Classification of conservative hydrodynamic chains. Vlasov type kinetic equation, Riemann mapping and the method of symmetric hydrodynamic reductions.

Maxim V. Pavlov\(^1\), Sergei Zykov\(^2,3\)

\(^1\)Department of Mathematical Physics, P.N. Lebedev Physical Institute of Russian Academy of Sciences, Moscow, Leninskij Prospekt, 53
\(^2\)Department of Mathematical Physics, SISSA, Trieste, Italy
\(^3\)Institute of Metal Physics, Ural branch of RAS, Ekaterinburg, Russia

Abstract

A complete classification of integrable conservative hydrodynamic chains is presented. These hydrodynamic chains are written via special coordinates – moments, such that right hand sides of these infinite component systems depend linearly on a discrete independent variable \(k\). All variable coefficients of these hydrodynamic chains can be expressed via modular forms with respect to moment \(A^0\), via hypergeometric functions with respect to moment \(A^1\); they depend polynomially on moment \(A^2\) and linearly on all other higher moments \(A^k\). A dispersionless Lax representation is found. Corresponding collisionless Boltzmann (Vlasov like kinetic) equation is derived. A Riemann mapping is constructed. A generating function of conservation laws and commuting flows is presented.

Contents

1 Introduction 3
2 Zakharov’s moment decomposition 4
3 Canonical variables 13
4 General solution in the “triangular” case 14
5 Conservative hydrodynamic chains 18
6 Egorov’s case 20
hydrodynamic chains, Riemann invariants, symmetric hydrodynamic type systems.

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1 Introduction

In past years (2003 up to now) significant results were obtained in the theory of integrable hydrodynamic chains (see [13], [14], [6], [16], [19]). A first integrable hydrodynamic chain

$$A_t^k = A_{x}^{k+1} + kA^{k-1}A_x^0, \quad k = 0, 1, 2, \ldots$$

(1)

was derived by D. Benney (see [2]) in 1973. An integrability of the Benney hydrodynamic chain can be illustrated by an existence of a generating function of conservation laws (see [2], [11])

$$p_t = \left( \frac{p^2}{2} + A^0 \right)_x,$$

(2)

where a generating function of conservation law densities is given by

$$p = \lambda - \frac{H_0}{\lambda} - \frac{H_1}{\lambda^2} - \frac{H_2}{\lambda^3} - \ldots,$$

(3)

whose all conservation law densities are polynomial functions with respect to moments $A^k$, i.e. $H_0 = A^0, H_1 = A^1, H_2 = A^2 + (A^0)^2, H_3 = A^3 + A^0A^1 \ldots$. It means that hydrodynamic chain (1) also can be written in the conservative form (see, for instance, [17])

$$\partial_t H_0 = \partial_x H_1, \quad \partial_t H_k = \left( H_{k+1} - \frac{1}{2}\sum_{m=0}^{k-1}H_mH_{k-1-m} \right)_x, \quad k = 1, 2, \ldots$$

(4)

We are interested in a description of integrable hydrodynamic chains written in the form (cf. (1))

$$A_t^k = f_1A_{x}^{k+1} + f_0A_x^k + A^{k+1}(s_0A_x^0 + s_1A_x^1) + A^k(r_0A_x^0 + r_1A_x^1) + k[A^{k+1}(w_0A_x^0 + w_1A_x^1 + w_2A_x^2) + A^k(v_0A_x^0 + v_1A_x^1 + v_2A_x^2) + A^{k-1}(u_0A_x^0 + u_1A_x^1 + u_2A_x^2)],$$

(5)

where coefficients $f_i, s_j, r_k$ depend on first two moments $A^0$ and $A^1$ only, while all other coefficients $w_m, v_n, u_p$ depend just on first three moments $A^0, A^1$ and $A^2$.

Recently, two particular cases of hydrodynamic chains (5) were completely investigated. The Hamiltonian hydrodynamic chains (here $H_{1,k} \equiv \partial H_1/\partial A^k, k = 0, 1$)

$$A_t^k = (\alpha + \beta)H_{1,1}A_{x}^{k+1} + \beta H_{1,0}A_x^k + [\alpha(k + 1) + 2\beta]A^{k+1}(H_{1,1})_x + (\alpha k + 2\beta)A^k(H_{1,0})_x$$

are associated with the Kupershmidt Poisson brackets (see [5] and [10]); while the Hamiltonian hydrodynamic chains (here $H_{2,k} \equiv \partial H_2/\partial A^k, k = 0, 1, 2$)

$$A_t^k = 2H_{2,2}A_{x}^{k+1} + H_{2,1}A_x^k + (k + 2)A^{k+1}(H_{2,2})_x + (k + 1)A^k(H_{2,1})_x + kA^{k-1}(H_{2,0})_x$$

are associated with the Kupershmidt–Manin Poisson bracket (see the second part in [6], [8] and [10]). These hydrodynamic chains are integrable if and only if all components of corresponding Haantjes tensors vanish. It means that the corresponding Hamiltonian densities $H_1(A^0, A^1)$ and $H_2(A^0, A^1, A^2)$ cannot be arbitrary. A full list of admissible expressions is given in [5] and [6], respectively.

In a general case, the coefficient $f_1$ in (5) is reducible to the unity, the coefficient $f_0$ can be eliminate by an appropriate change of moments $A^k$; while all other coefficients can
be simplified in the integrable case only. Following the approach based on an existence of first three conservation laws and vanishing of the Haantjes tensor (see the first part in [6]), one can extract the integrable case

\[ A'_t = A^{k+1}_x - k[(A^{k+1}_x + u_0 A^k + u_{-1} A^{k-1}_x)\ln(A^2 + \sigma)]_x - A^k(u_0)_x - \lambda A^{k-1}(u_{-1})_x, \]

(6)

where functions \( u_0, u_{-1}, \) \( \sigma \) satisfy to an overdetermined system in an involution (see (54)). The same result can be obtained by the method of hydrodynamic reductions established by J. Gibbons and S.P. Tsarev in [9] and developed by E.V. Ferapontov and K.R. Khusnutdinova in [4]. In this paper, we utilize the concept of the so-called symmetric hydrodynamic reductions (see [16], [18]). In this case, an existence of a Riemann mapping \( \lambda(q, A^0, A^1, A^2, \ldots) \) connecting the Vlasov type kinetic equation (see [7], [12], [21]) with hydrodynamic chain (6) leads to an overdetermined system in involution (36), (37), (38), (39), (40), (41), whose general solution can be parameterized by hypergeometric functions. Then a generating function of conservation laws can be found in quadratures. Thus, an infinite series of conservation laws densities \( H_k(A^0, A^1, \ldots, A^k) \) allows to rewrite hydrodynamic chain (6) in the conservative form\(^1\) (cf. (4))

\[ \partial_t H_k = \partial_x F_k(H_0, H_1, \ldots, H_{k+1}), \quad k = 0, 1, 2, \ldots \]

(7)

We prove that its first two conservation laws coincide with first two conservation laws found in [6]. In the general case, E.V. Ferapontov and D.G. Marshall found that \( F_2(H_0, H_1, H_2) = \ln H_2 + G(H_0, H_1) \) and functions \( F_0(H_0, H_1), G(H_0, H_1) \) satisfy to another overdetermined system in involution (see [6]). Moreover, we prove that this system in involution (55) is equivalent to system in involution (36), (37), (38), (39), (40), (41). Thus, a complete classification of integrable conservative hydrodynamic chains (7) is given in this paper.

This paper is organized in the following way. In Section 2, symmetric \( 2N \) component hydrodynamic reductions are extracted by virtue of Zakharov’s moment decomposition (see [18], [21]). Such \( 2N \) component hydrodynamic type systems contain \( N \) component symmetric sub-systems, which are still hydrodynamic reductions. These \( N \) component hydrodynamic type systems imply to the Vlasov type kinetic equation. We show that an existence of the Riemann mapping connecting this Vlasov type kinetic equation with hydrodynamic chain (6) allows to select all integrable hydrodynamic chains. In Section 3, canonical coordinates in a moment space are introduced. Then a further investigation simplifies. In Section 4, a special “triangular” case is completely integrated. Three variable coefficients in (6) can be parameterized by solutions of the so-called Halphen–Darboux system (see [1]). In Section 5, a generating function of conservation laws is found. In Conclusion, a generalization of the approach presented in this paper is discussed.

2 Zakharov’s moment decomposition

The moment decomposition approach developed in [18] (see also [16]) is based on a concept of an existence of symmetric hydrodynamic type systems

\[ a'_t = \partial_x F(a; p)|_{p=a}, \quad i = 1, 2, \ldots, N, \]

\(^1\)A problem of a description of integrable hydrodynamic chains (7) was formulated in [13]. A particular and important Egorov’s case \( F_0(H_0, H_1) \equiv H_1 \) was investigated in [14].
which are nothing but hydrodynamic reductions of the hydrodynamic chains, where in all known cases before (see, for instance, [5], [16], [17], [18], [19], [15]) each corresponding moment $A^k$ depends on $N$ functions of a single variable$^2$, i.e.

$$A^k = \sum_{m=0}^{N} f_{mk}(a^m), \quad k = 1, 2, 3, ...$$

Another moment decomposition (introduced by V.E. Zakharov, see [21])

$$A^k = \sum_{m=0}^{N} (a^m)^k b^m$$

also is applicable in all these known cases (see [18]). Benney hydrodynamic chain (1) under this moment decomposition reduces to the $2N$ component hydrodynamic type system

$$a_i^t = \left( \frac{(a^i)^2}{2} + A^0 \right)_x, \quad b_i^t = (a^i b^i)_x,$$

which possesses a formal reduction to the $N$ component case (cf. (2))

$$a_i^t = \left( \frac{(a^i)^2}{2} + A^0 \right)_x,$$

if all field variables $b^k$ vanish, and $A^0$ becomes a function of all rest field variables $a^m$ only. Let us replace $a^i$ by $q(x, t, \lambda)$, where $\lambda$ is a parameter. It means $a^i = q(x, t, \xi^i)$, where $\xi^i$ are arbitrary constants. Then (2)

$$q_t = qq_x + A^0_x$$

by a semi-hodograph transformation $q(x, t, \lambda) \to \lambda(x, t, q)$ reduces to the linear equation

$$\lambda_t = q\lambda_x - \lambda_0 A^0_x,$$

which is known as the Vlasov kinetic equation (see [21]; or the collisionless Boltzmann equation, see [7]). Suppose $\lambda(x, t, q)$ is a function $\lambda(q, A^0, A^1, ...)$, where all moments $A^k(x, t)$ satisfy Benney hydrodynamic chain (1). Since we suppose all moments $A^k$ are independent, one can obtain an infinite series of equations

$$\partial_k \lambda = q^{-k} \partial_0 \lambda, \quad k = 0, 1, 2, ...,$$

where $\partial_k \equiv \partial/\partial A^k$, and $(\partial_0 \equiv \partial/\partial q)$

$$\partial_0 \lambda = \left( q - \sum_{m=0}^{\infty} m A^{m-1} \right)^{-1} \partial_q \lambda.$$  

$^2$Let us emphasize that $N$ is an arbitrary natural number.
A solution of (9) is given by

$$\lambda = B_1(q) \sum_{m=0}^{\infty} \frac{A^m}{q^{m+1}} + B_2(q),$$

(11)

where $B_1(q)$ and $B_2(q)$ are not determined yet functions. However, a substitution (11) into (10) yields $B_1(q) = 1$ and $B_2(q) = q$. Then (11) becomes nothing else but an inverse series to (3). Thus, we conclude that a symptom of an integrability of hydrodynamic chains is an existence of a Riemann mapping $\lambda(q, A^0, A^1, ...)$ connecting with Vlasov type kinetic equation (see below). In this paper, we utilize this property for a classification of integrable hydrodynamic chains.

This moment decomposition approach can be extended on a wide class of hydrodynamic chains (cf. (5))

$$A^k_t = \sum_{n=0}^{K} f_n A^{k+n}_x + \sum_{m=0}^{M} \left( \sum_{n=0}^{K} A^{k+n} q_{nm} + k \sum_{n=-1}^{K} A^{k+n} w_{nm} \right) A^m_x$$

(12)

where $K$ and $M$ are arbitrary natural numbers, all functions $f_i, s_{jk}, w_{lp}$ depend on first $M + 1$ moments (if $K = 1, M = 2, s_{0,2} = 0, s_{1,2} = 0$ and $f_0, f_1, s_{0,0}, s_{0,1}, s_{1,0}, s_{1,1}$ depend just on two first moments $A^0, A^1$; these hydrodynamic chains reduce to (5)). Indeed, (12) reduces to $N$ separate expressions for each index $i$ (let remind that $N$ is arbitrary)

$$(a^i)^k b^i_t + k(a^i)^{k-1} b^i a^i_t = \sum_{n=0}^{K} f_n [(a^i)^{k+n} b^i_x + (k + n)(a^i)^{k+n-1} b^i a^i_x]
+ \sum_{m=0}^{M} \left( \sum_{n=0}^{K} (a^i)^{k+n} b^i q_{nm} + k \sum_{n=-1}^{K} (a^i)^{k+n} b^i w_{nm} \right) A^m_x,$$

due to a substitution (8) in moments equipped by the index $k$ only. Moreover, the above $N$ expressions (due to their linear explicit dependence on a discrete variable $k$) can be split on two parts

$$b^i_t = \sum_{n=0}^{K} f_n [(a^i)^n b^i_x + n(a^i)^{n-1} b^i a^i_x] + \sum_{m=0}^{M} \sum_{n=0}^{K} (a^i)^n b^i s_{nm} A^m_x,$$

$$a^i_t = \sum_{n=0}^{K} f_n \cdot (a^i)^n a^i_x + \sum_{m=0}^{M} \sum_{n=-1}^{K} (a^i)^{n+1} w_{nm} A^m_x.$$  

(13)

As in the previous case, this $2N$ component hydrodynamic type system possesses $N$ component reduction (13), where all moments $A^m$ and all variable coefficients $w_{ik}$ depend on $N$ field variables $a^n$ only.

$\text{It is well known that a general solution of the above linear equation is parameterized by one arbitrary function of a single variable } \lambda(\lambda). \text{ However, in this approach, an existence of any solution is essential.}$
Our main observation successfully utilized in this approach is that integrable hydrodynamic chain (12) is associated with the auxiliary equation

\[ q_t = \sum_{n=0}^{K} f_n q^n q_x + \sum_{m=0}^{M} \sum_{n=-1}^{K} q^{n+1} w_{nm} A_x^m, \quad (14) \]

which obtains due to a formal replacement \( a^i \rightarrow q \) in (13). It means, that equation (14) is compatible with hydrodynamic chain (12), where function \( q \) must depend on moments \( A^k(x, t) \) and the parameter \( \lambda \). The semi-hodograph transformation \( q(x, t, \lambda) \leftrightarrow \lambda(x, t, q) \) reduces (14) to the linear equation

\[ \lambda_t - \sum_{n=0}^{K} f_n q^n \lambda_x + \sum_{m=0}^{M} \sum_{n=-1}^{K} q^{n+1} w_{nm} A_x^m \lambda_q = 0, \quad (15) \]

which we call the Vlasov type kinetic equation (cf. [12]). The function \( \lambda(x, t, q) \) depends on \( x, t \) implicitly via an explicit dependence on moments \( A^k(x, t) \). We shall call hydrodynamic chain (12) integrable if a Riemann mapping \( \lambda(q, A^0, A^1, \ldots) \) connecting Vlasov type kinetic equation (15) with (12) exists.

Examples: Hamiltonian hydrodynamic chains associated with the Kupershmidt–Manin Poisson bracket (see [8]; \( h_n \equiv \partial h/\partial A^n, h_{nm} \equiv \partial^2 h/\partial A^n \partial A^m \))

\[ A^k_t = \sum_{n=0}^{M-1} (n+1)h_{n+1}A_x^{k+n} + \sum_{m=0}^{M} \left( \sum_{n=0}^{M-1} (n+1)A^{k+n}h_{n+1,m} + k \sum_{n=-1}^{M-1} A^{k+n}h_{n+1,m} \right) A_x^m \]

are connected with the Vlasov type kinetic equation

\[ \lambda_t - \sum_{n=0}^{M-1} (n+1)h_{n+1}q^n \lambda_x + \sum_{m=0}^{M} \sum_{n=-1}^{M-1} q^{n+1} h_{n+1,m} A_x^m \lambda_q = 0, \]

where the Hamiltonian is given by \( H = \int h(A^0, A^1, \ldots, A^M) dx \) and (see (12))

\[ f_n = (n+1)h_{n+1}, \ K = M - 1, \ s_{nm} = (n+1)h_{n+1,m}, \ w_{nm} = h_{n+1,m}. \]

Hamiltonian hydrodynamic chains associated with the Kupershmidt Poisson brackets (see [5] and [10])

\[ A^k_t = \sum_{n=0}^{M} (\alpha n + \beta) h_n A_x^{k+n} + \sum_{m=0}^{M} \left( \sum_{n=0}^{M} (\alpha n + 2\beta) A^{k+n} h_{nm} + \alpha k \sum_{n=0}^{M} A^{k+n} h_{nm} \right) A_x^m \]

are connected with the Vlasov type kinetic equation

\[ \lambda_t - \sum_{n=0}^{M} (\alpha n + \beta) h_n q^n \lambda_x + \alpha \sum_{m=0}^{M} \sum_{n=0}^{M} q^{n+1} h_{nm} A_x^m \lambda_q = 0 \]

where the Hamiltonian is given by \( H = \int h(A^0, A^1, \ldots, A^M) dx \) and (see (12))

\[ f_n = (\alpha n + \beta) h_n, \ K = M, \ s_{nm} = (\alpha n + 2\beta) h_{nm}, \ w_{nm} = \alpha h_{nm}, \ w_{-1,m} = 0. \]
Without loss of generality and for simplicity let us consider a hydrodynamic chain written in the form

\[ A^k_t = \sum_{n=0}^1 f_n A^{k+n} + \sum_{m=0}^M \left( \sum_{n=0}^1 A^{k+n} s_{nm} + k \sum_{n=-1}^1 A^{k+n} w_{nm} \right) A^m_x. \]  

(16)

If this hydrodynamic chain is integrable, then also all its higher commuting flows belong to the general class determined by (12) with appropriate choices natural numbers \( K \) and \( M \).

**Lemma:** The coefficient \( f_1 \) can be fixed to the unity by the invertible point transformation \( \tilde{A}^k = (f_1)^k A^k \), then the coefficient \( f_0 \) can be eliminated by the invertible point transformation

\[ \tilde{A}^k = \sum_{m=0}^k \binom{k}{m} (f_0)^{k-m} A^m, \]

where \( \binom{k}{m} \) is a binomial coefficient. If \( \partial_M f_1 = 0 \) and \( \partial_M f_0 = 0 \), then hydrodynamic chain (16) reduces to the canonical form

\[ A^k_t = A^{k+1}_x + \sum_{m=0}^M \left( \sum_{n=0}^1 A^{k+n} s_{nm} + k \sum_{n=-1}^1 A^{k+n} w_{nm} \right) A^m_x, \]

(17)

if \( \partial_M f_1 \neq 0 \) or \( \partial_M f_0 \neq 0 \), then (16) reduces to (17), but \( M \) replaces by \( M + 1 \), correspondingly.

**Proof:** Hydrodynamic chain (16) is associated with the reduced version of (14)

\[ q_t = (f_1 q + f_0) q_x + \sum_{m=0}^M \sum_{n=-1}^1 q^{n+1} w_{nm} A^m_x. \]

Thus, the transformation \( \tilde{q} = f_1 q + f_0 \) reduces the above equation to a more simple case with \( f_1 = 1 \) and \( f_0 = 0 \). Corresponding Zakharov’s moment decomposition (8) transforms accordingly

\[ \tilde{A}^k = \sum_{m=0}^N (f_1 a^m + f_0) b^m. \]

(18)

This is nothing else but a linear combination of aforementioned transformations. This point transformation \( \tilde{A}^0 = A^0 \), \( \tilde{A}^1 = f_1 A^0 + (f_0)^2 \), \( \tilde{A}^2 = (f_1)^2 A^2 + 2 f_0 f_1 A^1 + (f_0)^3 \), \( \ldots \), \( \tilde{A}^M = (f_1)^M A^M + M f_0 (f_1)^{M-1} A^{M-1} + \ldots + (f_0)^{M+1} \ldots \) cannot be inverted to a similar form due to complexity of functions \( f_0(A^0, A^1, \ldots, A^M) \) and \( f_1(A^0, A^1, \ldots, A^M) \). Just higher moments \( A^{M+k}(A^0, A^1, \ldots, A^{M+k}) \) became linear expressions with respect to higher moments \( \tilde{A}^{M+1}, \tilde{A}^{M+2}, \ldots \).

On the other hand, 2N component hydrodynamic type system (13)

\[ b_i^t = (f_1 a^i + f_0) b_x^i + f_1 b_x^i a_x^i + b^i \sum_{m=0}^M \sum_{n=0}^1 (a^i)^n s_{nm} A^m_x, \quad a_i^t = (f_1 a^i + f_0) a_x^i + \sum_{m=0}^M \sum_{n=-1}^1 (a^i)^{n+1} w_{nm} A^m_x. \]

\[ 4 \text{Similar transformations preserving the Kupershmidt–Manin Poisson bracket were considered in [6], but with constant coefficients } f_0 \text{ and } f_1. \]
under the aforementioned transformation $c^i = f_1 a^i + f_0$ reduces to

$$b^i = c^i b_x^i + b^i c_x^i + b^i \sum_{m=0}^{M} \sum_{n=0}^{1} (c^i)^m s_{nm} A_x^m, \quad c^i = c^i c_x^i + \sum_{m=0}^{M} \sum_{n=-1}^{1} (c^i)^{n+1} \tilde{w}_{nm} A_x^m, \quad (19)$$

where

$$s_{1m} = \frac{s_{1m}}{f_1} - \partial_m \ln f_1, \quad s_{0m} = s_{0m} - \frac{f_0}{f_1} s_{1m} + f_0 \partial_m \ln f_1 - \partial_m f_0,$$

$$\bar{w}_{1m} = \frac{w_{1m}}{f_1} - \partial_m \ln f_1, \quad \bar{w}_{0,M+1} = \partial_M f_1, \quad \bar{w}_{-1,M+1} = f_1 \partial_M f_0 - f_0 \partial_M f_1,$$

$$\bar{w}_{0m} = w_{0m} + (1 - \delta_m, 0) \partial_m f_1 - \partial_m f_0 + 2 f_0 \partial_m \ln f_1 - \frac{f_0 w_{1m}}{f_1} \frac{\partial_p \ln f_1}{\partial_p f_0}.$$  

Due to (18), (19) can be written in the final form

$$b^i = c^i b_x^i + b^i c_x^i + b^i \sum_{m=0}^{M} \sum_{n=0}^{1} (c^i)^m s_{nm} \tilde{A}_x^m, \quad c^i = c^i c_x^i + \sum_{m=0}^{M+1} \sum_{n=-1}^{1} (c^i)^{n+1} \tilde{w}_{nm} \tilde{A}_x^m,$$

where coefficients $\tilde{s}_{ij}$ and $\tilde{w}_{kl}$ are expressed via new moments $\tilde{A}^n$. This is nothing else but a hydrodynamic reduction of hydrodynamic chain (17)

$$\tilde{A}_x^k = \tilde{A}_x^{k+1} + \sum_{m=0}^{M+1} \left( \sum_{n=0}^{1} \tilde{A}_x^{k+n} s_{nm} + k \sum_{n=-1}^{1} \tilde{A}_x^{k+n} \tilde{w}_{nm} \right) \tilde{A}_x^m.$$

Thus, Lemma is proved.

**Example:** The remarkable Kupershmidt hydrodynamic chain (see [10], [15])

$$A_x^k = A_x^{k+1} + \beta A_x^0 A_x^k + (k + \gamma) A_x^k A_x^0, \quad k = 0, 1, ...$$

reduces to canonical form (17)

$$\tilde{A}_x^k = \tilde{A}_x^{k+1} + [(1 - \beta) k + \gamma - \beta] \tilde{A}_x^0 \tilde{A}_x^k + \beta k \tilde{A}_x^{k-1} \left( \tilde{A}_x^1 + \frac{\gamma - \beta - 1}{2} (\tilde{A}_x^0)^2 \right), \quad k = 0, 1, ...$$

Thus, we can investigate an integrability of hydrodynamic chain (16) written in a more convenient form (17) instead (16). In such a case, (14) reduces to

$$q_t = q q_x + \sum_{m=0}^{M} \sum_{n=-1}^{1} q^{n+1} w_{nm} A_x^m.$$  

(20)
A consistency of (20) with (17) leads to an infinite set of equations
\[ \partial_{M+k}q = q^{-k} \partial_M q, \quad k = 0, 1, 2, \ldots \] (21)
and \( M + 1 \) equations \((m = 0, 1, 2, \ldots, M)\) reduces by virtue of (21) to
\[ (1-\delta_{m,0})\partial_{m-1}q + \sum_{k=0}^{M-1} \left( \sum_{n=0}^{1} s_{nm} A^{k+n} + k \sum_{n=-1}^{1} w_{nm} A^{k+n} \right) \partial_k q + \Sigma_m \partial_M q = \sum_{n=-1}^{1} w_{nm} q^{n+1} + q \partial_m q, \]
where \( M + 1 \) infinite sums are determined by
\[ \Sigma_m = \sum_{k=0}^{\infty} \left( \sum_{n=0}^{1} s_{nm} A^{M+n+k} + (M+k) \sum_{n=-1}^{1} w_{nm} A^{M+n+k} \right) \frac{1}{q^k}. \] (22)
However, all these infinite sums can be reduced to a sole sum only (see below). A consistency of the Vlasov type kinetic equation (cf. (15), see also (20))
\[ \lambda_t = q\lambda_x - \sum_{m=0}^{M} \sum_{n=-1}^{1} q^{n+1} w_{nm} A^{m} \lambda_q \]
with (17) yields an infinite set of equations (which is equivalent to (21) due to the transformation \( \partial_k q = -\partial_k \lambda / \partial_\lambda \))
\[ \partial_{M+k} \lambda = q^{-k} \partial_M \lambda, \quad k = 0, 1, 2, \ldots, \]
whose solution is given by
\[ \lambda = B_1(q, A^0, A^1, \ldots, A^{M-1})[\Sigma + B_2(q, A^0, A^1, \ldots, A^{M-1})], \]
where
\[ \Sigma = \sum_{p=0}^{\infty} \frac{A^p}{q^{p+1}}, \] (23)
while other \( M + 1 \) equations
\[ (1-\delta_{m,0})\partial_{m-1}\lambda + \sum_{k=0}^{M-1} \left( \sum_{n=0}^{1} s_{nm} A^{k+n} + k \sum_{n=-1}^{1} w_{nm} A^{k+n} \right) \partial_k \lambda + \Sigma_m \partial_M \lambda + \sum_{n=-1}^{1} w_{nm} q^{n+1} \partial_q \lambda = q \partial_m \lambda, \]
reduce to a linear system\(^5\) \( G_m(q, A^0, A^1, \ldots, A^M) \Sigma + Q_m(q, A^0, A^1, \ldots, A^M) = 0 \) due to (22) expresses via (23)
\[ \Sigma_m = \left( \sum_{n=0}^{1} s_{nm} q^{M+n+1} - \sum_{n=-1}^{1} (n+1) w_{nm} q^{M+n+1} \right) \Sigma - \sum_{n=-1}^{1} w_{nm} q^{M+n+2} \partial_q \Sigma \]
\[ + \sum_{n=-1}^{1} w_{nm} \sum_{p=0}^{M+n-1} \frac{n-p}{q^{p-n-M}} A^p - \sum_{n=0}^{1} s_{nm} \sum_{p=0}^{M+n-1} \frac{A^p}{q^{p-n-M}}. \]
\(^5\)This linear system does not contain a part proportional to \( \partial_q \Sigma \), because its corresponding coefficient vanishes automatically.
Since both coefficients $G_m$ and $Q_m$ must vanish independently, a full $2M + 2$ component system can be split on the two $M + 1$ component sub-systems of linear equations

$$q\partial_m \ln B_1 + (\delta_{m,0} - 1)\partial_{m-1} \ln B_1 - \sum_{k=0}^{M-1} \left( \sum_{n=0}^{1} s_{nm} A^{k+n} + k \sum_{n=-1}^{1} w_{nm} A^{k+n} \right) \partial_k \ln B_1$$

$$- \sum_{n=-1}^{1} w_{nm} q^{n+1} \partial_q \ln B_1 = \sum_{n=0}^{1} [s_{nm} - (n + 1)w_{nm}] q^n$$

(24)

and

$$q\partial_mB_2 + (\delta_{m,0} - 1)\partial_{m-1}B_2 - \sum_{k=0}^{M-1} \left( \sum_{n=0}^{1} s_{nm} A^{k+n} + k \sum_{n=-1}^{1} w_{nm} A^{k+n} \right) \partial_k B_2 - \sum_{n=-1}^{1} w_{nm} q^{n+1} \partial_q B_2$$

$$+ \sum_{n=0}^{1} [s_{nm} - (n + 1)w_{nm}] q^n B_2 = A^0(s_{1m} - w_{1m}) - \delta_{m,0}.$$

All derivatives of functions $B_1$ and $B_2$ can be expressed from this linear system with variable coefficients $s_{ij}$ and $w_{kl}$. A consistency of these derivatives leads to an overdetermined system in partial derivatives on $s_{ij}$ and $w_{kl}$ with respect to moments $A^0, A^1, \ldots, A^{M-1}$ and $q$, while a dependence on the highest moment $A^M$ can be found by a straightforward differentiation of linear system (24) written in the matrix form

$$\begin{pmatrix}
q + & \ast & \ldots & \ast & -w_{1,1}q^2 - w_{0,1}q + \\
\ast & q + & \ldots & \ast & -w_{1,2}q^2 - w_{0,2}q + \\
\ldots & \ldots & \ldots & \ldots & \\
\ast & \ast & \ldots & q + & -w_{1,M-1}q^2 - w_{0,M-1}q + \\
\ast & \ast & \ldots & \ast & -w_{1,M}q^2 - w_{0,M}q + \\
\end{pmatrix}
\begin{pmatrix}
\partial_0 \ln B_1 \\
\partial_1 \ln B_1 \\
\ldots \\
\partial_{M-1} \ln B_1 \\
\partial_q \ln B_1 \\
\end{pmatrix} =
\begin{pmatrix}
\tilde{w}_0q + \ast \\
\tilde{w}_1q + \ast \\
\ldots \\
\tilde{w}_{M-1}q + \ast \\
\tilde{w}_Mq + \ast \\
\end{pmatrix},$$

where $\tilde{w}_k = s_{1k} - 2w_{1k}$ and the mark "*" means elements independent on $q$. Indeed, such a differential consequence is given by

$$\begin{pmatrix}
\ast & \ast & \ldots & \ast & -w'_{1,1}q^2 - w'_{0,1}q + \\
\ast & \ast & \ldots & \ast & -w'_{1,2}q^2 - w'_{0,2}q + \\
\ldots & \ldots & \ldots & \ldots & \\
\ast & \ast & \ldots & \ast & -w'_{1,M-1}q^2 - w'_{0,M-1}q + \\
\ast & \ast & \ldots & \ast & -w'_{1,M}q^2 - w'_{0,M}q + \\
\end{pmatrix}
\begin{pmatrix}
\partial_0 \ln B_1 \\
\partial_1 \ln B_1 \\
\ldots \\
\partial_{M-1} \ln B_1 \\
\partial_q \ln B_1 \\
\end{pmatrix} =
\begin{pmatrix}
\tilde{w}'_0q + \ast \\
\tilde{w}'_1q + \ast \\
\ldots \\
\tilde{w}'_{M-1}q + \ast \\
\tilde{w}'_Mq + \ast \\
\end{pmatrix},$$

where the mark "*" means a partial derivative with respect to the moment $A^M$.

**Lemma:** Any row of the above linear system

$$\begin{pmatrix}
* & * & \ldots & * & -w'_{1,m}q^2 - w'_{0,m}q + \\
* & * & \ldots & * & -w'_{1,M}q^2 - w'_{0,M}q + \\
\end{pmatrix}$$

is proportional to the last row from the previous linear system

$$\begin{pmatrix}
* & * & \ldots & * & -w_{1,m}q^2 - w_{0,m}q + \\
* & * & \ldots & * & -w_{1,M}q^2 - w_{0,M}q + \\
\end{pmatrix}.\tag{26}$$

\textsuperscript{6} A determinant of this $(M+1) \times (M+1)$ matrix is a polynomial of degree $M + 2$ with respect to $q$, except some special cases, like $w_{1,M} = 0$, which should be considered separately.
Proof: Indeed, let us consider the linear system

\[
\begin{pmatrix}
q + * & * & * & \cdots & * & -w_{1,1}q^2 - w_{0,1}q + * & -\tilde{w}_0q - * \\
* & q + * & * & \cdots & * & -w_{1,2}q^2 - w_{0,2}q + * & -\tilde{w}_1q - * \\
* & * & \cdots & * & \cdots & \cdots & \cdots \\
* & * & \cdots & * & -w_{1,M-1}q^2 - w_{0,M-1}q + * & -\tilde{w}_{M-1}q - * \\
* & * & \cdots & * & -w_{1,M}q^2 - w_{0,M}q + * & -\tilde{w}_Mq - * \\
* & * & \cdots & * & -w_m'q^2 - w_{0,m}q + * & -\tilde{w}_m'q - * \\
\end{pmatrix}
\begin{pmatrix}
\partial_0 B^1 \\
\partial_1 B^1 \\
\vdots \\
\partial_{M-1} B^1 \\
\partial_q B^1 \\
B^1 \\
\end{pmatrix}
= 0,
\]

determined by the \((M + 2) \times (M + 2)\) matrix incorporating all rows of the original linear system and any row from its differential consequence. A determinant of this matrix equals zero for nontrivial solutions \(B^1\). Thus, the last row (see (25)) must be a linear combination of all other rows. However, most of them \((m = 0, 1, \ldots, M - 1)\) contain an element \(q + *\), which does not exist in first \(M\) entries of this last row. Thus, the last row cannot be expressed via these higher flows except the row with the number \(M\) (see (26)). It means, that all elements of these two rows must be proportional to each other. Lemma is proved.

Thus, the full set of equations is given by \((n = 0, 1, \ldots, M - 2)\)

\[
\frac{\beta_M'}{\beta_M} = \frac{\delta_M'}{\delta_M} = \frac{(\epsilon_{M-1}^n)'}{1 + \epsilon_{M-1}^n} = \frac{(\epsilon_M^n)'}{\epsilon_M^n},
\]

(27)

\[
\frac{\beta_m'}{\beta_M} = \frac{\delta_m'}{\delta_M} = \frac{(\epsilon_{M-1}^m)'}{1 + \epsilon_{M-1}^m} = \frac{(\epsilon_m^n)'}{\epsilon_m^n}, \quad m = 0, 1, \ldots, M - 1,
\]

(28)

where

\[
\beta_m = s_{0,m} - w_{0,m} + (s_{1,m} - 2w_{1,m})q,
\]

\[
\epsilon_m^n = nw_{-1,m}A^{n-1} + (s_{0,m} + nw_{0,m})A^n + (s_{1,m} + nw_{1,m})A^{n+1},
\]

\[
\delta_m = w_{-1,m} + w_{0,m}q + w_{1,m}q^2.
\]

All these equations can be subsequently integrated. Indeed, the first ratio in (27)

\[
\frac{\beta_M'}{\beta_M} = \frac{\delta_M'}{\delta_M}
\]

is nothing else but a cubic polynomial with respect to \(q\). Since \(q\) is arbitrary, all four coefficients must vanish independently. A general solution of corresponding four ordinary differential equations (with respect to \(A^M\) only) is given by

\[
s_{0,M} = (r_0 - M + 1)w_{1,M}, \quad s_{1,M} = (r_1 - M + 1)w_{1,M},
\]

\[
w_{0,M} = u_0w_{1,M}, \quad w_{-1,M} = u_{-1}w_{1,M},
\]

(29)

where functions \(r_0, u_0, r_1, u_{-1}\) depend on first \(M\) moments \(A^0, A^1, \ldots, A^{M-1}\). An integration of the second ratio in (27)

\[
\frac{\delta_M'}{\delta_M} = \frac{(\epsilon_{M-1}^n)'}{1 + \epsilon_{M-1}^n}
\]
leads to
\[ w_{1,M} = -\frac{1}{\sigma + A_M r_1}, \]  
(30)
where the function \( \sigma \) depends on first \( M \) moments \( A^0, A^1, \ldots, A^{M-1} \). An integration of the first ratio in (28)
\[ \frac{\beta'_m}{\beta_M} = \frac{\delta'_m}{\delta_M} \]
leads to
\[ w_{-1,m} = u_{-1} w_{1,m} + \gamma_{-1,m}, \quad w_{0,m} = u_0 w_{1,m} + \gamma_{0,m}, \]  
(31)
\[ s_{1,m} = (r_1 - M + 1) w_{1,m} + \rho_{1,m}, \quad s_{0,m} = (r_0 - M + 1) w_{1,m} + \rho_{0,m}, \]
where functions \( \gamma_{-1,m}, \gamma_{0,m}, \rho_{0,m}, \rho_{1,m} \) depend on first \( M \) moments \( A^0, A^1, \ldots, A^{M-1} \). An integration of the second ratio in (28)
\[ \frac{\delta'_m}{\delta_M} = \frac{(\epsilon_{M-1})'}{1 + \epsilon_{M-1}} \]
leads to
\[ w_{1,m} = \frac{\omega_m - A_M \rho_{1,m}}{\sigma + A_M r_1}, \]  
(32)
where functions \( \omega_m \) depend on first \( M \) moments \( A^0, A^1, \ldots, A^{M-1} \). It is easy to see, that all other ratios in (27) and (28) are fulfilled by virtue of (29), (30), (31), (32).

In the next Section, a more deep analysis is presented for hydrodynamic chain (5) written in form (17).

### 3 Canonical variables

The function \( B_1 \) depends on first \( M \) moments \( A^0, A^1, A^{M-1} \) only, but coefficients of linear system (24) depend also on \( A^M \) explicitly via (29), (30), (31), (32). Thus, each derivative of \( \ln B_1 \) can be expressed as a ratio of two polynomials with respect to \( q \). In a general case (if \( w_{1,M} \neq 0 \)), the common denominator is a polynomial of a degree \( M + 2 \). All numerators are polynomials of the same degree, except a numerator of derivative \( \ln B_1 \) with respect to \( q \). Its degree is \( M + 1 \). Let us introduce roots \( q_k(A^0, A^1, \ldots, A^{M-1}) \) of this polynomial as **basic field variables** for further computations. In such a case,
\[ \partial_q \ln B_1 = -\sum_{m=1}^{M+2} \frac{\alpha_m}{q - q_m}, \]  
(33)
where \( \alpha_m(A^0, A^1, \ldots, A^{M-1}) \) are not yet determined functions. It means, that
\[ B_1 = \alpha_0 \prod_{m=1}^{M+2} (q - q_m)^{-\alpha_m}, \]  
(34)
where \( \alpha_0(A^0, A^1, \ldots, A^{M-1}) \) is not yet determined function. A substitution (34) back to the first derivative of \( \ln B_1 \) with respect to \( q \) allows to express few (not all) variable
coefficients \((r_0, u_0, r_1, u_1, \sigma, \gamma_{-1,m}, \gamma_{0,m}, \rho_{0,m}, \rho_{1,m}, \omega_m)\), see the previous Section) via new field variables \(q_k\). Moreover, \(\alpha_0\) and all other \(\alpha_m\) must be constant parameters (this is a consequence of an absence of logarithmic terms in derivatives of \(\ln B_1\) with respect to moments \(A^k\); \(r_1\) is constant due to the constraint

\[
\sum_{m=1}^{M+2} \alpha_m = \frac{1}{r_1} - M - 1,
\]

following from comparison of r.h.s. in (33) with a corresponding expression from linear system (24). Without loss of generality, one can fix \(\alpha_0\) on the unity. The compatibility conditions \(\partial_k(\partial_q \ln B_1) = \partial_q(\partial_k \ln B_1), \partial_k(\partial_n \ln B_1) = \partial_n(\partial_k \ln B_1)\) imply to explicit relationships between some coefficients as well as dependencies \(\partial_k q_n\) via \(q_m\) and rest of initial coefficients. Finally, the compatibility conditions \(\partial_k(\partial_m q_n) = \partial_k(\partial_m q_n)\) should lead to a parametrization of all coefficients via \(q_m\) and their derivatives with respect to moments \(A^0, A^1, ..., A^{M-1}\).

However, this is not precisely true. Hydrodynamic chain (16) possesses a large class of invertible transformations, allowing to significantly reduce a number of distinguish coefficients. For instance, hydrodynamic chain (5) contains 15 coefficients, while its integrable version (6) contains just 3 coefficients. It means, that transformation (18) is necessary but not sufficient for a most appropriate choice of reduced number of coefficients for a satisfactory investigation. To avoid complexity of this problem, in this Section, we restrict our consideration on the case \(M = 2\) associated with hydrodynamic chain (5).

### 4 General solution in the “triangular” case

In this Section, we restrict our consideration on a most important case determined by the choice \(r_0 = 1\) and \(r_1 = 1\) (see (5) and comments to (12), i.e. the restrictions \(s_{0,2} = 0, s_{1,2} = 0\), i.e. (see (35))

\[
\sum_{m=1}^{4} \alpha_m = -2.
\]

Moreover, we essentially can simplify further computations fixing all \(\rho_{kn} = 0\), where \(k,n = 0,1\). Nevertheless, this is not a particular case. A complete description of conservative integrable hydrodynamic chains (7) is given by (6). In general case (17), an infinite set of conservation laws can be written in the form (cf. (7))

\[
\partial_t H_k = \partial_x F_k(H_0, H_1, ..., H_M), \quad k = 0, 1, 2, ..., M - 1,
\]

\[
\partial_t H_{M+k} = \partial_x F_{M+k}(H_0, H_1, ..., H_{M+k+1}), \quad k = 0, 1, 2, ...
\]

Just hydrodynamic chain (5) possesses an infinite set of conservation laws given by (7). Such hydrodynamic chains we call “triangular” in comparison with all other hydrodynamic chains (17), whose conservation laws possess a deviation from this triangular case, i.e. first \(M\) conservation law fluxes depend simultaneously on first \(M\) conservation law densities \(H_k\).
As it was mentioned in the previous Section, the compatibility conditions \( \partial_k(\partial_q \ln B_1) = \partial_q(\partial_k \ln B_1), \partial_k(\partial_n \ln B_1) = \partial_n(\partial_k \ln B_1) \) lead to the system in involution

\[
\partial_1 q_k = \frac{q_k^2 + u_0 q_k + u_{-1}}{S}, \quad \partial_1 S = \sum_{m=1}^{4} (2\alpha_m + 1)q_m, \quad (36)
\]

\[
\partial_1 u_{-1} = \frac{1}{S} \left( u_{-1} \sum_{m=1}^{4} (\alpha_m + 1)q_m - \prod_{k=1}^{3} q_k \sum_{m=1}^{4} \frac{\alpha_m + 1}{q_m} \right), \quad (37)
\]

\[
\partial_0 q_k = \frac{(q_k^2 + u_0 q_k + u_{-1}) (\partial_0 S - u_{-1})}{q_k S} - \frac{\partial_0 u_{-1}}{q_k} - \partial_0 u_0, \quad (38)
\]

\[
\partial_0 S = - \sum_{m<k} (\alpha_k + \alpha_m + 1)q_m q_k, \quad (39)
\]

\[
\partial_0 u_{-1} = \frac{1}{S} \left( \prod_{k=1}^{3} q_k - u_{-1} \sum_{m<k} (\alpha_k + \alpha_m + 1)q_m q_k - (u_{-1})^2 \right), \quad (40)
\]

where

\[
u_0 = \sum_{m=1}^{4} \alpha_m q_m, \quad S = A_0 u_{-1} + A_1 u_0 - \sigma, \quad (41)
\]

and 6 variable coefficients are connected with 3 others by (here \( k = 0, 1 \) only)

\[
\gamma_{0,k} = \partial_k u_0, \quad \gamma_{-1,k} = \partial_k u_{-1}, \quad \omega_k = -\partial_k \sigma.
\]

In this case, all coefficients \((29), (30), (31), (32)\) significantly reduce, then hydrodynamic chain \((5)\) transforms to a more compact form given by \((6)\).

Let us introduce the auxiliary functions

\[
\tilde{q}_k = \frac{q_k - q_4}{S}, \quad k = 1, 2, 3.
\]

Then equations \((36)\) reduce to the form

\[
\partial_1 \tilde{q}_k = \tilde{q}_k \left( \tilde{q}_k - \sum_{m=1}^{3} (\alpha_m + 1)\tilde{q}_m \right), \quad k = 1, 2, 3, \quad (42)
\]

\[
\partial_1 \ln S = \sum_{m=1}^{3} (2\alpha_m + 1)\tilde{q}_m, \quad u_{-1} = S \left( \partial_1 q_4 - q_4 \sum_{m=1}^{3} \alpha_m \tilde{q}_m \right) + q_4^2. \quad (43)
\]

A substitution \( u_{-1} \) from the above system into \((37)\) leads to the simple equation of the second order

\[
\partial_1^2 q_4 + (\alpha_4 + 1)\tilde{q}_1 \tilde{q}_2 \tilde{q}_3 S = 0. \quad (44)
\]

Let us introduce an intermediate function \( z = \partial_1 q_4 \) and four functions \( c_k(A^0), k = 0, 1, 2, 3. \)
Lemma: A general solution of system (42) is given by
\[ \tilde{q}_k = -\partial_1 \ln(z - c_k), \quad k = 1, 2, 3, \] (45)
where
\[ \partial_1 z = c_0 \prod_{m=1}^{3} (z - c_m)^{\alpha_m + 1}, \quad S = \frac{c_0^{-2}}{\alpha_4 + 1} \prod_{m=1}^{3} (z - c_m)^{-(2\alpha_m + 1)} . \] (46)

Proof: A substitution above formulas into (42), (43), (44) yields identities. A substitution (45) and (46) in (38) and (39) determines
\[ \partial_0 q_4 = \frac{c_0^{-1}}{\alpha_4 + 1} \prod_{m=1}^{3} (z - c_m)^{-\alpha_m} - z q_4, \]
where \( c_0(A^0) \) can be found by quadratures
\[ (\ln c_0)' = - \sum_{m=1}^{3} (\alpha_m + \alpha_4 + 1) c_m, \]
while all other \( c_k(A^k) \) satisfy a new modification of well-known generalized Darboux–Halphen system (see detail in [1])
\begin{align*}
    c'_1 &= \frac{\alpha_1}{\alpha_4 + 1} [c_1 (c_2 + c_3) - c_2 c_3] - \frac{\alpha_1 + \alpha_4 + 1}{\alpha_4 + 1} c_1^2, \\
    c'_2 &= \frac{\alpha_2}{\alpha_4 + 1} [c_2 (c_1 + c_3) - c_1 c_3] - \frac{\alpha_2 + \alpha_4 + 1}{\alpha_4 + 1} c_2^2, \\
    c'_3 &= \frac{\alpha_3}{\alpha_4 + 1} [c_3 (c_1 + c_2) - c_1 c_2] - \frac{\alpha_3 + \alpha_4 + 1}{\alpha_4 + 1} c_3^2. 
\end{align*} \quad (47)

Remark: A sole function
\[ s(A^0) = \frac{c_2 - c_3}{c_1 - c_3} \] (48)
satisfies the so-called Schwarzian equation (see [1])
\[ \frac{s'''}{s'} - \frac{3(s'')^2}{2(s')^2} = \left( 2 \frac{\alpha_1 \alpha_2 + \alpha_3 a_4}{s(s-1)} - \frac{(\alpha_2 + \alpha_4)(\alpha_1 + \alpha_3)}{s^2} - \frac{(\alpha_2 + \alpha_3)(\alpha_1 + \alpha_4)}{(s-1)^2} \right) \frac{(s')^2}{2}. \]

Then a solution of system (47) is given by
\[ c_1 = -\frac{1}{2} \left( \ln \left( \frac{s^{\alpha_1+\alpha_3} s'}{(s-1)^{\alpha_1-\alpha_4}} \right) \right)', \quad c_2 = -\frac{1}{2} \left( \ln \left( \frac{(s-1)^{\alpha_2+\alpha_3} s'}{s^{\alpha_2-\alpha_4}} \right) \right)', \quad c_3 = -\frac{1}{2} \left( \ln \left( s^{\alpha_1+\alpha_3} (s-1)^{\alpha_2+\alpha_3} s' \right) \right)'. \]

Under the simple linear transformation
\begin{align*}
    c_1 &= (1 + \alpha_1 + \alpha_3) \omega_1 - (1 + \alpha_3 + \alpha_4) \omega_2 + (1 - \alpha_1 + \alpha_4) \omega_3, \\
    c_2 &= (1 + \alpha_2 + \alpha_4) \omega_1 - (1 + \alpha_3 + \alpha_4) \omega_2 + (1 + \alpha_2 + \alpha_3) \omega_3, \\
    c_3 &= (1 + \alpha_1 + \alpha_3) \omega_1 + (1 - \alpha_3 + \alpha_4) \omega_2 + (1 + \alpha_2 + \alpha_3) \omega_3, 
\end{align*}
the above formulas reduce to the form derived in [1], i.e.

\[ \omega_1 = \omega_2 \omega_3 - \omega_1 (\omega_2 + \omega_3) + \omega^2, \]
\[ \omega_2 = \omega_1 \omega_3 - \omega_2 (\omega_1 + \omega_3) + \omega^2, \]
\[ \omega_3 = \omega_1 \omega_2 - \omega_3 (\omega_1 + \omega_2) + \omega^2, \]

where

\[ \omega^2 = \beta_1^2 (\omega_1 - \omega_2) (\omega_3 - \omega_1) + \beta_2^2 (\omega_2 - \omega_3) (\omega_1 - \omega_2) + \beta_3^2 (\omega_3 - \omega_1) (\omega_2 - \omega_3), \]
\[ \omega_1 = -\frac{1}{2} \left( \ln \frac{s'}{s} - 1 \right), \quad \omega_2 = -\frac{1}{2} \left( \ln \frac{s'}{s(s-1)} \right), \quad \omega_3 = -\frac{1}{2} \left( \ln \frac{s'}{s} \right), \]
\[ \alpha_1 = \frac{1}{2} (\beta_1 + \beta_2 - \beta_3 - 1), \quad \alpha_2 = \frac{1}{2} (-\beta_1 + \beta_2 + \beta_3 - 1), \quad \alpha_3 = \frac{1}{2} (\beta_1 - \beta_2 + \beta_3 - 1) \]

and (48) reduces to

\[ s(A^0) = \frac{\omega_2 - \omega_1}{\omega_2 - \omega_3}. \]

The function \( q_4(A^0, z) \) can be found by the quadrature

\[ dq_4 = \frac{c_0^{-1}}{\omega_1} \prod_{m=1}^{3} (z - c_m)^{-\alpha_m - 1} dz + \frac{c_0^{-1}}{\omega_2} \prod_{m=1}^{3} (z - c_m)^{-\alpha_m - 1} P_2(z) dA^0, \]

where \( P_2(z) \) is a polynomial in \( z \) of the second degree, i.e.

\[ P_2(z) = z^2 \prod_{m=1}^{3} (\alpha_m + \alpha_4 + 1) c_m + z \prod_{n=1}^{3} c_n \sum_{m=1}^{3} \frac{\alpha_m + 1}{c_m} - \prod_{n=1}^{3} c_n. \]

Then the first moment \( A_1(A^0, z) \) is determined by another quadrature

\[ dA_1 = \frac{c_0^{-1}}{\omega_1} \prod_{m=1}^{3} (z - c_m)^{-\alpha_m - 1} dz + \left( \frac{c_0^{-1}}{\omega_2} \prod_{m=1}^{3} (z - c_m)^{-\alpha_m - 1} G_2(z) + q_4 \right) dA^0, \]

where \( G_2(z) \) is a polynomial in \( z \) of the second degree, i.e.

\[ G_2(z) = -z^2 + z \prod_{m=1}^{3} (\alpha_m + \alpha_4 + 2) c_m + \prod_{n=1}^{3} c_n \sum_{m=1}^{3} \frac{\alpha_m}{c_m}. \]

Thus, all functions \( u_0, u_{-1} \) and \( \sigma \) (see (41), the second formula in (43) and the second formula in (46)) are expressed via above implicit dependencies \( z(A^0, A^1), q_4(A^0, A^1), \) i.e.

\[ \sigma = A^0 u_{-1} + A^1 u_0 - \frac{c_0^{-2}}{\omega_1} \prod_{m=1}^{3} (z - c_m)^{-2(\alpha_m + 1)}, \]
\[ u_0 = -\frac{1}{(\alpha_4 + 1)} c_0 \sum_{m=1}^{3} \frac{\alpha_m}{z - c_m} \prod_{k=1}^{3} (z - c_k)^{-\alpha_k} - 2q_4, \]
\[ u_{-1} = \frac{z}{(\alpha_4 + 1)} c_0 \prod_{k=1}^{3} (z - c_k)^{-2(\alpha_k + 1)} + \frac{q_4}{(\alpha_4 + 1)} c_0 \sum_{m=1}^{3} \frac{\alpha_m}{z - c_m} \prod_{k=1}^{3} (z - c_k)^{-\alpha_k} + q_4^2. \]
5 Conservative hydrodynamic chains

We believe that integrable hydrodynamic chain (6) must possess an infinite set of conservation laws (7). It means, that an infinite series of invertible triangular transformations $H_k(A^0, A^1, ..., A^k)$ can be found. In this Section, we present a generating function of conservation laws and prove an equivalence of the system in involution derived by E.V. Ferapontov and D.G. Marshall for conservative hydrodynamic chains (7) with system in involution (36)–(40) derived in the previous Section.

Theorem: Integrable hydrodynamic chain (6) possesses a generating function of conservation laws

$$\partial_t p(q, A^0, A^1) = \partial_x Q(q, A^0, A^1).$$ (49)

Proof: Integrable hydrodynamic chain (6) is associated with the Vlasov type kinetic equation (see (17))

$$q_t = q q_x - (q^2 + u_0q + u_{-1}) [\log(A^2 + \sigma)]_x + q(u_0)_x + (u_{-1})_x.$$ 

First two equations of hydrodynamic chain (6) are given by

$$A_t^0 = A_x^1, \quad A_t^1 = A_x^2 - [(A^2 + u_0A^1 + u_{-1}A^0)[\ln(A^2 + \sigma)]_x - A^1(u_0)_x - A^0(u_{-1})_x].$$

Since r.h.s. of $q_t$ and $A_t^1$ contain derivative $A^2_x$, formally $Q$ should depend on $A^2$. Differentiation (49) with respect to $x$ and $t$

$$\partial_q p \cdot q_t + \partial_0 p \cdot A_t^0 + \partial_1 p \cdot A_t^1 = \partial_q Q \cdot q_x + \partial_0 Q \cdot A_x^0 + \partial_1 Q \cdot A_x^1 + \partial_2 Q \cdot A_x^2$$

leads to

$$\partial_q Q = q \partial_q p,$$

$$\partial_0 Q = \left( \partial_0 u_{-1} + q \partial_0 u_0 - \frac{(q^2 + qu_0 + u_{-1}) \partial_0 \sigma}{\sigma + A_2} \right) \partial_q p + \left( A_0 \partial_0 u_{-1} + A_1 \partial_0 u_0 - \frac{(A_2 + A_0u_{-1} + A_1u_0) \partial_0 \sigma}{\sigma + A_2} \right) \partial_1 p,$$

$$\partial_1 Q = \partial_0 p + \left( \partial_1 u_{-1} + q \partial_1 u_0 - \frac{(q^2 + qu_0 + u_{-1}) \partial_1 \sigma}{\sigma + A_2} \right) \partial_q p + \left( A_0 \partial_1 u_{-1} + A_1 \partial_1 u_0 - \frac{(A_2 + A_0u_{-1} + A_1u_0) \partial_1 \sigma}{\sigma + A_2} \right) \partial_1 p,$$

$$\partial_2 Q = \frac{\sigma - A_0 u_{-1} - A_1 u_0}{\sigma + A_2} \partial_1 p - \frac{q^2 + qu_0 + u_{-1}}{\sigma + A_2} \partial_q p.$$ 

If $\partial_2 Q = 0$, then

$$\partial_1 p = \frac{q^2 + qu_0 + u_{-1}}{\sigma - A_0 u_{-1} - A_1 u_0} \partial_q p.$$ (50)

---

7 as usual, we are looking for a general solution. It means, that all $A^k_x$ are considered independently. Thus, corresponding coefficients must vanish separately.
and all other above expressions simplify to the form

\begin{align*}
\partial_q Q &= q \partial_q p, \\
\partial_0 Q &= \left( \frac{q^2 + qu_0 + u_{-1}}{S} \left( u_{-1} - \partial_0 S \right) + \partial_0 u_{-1} + q \partial_0 u_0 \right) \partial_q p,
\end{align*}

\begin{align*}
\partial_1 Q &= \partial_0 p + \left( \frac{q^2 + qu_0 + u_{-1}}{S} \left( u_0 - \partial_1 S \right) + \partial_1 u_{-1} + q \partial_1 u_0 \right) \partial_q p.
\end{align*}

A compatibility conditions \( \partial_1 (\partial_0 Q) = \partial_0 (\partial_1 Q) \), \( \partial_1 (\partial_q Q) = \partial_q (\partial_1 Q) \), \( \partial_q (\partial_0 Q) = \partial_0 (\partial_q Q) \) lead to three equations containing four second order derivatives \( \partial_0 q p, \partial qq p, \partial_0 q p, \partial_00 p \) only. Taking into account (50), the derivative \( \partial_1 q p \) is proportional to \( \partial qq p \), and all other three derivatives \( \partial_0 q p, \partial qq p, \partial_00 p \) can be expressed. Moreover, a direct further computation leads to the correspondence

\[ \partial q p = \frac{1}{B_1}. \]  

(51)

Thus, (see (34))

\[ \partial_q p = \prod_{m=1}^{4} (q - q_m)^{\alpha_m}. \]

(52)

The generating function of conservation law densities can be found in two quadratures (see (50) and (51))

\[ dp = \prod_{m=1}^{4} (q - q_m)^{-\alpha_m} dq - \frac{q^2 + qu_0 + u_{-1}}{S} \prod_{m=1}^{4} (q - q_m)^{-\alpha_m} dA^1 + (\partial_0 p) dA^0, \]

where \( \partial_0 p \) also is determined by corresponding second derivatives

\[ d(\partial_0 p) = (\partial_0 q p) dq + (\partial_01 p) dA^1 + (\partial_00 p) dA^0. \]

Nevertheless, an infinite series of conservation law densities can be found directly from (52).

In contrary with the above approach, all conservation laws can be found iteratively. The zeroth conservation law is given by the zeroth equation

\[ \partial_t H_0 = \partial_x F_0(H_0, H_1), \]

such that \( A^0 = H_0 \) and \( A^1 = F_0(H_0, H_1) \) (see integrable hydrodynamic chain (6)). Let us introduce an intermediate notation \( h = \partial_0 H_1 \).

**Lemma:** Integrable hydrodynamic chain (6) possesses first conservation law

\[ \partial_t H_1 = \partial_x \ln H_2 + G(H_0, H_1), \]

such that the second conservation law density

\[ H_2 = \frac{1}{A^2 + \sigma}, \]

the first conservation law density \( H_1 \) can be found by two quadratures

\[ dH_1 = hdA^0 + \frac{1}{S} dA^1, \quad dh = \left( \partial_0 \frac{u_0}{S} - \partial_1 \frac{u_{-1}}{S} \right) dA^0 + \left( \partial_1 \frac{1}{S} \right) dA^1, \]  

(53)
and the function $G(H_0, H_1)$ is determined by the quadrature

$$
dG = \frac{\partial_0 S - u_{-1}}{S} dA^0 + \left( h + \frac{\partial_1 S - u_0}{S} \right) dA^1. $$

**Proof:** can be obtained by a straightforward computation.

The system in involution on third derivatives of functions $F(H_0, H_1)$ and $G(H_0, H_1)$ was derived (see (55) in the Appendix) in paper [6].

**Theorem:** These functions $F(H_0, H_1)$ and $G(H_0, H_1)$ can be found in quadratures

$$
dG = \left( \frac{\partial_0 S - u_{-1}}{S} + u_0 h - Sh^2 \right) dH_0 + \left( \tilde{\partial}_1 \ln S - u_0 + hS \right) dH_1,
$$

$$
dF = -hSdH_0 + SdH_1,$$

where $\tilde{\partial}_0 \equiv \partial_{H_0}$, $\tilde{\partial}_1 \equiv \partial_{H_1}$. An inverse transformation is given by

$$
u_0 = \frac{\tilde{\partial}_{1,1}F}{\tilde{\partial}_1F} - \tilde{\partial}_0 F - \tilde{\partial}_1 G, \quad S = \tilde{\partial}_1 F, \quad u_{-1} = \tilde{\partial}_{0,1} F - \frac{\tilde{\partial}_0 F \tilde{\partial}_{1,1} F}{\tilde{\partial}_1 F} + \tilde{\partial}_0 F \tilde{\partial}_1 G - \tilde{\partial}_1 F \tilde{\partial}_0 G.$$

**Proof:** can be obtained by a straightforward computation.

**Remark:** All higher commuting flows belong to (12) in a general case. Indeed, a first commuting flow to hydrodynamic chain (6) is given by

$$
A_y^k = [(A^{k+2} + A^{k+1} u_0 + A^k u_{-1}) H_2]_x + k \left( (A^{k+2} + A^{k+1} u_0 + A^k u_{-1}) (H_2)_x - (A^{k+1} + A^k u_0 + A^{k-1} u_{-1}) \left( \frac{H_3}{H_2} - \frac{H_2 u_0}{2} \right)_x + \frac{1}{2} \left[ A^k (u_{0,H_1} \partial_x + \partial_x u_{0,H_1}) + A^{k-1} (u_{-1,H_1} \partial_x + \partial_x u_{-1,H_1}) \right] H_2 \right)
$$

where the third conservation law density is determined by (here $u_{0,H_1} = \partial_{H_1} u_0$, $u_{-1,H_1} = \partial_{H_1} u_{-1}$, $\sigma_{H_1} = \partial_{H_1} \sigma$)

$$
H_3 = (A^3 - \sigma u_0 + A^4 u_{-1} + \frac{1}{2} \sigma_{H_1}) H_2^3 + \frac{3}{2} u_0 H_2^2.
$$

A compatibility condition $(\lambda_t)_y = (\lambda_y)_t$ of corresponding Vlasov type kinetic equations (15) (i.e. $K = 1$ and $K = 2$, respectively) leads to some 2+1 dimensional quasilinear equation of the second order (a general classification was presented in [3]), which will be considered in a separate paper.

### 6 Egorov’s case

A most important and interesting case is the Egorov hydrodynamic chain (see [14]) selected by the simple choice $H_1 = A^1$ (see (53)). In such a case, $S = 1$, then all $\alpha_k = -1/2$ and general hydrodynamic chain (6) reduces to the form

$$
A_t^k = A_x^{k+1} - k[(A^{k+1} + A^k \partial_1 F + A^{k-1} \partial_0 F)[\ln(A^2 + A^1 \partial_1 F + A^0 \partial_0 F - 1)]_x - A^k (\partial_1 F)_x - A^{k-1}(\partial_0 F)_x],
$$
where the function $F$ is given by

$$F = \frac{1}{4} \int \eta(A^0) dA^0 + \ln \theta_1(A^1, A^0).$$

Here $\eta(A^0)$ is a solution of the Chazy equation

$$\eta''' = 3\eta'^2 - 2\eta''$$

and the Jacobi theta-function

$$\theta_1(A^1, A^0) = 2 \sum_{n=0}^{\infty} (-1)^n e^{-(n+1/2)^2 A^0} \sin[(2n + 1) A^1]$$

is connected with the above solution of the Chazy equation via an involutive system (see [14])

$$\partial_1 \theta_1 = -\mu \theta_1, \quad \partial_0 \theta_1 = \frac{1}{4} (\mu^2 - l) \theta_1,$$

$$\partial_1 \mu = l, \quad \partial_0 \mu = \frac{1}{4} \sqrt{4l^3 - 4\eta l^2 - 8\eta' l - \frac{8}{3} \eta''} - \frac{1}{2} \mu l,$$

$$\partial_1 l = \sqrt{4l^3 - 4\eta l^2 - 8\eta' l - \frac{8}{3} \eta''}, \quad \partial_0 l = l^2 - \eta l - \eta' - \frac{1}{2} \mu \sqrt{4l^3 - 4\eta l^2 - 8\eta' l - \frac{8}{3} \eta''}.$$

### 7 Conclusion

The crucial observation made in [18] is that a substitution of Zakharov moment decomposition (8) is applicable for hydrodynamic chains, whose r.h.s. expressions depend linearly on a discrete variable $k$ and contain a finite number of common variable coefficients (see (12)).

In comparison with approaches established earlier (see [4], [6], [9]), the method presented in this paper is not universal but most effective. A complete classification of conservative hydrodynamic chains is given by virtue of their representation in a special form (19). All conservation law densities $H_m$ can be expressed explicitly via moments $A^k$; all fluxes of corresponding conservation laws can be expressed explicitly via $H_m$; all commuting flows can be constructed explicitly (and their conservation laws); infinitely many hydrodynamic reductions can be extracted. Thus, infinitely many particular solutions to integrable hydrodynamic chains (19) can be presented (by the generalized hodograph method, see [20]).

### 8 Appendix

The system in involution for functions $u_0$, $u_{-1}$ and $S$ describing a family of integrable hydrodynamic chains (6) possesses a general solution parameterized by 9 arbitrary con-
\[ \begin{align*}
\partial_{1,1}u_{-1} &= \partial_1 u_{-1} \cdot \partial_1 \ln S + 2 \frac{\partial_0 u_{-1} - u_{-1} \partial_1 u_0}{S}, \\
\partial_{1,1}u_0 &= \partial_1 u_0 \cdot \partial_1 \ln S + 2 \frac{\partial_1 u_{-1} + \partial_0 u_0 - u_0 \partial_1 u_0}{S}, \\
\partial_{0,1}u_{-1} &= \partial_1 u_{-1} \cdot \partial_0 \ln S - 2 \frac{u_{-1} \partial_1 u_{-1} + \partial_0 u_0 - u_0 \partial_{1,1} u_{-1}}{S}, \\
\partial_{0,1}u_0 &= \partial_1 u_0 \cdot \partial_0 \ln S + 2 \frac{\partial_0 u_{-1} - u_{-1} \partial_1 u_0}{S}, \\
\partial_{0,1}S &= \partial_1 u_{-1} - \partial_0 u_0 + \frac{\partial_0 S - 2 u_{-1} \partial_1 S + u_0 \partial_0 S}{S}, \\
\partial_{1,1}S &= \frac{\partial^2 S}{S} - \frac{u_0 \partial_1 S + 2 \partial_0 S}{S}, \quad (54)
\end{align*} \]

\[ \begin{align*}
\partial_{0,0}S &= \frac{2 \partial_0 S}{S} + \frac{(u_0^2 - 2 u_{-1}) \partial_0 S}{S} + u_{-1} \frac{\partial_1 S}{S} + (\partial_1 u_{-1} - \partial_0 u_0) u_0 \\
&\quad + \left( \partial_0 u_0 - \partial_1 u_{-1} - \frac{u_0}{S} \partial_0 S - \frac{u_{-1} u_0}{S} \right) \partial_1 S, \\
\partial_{0,0}u_0 &= (\partial_0 u_0 - \partial_1 u_{-1}) \partial_1 u_0 + \frac{u_{-1} \partial_1 u_0 - 2 \partial_0 u_{-1}}{S} \partial_1 S \\
&\quad - 2 \frac{u_{-1} \partial_1 u_{-1} + \partial_0 u_0 - u_0 \partial_{1,1} u_{-1}}{S} + 2 \frac{(\partial_1 u_{-1} + \partial_0 u_0) - u_0 \partial_1 u_0}{S} \partial_0 S, \\
\partial_{0,0}u_{-1} &= 2 \partial_0 u_{-1} \cdot \partial_1 u_0 - (\partial_1 u_{-1} + \partial_0 u_0) \partial_1 u_{-1} + \frac{u_{-1} (\partial_1 u_{-1} + 2 \partial_0 u_0) - 2 u_0 \partial_{1,1} u_{-1}}{S} \\
&\quad + \frac{2 (\partial_0 u_{-1} - u_{-1} \partial_1 u_0) + u_0 \partial_1 u_{-1}}{S} \partial_0 S + 2 u_{-1} \partial_1 u_0 - u_{-1} (\partial_0 u_{-1} + (\partial_1 u_{-1} + \partial_0 u_0) u_0) + u_0^2 \partial_0 u_{-1}.
\end{align*} \]

The system in involution describing conservative hydrodynamic chains (7) was derived in [6]:

\[ \begin{align*}
\tilde{\partial}_{0,0}G &= \frac{2 \tilde{\partial}_{0,0} G}{\tilde{\partial}_1 F_0} + \left( -4 \tilde{\partial}_0 G \cdot \tilde{\partial}_1 G + 4 \tilde{\partial}_0 G \cdot \tilde{\partial}_0 F_0 - \tilde{\partial}_0^2 F_0 \right) \tilde{\partial}_{0,1} G \\\n&\quad + \left( \frac{2 \tilde{\partial}_0^2 G}{\tilde{\partial}_1 F_0} - \frac{2 \tilde{\partial}_{0,0} G}{\tilde{\partial}_1 F_0} \right) \tilde{\partial}_{1,1} G - \left( \frac{\left( \tilde{\partial}_0 F_0 - \tilde{\partial}_1 G \right) \tilde{\partial}_{0,0} F_0 + 2 \tilde{\partial}_0 G \cdot \tilde{\partial}_{0,1} F_0}{\tilde{\partial}_1 F_0} \right) \tilde{\partial}_0 G \\
&\quad + \left( 2 \tilde{\partial}_0 G + \frac{2 \left( \tilde{\partial}_0^2 G - 2 \tilde{\partial}_0 F_0 \cdot \tilde{\partial}_1 G + \tilde{\partial}_0^2 F_0 + \tilde{\partial}_{0,1} F_0 \right)}{\tilde{\partial}_1 F_0} \right) \tilde{\partial}_{0,0} G, \\
\tilde{\partial}_{1,1} G &= \frac{\left( \tilde{\partial}_1 G - \tilde{\partial}_0 F_0 \right) \tilde{\partial}_{1,1} F_0}{\tilde{\partial}_1 F_0} + 4 \tilde{\partial}_{0,1} G \cdot \tilde{\partial}_1 F_0 - \tilde{\partial}_{0,0} F_0 \cdot \tilde{\partial}_1 F_0 \\
&\quad - 2 \tilde{\partial}_1 G \cdot \tilde{\partial}_{0,1} F_0 + 2 \tilde{\partial}_0 F_0 \cdot \tilde{\partial}_{0,1} F_0 - \tilde{\partial}_0 G \cdot \tilde{\partial}_{1,1} F_0 + \frac{2 \left( \tilde{\partial}_1 G - \tilde{\partial}_0 F_0 \right) + \tilde{\partial}_{1,1} F_0}{\tilde{\partial}_1 F_0} \tilde{\partial}_{1,1} G,
\end{align*} \]

22
\[ \partial_{0,1} G = 2 \partial_0 G \cdot \partial_{1,1} G + 2 \partial_{0,0} G \cdot \partial_1 F_0 - 2 \partial_0 G \cdot \partial_{0,1} F_0 \]
\[ + \frac{\partial_{0,1} G \cdot \partial_{1,1} F_0}{\partial_1 F_0} + \frac{(\partial_0 F_0 - \partial_1 G) \partial_0 G \cdot \partial_{1,1} F_0}{\partial_1 F_0}, \]
\[ \partial_{0,0} G = -\partial_0^2 G \frac{\partial_{1,1} F_0}{\partial_1 F_0} + 4 \partial_0 G \cdot \partial_0 F_0 \cdot \partial_0 G + \left( 2 \partial_0 F_0 - 2 \partial_1 G + \frac{\partial_{1,1} F_0}{\partial_1 F_0} \right) \partial_{0,0} G, \]
\[ \partial_{1,1} F_0 = \frac{\partial_{1,1}^2 F_0}{\partial_1 F_0} + \left( \partial_1 G - \partial_0 F_0 \right) \partial_{1,1} F_0 + 2 \partial_1 F_0 \cdot \partial_{0,1} F_0, \]
\[ \partial_{0,1} F_0 = \partial_{0,0} F_0 \cdot \partial_1 F_0 + \left( \partial_0 G + \frac{\partial_{0,1} F_0}{\partial_1 F_0} \right) \partial_{1,1} F_0, \]
\[ \partial_{0,0} F_0 = 2 \partial_0 G \cdot \partial_{0,1} F_0 + \left( \partial_0 F_0 - \partial_1 G + \frac{\partial_{1,1} F_0}{\partial_1 F_0} \right) \partial_{0,0} F_0, \]
\[ \partial_{0,0} F_0 = \left( \partial_0 G + \frac{\partial_{0,1}^2 G - 2 \partial_0 G \cdot \partial_1 G + \partial_0^2 G - \partial_{1,1} G}{\partial_1 F_0} \right) \partial_{0,0} F_0 + \left( \frac{\partial_{0,0} F_0}{\partial_1 F_0} - 2 \partial_0 G \cdot \partial_1 G - \partial_{0,0} G \cdot \partial_0 F_0 \right) \partial_{0,1} F_0 + \frac{\left( \partial_0^2 G - \partial_{0,0} G \right) \partial_{1,1} F_0}{\partial_1 F_0}. \]

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