TURING INSTABILITY AND DYNAMIC PHASE TRANSITION FOR THE BRUSSELOTER MODEL WITH MULTIPLE CRITICAL EIGENVALUES

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Abstract. In this paper, we study the dynamic phase transition for one dimensional Brusselator model. By the linear stability analysis, we define two critical numbers \( \lambda_0 \) and \( \lambda_1 \) for the control parameter \( \lambda \) in the equation. Motivated by [9], we assume that \( \lambda_0 < \lambda_1 \) and the linearized operator at the trivial solution has multiple critical eigenvalues \( \beta_0^N \) and \( \beta_1^{N+1} \). Then, we show that as \( \lambda \) passes through \( \lambda_0 \), the trivial solution bifurcates to an \( S^1 \)-attractor \( A_N \). We verify that \( A_N \) consists of eight steady state solutions and orbits connecting them. We compute the leading coefficients of each steady state solution via the center manifold analysis. We also give numerical results to explain the main theorem.

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1. Introduction. In 1950s, Belousov and Zhabotinsky discovered chemical systems exhibiting temporal oscillations. It is now known as the Belousov-Zhabotinsky reaction and regarded as a generic chemical reaction that describes the concentrations of the reactants exhibiting oscillating behavior([3]). This reaction can be observed in a mixture of potassium bromate, cerium(IV) sulfate, and citric acid in dilute sulfuric acid. While most chemical reactions reach a state of homogeneity and equilibrium quickly, the Belousov-Zhabotinsky reaction maintains a prolonged state of non-equilibrium leading to macroscopic temporal oscillations and spatial pattern formation([6]).

In the pattern formation of the Belousov-Zhabotinsky reaction, the diffusion is thought to play an important role. Inhomogeneity in the concentration of reactants makes the difference of compounds in space, and the patterns are formed in the diffusion of the reacted compounds. This can be understood as a phase transition dynamics. As a control parameter related to the diffusion term passes through a critical number, a homogeneous steady state solution is destabilized. Then, the system undergoes a bifurcation from the trivial solution to an attractor. The bifurcated attractor determines the final patterns and the long time dynamics of solutions starting near the homogeneous steady state.

Pattern formation due to the diffusion effect was discovered by Turing([18]) and is now called the Turing instability. As a control parameter related to the diffusion term passes through a critical number, the governing equations of the chemical system bifurcate to an attractor and final patterns are saturated in the bifurcated attractor. Therefore, the understanding of the pattern formation is deeply related to the bifurcation analysis and have been studied extensively. See [10, 13] for instance.

In this paper, we are interested in the phase transition and pattern formation of the Belousov-Zhabotinsky reaction by focusing the Brusselator model. The Brusselator model was proposed by Prigogine and Lefever in [16] and is known as one of the famous models for an autocatalytic and oscillating chemical reaction. The reaction consists of four steps:

\[
\begin{align*}
A \xrightarrow{k_1} X, \\
B + X \xrightarrow{k_2} Y + D, \\
2X + Y \xrightarrow{k_3} 3X, \\
X \xrightarrow{k_4} E.
\end{align*}
\] (1.1)

Here, the chemical reactants \(A\) and \(B\) are used to make products \(D\) and \(E\) with intermediate autocatalytic reactants \(X\) and \(Y\). The third step is autocatalytic since two \(X\) molecules make three \(X\) molecules. This causes chemical oscillations. The numbers \(k_1, k_2, k_3\) and \(k_4\) are the rate of reactions for each component reaction.

Let \(\alpha, \lambda, U\) and \(V\) be the concentrations of \(A, B, X\) and \(Y\), respectively. We assume that \(\alpha, \lambda\) are held positive constant during the reaction, and \(U, V\) have variables in time and space. The law of mass action says that the rate of a chemical reaction is directly proportional to the product of the concentration of reactant. By this law, we obtain the reaction-diffusion system for (1.1):

\[
\begin{align*}
\frac{\partial U}{\partial t} &= \mu \Delta U + k_1 \alpha - (k_2 \lambda + k_4)U + k_3 U^2 V \quad \text{in} \, \Omega, \\
\frac{\partial V}{\partial t} &= \nu \Delta V + k_2 \lambda U - k_3 U^2 V \quad \text{in} \, \Omega, \\
\frac{\partial U}{\partial n} = \frac{\partial U}{\partial n} &= 0 \quad \text{on} \, \partial \Omega.
\end{align*}
\] (1.2)
Here $\Omega \subset \mathbb{R}^n$ ($1 \leq n \leq 3$) is a smooth bounded domain and $\mu$ and $\nu$ are diffusion constants of $X$ and $Y$, respectively. If we make a transformation

$$t = k_4^{-1}t', \quad x = lx', \quad U = \sqrt{k_4/k_3}U', \quad V = \sqrt{k_4/k_3}V',$$

$$\alpha = \sqrt{k_4^3/k_3^2k_3}\alpha', \quad \lambda = (k_4^3/k_2)\lambda', \quad \mu = l^2k_4\mu', \quad \nu = l^2k_4\nu',$$

and omit the primes, then we have the nondimensionalized form of (1.2):

$$\begin{aligned}
\frac{\partial U}{\partial t} &= \mu \Delta U + \alpha - (\lambda + 1)U + U'^2V \quad \text{in} \quad \Omega, \\
\frac{\partial V}{\partial t} &= \nu \Delta V + \lambda U - U'^2V \quad \text{in} \quad \Omega, \\
\frac{\partial U}{\partial n} &= \frac{\partial V}{\partial n} = 0 \quad \text{on} \quad \partial \Omega.
\end{aligned}$$ (1.3)

In this article, we study the Turing instability arising from the Brusselator model (1.3) by investigating the dynamic bifurcation process. The system (1.3) has a constant steady state solution $(\alpha, \lambda/\alpha)$. It is not difficult to see that $(\alpha, \lambda/\alpha)$ is asymptotically stable for the associated ODE system of (1.3) if and only if

$$\lambda < \lambda_1 := \alpha^2 + 1.$$ (1.4)

See [1] for instance. Thus, it is reasonable to consider the Turing instability in the case that $\lambda \leq \lambda_1$. To this end, we consider the perturbation centered at $(\alpha, \lambda/\alpha)$ and dynamic transition around it. Set

$$U = \alpha + u \quad \text{and} \quad V = \frac{\lambda}{\alpha} + v.$$ 

Then the system (1.3) is written as

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \mu \Delta u + (\lambda - 1)u + \alpha^2v + \frac{\lambda}{\alpha}u + 2\alpha uv + u'^2v, \\
\frac{\partial v}{\partial t} &= \nu \Delta v - \lambda u - \alpha^2v - \frac{\lambda}{\alpha}u - 2\alpha uv - u'^2v, \\
\frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial \Omega.
\end{aligned}$$ (1.5)

In what follows, we will set up a control parameter for (1.5) and study the dynamic transition of (1.5). We will show that (1.5) bifurcates from the trivial solution to an attractor as the control parameter passes over a critical number. We will find the spatial patterns of (1.5) by making a detailed analysis on the bifurcated attractor. The main tool for our study is the attractor bifurcation theory ([8]) based on the center manifold analysis. This theory focuses on the bifurcated attractor rather than any of the steady states or any of the connecting orbits. Since we deal with the whole structure of the attractor, we can obtain not only the asymptotic stability of the bifurcated point attractors or limit cycles, but also the stability of different solutions, saddle points for instance, in the bifurcated attractor. So, we can depict the long time dynamics of solutions near the attractor as well as stability of steady state solutions in the attractor. To verify the structure of the bifurcated attractor, we employ the center manifold reduction. We compute the leading terms of solutions on the center manifold in very detail which are necessary to know the exact stability of steady states or limit cycles in the attractor.

This paper is organized as follows. In Section 2, we set up the bifurcation problem for (1.5) and state the main result, Theorem 2.2. We also mention the main
contribution of this paper, \textit{bifurcation at multiple critical eigenvalues}. In Section 3, we prove Theorem 2.2 that verifies how the bifurcation occurs near the critical point \( \lambda_0 \) and provides the exact values of leading coefficients of the bifurcated steady state solutions up to the order \( O(\sqrt{\lambda - \lambda_0}) \). In Section 4, we investigate the bifurcation phenomena by numerical analysis that illustrates the main result. In Section 5, we provide a detailed derivation of (3.13), the system of reduced equations on the center manifold, used in the proof of Theorem 2.2.

2. Main result. In this section, we set up an abstract formula of (1.5) following the argument of [9] and state the main result of this paper. First, we define function spaces

\[
H = L^2(\Omega, \mathbb{R}^2),
\]

\[
H_1 = \{ w = (u, v) \in H^2(\Omega, \mathbb{R}^2) \mid \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \},
\]

and the operators \( \mathcal{L}_\lambda, G : H_1 \to H \) by

\[
\mathcal{L}_\lambda = \begin{bmatrix}
\mu \Delta + (\lambda - 1) & \alpha^2 \\
-\lambda & \nu \Delta - \alpha^2
\end{bmatrix},
\]

\[
G(w) = \begin{bmatrix}
(\lambda/\alpha)u^2 + 2\alpha uv \\
-(\lambda/\alpha)u^2 - 2\alpha uv
\end{bmatrix} + \begin{bmatrix}
u^2v \\
-u^2v
\end{bmatrix} =: G_2(w) + G_3(w).
\]

Then the system (1.5) can be written as

\[
\begin{cases}
\frac{\partial w}{\partial t} = \mathcal{L}_\lambda w + G(w) & \text{in } \Omega, \\
\frac{\partial w}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(2.1)

In computational aspect, it is more convenient to regard the functions \( G_j (j = 2, 3) \) as \( j \)-multilinear functions defined as follows: for \( w_k = (u_k, v_k) \in H_1 \) with \( k = 1, 2, 3 \),

\[
\begin{cases}
G_2(w_1, w_2) = \left( \frac{\lambda}{\alpha} u_1 u_2 + 2\alpha u_1 v_2 \right) \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \\
G_3(w_1, w_2, w_3) = u_1 u_2 v_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\end{cases}
\]

(2.2)

Then, \( G_2(w) = G_2(w, w) \) and \( G_3(w) = G_3(w, w, w) \).

Let us consider the eigenvalue problem \( \mathcal{L}_\lambda w = \beta w \), that is,

\[
\begin{cases}
\mu \Delta u + (\lambda - 1)u + \alpha^2v = \beta u & \text{in } \Omega, \\
\nu \Delta v - \lambda u - \alpha^2v = \beta v & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(2.3)

Let \( \rho_n \) and \( e_n \) be the \( n \)-th eigenvalue and eigenvector of the Laplacian operator with the Neumann boundary condition:

\[
\begin{cases}
-\Delta e_n = \rho_n e_n & \text{in } \Omega, \\
\frac{\partial e_n}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(2.4)

We note that

\[0 = \rho_0 < \rho_1 < \rho_2 < \cdots \to \infty.\]
For \( n = 0, 1, 2, \cdots \), let \( \beta_n^\pm = \beta_n^\pm(\lambda) \) be eigenvalues and \( \xi_n^\pm = \xi_n^\pm(\lambda) \in \mathbb{R}^2 \) be corresponding eigenvectors of

\[
M_n = \begin{bmatrix}
\lambda - \mu \rho_n - 1 & \alpha^2 \\
-\lambda & -\nu \rho_n - \alpha^2
\end{bmatrix}.
\]

Then, \( \beta_n^\pm \) also become eigenvalues of \( L_\lambda \) and the corresponding eigenvectors are \( \phi_n^\pm = \xi_n^\pm e_n \).

Explicitly,

\[
\begin{aligned}
\beta_n^\pm(\lambda) &= \frac{1}{2} \left\{ \text{Tr} \ M_n \pm \sqrt{\left(\text{Tr} \ M_n\right)^2 - 4 \det M_n} \right\}, \\
\text{Tr} \ M_n &= \lambda - \sigma_n, \\
\sigma_n &= \mu \rho_n + \nu \rho_n + \alpha^2 + 1, \\
\det M_n &= (\mu \rho_n + 1)(\nu \rho_n + \alpha^2) - \lambda \nu \rho_n, \\
\xi_n^\pm(\lambda) &= \begin{bmatrix} \alpha^2 \\ 1 + \mu \rho_n + \beta_n^\pm - \lambda \end{bmatrix}.
\end{aligned}
\]

If \( n > 0 \), we can write \( \det M_n \) as

\[
\det M_n = -\nu \rho_n \left[ \lambda - f(\rho_n) \right] \quad \text{where} \quad f(\rho) = \frac{(\rho \mu + 1)(\rho \nu + \alpha^2)}{\nu \rho}.
\]

We note that

\[
\begin{aligned}
\beta_n^-(\lambda) < \beta_n^+(\lambda) &= 0 & \iff & \text{Tr} \ M_n < 0 \ \text{and} \ \det M_n = 0, \\
\text{Re} \ \beta_n^+(\lambda) &= 0 & \iff & \text{Tr} \ M_n = 0 \ \text{and} \ \det M_n > 0.
\end{aligned}
\]

It is natural to define a critical number as the smallest number satisfying \( \det M_n = 0 \) or \( \text{Tr} \ M_n = 0 \). The smallest value of \( \lambda \) for \( \det M_n = 0 \) is given by

\[
\lambda_0 := \min_{\rho_n} f(\rho_n),
\]

and the smallest value of \( \lambda \) for \( \text{Tr} \ M_n = 0 \) is \( \lambda_1 \) defined by (1.4). Since \( f \) attains a unique minimum at

\[
x = \hat{\rho} := \frac{\alpha}{\sqrt{\mu \nu}},
\]

there is a number \( N \) such that either

\[
\lambda_0 = f(\rho_N) \quad \text{and} \quad \lambda_0 < f(\rho_n) \ \forall n \neq N,
\]

or

\[
\lambda_0 = f(\rho_N) = f(\rho_{N+1}) \quad \text{and} \quad \lambda_0 < f(\rho_n) \ \forall n \neq N, N + 1.
\]

We note that

\[
f(\rho_{N+1}) - f(\rho_N) = \mu(\rho_{N+1} - \rho_N) \left( 1 - \frac{\hat{\rho}^2}{\rho_N \rho_{N+1}} \right).
\]

So, the latter case (2.8) happens if and only if

\[
\rho_N \rho_{N+1} = \hat{\rho}^2 = \frac{\alpha^2}{\mu \nu}.
\]

In this case, we can rewrite \( \lambda_0 \) as

\[
\lambda_0 = (\mu \rho_N + 1)(\mu \rho_{N+1} + 1).
\]

Furthermore, since \( \det M_0 = \alpha^2 > 0, N \neq 0 \) for both cases (2.7) and (2.8). By summing up the above argument, we are led to the following lemma.
Lemma 2.1. [Principle of Exchange of Stability, [9]]

(i) If \( \lambda_0 < \lambda_1 \) and (2.7) is valid, then for all \( \lambda \) sufficiently close to \( \lambda_0 \),

\[
\beta_N^+ (\lambda) = \begin{cases} 
< 0 & \text{if } \lambda < \lambda_0, \\
0 & \text{if } \lambda = \lambda_0, \\
> 0 & \text{if } \lambda > \lambda_0,
\end{cases}
\]

(2.11)

\[
\beta_N^- (\lambda_0) < 0 \text{ and } \text{Re } \beta_N^+ (\lambda_0) < 0 \quad \forall n \neq N.
\]

(ii) If \( \lambda_0 < \lambda_1 \) and (2.8) is valid, then for all \( \lambda \) sufficiently close to \( \lambda_0 \),

\[
\beta_N^+ (\lambda), \beta_{N+1}^+ (\lambda) = \begin{cases} 
< 0 & \text{if } \lambda < \lambda_0, \\
0 & \text{if } \lambda = \lambda_0, \\
> 0 & \text{if } \lambda > \lambda_0,
\end{cases}
\]

(2.12)

\[
\beta_N^- (\lambda_0), \beta_{N+1}^- (\lambda_0) < 0 \text{ and } \text{Re } \beta_N^+ (\lambda_0) < 0 \quad \forall n \neq N, N + 1.
\]

(iii) If \( \lambda_1 < \lambda_0 \), then for all \( \lambda \) sufficiently close to \( \lambda_1 \),

\[
\text{Re } \beta_N^+ (\lambda) = \text{Re } \beta_N^- (\lambda) = \begin{cases} 
< 0 & \text{if } \lambda < \lambda_1, \\
0 & \text{if } \lambda = \lambda_1, \\
> 0 & \text{if } \lambda > \lambda_1,
\end{cases}
\]

(2.13)

\[
\text{Re } \beta_N^+ (\lambda_1) < 0 \quad \forall n \geq 1.
\]

By Lemma 2.1, as \( \lambda \) moves, we may expect two types of phase transition that leads to the Turing instability. First, if \( \lambda_0 < \lambda_1 \), then as \( \lambda \) crosses over \( \lambda_0 \), the trivial solution \((u, v) = (0, 0)\) of (1.5) loses its stability and gives us an attractor bifurcated from it. The bifurcation happens from real eigenvalues. Second, if \( \lambda_1 < \lambda_0 \), then as \( \lambda \) passes over \( \lambda_1 \), the trivial solution bifurcates to an attractor. The bifurcation comes from complex eigenvalues. In both cases, the bifurcated attractor is in charge of long time dynamics of solutions near the \((u, v) = (0, 0)\) and spatial patterns are saturated in it.

In [9], Ma and Wang proved that such a scenario really happens in the case (i) or (iii). They assumed that \( \beta_N^+ (\lambda_0) \) is simple in the case (i), that is, there is only one eigenvector for \( \beta_N^+ (\lambda_0) \). Then, as a generic bifurcation, one may expect the pitchfork bifurcation for (i) and the Hopf bifurcation for (iii). They justified this conclusion via a center manifold reduction. The main results of them is the calculation of the leading coefficients of solutions in the long time dynamics and gave a criteria of stability analysis. However, it is usually very difficult to determine whether the criteria is true or false for given domain \( \Omega \) since it is almost impossible to detect the exact eigenvalues and eigenvectors for (2.4). So, as an example of such criteria, they carried out the calculation in detail for one dimensional case, i.e., on \( \Omega = (0, L) \).

In this article, we study the dynamic transition of (1.5) near the trivial solution \((u, v) = (0, 0)\) for the case (ii) in Lemma 2.1. This case was not dealt with in [9]. The main difference between (i) and (ii) is the dimension of the center manifold. Although the difference of dimension is just one, this causes very complicated situation regarding the bifurcation analysis. So, the main contribution of this paper is the verification of the bifurcation at multiple critical eigenvalues. This helps us understand the role of multiple critical eigenvalues in the Turing instability. Indeed, we will show that the Turing instability leads to an \( S^1 \)-attractor bifurcation. The bifurcated attractor consists of finite number of steady state solutions and orbits.
connecting them. We determine the stability of steady state solutions and compute the leading terms of asymptotically stable solutions. Through this analysis that verifies the structure of the bifurcated attractor as complete as possible, we can understand the long time dynamics of solutions near the trivial solutions very specifically. In this point of view, we may say that our analysis has a good advantage compared to the results for the steady state bifurcation ([2, 4, 14, 15]) and the existence of global attractor ([5, 19]).

For the efficiency of calculation, we restrict ourselves to one dimensional case. That is, we consider (1.5) on an interval \( \Omega = (0, L) \subset \mathbb{R} \). Although various patterns may arise in higher dimensions ([7]), one dimensional problem also happens often as an approximation of higher dimensional chemical reactions. For instance, the Belousov-Zhabotinsky reaction in capillary tubes can be understood in such a framework ([12, 17]). The main result is as follows.

**Theorem 2.2.** Let \( \lambda_1 \) and \( \lambda_0 \) be defined by (1.4) and (2.5) such that \( \lambda_0 < \lambda_1 \). Suppose that (2.8) holds for some \( N \geq 2 \). Then, as \( \lambda \) passes through \( \lambda_0 \) to the right, the perturbed Brusselator system (1.5) on \( \Omega = (0, L) \) bifurcates from the trivial solution \( (u, v) = (0, 0) \) to an attractor \( A_N = A_N(\mu, \nu, \alpha, L) \). The bifurcated attractor satisfies the following.

(i) For any bounded open set \( \mathcal{D} \subset H \) with \( (u, v) = (0, 0) \in \mathcal{D} \), there exists \( \delta > 0 \) such that for \( \lambda \in (\lambda_0, \lambda_0 + \delta) \), \( A_N \) attracts \( \mathcal{D} \setminus \Gamma \) in \( H \), where \( \Gamma \) is the stable manifold of \( (u, v) = (0, 0) \) with codimension 2.

(ii) \( A_N \) is homeomorphic to the unit circle \( S^1 \).

(iii) \( A_N \) consists of eight steady state solutions and orbits connecting them. The steady state solutions are given by \( w_i^\pm := (u_i^\pm, v_i^\pm) \) for \( i = 1, 2, 3, 4 \). Explicitly, up to the order \( o(\sqrt{\lambda - \lambda_0}) \),

\[
\begin{align*}
w_1^+ &= \pm y_0 \xi_N^+(\lambda_0) \frac{N \pi x}{L}, \\
w_2^+ &= \pm z_0 \xi_{N+1}^+(\lambda_0) \frac{(N+1) \pi x}{L}, \\
w_3^+ &= y_1 \xi_N^+(\lambda_0) \frac{N \pi x}{L} \pm z_1 \xi_{N+1}^+(\lambda_0) \frac{(N+1) \pi x}{L}, \\
w_4^+ &= -y_1 \xi_N^+(\lambda_0) \frac{N \pi x}{L} \pm z_1 \xi_{N+1}^+(\lambda_0) \frac{(N+1) \pi x}{L}.
\end{align*}
\]

Here,

\[
\xi_m^+(\lambda_0) = \left[ \frac{-\nu \rho_m}{\mu \rho_m + 1} \right] \quad \text{for } m = N \text{ or } N+1.
\]

Moreover, \( y_i \) and \( z_i \) for \( i = 0, 1 \) are positive numbers defined by

\[
\begin{align*}
y_0^2 &= \frac{4 \alpha}{c_{30}} (\lambda - \lambda_0), \\
z_0^2 &= \frac{4 \alpha}{c_{30}} (\lambda - \lambda_0), \\
y_1^2 &= \frac{4 \alpha (\rho_N c_{30} - 2 \rho_{N+1} c_{12})}{\rho_N (c_{30} c_{03} - 4 c_{12} c_{21})} (\lambda - \lambda_0), \\
z_1^2 &= \frac{4 \alpha (\rho_{N+1} c_{30} - 2 \rho_{N} c_{21})}{\rho_{N+1} (c_{30} c_{03} - 4 c_{12} c_{21})} (\lambda - \lambda_0).
\end{align*}
\]
The numbers $c_{ij}$’s are given specifically by (5.10) in Section 5 and
\[ \rho_m = \frac{m^2 \pi^2}{L^2} \quad \text{for} \quad m = N \text{ or } N + 1. \]

(iv) If $c_{30}c_{03} - 4c_{12}c_{21} < 0$, then $w_1^\pm$, $w_2^\pm$ are asymptotically stable and $w_3^\pm$, $w_4^\pm$ are saddle. If $c_{30}c_{03} - 4c_{12}c_{21} > 0$, then $w_1^\pm$, $w_2^\pm$ are saddle and $w_3^\pm$, $w_4^\pm$ are asymptotically stable.

Theorem 2.2 tells us that the bifurcated attractor $A_N$ is responsible for the long time dynamics of solutions whose initial condition are close to $(0,0)$. The detailed structure of $A$ is depicted in this theorem. We prove Theorem 2.2 in Section 3. We also exhibit numerical results regarding Theorem 2.2 in Section 4, which help us understand the situation.

3. Proof of Theorem 2.2. In this section, we prove Theorem 2.2. We assume that

- $\Omega = (0, L) \subset \mathbb{R}$; (3.1)
- Lemma 2.1 (ii) holds for some $N \geq 2$; (3.2)
- $\lambda_0 = f(\rho_N) = f(\rho_{N+1})$; (3.3)
- $\lambda_0 < \lambda_1 = \sigma_0$ and $0 < \varepsilon := \lambda - \lambda_0 \ll 1$. (3.4)

The proof of parts (i) and (ii) of Theorem 2.2 follows from the Attractor Bifurcation Theorem (Theorem 6.1 of [8]). As a consequence, as $\lambda$ passes through $\lambda_0$ to the right, the perturbed Brusselator system (1.5) bifurcates from the trivial solution $(u, v) = (0, 0)$ to an attractor $A_N = A_N(\mu, \nu, \alpha, L)$ that is homeomorphic to $S^1$. In what follows, we concentrate on the proof of (iii) and (iv) by verifying the structure of $A_N$ by use of the center manifold analysis. This helps us to understand the pattern selection when the phase transition happens near $\lambda_0$.

The eigenvalues $\rho_n$ and their corresponding eigenvectors $e_n$ are explicitly given by
\[ \rho_n = \frac{n^2 \pi^2}{L^2} \quad \text{and} \quad e_n = \cos \frac{n \pi x}{L} \quad \text{for} \quad n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}. \]

By (2.9), we can determine the number $N$ by other parameters:
\[ L^2 = \frac{N(N+1)\pi^2}{\hat{\rho}} \quad \text{such that} \quad N = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\hat{\rho} L^2}{\pi^2}}, \]
where $\hat{\rho}$ is defined by (2.6). Then,
\[ \rho_N = \frac{N}{N+1} \hat{\rho}, \quad \rho_{N+1} = \frac{N+1}{N} \hat{\rho}. \] (3.5)

Using (2.10) and (3.5), we can express the critical number $\lambda_0$ as
\[ \lambda_0 = \mu^2 \rho_N \rho_{N+1} + \mu(\rho_N + \rho_{N+1}) + 1 = \frac{\mu \alpha^2 + 2N^2 + 2N + 1}{N(N+1)} \sqrt{\frac{\mu}{\nu}} \alpha + 1. \] (3.6)

Consequently, $\lambda_0 < \lambda_1$ if and only if
\[ \mu < \nu \quad \text{and} \quad \alpha > \frac{\sqrt{\mu \nu}}{\nu - \mu} \frac{2N^2 + 2N + 1}{N(N+1)} > \frac{2\sqrt{\mu \nu}}{\nu - \mu}. \] (3.7)

To calculate the center manifold function, we need to know the eigenvalues and the eigenvectors of the conjugate $L^*_\lambda$ of $L_\lambda$ since $L_\lambda$ is not a symmetric operator.
In fact, $\mathcal{L}_\lambda^*$ has eigenvalues $\beta_n^{\pm*}$ and eigenvectors $\phi_n^{\pm*} = \xi_n^{\pm*} e_n$, where $\beta_n^{\pm*}$ and $\xi_n^{\pm*}$ are the eigenvalues and the corresponding eigenvectors of the Hermitian conjugate matrix $M_n^*(\lambda)$ of $M_n(\lambda)$. In fact,

$$M_n^* = \begin{bmatrix} \lambda - \mu \rho_n - 1 & -\lambda \\ \alpha^2 & -\nu \rho_n - \alpha^2 \end{bmatrix}.$$  

and $\beta_n^{\pm*}$ are the complex conjugates of $\beta_n^{\pm}$. Let us compute the eigenvectors at the critical value $\lambda_0$. If $m = N$ or $N + 1$, then

$$\xi_m^+(\lambda_0) = \begin{bmatrix} \alpha^2 \\ \mu \rho_m + 1 - \lambda_0 \end{bmatrix} = -\alpha^2 \nu \rho_m \begin{bmatrix} -\nu \rho_m \\ \mu \rho_m + 1 \end{bmatrix}.$$  

For computational convenience in what follows, we redefine

$$\xi_m^+(\lambda_0) = \begin{bmatrix} -\nu \rho_m \\ \mu \rho_m + 1 \end{bmatrix} \text{ for } m = N \text{ or } N + 1.$$  

Similarly, one can compute

$$\xi_m^{++}(\lambda_0) = \begin{bmatrix} \nu \rho_m + \alpha^2 \\ \alpha^2 \end{bmatrix} \text{ for } m = N \text{ or } N + 1.$$  

We will use the following formula frequently for normalization: for $m = N$ or $N + 1$

$$B_m = -\frac{2}{L} \langle \phi_m^+(\lambda_0), \phi_m^{++}(\lambda_0) \rangle = \nu^2 \rho_m^2 + \alpha^2[(\nu - \mu)\rho_m - 1] > 0. \quad (3.8)$$  

It comes from (3.5) and (3.7) that

$$(\nu - \mu)\rho_{N+1} > (\nu - \mu)\rho_N = \frac{N}{N + 1} \cdot \alpha(\nu - \mu) \sqrt{\mu \theta} > \frac{2N}{N + 1} > 1.$$  

Hence,

$$B_m > 0 \text{ for } m = N \text{ or } m = N + 1.$$  

Moreover, for $m = N$ or $N + 1$

$$\begin{cases} \det M_m(\lambda) = -\nu \rho_m \big[\lambda_0 + \varepsilon - f(\rho_m)\big] = -\nu \rho_m \varepsilon, \\
\text{Tr} M_m(\lambda) = (\lambda_0 - \sigma_m) + \varepsilon = -\frac{B_m}{\nu \rho_m} + \varepsilon, \quad (3.9) \\
\beta_m^+(\lambda) = \frac{\nu \rho_m^2}{B_m} \varepsilon + O(\varepsilon^2). \end{cases}$$  

We check the third identity in (3.9):

$$2\beta_m^+(\lambda) = \text{Tr} M_m + \sqrt{\text{Tr} M_m^2 - 4(\det M_m)}$$  

$$= (\lambda_0 - \sigma_m) + \varepsilon + \sqrt{(\lambda_0 - \sigma_m)^2 + 2(\lambda_0 - \sigma_m)\varepsilon + 4\nu \rho_m \varepsilon + O(\varepsilon^2)}$$  

$$= (\lambda_0 - \sigma_m) + \varepsilon - (\lambda_0 - \sigma_m) \left[1 + \frac{\varepsilon}{2} \frac{2(\lambda_0 - \sigma_m) + 4\nu \rho_m}{(\lambda_0 - \sigma_m)^2} + O(\varepsilon^2)\right]$$  

$$= -\frac{2\nu \rho_m}{\lambda_0 - \sigma_m} \varepsilon + O(\varepsilon^2) = \frac{2\nu \rho_m^2}{B_m} \varepsilon + O(\varepsilon^2).$$  

Now, we want to know the long time dynamics of solutions (1.5) when $\lambda$ is slightly bigger than $\lambda_0$. The main strategy is to reduce the perturbed Brusselator system (1.5) on the center manifold at the trivial solution $(0, 0)$ for $\lambda$ being near $\lambda_0$. We decompose $H = E_1^2 \oplus E_2^2$ where $E_1^2 = \text{span} \{\phi_N^+(\lambda), \phi_{N+1}^+(\lambda)\}$. We note that

$$E_2^2 = \{w \in H | \langle w, \phi_N^+(\lambda) \rangle = \langle w, \phi_{N+1}^+(\lambda) \rangle = 0\}.$$
Let $P_j^\lambda : H \to E_j^\lambda$ be the canonical projections for $j = 1, 2$. We express a solution $w$ of (2.1) as

$$w = y\phi_N^+(\lambda) + z\phi_{N+1}^+(\lambda) + \Phi^\lambda(y, z),$$  

(3.10)

where $\Phi^\lambda : E_1^\lambda \to E_2^\lambda$ is a center manifold function. For simplicity, we frequently write $o(k)$ instead of $o(|y|^k + |z|^k)$ in what follows. We also omit $\lambda_0$ in an expression of quantity evaluated at $\lambda = \lambda_0$ if there is no confusion: for instance, $\phi_N^+(\lambda_0)$ instead of $\phi_N^+(\lambda_0)$, and so on. Then the reduced equation of (2.1) on the center manifold reads

$$\begin{cases}
\frac{dy}{dt} = \beta_N^+(\lambda)y + g_{11}(y, z, \lambda) + g_{12}(y, z, \lambda), \\
\frac{dz}{dt} = \beta_{N+1}^+(\lambda)z + g_{21}(y, z, \lambda) + g_{22}(y, z, \lambda),
\end{cases}$$

(3.11)

where

\begin{align*}
&g_{11}(y, z, \lambda) = \frac{\langle G_2(w), \phi_N^+(\lambda) \rangle}{\langle \phi_N^+(\lambda), \phi_N^+(\lambda) \rangle} = -2\frac{\langle G_2(w), \phi_N^+(\lambda) \rangle}{LB_N}, \\
&g_{12}(y, z, \lambda) = \frac{\langle G_3(w), \phi_N^+(\lambda) \rangle}{\langle \phi_N^+(\lambda), \phi_N^+(\lambda) \rangle} = -2\frac{\langle G_3(w), \phi_N^+(\lambda) \rangle}{LB_N}, \\
&g_{21}(y, z, \lambda) = \frac{\langle G_2(w), \phi_{N+1}^+(\lambda) \rangle}{\langle \phi_{N+1}^+(\lambda), \phi_{N+1}^+(\lambda) \rangle} = -2\frac{\langle G_2(w), \phi_{N+1}^+(\lambda) \rangle}{LB_{N+1}}, \\
&g_{22}(y, z, \lambda) = \frac{\langle G_3(w), \phi_{N+1}^+(\lambda) \rangle}{\langle \phi_{N+1}^+(\lambda), \phi_{N+1}^+(\lambda) \rangle} = -2\frac{\langle G_3(w), \phi_{N+1}^+(\lambda) \rangle}{LB_{N+1}}.
\end{align*}

(3.12)

After some calculation, we can deduce that up to the order $o(3)$,

$$\begin{cases}
\frac{dy}{dt} = \beta_N^+(\lambda)y - \frac{\nu^2\rho_N}{4\alpha B_N} (\rho_N c_{30} y^2 + 2\rho_{N+1} c_{12} y z^2) =: h_1(y, z), \\
\frac{dz}{dt} = \beta_{N+1}^+(\lambda)z - \frac{\nu^2\rho_{N+1}}{4\alpha B_{N+1}} (2\rho_N c_{21} y^2 z + \rho_{N+1} c_{03} z^3) =: h_2(y, z),
\end{cases}$$

(3.13)

where each constant $c_{ij} = c_{ij}(N, \mu, \nu, \alpha, L)$ is given by (5.10). The detailed derivation of (3.13) is given in Section 5. Appendix.

We want to find static solutions of (3.13) and determine their stability. First, we obtain $(\pm y_0, 0)$ and $(0, \pm z_0)$ where $y_0$ and $z_0$ are positive. Explicitly, we obtain by (3.9) that up to the order $O(\varepsilon^2)$.

$$\begin{cases}
y_0^2 = \frac{4\alpha B_N \beta_N}{\nu^2 \rho_N c_{30}} = \frac{4\alpha}{c_{30} \varepsilon}, \\
z_0^2 = \frac{4\alpha B_{N+1} \beta_{N+1}}{\nu^2 \rho_{N+1} c_{03}} = \frac{4\alpha}{c_{03} \varepsilon},
\end{cases}$$

(3.14)

These solutions exist as long as $c_{30} > 0$ or $c_{03} > 0$. If $y \neq 0$ and $z \neq 0$, then we deduce from (3.9) that up to the order $O(\varepsilon^2)$

$$\begin{cases}
\rho_N c_{30} y^2 + 2\rho_{N+1} c_{12} y z^2 = 4\alpha \rho_N \varepsilon, \\
2\rho_N c_{21} y^2 + \rho_{N+1} c_{03} z^2 = 4\alpha \rho_{N+1} \varepsilon.
\end{cases}$$

(3.15)
Hence, we have another four static solutions \((y_1, \pm z_1)\) and \((-y_1, \pm z_1)\) where \(y_1\) and \(z_1\) are positive. Explicitly, up to the order \(O(\varepsilon^2)\),

\[
\begin{align*}
y_1^2 &= \frac{4\alpha(\rho_N c_{03} - 2\rho_N^{+1}c_{12})}{\rho_N(c_{30}c_{03} - 4c_{12}c_{21})} \varepsilon, \\
z_1^2 &= \frac{4\alpha(\rho_N^{+1}c_{30} - 2\rho_N c_{21})}{\rho_N^{+1}(c_{30}c_{03} - 4c_{12}c_{21})} \varepsilon.
\end{align*}
\] (3.16)

Of course, these solutions may not exist unless the right hand sides are positive. Furthermore, by applying the Poincaré-Bendixon Theorem to (3.13), we deduce that \(A_N\) consists of two, four, six or eight singular points and orbits that connect them. So, the structure of \(A_N\) is determined by the stability of static solutions \((\pm y_0, 0), (0, \pm z_0), (y_1, \pm z_1), (-y_1, \pm z_1)\). (3.17)

If \((a, b)\) is one of (3.17), then we can write the corresponding steady state solutions on \(A_N\) as

\[
\begin{bmatrix}
    u(x; a, b) \\
v(x; a, b)
\end{bmatrix} = a\phi_N^+(\lambda_0) + b\phi_N^{+1}(\lambda_0) + o(\varepsilon)
\]

\[
= a\xi_N^+(\lambda_0) \cos \frac{N\pi x}{L} + b\xi_N^{+1}(\lambda_0) \cos \frac{(N + 1)\pi x}{L} + o(\varepsilon).
\] (3.18)

We have shown that the bifurcated attractor \(A_N\) consists of steady state solutions \(w_i^\pm\) for \(i = 1, 2, 3, 4\) and their connecting orbits. Each \(w_i^\pm\) may exist or not. If \(k\) is the number of steady state solutions, then all possible cases are as follows.

(i) \(k = 2\). Then, only either \(w_1^\pm\) or \(w_2^\pm\) can exist. Table 1 shows the stability of each singular point for \(k = 2\). Here, “×” means “does not exist” in the following Tables.

(ii) \(k = 4\). Then, either \(w_1^\pm\) and \(w_2^\pm\), or \(w_3^\pm\) and \(w_4^\pm\) can exist. Table 2 shows the stability of each singular point for \(k = 4\).

(iii) \(k = 6\). Then, only one of pairs \(w_1^\pm\) or \(w_2^\pm\) exists. Both \(w_3^\pm\) and \(w_4^\pm\) exist. For instance, Table 3 shows the stability of each singular point for \(k = 6\).

(iv) \(k = 8\). Then, all \(w_i^\pm\) for \(i = 1, 2, 3, 4\) can exist. Table 4 shows the stability of each singular point for \(k = 8\).
Table 3. Stability for $k = 6$

| subcases | $w^+_1$ | $w^-_1$ | $w^+_2$ | $w^-_2$ | $w^+_3$ | $w^-_3$ |
|-----------|---------|---------|---------|---------|---------|---------|
| (iii-1)   | stable  | saddle  | ×       | ×       | saddle  | stable  |
| (iii-2)   | saddle  | stable  | ×       | ×       | saddle  | stable  |
| (iii-3)   | ×       | ×       | stable  | saddle  | saddle  | stable  |
| (iii-4)   | ×       | ×       | saddle  | stable  | stable  | saddle  |

Table 4. Stability for $k = 8$

| subcases | $w^+_1$ | $w^-_1$ | $w^+_2$ | $w^-_2$ |
|-----------|---------|---------|---------|---------|
| (iv-1)    | stable  | stable  | saddle  | saddle  |
| (iv-2)    | saddle  | saddle  | stable  | stable  |

Figure 1-2 shows the structure of $A_N$ for subcases (i-1), (ii-1), (ii-3), (iii-1), (iv-1) and (iv-2) in Table 1, 2, 3 and 4.

![Figure 1](image1.png)  
(a) Case (i-1)  
(b) Case (ii-1)  
(c) Case (ii-3)  
(d) Case (iii-1)

Figure 1. Examples of Structure of $A_N$ in Table 1, 2 and 3.

We claim that only $k = 8$ is possible. We prove the claim by the stability analysis. To study the stability of static solutions, we need to compute

$$D(h_1, h_2)(y, z) = \begin{bmatrix} h_{11}(y, z) & h_{12}(y, z) \\ h_{21}(y, z) & h_{22}(y, z) \end{bmatrix}.$$
where
\[
\begin{aligned}
\begin{cases}
    h_{11}(y, z) = \beta_N^+(\lambda) - \frac{\nu^2 \rho_N}{4 \alpha B_N} (3 \rho_N c_{30} y^2 + 2 \rho_{N+1} c_{12} z^2), \\
    h_{12}(y, z) = -\frac{\nu^2 \rho_N}{\alpha B_N} \rho_{N+1} c_{12} y z, \\
    h_{21}(y, z) = -\frac{\nu^2 \rho_{N+1}}{\alpha B_{N+1}} \rho_N c_{21} y z, \\
    h_{22}(y, z) = \beta_{N+1}^+(\lambda) - \frac{\nu^2 \rho_{N+1}}{4 \alpha B_{N+1}} (2 \rho_N c_{21} y^2 + 3 \rho_{N+1} c_{03} z^2).
\end{cases}
\end{aligned}
\] (3.19)

Suppose that \((a, b)\) is one of (3.17). If \(D(h_1, h_2)(a, b)\) has two negative eigenvalues, then \((a, b)\) is asymptotically stable. In other words, every solution of (1.5) starting near \((0, 0)\) will converge to \((u(x; a, b), v(x; a, b))\). On the other hand, if \(D(h_1, h_2)(a, b)\) has one negative and one positive eigenvalues, then \((a, b)\) is saddle such that the solution (3.18) is a saddle point in the phase space \(H\) and its stable manifold is of codimension one.

We note that if \(w_1^\pm\) exist,
\[
D(h_1, h_2)(\pm y_0, 0) = \begin{bmatrix}
-\frac{2\nu^2 \rho_N^2 \varepsilon}{B_N} & 0 \\
0 & \frac{\nu^2 \rho_{N+1}(\rho_{N+1} c_{30} - 2 \rho_{N+1} c_{21}) \varepsilon}{c_{30} B_{N+1}}
\end{bmatrix}.
\] (3.20)

This matrix has least one negative eigenvalue \(-2\nu^2 \rho_N^2 \varepsilon / B_N\). Hence, both \((y_0, 0)\) and \((-y_0, 0)\) are asymptotically stable or saddle at the same time. However, this contradicts any of the facts (i-1) or (i-2). Similarly, we have
\[
D(h_1, h_2)(0, \pm z_0) = \begin{bmatrix}
\nu^2 \rho_N (\rho_{N+1} c_{30} - 2 \rho_{N+1} c_{12}) \varepsilon & 0 \\
\frac{c_{30} B_N}{c_{03} B_{N+1}} & \frac{-2\nu^2 \rho_{N+1}^2 \varepsilon}{B_{N+1}}
\end{bmatrix}.
\] (3.21)

Then, we are led by the same argument to a contraction if only \(w_2^\pm\) exist. Hence, we conclude that \(k \neq 2\). A similar argument shows that \(k \neq 6\). See Figure 1 (d).
On the other hand, by using (3.15) and (3.19), we can calculate

\[ D(h_1, h_2)(y_1, \pm z_1) = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}, \]

where

\[ h_{11} = \frac{\nu^2 \rho_N c_{30}}{2\alpha B_N} y_1^2, \quad h_{12} = \frac{\nu^2 \rho_N \rho_{N+1} c_{12}}{2\alpha B_N} y_1 z_1, \]

\[ h_{21} = \frac{\nu^2 \rho_N \rho_{N+1} c_{21}}{2\alpha B_N} y_1 z_1, \quad h_{22} = -\frac{\nu^2 \rho_N^2 c_{30}}{2\alpha B_{N+1}} z_1^2. \]

Then,

\[
\begin{cases}
\det D(h_1, h_2)(y_1, \pm z_1) = \frac{\nu^4 \rho_N^2 \rho_{N+1}^2 c_{30}^2 c_{12}^2 (c_{30} c_{03} - 4c_{12} c_{21})}{4\alpha^2 B_N}, \\
\text{Tr } D(h_1, h_2)(y_1, \pm z_1) = -\frac{\nu^2 \rho_N^2 c_{30}}{2\alpha B_N} y_1^2 - \frac{\nu^2 \rho_N^2 \rho_{N+1} c_{30}}{2\alpha B_{N+1}} z_1^2.
\end{cases}
\]

(3.22)

Moreover, a similar computation yields that

\[
\begin{cases}
\det D(h_1, h_2)(-y_1, \pm z_1) = \det D(h_1, h_2)(y_1, \pm z_1), \\
\text{Tr } D(h_1, h_2)(y_1, \pm z_1) = \text{Tr } D(h_1, h_2)(-y_1, \pm z_1).
\end{cases}
\]

(3.23)

Hence, all \((y_1, \pm z_1)\) and \((-y_1, \pm z_1)\) have the same stability criterion. So, the subcases (ii-3) and (ii-4) cannot happen.

Next, we show that \(k \neq 4\). Suppose that (ii-1) happens. Then, neither \(y_1\) nor \(z_1\) exist. Thus, if \(c_{30} c_{03} - 4c_{12} c_{21} < 0\) and \(\rho_N c_{03} - 2\rho_N \rho_{N+1} c_{12} < 0\) and \(\rho_{N+1} c_{30} - 2\rho_N c_{21} < 0\). This implies by (3.20) and (3.21) that both \((\pm y_0, 0)\) and \((0, \pm z_0)\) are asymptotically stable, a contradiction to Table 2. Likewise, if \(c_{30} c_{03} - 4c_{12} c_{21} < 0\), then \(\rho_N c_{03} - 2\rho_N \rho_{N+1} c_{12} > 0\) and \(\rho_{N+1} c_{30} - 2\rho_N c_{21} > 0\). So, both \((\pm y_0, 0)\) and \((0, \pm z_0)\) are saddle by (3.20) and (3.21), which also violates Table 2. Similarly, if (ii-2) happens, then we are led to a contradiction. As a result, we can say that \(k \neq 4\).

By summing up the above discussion, we conclude that \(k = 8\) and the claim is justified. Moreover, the above investigation also implies the stability criteria of (iv) of Theorem 2.2. This completes the proof of parts (iii) and (iv) of Theorem 2.2. □

4. Numerical results. We know from Theorem 2.2 that final patterns of solutions for (1.5) are given by one of \(w_i^\pm\), \(i = 1, 2, 3, 4\), according to initial conditions. In this section, we give some numerical results that illustrate Theorem 2.2. The numerical results are obtained by PDE Solver in MATLAB[11]. We explain Theorem 2.2 by comparing the steady state solutions and numerical solutions obtained by MATLAB. We exhibit six examples, Figures 3-8, which help us see how the final patterns for (1.5) may happen.

The data for our numerical simulations are as follows. We use two initial conditions \(w_0 = (u_0, v_0)\) or \(w_1 = (u_1, v_1)\):

\[ w_0(x) = \begin{bmatrix} u_0(x) \\ v_0(x) \end{bmatrix} = \begin{bmatrix} 1 \\ c_1(x) \end{bmatrix}, \quad (4.1) \]

or

\[ w_1(x) = \begin{bmatrix} u_1(x) \\ v_1(x) \end{bmatrix} = \begin{bmatrix} 1 \\ c_2(x) \end{bmatrix}. \quad (4.2) \]
Numerical tests are carried out on four cases:

\[
\begin{cases}
\text{case (i)}: \mu\rho_N = 1 < \mu\rho_{N+1}, \\
\text{case (ii)}: \mu\rho_N < 1 \text{ and } \mu\rho_{N+1} = 1, \\
\text{case (iii)}: \mu^2\rho_N\rho_{N+1} = 1.
\end{cases}
\]

Therefore,

\[
\mu = \begin{cases}
1/\rho_N & \text{for case (i)}, \\
1/\rho_{N+1} & \text{for case (ii)}, \\
1/\sqrt{\rho_N\rho_{N+1}} & \text{for case (iii)}.
\end{cases}
\]

We set \( L = \pi \) and \( \nu = 1 \). Moreover, using (2.9), we put

\[
\alpha = \sqrt{\rho_N\rho_{N+1}\mu\nu}.
\]

And, from (3.6), we set

\[
\begin{cases}
\lambda_0 = \dfrac{\mu}{\nu}\alpha^2 + \dfrac{2N^2 + 2N + 1}{N(N+1)} \sqrt{\dfrac{\mu}{\nu}} \alpha + 1, \\
\lambda = \lambda_0 + 10^{-5} \text{ such that } \varepsilon = \lambda - \lambda_0 = 10^{-5}.
\end{cases}
\]

In Figures 3-5, \( N = 4 \) is chosen whereas we put \( N = 8 \) in Figures 6-8.

Using PDE Solver in MATLAB, we can get the graph of \( w_i^\pm = (u_i^\pm, v_i^\pm) \) together with numerical solutions \( w_h = (u_h, v_h) \) at enough large time for \( N = 4 \) or \( N = 8 \). In Figures 3-8, we can see that \( w_h \) approach \( w_1^\pm \) or \( w_2^\pm \) according to initial conditions \( w_0 \) or \( w_1 \).

It is worthwhile to notice that numerical solutions tend to \( w_1^\pm \) or \( w_2^\pm \) in various data of initial conditions and parameters. It was not found that numerical solutions approach either \( w_3^\pm \) or \( w_4^\pm \). So, one may guess that the only subcase (iv-1) in Table 4 may happen. If this conjecture is true, the statement Theorem 2.2 (iv) should be restated as follows: \( w_1^\pm \) and \( w_2^\pm \) are asymptotically stable whereas \( w_3^\pm \) and \( w_4^\pm \) are saddle. At this moment, we are not able to prove this conjecture analytically. We have seen that the sign of \( c_{30}c_{03} - 4c_{12}c_{21} \) is important in determining the stability of \( w_i^\pm \) for \( i = 1, 2, 3, 4 \). Since the exact calculation of \( c_{ij} \) is very complicated, it seems difficult to find a general rule that captures the sign of \( c_{30}c_{03} - 4c_{12}c_{21} \) definitely for all cases of parameters \( \alpha, \mu, \nu \) and \( L \).

In the remaining part of this section, we give an example of an analytical proof for the above conjecture. We will show \( c_{30}c_{03} - 4c_{12}c_{21} < 0 \) for the case (iii) in (4.3), which implies by Theorem 2.2 (iv) that \( w_1^\pm \) and \( w_2^\pm \) are asymptotically stable while \( w_3^\pm \) and \( w_4^\pm \) are saddle. By the condition \( \mu^2\rho_N\rho_{N+1} = 1 \), we obtain the following identities:

\[
\rho_1 = \dfrac{1}{N(N+1)\mu}, \quad \rho_N = \dfrac{N+1}{N(N+1)\mu}, \quad \rho_{N+1} = \dfrac{N+1}{N\mu}, \quad \alpha = \sqrt{\dfrac{\mu}{\nu}}.
\]

By using formula in Subsection 5.2, after a little bit long but tedious calculation, we obtain that \( a_{3i} = b_{3i} = 0 \) for \( i = 1, 2, 3, 4 \),

\[
a_{11} = 0, \quad a_{12} = \dfrac{N(2N + 1)\sqrt{\nu}}{(N + 1)^3\sqrt{\mu}},
\]

\[
b_{11} = -\dfrac{2N^2(2N + 1)\nu\sqrt{\nu}}{3(N+1)(3N+1)(N+1)^2\mu\sqrt{\mu}}, \quad b_{12} = \dfrac{N(5N + 1)(2N + 1)\sqrt{\nu}}{6(N - 1)(3N+1)(N+1)^2\sqrt{\mu}}.
\]
and

\[ a_{21} = 0, \quad a_{22} = -\frac{(N + 1)(2N + 1)\sqrt{\nu}}{2N^3 \sqrt{\mu}}, \]

\[ b_{21} = \frac{2(N + 1)^2(2N + 1)\nu \sqrt{\nu}}{3N^2(N + 2)(3N + 2)\mu \sqrt{\mu}}, \quad b_{22} = -\frac{(N + 1)(5N + 4)(2N + 1)\sqrt{\nu}}{6N^2(N + 2)(3N + 2)\sqrt{\mu}}. \]

Then,

\[ 2a_{12} + b_{12} = \frac{N(2N + 1)(13N^2 - 6N - 5)\sqrt{\nu}}{6(N - 1)(3N + 1)(N + 1)^3 \sqrt{\mu}}, \]

\[ 2a_{22} + b_{22} = -\frac{(N + 1)(2N + 1)(23N^2 + 52N + 24)\sqrt{\nu}}{6(N + 2)(3N + 2)N^3 \sqrt{\mu}}, \]

\[ a_{31} + b_{31} = a_{32} + b_{32} = 0. \]

And, we also have

\[ c_{30} = \frac{N(2N + 1)(27N^3 - N^2 - 7N + 1)\nu \sqrt{\nu}}{3(N - 1)(3N + 1)(N + 1)^3 \mu \sqrt{\mu}}, \]

\[ c_{12} = \frac{3(2N + 1)\nu \sqrt{\nu}}{N \mu \sqrt{\mu}}, \]

\[ c_{21} = \frac{(2N + 1)(27N^3 - 10N^2 - 33N - 8)\nu \sqrt{\nu}}{3N(N - 1)(N + 1)(3N + 1)\mu \sqrt{\mu}}, \]

\[ c_{30} = \frac{(N + 1)(2N + 1)(18N^2 + 5)\nu \sqrt{\nu}}{6N^3 \mu \sqrt{\mu}}. \]

Finally, we get

\[ \rho_N c_{30} - 2\rho_{N-1} c_{12} = -\frac{(2N + 1)(18N^2 + 31)\nu \sqrt{\nu}}{6N^2 \mu^2 \sqrt{\mu}}, \]

\[ \rho_{N-1} c_{30} - 2\rho_N c_{21} = -\frac{(2N + 1)(27N^3 - 19N^2 - 59N - 17)\nu \sqrt{\nu}}{3(N - 1)(3N + 1)(N + 1)^2 \mu^2 \sqrt{\mu}}, \]

\[ c_{30} c_{30} - 4c_{12} c_{21} = \frac{(2N + 1)^2(1458N^4 + 1107N^3 - 2965N^2 - 2935N - 581)\nu \sqrt{\nu}}{18N^2(N - 1)(3N + 1)(N + 1)^2 \mu^2 \sqrt{\mu}}. \]

Thus, for \( N \geq 2 \)

\[ \begin{cases} c_{30}, c_{30} > 0, \\ \rho_N c_{30} - 2\rho_{N-1} c_{12}, \rho_{N+1} c_{30} - 2\rho_N c_{21}, c_{30} c_{30} - 4c_{12} c_{21} < 0. \end{cases} \]

Hence, we conclude by (3.14) and (3.16) that \((\pm y_0, 0), (0, \pm z_0), (y_1, \pm z_1)\) and \((-y_1, \pm z_1)\) exist for any \( N \geq 2 \). Furthermore, \( c_{30} c_{30} - 4c_{12} c_{21} < 0 \) as desired.

5. Appendix: Derivation of (3.13). This section is devoted to the derivation of (3.13) by the center manifold reduction. Since \( g_{ij}(y, z, \lambda) = g_{ij}(y, z, \lambda_0) + o(2) \) for \( i, j = 1, 2 \), we compute \( g_{ij} \) at \( \lambda_0 \). By Theorem 3.8 in [8], the center manifold function \( \Phi^\lambda \) satisfies that

\[ \mathcal{L}_\lambda \Phi^\lambda(y, z) = -P_2^\lambda G_2(y \phi_N^+(\lambda) + z \phi_{N+1}^+(\lambda)) + o(|y|^2 + |z|^2). \]  

(5.1)
Figure 3. Case (i) of (4.3) and $N = 4$. With $w(x, 0) = w_0(x)$, (a) $u_h(x, t) \rightarrow u^+_0(x)$ and (b) $v_h(x, t) \rightarrow v^+_0(x)$. With $w(x, 0) = w_1(x)$, (c) $u_h(x, t) \rightarrow u^+_1(x)$ and (d) $v_h(x, t) \rightarrow v^+_1(x)$.

We will use the following elementary calculation frequently: for all $m, n \in \mathbb{N}$,

\[
\begin{align*}
\hat{L}_0 e^m e^n dx &= \begin{cases} 
0, & m \neq n, \\
\frac{L}{2}, & m = n,
\end{cases} & \hat{L}_2 e^m e^n dx &= \begin{cases} 
0, & m \neq n, \\
L, & m = n.
\end{cases} \\
\hat{L}_0 e^3 m e^n dx &= \begin{cases} 
0, & n \neq 3m, \\
\frac{L}{8}, & n = 3m,
\end{cases} & \hat{L}_4 e^2 m^2 e^n dx &= \begin{cases} 
0, & n \neq m, \\
\frac{3L}{4}, & n = m.
\end{cases}
\end{align*}
\]

We also often use the relation $\mathbf{p} \cdot \mathbf{\xi}^+_m = \nu \rho_m$ for $m = N, N + 1$, where we denote

\[
\mathbf{p} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]

To compute $g_{ij}$, we make some decomposition as follows. We see that

\[
G_2(w, w, \lambda_0) = y^2 G_2(\phi_N^+, \phi_N^+) + z^2 G_2(\phi_{N+1}^+, \phi_{N+1}^+) + y z \eta_0 + y \eta_1 + z \eta_2 + o(3),
\]
Figure 4. Case (ii) of (4.3) and \(N = 4\). With \(w(x,0) = w_0(x)\), (a) \(u_h(x,t) \to u_1(x)\) and (b) \(v_h(x,t) \to v_1^+(x)\). With \(w(x,0) = w_1(x)\), (c) \(u_h(x,t) \to u'_1(x)\) and (d) \(v_h(x,t) \to v'_1(x)\).

where

\[
\eta_0 = G_2(\phi^+_N, \phi^+_{N+1}) + G_2(\phi^+_N, \phi^+_N), \\
\eta_1 = G_2(\phi^+_N, \Phi^o_0) + G_2(\Phi^o_0, \phi^+_N), \\
\eta_2 = G_2(\phi^+_N, \Phi^o_0) + G_2(\Phi^o_0, \phi^+_N).
\]

We also have

\[
G_3(w, w, w, \lambda_0) = y^2G_3(\phi^+_N, \phi^+_N, \phi^+_N) + z^2G_3(\phi^+_N, \phi^+_N, \phi^+_N) + y^2z\eta_3 + y^2z\eta_4 + o(3),
\]

where

\[
\eta_3 = \{G_3(\phi^+_N, \phi^+_N, \phi^+_N) + G_3(\phi^+_N, \phi^+_N, \phi^+_N) + G_3(\phi^+_N, \phi^+_N, \phi^+_N)\}, \\
\eta_4 = \{G_3(\phi^+_N, \phi^+_N, \phi^+_N) + G_3(\phi^+_N, \phi^+_N, \phi^+_N) + G_3(\phi^+_N, \phi^+_N, \phi^+_N)\}.
\]

5.1. Computation of \(g_{12}\) and \(g_{22}\). By the definition of \(G_3\) in (2.2), we obtain

\[
G_3(\phi^+_N, \phi^+_N, \phi^+_N) = \nu^2\rho^2_N(m\rho_N + 1)e^3_Np.
\]
Figure 5. Case (iii) of (4.3) and $N = 4$. With $u(x,0) = w_0(x)$, (a) $u_h(x,t) \to u_1^+(x)$ and (b) $v_h(x,t) \to v_1^+(x)$. With $w(x,0) = w_1(x)$, (c) $u_h(x,t) \to u_1^+(x)$ and (d) $v_h(x,t) \to v_1^+(x)$.

Hence,

\[
\langle G_3(\phi_N^+, \phi_N^+, \phi_N^+), \phi_N^+ \rangle = \frac{3}{8} L \nu^2 \rho_N^2 (\mu \rho_N + 1) \mathbf{p} \cdot \xi_N^{+*} = \frac{3}{8} L \nu^3 \rho_N^3 (\mu \rho_N + 1), \\
\langle G_3(\phi_{N+1}^+, \phi_{N+1}^+, \phi_{N+1}^+), \phi_{N+1}^+ \rangle = \frac{3}{8} L \nu^3 \rho_{N+1}^3 (\mu \rho_{N+1} + 1), \\
\langle G_3(\phi_N^+, \phi_N^+, \phi_N^+), \phi_{N+1}^+ \rangle = \langle G_3(\phi_{N+1}^+, \phi_{N+1}^+, \phi_{N+1}^+), \phi_{N+1}^+ \rangle = 0.
\]

Furthermore, we can see that

\[
G_3(\phi_N^+, \phi_N^+, \phi_{N+1}^+) = \nu^2 \rho_N^2 (\mu \rho_N + 1) e_N^2 e_{N+1} \mathbf{p}, \\
G_3(\phi_{N+1}^+, \phi_{N+1}^+, \phi_N^+) = G_3(\phi_{N+1}^+, \phi_{N}^+, \phi_{N}^+) = \nu^2 \rho_N \rho_{N+1} (\mu \rho_N + 1) e_N^2 e_{N+1} \mathbf{p},
\]

and

\[
G_3(\phi_{N+1}^+, \phi_{N+1}^+, \phi_{N+1}^+) = G_3(\phi_{N+1}^+, \phi_{N}^+, \phi_{N}^+), \\
G_3(\phi_{N}^+, \phi_{N+1}^+, \phi_{N}^+) = \nu^2 \rho_N \rho_{N+1} (\mu \rho_N + 1) e_N e_{N+1}^2 \mathbf{p}, \\
G_3(\phi_{N+1}^+, \phi_{N}^+, \phi_{N}^+) = \nu^2 \rho_{N+1}^2 (\mu \rho_{N+1} + 1) e_N e_{N+1}^2 \mathbf{p}.
\]
Figure 6. Case (i) of (4.3) and $N = 8$. With $w(x, 0) = w_0(x)$, (a) $u_h(x, t) \to u^-_1(x)$ and (b) $v_h(x, t) \to v^-_1(x)$. With $w(x, 0) = w_1(x)$, (c) $u_h(x, t) \to u^+_1(x)$ and (d) $v_h(x, t) \to v^+_1(x)$.

Then, by using the relation (2.9), we have that

\[
\eta_3 = \nu\rho_N(3\alpha^2 + \nu\rho_N + 2\nu\rho_{N+1})e^2_N e_{N+1} p,
\]

\[
\eta_4 = \nu\rho_{N+1}(3\alpha^2 + 2\nu\rho_N + \nu\rho_{N+1})e^2_{N+1} p.
\]

So, we obtain

\[
\langle \eta_3, \phi_{N+1}^+ \rangle = \frac{L}{4} \nu^2 \rho_N \rho_{N+1}(3\alpha^2 + \nu\rho_N + 2\nu\rho_{N+1}),
\]

\[
\langle \eta_4, \phi_N^+ \rangle = \frac{L}{4} \nu^2 \rho_N \rho_{N+1}(3\alpha^2 + 2\nu\rho_N + \nu\rho_{N+1}),
\]

\[
\langle \eta_3, \phi_N^+ \rangle = \langle \eta_4, \phi_{N+1}^+ \rangle = 0.
\]

As a consequence, by plugging the above results in (3.12), we get

\[
g_{12}(y, z) = -\frac{p_{30}}{4B_N} y^3 - \frac{p_{12}}{2B_N} y z^2 + o(3),
\]

\[
g_{22}(y, z) = -\frac{p_{21}}{2B_{N+1}} y^2 z - \frac{p_{03}}{4B_{N+1}} z^3 + o(3),
\] (5.2)
Figure 7. Case (ii) of (4.3) and $N = 8$. With $w(x, 0) = w_0(x)$, (a) $u_h(x, t) \to u_1^+(x)$ and (b) $v_h(x, t) \to v_2^+(x)$. With $w(x, 0) = w_2(x)$, (c) $u_h(x, t) \to u_1^-(x)$ and (d) $v_h(x, t) \to v_2^-(x)$.

where

\[
\begin{align*}
p_{30} &= 3\nu^3 \rho^3_N (\mu \rho_N + 1), \\
p_{12} &= \nu^2 \rho_N \rho_{N+1} (3\alpha^2 + 2\nu \rho_N + \nu \rho_{N+1}), \\
p_{21} &= \nu^2 \rho_N \rho_{N+1} (3\alpha^2 + \nu \rho_N + 2\nu \rho_{N+1}), \\
p_{03} &= 3\nu^3 \rho_{N+1}^3 (\mu \rho_{N+1} + 1).
\end{align*}
\]

Here $B_m$ is defined by (3.8) for $m = N, N + 1$.

5.2. Computation of $\Phi^\lambda_0$. By using (5.1), we compute the explicit form of leading terms of $\Phi$. This is necessary to obtain $g_{11}$ up to cubic order of $y$ and $z$. We note that $\Phi^\lambda$ is approximated by $\Phi^\lambda_0$ up to the order $o(2)$ and it is reasonable to set

\[
\Phi^\lambda_0(y, z) = y^2 \psi_1 + z^2 \psi_2 + yz \psi_3 + o(2),
\]

where $\psi_1 = (\psi_{11}, \psi_{12}), \ \psi_2 = (\psi_{21}, \psi_{22}), \ \psi_3 = (\psi_{31}, \psi_{32}) \in H$. Since

\[
G_2(y \phi_N^+ + z \phi_{N+1}^+) = y^2 G_2(\phi_N^+) + z^2 G_2(\phi_{N+1}^+) \\
+ yz \{G_2(\phi_N^+, \phi_{N+1}^+) + G_2(\phi_{N+1}^+, \phi_N^+)\},
\]
we derive from (5.1) three linear systems on $(0, L)$:

\[
\begin{align*}
\mathcal{L}_{\lambda_0} \psi_1 &= -P_2 G_2(\phi_N^+), \\
\mathcal{L}_{\lambda_0} \psi_2 &= -P_2 G_2(\phi_{N+1}^+), \\
\mathcal{L}_{\lambda_0} \psi_3 &= -P_2 \left\{ G_2(\phi_N^+, \phi_{N+1}^+) + G_2(\phi_{N+1}^+, \phi_N^+) \right\}, \\
\psi_i &= 0 \text{ at } x = 0, L \quad \text{for } i = 1, 2, 3.
\end{align*}
\]

Explicitly, we have

\[
\begin{align*}
\mathcal{L}_{\lambda_0} \psi_1 &= -\frac{\nu \rho N (\mu \rho N + 1)(\nu \rho N - \alpha^2)}{2\alpha} (e_0 + e_{2N}) \mathbf{p}, \\
\mathcal{L}_{\lambda_0} \psi_2 &= -\frac{\nu \rho N+1 (\mu \rho N+1 + 1)(\nu \rho N+1 - \alpha^2)}{2\alpha} (e_0 + e_{2N+2}) \mathbf{p}, \\
\mathcal{L}_{\lambda_0} \psi_3 &= -\frac{\nu^2 \rho N \rho N+1 - \alpha^4}{\alpha} (e_1 + e_{2N+1}) \mathbf{p}.
\end{align*}
\] (5.4)

Here, we used the fact that for $N \geq 2$,

\[
e_N^2 = \frac{e_0 + e_{2N}}{2}, \quad e_{N+1}^2 = \frac{e_0 + e_{2N+2}}{2}, \quad e_N e_{N+1} = \frac{e_1 + e_{2N+1}}{2} \in E_2.
\]
For instance, by (2.2), (2.9), and (3.3), we can derive that
\[
G_2(\phi_N^+, \phi_{N+1}^+) = \left\{ \frac{\lambda_0}{\alpha} \nu^2 \rho_N \rho_{N+1} - 2\alpha \nu \rho_N (\mu \rho_{N+1} + 1) \right\} e_N e_{N+1} p
\]
\[
= \frac{\nu \rho_N (\mu \rho_{N+1} + 1)(\nu \rho_{N+1} - \alpha^2)}{\alpha} e_N e_{N+1} p
\]
\[
= \left( \frac{\nu \rho_N + \alpha^2}{\alpha} (\nu \rho_{N+1} - \alpha^2) \right) \cdot e_1 + e_{2N+1}
\]
Since \( L_{\lambda_0} \) is invertible on \( E_2 \) by (2.12), each of (5.4) allows a unique solution by the Fredholm alternatives. As a consequence, it is natural to set
\[
\begin{align*}
\psi_{11} &= a_{11} e_0 + b_{11} e_{2N}, & \psi_{12} &= a_{12} e_0 + b_{12} e_{2N}, \\
\psi_{21} &= a_{21} e_0 + b_{21} e_{2N+2}, & \psi_{22} &= a_{22} e_0 + b_{22} e_{2N+2}, \\
\psi_{31} &= a_{31} e_1 + b_{31} e_{2N+1}, & \psi_{32} &= a_{32} e_1 + b_{32} e_{2N+1}.
\end{align*}
\tag{5.5}
\]
We note that if \( \psi = [ae_k + be_n, ce_k + de_n]^T \) with \( k \neq n \), then
\[
L_{\lambda_0} \psi = \left[ \begin{array}{c} \left[ \begin{array}{c} \mu \phi_k^2 + (\lambda_0 - 1) \alpha^2 \\
\lambda_0 \nu \phi_k^2 - \alpha^2 \end{array} \right] e_k + \left[ \begin{array}{c} -e_2 \\
\rho_k \end{array} \right] e_n \end{array} \right]
\]
\[
= \left[ \begin{array}{c} [(-\mu \rho_k + \lambda_0 - 1) \alpha \alpha^2] e_k + [(-\mu \rho_k + \lambda_0 - 1) \alpha \alpha^2] e_n \end{array} \right]
\]
\[
= \left[ \begin{array}{c} -[\lambda_0 \alpha \alpha^2] e_k + [\lambda_0 \alpha \alpha^2] e_n \end{array} \right]
\]
By plugging (5.5) and (5.6) in (5.4), we can determine the coefficients \( a_{ij} \) and \( b_{ij} \). Indeed, we have the relations for \( \psi_1 = (\psi_{11}, \psi_{12}) \):
\[
\left[ \begin{array}{c} \left[ (\lambda_0 - 1) a_{11} + \alpha^2 a_{12} \right] e_0 + \left[ (-\mu \rho_{2N} + \lambda_0 - 1) b_{11} + \alpha^2 b_{12} \right] e_{2N} \end{array} \right]
\]
\[
= -\nu \rho_N (\mu \rho_N + 1)(\nu \rho_N - \alpha^2) \left( \frac{a_{11} + b_{11}}{2\alpha} \right) \left[ \begin{array}{c} 1 \\
-1 \end{array} \right],
\]
which give us
\[
\begin{align*}
a_{11} &= 0, \\
a_{12} &= -\frac{\nu \rho_N (\mu \rho_N + 1)(\nu \rho_N - \alpha^2)}{2\alpha^3}, \\
b_{11} &= -\frac{2\nu \rho_N^2 (1 + \mu \rho_N)(1 - \mu \rho_{N+1})}{3\alpha \mu (4\rho_N - \rho_{N+1})}, \\
b_{12} &= -\frac{\mu \rho_{2N} + 1}{\nu \rho_{2N}} b_{11} = -\frac{N^2 \nu (1 + 4 \mu \rho_N)(1 + \mu \rho_N)(1 - \mu \rho_{N+1})}{6(1 + 2)(N + 1) \alpha \mu}.
\end{align*}
\]
Here, we used the fact (2.9), \( \rho_{2N} = 4 \rho_N \), and \( \rho_n = n^2 \rho_1 \) for \( n \geq 1 \). Similarly, we get for \( \psi_2 = (\psi_{21}, \psi_{22}) \),
\[
\begin{align*}
a_{21} &= 0, \\
a_{22} &= \frac{(N + 1)^2 \nu (1 + \mu \rho_{N+1})(1 - \mu \rho_N)}{2N^2 \alpha \mu}, \\
b_{21} &= \frac{2(N + 1)^2 \nu \rho_{N+1}(1 + \mu \rho_{N+1})(1 - \mu \rho_N)}{3(N + 2)(3N + 2) \alpha \mu}, \\
b_{22} &= \frac{(N + 1)^2 \nu (1 + 4 \mu \rho_{N+1})(1 + \mu \rho_{N+1})(1 - \mu \rho_N)}{6(N + 2)(3N + 2) \alpha \mu}.
\end{align*}
\]
and for $\psi_3 = (\psi_{31}, \psi_{32})$,

$$
\begin{align*}
a_{31} &= \frac{N(N + 1)\nu^2 \rho_1(1 - \mu^2 \rho_N \rho_{N+1})}{(N - 1)(N + 2)\alpha \mu}, \\
a_{32} &= -\frac{N(N + 1)\nu(1 + \mu \rho_1)(1 - \mu^2 \rho_N \rho_{N+1})}{(N - 1)(N + 2)\alpha \mu}, \\
b_{31} &= \frac{N(N + 1)\nu^2 \rho_{2N+1}(1 - \mu^2 \rho_N \rho_{N+1})}{(3N + 1)(3N + 2)\alpha \mu}, \\
b_{32} &= -\frac{N(N + 1)\nu(1 + \mu \rho_{2N+1})(1 - \mu^2 \rho_N \rho_{N+1})}{(3N + 1)(3N + 2)\alpha \mu}.
\end{align*}
$$

Now, by plugging the above expressions in (5.5), we get an explicit formula of $\Phi^{\lambda_0}$ by (5.3) up to the order $o(2)$.

5.3. Computation of $\eta_1$ and $\eta_2$. By (5.3) and $\Phi^{\lambda_0}$, we have that for $m = N$ or $N + 1$,

$$
G_2(\phi^{\lambda_0}_m, \Phi^{\lambda_0}) + o(2) = y^2 G_2(\phi^{\lambda_0}_m, \psi_1) + z^2 G_2(\phi^{\lambda_0}_m, \psi_2) + yz G_2(\phi^{\lambda_0}_m, \psi_3)
$$

$$
= -y^2 \left[ \left( \frac{\lambda_0}{\alpha} \psi_{11} + 2\alpha \psi_{12} \right) + z^2 \left( \frac{\lambda_0}{\alpha} \psi_{21} + 2\alpha \psi_{22} \right) \\
+ yz \left( \frac{\lambda_0}{\alpha} \psi_{31} + 2\alpha \psi_{32} \right) \right] \nu \rho_m e_m \mathbf{p}
$$

$$
= -y^2 \left[ 2\alpha a_{12} + \left( \frac{\lambda_0}{\alpha} b_{11} + 2\alpha b_{12} \right) e_{2N} \right] \nu \rho_m e_m \mathbf{p}
$$

$$
- z^2 \left[ 2\alpha a_{22} + \left( \frac{\lambda_0}{\alpha} b_{21} + 2\alpha b_{22} \right) e_{2N+2} \right] \nu \rho_m e_m \mathbf{p}
$$

$$
- yz \left[ \left( \frac{\lambda_0}{\alpha} a_{31} + 2\alpha a_{32} \right) e_1 + \left( \frac{\lambda_0}{\alpha} b_{31} + 2\alpha b_{32} \right) e_{2N+1} \right] \nu \rho_m e_m \mathbf{p}.
$$

Similarly, for $m = N$ or $N + 1$,

$$
G_2(\Phi^{\lambda_0}_m, \Phi^{\lambda_0}_m) + o(2) = -y^2 b_{11} e_{2N} + z^2 b_{21} e_{2N+2} + yz (a_{31} e_1 + b_{31} e_{2N+1})
$$

$$
\times \frac{(\nu \rho_m + 1)(\nu \rho_m - \alpha^2)}{\nu} e_m \mathbf{p}.
$$

By adding these two identities, we obtain

$$
\eta_3 = G_2(\phi^+_m, \Phi^{\lambda_0}) + G_2(\Phi^{\lambda_0}_m, \phi^+_m).
$$

Here, $m = N$ for $j = 1$, and $m = N + 1$ for $j = 2$.

5.4. Computation of $g_{11}$ and $g_{21}$. It is easy to check that the following six terms vanish:

$$
\langle G_2(\phi^{\lambda_0}_N, \phi^{\lambda_0}_N), \phi^{\lambda_0}_N \rangle, \quad \langle G_2(\phi^{\lambda_0}_{N+1}, \phi^{\lambda_0}_{N+1}), \phi^{\lambda_0}_N \rangle, \quad \langle \eta_0, \phi^{\lambda_0}_N \rangle,
$$

$$
\langle G_2(\phi^{\lambda_0}_N, \phi^{\lambda_0}_{N+1}), \phi^{\lambda_0}_{N+1} \rangle, \quad \langle G_2(\phi^{\lambda_0}_{N+1}, \phi^{\lambda_0}_{N+1}), \phi^{\lambda_0}_{N+1} \rangle, \quad \langle \eta_0, \phi^{\lambda_0}_{N+1} \rangle.
$$

We note that for $m = N$ or $N + 1$,

$$
\langle f \mathbf{p}, \phi^+_m \rangle = \nu \rho_m \int f e_m dx.
$$
Using this formula and (2.8), we obtain that up to the order $o(2)$

$$
\langle \eta_1, \phi_N^{+*} \rangle = -LV^2\rho_N^2 \left[ \frac{B}{2} \left( 2\alpha a_{12} + \alpha b_{12} + \frac{\mu\rho_N + 1}{\alpha} b_{11} \right) + \alpha a_{22} z^2 \right],
$$

$$
\langle \eta_1, \phi_{N+1}^{+*} \rangle = -\frac{1}{2} LV^2\rho_N\rho_{N+1} \left[ \frac{(\mu\rho_N + 1)(a_{31} + b_{31})}{\alpha} + \alpha (a_{32} + b_{32}) \right] y z,
$$

$$
\langle \eta_2, \phi_N^{+*} \rangle = -\frac{1}{2} LV^2\rho_N\rho_{N+1} \left[ \frac{(\mu\rho_N + 1)(a_{31} + b_{31})}{\alpha} + \alpha (a_{32} + b_{32}) \right] y z,
$$

$$
\langle \eta_2, \phi_{N+1}^{+*} \rangle = -LV^2\rho_N^2 \left[ \alpha a_{12} y^2 + \frac{z^2}{2} \left( 2\alpha a_{22} + \alpha b_{22} + \frac{\mu\rho_N + 1}{\alpha} b_{21} \right) \right].
$$

Now, we can compute (3.12) up to the order $o(3)$. In fact, by plugging the above results in (3.12), we get

$$
g_{11}(y, z) = \frac{y}{\langle \phi_N^{+*}, \phi_N^{+*} \rangle} \langle \eta_1, \phi_N^{+*} \rangle + z \langle \eta_2, \phi_N^{+*} \rangle = \frac{q_{30}}{\alpha B_N} y^3 + \frac{q_{12}}{\alpha B_N} y z^2 + o(3),
$$

$$
g_{21}(y, z) = \frac{y}{\langle \phi_{N+1}^{+*}, \phi_{N+1}^{+*} \rangle} \langle \eta_1, \phi_{N+1}^{+*} \rangle + z \langle \eta_2, \phi_{N+1}^{+*} \rangle = \frac{q_{21}}{\alpha B_{N+1}^2} y^3 + \frac{q_{32}}{\alpha B_{N+1}^2} z^3 + o(3),
$$

where

$$
\begin{align*}
q_{30} &= \nu^2\rho_N^2 \left[ 2\alpha^2 a_{12} + (\mu\rho_N + 1)b_{11} + \alpha^2 b_{12} \right], \\
q_{12} &= \nu^2\rho_N\rho_{N+1} \left[ 2\mu\rho_N^2 a_{22} + (\mu\rho_{N+1} + 1)(a_{31} + b_{31}) + \alpha^2 (a_{32} + b_{32}) \right], \\
q_{21} &= \nu^2\rho_N\rho_{N+1} \left[ 2\mu\rho_{N+1} a_{12} + (\mu\rho_N + 1)(a_{31} + b_{31}) + \alpha^2 (a_{32} + b_{32}) \right], \\
q_{32} &= \nu^2\rho_{N+1}^2 \left[ 2\alpha^2 a_{22} + (\mu\rho_{N+1} + 1)b_{21} + \alpha^2 b_{22} \right].
\end{align*}
$$

5.5. Derivation of (3.13). Using (5.2) and (5.7), we can rewrite (3.11) as

$$
\begin{align*}
\frac{dy}{dt} &= \beta_N^{+*}(\lambda) y - \frac{\alpha p_{30} - 4q_{30}}{4\alpha B_N} y^3 - \frac{\alpha p_{12} - 2q_{12}}{2\alpha B_N} y z^2 + o(3), \\
\frac{dz}{dt} &= \beta_{N+1}^{+*} z - \frac{\alpha p_{21} - 2q_{21}}{2\alpha B_{N+1}} y^2 z - \frac{\alpha p_{32} - 4q_{32}}{4\alpha B_{N+1}} z^3 + o(3),
\end{align*}
$$

or equivalently,

$$
\begin{align*}
\frac{dy}{dt} &= \beta_N^{+*}(\lambda) y - \frac{\nu^2\rho_N}{4\alpha B_N} \left( \rho_N c_{30} y^3 + 2\rho_{N+1} c_{12} y z^2 \right) + o(3), \\
\frac{dz}{dt} &= \beta_{N+1}^{+*} z - \frac{\nu^2\rho_{N+1}}{4\alpha B_{N+1}} \left( 2\rho_N c_{21} y^2 z + \rho_{N+1} c_{32} z^3 \right) + o(3),
\end{align*}
$$

which is equal to (3.13). Here,

$$
\begin{align*}
c_{30} &= 3\alpha \nu \rho_N (\mu\rho_N + 1) - 4 \left( (\mu\rho_N + 1)b_{11} + \alpha^2 (2a_{12} + b_{12}) \right), \\
c_{12} &= \alpha (3\alpha^2 + 2\nu \rho_N + \nu \rho_{N+1}) \\
&- 2 \left[ 2\mu\rho_N^2 a_{22} + (\mu\rho_{N+1} + 1)(a_{31} + b_{31}) + \alpha^2 (a_{32} + b_{32}) \right], \\
c_{21} &= \alpha (3\alpha^2 + \nu \rho_N + 2\nu \rho_{N+1}) \\
&- 2 \left[ 2\mu\rho_{N+1} a_{12} + (\mu\rho_N + 1)(a_{31} + b_{31}) + \alpha^2 (a_{32} + b_{32}) \right], \\
c_{32} &= 3\alpha \nu \rho_{N+1} (\mu\rho_{N+1} + 1) - \left[ (\mu\rho_{N+1} + 1)b_{21} + \alpha^2 (2a_{22} + b_{22}) \right].
\end{align*}
$$
Furthermore, direct calculations also yield

\[
2a_{12} + b_{12} = \frac{\nu(1 + \mu \rho N)(1 - \mu \rho N)}{6(N - 1)(3N + 1)\alpha \mu \rho N + 1} \times \\
\left[ 6(N - 1)(3N + 1)\rho N + N^2 \rho N + 1)(1 + 4\mu \rho N) \right],
\]

\[
2a_{22} + b_{22} = \frac{\nu(1 + \mu \rho N)(1 - \mu \rho N)}{6(N + 2)(3N + 2)\alpha \mu \rho N} \times \\
\left[ 6(N + 2)(3N + 2)\rho N + 1)(1 + 4\mu \rho N) \right],
\]

\[
a_{31} + b_{31} = \frac{2N^2(1 + 1)^2(2N^2 + 2N + 1)\nu^2 \rho_1 (1 - \mu^2 \rho N \rho N + 1)}{(N - 1)(N + 2)(3N + 1)(3N + 2)\alpha \mu},
\]

\[
a_{32} + b_{32} = -\frac{2N^2(1 + 1)^2(2N^2 + 2N + 1)\nu^2 \rho_1 (1 - \mu^2 \rho N \rho N + 1)}{(N - 1)(N + 2)(3N + 1)(3N + 2)\alpha \mu}.
\]

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