Spectral asymptotics of Euclidean quantum gravity with diff-invariant boundary conditions

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Abstract
A general method is known to exist for studying Abelian and non-Abelian gauge theories, as well as Euclidean quantum gravity, at 1-loop level on manifolds with boundary. In the latter case, boundary conditions on metric perturbations $h$ can be chosen to be completely invariant under infinitesimal diffeomorphisms, to preserve the invariance group of the theory and BRST symmetry. In the de Donder gauge, however, the resulting boundary-value problem for the Laplace-type operator acting on $h$ is known to be self-adjoint but not strongly elliptic. The latter is a technical condition ensuring that a unique smooth solution of the boundary-value problem exists, which implies, in turn, that the global heat-kernel asymptotics yielding 1-loop divergences and 1-loop effective action actually exists. The present paper shows that, on the Euclidean 4-ball, only the scalar part of perturbative modes for quantum gravity is affected by the lack of strong ellipticity. Further evidence for lack of strong ellipticity, from an analytic point of view, is therefore obtained. Interestingly, three sectors of the scalar-perturbation problem remain elliptic, while lack of strong ellipticity is ‘confined’ to the remaining fourth sector. The integral representation of the resulting $\zeta$-function asymptotics on the Euclidean 4-ball is also obtained; this remains regular at the origin by virtue of a spectral identity here obtained for the first time.

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1. Introduction

The study of gauge theories and quantum gravity on manifolds with boundary is motivated by the problems of quantum cosmology [1] and quantum field theory under the influence of external conditions [2], and several efforts have been produced in the literature to study boundary conditions and 1-loop semiclassical properties within this framework [3]. In our paper we are interested in boundary conditions for metric perturbations that are completely invariant under infinitesimal diffeomorphisms, since they are part of the general scheme according to which the boundary conditions are preserved under the action of the symmetry group of the theory [4–6]. In field-theoretical language, this means setting to zero at the boundary that part $\pi A$ of the gauge field $A$ that lives on the boundary $B$ ($\pi$ being a projection operator):

$$[\pi A]_B = 0,$$

(1.1)

as well as the gauge-fixing functional,

$$[\Phi(A)]_B = 0,$$

(1.2)

and the whole ghost field

$$[\varphi]_B = 0.$$

(1.3)

For Euclidean quantum gravity, equation (1.1) reads

$$[h_{ij}]_B = 0,$$

(1.4)

where $h_{ij}$ are perturbations of the induced 3-metric. To arrive at the gravitational counterpart of equations (1.2) and (1.3), note first that, under infinitesimal diffeomorphisms, metric perturbations $h_{\mu\nu}$ transform according to

$$\hat{h}_{\mu\nu} \equiv h_{\mu\nu} + \nabla(\mu \varphi^\nu),$$

(1.5)

where $\nabla$ is the Levi-Civita connection on the background 4-geometry with metric $g$, and $\varphi^\nu \, dx^\nu$ is the ghost 1-form (strictly, our presentation is simplified: there are two independent ghost fields obeying Fermi statistics, and we will eventually multiply by $-2$ the effect of $\varphi^\nu$ to take this into account). In geometric language, the infinitesimal variation $\delta h_{\mu\nu} \equiv \hat{h}_{\mu\nu} - h_{\mu\nu}$ is given by the Lie derivative along $\varphi$ of the 4-metric $g$. For manifolds with boundary, equation (1.5) implies that [7, 8]

$$\hat{h}_{ij} = h_{ij} + \varphi_{(ij)} + K_{ij} \varphi_0,$$

(1.6)

where the stroke denotes three-dimensional covariant differentiation tangentially with respect to the intrinsic Levi-Civita connection of the boundary, while $K_{ij}$ is the extrinsic-curvature tensor of the boundary. Of course, $\varphi_0$ and $\varphi_i$ are the normal and the tangential components of the ghost, respectively. By virtue of equation (1.6), the boundary conditions (1.4) are ‘gauge invariant’, i.e.

$$[\hat{h}_{ij}]_B = 0,$$

(1.7)

if and only if the whole ghost field obeys homogeneous Dirichlet conditions, so that

$$[\varphi_0]_B = 0,$$

(1.8)

$$[\varphi_i]_B = 0.$$

(1.9)

Conditions (1.8) and (1.9) are necessary and sufficient since $\varphi_0$ and $\varphi_i$ are independent, and three-dimensional covariant differentiation commutes with the operation of restriction to the boundary. We are indeed assuming that the boundary $B$ is smooth and not totally geodesic,
i.e. $K_{ij} \neq 0$. However, for totally geodesic manifolds, having $K_{ij} = 0$, condition (1.8) is no longer necessary.

On imposing boundary conditions on the remaining set of metric perturbations, the key point is to make sure that the invariance of such boundary conditions under the infinitesimal transformations (1.5) is again guaranteed by (1.8) and (1.9), since otherwise one would obtain incompatible sets of boundary conditions on the ghost field. Indeed, on using the DeWitt–Faddeev–Popov formalism for the ⟨out|in⟩ amplitudes of quantum gravity, it is necessary to use a gauge-averaging term in the Euclidean action, of the form [9]

$$I_{g.a.} = \frac{1}{16\pi G} \int_M \Phi_v \Phi^v \sqrt{\det g} \, d^4x,$$  \hspace{1cm} (1.10)

where $\Phi_v$ is any functional which leads to self-adjoint (elliptic) operators on metric and ghost perturbations. One then finds that

$$\delta \Phi_v (h) \equiv \Phi_v (h) - \Phi_v (\hat{h}) = \mathcal{F}_\mu^\nu \varphi^\mu,$$  \hspace{1cm} (1.11)

where $\mathcal{F}_{\mu}^{\nu}$ is an elliptic operator that acts linearly on the ghost field. Thus, if one imposes the boundary conditions

$$[\Phi_v (h)]_B = 0,$$  \hspace{1cm} (1.12)

and if one assumes that the ghost field can be expanded in a complete orthonormal set of eigenfunctions $u_{\lambda}^{(\nu)}$ of $\mathcal{F}_\mu^\nu$ which vanish at the boundary, i.e.

$$\mathcal{F}_{\mu}^{\nu} u_{\lambda}^{(\nu)} = \lambda u_{\lambda}^{(\nu)},$$  \hspace{1cm} (1.13)

$$\varphi_{\nu} = \sum_{\lambda} C_{\lambda} u_{\nu}^{(\lambda)},$$  \hspace{1cm} (1.14)

$$[u_{\mu}^{(\lambda)}]_B = 0,$$  \hspace{1cm} (1.15)

the boundary conditions (1.12) are automatically gauge-invariant under the Dirichlet conditions (1.8) and (1.9) on the ghost.

Having obtained the general recipe expressed by equations (1.4) and (1.12), we can recall what they imply on the Euclidean 4-ball. This background is relevant for 1-loop quantum cosmology in the limit of small 3-geometry on one hand [10], and for spectral geometry and spectral asymptotics on the other hand [11]. As shown in [7], if one chooses the de Donder gauge-fixing functional

$$\Phi_{\mu} (h) = \nabla^\nu \left( h_{\mu \nu} - \frac{1}{2} g_{\mu \nu} g^{\rho \sigma} h_{\rho \sigma} \right),$$  \hspace{1cm} (1.16)

which has the virtue of leading to an operator of Laplace type on $h_{\mu \nu}$ in the 1-loop functional integral, equation (1.12) yields the mixed boundary conditions

$$\left[ \frac{\partial h_{00}}{\partial \tau} + \frac{3}{\tau} h_{00} - \frac{1}{\tau^2} (g^{ij} h_{ij}) + \frac{2}{\tau^2} h_{00} \right]_B = 0,$$  \hspace{1cm} (1.17)

$$\left[ \frac{\partial h_{0i}}{\partial \tau} + \frac{3}{\tau} h_{0i} - \frac{1}{\tau^2} \frac{\partial h_{00}}{\partial x^i} \right]_B = 0.$$  \hspace{1cm} (1.18)

In [3, 7], the boundary conditions (1.4), (1.17) and (1.18) were used to evaluate the full 1-loop divergence of quantized general relativity on the Euclidean 4-ball, including all $h_{\mu \nu}$ and all ghost modes. However, the meaning of such a calculation became unclear after the discovery in [6] that the boundary-value problem for the Laplacian $P$ acting on metric perturbations is not strongly elliptic by virtue of tangential derivatives in the boundary conditions (1.17) and (1.18). Strong ellipticity [11] is a technical requirement ensuring that a unique smooth
solution of the boundary-value problem exists which vanishes at infinite geodesic distance from the boundary (see appendix A). If it is fulfilled, this ensures that the $L^2$ trace of the heat semigroup $e^{-tP}$ exists, with the associated global heat-kernel asymptotics that yields 1-loop divergence and 1-loop effective action. However, when strong ellipticity does not hold, the $L^2$ trace of $e^{-tP}$ acquires a singular part [6], and hence it is unclear how to attach a meaning to $\zeta$-function calculations.

All of this has motivated our analysis, which therefore starts in section 2 with the mode-by-mode form of the boundary conditions (1.4), (1.8), (1.9), (1.17) and (1.18) with the resulting eigenvalue conditions. Section 3 studies the matrix for coupled scalar modes, while section 4 obtains the first pair of resulting scalar-mode $\zeta$-functions and section 5 studies the remaining elliptic and non-elliptic parts of spectral asymptotics. Results and open problems are described in section 6, while technical details are given in the appendices.

2. Eigenvalue conditions on the 4-ball

On the Euclidean 4-ball, which can be viewed as the portion of flat Euclidean 4-space bounded by a 3-sphere of radius $q$, metric perturbations $h_{\mu\nu}$ can be expanded in terms of hyperspherical harmonics as [12, 13]

\[
h_{00}(x, \tau) = \sum_{n=1}^{\infty} a_n(\tau) Q^{(n)}(x),
\]

\[
h_{0i}(x, \tau) = \sum_{n=2}^{\infty} \left[ b_n(\tau) \frac{Q^{(n)}_i(x)}{(n^2 - 1)} + c_n(\tau) S^{(n)}_i(x) \right],
\]

\[
h_{ij}(x, \tau) = \sum_{n=3}^{\infty} \left[ d_n(\tau) \frac{Q^{(n)}_{ij}(x)}{(n^2 - 1)} + \frac{c_{ij}}{3} Q^{(n)}(x) \right] + \sum_{n=1}^{\infty} \left[ e_n(\tau) \left( S^{(n)}_{ij}(x) + S^{(n)}_{ji}(x) \right) + f_n(\tau) G^{(n)}_{ij}(x) \right],
\]

where $\tau \in [0, q]$ and $Q^{(n)}(x)$, $S^{(n)}_i(x)$ and $G^{(n)}_{ij}(x)$ are scalar, transverse vector and transverse-traceless tensor hyperspherical harmonics, respectively, on a unit 3-sphere with metric $c_{ij}$. By insertion of expansions (2.1)–(2.3) into the eigenvalue equation for the Laplacian acting on $h_{\mu\nu}$, and by setting $\sqrt{E} \rightarrow iM$, which corresponds to a rotation of contour in the $\zeta$-function analysis [14], one finds the modes as linear combinations of modified Bessel functions of first kind according to [13]

\[
a_n(\tau) = \frac{1}{\tau} \left[ \gamma_1 I_n(M\tau) + \gamma_3 I_{n-2}(M\tau) + \gamma_4 I_{n+2}(M\tau) \right],
\]

\[
b_n(\tau) = \gamma_2 I_n(M\tau) + (n + 1) \gamma_3 I_{n-2}(M\tau) - (n - 1) \gamma_4 I_{n+2}(M\tau),
\]

\[
c_n(\tau) = \varepsilon_1 I_{n+1}(M\tau) + \varepsilon_2 I_{n-1}(M\tau),
\]

\[
d_n(\tau) = \tau \left[ -\gamma_2 I_n(M\tau) + \frac{(n + 1)}{(n - 2)} \gamma_3 I_{n-2}(M\tau) + \frac{(n - 1)}{(n + 2)} \gamma_4 I_{n+2}(M\tau) \right],
\]

\[
e_n(\tau) = \tau \left[ (3\gamma_1 - 2\gamma_2) I_n(M\tau) - \gamma_3 I_{n-2}(M\tau) - \gamma_4 I_{n+2}(M\tau) \right],
\]

\[
f_n(\tau) = \tau \left[ -\frac{\varepsilon_1}{(n + 2)} I_{n+1}(M\tau) + \frac{\varepsilon_2}{(n - 2)} I_{n-1}(M\tau) \right],
\]

\[
k_n(\tau) = \alpha_1 \tau I_n(M\tau).
\]
Modified Bessel functions of the second kind are not included to ensure regularity at the origin \( \tau = 0 \). Moreover, normal and tangential components of the ghost field admit the following expansion on the 4-ball:

\[
\varphi_0(x, \tau) = \sum_{n=1}^{\infty} I_n(\tau) Q^{(n)}(x),
\]

\[
\varphi_j(x, \tau) = \sum_{n=2}^{\infty} \left[ m_n(\tau) \frac{Q^{(n)}(x)}{(n^2 - 1)} + p_n(\tau) S^{(n)}_j(x) \right],
\]

where the ghost modes \( I_n(\tau), m_n(\tau) \) and \( p_n(\tau) \) are found to read \([13]\)

\[
I_n(\tau) = \frac{1}{\tau} [\kappa_1 I_{n+1}(M\tau) + \kappa_2 I_{n-1}(M\tau)],
\]

\[
m_n(\tau) = -(n-1)\kappa_1 I_{n+1}(M\tau) + (n+1)\kappa_2 I_{n-1}(M\tau),
\]

\[
p_n(\tau) = 0 I_n(M\tau).
\]

At this stage, the boundary conditions (1.4), (1.17), (1.18), (1.8) and (1.9) can be re-expressed in terms of metric and ghost modes as

\[
\frac{d}{d\tau} a_n + \frac{6}{\tau} a_n - \frac{1}{\tau^2} b_n = 0 \quad \text{on } S^3,
\]

\[
\frac{d}{d\tau} b_n + \frac{3}{\tau} b_n - \frac{1}{2} a_n = 0 \quad \text{on } S^3,
\]

\[
\frac{d}{d\tau} c_n + \frac{3}{\tau} c_n = 0 \quad \text{on } S^3,
\]

\[
d_n = e_n = f_n = k_n = l_n = m_n = p_n = 0 \quad \text{on } S^3.
\]

Furthermore, formulae (2.4)–(2.10) and (2.13)–(2.15) can be used to obtain homogeneous linear systems that yield, implicitly, the eigenvalues of our problem. The conditions for finding non-trivial solutions of such linear systems are given by the vanishing of the associated determinants; these yield the eigenvalue conditions \( \delta(E) = 0 \), i.e. the equations obeyed by the eigenvalues by virtue of the boundary conditions. For the purpose of a rigorous analysis, we need the full expression of such eigenvalue conditions for each set of coupled modes. Upon setting \( \sqrt{E} \to iM \), we denote by \( D(Mq) \) the counterpart of \( \delta(E) \), bearing in mind that, strictly, only \( \delta(E) \) yields the eigenvalues implicitly, while \( D(Mq) \) is more convenient for \( \zeta \)-function calculations \([14]\).

To begin, the decoupled vector mode \( c_2(\tau) = I_1(M\tau) \) obeys the Robin boundary condition (2.18), which yields

\[
D(Mq) = I_3(Mq) + I_5(Mq) + \frac{6}{Mq} I_1(Mq),
\]

with degeneracy 6. Coupled vector modes \( c_n(\tau) \) and \( f_n(\tau) \) obey the boundary conditions (2.18) and (2.19), and hence the corresponding \( D(Mq) \) reads

\[
D_n(Mq) = I_{n-1}(Mq) \left( I_n(Mq) + I_{n+1}(Mq) + \frac{6}{Mq} I_{n-1}(Mq) \right)
+ \frac{(n-2)}{(n+2)} I_{n+1}(Mq) \left( I_{n-2}(Mq) + I_n(Mq) + \frac{6}{Mq} I_{n-1}(Mq) \right),
\]

with degeneracy \( 2(n^2 - 1) \), for all \( n \geq 3 \).
The scalar modes $a_1, e_1$ obey the boundary conditions
\[ \frac{d a_1}{dr} + \frac{6}{r} a_1 - \frac{1}{r^2} \frac{d e_1}{dr} = 0 \quad \text{at} \quad r = q, \quad e_1(q) = 0, \]  
which imply
\[ D(Mq) = 20 I_1(Mq) I_3(Mq) - Mq(I_0(Mq) + I_2(Mq)) I_3(Mq) \]
\[ + 3 Mq I_1(Mq)(I_2(Mq) + I_4(Mq)), \]  
with degeneracy 1.

The scalar modes $a_2, b_2, e_2$ obey the boundary conditions (2.16), (2.17) and (2.19) with $n = 2$, and hence yield the determinant
\[ D(Mq) = \det \begin{pmatrix} I_2(Mq) & -Mq I_2'(Mq) & 4 I_3(Mq) + Mq I_3'(Mq) \\ 3 I_2(Mq) & -(2 Mq I_2'(Mq) + 6 I_2(Mq)) & 2 Mq I_3'(Mq) + 9 I_3(Mq) \\ 3 I_2(Mq) & -2 I_2(Mq) & -I_4(Mq) \end{pmatrix}, \]  
with degeneracy 4.

For all $n \geq 3$, coupled scalar modes $a_n, b_n, d_n, e_n$ obey the boundary conditions (2.16), (2.17), (2.19). The resulting determinant reads
\[ D_n(Mq) = \det \rho_{ij}(Mq), \]  
with degeneracy $n^2$, where $\rho_{ij}$ is a $4 \times 4$ matrix with entries
\[ \rho_{11} = I_n(Mq) - Mq I_n'(Mq), \quad \rho_{12} = Mq I_n'(Mq), \]  
\[ \rho_{13} = (2 - n) I_{n-2}(Mq) + Mq I_{n-2}'(Mq), \quad \rho_{14} = (2 + n) I_{n+2}(Mq) + Mq I_{n+2}'(Mq), \]  
\[ \rho_{21} = -(n^2 - 1) I_n(Mq), \quad \rho_{22} = 2 Mq I_n'(Mq) + 6 I_n(Mq), \]  
\[ \rho_{23} = 2(n + 1) Mq I_{n-2}'(Mq) - (n^2 - 6n - 7) I_{n-2}(Mq), \]  
\[ \rho_{24} = -2(n - 1) Mq I_{n+2}'(Mq) - (n^2 + 6n - 7) I_{n+2}(Mq), \]  
\[ \rho_{31} = 0, \quad \rho_{32} = -I_n(Mq), \]  
\[ \rho_{33} = \frac{(n + 1)}{(n - 2)} I_{n-2}(Mq), \quad \rho_{34} = \frac{(n - 1)}{(n + 2)} I_{n+2}(Mq), \]  
\[ \rho_{41} = 3 I_n(Mq), \quad \rho_{42} = -2 I_n(Mq), \quad \rho_{43} = -I_{n-2}(Mq), \quad \rho_{44} = -I_{n+2}(Mq). \]  

Transverse-traceless tensor modes $k_n(\tau)$ yield, by virtue of equations (2.10) and (2.19),
\[ D_n(Mq) = I_n(Mq), \quad \forall \ n \geq 3, \]  
with degeneracy $2(n^2 - 4)$.

As far as ghost modes are concerned, the decoupled mode $l_1(\tau) = \frac{1}{r} I_2(M \tau)$ vanishes at the 3-sphere boundary and hence yields
\[ D(Mq) = I_2(Mq), \]  
with degeneracy $2(n^2 - 4)$. #
with degeneracy 1, while scalar and vector ghost modes lead to
\[ D_n(Mq) = I_{n+1}(Mq), \quad \forall \ n \geq 2, \] (2.36)
and
\[ D_n(Mq) = I_n(Mq), \quad \forall \ n \geq 2, \] (2.37)
respectively, with degeneracy \( n^2 \) for equation (2.36) and \( 2(n^2 - 1) \) for equation (2.37).

Our \( D(Mq) \) equations can be re-expressed in a very helpful way by using repeatedly the identities for modified Bessel functions and their derivatives in appendix B. Hence we find, on setting \( w = Mq \), that
\[ \frac{D(w)}{2} = I_2(w) \] (2.38)
for the decoupled vector mode in equation (2.20), while coupled vector modes in equation (2.21) yield
\[ \frac{(n+2)}{4n} D_n(w) = I_n(w) \left( \frac{I_n'(w)}{w} + \frac{2}{w} I_n(w) \right) \] (2.39)
Moreover, the scalar modes \( a_1, e_1 \) ruled by equation (2.23) yield
\[ \frac{D(w)}{4w} = I_2(w) \left( \frac{I_2'(w)}{w} + \frac{4}{w} I_2(w) \right), \] (2.40)
and the scalar modes \( a_2, b_2, e_2 \) ruled by equation (2.24) lead to
\[ \frac{D(w)}{6w^2} = I_1(w) I_3(w) \left( \frac{I_3'(w)}{w} + \frac{5}{w} I_3(w) \right). \] (2.41)

3. Matrix for coupled scalar modes

The hardest part of our analysis is the investigation of equation (2.25). For this purpose, we first exploit the formulae in appendix B to find
\[ \rho_{11} = I_n(w) - wI_n'(w), \quad \rho_{12} = wI_n'(w), \quad \rho_{13} = wI_n'(w) + nI_n(w), \quad \rho_{14} = wI_n'(w) - nI_n(w), \] (3.1)
\[ \rho_{21} = -(n^2 - 1)I_n(w), \quad \rho_{22} = 2(wI_n'(w) + 3I_n(w)), \] (3.2)
\[ \rho_{23} = (n+1) \left\{ \frac{3(n+1) + 2n(n-1)(n+3)}{w^2} \right\} I_n(w) + 2 \left[ \frac{w + (n-1)(n+3)}{w} \right] I_n'(w), \] (3.3)
\[ \rho_{24} = (n-1) \left\{ \frac{3(n-1) + 2n(n+1)(n-3)}{w^2} \right\} I_n(w) - 2 \left[ \frac{w + (n+1)(n-3)}{w} \right] I_n'(w), \] (3.4)
\[ \rho_{31} = 0, \quad \rho_{32} = -I_n(w), \] (3.5)
\[ \rho_{33} = \frac{(n+1)}{(n-2)} \left[ \left( 1 + \frac{2n(n-1)}{w^2} \right) I_n(w) + \frac{2(n-1)}{w} I_n'(w) \right], \] (3.6)
\[ \rho_{44} = \frac{(n-1)}{(n+2)} \left( \left( 1 + \frac{2n(n+1)}{w^2} \right) I_n(w) - \frac{2(n+1)}{w} I'_n(w) \right), \] (3.7)

\[ \rho_{41} = 3I_n(w), \quad \rho_{42} = -2I_n(w), \] (3.8)

\[ \rho_{43} = -\left( 1 + \frac{2n(n-1)}{w^2} \right) I_n(w) - \frac{2(n-1)}{w} I'_n(w), \] (3.9)

\[ \rho_{44} = -\left( 1 + \frac{2n(n+1)}{w^2} \right) I_n(w) + \frac{2(n+1)}{w} I'_n(w). \] (3.10)

The resulting determinant, despite its cumbersome expression, can be studied by introducing the variable

\[ y \equiv \frac{I'_n(w)}{I_n(w)}, \] (3.11)

which leads to

\[ D_n(w) = \frac{48n(1 - n^2)}{(n^2 - 4)} I_n^2(w)(y - y_1)(y - y_2)(y - y_3)(y - y_4), \] (3.12)

where

\[ y_1 \equiv -\frac{n}{w}, \quad y_2 \equiv \frac{n}{w}, \quad y_3 \equiv -\frac{n}{w} - \frac{w}{2}, \quad y_4 \equiv \frac{n}{w} - \frac{w}{2}, \] (3.13)

and hence

\[ \frac{(n^2 - 4)}{48n(1 - n^2)} D_n(w) = \left( I'_n(w) + \frac{n}{w} I_n(w) \right) \left( I'_n(w) - \frac{n}{w} I_n(w) \right) \times \left( I'_n(w) + \left( \frac{w}{2} + \frac{n}{w} \right) I_n(w) \right) \left( I'_n(w) + \left( \frac{w}{2} - \frac{n}{w} \right) I_n(w) \right). \] (3.14)

4. First pair of scalar-mode \( \zeta \)-functions

Equations (2.34)–(2.41) and (3.14) are sufficient to obtain an integral representation of the \( \zeta \)-function, the residues of which yield all heat-kernel coefficients. This topic is described in great detail in the existing literature (see, for example, [11, 15]) and hence we limit ourselves to a very brief outline before presenting our results.

Given the elliptic operator \( P \) acting on physical fields defined on the \( m \)-dimensional Riemannian manifold \( \mathcal{M} \), one can build the associated heat kernel \( U(x,x,t) \) and the corresponding integrated heat kernel (bundle indices are not written explicitly, but the fibre trace \( \text{tr} \) takes them into account)

\[ \text{Tr}_{L^2} e^{-tP} = \int_{\mathcal{M}} \text{tr} U(x, x; t) \sqrt{g} \, d^m x, \] (4.1)

which has the asymptotic expansion, as \( t \to 0^+ \),

\[ \text{Tr}_{L^2} e^{-tP} \sim (4\pi t)^{-\frac{m}{2}} \sum_{k=0}^{\infty} A_k t^{\frac{k}{2}}. \] (4.2)

The \( A_k \) coefficients are said to describe the global asymptotics in that they are obtained by integration over \( \mathcal{M} \) and its boundary \( \mathcal{B} \) of local geometric invariants built from the Riemann curvature of \( \mathcal{M} \), gauge curvature, extrinsic curvature of \( \mathcal{B} \), potential terms in \( P \) and in the boundary operator expressing the boundary conditions. On the other hand, since in the
strongly elliptic case the ζ-function of \( P \) is related to the integrated heat kernel by a Mellin transform [1, 3, 11]:

\[
\zeta_P(s) = \text{Tr}_{L^2} P^{-s} = \sum_{\lambda \in \Phi} \lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left( \text{Tr}_{L^2} e^{-tP} \right) dt, \quad (4.3)
\]

the global heat-kernel coefficients in the asymptotic expansion (4.2) can also be obtained from the residues of \( \zeta_P(s) \) [15].

Moreover, since the function occurring in the equation obeyed by the eigenvalues by virtue of the boundary conditions admits a canonical-product representation [1, 3], one can also express \( \zeta_P(s) \) as a contour integral which is eventually rotated to the imaginary axis. The residues of the latter integral yield therefore the \( \zeta \) coefficients used in evaluating 1-loop effective action and 1-loop divergences.

In our problem the \( P \) operator is the Laplacian on the Euclidean 4-ball acting on metric perturbations. Equations (2.34)–(2.41) correspond to a familiar mixture of Dirichlet and Robin boundary conditions for which integral representation of the \( \zeta \)-function and heat-kernel coefficients are immediately obtained. New features arise instead from equation (3.14), that gives rise to four different \( \zeta \)-functions. On studying the first line of equation (3.14), we can exploit the work in [16] and the uniform asymptotic expansion of Bessel functions and their first derivatives (see appendix B) to say that the integral representation of the resulting \( \zeta \)-function reads

\[
\zeta^+_{\pm}(s) \equiv \frac{(\sin \pi s)}{\pi} \sum_{n=3}^{\infty} n^{-(2s-2)} \int_0^\infty dz \, z^{-2s} \frac{\partial}{\partial z} \log[z^{-\beta_{\pm}(n)}(znI_n(zn) \pm nI_n(zn))] \quad (4.4)
\]

With our notation, \( \beta_+(n) = n, \beta_-(n) = n + 2 \), where these factors are fixed by the leading behaviour of the eigenvalue condition as \( z \to 0 \) [15]; the uniform asymptotic expansion of modified Bessel functions and their first derivatives (see appendix B) can be used to find (hereafter \( \tau = \tau(z) \equiv (1 + z^2)^{-1/2} \) from equation (B8))

\[
znI_n(zn) \pm nI_n(zn) \sim \frac{n}{\sqrt{2 \pi n}} e^{\alpha n} \sqrt{\tau} \left( 1 + \sum_{k=1}^\infty \frac{p_{k,\pm}(\tau)}{n^k} \right), \quad (4.5)
\]

where

\[
p_{k,\pm}(\tau) \equiv (1 \pm \tau)^{-1}(v_k(\tau) \pm \tau u_k(\tau)), \quad (4.6)
\]

for all \( k \geq 1 \), and

\[
\log \left( 1 + \sum_{k=1}^\infty \frac{p_{k,\pm}(\tau)}{n^k} \right) \sim \sum_{k=1}^\infty \frac{T_{k,\pm}(\tau)}{n^k}. \quad (4.7)
\]

Thus, the \( \zeta \)-functions (4.4) obtain, from the first pair of round brackets in equation (4.5), the contributions (cf [16])

\[
A_+(s) \equiv \sum_{n=3}^{\infty} n^{-(2s-2)} \frac{(\sin \pi s)}{\pi} \int_0^\infty dz \, z^{-2s} \frac{\partial}{\partial z} \log \left( 1 + (1 + z^2)^{-1} \right), \quad (4.8)
\]

\[
A_-(s) \equiv \sum_{n=3}^{\infty} n^{-(2s-2)} \frac{(\sin \pi s)}{\pi} \int_0^\infty dz \, z^{-2s} \frac{\partial}{\partial z} \log \left( 1 - (1 + z^2)^{-1} \right), \quad (4.9)
\]

where \( z^2 \) in the denominator of the argument of the log arises, in equation (4.9), from the extra \( z^{-2} \) in the prefactor \( z^{-\beta_{\pm}(n)} \) in definition (4.4). Moreover, the second pair of round brackets in equation (4.5) contributes \( \sum_{j=1}^\infty A_{j,\pm}(s) \), having defined

\[
A_{j,\pm}(s) \equiv \sum_{n=3}^{\infty} n^{-(2s+j-2)} \frac{(\sin \pi s)}{\pi} \int_0^\infty dz \, z^{-2s} \frac{\partial}{\partial z} T_{j,\pm}(\tau(z)), \quad (4.10)
\]
where, from the formulae

\[ T_{1,\pm} = p_{1,\pm}, \quad (4.11) \]

\[ T_{2,\pm} = p_{2,\pm} - \frac{1}{2} p_{1,\pm}^2, \quad (4.12) \]

\[ T_{3,\pm} = p_{3,\pm} - p_{1,\pm} p_{2,\pm} + \frac{1}{2} p_{1,\pm}^3, \quad (4.13) \]

we find

\[ T_{1,\pm} = -\frac{3}{8} \tau \pm \frac{1}{2} \tau^2 - \frac{3}{256} \tau^3, \quad (4.14) \]

\[ T_{2,\pm} = -\frac{1}{16} \tau^2 \pm \frac{3}{2} \tau^3 + \frac{1}{8} \tau^4 + \frac{5}{32} \tau^5 + \frac{5}{128} \tau^6, \quad (4.15) \]

\[ T_{3,\pm} = -\frac{21}{128} \tau^3 \pm \frac{3}{8} \tau^4 + \frac{509}{640} \tau^5 + \frac{25}{128} \tau^6 + \frac{21}{128} \tau^7 \pm \frac{15}{128} \tau^8 - \frac{1105}{1152} \tau^9, \quad (4.16) \]

and hence, in general,

\[ T_{j,\pm}(\tau) = \sum_{a=j}^{3j} f_a^{(j,\pm)} \tau^a. \quad (4.17) \]

We therefore find, with the same algorithms as in [15],

\[ \zeta^+_A(0) = -\frac{5}{4} + \frac{1079}{240} - \frac{1}{2} \sum_{a=3}^{9} f_a^{(3,+)} = \frac{146}{45}, \quad (4.18) \]

\[ \zeta^-_A(0) = -\frac{5}{4} + \frac{1079}{240} + \frac{1}{2} \sum_{a=3}^{9} f_a^{(3,-)} = \frac{757}{90}. \quad (4.19) \]

These results have been double-checked by using also the powerful analytic technique in [14].

**5. Further spectral asymptotics: elliptic and non-elliptic parts**

As a next step, the second line of equation (3.14) suggests considering \( \zeta \)-functions having the integral representation (cf equation (4.4) and see further comments in section 6)

\[ \zeta^\pm_A(s) \equiv \frac{(\sin \pi s)}{\pi} \sum_{n=3}^{\infty} n^{-(2s-2)} \times \int_0^\infty dz \, z^{-2s} \frac{\partial}{\partial z} \log \left[ z^{-\beta_+(n)} \left( z n I'_n(zn) + \left( \frac{z^2 n^2}{2} \pm n \right) I_n(zn) \right) \right]. \quad (5.1) \]

To begin, we exploit again the uniform asymptotic expansion of modified Bessel functions and their first derivatives to find (cf equation (4.5))

\[ zn I'_n(zn) + \left( \frac{z^2 n^2}{2} \pm n \right) I_n(zn) \sim \frac{n^2}{2\sqrt{2\pi n \sqrt{\tau}}} \left( \frac{1}{\tau} - \tau \right) \left( 1 + \sum_{k=1}^{\infty} r_{k,\pm}(\tau) \frac{n^k}{n^k} \right), \quad (5.2) \]

where we have (bearing in mind that \( u_0 = v_0 = 1 \))

\[ r_{k,\pm}(\tau) \equiv u_k(\tau) + \frac{2\tau}{(1-\tau^2)} \left( (v_{k-1}(\tau) \pm \tau u_{k-1}(\tau) \right), \quad (5.3) \]

for all \( k \geq 1 \). Hereafter we set

\[ \Omega \equiv \sum_{k=1}^{\infty} \frac{r_{k,\pm}(\tau)}{n^k}. \quad (5.4) \]
and rely upon the formula

$$\log(1 + \Omega) \sim \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\Omega^k}{k}$$

(5.5)

to evaluate the uniform asymptotic expansion (cf equation (4.7))

$$\log \left( 1 + \sum_{k=1}^{\infty} \frac{r_{k,\pm}(\tau(z))}{n^k} \right) \sim \sum_{k=1}^{\infty} \frac{R_{k,\pm}(\tau(z))}{n^k}. $$

(5.6)

The formulae yielding $R_{k,\pm}$ from $r_{k,\pm}$ are exactly as in equations (4.11)–(4.13), with $T$ replaced by $R$ and $p$ replaced by $r$ (see, however, comments below equation (5.10)). Hence we find, bearing in mind equation (5.3),

$$R_{1,\pm} = (1 \mp \tau)^{-1} \left( \frac{17}{8} \tau \mp \frac{1}{8} \tau^2 \mp \frac{5}{24} \tau^3 \mp \frac{5}{24} \tau^4 \right),$$

(5.7)

$$R_{2,\pm} = (1 \mp \tau)^{-2} \left( -\frac{47}{16} \tau^2 \mp \frac{15}{8} \tau^3 \pm \frac{21}{16} \tau^4 \mp \frac{3}{16} \tau^5 \mp \frac{1}{16} \tau^6 \mp \frac{5}{8} \tau^7 \mp \frac{5}{8} \tau^8 \right),$$

(5.8)

$$R_{3,\pm} = (1 \mp \tau)^{-3} \left( \frac{1721}{384} \tau^3 \mp \frac{441}{128} \tau^4 \mp \frac{597}{320} \tau^5 \mp \frac{1033}{960} \tau^6 \mp \frac{339}{80} \tau^7 \right.$$

$$\left. \mp \frac{29}{2} \tau^8 \mp \frac{243}{370} \tau^9 \pm \frac{221}{192} \tau^{10} \mp \frac{241}{384} \tau^{11} \pm \frac{1105}{1152} \tau^{12} \right),$$

(5.9)

and therefore

$$R_{j,\pm}(\tau(z)) = (1 \mp \tau)^{-j} \sum_{a=0}^{j} \frac{4j}{a} C_{a}^{(j,\pm)} \tau^{a},$$

(5.10)

where, unlike what happens for the $T_{j,\pm}$ polynomials, the exponent of $(1 \mp \tau)$ never vanishes.

Note that, at $\tau = 1$ (i.e. $z = 0$), our $R_{k,\pm}(\tau)$ and $R_{k,\pm}(\tau)$ are singular. Such behaviour is not seen for any of the strongly elliptic boundary-value problems (see third item in [11]). This technical difficulty motivates our efforts below and is interpreted by us as a clear indication of the lack of strong ellipticity proved, on general grounds, in [6].

The $\xi_{\pm}(s)$ function is more easily dealt with. It indeed receives contributions from terms in round brackets in equation (5.2) equal to (cf equation (4.9) and bear in mind that $\beta_{-} - \beta_{+} = 2$ in equation (5.1))

$$B_{-}(s) \equiv \sum_{n=3}^{\infty} n^{-(2s-2)} \frac{\sin \pi s}{\pi} \int_{0}^{\infty} dz z^{-2s} \frac{\partial}{\partial z} \log \left( \frac{1}{z} - \tau(z) \right)$$

$$= \alpha_{0}(s) \frac{\sin \pi s}{\pi} \int_{0}^{\infty} dz z^{-2s} \frac{\partial}{\partial z} \log \frac{1}{\sqrt{1+z}} = -\frac{1}{2} \alpha_{0}(s),$$

(5.11)

and $\sum_{j=1}^{\infty} B_{j,-}(s)$, having defined, with $\lambda = 0$, $j$ (cf equation (4.10))

$$\omega_{j}(s) \equiv \sum_{a=0}^{\infty} n^{-(2s+2\lambda-2)} = \xi_{\pm}(2s + \lambda - 2; 3),$$

(5.12)

$$B_{j,-}(s) \equiv \omega_{j}(s) \frac{\sin \pi s}{\pi} \int_{0}^{\infty} dz z^{-2s} \frac{\partial}{\partial z} R_{j,-}(\tau(z)).$$

(5.13)

On using the same method as in section 4, formulae (5.1)–(5.13) lead to

$$\xi_{\pm}(0) = -\frac{5}{4} + \frac{1079}{240} + \frac{5}{2} = \frac{1}{16} \sum_{a=3}^{12} C_{a}^{(3,\pm)} = \frac{206}{45},$$

(5.14)

a result which agrees with a derivation of $\xi_{\pm}(0)$ relying upon the method of [14].
Although we have stressed after equation (5.10) the problems with the \( \zeta^+_{\mathcal{B}}(s) \) part, for the moment let us proceed formally in the same way as above. Thus we define, in analogy to equation (5.11),

\[
B_{\pm}(s) \equiv \omega_0(s) \left( \frac{\sin \pi s}{\pi} \right) \int_0^{\infty} dz \, z^{-2s} \frac{\partial}{\partial z} \log \left( \frac{1}{\tau(z)} - \tau(z) \right),
\]

and, in analogy to equation (5.13),

\[
B_{j,\pm}(s) \equiv \omega_j(s) \left( \frac{\sin \pi s}{\pi} \right) \int_0^{\infty} dz \, z^{-2s} \frac{\partial}{\partial z} R_{j,\pm}(\tau(z)).
\]

In order to make the presentation as transparent as possible, we write out the derivatives of \( R_{j,\pm} \). On changing integration variable from \( z \) to \( \tau \) we define

\[
C_j(\tau) \equiv \frac{\partial}{\partial \tau} R_{j,\pm}(\tau),
\]

and we find the following results:

\[
C_1(\tau) = (1 - \tau)^{-2} \left( \frac{17}{12} - \frac{1}{2} \tau - \frac{1}{3} \tau^2 + \frac{5}{3} \tau^3 - \frac{5}{6} \tau^4 \right),
\]

\[
C_2(\tau) = (1 - \tau)^{-3} \left( \frac{1271}{128} \tau^2 - \frac{441}{32} \tau^3 + \frac{1635}{128} \tau^4 - \frac{163}{16} \tau^5 + \frac{1545}{64} \tau^6 - \frac{227}{4} \tau^7 + \frac{4223}{64} \tau^8 - \frac{211}{16} \tau^9 - \frac{5083}{128} \tau^{10} + \frac{1105}{32} \tau^{11} - \frac{1105}{128} \tau^{12} \right),
\]

\[
C_3(\tau) = (1 - \tau)^{-4} \sum_{a=j-1}^{4j} K_a^{(j)} \tau^a, \quad \forall j = 1, \ldots, \infty.
\]

These formulae engender a \( \zeta^+_{\mathcal{B}}(0) \) which can be defined, after change of variable from \( z \) to \( \tau \), by splitting the integral with respect to \( \tau \), in the integral representation of \( \zeta^+_{\mathcal{B}}(s) \), according to the identity

\[
\int_0^1 d\tau = \int_0^\mu d\tau + \int_\mu^1 d\tau,
\]

and taking the limit as \( \mu \to 1 \) after having evaluated the integral. More precisely, since the integral on the left-hand side is independent of \( \mu \), we can choose \( \mu \) small on the right-hand side so that, in the interval \([0, \mu]\) (and only there!), we can use the uniform asymptotic expansion of the integrand where the negative powers of \((1 - \tau)\) are harmless. Moreover, independence of \( \mu \) also implies that, after having evaluated the integrals on the right-hand side, we can take the \( \mu \to 1 \) limit. Within this framework, the limit as \( \mu \to 1 \) of the second integral on the right-hand side yields vanishing contribution to the asymptotic expansion of \( \zeta^+_{\mathcal{B}}(s) \).

With this caveat, on defining

\[
Q_{\mu}(\alpha, \beta, \gamma) \equiv \int_0^\mu \tau^\alpha (1 - \tau)^\beta (1 + \tau)^\gamma d\tau,
\]

we obtain the representations

\[
B_{\pm}(s) = -\omega_0(s) \left( \frac{\sin \pi s}{\pi} \right) \left[ -Q_{\mu}(2s, -s - 1, -s) + Q_{\mu}(2s, -s, -s - 1) - Q_{\mu}(2s - 1, -s, -s) \right],
\]
\[ B_{j,+}(s) = -\omega_j(s) \frac{(\sin \pi s)}{\pi} \sum_{a=j-1}^{4j} K^{(j)}_a Q_\mu(2s + a, -s - j - 1, -s). \] (5.24)

The relevant properties of \( Q_\mu(\alpha, \beta, \gamma) \) can be obtained by observing that this function is nothing but a hypergeometric function of two variables [17], i.e.

\[ Q_\mu(\alpha, \beta, \gamma) = \frac{\mu^{\alpha+1}}{\alpha + 1} \binom{\alpha + 1}{-\beta, -\gamma, \alpha + 2; \mu, -\mu}. \] (5.25)

In detail, a summary of results needed to consider the limiting behaviour of \( \zeta_+^\mu(s) \) as \( s \to 0 \) is

\[ \omega_0(s) \frac{(\sin \pi s)}{\pi} \sim -5s + O(s^2), \] (5.26)

\[ \omega_j(s) \frac{(\sin \pi s)}{\pi} \sim \frac{1}{2} \delta_{j,3} + \hat{b}_{j,3} s + O(s^2), \] (5.27)

\[ \lim_{\mu \to 1} Q_\mu(2s, -s - 1, -s) \sim -\frac{1}{s} + O(s^0), \] (5.28)

\[ \lim_{\mu \to 1} Q_\mu(2s, -s, -s - 1) \sim \log(2) + O(s), \] (5.29)

\[ \lim_{\mu \to 1} Q_\mu(2s - 1, -s, -s) \sim \frac{1}{2s} + O(s), \] (5.30)

\[ \lim_{\mu \to 1} Q_\mu(2s + a, -s - j - 1, -s) \]
\[ = \frac{\Gamma(-j - s)\Gamma(a + 2s + 1)}{\Gamma(a - j + s + 1)} F(a + 2s + 1, s, a - j + s + 1; -1) \]
\[ \sim \frac{b_{j,1}(a)}{s} + b_{j,0}(a) + O(s), \] (5.31)

where

\[ \hat{b}_{j,1} = -1 - 2^{j-2} + \xi R(j - 2)(1 - \delta_{j,3}) + \gamma \delta_{j,3}, \] (5.32)

\[ b_{j,-1}(a) = \frac{(-1)^{a+1}}{j!} \frac{\Gamma(a + 1)}{\Gamma(a - j + 1)} (1 - \delta_{a,j-1}), \] (5.33)

and we only strictly need \( b_{3,0}(a) \) which, unlike the elliptic cases studied earlier, now depends explicitly on \( a \) and is given by (\( \psi \) being the standard notation for the logarithmic derivative of the \( \Gamma \)-function)

\[ b_{3,0}(a) = \frac{1}{6} \frac{\Gamma(a + 1)}{\Gamma(a - 2)} \left[ -\log(2) - \frac{1}{4} (6a^2 - 9a + 1) \frac{\Gamma(a - 2)}{\Gamma(a + 1)} + 2\psi(a + 1) - \psi(a - 2) - \psi(4) \right]. \] (5.34)

Remarkably, the coefficient of \( \frac{1}{s} \) in the small-\( s \) behaviour of the generalized \( \zeta \)-function \( \zeta_+^\mu(s) \) is zero because it is equal to

\[ \lim_{s \to 0} s \zeta_+^\mu(s) = \sum_{a=2}^{12} b_{3,-1}(a) K_a^{(3)} = \frac{1}{6} \sum_{a=3}^{12} \Gamma(a - 1)(a - 2) K_a^{(3)}, \] (5.35)

which vanishes by virtue of the rather peculiar general property

\[ \sum_{a=j}^{4j} \frac{\Gamma(a + 1)}{\Gamma(a - j + 1)} K_a^{(j)} = \sum_{a=j}^{4j} \prod_{l=0}^{j-1} (a - l) K_a^{(j)} = 0, \quad \forall \ j = 1, \ldots, \infty, \] (5.36)
and hence we find eventually
\[ \zeta_B^\alpha(0) = -\frac{5}{4} + \frac{1079}{240} + \frac{5}{2} - \frac{1}{2} \sum_{a=2}^{12} b_{3,0}(a) K_a^{(j)} - \sum_{j=1}^{\infty} \bar{b}_{j,1} \sum_{a=j-1}^{4j} b_{j-1}(a) K_a^{(j)} \]
\[ = -\frac{5}{4} + \frac{1079}{240} + \frac{599}{720} = \frac{296}{45}, \quad (5.37) \]
because the infinite sum on the first line of equation (5.37) vanishes by virtue of equations (5.33) and (5.36), and exact cancellation of \( \log(2) \) terms is found to occur by virtue of equation (5.36).

To cross-check our analysis we remark that, on applying the technique of [14], one finds
\[ \zeta_B^\alpha(0) = -\frac{15}{4} + \frac{1079}{240} - \frac{1}{720} = \frac{67}{90}, \quad (5.38) \]
where \( -\frac{1}{720} \) results from working in the \( n \to \infty \) and \( w \to 0 \) limits in
\[ \left( I'_n(w) + \frac{w}{2} I_n(w) \right) \]
on the second line of equation (3.14); such a term then reduces to \( \left( I'_n(w) + \frac{n}{w} I_n(w) \right) \). A possible interpretation of the discrepancy between (5.37) and (5.38) is that, when strong ellipticity is violated, prescriptions for defining a \( \zeta(0) \) value exist but are inequivalent (see section 6).

Remaining contributions to \( \zeta(0) \), being obtained from strongly elliptic sectors of the boundary-value problem, are instead found to agree with the results in [7], i.e.
\[ \zeta(0) \text{[transverse traceless modes]} = -\frac{278}{45}, \quad (5.39) \]
\[ \zeta(0) \text{[coupled vector modes]} = \frac{404}{45}, \quad (5.40) \]
\[ \zeta(0) \text{[decoupled vector mode]} = -\frac{47}{7}, \quad (5.41) \]
\[ \zeta(0) \text{[scalar modes \((a_1, e_1; a_2, b_2, e_2)\)]} = -17, \quad (5.42) \]
\[ \zeta(0) \text{[scalar ghost modes]} = -\frac{140}{45}, \quad (5.43) \]
\[ \zeta(0) \text{[vector ghost modes]} = \frac{77}{40}, \quad (5.44) \]
\[ \zeta(0) \text{[decoupled ghost mode]} = \frac{5}{7}. \quad (5.45) \]

6. Concluding remarks

We have obtained the analytically continued eigenvalue conditions for metric perturbations on the Euclidean 4-ball, in the presence of boundary conditions completely invariant under infinitesimal diffeomorphisms in the de Donder gauge and with \( \alpha \) parameter set to 1 in equation (1.10). Second, this has made it possible to prove, for the first time in the literature, that only one sector of the scalar-mode determinant is responsible for lack of strong ellipticity of the boundary-value problem (see second line of equation (3.14) and the analysis in sections 4 and 5). The first novelty with respect to the work in [6] is a better understanding of the elliptic and non-elliptic sectors of spectral asymptotics for Euclidean quantum gravity. Moreover, as far as we know, the detailed spectral asymptotics for \( \zeta \)-functions of sections 4 and 5 was missing in the literature. We have also shown that one can indeed obtain a regular \( \zeta \)-function
asymptotics at small $s$ in the non-elliptic case by virtue of the remarkable identity (5.36), here obtained for the first time. Our prescription for the $\zeta(0)$ value differs from the result first obtained in [7], where, however, neither the strong ellipticity issue [6] nor the non-standard spectral asymptotics of section 5 had been considered.

As far as we can see, the issues raised by our results are as follows.

(i) The integral representation (5.1) is legitimate because the second line of equation (3.14) corresponds to the eigenvalue conditions, for $n \geq 3$,

$$F_B^\pm(n, x) \equiv J'_n(x) + \left( -\frac{x}{2} \pm \frac{n}{x} \right) J_n(x) = 0. \tag{6.1}$$

For both choices of sign in front of $\frac{n}{x}$, if $x_i$ is a root, then so is $-x_i$, with positive eigenvalue $E_i = x_i^2$ (having set the 3-sphere radius $q = 1$ for simplicity). For any fixed $n$, there is a countable infinity of roots $x_i$ and they grow approximately linearly with $n$. The function $F_B^\pm$ admits therefore a canonical-product representation [18] which ensures that the integral representation (5.1) reproduces the standard definition of a generalized $\zeta$-function, i.e.

$$\zeta(s) = \sum_{E_k > 0} d(E_k) E_k^{-s},$$

where $d(E_k)$ is the degeneracy of the eigenvalue $E_k$.

(ii) Even though the lack of strong ellipticity implies that the functional trace of the heat semigroup no longer exists, and hence the Mellin transform (4.3) relating $\zeta$-function to integrated heat kernel cannot be exploited, it remains possible to define the functional determinant of the operator $P$ acting on metric perturbations. For this purpose, a weaker assumption provides a sufficient condition, i.e. the existence of a sector in the complex plane free of eigenvalues of the leading symbol of $P$ [19]. Note also that, if one looks at the $A_1$ heat-kernel coefficient for boundary conditions involving tangential derivatives [11], it is exactly for the ball that the potentially divergent pieces involving the extrinsic curvature in $A_1$ cancel. Thus, on the Euclidean ball cancellations take place that maybe could explain why $\zeta(0)$ is finite. This might be therefore a very particular result for the ball.

(iii) In the non-elliptic case, the point of departure of the method in [14] from our method is the replacement of equation (6.1) with + sign, for large $n$ and small $x$, by the equation

$$F_A^+(n, x) \equiv J'_n(x) + \frac{n}{x} J_n(x) = 0. \tag{6.2}$$

With the notation in [14], this leads to an $I_{pole}(0)$ contribution to $\zeta(0)$ which is not affected by the non-trivial problems resulting from negative powers of $(1 - \tau)$ in the spectral asymptotics and studied in detail in our section 5. At present it remains unclear whether both regularization methods are admissible.

(iv) The remarkable factorization of eigenvalue conditions, with resulting isolation of the elliptic part of spectral asymptotics (transverse-traceless, vector and ghost modes, all modes in finite-dimensional sub-spaces and three of the four equations for scalar modes), suggests trying to re-assess functional integrals on manifolds with boundary, with the hope of being able to obtain unique results from the non-elliptic contribution. If this cannot be achieved, the two alternatives below should be considered again.

(v) Luckock boundary conditions [20], which engender BRST-invariant amplitudes but are not diffeomorphism invariant [3]. They have already been applied by Moss and Poletti [21, 22].
(vi) Non-local boundary conditions that lead to surface states in quantum cosmology and pseudo-differential operators on metric and ghost modes [23]. Surface states are particularly interesting since they describe a transition from quantum to classical regime in cosmology entirely ruled by the strong ellipticity requirement, while pseudo-differential operators are a source of technical complications.

There is therefore encouraging evidence in favour of Euclidean quantum gravity being able to drive further developments in quantum field theory, quantum cosmology and spectral asymptotics (see early mathematical papers in [24, 25]) in the years to come.

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Appendix A. Strong ellipticity

For an operator of Laplace type, the boundary-value problem is strongly elliptic with respect to the cone $C - R_+$ if, for any cotangent vector $u$ on the boundary $B$, for any $\lambda \in C - R_+$, for any pair $(u, \lambda) \neq (0, 0)$, there exists a unique solution $\phi$ of the differential equation ($r$ being the geodesic distance to the boundary $B$)

$$\left[ -\frac{g^2}{\partial r^2} + u_k u^k - \lambda \right] \phi(r) = 0,$$

subject to the asymptotic condition

$$\lim_{r \to \infty} \phi(r) = 0,$$

and to the boundary conditions (here $\phi(r) = e^{-\sigma r}$ with $\sigma \equiv \sqrt{u_k u^k - \lambda}$)

$$\pi \phi(r = 0) = \psi_0, \quad IT \phi(r = 0) + (I - \pi) \phi'(r = 0) = \psi_1,$$

where $\pi$ is the same projector as in equation (1.1), $iT$ is the leading symbol of that part of the boundary operator which involves tangential derivatives, while $\psi_0$ and $\psi_1$ are arbitrary boundary data. Eventually, all this is equivalent to proving positivity of the matrix $I \sqrt{u_k u^k - iT}$ [6].

Appendix B. Bessel functions

In section 2 we exploit the following identities obeyed by modified Bessel functions of first kind:

$$I_{n+1}(w) = I'_n(w) - \frac{n}{w} I_n(w),$$

$$I_{n-1}(w) = I'_n(w) + \frac{n}{w} I_n(w),$$

(B1) (B2)
\[ I_{n+2}(w) = \left( 1 + \frac{2(n+1)}{w^2} \right) I_n(w) - \frac{2(n+1)}{w} I'_n(w), \tag{B3} \]
\[ I_{n-2}(w) = \left( 1 + \frac{2(n-1)}{w^2} \right) I_n(w) + \frac{2(n-1)}{w} I'_n(w), \tag{B4} \]
\[ I'_{n+2}(w) = -\frac{2(n+1)}{w} \left( 1 + \frac{n(n+2)}{w^2} \right) I_n(w) + \left( 1 + \frac{2(n+1)(n+2)}{w^2} \right) I'_n(w), \tag{B5} \]
\[ I'_{n-2}(w) = \frac{2(n-1)}{w} \left( 1 + \frac{n(n-2)}{w^2} \right) I_n(w) + \left( 1 + \frac{2(n-1)(n-2)}{w^2} \right) I'_n(w). \tag{B6} \]

In sections 4 and 5 we use the uniform asymptotic expansion of modified Bessel functions \( I_\nu \) first found by Olver [26]:
\[ I_\nu(z) \sim \frac{e^{\nu \eta}}{\sqrt{2\pi \nu (1+z^2)^{\frac{3}{2}}}} \left( 1 + \sum_{k=1}^{\infty} \frac{u_k(\tau)}{\nu^k} \right), \tag{B7} \]
where
\[ \tau = (1+z^2)^{-\frac{1}{2}}, \quad \eta = (1+z^2)^{\frac{3}{4}} + \log \left( \frac{1+z}{\sqrt{1+z^2}} \right). \tag{B8} \]
This holds for \( \nu \to \infty \) at fixed \( z \). The polynomials \( u_k(\tau) \) can be found from the recurrence relation [15]
\[ u_{k+1}(\tau) = \frac{1}{2} \tau \left( 1 - \tau^2 \right) u'_k(\tau) + \frac{1}{8} \int_0^\tau d\rho (1 - 5\rho^2) u_k(\rho), \tag{B9} \]
starting with \( u_0(\tau) = 1 \). Moreover, the first derivative of \( I_\nu \) has the following uniform asymptotic expansion at large \( \nu \) and fixed \( z \):
\[ I'_\nu(z) \sim \frac{e^{\nu \eta}}{\sqrt{2\pi \nu (1+z^2)^{\frac{3}{2}}}} \left( 1 + \sum_{k=1}^{\infty} \frac{v_k(\tau)}{\nu^k} \right), \tag{B10} \]
with the \( v_k \) polynomials determined from the \( u_k \) according to [15]
\[ v_k(\tau) = u_k(\tau) + \tau (\tau^2 - 1) \left[ \frac{1}{2} u_{k-1}(\tau) + \tau u'_{k-1}(\tau) \right], \tag{B11} \]
starting with \( v_0(\tau) = u_0(\tau) = 1 \).

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