Generalized Bäcklund–Darboux transformation (GBDT): conservation laws, rational extensions and bispectrality

Alexander Sakhnovich
Fakultät für Mathematik, Universität Wien,
Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria
E-mail: olexandr.sakhnovych@univie.ac.at

Abstract. Bäcklund–Darboux transformations are closely related to the integrability and symmetry problems. For the generalized Bäcklund–Darboux transformation (GBDT), we consider conservation laws, rational extensions and bispectrality. We use the case of the nonlinear optics equation (and its auxiliary linear system) as an example.

1. Introduction
Bäcklund–Darboux transformations and related commutation methods constitute one of the most fruitful approaches to the construction of explicit solutions of linear and nonlinear equations and to the explicit spectral theoretic results (see, e.g., [1–6] and numerous references therein). Similar to various symmetries studied in group analysis, Bäcklund–Darboux transformations (BDTs) transform solutions of linear differential equations of some fixed class into solutions of differential equations of the same (or another fixed) class and potentials into potentials, whereas potentials are often solutions of the corresponding integrable nonlinear equations. Some connections between Darboux transformations and potential symmetries are discussed in an interesting paper [7].

Though the idea of BDTs and commutation methods goes back to Bäcklund, Darboux and Jacobi, the notion of the so called Darboux matrix has a much later origin. According to J. Ciesiński [1], “all approaches to the construction of Darboux matrices originate in the dressing method”. In order to explain the notion of the Darboux matrix, we consider some initial linear system

\[ u_x = G(x, z)u, \quad u \in \mathbb{C}^m, \]  

(1.1)

where \( u_x := \frac{d}{dx} u \) (\( u_x = \frac{\partial}{\partial x} u \) in the case of several variables), \( G(x, z) \) is some \( m \times m \) matrix function (i.e., \( G(x, z) \in \mathbb{C}^{m \times m} \)) and \( \mathbb{C} \) stands for the complex plain. The Darboux matrix (matrix function) \( \hat{w} \) satisfies the equation

\[ \hat{w}_x = \tilde{G}\hat{w} - \hat{w}G, \quad \tilde{G}(x, z) \in \mathbb{C}^{m \times m}. \]  

(1.2)

It is easy to see that if (1.1) and (1.2) hold, the product

\[ \hat{u} = \hat{w}u \]
satisfies the transformed system
\[ \tilde{u}_x = \tilde{G}(x, z)\tilde{u}. \]

The cases, where the Darboux matrix \( \tilde{w} \) can be constructed explicitly, if \( u \) and \( G \) are given explicitly, are of special interest. Clearly, in those cases we construct explicitly \( \tilde{u} \) and also \( \tilde{G} \).

We are interested in a generalized version of the Bäcklund–Darboux transformation for which \( w \) can be constructed explicitly, if \( u \) and \( \tilde{G} \) are given explicitly, and \( \tilde{u} \) can be constructed explicitly, if \( w \) and \( G \) are given explicitly, are of special interest. Clearly, in those cases we construct explicitly \( \tilde{u} \) and also \( \tilde{G} \).

We use the acronym GBDT. In GBDT (for the case of one space variable) Darboux matrix is explicitly, are of special interest. Clearly, in those cases we construct explicitly \( \tilde{u} \) and also \( \tilde{G} \).

More precisely, equation (1.7) is equivalent to the compatibility condition
\[ G_t - F_x + [G, F] = 0 \]

of the auxiliary systems
\[ u_x = G(x, t, z)u, \quad G = izD - [D, \varrho], \quad D = \text{diag}\{d_1, d_2, \ldots, d_m\} = D^*; \]
\[ \varrho^* = B\varrho B, \quad B = \text{diag}\{b_1, b_2, \ldots, b_m\} \quad (b_k = \pm 1); \quad [D, \varrho] := D\varrho - \varrho D, \]

where \( \text{diag} \) denotes a diagonal matrix. System (1.5) is an auxiliary system for the well-known N-wave (nonlinear optics) equation:
\[ [D, \varrho_1] - [\hat{D}, \varrho_x] = [[D, \varrho], [\hat{D}, \varrho]], \]
\[ \hat{D} = \text{diag}\{\hat{d}_1, \ldots, \hat{d}_m\} = \hat{D}^*. \]

More precisely, equation (1.7) is equivalent to the compatibility condition
\[ G_t - F_x + [G, F] = 0 \]

of the auxiliary systems
\[ u_x = G(x, t, z)u, \quad G = izD - [D, \varrho], \]
\[ u_t = F(x, t, z)u, \quad F = iz\hat{D} - [\hat{D}, \varrho]. \]

We shall study conservation laws, rational extensions and bispectrality for the case of GBDT for system (1.5) and also for the N-wave equation (1.7).

The integrable model describing interaction of three wave packages [12] is the most well-known subcase of the N-wave equation (1.7). Putting, for instance, \( m = 3 \), \( B = I_3 \) and \( d_1 > d_2 > d_3 \), and using transformations [13, Chapter 3]
\[ \psi_1 = (\hat{d}_2 - \hat{d}_1)/(d_1 - d_2), \quad \psi_2 = (\hat{d}_3 - \hat{d}_2)/(d_2 - d_3), \quad \psi_3 = (\hat{d}_3 - \hat{d}_1)/(d_1 - d_3); \]
\[ \phi_1 = -i\sqrt{d_1 - d_2} \varrho_{12}, \quad \phi_2 = -i\sqrt{d_2 - d_3} \varrho_{23}, \quad \phi_3 = -i\sqrt{d_1 - d_3} \varrho_{13}, \]
we rewrite (1.7) in the standard form of the corresponding 3-wave interaction:
\[
(\phi_1)_t + \psi_1(\phi_1)_x = i\varepsilon\overline{\phi_2}\phi_3, \quad (\phi_2)_t + \psi_2(\phi_2)_x = i\varepsilon\overline{\phi_1}\phi_3, \quad (\phi_3)_t + \psi_3(\phi_3)_x = i\varepsilon\phi_1\phi_2,
\]
where \(\overline{\phi}(x,t)\) stands for the function, which takes values complex conjugate to \(\phi(x,t)\), and
\[
\varepsilon = (d_1\hat{d}_2 - d_2\hat{d}_1 + d_2\hat{d}_3 - d_3\hat{d}_2 + d_3\hat{d}_1 - d_1\hat{d}_3)((d_1 - d_2)(d_1 - d_3)(d_2 - d_3))^{-1/2}.
\]
If \(\alpha\) is a scalar value or matrix, the notation \(\overline{\alpha}\) stands for the scalar, which is complex conjugate to \(\alpha\), or the matrix with entries complex conjugate to the entries of \(\alpha\), respectively.

2. Preliminaries
GBDT for the system (1.5) and for the \(N\)-wave equation was described in [8] (see also [11, Subsection 1.1.3 and Section 7.1] and references therein). In this section, we give some necessary definitions and results from [8] and [11, Subsection 1.1.3 and Section 7.1]. We consider the case \(x \geq 0\) for the system (1.5) and the case \(x \geq 0, t \geq 0\) for the \(N\)-wave equation. We fix \(n \times n\) matrices \(A\) and \(S(0) = S(0)^*\) and an \(n \times m\) matrix \(\Pi(0)\) such that
\[
AS(0) - S(0)A^* = i\Pi(0)B\Pi(0)^*.
\]
We introduce the \(n \times m\) matrix function \(\Pi(x)\) and the \(n \times n\) matrix function \(S(x)\) via initial values \(\Pi(0)\) and \(S(0)\) and differential equations
\[
\Pi_x = -iA\Pi D + \Pi[D, \varrho], \quad S_x = \Pi DB\Pi^*.
\]
Relations (2.1) and (2.2) yield \(AS(x) - S(x)A^* = i\Pi(x)B\Pi(x)^*\). Then the following proposition describes GBDT determined by \(A\), \(S(0)\) and \(\Pi(0)\).

**Proposition 2.1** Let system (1.5) (such that (1.6) holds) be given. Then (in the points of invertibility of \(S(x)\)) the matrix function
\[
w_A(x, z) = I_m - iB\Pi(x)^*S(x)^{-1}(A - zI_n)^{-1}\Pi(x)
\]
is a Darboux matrix of (1.5) and satisfies the equation
\[
\frac{d}{dx}w_A(x, z) = (izD - [D, \tilde{\varrho}(x)])w_A(x, z) - w_A(x, z)(izD - [D, \varrho(x)]),
\]
where
\[
\tilde{\varrho} = \varrho - B\Pi^* S^{-1} \Pi, \quad \tilde{\varrho}^* = B\tilde{\varrho}B.
\]
Moreover, if \(\det S(x) \neq 0\) for \(x \geq 0\), a normalized fundamental solution \(\tilde{w}(x, z)\) of the transformed system
\[
\tilde{w}_x = (izD - [D, \tilde{\varrho}])\tilde{w}, \quad \tilde{w}(0, z) = I_m
\]
is given by the equality
\[
\tilde{w}(x, z) = w_A(x, z)w(x, z)w_A(0, z)^{-1},
\]
where \(w\) is the fundamental solution of the initial system (1.5) normalized by \(w(0, z) = I_m\).
Next, we add the variable $t$ and consider $\Pi(x,t)$, $S(x,t)$ and $w_A(x,t,z)$ which are determined by $A$, $\Pi(0,0)$ and $S(0,0) = S(0,0)^\ast$ via equations (2.2), (2.3) and

$$\Pi_t = -i\mathbf{A}\mathbf{D}\Pi + \Pi[D,\varrho]$, $S_t = \Pi\mathbf{D}\Pi^\ast$. \hspace{1cm} (2.8)$$

Instead of (2.1) we assume that

$$AS(0,0) - S(0,0)A^\ast = i\Pi(0,0)B\Pi(0,0)^\ast \hspace{1cm} (2.9)$$

which implies that $AS(x,t) - S(x,t)A^\ast = i\Pi(x,t)B\Pi(x,t)^\ast$. Proposition 2.1 yields:

**Proposition 2.2** Let an $m \times m$ matrix function $\varrho$ ($\varrho^\ast = B\varrho B$) be continuously differentiable and satisfy the N-wave equation (1.7). Then $\tilde{\varrho}$ of the form

$$\tilde{\varrho}(x,t) := \varrho(x,t) - B\Pi(x,t)^\ast S(x,t)^{-1}\Pi(x,t) \hspace{1cm} (2.10)$$

satisfies (in the points of invertibility of $S$) the equality $\tilde{\varrho}^\ast = B\tilde{\varrho}B$ and the N-wave equation.

Using [11, Therem 6.1] on wave functions, we easily obtain the next statement.

**Remark 2.3** If the conditions of Proposition 2.2 hold, $\det S(x,t) \neq 0$ on the semi-band $0 \leq x < \infty$, $0 \leq t \leq \varepsilon$ and $w(x,t,z)$ is the initial wave function (i.e., $w_x = Gw$, $w_t = Fw$ and $w(0,0,z) = I_m$), then the transformed wave function $\hat{w}(x,t,z)$ is given by the equality

$$\hat{w}(x,t,z) = w_A(x,t,z)w(x,t,z)w_A(0,0,z)^{-1}. \hspace{1cm} (2.11)$$

**Example 2.4** The real-valued case is of special interest [13, Section 3.4]. It is immediate from Proposition 2.2 (and formulas defining $\tilde{\varrho}$ considered in this proposition) that equalities

$$\varrho = \tilde{\varrho}, \hspace{0.5cm} A = -\mathbf{A}, \hspace{0.5cm} \Pi(0,0) = \Pi(0,0), \hspace{0.5cm} S(0,0) = S(0,0), \hspace{1cm} (2.12)$$

yield the equality $\tilde{\varrho} = \tilde{\varrho}$. Assume that (2.12) holds, and so $\tilde{\varrho} = \tilde{\varrho}$. Then, in the subcase $m = 3$, $B = I_3$, $d_1 > d_2 > d_3$ of the 3-wave equation, we can rewrite (1.14) in the real-valued form and obtain solutions of the corresponding exact resonance equations. Namely, we set

$$\varphi_1 = -i\varphi_1 = -\sqrt{d_1 - d_2} \tilde{g}_{12}, \hspace{0.5cm} \varphi_2 = -i\varphi_2 = -\sqrt{d_2 - d_3} \tilde{g}_{23}, \hspace{0.5cm} \varphi_3 = -i\varphi_3 = -\sqrt{d_1 - d_3} \tilde{g}_{13},$$

and, taking into account (1.14), derive

$$(\varphi_1)_t + \psi_1(\varphi_1)_x = \varepsilon\varphi_2\varphi_3, \hspace{0.5cm} (\varphi_2)_t + \psi_2(\varphi_2)_x = \varepsilon\varphi_1\varphi_3, \hspace{0.5cm} (\varphi_3)_t + \psi_3(\varphi_3)_x = -\varepsilon\varphi_1\varphi_2. \hspace{1cm} (2.13)$$

3. Conservation laws, rational extensions and bispectrality

3.1. Conservation laws

When $\varrho$ is differentiable with respect to $t$ (to $x$), we can differentiate both sides of the first relation in (2.2) (in (2.8)) with respect to $t$ (to $x$). If $\varrho$ is differentiable with respect to $x$ and $t$ and $\varrho_t$ is continuous, then, according to the so called Clairaut’s (or Schwarz’s) theorem, we have $\Pi_{xt} = \Pi_{tx}$. Thus $\Pi_{xt} = \Pi_{tx}$ is the necessary condition of the compatibility of the first relations in (2.2) and (2.8). In a similar way, $S_{xt} = S_{tx}$ is a necessary condition of the compatibility of the second relations in (2.2) and (2.8). In other words, the necessary compatibility conditions are

$$(-i\mathbf{A}\mathbf{D}\Pi + \Pi[D,\varrho])_t = (-i\mathbf{A}\mathbf{D}\Pi + \Pi[D,\varrho])_x, \hspace{1cm} (3.1)$$

$$\Pi DB\Pi^\ast)_x = \Pi DB\Pi^\ast)_x, \hspace{1cm} (3.2)$$

where $\Pi_x$ should be substituted by $-i\mathbf{A}\mathbf{D}\Pi + \Pi[D,\varrho]$ and $\Pi_t$ should be substituted by $-i\mathbf{A}\mathbf{D}\Pi + \Pi[D,\varrho]$. 


Proposition 3.1 Let $\rho$ be differentiable and satisfy the $N$-wave equation (1.7). Then the equalities (3.1) and (3.2) hold.

Proof. Substituting expressions for $\Pi_t$ and $\Pi_x$ we obtain

$$( -i A \Pi D + \Pi[D, \rho] )_t = -i A ( -i A \Pi \hat{D} + \Pi[\hat{D}, \rho]) D + ( -i A \Pi D + \Pi[\hat{D}, \rho]) [D, \rho],$$

$$( -i A \Pi D + \Pi[\hat{D}, \rho] )_x = -i A ( -i A \Pi D + \Pi[D, \rho]) \hat{D} + ( -i A \Pi D + \Pi[D, \rho]) \hat{D} + \Pi[\hat{D}, \rho_x].$$

Therefore, taking into account (1.7) we derive

$$( -i A \Pi D + \Pi[D, \rho] )_t - ( -i A \Pi D + \Pi[D, \rho] )_x = \Pi([D, \rho] - [\hat{D}, \rho_x] - [[D, \rho], [\hat{D}, \rho]]) = 0,$$

and so (3.1) is proved. Using the same substitutions as before, we have also

$$(\Pi D B \Pi^*)_t = ( -i A \Pi D + \Pi[\hat{D}, \rho]) D B \Pi^* + \Pi D B (i D \Pi^* A^* + [\rho^*, \hat{D}] \Pi^*),$$

$$(\Pi D B \Pi^*)_x = ( -i A \Pi D + \Pi[D, \rho]) \hat{D} B \Pi^* + \Pi \hat{D} B (i D \Pi^* A^* + [\rho^*, D] \Pi^*).$$

Equality (3.2) follows from (3.3), (3.4) and $B \rho^* = \rho B$. □

The sufficiency of the proved above equalities $\Pi_{xt} = \Pi_{tx}$ and $S_{xt} = S_{tx}$ for the compatibility of (2.2) and (2.8) is not self-evident but may be proved in a way similar to the proof of [11, Theorem 6.1].

Remark 3.2 We note that relations (3.1) and (3.2) may be considered as conservation laws for $\Pi(x, t)$.

3.2. Rational extensions

According to Propositions 2.1 and 2.2, in the case

$$\rho \equiv 0, \quad \sigma(A) = \{0\} \quad (\sigma \text{ stands for spectrum}),$$

the transformed potentials $\tilde{\rho}$ are rational matrix functions (rational extensions in the terminology of [14]). More precisely, if $\rho \equiv 0$ and $A$ is nilpotent, formulas (2.2), (2.3), (2.5), (2.7), (2.8), (2.10) and (2.11) imply the following proposition.

Proposition 3.3 Let (3.5) hold. Then $\Pi(x)$ and $S(x)$ are matrix polynomials with respect to $x$, $\Pi(x, t)$ and $S(x, t)$ are matrix polynomials with respect to $x$ and $t$; $\tilde{\rho}(x) = p(x)/\det S(x, t)$ and $\tilde{\rho}(x, t) = p(x, t)/\det S(x, t)$, where $p(x)$ is a matrix polynomial with respect to $x$, $p(x, t)$ is a matrix polynomial with respect to $x$ and $t$, det $S(x)$ is a polynomial with respect to $x$ and $t$. Moreover, for Darboux matrices $w_A$ we have $w_A(x, z) = P(x, z)/\det S(x)$ and $w_A(x, t, z) = P(x, t, z)/\det S(x, t)$, where $P(x, z)$ is a matrix polynomial with respect to $x$ and $1/z$ ($x, t$ and $1/z$). Thus, the transformed fundamental solutions $\tilde{w}(x, z)$ and wave functions $\tilde{w}(x, t, z)$ are expressed via matrix polynomials and exponents of diagonal matrices:

$$\tilde{w}(x, z) = w_A(x, z)e^{izx}Dw_A(0, 0, z)^{-1}, \quad \tilde{w}(x, t, z) = w_A(x, t, z)e^{iz(xD+t\hat{D})}w_A(0, 0, z)^{-1}. \quad (3.6)$$

Proof. Since $\rho \equiv 0$, the first equation in (2.2) yields that the $i$th column $f_i$ of $\Pi$ is given by

$$f_i(x) = e^{-id_i x A}f_i(0) = \sum_{k=0}^{n-1} \frac{1}{k!} (-id_i x A)^k f_i(0),$$

and so $\Pi(x)$ is a matrix polynomial. (Here we used the equality $A^n = 0$, which holds for all $n \times n$ nilpotent matrices.) Hence, the second equation in (2.2) yields that $S(x)$ is a matrix polynomial. In the same way we prove the polynomial
form of Π(x, t) and S(x, t), and the required properties of ˜ρ follow. According to formula (1.84) from [11, p. 24] we have \( w_A(z)Bw_A(\bar{z})^* = B \) and, in particular, the equalities

\[
 w_A(0, z)^{-1} = Bw_A(0, \bar{z})^* B, \quad w_A(0, 0, z)^{-1} = Bw_A(0, 0, \bar{z})^* B
\]

are valid. Finally, the representations of \( w_A(x, z) \) and \( w_A(x, t, z) \) follow from the definition (2.3), properties of Π and S and the expansion \((A - zI_n)^{-1} = -z^{-1} \sum_{k=0}^{n-1} (z^{-1} A)^k. \square\)

**Remark 3.4** System (1.5) includes important subclasses: self-adjoint and skew-self-adjoint Dirac-type systems (which are also called Zakharov–Shabat or AKNS systems). Namely, putting

\[
 D = j = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix} \quad (m_1 + m_2 = m), \quad V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}
\]

and \( B = j \), we rewrite (1.5), (1.6) in the form of the self-adjoint Dirac-type system

\[
 u_x = i(zj + jV)u, \quad v = 2i\varrho_{12}, \quad (3.8)
\]

where \( \varrho_{12} \) is the upper right \( m_1 \times m_2 \) block of \( \varrho \). Assuming (3.7) and \( B = I_m, \) we rewrite (1.5), (1.6) in the form of the skew-self-adjoint Dirac-type system

\[
 u_x = (izj + jV)u, \quad v = -2\varrho_{12}. \quad (3.9)
\]

### 3.3. Bispectrality

The notion of bispectrality was introduced in [15] (see also [16] and references therein). Matrix bispectrality for system (1.5) was studied in [17]. Namely, it was shown that for \( \varrho \) constructed in Proposition 3.3 and, correspondingly, for the solution \( W(x, z) = w_A(x, z)e^{izxD} \) of system

\[
 W_x = (izD - [D, \varrho])W, \quad (3.10)
\]

there is a non-degenerate matrix linear differential operator \( B \) with respect to the variable \( z \) such that \( B(z)W = 0 \). It was suggested in [16] (and in some earlier works by F. Grünbaum and coauthors) that the bispectrality requirement \( WB(z) = \Theta(x)W \) is more meaningful than the requirement \( B(z)W = \Theta(x)W \). Unfortunately, the approach from [17] does not work properly for that case. Therefore, we should consider the system

\[
 W_x = (izD - [D, \varrho])W - izWD \quad (3.11)
\]

instead of system (1.5). In view of (2.4), the matrix function \( w_A \) satisfies this system in the case \( \varrho \equiv 0 \). Thus, we can deal in the same way as in [18], where Dirac systems were treated. Namely, if \( \sigma(A) \) is concentrated at one point \( \lambda \in \mathbb{C} \) we write down the resolvent \((A - zI_n)^{-1} \) in the form

\[
 (A - zI_n)^{-1} = -\sum_{k=1}^{n} (z - \lambda)^{-k}(A - \lambda I_n)^{k-1}. \quad (3.12)
\]

By virtue of (2.3) and (3.12) the next proposition is immediate.

**Proposition 3.5** Assume that \( \varrho \equiv 0 \), the spectrum of \( A \) is concentrated at some point \( \lambda \in \mathbb{C} \) and \( \tilde{\varrho} \) is given by (2.5). Then system (3.11) is bispectral in the sense of [16].
Indeed, it is easy to choose some coefficients $c_s$ so that the operator $B(z) = \sum_{s=1}^{n+1} c_s (z - \lambda)^s \frac{\partial^s}{\partial z^s}$ is non-degenerate and satisfies the equalities

$$B(z)(z - \lambda)^{-k} = 0 \quad (1 \leq k \leq n).$$

(3.13)

According to formula (2.4) and identity $\varrho \equiv 0$, the matrix function $W = w_A(x, z)$ satisfies (3.11). Therefore, definition (2.3) of $w_A$ and equalities (3.12) and (3.13) imply $B(z)W = 0$.

If we want to consider system (3.10), where $\varrho \equiv 0$ and $\sigma(A) = \{\lambda\}, \quad \lambda \in \mathbb{R}$, we rewrite the first relation in (2.2) in the form

$$\left(\Pi(x)e^{i\lambda xD}\right)_x = -i(A - \lambda I_m)\left(\Pi(x)e^{i\lambda xD}\right)D.$$

(3.14)

Hence, taking into account the fact that $\sigma(A)$ is concentrated at $\lambda$, we see that $\Pi(x)e^{i\lambda xD} = P(x, \lambda) = P(x)$, where $P(x)$ is a matrix polynomial with respect to $x$. Substituting $\Pi(x) = P(x)e^{-i\lambda xD}$ into (2.2) and using the equalities $D = D^*$, $\lambda = \overline{\lambda}$, we obtain

$$S_x = P(x)e^{-i\lambda xD}DBe^{i\lambda xD}P(x)^* = P(x)DBP(x)^*, \quad \text{i.e.,} \quad S(x) = \tilde{P}(x),$$

(3.15)

where $\tilde{P}(x)$ a matrix polynomial with respect to $x$. Introduce an operator $B = \text{diag}\{B_1, \ldots, B_m\}$ such that $B_k$ has the form

$$f B_k = \sum_{\ell=1}^{N} c_{\ell k}(z) \frac{\partial^{\ell}}{\partial z^{\ell}} ((z - \lambda)^m f), \quad N \in \mathbb{N},$$

(3.16)

and notice that the equality

$$\sum_{\ell=1}^{N} c_{\ell k}(z) \frac{\partial^{\ell}}{\partial z^{\ell}} ((z - \lambda)^m w_A(x, z)e^{izx xD}e_k) = 0, \quad e_k := \{\delta_{ik}\}_{i=1}^{m} \quad (1 \leq k \leq m).$$

(3.17)

is equivalent to the relation

$$\sum_{\ell=1}^{N} c_{\ell k}(z) \left(\frac{\partial}{\partial z} + id_k x\right)^\ell ((z - \lambda)^m e^{i\lambda xD} \left(I_m - iBP(x)^*\tilde{P}(x)^{-1}(A - zI_n)^{-1}P(x)\right)e^{i\lambda xD}e_k) = 0,$$

which, in turn, is equivalent to

$$\sum_{\ell=1}^{N} c_{\ell k}(z) \left(\frac{\partial}{\partial z} + id_k x\right)^\ell \times \left((\det \tilde{P}(x))(z - \lambda)^m \left(I_m - iBP(x)^*\tilde{P}(x)^{-1}(A - zI_n)^{-1}P(x)\right)e_k\right) = 0.$$

(3.18)

**Open Problem.** Are there some cases, where matrices $D$, $A$, $S(0)$ and $\Pi(0)$ generate nontrivial $\varrho$ such that $WB(z) = 0$ for the solution $W(x, z) = w_A(x, z)e^{izx xD}$ of (3.10) and for $B \neq 0$ of the form given above? In other words, are there $D$, $A$, $S(0)$, $\Pi(0)$ and $\{c_{\ell k}\}$ satisfying (3.18)?

**Acknowledgments**
The research was supported by the Austrian Science Fund (FWF) under Grant No. P24301.
References
[1] Cieśliński J L 2009 Algebraic construction of the Darboux matrix revisited J. Phys. A 42:40 404003
[2] Gesztesy F 1993 A complete spectral characterization of the double commutation method J. Funct. Anal. 117:2 401
[3] Gesztesy F and Teschl G 1996 On the double commutation method Proc. Amer. Math. Soc. 124:6 1831
[4] Gu C H, Hu H and Zhou Z 2005 Darboux Transformations in Integrable Systems. Theory and their Applications to Geometry (Mathematical Physics Studies vol 26) (Dordrecht: Springer)
[5] Kostenko A, Sakhnovich A and Teschl G 2012 Commutation methods for Schrödinger operators with strongly singular potentials Math. Nachr. 285:4 392
[6] Matveev V B and Salle M A 1991 Darboux Transformations and Solitons (Berlin: Springer)
[7] Popovych R O, Kunzinger M and Ivanova N M 2008 Conservation laws and potential symmetries of linear parabolic equations Acta Appl. Math. 100:2 113
[8] Sakhnovich A L 1994 Dressing procedure for solutions of nonlinear equations and the method of operator identities Inverse Problems 10:3 699
[9] Sakhnovich A L 2001 Generalized Bäcklund–Darboux transformation: spectral properties and nonlinear equations J. Math. Anal. Appl. 262:1 274
[10] Sakhnovich A L 2010 On the GBDT version of the Bäcklund–Darboux transformation and its applications to linear and nonlinear equations and Weyl theory Math. Model. Nat. Phenom. 5:4 340
[11] Sakhnovich A L, Sakhnovich L A and Roitberg I Ya 2013 Inverse Problems and Nonlinear Evolution Equations. Solutions, Darboux Matrices and Weyl–Titchmarsh Functions (de Gruyter Studies in Mathematics vol 47) (Berlin: De Gruyter)
[12] Zakharov V E and Manakov S V 1975 The theory of resonance interaction of wave packets in nonlinear media Soviet Phys. JETP 42:5 842
[13] Novikov S, Manakov S V, Pitaevskii L P and Zakharov V E 1984 Theory of Solitons. The Inverse Scattering Method. Contemporary Soviet Mathematics (New York: Consultants Bureau [Plenum])
[14] Grandati Y 2011 Solvable rational extensions of the isotonic oscillator Ann. Physics 326:8 2074
[15] Duistermaat J J and Grünbaum F A 1986 Differential equations in the spectral parameter Commun. Math. Phys. 103:177
[16] Grünbaum F A 2014 Some noncommutative matrix algebras arising in the bispectral problem SIGMA Symmetry Integrability Geom. Methods Appl. 10 078
[17] Sakhnovich A L and Zubelli J P 2001 Bundle bispectrality for matrix differential equations Integral Equations Operator Theory 41 472
[18] Gohberg I, Kaashoek M A and Sakhnovich A L 1998 Pseudo-canonical systems with rational Weyl functions: explicit formulas and applications J. Differential Equations 146:2 375