On almost EGS theorems

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Abstract

We show assuming small data that massless solutions to the reflection symmetric Einstein-Vlasov system with Bianchi VII0 symmetry which are not locally rotational symmetric, can be arbitrarily close to and will remain close to isotropy as regards the shear. However in general the shear will not tend to zero and the Hubble normalised Weyl curvature will blow up. We thus have generalised the work [10, 16] which considered a non-tilted radiation fluid to the massless Vlasov case. This represents another example of the fact that almost-EGS theorems do not hold in general and that collisionless matter behaves differently than a perfect fluid.

1 Introduction

Since the microwave background is almost isotropic it is natural to consider an isotropic matter distribution. What are the consequences of this assumption for the space-time? The theorem of Ehlers, Geren and Sachs [5] gave an answer to this question proving that for collisionless matter the space-time has to be either stationary or Robertson-Walker. This was generalised later to the Boltzmann case [15].

What happens if the matter distribution is almost isotropic and the universe is non-stationary? One might think that the space-time has to be almost isotropic. Different results were obtained proving almost EGS-theorems following the research line initiated in [14].

However these results were obtained under certain assumptions which do not hold in general [11] [12].

If a cosmological constant is present non-linear stability and isotropisation of solutions has been shown for a variety of matter models cf. [13] [6] and references therein. What happens if no cosmological constant is present?

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In [16] it was shown that Bianchi VII\textsubscript{0} solutions which are not locally rotational symmetric (LRS) with a non-tilted perfect fluid isotropise as regards the shear but do not isotropise as regards the Hubble normalised Weyl curvature. They also showed that self-similarity breaking occurs for any non-LRS Bianchi VII\textsubscript{0} solution. This was proven for any non-tilted fluid except for a radiation fluid. The latter case was proven in [10]. Afterwards these results were extended to a tilted fluid in [4, 9].

One could argue that these results are special since the matter model is a perfect fluid, but more recently the massive case for reflection symmetric solutions to the Einstein-Vlasov system with Bianchi VII\textsubscript{0} symmetry was covered in [7]. Since the microwave background comes from massless particles it is of special interest to treat this case and in the present paper we obtain a similar result for the massless case assuming small data.

The massless case (radiation fluid) when treating a tilted or non-tilted fluid for a fluid with a linear equation of state \( P = (\gamma - 1)\rho \) was more complicated in the case \( \gamma = \frac{4}{3} \). The reason is that in this case some eigenvalues of the linear part vanish and center manifold theory has to be used. As a consequence the shear variables have a polynomial decay for \( \gamma = \frac{4}{3} \) while they have an exponential decay for \( \frac{2}{3} < \gamma < 2 \) with \( \gamma \neq \frac{4}{3} \).

In the Vlasov case the massive and the massless case are also different. In the massive case the solutions tend to the same behaviour as in the dust case forming a system of differential equations where the linear part has negative eigenvalues. In the massless case the linear part has an eigenvalue which vanishes, but now instead of an equilibrium point there is an equilibrium line.

Nevertheless in both massive and massless Vlasov case there is an exponential decay. The behaviour at late times thus differs from that of a radiation fluid where a polynomial decay was found [10].

Moreover in [10] it was shown that the shear tends to zero both in the non-tilted and the tilted case. We show here that for collisionless matter although the shear will always remain small, it nevertheless does not tend to zero, which is another difference with respect to a fluid.

Finally the result of this paper extends the known results concerning the late time behaviour of massless solutions to the Einstein-Vlasov system with Bianchi symmetries where there has been some recent progress [1, 2, 8].

### 2 The massless Einstein-Vlasov system

In this section we introduce the massless Einstein-Vlasov system with Bianchi symmetry. Consider a four-dimensional oriented and time oriented Lorentzian manifold \((\mathcal{M}, g, t)\) and a distribution function \(f\), then the massless Einstein-Vlasov system is written as

\[
G_{\alpha\beta} = T_{\alpha\beta},
\]

\[
\mathcal{L}f = 0,
\]

where \(G_{\alpha\beta}\) is the Einstein tensor and \(T_{\alpha\beta}\) is the energy-momentum tensor defined by

\[
T_{\alpha\beta} = \int_{\mathcal{H}\setminus\{0\}} \chi p_{\alpha} p_{\beta}.
\]
Here, the integration is over the future pointing light-cone $\mathcal{H}$ at a given spacetime point which is defined by

$$p_\alpha p_\beta \, 4^\alpha_\beta = 0, \quad p^0 > 0$$

with the apex removed, and $\chi$ is the distribution function multiplied by the Lorentz invariant measure and $\mathcal{L}$ the Liouville operator.

The basic equations we will use can be found in Sections 7.3–7.4 and Chapter 25 of [13]. We also refer to this book for an introduction to the Einstein-Vlasov system. Let $\Sigma$ be a spacelike hypersurface in $\mathcal{M}$ with $n$ its future directed unit normal. Let $g$ be the Riemannian metric induced on $\Sigma$ by $4^g$. We define the second fundamental form as $k(X,Y) = \langle \nabla_X n, Y \rangle$ for vectors $X$ and $Y$ tangent to $\Sigma$, where $\nabla$ is the Levi-Civita connection associated with $4^g$. The Hamiltonian and momentum constraint equations are as follows:

$$R - k_{ij}k^{ij} + k^2 = 2\rho,$$

$$\nabla^j k_{ji} - \nabla_i k = -J_i,$$

where $k = k_{ab}g^{ab}$ the trace of the second fundamental form, $R$ and $\nabla$ the scalar curvature and the Levi-Civita connection of $g$ respectively, and matter terms are given by $\rho = T_{\alpha\beta}n^\alpha n^\beta$ and $J_i X^i = -T_{\alpha\beta}n^\alpha X^\beta$ for $X$ tangent to $\Sigma$. Here and throughout the paper we assume that Greek letters run from 0 to 3, while Latin letters from 1 to 3, and also follow the sign conventions of [13].

2.1 The massless Einstein-Vlasov system with Bianchi symmetry

A Bianchi spacetime is defined to be a spatially homogeneous spacetime whose isometry group possesses a three-dimensional subgroup that acts simply transitively on spacelike orbits. A Bianchi spacetime admits a Lie algebra of Killing vector fields. These vector fields are tangent to the group orbits, which are the surfaces of homogeneity. Using a left-invariant frame, the metric induced on the spacelike hypersurfaces depends only on the time variable. Let $G$ be the three-dimensional Lie group, $e_i$ a basis of the Lie algebra, and $\xi^i$ the dual of $e_i$. The metric of the Bianchi spacetime in the left-invariant frame is written as

$$4^g = -dt \otimes dt + g_{ij} \xi^i \otimes \xi^j$$

on $\mathcal{M} = I \times G$ with $e_0$ future oriented. Define the structure constants by

$$[e_i, e_j] = C_{ij}^l e_l.$$

We will need equations (25.17)–(25.18) of [13] (without scalar field) with the notation $T_{ab} = S_{ab}$:

$$\dot{g}_{ab} = 2k_{ab},$$

$$\dot{k}_{ab} = -R_{ab} + 2k_{ai}k^{ai} - k k_{ab} + S_{ab},$$

where the dot means the derivative with respect to time $t$ and $R_{ab}$ is the Ricci tensor associated to the induced 3-metric. Note that in the massless case

$$g^{ab}S_{ab} = \rho.$$
Since $k$ does not depend on spatial variables, the constraint equations are as follows:

$$R - k_{ij}k^{ij} + k^2 = 2\rho,$$  \hspace{1cm} (4)

$$\nabla^i k_{ji} = -J_i,$$  \hspace{1cm} (5)

where we have dropped the bar on the covariant derivative by a slight abuse of notation. Following conventions of [13]

$$\nabla_{e_j} e_l = \Gamma^i_{jl} e_i$$

and the connection coefficients can be expressed in terms of the structure constants, cf. (25.3) of [13]:

$$\Gamma^i_{jl} = \frac{1}{2} g^{mi}(-C^n_{lm}g_{nj} + C^n_{mj}g_{ln} + C^n_{jl}g_{nm}).$$  \hspace{1cm} (6)

Note that as a consequence from last equation

$$\Gamma^i_{il} = C^n_{nl}, \quad \Gamma^i_{jj} = g^{mi}g^{nj}C^n_{mj}, \quad \Gamma^i_{jl}g^{jl} = g^{mi}C^n_{ml}.$$  \hspace{1cm} (7)

since $g^{mi}C^n_{mi}$ vanishes due to the symmetry of the metric and the antisymmetry of the structure constants. Moreover the only non-zero components of $\Gamma^i_{\beta\gamma}$ are $\Gamma^a_{\beta\gamma}$ and

$$\Gamma^0_{ab} = k_{ab}, \quad \Gamma^0_{a0} = \Gamma^0_{0a} = k^a_a,$$  \hspace{1cm} (8)

cf. Section 25.1 of [13].

We wish to express the momentum constraint in terms of the connection coefficients. We have

$$\nabla^a k_{bc} = g^{ad}\nabla_d k_{bc} = g^{ad}\left(\frac{\partial k_{bc}}{\partial x^d} - \Gamma^f_{db}k_{fc} - \Gamma^f_{dc}k_{bf}\right) = -g^{ad}(\Gamma^f_{da}k_{fc} + \Gamma^f_{df}k_{bf}),$$

which means

$$J_i = g^{ad}(\Gamma^f_{da}k_{fi} + \Gamma^f_{df}k_{af})$$

and that one can express $J_i$ in terms of the metric, second fundamental form and the structure constants. Using (7) and considering the Bianchi A case where $C^i_{ml} = 0$, the last equation turns to

$$J_i = k^m_i C^n_{mi}.$$  \hspace{1cm} (9)

Now denote by $\epsilon_{ijk}$ the standard permutation symbol. For Bianchi A spacetime we have that (cf. E.1 of [13] with $\alpha_j = 0$)

$$C^n_{mi} = \epsilon_{mi}n^l_n$$  \hspace{1cm} (10)

where $n^l_k$ is a symmetric matrix also called the structure constant matrix which characterises the Bianchi type. We thus have:

$$J_i = \epsilon_{mi}n^l_n k^m_n.$$  \hspace{1cm} (10)
Below, we collect and derive several useful equations. Using the fact that
\[ \dot{g}^{ab} = -2k^{ab} \]  
we obtain
\[ \dot{k}_b^a = -R_b^a - k k_b^a + S_b^a, \]  
and by taking the trace of \( \dot{k}_b^a \) we have
\[ \dot{k} = -R - k^2 + \rho. \]  
It is convenient to express the second fundamental form as
\[ k_{ab} = \sigma_{ab} + Hg_{ab}, \]  
where \( \sigma_{ab} \) is the trace free part and \( H = \frac{1}{3}k \) is the Hubble parameter. Then \( \rho \) becomes
\[ \dot{H} = -3H^2 - \frac{1}{3}R + \frac{1}{3}\rho, \]  
and \( \Omega \) becomes
\[ \Omega = \frac{\rho}{3H^2} = 1 + \frac{1}{6}R - \frac{1}{6}F, \]  
where \( R = \frac{\dot{R}}{H^2} \) and \( F = \frac{\sigma_{ab}\sigma^{ab}}{H^2} \). In terms of the trace free part \( \dot{\sigma}_b^a \) transforms into
\[ \dot{\sigma}_b^a = -3H\sigma_b^a - R_b^a + S_b^a - (3H^2 + \dot{H})\delta_b^a = -3H\sigma_b^a - R_b^a + S_b^a - (-\frac{1}{3}R + \frac{1}{3}\rho)\delta_b^a, \]  
or
\[ \dot{\sigma}_b^a = -3H\sigma_b^a - r_b^a + \pi_b^a, \]  
where \( r_b^a \) and \( \pi_b^a \) are the trace free part of \( R_b^a \) and \( S_b^a \) respectively.

By the constraint equation \( \Omega \) one can eliminate the energy density such that \( \rho \) reads:
\[ \dot{k} = -\frac{1}{2}R - \frac{1}{2}k^2 - \frac{1}{2}k_{ij}k^{ij}. \]  
Using the trace free part of \( k_{ab} \) and the Hubble variable we obtain
\[ \frac{d}{dt}(H^{-1}) = -\frac{\dot{H}}{H^2} = 2 + \frac{1}{6}\dot{R} + \frac{\Sigma_a^b\Sigma^a_b}{6}, \]  
where we have defined
\[ \Sigma^a_b = \sigma^a_b. \]  
It is convenient to introduce a dimensionless time variable \( \tau \) as follows:
\[ \frac{d\tau}{dt} = H^{-1}. \]
and denote derivation with respect to that variable by a prime. Sometimes it is also useful to use the variable \( q \):

\[
q = -1 - \frac{\dot{H}}{H^2} = 1 + \frac{1}{6} R + \frac{1}{6} F,
\]

where we have used (17) in the last equation. The evolution equation of \( \Sigma_b^a \) is then

\[
(\Sigma_b^a)' = -\left( 3 + \frac{\dot{H}}{H^2} \right) \Sigma_b^a + \frac{\pi_b^a - r_b^a}{H^2} = (q - 2) \Sigma_b^a + \frac{\pi_b^a - r_b^a}{H^2}.
\]

Using (17) and (14) we have

\[
(\Sigma^a_b)' = \left( -1 + \frac{1}{6} \dot{R} + \frac{1}{6} F \right) \Sigma^a_b + 3 \left( 1 + \frac{1}{6} \dot{R} - \frac{1}{6} F \right) \frac{\pi_b^a - r_b^a}{\rho}.
\]

Since \( \Sigma_b^a \) is trace free sometimes it is convenient to work with \( \Sigma_+^a \) and \( \Sigma_-^a \) as was done in [7] which are defined by

\[
\Sigma_+ = \frac{1}{2H} (\sigma_2^2 + \sigma_3^2), \quad \Sigma_- = \frac{1}{2\sqrt{3}H} (\sigma_2^2 - \sigma_3^2),
\]

so that

\[
(\Sigma_1^a, \Sigma_2^a, \Sigma_3^a) = (-2\Sigma_+, \Sigma_+ + \sqrt{3}\Sigma_-), \Sigma_+ - \sqrt{3}\Sigma_-).
\]

Using \( H^2 = \frac{\rho}{M} \) and define \( w_\pm \) analogously to \( \Sigma_\pm \) by

\[
w_+ = \frac{\pi_2^3 + \pi_3^3}{2\rho}, \quad w_- = \frac{\pi_2^3 - \pi_3^3}{2\sqrt{3}\rho}.
\]

Note that by definition \( \pi_2^3 = S_2^3 - \frac{1}{3} \text{tr} S, \pi_3^3 = S_3^3 - \frac{1}{3} \text{tr} S \) and using (3) we obtain

\[
w_+ = \frac{S_2^2 - \frac{1}{3}\rho + S_3^2 - \frac{1}{3}\rho}{2\rho}, \quad w_- = \frac{S_2^2 - S_3^2}{2\sqrt{3}\rho}.
\]

Using the fact that \( 0 \leq S_2^2 + S_3^2 \leq \text{tr} S = \rho \) and \( 0 \leq S_2^2 \leq \text{tr} S, 0 \leq S_3^2 \leq \text{tr} S \), we obtain the following bounds for \( w_\pm \):

\[
-\frac{1}{3} \leq w_+ \leq \frac{1}{6}, \quad -\frac{1}{2\sqrt{3}} \leq w_- \leq \frac{1}{2\sqrt{3}}.
\] (19)

Our evolution equations of \( \Sigma_\pm \) are thus

\[
\Sigma'_+ = (q - 2)\Sigma_+ + \frac{2R - 3(R_2^2 + R_3^2)}{6H^2} + 3w_+\Omega,
\] (20)

\[
\Sigma'_- = (q - 2)\Sigma_- + \frac{R_3^2 - R_2^2}{2\sqrt{3}H^2} + 3w_-\Omega.
\] (21)
2.2 Vlasov equation with Bianchi symmetry

Since we use a left-invariant frame, \( f \) will not depend on \( x^a \). We will assume that \( f \) has compact support for simplicity. Moreover, since \( g_{00} = g^{00} = -1 \) and \( g^{0a} = 0 \), we have \( p^0 = -p_0 = \sqrt{p_a p_b g^{ab}} \), \( p = T_{00} \), and \( J_a = -T_{0a} \). The frame components of the energy-momentum tensor are thus

\[
\rho = (\det g)^{-\frac{1}{2}} \int f p^0 dp,
\]

\[
J_i = (\det g)^{-\frac{1}{2}} \int f \frac{p_i p^0}{p^0} dp,
\]

\[
S_{ij} = (\det g)^{-\frac{1}{2}} \int f \frac{p_i p_j p^0}{p^0} dp,
\]

where the distribution function is understood as \( f = f(t, p) \) with \( p = (p_1, p_2, p_3) \).

We have that

\[
\frac{d}{dt}(\det g) = 6H(\det g),
\]

which means that

\[
\det g = (\det g(\tau_0))e^{6(\tau - \tau_0)}.
\]  

(22)

The particle current density is

\[
N^\alpha = (\det g)^{-\frac{1}{2}} \int f \frac{p^\alpha}{p^0} dp.
\]

The conservation of particle current density implies that

\[
\nabla_\alpha N^\alpha = 0,
\]

which for any Bianchi A spacetime, where

\[
C^m_{nl} = 0
\]

holds, implies with (7) and (8) that

\[
\dot{N}^0 = -3HN^0,
\]

and thus

\[
N^0(\tau) = N^0(\tau_0)e^{-3(\tau - \tau_0)}.
\]

Together with (22) we obtain that \( \int f dp \) is a conserved quantity:

\[
\int f(\tau, p) dp = \int f(\tau_0, p) dp.
\]

Using the expressions for the connection coefficients and the antisymmetry of the structure constants the Vlasov equations as expressed in (25.14) of [13] where \( f \) depends on \( p^i \) turns into

\[
p_0 \frac{\partial f}{\partial t} = (2k_b p^0 p^b + g^{mi} C_{ma} p^d p_d) \frac{\partial f}{\partial p^i}.
\]
Considering \( f \) as function of \( p_i \), i.e. \( f(t,p_i) = \tilde{f}(t,p_i) = \bar{f}(t,g_{ij} p_j) \), we have that

\[
\frac{\partial f}{\partial t} = \frac{\partial \tilde{f}}{\partial t} + \frac{\partial \bar{f}}{\partial p_i} \dot{g}_{ij} p_j,
\]

\[
\frac{\partial f}{\partial p_i} = \frac{\partial \tilde{f}}{\partial p_i} \dot{g}_{ij},
\]

from which follows that the equation for the distribution function in terms of \( p_i \) dropping the bar by a slight abuse of notation is

\[
p^0 \frac{\partial f}{\partial t} + C^d_{ba} p^b p^d \frac{\partial f}{\partial p_a} = 0. \tag{23}
\]

When treating the Bianchi I case in [8] we considered derivatives of the following quantity:

\[
w^i_j = \frac{S^i_j}{\rho} = \frac{\int f p_i p_a g^{aj} (p^0)^{-1} dp}{\int f p^0 dp} \tag{24}
\]

In the Bianchi I case \( f \) does not depend on \( t \), but in general it does. Let us thus consider the terms where the time derivative of \( f \) is involved. Consider first the term related coming from the energy density. If we substitute the Vlasov equation (23) and integrate by parts we have

\[
\int \frac{\partial f}{\partial t} p^0 dp = -C^d_{ba} \int p^b p_a \frac{\partial f}{\partial p_a} dp = C^d_{ba} \int f \frac{\partial}{\partial p_a} (p^b p_a) dp
\]

\[
= C^d_{ba} \int f (\delta^a_d p^b + p_d g^{ab}) dp = C^a_{ba} \int f p^b dp,
\]

where we have used again the fact that \( g^{ab} \) is symmetric while \( C^a_{ba} \) is antisymmetric which implies that summing over both of them gives zero. Moreover for all Bianchi A spacetimes \( C^a_{ba} = 0 \) which means that the whole term vanishes in that case.

Consider now the other term which appears when taking the time derivative of the distribution function in (24) and call it \( V^j_i \). Using again the Vlasov equation and integrating by parts

\[
V^j_i = \int \frac{\partial f}{\partial t} p_f g^{ij} (p^0)^{-1} dp = -C^d_{ba} \int \frac{\partial f}{\partial p_a} p^b p_d p_f g^{ij} (p^0)^2 dp
\]

\[
= C^d_{ba} g^{ij} g^{eb} \int f \frac{\partial}{\partial p_a} \left[ \frac{p_e p_d p_f}{(p^0)^2} \right] dp.
\]

If \( e \) is equal to \( a \) the term vanishes due to the antisymmetry of the structure constants and \( d = a \) the term vanishes for Bianchi A spacetimes. Let us assume we consider Bianchi A space times. In that case

\[
V^j_i = C^d_{ba} g^{ij} g^{eb} \int f p_e p_d \frac{\partial}{\partial p_a} \left[ \frac{p_e p_f}{(p^0)^2} \right] dp = C^d_{ba} g^{ij} \int f p^b p_d \frac{\partial}{\partial p_a} \left[ \frac{p_e p_f}{(p^0)^2} \right] dp.
\]

Now

\[
\frac{\partial (p^0)^2}{\partial p_a} = 2p^a,
\]
which also vanishes when summed over the rest due the antisymmetry of the structure constants. Thus

\[ V^j_i = C^d_{ba} g^{ij} \int f \frac{p^b p_d}{(p^0)^2} \frac{\partial}{\partial p_a} [p_i p_f] \, dp = C^d_{ba} g^{ij} \int f \frac{p^b p_d}{(p^0)^2} [\delta^a_i p_f + \delta^a_p] \, dp. \]  

(25)

Define

\[ W^j_i = \frac{V^j_i}{H \int f p^0 \, dp}. \]  

(26)

The derivative of \( w^j_i \) with respect to \( \tau \) is then:

\[ (w^j_i)' = -2 w^a_i \Sigma^j_a + w^b_a \Sigma^a_i w^j_i + \Sigma^d_{abc} \xi^{jd}_{ic} + W^j_i, \]  

(27)

where

\[ \xi^{jd}_{ic} = \frac{\int f p_i p^j p_d (p^0)^{-3} \, dp}{\int f p^0 \, dp}. \]  

(28)

The derivative of \( \xi^{jd}_{ic} \) is

\[ \left( \xi^{jd}_{ic} \right)' = -2 \Sigma^d_{jic} \xi^{jd}_{ic} - 2 \Sigma^j_{ic} \xi^{jd}_{ic} + 3 \Sigma^a_{ic} \int f p_i p^j p_d p_a p_b (p^0)^{-5} \, dp \frac{\partial}{\partial p_a} \] 

\[ + \Sigma^d_{abc} \left( \xi^{jd}_{ic} + \hat{\xi}^{jd}_{ic} \right) + \Sigma^d_{abc} \xi^{jd}_{ic} + Z^{jd}_{ic}, \]  

(29)

Consider the integral in the numerator and call it \( X^{jd}_{ic} \). Doing a similar procedure as for \( W^j_i \), i.e. using the Vlasov equation and integrating by parts:

\[ X^{jd}_{ic} = \int \partial_t f p_i p^j p_d (p^0)^{-3} \, dp = -C^a_{ba} \int \frac{\partial f}{\partial p_a} \frac{p^b p_a p^j p^d (p^0)^{-4} \, dp}{(p^0)^4} \] 

\[ = C^a_{ba} \int f \frac{\partial}{\partial p_a} \left[ \frac{p^b p_a p^j p^d}{(p^0)^4} \right] \, dp = C^a_{ba} \int f \frac{p^b p_d}{(p^0)^4} \frac{\partial}{\partial p_a} (p_i p^j p^d) \, dp, \]  

(30)

where the last equality is obtained by the same considerations as when treating \( W^j_i \).

We will consider that we are close to the case that \( f \) has some symmetries. In that case \( \xi^{jd}_{ic} \) takes some specific values which we will denote by \( \hat{\xi}^{jd}_{ic} \). Let us consider the trace free part \( \tilde{w}^j_i = w^j_i - \frac{1}{3} \delta^j_i \) and define

\[ \tilde{\xi}^{jd}_{ic} = \xi^{jd}_{ic} - \hat{\xi}^{jd}_{ic}. \]  

(31)

Then (27) turns into

\[ (\tilde{w}^j_i)' = -\frac{2}{3} \Sigma^j_i - 2 \tilde{w}^a_i \Sigma^j_a + \tilde{w}^b_a \Sigma^a_i (\tilde{w}^j_i + \frac{1}{3} \delta^j_i) + \Sigma^d_{abc} \left( \tilde{\xi}^{jd}_{ic} + \hat{\xi}^{jd}_{ic} \right) + W^j_i. \]  

(32)
In particular for the terms \( w_+ = \frac{1}{2}(\tilde{w}_2^2 + \tilde{w}_3^2), w_- = \frac{1}{2\sqrt{3}}(\tilde{w}_2^2 - \tilde{w}_3^2) \) we have
\[
\begin{align*}
w'_+ &= -\frac{2}{3} \Sigma_+ - \tilde{w}_2^a \Sigma_a^2 - \tilde{w}_3^a \Sigma_a^3 + \tilde{w}_a^b \Sigma_b^4 \left( w_+ + \frac{1}{3} \right) + \frac{1}{2} \Sigma_d^a \left( \tilde{\xi}_{2c}^d + \tilde{\xi}_{3c}^d + \tilde{\xi}_{3c}^d \right) + W_+, \\
w'_- &= -\frac{2}{3} \Sigma_- - \frac{1}{\sqrt{3}} \left( \tilde{w}_2^a \Sigma_a^2 - \tilde{w}_3^a \Sigma_a^3 \right) + \tilde{w}_a^b \Sigma_b^4 w_- + \frac{1}{2\sqrt{3}} \Sigma_d^a \left( \tilde{\xi}_{2c}^d + \tilde{\xi}_{3c}^d - \tilde{\xi}_{3c}^d \right) + W_-,
\end{align*}
\]
where we have used the notation
\[
\begin{align*}
W_+ &= \frac{1}{2} (W_2^2 + W_3^2), \\
W_- &= \frac{1}{2\sqrt{3}} (W_2^2 - W_3^2).
\end{align*}
\]

3 The equations for Bianchi VII\(_0\) with reflection symmetry for massless Vlasov particles close to the isotropic case

In the following we will deduce the relevant equations for Bianchi VII\(_0\) with reflection symmetry. We will exclude the LRS case, since that cases reduces to Bianchi I, which presents a completely different behaviour and was already studied in [8].

We will assume that we are close to the isotropic case. This means that as in [8] the only non-vanishing expressions for \( \hat{\xi}_{ab}^c \) are (suspending the Einstein summation convention for the next expressions):
\[
\begin{align*}
\hat{\xi}_{aa}^a &= 1/3, & a = 1, 2, 3 \\
\hat{\xi}_{ab}^b &= \hat{\xi}_{ba}^b = \hat{\xi}_{aa}^b = 1/15, & a \neq b, \quad a, b = 1, 2, 3.
\end{align*}
\]
Setting this into the equations for \( w_{\pm} \), we have
\[
\begin{align*}
w'_+ &= -\frac{8}{15} \Sigma_+ - \tilde{w}_2^a \Sigma_a^2 - \tilde{w}_3^a \Sigma_a^3 + \tilde{w}_a^b \Sigma_b^4 \left( w_+ + \frac{1}{3} \right) + \frac{1}{2} \Sigma_d^a \left( \tilde{\xi}_{2c}^d + \tilde{\xi}_{3c}^d \right) + W_+, \\
w'_- &= -\frac{8}{15} \Sigma_- - \frac{1}{\sqrt{3}} \left( \tilde{w}_2^a \Sigma_a^2 - \tilde{w}_3^a \Sigma_a^3 \right) + \tilde{w}_a^b \Sigma_b^4 w_- + \frac{1}{2\sqrt{3}} \Sigma_d^a \left( \tilde{\xi}_{2c}^d - \tilde{\xi}_{3c}^d \right) + W_-.
\end{align*}
\]
If \( f \) is reflection symmetric, i.e.
\[
f(p_1, p_2, p_3, t) = f(-p_1, -p_2, p_3) = f(p_1, -p_2, -p_3, t),
\]
the metric and the second fundamental form are diagonal, the evolution will preserve this symmetry.

Everything will be diagonal and also \( J_a = T_{0a} = 0 \) since it is an integral over an odd number of momenta which vanishes due to the reflection symmetry. Since \( J_a = 0 \) we have from [10] that \( k_i^j \) and \( n^{ij} \) commute, which means there exists a basis where one can simultaneously diagonalise both. If we thus choose eigenvectors of \( k_i^j(t_0) \) as the frame and choose \( g(t_0) \) to be diagonal since \( J_a = 0 \), everything will remain diagonal. Now if \( n^{ij} \) is diagonal we have that the right
hand side of (10) is always zero, which comes from the fact that due to (9), the structure constants cannot have two equal indices in that case. As a consequence the momentum constraint will be trivially be satisfied for all time.

Moreover due to this symmetry the quantity $W_{ij}$ defined in (24) and the quantity $Z_{jd}^{ic}$ defined in (29) vanish as well. The reason is the same as for $J_a$. The integrals involved in the numerator for $W_{ij}$ and $Z_{jd}^{ic}$ are $V_{ij}$ and $X_{jd}^{ic}$ which are integrals over an odd number of momenta which due to the reflection symmetry vanish.

As a consequence, for the reflection symmetric case, the equations for $\tilde{w}$ and $\xi_{ic}^{jd}$ will be identical to the equations for the Bianchi I case treated in [8]. In particular the equations for $w_{\pm}$ in the reflection symmetric case using

$$\bar{w}_2^2 \Sigma_2^2 + \bar{w}_3^2 \Sigma_3^2 = 2w_+ \Sigma_+ + 6w_- \Sigma_-,$$

$$\bar{w}_2^2 \Sigma_2^2 - \bar{w}_3^2 \Sigma_3^2 = 2\sqrt{3} \Sigma_- w_+ + 2\sqrt{3} \Sigma_+ w_-,$$

$$\bar{w}_a^b \Sigma_a^b = 6w_+ \Sigma_+ + 6w_- \Sigma_-,$$

are

$$w_+' = -\frac{8}{15} \Sigma_+ - 4w_- \Sigma_- + 6(w_+ \Sigma_+ + w_- \Sigma_-)w_+ + R_+,$$

$$w_-' = -\frac{8}{15} \Sigma_- - 2(w_+ \Sigma_- + w_- \Sigma_+) + 6(w_+ \Sigma_+ + w_- \Sigma_-)w_- + R_-,$$

with

$$R_+ = \frac{1}{2} \left[ \Sigma_+ \left( 2\bar{\xi}_{21}^2 - 2\bar{\xi}_{21}^{21} + 2\bar{\xi}_{31}^{2} + 2\bar{\xi}_{33}^{2} + \bar{\xi}_{33}^{3} + \sqrt{3} \Sigma_- \left( \bar{\xi}_{22}^{2} - \bar{\xi}_{33}^{3} \right) \right) \right],$$

$$R_- = \frac{1}{2\sqrt{3}} \left[ \Sigma_+ \left( -2\bar{\xi}_{21}^{2} + 2\bar{\xi}_{31}^{2} + \bar{\xi}_{22}^{2} - \bar{\xi}_{33}^{3} + \sqrt{3} \Sigma_- \left( \bar{\xi}_{22}^{2} - 2\bar{\xi}_{31}^{2} + \bar{\xi}_{33}^{3} \right) \right) \right].$$

The constraint equation is

$$\Omega = \frac{\rho}{3H^2} = 1 - \Sigma_+^2 - \Sigma_-^2 + \frac{R}{6H^2}.$$  

For the reflection symmetric massless Bianchi VII_0 case we have the same equations for the curvature variables as in the massive case [7]. Let $(ijk)$ denote a cyclic permutation of $(123)$ and let us suspend the Einstein summation convention for the next three formulas. Introduce $\nu_i$ as the signs depending on the Bianchi type [Table 1 of [3]]. Now define

$$n_i = \nu_i \sqrt{\frac{g_{ii} g_{jj} g_{kk}}{g_{jj} g_{kk}}},$$

where this means e.g. $n_1 = \nu_1 \sqrt{g_{11} g_{22} g_{33}}$. Using this notation the Ricci tensor is given by [(11a) of [3]]

$$R_i^j = \frac{1}{2} \left[ n_i^2 - (n_j - n_k)^2 \right].$$

The curvature variables are

$$N_{ii} = \frac{n_i}{H}.$$
In the Bianchi VII\(_0\) case we have \(\nu_1 = 0\) and \(\nu_2 = \nu_3 = 1\), which means that \(N_{22}\) and \(N_{33}\) are positive definite and the only non-vanishing structure constants are (cf. Appendix E, p. 695 of [13]):

\[
C_{31}^2 = 1 = -C_{13}^2, \quad C_{12}^3 = 1 = -C_{21}^3.
\] (33)

The curvature expressions are

\[
R_1^1 = R = -\frac{1}{2}(n_2 - n_3)^2, \\
R_2^2 = -R_3^3 = \frac{1}{2}(n_2^2 - n_3^2).
\]

The relevant equations are

\[
N_{22}' = N_{22}(q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-), \\
N_{33}' = N_{33}(q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-),
\]

where \(N_{22} > 0\) and \(N_{33} > 0\), which we transform into the following variables:

\[
N_+ = \frac{N_{22} + N_{33}}{2} > 0, \\
N_- = \frac{N_{22} - N_{33}}{2\sqrt{3}},
\]

which implies

\[
\frac{R}{6H^2} = -N_2^2,
\]

and

\[
N_+^2 - 3N_-^2 > 0.
\]

As a consequence

\[
\Omega = 1 - \Sigma_+^2 - \Sigma_-^2 - N_2^2.
\]

The relevant evolution equations are thus

\[
\Sigma_+ = (q - 2)\Sigma_+ - 2N_- + 3w_+\Omega, \\
\Sigma_- = (q - 2)\Sigma_- - 2N_+ N_- + 3w_-\Omega, \\
N_+ = (q + 2\Sigma_+)N_+ + 6\Sigma_- N_- , \\
N_- = (q + 2\Sigma_+)N_+ + 2\Sigma_- N_+, \\
\Omega' = 2\Omega (q - 1 - 3\Sigma_+ w_+ - 3\Sigma_- w_-) .
\] (34) (35) (36) (37) (38)

### 3.1 WHU variables

Making the same transformations as in WHU [16]

\[
M = \frac{1}{N_+} > 0, \\
N_- = X \sin \psi, \quad X > 0, \\
\Sigma_- = X \cos \psi, \quad X > 0,
\]

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so that
\[ \Omega = 1 - \Sigma_+^2 - X^2. \]  
(39)

Since \( \Omega = \frac{\rho}{3H^2} \geq 0 \), we have that
\[ \Sigma_+^2 + X^2 \leq 1. \]  
(40)

The relevant equations for \( \Sigma_+, M, X \) and \( \psi \) are obtained from equations (23)-(26) of [7] by making the following replacements in terms of the relevant quantities we use here:

\[
S_+ = \frac{1}{6H^2} (3S_+^2 + 3S_3^3 - 2S) = \frac{1}{6H^2} (3\pi_3^2 + 3\pi_3^3) = 3w_+ \Omega, \\
S_- = \frac{1}{2\sqrt{3}H^2} (S_2^2 - S_3^2) = \frac{1}{2\sqrt{3}H^2} (\pi_2^2 - \pi_3^3) = 3w_- \Omega, \\
q = 1 + \frac{R}{6H^2} + \frac{1}{6} F = 1 - N_+^2 + \Sigma_+^2 + \Sigma_3^2 = 1 + \Sigma_+^2 + X^2 \cos 2\psi, \\
Q = 1 + \Sigma_+^2.
\]

The relevant equations in our case are thus
\[
M' = -M[1 + \Sigma_+^2 + 2\Sigma_+ + X^2(\cos 2\psi + 3M \sin 2\psi)], \\
\Sigma_+ = -X^2 - \Sigma_+(1 - \Sigma_+^2) + (1 + \Sigma_+)X^2 \cos 2\psi + 3w_+ \Omega, \\
X' = [\Sigma_+(1 + \Sigma_+) + (X^2 - 1 - \Sigma_+) \cos 2\psi]X + 3w_- \Omega \cos \psi, \\
\psi' = 2M^{-1} + (1 + \Sigma_+) \sin 2\psi - X^{-1} 3w_- \Omega \sin \psi, \\
\Omega' = 2(\Sigma_+^2 + X^2 \cos 2\psi - 3\Sigma_+ w_+ - 3X w_- \cos \psi) \Omega, \\
w_+ = \left( -\frac{8}{15} + \alpha + 6w_+^2 \right) \Sigma_+ + [w_-(6w_+ - 4) + \beta] X \cos \psi, \\
w_-' = [w_-(2 + 6w_+) + \gamma] \Sigma_+ + \left( 6w_+^2 - \frac{8}{15} - 2w_+ + \delta \right) X \cos \psi,
\]  
(41)-(47)

where
\[
\alpha = \frac{1}{2} \left( 2\xi_{32}^2 - 2\xi_{21}^2 - 2\xi_{31}^3 + \xi_{22}^2 + \xi_{33}^3 \right), \\
\beta = \frac{1}{2} \sqrt{3} \left( \xi_{22}^2 - \xi_{33}^3 \right), \\
\gamma = \frac{1}{2} \sqrt{3} \left( -2\xi_{21}^2 + 2\xi_{31}^3 + \xi_{22}^2 - \xi_{33}^3 \right), \\
\delta = \frac{1}{2} \left( \xi_{22}^2 - 2\xi_{32}^2 + \xi_{33}^3 \right).
\]  
(48)-(51)

We have put the equations (41)-(45) in such a way that they are easy to compare with the equations in [10, 16]. They are identical to the equations in [16] if one sets in our equations \( w_+ = w_- = 0 \).

Define
\[
\Phi_{iaco}^{jdb} = \frac{\int f p_i p_j p_k p_l p_m p_n (p_0)^{-5} dp}{\int f p^0 dp}.
\]
The equations for $\xi_{ic}^{jd}$ are then

$$
\left(\xi_{ic}^{jd}\right)' = -2\Sigma_{f}^{d}\xi_{ic}^{jf} - 2\Sigma_{j}^{f}\xi_{ic}^{jd} + 3\Sigma_{a}^{b}\Phi_{ia}^{db} + \xi_{ic}^{jd}w_{ab}^{\gamma}w_{bc},
$$

(52)

In the reflection symmetric case and assuming that our metric is diagonal the relevant equations are

$$
\left(\tilde{\xi}_{22}^{22}\right)' = \left(\tilde{\xi}_{33}^{33}\right)' = \left(\tilde{\xi}_{21}^{21}\right)' = \left(\tilde{\xi}_{31}^{31}\right)' = \left(\tilde{\xi}_{32}^{32}\right)'
$$

\begin{align*}
&= \left(\tilde{\xi}_{22}^{22}\right)' = \left(\tilde{\xi}_{33}^{33}\right)' = \left(\tilde{\xi}_{21}^{21}\right)' = \left(\tilde{\xi}_{31}^{31}\right)' = \left(\tilde{\xi}_{32}^{32}\right)'
\end{align*}

\begin{align*}
&= \left[(-4 + 6w_+)\Sigma_+ + (6w_+ - 4\sqrt{3})X \cos \psi\right] \left(\tilde{\xi}_{22}^{22} + \frac{1}{5}\right) + 3\Sigma_{a}^{b}\Phi_{22a}^{22b},
&= \left[(-4 + 6w_+)\Sigma_+ + (6w_+ + 4\sqrt{3})X \cos \psi\right] \left(\tilde{\xi}_{33}^{33} + \frac{1}{5}\right) + 3\Sigma_{a}^{b}\Phi_{33a}^{33b},
&= \left[(2 + 6w_+)\Sigma_+ + (6w_+ - 2\sqrt{3})X \cos \psi\right] \left(\tilde{\xi}_{21}^{21} + \frac{1}{15}\right) + 3\Sigma_{a}^{b}\Phi_{21a}^{21b},
&= \left[(2 + 6w_+)\Sigma_+ + (6w_+ + 2\sqrt{3})X \cos \psi\right] \left(\tilde{\xi}_{31}^{31} + \frac{1}{15}\right) + 3\Sigma_{a}^{b}\Phi_{31a}^{31b},
&= \left[(-4 + 6w_+)\Sigma_+ + 6w_+ X \cos \psi\right] \left(\tilde{\xi}_{32}^{32} + \frac{1}{15}\right) + 3\Sigma_{a}^{b}\Phi_{32a}^{32b}.
\end{align*}

(53) – (57)

Note that the system is not closed, since we could consider the evolution equations of higher order terms such as $\Phi_{ic}^{jd}$ etc. However the considered evolution equations will be sufficient to obtain the desired future asymptotic behaviour.

4 Future asymptotic behaviour

In order to obtain the desired result it will be necessary to assume closeness to an equilibrium line given by the curve

$$
-\frac{1}{3}X^2 + w_+(1 - X^2) = 0.
$$

This will become clear when treating the truncated system in Section 4.3. This system is obtained by neglecting the oscillatory terms, the higher order momentum terms $\tilde{\xi}_{ij}^{kl}$ and focusing on the equations for $\Sigma_+, X$ and $w_+.$

Define

$$
Y = -\frac{1}{3}X^2 + w_+(1 - X^2).
$$

(58)

Then the following theorem holds:

**Theorem 1.** Consider any $C^\infty$-solution of the massless Einstein-Vlasov system with reflection and Bianchi VII$_0$ symmetry, which is not LRS, given by equations (23), (33) and (41)–(47) with $X(\tau_0) \neq 0, w_-(\tau_0) \neq 0$ and with $C^\infty$-initial data. Assume that $\tilde{M}(\tau_0), [\Sigma_+(\tau_0)], [X(\tau_0)], [w_+(\tau_0)], [\tilde{\xi}_{22}^{22}(\tau_0)], [\tilde{\xi}_{33}^{33}(\tau_0)], [\tilde{\xi}_{21}^{21}(\tau_0)], [\tilde{\xi}_{31}^{31}(\tau_0)],$
Let \( |\tilde{\xi}^{31}_{ij}(\tau_0)|, |\tilde{\xi}^{32}_{ij}(\tau_0)| \) be sufficiently small. Then at late times there exists a small constant \( \varepsilon \) such that the following estimates hold:

\[
M = O(\varepsilon e^{(-1+\varepsilon)\tau}), \\
Y = O(\varepsilon e^{(-\frac{1}{2}+\varepsilon)\tau}), \\
\Sigma_+ = O(\varepsilon e^{(-\frac{1}{2}+\varepsilon)\tau}), \\
w_+ = O(1).
\]

(59) \hspace{1cm} (60) \hspace{1cm} (61) \hspace{1cm} (62)

The proof we will be done using a bootstrap argument and will be finished in Section 4.4.

We begin by observing that the relevant quantities are bounded. The quantities \( \Sigma_+, X \) and \( \Omega \) are bounded due to (39)–(40), \( w_+ \) and \( w_- \) are bounded due to (19). Moreover \( \xi^{ij}_{ij} \) can be bounded by \( w^i_+ \) or \( w^j_- \) (no summation of the indices in the previous three quantities) which are bounded by 1. The quantities \( \tilde{\xi}^{ij}_{ij} \) just differ a constant from \( \xi^{ij}_{ij} \) according to (31). As a consequence \( \alpha, \beta, \gamma \) and \( \delta \) defined in (48)–(51) are bounded as well. Similarly \( \Phi^{ijk}_{ij} \) is bounded which implies that also the derivatives of \( \Sigma_+, X, \Omega, w_+, w_- \) and \( \tilde{\xi}^{ij}_{ij} \) are bounded.

### 4.1 Estimate of \( M \)

Since \( X \) and \( w_+ \) are small, we can assume that \( Y \) is small. We will use a bootstrap argument. Let us assume that there exists an interval \( [\tau_0, \tau_1] \) where the following estimates hold:

\[
M(\tau) \leq \varepsilon_M, \\
|\Sigma_+(\tau)| \leq \varepsilon_{\Sigma} e^{(-\frac{1}{2}+\varepsilon)\tau}, \\
|Y(\tau)| \leq \varepsilon_Y e^{(-\frac{1}{2}+\varepsilon)\tau}, \\
\left|\frac{w_+}{X}(\tau)\right| \leq C e^{\frac{1}{4}\tau},
\]

(63) \hspace{1cm} (64) \hspace{1cm} (65) \hspace{1cm} (66)

where \( \varepsilon_M, \varepsilon_\Sigma, \varepsilon_Y \) are some small quantities all smaller than \( \varepsilon \). The last assumption (66) will hold initially since \( X \) is assumed to be different from zero and \( C \) is some arbitrary constant, not necessarily small.

Define

\[
\bar{M} = \frac{M}{1 - \frac{1}{4}MX^2 \sin 2\psi}.
\]

(67)

Then

\[
\bar{M}' = \frac{-(1 + \Sigma_+)^2 + MXA}{\frac{1}{4}MX^2 \sin 2\psi},
\]

where

\[
A = \left(-3X + \frac{1}{2}X' + \frac{1}{2}X \cos 2\psi(1 + \Sigma_+)\right) \sin 2\psi - \frac{3}{2}\Omega w_- \cos 2\psi \sin \psi
\]
is a bounded quantity since \( X, X', \Sigma_+, w_-, \) and \( \Omega \) are bounded. Using the bound of the bootstrap assumptions (63)–(64) we obtain

\[-1 - C\varepsilon \leq \frac{M'}{M} \leq -1 + C\varepsilon,\]

which implies that

\[\bar{M}(\tau_0) e^{(-1-C\varepsilon)(\tau-\tau_0)} \leq \bar{M}(\tau) \leq \bar{M}(\tau_0) e^{(-1+C\varepsilon)(\tau-\tau_0)}, \quad (68)\]

and

\[M(\tau) \leq (1 - \frac{1}{4} M X^2 \sin 2\psi)(\tau) \frac{M(\tau_0)}{(1 - \frac{1}{4} MX^2 \sin 2\psi)(\tau_0)} e^{(-1+C\varepsilon)(\tau-\tau_0)} \leq M(\tau_0) \frac{1 + C\varepsilon M}{1 - CM(\tau_0)} e^{(-1+C\varepsilon)(\tau-\tau_0)}.\]

Choosing \( M(\tau_0) \) smaller than \( \frac{1}{2} \varepsilon_M \) and making \( \varepsilon_M \) smaller if necessary we obtain

\[M(\tau) \leq (1 + C\varepsilon M) e^{(-1+C\varepsilon)(\tau-\tau_0)} \leq \varepsilon_M e^{(-1+C\varepsilon)(\tau-\tau_0)},\]

which is an improvement of the bootstrap assumption (63) for any \( \tau > \tau_0 \) and which proves (59) provided we improve the remaining bootstrap assumptions (64)–(65).

### 4.2 Suppressing the oscillations

Let us make a similar change of variables as in [10] for the other variables (the change of variables for \( \Sigma_+, X, M, \Omega \) is the same as in [10] if we set \( w_-=0 \):

\[\bar{\Sigma}_+ = \Sigma_+ - \frac{1}{4} M (1 + \Sigma_+) X^2 \sin 2\psi, \quad (69)\]

\[\bar{X} = X + \frac{1}{4} M (X^2 - 1 - \Sigma_+) \sin 2\psi - \frac{3}{2} M w_- \Omega \sin \psi, \quad (70)\]

\[\bar{w}_+ = w_+ - \frac{1}{2} M [w_-(6w_+ - 4) + \beta] X \sin \psi, \quad (71)\]

\[\bar{\Omega} = \Omega + \frac{1}{2} M X^2 \sin 2\psi - 3 M w_- \sin \psi \]

\[\bar{w}_- = w_- - \frac{1}{2} M \left( 6w_-^2 - \frac{8}{15} - 2w_+ + \delta \right) X \sin \psi.\]

Note that all the relevant quantities are bounded. Also the derivatives of \( \alpha, \beta, \gamma, \delta \) are bounded using (52). Since we assume that \( M \) is small we have that \( \bar{\Sigma}_+, \bar{X}, \bar{M}, \bar{\Omega}, \bar{w}_+, \bar{w}_- \) are bounded.

Using (41)–(47) the evolution equations of all the barred quantities are as follows:
\[
\dot{M} = - \bar{M} \left[ (1 + \Sigma_+)^2 + O(M) \right], \\
\dot{\Sigma}_+ = - \bar{X}^2 - \bar{\Sigma}_+ (1 - \Sigma_+^2) + 3 \bar{w}_+ \Omega + O(M), \\
\dot{X} = [\bar{\Sigma}_+ (1 + \Sigma_+) + O(M)] \bar{X} + O(M) + O \left( \frac{M w_-^2}{X} \right), \\
\dot{\Omega} = 2 \left[ \Sigma_+^2 - 3 \Sigma_+ \bar{w}_+ + O(M) \right] \Omega, \\
\dot{\bar{w}}_+ = \left( - \frac{8}{15} + \alpha + 6 \bar{w}_+^2 \right) \Sigma_+ + O(M), \\
\dot{\bar{w}}_- = [\bar{w}_- (-2 + 6 \bar{w}_+) + \gamma] \Sigma_+ + O(M),
\]

where the expressions \( O(M) \) are quantities of order \( M \), i.e. there are constants \( C_1, C_2 \) such that
\[
- C_1 M \leq O(M) \leq C_2 M
\]
and similarly for \( O(M w_-^2 - X) \).

We also want to suppress the oscillations for the terms concerning \( \xi_{22}, \xi_{33}, \xi_{21}, \xi_{31}, \xi_{32} \).

The equations (53)–(57) are all of the form (no summation of the indices):
\[
(\bar{\xi}_{ij})' = f_1 (\bar{w}+, \bar{w}-, \bar{\xi}_{ij}, \Phi^{ij}_{kl} \Sigma_+ + f_2 (\bar{w}+, \bar{w}-, \bar{\xi}_{ij}, \Phi^{ij}_{kl} X \cos \psi.
\]
Thus making the change of variable
\[
\bar{\xi}_{ij} = \bar{\xi}_{ij} - \frac{1}{2} M f_2 X \sin \psi
\]
we obtain
\[
(\bar{\xi}_{ij})' = f_1 (\bar{w}+, \bar{w}-, \bar{\xi}_{ij}, \Phi^{ij}_{kl} \Sigma_+ + O(M).
\]

### 4.3 The truncated system

Before considering the full system we consider the truncated system where we neglect the oscillatory terms with \( M \) and the term \( \alpha \) which is related to higher order momenta. Since \( \bar{w}_- \) basically only appears in the evolution equation of \( \bar{w}_+ \) we ignore that equation for the moment.

The truncated system is
\[
\dot{\Sigma}_+ = - \bar{X}^2 - \bar{\Sigma}_+ (1 - \Sigma_+^2) + 3 \bar{w}_+ (1 - \bar{X}^2 - \bar{\Sigma}_+^2), \\
\dot{X} = [\bar{\Sigma}_+ (1 + \Sigma_+) \bar{X}, \\
\dot{\bar{w}}_+ = \left( - \frac{8}{15} + 6 \bar{w}_+^2 \right) \bar{\Sigma}_+.
\]

Note that the curve \(- \frac{1}{4} \bar{X}^2 + \bar{w}_+ (1 - \bar{X}^2) = 0\) at \( \bar{\Sigma}_+ = 0 \) is an invariant manifold since all derivatives of any order vanish there. This means that there is an equilibrium value for \( \bar{w}_+ \) which is
\[
\bar{w}_+ = \frac{\bar{X}^2}{3 (1 - \bar{X}^2)}
\]
or if we express \( \bar{X} \) in terms of \( \bar{w}_+ \):
\[
\bar{X}^2 = \frac{\bar{w}_+}{\frac{1}{3} + \bar{w}_+}.
\]

We prove that
Lemma 1. Consider the system of differential equations \(81\)–\(83\). Define
\[
\hat{Y} = -\frac{1}{3} \hat{X}^2 + \hat{w}_+ \left(1 - \hat{X}^2\right) = -\hat{X}^2 \left(\frac{1}{3} + \hat{w}_+\right) + \hat{w}_+, \tag{84}
\]
and consider the function
\[
\hat{u} = (2 \hat{X}^2 + 8) \hat{\Sigma}_+^2 - 15 \hat{\Sigma}_+ \hat{Y} + 45 \hat{Y}^2. \tag{85}
\]
Then \(\hat{u}' + \hat{u} = O(\hat{\Sigma}_+^3 + \hat{\Sigma}_+^2 + \hat{\Sigma}_+^2 + \hat{\Sigma}_+^2 + \hat{Y}_+^2).

Proof. The proof is a straightforward computation.

The evolution equation for \(\hat{\Sigma}_+\) using the variable \(\hat{Y}\) is then
\[
\hat{\Sigma}_+' = 3 \hat{Y} - \hat{\Sigma}_+(1 - \hat{\Sigma}_+^2) - 3 \hat{w}_+ \hat{\Sigma}_+^2.
\]
From \(84\) we obtain the evolution equation for \(\hat{Y}\) which is
\[
\hat{Y}' = \hat{\Sigma}_+ \left[-2 \hat{X}^2 (1 + \hat{\Sigma}_+) \left(\frac{1}{3} + \hat{w}_+\right) + (1 - \hat{X}^2) \left(-\frac{8}{15} + 6 \hat{w}_+^2\right)\right]
= \hat{\Sigma}_+ \left[-2 \hat{X}^2 \hat{\Sigma}_+ \left(\frac{1}{3} + \hat{w}_+\right) - \frac{2}{15} \hat{X}^2 - \frac{8}{15} + 6 \hat{Y}_+ \hat{w}_+\right].
\]
Note that \(\hat{u}\) is zero for \(\hat{\Sigma}_+ = \hat{Y} = 0\). In fact using Sylvester’s criterion we see it is positive definite for any other value.

The evolution equation for \(\hat{\Sigma}_+\) is using \(85\) as follows:
\[
\hat{u}' = 4 \hat{X}^2 (1 + \hat{\Sigma}_+) \hat{\Sigma}_+^3 + [2 (2 \hat{X}^2 + 8) \hat{\Sigma}_+ - 15 \hat{Y}] [3 \hat{Y} - \hat{\Sigma}_+(1 - \hat{\Sigma}_+^2) - 3 \hat{w}_+ \hat{\Sigma}_+^2]
+ (90 \hat{Y} - 15 \hat{\Sigma}_+) \hat{\Sigma}_+ \left[-2 \hat{X}^2 \hat{\Sigma}_+ \left(\frac{1}{3} + \hat{w}_+\right) - \frac{2}{15} \hat{X}^2 - \frac{8}{15} + 6 \hat{Y}_+ \hat{w}_+\right],
\]
from which follows:
\[
\hat{u}' + \hat{u} = [4 \hat{X}^2 (1 + \hat{\Sigma}_+) + 2 (2 \hat{X}^2 + 8) (\hat{\Sigma}_+ - 3 \hat{w}_+) - 15 \hat{Y} + 10 \hat{X}^2 + 30 \hat{X}^2 \hat{w}_+] \hat{\Sigma}_+^3
- \left[45 \hat{w}_+ + 180 \hat{X}^2 \left(\frac{1}{3} + \hat{w}_+\right)\right] \hat{Y} \hat{\Sigma}_+^2 + 540 \hat{w}_+ \hat{\Sigma}_+ \hat{Y}_+^2,
\]
which proves the lemma.

Using a Runge-Kutta-Fehlberg method one can solve easily the truncated system for some random numbers and from the graph one see that the variables converge rapidly to the equilibrium values.
4.4 Closing the bootstrap argument

In order to close the bootstrap argument we generalise Lemma 4 which corresponds to the truncated system to the general system using the estimate for $M$ and bounds for the matter terms.

Note that all the smallness assumptions for the different variables imply that the corresponding barred quantities are also small making $M$ smaller if necessary.

4.4.1 Bounds for the matter terms

For the matter terms it will be sufficient to show that they can be bounded by small quantities. From (80) and since $w_+, w_-, \xi^{ij}, \Phi^{ij}$ are bounded we have using (59) and (64)

$$(\tilde{\xi}^{ij})' \leq C(|\tilde{\Sigma}+| + M) \leq C[\varepsilon \Sigma e^{(\frac{3}{2}+\varepsilon)\tau} + \varepsilon e^{(-1+\varepsilon)\tau}],$$

which integrated gives

$$\tilde{\xi}^{ij}(\tau_0) - C\varepsilon \leq \tilde{\xi}^{ij}(\tau) \leq \tilde{\xi}^{ij}(\tau_0) + C\varepsilon$$

and also

$$\tilde{\xi}^{ij}(\tau_0) - C\varepsilon \leq \tilde{\xi}^{ij}(\tau) \leq \tilde{\xi}^{ij}(\tau_0) + C\varepsilon. \quad (86)$$

As a consequence $\alpha$ and $\gamma$ have a similar bound.

From (79) we obtain the same type of bound having in mind that $\gamma$ is bounded, obtaining:

$$\bar{w}-(\tau_0) - C\varepsilon \leq \bar{w}-(\tau) \leq \bar{w}-(\tau_0) + C\varepsilon. \quad (87)$$

4.4.2 Estimate of the quotient of $w_-$ and $X$

Now consider the evolution equation of $B = \frac{\bar{w}_-}{X}$ using the evolution equations for $\bar{X}$ (76) and $\bar{w}_-$ (79):

$$B' = B \left( -3 + 6\bar{w}_+ - \tilde{\Sigma}_+ + \frac{\gamma}{\bar{w}_-} \right) \tilde{\Sigma}_+ + O \left( \frac{BM}{\bar{w}_-} \right) + O(BM) + O \left( \frac{B^2M}{\bar{w}_-} \right) + O(MB^3),$$

Plot of the evolution of solutions to the truncated system
from which follows that

\[
B' = \left( -3 + 6\bar{w} + \bar{\Sigma} + \frac{\gamma}{\bar{w}} \right) \bar{\Sigma} + O\left( \frac{M}{\bar{w}} \right) + O(M) + O\left( BM \right) + O(MB^2).
\]

We show now that the right hand side of (88) is a small quantity, smaller then some \( \varepsilon \) using the bootstrap assumptions for \( \Sigma + \bar{w} \) and \( \Sigma + \bar{w} \) (64), (66), the estimates obtained for \( \gamma \) and \( w \) (86), (87) and the obtained estimate of \( M \) (59) and since

\[
\begin{align*}
\frac{\gamma}{\bar{w}} \Sigma &\leq \frac{(C\gamma(\tau_0) + C\varepsilon)\varepsilon(\gamma + \varepsilon)}{\bar{w}(\tau_0) - C\varepsilon}, \\
\frac{M}{\bar{w}} &\leq \frac{\varepsilon\varepsilon(1 + \varepsilon)}{\bar{w}(\tau_0) - C\varepsilon}, \\
BM &\leq \frac{C\varepsilon\varepsilon(\gamma + \varepsilon)}{\bar{w}(\tau_0) - C\varepsilon}, \\
MB^2 &\leq \frac{C\varepsilon\varepsilon(1 + \varepsilon)}{\bar{w}(\tau_0) - C\varepsilon}.
\end{align*}
\]

We have assumed that \( \bar{w}(\tau_0) \neq 0 \), so that \( \bar{w}(\tau_0) - C\varepsilon \) will be different from zero making \( \varepsilon \) smaller if necessary. The inverse of \( \bar{w}(\tau_0) - C\varepsilon \) might be a big quantity, but since it appears always multiplied with \( \varepsilon \) the terms on the right hand side of the inequalities (89)–(92) are all small. In fact they are integrable, so that integrating (88) we obtain

\[
|B| = ||\bar{w}| \leq ||\bar{w}(\tau_0)| C.
\]

Choosing \( \varepsilon \) smaller if necessary this improves the bootstrap assumption for \( \bar{w} \), giving

\[
\frac{w}{X}(\tau) = O(1).
\]

4.4.3 Estimate of \( Y \)

Let us define

\[
\bar{Y} = -\frac{1}{3}X^2 + \bar{w}(1 - X^2)
\]

and

\[
u = (2\bar{X}^2 + 8)\bar{\Sigma}^2 - 15\bar{\Sigma} + 45\bar{Y}^2.
\]

We want to estimate the derivative of \( u \). Since we can use Lemma \( \|$ we have to focus only on the terms which are not present in the truncated case. The result after some computations is that

\[
u' \leq (-1 + C\varepsilon)u + O\left[ \bar{\Sigma}^3 + \bar{\Sigma}^2 \bar{Y} + \bar{\Sigma} \bar{Y}^2 + M \left( \bar{\Sigma} + \bar{\Sigma}B + \bar{Y} + \bar{Y} \right) \right],
\]

where we have used the estimate (94) and (86) and the fact that we can choose the different initial data for the forth order momenta \( |\xi_{22}^2(\tau_0)|, |\xi_{33}^2(\tau_0)|, |\xi_{22}^1(\tau_0)|, |\xi_{33}^1(\tau_0)|, |\xi_{22}^3(\tau_0)|, |\xi_{33}^3(\tau_0)| \) to be smaller than \( \varepsilon \), so that \( a \leq C\varepsilon \).
Now using the bootstrap assumptions and the estimate (94) we obtain that
\[ u = O(e^{-\frac{1}{2} + C\varepsilon} \tau) \]
and
\[ Y = O(e^{-\frac{1}{2} + C\varepsilon} \tau) \]
Choosing \( \varepsilon \) smaller if necessary gives the estimates for \( \Sigma_+ \) and \( Y \) which closes that bootstrap argument and finishes the proof of Theorem 1.

4.5 The Weyl parameter and the shear at late times

The Weyl parameter \( W \) or Hubble normalised Weyl curvature has the following expression [cf. (3.39) of [16] and references therein for details]:
\[ W = \frac{2X}{M}[1 + O(M)] \] (95)

We conclude with the following corollary:

**Corollary 1.** Consider the same assumptions as in Theorem 1. Then
\[ W \geq Ce^\tau, \] (96)
and
\[ \lim_{\tau \to \infty} M = \lim_{\tau \to \infty} \Sigma_+ = 0, \]
\[ \lim_{\tau \to \infty} X = X_\infty, \lim_{\tau \to \infty} w_+ = (w_+)\infty, \]
\[ \lim_{\tau \to \infty} w_- = (w_-)\infty, \]
with
\[ (w_+)\infty = \frac{X_\infty^2}{3(1 - X_\infty^2)}. \] (97)

**Proof.** Using the estimate (95) and the estimate for \( \bar{w}_- \) (87) making \( \varepsilon \) smaller if necessary we obtain a lower bound for \( X \)
\[ X \geq CX(\tau_0) \left| \frac{w_-(\tau)}{w_- (\tau_0)} \right| \geq CX(\tau_0) \left| 1 + O \left( \frac{\varepsilon}{w_- (\tau_0)} \right) \right|. \] (98)
Together with the estimate for \( M \) gives the desired conclusion for the Hubble normalised Weyl curvature (96). For the second part, note that the second derivatives of the barred quantities \( \bar{X}, \bar{\Sigma}_+, \bar{M}, \bar{w}_+ \) and \( \bar{w}_- \) are bounded. The terms we have collected in \( O(M) \) in the differential equations for the barred quantities are polynomials of \( \bar{X}, \bar{\Sigma}_+, \bar{M}, \bar{w}_+, \bar{w}_-, \cos \psi \) and \( \sin \psi \), all multiplied by \( M \). All the possible derivatives of these quantities will be bounded, since the only problematic term could come from the derivative of \( \psi \). The quantity \( \frac{1}{M} \) can be absorbed by the factor \( M \) and the quantity \( \frac{1}{X} \) is now not problematic anymore since we have shown that it \( X \) has the lower bound (98). The limits for \( \bar{\Sigma}_+, \bar{M} \) are obtained from the estimates of the theorem. Taking the limit in the differential equations for \( \bar{X}, \bar{w}_+ \) and \( \bar{w}_- \) we obtain that these quantities also have a limit. Taking the limit of the definitions of \( \bar{X}, \bar{\Sigma}_+, \bar{M}, \bar{w}_+, \bar{w}_- \) and \( \bar{Y} \) in terms of the unbarred quantities together with the limit of \( M \) we obtain the desired conclusion. \( \square \)
5 Discussion

Our results show that one can have a homogeneous cosmological model with massless particles where the shear is arbitrarily small and remains small, and the anisotropy of the matter distribution is arbitrarily small and remains small, but where the space-time is far from and remains far from an isotropic spacetime as the corollary shows.

In particular we have shown the asymptotic behaviour of reflection symmetric solutions to the Einstein-Vlasov system with Bianchi VII\textsubscript{0} symmetry. The solutions tend to the equilibrium solution of what we have called the truncated system assuming small data.

In the massive case it was shown that reflection symmetric solutions to the Einstein-Vlasov system with Bianchi VII\textsubscript{0} symmetry behave as Einstein-dust solutions assuming small data, cf. Theorem 1 of [7].

In the massless case, the relation between pressure (trace of energy momentum tensor) and energy density is fixed and given by [3]. However here we have shown that in the massless case, the behaviour differs from that of a radiation fluid, in the sense that we obtain an exponential decay rate, while the decay rate for a radiation fluid, both tilted and non-tilted, is polynomial [9, 10, 16].

Another difference is that in the collisionless case the shear will always remain small, but will not tend to zero. If we come back to the question of what happens with the spacetime if the matter distribution is almost isotropic, the answer to this question would be in the case considered here given by our corollary. In particular the anisotropy of the Hubble normalised trace free part of the second fundamental form is related to the anisotropy of the energy momentum tensor by [97].

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