On the Theory of the Spectrum of 2 x 2 Hyperbolic Systems of the Second Order

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Abstract

The work is devoted to the investigation of the spectral characteristics of some boundary value problems for systems of linear differential equations in partial derivatives. The need to study the properties of the solvability of linear differential equation systems in partial derivatives occurs every time during the corresponding study of natural phenomena and processes. Important applications of the theory of systems of partial differential equations and problems associated with the study of the properties of the solvability of boundary value problems formulated to stimulate research relevant spectral problems. In the middle of the closed differential operator \( L : \mathcal{H}_{t,x} \to \mathcal{H}_{t,x} \) generated by the Dirichlet studied spectra: continuous \( \sigma_{L} \) and the residue \( R_{\sigma L} \) spectra of closed \( L : \mathcal{H}_{t,x} \to \mathcal{H}_{t,x} \) form an empty set: \( \sigma_{L} \cap R_{\sigma L} = \emptyset \). The point spectrum \( P_{\sigma L} \) of operator \( L : \mathcal{H}_{t,x} \to \mathcal{H}_{t,x} \) is located on the real axis of the complex plane \( C \). Own vector function of \( L \)-operator form a Riesz basis in the Hilbert space \( \mathcal{H}_{t,x} \).

Keywords: Basis, Boundary Value Problems, Closed Operators, Hyperbolic Systems, Orthogonal Basis, Riesz Basis, Spectrum

1. Introduction

The author’s work is devoted to the study of the spectral properties and characteristics of the two systems of linear partial differential equations of second order of a distinguished variable \( t \), considered in the closure \( V_{t,x} \) of the limited area \( \Omega_{t,x} \) of Euclidean space \( \mathbb{R}^{2} \). We study the system of equations are conveniently written in the form of so-called operator or an operator differential equation of the form (1):

\[
L(D_{t}^{2}, B) = a \cdot D_{t}^{2} + b \cdot Bu = f.
\]

Here, \( a, b \) - the size of the matrix 2 x 2; \( D_{t} \) operation of differentiation of variable \( t \). Closed operator \( B \) acts in a separable complex Hilbert space \( \mathcal{H} \) and meets certain requirements, formulated in terms of the spectral theory of operators. Adding system of conditions (2) to the Equation (1) is studied on a finite interval \( V_{t} = [0, \pi] \):

\[
G_{t} = 0 \quad (2)
\]

It describes the behaviour of the function \( u(t) \) at \( 0, \pi \) we obtain a boundary value problem for the solution of which we understand the strong solution. Having determined the (generalized) solution of the boundary problem (1), (2), we obtain a closed differential \( L \)-operator, acting appropriately chosen function space \( \mathcal{H}_{t,x} \). Under the spectral characteristics of the boundary value problem (1), (2) we understand the spectral properties of the closed differential operator \( L : \mathcal{H}_{t,x} \to \mathcal{H}_{t,x} \) generated by the corresponding boundary value problem under study. In the future, as a condition for the solvability and properties of solutions of boundary value problems studied or described in terms of the properties of the resolvent \( R_{\lambda} \) or the terms of the spectral properties of vector-functions of a closed \( L \)-operator, being mapped to the study of the boundary problem (1), (2).

The need for the study of systems of linear differential equations in partial derivatives arises in the study of a variety of physical, chemical, biological, social, economic (especially topical at the moment). For example, systems of S.L. Sobolev in the case of a compressible fluid - in...
hydrodynamics system of equations of mixed type in transonic gas dynamics occur. For the study of such systems of equations and also lead many topical problems of the theory of small deformations of the surface of revolution, membrane theory of shells with variable curvature of the plate, plate bending theory of variable thickness from an acute angle. Important applications of the theory of systems of partial differential equations and problems associated with the study of the properties of the solvability of boundary value problems stimulated the study of the corresponding spectral problems.

Spectral theory of closed operators generated by boundary value problems for equations and systems of partial differential equations, began to develop only recently in a number of works by Russian and foreign mathematicians. We studied at the same time as the asymptotic behaviour of the eigenvalues and the location of the spectrum on the complex plane, and the basic properties of systems composed of vector-functions. Investigation of the structure of the spectrum and the possibility of expanding solutions sets of vector-functions is now one of the main directions in the study of the spectrum of the theory of boundary value problems for systems of differential equations in partial derivatives. Despite considerable interest in the said problems, it has not yet developed a method to answer questions, even for the simplest systems of partial differential equations when more than two variables; the general problems of the spectral theory of boundary value problems for systems of linear differential equations in partial derivatives are not fully understood. Largely this applies to systems not related to the classic types: elliptical, hyperbolic, and parabolic. Given the importance of the properties of non-classical boundary problems for systems of linear differential equations in partial derivatives, the study of the spectral characteristics of the latter is quite important.

The theory of boundary value problems for systems of linear differential equations in partial derivatives, with a variety of applications based on multiple methods (asymptotic, variation, projection, numerical methods, methods of integral equations, functional and others) and forms (successive approximation compressive display various forms of integrated transforms, spectral, etc.) study. In this regard, we note that the study is based on methods that are called functional properties as described in terms of solvability of the spectral theory of linear closed operators.

2. Method

The idea of the research method belongs to A.A. Dezin and is as follows. Assume that \( H ; H'' \) is a pair of separable Hilbert spaces, each of which is given orthonormal basis \( \{ \varphi_k \}_{k=1}^{\infty} \) and \( \{ \psi_k \}_{k=1}^{\infty} \). Let us form the Hilbert space \( H \) as follows. The basis shall be \( H \) the set of ordered pairs \( \varphi_k \otimes \psi_j \) defined for these pairs of scalar product rule:

\[
( \varphi_k \otimes \psi_j , \varphi_l \otimes \psi_m ) = (\varphi_k, \varphi_l ) (\psi_j, \psi_m )
\]

(1*)

where the right - the scalar product in \( H ; H'' \) respectively. Thus, relative to the norm generated by the scalar product of (1) basis \( \{ \varphi_k \otimes \psi_j \} \) - is orthonormal. Product (1*) applies in the usual way on finite linear combinations

\[
\sum f_{ik} \varphi_k \otimes \psi_j
\]

(2*)

Replenishment of the introduced norm of the set of finite linear combinations (2*) to give (full) Hilbert space \( H = H' \otimes H'' \) - is tensor product of Hilbert spaces source.

In accordance with the design for any pair of elements of \( f = \sum f_k \varphi_k \in H' \) and \( g = \sum g_k \psi_k \in H'' \) determined by their tensor product

\[
f \otimes g = \sum f_k g_k \varphi_k \otimes \psi_k,
\]

(As \( \sum_{i,k} |f_k|^2 + |g_k|^2 < \infty \).

If now \( A' : H' \rightarrow H'' \) - closed linear operator with dense domain \( D(A') \otimes \varphi_k \in D(A') \) for any \( k \) and operator \( A' : H' \rightarrow H'' \) has similar properties to that of the dense \( H \) many elements of the form (2*) (on the set of finite linear combinations) the operator

\[
A' \otimes A'' \left( \sum f_{ik} \varphi_k \otimes \psi_j \right) = \sum f_{ik} A' \varphi_k \otimes A'' \psi_j
\]

Closure in \( H \) thus given operator \( A' \otimes A'' \) (with dense domain) is defined by the operator \( A' \otimes A'' : H \rightarrow H \).

If \( H = H' \otimes H'' \) and \( H', H'' \) - functional spaces, then \( H' \) can be naturally embedded into \( H \) by identifying a subset of \( H' \otimes 1 \) (composed of elements of the form \( f \otimes 1 \), \( f \in H' \)). Due to the above, the elements \( H' \) often regarded as elements of \( H \) without any reservations (and without going into the notation from \( f \) to \( f \otimes 1 \)). The situation is similar with operators \( A' : H' \rightarrow H'' \) : They identify with the operators of the form \( A' \otimes 1 \).

The above construction occurs naturally whenever \( H', H'' \) - our standard Hilbert space of functions over some
areas \( V' \subset \mathbb{R}^n \), \( V'' \subset \mathbb{R}^n \). Then H- adequate space above \( V' \times V'' \) At that operation \( L'(D) \otimes L''(D) \) are written into the corresponding operator usually just as 
\[
\sum a_i(x) b_i(y) D_i^\alpha D_j^\beta
\]
i.e. without the use of designations \( \otimes \) for the tensor product.

Since the transition in the Hilbert space from Riesz basis to an orthonormal basis is tantamount to replacing the back of the scalar product for an equivalent, it is clear that these are naturally subject to review on a case where \( \{\varphi_i\}, \{\psi_j\} \)- Riesz basis in \( H', H'' \).

Moving from \( H = H' \otimes H'' \) to the case of an arbitrary number of factors \( \mathcal{H} = \bigotimes_{k=1}^n \mathcal{H}^k \) automatically.

Convenience class operators, functions of which allow a very simple definition, which is M-operators. Indeed, if \( A : H \to H \) there is M-operator \( \{\varphi_k\} \)- the system of own functions of \( A \), form a Riesz basis and can, therefore, for any

\[
u \in D(A) \text{ record } u = \sum u_k \varphi_k, \quad Au = \sum u_k A \varphi_k = \sum \lambda_k u_k \varphi_k,
\]

then, assuming that, for example, that \( F(z) \) is analytic in \( \Omega \subset \mathbb{C} \) such that \( \lambda_k \in \Omega \) for all \( k \), it is enough to put 
\[
F(A)u = \sum F(\lambda_k) u_k \varphi_k
\]

Where in \( u \in D(F_k) \) whenever the series (3*) converges. The domain of the operator \( F(A) \) knowingly dense (it contains all finite linear combinations of elements of the basis).

The difficulties encountered when trying to use the reduced ideal scheme to specific situations that arise in the analysis of boundary value problems that are usually associated, on the one hand, the complex nature of the respective functions \( F(z) \), and the other - the desire to include consideration of operators and non-M-operators.

This scheme shall be distributed immediately to the case \( \mathcal{H} = \bigotimes_{k=1}^n \mathcal{H}^k \) and \( A^k : \mathcal{H}^k \to \mathcal{H}^k \) and \( F(z_1, \ldots, z_n) \) - Function of \( n \) complex variables, satisfies the relevant requirements. Operators \( A^k \) assumed that, of course, commuting.

### 3. Results and Conclusion

#### 3.1 Study of the Spectral Characteristics of the Two Boundary Value Problems

The paper presents a comparative study of the spectral characteristics of the two boundary value problems considered in a bounded domain finite-dimensional Euclidean space. The study is conducted from the standpoint of operator-differential equations of second order in a dedicated variable \( t \). The simplest examples of classical systems of differential equations in partial derivatives, falling within the field of our considerations can serve as hyperbolic systems of the form (3):

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} + \alpha &\frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^2 u}{\partial t} + \gamma \frac{\partial^2 u}{\partial x} = \lambda u + f^1 \\
\frac{\partial^2 u}{\partial t^2} + \alpha &\frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^2 u}{\partial t} + \gamma \frac{\partial^2 u}{\partial x} = \lambda u + f^2
\end{align*}
\]

The system (3) variables \( \alpha, \beta, \gamma \)- are constants. From the perspective of the study of the spectral properties of systems of differential equations in partial derivatives of the situation becomes more complicated when \( \alpha(t,x), \beta(t,x), \gamma(t,x) \)- functions can be degenerate\(^4\) or irregular\(^1\) in the terminology proposed by A.A. Dezin\(^1\).

#### 3.2 Tensor Product of Hilbert Spaces and Closed Operators

Suppose we are given \( H', H'' \)- a pair of separable Hilbert spaces, each with its own set orthonormal basis \( \{\varphi_k\}_{k=1}^{\infty}, \{\psi_k\}_{k=1}^{\infty} \). We construct an abstract Hilbert space \( \mathcal{H} \) as follows. As a basis of the original Hilbert space \( H \) choose a set of ordered pairs of the form \( \varphi_k \otimes \psi_n \), denoted for these pairs scalar product by the formula (4):

\[
\langle \varphi_k \otimes \psi_n, \varphi_l \otimes \psi_m ; \mathcal{H} \rangle = \langle \varphi_k, \varphi_l ; \mathcal{H}' \rangle \cdot \langle \psi_n, \psi_m ; \mathcal{H}'' \rangle
\]

Where the right are the scalar product in Hilbert spaces \( \mathcal{H}' \) and \( \mathcal{H}'' \) respectively. Thus, relative to the norm generated by the scalar product (4) basis \( \{\varphi_k \otimes \psi_n\}_{k, n} \) is orthonormal. The scalar product (4) applies in the usual way on finite linear combinations (5):

\[
\sum_{k,j} \varphi_k \otimes \psi_j
\]
Supplementing by the introduced norm (4) of the set of finite linear combinations (5) gives the (full) completely new Hilbert space \( H = H' \otimes H'' \) - Tensor product of Hilbert spaces source. In accordance with the design for any pair of elements \( f = \sum f_k \varphi_k \in H', \ g = \sum g_k \psi_k \in H'' \) naturally defined by their tensor product (6):

\[
f \otimes g = \sum f_k g_k \varphi_k \otimes \psi_k ,
\]

since \( \sum |f_k |g_k | < \infty \).

If now \( A' : H' \rightarrow H' \) - closed linear operator with close to \( H \) domain \( D(A') \varphi_k \in D(A') \) for any \( k \) and operator \( A'' : H'' \rightarrow H'' \) has similar properties to that of the dense \( H \) set of elements of the form (5) (on the set of finite linear combinations) the operator (7):

\[
A' \otimes A'' \left( \sum f_k \varphi_k \otimes \psi_k \right) = \sum f_k A' \varphi_k \otimes A'' \psi_k
\]

Closure in \( H \) of thus given operator \( A' \otimes A'' \) (with dense domain) defines the operator \( A' \otimes A'' : \mathcal{H} \rightarrow H' \). If \( H = H' \otimes H'' \) and \( H', H'' \) - functional spaces, the \( H' \). It can be naturally embedded into \( m \) due to the identification of a subset \( H' \otimes 1 \). Due to the above, the elements \( H' \) often regarded as elements \( H \) without any reservations (and without the transition from \( f \) to \( f \otimes 1 \)).

The situation is similar with operators \( A' : H' \rightarrow H' \); they identify with the operators of the form: \( A' \otimes 1 \). The above construction occurs naturally whenever \( H' , H'' \) - our standard Hilbert space over some areas \( V' \subset \mathbb{R}^n \), \( V'' \subset \mathbb{R}^n \). Then \( H \) - adequate space above \( V' \times V'' \) the corresponding operation \( L(D) \otimes L'(D) \) usually just decided to write in the form (8):

\[
\sum a_{ij}(x)b_{ij}(y)D_x^i D_y^j
\]

that is, without the use of designations \( \otimes \) for the tensor product.

Since the transition from the Hilbert space Riesz basis to an orthonormal basis and is tantamount to replacing the back of the scalar product for an equivalent, it is clear that consideration of the above subject naturally to the case when \( \{ \varphi_k \otimes \psi_j \} - \) Riesz basis in \( H', H'' \). Moving from \( H = H' \otimes H'' \) to the case of an arbitrary number of factors \( \mathcal{H} = \bigotimes^k \mathcal{H} \) automatically.

The work is dedicated to the study and description of the spectral properties of a closed and at the same time unlimited differential operator generated by the Dirichlet problem for the \( 2 \times 2 \) hyperbolic system of the form (9), considered in the closure \( V_{ux} \) restricted area \( \Omega_{ux} \) Euclidean space \( \mathbb{R}^2_{x,t} \):

\[
\frac{\partial^2 u}{\partial t^2} + Bu = \lambda u + f
\]

In a heterogeneous system (9) \( B \) symbol appears or some number or some closed operator. Where \( u \) is a vector function: \( u = u(t,x) \) \( u = (u^1, u^2)^T \), adding to the system of equations (9) Dirichlet conditions (10), we obtain the problem (9), (10)

\[
u|_{\partial \Omega_{ux}}.
\]

In the future, for the sake of simplicity, we assume \( \Omega_{ux} = (0,\pi)^2 \) the need to study the properties of the solvability of systems of linear differential equations in partial derivatives arises in the study of various economic, physical, chemical, biological, and social processes and phenomena. Studying the properties of the solvability of boundary value problems for systems of linear differential equations in partial derivatives is quite a challenge. Private and quite important aspect of this study is to describe the task of the spectral properties of the studied systems of linear differential equations in partial derivatives. Largely this applies to \( m \times m \) systems of partial differential equations of order \( n \) elliptic, parabolic, and hyperbolic.

It is known, for example, that the hyperbolic system (Godunov, 1979) of “the simplest form” (11)

\[
\frac{\partial u}{\partial t} + \frac{1}{\rho_0} \frac{\partial p}{\partial x} = 0
\]

\[
\frac{\partial p}{\partial t} + \rho_0 \sigma_0^2 \frac{\partial u}{\partial x} = 0
\]

It describes the propagation of plane sound waves (small perturbations) in the stationary medium. Where \( u \) is velocity of the disturbed environment, \( p \) is pressure in this environment. Constant \( \rho_0 \), \( c_0 \) are associated with the properties of a medium at rest: \( \rho_0 \) - Its density, and \( c_0 \) - constant characterizing the compressibility. Further, for example, an important role is played by the system of equations of mixed type arising naturally in transonic gas dynamics. For the study of such systems of equations, also lead many urgent problems of the theory of small deformations of surfaces of revolution, membrane theory of shells with variable curvature sign, as well as the theory of bending of a plate of variable thickness from an acute
angle. It naturally raises the question of the “correct” operator of boundary value problems for systems of linear differential equations in partial derivatives. The solution to this problem lies in the way the study of the spectral properties of the model boundary value problems is widely used in the work\textsuperscript{2,7,16}. Denote \( e_k = (\delta_k^1, \delta_k^2) \); \( k = 1,2 \); orthonormal basis of the Euclidean space \( \mathbb{E}^2 \); Consisting of the column vectors, and the symbol \( \mathcal{U}^2 \). Unitary space elements \( u = u^1 e_1 + u^2 e_2 \), scalar product which is given by: \( (u, v) = u^1 v^1 + u^2 v^2 \), Suppose also that, as is common, \( \mathcal{H}^2 \) - Hilbert space of complex vector-valued functions, the rate of which is given by (12):

\[
[u; \mathcal{H}^2] = \int \int [u(\tau, \xi), \mathcal{U}^2] d\tau d\xi
\]

Assume also that \( \mathcal{D} \) - Linear set of smooth complex vector functions \( u = u(t, x) \) belonging to the class \( C(V_{t,x}) \cap C^2(\Omega t,x) \) and satisfy the conditions (10). We describe, as agreed, the spectral properties of the Dirichlet problem for a hyperbolic system (9). Denote \( L \) operator region is the set \( \mathcal{D} \) and the set of values defined by the right side of the inhomogeneous system (9), we obtain a hyperbolic operator; this operator is not closed. Applying in a Hilbert space \( \mathcal{H}^2 \), the standard procedure, we get a closed closure extension \( L \) of the differential operator \( L \). In this case we say that the closed operator \( L: \mathcal{H}^2_{t,x} \rightarrow \mathcal{H}^2_{t,x} \) generated by the problem (9), (10). We will study its spectrum and the spectral properties of its own vector functions. Speaking of the spectrum and the spectral properties of its own vector-valued functions we adhere to the terminology used in the monographs\textsuperscript{10,8}. For each fixed \( x \in (0, \pi) \) the general solution of the homogeneous system belonging inhomogeneous system (9) can be represented in the form (13) (14):

\[
u^1(t) = C_1 \sin(\sqrt{B + \lambda_1} t) + C_2 \cos(\sqrt{B + \lambda_1} t) + C_3 \sin(\sqrt{B + \lambda_2} t) + C_4 \cos(\sqrt{B + \lambda_2} t)
\]

\[
u^2(t) = -C_1 \sin(\sqrt{B + \lambda_1} t) - C_2 \cos(\sqrt{B + \lambda_1} t) + C_3 \sin(\sqrt{B + \lambda_2} t) + C_4 \cos(\sqrt{B + \lambda_2} t)
\]

Based on the Dirichlet conditions (2) and from (13), (14) we immediately obtain: \( C_1 = C_2 = 0 \). Next, for each fixed \( x \in (0, \pi) \) general solution of system (13), (14) satisfies the condition: \( u(0) = 0 \) represented in the form: (15) (16)

\[
u^1(t) = C_3 \sin(\sqrt{B + \lambda_2} t) + C_4 \sin(\sqrt{B + \lambda_2} t)
\]

\[
u^2(t) = -C_3 \sin(\sqrt{B + \lambda_2} t) + C_4 \sin(\sqrt{B + \lambda_2} t)
\]

Solving \( 4 \times 4 \) system of linear algebraic equations: \( u(0) = u(\pi) = 0 \) we write the matrix \( M(\lambda) \) corresponding to the given system of equations. Matrix \( M(\lambda) \) has the form of (17):

\[
\begin{bmatrix}
0 & 1 & 0 & 1 \\
\sin \sqrt{B + \lambda_1} & \cos \sqrt{B + \lambda_1} & \sin \sqrt{B + \lambda_2} & \cos \sqrt{B + \lambda_2} \\
0 & -1 & 0 & 1 \\
-\sin \sqrt{B + \lambda_1} & -\cos \sqrt{B + \lambda_1} & \sin \sqrt{B + \lambda_2} & \cos \sqrt{B + \lambda_2}
\end{bmatrix}
\]

(17)

We now find the values of the parameter \( \lambda \) for which \( u(0) = u(\pi) = 0 \). We now note that the determinant \( \Delta(\lambda) \) of matrix \( M(\lambda) \) represented in the form: (18)

\[
\Delta(\lambda) = \sin(\sqrt{B + \lambda_1} \pi) \sin(\sqrt{B + \lambda_2} \pi) = 0
\]

(18)

Solving Equation (18) with respect to \( \lambda \) we can find the eigenvalues of \( L \).

We have:

\[
\lambda_{1,k} = k^2 - B; k \in N \quad \lambda_{2,k} = k^2 + B; \quad k \in N
\]

Own value \( \lambda_{1,k} \) of \( L \) operator belongs to a private vector function: \( u(t,x) = \sin(kt)(e_1 + e_2) \). Own value \( \lambda_{2,k} \) of \( L \) operator belongs to a private operator vector function: \( u(t,x) = \sin(kt)(e_1 + e_2) \). Further, assume that \( B \)-closed operator: \( B = \frac{d^2 u}{dx^2} + \frac{du}{dx} \) defined on \( (0, \pi) \) and satisfying the conditions of Dirichlet. We now formulate the final version in the form of the theorem (1).

**Theorem 1.** Spectrum \( \sigma(L) \) of \( L \) operator, generated by the (9), (10), consists of a closure \( \overline{\sigma(L)} \) in the complex plane of the point spectrum \( P = \sigma_L \) a plurality of \( \sigma(L) = \sigma(L) \setminus P \) \( L \) forms a continuous spectrum of the \( L \) operator. The point spectrum of the \( L \) operator is given by (19):

\[
\lambda_{m,k,s} = k^2 + (-1)^m \cdot B(s) \; B(s) = \frac{-s^2}{2}, \; m = 1,2; k \in N, s \in N.
\]

(19)

Vector-function of the \( L \) operator, owned his own value (19) can be represented in the form (20):

\[
u_{m,k,s}(t,x) = \sin(kt)(e_1 + (-1)^{m+1} e_2) e^{\frac{x^2}{2}} \sin(sx)
\]

(20)

Sequence \( \{\nu_{m,k,s}(t,x)\} \) of vector-functions forms a Riesz basis in the Hilbert space \( \mathcal{H}^2 \).

We now formulate the final version in the form of theorems (2). Further assume that \( B \) is closed operator:

\[
B = \frac{d^2 u}{dx^2} + \frac{du}{dx}
\]
Theorem 2. Spectrum $\sigma L$ of the $L$ operator, generated by the (9), (10), consists of a closure $\overline{\sigma L}$ on the complex plane of the point spectrum $P \sigma L$, plurality of $\sigma L = \sigma L \setminus \overline{\sigma L}$ forms a continuous spectrum of the $L$ operator. The point spectrum of the $L$ operator is given by (21):

$$
\lambda_{m,k,s} = k^2 + (-1)^m \cdot B(s); \quad B(s) = \left\{ -\frac{1}{4} \cdot s^2 \right\}; \quad m = 1, 2; k \in \mathbb{N}, s \in \mathbb{N}.
$$

Vector-function of the $L$ operator, owned his own value (21) can be represented in the form (22):

$$
u_{m,k,s}(t,x) = \sin(kt)(e_i + (-1)^m e_j)e^{-\frac{s}{2}} \sin(sx).$$

Theorem 3. Spectrum $\sigma L$ of the $L$ operator, generated by the (23) (3), consists of a closure $\overline{\sigma L}$ on the complex plane of the point spectrum $P \sigma L$. Plurality $\sigma L \setminus P \sigma L$ forms a continuous spectrum of the $L$ operator. The point spectrum of the $L$ operator is given by:

$$
\lambda_{m,k,s} = -k^2 + (-1)^m \cdot s^2; \quad m = 1, 2; k \in \mathbb{N}, s \in \mathbb{N}.
$$

Vector-function of the $L$ operator, owned his own value $\lambda_{m,k,s}$, represented in the form:

$$
u_{m,k,s}(t,x) = (e_i + (-1)^m e_j)\sin(kt)\sin(sx).$$

3.3 A Hyperbolic System without Lowest Terms

Denote with $L$ the operator whose domain is the set of $D$ and the set of values defined by the right part of the system (23), we obtain a hyperbolic differential operator; this operator is not closed. Applying $\mathcal{H}_{t,x}$ closure standard procedure, we obtain a closed extension $L$ of the operator $L$. In this case we say that the closed operator $L: \mathcal{H}_{t,x} \to \mathcal{H}_{t,x}$ generated by the problem (23) (3). We will study its spectrum and the spectral properties of its own vector functions. Resolvent set, spectrum, point spectrum, the continuous spectrum and residual spectrum of the $L$ operator we denote $\rho_L, \sigma_L, \sigma_P, \rho_P, \sigma_R, \rho_R$ respectively.

We describe the first spectral properties of the Dirichlet problem for a hyperbolic system (23) without the lowest terms.

Proof. It suffices to note that the sequence $u_{m,k,s}(t) = (e_i + (-1)^m e_j)\sin(kt)$ is complete and orthonormal $\mathcal{H}_t = \mathcal{H}_x \otimes \mathcal{H}_x$, $\mathcal{H}_t = L_2[0,2\pi]$, and use the representation $\mathcal{H}_t \otimes \mathcal{H}_x$ as a tensor product of Hilbert spaces $\mathcal{H}_t$ and $\mathcal{H}_x$ i.e. use formula $\mathcal{H}_{t,x} = \mathcal{H}_t^2 \otimes \mathcal{H}_x$, $\overline{\mathcal{H}_x} = L_2[0,2\pi]$.

Let us now consider a hyperbolic system of the form

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} &= \lambda u + f_1 \\
\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} &= \lambda u + f_2
\end{aligned}
$$

(25)

If the following Dirichlet conditions (26) are added to the system (25):

$$
u|_{x=0,\pi} = 0.$$

(26)

we will get the boundary problem (25), (26).

We are interested in the spectral properties of the
closed differential operator generated by the Dirichlet problem (25), (26). Let us build this operator. Denote \( \varepsilon_i = (\delta_i^1, \delta_i^2); i = 1, 2; \) orthonormal basis of the Euclidean space \( \mathcal{E}_2^1 \). Consisting of the column vectors, and the symbol \( \mathcal{U}_2^2 \) - Unitary space elements \( u = u^e_i + u^e_j; u^e \in \mathbb{C} \) with the scalar product \( \langle u, v, \mathcal{U}_2^2 \rangle \). Assume also that \( \mathcal{H}_{L}^{t,x} = L_2^2(\mathbb{C}^2) \) - Hilbert space of complex vector-valued functions \( u: V_{t,x} \rightarrow \mathbb{C}^2 \), the rate of which is given by:

\[
[u; \mathcal{H}_{L}^{t,x}]^2 = \int_{V_{t,x}} |u(t, \xi); \mathcal{U}_2^2|^2 \, d\tau d\xi
\]  

(27)

Assume also that \( D \) - Linear plurality of smooth complex vector functions \( u = u(t, x) \) belonging to the class \( C(\bar{\Omega}_{t,x}) \cap C^{(2)}(\Omega_{t,x}) \) and satisfying the Dirichlet conditions (26). We now describe the spectral properties of the hyperbolic system of the form (25). Denoting with the symbol \( L \) operator which domain is the set of smooth complex vector functions from plurality of \( D \), and the set of values defined by the right part of the system (25), we obtain a hyperbolic operator; this operator is not closed. Applying in \( \mathcal{H}_{L}^{t,x} \) standard procedure, we get a closed closure extension \( L \) of the differential operator \( L \). In this case, we say that the closed unbounded differential operator \( L: \mathcal{H}_{L}^{t,x} \rightarrow \mathcal{H}_{L}^{t,x} \) is generated by the problem (25), (26). We study the structure of its spectrum and the spectral properties of its own complex-vector functions.

**Theorem 4.** Spectrum \( \sigma \) of \( L \) operator, generated by the (25) (26), consists of a closure \( \mathcal{F}L \) on the complex plane of the point spectrum \( P \sigma L \). Plurality \( C \sigma L = \sigma L \) of \( P \sigma L \) forms a continuous spectrum of the \( L \) operator. The point spectrum of \( L \) operator is given by (28):

\[
\lambda_{m,k,s} = k^2 + (-1)^m \left( \frac{1}{4} + s^2 \right); m = 1, 2; k \in \mathbb{N}, s \in \mathbb{N}.
\]  

(28)

Vector-function of the \( L \) operator, owned by own value is represented in the form:

\[
u_{m,k,s}(t, x) = \sin(k t) \sin(s x) \left( e_i + (-1)^m e_j \right) e^{|x|^2} \]  

(29)

Sequence \( \{u_{m,k,s}(t, x) : m = 1, 2, k \in \mathbb{N}, s \in \mathbb{N} \} \) vector-functions of the \( L \) operator forms a Riesz basis in the Hilbert space \( \mathcal{H}_{L}^{t,x} \).

**Proof.** It is enough to note that the sequence \( \{u_{m,k,s}(t, x) : m = 1, 2, k \in \mathbb{N}, s \in \mathbb{N} \} \) vector functions \( u_{m,k,s}(t) = \sin(k t) e_i + (-1)^m e_j \) is complete and orthogonal in Hilbert space \( \mathcal{H}_{L}^{t,x} = \mathcal{H}_{L}^{t} \oplus \mathcal{H}_{L}^{x} \), \( \mathcal{H}_{L}^{t} = L_2^2(0, 2\pi) \) and take advantage of known representation \( \mathcal{H}_{L}^{t,x} \) as a tensor product of Hilbert spaces \( \mathcal{H}_{L}^{t}, \mathcal{H}_{L}^{x} \) i.e. by the formula \( \mathcal{H}_{L}^{t,x} = \mathcal{H}_{L}^{t} \otimes \mathcal{H}_{L}^{x} \), where \( \mathcal{H}_{L}^{x} = L_2^2(0, 2\pi) \).

We now carry out a comparative study of the spectral properties of the Dirichlet problem for a hyperbolic system of equations (30) without the “lowest term”:

\[
\begin{align*}
\frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^2 u}{\partial x^2} &= \lambda u + f^1 \\
\frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^2 u}{\partial x^2} &= \lambda u + f^2
\end{align*}
\]  

(30)

and for a hyperbolic system of Equations with the “lowest term” (31)

\[
\begin{align*}
\frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} &= \lambda u + f^1 \\
\frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} &= \lambda u + f^2
\end{align*}
\]  

(31)

Considered in the closure \( V_{t,x} \) restricted area \( \Omega_{t,x} = (0, \pi)^2 \) Euclidean space \( \mathbb{R}^2_{t,x} \). Join the system of equations (30), (31) the Dirichlet conditions (10), we obtain two boundary problems: the problem (30), (10) and the problem (31), (10).

For hyperbolic systems and more common so-called symmetric and asymmetric systems, there are a number of deep results related to the description of the correct boundary conditions. Description of regular boundary value problems for more general systems of equations of the first order on a distinguished variable \( t \) when the number of variables is dedicated to the work of more than two (Romanko, 1986). Study of the properties of solvability of the Cauchy problem for the simplest hyperbolic system of first order “lenticular second region” is devoted to the work (Dezin, 1959). However, the spectral properties of the boundary-value problems and boundary value problems of a different type with the number more than two almost unknown. Spectral properties of irregular boundary value problems have been studied in alignment (Kornienko, 2000). Spectral properties of degenerate operator-differential equations were studied in (Kornienko, 2000).

Denote \( e_i = (\delta_i^1, \delta_i^2); i = 1, 2; \) orthonormal basis of the Euclidean space \( \mathcal{E}_2^1 \). Consisting of the column vectors, and the symbol \( \mathcal{U}_2^2 \) - Unitary space elements \( u = u^e_i + u^e_j; u^e \in \mathbb{C} \) with the scalar product \( \langle u, v, \mathcal{U}_2^2 \rangle = u^e v^e + u^e v^e \). Assume also that \( \mathcal{H}_{L}^{t,x} = L_2^2(\mathbb{C}^2) \).
- Hilbert space of complex vector-valued functions \( u: V_{t,x} \to \mathbb{C}^2 \), the rate of which is given by (32):

\[
|u, \mathcal{H}_{t,x}^I| = \int_{V_{t,x}} |u(\tau, \xi); \mathcal{H}_{t,x}^I| d\tau d\xi
\]  

(32)

Assume also that D- Linear manifold of smooth complex vector-valued functions belonging to the class \( C(V_{t,x}) \cap C^1(\Omega t,x) \) and satisfying the Dirichlet conditions (26). We now describe the spectral properties of the hyperbolic system (30) without a “lowest term” on the complex plane \( u \).

### 3.4 A Hyperbolic System (30) without a Lowest Term \( \frac{\partial u}{\partial t} \)

Denote \( \hat{L} \) operator whose domain is the set of D and the set of values defined by the right part of the system (30), we obtain a hyperbolic differential operator; this operator is not closed. Applying in \( \mathcal{H}_{t,x}^I \) closure standard procedure, we obtain a closed extension \( L \) of the operator \( L \). In this case we say that the closed operator \( L: \mathcal{H}_{t,x} \to \mathcal{H}_{t,x} \), generated by the problem (10), (30). We will study its spectrum and the spectral properties of its own vector functions.

Resolvent set, spectrum, point spectrum, the continuous spectrum and residual spectrum of the L operator we denote \( \rho L, \sigma L, \rho_0 L, \sigma_0 L, \rho_0 L \) respectively.

Just as in the articles (Kornienko, 2013), (Kornienko, 2014) proved a theorem.

**Theorem 5.** Spectrum \( \sigma L \) L operator, generated by the (10) (30), consists of a closure \( \overline{\rho \sigma L} \) on the complex plane of the point spectrum \( P \sigma L \). Plurality \( \sigma L \setminus P \sigma L \) forms a continuous spectrum of the L operator. The point spectrum of the L operator is given by:

\[
\lambda_{m,k,s} = (-1)^m \left[ -k^2 - \frac{1}{4} \right], B(s); B(s) = s^2; m = 1,2; k \in \mathbb{N}, s \in \mathbb{N}.
\]

Vector-function of the L operator, owned by own value represented in the form:

\[
[\, u_{m,k,s}(t,x) = \sin(kt) \sin(sx) (e_i - (-1)^m e_j)]
\]

Sequence \( \{u_{m,k,s}(t,x): m = 1,2; k \in \mathbb{N}, s \in \mathbb{N}\} \) vector functions

It is complete and orthogonal in Hilbert space \( \mathcal{H}_{t,x}^I = \mathcal{H}_{t}^I \oplus \mathcal{H}_{x}^I \mathcal{H}_{t}^I = \mathcal{L}_2(0,\pi) \) and take advantage of known representation \( \mathcal{H}_{t,x}^I \) as a tensor product of Hilbert spaces \( \mathcal{H}_{x}^I \) and \( \mathcal{H}_{x}^I \) i.e. by the formula \( \mathcal{H}_{t,x}^I = \mathcal{H}_{t}^I \otimes \mathcal{H}_{x}^I \) and \( \mathcal{H}_{x}^I = \mathcal{L}_2(0,\pi) \).

### 3.5 Hyperbolic System with a Lowest Term \( \frac{\partial u}{\partial t} \)

Just like in the case of a hyperbolic system without the lowest terms, designated by the symbol \( L \) operator whose domain is the set of D and the set of values defined by the right part of the system (31), we obtain a hyperbolic differential operator; this operator is not closed. Applying in \( \mathcal{H}_{t,x}^I \) closure standard procedure, we obtain a closed extension \( L \) of the operator \( L \). In this case we say that the closed operator \( L: \mathcal{H}_{t,x} \to \mathcal{H}_{t,x} \), generated by the problem (10), (31). We will study its spectrum and the spectral properties of its own vector functions.

**Theorem 6.** Spectrum \( \sigma L \) L operator, generated by the (10) (31), consists of a closure \( \overline{\rho \sigma L} \) on the complex plane of the point spectrum \( P \sigma L \). Plurality \( \sigma L \setminus P \sigma L \) forms a continuous spectrum of the L operator. The point spectrum of the L operator is given by:

\[
\lambda_{m,k,s} = (-1)^m \left[ -k^2 - \frac{1}{4} \right], B(s); B(s) = s^2; m = 1,2; k \in \mathbb{N}, s \in \mathbb{N}.
\]

Vector-function of the L operator, owned by own value represented in the form:

\[
[\, u_{m,k,s}(t,x) = \sin(kt) \sin(sx) (e_i - (-1)^m e_j)]
\]

Sequence \( \{u_{m,k,s}(t,x): m = 1,2; k \in \mathbb{N}, s \in \mathbb{N}\} \) vector functions

It is complete and orthogonal in Hilbert space \( \mathcal{H}_{t,x}^I = \mathcal{H}_{t}^I \oplus \mathcal{H}_{x}^I \mathcal{H}_{t}^I = \mathcal{L}_2(0,\pi) \) and take advantage of known representation \( \mathcal{H}_{t,x}^I \) as a tensor product of Hilbert spaces \( \mathcal{H}_{x}^I \) and \( \mathcal{H}_{x}^I \) i.e. by the formula \( \mathcal{H}_{t,x}^I = \mathcal{H}_{t}^I \otimes \mathcal{H}_{x}^I \) and \( \mathcal{H}_{x}^I = \mathcal{L}_2(0,\pi) \).

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