SCHWARZ TYPE LEMMAS AND THEIR APPLICATIONS IN BANACH SPACES

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ABSTRACT. The main purpose of this paper is to develop some methods to investigate the Schwarz type lemmas of holomorphic mappings and pluriharmonic mappings in Banach spaces. Initially, we extend the classical Schwarz lemmas of holomorphic mappings to Banach spaces, and then we apply these extensions to establish a sharp Bloch type theorem for pluriharmonic mappings on homogeneous unit balls of $\mathbb{C}^n$ and to obtain some sharp boundary Schwarz type lemmas for holomorphic mappings in Banach spaces. Furthermore, we improve and generalize the classical Schwarz lemmas of planar harmonic mappings into the sharp forms of Banach spaces, and present some applications to sharp boundary Schwarz type lemmas for pluriharmonic mappings in Banach spaces. Additionally, using a relatively simple method of proof, we prove some sharp Schwarz-Pick type estimates of pluriharmonic mappings in JB$^*$-triples, and the obtained results provide the improvements and generalizations of the corresponding results in [9].

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1. Preliminaries

It is well known that the Schwarz lemma has become a crucial theme in lots of branches of mathematical research for more than a hundred years to date. We refer the reader to [1, 5, 10, 20, 21, 22, 25, 33, 44, 45, 47, 59, 67, 69] for more details on this topic. This paper continues the study of the classical Schwarz lemmas of holomorphic mappings and harmonic mappings (or complex-valued harmonic functions). First, we extend the classical Schwarz lemmas of holomorphic mappings to Banach spaces, and then we use the obtained results to establish a sharp Bloch type theorem for pluriharmonic mappings on homogeneous unit balls of $\mathbb{C}^n$ and obtain sharp boundary Schwarz type lemmas for holomorphic mappings in Banach spaces. In addition, we improve and generalize the classical Schwarz lemmas of planar harmonic mappings into the sharp forms of Banach spaces, and obtain some applications to sharp boundary Schwarz type lemmas for pluriharmonic mappings in Banach spaces. At last, we use a relatively simple method to prove some sharp Schwarz-Pick type estimates of pluriharmonic mappings in JB*-triples, and the obtained results are also the improvements and generalizations of the corresponding known results.

In order to state our main results, we need to recall some basic definitions and introduce some necessary terminologies.

Let $\mathbb{C}^n$ be the complex space of dimension $n \geq 1$, and $\| \cdot \|_e$ be the Euclidean norm on $\mathbb{C}^n$. For real or complex Banach spaces $X$ and $Y$ with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$, respectively, let $L(X,Y)$ be the space of all continuous linear operators from $X$ into $Y$ with the standard operator norm

$$
\|A\| = \sup_{x \in X \setminus \{0\}} \frac{\|Ax\|_Y}{\|x\|_X},
$$

where $A \in L(X,Y)$. Then $L(X,Y)$ is a Banach space with respect to this norm. Denote by $X^*$ the dual space of the real or complex Banach space $X$. For $x \in X \setminus \{0\}$, let

$$
T(x) = \{ l_x \in X^* : l_x(x) = \|x\|_X \text{ and } \|l_x\|_{X^*} = 1 \}.
$$

Then the well known Hahn-Banach theorem implies that $T(x) \neq \emptyset$.

Let $\psi$ be a mapping of a domain $\Omega \subset X$ into a real or complex Banach space $Y$, where $X$ is a complex Banach space. We say that $\psi$ is differentiable at $z \in \Omega$
if there exists a bounded real linear operator $D\psi : X \to Y$ such that
\[
\lim_{\|\tau\|_X \to 0^+} \frac{\|\psi(z + \tau) - \psi(z) - D\psi(z)\tau\|_Y}{\|\tau\|_X} = 0.
\]
Here $D\psi(z)$ is called the Fréchet derivative of $\psi$ at $z$. If $Y$ is a complex Banach space and $D\psi(z)$ is bounded complex linear for each $z \in \Omega$, then $\psi$ is said to be holomorphic on $\Omega$.

Also, for $z \in \Omega \setminus \{0\}$, if
\[
\lim_{r \to 1^-} \frac{\psi(rz) - \psi(z)}{r - 1}
\]
exists, then we call this the radial derivative of $\psi$ at $z$. If $\psi$ is differentiable at $z \in \Omega \setminus \{0\}$, then the radial derivative of $\psi$ at $z$ is equal to $D\psi(z)z$. So, in general, we denote the radial derivative of $\psi$ at $z$ by $D\psi(z)z$.

For a differentiable mapping $\psi : \Omega \to Y$ and for a point $z_0 \in \Omega$ which satisfies one of the following conditions:

(i) $\psi(z_0) = 0$;
(ii) $\psi(z_0) \neq 0$ and $\|\psi(z)\|_Y$ is differentiable at $z = z_0$,

we define
\[
|\nabla \|\psi\|_Y(z_0)| = \sup_{\|\beta\|_X = 1} \lim_{t \to 0^+} \frac{\|\psi(z_0 + t\beta)\|_Y - \|\psi(z_0)\|_Y}{t}.
\]

As in the proof of [68, eq.(3.1)], we obtain the following result.

**Proposition 1.1.**

\[
(1.1) \quad |\nabla \|\psi\|_Y(z_0)| = \begin{cases} 
\|D\psi(z_0)\| & \text{if } \psi(z_0) = 0; \\
\sup_{\|\beta\|_X = 1} |l_{\psi(z_0)}(D\psi(z_0)\beta)| & \text{if } \psi(z_0) \neq 0,
\end{cases}
\]

where $l_{\psi(z_0)} \in T(\psi(z_0))$.

Let $\Omega$ be a domain in a complex Banach space $X$. A mapping $f$ of $\Omega$ into a real or complex Banach space $Y$ is said to be pluriharmonic if the restriction of $l(f(\cdot))$ to every holomorphic curve is harmonic for any $l \in Y^*$ (cf. [9, 19, 39, 57, 58]). In particular, if $\Omega$ is a balanced domain, for a pluriharmonic mapping $f : \Omega \to Y$ and $w \in \partial \Omega$, then we let
\[
\Lambda_f(0; w) = \sup\{|\varphi[f, w, l_u](\zeta)| : l_u \in T(u), \|u\|_Y = 1\}
\]
\[
+ \sup\{|\varphi[f, w, l_u](\zeta)| : l_u \in T(u), \|u\|_Y = 1\},
\]

where
\[
\varphi[f, w, l_u](\zeta) = l_u(f(\zeta w)), \quad \zeta \in \mathbb{U}
\]
and $\mathbb{U}$ is the open unit disk of the complex plane $\mathbb{C}$. We note that the inequality $\Lambda_f(0; w) \leq \frac{4}{\pi}$ always holds for pluriharmonic mappings $f : \Omega \to Y$ with $\|f(x)\|_Y < 1$ for all $x \in \Omega$ by the harmonic Schwarz lemma on the unit disk. If $Y = \mathbb{C}^n$ or $Y = \ell_2$, where
\[
\ell_2 = \left\{ z = (z_1, z_2, \ldots) : z_j \in \mathbb{C}, \sum_{j=1}^{\infty} |z_j|^2 < \infty \right\},
\]

we consider the case of $\mathbb{C}^n$ or $\ell_2$.
then the mapping \( f = h + \overline{g} \) is pluriharmonic on \( \Omega \), where \( h \) and \( g \) are holomorphic in \( \Omega \). In this case,

\[
\Lambda_f(0; w) = \| Dh(0)w \|_Y + \| Dg(0)w \|_Y, \quad w \in \partial \Omega.
\]

Furthermore, if \( X = Y = \mathbb{C}^n \) and \( \Omega \) is a simply connected domain, then \( f : \Omega \to \mathbb{C}^n \) is pluriharmonic if and only if it has a representation \( f = h + \overline{g} \), where \( h \) and \( g \) are holomorphic in \( \Omega \). This representation is unique if \( g(0) = 0 \) (cf. \([19, 60, 65]\)). For a pluriharmonic mapping \( f = h + \overline{g} : \Omega \to \mathbb{C}^n \), it is an elementary exercise to see that the real Jacobian determinant of \( f \) can be written as

\[
\det J_f = \det \begin{bmatrix} Dh & Dg \\ Dg & Dh \end{bmatrix}
\]

and if \( h \) is locally biholomorphic in \( \Omega \), then the determinant of \( J_f \) has the form

\[
\det J_f = | \det Dh |^2 \det \left( I - Dg[Dh]^{-1}Dg[Dh]^{-1} \right),
\]

where \( I \) is the identity operator (see \([19]\)).

For an \( n \times m \) complex matrix \( A = (a_{ij}) \), the Frobenius norm of \( A \) is defined as follows:

\[
\| A \|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|^2}.
\]

Then we have

\[
(1.2) \quad \| A \|_F^2 \leq m \| A \|_2^2,
\]

where

\[
\| A \| = \sup_{\xi \in \mathbb{C}^m \setminus \{0\}} \frac{\| A\xi \|_e}{\| \xi \|_e}.
\]

Let \( \Omega \) be a domain in \( \mathbb{C}^m \). For a pluriharmonic mapping \( f : \Omega \to \mathbb{C}^n \), let

\[
\nabla f(z) = \left( \frac{\partial f}{\partial x_1}(z), \frac{\partial f}{\partial y_1}(z), \ldots, \frac{\partial f}{\partial x_m}(z), \frac{\partial f}{\partial y_m}(z) \right),
\]

where \( z = (z_1, \ldots, z_m) \in \Omega \) and \( z_j = x_j + iy_j \) for \( j = 1, \ldots, m \).

**Definition 1.2.** A complex Banach space \( X \) is called a \( JB^* \)-triple if there exists a triple product \( \{\cdot, \cdot, \cdot\} : X^3 \to X \) which is conjugate linear in the middle variable, but linear and symmetric in the other variables, and satisfies

1. \( \{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\} \);
2. the map \( a \triangledown a : x \in X \mapsto \{a, a, x\} \in X \) is hermitian with nonnegative spectrum;
3. \( \|\{a, a, a\}\|_X = \|a\|_X^3 \);

for \( a, b, x, y, z \in X \).
Let $\Omega$ be a domain in a complex Banach space $X$. Denote by $\text{Aut}(\Omega)$ the set of biholomorphic automorphisms of $\Omega$. A domain $\Omega \subset X$ is said to be homogeneous if for each $x, y \in \Omega$, there exists some mapping $f \in \text{Aut}(\Omega)$ such that $f(x) = y$. It is known that every bounded symmetric domain in a complex Banach space $X$ is homogeneous. Conversely, the open unit ball $B_X$ of $X$ admits a symmetry $s(z) = -z$ at 0 and if $B_X$ is homogeneous, then $B_X$ is a symmetric domain. It is well known that the Euclidean unit ball in $\mathbb{C}^n$, the polydisc in $\mathbb{C}^n$, and the classical Cartan domains are bounded symmetric domains in $\mathbb{C}^n$. Banach spaces with a homogeneous open unit ball are precisely the JB*-triples. We refer to [13, 14, 16, 51, 52, 53] for more details of JB*-triples and bounded symmetric domains.

Let $X$ be a JB*-triple. For every $z, w \in X$, the Bergman operator $B(z, w) \in L(X)$ is defined by

$$B(z, w)(\cdot) = I - 2\Box w + \{z, \{w, \cdot, w\}, z\},$$

where $z \Box w(x) = \{z, w, x\}$. Let $B_X$ be the unit ball of $X$. Then, for each $a \in B_X$, the Möbius transformation $g_a$ defined by

$$(1.3)\quad g_a(z) = a + B(a, a)^{1/2}(I + z \Box a)^{-1}z,$$

is a biholomorphic automorphism of $B_X$ with $g_a(0) = a, g_a(-a) = 0, g_{-a} = g_a^{-1}$ and $Dg_a(0) = B(a, a)^{1/2}$. By [41] Corollary 3.6, we have

$$(1.4)\quad \|Dg_{z_0}(0)^{-1}\| = \|Dg_{-z_0}(z_0)\| = \frac{1}{1 - \|z_0\|^2_X}.$$

Given JB*-triples $X_1, \ldots, X_n$, we can form the $\ell^\infty$-sum $X = X_1 \oplus \cdots \oplus X_n$ which becomes a JB*-triple equipped with the coordinatewise triple product:

$$\{x, y, z\} = \{(x_1, y_1, z_1), \ldots, (x_i, y_i, z_i), \ldots, (x_n, y_n, z_n)\}$$

for $x = (x_i), y = (y_i), z = (z_i) \in X$. Let $B_{X_j}$ be the open unit ball of $X_j$ for $j = 1, \ldots, n$. Then their product $B_{X_1} \times \cdots \times B_{X_n}$ is the open unit ball of $B_X$ of the JB*-triple $X$. Let $g_{a_j}$ ($a_j \in B_{X_j}$) be the Möbius transformation of $B_{X_j}$ for $j = 1, \ldots, n$. Then, for $a = (a_1, \ldots, a_n) \in B_X$, $g_a(z) = (g_{a_1}(z_1), \ldots, g_{a_n}(z_n))$, $z = (z_1, \ldots, z_n) \in B_X$, is the Möbius transformation of $B_X$.

2. SCHWARZ TYPE LEMMAS AND THEIR APPLICATIONS

The classical Schwarz lemma states that every holomorphic mapping $f$ of the unit disk $\mathbb{D}$ into itself with $f(0) = 0$ satisfies $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. Moreover, unless $f$ is a rotation, one has the strict inequality $|f'(0)| < 1$, and $f$ maps each disk $\mathbb{D}_r := \{z : |z| < r < 1\}$ into a strictly smaller one. Lindelöf removed the assumption “origin is a fixed point” and improved the classical Schwarz lemma of holomorphic mappings into the following form.
Theorem A. (Proposition 2.2.2) Let $f$ be a holomorphic mapping of $U$ into itself. Then, for $z \in U$, the following sharp estimate
\begin{equation}
|f(z)| \leq \frac{|f(0)| + |z|}{1 + |f(0)||z|}
\end{equation}
holds.

Remark 2.1. By the maximum modulus principle, Theorem A continuous to hold for holomorphic mapping $f$ from $U$ into $\overline{U}$.

Under the assumption of Theorem A, if “$|f(z)|$” in (2.1) is replaced by “$|f(z) - f(0)|$”, then Harris [32] obtained the following sharp estimate
\begin{equation}
|f(z) - f(0)| \leq |z| \frac{1 - |f(0)|^2}{1 - |f(0)||z|}.
\end{equation}
The extension of the estimate (2.2) is probably of independent interest. In particular, if the origin is a fixed point of $f$ in Theorem A, then Osserman obtained a better estimate as follows.

Theorem B. (Lemma 2) Let $f$ be a holomorphic mapping of $U$ into itself with $f(0) = 0$. Then, for $z \in U$, the following sharp estimate
\begin{equation}
|f(z)| \leq |z| \frac{|f'(0)| + |z|}{1 + |f'(0)||z|}
\end{equation}
holds.

As an application, by using (2.3), Osserman [54] established a version of the boundary Schwarz lemma which is as follows.

Theorem C. Let $f$ be a holomorphic mapping of $U$ into itself with $f(0) = 0$. If $f$ is holomorphic at $b \in \partial U$ (or more generally, if $f$ is differentiable at $b \in \partial U$) and $|f(b)| = 1$, then $|f'(b)| \geq 2/(1 + |f'(0)|)$. Moreover, the inequality is sharp.

In fact, Unkelbach [63] had established a similar result as follows: Let $f$ be a holomorphic mapping of $U$ into itself with $f(0) = 0$. If
$$D = \lim_{z \to 1} \frac{1 - f(z)}{1 - z}$$
exists, then $D \geq 1$. Moreover, if $f(0) = \rho e^{i\varphi}$, then $D \geq 2(1 - \rho \cos \varphi)/(1 - \rho^2)$. This inequality is also sharp. On the related investigation of the boundary Schwarz type lemmas of the Poisson-Stieltjes integral, we refer to [35].

In the following, we extend Theorems A and B to Banach spaces, and then we apply the obtained results to study the Bloch type Theorem and the boundary Schwarz type lemmas.

Theorem 2.2. Suppose that $B_X$ and $B_Y$ are the unit balls of the complex Banach spaces $X$ and $Y$, respectively. Let $f : B_X \to B_Y$ be a holomorphic mapping. Then
$$\|f(z)\|_Y \leq \frac{\|f(0)\|_Y + \|z\|_X}{1 + \|f(0)\|_Y \|z\|_X}$$
for $z \in B_X$. 
This estimate is sharp with equality possible for each value of \( \|f(0)\|_Y \) and for each \( z \in B_X \).

**Theorem 2.3.** Suppose that \( B_X \) and \( B_Y \) are the unit balls of the complex Banach spaces \( X \) and \( Y \), respectively. Let \( f : B_X \to B_Y \) be a holomorphic mapping with \( f(0) = 0 \). Then

\[
\|f(z)\|_Y \leq \frac{\|Df(0)\| + \|z\|_X}{1 + \|Df(0)\|\|z\|_X} \|z\|_X \leq \|z\|_X \text{ for } z \in B_X.
\]

The first estimate is sharp with equality possible for each value of \( \|Df(0)\| \) and for each \( z \in B_X \).

We use \( B \) to denote the homogeneous unit ball of \( X = \mathbb{C}^n \). It is easy to see that \( B \) is the unit ball of a finite dimensional JB*-triple \( X \). Let \( k \in [0, 1) \) be a constant. Denote by \( \mathcal{PH}(k) \) the set of all pluriharmonic mappings \( f = h + g \) of \( B \) into \( \mathbb{C}^n \) with \( h(0) = g(0) = 0 \) and

\[
\|\omega_f\| \leq k,
\]

where \( h \) is locally biholomorphic in \( B \), \( g \) is holomorphic in \( B \), \( \omega_f = Dg[Dh]^{-1} \) and

\[
\|\omega_f\| = \sup_{z \in B, \xi \in \mathbb{C}^n \setminus \{0\}} \frac{\|\omega_f(z)\|_e}{\|\xi\|_e}.
\]

For \( n \geq 2 \), \( f = h + g \in \mathcal{PH}(k) \) is a quasiregular mapping if and only if \( h \) is a quasiregular mapping (see [11]). In particular, if \( n = 1 \), then \( f \in \mathcal{PH}(k) \) is a \( K \)-quasiregular mapping, where \( K = (1 + k)/(1 - k) \) (cf. [12] [18] [64] [66]).

Denote by \( \mathcal{P} \) a set of mappings from \( B \) into \( \mathbb{C}^n \). For a mapping \( f \in \mathcal{P} \) and a point \( a \in B \), we write \( \mathcal{R}_f(a) \) as the radius of the largest univalent Euclidean ball centered at \( f(a) \) in \( f(B) \). Here a univalent ball in \( f(B) \) centered at \( f(a) \) means that \( f \) maps an open subset of \( B \) containing the point \( a \) univalently onto this ball. Let

\[
\mathcal{R} = \inf_{f \in \mathcal{P}} \sup_{a \in B} \mathcal{R}_f(a).
\]

If \( \mathcal{R} > 0 \) is finite, then we call \( \mathcal{R} \) the Bloch type constant of the set \( \mathcal{P} \). One of the long standing open problems of determining the precise value of Bloch type constant of holomorphic mappings with one variable has attracted much attention (see [3] [4] [24] [43] [49]). For holomorphic mappings of several complex variables, the Bloch type constant does not exist unless one considers the class of functions under certain constraints. For example, consider \( f_k(z) = (kz_1, z_2/k, z_3, \ldots, z_n) \) for \( k \in \mathbb{N} = \{1, 2, \ldots\} \), where \( n \geq 2 \) and \( z \) is in the Euclidean unit ball \( B^n \) of \( \mathbb{C}^n \). It is easy to see that each \( f_k \) is univalent and \( |f_k(0)| = \det Df_k(0) - 1 = 0 \). Moreover, each \( f_k(B^n) \) contains no ball with radius bigger than \( 1/k \). Hence, there does not exist an absolute constant \( r_0 \) which can work for all \( k \in \mathbb{N} \) such that \( \{z \in \mathbb{C}^n : \|z\|_e < r_0 \} \) is contained in \( f_k(B^n) \). For more details on studies of the Bloch type constant of holomorphic mappings with several complex variables, we refer to the works of Chen and Gauthier [6], Fitzgerald and Gong [24], Graham and Varolin [26], Hamada [23], Takahashi [61], and Wu [67]. On the studies of the Bloch type constant for the class of pluriharmonic mappings, we refer to [8] [30].
In the following, for $f = h + \mathbf{g} \in \mathcal{P}\mathcal{H}(k)$, we will use Theorems 2.2 and 2.3 to investigate the ratio $\mathcal{B}_f/\mathcal{B}_h$ and give a sharp estimate. For the related studies of the planar harmonic mappings, see [8, 10, 24].

**Theorem 2.4.** For $k \in [0, 1)$, let $f = h + \mathbf{g} \in \mathcal{P}\mathcal{H}(k)$. Then, for $z \in \mathbf{B}$,

\begin{equation}
1 - k \leq \frac{\mathcal{B}_f(z)}{\mathcal{B}_h(z)} \leq \mu_k \left( \frac{\|\omega_f(z)\|}{k} \right) \leq \mu_k(1) = 1 + k,
\end{equation}

where

\[
\mu_k(x) = \begin{cases} 
1 + k \left[ \frac{1}{x} + \left(1 - \frac{1}{x^2}\right) \log(1 + x) \right] & \text{for } x \in (0, 1], \\
\mu_k(x) = \lim_{x \to 0^+} \mu_k(x) = 1 + \frac{k}{2} & \text{for } x = 0.
\end{cases}
\]

Moreover, the left hand of (2.4) is sharp for all $z \in \mathbf{B}$, and the right hand of (2.4) is asymptotically sharp when $k$ tends to 0.

In [67], Wu generalized the classical Schwarz lemma of holomorphic mappings to higher dimension. Burns and Krantz [5] established a new version Schwarz lemma at the boundary, and obtained a new rigidity result for holomorphic mappings. Later, Huang [37] further strengthened the result of Burns-Krantz for holomorphic mappings with an interior fixed point. Recently, Liu and Tang [47] obtained a Schwarz lemma at the boundary of holomorphic mappings from a Levi strongly pseudoconvex domain into itself. See [25, 27, 38, 62] for more details on this line. In the following, we will apply Theorem 2.3 to establish a new version Schwarz lemma at the boundary, which is a generalization of Theorem C.

**Theorem 2.5.** Let $B_X$ and $B_Y$ be the unit balls of the complex Banach spaces $X$ and $Y$, respectively. Suppose that $f$ is a holomorphic mapping of $B_X$ into $B_Y$. If $f(0) = 0$ and $f$ is holomorphic at $b \in \partial B_X$ (or more generally, the radial derivative $Df(b)B$ exists at $b \in \partial B_X$) with $\|f(b)\|_Y = 1$, then

\[ \|Df(b)B\|_Y \geq \frac{2}{1 + \|Df(0)\|}. \]

This inequality is sharp with equality possible for each value of $\|Df(0)\|$.

In particular, if we replace $B_X$ and $B_Y$ by a balanced domain and a finite dimensional bounded symmetric domain in Theorem 2.5, respectively, then we obtain a better estimate (cf. [31]). Before we present the next result, let us recall some definitions.

Let $B_Y$ be a bounded symmetric domain realized as the open unit ball of a finite dimensional JB*-triple $Y$. We recall a constant $c(B_Y)$ defined in [29]. Let $h_0$ be the Bergman metric on $B_Y$ at 0 and let

\[ c(B_Y) = \frac{1}{2} \sup_{x, y \in B_Y} |h_0(x, y)|. \]
It follows from [15, Ineq. (2.3)] that
\[ \dim Y + r \leq c(B_Y) \leq \dim Y, \]
where \( r \) is the rank of \( Y \).

An element \( x \) in a JB*-triple \( Y \) is called a tripotent if \( x \) satisfies \( \{x, x, x\} = x \). If two tripotents \( x \) and \( y \) satisfy \( 2x \square y = 0 \), then \( x \) and \( y \) are said to be orthogonal. Obviously, orthogonality is a symmetric relation. A tripotent \( x \) is said to be maximal if any tripotent which is orthogonal to \( x \) is 0.

**Theorem 2.6.** Suppose that \( G \) is a balanced domain in a complex Banach space \( X \) and \( B_Y \) is a bounded symmetric domain realized as the open unit ball of a finite dimensional JB*-triple \( Y \). Let \( \Gamma \subset \partial B_Y \) be the set of maximal tripotents of \( Y \), and let \( f : G \to B_Y \) be a holomorphic mapping. Also let \( f \) be holomorphic at \( z = \alpha \in \partial G \) and \( f(\alpha) = \beta \in \Gamma \).

(i) We have
\[ \frac{1}{2c(B_Y)} h_0(Df(\alpha)\alpha, \beta) \geq \frac{2\left| 1 - \frac{1}{2c(B_Y)} h_0(f(0), \beta) \right|^2}{1 - \left| \frac{1}{2c(B_Y)} h_0(f(0), \beta) \right|^2 + \|Df(0)\alpha\|_Y}, \]
where \( h_0 \) is the Bergman metric on \( B_Y \) at 0.

(ii) Moreover, if \( f(0) = 0 \), then we have
\[ \frac{1}{2c(B_Y)} h_0(Df(\alpha)\alpha, \beta) \geq \frac{2}{1 + \|Df(0)\alpha\|_Y}. \]

(iii) In particular, if \( G = B_X \) is the unit ball of \( X \), then the inequalities (2.5) and (2.6) are sharp with equality possible for each values of
\[ a = \frac{1}{2c(B_Y)} h_0(f(0), \beta), \quad b = \frac{1}{2c(B_Y)} h_0(Df(0)\alpha, \beta) \]
with \( |b| \leq 1 - |a|^2 \).

We remark that Theorem 2.6 is an improvement and generalization of [46, Theorem 3.1] and [48, Theorem 3.1] (cf. [70, Theorem 1.5]). By using arguments similar to those in the proof of Theorem 2.6, we have the following theorem (cf. [70, Theorem 1.5]). We omit the proof.

**Theorem 2.7.** Suppose that \( G \) is a balanced domain in a complex Banach space \( X \) and \( B_H \) is unit ball of a complex Hilbert space \( H \) with inner product \( \langle \cdot, \cdot \rangle \). Let \( f : G \to B_H \) be a holomorphic mapping. If \( f \) is holomorphic at \( z = \alpha \in \partial G \) and \( f(\alpha) = \beta \in \partial B_H \), then
\[ \langle Df(\alpha)\alpha, \beta \rangle \geq \frac{2\left| 1 - \langle f(0), \beta \rangle \right|^2}{1 - \left| \langle f(0), \beta \rangle \right|^2 + \|Df(0)\alpha\|_H}. \]

Moreover, if \( f(0) = 0 \), then we have
\[ \langle Df(\alpha)\alpha, \beta \rangle \geq \frac{2}{1 + \|Df(0)\alpha\|_H}. \]
In particular, if $G = B_X$ is the unit ball of $X$, then the inequalities \((2.7)\) and \((2.8)\) are sharp with equality possible for each values of $\langle f(0), \beta \rangle$ and $\|Df(0)\alpha\|_H$ with $\|Df(0)\alpha\|_H \leq 1 - |\langle f(0), \beta \rangle|^2$.

The proofs of Theorems 2.2 \textendash{} 2.6 will be presented in part I of Section 4.

3. SCHWARZ TYPE LEMMAS OF PLURIHARMONIC MAPPINGS AND THEIR APPLICATIONS

Heinz in his classical paper [34] showed that the following version of Schwarz Lemma for harmonic mappings: If $f$ is a harmonic mapping of $U$ into itself with $f(0) = 0$, then

\begin{equation}
|f(z)| \leq \frac{4}{\pi} \arctan |z|
\end{equation}

for $z \in U$. In 2011, Chen and Gauthier generalized \((3.1)\) into the following form.

**Theorem D.** ([8, Theorem 4]) Let $f$ be a pluriharmonic mapping of the Euclidean unit ball $B^n$ into the Euclidean unit ball $B^m$ such that $f(0) = 0$, where $m$ is a positive integer. Then, for all $z \in B^n$,\n
$$\|f(z)\| \leq \frac{4}{\pi} \arctan \|z\|_e.$$\n
This estimate is sharp.

Hamada and Kohr [30] extended Theorem D to pluriharmonic mappings of the unit ball of a complex Banach space $X$ into the unit ball $B^n_a$ of $C^n$ with respect to an arbitrary norm $\| \cdot \|_a$ on $C^n$ as follows.

**Theorem E.** ([30, Theorem 4.1]) Let $B_X$ be the unit ball of a complex Banach space $X$, $B^n_a$ be the unit ball of $C^n$ with respect to an arbitrary norm $\| \cdot \|_a$ on $C^n$ and $f : B_X \to B^n_a$ be a pluriharmonic mapping such that $f(0) = 0$. Then, for $z \in B_X$, the following sharp inequality\n
$$\|f(z)\|_a \leq \frac{4}{\pi} \arctan \|z\|_X$$

holds.

We remove the assumption “$f(0) = 0$” in Theorem E and obtain the following result.

**Theorem 3.1.** Suppose that $B_X$ and $B_Y$ are the unit balls of the complex Banach spaces $X$ and $Y$, respectively, and $f : B_X \to B_Y$ is a pluriharmonic mapping. Then\n
$$\left\| f(z) - \frac{1 - \|z\|^2_X}{1 + \|z\|^2_X} f(0) \right\|_Y \leq \frac{4}{\pi} \arctan \|z\|_X \text{ for } z \in B_X.$$\n
In particular, if $f(0) = 0$, then this estimate is sharp.

In particular, if $f(0) = 0$ in Theorem 3.1, then we have a better estimate as follows (cf. [70, Theorem 1.7]).
Theorem 3.2. Assume the hypothesis of Theorem 3.1 and in addition let $f(0) = 0$. Then we have

$$
\|f(z)\|_Y \leq \frac{4}{\pi} \arctan \left( \frac{\|z\|_X + \frac{\pi}{4} \Lambda f(0; w)}{1 + \frac{\pi}{4} \Lambda f(0; w) \|z\|_X} \|z\|_X \right) \leq \frac{4}{\pi} \arctan \|z\|_X \text{ for } z \in B_X,
$$

where $w = z/\|z\|_X$.

Let $f$ be a one-to-one harmonic mapping of $U$ onto itself with $f(0) = 0$ which is $C^1$ up to the boundary. By using (3.1), Heinz [34, Ineq. (15)] proved that, for any $\theta \in [0, 2\pi]$,

$$
|f_\zeta(e^{i\theta})| + |f_{\bar{\zeta}}(e^{i\theta})| \geq \frac{2}{\pi}.
$$

In the following, we extend (3.2) into the following forms.

First, we will apply Theorem 3.1 and the Harnack principle to establish a new version Schwarz lemma at the boundary for pluriharmonic mappings.

Theorem 3.3. Assume the hypotheses of Theorem 3.1. In addition, assume that the radial derivative $Df(b)b$ exists at $b \in \partial B_X$ with $\|f(b)\|_Y = 1$. Then we have

$$
\|Df(b)b\|_Y \geq \max \left\{ \frac{2}{\pi} - \|f(0)\|_Y, \frac{1}{2} - \|f(0)\|_Y \right\}.
$$

Next, we will apply Theorem 3.2 to establish a new version Schwarz lemma at the boundary for pluriharmonic mappings. For the proof, it suffices to use arguments similar to those in the proof of Theorem 2.5. We omit the proof.

Theorem 3.4. Assume the hypothesis of Theorem 3.3. Then, if $f(0) = 0$, we have

$$
\|Df(b)b\|_Y \geq \frac{4}{\pi} \frac{1}{1 + \frac{\pi}{4} \Lambda f(0; b)}.
$$

In particular, we consider the cases such that $B_Y$ is a bounded symmetric domain in $C^n$ or is the complex Hilbert ball (cf. [70, Theorems 1.8 and 1.12]). In these cases, the domain of the definition of the mapping $f$ can be generalized to a balanced domain in a complex Banach space.

Theorem 3.5. Suppose that $G$ is a balanced domain in a complex Banach space $X$ and $B_Y$ is a bounded symmetric domain realized as the open unit ball of a finite dimensional JB*-triple $Y = C^n$. Let $\Gamma \subset \partial B_Y$ be the set of maximal tripotents of $Y$, and let $f : G \to B_Y$ be a pluriharmonic mapping with $f(0) = 0$. Assume that $f$ is differentiable at $z = \alpha \in \partial G$ and $f(\alpha) = \beta \in \Gamma$. Let

$$
\varphi(\zeta) = \frac{1}{2e(B_Y)} h_0(f(\zeta\alpha), \beta), \quad \zeta \in U,
$$

where $h_0$ is the Bergman metric on $B_Y$ at 0. Then $\varphi$ is harmonic mapping of $U$ into itself with $\varphi(0) = 0$ and we have

$$
\Re(\varphi(1) + \varphi_{\bar{\zeta}}(1)) \geq \frac{4}{\pi} \frac{1}{1 + \frac{\pi}{4} \Lambda f(0, \alpha)},
$$

where “Re” denotes the real part of a complex number.
In particular, if \( f = h + \overline{g} \), where \( h \) and \( g \) are holomorphic on \( G \), and \( f \) satisfies the above assumptions, then we have

\[
\frac{1}{2c(B_Y)} \Re \left( h_0(Dh(\alpha)\alpha + Dg(\alpha)\alpha, \beta) \right) \geq \frac{4}{\pi} \frac{1}{1 + \frac{\pi}{4} \Lambda_f(0, \alpha)}.
\]

If \( G = B_X \) is the unit ball of \( X \), then the inequalities (3.3) and (3.4) are sharp.

If \( B_Y \) is the complex Hilbert ball, then we have the following result. We omit the proof, since it is similar to that in the proof of Theorem 3.5.

**Theorem 3.6.** Suppose that \( G \) is a balanced domain in a complex Banach space \( X \) and \( B_H \) is the unit ball of a complex Hilbert space \( H \) with inner product \( \langle \cdot, \cdot \rangle \). Let \( f : G \to B_H \) be a pluriharmonic mapping with \( f(0) = 0 \). Assume that \( f \) is differentiable at \( z = \alpha \in \partial G \) and \( f(\alpha) = \beta \in \partial B_H \). Let

\[
\varphi(\zeta) = \langle f(\zeta\alpha), \beta \rangle, \quad \zeta \in \mathbb{U}.
\]

Then \( \varphi \) is a harmonic mapping of \( \mathbb{U} \) into itself with \( f(0) = 0 \) and

\[
\Re(\varphi(1) + \varphi(1)) \geq \frac{4}{\pi} \frac{1}{1 + \frac{\pi}{4} \Lambda_f(0, \alpha)}.
\]

In particular, if \( H = \ell_2 \) and \( f = h + \overline{g} \), where \( h \) and \( g \) are holomorphic on \( G \), and \( f \) satisfies the above assumptions, then we have

\[
\Re(Dh(\alpha)\alpha + Dg(\alpha)\alpha, \beta) \geq \frac{4}{\pi} \frac{1}{1 + \frac{\pi}{4} \Lambda_f(0, \alpha)}.
\]

The inequalities (3.5) and (3.6) are sharp.

The classical Schwarz-Pick lemma states that an analytic function \( f \) of \( \mathbb{U} \) into itself satisfies

\[
|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{U}.
\]

Chu et al. [16, Lemma 3.12] generalized (3.7) to holomorphic mappings between the unit balls of JB*-triples (cf. [2], [6], [30]).

In 1989, Colonna established an analogue of the Schwarz-Pick lemma for planar harmonic mappings.

**Theorem F.** ([17, Theorems 3 and 4]) Let \( f \) be a harmonic mapping of \( \mathbb{U} \) into itself. Then, for \( z \in \mathbb{U} \),

\[
|f_z(z)| + |f_{\overline{z}}(z)| \leq \frac{4}{\pi} \frac{1}{1 - |z|^2}.
\]

This estimate is sharp, and all the extremal functions are

\[
f(z) = \frac{2\gamma}{\pi} \arg \left( \frac{1 + \phi(z)}{1 - \phi(z)} \right),
\]

where \( |\gamma| = 1 \) and \( \phi \) is a conformal automorphism of \( \mathbb{U} \).
For real valued harmonic functions in $U$, Kalaj and Vuorinen [40] and Chen [7, Theorem 1.2] obtained a better estimate as follows.

**Theorem G.** Let $f : U \to (-1, 1)$ be a real-valued harmonic function. Then the following inequalities hold:

1. $|\nabla f(z)| \leq \frac{4}{\pi} \frac{1 - |f(z)|^2}{1 - |z|^2}, \text{ for } z \in U$. This inequality is sharp for each $z \in U$.
2. $|\nabla f(z)| \leq \frac{4 \cos(\frac{\pi}{2} f(z))}{\pi} \frac{1 - |z|^2}{1 - |z|^2}, \text{ for } z \in U$. This inequality is sharp for each $z \in U$.

Chen and Gauthier [8] generalized Theorem F to pluriharmonic mappings between the Euclidean unit balls. Later, Hamada and Kohr [30] generalized Theorem F to pluriharmonic mappings from the unit ball of a JB$^*$-triple into the unit ball $B^n$ of $C^n$ with respect to an arbitrary norm on $C^n$.

For mappings with values in higher dimensional spaces, Pavlović [55, 56] showed that the inequality (3.7) does not hold for analytic functions $f$ of $U$ into $B^n$, where $n \geq 2$ is an integer. However, Pavlović proved the following Schwarz-Pick type lemma for analytic functions $f$ of $U$ into $B^n$:

$$|\nabla \|f(z)\|_e| \leq \frac{1 - \|f(z)\|_e^2}{1 - |z|^2}, \text{ for } z \in U,$$

where $\nabla \|f(z)\|_e$ denotes the gradient of $\|f\|_e$.

In [71], Zhu established a Schwarz-Pick type estimate for pluriharmonic mappings $f$ of $B^n$ into itself as follows.

**Theorem H.** ([71, Theorem 1.1]) For $n \geq 1$, let $f$ be a pluriharmonic mapping of $B^n$ into itself. Then the following inequality

$$|\nabla \|f(z)\|_e| \leq \frac{4 \sqrt{n}}{\pi} \frac{1}{1 - \|f(z)\|_e},$$

holds for all $z \in B^n$.

In [9], Chen and Hamada improved Theorem H into the sharp form for pluriharmonic mappings of $B^n$ into the unit ball of the Minkowski space. In the following, we will present a relatively simple method of proof to improve Theorem H and [9, Theorem 2.2] to pluriharmonic mappings from the unit ball $B_X$ of a JB$^*$-triple $X$ to the unit ball of a complex Banach space $Y$.

**Theorem 3.7.** Let $B_X$ be a bounded symmetric domain realized as the unit ball of a JB$^*$-triple $X$. Let $B_Y$ be the unit ball of a complex Banach space $Y$. Also, let $f : B_X \to B_Y$ be a pluriharmonic mapping. Let $z_0 \in B_X$ be a point which satisfies one of the following conditions:

(i) $f(z_0) = 0$;
(ii) $f(z_0) \neq 0$ and $\|f(z)|_Y$ is differentiable at $z = z_0$. 

Then
\[ |\nabla \|f\|_Y(z_0)| \leq \frac{4 \pi}{1 - \|z_0\|^2_X}. \]

The estimate (3.8) is sharp for each \( z_0 \in \mathbb{B}_X \).

In particular, if \( f \) is a pluriharmonic mapping of \( \mathbb{B}_X \) into the unit ball \( \mathbb{B}_Y \) of a real Banach space \( Y \), then we have a better estimate as follows. Here we omit the proof because it suffices to use Theorem F instead of Theorem G and use arguments similar to those in the proof of Theorem 3.7.

**Theorem 3.8.** Let \( \mathbb{B}_X \) be a bounded symmetric domain realized as the unit ball of a JB*-triple \( X \). Let \( \mathbb{B}_Y \) be the unit ball of a real Banach space \( Y \). Also, let \( f : \mathbb{B}_X \to \mathbb{B}_Y \) be a pluriharmonic mapping. Let \( z_0 \in \mathbb{B}_X \) be a point which satisfies one of the following conditions:

(i) \( f(z_0) = 0 \);
(ii) \( f(z_0) \neq 0 \) and \( \|f(z)\|_Y \) is differentiable at \( z = z_0 \).

Then we have the following estimates.

(1) \[ |\nabla \|f\|_Y(z_0)| \leq \frac{4 \pi}{1 - \|z_0\|^2_X}. \]

(2) \[ |\nabla \|f\|_Y(z_0)| \leq \frac{4 \cos(\pi/2 \|f(z_0)\|_Y)}{1 - \|z_0\|^2_X}. \]

The above estimates are sharp for each \( z_0 \in \mathbb{B}_X \).

We remark that Theorem 3.8 is an improvement and generalization of [9, Theorem 2.3].

In [71], Zhu established some other Schwarz-Pick type estimates as follows.

**Theorem I.** ([71, Theorem 1.2]) Let \( f = h + \overline{g} \) be a pluriharmonic self-mapping of the unit ball \( \mathbb{B}^n \) of \( \mathbb{C}^n \), where \( h \) and \( g \) are holomorphic mappings of \( \mathbb{B}^n \) into \( \mathbb{C}^n \). Then

(3.9) \[ \|Dh(z)e_j\|^2_e + \|Dg(z)e_j\|^2_e \leq \frac{1 - \|f(z)\|^2}{(1 - \|z\|e)^2}, \quad j = 1, \ldots, n, \]

where \( e_1, \ldots, e_n \) is the usual orthonormal basis of \( \mathbb{C}^n \), and

(3.10) \[ \|\nabla f(z)\|^2_F \leq \frac{2n(1 - \|f(z)\|^2_e)}{(1 - \|z\|e)^2}. \]

**Theorem J.** ([71, Theorem 1.3]) Let \( f = h + \overline{g} \) be a pluriharmonic self-mapping of \( \mathbb{B}^n \), where \( h \) and \( g \) are holomorphic mappings of \( \mathbb{B}^n \) into \( \mathbb{C}^n \). Assume that \( h \) is locally biholomorphic on \( \mathbb{B}^n \) and \( \|\omega_f\| \leq k < 1 \) holds on \( \mathbb{B}^n \). Let \( K = (1 + k)/(1 - k) \). Then the following inequalities hold for all \( z \in \mathbb{B}^n \):

(3.11) \[ \|Dh(z)\| + \|Dg(z)\| \leq \frac{2K}{K + 1} \sqrt{n(1 - \|f(z)\|^2_e)} / (1 - \|z\|e), \]

(3.12) \[ |\nabla \|f\|_e(z)| \leq \sqrt{n}(\|Dh(z)\| + \|Dg(z)\|) \]
and

(3.13) \[ \|\nabla f(z)\|_F \leq \sqrt{2n(\|Dh(z)\|^2 + \|Dg(z)\|^2)}. \]

Chen and Hamada [9] generalized and improved Theorem I into the following sharp form on pluriharmonic mappings of the polydisc \( U^n \) in \( \mathbb{C}^n \) into the Euclidean unit ball \( B^n \) in \( \mathbb{C}^n \).

**Theorem K.** For \( n \geq 1 \), let \( f = (f_1, \ldots, f_m) : U^n \to B^m \) be a pluriharmonic mapping, where \( m \) is a positive integer. Then, for \( z = (z_1, \ldots, z_n) \in U^n \), we have

(3.14) \[ \sum_{j=1}^m \sum_{k=1}^n \left( \left| \frac{\partial f_j(z)}{\partial z_k} \right|^2 + \left| \frac{\partial f_j(z)}{\partial \overline{z}_k} \right|^2 \right) \leq \frac{1 - \|f(z)\|^2}{(1 - \|z\|^2)^2}, \]

where \( \|z\|_{\infty} = \max_{1 \leq j \leq n} |z_j| \). Moreover, the inequality (3.14) is sharp for each \( z \in U^n \) with \( |z_1| = \cdots = |z_n| \).

In the following, we will generalize and improve (3.12) into the following form. Moreover, we only need to assume that \( f \) is pluriharmonic.

**Proposition 3.9.** Let \( \Omega \) be a domain in a complex Banach space \( X \), and \( f = h + \overline{g} : \Omega \to \mathbb{C}^n \) be a pluriharmonic mapping, where \( h \) and \( g \) are holomorphic mappings of \( \Omega \) into the Euclidean space \( \mathbb{C}^n \). Then

\[ |\nabla ||f||_{\ell^2}(z)| \leq \|Dh(z)\| + \|Dg(z)\|, \quad z \in \Omega. \]

**Proof.** For a pluriharmonic mapping \( f = h + \overline{g} : \Omega \to \mathbb{C}^n \), we have (see e.g. [30, p.638]),

\[ \|Df(z)\| \leq \|Dh(z)\| + \|Dg(z)\|, \quad z \in \Omega. \]

Then, by (1.11), we have

\[ |\nabla ||f||_{\ell^2}(z)| \leq \|Df(z)\| \leq \|Dh(z)\| + \|Dg(z)\|, \quad z \in \Omega. \]

This completes the proof. \( \square \)

Let \( X \) be the \( \ell^\infty \)-sum \( X = X_1 \oplus \cdots \oplus X_m \) of JB*-triples \( X_1, \ldots, X_m \) and let \( B_X = B_{X_1} \times \cdots \times B_{X_m} \) be the unit ball of \( X \). By applying Theorem K, we obtain the following improvement of (3.9) to pluriharmonic mappings of the unit ball \( B_X = B_{X_1} \times \cdots \times B_{X_m} \) into the Euclidean unit ball \( B^n \) of \( \mathbb{C}^n \).

**Theorem 3.10.** Let \( X \) be the \( \ell^\infty \)-sum \( X = X_1 \oplus \cdots \oplus X_m \) of JB*-triples \( X_1, \ldots, X_m \) and \( B_X = B_{X_1} \times \cdots \times B_{X_m} \) be the unit ball of \( X \). Also, let \( f = h + \overline{g} : B_X \to B^n \) be a pluriharmonic mapping, where \( h \) and \( g \) are holomorphic mappings of \( B_X \) into \( \mathbb{C}^n \). Then we have

(3.15) \[ \sum_{j=1}^m \left( \|Dh(z)\|_{\ell^2}^2 + \|Dg(z)\|_{\ell^2}^2 \right) \leq \frac{1 - \|f(z)\|^2}{(1 - \|z\|_{X}^2)^2} \]

for all \( w_j \in X_j \) with \( \|w_j\|_{X_j} = 1 \) \( (1 \leq j \leq m) \), where \( \tilde{w}_j = I_j(w_j) \) and \( I_j : X_j \to X \) is the natural inclusion mapping for \( j = 1, \ldots, m \).

Moreover, the inequality (3.15) is sharp for each \( z = (z_1, \ldots, z_m) \in B_X \) with \( \|z_1\|_{X_1} = \cdots = \|z_m\|_{X_m} \).
In particular, for $\mathbb{B}_X = \mathbb{B}^{k_1} \times \cdots \times \mathbb{B}^{k_p}$, we have the following generalization of (3.14) to pluriharmonic mappings of $\mathbb{B}_X$ into $\mathbb{B}^n$, where $\mathbb{B}^{k_j}$ are the Euclidean unit balls in $\mathbb{C}^{k_j}$ for $j \in \{1, \ldots, p\}$.

**Corollary 3.11.** Let $\mathbb{B}_X = \mathbb{B}^{k_1} \times \cdots \times \mathbb{B}^{k_p}$. Also, let $f = h + \overline{g} : \mathbb{B}_X \to \mathbb{B}^n$ be a pluriharmonic mapping, where $h$ and $g$ are holomorphic mappings of $\mathbb{B}_X$ into $\mathbb{C}^n$. Then

\[
\sum_{j=1}^{n} \sum_{k=1}^{m} \left( \left| \frac{\partial f_j(z)}{\partial z_k} \right|^2 + \left| \frac{\partial f_j(z)}{\partial \overline{z}_k} \right|^2 \right) \leq \kappa(1 - \| f(z) \|_n^2) \left( 1 - \| z \|_X^2 \right)^2,
\]

where $m = k_1 + \cdots + k_p$ and $\kappa = \max\{k_1, \ldots, k_p\}$.

**Proof.** By using the natural inclusion map $\zeta_j \in \mathbb{B}^{k_j} \to (\zeta_j, 0) \in \mathbb{B}^\kappa$, we may assume that $\kappa = k_1 = \cdots = k_p$. Let $\epsilon_1, \ldots, \epsilon_\kappa$ be the usual orthonormal basis of $\mathbb{C}^\kappa$ and let $\overline{e}_{j,l} = I_j(\epsilon_l)$, where $I_j : X_j \to X$ is the natural inclusion mapping. Then we have $\| \epsilon_\nu \|_{X_j} = 1$ for each $\nu$ with $1 \leq \nu \leq \kappa$ and each $j$ with $1 \leq j \leq p$. By (3.15), we have

\[
\sum_{j=1}^{p} \left( \| Dh(z)\overline{e}_{j,l} \|_e^2 + \| Dg(z)\overline{e}_{j,l} \|_e^2 \right) \leq \frac{1 - \| f(z) \|_e^2}{(1 - \| z \|_X^2)^2}, \quad 1 \leq l \leq \kappa,
\]

which gives that

\[
\sum_{j=1}^{n} \sum_{k=1}^{m} \left( \left| \frac{\partial f_j(z)}{\partial z_k} \right|^2 + \left| \frac{\partial f_j(z)}{\partial \overline{z}_k} \right|^2 \right) = \sum_{l=1}^{\kappa} \sum_{j=1}^{p} \left( \| Dh(z)\overline{e}_{j,l} \|_e^2 + \| Dg(z)\overline{e}_{j,l} \|_e^2 \right) \leq \frac{\kappa(1 - \| f(z) \|_n^2)}{(1 - \| z \|_X^2)^2}.
\]

This completes the proof. $\Box$

The following proposition is a generalization of (3.13) to pluriharmonic mappings of a domain $\Omega$ in the Euclidean space $\mathbb{C}^m$ into the Euclidean space $\mathbb{C}^n$. Note that we only need to assume that $f$ is pluriharmonic.

**Proposition 3.12.** Let $\Omega$ be a domain in the Euclidean space $\mathbb{C}^m$. Let $f = h + \overline{g} : \Omega \to \mathbb{C}^n$ be a pluriharmonic mapping, where $h$ and $g$ are holomorphic mappings from $\Omega$ to the Euclidean space $\mathbb{C}^n$. Then, we have

\[
\| \nabla f(z) \|_F \leq \sqrt{2m(\| Dh(z) \|_F^2 + \| Dg(z) \|_F^2)}.
\]

**Proof.** The inequality (3.17) follows from the relation

\[
\| \nabla f(z) \|_F^2 = 2(\| Dh(z) \|_F^2 + \| Dg(z) \|_F^2)
\]

and the inequality (1.2). This completes the proof. $\Box$

The following corollary is a generalization of (3.10) to pluriharmonic mappings from $\mathbb{B}_X = \mathbb{B}^{k_1} \times \cdots \times \mathbb{B}^{k_p}$ to $\mathbb{B}^n$. Note that (3.19) an improvement of (3.10). In particular, the following result holds for $\mathbb{B}_X = \mathbb{B}^m$ or for $\mathbb{B}_X = \mathbb{U}^m$. 

Corollary 3.13. Assume the hypotheses of Corollary 3.11. Then
\[
\| \nabla f(z) \|_F^2 \leq \frac{2\kappa(1 - \| f(z) \|_X^2)}{(1 - \| z \|_X^2)^2},
\]
where \( \kappa = \max\{k_1, \ldots, k_p\} \).

Proof. The inequality (3.16) and the relation (3.18) imply (3.19). This completes the proof. \(\qed\)

The following theorem is a generalization of (3.11) to pluriharmonic mappings from the unit ball \(B_X\) of a finite dimensional JB\(^*\)-triple \(X\) to the Euclidean unit ball \(B^n\) of \(C^n\), where \(n = \dim X\). Note that the condition \(\| \omega f \| < 1\) in \(B^n\) implies that \(f\) is a sense-preserving and locally univalent mapping in \(B^n\) (see [30, p.637]). Also, the condition (3.20) is also an improvement of (3.11).

Theorem 3.14. Let \(B_X\) be a bounded symmetric domain realized as the unit ball of a JB\(^*\)-triple \(X\) with \(\dim X = n < \infty\). Also, let \(f = h + g : B_X \to B^n\) be a pluriharmonic mapping, where \(h\) and \(g\) are holomorphic mappings of \(B_X\) into \(C^n\).

Assume that \(h\) is locally biholomorphic in \(B_X\) and \(\| \omega_f \| \leq k < 1\) holds in \(B_X\). Then, for all \(z \in B_X\),
\[
\| Dh(z) \| + \| Dg(z) \| \leq \frac{2K}{\sqrt{2(K^2 + 1)}} \sqrt{\frac{1 - \| f(z) \|_X^2}{1 - \| z \|_X^2}},
\]
where \(K = (1 + k)/(1 - k)\).

The proofs of Theorems 3.11-3.14 will be given in part II of Section 4.

4. PROOFS OF THE MAIN RESULTS

Part I. Schwarz type lemmas of holomorphic mappings and their applications.

The proof of Theorem 2.2. Let \(z\) be any fixed point in \(B_X \setminus \{0\}\). Without loss of generality, we assume that \(f(z) \neq 0\). Let \(\eta = z/\| z \|_X \in \partial B_X\). For any fixed \(b \in \partial B_Y\), let
\[
F(\xi) = l_b(f(\xi \eta)), \quad \xi \in U,
\]
where \(l_b \in T(b)\). Then \(F\) is a holomorphic mapping of \(U\) into \(\overline{U}\). By Remark 2.1, we have
\[
|F(\xi)| \leq \frac{|F(0)| + |\xi|}{1 + |F(0)||\xi|}
\]
for \(\xi \in U\). Elementary calculation leads to
\[
|F(0)| = |l_b(f(0))| \leq ||l_b||_{y^\prime} \| f(0) \|_{y^\prime} = \| f(0) \|_{y^\prime}.
\]
Since, for any fixed \(a \in [0, 1]\), the function \(\varphi(x) = (a + x)/(1 + ax)\) is increasing with respect to the variable \(x \in [0, \infty)\), by (4.1) and (4.2), we see that
For each fixed $b$ and any fixed point $f$ in (4.4), we obtain
\[ \|f(z)\|_Y \leq \frac{\|f(0)\|_Y + \|z\|_X}{1 + \|f(0)\|_Y \|z\|_X}. \]

Finally, by letting $\xi = \|z\|_X$ and $b = f(z)/\|f(z)\|_Y$ in (4.3), we obtain
\[ \|f(z)\|_Y \leq \frac{\|f(0)\|_Y + \|z\|_X}{1 + \|f(0)\|_Y \|z\|_X}. \]

Next, we show the sharpness part. If there is a point $z_0 \in B_X$ such that $\|f(z_0)\|_Y = 1$, then $\|f(z)\|_Y = 1$ for all $z \in B_X$. In this case, the sharpness part is obvious. In the following, we assume that $\|f(z)\|_Y < 1$ for all $z \in B_X$. For any fixed point $z_0 \in B_X \setminus \{0\}$, let $w_0 \in T(w_0)$ be fixed, where $w_0 = z_0/\|z_0\|_X$. For each fixed $b \in \partial B_Y$ and any fixed $a \in [0, 1)$, let
\[ f(z) = \frac{a + l_{w_0}(z)}{1 + al_{w_0}(z)} b, \quad z \in B_X. \]

Then $f$ is a holomorphic mapping from $B_X$ into $B_Y$ and by taking $z = z_0$, we obtain
\[ \|f(z_0)\|_Y = \left| \frac{a + l_{w_0}(z_0)}{1 + al_{w_0}(z_0)} \right| = \frac{\|f(0)\|_Y + \|z_0\|_X}{1 + \|f(0)\|_Y \|z_0\|_X}, \]
which completes the proof.

**The proof of Theorem 2.3.** Let $z$ be any fixed point in $B_X \setminus \{0\}$. Without loss of generality, we assume that $f(z) \neq 0$. Let $\eta = z/\|z\|_X \in \partial B_X$. For any fixed $b \in \partial B_Y$, let

\[ F(\xi) = l_b(f(\xi\eta)), \quad \xi \in \mathbb{U}, \]
where $l_b \in T(b)$. Then $F$ is a holomorphic mapping of $\mathbb{U}$ into itself with $F(0) = 0$. By Theorem B, we have
\[ |F(\xi)| \leq \frac{|F'(0)| + |\xi|}{1 + |F'(0)| |\xi|} |\xi| \quad \text{for} \quad \xi \in \mathbb{U}. \]

Elementary calculation leads to
\[ |F'(0)| = |l_b(Df(0)\eta)| \leq \|l_b\|_Y \|Df(0)\eta\|_Y \leq \|Df(0)\|, \]
which together with (4.4) yields that
\[ |l_b(f(\xi\eta))| \leq \frac{\|Df(0)\| + |\xi|}{1 + \|Df(0)\| |\xi|} |\xi|. \]

Finally, by letting $\xi = \|z\|_X$ and $b = f(z)/\|f(z)\|_Y$ in (4.5), we obtain
\[ \|f(z)\|_Y \leq \frac{\|Df(0)\| + \|z\|_X}{1 + \|Df(0)\| \|z\|_X} \|z\|_X. \]
Next, we show the sharpness part. For any fixed point \( z_0 \in B_X \setminus \{0\} \), let \( l_{w_0} \in T(w_0) \) be fixed, where \( w_0 = z_0/\|z_0\|_X \). For any fixed \( b \in \partial B_Y \) and any fixed \( a \in [0, 1] \), let
\[
\|f(z)\|_Y = \left| l_{w_0}(z_0) \frac{a + l_{w_0}(z_0)}{1 + al_{w_0}(z_0)} b, \quad z \in B_X. \right|
\]
Then \( f \) is a holomorphic mapping from \( B_X \) into \( B_Y \) with \( f(0) = 0 \) and by letting \( z = z_0 \), we have
\[
\|f(z_0)\|_Y = \left| l_{w_0}(z_0) \frac{a + l_{w_0}(z_0)}{1 + al_{w_0}(z_0)} \right| = \|a\|_X \frac{\|Df(0)\| + \|z_0\|_X}{1 + \|Df(0)\| \|z_0\|_X},
\]
which completes the proof. \( \square \)

**The proof of Theorem 2.4.** We only need to prove (2.4) for \( k \in (0, 1) \), since \( \mathcal{B}_f / \mathcal{B}_h \equiv 1 \) for \( k = 0 \). In the following, we assume that \( k \in (0, 1) \) and we divide the proof into five steps.

**Step 1.** For \( f = h + \varphi \in \mathcal{PH}(k) \), we claim that \( \mathcal{B}_f(0) = d(0, \partial f(B)) \) and \( \mathcal{B}_h(0) = d(0, \partial h(B)) \), where \( d(0, \partial f(B)) \) and \( d(0, \partial h(B)) \) denote the Euclidean distances from 0 to \( \partial f(B) \) and \( \partial h(B) \), respectively. We only need to prove \( \mathcal{B}_f(0) = d(0, \partial f(B)) \) because the proof of another one is similar. From the definition of \( \mathcal{B}_f(0) \), we see that \( \mathcal{B}_f(0) \) is equal either to the Euclidean distance from \( f(0) \) to a boundary point of \( f(B) \) or to the Euclidean distance from \( f(0) \) to a critical value of \( f \). In the following, we will show that the critical value of \( f \) does not exist. Since \( \|\omega_f\| \leq k \), we see that
\[
\det J_f = \det Df = \det Dg[Dh]^{-1} \det [Dg[Dh]^{-1}]^{-1} \neq 0.
\]
Consequently, \( f \) is locally univalent in \( B \). Let \( V \) be a subdomain of \( B \) such that \( f(V) = \{w : \|w\|_e < \mathcal{B}_f(0)\} \). If there exist \( z_0 \in \partial V \) and \( z_1 \in V \) such that \( f(z_0) = f(z_1) \), then, by the condition that \( \det J_f \neq 0 \) in \( B \), there exist neighbourhoods \( U_0(z_0) \) of \( z_0 \), \( U_1(z_1) \) of \( z_1 \), and \( U_2(f(z_0)) \) of \( f(z_0) = f(z_1) \) such that \( f \) maps \( U_0(z_0) \) and \( U_1(z_1) \) onto \( U_2(f(z_0)) \) univalently, respectively. This contradicts the fact that \( f \) is univalent in \( V \). Therefore, the critical points do not exist. Hence \( \mathcal{B}_f(0) = d(0, \partial f(B)) \). Without loss of generality, we assume that there exists a boundary point \( \xi_0 \) of \( f(B) \) such that \( \xi_0 \in \{w \in \mathbb{C}^n : \|w\|_e = \mathcal{B}_f(0)\} \). Let \( \ell_{\xi_0} = f^{-1}(0, \xi_0) \) be the preimage of the semi-open segment \( [0, \xi_0] \) with the starting point 0 in the ball \( B \). Then
\[
\mathcal{B}_f(0) = \|\xi_0\|_e = \int_{\ell_{\xi_0}} \|df(z)\|_e = \inf_{\gamma} \int_{\gamma} \|df(z)\|_e,
\]
where the minimum is taken over all smooth paths \([0, 1] \ni t \mapsto \gamma(t) \in B \) with \( \gamma(0) = 0 \) and \( \lim_{t \to 1^-} \|\gamma(t)\|_X = 1 \). Similarly, we can also assume that
\[
\mathcal{B}_h(0) = \|\xi_1\|_e = \int_{\ell_{\xi_1}} \|dh(z)\|_e = \inf_{\gamma} \int_{\gamma} \|dh(z)\|_e,
\]
where \( \xi_1 \) is a boundary point of \( h(B) \) such that \( \xi_1 \in \{w \in \mathbb{C}^n : \|w\|_e = \mathcal{B}_h(0)\} \) and the simple smooth curve \( \ell_{\xi_1} = h^{-1}(0, \xi_1) \) is the preimage of the semi-open
segment $[0, \xi_1)$ under the mapping $h$. For $t \in [0, 1)$, let $\ell_{\xi_0} := \ell_{\xi_0}(t) = f^{-1}(\xi_0 t)$ and $\ell_{\xi_1} := \ell_{\xi_1}(t) = h^{-1}(\xi_1 t)$.

**Step 2.** We first establish the lower bound of $B_f(0)/B_h(0)$. Differentiation of the equation $f^{-1}(f(z)) = z$ yields the following two formulas:

$$DF^{-1}Dh + Df^{-1}Dg = I$$
and

$$Df^{-1}Dg + Df^{-1}Dh = O,$$

which imply that

(4.7) $$Df^{-1} = [Dh]^{-1} (I - \overline{\omega}_f)$$
and

(4.8) $$Df^{-1} = -[Dh]^{-1} (I - \overline{\omega}_f)\overline{\omega}_f,$$

where $f^{-1}(w) = (\sigma_1(w), \ldots, \sigma_n(w))$ and $Df^{-1}(w) = \left(\frac{\partial \sigma_j}{\partial w_k}\right)_{n \times n}$ for $k, j \in \{1, \ldots, n\}$. Then, by (4.7) and (4.8), we have

$$\left\| DhDf^{-1} \frac{\xi_0}{\|\xi_0\|_e} \right\|_e + \left\| DhDf^{-1} \frac{\overline{\xi}_0}{\|\overline{\xi}_0\|_e} \right\|_e \leq \left\| (I - \overline{\omega}_f)^{-1} \right\|$$

$$+ \left\| (I - \overline{\omega}_f)\overline{\omega}_f \right\|$$

$$\leq \frac{1}{1 - \|\omega_f\|^2} + \frac{\|\omega_f\|^2}{1 - \|\omega_f\|^2}$$

$$= \frac{1}{1 - \|\omega_f\|^2}.$$

Consequently,

$$B_h(0) = \int_0^1 \left\| dh(\ell_{\xi_1}(t)) \right\|_e \leq \int_0^1 \left\| dh(\ell_{\xi_0}(t)) \right\|_e$$

$$= \int_0^1 \left\| (Dh(\ell_{\xi_0}(t))Df^{-1}(\xi_0 t)\xi_0 + Dh(\ell_{\xi_0}(t))\overline{Df^{-1}(\xi_0 t)\overline{\xi}_0} \right\|_e$$

$$\leq \|\xi_0\|_e \int_0^1 \left( \left\| Dh(\ell_{\xi_0}(t))Df^{-1}(\xi_0 t) \frac{\xi_0}{\|\xi_0\|_e} \right\|_e + \left\| Dh(\ell_{\xi_0}(t))\overline{Df^{-1}(\xi_0 t)\overline{\xi}_0} \|\overline{\xi}_0\|_e \right\|_e \right) dt$$

$$\leq \frac{1}{1 - \|\omega_f(\ell_{\xi_0}(t))\|} \int_0^1 \frac{dt}{1 - \|\omega_f(\ell_{\xi_0}(t))\|},$$

which gives that

(4.9) $$\frac{B_f(0)}{B_h(0)} \geq \int_0^1 \frac{1}{1 - \|\omega_f(\ell_{\xi_0}(t))\|} \geq 1 - k.$$
Step 3. In this step, we will give the lower bound of $\mathcal{B}_f(z)/\mathcal{B}_h(z)$ for all $z \in \mathcal{B}$. Since $\mathcal{B}$ is homogeneous, we see that, for any fixed $\zeta \in \mathcal{B}$, there exists a $\phi \in \text{Aut}(\mathcal{B})$ such that $\phi(0) = \zeta$. For $z \in \mathcal{B}$, let

\[(4.10) \quad F(z) = f(\phi(z)) - f(\phi(0)) = H(z) + \overline{G(z)},\]

where $H(z) = h(\phi(z)) - h(\phi(0))$ and $G(z) = g(\phi(z)) - g(\phi(0))$. Then $H(0) = G(0) = 0$ and

$$\|\omega_F(z)\| = \|DG(z)[DH(z)]^{-1}\| = \|\omega_f(\phi(z))\| \leq k,$$

which imply that $F \in \mathcal{P}\mathcal{H}(k)$.

By (4.9) and (4.10), we have

\[(4.11) \quad \mathcal{B}_F(0) = \mathcal{B}_f(\zeta) \geq (1 - k)\mathcal{B}_H(0).\]

Note that $\mathcal{B}_H(0) = \mathcal{B}_h(\zeta)$, which, together with (4.11), implies that

\[(4.12) \quad \mathcal{B}_f(\zeta) \geq (1 - k)\mathcal{B}_h(\zeta).\]

Next we prove that (4.12) is sharp for all $\zeta \in \mathcal{B}$. Let

$$R = \inf\{\|z\|_e : z \in \partial\mathcal{B}\}.$$

Then there exists a point $z_0 \in \partial\mathcal{B}$ such that $\|z_0\|_e = R$. Let $U$ be a unitary transformation of $\mathbb{C}^n$ such that $Uz_0$ is a pure imaginary vector in $\mathbb{C}^n$. For $z \in \mathcal{B}$, let

$$f(z) = h(z) + \overline{g(z)} = Uz + kU\overline{z},$$

where $k \in [0, 1)$ is a constant. Then $\mathcal{B}_h(0) = R$. Also, $f$ is univalent on $\mathcal{B}$ and $\mathcal{B}_f(0) = R(1 - k)$, which gives that $\mathcal{B}_f(0)/\mathcal{B}_h(0) = 1 - k$. In the following, we will show that $\mathcal{B}_f(\zeta)/\mathcal{B}_h(\zeta) = 1 - k$ for all $\zeta \in \mathcal{B}$.

For any fixed $\zeta \in \mathcal{B}$, let

$$F(z) = f(\phi(z)) - f(\phi(0)) = H(z) + \overline{G(z)},$$

where $\phi \in \text{Aut}(\mathcal{B})$ with $\phi(0) = \zeta$. Then

$$\mathcal{B}_f(\zeta) = \mathcal{B}_F(0) = (1 - k)\mathcal{B}_H(0) = (1 - k)\mathcal{B}_h(\zeta).$$
Step 4. Now we estimate the upper bound of $\mathcal{B}_f(0)/\mathcal{B}_h(0)$. From (4.6) and the relation $Dh(\ell_{\xi_1}(t))\ell'_{\xi_1}(t) = \xi_1$, we have

$$
\mathcal{B}_f(0) = \int_0^1 \|df(\ell_{\xi_0}(t))\|_e \leq \int_0^1 \|df(\ell_{\xi_1}(t))\|_e \\
= \int_0^1 \left\| (Dh(\ell_{\xi_1}(t))\ell'_{\xi_1}(t) + Dg(\ell_{\xi_1}(t))\ell'_{\xi_1}(t)) dt \right\|_e \\
= \int_0^1 \left\| (\xi_1 + Dg(\ell_{\xi_1}(t))Dh(\ell_{\xi_1}(t))^{-1}Dh(\ell_{\xi_1}(t))\ell'_{\xi_1}(t)) dt \right\|_e \\
= \int_0^1 \left\| (\xi_1 + \omega_f(\ell_{\xi_1}(t))\ell_{\xi_1}(t)) dt \right\|_e \\
\leq \|\xi_1\|_e \int_0^1 (1 + \|\omega_f(\ell_{\xi_1}(t))\|) dt \\
(4.13)
= \mathcal{B}_h(0) \left(1 + \int_0^1 \|\omega_f(\ell_{\xi_1}(t))\| dt\right).
$$

Applying Theorem 2.2 to $\omega_f/k$, we have

$$
\|\omega_f(z)\|_k \leq \|z\|_X + \frac{\|\omega_f(0)\|}{k} \|z\|_X
$$

for $z \in \mathcal{B}$, where $X = \mathbb{C}^n$. Since $h^{-1}(w\|\xi_1\|_e)$ biholomorphically maps $\mathbb{B}^n$ onto some subdomain of $\mathcal{B}$ with $h^{-1}(0) = 0$, by Theorem 2.3 we see that $\|\ell_{\xi_1}(t)\|_X \leq t$. Consequently, by (4.13) and (4.14), we have

$$
(4.15) \quad \mathcal{B}_f(0) \leq \mathcal{B}_h(0) \left(1 + k \int_0^1 \frac{\|\omega_f(0)\| + k\|\ell_{\xi_1}(t)\|_X}{k + \|\omega_f(0)\| \|\ell_{\xi_1}(t)\|_X} dt\right) \\
\leq \mathcal{B}_h(0) \mu_k \left(\frac{\|\omega_f(0)\|}{k}\right).
$$

For any fixed $k \in [0, 1)$, it is not difficult to see that $\mu_k(x)$ is an increasing function of $x \in (0, 1]$.

Step 5. At last, we will establish the upper bound of $\mathcal{B}_f(z)/\mathcal{B}_h(z)$ for all $z \in \mathcal{B}$. For any fixed $\zeta \in \mathcal{B}$, let $\phi \in \text{Aut}(\mathcal{B})$ with $\phi(0) = \zeta$. It follows from (4.10) and (4.15) that

$$
(4.16) \quad \mathcal{B}_f(\zeta) = \mathcal{B}_f(0) \leq \mu_k \left(\frac{\|\omega_f(\phi(0))\|}{k}\right) \mathcal{B}_h(0),
$$

where $F$ and $H$ are defined in (4.10). Note that

$$
(4.17) \quad \mathcal{B}_h(0) = \mathcal{B}_h(\zeta).
$$

It follows from (4.16) and (4.17) that...
(4.18) \( \mathcal{B}_f(\zeta) \leq \mu_k \left( \frac{\| \omega f(\zeta) \|}{k} \right) \) \( \mathcal{B}_h(\zeta) \leq \mu_k(1)\mathcal{B}_h(\zeta) = (1 + k)\mathcal{B}_h(\zeta) \).

Furthermore, the estimate in (4.18) is asymptotically sharp because
\[
\lim_{k \to 0^+} \mu_k(x) = 1.
\]
The proof of the theorem is finished. \( \square \)

The proof of Theorem 2.5.

The triangle inequality leads to
\[
(4.19) \quad \left\| f(z) - f(b) \right\| \geq \frac{1 - \|f(z)\|_Y}{1 - \|z\|_X}
\]
for \( z \in B_X \). It follows from Theorem 2.3 that
\[
\frac{1 - \|f(z)\|_Y}{1 - \|z\|_X} \geq \frac{1 + \|z\|_X}{1 + \|Df(0)\|\|z\|_X},
\]
which, together with (4.19), implies that
\[
(4.20) \quad \liminf_{r \to 1^-} \left\| \frac{f(rb) - f(b)}{\|rb\|_X - \|b\|_X} \right\|_Y \geq \liminf_{r \to 1^-} \frac{1 + \|rb\|_X}{1 + \|Df(0)\|\|rb\|_X} = \frac{2}{1 + \|Df(0)\|},
\]
Since the radial derivative
\[
Df(b) = \lim_{r \to 1^-} \frac{f(rb) - f(b)}{r - 1}
\]
exists, the desired result follows from (4.20).

For a given value \( \|Df(0)\| = r \in [0, 1] \), the sharpness part follows from the mapping
\[
f(z) = \frac{l_b(z) + r}{1 + rl_b(z)}l_b(z)y, \quad z \in B_X,
\]
where \( l_b \in T(b) \), and \( y \in \partial B_Y \) is arbitrary. \( \square \)

Theorem L. (70, Theorem 1.1) Let \( f \) be a holomorphic self-mapping of \( \mathbb{U} \). If \( f \)
is holomorphic at \( z = 1 \) with \( f(1) = 1 \), then
\[
f'(1) \geq \frac{2|1 - f(0)|^2}{1 - |f(0)|^2 + |f'(0)|}.
\]
This estimate is sharp with equality possible for each value of \( f(0) \) and \( |f'(0)| \) with \( |f'(0)| \leq 1 - |f(0)|^2 \). The extreme function is
\[
f(z) = \frac{\gamma A(z) + f(0)}{1 + \gamma f(0)A(z)},
\]
where \( \gamma = (1 - f(0))/(1 - f(0)) \) and \( A(z) = z((1 - |f(0)|^2)z + |f'(0)|)/((1 - |f(0)|^2) + |f'(0)|z). \)
The proof of Theorem 2.6. For \( \zeta \in U \), let

\[
F(\zeta) = \frac{1}{2c(B_Y)} h_0(f(\zeta\alpha), \beta),
\]

where \( f : G \rightarrow B_Y \) is a holomorphic mapping with \( f(\alpha) = \beta \). By the assumption, and Loos [50, Theorem 6.5], we have \( F(1) = 1 \). Then \( F \) is a holomorphic mapping of \( U \) into itself such that \( F \) is holomorphic at \( \zeta = 1 \) and \( F(1) = 1 \). Elementary computations lead to

\[
F'(1) = \frac{1}{2c(B_Y)} h_0(Df(\alpha)\alpha, \beta) \quad \text{and} \quad F'(0) = \frac{1}{2c(B_Y)} h_0(Df(0)\alpha, \beta),
\]

which, together with Theorem L, yield that

\[
\frac{1}{2c(B_Y)} h_0(Df(\alpha)\alpha, \beta) \geq \frac{2 \left| 1 - \frac{1}{2c(B_Y)} h_0(f(0), \beta) \right|^2}{1 - \frac{1}{2c(B_Y)} h_0(f(0), \beta)^2 + ||Df(0)\alpha||_Y},
\]

where we have used the inequality \( |F'(0)| \leq ||Df(0)\alpha||_Y \). Next, we prove the sharpness part. Since \( G = B_X \) is the unit ball of \( X \), for any holomorphic function \( \varphi \) of \( U \) into itself and for any \( l_\alpha \in T(\alpha) \), the mapping \( f(z) = \varphi(l_\alpha(z)) \beta \) is a holomorphic mapping of \( B_X \) into \( B_Y \). Then, it follows from Theorem L that there exists a holomorphic mapping of \( B_X \) into \( B_Y \) with \( f(\alpha) = \beta \) such that

\[
h_0(f(\zeta\alpha), \beta) = 2c(B_Y) \frac{a + \epsilon \tau(\zeta)}{1 + \epsilon \tau(\zeta)},
\]

where

\[
a = \frac{1}{2c(B_Y)} h_0(f(0), \beta), \quad b = \frac{1}{2c(B_Y)} h_0(Df(0)\alpha, \beta)
\]

and

\[
\epsilon = 1 - \frac{a}{1 - \pi}, \quad \tau(\zeta) = \frac{(1 - |a|^2)\zeta + |b|}{(1 - |a|^2) + |b|\zeta}.
\]

The proof of this theorem is finished. \( \square \)

Part II. Schwarz type lemmas of pluriharmonic mappings and their applications.

The proof of Theorem 3.1. Let \( z \in B_X \setminus \{0\} \) be fixed. Without loss of generality, we may assume that \( a = f(z) - \frac{1 - ||z||_X^2}{1 + ||z||_X^2} f(0) \neq 0 \). Let \( w = z/||z||_X \in \partial B_X \) and let \( u \in \partial B_Y \) be arbitrarily fixed. Since, for each \( l_u \in T(u) \),

\[
\varphi(\zeta) = l_u(f(\zeta w)), \quad \zeta \in U,
\]

is a harmonic mapping in \( U \) such that \( \varphi(U) \subset U \), we obtain from [55, Theorem 3.6.1] (or [36, Theorem 1]) that, for all \( \zeta \in U \),

\[
\left| l_u(f(\zeta w)) - \frac{1 - |\zeta|^2}{1 + |\zeta|^2} l_u(f(0)) \right| \leq \frac{4}{\pi} \arctan|\zeta|.
\]
Especially, let $\zeta = \|z\|_X$. Since $\zeta w = z$, we have

$$|l_u(a)| = \left| l_u \left( f(z) - \frac{1 - \|z\|_X^2}{1 + \|z\|_X^2} f(0) \right) \right| \leq \frac{4}{\pi} \arctan \|z\|_X.$$  

Finally, if $u = a/\|a\|_Y$, then we obtain

$$\left\| f(z) - \frac{1 - \|z\|_X^2}{1 + \|z\|_X^2} f(0) \right\|_Y \leq \frac{4}{\pi} \arctan \|z\|_X.$$  

Next, we prove the sharpness part. For any fixed point $z_0 \in B_X \setminus \{0\}$, let $l_{w_0} \in T(w_0)$ be fixed, where $w_0 = z_0/\|z_0\|_X$. It follows from [55, Theorem 3.6.1] (or [36, Theorem 1]) that there exists a harmonic mapping $\Phi$ of $U$ into itself with $\Phi(0) = 0$ such that

$$|\Phi(l_{w_0}(z_0))| = \frac{4}{\pi} \arctan |l_{w_0}(z_0)|.$$  

For any fixed $b \in \partial B_Y$, let

$$f(z) = \Phi(l_{w_0}(z))b, \ z \in B_X.$$  

Then $f : B_X \to B_Y$ is harmonic and

$$\|f(z_0)\| = |\Phi(l_{w_0}(z_0))| = \frac{4}{\pi} \arctan |l_{w_0}(z_0)| = \frac{4}{\pi} \arctan \|z_0\|_X;$$

which completes the proof. \hfill \Box

**Theorem M.** ([70, Theorem 1.7]) *Let $f$ be a harmonic mapping of $U$ into itself with $f(0) = 0$. Then, for $z \in \mathbb{U}$, we have*

$$|f(z)| \leq \frac{4}{\pi} \arctan \left( \frac{|z| + \frac{4}{\pi} (|f_1(0)| + |f_2(0)|)}{1 + \frac{4}{\pi} (|f_1(0)| + |f_2(0)|)|z|} \right).$$

**The proof of Theorem 3.2** For any fixed $z \in B_X \setminus \{0\}$, let $w = z/\|z\|_X \in \partial B_X$. Without loss of generality, we assume $f(z) \neq 0$. Let $u \in \partial B_Y$ be arbitrarily fixed. Since, for each $l_u \in T(u)$, the function $\varphi$ defined by

$$\varphi(\zeta) = l_u(f(\zeta w))$$

is a harmonic mapping of $U$ into itself with $\varphi(0) = 0$, we obtain from Theorem M that, for all $\zeta \in \mathbb{U},$

$$|l_u(f(\zeta w))| = |\varphi(\zeta)| \leq \frac{4}{\pi} \arctan \left( \frac{|\zeta| + \frac{4}{\pi} (|\varphi_1(0)| + |\varphi_2(0)|)}{1 + \frac{4}{\pi} (|\varphi_1(0)| + |\varphi_2(0)|)|\zeta|} \right).$$

By the definition of $\Lambda_f(0; w)$, we have

$$|\varphi_1(0)| + |\varphi_2(0)| \leq \Lambda_f(0; w),$$

which, together with (4.21), implies that
(4.22) \[ |l_u(f(\zeta w))| \leq \frac{4}{\pi} \arctan \left( \frac{\|z\| X + \frac{\pi}{4} \Lambda f(0; w)}{1 + \frac{\pi}{4} \Lambda f(0; w)\|z\| X} \right). \]

By letting \( \zeta = \|z\| X \) in (4.22), we have
\[ |l_u(f(z))| \leq \frac{4}{\pi} \arctan \left( \frac{\|z\| X}{1 + \frac{\pi}{4} \Lambda f(0; w)\|z\| X} \right). \]

Finally, if \( u = f(z)/\|f(z)\| Y \), then we get the desired result. \( \square \)

The proof of Theorem 3.3. It follows from Theorem 3.1 that
\[ \frac{1 - \|f(z)\| Y}{1 - \|z\| X} \geq \frac{1 - \frac{4}{\pi} \arctan \|z\| X}{1 - \|z\| X} - \frac{1 + \|z\| X \|f(0)\| Y}{1 + \|z\| X}. \]

By using arguments similar to those in the proof of Theorem 2.5, we obtain
\[ \|Df(b)b\| Y \geq \liminf_{r \to 1^-} \left( \frac{1 - \frac{4}{\pi} \arctan \|rb\| X}{1 - \|rb\| X} - \frac{1 + \|rb\| X \|f(0)\| Y}{1 + \|rb\| X} \right) \]
\[ = \frac{2}{\pi} - \|f(0)\| Y. \]

Next, let \( l(f(b)) \in T(f(b)) \) and
\[ p(\zeta) = 1 - \text{Re}(l(f(b))(\zeta b)), \quad \zeta \in \mathbb{U}. \]

Then \( p \) is a positive real valued harmonic function on \( \mathbb{U} \) with \( p(1) = 0 \). By Harnack’s inequality, we have
\[ \frac{1 - \rho}{1 + \rho} p(0) \leq p(\zeta), \quad \rho = |\zeta| < 1. \]

Therefore, we have
\[ \frac{1 - \rho}{1 + \rho} p(0) \leq p(r) \leq \frac{p(r) - p(1)}{1 - r}, \quad 0 < r < 1. \]

Letting \( r \to 1^- \), we have
\[ \frac{1 - \|f(0)\| Y}{2} \leq \frac{1 - \text{Re}(l(f(b))(f(0)))}{2} \leq \text{Re}(l(f(b))(Df(b)b)) \leq \|Df(b)b\| Y. \]

Then combining (4.23) and (4.24) gives the desired result. This completes the proof. \( \square \)
The proof of Theorem 3.5. By the definition of \( c(\mathbb{B}_Y) \) and the assumption on \( f \), we see that \( \varphi \) is a harmonic mapping of \( U \) into itself with \( \varphi(0) = 0 \). Also, \( \varphi \) is differentiable at \( \zeta = 1 \) and \( \varphi(1) = 1 \). Then, by [70, Theorem 1.8], we have

\[
\Re(\varphi_\zeta(1) + \varphi_{\overline{\zeta}}(1)) \geq \frac{4}{\pi} \frac{1}{1 + \frac{4}{\pi} \Lambda_f(0; \alpha)} 
\]

which implies that (3.3). The mapping \( f(z) = \psi(l_\alpha(z)) \beta \) gives the sharpness, where

\[
\psi(\zeta) = \frac{2}{\pi} \arctan \frac{2 \Re(\zeta)}{1 - |\zeta|^2}, \quad \zeta \in U.
\]

This completes the proof. \( \square \)

The proof of Theorem 3.7. First, we consider the case \( z_0 = 0 \). By Proposition 1.1, it suffices to show that

\[
\left\{
\begin{aligned}
\|Df(0)\| &\leq \frac{4}{\pi} \quad \text{if } f(0) = 0; \\
\sup_{\|\beta\|_X = 1} |l_f(0)(Df(0)\beta)| &\leq \frac{4}{\pi} \quad \text{if } f(0) \neq 0.
\end{aligned}
\right.
\]

(i) If \( f(0) = 0 \), then let \( F(\zeta) = l_f(\zeta) \beta \) for \( \zeta \in U \), where \( \beta \in X \) with \( \|\beta\|_X = 1 \) and \( l \in Y^* \) with \( \|l\|_{Y^*} = 1 \) are arbitrarily fixed. Then \( F : U \to U \) is harmonic. By applying Theorem F to the harmonic mapping \( F \), we have

\[
|l(Df(0)\beta)| \leq \frac{4}{\pi}.
\]

Since \( \beta \in X \) with \( \|\beta\|_X = 1 \) and \( l \in Y^* \) with \( \|l\|_{Y^*} = 1 \) are arbitrary, we obtain that

\[
\|Df(0)\| \leq \frac{4}{\pi}.
\]

(ii) If \( f(0) \neq 0 \), then let \( F(\zeta) = l_f(0)(f(\zeta)\beta) \) for \( \zeta \in U \), where \( \beta \in X \) with \( \|\beta\|_X = 1 \) is arbitrarily fixed. Then \( F : U \to U \) is harmonic. By applying Theorem F to the harmonic mapping \( F \), we have

\[
|l_f(0)(Df(0)\beta)| \leq \frac{4}{\pi}.
\]

Therefore, we have proved (3.8) in the case \( z_0 = 0 \).

Next, we consider the case \( z_0 \neq 0 \). Let \( g_{z_0} \in \text{Aut}(\mathbb{B}_X) \) be the Möbius transformation of \( \mathbb{B}_X \) defined by (1.3). By Proposition 1.1, we have

\[
|\nabla f(0) - g_{z_0}| \leq |\nabla f \circ g_{z_0} - g_{z_0} - 1| \cdot \|Dg_{z_0}(0)^{-1}\|_X.
\]

Since \( f \circ g_{z_0} \) satisfies the assumptions of the theorem for \( z_0 = 0 \), by applying (3.8) in the case \( z_0 = 0 \) and using (1.4), we obtain that

\[
|\nabla f(0) - g_{z_0}| \leq \frac{4}{\pi} \frac{1}{1 - \|z_0\|_X^2}.
\]
Finally, we will show that the estimate (3.8) is sharp. Let \( z_0 \in \mathbb{B}_X \setminus \{0\} \) be fixed and let \( w_0 = z_0 / \| z_0 \|_X \). It follows from Theorem F that there exists a harmonic mapping \( \phi \) of \( U \) into itself such that \( \phi(\| z_0 \|_X) \in \mathbb{R} \setminus \{0\} \) and

\[
|\phi(\| z_0 \|_X)| + |\phi(\| z_0 \|_X)| = \frac{4}{\pi} \frac{1}{1 - \| z_0 \|_X^2}.
\]

For any fixed \( l_{w_0} \in T(w_0) \) and any fixed \( a \in \partial B_Y \), let

\[ f(z) = \phi(l_{w_0}(z))a, \quad z \in \mathbb{B}_X. \]

Then \( f \) is a pluriharmonic mapping from \( \mathbb{B}_X \) into \( B_Y \). Moreover,

\[
|\nabla \| f \|_Y (z_0)| = \sup_{\| \beta \|_X = 1} |\phi(\| z_0 \|_X) l_{w_0}(\beta) + \phi(\| z_0 \|_X) l_{w_0}(\beta)| \]

\[
= |\phi(\| z_0 \|_X)| + |\phi(\| z_0 \|_X)| \]

\[
= \frac{4}{\pi} \frac{1}{1 - \| z_0 \|_X^2}.
\]

If \( z_0 = 0 \), then for arbitrary \( w_0 \in \partial \mathbb{B}_X \), by using the above argument, we have

\[
|\nabla \| f \|_Y (0)| = \frac{4}{\pi}.
\]

This completes the proof.

**The proof of Theorem 3.10** First, we show that

\[
(4.25) \quad \sum_{j=1}^{m} \left( \| Dh(0)\bar{w}_j \|_e^2 + \| Dg(0)\bar{w}_j \|_e^2 \right) \leq 1 - \| f(0) \|_e^2
\]

for all \( w_j \in X_j \) with \( \| w_j \|_{X_j} = 1 \) (1 \( \leq j \leq m \)). Indeed, let \( w_j \in X_j \) with \( \| w_j \|_{X_j} = 1 \) (1 \( \leq j \leq m \)) be fixed and let \( F(\zeta_1, \ldots, \zeta_m) = f(\zeta_1 w_1, \ldots, \zeta_m w_m) \) for \( \zeta = (\zeta_1, \ldots, \zeta_m) \in U^m \). Applying Theorem K to \( F \), we obtain (4.25).

Next, let \( z \in \mathbb{B}_X \setminus \{0\} \) and \( w = (w_1, \ldots, w_m) \in X \) with \( \| w_j \|_{X_j} = 1 \) for \( j = 1, \ldots, m \) be fixed. Let \( g_z \) be the Möbius transformation defined by (1.3). Applying (4.25) to the pluriharmonic mapping \( f \circ g_z = h \circ g_z + g \circ g_z \) and the unit vector \( [Dg_z(0)]^{-1}w / \|[Dg_z(0)]^{-1}w\|_X \), we have

\[
\sum_{j=1}^{m} \left( \| Dh(z)\bar{w}_j \|_e^2 + \| Dg(z)\bar{w}_j \|_e^2 \right) \leq \| f(z) \|_e^2 / \|[Dg_z(0)]^{-1}w\|_X^2 \leq 1 - \| f(z) \|_e^2.
\]

Therefore, by using (4.4), we have

\[
\sum_{j=1}^{m} \left( \| Dh(z)\bar{w}_j \|_e^2 + \| Dg(z)\bar{w}_j \|_e^2 \right) \leq \|[Dg_z(0)]^{-1}w\|_X^2 (1 - \| f(z) \|_e^2) \leq \frac{1 - \| f(z) \|_e^2}{(1 - \| z \|_X^2)^2}
\]

as desired.

Next, we prove the sharpness part. Let \( a = (a_1, \ldots, a_m) \in \mathbb{B}_X \setminus \{0\} \) with \( \| a_1 \|_{X_1} = \cdots = \| a_m \|_{X_m} \) be arbitrarily fixed. For \( z \in \mathbb{B}_X \), let

\[ f(z) = (f_1(z), f_2(z), \ldots, f_n(z)) \in \mathbb{B}^n, \]
where
\[ f_1(z) = \frac{-\|a_1\|_{X_1} + f_a(z)}{1 - \|a_1\|_{X_1} \eta}(z), \]
and \( f_j(z) \equiv 0 \) for \( j \in \{2, \ldots, n\} \). Then this mapping gives the sharpness at \( z = a \).
This completes the proof. \( \square \)

The proof of Theorem 3.14 Let \( w \in X \) with \( \|w\|_X = 1 \) be fixed. Since
\[ \|Dg(z)w\|_e = \|\omega_f(z)Dh(z)w\|_e \leq k\|Dh(z)w\|_e \]
for all \( z \in B_X \), there exists a function \( \eta_w(z) \in [0, k] \) such that \( \|Dg(z)w\|_e = \eta_w(z)\|Dh(z)w\|_e \).
Then, by (3.15) with \( m = 1 \), we have
\[ \|Dh(z)w\|_e \leq \frac{1}{\sqrt{1 + \eta_w^2(z)}} \sqrt{1 - \|f(z)\|^2} \]
and
\[ \|Dg(z)w\|_e \leq \frac{\eta_w(z)}{\sqrt{1 + \eta_w^2(z)}} \sqrt{1 - \|f(z)\|^2}, \]
which, together with the monotonicity of function \( \chi(t) = (1 + t)/\sqrt{1 + t^2} \) for \( t \in [0, 1] \), give that
\[ \|Dh(z)\| + \|Dg(z)\| \leq \frac{1 + \eta_w(z)}{\sqrt{1 + \eta_w^2(z)}} \sqrt{1 - \|f(z)\|^2} \]
\[ \leq \frac{1 + k}{\sqrt{1 + k^2}} \sqrt{1 - \|f(z)\|^2} \]
\[ = \frac{2K}{\sqrt{2(K^2 + 1)}} \sqrt{1 - \|f(z)\|^2}. \]
This completes the proof. \( \square \)

5. A CONCLUDING REMARK

Let \( B_j \) be the unit ball of a complex Hilbert space \( H_j \) for \( j = 1, 2 \), respectively. Note that if \( f \) is \( C^1 \) at \( z_0 \in \partial B_1 \) with values in \( H_2 \), then the adjoint operator \( Df(z_0)^* \) is defined by
\[ \text{Re} \left( \langle Df(z_0)^* w, z \rangle_{H_1} \right) = \text{Re} \left( \langle w, Df(z_0)z \rangle_{H_2} \right) \]
for \( z \in H_1 \), \( w \in H_2 \), where \( \langle \cdot, \cdot \rangle_{H_j} \) is the inner product of \( H_j \), \( j = 1, 2 \). The following result was obtained in [25, Proposition 1.8].

Proposition 5.1. Let \( B_j \) be the unit ball of a complex Hilbert space \( H_j \) for \( j = 1, 2 \), respectively. Let \( f : B_1 \to B_2 \) be a pluriharmonic mapping. Assume that \( f \) is of class \( C^1 \) at some point \( z_0 \in \partial B_1 \) and \( f(z_0) = w_0 \in \partial B_2 \). Then there exists a constant \( \lambda \in \mathbb{R} \) such that \( Df(z_0)^* w_0 = \lambda z_0 \). Moreover, \( \lambda \geq 1 - \text{Re} \left( \frac{f(0), w_0}{2} \right) > 0 \).
By using Proposition 5.1 and the arguments similar to those in the proof of Theorem 3.3, we obtain a better estimate:

\[
\lambda \geq \max \left\{ \frac{2}{\pi} - \| f(0) \|_{H^2}, \frac{1 - \Re\left( \langle f(0), w_0 \rangle \right)}{2} \right\}.
\]

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