Preferred traces on C*-algebras of self-similar groupoids arising as fixed points

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Abstract
Recent results of Laca, Raeburn, Ramagge and Whittaker show that any self-similar action of a groupoid on a graph determines a 1-parameter family of self-mappings of the trace space of the groupoid C*-algebra. We investigate the fixed points for these self-mappings, under the same hypotheses that Laca et al. used to prove that the C*-algebra of the self-similar action admits a unique KMS state. We prove that for any value of the parameter, the associated self-mapping admits a unique fixed point, which is a universal attractor. This fixed point is precisely the trace that extends to a KMS state on the C*-algebra of the self-similar action.

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PREFERRED TRACES ON $C^*$-ALGEBRAS OF SELF-SIMILAR
GROUPOIDS ARISING AS FIXED POINTS

JOAN CLARAMUNT AND AIDAN SIMS

Abstract. Recent results of Laca, Raeburn, Ramagge and Whittaker show that any
self-similar action of a groupoid on a graph determines a 1-parameter family of self-
mappings of the trace space of the groupoid $C^*$-algebra. We investigate the fixed points
for these self-mappings, under the same hypotheses that Laca et al. used to prove that
the $C^*$-algebra of the self-similar action admits a unique KMS state. We prove that for
any value of the parameter, the associated self-mapping admits a unique fixed point,
which is a universal attractor. This fixed point is precisely the trace that extends to a
KMS state on the $C^*$-algebra of the self-similar action.

There has been a lot of recent interest in the structure of KMS states for the natural
gauge actions on $C^*$-algebras associated to algebraic and combinatorial objects (see, for
example, [1, 2, 3, 6, 8, 9, 10, 11, 17]). The theme is that there is a critical inverse tem-
perature below which the system admits no KMS states, and above this critical inverse
temperature the structure of the KMS simplex reflects some of the underlying combina-
torial data. For example, for $C^*$-algebras of strongly-connected finite directed graphs, the
critical inverse temperature is the logarithm of the spectral radius of the graph, there is a
unique KMS state at this inverse temperature, and at supercritical inverse temperatures
the extreme KMS states are parameterised by the vertices of the graph [5, 8].

A particularly striking instance of this phenomenon appeared recently in the context
of $C^*$-algebras associated to self-similar groups [14, 12] and, more generally, self-similar
actions of groupoids on graphs [13]. Roughly speaking a self-similar action of a groupoid
on a finite directed graph $E$ consists of a discrete groupoid $G$ with unit space identified
with $E^0$, and an action of $G$ on the left of the path-space of $E$ with the property that
for each groupoid element $g$ and each path $\mu$ for which $g \cdot \mu$ is defined, there is a unique
groupoid element $g|_{\mu}$ such that $g \cdot (\mu\nu) = (g \cdot \mu)(g|_{\mu} \cdot \nu)$ for any other path $\nu$.

In [13], the authors first show that at supercritical inverse temperatures, the KMS
states on the Toeplitz algebra $\mathcal{T}(G, E)$ of the self-similar action are determined by their
restrictions to the embedded copy of $C^*(G)$. They then show that the self-similar action
can be used to transform an arbitrary trace on $C^*(G)$ into a new trace that extends to a
KMS state, and that this transformation is an isomorphism of the trace simplex of $C^*(G)$
onto the KMS-simplex of $\mathcal{T}(G, E)$. The transformation is quite natural: given a trace

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\( \tau \) on \( C^*(\mathcal{G}) \) and given \( g \in \mathcal{G} \), the value of the transformed trace at the generator \( u_g \) is a weighted infinite sum of the values of the original trace on restrictions \( g|_\mu \) of \( g \) such that \( g \cdot \mu = \mu \); so the transformed trace at \( u_g \) reflects the proportion—as measured by the initial trace—of the path-space of \( E \) that is fixed by \( g \). Building on this analysis, Laca, Raeburn, Ramagge and Whittaker proved that if \( E \) is strongly connected and the self-similar action satisfies an appropriate finite-state condition, then \( \mathcal{T}(\mathcal{G}, E) \) admits a unique KMS state at the critical inverse temperature and this is the only state that factors through the quotient \( \mathcal{O}(\mathcal{G}, E) \) determined by the Cuntz–Krieger relations for \( E \). So the KMS structure picks out a “preferred trace” on the groupoid \( C^* \)-algebra \( C^*(\mathcal{G}) \). Some enlightening examples of this are discussed in [13, Section 9].

This paper is motivated by the observation that the transformation described in the preceding paragraph for a given inverse temperature \( \beta \) is a self-mapping \( \chi_\beta \) of the simplex of normalised traces of \( C^*(\mathcal{G}) \), and so can be iterated. This raises a natural question: for which initial traces \( \tau \) and at which supercritical inverse temperatures does the sequence \( (\chi_\beta^n(\tau))_{n=1}^\infty \) converge, and what information about the self-similar action do the limit traces—that is, the fixed points for \( \chi_\beta \)—encode? Our main result, Theorem 2.1, gives a very satisfactory answer to this question: the hypotheses of [13] (namely that \( E \) is strongly connected and the action satisfies the finite-state condition) seem to be exactly the hypotheses needed to guarantee that \( \chi_\beta \) admits a unique fixed point for every supercritical \( \beta \), that this fixed point is a universal attractor, and that it is precisely the preferred trace that extends to a KMS state at the critical inverse temperature.

1. Preliminaries

1.1. KMS states. Consider a \( C^* \)-algebra \( A \) together with a strongly continuous homomorphism \( \alpha : \mathbb{R} \to \text{Aut}(A) \). An element \( x \in A \) is called analytic if the function \( t \mapsto \alpha_t(x) \) extends to an analytic function from \( \mathbb{C} \) to \( A \). The set \( A^a \) of analytic elements is a dense \( * \)-subalgebra of \( A \) (see for example [15, Chapter 8]).

We say that a state \( \phi \) of \( A \) satisfies the Kubo–Martin–Schwinger (KMS) condition at inverse temperature \( \beta \in (0, \infty) \) if it satisfies

\[
\phi(xy) = \phi(y\alpha_\beta(x)) \quad \text{for all analytic } x, y \in A.
\]

We call such a state \( \phi \) a KMS\(_\beta\) state for \((A, \alpha)\). It is well-known that a state \( \phi \) is KMS\(_\beta\) if and only if there exists a set \( S \) of analytic elements such that \( \text{span } S \) is an \( \alpha \)-invariant dense subspace of \( A \), and \( \phi \) satisfies the KMS condition at all \( x, y \in S \).

1.2. Self-similar groupoids. A groupoid is a countable small category \( \mathcal{G} \) with inverses. In this paper, we will use \( d \) and \( t \) for the domain and terminus maps \( \mathcal{G} \to \mathcal{G}^{(0)} \) to distinguish them from the range and source maps on directed graphs. For \( u \in \mathcal{G}^{(0)} \), we write \( \mathcal{G}_u = \{ g \in \mathcal{G} : d(g) = u \} \) and \( \mathcal{G}^u = \{ g \in \mathcal{G} : t(g) = u \} \).

Consider a finite directed graph \( E = (E^0, E^1, r, s) \). For \( n \geq 2 \), write \( E^n \) for the paths of length \( n \) in \( E \); that is \( E^n = \{ e_1e_2 \ldots e_n : e_i \in E^1, r(e_{i+1}) = s(e_i) \} \). We write \( E^* := \bigcup_{n=1}^\infty E^n \). We can visualise the set \( E^* \) as indexing the vertices of a forest \( T = T_E \) given by \( T^0 = E^* \) and \( T^1 = \{ (\mu, \mu e) \in E^* : \mu \in E^*, e \in E^1 \text{ and } s(\mu) = r(e) \} \). Throughout this paper, we write \( A_E \) for the integer matrix with entries \( A_E(v, w) = |vE^1w| \).

We are interested in self-similar actions of groupoids on directed graphs \( E \) as introduced and studied in [13]. To describe these, first recall that a partial isomorphism of the forest
$T_E$ corresponding to a directed graph $E$ as above consists of a pair $(v, w) \in E^0 \times E^0$ and a bijection $g : vE^* \to wE^*$ such that

1. $g|_{vE^k} : vE^k \to wE^k$ is bijective for $k \geq 1$.
2. $g(\mu e) \in g(\mu)E^1$ for $\mu \in vE^*$ and $e \in E^1$ with $r(e) = s(\mu)$.

The set of partial isomorphisms of $T_E$ forms a groupoid $\text{Pliso}(T_E)$ with unit space $E^0$ [13, Proposition 3.2]: the identity morphism associated to $v \in E^0$ is the partial isomorphism $\text{id}_v : vE^* \to vE^*$ given by the identity map on $vE^*$; the inverse of $g : vE^* \to wE^*$ is the standard inverse map $g^{-1} : wE^* \to vE^*$; and the groupoid multiplication is composition.

Definition 1.1 ([12, Definition 3.3]). Let $E$ be a directed graph, and let $\mathcal{G}$ be a groupoid with unit space $E^0$. A faithful action of $\mathcal{G}$ on $T_E$ is an injective groupoid homomorphism $\theta : \mathcal{G} \to \text{Pliso}(T_E)$ that is the identity map on unit spaces. We write $g \cdot \mu$ rather than $\theta_g(\mu)$ for $g \in \mathcal{G}$ and $\mu \in E^*$ with $d(\mu) = r(\mu)$. The action $\theta$ is self-similar if for each $g \in \mathcal{G}$ and $\mu \in d(\mu)E^*$ there exists $|g|_\mu \in \mathcal{G}$ such that $d(g|_\mu) = s(\mu)$ and

\begin{equation}
 g \cdot (\mu \nu) = (g \cdot \mu)(g|_\mu \cdot \nu) \quad \text{for all } \nu \in s(\mu)E^*.
\end{equation}

The faithfulness condition ensures that for each $g \in \mathcal{G}$ and $\mu \in E^*$ with $d(g) = r(\mu)$, there is a unique element $g|_\mu \in \mathcal{G}$ satisfying (1.1). Throughout the paper, we will write $\mathcal{G} \rhd E$ to indicate that the groupoid $\mathcal{G}$ acts faithfully on the directed graph $E$.

By [13, Proposition 3.6], self-similar groupoid actions have the following properties, which we will use without comment henceforth: for $g, h \in \mathcal{G}$, $\mu \in d(g)E^*$, and $\nu \in s(\mu)E^*$,

1. $g|_{\nu \nu} = (g|_\mu)|_\nu$,
2. $\text{id}_{r(\mu)} = \text{id}_{s(\mu)}$,
3. if $(h, g) \in \mathcal{G}^{(2)}$, then $(h|_{g \mu}, g|_\mu) \in \mathcal{G}^{(2)}$, and $(hg)|_\mu = h|_{g \mu} g|_\mu$,
4. $(g^{-1})|_\mu = (g|_{g^{-1} \mu})^{-1}$.

We say that a self-similar action $\mathcal{G} \rhd E$ is finite-state if for every element $g \in \mathcal{G}$, the set $\{g|_\mu : \mu \in d(g)E^*\}$ is a finite subset of $\mathcal{G}$.

1.3. The $C^*$-algebras of a self-similar groupoid. The Toeplitz algebra of a self-similar action $\mathcal{G} \rhd E$ is defined in [13] as follows. A Toeplitz representation $(v, q, t)$ of $(\mathcal{G}, E)$ in a unital $C^*$-algebra $B$ is a triple of maps $v : \mathcal{G} \to B$, $q : E^0 \to B$, $t : E^1 \to B$ such that

1. $(q, t)$ is a Toeplitz–Cuntz–Krieger family in $B$ such that $\sum_{w \in E^0} q_w = 1_B$;
2. $\{v_g : g \in \mathcal{G}\}$ is a family of partial isometries in $B$ satisfying $v_gv_h = \delta_{d(g), t(h)} v_{gh}$ and $v_{g^{-1}} = v_g^*$ for all $g, h \in \mathcal{G}$, and $v_w = q_w$ for $w \in G^{(0)} = E^0$;
3. $v_g t_e = \delta_{d(g), t(e)} t_{g \cdot v_g|_e}$ for $g \in \mathcal{G}$ and $e \in E^1$; and
4. $v_g q_w = \delta_{d(g), w} q_w v_g$ for all $g \in \mathcal{G}$ and $w \in E^0$.

Standard arguments show that there exists a universal $C^*$-algebra $\mathcal{T}(\mathcal{G}, E)$ generated by a Toeplitz representation $\{u, p, s\}$. We have $\mathcal{T}(\mathcal{G}, E) = \text{span} \{s_\mu u_\mu s_\nu^* : \mu, \nu \in E^*, g \in G^{s(\mu)}\}$. We call $\mathcal{T}(\mathcal{G}, E)$ the Toeplitz algebra of the self-similar action $\mathcal{G} \rhd E$. The argument of the paragraph following [13, Theorem 6.1] applied with $\pi_{x}$ replaced by a faithful representation of $C^*(\mathcal{G})$ shows that $C^*(\mathcal{G})$ embeds in $\mathcal{T}(\mathcal{G}, E)$ as a unital $C^*$-subalgebra via an embedding satisfying $\delta_g \mapsto u_g$.

Following [13, Proposition 4.7], the Cuntz–Pimsner algebra of $(\mathcal{G}, E)$, denoted $\mathcal{O}(\mathcal{G}, E)$, is defined to be the quotient of $\mathcal{T}(\mathcal{G}, E)$ by the ideal $I$ generated by $\{p_v - \sum_{e \in E_1} s_e s_e^* : v \in E^0\}$. We have $1_{\mathcal{O}(\mathcal{G}, E)} = \sum_{\mu \in E^*} s_\mu s_\mu^*$ for any $n$. 
1.4. Dynamics on $\mathcal{T}(G, E)$ and $\mathcal{O}(G, E)$. The universal property of $\mathcal{T}(G, E)$ yields a dynamics $\sigma : \mathbb{R} \to \text{Aut}(\mathcal{T}(G, E))$ such that

$$\sigma_t(u_g) = u_g, \quad \sigma_t(q_w) = q_w, \quad \text{and} \quad \sigma_t(t_e) = e^{it}t_e$$

for all $t \in \mathbb{R}$, $g \in G$, $w \in E^0$, and $e \in E^1$. Since each $p_v - \sum_{e \in E^1} s_es^*_e$ is fixed by $\sigma$, the dynamics $\sigma$ descends to a dynamics, also denoted $\sigma$, on $\mathcal{O}(G, E)$.

Let $\rho(A_E)$ denote the spectral radius of the adjacency matrix $A_E$. Proposition 5.1 of [13] shows that there are no KMS$_\beta$ states on $(\mathcal{T}(G, E), \sigma)$ for $\beta < \log \rho(A_E)$. In [13, Theorem 6.1], given a trace $\tau$ on the groupoid algebra $C^*(G)$, the authors show that for $\beta > \log \rho(A_E)$, the series

$$Z(\beta, \tau) := \sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu \in E^k} \tau(u_{\sigma(\mu)})$$

converges to a positive real number, and that there is a KMS$_\beta$ state $\Psi_{\beta, \tau}$ on the Toeplitz algebra $\mathcal{T}(G, E)$ given by

$$\Psi_{\beta, \tau}(s_is_is^*_i) = \delta_{\mu, \nu} e^{-\beta |\mu|} Z(\beta, \tau)^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \left( \sum_{\lambda \in E^k} \tau(u_{g_{\lambda}}) \right).$$

They show that the map $\tau \mapsto \Psi_{\beta, \tau}$ is an isomorphism from the simplex of tracial states of $C^*(G)$ to the KMS$_\beta$-simplex of $\mathcal{T}(G, E)$.

2. A fixed-point theorem, and the preferred trace on $C^*(G)$

Consider a self-similar action $G \rtimes E$ and a number $\beta > \log \rho(A_E)$. As mentioned in Section 1.3, $C^*(G)$ is a unital $C^*$-subalgebra of $\mathcal{T}(G, E)$. The starting point for our analysis is that if $\tau$ is a trace on $C^*(G)$ and $\Psi_{\beta, \tau}$ is the associated KMS$_\beta$-state of $\mathcal{T}(G, E)$ given by (1.2), then $\Psi_{\beta, \tau}|_{C^*(G)}$ is again a trace on $C^*(G)$. So there is a mapping $\chi_{\beta} : \text{Tr}(C^*(G)) \to \text{Tr}(C^*(G))$ given by

$$\chi_{\beta}(\tau) = \Psi_{\beta, \tau}|_{C^*(G)}.$$

Our main theorem is the following; its proof occupies the remainder of the paper.

**Theorem 2.1.** Let $E$ be a finite strongly connected graph, suppose that $G \rtimes E$ is a faithful self-similar action of a groupoid $G$ on $E$, and suppose that $\beta > \log \rho(A_E)$. If $G \rtimes E$ is finite state, then

1. the map $\chi_{\beta} : \text{Tr}(C^*(G)) \to \text{Tr}(C^*(G))$ of (2.1) has a unique fixed point $\theta$;
2. for any $\tau \in \text{Tr}(C^*(G))$ we have $\chi_{\beta}(\tau) \mathop{\rightarrow}^{w^*} \theta$; and
3. $\theta$ is the unique trace on $C^*(G)$ that extends to a KMS$_{\log \rho(A_E)}$-state of $\mathcal{T}(G, E)$.

We start with a straightforward observation about the map $\chi_{\beta}$ of (2.1).

**Lemma 2.2.** Let $G \rtimes E$ be a faithful self-similar action of a groupoid on a finite strongly connected graph, and suppose that $\beta > \log \rho(A_E)$. Then the map $\chi_{\beta}$ is weak*-continuous. If $\tau \in \text{Tr}(C^*(G))$ and $(\chi_{\beta}(\tau))^n$ is weak*-convergent, then $\theta := \lim_n \chi_{\beta}(\tau)$ belongs to $\text{Tr}(C^*(G))$ and $\chi_{\beta}(\theta) = \theta$. 


Proof. The map $\tau \mapsto \Psi_{\beta,\tau}$ is a homeomorphism and hence continuous, and restriction of states to a subalgebra is clearly continuous, so $\chi_\beta$ is continuous. Hence if $\chi_\beta^\tau(n(\tau))^\ast \to \theta$, then $\theta \in \text{Tr}(C^*(G))$ because the trace simplex of a unital $C^*$-algebra is weak$^*$-compact, and then $\chi_\beta(\theta) = \chi_\beta(\lim_{n} \chi_\beta^\tau(n(\tau))) = \lim_{n} \chi_\beta^{n+1}(\tau) = \theta$. $\square$

Proposition 2.3. Let $G \to E$ be a faithful self-similar action of a groupoid on a finite graph, and fix $\beta > \log \rho(A_E)$. Let $\chi_\beta : \text{Tr}(C^*(G)) \to \text{Tr}(C^*(G))$ be the map (2.1). For $\tau \in \text{Tr}(C^*(G))$, define

$$N(\beta, \tau) := e^\beta (1 - Z(\beta, \tau)^{-1}).$$

(1) If $\tau \in \text{Tr}(C^*(G))$ is a fixed point for $\chi_\beta$, then for each $g \in G$, we have

$$N(\beta, \tau)^n \tau(u_g) = \sum_{\mu \in E^n, g \mu = \mu} \tau(u_{g \mu}) \quad \text{for all } n \geq 1.$$  

(2) If $E$ is strongly connected with adjacency matrix $A_E$, and $\tau \in \text{Tr}(C^*(G))$ satisfies (2.2), then $m := (\tau(u_v))_{v \in E^0}$ is the Perron–Frobenius eigenvector of $A_E$, and $N(\beta, \tau) = \rho(A_E)$.

Proof. (1) For each $g \in G$ we have

$$\tau(u_g) = \chi_\beta(\tau)(u_g) = Z(\beta, \tau)^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \left( \sum_{\mu \in E^k, g \mu = \mu} \tau(u_{g \mu}) \right) = Z(\beta, \tau)^{-1} \left[ \tau(u_g) + e^{-\beta} \sum_{k=0}^{\infty} e^{-\beta k} \left( \sum_{\mu \in E^{k+1}, g \mu = \mu} \tau(u_{g \mu}) \right) \right].$$

The map $(e, \nu) \mapsto e\nu$ is a bijection

$$\{(e, \nu) \in E^1 \times E^k : s(e) = r(\nu), g \cdot e = e \text{ and } g|_e \cdot \nu = \nu\} \to \{\mu \in E^{k+1} : g \cdot \mu = \mu\}.$$ 

So the definition of $\Psi_{\beta,\tau}$ yields

$$\tau(u_g) = Z(\beta, \tau)^{-1} \tau(u_g) + \sum_{e \in E^1, g \cdot e = e} \left( Z(\beta, \tau)^{-1} e^{-\beta} \sum_{k=0}^{\infty} e^{-\beta k} \left( \sum_{\nu \in E^k, g \nu = \nu} \tau(u_{g \nu}) \right) \right)$$

$$= Z(\beta, \tau)^{-1} \tau(u_g) + \sum_{e \in E^1, g \cdot e = e} \Psi_{\beta,\tau}(s_e u_{g|_e} s_e^*).$$

(2.3)

We have $\Psi_{\beta,\tau}(s_e u_{g|_e} s_e^*) = \delta_{s(e),r(g)} \delta_{s(e),d(g)} e^{-\beta} \Psi_{\beta,\tau}(u_{g|_e}) = e^{-\beta} \chi_\beta(\tau)(u_{g|_e})$. Applying this and rearranging (2.3) gives

$$e^\beta (1 - Z(\beta, \tau)^{-1}) \tau(u_g) = \sum_{e \in E^1, g \cdot e = e} \chi_\beta(\tau)(u_{g|_e}) = \sum_{e \in E^1, g \cdot e = e} \tau(u_{g|_e}).$$

Statement (1) now follows from an induction on $n$.

(2) Using (2.2) for $\tau$ with $n = 1$ at the first step, we see that for $v \in E^0$,

$$m_v = N(\beta, \tau)^{-1} \sum_{e \in E^1} \tau(u_{s(e)}) = N(\beta, \tau)^{-1} \sum_{w \in E^0} A_E(v, w) \tau(u_w) = N(\beta, \tau)^{-1} (A_E m)_v.$$ 

Hence, since $1 = \tau(1) = \sum_{v \in E^0} \tau(u_v)$, the vector $m$ is a unimodular nonnegative eigenvector for the irreducible matrix $A_E$ and has eigenvalue $N(\beta, \tau)$. So the Perron–Frobenius theorem [16, Theorem 1.6] shows that $m$ is the Perron–Frobenius eigenvector and $N(\beta, \tau) = \rho(A_E)$. $\square$
We now turn our attention to the situation where $E$ is strongly connected, and $G \curvearrowright E$ is finite-state, and aim to show that $\chi_\beta$ admits a unique fixed point. The strategy is to show that if $C^*(G)$ admits a trace $\theta$ satisfying (2.2), then for any other trace $\tau$ we have $\chi_\beta(\tau) \to \theta$. From this it will follow first that $\chi_\beta$ admits at most one fixed point, and second that a trace $\theta$ is fixed point if and only if it satisfies (2.2). We start with an easy result from Perron–Frobenius theory.

**Lemma 2.4.** Let $A \in M_n(\mathbb{R})$ be an irreducible matrix, and take $\beta > \log \rho(A)$.

1. The matrix $I - e^{-\beta}A$ is invertible, and $A_vN := (I - e^{-\beta}A)^{-1}$ is primitive; indeed, every entry of $A_vN$ is strictly positive.

2. Let $m^A$ be the Perron–Frobenius eigenvector of $A$. Then $m^A$ is also the Perron–Frobenius eigenvector of $A_vN$, and $\rho(A_vN) = (1 - e^{-\beta} \rho(A))^{-1}$.

**Proof.** (1) The matrix $I - e^{-\beta}A$ is invertible because $e^\beta > \rho(A)$ and so does not belong to the spectrum of $A$. As in, for example, [4, Section VII.3.1], we have

$$A_vN := (I - e^{-\beta}A)^{-1} = \sum_{k=0}^{\infty} e^{-k\beta} A^k.$$ 

Fix $i, j \leq n$. Since $A$ is irreducible, we have $A^k_{i,j} > 0$ for some $k \geq 0$, and since $A^l_{i,j} \geq 0$ for all $l$, we deduce that $(A_vN)_{i,j} \geq e^{-k\beta} A^k_{i,j} > 0$.

(2) We compute $A_vN^{-1}m^A = (I - e^{-\beta}A)m^A = (1 - e^{-\beta} \rho(A))m^A$. Multiplying through by $(1 - e^{-\beta} \rho(A))^{-1}A_vN$ shows that $m^A$ is a positive eigenvector of $A_vN$ with eigenvalue $(1 - e^{-\beta} \rho(A))^{-1}$, so the result follows from uniqueness of the Perron–Frobenius eigenvector of $A_vN$. □

**Notation 2.5.** Henceforth, given a self-similar action $G \curvearrowright E$ of a groupoid on a finite graph, and a trace $\tau \in \text{Tr}(C^*(G))$, we denote by $x_\tau \in [0, 1]^{E^0}$ the vector

$$x_\tau = (\tau(u_v))_{v \in E^0}.$$ 

**Proposition 2.6.** Let $G \curvearrowright E$ be a faithful self-similar action of a groupoid on a finite strongly connected graph. Fix $\beta > \log \rho(A_E)$, and let $A_vN := (I - e^{-\beta}A_E)^{-1}$. Let $\chi_\beta : \text{Tr}(C^*(G)) \to \text{Tr}(C^*(G))$ be the map (2.1). Fix $\tau \in \text{Tr}(C^*(G))$. Then

$$x^{\chi_\beta(\tau)} = \|A_vN x_\tau\|_1^{-1} A_vN x_\tau.$$ 

**Proof.** For $v \in E^0$, the definition of $\chi_\beta$ gives

$$\chi_\beta(\tau)(u_v) = Z(\beta, \tau)^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \left( \sum_{\mu \in E_k} \tau(u_{s(\mu)}) \right)$$

$$= Z(\beta, \tau)^{-1} \sum_{k=0}^{\infty} e^{-\beta k} (A_E^k x_\tau)_v = Z(\beta, \tau)^{-1} (A_vN x_\tau)_v.$$ 

So an induction gives $x^{\chi_\beta(\tau)} = Z(\beta, \chi_\beta^{-1}(\tau))^{-1} \cdots Z(\beta, \tau)^{-1} A_vN x_\tau$. Since $x^{\chi_\beta(\tau)}$ has unit 1-norm, we have $Z(\beta, \chi_\beta^{-1}(\tau))^{-1} \cdots Z(\beta, \tau)^{-1} = \|A_vN x_\tau\|_1^{-1}$, and the result follows. □

Our next result shows that for any $\tau \in \text{Tr}(C^*(G))$, the sequence $x^{\chi_\beta(\tau)}$ converges exponentially quickly to the Perron–Frobenius eigenvector of $A_E$. 

Theorem 2.7. Let $\mathcal{G} \rhd E$ be a faithful self-similar action of a groupoid on a finite strongly connected graph. Fix $\beta > \log \rho(A_E)$. Let $\chi_\beta : \text{Tr}(C^*(\mathcal{G})) \to \text{Tr}(C^*(\mathcal{G}))$ be the map (2.1). Fix $\tau \in \text{Tr}(C^*(\mathcal{G}))$. Let $m = m^E$ be the Perron–Frobenius eigenvector of $A_E$. Then $x^{\chi_\beta(\tau)} \to m^E$ exponentially quickly, and $Z(\beta, \chi_\beta(\tau)) \to \rho(A_{vN})$ exponentially quickly.

Proof. Since $E$ is strongly connected, Lemma 2.4 shows that $m$ is the (right) Perron–Frobenius eigenvector of $A_{vN} := (I - e^{-\beta}A_E)^{-1}$. Write $\tilde{m}$ for the left Perron–Frobenius eigenvector of $A_{vN}$ such that $\tilde{m} \cdot m = 1$.

Let $r := \tilde{m} \cdot x^\tau$. Then $r > 0$ because every entry of $\tilde{m}$ is strictly positive, and $x^\tau$ is a nonnegative nonzero vector.

Proposition 2.6 implies that

$$x^{\chi_\beta(\tau)} - m_v = \frac{\rho(A_{vN})^n}{\|A_{vN}x^\tau\|_1} \left[ \left( \rho(A_{vN})^{-n} A_{vN}^n x^\tau - rm \right)_v + \left( r - \| \rho(A_{vN})^{-n} A_{vN}^n x^\tau \|_1 \right) m_v \right].$$

By [16, Theorem 1.2], there exist a real number $0 < \lambda < 1$, a positive constant $C$, and an integer $s \geq 0$ such that for large $n$ we have $\rho(A_{vN})^{-n} A_{vN}^n - m \cdot \tilde{m} \leq C n^s \lambda^n$. In fact, since $Cn^s (\lambda'/\lambda)^n \to 0$ for any $0 < \lambda' < \lambda < 1$, by adjusting the value of $\lambda$, we can take $C = 1$ and $s = 0$. So for large $n$, we have

$$| \rho(A_{vN})^{-n} (A_{vN}^n x^\tau)_v - rm_v | \leq \lambda^n.$$

Since $v \in E^0$ was arbitrary, summing over $v \in E^0$ we deduce that

$$| r - \rho(A_{vN})^{-n} A_{vN}^n x^\tau |_1 \leq |E^0| \lambda^n.$$

Hence $\rho(A_{vN})^{-n} A_{vN}^n x^\tau \big|_1 \xrightarrow{n \to \infty} r$ exponentially quickly. Making this approximation twice in (2.5), we obtain

$$| x^{\chi_\beta(\tau)} - m_v | \leq \frac{(1 + |E^0|)}{\rho(A_{vN})^{-n} A_{vN}^n x^\tau \big|_1} \lambda^n,$$

which converges exponentially to 0. Hence $x^{\chi_\beta(\tau)} \to m$ exponentially quickly.

For the second statement, using Proposition 2.6 at the third equality, we calculate

$$Z(\beta, \chi_\beta(\tau)) = \sum_{k=0}^{\infty} e^{-\beta k} \sum_{u \in E^k} \chi_\beta(\tau)(u_{\mu(u)})$$

$$= \| A_{vN} x^{\chi_\beta(\tau)} \|_1 = \| A_{vN}^{n+1} x^\tau \|_1 = \frac{\rho(A_{vN})^{-(n+1)} \| A_{vN}^{n+1} x^\tau \|_1}{\rho(A_{vN})^{-n} \| A_{vN}^n x^\tau \|_1} \rho(A_{vN}).$$

We saw that $\rho(A_{vN})^{-(n+1)} \| A_{vN}^{n+1} x^\tau \|_1$ converges to $r > 0$ exponentially quickly, so the ratio $\frac{\rho(A_{vN})^{-(n+1)} \| A_{vN}^{n+1} x^\tau \|_1}{\rho(A_{vN})^{-n} \| A_{vN}^n x^\tau \|_1}$ converges exponentially quickly to 1. □

The following estimate is needed for our key technical result, Theorem 2.9.

Lemma 2.8. Let $\mathcal{G} \rhd E$ be a faithful finite-state self-similar action of a groupoid on a finite strongly connected graph. Let $A_{vN} := (I - e^{-\beta}A_E)^{-1}$, and let $m = m^E$ be the
unimodular Perron–Frobenius eigenvector of \( A \). For \( g \in \mathcal{G} \setminus E^0 \), \( v \in E^0 \), and \( k \geq 0 \), define
\[
\mathcal{G}_g^k(v) := \{ \mu \in d(g)E^k v : g \cdot \mu = \mu \} \quad \text{and} \quad \mathcal{F}_g^k(v) := \{ \mu \in \mathcal{G}_g^k(v) : g|_\mu = v \}.
\]
Then for \( \beta > \log \rho(A_E) \) and \( g \in \mathcal{G} \), we have
\[
\sum_{k=0}^\infty e^{-\beta k} \sum_{v \in E^0} |\mathcal{G}_g^k(v) \setminus \mathcal{F}_g^k(v)| m_v < \rho(A_{vN})m_{d(g)}.
\]

**Proof.** The argument of [13, Lemma 8.7] shows that there exists \( k(g) > 0 \) such that
\[
\sum_{v \in E^0} |\mathcal{G}_g^{nk(g)}(v) \setminus \mathcal{F}_g^{nk(g)}(v)| m_v \leq (\rho(A_E)^{k(g)} - 1)^n m_{d(g)}
\]
for all \( n \geq 0 \). For each \( k \in \mathbb{N} \) we also have
\[
\sum_{v \in E^0} |\mathcal{G}_g^k(v)| m_v \leq \sum_{v \in E^0} |d(g)E^k v| m_v = (A_{E}^k m)_{d(g)} = \rho(A_E)^{k} m_{d(g)}.
\]
Combining these estimates and using Lemma 2.4(2) at the final step, we obtain
\[
\sum_{k=0}^\infty e^{-\beta k} \sum_{v \in E^0} |\mathcal{G}_g^k(v) \setminus \mathcal{F}_g^k(v)| m_v
\]
\[
= \sum_{k \neq k(g)} e^{-\beta k} \sum_{v \in E^0} |\mathcal{G}_g^k(v) \setminus \mathcal{F}_g^k(v)| m_v + e^{-\beta k(g)} \sum_{v \in E^0} |\mathcal{G}_g^{k(g)}(v) \setminus \mathcal{F}_g^{k(g)}(v)| m_v
\]
\[
\leq \sum_{k \neq k(g)} e^{-\beta k} \rho(A_E)^{k} m_{d(g)} + e^{-\beta k(g)} (\rho(A_E)^{k(g)} - 1) m_{d(g)}
\]
\[
< \sum_{k=0}^\infty e^{-\beta k} \rho(A_E)^{k} m_{d(g)}
\]
\[
= \rho(A_{vN})m_{d(g)}.
\]

We are now ready to prove a converse to Proposition 2.3(1), under the hypotheses that \( E \) is strongly connected and the action of \( \mathcal{G} \) on \( E \) is finite-state.

**Theorem 2.9.** Let \( \mathcal{G} \curvearrowright E \) be a faithful finite-state self-similar action of a groupoid on a finite strongly connected graph. Fix \( \beta > \log \rho(A_E) \). Let \( \chi_\beta : \text{Tr}(C^*(\mathcal{G})) \to \text{Tr}(C^*(\mathcal{G})) \) be the map (2.1). Suppose that \( \theta \in \text{Tr}(C^*(\mathcal{G})) \) satisfies (2.2). Then for any \( \tau \in \text{Tr}(C^*(\mathcal{G})) \), we have \( \lim_{\nu} \chi_\beta^n(\tau) = \theta \). In particular, \( \theta \) is a fixed point for \( \chi_\beta \).

**Proof.** We will prove that for each \( g \in \mathcal{G} \) there are constants \( 0 < \lambda < 1 \) and \( K, D > 0 \) such that \( |\chi_\beta^n(\tau)(u_g) - \theta(u_g)| < (nK + D)K\lambda^{n-1} \) for all \( n \geq 0 \). Since \( (nK + D)\lambda^{n-1} \to 0 \) exponentially quickly in \( n \), the first statement will then follow from an \( \varepsilon/3 \)-argument.

To simplify notation, define \( \tau_0 := \tau \) and \( \tau_n := \chi_\beta^n(\tau) \) for \( n \geq 1 \). For \( g \in \mathcal{G} \) and \( n \geq 0 \), let
\[
\Delta_n(g) := \tau_n(u_g) - \theta(u_g).
\]
Fix \( g \in \mathcal{G} \); if \( t(g) \neq d(g) \), then \( \tau_n(u_g) = \theta(u_g) = 0 \) by [13, Proposition 7.2], so we may assume that \( t(g) = d(g) \). Since the action is finite-state, the set \( \{ g|_\mu : \mu \in d(g)E^* \} \) is
finite. By Lemma 2.8, there is a constant $\alpha < 1$ such that

$\sum_{k=0}^{\infty} e^{-\beta k} \sum_{v \in E^0} |G^k_{g|\mu}(v) \setminus F^k_{g|\mu}(v)| m_v < \alpha \rho(A_{vN})m_d(g|\mu)$

for all $\mu \in E^*$.

Since $\theta$ satisfies (2.2), we have

$$\theta(u_g) = N(\beta, \theta)^{-k} \sum_{\mu \in E^k, g \cdot \mu = \mu} \theta(u_{g|\mu}) \quad \text{for all } k \geq 0.$$

Consequently,

$$\sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu \in E^k, g \cdot \mu = \mu} \theta(u_{g|\mu}) = \sum_{k=0}^{\infty} e^{-\beta k} N(\beta, \theta)^k \theta(u_g) = (1 - e^{-\beta} N(\beta, \theta))^{-1} \theta(u_g).$$

Since $N(\beta, \theta) = e^\beta (1 - Z(\beta, \theta)^{-1})$ by definition, we can rearrange to obtain

$$\theta(u_g) = Z(\beta, \theta)^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu \in E^k, g \cdot \mu = \mu} \theta(u_{g|\mu}).$$

Using this, and applying the definition of $\chi_\beta$ at the third equality, we calculate

$$\Delta_{n+1}(g) = \tau_{n+1}(u_g) - \theta(u_g)$$

$$= \chi_\beta(\tau_n)(u_g) - Z(\beta, \theta)^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu \in E^k, g \cdot \mu = \mu} \theta(u_{g|\mu})$$

$$= Z(\beta, \tau_n)^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu \in E^k, g \cdot \mu = \mu} \tau_n(u_{g|\mu}) - Z(\beta, \theta)^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu \in E^k, g \cdot \mu = \mu} \theta(u_{g|\mu}).$$

Since the sums are absolutely convergent, we can rewrite each $\theta(u_{g|\mu})$ as $\tau_n(u_{g|\mu}) - \Delta_n(g|\mu)$ and rearrange to obtain

$$\Delta_{n+1}(g) = \left( Z(\beta, \tau_n)^{-1} - Z(\beta, \theta)^{-1} \right) \sum_{k=0}^{\infty} e^{-\beta k} \left( \sum_{\mu \in E^k, g \cdot \mu = \mu} \tau_n(u_{g|\mu}) \right)$$

$$+ Z(\beta, \theta)^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu \in E^k, g \cdot \mu = \mu} \Delta_n(g|\mu).$$

Since $\theta$ satisfies (2.2), Proposition 2.3(2) combined with the definition of $N(\beta, \theta)$ imply that $Z(\beta, \theta) = (1 - e^{-\beta} N(\beta, \theta))^{-1} = (1 - e^{-\beta} \rho(A))^{-1}$, and then Lemma 2.4(2) gives $Z(\beta, \theta) = \rho(A_{vN})$. Also, by definition of $\chi_\beta$, we have $\sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu \in E^k, g \cdot \mu = \mu} \tau_n(u_{g|\mu}) = Z(\beta, \tau_n)\tau_{n+1}(u_g)$. Making these substitutions in (2.7), we obtain

$$\Delta_{n+1}(g) = \left( Z(\beta, \tau_n)^{-1} - \rho(A_{vN})^{-1} \right) Z(\beta, \tau_n)\tau_{n+1}(u_g)$$

$$+ \rho(A_{vN})^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu \in E^k, g \cdot \mu = \mu} \Delta_n(g|\mu).$$
With \(G_g^k(v)\) and \(F_g^k(v)\) defined as in Lemma 2.8, the preceding expression for \(\Delta_{n+1}(g)\) becomes
\[
\Delta_{n+1}(g) = \left( Z(\beta, \tau_n) - \rho(A_{vN})^{-1}Z(\beta, \tau_n) \tau_{n+1}(u_g) \right) + \rho(A_{vN})^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{v \in E^0} \sum_{\mu \in G_g^k(v) \setminus F_g^k(v)} \Delta_n(g|\mu) + \sum_{\mu \in F_g^k(v)} \Delta_n(g|\mu).
\] (2.8)

The Cauchy–Schwarz inequality implies that for any \(h \in G\),
\[
|\tau_{n+1}(u_h)|^2 = |\tau_{n+1}(u_h^* u_{t(h)})|^2 \leq \tau_{n+1}(u_h^* u_h) \tau(u_{t(h)}^* u_{t(h)}) = \tau_{n+1}(u_{d(h)}) \tau_{n+1}(u_{t(h)}).
\]
Since our fixed \(g\) satisfies \(d(g) = t(g)\), taking square roots in the preceding estimate gives \(|\tau_{n+1}(u_g)| \leq \tau_{n+1}(u_{d(g)})\). Applying this combined with the triangle inequality to the right-hand side of (2.8), we obtain
\[
|\Delta_{n+1}(g)| \leq \left| Z(\beta, \tau_n) - \rho(A_{vN})^{-1}Z(\beta, \tau_n) \tau_{n+1}(u_{d(g)}) \right|
+ \rho(A_{vN})^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{v \in E^0} \sum_{\mu \in G_g^k(v) \setminus F_g^k(v)} |\Delta_n(g|\mu)| + \sum_{\mu \in F_g^k(v)} |\Delta_n(g|\mu)|,
\]
which, using that \(g|\mu = v\) for \(\mu \in F_g^k(v)\), becomes
\[
|\Delta_{n+1}(g)| \leq \left| Z(\beta, \tau_n) - \rho(A_{vN})^{-1}Z(\beta, \tau_n) \tau_{n+1}(u_{d(g)}) \right|
+ \rho(A_{vN})^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{v \in E^0} \sum_{\mu \in G_g^k(v) \setminus F_g^k(v)} |\Delta_n(g|\mu)|
+ \rho(A_{vN})^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{v \in E^0} \sum_{\mu \in G_g^k(v) \setminus F_g^k(v)} |\Delta_n(v)|.
\]
Since \((Z(\beta, \tau_n) - \rho(A_{vN})^{-1}Z(\beta, \tau_n) = \rho(A_{vN})^{-1}(\rho(A_{vN}) - Z(\beta, \tau_n))\), we obtain
\[
|\Delta_{n+1}(g)| \leq \rho(A_{vN})^{-1} |\rho(A_{vN}) - Z(\beta, \tau_n)| \tau_{n+1}(u_{d(g)})
+ \rho(A_{vN})^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{v \in E^0} \sum_{\mu \in G_g^k(v) \setminus F_g^k(v)} |\Delta_n(g|\mu)|
+ \rho(A_{vN})^{-1} \sum_{\mu \in d(g)E^*} e^{-\beta|\mu||\Delta_n(s(\mu))|}.
\]
By Theorem 2.7 there are positive constants \(\lambda_0\), \(K_1\) and \(K_2\) with \(\lambda_0 < 1\) such that \(|\rho(A_{vN}) - Z(\beta, \tau_n)| < K_1\lambda_0^n\) for all \(n\) and \(|\Delta_n(v)| = |\tau_n(u_v) - m_v| < K_2\lambda_0^n\) for all \(v \in E^0\) and \(n \geq 0\). Thus we obtain
\[
|\Delta_{n+1}(g)| \leq K_1\lambda_0^n \rho(A_{vN})^{-1} \tau_{n+1}(u_{d(g)})
+ \rho(A_{vN})^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{v \in E^0} \sum_{\mu \in G_g^k(v) \setminus F_g^k(v)} |\Delta_n(g|\mu)|
+ K_2\lambda_0^n \rho(A_{vN})^{-1} \sum_{\mu \in d(g)E^*} e^{-\beta|\mu|}.
\]
Theorem 3.1(a) of [8] shows that \( \sum_{\mu \in d(g)E^*} e^{-\beta|\mu|} \) converges, and since the entries of the Perron–Frobenius eigenvector \( m \) are strictly positive, \( l := \max_v m_v^{-1} \) is finite. So \( K := 2l \rho(A_{vN})^{-1} \max\{K_1, K_2 \sum_{\mu \in E^*} e^{-\beta|\mu|} \} \) satisfies

\[
|\Delta_{n+1}(g)| \leq K \lambda_0^n m_d(g) + \rho(A_{vN})^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{v \in E^0} \sum_{\mu \in G^k(v) \setminus F^k_h(v)} |\Delta_n(g|\mu)|.
\]

Since both \( \lambda_0 \) and the constant \( \alpha \) of (2.6) are less than 1, the quantity \( \lambda := \max\{\lambda_0, \alpha\} \) is less than 1.

Let \( D := l \max_{\mu \in d(g)E^*} (|\tau(u_{g|\mu})| + |\theta(u_{g|\mu}))|) \), which is finite because \( G \rhd E \) is finite state. Let \( g|E^* \) := \( \{g|\mu : \mu \in E^*\} \subseteq G \). We will prove by induction that \( |\Delta_n(h)| \leq (nK + D) \lambda^{n-1} m_d(h) \) for all \( n \) and for all \( h \in g|E^* \). The base case \( n = 0 \) is trivial because each \( |\Delta_0(h)| = |\tau(u_h) - \theta(u_h)| \leq |\tau(u_h)| + |\theta(u_h)| \leq Dl^{-1} \leq \lambda^{-1} Dm_d(h) \). Now suppose as an inductive hypothesis that \( |\Delta_n(h)| \leq (nK + D) \lambda^{n-1} m_d(h) \) for all \( h \in g|E^* \). Fix \( h \in g|E^* \). Applying the inductive hypothesis on the right-hand side of (2.9), and then using that \( h|E^* \subseteq g|E^* \) and invoking (2.6) gives

\[
|\Delta_{n+1}(h)| \leq K \lambda_0^n m_d(h) + (nK + D) \lambda^{n-1} \rho(A_{vN})^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{v \in E^0} \sum_{\mu \in G^k(v) \setminus F^k_h(v)} m_d(h|\mu) -
\]

\[
= K \lambda_0^n m_d(h) + (nK + D) \lambda^{n-1} \rho(A_{vN})^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{v \in E^0} |G^k_h(v) \setminus F^k_h(v)| m_v
\]

\[
\leq K \lambda_0^n m_d(h) + (nK + D) \lambda^{n-1} \alpha m_d(h),
\]

and since \( \lambda_0, \alpha < \lambda \) we deduce that

\[
|\Delta_{n+1}(h)| \leq ((n+1)K + D) \lambda^n m_d(h).
\]

The claim follows by induction. In particular we have \( |\Delta_n(g)| \leq (nK + D) \lambda^{n-1} m_d(g) \) for all \( n \) as claimed. This proves the first statement.

The second statement follows immediately from Lemma 2.2.

\[ \square \]

**Proof of Theorem 2.1.** (1) Let \( m = m^E \) be the Perron–Frobenius eigenvector of \( A_E \). For \( v \in G^{(0)} = E^0 \), let \( c_v := m_v \). Fix \( g \in G \setminus E^0 \). By [13, Proposition 8.2], the sequence

\[
\left( \rho(A_E)^{-n} \sum_{v \in E^0} |\{\mu \in E^n : g : g|\mu = \mu, g|\mu = v\}| m_v \right)_{n=1}^{\infty}
\]

converges to some \( c_g \in [0, m_d(g)] \). By [13, Theorem 8.3], there is a KMS\( \rho(A_E) \)-state \( \psi \) of \( \mathcal{T}(G, E) \) that factors through \( \mathcal{O}(G, E) \). This \( \psi \) satisfies

\[
\psi(s_{\mu} u_g s_{\nu}^*) = \begin{cases} 
\rho(A_E)^{-|\mu|} c_g & \text{if } \mu = \nu \text{ and } d(g) = t(g) = s(\mu) \\
0 & \text{otherwise.}
\end{cases}
\]

In particular, \( \theta := \psi|_{C_*G} \) belongs to \( \text{Tr}(C_*G) \).

We claim that \( \theta \) is a fixed point for \( \chi_\beta \). By the final statement of Theorem 2.9, it suffices to show that \( \theta \) satisfies (2.2). Proposition 8.1 of [13] shows that \( x^0 = (\theta(u_v))_{v \in E^0} \).
is equal to $m$. Using this, we see that

$$Z(\beta, \theta) = \sum_{v \in E^0} \sum_{k=0}^{\infty} e^{-k\beta} \sum_{\mu \in v E_k} \theta(s(\mu)) = \left\| \sum_{k=0}^{\infty} (e^{-k\beta} A_E^k x) \right\|_1$$

$$= \left\| \sum_{k=0}^{\infty} (e^{-k\beta} \rho(A_E)^k x) \right\|_1 = (1 - e^{-\beta} \rho(A_E))^{-1}.$$ 

Hence $N(\beta, \theta) = \rho(A_E)$.

Since $1_{O(G,E)} = \sum_{v \in E^0_\alpha} p_v = \sum_{e \in E^1} s_e s_e^*$, we have

$$\theta(u_g) = \psi(u_g) = \sum_{e \in E^1} \psi(u_g s_e s_e^*) = \sum_{e \in E^1} \delta_{d(g),r(e)} \psi(s_{g,e} u_{g,e} s_{e}^*)$$

$$= \sum_{e \in E^1} \delta_{d(g),r(e)} \delta_{g,e} \delta_{d(g,e),s(e)} \delta_{t(g,e),s(e)} \rho(A_E)^{-1} \theta(u_{g,e}) = N(\beta, \theta)^{-1} \sum_{e \in E^1} \theta(u_{g,e}).$$

Now an easy induction shows that $\theta$ satisfies relation (2.2).

It remains to prove that $\theta$ is the unique fixed point for $\chi_\beta$. For this, suppose that $\theta'$ is a fixed point for $\chi_\beta$, so $\theta' = \lim_{n \to \infty} \chi_\beta^n (\theta')$. Since $\theta$ satisfies (2.2), Theorem 2.9 shows that $\lim_{n \to \infty} \chi_\beta^n (\theta') = \theta$. So $\theta' = \theta$.

(2) This follows immediately from Theorem 2.9 because $\theta$ satisfies (2.2).

(3) The trace $\theta$ of part (1) extends to a $\text{KMS}_{\log \rho(A_E)}$ state of $T(G,E)$ by construction. If $\phi$ is any $\text{KMS}_{\log \rho(A_E)}$-state of $T(G,E)$, then it restricts to a $\text{KMS}_{\log \rho(A_E)}$-state of the subalgebra $T C^*(E)$, so it follows from [8, Theorem 4.3(a)] that $\phi$ agrees with $\psi$ on $T C^*(E)$, and in particular $(\phi(u_v))_{v \in E^0}$ is equal to the Perron–Frobenius eigenvector $m_E$. So [13, Proposition 8.1] shows that $\phi$ factors through $O(G,E)$. By construction, $\psi$ also factors through $O(G,E)$. By [13, Theorem 8.3(2)], there is a unique KMS state on $O(G,E)$, and we deduce that $\phi = \psi$. In particular, $\phi|_{C^*(G)} = \psi|_{C^*(G)} = \theta$. \qed

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