A WEAK GALERKIN-MIXED FINITE ELEMENT METHOD FOR THE STOKES-DARCY PROBLEM

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Abstract. In this paper, we propose a new numerical scheme for the coupled Stokes-Darcy model with Beavers-Joseph-Saffman interface condition. We use the weak Galerkin method to discretize the Stokes equation and the mixed finite element method to the Darcy equation. A discrete inf-sup condition is proved and optimal error estimates are also derived. Numerical experiments validate the theoretical analysis.

Key words. weak Galerkin finite element methods, mixed finite element methods, weak gradient, coupled Stokes-Darcy problems

AMS subject classifications. Primary, 65N30, 65N15, 65N12; Secondary, 35B45, 35J50

1. Introduction. The coupling of fluid flow and porous media flow has received an increasing attention during the last decade. This coupled flow arises in many fields, such as the transport of contaminants through steams in environment, the filtration of flood through vessel walls in physiology, and some technologies involving fluid filter in industrial. Interested readers may refer to [14, 17, 27, 32] and the reference therein.

The mathematical model of such a coupled problem consists of Stokes equations in the fluid region and the Darcy’s law in the porous medium. Appropriate interface conditions, namely mass conservation, balance of force and the Beavers-Joseph-Saffman condition [8, 20, 36] are imposed on the interface between the free flow region and porous medium flow region.

Early studies on numerical simulations and error analysis for the coupled Stokes-Darcy problem can be found in [15, 37]. In a comprehensive study presented in [14], Discacciati et al. analyze a standard velocity-pressure formulation in the Stokes region and a second order primal elliptic problem in the Darcy region. Continuous finite element methods are used in both space. In [26], Layton et al. consider a mixed formulation in Darcy region, which involves the velocity and pressure simultaneously. They prove the existence and uniqueness of a weak solution to the mixed Stokes-Darcy system. Continuous finite element method employed in Stokes region and the mixed finite element method used in Darcy region. Later, the discontinuous Galerkin(DG) methods are applied to this problem [33, 34]. The work combines DG method for the Stokes equations with the mixed finite element method for the Darcy...
equation is proposed in [33]. Analysis of the DG method for both Stokes and Darcy equations introduced in [34]. In addition, preconditioning techniques are also used for the coupled flow [12]. More recent studies concerning the Stokes-Darcy problem can be found in [2, 10, 11, 13, 22, 23, 31, 38, 39, 40, 44].

The weak Galerkin (WG) finite element method is proposed in [41] by Wang and Ye for the second order elliptic equation. They introduce totally discontinuous weak functions and corresponding weak differential operators. Numerical implementation of WG methods for different models with more general finite element partitions is discussed in [29]. The WG scheme is designed on arbitrary shape of polygons in 2D or polyhedra in 3D with certain shape regularity by introducing a stabilizer in [42]. Unified study for WG methods and other discontinuous Galerkin methods is presented in [18, 19]. In the past few years, the WG method is widely applied to many partial differential problems because of its flexibility and efficiency. The corresponding work can be found in [30, 45, 46, 48, 49].

Recently, WG methods are developed for solving the Stokes-Darcy model. In [23], the coupled system is described by Stokes equations in primal velocity-pressure formulation and the Darcy’s law in primal pressure formulation. The piecewise constant elements are used to approximate the velocity, hydraulic and pressure. Furthermore, the same formulation is discussed in [22], different choices of WG finite element spaces are investigated, the classical meshes in [23] are extended to general polygonal meshes. In [13], the authors consider the mixed formulation in the Darcy region, both the Stokes region and Darcy region involve the velocity and the pressure. Strong coupling of the Stokes-Darcy system is achieved in the discrete space by using the WG approach.

As mentioned above, we can see that WG methods show a high flexibility for dealing with the Stokes-Darcy problem. However, the decoupling of the elements leads to an increase in the total degrees of freedom, which limits the practical utility of WG methods, especially in high order approximations. The aim of this article is to introduce a new numerical scheme with fewer number of degrees of freedom for the same mixed Stokes-Darcy formulation as [13]. To this end, we use different finite element discretizations for the two regions. The WG method is still employed to approximate the velocity and the pressure in Stokes region. A summary for the features of WG methods to solve Stokes equation is provided in [33]. As for the Darcy region, the same unknowns are approximated by the mixed finite element (MFEM) method, which is different from the WG approximation in [13]. Readers may refer to, e.g. [24] for a comparison of degrees of freedom between WG methods and MFEM methods. Several standard mixed finite element spaces can be chosen, such as RT spaces [35], BDM spaces [7], BDFM spaces [6] and so on. The efficiency of the MFEM has been demonstrated in [3, 9, 28]. Lagrange multiplier is introduced to impose the continuity of the velocity. The benefit of our approach is the possibility of combining the efficiency of the MFEM methods for Darcy problem with the flexibility of WG methods for Stokes problem. However, the combination of these two different finite element methods makes the proof process more complex for the inf-sup condition than [13]. Inspired by the work in [33], we construct two local projection operators in different region to prove it.

The rest of the paper is organized as follows. In the next section, we present the model problem, some notations and function spaces. In Section 3, we introduce weak
Galerkin methods and construct WG-MFEM numerical scheme for the Stokes-Darcy problem. The well-posedness of the scheme is analyzed in Section 4. We derive the error estimates for the corresponding numerical approximations in Section 5. Finally, some numerical examples are presented to show the good performance of the developed algorithm in Section 6.

2. Model Problem and Weak Formulation. Let $\Omega$ be a bounded domain in $\mathbb{R}^2$, subdivided into a free fluid region $\Omega_s$ and a porous region $\Omega_d$. Denote by $\Gamma = \partial \Omega_s \cap \partial \Omega_d$ the interface, and by $\Gamma_s = \partial \Omega_s \setminus \Gamma$, $\Gamma_d = \partial \Omega_d \setminus \Gamma$ the outer boundary. Moreover, let $\mathbf{n}$ and $\mathbf{\tau}$ be the unit normal and tangential vectors to $\Gamma$, respectively, see Fig. 2.1.

![Fig. 2.1. Domain schematic for Stokes-Darcy coupled flow.](image)

In $\Omega_s$, the fluid flow is governed by Stokes equations.

\begin{align*}
\text{(2.1)} \quad & -\nabla \cdot \mathbf{T}(\mathbf{u}_s, p_s) = f_s \quad \text{in } \Omega_s, \\
\text{(2.2)} \quad & \nabla \cdot \mathbf{u}_s = 0 \quad \text{in } \Omega_s, \\
\text{(2.3)} \quad & \mathbf{u}_s = 0 \quad \text{on } \Gamma_s,
\end{align*}

where $\mathbf{T}$ is the stress tensor, $\mathbf{T}(\mathbf{u}_s, p_s) = 2\nu \mathbf{D}(\mathbf{u}_s) - p_s \mathbf{I}$ and $\mathbf{D}(\mathbf{u}_s) = \frac{1}{2}(\nabla \mathbf{u}_s + \nabla^T \mathbf{u}_s)$, $\nu$ is the kinematic viscosity of the fluid and $\mathbf{I}$ is the identity matrix. $f_s$ is a given external body force.

In $\Omega_d$, the porous media flow is governed by Darcy’s law.

\begin{align*}
\text{(2.4)} \quad & \nabla \cdot \mathbf{u}_d = f_d \quad \text{in } \Omega_d, \\
\text{(2.5)} \quad & \mathbf{u}_d = -\mathbf{K} \nabla p_d \quad \text{in } \Omega_d, \\
\text{(2.6)} \quad & \mathbf{u}_d \cdot \mathbf{n}_d = 0 \quad \text{on } \Gamma_d,
\end{align*}

where $\mathbf{K}$ is the symmetric positive-defined permeability tensor, $f_d$ is the source term and satisfies the following condition

$$
\int_{\Omega_d} f_d = 0.
$$
The interface conditions on $\Gamma$ consist of three parts.

\begin{align}
(2.7) & \quad u_s \cdot n = u_d \cdot n \quad \text{on } \Gamma, \\
(2.8) & \quad -\mathbf{T}(u_s, p_s) \cdot n = p_d \quad \text{on } \Gamma, \\
(2.9) & \quad -\mathbf{T}(u_s, p_s) \cdot \tau = \mu K^{1/2} u_s \cdot \tau \quad \text{on } \Gamma.
\end{align}

Condition (2.7) is the result of mass conservation across the interface, condition (2.8) represents the fact that normal force on the interface is balance, and condition (2.9) is the Beavers-Joseph-Saffman interface condition, in which $\mu \geq 0$ is a parameter depending on the properties of the porous medium.

Next, we recall some notations for Sobolev space [1]. Let $K$ be a polygon in $\mathbb{R}^2$, $H^m(K)$ stands for the Sobolev space. We denote by $\| \cdot \|_{m,K}$ and $| \cdot |_{m,K}$ the norm and semi-norm in $H^m(K)$, $m \geq 0$. When $m = 0$, $H^0(K)$ coincides with $L^2(K)$ and we shall drop the subscript $K$ in the norm and semi-norm notations.

We define the space $H(\text{div}; K)$ as follows.

$$H(\text{div}; K) = \{ v : v \in [L^2(K)]^d, \nabla \cdot v \in L^2(K) \},$$

with norm

$$\| v \|_{H(\text{div}, K)} = (\| v \|_K^2 + \| \nabla \cdot v \|_K^2)^{1/2}.$$ We also define

$$L^2_0(K) = \{ q \in L^2(K) : \int_K q \, dx = 0 \}.$$ Then the function space for the velocity and the pressure are defined as

$$V := \{ v \in H(\text{div}, \Omega), v|_{\Omega_s} \in H^1(\Omega_s), \ v = 0 \text{ on } \Gamma_s, \ v \cdot n_d = 0 \text{ on } \Gamma_d \},$$

and

$$M := L^2_0(\Omega).$$

Now we are ready to state the weak formulation of the Stokes-Darcy problem [2.1] - [2.9]. Find $(u, p) \in V \times M$ such that

\begin{align}
(2.10) & \quad a(u, v) + b(v, p) = (f_s, v)_{\Omega_s} \quad \forall \ v \in V, \\
(2.11) & \quad b(u, q) = (f_d, q)_{\Omega_d} \quad \forall \ q \in M,
\end{align}

where

$$a(u, v) = 2\nu(D(u), D(v))_{\Omega_s} + (K^{-1} u, v)_{\Omega_d} + \mu K^{1/2} \langle u_s \cdot \tau, v_s \cdot \tau \rangle_{\Gamma},$$

$$b(v, q) = - (\nabla \cdot v, q)_{\Omega}.$$ The existence and the uniqueness of the weak solutions have been proved in [26].

3. Discretization. In this section, we first introduce some basic definitions and preliminaries which will be used throughout the rest of this article. Then we construct numerical scheme for (2.10) - (2.11).
3.1. Notations for Partitions. In what follows, $\Omega_i$ refers to either $\Omega_s$ or $\Omega_d$, and it is the same for the other symbols with subscript $i$. Let $T_{s,h}$ be the partition of $\Omega_s$. Denote by $\mathcal{T}_h$ the union of $\mathcal{T}_{s,h}$ and $\mathcal{T}_{d,h}$, where $\mathcal{T}_{s,h}$ is a WG-regular partition and $\mathcal{T}_{d,h}$ consists of triangles or rectangles. $T_s$ represents the element of $\mathcal{T}_{s,h}$ and $T_d$ represents the element of $\mathcal{T}_{d,h}$. Denote the edges in $\mathcal{T}_h$ by $\mathcal{E}_h$, and define $\mathcal{E}_h$ to be the set of all edges in $\mathcal{T}_h \cap (\Omega_s \cup \Gamma_s)$, and $\mathcal{E}_d^i$ be the set of edges in $\mathcal{T}_h \cap (\Omega_d \cup \Gamma_d)$. The set of all edges in $\mathcal{T}_h \cap \Gamma$ is denoted by $\Gamma_h$. Especially, the partition $\mathcal{T}_{s,h}$ and $\mathcal{T}_{d,h}$ are not necessary to be consistent on the interface $\Gamma$. Denote the size of $T_i$ by $h_{T_i}$, the mesh size of $T_i$ and $s,h$ may not necessarily be related to the trace of $\mathcal{E}_h$ on $\partial T_s$. Note that $v_{s,b}$ takes single value on $e_s$. For convenience, we write $v_{s,b}$ as $\{v_{s,0}, v_{s,b}\}$ in short.

To define the WM method, we first give a brief introduction of weak function on $T_s$,

$$
\begin{align*}
\mathbf{v}_{s,h} = \begin{cases} 
\mathbf{v}_{s,0}, & \text{in } T_s, \\
\mathbf{v}_{s,b}, & \text{on } \partial T_s.
\end{cases}
\end{align*}
$$

The weak function is formed by the internal function $v_{s,0}$ and the boundary function $v_{s,b}$, where $v_{s,b}$ may not necessarily be related to the trace of $v_{s,0}$ on $\partial T_s$. Note that $v_{s,b}$ takes single value on $e_s$. For convenience, we write $v_{s,b}$ as $\{v_{s,0}, v_{s,b}\}$ in short.

In Stokes region, we define the following WG space for the velocity variable.

$$
V_h^s = \{ v_{s,h} = \{v_{s,0}, v_{s,b}\} \in [L^2(\Omega_s)]^2 \times [L^2(\mathcal{E}_h^s)]^2 : v_{s,0}|_{\mathcal{T}_s} \in [P_{\alpha_s}^s(T_s)]^2 \text{ for } T_s \in \mathcal{T}_{s,h}, \vspace{1mm} \\
v_{s,b}|_{e_s} \in [P_\beta(e_s)]^2 \text{ for } e_s \in \mathcal{E}_h^s \cup \Gamma_h, v_{s,b}|_{e_s} = 0 \text{ for } e_s \in \mathcal{E}_h^s \cap \Gamma_s, \}
$$

and the finite element space for the pressure variable as

$$
M_h^s = \{ q_{s,h} \in L^2(\Omega_s) : q_{s,h}|_{\mathcal{T}_s} \in P_{\gamma_s}(T_s), T_s \in \mathcal{T}_{s,h} \}
$$

where non-negative integers $\alpha_s$, $\beta$ and $\gamma_s$ satisfy

$$
\begin{align*}
\beta - 1 & \leq \gamma_s \leq \beta \leq \alpha_s \leq \beta + 1, \\
\alpha_s & \leq \gamma_s + 1, \\
1 & \leq \beta.
\end{align*}
$$

Remark 3.1. For $\alpha_s = 1$, $\beta = 0$, $\gamma_s = 0$, the situation is more complicated. Interested readers may refer to [42, 44] for details.

Then, we give the mixed finite element spaces corresponding to the Darcy region.

For the velocity variable

$$
V_d^\gamma = \{ \mathbf{v}_d \in H(\text{div}, \Omega_d) : \vspace{1mm} \\
\mathbf{v}_d|_T \in P_{\alpha_d}(T_d) \text{ for } T_d \in \mathcal{T}_{d,h}, \mathbf{v}_d \cdot \mathbf{n} = 0 \text{ for } \mathcal{E}_h^d \cap \Gamma_d, \}
$$

and for the pressure variable

$$
M_d^\gamma = \{ q_{d,h} \in L^2(\Omega_d) : q_{d,h}|_{T_d} \in P_{\gamma_d}(T_d) \text{ for } T_d \in \mathcal{T}_{d,h} \},
$$

where

$$
\begin{align*}
\gamma_d & \leq \alpha_d, \\
\alpha_d - 1 & \leq \gamma_d.
\end{align*}
$$
We assume that $\nabla \cdot V_h^d \subset M_h^d$.

In order to impose the continuity of the velocity on the interface, we introduce the discrete space for Lagrange multiplier.

$$A_h = V_h^d \cdot n.$$  

Now, we can define the global discrete velocity space $V_h$ and the discrete pressure space $M_h$ as follows.

$$V_h = \{ v_h = (v_{s,h}, v_{d,h}) \in V_h^s \times V_h^d : \sum_{e \in \Gamma_h} \int_{e} \eta (v_{s,h} - v_{d,h}) \cdot n = 0, \ \forall \ \eta \in \Lambda_h \},$$

$$M_h = M_h^s \times M_h^d.$$  

### 3.2. Discrete Weak Operators.

Next, we introduce some weak differential operators for $v_{s,h} \in V_h^s$.

**Definition 3.1.** For any $v_{s,h} \in V_h^s$, $T_s \in T_{s,h}$, the discrete weak gradient $\nabla_w v_{s,h}|_{T_s} \in [P_\beta(T_s)]^{d \times d}$ satisfies

$$\nabla_w v_{s,h}|_{T_s} = -(v_{s,0}, \nabla \cdot \tau)|_{T_s} + \langle v_{s,b}, \tau \cdot n \rangle_{\partial T_s}, \quad \forall \tau \in [P_\beta(T_s)]^{d \times d}.$$  

Analogously, we can define the discrete weak divergence.

**Definition 3.2.** For any $v_{s,h} \in V_h^s$, $T_s \in T_{s,h}$, the discrete weak gradient $\nabla_w v_{s,h}|_{T_s} \in [P_\beta(T_s)]^{d \times d}$ satisfies

$$\langle \nabla_w \cdot v_{s,h}, q_{s,h} \rangle_T = -(v_{s,0}, \nabla q_{s,h})_{T_s} + \langle v_{s,b}, q_{s,h} \rangle_{\partial T_s}, \quad \forall q_{s,h} \in P_\beta(T_s).$$  

Finally, denote by $D_w(v_{s,h})$ the weak strain tensor given by

$$D_w(v_{s,h}) = \frac{1}{2} (\nabla_w v_{s,h} + \nabla_w v_{s,h}^T).$$  

### 3.3. Numerical Scheme.

Define $Q_h = \{Q_0, Q_b\}$ the projection operator from $L^2(\Omega_s)$ onto $V_h^s$, where $Q_0$ is the $L^2$ projection onto $[P_\beta(T_s)]^2$, $\forall T_s \in T_{s,h}$, $Q_b$ is the $L^2$ projection onto $[P_\beta(e_s)]^2$, $\forall e_s \in \mathcal{E}_h$.

We are now in a position to give a numerical scheme for the coupled Stokes-Darcy problem. To this end, we define some bilinear forms in the discrete spaces. For any $u_h = (u_{s,h}, u_{d,h})$, $v_h = (v_{s,h}, v_{d,h}) \in V_h$, $p_h = (p_{s,h}, p_{d,h})$ and $q_h = (q_{s,h}, q_{d,h}) \in M_h$, define

$$a_{s,h}(u_{s,h}, v_{s,h}) = \sum_{T_s \in T_{s,h}} (2\nu D_w(u_{s,h}), D_w(v_{s,h}))_{T_s} + s(u_{s,h}, v_{s,h}),$$

$$s(u_{s,h}, v_{s,h}) = \sum_{T_s \in T_{s,h}} h_{T_s}^{-1} (Q_b u_{s,0} - u_{s,b}, Q_b v_{s,0} - v_{s,b})_{\partial T_s},$$

$$a_{i,h}(u_{s,h}, v_{s,h}) = \langle \mu K^{-\frac{1}{2}} u_{s,b}, \tau, v_{s,b} \rangle_{\Gamma_h},$$

$$b_{s,h}(v_{s,h}, q_{s,h}) = - (\nabla_w \cdot v_{s,h}, q_{s,h})_{\Omega_s},$$

$$b_{d,h}(v_{d,h}, q_{d,h}) = - (\nabla \cdot v_{d,h}, q_{d,h})_{\Omega_d},$$

$$a_h(u_h, v_h) = a_{s,h}(u_{s,h}, v_{s,h}) + a_{i,h}(u_h, v_h) + a_{d}(u_{d,h}, v_{d,h}),$$

$$b_h(v_h, q_h) = b_{s,h}(v_{s,h}, q_{s,h}) + b_{d,h}(v_{d,h}, q_{d,h}).$$
With these preparations, we give the numerical scheme as follows.

\textbf{WG-MFEM Scheme 1.} Seek \(u_h \in V_h, \ p_h \in M_h\) such that

\begin{align}
(3.4) & \quad a_h(u_h, v_h) + b_h(v_h, p_h) = (f_s, v_h)_{\Omega_s}, \\
(3.5) & \quad b_h(u_h, q_h) = (f_d, q_h)_{\Omega_d},
\end{align}

for all \(v_h = (v_{s,h}, v_{d,h}) \in V_h, \) and \(q_h \in M_h.\)

\section{4. Existence and Uniqueness.}

In this section, we prove two important properties of the numerical scheme: the boundedness of \(a_h(\cdot, \cdot)\) and the inf-sup condition of \(b_h(\cdot, \cdot)\). The existence and uniqueness of the approximate solutions then follow from the two properties.

We first define a discrete norm on \(V_h^s\) by

\[
\|v_h\|_{V_h^s}^2 = 2\nu\|D_w(v_{s,h})\|^2_{\Omega_s} + \sum_{T_s \in T_s} h_s^{-1}\|Q_b v_{s,0} - v_{s,b}\|^2_{\partial T_s} + \|\mu_{\Omega}^{-1/2}v_{s,b} \cdot \tau\|_{\Gamma}^2.
\]

It is obvious that \(\|\cdot\|_{V_h^s}\) is a semi-norm. In order to demonstrate \(\|\cdot\|_{V_h^s}\) is a well-defined norm on \(V_h^s\), we introduce the following estimate.

\textbf{Lemma 4.1.} For any \(v_{s,h} \in V_h^s\), we have

\[
\sum_{T_s \in T_s} \|\nabla v_{s,0}\|_{T_s} \leq C \|v_{s,h}\|_{V_h^s}, \quad \forall \ T_s \in T_s,h.
\]

\textbf{Proof.} From [9], we know the following discrete Korn's inequality holds.

\[
\sum_{T_s \in T_s,h} \|\nabla v_{s,0}\|_{T_s}^2 \leq C \left( \sum_{T_s \in T_s,h} \|D(v_{s,0})\|_{T_s}^2 + \sup_{m \in RM, \|m\|_{\Gamma} = 1} \left( \int_{T_s} v_{s,0} \cdot m ds \right)^2 + \sum_{e_s \in E_s \setminus T_s} \|\sigma_{e_s}[v_{s,0}]\|_{e_s}^2 \right),
\]

where \(RM\) is the space of rigid motions, \(\sigma_{e_s}\) is the \(L^2\) projection operator onto \([P_1(e_s)]^d, [\cdot]\) denotes the jump on edges. Each term on the left hand of the inequality can be handled as follows.

Using the integration by parts and the definition of \(\nabla w\) on each element \(T_s \in T_s,h\), we have that

\[
(D(v_{s,0}), D(v_{s,0}))_{T_s} = (-v_{s,0}, \nabla \cdot D(v_{s,0}))_{T_s} + \langle v_{s,0}, D(v_{s,0}) \cdot n \rangle_{\partial T_s}
\]

\[
= (-v_{s,0}, \nabla \cdot D(v_{s,0}))_{T_s} + \langle v_{s,b}, D(v_{s,0}) \cdot n \rangle_{\partial T_s}
\]

\[
+ (v_{s,0} - v_{s,b}, D(v_{s,0}) \cdot n)_{\partial T_s}
\]

\[
= (\nabla w, v_{s,h})_{T_s} + (Q_b v_{s,0} - v_{s,b}, D(v_{s,0}) \cdot n)_{\partial T_s}
\]

\[
= (D_w v_{s,h}, D(v_{s,0}))_{T_s} + (Q_b v_{s,0} - v_{s,b}, D(v_{s,0}) \cdot n)_{\partial T_s}.
\]
Summing over all element $T_s \in \mathcal{T}_s,h$ and applying the trace inequality (4.9), the inverse inequality (4.10), we obtain
\[
\|D(v_{s,0})\|^2_{T_s} \leq C(\|D_w v_{s,h}\|_{T_s} \|D(v_{s,0})\|_{T_s} + \|Q_b v_{s,0} - v_{s,h}\|_{\partial T_s} \|D(v_{s,0})\|_{\partial T_s})
\]
\[
\leq C(\|D_w v_{s,h}\|_{T_s} + h_s^{-1} \|Q_b v_{s,0} - v_{s,h}\|_{\partial T_s}) \|D(v_{s,0})\|_{T_s}.
\]
Therefore,
\[
\sum_{T_s \in \mathcal{T}_s,h} \|D(v_{s,0})\|_{T_s} \leq C\|v_{s,h}\|_{V^*_h}.
\]

For the second and the third terms, since $\beta \geq 1$ and $v_{s,b} = 0$ on $\Gamma_s$, we have
\[
\sup_{m \in RM, \|m\|_{\mathcal{G}_s} = 1} \left( \int_{\Gamma_s} v_{s,0} \cdot m ds \right) = \sup_{m \in RM, \|m\|_{\mathcal{G}_s} = 1} \left( \int_{\Gamma_s} (Q_b v_{s,0} - v_{s,b}) \cdot m ds \right)
\]
\[
\leq C\|v_{s,h}\|_{V^*_h}.
\]
and
\[
\sum_{e_s \in E^0_s} \|\pi e[v_{s,0}]\|_{e_s} \leq \sum_{e_s \in E^0_s} \|Q_b[v_{s,0}]\|_{e_s} \leq \sum_{T_s \in \mathcal{T}_s,h} \|Q_b v_{s,0} - v_{s,h}\|_{\partial T_s} \leq C\|v_{s,h}\|_{V^*_h}.
\]
The proof is completed. \(\square\)

**Lemma 4.2.** $\cdot \|V^*_h$ provides a norm in $V^*_h$.

**Proof.** It suffices to check the positivity property of the semi-norm $\cdot \|V^*_h$. To this end, assume that $\|v_{s,h}\|_{V^*_h} = 0$ for some $v_{s,h} \in V^*_h$. Then we obtain $D_w(v_{s,h}) = 0$ on all $T_s \in \mathcal{T}_s,h$. $Q_b v_{s,0} = v_{s,h}$ on $\partial T_s$. $v_{s,h} \cdot \tau = 0$ on $\Gamma$. From the Lemma 4.1, we have $\nabla v_{s,0} = 0$ on all $T_s$, which implies that $v_{s,0}$ is constant on $\Omega_s$. Moreover, $Q_b v_{s,0} = v_{s,b}$ yields $v_{s,h}$ is a constant in $\Omega_s$. Combining with the fact that $v_{s,b} = 0$ on $\Gamma_s$, we know that $v_{s,h} = 0$. \(\square\)

Now, we can define a discrete norm on $V_h$.
\[
\|v_h\|^2_{V^*_h} = \|v_{s,h}\|^2_{V^*_h} + \|v_{d,h}\|^2_{\Omega_d} + \|\nabla \cdot v_{d,h}\|^2_{\Omega_d}.
\]

It follows from the definition of norm (4.1) and the Cauchy Schwarz inequality that coercivity and boundedness hold true for the bilinear form $a_h(\cdot, \cdot)$.

**Lemma 4.3.** For any $u_h, v_h \in V_h$, we have
\[
a_h(v_h, v_h) = \|v_h\|^2_{V^*_h}, \quad \forall v_h \in V_h, \quad \nabla \cdot v_{d,h} = 0,
\]
\[
a_h(u_h, v_h) \leq C\|u_h\|_{V^*_h} \|v_h\|_{V^*_h}, \quad \forall u_h, v_h \in V_h.
\]

Besides the projection $Q_h = \{Q_0, Q_b\}$ defined in the previous section, we need another local $L^2$ projections, for each element $T_s \in \mathcal{T}_s,h$, denote by $Q_h$ the $L^2$ projection onto $[P_0(T_s)]^{2 \times 2}$ and by $Q_{h_b}$ the $L^2$ projection onto $P_0(T_s)$.

**Lemma 4.4.** The projection operators defined above satisfy
\[
\nabla w(\nabla v) = Q_h(\nabla v) \quad \forall v \in [H^1(\Omega_s)]^d,
\]
\[
\nabla w \cdot (\nabla v) = Q_{h_b}(\nabla v) \quad \forall v \in H(div, \Omega_s).
\]
The proof of this Lemma can be found in [43].

As for Darcy region, denote the velocity space \( V|_{\Omega_d} \) by \( V^d \). Then we define the MFEM interpolant \( \Pi^d_h : V^d \cap [H^\theta(\Omega_d)]^2 \to V^d_h \) with \( \theta > 0 \) satisfying [9], for any \( v_d \in V^d \cap (H^\theta(\Omega_d))^2 \),

\[
(\nabla \cdot \Pi^d_h v_d - v_d, q_{d,h}) = 0, \quad \forall q_{d,h} \in M^d_h, \tag{4.6}
\]

\[
\int_e ((\Pi^d_h v_d - v_d) \cdot n_e) w_{d,h} \cdot n_e \, ds = 0, \quad \forall e \in \Gamma^d_h \cup \Gamma_h, \quad \forall w_{d,h} \in V^d_h. \tag{4.7}
\]

In addition, we denote by \( R_s^h \) the \( L^2 \) projection onto \( M^s_h \), and by \( R_d^h \) the \( L^2 \) projection onto \( M^d_h \).

Next, we introduce the discrete inf-sup condition for the bilinear form \( b_h(\cdot, \cdot) \).

**Lemma 4.5.** (inf-sup) There exists a positive constant \( C \) independent of \( h \) such that

\[
\sup_{v_h \in V_h} \frac{b_h(v_h, q_h)}{\|v_h\|_{V_h}} \geq C \|q_h\|_{M_h}
\]

for all \( q_h \in M_h \).

**Proof.** According to [4], we know that for any \( q_h \in M_h \), there exists a \( v \in [H^1_0(\Omega)]^2 \) such that

\[
\nabla \cdot v = -q_h \quad \text{in} \ \Omega,
\]

and \( \|v\|_{1,\Omega} \leq C \|q_h\|_{0,\Omega} \).

Note that

\[
b_{s,h}(Q_h v, q_h) = -(\nabla_w \cdot Q_h v, q_h)_{\Omega_s} = -(Q_h(\nabla \cdot v), q_h)_{\Omega_s}
\]

\[
= -(\nabla \cdot v, q_h)_{\Omega_s} = \|q_h\|_{\Omega_s}^2,
\]

and

\[
b_{d,h}(v, q_h) = -(\nabla \cdot v, q_h)_{\Omega_d} = \|q_h\|_{\Omega_d}^2.
\]

Next, we construct an projection operator \( \pi_h : (V \cap [H^1(\Omega)]^2) \to V_h \) such that

\[
b_{s,h}(\pi_h v - Q_h v, q_h) = 0, \quad b_{d,h}(\pi_h v - v, q_h) = 0, \quad \forall q_h \in M_h.
\]

Let \( \pi_h v = (\pi^s_h v, \pi^d_h v) \in V^s_h \times V^d_h \). First, we take \( \pi^s_h v = Q_h v \). It is obvious that \( b_{s,h}(\pi_h v - Q_h v, q_h) = 0 \). In addition, the following estimate holds.

\[
\|Q_h v\|_{V^d_h} \leq C \|v\|_{1,\Omega_s}
\]

Readers may refer to [13] for the proof for this estimate. Next, we need to define the operator \( \pi^d_h v \). Consider the following auxiliary problem

\[
\nabla \cdot \nabla \phi = 0 \quad \text{in} \ \Omega_d,
\]

\[
\nabla \phi \cdot n = 0 \quad \text{on} \ \Gamma_d,
\]

\[
\nabla \phi \cdot n = (\pi^d_h v - v) \cdot n \quad \text{on} \ \Gamma.
\]
It follows from the definition of the projection operator $Q_h$ that
\[
\int_{\Gamma} (\pi_h v - v) \cdot n \, ds = \int_{\Gamma} (Q_h v - v) \cdot n \, ds = 0.
\]

So the auxiliary problem is well-posed. Let $z = \nabla \phi$, we notice that the function $\pi_h v \cdot n \in H^\theta(\Gamma)$ for any $0 \leq \theta \leq \frac{1}{2}$. By elliptic regularity \[25\],
\[
\|z\|_{\theta, \Omega_d} \leq C \|\pi_h v - v\|_{\theta - \frac{1}{2}, \Gamma}, \quad 0 \leq \theta \leq \frac{1}{2}.
\]

Let $w = v + z$. Then we have
\[
\nabla \cdot w = \nabla \cdot (v + z) = \nabla \cdot v \quad \text{in } \Omega_d,
\]
\[
\pi_h v \cdot n = v \cdot n + z \cdot n = \pi_h v \cdot n \quad \text{on } \Gamma.
\]

Define $\pi_h v := \Pi_h^d w$. From the definition of $\Pi_h^d$, we know that
\[
b_{d,h}(\pi_h v, q_{d,h}) = b_{d,h}(\Pi_h^d w, q_{d,h}) = b_{d,h}(w, q_{d,h})
= - (\nabla \cdot w, q_{d,h}) = -(\nabla \cdot v, q_{d,h}) = b_{d,h}(v, q_{d,h}), \quad \forall q_{d,h} \in M^d_h.
\]

So the interpolant operator $\pi_h v$ satisfies $b_{d,h}(\pi_h v - v, q_{d,h}) = 0$.

Next, we prove that $\pi_h v \in V_h$. For any $e \in \Gamma_h$ and $\eta \in \Lambda_h$, using (4.7), (4.9) and (4.10), we have
\[
\int_e \pi_h v \cdot n \eta \, ds = \int_e \Pi_h^d w \cdot n \eta \, ds
= \int_e w \cdot n \eta \, ds = \int_e \pi_h v \cdot n \eta \, ds.
\]

It remains to give the bound of the operator $\pi_h^d$. From Lemma (A.2) and (4.8),
\[
\|\pi_h^d v\|_{V_h} = \|\Pi_h^d w\|_{V_h}
\leq \|\Pi_h^d v\|_{V_h} + \|\Pi_h^d z\|_{V_h}
\leq C(\|v\|_{1,\Omega_d} + \|z\|_{\theta, \Omega_d})
\leq C(\|v\|_{1,\Omega_d} + \|\pi_h v - v\|_{\theta - \frac{1}{2}, \Gamma}).
\]

Using the trace inequality (A.9) and the projection inequality (A.2), we have
\[
\|(\pi_h^d v - v) \cdot n\|_e \leq \|Q_0 v - v\|_e
\leq Ch^{- \frac{1}{2}} \|Q_0 v - v\|_{T_*} + Ch^{\frac{1}{2}} \|\nabla(Q_0 v - v)\|_{T_*}
\leq Ch^{\frac{1}{2}} \|v\|_{1, T_*}.
\]

Thus, we obtain $\|\pi_h^d v\|_{V_h} \leq C\|v\|_{1, \Omega}$. Furthermore,
\[
\|\pi_h v\|_{V_h} \leq C\|v\|_{1, \Omega}.
\]
Combining with the above estimates, we get
\[
\frac{b_h(\pi_h v, q_h)}{\|\pi_h v\|_{V_h}} = \frac{b_{s,h}(Q_h v, q_h) + b_{d,h}(\Pi_h^d v, q_h)}{\|\pi_h v\|_{V_h}} \\
\geq C \frac{b_{s,h}(Q_h v, q_h) + b_{d,h}(\Pi_h^d v, q_h)}{\|\pi_h v\|_{1,\Omega}} \\
\geq C \frac{(\nabla \cdot v, q_h) + b_{d,h}(\Pi_h^d v, q_h)}{\|\pi_h v\|_{1,\Omega}} \\
\geq C \frac{\|q_h\|_1^2}{\|v\|_{1,\Omega}} \\
\geq C \|q_h\|_{\Omega},
\]
which completes the proof. \(\square\)

**Lemma 4.6.** For \( v \in [H^1(\Omega)]^2 \), such that \( v|_{\Omega_d} \in [H^\gamma_d(\Omega_d)]^2 \), there exists \( \tilde{v}_h \in V_h \) such that

\[
(4.11) \quad b_{d,h}(v - \tilde{v}_h, q_{d,h}) = 0, \quad \forall \, q_{d,h} \in M_h,
\]
\[
(4.12) \quad \|v - \tilde{v}_h\|_{V_h} \leq C (h_d^{\alpha_d+1} \|v|_{\alpha_d+1,\Omega_d} + h_d^{\gamma_d+1} \|\nabla \cdot v|_{\gamma_d+1,\Omega_d} + h_d^{\alpha_d+1/2} \|v\|_{\alpha_d+1,\Omega_d}).
\]

**Proof.** Recall the interpolant \( \pi_h^d v \) constructed in Lemma 4.5, then (4.11) can be deduced directly. We only need to prove (4.12). From the definition of \( \pi_h^d v \), we know

\[
(4.13) \quad \|v - \pi_h^d v\|_{V_h} = \|v - \Pi_h^d w\|_{V_h} \leq \|v - \Pi_h^d w\|_{V_h} + \|\Pi_h^d (w - v)\|_{V_h}.
\]

Using Lemma (4.2), the first term on the right-hand side of (4.13) can be estimated as follows

\[
\|v - \Pi_h^d w\|_{V_h} \leq C (h_d^{\alpha_d+1} \|v|_{\alpha_d+1,\Omega_d} + h_d^{\gamma_d+1} \|\nabla \cdot v|_{\gamma_d+1,\Omega_d}).
\]

For the second term, using estimate (4.8) and (4.1),

\[
\|\Pi_h^d (w - v)\|_{V_h} = \|\Pi_h^d (w - v)\|_{V_h} \leq \|w - v\|_{0,\Omega_d} \\
\leq \|((\pi_h^d v - v) \cdot \mathbf{n})_{\partial,\Omega} \leq C h_d^{\alpha_d+1/2} \|v\|_{\alpha_d+1,\Omega_d}.
\]

Combining the estimates above we complete the proof. \(\square\)

**Lemma 4.7.** The numerical scheme (3.4) – (3.5) has a unique solution.

**Proof.** Since the problem is finite dimensional, it suffices to show that the solution is unique. Set \( f_s = 0, \ f_d = 0 \). Then take \( v_h = u_h \) and \( q_h = p_h \), we have

\[
a_h(u_h, u_h) = 0,
\]
and

\[
b_h(u_h, q_h) = 0 \quad \forall \, q_h \in M_h.
\]

Combining with the results above, we know that \( a_h(u_h, u_h) = 0 \), which implies that \( u_h = 0 \). Furthermore, we derive that

\[
b(v_h, p_h) = 0 \quad \forall \, v_h \in V_h.
\]

From the inf-sup condition we know \( p_h = 0 \). \(\square\)
5. Error Estimates. In this section, we derive the optimal error estimates for the velocity in the energy norm and the pressure in the $L^2$ norm.

**Lemma 5.1.** For any $w_s \in [H^1(\Omega_s)]^2$, $\rho_s \in H^1(\Omega_s)$, and $v_{s,h} \in V_h$, it follows that

\begin{align}
(D_w(Q_h w_s), D_w(v_{s,h}))_{\Omega_s} &= (D(w_s), D(v_{s,0}))_{\Omega_s} - \sum_{T_s \in T_s,h} \langle v_{s,0} - v_{s,b}, Q_h D(w_s) \cdot n \rangle_{\partial T_s}, \\
\sum_{T_s \in T_s,h} (\nabla_v \cdot v_{s,h}, R_h \rho_s)_{\Omega_s} &= \sum_{T_s \in T_s,h} \langle \nabla_v v_{s,0}, (R_h \rho_s) \cdot n \rangle_{\partial T_s}.
\end{align}

**Proof.** According to the commutative property \[4.3\], we know that $D_w(Q_h u_s) = Q_h D(u_s)$ is symmetric. Thus,

$$
(D_w(Q_h w_s), D_w(v_{s,h}))_{T_s} = (Q_h D(w_s), D_w(v_{s,h}))_{T_s} = (Q_h D(w_s), \nabla_w v_{s,h})_{T_s}.
$$

It follows from the definition of weak gradient \[3.2\] and the integration by parts, we have

\begin{align*}
\sum_{T_s \in T_s,h} (D_w(Q_h w_s), \nabla_w v_{s,h})_{T_s} &= \sum_{T_s \in T_s,h} (-\langle \nabla \cdot (Q_h D(w_s)), v_{s,0} \rangle_{T_s} + \langle v_{s,b}, Q_h D(w_s) \cdot n \rangle_{\partial T_s}) \\
&= \sum_{T_s \in T_s,h} \langle Q_h D(w_s), \nabla v_{s,0} \rangle_{T_s} - \langle v_{s,0} - v_{s,b}, Q_h D(w_s) \cdot n \rangle_{\partial T_s} \\
&= \sum_{T_s \in T_s,h} \langle Q_h D(w_s), D(v_{s,0}) \rangle_{T_s} - \langle v_{s,0} - v_{s,b}, Q_h D(w_s) \cdot n \rangle_{\partial T_s}.
\end{align*}

The proof of \[5.2\] is similar, so we omit details here. \(\square\)

With the above lemma, we can establish the error equations.

**Lemma 5.2.** Let $(u, p)$ be the solutions of \[2.1\] - \[2.9\], and $(u_h, p_h)$ be the solutions of \[3.4\] - \[3.5\], we have

\begin{align}
&\forall s,h, (Q_h u_s - u_{s,h}, v_{s,h}) + a_{s,h}(Q_h u_s - u_{s,h}, v_{s,h}) + b_{s,h}(v_{s,h}, R_h^s p_s - p_{s,h}) \\
&= l_1(u_s, v_{s,h}) - l_2(p_s, v_{s,h}) - l_3(u_s, v_{s,h}) - \langle p_d, v_{s,b} \cdot n \rangle_{\Gamma_h} + s(Q_h u_s, v_{s,h}), \\
&\forall s,h, b(Q_h u_s - u_{s,h}, q_{s,h}) = 0, \\
&\forall s,h, b(u_d - u_{d,h}, q_{d,h}) = 0
\end{align}

for any $v \in V_h$ and $q_h \in M_h$, where

\begin{align*}
l_1(u_s, v_{s,h}) &= \sum_{T_s \in T_s,h} \langle 2\nu(v_{s,0} - v_{s,b}), D(u_s) \cdot n - (Q_h D(u_s)) \cdot n \rangle_{\partial T_s} \\
l_2(p_s, v_{s,h}) &= \sum_{T_s \in T_s,h} \langle v_{s,0} - v_{s,b}, (p_s - R_h^s p_s) n \rangle_{\partial T_s} \\
l_3(u_s, v_{s,h}) &= \sum_{e \in \Gamma_h} \langle \mu K^{-\frac{1}{2}}(u_s - Q_h u_s) \cdot \tau, v_{s,b} \cdot \tau \rangle_e.
\end{align*}
Proof. Multiplying the Stokes equation (2.1) with \( v_{s,0} \) in \( v_{s,h} = \{ v_{s,0}, v_{s,b} \} \in V_h^s \) and integrating by parts over every element \( T_s \),

\[
(f_s, v_{s,0})_{\Omega_s} = \sum_{T_s \in T_{s,h}} (2\nu D(u_s), \nabla v_{s,0})_{T_s} - \sum_{T_s \in T_{s,h}} (\nabla \cdot v_{s,0}, p_s)_{T_s}
- \sum_{T_s \in T_{s,h}} (2\nu (v_{s,0} - v_{s,b}), D(u_s) \cdot n)_{\partial T_s} + \sum_{T_s \in T_{s,h}} \langle v_{s,0} - v_{s,b}, p_s n \rangle_{\partial T_s}
- \sum_{e \in \Gamma_h} \langle v_{s,b}, \Gamma(u_s, p_s) n \rangle_e.
\]

By the regularity of the true solution \( u_s \) and \( p_s \), and the fact that \( v_{s,b} = 0 \) on \( \Gamma_h^s \),

\[
(f_s, v_{s,0})_{\Omega_s} = \sum_{T_s \in T_{s,h}} (2\nu D(u_s), D(v_{s,0}))_{T_s} - \sum_{T_s \in T_{s,h}} (\nabla \cdot v_{s,0}, p_s)_{T_s}
- \sum_{T_s \in T_{s,h}} (2\nu (v_{s,0} - v_{s,b}), D(u_s) \cdot n)_{\partial T_s} + \sum_{T_s \in T_{s,h}} \langle v_{s,0} - v_{s,b}, p_s n \rangle_{\partial T_s}
- \sum_{e \in \Gamma_h} \langle v_{s,b}, \Gamma(u_s, p_s) n \rangle_e.
\]

From the interface conditions, we know that

\[- \sum_{e \in \Gamma_h} \langle v_{s,b}, \Gamma(u_s, p_s) n \rangle_e = \langle p_d, v_{s,b} \cdot n \rangle_{\Gamma_h^s} + \langle \mu K^{-\frac{1}{2}} u_s \cdot \tau, v_{s,b} \cdot \tau \rangle_{\Gamma_h^s}.
\]

Applying Lemma (5.1) yields

\[
(f_s, v_{s,0})_{\Omega_s} = \sum_{T_s \in T_{s,h}} (2\nu D(w_s), D(v_{s,0}))_{T_s} - \sum_{T_s \in T_{s,h}} (\nabla w \cdot v_{s,h}, R_h^p p_s)_{T_s}
- \sum_{T_s \in T_{s,h}} (2\nu (v_{s,0} - v_{s,b}), D(u_s) \cdot n)_{\partial T_s}
+ \sum_{T_s \in T_{s,h}} \langle v_{s,0} - v_{s,b}, (p_s - R_h^p p_s) n \rangle_{\partial T_s} + \sum_{e \in \Gamma_h} \langle p_d, v_{s,b} \cdot n \rangle_e
+ \sum_{e \in \Gamma_h} \langle \mu K^{-\frac{1}{2}} u_s \cdot \tau, v_{s,b} \cdot \tau \rangle_e
= a_{s,h}(Q_h u_s, v_{s,h}) + a_{i,h}(Q_h u_s, v_{s,h}) + b_{s,h}(v_{s,h}, R_h^p p_s) - s(Q_h u_s, v_{s,h})
- \sum_{T_s \in T_{s,h}} (2\nu (v_{s,0} - v_{s,b}), D(u_s) \cdot n)_{\partial T_s}
+ \sum_{T_s \in T_{s,h}} \langle v_{s,0} - v_{s,b}, (p_s - R_h^p p_s) n \rangle_{\partial T_s} + \sum_{e \in \Gamma_h} \langle p_d, v_{s,b} \cdot n \rangle_e
+ \sum_{e \in \Gamma_h} \langle \mu K^{-\frac{1}{2}} (u_s - Q_h u_s) \cdot \tau, v_{s,b} \cdot \tau \rangle_e.
\]
Therefore, we have
\[
\begin{align*}
&\alpha_{s,h}(Q_h u_s, v_s, h) + \beta_{s,h}(Q_h u_s, v_s, h) + \gamma_{s,h}(v_s, h, R_h^p) \\
&= (f_s, v_s)_{\Omega_s} + s(Q_h u_s, v_s, h) \\
&\quad + \sum_{T \in \mathcal{T}_h} (2\nu(v_{s,0} - v_s, h) \cdot D(u_s) - (Q_h D(u_s)) \cdot n)_{\partial T_s} \\
&\quad - \sum_{T \in \mathcal{T}_h} (v_{s,0} - v_{s,b}, (p_s - R_h^p) n)_{\partial T_s} - \sum_{e \in \mathcal{E}_h} (p_d, v_{s,b} \cdot n)_{e} \\
&\quad - \sum_{e \in \mathcal{E}_h} (\mu K^{-1/2}(u_s - Q_h u_s) \cdot \tau, v_{s,b} \cdot \tau)_{e}.
\end{align*}
\]

Using the definition of $Q_h$ and $k_h$, we have
\[
b_{s,h}(Q_h u_s, q_h) = -(\nabla_w \cdot (Q_h u_s), q_h) = -(Q_h (\nabla \cdot u_s), q_h) = (\nabla \cdot u_s, q_h) = 0.
\]

As for the Darcy’s law \((2.5)\), multiplying a test function $v_{d,h} \in V_h^d$ and using integration by parts on the Darcy region yields
\[
0 = (K^{-1} u_d, v_{d,h}) + (\nabla p_d, v_{d,h}) \\
= (K^{-1} u_d, v_{d,h}) - (p_d, \nabla \cdot v_{d,h}) - (p_d, v_{d,h} \cdot n)_{\Gamma} \\
= a_{d,h}(u_d, v_{d,h}) + b_{d,h}(v_{d,h}, p_d) - (p_d, v_{d,h} \cdot n)_{\Gamma},
\]

which means that
\[
a_{d}(u_d, v_{d,h}) + b_{d,h}(v_{d,h}, p_d) = (p_d, v_{d,h} \cdot n)_{\Gamma}.
\]

It is obvious that
\[
b_{d,h}(u_d, q_h) = (f_d, q_h).
\]

Combining with \((3.4) - (3.5)\), we obtain equations \((5.3) - (5.6)\).

**Theorem 5.3.** Let $(u, p)$ be the solutions of the coupled problem \((2.1) - (2.9)\). Assume that $u_{i,h} \in [H^{\alpha_i+1}(\Omega)]^2$, $p_{i,h} \in H^{\gamma_i+1}(\Omega)$, $i = s, d$. Let $(u_i, p_i)$ be the discrete solutions of \((3.4) - (3.5)\). Then the following estimate holds.

\[
\|Q_h u_s - u_{s,h}\|_{V_h^d} + \|u_d - u_{d,h}\|_{V_h^d} \\
\leq C(h_s^{\beta+1}||u_s||_{\beta+2, \Omega} + h_s^{\gamma+1}||p_s||_{\gamma+1, \Omega}) + h_s^{\alpha_s}||u_s||_{\beta+1, \Omega} + h_s^{\gamma_s}||u_s||_{\beta+1, \Omega} \\
+ C(h_s^{\alpha_d}||u_d||_{\beta+1, \Omega} + h_s^{\gamma+1}||u_d||_{\beta+2, \Omega} + h_s^{\gamma_d+1/2}||p_d||_{\beta+1, \Omega}).
\]

**Proof.** Adding equation \((5.4)\) to \((5.3)\), we have
\[
\begin{align*}
&\alpha_{s,h}(Q_h u_s - u_{s,h}, v_s, h) + \beta_{s,h}(Q_h u_s - u_{s,h}, v_s, h) + \gamma_{s,h}(v_s, h, R_h^p) - (p_s - \tilde{p}_d, v_{s,h}) \\
&\quad + \alpha_{d,h}(Q_h u_s - u_{s,h}, v_d, h) + \beta_{d,h}(Q_h u_s - u_{s,h}, v_d, h) + \gamma_{d,h}(v_d, h, R_h^p) - (p_d, v_{d,h} \cdot n)_{\Gamma} \\
&= l_1(u_s, v_s, h) - l_2(p_s, v_{s,h}) - l_3(u_s, v_{s,h}) + s(Q_h u_s, v_s, h) \\
&\quad + a_{d}(u_d - u_{d,h}, v_d, h) + b_{d}(v_{d,h}, R_h^p - p_d) - (p_d, v_{d,h} \cdot n)_{\Gamma}.
\end{align*}
\]

From the Lemma \((4.6)\) and equation \((5.6)\), we get
\[
b_{d}(u_d, q_h) = 0, \quad \forall q_h \in M_h^d.
\]
Since $\nabla \cdot V_h^d \subset M_h^d$,
\[
\nabla \cdot (u_{d,h} - \tilde{u}_d) = 0, \text{ in } \Omega_d.
\]

Define $e_{s,h} = Q_h u_s - u_{s,h}, e_{d,h} = u_d - u_{d,h}, \epsilon_{s,h} = R_h^d p_s - p_{s,h}$ and $\epsilon_{d,h} = R_h^d p_d - p_{d,h}$. Taking $v_{s,h} = e_{s,h}, v_{d,h} = e_{d,h}, q_{s,h} = \epsilon_{s,h}$ and $q_{d,h} = \epsilon_{d,h}$ in (5.8) and combining with (5.5), we have
\[
\begin{align*}
&\quad a_{s,h}(e_{s,h}, e_{s,h}) + a_{i,h}(e_{s,h}, e_{s,h}) + a_d(e_{d,h}, e_{d,h}) \\
&= l_1(u_s, e_{s,h}) - l_2(p_s, e_{s,h}) - l_3(u_s, e_{s,h}) + s(Q_h u_s, e_{s,h}) \\
&\quad + a_d(\tilde{u}_d - u_d, e_{d,h}) - \langle p_d, (v_{s,h} - v_{d,h}) \cdot n \rangle_{\Gamma_h}.
\end{align*}
\]

We define $e_h = (e_{s,h}, e_{d,h})$. Making use of coercivity (4.2) and noting that $\nabla \cdot e_{d,h} = 0$ in $\Omega_d$, we obtain
\[
\|e_h\|^2_{V_h} = a_{s,h}(e_{s,h}, e_{s,h}) + a_{i,h}(e_{s,h}, e_{s,h}) + a_d(e_{d,h}, e_{d,h}) \\
= l_1(u_s, e_{s,h}) - l_2(p_s, e_{s,h}) - l_3(u_s, e_{s,h}) + s(Q_h u_s, e_{s,h}) \\
+ a_d(\tilde{u}_d - u_d, e_{d,h}) - \langle p_d, (e_{s,h} - e_{d,h}) \cdot n \rangle_{\Gamma_h}.
\]

Next, we are going to estimate each term on the right-hand side of the above equation one by one. It follows from (4.14 - 4.17) that
\[
\begin{align*}
l_1(u_s, e_{s,h}) - l_2(p_s, e_{s,h}) - l_3(u_s, e_{s,h}) + s(Q_h u_s, e_{s,h}) \\
\leq C(h_{d+1}^2 \|u_s\|_{\beta+2, \Omega_s} + H_{d+2}^2 \|p_s\|_{\gamma+1, \Omega_s} + H_{d+1}^2 \|u_s\|_{\beta+1, \Omega_s} + H_{d+1}^2 \|u_s\|_{\alpha+1, \Omega_s}) \|e_{s,h}\|_{V_h}.
\end{align*}
\]

Using the Cauchy-Schwarz inequality and (4.12), we have
\[
\begin{align*}
&\quad a_d(\tilde{u}_d - u_d, e_{d,h}) \\
&\leq C\|\tilde{u}_d - u_d\|_{V_h} \cdot \|e_{d,h}\|_{V_h} \\
&\leq C(h_d^2 \|u_d\|_{\alpha_d+1, \Omega_d} + H_{d+1}^{\alpha_d+1} \|u_d\|_{\gamma d+2} + H_{d+1}^{\alpha_d+1} \|u_s\|_{\alpha+1, \Omega_s}) \|e_{d,h}\|_{V_h}.
\end{align*}
\]

Finally, to estimate $\langle p_d, (e_{s,h} - e_{d,h}) \cdot n \rangle_{\Gamma_h}$, we define a $L^2$ projection $R_h^c$ onto $\Lambda_h$ as follows.
\[
\langle p_d, \lambda_h \rangle_{\Gamma_h} = \langle R_h^c p_d, \lambda_h \rangle_{\Gamma_h}, \quad \forall \lambda_h \in \Lambda_h.
\]

Since $e_h = (e_{s,h}, e_{d,h}) \in V_h$, from the definition of $V_h$, we know that
\[
\sum_{e \in \Gamma_h} \int_{\partial e} \eta (e_{s,h} - e_{d,h}) \cdot n = 0, \quad \forall \eta \in \Lambda_h.
\]

Combining with the fact that $R_h^c p_d \in \Lambda_h$,
\[
\sum_{e \in \Gamma_h} \langle p_d, (e_{s,h} - e_{d,h}) \cdot n \rangle_e = \sum_{e \in \Gamma_h} \langle p_d - R_h^c p_d, (e_{s,h} - e_{d,h}) \cdot n \rangle_e
\]

Noting that $e_{d,h} \cdot n \in \Lambda_h$, so we have
\[
\sum_{e \in \Gamma_h} \langle p_d, (e_{s,h} - e_{d,h}) \cdot n \rangle_e = \sum_{e \in \Gamma_h} \langle p_d - R_h^c p_d, (e_{s,h}) \cdot n \rangle_e
\]
For any constant vector $c_e$, using the property of $R^e_h$, the trace inequality (4.9) and Lemma (4.1), we obtain
\[
\sum_{e \in T_h} \langle p_d - R^e_h p_d, (e_{s,h} \cdot n) \rangle_e \\
= \sum_{e \in T_h} \langle p_d - R^e_h p_d, (e_{s,h} - c_e) \cdot n \rangle_e \\
\leq \sum_{e \in T_h} \|p_d - R^f_h p_d\|_e \|e_{s,h} - c_e\|_e \\
\leq \sum_{e \in T_h} \|p_d - R^f_h p_d\|_e \|e_{s,h} - c_e\|_e \\
\leq Ch_d^{\gamma+1/2} \|p_d\|_{\gamma+1,\Omega_d} \sum_{T_e \in T_h} (\|e_{s,h} - Q_h e_{s,0}\|_{\partial T_e} + \|Q_h e_{s,0} - c_e\|_{\partial T_e}) \\
\leq Ch_d^{\gamma+1/2} \|p_d\|_{\gamma+1,\Omega_d} \left( h_s^{1/2} \|e_{s,h}\|_{V_h^\gamma} + \sum_{T_e \in T_h} C(h_s^{-1/2} \|e_{s,0} - c_e\|_{T_e} + h_s^{1/2} \|\nabla e_{s,0}\|_{T_e}) \right) \\
\leq Ch_d^{\gamma+1/2} \|p_d\|_{\gamma+1,\Omega_d} h_s^{1/2} \|e_{s,h}\|_{V_h^\gamma}.
\]

Combining the above estimates, we obtain
\[
\|e_{s,h}\|_{V_h^\gamma} + \|u_d - u_{d,h}\|_{V_h^\gamma} \\
= \|e_{s,h}\|_{V_h^\gamma} + \|u_{d,h}\|_{V_h^\gamma} + \|u_d - u_{d,h}\|_{V_h^\gamma} \\
\leq C(h_s^{\gamma+1} \|u_s\|_{\beta+2,\Omega_s} + h_s^{\gamma+1} \|p_s\|_{\gamma+1,\Omega_s} + h_s^{\gamma+1} \|u_s\|_{\beta+1,\Gamma} + h_s^{\gamma+1} \|u_s\|_{\alpha_s+1,\Omega_s}) \\
+ C(h_s^{\gamma+1} \|u_d\|_{\alpha_s+1,\Omega_d} + h_s^{\gamma+1} \|u_d\|_{\gamma+2} + h_s^{\gamma+1} \|p_d\|_{\gamma+1,\Omega_d}).
\]

which completes the proof of the theorem. \(\Box\)

**Theorem 5.4.** Under the assumption of Theorem (5.3), we have
\[
(5.9) \quad \|R^e_h p_s - p_{s,h}\|_{\Omega_s} + \|p_d - p_{d,h}\|_{\Omega_d} \\
\leq C(h_s^{\gamma+1} \|u_s\|_{\beta+2,\Omega_s} + h_s^{\gamma+1} \|p_s\|_{\gamma+1,\Omega_s} + h_s^{\gamma+1} \|u_s\|_{\beta+1,\Gamma} + h_s^{\gamma+1} \|u_s\|_{\alpha_s+1,\Omega_s}) \\
+ C(h_s^{\gamma+1} \|u_d\|_{\alpha_s+1,\Omega_d} + h_s^{\gamma+1} \|u_d\|_{\gamma+2} + h_s^{\gamma+1} \|p_d\|_{\gamma+1,\Omega_d}).
\]

**Proof.** The error equation (5.8) can be written as
\[
b_{s,h}(v_{s,h}, R_h^e p_s - p_{s,h}) + b_d(v_{d,h}, R_h^d p_d - p_{d,h}) \\
= -a_{s,h}(Q_h u_s - u_{s,h}, v_{s,h}) - a_{i,h}(Q_h u_s - u_{s,h}, v_{s,h}) + a_d(u_{d,h} - u_{d,h}, v_{d,h}) + l_1(u_s, v_{s,h}) \\
- l_2(p_s, v_{s,h}) - l_3(v_{s,h}, u_{s,h}) + s(Q_h u_s, v_{s,h}) + b_d(v_{d,h}, R_h^d p_d - p_d) - \langle p_d, (v_{s,h} - v_{s,h}) \cdot n\rangle_{\Gamma_h}.
\]

From the definition of $R^e_h$, we know that
\[
b_d(v_{d,h}, R_h^d p_d - p_d) = 0.
\]

Thus,
\[
b_{s,h}(v_{s,h}, R_h^e p_s - p_{s,h}) + b_{d,h}(v_{d,h}, R_h^d p_d - p_{d,h}) \\
\leq C\|Q_h u_s - u_{s,h}\|_{V_h^\gamma} \|v_{s,h}\|_{V_h^\gamma} + C\|u_{d,h} - u_{d,h}\|_{V_h^\gamma} \|v_{d,h}\|_{V_h^\gamma} \\
+ C(h_s^{\gamma+1} \|u_s\|_{\beta+2,\Omega_s} + h_s^{\gamma+1} \|p_s\|_{\gamma+1,\Omega_s} + h_s^{\gamma+1} \|u_s\|_{\beta+1,\Gamma} + h_s^{\gamma+1} \|u_s\|_{\alpha_s+1,\Omega_s}) \|v_{s,h}\|_{V_h^\gamma} \\
+ Ch_d^{\gamma+1/2} h_s^{1/2} \|p_d\|_{\gamma+1,\Omega_d} \|v_{d,h}\|_{V_h^\gamma}.
\]
It follows from the inf-sup condition and the Theorem 5.3 that
\[
\| R_h^* p_s - p_{s,h} \|_{\Omega_s} + \| R_h^d p_d - p_{d,h} \|_{\Omega_d} \
\leq C(h_s^{s+1} \| u_s \|_{\beta+1,\Omega_s} + h_s^{d+1} \| u_d \|_{\beta+1,\Omega_d}) + C(h_s^{\alpha_s} \| u_d \|_{\alpha_s+1,\Omega_d} + h_d^{d+1} \| u_d \|_{\gamma_d+2} + h_d^{d+1} \| p_d \|_{\gamma_d+1,\Omega_d}).
\]
Finally, using the estimate (4.8), we have
\[
\| R_h^* p_s - p_{s,h} \|_{\Omega_s} + \| R_h^d p_d - p_{d,h} \|_{\Omega_d} \
\leq C(h_s^{s+1} \| u_s \|_{\beta+2,\Omega_s} + h_s^{d+1} \| u_s \|_{\gamma_s+1,\Omega_s} + h_s^{\beta+1} \| u_s \|_{\beta+1,\Omega_s} + h_s^{\alpha_s} \| u_s \|_{\alpha_s+1,\Omega_s}) + C(h_s^{\alpha_d} \| u_d \|_{\alpha_d+1,\Omega_d} + h_d^{d+1} \| u_d \|_{\gamma_d+2} + h_d^{d+1} \| p_d \|_{\gamma_d+1,\Omega_d}),
\]
which complete the proof. \( \square \)

6. Numerical Test. In this section, we use two examples to verify our theoretical results on the WG-MFEM scheme for the Stokes-Darcy problem.

In the first example, we solve the following coupled problem on \( \{\Omega_s = (0, \pi) \times (0, \pi)\} \cup \{\Omega_d = (0, \pi) \times (-\pi, 0)\} \) and the interface \( \Gamma = (0, \pi) \times \{0\} \):
\[
\begin{align*}
(6.1) \quad & \quad - \nabla \cdot (\nabla u_s + \nabla^T u_s) + \nabla p_s = f_s \quad \text{in} \ \Omega_s, \\
(6.2) \quad & \quad - \nabla \cdot (\nabla p_d) = 0 \quad \text{in} \ \Omega_d, \\
(6.3) \quad & \quad \begin{pmatrix} -\nabla u_s - \nabla^T u_s + p_s I \cdot n \\ -\nabla u_s - \nabla^T u_s + p_s I \cdot \tau \end{pmatrix} = \begin{pmatrix} u_d \cdot n \\ p_d \\ u_s \cdot \tau \end{pmatrix} \quad \text{on} \ \Gamma,
\end{align*}
\]
with outside boundary conditions
\[
\begin{align*}
u_s &= \begin{pmatrix} 2 \sin y \cos y \cos x \\ \sin^2 y - 2 \sin x \end{pmatrix} \quad \text{on} \ \Gamma_s, \\
v_d \cdot n &= \begin{pmatrix} (e^{-y} - e^y) \cos x \\ (e^{-y} - e^y) \sin x \end{pmatrix} \cdot n \quad \text{on} \ \Gamma_d.
\end{align*}
\]
The source functions in (6.1) \& (6.2) are defined by
\[
f_s = \begin{pmatrix} \sin y \cos x(5 \cos y + 1) \\ \sin x(- \cos^2 y + 4 \sin^2 y - 1 + \cos y) \end{pmatrix},
\]
f_d = 0.

The exact solutions are
\[
\begin{align*}
u_s &= \begin{pmatrix} 2 \sin y \cos y \cos x \\ \sin^2 y - 2 \sin x \end{pmatrix} \quad \text{in} \ \Omega_s, \\
p_s &= \sin x \sin y \quad \text{in} \ \Omega_s, \\
u_d &= \begin{pmatrix} -(e^y - e^{-y}) \cos x \\ -(e^y + e^{-y}) \sin x \end{pmatrix} \quad \text{in} \ \Omega_d, \\
p_d &= (e^y - e^{-y}) \sin x \quad \text{in} \ \Omega_d.
\end{align*}
\]
On the interface, (6.3) is satisfied as
\[
\begin{align*}
\begin{pmatrix} 2 \sin x \\ 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 2 \sin x \\ 0 \\ 0 \end{pmatrix} \quad \text{on} \ \Gamma.
\end{align*}
\]
We plot the velocity field \((u_s \& u_d)\) in Figure 6.1.

In the computation, the first level grid consists of four triangles, cutting each of two rectangles (see Figure 6.1) into two triangles by the north-west to south-east diagonal line. Then, each subsequent grid is a bi-sectional refinement. We apply the weak Galerkin \(P_k\) finite element method for \(u_s\) and \(p_s\) and the mixed BDM \(P_k\) finite element method for computing \(u_d\) and \(p_d\) in solving (6.4). The errors and numerical orders of convergence for the unknown functions in various norms are reported in Tables 6.1–6.4. We can see that all numerical solutions are convergent of optimal order, as proved in our two theorems. Because of coupling, the elliptic regularity for the Stokes-Darcy problem is not known. Partially for this reason, one order higher \(L^2\) convergence for the velocity cannot be proved, or may be proved under some unknown conditions. It does appear, for this example but not for next example, in Tables 6.1–6.4. We still call such a phenomenon one-order superconvergence for the velocity in \(L^2\) by the WG-BDM \(P_k\) elements \((k = 1, 2, 3, 4)\).

In the second numerical example, we solve the coupled problem (6.4) on domain \(\{\Omega_s = (0, 1) \times (1, 2)\} \cup \{\Omega_d = (0, 1) \times (0, 1)\}\). The exact solutions are

\[
\begin{align*}
\mathbf{u}_s &= \begin{pmatrix}
-\cos(\pi x) \sin(\pi y) \\
\sin(\pi x) \cos(\pi y)
\end{pmatrix} & \text{in } \Omega_s, \\
p_s &= \sin(\pi x) & \text{in } \Omega_s, \\
\mathbf{u}_d &= \begin{pmatrix}
-y \pi \cos(\pi x) \\
-\sin(\pi x)
\end{pmatrix} & \text{in } \Omega_d, \\
p_d &= y \sin(\pi x) & \text{in } \Omega_d.
\end{align*}
\]
On the interface $\Gamma = (0,1) \times \{1\}$, the condition (6.3) is reduced to

\[
\begin{pmatrix}
-\sin \pi x \\
\sin \pi x \\
0
\end{pmatrix}
= \begin{pmatrix}
-\sin \pi x \\
\sin \pi x \\
0
\end{pmatrix}
\]

on $\Gamma$.

The velocity field $(\mathbf{u}_s, \mathbf{u}_d)$ is plotted in Figure 6.2. The computational grids are same as those in last example, described above. We list the order of convergence in Tables 6.5–6.8 by $P_1$, $P_2$, $P_3$ and $P_4$ WG-BDM coupled finite element methods. The results confirm the two theorems proved here. Like the computation for the first example, one order superconvergence is obtained in the $P_1$ and $P_3$ WG-BDM element velocity solutions in $L^2$-norm. Unlike the first example, this example does not have an $L^2$-superconvergence for the $P_2$ and the $P_4$ WG–BDM coupled elements. To see the superconvergence in the other cases, we plot the solution and the error for the $P_3$ coupled element in Figures 6.3–6.5.

To see if we have different $L^2$-convergence, we compute the second example by the coupled $P_k$ WG vector and $P_k$ CG scalar elements with $k$ varying from 1 to 5. The corresponding results are recorded in Table 6.9. The observed $L^2$ convergence orders of the velocity are $k + 1$ for all polynomial degrees, as predicated by the theory of the $P_k$ WG elements for the Stokes equations. In particular, we have another order higher $L^2$-convergence for the $P_2$ element in the Darcy region. It behaves the same
Table 6.3
The errors and the order $O(h^k)$ of convergence by the $P_3$ WG elements and BDM$_3$ elements, for (6.3).

| level | $\|Q_0u_s - u_{s,h}\|_0$ | $k$ | $\|Q_hu_s - u_{s,h}\|_0$ | $k$ | $\|R_h^kp_s - p_{s,h}\|_0$ | $k$ |
|-------|----------------|-----|----------------|-----|----------------|-----|
| 3     | 0.1791E-01    | 3.5 | 0.1887E+00    | 2.7 | 0.2979E-01    | 2.6 |
| 4     | 0.1342E-02    | 3.7 | 0.2765E-01    | 2.8 | 0.3425E-02    | 3.1 |
| 5     | 0.9197E-04    | 3.9 | 0.3746E-02    | 2.9 | 0.3748E-03    | 3.2 |
| 6     | 0.5967E-05    | 3.9 | 0.4836E-03    | 3.0 | 0.4302E-04    | 3.1 |

| $\|I_hu_d - u_{d,h}\|_0$ | $k$ | $\|\text{div}(u_d - u_{d,h})\|_0$ | $k$ | $\|I_hp_d - p_{d,h}\|_0$ | $k$ |
|----------------|-----|----------------|-----|----------------|-----|
| 3              | 0.1207E-01 | 3.9 | 0.1281E+00 | 2.9 | 0.1222E+00 | 2.9 |
| 4              | 0.7753E-03 | 4.0 | 0.1640E-01 | 3.0 | 0.1562E-01 | 3.0 |
| 5              | 0.4877E-04 | 4.0 | 0.2062E-02 | 3.0 | 0.1964E-02 | 3.0 |
| 6              | 0.3051E-05 | 4.0 | 0.2582E-03 | 3.0 | 0.2458E-03 | 3.0 |

Table 6.4
The errors and the order $O(h^k)$ of convergence by the $P_4$ WG elements and BDM$_4$ elements, for (6.4).

| level | $\|Q_0u_s - u_{s,h}\|_0$ | $k$ | $\|Q_hu_s - u_{s,h}\|_0$ | $k$ | $\|R_h^kp_s - p_{s,h}\|_0$ | $k$ |
|-------|----------------|-----|----------------|-----|----------------|-----|
| 3     | 0.1925E-02    | 4.7 | 0.3302E-01    | 3.7 | 0.3850E-02    | 3.9 |
| 4     | 0.6334E-04    | 4.9 | 0.2141E-02    | 3.9 | 0.2499E-03    | 3.9 |
| 5     | 0.2007E-05    | 5.0 | 0.1349E-03    | 4.0 | 0.1579E-04    | 4.0 |

| $\|I_hu_d - u_{d,h}\|_0$ | $k$ | $\|\text{div}(u_d - u_{d,h})\|_0$ | $k$ | $\|I_hp_d - p_{d,h}\|_0$ | $k$ |
|----------------|-----|----------------|-----|----------------|-----|
| 3              | 0.6399E-03 | 4.9 | 0.8709E-02 | 3.9 | 0.7438E-02 | 3.9 |
| 4              | 0.1989E-04 | 5.0 | 0.5589E-03 | 4.0 | 0.4763E-03 | 4.0 |
| 5              | 0.6208E-06 | 5.0 | 0.3517E-04 | 4.0 | 0.2995E-04 | 4.0 |

as in solving a pure Darcy problem.

To the best of our knowledge, there exists no general analysis for optimal error estimates of the velocity in $L^2$ norm. Fortunately, some researchers have noticed this problem and made efforts for some specific scheme (such as a monolithic strongly conservative numerical scheme) [16, 21]. But these work still cannot explain the $L^2$ convergence in our study. We will explore the phenomenon in the future work.

7. Conclusion. In this paper, the weak Galerkin finite element method coupled with the mixed finite element method is introduced for the Stokes-Darcy problem. We designed the numerical scheme and derived the optimal error estimates in broken $H^1$ norm for velocity and in $L^2$ for pressure. We found that the convergence order of velocity in $L^2$ norm is not always optimal form the numerical experiments. This phenomenon is strange and it will be studied in the following work.

Appendix A. Some Technique Tools. In this Appendix, we are going to introduce some technical results which have been used in previous section to derive error estimates.

**Lemma A.1.** Let $T_{s,h}$ be a finite element partition of domain $\Omega_s$ satisfying the shape regularity assumptions as specified in [12], we assume $w$ and $p$ are sufficiently smooth. Then, for $0 \leq m \leq 1$ we have
Table 6.5
The errors and the order \(O(h^k)\) of convergence by the \(P_1\) WG and BDM\(_1\) coupled element, for (6.6).  

| Level | \[Q_0\mathbf{u}_s - \mathbf{u}_{s,h}\]\(0\) | \[Q_h\mathbf{u}_s - \mathbf{u}_{s,h}\]\(k\) | \[R_h^s p_s - p_{s,h}\]\(0\) | \[R_h^s p_s - p_{s,h}\]\(k\) |
|-------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 5     | 0.9163E-02 1.9                   | 0.2177E+00 1.0                  | 0.6146E-01 1.0                  | 0.6146E-01 1.0                  |
| 6     | 0.2312E-02 2.0                   | 0.1087E+00 1.0                  | 0.3057E-01 1.0                  | 0.3057E-01 1.0                  |
| 7     | 0.5790E-03 2.0                   | 0.5424E-01 1.0                  | 0.1525E-02 1.0                  | 0.1525E-02 2.0                  |
|       | \[I_h\mathbf{u}_d - \mathbf{u}_{d,h}\]\(0\) | \[\text{div}(\mathbf{u}_d - \mathbf{u}_{d,h})\]\(0\) | \[I_h p_d - p_{d,h}\]\(0\) | \[I_h p_d - p_{d,h}\]\(2\) |
| 5     | 0.3463E-02 2.0                   | 0.1960E+00 1.0                  | 0.1458E-02 2.0                  | 0.1458E-02 2.0                  |
| 6     | 0.8565E-03 2.0                   | 0.9811E-01 1.0                  | 0.3651E-03 2.0                  | 0.3651E-03 2.0                  |
| 7     | 0.2128E-03 2.0                   | 0.4907E-01 1.0                  | 0.9126E-04 2.0                  | 0.9126E-04 2.0                  |

(A.1) \[
\sum_{T \in T_h} h_{T_h}^{2m} \| \mathbf{w} - Q_h \mathbf{w} \|_{T_s}^2 \leq Ch_{T_s}^{2(r+1)} \| \mathbf{w} \|_{T_s,r+1}^2, \quad 1 \leq r \leq \alpha_s,
\]

(A.2) \[
\sum_{T \in T_h} h_{T_h}^{2m} \| \mathbf{w} - Q_h \mathbf{w} \|_{e_s}^2 \leq Ch_{T_h}^{2(r+1)} \| \mathbf{w} \|_{e_s,r+1}^2, \quad 1 \leq r \leq \beta,
\]

(A.3) \[
\sum_{T \in T_h} h_{T_h}^{2m} \| \nabla \mathbf{w} - Q_h (\nabla \mathbf{w}) \|_{T_s}^2 \leq Ch_{T_s}^{2r} \| \mathbf{w} \|_{T_s,r+1}^2, \quad 1 \leq r \leq \beta,
\]

(A.4) \[
\sum_{T \in T_h} h_{T_h}^{2m} \| \rho - Q_h \rho \|_{T_s}^2 \leq Ch_{T_s}^{2r} \| \rho \|_{T_s,r}^2, \quad 1 \leq r \leq \beta.
\]

Here \(C\) denotes a generic constant independent of the mesh size \(h\) and the functions.
The errors and the order $O(h^k)$ of convergence by the $P_2$ WG and BDM$_2$ coupled element, for (6.6).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
level & $\|Q_0 u_s - u_{s,h}\|_0$ & $k$ & $\|Q_h u_s - u_{s,h}\|_0$ & $k$ & $\|R_h^s p_s - p_{s,h}\|_0$ & $k$ \\
\hline
4 & 0.1293E-01 & 2.1 & 0.8738E-01 & 2.0 & 0.2327E-01 & 2.0 \\
5 & 0.3196E-02 & 2.0 & 0.2210E-01 & 2.0 & 0.5794E-02 & 2.0 \\
6 & 0.7967E-03 & 2.0 & 0.5555E-02 & 2.0 & 0.1444E-02 & 2.0 \\
\hline
$\|I_h u_d - u_{d,h}\|_0$ & $k$ & $\|\text{div}(u_d - u_{d,h})\|_0$ & $k$ & $\|I_h p_d - p_{d,h}\|_0$ & $k$ \\
\hline
4 & 0.1192E-01 & 2.0 & 0.5139E-01 & 2.0 & 0.4266E-02 & 2.0 \\
5 & 0.2995E-02 & 2.0 & 0.1292E-01 & 2.0 & 0.1070E-02 & 2.0 \\
6 & 0.7497E-03 & 2.0 & 0.3235E-02 & 2.0 & 0.2678E-03 & 2.0 \\
\hline
\end{tabular}
\end{table}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6_3.png}
\caption{The solution for $(u_s, u_d)_1$ and its error by $P_3$ elements on level 4, for (6.6).}
\end{figure}

in the estimates.

**Lemma A.2.** $\Pi_h^d$ satisfies the approximation properties

\begin{align}
\|v_d - \Pi_h^d v_d\|_{0,T} \leq C h_T^m |v_d|_{m,T}, & \quad 1 \leq m \leq \alpha_d + 1, \\
\|\nabla \cdot (v_d - \Pi_h^d v_d)\|_{0,T} \leq C h_T^m |\nabla v_d|_{m,T}, & \quad 0 \leq m \leq \gamma_d + 1.
\end{align}

**Lemma A.3.** Let $p|_{\Omega_s} \in H^{\gamma_s}(\Omega_s)$, $p|_{\Omega_d} \in H^{\gamma_d}(\Omega_d)$, then we have

\begin{align}
\|p - R_h p\|_{m,T} \leq C h_T^{-m} |p|_{\gamma_s,T} & \quad T \in \Omega_s, \ m = 0, 1, \\
\|p - R_h p\|_{m,T} \leq C h_T^{-m} |p|_{\gamma_d,T} & \quad T \in \Omega_d, \ m = 0, 1.
\end{align}

Let $T_s$ be an element satisfying the assumption verified in [42] with $e_s$ as a side. For any function $g \in H^1(T)$, the following trace inequality has been proved in [42]

\begin{align}
\|g\|_{T_s}^2 \leq C (h_{T_s}^{-1} |g|_{T_s}^2 + h_{T_s} |\nabla g|_{T_s}^2).
\end{align}
Table 6.7
The errors and the order $O(h^k)$ of convergence by the $P_3$ WG and BDM$_3$ coupled element, for (6.6).

| level | $\|Q_0u_s - u_{s,h}\|$ | $k$ | $\|Q_hu_s - u_{s,h}\|$ | $k$ | $\|R_h^sp_s - p_{s,h}\|$ | $k$ |
|-------|-----------------|-----|-----------------|-----|-----------------|-----|
| 3     | 0.1074E-02      | 3.9 | 0.3643E-01      | 2.9 | 0.5218E-02      | 3.0 |
| 4     | 0.6950E-04      | 3.9 | 0.4757E-02      | 2.9 | 0.6587E-03      | 3.0 |
| 5     | 0.4417E-05      | 4.0 | 0.6074E-03      | 3.0 | 0.8273E-04      | 3.0 |

$\|I_hu_d - u_{d,h}\|$ | $k$ | $\|\text{div}(u_d - u_{d,h})\|$ | $k$ | $\|I_hp_d - p_{d,h}\|$ | $k$ |
|-------|-----------------|-----|-----------------|-----|-----------------|-----|
| 3     | 0.1091E-02      | 4.0 | 0.1481E-01      | 2.9 | 0.1321E-02      | 2.7 |
| 4     | 0.6864E-04      | 4.0 | 0.1868E-02      | 3.0 | 0.1725E-03      | 2.9 |
| 5     | 0.4294E-05      | 4.0 | 0.2341E-03      | 3.0 | 0.2179E-04      | 3.0 |

Fig. 6.4. The solution for $(u_s, u_d)_2$ and its error by $P_3$ elements on level 4, for (6.6).

Particularly, if $g$ is polynomial in $T$ we have the inverse inequality

(A.10)
$$\|\nabla g\|_{T_s}^2 \leq Ch_{T_s}^{-2} \|g\|_{T_s}^2,$$

where $C$ is a constant only related to the degree of polynomial and the dimension. Combining with the trace inequality we can get further that

(A.11)
$$\|\nabla g\|_{e}^2 \leq Ch_{T_s}^{-1} \|g\|_{T_s}^2.$$

The vector version of the trace theorem and the inverse theorem are trivial.

**Lemma A.4.** For any $v_{s,h} \in V_h^\ast$, we have

(A.12)
$$\sum_{T_s \in T_{s,h}} \|v_{s,0} - v_{s,h}\|_{\partial T_s} \leq Ch_{T_s}^{\frac{1}{2}} \|v_{s,h}\|_{V_h^\ast}.$$

**Proof.** When $\alpha_s = \beta$, (A.12) is obvious. So we only need to discuss the case that $\alpha_s = \beta + 1$. We only consider the vector valued function $v_{s,h}$. From Lemma 4.1 we have

(A.13)
$$\sum_{T_s \in T_{s,h}} \|\nabla v_{s,0}\|_{T_s} \leq C \|v_{s,h}\|_{V_h^\ast}.$$
Fig. 6.5. The solution for \((p_s, p_d)\) and its error by \(P_4\) elements on level 4, for (6.6).

Table 6.8

The errors and the order \(O(h^k)\) of convergence by the \(P_4\) WG and BDM\(_4\) coupled element, for (6.6).

| level | \(\|Q_0 u_s - u_{s,h}\|_0\) | \(k\) | \(\|Q_h u_s - u_{s,h}\|\) | \(k\) | \(\|H^1_h p_s - p_{s,h}\|_0\) | \(k\) |
|-------|-----------------|-----|-----------------|-----|-----------------|-----|
| 3     | 0.1025E-03      | 4.7 | 0.4604E-02      | 3.9 | 0.5025E-03      | 4.0 |
| 4     | 0.4533E-05      | 4.5 | 0.2951E-03      | 4.0 | 0.3184E-05      | 4.0 |
| 5     | 0.2428E-06      | 4.2 | 0.1862E-04      | 4.0 | 0.1997E-05      | 4.0 |

Using the trace inequality [A.9] and Poincaré inequality, we can obtain that

\[
\sum_{T_s \in T_{s,h}} \|v_{s,0} - v_{s,h}\|_{\partial T_s} \leq C \sum_{T_s \in T_{s,h}} \|\nabla v_{s,0}\|_{\partial T_s} + \sum_{T_s \in T_{s,h}} \|Q_h v_{s,0} - v_{s,h}\|_{\partial T_s}
\]

\[
\leq C h_{s,h} T_s \|v_{s,0}\|_{T_s} + h_{s,h}^{\frac{1}{2}} \sum_{T_s \in T_{s,h}} \|Q_h v_{s,0} - v_{s,h}\|_{T_s}
\]

\[
\leq C h_{s,h}^{\frac{1}{2}} \|v_{s,h}\|_{V_s^h},
\]

which completes the proof. \(\square\)

Lemma A.5. Let \(w_{\Omega_s} \in [H^{\alpha_s}(\Omega_s)]^n\), \(\rho_{\Omega_s} \in H^{\gamma_s}(\Omega_s)\), \(i = s, d\), and \(v \in V_{s,h}\). Assume that the finite element partition \(T_{s,h}\) is shape regular. Then we have the following estimates

(A.14) \(l_1(w_s, v_{s,h}) \leq C h_{s,h}^{\beta+1} \|w_s\|_{\beta+2, \Omega_s} \|v_{s,h}\|_{V_s^h}\),

(A.15) \(l_2(p_s, v_{s,h}) \leq C h_{s,h}^{\gamma+1} \|\rho_s\|_{\gamma+1} \|v_{s,h}\|_{V_s^h}\),

(A.16) \(l_3(w_s, v_{s,h}) \leq C h_{s,h}^{\beta+1} \|w_s\|_{\beta+1, \Gamma} \|v_{s,h}\|_{V_s^h}\),

(A.17) \(s(Q_h w_s, v_{s,h}) \leq C h_{s,h}^{\alpha} \|w_s\|_{\alpha+1, \Gamma} \|v_{s,h}\|_{V_s^h}\).
Table 6.9
The errors and the orders O(h^k) of convergence, for (6.6).

| level | ∥Q_h u_s - u_s,h∥₀ | k | ∥Q_h p_s - p_s,h∥₀ | k | ∥I_h p_d - p_d,h∥₀ | k |
|-------|---------------------|---|---------------------|---|---------------------|---|
| 4     | 0.3625E-01          | 1.8 | 0.1223E+00          | 0.9 | 0.8650E-02          | 1.9 |
| 5     | 0.9363E-02          | 2.0 | 0.6141E-01          | 1.0 | 0.2181E-02          | 2.0 |
| 6     | 0.2361E-02          | 2.0 | 0.3059E-01          | 1.0 | 0.5441E-03          | 2.0 |

By coupled P₁ WLG vector and P₁ CG scalar element.

| level | ∥Q_h u_s - u_s,h∥₀ | k | ∥Q_h p_s - p_s,h∥₀ | k | ∥I_h p_d - p_d,h∥₀ | k |
|-------|---------------------|---|---------------------|---|---------------------|---|
| 4     | 0.1682E-02          | 2.9 | 0.1261E-01          | 2.0 | 0.4920E-04          | 3.7 |
| 5     | 0.2126E-03          | 3.0 | 0.3138E-02          | 2.0 | 0.3755E-05          | 3.7 |
| 6     | 0.2664E-02          | 3.0 | 0.7792E-03          | 2.0 | 0.2994E-06          | 3.6 |

By coupled P₂ WLG vector and P₂ CG scalar element.

| level | ∥Q_h u_s - u_s,h∥₀ | k | ∥Q_h p_s - p_s,h∥₀ | k | ∥I_h p_d - p_d,h∥₀ | k |
|-------|---------------------|---|---------------------|---|---------------------|---|
| 4     | 0.6905E-04          | 4.0 | 0.6543E-03          | 3.0 | 0.2390E-05          | 4.1 |
| 5     | 0.4374E-05          | 4.0 | 0.8262E-02          | 3.0 | 0.1435E-06          | 4.1 |
| 6     | 0.2748E-04          | 4.0 | 0.1037E-04          | 3.0 | 0.8798E-08          | 4.0 |

By coupled P₃ WLG vector and P₃ CG scalar element.

| level | ∥Q_h u_s - u_s,h∥₀ | k | ∥Q_h p_s - p_s,h∥₀ | k | ∥I_h p_d - p_d,h∥₀ | k |
|-------|---------------------|---|---------------------|---|---------------------|---|
| 3     | 0.8282E-04          | 4.9 | 0.4249E-03          | 3.8 | 0.3119E-05          | 5.0 |
| 4     | 0.2650E-05          | 5.0 | 0.2744E-04          | 4.0 | 0.9716E-07          | 5.0 |
| 5     | 0.8342E-07          | 5.0 | 0.1731E-05          | 4.0 | 0.3025E-08          | 5.0 |

By coupled P₄ WLG vector and P₄ CG scalar element.

| level | ∥Q_h u_s - u_s,h∥₀ | k | ∥Q_h p_s - p_s,h∥₀ | k | ∥I_h p_d - p_d,h∥₀ | k |
|-------|---------------------|---|---------------------|---|---------------------|---|
| 3     | 0.5416E-05          | 5.9 | 0.2996E-04          | 4.9 | 0.1474E-06          | 6.0 |
| 4     | 0.8596E-07          | 6.0 | 0.9595E-06          | 5.0 | 0.2286E-08          | 6.0 |
| 5     | 0.1349E-08          | 6.0 | 0.3020E-07          | 5.0 | 0.3557E-10          | 6.0 |

By coupled P₅ WLG vector and P₅ CG scalar element.

Proof. Using Cauchy-Schwarz inequality, (A.12) and (4.3), we have

\[
l_1(w_s, v_{s,h}) = 2\nu \sum_{T \in T_{s,h}} \langle v_{s,0} - v_{s,b}, D(w_s) \cdot n - (Q_h D(w_s)) \cdot n \rangle_{\partial T_s} \\
\leq C \left( \sum_{T \in T_{s,h}} h_{T_s}^{-1} \|v_{s,0} - v_{s,b}\|^2_{\partial T_s} \right)^{1/2} \left( \sum_{T \in T_{s,h}} h_{T_s} \|D(w_s) - QD(w_s)\|^2_{\partial T_s} \right)^{1/2} \\
\leq Ch_s^{\beta+1}\|w_s\|_{\beta+2,\Omega_s}\|v_{s,h}\|_{V_{s,h}}.
\]

The similarly technique can be applied to the following estimate,

\[
l_2(\rho_s, v_{s,h}) = \sum_{T_s \in T_{s,h}} \langle v_{s,0} - v_{s,b}, (\rho_s - R^s_h \rho_s) n \rangle_{\partial T_s} \\
\leq \left( \sum_{T \in T_{s,h}} h_{T_s}^{-1} \|v_{s,0} - v_{s,b}\|^2_{\partial T_s} \right)^{1/2} \left( \sum_{T \in T_{s,h}} h_{T_s} \|\rho_s - R^s_h \rho_s\|^2_{\partial T_s} \right)^{1/2} \\
\leq Ch_s^{\gamma_s+1}\|\rho_s\|_{\gamma_s+2,\Omega_s}\|v_{s,h}\|_{V_{s,h}}.
\]
By the definition of the norm \( \| \cdot \|_V \) and (A.2),
\[
I_3(w_s, v_{s,h}) = \sum_{e \in \Gamma_s} (\mu K_e^{-1/2} (u_s - Q_h w_s) \cdot \tau, v_{s,h} \tau)_e \\
\leq C \| w_s - Q_h w_s \|_1 \| v_{s,h} \|_V^h \\
\leq C h^{\frac{1}{2}} \| w_s \|_{H^{\frac{3}{2}}(\Omega)} \| v_{s,h} \|_V^h .
\]

Finally, from the property of \( Q_h \), trace inequality (A.9), we know that
\[
\begin{align*}
s(Q_h w_s, v_{s,h}) &= \sum_{T \in \mathcal{T}_s} h_{T,s}^{-1} (Q_0 w_s - w_s, Q_0 v_{s,0} - v_{s,b})_{\partial T_s} \\
&\leq C \left( \sum_{T \in \mathcal{T}_s} h_{T,s}^{-1} \| Q_0 w_s - w_s \|_{\partial T_s}^2 \right)^{1/2} \left( \sum_{T \in \mathcal{T}_s} h_{T,s}^{-1} \| Q_0 v_{s,0} - v_{s,b} \|_{\partial T_s}^2 \right)^{1/2} \\
&\leq C h_{s}^\alpha \| w_s \|_{\alpha+1} \| v_{s,h} \|_V^h .
\end{align*}
\]

\[ \square \]

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