Constructing new nonlinear evolution equations with supersymmetry

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Abstract
The factorisation method commonly used in linear supersymmetric quantum mechanics is extended, such that it can be applied to nonlinear quantum mechanical systems. The new method is distinguishable from the linear formalism, as the superpotential is forced to become eigenfunction-dependent. Example solutions are given for the nonlinear Schrödinger equation and its supersymmetric partner equation. This method allows new nonlinear evolution equations to be constructed from the solutions of known nonlinear equations, and has the potential to be a useful tool for mathematicians and physicists working in the field of nonlinear systems, allowing the discovery of previously unknown ‘dualities’ amongst nonlinear evolution equations.

Keywords: solitons, nonlinear Schrödinger equation, supersymmetry, nonlinear evolution equations

(Some figures may appear in colour only in the online journal)

1. Introduction
Supersymmetry (SUSY), a spacetime symmetry, was formulated in the context of quantum field theory as a relationship between fermions and bosons, which provided a means of fixing some crucial problems with the standard model of particle physics [1, 2]. The concepts of SUSY have since been applied to traditional linear quantum mechanics (QM), creating a new way of understanding relationships between the potentials of the Schrödinger equation [3]. Supersymmetry in QM, or SUSY-QM, has had significant impact in the field of optics in particular, where it is instead known as optical SUSY, and has applications in many areas such as mode conversion, transformation optics, and laser arrays [4–8]. However, the focus has not strayed from linear optical problems. Extending this formalism so that it will have potential applications in nonlinear optics seems a natural way to progress. While there is some previous work on the supersymmetry of the nonlinear Schrödinger equation (NLSE) [9, 10], it uses the
traditional formalism associated with bosons and fermions (by using Grassmann even or odd fields, respectively). Furthermore, research on ‘nonlinear SUSY’ focuses on what is known as N-fold or polynomial SUSY [11], and not on the application of SUSY-QM to nonlinear equations.

In this paper, we show for the first time that the concept of 1D SUSY-QM can be extended to Hamiltonians leading to nonlinear evolution equations, and in particular we present the application of this idea to the NLSE as a representative, important example. It was shown by Bernstein that the NLSE is factorisable and that its eigenstates are tied to the factorisation by a Miura transform [12], which, in the context of SUSY-QM, can be realised as the Riccati equation one needs to solve to construct supersymmetric Hamiltonians [3]. The traditional method of factorisation is reworked to account for the dependence of a nonlinear Hamiltonian on its own eigenstates. We show that this factorisation can be rearranged in analogy with linear SUSY-QM, to create a new, eigenstate-dependent nonlinear equation.

This method is shown to be applicable to any NLSE-type equation, where the existence of solutions is entirely dependent on whether one can solve the Riccati equation. Furthermore, the ‘SUSY-partner’ equation is only nonlinear if the equation it is derived from is nonlinear, and the superpotential is consequently forced to be eigenstate-dependent. Hence, this formalism is distinct from linear SUSY-QM. Lastly, it is shown that if we consider the nonlinear SUSY partner equation to have an additional solution, analogous to the solution corresponding to the additional ground state level granted to one of a pair of Hamiltonians in linear SUSY-QM, then for the ‘level’ corresponding to this solution, the equation reduces to a scale-free non-linear equation, which can be transformed into a linear equation with a simple substitution.

The aim of this work is to present a new method to construct pairs of nonlinear equations which can be considered supersymmetric, as they are related through factorisation. As there are known solutions to the NLSE, we can find solutions to the equation we have constructed from it. As an example, we find solutions of our nonlinear-SUSY partner equation, from the known fundamental light and dark soliton solutions of the NLSE.

2. The nonlinear SUSY transformation

Establishing a SUSY relationship between two Hamiltonians in QM, known as superpartners, relies on being able to factorise the Hamiltonian operator in question [3]. For Hamiltonians in linear QM, we can simply require (imposing $\hbar = 1, \, m = 1/2$):

$$H^{(1)} = -\frac{d^2}{dx^2} + V^{(1)}(x) = \hat{A}^\dagger \hat{A} + E_0,$$

and,

$$H^{(2)} = -\frac{d^2}{dx^2} + V^{(2)}(x) = \hat{A} \hat{A}^\dagger + E_0,$$

where $x$ is the spatial variable, $V^{(1,2)}$ are two (in general different) potentials, and $E_0$ is the ground state energy of the first system. The factorisation operators (in a sense analogous to the creation and annihilation operators of the harmonic oscillator) have the form $\hat{A} = d/dx + W(x)$ and $\hat{A}^\dagger = -d/dx + W(x)$, where $W(x)$ is known as the superpotential, a function that connects, and from which one can derive, both $V^{(1)}$ and $V^{(2)}$. If the above requirements are met, the two Hamiltonians will share a spectrum of energies denoted by $E_n$, with the exception
that an eigenstate corresponding to the ground state energy, $E_0$, will not exist for the system governed by $H_2$. To transform from one Hamiltonian to the other by means of a SUSY transformation, one must simply solve either of the following Riccati equations:

$$W^2(x) - W'(x) = V^{(1)}(x) - E_0,$$

or

$$W^2(x) + W'(x) = V^{(2)}(x) - E_0.$$  \hspace{1cm} (3)

If the eigenfunctions of $H^{(1)}$ corresponding to energy $E_n$ are denoted by $\phi_n$, and the eigenfunctions of $H^{(2)}$ are denoted $\psi_n$, then they are related by the equations $\psi_n = (E_n - E_0)^{-1/2}\hat{A}\phi_n$ and $\phi_n = (E_n - E_0)^{-1/2}\hat{A}^\dagger\psi_n$, when $n \geq 1$. In order for the supersymmetry to remain unbroken, the condition $\hat{A}\phi_0 = 0$ must be upheld, ensuring no eigenfunction corresponding to $E_0$ exists for $H^{(2)}$.

Let us now consider the nonlinear Hamiltonian $H_n = -d^2/dx^2 - \kappa|\psi_n|^2$, such that when it acts on $\psi_n$ we recover the stationary NLSE:

$$H_n\psi_n = -\frac{d^2}{dx^2}\psi_n - \kappa|\psi_n|^2\psi_n = E_n\psi_n.$$  \hspace{1cm} (5)

In order to factorise this Hamiltonian in its general form, we must require our factorisation operators to also be wavefunction-dependent, labelled by the integer $n$; the integer label does not indicate any discreteness, nor relates to a spectrum, as this concept does not apply to $H^{(2)}$. An $\hat{A}$ of the form

$$\hat{A}_n = \frac{d}{dx} + W_n(x),$$

and is dependent on the eigenfunctions, meaning in general $W_n \neq W_0$. It is convenient to take the NLSE as $H_n^{(1)}$, and use it to find a second nonlinear system with Hamiltonian $H_n^{(2)}$ and additional level $E_0$. The above procedure is implemented as follows. We require

$$H_n^{(2)} = -\frac{d^2}{dx^2} - \kappa|\psi_n|^2 = \hat{A}_n\hat{A}_n^\dagger + E_0,$$  \hspace{1cm} (6)

and solve

$$W_n^2(x) + W_n'(x) = -\kappa|\psi_n|^2 - E_0$$  \hspace{1cm} (7)

to find the level-dependent superpotential, $W_n(x)$. The choice of $E_0$ is arbitrary, as long as it is lower than the ground state energy of the second Hamiltonian, $E_1$. We can thus engineer the available solutions by making an informed choice for $E_0$. Our first Hamiltonian has the level dependent form

$$H_n^{(1)} = -\frac{d^2}{dx^2} + W_n^2(x) - W_n'(x) + E_0.$$  \hspace{1cm} (8)

Using the superpotential obtained from equation (7), we can find the eigenfunctions from the condition $\phi_n = (E_n - E_0)^{-1/2}(-d/\kappa + W_n(x))\psi_n$. Furthermore, we can obtain a definition of the superpotential in terms of the two eigenfunctions:
where the prime indicates a derivative in $x$.

This allows us to find the SUSY-QM partner to the NLSE purely in terms of the eigenfunctions of the two equations; we essentially have a coupled nonlinear system. There are various ways to write equation (8) such that the superpotential is eliminated, however too much substitution between $\psi_n$ and $\phi_n$ with the goal to eliminate $\psi_n$ terms will lead to the tautology $E_n \phi_n = E_n \phi_n$. Our preferred form of the nonlinear equation corresponding to $H_n^{(1)} \phi_n$, is thus

$$\phi''_n - 2 \left( \frac{\phi'_n}{\phi_n} \right)^2 - \kappa \phi_n |\psi_n|^2 + 2(E_n - E_0)^{1/2} (\psi_n \frac{\phi'_n}{\phi_n} - \psi'_n) = E_n \phi_n.$$  

This can alternatively be written as:

$$\frac{d^2}{dx^2} \phi_n - 8 \left( \frac{d}{dx} \sqrt{\phi_n} \right)^2 - \kappa \phi_n |\psi_n|^2 - 2 \sqrt{\Delta E_n} \frac{d}{dx} \left( \frac{\psi_n}{\phi_n} \right) = E_n \phi_n,$$

where $\Delta E_n = E_n - E_0$. We can think of the system described by equation (11) as symmetric—in the quantum mechanical sense—to the NLSE, equation (5). The concept of ‘energy levels’ is unnatural in the context of nonlinear equations, so instead of a relationship between two potentials which have identical spectra (bar the ground state of the first), we have a relationship between two nonlinear terms, which is defined by the nonlinear SUSY construction. We can think of the functions $\psi_n$ as distinct eigenfunctions of the NLSE, $\phi_n$ as distinct eigenfunctions of its SUSY partner, and $\phi_0$ as an eigenfunction corresponding to energy $E_0$, for which no corresponding eigenstate of the NLSE exists. It is clear that a function, $\psi_n$, which allows this equation to be solved for a given $\phi_n$, will solve the NLSE. The additional energy dependent term comes from the derivative of the superpotential, and although unusual, cannot be avoided.

It is now possible to generalise the above result. It should be clear that for any nonlinear equation which can be written in the form:

$$- \psi''_n - N(\psi_n) \psi_n = E_n \psi_n,$$

where $N(\psi_n)$ represents a nonlinear operator, there exists a SUSY-QM partner equation which will have the form,

$$\frac{d^2}{dx^2} \phi_n - 8 \left( \frac{d}{dx} \sqrt{\phi_n} \right)^2 - N(\psi_n) \phi_n - 2 \sqrt{\Delta E_n} \frac{d}{dx} \left( \frac{\psi_n}{\phi_n} \right) = E_n \phi_n,$$

given one can solve the Riccati equation, equation (7). This equation can be greatly simplified by making the substitution $\phi_n = 1/\psi_n$, and rearranging:

$$-u''_n - N(\psi_n) u_n - 2 \sqrt{\Delta E_n} (u'_n \psi_n + \psi'_n u_n^2) = E_n u_n.$$  

Equation (14) now maintains a more typical format, and is also clearly still nonlinear. It is important to observe that the nonlinear terms of this equation do not arise from the $N(\psi_n)$ term, but from the requirement that the superpotential, $W_n$, is forced to be eigenfunction-dependent. It may seem that one can set the nonlinear operator in equation (14) to zero, and receive a nonlinear equation partnered to an equation for a free particle; this is incorrect, as in this case the superpotential is clearly eigenfunction-independent. The $\psi_n$ in equation (14) must belong to a nonlinear equation, in order for our SUSY construction to be valid. It is important to clarify here that the above equations are valid for any $n$, corresponding to any of
the $n$ possible solutions of the original equation (in the case of the NLSE, $n$ can be anything from one to infinity). To successfully find solutions to the partner equations, one must be able to solve the Riccati equation for the original nonlinearity, for any given $n$th solution.

We shall now show that when a solution to equation (12) ceases to exist, equation (14) reduces to the Schrödinger equation for a free particle.

3. The ‘vacuum’ equation

From linear SUSY-QM, we have the condition $\hat{A}\phi_0 = 0$ (annihilation of the ground state), which gives us the simple relation $W(x) = -d/dx \ln \phi_0 = -\phi_0'/\phi_0$. For us, this condition is no longer universal, and only valid for some particular solution $\phi_0$, which exists for the case $\psi_0 = 0$ for unbroken SUSY. Making the standard quantum-mechanical substitution $E_0 \to i(d/dt)$, equation (13) reduces to:

$$i\dot{\phi}_0 - \phi_0'' + 2\frac{(\phi_0')^2}{\phi_0} = 0,$$

(15)

where the dot indicates a time derivative. This equation can be easily solved, and its bound state solutions include hyperbolic secants. Equation (15) can be written as the Lax equation seen in the work of Zakharov and Shabat [13],

$$i \frac{\partial \hat{L}}{\partial t} + [\hat{L}, \hat{M}] = 0,$$

(16)

for the following Lax pair:

$$\hat{L} = \frac{d}{dx} + \frac{1}{\phi_0},$$

(17)

and

$$\hat{M} = \frac{d}{dx} + \frac{\phi_0'}{\phi_0^2} + \frac{1}{\phi_0}.$$  

(18)

Interestingly, equation (15) is reminiscent of the equation for the propagation of an optical field seen in what is known as scale-free optics [14], although with some important differences in the dimensionality and the use of intensities instead of the ratio of envelope fields. The system is known as ‘scale-free’ due to the fact it is intensity independent. This property is mirrored in our equation by the fact that the amplitude of the solutions play no role in the dynamics. This is an indication that the equation is in fact a linear equation in disguise, and on making our earlier substitution $\phi_0 = 1/u_0$, equation (15) essentially becomes the Schrödinger equation for a free particle, and it becomes clear that the Lax pair above is ‘fake’; it says nothing about the integrability of the system, see for instance [15] for a full explanation of this point.

4. An example solution

It should be noted that knowledge of the solutions of the NLSE is not needed to construct a partner equation using the factorisation method, but one can find the solutions of the partner equation whenever a solution to the NLSE is known, as clarified earlier. Here, we will show how to find solutions of equation (11) using solutions of the NLSE. Note that to solve
equation (7) for some eigenstate $\psi_n$, any value of $E_0 \neq E_n$ can be chosen without ‘breaking’ SUSY [3], and causing the normalisation factor in $\phi_n = (E_n - E_0)^{-1/2} \hat{A}^\dagger \psi_n$ to become infinite.

For solutions of the NLSE (with $N(\psi_n) = -2|\psi_n|^2$, i.e. $\kappa = 2$) of the form $\psi_n = a \text{sech}(ax)$ (fundamental bright solitons), equation (7) can, in general, be solved. As the concept of a spectrum when talking about nonlinear equations is redundant, and $n$ is an arbitrary label, we are free to choose to label $\text{sech}(x)$ as $\psi_1$ and $2 \text{sech}(2x)$ as $\psi_2$, which allows us to distinguish the energies and superpotentials associated with them (of course, there is an infinite number of solutions to the NLSE, since the equation is integrable, but we will use these two as our first example). By fixing the parameter $E_0$, it is then easy to show that the superpotential becomes eigenstate-dependent in the nonlinear SUSY case, as predicted. By solving equation (7) for $\psi_1$ and $\psi_2$, when $E_0 = -16$, one can find the results displayed in table 1, and plotted in figures 1 and 2.
We can also solve equation (7) for solutions of the NLSE (with $N(\psi_n) = 2|\psi_n|^2$, i.e. $\kappa = -2$) of the form $\psi = a \tanh (ax)$. The results which follow for $\psi_3 = \tanh (x)$ and $\psi_4 = 2 \tanh (2x)$, when $E_0 = -2$, are shown in table 2, and plotted in figures 3 and 4.

From figures 1 and 2, we can see that a soliton solution of the NLSE will give us a soliton solution of equation (11). If one chooses to write the partner equation in the form of equation (14) then the solutions will become singular. We can see from figures 3 and 4 that the same is true if one starts with the dark soliton.

Hence, for a fixed value of the parameter $E_0$, it is clear the superpotential and the solution to the partner equation which follows is entirely dependent on the choice of $\psi_n$, demonstrating that the method discussed in this paper takes a solution of one nonlinear equation and uses it to find a solution of another nonlinear equation. This procedure is distinct from linear quantum mechanical SUSY, which relates the spectra of two potentials. Of course, new solutions can be found by simply changing the parameters or considering other solutions to the Riccati equation (equation (7)), making this a useful tool when it comes to finding solutions to apparently complex equations.

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**Table 2.** Example solutions of equations (5), (7), (11) and (14), connected by the nonlinear SUSY method described in this paper.

| $E_0 = -2, \kappa = -2$ | $E_0 = -2, \kappa = -2$ |
|--------------------------|--------------------------|
| $\psi_3 = \tanh (x)$    | $\psi_4 = 2 \tanh (2x)$ |
| $W_3 = 2 \tanh (x)$     | $W_4 = \sqrt{10} - \frac{4 \sech^2 (2x)}{\sqrt{10} - 2 \tanh (2x)}$ |
| $\phi_3 = 1 - \frac{1}{4} \sech^2 (x)$ | $\phi_4 = \frac{2(2 - \sqrt{10} \tanh (2x))}{\sqrt{10} - 2 \tanh (2x)}$ |
| $u_3 = \frac{1}{1 - \frac{1}{4} \sech^2 (x)}$ | $u_4 = \frac{\sqrt{10} - 2 \tanh (2x)}{2(2 - \sqrt{10} \tanh (2x))}$ |

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**Figure 2.** $\psi_2, W_2, \phi_2, \text{and} u_2$, as displayed in table 1, plotted in units of $\hbar = 2m = 1$. $E_0 = -16, \kappa = 2$. 
Figure 3. $\psi_3$, $W_3$, $\phi_3$, and $u_3$, as displayed in table 2, plotted in units of $\hbar = 2m = 1$. $E_0 = -2, \kappa = -2$.

Figure 4. $\psi_4$, $W_4$, $\phi_4$, and $u_4$, as displayed in table 2, plotted in units of $\hbar = 2m = 1$. $E_0 = -2, \kappa = -2$. 
5. Conclusions

We have presented a new method for constructing nonlinear evolution equations by extending the formalism of SUSY-QM to nonlinear systems. The result is an equation dependent on both the eigenstates of the original nonlinear equation and the eigenstates of the new equation. When the eigenstates of the original nonlinear equation vanish, the new equation can be reduced to a linear one using a simple transformation. Our scheme is easily extended to all NLSE-type evolution equations containing second order derivatives, and can be used to obtain SUSY-partner equations for a large variety of nonlinear models, establishing a web of previously unknown ‘dualities’ between soliton solutions and their respective equations. Last but not least, further evolutions of our theory could give a better understanding of the nonlinearities of unconventional materials such as the ones presented in [14, 16, 17]. In fact, it may happen that propagation equations of certain nonlinear physical systems (such as the one presented in [14]) have a simple but unconventional form, which is unfamiliar to standard soliton theory. In optics, such propagation equations are often based on the slowly-varying envelope approximation, and are therefore first order partial differential equations for the electric field envelope. In such cases, our method could elucidate connections between these seemingly bizarre equations with more conventional integrable equations, opening up the possibility to complete solve the system and understand it theoretically in a more thorough way, such as in the case of the scale-free equation of [14].

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References

[1] Ramond P 1971 Phys. Rev. D 3 2415
[2] Binetruy P 2006 Supersymmetry: Theory, Experiment and Cosmology (Oxford: Oxford University Press)
[3] Cooper F, Khare A and Sukhatme U 2001 Supersymmetry in QM (Singapore: World Scientific)
[4] Miri M-A, Heinrich M, El-Ganainy R and Christodoulides D N 2013 Phys. Rev. Lett. 110 233902
[5] Heinrich M, Miri M-A, Stützer S, El-Ganainy R, Nolte S, Szameit A and Christodoulides D N 2014 Nat. Commun. 5 3698
[6] Miri M-A, Heinrich M and Christodoulides D N 2014 Optica 1 89
[7] Miri M-A, Heinrich M and Christodoulides D N 2013 Phys. Rev. A 87 043819
[8] El-Ganainy R, Li Ge, Khajavikhan M and Christodoulides D N 2015 Phys. Rev. A 92 033818
[9] Kulish P P 1985 Lett. Math. Phys. 10 87
[10] Chowdhury R A and Naskar M 1987 J. Math. Phys. 28 1809
[11] Andrianov A A and Sokolov A V 2003 Nucl. Phys. B 660 25
[12] Bernstein S 2006 Complex Var. Elliptic Equa. 51 429
[13] Zakharov V E and Shabat A B 1972 *Sov. Phys.—JETP* **34** 62
[14] DelRe E, Spinozzi E, Agranat A J and Conti C 2011 *Nat. Photon.* **5** 39
[15] Butler S and Hay M 2015 *AIP Conf. Proc.* **1648** 180006
[16] Huang J et al 2016 *Nat. Nanotechnol.* **11** 60
[17] Tian Y et al 2017 *Adv. Mater.* **29** 1701165