Online Learning with Diverse User Preferences

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Abstract—In this paper, we investigate the impact of diverse user preference on learning under the stochastic multi-armed bandit (MAB) framework. We aim to show that when the user preferences are sufficiently diverse and each arm can be optimal for certain users, the $O(\log T)$ regret incurred by exploring the sub-optimal arms under the standard stochastic MAB setting can be reduced to a constant. Our intuition is that to achieve sub-linear regret, the number of times an optimal arm being pulled should scale linearly in time; when all arms are optimal for certain users and pulled frequently, the estimated arm statistics can quickly converge to their true values, thus reducing the need of exploration dramatically. We cast the problem into a stochastic linear bandits model, where both the users preferences and the state of arms are modeled as independent and identical distributed (i.i.d) $d$-dimensional random vectors. After receiving the user preference vector at the beginning of each time slot, the learner pulls an arm and receives a reward as the linear product of the preference vector and the arm state vector. We also assume that the state of the pulled arm is revealed to the learner once it is pulled. We propose a Weighted Upper Confidence Bound (W-UCB) algorithm and show that it can achieve a constant regret when the user preferences are sufficiently diverse. The performance of W-UCB under general setups is also completely characterized and validated with synthetic data.

I. INTRODUCTION

Real-time resource allocation in next-generation networked systems face a prominent challenge: allocation decisions must be made to meet the heterogeneous demands of diverse users; however, the outcome of a given allocation decision, such as the throughput or delay for a selected route in a communication network, may be imperfectly known or change quickly in time. This is particularly relevant for networks operating in dynamic environments, such as cognitive radio systems, cloud computing centers, crowdsourcing platforms, etc.

In this paper, we cast the online resource allocation problem into the stochastic multi-armed bandit (MAB) framework [1]–[5]. While MAB has been the predominant tool that isolates the tradeoff between exploration and exploitation in sequential learning and control problems, the impact of diverse user preferences on the learning regret performance has rarely been studied. That is the main focus of this paper.

Specifically, we consider a set of $K$ arms (decisions). The state of each arm (outcome of each decision) is represented by a $d$-dimensional vector, which varies according to an independent and identically distributed (i.i.d) random processes in time. We assume the statistics of the arm states are unknown a priori, and the the learner only observes the state of an arm after it is pulled. The user preference is modeled as another i.i.d. $d$-dimensional weight vector, and is available to the learner before it makes decision. Each time, the reward of pulling an arm is the inner product of the user preference vector and the state of the pulled arm. Due to the diverse user preferences, the expected reward obtained by pulling each arm varies for different users. Therefore, in contrast to conventional stochastic MAB setting where there exists a unique optimal arm, under this setting, the optimal arm changes for different group of users. Intuitively, such diversity creates an opportunity for the learner to exploit the user preference to reduce the learning regret: when an arm is pulled frequently as the optimal arm for a group of users, its statistics can be estimated accurately, thus the learner does not have to spend much time exploring it when current user preference is not favorable.

A. Main Contributions

The main contributions of this paper are three fold.

1) We study a novel stochastic linear bandits model, which captures diverse user preferences and has implications in many practical scenarios. Under this model, the random and heterogeneous user preference creates an opportunity for the learner to shorten its exploration process, thus achieving a better exploration-exploitation tradeoff and reducing the regret.

2) We propose a Weighted Upper Confidence Bound (W-UCB) algorithm when arm statistics are unavailable a priori. Although W-UCB is a simple and intuitive algorithm, it admits a unique structure of the regret. Through theoretical analysis, we show that the cumulative regret can be decomposed into two parts: a constant term depending on the arms that are optimal for certain users and an $O(\log T)$ term depending on the strictly sub-optimal arms. If all arms are optimal for certain users, the $O(\log T)$ term disappears.

3) We show that the W-UCB algorithm is order-optimal by establishing an order-matching lower bound on the regret.

B. Related Literature

The proposed bandit model is similar to the stochastic linear contextual bandits model in the literature [6]. In the contextual MAB setting, the learner repeatedly takes one of $K$ actions in response to the observed context [7]. Efficient exploration
according to instantaneous context is of critical importance in contextual bandit in order to achieve small learning regret. The strongest known results [7]–[12] achieve an optimal regret after $T$ rounds of $O(\sqrt{KT \log T})$ with high probability. The main difficulty in such setting is that there is no assumption on the reward for different contexts and actions, thus it is impossible to share information between different context or arms.

In [13], it considers a linear reward structure and proposes a LinUCB algorithm. A modified version of this algorithm, named SupLinUCB, is considered in [14], and shown to achieve $O(\sqrt{dT})$ regret, where $d$ is the dimension of the context. [15] mixes LinUCB and SupLinUCB with kernel functions and proposes an algorithm to further reduce the regret to $O(\sqrt{dT})$, where $d$ is the effective dimension of the kernel feature space.

Recently, a few works start to take the diversity in contexts into consideration. In [16], it proposes a concept called covariate diversity, which requires that the covariance matrix of the observed contexts conditioned on any half space is positive definite. Under this condition, it shows that the exploration-free greedy algorithms is near-optimal for a two-armed bandit under the stochastic setting and achieves regret in $O(\log T)$.

[17] investigates a perturbed adversarial setting with a similar notion of diversity, and shows that greedy algorithms can achieve regrets in $O(\sqrt{dT})$. It has been shown empirically in [18] that many contextual bandits problems can be solved purely via the implicit exploration imposed by the diversity of contexts.

A major difference between our model and the contextual linear bandits models studied in [13]–[17] is that, we assume the state of pulled arm is revealed to the learner, while in the contextual linear bandits setting, only the reward is observable. Our assumption enables the learner to easily share information under different user preferences, thus elucidating the impact of diverse user preferences on the learning performance.

II. Problem Formulation

We consider a set of $K$ arms denoted as $[K] = \{1, 2, \ldots, K\}$. At time slot $t$, the state of each item $i \in [K]$ is represented by a $d$-dimensional vector $x_{i,t} \in [0, 1]^d$, which evolves in an i.i.d. fashion. We denote the distribution of arm $i$ as $\nu_i$. Denote the mean vector of $x_{i,t}$ as $\mu_i \in (0, 1)^d$. Meanwhile, we assume at the beginning of each time slot $t$, a weight vector $\lambda_t \in \mathbb{R}^d_+$, representing the preference of the incoming user, becomes available to the learner. We assume $\|\lambda_t\|_1 = 1$, and the reward obtained by pulling arm $i$ at time $t$ is $\lambda_t^\top x_{i,t}$. We assume that the state of the pulled arm $x_{i,t}$ is revealed to the learner once it is pulled. We consider an online learning setting where the objective of the learner is to sequentially pull the arms in order to maximize the expected reward, based on the instantaneous user preference and historical observations of the arm states.

In order to analyze the impact of user preference diversity into subsets as follows:

$$\Lambda^j := \left\{ \lambda | \lambda^\top \mu_j > \lambda^\top \mu_i, \forall i \neq j, i \in [K] \right\},$$

i.e., the subset of user preferences for which arm $j$ yields the maximum expected reward. We assume for each $\lambda$, the optimal arm is unique. This is similar to the assumption under standard MAB setting that the reward gap between the optimal arm and the second best arm is bounded away from zero.

We consider a stochastic setting where $\lambda_t$ is i.i.d. Define $\rho_j := \mathbb{P}[\lambda_t \in \Lambda^j]$. Then $\sum_{j=1}^K \rho_j = 1$. Note that for some $j \in [K]$, we may have $\rho_j = 0$. We call such arms as strictly sub-optimal arms. We group the arms that are optimal under certain contexts in $S_1 := \{j \in [K] | \rho_j > 0\}$, and the remaining strictly sub-optimal arms in $S_2 := [K] \setminus S_1$. Intuitively, the size of $S_1$ indicates the diversity level in users: if $|S_1| = K$, every arm could be optimal; if $|S_1| = 1$, it reduces to the conventional MAB case where only one optimal arm exists. For ease of exposition, in the following, we assume $\rho_j = \frac{1}{|S_1|}, \forall j \in S_1$. Our method can be easily extended to the general situation that $\sum_{j \in S_1} \rho_j = 1, \forall \rho_j > 0$.

Denote $a_t \in [K]$ as the arm pulled at time $t$, and $a^*_t$ as the optimal arm that maximizes the expected reward at time $t$ if $\{\mu_i\}$ were given a priori. Then, if $\lambda_t \in \Lambda^j, a^*_t = j$. The per-slot regret is defined as $\lambda_t^\top x_{a^*_t,t} - \lambda_t^\top x_{a_t,t}$, and the expected accumulated regret up to time $T$ can then be defined as

$$\mathbb{E}[R(T)] := \mathbb{E}\left[ \sum_{t=1}^T \lambda_t^\top x_{a^*_t,t} - \lambda_t^\top x_{a_t,t} \right]. \quad (1)$$

Denote the observations up to time $t-1$ as $\mathcal{H}^{t-1}$, i.e., $\mathcal{H}^{t-1} := \cup_{\tau=1}^{t-1}\{x_{a_\tau,\tau}, \lambda_\tau\}$. Then, without a priori statistics about $\{x_{i,t}\}$ and $\{\lambda_t\}$, our objective is to design an online algorithm to decide $a_t$ based on $\mathcal{H}^{t-1}$ and $\lambda_t$, so that the $\mathbb{E}[R(T)]$ grows sublinearly in $T$.

III. Main Results

A. Algorithm

In this section, we propose a Weighted Upper Confidence Bound (W-UCB) algorithm to adaptively match the arms with users. W-UCB adopts the Optimistic Facing Uncertainty (OFU) principle where the learner always chooses the arm with the highest potential reward after adding a UCB term. Specifically, we define $N_i(t) := \sum_{\tau=1}^{t-1} I\{a_\tau = i\}$, i.e., the number of times that arm $i$ is pulled until time $t$, and denote $\hat{x}_{i,t}$ as the sample average of the state of arm $i$ right before time $t$, i.e., $\hat{x}_{i,t} = \frac{\sum_{\tau=1}^{t-1} 1\{a_\tau = i\} x_{a_\tau,\tau}}{N_i(t)}$. Then, we use $u_{i,t} := \frac{2\log N_i(t)}{N_i(t)}$ to control the width of the upper confidence interval on $x_{i,t}$. In each time $t$, the learner picks the best arm based on inner produce of the UCB padded sample average $\hat{x}_{i,t}$ and $\lambda_t$. The algorithm is presented in Algorithm 1.

Although W-UCB seems a straightforward extension of the standard UCB algorithm, the main technical difficulty and correspondingly our novel contribution, however, lies in the theoretical analysis. This is because under standard UCB for conventional MAB model, the optimal arm is fixed, and the
optimal $O(\log T)$ regret mainly comes from pulling the suboptimal arms during exploration. However, under our setting, the optimal arm is user-dependent: an arm in $S_1$ could be both optimal or sub-optimal, depending on the instantaneous user preference. Therefore, in order to analyze the regret, we need to carefully track the number of times that an arm is being pulled as the optimal arm, or as a sub-optimal arm.

B. Upper Bound on Regret under W-UCB

In order to facilitate our analysis, we assume that for each $\lambda \in \Lambda^j$ and each $i \in [K]\{j\}$, there exist two constants $l, h > 0$ such that $l \leq \lambda^i \mu_j - \lambda^j \mu_i \leq h$, i.e., the expected reward gap between the optimal arm and any sub-optimal arm under any context is bounded.

**Theorem 1** Under W-UCB algorithm, $\mathbb{E}[R(T)] \leq \frac{8h|S_1||S_2|}{\rho^2} \log T + C_1$, where $C_1$ is a constant.

**Remark:** When $S_1 = [K]$, $|S_2| = 0$, $\mathbb{E}[R(T)]$ reduces to a constant. This indicates that the UCB padding term under the W-UCB algorithm quickly shrinks to a small value, and the algorithm actually performs exploitation with high probability, thus reducing the regret incurred during exploration. This corroborates our intuition that the diversity of context can lead to regret improvement.

The proof of Theorem 1 is deferred to Section IV-A.

C. Lower Bound

First, we define $\alpha$-consistent policies as follows.

**Definition 1** A policy is said to be $\alpha$-consistent for fixed $\alpha \in (0,1)$ if and only if there exists a prefixed constant $C$ s.t. $\mathbb{E}\left[\sum_{t=1}^{T} \mathbb{I}\{a^*_t \neq a_t\}\right] \leq Ct^\alpha$ for all $t > 0$.

**Theorem 2** If $S_1 \neq [K]$, then for any $\alpha$-consistent policy, when $T$ is sufficiently large, we have

$$\mathbb{E}[R(T)] \geq \sum_{i \in S_2} 2(1-\alpha) \log T + 2 \log \left(16C|S_1|\right) l$$

Here $KL(\nu_i || \nu_j)$ is the KL divergence between arm state distributions $\nu_i$ and $\nu_j$.

The proof of Theorem 2 is provided in Section IV-B.

**Remark:** Theorem 2 indicates that when $S_1 \neq [K]$, the lower bound scales in $O(|S_2| \log T)$, which matches the order of the upper bound in Theorem 1 up to a coefficient. This indicates that the W-UCB algorithm is order-optimal for such scenario.

IV. THEORETICAL ANALYSIS

A. Upper Bound Analysis

We first introduce the following notations. Recall that we use $N_i(t)$ to denote the number of times that arm $i$ has been pulled up to time $t$. In the following, we denote $N_j^T(t) := \sum_{\tau=1}^{t} \mathbb{I}\{\lambda_{\tau} \in \Lambda^j\}$, i.e., the number of times that the user preferences fall in $\Lambda^j$ up to time $t$. We also define $N_j^T(t) := \sum_{\tau=1}^{t} \mathbb{I}\{a_{\tau} = i, \lambda_{\tau} \in \Lambda^j\}$, i.e., the number of times that arm $i$ has been pulled when the corresponding weight vector falls in $\Lambda^j$ up to $t$. We have the following observations.

**Lemma 1** For $j \in S_1$, when $t \geq \left(4(K-1)|S_1|\right)^2 \left(\frac{1}{\rho} + 1\right)^2$, we have

$$\mathbb{P}\left[N_j^T(t) < \frac{t}{4|S_1|}\right] \leq e^{-\frac{t}{4|S_1|}} + 2dK \left[(1-\rho)t + 1\right] \frac{1}{(\rho t)^2}$$

where $\rho := \frac{1}{4(K-1)|S_1|}$.

The proof of Lemma 1 and those of Lemma 2 and Lemma 3 can be found in the later version of this paper [19].

A sketch of the proof is as follows: First, we note that due to the i.i.d. assumption on the incoming weight vectors, $N_j^T(t)$ should be around $\frac{t}{4|S_1|}$ with high probability. Then, we show that if $N_j^T(t) < \frac{t}{4|S_1|}$, with high probability, there must exist at least one sub-optimal arm being pulled $\Omega(t)$ times over the first $t$ time slots with user preferences in $\Lambda^j$. Under the W-UCB algorithm, if an arm $i$ has been pulled $\Omega(\frac{t}{4|S_1|})$ times, its UCB padding term must be very small, and its sample average $\hat{x}_{i,t}$ should be very close to $\mu_i$ with high probability. Therefore, it is unlikely for the learner to mistakenly choose it as the optimal arm when $t$ is sufficiently large. Therefore, the probability to have a suboptimal arm being pulled $\Omega(\frac{t}{4|S_1|})$ times is small, which in turn, shows that the event $N_j^T(t) < \frac{t}{4|S_1|}$ happens with small probability.

**Lemma 2** For any $i, j \in S_1$, $i \neq j$.

$$\mathbb{E}\left[N_j^T(T)\right] \leq \frac{1}{|S_1|} \left(\frac{1024(K-1)|S_1|^2}{t^4} + 2|S_1|^2 + 6dK \frac{1}{\rho^3} + 3d\right)$$

where $\rho := \frac{1}{4(K-1)|S_1|}$.

Lemma 2 indicates that wrongly pulling an arm $i \in S_1$ when the context falls in $\Lambda^j$ does not happen frequently in expectation.

**Lemma 3** For any $j \in S_1$, $i \in S_2$.

$$\mathbb{E}\left[\sum_{j \in S_1} N_j^T(T)\right] \leq \frac{8 \log T}{t^2} + 3d.$$
Since
\[ \mathbb{E}[R(T)] \leq \sum_{i,j \in S_1} \mathbb{E} \left[ N_i^j(T) \right] + \sum_{i \in S_2} \sum_{j \in S_1} \mathbb{E} \left[ N_i^j(T) \right], \]
applying Lemma 2 and Lemma 3, we obtain the upper bound in Theorem 1.

B. Lower Bound Analysis

For any given a pair of arms \( j \in S_1, i \in S_2 \), we will construct an alternative distribution \( \nu \) on \( X_i \), so that arm \( i \) will become the optimal arm when \( \lambda_t \in \Lambda^j \). Specially, let \( \tilde{x}_{i,t} = (1 - \epsilon_j) x_j + \epsilon_j 1 \) be the status vector of arm \( i \) under the alternative distribution, where \( \epsilon_j \) is a small constant lying in \((0, 1)\).

We note that \( \tilde{x}_{i,t} \in (0, 1)^d \).

When \( \lambda_t \in \Lambda^j \), we have
\[
\mathbb{E}[\nu_i \tilde{x}_{i,t}] = \mathbb{E}[(1 - \epsilon_j) \lambda^j_i x_j + \epsilon_j \lambda^j_i 1] = (1 - \epsilon_j) \lambda^j_i \mu_j + \epsilon_j \lambda^j_i 1 \]
where \( \mu_j \in (0, 1)^d \). Thus, arm \( i \) becomes the optimal arm when \( \lambda_t \in \Lambda^j \) under this new distribution.

Besides, for any \( \lambda_t \in \Lambda^k, k \neq j \), we have
\[
\mathbb{E}[\nu_i \tilde{x}_{i,t}] = \mathbb{E}[(1 - \epsilon_j) \lambda^k_i x_j + \epsilon_j \lambda^k_i 1] = (1 - \epsilon_j) \lambda^k_i \mu_j + \epsilon_j < \lambda^j_i \mu_j + \epsilon_j < \lambda^k_i \mu_j \]
where the last inequality follows from bounds on \( \epsilon_j \). Inequality (5) indicates that under the alternative distribution, the optimal arm remains unchanged when \( \lambda_t \notin \Lambda^j \).

In the following, we use \( \mathbb{E}, \mathbb{E}, \mathbb{P}, \mathbb{P} \), \( \tilde{a}_t \), \( \tilde{a}_t \) to denote the expectation, the probability measure, and the optimal arm under the original and the alternative distributions, respectively.

Consider event \( I_t := \left\{ \sum_{t=1}^{T} I\{a_t = i\} > \frac{T}{2|S_1|} \right\} \) and its complement \( I_t^c \). We bound \( \mathbb{P}[I_T] \) and \( \mathbb{P}[I_T^c] \) as follows.

Under the original distribution, \( i \in S_2 \), which implies that if \( a_t = i \), we must have \( a_t^* \neq a_t \). Thus, \( I\{a_t = i\} \subseteq I\{a_t^* \neq a_t \} \). According to the definition of \( \alpha \)-consistent policy, we have
\[
\mathbb{P}[a_T = i] = \mathbb{P}[a_T^* \neq a_T] \leq CT^\alpha.
\]

Then, according to Markov’s inequality, we have
\[
\mathbb{P}[I_T] = \mathbb{P}\left[ \sum_{t=1}^{T} I\{a_t = i\} > \frac{T}{2|S_1|} \right] \leq \mathbb{E}\left[ \sum_{t=1}^{T} I\{a_t = i\} \right] \leq \frac{T}{2|S_1|} \leq 2|S_1| CT^{\alpha - 1}.
\]

Next, we try to bound \( \mathbb{P}[I_T^c] \). Denote \( H_T \) as the event that \( \sum_{t=1}^{T} I\{a_t \in \Lambda^j \} > \frac{T}{2|S_1|} \). Then,
\[
\mathbb{P}[I_T^c] \leq \mathbb{P}[I_T^c \cap H_T] + \mathbb{P}[H_T],
\]
\[
\leq \mathbb{P}\left[ \sum_{t=1}^{T} I\{a_t \in \Lambda^j, a_t \neq i\} \geq \frac{T}{6|S_1|} \right] + \mathbb{P}[H_T] \]
where \( (9) \) is due to the fact that if \( \sum_{t=1}^{T} I\{a_t = i\} > \frac{T}{2|S_1|} \), and \( \sum_{t=1}^{T} I\{a_t \in \Lambda^j \} \leq \frac{T}{2|S_1|} \), we must have \( \sum_{t=1}^{T} I\{a_t \in \Lambda^j \} \leq \frac{T}{2|S_1|} \); (10) is based on Markov’s inequality and Hoeffding’s inequality, and (11) comes from the definition of \( \alpha \)-consistent policy and the fact that \( a_t^* = i \) when \( \lambda \in \Lambda^j \) under the alternative distribution.

Then, according to Lemma 3.2 in [3] and Lemma 18 in [20], we have
\[
\mathbb{K}(\nu_i \| \tilde{\nu}_i) \mathbb{E}[N_i(T)] = \mathbb{K}(\mathbb{P}(Z^T) \| \mathbb{P}(\tilde{Z}^T)) \geq -2 \log \left( \mathbb{P}[I_T] + \mathbb{P}[I_T^c] \right)
\]
\[
\geq -2 \log \left( 2(1 - \alpha) \log T + 2 \log (16|S_1|) \right). \]

We note that when \( T \) is sufficiently large, \( \exp \left( -\frac{2T}{9|S_1|^2} \right) < 8CT^{\alpha - 1}|S_1| \). Thus,
\[
\mathbb{K}(\nu_i \| \tilde{\nu}_i) \mathbb{E}[N_i(T)] \geq -2 \log \left( 2(1 - \alpha) \log T + 2 \log (16|S_1|) \right).
\]

Letting \( \epsilon_j \rightarrow 0 \), we have \( \mathbb{K}(\nu_i \| \tilde{\nu}_i) \rightarrow \mathbb{K}(\nu_i \| \nu_i) \). Then,
\[
\mathbb{E}[N_i(T)] \geq \frac{2(1 - \alpha) \log T + 2 \log (16|S_1|)}{\mathbb{K}(\nu_i \| \nu_i)}.
\]

Since whenever \( i \in S_2 \) is pulled, it incurs a per-step regret at least \( 1 \), we have \( \mathbb{E}[R(T)] \geq \sum_{i \in S_2} \mathbb{E}[N_i(T)] \). Combining with (17), we have the lower bound in Theorem 2 established when \( T \) is sufficiently large.

V. EXPERIMENT RESULTS

We first assume the weight vector \( \lambda_i \in \mathbb{R}^5 \) takes a uniform distribution over \( \frac{1}{8} \mathbb{E}_{i=1}^{5} \), where \( \lambda \) is a 5-dimensional all-ones vector, and \( \mathbb{E}_i \) is a unit vector with the \( i \)-th entry to be one. For the distribution of the arm states, we first generate five basic arms with \( x_{i,t} = \frac{1}{2} \mathbb{E}_i + \frac{1}{2} n_t \mathbb{E}_i \) and \( n_t \) is an i.i.d. random variable uniformly distributed over \( [1/5, 3/5] \). Thus, we have \( \mu_i = \frac{3}{10} \mathbb{E} + \frac{1}{10} n_t \). We then generate the states of the rest arms \( x_{i,t} \) by mixing samples independently generated under distributions \( \nu_i \nu_i^{5-i} \). Specifically, we let \( \pi_0 := \{0, 0.1, 0.2, 0.3, 0.4\} \) be a weight vector over the samples. Each time, we randomly permute \( \pi_0 \) to get a different vector \( \pi_j \), \( j = 6, 7, \ldots \) and generate a new mean vector \( x_{j,i} \) by linearly combining the vectors sampled from \( \nu_i \nu_i^{5-i} \) with the weight vector \( \pi_j \).

Based on the construction, we can verify that \( S_1 \) includes the first five arms, each being the unique best arm under a context, while \( S_2 \) includes the rest arms. We change the total number of arms \( K \) by including different number of arms to \( S_2 \), and evaluate the regret performance of W-UCB through simulation. The results are plotted in Fig. 4. (a). For each value
of $K$, we run 20 sample paths and calculate the sample average of $R(T)$ and its stand deviation. As we can see, when $K = 5$, which corresponds to the case that $S_1 = [K]$, the regret quickly converges to a constant value. When $K = 10, 15, 20$, we have $|S_2| = 5, 10, 15$, respectively. The corresponding regrets increase sublinearly in $T$, and monotonically increase in $K$. This corroborates our theoretical results in Theorem 1.

Finally, we evaluate how the regret changes as the diversity level changes for fixed arm distributions. We set $K = 5$ and eliminate certain values of $\lambda_i$ in the set $\{ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5} \}$ in order to vary $|S_1|$. As shown in Fig 1(c), as $|S_1|$ increases, the regret decreases, which verifies our intuition that the diversity of user preferences can help reduce the learning regret.

VI. CONCLUSIONS

In this paper, we investigated the impact of the user preference diversity on the learning regret under a MAB setting. We showed that a straightforward extension of the standard UCB algorithm, named as W-UCB, can lead to $O(1) + O(|S_2||S_1| \log T)$ regret. When there exists sufficient diversity in the user preferences so that $|S_2| = 0$, the regret is bounded by a constant. We also established order-matching lower bounds, indicating the order-wise optimality of W-UCB.

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In order to simplify the notation, in the following, we let \( \eta_t := \frac{t}{4(K-1)S_{1T}} \). Then, we aim to show that \( N_j^j(t) < (K-1) \eta_t \) happens with small probability when \( t \) is sufficiently large.

First, we note that
\[
\mathbb{P} \left[ N_j^j(t) < (K-1) \eta_t \right] = \mathbb{P} \left[ N_j^j(t) < \eta_t(K-1), \ N_j^j(t) \leq 2(K-1) \eta_t \right] + \mathbb{P} \left[ N_j^j(t) < (K-1) \eta_t, \ N_j^j(t) > 2(K-1) \eta_t \right]
\]

For the first term in equation (18), based on Hoeffding’s inequality, we have
\[
\mathbb{P} \left[ N_j^j(t) < (K-1) \eta_t, \ N_j^j(t) \leq 2(K-1) \eta_t \right] \leq \mathbb{P} \left[ N_j^j(t) \leq 2(K-1) \eta_t \right] \leq e^{-\frac{2\eta_t^2}{N_j^j(t)}} \leq e^{-\frac{2\eta_t^2}{2(K-1) \eta_t}} = e^{-\frac{\eta_t}{K-1}}.
\]

Besides, when \( N_j^j(t) < (K-1) \eta_t \) and \( N_j^j(t) > 2(K-1) \eta_t \), we have
\[
\frac{1}{K-1} \sum_{i=1, i \neq j}^K N_i^i(t) = \frac{N_j^j(t) - N_i^i(t)}{K-1} > 2 \eta_t - \eta_t = \eta_t,
\]
which indicates that there must exist an arm \( k \neq j \) that has been pulled at least \( \eta_t \) times up to time \( t \). Mathematically, it indicates that \( \left\{ N_j^j(t) < (K-1) \eta_t, N_j^j(t) > 2(K-1) \eta_t \right\} \subseteq \bigcup_{i=1, i \neq j}^K \left\{ N_i^i(t) > \eta_t \right\} \). Applying the union bound, we have
\[
\mathbb{P} \left[ N_j^j(t) < (K-1) \eta_t, N_j^j(t) > 2(K-1) \eta_t \right] \leq \sum_{i=1, i \neq j}^K \mathbb{P} \left[ N_i^i(t) > \eta_t \right].
\]

In order to proceed, we introduce the following notations. For any \( i \neq j \), we define
\[
\ell_j^i = \max\{ \tau : \tau \leq t, a_{\tau} = i, \lambda_{\tau} \in \Lambda_j^j \},
\]
i.e., while the last time until \( t \), when arm \( i \) is pulled while the optimal arm is \( j \). If such event doesn’t happen before \( t \), we simply define \( \ell_j^i = 0 \). Besides we define
\[
\ell_j^j = \max\{ \tau : \tau \leq t, a_{\tau} \neq j, \lambda_{\tau} \in \Lambda_j^j \},
\]
i.e., the last time up to \( t \), when a suboptimal arm is pulled while the optimal arm is \( j \). We note that \( \ell_j^j = \max_{i \neq j} \ell_j^i \).

The definition of \( \ell_j^i \) implies that \( N_j^i(t) \geq N_j^j(t) \). Since we choose arm \( i \) at time \( \ell_j^i \), according to Algorithm [1], we must have
\[
\lambda_j^i \left( \hat{x}_{j,\ell_j^i} + u_j \ell_j^i \right) \geq \lambda_j^j \left( \hat{x}_{j,\ell_j^i} + u_j \ell_j^i \right),
\]
which implies that at least one of following two events must happen: 1) the UCB padded estimated expected reward by pulling arm \( j \) is below the actual mean, i.e., \( \lambda_j^j \left( \hat{x}_{j,\ell_j^i} + u_j \ell_j^i \right) \leq \lambda_j^j \mu_j \), or 2) the UCB padded estimated expected reward by pulling arm \( i \) is above the actual expected reward by pulling arm \( j \), i.e., \( \lambda_j^i \left( \hat{x}_{j,\ell_j^i} + u_j \ell_j^i \right) \geq \lambda_j^j \mu_j \). Therefore, we have
\[
\mathbb{P} \left[ N_j^i(t) > \eta_t \right] = \mathbb{P} \left[ N_j^i(\ell_j^i) > \eta_t \right] = \mathbb{P} \left[ N_j^j(\ell_j^i) > \eta_t, \ell_j^i > \eta_t \right] \leq \mathbb{P} \left[ \lambda_j^i \left( \hat{x}_{j,\ell_j^i} + u_j \ell_j^i \right) \leq \lambda_j^j \mu_j, N_j^j(\ell_j^i) > \eta_t, \ell_j^i > \eta_t \right]
\]
\[
+ \mathbb{P} \left[ \lambda_j^i \left( \hat{x}_{j,\ell_j^i} + u_j \ell_j^i \right) > \lambda_j^j \mu_j, N_j^j(\ell_j^i) > \eta_t, \ell_j^i > \eta_t \right],
\]
where (22) comes from the definition of \( \ell_j^i \) and the fact that \( \ell_j^i \geq N_j^j(t) \).

Let \( \bar{\mu}_{i,s} \) be the sample average state of arm \( i \) after pulling it for \( s \) times, and \( x(k) \) be the \( k \)-th entry in a vector \( x \). Then, for (23), we have
\[
\mathbb{P} \left[ \lambda_j^i \left( \hat{x}_{j,\ell_j^i} + u_j \ell_j^i \right) \leq \lambda_j^j \mu_j, N_j^j(\ell_j^i) > \eta_t, \ell_j^i > \eta_t \right] \leq \sum_{\tau = [\eta_t]} \mathbb{P} \left[ \lambda_j^i \left( \hat{x}_{j,\tau} + u_j \tau \right) \leq \lambda_j^j \mu_j, \ell_j^i = \tau \right]
\]

(25)
\[ (28) \] comes from the assumption \( \lambda_i(k) > 0, \forall k \in [d] \), and \((31)\) comes from Chernoff-Hoeffding inequality.

Next, we will bound \((24)\). We note that

\[
\begin{align*}
&\mathbb{P}\left[ \mathbf{x}_j(k) + u_j \geq \lambda_j(k), N_j^i(t^j) > \eta_i, t^j \right] \\
&\leq t \sum_{\tau=1}^{d} \mathbb{P}\left[ \mathbf{x}_j(k) + u_j \geq \lambda_j(k), N_j^i(t^j) > \eta_i, t^j = \tau \right] \\
&= \sum_{\tau=1}^{d} \mathbb{P}\left[ \mathbf{x}_j(k) + u_j \geq \lambda_j(k), t^j = \tau \right]
\end{align*}
\]

(29)  

When \( N_j^i(t^j) > \eta_i \), we have \( N_i(t^j) - 1 \geq N_j^i(t^j) - 1 > \eta_i - 1 \). Therefore,

\[
2u_j \lambda_j^T \geq 2 \sqrt{\frac{2 \log t^j}{N_i(t^j) - 1}} \leq 2 \sqrt{\frac{2 \log t}{t^j - 1}} \leq 2 \sqrt{\frac{2t^{1/2}}{4(K-1)|S_1| - 1}}
\]

(35)  

Thus, when \( t \geq c \triangleq \left( \frac{2(K-1)|S_1|}{4(K-1)|S_1| - 1} \right)^2 (\frac{a}{2} + 1) \), and \( N_j^i(t^j) > \eta_i \), we have \((35)\) upper bounded by \( l \), the lower bound on \( \lambda_j^T (\mu_j - \mu_i) \). Then, when \( t > c \), we have

\[
\begin{align*}
&\sum_{\tau=1}^{d} \mathbb{P}\left[ \mathbf{x}_j(k) + u_j \geq \lambda_j(k), \lambda_j^T (\mu_j - \mu_i), N_j^i(t^j) > \eta_i, t^j = \tau \right] \\
&\leq \sum_{\tau=1}^{d} \mathbb{P}\left[ \mathbf{x}_j(k) + u_j \geq \lambda_j(k), \eta_i, t^j = \tau \right]
\end{align*}
\]

(36)  

Following a similar approach as in \((25)-(32)\), we can upper bound \((36)\) by \( \sum_{\tau=1}^{d} \frac{d}{r^j} \). Then when \( t > c \), combining with \((32)\), we have

\[
\begin{align*}
&\sum_{i=1, i \neq j}^{K} \mathbb{P}\left[ N_j^i(t) > \eta_i \right]
\end{align*}
\]
= \sum_{i=1, t \neq j}^{K} \mathbb{P}\left[\lambda_0^T (\hat{x}_{j,t} + u_{j,t} 1) \leq \lambda_0^T \mu_j, N_i^j(\xi_j^j) > \eta, \xi_j^j > \eta_t\right] \

\leq \sum_{t=|\eta|}^{2d(K-1)/\tau^3} \left(1 - \rho\right)^{t+1} \left[\frac{2dK}{\rho t}\right] \leq \left(1 - \rho\right)^{t+1} \left[\frac{2dK}{\rho t}\right] 

Combining (19), (21), and (38), when \( t > c_j \), we have

\[ \mathbb{P}\left[N_j^j(t) < \frac{t}{4[S_t]}\right] \leq e^{2\frac{r^2}{\tau^3}} + \left[\frac{1 - \rho}{\rho t}\right] \left(1 - \rho\right)^{t+1} \frac{2dK}{\rho t} \]

A. Proof of Lemma 2 and Lemma 3

Before we proceed, we first introduce the following lemma.

**Lemma 4** Under the W-UCB algorithm, if \( a_t \neq a_i^* \), then, one of the following three events must happen:

\[ \mathcal{E}_{1}^i := \left\{ \lambda_1^T (\hat{x}_{a_i^*,t} + u_{a_i^*,t} 1) \leq \lambda_1^T \mu_{a_i^*} \right\}, \]
\[ \mathcal{E}_{2}^i := \left\{ \lambda_1^T \hat{x}_{a_i,t} \geq \lambda_1^T \left( \mu_{a_i} + u_{a_i,t} 1 \right) \right\}, \]
\[ \mathcal{E}_{3}^i := \left\{ 2u_{a_i,t} \lambda_1^T 1 > \lambda_1^T \left( \mu_{a_i^*} - \mu_{a_i} \right) \right\}. \]

**Proof:** We prove lemma 4 through contradiction. Assume the learner chooses arm \( a_t \neq a_i^* \), however, none of the three events happen, i.e., \( \mathbb{1}\{\mathcal{E}_{1}^i\} = 0, i = 1, 2, 3. \) Then, we have

\[ \lambda_1^T (\hat{x}_{a_i^*,t} + u_{a_i^*,t} 1) > \lambda_1^T \mu_{a_i^*} \]
\[ = \lambda_1^T \mu_{a_i} + \lambda_1^T (\mu_{a_i^*} - \mu_{a_i}) \]
\[ \geq \lambda_1^T \mu_{a_i} + 2u_{a_i,t} \lambda_1^T 1 \]
\[ > \lambda_1^T (\hat{x}_{a_i,t} + u_{a_i,t} 1) + u_{a_i,t} \lambda_1^T 1 \]
\[ \geq \lambda_1^T (\hat{x}_{a_i,t} + u_{a_i,t} 1) = \lambda_1^T (\hat{x}_{a_i,t} + u_{a_i,t} 1) \]

(42)

where (40), (41), (42) follow from the assumption that \( \mathbb{1}\{\mathcal{E}_{1}^i\} = 0, \mathbb{1}\{\mathcal{E}_{3}^i\} = 0, \) and \( \mathbb{1}\{\mathcal{E}_{2}^i\} = 0, \) respectively.

Thus, according to Algorithm 1, arm \( a_i^* \) should be pulled at time \( t \), instead of suboptimal arm \( a_i \), which contradicts with the initial assumption. Therefore, when arm \( a_i \) is pulled at time \( t \), at least one of those three events must happen. □

Lemma 4 indicates that when a suboptimal arm is pulled at time \( t \), either the estimated reward of the optimal arm is lying below the lower confidence bound, or estimated reward of the suboptimal arm is lying above its upper confidence bound, or the confidence interval is wider than the corresponding gap of the reward generated by the optimal and the suboptimal arm.

Denote \( \mathcal{E}_{i,j} \) := \( \{a_t = i, \lambda_t \in \Lambda^j\} \), i.e., the event that arm \( i \) is pulled at time \( t \) when the context is in \( \Lambda^j \). Based on the definition of \( \mathcal{E}_{i,j} \), \( \mathcal{E}_{k} \) and Lemma 4, we have, \( \mathcal{E}_{i,j} \cap \mathcal{E}_{k} = \{\mathcal{E}_{i,j} \cap \mathcal{E}_{k}\} \), \( k = 1, 2, 3, \) and \( \mathcal{E}_{i,j} \subseteq \cup_{k=1,2,3} \left( \mathcal{E}_{i,j} \cap \mathcal{E}_{k}\right) \).

Therefore,

\[ \mathbb{E}\left[\bigcup_{t=1}^{T} \mathbb{1}\{\mathcal{E}_{i,j} \cap \mathcal{E}_{k}\}\right] \]
\[ \leq \sum_{t=1}^{T} \mathbb{P}\left[\lambda_t \in \Lambda_j, 2u_{i,t} \lambda_1^T 1 \downarrow \lambda_1^T (\mu_j - \mu_i)\right] \]
\[ = \sum_{t=1}^{T} \mathbb{P}\left[\lambda_t \in \Lambda_j\right] \mathbb{P}\left[2u_{i,t} \lambda_1^T 1 > \lambda_1^T (\mu_j - \mu_i) \mid \lambda_t \in \Lambda_j\right] \]
\[ \leq \frac{1}{|S_1|} \sum_{t=1}^{T} \mathbb{P}\left[2u_{i,t} > l_i^j \mid \lambda_t \in \Lambda_j\right] \]
\[ \leq \frac{1}{|S_1|} \sum_{t=1}^{T} \mathbb{P}\left[2\sqrt{\frac{2 \log t}{N_i(t-1)}} > l^j_i\right] \]
We note that for any \( i \neq j \),

\[
E \left[ \sum_{t=1}^{T} \mathbb{1} \{ \mathcal{E}^i_{t,t} \cup \mathcal{E}^1_{t} \} \right] \\
\leq \sum_{t=1}^{T} \mathbb{P} \left[ \lambda_t^i \left( \hat{x}_{a^*_t,t} + u_{a^*_t,t} \right) \leq \lambda_t^i \mu_t, a^*_t = j, a_t = i \right] \\
= \frac{1}{|S_1|} \sum_{t=1}^{T} \mathbb{P} \left[ \lambda_t^j \left( \hat{x}_{j,t} + u_{j,t} \right) \leq \lambda_t^j \mu_j \mid \lambda_t \in \Lambda \right] \\
\leq \frac{1}{|S_1|} \sum_{t=1}^{T} \sum_{s=1}^{d} \mathbb{P} \left[ \hat{\mu}_{j,s}(k) + u_{j,t} \leq \mu_j(k) \mid \lambda_t \in \Lambda \right] \\
= \frac{1}{|S_1|} \sum_{t=1}^{T} \sum_{s=1}^{d} \mathbb{1} \\
\leq \frac{1.21d}{|S_1|} \quad \text{(50)}
\]

where (46) comes from the i.i.d. assumption on \( \lambda_t \) and the fact that \( \hat{x}_{j,t}(k) + u_{j,t} \leq \mu_j(k) \) is determined by \( \mathcal{H}^{t-1} \) and \( \lambda_t \), and (48) comes from Hoeffding’s inequality.

Similarly, we have

\[
E \left[ \sum_{t=1}^{n} \mathbb{1} \{ \mathcal{E}^i_{t,t} \cup \mathcal{E}^2_{t} \} \right] \leq \frac{1.21d}{|S_1|} \quad \text{(51)}
\]

Then, we consider the case when arm \( i \in S_1 \). Since \( \rho_i > 0 \), there exists a constant \( f \triangleq \frac{64}{|S_1|^2} t^4 + 1 \), such that when \( t > f, \frac{8 \log t}{t^2} < \frac{1}{|S_1|} \). Let \( T = \max\{f, c + 1\} \), where \( c \) is a constant defined in the proof of Lemma 1 that \( c \triangleq \left(4(K - 1)|S_1|\right)^2 \left(\frac{8}{t^2} + 1\right)^2 \). We have

\[
\sum_{t=1}^{T} \mathbb{P} \left[ N_i(t-1) < \frac{8 \log t}{t^2} \right] \\
\leq C + \sum_{t=T_i,j+1}^{T} \mathbb{P} \left[ N_i(t-1) < \frac{t - 1}{4|S_1|} \right] \\
\leq C + \sum_{t=C+1}^{T} \mathbb{P} \left[ N_i(t-1) < \frac{t - 1}{4|S_1|} \right] \\
\leq C + \sum_{t=C+1}^{\infty} \left( e^{-\frac{t-1}{2|S_1|^2}} + \left[ (1 - \rho)(t - 1) + 1 \right] \frac{2dK}{(t-1)^3 \rho^2} \right) \quad \text{(52)}
\]
\[ \leq C + \sum_{t=2}^{\infty} \left( e^{-\frac{t-1}{|S_1|}} + \left[ (1 - \rho)(t - 1) + 1 \right] \frac{2dK}{(t - 1)^3 \rho^3} \right) \]

\[ \leq C + 2|S_1|^2 + \frac{\pi^2(1 - \rho)dK}{3\rho^3} + \frac{2.42dK}{\rho^3} \]  (53)

where \( S_2 \) comes from the assumption that \( t \geq C + 1 > c + 1 \), thus Lemma 1 applies.

Combining (45), (53), we have \( \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{1}\{E_i^j \cap E_i^k \} \right] \leq \frac{1}{|S_1|} \left( C + 2|S_1|^2 + \frac{\pi^2(1 - \rho)dK}{3\rho^3} + \frac{2.42dK}{\rho^3} \right) \).

Therefore, by putting all three cases together, we have

\[ \mathbb{E} \left[ N_i^j(T) \right] = \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{1}\{E_i^j \} \right] \leq \rho \left( C + 2|S_1|^2 + \frac{\pi^2(1 - \rho)dK}{3\rho^3} + \frac{2.42dK}{\rho^3} + 2.42d \right) \]

The proof of Lemma 2 is thus completed.

Next, we consider the case when \( i \in S_2 \). Denote \( u_T := \frac{8 \log T}{l^2} \). We note that if \( N_i(t - 1) \geq u_T \), it implies \( N_i(t - 1) \geq \frac{8 \log T}{l^2} \), \( \forall t \in \{1, \ldots, T\} \). Thus, we have \( 2u_i, \lambda_i^j 1 \leq l \leq \lambda_i^j (\mu_j - \mu_i) \), which indicates that \( E_i^j \cap E_i^k = \emptyset \) when \( N_i(t - 1) \geq u_T \).

We note that

\[ \mathbb{E} \left[ \sum_{j \in S_1} N_i^j(T) \right] = \mathbb{E} \left[ \sum_{j \in S_1} \sum_{t=1}^{T} \mathbb{1}\{E_i^j \} \right] \]

\[ \leq \mathbb{E} \left[ \sum_{j \in S_1} \sum_{t=1}^{T} \mathbb{1}\{E_i^j \} \cdot \mathbb{1}\{N_i(t - 1) \geq u_T\} + \mathbb{1}\{N_i(t - 1) \leq u_T\} \right] \]

\[ \leq u_T + \mathbb{E} \left[ \sum_{j \in S_1} \sum_{t=1}^{T} \mathbb{1}\{E_i^j \} \cdot \mathbb{1}\{N_i(t - 1) \geq u_T\} \right] \]

\[ \leq u_T + \mathbb{E} \left[ \sum_{j \in S_1} \sum_{t=1}^{T} \mathbb{1}\{E_i^j \} \cdot \mathbb{1}\{E_i^k \cup E_i^j \} \right] \]

\[ \leq u_T + \mathbb{E} \left[ \sum_{j \in S_1} \sum_{t=1}^{T} \mathbb{1}\{E_i^j \} \cdot \mathbb{1}\{E_i^j \} \right] \]

\[ \leq \frac{8 \log T}{l^2} + 1 + 2.42d, \]  (59)

where (56) comes from the fact that \( E_i^j \cap E_i^k = \emptyset \) when \( N_i(t - 1) \geq u_T \). (57) comes from the fact that \( E_i^j \subseteq \cup_{k=1,2,3} (E_i^j \cap E_i^k) \), and (59) comes from (50) and (51).