Dynamic Programming Principle for Combined Optimal Stopping and Stochastic Control with \( f \)-conditional Expectation

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Abstract

In this paper, we study combined optimal stopping and stochastic control problems for \( f \)-conditional expectations with jumps. Our main contribution is to establish a dynamic programming principle. This requires some particular techniques due to the nonlinearity of the expectation. Using this result, we prove that the value function is a viscosity solution of an obstacle problem for an Hamilton-Jacobi-Bellman equation. Under additional assumptions, we provide an uniqueness result for the viscosity solution of this obstacle problem in the class of bounded continuous functions. Some examples in mathematical finance are given.

Key-words: Robust optimal stopping, reflected backward stochastic differential equations with jumps, dynamic programming principle, Hamilton-Jacobi-Bellman variational inequalities, viscosity solution, non linear expectation.

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1 Introduction

Many studies are devoted to stochastic control and optimal stopping problems, associated dynamic programming principles (DPP) as well as their links with Hamilton-Jacobi-Bellman PDEs (see e.g. [19, 15], [10] and recently [5]). Some papers also address combined stochastic control and optimal stopping problems (see e.g. [6]), which can generally be written as follows:

\[
\sup_{\tau \in \mathcal{T}} \sup_{\alpha \in \mathcal{A}} E \left[ \int_{0}^{\tau} g(\alpha_s, X^\alpha_s) \, ds + h(X^\alpha_\tau) \right], \tag{1.1}
\]

where \( \mathcal{T} \) denotes the set of stopping times, \( \mathcal{A} \) is a set of admissible controls and \( (X^\alpha_t) \) is a controlled diffusion process with jumps of the form

\[
X^\alpha_t = x + \int_{0}^{t} b(X^\alpha_s, \alpha_s) \, ds + \int_{0}^{t} \sigma(X^\alpha_s, \alpha_s) \, dW_s + \int_{0}^{t} \int_{\mathbb{R}^*} \beta(X^\alpha_s, \alpha_s, e) \tilde{N}(ds, de).
\]

The random variable \( h(X^\alpha_\tau) \) represents some terminal reward and \( g(\alpha_s, X^\alpha_s) \) can be interpreted as an instantaneous reward process. In these papers, the authors establish a DPP, from which they derive the associated HJB variational inequalities (HJBVI).

In the last years, there has been several studies on stochastic control and optimal stopping problems for \( f \)-conditional expectations induced by Backward stochastic differential equations (BSDEs) mainly in the Brownian case (see e.g. [3, 4, 17]). Recently, two of the authors have studied stopping time problems with jumps and irregular payoff [24]. A key property of these optimal stopping problems is that the value function can be characterized as the solution of a non linear reflected BSDE. In a Markovian framework, using this characterization and comparison theorems for BSDEs with jumps, we directly prove in [8] that the value function is a viscosity solution of a non linear variational inequality. Under additional assumptions, we provide an uniqueness result for the viscosity solution of this obstacle problem in the class of bounded continuous functions.

In this paper, we consider a combined optimal stopping and stochastic control problem for \( f \)-conditional expectations in the Markovian case, of the following form:

\[
\sup_{\tau \in \mathcal{T}} \sup_{\alpha \in \mathcal{A}} \mathcal{E}_0^{\alpha, \tau}[h(X^\alpha_\tau)], \tag{1.2}
\]

where \( \mathcal{E}^\alpha \) is the nonlinear conditional expectation associated with a BSDE with driver \( f(\alpha_t, X^\alpha_t, y, z, k) \).

Note that Problem (1.1) is a particular case of (1.2) when \( f(\alpha_t, X^\alpha_t, y, z, k) \equiv g(\alpha_t, X^\alpha_t) \).

The value function of Problem (1.2) cannot generally be characterized as the solution of a reflected BSDE. In particular, the dynamic programming principle (DPP) can no longer be
derived from the flow property of reflected BSDEs only. Moreover, it is not possible to prove as easily as in [8], that the value function is a viscosity solution of an associated obstacle problem.

The main contribution of our paper is to generalize the classical DPP for linear expectations to the nonlinear case. The proof of our DPP requires some sophisticated probabilistic techniques, due to the nonlinearity of the expectation and to some measurability issues.

Using the DPP and some BSDEs techniques, we can derive that the value function is a viscosity solution of a nonlinear obstacle problem for an Hamilton-Jacobi-Bellman equation. Under additional assumptions, we prove that there exists a unique viscosity solution of this HJBVI in the class of bounded continuous functions. These results generalize those associated to combined optimal stopping and stochastic control with linear expectation.

Finally, we give two examples. First, we consider a robust pricing problem of American options in the case of nonlinear evaluation and ambiguity on the model. Our formulation allows us to take into account the ambiguity on the drift, the volatility and the nonlinear price system. We also present a mixed optimization problem of recursive utility of terminal wealth.

The paper is organized as follows: in Section 2, we introduce our combined optimal stopping and stochastic control problem. Using reflected BSDEs results, we express this problem as a control problem for reflected BSDEs. In Section 3, we establish the dynamic programming principle, first at deterministic times and then at stopping times, as well as some properties of the value function such as polynomial growth and continuity. In Section 4, we study the links between the value function and HJB variational inequalities. We first prove that the value function is a viscosity solution of an HJBVI. Then, under additional assumptions, we provide an uniqueness result for the viscosity solution of this obstacle problem. In Section 5, we give two examples in mathematical finance. In the Appendix, we provide several useful properties and some auxiliary results.

2 Formulation of the problem

We consider the product space $$\Omega := \Omega_W \otimes \Omega_N$$, where $$\Omega_W$$ is the set of continuous functions from $$[0, T]$$ into $$\mathbb{R}$$ and $$\Omega_N$$ is the set of integer-valued measures on $$[0, T] \times \mathbb{R}$$. For $$\omega = (\omega^1, \omega^2) \in \Omega$$, we set $$W_t(\omega) = \omega^1_t$$ and $$N_t(\omega) = \omega^2_t$$ and define $$\mathbb{F}^W := (\mathcal{F}_t^W)_{t \leq T}$$ and $$\mathbb{F}^N := (\mathcal{F}_t^N)_{t \leq T}$$, the filtrations associated respectively with $$W$$ and $$N$$. Let $$\mathbb{P}^W$$ be the Wiener measure on $$(\Omega_W, \mathbb{F}^W)$$ and $$\mathbb{P}^N$$ be the probability measure on $$(\Omega_N, \mathbb{F}^N)$$ under which $$N$$ is a Poisson measure with intensity $$\bar{N}(dt, de) = \nu(de)dt$$, where $$\nu$$ is a $$\sigma$$-finite measure on $$\mathbb{R}^*$$ with $$\int_{\mathbb{R}^*} (1 \wedge e^2) \nu(de) < \infty$$.

We introduce some notation and definitions.
Let $\mathcal{A}$ be the set of controls, defined as the set of predictable processes valued in a compact subset $A$ of $\mathbb{R}$. For each $\alpha \in \mathcal{A}$, initial time $t \in [0, T]$ and initial condition $x$ in $\mathbb{R}$, let $\{X_s^{\alpha,t,x}, t \leq s \leq T\}$ be the unique $\mathbb{R}$-valued solution of the SDE with jumps:

$$X_s^{\alpha,t,x} = x + \int_t^s b(X_r^{\alpha,t,x}, \alpha_r)dr + \int_t^s \sigma(X_r^{\alpha,t,x}, \alpha_r)dW_r + \int_t^s \int_{\mathbb{R}^*} \beta(X_r^{\alpha,t,x}, \alpha_r, e)\tilde{N}(dr, de), \quad (2.1)$$

where $b, \sigma : \mathbb{R} \times A \to \mathbb{R}$, are Lipschitz continuous with respect to $x$ and $\alpha$, and $\beta : \mathbb{R} \times A \times \mathbb{R}^* \to \mathbb{R}$ is a measurable function such that for some constant $C \geq 0$, and for all $e \in \mathbb{R}$

$$|\beta(x, \alpha, e)| \leq C(1 \lor |e|), \quad x \in \mathbb{R}, \alpha \in A$$

$$|\beta(x, \alpha, e) - \beta(x', \alpha', e)| \leq C(|x - x'| + |\alpha - \alpha'|)(1 \lor |e|), \quad x, x' \in \mathbb{R}, \alpha, \alpha' \in A.$$ 

The control $\alpha$ may represent the ambiguity on the model. For example, in finance, it can take into account the uncertainty on the volatility and the drift of the stock price process.

Let $\mathcal{L}_2^\nu$ denote the set of Borelian functions $l : \mathbb{R}^* \to \mathbb{R}$ such that $\|l\|^2 := \int_{\mathbb{R}^*} l^2(e)\nu(de) < \infty$. The set $\mathcal{L}_2^\nu$ is a Hilbert space equipped with the scalar product $\langle l, l \rangle_\nu := \int_{\mathbb{R}^*} l(e)l(e)\nu(de)$ for all $\delta, l \in \mathcal{L}_2^\nu \times \mathcal{L}_2^\nu$.

The criterion, depending on $\alpha$, is defined via a BSDE as follows. We introduce two functions $\gamma$ and $f$ which satisfy the following hypothesis denoted by (H$_1$):

- $\gamma : \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^*)$-measurable,
  
  $|\gamma(x, e) - \gamma(x', e)| < C|x - x'|(1 \lor |e|), x, x' \in \mathbb{R}, e \in \mathbb{R}^*$

$|\gamma(x, e)| \leq C(1 \lor |e|)$ and $\gamma(x, e) \geq -1$, $e \in \mathbb{R}^*$

- $f : A \times [0, T] \times \mathbb{R}^3 \times \mathcal{L}_2^\nu \to \mathbb{R}$ is continuous in $t$ uniformly with respect to $x, y, z, k, \alpha$ and satisfies

(i) $|f(\alpha, t, x, 0, 0, 0)| \leq C(1 + |x|^p), x \in \mathbb{R}$

(ii) $|f(\alpha, t, x, y, z, k) - f(\alpha', t, x', y', z', k')| \leq C(|\alpha - \alpha'| + |x - x'| + |y - y'| + |z - z'| + \|k - k'\|_{\mathcal{L}_2}), \forall 0 \leq t \leq T, y, y' \in \mathbb{R}, z, z' \in \mathbb{R}, k, k' \in \mathcal{L}_2^\nu, \alpha, \alpha' \in A.$

(iii) $f(\alpha, t, x, y, z, k_1) - f(\alpha, t, x, y, z, k_2) \geq -\gamma(x, \cdot, k_1 - k_2) >_\nu, \forall t, x, y, z, k_1, k_2, \alpha.$

We denote by $\mathcal{E}$ the $f^\alpha$-conditional expectation, defined for each stopping time $S$ and for each $\eta \in \mathcal{L}^2(\mathcal{F}_S)$ as:

$$\mathcal{E}_{r,S}^{\alpha,t,x}[\eta] := X_r^{\alpha,t,x}, t \leq r \leq S$$
where \((X_t^{\alpha,t,x})_{t \leq r \leq T}\) is the solution of the BSDE associated with driver \(f^{\alpha,t,x}(r, y, z, k) := f(\alpha, r, X_t^{\alpha,t,x}, y, z, k)\), terminal time \(S\) and terminal condition \(\eta\), that is satisfying:

\[
\begin{cases}
-dX_t^{\alpha,t,x} = f(\alpha, r, X_t^{\alpha,t,x}, Z_t^{\alpha,t,x}, K_t^{\alpha,t,x}(\cdot))dr - \int_0^T Z_t^{\alpha,t,x}dW_r - \int_0^T \int_{\mathbb{R}} K_t^{\alpha,t,x}(e)\tilde{N}(dr, de) \\
X_S^{\alpha,t,x} = \eta,
\end{cases}
\]

where \((Z_t^{\alpha,t,x}), (K_t^{\alpha,t,x})\) are the associated processes, which belong respectively to \(\mathbb{H}^2\) and \(\mathbb{H}_v^2\). Here, \(\mathbb{H}^2\) (resp. \(\mathbb{H}_v^2\)) denotes the set of predictable processes \((Z_t)\) (resp. \((k_t(\cdot))\)) such that \(\mathbb{E}\int_0^T Z_t^2ds < \infty\) (resp. \(\mathbb{E}\int_0^T \|k_s\|_{\mathbb{L}^2}^2ds < \infty\)). Also, \(S^2\) denotes the set of real-valued RCLL adapted processes \((\varphi_s)\) with \(\mathbb{E}[\sup_s \varphi_s^2] < \infty\).

For each \(\alpha \in \mathcal{A}\), for each \((t, x) \in [0, T] \times \mathbb{R}\), the associated dynamic payoff is given by the process \((\xi_t^{\alpha,t,x})_{t \leq s \leq T}\), defined via the state process \((X_t^{\alpha,t,x})\) as follows. We introduce:

- \(g \in \mathcal{C}(\mathbb{R}), h : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}\) are Lipschitz continuous with respect to \(x\), uniformly in \(t\) and continuous with respect to \(t\), uniformly in \(x\). Also there exist \(p \in \mathbb{N}\) and a real constant, still denoted by \(C\), such that

\[
|g(x)| + |h(t, x)| \leq C(1 + |x|^p), \forall t \in [0, T], x \in \mathbb{R}. \tag{2.2}
\]

- \(h(T, x) \leq g(x), \forall x \in \mathbb{R}\)

For each initial condition \((t, x) \in [0, T] \times \mathbb{R}\) and each control \(\alpha \in \mathcal{A}\), the reward is defined by:

\[
\xi_t^{\alpha,t,x} := h(s, X_s^{\alpha,t,x})1_{s \leq T} + g(X_T^{\alpha,t,x})1_{s=T}. \tag{2.3}
\]

Let us define the map \(\overline{h}\) by \(\overline{h}(t, x) := h(t, x)\) if \(t < T\) and \(\overline{h}(T, x) := g(x)\). The reward can then be rewritten as

\[
\xi_t^{\alpha,t,x} := h(s, X_s^{\alpha,t,x}), \ s \leq T.
\]

Let \(\mathcal{T}\) be the set of stopping times with values in \([0, T]\). Suppose the initial time is equal to 0. For each initial condition \(x \in \mathbb{R}\), we consider the following robust optimal stopping problem:

\[
u(0, x) := \sup_{\tau \in \mathcal{T}} \sup_{\alpha \in \mathcal{A}} \mathcal{E}_{0,\tau}^{\alpha,0,x}[\xi_{\tau}^{\alpha,0,x}]. \tag{2.4}
\]

We now make the problem dynamic. Let us first introduce the following notation.

Let \(t \in [0, T]\). We define the \(t\)-translated canonical space denoted by \(\Omega_t\), that is the set of \(t\)-translated paths \(\omega_t^t = (\omega_s^t)_{s \geq t} := (\omega_s - \omega_t)_{s \geq t}\). Note that \((\omega_s^{1,t})_{s \geq t} := (\omega_s^1 - \omega_t^1)_{s \geq t}\) (resp. \((\omega_s^{2,t})_{s \geq t} := (\omega_s^2 - \omega_t^2)_{s \geq t}\)) corresponds to the realizations of the translated Brownian motion.
(\(W_s - W_t)_{s \geq t}\) (resp. of the Poisson random measure \((N_s - N_t)_{s \geq t}\)). Let \(\mathbb{F}' = (\mathcal{F}'_s)_{t \leq s \leq T}\) be the associated filtration. Let us denote by \(\mathcal{T}'_{[t, T]}\) the set of stopping times with respect to \(\mathbb{F}'\), and by \(\mathcal{A}'_t\) the set of controls \(\alpha : [t, T] \times \Omega' \mapsto A\), which are \(t\)-predictable, that is \(\mathcal{P}'\)-measurable, where \(\mathcal{P}'\) is the predictable \(\sigma\)-algebra on \(\Omega' \times [t, T]\).

For each initial time \(t\) and each initial condition \(x\), the associated value function is defined by:

\[
u(t, x) := \sup_{\tau \in \mathcal{T}'_{[t, T]}} \sup_{\alpha \in \mathcal{A}'_t} \mathcal{E}^{\alpha}_{t, \tau}[\zeta_{\tau}]. \tag{2.5}\]

Note that since \(\alpha\) and \(\tau\) depend only on \(\omega'\), the SDE satisfied by \(X^{\alpha, t, x}\) and the BSDE satisfied by \(\mathcal{E}^{\alpha, t, x}[\zeta_{\tau}]\) can be solved with respect to the translated Brownian motion \((W_s - W_t)_{s \geq t}\) and the translated Poisson random measure \(N([t, s], \cdot)_{s \geq t}\), and hence on the restricted space \(\Omega' \times [t, T]\). Hence the function \(\nu\) is well defined as a deterministic function of \(t\) and \(x\).

For each \(\alpha \in \mathcal{A}'_t\), we introduce the function \(\nu^{\alpha}\) defined as

\[
u^{\alpha}(t, x) := \sup_{\tau \in \mathcal{T}'_{[t, T]}} \sup_{\alpha \in \mathcal{A}'_t} \mathcal{E}^{\alpha, t, x}[\zeta_{\tau}]. \tag{2.6}\]

By Theorem 3.2 in [24], for each \(\alpha\), the value function \(\nu^{\alpha}\) corresponds to the solution of the reflected BSDE associated to driver \(f^{\alpha, t, x} := f(\alpha, \cdot, X^{\alpha, t, x}, y, z, k)\) and obstacle process \(\zeta^{\alpha, t, x}\), that is

\[
u^{\alpha}(t, x) = Y^{\alpha, t, x}_t, \tag{2.7}\]

where \((Y^{\alpha, t, x}, Z^{\alpha, t, x}, K^{\alpha, t, x}) \in S^2 \times \mathbb{H}^2 \times \mathbb{H}^2\) is the solution of the following RBSDE:

\[
\begin{dcases}
Y^{\alpha, t, x}_s = g(X^{\alpha, t, x}_T) + \int_s^T f(\alpha, r, X^{\alpha, t, x}_r, Y^{\alpha, t, x}_r, Z^{\alpha, t, x}_r, K^{\alpha, t, x}(\cdot))dr + A^{\alpha, t, x}_T - A^{\alpha, t, x}_s \\
- \int_s^T Z^{\alpha, t, x}_r dW_r - \int_s^T \int_{\mathbb{R}^*} K^{\alpha, t, x}(r, \epsilon) \tilde{N}(dr, de) \\
Y^{\alpha, t, x}_s \geq \zeta^{\alpha, t, x}_s, 0 \leq s \leq T \text{ a.s. ,}
\end{dcases} \tag{2.8}\]

\(A^{\alpha, t, x}\) is a nondecreasing, continuous predictable process with

\[
A^{\alpha, t, x}_t = 0 \text{ and such that}
\]

\[
\int_0^T (Y^{\alpha, t, x}_s - \zeta^{\alpha, t, x}_s) dA^{\alpha, t, x}_s = 0 \text{ a.s.}
\]

The continuity of the nondecreasing process \(A^{\alpha, t, x}\) comes from the assumptions made on \(h\) and \(g\). Indeed, they imply that the obstacle \((\zeta^{\alpha, t, x})_{s \geq t}\) is continuous except at the inaccessible
jump times of the Poisson measure, and at time $T$ with $\Delta\xi_{t}^{a,t,x} \leq 0$ a.s., and this ensures the continuity of $A_{t}^{a,t,x}$ by Th. 2.6 in [24].

Note that the above RBSDE can be solved on the restricted space $\Omega^{t} \times [t,T]$, with respect to the $t$-translated Brownian motion and the $t$-translated Poisson random measure.

Our initial mixed optimal stopping/control problem (2.5) can thus be reduced to a control problem for reflected BSDEs: $u(t,x) = \sup_{\alpha \in A_{t}} Y_{t}^{a,t,x}$. This key property will be used to solve our problem. We point out that even in the classical case of linear expectations, this approach allows us to provide alternative proofs of the DPP for mixed problems.

Remark 2.1. Contrary to the previous literature on robust optimal stopping with nonlinear expectations (see [3, 4, 24]), the value function of our problem is not a priori a solution of a reflected BSDE. This is linked to the fact that our reward process (2.3) depends on the control via the state process, which was not the case in the previous works.

We also underline that in [24, 3], the robustness formulation can be applied to model ambiguity via a set of probabilities parametrized by some control process $\alpha$. However, when there is an underlying state process, it does not allow us to take into account the uncertainty on its volatility and can only handle ambiguity on the drift term, contrary to our mixed problem (2.5).

3 Dynamic programming principle

3.1 Preliminary properties of the value function

Lemma 3.1. The function $u$ has at most polynomial growth at infinity.

Proof. By applying some estimates on the solution of an RBSDE (see Prop. 5.1 in [8]), we obtain:

$$|Y_{t}^{a,t,x}|^{2} = |Y_{0}^{a,t,x}|^{2} \leq K(\mathbb{E}(\int_{0}^{t} f(\alpha_{s}, s, X_{s}^{a,t,x}, 0, 0, 0)ds + \sup_{0 \leq s \leq T} h(s, X_{s}^{a,t,x}^{2}), \forall \alpha \in A_{t}^{t}, (3.1)$$

where $K$ is a real constant which depends only on $C$ and $T$. Using now the hypothesis of polynomial growth on $f, h, g$ and the standard estimate

$$\mathbb{E}[\sup_{0 \leq s \leq T} |X_{s}^{a,t,x}|^{2}] \leq C(1 + x^{2}),$$

we derive that there exist $\bar{C} \in \mathbb{R}$ and $p \in \mathbb{N}$ such that $|u^{a}(t,x)| \leq \bar{C}(1 + |x|^{p}), \forall t \in [0,T], \forall x \in \mathbb{R}$. We finally get that $|u(t,x)| \leq \sup_{a \in A_{t}} |u^{a}(t,x)| \leq \bar{C}(1 + |x|^{p})$. \qed
Let $\mathbb{H}_t^2$ be the set of predictable processes $\alpha$ on $\Omega^t \times [t, T]$ such that $\|\alpha\|_{\mathbb{H}_t^2} := E[\int_t^T \alpha_u^2 du] < \infty$. It is a Hilbert space, equipped with the scalar product $<\alpha, \alpha'>_{\mathbb{H}_t^2} = E[\int_t^T \alpha_u \alpha'_u du]$.

**Lemma 3.2.**
- The function $u$ is Lipschitz continuous with respect to $x$, uniformly in $t$. Also, the functions $u^\alpha$ are Lipschitz continuous with respect to $x$, uniformly in $\alpha, t$.
- Let $t \in [0, T]$. The map $\alpha \mapsto u^\alpha(t, x); A^t_1 \to \mathbb{R}$ is Lipschitz continuous for the norm $\| \cdot \|_{\mathbb{H}_t^2}$, uniformly in $t, x$.

**Proof.**
- We have for all $t$:
  \[
  \sup_{\alpha \in A^t_1} |u^\alpha(t, x) - u^\alpha(t, x')| = \sup_{\alpha \in A^t_1} |Y^\alpha_{t, t, x} - Y^\alpha_{t, t, x'}|.
  \]

  The a priori estimates on reflected BSDEs (see Prop. 5.1 in [8]) and the Lipschitz property of $h$ with respect to $\alpha, x$, uniformly in $s, y, z$ imply
  \[
  \sup_{t, \alpha \in A^t_1} |u^\alpha(t, x) - u^\alpha(t, x')| \leq C|x - x'|,
  \]
  where $C$ is a constant which depends only on $T$ and the Lipschitz constant of $f$ and $h$. We also have
  \[
  |u(t, x) - u(t, x')| = \sup_{\alpha \in A^t_1} u^\alpha(t, x) - \sup_{\alpha \in A^t_1} u^\alpha(t, x') \leq \sup_{t, \alpha \in A^t_1} |u^\alpha(t, x) - u^\alpha(t, x')| \leq C|x - x'|.
  \]

  The result follows.

- Let $\alpha^1, \alpha^2 \in \mathbb{H}_t^2$. By classical estimates on diffusion processes, and the assumptions made on the coefficients, we get $E[\sup_{s \geq t} |X^\alpha_{s, t, x} - X^\alpha_{s, t, x}|^2] \leq C\|\alpha^1 - \alpha^2\|^2_{\mathbb{H}_t^2}$. Hence, by a priori estimates on RBSDEs (Prop. 5.1 in [8]), the Lipschitz property of $f$ and $h$ with respect to $\alpha, x$, uniformly in $s, y, z$, we get that $|Y^\alpha_{t, t, x} - Y^\alpha_{t, t, x}| \leq C\|\alpha^1 - \alpha^2\|_{\mathbb{H}_t^2}$.

Continuity of $u$ with respect to time will be proved later in Section 3.3 because it will require the dynamic programing principle at deterministic times.

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3.2 Dynamic programming principle at deterministic times

For each $\alpha \in A^t$, we denote by $Y_{t,T}^{\alpha,t,x}[\overline{h}(u, X^\alpha_{t,u})_{t\leq u \leq T}]$ the solution of the RBSDE associated to driver $f^{\alpha,t,x}$ and obstacle $\overline{h}(u, X^\alpha_{t,u})_{t\leq u \leq T}$. Using this notation, equality (2.7) can be written:

$$Y_{t,T}^{\alpha,t,x} = Y_{t,T}^{\alpha,t,x}[\overline{h}(u, X^\alpha_{t,u})_{t\leq u \leq T}] = \sup_{\tau \in \mathcal{T}_t^T} \mathcal{E}^{\alpha,t,x}_{t,\tau}[\overline{h}(\tau, X^\alpha_{\tau,x})] = u^\alpha(t, x).$$

**Theorem 3.3.** The value function defined by (2.5) satisfies the following dynamic programming principle:

$$u(t, x) = \sup_{\alpha \in A^t} \sup_{s \in [t, T]} \mathcal{E}^{\alpha,t,x}_{t,s \wedge \tau}[h(\tau, X^\alpha_{\tau,x})1_{\tau < s} + u(s, X^\alpha_s)1_{s \leq \tau}] , \forall s \geq t. \quad (3.2)$$

**Proof.** Equality (3.2) is equivalent to

$$u(t, x) = \sup_{\alpha \in A^t} Y_{t,s}^{\alpha,t,x}[h(u, X^\alpha_{u,t})1_{t \leq u < s} + u(s, X^\alpha_s)1_{u = s}] , \forall s \geq t.$$  

The proof is divided in two steps.

**Step 1.** We first show that:

$$u(t, x) \leq \sup_{\alpha \in A^t} Y_{t,s}^{\alpha,t,x}[h(u, X^\alpha_{u,t})1_{t \leq u < s} + u(s, X^\alpha_s)1_{u = s}] , \forall s \geq t \quad (3.3)$$

Let $t \in [0, T]$, $s \in [t, T]$ and $\alpha \in A^t$. As $\alpha$ is $\mathbb{F}^t$-progressively measurable and $\mathbb{F}^t$ is the filtration associated to the canonical process on $\Omega^t$, we can express $\alpha$ in the following form:

$$\alpha = \alpha (\omega^t, \omega^s),$$

where $\omega^s = (\omega_u - \omega_u)_{u \geq s}$ and $\omega^t = (\omega_u^t)_{t \leq u \leq s}$ with $\omega_u^t = \omega_u - \omega_t$.

In the following, to simplify notation, $\omega^t$ will be also denoted by $\omega$.

We state a preliminary measurability result.

**Lemma 3.4.** Let $t \in [0, T]$, $s \in [t, T]$ and $\alpha \in A^t$. Let $\omega = \omega^t = \omega^s$. The map $\alpha^t: (\Omega, \mathcal{F}^t_s) \rightarrow (\mathbb{H}_s^2, \mathcal{B}(\mathbb{H}_s^2)); \omega \rightarrow \alpha(\omega, \cdot)$ is measurable.

**Proof.** Since $\mathbb{H}_s^2$ is a separable Hilbert space, it admits a countable orthonormal basis $\{e_i, i \in \mathbb{N}\}$. For each $\omega$,

$$\alpha_u(\omega, \cdot) = \sum_i \beta_i(\omega) e_i(\cdot) \, dP \otimes du \text{ a.s.} \quad (3.4)$$

where $\beta_i(\omega) = E[\int_s^T \alpha_u(\omega, \cdot) e_i(\cdot) du]$. In order to obtain the desired result, it is sufficient to show that $\beta^t$ is $\mathcal{F}_s^t$-measurable in the case of a simple process $\alpha$, that is $\alpha_u = h_j 1_{[t_j, t_{j+1}]}(u)$,
where \( h_j \) is \( \mathcal{F}_t^j \) measurable. The map \( \left( \omega, \omega^* \right) \mapsto h_j(\omega, \omega^*) \) is \( \mathcal{F}_t^j \otimes \mathcal{F}_t^j \)-measurable. Hence by the Fubini theorem the map \( \omega \mapsto h_j(\omega, \cdot) \) is \( \mathcal{F}_t^j \)-measurable, which ends the proof of the lemma. \( \square \)

**End of proof of Step 1.** By Lemma 3.4 the map \( \omega \mapsto \alpha(\omega, \cdot) \) is \( \mathcal{F}_t^j \)-measurable. Moreover, by Lemma 3.2 \( u^\alpha(s, x) = Y^{\alpha,s,x}_s \) is continuous with respect to \( \alpha \) and \( x \). At fixed \( s, \omega \), \( Y^{\alpha(\omega), s, X^{\alpha(\omega), t,x}} = \left[ h(u, X^{\alpha(\omega), s, X^{\alpha(\omega), t,x}}) \right] \) is the solution at time \( s \) of the RBSDE on \( \Omega^s \times [s, T] \) with respect to the \( s \)-translated Brownian motion and \( s \)-translated Poisson measure. By composition, for each \( s \geq t \), the map \( \omega \mapsto Y^{\alpha(\omega), s, X^{\alpha(\omega), t,x}} = \left[ h(u, X^{\alpha(\omega), s, X^{\alpha(\omega), t,x}}) \right] \) is \( \mathcal{F}_s^j \)-measurable. Hence, by applying the flow property for RBSDEs, we get that:

\[
Y^{\alpha,t,x}_{t,T} \left[ h(u, X^{\alpha,t,x}_{t,U}) \right]_{t \leq u \leq T} = Y^{\alpha,t,x}_{t,s} \left[ h(u, X^{\alpha,t,x}_{s,U}) 1_{u < s} + Y^{\alpha,s,T}_{s,u} \left[ h(r, X^{\alpha,s,T}_{r,U}) 1_{r \leq s} \right] \right] = Y^{\alpha,t,x}_{t,s} \left[ h(u, X^{\alpha,t,x}_{s,U}) 1_{u < s} + u(s, X^{\alpha,t,x}) 1_{u = s} \right].
\]

Note that at fixed \( s, \omega \), \( \alpha(\omega, \cdot) \in \mathcal{A}_s^j \). By using the definition of \( u^\alpha \), we get that:

\[
Y^{\alpha(\omega), s, X^{\alpha(\omega), t,x}} \left[ h(r, X^{\alpha(\omega), s, X^{\alpha(\omega), t,x}}) \right]_{s \leq r \leq T} = u^{\alpha(\omega)}(s, X^{\alpha(\omega), t,x}) \leq u(s, X^{\alpha(\omega), t,x}).
\]

Finally, the comparison theorem for RBSDEs with jumps (see [23]) leads to:

\[
Y^{\alpha,t,x}_{t,T} \left[ h(u, X^{\alpha,t,x}_{t,U}) \right]_{t \leq u \leq T} \leq Y^{\alpha,t,x}_{t,s} \left[ h(u, X^{\alpha,t,x}_{s,U}) 1_{u < s} + u(s, X^{\alpha,t,x}) 1_{u = s} \right].
\]

Since \( \alpha \in \mathcal{A}_t^j \) is arbitrary, we get inequality (3.3).

**Step 2.** It remains to show the following inequality:

\[
u(t, x) \geq \sup_{\alpha \in \mathcal{A}_t^j} Y^{\alpha,t,x}_{t,s} \left[ h(u, X^{\alpha,t,x}_{s,U}) 1_{u < s} + u(s, X^{\alpha,t,x}) 1_{u = s} \right], \quad \forall s \geq t. \tag{3.5}
\]

For each \( s \in [t, T] \), we introduce the set of restrictions to \([s, T]\) of the controls in \( \mathcal{A}_t^j \), which is denoted by \( \mathcal{A}_s^t \).

**Lemma 3.5.** Let \( t \in [0, T] \), \( s \in [t, T] \) and \( \xi \in L^2(\mathcal{F}_s^j) \). There exists a sequence \( (\alpha^n)_{n \in \mathbb{N}} \in \mathcal{A}_s^t \) such that:

\[
u(s, \xi(\cdot)) = \lim_{n \to \infty} u^{\alpha^n}(s, \xi(\cdot)) \text{ a.s.}
\]

**Proof.** Let \( 0 \leq t \leq s \leq T \). Recall that \( \Omega^s = \{(\omega_u - \omega_s)_{u \geq s}; \omega \in \Omega\} \) and \( \Omega^t = \{(\omega_u - \omega_t)_{t \leq u \leq s}; \omega \in \Omega\} \). For each \( \omega \in \omega^s \), by using the definition of the function \( u \) we have:

\[
u(s, \xi(\omega)) = \sup_{\alpha \in \mathcal{A}_s^t} u^{\alpha}(s, \xi(\omega)).
\]
For each \( *\omega \in \ast \Omega \), for all \( n \in \mathbb{N} \), there exists \( \alpha^n \in \mathcal{A}^t_s \) such that
\[
u(s, \xi(s, \omega)) \leq \nu^n(s, \xi(s, \omega)) - \frac{1}{n}, \tag{3.6}\]

Now, \( \mathcal{A}^t_s \) is a closed subspace of \( \mathbb{H}^2_s \), the separable Hilbert space of predictable processes on \( \Omega^* \times [s, T] \), equipped with the norm \( \| \cdot \|_{\mathbb{H}^2} \). Hence, \( \mathcal{A}^t_s \) is a Polish space. The section theorem (see Section 81 in the Appendix of Ch. III in [7]) can thus be applied. Hence, since \( \xi \) is \( \mathcal{F}^t_s \)-measurable, for each \( n \) there exists a map \( \alpha^n : \ast \omega \mapsto \alpha^n(\ast \omega, \cdot) : \ast \Omega \rightarrow \mathcal{A}^t_s, \mathcal{F}^t_s \)-measurable, such that for almost every \( \ast \omega \in \ast \Omega \),
\[
u(s, \xi(s, \omega)) \leq \nu^n(s, \xi(s, \omega)) - \frac{1}{n} \tag{3.7}\]
is satisfied. We now prove that the process \( \alpha^n : [t, T] \times \Omega^t \mapsto \mathbb{R} \) is \( \mathcal{P}^t \)-measurable, where \( \mathcal{P}^t \) is the predictable \( \sigma \)-algebra defined on \( ([t, T] \times \Omega^t, \mathcal{F}^t) \). To simplify notation, let us denote \( \alpha^n \) by \( \alpha \).

Recall that for each \( *\omega \), we have \( \alpha_u(*\omega, \cdot) = \sum_i \beta^i(*\omega) e^i(\cdot) \) where \( \beta^i(*\omega) = \langle \alpha(*\omega, \cdot), e^i(\cdot) \rangle_{\mathbb{H}^2} \) and \( \{e^i, i \in \mathbb{N}\} \) is a countable orthonormal basis of \( \mathbb{H}^2_s \). Note that \( \beta_i \) is \( \mathcal{F}^t_s \)-measurable by composition. Since \( e^i \) is \( \mathcal{P}^t \)-measurable, the process \( (\beta^i e^i)_{u \geq s} \) is \( \mathcal{P}^t \)-measurable. Indeed, if we take \( c_i = H1_{[r, T]} \) with \( r \geq s \) and \( H \) a random variable \( \mathcal{F}^s \)-measurable, then the random variable \( \beta^i H \) is \( \mathcal{F}^t \)-measurable and hence the process \( \beta^i H1_{[r, T]} \) is \( \mathcal{P}^t \)-measurable. The process \( \alpha \) is thus \( \mathcal{P}^t \)-measurable, which completes the proof of the lemma. \( \square \)

**Remark 3.6.** Note that even in the linear case, this result had not been stated in the previous literature. It allows us to avoid the techniques of "balls" which are classically used in the Markovian control literature.

**End of the proof of Step 2:** Fix \( s \in [t, T] \) and \( \alpha \in \mathcal{A}^t_s \). By the above lemma, there exists a sequence \( \alpha^n \in \mathcal{A}^t_s \) such that:
\[
u(s, X^\alpha_{s, t}, x) = \lim_{n \to \infty} \nu^n(s, X^\alpha_{s, t}, x) = \lim_{n \to \infty} \nu^n(s, X^\alpha_{s, t}, x) = \nu(s, X^\alpha_{s, t}, x) \tag{3.7}\]
where the second equality follows from Proposition A.1. We set:
\[
\tilde{\alpha}_u^n := \alpha_u 1_{u < s} + \alpha_u^n 1_{s \leq u \leq T}. \tag{3.8}\]
Note that $\tilde{\alpha}^n \in A^t_T$. The relations (3.7), (3.8), the continuity properties of RBSDEs (see Appendix in [8]) and the flow property lead to:

$$Y_{t,s}^{\alpha,t,x} \left[ h(u, X_u^{\alpha,t,x}) \mathbf{1}_{u < s} + \lim_{n \to \infty} Y_{s,T}^{\alpha^n,s,x^{\alpha,t,x}} \left[ \overline{h}(r, X_r^{\alpha^n,s,x^{\alpha,t,x}}) \mathbf{1}_{r \geq s} \right] \right] = \lim_{n \to \infty} Y_{t,s}^{\tilde{\alpha}^n,t,x} \left[ h(u, X_u^{\tilde{\alpha}^n,t,x}) \mathbf{1}_{u < s} + Y_{s,T}^{\tilde{\alpha}^n,s,x^{\tilde{\alpha}^n,t,x}} \left[ \overline{h}(r, X_r^{\tilde{\alpha}^n,s,x^{\tilde{\alpha}^n,t,x}}) \mathbf{1}_{r \geq s} \right] \right]$$

$$= \lim_{n \to \infty} Y_{t,T}^{\tilde{\alpha}^n,t,x} \left[ h(u, X_u^{\tilde{\alpha}^n,t,x}) \mathbf{1}_{t \leq u \leq T} \right] \leq u(t, x) \quad (3.9)$$

Now, by taking the supremum on $\alpha \in A^t_T$ in inequality (3.9), we get (3.5). \hfill $\Box$

### 3.3 Dynamic programming principle at stopping times

In this section, using the DPP at deterministic times, we provide the continuity of the value function with respect to $t$. We then show that the DPP still holds for stopping times.

**Lemma 3.7.** The function $u$ is continuous with respect to $t$, uniformly in $x$.

**Proof.** Let $0 \leq t \leq s \leq T$. We have the following equation:

$$|u(t, x) - u(s, x)| \leq |u(t, x) - \sup_{\alpha \in A^t_T} \mathcal{E}_{t,s}^{\alpha,t,x} [u(s, X_s^{\alpha,t,x})]| + |\sup_{\alpha \in A^t_T} \mathcal{E}_{t,s}^{\alpha,t,x} [u(s, X_s^{\alpha,t,x})] - u(s, x)|. \quad (3.10)$$

We start by estimating $|\sup_{\alpha \in A^t_T} \mathcal{E}_{t,s}^{\alpha,t,x} [u(s, X_s^{\alpha,t,x})] - u(s, x)|$.

$$|\sup_{\alpha \in A^t_T} \mathcal{E}_{t,s}^{\alpha,t,x} [u(s, X_s^{\alpha,t,x})] - u(s, x)| \leq \sup_{\alpha \in A^t_T} |\mathcal{E}_{t,s}^{\alpha,t,x} [u(s, X_s^{\alpha,t,x})] - u(s, x)|$$

$$\leq \sup_{\alpha \in A^t_T} |\mathcal{E}_{t,s}^{\alpha,t,x} [u(s, X_s^{\alpha,t,x})] - \mathcal{E}_{t,s}^0 [u(s, x)]| \leq CE \left[ \sup_{t \leq r \leq s} (X_r^{\alpha,t,x} - x)^2 \right]^\frac{1}{2} \leq C|s - t|(1 + x^{2p})^\frac{1}{2}. \quad (3.11)$$

Here, $\mathcal{E}^0$ denotes the conditional expectation associated to the driver equal to 0. By symmetry, the estimation still holds when $s \leq t$. In order to obtain the above relation, we have used the a priori estimates on BSDEs ([23]), the Lipschitz property of $u$ (see Lemma 3.2) and the polynomial growth of $u$ (see Lemma 3.1).

We estimate now $|u(t, x) - \sup_{\alpha \in A^t_T} \mathcal{E}_{t,s}^{\alpha,t,x} [u(s, X_s^{\alpha,t,x})]|$.

By using the DPP for deterministic times (see Theorem 3.3), we get that:

$$u(t, x) = \sup_{\alpha \in A^t_T} \lim_{n \to \infty} \mathcal{E}_{t,s}^{\alpha,t,x} \left[ h(\tau, X_{\tau}^{\alpha,t,x}) \mathbf{1}_{\tau < s} + u(s, X_s^{\alpha,t,x}) \mathbf{1}_{\tau \geq s} \right] \geq \sup_{\alpha \in A^t_T} \mathcal{E}_{t,s}^{\alpha,t,x} [u(s, X_s^{\alpha,t,x})],$$
by noticing that \( s \in \mathcal{T}'_{[t,T]} \). This DPP also yields:

\[
\begin{align*}
    u(t, x) - \sup_{a \in A_t} \mathcal{E}^{\alpha,t,x}_{t,s}[u(s, X_s^{\alpha,t,x})] & \leq \sup_{a \in A_t} \mathcal{E}^{\alpha,t,x}_{t,s}[h(\tau, X_\tau^{\alpha,t,x})1_{\tau<s} + u(s, X_s^{\alpha,t,x})1_{\tau\geq s}] \\
    & - \mathcal{E}^{\alpha,t,x}_{t,s}[u(s, X_s^{\alpha,t,x})1_{\tau<s} + u(s, X_s^{\alpha,t,x})1_{\tau<s}]
\end{align*}
\]

\[
\leq \sup_{a \in A_t} \sup_{\tau \in \mathcal{T}'_{[t,T]}} \mathcal{E}^{\alpha,t,x}_{t,s}[h(\tau, X_\tau^{\alpha,t,x})1_{\tau<s} + u(s, X_s^{\alpha,t,x})1_{\tau\geq s} - \mathcal{E}^{\alpha,t,x}_{t,s}[u(s, X_s^{\alpha,t,x})1_{\tau<s} + h(s, X_s^{\alpha,t,x})1_{\tau<s}]]
\]

where \( \tilde{f}(s, \cdot) = f^{\alpha,t,x}(s, \cdot)1_{s \leq \tau} \), because \( u \geq h \). By using a priori estimates for BSDEs \[23\], we get:

\[
|u(t, x) - \sup_{a \in A_t} \mathcal{E}^{\alpha,t,x}_{t,s}[u(s, X_s^{\alpha,t,x})]|^2 \leq C \sup_{a \in A_t} \sup_{\tau \in \mathcal{T}'_{[t,T]}} \mathbb{E}\left[(h(\tau, X_\tau^{\alpha,t,x})1_{\tau<s} - h(s, X_s^{\alpha,t,x})1_{\tau<s})^2\right] + \mathbb{E}\left[\int_{\tau<s} f^2(\alpha_r, r, h(\tau, X_\tau^{\alpha,t,x}), 0, 0)dr\right]
\]

\[
\leq C \sup_{t \leq \tau \leq s} \sup_{x} \left|h(r, x) - h(s, x)\right| + (s - t)^2 + (1 + x)^2\left|s - t\right|\left|s - t\right|(1 + x^2)^{\frac{3}{2}}. \tag{3.12}
\]

In order to obtain (3.12), we have used the Lipschitz property in \( x \) of \( h \), the polynomial growth of \( h \) in \( x \) and the standard estimates for SDEs. From (3.10), (3.11) and (3.12) we derive that:

\[
|u(t, x) - u(s, x)| \leq C \sup_{t \leq \tau \leq s} \sup_{x} \left|h(r, x) - h(s, x)\right| + (s - t)^2 + (1 + x)^2\left|s - t\right|\left|s - t\right|(1 + x^2)^{\frac{3}{2}}, \tag{3.13}
\]

which completes the proof of the lemma. \( \square \)

Using the continuity property of \( u \) with respect to \( t \), we show now that the dynamic programming principle also holds for stopping times.

**Theorem 3.8.** The value function defined by (2.5) satisfies the following dynamic programming principle: for each \( t \in [0, T] \) and for each stopping time \( \theta \in \mathcal{T}'_{[t,T]} \),

\[
    u(t, x) = \sup_{a \in A_t} \mathbb{E}\left[\int_{t\wedge\tau}^{\tau\wedge\theta} h(\tau, X_\tau^{\alpha,t,x})\,d\tau + u(\theta, X_\theta^{\alpha,t,x})1_{\theta \geq \tau}\right]. \tag{3.14}
\]

**Proof.**

Let us first show that:

\[
\sup_{a \in A_t} \mathbb{E}\left[\int_{t\wedge\tau}^{\tau\wedge\theta} h(\tau, X_\tau^{\alpha,t,x})\,d\tau + u(\theta, X_\theta^{\alpha,t,x})1_{\theta \geq \tau}\right] \leq u(t, x), \tag{3.15}
\]

13
or, equivalently, the following inequality:

$$
\sup_{\alpha \in \mathcal{A}_t^n} Y_{t,\theta}^{\alpha,t,x} \left[ h(s, X_s^{\alpha,t,x}) 1_{s<\theta} + u(\theta, X_{\theta}^{\alpha,t,x}) 1_{s=\theta} \right] \leq u(t, x). \tag{3.16}
$$

Fix $\theta \in T_{t,T}$. We define $\theta^n := \sum_{k=0}^{2n-1} t_k 1_{A_k} + T 1_{\theta=T}$, where $t_k := \frac{(k+1)T}{2^n}$ and $A_k := \{ \frac{kT}{2^n} \leq \theta < \frac{(k+1)T}{2^n} \}$. Note that $\theta^n \in T_{t,T}$ and $\theta^n \downarrow \theta$ a.s. The continuity of $u$ in $(t,x)$, the right continuity of $X^{\alpha,t,x}$ and a continuity property of RBSDEs (see Proposition 4.2) imply that (3.16) is equivalent to:

$$
\sup_{\alpha \in \mathcal{A}_t^n} \lim_{n \to \infty} Y_{t,\theta}^{\alpha,t,x} \left[ h(s, X_s^{\alpha,t,x}) 1_{s<\theta^n} + u(\theta^n, X_{\theta^n}^{\alpha,t,x}) 1_{s=\theta^n} \right] \leq u(t, x). \tag{3.17}
$$

By Lemma 3.5 for each $k$, for each $n$, there exists an "optimizing" sequence $(\alpha_{s,n,p,k})_{p \in \mathbb{N}}$ of controls in $\mathcal{A}_t^n$ for the value function $u(t_k, X_{t_k,k}^{\alpha})$. Setting $\alpha_{s,n,p,k} := \sum_{k=0}^{2n-1} \alpha_{s,n,p,k} 1_{A_k}$, we have:

$$
\begin{align*}
&u(\theta^n, X_{\theta^n}^{\alpha,t,x}) = \sum_{k=0}^{2n-1} u(t_k, X_{t_k,k}^{\alpha,t,x}) 1_{A_k} = \sum_{k=0}^{2n-1} \lim_{p \to \infty} Y_{t_k,T}^{\alpha_{n,p,k},t_k,X_{t_k,k}^{\alpha,t,x}} \left[ h(s, X_s^{\alpha_{n,p,k},t_k,X_{t_k,k}^{\alpha,t,x}}) 1_{t_k \leq s \leq T} \right] 1_{A_k} \\
&= \lim_{p \to \infty} Y_{t^n,T}^{\alpha_{n,p},t^n,x} \left[ h(s, X_s^{\alpha_{n,p},t^n,x}) 1_{s \leq t^n} \right] \text{ a.s.} \tag{3.18}
\end{align*}
$$

We set:

$$
\tilde{\alpha}_{s,n,p} := \alpha_{s} 1_{s<\theta^n} + \alpha_{n,p} 1_{s=\theta^n}.
$$

Note that $\tilde{\alpha}_{s,n,p} \in \mathcal{A}_t^n$.

The relations (3.17), (3.18), the continuity of the RBSDE with respect to the obstacle and the flow property of the solution of the RBSDE lead to:

$$
\begin{align*}
&Y_{t,\theta}^{\alpha,t,x} \left[ h(s, X_s^{\alpha,t,x}) 1_{s<\theta^n} + \lim_{p \to \infty} Y_{t^n,T}^{\alpha_{n,p},t^n,x} \left[ h(u, X_{u}^{\alpha_{n,p},t^n,x}) 1_{u \geq \theta^n} \right] 1_{s=\theta^n} \right] \\
&= \lim_{p \to \infty} Y_{t,\theta}^{\tilde{\alpha}_{n,p},t^n,x} \left[ h(s, X_s^{\tilde{\alpha}_{n,p},t^n,x}) 1_{s<\theta^n} + Y_{t^n,T}^{\tilde{\alpha}_{n,p},t^n,x} \left[ h(u, X_{u}^{\tilde{\alpha}_{n,p},t^n,x}) 1_{u \geq \theta^n} \right] 1_{s=\theta^n} \right] \\
&= \lim_{p \to \infty} Y_{t,T}^{\tilde{\alpha}_{n,p},t^n,x} \left[ h(s, X_s^{\tilde{\alpha}_{n,p},t^n,x}) 1_{t \leq s \leq T} \right] \leq u(t, x). \tag{3.19}
\end{align*}
$$

Hence, we have shown that for each $n$,

$$
Y_{t,\theta}^{\alpha,t,x} \left[ h(s, X_s^{\alpha,t,x}) 1_{s<\theta^n} + u(\theta^n, X_{\theta^n}^{\alpha,t,x}) 1_{s=\theta^n} \right] \leq u(t, x).
$$

Now, by letting $n$ tend to $\infty$ in this inequality, and by taking the supremum on $\alpha \in \mathcal{A}_t^n$, we get (3.17).

The inverse inequality in (3.15) can be shown by using similar arguments as above. The proof, which is even simpler, is omitted. \qed
4 Links between the value function and HJB variational inequalities

We introduce the following HJBVI:

\[
\begin{align*}
\min(u(t, x) - h(t, x), \\
\inf_{a \in A} (-\frac{\partial u}{\partial t}(t, x) - \lambda^\alpha u(t, x) - f(\alpha, t, x, u(t, x), (\sigma \frac{\partial u}{\partial x})(t, x), B^\alpha u(t, x))) = 0, (t, x) \in [0, T] \times \mathbb{R}\
\end{align*}
\]

\[u(T, x) = g(x), x \in \mathbb{R}
\]

where \(\lambda^\alpha := A^\alpha + K^\alpha\), and for \(\phi \in C^2(\mathbb{R})\),

- \(A^\alpha \phi(x) := \frac{1}{2}\sigma^2(x, \alpha) \frac{\partial^2 \phi}{\partial x^2}(x) + b(x, \alpha) \frac{\partial \phi}{\partial x}(x)\)
- \(K^\alpha \phi(x) := \int_{\mathbb{R}} \left( \phi(x + \beta(x, \alpha, e)) - \phi(x) - \frac{\partial \phi}{\partial x}(x) \beta(x, \alpha, e) \right) \nu(de)\)
- \(B^\alpha \phi(x) := \phi(x + \beta(x, \alpha, \cdot)) - \phi(x)\).

In the following, we prove that the solution of the robust problem \([2, 5]\) corresponds to the viscosity solution of HJBVI \([1, 1]\).

**Definition 4.1.** A continuous function \(u\) is said to be a \textit{viscosity subsolution} of \([1, 1]\) if \(u(T, x) \leq g(x), x \in \mathbb{R}\), and if for any point \((t_0, x_0) \in [0, T] \times \mathbb{R}\) and for any \(\phi \in C^{1,2}([0, T] \times \mathbb{R})\) such that \(\phi(t_0, x_0) = u(t_0, x_0)\) and \(\phi - u\) attains its minimum at \((t_0, x_0)\), we have

\[
\min(u(t_0, x_0) - h(t_0, x_0), \\
\inf_{a \in A} (-\frac{\partial \phi}{\partial t}(t_0, x_0) - \lambda^\alpha \phi(t_0, x_0) - f(\alpha, t_0, x_0, u(t_0, x_0), (\sigma \frac{\partial \phi}{\partial x})(t_0, x_0))(t_0, x_0), B^\alpha \phi(t_0, x_0))) \leq 0.
\]

In other words, if \(u(t_0, x_0) > h(t_0, x_0)\),

\[
\inf_{a \in A} (-\frac{\partial \phi}{\partial t}(t_0, x_0) - \lambda^\alpha \phi(t_0, x_0) - f(\alpha, t_0, x_0, u(t_0, x_0), (\sigma \frac{\partial \phi}{\partial x})(t_0, x_0))(t_0, x_0), B^\alpha \phi(t_0, x_0))) \leq 0.
\]

- A continuous function \(u\) is said to be a \textit{viscosity supersolution} of \([1, 1]\) if \(u(T, x) \geq g(x), x \in \mathbb{R}\), and if for any point \((t_0, x_0) \in [0, T] \times \mathbb{R}\) and for any \(\phi \in C^{1,2}([0, T] \times \mathbb{R})\) such that \(\phi(t_0, x_0) = u(t_0, x_0)\) and \(\phi - u\) attains its maximum at \((t_0, x_0)\), we have

\[
\min(u(t_0, x_0) - h(t_0, x_0), \\
\inf_{a \in A} (-\frac{\partial \phi}{\partial t}(t_0, x_0) - \lambda^\alpha \phi(t_0, x_0) - f(\alpha, t_0, x_0, u(t_0, x_0), (\sigma \frac{\partial \phi}{\partial x})(t_0, x_0), B^\alpha \phi(t_0, x_0))) \geq 0.
\]
In other words, we have both \( u(t_0, x_0) \geq h(t_0, x_0) \),

\[
\inf_{\alpha \in A} (-\frac{\partial \phi}{\partial t}(t_0, x_0) - L^\alpha \phi(t_0, x_0) - f(\alpha, t_0, x_0, u(t_0, x_0), (\sigma \frac{\partial \phi}{\partial x})(t_0, x_0), B^\alpha \phi(t_0, x_0))) \geq 0.
\]

### 4.1 The value function, solution of the HJBVI

In this section, we prove that the value function of our robust optimal stopping problem is a viscosity solution of the above HJBVI. We use the dynamic programming principle (DPP) established in the previous section (see Theorem 3.14).

**Theorem 4.2.** The function \( u \), defined by (4.2), is a viscosity solution (i.e. both a viscosity sub- and supersolution) of the obstacle problem (4.1).

**Proof.** We prove that \( u \) is a subsolution of (4.1).

Let \((t_0, x_0) \in [0, T] \times \mathbb{R} \) and \( \phi \in C^{1,2}([0, T] \times \mathbb{R}) \) be such that \( \phi(t_0, x_0) = u(t_0, x_0) \) and \( \phi(t, x) \geq u(t, x), \forall (t, x) \in [0, T] \times \mathbb{R} \). Without loss of generality, we can suppose that the minimum of \( u - \phi \) attained at \((t_0, x_0)\) is strict.

We suppose that \( u(t_0, x_0) > h(t_0, x_0) \) and that

\[
\inf_{\alpha \in A} (-\frac{\partial \phi}{\partial t}(t_0, x_0) - L^\alpha \phi(t_0, x_0) - f(\alpha, t_0, x_0, \phi(t_0, x_0), (\sigma \frac{\partial \phi}{\partial x})(t_0, x_0), B^\alpha \phi(t_0, x_0))) > 0.
\]

By uniform continuity of \( K^\alpha \phi \) and \( B^\alpha \phi : [0, T] \times \mathbb{R} \to L^2 \) with respect to \( \alpha \), we can suppose that there exists \( \epsilon > 0, \eta_k > 0 \) such that: \( \forall (t, x) \) such that \( t_0 \leq t \leq t_0 + \eta_k < T \) and \( |x - x_0| \leq \eta_k \), we have: \( u(t, x) \geq h(t, x) + \epsilon \) and

\[
-\frac{\partial}{\partial t}\phi(t, x) - L^\alpha \phi(t, x) - f(\alpha, t, x, \phi(t, x), (\sigma \frac{\partial \phi}{\partial x})(t, x), B^\alpha \phi(t, x)) \geq \epsilon, \forall \alpha \in A.
\]

Let \( \alpha \) be an arbitrary control of \( A^{t_0}_0 \) and \( X^{\alpha,t_0,x_0} \) the associated state process.

We define the stopping time \( \theta^\alpha \) as follows:

\[
\theta^\alpha := (t_0 + \eta_k) \land \inf\{s \geq t_0, |X^{\alpha,t_0,x_0}_s - x_0| \geq \eta_k\}
\]

Applying Itô’s lemma to \( \phi(t, X^{\alpha,t_0,x_0}_t) \), we obtain:

\[
\phi(t, X^{\alpha,t_0,x_0}_t) = \phi(\theta^\alpha, X^{\alpha,t_0,x_0}_{\theta^\alpha}) - \int_t^{\theta^\alpha} \nu^{\alpha_s}(s, X^{\alpha,t_0,x_0}_s)ds - \int_t^{\theta^\alpha} (\sigma \frac{\partial \phi}{\partial x})(s, X^{\alpha,t_0,x_0}_s)dW_s - \int_t^{\theta^\alpha} B^\alpha \phi(s, X^{\alpha,t_0,x_0}_s)N(ds, de)
\]

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where $\psi^\alpha(s, x) := \frac{\partial}{\partial s} \phi(s, x) + L^\alpha \phi(s, x)$.

Note that $\left( \phi(s, X_s^{\alpha_0, x_0}), (\sigma \frac{\partial \phi}{\partial x})(s, X_s^{\alpha_0, x_0}), B^{\alpha_0} \phi(s, X_s^{\alpha_0, x_0}); s \in [t_0, \theta^\alpha] \right)$ is the solution of the BSDE associated with the driver process $-\psi^\alpha(s, X_s^{\alpha_0, x_0})$ and terminal value $\phi(\theta^\alpha, X_{\theta^\alpha}^{\alpha_0, x_0})$. By inequality (4.5) on the drivers and Proposition A.4 in the Appendix imply:

$$\delta^\alpha$$

To simplify notation, set $B$ where $f$

and $g$

for each $s,X_t^{\alpha_0, x_0}$.

Note that $(\alpha, s, X_s^{\alpha_0, x_0}) = \phi(s, X_s^{\alpha_0, x_0})$, $(\sigma \frac{\partial \phi}{\partial x})(s, X_s^{\alpha_0, x_0}), B^{\alpha_0} \phi(s, X_s^{\alpha_0, x_0}) \geq \epsilon$.

By the definition of $\psi^\alpha$, inequality (4.4) can be written:

$$-\psi^\alpha(s, X_s^{\alpha_0, x_0}) \geq f \left( \alpha, s, X_s^{\alpha_0, x_0}, \phi(s, X_s^{\alpha_0, x_0}), (\sigma \frac{\partial \phi}{\partial x})(s, X_s^{\alpha_0, x_0}), B^{\alpha_0} \phi(s, X_s^{\alpha_0, x_0}) \right) + \epsilon$$

for each $s \in [t_0, \theta^\alpha]$. The above inequality gives a relation between the drivers $-\psi^\alpha(s, X_s^{\alpha_0, x_0})$ and $f(\alpha, \cdot)$ of two BSDEs. Now, since the minimum $(t_0, x_0)$ is strict, there exists $\gamma_\epsilon$ such that:

$$u(t, x) - \psi(t, x) \leq -\gamma_\epsilon$$

on $[0, T] \times \mathbb{R} \setminus B_{\eta}(t_0, x_0)$, where $B_{\eta}(t_0, x_0)$ is the open ball centered at $(t_0, x_0)$ with radius $\eta$. We have

$$\phi(\theta, X_{\theta}^{\alpha_0, x_0}) = \phi(t, X_t^{\alpha_0, x_0})1_{t < \theta^\alpha} + \phi(\theta, X_{\theta}^{\alpha_0, x_0})1_{t \geq \theta^\alpha}.$$

To simplify notation, set $\delta_\epsilon := \min(\epsilon, \gamma_\epsilon)$. Using the definition of $\theta^\alpha$, it follows that for each $t \in [t_0, \theta^\alpha]$

$$\phi(t, X_t^{\alpha_0, x_0}) \geq (h(t, X_t^{\alpha_0, x_0}) + \delta_\epsilon)1_{t < \theta^\alpha} + (u(\theta, X_{\theta}^{\alpha_0, x_0}) + \delta_\epsilon)1_{t = \theta^\alpha} \quad \text{a.s.}$$

(4.6)

This, together with inequality (4.5) on the drivers and Proposition A.4 in the Appendix imply:

$$\phi(t_0, x_0) \geq Y_{t_0, \theta^\alpha}^{\alpha_0, x_0} [h(t, X_t^{\alpha_0, x_0})1_{t < \theta^\alpha} + u(\theta, X_{\theta}^{\alpha_0, x_0})1_{t = \theta^\alpha}] + \delta_\epsilon K,$$

(4.7)

where $K$ is a positive constant which only depends on $T$ and the Lipschitz constant of $f$.

On the other hand, there exists an $\frac{\delta K}{2}$ - optimal control $\alpha^\varepsilon \in A_{t_0}^{\theta^\alpha}$ for $u(t_0, x_0) = \sup_{\alpha \in A_{t_0}^{\theta^\alpha}} Y_{t_0, T}^{\alpha, x_0}$, that is satisfying $u(t_0, x_0) \leq Y_{t_0, \theta^\alpha}^{\alpha^\varepsilon, x_0} + \frac{\delta K}{2}$. By the flow property of $Y_{t_0, T}^{\alpha_0, x_0}$ and Proposition A.1, we get:

$$\phi(t_0, x_0) = u(t_0, x_0) \leq Y_{t_0, \theta^\alpha}^{\alpha^\varepsilon, t_0} [h(s, X_s^{\alpha^\varepsilon, t_0})1_{s < \theta^\alpha} + u^{\alpha^\varepsilon}(\theta, X_{\theta}^{\alpha^\varepsilon, t_0})1_{s = \theta^\alpha}] + \frac{\delta_\epsilon K}{2}.$$

(4.8)
Hence, by using inequality $u^{a^x} \leq u$, and the comparison theorem for RBSDEs we derive:

$$
\phi(t, 0, x) = u(t, 0, x) \leq Y^{\alpha^x, t, 0} [h(s, X^{\alpha^x, t, 0}) 1_{s < \theta^{a^x}} + u(\theta^{a^x}, X^{\alpha^x, t, 0}) 1_{s = \theta^{a^x}}] + \frac{\delta_x K}{2}.
$$

(4.9)

As (4.7) holds for all $\alpha \in A^{V_0}$, and hence in particular for $\alpha = \alpha^x$, we get a contradiction.

- We now prove that $u$ is a viscosity supersolution of (4.7).

We need here the dynamic programming principle.

Let $(t_0, x_0) \in [0, T] \times \mathbb{R}$ and $\phi \in C^{1,2}([0, T] \times \mathbb{R})$ be such that $\phi(t_0, x_0) = u(t, 0, x_0)$ and $\phi(t, x) \leq u(t, x), \forall (t, x) \in [0, T] \times \mathbb{R}$.

Without loss of generality, we can suppose that the maximum is strict in $(t_0, x_0)$. Since the solution $(Y^{\alpha^x, t, 0})$ stays above the obstacle, for each $\alpha \in A$, we have:

$$
u(t_0, x_0) \geq h(t_0, x_0).
$$

Our aim is to show that:

$$
\inf_{\alpha \in A} \left( -\frac{\partial}{\partial t} \phi(t_0, x_0) - L^\alpha \phi(t_0, x_0) - f \left( \alpha, t_0, x_0, \phi(t_0, x_0), (\sigma \frac{\partial \phi}{\partial x})(t_0, x_0), B^\alpha \phi(t_0, x_0) \right) \right) \geq 0.
$$

Let us suppose that:

$$
\inf_{\alpha \in A} \left( -\frac{\partial}{\partial t} \phi(t_0, x_0) - L^\alpha \phi(t_0, x_0) - f \left( \alpha, t_0, x_0, \phi(t_0, x_0), (\sigma \frac{\partial \phi}{\partial x})(t_0, x_0), B^\alpha \phi(t_0, x_0) \right) \right) < 0.
$$

By continuity, we can suppose that there exists $\alpha \in A$, $\epsilon > 0$ and $\eta_\epsilon > 0$ such that:

$\forall (t, x)$ such that $t_0 \leq t \leq t_0 + \eta_\epsilon < T$ and $|x - x_0| \leq \eta_\epsilon$, we have:

$$
- \frac{\partial}{\partial t} \phi(t, x) - L^\alpha \phi(t, x) - f \left( \alpha, t, x, \phi(t, x), (\sigma \frac{\partial \phi}{\partial x})(t, x), B^\alpha \phi(t, x) \right) \leq -\epsilon.
$$

(4.10)

Now, we introduce the state process $X^{\alpha^x, t, 0}$ associated to the above constant control $\alpha$ and define the stopping time $\theta$ as:

$$
\theta := (t_0 + \eta_\epsilon) \wedge \inf \{ s \geq t_0, |X^{\alpha^x, t, 0} - x_0| \geq \eta_\epsilon \}.
$$

By Itô's lemma applied to $\phi(s, X^{\alpha^x, t, 0})$, we have that

$$
\left( \phi(s, X^{\alpha^x, t, 0}), (\sigma \frac{\partial \phi}{\partial x})(s, X^{\alpha^x, t, 0}), B^\alpha \phi(s, X^{\alpha^x, t, 0}); s \in [t_0, \theta] \right)
$$

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is the solution of the BSDE associated with terminal time \( \theta \), terminal value \( \phi(\theta, X_{\theta}^{\alpha,t_0,x_0}) \) and driver \( -\psi(\alpha, s, X_{\alpha}^{\alpha,t_0,x_0}) \). The definition of the stopping time \( \theta \) and inequality (4.10) lead to:

\[
- \psi(\alpha, s, X_{\alpha}^{\alpha,t_0,x_0}) \leq f(\alpha, s, X_{\alpha}^{\alpha,t_0,x_0}, \phi(\alpha, s, X_{\alpha}^{\alpha,t_0,x_0}), (\sigma \partial \phi / \partial x)(s, X_{\alpha}^{\alpha,t_0,x_0}), B^{\alpha}(s, X_{\alpha}^{\alpha,t_0,x_0})),
\]

for \( t_0 \leq s \leq \theta \) ds \( \otimes \) dP-a.s. Now, since the maximum \((t_0, x_0)\) is strict, there exists \( \gamma_\epsilon \) (which depends on \( \eta_\epsilon \)) such that:

\[
u(t, x) \geq \phi(t, x) + \gamma_\epsilon \text{ on } [0, T] \times \mathbb{R} \setminus B_{\eta_\epsilon}(t_0, x_0) \tag{4.12}
\]

which implies

\[
\phi(\theta, X_{\theta}^{\alpha,t_0,x_0}) \leq u(\theta, X_{\theta}^{\alpha,t_0,x_0}) - \gamma_\epsilon.
\]

Hence, using inequality (4.11) on the drivers, together with the comparison theorem for BSDEs and Proposition (A.3), we derive that:

\[
\phi(t_0, x_0) = \mathcal{E}^{\alpha} \phi(\theta, X_{\theta}^{\alpha,t_0,x_0}) \leq \mathcal{E}^{\alpha} u(\theta, X_{\theta}^{\alpha,t_0,x_0} - \gamma_\epsilon) \leq \mathcal{E}^{\alpha} u(\theta, X_{\theta}^{\alpha,t_0,x_0}) - \gamma_\epsilon K. \tag{4.13}
\]

Now, by the dynamic programming principle (Th. 3.8), we have:

\[
\phi(t_0, x_0) = u(t_0, x_0) \geq \mathcal{E}^{\alpha} u(\theta, X_{\theta}^{\alpha,t_0,x_0}). \tag{4.14}
\]

From (4.13) and (4.14), we get a contradiction. \( \square \)

### 4.2 Uniqueness of the viscosity solution

Now we provide a uniqueness result for (4.1). This result is obtained under more restrictive assumptions that the existence one, namely we assume:

**Assumption 4.3.**

1. \( f(\alpha, s, X^{\alpha,x_0}(\omega), y, z, k) := \overline{f}(\alpha, s, X^{\alpha,x_0}(\omega), y, z, \int_{\mathbb{R}} k(e) \gamma(x, e) \nu(de)) I_{s \geq t} \)

where \( \overline{f} : A \times [0, T] \times \mathbb{R}^4 \rightarrow \mathbb{R} \) satisfies:

\[
(i) \ |\overline{f}(\alpha, t, x, 0, 0, 0)| \leq C, \text{ for any } x \in \mathbb{R}, \ t \in [0, T], \ \alpha \in A.
\]

\[
(ii) \ |\overline{f}(\alpha, t, x, y, z, k) - \overline{f}(\alpha, t, x', y', z', k')| \leq C(|y - y'| + |z - z'| + |k - k'|), \text{ for any } x, x' \in \mathbb{R}, \ t \in [0, T], \ y, y' \in \mathbb{R}, \ z, z' \in \mathbb{R}, \ k, k' \in \mathbb{R}, \ \alpha \in A.
\]

\[
(iii) \ k \rightarrow \overline{f}(\alpha, t, x, y, z, k) \text{ is non-decreasing, for any } (\alpha, t, x, y, z, k) \in A \times [0, T] \times \mathbb{R}^4.
\]
2. For each \( R > 0 \), there exists a continuous function \( m_R : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( m_R(0) = 0 \) and 
\[
|f(\alpha, t, x, v, p, q) - f(\alpha, t, y, v, p, q)| \leq m_R(|x - y|(1 + |p|)),
\]
for any \( t \in [0, T] \), \( |x|, |y|, |v| \leq R, p, q \in \mathbb{R}, \alpha \in A \).

3. \(|\gamma(x, e) - \gamma(y, e)| \leq C|x - y|(1 + |e|^2)\) and \( \gamma(x, e) \geq 0 \), for any \( x, y \in \mathbb{R}, e \in \mathbb{R}^* \).

4. There exists \( r > 0 \) such that for any \( x \in \mathbb{R}, t \in [0, T], u, v \in \mathbb{R}, p \in \mathbb{R}, l \in \mathbb{R}, \alpha \in A \):
\[
|f(\alpha, t, x, v, p, l) - f(\alpha, t, u, p, l)| \geq r(u - v) \text{ when } u \geq v.
\]

5. The maps \( b, \sigma, \beta \) are continuous with respect to \( \alpha \).

6. \(|h(t, x)| + |g(x)| \leq C\), for any \( x \in \mathbb{R}, t \in [0, T] \).

**Theorem 4.4** (Comparison principle). Under the above hypothesis, if \( U \) is a bounded viscosity subsolution and \( V \) is a bounded viscosity supersolution of the obstacle problem \([4.1]\), then 
\( U(t, x) \leq V(t, x) \), for each \((t, x) \in [0, T] \times \mathbb{R}\).

The proof is given in Appendix.

**Corollary 4.5** (Uniqueness). Under the above hypothesis, there exists a unique solution of the obstacle problem \([4.1]\) in the class of bounded continuous functions.

Note first that the previous results still hold in the case of a multidimensional state process, Brownian motion and Poisson random measure. The proofs are exactly the same but the notation is more complicated.

## 5 Examples

**Pricing of American options with model ambiguity.** We consider a financial market with two risky assets. Here, \( \alpha \) represents some ambiguity parameter process and the associated state process \( X^\alpha = (X^\alpha_1, X^\alpha_2)' \) represents the price process of the two risky assets. It is supposed to follow the dynamics:
\[
\begin{align*}
dX^\alpha_{t,1} & = r(t, X^\alpha_t, \alpha_t)dt + \sigma^1(t, X^\alpha_t, \alpha_t)dW_t \\
\end{align*}
\[
\begin{align*}
dX^\alpha_{t,2} & = r(t, X^\alpha_t, \alpha_t)dt + \sigma^2(t, X^\alpha_t, \alpha_t)dW_t + \beta(t, X^\alpha_t, \alpha_t)\tilde{N}(dt),
\end{align*}
\]
where $r(t, x, \alpha)$ represents the interest rate and $\tilde{N}_t = N_t - \lambda t$ with $N$ a Poisson process of intensity $\lambda$.

We denote by $\varphi_t = (\varphi_t^1, \varphi_t^2)$ the number of risky assets held in the portfolio at time $t$ and by $(V_t^\alpha)$ the associated wealth process. The strategy is supposed to be self-financing, that is the wealth process satisfies the following dynamics:

$$dV_t^\alpha = r(t, X_t^\alpha, \alpha_t)V_t^\alpha dt + \varphi_t^1 \sigma(t, X_t^\alpha, \alpha_t)dW_t + \varphi_t^2 \beta(t, X_t^\alpha, \alpha_t)\tilde{N}(dt).$$

where $\sigma = (\sigma^1, \sigma^2)'$, $\sigma^1, \sigma^2 > 0$ and the coefficients $\sigma^1, \sigma^2, r, \beta, \beta^{-1}, (\sigma^1)^{-1}, (\sigma^2)^{-1}$ are bounded.

Let $T > 0$ and let $\xi \in L^2(\mathcal{F}_T)$. We consider a contingent claim with maturity $T$ and payoff $\xi$. There exists an unique process $(V^\alpha, \varphi^\alpha) \in \mathcal{S}^2 \times (\mathbb{H}^2)^2$ solution of the following BSDE:

$$
\begin{cases}
-dV_t^\alpha = f(\alpha_t, t, X_t^\alpha, V_t^\alpha, (\varphi_t^\alpha)' \sigma_t, \varphi_t^2 \beta_t)dt - \varphi_t^1 \sigma_t dW_t - \varphi_t^2 \beta_t \tilde{N}(dt) \\
V_T^\alpha = \xi,
\end{cases}
$$

where $V_t^\alpha$ is the price of claim $\xi$ and $\varphi_t^\alpha$ is the hedging strategy.

Suppose we consider some nonlinear constraints on the wealth such as a borrowing interest rate $R_t$ greater than the bond rate $r$, taxes (see [13]), or the presence of a large investor. Then, the dynamics of the wealth associated with strategy $\varphi$ are no longer linear and can be written as follows:

$$-dV_t^\alpha = f(\alpha_t, t, X_t^\alpha, V_t^\alpha, (\varphi_t^\alpha)' \sigma_t, \varphi_t^2 \beta_t)dt - \varphi_t^1 \sigma_t dW_t - \varphi_t^2 \beta_t \tilde{N}(dt).$$

For each maturity $T$ and terminal payoff $\xi \in L^2(\mathcal{F}_T)$, there exists an unique solution $(V^\alpha, \varphi^\alpha) \in \mathcal{S}^2 \times (\mathbb{H}^2)^2$ of the following BSDE:

$$
\begin{cases}
-dV_t^\alpha = f(\alpha_t, t, X_t^\alpha, V_t^\alpha, (\varphi_t^\alpha)' \sigma_t, \varphi_t^2 \beta_t)dt - (\varphi_t^\alpha)' \sigma_t dW_t - \varphi_t^2 \beta_t \tilde{N}(dt) \\
V_T^\alpha = \xi,
\end{cases}
$$

where $V_t^\alpha$ is the price of claim $\xi$ and $\varphi_t^\alpha$ is the hedging strategy.

In this framework, the robust price at time $t$ of an American option with payoff $\overline{h}(s, X_t^{\alpha, t, x})$, $s \geq t$, is given by

$$u(t, x) := \sup_{\tau \in T_t} \sup_{\alpha \in \mathcal{A}_t} \mathcal{E}^{\alpha, t, x}_t [\overline{h}(\tau, X_\tau^{\alpha, t, x})],$$

where $x$ denotes the price vector of the risky assets at time $t$. 

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A mixed optimization problem of recursive utility of terminal wealth. We consider a financial market with one risky asset. The control \( \alpha \) represents here the proportion of wealth invested in the risky asset. We suppose that \( h(t, x) = g(x) \), where \( g \) corresponds to a utility function. For each initial time \( t \in [0, T] \) and initial condition \( x \in \mathbb{R}^*_+ \), for each portfolio strategy \( \alpha \in A^t \), the state process \( X^{\alpha, t, x} \) defined by (2.1) represents the wealth process. If the agent decides to stop at time \( \tau \in \mathcal{T}^t \), the associated recursive utility function is defined as \( E^{\alpha, t, x}_t \left[ \xi^t \left[ g(X^{\alpha, t, x}_\tau) \right] \right] \). His aim is to maximize his recursive utility of wealth over all portfolio strategies \( \alpha \in A^t \) and stopping times \( \tau \in \mathcal{T}^t \). The value function at time \( t \) is then given by

\[
\sup_{\tau \in \mathcal{T}^t \left[ t, T \right]} \sup_{\alpha \in A^t} E^{\alpha, t, x}_t \xi^t \left[ g(X^{\alpha, t, x}_\tau) \right].
\]

A Appendix

We state below that the equality \( Y^{\alpha, t, x}_t \xi^t \xi^t \left[ h(s, X^{\alpha, t, x}_s) \right]_{t \leq s \leq T} = u^\alpha(t, x) \) (see (2.6)) still holds with \( x \) replaced by a random variable \( \xi \in L^2(F_t) \).

**Proposition A.1.** Let \( \xi \in L^2(F_t) \). Then, for all \( \alpha \in A^t \),

\[
Y^{\alpha, t, \xi}_t \xi^t \left[ h(s, X^{\alpha, t, \xi}_s) \right]_{t \leq s \leq T} = u^\alpha(t, \xi) \text{ a.s.} \quad (A.1)
\]

**Proof.** The proof is based on classical arguments. Let us consider a simple random variable \( \xi^{(n)} \in L^2(F_t) \), that is such that there exists \( n \in \mathbb{N}, A_1, A_2, ..., A_n \in F_t, a_1, a_2, ..., a_n \in \mathbb{R} \) such that

\[
\xi^{(n)} = \sum_{i=1}^n a_i 1_{A_i} \text{ a.s.}
\]

By using the definitions of \( \xi^{(n)} \) and \( u^\alpha(t, x) \), we obtain:

\[
Y^{\alpha, t, \xi^{(n)}}_t \xi^t \left[ h(s, X^{\alpha, t, \xi^{(n)}}_s) \right]_{t \leq s \leq T} = \sum_{i=1}^n 1_{A_i} Y^{\alpha, t, a_i}_t \xi^t \left[ h(s, X^{\alpha, t, a_i}_s) \right]_{t \leq s \leq T} = \sum_{i=1}^n 1_{A_i} u^\alpha(t, a_i) = u^\alpha(t, \xi^{(n)}) \text{ a.s.} \quad (A.2)
\]

The continuity of \( u^\alpha \) with respect to \( x \) implies that:

\[
u^\alpha(t, \xi^{(n)}) \rightarrow_{n \to \infty} u^\alpha(t, \xi) \text{ a.s.} \quad (A.3)
\]

Now, the estimates on \( X \) with respect to the initial condition and the a priori estimates on \( Y \) (Proposition 5.1 in [8]) lead to:

\[
\|u^\alpha(t, \xi^{(n)}) - Y^{\alpha, t, \xi}_t \xi^t \left[ h(s, X^{\alpha, t, \xi}_s) \right]_{t \leq s \leq T}\|^2 \leq C \|\xi^{(n)} - \xi\|^2 \text{ a.s.} \quad (A.4)
\]

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which implies that
\[
|u^n(t, \xi^{(n)}) - Y_{t}^{\alpha,t,\xi}[h(s, X_{s}^{\alpha,t,\xi})_{t \leq s \leq T}]|^2 \to \frac{n \to \infty}{0} \text{ a.s.} \tag{A.5}
\]
From (A.3), (A.5) and the uniqueness of the limit, we derive that:
\[
u^n(t, \xi) = Y_{t}^{\alpha,t,\xi}[h(s, X_{s}^{\alpha,t,\xi})_{t \leq s \leq T}] \text{ a.s.} \tag{A.6}
\]

Using the a priori estimates on RBSDEs and standard computations, one can show the following property:

**Proposition A.2** (A continuity result). Let \( T > 0 \). Let \((\xi_t)\) be an RCLL process in \( \mathcal{S}^2 \). Let \( f \) be a given standard driver. Let \( \{\theta^n, n \in \mathbb{N}\} \) be a family of stopping times in \( \mathcal{T}_0 \), converging a.s. to \( 0 \in \mathcal{T}_0 \) as \( n \) tends to \( \infty \). Let \((\xi^n, n \in \mathbb{N})\) be a sequence of random variables such that \( E[\text{ess} \sup_n (\xi^n)^2] < +\infty \), and for each \( n, \xi^n \) is \( \mathcal{F}_{\theta^n} \)-measurable. Suppose that \( \xi^n \) converges a.s. to an \( \mathcal{F}_\theta \)-measurable random variable \( \xi \) as \( n \) tends to \( \infty \). Let \( Y^n_t := Y_{t,\theta^n}(\xi^n_{s<\theta^n} + \xi^n_{s=\theta^n}) \); \( Y_t := Y_{t,\theta}(\xi_{s<\theta} + \xi_{s=\theta}) \) be the solutions of the RBSDEs associated with driver \( f \), terminal time \( \theta^n \) (resp. \( \theta \)) and obstacle \( \xi^n_{s<\theta^n} + \xi^n_{s=\theta^n} \) (resp. \( \xi_{s<\theta} + \xi_{s=\theta} \)). Then, for each \( t \), \( Y^n_t \) converges to \( Y_t \) a.s.

**Proof of Theorem 4.4** The proof follows the one of Th. 4.2 in [8]. The only difference appears in the last part of the proof. Using the same notation, Inequalities (4.11) in that paper are now replaced by

\[
\begin{aligned}
\inf_{\alpha \in A} F(\alpha, t^n, x^n, U(t^n, x^n)), a, \overline{p}, X, I_1^1[a, t^n, x^n, \varphi_x] \\
+ I_2^1[a, t^n, x^n, \overline{p}, U], I_2^1[a, t^n, x^n, \varphi_x] + I_2^2[a, t^n, x^n, U] \leq 0
\end{aligned}
\]
\[
\inf_{\alpha \in A} F(\alpha, s^n, y^n, V(s^n, y^n)), a, \overline{q}, Y, I_2^1[a, s^n, y^n, \varphi_y] \\
+ I_1^1[a, s^n, y^n, \overline{q}, V], I_2^1[a, s^n, y^n, \varphi_y] + I_2^2[a, s^n, y^n, V] \geq 0
\]

where
\[
F(\alpha, t, x, u, a, p, X, l_1, l_2) := -\alpha - \frac{1}{2}\sigma^2(x, \alpha)X - b(x, \alpha) * p - l_1 - f(\alpha, t, x, u, p\sigma(x), l_2) \tag{A.8}
\]
and \( \varphi_x \) denotes the function \((t, x) \mapsto \varphi(t, x, s^n, y^n)\) and \( \varphi_y \) the function \((s, y) \mapsto \varphi(t^n, x^n, s, y)\). Consequently, \( A \) compact together with (A.7) imply that there exists \( \alpha_{e,n} \in A \) such that:

\[
\begin{aligned}
\inf_{\alpha \in A} F(\alpha, t^n, x^n, U(t^n, x^n)), a, \overline{p}, X, I_1^1[\alpha, t^n, x^n, \varphi_x] \\
+ I_1^2[\alpha, t^n, x^n, \overline{p}, U], I_2^1[\alpha, t^n, x^n, \varphi_x] + I_2^2[\alpha, t^n, x^n, U] \leq 0
\end{aligned}
\]
\[
\inf_{\alpha \in A} F(\alpha, s^n, y^n, V(s^n, y^n)), a, \overline{q}, Y, I_1^1[\alpha, s^n, y^n, \varphi_y] \\
+ I_1^2[\alpha, s^n, y^n, \overline{q}, V], I_2^1[\alpha, s^n, y^n, \varphi_y] + I_2^2[\alpha, s^n, y^n, V] \geq 0
\]

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The rest of the proof can be adapted from the one of Th. 4.2 in [8].

Extensions of comparison results for BSDEs and RBSDEs with Jumps. We first provide an extension of the comparison theorem for BSDEs given in [23] which formally states that if two terminal conditions \( \xi_1, \xi_2 \) satisfy \( \xi_1 \geq \xi_2 + \varepsilon \), then the associated solutions \( X^1 \) and \( X^2 \) satisfy \( X^1 \geq X^2 + \varepsilon K \).

**Proposition A.3.** Let \( t_0 \in [0, T] \) and let \( \theta \in \mathcal{T}_{t_0} \). Let \( \xi_1 \) and \( \xi_2 \in L^2(\mathcal{F}_\theta) \). Let \( f_1 \) be a driver. Let \( f_2 \) be a Lipschitz driver with Lipschitz constant \( C > 0 \). For \( i = 1, 2 \), let \((X^i_t, \pi^i_t, l^i_t)\) be a solution in \( S^2 \times H^2 \times H^2 \) of the BSDE associated with driver \( f_i \), terminal time \( \theta \) and terminal condition \( \xi_i \):

\[-dX^i_t = f_i(t, X^i_t, \pi^i_t, l^i_t)dt - \pi^i_t dW_t - \int_{\mathbb{R}^*} l^i_t(u)\tilde{N}(dt, du); \quad X^i_\theta = \xi_i. \tag{A.10}\]

Assume that there exists a bounded predictable process \((\gamma_t)\) such that \( dt \otimes dP \otimes \nu(de) \)-a.s. \( \gamma_t(\varepsilon) \geq -1 \) and \( |\gamma_t(\varepsilon)| \leq C(1 \wedge |\varepsilon|) \), and such that

\[f_2(t, X^2_t, \pi^2_t, l^2_t) - f_2(t, X^1_t, \pi^1_t, l^1_t) \geq \langle \gamma_t, l^1_t - l^2_t \rangle_\nu, \quad t_0 \leq t \leq \theta, \quad dt \otimes dP \text{ a.s.} \tag{A.11}\]

Suppose also that

\[
\begin{align*}
\xi_1 &\geq \xi_2 + \varepsilon \text{ a.s.}\\
f_1(t, X^1_t, \pi^1_t, l^1_t) &\geq f_2(t, X^1_t, \pi^1_t, l^1_t) \quad t_0 \leq t \leq \theta, \quad dt \otimes dP \text{ a.s.}
\end{align*}
\]

where \( \varepsilon \) is a real constant. Then, for each \( t \in [t_0, \theta] \),

\[X^1_t - X^2_t \geq \varepsilon e^{-CT} \text{ a.s.}\]

**Proof.** From inequality (4.22) in the proof of the Comparison Theorem in [23], we derive that

\[X^1_{t_0} - X^2_{t_0} \geq e^{-CT}\mathbb{E}[H_{t_0, \theta} \varepsilon |\mathcal{F}_{t_0}] \text{ a.s.},\]

where \( C \) is the Lipschitz constant of \( f_2 \), and \((H_{t_0,s})_{s \in [t_0,T]}\) is the square integrable non negative martingale satisfying

\[dH_{t_0,s} = H_{t_0,s} \left[ \beta_s dW_s + \int_{\mathbb{R}^*} \gamma_s(u)\tilde{N}(ds, du) \right]; \quad H_{t_0,t_0} = 1, \]

\((\beta_s)\) being a predictable process bounded by \( C \). The result follows.
From this property and the characterization of the solution of the reflected BSDE as the value function of an optimal stopping problem (see Theorem 3.2 in [23]), we derive the following comparison result between the solution of a BSDE and the solution of an RBSDE:

**Proposition A.4.** Let $t_0 \in [0, T]$ and let $\theta \in \mathcal{T}_{t_0}$. Let $\xi_1 \in L^2(\mathcal{F}_\theta)$. Let $f_1$ be a driver and let $f_2$ be a Lipschitz driver with Lipschitz constant $C > 0$ satisfying Assumption 3.1 in [23]. Let $(\xi^2_t) \in \mathcal{S}^2$. Let $(Y^2_t)$ be the solution of the RBSDE associated to $f_2$ and obstacle $(\xi^2_t)$, and let $(X^1_t, \pi^1_t, l^1_t)$ be a solution of the BSDE associated to $f_1$, terminal time $\theta$ and terminal condition $\xi^1$. Suppose that

\[
\begin{aligned}
f_1(t, X^1_t, \pi^1_t, l^1_t) &\geq f_2(t, X^1_t, \pi^1_t, l^1_t), \quad t_0 \leq t \leq \theta, \ dt \otimes dP \text{ a.s.} \\
X^1_t &\geq \xi^2_t + \varepsilon, \quad t_0 \leq t \leq \theta \text{ a.s.,}
\end{aligned}
\]

(A.12)

where $\varepsilon$ is a real constant. Then for each $t \in [t_0, \theta]$, we have $X^1_t \geq Y^2_t + \varepsilon e^{-CT}$ a.s.

**Proof.** Let $t \in [t_0, \theta]$. Recall that $Y^2_t = \text{ess}\sup_{\tau \in \mathcal{T}_{[t, \theta]}} \mathcal{E}_{t, \tau}^2(\xi^2_\tau)$. Now, by Proposition A.3 for each $\tau \in \mathcal{T}_{[t, \theta]}$, $X^1_t \geq \mathcal{E}_{t, \tau}^2(\xi^2_\tau) + e^{-CT}\varepsilon$ a.s. By taking the supremum over $\tau \in \mathcal{T}_{[t, \theta]}$, the result follows. \qed

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