Asymptotic results for a multivariate version of the alternative fractional Poisson process*

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Abstract

A multivariate fractional Poisson process was recently defined in [3] by considering a common independent random time change for a finite dimensional vector of independent (non-fractional) Poisson processes; moreover it was proved that, for each fixed \( t \geq 0 \), it has a suitable multinomial conditional distribution of the components given their sum. In this paper we consider another multivariate process \( \{M^\nu(t) = (M^\nu_1(t), \ldots, M^\nu_m(t)) : t \geq 0\} \) with the same conditional distributions of the components given their sums, and different marginal distributions of the sums; more precisely we assume that the one-dimensional marginal distributions of the process \( \{\sum_{i=1}^m M^\nu_i(t) : t \geq 0\} \) coincide with the ones of the alternative fractional (univariate) Poisson process in [2]. We present large deviation results for \( \{M^\nu(t) = (M^\nu_1(t), \ldots, M^\nu_m(t)) : t \geq 0\} \), and this generalizes the result in [2] concerning the univariate case. We also study moderate deviations and we present some statistical applications concerning the estimation of the fractional parameter \( \nu \).

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1 Introduction

Fractional Poisson processes are widely studied in the literature by considering a version of some known equations for the probability mass functions with fractional derivatives and/or fractional difference operators (see [14], [15], [4], [5], [17], [20] and [21]). Typically these processes are often represented in terms of randomly time-changed and subordinated processes (see e.g. [13] and [16]) and appear in several applications (see e.g. [6], where the surplus process of an insurance company is modeled by a compound fractional Poisson process).

A multivariate (space and/or time) fractional Poisson process was recently defined in [3] by considering a common independent random time change in terms of the stable subordinator and/or its inverse for a finite dimensional vector of independent (non-fractional) Poisson processes. In the proof of Proposition 4 in [3] it was proved that, for each fixed \( t \geq 0 \), the conditional (joint) distribution of the components of this multivariate process given their sum is multinomial; moreover this conditional multinomial distribution does not depend on \( t \) and on the fractional parameters.

In this paper we consider another multivariate process \( \{M^\nu(t) = (M^\nu_1(t), \ldots, M^\nu_m(t)) : t \geq 0\} \) with the same conditional distributions of the components given their sums, but we change the distribution of the sums of the components. More precisely we assume that the one-dimensional

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marginal distributions of the process \( \{ \sum_{i=1}^{m} M_i^m(t) : t \geq 0 \} \) coincide with the ones of the alternative fractional (univariate) Poisson process in [2]; in other words we mean the alternative fractional Poisson processes in [4] with a deterministic time-change. Thus it is natural to define the process in this paper as the multivariate version of the alternative fractional Poisson process.

The alternative fractional Poisson process in [4] appears as the process which counts the number of changes of direction of a fractional telegraph process (see e.g. (4.7) in [10]), and of a reflected random flight on the surface of a sphere (see e.g. (4.24) in [8]). Some generalizations of the alternative fractional Poisson process in [4] can be found in [11] (see (3.5)) and in [19] (see Proposition 2.1). In all these cases we have a weighted Poisson process as in [1]; the concept of weighted Poisson process for \( \{ \sum_{i=1}^{m} M_i^m(t) : t \geq 0 \} \) is illustrated in Remark 1.

The aim of this paper is to present large deviation results for the multivariate version of the alternative fractional Poisson process. The theory of large deviations gives an asymptotic computation of small probabilities on exponential scale (see e.g. [9] as references on this topic). The main tool used in the proofs of Propositions 1 and 2 is the Gärtner Ellis Theorem (see e.g. Section 2.3 in [9]). We point out that in [2] we study large deviations only; in particular Proposition 1 in this paper reduces to Proposition 4.1 in [2] if we consider the univariate case \( m = 1 \) (see Remark 2).

The term moderate deviations is used for a class of large deviation principles governed by the same quadratic rate function which uniquely vanishes at the origin. Typically moderate deviations fill the gap between a convergence to zero and an asymptotic Normality result. We also recall that, as pointed out in some references (see e.g. [7] and the references cited therein), under certain conditions one can obtain the weak convergence to a centered Normal distribution whose variance is determined by a large deviation principle obtained by the Gärtner Ellis Theorem.

We conclude with the outline of the paper. We start with some preliminaries in Section 2. The multivariate process studied in this paper is defined in Section 3. Large and moderate deviation results are presented in Section 4. We conclude with some statistical applications in Section 5.

2 Preliminaries

We always set \( 0 \log 0 = 0 \). In general we deal with vectors in \( \mathbb{R}^m \) and we use the following notation: \( \underline{x} = (x_1, \ldots, x_m) \), and \( \underline{0} = (0, \ldots, 0) \) is the null vector; \( \underline{x} \geq \underline{0} \) means that \( x_1, \ldots, x_m \geq 0 \); we set \( s(\underline{x}) = \sum_{i=1}^{m} x_i \) and \( \langle \underline{x}, \underline{y} \rangle = \sum_{i=1}^{m} x_i y_i \).

2.1 Preliminaries on large (and moderate) deviations

We recall the basic definitions (see e.g. [9], pages 4-5). Let \( \mathcal{Z} \) be a Hausdorff topological space with Borel \( \sigma \)-algebra \( \mathcal{B}_\mathcal{Z} \). A speed function is a family of numbers \( \nu_t : t > 0 \) such that \( \lim_{t \to \infty} \nu_t = \infty \).

A lower semi-continuous function \( I : \mathcal{Z} \to [0, \infty) \) is called rate function. A family of \( \mathcal{Z} \)-valued random variables \( \{Z_t : t > 0\} \) satisfies the large deviation principle (LDP for short), as \( t \to \infty \), with speed function \( v_t \) and rate function \( I \) if

\[
\lim_{t \to \infty} \sup \frac{1}{v_t} \log P(Z_t \in F) \leq - \inf_{z \in F} I(z) \quad (\text{for all closed sets } F)
\]

and

\[
\lim_{t \to \infty} \inf \frac{1}{v_t} \log P(Z_t \in G) \geq - \inf_{z \in G} I(z) \quad (\text{for all open sets } G).
\]

A rate function \( I \) is said to be good if all the level sets \( \{ \{z \in \mathcal{Z} : I(z) \leq \gamma\} : \gamma \geq 0\} \) are compact.

The term moderate deviations is used when, for all positive numbers \( \{a_t : t > 0\} \) such that

\[
a_t \to 0 \text{ and } ta_t \to \infty \quad (\text{as } t \to \infty),
\]
we have a LDP for suitable centered random variables on $Z = \mathbb{R}^m$ (for some $m \geq 1$) with speed $1/a_t$ and the same quadratic rate function which uniquely vanishes at the origin of $\mathbb{R}^m$ (we mean that the rate function does not depend on the choice of $\{a_t : t > 0\}$). Typically moderate deviations fill the gap between two regimes (for the second one see Remark 7):

- a convergence (at least in probability) to zero of centered random variables (case $a_t = \frac{1}{t}$);
- a weak convergence to a centered Normal distribution (case $a_t = 1$).

Note that in both case one condition in (1) fails.

2.2 Preliminaries on (generalized) Mittag-Leffler functions

Let

$$E_{\alpha,\beta}(x) := \sum_{r \geq 0} \frac{x^r}{\Gamma(\alpha r + \beta)}$$

be the Mittag-Leffler function (see e.g. [18], page 17), and let

$$E_{\gamma,\alpha,\beta}(x) := \sum_{j \geq 0} \frac{(\gamma)^{(j)} x^j}{j! \Gamma(\alpha j + \beta)}$$

be the generalized Mittag-Leffler function (see e.g. (1.9.1) in [12]) where

$$(\gamma)^{(j)} := \begin{cases} \gamma(\gamma + 1) \cdots (\gamma + j - 1) & \text{if } j \geq 1 \\ 1 & \text{if } j = 0, \end{cases}$$

is the rising factorial, also called Pochhammer symbol (see e.g. (1.5.5) in [12]). Note that we have $E_{1,\alpha,\beta}^1$, i.e. $E_{\gamma,\alpha,\beta}^\gamma$ with $\gamma = 1$, coincides with $E_{\alpha,\beta}$ in (2). In view of what follows (see e.g. (1.8.27) in [12]) we recall that, if we use the symbol $\sim$ to mean that the ratio tends to 1, we have

$$E_{\nu,\beta}(z) \sim z^{(1-\beta)/\nu} e^{z^\nu/\nu}$$

as $z \to \infty$; (3)

actually we can say that

$$E_{\nu,\beta}(z) = \frac{1}{\nu} z^{(1-\beta)/\nu} e^{z^\nu/\nu} + r(z), \text{ where } r(z) \to 0 \text{ as } z \to \infty.$$ (4)

3 An alternative multivariate fractional Poisson process

Let $\mathbf{\lambda} \in (0, \infty)^m$ be arbitrarily fixed (actually we could consider $\mathbf{\lambda} \in [0, \infty)^m \setminus \{0\}$ with suitable modifications). We present a multivariate fractional Poisson process $\{M_{\nu}^\nu(t) : t \geq 0\}$ where, as in the proof of Proposition 4 in [3], for all $t \geq 0$ we consider the following conditional multinomial distribution of $(M_{\nu}^1(t), \ldots, M_{\nu}^m(t))$ given their sum $s(M_{\nu}^\nu(t)) = \sum_{i=1}^m M_{\nu}^i(t)$:

$$P(M_{\nu}^\nu(t) = \mathbf{k} | s(M_{\nu}^\nu(t)) = s(k)) = \frac{(s(k))!}{k_1! \cdots k_m!} \prod_{i=1}^m \left( \frac{\lambda_i}{s(\mathbf{\lambda})} \right)^{k_i}$$

for all integers $k_1, \ldots, k_m \geq 0$.

Moreover we assume that the one-dimensional marginal distributions of the sum process $\{s(M_{\nu}^\nu(t)) : t \geq 0\}$ coincide with the ones of the alternative fractional (univariate) Poisson process in [2] with parameter $s(\mathbf{\lambda})$ (in place of $\mathbf{\lambda}$), i.e.

$$P(s(M_{\nu}^\nu(t)) = h) = \frac{(s(\mathbf{\lambda}))^h}{\Gamma(\nu h + 1)} \cdot \frac{1}{E_{\nu,1}(s(\mathbf{\lambda}))^{\nu h}}$$

for all integer $h \geq 0$. 3
Remark 1 (Weighted Poisson process). For all $t \geq 0$ we have
\[
P(s(M^\nu(t)) = h) = \frac{w(h) (s(\nu)t^\nu)^h e^{-s(\nu)t^\nu}}{\sum_{j \geq 0} w(j) (s(\nu)t^\nu)^j e^{-s(\nu)t^\nu}} \text{ for all integer } h \geq 0,
\]
where $w(h) := \frac{h!}{\Gamma(vh+1)}$.

Thus, for each fixed $t \geq 0$, we consider the following multivariate probability mass function for the random variable $M^\nu(t)$:
\[
P(M^\nu(t) = k) = P(s(M^\nu(t)) = s(k)) P(s(M^\nu(t)) = s(k)) \\
= \frac{(s(k))!}{k_1! \cdots k_m!} \prod_{i=1}^m \left( \frac{\lambda_i}{s(\nu)} \right)^{k_i} \frac{(s(\nu)t^\nu)^{s(k)}}{\Gamma(s(\nu))} \cdot \frac{1}{E_{\nu,1}(s(\nu)t^\nu)} \\
= \frac{(s(k))!}{k_1! \cdots k_m!} \lambda_1^{k_1} \cdots \lambda_m^{k_m} \frac{(t^\nu)^{s(k)}}{\Gamma(s(\nu))} \cdot \frac{1}{E_{\nu,1}(s(\nu)t^\nu)} \text{ for all integers } k_1, \ldots, k_m \geq 0.
\]
The moment generating functions of the $(m$-variate$)$ random variables \( \{M^\nu(t) : t \geq 0\} \), with argument $\theta \in \mathbb{R}^m$, are
\[
\mathbb{E} \left[ e^{\theta M^\nu(t)} \right] = \sum_{k \geq 0} e^{\sum_{i=1}^m \theta_i k_i} P(M^\nu(t) = k) \\
= \frac{1}{E_{\nu,1}(s(\nu)t^\nu)} \sum_{k \geq 0} \frac{(s(k))!}{k_1! \cdots k_m!} \prod_{i=1}^m (e^{\theta_i} \lambda_i)^{k_i} \cdot \frac{(t^\nu)^{s(k)}}{\Gamma(s(\nu))} \\
= \frac{1}{E_{\nu,1}(s(\nu)t^\nu)} \sum_{r \geq 0} \left( \sum_{i=1}^m \lambda_i e^{\theta_i} \right)^r (t^\nu)^r
\]
and therefore
\[
\mathbb{E} \left[ e^{\theta M^\nu(t)} \right] = \frac{E_{\nu,1}(s(\nu)t^\nu)}{E_{\nu,1}(s(\nu)t^\nu)} \left( \sum_{i=1}^m \lambda_i e^{\theta_i} \right) \left( t^\nu \right)^r.
\]

Moreover the expected values are
\[
\mathbb{E}[M^\nu(t)] = \nabla \mathbb{E} \left[ e^{\theta M^\nu(t)} \right] \bigg|_{\theta = 0} = \frac{E_{\nu,1}(s(\nu)t^\nu)}{E_{\nu,1}(s(\nu)t^\nu)} t^\nu \sum_{i=1}^m \lambda_i = \frac{E_{\nu,1}(s(\nu)t^\nu)}{E_{\nu,1}(s(\nu)t^\nu)} \frac{t^\nu}{\nu}
\]
by (1.8.22) in [12] and some computations with generalized Mittag-Leffler functions; note that, if we set $m = 1$ and if we replace $t^\nu$ with $t$, formula [5] meets (4.6) in [3].

4 Large and moderate deviations

We start with large deviations.

Proposition 1. The family of random variables \( \left\{ \frac{M^\nu(t)}{t} : t > 0 \right\} \) satisfies the LDP with speed $v_t = t$ and good rate function $\Lambda^*$ defined by
\[
\Lambda^*(x) := \left\{ \begin{array}{ll}
\sum_{i=1}^m x_i \log \left( \frac{e^{\nu x_i}}{\lambda_i s(\nu)} \right) - \nu s(x) + (s(\nu))^{1/\nu} & \text{if } x \in [0, \infty)^m \\
\nu \log 0 = 0 & \text{otherwise}
\end{array} \right.
\]
(we recall that $0 \log 0 = 0$).
Proof. We apply Gärtner Ellis Theorem. Then, by taking into account (5) and (3), for all $\theta \in \mathbb{R}^m$ we have
\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left[ e^{\langle \theta, M^\nu(t) \rangle} \right] = \left( \sum_{i=1}^{m} \lambda_i e^{\theta_i} \right)^{1/\nu} - \left( \sum_{i=1}^{m} \lambda_i \right)^{1/\nu} = \left( \sum_{i=1}^{m} \lambda_i e^{\theta_i} \right)^{1/\nu} - (s(\bar{\Lambda}))^{1/\nu} =: \Lambda(\theta). \tag{7}
\]
So, since $\Lambda$ is finite everywhere and differentiable, the LDP holds with speed $v_t = t$ and good rate function $\Lambda^*$ defined by
\[
\Lambda^*(x) := \sup_{\theta \in \mathbb{R}^m} \{ \langle \theta, x \rangle - \Lambda(\theta) \}.
\]
We conclude the proof showing that this rate function coincides with the one in the statement.

- The case $x \not\in [0, \infty)^m$ is trivial; in fact, in this case, we have $x_i < 0$ for some $i \in \{1, \ldots, m\}$, and therefore $\Lambda^*(x) = \infty$ by taking $\theta_j = 0$ for $j \neq i$, and by letting $\theta_i \to -\infty$. On the other hand we have $\Lambda^*(x) = \infty$ because $P(M^\nu(t)/t \in [0, \infty)^m) = 1$ for all $t > 0$, and $[0, \infty)^m$ is a closed set.
- For $x \in (0, \infty)^m$ we consider the system of equations (for $i \in \{1, \ldots, m\}$)
\[
x_i = \frac{\partial}{\partial \theta_i} \Lambda(\theta), \text{ i.e. } x_i = \frac{1}{\nu} \left( \sum_{j=1}^{m} \lambda_j e^{\theta_j} \right)^{1/\nu - 1} \lambda_i e^{\theta_i},
\]
and we have a unique solution $\theta(x) = (\theta_1(x), \ldots, \theta_m(x))$ defined by
\[
\theta_i(x) = \log \left( \frac{\nu^\nu \cdot x_i}{\lambda_i (s(x))^{1-\nu}} \right);
\]
in fact
\[
\frac{1}{\nu} \left( \sum_{j=1}^{m} \lambda_j e^{\theta_j(x)} \right)^{1/\nu - 1} \lambda_i e^{\theta_i(x)} = \frac{1}{\nu} \left( \sum_{j=1}^{m} \lambda_j \cdot \nu^\nu \cdot \frac{x_j}{\lambda_j (s(x))^{1-\nu}} \right)^{1/\nu - 1} \lambda_i \cdot \nu^\nu \cdot \frac{x_i}{\lambda_i (s(x))^{1-\nu}} = x_i.
\]
Thus
\[
\Lambda^*(x) := \langle \theta(x), x \rangle - \Lambda(\theta(x))
\]
\[
= \sum_{i=1}^{m} x_i \log \left( \frac{\nu^\nu \cdot x_i}{\lambda_i (s(x))^{1-\nu}} \right) - \left( \sum_{i=1}^{m} \lambda_i \cdot \frac{\nu^\nu \cdot x_i}{\lambda_i (s(x))^{1-\nu}} \right)^{1/\nu} + (s(\bar{\Lambda}))^{1/\nu}
\]
\[
= \sum_{i=1}^{m} x_i \log \left( \frac{\nu^\nu \cdot x_i}{\lambda_i (s(x))^{1-\nu}} \right) - (\nu^\nu (s(x))^{\nu})^{1/\nu} + (s(\bar{\Lambda}))^{1/\nu}
\]
\[
= \sum_{i=1}^{m} x_i \log \left( \frac{\nu^\nu \cdot x_i}{\lambda_i (s(x))^{1-\nu}} \right) - \nu s(x) + (s(\bar{\Lambda}))^{1/\nu}.
\]
- The final case concerns $x \in [0, \infty)^m \setminus (0, \infty)^m$. For $x = 0$ we have
\[
\Lambda^*(0) = \sup_{\theta \in \mathbb{R}^m} \left\{ -\left( \sum_{i=1}^{m} \lambda_i e^{\theta_i} \right)^{1/\nu} + (s(\bar{\Lambda}))^{1/\nu} \right\} = (s(\bar{\Lambda}))^{1/\nu}.
\]
by letting \( \theta_1, \ldots, \theta_m \to -\infty \). For \( x \in [0, \infty)^m \setminus ((0, \infty)^m \cup \{0\}) \) we consider the set \( S(x) := \{ i \in \{1, \ldots, m\} : x_i > 0 \} \), and we have \( \emptyset \neq S(x) \neq \{1, \ldots, m\} \). Then

\[
\Lambda^*(x) := \sup_{\theta \in \mathbb{R}^m} \left\{ \sum_{i \in S(x)} \theta_i x_i - \left( \sum_{i=1}^m \lambda_ie^{\theta_i} \right)^{1/\nu} + (s(\lambda))^{1/\nu} \right\}
\]

and, after letting \( \theta_i \to -\infty \) for \( i \notin S(x) \), we can consider the system of equations

\[
x_i = \frac{1}{\nu} \left( \sum_{j=1}^m \lambda_j e^{\theta_j} \right)^{1/\nu-1} \lambda_i e^{\theta_i} \text{ (for } i \in S(x) \text{)}
\]

and we can adapt what we said above for \( x \in (0, \infty)^m \).

We can say that \( \Lambda^*(x) = 0 \) if and only if \( x = \nabla \Lambda(0) \), where

\[
\nabla \Lambda(0) = \frac{1}{\nu} \cdot (s(\lambda))^{1/\nu-1} \cdot \Lambda. \tag{8}
\]

In particular we can check that

\[
\Lambda^*(\nabla \Lambda(0)) = \sum_{i=1}^m \frac{1}{\nu} \cdot (s(\lambda))^{1/\nu-1} \lambda_i \log \left( \frac{\nu^\nu \cdot \frac{1}{\nu} \cdot (s(\lambda))^{1/\nu-1} \lambda_i}{\lambda_i \left( \frac{1}{\nu} \cdot (s(\lambda))^{1/\nu-1+1} \right)} \right)
\]

\[
- \nu \cdot \frac{1}{\nu} \cdot (s(\lambda))^{1/\nu-1+1} + (s(\lambda))^{1/\nu} = 0.
\]

The following remarks concern Proposition 1.

**Remark 2 (The case \( m = 1 \)).** Proposition 1 here reduces to Proposition 4.1 in [2] when \( m = 1 \) (actually some parts of the proof are simplified). In particular for the rate function \( \Lambda^* \) (with \( x \) in place of \( x \) and \( s(x) \), and \( \lambda \) in place of \( \Lambda \) and \( s(\lambda) \) ) we have

\[
\Lambda^*(x) := \begin{cases} 
\frac{x \log \left( \frac{(\nu x)^\nu}{\lambda} \right) - \nu x + \lambda^{1/\nu}}{\infty} & \text{if } x \geq 0 \\
\frac{x_1 \log \left( \frac{x_1}{\lambda_1} \right) - x_1 + \lambda_1}{\infty} & \text{if } x < 0
\end{cases} = I_{\nu,\lambda}^{(A)}(x),
\]

where \( I_{\nu,\lambda}^{(A)} \) is the rate function in Proposition 4.1 in [2].

**Remark 3 (The case \( \nu = 1 \)).** We have

\[
\Lambda^*_{(\nu=1)}(x) := \begin{cases} 
\sum_{i=1}^m \left\{ x_i \log \left( \frac{x_i}{\lambda_i} \right) - x_i + \lambda_i \right\} & \text{if } x \in [0, \infty)^m \\
\text{otherwise}.
\end{cases}
\]

Thus \( \Lambda^*_{(\nu=1)}(x) = \sum_{i=1}^m I_{1,\lambda_i}^{(A)}(x_i) \) for all \( x \in \mathbb{R}^m \), where \( I_{\nu,\lambda}^{(A)} \) is the rate function in Proposition 4.1 in [2] (as in Remark 2); this equality agrees the well-known independence of the one-dimensional marginal processes \( \{M^t_1(t) : t \geq 0\}, \ldots, \{M^t_m(t) : t \geq 0\} \).

**Remark 4 (An alternative expression of \( \Lambda^* \)).** If we consider the relative entropy of a probability measure \( p = (p_1, \ldots, p_m) \) on \( \{1, \ldots, m\} \) with respect to another one \( q = (q_1, \ldots, q_m) \), i.e.

\[
H(p; q) := \sum_{i=1}^m p_i \log \left( \frac{p_i}{q_i} \right),
\]
for $x \in [0, \infty)^m$ we have

$$
\Lambda^*(x) = \sum_{i=1}^{m} x_i \log \left( \frac{x_i/s(x)}{\lambda_i/s(\lambda)} \right) + \sum_{i=1}^{m} x_i \log \left( \frac{\nu^\lambda}{s(x)} \right) \frac{s(x)}{\lambda} - \nu s(x) + (s(\lambda))^{1/\nu}
$$

$$
= s(x) H \left( \frac{x}{s(x)}; \frac{\lambda}{s(\lambda)} \right) + s(x) \log \left( \frac{\nu^\lambda(s(x))^{1/\nu}}{s(\lambda)} \right) - \nu s(x) + (s(\lambda))^{1/\nu},
$$

where $I_{\nu,s(\lambda)}^{(A)}$ concerns the notation used for the rate function in Proposition 4.1 in [2] (see Remark 3). Obviously, for $x = 0$, we have $s(x) H \left( \frac{x}{s(x)}; \frac{\lambda}{s(\lambda)} \right) = 0$.

The next proposition concerns moderate deviations. In view of what follows we need to introduce the matrix $C = (c_{jk})_{j,k \in \{1, \ldots, m\}}$ defined by

$$
c_{jk}^{(\nu)} := \begin{cases} 
\frac{1}{\nu} \left( \frac{1}{\nu} - 1 \right) (s(\lambda))^{1/\nu-2} \lambda_j \lambda_k & \text{if } j \neq k \\
\frac{1}{\nu} \left( \frac{1}{\nu} - 1 \right) (s(\lambda))^{1/\nu-2} \lambda_j^2 + \frac{1}{\nu} (s(\lambda))^{1/\nu-1} \lambda_j & \text{if } j = k 
\end{cases}
$$

(9)

and the function $\tilde{\Lambda}$ defined by

$$
\tilde{\Lambda}(\theta) := \frac{1}{2} \theta^T C \theta.
$$

(10)

**Proposition 2.** For all families of positive numbers $\{a_t : t > 0\}$ such that (11) holds, the family of random variables $\left\{ \sqrt{ta_t} \cdot \frac{M^\nu(t) - \mathbb{E}[M^\nu(t)]}{t} : t > 0 \right\}$ satisfies the LDP with speed $1/a_t$ and good rate function $\tilde{\Lambda}^*$ defined by

$$
\tilde{\Lambda}^*(x) := \sup_{\theta \in \mathbb{R}^m} \{ \langle \theta, x \rangle - \tilde{\Lambda}(\theta) \}.
$$

Proof. We apply Gärtner Ellis Theorem and the desired LDP holds if we prove that

$$
\lim_{t \to \infty} \frac{1}{1/a_t} \log \mathbb{E} \left[ e^{\frac{1}{a_t} \sqrt{ta_t} \cdot \frac{M^\nu(t) - \mathbb{E}[M^\nu(t)]}{t} } \right] = \tilde{\Lambda}(\theta) \quad (\text{for all } \theta \in \mathbb{R}^m).
$$

We start with some manipulations where we take into account (5) and (6):

$$
\Lambda_t(\theta) = a_t \log \mathbb{E} \left[ e^{\frac{1}{a_t} \sqrt{ta_t} \cdot \frac{M^\nu(t) - \mathbb{E}[M^\nu(t)]}{t} } \right]
$$

$$
= a_t \left( \log \mathbb{E} \left[ e^{\frac{1}{a_t} \sqrt{ta_t} \cdot \frac{M^\nu(t)}{t} } \right] - \frac{1}{\sqrt{ta_t}} \langle \theta, \mathbb{E}[M^\nu(t)] \rangle \right)
$$

$$
= a_t \left( \log \frac{E_{\nu,1} \left( \sum_{i=1}^{m} \lambda_i e^{\theta_i/\sqrt{ta_t}} t^\nu \right)}{E_{\nu,1}(s(\lambda))^{1/\nu}} - \frac{1}{\sqrt{ta_t}} \frac{E_{\nu,1}(s(\lambda))^{1/\nu}}{E_{\nu,1}(s(\lambda))^{1/\nu}} \nu(\theta, \lambda) \right).
$$

Thus, after some computations, we get

$$
\Lambda_t(\theta) = A_1(t) + A_2(t)
$$

where

$$
A_1(t) := a_t \left( \log \frac{E_{\nu,1} \left( \sum_{i=1}^{m} \lambda_i e^{\theta_i/\sqrt{ta_t}} t^\nu \right)}{\frac{1}{\nu} e^{(\sum_{i=1}^{m} \lambda_i e^{\theta_i/\sqrt{ta_t}} t^\nu)}} - \log \frac{E_{\nu,1}(s(\lambda))^{1/\nu}}{\frac{1}{\nu} e^{(s(\lambda))^{1/\nu}}} \right)
$$

and

$$
A_2(t) := \frac{1}{\sqrt{ta_t}} \nu(\theta, \lambda).
$$
and, if we consider the function \( \Lambda \) in (7),
\[
A_2(t) := t a_t \left( \frac{1}{t} \log \frac{\frac{1}{\nu} \nu e^{\nu \sum_{i=1}^{m+1} \lambda_i e^{\nu t / \sqrt{\nu t}}}^{1/\nu}}{\nu^{1/\nu - 1}} - \frac{1}{\sqrt{t a_t}} \frac{E_{\nu,\nu}(s(\Lambda)t^\nu) t^{\nu - 1}}{E_{\nu,1}(s(\Lambda)t^\nu)} (\theta, \Lambda) \right)
\]
\[
= t a_t \left( \Lambda \left( \frac{1}{\sqrt{t a_t}} \cdot \theta \right) - \frac{1}{\sqrt{t a_t}} \frac{E_{\nu,\nu}(s(\Lambda)t^\nu) t^{\nu - 1}}{E_{\nu,1}(s(\Lambda)t^\nu)} (\theta, \Lambda) \right).
\]

Then, for all \( \theta \in \mathbb{R}^m \), we have \( A_1(t) \to 0 \) as \( t \to \infty \) (this is a consequence of \( a_t \to 0 \), stated in (1), and (3)), and we complete the proof showing that
\[
\lim_{t \to \infty} A_2(t) = \tilde{\Lambda}(\theta)
\]
where \( \tilde{\Lambda} \) is the function in (10). Now we consider the Taylor formula for \( \Lambda \), and we have
\[
\Lambda(\eta) = \Lambda(0) + \langle \nabla \Lambda(0), \eta \rangle + \frac{1}{2} \langle \eta, H_\Lambda(0) \eta \rangle + o(\|\eta\|^2) = \frac{1}{\nu} (s(\Lambda))^{1/\nu - 1} \langle \Lambda, \eta \rangle + \frac{1}{2} \langle \eta, C\eta \rangle + o(\|\eta\|^2),
\]
where \( \frac{o(\|\eta\|^2)}{\|\eta\|^2} \to 0 \) as \( \|\eta\| \to 0 \) (we have taken into account \( \Lambda(0) = 0 \), (5) and the equality \( H_\Lambda(0) = C \) which can be checked by inspection); then, after some computations where we take into account (10), we obtain
\[
A_2(t) = t a_t \left( \frac{1}{\sqrt{t a_t}} \cdot (s(\Lambda))^{1/\nu - 1} \langle \Lambda, \theta \rangle + \frac{1}{2} \langle \theta, C\theta \rangle + o \left( \frac{1}{t a_t} \right) - \frac{1}{\sqrt{t a_t}} \frac{E_{\nu,\nu}(s(\Lambda)t^\nu) t^{\nu - 1}}{E_{\nu,1}(s(\Lambda)t^\nu)} \langle \theta, \Lambda \rangle \right)
\]
\[
= \tilde{\Lambda}(\theta) + t a_t \left( \frac{1}{\sqrt{t a_t}} \right) + \frac{\theta \cdot \Lambda}{\nu} \cdot \sqrt{t a_t} \left( \frac{(s(\Lambda))^{1/\nu - 1} - E_{\nu,\nu}(s(\Lambda)t^\nu) t^{\nu - 1}}{E_{\nu,1}(s(\Lambda)t^\nu)} \right).
\]
Then we get (11) if we prove that
\[
\lim_{t \to \infty} \sqrt{t a_t} \left( \frac{(s(\Lambda))^{1/\nu - 1} - E_{\nu,\nu}(s(\Lambda)t^\nu) t^{\nu - 1}}{E_{\nu,1}(s(\Lambda)t^\nu)} \right) = 0.
\]
This is true because \( (s(\Lambda))^{1/\nu - 1} - E_{\nu,\nu}(s(\Lambda)t^\nu) t^{\nu - 1} \) goes to zero exponentially fast (as \( t \to \infty \)), and therefore it goes to zero faster than the possible divergence of \( \sqrt{t a_t} \); in fact, for two suitable remainder terms \( r_1(t) \) and \( r_2(t) \) concerning the expansions of Mittag-Leffler functions in (4), we have
\[
(s(\Lambda))^{1/\nu - 1} - E_{\nu,\nu}(s(\Lambda)t^\nu) t^{\nu - 1} = \frac{(s(\Lambda))^{1/\nu - 1} E_{\nu,1}(s(\Lambda)t^\nu) - E_{\nu,\nu}(s(\Lambda)t^\nu) t^{\nu - 1}}{E_{\nu,1}(s(\Lambda)t^\nu)}
\]
\[
= \frac{(s(\Lambda))^{1/\nu - 1} \left( \frac{1}{\nu} e^{(s(\Lambda))^{1/\nu} t} + r_1(t) \right) - \left( \frac{1}{\nu} e^{(s(\Lambda))^{1/\nu} t} + r_2(t) \right) t^{\nu - 1}}{E_{\nu,1}(s(\Lambda)t^\nu) + r_1(t)}
\]
\[
= \frac{(s(\Lambda))^{1/\nu - 1} r_1(t) - r_2(t) t^{\nu - 1}}{\frac{1}{\nu} e^{(s(\Lambda))^{1/\nu} t} + r_1(t)}.
\]
Thus (11) holds, and the proof of the proposition is complete.

The following remarks concern Proposition 2. In particular Remark 5 has some connections with Remark 3 presented above.

**Remark 5** (The rate function \( \tilde{\Lambda}^* \) when \( C \) is invertible). If \( C \) is invertible one can check that, for all \( \bar{\nu} \in \mathbb{R}^m \),
\[
\tilde{\Lambda}^*(\bar{\nu}) := \langle C^{-1} \bar{\nu}, \bar{\nu} \rangle - \tilde{\Lambda}(C^{-1} \bar{\nu}) = \frac{1}{2} \langle \bar{\nu}, C^{-1} \bar{\nu} \rangle.
\]
Remark 6 (The case \( \nu = 1 \)). We have
\[
\begin{align*}
\gamma_j^{(1)} := & \begin{cases} 
0 & \text{if } j \neq k \\
\lambda_j & \text{if } j = k 
\end{cases} 
\end{align*}
\]
by (9). Moreover \( C \) is invertible (since \( \theta \in (0, \infty)^m \)) and, by Remark 3, we have
\[
\tilde{\Lambda}^{*}_{(\nu = 1)}(x):= \begin{cases} 
\frac{1}{2} \sum_{i=1}^{m} \frac{x_i^2}{\infty} & \text{if } x \in [0, \infty)^m \\
\text{otherwise} 
\end{cases}
\]
We can also say that \( \tilde{\Lambda}^{*}_{(\nu = 1)}(x) = \sum_{i=1}^{m} \tilde{f}^{(A)}_{1, \lambda}(x_i) \) for all \( x \in \mathbb{R}^m \), where \( \tilde{f}^{(A)}_{\nu, \lambda} \) is the rate function \( \tilde{\Lambda}^{*}_{(\nu = 1)} \) for \( m = 1 \). This agrees with what we said in Remark 3 (in particular we mean the independence of the one-dimensional marginal processes \( \{ M^1_t(t): t \geq 0\}, \ldots, \{ M^m_m(t): t \geq 0\} \)).

Remark 7 (Asymptotic Normality). The computations in the proof of Proposition 3 still work even if \( a_t = 1 \) (a case in which the first condition in (11) fails). Then \( \frac{M^\nu(t) - \mathbb{E}[M^\nu(t)\]]}{\sqrt{t}} \) converges weakly (as \( t \to \infty \)) to the centered Normal distribution with covariance matrix \( C \).

5  Statistical applications

In this section we present an estimator \( \hat{V}_t \) of \( \nu \), and the vector \( \theta \) is assumed to be known. The aim is to present some asymptotic results (as \( t \to \infty \)).

In particular we also assume that \( s(\theta) \geq 1 \). In fact the function \( f_a: (0, \infty) \to (0, \infty) \) defined by \( f_a(x):= \frac{1}{x} \cdot a^{1/x} \) is invertible if \( a \geq 1 \) (this can be checked noting that
\[
f'_a(x) = a^{1/x} \left( \frac{1}{x^2} \log a - 1 \right),
\]
and therefore \( f'_a(x) < 0 \) on \( (0, \infty) \); then, since \( s(\theta) \geq 1 \), we consider the estimator defined by
\[
\hat{V}_t := g_{s(\theta)} \left( \frac{s(M^\nu(t))}{t} \right),
\]
where \( g_{s(\theta)} \) is the inverse of \( f_{s(\theta)} \). It is quite natural to consider this estimator because of its consistency; in fact \( \frac{s(M^\nu(t))}{t} \) converges to \( \frac{1}{\nu} (s(\theta))^{1/\nu} \) (as \( t \to \infty \)), which is the sum of the components of the vector in (8).

It is also worth noting that the argument of \( g_{s(\theta)} \) can be equal to zero; so we need to consider \( f_{s(\lambda)}, g_{s(\lambda)}: [0, \infty] \to [0, \infty] \) where \( f_{s(\lambda)}(0) = g_{s(\lambda)}(0) = \infty, f_{s(\lambda)}(0) = g_{s(\lambda)}(0) = \infty \) and \( [0, \infty] \) is endowed with a suitable topology (an extended version of the one on \( (0, \infty) \)) with respect to which \( f_{s(\lambda)}, g_{s(\lambda)}: [0, \infty] \to [0, \infty] \) are continuous functions between Hausdorff topological spaces. The continuity of \( g_{s(\lambda)} \) is required for the application of the contraction principle (see e.g. Theorem 4.2.1 in [8]) in the proof of the next proposition.

Proposition 3. Assume that \( s(\theta) \geq 1 \). Then the family of random variables \( \{ \hat{V}_t: t > 0 \} \) satisfies the LDP with speed \( t \) and good rate function \( J_\nu \) defined by
\[
J_\nu(\hat{v}) := \begin{cases} 
\frac{\hat{v}}{\nu} \cdot (s(\lambda))^{1/\nu} \log \left( \frac{\hat{v}}{\nu} \cdot (s(\lambda))^{1/\nu - 1/\nu} \right) - \frac{\hat{v}}{\nu} \cdot (s(\lambda))^{1/\nu} + (s(\lambda))^{1/\nu} & \text{if } \hat{v} \geq 0 \\
0 & \text{if } \hat{v} < 0.
\end{cases}
\]
Proof. If we combine Proposition 1 and the contraction principle, the desired LDP holds with speed $t$ and good rate function $J_{\nu}$ defined by

$$J_{\nu}(\tilde{\nu}) := \inf\{\Lambda^*(x) : s(s(\tilde{x})) = \tilde{\nu}\}.$$ 

So in what follows we manipulate the expression of $J_{\nu}$ here to meet its expression in the statement of the proposition. The case $\tilde{\nu} < 0$ is trivial because we have the infimum over the empty set; thus, from now on, we restrict the attention on the case $\tilde{\nu} \geq 0$. Firstly we take into account the expression of $\Lambda^*$ in Remark 1 and we have

$$J_{\nu}(\tilde{\nu}) = \inf\{\Lambda^*(x) : s(x) = s(s(\tilde{x})) = \tilde{\nu}\}$$

$$= s(s(\tilde{x})) \inf \left\{ H \left( \frac{x}{s(s(\tilde{x}))} : \frac{\lambda}{s(\tilde{x})} \right) : s(s(\tilde{x})) = \tilde{\nu} \right\} + I_{\nu,s(\tilde{x})}(s(s(\tilde{x}))).$$

Moreover the first term is equal to zero; in fact, if $s(s(\tilde{x})) > 0$, for $y = \frac{s(s(\tilde{x}))}{s(\tilde{x})} \cdot \lambda$ we have

$$\inf \left\{ H \left( \frac{x}{s(s(\tilde{x}))} : \frac{\lambda}{s(\tilde{x})} \right) : s(s(\tilde{x})) = \tilde{\nu} \right\} = H \left( \frac{y}{s(s(\tilde{x}))} : \frac{\lambda}{s(\tilde{x})} \right) = 0.$$ 

In conclusion we have

$$J_{\nu}(\tilde{\nu}) = I_{\nu,s(\tilde{x})}(s(s(\tilde{x}))) = s(s(\tilde{x})) \log \left( \frac{\nu^\nu s(s(\tilde{x}))^\nu}{s(s(\tilde{x}))} \right) - \nu \log (s(s(\tilde{x}))) + (s(s(\tilde{x})))^{1/\nu}$$

$$= \frac{\nu}{\nu} \cdot (s(s(\tilde{x})))^{1/\nu} \log \left( \frac{\nu}{\nu} \cdot (s(s(\tilde{x})))^{1/\nu - 1/\nu} \right) - \frac{\nu}{\nu} \cdot (s(s(\tilde{x})))^{1/\nu} + (s(s(\tilde{x})))^{1/\nu}$$

and this completes the proof. 

\[\square\]

Remark 8 (On the probability to have a bad estimate). The estimator $\hat{V}_t$ can provide a bad estimate of $\nu$ when is larger than 1. However we can say that the event $\{\hat{V}_t > 1\}$ occurs with an exponentially small probability; in fact we have $\lim_{t \to \infty} \frac{1}{t} \log P\{\hat{V}_t > 1\} = -J_{\nu}(1)$.

Remark 9 (An alternative expression of $J_{\nu}$). Let us consider the function $D(\cdot; \cdot)$ be defined by

$$D(\lambda_1; \lambda_2) := \lambda_1 \log \frac{\lambda_1}{\lambda_2} - \lambda_1 + \lambda_2$$

for $\lambda_1 \geq 0$ and $\lambda_2 > 0$. Then, for $\tilde{\nu} \geq 0$, we have

$$J_{\nu}(\tilde{\nu}) = D \left( \frac{\nu}{\nu} \cdot (s(s(\tilde{x})))^{1/\nu}; (s(s(\tilde{x})))^{1/\nu} \right).$$

The following corollary provides the asymptotic decay of the probability of first kind error for the hypothesis testing

$$H_0 : \nu = \nu_0 \text{ versus } H_1 : \nu = \nu_1,$$

with $\nu_0 \neq \nu_1$.

More precisely we mean $P_{H_0}(R_k)$ where $R_k$ is the critical region defined by

$$R_k := \begin{cases} \{\hat{V}_t \geq k\} & \text{if } \nu_0 < \nu_1, \text{ for some } k > \nu_0 \\ \{\hat{V}_t \leq k\} & \text{if } \nu_0 > \nu_1, \text{ for some } k < \nu_0. \end{cases}$$

Corollary 1. Assume that $s(s(\tilde{x})) \geq 1$. Then $\lim_{t \to \infty} \frac{1}{t} \log P_{H_0}(R_k) = -J_{\nu_0}(k)$. 

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Proof. We have
\[
\lim_{t \to \infty} \frac{1}{t} \log P_{H_0}(R_k) = - \left\{ \begin{array}{ll}
\inf \{ J_{\nu_0}(\hat{\nu}) : \hat{\nu} \geq k \} & \text{if } \nu_0 < \nu_1 \\
\inf \{ J_{\nu_0}(\hat{\nu}) : \hat{\nu} \leq k \} & \text{if } \nu_0 > \nu_1
\end{array} \right.
\]
Then, by taking into account the allowed range of values for \( k \), the proof is complete if we show that \( J_{\nu_0}(\hat{\nu}) \) is decreasing if \( \hat{\nu} < \nu_0 \) and is increasing if \( \hat{\nu} > \nu_0 \) (note that \( J_{\nu_0}(\nu_0) = 0 \)). In order to do that we recall that the monotonicity intervals for \( \lambda_1 \) (when \( \lambda_2 \) is fixed) of the function \( D(\lambda_1; \lambda_2) \) in Remark 9: it is decreasing for \( \lambda_1 \in (0, \lambda_2) \), is increasing for \( \lambda_1 \in (\lambda_2, \infty) \), and \( D(\lambda_2; \lambda_2) = 0 \). Then, since \( f_{s(\lambda)}(\hat{\nu}) = \frac{1}{\hat{\nu}} \cdot (s(\lambda))^{1/\hat{\nu}} \) is decreasing, \( J_{\nu_0}(\hat{\nu}) = D \left( \frac{\nu_0}{\hat{\nu}} \cdot (s(\lambda))^{1/\hat{\nu}} ; s(\lambda) \right) \) decreases (to zero) when \( \hat{\nu} \) moves from 0 to \( \nu_0 \), and increases (from zero) when \( \hat{\nu} \) moves from \( \nu_0 \) to infinity. \( \square \)

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