GENERIC IDEALS AND MORENO-SOCÍAS CONJECTURE

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Abstract. Let $f_1, \ldots, f_n$ be homogeneous polynomials generating a generic ideal $I$ in the ring of polynomials in $n$ variables over an infinite field. Moreno-Socías conjectured that for the graded reverse lexicographic term ordering, the initial ideal $\text{in}(I)$ is a weakly reverse lexicographic ideal. This paper contains a new proof of Moreno-Socías’ conjecture for the case $n = 2$.

1. Introduction

We begin by introducing the definitions necessary to understand the conjecture under study.

Let $R = K[x_1, \ldots, x_n] = K[\mathbf{x}]$ be the polynomial ring in $n$ variables over an infinite field $K$, which is an extension of a base field $F$. Then $R$ is an $\mathbb{N}$-graded ring such that for each $d \in \mathbb{N}$, $R_d$ is the set of all homogeneous polynomials (forms) of total degree equal to $d$, where $\deg x_1 = \cdots = \deg x_n = 1$.

Definition 1.1. Let $f = \sum_m \alpha_m m$ be a form of degree $d$ such that $m$ runs over all monomials of degree $d$ and $\alpha_m \in K$. Then $f$ is called generic if the coefficients $\alpha_m$ satisfy the following two conditions:

1. If $m \neq m'$, then $\alpha_m \neq \alpha_{m'}$.
2. The set of all $\alpha_m$ is algebraically independent over $F$.

Definition 1.2. An ideal $I \subset R$ is called generic if $I$ can be generated by generic forms $f_1, \ldots, f_r$, where $f_i = \sum_{m_i} \alpha_{m_i} m_i$, satisfying the following conditions:

1. $\alpha_{m_{i_1}} \neq \alpha_{m_{i_j}}$ for $i \neq j$.
2. The set $\{\alpha_{m_{i_1}} : 1 \leq i \leq r\}$ is algebraically independent over $F$.

Definition 1.3 (Graded reverse lexicographic order). Let $\mathbf{x}^\alpha, \mathbf{x}^\beta \in R$. Then let $\mathbf{x}^\beta > \mathbf{x}^\alpha$ if $\deg \mathbf{x}^\beta > \deg \mathbf{x}^\alpha$ or $\deg \mathbf{x}^\alpha = \deg \mathbf{x}^\beta$ and the right-most nonzero entry of $\beta - \alpha$ is negative.

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Definition 1.4. A monomial ideal \( J \subset R \) is *weakly reverse lexicographic* if, whenever \( x^\beta \) is a minimal generator of \( J \), then every monomial of the same degree which precedes \( x^\beta \) in the reverse lexicographic term order must be in \( J \). And \( J \) is *reverse lexicographic* if, whenever \( x^\beta \in J \), then every monomial of the same degree which precedes \( x^\beta \) must be in \( J \).

Remark 1.5. Every reverse lexicographic ideal is weakly reverse lexicographic.

Definition 1.6. Let \( f = \sum_{\alpha \in A} a_\alpha x^\alpha \) be a nonzero polynomial in \( R \) such that \( a_\alpha \neq 0 \) for all \( \alpha \in A \). Let \( \prec \) be a term order. The *leading monomial* of \( f \) is \( \text{LM}_\prec(f) = x^\beta \) if \( \beta \in A \) and \( x^\alpha < x^\beta \) for all \( \alpha \in A, \alpha \neq \beta \).

Definition 1.7. For an ideal \( I \subset R \), the *initial ideal* of \( I \) is the ideal \( \langle \text{LM}_\prec(f) : f \in I \rangle \).

We are now ready to state the Moreno-Socías Conjecture and some known facts about it.

**Conjecture** (Moreno-Socías [3, 4]). Let \( d_1, \ldots, d_n \in \mathbb{N} \) and \( I \subset R \) be a generic ideal generated by a sequence of polynomials \( f_1, \ldots, f_n \) of degrees \( d_1, \ldots, d_n \in \mathbb{N} \), and \( J = \text{in}(I) \), the initial ideal of \( I \) with respect to the graded reverse lexicographic order. Then \( J \) is weakly reverse lexicographic.

As pointed out in [4], this conjecture implies many other conjectures, in particular, Fröberg’s Conjecture [2], which gives a formula for the Hilbert series of generic ideals.

The Moreno-Socías Conjecture is trivial when \( n = 1 \). It was proven by Moreno-Socías for \( n = 2 \) in his thesis [3], as was pointed out to us by one of the referees. We learnt about the conjecture in [4], and this paper contains a proof for the case \( n = 2 \), which is quite different from the one in [3]. Our proof is quite elementary, and was discovered through extensive calculations using the computer algebra systems MAPLE and Singular. Unfortunately, it seems unlikely that our methods can be extended to deal with the case of more variables. Already for \( n = 3 \), computer calculations become impractical.

2. The two variable case

In this section we show that Moreno-Socías Conjecture is true for the case of two generic forms in the ring of polynomials in two variables. Let \( R = K[x,y] \) be the polynomial ring in 2 variables over an arbitrary infinite field \( K \), with a base field \( F \).

Let \( n \leq m, \mu = m - n \), and let

\[
\begin{align*}
f_1 &= a_{1,1}x^n + a_{1,2}x^{n-1}y + \cdots + a_{1,n}xy^{n-1} + a_{1,n+1}y^n, \\
f_2 &= b_{2,1}x^m + b_{2,2}x^{m-1}y + \cdots + b_{2,m}xy^{m-1} + b_{2,m+1}y^m
\end{align*}
\]

be generic forms of degree \( n \) and \( m \), respectively, generating the generic ideal \( I = \langle f_1, f_2 \rangle \) in \( R \). Since \( n \leq m \), one can divide \( f_2 \) by \( f_1 \):

\[
f_2 = qf_1 + r,
\]
where
\[ r = a_{2,1}x^{n-1}y^{\mu +1} + a_{2,2}x^{n-2}y^{\mu +2} + \cdots + a_{2,n}y^m \]
is a generic form of degree \( m \) with \( n \) terms and \( I = \langle f_1, f_2 \rangle = \langle f_1, r \rangle \). Thus, without loss of generality for Gröbner basis calculations, one can start with the following forms:
\[
\begin{align*}
  f_1 &= a_{1,1}x^n + a_{1,2}x^{n-1}y + \cdots + a_{1,n}xy^{n-1} + a_{1,n+1}y^n, \\
  f_2 &= a_{2,1}x^{n-1}y^{\mu +1} + a_{2,2}x^{n-2}y^{\mu +2} + \cdots + a_{2,n}y^m.
\end{align*}
\]

Construct the set \( G = \{f_1, f_2, \ldots, f_{n+1}\} \) from \( f_1 \) and \( f_2 \) by taking \( f_{t+2} = \overline{S(f_t, f_{t+1})}^{f_{t+1}} \) for \( 1 \leq t \leq n - 1 \), where \( S(f_t, f_{t+1}) \) is the S-polynomial of \( f_t \) and \( f_{t+1} \) (see, e.g., [1, §2.6, Definition 4]). Then it is not hard to see that
\[
\begin{align*}
  f_1 &= a_{1,1}x^n + a_{1,2}x^{n-1}y + \cdots + a_{1,n}xy^{n-1} + a_{1,n+1}y^n, \\
  f_2 &= a_{2,1}x^{n-1}y^{\mu +1} + a_{2,2}x^{n-2}y^{\mu +2} + \cdots + a_{2,n}y^m, \\
  f_3 &= a_{3,1}x^{n-2}y^{\mu +3} + a_{3,2}x^{n-3}y^{\mu +4} + \cdots + a_{3,n-1}y^{m+1}, \\
  &\vdots \\
  f_t &= a_{t,1}x^{n-(t-1)}y^{\mu +(2t-3)} + \cdots + a_{t,n-(t-2)}y^{m+(t-2)}, \\
  &\vdots \\
  f_n &= a_{n,1}xy^{\mu +(2n-3)} + \cdots + a_{n,2}y^{m+(n-2)}, \\
  f_{n+1} &= a_{n+1,1}y^{\mu +(2n-1)},
\end{align*}
\]

where, for \( 1 \leq t \leq n - 1 \),
\[
\begin{align*}
  a_{t+2,i} &= \left\{
  \frac{(a_{t+1,2} - a_{t+2,1})}{a_{t+1,1}} - \frac{a_{t+1,1} + 1}{a_{t+1,1}} \left( \frac{a_{t,2}}{a_{t+1,1}} - \frac{a_{t+1,2}}{a_{t+1,1}} \right) \right. \\
  &\quad \left. 1 \leq i \leq n - t - 1, \\
  &\quad i = n - t.
\end{align*}
\]

All the coefficients are nonzero, since the forms we started with are generic. The following theorem shows that \( G \) is a Gröbner basis for the ideal \( I \).

**Theorem 2.1.** Let \( I = \langle f_1, f_2 \rangle \), where \( f_1 \) and \( f_2 \) as in [3], and let \( G \) be the set in [2]. Then \( G \) is a Gröbner basis for the ideal \( I \) with respect to the graded reverse lexicographic order, with \( x > y \).

**Proof.** By using [1, §2.9 Theorem 9] and the fact that \( f_{t+2} = \overline{S(f_t, f_{t+1})}^{f_{t+1}} \) for \( 1 \leq t \leq n - 1 \), it is enough to show that the set of syzygies \( S = \{ S_{t,i+1} : 1 \leq t \leq n - 1 \} \) forms a homogeneous basis for the set of all syzygies \( \{ S_{i,j} : 1 \leq i < j \leq n + 1 \} \) among the elements of \( G \).

To simplify the calculations, we are going to make all the polynomials of \( G \) monic by dividing each one by its leading coefficient; let \( b_{x,t} = \frac{a_{t,1}}{a_{t,1}} \). So without loss of generality, we will assume that all \( f_i \) are monic. It is easy to see that \( S_{i,i+1} \) is a homogeneous syzygy. In fact,
\[
S_{i,i+1} = \left\{
\begin{array}{ll}
(y^{\mu +1}, -x, 0, \ldots, 0) & i = 1, \\
(0, \ldots, y^2, -x, 0, \ldots, 0) & 1 < i \leq n.
\end{array}
\right.
\]
Claim: For all $1 \leq i \leq n$ and $1 \leq t \leq n - i + 1$, $S_{i,i+t}$ is generated by elements of $S$. In fact

$$S_{i,i+t} = y^{2t-2}S_{i,i+1} + xy^{2t-4}S_{i+1,i+2} + \cdots + x^{t-1}S_{i+t-1,i+t}$$

(5)

Let $1 \leq i \leq n$. The proof will proceed by induction on $t$. For $t = 1$, $S_{i,i+t} = S_{i,i+1} \in S$. Now assume the claim is true for $t$, i.e., $S_{i,i+t}$ is a combination of elements of $S$.

We need to show (5) for $t + 1$. Hence, we need to show that $S_{i,i+t+1}$ can be written as a combination of elements of $S$. For $i = 1$,

$$S_{1,1+(t+1)} = (y^{\mu+2(t+1)-1}, 0, \ldots, 0, \underbrace{-x^{t+1}}_{t+2-th \ pos.}, 0, \ldots, 0)$$

$$= y^2(y^{\mu+2t-1}, 0, \ldots, 0, \underbrace{-x^t}_{t+1-th \ pos.}, 0, \ldots, 0) + x^t(0, \ldots, y^2, \underbrace{-x^{t+1}}_{t+2-th \ pos.}, 0, \ldots, 0)$$

$$= y^2S_{1,1+t} + x^tS_{1+t,(1+t)+1}.$$

And for $i > 1$,

$$S_{i,i+(t+1)} = (0, \ldots, 0, y^{2(t+1)}, 0, \ldots, 0, \underbrace{-x^{t+1}}_{i-th \ pos.}, \underbrace{0, \ldots, 0}_{i+t+1-th \ pos.})$$

$$= y^2(0, \ldots, 0, y^{2t}, 0, \ldots, 0, \underbrace{0, \ldots, 0}_{i-th \ pos.}, \underbrace{-x^t}_{i+t-th \ pos.}) + x^t(0, \ldots, 0, y^2, \underbrace{-x^t}_{i-th \ pos.}, 0, \ldots, 0)$$

$$= y^2S_{i,i+t+1} + x^tS_{i+t,(i+t)+1}.$$

Thus, the set $S$ is a homogeneous basis of the set of all syzygies on $G$. Therefore, by [1], §2.9 Theorem 9] and the way that we constructed $G$, $G$ is a Gröbner basis for the ideal $I$. \hfill \Box

**Corollary 2.2.** Let $I$, $G$ and $R$ be as above, then the initial ideal of $I$ with respect to the graded reverse lexicographic order is:

$$J = \langle x^n, x^{n-1}y^{\mu+1}, x^{n-2}y^{\mu+3}, \ldots, xy^{n+2n-3}, y^{\mu+2n-1} \rangle.$$

**Proof.** The proof follows from the fact that $G$ is a Gröbner basis for the ideal $I$ with respect to the graded reverse lexicographic order. Hence the initial ideal of $I$ with respect to this order is

$$\text{in}(I) = \langle LM(f) : f \in G \rangle$$

$$= \langle x^n, x^{n-1}y^{\mu+1}, x^{n-2}y^{\mu+3}, \ldots, xy^{n+2n-3}, y^{\mu+2n-1} \rangle.$$

\hfill \Box
Lemma 2.3. The monomial ideal $J$ in (6) is a reverse lexicographic ideal.

Proof. Let $x^\alpha y^\beta \in J$ and let $x^s y^t$ such that $s + t = \alpha + \beta$ and $s \geq \alpha$. We need to show that $x^s y^t \in J$. Since $x^\alpha y^\beta \in J$, there exist $\alpha_1, \beta_1 \geq 0$ and an $i$ such that $x^\alpha y^\beta = x^{\alpha_1 + i} y^{\beta_1} (x^{n-i} y^{\mu+2i-1})$. Thus $\alpha = \alpha_1 + n - i$. Since $s \geq \alpha$, there exists an $h \leq i$ such that $s = \alpha_1 + n - h$. But $s + t = \alpha + \beta = \alpha_1 + n + i + \beta_1 - \mu - 1$, so

$$t = \mu + h + i + \beta_1 - 1 \geq \mu + 2i + \beta_1 - 1 \geq \mu + 2i - 1.$$ 

Therefore, $x^s y^t$ is divisible by $x^{n-i} y^{\mu+2i-1} \in J$ and hence $x^s y^t \in J$.

Now we combine Corollary 2.2 and Lemma 2.3 to get a proof of the main theorem.

Theorem 2.4 (Moreno-Socías Conjecture for two variables). Let $R = K[x, y]$ be the polynomial ring over an infinite field $K$. Let $d_1, d_2 \in \mathbb{N}$ and let $f_1 \in R_{d_1}, f_2 \in R_{d_2}$ be generic forms generating a generic ideal $I$. Let $J = \text{in}(I)$ be the initial ideal of $I$ with respect to the graded reverse lexicographic order. Then $J$ is weakly reverse lexicographic.

As mentioned before, the main idea for the proof was to recognize the formula for the initial ideal from computations. We were not able to do enough computations for higher $n$. Moreover, even if one could carry out enough calculations to guess a formula like (6), we believe that it would be hard to prove it: one needs to get a
Gröbner basis like (3), which will be hard to find using the same methods as for $n = 2$.

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