TOWARDS OPTIMAL GRADIENT BOUNDS FOR
THE TORSION FUNCTION IN THE PLANE

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Abstract. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded, convex domain and let \( u \) be the solution of \( -\Delta u = 1 \) vanishing on the boundary \( \partial \Omega \). The estimate

\[
\| \nabla u \|_{L^\infty(\Omega)} \leq c |\Omega|^{1/2}
\]

is classical. We use the P-functional, the stability theory of the torsion function and Brownian motion to establish the estimate for a universal \( c < (2\pi)^{-1/2} \). We also give a numerical construction showing that the optimal constant satisfies \( c \geq 0.357 \). The problem is important in different settings: (1) as the maximum shear stress in Saint Venant Elasticity Theory, (2) as an optimal control problem for the constrained maximization of the lifetime of Brownian motion started close to the boundary and (3) and optimal Hermite-Hadamard inequalities for subharmonic functions on convex domains.

1. Introduction and result
We study a basic question for elliptic partial differential equations in the plane that arises naturally in a variety of contexts. Let \( \Omega \subset \mathbb{R}^2 \) be a convex domain and let

\[
\begin{align*}
-\Delta u &= 1 \quad \text{inside } \Omega \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

The purpose of this paper is to study a shape optimization problem and to establish bounds on the gradient. More precisely, our main result is as follows.

**Theorem 1 (Main Result).** Let \( \Omega \subset \mathbb{R}^2 \) be a convex, bounded domain and let \( u \) denote the solution of \( -\Delta u = 1 \) with Dirichlet boundary conditions. For some universal \( c < 1/\sqrt{2\pi} \sim 0.398 \),

\[
\| \nabla u \|_{L^\infty(\Omega)} \leq c \cdot |\Omega|^{1/2}
\]

and the constant \( c \) cannot be replaced by 0.357.

Though this is only a very slight improvement over the bound \( 1/\sqrt{2\pi} \) obtained in [11], our proof uses very different arguments, is somewhat robust and suggestive of future directions. One could presumably make the improvement explicit, though it would result in a very small number (something like \( \sim 10^{-10} \) or possibly even less as it depends on other constants for which only non-optimal estimates are available). Small improvements of this nature are not unusual, we refer to Bonk [16] or Bourgain [17] for other examples. As in those cases, the main advance is not the new

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constant but the new ideas that allow for an improvement. We also have high-
precision numerical results that imply an improved lower bound and possibly point
the way to a better understanding of the optimal shape.

Figure 1. A domain for which \( \| \nabla u \|_{L^\infty} \sim 0.357|\Omega|^{1/2} \). The point
on the boundary at which the derivative is maximal is marked.

We note that, in contrast to many other shape optimization problems, the optimal
shape for this functional is quite distinct from a disk. We have some numerical
examples for domains that we believe to be close to optimal and we consider a
more precise understanding of extremal domains to be an interesting problem. Is
their boundary smooth? Is it true that their curvature vanishes in exactly one
point? What is particularly noteworthy is that this shape optimization problem
appears (somewhat hidden) in a variety of different settings which we now survey.
Indeed, we believe it is one of the most fundamental shape optimization problems
that is far from being understood.

1.1. Saint Venant theory. The function \( u \) is also known as the torsion function
in the Saint Venant theory of elasticity \[72\] from 1856. In particular, the maximum
shear stress is known to occur on the boundary (a 1930 result of Polya \[66\])

\[
\tau = \left\| \frac{\partial u}{\partial n} \right\|_{L^\infty(\partial \Omega)} = \| \nabla u \|_{L^\infty(\Omega)}.
\]

This quantity has been of substantial interest in elasticity theory, see e.g. \[3, 58, 59, 62, 63, 64, 65, 66, 67, 69, 74\]. A classical inequality is \[74, \text{Eq. 6.12}\]

\[
\tau^2 \leq 2 \| u \|_{L^\infty(\Omega)}
\]

and many different bounds on the maximum of the function are known (some of
these bounds are discussed in §2.1). The best bound for arbitrary convex sets in
terms of the volume comes from \[11\]

\[
\tau \leq \frac{|\Omega|^{1/2}}{\sqrt{2\pi}}.
\]

An earlier result can be found in \[48\]. Many more results are available if one is
allowed to control more properties of \( \Omega \), in particular if one has bounds on the
curvature of \( \partial \Omega \), see e.g. \[58, 59, 62, 63, 64, 74\]. As for lower bounds, ellipses are a
particularly simple example because the torsion function can be computed in closed
form (this was already pointed out in \[11, 18\]). A simple computation shows that for any $0 < a < 1$, the function

$$u(x, y) = 1 - \frac{ax^2 + (1-a)y^2}{2}$$

is the torsion function on the set $\{(x, y) \in \mathbb{R}^2 : u(x, y) \geq 0\}$ which is an ellipse with

$$|\Omega| = \frac{2\pi}{\sqrt{a(1-a)}}.$$  

Evaluating $\nabla u$ in $x = \sqrt{2/a}$ shows that

$$\tau = \|\nabla u\|_{L^\infty(\Omega)} \geq \sqrt{2a}$$

and therefore the best constant $c$ in the estimate

$$\tau = \|\nabla u\|_{L^\infty(\Omega)} \leq c|\Omega|^{1/2}$$

has to satisfy

$$c \geq \max_{0 < a < 1} \frac{\sqrt{2a}}{\sqrt{2\pi}} \frac{2\pi}{\sqrt{a(1-a)}} = \frac{3^{3/4}}{4\sqrt{\pi}} \sim 0.321 \ldots$$

This narrows down the optimal constant to lie in the range $c \in (0.321, 0.398)$. A priori, this looks like a small range already – we emphasize that it is in the nature of the problem for the largest derivative to be relatively stable under perturbations of the domain. Indeed, there is still a large variety of different convex bodies for which the largest derivative lies in that range. Our actual understanding of the problem is still modest.

1.2. Large Gradients: Historical Remarks. We are interested in how big the largest gradient can be. Saint Venant himself tried to understand the location of the largest gradient since this is of interest in applications of elasticity theory: where is the maximal stress? Saint Venant writes

Les points dangereux sont donc, comme dans l’ellipse et le rectangle, les points du contour les plus rapprochés de l’axe de torsion, ou les extrémités des petits diamètre. (Saint Venant, \[72\])

More generally, on an ellipse the maximal derivatives are assumed on the short axis, a result considered ‘startling to many’ according to Thomson & Tait in 1867 \[81\] (Thomson would later be known as Lord Kelvin) who write in their Treatise on Natural Philosophy

M. de St. Venant also calls attention to a conclusion from his solutions which to many may be startling, that in the simpler cases the places of greatest distortion are those points of the boundary which are nearest to the axis [...] and the places of least distortion those farthest from it. (Thomson & Tait \[81, \S 710\])

Boussinesq \[18\] gave a heuristic explanation in 1871. In 1900, Filon \[27\] confirms that the ‘fail points’ (les points dangereux) for ellipses are along the shorter axis. Around 1920, Griffith & Sir G. I. Taylor \[31\] report on an apparatus using soap bubbles to compute torsion. The fact that the points with the largest gradient lie on the boundary was rigorously proven only in 1930 by Polya \[66\].
Saint Venant conjectured that the maximum of the gradient in a convex domain is assumed in the point on the boundary where the largest inscribed circle intersects the boundary. Sweers [71] showed that the conjecture fails for either the set

\[ \Omega = \{(x_1, x_2) \in \mathbb{R}^2 : (|x_1| + 1)^2 + x_2^2 < 4, |x_2| \leq 1\} \]

or for one of the superlevel sets \( \{x \in \Omega : u(x) \geq \varepsilon\} \) and thus fails in general (see also [43, 71, 77, 78]). However, the statement is known to hold under some additional assumptions (see e.g. Kawohl [39, 40]).

\[ \text{Figure 2. The ‘barrel’ domain of Sweers [71].} \]

1.3. A Benchmark PDE. The equation \(-\Delta u = 1\) with Dirichlet boundary conditions inside convex domains has for a long time been a ‘benchmark PDE’ for which new types of techniques and results are being developed: in a certain sense, it is the simplest elliptic partial differential equation after \(\Delta u = 0\). An example is the seminal result of Makar-Limanov [52] who showed that solutions of \(-\Delta u = 1\) in convex domains have the property that \(\sqrt{u}\) is concave. Many more results have since been established, we refer to the survey of Keady & McNabb [41]. Results with a similar degree of precision are really only available for the first eigenfunction of the Laplacian. A fundamental result by Brascamp & Lieb [13] for the ground state of the Laplacian \(-\Delta u = \lambda_1 u\) inside convex domains in the plane is convexity of the level set. Problems of this nature are of continued interest [1, 26, 34, 38, 51, 60, 74]. We also emphasize that there are very precise results obtained for the structure of the first Laplacian eigenfunction by Grieser and Jerison [29, 30, 47] with exciting recent developments by T. Beck [8, 9, 10]. The usual question is: how much of the insight gained for these types of special solutions can be carried over to more general solutions? We also mention the larger field of Shape Optimization (representative textbooks being Baernstein [2], Lieb & Loss [45], Henrot [34], Polya & Szegő [68]) concerned with the interplay of solutions of partial differential equations (or functionals of the solutions) with the geometry of the underlying domain. Both \(-\Delta u = \lambda_1 u\) and \(-\Delta u = 1\) have been actively investigated from that perspective and we see our contribution firmly aligned with this line of inquiry. In particular, our approach is sufficiently robust to yield nontrivial results for larger families of PDE’s. However, since we have not fully understood \(-\Delta u = 1\), we have not pursued this further at this time.
1.4. **Expected Lifetime of Brownian motion.** The solution of $-\Delta u = 1$ with Dirichlet boundary conditions describes (up to constants depending on normalization) the expected lifetime of Brownian motion. More precisely, $u(x)$ gives the expected lifetime of Brownian motion started in $x$ before hitting the boundary $\partial \Omega$. We refer to [4, 5, 6, 7] and references therein. There are many open problems. It is known, for example, that for simply connected $\Omega \subset \mathbb{R}^2$.

$$\frac{\text{inrad}(\Omega)^2}{2} \leq \text{maximum expected lifetime} \leq c \cdot \text{inrad}(\Omega)^2.$$ 

Here, the constant $c$ is known to be 1 in convex domains [74], the extreme case is that of an infinite strip. Such results are only possible due to the potential-theoretic rigidity of two dimensions, we refer to [28, 44, 46, 70] for substitute results in higher dimensions. A result of a similar flavor is the following: among all convex sets with fixed volume $|\Omega|$, the disk maximizes both the average expected lifetime (a 1948 result of Polya [67]) as well as the maximum expected lifetime (this follows essentially from a rearrangement principle of Talenti [80]). For this type of problem, we also refer to recent work of Hamel & Russ [33] (see also J. Lu and the second author [49]).

![Figure 3. A domain $\Omega$ satisfying $\|\nabla u\|_{L^\infty} \sim 0.34|\Omega|^{1/2}$. Brownian motion started in the point $x_0$ close to the boundary has a (relatively) large expected lifetime compared to other domains.](image)

The problem of sharp gradient estimates for the torsion function is equivalent to a natural and interesting problem in probability theory. As mentioned above, the shape maximizing the expected lifetime of Brownian motion among domains with fixed volume is the disk (and, unsurprisingly, we start Brownian motion in the center of the disk).

**Problem.** Among all convex domains of fixed volume $\Omega$, which one maximizes the expected lifetime of Brownian motion that starts in a point that is within $\varepsilon$ distance of the boundary?

The Blaschke selection theorem guarantees the existence of these extremal domains but it is not at all clear what sort of regularity properties they have. This problem depends on the size of $\varepsilon$, our paper is only concerned with the case $\varepsilon \to 0$. It is not too surprising that the expected lifetime is going to decay: starting close to the boundary makes it exceedingly likely to hit the boundary rather quickly. Nonetheless, there will always be Brownian paths that survive the initial dangerous phase and then explore the domain. Which shape guarantees the longest lifetime? It is intuitively clear that since we start close to the boundary, that part of the boundary
should be as flat as possible within the confines of being a convex domain of a fixed volume. This makes it seem natural to assume that the curvature of the boundary of the optimal domain vanishes in exactly one point. Our numerical construction of a lower bound suggests certain shapes that might be close to extremal, we believe this to be an interesting open problem.

![Figure 4. A domain for which \( \| \nabla u \|_{L^\infty} \sim 0.357 |\Omega|^{1/2} \). Brownian motion started in the point \( x_0 \) close to the boundary has a (relatively) large expected lifetime compared to other domains.](image)

We observe that these two examples (Fig. 3 and Fig. 4) have quite different shapes but have maximal gradients of a fairly similar size. As mentioned above, the problem comes with a great deal of stability which is echoed in these two numbers being rather similar. In particular, the difference of 1/50 in these maximal gradients is actually, considering the underlying stability, a big difference. This further illustrates the difficulty of the problem: understanding the extreme domains for a quantity that is very stable.

1.5. **Sharp Hermite-Hadamard inequalities.** The Hermite-Hadamard inequality [32, 35] is a (very) elementary fact: if \( f : [a, b] \to \mathbb{R} \) is a convex function, then

\[
\frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.
\]

There have been a very large number of variations on this inequality, we refer to [12, 19, 20, 21, 23, 25, 53, 54, 55, 56] and the textbook [24]. Somewhat surprisingly, given the large number of results and extensions, until recently there has been relatively little work on the general higher-dimensional case. The second author [79] proved that if \( \Omega \subset \mathbb{R}^n \) is bounded and convex and \( f : \Omega \to \mathbb{R} \) is convex and positive on \( \partial \Omega \), then

\[
\frac{1}{|\Omega|} \int_\Omega f(x) dx \leq \frac{c_n}{|\partial \Omega|} \int_{\partial \Omega} f(x) d\sigma,
\]

where \( c_n \) is a universal constant depending only on the dimension. The best known estimate for the constant is \( n - 1 \leq c_n \lesssim n^{3/2} \) obtained in [11]. In the special case of two dimensions, the best known estimate is \( c_2 \leq 3 \) (also in [11]). The second
author [79] also showed that, under the same assumptions on $\Omega$ and $f$, we also have
\[ \int_{\Omega} f(x)dx \leq c_n |\Omega|^{1/n} \int_{\partial \Omega} f(x)d\sigma. \]
This was then improved by Jianfeng Lu and the second author [48] who proved that $c_n \leq 1$ in all dimensions and that the inequality holds for the larger family of subharmonic functions $\Delta f \geq 0$ (every convex function is subharmonic). A slightly better estimate on the constant was given in [11] which proved that in two dimensions $c_2 \leq 1/\sqrt{2\pi} \sim 0.39$. A characterization of the optimal constant as the largest gradient term of the torsion function is due to Niculescu & Persson [55]. The argument is as follows: since $f$ is subharmonic, i.e. $\Delta f \geq 0$, and $u \geq 0$ vanishes on the boundary, integration by parts shows
\[ \int_{\Omega} f dx = \int_{\Omega} f(-\Delta u) dx = \int_{\partial \Omega} \frac{\partial u}{\partial \nu} f d\sigma - \int_{\Omega} (\Delta f) u dx \]
\[ \leq \int_{\partial \Omega} \frac{\partial u}{\partial \nu} f d\sigma \]
\[ \leq \max_{x \in \partial \Omega} \frac{\partial u}{\partial \nu}(x) \int_{\partial \Omega} f d\sigma, \]
where $\nu$ is the inward pointing normal vector and we used the condition $f|_{\partial \Omega} \geq 0$ in the last step. Moreover, by letting $f$ be the harmonic extension of a characteristic function in a small part of the domain where $\partial u/\partial \nu$ is close to maximal, we see that this constant is optimal. This characterization, combined with our result, implies the following improved Hermite-Hadamard inequality.

**Theorem 2.** Let $\Omega \subset \mathbb{R}^2$ be a convex domain and let $f : \Omega \to \mathbb{R}$ be a subharmonic function, $\Delta f \geq 0$, assuming positive values on $\partial \Omega$. Then, for some universal constant $c < 1/\sqrt{2\pi} \sim 0.398\ldots$,
\[ \int_{\Omega} f(x)dx \leq c \cdot |\Omega|^{1/2} \int_{\partial \Omega} f(x)d\sigma \]
and the constant $c$ cannot be replaced by 0.357.

We emphasize that this problem is completely equivalent to the other two problems: optimal gradient estimates for the torsion function and the largest lifetime of Brownian time constrained to starting close to the boundary. We do not have any knowledge about the extremal domain for this Hermite-Hadamard inequality. We do not know anything about the regularity of the boundary and we know nothing about the behavior of the curvature. However, courtesy of the underlying intuition coming from maximizing constrained Brownian motion, we have numerical examples of domains that we believe to be close to optimal.

1.6. **Our Approach.** We conclude by explaining our approach to prove Theorem 1. It is based on the notion of Fraenkel asymmetry: for any domain $\Omega \subset \mathbb{R}^n$, its Fraenkel asymmetry $A(\Omega)$ is defined by
\[ A(\Omega) = \inf_{|B|=|\Omega|} \frac{|B\Delta \Omega|}{|\Omega|}, \]
where $B$ ranges over all balls in $\mathbb{R}^n$ that have the same volume as $\Omega$ and $A\Delta B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of two sets. The Fraenkel asymmetry satisfies $0 \leq A(\Omega) \leq 2$ and is a quantitative measure of how close a set is to a
ball. Combining the P-functional of Sperb [74], an old inequality of Larry Payne [57] and a recent quantitative Saint Venant theorem due to Brasco, De Philippis and Velichkov [14], we can establish that for any convex \( \Omega \subset \mathbb{R}^2 \) with area \( |\Omega| = 1 \), the solution of \(-\Delta u = 1\) with Dirichlet boundary conditions, satisfies, for some universal constant \( \tau > 0 \),

\[
\|\nabla u\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} - \tau \cdot A(\Omega)^3.
\]

This establishes a uniform improvement for all domains satisfying \( A(\Omega) \in (\varepsilon_0, 2) \) for any \( \varepsilon_0 > 0 \). It remains understand the case where \( \Omega \) is very close to the disk in the sense of \( A(\Omega) \leq \varepsilon_0 \). Here we will establish the following stability result.

**Proposition.** There exists \( \varepsilon_0 > 0 \) such that if \( \Omega \subset \mathbb{R}^2 \) is convex, \( A(\Omega) \leq \varepsilon_0 \) and \( u \) solves \(-\Delta u = 1\) with Dirichlet boundary conditions, then

\[
\|\nabla u\|_{L^\infty(\Omega)} \leq 0.39 < \frac{1}{\sqrt{2\pi}}.
\]

This raises several interesting open problems: is there a form of stability theory for the maximal gradient of the torsion function in convex domains? More precisely, one could ask whether for all convex \( \Omega \) with \( |\Omega| = 1 \), the solution of \(-\Delta u = 1\) with Dirichlet boundary conditions satisfies

\[
\|\nabla u\|_{L^\infty(\Omega)} \leq \frac{1}{2\sqrt{2\pi}} + c \cdot A(\Omega)^\alpha
\]

for some fixed constants \( \alpha \) and \( c \). One might expect this quantity to be quite stable. Indeed, one of the reasons why understanding extremal domains for this problem might be difficult is precisely this form of stability that is also reflected in our numerical examples and, for example, in the quantitative Saint Venant Theorem (where the improvement is comparatively small, \( A(\Omega)^3 \)). A more general question would be whether, for any two convex domains \( \Omega_1, \Omega_2 \in \mathbb{R}^2 \) having the same measure, the solutions \( u_1, u_2 \) satisfy a general stability property along the lines of

\[
\|\nabla u_1\|_{L^\infty(\Omega_1)} - \|\nabla u_2\|_{L^\infty(\Omega_2)} \leq c \cdot \left( \frac{|(\Omega_1 \setminus \Omega_2) \cup (\Omega_2 \setminus \Omega_1)|}{|\Omega_1|} \right)^\beta.
\]

### 2. Observations and Remarks

We first summarize the existing arguments surrounding the P-functional approach, discuss a seemingly novel isoperimetric inequality and then proceed to establish the main result in §3.

#### 2.1. Summary of the existing argument

We will present the relevant material for the equation \(-\Delta u = 2\) with Dirichlet boundary conditions inside a convex domain. Our object of interest, gradient bounds for solutions of \(-\Delta u = 1\), then follow from a rescaling by a factor of 2. It is slightly more convenient to work with \(-\Delta u = 2\) since that is the scaling corresponding to the expected lifetime of Brownian motion. The P-functional associated with solutions of \(-\Delta u = 2\) is

\[
P(u) = |\nabla u|^2 + 4u.
\]

The important property is that \( P(u) \) assumes its maximum at the unique global maximum of \( u \). Moreover, it is known that \( |\nabla u| \) assumes its largest value on the
boundary (see [66, 74]). These two properties combined imply that
\[ \|\nabla u\|_{L^\infty(\Omega)}^2 \leq 4 \|u\|_{L^\infty}. \]
We repeat an argument from the book of Sperb [74]. We have, for all \( x \in \Omega \),
\[ |\nabla u(x)| \leq 2 \sqrt{\|u\|_{L^\infty} - u(x)}. \]
Moreover, the gradient is larger than any directional derivative. Integrating this identity along the shortest line connecting the point \( x_0 \) at which \( u \) assumes its maximum to the closest point on the boundary shows that
\[ \|u\|_{L^\infty} \leq d(x_0, \partial \Omega)^2 \leq \text{inrad}(\Omega)^2. \]
However, one does not necessarily need to integrate along a line leading to the boundary; integrating over lines leading to an arbitrary point in the domain shows that
\[ \|u\|_{L^\infty(\Omega)} \leq \|x - x_0\|^2 + u(x). \]
Integrating over the entire domain \( \Omega \), which we assume to have volume 1, leads to
\[ \|u\|_{L^\infty(\Omega)} \leq \int_{\Omega} \|x - x_0\|^2 \, dx + \int_{\Omega} u(x) \, dx. \]
This is a known result and can be found in [74, Eq. (6.14)]. The quantity
\[ \int_{\Omega} u(x) \, dx \]
and a well-studied object. Saint Venant [72] conjectured in the 1850s that among all domains of fixed area, the torsional rigidity is maximized by the circle (now known as Saint Venant’s Theorem). The first rigorous proof seems to have been given in a 1948 paper of Polya [67]. Davenport gave another proof (in [68]), a third proof is due to Makai [50]. Nowadays, it is often considered a consequence of Talenti’s rearrangement principle [80]. For the purpose of obtaining an upper bound, it thus suffices to assume \( \Omega \) is a disk of radius \( \rho = 1/\sqrt{\pi} \) in which case the torsion function is explicit:
\[ \frac{1}{2\pi} - \frac{1}{2} \|x\|^2 \]
and thus
\[ \int_{\Omega} u(x) \, dx \leq \int_0^{1/\sqrt{\pi}} \left( \frac{1}{2\pi} - \frac{y^2}{2} \right) 2\pi y \, dy = \frac{1}{4\pi}. \]
Moreover, as implied by Talenti’s rearrangement principle [80], the maximum value of \( u \) also increases under symmetrization and thus, for all domains of area \( |\Omega| = 1 \), we have
\[ \|u\|_{L^\infty} \leq \frac{1}{2\pi}. \]
Summarizing these ideas, we see that if \( \Omega \subset \mathbb{R}^2 \) is a convex set of unit area and \(-\Delta u = 2 \) in \( \Omega \) with Dirichlet boundary conditions, then
\[ \|\nabla u\|_{L^\infty(\Omega)} \leq 2 \|u\|_{L^\infty}^{1/2} \leq \sqrt{\frac{2}{\pi}}, \]
\[ \|\nabla u\|_{L^\infty(\Omega)} \leq 2 \text{inrad}(\Omega) \]
\[ \|\nabla u\|_{L^\infty(\Omega)}^2 \leq \frac{1}{\pi} + 4 \int_{\Omega} \|x - x_0\|^2 \, dx. \]
The first bound yields, after scaling by a factor of 2, the upper bound of \(1/\sqrt{2\pi} \sim 0.39\ldots\) that we wish to improve upon. The second inequality would imply an improvement as soon as \(\text{inrad}(\Omega) < 1/\sqrt{2\pi}\) (however, \(|\Omega| = 1\) only yields \(\text{inrad}(\Omega) < 1/\sqrt{\pi}\)). The third estimate, coupled with an isoperimetric estimate on the polar momentum, does not yield any improvement either. In the purpose of investigating it, we did come across an amusing isoperimetric principle for the polar momentum that we could not find in the literature.

2.2. An Isoperimetric Principle for Polar Momentum.

**Lemma 1.** Let \(0 < \rho \leq 1/\sqrt{\pi}\). Among all convex sets \(\Omega \subset \mathbb{R}^2\) with area 1 that contain a disk of radius \(\rho\) around the origin, the functional

\[
J(\Omega) = \int_{\Omega} \|x\|^2 dx
\]

is maximized if \(\Omega\) is the convex hull of a disk of radius \(\rho\) and a point (the point is uniquely defined by the volume constraint \(|\Omega| = 1\) up to rotation invariance).

There is nothing particularly special about the weight \(\|x\|^2\) and the statement holds if \(\|x\|^2\) is replaced by a strictly monotonically increasing function.

![Figure 5. The extremal configuration: the distance of \(x_0\) is uniquely determined by \(\rho\) and \(|\Omega| = 1\).](image)

The Blaschke selection theorem immediately implies the existence of a (not necessarily unique) extremizing convex domain for this problem which we call \(\Omega\). It remains to understand its properties. We introduce \(f : (0, \infty) \to \mathbb{R}\) via

\[
f(r) = |\{x \in \Omega : \|x\| = r\}|
\]

We have

\[
\int_0^\infty f(r) dr = 1 \quad \text{and want to maximize} \quad \int_0^\infty f(r)r^2 dr.
\]

We know that there exists a constant \(M(\rho)\) such that \(f(r) = 0\) for all \(r > M(\rho)\) (since \(\Omega\) is a convex set contains a disk of radius \(\rho\), it cannot have an arbitrarily large diameter since that would violate \(|\Omega| = 1\)). Integration by parts yields

\[
\int_0^{M(\rho)} f(r)r^2 dr = F(r)r^2|_0^{M(\rho)} - 2 \int_0^{M(\rho)} F(r)r dr
\]

\[
= M(\rho)^2 - 2 \int_0^{M(\rho)} F(r)r dr.
\]
This shows that instead of maximizing the integral over $f(r)r^2$, we could instead try to minimize the integral over $F(r)r$, where $F(r)$ is the amount of area of $\Omega$ of distance at most $r$ from the origin. This, however, turns out to have a relatively simple solution for each value of $r$ that happens to not depend on $r$. In particular, Lemma 1 is implied by a (stronger) geometric statement which we now prove.

**Lemma 2.** Let $0 < \rho \leq 1/\sqrt{\pi}$. For each $r > \rho$, among all convex sets $\Omega \subset \mathbb{R}^2$ with area 1 that contain a disk of radius $\rho$ around the origin, the area $|\{x \in \Omega : \|x\| \leq r\}|$ is minimized if $\Omega$ is the convex hull of a disk of radius $\rho$ and a point (the point is uniquely defined by the volume constraint $|\Omega| = 1$ up to rotational invariance).

**Proof.** There is a total of

$$0 < |\Omega| - \rho^2\pi = 1 - \rho^2\pi$$

area outside the disk of radius $\rho$ and we assume that

$$A = |\{x \in \Omega : \rho \leq \|x\| \leq r\}| \quad \text{and} \quad B = |\{x \in \Omega : \|x\| \geq r\}|.$$

We are interested in understanding which convex domain minimizes $A$ or, since $A + B = 1 - \rho^2\pi$ is fixed, which shape maximizes $B$. We introduce a third quantity, the length $\ell$ of the level set $r$,

$$\ell = |\{x \in \Omega : \|x\| = r\}|.$$

**Figure 6.** The circle of radius $\rho$, the set $\{x \in \Omega : \|x\| = r\}$ and the induced cones.

We first argue that, depending on the size of $B$, $\ell$ cannot be too small. This can be done as follows: a priori we have no clear idea about the structure of $\{x \in \Omega : \|x\| = r\}$. For example, it could be comprised of several intervals. We note that it could a priori also be comprised of an entire circle, however, we are interested in lower bounds on $\ell$ in terms of $B$ so this case is of less interest. Suppose now that $\{x \in \Omega : \|x\| = r\}$ is comprised of several intervals. Then the fact that the endpoints of the intervals are endpoints implies, together with convexity, the existence of cone structures in which the set $\{x \in \Omega : \|x\| > r\}$ must be contained, thus giving an upper bound on $B$. Moreover, we note that this upper bound is maximal if $\{x \in \Omega : \|x\| = r\}$ is comprised of a single interval of length $\ell$. This implies an upper bound on $B$ in terms of $\ell$ or, conversely, it implies a lower bound
on $\ell$ given $B$. Moreover, the convex hull of the disk and a point shows that this bound is sharp. We now proceed to argue in a different direction. Clearly, there must be a lower bound on $A$ given $\ell$ (see the same Figure). Each element $x \in \Omega$ with $\|x\| = r$ casts, by convexity, a shadow and the total volume of shadows is minimized if they overlap as much as possible which happens when $\{x \in \Omega: \|x\| = r\}$ is an interval. We see that this estimate is also sharp for the convex hull of a disk and a point outside. However, $A + B = 1 - \rho^2 \pi$. We can now summarize the argument as follows.

\[ B \text{ is of a certain size } \implies \ell \geq \text{function}(B) \]
\[ \implies A \geq \text{function}_2(\ell) \]
\[ \implies B \leq 1 - \rho^2 \pi - \text{function}_2(\text{function}_1(B)). \]

Phrased differently, if $B$ is very big, then so is $\ell$. However, then $\ell$ is big and $\{x \in \Omega: \|x\| = r\}$ casts a large shadow forcing $A$ to be very big. Then, however, since $A + B$ is fixed, this requires $B$ to be small. Since all these inequalities are sharp for the case of $\Omega$ being the convex hull of a disk and a point outside, we obtain the desired result. □

3. Proof of Theorem 1 and Theorem 2.

3.1. Outline of the Argument. The first part of the argument, an improved gradient estimate for domains that have a large Fraenkel asymmetry, is easy to describe. The first ingredient is something already used in prior approaches to the problem. We will, throughout the rest of the argument, work with $-\Delta u = 2$ to simplify exposition (and to have the right scaling for Brownian motion).

**Theorem 3 (Sperb’s P-function [74]).** Let $\Omega \subset \mathbb{R}^2$ be convex and assume that $u$ is the function satisfying $-\Delta u = 2$ with Dirichlet boundary conditions on $\partial \Omega$. Then

\[ P(u) = |\nabla u|^2 + 4u \]

assumes its maximum at a critical point of $u$.

Moreover, the gradient assumes its maximum on the boundary and thus

\[ \|\nabla u\|_{L^\infty(\Omega)} \leq 2\|u\|^{1/2}_{L^\infty(\Omega)}. \]

The next estimate is due to Larry Payne and provides an upper bound on the maximum of the torsion function in terms of the gradient energy. We also refer to Salakhudinov [73], Sperb [75] and Payne [58, 61].

**Theorem 4 (Payne [57]).** Let $\Omega \subset \mathbb{R}^2$ be simply connected and assume that $u$ is the function satisfying $-\Delta u = 2$ with Dirichlet boundary conditions on $\partial \Omega$. Then

\[ \|u\|_{L^\infty}^2 \leq \frac{1}{2\pi} \int_{\Omega} |\nabla u|^2 \, dx. \]

We recall that the torsion function on a disk of radius $1/\sqrt{\pi}$ in polar coordinates is given by $u(r) = 1/(2\pi) - r^2/2$ allowing us to compute

\[ \|u\|_{L^\infty}^2 = \frac{1}{4\pi^2} = \frac{1}{2\pi} \int_0^{1/\sqrt{\pi}} r^2 \cdot 2r \pi \, dr = \frac{1}{2\pi} \int_{\Omega} |\nabla u|^2 \, dx. \]
and thus Payne’s inequality is sharp on the disk. Integration by parts using \(-\Delta u = 2\) allows us to rewrite this integral as

\[
\|u\|_{L^\infty}^2 \leq \frac{1}{2\pi} \int_{\Omega} |\nabla u|^2 \, dx = \frac{1}{\pi} \int_{\Omega} u(x) \, dx.
\]

The third ingredient is a fairly recent stability statement for torsional rigidity. To state it, we recall the notion of Fraenkel asymmetry \(A\) of a set \(\Omega \subset \mathbb{R}^n\): in short, the Fraenkel asymmetry is defined by

\[
A(\Omega) = \inf_{|B| = |\Omega|} \frac{|B \Delta \Omega|}{|\Omega|},
\]

where \(B\) ranges over all balls in \(\mathbb{R}^n\) that have the same volume as \(\Omega\) and \(A \Delta B = (A \setminus B) \cup (B \setminus A)\) is the symmetric difference of two sets. The Fraenkel asymmetry satisfies \(0 \leq A(\Omega) \leq 2\) and is a quantitative measure of how close a set is to a ball.

**Theorem 5** (Brasco, De Philippis and Velichkov [14]). Let \(\Omega\) be an open set in \(\mathbb{R}^2\), let \(\Omega^*\) be the symmetrized open set with the same volume and let

\[
T(\Omega) = \int_{\Omega} u(x) \, dx,
\]

where \(u\) solves \(-\Delta u = 2\) in \(\Omega\) with Dirichlet boundary conditions. Then, for some universal \(\tau > 0\),

\[
T(\Omega^*) - T(\Omega) \geq \tau A(\Omega)^3.
\]

Explicit estimates on the size of \(\tau\) are available but presumably many orders of magnitude away from optimal [14] and we will not pursue an explicit quantitative estimate here. We also refer to the particularly nice survey [15]. Combining these three results implies an explicit quantitative improvement for \(A(\Omega) > 0\). In order to obtain a uniform estimate, we obtain, via a completely different line of reasoning, an estimate in the range \(A(\Omega) \in (0, \varepsilon_0)\) for some \(\varepsilon_0\) sufficiently small. This will be the remainder of the argument.

### 3.2. Estimating Brownian Motion.

We now arrive at the heart of the argument. Our goal is to estimate that largest derivative of

\[
-\Delta u = 2 \quad \text{inside } \Omega
\]

\[
u = 0 \quad \text{on } \partial \Omega
\]

on the boundary, where \(\Omega \subset \mathbb{R}^2\) is a convex domain normalized to have measure \(|\Omega| = 1\) that satisfies \(A(\Omega) \leq \varepsilon_0\) for some \(\varepsilon_0\) sufficiently small (but positive and universal).

The convex set \(\Omega\)

![Figure 7](image-url)  
**Figure 7.** An outline of the geometry.
We proceed by estimating the lifetime of Brownian motion itself, the beginning of our argument is adapted from [48]. We estimate the expected lifetime of Brownian motion started at distance $\varepsilon$ from the point $x_0 \in \partial \Omega$. Since $\Omega$ is convex, there is a supporting hyperplane (i.e. a line) in $x_0$ such that all of $\Omega$ is strictly contained on one side of the hyperplane (and possibly the hyperplane itself). We pick a time $T$ (to be chosen later) and consider Brownian motion running for $T$ units of time. Brownian motion in the plane can be decoupled into two independent (one-dimensional) Brownian motions in two coordinates. We consider those as separate. For now we focus only, using the orientation suggested in the Figure, on the Brownian motion acting in the $y$ coordinate. Clearly, if one starts rather close to the boundary, then the likelihood of hitting the boundary must be rather large. This quantity is known and classical in probability theory.

**Lemma 3** (Reflection principle; e.g. [37]). Let $\varepsilon > 0$ and let $T_\varepsilon$ be the hitting time for Brownian motion started at $B(0) = \varepsilon$, $T_\varepsilon = \inf \{t > 0 : B(t) = 0\}$. Then

$$P(T_\varepsilon \leq t) = 2 - 2\Phi\left(\frac{\varepsilon}{\sqrt{t}}\right),$$

where $\Phi$ is the cumulative distribution function of $\mathcal{N}(0,1)$.

Exactly as in [48], we can use this identity to compute the density $\psi$ of the stopping time

$$\psi(t) = \frac{d}{dt} P(T_\varepsilon \leq t) = \Phi'\left(\frac{\varepsilon}{\sqrt{t}}\right) \frac{\varepsilon}{t^{3/2}} = \frac{\varepsilon}{\sqrt{2\pi t^{3/2}}} e^{-\varepsilon^2/2t}.$$

This distribution decays like $\sim t^{-3/2}$ and does not have a finite mean. However, the expected lifetime of particles hitting the threshold before time $T$ can be computed and is bounded by

$$E(T_\varepsilon | T_\varepsilon \leq T) = \int_0^T \psi(t) dt \leq \int_0^T \frac{\varepsilon}{\sqrt{2\pi t^{1/2}}} dt = \frac{\varepsilon}{\sqrt{2} \sqrt{T}}.$$

We will work in the regime where $T > 0$ is fixed and $\varepsilon \to 0$. In that regime, we use

$$\Phi\left(\frac{\varepsilon}{\sqrt{T}}\right) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \frac{\varepsilon}{\sqrt{T}} + \text{l.o.t.}$$

and therefore

$$P(T_\varepsilon > T) = (1 + o(1)) \sqrt{\frac{2}{\pi}} \frac{\varepsilon}{\sqrt{T}}$$

as $\varepsilon \to 0$.

Here and henceforth, $o(1)$ will always refer to a quantity going to 0 as $\varepsilon \to 0$ for $T$ fixed. We will now introduce a new ingredient into the mix. Instead of merely trying to control the number of particles that do not collide with the line up to time $T$, we are also interested in the distribution of the survivors. For this purpose, we fix the interval $[0, \pi]$ and study the solution of the heat equation

$$(\partial_t - \Delta)u = 0$$

with Dirichlet boundary conditions at 0 and $\pi$. The stochastic interpretation of the heat equation implies that the survival distribution at time $T$ is given by the solution of $u(T,x)$ where $u(0,x) = \delta_x$ is a Dirac mass in $x$. We note that, in this formulation, we also introduce an additional barrier at $\pi$ — since we will only work in the setting of $\mathcal{A}(\Omega) \leq \varepsilon_0$, no part of $\Omega$ can cross the second barrier and we obtain a valid upper bound. The eigenfunctions of the Laplace operator $-\Delta$ with
Dirichlet boundary conditions on \([0, \pi]\) are given by \(\phi_k(x) = \sqrt{2/\pi} \sin (kx)\) and the eigenvalues are \(k^2\). This implies that the heat kernel can be written as (where we scale time a factor \(1/2\) to scale correctly with respect to classical Brownian motion)

\[
p_t(x, y) = \frac{2}{\pi} \sum_{k=1}^{\infty} e^{-k^2/2} \sin (kx) \sin (ky).
\]

We are again working in a favorable regime: since we will keep time \(T\) fixed and send \(\varepsilon \to 0\), we obtain the survivor density

\[
p_T(\varepsilon, y) = \frac{2}{\pi} \sum_{k=1}^{\infty} e^{-k^2T/2} \sin (k\varepsilon) \sin (ky) = (1 + o(1)) \frac{2\varepsilon}{\pi} \sum_{k=1}^{\infty} e^{-k^2T/2} k \sin (ky).
\]

Since we have the explicit distribution, we should be able to recover the Lemma cited above and will now do so. The total survival likelihood is given by

\[
\int_0^{\pi} p_T(\varepsilon, y) dy = (1 + o(1)) \frac{2\varepsilon}{\pi} \int_0^{\pi} \sum_{k=1}^{\infty} e^{-k^2T/2} k \sin (ky)
\]

\[
= (1 + o(1)) \frac{4\varepsilon}{\pi} \sum_{k \in \mathbb{N}, \text{ odd}} e^{-k^2T/2}
\]

\[
\sim \frac{2\varepsilon}{\pi} \int_0^{\infty} e^{-(x\sqrt{T})^2/2} dx \sim \sqrt{\frac{2}{\pi}} \frac{\varepsilon}{\sqrt{T}}.
\]

\text{Figure 8. The survivors when starting in } x = 0.01 \text{ and running until } T = 0.2. \text{ One observes that even though few survive, the survivors travel quite a distance. Moreover, the survival probability goes to 0 as } \varepsilon \to 0 \text{ but the profile converges.}

We would like this number to correspond to our survival estimate obtained above via the reflection principle. There is no reason it should since particles are being absorbed on both ends of the interval. However, when \(T\) is small, one would assume that the right-hand side of the interval has little to no effect. Indeed, for \(T \to 0\), we have

\[
(1 + o(1)) \frac{4\varepsilon}{\pi} \sum_{k \text{ odd}} e^{-k^2T/2} = (1 + o(1)) \frac{4\varepsilon}{\pi} \sum_{k \in \mathbb{N}, \text{ odd}} e^{-(k\sqrt{T})^2/2}
\]

\[
\sim \frac{2\varepsilon}{\pi} \int_0^{\infty} e^{-(x\sqrt{T})^2/2} dx \sim \sqrt{\frac{2}{\pi}} \frac{\varepsilon}{\sqrt{T}}.
\]
which recovers the estimate for the survival probability derived above for $T \to 0$. However, what is important for our successive argument is having direct access to the distribution of the surviving particles. Returning to the two-dimensional picture, we can think of running two-dimensional Brownian motion and then estimating the distribution of the survivors after $T$ units of time. We do not have any idea where $\Omega$ is located, so we initially work in the larger domain created by the supporting hyperplane. We just derived the distribution along the $y$-coordinate, the $x$-coordinate is an independent Brownian motion which has a Gaussian distribution after $T$ units of time. This, in turn, implies that we have an estimate on the distribution of surviving Brownian motions in a half-space.

So far, we haven’t made any use of the domain $\Omega$: clearly, it implies that our closed-form representation of survival probabilities is a strict upper bound on the actual distribution within $\Omega$ (since some of the particles lie outside and some of the paths will leave $\Omega$ before returning). Here, we use one more elementary Lemma (whose scaling could presumably be slightly improved but the simplest possible argument is enough for our purposes here).

**Lemma 4.** If $\Omega \subset \mathbb{R}^2$ is convex, has volume 1 and satisfies $A(\Omega) \leq \delta_0$ for some universal $\delta_0 > 0$, then $\Omega \subset B$ for some ball $B$ satisfying

$$|B| \leq 1 + c \cdot A(\Omega)^{1/2},$$

where $c$ is a universal constant.

**Proof.** Let $B$ be the ball minimizing the Fraenkel asymmetry and let us assume for ease of exposition that $B$ is centered in 0. Since $|\Omega| = 1$, we have $|B| = 1$ and $B$ has radius $1/\sqrt{\pi}$. Suppose now that $x_0 \in \Omega$ and $\|x_0\| = 1/\sqrt{\pi} + z$ for $z$ sufficiently small. The convex hull of the point $x_0$ with $\Omega$ that lies outside of $B$ has area at least $\sim z^2$ and thus $z \lesssim A(\Omega)^{1/2}$. Scaling up the ball $B$ by that factor will then do the job. \(\square\)

We can now summarize the argument as follows:

1. we bound the lifetime of Brownian motion inside $\Omega$ from above by bounding its expected lifetime within two lines up to time $T$; moreover, we compute the probability density function of the survivors
2. in the second step, we have to bound the expected further lifetime of the survivors: however, they are contained in a ball of radius $1/\sqrt{\pi} + c \cdot A(\Omega)^{1/2}$ and for this ball we can compute it in closed form.

We recall that the expected lifetime of Brownian motion in a disk of radius $1/\sqrt{\pi}$ centered at $(0, 1/\sqrt{\pi})$ is given by

$$v(x, y) = \left( \frac{1}{2\pi} - \frac{x^2 + (y - \pi^{-1/2})^2}{2} \right)_+, \tag{1}$$

where we use the abbreviation $A_+ = \max\{A, 0\}$. The solution of the equation $-\Delta u = 2$ on a disk is continuous in the radius, it thus suffices to show a sufficiently large gap for $A(\Omega) = 0$. Collecting all these estimates, we obtain that

$$\mathbb{E} \text{ lifetime} \leq \mathbb{E}(T_\epsilon | T_\epsilon \leq T) + \mathbb{E}(T_\epsilon | T_\epsilon > T) \mathbb{P}(T_\epsilon > T)$$

$$\leq \epsilon \sqrt{\frac{2}{\pi} \sqrt{T}} + (1 + o(1)) \frac{\sqrt{2}}{\pi \sqrt{T}} \epsilon \mathbb{E}(T_\epsilon | T_\epsilon > T)$$
We can write down the conditional density function for the survivors within the two plates as \( \varepsilon \to 0 \) as

\[
s(x, y) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}} \sqrt{\frac{2T}{\pi}} \sum_{k=1}^{\infty} e^{-k^2 t/2} k \sin (ky)
\]

\[
= \frac{e^{-\frac{x^2}{2}}}{\pi} \sum_{k=1}^{\infty} e^{-k^2 t/2} k \sin (ky).
\]

This is, finally, where we put the domain \( \Omega \) into play. Denoting the solution of \( -\Delta u = 2 \) inside \( \Omega \), we can use the Markov property of Brownian motion to write

\[
E(T_\varepsilon | T_\varepsilon > T) \leq T + \int_\Omega s(x, y)u(x, y)dx dy.
\]

This, a priori, seems tricky for two reasons: we do not know where exactly the domain \( \Omega \) is and we do not know what the torsion function \( u \) looks like. The second problem is easy to address: by surrounding the domain \( \Omega \) by a slightly larger ball (we refer to the Lemma above), we can simply bound the torsion function from above by the torsion function in the larger ball. We do not know where the ball should be located and bypass this issue by simply taking the maximum over all translations of the ball. Put differently, we set

\[
u(x, y) = \frac{1}{2\pi} - \frac{(x - x_c)^2 + (y - y_c)^2}{2}
\]

and write

\[
E(T_\varepsilon | T_\varepsilon > T) \leq T + \max_{x_c, y_c} \int_\Omega s(x, y)u(x, y)dx dy.
\]

If we can show that this quantity is, for some value \( T \) strictly smaller than \( 1/\sqrt{2\pi} \), then this is also true for slightly larger radii by the continuity of all the involved objects. Picking \( T = 0.13 \) leads to the upper bound 0.77 and thus, divided by 2, to the upper bound 0.385 that is strictly smaller than \( 1/\sqrt{2\pi} \).

4. A Comment on the Numerics

We conclude with a quick description of the way we did numerical computation. Since we are interested in very precise estimates for the derivative at the boundary, we opted for a potential-theoretic approach over classical finite element methods. Using classical potential theory, Poisson’s equation

\[
\Delta u = -1, \quad x \in \Omega,
\]

\[
u = 0, \quad x \in \partial \Omega,
\]

can be reduced to a boundary integral equation. In particular, we decompose the solution \( u \) into two functions \( \phi \), which satisfies the inhomogeneous partial differential equations, and \( \psi \), which satisfies homogeneous partial differential equation and ensures that the boundary conditions are satisfied. A suitable \( \phi \) can be obtained by convolving the right-hand side (the constant function “-1”) with the Green’s function

\[
G(x, x') = \frac{1}{2\pi} \log |x - x'|,
\]
which produces

\[ \phi(x) = -\frac{1}{2\pi} \int_{\Omega} \log |x - x'| \, dx'. \]

Integrating by parts in the above expression, we obtain

\[ \phi(x) = \frac{1}{4\pi} \int_{\partial\Omega} \left( \log |x - x'| - \frac{1}{2} \right) (x' - x) \cdot n(x') \, ds, \]

where \( n \) denotes the inward-facing normal. The above expression holds for any \( x \in \bar{\Omega} \). The gradient of \( \phi \) can also be computed as a boundary integral via the following expression

\[ \nabla \phi(x) = -\frac{1}{4\pi} \int_{\partial\Omega} \left( \log |x - x'| - \frac{1}{2} \right) n(x') \, ds + \frac{1}{4\pi} \int_{\partial\Omega} \frac{x - x'}{|x - x'|^2} (x' - x) \cdot n(x') \, ds, \]

for all \( x \in \text{int} \Omega \). For \( x \in \partial\Omega \) the above expression is also valid and the integrand is well-defined with a removable singularity when \( x = x' \) provided \( \partial\Omega \) is twice differentiable in a neighborhood of \( x \in \partial\Omega \). These expressions can be further simplified when \( \Omega \) is a polygon using the identities

\[ \int_{\alpha} \left( \log \sqrt{x^2 + \sigma^2} - \frac{1}{2} \right) \, dx = \frac{1}{2} \left( \beta \log \sqrt{\beta^2 + \sigma^2} - \alpha \log \sqrt{\alpha^2 + \sigma^2} \right) + \sigma \tan \frac{\beta}{\sigma} - \sigma \tan \frac{\alpha}{\sigma}, \]

and

\[ \int_{\alpha} \frac{(x, a)}{x^2 + \sigma^2} \, dx = \left( \log \left( \frac{\sqrt{\beta^2 + \sigma^2}}{\sqrt{\alpha^2 + \sigma^2}} \right), \tan \frac{\beta}{\sigma} - \tan \frac{\alpha}{\sigma} \right). \]

In particular, if \( \Gamma \subset \partial\Omega \) is an edge with vertices \( a = (x_0, y_0) \) and \( b = (x_1, y_1) \) then \( \alpha = (a - x) \cdot v, \beta = (b - x) \cdot v \) and \( \sigma = (x - a) \cdot n = (x - b) \cdot n \) where \( n \) is the normal to the line segment

\[ n = \frac{(y_0 - y_1, x_1 - x_0)}{\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}} \]

and \( v \) is the unit tangent vector

\[ v = \frac{(x_1 - x_0, y_1 - y_0)}{\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}}. \]

Moreover, \( (x' - x) \cdot n(x') = -n \cdot (x - b) = -n \cdot (x - a) \) for any \( x' \in \Gamma \) and

\[ x - x' = \sigma n - ((x' - x) \cdot v) v. \]

For \( \phi \) constructed above, if \( \psi \) satisfies the boundary value problem

\[ \Delta \psi(x) = 0, \quad x \in \Omega, \]
\[ \psi(x) = -\phi(x), \quad x \in \partial\Omega, \]

then \( u = \psi + \phi \) satisfies the equation \( \Delta u = -1 \) with Dirichlet boundary conditions.
The boundary value problem for $\psi$ can be reduced to a boundary integral equation using classical potential theory. In particular, we represent $\psi$ via the formula

$$
\psi(x) = \int_{\partial \Omega} \frac{x - x'}{|x - x'|^2} \cdot n(x') \sigma(x') \, ds',
$$

where $\sigma$ is an unknown function, which we will refer to as the density. We note that $\psi$ defined in this manner is harmonic in the interior of $\Omega$; all that is required is to choose $\sigma$ so that the boundary conditions are satisfied. Taking the limit as $x$ approaches a point $x_0$ on the boundary $\partial \Omega$, we obtain

$$
-\pi \sigma(x_0) + \int_{\partial \Omega} \frac{x_0 - x'}{|x_0 - x'|^2} \cdot n(x') \, ds' = -\phi(x_0),
$$

for all $x_0 \in \partial \Omega$. Existence and uniqueness of solutions to the above boundary integral equation for Lipschitz domains is a classical result. In this paper, when solving boundary integral equations of this form we use a method similar to that described in [22] when $\partial \Omega$ is twice-differentiable, and the approach described in [30] when $\Omega$ is a polygon.

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