Stochastic Volterra equations driven by fractional Brownian motion with Hurst parameter $H > 1/2$

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Abstract

In this note we prove an existence and uniqueness result of solution for stochastic Volterra integral equations driven by a fractional Brownian motion with Hurst parameter $H > 1/2$, showing also that the solution has finite moments. The stochastic integral with respect to the fractional Brownian motion is a pathwise Riemann-Stieltjes integral.

Keywords: stochastic Volterra equations, fractional Brownian motion, Riemann-Stieltjes integral

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Running head: Stochastic Volterra equations driven by fBm

0 Introduction

Consider the stochastic Volterra equation on $\mathbb{R}^d$

$$X(t) = X_0 + \int_0^t b(t, s, X(s))ds + \int_0^t \sigma(t, s, X(s))dW^H_s, \quad t \in (0, T],$$ (0.1)

where $W^H = \{W^H,j, j = 1, \ldots, m\}$ are independent fractional Brownian motions with Hurst parameter $H > 1/2$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In this paper, the integral with respect $W$ is a pathwise Riemann-Stieltjes integral. We will define it using a pathwise approach. Indeed, Young [13] proved that if we have a stochastic processes $\{u(t), t \geq 0\}$ whose trajectories are $\lambda$-Hölder continuous with $\lambda > 1 - H$, then the Riemann-Stieltjes integral $\int_0^T u(s)dW^H_s$ exists for each trajectory. Using the techniques of fractional calculus, Zähle [14] introduced a generalized Stieltjes integral that coincides with the Riemann-Stieltjes integral $\int_0^T f dg$ when the functions $f$ and $g$ are Hölder continuous of orders $\lambda$ and $\beta$, respectively, with $\lambda + \beta > 1$. Moreover, this generalized Stieltjes integral can be expressed in terms of a fractional derivative operator.

Following the ideas given by Zähle [14] and using the Riemann-Stieltjes integral, Nualart and Rascanu [9] proved a general result on the existence and
uniqueness of solution for a class of multidimensional time dependent stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H > 1/2$. Their proofs begin with a deterministic existence uniqueness theorem based on some a priori estimates on Lebesgue and the generalized Stieltjes integral.

In our paper, we extend the results in Nualart and Rascanu \cite{9} to multidimensional stochastic Volterra equations. Using the Riemann-Stieltjes integral, we give the existence and uniqueness of a solution to our equation (1.1). We also show that the solution has finite moments. Since we follow the methodology presented in Nualart and Rascanu \cite{9}, our main aim is to obtain precise estimates for Lebesgue and Riemann-Stieltjes Volterra integrals. Once we have obtained these estimates, the proofs of our existence and uniqueness results follow exactly the ones in Nualart and Rascanu \cite{9}. Let us note that our results include as a particular case the results of Nualart and Rascanu.

There are a lot of references on stochastic differential equations driven by a fractional Brownian motion and many papers about stochastic Volterra equations (see for instance \cite{1}, \cite{2}, \cite{3}, \cite{11}). Nevertheless the literature about Volterra equations driven by a fractional Brownian motion is scarce. As far as the authors know, the main references are the papers of Deya and Tindel \cite{4}, \cite{5}. In these papers they consider the case $H > 1/3$ using an algebraic integration setting. For the case $H > 1/2$ they also use a Young integral but they deal with $b \equiv 0$ and the set of hypothesis on the coefficients are different and stronger in some sense.

On the other hand, several references have followed the ideas of Nualart and Rascanu for different types of stochastic models (see \cite{9}, \cite{17}, \cite{8}, \cite{10}).

The structure of the paper is as follows: in the next section we state the main result of our paper. In Section 2 we study some estimates for Lebesgue integrals. Section 3 is devoted to obtain similar estimations for Riemann-Stieltjes integrals. In Section 4 we recall the results about deterministic equations and how we apply them to stochastic equations driven by fractional Brownian motion. Finally, in Section 5 we give some technical lemmas that we use throughout the paper.

1 Main results

Let $\frac{1}{2} < H < 1$, $\alpha \in (1-H, \frac{1}{2})$. Denote by $W^\alpha_0(0,T;\mathbb{R}^d)$ the space of measurable functions $f : [0,T] \rightarrow \mathbb{R}^d$ such that

$$\|f\|_{\alpha,\infty} := \sup_{t \in [0,T]} \left( |f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}}ds \right) < \infty.$$ 

For any $0 < \lambda \leq 1$, denote by $C^\lambda(0,T;\mathbb{R}^d)$ the space of $\lambda$--Hölder continuous functions $f : [0,T] \rightarrow \mathbb{R}^d$ such that

$$\|f\|_\lambda := \|f\|_\infty + \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t-s)^\lambda} < \infty,$$
where 

\[ \|f\|_\infty := \sup_{s \in [0,T]} |f(s)|. \]

Let us consider the following hypothesis:

- **(H1)** \( \sigma : [0, T]^2 \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m \) is a measurable function such that there exists the derivatives \( \partial_x \sigma(t, s, x) \), \( \partial_y \sigma(t, s, x) \) and \( \partial^2_{x,y} \sigma(t, s, x) \) and there exists some constants \( 0 < \beta, \mu, \delta \leq 1 \) for every \( N \geq 0 \) there exists \( K_N > 0 \) such that the following properties hold:
  
  1. \( |\sigma(t, s, x) - \sigma(t, s, y)| + |\partial_y \sigma(t, s, x) - \partial_y \sigma(t, s, y)| \leq K |x - y|, \forall x, y \in \mathbb{R}^d, \forall s, t \in [0, T], \)
  2. \( |\partial_x \sigma(t, s, x) - \partial_x \sigma(t, s, y)| + |\partial^2_{x,y} \sigma(t, s, x) - \partial^2_{x,y} \sigma(t, s, y)| \leq K_N |x - y|^{\delta}, \forall x, y \in \mathbb{R}^d, \forall s, t \in [0, T], \)
  3. \( |\sigma(t_1, s, x) - \sigma(t_2, s, x)| + |\partial_x \sigma(t_1, s, x) - \partial_x \sigma(t_2, s, x)| \leq K |t_1 - t_2|^{\mu}, \forall x \in \mathbb{R}^d, \forall t_1, t_2, s \in [0, T], \)
  4. \( |\sigma(t, s_1, x) - \sigma(t, s_2, x)| + |\partial_x \sigma(t, s_1, x) - \partial_x \sigma(t, s_2, x)| \leq K |s_1 - s_2|^{\beta}, \forall x \in \mathbb{R}^d, \forall s_1, s_2, t \in [0, T], \)
  5. \( |\partial^2_{x,y} \sigma(t, s_1, x) - \partial^2_{x,y} \sigma(t, s_2, x)| \leq K |s_1 - s_2|^{\gamma}, \forall x \in \mathbb{R}^d, \forall s_1, s_2, t \in [0, T], \)

- **(H2)** \( b : [0, T]^2 \times \mathbb{R}^d \to \mathbb{R}^d \) is a measurable function such that there exists \( b_0 \in L^\rho([0, T]^2; \mathbb{R}^d) \) with \( \rho \geq 2, 0 < \mu \leq 1 \) and \( \forall N \geq 0 \) there exists \( L_N > 0 \) such that:
  
  1. \( |b(t, s, x) - b(t, s, y)| \leq L_N |x - y|, \forall |x|, |y| \leq N, \forall s, t \in [0, T], \)
  2. \( |b(t_1, s, x) - b(t_2, s, x)| \leq L |t_1 - t_2|^{\mu}, \forall x \in \mathbb{R}^d, \forall t_1, t_2, s \in [0, T], \)
  3. \( |b(t, s, x)| \leq L_0|x| + b_0(t, x), \forall x \in \mathbb{R}^d, \forall s, t \in [0, T], \)
  4. \( |b(t_1, s, x_1) - b(t_2, s, x_2)| \leq L_N|t_1 - t_2|x_1 - x_2| \forall x_1, x_2 \leq N, \forall t, s \in [0, T]. \)

- **(H3)** There exists \( \gamma \in [0, 1] \) and \( K_0 > 0 \) such that
  
  \( |\sigma(t, s, x)| \leq K_0(1 + |x|^{\gamma}), \forall x \in \mathbb{R}^d, \forall s, t \in [0, T]. \)

**Remark 1.1** Actually, we can consider \( \sigma \) and \( b \) defined only in the set \( D \times \mathbb{R}^d \) with \( D = \{(t, s) \in [0, T]^2; s \leq t\} \).

Under these assumptions we are able to prove that our problem admits a unique solution. The result of existence and uniqueness reads as follows:
Theorem 1.2 Assume that $X_0$ is a $\mathbb{R}^d$-valued random variable and that $b$ and $\sigma$ satisfy hypothesis (H1) and (H2) respectively with $\beta > 1 - H$, $\delta > \frac{1}{H} - 1$, $\min\{\beta, \delta\} > 1 - \mu$. Set $\alpha_0 := \min\left\{\frac{1}{2}, \beta, \delta\right\}$.

Then if $\alpha \in ((1 - H) \lor (1 - \mu), \alpha_0)$ and $\rho \leq \frac{1}{\alpha}$, equation (0.1) has an unique solution $X \in L^0(\Omega, \mathcal{F}, \mathbb{P}; W^{\alpha,\infty}_0(0, T; \mathbb{R}^d))$ and for $P$–almost all $\omega \in \Omega$, $X(\omega, \cdot) \in C^{1-\alpha}(0, T; \mathbb{R}^d)$.

Moreover, if $\alpha \in ((1 - H) \lor (1 - \mu), \alpha_0 \lor (2 - \gamma)/4)$, $\rho \geq 1/\alpha$, $X_0 \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ and (H3) holds then $E(\|X\|_{p, \infty}) < \infty$, $\forall p \geq 1$.

In order to prove these results, we need to introduce a new norm in the space $W^{\alpha,\infty}_0(0, T; \mathbb{R}^d)$ that is for any $\lambda \geq 1$:

$$\|f\|_{\alpha,\lambda} := \sup_{t \in [0, T]} \exp(-\lambda t) \left( |f(t)| + \int_0^t |f(t) - f(s)| \frac{1}{(t - s)^{\alpha + 1}} ds \right).$$

It is easy to check that, for any $\lambda \geq 1$, this norm is equivalent to $\|f\|_{\alpha,\infty}$.

2 Lebesgue integral

Let us consider first the ordinary Lesbesgue integral. Given $f : [0, T]^2 \to \mathbb{R}^d$ a measurable function we define

$$F_t(f) = \int_0^t f(t, s) ds.$$ 

Proposition 2.1 Let $0 < \alpha < \frac{1}{2}$ and $f : [0, T]^2 \to \mathbb{R}^d$ a measurable function such that for all $0 < s \leq t_1, t_2$, $|f(t_1, s) - f(t_2, s)| \leq L|t_1 - t_2|^\mu$ with $\mu > \alpha$. If

$$\sup_{t \in [0, T]} \int_0^t \frac{|f(t, s)|}{(t - s)^\alpha} ds < \infty$$

then $F(f) \in W^{\alpha,\infty}_0(0, T; \mathbb{R}^d)$ and

$$|F_t(f)| + \int_0^t \frac{|F(t) - F_s(f)|}{(t - s)^{\alpha + 1}} ds \leq C^{(1)}_{\alpha, T} \int_0^t \frac{|f(t, s)|}{(t - s)^\alpha} ds + C^{(2)}_{\alpha, L, \mu} t^{1-\mu}.$$ (2.1)
Proof: We have that

\[
|F_t(f)| + \int_0^t \frac{|F_s(f) - F_s(f)|}{(t-s)^{\alpha+1}} ds \leq \int_0^t |f(t,u)| du \\
+ \int_0^t \int_0^s |f(t,u) - f(s,u)| du ds + \int_0^t \frac{|f(t,u)|}{(t-s)^{\alpha+1}} ds
\]

\[
\leq T^{\alpha} \int_0^t \frac{|f(t,u)|}{(t-u)^{\alpha+1}} du + L \int_0^t \frac{s}{(t-s)^{1-\mu+\alpha}} ds \\
+ \int_0^t \int_0^u \frac{|f(t,u)|}{(t-s)^{\alpha+1}} ds du
\]

\[
\leq \left( T^{\alpha} + \frac{1}{\alpha} \right) \int_0^t \frac{|f(t,u)|}{(t-u)^{\alpha}} du + \frac{L}{(\mu-\alpha)} t^{1+\mu-\alpha},
\]

so \( (2.1) \) holds with \( C_{a,T}^{(1)} = (T^{\alpha} + \frac{1}{\alpha}) \) and \( C_{a,L,\mu}^{(2)} = \frac{L}{(\mu-\alpha)} \) and we also have proved that \( F(f) \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d) \). \( \square \)

Given \( f : [0,T] \rightarrow \mathbb{R}^d \) let us now define

\[
F_t^{(b)}(f) = \int_0^t b(t,s,f(s)) ds.
\]

Proposition 2.2 Assume that \( b \) satisfies (H2) with \( \rho = \frac{1}{\alpha} \) and \( \mu > (1-\alpha) \lor \alpha \).

1. If \( f \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d) \) then \( F_t^{(b)}(f) \in C^{1-\alpha}(0,T;\mathbb{R}^d) \) and

\[
\left\| F_t^{(b)}(f) \right\|_{1-\alpha} \leq d^{(1)} (1 + \|f\|_{\infty}),
\]

\[
\left\| F_t^{(b)}(f) \right\|_{\alpha,\lambda} \leq \frac{d^{(2)}}{\lambda^{1-\alpha}} \left( 1 + \|f\|_{\alpha,\lambda} \right),
\]

for all \( \lambda \geq 1 \), where \( d^{(1)} \) and \( d^{(2)} \) are positive constants depending only on \( \mu,\alpha,T,L,L_0 \) and a constant \( B_{0,\alpha} \) that depends on \( b \).

2. If \( f, h \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d) \) are such that \( \|f\|_{\infty} \leq N, \|h\|_{\infty} \leq N \), then

\[
\left\| F_t^{(b)}(f) - F_t^{(b)}(h) \right\|_{\alpha,\lambda} \leq \frac{d_N}{\lambda^{1-\alpha}} \|f - h\|_{\alpha,\lambda},
\]

for all \( \lambda \geq 1 \), where \( d_N \) depends on \( \alpha, T \) and \( L_N \) from (H2).

Proof: In order to simplify the presentation we will assume \( d = 1 \). Let \( f \in W_0^{\alpha,\infty}(0,T) \), then for \( 0 \leq s < t \leq T \)

\[
\left| F_t^{(b)}(f) - F_s^{(b)}(f) \right| \leq \int_0^s |b(t,u,f(u)) - b(s,u,f(u))| du \\
+ \int_s^t |b(t,u,f(u))| du \\
\leq Ls(t-s)^{\mu} + \int_s^t (L_0|f(u)| + b_0(t,u)) du \\
\leq (t-s)^{1-\alpha} (LT^{\mu+\alpha} + L_0T^{\alpha} \|f\|_{\infty} + B_{0,\alpha}),
\]

(2.4)
where \( B_{0,\alpha} := \sup_{t \in [0, T]} \left( \int_0^t |b_0(t, u)|^{1/\alpha} du \right)^\alpha \). The same computations for \( s = 0 \) give us
\[
\left| F_t^{(b)}(f) \right| \leq L_0 t \| f \|_{\infty} + B_{0,\alpha} t^{1-\alpha}.
\]
Hence
\[
\left\| F^{(b)}(f) \right\|_{1-\alpha} \leq L_0 (T + T^\alpha) \| f \|_{\infty} + B_{0,\alpha} (1 + T^{1-\alpha}) + L\mu^{\alpha},
\]
and (2.2) holds with \( d^{(1)} = (1 + T^{1-\alpha})(B_{0,\alpha} + T^\alpha L_0) + L\mu^{\alpha} \). By (2.1) we have
\[
\left| F_t^{(b)}(f) \right| + \int_0^t \frac{\left| F_t^{(b)}(f) - F_s^{(b)}(f) \right|}{(t-s)^{\alpha+1}} ds \leq C_{\alpha,T}(1) \int_0^t \frac{|b(t, s, f(s))|}{(t-s)^{\alpha}} ds + C_{\alpha,L}{\alpha} \mu^{\alpha+1} t^{1-\alpha}
\leq C_{\alpha,T}(1) \int_0^t \frac{L_0 |f(s)| + b_0(t, s)}{(t-s)^{\alpha}} ds + C_{\alpha,L}{\alpha} \mu^{\alpha+1} t^{1-\alpha}
\leq C_{\alpha,T}(1) \left( L_0 \int_0^t \frac{|f(s)|}{(t-s)^{\alpha}} ds + B_{0,\alpha} \left( \frac{1-\alpha}{1-2\alpha} \right) t^{1-2\alpha} \right)
+ C_{\alpha,L}{\alpha} \mu^{\alpha+1} t^{1-\alpha}.
\]
Then using that
\[
\int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^{\alpha}} ds \leq \lambda^{\alpha-1} \Gamma(1-\alpha) \quad \text{and} \quad \sup_{t \in [0, T]} t^{\mu} e^{-\lambda t} \leq \left( \frac{\mu}{\lambda} \right)^\mu e^{-\mu},
\]
we get that
\[
\left\| F^{(b)}(f) \right\|_{\alpha,\lambda} \leq C_{\alpha,T}(1) L_0 \left\| f \right\|_{\alpha,\lambda} \sup_{t \in [0, T]} \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^{\alpha}} ds \leq C_{\alpha,L}{\alpha} \mu^{\alpha+1} t^{1-\alpha}
+ C_{\alpha,L}{\alpha} \mu^{\alpha+1} t^{1-\alpha}
+ C_{\alpha,T}(1) B_{0,\alpha} \left( \frac{1-\alpha}{1-2\alpha} \right)^{1-\alpha} \sup_{t \in [0, T]} e^{-\lambda t^{1-2\alpha}}
\leq C_{\alpha,T}(1) L_0 \Gamma(1-\alpha) \lambda^{\alpha-1} \left\| f \right\|_{\alpha,\lambda}
+ C_{\alpha,L}{\alpha} \mu^{\alpha-1} \left( 1 + \alpha - \mu \right)^{1-\mu} \lambda^{\alpha-1} - \mu
+ C_{\alpha,T}(1) B_{0,\alpha} \left( \frac{1-\alpha}{1-2\alpha} \right)^{2\alpha-1} \lambda^{2\alpha-1}
\leq d^{(2)} \lambda^{2\alpha-1} \left( 1 + \left\| f \right\|_{\alpha,\lambda} \right).
Finally, \( d^{(2)} = C^{(1)}_{\alpha,T} L_0 \Gamma(1-\alpha) + C^{(1)}_{\alpha,T} B_{0,\alpha} (1-\alpha)^{1-\alpha} (1 - 2\alpha)\alpha e^{2\alpha - 1} + C^{(2)}_{\alpha,T} e^{\alpha - \mu - 1} (1 + \mu - \alpha)^{1 + \mu - \alpha}. \)

Now, consider \( f, h \in W^{0,\infty}_0(0,T) \) such that \( \|f\|_\infty \leq N, \|h\|_\infty \leq N. \) We obtain that

\[
\left| F_t^{(b)}(f) - F_t^{(b)}(h) \right| \leq \int_0^t \left| b(t, u, f(u)) - b(t, u, h(u)) \right| du
\leq L_N \int_0^t |f(u) - h(u)| du
\] (2.7)

and

\[
|F_t^{(b)}(f) - F_t^{(b)}(h) - F_s^{(b)}(f) + F_s^{(b)}(h)| \leq \int_s^t |b(t, u, f(u)) - b(t, u, h(u))| du
+ \int_0^s |b(t, u, f(u)) - b(t, u, h(u)) - b(s, u, f(u)) + b(s, u, h(u))| du
\leq L_N \int_s^t |f(u) - h(u)| du + L_N |t - s| \int_0^s |f(u) - h(u)| du.
\] (2.8)

Then, using (2.7) and (2.8) we have

\[
\left| F_t^{(b)}(f) - F_t^{(b)}(h) \right| \leq \int_0^t \frac{|F_t^{(b)}(f) - F_t^{(b)}(h) - F_s^{(b)}(f) + F_s^{(b)}(h)|}{(t-s)^{\alpha + 1}} ds
\leq L_N \int_0^t |f(u) - h(u)| du + L_N \int_0^t \frac{|f(u) - h(u)| du}{(t-s)^\alpha} ds
+ L_N \int_0^t \frac{|f(u) - h(u)| du}{(t-s)^{\alpha + 1}} ds
\leq L_N \int_0^t |f(u) - h(u)| du + \frac{L_N}{1 - \alpha} \int_0^t \frac{|f(u) - h(u)|}{(t-u)^{\alpha - 1}} du
+ \frac{L_N}{\alpha} \int_0^t |f(u) - h(u)| du.
\]

Finally,

\[
\left\| F^{(b)}(f) - F^{(b)}(h) \right\|_{\alpha,\lambda} \leq \sup_{t \in [0,T]} \left( L_N \left( 1 + \frac{T^{1-\alpha}}{1-\alpha} \right) \int_0^t e^{-\lambda(t-u)} du \right.
+ \frac{L_N}{\alpha} \int_0^t \frac{e^{-\lambda(t-u)}}{(t-u)^{\alpha - 1}} du \left\| f - h \right\|_{\alpha,\lambda}
\leq L_N \left( \frac{1}{\lambda} \left( 1 + \frac{T^{1-\alpha}}{1-\alpha} \right) + \frac{\Gamma(1-\alpha)}{\alpha \lambda^{1-\alpha}} \right) \|f - h\|_{\alpha,\lambda}
\leq \frac{d_N}{\lambda^{1-\alpha}} \|f - h\|_{\alpha,\lambda},
\]

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where \( d_N = L_N \left( 1 + \frac{T^{1-\alpha}}{1-\alpha} + \frac{\Gamma(1-\alpha)}{\alpha} \right). \)

## 3 Riemann-Stieltjes integral

Let us now consider the Riemann-Stieltjes integral introduced by Zähle, which is based on fractional integrals and derivatives. We refer the reader to Zähle [14] and the references therein for a detailed account about this generalized integral and the relationships with fractional calculus. Here we just recall some basic results.

Fix a parameter \( 0 < \alpha < \frac{1}{2} \). Denote by \( W^{1-\alpha,\infty}_T(0,T) \) the space of measurable functions \( g : [0, T] \to \mathbb{R} \) such that

\[
\|g\|_{1-\alpha,\infty,T} := \sup_{0 < s < t < T} \left( |g(t) - g(s)| \left( \frac{1}{(t-s)^{1-\alpha}} + \int_s^t |g(y) - g(s)| \frac{1}{(y-s)^{2-\alpha}} dy \right) \right) < \infty.
\]

Moreover if \( g \) belongs to \( W^{1-\alpha,\infty}_T(0,T) \) we define

\[
\Lambda_\alpha(g) := \frac{1}{\Gamma(1-\alpha) \Gamma(\alpha)} \sup_{0 < s < t < T} \left| \left( D^{1-\alpha}_t g_{t-} \right)(s) \right| 
\leq \frac{1}{\Gamma(1-\alpha) \Gamma(\alpha)} \|g\|_{1-\alpha,\infty,T} < \infty,
\]

where \( (D^{1-\alpha}_t g_{t-})(s) \) is a Weyl derivative. We also denote by \( W^{\alpha,1}_0(0,T) \) the space of measurable functions \( f : [0, T] \to \mathbb{R} \) such that

\[
\|f\|_{\alpha,1} := \int_0^T \int_0^s \frac{|f(s) - f(y)|}{(s-y)^{\alpha+1}} dy ds < \infty.
\]

Then, if \( f \) is a function in the space \( W^{\alpha,1}_0(0,T) \) and \( g \) belongs to \( W^{1-\alpha,\infty}_T(0,T) \), the integral \( \int_0^t f(s) dg_s \) exists for all \( t \in [0, T] \) and we can define

\[
\int_0^t f(s) dg_s = \int_0^T f(s) \mathbf{1}_{(0,t)}(s) dg_s.
\]

Furthermore the following estimate holds

\[
\left| \int_0^t f(s) dg_s \right| \leq \Lambda_\alpha(g) \|f\|_{\alpha,1}.
\]

Given \( f : [0, T] \to \mathbb{R} \) such that for any \( t \in [0, T] \), \( f(t, \cdot) \in W^{\alpha,1}_0([0, T]) \) we can also consider the integral

\[
G_t(f) = \int_0^t f(t, s) dg_s = \int_0^T f(t, s) \mathbf{1}_{(0,t)}(s) dg_s,
\]

and the estimate

\[
\left| \int_0^t f(t, s) dg_s \right| \leq \Lambda_\alpha(g) \|f(t, \cdot)\|_{\alpha,1}.
\]
Proposition 3.1 Let $g \in W_{0}^{1_{-\alpha,\infty}}(0,T)$ and $f : [0,T]^2 \to \mathbb{R}^d$ such that $f(t,\cdot) \in W_{0}^{\alpha,1}(0,T)$ for all $t \in [0,T]$ and such that $|f(t_1,s) - f(t_2,s)| \leq K(s)|t_1 - t_2|^\mu$, with $\mu > \alpha$. Then for all $s < t$, the following estimates hold

\[ |G_t(f) - G_s(f)| \leq \Lambda_\alpha(g) \left( |t-s|^\mu \int_0^s \frac{K(u)}{u^\alpha} du + \int_s^t |f(t,u)| \right) + \alpha \int_0^s \int_0^u \frac{|f(t,u) - f(s,u) - f(t,y) + f(s,y)|}{(u-y)^{\alpha+1}} dy du + \alpha \int_s^t \int_s^u \frac{|f(t,u) - f(t,y)|}{(u-y)^{\alpha+1}} dy du \]  

for all $\alpha, T \in (\mathbb{R}, \cdot)$, the following estimates hold

\[ |G_t(f) - G_s(f)| + \alpha \int_0^t \frac{|G_t(f) - G_s(f)|}{(t-s)^{\alpha+1}} ds \leq \Lambda_\alpha(g) \left( C_{\alpha} \int_0^t \frac{K(u)}{u^\alpha} (t-u)^{\mu-\alpha} du + C_{\alpha,T} \int_0^t \frac{((t-u)^{-2\alpha} + u^{-\alpha})}{((u-y)^{\alpha+1})} dy du + \alpha \int_0^t \int_0^u \frac{|f(t,u) - f(t,y)|}{(u-y)^{\alpha+1}} dy du \right). \]

Proof: We have

\[ |G_t(f) - G_s(f)| \leq \Lambda_\alpha(g) \left( \int_0^s \frac{|f(t,u) - f(s,u)|}{u^\alpha} du + \int_s^t |f(t,u)| \right) + \alpha \int_0^s \int_0^u \frac{|f(t,u) - f(s,u) - f(t,y) + f(s,y)|}{(u-y)^{\alpha+1}} dy du + \alpha \int_s^t \int_s^u \frac{|f(t,u) - f(t,y)|}{(u-y)^{\alpha+1}} dy du \leq \Lambda_\alpha(g) \left( |t-s|^\mu \int_0^s \frac{K(u)}{u^\alpha} du + \int_s^t |f(t,u)| \right) + \alpha \int_0^s \int_0^u \frac{|f(t,u) - f(s,u) - f(t,y) + f(s,y)|}{(u-y)^{\alpha+1}} dy du + \alpha \int_s^t \int_s^u \frac{|f(t,u) - f(t,y)|}{(u-y)^{\alpha+1}} dy du \]

(3.1)
Proposition 3.2 Let \( g \in W^{1-\alpha, \infty}_T(0, T) \). Assume that \( \sigma \) satisfies (H1) with \( \beta > \alpha > 1 - \mu \).
1. If \( f \in W_0^{0,\infty}(0, T; \mathbb{R}^d) \) then
\[
G^{(\sigma)}(f) \in C^{1-\alpha}(0, T; \mathbb{R}^d) \subseteq W_0^{0,\infty}(0, T; \mathbb{R}^d).
\]

Moreover,
\[
\left\| G^{(\sigma)}(f) \right\|_{1-\alpha} \leq \Lambda_\alpha(g) d^{(3)}(1 + \|f\|_{\alpha, \infty}),
\]
\[
\left\| G^{(\sigma)}(f) \right\|_{\alpha, \lambda} \leq \frac{\Lambda_\alpha(g) d^{(4)}}{\lambda^{1-2\alpha}} (1 + \|f\|_{\alpha, \lambda}),
\]
for all \( \lambda \geq 1 \) where the constants \( d^{(i)} \) for \( i = 3, 4 \) depend only on \( \alpha, \beta, \mu, K, T \) and \( N \).

2. If \( f, h \in W_0^{0,\infty}(0, T; \mathbb{R}^d) \) are such that \( \|f\|_{\infty} \leq N, \|h\|_{\infty} \leq N \), then
\[
\left\| G^{(\sigma)}(f) - G^{(\sigma)}(h) \right\|_{\alpha, \lambda} \leq \frac{\Lambda_\alpha(g) d'_N}{\lambda^{1-2\alpha}} (1 + \Delta(f) + \Delta(h)) \|f - h\|_{\alpha, \lambda},
\]
for all \( \lambda \geq 1 \), where
\[
\Delta(f) = \sup_{u \in [0, T]} \int_0^u \frac{|f(u) - f(s)|^\beta}{(u - s)^{\alpha+1}} ds,
\]
and the constant \( d'_N \) depends only on \( \alpha, \beta, \mu, N, K, T \).

**Proof:** We will assume \( d = m = 1 \) in order to simplify the presentation of the proof. First we see that if \( f \in W_0^{0,\infty}(0, T) \) then \( \sigma(t, \cdot, f(\cdot)) \in W_0^{0,\infty}([0, T]) \) for all \( t \in [0, T] \). Indeed:
\[
|\sigma(t, r, f(r))| + \int_0^r \frac{|\sigma(t, r, f(r)) - \sigma(t, s, f(s))|}{(r - s)^{\alpha+1}} ds
\]
\[
\leq K(t^\mu + r^\beta + |f(r)|) + |\sigma(0, 0, 0)| + K \int_0^r \frac{|f(r) - f(s)|}{(r - s)^{\alpha+1}} ds + K \int_0^r \frac{|f(r)|}{(r - s)^{\alpha+1}} ds.
\]
\[
+ K \int_0^r \frac{|f(r) - f(s)|}{(r - s)^{\alpha+1}} ds + K \left( |f(r)| + \int_0^r \frac{|f(r) - f(s)|}{(r - s)^{\alpha+1}} ds \right).
\]

So, for all \( t \)
\[
\|\sigma(t, \cdot, f(\cdot))\|_{\alpha, \infty} \leq K^{(2)} + K \|f\|_{\alpha, \infty},
\]
with \( K^{(2)} = K \left( T^\mu + T^\beta + \frac{T^\beta - \alpha}{\beta - \alpha} \right) + |\sigma(0, 0, 0)|. \]
Now if \( f \in W^{\alpha, \infty}_0(0, T) \) under assumptions (H1) from (3.8) we have,

\[
\left\| G(\sigma)(f) \right\|_{\infty} \leq \sup_{t \in [0, T]} \Lambda_\alpha(g) \left( \int_0^t \left| \sigma(t, u, f(u)) \right| \frac{u^\alpha}{(u-s)^\alpha} du + \alpha \int_0^t \int_0^u \left| \sigma(t, u, f(u)) - \sigma(t, y, f(y)) \right| \frac{(u-y)^{\alpha+1}}{(u-s)^\alpha} dydu \right) \leq \Lambda_\alpha(g) \left( \frac{T^{1-\alpha}}{1-\alpha} + \alpha T \right) \sup_{t \in [0, T]} \|\sigma(t, \cdot, f(\cdot))\|_{\alpha, \infty}.
\]

If we come back to (3.1) with \( K(u) = K \) and using Lemma 5.2 we have,

\[
\left| G_t^{(\sigma)}(f) - G_s^{(\sigma)}(f) \right| \leq \Lambda_\alpha(g) \left( |t-s|^\mu K \int_0^t u^{-\alpha} du + \int_s^t \frac{\|\sigma(t, \cdot, f(\cdot))\|_{\infty}}{(u-s)^\alpha} du + \alpha \int_s^t \int_s^u \frac{|u-y|^\beta + |f(u) - f(y)|}{(u-y)^{\alpha+1}} dydu \right) \leq (t-s)^{1-\alpha} \Lambda_\alpha(g) K \left( \frac{T^u}{1-\alpha} + \frac{\|\sigma(t, \cdot, f(\cdot))\|_{\infty}}{1-\alpha} \right) + \alpha T^{1+\beta} \left( \frac{1}{(\beta - \alpha)(1 + \beta - \alpha)} + \|f\|_{\alpha, \infty} \right) + \alpha \left( \frac{T^{\beta}}{(\beta - \alpha)} + T^\alpha \|f\|_{\alpha, \infty} \right) \leq (t-s)^{1-\alpha} \Lambda_\alpha(g) K_{\alpha, T}^{(1)} \left( 1 + \|\sigma(t, \cdot, f(\cdot))\|_{\alpha, \infty} + \|f\|_{\alpha, \infty} \right),
\]

where

\[
K_{\alpha, T}^{(1)} = 5K \left( \frac{1}{1+\alpha} + \frac{1}{\beta - \alpha} + \frac{1}{(\beta - \alpha)(1 + \beta - \alpha)} \right) \left( 1 + \alpha T^{1+\beta} \right).
\]

So, using (3.12) we can deduce that \( G(\sigma)(f) \in C^{1-\alpha}(0, T) \), and (3.9) holds.

Let us study now the norm \( \| \cdot \|_{\alpha, \lambda} \). Set

\[
\Sigma(t, s, u, y, f) := \sigma(t, u, f(u)) - \sigma(s, u, f(u)) - \sigma(t, y, f(y)) + \sigma(s, y, f(y)).
\]
Using (3.2) with $K(u) = K$, we have
\[
\left\|G^{(\sigma)}(f)\right\|_{\alpha,\lambda} \leq \Lambda_\alpha(g) \sup_{t \in [0,T]} e^{-\lambda t} \left( C_\alpha^{(3)} K \int_0^t \frac{(t-u)^{\mu-\alpha}}{u^\alpha} du \right.
\]
\[
+ C_{\alpha,T}^{(4)} \int_0^t ((t-u)^{-2\alpha} + u^{-\alpha}) \left( |\sigma(t,u,f(u))| \right. 
\]
\[
+ \left. \left. \int_0^u \frac{|\sigma(t,u,f(u)) - \sigma(t,y,f(y))|}{(u-y)^{\alpha+1}} dy du \right) \right.
\]
\[
+ \alpha \int_0^t \int_0^s \int_0^u \frac{|\Sigma(t,s,u,y,f)|}{(u-y)^{\alpha+1}(t-s)^\alpha} dyduds 
\]
\[
\leq \Lambda_\alpha(g) \sup_{t \in [0,T]} \left( C_\alpha^{(3)} KB(1 - \alpha, 1 + \mu - \alpha) e^{-\lambda t^{1+\mu-2\alpha}} + C_{\alpha,T}^{(4)} A_1 + \alpha A_2 \right),
\]
where
\[
A_1 = e^{-\lambda t} \int_0^t ((t-u)^{-2\alpha} + u^{-\alpha}) \left( |\sigma(t,u,f(u))| \right.
\]
\[
+ \left. \left. \int_0^u \frac{|\sigma(t,u,f(u)) - \sigma(t,y,f(y))|}{(u-y)^{\alpha+1}} dy du \right),
\]
\[
A_2 = e^{-\lambda t} \int_0^t \int_0^s \int_0^u \frac{|\Sigma(t,s,u,y,f)|}{(u-y)^{\alpha+1}(t-s)^\alpha} dyduds.
\]

Indeed, we have used (see (3.7)) that
\[
\int_0^t (t-u)^q u^p du = t^{p+q+1} \int_0^1 (1-y)^q y^p dy = B(p+1,q+1) t^{p+q+1}. \tag{3.13}
\]

Moreover, using (H1) it holds that
\[
A_1 \leq e^{-\lambda t} \int_0^t ((t-u)^{-2\alpha} + u^{-\alpha}) \left( K(t^\mu + u^\beta + |f(u)|) + |\sigma(0,0,0)| \right)
\]
\[
+ K \int_0^u \frac{|f(u) - f(y)|}{(u-y)^{\alpha+1}} dy + K \int_0^u (u-y)^{\beta-\alpha-1} dy \right) du 
\]
\[
\leq A_{1,1} + A_{1,2}, \tag{3.14}
\]

where,
\[
A_{1,1} = e^{-\lambda t} \int_0^t ((t-u)^{-2\alpha} + u^{-\alpha})
\]
\[
\times \left( |\sigma(0,0,0)| + K \left( |f(u)| + \int_0^u \frac{|f(u) - f(y)|}{(u-y)^{\alpha+1}} dy \right) \right) du,
\]
\[
A_{1,2} = Ke^{-\lambda t} \int_0^t ((t-u)^{-2\alpha} + u^{-\alpha}) \left( t^\mu + u^\beta + \frac{1}{\beta-\alpha} u^{\beta-\alpha} \right) du.
\]
In proposition 4.2 in [9] it has been proved that
\[ \int_0^t e^{-\lambda(t-u)}((t-u)^{-2\alpha} + u^{-\alpha})du \leq C_\alpha \lambda^{2\alpha-1} \] (3.15)
where \( C_\alpha \leq \frac{1}{1-2\alpha} + 4 \). Then, the term \( A_1 \) can be treated as in [9]. We get that
\[
\sup_{t \in [0,T]} A_{1,1} \leq C_\alpha \lambda^{2\alpha-1} \sup_{u \in [0,T]} e^{-\lambda u} \left(|\sigma(0,0,0)| + K \left( |f(u)| + \int_0^u \frac{|f(u) - f(s)|}{(u-s)^{\alpha+1}}ds \right) \right) \\
\leq \lambda^{2\alpha-1} C_\alpha (|\sigma(0,0,0)| + K) (1 + \|f\|_{\alpha,\lambda}).
\] (3.16)
The term \( A_{1,2} \) can be computed easily using (3.13). Indeed, we get that
\[
A_{1,2} = Ke^{-\lambda t} \left( \frac{t^{1+\mu-2\alpha}}{1-2\alpha} + \frac{t^{1+\mu-\alpha}}{1-\alpha} + \frac{t^{\beta-\alpha+1}}{\beta-\alpha+1} \right) \\
+ B(1 + \beta - \alpha, 1 - 2\alpha) \frac{t^{\beta-3\alpha+1}}{\beta-\alpha} \\
+ t^{\beta-2\alpha+1} \left( \frac{1}{(\beta-\alpha)(\beta-2\alpha+1)} + B(\beta + 1, 1 - 2\alpha) \right) \\
\leq K_{(3)}^{(3)} e^{-\lambda t} \left( t^{1+\mu-2\alpha} + t^{1+\mu-\alpha} + t^{\beta-\alpha+1} + t^{\beta-3\alpha+1} + t^{\beta-2\alpha+1} \right),
\]
where,
\[
K_{(3)}^{(3)} = \frac{1}{1-2\alpha} + \frac{1}{1-\alpha} + \frac{1}{\beta-\alpha+1} + \frac{1}{\beta-\alpha} B(1 + \beta - \alpha, 1 - 2\alpha) \\
+ \frac{1}{(\beta-\alpha)(\beta-2\alpha+1)} + B(\beta + 1, 1 - 2\alpha).
\]
So, using (2.6) it holds that
\[
\sup_{t \in [0,T]} A_{1,2} \leq K_{(3)}^{(3)} K_{(4)}^{(4)} \lambda^{3\alpha-\beta-1} + \lambda^{2\alpha-\mu-1}.
\]
where
\[
K_{(4)}^{(4)} = \left( \frac{1 + \mu - 2\alpha}{e} \right)^{1+\mu-2\alpha} + \left( \frac{1 + \mu - \alpha}{e} \right)^{1+\mu-\alpha} + \left( \frac{\beta - \alpha + 1}{e} \right)^{\beta-\alpha+1} \\
+ \left( \frac{\beta - 2\alpha + 1}{e} \right)^{\beta-2\alpha+1} + \left( \frac{\beta - 3\alpha + 1}{e} \right)^{\beta-3\alpha+1}.
\]
Putting together (3.14) and (3.16) we finally get that
\[
\sup_{t \in [0,T]} A_1 \leq \lambda^{2\alpha-1} K_{(5)}^{(5)} K_{\alpha,\beta,K} \left( 1 + \|f\|_{\alpha,\lambda} \right),
\] (3.17)
From (3.17) and (3.18), (3.10) holds with
\begin{align*}
K^{(5)}_{\alpha, \beta, K} &= C_\alpha (|\sigma(0,0,0)| + K) + K^{(3)}_{\alpha, \beta} K^{(4)}.
\end{align*}

On the other hand, we will use lemma 5.2 to study the term $A_2$. We can write
\begin{align*}
A_2 &\leq e^{-\lambda t} \int_0^t \int_0^s \int_0^u K|t-s||u-y|^\beta + K|t-s||f(u) - f(y)| \, dy \, du \, ds \\
&\leq Ke^{-\lambda t} \int_0^t \int_0^s \int_0^u |t-s|^{-\alpha} |u-y|^\beta \alpha - 1 \, dy \, du \\
&\quad + Ke^{-\lambda t} \int_0^t \int_0^u \int_0^u (t-s)^{-\alpha} |f(u) - f(y)| \, ds \, dy \\
&\leq K e^{-\lambda t} \int_0^t \int_0^s (t-s)^{-\alpha} u^{\beta - \alpha} \, du \\
&\quad + K e^{-\lambda t} \int_0^t \int_0^u (t-u)^{1-\alpha} |f(u) - f(y)| \, du \\
&\leq KB(1-\alpha, \beta - \alpha + 1) e^{-\lambda t} \int_0^t (t-s)^{-\alpha} s^{\beta - \alpha + 1} \, ds \\
&\quad + K \frac{\|f\|_{\alpha, \lambda}}{1 - \alpha} \int_0^t e^{-\lambda (t-u)} (t-u)^{1-\alpha} \, du \\
&\leq \frac{KB(1-\alpha, \beta - \alpha + 1)}{(\beta - \alpha)(\beta - \alpha + 1)} e^{\beta - 2\alpha + 2} e^{-\lambda t} + \frac{KT^{1-\alpha}}{1 - \alpha} \lambda^{-1} \|f\|_{\alpha, \lambda}
\end{align*}
and
\begin{align*}
\sup_{t \in [0,T]} A_2 &\leq K^{(6)}_{\alpha, \beta} \lambda^{2\alpha - \beta - 2} + K^{(7)}_{\alpha, \beta, \lambda} \lambda^{-1} \|f\|_{\alpha, \lambda} \\
&\leq (K^{(6)}_{\alpha, \beta} + K^{(7)}_{\alpha, \beta}) \lambda^{-1} (1 + \|f\|_{\alpha, \lambda}),
\end{align*}
where
\begin{align*}
K^{(6)}_{\alpha, \beta} &= \frac{KB(1-\alpha, \beta - \alpha + 1)}{(\beta - \alpha)(\beta - \alpha + 1)} \left( \frac{\beta - 2\alpha}{e} \right)^{\beta - 2\alpha + 2} \quad \text{and} \quad K^{(7)}_{\alpha, \beta} = \frac{KT^{1-\alpha}}{1 - \alpha}.
\end{align*}

From (3.17) and (3.18), (3.10) holds with
\begin{align*}
d^{(4)} &= C^{(3)}_{\alpha, K} KB(1-\alpha, 2-\alpha)(2-2\alpha)^2 e^{2\alpha - 2} + C^{(4)}_{\alpha, T} K^{(5)}_{\alpha, \beta, K} + \alpha (K^{(6)}_{\alpha, \beta} + K^{(7)}_{\alpha, \beta}).
\end{align*}

Assume now that $\|f\|_{\infty} \leq N$ and $\|h\|_{\infty} \leq N$. Notice that from lemma 5.2 with $s_1 = s_2$ we obtain that
\begin{align*}
|\sigma(t, u, f(u)) - \sigma(t, u, h(u))| \leq K|f(u) - h(u)||t - s|.
\end{align*}
Furthermore, using \(K(u) = K|f(u) - h(u)|\), we can write,

\[
\|G^{(\sigma)}(f) - G^{(\sigma)}(h)\|_{\alpha, \lambda} \\
\leq \Lambda_\alpha(g) \sup_{t \in [0,T]} e^{-\lambda t} \left( C^{(3)}_\alpha K \int_0^t \frac{|f(u) - h(u)|}{u^\alpha} (t-u)^{1-\alpha} du \right) \\
+ C^{(4)}_{\alpha, T} \int_0^t ((t-u)^{2\alpha} + u^{-\alpha}) \left( |\sigma(t, u, f(u)) - \sigma(t, u, h(u))| \\
+ \int_0^u \frac{|\sigma(t, u, f(u)) - \sigma(t, u, h(u)) - \sigma(t, y, f(y)) + \sigma(t, y, h(y))|}{(u-y)^{\alpha+1}} dy \right) du \\
+ \alpha \int_0^t \int_0^s \int_0^u (u-y)^{-\alpha-1}(t-s)^{-\alpha-1} |\sigma(t, u, f(u)) - \sigma(t, u, h(u)) \\
- \sigma(s, u, f(u)) + \sigma(s, u, h(u)) - \sigma(t, y, f(y)) + \sigma(t, y, h(y)) \\
+ \sigma(s, y, f(y)) - \sigma(s, y, h(y))| dy du ds \\
= \Lambda_\alpha(g) \sup_{t \in [0,T]} \left( C^{(3)}_\alpha KB_0 + C^{(4)}_{\alpha, T}(B_1 + B_2) + \alpha B_3 \right),
\]

where

\[
B_0 = e^{-\lambda t} \int_0^t \frac{|f(u) - h(u)|}{u^\alpha} (t-u)^{1-\alpha} du,
\]

\[
B_1 = e^{-\lambda t} \int_0^t ((t-u)^{2\alpha} + u^{-\alpha})|\sigma(t, u, f(u)) - \sigma(t, u, h(u))| du,
\]

\[
B_2 = e^{-\lambda t} \int_0^t ((t-u)^{2\alpha} + u^{-\alpha}) \\
\times \int_0^u \frac{|\sigma(t, u, f(u)) - \sigma(t, u, h(u)) - \sigma(t, y, f(y)) + \sigma(t, y, h(y))|}{(u-y)^{\alpha+1}} dy du,
\]

\[
B_3 = e^{-\lambda t} \int_0^t \int_0^s \int_0^u (u-y)^{-\alpha-1}(t-s)^{-\alpha-1} |\sigma(t, u, f(u)) - \sigma(t, u, h(u)) \\
- \sigma(s, u, f(u)) + \sigma(s, u, h(u)) - \sigma(t, y, f(y)) + \sigma(t, y, h(y)) \\
+ \sigma(s, y, f(y)) - \sigma(s, y, h(y))| dy du ds.
\]
$B_0$ can be studied easily,
\[
\sup_{t \in [0,T]} B_0 \leq \|f - h\|_{\alpha, \lambda} \sup_{t \in [0,T]} t^{1-\alpha} \int_0^t e^{-\lambda(t-u)}u^{-\alpha}du
\]
\[
\leq \|f - h\|_{\alpha, \lambda} \sup_{t \in [0,T]} t^{1-\alpha} \int_t^{\lambda t} e^{-\lambda(t-x)}dx
\]
\[
\leq \|f - h\|_{\alpha, \lambda} \frac{T^{1-\alpha}}{\lambda^{1-\alpha}} \sup_{z > 0} \int_0^z e^{-\lambda(z-x)}dx
\]
\[
\leq \|f - h\|_{\alpha, \lambda} KT^{1-\alpha} \lambda^{\alpha-1}.
\]

(3.19)

Using (3.15) we can deal with $B_1$
\[
\sup_{t \in [0,T]} B_1 \leq K \sup_{t \in [0,T]} e^{-\lambda t} \int_0^t ((t-u)^{-2\alpha} + u^{-\alpha})|f(u) - h(u)|du
\]
\[
\leq K \|f - h\|_{\alpha, \lambda} \sup_{t \in [0,T]} \int_0^t e^{-\lambda(t-u)}((t-u)^{-2\alpha} + u^{-\alpha})du
\]
\[
\leq KC_\alpha \lambda^{2\alpha-1} \|f - h\|_{\alpha, \lambda},
\]

(3.20)

where $C_\alpha \leq \frac{1}{1-2\alpha} + 4$.

We will call lemma 5.1 and similar computations to those used to deal with $B_1$ in order to obtain an estimation for $B_2$
\[
\sup_{t \in [0,T]} B_2 \leq \sup_{t \in [0,T]} e^{-\lambda t} \int_0^t ((t-u)^{-2\alpha} + u^{-\alpha}) \left( K \int_0^u \frac{|f(u) - h(u)|}{(u-y)^{\alpha+1-\beta}}dy 
\right.
\]
\[
+KN \int_0^u \frac{|f(u) - h(u)|}{(u-y)^{\alpha+1}} \left( |f(u) - f(y)|^\delta + |h(u) - h(y)|^\delta \right) dy
\]
\[
+KN \int_0^u \frac{|f(u) - h(u) - f(y) + h(y)|}{(u-y)^{\alpha+1}}dy \left) \right. du
\]
\[
\leq \left[ (KN (\Delta(f) + \Delta(h)) + 1) \int_0^t e^{-\lambda(t-u)}((t-u)^{-2\alpha} + u^{-\alpha})du
\right.
\]
\[
+ \frac{K}{\beta-\alpha} \sup_{t \in [0,T]} \int_0^t e^{-\lambda(t-u)}((t-u)^{-2\alpha} + u^{-\alpha})u^{\beta-\alpha}du \left) \right. \|f - h\|_{\alpha, \lambda}
\]
\[
\leq K_{\alpha, \beta, N}^{(8)} \lambda^{2\alpha-1} \|f - h\|_{\alpha, \lambda}(1 + \Delta(f) + \Delta(h)),
\]

(3.21)

where $K_{\alpha, \beta, N}^{(8)} = C_\alpha \left( K + \frac{KT^{\beta-\alpha}}{\beta-\alpha} \right)$.

$B_3$ is bounded using lemma 5.3
\[
B_3 \leq KN B_{3,1} + KB_{3,2} + KN B_{3,3},
\]

(3.22)
where

\[ B_{3,1} = e^{-\lambda t} \int_0^t \int_0^s \int_0^u \frac{|f(u) - h(u) - f(y) + h(y)|}{|t - s|^\alpha(u - y)^{\alpha + 1}} \, dy \, ds \, du, \]

\[ B_{3,2} = e^{-\lambda t} \int_0^t \int_0^s \int_0^u \frac{|f(u) - h(u)|}{|t - s|^\alpha(u - y)^{\alpha - \beta + 1}} \, dy \, ds \, du, \]

\[ B_{3,3} = e^{-\lambda t} \int_0^t \int_0^s \int_0^u \frac{|f(u) - h(u)|}{|t - s|^\alpha(u - y)^{\alpha + 1}} \left( |f(u) - f(y)|^\delta + |h(u) - h(y)|^\delta \right) \, dy \, ds \, du. \]

Then,

\[ B_{3,1} \leq e^{-\lambda t} \frac{1}{1 - \alpha} \int_0^t \int_0^u (t - u)^{1 - \alpha} |f(u) - h(u) - f(y) + h(y)| \, dy \, du \]

\[ \leq \frac{1}{1 - \alpha} \|f - h\|_{\alpha, \lambda} \int_0^t e^{-\lambda(t - u)} (t - u)^{1 - \alpha} \, du \]

\[ \leq \frac{T^{1 - \alpha}}{1 - \alpha} \lambda^{-1} \|f - h\|_{\alpha, \lambda}, \tag{3.23} \]

\[ B_{3,2} \leq e^{-\lambda t} \frac{1}{1 - \alpha} \int_0^t \int_0^u |f(u) - h(u)|(u - y)^{\beta - \alpha - 1} (t - u)^{1 - \alpha} \, dy \, du \]

\[ \leq \frac{1}{(1 - \alpha) (\beta - \alpha)} \|f - h\|_{\alpha, \lambda} \int_0^t e^{-\lambda(t - u)} u^{\beta - \alpha} (t - u)^{1 - \alpha} \, du \]

\[ \leq \frac{T^{1 + \beta - 2\alpha}}{(1 - \alpha) (\beta - \alpha)} \lambda^{-1} \|f - h\|_{\alpha, \lambda}, \tag{3.24} \]

\[ B_{3,3} \leq e^{-\lambda t} \frac{1}{1 - \alpha} \int_0^t \int_0^u \frac{|f(u) - h(u)|}{(t - u)^{\alpha - 1}(u - y)^{\alpha + 1}} \]

\[ \times (|f(u) - f(y)|^\delta + |h(u) - h(y)|^\delta) \, dy \, du \]

\[ \leq \frac{1}{(1 - \alpha)} (\Delta(f) + \Delta(h)) e^{-\lambda t} \int_0^t \frac{|f(u) - h(u)|}{(t - u)^{\alpha - 1}} \, du \]

\[ \leq \frac{1}{(1 - \alpha)} (\Delta(f) + \Delta(h)) \|f - h\|_{\alpha, \lambda} \int_0^t e^{-\lambda(t - u)} (t - u)^{1 - \alpha} \, du \]

\[ \leq \frac{1}{(1 - \alpha)} \lambda^{-1} T^{1 - \alpha} (\Delta(f) + \Delta(h)) \|f - h\|_{\alpha, \lambda}. \tag{3.25} \]

Now using (3.22), (3.23), (3.24) and (3.25) we have

\[ \sup_{t \in [0, T]} B_3 \leq K^{(9)}_{\alpha, \beta, \lambda} \lambda^{-1} (1 + \Delta(f) + \Delta(h)) \|f - h\|_{\alpha, \lambda}, \tag{3.26} \]

where

\[ K^{(9)}_{\alpha, \beta, \lambda} = \frac{K_N}{1 - \alpha} + \frac{K_N T^{1 + \beta - 2\alpha}}{(\beta - \alpha)(1 - \alpha)} + \frac{K_N T^{1 - \alpha}}{(1 - \alpha)}. \]
So putting together (3.19), (3.20), (3.21) and (3.26) we obtain that (3.11) holds for
\[ d_N' = \left( C_{\alpha,T}^{(4)} + 1 \right) \left( C_{\alpha}^{(3)} KKN T^{1-\alpha} + KC_{\alpha} + \left( K_{\alpha,\beta,N}^{(8)} + K_{\alpha,\beta,N}^{(9)} \right) \right). \]

\[ \square \]

4 Deterministic and stochastic equations

Set \( 0 < \alpha < \frac{1}{2} \) and \( g \in W_{T}^{1-\alpha,\infty}(0, T; \mathbb{R}^m) \). Consider the deterministic differential equation on \( \mathbb{R}^d \)
\[ x(t) = x_0 + \int_0^t b(t, s, x(s))ds + \int_0^t \sigma(t, s, x(s))dg_s, \quad t \in [0, T]. \tag{4.1} \]
Using the notations introduced previously we can write the equation (4.1) as:
\[ x(t) = x_0 + F_t^{(b)}(x) + G_t^{(\sigma)}(x), \quad t \in [0, T]. \tag{4.2} \]
Then we can state the result of existence and uniqueness of solution

**Theorem 4.1** Assume that \( b \) and \( \sigma \) satisfy hypothesis \((H1)\) and \((H2)\) with \( \rho = 1/\alpha, \delta \leq 1, \min\{\beta, \frac{\delta}{1+\delta}\} > 1 - \mu \) and
\[ 0 < 1 - \mu < \alpha < \alpha_0 := \min \left\{ \frac{1}{2}, \frac{\delta}{1+\delta} \right\}. \]
Then equation (4.1) has a unique solution \( x \in W_0(0, T; \mathbb{R}^d) \cap C^{1-\alpha}(0, T; \mathbb{R}^d) \).

The proof of this theorem follows the same computations to those given in Theorem 5.1 in [9]. Indeed, notice that the estimates that we have obtained for Lebesgue integrals and Riemann-Stieltjes integrals in Propositions 2.2 and 3.2 are the same, with different constants, to those proved in Propositions 4.4 and 4.2 of [9].

We provide now an upper bound for the norm of the solution.

We define \( \varphi(\alpha, \gamma) \) as,
\[ \varphi(\alpha, \gamma) = \begin{cases} \frac{1}{1-\alpha} & \text{if } 0 \leq \gamma < \frac{1-2\alpha}{1-\alpha}, \\ \frac{1}{1-2\alpha} & \text{if } \frac{1-2\alpha}{1-\alpha} \leq \gamma < 1, \\ \frac{1}{1-2\alpha} & \text{if } \gamma = 1. \end{cases} \]

**Proposition 4.2** Assume that \( b \) and \( \sigma \) satisfy hypothesis of Theorem 4.1 and that \( \sigma \) satisfies also \((H3)\). Then, the unique solution of the equation (4.1) satisfies
\[ \|x\|_{\alpha,\infty} \leq C_{\alpha}^{(5)} e^{C_{\alpha}^{(6)}(g)\varphi(\alpha, \gamma)}, \]
where the constants \( C_{\alpha}^{(5)} \) and \( C_{\alpha}^{(6)} \) depend only on \( T, \alpha, \gamma \) and the constants that appear in conditions \((H1)\), \((H2)\) and \((H3)\).
\textbf{Proof:} We will denote by $C$ a positive constant, depending on $T$, $\alpha$, $\gamma$ and the constants that appear in conditions (H1), (H2) and (H3), that will change from line to line. Using (3.8) we have

\begin{align*}
\left| G_t^{(\sigma)}(x) \right| & \leq \Lambda_\alpha(g) \left( \int_t^1 \frac{\left| \sigma(t, s, x(s)) \right|}{s^\alpha} \, ds \\
& \quad + \alpha \int_0^t \int_0^s \frac{\left| \sigma(t, s, x(s)) - \sigma(t, r, x(r)) \right|}{(s - r)^{\alpha + 1}} \, dr \, ds \right) \\
& \leq \Lambda_\alpha(g) \left( K_0 \int_0^t \frac{1 + |x(s)|^\gamma}{s^\alpha} \, ds + \alpha K \int_0^t \int_0^s \frac{|x(s) - x(r)|}{(s - r)^{\alpha + 1}} \, dr \, ds \\
& \quad + \frac{\alpha K}{(\beta - \alpha)(\beta - \alpha + 1)} \right) \\
& \leq C\Lambda_\alpha(g) \left( 1 + \int_0^t \left( s^{-\alpha} |x(s)|^\gamma + s^{-\alpha} \int_0^s \frac{|x(s) - x(r)|}{(s - r)^{\alpha + 1}} \, dr \right) \, ds \right).
\end{align*}

Then using (3.4) for $K(u) = K$, (3.5) and (3.6), we have

\begin{align*}
\int_0^t \left| G_t(f) - G_s(f) \right| \frac{1}{(t - s)^{\alpha + 1}} \, ds & \leq \Lambda_\alpha(g) \left( K_B(1 - \alpha, 1 + \mu - \alpha) t^{1 + \mu - 2\alpha} \\
& \quad + \alpha \int_0^t \int_0^s \int_0^u \frac{|f(t, u) - f(s, u) - f(t, y) + f(s, y)|}{(u - y)^{\alpha + 1}(t - s)^{\alpha + 1}} \, dy \, du \, ds \\
& \quad + B(2\alpha, 1 - \alpha) \int_0^t \frac{|f(t, u)|}{(t - u)^{2\alpha}} \, du \\
& \quad + \int_0^t \int_0^u \frac{|f(t, u) - f(t, y)|}{(u - y)^{\alpha + 1}(t - y)^{-\alpha}} \, dy \, du \right).
\end{align*}
Following the same computations we have done for the study of $A_2$ and applying the previous result combined with lemma 5.2, it holds that
\[
\int_0^t \left| \frac{G_i^{(\alpha)}(x) - G_s^{(\alpha)}(x)}{(t - s)^{\alpha + 1}} \right| ds \leq C \Lambda_\alpha(g) \left( \frac{K B (1 - \alpha, \mu - \alpha + 1)}{(\mu - \alpha)} \right) t^{\mu - 2\alpha + 1} \\
+ K \alpha B (1 - \alpha, \beta - \alpha + 1) t^{\beta - 2\alpha + 2} \\
+ \frac{K}{1 - \alpha} \int_0^t \int_0^u (t - u)^{1-\alpha} \frac{|x(u) - x(y)|}{(u - y)^{\alpha + 1}} dy du \\
+ K_0 B(2\alpha, 1 - \alpha) \int_0^t \frac{1 + |x(u)|^\gamma}{(t - u)^{2\alpha}} du \\
+ K \int_0^t \int_0^u \frac{|x(u) - x(y)|}{(u - y)^{\beta + 1}} (t - y)^{-\alpha} dy du \\
+ \int_0^t \frac{|x(u)|^\gamma}{(t - u)^{2\alpha}} du + \int_0^t \int_0^u \frac{|x(u) - x(y)|}{(u - y)^{\alpha + 1}} (t - y)^{-\alpha} dy du \\
\leq C \Lambda_\alpha(g) \left( 1 + \int_0^t \frac{1 + |x(u) - x(y)|}{(t - u)^{\alpha + 1}} dy du \\
+ \int_0^t \frac{|x(u)|^\gamma}{(t - u)^{2\alpha}} du + \int_0^t \int_0^u \frac{|x(u) - x(y)|}{(u - y)^{\alpha + 1}} (t - y)^{-\alpha} dy du \right). \]

Finally using (2.5) we get
\[
\left| F_t^{(b)}(x) \right| + \int_0^t \left| F_t^{(b)}(x) - F_s^{(b)}(x) \right| ds \leq C \left( 1 + \int_0^t (t - s)^{-\alpha} |x(s)| ds \right). \]

The proof finishes following the same computations of Proposition 5.1 in [9]. Indeed, set
\[
h(t) = |x(t)| + \int_0^t \frac{|x(t) - x(s)|}{(t - s)^{\alpha + 1}} ds. \]

Then the following inequality holds
\[
h(t) \leq C(1 + \Lambda_\alpha(g)) \left( 1 + \int_0^t \left( (t - s)^{-(1 - 1/\varphi(\alpha, \gamma))} + s^{-\alpha} \right) h(s) ds \right). \]

Using a Gronwall inequality (see [9]) we finish the proof. \(\square\)

We will finish applying the results for deterministic equations to stochastic equations. The stochastic integral appearing throughout this paper is a path-wise Riemann-Stieltjes integral and it is well known that this integral exists if the process that we integrate has Hölder continuous trajectories of order larger than
Set $\alpha \in (1 - H, \frac{1}{2})$. For any $\delta \in (0, 2]$, by Fernique’s theorem it holds that

$$E(\exp(\Lambda_\alpha(W)\delta)) < \infty.$$  

Then if $u_t = \{u_t(s), s \in [0, T]\}$ is a stochastic process whose trajectories belong to the space $W_{T}^{\alpha,1}(0, T)$, the Riemann–Sieljes integral $\int_{0}^{T} u_t(s) dW_s$ exists and we have that

$$\left| \int_{0}^{T} u_t(s) dW_s \right| \leq G\|u_t\|_{\alpha,1},$$

where $G$ is a random variable with moments of all orders (see Lemma 7.5 in [9]). Moreover, if the trajectories of $u$ belong to $W_0^{\alpha,\infty}(0, T)$, then the indefinite integral $\int_{0}^{T} u_t(s) dW_s$ is Hölder continuous of order $1 - \alpha$ and with trajectories in $W_0^{\alpha,\infty}(0, T)$. As a simple consequence of these facts and our deterministic results, we obtain the proof of our main theorem.

## 5 Appendix

In this section we give some technical lemmas used in the estimates obtained in Section 3.

**Lemma 5.1** Let $\sigma : [0, T]^2 \times \mathbb{R} \to \mathbb{R}$ be a function satisfying the hypothesis (H1). Then for all $N > 0$, $t, s_1, s_2 \in \mathbb{R}$ and $|x_1|, |x_2|, |x_3|, |x_4| \leq N$

$$|\sigma(t, s_1, x_1) - \sigma(t, s_2, x_2) - \sigma(t, s_1, x_3) + \sigma(t, s_2, x_4)| \leq K_N |x_1 - x_2 - x_3 + x_4| + K |x_1 - x_3| |s_2 - s_1|^3$$

$$+K_N |x_1 - x_3| \left( |x_1 - x_2|^3 + |x_3 - x_4|^3 \right).$$  

**Proof:** By the mean value theorem we can write

$$\sigma(t, s_1, x_1) - \sigma(t, s_2, x_2) - \sigma(t, s_1, x_3) + \sigma(t, s_2, x_4)$$

$$= \int_{0}^{1} (x_1 - x_3) \partial_x \sigma(t, s_1, \theta x_1 + (1 - \theta) x_3) d\theta$$

$$- \int_{0}^{1} (x_2 - x_4) \partial_x \sigma(t, s_2, \theta x_2 + (1 - \theta) x_4) d\theta$$

$$= \int_{0}^{1} (x_1 - x_2 - x_3 + x_4) \partial_x \sigma(t, s_2, \theta x_2 + (1 - \theta) x_4) d\theta$$

$$+ \int_{0}^{1} (x_1 - x_3) \left( \partial_x \sigma(t, s_1, \theta x_1 + (1 - \theta) x_3) \right)$$

$$- \partial_x \sigma(t, s_2, \theta x_2 + (1 - \theta) x_4)) d\theta.$$  

We can obtain this using the hypothesis (H1). \qed
Lemma 5.2 Let $\sigma : [0, T]^2 \times \mathbb{R} \to \mathbb{R}$ be a function satisfying the hypothesis (H1). Then for all $t_1, t_2, s_1, s_2 \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}$

$$|\sigma(t_1, s_1, x_1) - \sigma(t_2, s_1, x_1) - \sigma(t_1, s_2, x_2) + \sigma(t_2, s_2, x_2)|$$

$$\leq K|t_1 - t_2| (|s_1 - s_2|^\beta + |x_1 - x_2|).$$

(5.2)

Proof: By the mean value theorem we can write

$$\sigma(t_1, s_1, x_1) - \sigma(t_2, s_1, x_1) - \sigma(t_1, s_2, x_2) + \sigma(t_2, s_2, x_2)$$

$$= \int_0^1 (t_1 - t_2) \partial_t \sigma(\theta t_1 + (1 - \theta) t_2, s_1, x_1) d\theta$$

$$- \int_0^1 (t_1 - t_2) \partial_t \sigma(\theta t_1 + (1 - \theta) t_2, s_2, x_2) d\theta$$

$$= \int_0^1 (t_1 - t_2) (\partial_t \sigma(\theta t_1 + (1 - \theta) t_2, s_1, x_1)$$

$$- \partial_t \sigma(\theta t_1 + (1 - \theta) t_2, s_2, x_1)) d\theta$$

$$+ \int_0^1 (t_1 - t_2) (\partial_t \sigma(\theta t_1 + (1 - \theta) t_2, s_2, x_1)$$

$$- \partial_t \sigma(\theta t_1 + (1 - \theta) t_2, s_2, x_2)) d\theta,$$

and we can obtain (5.2) using the hypothesis (H1). \qed

Lemma 5.3 Let $\sigma : [0, T]^2 \times \mathbb{R} \to \mathbb{R}$ be a function satisfying the hypothesis (H1). Then for all $N > 0$, $t_1, t_2, s_1, s_2 \in \mathbb{R}$ and $|x_1|, |x_2|, |x_3|, |x_4| \leq N$

$$|\sigma(t_1, s_1, x_1) - \sigma(t_1, s_1, x_2) - \sigma(t_2, s_1, x_1) + \sigma(t_2, s_1, x_2)$$

$$- \sigma(t_1, s_2, x_3) + \sigma(t_1, s_2, x_4) + \sigma(t_2, s_2, x_3) - \sigma(t_2, s_2, x_4)|$$

$$\leq K_N |t_1 - t_2| |x_1 - x_2 - x_3 + x_4| + K|x_1 - x_2||t_1 - t_2||s_1 - s_2|^{\beta}$$

$$+ K_N |x_1 - x_2||t_1 - t_2| (|x_1 - x_3|^{\delta} + |x_2 - x_4|^{\delta}).$$

(5.3)
**Proof:** By the mean value theorem we can write

\[
\begin{align*}
\sigma(t_1, s_1, x_1) - \sigma(t_1, s_1, x_2) - \sigma(t_2, s_1, x_1) + \sigma(t_2, s_1, x_2) - \sigma(t_1, s_2, x_3) \\
+ \sigma(t_1, s_2, x_4) + \sigma(t_2, s_2, x_3) - \sigma(t_2, s_2, x_4) \\
= (x_1 - x_2) \int_0^1 [\partial_x \sigma(t_1, s_1, \theta x_1 + (1 - \theta)x_2) \\
- \partial_x \sigma(t_2, s_1, \theta x_1 + (1 - \theta)x_2)] d\theta \\
- (x_3 - x_4) \int_0^1 [\partial_x \sigma(t_1, s_2, \theta x_3 + (1 - \theta)x_4) \\
- \partial_x \sigma(t_2, s_2, \theta x_3 + (1 - \theta)x_4)] d\theta \\
= (t_1 - t_2) \left[ (x_1 - x_2) \\
\int_0^1 \int_0^1 \partial_{x,t}^2 \sigma(\lambda t_1 + (1 - \lambda)t_2, s_1, \theta x_1 + (1 - \theta)x_2) d\lambda d\theta \\
- (x_3 - x_4) \int_0^1 \int_0^1 \partial_{x,t}^2 \sigma(\lambda t_1 + (1 - \lambda)t_2, s_2, \theta x_3 + (1 - \theta)x_4) d\lambda d\theta \right] \\
\leq (t_1 - t_2) \left[ (x_1 - x_2) \int_0^1 \int_0^1 \left( \partial_{x,t}^2 \sigma(\lambda t_1 + (1 - \lambda)t_2, s_1, \theta x_1 + (1 - \theta)x_2) \\
- \partial_{x,t}^2 \sigma(\lambda t_1 + (1 - \lambda)t_2, s_2, \theta x_3 + (1 - \theta)x_4) \right) d\lambda d\theta \\
+ (x_1 - x_2 - x_3 + x_4) \\
\int_0^1 \int_0^1 \partial_{x,t}^2 \sigma(\lambda t_1 + (1 - \lambda)t_2, s_2, \theta x_3 + (1 - \theta)x_4) d\lambda d\theta \right].
\end{align*}
\]

So we can obtain (5.3) using the hypothesis (H1). □

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