Algebra over generalized rings

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Abstract

For a commutative ring $A$, we have the category of (bounded-below) chain complexes of $A$-modules $Ch_+(A\text{-mod})$, a closed symmetric monoidal category with a compatible stable Quillen model structure. The associated homotopy category is the derived category $\mathbb{D}(A\text{-mod})$, where one inverts all the quasi-isomorphisms, and it has the good description as

$$\mathbb{D}(A\text{-mod}) = \text{Ch}_+(\text{proj-}A\text{-mod}) / \simeq$$

the chain complexes made up of projective $A$-module in each dimension, and chain maps taken up to chain homotopy. We give here the analogous theory for a (commutative) generalized ring in the sense of [4]. We use here a different concept of “$A$-module” than the one used in [4] (which was useful for the introduction of derivatives and the cotangent complex in the arithmetical settings). We refer to the new concept as “$A$-Set”.

For an ordinary commutative ring $A$, an $A$-set is just an $A$-module in the usual meaning, and our construction will be equivalent to $\mathbb{D}(A\text{-mod})$. For the initial object of the category of generalized rings $\mathbb{F}$ “the field with one element”, we obtain the category of symmetric spectra, and the associated stable homotopy category with its smash product (an $\mathbb{F}$-set is just a pointed set, i.e. a set $X$ with a distinguish element $O_X \in X$). Thus the analogous theories of stable homotopy and of chain complexes of modules over a commutative ring appear as two sides of the same coin, and moreover, they appear in a context where they interact (via the forgetful functor and its left adjoint - the base change functor). For the “real integers” $A = \mathbb{Z}_R$, the $\mathbb{Z}_R$-sets include the symmetric convex subsets of $\mathbb{R}$-vector spaces. We also give the global theory of the derived category of $O_X$-sets, for a generalized scheme $X$, in a way that is based on the local projective model structure rather than follow the path of Grothendieck [3] Jardine [7] Voevodsky [14], of using injective resolution (but inevitably uses $\infty$-categories).

1 Generalized Rings cf. [4]

Let $\text{Fin}_0$ denote the category of finite pointed sets. For a map $f \in \text{Fin}_0(X,Y)$, we have its kernel and cokernel

$$(i) \quad \text{Ker}(f) : f^{-1}(O_Y) \to X$$

$$(ii) \quad \text{Cok}(f) : Y \to Y/f(X)$$

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where $Y/f(X)$ is obtained from $Y$ by contracting $f(X)$ to a point. There is a canonical map

\[ \hat{f} : X/f^{-1}(O_Y) = \text{CokKer}(f) \rightarrow \text{KerCok}(f) = f(X) \]

which is not always an isomorphism, and we let $\mathcal{F}_0$ denote the subcategory of $\text{Fin}_0$ with the same objects, but with the maps $f$ for which $\hat{f}$ is an isomorphism, $X/f^{-1}(O_Y) \cong f(X)$, that is $f$ is bijective away from $O_Y$:

\[ f(x_1) = f(x_2) \neq O_Y \Rightarrow x_1 = x_2. \]

We view $\mathcal{F}_0$ as the category of “finite dimensional vector spaces over the field with one element”. We will omit the pointed element $O_X \in X$ for $X \in \text{Fin}_0$ and we have isomorphic category $\text{Fin}_*$, where the objects are finite sets and the maps are partially-defined functions

\[ \text{Fin}_0 \xrightarrow{\sim} \text{Fin}_* \]

\[ X \quad \mapsto \quad X \setminus \{O_X\} \]

\[ n_+ := n \coprod \{O_n\} \quad \mapsto \quad n \]

We denote by $\mathcal{F} \subseteq \text{Fin}_*$ the subcategory corresponding to $\mathcal{F}_0 \subseteq \text{Fin}_0$. Thus $\mathcal{F}$ has objects the finite sets, and the maps are the partially defined bijections

\[ \mathcal{F}(m, n) = \left\{ f : D(f) \xrightarrow{\sim} I(f), \; D(f) \subseteq m, \; I(f) \subseteq n \right\} \]

Note that the category $\mathcal{F}$ is self-dual, $\mathcal{F} \cong \mathcal{F}^{\text{op}}$, and we have an involution $f \mapsto f^t$, $\mathcal{F}(m, n) \xrightarrow{\sim} \mathcal{F}(n, m)$, taking $f = \left( f : D(f) \xrightarrow{\sim} I(f) \right)$ to $f^t = \left( f^{-1} : I(f) \xrightarrow{\sim} D(f) \right)$, satisfying

\[ f^{tt} = f, \quad (f_2 \circ f_1)^t = f_1^t \circ f_2^t, \quad (\text{Id}_n)^t = \text{Id}_n. \]

The initial and final object of $\mathcal{F}$ is thus the empty set $\emptyset$.

\[ \textbf{Definition}[3]: \] A generalized ring is a functor $A : \mathcal{F} \rightarrow \text{Set}_0$, pointed $A_0 = \{0\}$, and with operations of “multiplication” and “contraction” as follows. For $f \in \text{Fin}_*(m, n)$ put $A_f = \prod_{j \in n} A_{f^{-1}(j)}$. For $b = (b_j) \in A_f$, $b_j \in A_{f^{-1}(j)}$, we usually omit $f$ and write “$m \xrightarrow{b} n$”.

(i) \textbf{multiplication}: $A_n \times A_f \rightarrow A_m$

\[ a, (b_j) \mapsto a \triangleleft b \]

(ii) \textbf{contraction}: $A_m \times A_f \rightarrow A_n$

\[ c, (b_j) \mapsto c \parallel b \]

For a map $g \in \text{Fin}_*(n, q)$, we can extend these maps fiberwise to get

\[ \text{(i) multiplication}: A_q \times A_f \rightarrow A_{gp(f)} \]

\[ a, b \mapsto a \triangleleft b, \; (a \triangleleft b)_i = a_i \triangleleft (b_j)_{f(j)=i} \quad \text{for} \; i \in q \]
(ii) **Contraction:** \( A_g f \times A_f \to A_g \)
\[ c, b \mapsto c \parallel b, \quad (c \parallel b)_i = c_i \parallel (b_j)_{f(j)=i}, \quad i \in q \]

We require these operations to satisfy the following axioms.

(1.9) **Associativity:** \((a \triangleright b) \triangleright e = a \triangleright (b \triangleright e)\)

(1.10) **Unit:** We have \(1 \in A_{[1]},\) (with \([1]\) denoting the set with one element):

\[
\begin{align*}
1 \triangleright a &= a \\
1 \triangleright (1)_{j \in n} &= a = a \parallel (1)_{j \in n}, \quad a \in A_n.
\end{align*}
\]

Thus, forgetting the contraction, \(A\) is just an operad. We require the following identities to hold (in \(A_{k \to q},\) where we can take \(q = \{\ast\} = [1]\))

(1.11) **Left-Adjunction:** \((d \parallel c) \parallel a = d \parallel (a \triangleright c)\)

(1.12) **Right-Adjunction:** \(d \parallel (a \parallel c) = (d \triangleright c) \parallel a\)

(1.13) **Left-Linear:** \(d \triangleright (a \parallel c) = (d \triangleright a) \parallel c\)
Right-Linear: 
\[(d \sslash c) \triangleleft a = (d \triangleleft \tilde{a}) \sslash \tilde{c}\]

Here \(\tilde{a}\) (resp. \(\tilde{c}\)) is obtained from \(a\) (resp. \(c\)) by identifying the fibers; i.e. denoting by \(f : n \to m\) we have

\[(1.15) \quad \tilde{a} = (\bar{a}_j)\] with \(\bar{a}_j = a_{f(j)}\)

It follows from the axioms that for a generalized ring \(A\), the set \(A_{[1]}\) is an associative (1.9), unital (1.10), but also commutative (1.14), monoid with respect to multiplication. It has the involution

\[(1.16) \quad a^t := 1 \sslash a, \quad a \in A_{[1]}\]

We denote by \(A^+_{[1]}\) the sub-monoid of symmetric element.

\[(1.17) \quad A^+_{[1]} = \{a \in A_{[1]} \mid a^t = a\}\]

Note that the axioms (1.11-14) imply that any formula made up of multiplication \(\triangleleft\), and contraction \(\sslash\), is equivalent to a formula with one contraction of the form

\[(1.18) \quad (a_1 \triangleleft a_2 \triangleleft \ldots \triangleleft a_m) \sslash (b_1 \triangleleft b_2 \triangleleft \ldots \triangleleft b_n)\]

A homomorphism of generalized rings \(\varphi : B \to A\) is a natural transformation \(\varphi_n : B_n \to A_n\) preserving multiplication, contraction, and the unit 1 (the unit 0 is always preserved since for \(n = 0\), \(B_0 = \{0\} = A_0\)). We thus have a category of generalized rings \(\text{GR}\) (these are what we called “commutative-generalized-ring” in [4], because the axiom (1.14) imply the commutativity of \(A_{[1]}\)).

The category \(\text{GR}\) is complete and co-complete. Limits and filtered co-limits are formed in \(\text{Set}_0\). More general co-limits are more complicated, but we do have push-outs

\[(1.19) \quad \begin{array}{c}
B \rightarrow A_1 \\
\downarrow \quad \\
A_0 \rightarrow A_0 \otimes A_1
\end{array}\]
and in particular (categorical) sums \( A_0 \otimes_{\mathbb{F}} A_1 \).

We let \( \text{TGR} \subseteq \text{GR} \) denote the full subcategory with objects the generalized rings \( A \) satisfying the "total-commutativity", using the notation \( s \) of (1.15).

(1.20) \[ c \triangleleft \tilde{a} = a \triangleleft \tilde{c} \quad \text{in} \quad A_{n \prod_k \rightarrow m} \]

(we can take \( m = \{ \ast \} = [1] \)), so \( a \in A_k, \ c \in A_n, \ c \triangleleft \tilde{a} = a \triangleleft \tilde{c} \in A_{n \prod_k} \).

The inclusion \( \text{TGR} \hookrightarrow \text{GR} \) has a left-adjoint, \( A \mapsto A^T \), where \( A^T \) is the maximal totally commutative quotient of \( A \).

Here are some examples of totally-commutative generalized rings.

(1.21) \( \mathbb{F} \).

The initial object of \( \text{GR} \), "the field with one element", is denoted by \( \mathbb{F} \),

\[ F \sim 
\]

\[ \ni \rightarrow \mathbb{F} \quad \text{Set}_n \ni := \{ 0 \}_{\mathbb{n}} := \{ \delta_i \}_{i \in \mathbb{n}} \]

(1.22) \( \mathbb{F} \{ M \} \).

For a commutative monoid \( M \), we have \( \mathbb{F} \{ M \} \in \text{TGR} \)

\[ \mathbb{F} \{ M \}_n := (n \times M) \prod \{ 0_n \} \]

This give a full and faithful functor \( \mathbb{F} \{ M \} : \text{CMon} \rightarrow \text{TGR} \)

(1.23) A Commutative-rig (or a "semi-ring") is a set \( A \) with two associative and commutative operations of addition \( + \), with unit 0, and multiplication \( \cdot \), with unit 1, where \( x \cdot 0 = 0 \) for all \( x \in A \), and we have distributivity \( x \cdot (y_1 + y_2) = (x \cdot y_1) + (x \cdot y_2) \). These form a category \( \text{CRig} \), where the arrows are the set maps preserving the operations and the units 0,1.

For \( A \in \text{CRig} \), we have \( A \in \text{TGR} \) (denoted by the same letter!), with \( A_n := A^n \), and with the operations of multiplication and contraction defined as follows:

Note that \( A_f \equiv \prod_{j \in n} A_{f^{-1}(j)} = \prod_{i \in n} f^{-1}(j) = A_{D(f)} \),

(i) \( \triangleleft : A_n \times A_f \rightarrow A_m \)

\[ (a_j)_{j \in n}, \ (b_i)_{i \in D(f)} \mapsto (a \triangleleft b)_i := a_{f(i)} \cdot b_i, \quad i \in D(f), \]

\[ := 0, \quad i \in m \setminus D(f) \]

(ii) \( \parallel : A_m \times A_f \rightarrow A_n \)

\[ (c_i)_{i \in m}, \ (b_i)_{i \in D(f)} \mapsto (c \parallel b)_j := \sum_{i \in f^{-1}(j)} c_i \cdot b_i, \quad j \in I(f) \]

\[ := 0, \quad j \in n \setminus I(f) \]

This gives a full and faithful embedding \( \text{CRig} \hookrightarrow \text{TGR} \).

Examples of commutative-rigs, include commutative-rings, but there are
many more examples where addition is not invertible, such as the "tropical" rigs
\[\{0, 1\} \leftrightarrow [0, 1] \leftrightarrow [0, \infty) \equiv \mathbb{R}_\text{max}^+\]
with the usual multiplication, and with \(y_1 + y_2 := \max\{y_1, y_2\}\).

(1.24) \(\mathbb{Z}_R\) and \(\mathbb{Z}_C\)
We have the (maximal compact sub-topological-generalized ring), the "real integers" \(\mathbb{Z}_R \subseteq \mathbb{R}\) (resp. the "complex integers" \(\mathbb{Z}_C \subseteq \mathbb{C}\)), given by
\[
(\mathbb{Z}_R)_n := \left\{ (a_j) \in \mathbb{R}^n, \sum_{j \in n} |a_j|^2 \leq 1 \right\}
\]
resp.
\[
(\mathbb{Z}_C)_n := \left\{ (a_j) \in \mathbb{C}^n, \sum_{j \in n} |a_j|^2 \leq 1 \right\}
\]
the unit \(\ell_2\) - ball in \(\mathbb{R}^n\), resp \(\mathbb{C}^n\).

Remark 1.25. We prefer to work with \(\text{GR}\), rather than \(\text{TGR}\), since the categorical sum in \(\text{GR}\) of \(\mathbb{Z}_R\) with itself, \(\mathbb{Z}_R \otimes \mathbb{Z}_R\), is a very interesting object, while the categorical sum in \(\text{TGR}\) reduces to the ordinary integer: \((\mathbb{Z}_R \otimes \mathbb{Z}_R) \sim \to \mathbb{Z}\) (and so the "Arithmetical surface" reduces to its diagonal, as it does with ordinary rings).

Definition 1.26. We define the "commutative" generalized rings \(\text{CGR}\) to be the intermediate full subcategory
\[\text{TGR} \subseteq \text{CGR} \subseteq \text{GR}\]
where \(A \in \text{GR}\) is \underline{commutative} if for \(a, b \in A_n\), \(c, d \in A_m\) \((x_{i,j}) \in (A_{[1]})^m \prod n\), we have (in \(A_{[1]}\)):

\[
(a \triangleleft d \triangleleft (x_{i,j})) \not\parallel (b \triangleleft c) = (d \triangleleft \tilde{a} \triangleleft (x_{i,j})) \not\parallel (c \triangleleft \tilde{b})
\]

2 \(A\)-sets

Definition 2.1. For \(A \in \text{CGR}\), an \(A\)-set is a pointed set \(M \in \text{Set}_0\) together with an \(A\)-action: For each \(n \in \mathcal{F}\) we have maps

\[
A_n \times M^n \times A_n \longrightarrow M
\]
\[
(b, m, d) \longmapsto (b, m, d)_n
\]

These maps are required to satisfy the following axioms:

\textbf{Associativity:} For a map \(f \in \text{Set}_*(m, n)\), for \(b, d \in A_n\), \(b', d' \in A_f\),
\[
(b \triangleleft b', m_i, d \triangleleft d')_m = (b, (b'_j, m_i, d'_j)_{f-1(j)}, d)_n
\]
**Unit:** \((1, m, 1)[1] = m\)  
(i.e. we have an action of the operad \(A \times A\)).

**naturality:** For \(f \in \text{Set}_n(m, n), d \in A_n, b \in A_m, a \in A_f, (m_j) \in M^n\),
\[
\langle b, m f(i), d \triangleright a \rangle_m = \langle b, m j, d \rangle_n
\]
\[
\langle d \triangleright a, m f(i), b \rangle_m = \langle d, m j, b \triangleright a \rangle_n
\]

**Commutativity:** For \(b, d \in A_n, b', d' \in A_m, m_{j,i} \in M^{n \times m}\),
\[
\langle d \triangleright d', m_{j,i}, b \triangleright b' \rangle_{n \times m} = \langle d' \triangleright d, m_{j,i}, b' \triangleright b \rangle_{m \times n}
\]

A map \(\varphi : M \to N\) of \(A\text{-Set}\) is a set map \(\varphi \in \text{Set}_0(M, N)\), preserving the \(A\)-action,
\[
\varphi (\langle b, m_j, d \rangle_n) = \langle b, \varphi(m_j), d \rangle_n
\]
Thus we have the category \(A\text{-Set}\).

**Examples of \(A\text{-Set}\)**

(2.2) Given a homomorphism \(\varphi \in \text{CGR}(A, B), B[1]\) is an \(A\text{-Set}:
\[
\langle b, x_i, d \rangle_n := (\varphi(b) \triangleright (x_i)) \triangleright (\varphi(d)), \quad b, d \in A_n, x_i \in (B[1])^n.
\]

(2.3) A sub-\(A\)-set of \(A[1]\) is just an ideal, cf.[1].

(2.4) For \(A = \mathbb{F}\) we have \(\mathbb{F}\text{-set} \equiv \text{Set}_0\), and for \(M \in \text{Set}_0\) there is a unique \(\mathbb{F}\)-action:
\[
\langle b, m_j, d \rangle_n = \begin{cases} m_i & \text{if } b = d = \delta_i \\ 0 & \text{otherwise} \end{cases}
\]

(2.5) For a commutative ring \(A \in \text{CRing} \subseteq \text{CGR}\), we have
\(A\text{-Set} \equiv A\text{-mod}\)

the category of \(A\)-modules.

(2.6) For the real (resp. complex) integers \(A = \mathbb{Z}_R\) (resp. \(\mathbb{Z}_C\)), the "finite-dimensional-torsion-free" \(A\)-Sets are the convex subsets \(M \subseteq \mathbb{R}^n\) (resp. \(M \subseteq \mathbb{C}^n\)) which are symmetric \(u \cdot M \subseteq M\) for \(|u| \leq 1\).

(2.7) **Notation:** For \(M \in A\text{-Set},\) and for \(m \in M, a \in A[1]\) we write:
\[
a \cdot m := \langle a, m, 1 \rangle[1],
\]
\[
m \cdot a := \langle 1, m, a \rangle[1] = a^t \cdot m, \quad a^t = 1 \triangleright a
\]

We have
\[
a_1 \cdot (a_2 \cdot m) = (a_1 \triangleright a_2) \cdot m, \quad 1 \cdot m = m, \quad (a_1 \cdot m) \cdot a_2 = a_1 \cdot (m \cdot a_2)
\]
The category $A$-Set is complete and co-complete. Inverse limits, and filtered co-limits are formed in $\text{Set}_0$, while co-limits and sums are more complicated. Given a set $V$ the free $A$-Set on $V$, $A^V$, is given by

\[(2.8)\quad A^V = \left( \prod_{n \in F} A_n \times V^n \times A_n \right) / \sim\]

where $\sim$ is the equivalence relation generated by naturality and commutativity. The element of $A^V$ can be written, non-uniquely, as

\[\langle b, v, d \rangle_n, \quad b, d \in A_n, \quad v \in V^n.\]

For $M \in A$-Set, we have

\[(2.9)\quad \text{Set}(V, M) \equiv A\text{-Set}(A^V, M)\]

\[\varphi \mapsto \tilde{\varphi}(\langle b, v, d \rangle_n) = \langle b, \varphi(v), d \rangle_n.\]

When $V = \{v_0\}$ is a singleton, $A^{\{v_0\}} \equiv A_{[1]} \cdot v_0$, and the free $A$-Set on one generator is just $A_{[1]}$.

Given a homomorphism $\varphi \in \text{CGR}(B, A)$, we have an adjunction (we use "geometric" notation):

\[(2.10)\quad \begin{array}{ccc}
\varphi^* & \cong & \varphi_* \\
\text{A-set} & \downarrow & \text{B-set} \\
\end{array}\]

where $\varphi_*N \equiv N$ with the $B$-action $\langle b, m, b' \rangle_n := (\varphi(b), m, \varphi(b'))_n$. The left adjoint is $\varphi^*M = \left( \prod_n A_n \times M^n \times A_n \right) / \sim$, where $\sim$ is the equivalence relation generated by naturality, commutativity, and $B$-linearity:

\[\langle a, (b_j, m_i, b'_i)_{f^{-1}(j)}, a' \rangle_n = \langle a \triangleleft \varphi(b), m, a' \triangleleft \varphi(b') \rangle_m\]

for $f \in \text{Set}(m, n)$, $a, a' \in A_n, b, b' \in B_f$.

In particular, for $B = F$ and $\phi \in \text{CGR}(F, A)$ the unique homomorphism, $\phi_*$ is just the functor forgetting the $A$-action, and for $V \in \text{Set}_0$ $\phi^*V = A^V \setminus \{0\}$ is the free $A$-set on $V \setminus \{0\}$.

For $M, N, K \in A$-set, let the "bilinear maps" be defined by

\[(2.11)\quad \text{Bil}_A(M, N; K) = \left\{ \varphi : M \wedge N \to K, \quad \varphi((a, m_j, a'), n) = \langle a, \varphi(m_j \wedge n), a' \rangle, \right.\]

\[\left. \varphi(m, \langle a, n_j, a' \rangle) = \langle a, \varphi(m \wedge n_j), a' \rangle \right\}\]

It is a functor in $K$, and as such it is representable

\[(2.12)\quad \text{Bil}_A(M, N; K) \equiv A\text{-Set}(M \otimes_A N, K)\]

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Where $M \otimes_A N$ is the free $A$-Set on $M \wedge N$ modulo the equivalence relation generated by the $A$-bilinear relations, and where $\otimes : M \wedge N \to M \otimes_A N$ is the universal $A$-bilinear map. The elements of $M \otimes_A N$ can be written, non-uniquely, as $\langle a, m_j \otimes n_j, a' \rangle_n$. We thus get a bi-functor

$$(2.13) \quad - \otimes_A - : A\text{-Set} \times A\text{-Set} \to A\text{-Set}$$

giving a symmetric monoidal structure on $A\text{-Set}$, with unit $A[1]$. This symmetric monoidal structure on $A\text{-Set}$ is closed,

$$(2.14) \quad A\text{-Set}(M \otimes_A N, K) \equiv A\text{-Set}(M, \text{Hom}_A(N, K))$$

with the internal Hom functor

$\text{Hom}_A(-, -) : (A\text{-Set})^\text{op} \times A\text{-Set} \to A\text{-Set}$

where the $A$-action on $\text{Hom}_A(M, N)$ is given by

$\langle b, \varphi_j, d \rangle_n(m) := \langle b, \varphi_j(m), d \rangle_n, \quad b, d \in A_n, \quad \varphi_j \in \text{Hom}_A(M, N) \quad m \in M.$

This in itself is a map of $A\text{-Set}$ because of commutativity, and all the other properties ($\S(2.1)$: associativity, unit, naturality, commutativity) follow from their validity in $N$.

The tensor product commute with extension of scalars: for $\varphi \in \text{CGR}(B, A)$, and for $M, N \in B\text{-Set}$, we have

$$\varphi^*(M \otimes_B N) \equiv \varphi^* M \otimes_A \varphi^* N$$

$$\varphi^* B[1] \equiv A[1]$$

We have therefore also the adjunction formula

$$(2.15) \quad \varphi_* \text{Hom}_A(\varphi^* M, N) \cong \text{Hom}_B(M, \varphi_* N)$$

The tensor product is distributive over sums,

$$(2.16) \quad M \otimes_A \left( \bigsqcup_i N_i \right) \cong \bigsqcup_i (M \otimes_A N_i).$$

and more generally commutes with colimits.

3 Simplicial $A$-Set, $A$-Set$^\Box$

We denote by $\Delta$ the simplicial category of finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$, and monotone maps, and we let $\Box = \Delta^\text{op}$ denote
the opposite category. We denote by $A\text{-Set}^\vee \equiv (A\text{-Set})^{\Delta^{op}}$ the category of simplicial objects in $A\text{-Set}$. It has objects $M = (M_n)_{n \geq 0} \in (\text{Set}_0)^{\Delta^{op}}$, pointed simplicial sets, with an $A$-action in each dimension $n \geq 0$,

$$A_m \times (M_n)^m \times A_m \to M_n$$

$$(a, m_i, a') \mapsto \langle a, m_i, a' \rangle_m$$

compatible with the simplicial operations:

$$\ell^* : M_{n'} \to M_n$$

satisfies

$$\ell^* (\langle a, m_i, a' \rangle_{n'}) = \langle a, \ell^*(m_i), \ell^*(a') \rangle_{n}.$$

The category $A\text{-Set}^\vee$ is complete and co-complete.

**Remark 3.3.** More generally, we have the category of simplicial-commutative-generalized-rings $\text{CGR}^\vee$, and for $A = (A^n)_{n \geq 0} \in \text{CGR}^\vee$ we have the category of simplicial $A$-sets $A\text{-Set}^\vee$, with objects the pointed simplicial sets $M = (M_n)_{n \geq 0}$, with compatible $A^n$-action in dimension $n \geq 0$, i.e. for $m \in F$,

$$A^n_m \times (M_n)^m \times A^n_m \to M_n$$

$$(a, m_i, a') \mapsto \langle a, m_i, a' \rangle^n_m$$

and for $\ell \in \Delta(n, n')$,

$$\ell^* (\langle a, m_i, a' \rangle^n_{m'}) = \langle a, \ell^*(m_i), \ell^*(a') \rangle^n_m.$$

While $\text{CGR}^\vee$ are important for "derived-arithmetical-geometry", we shall concentrate here on $\text{CGR}$, to keep the notation simpler.

The category $A\text{-Set}^\vee$ inherits a closed symmetric monoidal structure,

$$\_ \otimes^A \_ : A\text{-Set}^\vee \times A\text{-Set}^\vee \to A\text{-Set}^\vee$$

$$(M, \otimes^A N)_n := M_n \otimes_A N_n$$

$$(\text{Hom}^A(\_ , \_)) : (A\text{-Set}^\vee)^{\text{op}} \times A\text{-Set}^\vee \to A\text{-Set}^\vee$$

$$\text{Hom}^A(M, N)_n := A\text{-Set}^\vee(M \otimes_A (\Delta(n)_+), N)$$

In particular, the category $A\text{-Set}^\vee$ is tensored, co-tensored and enriched over pointed simplicial sets. Denoting by $\phi_A \in \text{CGR}(F, A)$ the unique map, we have for $M, K \in A\text{-Set}^\vee$, $F, K, \in F\text{-Set}^\vee \equiv (\text{Set}_0)^{\Delta^{op}}$,

$$(i) \quad M \otimes K := M \otimes^A \phi_A K.$$

$$(ii) \quad M^K := \text{Hom}^A(\phi_A K, M).$$

$$(iii) \quad \text{Map}^A(N, M) := \phi_A \cdot \text{Hom}^A(N, M) \in \text{Set}_0^\vee$$
The category $A\text{-Set}^\mathbb{V}$ has a simplicial, cellular, Quillen model structure given
by cf. [9].

(i) **Fibrations**: $\mathcal{F}_A \equiv \phi^{-1}_* (\mathcal{F}_\mathbb{V})$, $\mathcal{F}_\mathbb{V} \equiv$ Kan fibrations.

(ii) **Weak-equivalence**: $\mathcal{W}_A \equiv \phi^{-1}_* (\mathcal{W}_\mathbb{V})$, $\mathcal{W}_\mathbb{V} \equiv$ [Weak-equivalence of (pointed) simplicial sets]

(iii) **Cofibrations**: $\mathcal{C}_A \equiv \mathcal{L}(\mathcal{W}_A \cap \mathcal{F}_A)$, the maps satisfying the left lifting property with respect to the trivial fibrations.

The cofibrations can also be characterized as the retracts of the free maps, where a map $f : M \to N$ is free if there exists subsets $V_n \subseteq N_n$ with $\ell^*(V_n') \subseteq V_n$ for all surjective $\ell \in \Delta(n, n')$, and such that $f_n$ induces isomorphism

$$M_n \bigsqcup \phi^*_A (V_n) \xrightarrow{\sim} N_n$$

where $\phi^*_A (V_n)$ is the free $A$-Set on $V_n$. This model structure is cofibrantly generated

(i) $\mathcal{F}_A \equiv \mathcal{R} \{ \phi^*_A (A^k_{n+n}) \to \phi^*_A (\Delta(n)_+) \}_{0 \leq k \leq n}^{n > 0}$

(ii) $\mathcal{W}_A \cap \mathcal{F}_A \equiv \mathcal{R} \{ \phi^*_A (\partial \Delta(n)_+) \to \phi^*_A (\Delta(n)_+) \}_{n > 0}$

Where $\mathcal{R} \{ \_ \}$ are the map satisfying the Right lifting property w.r.t.$\{ \_ \}$.

The model structure is compatible with the symmetric monoidal structure: For $i. : N. \to N.'$, $j. : M. \to M.'$ in $\mathcal{C}_A$, we have

$$i. \Box j. : (N. \otimes_{A} M.') \coprod_{N. \otimes_{A} M.} (N.' \otimes_{A} M.) \to N.' \otimes_{A} M.'$$

is also in $\mathcal{C}_A$, and moreover, if $i.$ is in $\mathcal{W}_A$, also $i. \Box j.$ is in $\mathcal{W}_A$. To see this compatibility of the monoidal and model structures, we may assume $i.$ and $j.$ are free

$$N_n \bigsqcup \phi^*_A (V_n) \xrightarrow{\sim} N'_n, \quad M_n \bigsqcup \phi^*_A (W_n) \xrightarrow{\sim} M'_n$$

and then

$$\left( N_n \otimes_A M'_n \bigsqcup_{N_n \otimes_A M_n} N'_n \otimes_A M_n \right) \coprod \phi^*_A (V_n) \otimes_A \phi^*_A (W_n) \xrightarrow{\sim} N'_n \otimes_A M'_n$$

and since $\phi^*_A (V_n) \otimes_A \phi^*_A (W_n) \equiv \phi^*_A (V_n \coprod W_n)$, we see that $i. \Box j.$ is also free.
A homomorphism $\varphi \in \text{CGR}(B, A)$ induces a Quillen adjunction,

$$
\varphi^* 
\begin{array}{c}
\downarrow \\
B-\text{Set}^\vee
\end{array}
\begin{array}{c}
A-\text{Set}^\vee
\end{array}
\varphi_*
$$

(3.12)

When $A$ is a commutative ring, we have by the Dold-Kan correspondence $A-\text{Set}^\vee \cong \text{Ch}_{\geq 0}(A\text{-mod})$, and the model structure on $A-\text{Set}^\vee$ corresponds to the projective model structure on $\text{Ch}_{\geq 0}(A\text{-mod})$, which embeds in the stable model structure $\text{Ch}_+(A\text{-mod})$. For general $A$, the model structure on $A-\text{Set}^\vee$ is not stable, and we shall stabilize it, (preserving the symmetric monoidal structure), using (symmetric) spectra.

4 Symmetric Spectra: $S_A^\cdot \text{-mod}$ (cf. [5])

Let $\Sigma_n$ denote the symmetric group on $n$ letters, and let $\Sigma = \coprod_{n \geq 0} \Sigma_n$ denote the category of finite bijection. The category $\Sigma$ is equivalent to the category

$$
\text{Iso}(\mathcal{T}) \equiv \text{Iso}(\text{Fin}_0) \equiv \text{Iso}(\text{Fin})
$$

(4.1)

The category of symmetric sequences in $A-\text{Set}^\vee$ is

$$
\Sigma(A) \equiv \left(\text{A-Set}^\vee\right)^\Sigma \equiv (A-\text{Set})^\Sigma \times \text{Fin} \cong (A-\text{Set})^{\text{Iso}(\text{Fin}) \times \Delta^\text{op}}
$$

(4.2)

It has objects $M = \{M^n\}_{n \geq 0}$, with $M^n \in \left(\text{A-Set}^\vee\right)^{\Sigma_n}$ a simplicial $A$-Set with an action of $\Sigma_n$. The category $\Sigma(A)$ is complete and co-complete. The category $\Sigma(A)$ has a closed symmetric monoidal structure

$$
\_ \otimes_{\Sigma(A)} \_ : \Sigma(A) \times \Sigma(A) \to \Sigma(A)
$$

(4.3)

$$(M\cdot \otimes_{\Sigma(A)} N\cdot)^n := \coprod_{p+q=n} \Sigma_n \times \Sigma_p \times \Sigma_q (M^p \otimes_A N^q)$$

Here the induction functor is the left adjoint of the forgetful functor

$$
(A-\text{Set})^\Sigma \times \Sigma_n \to \left(A-\text{Set}^\vee\right)^{\Sigma_p \times \Sigma_q},
$$

and is given by

$$
\Sigma_n \times_{\Sigma_p \times \Sigma_q} (M) := \coprod_{\Sigma_n/\Sigma_p \times \Sigma_q} M
$$
Equivalently, writing $M \cdot, N \cdot \in \Sigma(A)$ as functor $\text{Iso}(\text{Fin}) \to A\text{-Set}$ we have

$$\left( M \otimes_{\Sigma(A)} N \right)^n := \coprod_{n=n_0 \coprod n_1} M^{n_0} \otimes_A N^{n_1} \tag{4.4}$$

the sum over all decomposition of $n$ as a disjoint union of subsets $n_0, n_1 \subseteq n$. The unit of this monoidal structure is the symmetric sequence

$$\mathbb{1}_A := (A_{[1]}, 0, 0, 0, \ldots) \tag{4.5}$$

**Remark 4.6.** Note that this monoidal structure is symmetric,

$$\Sigma_{M \cdot, N \cdot} : M \otimes_{\Sigma(A)} N \cdot \cong N \otimes_{\Sigma(A)} M \cdot$$

This symmetry is clear in the formula (4.4), $M^{n_0} \otimes_A N^{n_1} \cong N^{n_1} \otimes_A M^{n_0}$ but in formula (4.3), the symmetry isomorphisms

$$\Sigma_n \times \Sigma_p \times \Sigma_q \left( M^p \otimes_A N^q \right) \cong \Sigma_n \times \Sigma_q \times \Sigma_p \left( N^q \otimes_A M^p \right)$$

involves the $(p, q)$-shuffle $\omega_{p,q} \in \Sigma_n$ that conjugates $\Sigma_p \times \Sigma_q$ to $\Sigma_q \times \Sigma_p$.

The internal Hom is given by

$$\text{Hom}_{\Sigma(A)}(-, -) : \Sigma(A)^{op} \times \Sigma(A) \to \Sigma(A)$$

$$\text{Hom}_{\Sigma(A)}(M \cdot, N \cdot)^n := \prod_{k \geq 0} \text{Hom}_A(M^k, N^{k+n}) \tag{4.7}$$

$$\Sigma(A)(M \cdot \otimes_{\Sigma(A)} N \cdot, K \cdot) \cong \Sigma(A)(M \cdot, \text{Hom}_{\Sigma(A)}(N \cdot, K \cdot))$$

For a homomorphism $\varphi \in \text{CGR}(B, A)$, we have adjunction

$$\varphi^* \left( \begin{array}{c} \Sigma(A) \\ \Sigma(B) \end{array} \right) \varphi_*$$

and $\varphi^*$ is strict-monoidal

$$\varphi^*(M \cdot \otimes_{\Sigma(B)} N \cdot) \cong \varphi^*(M \cdot) \otimes_{\Sigma(A)} \varphi^*(N \cdot), \quad \varphi^*(\mathbb{1}_B) \cong \mathbb{1}_A \tag{4.8}$$

The category $\Sigma(\mathcal{F}) \equiv (\text{Set}_0)^{\Sigma \times \Delta^{op}}$ is the usual category of symmetric sequence of pointed simplicial set, and in particular contains the sphere-spectrum:

$$S_{\mathcal{F}} := \left\{ S^n = S^1 \wedge \cdots \wedge S^1 \right\}_{n \geq 0} \tag{4.10}$$
with the permutation action of \( \Sigma_n \) on \( S^n \). The sphere-spectrum is a monoid object of \( \Sigma(F) \), with multiplication

\[
m : S_F \otimes_{\Sigma(F)} S_F \to S_F
\]

\[
m(S^n \otimes S^m) = m(S^n \wedge S^m) = S^{n+m}
\]

Note that it is a commutative monoid, \( m = m \circ \mathcal{I}_{S_F,S_F} \), cf. remark (4.6). The unit is given by the embedding

\[
\varepsilon : \mathbb{I}_F \equiv (F[1], 0, 0, \ldots) \equiv (S^0, 0, 0, \ldots) \hookrightarrow S_F
\]

We write \( S_A = \phi_A^* \Sigma_A \) for the corresponding commutative monoid object of \( \Sigma(A) \). We let \( S_A^{\text{mod}} \subseteq \Sigma(A) \) denote the sub-category of \( S_A \)-modules, this is the category of “symmetric spectra”. It has objects the symmetric sequences \( M = \{ M^n \} \in \Sigma(A) \), together with associative unital \( S_A \)-action \( S_A \otimes \Sigma(A) M \xrightarrow{m} M \), or equivalently, associative unital, \( \Sigma_p \times \Sigma_q \hookrightarrow \Sigma_{p+q} \) covariant, action \( S^p \wedge M^q \to M^{p+q} \). The maps in \( S_A^{\text{mod}} \) are the maps in \( \Sigma(A) \) that preserve the \( S_A \)-action. The category \( S_A^{\text{mod}} \) is complete and co-complete.

The category \( S_A^{\text{mod}} \) has a closed symmetric monoidal structure

\[
- \otimes_{S_A} - : S_A^{\text{mod}} \times S_A^{\text{mod}} \to S_A^{\text{mod}}
\]

\[
M \otimes_{S_A} N := \text{Cok} \left\{ M \otimes \Sigma(A) S_A \otimes \Sigma(A) N \xrightarrow{m \otimes \text{id}_N} M \otimes \Sigma(A) N \right\}
\]

The unit is

\[
S_A := \{ \phi_A^* S^0, \phi_A^* S^1, \ldots, \phi_A^* S^n, \ldots \}.
\]

The internal Hom is given by

\[
\text{Hom}_{S_A}(-, -) : (S_A^{\text{mod}})^{op} \times S_A^{\text{mod}} \to S_A^{\text{mod}}
\]

\[
\text{Hom}_{S_A}(M, N) := \text{Ker} \left\{ \text{Hom}_{\Sigma(A)}(M, N) \Rightarrow \text{Hom}_{\Sigma(A)} \left( S_A \otimes \Sigma(A) M, N \right) \right\}.
\]

\[
S_A^{\text{mod}}(M \otimes_{S_A} N, K) \equiv S_A^{\text{mod}}(M, \text{Hom}_{S_A}(N, K))
\]

The category \( S_A^{\text{mod}} \) is tensored, co-tensored, and enriched over pointed simplicial sets \( \mathcal{F}\text{-Set}^\vee \): For \( K \in \mathcal{F}\text{-Set}^\vee \), \( M \in S_A^{\text{mod}} \), we have \( M \otimes K \), \( (M^K)^n \in \mathcal{F}\text{-Set}^\vee \) where

\[
(M \otimes K)^n = M^n \otimes K = M^n \wedge K
\]

\[
\left(M^K\right)^n = (M^n)^K \equiv \mathcal{F}\text{-Set}^\vee (K \wedge \Delta(-)_+, M^n)
\]
The enrichment is given via the mapping space

\[(4.19) \quad \text{Map}_{S_A} (M', N)_n \equiv S_A\text{-mod} (M' \otimes \Delta(n)_+, N') \in \mathbb{F}\text{-Set}^\vee \equiv (\text{Set})^{\Delta^p} \]

We have the adjunctions,

\[(4.20) \quad S'_A\text{-mod} (M' \otimes K, N') \equiv S_A\text{-mod} \left(M', (N')^K\right) \equiv \mathbb{F}\text{-Set}^\vee \left(K, \text{Map}_{S_A} (M', N')\right)\]

\[(4.21) \quad \text{Map}_{S_A} (M' \otimes K, N') \equiv \text{Map}_{S_A} \left(M', (N')^K\right) \equiv \text{Map}_{S_A} (M', N')^K\]

Taking \(K = \text{Map}_{S_A} (M', N')\), the \(\text{id}_{\text{Map}(M' ,N')} \in \text{Map}_{S_A} (M', N')^K\), on r.h.s. \[(4.21)\] corresponds to the evaluation map on l.h.s \[(4.21)\]

\[(4.22) \quad \text{ev}_{M',N'} : M' \otimes_{\mathbb{F}} \text{Map}_{S_A} (M', N') \to N'\]

Taking \(K = \text{Map} (M', L') \otimes_{\mathbb{F}} \text{Map}(L', N')\), the map

\[\text{ev}_{L',N'} \circ (\text{ev}_{M',L'} \otimes \text{id}_{\text{Map}(L',N')})\]

on the l.h.s of \[(4.21)\] corresponds to the composition map, on r.h.s.

\[(4.23) \quad \text{comp}_{L'} : \text{Map}(M', L') \otimes_{\mathbb{F}} \text{Map}(L', N') \to \text{Map}(M', N')\]

which is associative and unital.

We have the embedding \(\left(A\text{-Set}^\vee\right)^{\Sigma_n} \hookrightarrow \Sigma(A), \, M \mapsto M[n]\), and the following
The "free-$S_A$-module of level-$n$" on $M \in \text{A-Set}^\vee$ is the composition of the left adjoints of (4.24),

$F_n(M) := S_A \otimes (\Sigma_n \times M) [n] = \left(0, \ldots, 0, \Sigma_n \times M, \ldots, \Sigma_{n+p} \times S^p \otimes M, \ldots\right)$.

We have the projective model structure on $S_A$-mod, compatible with the symmetric monoidal structure,

$\mathcal{F}_{\text{lev}}^S_A = \{f \in S_A\text{-mod}(M', N'), f^n \in \mathcal{F}_A \text{ all } n \geq 0\}$

$\mathcal{W}_{\text{lev}}^S_A = \{f \in S_A\text{-mod}(M', N'), f^n \in \mathcal{W}_A \text{ all } n \geq 0\}$

$\mathcal{C}S_A = \{\mathcal{W}_{\text{lev}}^S_A \cap \mathcal{F}_{\text{lev}}^S_A\}$

It is left proper, simplicial, cofibrantly generated by

$J = \coprod_{n \geq 1} F_n (\phi_A^* \partial \Delta(n)_+ \hookrightarrow \phi_A^* \Delta(n)_+)$

$I = \coprod_{0 \leq k \leq n \geq 1} F_n (\phi_A^* \Lambda^n_{k+} \hookrightarrow \phi_A^* \Delta(n)_+)$

A map $f \in S_A\text{-mod}(M', N')$ is a cofibration, $f \in \mathcal{C}S_A$, if and only if $Ev^n (f \square j) : M^n \coprod_{S_A} M \cdot \mathcal{L}^n N' \to N'$ is in $\mathcal{C}A$ and $\Sigma_n$ acts freely away from its image, for all $n \geq 0$, where the "latching functor" $\mathcal{L}^n$ is

$\mathcal{L}^n M = Ev^n (M \otimes (\Sigma_A/S_A)) = \prod_{0 \leq k < n} \Sigma_{\Sigma_k \times \Sigma_{n-k}} (M^k \otimes \Sigma_{n-k})$
We say $M \in S_A\text{-mod}$ is an $\Omega$-spectrum, or a fibrant object $M \in (S_A\text{-mod})_F$ if $M$ is levelwise fibrant, and if the adjoint of the action map

$$m^{1,n} : S^n_A \otimes_A M^n \rightarrow M^{n+1}$$

is a weak equivalence

$$(m^{1,n})^\natural : M^n \xrightarrow{\sim} \text{Hom}_{S_A} (S^n_A, M^{n+1}) = (M^{n+1})^{S^1} := \Omega M^{n+1}$$

The stable model structure on $S_A\text{-mod}$ is a Bousfield localization of the projective model structure, having the same cofibrations, but with the fibrant objects being $(S_A\text{-mod})_F$

$$(4.30) \quad \text{Cofibrations:} \quad \mathcal{C}_{S_A} = \mathcal{L} \left\{ \mathcal{W}_{S_A}^\text{lev} \cap \mathcal{F}_{S_A}^\text{lev} \right\}$$

$$(4.31) \quad \text{Weak equivalences:} \quad \mathcal{W}_{S_A} = \left\{ f \in S_A\text{-mod} (M', N'), \quad \text{Map}_{S_A} (N', X') \xrightarrow{\sim} \text{Map}_{S_A} (M', X) \in \mathcal{W}_F \right\}$$

$$(4.32) \quad \text{Fibrations:} \quad \mathcal{F}_{S_A} = \mathcal{R} \left\{ \mathcal{C}_{S_A} \cap \mathcal{W}_{S_A} \right\}$$

It is left proper, simplicial, cellular, Quillen Model structure compatible with its symmetric monoidal structure. It is moreover stable. We let

$$D(A) = \text{Ho} (S_A\text{-mod}) \cong S_A\text{-mod} \left[ \mathcal{W}_{S_A}^{-1} \right]$$

denote the associated homotopy category, this is the derived category of $A\text{-Set}$. It is a triangulated symmetric monoidal category.

We have the following Quillen adjunctions

$$(4.32) \quad \begin{array}{ccc}
S_A\text{-mod} & \xrightarrow{\sim} & S_A\text{-mod} \\
\Sigma_n S_A\text{-mod} & \xrightarrow{\sim} & S_A\text{-mod}
\end{array}$$

where

$$(rM)^n := M^{n+1}, \quad rM' := \text{Hom}_{S_A} (F_1 S_A^n, M')$$

$$(\ell M)^n := \Sigma_n \times M^{n-1}, \quad \ell M := F_1 S_A^n \otimes_A M'$$

$$\Omega M' := (M)^{S^1}$$

$$M' \otimes_A S_A^1 = M' \otimes_A S_A^1$$
They induce Quillen equivalence, and we have the inverse equivalences of $\mathcal{D}(A)$,
\begin{equation}
\mathbb{R}\Omega \simeq \mathbb{L}\ell, \quad \bigwedge_{p}^{\infty} S^{1} \simeq \mathbb{R}r
\end{equation}

Given a homomorphism $\varphi \in \text{CGR}(B, A)$ we get an induced adjunction
\[
\begin{array}{c}
\mathcal{D}(A) \\
\downarrow \varphi^{*} \\
\mathcal{D}(B)
\end{array}
\begin{array}{c}
\mathbb{L}\varphi^{*} \\
\uparrow \varphi_{*} \\
\mathbb{R}\varphi_{*}
\end{array}
\]
and $\mathbb{L}\varphi^{*}$ is a monoidal functor commuting with $\mathbb{L}\ell$ and $\mathbb{R}r$. We have the adjunction formula for $M \in \mathcal{D}(B)$, $N \in \mathcal{D}(A)$,
\begin{equation}
\mathbb{R}\varphi_{*} \left( \mathbb{R}\text{Hom}_{S_{A}}(\mathbb{L}\varphi^{*}M, N) \right) \equiv \mathbb{R}\text{Hom}_{S_{A}}(M, \mathbb{R}\varphi_{*}N)
\end{equation}

5 Global theory

For $A \in \text{GR}$, commutative or not, the sub-$A$-sets of $A_{[1]}$ are the ideals of $A$. Such an ideal $\mathfrak{J} \subseteq A_{[1]}$ is called a +ideal if it is generated as an ideal by its subset of symmetric elements $\mathfrak{J}^{+} = \mathfrak{J} \cap A_{[1]}^{+} = \{a \in \mathfrak{J}, a^{t} = a\}$. A +ideal $p \subseteq A_{[1]}$ is called +prime if the set $S_{p} = A_{[1]}^{+} \setminus p$ is multiplicatively closed. The set of +primes of $A$, $\text{spec}^{+}(A)$, is a compact (every open cover has a finite subcover), Sober or Zariski (every closed irreducible subset has a unique generic point) topological space with closed subsets $V^{+}(\mathfrak{J}) := \{p \supseteq \mathfrak{J}\}$. A +ideal, and basic open subset $D^{+}(s) := \{p \not\supset s\}$, $s = s^{t} \in A_{[1]}^{+}$. By localization one obtains a sheaf $\mathcal{O}_{A}$ of GR over $\text{spec}^{+}(A)$, with stalks $\mathcal{O}_{A}|_{p} \cong A_{p} = S_{p}^{-1}A$, and with sections $\mathcal{O}_{A}(D^{+}(s)) \cong A_{s}^{[1]}$, cf. [4].

A generalized scheme $(X, \mathcal{O}_{X}) \in \text{GSch}$, is a topological space $X$, a sheaf of generalized rings $\mathcal{O}_{X}$ over $X$, with local stalks, and we have an open cover $X = \bigcup_{i \in I} U_{i}$, with $(U_{i}, \mathcal{O}_{X}|_{U_{i}}) \cong \text{spec}^{+}(\mathcal{O}_{X}(U_{i}))$. We can take $I = \text{Aff}/X$, the collection of all affine open subsets of $X$, and we view $\text{Aff}/X$ as a category with respect to inclusions. We denote by $\phi_{V,U} : \mathcal{O}_{X}(U) \to \mathcal{O}_{X}(V)$ the restriction homomorphism for $V \subseteq U$, $\phi_{W,V} \circ \phi_{V,U} = \phi_{W,U}$, $\phi_{U,U} = \text{id}_{\mathcal{O}_{X}(U)}$.

We shall assume all $\mathcal{O}_{X}(U)$ are commutative (1.26).

**Definition 5.1.** A pre-$\mathcal{O}_{X}$-set (resp. pre-$\mathcal{O}_{X}$-$\text{Set}^{\Sigma}$, pre-$\Sigma(\mathcal{O}_{X})$, pre-$S_{\mathcal{O}_{X}}$-$\text{mod}$) is an assignment for each open $U \subseteq X$ of an $\mathcal{O}_{X}(U)$-$\text{Set}^{\Sigma}$, $\Sigma(\mathcal{O}_{X}(U))$, $S_{\mathcal{O}_{X}(U)}$-$\text{mod}$ and for $V \subset U$, an $\mathcal{O}_{X}(U)$-map $r_{V,U} : M(U) \to M(V)$, where $M(V)$ is acted on by $\mathcal{O}_{X}(U)$ via $\phi_{V,U}$, and $r_{W,V} \circ r_{V,U} = r_{W,U}$, $r_{U,U} = \text{id}_{M(U)}$. 18
The maps are the natural transformations commuting with the \( \mathcal{O}_X \)-action (resp. and with the simplicial \( \nabla \); symmetric \( \Sigma = \coprod \Sigma_n \); sphere \( S^\cdot = \{ S^n \} \); actions) so that we have categories:

\[
\text{pre-}\mathcal{O}_X\text{-Set}
\]

resp. \( \text{pre-}\mathcal{O}_X\text{-Set}^{\nabla} \equiv (\text{pre-}\mathcal{O}_X\text{-Set})^{\Delta^{op}} \)

\[
\text{pre-}S^\cdot_{\mathcal{O}_X}\text{-mod} \subseteq \text{pre-}\Sigma(\mathcal{O}_X) \equiv (\text{pre-}\mathcal{O}_X\text{-Set})^{\Delta^{op} \times \coprod \Sigma_n}
\]

All these categories are complete and co-complete (limits and colimits are created sectionwise over open subsets). They have a closed symmetric monoidal structure defined sectionwise which we shall denote by \( \hat{\otimes} \) (resp. \( \hat{\otimes}^{\nabla} \); \( \hat{\otimes}_{\Sigma(\mathcal{O}_X)} \); \( \hat{\otimes}_{S^\cdot_{\mathcal{O}_X}} \)),

\[
(M \hat{\otimes} \mathcal{O}_X N)(U) := M(U) \hat{\otimes}_{\mathcal{O}_X(U)} N(U)
\]

The internal Hom will be denoted by \( \text{Hom}_{\mathcal{O}_X} \) (resp. \( \text{Hom}_{\Sigma(\mathcal{O}_X)} \); \( \text{Hom}_{S^\cdot_{\mathcal{O}_X}} \)) and is defined as the equalizer

\[
\text{Hom}_{\mathcal{O}_X}(M, M')(U) \longrightarrow \prod_{V \subseteq U} \text{Hom}_{\mathcal{O}_X}(V) (M(V), M'(V))
\]

\[
\longrightarrow \prod_{W \subseteq V \subseteq U} \text{Hom}_{\mathcal{O}_X}(V) (M(V), M'(W))
\]

\[
\{ \varphi_V \} \longmapsto \varphi_W \circ r_{W,V} \longmapsto r'_{W,V} \circ \varphi_V
\]

We have the full subcategories of sheaves

\[
\mathcal{O}_X\text{-Set} \subseteq \text{pre-}\mathcal{O}_X\text{-Set}
\]

resp. \( \mathcal{O}_X\text{-Set}^{\nabla} \subseteq \text{pre-}\mathcal{O}_X\text{-Set}^{\nabla} \)

\[
\Sigma(\mathcal{O}_X) \subseteq \text{pre-}\Sigma(\mathcal{O}_X)
\]

\[
S^\cdot_{\mathcal{O}_X}\text{-mod} \subseteq \text{pre-}S^\cdot_{\mathcal{O}_X}\text{-mod}
\]

where a pre-sheaf \( M(U) \) is a sheaf if for all open covers \( U = \bigcup_{i \in I} U_i \),

\[
M(U) \longrightarrow \prod_{i} M(U_i) \longrightarrow \prod_{i_0, i_1} M(U_{i_0} \cap U_{i_1})
\]

\[
\{ m_i \} \longmapsto r_{U_{i_0} \cap U_{i_1}, U_{i_0}}(m_{i_0}) \longmapsto r_{U_{i_0} \cap U_{i_1}, U_{i_1}}(m_{i_1})
\]

is an equalizer. The above embeddings of categories (5.5) have left-adjoints \( M \mapsto M^2 \) the sheafification functor. These subcategories of sheaves are complete.
The categories of sheaves are closed symmetric monoidal via

\[ M \otimes_{\mathcal{O}_X} N := (\mathcal{O}_X \otimes \mathcal{O}_N)^\Sigma \; \text{(using sheafification)} \]

\[ \text{Hom}_{\mathcal{O}_X}(M, N) := \text{Hom}_{\mathcal{O}_X}(M, N) \]

Notice that the internal pre-sheaf Hom of two sheaves is also a sheaf.

Given a map of generalized schemes \( f \in \mathbf{GSch}(X, Y) \), we get adjunctions

\[ \begin{array}{ccc}
\mathcal{O}_X\text{-Set} & \xrightarrow{f^*} & \mathcal{O}_Y\text{-Set} \\
\mathcal{O}_Y\text{-Set} & \xleftarrow{f_*} & \mathcal{O}_X\text{-Set} \\
\Sigma(\mathcal{O}_X) & \xrightarrow{f^*} & \Sigma(\mathcal{O}_Y) \\
\Sigma(\mathcal{O}_Y) & \xleftarrow{f_*} & \Sigma(\mathcal{O}_X) \\
\end{array} \]

\[ \begin{array}{ccc}
\mathcal{O}_X\text{-mod} \quad \xrightarrow{f^*} \quad \mathcal{O}_Y\text{-mod} \\
\mathcal{O}_Y\text{-mod} \quad \xleftarrow{f_*} \quad \mathcal{O}_X\text{-mod} \\
S_{\mathcal{O}_X} \quad \xrightarrow{f^*} \quad S_{\mathcal{O}_Y} \\
S_{\mathcal{O}_Y} \quad \xleftarrow{f_*} \quad S_{\mathcal{O}_X} \\
\end{array} \]

satisfying the usual properties, such as the existence of canonical equivalences

\[ (g \circ f)_* \sim g_* \circ f_* \quad (g \circ f)^* \sim f^* \circ g^* \]

\[ f_\ast \text{Hom}_{\mathcal{O}_X}(f^* M, N) \sim \text{Hom}_{\mathcal{O}_Y}(M, f_* N) \]

Given \( A \in \text{CGR}, S \subseteq A[1] \) a multiplicative subset, and \( M \in A\text{-Set} \), we can localize \( M \) to obtain \( S^{-1}M \in S^{-1}A\text{-Set} \) in the usual way: \( S^{-1}M := (M \times S) / \sim \), with the equivalence relation \( \sim \) given by

\[ (m_1, s_1) \sim (m_2, s_2) \text{ if } s \cdot s_2 \cdot m_1 = s \cdot s_1 \cdot m_2 \text{ for some } s \in S. \]

We denote by \( m/s \) the equivalence class of \( (m, s) \).

The functor \( M \mapsto S^{-1}M \) commutes with colimits, finite-limits, tensor products and Hom’s. We obtain a functor

\[ \begin{array}{ccc}
A\text{-Set} & \longrightarrow & \mathcal{O}_A\text{-Set} \\
M & \longrightarrow & M^2 \\
\end{array} \]

\[ M^2 := \left\{ \sigma : U \rightarrow \coprod_{p \in U} S^{-1}_p M, \sigma(p) \in S^{-1}_p M, \text{ locally constant} \right\} \]
where \( \sigma \) is locally constant if for all \( p \in U \), there is a neighborhood \( p \in D^+(s) \subseteq U \), and \( m \in M \) such that for \( q \in D^+(s), \sigma(q) \equiv m/s \in S_q^{-1}M \).

This functor prolongs naturally to functors

\[
\begin{align*}
\text{A-Set}^\triangledown & \longrightarrow \mathcal{O}_A\text{-Set}^\triangledown \\
\Sigma(A) & \longrightarrow \Sigma(\mathcal{O}_A) \\
S_A\text{-mod} & \longrightarrow S_{\mathcal{O}_A}\text{-mod} \\
M & \longrightarrow M^2
\end{align*}
\]

We have an identification of the stalk of \( M^2 \) at \( p \in \text{spec}^+(A) \),

\[
M^2 \bigg|_p = \lim_{U \ni p} M^2(U) \longrightarrow S_p^{-1}M := M_p
\]

Moreover, we have the identification of the global sections of \( M^2 \) over an affine basic open subset \( D^+(s) \subseteq \text{spec}^+(A) \),

\[
\Psi : M[1/s] := \{ s^n \}^{-1} M \longrightarrow M^2(D^+(s)) \longrightarrow \sigma(p) \equiv \frac{m}{s^n} \in S_p^{-1}M
\]

**Proof.** The map \( \Psi \) which takes \( m/s^n \in M[1/s] \) to the constant section \( \sigma \) with \( \sigma(p) \equiv m/s^n \) for all \( p \in D^+(s) \) is well defined and is a map of \( A[1/s]\text{-Sets} \). \( \Psi \) is injective: Assume \( \Psi(m_1/s^{n_1}) = \Psi(m_2/s^{n_2}) \). Let \( \mathfrak{I} \) denote the \( + \)ideal generated by \( \{ a \in A^+_1, a \cdot s^{n_2} \cdot m_1 = a \cdot s^{n_1} \cdot m_2 \} \). We have

\[
m_1/s^{n_1} = m_2/s^{n_2} \in A_p \quad \text{for all} \quad p \in D^+(s)
\]

\[
\Rightarrow s_p \cdot s^{n_2} \cdot m_1 = s_p \cdot s^{n_1} \cdot m_2 \quad \text{with} \quad s_p \in A^+_1 \setminus p, \text{ all} \quad p \in D^+(s)
\]

\[
\Rightarrow \mathfrak{I} \not\subseteq p, \text{ all} \quad p \in D^+(s)
\]

\[
\Rightarrow V^+(\mathfrak{I}) \cap D^+(s) = \emptyset
\]

\[
\Rightarrow V^+(\mathfrak{I}) \subseteq V^+(s)
\]

\[
\Rightarrow s_n \in \mathfrak{I} \text{ for some} \quad n > 0
\]

\[
\Rightarrow s_n \cdot s^{n_2} \cdot m_1 = s_n \cdot s^{n_1} \cdot m_2
\]

\[
\Rightarrow m_1/s^{n_1} = m_2/s^{n_2} \text{ in} \quad M[1/s]
\]
\( \Psi \) is surjective: Fix \( \sigma \in M^2(D^+(s)) \). We get a covering by basic open subsets on which \( \sigma \) is constant,

\[
D^+(s) = D^+(s_1) \cup \ldots \cup D^+(s_N), \quad \sigma(p) \equiv m_i/s_i \text{ for } p \in D^+(s_i).
\]

On the subsets \( D^+(s_is_j) = D^+(s_i) \cap D^+(s_j) \) we have \( m_i/s_i = m_j/s_j \), and by the injectivity of \( \Psi : (s_is_j)^n \cdot s_j \cdot m_i = (s_is_j)^n \cdot s_i \cdot m_j \) (with one \( n \) which works for all \( i, j \leq N \)). Replacing \( s_i^n \cdot m_i \) by \( m_i \), and \( s_{i+1} \) by \( s_i \), we may assume \( \sigma \) is given by \( m_i/s_i \) on \( D^+(s_i) \), and moreover for all \( i, j \leq N \):

\[
s_j \cdot m_i = s_i \cdot m_j.
\]

Since \( D^+(s) \subseteq \bigcup_i D^+(s_i) \), we have for some \( k > 0 \), \( b, d \in A_\ell, j : \ell \to \{1, \ldots, N\} \),

\[
s^k = (b \triangleleft s_j(x)) \parallel d
\]

Define \( m \in M \) by

\[
m := \langle b, m_{j(x)}, d \rangle
\]

We have for \( i = 1, \ldots, N \)

\[
s_i \cdot m = s_i \cdot \langle b, m_{j(x)}, d \rangle
= \langle b, s_i \cdot m_{j(x)}, d \rangle
= \langle b, s_{j(x)} \cdot m_i, d \rangle
= \langle b \triangleleft s_{j(x)}, m_i, d \rangle
= \langle (b \triangleleft s_{j(x)}) \parallel d, m_i, 1 \rangle
= s^k \cdot m_i
\]

Thus \( m_i/s_i = m/s^k \) in \( M \{1/s\} \) and the section \( \sigma \) is constant \( \sigma = \Psi(m/s^k) \), and \( \Psi \) is surjective.

Given a generalized scheme \( X \), we have the full subcategories of "quasi-coherent" sheaves

\[
q.c.\mathcal{O}_X\text{-Set} \subseteq \mathcal{O}_X\text{-Set}
\]

resp., \( q.c.\mathcal{O}_X\text{-Set}^\vee \subseteq \mathcal{O}_X\text{-Set}^\vee \)

\[
q.c.\Sigma(\mathcal{O}_X) \subseteq \Sigma(\mathcal{O}_X)
\]

\[
q.c.\mathcal{S}_{\mathcal{O}_X}\text{-mod} \subseteq \mathcal{S}_{\mathcal{O}_X}\text{-mod}
\]

where an object \( M \) is quasi-coherent if there is a covering of \( X \) by open affines

\( X = \cup_i \text{spec}^+(A_i) \), and there are \( M_i \in A_i\text{-Set} \) (resp. \( A_i\text{-Set}^\vee \), \( \Sigma(A_i) \), \( \mathcal{S}_{A_i}\text{-mod} \)).
with \( M|_{\text{spec}^+(A)} \cong M_i^2 \). Equivalently, for all open affine \( \text{spec}^+(A) \subseteq X \), we have

\[
M|_{\text{spec}^+(A)} \cong M (\text{spec}^+(A))^2.
\]

For an affine scheme \( X = \text{spec}^+(A) \) we get equivalences of categories

\[
\begin{array}{ccc}
M^2 & \cong & M \\
\downarrow & & \downarrow \\
M & \cong & M \text{spec}^+(A) \subseteq \text{spec}^+(A) = M (\text{spec}^+(A))^2.
\end{array}
\]

For a mapping of generalized schemes \( f : X \to Y \), the functor \( f^* \) takes quasi-coherent \( \mathcal{O}_Y \)-Sets to quasi-coherent \( \mathcal{O}_X \)-Set and we get strict-monoidal functors

\[
\begin{array}{ccc}
f^* : q.c.\mathcal{O}_Y \text{-Set} & \longrightarrow & q.c.\mathcal{O}_X \text{-Set} \\
f^* : q.c.\mathcal{O}_Y \text{-Set}^\nabla & \longrightarrow & q.c.\mathcal{O}_X \text{-Set}^\nabla \\
f^* : q.c.\Sigma(\mathcal{O}_Y) & \longrightarrow & q.c.\Sigma(\mathcal{O}_X) \\
f^* : q.c.\mathcal{S}_{\mathcal{O}_Y} \text{-mod} & \longrightarrow & q.c.\mathcal{S}_{\mathcal{O}_X} \text{-mod}
\end{array}
\]
Moreover, for affine open subsets \( U = \text{spec}^+(A) \subseteq X, V = \text{spec}^+(B) \subseteq Y \), with \( f(U) \subseteq V \) and with \( \varphi = f^\#: B \to A \) the associated homomorphism, we have for quasi-coherent sheaf \( M \) on \( Y \),

\[
(5.25) \quad f^*M \bigg|_U \cong (\varphi^*M(V))^\sharp.
\]

**Definition 5.26.** A mapping of generalized schemes \( f : X \to Y \) is called a \( c \)-map (“compact and quasi-separated”) if we can cover \( Y \) by open affine subsets (or equivalently, if for all open affine) \( U = \text{spec}^+(B) \subseteq Y \), we have that \( f^{-1}(U) \) is compact, so that

\[
f^{-1}(U) = \bigcup_{i=1}^{N} V_i, \quad V_i = \text{spec}^+(A_i),
\]

is a finite union of open affine subsets, and if moreover for \( i, j \leq N \), we have \( V_i \cap V_j \) also compact, so that

\[
V_i \cap V_j = \bigcup_{k=1}^{N_{ij}} W_{ijk}, \quad W_{ijk} = \text{spec}^+(A_{ijk}).
\]

For a \( c \)-map \( f : X \to Y \), the functor \( f_* \) takes quasi-coherent sheaves to quasi-coherent sheaves, so we get adjunctions

\[
\begin{array}{ccc}
q.c.\mathcal{O}_X\text{-Set} & \xleftarrow{\quad f^*} & q.c.\mathcal{O}_Y\text{-Set} \\
\quad & \text{and} & \\
q.c.\mathcal{O}_X\text{-Set}^{\triangledown} & \xrightarrow{\quad f_*} & q.c.\mathcal{O}_Y\text{-Set}^{\triangledown}
\end{array}
\]

\[
(5.27)
\]

\[
\begin{array}{ccc}
q.c.\Sigma(\mathcal{O}_X) & \xleftarrow{\quad f^*} & q.c.\mathcal{O}_X\text{-mod} \\
\quad & \text{and} & \\
q.c.\Sigma(\mathcal{O}_Y) & \xrightarrow{\quad f_*} & q.c.\mathcal{O}_Y\text{-mod}
\end{array}
\]

Indeed, for a quasi-coherent sheaf \( M \) on \( X \), we have (using the above notations) that \( f_*M|_U = f_*M|_{\text{spec}^+(U)} \) is the equalizer

\[
(5.28) \quad f_*M|_U \longrightarrow \prod_{i=1}^{N} f_*M|_{V_i} \longrightarrow \prod_{i,j \leq N} \prod_{k=1}^{N_{ij}} f_*M|_{W_{ijk}}
\]

and

\[
\begin{align*}
& f_*\left(M|_{V_i}\right) = \left(f^\#_iM(V_i)\right)^\sharp, \\
& f_*\left(M|_{W_{ijk}}\right) = \left(f^\#_{ijk}M(W_{ijk})\right)^\sharp
\end{align*}
\]
are quasi-coherent.

For a generalized scheme $X$, let $\mathcal{D}(X) = \text{Aff}/X \times S'$ denote the simplicial category with objects pairs $(\text{spec}^+(A), M')$, with $\text{spec}^+(A) \subseteq X$ open affine, and $M' \in S'_A$-mod. The maps are given by

$$\mathcal{D}(X) \left((\text{spec}^+(A_0), M_0), (\text{spec}^+(A_1), M_1)\right) = \mathbb{R}\text{Map}_{S_A} (L\phi^* M_1, M_0)$$

where $\phi : \text{spec}^+(A_0) \hookrightarrow \text{spec}^+(A_1)$ denotes the inclusion (and are empty if $\text{spec}^+(A_0) \not\subseteq \text{spec}^+(A_1)$). We have a forgetful functor

$$\rho : \mathcal{D}(X) \longrightarrow \text{Aff}/X.$$ 

The coherent nerve of $\mathcal{D}(X)$ is the simplicial set $N_{\Delta} (\mathcal{D}(X)) \in \text{Set}^\Delta$ with $n$ simplices

$$N_{\Delta} (\mathcal{D}(X))_n = \text{Cat}_{\Delta} (\mathcal{C}(\Delta^n), \mathcal{D}(X))$$

Explicitly, its elements are given by the data of

(i) $(U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n) \in N_n (\text{Aff}/X), \quad U_i = \text{spec}^+(A_i) \subseteq X$ open,

(ii) $M_i \in S'_A$-mod

(iii) For $0 \leq i < j \leq n$, $\varphi_{ij} \in S'_A$-mod $(L\phi_{ij}^* M_j, M_i)$

where $\phi_{ij} : U_i \hookrightarrow U_j$ denotes the inclusion, and higher coherent homotopies given by

$$\varphi_{\tau} \in S'_A$-mod $(L\phi_{ij}^* M_j \otimes \Delta(\ell)_+ M_i)$

(5.33)

for all $\tau = (\tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_\ell) \in N_\ell (P_{ij})$

(Where $P_{ij}$ is the set of subsets of $[i, j]$ containing the end points). These are required to satisfy:

$$d_k \varphi_\tau = \varphi_{(\tau_0 \subseteq \cdots \hat{\tau}_k \cdots \subseteq \tau_\ell)}$$

and

$$\varphi_{\tau_0} = \varphi_{ik_1} \circ L\phi_{i k_1}^* \left( \varphi_{k_1, k_2} \circ L\phi_{k_1, k_2}^* \left( \cdots \varphi_{k_{\ell-1}, k_\ell} \circ L\phi_{k_{\ell-1}, k_\ell}^* \left( \varphi_{k_\ell, j} \right) \right) \right)$$

(5.35)

Forgetting all the data but the $n$ simplex $(U_0 \subseteq \cdots \subseteq U_n)$ of $\text{Aff}/X$ gives an inner, Cartesian and co-Cartesian, fibration of quasi-categories

$$N_{\Delta} (\rho) : N_{\Delta} (\mathcal{D}(X)) \longrightarrow N (\text{Aff}/X).$$
The derived category of quasi-coherent symmetric spectra $\mathcal{D}(X) = \operatorname{Ho}D(X)_\infty$ is the homotopy category of $D(X)_\infty$ the $(\infty$-categorical) limit of $D(A)_\infty$, where $D(A)_\infty$ is the $\infty$-category of fibrant co-fibrant $S_A$-mod associated to $\mathcal{D}(A) = \operatorname{Ho}(S_A$-mod), and the limit taken over all $A$ with $\text{spec}^+(A) \in \text{Aff}/X$.

By Lurie’s straightening/unstraightening functors, [8], Corollary 3.3.3.2, this limit $D(X)_\infty$ is equivalent to the full sub-$\infty$-category on Cartesian sections of the $\infty$-category of sections of $\mathcal{D}(X)_\infty$.

where a section $m : N(\text{Aff}/X) \longrightarrow M(\mathcal{D}(X))$

(5.37) $N_\Delta(\rho) \circ m = \text{id}$

is Cartesian iff for all $n$-simplex $(U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n) \in N_n(\text{Aff}/X)$, $U_i = \text{spec}^+(A_i)$, the data $m(U_i)$ in the notions of (5.32) satisfies that all the arrows (iii) $\varphi_{ij} \in S_A$-mod($(\mathcal{L}\varphi_{ij}(M_j), M_i), 0 \leq i < j \leq n$, are weak-equivalences.

The category $D(X)_\infty$ is symmetric monoidal stable quasi-category, so $\mathcal{D}(X)$ is symmetric monoidal triangulated category.

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