Distribution of Points of Interpolation and of Zeros of Exactly Maximally Convergent Multipoint Padé Approximants

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Abstract
Given a regular compact set $E$ in $\mathbb{C}$, a unit measure $\mu$ supported by $\partial E$, a triangular point set $\beta := \{ (\beta_{n,k}^m)_{n=1}^{m} \}$, $\beta \in \partial E$ and a function $f$, holomorphic on $E$, let $\pi_{n,m}^{\beta,f}$ be the associated multipoint $\beta$-Padé approximant of order $(n,m)$. We show that if the sequence $\pi_{n,m}^{\beta,f}$, $n \in \Lambda$, $\Lambda \subseteq \mathbb{N}$, $m$-fixed, converges exactly $\mu$-maximally to $f$ with respect to the $m$-meromorphy, then the points $\beta_{n,k}^m$ are uniformly distributed on $\partial E$ with respect to $\mu$ as $n \in \Lambda$. Furthermore, a result about the behavior of the zeros of the exact maximally convergent sequence $\Lambda$ is provided, under the condition that $\Lambda$ is “dense enough”.

Keywords
Multipoint Padé Approximants, Maximal Convergence, Domain of m-Meromorphy

1. Introduction
We first introduce some needed notations.

Let $\Pi_n$, $n \in \mathbb{N}$ be the class of the polynomials of degree $\leq n$ and $R_{r,s} := \{ r = p/q, p \in \Pi_s, q \in \Pi_r, q \neq 0 \}$. Given a compact set $E$, we say that $E$ is regular, if the unbounded component of the complement $E^c := \mathbb{C} \setminus E$ is solvable with respect to Dirichlet problem. We will assume throughout the paper that $E$ possesses a connected complement $E^c$. In what follows, we will be working with the max-norm $\| \cdot \|_E$ on $E$, that is $\| \cdot \|_E := \max_{z \in E} | \cdot (z) |$.

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Let \( B(E) \) be the class of the unit measures supported on \( E \), that is \( \text{supp}(\cdot) \subseteq E \). We say that the infinite sequence of Borel measures \( \{\mu_n\} \in B(E) \) converges in the weak topology to a measure \( \mu \) and write \( \mu_n \to \mu \), if
\[
\int g(t) \, d\mu_n \to \int g(t) \, d\mu
\]
for every function \( g \) continuous on \( E \). We associate with a measure \( \mu \in B(E) \), the logarithmic potential
\[
U_\mu(z) := \int \log \frac{1}{|z-t|} \, d\mu.
\]
Recall that \( U_\mu \) ([1]) is a function superharmonic in \( \mathbb{C} \), subharmonic in \( \mathbb{C} \setminus \text{supp}(\mu) \), harmonic in \( \text{supp}(\mu) \) and
\[
U_\mu(z) = \ln \frac{1}{|z|} + o(1), \quad z \to \infty.
\]
We associate with a polynomial \( p \in \Pi_n \), the normalized counting measure \( \mu_p \) of \( p \), that is
\[
\mu_p(F) := \frac{\text{number of zeros of } p \text{ on } F}{\deg p},
\]
where \( F \) is a point set in \( \mathbb{C} \).

Given a domain \( B \subset \mathbb{C} \), a function \( g \) and a number \( m \in \mathbb{N} \), we say that \( g \) is \( m \)-meromorphic in \( B \) \( \{g \in \mathcal{M}_m(B)\} \) if \( g \) has no more than \( m \) poles in \( B \) (poles are counted with their multiplicities). We say that a function \( f \) is holomorphic on the compactum \( E \) and write \( f \in \mathcal{A}(E) \), if it is holomorphic in some open neighborhood of \( E \).

Let \( \beta \) be an infinite triangular table of points , \( \beta := \{(\beta_{n,k})_{k=1}^n\} \), \( \beta_{n,k} \in E \), with no limit points outside \( E \) (we write \( \beta \in E \)). Set
\[
\omega_{\beta}(z) := \prod_{k=1}^n (z - \beta_{n,k}).
\]

Let \( f \in \mathcal{A}(E) \) and \( (n,m) \) be a fixed pair of nonnegative integers. The rational function \( p_{n,m}^{\beta,f} := p/q \) where the polynomials \( p \in \Pi_n \) and \( q \in \Pi_m \) are such that
\[
\frac{fq - p}{\omega_{n,m+1}} \in \mathcal{A}(E)
\]
is called a \( \beta \)-multipoint Padé approximant of \( f \) of order \( (n,m) \). As is well known, the function \( p_{n,m}^{\beta,f} \) always exists and is unique [3] [4]. In the particular case when \( \beta \equiv 0 \), the multipoint Padé approximant \( p_{n,m}^{\beta,f} \) coincides with the classical Padé approximant \( \pi_{n,m}^{f} \) of order \( (n,m) \) ([5]).

Set
\[
\pi_{n,m}^{\beta,f} = \frac{p_{n,m}^{\beta,f}}{q_{n,m}^{\beta,f}},
\]
where the polynomials \( p_{n,m}^{\beta,f} \) and \( q_{n,m}^{\beta,f} \) do not have common divisors. The zeros of \( Q_{n,m}^{\beta,f} \) are called free zeros of \( \pi_{n,m}^{\beta,f} \); \( \deg Q_{n,m}^{\beta,f} \leq m \).

We say that the points \( \beta_{n,k} \) are uniformly distributed relatively to the measure \( \mu \), if
\[
\mu_{\beta_{n,k}} \to \mu, \quad n \to \infty.
\]

We recall the notion of \( m_t \)-Hausdorff measure (cf. [6]). For \( \Omega \subset \mathbb{C} \), we set
\[ m_1(\Omega) := \inf \left\{ \sum \nu |V_\nu| \right\} \]

where the infimum is taken over all coverings \( \{\sum \nu |V_\nu| \} \) of \( \Omega \) by disks and \( |V_\nu| \) is the radius of the disk \( V_\nu \).

Let \( D \) be a domain in \( \mathbb{C} \) and \( \varphi \) a function defined in \( D \) with values in \( \overline{\mathbb{C}} \). A sequence of functions \( \{\varphi_\nu\} \), meromorphic in \( D \), is said to converge to a function \( \varphi \) \( m_1 \)-almost uniformly inside \( D \) if for any compact subset \( K \subset D \) and every \( \varepsilon > 0 \) there exists a set \( K_\varepsilon \subset K \) such that \( m_1(K \setminus K_\varepsilon) < \varepsilon \) and the sequence \( \{\varphi_\nu\} \) converges uniformly to \( \varphi \) on \( K_\varepsilon \).

For \( \mu \in \mathcal{B}(E) \), define

\[ \rho_{\min} := \inf_{z \in E} e^{-U^\mu(z)} , \]

and

\[ \varrho_{\max} := \max_{z \in E} e^{-U^\mu(z)} ; \]

(\( U^\mu \) is superharmonic on \( E \); hence, it attains its minimum (on \( E \)). As is known ([1] [7]),

\[ e^{-U^\mu(z)} \geq \rho_{\min} , \quad z \in E^c , \]

Set, for \( r > \rho_{\min} \),

\[ E_\mu(r) := \left\{ z \in \mathbb{C} , e^{-U^\mu(z)} < r \right\} . \]

Because of the upper semicontinuity of the function \( \chi(z) := e^{-U^\mu(z)} \), the set \( E_\mu(r) \) is open; clearly \( E_\mu(r_1) \subset E_\mu(r_2) \) if \( r_1 \leq r_2 \) and \( E_\mu(r) \supset E \) if \( r > \varrho_{\max} \).

Let \( f \in \mathcal{A}(E) \) and \( m \in \mathbb{N} \) be fixed. Let \( R_{m,\mu}(f) = R_{m,\mu} \) and \( D_{m,\mu}(f) = D_{m,\mu} := E_\mu(R_{m,\mu}) \) denote, respectively, the radius and domain of \( m \)-meromorphy with respect to \( \mu \), that is

\[ R_{m,\mu} := \sup \left\{ r , f \in \mathcal{M}_m \left( E_\mu(r) \right) \right\} . \]

Furthermore, we introduce the notion of a \( \mu \)-maximal convergence to \( f \) with respect to the \( m \)-meromorphy of a sequence of rational functions \( \{r_{n,\nu}\} \) (a \( \mu \)-maximal convergence), that is, for any \( \varepsilon > 0 \) and each compact set \( K \subset D_m \), there exists a set \( K_\varepsilon \subset K \) such that \( m_1(K \setminus K_\varepsilon) < \varepsilon \) and

\[ \limsup_{n \to \infty} \left\| f - r_{n,\nu} \left( 0 , f \right) \right\|_{K_\varepsilon} \leq \frac{e^{-U^\mu}}{R_{m,\mu}(f)} . \]

Hernandez and Calle Ysern proved the followings:

**Theorem A [8]:** Let \( E , \mu , \beta \) and \( \omega_\alpha , n = 1,2, \ldots \) be defined as above. Suppose that \( \mu_\omega_\alpha \to \mu \) as \( n \to \infty \) and \( f \in \mathcal{A}(E) \). Then, for each fixed \( m \in \mathbb{N} \), the sequence \( n_{\mu_\omega_\alpha} \) converges to \( f \) \( \mu \)-maximally with respect to the \( m \)-meromorphy.

Theorem A generalizes Saft’'s theorem of Montessus de Ballore’s type about multipoint Padé approximants (see [3]).

We now utilize the normalization of the polynomials \( Q_{n,m}(z) \) with respect to a given open set \( D_{m,\mu} \), that is,

\[ Q_{n,m}(z) = \prod \left( z - \alpha_{n,k} \right) \prod \left( 1 - \frac{1}{\alpha_{n,k}} \right) , \]

where \( \alpha_{n,k} \), \( \alpha_{n,k}^* \) are the zeros lying inside, resp. outside \( D_{m,\mu} \). Under this normalization, for every compact set \( K \) and \( n \) large enough there holds

\[ \left\| Q_{n,m}^{R/f} \right\|_{K} \leq C_1 , \]

where \( C_1 = C_1(K) \) is a positive constant, depending on \( K \). In the sequel, we denote by \( C_1 \) positive constant, independent on \( n \) and different at different occurrences.

In [8], the set \( K_\varepsilon \) (look at the definition of a \( \mu \)-maximal convergence) is explicitly written, namely
For \( \Omega(\epsilon) \) we have
\[
m_{i}(\Omega(\epsilon)) \leq \epsilon.
\]
For points \( z \in K \setminus \Omega(\epsilon) \), we have
\[
\left| Q_{n,m}^{\beta,f}(z) \right| \geq C_{2}(\epsilon/mn^{2})^{k_{n}},
\]
where \( k_{n} \) stands for the number of the zeros of \( Q_{n,m}^{\beta,f} \) in \( D_{m,\mu} \) : \( k_{n} \leq m \).

Let \( Q \) be the monic polynomial, the zeros of which coincide with the poles of \( f \) in \( D_{m,\mu} \); \( \deg Q \leq m \). It was proved in [8] (Proof of Lemma 2.3) that for every compact subset \( K \) of \( D_{m,\mu} \)
\[
\limsup_{n \to \infty} \left\| f Q_{n,m}^{\beta,f} - Q_{n,m}^{\beta,f} \right\|_{K}^{\mu} \leq \frac{e^{-\mu R}}{R_{m,\mu}}.
\]

Hence, \(-U^{\mu}(z) - \log R_{m,\mu} \) is a harmonic majorant in \( D_{m,\mu} \) of the family \( \left\{ (f Q_{n,m}^{\beta,f} - Q_{n,m}^{\beta,f})(z) \right\}_{n=1}^{\infty} \).

**Theorem B [8]:** With \( E, \mu, m, \omega, \) and \( f \) as in Theorem A, assume that \( K \) is a regular compact set for which \( \log R_{m,\mu} \) is not attained at a point on \( E \). Suppose that the function \( f \) is defined on \( K \) and satisfies
\[
\limsup_{n \to \infty} \left\| f - \pi_{n,m}^{\beta,f} \right\|_{K}^{\mu} \leq \frac{e^{-\mu R}}{R_{m,\mu}} \quad R < 1.
\]
Then \( R \leq R_{m,\mu}(f) \).

Suppose that \( \infty > R_{m,\mu} > \sigma_{\max} \) and \( D_{m,\mu} \) is connected. Let \( V \) be a disk in \( D_{m,\mu} \setminus \{ z \in E \} \), centered at a point \( z_{0} \) of radius \( r > 0 \) and such that \( f \) is analytic on \( V \). Fix \( r_{1} \), \( 0 < r_{1} < r \) and set \( A := \{ z, r_{1} \leq |z - z_{0}| \leq r \} \). Fix a number \( \epsilon < (r - r_{1})/4 \). Introduce, as before, the set \( \Omega(\epsilon) \). Recall that
\[
m_{i}(\Omega(\epsilon)) \leq \epsilon.
\]
It is clear that the set \( A \setminus \Omega(\epsilon) \) contains a concentric circle \( \Gamma \) (otherwise we would obtain a contradiction with \( m_{i}(\Omega(\epsilon)) < (r - r_{1})/4 \)). We note that the function \( f \) and the rational functions \( \pi_{n,m}^{\beta,f} \) are well defined on \( \Gamma \). Viewing (3), we may write
\[
\limsup_{n \to \infty} \left\| f - \pi_{n,m}^{\beta,f} \right\|_{\Gamma}^{\mu} \leq \frac{e^{-\mu R}}{R_{m,\mu}}.
\]
Suppose that
\[
\limsup_{n \to \infty} \left\| f - \pi_{n,m}^{\beta,f} \right\|_{\Gamma}^{\mu} < \frac{e^{-\mu R}}{R_{m,\mu}}.
\]
or, what is the same,
\[
\limsup_{n \to \infty} \left\| f - \pi_{n,m}^{\beta,f} \right\|_{\Gamma}^{\mu} \leq \frac{e^{-\mu R}}{R_{m,\mu} + \sigma} < 1.
\]
for an appropriate \( \sigma > 0 \). Then,
\[
\left( f - \pi_{n,m}^{\beta,f} \right)(z) \leq C_{3}(n^{2} \frac{m}{\epsilon})^{w} \left( \frac{e^{-\mu R}}{R_{m,\mu} + \sigma} \right)^{n}
\]
for all \( z \in \Gamma \) and \( n \) large enough. This leads to
\[
\limsup_{n \to \infty} \left\| f - \pi_{n,m}^{\beta,f} \right\|_{\Gamma}^{\mu} \leq \frac{e^{-\mu R}}{R_{m,\mu} + \sigma}.
\]
using Theorem B, we arrive at $R_{n,m} + \sigma < R_{n,m}$. The contradiction yields
\[
\limsup_{n \to \infty} \left\| Q^n_{\alpha,m} f - Q^n_{\beta,m} f \right\|_{L^p} = \left\| e^{-U^s} \right\|_{L^p} / R_{n,m},
\]
where $V_{\Gamma}$ is the disk bounded by $\Gamma$.

Then the function $-U^s - \ln R_{n,m}$ is an exact harmonic majorant of the family $\left\{ \left. f Q^n_{\alpha,m} - P^n_{\beta,m}, \right|_{E} \right\}^{\infty}$ in $D_{n,m}$ (see (3)). Therefore, there exists a subsequence $\Lambda$ such that for every compact subset $K \subset D_{n,m} \setminus E$
\[
\lim_{n \to \infty} \left\| Q^n_{\alpha,m} f - P^n_{\beta,m} / Q^n_{\alpha,m} \right\|_{L^p} = \left\| e^{-U^s} \right\|_{L^p} / R_{n,m}.
\]
(see [9] [10] for a discussion of exact harmonic majorant). We will refer to this sequences as to an exact $\mu$-maximal convergent sequence to $f$ with respect to the $m$-meromorphy.

It is clear that for any $\varepsilon > 0$ and each compactum $K \subset D_{n,m}$ there exists a set $K_\varepsilon \subset K$ such that $m_1 (K \setminus K_\varepsilon) < \varepsilon$ and
\[
\lim_{n \to \infty} \left\| f - \pi^n_{\alpha,m} \right\|_{L^p_{K_\varepsilon}} = \left\| e^{-U^s} \right\|_{L^p} / (R_{n,m}).
\]

2. Main Results and Proofs

The main result of the present paper is

Theorem 1: Under the same conditions on $E$, assume that $\mu \in \mathcal{B}(\partial E)$ and that $\beta \subset \partial E$ is a triangular set of points. Let $m \in \mathbb{N}$ be fixed, $f \in \mathcal{A}(E)$ and $V_{\beta} \max < R_{n,m} < \infty$. Suppose that $D_{n,m}$ is connected. If for a subsequence $\Lambda$ of the multipoint Padé approximants $\pi^n_{\alpha,m}$ condition (4) holds, then $\mu_{n} \rightarrow \mu$ as $n \rightarrow \infty$, $n \in \Lambda$.

The problem of the distribution of the points of interpolation of multipoint Padé approximants has been investigated, so far, only for the case when the measure $\mu$ coincides with the equilibrium measure $\mu_{e}$ of the compact set $E$. It was first raised by Walsh ([11], Chp. 3) while considering maximally convergent polynomials with respect to the equilibrium measure. He showed that the sequence $\mu_{n}$ converged weakly to $\mu_{e}$ through the entire set $\mathbb{N}$ (respectively their associated balayage measures onto the boundary of $E$) iff the interpolating polynomials at the points of $\beta$ of every function $f_n(z)$ of the form $f_n(z) := 1 / (1 - z)$, $t$-fixed, $t \in E$, converged $\mu_{e}$-maximally to $f$. Walsh’s result was extended to multipoint Padé approximants with a fixed number of the free poles by Ikonomov in [12], as well as to generalized Padé approximants, associated with a regular condenser [13]. The case of polynomial interpolation of an arbitrary function $f \in \mathcal{A}(E)$ was considered by Grothmann [14]; he established the existence of an appropriate sequence $\Lambda$ such that $\mu_{n} \rightarrow \mu_{e}$, $n \rightarrow \infty$, $n \in \Lambda$, respectively the balayage measures onto $\partial E$. Grothmann’s result was extended to multipoint Padé approximants $\pi^n_{\alpha,m}$ with a fixed number of the free poles (see [15]). Finally, in [16] the case was considered, when the degrees of the denominators tended slowly to infinity, namely, $m_\alpha = o(n / \ln n)$, $n \rightarrow \infty$.

As a consequence of Theorem 1, we derive

Theorem 2: Under the conditions of Theorem 1, suppose that the $\mu$-exact maximally convergent sequence $\Lambda := \left\{ n_k \right\}_{k=1}^{\infty}$ satisfies the condition to be “dense enough”, that is
\[
\limsup_{n_k \to \infty} \frac{H_{n_k}}{n_k} < \infty.
\]

Then, there is at least one point $z_0 \in \partial D_{n,m}(f)$ such that
\[
\limsup_{n \to \infty} \mu_{n,m} \left( V_{z_0}(r) \right) > 0.
\]

Proof of Theorem 1: Set $Q^n_{\alpha,m} := Q_\alpha$, $P^n_{\alpha,m} := P_\alpha$ and $F := fQ$. Fix numbers $R$, $\tau$, $r$ such that $\theta_{\max} < R < \tau < r < R_{n,m}$ and $E_{\tau}(R)$ is connected. Then, by the conditions of the theorem, for every compactum $K \subset D_{n,m}$ (comp. (4))
\[
\lim_{n \to \infty} \left\| FQ_\alpha - PQ_\alpha \right\|_{L^p_{K_\varepsilon}} = \left\| e^{-U^s} \right\|_{L^p} / (R_{n,m}).
\]
(5)

Select a positive number $\eta$ such that $R + \eta < \tau < \tau + \eta < r < R_{n,m}$. Let $V_\Gamma$ be an analytic curve in
such that $\Gamma$ winds around every point in $E_\mu(R)$ exactly once. In an analogous way, we select a curve $\gamma \subset E_\mu(R+N)\setminus E_\mu(R)$. Additionally, we require that $U^\mu$ is constant on $\Gamma$ and $\gamma$. Set

$$F_\mu(z) := \frac{1}{n}\ln|FQ_\mu(z)| + U^\mu(z) + \ln R_{m,\mu}, \quad n \in \Lambda.$$  \hfill (6)

Let $\sigma > 0$ be arbitrary. The functions $F_\mu$ are subharmonic in $E_\mu(R)\setminus E_\mu(R)$. By (5) and the choice of $\Gamma$, 

$$\max_{i \in I} F_\mu(t) \leq -\min_{i \in I} + \max_{i \in I} + \sigma \leq \sigma, \quad N \in \Lambda, \quad n \geq n_1(\sigma),$$

and, analogously,

$$\max_{i \in I} F_\mu(t) \leq -\min_{i \in I} + \max_{i \in I} \leq \sigma, \quad N \in \Lambda, \quad n > n_1.$$  \hfill (7)

Then, by the max-principle of subharmonic functions,

$$\max_{z \in A_{\Gamma,\gamma}} F_\mu(z) \leq \sigma, \quad n \in \Lambda, \quad n \geq n_2, \quad N \in \Lambda,$$

where $A_{\Gamma,\gamma}$ is the “annulus”, bounded by $\Gamma$ and $\gamma$.

On the other hand, by (5), there exists, for every compact set $K \subset E_\mu(R)$ and $n$ large enough, a point $z_{n,K} \in K$ such that

$$-U^\mu(z_{n,K}) - \ln R_{m,\mu} \leq \frac{1}{n}\ln|FQ_\mu(z_{n,K})| - QP_\mu(z_{n,K}), \quad n \geq n_3(K), \quad n \in \Lambda.$$

Therefore,

$$-\sigma \leq F_\mu(z_{n,K}), \quad n \geq n_2(K,\sigma).$$  \hfill (8)

Further, by the formula of Hermite-Lagrange, for $z \in \gamma$ we have

$$FQ_\mu(z) - QP_\mu(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{\omega_{\mu+1}^{m}(z)}{\omega_{\mu+1}^{m+1}(t)} \frac{FQ_\mu(t) - QP_\mu(t)}{t - z} \mathrm{d}t.$$

Hence, by (5),

$$\frac{1}{n}\ln|FQ_\mu(z)| - QP_\mu(z) \leq \max_{i \in I} U_{\mu+1}^{m\infty}(t) - U_{\mu+1}^{m\infty}(z) + \frac{1}{n}\ln|FQ_\mu| - QP_\mu| + \frac{1}{n}\text{const}$$

$$\leq \max_{i \in I} U_{\mu+1}^{m\infty}(t) - U_{\mu+1}^{m\infty}(z) - \min_{i \in I} U^\mu(t) - \ln R_{m,\mu} + \sigma,$$

$$n \in \Lambda, \quad n \geq n_4 = n_4(\sigma) > n_1,$$

where $U_{\mu+1}^{m\infty} := U_{\mu+1}^{m\infty}$. To simplify the notations, we set $U_{\mu+1}^{m\infty} := U_{\mu+1}^{m\infty}$. (the correctness will be not lost, since $m \in N$ is fixed). Invoking into consideration the functions $F_\mu$ (see (6)), we get for $z \in \gamma$

$$F_\mu(z) \leq \max_{i \in I} \left( U^\mu(t) - U^\mu(z) \right) + \min_{i \in I} U^\mu(t) + \left(U^\mu(z) - U^\mu(z) \right), \quad n \in \Lambda, \quad n \geq n_2 \geq n_1.$$

By Helly’s selection theorem [1], there exists a subsequence of $\Lambda$ which we denote again by $\Lambda$ such that $\mu_{\mu+1}^{\infty} \Rightarrow \omega, \quad n \in \Lambda$. Passing to the limit, we obtain

$$\limsup_{n \in \Lambda} F_\mu(z) \leq \max_{i \in I} \left( U^\mu(t) - U^\mu(z) \right) + \left(U^\mu(z) - U^\mu(z) \right), \quad z \in \gamma. \hfill (9)$$

Consider the function $\phi$, harmonic in $A_{\Gamma,\gamma}$ and

$$\phi := \begin{cases} \Gamma, & \min \left(0, -\min_{i \in \gamma} \left(U^\mu(t) - U^\mu(z) \right) + \left(U^\mu(z) - U^\mu(z) \right) \right) \end{cases}, \quad \gamma.$$
From (7) and (9), we arrive at
\[ \limsup F_n(z) \leq \phi, \]
for \( z \) in \( A_{1,\gamma} \). Being harmonic, \( \phi \) obeys the maximum and the minimum principles in this region. The definition yields
\[ \phi(z) \leq 0, \quad z \in A_{1,\gamma}, \]
We will show that
\[ \phi(z) \equiv 0, \quad (10) \]
Suppose that (10) is not true. Let \( \Upsilon \) be a closed curve in the set \( E_{\gamma+\sigma} - \gamma' \), where \( \gamma' \) stands for the interior of \( \gamma \). Then there exists a number \( \theta > 0 \) such that \( \phi \leq -\Theta \) for every \( z \in \Upsilon \). This inequality contradicts (8), for \( \sigma \) close enough to the zero and \( n \in \Lambda \) sufficiently large.
Hence, \( \phi \equiv 0 \). Then the definition of \( \phi \) yields
\[ U(z) - U'(z) = \min_{\iota \in \gamma} (U(z) - U'(z)), \quad z \in \gamma. \]
The function \( U(z) - U'(z) \) is harmonic in the unbounded complement \( G \) of \( \gamma \), and by the maximum principle,
\[ U(z) - U'(z) \equiv \text{Constant}, \quad z \in G, \]
consequently,
\[ U(z) - U'(z) \equiv \text{Constant}, \quad z \in E^c. \]
On the other hand, \( (U(z) - U'(z))(\infty) = 0 \), which yields \( U(z) \equiv U'(z) \) in \( E^c \). By Carleson’s Lemma, \( \mu = \omega \). On this, Theorem 1 is proved.

The proof of Theorem 2 will be preceded by an auxiliary lemma

**Lemma 1** [17]: Given a domain \( U \), a regular compact subset \( S \) and a sequence \( \mathcal{A} := \{n_k\} \) of positive integers, \( n_k < n_{k+1} \), \( k = 1,2,\ldots \), such that
\[ \limsup_{n_k \to \infty} \frac{n_{k+1}}{n_k} < \infty, \]
Suppose that \( \{\phi_n\} \) is a sequence of rational functions, \( \phi_n = R_n \mu_n \), \( k = 1,2,\ldots, \), \( \phi_n = \phi_n' / \phi_n'' \) having no more than \( m \) poles in \( U \) and converging uniformly of \( \partial S \) to a function \( \phi \neq 0 \) such that
\[ \limsup_{n_k \to \infty} \frac{\|\phi_n - \phi\|^2_{H(S)}}{n_k} < 1. \]
Assume, in addition, that on each compact subset of \( U \)
\[ \lim_{n_k \to \infty} \mu_{\kappa_{n_k}}(K) = 0. \]
Then the function \( \phi \) admits a continuation into \( U \) as a meromorphic function with no more than \( m \) poles.

**Proof of Theorem 2**: We preserve the notations from the proof of Theorem 1.

The proof of Theorem 2 follows from Lemma 1 and Theorem 1. Indeed, under the conditions of the theorem the sequence \( \{\pi_n\}_{n \in \Lambda} \) converges maximally to \( f \) with respect to the measure \( \mu \) and the domain \( D_{m,\mu} \) . Hence, inside \( D_{m,\mu} \) (on compact subsets) condition (11) if fulfilled. From the proof of Theorem 1, we see that there is a regular compact subset \( S \) of \( D_{m,\mu} \) such that \( \limsup_{n \in \Lambda} \| \pi_n - \|f\|_S < 1. \)
Suppose now that the statement of Theorem 2 is not true. Then there is, for every \( z \in \partial D_{m,\mu} \) a disk \( V_z \) with \( \lim_{n \to \infty} \mu_n(V_z) = 0 \). We select a finite covering of disks \( V_{\zeta_j} \) such that \( W := \bigcup_{\zeta_j} \partial D_{m,\mu} \) . Condition (11) holds inside \( W \). Applying Lemma 1 with respect to the sequence \( \pi_n \) and to the domain \( D_{m,\mu} \bigcup W \), we conclude that \( f \in \mathcal{M}_{\mu}(\mathcal{D}_{m,\mu}) \). This contradicts the definition of \( D_{m,\mu} \).

On this, the proof of Theorem 2 is completed. Q.E.D.
Using again Lemma 1 and applying Theorem A, we obtain a more general result about the zero distribution of the sequence \( \{ \pi_{n, \mu} \} \).

**Theorem 3:** Let \( E \) be a regular compactum in \( \mathbb{C} \) with a connected complement, let \( \mu \in B(E) \) and \( \beta \in E \) be a triangular point set. Let the polynomials \( \omega_n, \ n = 1, 2, \ldots \), be defined as above. Suppose that \( \mu_n \to \mu \) as \( n \to \infty \) and \( f \in A(E) \). Let \( m \in \mathbb{N} \) be fixed, and suppose that \( R_{w, \mu} < \infty \). Then there is at least one point \( z_0 \in \partial D_{\mu, \mu} \) such that \( \limsup_{n \to \infty} \mu_n \left( \left( \frac{1}{n} \sum_{\beta \in E} \right) f(z_0) \right) > 0 \) for every positive \( r \).

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