On metric structures of normed gyrogroups

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Abstract

In this article, we indicate that the open unit ball in \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) admits norm-like functions compatible with the Poincaré and Beltrami–Klein metrics. This leads to the notion of a normed gyrogroup, similar to that of a normed group in the literature. We then examine topological and geometric structures of normed gyrogroups. In particular, we prove that the normed gyrogroups are homogeneous and form left invariant metric spaces and derive a version of the Mazur–Ulam theorem. We also give certain sufficient conditions, involving the right-gyrotranslation inequality and Klee’s condition, for a normed gyrogroup to be a topological gyrogroup.

Keywords. Topological gyrogroup, normed gyrogroup, gyronorm, left invariant metric, Mazur–Ulam theorem.

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1 Introduction

Roughly speaking, a normed group is a group that comes with a compatible norm (also called a length function), similar to the case of normed linear spaces. A prominent example of a normed group is a finitely generated group with the word metric, which is one of the main ingredients in geometric group theory. The normed groups abound as an integral part in the theory of topological groups [5]. They are blended objects that have significance in group theory, geometry, analysis, and topology, to name a few.

In [27], Ungar studies a parametrization of the Lorentz transformation group. This leads to the formation of gyrogroup theory, a rich subject in mathematics [3, 9, 23–26, 30]. Loosely speaking, a gyrogroup is a group-like structure in which
the associative law fails to satisfy. However, it obeys the gyroassociative law, a weak form of associativity, as well as the loop property, an algebraic rule equivalent to the Bol identity in loop theory. One of the virtues of studying gyrogroups is an application in non-Euclidean geometry [28], where Ungar examines analytic hyperbolic geometry using the gyrolanguage.

Atiponrat [4] and Cai et al. [6] pave the way for studying topological gyrogroups. In particular, Atiponrat proves that in the class of topological gyrogroups, being a $T_0$-space is equivalent to being a $T_3$-space [4, Theorem 3]. Further, she attempts to extend the famous Birkhoff–Kakutani theorem by proving that every first-countable Hausdorff topological gyrogroup is premetrizable [4, Theorem 4]. The latter result is strengthened when Cai et al. prove that every first-countable Hausdorff topological gyrogroup is metrizable [6, Theorem 2.3]. The achieved results inspire us to investigate topological properties of gyrogroups, which eventually bring us to the notion of a normed gyrogroup. This provides a large class of gyrogroups with a left invariant metric, and some of them are indeed topological gyrogroups.

2 Preliminaries

Standard terminology and notation in algebra, topology, and geometry used throughout the article are defined as usual. In this section, we collect relevant definitions and elementary properties of gyrogroups for reference [22, 28]. The reader familiar with gyrogroup theory may skip this section.

Let $G$ be a nonempty set equipped with a binary operation $\oplus$ on $G$. Denote by $\text{Aut} G$ the group of automorphisms of $(G, \oplus)$.

Definition 2.1 (Gyrogroups). A nonempty set $G$, together with a binary operation $\oplus$ on $G$, is called a gyrogroup if it satisfies the following axioms.

(G1) There exists an element $e \in G$ such that $e \oplus a = a$ for all $a \in G$.

(G2) For each $a \in G$, there exists an element $b \in G$ such that $b \oplus a = e$.

(G3) For all $a, b \in G$, there is an automorphism $\text{gyr}[a,b] \in \text{Aut} G$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a,b]c$$

(left gyroassociative law)

for all $c \in G$.

(G4) For all $a, b \in G$, $\text{gyr}[a \oplus b, b] = \text{gyr}[a, b]$.

(left loop property)
Note that the axioms in Definition 2.1 imply the right counterparts. In particular, any gyrogroup has a unique two-sided identity $e$, and an element $a$ of the gyrogroup has a unique two-sided inverse $\ominus a$. The automorphism $\text{gyr}[a, b]$ is called the gyroautomorphism generated by $a$ and $b$. It is clear that every group satisfies the gyrogroup axioms (the gyroautomorphisms are the identity map) and hence is a gyrogroup. Conversely, any gyrogroup with trivial gyroautomorphisms forms a group. From this point of view, gyrogroups naturally generalize groups.

The table below summarizes some algebraic properties of gyrogroups [22,28], which will prove useful in studying topological and geometric aspects of gyrogroups in Sections 3 and 4. We remark that gyroautomorphisms play an essential role in gyrogroup theory; for example, they appear as part of generic algebraic rules extended from group-theoretic identities.

| GYROGROUP IDENTITY | NAME/REFERENCE |
|--------------------|----------------|
| $\ominus(\ominus a) = a$ | Involution of inversion |
| $\ominus a \ominus (a \ominus x) = x$ | Left cancellation law |
| $\text{gyr}[a, b]c = \ominus(a \ominus b) \ominus (a \ominus (b \ominus c))$ | Gyrorator identity |
| $\ominus(a \ominus b) = \text{gyr}[a, b](\ominus b \ominus a)$ | cf. $(ab)^{-1} = b^{-1}a^{-1}$ |
| $(\ominus a \ominus b) \ominus \text{gyr}[\ominus a, b](\ominus b \ominus c) = \ominus a \ominus c$ | cf. $(a^{-1}b)(b^{-1}c) = a^{-1}c$ |
| $\text{gyr}[\ominus a, \ominus b] = \text{gyr}[a, b]$ | Even property |
| $\text{gyr}[b, a] = \text{gyr}^{-1}[a, b]$, the inverse of $\text{gyr}[a, b]$ | Inversive symmetry |
| $\varphi(\text{gyr}[a, b]c) = \text{gyr}[\varphi(a), \varphi(b)]\varphi(c)$ | Gyration preserving under a gyrogroup homomorphism $\varphi$ |
| $L_a \circ L_b = L_{a \ominus b} \circ \text{gyr}[a, b]$ | Composition law for left gyrotranslations |

Table 1: Algebraic properties of gyrogroups (cf. [22,28]).

3 Normed gyrogroups and concrete examples

In this section, we establish that any gyrogroup with an appropriate length function, called a gyronorm, has the metric structure and hence is a Hausdorff space. We also exhibit a few examples of well-known gyrogroups that have gyronorms.

3.1 The definition and basic properties

**Definition 3.1 (Gyronorms).** Let $G$ be a gyrogroup. A function $\| \cdot \| : G \rightarrow \mathbb{R}$ is called a gyronorm on $G$ if the following properties hold:
Any gyrogroup with a gyronorm is called a normed gyrogroup. We remark that the term “gyronorm” is quite different from what Ungar used in Chapter 4 of [28]. In Section 3.2, we give several concrete examples of gyrogroups with a gyronorm. Clearly, Definition 3.1 is a generalization of the notion of a group-norm [5, p. 8], which in turn is motivated by norms on linear spaces. Furthermore, any gyrogroup may be viewed as a normed gyrogroup with a gyronorm defined by

\[
\|x\| = \begin{cases} 
0 & \text{if } x = e; \\
1 & \text{if } x \neq e.
\end{cases}
\]

**Theorem 3.2.** Let \( G \) be a normed gyrogroup. Define

\[
d(x, y) = \| \ominus x \ominus y \|
\]

for all \( x, y \in G \). Then \( d \) is a metric on \( G \) and so \((G, d)\) forms a metric space.

**Proof.** By definition, \( d(x, y) = \| \ominus x \ominus y \| \geq 0 \) for all \( x, y \in G \). Clearly, \( d(x, x) = \| e \| = 0 \) for all \( x \in G \). Suppose that \( d(x, y) = 0 \). Then \( \ominus x \ominus y = e \). By definition, \( \ominus x \ominus y = e \). Hence, \( x = y \) by the left cancellation law.

Let \( x, y, z \in G \). Using appropriate properties of gyrogroups in Table 1, together with the defining properties of a gyronorm, we obtain

\[
d(y, x) = \| \ominus y \ominus x \| = \| (\ominus y \ominus x) \| = \| \ominus y \ominus [\ominus y \ominus x] \| = \| \ominus x \ominus y \| = d(x, y).
\]

Furthermore, we obtain

\[
d(x, z) = \| \ominus x \ominus z \|
\]

\[
= \| (\ominus x \ominus y) \ominus \text{gyr}[\ominus y] (\ominus y \ominus z) \|
\]

\[
\leq \| \ominus x \ominus y \| + \| \text{gyr}[\ominus y] (\ominus y \ominus z) \|
\]

\[
= \| \ominus x \ominus y \| + \| \ominus y \ominus z \|
\]

\[
= d(x, y) + d(y, z).
\]

This proves that \( d \) satisfies the defining properties of a metric. \( \square \)
The metric $d$ induced by a gyronorm on $G$ in Theorem 3.2 is called a gyronorm metric. Whenever we say that $G$ is a normed gyrogroup, we assume that $G$ is endowed with the corresponding gyronorm metric and that $G$ carries the topology induced by this metric, unless mentioned otherwise. It is clear that every isometry of a normed gyrogroup to itself is a homeomorphism for the inverse of an isometry is again an isometry. Next, we show that known gyrogroups in the literature possess gyronorms.

### 3.2 Concrete examples

#### 3.2.1 An $n$-dimensional Euclidean version of the Einstein gyrogroup

Let $B$ denote the open unit ball in $n$-dimensional Euclidean space $\mathbb{R}^n$, that is,

$$
B = \{ v \in \mathbb{R}^n : \|v\| < 1 \},
$$

(3.2)

where $\| \cdot \|$ denotes the Euclidean norm on $\mathbb{R}^n$. The Einstein gyrogroup consists of $B$, together with Einstein addition $\oplus_E$ given by

$$
 u \oplus_E v = \frac{1}{1 + \langle u, v \rangle} \left( u + \frac{1}{\gamma_u} v + \frac{\gamma_u}{1 + \gamma_u} \langle u, v \rangle u \right)
$$

(3.3)

for all $u, v \in B$, where $\gamma_u$ is the Lorentz factor given by $\gamma_u = \frac{1}{\sqrt{1 - \|u\|^2}}$. The zero vector $0$ acts as the identity of $B$ under $\oplus_E$. For each $v \in B$, the negative vector $-v$ acts as the inverse of $v$ with respect to Einstein addition. By Proposition 2.4 of [14], the gyroautomorphisms of $(B, \oplus_E)$ are orthogonal in the sense that

$$
\| \text{gyr}[u, v]w \| = \|w\|
$$

for all $u, v, w \in B$.

Define a function $\| \cdot \|_E$ on $B$ by the equation

$$
\|v\|_E = \tanh^{-1} \|v\|, \quad v \in B,
$$

(3.4)

where $\tanh^{-1}$ denotes the inverse of the hyperbolic tangent function on $\mathbb{R}$.

**Theorem 3.3.** The function $\| \cdot \|_E$ defined by (3.4) is a gyronorm on the Einstein gyrogroup.

**Proof.** Since $\tanh r \geq 0$ if and only if $r \geq 0$, it follows that $\|v\|_E \geq 0$ for all $v \in B$. Note that $\|0\|_E = \tanh^{-1} 0 = 0$. Suppose that $\|v\|_E = 0$. Then $0 = \tanh^{-1} \|v\|$, which implies $\|v\| = 0$. Hence, $v = 0$. Let $v \in B$. Then

$$
\| \ominus v \|_E = \| -v \|_E = \tanh^{-1} \| -v \| = \tanh^{-1} \|v\| = \|v\|_E.
$$
Let \( u, v \in B \). Applying Proposition 3.3 of [14] and Lemma 3.2 (iv) of [14] gives
\[
\|u \oplus_E v\|_E \leq \|u\|_E + \|v\|_E.
\]

Let \( u, v, w \in B \). Then
\[
\|\text{gyr}[u, v]w\|_E = \tanh^{-1}\|\text{gyr}[u, v]w\| = \tanh^{-1}\|w\| = \|w\|_E.
\]

It follows from Theorems 3.2 and 3.3 that
\[
d_E(u, v) = \tanh^{-1}\| - u \oplus_E v\|
\]
defines a metric on \( B \). This metric is called the rapidity metric on the Einstein gyrogroup [14, 28]. It is known that the rapidity metric on the Einstein gyrogroup agrees with the Cayley–Klein metric on the Beltrami–Klein model of \( n \)-dimensional hyperbolic geometry [14, p. 1233].

Note that the Euclidean norm is indeed a gyronorm on the Einstein gyrogroup. This follows from the fact that
\[
\|u \oplus_E v\| \leq \|u\| + \|v\| = \frac{\|u\| + \|v\|}{1 + \|u\|\|v\|} \leq \|u\| + \|v\|,
\]
where the first inequality is worked out in Proposition 3.3 of [14] and \( \oplus \) is the restricted Einstein addition on the open interval \((-1, 1)\) given by \( r \oplus s = \frac{r + s}{1 + rs} \) for all \( r, s \in (-1, 1) \). Denote by \( d_e \) the gyronorm metric induced by the Euclidean norm. That is,
\[
d_e(u, v) = \| - u \oplus_E v\|
\]
for all \( u, v \in B \). In fact, \( d_e \) is known as the Einstein gyrometric [28, p. 222]. Our results provide an elegant proof that the Einstein gyrometric is indeed a metric on the open unit ball of \( \mathbb{R}^n \).

Note that \( d_e(u, v) \leq d_E(u, v) \) for all \( u, v \in B \) because \( x \mapsto x - \tanh^{-1}x \) defines a strictly decreasing function on the open interval \((0, 1)\). This implies that the topology generated by \( d_E \) is finer than the topology generated by \( d_e \). Next, we prove that the topology generated by \( d_e \) is finer than the topology generated by \( d_E \). Let \( u \in B \) and let \( \varepsilon > 0 \). Choose \( \delta = \tanh \varepsilon \). Let \( v \in B_{d_e}(u, \delta) \). Then \( d_e(u, v) < \delta \), that is, \( \| - u \oplus_E v\| < \tanh \varepsilon \). It follows that
\[
d_E(u, v) = \tanh^{-1}\| - u \oplus_E v\| < \varepsilon
\]
for \( \tanh^{-1} \) is a strictly increasing function on its domain. Thus, \( v \in B_{d_E}(u, \varepsilon) \). This proves that \( B_{d_e}(u, \delta) \subseteq B_{d_E}(u, \varepsilon) \). Therefore, \( d_e \) and \( d_E \) generate the same topology on \( B \).
3.2.2 An $n$-dimensional Euclidean version of the Möbius gyrogroup

The Möbius gyrogroup consists of the same underlying set as the Einstein gyrogroup, but its binary operation, called Möbius addition, is defined by

$$u \oplus_M v = \frac{(1 + 2(u, v) + \|v\|^2)u + (1 - \|u\|^2)v}{1 + 2(u, v) + \|u\|^2\|v\|^2}$$

(3.7)

for all $u, v \in B$. The zero vector acts as the identity of $B$ under $\oplus_M$. For each $v \in B$, the negative vector $-v$ acts as the inverse of $v$ with respect to Möbius addition. By Proposition 2.4 of [14], the gyroautomorphisms of $(B, \oplus_M)$ are orthogonal in the sense that

$$\|\text{gyr}[u, v]w\| = \|w\|$$

for all $u, v, w \in B$.

By Proposition 2.3 of [14], the map $\Phi$ defined by

$$\Phi(v) = \frac{2}{1 + \|v\|^2} v, \quad v \in B$$

(3.8)

is a gyrogroup isomorphism from $(B, \oplus_M)$ to $(B, \oplus_E)$. In particular, $\Phi$ is a bijection from $B$ to itself. Let $\| \cdot \|_E$ be the gyronorm on the Einstein gyrogroup defined by (3.4). Define a function $\| \cdot \|_M$ by the equation

$$\|v\|_M = \frac{1}{2} \|\Phi(v)\|_E, \quad v \in B.$$  

(3.9)

**Theorem 3.4.** The function $\| \cdot \|_M$ defined by (3.9) is a gyronorm on the Möbius gyrogroup.

*Proof.* The theorem follows directly from the fact that $\| \cdot \|_E$ defines a gyronorm on $(B, \oplus_E)$ and that $\Phi$ is a gyrogroup isomorphism. \qed

By Theorems 3.2 and 3.4,

$$d_M(u, v) = \frac{1}{2} \text{tanh}^{-1} \|\Phi(-u \oplus_M v)\|$$

(3.10)

defines a metric on $B$. This metric is called the rapidity metric on the Möbius gyrogroup [14, 28]. It is known that the rapidity metric on the Möbius gyrogroup is half of the Poincaré metric with curvature $-1$ on the Poincaré model of $n$-dimensional hyperbolic geometry [14, Theorem 3.7].
4 Topological and geometric structures

Normed gyrogroups have nice topological and geometric structures. Further, they share certain remarkable analogies with normed groups. In fact, several of the results proved in this section are inspired by the expository article of Bingham and Ostaszewski [5] that treats normed and topological groups. Roughly speaking, normed gyrogroups are left invariant, homogeneous, and isotropic.

4.1 Topological and geometric properties

Theorem 4.1. Let $G$ be a normed gyrogroup. Then the gyronorm metric is invariant under left gyrotranslation:

$$d(a \oplus x, a \oplus y) = d(x, y)$$

(4.1)

for all $a, x, y \in G$. Hence, every left gyrotranslation of $G$ is an isometry of $G$ with respect to the gyronorm metric.

Proof. Let $a \in G$. Recall that the left gyrotranslation by $a$, denoted by $L_a$, is defined by $L_a(x) = a \oplus x$ for all $x \in G$. By Theorem 18 (1) of [22], $L_a$ is a bijection from $G$ to itself.

Next, we prove that the gyronorm metric $d$ is invariant under $L_a$. Let $x, y \in G$. Using appropriate properties of gyrogroups in Table 1, together with the defining properties of a gyronorm, we obtain

$$d(L_a(x), L_a(y)) = ||\ominus(a \oplus x) \ominus(a \oplus y)||$$

$$= ||\text{gyr}[a, x](\ominus x \ominus a) \ominus(a \oplus y)||$$

$$= ||(\ominus x \ominus a) \ominus\text{gyr}[x, a](a \oplus y)||$$

$$= ||(\ominus x \ominus a) \ominus\text{gyr}[x, a](a \oplus y)||$$

$$= ||\ominus x \ominus y||$$

$$= d(x, y).$$

Corollary 4.2. If $G$ is a normed gyrogroup, then every left gyrotranslation of $G$ is a homeomorphism.

Theorem 4.3. Let $G$ be a normed gyrogroup. If $\tau \in \text{Aut} G$ and $||\tau(x)|| = ||x||$ for all $x \in G$, then $\tau$ is an isometry of $G$ with respect to the gyronorm metric.

Proof. By assumption,

$$d(\tau(x), \tau(y)) = ||\ominus \tau(x) \ominus \tau(y)|| = ||\tau(\ominus x \ominus y)|| = ||\ominus x \ominus y|| = d(x, y).$$

and so $\tau$ defines an isometry of $G$. 

$\Box$
Corollary 4.4. If $G$ is a normed gyrogroup, then the gyroautomorphisms of $G$ are isometries (and also homeomorphisms) of $G$.

**Theorem 4.5 (Homogeneity).** If $G$ is a normed gyrogroup, then $G$ is homogeneous in the sense that if $x$ and $y$ are arbitrary points of $G$, then there is an isometry $T : G \to G$ (and also a homeomorphism of $G$) such that $T(x) = y$.

**Proof.** Let $x, y \in G$. Define $T = L_y \circ L_{\ominus x}$. By Theorem 4.1, $T$ is an isometry of $G$. Further, $T(x) = (L_y \circ L_{\ominus x})(x) = L_y(L_{\ominus x}(x)) = L_y(\ominus x \oplus x) = L_y(e) = y \oplus e = y$.

Thus, $G$ is homogeneous. □

**Theorem 4.6 (Isotropy).** If $G$ is a nondegenerate normed gyrogroup; that is, $G$ has a nonidentity gyroautomorphism, then $G$ is isotropic in the sense that for each point $p \in G$, there exists a nonidentity isometry $T$ of $G$ such that $T(p) = p$.

**Proof.** Let $p$ be an arbitrary point of $G$. Let $\tau$ be a nonidentity gyroautomorphism of $G$. Then $\tau(e) = e$. Define $T = L_p \circ \tau \circ L_{\ominus p}$. Note that $T$ is an isometry of $G$, being the composite of isometries of $G$. Further, $T(p) = p$. Note that $T$ is not the identity transformation of $G$; otherwise, we would have $I = L_p \circ \tau \circ L_{\ominus p} = L_p \circ \tau \circ L_{\ominus p}^{-1}$ and would have $\tau = I$, a contradiction. □

Recall that the famous Mazur–Ulam theorem states that any isometry between normed linear spaces over $\mathbb{R}$ that fixes the zero vector must be linear; see, for instance, [10, Theorem 1.3.5]. Extensions of the Mazur–Ulam theorem are studied by Rassias [18, 19] and by Rassias et al. [20, 21]. Further, the Mazur–Ulam theorem is examined in the setting of **gyrovector spaces** by Abe [1] and by Abe and Hatori [2]. Here, we prove a normed-gyrogroup version of the Mazur–Ulam theorem.

**Theorem 4.7.** Let $G$ be a normed gyrogroup. If $f$ is an isometry of $G$ with respect to the gyronorm metric, then

$$f = L_{f(e)} \circ \rho,$$

where $\rho$ is an isometry of $G$ that leaves the gyrogroup identity fixed.

**Proof.** Suppose that $f$ is an isometry of $G$. By definition, $f$ is a permutation of $G$. By Proposition 19 of [22], $f = L_{f(e)} \circ \rho$, where $\rho$ is a permutation of $G$ that fixes $e$. As in the proof of Theorem 18 of [22], $L_{f(e)}^{-1} = L_{\ominus f(e)}$ and so $\rho = L_{\ominus f(e)} \circ f$. Hence, $\rho$ is an isometry of $G$, being the composite of isometries of $G$. □
4.2 A characterization of normed gyrogroups

Note that the gyronorm of an arbitrary normed gyrogroup can be recovered by its corresponding metric:

\[ \|x\| = d(e, x), \quad x \in G. \]  \hspace{1cm} (4.3)

It turns out that Theorem 4.1 provides a characterizing property of normed gyrogroups, as shown in the following theorem.

**Theorem 4.8.** Let \( G \) be a gyrogroup with a metric \( d \). If \( d \) is invariant under left gyrotranslation, that is,

\[ d(a \oplus x, a \oplus y) = d(x, y) \]

for all \( a, x, y \in G \), then \( \|x\| = d(e, x) \) defines a gyronorm on \( G \) that generates the same metric.

**Proof.** It is clear that \( \|x\| \geq 0 \) and \( \|x\| = 0 \) if and only if \( x = e \). Let \( x \in G \). By the left gyrotranslation invariant, \( \|\ominus x\| = d(e, \ominus x) = d(x \oplus e, x \ominus x) = d(x, e) = \|x\| \).

Let \( x, y \in G \). Direct computation shows that

\[ \|x \oplus y\| = d(x \oplus y, e) = d(y, \ominus x) \leq d(y, e) + d(e, \ominus x) = \|x\| + \|y\|. \]

Let \( a, b, x \in G \). By the gyrator identity,

\[ \|\text{gyr}[a, b]x\| = d(\text{gyr}[a, b]x, e) \]
\[ = d(\ominus(a \oplus b) \oplus (a \oplus (b \oplus x)), e) \]
\[ = d(a \oplus (b \oplus x), a \oplus b) \]
\[ = d(b \oplus x, b) \]
\[ = d(x, e) \]
\[ = \|x\|. \]  \hspace{1cm} \Box

In view of Theorems 3.2, 4.1, and 4.8, there is a one-to-one correspondence between the class of normed gyrogroups and the class of gyrogroups with a left-gyrotranslation-invariant metric:

\{Normed gyrogroups\} \leftrightarrow \{(G, d), d \text{ left-gyrotranslation-invariant metric}\}.

One of the advantages of Theorem 4.8 is illustrated in the example below.

**Example 4.9 (The complex Möbius gyrogroup).** The Poincaré disk model consists of the open unit disk in the complex plane,

\[ \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}, \]  \hspace{1cm} (4.4)
and the (complex version of) Poincaré metric defined by

\[ d_P(w, z) = 2 \tanh^{-1} \left| \frac{w - z}{1 - wz} \right| \]  (4.5)

for all \( w, z \in \mathbb{D} \). Here, the factor 2 is added to (4.5) so that the metric corresponds to a curvature of \(-1\).

A complex version of Möbius addition is defined by

\[ a \oplus_M b = \frac{a + b}{1 + \overline{ab}}, \quad a, b \in \mathbb{D}, \]  (4.6)

which gives \( \mathbb{D} \) the gyrogroup structure [29]. It is not difficult to check that 0 is the identity of \( \mathbb{D} \), that the inverse of \( a \) is \(-a\), and that the gyroautomorphism generated by \( a \) and \( b \) is a disk rotation corresponding to the unimodular complex \( \frac{1 + ab}{1 + \overline{ab}} \).

Using (4.5) and (4.6), we have by inspection that

\[ d_P(a \oplus_M w, a \oplus_M z) = d_P(w, z) \]

for all \( a, w, z \in \mathbb{D} \). Hence, by Theorem 4.8, \( \mathbb{D} \) forms a normed gyrogroup whose gyronorm is given by \( \|z\| = 2 \tanh^{-1} |z| \) for all \( z \in \mathbb{D} \). This leads to the well-known fact that any Möbius transformation (also called a conformal self-map) of \( \mathbb{D} \) of the form

\[ z \mapsto \frac{a + z}{1 + \overline{az}}, \quad z \in \mathbb{D}, \]

where \( a \) is a fixed element in \( \mathbb{D} \), is an isometry of \( \mathbb{D} \) with respect to the Poincaré metric. The study of Möbius transformations is an important topic in mathematics. This is evidenced by characterizations of Möbius transformations found in the literature; see, for instance, [7, 8, 11–13, 16, 17].

### 4.3 Sufficient conditions to be a topological gyrogroup

In [15], Klee shows that in the class of groups with a metric \( d \) the condition that \( d(xy, ab) \leq d(x, a) + d(y, b) \) is equivalent to the bi-invariant of \( d \). This motivates the following theorem for normed gyrogroups:

**Theorem 4.10.** Let \( G \) be a normed gyrogroup with the corresponding metric \( d \). Then the following conditions are equivalent:

(I) Right-gyrotranslation inequality: \( d(x \oplus a, y \oplus a) \leq d(x, y) \) for all \( a, x, y \in G \);

(II) Klee’s condition: \( d(x \oplus y, a \oplus b) \leq d(x, a) + d(y, b) \) for all \( a, b, x, y \in G \).
Proof. Assume that the right-gyrotranslation inequality holds. Then we have
\[
d(x \oplus y, a \oplus b) \leq d(x \oplus y, x \oplus b) + d(x \oplus b, a \oplus b) \\
= d(y, b) + d(x \oplus b, a \oplus b) \\
\leq d(y, b) + d(x, a) \\
= d(x, a) + d(y, b)
\]
for all \(a, b, x, y \in G\). Conversely, if Klee’s condition holds, then
\[
d(x \oplus a, y \oplus a) \leq d(x, y) + d(a, a) = d(x, y)
\]
for all \(a, x, y \in G\). \(\square\)

Recall that a gyrogroup \(G\) endowed with a topology is called a topological gyrogroup if (i) the gyroaddition map \((x, y) \mapsto x \oplus y\) is jointly continuous and (ii) the inversion map \(x \mapsto \ominus x\) is continuous [4, Definition 1]. In general, a normed gyrogroup need not be a topological gyrogroup. The conditions mentioned in Theorem 4.10 are sufficient conditions for a normed gyrogroup to be a topological gyrogroup. It is still an open question whether these conditions are necessary.

Theorem 4.11. Let \(G\) be a normed gyrogroup. If one of the conditions in Theorem 4.10 holds, then \(G\) is a topological gyrogroup with respect to the topology induced by the gyronorm metric.

Proof. Denote by \(A\) the gyroaddition map: \(A(x, y) = x \oplus y\). Let \((x, y)\) be an arbitrary point of \(G \times G\) and let \(V\) be a neighborhood of \(A(x, y) = x \oplus y\). By definition, there is an \(\varepsilon > 0\) such that \(B(x \oplus y, \varepsilon) \subseteq V\). Define \(S = B(x, \varepsilon/2)\) and \(T = B(y, \varepsilon/2)\). Set \(U = S \times T\). Since \(S\) and \(T\) are open in \(G\), it follows that \(U\) is a neighborhood of \((x, y)\) in \(G \times G\). Let \((a, b) \in U\). Then \(a \in S\) and \(b \in T\). By assumption,
\[
d(x \oplus y, a \oplus b) \leq d(x, a) + d(y, b) < \varepsilon.
\]
Hence, \(A(a, b) = a \oplus b \in B(x \oplus y, \varepsilon) \subseteq V\) and so \(A(U) \subseteq V\). This proves that \(A\) is continuous.

Denote by \(t\) the inversion map of \(G\). By the right-gyrotranslation inequality,
\[
d(t(x), t(y)) = d(\ominus x, \ominus y) \\
\leq d(\ominus e, \ominus y \oplus x) \\
= d(y \oplus e, y \oplus (\ominus y \oplus x)) \\
= d(x, y)
\]
for all \(x, y \in G\). This also implies that \(d(x, y) \leq d(t(x), t(y))\) for all \(x, y \in G\) because \(t = t^{-1}\). Thus, \(d(t(x), t(y)) = d(x, y)\) for all \(x, y \in G\) and so \(t\) is an isometry of \(G\). Hence, \(t\) is continuous. This proves that \(G\) is a topological gyrogroup. \(\square\)
According to Theorem 2.18 of [5], the group-norm of a group $\Gamma$ is abelian, that is,

$$\|gh\| = \|hg\|$$

for all $g, h \in \Gamma$ if and only if the metric induced by this group-norm is bi-invariant. This motivates the following theorem for normed gyrogroups:

**Theorem 4.12.** Let $G$ be a normed gyrogroup with the corresponding metric $d$. Then the following conditions are equivalent:

1. **Commutative-like condition:** $\|(a \oplus x) \oplus \text{gyr}[a, x](y \ominus a)\| = \|x \oplus y\|$ for all $a, x, y \in G$.

2. **Bi-gyrotranslation invariant:** $d(x \oplus a, y \oplus a) = d(x, y) = d(a \oplus x, a \oplus y)$ for all $a, x, y \in G$.

**Proof.** Let $a, x, y \in G$. Direct computation shows that

$$d(x \oplus a, y \oplus a) = \|\ominus (x \oplus a) \ominus (y \oplus a)\|$$

$$= \|\text{gyr}[x, a](\ominus a \ominus x) \ominus (y \oplus a)\|$$

$$= \|\ominus (a \ominus x) \ominus \text{gyr}[a, x](y \ominus a)\|$$

$$= \|\ominus a \ominus x \ominus \text{gyr}[a, x](y \ominus (\ominus a))\|$$

$$= \|\ominus x \ominus y\|$$

$$= d(x, y). \quad (4.7)$$

Hence, $d$ is invariant under right gyrotranslation. By Theorem 4.1, $d$ is invariant under left gyrotranslation as well. Conversely, computation as in (4.7) with $\ominus a$ in place of $a$ and $\ominus x$ in place of $x$ gives

$$\|x \ominus y\| = d(\ominus x, y) = d(\ominus x \ominus a, y \ominus a) = \|(a \ominus x) \ominus \text{gyr}[a, x](y \ominus a)\|. \quad \square$$

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**References**

[1] T. Abe, *Gyrometric preserving maps on Einstein gyrogroups, Möbius gyrogroups and proper velocity gyrogroups*, Nonlinear Funct. Anal. Appl. **19** (2014), 1–17.

[2] T. Abe and O. Hatori, *Generalized gyrovector spaces and a Mazur–Ulam theorem*, Publ. Math. Debrecen **87** (2015), no. 3-4, 393–413.
[3] T. Abe and K. Watanabe, \textit{Finitely generated gyrovector subspaces and orthogonal gyrodecomposition in the Möbius gyrovector space}, J. Math. Anal. Appl. \textbf{449} (2017), 77–90.

[4] W. Atiponrat, \textit{Topological gyrogroups: Generalization of topological groups}, Topology Appl. \textbf{224} (2017), 73–82.

[5] N. Bingham and A. Ostaszewski, \textit{Normed versus topological groups: Dichotomy and duality}, Dissertationes Math. \textbf{472} (2010), 1–138.

[6] Z. Cai, S. Lin, and W. He, \textit{A note on paratopological loops}, Bull. Malays. Math. Sci. Soc., DOI: 10.1007/s40840-018-0616-y.

[7] O. Demirel, \textit{A characterization of Möbius transformations by use of hyperbolic triangles}, J. Math. Anal. Appl. \textbf{398} (2013), no. 2, 457–461.

[8] O. Demirel and E. S. Seyrantepe, \textit{A characterization of Möbius transformations by use of hyperbolic regular polygons}, J. Math. Anal. Appl. \textbf{374} (2011), no. 2, 566–572.

[9] M. Ferreira, \textit{Harmonic analysis on the Möbius gyrogroup}, J. Fourier Anal. Appl. \textbf{21} (2015), no. 2, 281–317.

[10] R. Fleming and J. Jamison, \textit{Isometries on Banach spaces: Function spaces}, Monographs and Surveys in Pure and Applied Mathematics, vol. 129, Chapman & Hall/CRC, Boca Raton, FL, 2003.

[11] H. Haruki and Th. M. Rassias, \textit{A new characteristic of Möbius transformations by use of Apollonius points of triangles}, J. Math. Anal. Appl. \textbf{197} (1996), 14–22.

[12] \textit{A new characteristic of Möbius transformations by use of Apollonius quadrilaterals}, Proc. Amer. Math. Soc. \textbf{126} (1998), no. 10, 2857–2861.

[13] \textit{A new characterization of Möbius transformations by use of Apollonius hexagons}, Proc. Amer. Math. Soc. \textbf{128} (2000), no. 7, 2105–2109.

[14] S. Kim and J. Lawson, \textit{Unit balls, Lorentz boosts, and hyperbolic geometry}, Results Math. \textbf{63} (2013), 1225–1242.

[15] V. Klee, \textit{Invariant metrics in groups (solution of a problem of Banach)}, Proc. Amer. Math. Soc. \textbf{3} (1952), 484–487.

[16] P. Niamsup, \textit{A note on the characteristics of Möbius transformations}, J. Math. Anal. Appl. \textbf{248} (2000), 203–215.
[17] _______, A note on the characteristics of Möbius transformations. II, J. Math. Anal. Appl. 261 (2001), 151–158.

[18] Th. M. Rassias, On the A. D. Aleksandrov problem of conservative distances and the Mazur-Ulam theorem, Proceedings of the Third World Congress of Nonlinear Analysts, Part 4 (Catania, 2000), vol. 47, 2001, pp. 2597–2608.

[19] _______, On the Aleksandrov problem for isometric mappings, Appl. Anal. Discrete Math. 1 (2007), 18–28.

[20] Th. M. Rassias and P. Šemrl, On the Mazur-Ulam theorem and the Aleksandrov problem for unit distance preserving mappings, Proc. Amer. Math. Soc. 118 (1993), no. 3, 919–925.

[21] Th. M. Rassias and S. Xiang, On mappings with conservative distances and the Mazur-Ulam theorem, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. 11 (2000), 1–8.

[22] T. Suksumran, Essays in mathematics and its applications: In honor of Vladimir Arnold, Th. M. Rassias and P. M. Pardalos (eds.), ch. The Algebra of Gyrogroups: Cayley’s Theorem, Lagrange’s Theorem, and Isomorphism Theorems, pp. 369–437, Springer, Switzerland, 2016.

[23] _______, Gyrogroup actions: A generalization of group actions, J. Algebra 454 (2016), 70–91.

[24] _______, Involutive groups, unique 2-divisibility, and related gyrogroup structures, J. Algebra Appl. 16 (2017), no. 6, 1750114 (22 pages).

[25] _______, Modern discrete mathematics and analysis, N. J. Daras and Th. M. Rassias (eds.), Springer Optimization and Its Applications, vol. 131, ch. Cauchy’s Functional Equation, Schur’s Lemma, One-Dimensional Special Relativity, and Möbius’s Functional Equation, pp. 389–396, Springer, Cham, 2018.

[26] T. Suksumran and K. Wiboonton, Möbius’s functional equation and Schur’s lemma with applications to the complex unit disk, Aequat. Math. 91 (2017), no. 3, 491–503.

[27] A. Ungar, Thomas rotation and the parametrization of the Lorentz transformation group, Found. Phys. Lett. 1 (1988), 57–89.

[28] _______, Analytic hyperbolic geometry and Albert Einstein’s Special Theory of Relativity, World Scientific, Hackensack, NJ, 2008.
[29] ______, From Möbius to gyrogroups, Amer. Math. Monthly 115 (2008), no. 2, 138-144.

[30] ______, Parametric realization of the Lorentz transformation group in pseudo-Euclidean spaces, J. Geom. Symmetry Phys. 38 (2015), 39-108.