COMBINATORICS OF LABELLING IN HIGHER DIMENSIONAL AUTOMATA

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Abstract. The main idea for interpreting concurrent processes as labelled precubical sets is that a given set of \( n \) actions running concurrently must be assembled to a labelled \( n \)-cube, in exactly one way. The main ingredient is the non-functorial construction called labelled directed coskeleton. It is defined as a subobject of the labelled coskeleton, the latter coinciding in the unlabelled case with the right adjoint to the truncation functor. This non-functorial construction is necessary since the labelled coskeleton functor of the category of labelled precubical sets does not fulfil the above requirement. We prove in this paper that it is possible to force the labelled coskeleton functor to be well-behaved by working with labelled transverse symmetric precubical sets. Moreover, we prove that this solution is the only one. A transverse symmetric precubical set is a precubical set equipped with symmetry maps and with a new kind of degeneracy map called transverse degeneracy. Finally, we also prove that the two settings are equivalent from a directed algebraic topological viewpoint. To illustrate, a new semantics of CCS, equivalent to the old one, is given.

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References

Part I. Introduction

I.1. Presentation of the results

Directed algebraic topology is a field of research aiming at modelling time flows of concurrent processes and their properties by various algebraic topological models \cite{FGR98, Gau03, GG03, Gra03, Kri08, FR08, Gou03} (cf. \cite{vG06} for other references). In this work, we are interested in concurrent processes arising from process algebras \cite{WN95, Mil89, BHR84}, and more precisely in the labelling process of these objects, which is related to combinatorics in a non-trivial way. By borrowing several ideas coming from \cite{Wor04} and \cite{Gou02} (see also \cite{Pra91, CS96, vG06, Fah05a, Fah05b}), with several slight modifications, the paper \cite{Gau08} presented a semantics of process algebras in terms of \textit{labelled precubical sets}. We consider in this paper only the case of Milner’s calculus of communicating systems (CCS). The adaptation to other synchronization algebras and therefore to other process algebras is straightforward and is left to the reader.

The principle of this semantics is that the concurrent execution of \( n \) actions is abstracted by a full labelled \( n \)-cube. Each coordinate corresponds to one of the \( n \) actions, and therefore two opposite faces are labelled by the same action (e.g., Figure \ref{fig:Cub} represents the concurrent execution of two actions \( a \) and \( b \)). The core of the construction of \cite{Gau08} is the non-functorial notion of \textit{labelled directed coskeleton}. It is applied to the fibered product of the 1-dimensional parts of two full labelled cubes representing two higher dimensional transitions. This construction is the key ingredient to defining the parallel composition with synchronization of CCS in \cite{Gau08}. It is defined as a subobject of the labelled (1-dimensional) coskeleton. The latter coincides with the usual coskeleton, i.e. the right adjoint to the truncation functor, when the set of labels is a singleton. The labelled directed coskeleton construction \( \cosk{\Sigma} \) takes a particular kind of 1-dimensional labelled precubical set \( K \) (the set of vertices \( K_0 \) must be a cube) to a higher dimensional labelled precubical set \( \cosk{\Sigma}(K) \) such that each set of \( n \) actions running concurrently is assembled to an \( n \)-cube, in exactly one way. This role cannot be played by the full labelled 1-dimensional coskeleton functor \( \cosk{1}\Sigma \) (see Proposition \ref{prop:cosk1}) of the category of labelled precubical sets since the latter may add several different \( n \)-cubes for the same set of \( n \) actions running concurrently.

The purpose of this paper is to introduce the notion of \textit{labelled transverse symmetric precubical set}. A transverse symmetric precubical set is a precubical set equipped with symmetry maps as in \cite{GM03} and with a new kind of degeneracy map called transverse degeneracy. To the best of our knowledge, the latter maps seem to be new. In this new category of precubical sets, the labelled coskeleton functor is well behaved, as explained in Theorem \ref{thm:cosk} and Theorem \ref{thm:cosk_free}. Indeed, the labelled transverse symmetric precubical set \( \mathcal{L}(\cosk{\Sigma}(K)) \) freely generated by the labelled directed coskeleton \( \cosk{\Sigma}(K) \) of \( K \) is isomorphic to the labelled coskeleton functor \( \cosk{1}\Sigma(K) \) of the category of labelled transverse symmetric precubical
sets applied to $K$ if $K$ is the 1-dimensional part of an $n$-cube or a fibered product over a synchronization algebra. Since the labelled transverse symmetric precubical set $L(\cosk(K))$ and the labelled precubical set $\cosk(K)$ generate the same topological space of execution paths by Proposition II.2.10 and Figure 3 this result gives a functorial interpretation of the labelled directed coskeleton construction which is equivalent to the non-functorial construction from a directed algebraic topological point of view.

The labelled coskeleton functor in the category of labelled transverse symmetric precubical sets is therefore a categorical machinery allowing the understanding of the combinatorics of the labelling process in the parallel composition with synchronization of CCS. The advantage of this labelled coskeleton construction is twofold: 1) it is a functorial construction; 2) it is defined for any labelled 1-dimensional [transverse symmetric] precubical set, allowing future generalizations.

This enables us to give a semantics of CCS in terms of labelled transverse symmetric precubical sets which is equivalent to the one of [Gau08] in terms of labelled precubical sets from a directed algebraic topological point of view: see Theorem IV.3.3 and Figure 3.

I.2. Outline of the paper and reading guide

The paper is divided in three parts:

(1) Section II.1, Section II.2 and Section II.3 generalize notions previously introduced in [Gau08] to any category of cubes.

(2) Section III.1 contains the mathematical treatment. The reader only interested in the computer-scientific applications will only have to read the statement of Theorem III.1.24.

(3) Section IV.1, Section IV.2 and Section IV.3 are the computer-scientific part of the paper.

The core of the paper is the categorical interpretation of the non-functorial labelled directed coskeleton construction using a generalization of the notion of labelled precubical set. The notion of category of cubes, and the generalized notion of labelled precubical sets are presented in Section II.1. The main difficulty is the definition of the generalized precubical set of labels. Section II.2 proves that all the notions of labelled precubical sets are equivalent from a directed algebraic topological point of view, in particular that they generate the same path space of execution paths. This section is the only topological one of the paper. Proposition II.2.10 is only used in Theorem IV.3.3 to conclude that the two semantics of CCS generate the same spaces of execution paths. There is also a small application (Proposition II.2.13) which is used inside the proof of Theorem IV.1.8. The topological material of Section II.2 is not necessary for the proof of Proposition II.2.13 but a pure combinatorial proof would be far more complicated. Section II.3 generalizes to all categories of precubical sets the labelled coskeleton functor. It is defined as a right adjoint of a truncation functor, as in the setting of labelled precubical sets.

Section III.1 is the mathematical core of the paper. It proves that all labelled coskeleton functors but one are defective. Indeed, the labelled coskeleton of the 1-dimensional part of the $n$-cube is never contractible in a directed algebraic topological sense, except for the unique shell-complete category of cubes, the maximal one containing all adjacency-preserving maps. This is the key property to obtaining a well-behaved labelled coskeleton functor (see

1The words “transverse symmetric” can be omitted here by Proposition II.1.19 and Proposition II.1.28.
A presheaf over the unique shell-complete category of cubes is called a *transverse symmetric precubical set*.

Section [IV.1] is the first section of the computer-scientific part of the paper. It explains how one can use the preceding constructions to represent the parallel composition in CCS of a \(m\)-transition with a \(n\)-transition, modelled by a full labelled \(m\)-cube and a full labelled \(n\)-cube respectively. In other terms, it studies parallel composition in the local case. It is shown that the definition of the fibered product in CCS must be slightly modified to allow the use of the labelled coskeleton functor of the category of labelled transverse symmetric precubical sets.

Section [IV.2] then studies parallel composition in CCS in the global case. It compares the two notions of synchronized tensor products in the category of labelled precubical sets and in that of labelled transverse symmetric ones. It is then proved in Section [IV.3] that the two semantics of CCS in terms of labelled precubical sets and labelled transverse symmetric ones are equivalent from a directed algebraic topological point of view.

Finally, Section [V.1] is an additional section treating the particular case of labelled symmetric precubical sets. This formalism will enable us to establish a link between concurrent processes viewed as precubical sets and Cattani-Sassone higher dimensional transition systems in \([\text{Gau}09]\).

### I.3. Prerequisites

The paper \([\text{Gau}08]\) contains an introduction to CCS for mathematicians which is enough to understand Section [IV.2] and [IV.3] of this paper. Computer scientists might prefer \([\text{Mil}89]\) and \([\text{WN}95]\). For the rest of the paper, only general knowledge in category theory \([\text{ML}98]\) [\text{MLM}94] is required, in particular in presheaf theory and in the theory of locally presentable categories \([\text{AR}94]\). A few model category techniques are used in Section [II.2]. In fact, except for Section [II.2], the rest of the paper is purely combinatorial. Possible references for model categories are \([\text{DS}95]\) \([\text{Hov}99]\) and \([\text{Hir}03]\).

### Part II. About labelled precubical sets over categories of cubes

#### II.1. Labelled precubical set over a category of cubes

We want to generalize the notion of labelled precubical set introduced in \([\text{Gau}08]\) by working on a category of cubes \(\mathcal{A}\) (see Definition II.1.7) instead on the reduced box category \(\Box\) (see Definition II.1.2) as in \([\text{Gau}08]\). The particular case \(\mathcal{A} = \Box\) will give back the notion of labelled precubical set.

**Category of cubes (definition and examples).** The category of partially ordered sets or posets together with the strictly increasing maps \((x < y \implies f(x) < f(y))\) is denoted by \(\text{PoSet}\). It is worth noting that it is not the usual category of partially ordered sets since we restrict to strictly increasing maps. Let \([0] = \{()\}\) and \([n] = \{0, 1\}^n\) for \(n \geq 1\). By convention, one has \(\{0, 1\}^0 = [0] = \{()\}\). The set \([n]\) is equipped with the product ordering \((\epsilon_1, \ldots, \epsilon_n) \leq (\epsilon_1', \ldots, \epsilon_n')\) if and only if for every \(1 \leq i \leq n\), one has \(\epsilon_i \leq \epsilon_i'\). The poset \([n]\) is also called the \(n\)-cube.

**II.1. Definition.** Let \(\delta^\alpha_i : [n - 1] \to [n]\) be the set map defined for \(1 \leq i \leq n\) and \(\alpha \in \{0, 1\}\) by \(\delta^\alpha_i(\epsilon_1, \ldots, \epsilon_{n-1}) = (\epsilon_1, \ldots, \epsilon_{i-1}, \alpha, \epsilon_i, \ldots, \epsilon_{n-1})\). These maps are called the face maps.

They satisfy the cocubical relations \(\delta^\beta_j \delta^\alpha_i = \delta^\alpha_i \delta^\beta_{j-1}\) for \(i < j\) and for all \((\alpha, \beta) \in \{0, 1\}^2\).
II.1.2. **Definition.** The reduced box category, denoted by □, is the subcategory of PoSet with the set of objects \([n], n \geq 0\) and generated by the morphisms \(\delta^n_i\).

It is well-known that the face maps together with the cocubical relations give a presentation by generators and relations of the small category □ [GM03].

II.1.3. **Proposition.** Let \(n \geq 1\). Let \((\epsilon_1, \ldots, \epsilon_n)\) and \((\epsilon'_1, \ldots, \epsilon'_n)\) be two elements of the poset \([n]\) with \((\epsilon_1, \ldots, \epsilon_n) < (\epsilon'_1, \ldots, \epsilon'_n)\). Then there exist \(i_1 > \cdots > i_{n-r}\) and \(\alpha_1, \ldots, \alpha_{n-r} \in \{0, 1\}\) such that \((\epsilon_1, \ldots, \epsilon_n) = \delta^\alpha_{i_1} \cdots \delta^{\alpha_{n-r}}_{i_{n-r}} (0 \ldots 0)\) and \((\epsilon'_1, \ldots, \epsilon'_n) = \delta^\alpha_{i_1} \cdots \delta^{\alpha_{n-r}}_{i_{n-r}} (1 \ldots 1)\) where \(r \geq 0\) is the number of 0 (resp. 1) in the arguments 0...0 (resp. 1...1). In other terms, \((\epsilon_1, \ldots, \epsilon_n)\) is the bottom element and \((\epsilon'_1, \ldots, \epsilon'_n)\) the top element of a \(r\)-dimensional subcube of \([n]\).

**Proof.** The set \(\{1, \ldots, n\}\) is equal to the disjoint union

\[
\{i \in \{1, \ldots, n\}, \epsilon_i = \epsilon'_i\} \cup \{i \in \{1, \ldots, n\}, \epsilon_i < \epsilon'_i\}.
\]

In the latter case, one necessarily has \(\epsilon_i = 0\) and \(\epsilon'_i = 1\). □

II.1.4. **Definition.** Let \(n \geq 1\). Let \((\epsilon_1, \ldots, \epsilon_n)\) and \((\epsilon'_1, \ldots, \epsilon'_n)\) be two elements of the poset \([n]\). The integer \(r\) of Proposition II.1.3 is called the distance between \((\epsilon_1, \ldots, \epsilon_n)\) and \((\epsilon'_1, \ldots, \epsilon'_n)\). Let us denote this situation by \(r = d((\epsilon_1, \ldots, \epsilon_n), (\epsilon'_1, \ldots, \epsilon'_n))\). By definition, one has

\[
r = \sum_{i=1}^{n} |\epsilon_i - \epsilon'_i|.
\]

II.1.5. **Definition.** A set map \(f : [m] \to [n]\) is adjacency-preserving if it is strictly increasing and if \(d((\epsilon_1, \ldots, \epsilon_m), (\epsilon'_1, \ldots, \epsilon'_m)) = 1\) implies \(d(f(\epsilon_1, \ldots, \epsilon_m), f(\epsilon'_1, \ldots, \epsilon'_m)) = 1\).

An adjacency-preserving map does not necessarily preserve distance. For example, the map \(\gamma_1 : [2] \to [2]\) defined by \(\gamma_1(\epsilon_1, \epsilon_2) = (\max(\epsilon_1, \epsilon_2), \min(\epsilon_1, \epsilon_2))\) is adjacency-preserving and not distance-preserving because \(\gamma_1(1, 0) = \gamma_1(0, 1) = (1, 0)\). We shall later see that \(\gamma_1\) is an example of transverse degeneracy map (cf. Definition II.1.1).

II.1.6. **Proposition.** For any \(n \geq 1\), the set map \(\delta^n_0 : [n-1] \to [n]\) is adjacency-preserving. Any strictly increasing map from \(\{0 < 1\}^n\) to itself is adjacency-preserving as well.

**Proof.** That the set map \(\delta^n_0 : [n-1] \to [n]\) is adjacency-preserving is clear. Let \(f\) be a strictly increasing map from \(\{0 < 1\}^n\) to itself. Let \(x\) and \(y\) be two elements of \(\{0 < 1\}^n\) with \(d(x, y) = 1\) and, for example, \(x < y\). Then there exists a strictly increasing chain \((0, \ldots, 0) = x_0 < x_1 < \cdots < x_n = (1, \ldots, 1)\) of \(\{0 < 1\}^n\) with \(x = x_{i-1}\) and \(y = x_i\) for some \(i \geq 1\). Then \(f(x_0) < f(x_1) < \cdots < f(x_n)\) is a strictly increasing chain of \(\{0 < 1\}^n\). Therefore one has \(f(x_0) = x_0\) and \(f(x_n) = x_n\). It is easy to see that \(n = d(f(x_0), f(x_n)) = \sum_{i=1}^{n} d(f(x_{i-1}), f(x_i))\). So for all \(i \geq 1\), one has \(d(f(x_{i-1}), f(x_i)) = 1\). Thus, \(f\) is adjacency-preserving. □

II.1.7. **Definition.** A category of cubes \(A\) is a subcategory of PoSet such that:

- the set of objects is \([n], n \geq 0\)
- there is the inclusion \(\Box \subset A\)
- every morphism of \(A\) is adjacency-preserving.

The minimal category of cubes for inclusion is the reduced box category □.
II.1.8. **Notation.** Let us denote by $\square$ the subcategory of $\text{PoSet}$ containing all adjacency-preserving maps.

The category $\square$ is the maximal category of cubes for inclusion. In other terms, a small category $\mathcal{C}$ is a category of cubes if and only there are the inclusions $\square \subset \mathcal{C} \subset \widehat{\square}$.

II.1.9. **Notation.** For the sequel, $\mathcal{A}$ always denotes a category of cubes.

II.1.10. **Definition.** [GM03] Let $\sigma_i : [n] \to [n]$ be the set map defined for $1 \leq i \leq n - 1$ and $n \geq 2$ by $\sigma_i(\epsilon_1, \ldots, \epsilon_n) = (\epsilon_1, \ldots, \epsilon_{i-1}, \epsilon_i, \epsilon_{i+1}, \epsilon_{i+2}, \ldots, \epsilon_n)$. These maps are called the symmetry maps.

The symmetry maps are clearly adjacency-preserving.

II.1.11. **Notation.** Let us denote by $\square_\mathcal{A}$ the smallest category of cubes containing the symmetry maps.

We have the inclusions of categories of cubes $\square \subset \square_\mathcal{A} \subset \widehat{\square}$.

### Unlabelled $\mathcal{A}$-set.

II.1.12. **Definition.** An (unlabelled) $\mathcal{A}$-set is a presheaf over $\mathcal{A}$. The corresponding category is denoted by $\mathcal{A}^{\text{op}}\text{Set}$.

Let $K$ be an object of $\mathcal{A}^{\text{op}}\text{Set}$. The set $K([n])$ will be also denoted by $K_n$. A map $f : K \to L$ of $\mathcal{A}^{\text{op}}\text{Set}$ will be also denoted by $(f_n)_{n \geq 0}$ where $f_n : K_n \to L_n$ is the corresponding set map. For any map $k : [m] \to [n]$ of $\mathcal{A}$ and any $\mathcal{A}$-set $K$, denote by $k^* : K_n \to K_m$ the set map induced by $k$.

Let $p \geq 0$. The $p$-dimensional $\mathcal{A}$-cube or $p$-cube $\mathcal{A}[p]$ is by definition the presheaf $\mathcal{A}(-, [p])$. In other terms, $\mathcal{A}[p]_k$ is the set of maps from $[k]$ to $[p]$ in the category of cubes $\mathcal{A}$. The boundary $\partial\mathcal{A}[p]$ of the $p$-dimensional $\mathcal{A}$-cube is the presheaf defined by $\partial\mathcal{A}[p]_k = \mathcal{A}[p]_k$ if $k < p$ and $\partial\mathcal{A}[p]_p = \emptyset$ otherwise. In particular, the boundary of the 0-dimensional $\mathcal{A}$-cube is the empty presheaf.

Let $\mathcal{A}_n \subset \mathcal{A}$ be the full subcategory of $\mathcal{A}$ whose set of objects is $\{[k], k \leq n\}$. The category of presheaves over $\mathcal{A}_n$ is denoted by $\mathcal{A}^n_{\text{op}}\text{Set}$. Its objects are called the $n$-dimensional $\mathcal{A}$-sets. The category of $n$-dimensional $\mathcal{A}$-sets can be identified with the full subcategory of the category of $\mathcal{A}$-sets $K$ such that $K_p = \emptyset$ for $p > n$.

Let $K$ be an $\mathcal{A}$-set. Let $K_{\leq n}$ be the $\mathcal{A}$-set obtained from $K$ by keeping the $p$-dimensional cubes of $K$ only for $p \leq n$. In particular, $K_{\leq 0} = K_0$. Note that one has $\partial\mathcal{A}[n] = \mathcal{A}[n]_{\leq n-1}$ for every $n \geq 0$ since our precubical sets contain no degeneracy maps in the usual sense.

II.1.13. **Definition.** A $\square$-set is called a precubical set [BH81]. A $\square_\mathcal{A}$-set is called a symmetric precubical set [GM03]. A $\widehat{\square}$-set is called a transverse symmetric precubical set.

The inclusion functor $\square \subset \mathcal{A}$ induces a forgetful functor $\omega : \mathcal{A}^{\text{op}}\text{Set} \to \square^{\text{op}}\text{Set}$ which has both a left and a right adjoint obtained respectively as a left and a right Kan extension along the inclusion $\square^{\text{op}} \subset \mathcal{A}^{\text{op}}$. The right adjoint is denoted by $\mathcal{R}_\mathcal{A} : \square^{\text{op}}\text{Set} \to \mathcal{A}^{\text{op}}\text{Set}$. The left adjoint $\mathcal{L}_\mathcal{A} : \square^{\text{op}}\text{Set} \to \mathcal{A}^{\text{op}}\text{Set}$ is of special interest since it formally adds all additional operators defining an $\mathcal{A}$-set. The two following propositions state some elementary remarks about $\mathcal{L}_\mathcal{A}$ which will be reused later.

\footnote{Note that the last notion is new.}
II.1.14. **Proposition.** Let $K$ be a precubical set. Then one has the isomorphism

$$\mathcal{L}_A(K) \cong \lim_{\square[n] \rightarrow K} A[n].$$

In particular, there is the isomorphism of $A$-sets $\mathcal{L}_A(\square[n]) \cong A[n]$.

**Proof.** For every $A$-set $K$, one has $K_n = (\omega_A K)_n$ for all $n \geq 0$ since the inclusion functor $\square \subset A$ is the identity on objects. So one has the bijections of sets

$$\mathcal{A}^{op}\text{-Set}(\mathcal{L}_A(\square[n]), K) \cong \mathcal{A}^{op}\text{-Set}(\square[n], \omega_A K) = (\omega_A K)_n = K_n = \mathcal{A}^{op}\text{-Set}(A[n], K).$$

By the Yoneda lemma, one obtains the isomorphism $\mathcal{L}_A(\square[n]) \cong A[n]$ for all $n \geq 0$. Since $\mathcal{L}_A$ is a left adjoint, it preserves colimits. So one obtains for every precubical set $K$

$$\mathcal{L}_A(K) = \mathcal{L}_A \left( \lim_{\square[n] \rightarrow K} \square[n] \right) \cong \lim_{\square[n] \rightarrow K} A[n].$$

II.1.15. **Proposition.** Let $K$ be a precubical set. The identity map $\text{Id}_{\mathcal{L}_A(K)}$ induces by adjunction an inclusion of presheaves $i_K : K \subset A\mathcal{L}_A(K)$.

**Proof.** Since the functor $K \mapsto K_p$ from precubical sets to sets is colimit-preserving for every $p \geq 0$, one has the bijections

$$K_p \cong \lim_{\square[n] \rightarrow K} \square[n]_p$$

and

$$\omega_A\mathcal{L}_A(K)_p \cong \lim_{\square[n] \rightarrow K} \omega_A A[n]_p = \lim_{\square[n] \rightarrow K} A([p], [n]).$$

Each set map $\square[n]_p \rightarrow \omega_A A[n]_p$ is one-to-one because of the inclusions of sets $\square([p], [n]) \subset A([p], [n])$ for every $p \geq 0$. For any map $g : [n] \rightarrow [n']$ of $\square$, one has the commutative diagram of sets

$$\begin{array}{ccc}
\square[n]_p & \xrightarrow{\subset} & \square[n']_p \\
\downarrow & & \downarrow \\
\omega_A A[n]_p = A([p], [n]) & \xrightarrow{\subset} & \omega_A A[n']_p = A([p], [n']).
\end{array}$$

The bottom map is one-to-one since it consists of composing by $g$ which is one-to-one as any map of $\square$. So each set map $\omega_A A[n]_p \rightarrow \omega_A A[n']_p$ of the diagram calculating $\omega_A\mathcal{L}_A(K)_p$ is one-to-one as well. One deduces that the map $K_p \rightarrow \omega_A\mathcal{L}_A(K)_p$ is one-to-one.

II.1.16. **Notation.** Since $K \subset \omega_A\mathcal{L}_A(K)$ is an inclusion, $i_K(y)$ will be simply denoted by $y$ for any $y \in K$. 


The 1-dimensional case. This paragraph proves that the 1-dimensional case does not depend on the choice of the category of cubes. The crucial facts are that a category of cubes contains all face maps and that all morphisms are adjacency-preserving.

II.1.17. Proposition. For every $m > n$, one has $A([m],[n]) = \emptyset$. For every $n \geq 0$, the inclusion $\square \subset A$ implies the bijections $[n] \cong \square([0],[n]) \cong A([0],[n])$ and $\square([1],[n]) \cong A([1],[n])$.

Note that this implies that $A$ cannot have any degeneracies.

Proof. It is clear that $A([m],[n]) \neq \emptyset$ implies $m \leq n$. One has the inclusions

$$\square([0],[n]) \subset A([0],[n]) \subset \overset{\sim}{\square}([0],[n]) = \square([0],[n]) = \{ \delta^n_1 \ldots \delta^n_1, (\epsilon_1, \ldots, \epsilon_n) \in [n] \}$$

hence the second assertion. For every $n \geq 0$, the inclusion $\square([1],[n]) \subset A([1],[n])$ is a bijection since every map of $A$ is adjacency-preserving by definition of a category of cubes, hence the third assertion. □

II.1.18. Proposition. Let $K$ be a precubical set. Then the inclusion of precubical sets $K \subset \omega_A \mathcal{L}_A(K)$ induces the isomorphism of 1-dimensional precubical sets $K_{\leq 1} \cong \omega_A \mathcal{L}_A(K)_{\leq 1}$.

Proof. We already know by Proposition II.1.17 that for every $n \geq 0$, the inclusions $\square([0],[n]) \subset A([0],[n])$ and $\square([1],[n]) \subset A([1],[n])$ are bijective. So the inclusion of presheaves $\square[n]_{\leq 1} \subset \omega_A [n]_{\leq 1}$ is an isomorphism for every $n \geq 0$. Since the forgetful functor $\omega_A : A^{op}Set \to \square^{op}Set$ is a left adjoint, it is colimit-preserving. Hence the proof is complete. □

II.1.19. Proposition. The category of 1-dimensional precubical sets is equivalent to the category of 1-dimensional $A$-sets.

Proof. The adjunction $\mathcal{L}_A : \square^{op}Set \rightleftharpoons A^{op}Set : \omega_A$ induces an adjunction $(\mathcal{L}_A)_{\leq 1} : \square_1^{op}Set \rightleftharpoons A_1^{op}Set : (\omega_A)_{\leq 1}$ by Proposition II.1.13. We already know by Proposition II.1.18 that there is the isomorphism $(\omega_A)_{\leq 1}(\mathcal{L}_A)_{\leq 1} \cong \text{Id}_{\square_1^{op}Set}$. One has $(\mathcal{L}_A)_{\leq 1}(\omega_A)_{\leq 1}(A[0]) \cong A[0]$ by Proposition II.1.14 and $(\mathcal{L}_A)_{\leq 1}(\omega_A)_{\leq 1}(A[1]) \cong A[1]$ by Proposition II.1.17 and Proposition II.1.14. Hence the isomorphism of functors $(\mathcal{L}_A)_{\leq 1}(\omega_A)_{\leq 1} \cong \text{Id}_{A_1^{op}Set}$. □

Labelled $A$-set. We fix a non-empty set $\Sigma$ of labels or of actions. It always contains a distinguished label denoted by $\tau$. We want to label the cubes of a $A$-set with the elements of $\Sigma$. A labelled $A$-set will be a map of $A$-sets $K \rightarrow L$ where $L$ is the $A$-set of labels. Let us start by recalling the construction of the precubical set of labels.

II.1.20. Proposition. (Variant of Goubault’s construction [Gou02]) Let

- $(!\Sigma)_0 = \{ () \}$ (the empty word)
- for $n \geq 1$, $(!\Sigma)_n = \Sigma^n$
- $\partial^n_i(a_1, \ldots, a_n) = \partial^n_i(a_1, \ldots, a_n) = (a_1, \ldots, \widehat{a_i}, \ldots, a_n)$ where the notation $\widehat{a_i}$ means that $a_i$ is removed.

Then these data generate a precubical set denoted by $!\Sigma$.

II.1.21. Definition. Let $K$ be an $A$-set. Let $x \in K_p$ with $p \geq 1$. The boundary of $x$ is the composite map $\partial x : \partial A[p] \subset A[p] \xrightarrow{x} K$. 
The main feature of the precubical set $\pi$ is that for every $p \geq 2$, a $p$-cube of $\pi$, which labels the concurrent execution of $p$ actions like in Figure 1, is determined by its boundary. In other terms, a commutative square of precubical sets of the form

$$
\begin{array}{ccc}
\partial [p] & \rightarrow & \pi \\
\subseteq & & k \\
\sqcup & & \subseteq \\
[p] & \rightarrow & 1
\end{array}
$$

with $p \geq 2$ where 1 is the terminal precubical set admits at most one lift $k$. An equivalent mathematical formulation of the preceding condition is that for every commutative square of precubical sets of the form

$$
\begin{array}{ccc}
\partial [p] & \rightarrow & [p] \\
\subseteq & f & \\
[p] & \rightarrow & \pi
\end{array}
$$

one has $f = g$. So every commutative square of precubical sets of the form

$$
\begin{array}{ccc}
[p] \cup \partial [p] & \rightarrow & \pi \\
\subseteq & k & \\
[p] & \rightarrow & 1
\end{array}
$$

with $p \geq 2$ admits exactly one lift $k = f = g$. In other terms, the precubical set $\pi$ turns out to be orthogonal to the set of maps $\{[p] \cup \partial [p] \rightarrow [p], p \geq 2\}$ in the sense of [AR94, Definition 1.32].

Because of the inclusion $K \subset \omega A \mathcal{L}_A(K)$ for every precubical set $K$, we need more cubes for the $A$-set of labels as soon as the inclusion $\Box \subset A$ is strict. Indeed, we must be able to label all cubes of $\mathcal{L}_A(K)$ for every labelled precubical set $K \rightarrow !\pi$. The first candidate for the $A$-set
of labels is then the $A$-set $\mathcal{L}_A(!\Sigma)$ freely generated by $!\Sigma$. However, it is not well-behaved. Consider the two set involutions $\sigma_1 : [2] \to [2]$ and $\text{Id}_{[2]} : [2] \to [2]$. Let us suppose that $\sigma_1 \in A$. Then the two 2-cubes $(\sigma_1)^*(\tau, \tau)$ and $(\tau, \tau)$ of $\mathcal{L}_A(!\Sigma)$ have the same boundary. This means that the commutative square of $A$-sets

\[
\begin{array}{ccc}
\partial A[p] & \xrightarrow{\partial(\sigma_1)^*(\tau, \tau) = \partial(\tau, \tau)} & \mathcal{L}_A(!\Sigma) \\
\downarrow & & \downarrow \\
A[p] & \xrightarrow{\mathcal{L}_A(!\Sigma)} & 1
\end{array}
\]

has two distinct lifts $k = (\sigma_1)^*(\tau, \tau)$ and $k = (\tau, \tau)$. In other terms, the $A$-set $\mathcal{L}_A(!\Sigma)$ is never orthogonal to the set of morphisms \{A\}, orthogonal to the set of morphisms $\mathcal{L}_A(!\Sigma)$ belongs to $A$. In fact, the $A$-set $\mathcal{L}_A(!\{\tau\})$ is even not the terminal $A$-set in this case. Yet, the notion of $A$-set must coincide with the unlabelled notion if the set of labels is equal to \{\tau\}. The full subcategory \{A[p]|_{\partial A[p]} A[p] \to A[p], p \geq 2\} of $A$-sets orthogonal to the set of maps \{A[p]|_{\partial A[p]} A[p] \to A[p], p \geq 2\} is a full reflective subcategory of the locally presentable category of $A$-sets by [AR94, Theorem 1.39]. Let

\[
\mathcal{L}_A : A^{op}\text{Set} \to \{A[p]|_{\partial A[p]} A[p] \to A[p], p \geq 2\}^{\perp}
\]

be the left adjoint to the inclusion functor \{A[p]|_{\partial A[p]} A[p] \to A[p], p \geq 2\}^{\perp} \subset A^{op}\text{Set}.

II.1.22. Definition. The $A$-set of labels is the $A$-set $\mathcal{L}_A(!\Sigma)$.

In $\mathcal{L}_A(!\Sigma)$, the two 2-cubes $(\sigma_1)^*(\tau, \tau)$ and $(\tau, \tau)$ are forced to be equal. Note that there is the isomorphism of precubical sets $\mathcal{L}_A(!A) \cong !\Sigma$.

II.1.23. Definition. A labelled $A$-set (over $\Sigma$) is an object of the comma category $A^{op}\text{Set} \downarrow \mathcal{L}_A(!\Sigma)$.

That is, an object is a map of $A$-sets $\ell : K \to \mathcal{L}_A(!\Sigma)$ and a morphism is a commutative diagram

\[
\begin{array}{ccc}
K & \xrightarrow{\ell} & \mathcal{L}_A(!\Sigma) \\
\downarrow & & \downarrow \\
\mathcal{L}_A(!\Sigma) & \xrightarrow{\mathcal{L}_A(!\Sigma)} & \mathcal{L}_A(!\Sigma)
\end{array}
\]

The $\ell$ map is called the labelling map. The $A$-set $K$ is sometimes called the underlying $A$-set of the labelled $A$-set.

The functor $\mathcal{L}_A : \square^{op}\text{Set} \to A^{op}\text{Set}$ induces a functor (denoted in the same way)

\[
\mathcal{L}_A : \square^{op}\text{Set} \downarrow !\Sigma \to A^{op}\text{Set} \downarrow \mathcal{L}_A(!\Sigma)
\]

which takes $\ell : K \to !\Sigma$ to the composite $\mathcal{L}_A(K) \xrightarrow{\mathcal{L}_\ell} \mathcal{L}_A(!\Sigma) \to \mathcal{L}_A(!\Sigma)$.

II.1.24. Proposition. Let $K$ be an $A$-set. Then the map of $A$-sets $K \to \mathcal{L}_A(K)$ induces the isomorphism of 1-dimensional $A$-sets $K_{\leq 1} \cong \mathcal{L}_A(K)_{\leq 1}$.
Proof. For every $p \geq 2$ and for every commutative diagram of solid arrows
\[
\begin{array}{ccc}
\mathcal{A}[p] \sqcup_{\partial \mathcal{A}[p]} \mathcal{A}[p] & \to & K \\
\downarrow & & \downarrow \\
\mathcal{A}[p] & \to & 1,
\end{array}
\]
there exists at most one lift $k$. So an $\mathcal{A}$-set $K$ is orthogonal to the set of morphisms \{\mathcal{A}[p] \sqcup_{\partial \mathcal{A}[p]} \mathcal{A}[p] \to \mathcal{A}[p], p \geq 2\} if and only if the canonical map $K \to 1$ satisfies the right lifting property with respect to the same set of morphisms. So the $\mathcal{A}$-set $\mathbf{Sh}_\mathcal{A}(K)$ can be obtained by a small object argument by factoring the map $K \to 1$ as a composite $K \to \mathbf{Sh}_\mathcal{A}(K) \to 1$ where $K \to \mathbf{Sh}_\mathcal{A}(K)$ is a relative \{\mathcal{A}[p] \sqcup_{\partial \mathcal{A}[p]} \mathcal{A}[p] \to \mathcal{A}[p], p \geq 2\}-cell complex and where the map $\mathbf{Sh}_\mathcal{A}(K) \to 1$ satisfies the right lifting property with respect to the same set of morphisms. The small object argument is possible by [Bek00, Proposition 1.3] since the category of $\mathcal{A}$-sets is locally presentable, as every presheaf category. Since for every $p \geq 2$, the map of $\mathcal{A}$-sets $\mathcal{A}[p] \sqcup_{\partial \mathcal{A}[p]} \mathcal{A}[p] \to \mathcal{A}[p]$ induces an isomorphism
\[(\mathcal{A}[p] \sqcup_{\partial \mathcal{A}[p]} \mathcal{A}[p])_\leq 1 \cong \mathcal{A}[p]_\leq 1,
\]
one deduces that the canonical map $K \to \mathbf{Sh}_\mathcal{A}(K)$ induces an isomorphism $K_\leq 1 \cong \mathbf{Sh}_\mathcal{A}(K)_\leq 1$. \hfill $\square$

II.1.25. Proposition. There is the isomorphism $\mathbf{Sh}_\mathcal{A} \mathcal{L}_\mathcal{A}(\{\tau\}) \cong 1$. Therefore when $\Sigma = \{\tau\}$, the category of labelled $\mathcal{A}$-sets is equivalent to the category of unlabelled $\mathcal{A}$-sets.

Proof. Indeed, both the functors $\mathbf{Sh}_\mathcal{A}$ and $\mathcal{L}_\mathcal{A}$ do not modify the set of 0-cubes and the set of 1-cubes by Proposition II.1.18 and Proposition II.1.24. Moreover, for any $\mathcal{A}$-set $K$ such that $K_1$ is a singleton, it is clear by induction on $p \geq 1$ that the set $(\mathbf{Sh}_\mathcal{A} K)_p$ is a singleton. So $\mathbf{Sh}_\mathcal{A} \mathcal{L}_\mathcal{A}(\{\tau\}) \cong 1$ (the terminal object of $\mathcal{A}^{op}\mathbf{Set}$). \hfill $\square$

II.1.26. Notation. Let $(a_1, \ldots, a_n) \in \Sigma^n$ with $n \geq 1$. The labelled precubical set $\Box[a_1, \ldots, a_n]$ denotes the map $\ell : [n] \to \Sigma$ such that $\ell(1_{[a_i]}) = (a_1, \ldots, a_n)$.

Figure II gives the example of the labelled 2-cube $\Box[a, b]$. It represents the concurrent execution of $a$ and $b$. It is important to notice that two opposite faces of Figure II have the same label.

II.1.27. Notation. Let $(a_1, \ldots, a_n) \in \Sigma^n$ with $n \geq 1$. The labelled $\mathcal{A}$-set $\mathcal{A}[a_1, \ldots, a_n]$ denotes the labelled $\mathcal{A}$-set $\mathcal{L}_\mathcal{A}(\Box[a_1, \ldots, a_n])$.

II.1.28. Proposition. The category of labelled 1-dimensional precubical sets is equivalent to the category of labelled 1-dimensional $\mathcal{A}$-sets.

Proof. This is a consequence of Proposition II.1.19 and Proposition II.1.24. \hfill $\square$

II.2. Geometric realization of labelled $\mathcal{A}$-set

The purpose of this section is to prove that the geometric realization functor $\Box^{op}\mathbf{Set} \dashv \mathbf{Flow}$ of $\mathcal{A}^{op}\mathbf{Set}$ which takes a labelled precubical set to the corresponding labelled flow factors as a composite $\Box^{op}\mathbf{Set} \dashv \mathbf{Flow}$ of $\mathcal{A}^{op}\mathbf{Set} \dashv \mathbf{Sh}_\mathcal{A} \mathcal{L}_\mathcal{A}(\Sigma) \to \mathbf{Flow}$, where the left-hand
functor is induced by $\mathcal{L}_A$. This result ensures that all the notions of labelled $\mathcal{A}$-sets are equivalent from a directed algebraic topological point of view. The results of this section are used only in Theorem [IV.3.3] and in Theorem [IV.1.8].

Unlabelled flow. The category $\text{Top}$ of compactly generated topological spaces (i.e. of weak Hausdorff $k$-spaces) is complete, cocomplete and cartesian closed (more details for these kinds of topological spaces are in [Bro06, May99], the appendix of [Lew78] and also in the preliminaries of [Gau03]). For the sequel, all topological spaces will be supposed to be compactly generated. A compact space is always Hausdorff.

II.2.1. Definition. $\text{Gau03}$ A (time) flow $X$ is a small topological category without identity maps. The set of objects is denoted by $X^0$. The topological space of morphisms from $\alpha$ to $\beta$ is denoted by $\mathbb{P}_{\alpha,\beta}X$. The elements of $X^0$ are also called the states of $X$. The elements of $\mathbb{P}_{\alpha,\beta}X$ are called the (non-constant) execution paths from $\alpha$ to $\beta$. A flow $X$ is loopless if for every $\alpha \in X^0$, the space $\mathbb{P}_{\alpha,\alpha}X$ is empty.

II.2.2. Notation. Let $\mathbb{P}X = \bigsqcup_{(\alpha,\beta) \in X^0 \times X^0} \mathbb{P}_{\alpha,\beta}X$. The topological space $\mathbb{P}X$ is called the path space of $X$. The source map (resp. the target map) $\mathbb{P}X \to X^0$ is denoted by $s$ (resp. $t$).

II.2.3. Definition. Let $X$ be a flow, and let $\alpha \in X^0$ be a state of $X$. The state $\alpha$ is initial if $\alpha \notin t(\mathbb{P}X)$, and the state $\alpha$ is final if $\alpha \notin s(\mathbb{P}X)$.

II.2.4. Definition. A morphism of flows $f : X \to Y$ consists in a set map $f^0 : X^0 \to Y^0$ and a continuous map $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ such that $s(\mathbb{P}f(x)) = f^0(s(x))$, $t(\mathbb{P}f(x)) = f^0(t(x))$, and $\mathbb{P}f(x \ast y) = \mathbb{P}f(x) \ast \mathbb{P}f(y)$ for every $x, y \in \mathbb{P}X$. The corresponding category is denoted by $\text{Flow}$.

The strictly associative composition law

$$\begin{cases}
\mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\beta,\gamma}X \to \mathbb{P}_{\alpha,\gamma}X \\
(x, y) \mapsto x \ast y
\end{cases}$$

models the composition of non-constant execution paths. The composition law $\ast$ is extended in a usual way to states, that is to constant execution paths, by $x \ast t(x) = x$ and $s(x) \ast x = x$ for every non-constant execution path $x$.

Here are two fundamental examples of flows:

(1) Let $S$ be a set. The flow associated with $S$, still denoted by $S$, has $S$ as a set of states and the empty space as path space. This construction induces a functor $\text{Set} \to \text{Flow}$ from the category of sets to that of flows. The flow associated with a set is loopless.

(2) Let $(P, \leq)$ be a poset. The flow associated with $(P, \leq)$, and still denoted by $P$ is defined as follows: the set of states of $P$ is the underlying set of $P$; the space of morphisms from $\alpha$ to $\beta$ is empty if $\alpha \geq \beta$ and equal to $\{(\alpha, \beta)\}$ if $\alpha < \beta$ and the composition law is defined by $(\alpha, \beta) \ast (\beta, \gamma) = (\alpha, \gamma)$. This construction induces a functor $\text{PoSet} \to \text{Flow}$ from the category of posets together with the strictly increasing maps to the category of flows. The flow associated with a poset is loopless as well.\footnote{and must be loopless! This is one of the reasons for working with small categories without identity maps.}

There is an important model structure on $\text{Flow}$ which is characterized as follows \text{Gau03}:

- The weak equivalences are the weak S-homotopy equivalences, i.e. the morphisms of flows $f : X \to Y$ such that $f^0 : X^0 \to Y^0$ is a bijection of sets and such that $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ is a weak homotopy equivalence.
• The fibrations are the morphisms of flows \( f : X \rightarrow Y \) such that \( \mathbb{P}f : \mathbb{P}X \rightarrow \mathbb{P}Y \) is a Serre fibration\(^4\).

This model structure is cofibrantly generated. The cofibrant replacement functor is denoted by \((-)\text{cof}\).

**Labelled flow.**

**II.2.5. Definition.** \([\text{Gau08}]\) The flow of labels \(?\Sigma\) is defined as follows: \((?\Sigma)^0 = \{\emptyset\}\) and \(\mathbb{P}?\Sigma\) is the discrete free commutative semigroup generated by the elements of \(\Sigma\).

**II.2.8. Theorem.** \([\text{Gau08}]\) A labelled flow is an object of the comma category \(\text{Flow}\downarrow ?\Sigma\). That is an object is a map of flows \(\ell : X \rightarrow ?\Sigma\) and a morphism is a commutative diagram

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\ ?\Sigma & \longrightarrow & \\
\end{array}
\]

The \(\ell\) map is called the labelling map. The flow \(X\) is sometimes called the underlying flow of the labelled flow.

**Geometric realization of a labelled precubical set.** A state of the flow associated with the poset \(\{0 < \hat{1}\}^n\) (i.e. the product of \(n\) copies of \(\{0 < \hat{1}\}\)) is denoted by an \(n\)-tuple of elements of \(\{0, \hat{1}\}\). By convention, \(\{0 < \hat{1}\}^0 = \{()\}\). The unique morphism/execution path from \((x_1, \ldots, x_n)\) to \((y_1, \ldots, y_n)\) is denoted by an \(n\)-tuple \((z_1, \ldots, z_n)\) of \(\{0, \hat{1}, \ast\}\) with \(z_i = x_i\) if \(x_i = y_i\) and \(z_i = \ast\) if \(x_i < y_i\). For example in the flow \(\{0 < \hat{1}\}^2\) (cf. Figure 2), one has the algebraic relation \((\ast, \ast) = (\ast, \hat{1}) = (\ast, \hat{1}) = (\ast, 0) = (\ast, \hat{1})\).

Let \(\square \rightarrow \text{PoSet} \subset \text{Flow}\) be the functor defined on objects by the mapping \([n] \mapsto \{0 < \hat{1}\}^n\) and on morphisms by the mapping

\[
\delta_i^0 : (\epsilon_1, \ldots, \epsilon_{n-1}) \mapsto (\epsilon_1, \ldots, \epsilon_{i-1}, \alpha, \epsilon_i, \ldots, \epsilon_{n-1}),
\]

where the \(\epsilon_i\)'s are elements of \(\{0, \hat{1}, \ast\}\). The functor \([n] \mapsto \{0 < \hat{1}\}^n\) from \(\square\) to \(\text{Flow}\) induces a bad realization functor from \(\square^{op}\text{Set}\) to \(\text{Flow}\) defined by

\[
|K|_{\text{bad}} := \lim_{\square[n] \to K} \{0 < \hat{1}\}^n.
\]

**II.2.7. Theorem.** \([\text{Gau08}, \text{Theorem 7.1}]\) and \([\text{Gau08}, \text{Proposition 8.1}]\) For all \(n \geq 3\), the inclusion \(\partial \square[n] \subset \square[n]\) induces an isomorphism of flows \(|\partial \square[n]|_{\text{bad}} \cong |\square[n]|_{\text{bad}}\). One has the isomorphism of flows \(|\Sigma|_{\text{bad}} \cong ?\Sigma\).

**II.2.8. Definition.** \([\text{Gau08}]\) Let \(K\) be a precubical set. By definition, the geometric realization of \(K\) is the flow

\[
|K| := \lim_{\square[n] \to K} \{0 < \hat{1}\}^{n\text{cof}}
\]

The natural trivial fibrations \(\{0 < \hat{1}\}^{n\text{cof}} \rightarrow \{0 < \hat{1}\}^n\) for \(n \geq 0\) induce a natural map \(|K| \rightarrow |K|_{\text{bad}}\) for any precubical set \(K\). Let \(K \rightarrow !\Sigma\) be a labelled precubical set. Then the composition \(|K| \rightarrow |\Sigma| \rightarrow |\Sigma|_{\text{bad}} \cong ?\Sigma\) gives rise to a labelled flow.
Geometric realization of a labelled $A$-set. Let $A \to \text{PoSet} \subset \text{Flow}$ be the functor defined on objects by the mapping $[n] \mapsto \{0 < 1\}^n$ and on morphisms as follows. Let $f : [m] \to [n]$ be a map of $A$ with $m, n \geq 0$. Let $(\epsilon_1, \ldots, \epsilon_m) \in \{0, 1, *\}^m$ be a $r$-cube. Since $f$ is adjacency-preserving, the two elements $f(s(\epsilon_1, \ldots, \epsilon_m))$ and $f(t(\epsilon_1, \ldots, \epsilon_m))$ are respectively the initial and final states of a unique $r$-dimensional subcube denoted by $f(\epsilon_1, \ldots, \epsilon_m)$ of $[n]$ with $f(\epsilon_1, \ldots, \epsilon_m) \in \{0, 1, *\}^n$. Note that the composite functor $\Box \subset A \to \text{PoSet} \subset \text{Flow}$ is the functor defined above. The functor $[n] \mapsto \{0 < 1\}^n$ from $A$ to $\text{Flow}$ induces a bad realization functor from $A^{op}\text{Set}$ to $\text{Flow}$ defined by

$$|K|_{bad} := \lim_{A[n] \to K} \{0 < 1\}^n.$$  

II.2.9. Definition. Let $K$ be an $A$-set. By definition, the geometric realization of $K$ is the flow

$$|K| := \lim_{A[n] \to K} (\{0 < 1\}^n)^{cof}$$

Note that the two geometric realizations of $A$-sets are colimit-preserving. In fact, it is easy to prove that both are left adjoints.

II.2.10. Proposition. Let $K$ be a precubical set. Then there are the natural isomorphisms of flows $|\mathcal{L}_A(K)|_{bad} \cong |K|_{bad}$ and $|\mathcal{L}_A(K)| \cong |K|$.

Proof. Since all functors involved in the statement of the proposition are left adjoint and therefore colimit-preserving, it suffices to check the isomorphism for $K = \Box [n]$. The proof is complete after Proposition II.1.14.

II.2.11. Corollary. For all $n \geq 3$, the inclusion $\partial A[n] \subset A[n]$ induces an isomorphism of flows $|\partial A[n]|_{bad} \cong |A[n]|_{bad}$.

Proof. Since $\mathcal{L}_A$ is colimit-preserving, one has $\mathcal{L}_A(\partial [n]) \cong \partial A[n]$. So by Proposition II.2.10 and Theorem II.2.7 one obtains $|\partial A[n]|_{bad} \cong |\partial [n]|_{bad} \cong |[n]|_{bad} \cong |A[n]|_{bad}$.

II.2.12. Proposition. Let $K$ be an $A$-set. The canonical map $K \to \text{Sh}_A(K)$ induces an isomorphism of flows $|K|_{bad} \cong |\text{Sh}_A(K)|_{bad}$.

Proof. We already know that the map $K \to \text{Sh}_A(K)$ is obtained by factoring the canonical map $K \to 1$ as the composite $K \to \text{Sh}_A(K) \to 1$ where $K \to \text{Sh}_A(K)$ is a relative $\{A[p] \sqcup_{\partial A[p]} A[p] \to A[p], p \geq 2\}$-cell complex and the map $\text{Sh}_A(K) \to 1$ satisfies the right lifting property.
with respect to the same set of morphisms. So the map \(|K|_{bad} \to |Sh_A(K)|_{bad}\) is a relative 
\(|\mathcal{A}[2]|_{bad} ∩ |\mathcal{A}[2]|_{bad} \to |\mathcal{A}[2]|_{bad}\)-cell complex by Corollary II.2.11. Figure 2 explains why the map of flows
\(|\mathcal{A}[2]|_{bad} \sqcup |\mathcal{A}[2]|_{bad} \to |\mathcal{A}[2]|_{bad}\) is in fact an isomorphism. Hence the proof is complete.

The commutative diagram of flows of Figure 3 concludes the section. It proves that labelled precubical sets and labelled \(\mathcal{A}\)-sets are equivalent from a directed algebraic topological point of view, \(K\) being any labelled precubical set.

An application. We give now a small application of the notion of geometric realization of labelled \(\mathcal{A}\)-set which will be reused later. The following proposition could of course be proved without using the topological material of this section. However, the proof would be more complicated (see the proof of [Gau08, Theorem 7.1]).

II.2.13. Proposition. Let \(\ell : \mathcal{A}[p] \to Sh_A\mathcal{L}_A(\Sigma)\) be a full labelled \(p\)-dimensional \(\mathcal{A}\)-cube with \(p ≥ 2\). Then there exists \((a_1, \ldots, a_p) ∈ \Sigma^p\) such that for every maximal path \((c_1, \ldots, c_p)\) of \(\mathcal{A}[p]\), i.e. for any \(p\)-tuple of 1-cubes of \(\mathcal{A}[p]\) with \(\partial^0_1(c_i) = (0, \ldots, 0), \partial^1_1(c_i) = \partial^0_1(c_{i+1})\) for \(1 ≤ i ≤ p - 1\) and \(\partial^1_1(c_p) = (1, \ldots, 1)\), one has \(\ell(c_1) * \cdots * \ell(c_p) = a_1 * \cdots * a_p\).

Proof. Let \((c_1, \ldots, c_p)\) and \((c_1', \ldots, c_p')\) be two maximal paths. Since there is a unique morphism from \((\vec{0}, \ldots, \vec{0})\) to \((\vec{1}, \ldots, \vec{1})\) in \(|\mathcal{A}[p]|_{bad}\) (this is the key point!), one has \(\ell(c_1) * \cdots * \ell(c_p) = \ell(c_1') * \cdots * \ell(c_p')\) in the flow \(|Sh_A\mathcal{L}_A(\Sigma)|_{bad}\). But the semigroup \(\mathbb{P}(\mathcal{L}_A(\Sigma)|_{bad}) \cong \mathbb{P}(\Sigma)\) is the free commutative semigroup generated by the elements of \(\Sigma\). Hence the result. □

II.3. Labelled coskeleton over a category of cubes

We give in this section the generalization of the notion of labelled coskeleton to any category of labelled precubical sets. The particular case \(\mathcal{A} = \square\) will give back the situation of [Gau08].

The unlabelled version, i.e. when \(\Sigma = \{\tau\}\) is the classical coskeleton functor, right adjoint to the truncation functor \([\mathcal{B}H81]\).

\(^5\)Intuitively, adding an algebraic relation is an idempotent operation.
The unlabelled case.

II.3.1. Proposition. Let \( n \geq 0 \).

1. The functor \( K \mapsto K_{\leq n} \) from \( \mathcal{A}_{n+1}^{\text{op}} \text{Set} \) to \( \mathcal{A}_n^{\text{op}} \text{Set} \) has a right adjoint denoted by \( \cosk_{n,n+1}^A : \mathcal{A}_n^{\text{op}} \text{Set} \to \mathcal{A}_{n+1}^{\text{op}} \text{Set} \). There is an inclusion of presheaves

\[
K \subseteq \cosk_{n,n+1}^A(K)
\]

natural with respect to the \( n \)-dimensional \( A \)-set \( K \). This inclusion induces the isomorphism \( K \cong \cosk_{n,n+1}^A(K)_{\leq n} \).

2. The functor \( K \mapsto K_{\leq n} \) from \( \mathcal{A}_n^{\text{op}} \text{Set} \) to \( \mathcal{A}_n^{\text{op}} \text{Set} \) has a right adjoint denoted by \( \cosk_{n}^A : \mathcal{A}_n^{\text{op}} \text{Set} \to \mathcal{A}_n^{\text{op}} \text{Set} \). There is an inclusion of presheaves \( K \subseteq \cosk_{n}^A(K) \) natural with respect to the \( n \)-dimensional \( A \)-set \( K \). This inclusion induces the isomorphism \( K \cong \cosk_{n}^A(K)_{\leq n} \).

3. Let \( \cosk_{n,n+1}^{A} = \cosk_{n+1,n+2}^{A} \circ \cdots \circ \cosk_{n,n+1}^{A} \circ \cosk_{n,n}^{A} \) where the functor

\[
\cosk_{n,n}^A : \mathcal{A}_n^{\text{op}} \text{Set} \to \mathcal{A}_n^{\text{op}} \text{Set}
\]

denotes the identity functor. Then there is an isomorphism of functors

\[
\cosk_{n}^A \cong \lim_{\to} \cosk_{n,n+p}^A.
\]

Proof. Let us prove the first assertion. The functor \( K \mapsto K_{\leq n} \) from \( \mathcal{A}_{n+1}^{\text{op}} \text{Set} \) to \( \mathcal{A}_n^{\text{op}} \text{Set} \) is induced by the inclusion of categories \( \mathcal{A}_{n+1}^{\text{op}} \subseteq \mathcal{A}_n^{\text{op}} \). Thus, the right adjoint is obtained by taking the right Kan extension along \( \mathcal{A}_{n+1}^{\text{op}} \subseteq \mathcal{A}_n^{\text{op}} \). The isomorphism of presheaves \( K_{\leq n} \cong K \) for an \( n \)-dimensional \( A \)-set \( K \) induces by adjunction a natural map \( K \to \cosk_{n,n+1}^A(K) \). Let \( p \leq n \). There is a bijection \( \mathcal{A}_n^{\text{op}} \text{Set}(\mathcal{A}[p],K) \cong \mathcal{A}_n^{\text{op}} \text{Set}(\mathcal{A}[p],\cosk_{n,n+1}^A(K)) \) because of the isomorphism \( \mathcal{A}[p]_{\leq n} \cong \mathcal{A}[p] \). Hence we obtain the desired inclusion. The proof of the second assertion is similar to the above proof. The third assertion is obvious.

\[\square\]

II.3.2. Definition. Let \( K \) be an \( A \)-set. An \((n+1)\)-cube of \( \cosk_{n,n+1}^A(K_{\leq n}) \), i.e. a map \( \partial \mathcal{A}[n+1] \to K \) is called a \( n \)-dimensional shell or \( n \)-shell of \( K \).

The labelled case. Before giving the labelled version of Proposition II.3.1 let us prove the following general categorical fact.

II.3.3. Proposition. Let \( L : C \Rightarrow D : R \) be a categorical adjunction where \( L \) is the left adjoint and \( R \) the right one. Let us suppose that \( C \) has all pullbacks. Let \( A \) be an object of \( C \). Then the functor \( L_A : C \downarrow A \to D \downarrow L(A) \) defined by \( L_A(X \to A) := L(X) \to L(A) \) has a right adjoint \( R_A \) defined by the following pullback diagram of \( C \):

\[
\begin{array}{ccc}
R_A(Y) & \to & R(Y) \\
\downarrow & & \downarrow \\
A & \to & R(L(A))
\end{array}
\]

where the map \( A \to R(L(A)) \) is the unit of the adjunction.
Note that we are going to use Proposition II.3.3 with \( C \) and \( D \) locally presentable. In this situation, the categories \( C\downarrow A \) and \( D\downarrow L(A) \) are both locally presentable as well by [AR94, Proposition 1.57]. In particular, the category \( C\downarrow A \) has a generator and is co-wellpowered. The functor \( L_A : C\downarrow A \to D\downarrow L(A) \) is colimit-preserving since \( L \) is colimit-preserving. So by the opposite of the Special Adjoint Functor Theorem, the functor \( L_A \) has a right adjoint.

**Proof.** Let \( X \to A \) be an object of \( C\downarrow A \). Let \( Y \to L(A) \) be an object of \( D\downarrow L(A) \). There is a bijection between the commutative diagrams of the form

\[
\begin{array}{ccc}
X & \rightarrow & R_A(Y) \\
\downarrow & & \downarrow \\
A & \rightarrow & A
\end{array}
\]

and the commutative diagrams of the form

\[
\begin{array}{ccc}
X & \rightarrow & R(Y) \\
\downarrow & & \downarrow \\
A & \rightarrow & R(L(A))
\end{array}
\]

because of the universal property of pullback. And there is a bijection between the latter diagrams and the commutative diagrams of the form

\[
\begin{array}{ccc}
L(X) & \rightarrow & Y \\
\downarrow & & \downarrow \\
L(A) & \rightarrow & L(A)
\end{array}
\]

by universality of adjunction. Hence the result.

Here is now the labelled analogue of Proposition II.3.1.

**II.3.4. Proposition.** Let \( n \geq 0 \).

(1) The functor \( K \mapsto K_{\leq n} \) from \( A_{n+1}^{op}\Set\downarrow\Sh_A\mathcal{L}_A(\Sigma) \) to \( A_{n}^{op}\Set\downarrow\Sh_A\mathcal{L}_A(\Sigma) \) has a right adjoint denoted by \( \cosk^{A,\Sigma}_{n,n+1} : A_{n+1}^{op}\Set\downarrow\Sh_A\mathcal{L}_A(\Sigma) \to A_{n}^{op}\Set\downarrow\Sh_A\mathcal{L}_A(\Sigma) \). There is an inclusion of presheaves

\[
K \subset \cosk^{A,\Sigma}_{n,n+1}(K)
\]
	natural with respect to the \( n \)-dimensional labelled \( A \)-set \( K \). This inclusion induces the isomorphism \( K \cong \cosk^{A,\Sigma}_{n,n+1}(K)_{\leq n} \).
(2) The functor $\mathcal{K} \mapsto \mathcal{K} \leq n$ from $\mathcal{A}^{\text{op}} \mathbf{Set}|_{\mathbf{Sh}_A \mathcal{L}_A(\{\Sigma\})}$ to $\mathcal{A}^{\text{op}} \mathbf{Set}|_{\mathbf{Sh}_A \mathcal{L}_A(\{\Sigma\})}$ has a right adjoint denoted by $\cosk^{\mathcal{A},\Sigma}_n$. There is an inclusion of presheaves $\mathcal{K} \subset \cosk^{\mathcal{A},\Sigma}_n(\mathcal{K})$ natural with respect to the $n$-dimensional labelled $\mathcal{A}$-set $\mathcal{K}$. This inclusion induces the isomorphism $\mathcal{K} \cong \cosk^{\mathcal{A},\Sigma}_n(\mathcal{K}) \leq n$.

(3) Let $\cosk^{\mathcal{A},\Sigma}_n : \mathcal{A}^{\text{op}} \mathbf{Set}|_{\mathbf{Sh}_A \mathcal{L}_A(\{\Sigma\})} \to \mathcal{A}^{\text{op}} \mathbf{Set}|_{\mathbf{Sh}_A \mathcal{L}_A(\{\Sigma\})}$ denote the identity functor. Then there is an isomorphism of functors $\cosk^{\mathcal{A},\Sigma}_n \cong \lim_{\rightarrow} \cosk^{\mathcal{A},\Sigma}_{n,p}$. 

Proof. We note that the categories $\mathcal{A}^{\text{op}} \mathbf{Set}|_{\mathbf{Sh}_A \mathcal{L}_A(\{\Sigma\})}$ and $\mathcal{A}^{\text{op}} \mathbf{Set}|_{\mathbf{Sh}_A \mathcal{L}_A(\{\Sigma\})}$ are isomorphic. So the theorem is a consequence of Proposition 11.3.1 and Proposition 11.3.3. □

Note that for every $n \geq 1$ and for every $n$-dimensional labelled $\mathcal{A}$-set $\mathcal{K}$, one has the pullback diagram of $\mathcal{A}$-sets

\[
\begin{array}{ccc}
\cosk^{\mathcal{A},\Sigma}_n(\mathcal{K}) & \rightarrow & \cosk^{\mathcal{A}}_n(\mathcal{K}) \\
\downarrow & & \downarrow \\
\mathbf{Sh}_A \mathcal{L}_A(\{\Sigma\}) & \rightarrow & \cosk^{\mathcal{A}}_n((\mathbf{Sh}_A \mathcal{L}_A(\{\Sigma\}))_{\leq n}).
\end{array}
\]

Intuitively, this means that the labelled coskeleton functor keeps from the unlabelled one only the shells which are compatibly labelled. For example, the boundary of a square is compatibly labelled if and only if opposite sides are labelled in the same way.

II.3.5. Definition. Let $\mathcal{K}$ be a labelled $\mathcal{A}$-set. An $(n + 1)$-cube of $\cosk^{\mathcal{A},\Sigma}_{n,n+1}(\mathcal{K}_{\leq n})$ is called a labelled $n$-dimensional shell or $n$-shell of $\mathcal{K}$.

The following proposition generalizes [Gau08, Definition 3.12 and Proposition 3.13].

II.3.6. Proposition. Let $\mathcal{K}$ be a labelled $\mathcal{A}$-set. The set of labelled $n$-dimensional shells of $\mathcal{K}$ is in bijection with the set of commutative diagrams of the form

\[
\begin{array}{ccc}
\partial \mathcal{A}[n+1] & \rightarrow & \mathcal{K} \\
\downarrow & & \downarrow \\
\mathcal{A}[n+1] & \rightarrow & \mathbf{Sh}_A \mathcal{L}_A(\{\Sigma\}).
\end{array}
\]

Since $\mathbf{Sh}_A \mathcal{L}_A(\{\tau\})$ is the terminal $\mathcal{A}$-set by Proposition 11.25, the case $\Sigma = \{\tau\}$ coincides with the unlabelled notion of Definition 11.3.2.
Proof. Let $A[n+1] \to \cosk_{n+1}^A(K_{\leq n})$ be a labelled $n$-shell of $K$. By adjunction, one obtains the commutative diagram of labelled $A$-sets

$$
\begin{array}{ccc}
\partial A[n+1] & \longrightarrow & K_{\leq n} \\
\downarrow & & \downarrow \\
A[n+1] & \longrightarrow & \cosk_{n+1}^A(K_{\leq n}).
\end{array}
$$

By composing with the labelling map $\cosk_{n+1}^A(K_{\leq n}) \to Sh_A L_A(!\Sigma)$, one obtains the commutative diagram of $A$-sets

$$
\begin{array}{ccc}
\partial A[n+1] & \longrightarrow & K \\
\downarrow & & \downarrow \\
A[n+1] & \longrightarrow & Sh_A L_A(!\Sigma).
\end{array}
$$

Conversely, from such a diagram, one obtains the commutative diagram of $A$-sets

$$
\begin{array}{ccc}
A[n+1]_{\leq n} & \longrightarrow & K_{\leq n} \\
\downarrow & & \downarrow \\
Sh_A L_A(!\Sigma) & \longrightarrow & Sh_A L_A(!\Sigma),
\end{array}
$$

hence the result by adjunction. \qed

---

### Part III. Mathematical treatment

#### III.1. Shell-complete category of cubes

The purpose of this combinatorial section is to address the following question. Is it possible to find a category of cubes $A$ such that $\cosk_{A}^1(A[a_1,\ldots,a_n]_{\leq 1})$ is exactly the labelled $n$-cube $A[a_1,\ldots,a_n]$ for every $n \geq 0$ and every $a_1,\ldots,a_n \in \Sigma$? Let us repeat one more time that there is always a strict inclusion $\Box[a_1,\ldots,a_n] \subset \cosk_{1}^\Sigma(\Box[a_1,\ldots,a_n]_{\leq 1})$ for every $n \geq 2$ by [Gau08, Proposition 3.15] and that this is the reason for introducing in [Gau08] the non-functorial subobject of $\cosk_{1}^\Sigma(\Box[a_1,\ldots,a_n]_{\leq 1})$ called the labelled directed coskeleton of $\Box[a_1,\ldots,a_n]_{\leq 1}$ (see Definition III.1.2). For $\Sigma = \{\tau\}$, i.e. for the unlabelled case, the previous equality reduces to finding a category of cubes $A$ such that $\cosk_{A}^1(A[n]_{\leq 1}) \cong A[n]$ for every $n \geq 0$. Such a category $A$ will be called a shell-complete category of cubes. We will see in Theorem III.1.15 that such a category of cubes answers the question above. We will see in Theorem III.1.22 that there exists one and only one such a category of cubes.
Definition and elementary properties.

III.1.1. Proposition. Let \( p, q \geq 0 \). The natural bijection

\[
\mathcal{A}^{op}\text{-Set}(\mathcal{A}[p], \mathcal{A}[q]) \cong \mathcal{A}([p], [q])
\]

induced by the mapping \( f \mapsto f_{\ast} \) given by the Yoneda lemma takes \( f : \mathcal{A}[p] \to \mathcal{A}[q] \) to \( f_{0} : \mathcal{A}([0], [p]) \to \mathcal{A}([0], [q]) \cong [q] \).

Proof. Let \( f : \mathcal{A}[p] \to \mathcal{A}[q] \) be a map of \( \mathcal{A}^{op}\text{-Set} \). The map \( \delta_{p}^{\epsilon_{p}} \ldots \delta_{1}^{\epsilon_{1}} : [0] \to [p] \) induces a commutative square of sets

\[
\begin{array}{ccc}
\mathcal{A}([p], [p]) & \xrightarrow{f_{p}} & \mathcal{A}([p], [q]) \\
\downarrow \downarrow & & \downarrow \downarrow \\
\mathcal{A}([0], [p]) & \xrightarrow{f_{0}} & \mathcal{A}([0], [q])
\end{array}
\]

for any \( \epsilon_{1}, \ldots, \epsilon_{p} \in \{0, 1\} \) since \( \mathcal{A} \) is a category of cubes. So

\[
f_{0}(\delta_{p}^{\epsilon_{p}} \ldots \delta_{1}^{\epsilon_{1}})^{\ast}(\text{Id}_{[p]}) = (\delta_{p}^{\epsilon_{p}} \ldots \delta_{1}^{\epsilon_{1}})^{\ast}(f_{p}(\text{Id}_{[p]})).
\]

Therefore \( f_{0} = f_{p}(\text{Id}_{[p]}) \).

The following proposition motivates the notion of shell-complete category of cubes.

III.1.2. Proposition. For any \( q \geq 0 \), the canonical map \( \mathcal{A}[q] \to \cosk_{1}^{A}(\mathcal{A}[q] \leq 1) \) induced by the isomorphism \( \mathcal{A}[q] \leq 1 \cong \mathcal{A}[q]_{\leq 1} \) is an inclusion of presheaves. For \( q = 0 \) or \( q = 1 \), this inclusion is always an equality for any category of cubes \( \mathcal{A} \).

Proof. Let \( x \) and \( y \) be two \( k \)-cubes of \( \mathcal{A}[q] \) having the same image by the map

\[
\mathcal{A}[q] \longrightarrow \cosk_{1}^{A}(\mathcal{A}[q] \leq 1).
\]

So one has the commutative diagram of \( \mathcal{A} \)-sets

\[
\begin{array}{ccc}
\mathcal{A}[k] & \xrightarrow{x} & \mathcal{A}[q] \\
y & \searrow & \cosk_{1}^{A}(\mathcal{A}[q] \leq 1) \\
\end{array}
\]

By adjunction, one obtains the commutative diagram of \( \mathcal{A} \)-sets

\[
\begin{array}{ccc}
\mathcal{A}[k] \leq 1 & \xrightarrow{x_{\leq 1}} & \mathcal{A}[q] \leq 1 \\
y_{\leq 1} & \searrow & \cong \\
\end{array}
\]

In particular, the two set maps \( x_{0}, y_{0} : [k] \Rightarrow [q] \) are equal. Thus, by Proposition III.1.1 one obtains \( x = y \). The last assertion is a consequence of Proposition II.3.1.

Hence the definition:

III.1.3. Definition. A category of cubes \( \mathcal{A} \) is shell-complete if for every \( p \geq 2 \), the canonical inclusion of presheaves \( \mathcal{A}[p] \subset \cosk_{1}^{A}(\mathcal{A}[p] \leq 1) \) is an isomorphism.
The category of cubes $\square$ is of course not shell-complete by [Gan08 Proposition 3.15]. For example, the precubical set $\cosk_1^\square([2]_{\leq 1})$ contains the 2-cube $x : \square[2] \to \cosk_1^\square([2]_{\leq 1})$ corresponding by adjunction to the map $\partial x : \partial [2] = [2]_{\leq 1} \to [2]_{\leq 1}$ characterized by $x_0(\epsilon_1, \epsilon_2) = (\epsilon_2, \epsilon_1)$. It is not a 2-cube of $\square[2]$ since the only 2-cube of the precubical set $\square[2]$ is the identity of $[2]$.

In general, for any $p, q \geq 2$, there exists at most one lift $\overline{\pi}$ in the commutative diagram of solid arrows

$$
\begin{array}{ccc}
\partial A[p] & \xrightarrow{x} & A[q] \\
\downarrow & & \downarrow \\
A[p] & \xrightarrow{\overline{\pi}} & 1,
\end{array}
$$

where 1 is the terminal object. Indeed, by Proposition III.1.1, the bijection of sets

$$\mathcal{A}^{op}(A[p], A[q]) \cong A([p], [q])$$

takes $\overline{\pi}$ to $x_0$. Shell-completeness means that this lift always exists.

**III.1.4. Theorem.** Let $A$ be a category of cubes. The following conditions are equivalent:

1. The category $A$ is shell-complete.
2. For any $p, q \geq 2$, for any map $x : \partial A[p] \to A[q]$, the set map $x_0 : [p] \to [q]$ belongs to $A$.
3. For any $p, q \geq 2$, any map $x : \partial A[p] \to A[q]$ factors uniquely as a composite $x : \partial A[p] \to A[p] \to A[q]$.

**Proof.** Let us prove the implication (1) $\implies$ (2). Let $x : \partial A[p] \to A[q]$ be a map of $A$-sets with $p, q \geq 2$. One can suppose that $p \leq q$ by Proposition II.1.17. Then $x$ factors (uniquely) as a composite

$$x : \partial A[p] = A[p]_{\leq p-1} \to A[q]_{\leq p-1} \to A[q].$$

One has the isomorphisms

$$A[q]_{\leq p-1} \cong \cosk^A_1(A[q]_{\leq 1})_{\leq p-1} \cong \cosk^A_{1,p-1}(A[q]_{\leq 1})$$

since $A$ is shell-complete and by Proposition III.1.1. So $x$ factors as a composite

$$x : \partial A[p] \to A[p] \to \cosk^A_{1,p-1}(\cosk^A_{1,p-1}(A[q]_{\leq 1})) = A[q]_{\leq p} \to A[q].$$

So $x_0 : [p] \to [q]$ is a morphism of $A$ by Proposition III.1.1.

Let us prove now the implication (2) $\implies$ (1). Proposition II.3.1 and Proposition III.1.2 imply that there is an inclusion of presheaves

$$A[q]_{\leq p} \subset (\cosk^A_1(A[q]_{\leq 1}))_{\leq p} = \cosk^A_{1,p}(A[q]_{\leq 1})$$

for any $p \geq 1$. This inclusion is trivially an equality for $p = 1$. Let us prove by induction on $p$ that this inclusion is an equality. This will establish the shell-completeness of $A$. Let us suppose the equality proved for $p \geq 1$. Let $x : A[p+1] \to \cosk^A_{1,p+1}(A[q]_{\leq 1})$ be a $(p+1)$-dimensional $A$-cube of $\cosk^A_{1,p+1}(A[q]_{\leq 1})$. By adjunction and by induction hypothesis, one obtains a map

$$\partial x : A[p+1]_{\leq p} = \partial A[p+1] \to \cosk^A_{1,p}(A[q]_{\leq 1}) \cong A[q]_{\leq p} \subset A[q].$$
By hypothesis, the map \(x_0 : [p + 1] \to [q]\) belongs to \(\mathcal{A}\). Thus, by Proposition III.1.1, there is a commutative diagram of \(\mathcal{A}\)-sets

\[
\begin{array}{ccc}
\partial \mathcal{A}[p + 1] & \longrightarrow & \cosk_{p}^{\mathcal{A}}(\mathcal{A}[q]_{\leq 1}) \\ & & \cong \mathcal{A}[q]_{\leq p} \\
\longrightarrow & & \downarrow \mathcal{C} \\
\mathcal{A}[p + 1] & \longrightarrow & \mathcal{A}[q]_{\leq p + 1}.
\end{array}
\]

Hence the equality for \(p + 1\).

The equivalence (2) \(\iff\) (3) is a consequence of Proposition III.1.1. \(\square\)

### Examples of shell-complete categories of cubes.

#### III.1.5. Theorem. The category of cubes \(\mathcal{\hat{D}}\) (i.e. the maximal category of cubes containing all adjacency-preserving maps) is shell-complete.

**Proof.** Let \(x : \partial \mathcal{\hat{D}}[p] \to \mathcal{\hat{D}}[q]\) be a morphism of \(\mathcal{\hat{D}}^{\text{op}}\text{Set}\) with \(p, q \geq 2\). For all \(k\) such that \(1 \leq k \leq p - 1\), one has the commutative diagram of sets

\[
\begin{array}{ccc}
\mathcal{\hat{D}}([k], [p]) & x_k & \mathcal{\hat{D}}([k], [q]) \\
& & \downarrow \mathcal{\hat{D}} \delta_1^{k} \ldots \delta_k^{1} \delta_1^{k} \ldots \delta_k^{1} \\
\mathcal{\hat{D}}([0], [p]) & x_0 & \mathcal{\hat{D}}([0], [q])
\end{array}
\]

for all \(\epsilon_1, \ldots, \epsilon_k \in \{0, 1\}\) since \(x : \partial \mathcal{\hat{D}}[p] \to \mathcal{\hat{D}}[q]\) is a map of \(\mathcal{\hat{D}}^{\text{op}}\text{Set}\) and where the set map \(\mathcal{\hat{D}} \delta_1^{k} \ldots \delta_k^{1}\) is induced by the morphism \(\delta_1^{k} \ldots \delta_k^{1} : [0] \to [k]\) of \(\mathcal{\hat{D}}\). With \(\phi \in \mathcal{\hat{D}}([k], [p])\), that means that \(x_k(\phi)(\delta_1^{k} \ldots \delta_k^{1}) = x_0(\phi(\delta_1^{k} \ldots \delta_k^{1}))\). Thus, one obtains \(x_k(\phi)(\epsilon_1, \ldots, \epsilon_k) = x_0(\phi(\epsilon_1, \ldots, \epsilon_k))\). So \(x_k(\phi) = x_0(\phi)\) with the identification \(\mathcal{\hat{D}}([0], [p]) \cong [p]\). Let \((\epsilon_1, \ldots, \epsilon_p)\) and \((\epsilon'_1, \ldots, \epsilon'_p)\) be two elements of \([p]\) with \(\epsilon_i = \epsilon'_i\) for all \(i\) but one denoted by \(i_0\). Suppose moreover that \(\epsilon_{i_0} = 0\) and \(\epsilon'_{i_0} = 1\). Since \(p \geq 2\), there exists \(i_1 \in \{1, \ldots, p\}\) \(\backslash\{i_0\}\). Consider \(\delta_{i_1}^{i_1} : [p - 1] \to [p]\). Then \(x_{p - 1}(\delta_{i_1}^{i_1}) = x_0(\epsilon_{i_1})\). The preceding equality applied to \((\epsilon_1, \ldots, \epsilon_{i_1}, \ldots, \epsilon_p)\) gives

\[
x_0(\epsilon_1, \ldots, \epsilon_p) = x_{p - 1}(\delta_{i_1}^{i_1})(\epsilon_1, \ldots, \epsilon_{i_1}, \ldots, \epsilon_p) < x_{p - 1}(\delta_{i_1}^{i_1})(\epsilon'_1, \ldots, \epsilon_{i_1}, \ldots, \epsilon'_p) = x_0(\epsilon'_1, \ldots, \epsilon'_p)
\]

since the map \(x_{p - 1}(\delta_{i_1}^{i_1}) : [p - 1] \to [q]\) is a morphism of the small category \(\mathcal{\hat{D}}\). So the set map \(x_0 : [p] \to [q]\) is adjacency-preserving, i.e. it belongs to the small category \(\mathcal{\hat{D}}\). Thus, the small category \(\mathcal{\hat{D}}\) is shell-complete by Theorem III.1.4. \(\square\)

#### III.1.6. Proposition. Let \(\mathcal{A}\) be a category of cubes. Let \(p, q \geq 2\). The set \(\mathcal{A}^{\text{op}}\text{Set}(\partial \mathcal{A}[p], \mathcal{A}[q])\) is equal to the set of families \((f_i^\alpha) : \mathcal{A}[p - 1] \to \mathcal{A}[q])\) of morphisms of \(\mathcal{A}^{\text{op}}\text{Set}\) with \(1 \leq i \leq p\) and \(\alpha \in \{0, 1\}\) with \((f_j^\beta)_{0}^{\delta_1} = (f_j^\beta)_{0}^{\delta_1}\) for any \(i < j\) and any \(\alpha, \beta \in \{0, 1\}\).
Proof. Let \( f : \partial A[p] \to A[q] \) be a morphism of \( \mathcal{A}^{\text{op}}\text{Set} \). The 2p inclusions \( A[p-1] \subset \partial A[p] \) with \( 1 \leq i \leq p \) and \( \alpha \in \{0,1\} \) induce 2p maps \( f^\alpha_i : A[p-1] \to \partial A[p] \to A[q] \) such that \( (f^\alpha_i)_0 = f_0 \delta^\alpha_i \) with \( 1 \leq i \leq p \) and \( \alpha \in \{0,1\} \). The equalities \( (f^\beta_j)_0 \delta^\alpha_i = (f^\alpha_i)_0 \delta^\beta_j \) for any \( i < j \) and any \( \alpha, \beta \in \{0,1\} \) are then a consequence of the cocubical relations.

Conversely, let \( (f^\alpha_i : A[p-1] \to A[q]) \) be a family of morphisms of \( \mathcal{A}^{\text{op}}\text{Set} \) with \( 1 \leq i \leq p \) and \( \alpha \in \{0,1\} \) such that \( (f^\beta_j)_0 \delta^\alpha_i = (f^\alpha_i)_0 \delta^\beta_j \) for any \( i < j \) and any \( \alpha, \beta \in \{0,1\} \). Consider the set map \( g : [p] \to [q] \) defined by \( g(\epsilon_1, \ldots, \epsilon_p) := (f^\beta_p)_0(\epsilon_1, \ldots, \epsilon_{p-1}) \). Then \( g \delta^\alpha_i = (f^\alpha_i)_0 \) by definition of \( g \) and for any \( 0 \leq i < p \), one has

\[
g^\delta^\alpha_i (\epsilon_1, \ldots, \epsilon_{p-1}) = (f^\beta_{p-1})_0^\alpha (\epsilon_1, \ldots, \epsilon_{p-2}) = (f^\beta_{p-1})_0^\alpha (\epsilon_1, \ldots, \epsilon_{p-2}) = (f^\alpha_i)_0 (\epsilon_1, \ldots, \epsilon_{p-1})
\]

for any \( \alpha, \epsilon_{p-1} \in \{0,1\} \) thanks to the cocubical relations. So one obtains \( g \delta^\alpha_i = (f^\alpha_i)_0 \) for \( 0 \leq i < p \) and \( \alpha \in \{0,1\} \). The mapping \( \phi \to g \phi \) gives rise for each \( 0 \leq k \leq p-1 \) to a set map \( g_k : \partial A[p] \to A(k] \to A[q]_k := A([k], [q]) \). For any morphism \( \psi : [k'] \to [k] \) of \( A \) with \( 0 \leq k' \leq k \leq p-1 \), one obtains a diagram of sets

\[
\begin{array}{ccc}
\partial A[p] & \xrightarrow{g} & A[q]_k \\
\downarrow & & \downarrow \\
\partial A[p][k'] & \xrightarrow{g} & A[q][k']
\end{array}
\]

which is commutative since the two boundaries of the square takes \( \phi \in A[p]_k \) to \( g \phi \psi \).

\[\square\]

III.1.7. Proposition. Let \( \mathcal{A} \) and \( \mathcal{B} \) be two categories of cubes such that \( \mathcal{A} \subset \mathcal{B} \). Let \( p, q \geq 2 \). Then one has the inclusion \( \mathcal{A}^{\text{op}}\text{Set}(\partial A[p], A[q]) \subset \mathcal{B}^{\text{op}}\text{Set}(\partial B[p], B[q]) \) by identifying the maps \( f \) with the corresponding set maps \( f_0 \) from \( [p] \) to \( [q] \).

Proof. This is a corollary of Proposition III.1.6 and of the fact that \( \mathcal{A}([p-1], [q]) \subset \mathcal{B}([p-1], [q]) \).

\[\square\]

III.1.8. Theorem. There exists a smallest shell-complete category, denoted by \( \square \).

Proof. Let \( (\square (i))_{i \in I} \) be the class of all shell-complete small categories of cubes. This class is non-empty by Theorem III.1.5 and small since for any \( i \), there is the inclusion \( \square (i) \subset \text{PoSet} \). Consider the small category \( \square = \bigcap_{i \in I} \square (i) \). Let \( f : \partial \square [p] \to \square [q] \) be a map of \( \square^{\text{op}}\text{Set} \) with \( p, q \geq 2 \). By Proposition III.1.7 the morphism of presheaves \( f \) gives rise for each \( i \in I \) to a morphism of presheaves \( f (i) : \partial \square (i)[p] \to \square (i)[q] \). By Theorem III.1.3 \( f_0 = (f (i))_0 \) is a morphism of \( \square (i) \) for each \( i \in I \). So by Theorem III.1.4 again, the category \( \square \) is shell-complete.

\[\square\]

Some combinatorial lemmas. Let us recall that \( \sigma_i : [n] \to [n] \) is the set map defined for \( 1 \leq i \leq n-1 \) and \( n \geq 2 \) by \( \sigma_i(\epsilon_1, \ldots, \epsilon_n) = (\epsilon_1, \ldots, \epsilon_{i-1}, \epsilon_{i+1}, \epsilon_i, \epsilon_{i+2}, \ldots, \epsilon_n) \) (cf. Definition II.1.10).

III.1.9. Proposition. (GM03 p195) Let \( \sigma_i : [n] \to [n] \) be the set map defined for \( 1 \leq i \leq n-1 \) and \( n \geq 2 \) by \( \sigma_i(\epsilon_1, \ldots, \epsilon_n) = (\epsilon_1, \ldots, \epsilon_{i-1}, \epsilon_{i+1}, \epsilon_i, \epsilon_{i+2}, \ldots, \epsilon_n) \). One has the relations

\[
\sigma_i \delta^\alpha_j = \delta^\alpha_j \sigma_i-1 \text{ for } j < i, \quad \sigma_i \delta^\alpha_j = \delta^\alpha_{i+1} \text{ for } j = i, \quad \sigma_i \delta^\alpha_j = \delta^\alpha_i \text{ for } j = i+1 \text{ and } \sigma_i \delta^\alpha_j = \delta^\alpha_j \sigma_i \text{ for } j > i+1.
\]
III.1.10. **Proposition.** \( \sigma_i \in \bar{\Box} \).

**Proof.** Let us prove by induction on \( n \geq 2 \) that the set maps \( \sigma_i : [n] \to [n] \) for \( 1 \leq i \leq n - 1 \) belong to \( \bar{\Box} \). The composite map \( \partial \sigma_1 : \partial [2] \subset [2] \to [2] \) induces a map \( \partial \sigma_1 : \partial [2] \to \Box [2] \) by Proposition III.1.7 since \( \Box \subset \bar{\Box} \). So \( \sigma_1 : [2] \to [2] \) is a map of \( \Box \) by Theorem III.1.3 since \( \Box \) is shell-complete. Hence the proof is complete for \( n = 2 \). Now assume that \( n > 2 \). By Proposition III.1.9 and by induction hypothesis, the \( 2n \) set maps \( \sigma_i \delta^j_\alpha : [n - 1] \subset [n] \to [n] \) belong to \( \bar{\Box} \). These \( 2n \) morphisms of \( \Box \) induce a morphism \( \partial \Box [n] \to \Box [n] \) by Proposition III.1.6. So \( \sigma_i : [n] \to [n] \) belongs to \( \bar{\Box} \) by shell-completeness. \( \square \)

To our knowledge, the structure maps introduced below are new. They are related to the notion of *connection* in the setting of *cubical* sets, see [AA89] [AABS02]; indeed, with their notation of \( \varepsilon \) for degeneracies and \( \Gamma^\circ \) for connections, one has \( \Gamma^+ = \varepsilon_i \gamma_i \) and \( \Gamma^- = \varepsilon_i \gamma_i \). An example of use of these connections in directed algebraic topology can be found in [Gau01] and [Gau01].

III.1.11. **Definition.** Let \( \gamma_i : [n] \to [n] \) be the set map defined for \( 1 \leq i \leq n - 1 \) and \( n \geq 2 \) by

\[
\gamma_i(e_1, \ldots, e_n) = (e_1, \ldots, e_{i-1}, \max(e_i, e_{i+1}), \min(e_i, e_{i+1}), e_{i+2}, \ldots, e_n).
\]

These maps are called the *transverse degeneracy maps*.

III.1.12. **Proposition.** One has the relations \( \gamma_j \delta^\alpha_i = \delta^\alpha_i \gamma_j \) for \( j < i - 1 \), \( \gamma_j \delta^\alpha_i = \delta^\alpha_i \gamma_{j-1} \) for \( j \geq i + 1 \), \( \gamma_j \delta^\alpha_i = \delta^\alpha_i \gamma_{i-\alpha} \) for \( j = i - 1 \) and \( \gamma_j \delta^\alpha_i = \delta^\alpha_i \gamma_{i+1-\alpha} \) for \( j = i \).

**Proof.** The relation \( \gamma_j \delta^\alpha_i = \delta^\alpha_i \gamma_j \) for \( j < i - 1 \) is obvious. One has

\[
\gamma_j \delta^\alpha_i (e_1, \ldots, e_{i-1}, \alpha, e_i, \ldots, e_{n-1}) = \delta^\alpha_i \gamma_{j-1} (e_1, \ldots, e_{n-1})
\]

for \( j \geq i + 1 \). For \( j = i - 1 \), one has

\[
\gamma_j \delta^1_i (e_1, \ldots, e_{n-1}) = \gamma_j (e_1, \ldots, e_{i-1}, 1, e_i, \ldots, e_{n-1}) = \delta^1_i (e_1, \ldots, e_{n-1})
\]
and

\[
\gamma_j \delta^0_i (e_1, \ldots, e_{n-1}) = \gamma_j (e_1, \ldots, e_{i-1}, 0, e_i, \ldots, e_{n-1}) = \delta^0_i (e_1, \ldots, e_{n-1}).
\]

Finally for \( j = i \), one has

\[
\gamma_j \delta^1_i (e_1, \ldots, e_{n-1}) = \gamma_j (e_1, \ldots, e_{i-1}, 1, e_i, \ldots, e_{n-1}) = \delta^1_i (e_1, \ldots, e_{n-1})
\]
and

\[
\gamma_j \delta^0_i (e_1, \ldots, e_{n-1}) = \gamma_j (e_1, \ldots, e_{i-1}, 0, e_i, \ldots, e_{n-1}) = \delta^0_i (e_1, \ldots, e_{n-1}).
\]
\( \square \)

III.1.13. **Proposition.** \( \gamma_i \in \bar{\Box} \).

**Proof.** The proof is mutatis mutandis the one of Proposition III.1.10 \( \square \)

III.1.14. **Proposition.** Let \( 0 \leq m \leq n \). Every adjacency-preserving (resp. adjacency-preserving one-to-one) map \( f : [m] \to [n] \) factors uniquely as a composite \( [m] \overset{\psi}{\to} [m] \overset{\phi}{\to} [n] \) with \( \phi \in \Box \) and \( \psi \) adjacency-preserving (resp. adjacency-preserving one-to-one).

Note that by a cardinality argument, if \( \psi : [m] \to [m] \) is one-to-one, then it is bijective.

**Proof.** One has \( d(f((0, \ldots, 0), f(1, \ldots, 1))) = m \). So by Proposition III.1.3 \( f([m]) \) is an \( m \)-subcube of \( [n] \). So the assertion is a consequence of Proposition III.1.6 \( \square \)
The uniqueness and a negative result.

III.1.15. Theorem. The category of cubes $\square$ is the unique shell-complete category of cubes.

Proof. It suffices to prove that the inclusion $\square \subset \hat{\square}$ is an equality since the category of cubes $\hat{\square}$ is the maximal category of cubes.

For any $p, q \geq 0$, there is the inclusion $\square([p],[q]) \subset \hat{\square}([p],[q]) \subset \hat{\square}([p],[q])$ and one wants to prove the equality $\hat{\square}([p],[q]) = \hat{\square}([p],[q])$.

For $p > q$, one has $\square([p],[q]) = \square([p],[q]) = \varnothing$ by Proposition III.1.17. One has $\square([0],[q]) = \square([0],[q]) = \varnothing$ by Proposition III.1.17 again. It remains to prove the equality $\hat{\square}([p],[q]) = \hat{\square}([p],[q])$ for $2 \leq p \leq q$ for a fixed $q$ by induction on $p$.

First of all, let us treat the case $p = 2$. Let $f \in \hat{\square}([2],[q])$. By Proposition III.1.14, the set map $f$ factors uniquely as a composite of set maps $f : [2] \xrightarrow{\phi} [2] \xrightarrow{\psi} [q]$ with $\phi \in \hat{\square}$ and $\psi \in \square$. It is easy to see that the set $\hat{\square}([2],[2])$ consists of the four set maps $\text{Id}_{[2]}: (e_1,e_2) \mapsto (e_1,e_2)$, $\sigma_1: (e_1,e_2) \mapsto (e_2,e_1)$, $\gamma_1: (e_1,e_2) \mapsto (\text{min}(e_1,e_2), \text{max}(e_1,e_2))$ and $\sigma_1\gamma_1: (e_1,e_2) \mapsto (\text{max}(e_1,e_2), \text{min}(e_1,e_2))$. So $\hat{\square}([2],[2]) = \hat{\square}([2],[2])$ by Proposition III.1.10 and Proposition III.1.13. Therefore one obtains $\hat{\square}([2],[q]) = \hat{\square}([2],[q])$ for any $q \geq 0$.

Let us now treat the case $p \geq 3$. Every set map $f \in \hat{\square}([p],[q])$ for $p \geq 3$ gives rise to a map $x : \hat{\square}([p]) \to \hat{\square}([q])$ such that $x_0 = f$ by Proposition III.1.1. By composition, one obtains a map $\partial x : \partial \hat{\square}([p]) \subset \hat{\square}([p]) \to \hat{\square}([q])$. By Proposition III.1.6, one obtains $2p$ maps $x_i^\alpha : [p-1] \to [q]$ with $1 \leq i \leq p$ and $\alpha \in \{0,1\}$ such that $(x_i^\alpha)_0 = (x_i^\beta)_0 = (x_i^\gamma)_0 = (x_i^\delta)_0$ for any $i < j$ and any $\alpha, \beta, \gamma, \delta \in \{0,1\}$. By Proposition III.1.1, the $2p$ set maps $(x_i^\alpha)_0 : [p-1] \to [q]$ for $1 \leq i \leq p$ and $\alpha \in \{0,1\}$ belong to $\hat{\square}$. So by induction hypothesis, the latter set maps belong to $\square$ as well. By Proposition III.1.1 again, one obtains $2p$ maps $y_i^\alpha : [p-1] \to [q]$ with $1 \leq i \leq p$ and $\alpha \in \{0,1\}$ such that $(y_i^\alpha)_0 = (y_i^\beta)_0 = (y_i^\gamma)_0 = (y_i^\delta)_0$ for any $i < j$ and any $\alpha, \beta, \gamma, \delta \in \{0,1\}$ and such that $(y_i^\alpha)_0 = (x_i^\alpha)_0$ for all $1 \leq i \leq p$ and $\alpha \in \{0,1\}$. So by Proposition III.1.6 one obtains a map $\partial y : \partial \hat{\square}([p]) \to \hat{\square}([q])$ such that $(\partial y)_0 = y_i^\alpha$ for all $1 \leq i \leq p$ and $\alpha \in \{0,1\}$. By Theorem III.1.4 and since $\hat{\square}$ is shell-complete, the set map $(\partial y)_0 = x_0 = f : [p] \to [q]$ then belongs to $\square$. The induction on $p$ is complete.

III.1.16. Theorem. The category of cubes $\square$ generated by the $\delta_i^0$, $\sigma_i$ and $\gamma_i$ operators is not shell-complete. In other terms, the inclusion of small categories $\square \subset \hat{\square}$ is strict.

Proof. It suffices to find a morphism of $\hat{\square}$ which does not belong to $\square$. Consider the set map $f : [3] \to [3]$ sending the poset of vertices of the 3-cube (Figure 3) to the poset depicted in Figure 3.

It is clear that $f$ is adjacency-preserving, i.e. $f \in \hat{\square}$. One has

- $f(0,0,0) = (0,0,0), f(0,1,0) = (0,0,1), f(0,0,1) = (0,0,1), f(0,1,1) = (0,1,1), f(1,0,0) = (0,0,0), f(1,0,1) = (0,0,1), f(0,1,0) = (0,1,1), f(1,1,1) = (1,1,1)$, so $f\delta_0 = \delta_0^0\sigma_1\gamma_1.$
- $f(0,0,0) = (0,0,0), f(1,0,0) = (1,0,1), f(1,1,0) = (0,1,1), f(1,1,1) = (1,1,1), f(1,0,1) = (1,0,1), f(1,1,0) = (1,1,1), f(0,1,0) = (0,1,1), f(1,0,0) = (0,0,0), f(1,0,1) = (0,0,1), f(0,1,1) = (0,1,1), f(1,1,1) = (1,1,1), so $f\delta_1 = \delta_1^0\sigma_1.$
- $f(0,0,0) = (0,0,0), f(1,0,0) = (0,0,1), f(0,0,1) = (0,0,1), f(0,1,0) = (0,1,1), f(1,0,1) = (1,0,1), f(1,1,1) = (1,1,1), so $f\delta_2 = \delta_2^0\sigma_1\gamma_1.$
- $f(0,0,0) = (0,0,0), f(0,1,0) = (0,1,1), f(1,0,0) = (1,0,1), f(0,1,1) = (0,1,1), f(1,0,1) = (1,1,1), f(1,1,0) = (1,1,1), so $f\delta_3 = \delta_3^0\sigma_1.$


The six set maps $\delta_0^1 = \delta_0^0 \pi_1 \gamma_1$.

- $f(0, 0, 0) = (0, 0, 0)$, $f(1, 0, 0) = (0, 0, 1)$, $f(0, 1, 0) = (0, 0, 1)$, $f(1, 1, 0) = (0, 1, 1)$, so $f\delta_0^1 = \delta_0^0 f_0$.
- $f(0, 0, 1) = (0, 0, 1)$, $f(1, 0, 1) = (1, 0, 1)$, $f(0, 1, 1) = (0, 1, 1)$, $f(1, 1, 1) = (1, 1, 1)$, so $f\delta_0^1 = \delta_0^1$.

The six set maps $f\delta_0^1$, $f\delta_1^1$, $f\delta_2^0$, $f\delta_0^1$, $f\delta_0^0$ and $f\delta_0^1$ belong to $\square$, giving rise to a morphism of presheaves $\partial \square [3] \to \square [3]$ by Proposition III.1.6. Any set map $g : [2] \to [3]$ of $\square$ factors uniquely as a composite $[2] \xrightarrow{g_1} [2] \xrightarrow{g_2} [3]$ with $g_1 \in \square$ and $g_2 \in \square$ by Proposition III.1.14.

So the set map $f_0 = \sigma_1 \gamma_1 : [2] \to [2]$ is the unique set map of $\square$ such that $f\delta_0^1 = \delta_0^0 f_0$. And the set map $f_1 = \sigma_1 : [2] \to [2]$ is the unique set map of $\square$ such that $f\delta_1^1 = \delta_1^1 f_1$. Since $f_0 \neq f_1$, the set map $f : [3] \to [3]$ cannot be a composite of $\sigma_i : [3] \to [3]$ and $\gamma_i : [3] \to [3]$ with $i = 1, 2$ by Proposition III.1.9 and Proposition III.1.12. Therefore $f \notin \square$.

In fact, we do not know any “small” presentation by generators and relations of the small category $\square$. This is an interesting and open question. It seems to be related to similar questions arising in combinatorics:

III.1.17. Definition. An extremal path of $[n]$ is a $n$-tuple $(A_1, \ldots, A_n)$ of $[n]$ such that $A_1 = 0_1 < A_2 < \cdots < A_{n-1} < A_n = 1_n$. The set of extremal paths of $[n]$ is denoted by $P_n$.

III.1.18. Notation. Let $e_i$ be the element $(\epsilon_1, \ldots, \epsilon_n)$ of $[n]$ such that $\epsilon_i = 1$ if and only if $i \in I$.

There is a bijection $p : \Sigma_n \to P_n$ from the $n$-th symmetric group $\Sigma_n$ to $P_n$ defined by $p(\sigma) = (e_\varnothing, e_{\{\sigma(1)\}}, e_{\{\sigma(1), \sigma(2)\}}, \ldots, e_{\{1, \ldots, n\}})$. 

III.1.19. Proposition. Let $\sigma \in \Sigma_n$. Let $s_i$ be the transposition $(i \ i + 1)$. Then one has the equalities
\[ p(\sigma s_i) = (\sigma_i(e_{\sigma}), \sigma_i(e_{\{\sigma(1)\}}), \sigma_i(e_{\{\sigma(1), \sigma(2)\}}), \ldots, \sigma_i(e_{\{1, \ldots, n\}})) \]
and
\[ p(\sigma, \pi_i) = (\gamma_i(e_{\sigma}), \gamma_i(e_{\{\sigma(1)\}}), \gamma_i(e_{\{\sigma(1), \sigma(2)\}}), \ldots, \gamma_i(e_{\{1, \ldots, n\}})) \]
where $\pi_i$ are the elementary increasing bubble sort operators (cf. [GM03, III.1.20]. It suffices to prove the identity
\[ (\max((\pi_i, \pi_1)^{\gamma_i(1)}(e_1, e_2, e_3) = 1) = \max((\pi_i, \pi_1)^{\gamma_i(2)}(e_1, e_2, e_3) = 1) \]
and
\[ (\pi_i, \pi_1)^{\gamma_i(1)}(e_1, e_2, e_3) = \max((\pi_i, \pi_1)^{\gamma_i(2)}(e_1, e_2, e_3) = 1) \]
for any $(e_1, e_2, e_3) \in \{0, 1\}^3$. By distributivity of min and max over each other, one has:
\[ \min((\max(e_1, e_2), \max(e_1, e_2)), \min(e_2, e_3)) \]
\[ = \max(\min(e_1, e_2), \min(e_1, e_2), \min(e_2, e_3)) \]
\[ = \max(\min(e_1, e_2), \min(e_1, e_2), \min(e_2, e_3)). \]
Proof. Trivial. □

As a corollary, the monoid $\pi([n], [n])$ is isomorphic to the monoid $\langle s_i, \pi_i \rangle_{i=1, \ldots, n}$ of set maps from $\Sigma_n$ to itself generated by the operators $s_i$ and $\pi_i$. In particular, it satisfies the relations:

III.1.20. Proposition. Let $n \geq 1$. The monoid of set maps from $[n]$ to itself generated by the $\sigma_i$ and $\gamma_i$ operators satisfies the relations:

- $\sigma_i \sigma_i = \text{Id}$, $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $i = j - 1$ and $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $i < j - 1$ (the Moore relations for symmetry operators)
- $\gamma_i \gamma_i = \gamma_i$, $\gamma_i \gamma_j \gamma_i = \gamma_j \gamma_i \gamma_j$ for $i = j - 1$ and $\gamma_i \gamma_j = \gamma_j \gamma_i$ for $i < j - 1$ (the Moore relations for transverse degeneracy)
- $\gamma_j \gamma_i = \gamma_i \gamma_j$ for $j > i + 1$ and $j < i - 1$, $\gamma_i \gamma_i = \gamma_i$, $\gamma_i \gamma_j \gamma_i = \gamma_i \gamma_j \gamma_i$.

Proof. The Moore relations for symmetry operators are explained for example in [GM03, Theorem 8.1]. Let us prove the Moore relations for transverse degeneracy maps. The relations $\gamma_i \gamma_i = \gamma_i$ and $\gamma_i \gamma_j = \gamma_j \gamma_i$ for $i = j - 1$ are obvious. One wants to prove that $\gamma_i \gamma_{i+1} \gamma_i = \gamma_{i+1} \gamma_i \gamma_i$. It suffices to prove the identity $\gamma_1 \gamma_2 \gamma_1(e_1, e_2, e_3) = \gamma_2 \gamma_1 \gamma_2(e_1, e_2, e_3)$. One has
\[ \gamma_1 \gamma_2 \gamma_1(e_1, e_2, e_3) = \gamma_1 \gamma_2(\max(e_1, e_2), \min(e_1, e_2), e_3) \]
\[ = \gamma_1(\max(e_1, e_2), \max(\min(e_1, e_2), e_3), \min(e_1, e_2, e_3)) \]
\[ = (\max(e_1, e_2, e_3), \min(\max(e_1, e_2, e_3), \min(e_1, e_2, e_3)), \min(e_1, e_2, e_3)) \]
and
\[ \gamma_2 \gamma_1 \gamma_2(e_1, e_2, e_3) = \gamma_2 \gamma_1(e_1, \max(e_2, e_3), \min(e_2, e_3)) \]
\[ = \gamma_2(\max(e_1, e_2, e_3), \min(e_1, \max(e_2, e_3)), \min(e_2, e_3)) \]
\[ = (\max(e_1, e_2, e_3), \min(\max(e_1, e_2, e_3), \min(e_2, e_3)), \min(e_1, e_2, e_3)) \]
It remains to check the equality
\[ \min(\max(e_1, e_2), \max(\min(e_1, e_2), e_3)) = \max(\min(e_1, \max(e_2, e_3)), \min(e_2, e_3)) \]
for any $(e_1, e_2, e_3) \in \{0, 1\}^3$.
The proof will be complete by establishing the relations between transverse degeneracy maps and symmetry operators. The equalities \( \gamma_j \sigma_i = \sigma_i \gamma_j \) for \( j > i + 1 \) and \( j < i - 1 \) and \( \gamma_i \sigma_i = \gamma_i \) are obvious. One wants to prove that \( \sigma_{i+1} \gamma_i \sigma_{i+1} = \sigma_i \gamma_{i+1} \sigma_i \). It suffices to prove the identity \( \sigma_2 \gamma_1 \sigma_2 (\epsilon_1, \epsilon_2, \epsilon_3) = \sigma_1 \gamma_2 \sigma_1 (\epsilon_1, \epsilon_2, \epsilon_3) \). One has
\[
\sigma_2 \gamma_1 \sigma_2 (\epsilon_1, \epsilon_2, \epsilon_3) = \sigma_2 (\max (\epsilon_1, \epsilon_3), \min (\epsilon_1, \epsilon_3), \epsilon_2) = (\max (\epsilon_1, \epsilon_3), \epsilon_2, \min (\epsilon_1, \epsilon_3))
\]
and
\[
\sigma_1 \gamma_2 \sigma_1 (\epsilon_1, \epsilon_2, \epsilon_3) = \sigma_1 (\epsilon_2, \max (\epsilon_1, \epsilon_3), \min (\epsilon_1, \epsilon_3)) = (\max (\epsilon_1, \epsilon_3), \epsilon_2, \min (\epsilon_1, \epsilon_3)).
\]

\[\text{[HT09]}\] Conjecture 3.5 and Paragraph 3.1.1 suggest the following conjecture:

**III.1.21. Conjecture.** Proposition [III.1.20](#) gives a presentation by generators and relations of the monoid \( \square ([n], [n]) \) for every \( n \geq 2 \).

**Functorial interpretation of the labelled directed coskeleton.** For \( n \geq 2 \), and for every \( a_1, \ldots, a_n \in \Sigma \), the inclusion \( \square [a_1, \ldots, a_n] \subset \cosk_1 \square (\square [a_1, \ldots, a_n]) \) is strict by [Gau08 Proposition 3.15](#). The strictness of the latter inclusion means that the concurrent execution of \( n \) actions always assemble in \( \cosk_1 \square (\square [a_1, \ldots, a_n]) \) to several labelled \( n \)-cubes. To remedy this problem, the labelled directed coskeleton construction is introduced in [Gau08](#). Its main feature is to select one \( n \)-cube (the *non-twisted* one) for each multiset of \( n \) actions running concurrently:

**III.1.22. Definition.** Let \( K \) be a 1-dimensional labelled precubical set with \( K_0 = [p] \) for some \( p \geq 0 \). The labelled directed coskeleton of \( K \) is the labelled precubical set \( \cosk_1 \square (K) \) defined as the subobject of \( \cosk_1 \square (K) \) such that:

- \( \cosk_1 \square (K)_{\leq 1} = \cosk_1 \square (K)_{\leq 1} \)
- for every \( n \geq 2 \), \( x \in \cosk_1 \square (K)_n \) is an \( n \)-cube of \( \cosk_1 \square (K) \) if and only if the set map \( x_0 : [n] \to [p] \) is non-twisted, i.e. \( x_0 : [n] \to [p] \) is a composite\(^6\)
  \[
  x_0 : [n] \xrightarrow{\phi} [q] \xrightarrow{\psi} [p],
  \]
  where \( \psi \) is a morphism of the small category \( \square \) and where \( \phi \) is of the form \( (\epsilon_1, \ldots, \epsilon_n) \mapsto (\epsilon_{i_1}, \ldots, \epsilon_{i_q}) \) such that \( \{1, \ldots, n\} \subset \{i_1, \ldots, i_q\} \) and such that the first appearance of \( \epsilon_i \) is before the first appearance of \( \epsilon_{i+1} \) in \( (\epsilon_{i_1}, \ldots, \epsilon_{i_q}) \) for any \( 1 \leq i \leq n \) by reading from the left to the right.

The fundamental property of the labelled directed coskeleton is then:

\(^6\)The factorization is necessarily unique.
III.1.23. Theorem. \[\text{[Gau08, Proposition 3.21]}\] Let \( n \geq 1 \). Let \((a_1, \ldots, a_n) \in \Sigma^n\). Then one has the isomorphism of labelled precubical sets
\[
\cosk^\Sigma (\Box[a_1, \ldots, a_n]_{\leq 1}) \cong [a_1, \ldots, a_n].
\]

The following theorem gives the functorial interpretation of the labelled directed coskeleton construction:

III.1.24. Theorem. The category of cubes \( \mathcal{A} = \hat{\Box} \) (i.e. the maximal category of cubes containing all adjacency-preserving maps) is the only category of cubes such that for every \( n \geq 1 \) and every \((a_1, \ldots, a_n) \in \Sigma^n\), there is the isomorphism of labelled \( \mathcal{A} \)-sets
\[
\mathcal{L}_\mathcal{A}(\cosk^\Sigma (\Box[a_1, \ldots, a_n]_{\leq 1})) \cong \cosk^\mathcal{A}_1(\mathcal{A}[a_1, \ldots, a_n]) (\cong \mathcal{A}[a_1, \ldots, a_n]).
\]

Proof. This is a consequence of Theorem III.1.23, Proposition II.1.14 and Theorem III.1.15. \( \square \)

The commutative diagram of Figure 3 proves that the labelled directed coskeleton construction and the labelled transverse symmetric coskeleton functor are equivalent from a directed algebraic topological point of view.

Part IV. Computer-scientific application

A short introduction to process algebra can be found in [WN95]. An introduction to CCS (Milner’s calculus of communicating systems [Mil89]) for mathematicians is available in Gau08.

IV.1. Parallel composition (local case)

We want to explain in this section how it is possible to use the labelled transverse symmetric coskeleton functor to model the parallel composition in CCS of two labelled cubes representing two higher dimensional transitions.

The fibered product in CCS.

IV.1.1. Notation. \( \mathcal{L} := \mathcal{L}_\Box \), \( \text{Sh} := \text{Sh}_\Box \), and \( \omega = \omega_\Box \).

The set \( \Sigma \setminus \{\tau\} \), which may be empty, is now supposed to be equipped with an involution \( a \mapsto \overline{a} \). In Milner’s calculus of communicating systems (CCS) [Mil89], which is the only case treated of this paper, one has \( a \neq \overline{a} \). However, this mathematical hypothesis is not used in this paper. The involution on \( \Sigma \setminus \{\tau\} \) is used only in Definition IV.1.2 of the fibered product (and in the new definition given with the proof of Theorem IV.1.8 of two 1-dimensional labelled (transverse symmetric) precubical sets over \( \Sigma \)). For other examples of fibered products over other synchronization algebras than the one of CCS, cf. [Mil89] and [WN95].

IV.1.2. Definition. Let \( K \) and \( L \) be two 1-dimensional labelled (transverse symmetric) precubical sets. The fibered product of \( K \) and \( L \) over \( \Sigma \) is the 1-dimensional labelled precubical set \( K \times_\Sigma L \) defined as follows:

- \( (K \times_\Sigma L)_0 = K_0 \times L_0 \),
- \( (K \times_\Sigma L)_1 = (K_1 \times L_0) \cup (K_0 \times L_1) \cup \{(x, y) \in K_1 \times L_1, \ell(x) = \ell(y)\} \),
- \( \partial_0^\Sigma (x, y) = (\partial_0^\Sigma (x), y) \) for any \( (x, y) \in K_1 \times L_0 \),
- \( \partial_1^\Sigma (x, y) = (x, \partial_1^\Sigma (y)) \) for any \( (x, y) \in K_0 \times L_1 \),
The 1-dimensional labelled precubical set $K \times_{\Sigma} L$ is the key ingredient in the definition of the synchronized tensor product of labelled precubical sets given in [Gau08], and recalled in Section IV.2. Figure 6 describes $\square[a,b]_{\leq 1} \times_{\Sigma} \square[\tau]$. 

We want to prove in this section that for every $a_1, \ldots, a_{m+n} \in \Sigma$, the labelled precubical set $\cosk_{\Sigma}^1(\square[a_1,\ldots,a_m]_{\leq 1} \times_{\Sigma} \square[a_{m+1},\ldots,a_{m+n}]_{\leq 1})$ can be interpreted as a full labelled coskeleton in the category of labelled transverse symmetric precubical sets.

IV.1.3. Proposition. Let $K$ be a precubical set. For any $p$-cube $x$ of $\omega L(K)$ with $p \geq 0$, there exists a $p$-cube $y$ of $K \subset \omega L(K)$ and a map $\mu \in \widehat{\square}(\square[p],[p])$ such that $x = \mu^*(y)$ where $\mu^*: \mathcal{L}(K)_p \to \mathcal{L}(K)_p$ is the image of $\mu$ by the presheaf $\mathcal{L}(K) \in \widehat{\square}^{op}\text{Set}$.

Proof. Let

$$x \in \omega L(K)_p \cong \lim_{\square[n] \to K} \widehat{\square}[n]_p \cong \lim_{\square[n] \to K} \widehat{\square}(\square[p],[n]).$$

Then there exists an $n$-cube $z: \square[n] \to K$ and $\overline{\square} \in \widehat{\square}(\square[p],[n])$ (the copy corresponding to $z$) such that $z \circ \overline{\square} = x$. By Proposition III.1.14, $\overline{\square} = [p] \overset{\mu}{\to} [p] \overset{\phi}{\to} [n]$ with $\mu \in \square$ and $\phi \in \square$. Then $\phi^*(z)$ is a $p$-cube of $K$ and $\mu^*(\phi^*(z)) = x$. So $y = \phi^*(z)$ is a solution. $\square$

Note that the decomposition $x = \mu^*(y)$ is unique. But this fact will not be used in the sequel. Indeed, let us consider another decomposition $x = \mu'^*(y')$, $z': \square[n'] \to K$, $z' \circ \overline{\square}' = x$ where $\overline{\square}' : [p] \overset{\mu'}{\to} [p] \overset{\phi'}{\to} [n']$ belongs to the copy of $\widehat{\square}(\square[p],[n'])$ corresponding to $z'$ and $y' = \phi'^*(z')$. Since $\overline{\square} \in \widehat{\square}(\square[p],[n])$ and $\overline{\square}' \in \widehat{\square}(\square[p],[n'])$ are equal in the colimit calculating $\omega \mathcal{L}(K)_p$, the two sets $\widehat{\square}(\square[p],[n])$ and $\widehat{\square}(\square[p],[n'])$ are related in the colimit by a zig-zag sequence of maps of $\square$ (this is the crucial point) relating $\overline{\square}$ and $\overline{\square}'$. We can suppose that there exists a map $h : [n] \to [n']$ such that $h \circ \overline{\square} = \overline{\square}'$ and such that $z = z' \circ h$. Then the composite $[p] \overset{\mu}{\to} [p] \overset{\phi}{\to} [n] \overset{h}{\to} [n']$ gives the unique decomposition of $\overline{\square}'$ as the composite of a map of $\widehat{\square}(\square[p],[p])$ followed by a map of $\square$ by Proposition III.1.14. Thus, $\mu = \mu'$ and $h \circ \phi = \phi'$. Therefore $y' = \phi'^*(z') = z' \circ h \circ \phi = z \circ \phi = y$. 

**Figure 6.** Representation of $\square[a,b]_{\leq 1} \times_{\Sigma} \square[\tau]$, labelled over $\Sigma$.
We will need the following combinatorial lemma twice in the sequel:

**IV.1.4. Proposition.** Let \( x : [p] \to [r] \) be a strictly increasing set map. Then there exists a unique decomposition of \( x \) as
\[
[p] \xrightarrow{\mu} [p'] \xrightarrow{\phi} [q] \xrightarrow{\psi} [r]
\]
such that \( \phi \) is non-twisted, \( \psi \in \Box \) and \( \mu = (g_1, \ldots, g_{p'}) \) where the \( g_i : [p] \to [1] \) are non-constant and mutually distinct (i.e. \( g_i \neq g_j \) implies \( i = j \)). Moreover, \( p \leq p' \), \( \mu \) is strictly increasing, and it is also adjacency-preserving if and only if \( p = p' \).

**Proof.** Let \( x = (x^{(1)}, \ldots, x^{(r)}) \) where the maps \( x^{(i)} : [p] \to [1] \) are the \( r \) projection maps. The map \( \psi \) is necessarily the composite \( \delta^\alpha_1 \cdots \delta^\alpha_r \) where \( \{i_1 > \cdots > i_s\} = \{i \in \{1, \ldots, r\} \mid x^{(i)} = 0 \text{ or } x^{(i)} = 1\} \) and where \( x^{(i_s)}(\epsilon_1, \ldots, \epsilon_p) = \alpha_k \) for all \( (\epsilon_1, \ldots, \epsilon_p) \in [p] \). Let \( A \subset \{1, \ldots, r\} \) be the subset of \( i \) such that \( x^{(i)} \) is a non-constant map. Consider the equivalence relation on the set \( A \) defined by \( i \sim j \) if and only if \( x^{(i)} = x^{(j)} \). Let \( p' = \text{card}(A/\sim) \) where \( \text{card}(S) \) denotes the cardinality of the set \( S \). The map \( \mu = (x^{(j_1)}, \ldots, x^{(j_{p'})}) \) is obtained by taking in each equivalence class of \( \sim \) the representative \( x^{(j)} \) with the smallest \( j \) and by imposing \( j_1 < \cdots < j_{p'} \). The non-twisted map \( \phi \) is then defined so that the repetitions encode the equivalence relation \( \sim \). Since \( x \) is strictly increasing, the set map \( \mu \) is also strictly increasing. Therefore \( p \leq p' \). Since none of the set maps \( x^{(j_i)} \) are constant, one has \( \mu (0, \ldots, 0) = (0, \ldots, 0) \) and \( \mu (1, \ldots, 1) = (1, \ldots, 1) \). Thus, if \( p < p' \), then \( \mu \) cannot be adjacency-preserving. And if \( p = p' \), then \( \mu \) is adjacency-preserving by Proposition II.1.16 This decomposition is clearly unique. \( \square \)

**IV.1.5. Theorem.** Let \( \Box[a_1, \ldots, a_m] \) and \( \Box[a_{m+1}, \ldots, a_{m+n}] \) be two labelled cubes with \( m \geq 0 \) and \( n \geq 0 \). Then there is an inclusion of presheaves
\[
\mathcal{L}
(\cosk^\Sigma (\Box[a_1, \ldots, a_m]_{\leq 1} \times \Sigma \Box[a_{m+1}, \ldots, a_{m+n}]_{\leq 1}))
\subset \cosk_1^\Sigma (\hat{\Box}[a_1, \ldots, a_m]_{\leq 1} \times \Sigma \hat{\Box}[a_{m+1}, \ldots, a_{m+n}]_{\leq 1}).
\]
Moreover, when \( \Sigma \setminus \{\tau\} \) is non-empty, there exist two labelled cubes such that the above inclusion is strict.

**Proof.** Let \( K \) be a labelled precubical set. Consider the composite set map, natural with respect to \( K \),
\[
(\Box^{op}\mathbf{Set}|!\Sigma) \left( K, \cosk^\Sigma (\Box[a_1, \ldots, a_m]_{\leq 1} \times \Sigma \Box[a_{m+1}, \ldots, a_{m+n}]_{\leq 1}) \right)
\to (\Box^{op}\mathbf{Set}|!\Sigma) \left( K, \cosk_1^\Sigma (\Box[a_1, \ldots, a_m]_{\leq 1} \times \Sigma \Box[a_{m+1}, \ldots, a_{m+n}]_{\leq 1}) \right)
\cong (\Box^{op}\mathbf{Set}|!\Sigma) \left( K_{\leq 1}, \Box[a_1, \ldots, a_m]_{\leq 1} \times \Sigma \Box[a_{m+1}, \ldots, a_{m+n}]_{\leq 1}) \right)
\cong (\Box^{op}\mathbf{Set}|!\Sigma) \left( \mathcal{L}(K)_{\leq 1}, \Box[a_1, \ldots, a_m]_{\leq 1} \times \Sigma \Box[a_{m+1}, \ldots, a_{m+n}]_{\leq 1}) \right)
\cong (\Box^{op}\mathbf{Set}|!\Sigma) \left( \mathcal{L}(K), \cosk_1^\Sigma (\hat{\Box}[a_1, \ldots, a_m]_{\leq 1} \times \Sigma \hat{\Box}[a_{m+1}, \ldots, a_{m+n}]_{\leq 1}) \right),
\]
the first and last isomorphisms by adjunction and the second one by Proposition II.1.28. Take \( K = \cosk^\Sigma (\Box[a_1, \ldots, a_m]_{\leq 1} \times \Sigma \Box[a_{m+1}, \ldots, a_{m+n}]_{\leq 1}) \). The identity of \( K \) yields a map
of labelled transverse symmetric precubical sets

\[
\begin{align*}
    f : \mathcal{L} \left( \cosk^\Sigma \left( \square [a_1, \ldots, a_m] \leq 1 \times \Sigma \square [a_{m+1}, \ldots, a_{m+n}] \leq 1 \right) \right) & \longrightarrow \cosk^\Sigma \left( \hat{\square} [a_1, \ldots, a_m] \leq 1 \times \Sigma \hat{\square} [a_{m+1}, \ldots, a_{m+n}] \leq 1 \right).
\end{align*}
\]

The case \( K = \square [p] \) for \( p \geq 0 \) gives the injection of sets

\[
\left( \cosk^\Sigma \left( \square [a_1, \ldots, a_m] \leq 1 \times \Sigma \square [a_{m+1}, \ldots, a_{m+n}] \leq 1 \right) \right)_p \subseteq \left( \cosk^\Sigma \left( \hat{\square} [a_1, \ldots, a_m] \leq 1 \times \Sigma \hat{\square} [a_{m+1}, \ldots, a_{m+n}] \leq 1 \right) \right)_p.
\]

The set map \( f_p \) is therefore one-to-one for every \( p \geq 0 \) by Proposition \( \text{[IV.1.3]} \) and Proposition \( \text{[IV.1.4]} \) Suppose now that \( \Sigma \{\tau\} = \Sigma \) is the only new part):}
• The labelling map is defined by:
  - \( \ell(x, y) = \ell(x) \) for any \((x, y) \in \textit{Sh}[a_1, \ldots, a_m] \times \textit{Sh}[a_{m+1}, \ldots, a_{m+n}] \)
  - \( \ell(x, y) = \ell(y) \) for any \((x, y) \in \textit{Sh}[a_1, \ldots, a_m] \times \textit{Sh}[a_{m+1}, \ldots, a_{m+n}] \)
  - \( \ell(x, y) = (r, s) \in \mathbb{N}^\ast \times \mathbb{N}^\ast \) for any \((x, y) \in \textit{Sh}[a_1, \ldots, a_m] \times \textit{Sh}[a_{m+1}, \ldots, a_{m+n}] \)

IV.1.7. Lemma. Let \( c : \textit{Sh}[1] \to \textit{Sh}[a_1, \ldots, a_m] \times_{\Sigma} \textit{Sh}[a_{m+1}, \ldots, a_{m+n}] \) be a 1-cube of \( \textit{Sh}[a_1, \ldots, a_m] \times_{\Sigma} \textit{Sh}[a_{m+1}, \ldots, a_{m+n}] \). Then the set map \( c_0 : [1] \to [m+n] \) satisfies \( c_0(0) < c_0(1) \) and there are two mutually exclusive possibilities:

  1. \( d(c_0(0), c_0(1)) = 1 \) and \( c_0(\alpha) = \delta^\alpha_r(X) \) for some \( X \in [m,n-1] \) with \( 1 \leq r \leq m+n \) and for \( \alpha = 0,1 \). In this case, \( \ell(c) = a_r \in \Sigma \).

  2. \( d(c_0(0), c_0(1)) = 2 \) and \( c_0(\alpha) = \delta^\alpha_r \delta^\alpha_s(X) \) for some \( X \in [m,n-2] \) with \( 1 \leq r \leq m \) and \( m+1 \leq s \leq m+n \) and for \( \alpha = 0,1 \). In this case, \( \ell(c) = (r,s) \in \mathbb{N}^\ast \times \mathbb{N}^\ast \).

Proof. Obvious.

Note that Lemma IV.1.7 holds for \( \textit{Sh}[a_1, \ldots, a_m] \times_{\Sigma} \textit{Sh}[a_{m+1}, \ldots, a_{m+n}] \) as well by replacing in the last sentence \( \ell(c) = (r,s) \in \mathbb{N}^\ast \times \mathbb{N}^\ast \) by \( \ell(c) = \tau \).

We are now ready to give the categorical interpretation of the labelled directed coskeleton construction when applied to the fibered product of two 1-dimensional labelled precubical sets.

IV.1.8. Theorem. Let \( \textit{Sh}[a_1, \ldots, a_m] \) and \( \textit{Sh}[a_{m+1}, \ldots, a_{m+n}] \) be two labelled cubes with \( m \geq 0 \) and \( n \geq 0 \). Then one has the isomorphism of labelled transverse symmetric precubical sets

\[
\mathcal{L} \left( \cosk_{1}^{\Sigma} \left( \textit{Sh}[a_1, \ldots, a_m] \times_{\Sigma} \textit{Sh}[a_{m+1}, \ldots, a_{m+n}] \right) \right) \cong \cosk_{1}^{\Sigma} \left( \textit{Sh}[a_1, \ldots, a_m] \times_{\Sigma} \textit{Sh}[a_{m+1}, \ldots, a_{m+n}] \right)
\]

where the right-hand labelled transverse symmetric precubical set over \( \Sigma \) is viewed as labelled over \( \Sigma \) by composing its labelling map with the morphism of transverse symmetric precubical sets \( \text{Sh} \mathcal{L}(\Sigma) \to \text{Sh} \mathcal{L}(\Sigma) \), where the set map \( \Sigma \to \Sigma \) is defined as the identity on \( \Sigma \) and by the mapping \( (p, q) \mapsto \tau \) on the complement.

Note that with \( m = 0 \) or \( n = 0 \), we have the isomorphism of Theorem III.1.24

Proof. \textbf{Injectivity} There is an inclusion of presheaves

\[
\cosk_{1}^{\Sigma} \left( \textit{Sh}[a_1, \ldots, a_m] \times_{\Sigma} \textit{Sh}[a_{m+1}, \ldots, a_{m+n}] \right) \subset \cosk_{1}^{\Sigma} \left( \textit{Sh}[a_1, \ldots, a_m] \times_{\Sigma} \textit{Sh}[a_{m+1}, \ldots, a_{m+n}] \right)
\]

\[\text{7 instead of } \ell(x, y) = \tau\]
since less shells are filled in the left-hand term than in the right-hand term because of the labelling over $\Sigma$. Moreover, one has the equalities

$$ \left( \mathcal{L} \left( \cosk^{\Sigma} \left( [a_1, \ldots, a_m] \times_{\Sigma} [a_{m+1}, \ldots, a_{m+n}] \right) \right) \right)_{\leq 1} $$

$$ = \left( \cosk^{\hat{\Sigma}} \left( [a_1, \ldots, a_m] \times_{\Sigma} \hat{[a_{m+1}, \ldots, a_{m+n}]]} \right) \right)_{\leq 1} $$

by Proposition IV.21 and Proposition IV.34. Let

$$ x \in \left( \mathcal{L} \left( \cosk^{\Sigma} \left( [a_1, \ldots, a_m] \times_{\Sigma} [a_{m+1}, \ldots, a_{m+n}] \right) \right) \right)_p $$

with $p \geq 2$. Then $x = \mu^*(y)$ where $\mu : [p] \to [p]$ is an adjacency-preserving map and where $y$ is a $p$-cube of the labelled precubical set

$$ \cosk^{\Sigma} \left( [a_1, \ldots, a_m] \times_{\Sigma} [a_{m+1}, \ldots, a_{m+n}] \right) $$

by Proposition IV.18. The map of 1-dimensional precubical sets

$$ y_{\leq 1} : [p] \to (\cosk^{\Sigma} \left( [a_1, \ldots, a_m] \times_{\Sigma} [a_{m+1}, \ldots, a_{m+n}] \right))_{\leq 1} $$

induces by Proposition IV.28 a map of transverse symmetric 1-dimensional precubical sets

$$ y_{\leq 1} : \hat{[p]} \to [a_1, \ldots, a_m] \times_{\Sigma} \hat{[a_{m+1}, \ldots, a_{m+n}]}_{\leq 1} $$

The latter induces a unique map $\overline{\mu} : \hat{[p]} \to \hat{[a_1, \ldots, a_m]} \times_{\Sigma} \hat{[a_{m+1}, \ldots, a_{m+n}]}_{\leq 1}$ of 1-dimensional precubical sets which is this time labelled over $\Sigma$ since the underlying precubical sets of $[a_1, \ldots, a_m] \times_{\Sigma} [a_{m+1}, \ldots, a_{m+n}]_{\leq 1}$ and $[a_1, \ldots, a_m] \times_{\Sigma} [a_{m+1}, \ldots, a_{m+n}]_{\leq 1}$ are equal. The map $\overline{\mu}$ induces by adjunction a unique $p$-dimensional transverse symmetric cube of

$$ \cosk^{\hat{\Sigma}} \left( [a_1, \ldots, a_m] \times_{\Sigma} \hat{[a_{m+1}, \ldots, a_{m+n}]}_{\leq 1} \right) $$

Thus the inclusion

$$ \mathcal{L} \left( \cosk^{\Sigma} \left( [a_1, \ldots, a_m] \times_{\Sigma} [a_{m+1}, \ldots, a_{m+n}] \right) \right) $$

$$ \subseteq \cosk^{\hat{\Sigma}} \left( [a_1, \ldots, a_m] \times_{\Sigma} \hat{[a_{m+1}, \ldots, a_{m+n}]}_{\leq 1} \right) $$
factors uniquely as a composite of inclusions

\[ \mathcal{L} \left( \cosk_{\Sigma} \left( \Box [a_1, \ldots, a_m] \times \Sigma \Box [a_{m+1}, \ldots, a_{m+n}] \right) \right) \]

\[ \subset \cosk_{\Sigma} \left( \hat{\Box} [a_1, \ldots, a_m] \times \hat{\Sigma} \Box [a_{m+1}, \ldots, a_{m+n}] \right) \]

\[ \subset \cosk_{\Sigma} \left( \hat{\Box} [a_1, \ldots, a_m] \times \hat{\Sigma} \Box [a_{m+1}, \ldots, a_{m+n}] \right) . \]

Let us call \( f \) again the inclusion

\[ \mathcal{L} \left( \cosk_{\Sigma} \left( \Box [a_1, \ldots, a_m] \times \Sigma \Box [a_{m+1}, \ldots, a_{m+n}] \right) \right) \]

\[ \subset \cosk_{\Sigma} \left( \hat{\Box} [a_1, \ldots, a_m] \times \hat{\Sigma} \Box [a_{m+1}, \ldots, a_{m+n}] \right) . \]

It then remains to prove that for every \( p \geq 2 \), the set map \( f_p \) is onto.

**Surjectivity** Let \( x : \Box [p] \to \cosk_{\Sigma} \left( \Box [a_1, \ldots, a_m] \times \Sigma \Box [a_{m+1}, \ldots, a_{m+n}] \right) \) be a \( p \)-dimensional transverse symmetric cube of \( \cosk_{\Sigma} \left( \Box [a_1, \ldots, a_m] \times \Sigma \Box [a_{m+1}, \ldots, a_{m+n}] \right) \) with \( p \geq 2 \). Let \( x_0 = (x^{(1)}, \ldots, x^{(m+n)}) \) where the \( x^{(i)} : [p] \to [1] \) are the \( m + n \) projections. Let us apply the decomposition of Proposition \[ IV.1.4 \]

Let \( \mu = (x^{(j_1)}, \ldots, x^{(j_p)}) \). If one had \( p < p' \), then there would exist a 1-cube \( c : [1] \to [p] \) such that \( d(\mu(c(0)), \mu(c(1))) \approx [p] \approx [1] \). By Lemma \[ IV.1.7 \] one would have

\[ 1 < d(\mu(c(0)), \mu(c(1))) \leq d(\psi \mu(c(0)), \psi \mu(c(1))) \leq 2, \]

and therefore \( d(\mu(c(0)), \mu(c(1))) = 2 \). Thus, one would have \( \mu c(\alpha) = \delta^\alpha_\mu(X) \) for some \( u < v \), \( X \in [p'] - [2] \) and for \( \alpha = 0, 1 \). By Lemma \[ IV.1.7 \] one obtains \( \psi \mu c(\alpha) = \delta^\alpha_\mu c (Z) \) for some cube \( Z \in [m + n - 2] \) and for \( \alpha = 0, 1 \). Finally, \( \ell(\psi \mu c) = (j_u, j_v) \).

Use of the particular labelling of \( \Sigma_{\Sigma} \). The crucial point is that the labelling of \( \Sigma_{\Sigma} \) implies \( x^{(j_u)} = x^{(j_v)} \), which contradicts the definition of \( \mu \). By Proposition \[ IV.2.13 \] the commutative word \( W = \ell(x c_1) \ldots \ell(x c_p) \) of the free commutative monoid without unit generated by \( \Sigma \) does not depend on the maximal path \( (c_1, \ldots, c_p) \) of \( \Box [p] \). And one of the labels is necessarily \( (j_u, j_v) \). If \( (\epsilon_1, \ldots, \epsilon_p) = (0, \ldots, 0) \), then \( x^{(j_u)}(\epsilon_1, \ldots, \epsilon_p) = x^{(j_v)}(\epsilon_1, \ldots, \epsilon_p) = 0 \). Let us suppose now that \( (\epsilon_1, \ldots, \epsilon_p) \neq (0, \ldots, 0) \). By Proposition \[ IV.2.13 \] for every maximal path \( (c_1, \ldots, c_r) \) of the \( r \)-subcube from \( (0, \ldots, 0) \) to \( (\epsilon_1, \ldots, \epsilon_p) \), the commutative word \( W' = \ell(x c_1) \ldots \ell(x c_r) \) is a subword of \( W \) which does not depend on \( (c_1, \ldots, c_r) \). If \( (j_u, j_v) \) belongs to \( W' \), then \( x^{(j_u)}(\epsilon_1, \ldots, \epsilon_p) = x^{(j_v)}(\epsilon_1, \ldots, \epsilon_p) = 1 \). If \( (j_u, j_v) \) does not belong to \( W' \), then it belongs to the complement of \( W' \) in \( W \). So \( x^{(j_u)}(\epsilon_1, \ldots, \epsilon_p) = x^{(j_v)}(\epsilon_1, \ldots, \epsilon_p) = 0 \). Hence, \( x^{(j_u)} = x^{(j_v)} \), which is the desired contradiction.

End of the proof. Hence, one obtains the equality \( p = p' \) thanks to the particular labelling of \( \Sigma_{\Sigma} \). The map \( \mu \) is therefore adjacency-preserving by Proposition \[ IV.1.6 \] Note that \( x_0 \) has no reason to be adjacency-preserving. By definition of the labelled directed coskeleton, there exists a \( p \)-cube

\[ y : \Box [p] \to \cosk_{\Sigma} \left( \Box [a_1, \ldots, a_m] \times \Sigma \Box [a_{m+1}, \ldots, a_{m+n}] \right) \]

So far, the particular labelling of \( \Sigma_{\Sigma} \) has not been used in the surjectivity part of the proof. In the counterexample of Theorem \[ IV.1.3 \] one has \( p = 2 \) and \( p' = 4 \). So we cannot yet conclude that \( p = p' \).
such that \( y_0 = \psi \phi \). Then \( \mathcal{L}(y) \) is a \( p \)-cube of the labelled transverse symmetric precubical set

\[
\mathcal{L} \left( \cosk_\Sigma (\boxtimes_m [a_1, \ldots, a_m]^{\leq 1} \times_{\Sigma} \boxtimes_m [a_{m+1}, \ldots, a_{m+n}]^{\leq 1}) \right)
\]

such that \( \mathcal{L}(y)_0 = \psi \phi \). Then \( \mu^*(\mathcal{L}(y)) \) is a \( p \)-cube of the labelled transverse symmetric precubical set

\[
\mathcal{L} \left( \cosk_\Sigma (\boxtimes_m [a_1, \ldots, a_m]^{\leq 1} \times_{\Sigma} \boxtimes_m [a_{m+1}, \ldots, a_{m+n}]^{\leq 1}) \right)
\]

such that \( (\mu^*(\mathcal{L}(y)))_0 = \psi \phi \mu \). By construction of \( f \), the \( p \)-cube \( f(\mu^*(\mathcal{L}(y))) \) of the labelled transverse symmetric precubical set \( \cosk_1^\Sigma ([a_1, \ldots, a_m]^{\leq 1} \times_{\Sigma} [a_{m+1}, \ldots, a_{m+n}]^{\leq 1}) \) satisfies \( (f(\mu^*(\mathcal{L}(y))))_0 = \psi \phi \mu = x_0 \). Since there is at most one 1-cube between two vertices of \( \boxtimes_m [a_1, \ldots, a_m]^{\leq 1} \times_{\Sigma} \boxtimes_m [a_{m+1}, \ldots, a_{m+n}]^{\leq 1} \), this implies \( (f(\mu^*(\mathcal{L}(y))))^{\leq 1} = x^{\leq 1} \) and therefore \( f(\mu^*(\mathcal{L}(y))) = x \) by adjunction. So \( f \) is an isomorphism of labelled transverse symmetric precubical sets. \( \square \)

Theorem IV.1.8 is of course false for any other category of cubes than \( \widehat{\boxtimes} \). Indeed, the particular case \( n = 0 \) and \( a_1 = \cdots = a_m = \tau \) gives back the inclusion of presheaves

\[
\mathcal{A}[m] \cong \mathcal{L}_A \left( \cosk_\Sigma (\boxtimes_m [m]^{\leq 1}) \right) \subset \cosk_1^\Sigma (\mathcal{A}[m]^{\leq 1})
\]

which is an equality if and only if the category of cubes \( \mathcal{A} \) is shell-complete, so if and only if \( \mathcal{A} = \widehat{\boxtimes} \) by Theorem I[1.15]. The crucial point in the proof of Theorem IV.1.8 is that the map \( \mu : [p] \to [p] \) must belong to \( \mathcal{A} \). Therefore, it is really needed to work with the whole category \( \boxtimes \) of all adjacency-preserving maps.

IV.2. Parallel composition (global case)

We can now relate the synchronized tensor product of labelled precubical sets with the synchronized tensor product of labelled transverse symmetric precubical sets. First of all, let us give the definition of these two synchronized tensor products.

**Definition.**

IV.2.1. **Definition.** [Gau08] Let \( K \) and \( L \) be two labelled precubical sets. The tensor product with synchronization (or synchronized tensor product) of \( K \) and \( L \) is

\[
K \otimes_\Sigma L := \lim_{\boxtimes_m \to K} \lim_{\boxtimes_m \to L} \cosk_\Sigma (\boxtimes_m^{\leq 1} \times_{\Sigma} \boxtimes_m^{\leq 1})
\]

IV.2.2. **Definition.** Let \( K \) and \( L \) be two labelled transverse symmetric precubical sets. The tensor product with synchronization (or synchronized tensor product) of \( K \) and \( L \) is

\[
K \otimes_\Sigma L := \lim_{\boxtimes_m \to K} \lim_{\boxtimes_m \to L} \cosk_1^\Sigma (\boxtimes_1^{\leq 1} \times_{\Sigma} \boxtimes_1^{\leq 1})
\]

The two constructions coincide. For the sequel, the category of small categories is denoted by \( \textbf{Cat} \). Let \( H : I \to \textbf{Cat} \) be a functor from a small category \( I \) to \( \textbf{Cat} \). The Grothendieck construction \( I \int H \) is the category defined as follows [Tho79]: the objects are the pairs \((i, a)\) where \( i \) is an object of \( I \) and \( a \) is an object of \( H(i) \); a morphism \((i, a) \to (j, b)\) consists in a map \( \phi : i \to j \) and in a map \( h : H(\phi)(a) \to b \).
IV.2.3. **Lemma.** Let $\mathcal{A}$ be a category of cubes. Let $I$ be a small category, and $i \mapsto K^i$ be a functor from $I$ to the category of labelled $\mathcal{A}$-sets. Let $K = \varinjlim K^i$. Let $H : I \to \text{Cat}$ be the functor defined by $H(i) = \mathcal{A}\downarrow K^i$. Then the functor $\iota : I \int H \to \mathcal{A}\downarrow K$ defined by $\iota(i, \mathcal{A}[m] \to K^i) = (\mathcal{A}[m] \to K)$ is final in the sense of [ML98]: that is to say the comma category $k \downarrow \iota$ is nonempty and connected for all objects $k$ of $\mathcal{A}\downarrow K$.

**Proof.** The proof is similar to the proof of [Gau08 Lemma A.1]. \qed

IV.2.4. **Proposition.** Let $\mathcal{A}$ be a category of cubes. Let $F : \mathcal{A} \times \mathcal{A} \to C$ be a functor where $C$ is a cocomplete category. Let $\hat{F} : (\mathcal{A}^\op \downarrow \mathcal{L}_{\mathcal{A}_L}(!\Sigma)) \times (\mathcal{A}^\op \downarrow \mathcal{L}_{\mathcal{A}_L}(!\Sigma)) \to C$ be the functor defined by

$$
\hat{F}(K, L) := \varinjlim_{A[m] \to K} \varinjlim_{A[n] \to L} F([m], [n]).
$$

Then for any labelled $\mathcal{A}$-set $L$, the two functors

$$
\hat{F}(L, -) : \mathcal{A}^\op \downarrow \mathcal{L}_{\mathcal{A}_L}(!\Sigma) \to C
$$

and

$$
\hat{F}(-, L) : \mathcal{A}^\op \downarrow \mathcal{L}_{\mathcal{A}_L}(!\Sigma) \to C
$$

are colimit-preserving.

**Proof.** The proof is similar to the proof of [Gau08 Proposition A.2]. Let $K = \varinjlim K^i$ be a colimit of labelled $\mathcal{A}$-sets. By definition, one has the isomorphism

$$
\varinjlim_i \hat{F}(K^i, L) \cong \varinjlim_i \varinjlim_{A[m] \to K^i} \varinjlim_{A[n] \to L} F([m], [n]).
$$

Consider the functor $H : I \to \text{Cat}$ defined by $H(i) = \mathcal{A}\downarrow K^i$. Consider the functors $F_i : H(i) \to C$ defined by

$$
F_i(\mathcal{A}[m] \to K^i) = \varinjlim_{A[n] \to L} F([m], [n]).
$$

Consider the functor $\overline{F} : I \int H \to C$ defined by

$$
\overline{F}(i, \mathcal{A}[m] \to K^i) = \varinjlim_{A[n] \to L} F([m], [n]).
$$

Then the composite $H(i) \subset I \int H \to C$ is exactly $F_i$. Therefore one has the isomorphism

$$
\varinjlim_i \varinjlim_{A[m] \to K^i} \varinjlim_{A[n] \to L} F([m], [n]) \cong \varinjlim_{(i, A[m] \to K^i)} \varinjlim_{A[n] \to L} F([m], [n])
$$

by [CS02 Proposition 40.2]. The functor $\iota : I \int H \to \mathcal{A}\downarrow K$ defined by $\iota(i, \mathcal{A}[m] \to K) = (\mathcal{A}[m] \to K)$ is final in the sense of [ML98] by Lemma 12.2.3. Therefore by [ML98 p. 213, Theorem 1] or [Hir03 Theorem 14.2.5], one has the isomorphism

$$
\varinjlim_{(i, A[m] \to K^i)} \varinjlim_{A[n] \to L} F([m], [n]) \cong \varinjlim_{A[m] \to K} \varinjlim_{A[n] \to L} F([m], [n]) =: \hat{F}(K, L).
$$

\qed
IV.2.5. Corollary. Let $I$ be a small category. Let $i \mapsto K^i$ be a functor from $I$ to the category of labelled transverse symmetric precubical sets, and let $L$ be a labelled transverse symmetric precubical set. Then one has the natural isomorphism

$$(\lim_i K^i) \otimes L \cong \lim_i (K^i \otimes L).$$

IV.2.6. Theorem. Let $K$ and $L$ be two labelled precubical sets. Then there is the natural isomorphism of labelled transverse symmetric precubical sets

$$\mathcal{L}(K \otimes L) \cong \mathcal{L}(K) \otimes \mathcal{L}(L).$$

Proof. One has

$$\mathcal{L}(K \otimes L) \cong \mathcal{L}\left(\lim_{\square \to K} \lim_{\square \to L} \cosk^\Sigma (\square[m] \times_\Sigma \square[n])\right),$$

$$\cong \lim_{\square \to K} \lim_{\square \to L} \mathcal{L}\left(\cosk^\Sigma (\square[m] \times_\Sigma \square[n])\right) \text{ since } \mathcal{L} \text{ is a left adjoint},$$

$$\cong \lim_{\square \to K} \lim_{\square \to L} \cosk^\Sigma (\square[m] \times_\Sigma \square[n]),$$

$$\cong \lim_{\square \to K} \lim_{\square \to L} \mathcal{L}(\square[m] \otimes \Sigma \square[n]),$$

$$\cong \lim_{\square \to K} \lim_{\square \to L} \mathcal{L}(\square[m]) \otimes \Sigma \mathcal{L}(\square[n]) \text{ by Proposition [II.1.14]},$$

$$\cong \left(\lim_{\square \to K} \mathcal{L}(\square[m])\right) \otimes \Sigma \left(\lim_{\square \to L} \mathcal{L}(\square[n])\right) \text{ by Corollary [IV.2.5]},$$

$$\cong \mathcal{L}(K) \otimes \Sigma \mathcal{L}(L) \text{ since } \mathcal{L} \text{ is a left adjoint.}$$

Associativity. As in [Gau08], it is also possible to prove that the synchronized tensor product of labelled transverse symmetric precubical sets is associative.

IV.2.7. Theorem. Let $K$, $L$ and $M$ be three labelled transverse symmetric precubical sets. Then there is a canonical isomorphism of labelled transverse symmetric precubical sets

$$(K \otimes L) \otimes \Sigma M \cong K \otimes (L \otimes \Sigma M).$$

Proof. One has
As already pointed out in [Gau08], it is false in general that the two labelled precubical sets $K \otimes \Sigma L$ and $L \otimes \Sigma K$ are isomorphic as labelled precubical sets. Indeed, let us suppose that $\Sigma \setminus \{\tau\}$ contains an element $a$. Then $\boxdot[a] \otimes \Sigma \boxdot[\tau] \cong \boxdot[a, \tau]$ and $\boxdot[\tau] \otimes \Sigma \boxdot[a] \cong \boxdot[\tau, a]$. Because of the lack of symmetry operators, the two labelled 2-cubes $\boxdot[a, \tau]$ and $\boxdot[\tau, a]$ cannot be isomorphic as labelled precubical sets. However, the two underlying precubical sets are of course isomorphic, as already pointed out in [Gau08]. In the category of transverse symmetric precubical sets, the situation is much better. Indeed, one has the isomorphisms of labelled transverse symmetric precubical sets

$$\boxdot[a, \tau] \cong \mathcal{L}(\boxdot[a, \tau]) \cong \hat{\boxdot}[\tau, a] \cong \mathcal{L}(\hat{\boxdot}[\tau, a]).$$

**IV.2.8. Proposition.** Let $K$ and $L$ be two labelled transverse symmetric precubical sets. Then there is a natural isomorphism of labelled transverse symmetric precubical sets $K \otimes \Sigma L \cong L \otimes \Sigma K$.

**Sketch of proof.** It suffices to use Corollary [IV.2.5] together with the isomorphism

$$\boxdot[a_1, \ldots, a_m] \otimes \Sigma \hat{\boxdot}[b_1, \ldots, b_n] \cong \hat{\boxdot}[b_1, \ldots, b_n] \otimes \Sigma \hat{\boxdot}[a_1, \ldots, a_m]$$

(built using the symmetry operators) for all labelled full cubes $\boxdot[a_1, \ldots, a_m]$ and $\hat{\boxdot}[b_1, \ldots, b_n]$. 

**IV.3. Comparison of the two semantics of CCS**

**Interpreting CCS as labelled precubical sets.** The CCS *process names* are generated by the following syntax:

$$P ::= nil \mid a.P \mid (va)P \mid P + P \mid P|P \mid \text{rec}(x)P(x)$$

where $P(x)$ means a process name with one free variable $x$. The variable $x$ must be guarded, that is it must lie in a prefix term $a.P'(x)$ for some $a \in \Sigma$.

**IV.3.1. Definition.** A *labelled precubical set* $\ell : K \rightarrow !\Sigma$ decorated by process names is a labelled precubical set together with a set map $d : K_0 \rightarrow \text{Proc}_\Sigma$ called the decoration.
Let us define by induction on the syntax of the CCS process name $P$ the decorated labelled precubical set $\Box[P]$ (see [Gau08] for further explanations). The labelled precubical set $\Box[P]$ has a unique initial state canonically decorated by the process name $P$, and its other states will be decorated in an inductive way. Therefore for every process name $P$, $\Box[P]$ is an object of the double comma category $\{i\}↓\Box Set↓!\Sigma$. One has $\Box[\text{nil}] := \Box[0]$, $\Box[\mu.\text{nil}] := \mu.\text{nil} \rightarrow \text{nil}$, $\Box[P + Q] := \Box[P] \oplus \Box[Q]$ with the binary coproduct taken in $\{i\}↓\Box Set↓!\Sigma$, the pushout diagram of precubical sets

$$
\begin{array}{ccc}
\Box[0] = \{0\} & \xrightarrow{0\rightarrow P} & \Box[\mu.\text{nil}] \\
\Box[P] & \rightarrow & \Box[\mu.P]
\end{array}
$$

the pullback diagram of precubical sets

$$
\begin{array}{ccc}
\Box[(\nu a)P] & \rightarrow & \Box[P] \\
!\Sigma \rightarrow & !\Sigma,
\end{array}
$$

the formula giving the interpretation of the parallel composition with synchronization

$$
\Box[P||Q] := \Box[P] \otimes !\Sigma \Box[Q]
$$

and finally $\Box[\text{rec}(x)P(x)]$ defined as the least fixed point of $P(−)$.

The prefix operator, the direct sum and the restriction operator are $\omega$-continuous, that is to say they preserve the upper bounds of ascending $\omega$-chains of labelled precubical sets $K^0 \subset K^1 \subset K^2 \subset \ldots$, since they are finitely accessible and since the upper bound is given by the colimit of the chain. The synchronized tensor product is also $\omega$-continuous since it is colimit-preserving by [Gau08, Proposition A.2]. Moreover, the condition imposed on $P(x)$ implies that for all process names $Q_1$ and $Q_2$ with $\Box[Q_1] \subset \Box[Q_2]$, one has $\Box[P(Q_1)] \subset \Box[P(Q_2)]$. Therefore the mapping $P(−)$ is $\omega$-continuous and non-decreasing. Thus, the labelled precubical set

$$
\Box[\text{rec}(x)P(x)] := \lim_n \Box[P^n(\text{nil})] \cong \bigcup_{n\geq 0} \Box[P^n(\text{nil})]
$$

will be equal to the least fixed point of $P(−)$. This is a particular case of the Kleene fixed-point theorem on a directed complete partial order.

**Interpreting CCS as labelled transverse symmetric precubical sets.** Let us give now the new semantics of CCS in terms of labelled transverse symmetric precubical sets.

**IV.3.2. Definition.** A labelled transverse symmetric precubical set $\ell : K \rightarrow \text{Sh} L(!\Sigma)$ decorated by process names is a labelled transverse symmetric precubical set together with a set map $d : K_0 \rightarrow \text{Proc}_\Sigma$ called the decoration.

The interpretation of a CCS process name $P$ in terms of a decorated labelled transverse symmetric precubical set $\hat{\Box}[P]$ is defined by induction on the syntax of $P$, as for the case of
labelled precubical sets. The only differences with the latter case are the pullback diagram

\[
\begin{array}{ccc}
\hat{\square}[\nu a \mathcal{P}] & \longrightarrow & \hat{\square}[\mathcal{P}] \\
\downarrow & & \downarrow \\
\text{Sh } \mathcal{L}(\!(\Sigma \backslash \{a, \overline{a}\})\!) & \longrightarrow & \text{Sh } \mathcal{L}(\!\Sigma) \\
\end{array}
\]

and the equation \( \hat{\square}[\mathcal{P}] |\rangle :\rightarrow \hat{\square}[\mathcal{P}] \otimes_{\Sigma} \hat{\square}[\mathcal{Q}] \) where \( \otimes_{\Sigma} \) is now the synchronized tensor product of labelled transverse symmetric precubical sets. Corollary IV.2.5 enables us to construct the least fixed point of \( \underline{P}(\mathcal{P}) \) in the same way as in the case of labelled precubical sets.

**The two semantics have same geometric realization.**

**IV.3.3. Theorem.** For every CCS process name \( \mathcal{P} \), there is an isomorphism of labelled transverse symmetric precubical sets \( \hat{\square}[\mathcal{P}] \cong \mathcal{L}(\hat{\square}[\mathcal{P}]) \) and an isomorphism of (labelled) flows \( |\rangle \hat{\square}[\mathcal{P}] \cong |\rangle \mathcal{L}(\hat{\square}[\mathcal{P}]) \).

**Proof.** Let \( K \) be a labelled precubical set. Let \( a \in \Sigma \backslash \{\tau\} \). Let \((\nu a)K\) be the labelled precubical set defined by the pullback diagram

\[
\begin{array}{ccc}
(\nu a)K & \xrightarrow{\subset} & K \\
\downarrow & & \downarrow \\
\!(\Sigma \backslash \{a, \overline{a}\}) & \longrightarrow & \!\Sigma. \\
\end{array}
\]

One obtains the commutative diagram of labelled transverse symmetric precubical sets

\[
\begin{array}{ccc}
\mathcal{L}(\!(\nu a)K\!) & \xrightarrow{\subset} & \mathcal{L}(K) \\
\downarrow & & \downarrow \\
\mathcal{L}(\!(\Sigma \backslash \{a, \overline{a}\})\!) & \longrightarrow & \mathcal{L}(\!\Sigma) \\
\downarrow & & \downarrow \\
\text{Sh } \mathcal{L}(\!(\Sigma \backslash \{a, \overline{a}\})\!) & \longrightarrow & \text{Sh } \mathcal{L}(\!\Sigma). \\
\end{array}
\]

The map \((\nu a)K \rightarrow K\) is an inclusion of presheaves: the labelled precubical set \((\nu a)K\) is the subobject of \( K \) containing the labelled cubes of \( K \) not containing \( a \) nor \( \overline{a} \) as label. By Proposition I.1.14 the transverse symmetric precubical set \( \mathcal{L}(\!(\nu a)K\!) \) is the subobject of \( \mathcal{L}(K) \) containing the \( p \)-cubes of \( \mathcal{L}(K) \) of the form \( \mu^*(x) \) where \( \mu : [p] \rightarrow [p] \) is a map of \( \hat{\square} \) and \( x \) is a \( p \)-cube of \((\nu a)K\). Therefore the map \( \mathcal{L}(\!(\nu a)K\!) \rightarrow \mathcal{L}(K) \) is an inclusion of presheaves as
well. Consider now a commutative diagram of labelled transverse symmetric precubical sets

\[
\begin{array}{ccc}
X & \xrightarrow{k} & L((\nu a)K) \\
\downarrow f & & \downarrow c \\
L(K) & \xrightarrow{c} & Sh L(\{\Sigma \setminus \{a, \overline{a}\}\}) \\
\end{array}
\]

Every \( p \)-cube \( x \) of \( Z \) is taken to a \( p \)-cube \( f(x) \) of \( L(K) \). By Proposition [IV.1.3] \( f(x) = \mu^*(y) \) for some \( p \)-cube \( y \in K \) and for some map \( \mu : [p] \to [\hat{\square}] \) of \( \hat{\square} \). By construction, \( y \) does not use the labels \( a \) or \( \overline{a} \). Thus \( y \in (\nu a)K \). Therefore \( f(x) \) is a \( p \)-cube of \( L((\nu a)K) \). Hence \( k \) exists and is unique since the map \( L((\nu a)K) \to L(K) \) is an inclusion of presheaves. Thus, the diagram of labelled transverse symmetric precubical sets

\[
\begin{array}{ccc}
L((\nu a)K) & \xrightarrow{c} & L(K) \\
\downarrow & & \downarrow \\
Sh L(\{\Sigma \setminus \{a, \overline{a}\}\}) & \xrightarrow{\sim} & Sh L(\{\Sigma\}) \\
\end{array}
\]

is a pullback. So the isomorphism \( \hat{\square}[P] \cong L(\square[P]) \) implies the isomorphism \( \hat{\square}[(\nu a)P] \cong L(\square[(\nu a)P]) \). Therefore, the isomorphism of labelled transverse symmetric precubical sets \( \hat{\square}[P] \cong L(\square[P]) \) is proved by induction on the syntax of the process name \( P \), using Theorem [IV.2.6] and the fact that the functor \( L \) preserves colimits since it is a left adjoint. The isomorphism of labelled flows \( \hat{\square}[P] \cong |\square[P]| \) is a consequence of Proposition [II.2.10] \( \square \)

Part V. Appendix

V.1. The case of labelled symmetric precubical sets

By Theorem [III.1.16] the category of cubes \( \square_S \) is not shell-complete. It is interesting anyway for the three following reasons. 1) It is possible to give an explicit description of the symmetric precubical sets of labels with Proposition [V.1.3]. Such a description is still an open problem for the transverse symmetric precubical set of labels (cf. Conjecture [III.1.21]). 2) The category of cubes \( \square_S \) is the smallest category of cubes \( \mathcal{A} \) such that the labelled cubes \( \mathcal{A}[a_{\sigma(1)}, \ldots, a_{\sigma(n)}] \) with \( a_1, \ldots, a_n \in \Sigma \) for \( \sigma \) running over the set of permutations of \( \{1, \ldots, n\} \) belong to the same isomorphism class. Let us recall that the labelled precubical sets \( \square[a_{\sigma(1)}, \ldots, a_{\sigma(n)}] \) and \( \square[a'_{\sigma(1)}, \ldots, a'_{\sigma(n)}] \) are not isomorphic as soon as \( (a_{\sigma(1)}, \ldots, a_{\sigma(n)}) \neq (a'_{\sigma(1)}, \ldots, a'_{\sigma(n)}) \). 3) There is a strong link between labelled symmetric precubical sets and higher dimensional transition systems in the sense of Cattani and Sassone [CS96], see [Gau09]. Indeed, it turns out that the category of higher dimensional transition systems in the sense of Cattani and Sassone is equivalent to a full reflective subcategory of that of labelled symmetric precubical sets.
Description of the symmetric precubical set of labels. The following combinatorial lemma is well-known (see [Wac07] for a survey).

V.1.1. Lemma. Let \( p \geq 1 \). The group of automorphisms of the poset \([p]\) is isomorphic to the symmetric group on \( \{1, \ldots, p\} \). In other terms, let \( f \) be an automorphism of the poset \([p]\). Then there exists a permutation \( \pi \) of the set \( \{1, \ldots, p\} \) such that \( f(e_1, \ldots, e_p) = (\pi(1), \ldots, \pi(p)) \).

\[ \square \]

\[ \square \]

Proof. Let \( I \subset \{1, \ldots, p\} \). Let \( e_I \) be the element \((e_1, \ldots, e_p)\) of \([p]\) such that \( e_i = 1 \) if and only if \( i \in I \). Since \( f \) is bijective and strictly increasing, it preserves the distance of Proposition III.1.3. The distance between \( e_\varnothing \) and \( f(e_{\{i\}}) \) is 1. So there exists a permutation \( \pi \) of \( \{1, \ldots, p\} \) such that \( f(e_{\{i\}}) = e_{\{\pi(i)\}} \). Let \( g(e_1, \ldots, e_p) = (\pi^{-1}(1), \ldots, \pi^{-1}(p)) \). Then \( g(f(e_{\{i\}})) = g(e_{\{\pi(i)\}}) = e_{\{i\}} \). It then suffices to prove by induction on the cardinality \( c \) of \( I \) that \( g(f(e_I)) = e_I \). Let \( c \geq 2 \) with \( c \leq p \). Assume that \( g(f(e_J)) = e_J \) for all subsets \( J \) of \( \{1, \ldots, p\} \) of cardinality \( c-1 \). Let \( I \) be a subset of \( \{1, \ldots, p\} \) of cardinality \( c \). Then the distance between \( g(f(e_{I\setminus\{i\}})) \) and \( g(f(e_I)) \) is 1 for all \( i \in I \). By induction hypothesis, one has \( g(f(e_{I\setminus\{i\}})) = e_{I\setminus\{i\}} \). So the only possibility is \( g(f(e_I)) = e_I \).

\[ \square \]

V.1.2. Proposition. An adjacency-preserving map \( f : [m] \to [n] \) belongs to \( \square_S \) if and only if \( f \) is one-to-one.

\[ \square \]

\[ \square \]

Proof. It is clear that any map of \( \square_S \) is one-to-one. Conversely, let \( f : [m] \to [n] \) be a one-to-one adjacency-preserving map. Then by Proposition III.1.14, \( f \) factors uniquely as a composite \( [m] \xrightarrow{\psi} [m] \xrightarrow{\phi} [n] \) with \( \phi \in \square \) and \( \psi \) adjacency-preserving one-to-one. A cardinality argument implies that \( \psi \) is a bijection. Therefore \( f \in \square_S \) by Lemma V.1.1.

As for precubical sets, let \( \partial^a_i = (\delta^a_i)^* \). And let \( s_i = (\sigma_i)^* \).

V.1.3. Proposition. The symmetric precubical set of labels \( \text{Sh}_{\square_S} L_{\square_S}(\Sigma) \) is isomorphic to the following symmetric precubical set, denoted by \( 1^S \Sigma \):

- \( (1^S \Sigma)_0 = \{()\} \) (the empty word)
- \( (1^S \Sigma)_n = \Sigma^n \)
- \( \partial^a_i(a_1, \ldots, a_n) = \partial^a_i(a_1, \ldots, a_n) = (a_1, \ldots, \hat{a_i}, \ldots, a_n) \) where the notation \( \hat{a_i} \) means that \( a_i \) is removed.
- \( s_i(a_1, \ldots, a_n) = (a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_n) \) for \( 1 \leq i \leq n \).

\[ \square \]

\[ \square \]

Proof. The category of cubes \( \square_S \) is the small category freely generated by the \( \delta^a_i \) and \( \sigma_i \) operators and by the cocubical relations, the algebraic relations of Proposition III.1.2 and the Moore relations for symmetry operators \( \sigma_i \sigma_i = \text{Id} \), \( \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \) for \( i = j + 1 \) and \( \sigma_i \sigma_j = \sigma_j \sigma_i \) for \( i < j \) by [CM03] Theorem 8.1.

It is easy to prove that the \( s_i \) and \( \partial^a_i \) operators of \( 1^S \Sigma \) satisfy the dual of these algebraic relations. So \( 1^S \Sigma \) together with the \( \partial^a_i \) and \( s_i \) operators is a well-defined symmetric precubical set.

The identity of \( \Sigma \) yields a map of precubical sets \( \Sigma \to \omega_{\square_S}(1^S \Sigma) \). Hence by adjunction, one obtains a map \( L_{\square_S}(\Sigma) \to 1^S \Sigma \). The symmetric precubical set \( 1^S \Sigma \) is orthogonal to the set of morphisms \( \{\square_S[p] \to \square_S[p] \} \) for the same reason as \( 1^S \Sigma \) is orthogonal to the set of morphisms \( \{[p] \to [p] \} \). Hence by adjunction, one obtains a map of symmetric precubical sets \( f : \text{Sh}_{\square_S} L_{\square_S}(\Sigma) \to 1^S \Sigma \) which is clearly onto: an inverse image of \((a_1, \ldots, a_n) \in (1^S \Sigma)_n \) for \( n \geq 1 \) is given by the image of \((a_1, \ldots, a_n) \in L_{\square_S}(\Sigma)_n \) by the canonical map \( L_{\square_S}(\Sigma) \to \text{Sh}_{\square_S} L_{\square_S}(\Sigma) \).
Let us prove by induction on \( p \geq 1 \) that the map \( f_{\leq p} : (\text{Sh}_{\square S} \mathcal{L}_{\square S}(\Sigma))_{\leq p} \to (\Sigma_{\leq p})_{\leq p} \) is one-to-one. The map induces the isomorphism \( f_{\leq 1} : (\text{Sh}_{\square S} \mathcal{L}_{\square S}(\Sigma))_{\leq 1} \to (\Sigma_{\leq 1})_{\leq 1} \) by Proposition IV.1.18 and Proposition II.1.24. Hence the proof is complete for \( p = 1 \). Let us suppose that the map \( f_{\leq p} : (\text{Sh}_{\square S} \mathcal{L}_{\square S}(\Sigma))_{\leq p} \to (\Sigma_{\leq p})_{\leq p} \) is an isomorphism for \( p \geq 1 \). Let \( x, y \in (\text{Sh}_{\square S} \mathcal{L}_{\square S}(\Sigma))_{p+1} \) be two \((p+1)\)-cubes having the same image in \( \Sigma_p \). Then they have the same boundary in \((\Sigma_{\leq p})_{\leq p}\), and therefore \( x \) and \( y \) have the same boundary \( \partial x = \partial y \) by induction hypothesis. One obtains a commutative square of solid arrows

\[
\begin{array}{c}
\boxtimes_S[p+1] \sqcup \boxtimes_S[p+1] \boxtimes_S[p+1] \\
\boxtimes_S[p+1] \rightarrow \text{Sh}_{\square S} \mathcal{L}_{\square S}(\Sigma) \\
\text{Sh}_{\square S} \mathcal{L}_{\square S}(\Sigma) \rightarrow 1.
\end{array}
\]

The lift \( k \) exists and is unique. So \( x = k = y \). The induction hypothesis is therefore proved for \( p + 1 \).

### The labelled directed symmetric coskeleton construction

The following proposition is similar to Proposition IV.1.3.

**Proposition.** Let \( K \) be a precubical set. For any \( p \)-cube \( x \) of \( \omega \square S \mathcal{L}_{\square S}(K) \) with \( p \geq 0 \), there exists a \( p \)-cube \( y \) of \( K \subset \omega \square S \mathcal{L}_{\square S}(K) \) and a map \( \mu \in \square_S(p, p) \) such that \( x = \mu^*(y) \) where \( \mu^* : \mathcal{L}_{\square S}(K)_p \to \mathcal{L}_{\square S}(K)_p \) is the image of \( \mu \) by the presheaf \( \mathcal{L}_{\square S}(K) \in \mathcal{S}\text{Set} \).

Note that as in Proposition IV.1.3 the decomposition is actually unique.

**Proof.** With the notations of the proof of Proposition IV.1.3 By Proposition III.1.14 and Proposition V.1.2 the set map \( \overline{\pi} : [p] \to [n] \) factors as a composite \( [p] \xrightarrow{\mu} [p] \xrightarrow{\phi} [n] \) with \( \mu \in \square_S \) and \( \phi \in \square \).

By Proposition III.3.4 the truncation functor

\[
\mathcal{S}\text{Set} \downarrow \mathcal{S}\text{Set} \mathcal{L}_{\square S}(\Sigma) \to \mathcal{S}\text{Set} \downarrow \mathcal{S}\text{Set} \mathcal{L}_{\square S}(\Sigma)
\]

has a right adjoint

\[
\text{cosk}_{\square S}^{\Sigma} : (\square_S)_n^{\text{op}} \mathcal{S}\text{Set} \downarrow \mathcal{S}\text{Set} \mathcal{L}_{\square S}(\Sigma) \to \mathcal{S}\text{Set} \downarrow \mathcal{S}\text{Set} \mathcal{L}_{\square S}(\Sigma).
\]

**Definition.** (Compare with Definition III.1.22) Let \( K \) be a 1-dimensional labelled symmetric precubical set with \( K_0 = [p] \) for some \( p \geq 0 \). The labelled symmetric directed coskeleton of \( K \) is the labelled precubical set \( \text{cosk}_{\square S}^{\Sigma}(K) \) defined as the subobject of \( \text{cosk}_{\square S}^{\Sigma}(K) \) such that:

- \( \text{cosk}_{\square S}^{\Sigma}(K)_{\leq 1} = \text{cosk}_{\square S}^{\Sigma}(K)_{\leq 1} \)
- for every \( n \geq 2 \), \( x \in \text{cosk}_{\square S}^{\Sigma}(K)_n \) is an \( n \)-cube of \( \text{cosk}_{\square S}^{\Sigma}(K) \) if and only if the set map \( x_0 : [n] \to [p] \) is non-twisted, i.e. \( x_0 : [n] \to [p] \) is a composite of \( \phi \).

\[x_0 : [n] \xrightarrow{\phi} [q] \xrightarrow{\psi} [p],\]

\[\text{The factorization is necessarily unique.}\]
where \( \psi \) is a morphism of the small category \( \square \) and where \( \phi \) is of the form
\[
(\epsilon_1, \ldots, \epsilon_n) \mapsto (\epsilon_{i_1}, \ldots, \epsilon_{i_q})
\]
such that \( \{1, \ldots, n\} \subset \{i_1, \ldots, i_q\} \).

The link with labelled precubical sets is:

**V.1.6. Proposition.** Let \( K \) be a 1-dimensional labelled (symmetric) precubical set with \( K_0 = [p] \) for some \( p \geq 0 \). Then there is the isomorphism of labelled symmetric precubical sets
\[
\mathcal{L}_{\square S}(\cosk^\Sigma (K)) \cong \cosk_S^\Sigma(K).
\]

**Proof.** By a proof similar to the one of Theorem [IV.1.5] one obtains the inclusion of presheaves
\[
\mathcal{L}_{\square S}(\cosk^\Sigma (K)) \subset \cosk_{1, \Sigma}^\square S(K).
\]
It is clear that the inclusion above factors as the composite of inclusions
\[
\mathcal{L}_{\square S}(\cosk^\Sigma (K)) \subset \cosk_1^\Sigma S(K) \subset \cosk_{1, \Sigma}^\square S(K).
\]
The left-hand inclusion is an equality by Proposition [V.1.4].

Thanks to Proposition [II.2.10], one obtains the isomorphism of flows
\[
|\cosk^\Sigma (K)| \cong |\cosk^\Sigma_S(K)|.
\]

**Interpreting CCS as labelled symmetric precubical sets.**

**V.1.7. Definition.** Let \( K \) and \( L \) be two labelled symmetric precubical sets. The tensor product with synchronization (or synchronized tensor product) of \( K \) and \( L \) is
\[
K \otimes_\Sigma L := \lim_{\square S[m] \to K} \lim_{\square S[n] \to L} \cosk_S^\Sigma(\square S[m] \times_\Sigma \square S[n] \leq 1).
\]

One can then easily adapt the semantics of CCS to the case of labelled symmetric precubical sets. The interest of this setting is that it is simpler than the one of transverse symmetric precubical sets, and that, as in Proposition [IV.2.8], there is an isomorphism of labelled symmetric precubical sets \( K \otimes_\Sigma L \cong L \otimes_\Sigma K \) for all labelled symmetric precubical sets \( K \) and \( L \). The synchronized tensor product of symmetric precubical sets is also colimit-preserving by Proposition [IV.2.4] and therefore associative.

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