Ghost-free Gauss-Bonnet Theories of Gravity

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In this work we develop a theoretical framework for Gauss-Bonnet modified gravity theories, in which ghost modes can be eliminated at the equations of motion level. Particularly, after we present how the ghosts can occur at the level of equations of motion, we employ the Lagrange multipliers technique, and by means of constraints we are able to eliminate the ghost modes from Gauss-Bonnet theories of the form \( f(G) \) and \( F(R,G) \) types. Some cosmological realizations in the context of the ghost free \( f(G) \) gravity are presented, by using the reconstruction technique we developed. Finally, we explore the modifications to the Newton law of gravity generated by the ghost-free \( f(G) \) theory.

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I. INTRODUCTION

Undoubtedly one of the mysteries in theoretical physics is to find a consistent way to describe all the observed interactions under the same theoretical framework. This would require gravity to be quantized in some way and up to date, only string theory seems to provide a complete UV completion of all known particle physics theories. In cosmology, the quantum gravity era controls the pre-inflationary era, during which gravity is expected to be unified with all the other three interactions. It is evident that during this pre-inflationary era, string theory would be the most appropriate theory to describe the physical laws of our Universe, however it is not easy to prove that this is indeed the case. However some string theory effects could have their impact on the inflationary era, and this impact may be in fact measurable. There exist many theories in modern theoretical cosmology which take into account string theory motivated terms in the interaction Lagrangian of the model, such as the scalar-Einstein-Gauss-Bonnet gravity theory \([1,2]\), in which case the Lagrangian is of the form,

\[
S = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} R - \frac{1}{2} \partial_\mu \chi \partial^\mu \chi + h(\chi) G - V(\chi) + \mathcal{L}_{\text{matter}} \right),
\]

where \( G \) is the Gauss-Bonnet invariant defined as follows,

\[
G \equiv R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}.
\]

The scalar-Einstein-Gauss-Bonnet models are motivated by \( \alpha' \) corrections in superstring theories \([3]\), and they serve as a consistent example of how string theory may leave its impact on the primordial acceleration era of the Universe. Another very well studied class of theories in the same context, is that of \( f(G) \) gravity \([4,5]\), in which case the Lagrangian is of the form,

\[
S = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} R + f(G) + \mathcal{L}_{\text{matter}} \right).
\]

These theories contain a function of the Gauss-Bonnet invariant, and therefore the presence of this function generates non-trivial effects in the theory, due to the fact that the effect of the Gauss-Bonnet term does not appear as a total
derivative anymore, as in the linear theory of the Gauss-Bonnet scalar. Both these theories belong to a wider class of cosmological models which are known as modified gravity models \( \mathcal{F} \), and which generalize the standard Einstein-Hilbert theory. The motivation for studying such theories comes from the fact that in the context of these, several cosmological eras may be described by the same theory in a unified way, see for example Ref. [16] in which the unified description of the inflationary and of the dark energy eras was given in terms of \( f(R) \) gravity. In addition, similar studies were presented in terms of scalar Einstein Gauss-Bonnet models [17] and \( f(\mathcal{G}) \) models.

Due to the importance of the models containing or involving the Gauss-Bonnet scalar, which are string theory motivated in most cases, in this paper we shall address an important shortcoming of these theories, namely the existence of ghosts. Usually, higher-derivative theories contain ghost degrees of freedom due to the Ostrogradsky’s instability, see for example [18]. As was pointed out in [19], ghost degrees of freedom may occur at various levels of the theory, even at the cosmological perturbations level of \( F(R, \mathcal{G}) \) theories, where superluminal modes \( \sim k^4 \) occur, where \( k \) is the associated wavenumber. Having these issues in mind, in this paper we shall investigate how the ghosts may be eliminated from \( f(\mathcal{G}) \) and \( F(R, \mathcal{G}) \) theories. Particularly, by using an appropriate constraint used firstly in the context of mimetic gravity [20, 21], we shall demonstrate that the resulting theories are ghost-free. Similar constrained Gauss-Bonnet theories in the context of mimetic gravity were studied in [22]. Also ghost-free theories were also developed in Refs. [24, 25], but in a different context. In this work we shall also consider the cosmological evolution of the resulting theories, and we shall investigate how several cosmological evolutions may be realized by the ghost-free models we will develop, emphasizing on the dark energy era and inflationary era. Finally, we shall investigate how the Newton law is modified in the context of the ghost-free \( f(\mathcal{G}) \) gravity.

This paper is organized as follows: In section II we address the ghost issue in the context of \( f(\mathcal{G}) \) gravity. We firstly demonstrate how ghosts may occur in this theory and we provide two remedy theories, which are ghost-free extensions of \( f(\mathcal{G}) \) gravity. In section III we investigate how several cosmological evolutions may be realized in the context of the proposed ghost-free \( f(\mathcal{G}) \) theory. In section IV we discuss how the Newton law becomes in the context of the ghost-free \( f(\mathcal{G}) \) gravity, and finally in section V we briefly investigate how a general \( F(R, \mathcal{G}) \) theory may be rendered ghost free.

II. GHOST-FREE \( f(\mathcal{G}) \) GRAVITY

In this section we shall investigate how to obtain a ghost free \( f(\mathcal{G}) \) gravity, and we shall employ the Lagrange multipliers formalism in order to achieve this. Before getting into the details of our formalism, we will start the presentation by showing explicitly how ghost modes may occur in \( (\mathcal{G}) \) gravity at the equations of motion level, and the ghost-free version construction of the theory follows.

A. Ghosts in \( f(\mathcal{G}) \) Gravity

In order to investigate if any ghost modes could appear in \( f(\mathcal{G}) \) gravity model [3], we investigate the equations of motion, by considering a general variation of the metric of the following form,

\[
g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}.
\] (4)

Effectively, the variations of \( \delta \Gamma_{\mu\nu}^\kappa, \delta R_{\mu\nu\lambda\sigma}, \delta R_{\mu\nu}, \) and \( \delta R \) read,

\[
\begin{align*}
\delta \Gamma_{\mu\nu}^\kappa &= \frac{1}{2} g^{\kappa\lambda} \left( \nabla_\mu \delta g_{\nu\lambda} + \nabla_\nu \delta g_{\mu\lambda} - \nabla_\lambda \delta g_{\mu\nu} \right), \\
\delta R_{\mu\nu\lambda\sigma} &= \frac{1}{2} \left[ \nabla_\lambda \nabla_\nu \delta g_{\mu\sigma} - \nabla_\mu \nabla_\nu \delta g_{\lambda\sigma} - \nabla_\sigma \nabla_\nu \delta g_{\mu\lambda} + \nabla_\sigma \nabla_\mu \delta g_{\lambda\nu} + \delta g_{\mu\nu} R^\rho_{\nu\lambda\sigma} - \delta g_{\rho\sigma} R^\rho_{\mu\lambda\nu} \right], \\
\delta R_{\mu\nu} &= \frac{1}{2} \left[ \nabla_\nu \nabla^\nu \delta g_{\mu\rho} + \nabla_\rho \nabla^\rho \delta g_{\mu\nu} - \square \delta g_{\mu\nu} - \nabla_\mu \nabla_\nu \left( g^{\rho\lambda} \delta g_{\rho\lambda} \right) - 2 R^\lambda_{\nu\rho} \delta g_{\lambda\mu} + R^\rho_{\mu} \delta g_{\rho\nu} + R^\rho_{\nu} \delta g_{\rho\mu} \right], \\
\delta R &= - \delta g_{\mu\nu} R^{\mu\nu} + \nabla^\mu \nabla^\nu \delta g_{\mu\nu} - \square \left( \delta g_{\mu\nu} \right).
\end{align*}
\] (5)

Accordingly the variation of the Gauss-Bonnet scalar \( \delta \mathcal{G} \) reads,

\[
\begin{align*}
\delta \mathcal{G} &= 2R \left( -\delta g_{\mu\nu} R^{\mu\nu} + \nabla^\mu \nabla^\nu \delta g_{\mu\nu} - \nabla^2 \left( g^{\mu\nu} \delta g_{\mu\nu} \right) \right) + 8R^{\rho\sigma} R^\mu_{\rho\nu} \delta g_{\mu\nu} - 4 \left( R^{\rho\sigma} \nabla_\rho \nabla^\nu + R^\rho_{\mu} \nabla_\rho \nabla^\nu \right) \delta g_{\mu\nu} \\
&\quad + 4R^{\mu\nu} \nabla^2 \delta g_{\mu\nu} + 4R^{\rho\sigma} \nabla_\rho \nabla_\sigma \left( g^{\mu\nu} \delta g_{\mu\nu} \right) - 2R^{\mu\nu\sigma\tau} R^\rho_{\mu\sigma\tau} \delta g_{\mu\nu} - 4R^{\rho\mu\nu\sigma} \nabla_\rho \nabla_\sigma \delta g_{\mu\nu}.
\end{align*}
\] (6)
Then for the $f(G)$ gravity model \((3)\), by varying the action with respect to the metric tensor $g_{\mu\nu}$, we obtain the following equations of motion,

\[
0 = \frac{1}{2\kappa^2} \left( -R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R \right) + T^{\mu\nu}_{\text{matter}} + \frac{1}{2} g^{\mu\nu} f(G) + \left( -2RR^{\mu\nu} + 8R^{\rho\sigma} R^{\mu\rho}_{\phantom{\mu\rho} \nu\sigma} - 2R^{\mu\rho\sigma\tau} R^{\nu}_{\rho\sigma\tau} \right) f' (G)
+ 2(\nabla^{\mu} \nabla^{\nu} - g^{\mu\nu} \Box) (R f' (G)) - 4 \nabla^{\mu} \nabla_{\rho} (R^{\nu}_{\rho} f' (G)) - 4 \nabla^{\nu} \nabla_{\rho} (R^{\mu}_{\rho} f' (G)) + 4 \Box (R^{\mu\nu} f' (G))
+ 4g^{\mu\nu} \nabla_{\rho} \nabla_{\sigma} (R^{\rho\sigma}_{\mu\nu} f' (G)) - 4 \nabla_{\rho} \nabla_{\sigma} (R^{\rho\sigma}_{\mu\nu} f' (G)).
\]  

(7)

By using the Bianchi identities,

\[
\nabla^{\rho} R^{\rho\mu\nu} = \nabla_{\mu} R_{\nu\tau} - \nabla_{\nu} R_{\mu\tau}, \quad \nabla^{\rho} R_{\mu\nu} = \frac{1}{2} \nabla_{\mu} R,
\]

\[
\nabla_{\rho} \nabla_{\sigma} R^{\rho\mu\nu} = \Box R^{\mu\nu} - \frac{1}{2} \nabla^{\mu} \nabla^{\nu} R + R^{\rho\sigma\mu\nu} R_{\rho\sigma} - R^{\rho}_{\phantom{\rho} \rho} R^{\mu\nu}, \quad \nabla_{\rho} \nabla_{\sigma} R_{\rho\sigma} = \frac{1}{2} \Box R,
\]

we can rewrite Eq. \((6)\) as follows,

\[
0 = \frac{1}{2\kappa^2} \left( -R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R \right) + T^{\mu\nu}_{\text{matter}} + \frac{1}{2} g^{\mu\nu} f(G) + \left( -2RR^{\mu\nu} - 2R^{\mu\rho\sigma\tau} R^{\nu}_{\rho\sigma\tau} + 4R^{\mu}_{\rho} R^{\nu\rho} + 4R^{\rho\sigma} R^{\mu\nu}_{\rho\sigma} \right) f' (G)
+ 2R \nabla^{\mu} \nabla^{\nu} f' (G) - 2g^{\mu\nu} R \Box f' (G) - 4 \nabla^{\mu} \nabla^{\nu} \nabla_{\rho} f' (G) - 4 \nabla^{\mu} \nabla^{\nu} \nabla_{\rho} f' (G)
+ 4R^{\rho\mu\nu} \Box f' (G) + 4g^{\mu\nu} R^{\rho\sigma\mu\nu} \nabla_{\rho} \nabla_{\sigma} f' (G) - 4R^{\rho\sigma\mu\nu} \nabla_{\rho} \nabla_{\sigma} f' (G).
\]

(9)

Also in four dimensions, we have the following identity,

\[
0 = \frac{1}{2} g^{\mu\nu} G - 2RR^{\mu\nu} - 2R^{\mu\rho\sigma\tau} R^{\nu}_{\rho\sigma\tau} + 4R^{\mu}_{\rho} R^{\nu\rho} + 4R^{\rho\sigma} R^{\mu\nu}_{\rho\sigma}.
\]

(10)

Then Eq. \((9)\) takes the following form,

\[
0 = \frac{1}{2\kappa^2} \left( -R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R \right) + T^{\mu\nu}_{\text{matter}} + \frac{1}{2} g^{\mu\nu} f(G) - 2 \nabla^{\mu} \nabla^{\nu} f' (G) - 2g^{\mu\nu} R \Box f' (G)
- 4R^{\rho\mu\nu} \Box f' (G) - 4R^{\rho\mu\nu} \nabla_{\rho} f' (G) + 4R^{\rho\mu\nu} \nabla_{\rho} f' (G) + 4g^{\mu\nu} R^{\rho\sigma\mu\nu} \nabla_{\rho} \nabla_{\sigma} f' (G) - 4R^{\rho\sigma\mu\nu} \nabla_{\rho} \nabla_{\sigma} f' (G).
\]

(11)

We now rewrite Eq. \((11)\) in the following form,

\[
0 = \frac{1}{2\kappa^2} \left( -R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R \right) + T^{\mu\nu}_{\text{matter}} + \frac{1}{2} g^{\mu\nu} f(G) - 2 \nabla^{\mu} \nabla^{\nu} f' (G) - 2g^{\mu\nu} R \Box f' (G)
- 4R^{\rho\mu\nu} \Box f' (G) - 4R^{\rho\mu\nu} \nabla_{\rho} f' (G) + 4R^{\rho\mu\nu} \nabla_{\rho} f' (G) + 4g^{\mu\nu} R^{\rho\sigma\mu\nu} \nabla_{\rho} \nabla_{\sigma} f' (G) - 4R^{\rho\sigma\mu\nu} \nabla_{\rho} \nabla_{\sigma} f' (G).
\]

(12)

Having in mind that,

\[
g^{\mu\nu} D_{\mu\nu}^{\gamma\eta} = 4 \left( \frac{-1}{2} g^{\gamma\eta} R + R^{\gamma\eta} \right),
\]

(13)

we find in component form,

\[
D_{\mu\nu}^{00} = 2R - 2g_{\mu\nu}g^{00} R - 8R^{0}_{\phantom{0}0} + 4g_{\mu\nu} R^{00} + 4g^{00} R_{00} - 4R^{0}_{\phantom{0}0} 0,
\]

\[
D_{\mu\nu}^{ij} = 4g_{ij} R^{00} - 4R^{0}_{\phantom{0}i j} - 2g_{ij} R^{00} R + 4R_{ij} g^{00}.
\]

(14)

If we choose the gauge in which $g_{0i} = 0$, then the quantity $D_{\mu\nu}^{00}$ vanishes but $D_{ij}^{00}$ does not vanish in general. This indicates that Eq. \((11)\) includes the fourth derivative of metric with respect to the cosmic time coordinate and therefore ghost modes might appear. We may see the existence of ghost modes explicitly, by considering perturbations. Let a solution of \((11)\) be $g_{\mu\nu} = g_{\mu\nu}^{(0)}$ and we denote the curvatures and connections given by $g_{\mu\nu}^{(0)}$ by using the indexes \(\prime(0)\). Then in order to investigate if any ghost could exist, we may consider the variation of \((11)\) around the solution $g_{\mu\nu}^{(0)}$ as follows $g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}$. For the variation of $\delta g_{\mu\nu}$, we may impose the following gauge condition,

\[
0 = \nabla^{\mu} \delta g_{\mu\nu}.
\]

(15)
Then Eq. (6) reduces to,

\[
\delta G = 2R \left( -\delta g_{\mu\nu} R^{\mu\nu} - \nabla^2 \left( g^{\mu\nu} \delta g_{\mu\nu} \right) \right) + 8R^{\rho\sigma \rho} \rho \sigma \delta g_{\mu\nu} + 4R^{\rho\mu \nu} \nabla^2 \delta g_{\mu\nu} + 4R^{\rho\mu \sigma} \nabla_\sigma (g^{\mu\nu} \delta g_{\mu\nu}) \\
- 2R^{\rho\sigma \tau} R^{\rho\sigma \tau} \delta g_{\mu\nu} - 4R^{\rho\mu\sigma\nu} \nabla_\rho \nabla_\sigma \delta g_{\mu\nu} .
\] (16)

Even if we impose the condition \( \delta g_{\mu\nu} = 0 \), Eq. (16) has the following form,

\[
\delta G = -2R^{\mu\nu} \delta g_{\mu\nu} + 8R^{\rho\sigma \rho} \rho \sigma \delta g_{\mu\nu} + 4R^{\rho\mu \nu} \nabla^2 \delta g_{\mu\nu} - 2R^{\rho\sigma \tau} R^{\rho\sigma \tau} \delta g_{\mu\nu} - 4R^{\rho\mu\sigma\nu} \nabla_\rho \nabla_\sigma \delta g_{\mu\nu} ,
\] (17)

which also contains the second derivative of the metric \( g_{\mu\nu} \) with respect to the cosmic time coordinate. Under the perturbation \( g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu} \), the term \( D_{\mu\nu}^{\tau\tau} \nabla_\tau \nabla_\eta f' (G) \) takes the following form,

\[
D_{\mu\nu}^{\tau\tau} \nabla_\tau \nabla_\eta f' (G) \rightarrow D_{\mu\nu}^{\tau\eta} \nabla_\tau \nabla_\eta f' \left( G^{(0)} \right) + D_{\mu\nu}^{\tau\eta} \nabla_\tau \nabla_\eta \left( f'' \left( G^{(0)} \right) \delta G \right) + \ldots ,
\] (18)

which contains the fourth derivative of the metric \( g_{\mu\nu} \) with respect to the cosmic time coordinate, and therefore the perturbed equation (12) may have a ghost mode. Note that in Eq. (18), the “\( \ldots \)” expresses the terms occurring from the variation of \( D_{\mu\nu}^{\tau\eta} \nabla_\tau \nabla_\eta \). The propagating mode is a scalar expressed by the Gauss-Bonnet invariant as it is clear from Eq. (12). Having presented explicitly how a ghost mode may occur in \( f(G) \) gravity, we now demonstrate how the ghost modes may be eliminated or avoided in this theory. This is the subject of the next subsection.

### B. Development of a Ghost-free \( f(G) \) Gravity

In this subsection, we consider how we can avoid the ghost in \( f(G) \) gravity. To this end, we rewrite the action of Eq. (9) by introducing an auxiliary field \( \chi \) as follows,

\[
S = \int d^4 x \sqrt{-g} \left( \frac{1}{2\kappa^2} R + h (\chi) G - V (\chi) + \mathcal{L}_{\text{matter}} \right) .
\] (19)

Then by varying the action (19) with respect to the auxiliary field \( \chi \), we obtain the following equation,

\[
0 = h' (\chi) G - V' (\chi) ,
\] (20)

which can be solved with respect to \( \chi \) as a function of the Gauss-Bonnet invariant \( G \) as follows, \( \chi = \chi (G) \). Then by substituting the obtained expression into Eq. (20), we reobtain the action of Eq. (3) with \( f(G) \) being equal to,

\[
f (G) = h (\chi (G)) G - V (\chi (G)) .
\] (21)

On the other hand, by varying the action (20) with respect to the metric tensor, we obtain,

\[
0 = \frac{1}{2\kappa^2} \left( -R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R \right) + \frac{1}{2} T_{\text{matter} \mu\nu} - \frac{1}{2} g_{\mu\nu} V (\chi) + D_{\mu\nu}^{\tau\eta} \nabla_\tau \nabla_\eta h (\chi) ,
\] (22)

with \( D_{\mu\nu}^{\tau\eta} \) being defined in Eq. (12). Since \( \chi \) can be given by a function of the Gauss-Bonnet invariant \( G \), Eq. (22) is the fourth order differential equation for the metric, which may actually generate the ghost modes. Eq. (22) indicates that the propagating scalar mode is quantified in terms of \( \chi \). Then in order to make the scalar mode not to be ghost, we may add a canonical kinetic term of \( \chi \) in the action (19) as in the model of Eq. (11) (11), where we have chosen the mass dimension of \( \chi \) to be unity. Then instead of Eqs. (20) and (22), we obtain,

\[
0 = \Box \chi + h' (\chi) G - V' (\chi) ,
\] (23)

\[
0 = \frac{1}{2\kappa^2} \left( -R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R \right) + \frac{1}{2} T_{\text{matter} \mu\nu} + \frac{1}{2} \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} g_{\mu\nu} \left( \frac{1}{2} \partial_\rho \chi \partial^\rho \chi + V (\chi) \right) + D_{\mu\nu}^{\tau\eta} \nabla_\tau \nabla_\eta h (\chi) .
\] (24)

Since the equations derived above do not contain higher than second order derivatives, if we impose initial conditions for the following quantities \( g_{\mu\nu}, \chi \), and \( h \) on a spatial hypersurface of constant cosmic time, the evolution of \( g_{\mu\nu} \) and \( \chi \) is uniquely determined, and as it is clear from Eq. (24), these could not be ghosts. In the model of Eq. (11), we have introduced a new dynamical degree of freedom, namely \( \chi \), but if we like to reduce the dynamical degrees of
freedom, we may impose a constraint as in the mimetic gravity case \cite{20,22}, by introducing the Lagrange multiplier field $\lambda$, as follows,

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} R + \lambda \left( \frac{1}{2} \partial_\mu \chi \partial^{\mu} \chi + \frac{\mu^4}{2} \right) - \frac{1}{2} \partial_\mu \chi \partial^{\mu} \chi + h(\chi) G - V(\chi) + \mathcal{L}_{\text{matter}} \right),$$  \hspace{1cm}  \text{(25)}

where $\mu$ is a constant with mass-dimension one. Then, by varying the above action \text{(25)} with respect to $\lambda$, we obtain the constraint,

$$0 = \frac{1}{2} \partial_\mu \chi \partial^{\mu} \chi + \frac{\mu^4}{2}. \hspace{1cm}  \text{(26)}$$

Then due to the fact that the kinetic term becomes a constant, the kinetic term in the action of Eq. \text{(25)} can be absorbed into the redefinition of the scalar potential $V(\chi)$ as follows,

$$\tilde{V}(\chi) \equiv \frac{1}{2} \partial_\mu \chi \partial^{\mu} \chi + V(\chi) = -\frac{\mu^4}{2} + V(\chi), \hspace{1cm}  \text{(27)}$$

and we can rewrite the action of Eq. \text{(25)} as follows,

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} R + \frac{1}{2} \partial_\mu \chi \partial^{\mu} \chi + \frac{\mu^4}{2} \right) + h(\chi) G - \tilde{V}(\chi) + \mathcal{L}_{\text{matter}} \right). \hspace{1cm}  \text{(28)}$$

For the model of Eq. \text{(28)}, in addition to Eq. \text{(28)}, we have the following two equations of motion,

$$0 = - \frac{1}{\sqrt{-g}} \partial_\mu \left( \lambda g^{\mu\nu} \sqrt{-g} \partial_\nu \chi \right) + h'(\chi) G - \tilde{V}'(\chi), \hspace{1cm}  \text{(29)}$$

$$0 = \frac{1}{2\kappa^2} \left( -R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R \right) + \frac{1}{2} T_{\text{matter} \mu\nu} - \frac{1}{2} \lambda \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} g_{\mu\nu} \tilde{V}(\chi) + D_{\mu\nu} \nabla^\tau \nabla_\eta h(\chi), \hspace{1cm}  \text{(30)}$$

where we have also used Eq. \text{(26)}. By multiplying Eq. \text{(30)} with $g^{\mu\nu}$, we obtain,

$$0 = \frac{R}{2\kappa^2} + \frac{1}{2} T_{\text{matter}} + \frac{\mu^4}{2} \lambda - 2 \tilde{V}(\chi) - 4 \left( -R^{\eta\eta} + \frac{1}{2} g^{\eta\eta} R \right) \nabla_\tau \nabla_\eta h(\chi), \hspace{1cm}  \text{(31)}$$

where we used Eq. \text{(26)} and $T_{\text{matter}} \equiv g^{\mu\nu} T_{\text{matter} \mu\nu}$. Eq. \text{(31)} can be solved with respect to the Lagrange multiplier field $\lambda$, and the result is,

$$\lambda = -\frac{2}{\mu^4} \left( \frac{R}{2\kappa^2} + \frac{1}{2} T_{\text{matter}} - 2 \tilde{V}(\chi) - 4 \left( -R^{\eta\eta} + \frac{1}{2} g^{\eta\eta} R \right) \nabla_\tau \nabla_\eta h(\chi) \right). \hspace{1cm}  \text{(32)}$$

We expect that the model \text{(28)} could not contain a ghost mode. And actually by using perturbations of the metric, we now show explicitly that indeed the model \text{(28)} is ghost free. Let the general solutions of Eqs. \text{(26)}, \text{(29)}, and \text{(30)} be $g^{(0)}_{\mu\nu}$, $\chi^{(0)}$, and $\lambda^{(0)}$ and we consider the following perturbation,

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}, \hspace{1cm} \chi = \chi^{(0)} + \delta \chi, \hspace{1cm} \lambda = \lambda^{(0)} + \delta \lambda. \hspace{1cm}  \text{(33)}$$

Then Eqs. \text{(26)}, \text{(29)}, and \text{(30)} can be written,

$$0 = \partial_\mu \chi^{(0)} \partial_\mu \delta \chi - \delta g_{\mu\nu} \partial^\mu \chi^{(0)} \partial^\nu \chi^{(0)}, \hspace{1cm}  \text{(34)}$$

$$0 = \frac{g^{(0)}_{\rho\sigma}}{2\sqrt{-g^{(0)}}} \partial_\mu \left( \chi^{(0)} g^{(0)}_{\mu\nu} \sqrt{-g^{(0)}} \partial_\nu \chi^{(0)} \right) - \frac{1}{\sqrt{-g^{(0)}}} \partial_\mu \left( \lambda g^{(0)}_{\mu\nu} \partial_\nu \chi^{(0)} \right)$$

$$+ \frac{1}{\sqrt{-g^{(0)}}} \partial_\mu \left( \lambda g^{(0)}_{\mu\nu} \delta g^{(0)}_{\rho\sigma} \sqrt{-g^{(0)}} \partial_\nu \chi^{(0)} \right) - \frac{1}{2\sqrt{-g^{(0)}}} \partial_\mu \left( \lambda g^{(0)}_{\mu\nu} g^{(0)}_{\rho\sigma} \delta g^{(0)}_{\rho\sigma} \sqrt{-g^{(0)}} \partial_\nu \chi^{(0)} \right)$$

$$- \frac{1}{\sqrt{-g^{(0)}}} \partial_\mu \left( \chi^{(0)} g^{(0)}_{\mu\nu} \sqrt{-g^{(0)}} \partial_\nu \delta \chi \right) + h'' \left( \chi^{(0)} \right) \delta \chi G^{(0)} - \tilde{V}'' \left( \chi^{(0)} \right) \delta \chi$$

$$+ h' \left( \chi^{(0)} \right) \left( 2 R^{(0)} - \delta g_{\mu\nu} R^{(0)}_{\mu\nu} + \nabla^{(0)} \mu \nabla^{(0)} \nu \delta g_{\mu\nu} - \Box^{(0)} \left( g^{(0)}_{\mu\nu} \delta g_{\mu\nu} \right) \right) + 8 R^{(0)}_{\rho\sigma} R^{(0)}_{\mu\nu} \delta g^{(0)}_{\rho\sigma \mu\nu}.$$
By substituting Eq. (37) in Eq. (36), we may eliminate the obtained equation contains first and second derivatives of \( \delta g_{\mu\nu} \) and \( \chi \), especially the first and second derivatives with respect to the cosmic time \( t \). We can choose \( \chi^{(0)} \) to be,

\[
\chi^{(0)} = \mu^2 t.
\]

Then Eq. (33) takes the following form,

\[
0 = \delta \dot{\chi} - \mu^2 \delta g_{tt},
\]
and we also have $\delta \chi = \mu^2 \delta g_{tt}$. Then we can further eliminate the variation terms $\delta \chi$ and $\delta \dot{\chi}$, and the obtained equation contains the first and second derivatives of $\delta g_{\mu\nu}$ with respect to the cosmic time $t$, but does not include the first and second derivative terms $\delta \chi$ again with respect to time $t$. Then by providing the initial conditions for $\delta g_{\mu\nu}$, $\delta \dot{g}_{\mu\nu}$, and $\chi$ on a spatial hypersurface, we can determine the time evolution of $\delta g_{\mu\nu}$ uniquely up to the gauge invariance corresponding to the general covariance of the model, and the corresponding constraints. This indicates that the number of the physical degrees of freedom is only two. Eq. (39) also indicates that $\chi$ is not dynamical and the time evolution of $\chi$ is given by Eq. (40). Therefore, no additional degrees of freedom occur, compared to the standard Einstein-Hilbert gravity, and in effect, no ghost modes actually occur in the theory. Having demonstrated that the modified $f(\mathcal{G})$ gravity theory can be rendered ghost-free, let us consider several examples of cosmological evolutions which can be realized in the context of this theory. This is the subject of the next subsection.

III. FRW COSMOLOGY IN GHOST-FREE $f(\mathcal{G})$ GRAVITY

In this section, we consider the cosmology produced by the ghost-free $f(\mathcal{G})$ gravity model of Eq. (28). Especially we show that it is possible to realize any cosmological era of the Universe, by using the model under consideration. We will particularly try to realize the late and early-time acceleration eras.

A. A Reconstruction Technique for Model Building

Let us firstly demonstrate how the equations of motion of the model (28) become in the case the metric is a flat Friedman-Robertson-Walker metric (FRW) with line element,

$$ ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2. $$

(40)

For this metric, we have,

$$ \Gamma^i_{ij} = a^2 H \delta_{ij}, \quad \Gamma^t_{ij} = \Gamma^i_{tj} = H \delta^{i}, \quad \Gamma^t_{jk} = \tilde{\Gamma}^i_{jk}, \quad R_{tt} = - \left( \dot{H} + H^2 \right) a^2 \delta_{ij}, \quad R_{ij} = a^4 H^2 \left( \delta_{ik} \delta_{lj} - \delta_{il} \delta_{kj} \right), \quad R_{tt} = -3 \left( \dot{H} + H^2 \right), \quad R_{ij} = a^2 \left( \dot{H} + 3H^2 \right) \delta_{ij}, \quad R = 6\dot{H} + 12H^2, \quad \text{other components} = 0, $$

(41)

$$ \mathcal{G} = 2AH^2 \left( \dot{H} + H^2 \right), $$

where $H \equiv \frac{\dot{a}}{a}$. We also assume that $\lambda$ and $\chi$ depend solely on the cosmic time $t$, that is, $\lambda = \lambda(t)$ and $\chi = \chi(t)$. We also assume $T_{\text{matter},\mu\nu} = 0$ just for simplicity. Then a solution of Eq. (26) is given below,

$$ \chi = \mu^2 t. $$

(42)

In effect, the $(t, t)$ component and $(i, j)$ component of (30) yield,

$$ 0 = - \frac{3H^2}{2\kappa^2} - \mu^4 \frac{\lambda}{2} + \frac{1}{2} \tilde{V} \left( \mu^2 t \right) - 12\mu^2 H^3 h' \left( \mu^2 t \right), $$

(43)

$$ 0 = \frac{1}{2\kappa^2} \left( 2\dot{H} + 3H^2 \right) - \frac{1}{2} \tilde{V} \left( \mu^2 t \right) + 4\mu^4 H^2 h'' \left( \mu^2 t \right) + 8\mu^2 \left( \dot{H} + H^2 \right) H_h' \left( \mu^2 t \right). $$

(44)

On the other hand, Eq. (29) gives,

$$ 0 = \mu^2 \dot{\chi} + 3\mu^2 H \lambda + 24H^2 \left( \dot{H} + H^2 \right) h' \left( \mu^2 t \right) - \tilde{V}' \left( \mu^2 t \right), $$

(45)

Eq. (43) can be solved with respect to $\lambda$ as follows,

$$ \lambda = - \frac{3H^2}{\mu^4 \kappa^2} + \frac{1}{\mu^4} \tilde{V} \left( \mu^2 t \right) - \frac{24}{\mu^2} H^3 h' \left( \mu^2 t \right). $$

(46)

Then by substituting Eq. (46) into Eq. (45), we reobtain Eq. (44). On the other hand, Eq. (44) can be solved with respect to $\tilde{V} \left( \mu^2 t \right)$ as follows,

$$ \tilde{V} \left( \mu^2 t \right) = \frac{1}{\kappa^2} \left( 2\dot{H} + 3H^2 \right) + 8\mu^4 H^2 h'' \left( \mu^2 t \right) + 16\mu^2 \left( \dot{H} + H^2 \right) H_h' \left( \mu^2 t \right), $$

(47)
which tells that for arbitrary $h(\chi)$, if the potential $\tilde{V}(\chi)$ is assumed to be equal to,

$$\tilde{V}(\chi) = \left[ \frac{1}{\kappa^2} \left( 2\dot{H} + 3H^2 \right) + 8\mu^4H^2 h''(\mu^2 t) + 16\mu^2 \left( \dot{H} + H^2 \right) H h' (\mu^2 t) \right]_{t=\frac{\lambda}{\mu^2}},$$

then an arbitrary cosmological evolution of the Universe with Hubble rate $H = H(t)$ can be realized. By combining Eqs. (46) and (47), we also obtain,

$$\lambda = \frac{2\dot{H}}{\mu^4 \kappa^2} + 8H^2 h''(\mu^2 t) + \frac{8}{\mu^2} \left( 2\dot{H} - H^2 \right) H h' (\mu^2 t).$$

Basically the above procedure is a reconstruction method for the model (28) and by using this method it is possible to realize an arbitrarily given cosmological evolution. In the next subsection we shall use this reconstruction method.

B. Early and Late-time Accelerating Universe Cosmologies with Ghost-free $f(G)$ Gravity

In this subsection, we consider some examples of models which describe an accelerating Universe. As a first example, we consider a de Sitter space-time realization, in which case the Hubble rate $H = H_0$. Then by using Eq. (48), for an arbitrarily chosen function $h(\chi)$, the corresponding scalar potential is given by,

$$\tilde{V}(\chi) = \frac{3H_0^2}{\kappa^2} + 8\mu^4H_0^2 h''(\chi) + 16\mu^2H_0^3 h'(\chi).$$

Eq. (49) also indicates how the Lagrange multiplier $\lambda(t)$ in this model behaves, and it is equal to,

$$\lambda(t) = 8H_0^2 h''(\mu^2 t) - \frac{8}{\mu^2} H_0^3 h'(\mu^2 t).$$

Then by appropriately choosing the functional form of $h(\chi)$, we can obtain several different ghost-free $f(G)$ models which can realize a de Sitter evolution. Next we consider the model which mimics the ΛCDM model, in which case the Hubble rate $H$ is given by,

$$H = H_0 \coth \left( \frac{3}{2} H_0 t \right).$$

At late times, that is in the limit $t \to +\infty$, $H$ in Eq. (52) behaves as follows,

$$H \to H_0,$$

which corresponds to an asymptotic de Sitter spacetime. On the other hand, at early times, which era is reached in the limit $t \to 0$, the Hubble rate behaves as follows,

$$H \to \frac{3}{2t},$$

which corresponds to a matter or dust dominated Universe. Then by using Eq. (48), we find,

$$\tilde{V}(\chi) = \frac{3H_0^2}{\kappa^2} + 8\mu^4H_0^2 \coth^2 \left( \frac{3H_0}{2\mu^2} \chi \right) h''(\chi) + 16\mu^2H_0^3 \left( 1 - \frac{1}{2 \sinh^2 \left( \frac{3H_0}{2\mu^2} \chi \right) } \right) \coth \left( \frac{3H_0}{2\mu^2} \chi \right) h'(\chi),$$

and from Eq. (49) we can determine the functional form of the Lagrange multiplier $\lambda$, which is,

$$\lambda = \frac{3H_0^2}{\mu^4 \kappa^2 \sinh^2 \left( \frac{3H_0}{2H_0 t} \right)} + 8H_0^3 \coth^2 \left( \frac{3}{2} H_0 t \right) h''(\mu^2 t) - \frac{8H_0^3}{\mu^2} \left( 1 + \frac{4}{\sinh^2 \left( \frac{3}{2} H_0 t \right) } \right) \coth \left( \frac{3}{2} H_0 t \right) h'(\mu^2 t).$$

The model of Eq. (52), which is generated in the context of ghost-free $f(G)$ gravity by the scalar potential of Eq. (55), realizes the ΛCDM model without introducing any dark matter perfect fluid. Therefore, the model incorporates the cosmological constant part, corresponding to an equation of state (EoS) parameter being equal to $w = -1$, and
also incorporates the cold dark matter (CDM) part, corresponding to and EoS parameter exactly equal to \( w = 0 \). Thus we have succeeded to realize the present accelerating expansion of the Universe by using the ghost-free \( f(G) \) gravity model. Notably, the cosmological evolution \((52)\) can be realized in the context of the ghost-free \( f(G) \) by using a function \( h(\chi) \) and an arbitrary parameter \( \mu^2 \). In the case of the standard Einstein-Hilbert gravity, the FRW equations have the following form,

\[
\frac{3}{\kappa^2}H^2 = \rho_{\text{total}}, \quad -\frac{1}{\kappa^2} \left( 2\dot{H} + 3H^2 \right) = p_{\text{total}}, \tag{57}
\]

where \( \rho_{\text{total}} \) and \( p_{\text{total}} \) are the total energy density and the total pressure. In effect, the total equation of state (EoS) parameter \( w_{\text{total}} \) defined by \( w_{\text{total}} = \frac{p_{\text{total}}}{\rho_{\text{total}}} \) is equal to,

\[
w_{\text{total}} = -1 - \frac{2\dot{H}}{3H^2}. \tag{58}
\]

We should note that the effective total EoS parameter \( w_{\text{total}} \) includes the contributions of all the fluid components of the Universe like the dark energy, dark matter, and so on. The Planck 2018 results \([26]\), constrain the Hubble constant, which is the present value of the Hubble rate, as follows \( H_{\text{present}} = (67.4 \pm 0.5) \text{ km s}^{-1} \text{ Mpc}^{-1} \). Also the matter density parameter is constrained as \( \Omega_m = 0.315 \pm 0.007 \) and finally, the dark energy EoS parameter is constrained as \( w_0 = -1.03 \pm 0.03 \) although \( w_0 \) is different from \( w_{\text{total}} \). Since \( p_{\text{total}} = (1 - \Omega_m) w_0 \rho_{\text{total}} \), the Planck 2018 results indicate that,

\[
w_{\text{total}} = (1 - \Omega_m) w_0 \sim -0.705. \tag{59}
\]

Even for a general modified gravity theory, in the case of the ghost-free \( f(G) \) gravity case we developed in this paper, the effective total EoS parameter \( w_{\text{total}}^{\text{eff}} \) is defined in Eq. \(\text{(58)}\), that is,

\[
w_{\text{total}}^{\text{eff}} = -1 - \frac{2\dot{H}}{3H^2}. \tag{60}
\]

Then in the case of the de Sitter space as in the model \([18]\) in this paper, since the Hubble rate is a constant, \( H = H_0 \), we find \( w_{\text{total}}^{\text{eff}} = -1 \). On the other hand, in the case of the model mimicking the \( \Lambda \)CDM model, namely model \([52]\), we find,

\[
w_{\text{total}}^{\text{eff}} = -1 - \frac{1}{\cosh^2 \left( \frac{2}{3} H_0 t_{\text{present}} \right)}, \tag{61}
\]

where \( t_{\text{present}} \) is the value of the cosmic time today. In the model \([52]\), the dark matter contribution to the evolution is effectively included. Then the Planck 2018 results \([53]\) constrain the parameters of the model \([52]\). Due to the fact that the observed Hubble constant is \( H_{\text{present}} = (67.4 \pm 0.5) \text{ km s}^{-1} \text{ Mpc}^{-1} \), by using \([52]\) we find,

\[
H_0 \coth \left( \frac{2}{3} H_0 t_{\text{present}} \right) = (67.4 \pm 0.5) \text{ km s}^{-1} \text{ Mpc}^{-1}. \tag{62}
\]

On the other hand, combined with Eq. \(\text{(61)}\), the Planck 2018 results \([53]\) indicate that,

\[
\frac{1}{\cosh^2 \left( \frac{2}{3} H_0 t_{\text{present}} \right)} \sim 0.294. \tag{63}
\]

Then Eqs. \(\text{(62)}\) and \(\text{(63)}\) actually constrain the parameters \( H_0 \) and \( t_{\text{present}} \) of the model, so these can appropriately be chosen so that the constraints are satisfied.

In addition, since the \( \Lambda \)CDM model is still consistent with any constraint obtained from the observations on the current expansion of the Universe, the model \([52]\) mimicking the \( \Lambda \)CDM model should be consistent with the current observational data. In the future, perhaps some deviations from the standard \( \Lambda \)CDM model may be observed. Then by using the formulation of ghost-free \( f(G) \) gravity model which we presented in this paper, we can always construct a more realistic model than the \( \Lambda \)CDM model, according to future observations.

As another model, we shall consider the following cosmological model with parameters, \( \delta, H_0, H_i, t_s, \mu, \) and \( \Lambda \),

\[
H(t) = \delta e^{H_0 - H_i t} \tanh \left( \frac{t_s - t}{\mu} \right) + \Lambda, \tag{64}
\]
where the parameters $\mu$ and $H_t$ are measured in seconds in natural units, while the parameter $\delta$ has dimensions sec$^{-2}$ in natural units. In addition, the parameter $H_0$ is considered to be dimensionless. The above model has quite interesting early and late-time phenomenology if the free parameters are appropriately chosen, since it can qualitatively describe a quasi-de Sitter cosmological evolution at early times and an accelerating era of de Sitter form at late times. Indeed, if the parameter $t_s$ is chosen to be the age of the present Universe, and also if the parameter $\Lambda$ is chosen to be the present time cosmological constant, then at early times when $t \ll t_s$, the first term is approximated as follows,

$$H(t) \sim \delta (e^{H_0} - e^{H_0 H_t}) - \Lambda,$$

(65)
due to the fact that at early times,

$$\tanh \left( \frac{t_s - t}{\mu} \right) \sim 1.$$  

(66)

Hence, if $H_0$ and $H_t$ are appropriately chosen so that $e^{H_0}, H_t \gg \Lambda$, the early-time evolution is a quasi-de Sitter evolution of the form,

$$H(t) \sim \delta (e^{H_0} - e^{H_0 H_t}),$$

(67)
and the effective EoS parameter is nearly $w_{\text{total}}^{\text{eff}} \sim -1$. Accordingly, at late-times when $t \sim t_s$, the exponential in Eq. (64) tends to zero, and also we have,

$$\tanh \left( \frac{t_s - t}{\mu} \right) \sim 0,$$

(68)
in effect, the Hubble rate is again approximated by an exact de Sitter evolution,

$$H(t) \sim \Lambda.$$  

(69)

The realization of the model (65) in the context of the ghost free $f(G)$ is possible, if the scalar potential is equal to,

$$V(\chi(t)) = \frac{3}{\kappa^2} \left( e^{H_0-H_t t} \tanh \left( \frac{t_s - t}{\mu} \right) + \Lambda \right)^2 - \frac{2e^{H_0-H_t t} (H_t \mu \tanh(\frac{t_s - t}{\mu}) + \text{sech}^2(\frac{t_s - t}{\mu}))}{\mu^2}$$

$$+ 16\mu^2 \left( e^{H_0-H_t t} \tanh \left( \frac{t_s - t}{\mu} \right) + \Lambda \right)$$

$$\times \h'(\chi) \left( \left( e^{H_0-H_t t} \tanh \left( \frac{t_s - t}{\mu} \right) + \Lambda \right)^2 - H_t e^{H_0-H_t t} \tanh \left( \frac{t_s - t}{\mu} \right) - \frac{e^{H_0-H_t t} \text{sech}^2 \left( \frac{t_s - t}{\mu} \right)}{\mu} \right)$$

$$+ 8\mu^4 \h''(\chi) \left( e^{H_0-H_t t} \tanh \left( \frac{t_s - t}{\mu} \right) + \Lambda \right)^2,$$

(70)
and the Lagrange multiplier function $\lambda(\chi(t))$ is chosen as,

$$\lambda(\chi(t)) = \frac{2}{\kappa^2 \mu^4} \left( -H_t e^{H_0-H_t t} \tanh \left( \frac{t_s - t}{\mu} \right) + \frac{e^{H_0-H_t t} \text{sech}^2 \left( \frac{t_s - t}{\mu} \right)}{\mu} \right)$$

$$+ \frac{8\h'(\chi)}{\mu^2} \left( e^{H_0-H_t t} \tanh \left( \frac{t_s - t}{\mu} \right) + \Lambda \right)$$

$$\times \left( 2 \left( -H_t e^{H_0-H_t t} \tanh \left( \frac{t_s - t}{\mu} \right) - \frac{e^{H_0-H_t t} \text{sech}^2 \left( \frac{t_s - t}{\mu} \right)}{\mu} \right) \right) - \left( e^{H_0-H_t t} \tanh \left( \frac{t_s - t}{\mu} \right) + \Lambda \right)^2$$

$$+ 8\h''(\chi) \left( e^{H_0-H_t t} \tanh \left( \frac{t_s - t}{\mu} \right) + \Lambda \right)^2.$$

(71)

By appropriately choosing the function $h(\chi)$, one may obtain different models which can realize the same cosmological evolution (65), so a rich phenomenology can be obtained. The scalar potential at early times is much more simplified, since it takes the form,

$$V(\chi(t)) \sim \frac{3 (e^{H_0} - e^{H_0 H_t} t)^2 - 2e^{H_0 H_t}}{\kappa^2}$$
while at late-times it is approximated by,

\[ V(\chi(t)) \sim 8\Lambda^2 \mu^4 h''(\chi) + 16\Lambda^3 \mu^2 h'(\chi) + \frac{3\Lambda^2}{\kappa^2}. \]  

The most interesting feature of the ghost-free model can be seen by looking Eqs. (72) and (73), due to the presence of the function \( h(\chi) \) in both equations. This means that by appropriately choosing the function \( h(\chi) \) so that a viable early-time phenomenology is obtained, this choice will affect the late-time phenomenology to some extent, not via the late-time Hubble rate, but certainly through the scalar potential and the Lagrange multiplier function \( \lambda \). Therefore, quite interesting phenomenologies may be obtained, due to the fact that during the two eras the EoS parameter is nearly \( w_{\text{eff}} \sim -1 \), hence the potential and the Lagrange multiplier function may affect other observable quantities and render the model more compatible with the observational data. Work is in progress towards this direction.

Before closing this section we should note that other cosmological evolutions can be realized in the context of the ghost-free \( f(G) \) theory which developed. For example, consider the symmetric bounce with Hubble rate,

\[ H(t) = e^{\alpha t^2}, \]  

which a well known bounce cosmology \[27, 28\]. The symmetric bounce has interesting phenomenology, since in the limit \( t \to -\infty \), the EoS parameter is approximately, \( w_{\text{eff}} \sim -1 \), which is a nearly de Sitter phase. After that and as the bouncing point at \( t = 0 \) is approached, the Universe experiences quintessential acceleration which gradually turns to a decelerating expansion, with gradually negative and positive EoS parameter. Near the bouncing point, the Universe experiences another nearly de Sitter accelerating era, and as the cosmic time grows it is followed by a phantom accelerating era, which eventually tends to a nearly de Sitter expansion at \( t \to \infty \). It is conceivable that the most interesting part of this bounce cosmology, from a phenomenological point of view, is the contracting phase. This cosmological evolution can be realized by the scalar potential,

\[ V(\chi) = \frac{e^{\alpha t^2} \left( 8\kappa^2 \mu^4 e^{2\alpha t^2} h''(\chi) + 16\kappa^2 \mu^2 e^{2\alpha t^2} \left( e^{\alpha t^2} + 2\alpha t \right) h'(\chi) + 3e^{2\alpha t^2} + 4\alpha t \right)}{\kappa^2}, \]  

where \( \chi = t \mu^2 \), and also by the Lagrange multiplier function \( \lambda(\chi) \),

\[ \lambda(\chi) = 8e^{2\alpha t^2} h''(\chi) + \frac{8e^{\alpha t^2} \left( 4\alpha t e^{2\alpha t^2} - e^{2\alpha t^2} \right) h'(\chi)}{\mu^2} + \frac{4\alpha t e^{2\alpha t^2}}{\kappa^2 \mu^4}, \]  

where in both Eqs. (75) and (76), the function \( h(\chi) \) is arbitrary. Thus in the context of the formalism we developed, we do not have a single model realizing the symmetric bounce, but a class of models which can realize this cosmological evolution. In principle, the choice of the function \( h(\chi) \) can be done in such a way so that the phenomenological constraints can be satisfied. We do not further discuss this topic for brevity, but it is conceivable that there is much room for realizing interesting phenomenologies.

**IV. NEWTON LAW IN GHOST-FREE \( f(G) \) GRAVITY**

In this section we shall consider the Newton law in the context of ghost-free \( f(G) \) and we shall investigate how this becomes in the ghost free theory. In order to consider the correction to the Newton law, we assume the geometric background is flat, by considering the limit of \( H \to 0 \) in the last section. This is because we like to consider the Newton law at scales much smaller in comparison to the cosmological scales, which are of the order \( \sim \frac{1}{H} \) in an asymptotically de Sitter spacetime during the present time era of the Universe. Then Eq. (47) or Eq. (48) indicate that \( \dot{V}(\chi) = 0 \) although \( h(\chi) \) can be an arbitrary function in general. Therefore Eq. (40) suggests that \( \lambda = \lambda^{(0)} = 0 \). We also assume that the gauge condition (15) holds true. Then by using Eqs. (68), (69), (71), (72), and (73), we obtain,
By substituting Eq. (80) in (79), we obtain,

\[
\delta \lambda = - \frac{2}{\mu^4} \left( - \frac{1}{2 \kappa^2} \Box^{(0)} (\eta^{\rho\sigma} \delta g_{\rho\sigma}) + \frac{1}{2} \partial T_{\text{matter}} - 2 \mu^4 \left( \Box^{(0)} \delta g_{tt} + \partial_t^2 (\eta^{\rho\lambda} \delta g_{\rho\lambda}) + \Box^{(0)} (\eta^{\rho\sigma} \delta g_{\rho\sigma}) \right) h'' (\chi^{(0)}) \right) .
\]

(80)

We shall consider a static point gravitational source for the matter at the spatial origin, that is,

\[
\delta T_{\text{matter}, tt} = M \delta^{(3)} (x) , \quad \text{other components of } \delta T_{\text{matter}, \mu\nu} = 0 ,
\]

(82)

where \((x) = (x^4)\). In the following two subsections, we shall investigate how the Newton law is modified in the context of Lagrange multiplier constrained Einstein-Hilbert gravity and in the context of ghost-free \(f(G)\) gravity.

A. Newton Law for Lagrange Multiplier Constrained Einstein-Hilbert gravity

Let us first consider the constrained Einstein-Hilbert gravity case, in which case \(h (\chi) = 0\). Then Eq. (81) reduces to,

\[
0 = - \frac{1}{4 \kappa^2} \left\{ - \Box^{(0)} \delta g_{\mu\nu} - \partial_{\mu} \partial_{\nu} (\eta^{\rho\lambda} \delta g_{\rho\lambda}) + \partial^{\mu\nu} (\eta^{\rho\sigma} \delta g_{\rho\sigma}) \right\} + \frac{1}{2} \delta T_{\text{matter}, \mu\nu} \\
+ \delta_{\mu\nu} \delta_{t \nu} \left\{ - \frac{1}{2 \kappa^2} \Box^{(0)} (\eta^{\rho\sigma} \delta g_{\rho\sigma}) + \frac{1}{2} \partial T_{\text{matter}} - 2 \mu^4 \left( \Box^{(0)} \delta g_{tt} + \partial_t^2 (\eta^{\rho\lambda} \delta g_{\rho\lambda}) + \Box^{(0)} (\eta^{\rho\sigma} \delta g_{\rho\sigma}) \right) h'' (\chi^{(0)}) \right\} .
\]

(83)

The \((t, t), (i, j), \) and \((t, i)\) components of (83) yield,

\[
0 = - \frac{1}{4 \kappa^2} \left\{ - \Box^{(0)} \delta g_{tt} - \partial_t^2 (\eta^{\rho\lambda} \delta g_{\rho\lambda}) \right\} ,
\]

(84)

\[
0 = - \frac{1}{4 \kappa^2} \left\{ - \Box^{(0)} \delta g_{ij} - \partial_i \partial_j (\eta^{\rho\lambda} \delta g_{\rho\lambda}) + \delta g_{ij} \Box^{(0)} (\eta^{\rho\sigma} \delta g_{\rho\sigma}) \right\} ,
\]

(85)

\[
0 = - \frac{1}{4 \kappa^2} \left\{ - \Box^{(0)} \delta g_{ti} - \partial_t \partial_{i\nu} (\eta^{\rho\lambda} \delta g_{\rho\lambda}) \right\} .
\]

(86)

and Eq. (80) has the following form,

\[
\delta \lambda = - \frac{2}{\mu^4} \left( - \frac{1}{2 \kappa^2} \Box^{(0)} (\eta^{\rho\sigma} \delta g_{\rho\sigma}) + \frac{1}{2} \partial T_{\text{matter}} \right) .
\]

(87)

We now assume that,

\[
\delta g_{tt} = A (r) , \quad \delta g_{ij} = B (r) \delta_{ij} + C (r) x^i x^j , \quad \delta g_{ti} = 0 ,
\]

(88)

where \(r = \sqrt{\sum_{i=1,2,3} (x^i)^2} \). Then Eq. (80) is trivially satisfied and since,

\[
\eta^{\rho\lambda} \delta g_{\rho\lambda} = - A + 3 B + r^2 C ,
\]

\[
\triangle (x^i x^j C (r)) = 2 \delta_{ij} C (r) + \frac{6 x^i x^j}{r} C' (r) + x^i x^j C'' (r)
\]


\[ \partial_i \partial_j (-A + 3B + r^2C) = \frac{\delta_{ij}}{r} (-A' + 3B' + 2rC + r^2C') + \frac{x^i x^j}{r^3} (A' - rA'' - 3B' + 3rB'' + 3r^2C' + r^3C'') \]

\[ \triangle(-A + 3B + r^2C) = \frac{1}{r} \left(-2A' - rA'' + 6B' + 3rB'' + 6rC + 6r^2C' + r^3C''' \right) \]

Eq. (78) indicates that \( 0 = \delta\lambda \) as an initial condition, then the term \( B \), which reproduces the standard Newtonian potential \( \phi \), is an infinite number of solutions, which do not always reproduce the standard Newton law if \( \delta\lambda \neq 0 \). In addition, Eq. (77) also suggests that, \( \delta\lambda = \frac{2}{\mu^2} \left( -\frac{4\pi A_0}{2\kappa^2 \delta(3)(x)} + \frac{1}{2} M \delta(3)(x) \right) \).

If we put \( \delta\lambda = 0 \), we find,

\[ A_0 = \frac{\kappa^2 M}{4\pi} \]

which reproduces the standard Newtonian potential \( \phi_{\text{Newton}} \), that is,

\[ \phi_{\text{Newton}} = \frac{A}{2} = \frac{\kappa^2 M}{8\pi} = \frac{GM}{r} \]

where \( G = \frac{\mu^2}{4\pi} \) is the Newton gravitational constant. We should note, however, that Eq. (102) indicates that there is an infinite number of solutions, which do not always reproduce the standard Newton law if \( \delta\lambda \neq 0 \). In addition, Eq. (77) indicates that \( 0 = \partial_0 \delta\lambda \) if \( h = 0 \), which corresponds to the Einstein-Hilbert gravity case. Therefore, if we put \( \delta\lambda = 0 \) as an initial condition, then the term \( \delta\lambda \) always vanishes, and the model reproduces the standard Newton law.
B. Newton Law in Ghost-free \( f(G) \) Gravity

Let us now investigate how the Newton law becomes in the context of the ghost-free \( f(G) \) gravity model \[28\]. First we assume that Eq. \[85\] holds true in this case too. Then the general solutions of Eqs. \[77\] and \[78\] are given by,

\[
\delta \chi = -\mu^2 t A(r) + c_1(\mathbf{x}) , \quad \delta \lambda = \frac{1}{\mu^2 t^2} h \left( \mu^2 t \right) \left( r^2 \left( -A(r) + 3B(r) + r^2C(r) \right) \right) , \quad c_2(\mathbf{x}) ,
\]

where \( c_1(\mathbf{x}) \) and \( c_2(\mathbf{x}) \) appear by integrating with respect to \( t \), and these can be determined by Eq. \[79\]. However by assuming a spherical symmetry, then \( c_1(\mathbf{x}) \) and \( c_2(\mathbf{x}) \) should depend on \( \mathbf{x} \) via the radial coordinate \( r \), that is, \( c_1(\mathbf{x}) = c_1(r) \) and \( c_2(\mathbf{x}) = c_2(r) \). On the other hand, Eq. \[80\] has the following form,

\[
\delta \lambda = -\frac{2}{\mu^2} \left\{ -\frac{1}{2\kappa^2} \left( r^2 \left( -A(r) + 3B(r) + r^2C(r) \right) \right) - \frac{M}{2} \delta^{(3)}(\mathbf{x}) - \frac{2\mu^4}{r^2} \left( r^2 \left( 3B(r) + r^2C(r) \right) \right) h''(\mu^2 t) \right\} .
\]

By comparing \( \delta \lambda \) from Eq. \[105\] with \[106\], for arbitrary \( h(\chi) \), we find \( A(r) = 3B(r) + r^2C(r) = 0 \) and \( c_2(r) = -\frac{4\mu^4}{r^2} \delta^{(3)}(\mathbf{x}) \). If surely \( A(r) = 0 \), the result is in conflict with the resulting Newton law of the constrained Einstein gravity case, given in Eq. \[114\]. This indicates that the assumption \[85\] is not satisfied and the correction to the Newton law should be time-dependent, which could constrain \( \mu^2 \), \( h(\chi) \), and/or \( V(\Phi, \Theta) \), so that the correction could be consistent with any experiment or observation. Eq. \[81\] indicates that the correction to the Newton law in the case of Einstein-Hilbert gravity is proportional to the parameter \( \mu^4 \) and the function \( h(\chi) \), and therefore if \( \mu^4 \) or \( h(\chi) \) are small enough, the constraint for the Newton law is always satisfied. For the case of the model \[52\] which mimics the \( \Lambda \)CDM model, as long as we consider the Newton law at scales much smaller than the cosmological scales \( \sim \frac{1}{\mu} \), and as long as \( h(\chi) \) is small, the constraint for the Newton law is independent on the cosmological constraints. So the constraints \[52\] and \[63\] can be imposed without restricting \( \mu \) and \( h(\chi) \).

V. GHOST-FREE \( F(R, G) \) GRAVITY

As a final task we shall demonstrate how to obtain a ghost-free \( F(R, G) \) theory of gravity. The vacuum \( F(R, G) \) gravity action is,

\[
S = \int d^4x \sqrt{-g} F(R, G) ,
\]

where \( F(R, G) \) is a function of the scalar curvature \( R \) and \( G \) stands for the Gauss-Bonnet invariant given in Eq. \[2\]. It was claimed that this model \[107\] has ghost instabilities \[19\], so let us see how ghost degrees of freedom are manifested at the equations of motion level. By introducing two auxiliary fields \( \Phi \) and \( \Theta \), the action of Eq. \[107\] can be rewritten as follows,

\[
S = \int d^4x \sqrt{-g} \left\{ \frac{R^2}{2\kappa^2} + \Theta G - V(\Phi, \Theta) \right\} ,
\]

where we have introduced the gravitational coupling \( \kappa \) in order to make \( \Phi \) and \( \Theta \) dimensionless. By varying the action \[108\] with respect to \( \Phi \) and \( \Theta \), we obtain,

\[
\frac{R}{2\kappa^2} = \frac{\partial V(\Phi, \Theta)}{\partial \Phi} , \quad G = \frac{\partial V(\Phi, \Theta)}{\partial \Theta} ,
\]

which can be algebraically solved with respect \( \Phi \) and \( \Theta \), that is, \( \Phi = \Phi(R, G) \) and \( \Theta = \Theta(R, G) \). Then by substituting the obtained expressions for \( \Phi = \Phi(R, G) \) and \( \Theta = \Theta(R, G) \) in Eq. \[108\], we obtain the action \[107\] with,

\[
F(R, G) = \Phi(R, G) R + \Theta(R, G) G - V(\Phi(R, G), \Theta(R, G)) .
\]

In order to investigate the properties of the action \[108\], we work in the Einstein frame, so under a conformal transformation of the form \( g_{\mu\nu} \rightarrow e^{\phi} g_{\mu\nu} \), the curvatures are transformed as follows \[16\], \[20\],

\[
R_{\mu\nu} \rightarrow \left\{ \begin{array}{l}
R_{\mu\nu} - \frac{1}{2} \left( g_{\zeta\nu} \nabla_{\mu} \zeta + g_{\mu\nu} \nabla_{\zeta} \phi - g_{\mu\zeta} \nabla_{\nu} \phi - g_{\nu\zeta} \nabla_{\mu} \phi \right) \\
+ \frac{1}{4} \left( g_{\zeta\nu} \partial_{\mu} \phi \partial_{\rho} \phi + g_{\mu\nu} \partial_{\rho} \phi \partial_{\zeta} \phi - g_{\mu\zeta} \partial_{\nu} \phi \partial_{\rho} \phi - g_{\nu\zeta} \partial_{\mu} \phi \partial_{\rho} \phi \right)
\end{array} \right.
\]
\[-\frac{1}{4} (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma}) \partial^\rho \phi \partial^\sigma \phi \}\),

\[R_{\mu\nu} \rightarrow R_{\mu\nu} - \frac{1}{2} (2 \nabla_\mu \nabla_\nu \phi + g_{\mu\nu} \Box \phi) + \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\rho \phi \partial_\rho \phi \},

\[R \rightarrow \left( R - 3 \Box \phi - \frac{3}{2} \partial^\rho \phi \partial_\rho \phi \right) e^{\phi} \cdot \] \tag{111}

Therefore the Gauss-Bonnet invariant \( G \) in Eq. 2 is transformed in the following way,

\[G \rightarrow e^{-2\phi} \left[ G + \nabla_\mu \left\{ 4 \left( R^{\mu\nu} - \frac{1}{2} \partial^\rho \phi \partial_\rho \phi \right) \partial_\nu \phi + 2 \left( \partial^\mu \phi \partial_\nu \phi - \left( \nabla_\nu \nabla^\mu \phi \right) \partial^\rho \phi \right) + \partial_\nu \phi \partial^\rho \phi \partial_\rho \phi \right\} \right] \]. \tag{112}

Then by writing \( \Phi = e^{-\phi} \), the action of Eq. 108 can be rewritten by taking into account the conformal transformation \( g_{\mu\nu} \rightarrow e^\theta g_{\mu\nu} \) as follows,

\[S = \int d^4 x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left( R - \frac{3}{2} \partial^\rho \phi \partial_\rho \phi \right) + \Theta G - \partial_\nu \Theta \left\{ 4 \left( R^{\mu\nu} - \frac{1}{2} \partial^\rho \phi \partial_\rho \phi \right) \partial_\nu \phi + 2 \left( \partial^\mu \phi \partial_\nu \phi - \left( \nabla_\nu \nabla^\mu \phi \right) \partial^\rho \phi \right) + \partial_\nu \phi \partial^\rho \phi \partial_\rho \phi \right\} - e^{2\phi} V \left( e^{-\phi}, \Theta \right) \right\}. \tag{113}

This action (113) may have ghost degrees of freedom due to the existence of \( \Theta \). As in the last section, we might eliminate the ghost degrees of freedom by writing \( \Theta = e^\theta \) and add a constraint to the action (113) by using the Lagrange multiplier field \( \lambda \), in the following way,

\[S = \int d^4 x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left( R - \frac{3}{2} \partial^\rho \phi \partial_\rho \phi - \lambda \left( \partial_\nu \theta \partial^\mu \theta + \mu^2 \right) \right) + e^\theta G - e^\theta \partial_\mu \theta \left\{ 4 \left( R^{\mu\nu} - \frac{1}{2} \partial^\rho \phi \partial_\rho \phi \right) \partial_\nu \phi + 2 \left( \partial^\mu \phi \partial_\nu \phi - \left( \nabla_\nu \nabla^\mu \phi \right) \partial^\rho \phi \right) + \partial_\nu \phi \partial^\rho \phi \partial_\rho \phi \right\} - e^{2\phi} V \left( e^{-\phi}, e^\theta \right) \right\}. \tag{114}

As in the previous section, the scalar fields \( \theta \) and \( \lambda \) are not dynamical degrees of freedom and the dynamical degrees of freedom are actually the metric and the scalar field \( \phi \), as in the standard \( F(R) \) gravity, therefore no ghost degrees of freedom occur in the theory.

VI. CONCLUSIONS

The focus in this work was to enlighten the ghost problem of the modified gravity theories containing the Gauss-Bonnet scalar \( G \). Particularly, we studied two kind of theories, namely \( f(G) \) gravity and \( F(R, G) \) gravity. In both cases we investigated how the ghost degrees of freedom may appear even at the equations of motion level, by using perturbations of the metric, and as we demonstrated, ghost degrees of freedom haunt both the aforementioned modified gravity theories. In both cases, we provided a theoretical remedy by using the Lagrange multiplier formalism which materializes constraints in terms of the Lagrange multipliers. As we demonstrated, our formalism leads to the elimination of the ghost degrees of freedom in both the \( f(G) \) gravity and \( F(R, G) \) gravity theories, and thus the resulting theories can in principle produce ghost free primordial curvature perturbations. Especially, in the \( F(R, G) \) gravity case, this was a serious issue due to the fact that modes \( \sim k^4 \) occurred in the master equation which governed the evolution of the primordial curvature perturbations. For the case of the ghost-free \( f(G) \) gravity theory, we investigated how accelerating cosmologies can be realized by these theories. The formalism which we presented can be used as a reconstruction technique, and as we demonstrated there is room for rich model building, since in principle any cosmological evolution can be realized by a number of different ghost-free \( f(G) \) theories, due to the freedom provided by the Lagrange multiplier formalism. A future step of the results we presented, is to provide a concrete formalism to study the inflationary period which can be technically difficult, due to the presence of the Lagrange multiplier. Work is in progress towards this research line.

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