Teaching and Learning Hyperbolic Functions (II): Other Trigonometric Properties and Their Inverses

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Abstract

As part of a larger project entitled "Training and developing the competences of children, students and teachers to solve problems / exercises in Mathematics", in a recent paper with the same generic name as this one and numbered with (I), I presented the definitions, the consequences immediate resulting from these and a series of 38 properties of hyperbolic functions, properties that we divided into four groups, as follows: A) "Trigonometric" properties - nine properties; B) The derivatives of hyperbolic functions - six properties; C) The primitives (indefinite integrals) of hyperbolic functions - six properties and D) The monotony and the invertibility of hyperbolic functions - 17 properties. In this paper we will continue this approach and will present and prove another 54 properties of these functions, properties that we will divide into three groups, as follows: E) Other properties "trigonometric" - 42 properties; F) Immediate properties of the inverse of hyperbolic functions - six properties and G) The derivatives of the inverse of hyperbolic functions - six properties. These properties, as well as others that we will present and prove later, will be used in various applications in Algebra or Mathematical Analysis.

Keywords: hyperbolic sine, hyperbolic cosine, hyperbolic tangent, hyperbolic cotangent, hyperbolic secant, hyperbolic cosecant.

1. Introduction

In (Vâlcan, 2016) we started the presentation of a didactic exposition of fundamental properties of so-called "hyperbolic functions" in many aspects analogous to the usual trigonometric functions. Also there I said that hyperbolic functions meet together many times in different physical and technical research, having a very important role in non-Euclidean geometry of Lobacevski participated in all relationships (interdependences) this geometry. But independently of these annexes, the theory of hyperbolic functions can present a significant interest to a student or a teacher of Mathematics in secondary education because the analogy between the hyperbolic and trigonometric functions clarifies in a new face many problems of trigonometry. As I said hyperbolic functions occur naturally as simple combinations of exponential function, $e^x$, a function that is much studied in School Mathematics. Indeed, the two main functions, hyperbolic cosine, and hyperbolic sine is semisum or semi difference of $e^x$ and $e^{-x}$, see the following equalities (2.1) and (2.2). We have to admit that in undergraduate education in Romania, these functions are almost unknown, so students and teachers, despite the fact that they present many similarities to the trigonometric functions and, in addition, have numerous applications in integral calculus. I must admit that no students from the Faculty of Mathematics do not really know these things. In conclusion, we can say that literature the domain in Romania is very poor in providing information about the hyperbolic functions and their applications. Not even abroad things are not so good – see (Vâlcan, 2016, Introduction). In this second paper, continuing with the ideas from (Vâlcan, 2016) we will present another 54 properties of these functions.

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To the reader interested in these issues we also recommend reading the bibliographic sources (Abramowitz & Stegun, 1973), (Anderson, 1999), (Beyer, 1987), (Harris & Stocker, 1998), (Jeffrey, 2000), (Yates, 1953), (Zwillinger, 1995) and but also other bibliographic sources that are available online. To give coherence and consistency to the ideas of these two works, we will recall the results of the first work; that and because we will always refer to them.

2. The definitions of hyperbolic functions

In this first section we have defined the hyperbolic functions and we have presented the first their properties.

**Definition 2.1:** The function \( sh : \mathbb{R} \rightarrow \mathbb{R} \), given by law, for every \( x \in \mathbb{R} \),

\[
sh(x) = \frac{e^x - e^{-x}}{2},
\]

is called **hyperbolic sine** (in latin, **sinus hyperbolus**) by argument \( x \).

**Definition 2.2:** The function \( ch : \mathbb{R} \rightarrow [1, +\infty) \), given by law, for every \( x \in \mathbb{R} \),

\[
ch(x) = \frac{e^x + e^{-x}}{2},
\]

is called **hyperbolic cosine** (in latin, **cosinus hyperbolus**) by argument \( x \).

**Definition 2.3:** The function \( th : \mathbb{R} \rightarrow (-1, 1) \), given by law, for every \( x \in \mathbb{R} \),

\[
th(x) = \frac{sh(x)}{ch(x)},
\]

is called **hyperbolic tangent** by argument \( x \).

**Definition 2.4:** The function \( cth : \mathbb{R}^* \rightarrow (-\infty, -1) \cup (1, +\infty) \), given by law, for every \( x \in \mathbb{R}^* \),

\[
cth(x) = \frac{ch(x)}{sh(x)},
\]

is called **hyperbolic cotangent** by argument \( x \).

**Definition 2.5:** The function \( sch : \mathbb{R} \rightarrow (0, 1] \), given by law, for every \( x \in \mathbb{R} \),

\[
sch(x) = \frac{1}{ch(x)},
\]

is called **hyperbolic secant** by argument \( x \).

**Definition 2.6:** The function \( csh : \mathbb{R}^* \rightarrow \mathbb{R}^* \), given by law, for every \( x \in \mathbb{R}^* \),

\[
csh(x) = \frac{1}{sh(x)},
\]

is called **hyperbolic cosecant** by argument \( x \).

**Remarks 2.7:** From the above definitions it follows that:

1) The functions \( sh \) and \( ch \) are linear combinations of exponential functions:
\[
x \mapsto e^x, \quad x \mapsto e^{-x}, \quad x \in \mathbb{R},
\]
and vice versa, that is, the functions \( e^x \) and \( e^{-x} \) are linear combinations of the functions:
\[
x \mapsto sh(x), \quad x \mapsto ch(x), \quad x \in \mathbb{R},
\]
because, for every \( x \in \mathbb{R} \):
\[
e^x = ch(x) + sh(x)
\]
and
\[
e^{-x} = ch(x) - sh(x).
\]

2) For every \( x \in \mathbb{R} \),
\[
sh(x) = \frac{e^{2x} - 1}{2e^x} = \frac{1 - e^{-2x}}{2e^{-x}}.
\]

3) For every \( x \in \mathbb{R} \),
\[
ch(x) = \frac{e^{2x} + 1}{2e^x} = \frac{1 + e^{-2x}}{2e^{-x}}.
\]
4) For every $x \in \mathbb{R}$
$$\text{th}(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{-2x}}.$$  

(2.3')

5) For every $x \in \mathbb{R}$,
$$\text{ch}(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1} = \frac{1 + e^{-2x}}{1 - e^{-2x}}.$$  

(2.4')

6) For every $x \in \mathbb{R}$,
$$\text{sch}(x) = \frac{2}{e^x + e^{-x}} = \frac{2e^x}{e^{2x} + 1} = \frac{2e^{-x}}{1 + e^{-2x}}.$$  

(2.5')

5) For every $x \in \mathbb{R}^*$,
$$\text{csh}(x) = \frac{2}{e^x - e^{-x}} = \frac{2e^x}{e^{2x} - 1} = \frac{2e^{-x}}{1 - e^{-2x}}.$$  

(2.6')

6) To make analogies with the trigonometric functions, but for abbreviations, we use the following notations further:

a) for every $x \in \mathbb{R}$,
$$\text{not.} \quad \text{sh}(x) = \text{sh}x, \quad \text{not.} \quad \text{ch}(x) = \text{ch}x, \quad \text{not.} \quad \text{th}(x) = \text{th}x; \quad \text{not.} \quad \text{sch}(x) = \text{sch}x,$$

b) for every $x \in \mathbb{R}^*$,
$$\text{not.} \quad \text{cth}(x) = \text{cth}x, \quad \text{not.} \quad \text{csh}(x) = \text{csh}x;$$

but any expression including simple fraction, which will be the argument of one of these functions will be put between brackets. Also here we specify that in some papers functions: sh, ch, th, cth, sch, csh, they are denoted, respectively: sinh, cosh, tanh, cotanh, sech, cosech. □

Remarks 2.8: From the above definitions and remarks it follows that:

1) The function sh is odd, i.e.: for every $x \in \mathbb{R}$,
$$\text{sh}(x) = \text{sh}x; \quad \text{sh}(-x) = -\text{sh}x.$$  

(2.9)

5) The function sch is even, i.e.: for every $x \in \mathbb{R}$,
$$\text{sch}(x) = \text{sch}x; \quad \text{sch}(-x) = \text{sch}x.$$  

(2.13)

2) The function ch is even, i.e.: for every $x \in \mathbb{R}$,
$$\text{ch}(x) = \text{ch}x; \quad \text{ch}(-x) = \text{ch}x.$$  

(2.10)

6) The function csh is odd, i.e.: for every $x \in \mathbb{R}^*$,
$$\text{csh}(x) = \text{csh}x; \quad \text{csh}(-x) = -\text{csh}x.$$  

(2.14)

3) The function th is odd, i.e.: for every $x \in \mathbb{R}$,
$$\text{th}(x) = \text{th}x; \quad \text{th}(-x) = -\text{th}x.$$  

(2.11)

7) For every $x \in \mathbb{R}$,
$$\text{cth}(x) = \text{cth}x; \quad \text{cth}(-x) = \text{cth}x.$$  

(2.15)

4) The function cth is odd, i.e.: for every $x \in \mathbb{R}^*$,
$$\text{csh}(x) = \text{csh}x; \quad \text{csh}(-x) = -\text{csh}x.$$  

(2.12)

8) For every orice $x \in \mathbb{R}$,
$$\text{th}(x) \in (-1,1).$$  

(2.16)

3. Fundamental properties of hyperbolic functions

In this section of the paper (Vălcan, 2017) we have presented the fundamental properties of hyperbolic functions; we referred here to the first 38 of these properties, divided into four groups:

A. "Trigonometric" properties – nine properties;

B. The derivatives of hyperbolic functions – six properties;

C. The primitives (indefinite integrals) of hyperbolic functions – six properties;

D. The monotony and the invertibility of hyperbolic functions – 17 properties.

All of these properties have been proveed at least in a way.

Proposition 3.1: The following statements hold:

A. "Trigonometric" properties

1) For every $x \in \mathbb{R}$,
$$d^2x = 1.$$  

(3.1)

2) For every $x, y \in \mathbb{R}$,
$$\text{sh}(x+y) = \text{sh}x \cdot \text{sh}y + \text{ch}x \cdot \text{ch}y.$$  

(3.2)

3) For every $x, y \in \mathbb{R}$,
$$\text{sh}(x+y) = \text{sh}x \cdot \text{sh}y.$$  

(3.3)

4) For every $x, y \in \mathbb{R}$,
$$\text{ch}(x+y) = \text{ch}x \cdot \text{ch}y + \text{sh}x \cdot \text{sh}y.$$  

(3.4)

5) For every $x, y \in \mathbb{R}$,
$$\text{ch}(x+y) = \text{ch}x \cdot \text{sh}y + \text{sh}x \cdot \text{ch}y.$$  

(3.5)

6) For every $x, y \in \mathbb{R}$,
$$\text{ch}(x+y) = \text{sh}x \cdot \text{ch}y - \text{sh}x \cdot \text{ch}y.$$  

(3.6)
\[ \text{th}(x+y) = \frac{\text{th}x + \text{th}y}{1 + \text{th}x \cdot \text{th}y}. \]  
\[ \text{cth}(x+y) = \frac{\text{cth}x \cdot \text{cth}y + 1}{\text{cth}x + \text{cth}y}. \]  
\[ \text{th}(x+y) = \frac{\text{th}x - \text{th}y}{1 - \text{th}x \cdot \text{th}y}. \]  
\[ \text{cth}(x+y) = \frac{\text{cth}x \cdot \text{cth}y - 1}{\text{cth}y - \text{cth}x}. \]

7) For every \( x, y \in \mathbb{R} \),

8) For every \( x, y \in \mathbb{R}^* \), with the property that \( x + y \neq 0 \).

B. The derivatives of hyperbolic functions

10) For every \( x \in \mathbb{R} \),

\[ (\text{sh}x)' = \text{ch}x. \]  
\[ (\text{ch}x)' = \text{sh}x. \]  
\[ (\text{th}x)' = \frac{1}{\text{ch}^2 x} = \frac{1}{\text{ch}x}. \]  
\[ (\text{cthx})' = -\frac{\text{sh}x}{\text{ch}^2 x} = -\text{th}x \cdot \text{sh}x. \]  
\[ (\text{cthx})' = -\frac{\text{ch}x}{\text{sh}^2 x} = -\text{cth}x \cdot \text{sh}x. \]

C. The primitives (indefinite integrals) of hyperbolic functions

16) For every \( x \in \mathbb{R} \),

\[ \int \text{sh}x \cdot dx = \text{ch}x + C. \]  
\[ \int \text{ch}x \cdot dx = \text{sh}x + C. \]  
\[ \int \text{th}x \cdot dx = \text{ln}(\text{ch}x) + C. \]  
\[ \int \text{csh}x \cdot dx = \frac{1}{2} \ln \left( \frac{\text{ch}x - 1}{\text{ch}x + 1} \right) + C. \]

19) For every \( x \in \mathbb{R}^* \).

D. The monotonity and the invertibility of hyperbolic functions

22) The function \( \text{sh} \) \( x \) is strictly increasing on \( \mathbb{R} \).

23) The function \( \text{sh} \) \( x \) is invertible and its inverse is the function:

\[ \text{sh}^{-1} : \mathbb{R} \rightarrow \mathbb{R}, \]

\[ \text{sh}^{-1}(x) = \ln(x + \sqrt{x^2 + 1}). \]

24) The function \( \text{ch} \) \( x \) is strictly decreasing on \((-\infty, 0)\) and strictly increasing on \((0, +\infty)\).

25) The function \( \text{ch} \) \( x \) - the restriction of function \( \text{ch} \) to the interval \((-\infty, 0)\), is invertible and its inverse is the function:

\[ \text{ch}^{-1} : (-1, +\infty) \rightarrow (-\infty, 0), \]

where, for every \( x \in (-1, +\infty) \),

\[ \text{ch}^{-1}(x) = \ln(x + \sqrt{x^2 - 1}). \]

26) The function \( \text{ch} \) \( x \) - the restriction of function \( \text{ch} \) to the interval \((0, +\infty)\), is invertible and its inverse is the function:

\[ \text{ch}^{-1} : (1, +\infty) \rightarrow [0, +\infty), \]

where, for every \( x \in (1, +\infty) \),

\[ \text{ch}^{-1}(x) = \ln(x + \sqrt{x^2 - 1}). \]

27) The function \( \text{th} \) \( x \) is strictly increasing on \( \mathbb{R} \).

28) The function \( \text{sh} \) \( x \) is invertible and its inverse is the function:

\[ \text{sh}^{-1} : (-1, 1) \rightarrow \mathbb{R}, \]

where, for every \( x \in (-1, 1) \),
(3.25) \[ \text{th}^{-1}(x) = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right). \]

29 The function \( \text{th} \) is strictly decreasing both on \((-\infty, 0)\), as well as on \((0, +\infty)\).

30 The function \( \text{th}_1 \) - the restriction of function \( \text{th} \) to the interval \((-\infty, 0)\), is invertible and its inverse is the function:
\[
\text{th}_1^{-1} : (-\infty, 1) \to (-\infty, 0),
\]
where, for every \( x \in (-\infty, -1) \),
\[
\text{th}_1^{-1}(x) = \frac{1}{2} \ln \left( \frac{x + 1}{x - 1} \right). \quad (3.26)
\]

31 The function \( \text{th}_2 \) - the restriction of function \( \text{th} \) to the interval \((0, +\infty)\), is invertible and its inverse is the function:
\[
\text{th}_2^{-1} : (1, +\infty) \to (0, +\infty),
\]
where, for every \( x \in (1, +\infty) \),
\[
\text{th}_2^{-1}(x) = \frac{1}{2} \ln \left( \frac{x + 1}{x - 1} \right). \quad (3.26')
\]

32 The function \( \text{ch} \) is invertible and its inverse is the function:
\[
\text{ch}^{-1} : (-\infty, 1) \cup (1, +\infty) \to \mathbf{R}^*,
\]
where, for every \( x \in (-\infty, -1) \cup (1, +\infty) \),
\[
\text{ch}^{-1}(x) = \frac{1}{2} \ln \left( \frac{x + 1}{x - 1} \right). \quad (3.26'')
\]

33 The function \( \text{sch} \) is strictly increasing on \((-\infty, 0)\) and strictly decreasing on \([0, +\infty)\).

34 The function \( \text{sch}_1 \) - the restriction of function \( \text{sch} \) to the interval \((-\infty, 0)\), is invertible and its inverse is the function:
\[
\text{sch}_1^{-1} : (0, 1] \to (-\infty, 0),
\]
where, for every \( x \in (0, 1] \),
\[
\text{sch}_1^{-1}(x) = \ln \left( \frac{1 - \sqrt{1 - x^2}}{x} \right). \quad (3.27)
\]

35 The function \( \text{sch}_2 \) - the restriction of function \( \text{sch} \) to the interval \([0, +\infty)\), is invertible and its inverse is the function:
\[
\text{sch}_2^{-1} : (0, 1] \to (0, +\infty),
\]
where, for every \( x \in (0, 1] \),
\[
\text{sch}_2^{-1}(x) = \ln \left( \frac{1 + \sqrt{1 - x^2}}{x} \right). \quad (3.28)
\]

36 The function \( \text{csch} \) is strictly decreasing both on \((-\infty, 0)\), as well as on \((0, +\infty)\).

37 The function \( \text{csch}_1 \) - the restriction of function \( \text{csch} \) to the interval \((-\infty, 0)\) is invertible and its inverse is the function:
\[
\text{csch}_1^{-1} : (-\infty, 0) \to (-\infty, 0),
\]
where, for every \( x \in (-\infty, 0) \),
\[
\text{csch}_1^{-1}(x) = \ln \left( \frac{1 - \sqrt{1 + x^2}}{x} \right). \quad (3.29)
\]

38 The function \( \text{csch}_2 \) - the restriction of function \( \text{csch} \) to the interval \((0, +\infty)\), is invertible and its inverse is the function:
\[
\text{csch}_2^{-1} : (0, +\infty) \to (0, +\infty),
\]
where, for every \( x \in (0, +\infty) \),
\[
\text{csch}_2^{-1}(x) = \ln \left( \frac{1 + \sqrt{1 + x^2}}{x} \right). \quad (3.30)
\]

4. Other trigonometric properties and their inverses

First, before exposing new results, it is necessary here to make some remarks, analogous to those of Remarks 2.7, point 6):
Remark 4.1: To make analogies with the inverse of trigonometric functions, but for abbreviations, we use the following notations further.

\textbf{a}) for every } x \in \mathbb{R},
\begin{equation}
\text{sh}^{-1}(x) = \text{sh}^{-1}x,
\end{equation}

\textbf{b}) for every } x \in (1, +\infty),
\begin{equation}
\text{ch}^{-1}(x) = \text{ch}^{-1}x \quad \text{and} \quad \text{ch}^{-1}(x) = \text{ch}^{-1}x,
\end{equation}

\textbf{c}) for every } x \in (-1, 1),
\begin{equation}
\text{th}^{-1}(x) = \text{th}^{-1}x,
\end{equation}

\textbf{d}) for every } x \in (-\infty, 1),
\begin{equation}
\text{cth}^{-1}(x) = \text{cth}^{-1}x,
\end{equation}

\textbf{e}) for every } x \in (1, +\infty),
\begin{equation}
\text{cth}^{-1}(x) = \text{cth}^{-1}x,
\end{equation}

but any expression including simple fraction, which will be the argument of one of these functions, will be put between brackets. Here we mention that in some papers the functions: \text{sh}^{-1}, \text{ch}^{-1}, \text{th}^{-1}, \text{cth}^{-1}, \text{sh}^{-1}, \text{ch}^{-1}, \text{cth}^{-1}, \text{csh}^{-1}, \text{csh}^{-1}, \text{csh}^{-1}, \text{arcsech} and \text{arcsech}. □

Proposition 4.2: The following statements hold.

E. Other properties „trigonometric”

\textbf{1) For every } x \in \mathbb{R},
\begin{equation}
\text{sh}x = 2\text{sh}\left(\frac{x}{2}\right) - \text{ch}\left(\frac{x}{2}\right), \quad \text{(4.1)}
\end{equation}

\textbf{2) For every } x \in \mathbb{R},
\begin{equation}
\text{sh}(2x) = 2\text{sh}x\text{ch}x, \quad \text{(4.2)}
\end{equation}

\textbf{3) For every } x \in \mathbb{R},
\begin{equation}
\text{ch}x = \text{sh}^{2}\left(\frac{x}{2}\right) + \text{ch}^{2}\left(\frac{x}{2}\right), \quad \text{(4.3)}
\end{equation}

\textbf{4) For every } x \in \mathbb{R},
\begin{equation}
\text{ch}(2x) = \text{sh}^{2}x + \text{ch}^{2}x, \quad \text{(4.4)}
\end{equation}

\textbf{5) For every } x \in (-\infty, 0),
\begin{equation}
\text{sh}\left(\frac{x}{2}\right) = \sqrt{\frac{\text{ch}x - 1}{2}}, \quad \text{(4.5)}
\end{equation}

\textbf{6) For every } x \in \mathbb{R},
\begin{equation}
\text{ch}\left(\frac{x}{2}\right) = \sqrt{\frac{\text{ch}x + 1}{2}}, \quad \text{(4.6)}
\end{equation}

\textbf{7) For every } x \in \mathbb{R},
\begin{equation}
\text{th}x = \frac{2 \cdot \text{th}\left(\frac{x}{2}\right)}{1 + \text{th}^{2}\left(\frac{x}{2}\right)}, \quad \text{(4.7)}
\end{equation}

\textbf{8) For every } x \in \mathbb{R},
\begin{equation}
\text{cth}(2x) = \frac{2 \cdot \text{cth}x}{1 + \text{cth}^{2}x}, \quad \text{(4.8)}
\end{equation}

\textbf{9) For every } x \in \mathbb{R},
\begin{equation}
\text{th}\left(\frac{x}{2}\right) = \begin{cases} \sqrt{\frac{\text{ch}x - 1}{\text{ch}x + 1}}, & \text{if } x < 0 \\ \sqrt{\frac{\text{ch}x - 1}{\text{ch}x + 1}}, & \text{if } x \geq 0 \end{cases}, \quad \text{(4.9)}
\end{equation}

\textbf{10) For every } x \in \mathbb{R}^\prime,
\begin{equation}
\text{ch}(2x) = \frac{\text{cth}^{2}x + 1}{2 \cdot \text{cth}x}, \quad \text{(4.11)}
\end{equation}

\textbf{11) For every } x \in \mathbb{R}^\prime,
12) For every $x \in \mathbb{R}$,
\[
\text{cthx} = \begin{cases} 
\frac{\text{ch}x + 1}{\text{ch}x - 1}, & \text{if } x < 0 \\
\frac{\text{ch}x + 1}{\text{ch}x - 1}, & \text{if } x \geq 0
\end{cases}
= \frac{\text{sh}x}{\text{ch}x - 1} = \text{cthx} + \text{shx}.
\] (4.12)

13) For every $x, y, z \in \mathbb{R}$,
\[
\text{sh}(x + y + z) = \text{shx} \cdot \text{ch}y + \text{chx} \cdot \text{shy} + \text{cthx} \cdot \text{cthy} + \text{cthx} \cdot \text{shy} + \text{chx} \cdot \text{shy}.
\] (4.13)

14) For every $x \in \mathbb{R}$,
\[
\text{sh}(3x) = \text{sh}x \cdot (4 - \text{sh}^2x) = \text{sh}x \cdot (4 - \text{sh}^2x + 1).
\] (4.14)

15) For every $x, y, z \in \mathbb{R}$,
\[
\text{sh}(x + y + z) = \text{shx} \cdot \text{ch}y + \text{shy} \cdot \text{chx} + \text{chx} \cdot \text{shy} + \text{shx} \cdot \text{shy} + \text{chx} \cdot \text{shy}.
\] (4.15)

16) For every $x \in \mathbb{R}$,
\[
\text{sh}(3x) = \text{sh}x \cdot (4 - \text{sh}^2x - 3) = \text{sh}x \cdot (4 - \text{sh}^2x + 1).
\] (4.16)

17) For every $x, y, z \in \mathbb{R}$,
\[
\text{th}(x + y + z) = \frac{\text{thx} + \text{thy} + \text{thz} + \text{thx} \cdot \text{thy} \cdot \text{thz}}{1 + \text{thx} \cdot \text{thy} + \text{thz} + \text{thx} \cdot \text{thz}}.
\] (4.17)

18) For every $x \in \mathbb{R}$,
\[
\text{th}(3x) = \frac{3 \cdot \text{thx} + \text{th}^3x}{1 + 3 \cdot \text{th}^3x}.
\] (4.18)

19) For every $x, y, z \in \mathbb{R}$, such that $x + y, x + z, x + y + z \in \mathbb{R}$,
\[
\text{cthx} \cdot \text{cthy} \cdot \text{cthz} + \text{cthx} + \text{cthy} + \text{cthz}.
\] (4.19)

20) For every $x \in \mathbb{R}$,
\[
\text{cthx} = \frac{\text{ch}^3x + 3 \cdot \text{ch}x}{1 + 3 \cdot \text{ch}^3x}.
\] (4.20)

21) For every $x \in \mathbb{R}$,
\[
1 - \text{thx}^2 = \frac{1}{\text{ch}^2x} = \text{sh}^{2}x;
\] i.e.,
\[
\text{sh}^{2}x + \text{th}^{2}x = 1;
\] or equivalent:
\[
\text{sh}^{2}x = \frac{1}{1 - \text{th}^{2}x} = \frac{1}{\text{sh}^{2}x}.
\] (4.21)

22) For every $x \in \mathbb{R}$,
\[
\text{thx} = \frac{1}{\text{sh}^2x} = \text{cthx};
\] i.e.,
\[
\text{th}^{2}x = \text{sh}^{2}x - 1;
\] or equivalent:
\[
\text{sh}^{2}x = \frac{1}{\text{ch}^2x - 1} = \frac{1}{\text{sh}^2x}.
\] (4.21)

23) For every $x, y \in \mathbb{R}$,
\[
\text{shx} \cdot \text{shy} = 2 \cdot \text{sh} \left( \frac{x + y}{2} \right) \cdot \text{ch} \left( \frac{x - y}{2} \right).
\] (4.22)

24) For every $x, y \in \mathbb{R}$,
\[
\text{shx} \cdot \text{shy} = 2 \cdot \text{sh} \left( \frac{x + y}{2} \right) \cdot \text{ch} \left( \frac{x - y}{2} \right).
\] (4.23)

25) For every $x, y \in \mathbb{R}$,
\[
\text{shx} \cdot \text{shy} = 2 \cdot \text{sh} \left( \frac{x + y}{2} \right) \cdot \text{ch} \left( \frac{x - y}{2} \right).
\] (4.24)

26) For every $x, y \in \mathbb{R}$,
\[
\text{shx} \cdot \text{shy} = \text{sh} \left( \frac{x + y}{2} \right) \cdot \text{ch} \left( \frac{x - y}{2} \right).
\] (4.25)

27) For every $x, y \in \mathbb{R}$,
\[
\text{shx} \cdot \text{shy} = \text{shy} \left( \frac{x + y}{2} \right) \cdot \text{chx} \cdot \text{chy}.
\] (4.26)

28) For every $x, y \in \mathbb{R}$,
\[
\text{shx} \cdot \text{shy} = \text{sh} \left( \frac{x - y}{2} \right) \cdot \text{chx} \cdot \text{chy}.
\] (4.27)

29) For every $x, y \in \mathbb{R}$,
\[
\text{shx} \cdot \text{shy} = \text{sh} \left( \frac{x - y}{2} \right) \cdot \text{shx} \cdot \text{shy}.
\] (4.28)

30) For every $x, y \in \mathbb{R}$,
\[
\text{shx} \cdot \text{shy} = \text{sh} \left( \frac{x - y}{2} \right) \cdot \text{shx} \cdot \text{shy}.
\] (4.29)

31) For every $x, y \in \mathbb{R}$,
\[
sh(x + y) + sh(x - y) = \frac{1}{2} \left( 2 \cdot th(x) \right) = \frac{1}{2} - \frac{1}{2} \cdot th(x).
\]

For every \( x, y \in \mathbb{R} \),
\[
sh(x + y) = \frac{1}{2} \left( 2 \cdot th(x) + \frac{1}{2} \right).
\]

For every \( x \in \mathbb{R} \),
\[
sh(x) = \frac{1}{4} \left( 3 \cdot th(x) + \frac{3}{4} \cdot sh(x) \right) + \frac{3}{8}.
\]

For every \( x \in \mathbb{R} \),
\[
ch(x) = \frac{1}{4} \left( ch(2x) + \frac{3}{4} \cdot ch(x) \right) + \frac{3}{8}.
\]

Properties immediate of the inverses of hyperbolic functions

For every \( x \in \mathbb{R} \),
\[
ch^{-1}(x) = \frac{1}{2} \left( 2 \cdot th^{-1}(x) \right).
\]

For every \( x \in [1, +\infty) \),
\[
ch^{-1}(x) = \frac{1}{2} \left( 1 - \frac{1}{x} \right).
\]

For every \( x \in (-1, 1) \),
\[
ch^{-1}(x) = th^{-1}(x).
\]

For every \( x \in (-\infty, -1) \cup (1, +\infty) \),
\[
ch^{-1}(x) = th^{-1}(x).
\]

The derivatives of the inverses of hyperbolic functions

For every \( x \in \mathbb{R} \),
\[
\left( ch^{-1}(x) \right)' = \frac{1}{\sqrt{x^2 + 1}}.
\]

For every \( x \in (1, +\infty) \),
\[
\left( ch^{-1}(x) \right)' = \frac{1}{\sqrt{x^2 - 1}}.
\]

For every \( x \in (-1, 1) \),
\[
\left( ch^{-1}(x) \right)' = \frac{1}{x \cdot \sqrt{1 - x^2}}.
\]

For every \( x \in (-\infty, -1) \),
\[
\left( ch^{-1}(x) \right)' = \frac{1}{x \cdot \sqrt{x^2 + 1}}.
\]

For every \( x \in (0, 1) \),
\[
\left( ch^{-1}(x) \right)' = \frac{1}{x \cdot \sqrt{1 - x^2}}.
\]

For every \( x \in (0, +\infty) \),
\[
\left( ch^{-1}(x) \right)' = \frac{1}{x \cdot \sqrt{x^2 + 1}}.
\]
Proof 1) According to the equality (3.2), for every \( x \in \mathbb{R} \),
\[
shx = sh\left(\frac{x}{2} + \frac{x}{2}\right) = sh\left(\frac{x}{2}\right) \cdot ch\left(\frac{x}{2}\right) + sh\left(\frac{x}{2}\right) = 2 - sh\left(\frac{x}{2}\right) \cdot ch\left(\frac{x}{2}\right);
\]
so, the equality (4.1) holds.

2) According to the equality (3.2), for every \( x \in \mathbb{R} \),
\[
sh(2x) = sh(x + x) = shx \cdot chx + shx = 2 - shx \cdot chx;
\]
so, the equality (4.2) holds.

3) According to the equality (3.4), for every \( x \in \mathbb{R} \),
\[
chx = ch\left(\frac{x}{2} + \frac{x}{2}\right) = sh\left(\frac{x}{2}\right) \cdot ch\left(\frac{x}{2}\right) + ch\left(\frac{x}{2}\right) = sh^2\left(\frac{x}{2}\right) + ch^2\left(\frac{x}{2}\right);
\]
so, the equality (4.3) holds.

4) According to the equality (3.4), for every \( x \in \mathbb{R} \),
\[
ch(2x) = ch(x + x) = chx \cdot chx + shx = sh^2x + ch^2x;
\]
so, the equality (4.4) holds. The equalities (4.4') and (4.4'') are obtained from the equalities (4.4) and (3.1), replacing one after the other, \( sh^2x \), respective \( ch^2x \).

5) According to the equality (4.3) or to the equality (4.4''), for every \( x \in \mathbb{R} \),
\[
chx = 2 - shx \cdot \left(\frac{x}{2}\right) + 1,
\]
whence, according to Definition 2.1, obtain both equalities (4.5) – for \( x < 0 \), respective (4.5') – for \( x \geq 0 \).

6) According to the equality (4.3) or to the equality (4.4''), for every \( x \in \mathbb{R} \),
\[
chx = 2 - chx \cdot \left(\frac{x}{2}\right) - 1,
\]
whence, according to the inequality (2.15), obtain the equality (4.6).

7) According to the equality (3.6), for every \( x \in \mathbb{R} \),
\[
\frac{thx}{2} = \frac{th\left(\frac{x}{2}\right) + th\left(\frac{x}{2}\right)}{1 + th\left(\frac{x}{2}\right) \cdot th\left(\frac{x}{2}\right)} = \frac{2 \cdot th\left(\frac{x}{2}\right)}{1 + th^2\left(\frac{x}{2}\right)};
\]
so, the equality (4.7) holds.

8) According to the equality (3.6), for every \( x \in \mathbb{R} \),
\[
\frac{th(2x)}{2} = \frac{thx + thx}{1 + thx \cdot thx} = \frac{2 \cdot thx}{1 + th^2x};
\]
so, the equality (4.8) holds.

9) For every \( x \in \mathbb{R} \), we have the equalities:
\[
\frac{th\left(\frac{x}{2}\right)}{ch\left(\frac{x}{2}\right)} = \frac{sh\left(\frac{x}{2}\right)}{ch\left(\frac{x}{2}\right)} \quad \text{(according to the equality (2.3))}
\]
\[
= \begin{cases} 
- \sqrt{\frac{chx - 1}{chx + 1}}, & \text{if } x < 0 \\
\sqrt{\frac{chx - 1}{chx + 1}}, & \text{if } x \geq 0
\end{cases} \quad \text{(according to the equalities (4.5), (4.5') and (4.6))}
\]
\[
= \frac{2 \cdot sh\left(\frac{x}{2}\right) \cdot ch\left(\frac{x}{2}\right)}{2 \cdot ch^2\left(\frac{x}{2}\right)}
\]
\[ = \frac{\text{sh}x}{\text{ch}x + 1} \quad \text{(according to the equalities (4.1) and (4.4'))} \]
\[ = \frac{\text{sh}^2x}{\text{sh}x \cdot (\text{ch}x + 1)} \]
\[ = \frac{\text{ch}^2x - 1}{\text{sh}x \cdot (\text{ch}x + 1)} \quad \text{(according to the equality (3.1))} \]
\[ = \frac{(\text{ch}x - 1) \cdot (\text{ch}x + 1)}{\text{sh}x \cdot (\text{ch}x + 1)} \quad \text{(according to the equality (3.1))} \]
\[ = \frac{\text{ch}x - 1}{\text{sh}x} \cdot \frac{1}{\text{sh}x} \]
\[ = \text{cthx} - \text{cshx} \quad \text{(according to the equalities (2.4) and (2.6)).} \]

Therefore, the equalities (4.9) hold.

10) According to the equality (3.8), for every \( x \in \mathbb{R}^* \),
\[ \text{cthx} = \text{cth} \left( \frac{x}{2} + \frac{x}{2} \right) = \frac{\text{cth} \left( \frac{x}{2} \right) \cdot \text{cth} \left( \frac{x}{2} \right) + 1}{\text{cth} \left( \frac{x}{2} \right) + \text{cth} \left( \frac{x}{2} \right)} = \frac{\text{cth} \left( \frac{x}{2} \right) + 1}{2 \cdot \text{cth} \left( \frac{x}{2} \right)}; \]
which shows that the equality (4.10) holds.

11) According to the equality (3.8), for every \( x \in \mathbb{R}^* \),
\[ \text{cth}(2x) = \text{cth}(x + x) = \frac{\text{cthx} \cdot \text{cthx} + 1}{\text{cthx} + \text{cthx} \cdot \text{cthx} + 1} = \frac{\text{cth}^2x + 1}{2 \cdot \text{cth}x}; \]
which shows that the equality (4.11) holds.

12) For every \( x \in \mathbb{R}^* \), we have the equalities:
\[ \text{cth} \left( \frac{x}{2} \right) = \frac{\text{ch} \left( \frac{x}{2} \right)}{\text{sh} \left( \frac{x}{2} \right)} \quad \text{(according to the equality (2.4))} \]
\[ = \begin{cases} 
- \sqrt{\frac{\text{ch}x + 1}{\text{ch}x - 1}}, & \text{if } x < 0 \\
\sqrt{\frac{\text{ch}x + 1}{\text{ch}x - 1}}, & \text{if } x \geq 0 
\end{cases} 
\quad \text{(according to the equalities (4.5), (4.5') and (4.6))} \]
\[ = \frac{2 \cdot \text{ch} \left( \frac{x}{2} \right)}{2 \cdot \text{sh} \left( \frac{x}{2} \right) \cdot \text{ch} \left( \frac{x}{2} \right)} \]
\[ = \frac{\text{ch}x + 1}{\text{sh}x} \quad \text{(according to the equalities (4.1) and (4.4'))} \]
\[ = \frac{(\text{ch}x + 1) \cdot (\text{ch}x - 1)}{\text{sh}x \cdot (\text{ch}x - 1)} = \frac{\text{ch}^2x - 1}{\text{sh}x \cdot (\text{ch}x - 1)} \]
\[ = \frac{\text{sh}^2x}{\text{sh}x \cdot (\text{ch}x - 1)} \quad \text{(according to the equality (3.1))} \]
\[ = \frac{\text{sh}x \cdot (\text{ch}x - 1)}{\text{sh}x \cdot (\text{ch}x - 1)} \]
\[ = \text{cthx} + \text{cshx} \quad \text{(according to the equalities (2.4) and (2.6)).} \]

Therefore, the equalities (4.12) hold.
13) According to the equalities (3.2) and (3.4), for every \( x, y, z \in \mathbb{R} \),
\[ sh(x+y+z) = sh[(x+y)+z] = sh(x+y)chz + shzch(x+y) = (shxchz + chzshx)chz + chxchzchz. \]
whence it follows that the equality (4.13) holds.

14) According to the equalities (4.13) and (3.1), for every \( x \in \mathbb{R} \),
\[ sh(3x) = sh(x+x+x) = shxchxchx + chxchxshx + shxshxshx = 3shxchxchx + shx(chxchx)chx + shxchxshx + shxshxshx \]
\[ = shx(4shx+3) = shx(4chx) + 3 = shx(4chx+3); \]
i.e., we have shown that the equalities (4.14) hold.

15) According to the equalities (3.4) and (3.2), for every \( x, y, z \in \mathbb{R} \),
\[ ch(x+y+z) = ch[(x+y)+z] = ch(x+y)chz + chzsh(x+y) = (chxchzh + chzshx)(chxchzh + chzshx)chxchzh + chzshxchxchz. \]
which shows that the equality (4.15) holds.

16) According to the equalities (4.15) and (3.1), for every \( x \in \mathbb{R} \),
\[ ch(3x) = ch(x+x+x) = chxchxchx + shxshxshxchx + shxshxshxchx + chxchxshxshx = chx(chx+3shxchx + shxshxshxchx + shxshxshxchx + chxchxshxshx) \]
\[ = chx(4chx+3) = chx(4chx+3); \]
i.e., we have shown that the equalities (4.16) hold.

17) According to the equalities (3.6) and (2.3), for every \( x, y, z \in \mathbb{R} \),
\[ th(x+y+z) = th[(x+y)+z] = \frac{th(x+y) + thz}{1 + th(x+y)thz} = \frac{thx + thy + thz + thx \cdot thy \cdot thz}{1 + thx \cdot thy + thy \cdot thz + thx \cdot thz}. \]

\[ \text{Otherwise.} \quad \text{According to the equalities (2.3), (4.13) and (4.15), for every \( x, y, z \in \mathbb{R} \),} \]
\[ th(x+y+z) = \frac{sh(x+y+z)}{ch(x+y+z)} = \frac{shxchzh + chzshxchz + chxchzhshxchzh + chzchzshxchzh}{chxchzhchz + chzchzshxchzh + chxchzhshxchzh + chzchzshxchzh} = \frac{thx + thy + thz + thx \cdot thy \cdot thz}{1 + thx \cdot thy + thy \cdot thz + thx \cdot thz}, \]
which shows that the equality (4.17) holds. Because, for every \( x, y, z \in \mathbb{R} \), \( th(x+y) \) and \( thz \in (-1,1) \), it follows that:
\[ th(x+y) \cdot thz + 1 \neq 0, \]
i.e.,
\[ 1 + thx \cdot thy + thy \cdot thz + thx \cdot thz \neq 0. \]
18) According to the equality (4.17), for every \( x \in \mathbb{R} \),
\[
\text{th}(3x) = \text{th}(x+x+x) = \frac{\text{th}x + \text{th}x + \text{th}x - \text{th}x \cdot \text{th}x}{1 + \text{th}x \cdot \text{th}x + \text{th}x \cdot \text{th}x + \text{th}x \cdot \text{th}x} = 3 \cdot \text{th}x + \text{th}^3x.
\]

**Otherwise.** According to the equalities (2.3), (4.14) and (4.16), for every \( x \in \mathbb{R} \),
\[
\text{th}(3x) = \frac{\text{sh}(3x)}{\text{ch}(3x)} = \frac{\text{shx} \cdot (3 \cdot \text{ch}^2x + \text{sh}^2x)}{\text{chx} \cdot (\text{ch}^4x + 3 \cdot \text{sh}^2x)} = \frac{\text{shx} \cdot (3 \cdot \text{ch}^2y + \text{sh}^2x)}{\text{chx} \cdot (\text{ch}^4y + 3 \cdot \text{sh}^2x)} = \frac{\text{sh}x \left(3 \cdot \frac{\text{ch}^2x + \text{sh}^2x}{\text{ch}^2x + \text{sh}^2x}\right)}{\text{chx} \left(\frac{\text{ch}^2x + 3 \cdot \text{sh}^2x}{\text{ch}^2x + 3 \cdot \text{sh}^2x}\right)}
\]
\[
= \frac{\text{th}x \cdot (3 + \text{th}^2x)}{1 + 3 \cdot \text{th}^2x};
\]
which shows that the equality (4.18) holds.

19) According to the equalities (3.8) and (2.4), for every \( x, y, z \in \mathbb{R}^* \), such that \( x+y, y+z, x+z, x+y+z \in \mathbb{R}^* \),
\[
\text{cth}(x+y+z) = \text{cth}[(x+y)+z] = \frac{\text{cth}(x+y) \cdot \text{cthz} + \text{cthx} \cdot \text{cthy} \cdot \text{cthz}}{\text{cth}(x+y) + \text{cthx} \cdot \text{cthy}} = \frac{\text{cth}(x+y) \cdot \text{cthz} + \text{cthx} \cdot \text{cthy} \cdot \text{cthz} + \text{cth}(x+y) + \text{cthx} \cdot \text{cthy}}{\text{cth}(x+y) + \text{cthx} \cdot \text{cthy}} = \frac{\text{cthx} \cdot \text{cthy} \cdot \text{cthz} + \text{cthx} \cdot \text{cthy} \cdot \text{cthz} + \text{cthx} \cdot \text{cthy} \cdot \text{cthz}}{\text{cthx} \cdot \text{cthy} + \text{cthz}}
\]
\[
= \frac{\text{cthx} \cdot \text{cthy} \cdot \text{cthz} + \text{cthx} \cdot \text{cthy} \cdot \text{cthz} + \text{cthx} \cdot \text{cthy} \cdot \text{cthz} + \text{cthx} \cdot \text{cthy} \cdot \text{cthz}}{\text{cthx} \cdot \text{cthy} + \text{cthz}} = \frac{\text{cthx} \cdot \text{cthy} \cdot \text{cthz} + \text{cthx} \cdot \text{cthy} \cdot \text{cthz} + \text{cthx} \cdot \text{cthy} \cdot \text{cthz} + \text{cthx} \cdot \text{cthy} \cdot \text{cthz}}{\text{cthx} \cdot \text{cthy} + \text{cthz}}.
\]

**Otherwise.** According to the equalities (2.4) and (4.13) and (4.15), for every \( x, y, z \in \mathbb{R}^* \), such that \( x+y, y+z, x+z, x+y+z \in \mathbb{R}^* \),
\[
\text{cth}(x+y+z) = \frac{\text{ch}(x+y+z)}{\text{sh}(x+y+z)} = \frac{\text{ch}(x+y) \cdot \text{shz} + \text{sh}(x+y) \cdot \text{chz}}{\text{sh}(x+y) + \text{ch}(x+y) \cdot \text{shz}}
\]
\[
= \frac{\text{chx} \cdot \text{chy} \cdot \text{chz} + \text{shx} \cdot \text{chy} \cdot \text{chz} + \text{shx} \cdot \text{chy} \cdot \text{shz} + \text{shx} \cdot \text{chy} \cdot \text{chz} + \text{chx} \cdot \text{chy} \cdot \text{shz} + \text{chx} \cdot \text{chy} \cdot \text{chz}}{\text{shx} \cdot \text{chy} \cdot \text{chz} + \text{shx} \cdot \text{chy} \cdot \text{chz} + \text{shx} \cdot \text{chy} \cdot \text{shz} + \text{shx} \cdot \text{chy} \cdot \text{chz}}
\]
\[
= \frac{\text{cth}(x+y+z) + \text{cthx} \cdot \text{cthy} \cdot \text{cthz} + \text{cthx} \cdot \text{cthy} \cdot \text{cthz}}{\text{cthx} \cdot \text{cthy} + \text{cthz}} = \frac{\text{cthx} \cdot \text{cthy} \cdot \text{cthz} + \text{cthx} \cdot \text{cthy} \cdot \text{cthz} + \text{cthx} \cdot \text{cthy} \cdot \text{cthz} + \text{cthx} \cdot \text{cthy} \cdot \text{cthz}}{\text{cthx} \cdot \text{cthy} + \text{cthz}}.
\]

**Otherwise.** According to the equalities (2.4) and (4.17), for every \( x, y, z \in \mathbb{R}^* \), such that \( x+y, y+z, x+z, x+y+z \in \mathbb{R}^* \),
\[
\text{cth}(x+y+z) = \frac{1}{\text{th}(x+y+z)} = \frac{1}{\text{th}(x+y) + \text{th}x \cdot \text{th}y + \text{th}x \cdot \text{th}y + \text{th}x \cdot \text{th}y} = \frac{1}{\text{cthx} \cdot \text{cthy} + \text{cthz} \cdot \text{cthy} + \text{cthx} \cdot \text{cthy} \cdot \text{cthz} + \text{cthx} \cdot \text{cthy} \cdot \text{cthz}}
\]
\[
\frac{\operatorname{cthx} \cdot \operatorname{cthy} \cdot \operatorname{cthz} + \operatorname{cthx} + \operatorname{cthy} + \operatorname{cthz}}{\operatorname{cthx} \cdot \operatorname{cthy} \cdot \operatorname{cthz}} = \frac{1 + \operatorname{cthx} \cdot \operatorname{cthy} + \operatorname{cthy} \cdot \operatorname{cthz} + \operatorname{cthx} + \operatorname{cthy} + \operatorname{cthz}}{1 + \operatorname{cthx} \cdot \operatorname{cthy} + \operatorname{cthy} \cdot \operatorname{cthz} + \operatorname{cthx} \cdot \operatorname{cthy} \cdot \operatorname{cthz}};
\]
therefore, the equality (4.19) holds. Then, since, according to the hypothesis, \(\operatorname{cthx}(x+y)+\operatorname{cthz} \neq 0\), it follows that:

\(1 + \operatorname{cthx} \cdot \operatorname{cthy} + \operatorname{cthy} \cdot \operatorname{cthz} + \operatorname{cthx} \cdot \operatorname{cthy} \cdot \operatorname{cthz} \neq 0\).

20) According to the equality (4.17), for every \(x \in \mathbb{R}^*\),
\[
\operatorname{cth}(3x) = \frac{\operatorname{cthx} \cdot \operatorname{cthx} \cdot \operatorname{cthy} + \operatorname{cthx} + \operatorname{cthx} \cdot \operatorname{cthy} + \operatorname{cthx} \cdot \operatorname{cthy} \cdot \operatorname{cthz}}{1 + \operatorname{cthx} \cdot \operatorname{cthy} + \operatorname{cthy} \cdot \operatorname{cthz} + \operatorname{cthx} \cdot \operatorname{cthy} \cdot \operatorname{cthz}} = \frac{\operatorname{cthx}^3 x + 3 \cdot \operatorname{cthx}}{1 + 3 \cdot \operatorname{cthx}^2 x}.
\]

Otherwise: According to the equalities (2.4), (4.14) and (4.16), for every \(x \in \mathbb{R}^*\),
\[
\frac{\operatorname{cth}(3x)}{\operatorname{sh}(3x)} = \frac{\operatorname{chx} \cdot (\operatorname{ch}^2 x + 3 \cdot \operatorname{sh}^2 x)}{\operatorname{shx} \cdot (\operatorname{sh}^2 x + 3 \cdot \operatorname{ch}^2 x)} = \frac{\operatorname{chx} \cdot \left(\frac{\operatorname{ch}^2 x + 3 \cdot \operatorname{sh}^2 x}{\operatorname{sh}^2 x + 3 \cdot \operatorname{ch}^2 x}\right)}{\operatorname{shx} \cdot \left(\frac{\operatorname{sh}^2 x + 3 \cdot \operatorname{ch}^2 x}{\operatorname{sh}^2 x + 3 \cdot \operatorname{ch}^2 x}\right)} = \frac{\operatorname{chx} \cdot \left(\frac{\operatorname{ch}^2 x + 3 \cdot \operatorname{sh}^2 x}{\operatorname{sh}^2 x + 3 \cdot \operatorname{ch}^2 x}\right)}{\operatorname{shx} \cdot \left(\frac{\operatorname{sh}^2 x + 3 \cdot \operatorname{ch}^2 x}{\operatorname{sh}^2 x + 3 \cdot \operatorname{ch}^2 x}\right)} = \frac{\operatorname{chx} \cdot (\operatorname{ch}^2 x + 3)}{1 + 3 \cdot \operatorname{ch}^2 x}.
\]

Otherwise: According to the equalities (2.4) and (4.18), for every \(x \in \mathbb{R}^*\),
\[
\frac{\operatorname{cth}(3x)}{\operatorname{th}(3x)} = \frac{1 + 3 \cdot \operatorname{ch}^2 x}{\operatorname{thx} \cdot (3 + \operatorname{th}^2 x)} = \frac{1 + 3 \cdot \frac{1}{\operatorname{cthx}^3}}{\operatorname{cthx} \cdot \left(3 + \frac{1}{\operatorname{cthx}^2}\right)} = \frac{\operatorname{cthx}^3 x + 3 \cdot \operatorname{cthx}}{1 + 3 \cdot \operatorname{cthx}^2 x};
\]
therefore, the equality (4.20) holds.

21) According to the equalities (2.3), (3.1) and (2.5), for every \(x \in \mathbb{R}\),
\[
1 - \operatorname{th}^2 x = \frac{\operatorname{ch}^2 x - \operatorname{sh}^2 x}{\operatorname{ch}^2 x} = \frac{1}{\operatorname{sh}^2 x};
\]
whence, according to the same equalities, it follows that, for every \(x \in \mathbb{R}\):
\[
\frac{1}{1 - \operatorname{th}^2 x} = \frac{1}{\operatorname{sh}^2 x};
\]
therefore, the equalities (4.21), (4.21') and (4.21'') hold.

22) According to the equalities (2.4), (3.1) and (2.6), for every \(x \in \mathbb{R}^*\),
\[
\frac{\operatorname{cth}^2 x - 1}{\operatorname{sh}^2 x} = \frac{\operatorname{ch}^2 x - \operatorname{sh}^2 x}{\operatorname{sh}^2 x} = \frac{1}{\operatorname{sh}^2 x};
\]
whence, according to the same equalities, it follows that, for every \(x \in \mathbb{R}^*\):
\[
\frac{1}{\operatorname{cth}^2 x - 1} = \frac{1}{\operatorname{csh}^2 x};
\]
therefore, the equalities (4.22), (4.22') and (4.22'') hold.

23) According to the equalities (3.2) and (3.3), for every \(a, b \in \mathbb{R}\),
\[
\operatorname{sh}(a + b) = \operatorname{sha} + \operatorname{cb} + \operatorname{sh} - \operatorname{cha} \quad \text{and} \quad \operatorname{sh}(a - b) = \operatorname{sha} - \operatorname{cb} - \operatorname{sh} - \operatorname{cha}.
\]
By adding together these two relationships, obtain:
\[
\text{(I)} \quad \operatorname{sh}(a + b) + \operatorname{sh}(a - b) = 2 \cdot \operatorname{sha} - \operatorname{cb}.
\]
Denoting:
\[
a + b = x \quad \text{and} \quad a - b = y,
\]
Denoting:
then:
\[ a = \frac{x + y}{2} \quad \text{and} \quad b = \frac{x - y}{2}, \]
and the equality (1) becomes the equality (4.23).

24) According to the equalities (3.2) and (3.3), for every \( a, b \in \mathbb{R} \),
\[ sh(a+b) = sha \cdot chb + shb \cdot cha \quad \text{and} \quad sh(a-b) = sha \cdot chb - shb \cdot cha. \]
By decreasing these two relationships, obtain:
\[ (1) \quad sh(a+b)-sh(a-b) = 2 \cdot shb \cdot cha. \]
Denoting, again:
\[ a+b=x \quad \text{and} \quad a-b=y, \]
then:
\[ a = \frac{x + y}{2} \quad \text{and} \quad b = \frac{x - y}{2}, \]
and the equality (1) becomes the equality (4.24).

25) According to the equalities (3.4) and (3.5), for every \( a, b \in \mathbb{R} \),
\[ ch(a+b) = cha \cdot chb + sha \cdot shb \quad \text{and} \quad ch(a-b) = cha \cdot chb - sha \cdot shb. \]
By adding together these two relationships, obtain:
\[ (1) \quad ch(a+b)+ch(a-b) = 2 \cdot cha \cdot chb. \]
Denoting, again:
\[ a+b=x \quad \text{and} \quad a-b=y, \]
then:
\[ a = \frac{x + y}{2} \quad \text{and} \quad b = \frac{x - y}{2}, \]
and the equality (1) becomes the equality (4.25).

26) According to the equalities (3.4) and (3.5), for every \( a, b \in \mathbb{R} \),
\[ ch(a+b) = cha \cdot chb + sha \cdot shb \quad \text{and} \quad ch(a-b) = cha \cdot chb - sha \cdot shb. \]
By decreasing these two relationships, obtain:
\[ (1) \quad ch(a+b)-ch(a-b) = 2 \cdot sha \cdot shb. \]
Denoting, again:
\[ a+b=x \quad \text{and} \quad a-b=y, \]
then:
\[ a = \frac{x + y}{2} \quad \text{and} \quad b = \frac{x - y}{2}, \]
and the equality (1) becomes the equality (4.26).

27) According to the equalities (2.3) and (3.2), for every \( x, y \in \mathbb{R} \),
\[ thx+thy = \frac{shx}{chx} + \frac{shy}{chy} = \frac{shx \cdot chy + shy \cdot chx}{chx \cdot chy} = \frac{sh(x + y)}{chx \cdot chy}; \]
so, the equality (4.27) holds.

28) According to the equalities (2.3) and (3.3), for every \( x \in \mathbb{R} \),
\[ thx-thy = \frac{shx}{chx} - \frac{shy}{chy} = \frac{shx \cdot chy - shy \cdot chx}{chx \cdot chy} = \frac{sh(x - y)}{chx \cdot chy}; \]
so, the equality (4.28) holds.

29) According to the equalities (2.4) and (3.2), for every \( x, y \in \mathbb{R}^* \),
\[ cthx+cthy = \frac{chx}{shx} + \frac{chy}{shy} = \frac{shx \cdot chy + shx \cdot chy}{shx \cdot shy} = \frac{sh(x + y)}{shx \cdot shy}; \]
so, the equality (4.29) holds.

30) According to the equalities (2.4) and (3.3), for every \( x, y \in \mathbb{R}^* \),
\[ cthx-chty = \frac{chx}{shx} - \frac{chy}{shy} = \frac{shy \cdot chx - shx \cdot chy}{shx \cdot shy} = \frac{sh(y - x)}{shx \cdot shy} = \frac{sh(x - y)}{shx \cdot shy}; \]
so, the equality (4.30) holds.
31) The assertion from the statement follows from the proof of point 23) - the equality (1).
32) The assertion from the statement follows from the proof of point 24) - the equality (1).
33) The assertion from the statement follows from the proof of point 25) - the equality (1).
34) The assertion from the statement follows from the proof of point 26) - the equality (1).
35) For every \( x \in \mathbb{R} \), we have the equalities:
\[
\text{sh}(2x) = 2 \cdot \text{sh}x \cdot \text{ch}x \quad \text{(according to the equality (4.2))}
\]
\[
= 2 \cdot \frac{\text{sh}x}{\text{ch}x} \cdot 1 \quad \text{(according to the first equality from (4.21))}
\]
\[
= \frac{2 \cdot \text{th}x}{1 - \text{th}^2x} \quad \text{(according to the equality (2.3))};
\]
which shows that the equality (4.35) holds also.
36) For every \( x \in \mathbb{R} \), we have the equalities:
\[
\text{ch}(2x) = \frac{\text{ch}^2x + \text{sh}^2x}{1} \quad \text{(according to the equality (4.4))}
\]
\[
= \frac{\text{ch}^2x + \text{sh}^2x}{\text{ch}^2x - \text{sh}^2x} \quad \text{(according to the equality (3.1))}
\]
\[
= \frac{\text{ch}^2x + \text{sh}^2x}{\text{ch}^2x} = 1 + \frac{\text{sh}^2x}{\text{ch}^2x}
\]
\[
= \frac{\text{ch}^2x - \text{sh}^2x}{\text{ch}^2x} = 1 - \frac{\text{sh}^2x}{\text{ch}^2x}
\]
\[
= \frac{1 + \text{th}^2x}{1 - \text{th}^2x} \quad \text{(according to the equality (2.3))};
\]
which shows that the equality (4.36) holds also.
37) For every \( x \in \mathbb{R} \), the equality (4.37) follows from the equality (4.41).
38) For every \( x \in \mathbb{R} \), the equality (4.38) follows from the equality (4.41).
39) For every \( x \in \mathbb{R} \), the equality (4.39) follows from first equality from (4.14).
40) For every \( x \in \mathbb{R} \), the equality (4.40) follows from first equality from (4.16).
41) For every \( x \in \mathbb{R} \), we have the equalities:
\[
\text{ch}(4x) = 2 \cdot \text{sh}^2(2x) + 1 \quad \text{(according to the equality (4.41))}
\]
\[
= 2 \cdot (2 \cdot \text{sh}x \cdot \text{ch}x)^2 + 1 \quad \text{(according to the equality (4.2))}
\]
\[
= 2 \cdot 4 \cdot \text{sh}^2x \cdot \text{ch}^2x + 1
\]
\[
= 8 \cdot \text{sh}^2x \cdot (\text{sh}^2x + 1) + 1 \quad \text{(according to the equality (3.1))}
\]
\[
= 8 \cdot \text{sh}^4x + 8 \cdot \text{sh}^2x + 1
\]
\[
= 8 \cdot \text{sh}^4x + 8 \left( \frac{1}{2} \cdot \text{ch}(2x) - \frac{1}{2} \right) + 1 \quad \text{(according to the equality (4.37))}
\]
\[
= 8 \cdot \text{sh}^4x + 8 \cdot \text{ch}(2x) - 3;
\]
whence it follows that, for every \( x \in \mathbb{R} \),
\[
\text{sh}^4x = \frac{1}{8} \cdot \text{ch}(4x) - \frac{1}{2} \cdot \text{ch}(2x) + \frac{3}{8};
\]
so the equality (4.41) holds.
42) For every \( x \in \mathbb{R} \), we have the equalities:
\[
\text{ch}(4x) = 2 \cdot \text{ch}^2(2x) - 1 \quad \text{(according to the equality (4.41))}
\]
\[
= 2 \cdot (2 \cdot \text{ch}x - 1)^2 - 1 \quad \text{(according to the equality (4.41))}
\]
\[
= 2 \cdot (4 \cdot \text{ch}^2x - 4 \cdot \text{ch}x + 1) - 1 = 8 \cdot \text{ch}^4x - 8 \cdot \text{ch}^2x + 1
\]
\[
= 8 \cdot \text{ch}^4x - 8 \left( \frac{1}{2} \cdot \text{ch}(2x) + \frac{1}{2} \right) + 1 \quad \text{(according to the equality (4.38))}
\]
=8\cdot\text{ch}^4x-4\cdot\text{ch}(2x)\cdot3;

whence it follows that, for every \(x \in \mathbb{R}\),
\[
\text{ch}^4x = \frac{1}{8} \cdot \text{ch}(4x) + \frac{1}{2} \cdot \text{ch}(2x) + \frac{3}{8};
\]
s so, the equality (4.42) holds.

43) According to the equality (3.22), for every \(x \in \mathbb{R}\),
\[
\text{sh}^{-1}(x) = \ln(x + \sqrt{(-x)^2 + 1}) = \ln(x + \sqrt{x^2 + 1}) = \ln \left( \frac{1}{\sqrt{x^2 + 1} + x} \right) = -\ln(x + \sqrt{x^2 + 1})
\]

which shows that the equality (4.43) holds.

44) According to the equalities (3.23) and (3.24), for every \(x \in [1, +\infty)\),
\[
\text{ch}^{-1}_1(x) = \ln(x - \sqrt{x^2 - 1}) = \ln \left( \frac{x^2 - x^2 + 1}{x + \sqrt{x^2 - 1}} \right) = \ln \left( \frac{1}{x + \sqrt{x^2 - 1}} \right) = -\ln(x + \sqrt{x^2 - 1})
\]

which shows that the equality (4.44) holds.

45) According to the equality (3.25), for every \(x \in (-1, 1)\),
\[
\text{th}^{-1}(x) = \frac{1}{2} \cdot \ln \left( \frac{1 + x}{1 - x} \right) = \frac{1}{2} \cdot \ln \left( \frac{1}{1 - x} \right) = \frac{1}{2} \cdot \ln \left( \frac{1 + x}{1 - x} \right) = -\text{th}^{-1}x;
\]

which shows that the equality (4.45) holds.

46) According to the equality (3.26), for every \(x \in (-\infty, -1) \cup (1, +\infty)\),
\[
\text{cth}^{-1}(x) = \frac{1}{2} \cdot \ln \left( \frac{x + 1}{-x} \right) = \frac{1}{2} \cdot \ln \left( -x + 1 \right) = \frac{1}{2} \cdot \ln \left( -x - 1 \right) = \frac{1}{2} \cdot \ln \left( \frac{x + 1}{-x - 1} \right) = -\text{cth}^{-1}x;
\]

which shows that the equality (4.46) holds.

47) According to the equalities (3.27) and (3.28), for every \(x \in (0, 1]\),
\[
\text{sch}^{-1}_1(x) = \ln \left( \frac{1 - \sqrt{1 - x^2}}{-x} \right) = \ln \left( \frac{1 - \sqrt{1 - x^2}}{x} \right) = \ln \left( \frac{x}{1 + \sqrt{1 - x^2}} \right) = -\ln \left( \frac{x}{1 + \sqrt{1 - x^2}} \right)
\]

which shows that the equality (4.47) holds.

48) According to the equalities (3.29) and (3.30), for every \(x \in (-\infty, 0)\),
\[
\text{ch}^{-1}_1 x = \ln \left( \frac{1 - \sqrt{1 + x^2}}{x} \right) = \ln \left( \frac{-x}{1 + \sqrt{1 + x^2}} \right) = \ln \left( \frac{-x}{1 + \sqrt{(-x)^2 + 1}} \right) = -\ln \left( \frac{x}{1 + \sqrt{(-x)^2 + 1}} \right)
\]

which shows that the equality (4.48) holds. On the other hand, (again) according to the equalities (3.29) and (3.30), for every \(x \in (0, +\infty)\),
\[
\text{ch}^{-1}_2 x = \ln \left( \frac{1 + \sqrt{1 + x^2}}{x} \right) = \ln \left( \frac{-x}{1 - \sqrt{1 + x^2}} \right) = \ln \left( \frac{-x}{1 - \sqrt{(-x)^2 + 1}} \right) = -\ln \left( \frac{x}{1 - \sqrt{(-x)^2 + 1}} \right)
\]

which shows that the equality (4.48') holds.

49) According to the equality (3.22), for every \(x \in \mathbb{R}\),
\[
(\text{sh}^{-1}x)' = (\ln(x + \sqrt{x^2 + 1}))' = \frac{1 + \frac{x}{\sqrt{x^2 + 1}}}{x + \sqrt{x^2 + 1}} = \frac{x + \sqrt{x^2 + 1}}{x + \sqrt{x^2 + 1}} = \frac{1}{\sqrt{x^2 + 1}};
\]

so, the equality (4.49) holds.
50) According to the equality (3.23), for every $x \in (1, +\infty)$,

$$(ch_1^{-1} x)' = (\ln(x - \sqrt{x^2 - 1}))' = \frac{1 - \frac{x}{\sqrt{x^2 - 1}}}{x - \sqrt{x^2 - 1}} = \frac{1}{\sqrt{x^2 - 1}};$$

so, the equality (4.50) holds. On the other hand, according to the equality (3.24), for every $x \in (1, +\infty)$,

$$(ch_2^{-1} x)' = (\ln(x + \sqrt{x^2 - 1}))' = \frac{1 + \frac{x}{\sqrt{x^2 - 1}}}{x + \sqrt{x^2 - 1}} = \frac{1}{\sqrt{x^2 - 1}};$$

which shows that the equality (4.50') holds also.

51) According to the equality (3.25), for every $x \in (-1, 1)$,

$$(th^{-1} x)' = \frac{1}{2} (\ln(1 + x) - \ln(1 - x))' = \frac{1}{2} \left( \frac{1 + x}{1 - x} \right)' = \frac{1}{2} \cdot \frac{1 - x + x}{(1 - x)^2} = \frac{1}{2} \cdot \frac{1 - x}{1 + x} \cdot \frac{2}{(1 - x)^2} = \frac{1}{1 - x^2};$$

so, the equality (4.51) holds.

52) According to the equalities (3.26), for every $x \in (-\infty, -1)$,

$$(th_1^{-1} x)' = \frac{1}{2} (\ln(1 + x) - \ln(1 - x))' = \frac{1}{2} \left( \frac{1 + x}{1 - x} \right)' = \frac{1}{2} \cdot \frac{x - 1}{x + 1} \cdot \frac{x - 1 - x - 1}{(x - 1)^2} = \frac{1}{2} \cdot \frac{x - 1}{x + 1} \cdot \frac{-2}{(x - 1)^2} = \frac{1}{1 - x^2};$$

so, the equality (4.52) holds. The other equalities - (4.52') and (4.52'') - are obtained analogously.

53) According to the equality (3.27), for every $x \in (0, 1)$,

$$(sch_1^{-1} x)' = \left( \ln\left( \frac{1 - \sqrt{1 - x^2}}{x} \right) \right)' = \frac{x}{1 - \sqrt{1 - x^2} \cdot x} = \frac{x}{\sqrt{1 - x^2} \cdot x - 1 + \sqrt{1 - x^2}} = \frac{1}{1 - \sqrt{1 - x^2}} \cdot \frac{x - 1}{x \cdot \sqrt{1 - x^2}} = \frac{1}{x \sqrt{1 - x^2}};$$

so, the equality (4.53) holds. On the other hand, according to the equality (3.28), for every $x \in (0, 1)$,

$$(sch_2^{-1} x)' = \left( \ln\left( \frac{1 + \sqrt{1 - x^2}}{x} \right) \right)' = \frac{x}{1 + \sqrt{1 - x^2} \cdot x} = \frac{x}{\sqrt{1 - x^2} \cdot x - 1 - \sqrt{1 - x^2}} = \frac{1}{1 + \sqrt{1 - x^2}} \cdot \frac{-1 + \sqrt{1 - x^2}}{x \cdot \sqrt{1 - x^2}} = \frac{1}{x \sqrt{1 - x^2}};$$

which shows that the equality (4.53') holds also.

54) According to the equality (3.29), for every $x \in (-\infty, 0)$,

$$(csch^{-1} x)' = \left( \ln\left( \frac{1 - \sqrt{1 + x^2}}{x} \right) \right)' = \frac{x}{1 - \sqrt{1 + x^2} \cdot x} = \frac{x}{\sqrt{1 + x^2} \cdot x - 1 + \sqrt{1 + x^2}} = \frac{1}{1 - \sqrt{1 + x^2}} \cdot \frac{x - 1}{x \cdot \sqrt{1 + x^2}} = \frac{1}{x \sqrt{1 + x^2}};$$

so, the equality (4.54) holds. On the other hand, according to the equality (2.30), for every $x \in (0, +\infty)$,
\begin{align*}
\left( \text{sech}^{-1}(x) \right)' &= \left( \ln \left( \frac{1 + \sqrt{1 + x^2}}{x} \right) \right)' = \frac{x}{\sqrt{1 + x^2}} \cdot \frac{x - 1 - \sqrt{1 + x^2}}{x^2} \\
&= \frac{1}{1 + \sqrt{1 + x^2}} \cdot \frac{x^2 - \sqrt{1 + x^2} - 1 - x^2}{x \cdot \sqrt{1 + x^2}} = \frac{1}{1 + \sqrt{1 - x^2}} \cdot \frac{-1 - \sqrt{1 + x^2}}{x \cdot \sqrt{1 + x^2}} = \frac{1}{x \cdot \sqrt{x^2} + 1};
\end{align*}

which shows that the equality (4.54) holds also.

5. Conclusions

As you can see, in this paper we presented 54 other properties of the hyperbolic functions, divided into three groups. And here, as in (Vălcăn, 2016), the aim was to form the reader's attention and interest in these issues, developing their global image about these features and their inversions. Precisely why the demonstrations are presented in full, in detail, so that it can be used in the classroom.

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