Volumes of Zariski chambers

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Abstract. Zariski chambers are natural pieces into which the big cone of an algebraic surface decomposes. They have so far been studied both from a geometric and from a combinatorial perspective. In the present paper we complement the picture with a metric point of view by studying a suitable notion of chamber sizes. Our first result gives a precise condition for the nef cone volume to be finite and provides a method for computing it inductively. Our second result determines the volumes of arbitrary Zariski chambers from nef cone volumes of blow-downs. We illustrate the applicability of this method by explicitly determining the chamber volumes on Del Pezzo and other anti-canonical surfaces.

Introduction

In this note we study the natural decomposition of the big cone on a smooth projective surface into Zariski chambers as introduced in [2]. Being convex cones (and therefore non-compact) the chambers cannot a priori be compared in terms of size. The purpose of this note is to introduce a notion of volume of Zariski chambers, find criteria for finiteness of chamber volumes, and to show how chamber volumes can be calculated explicitly.

Let $X$ be a smooth projective surface. We consider the convex cone Big$(X)$ in the Néron-Severi vector space $N^1_\mathbb{Z}(X) := N^1(X) \otimes \mathbb{R}$ spanned by the classes of big divisors on $X$. By the main result of [2], it admits a locally finite decomposition into locally polyhedral subcones with the following properties:

- the support of the negative part in the Zariski decomposition is constant on each subcone,
- the volume function is given by a quadratic polynomial on each subcone, and
- the stable base loci are constant in the interior of each subcone.

On account of the first listed property the subcones are called Zariski chambers. For a big and nef divisor $P$ on $X$ we consider the set Null$(P)$ of irreducible curves having intersection zero with $P$. The chamber $\Sigma_P$ corresponding to $P$ consists of all big divisors whose negative part in the Zariski decomposition has support Null$(P)$. For example, if $P$ is ample, then Null$(P)$ is empty, thus $\Sigma_P$ is the intersection of the big cone with the nef cone, the nef chamber.

Zariski chambers have first been studied with respect to geometric aspects in [2]. A combinatorial point of view has been taken in [1], where a method for determining

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the number of chambers was presented. In the present paper we would like to complement the picture with a metric point of view: we ask whether one can measure the ‘size’ of a chamber – with the aim of introducing a quantity that tells, intuitively, ‘how far’ a line bundle can be moved without changing its stable base locus. A natural starting point is an invariant that was introduced in [12] to measure the nef cones of Del Pezzo surfaces. It has recently been studied by Derenthal in a series of papers (see [4], [8], and also [6], [7], and [5]). We extend this notion in order to measure arbitrary Zariski chambers on arbitrary surfaces.

Note to begin with that the Néron-Severi vector space $N^1(X)$ can be equipped with a canonical Lebesgue measure $ds$ (not depending on the choice of a basis or on an isomorphism with $\mathbb{R}^n$) by requiring that the lattice $N^1(X)$ has covolume 1, i.e., by normalizing it in such a way such that the fundamental parallelootope of $N^1(X)$ with respect to a fixed basis has $ds$-volume 1. The transformation formula together with the fact that a matrix transforming lattice bases into lattice bases has determinant $\pm 1$ guarantees the independence of the choice of a lattice basis.

Consider then for a convex cone $C \subset N^1(X)$ in the Néron-Severi vector space of a smooth projective surface $X$ the set

$$C_C := \overline{C} \cap (-K_X)^{\leq 1},$$

where $(-K_X)^{\leq 1}$ denotes the half space of divisors having intersection at most 1 with the anticanonical divisor $-K_X$ on $X$. The cone volume $\text{Vol}(C)$ is defined to be the $ds$-volume of the set $C_C$. Note that the set $C_C$ need not be compact, hence infinite cone volume can occur. Our first main result gives a necessary and sufficient condition for the nef cone to have finite volume, and moreover states that in this case the volume can be computed inductively:

**Theorem 1.** Let $X$ be a smooth projective surface with Picard number $\rho$. The nef cone volume $\text{Vol}(\text{Nef}(X))$ is finite if and only if the anticanonical divisor $-K_X$ on $X$ is big. In this case there exists a divisor $D \in N^1_R(X)$ such that the nef cone volume is given by

$$\text{Vol}(\text{Nef}(X)) = \frac{1}{\rho} \cdot \sum E (D \cdot E) \cdot \text{Vol}(\pi_E(\text{Nef}(X))),$$

with the sum taken over all $(-1)$-curves $E$ in $X$ and $\pi_E$ denoting the contraction of $E$.

When applied to del Pezzo surfaces, one obtains the explicit values computed by Derenthal in [4] (see Example 3.6 below). Other examples will be the calculation of nef cone volumes on surfaces obtained by blowing up the projective plane in points on a line and in infinitely near points (see Section 3).

Note that the nef cone is always the closure of the nef chamber (non-big nef divisors have self-intersection zero, hence lie on the boundary of the nef cone). Therefore, the volume of the nef cone equals the volume of the Zariski chamber $\Sigma_H$ for any ample divisor $H$. The second main result of this note deals with the volumes of the remaining Zariski chambers. We show that in fact knowledge of nef chamber volumes on the surfaces resulting from the contraction $\pi_S : X \to Y$ of sets $S$ of pairwise disjoint $(-1)$-curves suffices to calculate the volumes of arbitrary chambers:

**Theorem 2.** Let $X$ be a smooth projective surface with Picard number $\rho$, and let $P$ be a big and nef divisor on $X$ and $S = \{E_1, \ldots, E_s\} = \text{Null}(P)$. Either $S$ contains a
curve of self-intersection less than $-1$ and $\text{Vol}(\Sigma_P) = \infty$, or $S$ consists of $s$ pairwise disjoint $(-1)$-curves and
\[
\text{Vol}(\Sigma_P) = \frac{(\rho - s)!}{\rho!} \text{Vol}(\pi_* S(X)).
\]

We give two applications of Theorem 2: first, we use it in Section 2 to determine all chamber volumes on del Pezzo surfaces. An interesting aspect here is that chambers of the same support size (the number $s$ appearing in Theorem 2) can lead to non-isomorphic surfaces by blow-down – and precisely this geometric difference can be detected from the chamber volumes. As a second application, we study chambers on certain surfaces with big but non-ample anticanonical divisor (Sections 3.2 and 3.3).

Throughout this paper we work over the complex numbers. We would like to thank the referee for his valuable comments.

1. Zariski chamber volumes

For the Zariski chamber decomposition we follow the notation from [2]: for a big and nef divisor $P$ on $X$ we consider the set $\text{Null}(P)$ of irreducible curves having intersection zero with $P$. Note that by the index theorem the intersection matrix of the curves in $\text{Null}(P)$ must be negative definite. The chamber $\Sigma_P$ corresponding to $P$ is defined as the set of all big divisor classes $D$ such that the support of the negative part in the Zariski decomposition of $D$, denoted by $\text{Neg}(D)$, equals $\text{Null}(P)$. In [2] it is shown that two chambers $\Sigma_P$ and $\Sigma_P'$ either coincide or are disjoint, and that all of the big cone is covered by the union of all Zariski chambers. Furthermore we consider the set $\text{Face}(P)$ defined as the intersection of the nef cone with $\text{Null}(P)$. If $\text{Face}(P)$ is contained in $\text{Big}(X)$, then $\text{Face}(P)$ turns out to be the lowest dimensional face of the nef cone containing $P$ (see [2, Remark 1.5]). We will frequently use the following

**Proposition 1.1.** The closure $\Sigma_P$ of the Zariski chamber corresponding to a big and nef divisor $P$ is the convex cone spanned by $\text{Face}(P)$ and the curves in $\text{Null}(P)$.

**Proof.** This follows from [2, Proposition 1.10] by taking the closure. \qed

Upon choosing a lattice basis of $N^1(X)$, the euclidean vector space $N^{1}_{\mathbb{R}}(X)$ is equipped with a norm $|| \cdot ||_2$ coming from the scalar product. For the contraction $\pi_S : X \to Y$ of a set $S$ of disjoint $(-1)$-curves we consider the map $\pi^*_S : N^{1}_{\mathbb{R}}(Y) \to N^{1}_{\mathbb{R}}(X)$ given by pulling back divisors. The pull-back of any fixed lattice basis $B$ of $N^1(Y)$ is extended to a lattice basis $B'$ of $N^1(X)$ by the elements of $S$. The map $\pi^*_S$ embeds $N^{1}_{\mathbb{R}}(Y)$ into $N^{1}_{\mathbb{R}}(X)$ isometrically with regard to the bases $B$ and $B'$. Note furthermore that for any $E \in S$ the hyperplane $E^\perp$ given by divisor classes having intersection zero with $E$ coincides with the hyperplane of vectors orthogonal to $E$ with respect to the scalar product once $B'$ has been fixed as lattice basis.

**Proposition 1.2.** Let $S = \{E_1, \ldots, E_s\}$ be a set of pairwise disjoint $(-1)$-curves on a smooth projective surface $X$. Then
\[
\text{Nef}(X) \cap S^\perp = \pi^*_S(\text{Nef}(\pi_S(X))).
\]
Proof. We prove the result for the case \( s = 1 \) and the assertion follows inductively. Consider the surjective morphism of smooth surfaces

\[
\pi_E : X \to Y
\]
given by the contraction of the \((-1)\)-curve \( E \in S \). Any divisor \( D \in E^\perp \) on \( X \) is the pull-back of a divisor \( \overline{D} \) on \( \pi_E(X) \), and for all divisors \( F \) on \( X \) we have the projection formula

\[
D \cdot F = \overline{D} \cdot \pi_E(F),
\]
implying that \( D \) is nef if and only if \( \overline{D} \) is. Furthermore, the pull-back of any curve in \( Y \) obviously lies in the hyperplane \( E^\perp \). \( \square \)

Corollary 1.3. Let \( P \) be a big and nef divisor such that all curves in \( \text{Null}(P) \) are \((-1)\)-curves and let \( S = \{ E_1, \ldots, E_s \} \) be a subset of \( \text{Null}(P) \). Then

\[
\Sigma_P \cap S^\perp = \pi_S^*(\Sigma_{\pi_S(P)}).
\]

Proof. As above it suffices to consider the case \( s = 1 \). Remember that \( \text{Face}(P) \) is given as the intersection of \( \text{Null}(P)^\perp \) and the nef cone \( \text{Nef}(X) \). Now, the intersection of the nef cone on \( X \) with the hyperplane \( E^\perp \) corresponds to the nef cone on \( \pi_E(X) \) via \( \pi_E^* \) by virtue of the proposition above. On the other hand

\[
\text{Null}(P) \cap E^\perp = \text{Null}(P) - \{ E \} = \pi_E^*(\text{Null}(\pi_E(P))),
\]
which implies the identity \( \text{Face}(P) = \pi^*(\text{Face}(\pi_E(P))) \). This, together with Proposition \[1.1\] completes the proof. \( \square \)

Let us now prove our first main result, which shows that the calculation of volumes of Zariski chambers can be reduced to the calculation of nef cone volumes.

Theorem 1.4. Let \( X \) be a smooth projective surface with Picard number \( \rho \), and let \( P \) be a big and nef divisor on \( X \) and \( S = \{ E_1, \ldots, E_s \} = \text{Null}(P) \). Either \( S \) contains a curve of self-intersection less than \(-1\) and \( \text{Vol}(\Sigma_P) = \infty \), or \( S \) consists of \( s \) pairwise disjoint \((-1)\)-curves and

\[
\text{Vol}(\Sigma_P) = \frac{(\rho - s)!}{\rho!} \text{Vol}(\text{Nef}(\pi_S(X))).
\]

Proof. The case \( s = 0 \) is trivial, so assume that \( S \) is non-empty. Note that since \( S = \text{Null}(P) \) has negative definite intersection matrix, the alternatives really constitute a dichotomy. Now, suppose there exists an irreducible curve \( C \in S \) with \( C^2 < -1 \). By adjunction we have

\[
-K_X \cdot C \leq 0,
\]
i.e., the hyperplane \( -(K_X) = 1 \) does not intersect the ray \( \mathbb{R}^+ \cdot [C] \) which is contained in \( \Sigma_P \). Therefore, \( \Sigma_P \) has infinite volume.

If \( S \) consists of pairwise disjoint \((-1)\)-curves, we know by Proposition \[1.1\] that \( \Sigma_P \) is the convex cone spanned by \( \text{Face}(P) \) and the curves of \( S \). By Proposition \[1.2\] we have

\[
\text{Face}(P) = \text{Nef}(X) \cap S^\perp = \pi_S^*(\text{Nef}(\pi_S(X))).
\]
Additionally, for $E \in S$ and for a divisor $D$ on $X$ with $D \cdot E = 0$ we have $-K_X \cdot D = -K_{\pi_S(X)} \cdot \overline{D}$, where $D = \pi_S^*D$, implying that $C_{\text{Nef}(X)} \cap S^\perp$ can be identified via $\pi_S^*$ with $C_{\text{Nef}(\pi_S(X))}$. For a lattice basis $C_1, \ldots, C_{\rho-s}$ of $N^1(\pi_S(X))$ the vectors
\[ \pi_S^*(C_1), \ldots, \pi_S^*(C_{\rho-s}), E_1, \ldots, E_s \]
form a basis of the lattice $N^1(X)$. Consider the polytopes
\[ P_1 := \text{conv}(C_{\text{Face}(P_1)}, E_1) \]
\[ P_j := \text{conv}(P_{j-1}, E_j) \quad 2 \leq j \leq s, \]
where conv denotes the convex hull. Each $P_j$ is a $(\rho-s+j)$-dimensional pyramid with base $P_{j-1}$ and vertex $E_j$. The vector $E_j$ is perpendicular to the subspace $E_1^\perp \cap \ldots \cap E_{j-1}^\perp$ containing $P_{j-1}$. Furthermore, due to the choice of the basis we have $\|E_j\|_2 = 1$. Therefore the pyramid $P_j$ has volume
\[ \text{Vol}(P_j) = \frac{1}{\rho-s+j} \text{Vol}(P_{j-1}). \]
Iterating this calculation eventually yields
\[ \text{Vol}(\Sigma_P) = \text{Vol}(P_0) = \frac{1}{\rho-s+1} \cdot \ldots \cdot \frac{1}{\rho-s+s} \text{Vol}(\text{Face}(P)) \]
\[ = \frac{(\rho-s)!}{\rho!} \text{Vol}(\text{Nef}(\pi_S(X))). \]

2. Del Pezzo surfaces

We will now show that Theorem 1.4 enables us to compute the volumes of all Zariski chambers on del Pezzo surfaces, i.e., on surfaces $X$ with ample anticanonical divisor $-K_X$. The classification of del Pezzo surfaces is well known: either $X$ is the projective plane $\mathbb{P}^2$, or $\mathbb{P}^1 \times \mathbb{P}^1$, or a blow-up $S_r$ of $\mathbb{P}^2$ in $1 \leq r \leq 8$ points in general position\(^1\). The degree of a del Pezzo surface is defined as the self-intersection of the anticanonical divisor. We have
\[ (-K_{\mathbb{P}^2})^2 = 9, \quad (-K_{\mathbb{P}^1 \times \mathbb{P}^1})^2 = 8, \quad (-K_{S_r})^2 = 9 - r. \]

**Lemma 2.1.** Let $S_r$ be a del Pezzo surface with $1 \leq r \leq 8$ and $E$ a $(-1)$-curve on $S_r$. Then $E$ is contracted to a point on a del Pezzo surface $Y$ of degree $9 - r + 1$ by a birational morphism
\[ \pi_E : S_r \to Y. \]
In particular
\[ N^1(S_r) = N^1(Y) \oplus \mathbb{Z}[E], \quad (2.1.1) \]
\[ -K_{S_r} = -K_Y - E. \quad (2.1.2) \]

\(^1\)In this case *in general position* means that no three of the points are collinear, no six lie on a conic and no eight on a cubic with one of them a double point.
Lemma 2.2. For a curve $E$ of self-intersection $E^2 = -1$ the adjunction formula combined with the ampleness of $-K_S$, reads

$$0 \leq g(E) = 1 + \frac{1}{2}(E^2 + EK_S) \leq 0,$$

implying that $E$ must be rational. By Castelnuovo’s Contractibility Criterion, $E$ is contracted by a birational morphism $\pi_E$ to a point on a smooth surface $Y$. Regarding $S_r$ as the blow-up of $Y$ in a point with exceptional divisor $E$ renders the asserted identities obvious. It is now left to prove that $Y$ is del Pezzo of degree $9 - r + 1$. Consider the self-intersection

$$(-K_Y)^2 = (-K_S + E)^2 = 9 - r + 1 > 0.$$

Furthermore, for any irreducible curve $C$ on $Y$ we have

$$(-K_Y \cdot C) = (\pi_E^*(-K_Y) \cdot \pi_E^*(C)) = ((-K_S - E) \cdot \pi_E^*(C)) = (-K_S \cdot \pi_E^*(C)) > 0.$$

Consequently, $-K_Y$ is ample by the Nakai criterion. □

Lemma 2.2. For $r \geq 3$, contracting a $(-1)$-curve on $S_r$ results in the surface $S_{r-1}$. For a $(-1)$-curve $E$ on $S_2$, we have $\pi_E(S_2) = S_1$, if there exists a $(-1)$-curve $E^\prime$ on $S_2$ such that $(E \cdot E^\prime) = 0$. Otherwise $\pi_E(S_2) = \mathbb{P}^1 \times \mathbb{P}^1$.

Proof. The assertion for $r \geq 3$ follows immediately from Lemma 2.2 and the classification of del Pezzo surfaces, since any del Pezzo surface of degree $9 - r + 1$ is a surface $S_{r+1}$.

Let now $r = 2$ and consider $Y := \pi_E(S_2)$. By the classification of del Pezzo surfaces, $Y$ is either $\mathbb{P}^1 \times \mathbb{P}^1$ or $S_1$. Suppose there is a $(-1)$-curve $E^\prime$ on $S_2$ disjoint from $E$. Then $E^\prime$ is the pull-back of a $(-1)$-curve on $Y$. Since $\mathbb{P}^1 \times \mathbb{P}^1$ is minimal, $Y$ must be a surface $S_1$.

If, on the other hand, there is no $(-1)$-curve on $S_2$ disjoint from $E$, then $Y$ cannot contain a $(-1)$-curve either: for such a curve $C$ on $Y$, the transform $\tilde{C}$ on a blow-up $X$ of $Y$ in a point $p$ is an irreducible curve with self-intersection $\tilde{C}^2 = C^2 - s^2$, where $s$ denotes the order of $C$ in the point $p$. Now, if $X$ is del Pezzo, then $\tilde{C}^2$ is at least $-1$, hence $s = 0$, and $\tilde{C}$ is a $(-1)$-curve not intersecting the exceptional curve $E$. Therefore, contracting a curve $E$ having positive intersection with the other $(-1)$-curves on $S_2$ results in $\mathbb{P}^1 \times \mathbb{P}^1$. □

We now apply our knowledge about the behaviour of del Pezzo surfaces under contractions to calculate the chamber volumes.

Proposition 2.3. Let $P$ be a big and nef divisor on a del Pezzo surface $S_r$, $1 \leq r \leq 8$, and let $\text{Null}(P) = \{E_1, \ldots, E_k\}$. If $k \neq r - 1$, the Zariski chamber $\Sigma_P$ corresponding to $P$ has the volume

$$\text{Vol}(\Sigma_P) = \frac{(r-k+1)!}{(r+1)!} \text{Vol}(\text{Nef}(S_{r-k})), \quad (2.3.1)$$

where $S_0 := \mathbb{P}^2$. Otherwise, i.e., for $k = r - 1$,

$$\text{Vol}(\Sigma_P) = \begin{cases} 
\frac{1}{4(r+1)!}, & \text{if } E_1, \ldots, E_k \text{ form a maximal negative definite system} \\
\frac{1}{6(r+1)!}, & \text{otherwise.} 
\end{cases} \quad (2.3.2)$$
Remark 2.4. The following nef cone volumes \( \text{Vol}(\text{Nef}(S_r)) \) are calculated in [4]. We will show in Example 3.6 how to obtain these values as an application of Theorem 1. Note that Derenthal considers numbers \( \alpha(S_r) \) which equal the nef cone volume multiplied by a dimensional factor \( r + 1 \). In fact, \( \alpha(S_r) \) is defined as the volume of the topmost 'slice' \( \text{Nef}(S_r) \cap K_{S_r}^{\leq 1} \) of the polytope \( \text{Nef}(S_r) \cap K_{S_r} \) considered here.

\[
\begin{array}{cccccccccc}
 r & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 \text{Vol}(C_{\text{Nef}(S_r)}) & 1/12 & 1/72 & 1/288 & 1/720 & 1/1080 & 1/840 & 1/240 & 1/9 \\
\end{array}
\]

Furthermore, \( \text{Vol}(\text{Nef}(\mathbb{P}^2)) = \frac{1}{3} \) and \( \text{Vol}(\text{Nef}(\mathbb{P}^1 \times \mathbb{P}^1)) = \frac{1}{8} \). The proposition thus gives sufficient information to calculate the volumes of all Zariski chambers on del Pezzo surfaces (see below).

Proof of Proposition 2.3. For \( k > r - 1 \) the asserted volume formula is a direct consequence of Theorem 1.4 together with Lemma 2.2 applied \( k \) times. The contraction of \( k = r \) pairwise disjoint \((-1)\)-curves on \( S_r \) results in a del Pezzo surface with Picard number 1, i.e., in \( \mathbb{P}^2 \). Thus in this case the result, again, follows immediately from Theorem 1.4. In case \( k = 0 \) the chamber \( \Sigma_P \) is the nef chamber, whereby the assertion turns out to be trivial.

Now, let us consider the remaining case \( k = r - 1 \). Again, the formula essentially follows from Theorem 1.4. What is still left to do is to establish whether \( E_1^+ \cap \ldots \cap E_{r-1}^+ \cap \text{Nef}(S_r) \) is identified by \( \pi^* \) with \( \text{Nef}(S_1) \) or with \( \text{Nef}(\mathbb{P}^1 \times \mathbb{P}^1) \). The proof of Lemma 2.2 implies that the transform of every \((-1)\)-curve on the surface \( \pi_{E_1,\ldots,E_{r-1}}(S_r) \) resulting from the contraction of \( E_1,\ldots,E_{r-1} \) is itself a \((-1)\)-curve on \( S_r \) not intersecting any of the \( E_i \). It therefore forms a negative definite system together with the curves \( E_i \). So, \( \pi_{E_1,\ldots,E_{r-1}}(S_r) \) equals \( \mathbb{P}^1 \times \mathbb{P}^1 \) if and only if \( E_1,\ldots,E_{r-1} \) form a maximal negative definite system and otherwise equals \( S_1 \). □

In [1] an algorithm was introduced that computes the number of Zariski chambers on a smooth surface with known negative curves by determining the number of negative definite principal submatrices of the intersection matrix of all negative curves. This algorithm can easily be modified in such a way that it returns the number of negative definite principal submatrices of a given size \( s \). This number evidently equals the number of Zariski chambers \( \Sigma_P \) whose support \( \text{Null}(P) \) contains \( s \) curves.

Note that for the chambers \( \Sigma_P \) of support size \( r - 1 \) the volume varies depending on whether the contraction of the curves in \( \text{Null}(P) \) yields the surface \( \mathbb{P}^1 \times \mathbb{P}^1 \), or the surface \( S_1 \). From the algorithm we only obtain the overall number of chambers of a given support size. However, it is easy to show that on any of the surfaces \( S_r \) exactly one third of the occurring chambers of support size \( r - 1 \) are of the first type: contracting any \( r - 2 \) curves from \( \text{Null}(P) \) results in a surface \( S_2 \), whose \((-1)\)-curves have intersection matrix

\[
\begin{pmatrix}
-1 & 1 & 1 \\
1 & -1 & 0 \\
1 & 0 & -1
\end{pmatrix}.
\]

Consequently the contraction of the first curve results in \( \mathbb{P}^1 \times \mathbb{P}^1 \) and contracting either of the other two yields \( S_1 \). Since the surface resulting from iterated contraction of several \((-1)\)-curves is independent of the order of contractions, the ratio between chambers of first to second type is one to two.
The numbers and volumes of Zariski chambers on del Pezzo surfaces are displayed in tables 1 to 8 where the first and second columns indicate the support size $k$ and the surface $\pi_k(S_r)$ obtained by contracting the curves in Null($P$).

| $k$ | $\pi_k(S_r)$ | number | Vol ($\Sigma^P$) |
|-----|--------------|--------|------------------|
| 0   | $S_1$        | 1      | 1/12             |
| 1   | $\mathbb{P}^2$ | 1      | 1/6              |

Table 1: Zariski chamber volumes on $S_1$

| $k$ | $\pi_k(S_r)$ | number | Vol ($\Sigma^P$) |
|-----|--------------|--------|------------------|
| 0   | $S_2$        | 1      | 1/72             |
| 1   | $S_1$        | 2      | 1/36             |
| 1   | $\mathbb{P}^1 \times \mathbb{P}^1$ | 1      | 1/24             |
| 2   | $\mathbb{P}^2$ | 1      | 1/18             |

Table 2: Zariski chamber volumes on $S_2$

| $k$ | $\pi_k(S_r)$ | number | Vol ($\Sigma^P$) |
|-----|--------------|--------|------------------|
| 0   | $S_3$        | 1      | 1/288            |
| 1   | $S_2$        | 6      | 1/288            |
| 2   | $S_1$        | 6      | 1/144            |
| 2   | $\mathbb{P}^1 \times \mathbb{P}^1$ | 3      | 1/96             |
| 3   | $\mathbb{P}^2$ | 2      | 1/72             |

Table 3: Zariski chamber volumes on $S_3$

3. Big anticanonical surfaces

3.1. Finiteness of nef chamber volume

As we have seen, the calculation of Zariski chamber volumes $\text{Vol}($$\Sigma^P$) reduces to calculations of nef cone volumes on surfaces resulting from contraction of curves in Null($P$). For that reason we for now turn our attention to nef chamber volumes. Our first question is: which surfaces have finite nef cone volume?

First note that $\kappa(X) = -\infty$ is a necessary condition for the nef cone on a surface $X$ to have finite volume. Otherwise the anticanonical divisor on the (in this case unique) minimal model $X'$ for $X$ would be nef with non-negative self-intersection by virtue of the well known classification of smooth algebraic surfaces. But then $-K_X \cdot K_X \leq 0$, hence $X'$ (and thus $X$ itself) would have infinite nef cone volume. Our aim is now to show:

**Proposition 3.1.** A smooth projective surface $X$ has finite nef cone volume if and only if its anticanonical divisor $-K_X$ is big.

For the proof we first need a statement on convex cones, which may be seen as an “in vitro” version of Kleiman’s ampleness criterion:
Lemma 3.2. Let $C \subset \mathbb{R}^n$ be a closed cone, and let

$$C^* = \{ x \in \mathbb{R}^n \mid x \cdot c \geq 0 \text{ for all } c \in C \}$$

be its dual cone (with respect to a fixed non-degenerate bilinear form). We have the following characterization of its interior:

$$\text{int}(C^*) = \{ x \in \mathbb{R}^n \mid x \cdot c > 0 \text{ for all } c \in C \setminus \{0\} \}$$

Proof. Denote by $D$ the set on the right-hand side. We show first that $\text{int}(C^*) \subset D$. Suppose to the contrary there exists a point $x_0 \in \text{int}(C^*)$ not in the set $D$, i.e., $x_0 \cdot c = 0$ for some non-zero $c \in C$. Consider the non-zero linear function

$$\varphi_c : \mathbb{R}^n \to \mathbb{R}$$

$$x \mapsto x \cdot c.$$

It has a zero in $x_0$, hence must take negative values on points in any neighbourhood $U$ of $x_0$. However, if we choose $U$ sufficiently small, then it is contained in $\text{int}(C^*) \subset C^*$, and hence $x \cdot c \geq 0$ for $x \in U$. This is a contradiction.

We now show that $D$ is an open set. As we already know that $\text{int}(C^*) \subset D \subset C^*$, this will conclude the proof. Let then $S \subset \mathbb{R}^n$ be the 1-sphere (with respect to any fixed norm). We are done if either $D$ or $C \cap S$ are empty. Otherwise consider for $d \in D$ the linear function

$$\psi_d : C \cap S \to \mathbb{R}$$

$$x \mapsto x \cdot d.$$ 

It has only positive values and assumes a minimum on the compact set $C \cap S$, hence there is a $\delta > 0$ such that $x \cdot d \geq \delta$ for all $x \in C \cap S$. Consequently there is a neighbourhood of $d$ in $\mathbb{R}^n$, all of whose elements have positive product with every $x \in C \cap S$, and hence with every $x \in C$. \qed
If \( C \) is the Mori-cone \( \overline{\text{NE}}(X) \) of a smooth complex variety \( X \), then its dual is by definition the nef cone \( \text{Nef}(X) \). Using now that by Kleiman’s theorem \cite[Theorem 1.4.23]{10} the ample cone is the interior of the nef cone, Lemma 3.2 recovers Kleiman’s ampleness criterion \cite[Theorem 1.4.29]{10}:

\[
\text{Amp}(X) = \{ D \in N^1_{\mathbb{R}}(X) \mid D \cdot \xi > 0 \text{ for all non-zero } \xi \in \text{NE}(X) \}.
\]

For our present purposes we will need the dual statement: If \( C \) is the nef cone of a smooth projective surface, then its dual is by definition the Mori cone, whose interior is by \cite[Theorem 2.2.26]{10} the big cone, and Lemma 3.2 yields:

**Corollary 3.3.** For any smooth projective surface \( X \) we have

\[
\text{Big}(X) = \{ D \in N^1_{\mathbb{R}}(X) \mid D \cdot D' > 0 \text{ for all non-zero } D' \in \text{Nef}(X) \}.
\]

**Proof of Proposition 3.1.** The nef cone on \( X \) has finite volume if and only if the hypersurface \( (−K_X)^{-1} \) intersects each of its rays. This is the case if and only if for each nef divisor \( D \) there exists a positive rational number \( d \) such that \( dD \cdot (−K_X) = 1 \), i.e., every nef divisor must have positive intersection with \( −K_X \). The assertion follows now from Corollary 3.3. \( \square \)

**Lemma 3.4.** Let \( X \) be a smooth projective surface and let \( −K_X \) be big. Then \( \overline{\text{NE}}(X) \) is finitely generated.

**Proof.** This is shown in \cite[Lemma 6]{3} for rational surfaces. Note however that the given proof of the finite generation does not depend on rationality: for any \( \varepsilon > 0 \) and
any ample divisor $H$ by the cone theorem we find finitely many irreducible curves $C_i$ such that
$$\overline{\text{NE}}(X) = \overline{\text{NE}}(X\langle -K_X - \varepsilon H \rangle_{\leq 0}) + \sum \mathbb{R}_0^+ \cdot [C_i].$$
Furthermore, for a sufficiently small $\varepsilon > 0$ the stable base locus $B(-K_X - 2\varepsilon H)$ equals the augmented base locus $B_+(-K_X)$, which is just $\text{Null}(P)$, where $P$ denotes the positive part in the Zariski decomposition of $-K_X$, see [9] Example 1.11. (The notion of augmented base locus was introduced in [9] and is motivated by [11]; we recommend [9] or [10, Sect. 10.3] for an exposition.)

Now, any irreducible curve $C$ in $(-K_X - \varepsilon H)^{\leq 0}$ has negative intersection with $-K_X - 2\varepsilon H$, thus is an element of $B(-K_X - 2\varepsilon H) = \text{Null}(P)$. Since the intersection matrix of the curves in $\text{Null}(P)$ is negative definite, there can be at most $\rho - 1$ such curves.

**Theorem 3.5.** Let $X$ be a smooth surface with big anticanonical divisor $-K_X$ and Picard number $\rho$. There exists a divisor $D \in N_1^1(X)$ such that the nef cone volume is given by
$$\text{Vol}(\text{Nef}(X)) = \frac{1}{\rho} \sum_E (D \cdot E) \cdot \text{Vol}(\text{Nef}(\pi_E(X))),$$
where the sum is taken over all $(-1)$-curves $E$ in $X$.

Note that together with Proposition 3.1 this yields Theorem 1 from the introduction.

**Proof.** The proof consists of two parts. First we argue that there is a divisor class $D \in N_1^1(X)$ with $-K_X \cdot D = 1$ such that $D$ has intersection zero with all irreducible curves whose self-intersection is strictly less than $-1$. In the second part we show that for such an element $D$ the claimed identity holds.

By assumption $-K_X$ is big, thus there exists a representation
$$-K_X = A + B$$
with $A$ an ample $\mathbb{Q}$-divisor and $B$ an effective $\mathbb{Q}$-divisor. Now, let $C$ be an irreducible curve on $X$ with $C^2 \leq -2$. By adjunction we have
$$0 \geq -K_X \cdot C = AC + BC.$$
The ampleness of $A$ implies that $A \cdot C$ is strictly positive, showing that $B \cdot C$ must be strictly negative. Now, being effective, $B$ admits a Zariski decomposition

$$B = P_B + N_B$$

in a nef part $P_B$ and a divisor $N_B$, whose components have negative definite intersection matrix. Any curve with $C \cdot B < 0$ thus must be one of the components of $N_B$, in other words, $C$ must be an element of Neg$(B)$. Since the intersection matrix of the curves in Neg$(B)$ is negative definite, Neg$(B)$ can contain at most $\rho - 1$ curves. By the same token, there exists a big and nef divisor $P$ on $X$ with Null$(P) = \text{Neg}(B)$ (see [1, Proposition 1.1]). In particular we have $P \cdot C = 0$ for all curves $C$ with $C^2 \leq -2$. Note that the nefness of $P$ together with the bigness of $-K_X$ implies the inequality $-K_X \cdot P > 0$. We can therefore set

$$D := \frac{1}{-K_X \cdot P} \cdot P,$$

obtaining a divisor with the desired properties.

We prove the volume formula by decomposing the polytope $\mathcal{P}_X := \text{Nef}(X) \cap (-K_X)^{\perp 1}$ into pyramids with vertex $D$ and the facets of $\mathcal{P}_X$ as bases. Since $D$ is nef by construction and contained in the hypersurface $(-K_X)^{\perp 1}$, the polytope’s volume is just the sum of the volumes of all the pyramids in the decomposition. Note that $D$ in addition lies inside all the hypersurfaces $C^\perp$ for curves with self-intersection less than $-1$. The corresponding pyramids thus have volume 0, hence the nef cone volume is just the sum of volumes of the pyramids with bases $\mathcal{P}_X \cap E^\perp$ for $(-1)$-curves $E$. As we have seen, these bases correspond to the $(\rho - 1)$-dimensional polytopes $\mathcal{P}_{\pi_E(X)}$, thus have the same volume as the nef cone on the surface $\pi_E(X)$ resulting from the contraction of $E$. The asserted formula follows once we have shown that the factor $D \cdot E$ represents the height of the pyramid corresponding to $E^\perp$. This is indeed the case: for a vector space basis $E_1, \ldots, E_{\rho-1}$ of $N^1(\pi_E(X))$, the vectors $\pi^*(E_1), \ldots, \pi^*(E_{\rho-1}), -E$ form a basis of $N^1(X)$. In this basis the vector $(0, \ldots, 0, 1)$ is a normal vector to the hypersurface $E^\perp$. Let $D$ have a representation $(\alpha_1, \ldots, \alpha_{\rho-1}, \alpha)$ in this basis. Then, since $E \cdot \pi^*(E_i) = 0$, the number $\alpha$ on the one hand is just the intersection product $D \cdot E$ and on the other hand its absolute value $|\alpha|$ is the distance of the point $D$ to the hypersurface $E^\perp$, i.e., the height of the pyramid in question. By our construction, the divisor $D$ is nef, therefore $|\alpha| = \alpha$. 

**Example 3.6.** Let $S_r$ be a del Pezzo surface with $3 \leq r \leq 8$. The decomposition into ample and effective part in the proof is just the trivial decomposition

$$-K_X = A + B = -K_X + 0,$$

hence Neg$(B)$ is empty. Therefore,

$$\text{Null}(-K_{S_r}) = \text{Neg}(B) = \emptyset.$$ 

Following the proof above, we set $D := \frac{1}{g-r}(-K_{S_r})$ and obtain

$$\text{Vol}(\text{Nef}(S_r)) = \frac{1}{\rho} \sum_E DE \cdot \text{Vol} (\text{Nef}(\pi_E(S_r)))$$

$$= \frac{1}{r+1} \sum_E \frac{1}{g-r} \cdot \text{Vol}(\text{Nef}(S_{r-1}))$$

$$= \frac{N_r}{(r+1)(g-r)} \text{Vol}(\text{Nef}(S_{r-1})).$$
where \( N_r \) denotes the number of \((-1\))-curves on the del Pezzo surface \( X_r \). This formula for the nef cone volume turns out to be the same as calculated in \([4]\).

The existence of a formula for the nef cone volume does not necessarily imply that it is easy to calculate for any given big anticanonical surface \( X \). Knowledge of the negative curves on \( X \) and on the surfaces resulting from contracting \((-1\))-curves is key to the calculation: with this knowledge, an inductive calculation is possible, since successive contraction of \((-1\))-curves eventually yields a minimal surface with \( \kappa = -\infty \), and the nef cone volumes on these surfaces are easy to compute.

Testa, Várilly-Alvarado, Velasco in \([13]\) list surfaces known to have big anticanonical divisor, e.g. rational surfaces with \( K_X^2 > 0 \) or blow-ups of Hirzebruch surfaces \( X_e, e \geq 1 \), in points that lie on the union of the section \( C \) with \( e + 1 \) fibers. Furthermore, they give a classification for surfaces which are obtained as blow-ups of \( \mathbb{P}^2 \) in \( r \) points and have big and effective anticanonical divisor (see \([13\text{, Theorem 3.4}]\)). For such a surface \( X \) one has either

- \( K_X^2 > 0 \), i.e., \( r \leq 8 \), or
- \( a \) of the blown-up points lie on a line, the other \( b = r - a \) points lie on a irreducible conic, and either \( ab = 0 \) or \( \frac{1}{a} + \frac{1}{b} > 1 \), or
- the blown-up points lie on the union of three lines \( L_1, L_2, L_3 \) with \( a_i \) of them exclusively on \( L_i \), and either \( a_1a_2a_3 = 0 \) or \( \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} > 1 \).

Note that these surfaces are no longer del Pezzo as soon as curves with self-intersection less than \(-1\) occur. In the first case this happens if the blown-up points are not in general position, in the second case if either \( a \geq 3 \) or \( b \geq 6 \), and in the third case if one of the lines \( L_i \) contains at least three of the blown-up points, or if three of the blown-up points on different lines \( L_i \) are collinear. Such surfaces thus provide interesting examples for the application of theorem 3.5. In order to illustrate our method we consider non-del Pezzo surfaces from the second class: we determine the nef cone volume and the volumes of the Zariski chambers for blow-ups in \( r \) points on a line – among all blow-ups of \( \mathbb{P}^2 \) these are in a sense the other extreme to del Pezzo surfaces (see Sect. 3.2). Finally, we do the analogous computations for blow-ups at infinitely near points (see Sect. 3.3). We plan to study further surfaces with big anticanonical class in a subsequent paper.

### 3.2. Blow-ups of points on a line in \( \mathbb{P}^2 \)

Let \( L \) be a line in \( \mathbb{P}^2 \) and \( p_1, \ldots, p_r \) points on \( L \). We consider the blow-up

\[ \pi : X^r_L \to \mathbb{P}^2 \]

in these points and denote the strict transform of \( L \) by \( \tilde{L} \). Furthermore, let \( L' \) denote the transform of a general line in \( \mathbb{P}^2 \).

**Proposition 3.7.** The negative curves on \( X^r_L \) are \( \tilde{L} \) and the exceptional curves \( E_1, \ldots, E_r \). Contracting any of the curves \( E_i \) results in a surface \( X^r_L \).  

**Proof.** Suppose there exists a curve \( C = dL' - \sum_{i=1}^r m_i E_i \) with negative self-intersection neither equal to \( \tilde{L} \) nor to one of the exceptional curves. By adjunction we have \(-K_{X^r_L} \cdot C \leq 1\), or

\[ 3d - \sum m_i \leq 1. \]
Since $\tilde{L}$ corresponds to the class $L' - \sum m_i E_i$ and has non-negative intersection $d - \sum m_i$ with $C$ we have
\[1 \geq 3d - \sum m_i = 2d + (d - \sum m_i) \geq 2d \geq 2,\]
a contradiction. The second assertion is obvious. \hfill\qedsymbol

We now determine the nef cone volume of $X^r_L$ and the volumes of all Zariski chambers on this surface.

**Proposition 3.8.** For any $r \geq 1$, the nef cone volume on $X^r_L$ is given by
\[
\Vol(Nef(X^r_L)) = \frac{1}{2r+2} \Vol(Nef(X^{r-1}_L)) = \left(\frac{1}{2}\right)^r \cdot \frac{1}{(r+1)!} \cdot \frac{1}{3}.
\]

**Proof.** Following the proof of Theorem 3.5 we need to determine a divisor $D$ on $X^r_L$ with $L \cdot D = 0$ and $-K_{X^r_L} \cdot D = 1$. These are the only conditions since $\tilde{L}$ is the only curve that can have self-intersection less than $-1$. The divisor class
\[D = \frac{1}{2} \cdot (L' - E_1)\]
satisfies these conditions, and moreover lies in the hyperplanes $E_i^\perp$ for all $i = 2, \ldots, r$. Therefore, by Proposition 3.7
\[
\Vol(Nef(X^r_L)) = \frac{1}{\rho} \sum_{i=1}^{r} (E_i \cdot D) \Vol(Nef(X^{r-1}_L)) = \frac{1}{r+1} \Vol(Nef(X^{r-1}_L)).
\]
Now, the second identity follows inductively using the fact that $\Vol(Nef(\mathbb{P}^2)) = \frac{1}{3}$. \hfill\qedsymbol

The following statements about the remaining Zariski chambers are immediate consequences of Theorem 1.4.

**Proposition 3.9.** If $r \geq 3$, then for a big and nef divisor $P$ on $X^r_L$ the set $\Null(P)$ either contains $\tilde{L}$ and $\Vol(\Sigma P) = \infty$, or $\Null(P)$ consists of $s$ exceptional curves and
\[
\Vol(\Sigma P) = \frac{(r+1-s)!}{(r+1)!} \Vol(Nef(X^{r-s}_L)) = \left(\frac{1}{2}\right)^{r-s} \cdot \frac{1}{(r+1)!} \cdot \frac{1}{3}.
\]
If $1 \leq r \leq 2$, then $X^r_L$ is the del Pezzo surface $S_r$ with chamber volumes according to Proposition 2.5.
3.3. Blow-ups of infinitely near points

As a final illustration of the applicability of our technique we consider surfaces obtained by iteratively blowing up points infinitely near to \( \mathbb{P}^2 \): we start by picking a line \( L \) in \( \mathbb{P}^2 \) and a point \( p_1 \) on \( L \). Blowing up \( p_1 \) yields the del Pezzo surface \( S_1 \) on which the strict transform \( L_1 \) of \( L \) is an irreducible curve of self-intersection zero. On the exceptional curve \( E_1 \) pick the point \( p_2 \) corresponding to the tangential direction of \( L \) in \( p_1 \). Blowing up \( p_2 \) by \( S_1 \) yields \( X_1 \). Now, on the exceptional curve \( E_2 \) of the second blow-up pick the point \( p_3 \) corresponding to the tangential direction of \( L_1 \) in \( p_2 \), blow it up, and denote the resulting surface by \( X_2 \). Repeating this process yields surfaces \( X_r \) for all natural numbers \( r \geq 2 \). Note that on these surfaces the anticanonical class decomposes as \(-K_X = (2L) + (L - E_1 - \ldots - E_r)\) into a big and an effective divisor, hence is big.

**Proposition 3.10.** For \( r \geq 2 \) the classes of negative curves on \( X_r \) are

- \( E_k - E_{k+1} \) for \( 1 \leq k \leq r - 1 \),
- \( E_r \), and
- \( L - E_1 - \ldots - E_r \).

**Proof.** By the construction of \( X_r \), the class \( L - E_1 - \ldots - E_r \) contains an irreducible curve \( L' \). Its self-intersection is \( 1 - r \). Again by construction, the classes \( E_k - E_{k+1} \) contain irreducible curves of self-intersection \(-2 \). Suppose there exists a negative curve \( E \) on \( X_r \) not listed above. Then \( E \) has a representation \( E = dL - \sum_{i=1}^{r} m_i E_i \) and by adjunction the intersection with the anticanonical divisor is at most 1. By the irreducibility of \( L' \), we obtain

\[
1 \geq -K_X E = 3d - \sum m_i = 2d + L'E \geq 2,
\]

a contradiction. \( \square \)

**Proposition 3.11.** For any \( r \geq 2 \), the nef cone volume on \( X_r \) is given by

\[
\text{Vol}(\text{Nef}(X_r)) = \frac{1}{2r(r+1)} \text{Vol}(\text{Nef}(X_{r-1}))
\]

\[
= \frac{1}{2^r \cdot r!(r+1)!} \cdot \frac{1}{3}.
\]

There is exactly one additional chamber having finite volume, namely the chamber \( \Sigma_P \) with \( \text{Null}(P) = \{E_r\} \). Its volume is

\[
\text{Vol}(\Sigma_P) = \frac{(r+1)!}{(r+1)!} \text{Vol}(\text{Nef}(\pi_{E_r}(X_r)))
\]

\[
= \frac{1}{r+1} \text{Vol}(\text{Nef}(X_{r-1})).
\]

**Proof.** The equations defining the required divisor \( D = dL - \sum a_i E_i \) in Theorem 3.5 in this setting are

\[
3d - a_1 - \ldots - a_r = 1,
\]

\[
d - a_1 - \ldots - a_r = 0,
\]

\[
a_j - a_{j+1} = 0 \quad \text{for } 1 \leq j \leq r - 1.
\]
Consequently we set \( D := \frac{1}{2}(L - \frac{1}{3} \sum_{i=1}^{r} E_i) \). Then by the theorem, the nef chamber volume is given by

\[
\begin{align*}
\text{Vol}(\text{Nef}(X_r^\infty)) &= \frac{1}{r+1} \cdot (D \cdot E_r) \cdot \text{Vol}(\text{Nef}(\pi_{E_r}(X_r^\infty))) \\
&= \frac{1}{2r(r+1)} \cdot \text{Vol}(\text{Nef}(X_r^{\infty}_{r-1})),
\end{align*}
\]

and the second asserted identity follows inductively.

The statement about the remaining Zariski chambers follows using Theorem 1.4.

\[\square\]

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