Two-point free energy distribution function in (1+1) directed polymers

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Abstract
In this brief communication it is demonstrated how by using the Bethe ansatz technique the explicit expression for the two-point free energy distribution function in (1+1) directed polymers with fixed end-point boundary conditions can be derived in a rather simple way. The obtained result is equivalent to the one previously derived by Prolhac and Spohn (2011 J. Stat. Mech. P01031).

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1. Introduction
Directed polymers in a quenched random potential as well as the equivalent problem of the Kardar–Parisi–Zang equation [1] have been the subject of intense investigations during almost the past three decades [2–13]. In the case of the so-called (1+1)-dimensional system, one studies the statistical properties of an elastic string $\phi(\tau)$ located in the 2D plane $(\phi, \tau)$ and directed along the $\tau$-axis within an interval $[0, t]$. At every point of the plane there is a quenched random potential $V[\phi, \tau]$, which competes against the elastic energy. The problem is defined by the Hamiltonian

$$H[\phi(\tau), V] = \int_0^t d\tau \left\{ \frac{1}{2} [\partial_\tau \phi(\tau)]^2 + V[\phi(\tau), \tau] \right\}.$$  (1)

Here the random potential $V[\phi, \tau]$ is described by the independent Gaussian distributions with the zero mean, $\langle V[\phi, \tau] \rangle = 0$, and

$$V[\phi, \tau]V[\phi', \tau'] = u \delta(\tau - \tau') \delta(\phi - \phi'),$$  (2)

where the parameter $u$ defines the strength of the disorder.

Such a system exhibits numerous non-trivial features deriving from the interplay between elasticity and disorder. In particular, in the limit $t \to \infty$ the polymer mean-squared displacement exhibits a universal scaling form $\langle \phi^2 \rangle \propto t^{4/3}$ (where $\langle \ldots \rangle$ and $(\ldots)$ denote the thermal and disorder averages) while the typical value of the free energy fluctuations scales as $t^{1/3}$ [5–8].
A more general problem is the derivation of the entire probability distribution function (PDF) of the free energy fluctuations. By definition, the partition function of the system (1)–(2) with the fixed boundary condition, \( \phi(t) = x \), is

\[
Z(x) = \int_{\phi(0) = 0}^{\phi(t) = x} D\phi(\tau) \exp[-\beta H(\phi)] = \exp[-\beta F(x)]
\]

(3)

where \( F(x) \) is the total free energy of the polymer which at time \( t \) arrives at the point \( x \). In the limit of large \( t \) random free energy scales as

\[
\beta F(x) = \beta f_0 + \lambda f(x)
\]

(4)

where \( f_0 \) is the trivial self-averaging contribution and

\[
\lambda = \frac{1}{2}(\beta^2 \tau^2 t)^{1/3} \propto t^{1/3}.
\]

A few years ago it was shown that in the limit \( t \to \infty \) the random quantity \( f \sim 1 \) is described by the Tracy–Widom (TW) distribution of the largest eigenvalue of the Gaussian Unitary Ensemble (GUE) [14–21]. Later on it was proved that the free energy PDF of the directed polymers with free boundary conditions is given by the Gaussian Orthogonal Ensemble TW distribution [22, 23], while in the presence of a ‘wall’ (\( \phi(\tau) \geq 0, \ 0 \leq \tau \leq t \)) such a PDF is given by the Gaussian Symplectic Ensemble TW distribution [24].

The aim of this brief communication is to study the two-point free energy PDF

\[
W(f_1, f_2; x_1, x_2) = \lim_{t \to \infty} \text{Prob}[f(x_1) > f_1; \ f(x_2) > f_2]
\]

(6)

which describes joint statistics of the free energies of two directed polymers coming to two different endpoints. Some time ago the result for this function was derived in terms of the Bethe ansatz replica technique under a particular decoupling assumption [25]. Here I am going to recompute this function using somewhat different computational tricks which do not require any supplementary assumptions. Since this function depends only on the distance between the two points \( x = |x_2 - x_1| \), to simplify formulas I will consider the particular case: \( x_1 = -\frac{1}{2}x \) and \( x_2 = +\frac{1}{2}x \). In other words, instead of (6) I will concentrate on the PDF

\[
W(f_1, f_2; x) = \lim_{t \to \infty} \text{Prob}[f(-x/2) > f_1; \ f(x/2) > f_2].
\]

(7)

Only recently was it proven [26] that the result of the present calculations (see equations (52) and (53) below) is equivalent to those obtained earlier [25].

### 2. Two-point distribution function

In terms of the partition function, equation (3), above the PDF, equation (7) can be defined as follows:

\[
W(f_1, f_2; x) = \lim_{\lambda \to \infty} \sum_{L=0}^{\infty} \sum_{R=0}^{\infty} \frac{(-1)^L (-1)^R}{L! R!} \times \exp(\lambda L f_1 + \lambda R f_2) \times Z(-x/2) \exp[\beta f_2 t] \times Z(x/2) \exp[\beta f_0 t] R!
\]

(8)

where \( \langle \ldots \rangle \) denotes the average over the random potentials (2). Performing the standard averaging of the \( (L + R) \)th power of the partition function, equation (3), one gets

\[
W(f_1, f_2; x) = \lim_{\lambda \to \infty} \sum_{L=0}^{\infty} \sum_{R=0}^{\infty} \frac{(-1)^L (-1)^R}{L! R!} \times \exp(\lambda L f_1 + \lambda R f_2 + \beta(L + R) f_0 t) \times \Psi \left( \frac{x/2, \ldots, x/2, x/2, \ldots, x/2}{L, R} \right)
\]

(9)
where the time-dependent \( N \)-point wave function \( \Psi(x_1, \ldots, x_N; t) \) is the solution of the imaginary time Schrödinger equation

\[
\beta \partial_t \Psi(x; t) = \left[ -\frac{1}{2} \sum_{a=1}^{N} \partial^2_{x_a} + \frac{1}{2} \sum_{a \neq b}^{N} \kappa \delta(x_a - x_b) \right] \Psi(x; t)
\]

(10)

with \( \kappa = \beta^3 u \) and the initial condition

\[
\Psi(x; t = 0) = \prod_{a=1}^{N} \delta(x_a).
\]

(11)

A generic eigenstate of such a system is characterized by \( N \) momenta \( \{Q_a\} (a = 1, \ldots, N) \) which split into \( M \) (\( 1 \leq M \leq N \)) clusters described by continuous real momenta \( q_\alpha (\alpha = 1, \ldots, M) \) and having \( n_a \) discrete imaginary parts

\[
Q_a \equiv q_\alpha^r = q_\alpha - \frac{i\kappa}{2}(n_a + 1 - 2r); \quad (r = 1, \ldots, n_a)
\]

(12)

with the global constraint

\[
\sum_{a=1}^{M} n_a = N.
\]

(13)

The time-dependent solution \( \Psi(x, t) \) of the Schrödinger equation (10) with the initial conditions, equation (11), can be represented in the form of the linear combination of the eigenfunctions \( \Psi^{(M)}(x) \):

\[
\Psi(x_1, \ldots, x_N; t) = \sum_{M=1}^{N} \frac{1}{M!} \left[ \int \mathcal{D}^{(M)}(q, n) \right] |C_M(q, n)|^2 \Psi^{(M)}(x) \Psi^{(M)^*}(0) \exp[-E_M(q, n)t]
\]

(14)

where we have introduced the notation

\[
\int \mathcal{D}^{(M)}(q, n) \equiv \prod_{a=1}^{M} \left[ \int_{-\infty}^{+\infty} \frac{dq_a}{2\pi} \sum_{n_a=1}^{\infty} \delta \left( \sum_{a=1}^{M} n_a, N \right) \right]
\]

(15)

and \( \delta(k, m) \) is the Kronecker symbol. For a given set of integers \( \{M, n_1, \ldots, n_M\} \) the eigenfunctions \( \Psi^{(M)}(x) \) can be represented as follows (for details see [27–31]):

\[
\Psi^{(M)}(x) = \sum_{P} N! \prod_{a=1}^{M} \left[ 1 + i\kappa \frac{\text{sgn}(x_a - x_b)}{Q_{\alpha_a} - Q_{\alpha_b}} \right] \exp \left[ i \sum_{a=1}^{M} Q_{\alpha_a} x_a \right]
\]

(16)

where the summation goes over \( N! \) permutations \( P \) of \( N \) momenta \( Q_a \), equation (12), over \( N \) particles \( x_a \); the normalization factor

\[
|C_M(q, n)|^2 = \frac{\kappa^N}{N! \prod_{a=1}^{M} (\kappa n_a)} \prod_{a < \beta}^{M} \left| q_a - q_\beta - \frac{i\kappa}{2}(n_a - n_\beta) \right|^2
\]

(17)

and the eigenvalues

\[
E_M(q, n) = \sum_{a=1}^{M} \left[ \frac{1}{2\beta} n_a q_a^2 - \frac{\kappa^2}{24\beta} n_a^3 \right] + \frac{\kappa^2}{24\beta} N.
\]

(18)

The last term in the above expression yields the self-averaging part of the free energy; choosing \( f_0 = \kappa^2/(24\beta^2) \) this term drops out of further calculations. Note also that according to the definition, equation (16), \( \Psi^{(M)}(0) = N! \).
Substituting equations (14)–(18) into equation (9), we get:

\[
W(f_1, f_2; x) = 1 + \lim_{\lambda \to \infty} \sum_{L_1 R \geq 1}^{\infty} \frac{(-1)^{L_1 + R}}{L_1^! R^!} e^{\lambda f_1 + \lambda f_2} \times \sum_{M=1}^{L_1 R} \frac{1}{M!} \prod_{a=1}^{\infty} \int_{-\infty}^{+\infty} \frac{dq_a}{2\pi \kappa} \frac{1}{q_a} e^{-\frac{\pi}{\kappa} q_a^2 + \frac{\pi}{\kappa} q_a^0} \times \delta \left( \sum_{a=1}^{L_1 R} q_a^0, L + R \right) |\tilde{C}_M(q, n)|^2 \sum_{\mathcal{P}(L,R)} \prod_{\alpha=1}^{L} \prod_{\beta=1}^{R} \sum_{a=1}^{L_1 \mathcal{P}(\alpha)} \sum_{b=1}^{R_\mathcal{P}(\beta)} \prod_{c=1}^{R_\mathcal{P}(\beta)} \left[ \frac{Q_{\mathcal{P}(\alpha)} - Q_{\mathcal{P}(\beta)} - i\kappa}{Q_{\mathcal{P}(\alpha)} - Q_{\mathcal{P}(\beta)}} \right] \times \exp \left[ -\frac{i}{2} x \sum_{a=1}^{L} q_{\mathcal{P}(\alpha)}^a + \frac{i}{2} \sum_{c=1}^{R} q_{\mathcal{P}(\beta)}^c \right] \tag{19}
\]

where

\[
|\tilde{C}_M(q, n)|^2 = \prod_{a < b} \left| q_a - q_b - \frac{i\kappa}{2} (n_a - n_b) \right|^2 \left| q_a - q_b - \frac{i\kappa}{2} (n_a + n_b) \right|^2. \tag{20}
\]

In equation (19) the summation over all permutations \( \mathcal{P} \) of \((L + R)\) momenta \( \{Q_1, \ldots, Q_{L+R} \} \) over \( L \) ‘left’ particles \( \{x_1, \ldots, x_L \} \) and \( R \) ‘right’ particles \( \{y_1, \ldots, y_R \} \) split into three parts: the permutations \( \mathcal{P}^{(L)} \) of \( L \) momenta (taken at random out of the total list \( \{Q_1, \ldots, Q_{L+R} \} \)) over \( L \) ‘left’ particles, the permutations \( \mathcal{P}^{(R)} \) of the remaining \( R \) momenta over \( R \) ‘right’ particles, and finally the permutations \( \mathcal{P}^{(L,R)} \) (or the exchange) of the momenta between the group ‘L’ and the group ‘R’. It is evident that due to the symmetry of the expression in equation (19) with respect to the permutations \( \mathcal{P}^{(L)} \) and \( \mathcal{P}^{(R)} \) the summations over these permutations give just the factor \( L! R! \).

Further simplification comes from the following general property of the Bethe ansatz wave function, equation (16). It has such a structure that for ordered particle positions (e.g. \( x_1 < x_2 < \cdots < x_N \)) in the summation over permutations, the momenta \( q_a \) belonging to the same cluster also remain ordered. In other words, if we consider the momenta, equation (12), of a cluster \( \alpha \), \( \{q_{\alpha}^1, q_{\alpha}^2, \ldots, q_{\alpha}^n \} \), belonging to the particles \( \{x_1, x_2, \ldots, x_n \} \), the permutation of any two momenta \( q_{\alpha}^a \) and \( q_{\alpha}^b \) of this ordered set gives zero contribution. Thus, in order to perform the summation over the permutations \( \mathcal{P}^{(L,R)} \) in equation (19) it is sufficient to split the momenta of each cluster into two parts: \( \{q_{\alpha}^1, \ldots, q_{\alpha}^m \} \mid \{q_{\alpha}^{m+1}, \ldots, q_{\alpha}^n \} \), where \( m_a = 0, 1, \ldots, n_a \) and where the momenta \( q_{\alpha}^1, \ldots, q_{\alpha}^m \) belong to the particles of the sector ‘L’ (whose coordinates are all equal to \(-x/2\)), while the momenta \( q_{\alpha}^{m+1}, \ldots, q_{\alpha}^n \) belong to the particles of the sector ‘R’ (whose coordinates are all equal to \(+x/2\)).

Let us introduce the numbering of the momenta of the sector ‘R’ in reverse order:

\[
q_{a}^\alpha, \quad q_{a}^\alpha, \quad \ldots, \quad q_{s_a}^\alpha \rightarrow \ q_{a}^\alpha, \quad q_{a}^\alpha, \quad \ldots, \quad q_{s_a}^\alpha, \tag{21}
\]

where \( m_a + s_a = n_a \) and (see equation (12))

\[
q_{s_a}^\alpha = q_a + \frac{i\kappa}{2} (n_a + 1 - 2r) = q_a + \frac{i\kappa}{2} (m_a + s_a + 1 - 2r). \tag{22}
\]

By definition, the integer parameters \( \{m_a\} \) and \( \{s_a\} \) fulfil the global constraints

\[
\sum_{a=1}^{M} m_a = L, \tag{23}
\]
\[
\sum_{a=1}^{M} s_a = R. \quad (24)
\]

In this way the summation over permutations \( \mathcal{P}^{(L,R)} \) in equation (19) is changed by the summations over the integer parameters \( \{m_a\} \) and \( \{x_a\} \), which allows us to lift the summations over \( L, R \), and \( \{n_a\} \). Straightforward calculations result in the following expression:

\[
W(f_1, f_2; x) = \lim_{\lambda \to -\infty} \left\{ 1 + \sum_{M=1}^{\infty} \left( \frac{-1}{M!} \sum_{m_a+s_a \geq 1} \int_{-\infty}^{+\infty} \frac{dq_a}{2\pi\kappa(m_a+s_a)} \right) \times \exp \left\{ \lambda m_a f_1 + \lambda s_a f_2 - \frac{i}{2} \frac{\lambda m_a}{\kappa s_a} q_a + \frac{i}{2} \frac{\lambda s_a}{\kappa m_a} s_a - \frac{1}{2} \frac{\kappa x m_a s_a}{\left( m_a + s_a \right)^{2/3}} \right\} \right\}
\]

\[
\times \exp \left\{ \frac{t}{2\beta} (m_a + s_a)^{2/3} + \frac{k^2 t}{24\beta} (m_a + s_a)^{1/3} \right\} \Bigg[ \tilde{C}_{M}(q, m + s) \Bigg] \Bigg[G_M(q, m, s) \Bigg] \quad (25)
\]

where

\[
G_M(q, m, s) = \prod_{a=1}^{M} \prod_{r=1}^{m_a+s_a} \left( \frac{q_{\alpha} - q_{\alpha}^{*}}{q_{\alpha}^{*} - q_{\alpha}} \right) \times \prod_{a < b} \prod_{r=1}^{m_a+s_a} \left( \frac{q_{\alpha} - q_{\beta}^{*}}{q_{\beta}^{*} - q_{\alpha}} \right)
\]

\[
= \prod_{a=1}^{M} \frac{\Gamma(1+m_a+s_a)}{\Gamma(1+m_a)\Gamma(1+s_a)} \times \prod_{a < b} \frac{\Gamma\left[ 1 + \frac{m_a + m_b + s_a + s_b}{2} + \frac{i}{2} (q_a - q_b) \right]}{\Gamma\left[ 1 + \frac{m_a + m_b + s_a - s_b}{2} + \frac{i}{2} (q_a - q_b) \right]}
\]

\[
\times \prod_{a < b} \frac{\Gamma\left[ 1 + \frac{m_a + m_b + s_a - s_b}{2} + \frac{i}{2} (q_a - q_b) \right]}{\Gamma\left[ 1 + \frac{m_a + m_b + s_a + s_b}{2} + \frac{i}{2} (q_a - q_b) \right]}
\]

(26)

After rescaling

\[
g_a \to \frac{\kappa}{2\lambda} q_a \quad (27)
\]

\[
x \to \frac{2\lambda^2}{\kappa} x \quad (28)
\]

with

\[
\lambda = \frac{1}{2} \left( \frac{k^2 t}{\beta} \right)^{1/3} = \frac{1}{2} (\beta^{-5/2} t^{-1/2})^{1/3} \quad (29)
\]

the normalization factor \( |\tilde{C}_{M}(q, m + s)|^2 \), equation (20), can be represented as follows:

\[
|\tilde{C}_{M}(q, m + s)|^2 = \prod_{a < b} \left[ \frac{\lambda(m_a + s_a) - \lambda(m_b + s_b) - iq_a + iq_b}{\lambda(m_a + s_a) + \lambda(m_b + s_b) - iq_a + iq_b} \right]^2
\]

\[
= \left( \prod_{a=1}^{M} [2\lambda(m_a + s_a)] \right) \times \det \left[ \frac{\lambda(m_a + s_a) - iq_a + \lambda(m_b + s_b) + iq_b}{\lambda(m_a + s_a) - iq_a + \lambda(m_b + s_b) + iq_b} \right] \quad (30)
\]

Substituting equations (27)–(30) into equation (25) and using the Airy function relation

\[
\exp \left[ \frac{1}{3} \lambda^3 (m_a + s_a)^3 \right] = \int_{-\infty}^{+\infty} dy \, \text{Ai}(y) \exp[\lambda(m_a + s_a) y] \quad (31)
\]
we get
\[ W(f_1, f_2; x) = \lim_{\lambda \to \infty} \left\{ 1 + \sum_{M=1}^{\infty} \frac{(-1)^M}{M!} \prod_{a=1}^{M} \left[ \int_{-\infty}^{+\infty} \frac{dy_a \, dq_a}{2\pi} \, \Lambda_i(y_a + q_a^2) \sum_{m_a + s_a \geq 1} (-1)^{m_a + s_a - 1} \right. \\
\times \exp \left\{ \lambda m_a \left( y_a + f_1 - \frac{1}{2} i q_a \right) + \lambda s_a \left( y_a + f_2 + \frac{1}{2} i q_a \right) - \lambda^2 m_a s_a x \right\} \\
\left. \right] \times \det K\left( \lambda m_a, \lambda s_a, q_a; (\lambda m_\beta, \lambda s_\beta, q_\beta)_{a, \beta = 1, \ldots, M} \right) G_{M} \left( \frac{x \lambda}{2}, m, s \right) \right\} \] (32)

where
\[ K(\lambda m, \lambda s; q) = \left( \lambda m', \lambda s'; q' \right) = \frac{1}{\lambda m + \lambda s - i q + \lambda m' + \lambda s' + i q'}. \] (33)

Using the relation
\[ \exp(-\lambda^2 mx) = \int_{-\infty}^{+\infty} \frac{d\xi_1 \, d\xi_2 \, d\xi_3}{(2\pi)^{3/2}} \times \exp \left\{ -\frac{1}{2} \left( \xi_1^2 + 2 \, i \lambda x \right) - \frac{1}{2} \left( \xi_2^2 + 2 \, i \lambda x \right) \right\} \] (34)
the expression in equation (32) can be represented as follows:
\[ W(f_1, f_2; x) = 1 + \sum_{M=1}^{\infty} \frac{(-1)^M}{M!} \prod_{a=1}^{M} \left[ \int_{-\infty}^{+\infty} \frac{dy_a \, dq_a \, d\xi_a \, d\xi_a \, dq_{\alpha} \, d\xi_{\alpha}}{(2\pi)^{3/2}} \, \Lambda_i(y_a + q_{\alpha}^2 - i \xi_{\alpha} \sqrt{x}) \right. \\
\times \exp \left\{ -\frac{1}{2} \left( \xi_{1\alpha}^2 + \frac{1}{2} i q_{\alpha} \sqrt{x} \right) - \frac{1}{2} \left( \xi_{2\alpha}^2 + \frac{1}{2} i q_{\alpha} \sqrt{x} \right) \right\} \\
\left. \right] \times \mathcal{S}_M(q, y, \xi_1, f_1, f_2, x) \right\} \] (35)

where
\[ \mathcal{S}_M(q, y, \xi_1, f_1, f_2, x) = \lim_{\lambda \to \infty} \prod_{a=1}^{M} \left[ \sum_{m_a + s_a \geq 1} (-1)^{m_a + s_a - 1} \right. \\
\times \exp\left( \lambda m_a (y_a + f_1 + \xi_{1a} \sqrt{x}) + \lambda s_a (y_a + f_2 + \xi_{2a} \sqrt{x}) \right) \\
\left. \right] \times \det K\left( \lambda m_a, \lambda s_a, q_a; (\lambda m_\beta, \lambda s_\beta, q_\beta)_{a, \beta = 1, \ldots, M} \right) G_{M} \left( \frac{x \lambda}{2}, m, s \right). \] (36)

To demonstrate how the summations over \( \{m_a\} \) and \( \{s_a\} \) are performed in the limit \( \lambda \to \infty \) let us consider the example of a general type:
\[ R(y) = \lim_{\lambda \to \infty} \prod_{a=1}^{M} \left[ \sum_{n_a \geq 1} (-1)^{n_a - 1} \exp(\lambda n_a y_a) \right] \Phi(\lambda; n_1, \ldots, n_M) \] (37)
where \( \Phi \) is a function which depends both on \( \lambda \) and on all summation parameters \( \{n_1, \ldots, n_M\} \). The above summations can be represented in terms of the integrals in the complex plane:
\[ R(y) = \lim_{\lambda \to \infty} \prod_{a=1}^{M} \left[ \frac{1}{2\pi i} \int_{C} \frac{dz_a}{\sin(\pi z_a)} \exp(\lambda z_a y_a) \right] \Phi(\lambda; z_1, \ldots, z_M) \] (38)
where the integration goes over the contour \( C \) shown in figure 1. Redefining \( z_a \to z_a/\lambda \), in the limit \( \lambda \to \infty \) we get:
\[ R(y) = \prod_{a=1}^{M} \left[ \frac{1}{2\pi i} \int_{C} \frac{dz_a}{z_a} \exp(z_a y_a) \right] \lim_{\lambda \to \infty} \Phi(\lambda; z_1/\lambda, \ldots, z_M/\lambda) \] (39)
where the parameters \( y_a \) and \( z_a \) remain finite in the limit \( \lambda \to \infty \).
The double summations over \(m_\alpha\) and \(s_\alpha\) in equation (36) can be represented as follows:

\[
\sum_{m_\alpha + s_\alpha \geq 1} (-1)^{m_\alpha + s_\alpha - 1} = \sum_{s_\alpha = 0}^{\infty} \delta_{s_\alpha,0} \sum_{m_\alpha = 1}^{\infty} (-1)^{m_\alpha - 1} + \sum_{m_\alpha = 0}^{\infty} \sum_{s_\alpha = 1}^{\infty} (-1)^{s_\alpha - 1}
\]

Thus in the integral representation, equations (37)–(39), for the function in equation (36), we get

\[
S_M(q, y, \xi_1, f_1, f_2; x) = \prod_{\alpha = 1}^{M} \left[ \int_C \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\xi_1 \alpha d\xi_2 \alpha d\xi_3 \alpha \right] (2\pi)^{3/2} Ai(\xi_1 \alpha + \xi_2 \alpha \sqrt{x}) \exp\left\{-\frac{1}{2} \left(\frac{\xi_3 \alpha}{\sqrt{x}}\right)^2 - \frac{1}{2} \left(\frac{\xi_2 \alpha}{\sqrt{x}}\right)^2 - \frac{1}{2} \xi_1 \alpha^2\right\}
\]

Using the explicit form of the factor \(G_M\), equation (26), and taking into account the gamma function property \(\lim_{|z| \to 0} \Gamma(1 + z) = 1\), we find

\[
\lim_{\lambda \to \infty} G_M\left(\frac{\kappa q}{2\lambda}, \frac{z_1}{\lambda}, \frac{z_2}{\lambda}\right) = 1.
\]
In the exponential representation of this determinant we get

\[ \text{Tr} \hat{\mathcal{M}} = \exp \left[ - \sum_{M=1}^{M=M} \frac{1}{M} \text{Tr} \hat{\mathcal{M}} \right] \quad (46) \]

where

\[
\text{Tr} \hat{\mathcal{M}} = \prod_{\alpha=1}^{M} \int_{-\infty}^{+\infty} \frac{dy}{2\pi} \int_{-\infty}^{+\infty} \frac{d\xi_1}{2\pi} \int_{-\infty}^{+\infty} \frac{d\xi_2}{2\pi} \int_{-\infty}^{+\infty} \frac{d\xi_3}{2\pi} \text{Ai}(y + q_a^2 - i\sqrt{q} \xi) \times \exp \left\{ - \frac{1}{2} \left( \xi_1 + \frac{i}{2} \sqrt{q} \xi \right)^2 - \frac{1}{2} \left( \xi_2 - \frac{i}{2} \sqrt{q} \xi \right)^2 - \frac{1}{2} \xi_3^2 \right\} \]

\[
\times \int_{\mathcal{C}} \frac{dz_1 a dz_2 a}{(2\pi i)^2} \left( \frac{2\pi i}{z_1 a} \delta(z_2 a) + \frac{2\pi i}{z_2 a} \delta(z_1 a) - \frac{1}{z_1 a z_2 a} \right) \times \exp \{z_1 a (y + f_1 + \xi_1 \sqrt{x}) + z_2 a (y + f_2 + \xi_2 \sqrt{x}) \} \]

\[
\times \prod_{\alpha=1}^{M} \left[ \frac{1}{z_{1\alpha} + z_{2\alpha} - i q_a + z_{1\alpha+1} + z_{2\alpha+1} + i q_{a+1}} \right] . \quad (47) \]

Here, by definition, it is assumed that \( z_{a+1} \equiv z_i (i = 1, 2) \) and \( q_{M+1} \equiv q_1 \). Substituting

\[
z_{1\alpha} + z_{2\alpha} - i q_a + z_{1\alpha+1} + z_{2\alpha+1} + i q_{a+1} \]

\[
= \int_0^\infty dw_a \exp[-(z_{1\alpha} + z_{2\alpha} - i q_a + z_{1\alpha+1} + z_{2\alpha+1} + i q_{a+1}) \omega_a] \quad (48) \]

into equation (47), we obtain

\[
\text{Tr} \hat{\mathcal{M}} = \int_0^\infty dw_{1} \ldots dw_{M} \prod_{\alpha=1}^{M} \Lambda(\omega_a, \omega_{a+1})] \quad (49) \]
where
\[
A(\omega, \omega') = \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{dq_1 dq_2 dq_3}{(2\pi)^{3/2}} \text{Ai}(y + q^2 + \omega + \omega' - i\xi_3 \sqrt{x}) \\
\times \exp \left\{ \left\{ -\frac{1}{2} \left( \xi_1 + \frac{1}{2} i q \sqrt{x} \right)^2 - \frac{1}{2} \left( \xi_2 - \frac{1}{2} i q \sqrt{x} \right)^2 \right\} - \frac{1}{2} \xi_3^2 - i q (\omega - \omega') \right\} \\
\times \exp \left\{ \int_{c} \frac{dz_1 dz_2}{(2\pi i)^2} \frac{2\pi i z_1}{z_1} \delta(z_1) + \frac{2\pi i z_2}{z_2} \delta(z_2) - \frac{1}{z_1 z_2} \right\} \\
\times \exp \{z_1 (y + f_1 + \xi_1 \sqrt{x}) + z_2 (y + f_2 + \xi_2 \sqrt{x}) \}. \quad (50)
\]

Integrating over \(z_1\) and \(z_2\) we get:
\[
\begin{align*}
A(\omega, \omega') &= \int_{0}^{+\infty} \frac{dy}{2\pi} \frac{dq_1 dq_2}{(2\pi)^{3/2}} \text{Ai} \left( y + q^2 - f_1 + \omega + \omega' + \frac{1}{2} i q x \right) \exp \left\{ -i q (\omega - \omega') \right\} \\
&\quad + \int_{0}^{+\infty} \frac{dy}{2\pi} \frac{dq_1 dq_2}{(2\pi)^{3/2}} \text{Ai} \left( y + q^2 - f_2 + \omega + \omega' - \frac{1}{2} i q x \right) \exp \left\{ -i q (\omega - \omega') \right\} \\
&\quad - \int_{-\infty}^{+\infty} \frac{dy}{2\pi} \frac{dq_1 dq_2}{(2\pi)^{3/2}} \text{Ai} \left( y + q^2 + \omega + \omega' - i\xi_3 \sqrt{x} \right) \exp \left\{ -\frac{1}{2} \left( \xi_1 + \frac{1}{2} i q \sqrt{x} \right)^2 - \frac{1}{2} \left( \xi_2 - \frac{1}{2} i q \sqrt{x} \right)^2 \right\} - \frac{1}{2} \xi_3^2 - i q (\omega - \omega') \right\} \\
&\quad \times \theta (y + f_1 + \xi_1 \sqrt{x}) \theta (y + f_2 + \xi_2 \sqrt{x}) \quad (51)
\end{align*}
\]

where \(\theta(y)\) is the step function. Redefining \(\xi_1 = (t - \eta)/\sqrt{2}, \xi_2 = (t + \eta)/\sqrt{2}, \xi_3 = (\iota + \zeta)/\sqrt{2}\), and integrating over \(q, t\) and \(\xi\), we find the following result:
\[
A(\omega, \omega') = 2^{1/3} K[2^{1/3} (\omega - \tilde{f}_1), 2^{1/3} (\omega' - \tilde{f}_1)] \exp \left\{ -\frac{1}{4} (\omega - \omega') x \right\} \\
+ 2^{1/3} K[2^{1/3} (\omega - \tilde{f}_2), 2^{1/3} (\omega' - \tilde{f}_2)] \exp \left\{ -\frac{1}{4} (\omega - \omega') x \right\} \\
- 2^{1/3} \int_{-\infty}^{+\infty} \frac{dy}{\sqrt{2\pi}} \frac{dq}{\sqrt{2\pi}} \text{Ai} \left( 2^{1/3} \left( y + \omega - \eta \sqrt{\frac{x}{8}} \right) \right) \exp \left\{ -\frac{1}{2} \eta^2 - \frac{1}{2} xy - \frac{1}{4} (\omega + \omega') + \frac{1}{3} \left( \frac{x}{4} \right)^3 \right\} \\
\times \theta \left( y + \tilde{f}_1 - \eta \sqrt{\frac{x}{8}} \right) \theta \left( y + \tilde{f}_2 + \eta \sqrt{\frac{x}{8}} \right) \quad (52)
\]

where \(\tilde{f}_1, \tilde{f}_2 = \frac{1}{4} (f_1, f_2 - x^2/16)\) and \(K(\omega, \omega') = \int_{0}^{+\infty} \frac{dy}{\sqrt{2\pi}} \text{Ai}(y + \omega) \text{Ai}(y + \omega')\) is the Airy kernel.

Thus the distribution function \(W(f_1, f_2; x)\), equation (7), is given the Fredholm determinant
\[
W(f_1, f_2; x) = \det[1 - \hat{A}] \quad (53)
\]

where \(\hat{A}\) is the integral operator with the kernel \(A(\omega, \omega')(\omega, \omega' \geq 0)\) given in equation (52).

Note that using the explicit expression (52) one can easily test the obtained result for three limit cases:
\[
\lim_{f_1 \to -\infty} A(\omega, \omega') = 2^{1/3} K[2^{1/3} (\omega - \tilde{f}_2), 2^{1/3} (\omega' - \tilde{f}_2)] \exp \left\{ -\frac{1}{4} (\omega - \omega') x \right\} \quad (54)
\]
\[
\lim_{f_2 \to -\infty} A(\omega, \omega') = 2^{1/3} K[2^{1/3} (\omega - \tilde{f}_1), 2^{1/3} (\omega' - \tilde{f}_1)] \exp \left\{ -\frac{1}{4} (\omega - \omega') x \right\} \quad (55)
\]
\[
\lim_{x \to 0} A(\omega, \omega') = 2^{1/3} K [2^{1/3}(\omega - f_1/2), 2^{1/3}(\omega' - f_1/2)] \theta(f_1 - f_2) + 2^{1/3} K [2^{1/3}(\omega - f_2/2), 2^{1/3}(\omega' - f_2/2)] \theta(f_2 - f_1)
\]

which demonstrate that in the case \( f_{1,2} \to -\infty \) we recover the usual GUE TW distribution for \( f_{2,1} \) correspondingly, while in the limit case \( x \to 0 \) we find the usual GUE TW distribution for \( f_1 \) (in the case \( f_1 > f_2 \)) and for \( f_2 \) (in the case \( f_2 > f_1 \)), as should be the case.

3. Conclusions

In view of the recent proof [26] that the result of the present calculations is equivalent to that obtained earlier by Prolhac and Spohn [25] we can conclude that the Bethe ansatz replica technique has demonstrated (once again) that it is the efficiency and robustness which allows one to perform computations of sufficiently complicated objects in a rather simple way.

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