Nonlinear spectral instability of steady-state flow of a viscous liquid past a rotating obstacle

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Abstract

We show that a steady-state solution to the system of equations of a Navier-Stokes flow past a rotating body is nonlinearly unstable if the associated linear operator $L$ has a part of the spectrum in the half-plane $\{ \lambda \in \mathbb{C}; \text{Re} \lambda > 0 \}$. Our result does not follow from known methods, mainly because the basic nonlinear operator is not bounded in the same space in which the instability is studied. As an auxiliary result of independent interest, we also show that the uniform growth bound of the $C_0$–semigroup $e^{Lt}$ is equal to the spectral bound of operator $L$.

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1 Introduction

1.1. The initial–boundary valued problem. Suppose a compact body $\mathcal{B}$ moves rigidly in an otherwise quiescent Navier-Stokes liquid translating and rotating about the $x_1$–axis with a constant angular velocity $\omega$ and a constant velocity $u_\infty$. In order to avoid that the region of flow be time-dependent, instead of referring the motion of the liquid to an inertial frame, it is convenient to describe it from a coordinate system attached to the body. In such a system, the relevant equations then take the following form

$$\begin{align*}
\partial_t u - (\omega e_1 \times x + u_\infty e_1) \cdot \nabla u + \omega e_1 \times u + u \cdot \nabla u &= -\nabla p + \nu \Delta u + f, \\
\text{div } u &= 0,
\end{align*}$$

(1.1)

(1.2)

where $u$, respectively $p$, are the transformed velocity, respectively the pressure, $e_1$ denotes the unit vector in the direction of the $x_1$–axis, $f$ is the transformed external body force and $x$ is the transformed spatial variable. The system (1.1), (1.2) is considered for $(x, t) \in \Omega \times (0, \infty)$, with $\Omega$ a fixed domain. The no–slip boundary condition for the velocity on the surface of the body transforms to

$$u(x, t) = u_\infty e_1 + \omega e_1 \times x \quad \text{for } x \in \partial \Omega,$$

(1.3)

and the information that the fluid is at rest in infinity leads to the condition

$$u(x, t) \to 0 \quad \text{for } |x| \to \infty.$$

(1.4)

The details on the used transformation and the way one can obtain equation (1.1) from the “classical” Navier–Stokes equation are described in many previous papers, see e.g. [31], [11] or [24].
The global in time existence of weak solutions to the problem (1.1)–(1.4), with a prescribed initial velocity in \( L^2(\Omega) \), which is divergence free in the sense of distributions and such that its normal component coincides with the normal component of \( u_\infty e_1 + \omega e_1 \times x \) for \( x \in \partial \Omega \) (in a certain sense of traces), was proven by Borchers in [1]. (It also follows from paper [40] on the weak solvability of the Navier–Stokes equations in a domain with generally moving boundaries.) The existence of strong solutions on a “short” time interval \((0, T)\), under the condition that \( u_\infty = 0 \), has been proven by Hishida [31], Galdi, Silvestre [22] and Cumsille, Tucsnak [3]. The latter formulate the problem in an inertial frame, where the region of flow is time-dependent. They consider a body force \( f \) locally square integrable from \((0, \infty)\) to \( W^{1,\infty}(\mathbb{R}^3) \) and the no–slip boundary condition for the velocity on the body \( \mathcal{B} \). Their main result, reformulated in terms of the transformed velocity \( \tilde{u} \) satisfying (1.1)–(1.4), states that for a given \( u_0 \in L^2_0(\Omega) \cap W^{1,2}_0(\Omega) \), there exists \( T > 0 \) and a unique solution \( \tilde{u} \) of the problem (1.5) such that \( \tilde{u}_{t=0} = u_0 \) and

\[
\tilde{u} \in L^2(0, T_0; W^{2,2}(\Omega)) \cap C([0, T_0]; W^{1,2}(\Omega)),
\]

\[
\partial_t \tilde{u} - (\omega e_1 \times \tilde{x} + u_\infty e_1) \cdot \nabla \tilde{u} + \tilde{u} \times \tilde{u} \in L^2((0, T_0); L^2(\Omega))
\]

for every \( T_0 \in (0, T) \). Moreover, either \( T = \infty \) or else the norm of \( u \) in \( W^{1,2}(\Omega) \) tends to infinity for \( t \to T^- \). Although a translational motion of \( \mathcal{B} \) is not considered in [3] (which corresponds to \( u_\infty = 0 \)), the results can be extended to the case \( u_\infty \neq 0 \) by means of standard arguments similar to the case \( \omega = 0 \), \( u_\infty \neq 0 \), studied e.g. in [29].

Galdi and Silvestre [21] studied a more general problem, i.e. the motion of a rigid body in a viscous incompressible fluid under the action of a given force and torque. They considered the body force acting on the fluid to be potential, which corresponds to \( f = 0 \) in equation (1.1), but, by the same method, one to extend their result to a large class of non-zero body forces \( f \). In [21], in addition to velocity and pressure fields, also the translational velocity and the angular velocity of the body become unknown. However, the main findings can be used also for the problem at hand, by prescribing translational and angular velocities, and then calculating the force and torque needed for the body to perform the requested motion. Thus, considering the particular case where translational velocity and angular velocity have the same direction \( e_1 \), one obtains an existence result for problem (1.1)–(1.4) in the class (1.5), entirely analogous to that following from [3].

Finally, among the many other works studying general qualitative properties of the problem (1.1)–(1.4), we wish to mention e.g. the papers [6, 7] (by Deuring, Kračmar and Nečasová), [10] (by Farwig), [13] (by Farwig, Krbeč and Nečasová), [19], [20], (by Galdi), [23] (by Galdi and Silvestre), [26] (by Geissert, Heck and Hieber), [30], [32] (by Hishida) and [33] (by Hishida and Shibata).

1.2. Steady-state solution and perturbation equations. We further suppose that \( \Omega \) is (an exterior) domain in \( \mathbb{R}^3 \) with a \( C^{2+\mu} \) boundary \( \partial \Omega \), for some \( \mu \in (0, 1) \), and let \( \mathbf{U}, \Pi \) be velocity and pressure field of a steady-state solution to problem (1.1)–(1.4) with the following properties: there exists \( r_0 \in (1, 3) \) such that

\[
\nabla \mathbf{U} \in L^{r_0}(\Omega)^{3 \times 3} \cap L^3(\Omega)^{3 \times 3}
\]

and there exist \( c_1, c_2 > 0 \) such that

\[
|\mathbf{U}(x)| \leq c_1, \quad |\nabla \mathbf{U}(x)| \leq \frac{c_2}{1 + |x|} \quad \text{for} \quad x \in \Omega.
\]
The existence of a steady-state solution with these properties for a large class of body forces follows e.g. from [18, Sec. XI.6] (for \( u_\infty \neq 0 \)) and [18, Sec. XI.7] or [20] (for \( u_\infty = 0 \) and \( \omega \) “sufficiently small”).

As we are mainly interested in instability of solution \((U, \Pi)\), it is useful to write the solutions of (1.1)–(1.4) in the form

\[
 u = U + v, \quad p = \Pi + q,
\]

where the perturbations \( v \) and \( q \) satisfy the equations

\[
 \partial_t v - (\omega e_1 \times x + u_\infty e_1) \cdot \nabla v + \omega e_1 \times v + U \cdot \nabla v + v \cdot \nabla U + v \cdot \nabla v = -\nabla q + \Delta v, \quad (1.8)
\]

\[
 \text{div} \ v = 0 \quad (1.9)
\]
in \( \Omega \times (0, \infty) \), and the conditions

\[
 v(x, t) = 0 \quad \text{for} \ x \in \partial \Omega, \quad (1.10)
\]

\[
 v(x, t) \to 0 \quad \text{for} \ |x| \to \infty. \quad (1.11)
\]

1.3. On steady-state stability. Although stability is not the main subject of this paper, we would like to recall some corresponding relevant works. Most of them concern the case \( \omega = 0 \), which describes the situation when \( \mathcal{B} \) does not rotate and just moves with the translational velocity \( u_\infty e_1 \).

The time–decay of perturbations of solution \((U, \Pi)\) in appropriate norms and under various conditions of smallness imposed on \( U \) was proven in the works [28], [29] (by Heywood), [36] (by Masuda) and further on in a series of other papers, see e.g. [25] for a detailed list of corresponding references. The assumption of “sufficient smallness” on \( U \) is avoided in the papers [39], [5], [8], [41] and [43]. It is worth remarking that, in these last four papers, the stability of \((U, \Pi)\) is shown to be determined just by the location of the eigenvalues of the associated relevant linear operator \( \mathcal{L} \), disregarding the presence of an essential spectrum \( \sigma_{\text{ess}}(\mathcal{L}) \), which is non-empty and touches the imaginary axis from the left at point 0. The stability of a steady-state solution \((U, \Pi)\) in the case when \( \omega \neq 0 \), under some conditions of smallness of \( U \), was proved in the papers [22], [33] (in the case \( u_\infty = 0 \)) and [48].

Sufficient conditions for stability of solution \( U \) without the condition of smallness of \( U \) have been derived in [24]. Here, in analogy with [39], the authors use the assumption of an appropriate time–decay property of the semigroup \( e^{\mathcal{L}t} \), applied to a finite family of certain functions.

1.4. Brief overview of available methods on spectral instability. In view of their intrinsic pertinence to our main finding, we wish to recall some of the most relevant and currently available methods on instability of steady solutions to various evolutionary differential equations.

The classical result of Coddington and Levinson [2] concerns the equation

\[
 \frac{dx}{dt} = \mathcal{L}x + \mathcal{N}(x, t) \quad (1.12)
\]

in \( \mathbb{R}^n \), where \( \mathcal{L} \) is a real \( n \times n \) matrix and \( |\mathcal{N}(x, t)| = o(|x|) \) for \( |x| \to 0 \) uniformly with respect to \( t \in (0, \infty) \). Theorem XIII.1.2 in [2] says that the zero solution \( x(t) \equiv 0 \) is unstable under the assumption that matrix \( \mathcal{L} \) has at least one eigenvalue with positive real part. This classical result was successively generalized by different authors to systems of PDE’s, by regarding (1.12) as an abstract evolution equation in Hilbert or Banach spaces; see the books [4], [27] and the papers [35], [45]. In these works it is assumed that \( \mathcal{L} \) and \( \mathcal{N} \) are appropriate linear and nonlinear operators, respectively, and that the spectrum of \( \mathcal{L} \) has a nonempty intersection with the half plane.
\( \{ \lambda \in \mathbb{C}; \Re \lambda > 0 \} \). Concerning further properties requested on \( L \), we begin to recall that Daleckij and Krejn [4] considered equation (1.12) in a Banach space \( X \) and assumed \( L \) bounded and closed in \( X \). (See [4 Theorem VII.2.3].) Kielhöfer [35] studied (1.12) as an equation in a Hilbert space \( H \) and assumed that \( L = A + M \), where \( A \) is a (linear) self-adjoint and positive definite with compact inverse, while \( M \) satisfies \( D(M) \supset D(A^\beta) \) for some \( \beta \in [0, \frac{1}{2}) \) and \( \| M u \| \leq c \| A^\beta u \| \) for \( u \in D(A^\beta) \). Henry [27] considered (1.12) as an equation in a Banach space \( X \) and assumed that \( L \) is a sectorial operator in \( X \). (See Theorem 5.3.1 in [27].) It should be remarked that, directly or indirectly, in all works [4, 35, 27] the operator \( L \) is supposed to be the generator of an analytic semigroup. Particularly significant, in this sense, becomes then the contribution by Shatah and Strauss [45] who only require \( L \) to be the generator of a \( C_0 \)-semigroup in the Banach space \( X \) where (1.12) is studied. Concerning the assumptions on the operator \( N \), it must be emphasized that in all the mentioned works [4, 35, 27] and [45], it is supposed that, in a neighborhood of 0, \( N(x, t) \) is “sufficiently small” compared to \( x \) in the norm of the space \( X \), with respect to which instability is investigated, a condition that is not met by the problem studied in this paper.

In addition to these general results, we would like to mention also those proved directly for Navier-Stokes equations. The problem (1.1)-(1.4) with \( \omega = u_{\infty} = 0 \) was studied in a bounded domain \( \Omega \subset \mathbb{R}^3 \) by Sattinger [42]. The author reformulated the question of stability and instability of a steady-state solution as the same question concerning the zero solution of an equation of the type (1.12) in \( L^2(\Omega) \) (see subsection 2.1 for the definition of this function space). Moreover, he showed that the operator \( L \) has a compact resolvent and proved, in particular, instability in the norm of \( L^2(\Omega) \) under the assumption that some of the eigenvalues of \( L \) have positive real part. An analogous result can be found in the book [50] by Yudovich and in the paper [16] by Friedlander et al., who proved the instability of a steady flow in an \( n \)-dimensional finite domain in the norm of \( H^s(\Omega) \) for \( s > \frac{1}{2}n + 1 \). Sazonov [43] treated problem (1.1)-(1.4) with \( \omega = 0 \) in an exterior domain \( \Omega \subset \mathbb{R}^3 \). He proved the instability of a steady-state flow in the norm of \( L^3(\Omega) \), assuming that at least one eigenvalue of the associated linearized operator operator \( L \) has positive real part and that \( U(x) \) has suitable summability properties for large \( |x| \). This question has been reconsidered independently by Friedlander et al. in [17], where the authors deal with a steady flow \( U \) in a bounded or unbounded domain \( \Omega \) of \( \mathbb{R}^n \), assuming that \( U \) has derivatives of all orders bounded in \( \Omega \) and the associated linear operator \( L \) has a part of the spectrum in the right half-plane of \( \mathbb{C} \). Instability is proven in the norm of \( L^r(\Omega) \) for any \( r \in (1, \infty) \). We wish to emphasize that for the proof of all the above results it is crucial that the operator \( L \) be the generator of an analytic semigroup.

5. Main results of this paper. We treat problem (1.1)-(1.4) in an exterior domain \( \Omega \), and, in contrast to [43] and [17], we consider the case \( \omega \neq 0 \). This implies that the semigroup \( e^{Lt} \) generated by the associated linear operator \( L \) is no longer analytic but only of class \( C_0 \) (see subsection 2.5), which forces us to employ different estimates and technique than those, for example, of [43] or [17]. We prove the instability of solution \((U, \Pi)\) in the norms of \( L^2(\Omega) \) and \( W^{1,2}(\Omega) \), under the assumption that \( L \) has at least one eigenvalue in the half-plane \( \{ \lambda \in \mathbb{C}; \Re \lambda > 0 \} \). (Although the operator \( L \) has a non–empty essential spectrum, we show that the right half-plane in \( \mathbb{C} \) may contain only eigenvalues of \( L \) and the number of eigenvalues with real parts exceeding any given \( \xi > 0 \) is finite, see subsection 2.8.) It is necessary to stress that none of the abstract general results from [4], [35], [27] or [45] can be directly applied to our problem. In addition to the fact that our operator \( L \) does not satisfy the assumptions from [4], [35] or [27], the main reason is that the nonlinear operator \( N \) is not bounded in the same space in which the stability or instability is considered. (See subsection 2.2.) The statement on the instability is formulated in Theorem 2.
As an important auxiliary result, we also prove that the uniform growth bound \( \gamma(e^{Lt}) \) of the \( C_0 \)-semigroup \( e^{Lt} \), as a semigroup in \( L^2(\Omega) \), is equal to the spectral bound \( s(L) \) of operator \( L \). We wish to point out that, as is well known, such an equality does not generally hold for \( C_0 \)-semigroups and the question under which additional conditions the equality holds belongs to the most interesting and challenging problems in the theory of the \( C_0 \)-semigroups. The result is formulated in Theorem 1.

2 The associated linear and nonlinear operators and the operator form of (1.8)–(1.11)

2.1. Notation. We denote vector functions and spaces of vector functions by boldface letters. We also denote by \( c \) a generic constant whose value may change from line to line. On the other hand, \( c \)'s with indices denote constants with fixed values.

- For \( R > 0 \), we set \( \Omega_R := \Omega \cap B_R(0) \) and \( \Omega^R := \Omega \cap \{ x \in \mathbb{R}^3; |x| > R \} \).
- For \( 1 < r \leq \infty \) and \( k \in \mathbb{N} \), we denote by \( \| \cdot \|_r \) or \( \| \cdot \|_{k,r} \) the norm of scalar–, vector– or tensor–valued function with components in \( L^r(\Omega) \) or \( W^{k,r}(\Omega) \), respectively. If the norm is considered on another domain than \( \Omega \) then we use, e.g., the notation \( \| \cdot \|_{r;\Omega_R} \), etc.
- \( (\cdot, \cdot)_2 \) is the scalar product in \( L^2(\Omega) \).
- \( C^\infty_{0,\sigma}(\Omega) \) denotes the space of infinitely differentiable divergence–free vector functions with a compact support in \( \Omega \). For \( 1 < r < \infty \) and \( k \in \{0\} \cup \mathbb{N} \), we denote by \( L^r_0(\Omega) \), respectively \( W^{k,r}_0(\Omega) \), the closure of \( C^\infty_{0,\sigma}(\Omega) \) in \( L^r(\Omega) \), respectively in \( W^{k,r}(\Omega) \).
- The orthogonal projection of \( L^2(\Omega) \) onto \( L^2_\sigma(\Omega) \) is denoted by \( P_\sigma \).
- \( \mathcal{L}(L^2_\sigma(\Omega)) \) denotes the space of bounded linear operators in \( L^2_\sigma(\Omega) \).

- We set \( A\phi := P_\sigma \Delta \phi \) for \( \phi \in D(A) := W^{1,2}_{0,\sigma}(\Omega) \cap W^{2,2}(\Omega) \). The operator \( A \) (Stokes operator) is non–positive and selfadjoint in \( L^2_\sigma(\Omega) \). Its domain \( D(A) \) is a Banach space with the norm \( \| A \cdot \|_2 + \| \cdot \|_2 \). Further, for \( \phi \in D(A) \) we define

\[
B^0 \phi := (e_1 \times x) \cdot \nabla \phi - e_1 \times \phi,
B^1 \phi := \partial_1 \phi,
B^2 \phi := -P_\sigma [U \cdot \nabla \phi + \phi \cdot \nabla U]
\]

By using (1.6) and (1.7), one can easily verify that the range of \( B^2 \) is in \( L^2_\sigma(\Omega) \). In fact, in [12] or [39] it is shown that \( B^1 \phi \) is in \( W^{1,2}(\Omega) \cap L^2_\sigma(\Omega) \) for \( \phi \in D(A) \), hence \( B^1 \) can also be considered to be an operator in \( L^2_\sigma(\Omega) \). Also, it is clear that \( B^0 \phi \in W^{1,2}_{loc}(\Omega) \cap L^2_\sigma(\Omega) \).

We next define

\[
D(L^0) := D(L) := \{ \phi \in D(A); (\omega e_1 \times x) \cdot \nabla \phi \in L^2(\Omega) \},
\]

with operators \( L_0 \) and \( L \) given by

\[
L^0 \phi := \nu A \phi + \omega B^0 \phi + u_\infty B^1 \phi,
L \phi := \nu A \phi + \omega B^0 \phi + u_\infty B^1 \phi + B^2 \phi.
\]
The symmetric part $L_s$ and the skew–symmetric part (= anti-symmetric part) $L_a$ of the operator $L$ are

$$L_s = \nu A + B_s^2, \quad L_a = \omega B^0 + u_\infty B^1 + B_a^2,$$

where

\begin{align*}
B_s^2 \phi &= -P_\sigma [\phi \cdot (\nabla U)_s], \\
B_a^2 \phi &= -P_\sigma [U \cdot \nabla \phi + \phi \cdot (\nabla U)_a].
\end{align*}

\begin{itemize}
  \item Finally, we denote by $N$ the operator defined as follows:
  \begin{equation*}
  N\phi := -P_\sigma (\phi \cdot \nabla \phi) \quad \text{(for $\phi \in D(A)$).}
  \end{equation*}
\end{itemize}

2.2. Important inequalities. Conditions (1.6), (1.7) and Sobolev’s inequality (see e.g. [18, p. 54]) imply that $U \in L^2(\Omega)$ for all $3r_0/(3-r_0) \leq a < \infty$. By using the latter along with (1.6), (1.7), and Hölder and Sobolev inequalities, one shows that $B^1, B^2$ and $N$ satisfy the following bounds for all $\phi \in D(A)$

\begin{equation*}
\begin{align*}
|B^1 \phi|_2 & \leq |\phi|_{1,2}, \\
|B^2 \phi|_2 + |B_a^2 \phi|_2 & \leq c_3 |\phi|_{1,2}, \\
|N \phi|_2 & \leq c_4 \|A\phi\|_{1/2}^2 |\phi|_{1,2}^{3/2}.
\end{align*}
\end{equation*}

It is proven in [24] that the operator $B^2$ is relatively $A$–compact, relatively $(\nu A + \omega B^0)$–compact and relatively $L^0$–compact in the space $L^2_0(\Omega)$.

2.3. An operator form of equations (1.8), (1.9). Applying standard arguments, one can show that the system of equations (1.8), (1.9) can be written as an operator equation

\begin{equation}
\frac{dv}{dt} = Lv + Nv
\end{equation}

in the space $L^2_0(\Omega)$. Here and further on, we mostly consider $v$ to be a function of one variable $t$ with values in an appropriate function space. (This justifies writing the derivative with respect to time as $dv/dt$ and not $\partial_t v$.)

From the results of papers [21] and [3] one deduces that for a given $v_0 \in W^{1,2}_{00,\sigma}(\Omega)$ there exists $T \in (0, \infty]$ and a solution $v$ of equation (2.4) on the time interval $(0, T)$ such that $v(0) = v_0$ and

\begin{equation}
\begin{align*}
v \in L^2(0, T_0; D(A)) \cap C([0, T_0); \ W^{1,2}_{00,\sigma}(\Omega)), \\
\frac{dv}{dt} - \omega B^0 v - u_\infty B^1 v \in L^2(0, T_0; \ L^2_0(\Omega))
\end{align*}
\end{equation}

for each $T_0 \in [0, T)$. Moreover, if $T < \infty$ then $\|v(t)\|_{1,2} \to \infty$ for $t \to T$. Note that such a solution is not classical, but it is more than just a mild solution. The main reason is that, due to (2.5), equation (2.4) makes sense, as an equation in $L^2_0(\Omega)$, for a.a. $t \in (0, T)$.

2.4. Spectra of the operators $L^0$ and $L$. Recall that a closed densely defined operator $S$ in a Banach space $X$ is said to be Fredholm if its range is closed and both $\text{nul}(S)$ (the nullity of $S$) and $\text{def}(S)$ (the deficiency of $S$) are finite. Operator $S$ is called semi-Fredholm if its range is closed and at least one of the numbers $\text{nul}(S)$, $\text{def}(S)$ is finite. According to the definition from [34, p. 243], the essential spectrum of $S$ is the set of those $\lambda \in \mathbb{C}$, for which the operator $S - \lambda I$ is
not semi-Fredholm. We denote by \( \sigma(S) \) the whole spectrum, by \( \sigma_p(S) \) the point spectrum and by \( \sigma_{ess}(S) \) the essential spectrum of \( S \). Note that one can also find a different definition of the essential spectrum in the literature, according to which \( \lambda \) belongs to the essential spectrum of \( S \) if the operator \( S - \lambda I \) is not Fredholm. (See e.g. the footnote on p. 243 in [34] or [44].) In order to distinguish between the two definitions, we denote by \( \tilde{\sigma}_{ess}(S) \) the essential spectrum of operator \( S \), satisfying the second definition. (We shall also denote by tilde some other quantities, related to the second definition of the essential spectrum.) Obviously, \( \sigma_{ess}(S) \subset \tilde{\sigma}_{ess}(S) \). Recall that while \( \sigma_{ess}(S) \) is preserved under relatively compact additive perturbations of \( S \), see [34, Theorem IV.5.35], \( \tilde{\sigma}_{ess}(S) \) is preserved under compact additive perturbations of \( S \), see [9, p. 248] or [44, Corollary 2.2].

The next two formulas provide a characterization of the essential spectrum of operator \( L^0 \); see [11, 12]:

\[
\sigma_{ess}(L^0) = \{ \lambda = \alpha + ik\omega \in \mathbb{C}; \alpha \leq 0, k \in \mathbb{Z} \} \quad \text{if } u_\infty = 0, \\
\sigma_{ess}(L^0) = \left\{ \lambda = \alpha + i\beta + ik\omega \in \mathbb{C}; \alpha, \beta \in \mathbb{R}, k \in \mathbb{Z}, \alpha \leq -\frac{\nu \beta^2}{u_\infty^2} \right\} \quad \text{if } u_\infty \neq 0. \tag{2.6}
\]

We observe that if \( u_\infty = 0 \) then \( \sigma_{ess}(L^0) \) is a union of infinitely many equidistant half-lines parallel to the real axis. If \( u_\infty \neq 0 \) then \( \sigma_{ess}(L^0) \) consists of a union of equally shifted filled in parabolas in the left half-plane of \( \mathbb{C} \). Similar results were obtained in [14] and [15] in the general \( L^q \)-setting. It follows from [14, Theorem 1.2] that all \( \lambda \in \sigma(L^0) \setminus \sigma_{ess}(L^0) \) are isolated eigenvalues of \( L \) with negative real parts and finite algebraic multiplicities. (This set may also be empty.) Thus, since \( \text{ind}(L^0 - \lambda I) \equiv \text{mul}(L^0 - \lambda I) - \text{def}(L^0 - \lambda I) \) is constant in \( \mathbb{C} \setminus \sigma_{ess}(L^0) \) (by [34, Theorem IV.5.17]), we deduce that \( \text{mul}(L^0 - \lambda I) = \text{def}(L^0 - \lambda I) < \infty \) for all \( \lambda \in \mathbb{C} \setminus \sigma_{ess}(L^0) \). Thus, \( L^0 - \lambda I \) is a Fredholm operator for all \( \lambda \in \mathbb{C} \setminus \sigma_{ess}(L^0) \). Consequently, \( \sigma_{ess}(L^0) = \tilde{\sigma}_{ess}(L^0) \).

As the operators \( L \) and \( L^0 \) differ only by operator \( B^2 \), which is relatively \( L^0 \)-compact, we have \( \sigma_{ess}(L) = \sigma_{ess}(L^0) \) (by [34, Theorem IV.5.35]). Moreover, the operator \( L - \alpha I \) is dissipative for large positive \( \alpha \), which can be easily verified by means of estimate (2.2). Hence all \( \lambda \in \mathbb{C} \) with positive and sufficiently large real parts belong to the resolvent set of \( L \). Since the open set \( \mathbb{C} \setminus \sigma_{ess}(L) \) has just one component, we deduce by means of the same arguments as in the previous paragraph that the set \( \sigma(L) \setminus \sigma_{ess}(L) \), if it is not empty, consists of an at most countable family of isolated eigenvalues of \( L \) with finite algebraic multiplicities (which can possibly cluster only at points of \( \sigma_{ess}(L) \)) and \( \text{mul}(L - \lambda I) = \text{def}(L - \lambda I) < \infty \) for all \( \lambda \in \mathbb{C} \setminus \sigma_{ess}(L) \). Hence \( L - \lambda I \) is a Fredholm operator for all \( \lambda \in \mathbb{C} \setminus \sigma_{ess}(L) \) and \( \sigma_{ess}(L) = \tilde{\sigma}_{ess}(L) \).

Thus, we observe that the operators \( L^0 \) and \( L \) have the same essential spectra described by (2.6), no matter which one of the two aforementioned definitions of the essential spectrum we use. Moreover, recall that \( \sigma(L^0) \setminus \sigma_{ess}(L^0) \) and \( \sigma(L) \setminus \sigma_{ess}(L) \) can also contain isolated eigenvalues with finite algebraic multiplicities, which may cluster in \( \mathbb{C} \) only at the boundary of the essential spectrum. While all such eigenvalues of \( L^0 \) have negative real parts, the eigenvalues of \( L \) may also lie in the half-plane \( \{ \lambda \in \mathbb{C}; \Re \lambda > 0 \} \).

**2.5. Semigroups generated by the operators \( L^0 \) and \( L \).** If \( \omega = 0 \) then operator \( L^0 \) generates an analytic semigroup \( e^{\omega t} \) in \( L^1_{q}(\Omega) \) for \( u_\infty = 0 \) (e.g. [37]), and for \( u_\infty \neq 0 \) [39]. However, if \( \omega \neq 0 \) (which we assume) then the same operator generates only a \( C_0 \)-semigroup in \( L^1_{q}(\Omega) \), see [31] or [26] for \( u_\infty = 0 \) and [46] for \( u_\infty \in \mathbb{R} \). As showed in [24], the operator \( L \) generates a \( C_0 \)-semigroup also in \( L^1_{q}(\Omega) \). (This is obtained by a relatively easy application of the Lumer–Phillips theorem.) Both semigroups \( e^{\omega t} \) and \( e^{\lambda t} \) are strongly continuous, but none of them is
2.6. Some estimates of the semigroup $e^{c_1 t}$. Although we need just the $L^2$-$L^2$ estimates of $e^{c_1 t}$ and $\nabla e^{c_1 t}$ in this paper, we recall, for completeness, the more general $L^r$-$L^s$ estimates, satisfied for $\phi \in L^2_0(\Omega)$ and $t > 0$:

$$
\|e^{c_1 t} \phi\|_r \leq c_5 t^2 \left( \frac{1}{r} - \frac{1}{s} \right) \|\phi\|_s \quad \text{for } 1 < s \leq r < \infty, \quad (2.7)
$$

$$
|e^{c_1 t} \phi|_{1,r} \leq c_6 t^{-\frac{1}{2} + \frac{3}{2} \left( \frac{1}{s} - \frac{1}{r} \right)} \|\phi\|_s \quad \text{for } 1 < s \leq r \leq 3. \quad (2.8)
$$

These inequalities are proved in [33] for the case $u_\infty = 0$, and in [46] for the case $u_\infty \neq 0$. Moreover, one can also deduce from [47, Theorem 1.1] that there exists $\rho > 0$ such that

$$
|e^{c_1 t} \phi|_{2,2} \leq \frac{c_7}{t} e^{\rho t} \|\phi\|_2
$$

(2.9)

for $\phi \in L^2_0(\Omega)$ and $t > 0$.

2.7. The uniform growth bound of a general $C_0$-semigroup. Assume, for a while, that $T = T(t)$ is a general $C_0$-semigroup in a Banach space $X$ (with the norm $\|\|$, $\gamma$), generated by operator $S$. The spectral bound $s(S)$ of $S$ and the uniform growth bound $\gamma(T)$ of the semigroup $T$ are defined by the formulas

$$
s(S) := \sup \{ \text{Re } \lambda; \lambda \in \sigma(S) \},
$$

$$
\gamma(T) := \inf \{ \mu \in \mathbb{R}; \exists M_\mu > 0 \ \forall t > 0 \ \forall x \in X : \|T(t)x\| \leq M_\mu e^{\mu t} \|x\| \},
$$

respectively. It is known that generally $s(S) \leq \gamma(T)$. The question of “under which conditions the equality $s(S) = \gamma(T)$ holds” is a classical problem in the theory of $C_0$-semigroups. It is known that $s(S) = \gamma(T)$ if the semigroup $T$ satisfies the spectral mapping theorem, i.e. if $\sigma(T(t)) \setminus \{0\} = \exp(t \sigma(S))$ holds for some $t > 0$. (See [38, Proposition 1.2.2] or [9, Proposition 2.2.2],) However, while the identities $\sigma_p(T(t)) \setminus \{0\} = \exp(t \sigma_p(S))$ and $\sigma_t(T(t)) \setminus \{0\} = \exp(t \sigma_t(S))$ hold for the point spectrum and the residual spectrum (see Theorems 2.1.2 and 2.1.3 in [38] or [9, Theorem IV.3.7]), the approximate point spectrum generally satisfies only the inclusion $\exp(t \sigma_a(S)) \subset \sigma_a(T(t)) \setminus \{0\}$. The identity $\exp(t \sigma_a(S)) = \sigma_a(T(t)) \setminus \{0\}$ (which implies the validity of the spectral mapping theorem and consequently also the equality $s(S) = \gamma(T)$), is satisfied e.g. if the semigroup $T$ is eventually uniformly continuous and, in particular, if it is analytic. (See [9, Theorem IV.3.10] or [38, Theorem 2.3.2].)

Further in this subsection, we present some useful notions and assertions, which are described and discussed in greater detail e.g. in [9, pp. 249–258]. The essential spectral radius of the semigroup $T$ at time $t$ is defined by the formula

$$
\overline{\rho}_{\text{ess}}(T(t)) := \sup \{ |\lambda|; \lambda \in \sigma_{\text{ess}}(T(t)) \},
$$

The essential spectral radius of $T(t)$ can also be characterized as the infimum of the set of $\rho > 0$ such that the implication “$\zeta \in \sigma(T(t)), |\zeta| > \rho \implies \zeta$ is an eigenvalue of $T(t)$ with a finite algebraic multiplicity” holds. The essential growth bound of the semigroup $T$ is defined to be

$$
\overline{\rho}_{\text{ess}}(T) := \frac{1}{t_0} \ln \overline{\rho}_{\text{ess}}(T(t_0)) \quad \text{(for any } t_0 > 0\text{).}$$

proven that $B$ that operator of $e^{(2.9)}$, and applying the same arguments as in [9], one also obtains $(2.8)$ for the concrete unbounded $m, n$.

2.8. The uniform growth bounds of the semigroups $e^{L\dot{t}}$ and $e^{C\dot{t}}$. Our main objective in this subsection is to prove the identities $\gamma(e^{L\dot{t}}) = s(L^0) = 0$ and $\gamma(e^{C\dot{t}}) = \bar{s}(L)$.

It follows immediately from formulas $(2.6)$ and from the inclusion $\sigma(L^0) \subset \{ \lambda \in \mathbb{C}; \text{Re} \lambda \leq 0 \}$ that $s(L^0) = 0$. Inequality $(2.7)$ (which implies $\gamma(e^{L\dot{t}}) \leq 0$) and the general inequality $\gamma(e^{C\dot{t}}) \geq s(L^0)$ yield $\gamma(e^{L\dot{t}}) = 0$. Furthermore, $\tilde{\gamma}(e^{L\dot{t}}) \geq 1$ for each $t > 0$, because $\tilde{\gamma}(e^{L\dot{t}})$ is the infimum of the set of $\rho > 0$ such that $\{ \zeta \in \sigma(e^{C\dot{t}}); |\zeta| > \rho \}$ consists of isolated eigenvalues of $e^{C\dot{t}}$ with finite algebraic multiplicities, and each set $\{ \zeta \in \sigma(e^{C\dot{t}}); |\zeta| > \rho \}$ (for $0 < \rho < 1$) contains $\{e^{\lambda t}; \lambda \in \sigma(L^0), \text{ln} \rho < \text{Re} \lambda \}$ (due to the spectral inclusion theorem), which is not isolated. On the other hand, since $\tilde{\gamma}(e^{C\dot{t}}) \leq \gamma(e^{L\dot{t}}) = 0$, we obtain $\tilde{\gamma}(e^{C\dot{t}}) = 0$. We have proven that

$$s(L^0) = \gamma(e^{L\dot{t}}) = \tilde{\gamma}(e^{C\dot{t}}) = 0. \quad (2.10)$$

Let us now focus on operator $L$ and the semigroup $e^{C\dot{t}}$. If $\omega = 0$ then $s(L) = \gamma(e^{C\dot{t}})$, because the semigroup $e^{C\dot{t}}$ is analytic. The validity of the same equality in the case $\omega \neq 0$ is, however, a subtler problem. Nevertheless, since $L = L^0 + B^2$, the semigroup $e^{C\dot{t}}$ satisfies the variation of parameters formula

$$e^{C\dot{t}} = e^{L\dot{t}} + \int_0^t e^{C(\tau - t)} B^2 e^{L\dot{t}} d\tau \quad (2.11)$$

for $t \geq 0$, see e.g. [9] p. 161]. (The formula has been in fact derived in [9] under the assumption that operator $B^2$ is bounded, but using the “smoothing” properties of $e^{C\dot{t}}$, following from $(2.8)$ and $(2.9)$, and applying the same arguments as in [9], one also obtains $(2.8)$ for the concrete unbounded operator $B^2$ we deal with.) The integral on the right hand side of $(2.11)$ converges in the topology of $L(L_0^2(\Omega))$, because

$$\left\| e^{C\dot{t}} B^2 e^{L(\tau - t)} \phi \right\|_2 \leq M_\mu e^{\mu \tau} \left\| B^2 e^{L(\tau - t)} \phi \right\|_2 \leq M_\mu e^{\mu \tau} c_3 \left\| e^{L(\tau - t)} \phi \right\|_{1,2}$$

$$\leq M_\mu e^{\mu \tau} \frac{c_3 c_6}{\sqrt{1 - \tau}} \left\| \phi \right\|_2$$

for all $\phi \in L_0^2(\Omega)$ and $\mu > \gamma(e^{C\dot{t}})$. We shall further need the next two lemmas:

**Lemma 1.** The operator $B^2 e^{L(\tau - t)}$ is compact in $L_0^2(\Omega)$ for each $t > 0$ and $0 \leq \tau < t$.

**Proof.** Let $t > 0$ and $0 \leq \tau < t$ be fixed. It follows from the inequalities $(2.7)$--$(2.9)$ that $e^{L(\tau - t)}$ is a bounded operator from $L_0^2(\Omega)$ to $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. In order to complete the proof, we show that $B^2$ is a compact operator from $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ to $L_0^2(\Omega)$. Thus, let $\{\phi_n\}$ be a bounded sequence in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and $R > 0$. Then

$$\left\| B^2 \phi_n - B^2 \phi_m \right\|_2 \leq \left\| U \cdot \nabla (\phi_n - \phi_m) + (\phi_n - \phi_m) \cdot \nabla U \right\|_2$$

$$\leq \left\| U \cdot \nabla (\phi_n - \phi_m) + (\phi_n - \phi_m) \cdot \nabla U \right\|_{2;\Omega_R}$$

$$+ \left\| U \cdot \nabla (\phi_n - \phi_m) + (\phi_n - \phi_m) \cdot \nabla U \right\|_{2;\Omega_R} \quad (2.12)$$

for $m, n \in \mathbb{N}$. Let $\epsilon > 0$ be given. Due to the assumptions (1.6) on function $U$, there exists $R > 0$ so large that the second term on the right hand side is less than or equal to $\epsilon/2$, independently of $m$ and $n$. Applying the compact imbedding $W^{2,2}(\Omega_R) \hookrightarrow W^{1,2}(\Omega_R)$ and the boundedness of
the operator \( \phi \mapsto [U \cdot \nabla \phi + \phi \cdot \nabla U] \) from \( W^{1,2}(\Omega_R) \) to \( L^2(\Omega_R) \) (which follows from (17)), one can show that there exists a subsequence of \( \{\phi_n\} \) (which we denote again by \( \{\phi_n\} \)), such that the sequence \( \{U \cdot \nabla \phi_n + \phi_n \cdot \nabla U\} \) converges in \( L^2(\Omega_R) \). Hence the first term on the right hand side of (2.12) is also less than or equal to \( \epsilon/2 \) for \( m \) and \( n \) sufficiently large. This shows that \( \{B^2\phi_n\} \) is a Cauchy sequence in \( L^2(\Omega) \). The proof is completed. \( \square \)

**Lemma 2.** Let \( t > 0 \). Then the operator \( \int_0^t e^{Lt} B^2 e^{C_0(t-\tau)} \ d\tau \) is compact in \( L^2(\Omega) \).

**Proof.** Let us at first show that for each given \( \phi \in L^2(\Omega) \), the function \( e^{Lt} B^2 e^{C_0(t-\tau)} \phi \) is continuous (in the norm of \( L^2(\Omega) \)) in dependence on \( \tau \) in the interval \([0, t]\). Thus, let \( \tau \in [0, t] \) be fixed and let \( \delta \) satisfy \(-\delta < \delta < t - \tau\). We have

\[
\left\| e^{L(t+\delta)} B^2 e^{C_0(t-\tau-\delta)} \phi - e^{L(t)} B^2 e^{C_0(t-\tau)} \phi \right\|_2 \\
\leq \left\| e^{L(t+\delta)} B^2 (e^{C_0(t-\tau-\delta)} - e^{C_0(t-\tau)}) \phi \right\|_2 + \left\| (e^{L(t+\delta)} - e^{L(t)}) B^2 e^{C_0(t-\tau)} \phi \right\|_2.
\]

The second term on the right hand side tends to zero for \( \delta \to 0 \) due to the strong continuity of the semigroup \( e^{Lt} \). The first term on the right hand side is less than or equal to

\[
M_\mu e^{\mu(t+\delta)} \left\| B^2 (e^{C_0(t-\tau-\delta)} - e^{C_0(t-\tau)}) \phi \right\|_2 \\
\leq M_\mu e^{\mu(t+\delta)} c_3 \left| (e^{C_0(t-\tau-\delta)} - e^{C_0(t-\tau)}) \phi \right|_{1,2} \\
\leq M_\mu e^{\mu(t+\delta)} c_3 \frac{c_6 \sqrt{2}}{\sqrt{t-\tau}} \left\| (e^{C_0((t-\tau)/2-\delta)} - e^{C_0((t-\tau)/2)}) \phi \right\|_2,
\]

which tends to zero for \( \delta \to 0 \) due to the strong continuity of the semigroup \( e^{C_0t} \). The continuity of \( e^{Lt} B^2 e^{C_0(t-\tau)} \phi \) in dependence on \( \tau \) is proven.

Thus, \( e^{Lt} B^2 e^{C_0(t-\tau)} \) is a family of compact linear operators in \( L^2(\Omega) \), strongly continuous in dependence on \( \tau \) for \( \tau \in [0, \xi] \) for every \( 0 < \xi < t \). This information enables us to apply Theorem C.7 from [9, p. 525] and conclude that \( \int_0^t e^{Lt} B^2 e^{C_0(t-\tau)} \ d\tau \) is a compact operator in \( L^2(\Omega) \) for every \( \xi \in (0, t) \). Since

\[
\int_0^t e^{Lt} B^2 e^{C_0(t-\tau)} \ d\tau = \lim_{\xi \to 0^+} \int_0^{t-\xi} e^{Lt} B^2 e^{C_0(t-\tau)} \ d\tau
\]

in the topology of \( L^2(\Omega) \), and the subspace of compact linear operators in \( L^2(\Omega) \) is closed in \( L^2(\Omega) \), we observe that the operator \( \int_0^t e^{Lt} B^2 e^{C_0(t-\tau)} \ d\tau \) is compact, too. \( \square \)

Formula (2.11) and Lemma 2 show that, for any \( t > 0 \), the operators \( e^{Lt} \) and \( e^{C_0t} \) differ just by an additive compact operator. Thus, \( \bar{\sigma}_{\text{ess}}(e^{Lt}) = \bar{\sigma}_{\text{ess}}(e^{C_0t}) \). Consequently, \( \bar{\gamma}_{\text{ess}}(e^{Lt}) = \bar{\gamma}_{\text{ess}}(e^{C_0t}) \) for each \( t > 0 \) and \( \bar{\gamma}_{\text{ess}}(e^{Lt}) = \bar{\gamma}_{\text{ess}}(e^{C_0t}) = 0 \). (The last identity is a part of (2.10).) Since \( \gamma(e^{Lt}) = \max\{\bar{\gamma}_{\text{ess}}(e^{Lt}); s(\mathcal{L})\} = \max\{0; s(\mathcal{L})\} \), we obtain the equality

\[
\gamma(e^{Lt}) = s(\mathcal{L}). \tag{2.13}
\]

**Theorem 1.** The uniform growth bound \( \gamma(e^{Lt}) \) of the semigroup \( e^{Lt} \) and the spectral bound \( s(\mathcal{L}) \) of operator \( \mathcal{L} \) are equal. Moreover, for every \( \xi > 0 \), the set \( \Gamma_\xi := \sigma(\mathcal{L}) \cap \{\lambda \in \mathbb{C}; \Re \lambda \geq \xi\} \) consists of at most a finite number of eigenvalues of \( \mathcal{L} \) with finite algebraic multiplicities.

**Proof.** The equality of \( \gamma(e^{Lt}) \) and \( s(\mathcal{L}) \) has already been proven. Let \( \xi > 0 \). We may suppose without loss of generality that \( s(\mathcal{L}) > 0 \) and \( \xi < s(\mathcal{L}) \). Assume, by contradiction, that the set \( \Gamma_\xi \)
is infinite. The elements of $\Gamma_\xi$ cannot accumulate at any point of $\mathbb{C}$, because it would contradict the description of $\sigma(\mathcal{L})$ given in subsection 2.4. Their real parts are in a bounded interval $[\xi, s(\mathcal{L})]$, so the real parts have a cluster point $\xi_0 \in [\xi, s(\mathcal{L})]$. Thus, the set $\exp(t \Gamma_\xi)$ (for some $t > 0$) has a cluster point on the circle $|z| = e^{\xi_0 t}$ in $\mathbb{C}$. This is, however, impossible, because the cluster point of $\exp(t \Gamma_\xi)$ is also a cluster point of $\exp[t \sigma(\mathcal{L})]$, i.e., also a cluster point of $\sigma(e^{t \xi})$ (due to the inclusion $\exp[t \sigma(\mathcal{L})] \subset \sigma(e^{t \xi})$) and since $e^{\xi_0 t} > 1 = \tilde{r}_{\text{ess}}(e^{t \xi}) = \tilde{r}_{\text{ess}}(e^{t \xi})$, the set $\sigma(e^{t \xi})$ cannot have a cluster point on the circle $|z| = e^{\xi_0 t}$. \hfill $\square$

### 3 Spectral instability of the zero solution of equation (2.4)

The purpose of this section is to prove the next theorem:

**Theorem 2.** Assume that $\sigma(\mathcal{L}) \cap \{ \lambda \in \mathbb{C}; \Re \lambda > 0 \} \neq \emptyset$. Then the zero solution of equation (2.4) is unstable in the sense that there exists $\epsilon > 0$ such that to any $\delta > 0$ there exists $T > 0$, $t^* \in (0, T)$ and a solution $v^*$ of equation (2.4) on the time interval $(0, T)$ such that $\|v^*(0)\|_{1,2} \leq \delta$ and $\|v^*(t^*)\|_{2} \geq \epsilon$. Consequently, the steady solution $U$ of the problem (1.1)–(1.4) (satisfying (1.6) and (1.7)) is unstable in the same sense.

We present the proof in eight steps, which are explained in subsections 3.1–3.8. For reader’s convenience, we focus just on the main ideas in this section, leaving the detailed explanation and derivation of technical arguments and estimates to Section 4.

**3.1. Concrete solutions $v_t$ and $v_1$ of equation (2.4) and function $v$.** It follows from Theorem 1 that $\sigma(\mathcal{L}) \cap \{ \lambda \in \mathbb{C}; \Re \lambda > 0 \}$ consists only of eigenvalues of $\mathcal{L}$ and that one can choose an eigenvalue with the largest real part, equal to $s(\mathcal{L})$. Let $a + ib$ be such an eigenvalue. Let $\zeta + i\eta$ be a corresponding eigenfunction. (The numbers $a$, $b$ and the functions $\zeta$, $\eta$ are supposed to be real.) The eigenfunction can be normalized so that $\|\zeta + i\eta\|_{1,2} = 1$. Obviously,

$$\|e^{t \xi} (\zeta + i\eta)\|_{1,2} = \|e^{(a+ib)t} (\zeta + i\eta)\|_{1,2} = e^{at} \|\zeta + i\eta\|_{1,2} = e^{at}.$$  

Let $\delta > 0$ and $v_t$ and $v_1$ be solutions of the equation (2.4), satisfying the initial conditions $v_t(0) = \delta \zeta$ and $v_1(0) = \delta \eta$, respectively. It follows from the existential results, cited in subsection 2.3, that both the solutions $v_t$ and $v_1$ exist on the time interval $[0, T]$ (for some $T > 0$), belong to the class (2.3) and either $T = \infty$ or $\|v_t(t)\|_{1,2} + \|v_1(t)\|_{1,2} \to \infty$ for $t \to T^-$. Put $v := v_t + iv_1$. Function $v$ satisfies the equation

$$\frac{dv}{dt} = \mathcal{L}v + \mathcal{N}v_t + i\mathcal{N}v_1$$  

and the initial condition

$$v(0) = \delta (\zeta + i\eta).$$  

**3.2. Numbers $K$ and $T_1$.** Let $K > 1$. Since $\|v(0)\|_{1,2} = \delta$, the inequality

$$\|v(t)\|_{1,2} \leq \delta K e^{at}$$  

holds for $t$ in some right neighborhood of 0. Denote by $T_1$ the maximum number such that (3.3) holds for all $t \in (0, T_1)$. ($T_1 = \infty$ is also admitted.)

From now on, all estimates, cited or derived in subsections 3.5 and 4.3–4.5, are related to $t$ in the time interval $[0, T_1]$ (if $T_1 < \infty$) or $[0, \infty)$ (if $T_1 = \infty$).
3.3. Decomposition of the space \( \mathbb{L}^2_\sigma(\Omega) \). Let \( \kappa > 0 \) be fixed. It is proven in [39] that the number of positive eigenvalues of the self-adjoint operator \( \nu A + (1 + \kappa)B^2_s \) is finite. Let us denote these eigenvalues by \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \), each of them being repeated as many times as is its multiplicity. Let \( \phi_1, \ldots, \phi_N \) be associated eigenfunctions. We can assume that the eigenfunctions have been chosen so that they constitute an orthonormal system in \( \mathbb{L}^2_\sigma(\Omega) \). Denote by \( \mathbb{L}^2_\sigma(\Omega)' \) the linear hull of \( \phi_1, \ldots, \phi_N \) and by \( P' \) the orthogonal projection of \( \mathbb{L}^2_\sigma(\Omega) \) onto \( \mathbb{L}^2_\sigma(\Omega)' \). The orthogonal complement to \( \mathbb{L}^2_\sigma(\Omega)' \) in \( \mathbb{L}^2_\sigma(\Omega) \) is denoted by \( \mathbb{L}^2_\sigma(\Omega)'' \) and the orthogonal projection of \( \mathbb{L}^2_\sigma(\Omega) \) onto \( \mathbb{L}^2_\sigma(\Omega)'' \) is denoted by \( P'' \). Then we have

\[
\mathbb{L}^2_\sigma(\Omega) = \mathbb{L}^2_\sigma(\Omega)' \oplus \mathbb{L}^2_\sigma(\Omega)''
\]

and the operator \( \nu A + (1 + \kappa)B^2_s \) is reduced on each of the subspaces \( \mathbb{L}^2_\sigma(\Omega)' \) and \( \mathbb{L}^2_\sigma(\Omega)'' \). Using the negative definiteness of \( \nu A + (1 + \kappa)B^2_s \) in \( \mathbb{L}^2_\sigma(\Omega)'' \), one can easily derive that

\[
\left( (\nu A + B^2_s) \phi, \phi \right)_{L_2} \leq -c_8 |\phi|^2_{L_2}
\]

for all \( \phi \in \mathbb{L}^2_\sigma(\Omega)'' \cap D(A) \), where \( c_8 = \kappa \nu / (1 + \kappa) \). Inequality (3.4) shows that the operator \( \nu A + B^2_s \) is essentially dissipative in \( \mathbb{L}^2_\sigma(\Omega)'' \).

3.4. Splitting of the problem (3.1), (3.2). We show in subsection 3.2 that the solution \( v \) of (3.1), (3.2) can be expressed in the form \( v = w + z \), where \( w, z \) are solutions of the equations

\[
\begin{align*}
\frac{dw}{dt} - \omega B^0 w - u \infty B^1 w &= \nu A w + B^2_s w - P'[\omega B^0 w + u \infty B^1 w - \kappa B^2_s w] \\
&\quad + P'' B^2_s w + P''[N v_t + i N v_i], \\
\frac{dz}{dt} - \omega B^0 z - u \infty B^1 z &= \nu A z + B^2 z + P' [\omega B^0 w + u \infty B^1 w - \kappa B^2_s w + B^2_s w] \\
&\quad + P'[N v_t + i N v_i]
\end{align*}
\]

with the initial conditions

\[
w(0) = 0, \quad z(0) = \delta(\zeta + i \eta)
\]

on the interval \( (0, T) \) in the class (2.5). We also show in subsection 3.2 that (3.5) is an equation in the space \( \mathbb{L}^2_\sigma(\Omega)'' \), where the operator \( \nu A + B^2_s \) is essentially dissipative. On the other hand, all terms on the right hand side of equation (3.6), except for \( \nu A z + B^2 z \), belong to the finite-dimensional space \( \mathbb{L}^2_\sigma(\Omega)' \), where all norms are equivalent. These properties play an important role in the estimates of \( w \) and \( z \) (and therefore also estimates of \( v \)), derived in subsections 4.3–4.5.

Note that the belonging of \( w \) and \( z \) to the class (2.5) is not sufficient to guarantee that the term \( P'' B^0 w \) in equation (3.6) has a sense, because \( w(t) \) need not generally be in \( D(\mathcal{L}^0) \) for a.a. \( t \in (0, T) \). Nevertheless, we also explain in subsection 4.1 how one should understand the meaning of \( P'' B^0 w \).

3.5. Estimates of functions \( z, w \) and \( v \). Equation (3.6) can also be written in the form

\[
\frac{dz}{dt} = \mathcal{L} z + P' [\omega B^0 w + u \infty B^1 w - \kappa B^2_s w + B^2_s w] + P'[N v_t + i N v_i].
\]

Using the integral representation of \( z \) by means of the variation of parameters formula, we get
\[ z = z_1 + z_2, \] where

\[ z_1(t) = e^{Lt}z(0) = \delta e^{(a+ib)t} (\zeta + i\eta), \]

\[ z_2(t) = \int_0^t e^{L(t-\tau)} P'[\omega B^0 w(\tau) + u_\infty B^1 w(\tau) - \kappa B^2 w(\tau) + B^2 w(\tau)] d\tau \]

\[ + \int_0^t e^{L(t-\tau)} P' [\mathcal{N} v_1(\tau) + \mathcal{N} v_1(\tau)] d\tau. \] (3.8)

The function \( z_1 \) satisfies the obvious equality

\[ \|z_1(t)\|_2 = \delta \|\zeta + i\eta\|_2 e^{at}. \] (3.9)

Let \( \mu \in (a, 2a) \). Since \( a = s(L) = \gamma(e^{Lt}) \), there exists \( M_\mu > 0 \) such that

\[ \|e^{Lt}\phi\|_2 \leq M_\mu e^{at} \|\phi\|_2 \] (3.10)

for all \( \phi \in L^2_\mu(\Omega) \) and \( t \geq 0 \). Applying in (3.8) this inequality, along with (3.10) and the estimates from subsection 4.1, we obtain

\[ \|z_2(t)\|_2 \leq \int_0^t M_\mu e^{\mu(t-\tau)} c_9 \|w(\tau)\|_2 d\tau + \frac{M_\mu c_{10} \delta^2 K^2}{2a - \mu} e^{2at}, \] (3.11)

where \( c_9 \) and \( c_{10} \) are appropriate positive constants. The function \( w \) can be estimated from equation (3.5), multiplying (3.5) by \( w \) and integrating in \( \Omega \), see subsection 4.3. We get:

\[ \|w(t)\|_2^2 + c_8 \int_0^t |w(\tau)|_{1,2}^2 d\tau \leq \frac{c_{11} \delta^4 K^4}{4a} e^{4at}. \] (3.12)

From (3.11) and (3.12), we get, in particular,

\[ \|z_2(t)\|_2 \leq c_{12} (\delta K e^{at})^2, \] (3.13)

which, combined with (3.8), yields

\[ \|z(t)\|_2 \leq \delta \|\zeta + i\eta\|_2 e^{at} + c_{12} (\delta K e^{at})^2. \] (3.14)

Similarly, we also derive the inequality

\[ \|z(t)\|_{1,2} \leq \delta \|\zeta + i\eta\|_{1,2} e^{at} + c_{13} (\delta K e^{at})^2 \equiv \delta e^{at} + c_{13} (\delta K e^{at})^2; \] (3.15)

see Subsection 4.4 for the details.

We also obtain an estimate of \( |w|_{1,2} \), by multiplying equation (3.5) by \((-Aw)\), integrating in \( \Omega \) and using inequalities (2.1), (2.2), (2.3) and (3.3). Summing then appropriately the estimates of \( \|z\|_2, \|z\|_{1,2}, \|w\|_2 \) and \( \|w\|_{1,2} \), we get

\[ \|v(t)\|_{1,2} \leq c_{14} (\delta e^{at}) + (c_{15} + c_{16} K) K^2 (\delta e^{at})^2 + c_{17} K^3 (\delta e^{at})^3, \] (3.16)

where the constants \( c_{14} - c_{17} \) are independent of \( \delta \) and \( K \). (See subsection 4.5.) Note that \( c_{14} > 1 \).

**3.6. Choice of the Number \( t^* \).** The right hand side of (3.16) is less than the right hand side of (3.3) at the initial time \( t = 0 \) if

\[ c_{14} + (c_{15} + c_{16} K) K^2 \delta + c_{17} K^3 \delta^2 < K, \]
which is satisfied if $K$ and $\delta$ are chosen so that

$$
c_{14} < K, \quad (c_{15} + c_{16} K) K^2 \delta + c_{17} K^3 \delta^2 < K - c_{14}.
$$

(3.17)

Assuming that (3.17) holds and using the fact that the right hand side of (3.16) grows with increasing $t$ faster than the right hand side of (3.3), we deduce that there exists $0 < t^* \leq T_1$ such that the right hand sides of (3.16) and (3.3) coincide at the time $t^*$. It means that

$$
c_{14} (\delta e^{at^*}) + (c_{15} + c_{16} K) K^2 (\delta e^{at^*})^2 + c_{17} K^3 (\delta e^{at^*})^3 = \delta K e^{at^*}.
$$

This yields

$$
\delta e^{at^*} = c_{18}(K) := \frac{2(K - c_{14})}{c_{17} K^2 [(c_{15} + c_{16} K) + \sqrt{(c_{15} + c_{16} K)^2 + 4(K - c_{14})}]}. \quad (3.18)
$$

3.7. Lower estimates of $\|v(t^*)\|_2$. (3.9) and (3.13) yield

$$
\|z(t^*)\|_2 \geq \|z_1(t^*)\|_2 - \|z_2(t^*)\|_2 \geq \delta \|\zeta + i \eta\|_2 e^{at^*} - c_{12} \delta^2 K^2 2 e^{2at^*}.
$$

Hence, due to (3.12), we also have

$$
\|v(t^*)\|_2 \geq \|z(t^*)\|_2 - \|w(t^*)\|_2 \geq \delta \|\zeta + i \eta\|_2 e^{at^*} - c_{19} \delta^2 K^2 2 e^{2at^*},
$$

where $c_{19} = c_{12} + \sqrt{c_{11}/(2 \sqrt{a})}$. Expressing $\delta e^{at^*}$ from (3.18), we obtain

$$
\|v(t^*)\|_2 \geq c_{18}(K) \left[ \|\zeta + i \eta\|_2 - c_{19} K^2 c_{18}(K) \right] =: c_{20}(K). \quad (3.19)
$$

If $K > c_{14}$ is chosen sufficiently close to $c_{14}$ then $\|\zeta + i \eta\|_2 > c_{19} K^2 c_{18}(K)$, which means that $c_{20}(K)$ is positive. It is remarkable that it is independent of $\delta$.

3.8. Completion of the proof. Recall that $v = v_r + iv_i$, where $v_r$ and $v_i$ are real solutions of equation (2.4), satisfying the initial conditions $v_r(0) = \delta \zeta$ and $v_i(0) = \delta \eta$, respectively. Put $\epsilon := c_{20}(K)/\sqrt{2}$. Inequality (3.19) implies that either $\|v_r(t^*)\|_2 \geq \epsilon$ or $\|v_i(t^*)\|_2 \geq \epsilon$. Thus, given $\delta > 0$ arbitrarily small (satisfying (3.17)), there exists a real solution $v^*$ of equation (2.4) (i.e. $v^* = v_r$ or $v^* = v_i$) whose initial $W^{1,2}$-norm is less than or equal to $\delta$ and the $L^2$-norm at the time $t = t^*$ is greater than or equal to $\epsilon$. This completes the proof of Theorem 2.

□

4 Appendix

4.1. Estimate (3.11). The crucial point in the derivation of (3.11) are the inequalities

$$
\int |x|^2 |\Delta \phi_k|^2 \, dx + \int |x|^2 |\nabla \phi_k|^2 \, dx < \infty \quad (k = 1, \ldots, N),
$$

(4.1)

see [24] Lemma 7. They enable one to show that

$$
\|P' B^0 \phi\|_2 + \|P' B^1 \phi\|_2 + \|P' B^2 \phi\|_2 + \|P' B^3 \phi\|_2 \leq c_{21} |\phi|_{1,2}
$$

(4.2)

for all $\phi \in D(A)$, see [24]. Using especially the fact that $P'$ is the projection onto the $N$-dimensional space $L^2_\Omega(\Omega)'$, where the norms $\|\cdot\|_2$ and $|\cdot|_{1,2}$ are equivalent, and mainly copying the procedure from [24], one can show that (4.2) is also satisfied with $c_{22} \|\phi\|_2$ on the right hand
side instead of \( c_{21} |\phi|_{1,2} \). Inequalities \( (4.1) \) imply that the term \( P' B^0 \phi \) is well defined, although \( B^0 \phi \) is not necessarily in \( L^2_0(\Omega) \) for \( \phi \in D(A) \). (Recall that the inclusion \( B^0 \phi \in L^2_0(\Omega) \) is guaranteed if \( \phi \in D(L^0) \).) Nevertheless, even for \( \phi \in D(A) \), one can put

\[
P' B^0 \phi := \sum_{k=1}^N \left( \int_{\Omega} \left[ (e_1 \times x) \cdot \nabla \phi - e_1 \times \phi \right] \cdot \phi_k \, dx \right) \phi_k, \tag{4.3}
\]

where the integral over \( \Omega \) equals

\[
\lim_{R \to \infty} \int_{\partial \Omega_R} \left[ (e_1 \times x) \cdot n \right] \phi \cdot \phi_k \, dS - \int_{\Omega_R} \left[ (e_1 \times x) \cdot \nabla \phi_k - e_1 \times \phi_k \right] \cdot \phi \, dx
\]

\[
= \lim_{R \to \infty} \left( \int_{\partial \Omega_R} \left[ (e_1 \times x) \cdot n \right] \phi \cdot \phi_k \, dS - \int_{\Omega_R} \left[ (e_1 \times x) \cdot \nabla \phi_k - e_1 \times \phi_k \right] \cdot \phi \, dx \right)
\]

\[
= \int_{\Omega} \left[ (e_1 \times x) \cdot \nabla \phi_k - e_1 \times \phi_k \right] \cdot \phi \, dx.
\]

(The surface integral over \( \partial \Omega_R \) equals zero because the integrand is equal to zero a.e. in \( \partial \Omega_R \).) Inequalities \( (4.1) \) guarantee the convergence of the last integral.

The term involving the nonlinear operator \( N \) can be estimated as follows:

\[
\| P' N \phi \|_2 = \sup_{\psi \in L^2_0(\Omega)} \frac{|(P' N \phi, \psi)_2|}{\| \psi \|_2} = \sup_{\psi \in L^2_0(\Omega)'} \frac{|(N \phi, \psi)_2|}{\| \psi \|_2}
\]

\[
= \sup_{\psi \in L^2_0(\Omega)'} \frac{1}{\| \psi \|_2} \left| \int_{\Omega} \phi \cdot \nabla \psi \cdot \phi \, dx \right| \leq \sup_{\psi \in L^2_0(\Omega)'} \frac{\| \phi \|^2_2}{\| \psi \|_2} \| \phi \|_1, \tag{4.4}
\]

(We use Hölder’s and Sobolev’s inequalities and the inclusion \( \psi \in L^2_0(\Omega)' \).) Applying these inequalities to the integrals in the formula for \( z_2(t) \) and estimating the norms \( \| v_t \|_{1,2} \) and \( \| v \|_{1,2} \) by means of \( (3.2) \), we obtain \( (3.1) \).

4.2. The system \( (3.5), (3.6) \). Let us first show that \( (3.5) \) is an equation in \( L^2_0(\Omega)'' \). Denote \( w^t := P^t w \) and \( w^{''} := P'' w \). We claim that \( w^t \equiv 0 \). Since \( dw'/dt \equiv P'(dw/dt) \) for a.a. \( t \in (0, T) \) and, from \( (2.5) \) and \( (4.3) \), we have

\[
P_t \frac{dw}{dt} = P' \left( \frac{dw}{dt} - \omega B^0 w - u_\infty B^1 w \right) + P' \left( \omega B^0 w + u_\infty B^1 w \right),
\]

equation \( (3.5) \) can be rewritten as follows

\[
\frac{dw^t}{dt} + \left( \frac{dw}{dt} - \omega B^0 w - u_\infty B^1 w \right) = \nu A w + B^2_s w + P' \left( \frac{dw}{dt} - \omega B^0 w - u_\infty B^1 w \right)
\]

\[
+ \kappa \left( P'' B^2_s w + P'' B^2_a w + P'' |N v_t + i N v_i|, \right)
\]

\[
\frac{dw^t}{dt} + P'' \left( \frac{dw}{dt} - \omega B^0 w - u_\infty B^1 w \right) = \left[ \nu A w^t + (1 + \kappa) B^2_s w \right] + \left[ \nu A w^{''} + (1 + \kappa) B^2_s w^{''} \right]
\]

\[
- \kappa \left( P'' B^2_s w + P'' B^2_a w + P'' |N v_t + i N v_i|, \right).
\]

Projecting the last equation on \( L^2_0(\Omega)' \) and using the fact that the operator \( \nu A + (1 + \kappa) B^2_s \) is reduced on \( L^2_0(\Omega)' \) and \( L^2_0(\Omega)'' \), we obtain

\[
\frac{dw'}{dt} = \nu A w' + (1 + \kappa) B^2_s w'.
\]
This, together with the initial condition \( w'(0) = 0 \), yields \( w' \equiv 0 \).

Assume that \((w, z)\) is a solution of (3.5)–(3.6). Summing the equations (3.5), (3.6) we observe that \( v \equiv w + z \) satisfies equation (3.1). Similarly, the sum of the initial conditions in (3.7) yields (3.2).

On the other hand, if \( v \) is a solution of (3.1), (3.2) on the time interval \((0, T)\) then, applying the same method as in [3], one can at first solve equation (3.5) with the initial condition \( w(0) = 0 \) as a linear problem for the unknown \( w \), and afterwards equation (3.6) with the initial condition \( z(0) = \delta(\zeta + i\eta) \) as a linear problem for the unknown \( z \). Both problems are uniquely solvable on the same interval \((0, T)\).

4.3. Estimate (3.12). We multiply equation (3.5) by \( w \), integrate in \( \Omega \) and apply the next identity, which comes from [24] Lemma 1 and holds for a.a. \( t \in (0, T) \):

\[
\frac{1}{2} \frac{d}{dt} \|w\|^2_2 + \int_\Omega (\frac{d}{dt}w - \omega B^0 w - u_\infty B^1 w) \cdot w \, dx = \frac{1}{2} \frac{d}{dt} \|w\|^2_2 \tag{4.5}
\]

(Lemma 1 in [24] is in fact formulated for a solution \( v \) of a concrete equation, but the equation is not used in the proof.) Thus, applying (4.5), (3.4) and the identities \( (P'' B^2 w, w)_2 = 0 \) and \( (P'(\omega B^0 w + u_\infty B^1 w + \kappa B^2_s w), w)_2 = 0 \) (following from the inclusion \( w \in L^2_\sigma(\Omega)'' \) and the fact that \( B^2_s \) is skew symmetric), we obtain

\[
\frac{d}{dt} \|w\|^2_2 + c_8 \|w\|^2_{1,2} + (\mathcal{N}v_r, w)_2 + (\mathcal{N}v_i, w)_2. \tag{4.6}
\]

Using Hölder’s and Sobolev’s inequalities, we get

\[
(\mathcal{N}v_r, w)_2 = \int_\Omega v_r \cdot \nabla v_r \cdot w \, dx = \int_\Omega v_r \cdot \nabla w \cdot v_r \, dx \\
\leq \|v_r\|_4^2 \|w\|_{1,2} \leq \iota \|w\|^2_{1,2} + c(\iota) \|v_r\|_4 \|v_r\|_2 \|v_r\|^3_{1,2}.
\]

The term \((\mathcal{N}v_i, w)_2\) can be estimated in the same way. Applying these estimates of \((\mathcal{N}v_r, w)_2\) and \((\mathcal{N}v_i, w)_2\) to (4.6), choosing \( \iota \) sufficiently small and also applying inequality (3.3), we obtain

\[
\frac{d}{dt} \|w\|^2_2 + c_8 \|w\|^2_{1,2} \leq c_{11} (\delta K e^{\alpha t})^4. \tag{4.7}
\]

Integrating (4.7) with respect to \( t \) and using the initial condition \( w(0) = 0 \), we derive (3.12).

4.4. Estimate (3.15). In order to estimate the norm \( \|z(t)\|_{1,2} \), we use the inequality

\[
\|\phi\|_{2,2} + \|B^0 \phi\|_2 \leq c_{24} \left( \|\mathcal{L} \phi\|_2 + \|\phi\|_2 \right) \tag{4.8}
\]

for \( \phi \in D(\mathcal{L}) \), which follows from [30] or [12]. (The inequality is proven for \( \mathcal{L}^0 \) instead of \( \mathcal{L} \) on the right hand side in [30] and [12], but it can be easily modified to the form (4.8).) As the integrands in the formula for \( z_2 \) lie in \( D(\mathcal{L}) \) for a.a. \( \tau \in (0, t) \), we obtain

\[
\|z(t)\|_{1,2} \leq \delta \|\zeta + i\eta\|_{1,2} e^{\alpha t} \\
+ c_{24} \int_0^t \|\mathcal{L} e^{\mathcal{L}(t-\tau)} P' [\omega B^0 w(\tau) + u_\infty B^1 w(\tau) - \kappa B^2_s w(\tau) + B^2_s w(\tau)] \|_2 \, d\tau \\
+ c_{24} \int_0^t \|\mathcal{L} e^{\mathcal{L}(t-\tau)} P' [\mathcal{N}v_r(\tau) + \mathcal{N}v_i(\tau)] \|_2 \, d\tau
\]
Since \( \mathcal{L} \) commutes with \( e^{\mathcal{L}(t-\tau)} \) and \( P' \) is a projection onto a finite-dimensional space, we derive (3.15) in the same way as (3.14).

### 5. Estimate (3.16).

In order to derive an estimate of \( |w|_{1,2} \), we multiply equation (3.5) by \((-A\omega)\) and use the formula

\[
\int_\Omega \left( \frac{d}{dt} - \omega B^0 w - u_\infty B^1 w \right) \cdot (-A\omega) \, dx = \frac{d}{dt} \frac{1}{2} |w|_{1,2}^2,
\]

which follows from [24 Lemma1], similarly as (4.5). If we also apply the inequalities (2.2), (2.3) and (4.2), we get

\[
\frac{d}{dt} \frac{1}{2} |w|_{1,2}^2 + \nu \| Aw \|_2^2 = -(B^2 z, Aw)_2 + (P'[\omega B^0 w + u_\infty B^1 w - \kappa B_s^2 w], Aw)_2 \\
- (P'[\omega B^0 w + u_\infty B^1 w - \kappa B_s^2 w], Aw)_2 \\
\leq c |w|_{1,2} \| Aw \|_2 + c \| Nv_t + Nv_i \|_2 \| Aw \|_2 \\
\leq \frac{\nu}{4} |w|_2^2 + c(\nu) |w|_{1,2}^2 + c \| Aw \|_2 \| v \|_{1,2}^{3/2} \| Aw \|_2 \\
\leq \frac{\nu}{4} |w|_2^2 + c(\nu) |w|_{1,2}^2 + c(\nu) |w|_2^2 + c(\nu) \delta K e^{at}^6, \\
\frac{d}{dt} |w|_{1,2}^2 + \nu \| Aw \|_2^2 \leq c_{25} |w|_{1,2}^2 + \nu \| Az \|_2^2 + c_{26} \delta K e^{at}^6, \tag{4.10}
\]

where \( c_{25} = c_{25}(\nu) \) and \( c_{26} = c_{26}(\nu) \).

In order to get rid of the term \( |Az|_{1,2}^2 \) on the right hand side of (4.10), we multiply equation (3.6) by \((-Az)\) and integrate in \( \Omega \). Applying formula (4.9) and inequalities (3.3), (3.15), (4.2) and (4.3), we obtain

\[
\frac{d}{dt} \frac{1}{2} |z|_{1,2}^2 + \nu \| Az \|_2^2 = -(B^2 z, Az)_2 - (P'[\omega B^0 w + u_\infty B^1 w - \kappa B_s^2 w + B_s^2 w], Az)_2 \\
- (P'[\omega B^0 w + u_\infty B^1 w - \kappa B_s^2 w], Az)_2 \\
\leq c_3 |z|_{1,2} \| Az \|_2 + c |w|_{1,2} \| Az \|_2 + c \| v \|_{1,2}^{3/2} \| Az \|_2 \\
\leq \frac{\nu}{2} \| Az \|_2^2 + c(\nu) |z|_{1,2}^2 + c(\nu) |w|_{1,2}^2 + c(\nu) \| v \|_2 \| v \|_{1,2}^2, \\
\frac{d}{dt} |z|_{1,2}^2 + \nu \| Az \|_2^2 \leq c_{27} \left[ \delta^2 e^{2at} + 2 c_{13} \delta e^{at} (\delta K e^{at})^2 + c_{13}^2 (\delta K e^{at})^4 \right] \\
+ c_{28} |w|_{1,2}^2 + c_{29} (\delta K e^{at})^4 \\
\leq c_{28} |w|_{1,2}^2 + c_{27} \delta^2 e^{2at} + 2 c_{13} (\delta K e^{at})^3 + (c_{13}^2 + c_{29}) (\delta K e^{at})^4, \tag{4.11}
\]
where $c_{27}$, $c_{28}$ and $c_{29}$ depend only on $\nu$. Multiplying inequality (4.7) by $c_{30} := (c_{25} + c_{28})/c_8$ and summing it with (4.11) and (4.10), we obtain

$$\frac{d}{dt} \left( |w(t)|^2_{1,2} + |z(t)|^2_{1,2} + c_{30} \|w(t)\|^2_2 + \|z(t)\|^2_2 \right) \leq c_{27} \delta^2 e^{2at} + 2c_{13} (\delta K e^{at})^3 + c_{31} (\delta K e^{at})^4 + c_{26} (\delta K e^{at})^6,$$

where $c_{31} := c_{13} + c_{29} + c_{11} c_{30}$. Integrating this inequality from 0 to $t$ and summing with (3.14) squared, we obtain

$$|w(t)|^2_{1,2} + |z(t)|^2_{1,2} + c_{30} \|w(t)\|^2_2 + \|z(t)\|^2_2 \leq \left[ \delta^2 |\zeta + i\eta|^2_{1,2} + \frac{c_{27}}{2a} \delta^2 e^{2at} + \frac{2c_{13}}{3a} (\delta K e^{at})^3 + \frac{c_{31}}{4a} (\delta K e^{at})^4 + \frac{c_{26}}{6a} (\delta K e^{at})^6 \right] + \left[ \delta^2 |\zeta + i\eta|^2_{2} e^{2at} + \frac{2c_{12}}{K} \|\zeta + i\eta\|^2_{2} (\delta K e^{at})^3 + c_{12} (\delta K e^{at})^4 \right] \leq \left( 1 + \frac{c_{27}}{2a} \right) \delta^2 e^{2at} + \frac{2(3ac_{12} + c_{13})}{3a} (\delta K e^{at})^3 + \left( \frac{c_{31}}{4a} + c_{12} \right) (\delta K e^{at})^4 + \frac{c_{26}}{6a} (\delta K e^{at})^6.$$

We may assume without loss of generality that $c_{30} > 1$. Then the left hand side is greater than or equal to $\|w(t)\|^2_{1,2} + \|z(t)\|^2_{1,2}$. Thus, using the inequality $\|w(t)\|^2_{1,2} \leq 2 \|w(t)\|^2_2 + 2 \|z(t)\|^2_2$, we get

$$\|w(t)\|^2_{1,2} + \|z(t)\|^2_{1,2} \leq \left( 1 + \frac{c_{27}}{2a} \right) \delta^2 e^{2at} + \frac{2(3ac_{12} + c_{13})}{3a} (\delta K e^{at})^3 + \left( \frac{c_{31}}{4a} + c_{12} \right) (\delta K e^{at})^4 + \frac{c_{26}}{6a} (\delta K e^{at})^6.$$

From this, we derive (3.16) by standard manipulations. The constants $c_{14} - c_{17}$ in (3.16) depend only on $a$, $c_{12}$, $c_{13}$, $c_{26}$, $c_{27}$ and $c_{31}$. 

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