Abstract

We briefly report on our recent results regarding the introduction of a notion of a q-quaternion and the construction of instanton solutions of a would-be deformed su(2) Yang-Mills theory on the corresponding SO_q(4)-covariant quantum space. As the solutions depend on some noncommuting parameters, this indicates that the moduli space of a complete theory will be a noncommutative manifold.

1 Introduction

The search for instantonic solutions has become a key point of investigation of Yang-Mills gauge theories on noncommutative manifolds after the discovery [26] that deforming \( \mathbb{R}^4 \) into the Moyal-Weyl noncommutative Euclidean space \( \mathbb{R}_\theta^4 \) regularizes the zero-size singularities of the instanton moduli space (see e.g. [30] [7] [4] [8] [23]). Among the available deformations of \( \mathbb{R}^4 \) there is also the Faddeev-Reshetikhin-Takhtadjan noncommutative Euclidean space \( \mathbb{R}_q^4 \) covariant under SO_q(4) [10], and it is therefore tempting to investigate this issue on it. There is still no satisfactory formulation [21] of gauge field theory on quantum group covariant noncommutative spaces (shortly: quantum spaces) like \( \mathbb{R}_q^4 \). One main reason is the lack of a proper (i.e. cyclic) trace to define gauge invariant observables (action,

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Another one is the \(\ast\)-structure of the differential calculus, which for real \(q\) is problematic. Probably a satisfactory formulation will be possible within a generalization of the standard framework of noncommutative geometry \([6]\). Here we leave these two issues aside and just ask for nontrivial solutions of the deformed (anti)selfduality equations.

As known, great simplifications in the search and classification of instantons in Yang-Mills theory on \(\mathbb{R}^4\) occur when the latter is promoted to the quaternion algebra \(\mathbb{H}\). We have recently introduced \([17]\) the notion of a \(q\)-deformed quaternion as the defining matrix of a copy of \(SU_q(2) \times \mathbb{R}_{\geq}\) (\(\mathbb{R}_{\geq}\) denoting the semigroup of nonnegative real numbers), showing that its entries are the coordinates of \(\mathbb{R}_{q}^4\). More details will be given in \([18]\). Then adopting the \(SO_q(4)\)-covariant differential calculus on \(\mathbb{R}_q^4\) \([5]\) and the corresponding Hodge duality map \([14, 15]\) in \(q\)-quaternion language we have found \([17]\) solutions \(A\) of the (anti)self-duality equations, in the form of 1-form valued \(2 \times 2\) matrices, that closely resemble their undeformed counterparts (instantons) in \(su(2)\) Yang-Mills theory on \(\mathbb{R}^4\). [The (still missing) complete gauge theory might be however a deformed \(u(2)\) rather than \(su(2)\) Yang-Mills theory.] The “coordinates of the center” of the instanton are nevertheless noncommuting parameters, differently from the Nekrasov-Schwarz theory. We have also found multi-instantons solutions: they are again parametrized by noncommuting parameters playing the role of “size” and “coordinates of the center” of the (anti)instantons. This indicates that the moduli space of a complete theory will be a noncommutative manifold. This is similar to what was proposed in \([20]\) for \(\mathbb{R}_{\theta}^4\) for selfdual deformation parameters \(\theta_{\mu\nu}\).

Here we briefly report on these results.

### 2 The \(q\)-quaternion bialgebra \(C(\mathbb{H}_q)\)

Any element \(X\) in the (undeformed) quaternion algebra \(\mathbb{H}\) is given by

\[
X = x_1 + x_2i + x_3j + x_4k,
\]

with \(x \in \mathbb{R}^4\) and imaginary \(i, j, k\) fulfilling

\[
i^2 = j^2 = k^2 = -1, \quad ijk = -1.
\]

Replacing \(i, j, k\) by Pauli matrices \(\times\) imaginary unit \(i\),

\[
X \leftrightarrow x = \begin{pmatrix} x_1 + x_4i & x_3 + x_2i \\ -x_3 + x_2i & x_1 - x_4i \end{pmatrix} = : \begin{pmatrix} \alpha & \gamma \\ -\gamma^* & \alpha^* \end{pmatrix}
\]

(where \(\alpha, \gamma \in \mathbb{C}\), and the quaternionic product becomes represented by matrix multiplication. Therefore \(\mathbb{H}\) essentially consists of all complex \(2 \times 2\) matrices of this form.

This can be \(q\)-deformed as follows. We just pick the pioneering definition of the (Hopf) \(\ast\)-algebra \(C(SU_q(2))\) \([35, 34]\) without imposing the
\[ \text{det}_q=1 \text{ condition: for } q \in \mathbb{R} \text{ consider the unital associative } \star \text{-algebra } \mathcal{A} \equiv C(\mathbb{H}_q) \text{ generated by elements } \alpha, \gamma, \alpha^*, \gamma^* \text{ fulfilling the commutation relations} \]
\[ \begin{align*}
\alpha \gamma &= q \gamma \alpha, & \alpha \gamma^* &= q \gamma^* \alpha, & \gamma \alpha^* &= q \alpha^* \gamma, \\
\gamma^* \alpha^* &= q \alpha^* \gamma^*, & [\alpha, \gamma] &= (1-q^2) \gamma \gamma^* & [\gamma^*, \gamma] &= 0. 
\end{align*} \tag{1} \]

Introducing the matrix
\[ x \equiv \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} := \begin{pmatrix} \alpha & -q \gamma^* \\ \gamma & \alpha^* \end{pmatrix} \]
we can rewrite these commutation relations as
\[ \hat{R} x_1 x_2 = x_1 x_2 \hat{R} \tag{2} \]
and the conjugation relations as \[ x_{\alpha \beta}^* = \epsilon_{\alpha \beta} x_{\delta \gamma}^* \epsilon_{\delta \alpha}, \text{ i.e.} \]
\[ x^\dagger = \bar{x} \quad \text{where } \bar{a} := \epsilon^{-1} a^T \epsilon \quad \forall a \in M_2. \tag{3} \]

Here we have used the braid matrix and the \( \epsilon \)-tensor of \( M_q(2), GL_q(2), SU_q(2) \),
\[ \epsilon = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} = -q \epsilon^{-1} \quad \hat{R}_{\gamma \delta}^\alpha = q \delta_{\gamma}^\alpha \delta_{\delta}^\beta + \epsilon^\alpha_{\beta} \epsilon_{\gamma \delta}. \tag{4} \]

with \( \epsilon \equiv (\epsilon_{\alpha \beta}) \) and \( \epsilon^{-1} \equiv (\epsilon^\alpha_{\beta}) \). So \( \mathcal{A} := C(\mathbb{H}_q) \) can be endowed also with a bialgebra structure (we are not excluding the possibility that \( x \equiv 0_2 \)), more precisely a real section of the bialgebra \( C(M_q(2)) \) of \( 2 \times 2 \) quantum matrices \([9, 35, 10]\). Since the coproduct
\[ \Delta(x_{\alpha \gamma}) = (ax)^{\alpha \gamma} \]
is an algebra map, the matrix product \( ax \) of any two matrices \( a, x \) with mutually commuting entries and fulfilling \((2, 3)\) again fulfills the latter. Therefore we shall call any such matrix \( x \) a \( q \)-quaternion, and \( \mathcal{A} := C(\mathbb{H}_q) \) the \( q \)-quaternion bialgebra.

As well-known, the so-called ‘\( q \)-determinant’ of \( x \)
\[ |x|^2 \equiv \text{det}_q(x) := x_{11} x_{22} - qx_{12} x_{21} = \alpha^* \alpha + \gamma^* \gamma \sim x_{\alpha \alpha'} x_{\beta \beta'}^* \epsilon_{\alpha \beta} \epsilon_{\alpha' \beta'}, \tag{5} \]
is central, manifestly nonnegative-definite and group-like. It is zero iff \( x \equiv 0_2 \). Relations \((2)\) can be also equivalently reformulated as
\[ x \bar{x} = \bar{x} x = |x|^2 I_2 \tag{6} \]
\( (I_2 \text{ denotes the unit } 2 \times 2 \text{ matrix}). \) If we assume \( x \neq 0_2 \) and extend \( C(\mathbb{H}_q) \) by the new (central, positive-definite) generator \( |x|^{-1} \) one finds that \( x \) is invertible with inverse
\[ x^{-1} = \frac{\bar{x}}{|x|^2}. \tag{7} \]
\(C(\mathbb{H}_q)\) becomes a Hopf \(\star\)-algebra [a real section of \(C(GL_q(2))\)]. The matrix elements of \(T := \frac{x}{|x|}\) fulfill the relations \(\mathbf{2}\) and

\[
T^\dagger = T^{-1} = T, \quad \det_q(T) = 1,
\]

namely generate as a quotient algebra \(C(SU_q(2))\) \([35, 34]\), therefore in this case the entries of \(x\) generate the (Hopf) \(\star\)-algebra of functions on the quantum group \(SU_q(2) \times GL^+(1)\), in analogy with the \(q = 1\) case.

As a \(\star\)-algebra, \(A := C(\mathbb{H}_q)\) coincides with the algebra of functions on the \(SO_q(4)\)-covariant quantum Euclidean Space \(\mathbb{R}_q^4\) of \([10]\), identifying their generators as

\[
x^1 = qx^{11}, \quad x^2 = x^{12}, \quad x^3 = -qx^{21}, \quad x^4 = x^{22}.
\]

The commutation relations are preserved by the (left) coactions of both \(SO_q(4) = SU_q(2) \otimes SU_q(2)'/\mathbb{Z}_2\) and of the extension \(\widetilde{SO}_q(4) := SO_q(4) \times GL^+(1) = \mathbb{H}_q \times \mathbb{H}_q'/GL(1)\) (the quantum group of rotations and scale transformations in 4 dimensions), which take the form

\[
x \rightarrow a x b^T.
\]

Here \(a, b\) are the defining matrices of \(SU_q(2), SU_q(2)\) in the first case and of \(\mathbb{H}_q, \mathbb{H}_q'\) in the second (with entries commuting with each other and with those of \(x\)), \(b^T\) means the transpose of \(b\), and matrix product is understood.

A different matrix version (with no interpretation in terms of \(q\)-deformed quaternions) of a \(SU_q(2) \times SU_q(2)\) covariant quantum Euclidean space was proposed in \([24]\).

### 3 Other preliminaries

The \(SO_q(4)\)-covariant differential calculus \((d, \Omega^*)\) on \(\mathbb{R}_q^4 \sim \mathbb{H}_q\) \([5]\) is obtained imposing covariant homogeneous bilinear commutation relations \([12]\) between the \(x^i\) and the differentials \(\xi^i := dx^i\). Partial derivatives are introduced through the decomposition \(d = \xi^a \partial_a = \xi^{a'0} \partial_{a'}\). All other commutation relations are derived by consistency. The complete list is given by

\[
P_{i\ell}^\ell_h x^h x^k = 0, \quad x^h \xi^i = q \hat{R}_{jki} \xi^j x^k, \quad (P_s + P_t)^{ij}_{hk} \xi^h \xi^k = 0, \quad P_{i\ell}^{ij} \partial_j \partial_\ell = 0, \quad \partial_{\ell} x^j = \delta^j_\ell + q \hat{R}_{jki} x^k \partial_\ell, \quad \partial^h \xi^i = q^{-1} \hat{R}_{jki} \xi^j \partial^h.
\]
\( \hat{R} \equiv \text{braid matrix of } SO_q(4); \ P_s, P_a, P_t \equiv \text{deformations of the symmetric trace-free, antisymmetric and trace projectors appearing in the projector decomposition of } \hat{R}. \) Up to the linear transformation (9)

\[ q\hat{R} = \hat{R} \otimes \hat{R}. \]

The Laplacian \( \Box \equiv \partial \cdot \partial := \partial_k g^{hk} \partial_h \) is \( SO_q(4) \)-invariant and commutes the \( \partial_i \). In \( \mathcal{H} \) there exists a special invertible element \( \Lambda \) such that

\[ \Lambda x^i = q^{-1} x^i \Lambda, \quad \Lambda \partial^i = q \partial^i \Lambda, \quad \Lambda \xi^i = \xi^i \Lambda. \]

Definitions:

- \( \bigwedge^* \equiv z\)-graded algebra generated by the \( \xi^i \), where grading \( z \equiv \text{degree in } \xi^i \); any component \( \bigwedge^p \) with \( z = p \) carries an irreducible representation of \( U_q so(4) \) and has the same dimension as in the \( q = 1 \) case.
- \( \mathcal{D}C^* \equiv z\)-graded algebra generated by \( x^i, \xi^i, \partial_i \). Elements of \( \mathcal{D}C^p \) are differential-operator-valued \( p \)-forms.
- \( \Omega^* \equiv z\)-graded subalgebra generated by \( x^i, \xi^i, \partial_i \). By definition \( \Omega^0 = \mathcal{A} \) itself, and both \( \Omega^* \) and \( \Omega^p \) are \( \mathcal{A} \)-bimodules. Also, we shall denote \( \Omega^* \) enlarged with \( \Lambda^{\pm 1} \) as \( \Omega^*_\Sigma \) (the latter is 4-dim).
- \( \mathcal{H} \equiv \text{subalgebra generated by the } x^i, \partial_i \). By definition, \( \mathcal{D}C^0 = \mathcal{H} \), and both \( \mathcal{D}C^* \) and \( \mathcal{D}C^p \) are \( \mathcal{H} \)-bimodules.

The restricted (but still 4-dimensional!) differential calculus \( (\Omega^*_\Sigma, d) \) coincides with the Woronowicz 4D- on \( C(SU_q(2)) \).

The special 1-form

\[ \theta := \frac{1}{1 - q^{-2}} |x|^{-2} d|x|^2 = \frac{q^{-2}}{q^2 - 1} \xi^{\alpha \alpha'} x^{\beta \beta'} |x|^2 \epsilon_{\alpha \beta} \epsilon_{\alpha' \beta'} \]

plays the role of "Dirac Operator" [6] of the differential calculus,

\[ d\omega_p = [-\theta, \omega_p], \quad \omega_p \in \Omega^p, \]

However, \( d(f^*) \neq (df)^* \), and moreover there is no \( * \)-structure \( * : \Omega^* \rightarrow \Omega^* \), but only a \( * \)-structure

\[ * : \mathcal{D}C^* \rightarrow \mathcal{D}C^* \]

[28], with a rather nonlinear character (the latter has been recently [16] recast in a much more suggestive form).

The Hodge map [14, 15] is a \( SO_q(4) \)-covariant, \( \mathcal{A} \)-bilinear map \( * : \Omega^p \rightarrow \Omega^{4-p} \) such that \( *^2 = \text{id} \), defined by

\[ *(\xi_1 ... \xi_p) = q^{-(p-2)} c_p \xi^{i_{p+1} ... i_p} \epsilon_{i_{p+1} ... i_p} \epsilon_{i_1 ... i_p} \Lambda^2 p - 4, \]
where $\varepsilon^{hijk} \equiv q$-epsilon tensor and $c_p$ are suitable normalization factors. Actually this extends to a $\mathcal{H}$-bilinear map $\ast : \mathcal{D}C^p \to \mathcal{D}C^{4-p}$ with the same features. For $p = 2$ $\Lambda$-powers disappear and one even gets a map $\ast : \Omega^2 \to \Omega^2$ defined by

$$\ast \xi^i \xi^j = \frac{1}{[2]_q} \xi^b \xi^c \varepsilon_{khij} \omega_{ji}. \quad (17)$$

$\Omega^2$ (resp. $\mathcal{D}C^2$) splits into the direct sum of $\mathcal{A}$- (resp. $\mathcal{H}$-) bimodules

$$\Omega^2 = \tilde{\Omega}^2 \oplus \tilde{\Omega}^{2'} \quad \text{ (resp. } \mathcal{D}C^2 = \tilde{\mathcal{D}}C^2 \oplus \tilde{\mathcal{D}}C^{2'})$$

of the eigenspaces of $\ast$ with eigenvalues $1, -1$ respectively, whose elements are “self-dual and anti-self-dual 2-forms”. $\tilde{\Omega}^2$ (resp. $\tilde{\mathcal{D}}C^2$) is generated by the self-dual exterior forms $(\xi \bar{\xi})_{\alpha\beta}$, or equivalently by the ones

$$f^{\alpha\beta} := (\xi \xi^\epsilon)^{\alpha\beta} \quad (18)$$

through (left or right) multiplication by elements of $\mathcal{A}$ (resp. $\mathcal{H}$). $f^{\alpha\beta}$ span a $(3,1)$ corepresentation space of $SU_q(2) \otimes SU_q(2)'$. One can find 1-form-valued matrices $a$ such that

$$d a^{\alpha\beta} = f^{\alpha\beta}; \quad (19)$$

$a$ is uniquely determined to be

$$a^{\alpha\beta} = \mathcal{P}_s^{\alpha\beta}(\xi^\epsilon x^T)^{\gamma\delta}, \quad (20)$$

where $\mathcal{P}_s$ is the $SU_q(2)$-covariant symmetric projector, if we require $a^{\alpha\beta}$ to transform as $f^{\alpha\beta}$, i.e. in the $(3,1)$ dimensional corepresentation of $SU_q(2) \times SU_q(2)'$, whereas will be defined up to $d$-exact terms of the form

$$\tilde{a} = a + \mathbf{1}_2 dM(|x|^2)$$

if we just require $\tilde{a}^{\alpha\beta}$ to be in the $(3,1) \oplus (1,1)$ reducible representation. In particular, the 1-form valued matrix $(dT)^T$ belongs to the latter. In the $q = 1$ limit $[20]$ becomes

$$a^{\alpha\beta} = \left( \xi^\epsilon x^T \right)^{(\alpha\beta)} = - \left\{ Im(\xi \bar{x}^\epsilon) \right\}^{\alpha\beta}.$$

Similarly, antiself-dual $\tilde{\Omega}^{2'}$, $\tilde{\mathcal{D}}C^{2'}$ are generated by $(\xi \bar{\xi})^{\alpha'\beta'}$, or equivalently by

$$f^{\alpha'\beta'} := (\xi \xi^\epsilon)^{\alpha'\beta'}, \quad (21)$$

and one can find 1-forms $a^{\alpha'\beta'}$ such that $d a^{\alpha'\beta'} = f^{\alpha'\beta'}$, etc.

Integration over $\mathbb{R}_q^4$ [31][12][13] can be introduced by the decomposition

$$\int_{\mathbb{R}_q^4} d^4 x = \int_0^\infty d|x| \int_{|x| \cdot S_q^3} dT^3$$

6
Integration over the radial coordinate has to fulfill the scaling property
\[ \int_0^\infty d|\mathbf{x}| \, g(|\mathbf{x}|) = \int_0^\infty d|q\mathbf{x}| \, g(|q\mathbf{x}|). \]
Integration over the quantum sphere \( S_q^3 \) is determined up to normalization by the requirement of \( SO_q(4) \)-invariance. The algebra of functions on the quantum sphere \( S_q^3 \) is determined up to normalization by the requirement of \( SO_q(4) \)-invariance.

This integration over \( \mathbb{R}_q^4 \) fulfills all the main properties of Riemann integration over \( \mathbb{R}^4 \), including Stokes' theorem, except the cyclic property.

## 4 Noncommutative gauge theories: standard framework

The standard framework \[6, 11, 22\] for noncommutative gauge theories (i.e. gauge theories on noncommutative manifolds) closely mimics that for commutative ones. In \( U(n) \) gauge theory the gauge transformations \( U \) are unitary \( A \)-valued (\( A \) being the algebra of functions on the noncommutative manifold) \( n \times n \) unitary matrices, \( U \in M_n(A) \equiv M_n(\mathbb{C}) \otimes \mathbb{C} A \). The gauge potential \( A \equiv (A^\alpha_\beta) \) is an antihermitean 1-form-valued \( n \times n \) matrix, \( A \in M_n(\Omega^1(A)) \). The definition of the field strength \( F \in M_n(\Omega^2(A)) \) associated to \( A \) is as usual \( F := dA + AA \). At the right-hand side the product \( AA \) has to be understood both as a (row by column) matrix product and as a wedge product. Even for \( n = 1 \), \( AA \neq 0 \), contrary to the commutative case. The Bianchi identity \( DF := dF + [A, F] = 0 \) is automatically satisfied and the Yang-Mills equation reads as usual \( D^* F = 0 \). Because of the Bianchi identity, the latter is automatically satisfied by any solution of the (anti)self-duality equations
\[ *F = \pm F. \tag{22} \]

The Bianchi identity, the Yang-Mills equation, the (anti)self-duality equations, the flatness condition \( F = 0 \) are preserved by gauge transformations

\[ A^U = U^{-1}(AU + dU), \quad \Rightarrow \quad F^U = U^{-1}FU. \]

As usual, \( A = U^{-1}dU \) implies \( F = 0 \). Up to normalization factors, the gauge invariant ‘action’ \( S \) and ‘Pontryagin index’ \( Q \) are defined by

\[ S = \text{Tr}(F^*F), \quad Q = \text{Tr}(FF) \tag{23} \]

where \( \text{Tr} \) stands for a positive-definite trace combining the \( n \times n \)-matrix trace with the integral over the noncommutative manifold (as such, \( \text{Tr} \) has to fulfill the cyclic property). If integration \( \int \) fulfills itself the cyclic property then this is obtained by simply choosing \( \text{Tr} = \int \text{tr} \), where \( \text{tr} \) stands for the ordinary matrix trace. \( S \) is automatically nonnegative.

In the present \( A \equiv C(\mathbb{R}_q^4) = C(\mathbb{H}_q) \) case there are 2 main problems:
1. Integration over $\mathbb{R}^4_q$ fulfills a deformed cyclic property \[31\].

2. $d(f^*) \neq (df)^*$, and there is no $*$-structure $\star : \Omega^* \to \Omega^*$, but only a $*$-structure $\star : \mathcal{D}C^* \to \mathcal{D}C^*$ \[28\], with a nonlinear character.

A solution to both problems might be obtained

1. allowing for $\mathcal{D}C^1$-valued $A (\Rightarrow \mathcal{D}C^2$-valued $F$’s), and/or

2. realizing $\text{Tr}(\cdot)$ by in the form $\text{Tr}(\cdot) := \int \text{tr}(W \cdot)$, with $W$ some suitable positive definite $\mathcal{H}$-valued (i.e. pseudo-differential-operator-valued) $n \times n$ matrix (this implies a change in the hermitean conjugation of differential operators), or even a more general form.

This hope is based on our results \[16\]: 1) the $\star$-structure $\star : \mathcal{D}C^* \to \mathcal{D}C^*$ can be recast in a more suggestive form of similarity transformations (involving the realization as pseudodifferential operators of the ribbon element $\tilde{w}$ and of the "vector field generators" $\tilde{Z}_j$ of the central extension of $U_q so(4)$ with dilatations); 2) $d$ and the exterior coderivative $\delta := - d\star$ become conjugated of each other

$$(\alpha_p, d\beta_{p-1}) = (\delta \alpha_p, \beta_{p-1}), \quad (d\beta_{p-1}, \alpha_p) = (\beta_{p-1}, \delta \alpha_p)$$

if one defines

$$(\alpha_p, \beta_p) = \int_{\mathbb{R}^4_q} \alpha_p^* \beta_p, \quad \beta_p = \frac{\tilde{w}'^{1/2}}{2} \beta_p$$

where $\tilde{w}'$ is the realization of $\tilde{w}$ as a pseudodifferential operator.

5 The (anti)instanton solution

We first recall the commutative ($q = 1$) solution of the self-duality eq. $*F = F$: the instanton solution of \[3\] in t’ Hooft \[32\] and in ADHM \[2\] quaternion notation (see \[1\] for an introduction) reads:

$$A = dx^i \sigma^0 \eta^a_{ij} x^j \frac{1}{\rho^2 + r^2/2},$$

$$= -\text{Im} \left\{ \frac{\xi}{|x|^2} \right\} \frac{1}{1 + \rho^2 |x|^2}$$

$$= - (dT) T \frac{1}{1 + \rho^2 |x|^2}$$

$$F = \xi \bar{\xi} \rho^2 \frac{1}{(\rho^2 + |x|^2)^2},$$

where $r^2 := x \cdot x = 2|x|^2$, $\eta^a_{ij}$ are the so-called ’t Hooft $\eta$-symbols and $\rho$ is the size of the instanton (here centered at the origin). The third equality is based on the identity

$$\frac{\xi}{|x|^2} = (dT) T + I_2 \frac{d|x|^2}{2|x|^2}$$
and the observation that the first and second term at the rhs are respectively antihermitean and hermitean, i.e. the imaginary and the real part of the quaternion at the lhs.

Noncommutative \((q\neq 1)\) solutions of \(\ast F = F\). Looking for \(A\) directly in the form \(A = \xi \bar{x} t/|x|^2 + \theta I_2 n\), where \(l, n\) are functions of \(x\) only through \(|x|\), one finds a family of solutions parametrized by \(\rho^2\) (a nonnegative constant, or more generally a further generator of the algebra) and by the function \(l\) itself. The freedom in the choice of \(l\) should disappear upon imposing the proper (and still missing) antihermiticity condition on \(A\), as it occurs in the \(q = 1\) case. For the moment, out of this large family we just pick one which has the right \(q \to 1\) limit and closely resembles the undeformed solutions \((24-25)\).

\[
\begin{align*}
A &= -(dT)\bar{T} \frac{1}{1 + \rho^2/|x|^2}, \\
F &= q^{-1}\xi \bar{\xi} T \frac{1}{q^2 |x|^2 + \rho^2} T \frac{1}{q^2 |x|^2 + \rho^2}.
\end{align*}
\]

(26)

Of course we have to extend the algebras so that they contain the rational functions at the rhs. The matrix elements \(A^\alpha{}\beta\) span a \((3, 1) \oplus (1, 1)\) dimensional corepresentation of \(SU_q(2) \times SU_q(2)'\), suggesting as the ‘fiber’ of the gauge group in the complete theory a (possibly deformed) \(U(2)\) [instead of a \(SU(2)\)].

One can shift the ‘center of the instanton’ away from the origin by the replacement (or ‘braided coaddition’ \([25]\))

\[x \rightarrow x - y,\]

where the ‘coordinates of the center’ \(y^i\) generate a new copy of \(A\), ‘braided’ with the original one (see below). Therefore the instanton moduli space must be a noncommutative manifold, with coordinates \(\rho, y^i\)! This is similar to what was proposed in \([20]\) for the instanton moduli space on \(\mathbb{R}^4_q\).

By the scaling and translation invariance of integration over \(\mathbb{R}^4_q\), if we could find a ‘good’ pseudodifferential operator \(W\) to define gauge invariant “action” and “topological charge” by

\[
Q := \int_{\mathbb{R}^4_q} \text{tr}(WF \ast F) = \int_{\mathbb{R}^4_q} \text{tr}(WF F) = S
\]

the latter would, as in the commutative case, equal a constant independent of \(\rho, y\) (which by the choice of the normalization of the integral we can make 1).

In the \(q = 1\) case multi-instanton solutions are explicitly written down in the socalled ‘singular gauge’. Note that as in the \(q = 1\) case \(T = x/|x|\) is unitary and singular at \(x = 0\). So it can play the role of a ‘singular gauge transformation’. In fact \(A\) can be obtained through the gauge
transformation \( A = T(\hat{A} T + d\hat{T}) \) from the singular gauge potential

\[
\hat{A} = T \frac{d T}{1 + \frac{|x|^2}{\rho^2}} = -\frac{1}{1 + |x|^2} (d T) T
\]

\[
= -\frac{1}{1 + |x|^2} \left[ q^{-1} \hat{\xi} \frac{x}{|x|^2} - \frac{q^{-3} I_2}{1 + q} \left( \frac{\xi^{\alpha\alpha'} x^{\beta\beta'}}{|x|^2} \epsilon_{\alpha\beta} \epsilon_{\alpha'\beta'} \right) \right].
\] (27)

\( \hat{A} \) can be expressed also in the form

\[
\hat{A} = \phi^{-1} \hat{D} \phi, \quad \phi := 1 + q^2 \rho^2 \frac{1}{|x|^2},
\]

where \( \hat{D} \) is the first-order-differential-operator-valued \( 2 \times 2 \) matrix obtained from the square bracket in (27) by the replacement \( x^{\alpha\alpha'}/|x|^2 \to q^2 \partial^{\alpha\alpha'} \):

\[
\hat{D} := q \hat{\xi} \partial - \frac{q^{-1} I_2}{q + 1} d
\]

(for simplicity we are here assuming that \( \rho^2 \) commutes with \( \xi^{\alpha\alpha'} \partial^{\beta\beta'} \)). \( \phi \) is harmonic:

\[
\Box \phi = 0.
\]

This is the analog of the \( q = 1 \) case, and is useful for the construction of multi-instanton solutions.

The anti-instanton solution is obtained just by converting unbarred into barred matrices, and conversely, as in the \( q = 1 \) case. For instance, from (26) we obtain the anti-instanton solution in the regular gauge

\[
A' = -(d T) T \frac{1}{1 + \rho^2 |x|^2},
\]

\[
F' = q^{-1} \hat{\xi} \frac{1}{|x|^2} \rho^2 \frac{1}{q^2 |x|^2 + \rho^2}.
\] (29)

6 Multi-instanton solutions

We have found solutions of the self-duality equation corresponding to \( n \) instantons in the “singular gauge” \([32, 33]\) in the form

\[
\hat{A} = \phi^{-1} \hat{D} \phi,
\]

where \( \phi \) is the harmonic scalar function

\[
\phi = 1 + \rho_1^2 \frac{1}{(x-y_1)^2} + \rho_2^2 \frac{1}{(x-y_1-y_2)^2} + \ldots + \rho_n^2 \frac{1}{(x-y_1-\ldots-y_n)^2}
\] (31)

as in the commutative case. In the commutative limit

\[
\rho_\mu \equiv \text{size of the } \mu \text{-th instanton},
\]

\[
v^i_\mu := \sum_{\nu=1}^\mu y^i_\nu \equiv \text{i-th coordinate of the } \mu \text{-th instanton}.
\]
are constants ($\mu = 1, 2, \ldots, n$). In the noncommutative setting the new generators $\rho_\mu^2, y_\nu^i$ have to fulfill the following nontrivial commutation relations:

\[
\rho_\nu^2 \rho_\mu^2 = q^2 \rho_\mu^2 \rho_\nu^2 \quad \nu < \mu \\
\rho_\nu^2 y_\mu^i = y_\mu^i \rho_\nu^2 \quad \nu < \mu \\
\rho_\mu^2 \xi_i = \xi_i \rho_\mu^2 \\
y_\mu^i y_\nu^j = q \hat{R}_{hkh}^j y_h^j y_k^i \quad \nu < \mu, \\
P_{A_hh} y_h^k y_k^i = 0.
\]

($\mu, \nu = 0, 1, \ldots, n$, and we have set $x_i = y_0^i$).

The last relation states that for any fixed $\nu$ the 4 coordinates $y_\nu^i$ generate a copy of $A$. The last but one states that the various copies of $A$ are braided w.r.t. each other (this is necessary for the $SO_q(4)$ covariance of the overall algebra).

The obvious consequence of the nontrivial commutation relations (32) is that in a complete theory the instanton moduli space must be a noncommutative manifold.

At least for low $n$, we have been able to go to a gauge potential $A$ ‘regular’ in $x_\mu^i = x^i - v_\mu^i$ by a ‘singular gauge transformation’ (as in the $q = 1$ case [19, 29, 33]), which also depends on $\rho_\mu^2, y_\nu^i$.

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