Evidence of entanglement, or:
Is Schrödinger’s cat really entangled?

Kedar S. Ranade$^1$ and Kaled Dechoum$^2$

$^1$Institut für Quantenphysik, Universität Ulm,
and Center for Integrated Quantum Science and Technology (IQST),
Albert-Einstein-Allee 11, D-89081 Ulm, Deutschland (Germany)

$^2$Instituto de Física, Universidade Federal Fluminense, 24210-340 Niterói RJ, Brazil

(Dated: August 14th, 2013)

Quantum effects and, in particular, entanglement are by now widely recognized in all areas of physics and related fields. However, we feel that the precise notion of entanglement—though mathematically well-defined—still generates confusion even in the academic environment. Our aim in this article is to point out and clarify some aspects of the notion of entanglement, in particular with respect to observable quantities. We do this by illustrating common misconceptions with Schrödinger’s cat paradox, explain them in mathematical terms and provide examples where misunderstandings arise and how they can be solved.

Keywords: Entanglement, observables, Schrödinger’s cat

I. INTRODUCTION

The notion of entanglement has by now grown to one of the most prominent features in quantum mechanics and, in particular, in quantum information theory. The power of quantum computers and the security of quantum cryptography, which belong to the most popularized fields of quantum theory, rely on or are closely related to the concept of entanglement. Nevertheless, students (and researchers) often struggle with conceptually different aspects of entanglement used in literature. On the one hand, there are foundational aspects of physics like the EPR paradox or Bell’s inequality (there are several works on these topics, e.g. Ref. [1]). On the other hand there appear more “applied” results of entanglement theory, including quantum cryptography, quantum teleportation, quantum computing and the like. A point which is often missed and which we want to address here, is that in order to observe
entanglement we must be specific on which quantities can actually be measured—and we could be even more general: in quantum mechanics, we always should specify and take into account which quantities can be measured and thus are observables.

Although mathematically well-defined, there seem to exist common misconceptions about the nature of entanglement. We shall therefore refer to entanglement in two related, but different ways, namely

- entanglement as a mathematical concept and
- entanglement as a resource in quantum processing.

On the one hand, with entanglement as a purely mathematical concept there is nothing mysterious or arcane about it: it is just a property of certain states on a tensor product of Hilbert spaces (i.e. density matrices or vectors). The principal difficulties in physically understanding entanglement arise only by invoking the notion of (space-time) locality, which result in peculiar features of quantum mechanics exhibited by the violation of Bell’s inequality. On the other hand, in any physical situation where we want to make use of quantum entanglement, we must, at least in principle, be able to prove the existence of entanglement (in the mathematical sense), which implies proving the absence of any classical explanations for the measurements observed. This also means that entanglement can only be described in terms of measurements, or to state it more dramatically: if there is no way of measuring it, there is no (physically usable) entanglement.

In this article, we shall clarify, without any attempt to claim novelty, these aspects of the notion of entanglement. We note that this can be taken even further: if, for any reason, we are restricted to observables which can jointly be measured (they are called commensurable and are represented by commuting operators), there always exist a classical description of the experiment. To summarize the main statements proposed in this article:

- By measuring commuting observables only, one can never prove the existence of entanglement.
- One needs at least one pair of non-commuting observables to be able to detect any quantum effect.

Since the notion of entanglement was introduced by Schrödinger in the very same paper where he introduced his well-known cat paradox, we take this as the starting point for our
survey and start by illustrating Schrödinger’s cat and explain why there is no evidence for entanglement there. Then, we will give the basic mathematical formalism of quantum mechanics and entanglement. After that, we will give some illustrations and finally summarize and conclude our results.

II. SHORT HISTORY OF SCHRÖDINGER’S CAT

In reply to the Einstein-Podolsky-Rosen paradox of 1935, a now famous german-language article “Die gegenwärtige Situation in der Quantenmechanik” (The present situation in quantum mechanics) appeared in three parts in the journal Die Naturwissenschaften (Natural sciences) in the same year where Erwin Schrödinger coined the german term “Verschränkung”. (The english equivalent “entanglement” also appeared in a paper by Schrödinger at the same time.) At the end of the first part an apparent paradox, now known as “Schrödinger’s cat”, was introduced. Let us, for the sake of completeness, recall this Gedankenexperiment: We put a living cat into a box together with an apparatus which is secured from the cat: In the apparatus, there is a small portion of a radioactive substance (say, a single nucleus), such that the probability of a single decay within one hour from the closing the box is one half. A decay would trigger a little hammer which destroys a little flask containing prussic acid that would immediately kill the cat. We open the box after one hour and check, whether the cat is alive or dead and whether the nucleus is decayed or not. If we perform the experiment several times, we find that the cat is dead half of the times and alive half of the times, and whenever the cat is dead, the nucleus has decayed, and whenever it is alive, the nucleus is intact; this is a classical correlation between the cat and the nucleus.

A. Quantum mechanics of the cat experiment

The usual analysis of Schrödinger’s cat is the following: The (classical) cat has two states, dead and alive, and so has the radioactive nucleus, decayed or not. Let us write this in the so-called Dirac notation: the states of the cat are |alive⟩ and |dead⟩, those of the nucleus |intact⟩ and |decayed⟩. The joint system of cat and nucleus now has—by classical reasoning—four possible states, which we may write as 1. |alive⟩|intact⟩, 2. |alive⟩|decayed⟩, 3. |dead⟩|intact⟩ and 4. |dead⟩|decayed⟩. In our model—the poison kills the cat and the
cat does not die of other causes—we have correlations (in the statistical sense) between the state of the cat and the state of the nucleus, namely that possibilities 2 and 3 are excluded. Note that up to now there is nothing “quantum” here, in particular, these correlations are not entanglement!

Any (even popular-science) introduction to quantum mechanics tells us that we can have superpositions of states. In particular, we could have a superposition of the possibilities 1 and 4 from above. As we do not know anything about the system, except that the nucleus has decayed with probability $1/2$, we may be tempted to conclude that the system, consisting of the cat and the nucleus, is in the superposition state

$$|\Psi\rangle = \frac{|\text{alive}\rangle|\text{intact}\rangle + |\text{dead}\rangle|\text{decayed}\rangle}{\sqrt{2}},$$

(1)

where the normalization $\frac{1}{\sqrt{2}}$ represents the square root of the probabilities as the wave function is the probability amplitude, and only its square root the probability density. This is—in quantum mechanics—a perfectly valid statement; in popular science one would say that, before the measurement, the cat is dead and alive at the same time. If we did not wait for precisely one half-life time of the nucleus, we would get an asymmetry for the two possibilities, since the probability of the decay or non-decay of the nucleus is not 50% each. We could incorporate this asymmetry, if not either probability is zero or one, but this will not significantly change our line of argument.

Nevertheless, up to now our reasoning to infer the state of the system is incorrect. The first thing, which may appear strange, is the lack of a phase. We know that quantum theory “lives” in a complex Hilbert space, where we have the choice of a complex phase $e^{i\phi}$. Global phases are irrelevant, but relative phases are not. So, apparently, for any choice of the angle $\phi \in (0; 2\pi)$, another equivalent state describing the system would be

$$|\Psi(\phi)\rangle = \frac{|\text{alive}\rangle|\text{intact}\rangle + e^{i\phi}|\text{dead}\rangle|\text{decayed}\rangle}{\sqrt{2}}.$$  

(2)

If we repeat the experiment several times, we will see that the cat is dead half of the time and alive half of the time, which completely reproduces our results from above. So in terms of our observations we cannot distinguish a state with some $\phi$ from another one with $\phi'$. Now this is not a problem, since quantum mechanics tells us that any of these states is entangled.

However, up to now we have dealt with pure-state quantum mechanics only. But a complete quantum-mechanical description should also take into account the possibility of mixed
states. In eqs. (1) and (2) we used coherent mixtures of quantum states, i.e. superpositions of state vectors. We can also consider classical or incoherent mixtures of states, i.e. convex sums of density matrices. This comes in quite naturally, if we say, that we do not know the actual state of the cat—it may be any of those mentioned in eq. (2)—so that we take the (statistical) “average” over all possible angles $\varphi$. This averaging procedure is mathematically well-defined and results in a mixed state

$$\rho = \frac{|\text{alive}\rangle \langle \text{alive}| \otimes |\text{intact}\rangle \langle \text{intact}| + |\text{dead}\rangle \langle \text{dead}| \otimes |\text{decayed}\rangle \langle \text{decayed}|}{2},$$

which once again exactly reproduces our observations. But this state is not entangled! This shows that from the description of Schrödinger’s cat experiment, we cannot conclude that the cat and the nucleus are entangled. We do not have enough evidence of entanglement in this system.

At this point we have shown that just checking the $2 \times 2 = 4$ states is insufficient to prove entanglement in a quantum state. In theoretical quantum physics, it is easy to specify a measurement which could provide evidence of entanglement as this is just two two-level systems, and such measurements are well-known for qubits. However, we cannot think of a practical implementation of such measurements if a cat is to be used.

**B. A modified cat paradox**

To give an example of how the state in eq. (3) may appear, we can slightly modify the experiment. Assume that the nucleus exhibits a $\beta^-$-decay and emits an electron and an antineutrino. The antineutrino transcends the box and is collected in some big neutrino detector. So the operator of that detector knows the state of the cat and may tell the state to our experimentalist. In this case the system is composed of three subsystems: the cat, the nucleus and the existence of the antineutrino (with two states: it exists or it does not exist). As we only consider the two systems of cat and nucleus, we have to trace out the third system, which results in the state described in eq. (3). If the neutrino detector did not exist, the neutrino would leave the earth undetected, but this cannot alter the state. So the observer cannot distinguish between the states of eq. (2) and eq. (3).

The deeper reason for this effect is that it is not possible to prove that a quantum system is entangled by a single measurement whatsoever (in the statistical sense, i.e. repeating the
experiment several times). In the example of the cat, the only measurement we perform is
the dead-alive-measurement of the cat and, accordingly, the intact-decayed-measurement of
the nucleus. To phrase it even more generally: No system can be shown to exhibit genuine
quantum phenomena, if the observer is restricted to a single measurement. All quantum
phenomena rely on measurements which are non-commensurable.

III. MATHEMATICAL FORMALISM

In order to present our statements in a more general way, we should introduce some
formalism. We start with a classical system, on which we can perform measurements. In our
case, we use a reduced model of the cat, i.e., we do not consider its position or momentum,
its color, its sex or whatever feature a cat may have. We only consider, whether it is dead
or alive, and our measurement thus has a binary outcome. We must assume that the other
features are unrelated to whether the cat is dead or alive.

As an axiom, for a quantum system there is always a Hilbert space associated with the
system. The dimension of that space is the number of classical outcomes, which may be
finite or infinite. In case we have two classical outcomes, like in our description of the cat
and the nucleus, respectively, the Hilbert space is the two-dimensional complex Hilbert space
\( \mathcal{H} = \mathbb{C}^2 \). (Note that two Hilbert spaces of the same dimension are isomorphic, and that
for every such dimension a Hilbert space can be constructed. For quantum mechanics, we
have \( \mathcal{H} = \mathbb{C}^n, \ n \in \mathbb{N} \), if the number of outcomes is finite. When we have a continuous
measurement, like the position, we have an infinite-dimensional space and things get much
more difficult, but we shall ignore that.) Combining two systems with \( n \) and \( m \) classical
outcomes (in our case \( n = m = 2 \)), the joint system may have \( n \cdot m \) outcomes, and the
associated mathematical construction is the tensor product of Hilbert spaces. In our case,
the Hilbert space of the joint system is thus four-dimensional, i.e. \( \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4 \).

Another axiom of quantum mechanics is that observables correspond to operators on
Hilbert space with their eigenvalues being the possible measurement outcomes. As we re-
quire that the outcome of a measurement be real (and not imaginary or complex), these
operators need to be self-adjoint. A state now is the mathematical object which maps to
each measurable operator its expectation value as a weighted mean of all possible outcomes.
So an operator \( \hat{A} \) is mapped to \( \langle \hat{A} \rangle \), and all we may know about the state is this expectation
for any measurable operator. How to specify this mapping? This can be done in terms of density matrices \( \rho \), where we have \( \hat{A} \mapsto \langle \hat{A} \rangle = \text{Tr}(\rho \hat{A}) \). If (and only if) \( \rho \) is a one-dimensional projector, there is a state vector \( |\Psi\rangle \in \mathcal{H} \), such that \( \rho = |\Psi\rangle\langle \Psi | \), and \( |\Psi\rangle \) is unique up to a phase. Note now that we cannot directly measure the state of a quantum system. We can only measure expectation values of operators. For two different density matrices, \( \rho \) and \( \rho' \), there always exist an operator \( \hat{A} \), such that \( \text{Tr}(\rho \hat{A}) \neq \text{Tr}(\rho' \hat{A}) \), i.e., we can distinguish \( \rho \) and \( \rho' \) by their statistical properties, if indeed \( \hat{A} \) represents a measurable quantity. In the best case, we may know the complete mapping \( \hat{A} \mapsto \langle \hat{A} \rangle \) for all self-adjoint operators \( \hat{A} \), so that \( \rho \) can be reconstructed uniquely. But if, for physical, mathematical or practical reasons, not all self-adjoint operators \( \hat{A} \) on the Hilbert space correspond to measurable quantities, it may happen that there are two (or more) density operators, say \( \rho \) and \( \rho' \), which yield the same result for all measurable quantities. Now we may take two different points of view (regarding the question of the nature of a quantum state, see also Ref. 5):

- The density operators \( \rho \) and \( \rho' \) are different states, but we do not have means to distinguish them (e.g. we can measure position and momentum of an atom, but do not have a magnetic field to determine spin).

- Since all measurable quantities have the same statistical properties, \( \rho \) and \( \rho' \) are actually the same physical state.

While the second interpretation is more physically motivated, we take the first, which is easier to handle in our discussion, but the results will be the same. So for a quantum system with a certain set of measurable quantities, to which (by assumption or measurement) we know that there are assigned certain outcomes, we find a set of compatible density operators. In the following section we shall define the mathematical notion of entanglement for a single density operator; for our system, the entanglement then could be the entanglement of any of these compatible density operators. In quantum information, entanglement is used as a resource, and it turns out that we have to minimize the entanglement over all compatible operators in order to find the usable entanglement for physical tasks.
A. Entanglement

There is a vast amount of literature on entanglement, e.g. the review by the Horodeckis, and we do not want to go into too many details here. As such, entanglement is a property of Hilbert spaces and their tensor products. The tensor product of two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ is the Hilbert space $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B$ of all superpositions of vectors $|\psi\rangle_A \otimes |\varphi\rangle_B$, where $|\psi\rangle_A \in \mathcal{H}_A$ and $|\varphi\rangle_B \in \mathcal{H}_B$. Now, any state on such a system, described by a state vector or a density matrix, is either separable or not; inseparable states are called entangled. (Note that we only consider bipartite entanglement here, i.e. we only have two subsystems; otherwise the theory gets more complicated.) The mathematical definition of entanglement is the following: A (state) vector $|\Psi\rangle_{AB} \in \mathcal{H}$ is separable, if it can be written in the form $|\psi\rangle_A \otimes |\varphi\rangle_B$, i.e., without the need of any superposition.

How to define entanglement for mixed states? Any mixed state can (in general, non-uniquely) be written in the form $\rho_{AB} = \sum_{i=1}^{n} p_i |\Psi_i\rangle_{AB} \langle \Psi_i|$, for some state vectors $|\Psi_i\rangle_{AB} \in \mathcal{H}$ and $p_i \geq 0$ with $\sum_{i=1}^{n} p_i = 1$. By definition, a state is separable, if there exists such a decomposition, where all the $|\Psi_i\rangle_{AB}$ are separable. This is the definition of entanglement, which we want to invoke, and all other concepts which appear in literature are derived from that.

We may also note the Schmidt decomposition (cf. e.g. Refs. 7 or 8): for any vector $|\Psi\rangle_{AB}$, there exist orthonormal systems $(|\psi_i\rangle_A)_{i=1}^{n}$ and $(|\psi_i\rangle_B)_{i=1}^{n}$ and $(\lambda_i)_{i=1}^{n}$ with $\sum_{i=1}^{n} |\lambda_i|^2 = 1$, such that

$$|\Psi\rangle_{AB} = \sum_{i=1}^{n} \lambda_i |\psi_i\rangle_A \otimes |\psi_i\rangle_B. \quad (4)$$

The state is thus separable, if and only if only one of the $\lambda_i$ is non-zero, and more generally it is possible to build entanglement measures based on the distribution of the $\lambda_i$.

B. State determination and tomography

Now consider the following task: we are given a quantum state $\rho$, whether pure or not, and we have to determine that state. (This question in itself gives rise to a large amount of problems, cf. e.g. Ref. 9.) As we can only perform measurements and measurements, in general, alter the state, there is, by principle, no way to determine the state. If we relax the requirements, we can move on to a related task: given $n$ copies of identically prepared
states, can we recover the state? Quantum information theory tells us that we can determine the state to arbitrary precision, provided \( n \to \infty \); this is known as state tomography. But there is always the necessity to perform different measurements. A measurement in quantum information corresponds to a basis of a Hilbert space and thus to possible observables. Such a measurement will thus only give us information on the diagonal elements of a density matrix in that basis, but not on the off-diagonal elements. We will explain in the following that true quantum effects can be observed, if and only if at least two different bases are used for measurements or, in other words, at least two observables do not commute.

**IV. APPLICATIONS AND EXAMPLES**

In this section we use the mathematical formalism to analyze the cat paradox and other applications in physical situations.

**A. Cat-nucleus system**

We start by examining Schrödinger’s cat. The joint system of the cat and the nucleus has the four classical states of section III. It is therefore described by a \( 4 \times 4 \)-density matrix, whose rows and columns we order as before: 1. \(|\text{alive}\rangle|\text{intact}\rangle\), 2. \(|\text{alive}\rangle|\text{decayed}\rangle\), 3. \(|\text{dead}\rangle|\text{intact}\rangle\) and 4. \(|\text{dead}\rangle|\text{decayed}\rangle\). From our repeated measurements we can try to infer the density matrix of the cat-nucleus system:

\[
\rho = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{pmatrix}.
\]

(5)

Which observables do we have? We have the cat dead/alive-observable \( \sigma_z^{(1)} \) and the nucleus decayed/intact-observable \( \sigma_z^{(2)} \) resulting in the joint observable \( \sigma_z^{(1)} \otimes \sigma_z^{(2)} \) with four possible outcomes. The probabilities that the four possibilities appear, are now \( a_{11}, a_{22}, a_{33} \) and \( a_{44} \). From our observations we know \( a_{11} = a_{44} = 1/2 \) and \( a_{22} = a_{33} = 0 \).

Now the density matrix must be positive semidefinite and, in particular, hermitian. Positive semidefiniteness can be checked by the Hurwitz-Sylvester (principal minor) criterion.
an $m \times m$ matrix is positive semidefinite, if and only if all its principal minors are non-negative. The principal minors are the determinants of the $2^n - 1$ submatrices (including the matrix itself), where a set of corresponding rows and columns of the original matrix is left out. In our case this immediately implies that all non-diagonal elements except $a_{14}$ and $a_{41}$ must vanish. By hermiticity, we are left with a parameter $a_{41} = a_{41}^* =: x$.

From our observations we have found the complete matrix up to a single complex parameter which must fulfill $a_{11}a_{44} - a_{14}a_{41} = 1/4 - |x|^2 \geq 0$ or $|x| < 1/2$ for positivity. But our measurement by no means reveals $x$. How to check whether $\rho$ is entangled? We make use of the Peres-Horodecki criterion for $2 \times 2$ systems\textsuperscript{11,12} stating that the partial transpose of a density matrix—the transpose with respect to either subsystem—is positive semidefinite, if and only if the state is separable. We calculate the non-zero eigenvalues of the partially transposed density matrix and find $+|x|$ and $-|x|$. So our observations are consistent with no entanglement $x = 0$, maximal entanglement $|x| = 1$ and any amount of partial entanglement $|x| \in (0;1)$.

To conclude, we can only determine the diagonal elements of the density matrix. But the quantum properties and, in particular, the entanglement of any system are related to the off-diagonal elements of the density matrix. \textit{We have learned nothing about the entanglement of the system!} It turns out that in order to have evidence for entanglement and, more generally, any “quantum” property as superposition, we must have at least two non-commuting observables. Moreover, this is also sufficient as the examples of Bell inequalities or quantum cryptography show, since in this case, we can always find off-diagonal values which indicate true quantum behavior.

**B. Quantum cryptography**

The cat-nucleus system may seem somewhat artificial. Now we want to point out that evidence of entanglement is of practical value in quantum information theory. One of the applications of quantum technology is quantum cryptography. In quantum cryptography two distant parties, Alice and Bob, want to share a message in such a way that an eavesdropper, Eve, cannot infer the content of the message. The use of the one-time pad enables them to do so, provided they share a key, i.e., a large enough random number Eve does not know. Suppose Alice and Bob share two qubits (e.g. photons with their polarization) in the pure
entangled state\(^{13}\)

\[ |\Phi^{+}\rangle_{AB} = \frac{1}{\sqrt{2}} \left[ |0\rangle_{A}|0\rangle_{B} + |1\rangle_{A}|1\rangle_{B} \right], \]  

which is sufficient to prove security. If Alice and Bob measure that state in the basis \{\langle 0 |, \langle 1 | \}\), they get with equal probability as outcome both zero or both one, i.e. a random bit. Since the state is pure, it cannot have any correlations with other systems, in particular, not with Eve’s system. But only from measurement in the basis \{\langle 0 |, \langle 1 | \}\}, Alice and Bob cannot say that Eve does not know the bit. Consider the alternative three-qubit state, the GHZ state

\[ |\Phi_{\text{GHZ}}\rangle_{ABB'} = \frac{1}{\sqrt{2}} \left[ |0\rangle_{A}|0\rangle_{B}|0\rangle_{B'} + |1\rangle_{A}|1\rangle_{B}|1\rangle_{B'} \right] \]  

with Eve possessing \(B'\). If Eve measures her state in the \{\langle 0 |_{B'}, \langle 1 |_{B'} \}\) basis she exactly knows what the random bit is, and this state obviously cannot be used for quantum cryptography. What is the difference between these two states? Alice and Bob would have to determine their reduced state, which in the latter case turns out to be the separable mixed state

\[ \rho_{AB} = \frac{|0\rangle_{A}\langle 0| \otimes |0\rangle_{B}\langle 0| + |1\rangle_{A}\langle 1| \otimes |1\rangle_{B}\langle 1|}{2}. \]  

As the reader may have noted by now, this is mathematically completely equivalent to Schrödinger’s cat and its modification (sec. II), and thus all conclusions apply here. So in order to be secure the parties in quantum cryptography need to check their entanglement, and this is where the measurement of non-commuting operators appear in all quantum cryptographic protocols.

One may ask how such ambiguity between two states appears in a real experiment. In a theoretical description of quantum cryptography, Alice prepares the two-photon state of eq. (6) and immediately measures her part in order to prepare the single-photon state to be sent to Bob. However, in a quantum-optical setting, a weak laser pulse is commonly used to produce photons. If the photon number of the laser is Poisson-distributed, by lowering the average number of photons, the fraction of pulses containing two or more photons may go down, but never reaches zero. The case of two photons is then theoretically described by the state of eq. (7), where photons \(B\) and \(B'\) are sent to Bob. But then Eve may just keep the photon of \(B'\); while Bob will correctly measure photon \(B\) (but does not measure the number of photons), Eve may infer his outcome from photon \(B'\). (In quantum cryptography, this is known as the photon-number splitting attack.)\(^{14}\)
C. Entanglement with the vacuum

Another question, which is sometimes raised, is the following: consider a single photon which crosses a 50:50 beam splitter into two channels, namely the reflected one (channel 1) and the transmitted (channel 2). So the input field is $|1\rangle$ and classically the output fields are either $|0\rangle|1\rangle$ (transmission) or $|1\rangle|0\rangle$ (reflection). If the transformation is unitary, the state after passing the beam splitter is something like $\frac{|0\rangle|1\rangle + |0\rangle|1\rangle}{\sqrt{2}}$. But how can it be that the vacuum, “nothing”, is entangled? This question is resolved by the following. First we have to distinguish the vacuum state $|0\rangle$ from the Hilbert space “nothing”, the nullvector. So vacuum means “no photon”, but not “no state”. Even if it is not populated the mode exists, and it is the entanglement between the states of the mode. (For further details on such questions, cf. e.g. Ref. 15.)

To phrase it differently: Consider two harmonic oscillators (two modes). On the one hand, we may consider the ground state, which in the Fock basis is written as $|0\rangle$ and interpreted as “no photons”. However, this state obviously has a non-vanishing wavefunction, and a change of basis should not affect a physical quantity like entanglement. It is not the state which is entangled, but the system in a specific state.

V. SUMMARY

In this article we have worked out what is needed for a state so that it can be considered to be entangled and illustrated this by Schrödinger’s cat paradox. We pointed out that the physically usable entanglement must take into account the possible observables and that in order to observe quantum phenomena we need at least two non-commuting observables. The examples we gave can always be resolved by a careful inspection of the things we look at: what is the system, how is it (and its measurements) described in Hilbert space and which information we can mathematically infer by measurements? This is all what is needed to evade seemingly strange observations.

Finally we wish to remark that we should always consider density operators, since—to modify Schrödinger’s statement—it is the density matrix (and not the wavefunction) which is the catalog of expectations.
ACKNOWLEDGMENTS

The authors thank Nathan Harshman (American University, Washington D.C.) for several helpful comments. K. S. Ranade acknowledges financial support by BMBF/QuOReP.

1 D. M. Greenberger, M. A. Horne, A. Shimony and A. Zeilinger, *Bell’s Theorem without inequalities*, American Journal of Physics 58 (1990), pp. 1131–1143

2 A. Einstein, B. Podolsky and N. Rosen, *Can quantum-mechanical description of physical reality be considered complete?*, Physical Review 47 (1935), pp. 777–780

3 E. Schrödinger, *Die gegenwärtige Situation in der Quantenmechanik* Die Naturwissenschaften 23 (1935), pp. 807–812, pp. 823–828, pp. 844–849; english translation by JOHN D. TRIMMER: *The Present Situation in Quantum Mechanics: A Translation of Schrödinger’s “Cat Paradox” Paper*, Proceedings of the American Philosophical Society 124 (1980), pp. 323–338; reprinted in *Quantum Theory and Measurement* (eds. J. A. Wheeler and W. H. Zurek), Princeton University Press 1983

4 E. Schrödinger, *Discussion of probability relations between separated systems*, Proceedings of the Cambridge Philosophical Society 31 (1935), pp. 555–563; *Probability relations between separated systems*, ibid. 32 (1936), pp. 446–452

5 R. G. Newton, *What is a state in quantum mechanics?*, American Journal of Physics 72 (2004), pp. 348–350

6 R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, *Quantum entanglement*, Reviews of Modern Physics 81 (2009), pp. 865–942

7 A. K. Ekert and P. L. Knight, *Entangled quantum systems and the Schmidt decomposition*, American Journal of Physics 63 (1995), pp. 415–423

8 M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2000)

9 M. Paris and J. Řeháček, *Quantum State Estimation*, Springer (2004)

10 F. R. Gantmacher, *The theory of matrices* (vol. 1), Chelsea (1960)

11 A. Peres, *Separability criterion for density matrices*, Physical Review Letters 77 (1996), pp. 1413–1415
12 M. Horoecki, P. Horoecki and R. Horoecki, *Separability of mixed states: necessary and sufficient conditions*, Physics Letters A 223 (1996), pp. 1–8

13 A. K. Ekert, *Quantum cryptography based on Bell’s Theorem*, Physical Review Letters 67 (1991), pp. 661–663

14 G. Brassard, N. Lütkenhaus, T. Mor and B. C. Sanders, *Limitations on Practical Quantum Cryptography*, Physical Review Letters 85 (2000), pp. 1330–1333

15 S. J. van Enk, *Single-particle entanglement*, Physical Review A 72 (2005), 064306