GLUCK TWIST ALONG SATELLITE 2-KNOTS

SEUNGWON KIM

Abstract. In this paper, we show that the Gluck twist of certain satellite 2-knots in a 4-manifold do not change the diffeomorphism type.

1. Introduction

Let $K$ be an embedded 2-sphere in a 4-manifold $X$ with a product neighborhood. Consider an operation which cuts a neighborhood $N(K)$ of $K$ and glues it back in a different way. By Gluck [1], there are only two ways to glue it back, one is just the trivial gluing, and the other is called the Gluck twist. Gluck [1] showed that the Gluck twist of $S^4$ is a homotopy 4-sphere, which might give a potential counterexample to the smooth 4-dimensional Poincaré conjecture. (See Kirby’s problem 4.23 [5].)

Many knotted spheres (i.e., 2-knots) in $S^4$ are known to be Gluck twist trivial, such as ribbon 2-knots [1, 11], twist spun knots [2, 9], certain union of two ribbon disks [7], twist roll spun knots [8], band sums of all 2 such knots [3], and 2-knots 0-concordant to all such knots [6, 10].

In this paper, we consider the Gluck twist problem of a satellite 2-knot, which is defined below:

Definition 1.1. Let $P$ and $C$ be 2-knots embedded in $S^4$ and a 4-manifold $X$ respectively. Assume $C$ has a product neighborhood in $X$. Consider a simple loop $\gamma \subset S^4 - \nu(P)$. Then there exists a diffeomorphism $\rho : S^4 - \nu(\gamma) \to \nu(C)$, where $\nu(\cdot)$ denotes a neighborhood of $\cdot$ in a 4-manifold. Let $K = \rho(P) \subset X$. We call $K$ the satellite 2-knot in $X$ of companion $C$ with pattern $(P, V)$. Equivalently,

$$ (X, K) = \left( \left( \overline{X - \nu(C)} \right) \bigcup_{\partial \rho} (S^4 - \nu(\gamma)), P \right), $$

where

$$ \partial \rho = \rho |_{\partial(S^4 - \nu(\gamma))} : \partial(S^4 - \nu(\gamma)) \to \partial \nu(C) \simeq \partial(X - \nu(C)) $$

and

$$ P \subset S^4 - \nu(\gamma) \subset \left( \overline{X - \nu(C)} \right) \bigcup_{\partial \rho} (S^4 - \nu(\gamma)) \simeq X. $$

In [4], Hughes, Miller and the author studied the Gluck twist of a satellite 2-knot and showed that when the pattern is 0-concordant to a band sum of twist spun knots and the degree in $H_2(S^4 - \nu(\gamma)) \simeq H_2(S^2 \times D^2) \simeq \mathbb{Z}$ is zero, then the Gluck twist along the satellite 2-knot is trivial. Furthermore, if the degree is one, then the Gluck twist of the satellite 2-knot is the same as the Gluck twist along its companion.

In this paper, we extend the above result to a more general setting:
Theorem 1.2. Let $K$ be a satellite 2-knot in a 4-manifold $X$ of companion $C$ with pattern $P$. Suppose that the Gluck twist of $S^4$ along $P$ is trivial. Then the following holds:

(i) If $[P] \in H_2(S^4 - \nu(\gamma)) \simeq \mathbb{Z}$ is even, then the Gluck twist of $X$ along $K$ is diffeomorphic to $X$.

(ii) If $[P] \in H_2(S^4 - \nu(\gamma)) \simeq \mathbb{Z}$ is odd then the Gluck twist of $X$ along $K$ is diffeomorphic to the Gluck twist of $X$ along $C$.

Theorem 1.2 has the following immediate corollary:

Theorem 1.3. Let $K, P, C, X$ be as in Theorem 1.2. Suppose that $X \simeq S^4$. Then the following holds:

(i) If $[P]$ is even, then the Gluck twist along $K$ is diffeomorphic to $S^4$.

(ii) If $[P]$ is odd and the Gluck twist of $S^4$ along $C$ is trivial, then the Gluck twist of $S^4$ along $K$ is diffeomorphic to $S^4$.

Acknowledgements. The author would like to thank Hongtaek Jung, Maggie Miller, Jason Joseph and Hannah Schwartz for many helpful conversations about the earlier draft. The author was supported by the Institute for Basic Science (IBS-R003-D1) at the time of this project.

2. Proof of Main Theorem

Proof. Consider a banded unlink diagram of $P$. $\gamma$ can be isotoped so that it can be seen as the unknot in the banded unlink diagram of $P$. Note that the obvious disk bounded by $\gamma$ intersects the banded unlink diagram in $n$ times where $n \equiv [P] \pmod{2}$. Without loss of generality, we also can assume that the obvious disk does not intersect the bands.

We can think of the Gluck twist in the following way: First, get the natural handle decomposition from a banded unlink diagram of a 2-knot in a 4-manifold. Then, we add the +1-framed circle to a meridian of one of the dotted circles. Then, this Kirby diagram represents the Gluck twist of the given 4-manifold along the 2-knot with the given banded unlink diagram. See Figure 1 for an example.

Also, the Gluck twist along $K$ can be thought as follows: We first do the Gluck twist $S^4 - \nu(\gamma)$ along $P$ and glue it to $X - \nu(C)$ without twist. See the below equations.

\[
(X - \nu(K)) \cup (S^2 \times D^2) = ((X - \nu(C)) \cup (S^4 - \nu(\gamma)) - \nu(P)) \cup (S^2 \times D^2) \\
= ((X - \nu(C)) \cup (S^4 - \nu(\gamma) - \nu(P))) \cup (S^2 \times D^2) \\
= (X - \nu(C)) \cup ((S^4 - \nu(\gamma) - \nu(P)) \cup (S^2 \times D^2)).
\]
Figure 1: First figure is a banded unlink diagram of a spun trefoil in $S^4$. From this banded unlink, we can get the natural handle decomposition of its complement, by putting a dot on each unlink, and changing each band to a 0-framed circle as in the second figure. If we put a +1-framed circle to a meridian of one of the dotted circles, then we get a Kirby diagram of the Gluck twist of $S^4$ along the spun trefoil knot.

Figure 2: We slide the curve $\gamma$ through the +1-framed circle to get the figure in the middle from the figure in the left. Then, we push $\gamma$ below the index-2 critical points to sit on the 1-handlebody. Then, we can do small isotopy which does not intersect the dotted circles. Then, we push $\gamma$ back to the original level to get the figure in the right.

Here, $\phi$ is the self-diffeomorphism of $S^2 \times S^1$ which gives the Gluck twist. Note that $\partial \rho$ is the identity map of $S^2 \times S^1$.

Consider an isotopy of $\gamma$ through the +1-framed circle. This isotopy will link $\gamma$ and the +1-framed circle. However, we can push $\gamma$ down along the gradient flow of the Morse function so that $\gamma$ is sitting inside the 1-handle body, do an isotopy of $\gamma$ in the 1-handlebody which does not touch the dotted circles, and push it back to the level of the Kirby diagram so that $\gamma$ is unlinked from +1-framed circle. See Figure 2 for the actual moves in a Kirby diagram, and Figure 3 for an example. The +1-framed circle can be moved to any meridian of the dotted circles since it is isotopic to a meridian of $P$. Therefore, we can keep doing it so that $\gamma$ is unlinked from the every dotted circle and every attaching circle of the 2-handles.

Since the Gluck twist along $P$ is trivial, we can figure out the result of the Gluck twist if we can specify the framing of $\gamma$. Each time we push $\gamma$ through the +1-framed circle, framing changes by 1. All the other isotopies such as pushing up and down along the gradient flow, and the isotopy in the level below the index-2 critical points.
Figure 3: In the first figure, we slide the blue curve (γ) along the +1-framed circle to get the second figure. In the second figure, we push γ down to the level below the index-2 critical points, do small isotopy in that level, and push it back to the original level to get the third figure. We always can move the +1-framed circle to a meridian of any dotted circles, so we can move it like the fourth figure. Then we do the same moves to get the last figure.

which does not touch the dotted circles do not change the framing. Hence, the final framing of γ differs from the original framing by the number of times that it passes through the +1-framed circle mod 2, since γ’s framing is either 0 or 1. This number is same as the parity of \([P] \in H_2(S^2 \times D^2) \simeq \mathbb{Z}\). Hence, when \([P]\) is the even class, the framing is 0, so the gluing is trivial, i.e., we get X. Otherwise, the framing is 1, and the resultant manifold is diffeomorphic to the Gluck twist of X along C. This completes the proof of our main theorem.

References
1. Herman Gluck, The embedding of two-spheres into the four-sphere, Bulletin of the American Mathematical Society 67 (1961), 568–589.
2. Cameron McA. Gordon, Knots in the 4-sphere, Commentarii Mathematici Helvetici 51 (1976), 585–596.
3. Kazuo Habiro, Yoshihiko Marumoto, and Yuichi Yamada, Gluck surgery and framed links in 4-manifolds, Knots In Hellas’ 98, World Scientific, 2000, pp. 80–93.
4. Mark C. Hughes, Seungwon Kim, and Maggie Miller, Isotopies of surfaces in 4-manifolds via banded unlink diagrams, arXiv e-prints (2018), arXiv:1804.09169, To appear in Geometry & Topology.
5. Rob Kirby, Problems in low-dimensional topology, Proceedings of Georgia Topology Conference, Part 2, Citeseer, 1995.
6. Paul Michael Melvin, *Blowing Up and Down in 4-Manifolds*, ProQuest LLC, Ann Arbor, MI, 1977, Thesis (Ph.D.)–University of California, Berkeley. MR 2627246
7. Daniel Nash and András I. Stipsicz, *Gluck twist on a certain family of 2-knots*, Michigan Math. J. 61 (2012), no. 4, 703–713. MR 3049286
8. Patrick Naylor and Hannah Schwartz, *Gluck twisting roll spun knots*, arXiv e-prints (2020), arXiv:2009.05703.
9. Peter Sie Pao, *Nonlinear circle actions on the 4-sphere and twisting spun knots*, Topology 17 (1978), no. 3, 291–296. MR 508892
10. Nathan S. Sunukjian, *Surfaces in 4-manifolds: concordance, isotopy, and surgery*, Int. Math. Res. Not. IMRN (2015), no. 17, 7950–7978. MR 3404006
11. Takaaki Yanagawa, *On ribbon 2-knots*, Osaka Journal of Mathematics 6 (1969), 446–464.