Endomorphisms of spaces of virtual vectors fixed by a discrete group

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Abstract. A study is made of unitary representations π of a discrete group G that are of type II when restricted to an almost-normal subgroup Γ ⊆ G. The associated unitary representation π₀ of G on the Hilbert space of ‘virtual’ Γ₀-invariant vectors is investigated, where Γ₀ runs over a suitable class of finite-index subgroups of Γ. The unitary representation π₀ of G is uniquely determined by the requirement that the Hecke operators for all Γ₀ are the ‘block-matrix coefficients’ of π₀. If π|Γ is an integer multiple of the regular representation, then there is a subspace L of the Hilbert space of π that acts as a fundamental domain for Γ. In this case the space of Γ-invariant vectors is identified with L. When π|Γ is not an integer multiple of the regular representation (for example, if G = PGL(2, Z[1/p]), Γ is the modular group, π belongs to the discrete series of representations of PSL(2, R), and the Γ-invariant vectors are cusp forms), π is assumed to be the restriction to a subspace H₀ of a larger unitary representation having a subspace L as above. The operator angle between the projection P₀ onto H₀ (typically, a Bergman projection onto a space of analytic functions) and the projection P onto the subspace H₀ (typically, the characteristic function of the fundamental domain) and the projection P₀ onto the subspace H₀ (typically, a Bergman projection onto a space of analytic functions) is the analogue of the space of Γ-invariant vectors. It is proved that the character of the unitary representation π₀ is uniquely determined by the character of the representation π.

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1. Introduction and main results

Let $G$ be a countable discrete group, let $\Gamma$ be an almost-normal subgroup, and let $\mathcal{S}$ be the minimal intersection lattice of finite-index subgroups of $\Gamma$ that is closed with respect to the operation

$$\Gamma_0 \to (\Gamma_0)_\sigma = \Gamma_0 \cap \sigma \Gamma_0 \sigma^{-1}, \quad \sigma \in G.$$  \hspace{1cm} (1)

We assume throughout this paper that the intersection of the subgroups in $\mathcal{S}$ is the identity element.

Let us consider a unitary (projective) representation $\pi$ of $G$ on a Hilbert space $H$. Throughout this paper we make the assumption that $\pi|\Gamma$ is a multiple of the left regular representation $\lambda_\Gamma$ ([50], [44]) of $\Gamma$ (possibly skewed with a cocycle if $\pi$ is projective). Under this hypothesis, we construct Hilbert spaces $H^\Gamma_0$ of $\Gamma_0$-invariant vectors. More generally, we construct Hilbert spaces $H^\Gamma_0$ of $\Gamma_0$-invariant vectors, where $\Gamma_0$ runs over $\mathcal{S}$.

We will refer to the above vectors for $\Gamma_0 \in \mathcal{S}$ as ‘virtual’ $\Gamma_0$-invariant vectors, since they generally correspond to $\Gamma_0$-invariant, densely defined linear functionals on $H$. The Hilbert spaces $H^\Gamma_0$, $\Gamma_0 \in \mathcal{S}$, are not really subspaces of $H$; they are located in an enlargement, in the sense of Gelfand triples (see [18]), of the given Hilbert space (see Definition 14). For all subgroups $\Gamma_0, \Gamma_1 \in \mathcal{S}$ such that $\Gamma_1 \subseteq \Gamma_0$, there is a canonical isometric embedding $H^\Gamma_0 \subseteq H^\Gamma_1$.

We illustrate this in the case of the left regular representation $\lambda_\Gamma$ of $\Gamma$ into the unitary group $U(l^2(\Gamma))$ of the Hilbert space $l^2(\Gamma)$. In this case the space of $\Gamma$-invariant vectors is the one-dimensional space generated by the constant functions on $\Gamma$. This space can be regarded as a space of densely defined linear functionals on $l^2(\Gamma)$. In the general case $l^2(\Gamma)^{\Gamma_0} = \ell^2(\Gamma_0 \setminus \Gamma)$ in this formalism.

In general, if the unitary representation $\pi$ on the Hilbert space $H$ is an integer multiple of the left regular representation $\lambda_\Gamma$ of the discrete group $\Gamma$, then there is a subspace $L$ such that, $\Gamma$-equivariantly, we have

$$H \cong l^2(\Gamma) \otimes L \quad \text{and} \quad \pi|\Gamma \cong \lambda_\Gamma \otimes \text{Id}_L.$$

In this case too, it is obvious that one can identify the space of $\Gamma$-invariant vectors with the Hilbert space $L$. In this identification $L$ is no longer a true subspace of $H$. Moreover, $L$ is not unique.

If the unitary representation $\pi$ is not an integer multiple of the left regular representation $\lambda_\Gamma$, then we will make use of a subspace $L$ from a larger representation of $G$ which contains $\pi$ as a subrepresentation. In this case we will prove below that the operator angle between the projection onto the subspace of the subrepresentation and the projection onto $L$ can be used to construct the space of $\Gamma$-invariant vectors and its scalar product.

This is analogous to the Petersson scalar product formula for automorphic forms in [34], where he used a ‘measuring scale’ consisting of a fundamental domain $F$.
for the action of $\text{PSL}(2, \mathbb{Z})$ on the upper half-plane in order to introduce a scalar product on automorphic forms which are ‘virtual’ $\text{PSL}(2, \mathbb{Z})$-invariant vectors for the restrictions to $\text{PSL}(2, \mathbb{Z})$ of the representations $\pi_n \ (n \geq 2)$ in the discrete series of $\text{PSL}(2, \mathbb{R})$. The larger unitary representation containing $\pi_n$ as a subrepresentation is obtained as follows: if $\pi_n$ is realized as a unitary representation on the space $H^2(\mathbb{H}, \nu_n)$ of square-integrable analytic functions with respect to a measure $\nu_n$ on $\mathbb{H}$, then the larger unitary representation containing $\pi_n$ as a subrepresentation is realized on the corresponding Hilbert space $L^2(\mathbb{H}, \nu_n)$. The space of square-integrable functions supported in $\mathbb{F}$ plays the role of the subspace $L$ as above.

Let $\sigma$ be a group element in $G$. We use the notation introduced before the formula (1). It is obvious that $\sigma \Gamma \sigma^{-1} \Gamma \sigma^{-1} = \Gamma \sigma$. This implies that the representation $\pi$ induces a unitary transformation, denoted by $\pi^p(\sigma)$, that maps $H^\Gamma \sigma^{-1}$ onto $H^\Gamma \sigma$.

The main problem we consider in this paper is the analysis of the unitary representation $\pi^p$ induced by $\pi$ on the Hilbert space $\overline{H}$ obtained by taking the inductive limit of the Hilbert spaces $H^\Gamma \sigma_0$, $\sigma_0 \in \mathcal{S}$. The content of the classical Ramanujan–Petersson problem [24], [47], [6] is transformed into a harmonic analysis problem on the weak unitary containment of the representation $\pi^p$ in the unitary representation obtained by restricting to $G$ the left regular representation of the Schlichting completion [48], [26] of $G$ with respect to the subgroups in the family $\mathcal{S}$.

The representation $\pi^p$ is determined by the associated Hecke operators ([23], [24], [46]) on all levels $\Gamma_0$. The Hecke operators are the ‘block-matrix coefficients’ of the associated unitary representation $\pi^p$. We are using the notation $\pi^p$ for the representation on the space of all ‘virtual’ $\Gamma_0$-invariant vectors, $\Gamma_0 \in \mathcal{S}$, in order to emphasize the profinite completion procedure that is used in the construction of this representation. When the initial representation $\pi$ is projective with a cocycle $\epsilon$, and $\epsilon$ admits an extension to the Schlichting completion $\overline{G}$ introduced below, the construction in this paper works exactly as in the projective case.

The representation $\pi^p$ is widely used in a slightly different form in the literature (see, for example, [6], [13], [21]). In this paper we use an operator algebra construction of $\pi^p$. This construction enables us to obtain unitarily equivalent forms of $\pi^p$ that are more suitable for the computation of traces.

The main result of this paper is the correspondence between the two representations $\pi$ and $\pi^p$. The representation $\pi^p$ has a canonical block-matrix structure associated with a given choice of coset representatives for subgroups in $\mathcal{S}$. Consequently, we represent the associated Hecke operators as block matrices whose entries are ‘localized sums’ over cosets of the values of the original representation $\pi$ restricted to the space $L$ introduced above (which is the analogue of a fundamental domain for the action of the group $\Gamma$). As a corollary we obtain a precise formula relating the characters of $\pi$ and $\pi^p$.

To establish a relation between the two representations, we introduce an integrability condition which ensures the convergence of the sums for the matrix entries in the representations obtained for the Hecke operators.

**Definition 1.** Let $\pi_0$ be a unitary representation of $G$ such that $\pi_0|_\Gamma$ is a (not necessary integer) multiple of the left regular representation $\lambda_\Gamma$. We assume that
there is a unitary representation $\pi$ of $G$ into the unitary group of a Hilbert space $H$ containing $H_0$, with the following properties.

(i) $\pi|_\Gamma$ is an integer multiple of the left regular representation $\lambda_\Gamma$. Consequently, there is a Hilbert subspace $L$ of $H$ such that

$$H \cong l^2(\Gamma) \otimes L \quad \text{and} \quad \pi|_\Gamma \cong \lambda_\Gamma \otimes \text{Id}_L.$$ 

(ii) Let $e$ be the identity element of $G$. Denote the orthogonal projection from $H$ onto $L \cong \mathbb{C}e \otimes L$ by $P_L$. Assume that the projections in the family $\{\pi(g)P_L\pi(g^{-1}) \mid g \in G\}$ form a commutative family.

(iii) Assume that $\pi_0$ is a subrepresentation of $\pi$. Then $H_0 \subseteq H$ is $\pi(G)$-invariant, and $\pi_0$ is the restriction of $\pi$ to $H_0$. Denote by $P_0$ the orthogonal projection from $H$ onto $H_0$. Clearly, $\pi_0(g) = P_0\pi(g)P_0$, $g \in G$ in this case.

(iv) The product of the operators $P_0$ and $P_L$ is a trace class operator.

(v) For every $g$ in $G$ and $\Gamma_0$ in $S$ the sum

$$\sum_{\theta \in \Gamma_0g} P_L\pi_0(\theta)P_L$$

over the coset $\Gamma_0g$ is convergent in the space $C_2(L)$ of Hilbert–Schmidt operators.

(vi) The sum of the traces of the operators in the sum (2) is absolutely convergent. The sum in (2) is a trace class operator whose trace is equal to the sum of the traces of the operators in the sum.

(vii) The Murray–von Neumann dimension $\dim_{\{\pi_0(\Gamma)\}''}H_0$ ([50], [44], [25]) is a finite, strictly positive number.

The condition (vii) above can be interpreted as saying that $H_0$ is a finitely generated left Hilbert module over the representation $\pi_0|_\Gamma$, which is a multiple of the left regular representation of $\Gamma$.

In the main examples considered in this paper, where the representations $\pi_0$ are obtained from unitary representations in the discrete series of (projective) unitary representations of $\text{PSL}(2, \mathbb{R})$ by restriction to $G = \text{PGL}(2, \mathbb{Z}[1/p])$, the conditions (iii) and (iv) follow from computations in [53]. The condition (ii) is automatic if the representation $\pi$ comes from a Koopman unitary representation. Indeed, in this case the projection $P_L$ is the operator of multiplication by the characteristic function of a fundamental domain for $\Gamma$. Furthermore, the projections in the family are the operators of multiplication by the characteristic functions of translates of the fundamental domain by elements in $G$.

A unitary representation $\pi_0$ of $G$ as above determines, in a canonical way, a unitary representation $\pi_0^p$ of the Schlichting completion [48] $\overline{G}$ of $G$ with respect to the subgroups in the family $S$. The unitary representation $\pi_0^p$ acts on $\overline{H}^p$, the Hilbert space completion of the inductive limit of the Hilbert spaces $H^{\Gamma_0}$, $\Gamma_0 \in S$. This construction is rigorously presented in Theorem 33. The following result gives an explicit description of the block-matrix coefficients of the representation $\pi_0^p$. These are Hecke operators corresponding to subgroups $\Gamma_0 \in S$ and associated with the original unitary representation $\pi_0$. This representation is used to compute the character of the representation $\pi_0^p$. In the following theorem we use the notation $(\Gamma_0)_\sigma = \sigma\Gamma_0\sigma^{-1} \cap \Gamma_0$ for $\Gamma_0 \in S$ and $\sigma \in G$. 

Theorem 2. Let $\pi_0$ be a unitary representation of $G$ into the unitary group $\mathcal{U}(H_0)$ of a Hilbert space $H_0$. Assume that $\pi_0$ satisfies the technical conditions (i)–(vii) in Definition 1. Let $\pi_0^p$ be the unitary representation of $G$ introduced above.

For every $\Gamma_0$ in $\mathcal{S}$, fix a family of coset representatives for $\Gamma_0$ in $\Gamma$ such that $\Gamma = \bigcup_{i=1}^{[\Gamma:\Gamma_0]} \Gamma_0 s_i$. Represent $B(l^2(\Gamma_0 \setminus \Gamma))$ by the matrix unit

$$(e_{\Gamma_0 s_i}, e_{\Gamma_0 s_j})_{i,j=1,2,\ldots,[\Gamma:\Gamma_0]}.$$ 

For $\Gamma_0 \in \mathcal{S}$ and $\sigma \in G$ the Hecke operators associated with $\pi_0$ that correspond to the double coset $\Gamma_0 \sigma \Gamma_0$ are

$$[\Gamma_0 : (\Gamma_0)\sigma] P_{H_0^{\Gamma_0}} \pi_0^p(\sigma) P_{H_0^{\Gamma_0}}. \quad (3)$$

In view of the unitary equivalence

$$B(l^2(\Gamma_0 \setminus \Gamma)) \otimes B(L) \cong B(H_0^{\Gamma_0})$$

and the above choice of a matrix unit, the Hecke operators in (3) are unitarily equivalent to the operators in $B(l^2(\Gamma_0 \setminus \Gamma)) \otimes B(L)$ given by

$$\sum_{i,j} \sum_{\theta \in s_i^{-1} \Gamma_0 \sigma \Gamma_0 s_j} P_L \pi_0(\theta) P_L \otimes e_{\Gamma_0 s_i, \Gamma_0 s_j}. \quad (4)$$

The formula (4) lets us immediately deduce the following.

Corollary 3. For $\sigma \in G$ and $\Gamma_0 \in \mathcal{S}$ the trace of the Hecke operator in (3) is computed by the formula

$$\text{Tr}[\Gamma_0 : (\Gamma_0)\sigma] P_{H_0^{\Gamma_0}} \pi_0^p(\sigma) P_{H_0^{\Gamma_0}} = \sum_{s_i} \sum_{\theta \in s_i \Gamma_0 \sigma \Gamma_0 s_i^{-1}} \text{Tr}(P_L \pi_0(\theta) P_L). \quad (5)$$

When we specialize, for example, to the case $\Gamma_0 = \Gamma$, the formula (4) gives the following.

Assume that $\dim_{\{\pi_0(\Gamma)\}''} H_0$ is finite. The space $H_0^{\Gamma}$ of ‘virtual’ vectors invariant with respect to the unitary representation $\pi_0|_{\Gamma}$ acting on $H_0$ is unitarily equivalent to the range of the projection

$$\mathcal{P}_{\Gamma,L} = \sum_{\gamma \in \Gamma} P_L \pi_0(\gamma) P_L \in B(L),$$

which is a finite-dimensional subspace of $L$. The same unitary equivalence transforms the Hecke operator corresponding to a coset $\Gamma \sigma \Gamma$ and acting on ‘virtual’ $\Gamma$-invariant operators into the operator

$$\sum_{\gamma \in \Gamma \sigma \Gamma} P_L \pi_0(\gamma) P_L \in B(L), \quad \sigma \in G.$$
As a corollary of (5), we obtain a formula for the character ([22], [45], [12], [17]) of the representation \( \pi_0^p \). This proves that the character of \( \pi_0^p \) is determined by the positive-definite function \( \phi_0 \) on \( G \) given by
\[
\phi_0(g) = \text{Tr}(P_L \pi_0(g) P_L), \quad g \in G.
\] (6)

We use the notation and definitions in Theorem 2. Assume that the character \( \theta_{\pi_0^p} \) of the representation \( \pi_0^p \) of \( G \) is locally integrable with respect to Haar measure on \( G \). We recall that by [12] and (13) (for a different proof see also Lemma 36) we have
\[
\theta_{\pi_0^p}(\sigma) = \lim_{\Gamma_0 \downarrow \epsilon, \Gamma_0 \in \mathcal{S}} \text{Tr}(P_{H_0} \pi_0^p(\sigma) P_{H_0}).
\] (7)

For every \( g \in G \) let \( \Gamma^g \) be the stabilizer group of \( g \) in \( \Gamma \), defined by
\[
\Gamma^g = \{ \gamma \in \Gamma \mid \gamma g = g \gamma \}.
\]

Using the formulae (5) and (7), we obtain the following result.

**Corollary 4.** The value of the character \( \theta_{\pi_0^p} \) on an element \( \sigma \in G \) is computed by the formula
\[
\theta_{\pi_0^p}(\sigma) = \lim_{\Gamma_0 \downarrow \epsilon, \Gamma_0 \in \mathcal{S}} \frac{1}{[\Gamma_0 : (\Gamma_0)\sigma]} \sum_{\Gamma = \bigcup_{i=1}^{|\Gamma_0|} \Gamma_0 \sigma_i} \sum_{\theta \in s_t^{-1} \Gamma_0 \sigma \Gamma_0 s_i} \text{Tr}(P_L \pi_0(\theta) P_L).
\] (8)

If the group \( \Gamma^g \) is trivial, then
\[
\theta_{\pi_0^p}(g) = \lim_{\Gamma_0 \downarrow \epsilon, \Gamma_0 \in \mathcal{S}} \frac{1}{[\Gamma_0 : (\Gamma_0)g]} \sum_{\gamma \in \Gamma} \text{Tr}(P_L \pi_0(\gamma g \gamma^{-1})), \quad g \in G.
\] (9)

Under suitable conditions, the term on the right-hand side of (9) coincides with the character of the unitary representation \( \pi_0^p \). Assume that there is a unitary representation \( \pi_0^R \) extending \( \pi_0 \) to a locally compact group \( \overline{G}^R \) that contains \( G \) as a dense subgroup. Then the formula for the trace of the character of the representation \( \pi_0^p \) depends only on the trace of the character of \( \pi_0^R \).

**Lemma 5.** Consider a unitary representation \( \pi_0 \) satisfying the conditions in Definition 1. Let \( \overline{G}^R \) be a larger locally compact group containing \( G \) as a dense subgroup and such that \( \Gamma \) is a lattice in \( \overline{G}^R \). Assume the following conditions:

(a) \( \pi_0 \) extends to a unitary representation \( \pi_0^R \) of \( \overline{G}^R \) into the unitary group of \( H \), and moreover, \( \pi_0^R \) has a locally integrable (distributional) character [17] \( \text{Tr}(\pi_0^R(\cdot)) \);

(b) the representation \( \pi \) in Definition 1 also extends to a unitary representation \( \pi^R \) of \( \overline{G}^R \), and the equality \( \pi_0^R(g) = P_0 \pi^R P_0 \) holds for all \( g \in \overline{G}^R \);

(c) the set of elements \( g \in \overline{G}^R \) with non-trivial group \( \Gamma^g \) has zero measure with respect to Haar measure on \( \overline{G}^R \).
Then:

(i) for any element $g \in G$ as in assumption (c),

$$
\theta_{\pi_0^R}(g) = \sum_{\gamma \in \Gamma} \text{Tr}_{B(L)}(P_L \pi_0(\gamma g \gamma^{-1}) P_L);
$$

(ii) the character of $\pi_0^P$ when restricted to $G$ (with the exception of the points $g \in G$ such that $\Gamma^g$ is non-trivial) depends only on the character of $\pi_0^R$.

Remark 6. We consider the sum appearing in (8):

$$
\phi_{\Gamma_0}(\sigma) = \sum_{\Gamma \in [\Gamma : \Gamma_0]} \sum_{\theta \in \Gamma_0^{-1} \Gamma_0 \sigma \Gamma_0 \Gamma_0} \text{Tr}(P_L \pi_0(\theta) P_L), \quad \sigma \in G.
$$

When $\sigma = e$ and $\Gamma_0$ is a normal subgroup, this term is equal to

$$
[\Gamma : \Gamma_0] \sum_{\Gamma = \bigcup_{i=1}^{[\Gamma : \Gamma_0]} \Gamma_i \Gamma_0} \sum_{\theta \in \Gamma_0} \text{Tr}(P_L \pi_0(\theta) P_L) = [\Gamma : \Gamma_0] \sum_{\gamma \in \Gamma_0} \text{Tr}(P_L \pi_0(\gamma) P_L).
$$

Normalizing and taking the limit

$$
\Phi_{\pi_0}(\sigma) = \lim_{\Gamma_0 \downarrow \mathcal{G}, \dim H_{\Gamma_0}} \frac{\phi_{\Gamma_0}(\sigma)}{\dim H_{\Gamma_0}}, \quad \sigma \in G,
$$

we obtain a character of $G$ of the type considered in [33], [16], [9], [8], and [52].

The characters in Corollary 4 are of a different nature: they take an infinite value at the identity and possibly take an infinite value at other elements of the group.

Below we illustrate Theorem 2 and Corollary 4 in the particular case where the $\Gamma$-invariant vectors are automorphic forms. Let $G = \text{PGL}(2, \mathbb{Z}[1/p])$, let $p$ be a prime number, let $\Gamma$ be the modular group, and let the representation $\pi_0 = \pi_n|_G$, $n \in \mathbb{N}$, $n \geq 2$, be obtained by restricting to $G$ a (projective) unitary representation in the analytic discrete series $(\pi_n)_{n \geq 2}$ of the semisimple Lie group $\text{PSL}(2, \mathbb{R})$. Let $F$ be a fundamental domain for the action of the modular group on the upper half-plane. Let $d\nu_0(z) = (\text{Im } z)^{-2} d\bar{z} dz$ be the canonical measure on $\mathbb{H}$ which is invariant under the action of $\text{PSL}(2, \mathbb{R})$ by Möbius transformations. Then the projection $P_L$ in Definition 1 is the operator $M_{\chi_F}$ of multiplication by the characteristic function of $F$ on $H = L^2(\mathbb{H}, (\text{Im } z)^{-2} d\bar{z} dz)$. The unitary representation $\pi$ is the Koopman unitary representation of $G$ on

$$
L^2(\mathbb{H}, (\text{Im } z)^{-2} d\bar{z} dz)
$$

corresponding to the action of $G$ on $\mathbb{H}$ (see Examples 28 and 29).

In this case the projection $P_0$ is the Bergman projection onto the Hilbert space of $\pi_n$, which is the space of analytic functions on $\mathbb{H}$ that are square-integrable with respect to the measure $d\nu_n(z) = (\text{Im } z)^{n-2} d\bar{z} dz$. Moreover, the technical condition (v) in Definition 1 is equivalent to the $L^2$-convergence condition for the Berezin reproducing kernels [3] of the operators in the sum in (4). This condition holds in the particular case described here, by virtue of computations in [53] and [19], §3.3.

For a bounded operator $A$ on the Hilbert space $H_n$ we denote its Berezin symbol [3] by $\hat{A}(\bar{z}, \zeta)$, $z, \zeta \in \mathbb{H}$. The formula (9) then gives us the following result.
Corollary 7. Let $\sigma$ be an element in $G$ with trivial group $\Gamma^\text{st}_\sigma$. Then

$$
\theta_{\pi_n}(\sigma) = \left[ \lim_{\Gamma_0 \to 1, \Gamma_0 \in \mathcal{G}} \frac{1}{[\Gamma_0 : (\Gamma_0)_g]} \right] \int_{\mathbb{H}} \pi_n(\sigma)(z) \, d\nu_0(z)
$$

(12)

$$
= \left[ \lim_{\Gamma_0 \to 1, \Gamma_0 \in \mathcal{G}} \frac{1}{[\Gamma_0 : (\Gamma_0)_\sigma]} \right] \int_{\mathbb{H}} \frac{1}{\sigma z - z} \, d\nu_0(z)
$$

$$
= \left[ \lim_{\Gamma_0 \to \infty, \Gamma_0 \in \mathcal{G}} \frac{1}{[\Gamma_0 : (\Gamma_0)_\sigma]} \right] \theta_{\pi_n}(\sigma).
$$

The fact that the integral $\int_{\mathbb{H}} \pi_n(\sigma)(z) \, d\nu_0(z)$ in (12) is the character $\theta_{\pi_n}$ of the representation $\pi_n$ is proved in [32] using the Berezin quantization. In this context, the formula for the sum in (9) is computed by a different method in [53].

In §7 we apply the construction of $\Gamma$-invariant ‘virtual’ vectors in Theorem 2 to diagonal representations of $G$ of the form $\pi_0 \otimes \pi_0^{\text{op}}$ acting on the Hilbert space $H_0$. Here $\pi_0^{\text{op}}$ is the complex-conjugate representation associated with the representation $\pi_0$. We use a unitarily equivalent representation on the Hilbert spaces consisting of $\Gamma$-invariant vectors.

This representation is unitarily equivalent to the unitary representation $\text{Ad} \pi_0$ acting on the Hilbert space that is the ideal $\mathcal{C}_2(H_0) \subseteq B(H_0)$ of Hilbert–Schmidt operators. The space $\mathcal{H}^{\Gamma_0}$ of ‘virtual’ $\Gamma_0$-invariant vectors is the von Neumann algebra of operators $X \in B(H_0)$ that commute with $\pi_0(\Gamma_0)$. This algebra is (see Example 30) the commutant algebra $\mathcal{A}_0 = \{\pi_0(\Gamma_0)\}^\prime$.

For a type II factor $M$ with trace $\tau$, we denote by $L^2(M, \tau)$ the Hilbert space associated with $\tau$ through the Gelfand–Naimark–Segal construction [50]. This Hilbert space is obtained via hilbertian completion from the scalar product induced by the trace on the vector space $M$. By assumption, $\dim_{\{\pi_0(\Gamma_0)\}} H_0$ is finite, hence we conclude [50] that $\mathcal{A}_0$ is a type II von Neumann algebra. Then the Hilbert space of $\Gamma_0$-invariant vectors is the $L^2$-space $L^2(\mathcal{A}_0, \tau)$ associated with the type II von Neumann factor $\mathcal{A}_0$. The inductive limit of these Hilbert spaces when $\Gamma_0$ runs over $\mathcal{G}$ has a natural interpretation in terms of the Jones basic construction (Example 30; see also the construction in [39]).

This is particularly interesting when $\pi_0$ is the representation $\pi_n$ mentioned above, $(G = \text{PGL}(2, \mathbb{Z}[1/p]), p$ a prime, $\Gamma$ the modular group, $\pi_n$ obtained by restriction to $G$ of the discrete series of unitary representations of $\text{PSL}(2, \mathbb{R}))$. Consider the unitary Koopman representation $\pi_{\text{Koop}}$ (see Example 28) associated with the measure-preserving action of $G$ on $\mathbb{H}$, endowed with the measure $\nu_0$ introduced above.

Then by [42] or by the Berezin quantization theory ([3]; see also [37]), up to a unitary conjugation we have

$$
\pi_{\text{Koop}} \cong \pi_n \otimes \pi_n^{\text{op}}, \quad n \geq 2.
$$

(13)

Because of the above equivalence, the analysis of the representation of $G$ on the spaces of ‘virtual’ $\Gamma_0$-invariant vectors for the representation $\pi_n \otimes \pi_n^{\text{op}}$ leads to an analysis of the spaces of $\Gamma_0$-invariant vectors, $\Gamma_0 \in \mathcal{G}$, for the Koopman representation, and of the corresponding unitary action of $G$ on these spaces. The
latter representation corresponds to the action of the Hecke operators on Maass forms \[31\]. Using this method, we obtain in §7 (see also \[37\]) concrete algebraic formulae relating the matrix coefficients of the representation \(\pi_n\) with the expression for the Hecke operators associated with \(\pi_{\text{Koop}}\).

This method is useful in understanding the Koopman unitary representation \(\pi_{\text{Koop}}\), from which the action of the Hecke operators on Maass forms is derived. For the representation \(\pi_{\text{Koop}}\) we are exploiting a natural ‘square root’ given by the representation in the discrete series as in \(13\). The representation \(\pi_n \otimes \pi_n^{op}\) is easier to work with, because with the help of the operator-algebra interpretation the Hilbert spaces of ‘virtual’ \(\Gamma_0\)-invariant vectors, \(\Gamma_0 \in \mathcal{S}\), are canonically defined. We explain this construction below.

Let \(\mathcal{R}(G)\) be the von Neumann algebra generated by the right convolution operators \(\rho(g), g \in G\), acting on \(L^2(G)\). This is the commutant algebra of the algebra \(L(G)\) of left convolutors generated by the left convolution operators \(\lambda_g, g \in G\), acting on the same Hilbert space.

We consider the following algebras associated with the inclusions \(\Gamma \subseteq G, K \subseteq \mathcal{G}\).

- Let \(\mathcal{H}_0(K, \mathcal{G}) = \mathbb{C}(K \setminus \mathcal{G}/K)\) be the algebra of double cosets of the subgroup \(K\) of \(\mathcal{G}\) (\[1\], \[28\]). Denote by \(\mathcal{H}_0(G, \Gamma)\) the Hecke algebra \(\mathbb{C}(\Gamma \setminus G/\Gamma)\) generated by double cosets of the subgroup \(\Gamma\) of \(G\). The latter algebra is isomorphic to the Hecke algebra \(\mathcal{H}_0(K, \mathcal{G})\). The Hecke algebra has a natural involution operation which induces a \(*\)-algebra structure.

The Hecke algebra has a canonical faithful \(*\)-representation \[7\] into \(B(L^2(\Gamma \setminus G))\). We denote the elements of the standard basis of \(L^2(\Gamma \setminus G)\) by \([\Gamma\sigma], \sigma \in G\). The hypothesis \([\Gamma : \Gamma_\sigma] = [\Gamma : \Gamma_{\sigma \cdot 1}], \sigma \in G\), that we assumed on the equality of the indices has the effect that the state \(\langle \cdot, [\Gamma], [\Gamma] \rangle\) defined on \(B(L^2(\Gamma \setminus G))\) restricts to a trace on the image of the Hecke algebra. The reduced \(C^*\)-algebra of the Hecke algebra is the norm closure in the representation into \(B(L^2(\Gamma \setminus G))\) of the Hecke algebra \(\mathcal{H}_0(G, \Gamma)\). It will be denoted by \(\mathcal{H}_{\text{red}}(G, \Gamma)\). Clearly, \(\mathcal{H}_0(G, \Gamma) \cong \mathcal{H}_0(K, \mathcal{G})\).

The \(C^*\)-algebra \(\mathcal{A}(\mathcal{G})\) contains a canonical operator system \[35\] (which we introduce in Definition 8) associated with the maximal compact subgroup \(K\). We will explain below that \(*\)-representations of this operator system (see \(18\) in Lemma 9) contain all the information about the representation \(\pi_0\). These morphisms are the basic building blocks of the Hecke algebra representation associated with a representation of the form \(\pi_0 \otimes \pi_0^{op}\) (see Theorems 10 and 12). The essential tool for studying unitary representations of \(G\) of the form \(\pi_0 \otimes \pi_0^{op} \cong \text{Ad } \pi_0\) is a canonical representation (to be described below) of the Hecke algebra \(\mathcal{H}_0(G, \Gamma)\) into \(\mathcal{R}(G) \otimes B(L)\).

**Definition 8.** Let \(L(K, \mathcal{G}) = \mathbb{C}(\chi_{\sigma K} \mid \sigma \in G)\) be the linear subspace of \(C^*(\mathcal{G})\) generated by the characteristic functions of right cosets. Then \(L(K, \mathcal{G})^* = \mathbb{C}(\chi_{K\sigma} \mid \sigma \in G)\). We also consider the space

\[
\tilde{L}(K, \mathcal{G}) = L^\infty(\mathcal{G}, \mu)L(K, \mathcal{G}) \subseteq C^*(\mathcal{G} \rtimes L^\infty(\mathcal{G}, \mu)).
\]

Consider the following operator systems \[35\]:

\[
\mathcal{O}(K, \mathcal{G}) = L(K, \mathcal{G})(L(K, \mathcal{G}))^* \subseteq C^*(\mathcal{G}),
\]

\[
\tilde{\mathcal{O}}(K, \mathcal{G}) = \tilde{L}(K, \mathcal{G})(\tilde{L}(K, \mathcal{G}))^* \subseteq C^*(\mathcal{G} \rtimes L^\infty(\mathcal{G}, \mu)).
\]
In the above formulae the product is calculated in the $C^*$-algebras $C^*(\mathcal{G})$ and $C^*(\mathcal{G} \rtimes L^\infty(\mathcal{G}, \mu))$, respectively.

Then, clearly:

(i) $\mathcal{O}(K, \mathcal{G})$ is linearly generated by the characteristic functions of the form $\chi_{\sigma_1 K \sigma_2}$, $\sigma_1, \sigma_2 \in G$;

(ii) $L(K, \mathcal{G})$ and $(L(K, \mathcal{G}))^*$ are subspaces of $\mathcal{O}(K, \mathcal{G})$;

(iii) there is a canonical pairing

$$L(K, \mathcal{G}) \times L(K, \mathcal{G}) \to \mathcal{O}(K, \mathcal{G})$$

that maps $x, y \in L(K, \mathcal{G})$ into $xy^* \in \mathcal{O}(K, \mathcal{G})$.

Let $\mathcal{E}$ be a $C^*$-algebra. We call a linear map $\Phi : \mathcal{O}(K, \mathcal{G}) \to \mathcal{E}$ a $*$-representation of the operator system $\mathcal{O}(K, \mathcal{G})$ if

$$\Phi(xy^*) = \Phi(x)(\Phi(y))^*, \quad x, y \in L(K, \mathcal{G}).$$

A $*$-representation for the operator system $\tilde{\mathcal{O}}(K, \mathcal{G})$ is linear as a bimodule over the algebra $L^\infty(\mathcal{G}, \mu)$.

The Hecke algebra $\mathcal{H}_0(K, \mathcal{G})$ is the intersection $L(K, \mathcal{G}) \cap (L(K, \mathcal{G}))^*$. It is clearly closed with respect to the above pairing operation. Obviously, a $*$-representation of the operator system $\mathcal{O}(K, \mathcal{G})$ becomes a $*$-algebra representation when restricted to the Hecke algebra.

Let $G, \Gamma, \pi, \pi_0, P_0$, and $P_L$ be as in Definition 1. For the statements in the rest of the Introduction we assume for simplicity that the groups $\Gamma$ and $G$ have infinite non-trivial conjugacy classes. This assumption implies that the von Neumann algebras associated below with the above groups have unique traces.

We construct $*$-representations of the operator system $\tilde{\mathcal{O}}(K, \mathcal{G})$ that take values in the associated von Neumann algebras described below. Let us consider the following von Neumann algebras:

$$\mathcal{A} = \{\pi(\Gamma)\}' \cong \mathcal{B}(\Gamma) \bigotimes B(L) \subseteq \mathcal{B} = \mathcal{B}(G) \bigotimes B(L),$$

$$\mathcal{A}_0 = \pi_0(\Gamma)' = P_0 \mathcal{A} P_0.$$

We note that $\mathcal{A}$ and $\mathcal{B}$ are type II von Neumann algebras. If the space $L$ is infinite, then $\mathcal{A}$ and $\mathcal{B}$ are type $\text{II}_\infty$ von Neumann algebras. In the representation for $\{\pi(\Gamma)\}'$ in (16), the projection $P_0$, which by hypothesis commutes with $\pi(\Gamma)$ and thus belongs to $\mathcal{A}$, has the formula

$$P_0 = \sum_{\gamma \in \Gamma} \rho(\gamma) \otimes P_L \pi_0(\gamma) P_L \in \mathcal{C}_1(\mathcal{A}).$$

For a given von Neumann algebra $\mathcal{M}$ endowed with a faithful semifinite trace $T = T_{\mathcal{M}}$, we denote by $\mathcal{C}_1(\mathcal{M})$ the ideal of trace class operators associated with $\mathcal{M}$ [50].

For the operator system we associate with the unitary representation $\pi_0$ the $*$-representation constructed in Definition 8.

**Lemma 9.** Assume that $G, \Gamma, \pi_0, P_0$, and $P_L$ are as in Definition 1. For a coset $C = g \Gamma_0$ in $\mathcal{G}$ let

$$\tilde{\Phi}_{\pi_0, L}(C) = \sum_{\theta \in C} \rho(\theta) \otimes P_L \pi_0(\theta) P_L \in \mathcal{C}_1(\mathcal{B}),$$
by analogy with the formula (17). Then the following assertions hold.

(i) The restriction of the map $\tilde{\Phi}_{\pi_0,L}$ to $\mathscr{O}(K,G)$ is a $*$-representation of the operator system $\mathscr{O}(K,G)$. In particular,

$$\tilde{\Phi}_{\pi_0,L}(\chi_{\sigma_1,K})[\tilde{\Phi}_{\pi_0,L}(\chi_{\sigma_2,K})]^* = \tilde{\Phi}_{\pi_0,L}(\chi_{\sigma_1,K}\sigma_2^{-1}), \quad \sigma_1, \sigma_2 \in G. \quad (18)$$

(ii) By (17), $\tilde{\Phi}_{\pi_0,L}(\chi_K) = P_0$, which in view of (i) implies that

$$\tilde{\Phi}_{\pi_0,L}(\chi_{\sigma_1,K}) = \tilde{\Phi}_{\pi_0,L}(\chi_{\sigma_1,K}P_0), \quad \tilde{\Phi}_{\pi_0,L}(\chi_0K) = P_0\tilde{\Phi}_{\pi_0,L}(\chi_0K), \quad \sigma_1 \in G.$$

(iii) By (ii), the restriction of $\tilde{\Phi}_{\pi_0,L}$ to the Hecke algebra $\mathscr{H}_0(K,G) \cong \mathscr{H}_0(\Gamma,G)$ takes values in the algebra $\mathscr{A}_0 = P_0\mathscr{B}P_0$.

(iv) $\tilde{\Phi}_{\pi_0,L}|_{\mathscr{H}_0(K,G)}$ is trace preserving, and hence it extends continuously to a $C^*$-representation of the reduced $C^*$-Hecke algebra $\mathscr{H}_{\text{red}}(\Gamma,G)$ with values in $\mathscr{A}_0$.

The $*$-algebra representation of the Hecke algebra in (iii) is used to describe the Hecke operators associated with the diagonal unitary representation $\pi_0 \otimes \pi_0^\text{op}$ of $G$, on $\Gamma$-invariant vectors. As explained above, the latter representation is unitarily equivalent to $\text{Ad} \pi_0$, and the Hilbert space of $\Gamma$-invariant vectors can be canonically identified with the $L^2$-space [50] associated with the commutant $\{\pi_\sigma(\Gamma)\}^\prime$ of the von Neumann algebra. The formula for the Hecke operators is computed in the next theorem. The case where $\dim_{\mathbb{C}} L = 1$ was used in [37] to obtain estimates of the essential spectrum of Hecke operators acting on Maass forms. The general case of an arbitrary Murray–von Neumann dimension is treated in [38].

The Hecke operators are automatically completely positive maps. They are obtained using von Neumann algebra expectations [44], [50]. Let

$$E_{P_0(\mathscr{A}(G) \otimes B(L))P_0}P_0(\mathscr{A}(\Gamma) \otimes B(L))P_0$$

be the canonical normal conditional expectation mapping the type $\Pi_1$ factor $P_0(\mathscr{A}(G) \otimes B(L))P_0$ onto the subfactor $P_0(\mathscr{A}(\Gamma) \otimes B(L))P_0$.

**Theorem 10.** The Hecke operator $\Psi([\Gamma\sigma\Gamma])$ associated with the representation $\text{Ad} \pi_0$ and corresponding to the coset $[\Gamma\sigma\Gamma]$ for $\sigma$ in $G$ is a selfadjoint operator acting on the space $L^2(\mathscr{A}_0, \tau) = L^2(\{\pi_0(\Gamma)\}^\prime, \tau)$.

Then $\Psi([\Gamma\sigma\Gamma])$ is determined by its values on the algebra $\mathscr{A}_0$. For $\sigma \in G$ the Hecke operator $\Psi([\Gamma\sigma\Gamma])$ associates with

$$X \in \{\pi_0(\Gamma)\}^\prime = \mathscr{A}_0 = P_0\mathscr{B}P_0 = P_0(\mathscr{A}(\Gamma) \otimes B(L))P_0$$

the operator

$$\Psi([\Gamma\sigma\Gamma])(X) = E_{P_0(\mathscr{A}(G) \otimes B(L))P_0}P_0(\tilde{\Phi}_{\pi_0,L}(\Gamma\sigma\Gamma)X(\tilde{\Phi}_{\pi_0,L}(\Gamma\sigma\Gamma))^*). \quad (19)$$

This is Theorem 3.2 in [38], which generalizes results in [37]. The statement is adapted to the framework of the present paper. The present formalism proves that once the space $L$ has been chosen, the Hecke algebra representation is canonical.

Below we construct a canonical $*$-representation $\overline{\mathcal{D}\Phi}$ of the Hecke algebra $\mathscr{H}_0(K,G)$ into a canonical ‘double’ algebra (see below) such that all the
representations of the Hecke algebra as in (19) are obtained by composing \( \mathcal{D} \Phi \) with a quotient map.

We will perform the construction of the representation \( \mathcal{D} \Phi \) only in the case where the von Neumann dimension \( \dim \{ \pi_0(\Gamma) \}' \) \( H_0 \) is equal to 1. In this context \( \mathcal{A}_0 \) is simply the algebra \( \mathcal{A}(\Gamma) \). The argument can easily be extended to cover the case of von Neumann dimension greater than 1, but the formulae become more complicated.

The fact that the von Neumann dimension of the type II von Neumann algebra generated by \( \pi_0(\Gamma) \) is 1 means that the representation \( \pi_0 |_{\Gamma} \) admits a cyclic trace vector \( \zeta \). In this case, the \( * \)-representation \( e \Phi \pi_0, L \) in Lemma 9 is replaced by a representation \( t \) (constructed below) of the canonical system in Definition 8, which takes values in \( \mathcal{L}(G) \). We recall that the algebra \( \mathcal{L}(G) \) is anti-isomorphic to \( \mathcal{R}(G) \).

For \( \sigma_1, \sigma_2 \in G \) and for a subset of \( G \) of the form \( A = \sigma_1 \Gamma \sigma_2 \), we define
\[
(t(\sigma_1 K \sigma_2))(X) = \sum_{\theta \in A} \langle \pi_0(\theta) \zeta, \zeta \rangle \lambda_\theta \in \mathcal{L}(G).
\] (20)

Then letting the space \( L \) be \( \mathbb{C} \zeta \), we deduce from Lemma 9 (see Remark 42) that \( t \) defines a \( * \)-representation of the operator system \( \mathcal{O}(K, G) \) with values in the von Neumann algebra \( \mathcal{L}(G) \) (the complex conjugation in (20) is due to the fact that, via the canonical anti-isomorphism, we are switching from the algebra \( \mathcal{R}(G) \) to the algebra \( \mathcal{L}(G) \)).

The Hecke algebra in (19) of the \( * \)-representation \( \Psi \) now takes values in \( \mathcal{L}(\Gamma) \) and is given by the formula
\[
\Psi([\Gamma \sigma \Gamma])(X) = E_{\mathcal{L}(\Gamma)}(t(\Gamma \sigma \Gamma)X(t(\Gamma \sigma \Gamma))^*), \quad \sigma \in G.
\] (21)

The linear operator \( \Psi([\Gamma \sigma \Gamma]) \), \( \sigma \in G \), obviously extends to the Gelfand–Naimark–Segal space \( \ell^2(\Gamma) \) (see, for example, [50]) taken with respect to the canonical trace.

The Hecke operators \( \Psi([\Gamma \sigma \Gamma]) \), \( \sigma \in G \), are all completely positive and unital. Hence, the representation in (21) extends, by forgetting the algebra structure on the range space, to a \( * \)-representation \( \Psi_0 \) of \( \mathcal{A}_0(G, \Gamma) \) with values in \( B(\ell^2(\Gamma) \ominus \mathbb{C}1) \). As explained in Example 29 (see also [37], [39]), the validity of the Ramanujan–Petersson conjecture on Maass forms, in the particular example of \( G = \text{PGL}(2, \mathbb{Z}[1/p]) \), is equivalent to the representation \( \Psi_0 \) being extendible to a representation of the reduced \( C^* \)-Hecke algebra \( \mathcal{H}_{\text{red}}(G, \Gamma) \).

We will prove that the representation \( \Psi \) in (21) of the Hecke algebra is obtained from a canonical representation \( \mathcal{D} \Phi \) of the Hecke algebra.

We introduce the following two crossed product \( C^* \)-algebras. They are two variable extensions of the Roe algebras [43], [6].

**Definition 11.** Consider the following crossed product \( C^* \)-algebras:
\[
\mathcal{D} \mathcal{R}(G) = \chi_K(C^*(((G \times G^{\text{op}}) \rtimes L^\infty(\mathcal{G}, \mu)))\chi_K),
\] (22)
\[
\mathcal{D} \mathcal{R}(G) = \chi_K(C^*(((G \times \overline{G}^{\text{op}}) \rtimes L^\infty(\mathcal{G}, \mu)))\chi_K).
\] (23)

The \( C^* \)-algebra \( \mathcal{D} \mathcal{R}(G) \) has a canonical, Koopman-type, representation \( \alpha \) into \( B(\ell^2(\Gamma)) \) obtained as follows: let \( G \times G^{\text{op}} \) act as a groupoid on \( \Gamma \) by left and right
multiplication, the operation being defined whenever the result of the multiplication belongs to $\Gamma$. On the other hand, $C(K)$ acts by multiplication on $l^\infty(\Gamma)$ and hence on $\ell^2(\Gamma)$.

We observe that the two algebras in the above definition are corners (reduced by the projection $\chi_K \in L^\infty(\overline{G}, \mu)$) in the larger crossed product $C^*$-algebras corresponding to the measure-preserving actions of $G \times G^{\text{op}}$ and $\overline{G} \times \overline{G}^{\text{op}}$ on $\overline{G}$, respectively. Since the action is measure preserving, these two algebras have obvious reduced crossed product $C^*$-algebra counterparts.

In Theorem 12 below we prove that the Hecke algebra representation $\Psi$ in (21), which is in fact the Hecke algebra representation associated with the representation $\pi_0 \otimes \pi_0^{\text{op}}$, is obtained from an intrinsic representation of the Hecke algebra, denoted by $\mathcal{D}\Phi$, with values in the ‘double’ algebra $\mathcal{D}\mathcal{B}(G)$. To obtain the Hecke algebra representation $\Psi$, one composes the $*$-algebra representation $D\Phi$ with the Koopman-type representation $\alpha$ of the algebra $\mathcal{D}R(G)$ in Definition 11.

Since the $*$-algebra representation $D\Phi$ extends (because it preserves the trace) to the reduced $C^*$-Hecke algebra, the obstruction (if any) to extending the representation $\Psi_0$ to the reduced $C^*$-Hecke algebra $\mathcal{H}\text{red}(G, \Gamma)$ (which, as explained above, is the obstruction to the validity of the Ramanujan–Petersson conjecture on Maass forms in the case of the representation $\pi_n$ [37]) lies in the intersection of the kernel of the above canonical representation $\alpha$ with the image of the representation $D\Phi$.

For a function $f$ on $G$ we denote the corresponding convolution element in $C^*(G)$ by $L(f)$.

**Theorem 12.** (i) The correspondence
\[
[K\sigma K] \rightarrow \chi_K(L(\chi_{K\sigma K}) \otimes L(\chi_{K\sigma K}^{\text{op}}))\chi_K, \quad \sigma \in G,
\]
extends by linearity to a $*$-algebra representation $\overline{D\Phi}$ of the Hecke algebra $\mathcal{H}_0(K, G)$ into $\mathcal{D}\mathcal{B}(\overline{G})$. This representation is trace preserving, and thus obviously extends to $\mathcal{H}\text{red}(\Gamma, G)$ when the values are considered in the reduced crossed product $C^*$-algebra associated with $\mathcal{D}\mathcal{B}(\overline{G})$.

(ii) The $*$-representation $t$ in (20) of the operator system $\mathcal{O}(K, \overline{G})$ extends to a $\mathcal{D}\mathcal{B}(G)$-valued $*$-representation $t_2$ of an operator system $\mathcal{O}$ contained in $\mathcal{D}\mathcal{B}(\overline{G})$. The operator system $\mathcal{O}$ contains the image of $D\Phi$.

(iii) The composition $D\Phi = t_2 \circ \overline{D\Phi}$, which acts according to the formula
\[
[\Gamma\sigma\Gamma] \rightarrow \chi_K(t(\Gamma\sigma\Gamma) \otimes (t(\Gamma\sigma\Gamma))^{\text{op}})\chi_K,
\]
extends to a $*$-algebra representation of the Hecke algebra $\mathcal{H}\text{red}(\Gamma, G)$ into the reduced crossed product $C^*$-algebra associated with $\mathcal{D}\mathcal{B}(G)$.

(iv) Let $\alpha$ be as in Definition 11, and let $\Psi$ be the Hecke algebra representation in (21). Then
\[
\Psi = \alpha \circ D\Phi = \alpha \circ t_2 \circ \overline{D\Phi}.
\]

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2. Outline of the paper

We outline the construction of Hilbert spaces of ‘virtual’ $\Gamma$-invariant vectors and of the unitary action of $G$ on the inductive limit of these spaces. Recall from the Introduction that $\Gamma$ is an almost-normal subgroup of $G$: that is, the group $\Gamma_\sigma = \sigma \Gamma \sigma^{-1} \cap \Gamma$ has finite index in $\Gamma$ for all $\sigma \in G$. We assume in this paper that the values of the group indices $[\Gamma : \Gamma_\sigma]$ and $[\Gamma : \Gamma_\sigma^{-1}]$ are equal for all $\sigma \in G$. This assumption is automatic if $\Gamma$ is a group with infinite non-trivial conjugacy classes when there is a representation $\pi_0$ as in Definition 1 (for example, by the Jones index theory [25]). We also assume that the family $\mathcal{S}$ generated through the intersection operation by the subgroups $\Gamma_\sigma$ for $\sigma \in G$ separates points of $\Gamma$. Let $K$ be the profinite completion of $\Gamma$ with respect to the family $\mathcal{S}$.

We first consider a unitary representation $\pi$ of $G$ having the properties (i) and (ii) of Definition 1. In this case $\pi|_\Gamma$ is an integer multiple of the regular representation. To construct $\Gamma$-invariant vectors, we assume that $H$ is contained in a larger vector space $\mathcal{Y}$, and that $\pi$ extends to a representation $\pi_\mathcal{Y}$ of $G$ into the linear isomorphism group of $\mathcal{Y}$. We assume that the space $\mathcal{Y}^\Gamma$ consisting of vectors in $\mathcal{Y}$ fixed by $\pi_\mathcal{Y}(\Gamma)$ is non-trivial. To construct the representation $\pi^P$ one considers simultaneously the spaces $\mathcal{Y}^{\Gamma_0}$ of vectors in $\mathcal{Y}$ fixed by the action of $\pi_\mathcal{Y}(\Gamma_0)$, where $\Gamma_0$ is any group conjugate in $G$ to a subgroup in $\mathcal{S}$. We note that $\pi_\mathcal{Y}(\sigma)$ takes the vector space $\mathcal{Y}^{\Gamma_0}$ into $\mathcal{Y}^{\sigma \Gamma_0 \sigma^{-1}}$ for $\sigma \in G$.

On the spaces $\mathcal{Y}^{\Gamma_0}$ we construct a Hilbert space structure which determines Hilbert spaces $H^{\Gamma_0}$ of $\Gamma_0$-fixed vectors for $\Gamma_0 \in \mathcal{S}$. We impose the following condition on the scalar product in the spaces of invariant vectors: the inclusions

$$
H^{\Gamma_0} \subseteq H^{\Gamma_1}, \quad \Gamma_0, \Gamma_1 \in \mathcal{S}, \quad \Gamma_1 \subseteq \Gamma_0,
$$

are isometric, and $\pi_\mathcal{Y}(\sigma)$ maps $H^{\Gamma_0 \cap \Gamma_\sigma^{-1}}$ isometrically onto $H^{\sigma \Gamma_0 \sigma^{-1} \cap \Gamma_\sigma}$ for $\sigma \in G$.

By this procedure we get a prehilbertian space structure on the union $\bigvee_{\Gamma_0 \in \mathcal{S}} H^{\Gamma_0}$ in (35). Let $\overline{H^P}$ be the Hilbert space completion of this union. Clearly, $\pi_\mathcal{Y}$ induces a representation of $G$ into the isomorphisms of $\bigvee_{\Gamma_0 \in \mathcal{S}} \mathcal{Y}^{\Gamma_0}$.

The conditions we are imposing on the scalar products in the Hilbert spaces $H^{\Gamma_0}$, $\Gamma_0 \in \mathcal{S}$, imply that $\pi_\mathcal{Y}$ induces a unitary representation $\pi^P$ of $G$ into the unitary group $\mathcal{U}(\overline{H^P})$ of the Hilbert space $\overline{H^P}$. The notation $\pi^P$ stands for the above completion operation, involving passing from $\pi$ to the unitary representation $\pi^P$ on the Hilbert space completion of the union of the spaces of $\Gamma_0$ invariant vectors, $\Gamma_0 \in \mathcal{S}$.

We recall that the Schlichting completion $\overline{\mathcal{G}}$ (see, for example, [48], [51], [26], [29]) of the discrete group $G$ with respect to the subgroups in $\mathcal{S}$ is the locally compact, totally disconnected group obtained as the disjoint union of the double
cosets $K\sigma K$ with the obvious multiplication relation, where $\Gamma \sigma \Gamma$ runs over a set of representatives of double cosets for $\Gamma$ in $G$ (see also [4], [7], [21]).

Let $d_\pi = \dim (\pi(\Gamma))'' H_0 \in \mathbb{N}$ be the von Neumann dimension of $H_0$ as a module over the type II factor $\{\pi(\Gamma)\}''$ (see [19], § 3.3, for definitions and notation). We prove (Lemma 26) that the left regular representation $\lambda_K$ of $K$ into the unitary group $\mathcal{U}(L^2(K, \mu))$ has multiplicity $d_\pi$ in $\pi^p|_K$.

Let $\mu$ be the normalized Haar measure on $K$ and extend it on the locally compact group $\overline{G}$ to the Haar measure $\mu$, normalized by $\mu(K) = 1$. The assumption that $[\Gamma : \Gamma_\sigma] = [\Gamma : \Gamma_{\sigma^{-1}}]$ for all $\sigma$ in $G$, implies that the measure $\mu$ is bivariant under left and right translations by elements in the group $\overline{G}$.

We extend the construction of the above unitary representation to the case where $\pi_0$ is a unitary representation as in Definition 1. In this case the unitary representation $\pi_0|_{\Gamma}$ is no longer a multiple of the left regular representation of $\Gamma$, but it is a subrepresentation of a larger representation $\pi$ with the above properties.

We prove that the construction described above to obtain the unitary representation $\pi^p$ can be repeated for $\pi_0$. We get a unitary representation $\pi_0^p$ of $\overline{G}$ that is associated with the action of the original group $G$ on the spaces of $\Gamma_0$-invariant vectors, $\Gamma_0 \in \mathcal{S}$. The representation $\pi_0^p$ extends naturally to the $C^*$-algebra of the locally compact group $\overline{G}$. It also extends to a $C^*$-algebra representation of the full $C^*$-algebra associated simultaneously with the groups $G$ and $\overline{G}$ (see Definition 16).

In Theorem 34 we note that the ‘block-matrix coefficients’ of the representation $\pi_0^p$ are the Hecke operators associated with $\pi_0^p$ and corresponding to the level $\Gamma_0$, where the group $\Gamma_0 \in \mathcal{S}$ is determined by the size of the ‘block’. We use this in Theorem 2 to determine the explicit formula (see (4)) for the Hecke operators in terms of the data from the representation $\pi_0$.

One outcome of this paper is the relation between the representations $\pi_0$ and $\pi_0^p$. The representation $\pi_0^p$ is a type I representation of the group $\overline{G}$. Hence, it has an associated character ([22], [45], [17]), which we denote by $\theta_{\pi_0}^p = \text{Tr}\pi_0^p$. In Corollary 4 we prove that the values $\theta_{\pi_0^p}(g)$ computed on $g \in G$ are determined by summing the values of the positive-definite function $\phi_0$ on $G$ introduced in (6) over cosets. The same positive-definite function also determines the formula for ‘$\text{Tr} \pi_0(g)$’ for the original representation $\pi_0$.

The basic model for the unitary representations considered above is a representation of the spaces of $\Gamma_0$-invariant vectors in the case of the left regular representation $\lambda_G$ of $G$ into $\mathcal{U}(l^2(G))$ (see a more detailed exposition in Example 27 in § 4). In this case the Hilbert spaces $H^1_0$, $\Gamma_0 \in \mathcal{S}$, are the Hilbert spaces $l^2(\Gamma_0 \setminus G)$ with scalar product normalized so that the inclusions $l^2(\Gamma_0 \setminus G) \subseteq l^2(\Gamma_1 \setminus G)$ are isometric for $\Gamma_1 \subseteq \Gamma_0$. The Hilbert space $H^0_0$ is simply $L^2(\overline{G}, \mu)$, and in this particular case the representation $\pi^p$ is simply the left regular representation of $\overline{G}$ into the unitary group $\mathcal{U}(L^2(\overline{G}, \mu))$.

We recall (see § 1) that the Hecke algebra $\mathcal{H}_0(\Gamma, G)$ of double cosets of $\Gamma$ in $G$ has a canonical $*$-algebra embedding into $B(l^2(\Gamma \setminus G))$. Its closure in the uniform norm is a $C^*$-algebra $\mathcal{H}_\text{red}(\Gamma, G)$, called the reduced $C^*$-Hecke algebra by analogy with the reduced $C^*$-algebra of a discrete group ([7], [20], [51], [2], [14]). The representation

$$\mathcal{H}_\text{red}(\Gamma, G) \subseteq B(l^2(\Gamma \setminus G))$$
is called the left regular representation of the Hecke algebra. The commutant is generated by the right quasi-regular representation $\rho r_\Gamma G$ of $G$ into the unitary group of $L^2(\Gamma \setminus G)$ (see, for example, [7]).

The main thread of the Ramanujan–Petersson problem is the determination of bounds on the growth of the matrix coefficients and eigenvalues for Hecke algebra representations associated with a unitary representation $\pi$ of $G$ as above on Hilbert spaces of $\Gamma$-invariant vectors. The Ramanujan–Petersson conjecture is asking in fact (see [37]) when the matrix coefficients of the representation of $\mathcal{H}_0(\Gamma, G)$ associated with the unitary representation $\pi$ are weak limits of convex combinations of matrix coefficients of the Hecke algebra coming from its left regular representation.

In our terminology this is equivalent to determining when the spherical functions (matrix coefficients corresponding to vectors fixed by $K$) associated with the representation $\pi_0^p$ of $\mathcal{G}$ are weakly contained in the left regular representation of the Hecke algebra. Clearly, this is equivalent to $\pi_0^p$ being weakly contained in the left regular representation of the group $\mathcal{G}$ acting on $L^2(\mathcal{G}, \mu)$ by left translations. Deligne [15] proved the Ramanujan–Petersson conjecture for automorphic forms. For Maass forms [31] the general problem is open (see [49], [47], [24], [11], [37]). We formulate the following problem.

**Problem** (generalized Ramanujan–Petersson problem). Find conditions on the representation $\pi_0^p$ such that $\pi_0^p$ is weakly contained in the left regular representation $\lambda_{\mathcal{G}}$ of $\mathcal{G}$ on $L^2(\mathcal{G}, \mu)$. (It is enough to find conditions on the unitary representation $\pi$ of $G$ such that the unitary representation $\pi_0^p|_G$ of $G$ is weakly contained in the unitary representation $\lambda_{\mathcal{G}}|_G$.)

In the case of the unitary representation $\pi_n$ of $G = \text{PGL}(2, \mathbb{Z}[1/p])$ ($p$ a prime number, $n \in \mathbb{N}$) obtained by restriction from the discrete series of representations of $\text{PSL}(2, \mathbb{R})$, the representations $\pi_n^p$ contain all the group harmonic analysis information about the spaces of automorphic forms (here the group $\Gamma$ is $\text{PSL}(2, \mathbb{Z})$). The spherical matrix coefficients of $\pi_n^p$ encode the information about the eigenvalues of the Hecke operators.

Below we present examples of representations $\pi$ and the associated representations $\pi^p$. For a more detailed exposition see §4. The easiest case is when the unitary representation $\pi$ indicated above also has the property that $\pi|_{\Gamma}$ is an integer multiple of the left regular representation of $\Gamma$. In the terminology of the Murray–von Neumann dimension (see [19], §3.3), this is the case when $\dim(\pi(\Gamma))'' H$ is an integer. In this case, as explained above, there is a Hilbert subspace $L$ of $H$ with the property in the following definition.

**Definition 13.** Let $\pi$ be a unitary representation of $G$ as above, and consider a subspace $L$ of $H$ such that $\pi(\gamma)L$ is orthogonal to $L$ for $\gamma \neq e$. We call such a space a $\Gamma$-wandering subspace for $\pi$. If, in addition, $H = \bigvee_{\gamma \in \Gamma} \pi(\gamma)L$, then we call such a subspace a $\Gamma$-wandering generating subspace.

In the above context, the Hilbert spaces of $\Gamma_0$-invariant vectors are canonically identified with the Hilbert spaces $L \otimes L^2(\Gamma_0 \setminus G)$, $\Gamma_0 \in \mathcal{S}$. Recall that $K$ is the profinite completion of $\Gamma$ with respect to the subgroups in $\mathcal{S}$. The space $\overline{H}^p$ is identified with $L \otimes L^2(K, \mu)$. The problem is to identify the representation of the groups $G$ and $\mathcal{G}$ on $\overline{H}^p$. 
One case (see Example 28) in which the above situation occurs is when $(\mathcal{X}, \nu)$ is an infinite measure space on which $G$ acts by measure-preserving transformations. In this case $H = L^2(\mathcal{X}, \nu)$ and $\pi$ is the Koopman representation $\pi_{\text{Koop}}$ (see, for example, [27]):

\[(\pi_{\text{Koop}}(g)f)(x) = f(g^{-1}x), \quad x \in \mathcal{X}, \quad g \in G, \quad f \in L^2(\mathcal{X}, \nu). \quad (25)\]

We assume that the restriction of the action of $G$ to $\Gamma$ admits a fundamental domain $F$. Then we can take $L = L^2(F, \nu|_F)$. This is a $\Gamma$-wandering generating subspace, associated with the representation $\pi_{\text{Koop}}$. Then $\dim_{\{\pi_{\text{Koop}}(\Gamma)\}''} H = \infty$. In this case $H^p = L^2(F, \nu|_F) \otimes L^2(K, \mu)$. The representation $\pi_{\text{Koop}}^p$ is determined by the $\Gamma$-valued cocycle on $G \times F$ determined by the action of $G$ on $\mathcal{X}$ under the identification $\mathcal{X} \cong F \times \Gamma$. To obtain the representation $\pi_{\text{Koop}}^p$, one views this cocycle as having values in $K$.

Assume that $\pi_0$ is a representation of $G$ with the properties in Definition 1. This is typically the case when $\dim_{\{\pi_0(\Gamma)\}''} H$ is not an integer. Such a situation occurs when the $\Gamma$-invariant vectors are automorphic forms.

In this case $G = \text{PGL}(2, \mathbb{Z}[1/p])$, $p$ is a prime, $\Gamma$ is the modular group, and $H_n = H^2(\mathbb{H}, \nu_n)$. Here $\nu_n$ is the measure $(\text{Im} z)^{n-2} d\bar{z} dz$ on the upper half-plane $\mathbb{H}$. The representations $\pi_0 = \pi_n$ are obtained by restricting to $G$ the representations in the discrete series $(\pi_n)_{n \in \mathbb{N}, n \geq 2}$ of unitary representations of $\text{PSL}(2, \mathbb{R})$ (see, for example, [30]). In this case there is no canonical wandering subspace $L$, since

\[\dim_{\{\pi_0(\Gamma)\}''} H_n = \dim_{\Gamma} H_n = \frac{n-1}{12},\]

as proved in [19], § 3.3.d. The reason for the previous assertion about non-existence is the fact that if such a space $L$ exists, then

\[\dim_{\mathbb{C}} L = \dim_{\Gamma} H_n,\]

which is impossible if $(n - 1)/12$ is not an integer.

In the theory of automorphic forms, the problem of constructing Hilbert spaces of $\Gamma$-invariant vectors is solved by using a fundamental domain and the Petersson scalar product [34], which involves integration over the fundamental domain. In the framework of this paper, we replace the integration over a fundamental domain by the action of the projection $P_L$ onto the space $L$.

One assumes that there is a unitary representation $\hat{\pi}_n$ on a larger Hilbert space and that it contains $\pi_n$ as a subrepresentation as in Definition 1. In this case (see Example 29) the larger Hilbert space is $H = L^2(\mathbb{H}, \nu_n)$. The unitary representation $\hat{\pi}_n$ acts on functions on $\mathbb{H}$ by the same formula as $\pi_n$. The invariance property of the measure $\nu_n$ implies, as in the case of $\pi_n$, that $\hat{\pi}_n$ is a unitary representation of $G$.

We use the notation of Definition 1. Let $P_0$ be the orthogonal projection from $L^2(\mathbb{H}, \nu_n)$ onto the space $H_n$ of square-integrable analytic functions. We have $[P_0, \hat{\pi}_n(g)] = 0$ for all $g \in G$. Hence $\pi_n(g) = P_0 \hat{\pi}_n(g) P_0$ for $g \in G$. We take the space $L = L^2(F, \nu_n)$ as a canonical choice for the $\Gamma$-wandering generating subspace $L$ for $\hat{\pi}_n$. Recall that $P_L$ is the orthogonal projection onto $L$. It is obvious
that in this case \( P_L \) is precisely \( M_{\chi_F} \), the operator of multiplication by the characteristic function of the fundamental domain \( F \), acting on \( L^2(\mathbb{H}, \nu_n) \).

Computations in §3.3 of [19] imply that the product \( P_0 M_{\chi_F} \) is a trace class operator, with trace equal to the Murray–von Neumann dimension \( \dim \{ \pi_n(\Gamma) \}' H_n \).

To define the space of \( \Gamma \)-invariant vectors abstractly, we make use of the relative position (the operator angle) of the projections \( P_0 \) and \( M_{\chi_F} \). In this situation the technical condition (Definition 1) is the convergence of the series

\[
\sum_{\theta \in \Gamma \sigma \Gamma} P_L \pi_n(\theta) P_L, \quad \sigma \in G,
\]

in the space of Hilbert–Schmidt operators.

In the present example, this condition holds because the reproducing kernel for the projection onto the space of automorphic forms and the reproducing kernels of the associated Hecke operators are sums of the operator kernels (Berezin reproducing kernels [3], [36]) of the operators

\[
\sum_{\theta \in \Gamma \sigma \Gamma} M_{\chi_F} \pi_n(\theta) M_{\chi_F}, \quad \sigma \in G. \tag{26}
\]

Moreover, the sum of the traces of the corresponding operators is also absolutely convergent. This follows from computations in [53] and [19], §3.3.

We prove in Theorem 34 that the sum in (26) is a projection when the sum is taken over \( \Gamma \) (for example, with \( \sigma \) the identity element). We prove that the range of this projection (which is a subspace of \( L \)) is unitarily equivalent to the Hilbert space of \( \Gamma \)-invariant vectors. Moreover, the same unitary equivalence will transform the Hecke operator corresponding to a double coset \( [\Gamma \sigma \Gamma] \) into the sum in (26).

The advantage of this point of view on spaces of automorphic forms is that the formula (26) allows a direct computation of the traces of the Hecke operators on any level \( \Gamma_0 \in \mathcal{S} \). This is used to compute (Corollary 7) the values of the character \( \theta_{\pi_n}(g) \), \( g \in G \). These values are partial sums of the traces of operators as in (26) (see Theorem 34 and Remark 35).

The construction of the spaces of \( \Gamma \)-invariant vectors for the unitary representation \( \pi_{\text{Koop}}^p \) can be obtained in an alternative manner if the above representation admits a ‘square root’ as described below (see Example 30 in §4). Let the infinite measure space be \( \mathbb{H} \), endowed with the PSL\((2, \mathbb{R})\)-invariant measure \( d\nu_0 = (\text{Im } z)^{-2} d\bar{z} dz \). Let \( \pi = \pi_{\text{Koop}} \) be the corresponding Koopman unitary representation of PSL\((2, \mathbb{R})\) into the unitary group of \( L^2(\mathbb{H}, \nu_0) \). We denote by \( \pi_{n}^{\text{op}} \) the conjugate representation of \( \pi_n \). By the Berezin quantization method (see [42], [3]), the representation \( \pi_{\text{Koop}} \) factorizes as

\[
\pi_{\text{Koop}} \cong \pi_n \otimes \pi_n^{\text{op}}.
\]

As in the previous example, we let \( G = \text{PGL}(2, \mathbb{Z}[1/p]) \), \( p \) a prime, and let \( \Gamma \) be the modular group. The factorization of the representation \( \pi_{\text{Koop}} \) gives a canonical choice of the Hilbert spaces of \( \Gamma_0 \)-invariant vectors, \( \Gamma_0 \in \mathcal{S} \). Indeed, the representation \( \pi_n \otimes \pi_n^{\text{op}} \) is unitarily equivalent to the adjoint representation \( \text{Ad } \pi_n(g) \) into the unitary group of the Hilbert–Schmidt operators \( \mathcal{C}_2(H_n) \cong H_n \otimes \overline{H_n}^p \).
The larger vector space containing $\mathcal{C}_2(H_n)$ is $\mathcal{V} = B(H_n)$, the space of bounded linear operators on $H_n$. Then the adjoint representation $\text{Ad} \pi_n(g)$ extends to a representation into the inner automorphism group of $B(H_n)$. In this situation the space $\mathcal{V}^\Gamma_0$ of $\Gamma_0$-invariant vectors is the type II$_1$ factor $A_n(\Gamma_0) = \{ \pi_n(\Gamma_0) \}' = \{ X \in B(H_n) \mid [X, \pi_n(\gamma)] = 0, \ \gamma \in \Gamma_0 \}.

The fact that the commutant algebra $A_n(\Gamma_0)$ is a type II$_1$ factor is a consequence of the fact that $\dim_\Gamma H_n$ is finite (see [19], §3.3.d).

Then the Hilbert space $H^{\Gamma_0}$ is simply $L^2(A_n(\Gamma_0), \tau)$, the Gelfand–Naimark–Segal Hilbert space associated with the unique trace $\tau$ on $A_n(\Gamma_0)$. The family

$$\{ A_n(\Gamma_0) \}_{\Gamma_0 \in \mathcal{S}}$$

is a directed family of type II$_1$ factors. Let $A_n^\infty$ be the type II$_1$ factor obtained as the inductive limit of this directed family. We also denote by $\tau$ the unique trace on $A_n^\infty$.

Then the space $H^p$ is $L^2(A_n^\infty, \tau)$, and $\overline{\text{Ad} \pi_n^p}$ is the extension of $\text{Ad} \pi_n(g)$. In Theorem 10 (Theorem 3.2 in [38]) we proved that the $K$-spherical matrix coefficients for $\overline{\text{Ad} \pi_n^p}$ are explicitly computed from a $C^*$-representation determined by the $K$-spherical matrix coefficients for $\pi_n^p$. This representation is in fact the main algebraic tool in [37].

Again let $\pi_0$ be a representation of $G$ as in Definition 1. In all the constructions above, the main building block for the representations $\pi_0^p$ is a completely positive map $\Phi$ (see Theorem 37) supported on $C^*(G)$ with values in $B(L)$, and extending to $C^*(\tilde{G})$. The map $\Phi$ encodes the sums in (26). We extend $\Phi$ to $C^*(\tilde{G})$ by defining, for the characteristic function of a closed subset $C$ of $\tilde{G}$,

$$\Phi(\chi_C) = \sum_{\theta \in C} P_L \pi(\theta) P_L.$$

Then $\Phi$ is a completely positive map on $C^*(\tilde{G})$ with values in $B(L)$, and $\Phi$ is a $*$-preserving multiplicative representation of the operator system (Definition 13)

$$\mathcal{O}(K, G) = [C(\chi_{\sigma K} \mid \sigma \in G)] \cdot [C(\chi_{\sigma K} \mid \sigma \in G)]^* \subseteq C^*(\tilde{G});$$

the $*$-preserving, multiplicative property means that for any two $K$-cosets $K\sigma_1$ and $K\sigma_2$ in $\tilde{G}$ we have

$$\Phi(\chi_{K\sigma_1})^* \Phi(\chi_{K\sigma_2}) = \Phi(\chi_{\sigma_1 K}) \Phi(\chi_{K\sigma_2}) = \Phi(\chi_{\sigma_1 K\sigma_2}).$$

We prove in Proposition 40 that the representation $\pi_0^p$ is entirely reconstructible from the completely positive map $\Phi$.

Then $\Phi$ is an ‘operator-valued eigenvector’ for the Hecke algebra. Indeed, by the multiplicitivity property we get that

$$\Phi(\chi_{K\sigma_1 K}) \Phi(\chi_{K \sigma_2}) = \Phi(\chi_{K \sigma_1 K} \cdot \chi_{K \sigma_2}), \quad \sigma_1, \sigma_2 \in G,$$

where $\cdot$ denotes the convolution operation on functions on $\tilde{G}$. 

3. Axioms for constructing the Hilbert spaces of $\Gamma$-invariant vectors

Let $\Gamma \subseteq G$ be an almost-normal subgroup as in the Introduction, satisfying the condition $[\Gamma : \Gamma_\sigma] = [\Gamma : \Gamma_{\sigma^{-1}}]$ for all $\sigma$ in $G$. Let $\pi$ be a (projective) unitary representation of $G$ into the unitary group $U(H)$ of a Hilbert space $H$, with the properties (i) and (ii) in Definition 1. In particular, $\dim \{\pi(\Gamma)^*\} H$ is an integer or $\infty$. As observed in the previous section, this implies the existence of a $\Gamma$-wandering generating subspace $L$ for $\pi|_{\Gamma}$ (Definition 13). We recall that this means that $L$ is orthogonal to $\pi(\gamma)L$ for $\gamma \in \Gamma$ with $\gamma \neq e$, and that $H = \bigvee_{\gamma \in \Gamma} \pi(\gamma)L$.

We construct the Hilbert spaces $H^{T_0}$ of $\Gamma_0$-invariant vectors, $\Gamma_0 \in \mathcal{S}$. These spaces $H^{T_0}$ will be isometrically isomorphic to $l^2(\Gamma_0 \setminus \Gamma) \otimes L$ for $\Gamma_0 \in \mathcal{S}$. The main problem that we first consider in this section is to construct the representation of $G$ on the union of the spaces of $\Gamma_0$-invariant vectors.

The particular examples presented in the previous section suggest that one possible way of addressing this problem is to find an embedding of the Hilbert space $H$ into a larger vector space $V$ such that the representation $\pi$ extends to a representation $\pi_V$ of $G$ into the group of linear isomorphisms of $V$ that leaves the subspace $H$ of $V$ invariant. We explain the construction first in this case, then perform the construction based on properties (i) and (ii) in Definition 1.

We will work with subgroups $\Gamma_0$ that are conjugate in $G$ to subgroups in $\mathcal{S}$. We denote this enlarged class of subgroups of $G$ by $\mathcal{S}'$.

For $\Gamma_0 \in \mathcal{S}'$ we consider the spaces $\mathcal{V}^{T_0}$ of $\pi_{\mathcal{V}}(\Gamma_0)$-invariant vectors in $\mathcal{V}$. Then $\pi_{\mathcal{V}}$ has an obvious extension to $\mathcal{V}_\infty = \bigvee_{\Gamma_0 \in \mathcal{S}'} \mathcal{V}^{T_0} = \bigvee_{\Gamma_0 \in \mathcal{S}} \mathcal{V}^{T_0}$.

The remaining problem is to identify a $\pi_{\mathcal{V}}(G)$-invariant subspace of $\mathcal{V}_\infty$ that is endowed with a $\pi_{\mathcal{V}}(G)$-invariant prehilbertian scalar product. Using this scalar product, we define the Hilbert spaces $H^{T_0} \subseteq \mathcal{V}^{T_0}$, $\Gamma_0 \in \mathcal{S}'$. Then $\pi_{\mathcal{V}}$ induces a unitary representation of $G$ on $\bigvee_{\Gamma_0 \in \mathcal{S}'} H^{T_0}$.

The construction in this section is certainly similar to other constructions in the literature (see, for example, [6], [21]). However, in Theorem 22 we employ this construction to introduce specific $*-$representations of the Hecke algebra that involve expressions as in (26). These are generalized in the next section to the case when $\dim \pi(\Gamma) H$ is not an integer and hence when there is no $\Gamma$-wandering generating subspace.

In the following definition, we introduce a general formalism which is used to construct the representation $\pi^p$, starting with an extension of the given representation $\pi$ to a larger vector space that contains vectors invariant under the action of the subgroups in $\mathcal{S}$. In practice, as will be done later in this section, we construct directly the Hilbert spaces corresponding to ‘virtual’ $\Gamma_0$-invariant vectors, $\Gamma_0 \in \mathcal{S}$.

Definition 14 (formalism of $\Gamma$-invariant vectors). Let $\Gamma \subseteq G$ be as in the Introduction, and consider a (possibly projective) representation $\pi$ of $G$ into the unitary group of a Hilbert space $H$. We make the following assumptions.

(i) There exist a larger vector space $\mathcal{V}$ containing $H$ and a representation $\pi_{\mathcal{V}}$ of $G$ into the linear isomorphisms of $\mathcal{V}$ such that $\pi_{\mathcal{V}}(g)$ leaves $H$ invariant and
\[ \pi_V(g)|_H = \pi(g) \text{ for all } g \in G. \]  As above, for \( \Gamma_0 \) in \( \mathcal{S} \) we denote by \( \mathcal{Y}^{\Gamma_0} \) the subspace of \( \mathcal{Y} \) consisting of vectors fixed by the action of \( \Gamma_0 \). We consider the vector space \( \mathcal{Y}_\infty \) in (28).

(ii) There exist a dense \( \pi(G) \)-invariant subspace \( \mathcal{D}_\mathcal{Y} \subseteq H \), and a complex-valued bilinear form \( \langle \cdot, \cdot \rangle_\infty \) on

\[ \mathcal{Y}_\infty \times (\mathcal{Y}_\infty \lor \mathcal{D}_\mathcal{Y}) \]

with the following properties:

(ii.1) the restriction of \( \langle \cdot, \cdot \rangle_\infty \) to \( \mathcal{Y}_\infty \times \mathcal{Y}_\infty \) is a positive-definite prehilbertian scalar product;

(ii.2) for every \( \Gamma_0 \in \mathcal{S} \) and \( v \in \mathcal{Y}^{\Gamma_0} \) the linear map on \( \mathcal{D}_\mathcal{Y} \) defined by the restriction of the linear form \( \langle v, \cdot \rangle_\infty \) to \( \mathcal{D}_\mathcal{Y} \) is \( \Gamma_0 \)-invariant;

(ii.3) \( \langle \cdot, \cdot \rangle_\infty \) is \( \pi_V(G) \)-invariant, that is,

\[ \langle \pi_V(g)v_1, \pi_V(g)v_2 \rangle_\infty = \langle v_1, v_2 \rangle_\infty, \quad g \in G, \quad v_1 \in \mathcal{Y}_\infty, \quad v_2 \in \mathcal{Y}_\infty \lor \mathcal{D}_\mathcal{Y}. \]

If the above assumptions hold, then we denote by \( H^{\Gamma_0} \) the Hilbert space completion of \( \mathcal{Y}^{\Gamma_0} \) with respect to the scalar product \( \langle \cdot, \cdot \rangle_\infty \), and by \( \overline{H}^\Gamma \) the Hilbert space completion of \( \mathcal{Y}_\infty \) with respect to this scalar product.

The following lemma is an obvious consequence of the assumptions in Definition 14.

**Lemma 15.** Assume the conditions of Definition 14. Then the restriction of \( \pi_\mathcal{Y} \) to \( \mathcal{Y}_\infty \) extends to a unitary representation \( \pi^\mathcal{P} \) of \( G \) into the unitary group of \( \overline{H}^\Gamma \). Moreover, \( \pi^\mathcal{P} \) maps \( H^{\Gamma_0} \) isometrically onto \( H^{\Gamma_0\sigma \Gamma_0^{-1}} \) for \( \Gamma_0 \in \mathcal{S} \) and \( \sigma \in G \).

If \( \Gamma_1 \subseteq \Gamma_0, \Gamma_1, \Gamma_0 \in \mathcal{S} \), then the inclusion of \( H^{\Gamma_0} \) into \( H^{\Gamma_1} \) is isometric by construction. The orthogonal projection from \( H^{\Gamma_1} \) onto \( H^{\Gamma_0} \) is obtained by averaging over the cosets of \( \Gamma_0 \) in \( \Gamma_1 \).

If the original representation \( \pi \) is projective (see, for example, [5] and the references therein) with a cocycle \( \epsilon \in H^2(G, \mathbb{T}) \), then assuming that the extension \( \pi_\mathcal{Y} \) has the same cocycle, the above construction still works.

In what follows we will work with simultaneous representations of the group \( G \) and of its Schlichting completion \( \overline{G} \) [48]. Consequently, we introduce a universal \( C^* \)-algebra containing both \( C^*(G) \) and \( C^*(\overline{G}) \) as \( C^* \)-subalgebras.

**Definition 16.** With \( G \) and \( \overline{G} \) as above, let \( \mathcal{A}(G, \overline{G}) \) be the quotient of the universal crossed product \( C^* \)-algebra \( C^*(G \rtimes C^*(\overline{G})) \), where \( G \) acts by conjugation on \( C^*(\overline{G}) \) by the norm-closed ideal generated by the relations of the form

\[ g\chi_{K_0} = \chi_{gK_0}, \quad g \in G, \quad K_0 = \overline{\Gamma_0}, \quad \Gamma_0 \in \mathcal{S}. \]

Here \( \chi_{gK_0} \) denotes the characteristic function of the coset

\[ gK_0 = \overline{g\Gamma_0}, \]

where the closure operation is in \( \overline{G} \).

Then \( \mathcal{A}(G, \overline{G}) \) is the norm closure of the span

\[ \text{Sp}\{g\chi_{K_0} \mid g \in G, \quad K_0 = \overline{\Gamma_0}, \quad \Gamma_0 \in \mathcal{S}\}. \]
Assume that $\epsilon \in H^2(G, \mathbb{T})$ is a cocycle that also extends to $H^2(\overline{G}, \mathbb{T})$. Then, working with crossed products with the cocycle, we obtain a similar $C^*$-algebra, which we denote by $\mathcal{A}_\epsilon(G, \overline{G})$.

Using the previous two definitions, we prove that the representation $\pi^p$ extends simultaneously to $G$ and $\overline{G}$.

**Proposition 17.** For a representation $\pi$ as in Definition 14, the corresponding representation $\pi^p$ in Lemma 15 extends to a representation, also denoted by $\pi^p$, of the $C^*$-algebra $\mathcal{A}_\epsilon(G, \overline{G})$ into $B(H^p)$.

**Proof.** Let $\Gamma_0$ be a subgroup of $G$ belonging to the class $\widetilde{\mathcal{S}}$. Let $K_0 = \Gamma_0$, where the closure is taken in the topology of $\overline{G}$. Let $P_{H^p_{\Gamma_0}}$ be the orthogonal projection of $H^p$ onto the Hilbert space $H^p_{\Gamma_0}$. The extended representation $\pi^p$ is constructed by the map $\frac{1}{\mu(K_0)} \chi_{K_0}$. The normalization is necessary because in $C^*(\overline{G})$ the convolutor with a subgroup $K_0$ of $K$ is a non-trivial scalar multiple of a projection, since $(\chi_{K_0})^2 = \mu(K_0) \chi_{K_0}$. The elements in $G$ are represented on $H^p$ as unitary operators via the representation $\pi^p$ in Lemma 15. With this choice, all the relations defining the universal $C^*$-algebra $\mathcal{A}_\epsilon(G, \overline{G})$ obviously hold. $\square$

Given a representation $\pi$ of $G$ such that $\pi|_\Gamma$ admits a $\Gamma$-wandering generating subspace, we construct a representation as in Definition 14. We will construct the Hilbert spaces of $H^p_{\Gamma_0}$-invariant vectors directly, without constructing the space $\mathcal{V}$ in Definition 14.

**Lemma 18.** Let $\pi$ be a unitary representation of $G$ with the properties (i) and (ii) in Definition 1. In particular, $\pi|_{\Gamma}$ is an integer multiple of the left regular representation $\lambda_{\Gamma}$. With the notation in that definition the following assertions hold.

(i) For all $\Gamma_0 \in \mathcal{S}$ and $g \in G$, the sum

$$\sum_{\theta \in \Gamma_0 g} P_L \pi(\theta) P_L$$

over the coset $\Gamma_0 g$ is so-convergent (convergent in the strong-operator topology) in $B(L)$.

(ii) The subspace

$$\mathcal{D}_{L, \pi} = \left\{ h \in H \left| \sum_{\gamma \in \Gamma_0} P_L \pi(\gamma) h \text{ is so-convergent for all } \Gamma_0 \in \mathcal{S} \right. \right\}$$

is a dense $\pi(G)$-invariant subspace of $H$ containing $L$.

**Proof.** To prove (i) we note that it is sufficient to prove the statement for $\Gamma_0 = \Gamma$, since $\pi|_{\Gamma_0}$ remains an integer multiple of the left regular representation $\lambda_{\Gamma_0}$ of $\Gamma_0$. If we sum over the double coset $\Gamma \sigma \Gamma$, then we obtain exactly the Hecke operator associated with the double coset, acting on the space $L$, which is unitarily equivalent to the space of $\Gamma$-invariant vectors.

In the case where the representation $\pi$ is as in Example 28 in §4, the sum in (29) coincides with the representation in the Koopman unitary representation...
of the piecewise bijective transformation $\hat{\Gamma}g$ in Lemma 4, (i) of [41]. The sum is so-convergent because we are adding partial isometries corresponding to transformations with disjoint domains. The operator associated with the piecewise bijective transformation $\hat{\Gamma}g$ is consequently [41] a finite sum (of cardinality $[\Gamma : \Gamma_g]$) of partial isometries with orthogonal initial spaces. This argument carries over word-for-word to our case because of the assumption (ii) in Definition 1. The key feature that makes the argument work is that $\pi(\sigma)L$ is a $\sigma\Gamma\sigma^{-1}$-wandering subspace.

For (ii) we note that to prove the density assumption it suffices to assume that the equality $H = \ell^2(\Gamma)$ holds in a $\Gamma$-equivariant way. In this case the domain $D_{\mathcal{V}}$ is simply $\ell^1(\Gamma) \cap \ell^2(\Gamma)$. The $\pi(G)$-invariance of $H$ is now a consequence of (i).

**Definition 19.** We use the notation and definitions introduced above. Let $\Gamma_0$ be a subgroup in $\mathcal{S}$. Assume that $\Gamma$ is decomposed into cosets with respect to $\Gamma_0$ as $\Gamma = \bigcup \Gamma_0 r_j$, where the $r_j$ are coset representatives for $\Gamma_0$. Let $L^{\Gamma_0}$ be the subspace of $H$ obtained as the following sum of orthogonal subspaces of $H$:

$$L^{\Gamma_0} = \sum \pi(r_j)L.$$  \(31\)

Denote the orthogonal projection from $H$ onto $L^{\Gamma_0}$ by $P_{L^{\Gamma_0}}$.

We define the Hilbert space $H^{\Gamma_0} = \mathcal{V}^{\Gamma_0}$ as the space of formal sums

$$\left\{ \sum_{\gamma_0 \in \Gamma_0} \pi(\gamma_0)h \mid h \in \mathcal{D}_{L,\pi} \right\}$$  \(32\)

subject to the identification

$$\sum_{\gamma_0 \in \Gamma_0} \pi(\gamma_0)h = \sum_{\gamma_0 \in \Gamma_0} \pi(\gamma_0)l_0,$$  \(33\)

if $h \in \mathcal{D}_{L,\pi}$ and $l_0$ is the vector in $L^{\Gamma_0}$ given by

$$l_0 = \sum_{\gamma_0 \in \Gamma_0} P_{L^{\Gamma_0}}(\pi(\gamma_0)h).$$  \(34\)

The infinite sum in (34) is convergent since $h$ belongs to $\mathcal{D}_{L,\pi}$.

The condition in (33) corresponds to the fact that the sum over $\Gamma_0$ is invariant under the change of the summation variable from $\gamma$ to $\gamma\gamma_0$ for a fixed $\gamma_0$ in $\Gamma_0$. This condition must necessarily hold if the vector $h$ is a sum of translates of vectors in $L^{\Gamma_0}$ by elements in $\Gamma_0$. Using the above construction, we can introduce a unitary representation of $G$ acting on vectors invariant with respect to subgroups in $\mathcal{S}$.

**Proposition 20.** The positive-definite prehilbertian scalar product $\langle \cdot, \cdot \rangle_\infty$ on

$$\mathcal{V}_\infty = \bigvee_{\Gamma_0 \in \mathcal{S}} H^{\Gamma_0}$$

is defined for $\Gamma_0 \in \mathcal{S}$, $l_1 \in \mathcal{D}_{\Gamma,\pi}$, and $l_2 \in L^{\Gamma_0}$ by the formula

$$\left\langle \sum_{\gamma_0 \in \Gamma_0} \pi(\gamma_0)l_1, \sum_{\gamma_0 \in \Gamma_0} \pi(\gamma_0)l_2 \right\rangle_\infty = \frac{1}{[\Gamma : \Gamma_0]} \left\langle \sum_{\gamma_0 \in \Gamma_0} \pi(\gamma_0)l_1, l_2 \right\rangle,$$  \(35\)
where the scalar product in $H$ is used on the right-hand side of this equality. Clearly, $\mathcal{V}^\Gamma_0$ is embedded isometrically into $\mathcal{V}^\Gamma_1$ for $\Gamma_1 \subseteq \Gamma_0$. Let $\overline{H}^p$ be the Hilbert space completion of $\mathcal{V}_\infty$ with respect to the scalar product $\langle \cdot , \cdot \rangle_\infty$.

Then the unitary representation $\pi$ determines a unitary representation $\pi^p$ into the unitary group of $\overline{H}^p$, and it has the properties in Lemma 15.

Before proving the proposition, we note that the formula (35), which is equivalent to (36) below, is a generalization of the Petersson scalar product formula [34]. Indeed, with the notation in the Introduction consider the case of two automorphic forms of weight $n \in \mathbb{N}$, which are thus $\Gamma$-invariant vectors, as above, for the representation $\pi_n$. To obtain the scalar product of the two automorphic forms, we multiply one of them by the characteristic function $\chi_F$ of a fundamental domain, and then use the usual scalar product in $L^2(\mathbb{H}, \nu_n)$, which extends the scalar product in $H_n$. This is exactly what is implemented in the next formula by replacing the operator $M_{\chi_F}$ (in (26)) by the projection $P_L$, which has properties similar to those of $M_{\chi_F}$.

One can establish an equivalent expression for (35) analogous to the Petersson scalar product formula. For $h_1, h_2 \in \mathcal{D}_{L, \pi}$ let

$$l_i = \sum_{\gamma \in \Gamma} P_L \pi(\gamma) h_i, \quad i = 1, 2.$$  

Using the identification in (33), we get that (35) is equivalent to

$$\left\langle \sum_{\gamma \in \Gamma} \pi(\gamma) h_1, \sum_{\gamma' \in \Gamma} \pi(\gamma') h_2 \right\rangle_{\infty} = \left\langle P_L \left( \sum_{\gamma \in \Gamma} \pi(\gamma) \right) h_1, \sum_{\gamma' \in \Gamma} \pi(\gamma') h_2 \right\rangle_{\infty}. \quad (36)$$

This is further equal to

$$\left\langle P_L \left( \sum_{\gamma \in \Gamma} \pi(\gamma) \right) h_1, P_L \left( \sum_{\gamma' \in \Gamma} \pi(\gamma') h_2 \right) \right\rangle = \langle l_1, l_2 \rangle.$$

For more general subgroups $\Gamma_0 \in \mathcal{S}$, the formula (36) for the scalar product is similar, with the difference that instead of $P_L$ one uses the projection $P_{L, \Gamma_0}$ onto a $\Gamma_0$-wandering generating subspace of $\pi$.

**Proof of Proposition 20.** Let $\Gamma_0, \Gamma_1$ be two subgroups in $\mathcal{S}$ such that $\Gamma_1 \subseteq \Gamma_0$. We first split $\Gamma_0$ into cosets with respect to $\Gamma_1$. Using coset representatives, we split the sum in the formula (32) for vectors in $\mathcal{V}^\Gamma_0$ into $[\Gamma_0 : \Gamma_1]$ vectors which all belong to $\mathcal{V}^\Gamma_1$. Then $\mathcal{V}^\Gamma_0$ is embedded into $\mathcal{V}^\Gamma_1$. Indeed, if $\Gamma_0 = \bigcup_j r_j \Gamma_1$, then the embedding is realized as follows: if

$$\eta = \sum_{\gamma_0 \in \Gamma_0} \pi(\gamma_0) l_0, \quad l_0 \in L^\Gamma_0,$$  

is a generic vector in $\mathcal{V}^\Gamma_0$, then we identify $\eta$ with the element

$$\eta_1 = \sum_j \sum_{\gamma_1 \in \Gamma_1} \pi(\gamma_1) l_0 = \sum_{\gamma_1 \in \Gamma_1} \pi(\gamma_1) \left[ \sum_j \pi(r_j) l_0 \right] \in \mathcal{V}^\Gamma_1$$

of $\mathcal{V}^\Gamma_1$. 

The embedding $\mathcal{W}^{\Gamma_0} \subseteq \mathcal{W}^{\Gamma_1}$ is isometric. Indeed, for $l_0$ in $L^{\Gamma_0}$ the square of the norm of the vector $\eta \in H^{\Gamma_0}$ in (37) is determined according to the formula (35) and is equal to

$$\frac{1}{[\Gamma : \Gamma_0]} \langle l_0, l_0 \rangle,$$

where the scalar product is computed in $H$.

On the other hand, according to the same formula the norm of the vector $\eta_1$ in $\mathcal{W}^{\Gamma_1}$ is

$$\frac{1}{[\Gamma : \Gamma_1]} \left[ \sum_j \pi(r_j) l_0, \sum_k \pi(r_k) l_0 \right].$$

The set $\{r_j\}$ has cardinality $[\Gamma_0 : \Gamma_1]$. Moreover, the vectors $\pi(r_j) l_0$ are pairwise orthogonal. Hence, the square of the norm of $\eta_1$ is

$$\frac{1}{[\Gamma : \Gamma_1]} [\Gamma_0 : \Gamma_1] \langle l_0, l_0 \rangle = [\Gamma : \Gamma_1] \langle l_0, l_0 \rangle,$$

and thus the embedding $\mathcal{W}^{\Gamma_0}$ into $\mathcal{W}^{\Gamma_1}$ is isometric.

The representation $\pi_{\mathcal{W}}$ is defined as follows. Let $g \in G$, $\Gamma_0 \in \mathcal{S}$, and $l \in L$, and consider the vector

$$\eta = \sum_{\gamma_0 \in \Gamma_0} \pi(\gamma_0) l \in \mathcal{W}^{\Gamma_0}.$$

Then we split the coset $g\Gamma_0$ as a disjoint union

$$g\Gamma_0 = \bigcup_j \Gamma_j^0 y_j$$

of cosets of smaller subgroups $\Gamma_j^0$ in $\mathcal{S}$ such that

$$g\Gamma_j^0 g^{-1} = \Gamma_1^j \subseteq \Gamma,$$

$\Gamma_1^j \in \mathcal{S}.$

This is always possible by considering cosets of $\Gamma_0$ with respect to subgroups of $\Gamma_0 \cap \Gamma g^{-1}$. Then we define

$$\pi_{\mathcal{W}}(g) \eta = \sum_j \sum_{\gamma_i \in \Gamma_1^j} \pi(\gamma_i) (\pi(g y_j) l).$$

(38)

By the assumptions on the domain in (30), it follows that $\pi_{\mathcal{W}}(g) \eta$ belongs to $\mathcal{W}_\infty$. This is because $\pi_{\mathcal{W}}(g)$ maps $\mathcal{W}_{\Gamma_0}^{\Gamma_j^0 \cap \Gamma g^{-1}}$ onto $\mathcal{W}_{\Gamma_4}^{\Gamma_j^0 \cap \Gamma g^{-1}}$. Since for all $g \in G$ the indices of the subgroups $\Gamma g^{-1}$ and $\Gamma g$ are equal, the definition of the scalar product proves that $\pi_{\mathcal{W}}$ maps $\mathcal{W}^{\Gamma_0}$ isometrically into $\mathcal{W}_\infty$. Obviously, if $g \in G$ and $\Gamma_0 \in \mathcal{S}$, then $\mathcal{W}_{\Gamma_0}^{\Gamma g^{-1}}$ is contained in $\mathcal{W}_{\Gamma g^{-1} \cap \Gamma g}$. But $g\Gamma_0 g^{-1} \cap \Gamma_0$ is a subgroup in $\mathcal{S}$, and hence we have the alternative formula $\mathcal{W}_\infty = \bigvee_{\Gamma_0 \in \mathcal{S}} H^{\Gamma_0}$.

Consequently, we obtain a unitary representation $\pi_{\mathcal{V}}$ into the unitary group of the Hilbert space $H^{\Gamma_0}$ as in Definition 14. $\square$
Recall that for $\Gamma_0$ in $\mathcal{S}$ and $\sigma \in G$, we use the notation
\[(\Gamma_0)_\sigma = \sigma \Gamma_0 \sigma^{-1} \cap \Gamma_0.\]
The index $[\Gamma_0 : (\Gamma_0)_\sigma]$ will be used in the following computations.

For $\Gamma_0$ as above let $K_0 = \overline{\Gamma_0}$ be the closure of $\Gamma_0$ in the profinite completion $K$ of $\Gamma$. The next statement describes the explicit form of the matrix representation of the image under $\overline{\pi}^p$ of the operator of convolution with the characteristic function of the double coset $K_0 \sigma K_0$. On $\Gamma_0$-invariant vectors this is obviously the Hecke operator associated with the double coset $\Gamma_0 \sigma \Gamma_0$ and normalized by a constant. We first determine the precise normalization constants required for the Hecke operators. The normalization factor obtained is the index of the subgroup $(\Gamma_0)_\sigma$ in $\Gamma_0$. It is necessary because in the $C^*$-algebra $C^*(\overline{G},G)$ if $K_0$ is a subgroup of $K$, then the operator of convolution with $\chi_{K_0 \sigma K_0}$ is the scalar multiple, by the factor $[K_0 : (K_0)_\sigma]$, of the ordered product of the operators of convolution with $\chi_{K_0}$, $\sigma$, and $\chi_{K_0}$.

**Lemma 21.** In the notation introduced above, for each $\sigma \in G$ and each subgroup $K_0$ as above,
\[\overline{\pi}^p(\chi_{K_0 \sigma K_0}) = [\Gamma : (\Gamma_0)_\sigma] \left( \overline{\pi}^p(\chi_{K_0}) \overline{\pi}^p(\sigma) \overline{\pi}^p(\chi_{K_0}) \right).\]  

**Proof.** We work in the universal algebra $\mathcal{A} = C^*(G,\overline{G})$. For $\sigma \in \overline{G}$ denote the convolution with $\sigma$ by $L_\sigma$. Denote the operator of convolution with a continuous function $f$ on $\overline{G}$ by $L(f)$. Clearly, for every measurable subset $A$ of $\overline{G}$ we have
\[L(\chi_A) L_\sigma = L(\chi_{A \sigma}), \quad L_\sigma L(\chi_A) = L(\chi_{A \sigma}), \quad \sigma \in G.\]
Then, obviously,
\[L(f) = \int_{\overline{G}} f(g) L_\sigma \, dg\]  
and
\[(L(\chi_{K_0}))^2 = \nu(K_0) L(\chi_{K_0}).\]
In particular, $\nu(K_0)^{-1} L(\chi_{K_0})$ is a projection.

The same argument gives us that if $K_0$ and $K_1$ are subgroups as above, with $K_1$ a subgroup of $K_0$, then
\[L(\chi_{K_1}) L(\chi_{K_0}) = L(\chi_{K_0}) L(\chi_{K_1}) = \nu(K_1) L(\chi_{K_0}).\]  
We decompose $K_0$ as a union of right cosets for the subgroups $(\Gamma_0)_\sigma = K_0 \cap \sigma K_0 \sigma^{-1}$, with coset representatives $s_i \in K_0$, $i = 1, 2, \ldots, [K_0 : (K_0)_\sigma]$. These are the same as the coset representatives for $(\Gamma_0)_\sigma$ in $\Gamma_0$. Using the formula (42), we find that
\[L(\chi_{K_0}) L_\sigma L(\chi_{K_0}) = \sum_i L(s_i \chi_{(K_0)_\sigma}) L_\sigma L(\chi_{K_0}) = \sum_i L(s_i \sigma) L(\chi_{(K_0)_\sigma^{-1}}) L(\chi_{K_0})
= \nu((K_0)_{\sigma^{-1}}) \sum_i L(s_i \sigma) L(\chi_{K_0}) = \nu((K_0)_{\sigma^{-1}}) L(\chi_{K_0 \sigma K_0}).\]  
\[\square\]
As previously noted, the Hilbert spaces $H^{\Gamma_0}$, $\Gamma_0 \in \mathcal{S}$, are isometrically isomorphic to $l^2(\Gamma_0 \setminus \Gamma) \otimes L$, where the scalar product in $l^2(\Gamma_0 \setminus \Gamma)$ is chosen so that the embeddings $l^2(\Gamma_0 \setminus \Gamma) \subseteq l^2(\Gamma_1 \setminus \Gamma)$ are isometric for all $\Gamma_1 \subseteq \Gamma_0$. It turns out that the entries of the matrices representing Hecke operators are sums of the form (29). Lemma 21 indicates clearly that, after normalization, the formula for the classical Hecke operator corresponding to the sum over cosets is

$$[\Gamma : \Gamma_\sigma] P_{H^{\Gamma_0}} \pi^p(\sigma) P_{H^{\Gamma_0}}, \quad \sigma \in G, \quad \Gamma_0 \in \mathcal{S}. $$

**Theorem 22.** With the notation and definitions previously introduced in this section, fix a subgroup $\Gamma_0$ in $\mathcal{S}$, and choose a family $(s_i)$ of right coset representatives for $\Gamma_0 \subseteq \Gamma$. As in (31), consider the Hilbert space

$$L^{\Gamma_0} = \bigoplus_{i=1}^{[\Gamma : \Gamma_0]} \pi(s_i)L. \tag{43}$$

The Hilbert space norm on $L^{\Gamma_0}$ is normalized so that the embedding of $L$ into $L^{\Gamma_0}$ defined by the correspondence

$$l \in L \rightarrow \bigoplus_{i=1}^{[\Gamma : \Gamma_0]} \pi(s_i)l \in L^{\Gamma_0} = \bigoplus_{i=1}^{[\Gamma : \Gamma_0]} \pi(s_i)L, \tag{44}$$

is isometric. The space $L^{\Gamma_0}$ is identified in an obvious way with a subspace of $H$. In this case $L^{\Gamma_0}$ is endowed with a non-normalized scalar product inherited from $H$. Let $P_{L^{\Gamma_0}}$ be the orthogonal projection from $H$ onto $L^{\Gamma_0}$.

Then for $\sigma \in G$ the Hecke operator $[\Gamma : \Gamma_\sigma] P_{H^{\Gamma_0}} \pi^p(\sigma) P_{H^{\Gamma_0}}$ is unitarily equivalent to the bounded operator

$$A(\Gamma_0 \sigma \Gamma_0) = \sum_{\theta \in \Gamma_0 \sigma \Gamma_0} P_{L^{\Gamma_0}} \pi(\theta) P_{L^{\Gamma_0}}. \tag{45}$$

**Proof.** For a given $\Gamma_0 \in \mathcal{S}$ and a choice of coset representatives,

$$\Gamma = \bigcup \Gamma_0 s_i,$$

we construct a unitary operator $W^{\Gamma_0}$ from $L^{\Gamma_0} = \bigoplus \pi(s_i)L$ into $H^{\Gamma_0}$ as follows. For vectors $l_i \in L$ we define the isometry

$$W^{\Gamma_0} \left( \bigoplus \pi(s_i)l_i \right) = \sum_{i} \sum_{\gamma \in \Gamma_0} \pi(\gamma_0) \pi(s_i)l_i. $$

We prove that the diagram

$$\begin{array}{ccc}
H^{\Gamma_0} & \xrightarrow{W^{\Gamma_0}} & \bigoplus \pi(s_i)L \\
\downarrow [\Gamma_0 : (\Gamma_0)_\sigma] P_{H^{\Gamma_0}} \pi^p(\sigma) P_{H^{\Gamma_0}} & & \downarrow \sum_{\theta \in \Gamma_0 \sigma \Gamma_0} P_{L^{\Gamma_0}} \pi(\theta) P_{L^{\Gamma_0}} \\
H^{\Gamma_0} & \xleftarrow{W^{\Gamma_0}} & \bigoplus \pi(s_i)L
\end{array}$$
is commutative for \( \sigma \in G \). To do this we use the formula for the unitary operators \( \pi^p(\theta), \theta \in G \), defined in the proof of Proposition 20.

It is sufficient to verify the commutativity of the diagram in the case \( \Gamma = \Gamma_0 \); the cases corresponding to other subgroups \( \Gamma_0 \in \mathcal{S} \) are a consequence. Consider a vector \( l \in L \). We have

\[
W^\Gamma l = \sum_{\gamma \in \Gamma} \pi(\gamma)l \in V^\Gamma.
\]

If the decomposition of \( \Gamma \) into right cosets with respect to \( \Gamma_{\sigma^{-1}} \) is \( \Gamma = \bigcup \Gamma_{\sigma^{-1}}r_j \), then

\[
W^\Gamma l = \sum_j \sum_{\gamma \in \Gamma_{\sigma^{-1}}} \pi(\gamma)\pi(r_j)l.
\]

Applying \( \pi^p(\sigma) \), we get

\[
\sum_j \sum_{\gamma_1 \in \Gamma_{\sigma}} \pi(\gamma_1)\pi(\sigma r_j)l.
\]

Projecting on \( H^\Gamma \), we obtain

\[
\frac{1}{[\Gamma : \Gamma_{\sigma}]} \sum_j \sum_{\gamma \in \Gamma} \pi(\gamma)\pi(\sigma r_j)l,
\]

and this is equal to

\[
\frac{1}{[\Gamma : \Gamma_{\sigma}]} \sum_{\theta \in \Gamma \sigma \Gamma} \pi(\theta)l.
\]

On the other hand, the sum

\[
\sum_{\theta \in \Gamma \sigma \Gamma} P_L \pi(\theta)P_L,
\]

applied to the vector \( l \), gives us that

\[
\sum_{\theta \in \Gamma \sigma \Gamma} P_L \pi(\theta)l = \sum_j \sum_{\gamma \in \Gamma} \pi(\gamma)\pi(\sigma r_j)l.
\]

We apply the isometry \( W^\Gamma \) to this vector and use the identifications assumed in the structure of the space \( H^\Gamma \) (see the formula (33) in Definition 19). Then the above sum corresponds to the vector

\[
\sum_{r_j} \sum_{\gamma \in \Gamma} \pi(\gamma r_j)l = \sum_{\theta \in \Gamma \sigma \Gamma} \pi(\theta)l.
\]

Below we describe the straightforward inclusions between spaces of vectors invariant with respect to subgroups in \( \mathcal{S} \).

**Lemma 23.** Consider the family of Hilbert spaces \( l^2(\Gamma_0 \setminus \Gamma), \Gamma_0 \in \mathcal{S} \). On this family the scalar product is normalized so that the embeddings \( l^2(\Gamma_0 \setminus \Gamma) \subseteq l^2(\Gamma_1 \setminus \Gamma) \) are isometric for all \( \Gamma_0 \subseteq \Gamma_1 \). Then the following statements hold.
(i) The Hilbert spaces $H^{\Gamma_0}$ are isometrically isomorphic to $l^2(\Gamma_0 \setminus \Gamma) \otimes L$ for all $\Gamma_0 \in \mathcal{S}$, with the inclusion 
\[ H^{\Gamma_0} \subseteq H^{\Gamma_1} \]
obtained for $\Gamma_1 \subseteq \Gamma_0$ by tensoring with $L$, and with the isometric inclusion 
\[ l^2(\Gamma_0 \setminus \Gamma) \subseteq l^2(\Gamma_1 \setminus \Gamma). \]

(ii) Let $\pi$ be a representation with the properties (i) and (ii) in Definition 1. Then the Hilbert space $H^p$ is $\Gamma$-equivariantly isometrically isomorphic to $L^2(K, \mu) \otimes L$. Hence, $\pi^p|_K$ is a multiple of the left regular representation.

Using the above identification, we construct a block-matrix representation for the Hecke operators.

**Corollary 24.** In the notation from Lemma 23, denote the canonical matrix unit of the algebra $B(l^2(\Gamma_0 \setminus \Gamma))$ by 
\[ (e_{\Gamma_0 s_i, \Gamma_0 s_j})_{i,j=1,2,...,[\Gamma:\Gamma_0]}. \]
In view of the isomorphism 
\[ B(H^{\Gamma_0}) \cong B(l^2(\Gamma_0 \setminus \Gamma)) \otimes B(L) \]
defined above, the Hecke operator $[\Gamma_0 : (\Gamma_0)_\sigma] P_{H^{\Gamma_0}} \pi^p(\sigma) P_{H^{\Gamma_0}}$ can be represented as 
\[ \sum_i \sum_{\theta \in s_i^{-1} \Gamma_0 \sigma \Gamma_0 s_j} P_L \pi(\theta) P_L \otimes e_{\Gamma_0 s_i, \Gamma_0 s_j}. \] (46)

**Proof.** This statement is a consequence of the fact that the projection onto $L^{\Gamma_0} = \bigoplus \pi(s_i)L$ is $\sum \pi(s_i)P_L \pi(s_i)^*,$ where the $s_i$ are the coset representatives in the statement of Theorem 22. We use the coset representatives to construct a unitary operator mapping $L^{\Gamma_0}$ onto $l^2(\Gamma_0) \otimes L$. This will map $\bigoplus \pi(s_i)l_i$ to $\bigoplus [\Gamma_0 s_i] \otimes l_i$ for all $l_i \in L$. Conjugating the operator in (45) by this unitary operator, we get (46).

No further renormalization in the formula (46) is needed, as can be directly checked by letting $\sigma$ be the identity element in $G$. Then for $\Gamma_0 \in \mathcal{S}$ the left-hand side of (47) below is $\dim H^{\Gamma_0}$. Since $L$ is a $\Gamma$-wandering subspace of $H$, the right-hand side of (47) counts how many times the identity element falls in $s_i^{-1} \Gamma_0 s_i$, and multiplies the result by the dimension of $L$. $\square$

**Remark 25.** In particular, if $L$ is of finite dimension, then we have the following formula for the traces of the Hecke operators:
\[ \text{Tr}([\Gamma_0 : (\Gamma_0)_\sigma][P_{H^{\Gamma_0}} \pi(\sigma) P_{H^{\Gamma_0}}]) = \sum_i \sum_{\theta \in s_i^{-1} \Gamma_0 \sigma \Gamma_0 s_i} \text{Tr}(P_L \pi(\theta) P_L). \] (47)

We note that the correspondence $\pi \to \pi^p$ also preserves the dimension function $\dim_{(\pi(\Gamma))_\nu} H$. This is explained in the next result.
Lemma 26. Let \( \pi \) be a unitary representation as above. Then the following statements hold.

(i) The unitary representation \( \pi^p \) extends to a \( C^* \)-algebra representation of the full amalgamated free product \( C^* \)-algebra

\[
C^*(G) *_{C^*(K)} B(L^2(K))
\]

into \( B(H^p) \).

(ii) The unitary representation \( \pi^p \) of \( G \) has the property that \( \pi^p_0 |_K \) is a finite multiple of the left regular representation of \( K \) on \( L^2(K, \mu) \).

Proof. To prove (i), we must construct a representation of \( B(l^2(K)) \) into \( B(H^p) \).

Since \( B(l^2(K)) \) is generated by the algebra of convolutors with continuous functions on \( K \) and by the functions in \( C(K) \), viewed as multiplication operators, it is sufficient to describe a representation of the algebra \( C(K) \) into \( B(H^p) \) that is compatible with the representation of the algebra \( C^*(K) \) into \( B(H^p) \) obtained by restricting \( \pi^p \) to \( C^*(K) \). By translation invariance, it is sufficient to consider the following case. Let \( K_0 \) be the closure, in the profinite topology, of a subgroup \( \Gamma_0 \in \mathcal{S} \). Then \( \chi_{K_0} \), viewed as an element of the algebra \( C(K) \), will act on a vector of the form \( \sum_{\gamma_0 \in \Gamma_1} \pi(\gamma_0)l, \Gamma_1 \in \mathcal{S}, l \in L \), by mapping it to \( \sum_{\gamma_0 \in \Gamma_0 \cap \Gamma_1} \pi(\gamma_0)l \). It is easy to check that this defines a \( C^* \)-algebra representation of the \( C^* \)-algebra \( C^*(G) *_{C^*(K)} B(L^2(K)) \).

To prove (ii), we note that by restricting \( \pi^p \) to \( K \), the operators \( W^{\Gamma_0}, \Gamma_0 \in \mathcal{S} \), become intertwining operators between \( \pi^p |_K \) and the representation \( \lambda_K \otimes \text{Id}_L \) acting on \( L^2(K) \otimes L \). \( \square \)

4. Examples of representations and associated spaces of \( \Gamma \)-invariant vectors

In this section we discuss concrete examples which satisfy the axioms in Definition 14 and to which the construction in Proposition 20 can be applied. Here we will make free use of the notation in that definition and of the subsequent statements.

For unitary representations on spaces of \( \Gamma \)-invariant vectors, the following example plays the role that the left regular representation of a discrete group plays in the group’s space of representations. It is tempting to call this representation ‘regular’.

Example 27. Let \( \pi = \lambda_G \) be the left regular representation of \( G \) acting on \( H = l^2(G) \). In this case the vector space \( \mathcal{V} \) is the linear space of functions on \( G \). For \( \Gamma_0 \in \mathcal{S}, \mathcal{V}^{\Gamma_0} \) is the space of left \( \Gamma_0 \)-invariant functions on \( G \), that is, functions on \( \Gamma_0 \setminus G \). Then the Hilbert space \( H^{\Gamma_0} \) is \( l^2(\Gamma_0 \setminus G) \). The scalar product is defined so that if \( \Gamma_1 \subseteq \Gamma_0, \Gamma_0, \Gamma_1 \in \mathcal{S} \), then the inclusions

\[
l^2(\Gamma_0 \setminus G) \subseteq l^2(\Gamma_1 \setminus G)
\]

are isometric.

Clearly, in this case \( \mathcal{V}_\infty \) is the Hilbert space completion of the space

\[
\bigvee_{\Gamma_0 \in \mathcal{S}} l^2(\Gamma_0 \setminus G)
\]
and coincides with $L^2(\mathcal{G}, \mu)$. Then the representation $\pi^p$ is the left regular representation $\lambda_{\mathcal{G}}$ acting on $L^2(\mathcal{G}, \mu)$.

Note that here we implicitly use an identity in the Hilbert spaces having as a basis the set of cosets. This identity is the following:

$$[\Gamma_0 g] = \sum_i [\Gamma_1 s_i g], \quad g \in G.$$ 

After renormalization of the scalar product, this equality implies the existence of the isometric embedding

$$l^2(\Gamma_0 \setminus G) \subseteq l^2(\Gamma_1 \setminus G).$$

In particular, all quasi-regular representations $\lambda_{G/\Gamma_0}$ of $G$ on spaces of cosets are subrepresentations of $\lambda_{\mathcal{G}} |_{\Gamma_0}$. Indeed, by using the equality in the above formula we get that for all $\Gamma_1 \in \mathcal{F}$

$$l^2(\Gamma_1 / \Gamma_0) \subseteq \bigvee_{\Gamma_0 \in \mathcal{F}} l^2(\Gamma_0 \setminus G).$$

The quasi-regular representations occur with infinite multiplicity in the left regular representation $\lambda_{\mathcal{G}} |_{\Gamma_0}$, since they commute with the right action of $G$.

A standard choice of a $\Gamma$-wandering generating subspace of $l^2(G)$ will involve choosing representatives $\mathcal{C} \subseteq G$ of right cosets of $\Gamma$ in $G$ (so that $G$ is the disjoint union $\bigcup_{\sigma \in \mathcal{C}} \Gamma \sigma$). Since $P_{H r \nu}$ is the projection onto $l^2(\Gamma_0 \setminus G)$, we obtain by using the above construction a standard representation of the Hecke operators and Hecke algebras ([7], [4], [21], [51], [29]).

We describe a second standard example of the construction in Proposition 20 corresponding to the case $\dim_{\{\pi(\Gamma)\}^\nu} H = \infty$.

**Example 28.** Let $(\mathcal{X}, \nu)$ be an infinite measure space on which $G$ acts by measure-preserving transformations, and assume that the restriction of the action of $G$ to $\Gamma$ admits a fundamental domain $F$ in $\mathcal{X}$ with measure $\nu(F) = 1$. For each $\Gamma_0$ in $\mathcal{F}$ we fix a system of representatives of cosets:

$$\Gamma = \bigcup \Gamma_0 s_i.$$ 

Let

$$F_{\Gamma_0} = \bigcup s_i F.$$ 

Then $F_{\Gamma_0}$ is a fundamental domain for $\Gamma_0$. We renormalize the measure $\nu$ on $F_{\Gamma_0}$, and consider

$$\nu_{\Gamma_0} = \frac{1}{[\Gamma : \Gamma_0]} \nu.$$ 

The choice of representatives induces a projection $\pi_{\Gamma_0}: F_{\Gamma_0} \rightarrow F$, which simply maps $s_i f$ to $f$ for $f$ in $F$. Taking the dual spaces, we obtain an isometric inclusion

$$L^2(F, \nu) \subseteq L^2(F_{\Gamma_0}, \nu_{\Gamma_0}).$$
The unitary representation of $G$ on $L^2(\mathcal{X}, \nu)$ is simply the Koopman representation

$$\pi_{\text{Koop}}(g)f(x) = f(g^{-1}x), \quad x \in \mathcal{X}, \quad g \in G, \quad f \in L^2(\mathcal{X}, \nu).$$

We use the formalism in Definition 14 and let $\mathcal{V}$ be the linear space of measurable functions on $\mathcal{X}$. The subspace $\mathcal{V}^{\Gamma_0}$ obviously consists of the functions in $\mathcal{V}$ that are $\Gamma_0$-equivariant. Then $H^{\Gamma_0}$ is canonically identified with the Hilbert space

$$L^2(F_{\Gamma_0}, \nu_{\Gamma_0}).$$

This space is also identified with a subspace of the $\Gamma_0$-invariant functions on $\mathcal{X}$.

It is clear that in this case the Hilbert space $H^p$ is isometrically isomorphic to

$$L^2(K, \mu) \otimes L^2(F, \nu) = L^2(K \times F, \mu \times \nu).$$

The representation $\pi|_K^p$ is simply $\text{Id}_{L^2(F)} \otimes \lambda_K$, where $\lambda_K$ is the left regular representation of $K$ on $L^2(K, \mu)$.

The above construction also proves that the representation $\pi^p_{\text{Koop}|_G}$ itself is a Koopman unitary representation. It is easily recovered from the initial representation $\pi$. Indeed, taking the counting measure $\varepsilon$ on $\Gamma$, one has an isomorphism of measure spaces

$$(\mathcal{X}, \nu) \cong (\Gamma, \varepsilon) \times (F, \nu).$$

The action of $G$ on $\mathcal{X}$ in the above identification is described in terms of a cocycle on $G \times F$ with values in $\Gamma$, where $\Gamma$ acts by left multiplication on the factor $\Gamma$ in the product $\Gamma \times F$. When replacing the factor $\Gamma$ in this product by the factor $K$ in the product $K \times F$, we obtain on the measure space $(K \times F, \mu \times \nu)$ a measure-preserving action of $G$ having the same cocycle as the action of $G$ on $\Gamma \times F$. Then the unitary representation $\pi^p_{\text{Koop}|_G}$ is in fact the unitary Koopman representation corresponding to the action of $G$ on $(K \times F, \mu \times \nu)$.

In the above construction, the projection $P_L$ is the operator of multiplication by the characteristic function of $\chi_F$. The convergence condition requiring that the sum

$$\sum_{\theta \in \Gamma \sigma \Gamma} P_L \pi_{\text{Koop}}(\theta) P_L$$

be so-convergent is obvious in this case, since this sum is the Hecke operator (see, for example, [41]).

We briefly describe below how the scheme in Proposition 20 can be used for spaces of automorphic forms. This will also be studied in detail in the next section in a more general setting.

Example 29. Consider the measures $d\nu_n(z) = (\text{Im } z)^{n-2} d\varepsilon dz$ on $\mathbb{H}$. Let $\pi_n$, $n > 1$, $n \in \mathbb{N}$, be the discrete series [30] of (projective) unitary representations of $\text{PSL}(2, \mathbb{R})$. These representations act on the Hilbert space $H_n = H^2(\mathbb{H}, \nu_n)$. Their expression is the Koopman unitary representation with respect to the Möbius transformations of $\text{PSL}(2, \mathbb{R})$ on the upper half-plane, except for an additional multiplication by a modularity factor corresponding to the fact that the measure
\( \nu_n \) is not invariant under Möbius transformations (see [30]). We also consider the larger Hilbert space \( \hat{H}_n = L^2(\mathbb{H}, \nu_n) \), \( n \geq 1 \).

Let \( G = \text{PGL}(2, \mathbb{Z}[1/p]) \), where \( p \) is a prime, and let \( \Gamma = \text{PSL}(2, \mathbb{Z}) \). If \( \pi_n \) is a projective unitary representation, that is, if \( n \) is odd, then we are also given a 2-cocycle (expressing the projectivity of \( \pi_n \)). In this case it is \( \mathbb{Z}_2 \)-valued, and hence it extends to a 2-cocycle on the Schlichting completion, which is \( \text{PGL}(2, \mathcal{Q}_p) \), where \( \mathcal{Q}_p \) is the \( p \)-adic field.

By the results in [19], §3.3, it follows that \( \pi_n \big| \Gamma \) is a (not necessarily integer) multiple of the left regular representation \( \lambda \Gamma \). Indeed,
\[
\dim \{\pi_n(\Gamma)\}'' H_n = \frac{n - 1}{12}.
\]
Consequently, if \( (n - 1)/12 \) is not an integer, then there is no Hilbert space \( L \) such that \( H_n \) and \( l^2(\Gamma) \otimes L \) are isomorphic as \( \Gamma \)-modules. Moreover, even if \( (n - 1)/12 \) is an integer, there is no canonical choice of \( L \) which would enable one to proceed as in Proposition 20.

To overcome this problem we use a \( \Gamma \)-wandering generating subspace of a representation \( \hat{\pi}_n \) containing \( \pi_n \) as a subrepresentation. We let \( \hat{\pi}_n \) be the unitary representation of \( \text{PSL}(2, \mathbb{R}) \) given by the same algebraic formula on functions on \( \mathbb{H} \) as the algebraic formula that determines the representation \( \pi_n \). The same computation showing that \( \pi_n \) is a unitary representation also plainly proves that \( \hat{\pi}_n \) is a unitary representation.

It is a well-known fact [23], [34] that the associated Hilbert space \( H_n^\Gamma \) is the finite-dimensional Hilbert space consisting of automorphic forms of weight \( n \) for the group \( \Gamma = \text{PSL}(2, \mathbb{Z}) \). To apply the formalism in Definition 14, we let \( \mathcal{V} \) be the space of analytic functions on \( \mathbb{H} \). In the next section we use this scheme to compute the traces of Hecke operators.

Then to describe the scalar product in \( H_n^\Gamma \), we use the Hilbert space scalar product in the previous example for the unitary representation \( \hat{\pi}_n \). If \( F \) is a fundamental domain for the action of \( \Gamma \) on \( \mathbb{H} \), then we let \( P_L \) be the multiplication operator \( M_{\chi_F} \) by the characteristic function \( \chi_F \), acting on \( L^2(\mathbb{H}, \nu_n) \). The scalar product is explicitly constructed in the formula (50) in the next section. In the particular case treated in the present example, this formula for the scalar product turns out to be the canonical Petersson scalar product [34]. The procedure described above will be used later to obtain an explicit description of the Hecke operators and to compute their traces.

We give one more example of a case when the construction in Proposition 20 is used to construct \( \Gamma \)-invariant vectors. This example corresponds to representations of the form \( \pi \otimes \pi^\text{op} \), where \( \pi^\text{op} \) is the complex conjugate.

**Example 30.** Let \( \Gamma \subseteq G \), \( \pi \), \( \pi_0 \), \( P_0 \), and \( P_L \) be as in Definition 1. Consider the diagonal unitary representation \( \hat{\pi} = \pi_0 \otimes \pi_0^\text{op} \) of \( G \). In this example we assume for simplicity that all the subgroups in \( \mathcal{S} \) have infinite non-trivial conjugacy classes. This corresponds to all the associated von Neumann algebras having unique traces (being factors, that is, having no centre).

We note that even if \( \pi_0 \) is projective, the representation \( \pi_0 \otimes \pi_0^\text{op} \) is unitary, with no cocycle. Moreover, since in this case the Murray–von Neumann dimension is
infinite, it follows that the unitary representation \( \pi_0 \otimes \pi_0^{\text{op}} \) satisfies the conditions of Theorem 20. Since we reserved the notation \( \overline{H}_0^p \) for other purposes, we use here the notation \( H_0^{\text{op}} \) for the conjugate Hilbert space of \( H_0 \).

Then the representation \( \pi_0 \otimes \pi_0^{\text{op}} \) is unitarily equivalent to the representation \( \text{Ad} \pi_0 \) defined on \( G \) and with values in the unitary group of the Hilbert space

\[
H_0 \otimes H_0^{\text{op}} \cong \mathcal{C}_2(H_0) \subseteq B(H_0).
\]

Here \( \mathcal{C}_2(H_0) \) is the ideal of Hilbert–Schmidt operators acting on \( H_0 \) [50]. The formula for the representation \( \text{Ad} \pi_0 \) is

\[
\text{Ad} \pi_0(g)(X) = \pi_0(g)X\pi_0(g)^{-1}, \quad X \in \mathcal{C}_2(H_0), \quad g \in G.
\]

It extends in an obvious way to a (non-unitary) representation of \( G \) into the (inner) automorphisms of \( B(H_0) \).

In the setting of Definition 14 we let the space \( \mathcal{V} \) be \( B(H_0) \). Then

\[
\mathcal{V}^\Gamma = \{ \pi_0(\Gamma) \}' = \{ X | [X, \pi(\gamma)] = 0 \text{ for all } \gamma \in \Gamma \} \subseteq B(H_0).
\]

More generally, for \( \Gamma_0 \in \mathcal{I} \) we have

\[
\mathcal{V}^\Gamma_0 = \{ \pi(\Gamma_0) \}' \subseteq B(H_0).
\]

Since we assumed that all the groups \( \Gamma_0 \) in \( \mathcal{I} \) have infinite non-trivial conjugacy classes, it follows that the algebras \( \{ \pi_0(\Gamma_0) \}' \), \( \Gamma_0 \in \mathcal{I} \), are type II\(_1\) factors, and consequently each of them is endowed with a unique normalized trace \( \tau_{\Gamma_0} \).

We let \( \mathcal{A}_\infty \) be the type II\(_1\) factor obtained as the inductive, trace-preserving, directed limit of the factors \( \{ \pi_0(\Gamma_0) \}' \), \( \Gamma_0 \in \mathcal{I} \). Then \( \mathcal{A}_\infty \) has a unique trace \( \tau \) defined by the requirement that

\[
\tau|_{\{\pi_0(\Gamma_0)\}'} = \tau_{\Gamma_0}, \quad \Gamma_0 \in \mathcal{I}.
\]

For \( \sigma \in G \) and \( \Gamma_0 \in \mathcal{I} \), \( \text{Ad} \pi_0(\sigma) \) maps \( \{ \pi_0(\Gamma_0 \cap \Gamma_{\sigma^{-1}}) \}' \) into \( \{ \pi_0(\sigma\Gamma_0\sigma^{-1} \cap \Gamma_{\sigma}) \}' \). It follows that \( \text{Ad} \pi_0(\sigma) \) also maps \( \mathcal{A}_\infty \) onto \( \mathcal{A}_\infty \). Thus, \( \text{Ad} \pi_0(\sigma), \sigma \in G \), extends to an element in the automorphism group \( \text{Aut}(\mathcal{A}_\infty) \) of the factor \( \mathcal{A}_\infty \).

To obtain the Hilbert space of \( \Gamma_0 \)-invariant vectors, we use the standard \( L^2 \)-spaces associated with the corresponding type II\(_1\) factors (for the notation see, for example, [50]). Then

\[
(H_0 \otimes H_0^{\text{op}})^\Gamma_0 = L^2(\{ \pi(\Gamma_0) \}', \tau_{\Gamma_0}) \quad \text{and} \quad (H_0 \otimes H_0^{\text{op}})^p = L^2(\mathcal{A}_\infty, \tau).
\]

In particular, if \( \dim \{ \pi(\Gamma) \}'' H_0 = 1 \), then

\[
(H_0 \otimes H_0^{\text{op}})^\Gamma = L^2(\mathcal{L}(\Gamma), \tau) \cong \ell^2(\Gamma).
\]

Consequently, the unitary representation \( \text{Ad} \pi(\sigma), \sigma \in G \), induces the unitary representation

\[
\overline{\text{Ad} \pi}^{\text{op}} = \frac{\pi_0 \otimes \pi_0^{\text{op}}}{\pi_0 \otimes \pi_0^{\text{op}}}
\]

corresponding to \( \pi_0 \otimes \pi_0^{\text{op}} \), as defined in Proposition 20.
Although this is not needed in this paper, we note that by the Jones index theory [25], if we identify the Jones projection for the inclusion
\[ \pi_0(\Gamma_0)'' \subseteq \{ \pi_0(\Gamma)'' \] with the characteristic function of the closure of the subgroup \( \Gamma_0 \) in \( K \), then we get (see [39]) that \( A_\infty \) is isomorphic to the von Neumann crossed product algebra \( L(\pi_0L_\infty(K, \nu)) \), where \( \pi_0 \) acts by left translations on \( K \) and by right translations on \( K \).

5. Construction of the representation \( \pi_0^0 \) in the absence of a \( \Gamma \)-wandering generating subspace

In this section we analyse the case of a unitary representation \( \pi_0 \) of \( G \) on a Hilbert space \( H_0 \) such that \( \dim\{ \pi_0(\Gamma)'' \} \) is not necessarily an integer. We assume that this dimension is a finite positive number. Thus, there might be no \( \Gamma \)-wandering generating subspace \( L \subseteq H_0 \), as in Definition 13.

We assume the conditions in Definition 1. Recall that the assumptions in that definition require that there be a unitary representation \( \pi \) of \( G \) on a larger Hilbert space \( H \) and having a \( \Gamma \)-wandering generating subspace \( L \). The initial representation \( \pi_0 \) is required to be a subrepresentation of \( \pi \). Recall also that, as in Definition 1, \( P_L \) and \( P_0 \) denote the orthogonal projections from \( H \) onto \( L \) and \( H_0 \), respectively.

The spaces of \( \Gamma_0 \)-invariant vectors are constructed (see Proposition 20) as spaces of formal sums over the group \( \Gamma_0 \in \mathcal{S} \). The \( \Gamma_0 \)-invariant vectors are thus identified, as in §3, with \( \Gamma_0 \)-invariant unbounded linear forms on the Hilbert space \( H_0 \).

The use of an auxiliary representation \( \pi \) having a \( \Gamma \)-wandering generating subspace and containing the original representation \( \pi_0 \) as a subrepresentation is suggested by the case of automorphic forms (see the description in Example 29). Formally, automorphic forms are vectors fixed by \( \pi_n \big|_\Gamma \) acting on the Hilbert space \( H_n \). It is impossible to find genuine vectors with this property since square integrability fails. On the other hand, the algebraic formula defining the representation \( \pi_n \) admits \( \Gamma \)-invariant analytic functions. These are the automorphic forms [23]. To construct a Hilbert space structure on the space of automorphic forms of a given weight one uses the Petersson scalar product. To define the Petersson scalar product [34] one uses a fundamental domain \( F \) for the action of \( \Gamma \) on \( \mathbb{H} \). The space \( L^2(F, \nu_n) \) is a \( \Gamma \)-wandering generating subspace for the larger unitary representation containing \( \pi_n \) as a subrepresentation and acting on \( L^2(\mathbb{H}, \nu_n) \).

In the framework of this section, the unitary representation \( \pi_0 \) is \( \pi_n \), and the larger representation having a \( \Gamma \)-wandering generating subspace acts on \( L^2(\mathbb{H}, \nu_n) \). The projection \( P_L \) is the multiplication operator on \( L^2(\mathbb{H}, \nu_n) \) by the characteristic function \( \chi_F \) of the fundamental domain. Also, the projection \( P_0 \) is the (Bergman) projection onto the space \( H^2(\mathbb{H}, \nu_n) \) of analytic functions.

Below we describe the changes from the procedure in Proposition 20 and Definition 14 that are needed to address the present situation. We start with the construction of the space of \( \Gamma_0 \)-invariant vectors.

**Lemma 31.** Consider the groups \( \Gamma \subseteq G \) and a representation \( \pi_0 \) of \( G \) with the properties in Definition 1, where the notation is set.
For \( \Gamma_0 \) in \( \mathscr{S} \), fix a system of right coset representatives \( s_i \) for \( \Gamma_0 \) in \( \Gamma \) so that 
\[ \Gamma = \bigcup \Gamma_0 s_i. \]
Let \( L^{\Gamma_0} \) be the Hilbert space in (43) with the norm subject to the renormalization condition in (44) of Theorem 22. Let \( P_{L^{\Gamma_0}} \) be the orthogonal projection from \( H \) onto \( L^{\Gamma_0} \).

Then the formula
\[
\mathcal{D}_{\Gamma_0, L} = \sum_{\gamma \in \Gamma_0} P_{L^{\Gamma_0}} \pi_0(\gamma) P_{L^{\Gamma_0}} \in B(L^{\Gamma_0})
\]
defines a projection in \( B(L^{\Gamma_0}) \).

**Proof.** The fact that \( \mathcal{D}_{\Gamma_0, L} \) is a projection, and more generally the fact that the formula (55) in Theorem 34 defines a representation of the Hecke algebra of double cosets for \( \Gamma_0 \in \mathscr{S} \), is a straightforward consequence of the following identity, which is valid for \( \sigma_1, \sigma_2 \in G \) and \( \Gamma_0 \in \mathscr{S} \):
\[
\sum_{\gamma \in \Gamma_0} P_{L^{\Gamma_0}} \pi_0(\sigma_1 \gamma) P_{L^{\Gamma_0}} \pi_0(\gamma^{-1} \sigma_2) P_{L^{\Gamma_0}} = P_{L^{\Gamma_0}} \pi_0(\sigma_1 \sigma_2) P_{L^{\Gamma_0}}.
\]

The convergence of the series in this equality follows from the technical condition (v) assumed in Definition 1. Here we are taking the pointwise operator product of two operator series that are convergent in \( C_2(L) \). The formula (49) is then a direct consequence of the fact that
\[
\sum_{\gamma_0 \in \Gamma_0} \pi(\gamma) P_{L^{\Gamma_0}} \pi(\gamma^{-1})
\]
is the identity operator on \( H \).  \( \square \)

In the following, we adapt the content of Proposition 20 to the context of a representation \( \pi_0 \) with the properties in Definition 1. In the next lemma we construct the Hilbert space of ‘virtual’ \( \Gamma_0 \)-invariant vectors that is associated with \( \pi_0 \) (we assume the notation and definitions in Lemma 31).

**Lemma 32.** Let \( H_0^{\Gamma_0} \) be the space of formal series of the form
\[
H_0^{\Gamma_0} = \left\{ \sum_{\gamma \in \Gamma_0} \pi_0(\gamma) l \ \bigg| \ l \in L^{\Gamma_0} \right\}.
\]

Let \( \mathcal{D}_{L, \pi} \) be defined as in (30) in Lemma 18. The space \( H_0^{\Gamma_0} \) is subject to the same identification as in the formula (33) in Definition 19. Hence:

(i) \( H_0^{\Gamma_0} = \left\{ \sum_{\gamma \in \Gamma_0} \pi_0(\gamma) h \ \bigg| \ h \in \mathcal{D}_{L, \pi} \right\}; \)

(ii) the representation \( \pi_0 \) extends to a canonical representation \( \pi_0^p \) of \( G \) into the linear isomorphism group of the vector space \( \bigvee_{\Gamma_0 \in \mathscr{S}} H^{\Gamma_0} \).

**Proof.** Part (i) is a straightforward consequence of the identification in (33). The representation \( \pi_0^p \) in (ii) is constructed using the same procedure as for Proposition 20 (see (38)).  \( \square \)
Now we define a compatible scalar product in the spaces of $\Gamma_0$-invariant vectors introduced above, $\Gamma_0 \in \mathcal{S}$. We also construct a unitary representation on the inductive limit of the Hilbert spaces of $\Gamma_0$-invariant vectors.

**Theorem 33.** In the above context the following hold.

(i) Let $\Gamma_0 \in \mathcal{S}$ and let $h_1, h_2 \in \mathcal{D}_{L,\pi}$. By analogy with the formula (36) in Proposition 20 define

$$
\left\langle \sum_{\gamma_0 \in \Gamma_0} \pi_0(\gamma_0)h_1, \sum_{\gamma_0' \in \Gamma_0} \pi_0(\gamma_0')h_2 \right\rangle_\infty = \left\langle P_{L_{\Gamma_0}} \left( \sum_{\gamma_0 \in \Gamma_0} \pi_0(\gamma_0)h_1 \right), \sum_{\gamma_0 \in \Gamma_0} \pi_0(\gamma_0')h_2 \right\rangle.
$$

(50)

For subgroups $\Gamma_1 \subseteq \Gamma_0$ in $\mathcal{S}$ the inclusions $H_{\Gamma_0}^{\Gamma_1} \subseteq H_{\Gamma_1}^{\Gamma_1}$ are isometric. Hence, (50) extends to a scalar product $\langle \cdot, \cdot \rangle_\infty$ on $\bigvee_{\Gamma_0 \in \mathcal{S}} H_{\Gamma_0}^{\Gamma_0}$.

(ii) The scalar product $\langle \cdot, \cdot \rangle_\infty$ on the space $\bigvee_{\Gamma_0 \in \mathcal{S}} H_{\Gamma_0}^{\Gamma_0}$ satisfies the conditions in Definition 14.

(iii) Recall that the space $L_{\Gamma_0}^{\Gamma_0}$ in (43) is a $\Gamma_0$-wandering generating subspace for the representation $\pi|_{\Gamma_0}$. Let $\bar{L}_{\Gamma_0}^{\Gamma_0}$ be another $\Gamma_0$-wandering generating subspace of $H$ for the representation $\pi|_{\Gamma_0}$. Assume that there are two orthogonal decompositions

$$
\bar{L}_{\Gamma_0}^{\Gamma_0} = \bigoplus_{\gamma \in \Gamma_0} \bar{L}_\gamma \text{ and } L_{\Gamma_0}^{\Gamma_0} = \bigoplus_{\gamma \in \Gamma_0} L_\gamma
$$

such that $\bar{L}_\gamma = \pi(\gamma)L_\gamma$ for $\gamma \in \Gamma_0$. Then replacing $L_{\Gamma_0}^{\Gamma_0}$ by $\bar{L}_{\Gamma_0}^{\Gamma_0}$ in (50) does not change the value of the scalar product.

(iv) Let $\bar{H}_0^p$ be the Hilbert space completion of the inductive limit $\bigvee_{\Gamma_0 \in \mathcal{S}} H_{\Gamma_0}^{\Gamma_0}$ of the spaces constructed in (i). For $g \in G$ define the unitary transformation $\pi_0^p(g)$ on $\bar{H}_0^p$ by the formula (38) in the proof of Proposition 20. Then $\pi_0^p$ is a unitary representation of $G$ into the unitary group of the Hilbert space $\bar{H}_0^p$.

Before proving the theorem, we make a few remarks. The equality (50) in the definition of the scalar product can also be continued to the formula

$$
\left\langle \sum_{\gamma_0 \in \Gamma_0} \pi(\gamma_0)P_0h, \sum_{\gamma_0' \in \Gamma_0} \pi(\gamma_0')P_0l \right\rangle_\infty.
$$

This proves that the scalar product that we are defining on $H_0^{\Gamma_0}$ is consistent with the scalar product in the space $H^{\Gamma_0}$ in Proposition 20.

In the case of automorphic forms, when $\pi_0$ is the representation $\pi_n$ and $P_0$ is the Bergman projection onto the associated space of square-integrable analytic functions, the technical conditions (v) and (vi) in Definition 1 follow from the fact that the reproducing kernel for the space of automorphic forms is the sum over $\Gamma$ of the reproducing kernels restricted to the fundamental domain for the operators $\chi_F \pi_0(\gamma) \chi_F$, $\gamma \in \Gamma$. The same is true for the sum over any double coset, the sum of the kernels being equal in this case to the reproducing kernel for the Hecke operator associated with the double coset. The convergence of the reproducing kernels holds in the Hilbert–Schmidt norm [53]. We remark that the absolute convergence for the sum of traces is proved in the same paper [53].
The similarity to the Petersson scalar product formula follows from the fact that in the particular case corresponding to automorphic forms, the projection $P_L$ is replaced by the projection operator $M_{\chi_F}$ obtained by multiplication by the characteristic function $\chi_F$ of the fundamental domain $F$. The fact that $P_0M_{\chi_F}$ is a trace class operator was checked in [19], §3.3. The formula (50) is reminiscent of the Petersson scalar product. Indeed, to define the Petersson scalar product of two automorphic forms $f$ and $g$, one proceeds with the $L^2$-scalar product of $\chi_F f$ and $g$.

We note that we could have used the formula (55) in Theorem 34 below to define the Hecke operators directly. There (see also [40]) we give a direct proof that (55) is a representation of the Hecke algebra of double cosets of $\Gamma_0$ in $G$. On the other hand, using the space $H_{\Gamma_0}^{\Gamma_0}$ as a space of averaged sums over $\Gamma$ implies that the spaces of $\Gamma$-invariant vectors that we are considering in Theorem 33 correspond to the spaces of automorphic forms.

The advantage of the approach in Theorem 33 is that we have concrete formulæ for $\pi_0^p$ that are directly described in terms of the original representations $\pi_n$ and their interaction with $P_0M_{\chi_F}$. This will be used later in our paper for computing the traces and characters of the associated unitary representations.

**Proof of Theorem 33.** We use the fact that the sum in the formula (48) from Lemma 31 determines a finite-dimensional projection. Then the scalar product in (50) is further equal to

$$\left\langle P_{L^\Gamma_0} \left( \sum_{\gamma_0 \in \Gamma_0} \pi_0(\gamma_0) h_1 \right), P_{L^\Gamma_0} \left( \sum_{\gamma'_0 \in \Gamma_0} \pi_0(\gamma'_0) h_2 \right) \right\rangle = \langle \mathcal{P}_{\Gamma_0,L} h_1, \mathcal{P}_{\Gamma_0,L} h_2 \rangle = \langle \mathcal{P}_{\Gamma_0,L} h_1, h_2 \rangle. \quad (51)$$

Since $\mathcal{P}_{\Gamma_0,L}$ is a finite-dimensional projection on $L^{\Gamma_0}$, this is a well-defined scalar product in the space $H_{\Gamma_0}^{\Gamma_0}$.

The fact that the inclusions $H_{\Gamma_0}^{\Gamma_0} \subseteq H_{\Gamma_1}^{\Gamma_1}$ are isometric for $\Gamma_0, \Gamma_1 \in \mathcal{S}$ with $\Gamma_1 \subseteq \Gamma_0$ is proved exactly as in Proposition 20. Part (ii) is a straightforward consequence of the definition (50) of the scalar product on $\Gamma_0$-invariant vectors. It corresponds to the independence of the choice of the fundamental domain in the formula for the Petersson scalar product.

Let us prove the invariance assumption in (ii.3) of Definition 14. We will prove that $\pi_0^p(\sigma), \sigma \in G$, maps $H_{\Gamma_0}^{\sigma^{-1}}$ isometrically onto $H_{\Gamma_0}^{\sigma}$. As above, fix $h_1$ and $h_2$ in $\mathcal{D}_{L,\pi}$. Then, using the formula (51) and the construction of the representation $\pi_0^p$, we have

$$\left\langle \pi_0^p(\sigma) \left( \sum_{\gamma_0 \in \Gamma_{\sigma^{-1}}} \pi_0(\gamma_0) h_1 \right), \pi_0^p(\sigma) \left( \sum_{\gamma'_0 \in \Gamma_{\sigma^{-1}}} \pi_0(\gamma'_0) h_2 \right) \right\rangle \infty = \left\langle \sum_{\gamma_0 \in \Gamma_{\sigma}} \pi_0(\gamma_0)(\pi_0(\sigma) h_1), \sum_{\gamma'_0 \in \Gamma_{\sigma}} \pi_0(\gamma'_0)(\pi_0(\sigma) h_2) \right\rangle \infty = \left\langle P_{L^{\sigma}} \left[ \sum_{\gamma_0 \in \Gamma_{\sigma}} \pi_0(\gamma_0)(\pi_0(\sigma) h_1) \right], (\pi_0(\sigma) h_2) \right\rangle,$$
which, since $\pi_0(\sigma)$ is unitary on $H_0$, is further equal to
\[
\left\langle \left[ \pi(\sigma^{-1}) P_L \pi(\sigma) \right] \pi_0(\sigma^{-1}) \left( \sum_{\gamma_0 \in \Gamma_0} \pi_0(\gamma_0) (\pi_0(\sigma) h_1) \right), h_2 \right\rangle.
\] (52)

In the preceding formula $\pi(\sigma^{-1}) P_L \pi(\sigma)$ is the projection onto the space $\tilde{L} = \pi(\sigma^{-1}) L^\Gamma_\sigma$. This space is a $\Gamma_{\sigma^{-1}}$-wandering generating subspace for the representation $\pi|_{\Gamma_{\sigma^{-1}}}$. Denote by $P_{\tilde{L}}$ the orthogonal projection onto $\tilde{L}$. Then (52) is further equal to
\[
\left\langle P_{\tilde{L}} \left( \sum_{\gamma_0 \in \Gamma_{\sigma^{-1}}} \pi_0(\gamma_0) h_1 \right), h_2 \right\rangle.
\] (53)

By the technical condition (ii) in Definition 1, the subspace $\tilde{L}$ is equivalent to the subspace $L_{\Gamma_{\sigma^{-1}}}$ in the sense of (iii) in the present theorem. Hence by (iii) the expression in (53) is further equal to
\[
\left\langle P_{L}^{\Gamma_\sigma^{-1}} \left( \sum_{\gamma_0 \in \Gamma_{\sigma^{-1}}} \pi_0(\gamma_0) h_1 \right), h_2 \right\rangle = \left\langle \sum_{\gamma_0 \in \Gamma_{\sigma^{-1}}} \pi_0(\gamma_0) h_1, \sum_{\gamma_0' \in \Gamma_{\sigma^{-1}}} \pi_0(\gamma_0') h_2 \right\rangle_\infty.
\]

By the definition of the scalar product in (i) this chain of equalities proves that $\pi^p_0(\sigma)$ maps $H_{0}^{\Gamma_{\sigma^{-1}}}$ isometrically onto $H_{0}^{\Gamma_\sigma}$. More generally, for $\Gamma_0 \in \mathcal{S}$ the same type of argument gives us that $\pi^p_0(\sigma)$ maps $H_{0}^{\Gamma_{\sigma^{-1}} \cap \Gamma_0}$ onto $H_{0}^{\Gamma_\sigma \cap \sigma \Gamma_0 \sigma^{-1}}$. Hence, the invariance condition in (ii.3) of Definition 14 holds in the present case. Part (iv) is a straightforward consequence of the previous argument. \(\square\)

In the next theorem, we describe a unitarily equivalent representation of the Hecke operators acting on the spaces of $\Gamma_0$-invariant vectors in Theorem 33. Below we will prove that the projection $\mathcal{P}_{\Gamma_0, L} \in B(L^{\Gamma_0})$ in (48) is unitarily equivalent to a different projection introduced in (56). This is useful for computing the dimensions of the $\Gamma_0$-invariant vectors. The point is that we are proving that the range of $\mathcal{P}_{\Gamma_0, L}$, which is a subspace of $L^{\Gamma_0}$, is unitarily equivalent to the space of vectors associated with $H_0$ that are fixed by $\Gamma_0$.

The reproducing kernel formula for the projection onto the space of automorphic forms and the reproducing kernel formula in [53] for the Hecke operators prove that the spaces $H_0^{\Gamma_0}$ and the corresponding action of the Hecke operators on the spaces of $\Gamma_0$-invariant vectors in Lemma 32 and in the next theorem are the same (in the case of the upper half-plane) as those in the classical case.

**Theorem 34.** Under the conditions of Theorem 33 let $\Gamma_0 \in \mathcal{S}$. By Lemma 21, the representation of the Hecke algebra $\mathcal{H}_0(\Gamma_0, G)$ associated with the representation $\pi_0$ is defined by the correspondence
\[
[\Gamma_0 \sigma \Gamma_0] \rightarrow [\Gamma_0 : (\Gamma_0) \sigma] P_{H_{0}^{\Gamma_0}} \pi^p_0(\sigma) P_{H_{0}^{\Gamma_0}}, \quad \sigma \in G.
\] (54)

Then the following statements hold.

(i) Consider the following infinite sums over cosets of $\Gamma_0$, which are convergent because of conditions (v) and (vi) in Definition 1:
\[
A_0(\Gamma_0 \sigma \Gamma_0) = \sum_{\theta \in \Gamma_0 \sigma \Gamma_0} P_{L^{\Gamma_0}} \pi_0(\theta) P_{L^{\Gamma_0}}, \quad \sigma \in G;
\] (55)
(ii) The correspondence

\[ [\Gamma_0 \sigma \Gamma_0] \rightarrow A_0(\Gamma_0 \sigma \Gamma_0), \quad \sigma \in G, \]

determines a \(*\)-representation of \( \mathcal{H}_0(\Gamma_0, G) \) into \( B(P_{\Gamma_0, L} L^{\Gamma_0}) \). This representation is unitarily equivalent to the \(*\)-representation (see (54)) of the Hecke algebra by means of Hecke operators associated with the representation \( \pi_0 \).

Note that, in particular, the operator \( A(\Gamma_0) = \mathcal{P}_{\Gamma_0, L} \) is unitarily equivalent to the projection onto the \( \Gamma_0 \) invariant vectors.

Proof. This is similar to the proof of Theorem 22. As in Theorem 22 and its proof, with the formula (50) for the scalar product we can define partial isometries \( W_{\Gamma_0} : L^{\Gamma_0} \rightarrow H^{\Gamma_0} \) for \( \Gamma_0 \in \mathcal{S} \) by

\[
W^{\Gamma_0} l = \sum_{\gamma \in \Gamma_0} \pi_0(\gamma) l, \quad l \in L^{\Gamma_0}.
\]

Unlike in the case considered in Theorem 22, the operators \( W^{\Gamma_0} \) are partial isometries with initial space the projection \( \mathcal{P}_{\Gamma_0, L} \) in (48) and images equal to the spaces \( H^{\Gamma_0}, \Gamma_0 \in \mathcal{S} \).

We use the partial isometry \( W^{\Gamma_0} \) with initial space the projection \( \mathcal{P}_{\Gamma_0, L} \) in \( B(L^{\Gamma_0}) \).

Then \( W^{\Gamma_0} \) transforms the Hecke operator \( P_{H^{\Gamma_0}} \pi^p(\sigma) P_{H^{\Gamma_0}}, \sigma \in G \), unitarily into the expression in (55).

The formula (49) proves that for all \( \Gamma_0 \in \mathcal{S} \) and \( \sigma \in G \) we have

\[
A_0(\Gamma_0 \sigma \Gamma_0) = A_0(\Gamma_0 \sigma \Gamma_0) \mathcal{P}_{\Gamma_0, L} = \mathcal{P}_{\Gamma_0, L} A_0(\Gamma_0 \sigma \Gamma_0).
\]

The same formula proves that the operators

\[
A_0(\Gamma_0 \sigma \Gamma_0), \quad \sigma \in G,
\]

determine a representation of the Hecke algebra of double cosets of \( \Gamma_0 \) in \( G \). \( \square \)

Theorem 34 gives an explicit representation of the Hecke operators associated with the representation \( \pi^p_0 \) of \( \tilde{G} \) on \( \overline{H}^p_0 \), by directly using the information from the original representation \( \pi_0 \). We summarize this in Theorem 2.

Proof of Theorem 2. Let \( L^{\Gamma_0} \) be the subspace defined in (43) and endowed with the normalized scalar product in (44). With the use of Theorem 34 the proof of (4) becomes identical to that of the corresponding formula (46) in Corollary 24. We choose coset representatives as indicated in the statement of the theorem. In passing from (55) to (4) one simply needs to use the unitary operator

\[
\ell^2(\Gamma_0 \setminus \Gamma) \otimes L \cong \bigoplus [\Gamma_0 s_i] \otimes L, \quad L^{\Gamma_0} = \bigoplus \pi(s_i)L,
\]
(mapping \( \bigoplus [\Gamma_0 s_i] \otimes l_i \) to \( \bigoplus \pi(s_i)l_i \) for \( l_i \in L \). \( \square \)
When $\sigma$ is the identity in (4), we obtain the projection

$$\widehat{T}_{0,L} = \sum_{i,j} \sum_{\gamma \in s_i^{-1} \Gamma_0 s_j} P_L \pi_0(\gamma) P_L \otimes e_{\Gamma_0 s_i, \Gamma_0 s_j}.$$  \hfill (56)

Consequently, the operators in (4) belong to the algebra

$$\widehat{T}_{0,L} [B(l^2(\Gamma_0 \setminus \Gamma)) \otimes B(L)] \widehat{T}_{0,L}.$$

**Remark 35.** Assume the conditions of Theorem 34. By (4), one equivalent method for constructing a representation of the Hecke operators is as follows. As in Example 27, consider the vector space

$$\mathcal{V}_{\Gamma_0} = l^2(\Gamma_0 \setminus \Gamma), \quad \Gamma_0 \in \mathcal{S}.$$  

On $\mathcal{V}_{\Gamma_0}$ we take the scalar product defined as the linear extension of the bilinear form

$$\langle \Gamma_0 \sigma_1, \Gamma_0 \sigma_2 \rangle_{\pi_0} = \frac{1}{[\Gamma : \Gamma_0]} \sum_{\theta \in \sigma_1^{-1} \Gamma_0 \sigma_2} \text{Tr}(P_L \pi_0(\theta) P_L), \quad \Gamma_0 \in \mathcal{S}, \quad \sigma_1, \sigma_2 \in G.$$

For $\Gamma_0 \in \mathcal{S}$ we consider the usual algebraic representation of the Hecke operators on $\mathbb{C}(\Gamma_0 \setminus \Gamma)$. Using the scalar product $\langle \cdot, \cdot \rangle_{\pi_0}$ in the above formula on $\mathbb{C}(\Gamma_0 \setminus \Gamma)$, we obtain for the Hecke algebra a representation unitarily equivalent to the representation in (4).

This corresponds to considering the state $\varepsilon$ on $C^*(G)$ defined by

$$\varepsilon(\chi_{\sigma_1^{-1} \Gamma_0 \sigma_2}) = \frac{1}{[\Gamma : \Gamma_0]} \sum_{\theta \in \sigma_1^{-1} \Gamma_0 \sigma_2} \text{Tr}(P_L \pi_0(\theta) P_L), \quad \Gamma_0 \in \mathcal{S}, \quad \sigma_1, \sigma_2 \in G.$$

The state $\varepsilon$ cannot be used simultaneously on all levels $\Gamma_0 \in \mathcal{S}$ because of the renormalization factor $1/[\Gamma : \Gamma_0]$. Note that $\varepsilon$ is in fact the composition of the trace with the family of completely positive maps constructed in Theorem 37.

**6. The values of the character $\theta_{\pi_0}$ associated with the representation $\pi_0$**

In this section we derive a trace formula for the representation $\pi_0^P$. We note that $\pi_0^P$ is a type I representation of the $C^*$-algebra $C^*(G)$. As we noted in Proposition 17, $\pi_0^P$ extends to a representation of $\mathcal{A}(G, \overline{G})$, (or $\mathcal{A}^c(G, \overline{G})$ if a 2-cocycle is present). By Theorem 2, we have a formula for the Hecke operator $P_{H_0^p} \pi_0^P(\sigma) P_{H_0^p}$ associated with $\pi_0^P$. As explained below, this connects the trace formula for $\pi_0^P$ with the trace formula for the representation $\pi_0$.

The following lemma is proved in [12] (see the formula (13) there). Here we give a different proof.

**Lemma 36.** The character $\theta_{\pi_0^P} = \text{Tr} \pi_0^P$ of the representation $\pi_0^P$ is computed using (7).
Proof. Using the fact that the character is locally integrable [45], we have the following formula:

\[ \theta_{\pi_0}^{\sigma}(\sigma) = \lim_{\Gamma_0 \downarrow e, \Gamma_0 \in S} \frac{1}{\mu(\Gamma_0 \sigma \Gamma_0)} \text{Tr}(\pi_0^P(\chi_{\Gamma_0 \sigma \Gamma_0})), \quad \sigma \in G. \]  

(57)

Clearly, using the notation introduced before Lemma 21, we have

\[ \mu(\Gamma_0 \sigma \Gamma_0) = [\Gamma_0 : (\Gamma_0)_{\sigma}] \mu(\Gamma_0 \sigma), \quad \Gamma_0 \in S, \quad \sigma \in G. \]

Since the measure \( \mu \) is obtained from the Haar measure on the profinite completion of \( \Gamma \) and since, by the general assumptions, \( \mu \) is bivariant on \( \Gamma \), this is further equal to

\[ [\Gamma_0 : (\Gamma_0)] \frac{1}{[\Gamma : \Gamma_0]} \]

Hence, we continue the equality in (57) with

\[ \lim_{\Gamma_0 \downarrow e, \Gamma_0 \in S} \frac{[\Gamma : \Gamma_0]}{[\Gamma_0 : (\Gamma_0)_{\sigma}]} \text{Tr}(\pi_0^P(\chi_{\Gamma_0 \sigma \Gamma_0})). \]

Using the formula (39) in Lemma 21, we get that the last expression is equal to

\[ \lim_{\Gamma_0 \downarrow e, \Gamma_0 \in S} [\Gamma : \Gamma_0]^2 \text{Tr}(\pi_0^P(\chi_{\Gamma_0}) \pi_0^P(\sigma) \pi_0^P(\chi_{\Gamma_0})). \]

(58)

If \( K_0 \) is the closure of a subgroup in \( S \), then using the product of convolutor operators in \( C^*(\Gamma) \), we have \( (\chi_{K_0})^2 = \mu(K_0) \chi_{K_0} \). Hence, \( \frac{1}{\mu(K_0)} \chi_{K_0} \) is a projection.

We denote by \( \tilde{\chi}_{K_0} \) the renormalized convolutor operator \( \frac{1}{\mu(K_0)} \chi_{K_0} \). Thus, the right-hand side of (58) takes the form

\[ \lim_{\Gamma_0 \downarrow e, \Gamma_0 \in S} \text{Tr}(\pi_0^P(\tilde{\chi}_{\Gamma_0}) \pi_0^P(\sigma) \pi_0^P(\tilde{\chi}_{\Gamma_0})). \]

Since \( \pi_0^P \) is a representation of the algebra \( C^*(G, \Gamma) \), this is equal to

\[ \lim_{\Gamma_0 \downarrow e, \Gamma_0 \in S} \text{Tr}(P_{H_0}^\Gamma \pi_0^P(\sigma) P_{H_0}^\Gamma). \]

Proof of Corollary 4. The formula (8) follows immediately from (7) and (5). We group the terms of the sum in (8) according to conjugation classes, and take the limit. The fact that the group \( \Gamma_{st}^g \) is trivial prevents any \( \gamma \in \Gamma \) from appearing more than once in the sum. The absolute convergence of the sums involved in the limit process (Definition 1) implies the second assertion of the corollary.

Proof of Corollary 7. Recall that for a group element \( g \in G \), the stabilizer group \( \Gamma_{st}^g \) is \( \{ \gamma \mid \gamma g = g \gamma \} \). Let \( \sigma \) be an element in \( G = \text{PGL}(2, \mathbb{Z}[\frac{1}{p}]) \) such that \( \Gamma_{st}^\sigma \) is trivial. Then in (9) the sum

\[ \sum_{\gamma \in \Gamma} \text{Tr}(P_L \pi_n(\gamma \sigma \gamma^{-1})) \]

(59)

can be computed directly using the Berezin symbol function [3].
Indeed, this sum has the following expression:

$$
\sum_{\gamma \in \Gamma} \text{Tr}_{B(H_n)} (M_{X_F} \pi_0(\gamma) \pi_0(\sigma) \pi_0(\gamma^{-1})) = \sum_{\gamma \in \Gamma} \text{Tr}_{B(H_n)} (M_{X_F} \pi(\gamma) \pi_0(\sigma) \pi(\gamma^{-1}))
$$

$$
= \sum_{\gamma \in \Gamma} \text{Tr}_{B(H_n)} (\pi(\gamma^{-1}) M_{X_F} \pi(\gamma) \pi_0(\sigma)) = \sum_{\gamma \in \Gamma} \text{Tr}_{B(H_n)} (M_{X_F} \pi_0(\sigma)).
$$

Let $d\nu_0(z) = (\text{Im} z)^{-2} d\bar{z} dz$ be the canonical $\text{PSL}(2, \mathbb{R})$-invariant measure on $\mathbb{H}$ and let $\pi_n(\theta)(\bar{z}, z)$, $z \in \mathbb{H}$, be the Berezin contravariant symbol [3] of the unitary operator $\pi_n(\theta)$. Then the above chain of equalities can be continued with equality to

$$
\sum_{\gamma \in \Gamma} \int_{G} \pi_n(\theta)(\bar{z}, z) d\nu_0(z),
$$

which in turn is equal to

$$
\int_{\mathbb{H}} \pi_n(\theta)(\bar{z}, z) d\nu_0(z).
$$

This last expression is the character ‘$\text{Tr} \pi_n(\sigma)$’ of the representation $\pi_n$ (see [32]). (The same formula as above for the sum (59) was also computed differently in [53].)

Proof of Lemma 5. Denote by $H_0^R$ the Hilbert space on which the representation $\pi_0^R$ acts. Then

$$
\text{Tr}_{B(H_0^R)} \left( \int_{G^R} f(g) \pi_0^R(g) \, dg \right) = \sum_{\gamma} \text{Tr}_{B(H_0^R)} \left( P_{\pi(\gamma)L} \int_{G^R} f(g) \pi_0^R(g) \, dg \right)
$$

$$
= \sum_{\gamma} \int_{G^R} f(g) \text{Tr}_{B(H_0^R)} (P_{\pi(\gamma)L} \pi_0^R(g)) \, dg
$$

$$
= \sum_{\gamma \in \Gamma} \int_{G^R} f(g) \text{Tr}_{B(L)} (P_L \pi_0^R(\gamma g \gamma^{-1}) P_L) \, dg
$$

$$
= \int_{G^R} f(g) \sum_{\gamma} \text{Tr}_{B(L)} (P_L \pi_0^R(\gamma g \gamma^{-1}) P_L) \, dg.
$$

The second part of the lemma is a consequence of Corollary 4.

7. The case when the representation $\pi$ admits a ‘square root’ $\pi_0 \otimes \pi_0^{\text{op}}$

In this section we analyse the case where a unitary representation $\pi$ as in §3 admits a square root $\pi_0 \otimes \pi_0^{\text{op}}$, where $\pi_0$ is a (projective) unitary representation as in §5. Since the notation $\pi^\text{op}$ is reserved for the extension of $\pi$ to the Schlichting completion, we will use the notation $\pi_0^{\text{op}}$ in this section to denote the conjugate representation of $\pi_0$.

This is the situation of Example 28 in §4 when $G = \text{PGL}(2, \mathbb{Z}[1/p])$, $p$ is a prime, $\Gamma$ is the modular group, $\mathfrak{X} = \mathbb{H}$, and $\pi_{\text{Koop}}$ is the Koopman representation on $L^2(\mathbb{H}, \nu_0)$ corresponding to the action of $\text{PSL}(2, \mathbb{R})$ by Möbius transformations.
on the upper half-plane. By the Berezin quantization techniques [3], independently
noted in [42] (see also [36]), we have
\[ \pi_{\text{Koop}} = \pi_n \otimes \pi_n^\text{op}, \quad n \geq 1, \]
where \( \pi_n \) is any representation in the discrete series of the group \( \text{PSL}(2, \mathbb{R}) \). We
use this as a motivation for directly studying representations of the form \( \pi_0 \otimes \pi_0^\text{op} \),
where \( \pi_0 \) is as in the previous section.

Before proceeding to this study, we note one additional property common to all
the representations \( \pi_0 \) and \( \pi_0^\text{op} \) constructed in §§5 and 6. We will prove that these
representations are in a one-to-one correspondence with the completely positive
maps \( \Phi \) in Definition 14 that are the \( \ast \)-representations of the operator system.

We introduce the following notation. If \( g \in G \), then \( L_g \in \mathcal{C}^\ast(G) \) denotes the
convolutor with \( g \). For a function \( f \) in \( \mathcal{C}(G) \) let \( L_f \in \mathcal{C}^\ast(G) \) denote the operator
of convolution with \( f \). Such a representation plays the role of an ‘operator-valued
eigenvector’ for the Hecke algebra.

Indeed, we prove, in particular, the following property of the map \( \Phi \). If \( \sigma_1, \sigma_2 \in G \) and
\[ [\Gamma \sigma_1 \Gamma]\,[\Gamma \sigma_2] = \sum_j [\Gamma \theta_j], \]
then
\[ \Phi(L_{\chi[\Gamma \sigma_1 \Gamma]}) \Phi(L_{\chi[\Gamma \sigma_2]}) = \sum_j \Phi(L_{\chi[\Gamma \theta_j]}). \]

If \( \Phi \) were to take scalar values, then the above property would be exactly the
property an eigenvalue for the action of the Hecke algebra on \( \ell^2(\Gamma \setminus G) \) would have.

Let \( \mathcal{A}_0(G, \mathcal{G}) \subseteq \mathcal{A}(G, \mathcal{G}) \) be the dense subalgebra generated by convolution operators
with elements in \( G \) and by convolution operators with characteristic functions
of cosets of subgroups in \( \mathcal{G} \). In this section we denote by \( \mathcal{P}_{\pi_0,L} \) the projection
in (48) corresponding to \( \Gamma_0 = \Gamma \).

**Theorem 37.** Assume that \( \pi_0 \) is a representation of \( G \) as in Definition 1,
and define a linear map \( \Phi: \mathcal{A}_0(G, \mathcal{G}) \to \mathfrak{B}(L) \) as follows. Let \( g \in G \) and \( \Gamma_0 \in \mathcal{G} \), and let \( g\Gamma_0 \) be the corresponding coset. Define
\[ \Phi(L_{\chi[g\Gamma_0]}) = \sum_{\theta \in g\Gamma_0} P_L \pi_0(\theta) P_L, \]
\[ \Phi(L_g) = P_L \pi(g) P_L. \] (60)

Then \( \Phi \) has the following properties.

(i) \( \Phi|_{\mathcal{G}(K,G)} \) is a \( \ast \)-representation of the operator system \( \mathcal{G}(K,G) \) in Definition 8
and takes values in \( \mathfrak{B}(L) \).

(ii) Consider the vector space \( L(K, \mathcal{G}) \subseteq \mathcal{G}(K,G) \) in Definition 8. Then \( \Phi|_{L(K, \mathcal{G})} \)
has values in \( \mathfrak{B}(L) \mathcal{P}_{\pi_0,L} \), and \( \Phi|_{\mathcal{A}_0(K, \mathcal{G})} \) is a \( \ast \)-algebra representation of \( \mathcal{A}_0(K, \mathcal{G}) \)
into \( \mathcal{P}_{\pi_0,L} \mathfrak{B}(L) \mathcal{P}_{\pi_0,L} \). Consequently, \( \Phi|_{\mathcal{G}(K,G)} \) takes values in \( \mathfrak{B}(L) \mathcal{P}_{\pi_0,L} \mathfrak{B}(L) \).

(iii) \( \Phi \) is a completely positive map on \( \mathcal{A}_0(G, \mathcal{G}) \), that is, it maps positive elements
of the form \( X^* X \) with \( X \in \mathcal{A}_0(G, \mathcal{G}) \) into positive elements.
Proof. Part (i) is a consequence of (49). The formula (18) in Lemma 9, combined with Lemma 41, also provides a proof of (i).

Part (ii) is a consequence of (i).

The proof of (iii) is as follows. We use the fact that (i) proves the positivity of $\Phi$ on positive elements $X^*X$ with $X \in \mathbb{C}(\chi_{\sigma K} \mid \sigma \in G)$. For subgroups $\Gamma_0 \in \mathcal{S}$ we argue as follows. Let $K_0$ be the closure of $\Gamma_0$ in $\overline{G}$. We use Lemma 38 first to establish the positivity of $\Phi_{\Gamma_0}(X_0^*X_0)$ for $X_0 \in \mathbb{C}(\chi_{\sigma K_0} \mid \sigma \in G)$. The reduction formula (61) implies that $\Phi(X_0^*X_0)$ is also positive. □

This theorem can easily be generalized by replacing $\Gamma$ by a subgroup $\Gamma_0 \in \mathcal{S}$. In this case there is an obvious relation between the corresponding operator system representations (we use the definitions and notation in Theorem 37).

**Lemma 38.** Let $\Gamma_0$ be any subgroup in $\mathcal{S}$ and let $L^\Gamma_0$ be as in Theorem 33. Then, using $L^\Gamma_0$ instead of $L$, one can repeat the above construction for $\Gamma_0$ instead of $\Gamma$. The corresponding completely positive map $\Phi_{\Gamma_0}$ constructed in Theorem 37 will have the same properties as $\Phi$, with $\Gamma_0$ and $\Gamma_0$ replaced by $\Gamma$ and $K$.

Let $L$ be embedded into $L^\Gamma_0 = \bigoplus_i \pi(s_i)L \subseteq H_0$ by mapping $l$ in $L$ to the vector $l \oplus 0 \oplus 0 \oplus \cdots$. (Note that this is not the diagonal embedding of $L$ into $L^\Gamma$ used in Theorem 33.) Let $\tilde{P}_L$ be the projection of $L^\Gamma_0$ onto $L$. Then

$$\Phi = \tilde{P}_L \Phi_{\Gamma_0} \tilde{P}_L.$$ \hspace{1cm} (61)

**Proof.** This is straightforward from the formula (60). □

**Remark 39.** The operators $\Phi(\chi_{\sigma K}) \in B(L)$, $\sigma \in G$, are not isometries, since $\Phi$ is not a $*$-algebra representation. However, as we show below, the operators $\Phi(\chi_{\sigma K})$ are products of a projection and an isometry.

Indeed, for $\sigma \in G$, the partial isometry $L(\chi_{\sigma K})$ has as initial space the projection $L(\chi_{\sigma K\sigma^{-1}})$ and as range $L(\chi_K)$.

Consider the spaces $L^\Gamma_0$, $\Gamma_0 \in \mathcal{S}$, in Definition 19. These were defined there only for $\Gamma_0$ a subgroup of $\Gamma$. We define $L^\sigma \Gamma \sigma^{-1} \subseteq L^\Gamma_\sigma$ by the formula

$$L^\sigma \Gamma \sigma^{-1} = \pi_0(\sigma)L.$$ Then

$$\pi^p(L(\chi_{\sigma K})) = P_{L^\sigma \Gamma \sigma^{-1}} \pi^p_0(L(\chi_{\sigma K})) P_L.$$ On the other hand, using the skewed embedding of $L$ in $L^\Gamma_\sigma$ in Lemma 38, we find that

$$\Phi(\chi_{\sigma K}) = \tilde{P}_L \pi^p_0(L(\chi_{\sigma K})) P_L.$$ Here the projection $P_L$ corresponds to the standard embedding of $L$ into $L^\Gamma_\sigma$ as described in Definition 19, and $\tilde{P}_L$ is the projection in Lemma 38.

The completely positive maps constructed in Theorem 37 are the building blocks of the Hecke operators. In the next result we prove that a representation as in Theorem 37 encodes all the properties of the representation $\pi^p_0$. Hence, given $\Phi$, we can recover the representation $\pi^p_0$ and hence the representation $\pi$. 


Proposition 40. In the context of Theorem 37, the formula for the Hecke operators in Theorem 33 is as follows. Fix $\Gamma_0$ in $\mathcal{S}$ and choose a coset decomposition $\Gamma = \bigcup s_i \Gamma_0$. Then

$$[\Gamma_0 : (\Gamma_0)_{\sigma}] P_{H_0^{r_0}} \pi_0^r(\sigma) P_{H_0^{r_0}} = \sum_{i,j} \Phi\left(\chi_{s_i^{-1} \Gamma_0 \sigma \Gamma_0 s_j}\right) \otimes e_{\Gamma_0 s_i, \Gamma_0 s_j}, \quad \sigma \in G. \quad (62)$$

Therefore, there is a one-to-one correspondence between the representations $\pi_0$ as in Definition 1 and the completely positive maps $\Phi$ with the properties (i)–(iii) in Theorem 37.

Proof. The formula (62) is simply the formula (55) in Theorem 34, rewritten in the new context by using the formula (60) in Theorem 37.

Assume the properties (i)–(iii) in Theorem 37. We prove that the operators in (62) define a representation of the Hecke algebra of double cosets for all $\Gamma_0 \in \mathcal{S}$.

Ultimately, verification of the multiplicativity of the Hecke operators in (62) reduces to verification of identities of the form

$$\sum_{\gamma_0 \in \Gamma_0} P_{L^{r_0}} \pi_0(\sigma_1 \gamma_0) P_{L^{r_0}} \left(\pi_0(\gamma_0^{-1} \sigma_2)\right) P_{L^{r_0}} = P_{L^{r_0}} \left(\pi_0(\sigma_1 \sigma_2)\right) P_{L^{r_0}}. \quad (63)$$

The reason this equality holds is that $P_{L^{r_0}}$ is the projection onto a $\Gamma_0$-wandering generating subspace of $H$ (see also the proof below of Lemma 9). Thus, the only identity because of which the representation in (62) is a representation of the Hecke algebra is the identity

$$\sum_{\gamma_0 \in \Gamma_0} \pi(\gamma_0) P_{L^{r_0}} \pi(\gamma_0^{-1}) = \text{Id}_{H_0}.$$

In view of the decomposition $P_{L^{r_0}} = \sum_i \pi(s_i) P_L \pi(s_i)$, this is implied by the identity

$$\sum_{\gamma \in \Gamma} \pi(\gamma) P_L \pi(\gamma^{-1}) = \text{Id}_{H_0}.$$

But this is exactly the identity proving the multiplicativity property (ii) in Theorem 37.

Thus, if we know that $\Phi$ has the multiplicativity property (ii) in Theorem 37, then we automatically get that the completely positive maps $\Phi_{\Gamma_0}$ have the corresponding multiplicativity property in (ii) on the corresponding operator systems $B(L^{r_0}) \mathcal{P}_{\Gamma_0,L}$ for $\Gamma_0 \in \mathcal{S}$. Let $\mathcal{P}_{\Gamma_0,L}$ be the projection in (48).

Since the operator systems contain the corresponding Hecke algebras $\mathcal{H}_{\Gamma_0}(\Gamma_0, \overline{G})$, it follows that the representation in (62) is a $\ast$-algebra representation of the inductive limit of all the above Hecke algebras into the inductive limit of the spaces $\mathcal{P}_{\Gamma_0,L} B(L^{r_0}) \mathcal{P}_{\Gamma_0,L}$. But this inductive limit is exactly the space of bounded operators acting on the Hilbert space $\mathcal{H}_{\Gamma_0}^p$. Since, along with the Hecke algebras, we also have a representation of the spaces of cosets, it follows that we have reconstructed the unitary representation $\pi_0^p$ of the algebra $C^*(\overline{G})$. Hence, we can recover $\pi_0$ because of Theorem 37. □
In the statements discussed below (Lemmas 9 and 41 and Theorem 10) we recall results in [38]; see also [37] and [40] (we adapt the statement of these results to the present framework). In Lemma 9 we prove a result complementing the result in Theorem 37. We prove that the completely positive maps in Theorem 37 have a natural lifting to the algebra $L(G) \otimes B(L)$. This lifting was an important tool in the proofs in [37] of the essential norm estimates of the spectrum of the Hecke operators. In particular, we give an alternative interpretation for property (ii) in Theorem 37.

Proof of Lemma 9 (this was also proved in [38], Proposition 2.2 and Lemma 3.1).

The main step of the proof of the multiplicativity property in (18) is the following: by identifying the coefficients of the elements $\rho(g), g \in G$, on both sides of the equation, one reduces the proof of multiplicativity to a proof of the equality (also used in the proof of Proposition 40)

$$
\sum_{\gamma \in \Gamma} P_L \pi_0(\sigma_1 \gamma) P_L \pi_0(\gamma^{-1} \sigma_2) P_L = \sum_{\gamma \in \Gamma} P_L \pi_0(\sigma_1)\pi(\gamma) P_L \pi(\gamma^{-1})\pi_0(\sigma_2) P_L
$$

$$
= P_L \pi_0(\sigma_1)\pi_0(\sigma_2) P_L = P_L \pi_0(\sigma_1 \sigma_2) P_L, \quad \sigma_1, \sigma_2 \in G.
$$

The operators $\tilde{\Phi}_{\pi_0,L}$ in Lemma 9 have an interpretation analogous to that in Remark 39. Moreover, because of the convergence assumptions in Theorem 33, for the sums of the form $\sum_{\theta \in C} P_L \pi_0(\theta) P_L$ with cosets $C$ in $\overline{G}$, the operators $\tilde{\Phi}_{\pi_0,L}(C)$ are liftings of the operators $\Phi(C)$ in Theorem 37.

This is proved in the following result, which is a straightforward consequence of Theorem 37 and Lemma 9.

Lemma 41. Let $\varepsilon$ be the unbounded character on $\ell^1(G) \subseteq L(G)$ which associates with an element $x$ in $\ell^1(G)$ the sum of its coefficients, and extend $\varepsilon$ to an unbounded character $\tilde{\varepsilon} = \varepsilon \otimes \text{Id}_{B(L)}$:

$$
\tilde{\varepsilon}: \ell^1(G) \otimes B(L) \subseteq \mathcal{A}(G) \otimes B(L) \to B(L).
$$

The convergence assumption (v) in Definition 1 implies that the image of the *-representation $\tilde{\Phi}_{\pi_0,L}$ constructed in Lemma 9 is contained in the domain of $\tilde{\varepsilon}$. Consequently, the map $\Phi$ in Theorem 37 satisfies the commutative diagram

$$
\tilde{\varepsilon} \circ \tilde{\Phi}_{\pi_0,L}|_{\mathcal{O}(K,\overline{G})} = \Phi|_{\mathcal{O}(K,\overline{G})}.
$$

Proof. This is straightforward from the formulae for $\Phi$ and $\tilde{\Phi}_{\pi_0,L}$ in Theorem 37 and Lemma 9, respectively. □

The operators $\tilde{\Phi}_{\pi_0,L}$ are used to construct a unitarily equivalent representation (see the formula (19) for the Hecke operators associated with the unitary diagonal representation $\pi_0 \otimes \pi_0^{\text{op}}$ of $G$, where $\pi_0$ is as in Definition 1). This was first proved (in the case of Murray–von Neumann dimension equal to 1) in Theorem 22 of [37] (see also [40] for a more concise exposition), and then generalized to arbitrary dimension in Theorem 3.2 of [38]. For the convenience of the reader (since we are explaining the example of the Hecke algebra representation associated with the unitary representation $\pi_0 \otimes \pi_0^{\text{op}}$ of $G$) we recall in Theorem 10 the statement of
Theorem 3.2 in [38]. In Theorem 12 we provide an alternative proof that (19) gives a representation of the Hecke algebra of double cosets of the subgroup $\Gamma$ in $G$.

We use the identifications proved in Example 30 and the operators $\widetilde{\Phi}_{\pi_0, L}$ in Lemma 9 to explicitly describe the Hecke operators associated with the unitary diagonal representation $\pi_0 \otimes \pi_0^{\text{op}}$ of $G$, on $\Gamma$-invariant vectors. In this case the Hilbert spaces of $\Gamma$-invariant vectors are easier to handle, since we can canonically identify these spaces with the $L^2$-spaces of the von Neumann algebra of operators commuting with the image of the representation of the group $\Gamma$.

Using the Berezin quantization methods [3], or alternatively the results in [42], we proved in [37] that the above model for the Hecke operators acting on $\Gamma$-invariant vectors for $\pi_0 \otimes \pi_0^{\text{op}}$ is unitarily equivalent to the representation of the Hecke operators of the associated von Neumann algebra of Maass forms.

The content of Theorem 10 is an explicit operator algebra model of the previous representation associated with $\pi_0 \otimes \pi_0^{\text{op}}$.

We give a direct proof, in the particular case where $\dim \{\pi_0(\Gamma)\}'' H = 1$ (Theorem 12), of the fact that the Hecke operators in (19) of Theorem 10 define a multiplicative representation of the Hecke algebra of double cosets of $\Gamma$ in $G$. The proof will show that the Hecke algebra representation determined by the Hecke operators for $\pi_0 \otimes \pi_0^{\text{op}}$ is obtained from a canonical Hecke algebra representation, which is further composed with a quotient map. For simplicity of exposition we assume in the rest of the paper that the groups $\Gamma$ and $G$ have infinite non-trivial conjugacy classes, and hence that the associated von Neumann algebras have unique traces.

Remark 42. Assume that $\dim \{\pi_0(\Gamma)\}'' H = 1$. In the setting of Theorem 22 in §3, we take $L = L_0 = \mathbb{C} \xi$ with a cyclic trace vector $\xi \in H_0$ for $\pi_0|_1$. The construction in Lemma 9 gives a linear map $\widetilde{\Phi}_{\pi_0, L_0}$, which we now denote by $t$:

$$t: \mathcal{H}_{\text{red}}(\Gamma, G) \rightarrow \mathcal{R}(G) \otimes B(L_0) \cong \mathcal{R}(G).$$

Since $L_0$ is one-dimensional, we can use the vector $\xi$ to identify $L_0 = \mathbb{C} \xi$ with $\mathbb{C}$ and we can replace $P_{L_0} \pi_0(\theta) P_{L_0}$ by

$$\text{Tr}(P_{L_0} \pi_0(\theta) P_{L_0}) = \langle \pi_0(\theta) \xi, \xi \rangle, \quad \theta \in G.$$

For a coset $C$ of a subgroup in $\mathcal{S}$, the formula for $\widetilde{\Phi}_{\pi_0, L_0}(\chi_C)$ in Lemma 9 now takes the form

$$t(\chi_C) = \sum_{\theta \in C} \langle \pi_0(\theta) \xi, \xi \rangle \rho(\theta).$$

Consider the composition of $t$ with the canonical anti-isomorphism between $\mathcal{L}(G)$ and $\mathcal{R}(G)$ (for simplicity we denote the composition map also by $t$). Thus, $t$ is a linear map from $C^*(G)$ with values in $\mathcal{L}(G)$. Let $\lambda_g$ be the left convolutor with an element $g \in G$. Then $t$ is given by the formula

$$t(\chi_C) = \sum_{\theta \in C} \overline{\langle \pi_0(\theta) \xi, \xi \rangle} \lambda(\theta).$$

By Lemma 9, $t$ is a *-preserving multiplicative representation of the operator system $\mathcal{O}_{\Gamma, G} = \mathcal{O}(K, \overline{G})$ in Definition 8:

$$\mathcal{O}_{\Gamma, G} = \left[ \text{Sp}\{\chi_{\sigma_1 K} \mid \sigma_1 \in G\} \right] \left[ \text{Sp}\{\chi_{\sigma_2 K} \mid \sigma_2 \in G\} \right]^*.$$
We use the correspondence between the characteristic functions $\chi_{\sigma_1 K}$, $\chi_{K \sigma_2}$, and $\chi_{\sigma_1 K \sigma_2}$ and the respective cosets $\sigma_1 \Gamma$, $\Gamma \sigma_2$, and $\sigma_1 \Gamma \sigma_2$ in $G$. Then the $*$-preserving multiplicativity property for $t|_{\sigma_1 \Gamma, G}$ takes the form

$$t(\sigma_1 \Gamma) t(\sigma_2^{-1} \Gamma)^* = t(\sigma_1 \Gamma) t(\Gamma \sigma_2) = t(\sigma_1 \Gamma \sigma_2), \quad \sigma_1, \sigma_2 \in G.$$  

Remark 43. In practice it is difficult to find a cyclic trace vector $\xi$ as above. So it is preferable to use the construction in §5, Theorem 33. Thus, $\pi_0$ comes from a larger representation $\pi$ of $G$ into the unitary group of a Hilbert space $H$ by restricting to a space $H_0 \subseteq H$ that is invariant under $\pi(G)$. In this case we use a choice of the $\Gamma$-wandering generating subspace $L$ for $\pi|_{\Gamma}$. In the analytic discrete series of unitary representations $\pi_n$ ($n \geq 1$) of $\text{PSL}(2, \mathbb{R})$, such a choice is almost canonical, since it involves the selection of a fundamental domain for the action of $\Gamma$ on $\mathbb{H}$.

To obtain the representation $t$ from $\tilde{\Phi}_{\pi_0, L}$ in a straightforward manner, one proceeds directly as follows [38]. Consider the conditional expectations $E_{\mathcal{B}(G) \otimes B(L)}$, $E_{\mathcal{A}(\Gamma) \otimes B(L)}$ from $\mathcal{B} = \mathcal{B}(G) \otimes B(L)$, and $\mathcal{A} = \mathcal{A}(\Gamma) \otimes B(L)$ onto the respective algebras $\mathcal{B}(G) \otimes \mathbb{C} \text{Id}_B(L)$ and $\mathcal{A}(\Gamma) \otimes \mathbb{C} \text{Id}_B(L)$. The conditional expectations are simply computed by taking the operator trace on the tensor factor corresponding to $B(L)$.

For a coset $C$ as in Remark 42 we have

$$\tilde{t}(\chi_C) = E_{\mathcal{A}(\Gamma) \otimes \text{Id}_B(L)}(\tilde{\Phi}_{\pi_0, L}(\chi_C)) = \sum_{\theta \in C} \text{Tr}(P_L \pi_0(\theta)) \rho(\theta). \quad (64)$$

We use the formula (17) for the projection $P_0$ and define

$$\xi_0 = E_{\mathcal{A}(\Gamma) \otimes \text{Id}_B(L)}(P_0) = \sum_{\gamma \in \Gamma} \text{Tr}(P_L \pi_0(\gamma)) \rho(\gamma).$$

Since $\dim(\pi_0(\Gamma)^\vee) H_0 = 1$, and $P_0$ is a projection in $\mathcal{A} \subseteq \mathcal{B}$ of trace 1, it follows that $\xi_0$ has zero kernel. Moreover ([38], Proposition 3.3), the conditional expectation map, corrected with the inverse of the square root of $\xi_0$, is a von Neumann algebra isomorphism when restricted to $P_0 \mathcal{B} P_0$. Thus,

$$\tilde{E} = (\xi_0)^{-1/2} E_{\mathcal{A}(\Gamma) \otimes \text{Id}_B(L)}|_{P_0 \mathcal{B} P_0} (\xi_0)^{-1/2} \quad (65)$$

is a von Neumann algebra isomorphism from $P_0 \mathcal{B} P_0$ onto $\mathcal{B}(G)$. We define

$$t(\chi_C) = \tilde{E}(\tilde{t}(C)).$$

Then $t|_{\sigma_1 \Gamma, G}$ is an isomorphism from $\mathcal{A}(\Gamma, G)$ into $\mathcal{B}(G)$ (see Lemma 3.3 in [38] for the proof). Combining (64) and (65), we obtain the following alternative formula for the representation $t$ in Remark 42:

$$t(\chi_C) = (\xi_0)^{-1/2} \left[ \sum_{\theta \in C} \text{Tr}(P_L \pi_0(\theta)) \rho(\theta) \right] (\xi_0)^{-1/2}.$$
Proof of Theorem 12. Fix \( \sigma \in G \). Then the double coset \( \Gamma \sigma \Gamma \) decomposes as \( \Gamma \sigma \Gamma = \bigcup \Gamma \sigma \Gamma s_i = \bigcup \Gamma \sigma \Gamma r_j \). Hence,

\[
\chi_K(L(\chi_K \sigma K) \otimes L(\chi_K \sigma K)^\text{op}) \chi_K = \sum_{i,j} [L(\chi_K \sigma s_i) \otimes L(\chi_K s_j)^\text{op}] \chi_{s_i \sigma^{-1} K s_j^{-1} \cap K}
\]

\[
= \sum_{a,b} \chi_{r_a \sigma K \sigma^{-1} r_b \cap K} [L(\chi_{r_a \sigma K}) \otimes L(\chi_{r_b \sigma K})^\text{op}]
\]

\[
= \sum_{i,j,a,b} \chi_{r_a \sigma K \sigma^{-1} r_b \cap K} [L(\chi_{r_a \sigma K \sigma^{-1} s_j}) \otimes L(\chi_{r_b \sigma K \sigma^{-1} s_j})^\text{op}] \chi_{s_i \sigma^{-1} K s_j \cap K}.
\]

(66)

Here \( K_{\sigma^{-1}} \) is the closure in \( K \) of the subgroup \( \Gamma_{\sigma^{-1}} = \sigma^{-1} \Gamma \sigma \cap \Gamma \).

Using (66), one proves immediately (see, for example, the computations in [37], §5 or [40]) that the linear map in the statement is multiplicative. This completes the proof of part (i).

The representation \( t \) of the operator system \( \mathcal{O}_{\Gamma, G} = \mathcal{O}(K, G) \) extends in an obvious way to a representation \( \tilde{t} \) of the operator system \( L^\infty(G, \mu) \text{Sp}\{L(\chi_{\sigma K}) | \sigma \in G\} \) extends in an obvious way to a representation \( \tilde{t} \) of the operator system \( L^\infty(G, \mu) \text{Sp}\{L(\chi_{\sigma K}) | \sigma \in G\}^\ast \).

Then \( \tilde{t} \) extends to a ‘double’ representation \( t_2 \) of an operator system contained in \( C^\ast((G \times G) \text{op}) \times L^\infty(G, \mu) \) and containing the image of the Hecke algebra constructed in (i). This concludes the proof of part (ii).

An important observation for the proof of part (iii) is that all the operations that are involved in multiplication of two elements of the form \( \chi_K(L(\chi_K \sigma_1 K) \otimes L(\chi_K \sigma_2 K)^\text{op}) \chi_K \) remain inside the domain of the representation \( t_2 \) (see the first equality in the chain of equalities in (66)). Indeed, these operations involve only convolutions of the form

\[
L(\chi_{\sigma_1 K}) L(\chi_{\sigma_2 K}), \quad \sigma_1, \sigma_2 \in G,
\]

or their opposites.

Consequently, the composition of \( t_2 \) with the map \( \mathcal{D} \Phi \) in part (i) of the theorem gives a representation of the Hecke algebra. \( \square \)

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