Space of spaces as a metric space

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Abstract

In spacetime physics, we frequently need to consider a set of all spaces ('universes') as a whole. In particular, the concept of 'closeness' between spaces is essential. However there has been no established mathematical theory so far which deals with a space of spaces in a suitable manner for spacetime physics.

Based on the scheme of the spectral representation of geometry, we construct a space $S_N$, which is a space of all compact Riemannian manifolds equipped with the spectral measure of closeness. We show that $S_N$ can be regarded as a metric space. We also show other desirable properties of $S_N$, such as the partition of unity, locally-compactness and the second countability. These facts show that the space $S_N$ can be a basic arena for spacetime physics.

Running Title: 'Space of spaces as a metric space' or 'Space of spaces'

1 Introduction

In the course of the development of theoretical physics, mathematics has been an efficient language for a precise formulation and analysis of the problems. However sometimes physics goes ahead, namely we are occasionally
forced to face with a problem in physics for which appropriate mathematical language has not yet been established. In this case, we cannot even state the problem properly, though its importance is obvious. Though troublesome, it is such a situation that can be a strong motivation for a new development of mathematics, which in turn would help further progress in theoretical physics.

The theory presented here can be regarded as of this kind. In spacetime physics, it often happens that not a single space (‘universe’), but a set of spaces should be considered as a whole. However there is no workable mathematical theory suitable for handling such a situation.

Perhaps the most famous example of this category is the ‘spacetime foam’ picture due to Wheeler [13]: At near the Planck scale \( l_{pl} = \left( \frac{G \hbar}{c^3} \right)^{1/2} \approx 10^{-33} \text{ cm} \), it is anticipated that there are drastic topological fluctuations of spacetime taking place (‘spacetime foam’), because of the quantum effect on spacetime. Now as the observational energy scale \( E \) goes down, the finer topological structure of scale less than \( E^{-1} \) would be averaged in some manner, and the effective geometry would seem simpler than the original one. If \( E \) is decreased to the much lower energy scale, say \( 10^{15} \text{ GeV} \), almost all of the topological handles would be smoothed out, resulting in the simplest structure as we experience usually.

In the ‘spacetime foam’ picture briefly described above, we find the concept of a ‘set of spaces’ appears twice:

1. Topological fluctuations. We are tacitly considering a set of spaces with various topologies.

2. Scale-dependent topology [12, 8, 9]

It is appropriate to explain here the scale-dependent topology in some detail: In the standard mathematical context, topology by definition is a scale-free concept, i.e. it represents the geometrical properties that are independent from the sizes of the structures. However topology in spacetime physics is quite different: Here the observational energy scale \( E \) naturally enters into a discussion, so that geometrical structures of scales less than \( E^{-1} \) are of no significant meaning. Thus we need to consider effective topology as a function of \( E \). In other words, we need to establish a procedure of ‘topological approximation’ at the energy scale \( E \). In this manner, not only the topology (in the ordinary sense) of a handle, but also its size should be taken into
account in spacetime physics. Hence it is desirable to establish a suitable framework in which local geometry and global topology are treated on the same footing [10].

As another example which requires the concept of a ‘set of spaces’ we mention a fundamental problem in cosmology. The real geometry of our universe is very complicated so that we can perceive how our universe is only through cosmological models. Thus cosmology in principle requires a mapping procedure from reality to a model. By analyzing the observational data, we choose the optimal model among a set of models which is most compatible with the data. Here a mapping from reality to the model, or smoothing out the reality to get the model is taking place, but so far, this process is not understood well. We need a scheme for analyzing the mapping procedure itself quantitatively and for judging the validity of the model choice. For this purpose, we need to establish a suitable language which describes the ‘closeness’ between two spaces (reality and a model). Thus we should give a definite meaning to a space of all spaces, supplying a suitable distance between spaces, for further understanding of the problem of model fitting in cosmology [11].

The fundamental problems in spacetime physics like the examples mentioned above have been frequently discussed so far, without firm foundations for the space of spaces, effective topology, topological approximation, and so on.

As perhaps the first attempt to describe the scale-dependent topology quantitatively, the scattering cross-sections of a small handle of various topologies in 2-dimensional space have been investigated and it has been analyzed how the cross-sections are influenced by the topology of a handle and the energy scale of a probe [8, 9].

Based on this preliminary investigation, the more systematic scheme for handling space structures has been introduced: The spectral representation of geometrical structures along with the spectral distance between spaces [10]. It echoes a famous question in Riemannian geometry, ‘Can one hear the shape of a drum?’ [3]. The basic idea behind the spectral representation is very clear: We use the ‘sounds’ of a space to characterize the geometrical structures of the space. To be more specific, we use the set of eigenvalues of an elliptic operator (typically, the Laplacian) \( \{ \lambda_n \}_{n=1}^{\infty} \) to characterize geometry. (We confine ourselves to the case of spatially compact spaces for definiteness.) Then one can also introduce a measure of closeness \( d_N(\mathcal{G}, \mathcal{G}') \) between two
geometries $G$ and $G'$ by comparing $\{\lambda_n\}_{n=1}^N$ for $G$ and $\{\lambda'_n\}_{n=1}^N$ for $G'$. (We can treat the cut-off number $N$ as a running parameter.)

There are several advantages for the spectral representation:

1. The spectra $\{\lambda_n\}_{n=1}^\infty$ are countable number positive quantities and they are easy to handle.

2. The spectra $\{\lambda_n\}_{n=1}^\infty$ contain the information on both the local geometry and global topology of a space. In other words, $\{\lambda_n\}_{n=1}^\infty$ are suitable quantities for treating the local and global geometry in a unified manner.

3. On dimensional grounds, the lower (higher) spectrum corresponds to the larger (smaller) scale behavior of geometry. For instance, if we compare two sets of spectra up to the $N$-th eigenvalue, $\{\lambda_n\}_{n=1}^N$ and $\{\lambda'_n\}_{n=1}^N$, we are in effect comparing the corresponding two geometries, $G$ and $G'$, neglecting the small-scale behavior of order $o(\lambda_n^{-1/2})$. In this manner, the spectra are suitable for describing the scale-dependent behavior of geometry.

4. The spectra $\{\lambda_n\}_{n=1}^\infty$ are spatially diffeomorphism invariant quantities, and the spectral distance $d_N$ also possesses this property.

5. The spectral distance $d_N$ between spaces is constructed from purely ‘internal’ concepts, i.e. the spectra are defined within spaces themselves, and they are independent from the way of embeddings into some other space. In this sense $d_N$ is a physical measure of closeness between spaces.

The basic properties of the spectral distance have been investigated. Among several possibilities, one particular choice of the spectral distance $d_N$ is especially important since it can be derived from quite a general argument of introducing a distance, and since it can be related to the reduced density matrix element in quantum cosmology under some circumstances. At the same time, however, it turned out that this form of $d_N$ does not satisfy the triangle inequality \[14\]. Though the triangle inequality is not of absolute necessity, its utility clearly makes the arguments efficient and compact, and furthermore it is compatible with our intuitive notion of ‘closeness’. Thus further investigations on this point have been awaited.
In this paper we will show that the failure of the triangle inequality of the spectral distance $d_N$ is only a mild one, and in fact we will see that its slight modification, $\tilde{d}_N$, recovers the inequality. With the help of $\tilde{d}_N$, which is a distance in a rigorous sense, we can investigate the properties of the space $\mathcal{S}_N$, the space of all spaces equipped with $d_N$. Then we will see that, as topological spaces, the space $\mathcal{S}_N$ is equivalent to the space of all spaces equipped with $\tilde{d}_N$, which is a metric space. Thus $\mathcal{S}_N$ is a metrizable space and it provides us with a notion of ‘closeness’ between spaces in a manner that is compatible with our intuitive notion of ‘closeness’. In this way it is justified to treat $\mathcal{S}_N$ as a metric space, provided that we resort to $\tilde{d}_N$ whenever the triangle inequality is needed in the arguments. We will also show that the space $\mathcal{S}_N$ possesses several desirable properties, such as the partition of unity, locally-compactness, the second countability. These observations support us to regard the space $\mathcal{S}_N$ as a basic arena for spacetime physics.

In Section 2, we will introduce the spectral representation and the spectral distance $d_N$. In Section 3, we will consider the space $\mathcal{S}_N$, the space of all spaces equipped with $d_N$, and will see that $\mathcal{S}_N$ can be regarded as a metric space. We will also investigate several other properties of $\mathcal{S}_N$. Section 4 is devoted to several discussions.

2 The Spectral Distance

Let us first recollect the attempts in Riemannian geometry to define ‘closeness’ between spaces.

There is the Gromov-Hausdorff distance $d_{GH}(X, Y)$ between two compact metric spaces $X$ and $Y$ [2, 7]. It is defined by means of isometric embeddings of $X$ and $Y$ into another metric space $Z$ such as

$$d_{GH}(X, Y) := \inf_{\varphi_1, \varphi_2} d_H(\varphi_1(X), \varphi_2(Y)) .$$

Here $d_H$ denotes the Hausdorff distance on $Z$; $\varphi_1 : X \hookrightarrow Z$ and $\varphi_2 : Y \hookrightarrow Z$ are isometric embeddings of $X$ and $Y$ respectively, into $Z$. In other words,

\footnote{Let $(Z, d)$ be a metric space. For $A \subset Z$ and $\epsilon > 0$, we define the $\epsilon$-neighborhood of $A$ as $B(A, \epsilon) := \{x \in Z | d(x, A) < \epsilon\}$, where $d(x, A) := \inf_{y \in A} d(x, y)$. Then the Hausdorff distance between two subsets of $Z$, $A_1$ and $A_2$, is defined as $d_H(A_1, A_2) := \inf\{\epsilon | \epsilon > 0, A_1 \subset B(A_2, \epsilon), A_2 \subset B(A_1, \epsilon)\}$.}
we search for the optimal isometric embeddings such that $X$ and $Y$ overlap ‘as close as possible’ in $Z$, and the distance is defined as the order of failure in overlapping.

Another quantity related to ‘closeness’ between spaces is a norm of Riemannian manifolds due to Petersen [7]. Let $(U_{\lambda}, \phi_{\lambda})_{\lambda \in \Lambda}$ be an atlas of a Riemannian manifold $(\Sigma, h)$. If each patch $U_{\lambda}$ is chosen to be sufficiently small, the metric tensors w.r.t. (with respect to) the atlas do not change so much within each chart $(U_{\lambda}, \phi_{\lambda})$, namely, locally it looks like Euclidean to some extent. Now the larger the size of charts becomes the more the metric tensors vary within each chart. The norm due to Petersen can be explained as the maximum size of the admissible charts under the condition that the variation of the metric tensors within each chart lies within a given range. Its precise definition is very complicated and we do not go further here. This norm measures how close a Riemannian manifold is to the Euclidean space, but it does not provide us with the distance between two manifolds.

Though these quantities play a significant role in the convergence theory of Riemannian geometry [7], it seems too abstract and complicated to be directly applied to spacetime physics.

Thus let us focus on another measure of closeness between spaces which would be suitable for spacetime physics. We make use of the eigenvalues of an elliptic operator on a space (or ‘spectra’ hereafter). The spectra contain the information of both local geometry and global topology, and the difference in geometry reflects on the difference in the spectra. Thus, to state symbolically, “we ‘hear’ the shape of the universe”. Let us call such a representation of geometry in terms of the spectra the spectral representation, for brevity [10].

Now let $\text{Riem}$ be a space of all $D$-dimensional, compact Riemannian manifolds without boundaries. Let $\mathcal{G} = (\Sigma, h), \mathcal{G}' = (\Sigma', h') \in \text{Riem}$. (We regard them as models of spaces and not spacetimes.) For definiteness we consider only the Laplacian operator $\Delta$ here as an elliptic operator, though the arguments below are quite universal.

2 Geometrical structures are classified into two categories: local geometry and global geometry (global topology), though they are related to each other and there is no clear separation between them. Throughout this paper ‘geometry’ indicates both, i.e. the integrated geometrical properties. We use the symbols $\mathcal{G}, \mathcal{G}'$, etc. to represent geometry as a whole in this broad sense.
Setting the eigenvalue problem on each manifold
\[ \Delta f = -\lambda f , \]
the set of eigenvalues (numbered in increasing order) is obtained; \( \{ \lambda_m \}_{m=0}^{\infty} \) for \( G \) and \( \{ \lambda'_n \}_{n=0}^{\infty} \) for \( G' \).

The first option that one can imagine easily is probably
\[ d_{\text{Euclid}}(G, G') := \sqrt{\sum_{n=1}^{N} (\lambda_n - \lambda'_n)^2} , \]
which is similar to the Euclidean distance on \( \mathbb{R}^N \). However from the viewpoint of physics, it is unsatisfactory for two reasons.

(1) The spectra \( \{ \lambda_n \}_{n=1}^{\infty} \) have the physical dimension \([\text{Length}^{-2}]\), so that \( d_{\text{Euclid}} \) also has a dimension \([\text{Length}^{-2}]\). It introduces a scale into the theory, which is not desirable. For instance, the statement \( d_{\text{Euclid}} << 1 \) becomes meaningless, and rather we should say \( d_{\text{Euclid}} << l_{\text{pl}}^2 \), if we choose \( l_{\text{pl}} \) as a typical length scale. In this way a particular scale (e.g. \( l_{\text{pl}} \)) enters into the discussion. This is unsatisfactory, considering the fundamental nature of the theory of ‘closeness’ between spaces.

(2) Remembering the arguments of scale-dependent topology (see §1), it is clear that, in spacetime physics, the larger scale behavior of geometry is of more importance than the smaller scale behavior. However, looking at the expression of \( d_{\text{Euclid}} \), the measure \( d_{\text{Euclid}} \) counts the smaller scale behavior (i.e. \( \lambda_n \) with larger \( n \)) with more importance, which is unsatisfactory.

Thus a simple difference \( \lambda_n - \lambda'_n \) is not appropriate for our purpose. Rather we should take the ratio \( \frac{\lambda'_n}{\lambda_n} = 1 + \frac{\delta \lambda_n}{\lambda_n} \), which implies that the difference \( \delta \lambda_n := \lambda'_n - \lambda_n \) in the lower spectrum is counted with more importance.

Hence we introduce a measure of closeness between \( G \) and \( G' \) as
\[ d_N(G, G') = \sum_{n=1}^{N} F \left( \frac{\lambda'_n}{\lambda_n} \right) , \tag{1} \]
where \( F(x) \) \((x > 0)\) is a suitably chosen function which satisfies \( F \geq 0 \), \( F(1/x) = F(x) \), \( F(y) > F(x) \) if \( y > x \geq 1 \). Most of the cases we also require
\( \mathcal{F}(1) = 0 \). (However, see an exceptional case below (\( \mathcal{F}_2 \)).) Note that the zero modes \( \lambda_0 = \lambda_0' = 0 \) are excluded from the summation in Eq.(1). On dimensional grounds, \( \lambda_n \) with the larger number \( n \) reflects the smaller scale behavior of the geometry. Therefore the cut-off number \( N \) indicates the scale up to which \( \mathcal{G} \) and \( \mathcal{G}' \) are compared. In other words the difference between the two geometries in the scale of order \( o \left( \frac{1}{\sqrt{\lambda_N}} \right) \) is neglected. Treating \( N \) as a running parameter, \( d_N(\mathcal{G}, \mathcal{G}') \) as a function of \( N \) indicates the coarse grained similarity between \( \mathcal{G} \) and \( \mathcal{G}' \) at each scale. In this way the spectral measure of closeness gives a natural basis for analyzing the scale-dependent behavior of geometries. Here we remember that such a quantitative, scale-dependent description of the geometrical structures (the scale-dependent topology in particular) is very essential for developing spacetime physics (see §1) [8, 9].

Now there are several possibilities for the choice of \( \mathcal{F} \) in Eq.(1). Its detailed form should be determined according to the features of geometry that we are interested in. Among several possibilities, however, there is one especially interesting choice for \( \mathcal{F} \), which is \( \mathcal{F}_1(x) = \frac{1}{2} \ln \frac{1}{2} \left( \sqrt{x} + 1/\sqrt{x} \right) \). By this choice, \( d_N \) can be related to the reduced density matrix element in quantum cosmology. For the details on the derivation and physical interpretation of this choice, we refer the reader to Ref.[10]. Thus we get [10]

\[
\begin{align*}
d_N(\mathcal{G}, \mathcal{G}') &= \frac{1}{2} \sum_{n=1}^{N} \ln \frac{1}{2} \left( \sqrt{\frac{\lambda_n'}{\lambda_n}} + \sqrt{\frac{\lambda_n}{\lambda_n'}} \right),
\end{align*}
\]

This measure of closeness\(^3\) possesses the following properties:

(I) \( d_N(\mathcal{G}, \mathcal{G}') \geq 0 \), and \( d_N(\mathcal{G}, \mathcal{G}') = 0 \iff \mathcal{G} \sim \mathcal{G}' \), where \( \sim \) means equivalent in the sense of isospectral manifolds\(^4\).

(II) \( d_N(\mathcal{G}, \mathcal{G}') = d_N(\mathcal{G}', \mathcal{G}) \)

but it does not satisfy the triangle inequality [10],

(III) \( d_N(\mathcal{G}, \mathcal{G}') + d_N(\mathcal{G}', \mathcal{G}'') \not\geq d_N(\mathcal{G}, \mathcal{G}'') \).

\(^3\) We will see below that it is justified to regard \( d_N \) as a distance if a suitable care is taken. Until then, however, let us call it a ‘measure of closeness’ for safety.

\(^4\) Two non-isometric Riemannian manifolds \( \mathcal{G} \) and \( \mathcal{G}' \) are called isospectral manifolds when \( \{\lambda_m\}_{m=0}^\infty = \{\lambda'_n\}_{n=0}^\infty \) [3]. However, the weaker condition \( \{\lambda_m\}_{m=0}^N \equiv \{\lambda'_n\}_{n=0}^N \) is enough instead of \( \{\lambda_m\}_{m=0}^\infty \equiv \{\lambda'_n\}_{n=0}^\infty \) for the present purpose.
However, it turns out that the breakdown of the triangle inequality is only a mild one in the following sense: A universal constant $a > 0$ can be chosen such that $d_N(G, G') := d_N(G, G') + a$ satisfies the triangle inequality. Here $a$ is independent of $G$ and $G'$.

Indeed there is another option for $F$ as $F_2(x) := \log_2(\sqrt{x} + 1/\sqrt{x})$. Here $c$ is an arbitrary positive constant. Note that $F_2 = \frac{2c}{N\ln 2}(F_1 + \frac{1}{2}\ln 2)$. Then a modified measure of closeness becomes

$$\tilde{d}_N(G, G') = \sum_{n=1}^{N} \log_2 \left( \sqrt{\frac{\lambda_n'}{\lambda_n}} + \sqrt{\frac{\lambda_n}{\lambda_n'}} \right).$$

This measure $\tilde{d}_N$ satisfies

(I') $\tilde{d}_N(G, G') \geq c$, and $\tilde{d}_N(G, G') = c \iff G \sim G'$, where $\sim$ means equivalent in the sense of isospectral manifolds,

(II') $\tilde{d}_N(G, G') = \tilde{d}_N(G', G)$,

(III') $\tilde{d}_N(G, G') + \tilde{d}_N(G', G'') \geq \tilde{d}_N(G, G'')$.

In this case the triangle inequality holds ((III')) but the lower bound of $\tilde{d}_N$ is $c(>0)$ and not zero ((I')). Note that $\tilde{d}_N = \frac{2c}{N\ln 2}d_N + c$. Thus, the precise form of (III) turns out to be

$$d_N(G, G') + d_N(G', G'') + \frac{N}{2}\ln 2 \geq d_N(G, G'').$$

Looking at (I)-(III) and (I')-(III'), we see that the measures of closeness introduced here are generalizations of an ordinary distance.\footnote{It is desirable to construct the theory of a generalized metric space which is characterized by the generalized axioms of distance:}

\begin{enumerate}
    \item[(a)] $d(p, q) \geq 0$, and $d(p, q) = 0 \iff p = q$.
    \item[(b)] $d(p, q) = d(q, p)$.
    \item[(c)] $d(p, q) + d(q, r) + c \geq d(p, r)$, where $c > 0$ is a universal constant independent of $p, q$ and $r$.
\end{enumerate}
For later use, we also pay attention to another choice for $F$ in Eq.(1): we can choose $F_0(x) := \frac{1}{2} \ln \max(\sqrt{x}, 1/\sqrt{x})$, which is a slight modification of $F_1$. Then Eq.(1) becomes

$$\bar{d}_N(G, G') = \frac{1}{2} \sum_{n=1}^{N} \ln \max \left( \sqrt{\lambda_n}, \sqrt{\lambda_n'}, \sqrt{\lambda_n'/\lambda_n} \right).$$

(3)

It is clear that $\bar{d}_N$ satisfies all of the axioms of a distance. In particular it satisfies the triangle inequality

$$(\text{III}^\prime\prime) \quad \bar{d}_N(G, G') + \bar{d}_N(G', G'') \geq \bar{d}_N(G, G''),$$

because of the relation $\max(x, 1/x) \max(y, 1/y) \geq \max(xy, 1/xy)$ for $x, y > 0$. Therefore $\bar{d}_N$ is a distance.

3 The space of all spaces equipped with the spectral distance

We introduce an $r$-ball centered at $G$ defined by $d_N$ as

$$B(G, r; d_N) := \{ G' \in \text{Riem}/\sim | d_N(G, G') < r \}.$$

Here $d_N$ is the one defined by Eq.(2) and $\sim$ indicates the identification of isospectral manifolds.

We also consider an $r$-ball centered at $G$ defined by $\bar{d}_N$ as

$$B(G, r; \bar{d}_N) := \{ G' \in \text{Riem}/\sim | \bar{d}_N(G, G') < r \}.$$

Below we will show that the set of all balls defined by $d_N$ forms a basis of topology (Lemma 6 below), and that the topology generated by this set of balls is equivalent to the topology generated by the set of all balls defined by $\bar{d}_N$ (Theorem 1 below). Here we note that the latter topology makes the space $\text{Riem}/\sim$ a metric space. Thus the space

$$\mathcal{S}_N^0 := (\text{Riem}, d_N)/\sim$$

turns out to be a metrizable space, which is an idealistic property. (We will consider its completion $\mathcal{S}_N$ after establishing Theorem 1.)

Now we prepare a series of Lemma’s before showing Theorem 1.
Lemma 1

(1) $d_N(\mathcal{G}, \mathcal{G}') \leq \bar{d}_N(\mathcal{G}, \mathcal{G}')$.

(2) For $\forall B(\mathcal{G}, r; d_N)$ there exists $r'(> 0)$ s.t. $B(\mathcal{G}, r'; \bar{d}_N) \subset B(\mathcal{G}, r; d_N)$.

(3) For $\forall B(\mathcal{G}, r; \bar{d}_N)$ there exists $r'(> 0)$ s.t. $B(\mathcal{G}, r'; d_N) \subset B(\mathcal{G}, r; \bar{d}_N)$.

Proof:

(1): It immediately follows from the inequality $\frac{1}{2}(p+1/p) \leq \max(p, 1/p)$ for $p > 0$.

(2): Indeed, it follows that $B(\mathcal{G}, r; \bar{d}_N) \subset B(\mathcal{G}, r; d_N)$ due to Lemma 1 (1).

(3): Suppose there exist $\mathcal{G}$ and $r > 0$ s.t. $B(\mathcal{G}, \epsilon; d_N) \not\subset B(\mathcal{G}, r; \bar{d}_N)$ for $\forall \epsilon > 0$. For a fixed $\epsilon > 0$, take $\mathcal{G}'$ s.t. $\mathcal{G}' \in B(\mathcal{G}, \epsilon; d_N) \setminus B(\mathcal{G}, r; \bar{d}_N)$. Then $d_N(\mathcal{G}, \mathcal{G}') < \epsilon$, which implies that $\frac{1}{2} \ln \left(\frac{\sqrt{\lambda_n}}{\lambda_n} + \frac{\sqrt{\lambda_n}}{\lambda_n}\right) < \epsilon$ for $n = 1, 2, \cdots, N$. (Here $\{\lambda_n\}_{n=0}^\infty$ and $\{\lambda_n'\}_{n=0}^\infty$ are the spectra corresponding to $\mathcal{G}$ and $\mathcal{G}'$, respectively.) Then it easily follows that $1 \leq \max(\sqrt{\lambda_n}, \sqrt{\lambda_n'}) < \exp 2\epsilon + \sqrt{\exp 4\epsilon - 1}$ for $n = 1, 2, \cdots, N$. Thus we get a relation $r \leq \bar{d}_N(\mathcal{G}, \mathcal{G}') < \frac{N}{2} \ln(\exp 2\epsilon + \sqrt{\exp 4\epsilon - 1})$. However, this inequality cannot hold for $\epsilon$ s.t. $0 < \epsilon < \frac{1}{2} \ln(\cosh \frac{2\epsilon}{N})$, which is a contradiction. $\square$

Lemma 2

For $\forall \mathcal{G}'' \in B(\mathcal{G}', \epsilon; \bar{d}_N)$, it follows that $0 \leq \frac{|\lambda_n'' - \lambda_n'|}{\lambda_n'} < \exp 4\epsilon - 1$ ($n = 1, 2, \cdots, N$). Here $\{\lambda_n'\}_{n=0}^\infty$ and $\{\lambda_n''\}_{n=0}^\infty$ are the spectra corresponding to $\mathcal{G}'$ and $\mathcal{G}''$, respectively.

Proof:

The assumption implies that $0 \leq \frac{1}{2} \ln \prod_{n=1}^N \max \left(\sqrt{\lambda_n'}, \sqrt{\lambda_n''}\right) < \epsilon$, or $1 \leq \prod_{n=1}^N \max \left(\sqrt{\lambda_n'}, \sqrt{\lambda_n''}\right) < \exp 2\epsilon$. Thus $1 \leq \max \left(\sqrt{\lambda_n'}, \sqrt{\lambda_n''}\right) < \exp 2\epsilon$ ($n = 1, 2, \cdots, N$). From this inequality, either $0 \leq \frac{\lambda_n'' - \lambda_n'}{\lambda_n'} < \exp 4\epsilon - 1$ or $0 \leq \frac{\lambda_n - \lambda_n'}{\lambda_n'} < \exp 4\epsilon - 1$ follows. Then, it is straightforward to get $0 \leq \frac{|\lambda_n'' - \lambda_n'|}{\lambda_n'} < \exp 4\epsilon - 1$. $\square$
Lemma 3

Let $\{\lambda_n\}_{n=0}^{\infty}$, $\{\lambda'_n\}_{n=0}^{\infty}$ and $\{\lambda''_n\}_{n=0}^{\infty}$ are the spectra for $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}''$, respectively.

Then $d_N(\mathcal{G},\mathcal{G}'') = d_N(\mathcal{G},\mathcal{G}') + \frac{1}{4} \sum_{n=1}^{N} \frac{\lambda'_n - \lambda_n}{\lambda_n + \lambda''_n} \cdot \frac{\lambda''_n - \lambda'_n}{\lambda'_n} + R$. Here $R = \sum_{n=1}^{N} c_n(\frac{\lambda'_n - \lambda_n}{\lambda'_n})^2$, with $c_n$'s being finite constants. Furthermore $R$ is bounded as $|R| < \sum_{n=1}^{N} (\frac{\lambda'_n - \lambda_n}{\lambda'_n})^2$.

Proof:

(1°) Let $f(x) := \frac{1}{2} \ln \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ ($x > 0$). By Taylor-Maclaurin theorem, it follows that $f(x + \delta x) = f(x) + \frac{1}{2} \sqrt{x} \frac{1}{\sqrt{x}} \frac{\delta x}{x} + c(x + \xi \delta x)(\frac{\delta x}{x})^2$ with $0 < \xi < 1$. Here $c(x)$ is a smooth function for $x > 0$.

Indeed, setting $\sqrt{x} = \exp(\theta)$, $f(x)$ can be represented as $f(x) = \frac{1}{2} \ln \cosh(\theta)$.

Then $\frac{df(x)}{dx} = \frac{1}{4 \cosh^2(\theta)}$, and $\frac{d^2f(x)}{dx^2} = \frac{1}{x^2} F_2(\theta)$, where $F_2(\theta) = \frac{1}{4} F_2'(\theta) - F_1(\theta) = \frac{1}{8} \left(\frac{1}{\cosh^2(\theta)} - 2 \tanh(\theta)\right)$. Thus $\tilde{F}_2(x)$, which is $F_2$ regarded as a function of $x$, is a well-defined function of $\sqrt{x} + 1/\sqrt{x}$ and $\sqrt{x} - 1/\sqrt{x}$ and it is smooth for $x > 0$. Then we can set $c(x) = \frac{1}{2 \sqrt{x}} \tilde{F}_2(x)$.

We also note that $|c(x)| = \frac{1}{2} |F_2(\theta)| < \frac{1}{16} \left(\frac{1}{\cosh^2(\theta)} + 2 |\tanh(\theta)|\right) < \frac{3}{16} < 1$.

(2°) Applying the result of (1°) to $d_N(\mathcal{G},\mathcal{G}'') = \sum_{n=1}^{N} f(\frac{\lambda'_n}{\lambda_n}) = \sum_{n=1}^{N} f(\frac{\lambda''_n}{\lambda'_n} + \frac{\lambda'_n - \lambda_n}{\lambda'_n})$, the claim follows. □

Lemma 4

For $\forall \mathcal{G}' \in B(\mathcal{G}, r; d_N)$, there exists $\epsilon > 0$ s.t. $B(\mathcal{G}', \epsilon; \tilde{d}_N) \subset B(\mathcal{G}, r; d_N)$.

Proof:

Let $\rho := d_N(\mathcal{G}, \mathcal{G}')$. (Then $0 \leq \rho < r$.) For a fixed $\epsilon$, take $\forall \mathcal{G}'' \in B(\mathcal{G}', \epsilon; \tilde{d}_N)$. Then,

$$d_N(\mathcal{G}, \mathcal{G}'') = d_N(\mathcal{G}, \mathcal{G}') + \frac{1}{4} \sum_{n=1}^{N} \frac{\lambda'_n - \lambda_n}{\lambda_n + \lambda''_n} \cdot \frac{\lambda''_n - \lambda'_n}{\lambda'_n} + R \quad \text{(Lemma 3)}$$

$$< \rho + \frac{1}{4} \sum_{n=1}^{N} \frac{|\lambda''_n - \lambda'_n|}{\lambda'_n} + |R| .$$

Now let us pay attention to the last line. The last term is bounded as $|R| < \sum_{n=1}^{N} (\frac{\lambda'_n - \lambda_n}{\lambda'_n})^2$ (Lemma 3). Due to Lemma 2, thus, one can
choose \( \epsilon \) sufficiently small s.t. the last term \(|R|\) is less in magnitude than the middle term. Let \( \bar{\epsilon}(>0) \) be such \( \epsilon \). Then we can continue the estimation of \( d_N(G,G'') \) as

\[
\begin{align*}
d_N(G,G'') &< \rho + \frac{1}{2} \sum_{n=1}^{N} \frac{|\lambda_n'' - \lambda_n'|}{\lambda_n} \\
&< \rho + \frac{N}{2} (\exp 4\bar{\epsilon} - 1) \quad \text{(Lemma 2)}.
\end{align*}
\]

By choosing \( \bar{\epsilon} \) again if necessary, we can assume that \( 0 < \bar{\epsilon} < \frac{1}{4} \ln(1 + \frac{2}{N}(r - \rho)) \). Then it follows that \( d_N(G,G'') < r \) for \( \forall G'' \in B(G', \bar{\epsilon}; \bar{d}_N). \) Hence there exists \( \bar{\epsilon}(>0) \) s.t. \( B(G', \bar{\epsilon}; \bar{d}_N) \subset B(G, r; \bar{d}_N) \). \( \square \)

**Lemma 5**

For \( \forall G' \in B(G, r; \bar{d}_N) \), there exists \( \epsilon > 0 \) s.t. \( B(G', \epsilon; d_N) \subset B(G, r; \bar{d}_N) \).

**Proof:**

Since \( \bar{d}_N \) is a distance, there exists \( \epsilon_0(>0) \) s.t. \( B(G', \epsilon_0; \bar{d}_N) \subset B(G, r; \bar{d}_N). \)

However, according to **Lemma 1** (3), there exists \( \epsilon_1(>0) \) s.t. \( B(G', \epsilon_1; d_N) \subset B(G', \epsilon_0; \bar{d}_N). \) Hence \( B(G', \epsilon_1; d_N) \subset B(G, r; \bar{d}_N) \). \( \square \)

**Lemma 6**

The set of balls \( \{B(G, r; d_N) \mid G \in \text{Riem/} \}, \ r > 0 \} \) can form a basis of topology.

**Proof:**

Let \( B_1 \) and \( B_2 \) are two balls defined by \( d_N \), and suppose \( B_1 \cap B_2 \neq \emptyset \). Because of **Lemma 4**, for \( \forall G \in B_1 \cap B_2 \), there exists \( r(>0) \) s.t. \( B(G, r; \bar{d}_N) \subset B_1 \cap B_2 \). However, due to **Lemma 1** (3), there exists \( r'(>0) \) s.t. \( B(G, r'; d_N) \subset B(G, r; \bar{d}_N) \). Hence \( B(G, r'; d_N) \subset B_1 \cap B_2 \). Thus the set of all balls defined by \( d_N \) satisfies the condition for a basis of topology. \( \square \)

Now we can show

**Theorem 1**

The set of balls \( \{B(G, r; d_N) \mid G \in \text{Riem/} \}, \ r > 0 \} \) and the set of balls \( \{B(G, r; \bar{d}_N) \mid G \in \text{Riem/} \}, \ r > 0 \} \) generate the same topology on \( \text{Riem/} \).
Proof:
\( \forall B(\mathcal{G}, r; \tilde{d}_N) \) is open in \( \tilde{d}_N \) topology due to Lemma 4. On the other hand, \( \forall B(\mathcal{G}, r; \tilde{d}_N) \) is open in \( d_N \) topology due to Lemma 5. \( \Box \)

Corollary

The space \( \mathcal{S}_N \) is a metrizable space. The distance function for metrication is provided by \( \tilde{d}_N \).

Hence it is appropriate to extend \( \mathcal{S}_N \) to its completion \( \mathcal{S}_N \). We understand \( \{\lambda_n\}_{n=0}^{\infty} \), \( d_N \) and \( \tilde{d}_N \) are extended on \( \mathcal{S}_N \) accordingly.

Now the metrizable space \( \mathcal{S}_N \) is a normal space, not to mention a Hausdorff space.

Due to Theorem 1, it is justified to regard \( \mathcal{S}_N \) as a metric space, provided that we resort to the distance function \( \tilde{d}_N \) whenever the triangle inequality is needed in a discussion. From now on we call \( d_N \) in Eq.(1) (the form of \( d_N \) in Eq.(2) in particular) a spectral distance for brevity.

Here it may be appropriate to add some comments on the applications of \( \mathcal{S}_N \) to physics. From the viewpoint of the practical applications in spacetime physics, \( d_N \) is more convenient than \( \tilde{d}_N \) since the former is easier to handle than the latter, which contains \( \max \) in the expression. Furthermore \( d_N \) can be related to the reduced density matrix element for the universe in the context of quantum cosmology [10]: It is possible to state that, under some circumstances, two universes \( \mathcal{G} \) and \( \mathcal{G}' \) are separated far in \( d_N \) when their quantum decoherence is strong.

With these comments in mind let us now turn back to the mathematical aspects of the space \( \mathcal{S}_N \).

Since \( \mathcal{S}_N \) is a metrizable space, it follows that [4, 14]

Theorem 2

The space \( \mathcal{S}_N \) is paracompact.

Since \( \mathcal{S}_N \) is Hausdorff and paracompact, it follows that [4, 14]

Corollary

There exists a partition of unity subject to any open covering of \( \mathcal{S}_N \).

Now we prepare two Lemma’s to show that \( \mathcal{S}_N \) is locally compact (Theorem 3 below).

\[ \text{[6]} \text{ For the basics of point set topology, see e.g. Ref. [5, 4, 14] } \]
Lemma 7

The set $D(\mathcal{G}, r; d_N) := \{ \mathcal{G}' \in \mathcal{S}_N | d_N(\mathcal{G}, \mathcal{G}') \leq r \}$ is closed and compact in $\mathcal{S}_N$.

Proof:

(1') Note that the map $d_N(\mathcal{G}, \cdot): \mathcal{S}_N \to [0, \infty)$ is continuous and that $[0, r]$ is closed in $[0, \infty)$. Since $D(\mathcal{G}, r; d_N)$ is the inverse image of the closed set $[0, r]$ by the continuous map $d_N(\mathcal{G}, \cdot)$, it is closed in $\mathcal{S}_N$.

(2') We now show that $D(\mathcal{G}, r; d_N)$ is sequentially compact.

Any sequence $\{\mathcal{G}_n\}_{n=1}^{\infty} \subset D(\mathcal{G}, r; d_N)$ can be embedded into an $N$-cube in $\mathbb{R}^N$, $\{\mathcal{G}_n\}_{n=1}^{\infty} \hookrightarrow [0, L]^N$ for some $L > 0$. Indeed let $\{\lambda_k^{(n)}\}_{k=1}^{\infty}$ be the spectra (zero-mode is excluded) for $\mathcal{G}_n$. Then a map $\{\lambda_k^{(n)}\}_{k=1}^{N} \mapsto \bar{\mu}^{(n)} := \left( \sqrt{\frac{\lambda_1^{(n)}}{\lambda_1}}, \sqrt{\frac{\lambda_2^{(n)}}{\lambda_2}}, \ldots, \sqrt{\frac{\lambda_N^{(n)}}{\lambda_N}} \right)$ provides the embedding. ($(\{\lambda_k\}_{k=1}^{\infty}$ is the spectra for $\mathcal{G}$.) Here we note that $\sqrt{\lambda_k^{(n)}} (k = 1, 2, \cdots, N)$ is bounded as $\sqrt{\lambda_k^{(n)}} \leq (\exp 2r + \sqrt{\exp 4r - 1})\sqrt{\lambda_k}$ because $d_N(\mathcal{G}, \mathcal{G}_n) \leq r$. Hence we can set $L = \exp 2r + \sqrt{\exp 4r - 1}$.

Now $\{\bar{\mu}^{(n)}\}_{n=1}^{\infty}$ is a sequence in a compact set $[0, L]^N$, so that there exists its subsequence $\{\bar{\mu}^{(n')}\}_{n'=1}^{\infty}$ which converges to a point in $[0, L]^N$ (in the sense of $\mathbb{R}^N$ topology). Let this convergent point be $\bar{\mu}^{(\infty)} = (\mu_1^{(\infty)}, \mu_2^{(\infty)}, \cdots, \mu_N^{(\infty)})$.

Then for any $\epsilon$ $(0 < \epsilon < 1)$ there exists $M \in \mathbb{N}$ s.t. for all $m \geq M$, it follows that $1 - \epsilon < \frac{\mu_1^{(n)}}{\mu_k^{(\infty)}} < 1 + \epsilon$. Then it follows that $\frac{1}{\epsilon} \sum_{k=1}^{N} \ln(\frac{\mu_k^{(n)}}{\mu_k^{(\infty)}}) < \frac{N}{\epsilon} \ln(\frac{1}{1-\epsilon})$. This implies that, for any $\epsilon$ $(0 < \epsilon < 1)$, there exists $M \in \mathbb{N}$ s.t. for all $m, m' \geq M$, $d_N(\mathcal{G}_m, \mathcal{G}_m') < N \ln(\frac{1}{1-\epsilon})$. Namely $\{\mathcal{G}_n\}_{n=1}^{\infty}$ corresponding to $\{\bar{\mu}^{(n')}\}_{n'=1}^{\infty}$ is a Cauchy sequence w.r.t. $d_N$.

Thus $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a Cauchy sequence w.r.t. $d_N$ also, due to Lemma 1 (1). However $D(\mathcal{G}, r; d_N)$ is closed in the complete space $\mathcal{S}_N$, so that $\{\mathcal{G}_n\}_{n=1}^{\infty}$ converges to a point $\exists \mathcal{G}_\infty \in D(\mathcal{G}, r; d_N)$. Hence any sequence $\{\mathcal{G}_n\}_{n=1}^{\infty} \subset D(\mathcal{G}, r; d_N)$ contains a subsequence which converges to a point in $D(\mathcal{G}, r; d_N)$, i.e. $D(\mathcal{G}, r; d_N)$ is sequentially compact.

(3') Since $D(\mathcal{G}, r; d_N)$ is a set in the metrizable space $\mathcal{S}_N$ as well as sequentially compact, it is compact. $\square$
Lemma 8
The set \( D(\mathcal{G}, r; \bar{d}_N) := \{ \mathcal{G}' \in S_N | \bar{d}_N(\mathcal{G}, \mathcal{G}') \leq r \} \) is closed and compact in \( S_N \).

Proof:
Since \( \bar{d}_N \) is a distance, it is clear that \( D(\mathcal{G}, r; \bar{d}_N) \) is closed in \( \bar{d}_N \)-topology. Thus \( D(\mathcal{G}, r; \bar{d}_N) \) is closed in \( S_N \) due to Theorem 1.

The rest goes almost in the same manner as the Proof of Lemma 7: Any sequence \( \{ \mathcal{G}_n \}_{n=1}^{\infty} \subset D(\mathcal{G}, r; \bar{d}_N) \) can be embedded into an \( N \)-cube in \( \mathbb{R}^N \), \( \{ \mathcal{G}_n \}_{n=1}^{\infty} \mapsto [0, L]^N \) for some \( L > 0 \). The embedding map \( \{ \lambda_k^{(n)} \}_{k=1}^{N} \mapsto \bar{\mu}^{(n)} \) is the same as in the Proof of Lemma 7. Since \( \bar{d}_N(\mathcal{G}, \mathcal{G}_n) \leq r \), it follows that \( \sqrt{\lambda_k^{(n)}} \leq \exp 2r \sqrt{k} (k = 1, 2, \ldots, N) \) due to Lemma 2. Thus we can set \( L = \exp 2r \). Because \( \{ \bar{\mu}^{(n)} \}_{n=1}^{\infty} \) is a sequence in a compact set \( [0, L]^N \), there exists its subsequence \( \{ \bar{\mu}^{(n')} \}_{n'=1}^{\infty} \) which converges to a point \( \bar{\mu}^{(\infty)} \) in \( [0, L]^N \) (in the sense of \( \mathbb{R}^N \) topology). Repeating the same argument as in the Proof of Lemma 7, we conclude that \( \{ \mathcal{G}_{n'} \}_{n'=1}^{\infty} \) corresponding to \( \{ \bar{\mu}^{(n')} \}_{n'=1}^{\infty} \) is a Cauchy sequence w.r.t. \( d_N \). However \( D(\mathcal{G}, r; \bar{d}_N) \) is closed in the complete space \( S_N \), then \( \{ \mathcal{G}_{n'} \}_{n'=1}^{\infty} \) converges to a point \( \exists \mathcal{G}_\infty \in D(\mathcal{G}, r; \bar{d}_N) \). Hence \( D(\mathcal{G}, r; \bar{d}_N) \) is sequentially compact. Since \( D(\mathcal{G}, r; d_N) \) is a set in the metrizable space \( S_N \) as well as sequentially compact, it is compact. \( \square \)

Theorem 3
The space \( S_N \) is locally compact.

Due to this property of \( S_N \), an integral can be constructed on \( S_N \), which is important to introduce, e.g., probability distributions over \( S_N \).

Proof:
For any \( \mathcal{G} \in S_N \), one can take \( D(\mathcal{G}, r; d_N) \) or \( D(\mathcal{G}, r; \bar{d}_N) \) as its compact neighborhood because of Lemma 7 and Lemma 8. \( \square \)

Corollary
If a sequence of continuous functions on \( S_N \), \( \{ f_n \}_{n=1}^{\infty} \), pointwise converges to a function \( f_\infty \), then \( f_\infty \) is continuous on a dense subset of \( S_N \).

Proof:
The set of discontinuous points of $f_\infty$ is a set of first category\footnote{A set of first category is defined as a set which can be expressed as a union of at most countable number of sets that are nowhere dense.}. Since $S_N$ is Hausdorff and locally compact, it becomes a Baire space, i.e. a space in which the complement of any set of first category becomes dense \cite{4, 14}. Thus the claim follows. \hfill $\Box$

We also note that, since $S_N$ is Hausdorff and locally compact, one can consider its one-point compactification $S_N \cup \{\infty\}$, which is Hausdorff. Moreover, $S_N$ is metrizable so that it is completely regular, then one can construct its Stone-\v{C}ech compactification \cite{4, 14}.

Furthermore we can show

**Theorem 4**

The space $S_N$ satisfies the second countability axiom.

**Proof:**

(1°) Since $S_N$ is a metrizable space, it suffices to show that $S_N$ is separable \cite{4, 14}. First we choose a suitable countable subset of $S_N$. For a fixed $M \in \mathbb{N}$, the label $(m_1 \leq m_2 \leq \cdots \leq m_N)$ is uniquely assigned to the spectra $\{\lambda_n\}_{n=1}^\infty$, where $m_1, m_2, \cdots, m_N \in \mathbb{N}$: This can be achieved by choosing $m_1, m_2, \cdots, m_N$ s.t. $\frac{m_n-1}{M} < \lambda_n \ell^2 < \frac{m_n}{M}$ ($n = 1, 2, \cdots N$). (Here $\ell$ is any constant of physical dimension [Length]. It has been introduced only for the physical comfort, and it is not essential for the arguments below.) For a given $M$, thus, the space $S_N$ is uniquely decomposed into classes labeled by $(M; \vec{m})$. (Some of the classes can be empty.) Then we can choose a representative $G_{(M; \vec{m})}$ in the class $(M; \vec{m}) \neq \emptyset$. Thus we obtain a countable subset $C := \{G_{(M; \vec{m})} \mid M, m_1, m_2, \cdots, m_N \in \mathbb{N}\}$ (For notational simplicity let $(M; \vec{m})$ denote $(M; m_1 \leq \cdots \leq m_N)$ hereafter.)

(2°) Take $\forall G \in S_N$. For $\forall M \in \mathbb{N}$, there uniquely exists a class $(M; \vec{m})$ s.t. $G \in (M; \vec{m})$. Let $\{\lambda_n\}_{n=0}^\infty$ and $\{\lambda_n^*\}_{n=0}^\infty$ are, respectively, the spectra for $G$ and $G_{(M; \vec{m})}$, the representative of the class $(M; \vec{m})$. Then $|\lambda_n^* - \lambda_n|^2 \leq \frac{1}{M}$. Hence $\sqrt{1 - \frac{1}{M\lambda_n^*}} \leq \sqrt{1 - \frac{1}{M\lambda_n}} \leq \sqrt{1 + \frac{1}{M\lambda_n^*}}$. Thus $d_N(G_{(M; \vec{m})}, G) < \frac{\sqrt{N}}{2} \ln \left(1/\sqrt{1 - \frac{1}{M\lambda_n^*}}\right)$. Due to Lemma 1 (1), thus, it
follows that \( \forall \epsilon > 0 \), there exists \( g_{(M;\vec{m})} \in C \) s.t. \( g_{(M;\vec{m})} \in B(g, \epsilon; d_N) \). Hence \( S_N \) is separable, so that the claim follows immediately. \( \square \)

**Theorem 1-Theorem 4** and their **Corollary**’s indicate that one can construct calculus theory on \( S_N \) to a great extent, which makes \( S_N \) a basic arena for spacetime physics. Thus it is essential to investigate the mathematical structures of \( S_N \) in detail.

## 4 Discussion

We have introduced the space \( S_N \) and have shown that it has several desirable properties. In particular we have shown that \( S_N \) is a metrizable space and in effect it can be regarded as a metric space provided that care is taken with regard to the triangle inequality: Whenever we need the arguments linked with the triangle inequality, it is safer to resort to \( \bar{d}_N \), which is a slight modification of \( d_N \) and which defines the same point set topology as \( d_N \) (**Theorem 1**). However in most of the cases, \( d_N \) is of more importance as well as easier to handle in practical applications. Therefore it is significant that it has been justified to treat \( S_N \) as a metric space.

Several properties of \( S_N \) that we have shown indicate that the space of spaces \( S_N \) provides us with a firm platform for pursuing meaningful investigations in spacetime physics (Recall the arguments in §1). Hence it is awaited that more detailed investigation on the properties of \( S_N \) would be performed.

Since the spectral distance is explicitly defined in terms of the spectra, that are of definite physical meaning, it possesses direct applicability to physics as well as theoretical firmness. Explicit applications of the spectral formalism would be discussed elsewhere. (See e.g. Ref.[14].)

Finally we make some comments on the isospectral manifolds [6, 3, 1]. It is no surprise that there exist Riemannian manifolds with identical spectra of the Laplacian even though they are non-isometric to each other: In this case we are comparing ‘sounds’ produced by a single type of oscillation corresponding to the Laplacian. If we change a type oscillation, namely if we use a different elliptic operator, the difference in sound would make a distinction between such spaces. If by any chance there were non-isometric Riemannian manifolds, s.t. the spectra are identical for any elliptic operator, they should have been regarded as identical from the physical point of view.
We can even imagine a new picture of ‘space’ suggested by the spectral formalism, in which one regards the whole of the geometrical information of a space as a collection of all spectral information such as

\[ \text{Space} = \bigcup_k \left( \mathcal{D}_k, \left\{ \lambda_n^{(k)} \right\}_{n=0}^{\infty}, \left\{ f_n^{(k)} \right\}_{n=0}^{\infty} \right), \]

where \( \mathcal{D}_k \) denotes an elliptic operator, \( \left\{ \lambda_n^{(k)} \right\}_{n=0}^{\infty} \) and \( \left\{ f_n^{(k)} \right\}_{n=0}^{\infty} \) are its spectra and eigenfunctions. Here the index \( k \) runs over all possible elliptic operators. Any observation selects out a subclass of elliptic operators related to the observational apparatus so that only a small portion of the whole geometrical information is obtained by a single observation. In some cases, such incomplete information is not enough to distinguish some class of manifolds. (This is the physical interpretation of isospectral manifolds.) Then one should perform other type of observations (corresponding to other elliptic operators) to make finer distinctions. It is also tempting to regard the spectral information as most fundamental. Further investigations are required as to whether this viewpoint of spacetime makes sense.

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