Parametric amplification in quasi-PT symmetric coupled waveguide structures

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Abstract

The concept of non-Hermitian parametric amplification was recently proposed as a means to achieve an efficient energy conversion throughout the process of nonlinear three wave mixing in the absence of phase matching. Here we investigate this effect in a waveguide coupler arrangement whose characteristics are tailored to introduce passive PT symmetry only for the idler component. By means of analytical solutions and numerical analysis, we demonstrate the utility of these novel schemes and obtain the optimal design conditions for these devices.

1. Introduction

The introduction of parity-time reversal (PT) symmetry [1] in optics [2–6] has motivated a new line of research that aims at utilizing the peculiar effects associated with PT symmetry in order to achieve novel photonic functionalities ranging from optical isolation [7, 8] and unidirectional invisibility [9] to PT lasers [10–16] and optomechanical oscillators [17, 18]. In addition, these developments have stimulated further work in the general field of non-Hermitian photonics that do not necessarily respect PT symmetry such as exceptional points-based sensors [17, 18], microlasers that emit in an angular momentum mode [19], and supersymmetric laser arrays [13, 20].

Inspired by this progress, we have recently proposed a novel application of non-Hermitian photonic to control the nonlinear interactions inside optical parametric amplifiers (OPAs) [21]. For OPAs to operate properly, a phase matching condition between the interacting components must be satisfied. In the absence of this condition, the amplification process does not take place and energy oscillates back and forth between the different wave components. This stringent constraint has long posed a problem for utilizing the power of certain nonlinear material systems such as semiconductors [22].

In our recent publications, we have shown that by carefully engineering the dissipative spectral features of the nonlinear optical system under considerations, it is possible for the process of parametric amplification to take place even in the absence of Hermitian phase matching [21]. This is done by introducing optical losses to the idler component.

In this work, we investigate in detail a particular implementation of non-Hermitian parametric amplifiers based on quasi-PT symmetric coupled waveguide structures (a system where the idler component experiences passive PT symmetry as will be explained in details later). As shown in figure 1, the system consists of two coupled asymmetric waveguides. The nonlinear interaction takes place in one of them (say the left one) which we will call the non-linear guide. The second channel is designed to support a mode that matches only the idler component. Thus by introducing optical loss to this second waveguide, one can provide a dissipative coupling to the idler wave. We will call this second waveguide a reservoir channel. Thus effectively, the nonlinear waveguide together with the reservoir channel constitute a passive PT symmetric structure [5] for the idler component. Here we refer to this frequency selective passive PT symmetry as quasi-PT symmetry.
2. Analysis and results

Within the context of linear coupled mode theory between two waveguides and the nonlinear coupled wave analysis between the different components of the three wave mixing process, the system can be described by \[23\text{–25}]:

\[
\begin{align*}
\frac{dE_S}{dz} &= i\kappa_{SP}E_P^*E_S^*\exp(i\Delta \beta z), \\
\frac{dE_I}{dz} &= i\kappa_{IP}E_P^*E_I^*\exp(i\Delta \beta z) + i\kappa_D E_D, \\
\frac{dE_P}{dz} &= i\kappa_P E_I E_S \exp(-i\Delta \beta z), \\
\frac{dE_D}{dz} &= -\gamma_D E_D + i\kappa_D E_P,
\end{align*}
\]

(1)

where \(E_{S,I,P}\) are the signal, idler and pump field amplitudes respectively, while \(E_D\) is the field amplitude of the dissipative optical mode; \(\kappa_{S,I,P}\) are the nonlinear coupling coefficients, which are functions of the second order nonlinearity as well as the optical frequencies \(\omega_{S,I,P}\), whereas \(\kappa_D\) is the linear coupling between the idler and the dissipative modes. Finally, \(\Delta \beta = \beta_P - \beta_S - \beta_I\) is the propagation constant mismatch between the three different beams \([24]\), \(\gamma_D\) is the linear loss coefficient associated with the dissipative mode. In equation (1), \(z\) is the propagation distance, and asterisks denote complex conjugation.

We first consider the undepleted pump approximation (i.e. \(E_P\) can be taken as a constant). Under this condition, and by using the substitutions \(E_S(z) = A_S(z)\exp(i\Delta \beta z/2)\) and \(E_{I,D}(z) = A_{I,D}(z)\exp(i\Delta \beta z/2)\), we obtain:

\[
\begin{align*}
\frac{dA_S}{dz} &= i(\Delta \beta /2)A_S - i\kappa_S A_I, \\
\frac{dA_I}{dz} &= i\kappa_I A_S - i(\Delta \beta /2)A_I + i\kappa_D A_D, \\
\frac{dA_D}{dz} &= i\kappa_D A_I - [i(\Delta \beta /2) + \gamma_D] A_D,
\end{align*}
\]

(2)

where in the above we have also used \(\kappa_{S,I} = \kappa_{S,I}E_P\).

The behavior of equation (2) can be investigated by studying its eigenvalues \(\lambda\) that are associated with stationary solutions of the form \(A_{S,I,D}(z) = B_{S,I,D}\exp(i\lambda z)\), where \(B_{S,I,D}\) are constants. Particularly, the imaginary part of \(\lambda\) indicates amplification or decay depending on its sign.

The values of normalized amplitude amplification coefficients, defined as \(g = -\Im \{\lambda_N\} = -\Im \{\lambda\} / \sqrt{\kappa_S\kappa_I}\) (for the particular eigenvector that exhibits amplification), are plotted in figures 2(a) and (b) as a function of normalized phase mismatch \(\Delta \beta_N = \Delta \beta / \sqrt{\kappa_S\kappa_I}\) and the normalized coupling coefficient between the two asymmetric waveguides \(\kappa_D = \kappa_D / \sqrt{\kappa_S\kappa_I}\) (calculated at the idler frequency) for two different values of the normalized optical loss factor of the dissipation channel \(\gamma_D = \gamma_D / \sqrt{\kappa_S\kappa_I}\).
Interestingly, as we can see in figure 2(a), in the case when $g_{DN}=0$, one can still find a regime where amplification can occur even when $b_D N$ is outside the phase matching region ($b_D N < 2 \pi N$). This can be understood by thinking in terms of supermodes of the two asymmetric waveguides at the idler frequency, i.e., $k = \pm \mu (E_{EO}D + E_{ID}D)$. As a result of this hybridization, these supermodes exhibit shifted propagation constants that can satisfy the phase matching condition and eventually lead to amplification. We note however that this occurs only at a finite region bounded by two asymptotic lines. Outside this region, the dynamics exhibit only energy oscillations between the modes without any change in average powers. The detailed mathematical treatment of this case is presented in the appendix.

We now focus on the situation where losses are involved. As an example, we consider the case of $g_{DN} = 1$ as shown in figure 2(b). Here we note that amplification is not pinned to a finite region in the parameter space but rather occur for all points in the $(\Delta \beta_N, \kappa_{DN})$ plane. The exact value of $g$ is of course a function of the parameters $(\Delta \beta_N, \kappa_{DN})$. In particular, for a fixed value of $\Delta \beta_N$, one can observe an optimal regime where the values of $\kappa_{DN}$ can be chosen to achieve maximum amplification. In practice, this can be done by carefully engineering the separation between the two waveguides.

Figure 2(c) on the other hand shows a plot of the amplification factor as a function of $\gamma_{DN}$ and $\kappa_{DN}$ for a particular mismatch value of $\Delta \beta_N = 3$. Evidently, amplification can be achieved without any loss if the coupling

![Figure 2](image-url)
values are within the region discussed in figure 2(a). Outside this regime, amplification can only be attained by introducing optical dissipation to the auxiliary waveguide.

In order to verify our predictions based on the eigenstate analysis, we perform numerical integration of equation (1) for several different scenarios under the appropriate initial conditions. In doing so, we first rescale equation (1) by using $\xi = z/\delta_0$, $a_{S,L,P} = \eta_{S,L,P} E_{S,L,P} / E_0$ and $a_D = \eta_1^{-1} E_D / E_0$, where $\delta_0$ and $E_0$ are arbitrary length scale and reference electric field parameters, respectively, with the proportionality constants given by the dimensionless quantities $\eta_{S,L,P} = \sqrt{\kappa_{S,L,P}} \delta_0 E_0$. The rescaled equations, expressed in terms of scaled quantities $a_{S,L,P}$, are then similar to those of equation (1) with the following scaled values of the nonlinear couplings, linear coupling, phase mismatch, and optical loss: $\tilde{\kappa} = \eta_{S,L,P} \kappa_{S,L,P}$, $\tilde{\kappa}_D = \kappa_D \delta_0$, $\Delta \tilde{\beta} = \Delta \beta \delta_0$, and $\tilde{\gamma}_D = \gamma_D \delta_0$.

We first consider the undepleted pump approximation, where the normalized pump field $a_P$ is treated as a constant throughout the evolution. In all simulations, the Hermitian phase-matching is strongly violated with $\Delta \tilde{\beta} = 3$, and the initial conditions were taken to be $a_S(\xi = 0) = 1$, $a_{I,D}(\xi = 0) = 0$. Additionally, we choose the scaling parameters such that $\tilde{\kappa}_D = 1$. Figure 3 depicts the normalized signal, idler and auxiliary (propagating in the dissipative channel) optical beam powers $P_S = |a_S|^2$ (blue line), $P_I = |a_I|^2$ (red line) and $P_D = |a_D|^2$ (green line), respectively as a function of the normalized propagation distance $\xi$ with normalized linear loss coefficient $\tilde{\gamma}_D = 0$ for different values of the linear coupling $\tilde{\kappa}_D$.

As expected from the discussion of figure 2(a), when $\tilde{\gamma}_D = 0$, and $\tilde{\kappa}_D = 1.3$ or 4.5, the system is outside the amplification regime and the optical power associated with signal, idler and auxiliary beams oscillate without any change in average powers as shown in figures 3(a) and (d). By changing the above values of $\tilde{\kappa}_D$ to 1.45 or 4.35, we observe that the different components grow exponentially as a function of the propagation distance, as shown in figures 3(b), (c). Interestingly we observe some fast and small oscillatory behavior in figure 3. This can be attributed to the beating effects between the different eigenvectors associated with equation (2).

Figure 4 depicts the dynamical behavior in the presence of optical loss in the dissipative mode where for completeness we also present the oscillatory behavior of the system in the absence of any linear coupling (figure 3(a)). Figures 4(b)–(e) show the dynamics when $\tilde{\gamma}_D = 1$ for different value of $\tilde{\kappa}_D$. Evidently, in this regime, as the coupling is first increased from 0.5 to 1.5, the energy conversion occurs at a faster rate. As the coupling is further increased to 6, the amplification rate becomes less pronounced—in perfect agreement with the eigenvalue analysis.
We now treat the general case where the undepleted pump approximation no longer holds. Figure 5 depicts the numerical results for this scenario for different design parameters. In our simulations, we assumed that \( k = \tilde{\kappa} = 0.1 \) and \( x = a_0 + P \). In the phase-matching condition \((\Delta \tilde{\beta} = 0, \gamma_D = 0)\), oscillatory power transfer between the pump and both signal and idler beams is observed in the absence of linear coupling and loss, as shown in figure 5(a). When the dissipative mode is introduced without losses \((\gamma_D = 0)\), the phase-matching condition will be changed. Choosing a proper linear coupling \( \tilde{\kappa} \) can still obtain parametric amplification, as shown in figure 5(d) where \( \Delta \tilde{\beta} = 3 \) and \( \tilde{\kappa}_D = 3 \). An improper linear coupling \( \tilde{\kappa}_D \), too large or too small, will decrease the efficiency of power transfer, as shown in figures 5(b)–(c) where \( \tilde{\kappa}_D = 1.5 \) and 4.5 respectively. However, when the dissipative mode is introduced and losses are added to the dissipative components, the dynamical quantities asymptotically approach the steady state solution: \( P_{P_P}^{ss} = P_{P_D}^{ss} = P_{P_D}^{ss} = 0 \) and \( P_{P_S}^{ss} = P_{P} (\xi = 0) + P_{S} (\xi = 0) \approx P_{P} (\xi = 0) = 100 \), as shown in figures 5(e)–(f). These interesting results show

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**Figure 4.** Behavior of normalized power of the signal (blue curve starting at one), idler (red curve starting at zero) and auxiliary wave (green curve starting at zero) components as a function of the normalized distance when \( \Delta \tilde{\beta} = 3, \gamma_D = 1 \) for the following different linear coupling coefficients: (a)–(e) \( \kappa_D = 0, 0.5, 1, 1.5 \) and 6. Similar to figure 3, in all cases, the initial conditions were taken to be \( a_0 (\xi = 0) = 1, a_0 (\xi = 0) = 0 \).

**Figure 5.** Dynamics of normalized power of pump (black curve starting at 100), signal (blue curve starting at one), idler (red curve starting at zero) beams and the auxiliary waves (green curve starting at zero) as a function of the normalized distance when \( \tilde{\kappa} = 0.1 \) and under the different conditions indicated on the plots. In all cases, the initial conditions were taken to be \( a_0 (\xi = 0) = 10, a_0 (\xi = 0) = 1, a_0 (\xi = 0) = 0 \). Evidently, adding loss changes the dynamics drastically by allowing unidirectional energy conversion from the pump beam to the signal component.
that adding a dissipative mode can force the system to undergo a unidirectional energy conversion from the pump beam to the signal component—a task that could not be otherwise achieved in Hermitian systems.

3. Discussion and concluding remarks

In summary, we have investigated the dynamical behavior of non-Hermitian parametric amplifiers based on quasi-PT symmetric coupled waveguide channels. In these configurations, the two waveguides are asymmetric and their modal structure is assumed to be engineered in order to induce crosstalk only between the idler components. Our theoretical and analytical study revealed the existence of different regimes of operation depending on the interplay between the phase mismatch of the different frequency components from one side and the optical losses and coupling coefficient between the two channels on the other. In particular, we have shown in a certain regime of operation, introducing the auxiliary channel was enough to achieve phase matching even without introducing any loss. This is rather a form of dispersion engineering. On the other hand, outside this domain, amplification takes place only if optical loss is introduced to the idler component. These results might provide an alternative solution for utilizing the high nonlinearity of semiconductor materials which have been so far hindered by complex dispersion engineering requirements [22].

We note that in this work we have not discussed the details of how the optical loss can be introduced into the optical mode of the auxiliary waveguide. One possibility is to use metal strips on the waveguide surface as has been done in [5]. Another option is to use dopants that exhibit frequency selective absorption spectrum in the bandwidth of interest. These two strategies are material dependent and might not provide enough flexibility to operate at any desired frequency. Another attractive alternative is thus to employ structures whose effective complex index can be tailored at will by means of geometric designs. These include plasmonics, metamaterial and nano-antennas that, if designed properly, can function as meta-absorbers [26]. Another advantage of using meta-absorbers is also the possibility of engineering the spatial profile of the optical absorption by controlling the spatial distribution of the density and geometric design of these configurations. We carry out this study in future work.

Finally we note that other recent works that studied wave mixing processes in the presence of gain and loss can be found in [27, 28]. Additionally, we mention in passing that the similarity between PT phase transition and the crossover between the oscillatory and amplifying behavior in parametric down conversion in Hermitian systems was noted in [21] and soon after investigated in details in [29, 30].

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Appendix

In this appendix we provide further analysis for the effect observed in figure 2(a) where parametric amplification can take place by just adjusting the coupling between the two asymmetric waveguides even without introducing losses. In particular, in the absence of any dissipation, the eigenvalue spectrum of equation (2) can be obtained from the following condition:

\[
\begin{vmatrix}
\lambda - \Delta \beta/2 & \kappa_s & 0 \\
-\kappa_s & \lambda + \Delta \beta/2 & -\kappa_D \\
0 & -\kappa_D & \lambda + \Delta \beta/2
\end{vmatrix} = 0. \quad (A.1)
\]

By using the normalized quantities \(\lambda_N, \Delta \beta_N\), and \(\kappa_D N\), and substituting \(x, y\) for \(\Delta \beta N\) and \(\kappa_D N\), equation (A.1) reduces to:

\[
8x^4 + 4x^2y^2 + (8 - 2x^2 - 8y^2) \lambda_N + 4x - x^3 + 4xy^2 = 0. \quad (A.2)
\]

We now seek solutions of equation (2) that imply amplification, i.e. solutions that exhibit imaginary parts of \(\lambda\). We do so by noting that the solutions of equation (A.2) can be classified according to its discriminant \(\Delta = 18abcd - 4b^2d + b^2c - 4ac^3 - 27a^2d^2\), where \(a, b, c, d\) are the coefficients of \(\lambda_N^4, \lambda_N^3, \lambda_N^2, \lambda_N\), respectively. When \(\Delta > 0\), the solutions are all real and the system exhibits only oscillatory behavior whereas for \(\Delta < 0\), two complex conjugate solutions exist and thus amplification takes place. Clearly the boundary between the two regimes can be obtained by solving \(\Delta = 0\) to get:
\[ 8x^2y^2 = 8y^4 \pm (8y^2 + 1)^{3/2} + 20y^2 - 1. \]  \hspace{1cm} (A.3)

Equation (A.3) defines the two boundary curves that separate the two different regimes observed in figure 2(a). Interestingly, up to a small error margin, these two curves can be approximated by the simplified equations:

\[ y^2 = (x - \sqrt{2})^2 - (2 - \sqrt{2})^2, \]  \hspace{1cm} (A.4)
\[ [y - (3 - \sqrt{2})/2]^2 = [x + 3(\sqrt{2} - 1)/2]^2 - (2 - \sqrt{2})^2. \]

Asymptotically, for large \( x \), the above equations reduce to the linear relation \( y = x \pm \sqrt{2} \). The simplicity of this last expression begs for an intuitive explanation.

In order to gain more insight into this behavior, we rewrite equation (1) in a different set of bases within the undepleted pump approximation. By using the transformation \( C_{E,O} = (E_I \pm E_D)z/2 \) and substituting \( C_S = E_S^j \), we obtain:

\[
\begin{align*}
\frac{dC_S}{dz} &= -i\kappa_S C_E \exp(-i\Delta \beta^{-}z) - i\kappa_S C_O \exp(-i\Delta \beta^{+}z), \\
\frac{dC_E}{dz} &= i\kappa_I C_E \exp(i\Delta \beta^{-}z), \\
\frac{dC_O}{dz} &= i\kappa_I C_O \exp(i\Delta \beta^{+}z),
\end{align*}
\]

where \( \Delta \beta^{\pm} = \Delta \beta \pm \kappa_D \). In the regime where both \( \Delta \beta^{-} \) and \( \kappa_S \) are small relative to \( \Delta \beta^{+} \), the contribution from the \( C_O \) term becomes off-resonant and can be neglected. One can expect a small contribution from the odd term. Under these conditions, the eigenvalues of the reduced system (after dropping the \( C_O \) term) becomes:

\[ \lambda = \pm \sqrt{(\Delta \beta^{-}/2)^2 - \kappa_S \kappa_I}/2. \]

Therefore the phase matching condition reduces to \( |\Delta \beta| < \sqrt{2\kappa_S \kappa_I} \) —in agreement with the plot of figure 2(a) and the results of equation (A.3) or equation (A.4).

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