Supplementary Information for

Omnimodal topological polarization of bilayer networks: analysis in the Maxwell limit and experiments on a 3D-printed prototype

Mohammad Charara, James McInerney, Kai Sun, Xiaoming Mao and Stefano Gonella

Corresponding Author: James McInerney or Stefano Gonella.
E-mail: jmcinern@umich.edu or sgonella@umn.edu

This PDF file includes:

- Supplementary text
- Figs. S1 to S5
- Legends for Movies S1 to S3
- SI References

Other supplementary materials for this manuscript include the following:

- Movies S1 to S3
Supporting Information Text

Discussion of notation

The main text studies topological polarization in the context of dynamics, and hence requires bridging two disparate notation systems: one from the physics community and one from the mechanics community. This is achieved by utilizing the mechanics notation wherever possible. This appendix discusses the distinctions between the two notations.

In the physics literature, when we have a mechanical systems composed of point masses connected by harmonic springs, the matrix capturing the kinetic interaction between the masses is referred to as the dynamical matrix, typically represented by the symbol $D$. This matrix maps the displacements of the lattice sites to the forces balancing at the sites, $F = DU$, where $F$ is an array of forces and $U$ is an array of displacements. In the mechanics literature, this object is more typically called a stiffness matrix, typically represented by the symbol $K$. Moreover, in certain dynamics contexts, the dynamical matrix refers to the quantity $D = -\omega^2M + K$, which is the matrix that links forces and displacements in the frequency domain (Fourier-transformed equations of motion) for a dynamical problem. This discrepancy is reinforced by the fact that the matrix of spring constants (i.e., the coefficients of Hooke’s law) are written $K$ in the physics literature. This typically corresponds to the constitutive matrix, written as $C$, in the mechanics literature which is not to be confused with the symbol for the compatibility matrix that maps from vertex displacements to bond extensions, $E = CU$, in the physics literature. Thus, this compatibility matrix is instead written as $B$, as used in mechanics, and the spring constants are written as $E$ to signify their connection to the Young’s modulus of beam elements. Finally, the wavenumbers in the physics literature are typically written as $q$ or $k$, whereas the main text uses $\xi$ to accomodate the mechanics notation.

Topological polarization of planar kagome lattices

The main text leverages analytical results for the topological polarization of planar kagome lattices to determine the omnimodal topological polarization of mirror-symmetric kagome bilayers. This appendix reviews the known results for the planar kagome lattice.

The planar kagome lattice consists of cells that contain three vertices and six edges so that it is critically-coordinated in the bulk. These vertices are labeled in Fig. S1A as $r_1$, $r_2$, and $r_3$ along with the two lattice vectors $\ell_1$ and $\ell_2$. The six total edges of the unit cell consist of the three that connect the vertices within the cell to one another and edges that connect the vertices to adjacent cells: $r_2(0, 0)$ connects to $r_1(1, 0)$, $r_3(0, 0)$ connects to $r_2(-1, 1)$, and $r_1(0, 0)$ connects to $r_2(0, -1)$. These particular intercellular connections, in contrast to the dashed lines shown in Fig. S1A which belong to adjacent cells, are chosen to achieve a symmetric gauge.

The corresponding Bloch-periodic compatibility matrix has six rows and six columns, and its determinant is a second-order polynomial in the two Bloch factors $z_{1,2} = e^{\pm i2\pi}$. Thus, the planar kagome lattice generically exhibits two topological edge modes, which can be understood by the fact that two edges (constraints) are removed from each cell on the boundary in a finite-size kagome geometry. The localization of these zero modes is shown via bulk analysis by the strictly negative decay rates, $\kappa_2 \leq 0$, along the $\ell_2$-direction, in Fig. S1B and the constant winding number, $\nu_2$, computed by integrating over the wavenumber, $\xi$, in the $\ell_2$-direction from $-\pi$ to $\pi$, in Fig. S1C for all values of the wavenumber, $\xi$, in the $\ell_1$-direction. Importantly, the topological phase space of kagome lattices can be characterized analytically, as shown in Ref. (1).

Flat bands in coplanar Maxwell bilayers

The main text presents mirror-symmetric kagome bilayers with a choice of interlayer connections that exhibit lines of bulk modes in the absence of height modulations within the layers. Here, we present an alternative choice of interlayer connections that instead yields surfaces of bulk modes known as flat bands.

The key distinction between coplanar bilayers with lines of bulk zero modes and those with surfaces of bulk zero modes is that the interlayer connections in the former constrain an infinite number of vertices whereas in the latter they constrain a finite number of vertices. For the kagome bilayer, this second condition arises for the following interlayer connections: $r_1(0, 0)$ connects to $r_2^1(0, 0)$ connects to $r_2^1(1, -1)$ connects to $r_1^2(1, -1)$ connects to $r_2(0, -1)$ connects to $r_3(0, -1)$ connects to $r_1(0, 0)$. Thus, the vertices can simultaneously displace out-of-plane at any wavevector provided that their displacements are identical to one another:

$$\delta_1^2 = \delta_2^1 = \delta_3^1 e^{-i(\xi_1-\xi_2)} = \delta_4^1 e^{-i(\xi_1-\xi_2)} = \delta_5^2 e^{i\xi_2} = \delta_6^1 e^{i\xi_2}. \tag{1}$$

Since such zero modes exist at arbitrary wavevectors, the lowest band is entirely flat. Importantly, the introduction of small height modulations to make the bilayer noncoplanar yields decay rates which cannot be described as a perturbation to the flat band, as shown in Fig. S2, which makes it more difficult to use this type of connectivity to achieve fully polarized Maxwell bilayers.

Computational Beam Model: Details

In this work, we choose to work with Timoshenko beams to ensure that the results are robust even for short connections whose slenderness ratio may exceed the bounds for which Euler-Bernoulli beam theory is acceptable.

In 3D, the stiffness matrix for a single beam, $K_b$, encompasses stiffness contributions associated with axial, shear, bending, and torsional deformation. This is captured by reordering rows and columns and partitioning $K_b$ into minors that highlight the
Fig. S1. Topological polarization of the planar kagome lattice. (A) View of the lattice and specification of the unit cell with vertex positions $r_1 = (0, 0)$, $r_2 = (1, 0.3)$, $r_3 = (0.5, \sqrt{3}/2 + 0.6)$, and lattice vectors $\ell_1 = (2, 0)$, $\ell_2 = (1, \sqrt{3})$. The dashed lines signify edges that belong to adjacent cells. (B) The decay rate of zero modes, $\kappa_2$, along the $\ell_2$-direction as a function of the wavenumber, $\xi_1$, in the $\ell_1$-direction. (C) The winding number, $\nu_2$, computed by integrating over the wavenumber, $\xi_2$, in the $\ell_2$-direction from $-\pi$ to $\pi$ as a function of the wavenumber, $\xi_1$, in the $\ell_1$-direction.
Fig. S2. Decay rates for a non-coplanar bilayer whose interlayer connections exhibit a flat band in the coplanar limit, which inhibits prediction of where the two remaining zero modes localize.
contributions of different mechanisms. Recalling that \( K_b \) links an array \( f_m \) of nodal forces and moments, to an array \( [u_m \ \theta_m]^T \) of nodal displacements and rotations, the matrix partition can be written as

\[
K_{b12 \times 12} = \begin{pmatrix} K_{A2 \times 2} & 0 & 0 \\ 0 & K_{S4 \times 4} & K_{M4 \times 6} \\ 0 & K_{T} & K_{B6 \times 6} \end{pmatrix}.
\]  

where \( K_A \) is the contribution governing axial stiffness, \( K_S \) and \( K_B \) are square matrices controlling shear, and torsion and bending, respectively, and \( K_M \) is a rectangular mixed matrix coupling shear and bending:

\[
K_A = \begin{pmatrix} \frac{E A}{L} & -\frac{E A}{L} \\ -\frac{E A}{L} & \frac{E A}{L} \end{pmatrix}
\]  

\[
K_S = \begin{pmatrix} \frac{12EI_y \mu_y}{L^3} & 0 & \frac{12EI_y \mu_z}{L^3} \\ 0 & -\frac{12EI_y \mu_z}{L^3} & 0 \\ \frac{12EI_y \mu_z}{L^3} & 0 & -\frac{12EI_y \mu_z}{L^3} \end{pmatrix}
\]  

\[
K_B = \begin{pmatrix} \frac{GJ}{L} & 0 & 0 & \frac{GJ}{L} \\ 0 & \frac{EI_z (3\mu_y + 1)}{L} & 0 & \frac{EI_z (3\mu_y - 1)}{L} \\ \frac{GJ}{L} & 0 & \frac{EI_z (3\mu_y - 1)}{L} & 0 \\ 0 & \frac{EI_z (3\mu_y - 1)}{L} & 0 & \frac{EI_z (3\mu_y + 1)}{L} \end{pmatrix}
\]  

\[
K_M = \begin{pmatrix} 0 & \frac{6EI_y \mu_z}{L^2} & 0 & 0 & \frac{6EI_y \mu_y}{L^2} \\ \frac{6EI_y \mu_z}{L^2} & 0 & 0 & \frac{6EI_y \mu_z}{L^2} & 0 \\ 0 & \frac{6EI_y \mu_z}{L^2} & 0 & \frac{6EI_y \mu_y}{L^2} & 0 \\ 0 & 0 & -\frac{6EI_y \mu_y}{L^2} & 0 & \frac{6EI_y \mu_z}{L^2} \\ 0 & 0 & \frac{6EI_y \mu_z}{L^2} & 0 & \frac{6EI_y \mu_y}{L^2} \end{pmatrix}
\]  

where \( E \) is the Young's modulus, \( A = \pi R^2 \) is the cross-sectional area of the beam, \( L \) is the length of the beam element, \( I \) is the second moment of area of the cross section about the \( \hat{y} \) and \( \hat{z} \) axes \( (I = \pi R^4 / 4 \text{ with } R \text{ the cross section radius}) \), \( \mu \) is the shear correction factor for Timoshenko beams \( (2) \), \( G \) is the shear modulus, \( J \) is the polar second moment of area of the cross section \( (J = \pi R^4 / 2) \), \( A \) is the cross-sectional area of the beam. Note that because the beam element matrices are calculated in a local coordinate system where the \( \hat{x} \) axis is along the length of the beam, they must be rotated upon assembly into a global system matrix.

As previously mentioned, the stiffness matrix of a beam element is calculated by carrying out the integral \( K_b = \int_L B_b^T k B_b dL \) over the length of the element. \( k \) is diagonal constitutive matrix of elastic coefficients relating an array \( \sigma \) of stresses and moments to an array \( \epsilon \) of strains, shears, and curvatures \( (\text{i.e. } \sigma = k \epsilon) \) and is given by

\[
k = \begin{pmatrix} \frac{E A}{L} & 0 & 0 & 0 & 0 \\ 0 & GAk & 0 & 0 & 0 \\ 0 & 0 & GAk & 0 & 0 \\ 0 & 0 & 0 & \frac{GJ}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{EI}{3} \end{pmatrix}.
\]  

where \( k \) is the area shear correction factor. This matrix features elastic constants for axial deformation (row 1), shear deformation (rows 2 and 3), torsion (row 4), and bending (rows 5 and 6). \( B_b \) is an elemental matrix which captures a similar role in a beam element as the compatibility matrix does in a spring-and-mass bond. \( B_b \) contains derivatives of shape functions [that interpolate the elemental displacements and rotations between nodes]. These shape functions are linear in axial and torsional deformation, cubic in flexural deformation, and quadratic in rotations.

As we change the radius of the beam's cross section, the elastic coefficients for axial and flexural deformation scale at different rates – the ratio of the former to the latter is \( \propto 1/R^2 \). Thus, even though we never lose the inherent effects of clamped boundary conditions and the storage of bending energy of the beams, as we reduce the cross sectional radius \( R \), we asymptotically approach the dominance of axial deformability typical of the spring-mass case.

In the Timoshenko beam framework, the mass matrix of the beam elements is a non-diagonal consistent matrix, coupling degrees of freedom in the same manner as \( K_b \). This matrix is calculated by integrating the same elemental shape functions as \( K_b \), multiplied by material and cross-sectional properties, over the length of the element. The resulting matrix includes contribution from axial, lateral, rotatory (associated with the tilt of the beam cross-sections), and polar (twist about the beam axis) inertial effects. A detailed account of this derivation, as well as that for the beam's stiffness matrix, is provided in Ref. (2).
Wave Propagation: Experiments and Full-Scale Simulations at frequencies away from the polarized modes

We visualize the frequency selectivity of the polarized behavior by performing experiments and simulations at carrier frequencies away from the floppy modes at \( \xi = \pi \). Results are shown in Figs. S4 and S5 for simulations and experiments carried out at carrier frequencies of 300 and 1300 Hz, respectively. Excitations at 300 Hz reveal nearly identical wavefields when we switch the excitation edge from floppy to nonfloppy, with bulk-like characteristics activated from either side. Although the DFTs from the floppy edge do show some activation of the mode branches endowed with polarized character, the activation occurs at much longer wavelengths, where the decay rate of these modes is negligible (making them resemble the bulk modes), overall resulting in bulk-like behavior dominating the response. Experiments performed with a carrier frequency at 1300 Hz also show largely symmetric behavior, although we still see the persistence of some localization when exciting from the floppy edge. This is likely due to the fact that the excitation energy is spread over a band of frequencies due to the windowing applied to the burst, which causes the edge of the main lobe and the side lobes to excite modes in the topological region.
Fig. S3. Displacement vs cell index for 100-cell beam-discretized supercell, showing that, even at much longer supercell lengths, beams with the AR used in the lattice for this work do not produce six modes exhibiting floppy edge localization. Unlike the low-AR or spring and mass counterparts, modes 5 and 6 remain bulk-like when we increase the supercell to much longer length, revealing that these are not just long decay length modes that simply appear bulk-like due to the relatively short size of the 10-cell supercell. Thus, their residence in the bulk band, as shown in the band diagram, is confirmed.
Fig. S4. Wavefield and DFT plots for: experiments (A-B) and simulations (C-D) with an in-plane excitation at the rigid edge, experiments (E-F) and simulations (G-H) with an in-plane excitation at the floppy edge, experiments (I-J) and simulations (K-L) with an out-of-plane excitation at the rigid edge, and experiments (M-N) and simulations (O-P) with an out-of-plane excitation at the floppy edge, all carried out at a carrier frequency of 300 Hz (dashed blue lines). Colorbars in panel M and N apply to all wavefields and contours in the figure, respectively, with the former representing the displacement intensity in the wavefield and the latter highlighting modal activation on the excited edge. While the wavefields capture the signature of wave propagation on the lattice surface, the insets of D, H, L, and P allow to appreciate propagation through the 3D structure of the bilayer. Note that the wavefields and contours are normalized by the highest value in their respective data sets. While the main text includes experimental figures highlighting edge-selectivity in the topologically polarized lattice, this image highlights frequency selectivity, an equally important ingredient in ensuring this behavior is not spectrally ubiquitous, providing evidence for the topological nature of the activated modes.
Fig. S5. Wavefield and DFT plots for: experiments (A-B) and simulations (C-D) with an in-plane excitation at the rigid edge, experiments (E-F) and simulations (G-H) with an in-plane excitation at the floppy edge, experiments (I-J) and simulations (K-L) with an out-of-plane excitation at the rigid edge, and experiments (M-N) and simulations (O-P) with an out-of-plane excitation at the floppy edge, all carried out at a carrier frequency of 1300 Hz (dashed blue lines). Colorbars in panel M and N apply to all wavefields and contours in the figure, respectively, with the former representing the displacement intensity in the wavefield and the latter highlighting modal activation on the excited edge. While the wavefields capture the signature of wave propagation on the lattice surface, the insets of D, H, L, and P allow to appreciate propagation through the 3D structure of the bilayer. Note that the wavefields and contours are normalized by the highest value in their respective data sets. While the main text includes experimental figures highlighting edge-selectivity in the topologically polarized lattice, this image highlights frequency selectivity, an equally important ingredient in ensuring this behavior is not spectrally ubiquitous, providing evidence for the topological nature of the activated modes.
Movie S1. Wavefield evolutions for in-plane and out-of-plane excitation at 900 Hz on floppy and nonfloppy edges.

Movie S2. Wavefield evolutions for in-plane and out-of-plane excitation at 300 Hz on floppy and nonfloppy edges.

Movie S3. Wavefield evolutions for in-plane and out-of-plane excitation at 1300 Hz on floppy and nonfloppy edges.

References

1. C Kane, T Lubensky, Topological boundary modes in isostatic lattices. Nat. Phys. 10, 39–45 (2014).

2. H Karadeniz, MP Saka, V Togan, Finite element analysis of space frame structures in Stochastic Analysis of Offshore Steel Structures. (Springer), pp. 1–119 (2013).