2 d\sigma' + \text{const.}
\end{align*}
\] (69)
For consistency we also have to insure that $\partial_a \dot{X}_0^+ = 0$ is invariant under the transformation. That is, we have to check whether $\delta^{l.c.}_0 \dot{X}^+$ and $\delta^{l.c.}_0 \dot{X}'^+$ vanish on-shell. The first of these can be shown to vanish identically and yields nothing new, while the latter condition will determine the coefficient $r_2$.

To accomplish this we first note that the second derivative of a field $\phi$ does not transform as a tensor but has the transformation law

$$\delta_e \partial_a \partial_\beta \phi = \varepsilon^\gamma \partial_\gamma \partial_a \partial_\beta \phi + (\partial_a \varepsilon^\gamma) \partial_\gamma \partial_\beta \phi + (\partial_\beta \varepsilon^\gamma) \partial_\gamma \partial_a \phi + (\partial_a \partial_\beta \varepsilon^\gamma) \partial_\gamma \phi$$

under reparametrizations. In the light-cone gauge this takes a particularly simple form for the transformations of $\partial_\alpha X^+$:

$$\delta_e \partial_a \partial_\alpha X^+ = (\partial_\alpha \varepsilon^0) \dot{X}^+$$

$$\delta_0 \partial_a \partial_\alpha X^+ = \partial_\alpha (b \cdot \dot{X}^+ X^+ + b \cdot X^+ X^+ - X \cdot \dot{X}^+ b^+).$$

Using this we obtain the combined on-shell transformation properties:

$$\delta^{l.c.}_0 \dot{X}^+ = b \cdot \dot{X}^+ X^+ + b \cdot \dot{X}^+ X^+ + \varepsilon^0 \dot{X}^+ = 0$$

$$\delta^{l.c.}_0 \dot{X}'^+ = b \cdot \dot{X}'^+ X^+ + b \cdot X' \dot{X}^+ - X \cdot \dot{X}' b^+ + \varepsilon^0 \dot{X}^+ =$$

$$= b \cdot \dot{X}' X^+ + b \cdot X' \dot{X}^+ - X \cdot \dot{X}' b^+ - 2b \cdot \dot{X}_0' \dot{X}^+ + s' \dot{X}^+ =$$

$$= b \cdot X_0' \dot{X}^+ - X_0' \dot{X}^+ b^+ + r_2' \dot{X}^+. $$

The first of these vanish identically as promised and for the second to vanish we get

$$r_2' = X_0 \cdot \dot{X}_0' \frac{b^+}{X_0^+} - b \cdot X_0'.$$  \hspace{1cm} (73)

This can be integrated explicitly since $\dot{X}_0 \cdot X_0'$ vanishes:

$$r_2 = X_0 \cdot \dot{X}_0 \frac{b^+}{X_0^+} - b \cdot X_0 + C_1$$

where $C_1$ is some arbitrary integration constant. Plugging this value of $r_2$ into (69) then yields

$$\varepsilon^1(\sigma) = \int_0^\sigma d\sigma' \left( b \cdot X_0(\sigma') + X_0(\sigma') \cdot \dot{X}_0(\sigma') \frac{b^+}{X_0^+} \right) + C_1 \sigma + C_2.$$  \hspace{1cm} (75)

Since we are dealing with the closed string the periodicity condition $\varepsilon^1(0) = \varepsilon^1(1)$ must be satisfied, and this can be used to determine $C_1$:

$$\varepsilon^1 = \int_0^\sigma d\sigma' \left( b \cdot X_0 + X_0 \cdot \dot{X}_0 \frac{b^+}{X_0^+} \right)$$

$$- \sigma \int_0^1 d\sigma' \left( b \cdot X_0 + X_0 \cdot \dot{X}_0 \frac{b^+}{X_0^+} \right) + C_2.$$  \hspace{1cm} (76)
This, however, can be written in a simpler and manifestly periodic form

\[ \varepsilon^1(\sigma) = \oint d\sigma' \left( b \cdot X_0(\sigma') + X_0(\sigma') \cdot \dot{X}_0(\sigma') \frac{b^+}{X^{+0}_0} \right) h(\sigma' - \sigma) + C_2 \]  

(77)

where

\[ h(\sigma - \tilde{\sigma}) \equiv \sigma - \tilde{\sigma} - \frac{1}{2} \text{sign}(\sigma - \tilde{\sigma}). \]  

(78)

Thus we conclude that the special conformal transformations plus compensating reparametrizations that preserve both the transverse and the light cone gauge act as follows:

\[ \delta_{\text{l.c.}} X^m(\tau, \sigma) = b \cdot X(\tau, \sigma) X_m(\tau, \sigma) - \frac{1}{2}X^2(\tau, \sigma)b_m + \varepsilon^0(\tau, \sigma) X^m(\tau, \sigma) + \varepsilon^1(\sigma) X^m(\tau, \sigma) \]  

(79)

\[ \varepsilon^0(\tau, \sigma) = \frac{1}{2X^{+0}_0} b^+ X^2_0(\sigma) + r_2(\sigma) \tau - b \cdot \dot{X}_0(\sigma) \tau^2 \]

\[ r_2(\sigma) = X_0(\sigma) \cdot \dot{X}_0(\sigma) \frac{b^+}{X^{+0}_0} - b \cdot X_0(\sigma) - \oint d\sigma' \left( b \cdot X_0(\sigma') + X_0(\sigma') \cdot \dot{X}_0(\sigma') \frac{b^+}{X^{+0}_0} \right) \]

(80)

The constant \( C_2 \) is not fixed by these considerations. However, we recall that gauge fixing to the light-cone gauge leaves a rigid \( \sigma \)-translation unspecified. The \( C_2 \)-term generates precisely this remaining gauge symmetry.

\section{3 The quantum theory}

It is only possible to consistently quantize the tensile string in certain critical dimensions. This result can be arrived at in a number of different ways: By demanding that the Weyl invariance holds at the quantum level (absence of the conformal anomaly); by demanding Lorentz-symmetry in the light cone gauge or by demanding nilpotency of the BRST charge. For the tensile bosonic string all these methods (and a few other) lead to the critical dimension \( D=26 \).

The question of whether the quantization of the tensionless string leads to similar restrictions may likewise be investigated via various routes. The first of the above alternatives is not available, though, since the action has no Weyl symmetry. In the main part of the remainder of this article we shall be concerned with light-cone gauge quantization and the consequences of requiring the full space-time symmetry of the classical model (the conformal symmetry) to be preserved by quantization. We will also confirm the known result that the weaker requirement of quantum
Poincaré symmetry does not lead to any quantization problems. However, we first comment on BRST-quantization.

### 3.1 BRST-Quantization:

BRST-quantization may be applied either in a Hamiltonian or in a Lagrangian context.

A Hamiltonian BRST-quantization of the bosonic tensionless string was carried out in [25]. In phase space the second pair of equations in (36) read

\[ P^2 = P \cdot X' = 0 \]  

(80)

Starting from the algebra of these constraints the authors of [25] construct the Hamiltonian BRST-charge \( Q_H \), following the procedure used in, e.g., [36]. Nilpotency of \( Q_H \) is then checked in the quantized theory and found to hold independent of the dimension. This procedure says nothing about the space-time conformal symmetry, of course. It has been extended to include this symmetry in a mode-expansion approach in [35]. There obstructions to quantization are found.

As an alternative, we here consider the Lagrangian BRST quantization, following [37]. The 2D diffeomorphisms transformations of the fields are given by (20,21) where \( \varepsilon^\alpha \) is the transformation parameter. Following the standard BRST procedure we then introduce anticommuting ghosts \( c^\alpha \), antighosts \( \bar{c}^\alpha \) and auxiliary fields \( B^\alpha \).

The BRST transformations are

\[
\begin{align*}
    sX^m &= c\partial X^m, \\
    sV^\alpha &= c\partial V^\alpha - V\partial c^\alpha + \frac{i}{2}V^\alpha \partial c \\
    sc^\alpha &= -c\partial c^\alpha, \\
    s\bar{c}^\alpha &= iB^\alpha, \\
    sB^\alpha &= 0
\end{align*}
\]  

(81)

The gauge fixing fermion that implements the transverse gauge (22) is

\[
\Psi = -i \left( \bar{c}_0(V^0 - v) + \bar{c}_1 V^1 \right)
\]  

(82)

with \( v \) a constant. The gauge fixing and ghost Lagrangian is obtained as \( s\Psi \). The total gauge fixed action is

\[
S_{GF} = S_0^0 + \int d^2 \xi s\Psi
\]  

(83)

Written out in detail it is rather complicated, but after a redefinition of the auxiliary fields, \( B \rightarrow \hat{B} \),

\[
\hat{B}^\alpha \equiv B^\alpha + i\bar{c}_\beta \partial_\alpha c^\beta - \frac{i}{2}\bar{c}_\alpha (\partial \cdot c) + V^\alpha + i\partial_\beta (\bar{c}_\alpha c^\beta) + V^\beta \gamma^\prime_{\beta\alpha}
\]  

(84)

it simplifies to

\[
S_{GF} = \int d^2 \xi \left[ v^2 \partial_0 X^m \partial_0 X_m + \hat{B}_0 (V^0 - v) + \hat{B}_1 V^1 + i\bar{c}_\alpha D^\alpha_\beta c^\beta \right]
\]  

(85)
where
\[ D_\beta^\alpha = v \left( \frac{1}{2} \partial_0 - \frac{1}{2} \partial_1 \right). \] (86)

From this action we derive the momenta
\[ \Pi_m = 2v^2 \dot{X}_m, \quad \Pi_0 = \frac{iv}{2} \bar{c}_0, \quad \Pi_1 = iv \bar{c}_1, \] (87)
conjugate to \( X^m, c^0 \) and \( c^1 \), and all other momenta vanish. The Hamiltonian we find is
\[ H = \int d\sigma \left[ \frac{1}{4v^2} \Pi_m \Pi^m + \Pi_0 \Pi^c \right] \] (88)
and the BRST charge is
\[ Q = \int d\sigma \left[ -\frac{1}{4v^2} c^0 \Pi_m \Pi^m - \Pi_m X^m c^1 + \Pi_0 \dot{c}^0 + \Pi_1 \dot{c}^1 \right] \] (89)
One may check that \( Q \) generates the correct BRST transformations on \( X^m, c^\alpha \) and \( \bar{c}_\alpha \). Note that, since the corresponding momenta vanish, we have eliminated \( V^\alpha \) and \( B_\alpha \) from the theory without affecting the Poisson brackets between the remaining fields.

We note that our Hamiltonian (88) is the same as the one in (25), modulo a total divergence) and consequently generate the same equations of motion; those derivable from the action (85). Furthermore, the BRST-charge (89) equals that in (25), modulo equations of motion.

We may now quantize by making \( X, c \), and \( \Pi \) operators with canonical commutation relations and introducing a Hermitean ordering, all of which will be discussed in great detail below for the light-cone gauge. Finally, using the results of (25), we have
\[ \{ \hat{Q}, \hat{Q} \} = 0, \] (90)
independent of the dimension \( D \).

The straight-forward BRST-quantization described above is less restrictive than requiring quantum conformal invariance in the light-cone gauge. The consequences of the latter will be discussed in great detail in the next section, but we want to point out that an alternative would be to study the BRST quantization and implement the requirement of conformal invariance there. This could be done in several ways. In a covariant (as opposed to a light cone) formulation one should check that there are BRST-invariant generators generating the correct symmetry, and that \( Q^2 = 0 \). In a non-covariant formulation the algebra proper has to be checked. Finally one might study the BRST-quantization of a conformally covariant formulation of the tensionless string. This would essentially follow the lines presented above, but with some additional constraints and corresponding ghost/anti-ghost system.

3.2 The Light-cone Operator Algebra
**Strategy:** Our aim is to determine whether it is possible to realize the space-time conformal algebra in terms of quantum operators, acting on a Hilbert space corresponding to physical light-cone degrees of freedom only. Since the classical conformal generators (58) are polynomials of $X^i(\sigma)$ and $P^i(\sigma)$, products of such fundamental operators are needed in the quantum algebra, and we should regularize these products to make them meaningful. In general the regularization involves some arbitrariness, which hopefully can be parametrized by a finite number of constants in the limit where the regulator is removed. For each such constant there is a possible correction term to the composite operator. The study of the relevant generalizations of the generators can be organised by taking into account how they scale with the regulator, $\varepsilon$, i.e. by power counting. We will demonstrate how an anomaly in the space-time conformal algebra is unavoidable under quite general and reasonable assumptions.

**PRINCIPLES AND ASSUMPTIONS**

**Locality:** Locality means that physical quantities at different $\sigma$ contribute additively to the total observable, at least classically implying that different space-time points also contribute additively. If one ultimately wishes to construct an interacting theory with some notion of space-time locality one should thus try to respect world-sheet locality. It is therefore desirable to keep this locality as manifest as possible in all arguments. One may consider two locality concepts: "covariant locality", which treats all coordinates on an equal footing, or "transverse locality", which treats only the transverse coordinates as fundamental. The classical conformal generators are integrals along the string of local expressions depending on $X^i(\sigma)$, $P^i(\sigma)$ and $X^-(\sigma)$. In fact, the integrands do not even contain derivatives. They are clearly covariantly local. However, as seen in formula (50,53), $X^-(\sigma)$ depends on the whole string if regarded as a function of $X^i(\sigma)$ and $P^i(\sigma)$, and thus functions of $X^-(\sigma)$ are not in general transversely local.

We shall assume that the quantized conformal generators are covariantly local, since this is the locality concept satisfied classically. We write all expressions as functions of $X^i(\sigma)$, $P^i(\sigma)$ and $X^-(\sigma)$, so that one can determine covariant locality by inspection. The interesting conformal generators are cubic or quartic polynomials, so there is also a significant practical advantage in keeping this form of the generators rather than Fourier-transforming and thus obtaining multiple convolutions. The interpretation of regularization will also be much clearer in our approach. We require that any correction terms to the generators are also covariantly local.

**Regularization:** In the classical case the physics can be studied through functions on phase space, which in the light-cone gauge is parametrized by canonical coordinates $X^i(\sigma)$, $P^i(\sigma)$ and $x^-, p^+$. In quantum mechanics the classical canonical Poisson brackets are simply replaced by commutators, but for non-linear functions of the coordinates one also has to worry about ordering problems, which for systems
with an infinite number of degrees of freedom may even involve divergencies. In
our case the source of such divergence problems is easily traced to the canonical
commutation relations

\[ [x^-, p^+] = -i, \quad [X^i(\sigma), P^j(\sigma')] = i\delta^{ij}\delta(\sigma - \sigma') \]  

and to our interest (for physical reasons) in local operators. For example, the simple
Hermitean operator \( iX^i(\sigma)P^i(\sigma) - iP^i(\sigma)X^i(\sigma) \) is directly seen to diverge. Taking
into account that observables generally do not involve a precise value of \( \sigma \), but an
integral over some region, this divergence may be side-stepped by smearing \( X^i(\sigma) \)
and \( P^i(\sigma) \), i.e. by convoluting each of them with an approximate delta function.
The details of the approximate delta function will not matter, we only assume

\[
\lim_{\varepsilon \to 0} \int d\sigma f(\sigma)\delta_\varepsilon(\sigma) = f(0) \quad \delta_\varepsilon(-\sigma) = \delta_\varepsilon(\sigma) \quad (92)
\]

\[
\int d\sigma \delta_\varepsilon(\sigma) = 1 \quad \delta_\varepsilon(\sigma) = \frac{1}{\varepsilon} \delta_\varepsilon\left(\frac{\sigma}{\varepsilon}\right), \quad (93)
\]

where the regulator \( \varepsilon \) is seen to measure the scale of smearing. The limit \( \varepsilon \to 0 \) is
the local limit, which defines the physical system.

The regularization proposed above yields finite answers for all commutators as
long as \( \varepsilon > 0 \), but it is not the most general kind of regularization even for monomial
operators, since each canonical coordinate factor is smeared independently of the
others. A general smearing function depending on the positions of all operators
correlate these positions, unless it is factorized into functions depending only on one
position each. For such general regularizations one may still measure the scale of
smearing in terms of \( \varepsilon \).

The quantity that defines the quantum algebra is the canonical commutator, so
regularizations of the special factorizable kind, smears of canonical coordinates,
are easily handled. One just calculates their effect on the canonical commutator, and
proceeds from there on, using a modified delta function in a regularized canonical
commutator:

\[ [X^i(\sigma), P^j(\sigma')] = i\delta^{ij}\delta_\varepsilon(\sigma - \sigma') \]  

The smearing does not affect any other quantity. Obviously, a physically sensible
system can only be obtained in the local limit, when the regulator is removed, but
the presence of the regulator is essential for making sense of intermediate steps
in the calculations. The more general kind of non-factorizable regularization is
trickier, but can be treated as giving rise to correction terms, small in the parameter
\( \varepsilon \) that measures the regularization scale. This is done by power counting. We
shall in fact find that such corrections cannot affect the conclusions we draw from
simply regularizing the canonical commutator. Before describing our power counting
arguments we list the other essentials of our procedure.

**Reference ordering:** Since the canonical commutators relate the values of opera-
tor products with different orderings, there are numerous ways of rewriting one and
the same expression. In an algebraic calculation one wants to know whether further cancellations are possible or not. By defining a standard reference ordering of operators in all monomials, expressions can be directly compared, and after cancellations the result is unique. In ordinary oscillator calculations normal ordering is used, but since we do not have oscillators, a different prescription has to be applied.

Defining a non-Hermitean version of the $M^+-(\sigma)$ current

$$\mathcal{M}(\sigma) \equiv p^+X^-(\sigma)$$

and symbolically representing an operator and all its derivates raised to an arbitrary power with the same letter, we have found the ordering

$$p^+XMP$$

particularly convenient. It will be called the ”reference ordering”. Note that $P^i(\sigma)$ annihilates the ground state, so that some properties of normal ordering are retained in the tensionless limit (cf. Appendix). The reference ordering is however used solely as a book-keeping device, and our results will not be sensitive to the choice of vacuum (unless one assumes a broken $\sigma \to -\sigma$ symmetry like in refs. [26, 35]).

**Hermiticity:** Physical observables have real expectation values and should be represented by Hermitean operators. Thus we should require that the conformal generators (58) are Hermitean. Hardly any monomial operators ordered according to our reference ordering (96) are Hermitean by themselves, so an ordered expression for a generator contains correction terms that ensure Hermiticity. Consider for example the operator $X^i(\sigma)P^i(\sigma) + P^i(\sigma)X^i(\sigma)$ which is manifestly Hermitean, but not ordered. Using the regularized canonical commutation relations (94) we find the ordered form $2X^i(\sigma)P^i(\sigma) - iC$, where the ordering constant $C$ diverges as $\varepsilon^{-1}$ when $\varepsilon \to 0$ due to the scaling behaviour (92). Such singular ordering terms are typical, but as will be shown below, there will be only one independent ordering constant in the conformal algebra, due to the algebraic relations between conformal generators. For the power counting arguments we still have to keep track of the $\varepsilon \to 0$ singularities from these ordering terms.

**Conformal recursion:** The conformal algebra in light-cone coordinates is

\[
\begin{align*}
[p^i, k^j] &= m^{ij} - \delta^{ij}s \\
[s, p^i] &= p^i \\
[m^{ij}, p^k] &= \delta^{jk}p^i - \delta^{ik}p^j \\
[m^{ij}, m^{kl}] &= \delta^{ik}m^{jl} - \delta^{ij}m^{kl} + \delta^{ij}m^{jk} - \delta^{lk}m^{ij} \\
[k^i, p^\pm] &= p^\pm \mp \delta^{ij}p^j \\
[p^\pm, m^\pm] &= -\delta^{ij}p^j \\
[s, k^\pm] &= -k^\pm \\
[p^i, k^\pm] &= m^\pm \\
\end{align*}
\]
\[ [m^{ij}, m^{kl}] = \delta^{ki}m^{jl} - \delta^{kj}m^{il} \]
\[ [k^{i}, m^{kl}] = -\delta^{ij}k^{l} \]

\[ [p^{\pm}, m^{i\mp}] = -p^{i} \quad [k^{\mp}, m^{i\mp}] = -k^{i} \]
\[ [m^{\pm}, m^{i\mp}] = \delta^{ij}m^{\mp\mp} - m^{ij} \]
\[ [p^{\pm}, k^{\mp}] = s \pm m^{+-} \]

\[ [p^{\pm}, m^{+-}] = \mp p^{\pm}, \]
\[ [k^{\mp}, m^{+-}] = \mp k^{\mp} \]
\[ [m^{i\mp}, m^{+-}] = \mp m^{i\pm} \]

(97)

with all other brackets vanishing. From this we can make several interesting observations.

(i) The transverse generators form a (D-2)-dimensional Euclidean conformal algebra by themselves. The translation generators \( p^{i} \) and the conformal boosts \( k^{j} \) are sufficient to span this algebra by repeated commutators.

(ii) Adding \( p^{+} \) and its repeated commutators with the transverse generators produces \( m^{i+}, m^{+i}, \) and \( k^{+}. \)

(iii) Finally adding \( p^{-} \) then produces all remaining generators.

In following this three-step procedure of generating the full algebra, one also encounters conditions relating the generators, so that an erroneous ansatz for one of the initial generators \( p^{i}, p^{+}, p^{-} \) or \( k^{i} \) will be revealed. If, on the other hand, all such conditions are fulfilled, a realization of the conformal algebra has been constructed.

In the present case we know the form of the classical generators, and that their Poisson brackets satisfy the conformal algebra. The quantized generators should be closely related to the classical, since we expect there to be a classical limit when \( \hbar \to 0. \) However, the quantum generators may deviate from the classical due to ordering problems, due to the non-locality introduced via the regularization or due to the renormalization of some quantities.

We may now take advantage of the structure of the algebra described above and the form of the classical generators (58). \( p^{i}, p^{+} \) and \( p^{-} \) are so simple that no ordering problems can affect them, and \( p^{i} \) and \( p^{+} \) are linear so they are not even modified by regularization. The ordered form of the Hermitian conformal boost \( k^{i} \) contains an ordering constant of order \( \varepsilon^{-1}. \) This is the only independent ordering constant, since the commutator of two Hermitian operators gives \( i \) times a Hermitian operator, and all other generators can be obtained in this way. It remains to study the possible consequences of the regularization of \( p^{-} \) and \( k^{i}. \)

Above we have discussed how ordering terms may appear in the definition of the generators of the algebra. In addition, truly serious ordering problems can arise because commutators of non-linear operators need to be reordered before they comply with the fixed reference ordering. This process may generate anomalous terms.
By constructing all generators in the recursive way described above we disentangle the two problems: *Any deviation from the algebra which cannot be absorbed in the redefinition of the generators is an anomaly.*

**Conformal action on X:** Our object is to study whether the conformal algebra works for the quantized tensionless string, and we want to keep the geometrical picture of the transformations, not only their algebra. To what degree can the conformal generators be modified without jeopardizing their geometrical interpretation? Classically the conformal transformations are uniquely determined once a gauge has been fixed completely, i.e. the action of the conformal group on the coordinates of the string is precisely known when one has specified these coordinates exactly. One such choice of gauge is the light-cone gauge (though a rigid $\sigma$-translation is conveniently left unfixed). The philosophy of light-cone gauge quantization is to fix the gauge classically and quantize the gauge fixed description of the theory. This procedure should work whenever the geometry of the theory is left untouched by quantization. In such a case the action of the conformal generators is given by the classical expression except possibly for correction terms vanishing as $\varepsilon \to 0$.

We assume that the geometry is left intact by quantization, and thus that the action of the conformal group on $X^i(\sigma)$ approaches the classical result when the regulator is removed.

**Power counting:** We assume that the quantum conformal generators can be expanded in powers of the regulator $\varepsilon$ which has dimension of world-sheet length. Higher powers of the regulator will therefore combine with higher derivatives of the basic operators. As an example of how this works consider a smeared version of $X^i(0)$:

$$X^i_\varepsilon(\sigma) = \int d\sigma' \delta_\varepsilon(\sigma - \sigma') X^i(\sigma')$$

$$\approx X^i(\sigma) + \frac{\varepsilon^2}{2} X^{ii}(\sigma) \int d\left( \frac{\tilde{\sigma}}{\varepsilon} \right) \left( \frac{\tilde{\sigma}}{\varepsilon} \right)^2 \delta_1 \left( \frac{\tilde{\sigma}}{\varepsilon} \right) \tag{98}$$

There may also be singular terms in the conformal generators, but in order to preserve the classical transformations of $X^i(\sigma)$, we only allow such terms if they are required by Hermiticity and by the reference ordering, as discussed above.

**Possible "counterterms":** We are now ready to state concretely what correction terms to the conformal generators our principles allow. It is sufficient to describe the possible corrections to $p^-$ and $k^i$, since they together with the trivial operators $p^+$ and $p^j$ span the whole algebra.

In terms of reference ordered expressions the allowed correction terms are:

(i) The singular term from ordering in $k^i$, proportional to $ih\varepsilon^{-1}X^i$.

(ii) A similar, but real and regular correction to $k^i$, proportional to $hX^i$. Such a term also induces a c-number shift in the dilatation operator $s$.

(iii) Various $n$'th-derivative terms proportional to $\varepsilon^n$. The number of such possibilities is constrained by dimensional analysis and by the requirement that they
should disappear in the classical limit. We refrain from classifying them since we can instead show that their presence will not change our conclusions.

CONFORMAL RECURSION RESULTS

Given an ansatz for the basic generators and a regularization we can calculate the algebra, identify terms, and see what part of the result survives in the local limit. Note that there are terms in the generators diverging as \( \varepsilon \to 0 \). One has to consider also terms vanishing in this limit, since they can combine with the divergent terms. The most important and most difficult part of the calculation depends on the properties of \( X^-(\sigma) \). This should not come as a surprise, since the composite operator \( X^-(\sigma) \) is obtained by solving the non-abelian constraint (49,50). (It generates the residual Virasoro symmetry (24).) Thus, after the gauge has been fixed, most of the non-trivial gauge symmetries of a covariant formulation are encoded in this operator. In order to complete the technical description of our method we therefore have to describe how the properties of the operator \( X^-(\sigma) \) are derived and how it is regularized. Then we can finish the investigation by deriving an anomalous term in the algebra spanned by uncorrected, but quantized generators, and finally proving that no combination of correction terms can conspire to cancel the anomalous contribution.

\( X^- \): The constraint (18,24) which allows us to solve for \( X^- (\sigma) \) actually contains the derivative of \( X^- (\sigma) \), so we cannot get a closed expression for \( X^- (\sigma) \) unless we determine the integration constant. However, for the conformal algebra only the commutation rules of \( X^- (\sigma) \) are needed, not its precise form. The commutation rules can be found from integrating the commutation relations of \( X'^-(\sigma) \) calculable from (54), and imposing the "sum-rule" that the zero-mode \( x^- \) should be canonically conjugate to \(-p^+\) and commute with the transverse degrees of freedom:

\[
\int d\sigma \left[ X^-(\sigma), p^+ \right] = -i
\]
\[
\int d\sigma \left[ X^-(\sigma), X^i(\sigma') \right] = \int d\sigma \left[ X^-(\sigma), P_i(\sigma') \right] = 0 \quad (99)
\]

In addition one wants the commutation relations to be manifestly periodic, since the physics is. Surprisingly, periodicity of the commutation relations is not automatic, but it can be achieved by making use of the one gauge symmetry that has been left unfixed in the light-cone gauge, the rigid \( \sigma \)-translation \( \Delta \). In the following it will be convenient to phrase the discussion in terms of \( \mathcal{M}(\sigma) = p^+ X^-(\sigma) \). The remaining symmetry means that

\[
\mathcal{M}(1) - \mathcal{M}(0) = \int d\sigma \mathcal{M}'(\sigma) = \int d\sigma X^i(\sigma) P_i(\sigma) \equiv \Delta = 0 \quad (100)
\]

in the physical phase space. In quantum mechanics the corresponding statement is that \( \Delta \) annihilates physical states, but this "vanishing" of \( \Delta \) is of course only
effective if $\Delta$ has been commuted to the right (or left) of any other operators. If one reinterprets $\mathcal{M}'(\sigma)$ as
\[
\mathcal{M}'(\sigma) \rightarrow \mathcal{M}'(\sigma) - \Delta,
\]
equation (101)
one automatically obtains
\[
\mathcal{M}(1) - \mathcal{M}(0) = 0
\]
equation (102)
as well as manifestly periodic commutation relations of $\mathcal{M}(\sigma)$
\[
[X^i(\sigma), X^- (\sigma')] = \frac{i}{p^+} X'^i(\sigma) h(\sigma - \sigma')
\]
\[
[P^i(\sigma), X^- (\sigma')] = \frac{i}{p^+} \left\{ P'^i(\sigma) h(\sigma - \sigma') + P^i(\sigma) [1 - \delta \varepsilon (\sigma - \sigma')] \right\}
\]
\[
[X^-(\sigma), X^- (\sigma')] = \frac{i}{p^+} \left\{ \left[ X'^-(\sigma) + \frac{\Delta}{p^+} \right] h(\sigma - \sigma') - \left[ X'^-(\sigma') + \frac{\Delta}{p^+} \right] h(\sigma' - \sigma) \right\}
\]
equation (103)
$h$ is given in eq. (78). It is also necessary to understand the effects of regularization on $\mathcal{M}(\sigma)$. Following the principle of light-cone quantization we should keep the relation to the transverse degrees of freedom as close as possible to the classical relation. Although it is not manifest, the regularized $\mathcal{M}'(\sigma)$ is in fact Hermitean, due to the $\sigma \rightarrow -\sigma$ symmetry assumption in (92). Other possible regularization effects can be taken into account by allowing correction terms in $\mathcal{M}'(\sigma)$, containing derivatives and correspondingly being of higher order in $\varepsilon$. When integrated to give $\mathcal{M}(\sigma)$ such terms could give contributions to generators that do not necessarily appear to be local. However, we should recall that we assumed covariant locality, so that these correction terms should not appear isolated, but always as parts of $X^-$. Therefore they can all be handled by determining how they modify the commutation rules of $\mathcal{M}(\sigma)$, which will include new terms with more derivatives than the classical terms.

**Anomaly:** Having fixed above what quantized generators are physically acceptable, and how to commute them, we are now ready to discuss the actual calculation of the quantum algebra. The reference ordering (96) is constructed so as to minimize the number of times one has to reorder the result of a commutation in order to get an ordered result. In many cases one can exclude any deviations from the classical case, simply by checking that no new terms can appear from reordering. In other cases, the procedure of defining the more complicated conformal generators recursively from the algebra absorbs ordering terms that would otherwise have appeared to be dangerous. Only a few commutators of highly nonlinear generators remain as possible sources of anomalies.
Reordering is necessary whenever an operator quadratic in position is commuted with an operator quadratic in the conjugate momenta. Due to its construction as an integral of $X'^i(\sigma)P^i(\sigma)$ we may regard $\mathcal{M}(\sigma)$ as linear in position and linear in momentum. From the form of the generators (58) and the algebra (97) we then read off which commutators can cause problems. One should note that the notorious Lorentz commutator $[m^i, m^j]$, which harbours the tensile string anomaly, not even appears in the list of potential dangers. The reason is simply that ordering problems for Hermitean operators arise at higher order in $\hbar$ when the ordering is defined in terms of positions and momenta, than when it is defined in terms of annihilation and creation operators, a fact that is familiar from the one-dimensional harmonic oscillator. Postponing for a while the discussion of commutators with the quartic generator $k^-$ we are left with only $[m^-, k^+]$ and $[m^-, k^j]$.

The first of the above commutators can be checked to give the correct answer, while the second splits naturally in two parts. From the conformal algebra (97) we see that the trace part defines $k^-$, but a trace-less contribution would necessarily be anomalous. Indeed, this is what happens. A lengthy commutator calculation, involving also replacements $X'^i(\sigma)P^i(\sigma) \rightarrow M'_i(\sigma)$ and integrations by parts, finally yields an anomaly:

$$[k^i, m^j] + \delta^{ij}k^- \propto \int d\sigma \left( \frac{1}{\varepsilon p^+} \right) \left\{ X'^i(\sigma)P^j(\sigma) - x^i p^j + h.c. \right\}_{t.l.}$$

where the subscript $t.l.$ denotes the trace-less part. We note that the anomaly (104) vanishes for $\sigma$-independent coordinates and momenta. This feature is shared by all the other anomalies, generated by commutators with $k^-$, and therefore all quantization problems are caused by the extendedness of the string. In contrast, we find that massless scalar point-particles, described by the constant zero-modes, do admit a quantum conformal symmetry.

The anomalous term is divergent, a state of affairs which is not familiar from relativistic field theory, but the present framework is quite different. One might still ask whether there is any possibility of removing the anomaly by modifying the short-distance behaviour of the theory. In the next section we will argue that there is no way of doing this while respecting the natural locality assumptions we have discussed at length above.

Anomaly cancellation?: There are basically two ways of modifying the generators when taking quantum short-distance effects into account. Either generators are simply rescaled (renormalized) due to short-distance effects, or the generators receive corrections from derivative terms.

Rescaling the generators is quite unnatural, since there is no dimensionless coupling in the present problem, and thus no limit where the rescaling could be small. There is also an algebraic reason why this alternative is ruled out. One may check
that the only rescalings of the generators that preserve the conformal algebra and the definition of the Hamiltonian \( m^+ \) are those that are generated inside the algebra by \( m^+ \). Hence the anomaly does not have a form which allows it to be scaled away.

One could also envisage that the anomaly could be absorbed by changing some string model parameters. At our disposal we have, at most, a string tension \( T \) and the speed of light \( c \), but changing these parameters deform the algebra too drastically, already at the classical level.

It remains to consider derivative corrections, together with appropriate powers of \( \varepsilon \). Since also negative powers of \( \varepsilon \) appear in the generators, such terms can in principle correct the algebra in the local limit. However, among the commutation rules for the basic operators \((91, 103)\) there are none that decrease the number of derivatives, and thus the anomaly \((104)\) can never be compensated by local modifications of generators.

*Open strings:* We have checked that an analogous structure of anomalies appears for the open string with boundary conditions \( V^1(0) = V^1(1) = 0 \). The commutators of \( X^\sigma - (\sigma) \) are modified since they should no longer be periodic, but the end result and the arguments excluding compensating terms are unchanged.

**DISCUSSION**

Faced with the anomaly \((104)\) we have a number of alternatives.

(i) We simply accept that space-time conformal symmetry is not a symmetry of the quantum theory. Global symmetries can be anomalous without ruining the consistency of the theory, and the Poincaré subgroup is still a symmetry. But it is then quite mysterious why the space-time symmetry group of the tensile string should happen to give the right criterion for the critical dimension \( 5 \). The fact that the conformal symmetry seems to be present in lieu of the 2-D Weyl symmetry of the tensile string also suggests that it should be taken more seriously.

(ii) Something has been overlooked in going from the geometrical action to the quantized gauge-fixed theory. Perhaps purely geometrical and auxiliary fields as \( V^\alpha \) get a life of their own, like the conformal factor of the metric for non-critical tensile strings. Such a state of affairs is quite possible, but it is also hard to reconcile with the picture of tensionless strings being the \( T \to 0 \) limit of the ordinary string. How could new degrees of freedom suddenly appear? In any case, we cannot discuss this alternative in more detail with the tools of the present article.

(iii) The anomalies actually vanish. This may happen because their operator form is not the whole story. One also has to take the Hilbert space they act on into account. Previously we have only demanded that the states should be invariant under the rigid \( \sigma \)-translation gauge symmetry, and in the appendix we discuss what

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\(^5\)Recall that there is a formulation \((14, 55)\) of the tensionless string where conformal symmetry is manifest before fixing the gauge, just as the Lorentz symmetry before fixing the light-cone gauge in the tensile string.
conditions the vacuum should satisfy in quite general terms. We also know that a Fock space is not appropriate, since oscillators and decomposition into positive and negative frequencies does not describe free particles, which is roughly what the tensionless string consists of.

We should thus ask whether we can regard the vanishing of the anomalies as conditions on a Hilbert space yet to be constructed. In a relativistic field theory we would know the free Hilbert space and could thus not take such an attitude, but here we are faced with a new situation and should at least check the implications of such an unconventional treatment. There is however reason to fear that regarding the anomalies as constraints on the Hilbert space reduces its "size" considerably. We shall return to this question in the conclusions, but we first describe the consequences of our assumptions.

The anomaly (104) is proportional to the generators of \( \sigma \)-dependent special linear transformations of the transverse coordinates (\( L_{ij}^{t.l.} \) is trace-less). The constraint

\[
\tilde{L}_{ij}^{t.l.} |PHY S >= 0,
\]

is imposed to restore quantum conformal symmetry, so one should also require its variation under conformal transformations to annihilate physical states. The full set of constraints is given by the anomaly (104) and the additional anomalies from \([k^+, k^-], [k^i, k^-]\) and \([m^{-}, k^-]\), together with their repeated commutators with each other and with the conformal generators. We have checked part of this algebra using Mathematica, and found that one obtains all the generators of \( \sigma \)-dependent special linear transformations

\[
X^m(\sigma)P^n(\sigma) - x^m p^n + h.c.
\]

not just the transverse components. Some automatically vanish in the light-cone gauge, but the fact that all non-zero generators appear ensures that the constraints have a covariant interpretation. (This would have been evident already from the construction, were it not for the special role of zero-modes.)

Phrased covariantly, we could thus state that the Hilbert space of tensionless string states should be invariant with respect to special linear transformations on the canonical coordinates, zero-modes modes excluded, and invariant with respect to all conformal transformations of these linear transformations. The full algebra of constraints is then generated by repeated commutators.

At this point it is illuminating to recall a result due to Ogievetsky [34]: The finite set of special linear transformations and conformal transformations together generate all (analytic) diffeomorphisms. Along the lines suggested by this result one can speculate that physical states annihilated by the constraints have a non-zero mode dependence that is invariant under analytic space-time diffeomorphisms. They should then be characterized by the topology of the loop they trace in space, e.g. by the number of self-intersections, and possibly also by discontinuities in derivatives, kinks and cusps.
The algebra of constraints could give this result, and it does give something very similar, but the separation of zero-modes is quite subtle, and it may turn out that some additional non-topological structure survives. The problem is that the conformal transformations couple zero-modes and $\sigma$-dependent modes. Most of this coupling seems to disappear when all constraints are taken into account simultaneously, but we have not yet found a conclusive answer to how the zero-modes affect the precise symmetry of the Hilbert space.

While the above considerations could enlarge the Hilbert space compared to a purely topological theory, some of the anomalies have a form that seem to reduce it even more. In particular, $[k^+, k^-] = 0$ gives rise to a constraint

$$\int d\sigma (X^i(\sigma) - x^i)(X^i(\sigma) - x^i)|PHYS > = 0 \quad (107)$$

which in its turn, through commutators with $p^-$, generates

$$\int d\sigma (X^i(\sigma) - x^i)(P^i(\sigma) - p^i)|PHYS > = 0 \quad (108)$$

Since two of the above equations look like sums of squares, it is tempting to draw the conclusion that $X^i(\sigma) - x^i = P^i(\sigma) - p^i = 0$, i.e. that the physical Hilbert space should only consist of zero-modes. However, since the operators are regulated, the factorization tacitly assumed in this argument is not strictly true. In any case, the state space is severely restricted. In fact, the constraints (107,108) illustrate how one reobtains some of the constraints lost through the zero-mode subtleties. Namely, by supplying them with appropriate powers of $p^+$, they are identical with the contributions from the $\sigma$-dependent modes to $k^+, s$ and $p^-$. 

4 Conclusions

We have discussed classical and quantum aspects of the tensionless string. At the classical level we have reported on a particular method for deriving tensionless strings (and tensionless $p$-branes) which leads to an action with auxiliary fields that have a geometric meaning, just as for the tensile Weyl-invariant string.

In describing the symmetries of the model we have placed special emphasis on the space-time conformal symmetry, a symmetry that we think of as replacing Weyl-invariance in the $T \to 0$ limit. In this context it is interesting to note that in quantum field theory it is often useful to understand Poincaré symmetry as a broken conformal symmetry. In the early days of supergravity, e.g., the appearance of the minimal set of auxiliary fields $N = 1$ Poincaré supergravity became clear after when that theory was viewed as broken conformal supergravity. From this
point of view it is thus natural to try to gain insight into the Poincaré invariant tensile string theory by comparing it to a tensionless conformally invariant string theory. This complements the motivation from high-energy string theory presented in the introduction.

Quantizing the theory we find that the quantum theory differs from the classical theory drastically. Either the symmetries are changed or the degrees of freedom are changed. This conclusion should be independent of our method, and it is certainly independent of the choice we then make: For reasons described in the discussion section we have pursued the idea that the classical symmetries should survive as quantum symmetries. A careful analysis leads us to conclude that this is only possible if the physical Hilbert space is drastically constrained. We seem to be left with only diffeomorphism singlets as physical states. This points to a topological theory in quite an unsuspected way.

Note however that we have not constructed the Hilbert space, only derived restrictions on it.

Some of the constraints \([103,108]\) indicate that the physical states should be massless, spinless and have zero scaling dimension. These constraints remove the model very far from what is expected from the classical theory, but there is circumstantial evidence that we are still on the right track.

Our result is consistent with the selection rules found by Gross [3] for high-energy tensile string scattering. He found that scattering amplitudes for all external states where directly related to the tachyon amplitudes by kinematic factors in the \(T \rightarrow 0\) limit. Thus a description in terms of very few states may capture the essential physics. In addition these amplitudes are given by polarizations (spin) in the scattering plane, i.e. the plane defined by the relative momenta. Other polarization directions do not affect the amplitudes. The constraints \([103]\) imply that no spin is allowed for a single tensionless string, but on the other hand spin does not seem to make a difference unless it is combined with a relative momentum. The ultimate test of the constraints must come from how they affect multi-string Hilbert spaces in an interacting theory of tensionless strings.

Another sign of a drastic reduction of the spectrum is the scaling argument of Atick and Witten [10], which indicates that the short-distance degrees of freedom of string theory are much fewer than in particle theory. Finally, we can get an idea of a physical origin of the constraints by comparing with the study of Karliner et al. [18], on the wavefunction of the ordinary string. At distances below a fundamental length \(T^{-1/2}\), fluctuations completely dominate the wavefunction, and it makes little sense to specify a particular string configuration. In the tensionless limit this behaviour should extend to all of space-time.

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5 Appendix

The Vacuum

We define the vacuum $|0\rangle$ for the $T \to 0$ theory by the condition that

$$P^m |0\rangle_0 = 0.$$  (109)

$X^m(\sigma)$ and $P^m(\sigma)$ are the position and momentum operators at $\tau = 0$ used to quantize the theory. In terms of their Fourier components the condition (109) reads

$$P^m |0\rangle_0 = 0$$  (110)

while

$$X^m |0\rangle_0 \neq 0.$$  (111)

At $\tau = 0$ the position and momentum operators can be identified with those of the tensile theory, although the models have different dynamics. We have arrived at the definition (109) guided by a wish to keep a relation to the $T \neq 0$ Hilbert space and vacuum. We have argued as follows:

The vacuum $|0\rangle_T$ for the tensile theory is annihilated by the oscillators

$$\alpha^m_n(T) |0\rangle = \tilde{\alpha}^m_n(T) |0\rangle = 0 \quad \forall n > 0.$$  (112)

The oscillators may be expressed in terms of of the Fourier components of the coordinate and momentum operators as follows:

$$\alpha^m_n(T) = -in\sqrt{T}X^m_n + \frac{1}{2\sqrt{T}}P^m_n$$

$$\tilde{\alpha}^m_n(T) = in\sqrt{T}X^m_n + \frac{1}{2\sqrt{T}}P^m_n.$$  (113)

We want to maintain a connection to $|0\rangle_T$ when we define the vacuum $|0\rangle_0$ for the tensionless theory. As a first attempt we try defining $|0\rangle_0$ in analogy to (112), i.e.,

$$\alpha^m_n(T) |0\rangle_0 = \tilde{\alpha}^m_n(T) |0\rangle_0 = 0 \quad \forall n > 0.$$  (114)

with perhaps a different range of $m$ and $n$. If it is to hold in the limit $T \to 0$, (113) implies

$$P^m_n |0\rangle_0 = P^m_{-n} |0\rangle_0 = 0$$  (115)

since $X^m, P_m$ and $|0\rangle_0$ are $T$-independent by assumption. But then by (114) we also have

$$X^m_n |0\rangle_0 = X^m_{-n} |0\rangle_0 = 0.$$  (116)
If we assume \((114)\) to hold for all \(m, n > 0\), we thus have \(X^m_n |0\rangle_0 = P^m_n |0\rangle_0 = 0\) for all \(n \neq 0\). This is inconsistent with the commutation relations.

If we assume instead that \((114)\) holds for all \(n > 0\) and for all \(m < 0\), we have \(X^m_n |0\rangle_0 = P^m_n |0\rangle_0 = 0\) for all \(n > 0\). This is the choice made in \[26\]. One expects, however, that an asymmetric treatment of \(\alpha\) and \(\tilde{\alpha}\) should lead to a breakdown of the global \(\sigma\)-translational symmetry and hence to a non-zero two-momentum \(P_{\sigma}\) for the closed tensionless string. This is indeed what is reported in \[26\].

We further regard the possibility of "mixed choices", i.e., some positive and some negative \(n\)'s and \(m\)'s as completely unnatural.

Having thus discussed and discarded the possibility of requiring \((114)\) to hold for \(T \neq 0\), we turn to the remaining option; that this is satisfied in the limit \(T \to 0\) only. We still find the restriction \((115)\), of course, but \(X^m_{m,n} |0\rangle\) is not determined. This leaves us with \((109)\).
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