UNIFORM GLOBAL EXISTENCE AND CONVERGENCE OF EULER-MAXWELL SYSTEMS WITH SMALL PARAMETERS

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Abstract. The Euler-Maxwell system with small parameters arises in the modeling of magnetized plasmas and semiconductors. For initial data close to constant equilibrium states, we prove uniform energy estimates with respect to the parameters, which imply the global existence of smooth solutions. Under reasonable assumptions on the convergence of initial conditions, this allows to show the global-in-time convergence of the Euler-Maxwell system as each of the parameters goes to zero.

1. Introduction. In this paper we consider a compressible one-fluid Euler-Maxwell system, which arises in the modeling of plasmas consisting of electrons, of charge $q_e = -1$, and a single species of ions, of charge $q_i = 1$. More precisely, we want to prove the uniform global existence of smooth solutions near constant equilibrium states. We denote by $n$ and $u$ the scaled density and the velocity of the electrons, respectively, and by $E$ and $B$ the electric and the magnetic fields, respectively. These variables depend on the time $t \geq 0$ and the position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Let $\tau > 0$ be the momentum relaxation time, and let $\gamma > 0$ be the inverse of the speed of light. We consider the problem in $\mathbb{K}^3$, where $\mathbb{K} = \mathbb{R}$ for the Cauchy problem and $\mathbb{K} = T$, with $T = \mathbb{R}/\mathbb{Z}$, for the periodic problem. In plasma physics, ions and electrons are considered as two independent fluids. Let $n_i, u_i, \tau_i$ denote the scaled density, the velocity, and the relaxation time of ions, respectively, then the rescaled two-fluid Euler-Maxwell model reads (see [8, 3, 15]):

$$
\begin{align*}
\partial_t n + \text{div}(nu) &= 0, \\
\partial_t n_i + \text{div}(n_i u_i) &= 0, \\
\partial_t (n u) + \text{div}(nu \otimes u) + \nabla p(n) &= -n(E + \gamma u \times B) - \frac{nu}{\tau}, \\
\partial_t (n_i u_i) + \text{div}(n_i u_i \otimes u_i) + \nabla p(n_i) &= -n_i(E + u_i \times B) - \frac{n_i u_i}{\tau_i}, \\
\gamma \partial_t E - \text{curl}(B) &= \gamma(n u - n_i u_i), \\
\text{div}(E) &= n_i - n, \\
\gamma \partial_t B + \text{curl}(E) &= 0, \\
\text{div}(B) &= 0.
\end{align*}
$$

(1)

For convenience, we consider each term of order $O(1)$ as equal to 1. The dissipation terms $-\frac{nu}{\tau}$ and $-\frac{n_i u_i}{\tau_i}$ in the momentum equations are justified in [32].
When the ions are non-moving and become a uniform background with a fixed unit density, we obtain a one-fluid Euler-Maxwell model for electrons (see [9, 14, 18, 32, 35]):

$$
\begin{cases}
\partial_t n + \text{div}(nu) = 0, \\
\partial_t (nu) + \text{div}(nu \otimes u) + \nabla p(n) = -n(E + \gamma u \times B) - \frac{n u}{\tau}, \\
\gamma \partial_t E - \text{curl}(B) = \gamma nu, \quad \text{div}(E) = 1 - n, \\
\gamma \partial_t B + \frac{1}{\gamma} \text{curl}(E) = 0, \quad \text{div}(B) = 0,
\end{cases}
$$

for $t > 0$ and $x \in \mathbb{R}^3$. Here $p$ is the pressure function, supposed to be strictly increasing on $(0, +\infty)$. This system, when $n > 0$, is equivalent to

$$
\begin{cases}
\partial_t n + \text{div}(nu) = 0, \\
\partial_t u + (u, \nabla)u + \nabla h(n) = -(E + \gamma u \times B) - \frac{u}{\tau}, \\
\partial_t E - \frac{1}{\gamma} \text{curl}(B) = nu, \quad \text{div}(E) = 1 - n, \\
\partial_t B + \frac{1}{\gamma} \text{curl}(E) = 0, \quad \text{div}(B) = 0,
\end{cases}
$$

(2)

Here $h$ is the enthalpy function, defined by

$$
h(n) = \int_1^n \frac{p'(y)}{y} dy.
$$

Since $p$ is strictly increasing on $(0, +\infty)$, so is $h$. Thus, there exists a constant $\bar{\kappa} > 0$ such that $h'(n) \geq \bar{\kappa}$ for all $n \geq 1/2$.

The physical parameters $\tau$ and $\gamma$ can be chosen independently of each other. They are often very small compared to the physical size of the other quantities. Therefore, it is important to study the limits of system (2) as these parameters go to zero. The limit $\tau \to 0$ is called the zero-relaxation limit, and the limit $\gamma \to 0$ is called the non-relativistic limit.

We assume initial data for (2) can depend on the parameters. For the sake of simplicity, we denote them by

$$
(n, u, E, B)(t = 0) = (n_0^\varepsilon, u_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon),
$$

with $\varepsilon = \tau$ and $\varepsilon = \gamma$ in each of the studies, respectively.

For $n > 0$, the Euler-Maxwell system (2) is a symmetrizable hyperbolic system. Then the problem (2)-(3) has a unique local smooth solution when the initial data are smooth (see [21] and [26]). This solution is defined on a time interval $[0, T_*]$, with $T_* > 0$, but in general $T_*$ depends on the parameters $\tau$ and $\gamma$.

The asymptotic analysis at zero of the parameters are well known problems. The zero-relaxation limit $\tau \to 0$ and the non-relativistic limit $\gamma \to 0$ have been justified in [32] and [29], respectively. The results show that the limits of (2) are, respectively, a drift-diffusion system, and a compressible Euler-Poisson system. For each of these limits, the authors of [32] and [29] proved the validity of a local-in-time convergence of system (2) to the limit system, on a time interval independent of the parameter.

Another important problem concerns the global existence of smooth solutions when the parameters are fixed and the initial data are in a small neighborhood of the constant equilibrium state $(1, 0, 0, B_e)$, which is a particular solution of system (2), where $B_e \in \mathbb{R}^3$ is any given constant. In [32] (see also [12, 41, 33]), the authors
established a global existence result near such a constant equilibrium. This result did not take into account the dependence with respect to the parameters $\tau$ and $\gamma$.

The aim of the present paper is to establish, near a constant equilibrium state, the uniform global existence of a smooth solution and the convergence of (2)-(3) to these limit systems, with respect to each of the parameters $\tau$ and $\gamma$. For this purpose, we suppose that $(n_0, u_0, E_0, B_0)$ stays in a small neighborhood of $(1, 0, 0, B_{e})$, independent of the parameter $\varepsilon$. Under these assumptions, we establish uniform energy and time dissipation estimates with respect to the time and to the parameters. On one hand, these estimates imply the global existence of solutions staying in uniform neighborhoods of $(1, 0, 0, B_{e})$. On the other hand, the uniform estimates also imply the compactness of solution sequences. Hence, as one of the parameters goes to zero, we can pass to the limit globally in time in (2) to recover the limit equations.

For isentropic Euler equations with damping, the uniform global existence of smooth solutions and the zero-relaxation limit have been studied in [10, 25].

For the Euler-Poisson system (see [8, 28]), the zero-relaxation limit has been investigated by many authors. See, for instance, [27, 20] for global entropy solutions by the compensated compactness method in one space dimension, and [24, 23, 42] for local smooth solutions by the energy estimates in several space dimensions. Recently, in [34], the author justified the global-in-time convergence of the Euler-Poisson system in the zero-relaxation limit. For this purpose, the author proved a general energy estimate, including three parameters, before analyzing the consequence of this estimate separately for each parameter.

In the present paper, as in [34], we prove a key energy estimate (see (21)), including both parameters $\tau$ and $\gamma$, before showing the uniform and global-in-time existence of solutions for each of them. These main results are stated in theorems 2.1 and 2.3.

As a consequence, using a classical compactness argument (see [38]), we obtain a limit system as each of the parameters $\tau$ and $\gamma$ goes to zero. These results are described in theorems 2.2 and 2.4.

We remark that these four results are not trivial because system (2) does not satisfy the Shizuta-Kawashima condition (see [37]), which is a usual assumption to yield the global existence and asymptotic stability of smooth solutions to hyperbolic systems of balance laws (see [16, 43, 4, 36]). This condition is sufficient but not necessary to the global existence of solutions, as shown by examples given in [44, 6, 32].

Now let’s discuss the possible extension of the results 2.1-2.4 to the two-fluid Euler-Maxwell system (1). From the proof of theorem 2.1, it’s not obvious that energy estimates for Euler equations and Maxwell equations can be treated separately. Then, due to coupling terms of the system (1), the proof of an estimate similar to (21) seems lengthy and uses very different techniques from those used here. Therefore, we prefer to leave the problem of the global existence to the two-fluid Euler-Maxwell system for a future work.

An extension of theorems 2.1-2.4 to zero-electron-mass limit for Euler-Maxwell system is still an open problem (see [2] for a study on zero-electron-mass limit for Euler-Poisson system). We could also analyse the quasineutral limit, introduced in [8] (see also [31, 11] for various works on quasineutral limit), and a combination of the quasineutral and non-relativistic limits, justified in [30]. In these extensions,
the proof of an estimate similar to (21) seems to require further calculus. These extensions will be studied in a future work. We also refer to [39, 40, 32] and references therein for various analysis of Euler-Maxwell system with small parameters.

For later use in this paper, we recall some results on Moser-type calculus inequalities in Sobolev spaces, and the local existence of smooth solutions for symmetrizable hyperbolic systems. For any integer \( s \), we denote the usual spaces \( H^s(\mathbb{R}^3) \), \( L^2(\mathbb{R}^3) \) and \( L^\infty(\mathbb{R}^3) \), by \( H^s, L^2 \) and \( L^\infty \), respectively. Furthermore, we denote by \( \| \cdot \|_s \) the usual norm of \( H^s \), and by \( \| \cdot \| \) and \( \| \cdot \|_\infty \) the norms of \( L^2 \) and \( L^\infty \), respectively. For the sake of simplicity, we further denote that \( \| \cdot \| = \| \cdot \|_0 \). And for a multi-index \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3 \), we denote

\[
\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} \quad \text{with} \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3.
\]

**Lemma 1.1** (see [22, 26]). Let \( s \geq 1 \) be an integer. Suppose \( u \in H^s, \nabla u \in L^\infty \), and \( v \in H^{s-1} \cap L^\infty \). Then for every \( \alpha \in \mathbb{N}^3 \) with \( |\alpha| \leq s \), we have \( \partial^\alpha (uv) - u \partial^\alpha v \in L^2 \), and

\[
\| \partial^\alpha (uv) - u \partial^\alpha v \| \leq c_s \left( \| \nabla u \|_\infty \| D^{s-1} v \| + \| D^s u \| \| v \|_\infty \right),
\]

where \( c_s \) denotes a constant only depending on \( s \), and

\[
\| D^s u \| = \sum_{|\alpha|=s} \| \partial^\alpha u \|.
\]

In particular, when \( s \geq 3 \), a Sobolev inequality yields

\[
\| \partial^\alpha (uv) - u \partial^\alpha v \| \leq c_s \| \nabla u \|_{s-1} \| v \|_{s-1} \tag{4}
\]

**Proposition 1** (local existence of smooth solutions ; see [21, 26]). Let \( s \geq 3 \) and \( n_0^\varepsilon - 1, u_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon - B_e \in H^s \) with \( n_0^\varepsilon \geq 1/2 \). Then there exists \( T_\varepsilon > 0 \) such that the problem (2)-(3) has a unique smooth solution \( (n^\varepsilon, u^\varepsilon, E^\varepsilon, B^\varepsilon) \) satisfying

\[
n^\varepsilon - 1, u^\varepsilon, E^\varepsilon, B^\varepsilon - B_e \in C ([0, T_\varepsilon); H^s) \cap C^1 ([0, T_\varepsilon); H^{s-1}).
\]

This paper is organized as follows. In the next section, we state the main results of the paper, theorems 2.1-2.4, which consist of the uniform global existence of smooth solutions and their convergence as each of the parameters \( \tau \) and \( \gamma \) tends to zero. Section 3 is devoted to a key estimate, obtained by a series of energy estimates. The proof of theorems 2.1-2.4 are given in section 4. The last section includes the proof of two technical results.

2. **Main results.** For each of the parameters \( \tau, \gamma \in (0,1] \), we associate two results : one on the uniform global existence, and another one on the convergence of the solution as the parameter tends to zero. We state these results below, in order of the parameters \( \tau \) and \( \gamma \). A basic assumption on the initial data is

\[
n_0^\varepsilon - 1, u_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon - B_e \in H^s, \quad \text{with} \quad n_0^\varepsilon \geq 1/2,
\]

for \( \varepsilon = \tau \) or \( \gamma \).

2.1. **The case of the zero-relaxation limit \( \tau \to 0 \).**

**Theorem 2.1.** Let \( \gamma = 1 \) and \( s \geq 3 \) be an integer. There exist three constants \( \tau_0 > 0, \omega_0 > 0 \) and \( C_0 > 0 \), independent of \( \tau \) and any time, such that, for all \( \tau \in (0, \tau_0] \), if

\[
\| n_0^\varepsilon - 1 \|_s + \| u_0^\varepsilon \|_s + \| E_0^\varepsilon \|_s + \| B_0^\varepsilon - B_e \|_s \leq \omega_0,
\]
then the problem (2)-(3) admits a unique global solution \((n^\tau, u^\tau, E^\tau, B^\tau)\), satisfying
\[ n^\tau - 1, u^\tau, E^\tau, B^\tau - B_c \in C(\mathbb{R}^+; H^s) \cap C^1(\mathbb{R}^+; H^{s-1}). \]
Moreover, it holds,
\[
\|n^\tau(t) - 1\|_s^2 + \|u^\tau(t)\|_s^2 + \|E^\tau(t)\|_s^2 + \|B^\tau(t) - B_c\|_s^2 \\
+ \int_0^t \left[ \frac{1}{\tau} \|u^\tau(\xi)\|_s^2 + \tau \|n^\tau(\xi) - 1\|_s^2 \right] d\xi \\
\leq C_0 \left( \|n_0^\tau - 1\|_s^2 + \|u_0\|_s^2 + \|E_0^\tau\|_s^2 + \|B_0^\tau - B_c\|_s^2 \right), \quad \forall t \geq 0.
\]

**Theorem 2.2.** Let \((n^\tau, u^\tau, E^\tau, B^\tau)\) be the global solution given by theorem 2.1. We define a sequence with slow time by
\[
(\rho^\tau, v^\tau, F^\tau, Z^\tau)(t,x) = \left( n^{\tau, \frac{t}{\tau}}, E^\tau, B^\tau \right) \left( \frac{t}{\tau}, x \right).
\]
Assume that there exists \(n_0\) independent of \(\tau\) satisfying, as \(\tau \to 0\),
\[
n_0^\tau \to n_0, \quad \text{weakly in } H^s.
\]
Then there exist functions \(\bar{\rho}, \bar{F}, \bar{Z}\), with \(\bar{\rho} - 1, \bar{F}, \bar{Z} - B_c \in L^\infty(\mathbb{R}^+; H^s)\) and \(\bar{v} \in L^2(\mathbb{R}^+; H^s)\) such that, as \(\tau \to 0\), we have
\[
(\rho^\tau - 1, F^\tau, Z^\tau - B_c) \to (\bar{\rho} - 1, \bar{F}, \bar{Z} - B_c), \quad \text{weakly-* in } L^\infty(\mathbb{R}^+; H^s),
\]
\[
v^\tau \to \bar{v}, \quad \text{weakly in } L^2(\mathbb{R}^+; H^s),
\]
Moreover, for any \(T > 0\) and any \(s_1 \in (0, s)\), we have, as \(\tau \to 0\),
\[
\rho^\tau \to \bar{\rho}, \quad \text{strongly in } C([0,T]; H^{s_1}),
\]
and \((\bar{\rho}, \bar{\phi})\) is the unique global smooth solution of the following drift-diffusion system:
\[
\begin{cases}
\partial_t \bar{\rho} - \text{div} \left[ \bar{\rho} \left( \nabla h(\bar{\rho}) - \nabla \bar{\phi} \right) \right] = 0, \\
- \Delta \bar{\phi} = 1 - \bar{\rho},
\end{cases}
\]
with an initial condition
\[
\bar{\rho}(t = 0) = n_0.
\]
Note that \(\bar{\phi}\) is unique up to addition by a constant. In addition,
\[
\bar{Z} = \text{cst}, \quad \bar{F} = -\nabla \bar{\phi}, \quad \bar{v} = -\nabla \left[ h(\bar{\rho}) - \bar{\phi} \right].
\]

**2.2. The case of the non-relativistic limit \(\gamma \to 0\).**

**Theorem 2.3.** Let \(\tau = 1\) and \(s \geq 3\) be an integer. There exist two constants \(\varpi_1 > 0\) and \(C_1 > 0\), independent of \(\gamma\) and any time, such that, for all \(\gamma \in (0,1]\), if
\[
\|n_0^\gamma - 1\|_s + \|u_0\|_s + \|E_0^\gamma\|_s + \|B_0^\gamma - B_c\|_s \leq \varpi_1,
\]
then the problem (2)-(3) admits a unique global solution \((n^\gamma, u^\gamma, E^\gamma, B^\gamma)\) satisfying
\[ n^\gamma - 1, u^\gamma, E^\gamma, B^\gamma - B_c \in C(\mathbb{R}^+; H^s) \cap C^1(\mathbb{R}^+; H^{s-1}). \]
Moreover, it holds
\[
\|n^\gamma(t) - 1\|_s^2 + \|u^\gamma(t)\|_s^2 + \|E^\gamma(t)\|_s^2 + \|B^\gamma(t) - B_c\|_s^2 \\
+ \int_0^t \left[ \|u^\gamma(\xi)\|_s^2 + \|n^\gamma(\xi) - 1\|_s^2 \right] d\xi \\
\leq C_1 \left( \|n_0^\gamma - 1\|_s^2 + \|u_0\|_s^2 + \|E_0^\gamma\|_s^2 + \|B_0^\gamma - B_c\|_s^2 \right), \quad \forall t \geq 0.
\]
Theorem 2.4. Let \((n^\gamma, u^\gamma, E^\gamma, B^\gamma)\) be the global solution given by theorem 2.3. Assume that there exists \(n_0\) and \(u_0\) independent of \(\gamma\) satisfying, as \(\gamma \to 0\),
\[
(n_0^\gamma, u_0^\gamma) \to (n_0, u_0), \quad \text{weakly in } [H^s]^2.
\] (15)

Then there exist functions \(\bar{n}, \bar{u}, \bar{E}, \bar{B}\), with \(\bar{n} - 1, \bar{u}, \bar{E}, \bar{B} - B_e \in L^\infty(\mathbb{R}^+; H^s)\) such that, as \(\gamma \to 0\), we have
\[
(n^\gamma - 1, u^\gamma, E^\gamma, B^\gamma - B_e) \to (\bar{n} - 1, \bar{u}, \bar{E}, \bar{B} - B_e), \quad \text{weakly-ast in } [L^\infty(\mathbb{R}^+; H^s)]^4.
\] (16)

Moreover, for any \(T > 0\) and any \(s_1 \in [0, s]\), we have, as \(\gamma \to 0\),
\[
(n^\gamma, u^\gamma) \to (\bar{n}, \bar{u}), \quad \text{strongly in } [C([0, T]; H^{s_1})]^2,
\] (17)

and \((\bar{n}, \bar{u}, \bar{\phi})\) is the unique global smooth solution of the following Euler-Poisson system:
\[
\begin{cases}
\partial_t \bar{n} + \text{div}(\bar{n}\bar{u}) = 0, \\
\partial_t \bar{u} + (\bar{u}, \nabla)\bar{u} + \nabla h(\bar{n}) = \nabla \bar{\phi} - \bar{u}, \\
- \Delta \bar{\phi} = 1 - \bar{n},
\end{cases}
\] (18)

with an initial condition
\[
(\bar{n}, \bar{u})(t = 0) = (n_0, u_0).
\] (19)

Note that \(\bar{\phi}\) is unique up to addition by a constant. Furthermore, we have
\[
\bar{E} = -\nabla \bar{\phi} \quad \text{and} \quad \begin{cases}
\text{div}(\bar{B}) = 0, \\
\text{curl}(\bar{B}) = -\bar{n}\bar{u}.
\end{cases}
\] (20)

3. Uniform global estimates with respect to the parameters. Let \(\varepsilon = \tau\) or \(\varepsilon = \gamma\), with \(\varepsilon \in (0, 1]\). For simplicity, we drop the superscript \(\varepsilon\) in \((n^\varepsilon, u^\varepsilon, E^\varepsilon, B^\varepsilon)\). Set
\[
N = n - 1, \quad G = B - B_e, \quad W^I = \begin{bmatrix} N \\ u \end{bmatrix}, \quad W = \begin{bmatrix} N \\ u \\ E \\ G \end{bmatrix}.
\]

System (2) is symmetrizable hyperbolic, then proposition 1 can be applied. Let \(T > 0\) and \(W\) be a smooth solution of (2), defined on time interval \([0, T]\). Let \(s \geq 3\) be an integer. From now, we will denote by \(c > 0\) a generic constant independent of any time and the parameters \((\tau, \gamma)\).

In Section 2 we have presented four theorems, two for each limit study. For each limit, we need the following energy estimate to conclude the results.

For any \(\tau, \gamma \in (0, 1]\):
\[
\|W(t)\|^2_s + \int_0^t \left(\tau\|N(\xi)\|^2_s + \frac{1}{\tau}\|u(\xi)\|^2_s\right) d\xi \leq c\|W_0\|^2_s, \quad \forall t \in [0, T].
\] (21)

This estimate is similar to the one we obtain with Euler-Poisson system (see [34]) ; it is mostly due to the fact \(E\) and \(n\) are linked with \(\text{div}(E) = 1 - n\) in (2).

In the proof we use the following identity, satisfied for any vectors \(f, g \in \mathbb{R}^3\) :
\[
\text{curl}(f) \cdot g - \text{curl}(g) \cdot f = \text{div}(f \times g);
\] (22)

and we repeatedly use the continuous embedding \(H^{s-1} \hookrightarrow L^\infty\) and the following inequality
\[
\|z\|_{\infty} \leq c_{em}\|z\|_{s-1}, \quad \forall z \in H^s,
\] (23)

where \(c_{em} > 0\) is a constant.
In order to prove estimate (21), we introduce
\[ N_T = \sup_{0 \leq t \leq T} \left( \| N(t) \|_{s}^{2} + \| u(t) \|_{s}^{2} + \| E(t) \|_{s}^{2} + \| G(t) \|_{s}^{2} \right)^{1/2}, \]
and
\[ K_{\tau}(W^I) = \tau \| N \|_{s}^{2} + \frac{1}{\tau} \| u \|_{s}^{2}. \]
We also suppose that \( N_T \leq c \). The two following lemmas imply (21).

**Lemma 3.1.** We have, for any \( \tau, \gamma \in (0, 1] \),
\[
\frac{d}{dt} \left[ \sum_{0 \leq |\alpha| \leq s} \langle A_0(N) \partial^\alpha W^I, \partial^\alpha W^I \rangle + \| E \|_{s}^{2} + \| G \|_{s}^{2} \right] + \frac{1}{\tau} \| u \|_{s}^{2} \leq cN_T K_{\tau}(W^I). \]

**Lemma 3.2.** There exists a constant \( c_0 \), independent of \( (\tau, \gamma) \) and any time, satisfying
\[
\frac{d}{dt} \left[ \| N \|_{s-1}^{2} - c_0 \tau \sum_{0 \leq |\beta| \leq s-1} \langle \partial^\beta (\text{div} u), \partial^\beta N \rangle \right] + \tau \| \nabla N \|_{s-1}^{2} - c_0 \tau \| u \|_{s}^{2} \leq cN_T K_{\tau}(W^I). \]

The proofs of lemmas 3.1 and 3.2 are quite technical. There are developed in the last section. Multiplying the second result by any constant \( \kappa > 0 \) to be chosen, and adding it to the first result, we then integrate over \([0, t]\) to obtain
\[
\sum_{0 \leq |\alpha| \leq s} \langle A_0[N(t)] \partial^\alpha W^I(t), \partial^\alpha W^I(t) \rangle + \| E(t) \|_{s}^{2} + \| G(t) \|_{s}^{2} + \kappa \| N(t) \|_{s-1}^{2}
- c_0 \kappa \tau \sum_{0 \leq |\beta| \leq s-1} \langle \partial^\beta ([\text{div}(u(t))], \partial^\beta [N(t)] \rangle
+ \frac{1}{\tau} \int_{0}^{t} \left[ (1 - c_0 \kappa \tau^2) \| u(\xi) \|_{s}^{2} + \kappa \tau \| \nabla N(\xi) \|_{s}^{2} \right] d\xi
\leq c\| W(0) \|_{s}^{2} + cN_T \int_{0}^{t} K_{\tau} [W^I(\xi)] d\xi.
\]

Now, we use the fact \( A_0(N) \) is symmetric positive definite, and we take \( \kappa \) small enough to obtain
\[
\| W(t) \|_{s}^{2} + \int_{0}^{t} K_{\tau} [W^I(\xi)] d\xi \leq c \left[ \| W(0) \|_{s}^{2} + N_T \int_{0}^{t} K_{\tau} [W^I(\xi)] d\xi \right].
\] (24)

Then, taking \( N_T \) small enough, the last term on the right-hand side of (24) can be controlled by the left-hand side of (24), and we obtain (21).

4. **Consequences of estimate (21)**. Now we prove theorems 2.1-2.4. We consider each of the limits separately.

4.1. **The zero-relaxation limit.**

**Proof of theorem 2.1.** Recall that \( \gamma = 1 \) and \( \tau \in (0, 1] \) in the zero-relaxation limit. Then (5) is a direct consequence of estimate (21). It implies the global existence of solutions. \( \square \)
Proof of theorem 2.2. We first remark that, with the new scaling time (6), the system satisfied by $\rho^\tau$, $v^\tau$, $F^\tau$ and $Z^\tau$ is
\[
\begin{aligned}
\partial_t \rho^\tau + \divergence (\rho^\tau v^\tau) &= 0, \\
\tau^2 \left[ \partial_t v^\tau + (v^\tau, \nabla) v^\tau \right] + \tau (v^\tau \times Z^\tau) + F^\tau + \nabla h(\rho^\tau) &= -v^\tau, \\
\tau(\partial_t F^\tau - \rho^\tau v^\tau) &= \curl (Z^\tau), \\
\divergence (F^\tau) &= 1 - \rho^\tau, \\
\tau \partial_t Z^\tau + \curl (F^\tau) &= 0, \\
\divergence (Z^\tau) &= 0,
\end{aligned}
\]  
(25)
and estimate (5) gives, in particular,
\[
\sup_{t \geq 0} (\|\rho^\tau(t) - 1\|_s^2 + \|F^\tau(t)\|_s^2 + \|Z^\tau(t) - B_\epsilon\|_s^2) + \int_0^{+\infty} \|v^\tau(\xi)\|_s^2 d\xi \leq c \varepsilon_0^2.
\]

Thus, the sequences $(\rho^\tau - 1)_{\tau > 0}$, $(F^\tau)_{\tau > 0}$ and $(Z^\tau - B_\epsilon)_{\tau > 0}$ are bounded in $L^\infty (\mathbb{R}_+; H^s)$, and $(v^\tau)_{\tau > 0}$ is bounded in $L^2 (\mathbb{R}_+; H^s)$. It follows that, in $\mathcal{D}' (\mathbb{R}_+ \times \mathbb{R}^3)$,
\[
\tau^2 \left[ \partial_t v^\tau + (v^\tau, \nabla) v^\tau \right] + \tau (v^\tau \times Z^\tau) \rightharpoonup 0, \\
\tau(\partial_t F^\tau - \rho^\tau v^\tau) \rightharpoonup 0, \\
\tau \partial_t Z^\tau \rightharpoonup 0,
\]
and there exist functions $\bar{\rho} - 1$, $\bar{F}$, $\bar{Z} - B_\epsilon \in L^\infty (\mathbb{R}_+; H^s)$ and $\bar{v} \in L^2 (\mathbb{R}_+; H^s)$ such that, up to subsequences, the convergences (8)-(9) hold.

Moreover, using the first equation of (25), we deduce that $(\partial_t \rho^\tau)_{\tau > 0}$ is bounded in $L^2 (0, T; H^{s-1})$ for any $T > 0$.

Let $T > 0$. The sequence $(\rho^\tau)_{\tau > 0}$ is also bounded in $L^2 (0, T; H^s)$, then, by a classical compactness theorem (see [38]), for any $s_1 \in [0, s)$, the sequence $(\rho^\tau)_{\tau > 0}$ is relatively compact in $\mathcal{C} ([0, T]; H^{s_1})$, and, up to a subsequence, we have the strong convergence (10) by the uniqueness of the limit.

Consequently, we can pass to the limit in each of the nonlinear terms of (25) to obtain
\[
\begin{aligned}
\partial_t \bar{\rho} + \divergence (\bar{\rho} \bar{v}) &= 0, \\
\bar{F} + \nabla h(\bar{\rho}) &= -\bar{v}, \\
\curl (\bar{Z}) &= 0, \\
\divergence (\bar{F}) &= 1 - \bar{\rho}, \\
\curl (\bar{E}) &= 0, \\
\divergence (\bar{Z}) &= 0.
\end{aligned}
\]  
(26)

This directly gives the existence of $\bar{\rho}$ satisfying $\bar{F} = -\nabla \bar{\phi}$, and, replacing $\bar{v}$ in the first equation of (26) by its expression in (13), we obtain the drift-diffusion system (11).

Now we consider the initial condition of $\bar{\rho}$. The strong convergence (10) is uniform with respect to $t \in [0, T]$. Therefore,
\[
\rho^\tau (0, .) \rightharpoonup \bar{\rho}(0, .), \quad \text{in } H^{s_1}.
\]
Since $\rho^\tau (0, .) = \eta_0$, equality (12) follows from (7) and the uniqueness of the limit.

Finally, it is well known that the solution of the drift-diffusion system (11) with smooth initial data (12) is unique (see [13, 19]). This implies the convergence of the whole sequence $((\rho^\tau, v^\tau, F^\tau, Z^\tau))_{\tau > 0}$. \hfill \Box
4.2. The non-relativistic limit.

Proof of theorem 2.3. Recall that $\tau = 1$ and $\gamma \in (0,1]$ in the non-relativistic limit. Then (14) is a direct consequence of estimate (21). It implies the global existence of solutions. \hfill \Box

Proof of theorem 2.4. We first remark that the system satisfied by $(n^\gamma, u^\gamma, E^\gamma, B^\gamma)$ is

$$
\begin{cases}
\partial_t n^\gamma + \text{div}(n^\gamma u^\gamma) = 0, \\
\partial_t u^\gamma + (u^\gamma, \nabla)u^\gamma + \nabla h(n^\gamma) = -E^\gamma - \gamma(u^\gamma \times B^\gamma) - u^\gamma, \\
\gamma \partial_t E^\gamma - \text{curl}(B^\gamma) = \gamma n^\gamma u^\gamma, \quad \text{div}(E^\gamma) = 1 - n^\gamma, \\
\gamma \partial_t B^\gamma + \text{curl}(E^\gamma) = 0, \quad \text{div}(B^\gamma) = 0,
\end{cases}
$$

(27)

and estimate (14) gives, in particular,

$$
\sup_{t \geq 0} \left( \|n^\gamma(t) - 1\|^2_s + \|u^\gamma(t)\|^2_s + \|E^\gamma(t)\|^2_s + \|B^\gamma(t) - B_c\|^2_s \right) \leq c\varepsilon_0^2.
$$

This result implies the sequences $(n^\gamma - 1)_{\gamma > 0}$, $(u^\gamma)_{\gamma > 0}$, $(E^\gamma)_{\gamma > 0}$ and $(B^\gamma - B_c)_{\gamma > 0}$ are bounded in $L^\infty(\mathbb{R}_+; H^s)$.

It follows that, in $D'(\mathbb{R}_+ \times \mathbb{K}^3)$,

$$
\begin{align*}
\gamma(u^\gamma \times B^\gamma) & \xrightarrow{\gamma \to 0} 0, \\
\gamma \partial_t E^\gamma & \xrightarrow{\gamma \to 0} 0, \\
\gamma \partial_t B^\gamma & \xrightarrow{\gamma \to 0} 0, \\
\gamma n^\gamma u^\gamma & \xrightarrow{\gamma \to 0} 0,
\end{align*}
$$

and there exist functions $\bar{n} - 1$, $\bar{u}$, $\bar{E} - B_c \in L^\infty(\mathbb{R}_+; H^s)$ such that, up to subsequences, the convergences (16) hold.

Moreover, using the first equation of (27), we deduce that $(\partial_t n^\gamma)_{\gamma > 0}$ is bounded in $L^2(0, T; H^{s-1})$ for any $T > 0$.

Let $T > 0$. The sequence $(n^\gamma)_{\gamma > 0}$ is also bounded in $L^2(0, T; H^s)$, then, by a classical compactness theorem (see [38]), for any $s_1 \in [0, s)$, the sequence $(n^\gamma)_{\gamma > 0}$ is relatively compact in $C([0, T]; H^{s_1})$, and, up to a subsequence, we have the strong convergence for $n^\gamma$ in (17), by the uniqueness of the limit.

Similarly, since $(u^\gamma)_{\gamma > 0}$ is bounded in $L^2(0, T; H^s)$ and $(u^\gamma, \nabla u^\gamma)_{\gamma > 0}$ is bounded in $L^2(0, T; H^{s-1})$ for any $T > 0$, the second equation of (27) implies that $(\partial_t u^\gamma)_{\gamma > 0}$ is bounded in $L^2(0, T; H^{s-1})$. Then, by the same compactness theorem (see [38]), for any $s_1 \in (0, s)$, the sequence $(u^\gamma)_{\gamma > 0}$ is relatively compact in $C([0, T]; H^{s_1})$, and, up to a subsequence, we have the strong convergence for $u^\gamma$ in (17), by the uniqueness of the limit.

Consequently, we can pass to the limit in each of the nonlinear terms of (27) to obtain

$$
\begin{cases}
\partial_t \bar{n} + \text{div}(\bar{n}\bar{u}) = 0, \\
\partial_t \bar{u} + (\bar{u}, \nabla)\bar{u} + \nabla h(\bar{u}) = -\bar{E} - \bar{u}, \\
\text{curl}(\bar{B}) = \bar{n}\bar{u}, \quad \text{div}(\bar{E}) = 1 - \bar{n}, \\
\text{curl}(\bar{E}) = 0, \quad \text{div}(\bar{B}) = 0.
\end{cases}
$$

(28)

This directly gives the existence of $\bar{\phi}$ satisfying $\bar{E} = -\nabla \bar{\phi}$, and, by replacing $\bar{E}$ in (28), we obtain the Euler-Poisson system (18) with the conditions on $\bar{E}$ and $\bar{B}$ in (20).
Now we consider the initial condition of \((\bar{n}, \bar{u})\). The strong convergence (17) is uniform with respect to \(t \in [0, T]\). Therefore, in \(H^s\),
\[
n^\gamma(0, .) \xrightarrow{\gamma \to 0} \bar{n}(0, .) \quad \text{and} \quad u^\gamma(0, .) \xrightarrow{\gamma \to 0} \bar{u}(0, .).
\]
Since \((n^\gamma(0, .), u^\gamma(0, .)) = (n_0^\gamma, u_0^\gamma)\), equality (19) follows from (15) and the uniqueness of the limit.

Finally, it is well known that the solution of Euler-Poisson system (18) with initial conditions (19) is unique (see [1, 17]). This implies the convergence of the whole sequence \(((n^\gamma, u^\gamma, E^\gamma, B^\gamma))_{\gamma > 0}\).

5. Proof of lemmas 3.1 and 3.2.

5.1. Proof of lemma 3.1. The energy estimate in lemma 3.1 comes from two separated studies of (2). First, from the first two equations, we obtain the following inequality

**Lemma 5.1.**
\[
\frac{d}{dt} \left( \left\| \partial^\alpha E \right\|^2 + \left\| \partial^\alpha G \right\|^2 \right) \leq 2 \left\langle n \partial^\alpha u, \partial^\alpha E \right\rangle + \frac{1}{\tau} \left\| \partial^\alpha u \right\|^2 + c N_T K_T (W^I). \tag{29}
\]

Then, we study the Maxwell equations to obtain the following result :

**Lemma 5.2.**
\[
\frac{d}{dt} \left( \left\| \partial^\alpha E \right\|^2 + \left\| \partial^\alpha G \right\|^2 \right) \leq 2 \left\langle n \partial^\alpha u, \partial^\alpha E \right\rangle + c N_T K_T (W^I). \tag{30}
\]

Adding (29) and (30) yields
\[
\frac{d}{dt} \left( \left\| \partial^\alpha E \right\|^2 + \left\| \partial^\alpha G \right\|^2 \right) + \frac{1}{\tau} \left\| \partial^\alpha u \right\|^2 \leq c N_T K_T (W^I).
\]

Summing up this result over \(0 \leq \left| \alpha \right| \leq s\), we obtain lemma 3.1. \(\square\)

5.1.1. Proof of lemma 5.1. The first two equations in (2) can be written in the form
\[
\partial_t W^I + \sum_{j=1}^3 A_j(W^I) \partial x_j W^I = \begin{bmatrix}
0 \\
-E - \gamma u \times (G + B_e) - \frac{u}{\tau}
\end{bmatrix}, \tag{31}
\]
with
\[
A_j(W^I) = \begin{bmatrix}
u_j & 0 \\
h'(1 + N)e_j' & u_j I_3
\end{bmatrix}.
\]

Here \((e_1, e_2, e_3)\) is the canonical basis of \(\mathbb{R}^3\), \(I_3\) is the \(3 \times 3\) unit matrix, and \(y_j\) denotes the \(j\)-th component of \(y \in \mathbb{R}^3\).

Since \(n \geq 1/2\), we remark that the matrix
\[
A_0(N) = \begin{bmatrix}
h'(1 + N) & 0 \\
0 & (1 + N)I_3
\end{bmatrix}
\]
is symmetric positive definite, and the matrix
\[
\tilde{A}_j(W^I) = A_0(N) A_j(W^I) = \begin{bmatrix}
h'(1 + N)u_j & p'(1 + N)e_j' \\
p'(1 + N)e_j' & (1 + N)u_j I_3
\end{bmatrix}
\]
is symmetric for all \(1 \leq j \leq 3\).
Let $\alpha \in \mathbb{N}^3$ with $0 \leq |\alpha| \leq s$. Applying $\partial^\alpha$ to (31), and taking the inner product with $2A_0(N)\partial^\alpha W^I$ in $L^2(\mathbb{R}^3)$, lead to

$$\frac{d}{dt} \langle A_0(N)\partial^\alpha W^I, \partial^\alpha W^I \rangle$$

$$= \langle \partial_t [A_0(N)] \partial^\alpha W^I, \partial^\alpha W^I \rangle + \sum_{j=1}^3 \langle \partial_{x_j} \left[ A_j(W^I) \right] \partial^\alpha W^I, \partial^\alpha W^I \rangle$$

$$- 2 \langle n\partial^\alpha E, \partial^\alpha u \rangle - 2\gamma \langle (1 + N)\partial^\alpha u, \partial^\alpha (u \times G) \rangle$$

$$- \frac{2}{\tau} \langle (1 + N)\partial^\alpha u, \partial^\alpha u \rangle + 2 \langle J^0, \partial^\alpha W^I \rangle,$$

with $J^0 = 0$, and, for any $\alpha$ such that $1 \leq |\alpha| \leq s$, $J^\alpha = A_0(N) \sum_{j=1}^3 [A_j(W^I)] \partial^\alpha(\partial_{x_j} W^I) - \partial^\alpha (A_j(W^I)\partial_{x_j} W^I)] = \sum_{j=1}^3 J^\alpha_j,$

with the natural correspondence for $J^1_1, J^1_2, J^1_3$.

Let us write equality (32) as

$$\frac{d}{dt} \langle A_0(N)\partial^\alpha W^I, \partial^\alpha W^I \rangle = I_1 + \sum_{j=1}^3 I_{2,j} + I_3 + I_4 + I_5 + \sum_{j=1}^3 I_{6,j},$$

with the natural correspondence for $I_1, ..., I_{6,3}$. Now we analyze each of these terms.

* Developing $I_1, I_{2,j}$ and $I_{6,j}$, using repeatedly inequalities (4) and (23), we obtain

$$I_1 \leq cN_7K_r(W^I), \quad I_{2,j} \leq cN_7K_r(W^I), \quad I_{6,j} \leq cN_7K_r(W^I)$$

* Since $n \in [1/2, 3/2]$, we have

$$I_5 = -\frac{2}{\tau} \langle (1 + N)\partial^\alpha u, \partial^\alpha u \rangle \leq -\frac{1}{\tau} \|\partial^\alpha u\|^2.$$ (34)

* We remark that $I_4$ can be written as

$$I_4 = -2\gamma \langle (1 + N)\partial^\alpha (u \times G), \partial^\alpha u \rangle = -2\gamma \langle (1 + N) \left[ \partial^\alpha (u \times G) - \partial^\alpha G \times (u \times G) \right], \partial^\alpha u \rangle,$$

then inequality (4) leads to

$$I_4 \leq c\|\nabla G\|_{s-1} \|u\|_{s-1} \|\partial^\alpha u\| \leq c\|G\|_{s} \|u\|_{s}^2 \leq cN_7K_r(W^I).$$ (35)

Adding (33) and (35), and using (32), we obtain lemma 5.1. \qed

5.1.2. Proof of lemma 5.2. We start with the third equation in (2)

$$\partial_t E - \frac{1}{\gamma} \text{curl}(G) = nu.$$ We apply $\partial^\alpha$ and take the inner product with $\partial^\alpha E$ to obtain

$$\frac{d}{dt} \|\partial^\alpha E\|^2 - \frac{2}{\gamma} \langle \text{curl}(\partial^\alpha G), \partial^\alpha E \rangle = 2 \langle \partial^\alpha (nu), \partial^\alpha E \rangle,$$ (36)

Then we take the fourth equation in (2)

$$\partial_t G + \frac{1}{\gamma} \text{curl}(E) = 0.$$ We apply $\partial^\alpha$ and take the inner product with $\partial^\alpha G$ to obtain

$$\frac{d}{dt} \|\partial^\alpha G\|^2 + \frac{2}{\gamma} \langle \text{curl}(\partial^\alpha E), \partial^\alpha G \rangle = 0.$$ (37)
Applying Proof of Lemma 5.3. We treat each of them in the following lemma. This equality contains eight terms: two on the left-hand side, and six on the right-hand side. We use the first compatibility equation in (2) to obtain (41).

\begin{align}
\langle h'(n)\partial^3(\nabla N), \partial^3(\nabla N) \rangle &\geq \bar{\kappa}||\partial^3(\nabla N)||^2 \quad \text{(40)} \\
\langle \partial^3 E, \partial^3(\nabla N) \rangle &\geq ||\partial^3 N||^2 \quad \text{(41)} \\
\langle h'(n)\partial^3(\nabla N) - \partial^3 [h'(n)\nabla N], \partial^3(\nabla N) \rangle &\leq c_{N_T}||N||^2_s \quad \text{(42)} \\
\langle \partial_t(\partial^3 u), \partial^3(\nabla N) \rangle &\leq \frac{d}{dt} \langle \partial^3 (\text{div}(u)), \partial^3 N \rangle + c||u||^2_s \quad \text{(43)} \\
\langle \partial^3 [(u,\nabla)u], \partial^3(\nabla N) \rangle &\leq c||u||^2 \quad \text{(44)} \\
-\gamma \langle \partial^3 (u \times G), \partial^3(\nabla N) \rangle &\leq c||u||^2 + c_{N_T}||N||^2_s \quad \text{(45)} \\
\gamma \langle (\partial^3 u) \times B_e, \partial^3(\nabla N) \rangle &\leq c||u||_s||\nabla N||_{s-1} + \delta ||\nabla N||^2_{s-1} + c_\delta ||u||^2_s, \quad \forall \delta > 0, \quad \text{(46)}
\end{align}

where \( c_\delta > 0 \) denotes a generic constant only depending on \( \delta \).

\begin{align}
-\frac{1}{\tau} \langle \partial^3 u, \partial^3(\nabla N) \rangle + \frac{1}{2\tau} \frac{d}{dt} \left\langle \frac{1}{1+N} \partial^3 N, \partial^3 N \right\rangle &\leq c_{N_T}K_T(W^I) \quad \text{(47)}
\end{align}

Proof of Lemma 5.3.

1. We use the fact \( h'(n) \geq \bar{\kappa} > 0 \) to obtain (40).
2. We use the first compatibility equation in (2) to obtain (41).
3. We use inequality (4) to obtain (42).
4. Integrating by parts lead to (43).
5. We directly obtain (44) thanks to (23).
6. We obtain (45) thanks to

\begin{align}
-\gamma \langle \partial^3 (u \times G), \partial^3(\nabla N) \rangle &\leq c||G||_{s}||\partial^3 u||_s||N||_s + c||\nabla u||_{s}||\partial G||_s||N||_s \\
&\leq c||G||_{s}||u||_s||N||_s \leq c||u||^2_s + c_{N_T}||N||^2_s.
\end{align}
* A direct bounding gives
\[
\gamma \left\langle \partial^\beta u \times B_e, \partial^\beta (\nabla N) \right\rangle \leq c \|u\|_s \|\nabla N\|_{s-1} \leq \delta \|\nabla N\|_{s-1}^2 + c \delta \|u\|_s^2,
\]
for any \( \delta > 0 \). Then we have (46).

* The proof of (47) needs some calculus. Let us remark that
\[
-\frac{1}{\tau} \langle \partial^\beta u, \partial^\beta (\nabla N) \rangle = \frac{1}{\tau} \langle \partial^\beta [\text{div}(u)] , \partial^\beta N \rangle.
\]

Then we use the first equation in (2) to obtain
\[
-\frac{1}{\tau} \langle \partial^\beta u, \partial^\beta (\nabla N) \rangle = \frac{1}{\tau} \left\langle \partial^\beta (\partial_t N) \frac{1}{1 + N} , \partial^\beta N \right\rangle + \frac{1}{\tau} \left\langle \frac{\partial^\beta (\partial_t N)}{1 + N} - \partial^\beta \left( \frac{\partial_t N}{1 + N} \right) , \partial^\beta N \right\rangle + \frac{1}{\tau} \left\langle u . \partial^\beta (\nabla N) \frac{1}{1 + N} - \partial^\beta \left( u . \nabla N \right) \frac{1}{1 + N} , \partial^\beta N \right\rangle.
\]

We have to treat each term in the right-hand side of (48). The first one satisfies
\[
-\frac{1}{\tau} \left\langle \partial^\beta (\partial_t N) \frac{1}{1 + N} , \partial^\beta N \right\rangle + \frac{1}{\tau} \frac{d}{dt} \left\langle \frac{1}{1 + N} \partial^\beta N, \partial^\beta N \right\rangle = \frac{1}{2\tau} \left\langle \partial_t N \frac{1 + N}{(1 + N)^2} \partial^\beta N, \partial^\beta N \right\rangle \leq \frac{c}{\tau} \|\partial_t N\|_\infty \|\partial^\beta N\|^2 \leq \frac{c}{\tau} \|\text{div}(u)\|_\infty \|N\|_s^2 \leq \frac{c}{\tau} N_T K_r(W^I).
\]

The second term on the right-hand side of (48) satisfies
\[
\frac{1}{\tau} \left\langle u . \partial^\beta (\nabla N) \frac{1}{1 + N} , \partial^\beta N \right\rangle \leq \frac{c}{\tau} N_T K_r(W^I).
\]

Using inequality (4), the third one satisfies
\[
\frac{1}{\tau} \left\langle \partial^\beta (\partial_t N) \frac{1}{1 + N} - \partial^\beta \left( \frac{\partial_t N}{1 + N} \right) , \partial^\beta N \right\rangle \leq \frac{c}{\tau} N_T K_r(W^I),
\]
and the last one satisfies
\[
\frac{1}{\tau} \left\langle u . \partial^\beta (\nabla N) \frac{1}{1 + N} - \partial^\beta \left( u . \nabla N \right) \frac{1}{1 + N} , \partial^\beta N \right\rangle \leq \frac{c}{\tau} N_T K_r(W^I).
\]

Combining (48)-(52), we obtain (47).

End of the proof of lemma 3.2. Summing up (40)-(47), we obtain, for any \( \delta > 0 \),
\[
\kappa \|\partial^\beta (\nabla N)||^2 + \|\partial^\beta N\|^2 + \frac{1}{2\tau} \frac{d}{dt} \left\langle \frac{1}{1 + N} \partial^\beta N, \partial^\beta N \right\rangle \leq \frac{d}{dt} \langle \partial^\beta [\text{div}(u)] , \partial^\beta N \rangle + c \delta \|u\|_s^2 + \delta \|\nabla N\|_{s-1}^2 + \frac{c}{\tau} N_T K_r(W^I).
\]
Multiplying this inequality by $\tau$, summing over $\beta$, $0 \leq |\beta| \leq s - 1$, taking $\delta$ small enough, using $1/2 \leq n \leq 3/2$, we finally obtain

$$\tau \|\nabla N\|_{s-1}^2 + \tau \|N\|_{s-1}^2 + \frac{d}{dt} (\|N\|_{s-1}^2) \leq c\tau \sum_{0 \leq |\beta| \leq s-1} \frac{d}{dt} \langle \partial^\beta [\text{div}(u)], \partial^\beta N \rangle + c\tau \|u\|_{s}^2 + cN_K(W^I).$$

Finally, we remark that there exists a constant $c_1 > 0$ satisfying

$$\|\nabla N\|_{s-1}^2 + \|N\|_{s-1}^2 \geq c_1 \|N\|_{s}^2.$$

It yields lemma 3.2. □

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