GENERICITY OF WELL-POSED VECTOR OPTIMIZATION
PROBLEMS

MATTEO ROCCA

ABSTRACT. In this paper we consider well-posedness properties of vector optimization problems with objective function \( f : X \rightarrow Y \) where \( X \) and \( Y \) are Banach spaces and \( Y \) is partially ordered by a closed convex pointed cone with nonempty interior. The vector well-posedness notion considered in this paper is the one due to Dentcheva and Helbig \[5\], which is a natural extension of Tykhonov well-posedness for scalar optimization problems. When a scalar optimization problem is considered it is possible to prove (see e.g. \[22\]) that under some assumptions the set of functions for which the related optimization problem is well-posed is dense or even more in "big" i.e. contains a dense \( G_\delta \) set (these results are called genericity results). The aim of this paper is to extend these genericity results to vector optimization problems.

1. Introduction

Well-posedness properties are important qualitative characterizations for scalar and vector optimization problems. In particular, the notion of well-posedness plays a central role in stability theory for scalar optimization (see e.g. \[5\]). The well-posedness notion for scalar functions dates back to Hadamard \[12\] and to Tykhonov in \[27\]. Extensions to vector and set-valued cases are presented in several papers and are still a topic of research (see e.g. \[1\], \[2\], \[20\], \[5\], \[6\], \[22\], \[24\], \[15\], \[16\], \[11\], \[4\]). In \[24\] a classification of vector well-posedness notions in two classes is given: pointwise and global notions. The definitions of the first group consider a fixed efficient point (or the image of an efficient point) and deal with well-posedness of the vector optimization problem at this point. This approach imposes that the minimizing sequences related to the considered point are well-behaved. Since in the vector case the solution set is typically not a singleton, there is also a class of definitions, the so-called global notions, that involve the efficient frontier as a whole.

In the scalar optimization, a crucial point is the identification of classes of objective functions for which the related optimization problem enjoys well-posedness properties. It is known (see e.g. \[6\]) that scalar optimization problems with convex objective function \( f : X \rightarrow \mathbb{R} \), \( X \) finite-dimensional,
enjoy well-posedness properties. Similarly, it is known that vector optimization problems with cone-convex objective function $f : X \to Y$ with $X$ and $Y$ finite-dimensional, enjoy well-posedness properties (see e.g. \cite{24}).

When functions $f : X \to \mathbb{R}$ with $X$ infinite-dimensional are considered, it is known that convexity does not guarantee well-posedness (see e.g. \cite{6}). In this case it is interesting to find sets of functions for which the subset of well-posed functions is dense (when a suitable topology on the considered set of functions is introduced). A stronger version of these results leads to find sets of functions for which the subset of well-posed functions is "big" in the sense of Baire category, i.e. contains a dense $G_δ$ set (see e.g. \cite{17}, \cite{22} and references therein).

The aim of this paper is to extend this kind of results, called genericity results, to vector optimization problems with objective function $f : X \to Y$ where $X$ and $Y$ are Banach spaces. In our investigation we will focus on the pointwise well-posedness notion for vector functions due to Dentcheva and Helbig \cite{5}.

The outline of the paper is the following. In Section 2 we introduce the notation and we recall some preliminary notions. In Section 3 we recall some scalar and vector well-posedness notions. In Section 4 we give results concerning density of well-posed vector optimization problems, without convexity assumptions. Section 5 is devoted to genericity results under cone-convexity assumptions.

2. Preliminaries

Let $X$ and $Y$ be Banach spaces, $f : X \to Y$ and $C \subseteq Y$ a closed, convex, pointed cone with nonempty interior, endowing $Y$ with a partial order in the following way

\begin{align}
 y_1 \leq_C y_2 & \iff y_2 - y_1 \in C \\
 y_1 <_C y_2 & \iff y_2 - y_1 \in \text{int } C
\end{align}

(2.1)

In the following for a set $A \subseteq X$ we denote by $\text{diam } A$ the diameter of $A$, i.e.

$$\text{diam } A = \sup \{ \| x - y \| : x, y \in A \}$$

We denote respectively by $B$ the closed unit ball both in $X$ and $Y$ (from the context it will be clear to which space we refer).

We denote by $Y^*$ the topological dual space of $Y$ and by $C^*$ the positive polar cone of $C$, i.e.

$$C^* = \{ \xi \in Y^* : \langle \xi, c \rangle \geq 0, \forall c \in C \}$$

Consider the vector optimization problem

$$(X, f) \min f(x), \ x \in X.$$
A point $\bar{x} \in X$ is called an efficient solution for problem $(X, f)$ when
\[(f(\bar{x}) - f(x)) \cap (-C) = \{0\}\]
We denote by $\text{Eff}(X, f)$ the set of all efficient solutions for problem $(X, f)$. A point $\bar{x} \in X$ is called a weakly efficient solution for problem $(X, f)$ when
\[(f(\bar{x}) - f(x)) \cap (-\text{int } C) = \emptyset.\]
We denote by $\text{WEff}(X, f)$ the set of all weakly efficient solutions for problem $(X, f)$. We recall also (see e.g.\[3\]) that a point $\bar{x} \in X$ is said to be a strictly efficient solution for problem $(X, f)$ when, for every $\varepsilon > 0$, there exists $\delta > 0$ such that
\[(f(\bar{x}) - f(x)) \cap (\delta B - C) \subseteq \varepsilon B.\]
We denote by $\text{StEff}(X, f)$ the set of all strictly efficient solutions for problem $(X, f)$. Clearly $\text{StEff}(X, f) \subseteq \text{Eff}(X, f) \subseteq \text{WEff}(X, f)$.

**Definition 2.1.** \[21\] A function $f : X \to Y$, is said to be $C$-convex if $\forall x, z \in X$ and $t \in [0, 1]$ it holds
\[f(tx + (1 - t)z) \in tf(x) + (1 - t)f(z) - C\]

**Proposition 2.2.** (see e.g. \[21\]) $f : X \to Y$ is $C$-convex if and only if functions $g_{\xi}(x) = \langle \xi, f(x) \rangle$ are convex for every $\xi \in C^*$. We recall also that a function $f : X \to Y$ is said to be $\ast$-quasiconvex when functions $g_{\xi}(x) = \langle \xi, f(x) \rangle$ are quasiconvex for every $\xi \in C^*$ (see e.g. \[19\]).

For $y \in Y$, $L_f^{C}(y) := \{ x \in X : f(x) \in y - C \}$ is the level set of $f$. We say that $f : X \to Y$ is $C$-lower semicontinuous ($C$-lsc for short) when $L_f^{C}(y)$ is closed for every $y \in Y$ \[21\].

Now, we recall, the notion of oriented distance between a point $y \in Y$ and a set $A \subseteq Y$, denoted by $D_A(y)$.

**Definition 2.3.** For a set $A \subseteq Y$ the oriented distance is the function $D_A : Y \to \mathbb{R} \cup \{ \pm \infty \}$ defined as
\[(2.3) \quad D_A(y) = d_A(y) - d_{Y \setminus A}(y)\]
with $d_0(y) = +\infty$.

Function $D_A$ was introduced in \[13, 14\] to analyze the geometry of non-smooth optimization problems and obtain necessary optimality conditions. The next result summarizes some basic properties of function $D_A$.

**Proposition 2.4.** \[28, 8\] If the set $A$ is nonempty and $A \neq Y$, then
1. $D_A$ is real valued;
2. $D_A$ is 1-Lipschitzian;
3. $D_A(y) < 0$ for every $y \in \text{int } A$, $D_A(y) = 0$ for every $y \in \partial A$ and $D_A(y) > 0$ for every $y \in \text{int } (Y \setminus A)$;
4. if $A$ is closed, then it holds $A = \{ y : D_A(y) \leq 0 \}$.
5. If $A$ is convex, then $D_A$ is convex;
6. If $A$ is a cone, then $D_A$ is positively homogeneous;
7. If $A$ is a closed convex cone, then $D_A$ is nonincreasing with respect to the ordering relation induced on $Y$, i.e. the following is true: if $y_1, y_2 \in Y$ then $y_1 - y_2 \in A \Rightarrow D_A(y_1) \leq D_A(y_2)$; if $A$ has nonempty interior, then $y_1 - y_2 \in \text{int } A \Rightarrow D_A(y_1) < D_A(y_2)$;
8. It holds
   \begin{equation}
   (2.4) \quad D_A(y) = \max_{\xi \in C^* \cap \partial B} \langle \xi, y \rangle
   \end{equation}
   where $\partial A$ denotes the boundary of the set $A$.

**Theorem 2.5.** [24] If $f : X \to Y$ is $C$-convex, then for every $y \in Y$, function $D_{-C}(f(x) - y)$ is convex.

We associate to problem $(X, f)$ the scalar problem

$$(X, D_{-C}) \quad \min D_{-C}(f(x) - f(\bar{x})) , \, x \in X$$

with $\bar{x} \in X$. The relations of the solutions of this problem with those of problem $(X, f)$ are investigated in [28, 24, 8]. For the convenience of the reader, we quote the characterization of efficient points and weakly efficient points.

**Theorem 2.6.** [28, 24, 8] Let $f : X \to Y$.

1. $\bar{x} \in \text{WEff } (X, f)$ if and only if $\bar{x}$ is a solution of problem $(X, D_{-C})$.
2. If $\bar{x}$ is the unique solution of problem $(X, D_{-C})$, then $\bar{x} \in \text{Eff } (X, f)$.

3. Well-posedness for scalar and vector optimization problems

3.1. **Well-posedness for scalar optimization problems.** In this section we recall the notion of well-posedness for functions $f : X \to \mathbb{R}$ introduced by Tykhonov [27]. For a complete treatment of this notion and of its generalizations one can refer to [8, 22]. Clearly in this case problem $(X, f)$ reduces to a scalar minimization problem.

**Definition 3.1.** Let $f : X \to \mathbb{R}$. Problem $(X, f)$ is said to be Tykhonov well-posed (T-wp for short) if:

1. there exists a unique $\bar{x} \in X$ such that $f(\bar{x}) \leq f(x)$, $\forall x \in X$;
2. every sequence $x_n$ such that $f(x_n) \to \inf_X f$ is such that $x_n \to \bar{x}$.

Next proposition provides a useful characterization of Tykhonov well-posedness. It is called the Furi-Vignoli criterion [7].

**Proposition 3.2.** Let $f : X \to \mathbb{R}$ be lsc. The following alternatives are equivalent:

1. Problem $(X, f)$ is T-wp;
2. $\inf_{a \geq \inf_X f} \text{diam } L_f(a) = 0$, where $L_f(a) = \{ x \in X : f(x) \leq a \}$. 
The following result regarding well-posedness of convex functions defined on a finite-dimensional space is well-known.

**Theorem 3.3.** (see e.g. [6]) Let $X$ be finite-dimensional and $f : X \rightarrow \mathbb{R}$ be a convex function with a unique minimizer. Then problem $(X, f)$ is T-wp.

Theorem 3.3 does not hold when $X$ is infinite-dimensional as the following example shows (see e.g. [6]).

**Example 3.4.** Let $X$ be a separable Hilbert space with orthonormal basis $\{e_n, n \in \mathbb{N}\}$. Let $f(x) = \sum_{n=1}^{+\infty} \frac{(x,e_n)^2}{n^2}$. Then $f$ is continuous, convex and has $\bar{x} = 0$ as unique minimizer, but problem $(X, f)$ is not T-wp. Indeed the sequence $\sqrt{n}e_n$ is an unbounded minimizing sequence.

Consider now the space

$$\Gamma := \{ f : X \rightarrow \mathbb{R} : f \text{ is convex and lower semicontinuous} \}.$$ 

We endow $\Gamma$ with a distance compatible with the uniform convergence on bounded sets. Fix $\theta \in X$ and set for any two functions $f, g \in \Gamma$ and $i \in \mathbb{N}$,

$$\| f - g \|_i = \sup_{\| x - \theta \| \leq i} |f(x) - g(x)|.$$ 

If $\| f - g \|_i = \infty$ for some $i$, then set $d(f, g) = 1$, otherwise

$$d(f, g) = \sum_{i=1}^{\infty} 2^{-i} \frac{\| f - g \|_i}{1 + \| f - g \|_i}.$$ 

When $X$ is infinite-dimensional, it can be shown that the set of functions $f \in \Gamma$ such that problem $(X, f)$ is T-wp is "big" in the sense that contains a dense $G_δ$ set (see e.g. [22]).

**Theorem 3.5.** [22] Let $X$ be a Banach space and consider the set $\Gamma$, equipped with the topology of uniform convergence. Then the set of functions $f \in \Gamma$ such that problem $(X, f)$ is T-wp contains a dense $G_δ$ set.

If the convexity assumption is dropped weaker variants of Theorem 3.5 hold, in which density of the class of functions $f \in \Gamma$ such that problem $(X, f)$ is T-wp is proven. Next results (see e.g. [22]) will be useful in the following.

**Proposition 3.6.** Let $f : X \rightarrow \mathbb{R}$, assume $f$ has a minimum point $\bar{x} \in X$ and let $g(x) = f(x) + a\|x - \bar{x}\|$ with $a > 0$. Then problem $(X, g)$ is T-wp.

**Theorem 3.7.** (Ekeland’s Variational Principle) Let $f : X \rightarrow \mathbb{R}$ be a lsc, lower bounded function. Let $\varepsilon > 0, r > 0$ and $\bar{x} \in X$ be such that $f(\bar{x}) < \inf_X f + r\varepsilon$. Then, there exists $\hat{x} \in X$ enjoying the following properties:

1. $\| \hat{x} - \bar{x} \| < r$;
2. $f(\hat{x}) < f(\bar{x}) - \varepsilon\|\bar{x} - \hat{x}\|$;
3. Problem $(X, g)$ with $g(x) = f(x) + \varepsilon\|\hat{x} - x\|$ is T-wp.
3.2. Well-posedness for vector optimization problems. Several generalizations of the notion of well-posedness to vector functions have been proposed. We refer to [24] for a survey on the topic and a study of the relations among different well-posedness concepts. In that paper vector well-posedness notions have been divided in two classes: pointwise and global notions. Notions in the first class define the well-posedness of a vector problem with respect to a fixed efficient solution, while in the global notions the set of efficient solutions or weakly efficient solutions is considered as a whole.

In this paper we focus on the notion of well-posedness due to Dentcheva and Helbig [5] (DH-well-posedness) which is a pointwise notion according to [24].

**Definition 3.8.** Let \( f : X \to \mathbb{R} \). Problem \((X, f)\) is said to be DH-well-posed (DH-wp for short) at \( \bar{x} \in \text{Eff}(X, f) \) if

\[
\inf_{\alpha > 0} \text{diam} L^C_f(f(\bar{x}) + \alpha c) = 0, \quad \forall c \in C,
\]

where \( L^C_f(f(\bar{x}) + \alpha c) = \{ x \in X : f(x) \in f(\bar{x}) + \alpha c - C \} \).

In [24] it has been proven that DH-well-posedness is the strongest among the pointwise well-posedness notions, that is if problem \((X, f)\) is DH-wp at \( \bar{x} \in X \) then it is well-posed at \( \bar{x} \) according to the other pointwise well-posedness notions known in the literature. The next result gives a useful characterization of DH-well-posedness.

**Theorem 3.9.** [11, 24] Problem \((X, f)\) is DH-well-posed at \( \bar{x} \in \text{Eff}(X, f) \) if and only if problem \((X, D-C)\) is T-wp.

The following theorem (see [24]) gives a generalization of Theorem 3.3.

**Theorem 3.10.** Let \( X \) and \( Y \) be finite-dimensional. Assume \( f : X \to Y \) is a \( C \)-convex function, \( \bar{x} \in \text{Eff}(X, f) \) and \( f^{-1}(f(\bar{x})) = \{ \bar{x} \} \). Then problem \((X, f)\) is DH-wp at \( \bar{x} \).

DH-well-posedness imposes some restrictions on the set \( \text{Eff}(X, f) \). Indeed, if problem \((X, f)\) is DH-well-posed at \( \bar{x} \in \text{Eff}(X, f) \) then \( \bar{x} \in \text{StEff}(X, f) \). This property is typical of the vector case and shows that most of the vector well-posedness notions require implicitly stronger properties than the simple good behavior of minimizing sequence

**Theorem 3.11.** [24] If \( F : X \to Y \) is continuous and problem \((X, f)\) is DH-wp at \( \bar{x} \in \text{Eff}(X, f) \), then \( \bar{x} \in \text{StMin}(X, f) \).

4. Density of DH-well-posed functions

The first result in this section shows that if the set of functions

\[
\mathcal{H} = \{ f : X \to Y : \text{Eff}(X, f) \neq \emptyset \}
\]

is endowed with the topology of uniform convergence on bounded sets, then the set of functions \( g \in \mathcal{H} \) enjoying DH-wp properties is dense in \( \mathcal{H} \).
Theorem 4.1. Let $f \in \mathcal{H}$. Then, for every $\bar{x} \in \text{Eff}(X,f)$, there exists a sequence of functions $f_n : X \to Y$ such that $f_n \to f$ in the uniform convergence on bounded sets, $\bar{x} \in \text{Eff}(X,f_n)$ for every $n$ and problem $(X,f_n)$ is DH-up at $\bar{x}$. Further, if $f$ is continuous then $\bar{x} \in \text{StEff}(X,f_n)$ for every $n$.

Proof. Let $k^0 \in \text{int} C$ be fixed and consider the sequence of functions

$$f_n(x) = f(x) + \frac{1}{n}\|x - \bar{x}\|k^0$$

Since $\bar{x} \in \text{Eff}(X,f)$, it holds

(4.1) $$f(x) - f(\bar{x}) \not\in -C, \forall x \in X, x \neq \bar{x}$$

Hence

$$f(x) - f_n(\bar{x}) = f(x) - f(\bar{x}) + \frac{1}{n}\|x - \bar{x}\|k^0 \not\in -C, \forall x \in X, x \neq \bar{x}$$

since [24] holds. Hence $\bar{x} \in \text{Eff}(X,f_n) \forall n$. Since $\text{Eff}(X,f_n) \subseteq \text{WEff}(X,f_n)$, Theorem 2.6 implies $D_C(f_n(x) - f_n(\bar{x})) \geq 0$ for every $x \in X$. Now we prove problem $(X,f_n)$ is DH-wp at $\bar{x}$ for every $n$. From Theorem 3.9 we know that problem $(X,f)$ is DH-wp at $\bar{x}$ if and only if the scalar problem $(X,D_{\cdot C})$ is T-wp at $\bar{x}$. Since $\text{int} C \neq \emptyset$, $C^*$ has a closed convex weak*-compact base

(4.2) $$G = \{ \xi \in C^*: \langle \xi, k^0 \rangle = 1 \}$$

(see e.g. [18]). According to [24] there exists a constant $\alpha > 0$ such that

$$D_{\cdot C}(f_n(x) - f_n(\bar{x})) \geq \alpha \max_{\xi \in G} \langle \xi, f_n(x) - f_n(\bar{x}) \rangle$$

$$= \alpha \max_{\xi \in G} \langle \xi, f(x) - f(\bar{x}) + \frac{1}{n}\|x - \bar{x}\|k^0 \rangle$$

$$= \alpha \max_{\xi \in G} \langle \xi, f(x) - f(\bar{x}) \rangle + \frac{1}{n}\|x - \bar{x}\|$$

For a fixed $n$, let $x_k$ be a minimizing sequence for $D_{\cdot C}(f_n(x) - f_n(\bar{x}))$, that is $D_{\cdot C}(f_n(x_k) - f_n(\bar{x})) \to 0$. If $x_k \not\to \bar{x}$ we get

$$D_{\cdot C}(f_n(x_k) - f_n(\bar{x})) \geq \alpha \max_{\xi \in G} \langle \xi, f(x_k) - f(\bar{x}) \rangle + \frac{1}{n}\|x_k - \bar{x}\|$$

$$\geq \inf_{k \in N} \frac{1}{n}\|x_k - \bar{x}\| > 0$$

which contradicts to $x_k$ minimizing sequence for $D_{\cdot C}(f_n(x) - f_n(\bar{x}))$ (the last inequality follows since $\bar{x} \in \text{Eff}(X,f)$ implies $\max_{\xi \in G} \langle \xi, f(x_k) - f(\bar{x}) \rangle \geq 0 \forall x \in X$). Hence $x_k \to \bar{x}$ and problem $(X,f_n)$ is DH-wp at $\bar{x}$. Finally, we get the desired result observing that $f_n \to f$ in the uniform convergence on bounded sets. If $f$ is continuous, then apply Theorem 3.11 to conclude the proof. \qed

To prove the second density result in this section we need the following definition and the next lemma.
Definition 4.2. \([11]\) We say that \(f : X \to Y\) is \(C\)-bounded from below by \(\xi \in C^*\setminus \{0\}\) when \(\inf_{x \in X} \langle \xi, f(x) \rangle > -\infty\).

Let \(\bar{x} \in \text{Eff}(X, f)\), consider function
\[
h_{\bar{\xi}}(x) = \langle \bar{\xi}, f(x) \rangle
\]
and the associated scalar minimization problem
\[
(X, h_{\bar{\xi}}) \quad \min h_{\bar{\xi}}(x), \ x \in X
\]

Lemma 4.3. Let \(\bar{x} \in \text{Eff}(X, f)\) and \(\bar{\xi} \in C^*\setminus \{0\}\). If problem \((X, h_{\bar{\xi}})\) is T-wp at \(\bar{x}\), then problem \((X, f)\) is DH-wp at \(\bar{x}\).

Proof. Without loss of generality let \(\bar{\xi} \in C^* \cap \partial B\). Assume problem \((X, f)\) is not DH-wp at \(\bar{x}\). Since \(\bar{x} \in \text{Eff}(X, f)\), by Theorem 2.6 it holds \(D_{-C}(f(x) - f(\bar{x})) \geq 0\), for every \(x \in X\) and by Theorem 3.9 problem \((X, D_{-C})\) is not T-wp. Then there exists a sequence \(x_n \in X\) such that \(D_{-C}(f(x_n) - f(\bar{x})) \to 0\) but \(x_n \not\to \bar{x}\). Since, by Proposition 2.4
\[
D_{-C}(f(x_n) - f(\bar{x})) = \max_{\xi \in C^* \cap \partial B} \langle \xi, f(x_n) - f(\bar{x}) \rangle
\]
it follows
\[
D_{-C}(f(x_n) - f(\bar{x})) \geq \langle \bar{\xi}, f(x_n) - f(\bar{x}) \rangle = h_{\bar{\xi}}(x_n) - h_{\bar{\xi}}(\bar{x})
\]
Since problem \((X, h_{\bar{\xi}})\) is T-wp at \(\bar{x}\), it follows that \(\bar{x}\) is a minimum point for \(h_{\bar{\xi}}\) and hence \(h_{\bar{\xi}}(x_n) - h_{\bar{\xi}}(\bar{x}) = \langle \xi, f(x) - f(\bar{x}) \rangle \geq 0 \ \forall n\). From \(D_{-C}(f(x_n) - f(\bar{x})) \to 0\) it follows \(h_{\bar{\xi}}(x_n) \to h_{\bar{\xi}}(\bar{x})\) which contradicts problem \((X, h_{\bar{\xi}})\) is T-wp since \(x_n \not\to \bar{x}\). \(\square\)

In the next result we drop the assumption \(\text{Eff}(x, f) \neq \emptyset\) and we show that if the set of functions
\[
\mathcal{H}' = \{ f : X \to Y : \exists \xi \in C^* \text{ such that } f \text{ is } C^- \text{ bounded from below by } \xi \}
\]
is endowed with the topology of uniform convergence on bounded sets, then the set of functions \(g \in \mathcal{H}'\) enjoying DH-wp properties is dense in \(\mathcal{H}'\). We endow \(\mathcal{H}'\) with a distance compatible with the uniform convergence on bounded sets (see e.g. \([22]\)). Fix \(\theta \in X\) and set for any two functions \(f, g \in \mathcal{H}'\) and \(i \in \mathbb{N}\),
\[
\|f - g\|_i = \sup_{\|x - \theta\| \leq i} \|f(x) - g(x)\|.
\]
If \(\|f - g\|_i = \infty\) for some \(i\), then set \(d(f, g) = 1\), otherwise
\[
d(f, g) = \sum_{i=1}^{\infty} 2^{-i} \frac{\|f - g\|_i}{1 + \|f - g\|_i}.
\]

Theorem 4.4. Assume there exists \(\bar{\xi} \in C^*\setminus \{0\}\) such that \(f : X \to Y\) is \(C\)-bounded from below by \(\bar{\xi}\) and \(\langle \bar{\xi}, f(x) \rangle\) is lsc with respect to \(x \in X\). Then, there exists a sequence of functions \(f_n : X \to Y\) uniformly converging to
f on the bounded sets, such that $\text{Eff}(X, f_n) \neq \emptyset$ for every $n$ and problem $(X, f_n)$ is DH-wp at some $\bar{x}_n \in \text{Eff}(X, f_n)$.

**Proof.** Fix $\sigma > 0$ and take $j$ so large that setting $g(x) = f(x) + \frac{1}{j} \|x - \theta\|^k$ with $h^0 \in \text{int} C$ such that $\langle \xi, h^0 \rangle = 1$, it holds $d(f, g) < \frac{\sigma}{j}$. Now set

(4.4) \[ g_\xi(x) = \langle \xi, g(x) \rangle = \langle \xi, f(x) \rangle + \frac{1}{j} \|x - \theta\|^k \]

and observe that $g_\xi(x)$ is lower bounded by Definition 4.2. Hence, we can find $M > 0$ such that

$$\{x \in X : g_\xi(x) \leq \inf_{x \in X} g_\xi(x) + 1\} \subseteq B(\theta, M)$$

Let $s = \sum_{k=0}^{\infty} \frac{1}{2^k} (k + M)\|h^0\|$ and apply Theorem 3.7 with $\epsilon = \frac{\sigma}{2^s}$ and arbitrary $r$ to find a point $\bar{x} = \bar{x}_\sigma \in X$ such that $\|\bar{x} - \theta\| \leq M$, $\bar{x}$ is the unique minimizer of

$$h_\xi(x) = \langle \xi, g(x) \rangle + \epsilon \|x - \bar{x}\|$$

and problem $(X, h_\xi)$ is T-wp at $\bar{x}$. Let

$$h(x) = g(x) + \epsilon \|x - \bar{x}\| h^0$$

and observe that since $\bar{x}$ minimizes $h_\xi(x)$, it holds

$$h_\xi(x) - h_\xi(\bar{x}) = \langle \xi, h(x) - h(\bar{x}) \rangle > 0, \forall x \in X \setminus \{\bar{x}\}$$

which implies

$$D_{-C}(h(x) - h(\bar{x})) = \max_{\xi \in C, h(x) - \partial B} \langle \xi, h(x) - h(\bar{x}) \rangle \geq 0, \forall x \in X \setminus \{\bar{x}\}$$

Hence, Theorem 2.6 implies $\bar{x} \in \text{Eff}(X, h)$. Combining Theorem 2.6 and Lemma 4.3 we obtain that problem $(X, h)$

$$\min h(x), \ x \in X$$

is DH-wp at $\bar{x}$. Now observe that

$$\|h(x) - g(x)\| i \leq \epsilon \|k^0\| (i + M)$$

It follows $d(h, g) \leq \epsilon s = \frac{\sigma}{2}$ and then $d(f, h) < \sigma$. Take now $\sigma = \frac{1}{n}, n = 1, 2, \ldots$ and set $\bar{x}_n = \bar{x}_\sigma$ to complete the proof. \qed

The next result shows that under some hypotheses, the assumption in Theorem 4.4 is weaker than the assumption in Theorem 4.1. We recall the following fundamental result.

**Theorem 4.5. (Sion’s Minimax Theorem [25, 26])** Let $Z$ be a compact convex subset of a linear topological space and $W$ a convex subset of a linear topological space. Let $g$ be a real-valued function on $Z \times W$ such that

i) $g(\cdot, w)$ is upper semicontinuous and quasi-concave on $Z \forall w \in W$;

ii) $g(z, \cdot)$ is lower semicontinuous and quasi-convex on $W \forall z \in Z$. 
Then
\[ \sup_{z \in Z} \inf_{w \in W} f(z, w) = \inf_{w \in W} \sup_{z \in Z} f(z, w) \]

**Proposition 4.6.** Let \( f : X \to Y \) be \( * \)-quasiconvex and \( C \)-lsc with respect to \( x \in X \), for every \( \xi \in C^* \). Then, if \( \text{Eff}(X, f) \neq \emptyset \), there exists \( \xi \in C^* \setminus \{0\} \) such that \( f \) is \( C \)-bounded from below by \( \xi \).

**Proof.** Assume \( \text{Eff}(X, f) \neq \emptyset \) and let \( \bar{x} \in \text{Eff}(X, f) \). Ab absurdum assume that for every \( \xi \in C^* \setminus \{0\} \) it holds
\[
\inf_{x \in X} \langle \xi, f(x) \rangle = \inf_{x \in X} \langle \xi, f(x) - f(\bar{x}) \rangle = -\infty
\]
Since \( \text{int} C \neq \emptyset \), \( C^* \) has a weak*-compact base \( G \). Function \( g(\xi, x) = \langle \xi, f(x) - f(\bar{x}) \rangle \), \( \xi \in G \), \( x \in X \), is linear and continuous with respect to \( \xi \) and quasiconvex with respect to \( x \). Further, since \( f \) is \( C \)-lsc with respect to \( x \in X \), \( g(\xi, x) \) is lsc with respect to \( x \in X \). Since \( \bar{x} \in \text{Eff}(X, f) \), it holds \( \max_{\xi \in G} g(\xi, f(x) - f(\bar{x})) \geq 0 \) for every \( x \in X \). Apply Sion’s Minimax Theorem to get the following chain of equalities
\[
-\infty = \sup_{\xi \in G} \inf_{x \in X} \langle \xi, f(x) - f(\bar{x}) \rangle = \inf_{x \in X} \sup_{\xi \in G} \langle \xi, f(x) - f(\bar{x}) \rangle
\]
which implies there exists \( \tilde{x} \in X \) such that \( \sup_{\xi \in G} \langle \xi, f(\tilde{x}) - f(\bar{x}) \rangle < 0 \). A contradiction to \( \bar{x} \in \text{Eff}(X, f) \). \( \square \)

Generalized convexity assumptions in the previous result cannot be removed as the following example shows.

**Example 4.7.** Let \( X = \mathbb{R} \), \( Y = \mathbb{R}^2 \), \( C = C^* = \mathbb{R}_+^2 \), \( f : X \to Y \) defined as \( f(x) = (x, -x^2) \) is not \( * \)-quasiconvex. We have \( \text{Eff}(X, f) = [0, +\infty) \neq \emptyset \) but for any \( \xi \in C^* \setminus \{0\} \) we have \( \inf_{x \in X} \langle \xi, f(x) \rangle = -\infty \). Hence does not exist \( \xi \in C^* \setminus \{0\} \) such that \( \langle \xi, f(x) \rangle \) is bounded from below.

5. Genercity of DH-well-posedness for C-convex functions

In this section we show that the set of \( C \)-convex and \( C \)-lsc functions enjoying DH-well-posedness properties contains a dense \( G_\delta \) set. To prove the main theorem in this section we need some preliminary results.

**Proposition 5.1.** Let \( f : X \to \mathbb{R} \) a convex and lsc function, \( \bar{x} \in X \) and set \( g(x) = f(x) + a\|x - \bar{x}\|^\alpha \), \( a > 0, \alpha \geq 1 \). Then \( \lim_{\|x\| \to +\infty} g(x) = +\infty \). Furthermore \( g(x) \) is lower bounded.

**Proof.** We prove that for every sequence \( x_n \in X \) with \( \|x_n\| \to +\infty \) it holds \( \lim_{n \to +\infty} g(x_n) = +\infty \). Denote by \( X^* \) the topological dual space of \( X \). Since \( f(x) \) is convex, the set \( \partial f(\bar{x}) \subseteq X^* \) of all subgradients of \( f \) at \( \bar{x} \) is nonempty and by definition of subgradient \([10]\), for every continuous linear
functional \( v \in \partial f(\bar{x}) \) it holds \( f(x) \geq f(\bar{x}) + v(x - \bar{x}), \forall x \in X \). Hence,

\[
\lim_{n \to +\infty} g(x_n) = \lim_{n \to +\infty} [f(x_n) + a \|x_n - \bar{x}\|^\alpha] \\
\geq \lim_{n \to +\infty} (f(\bar{x}) + v(x_n - \bar{x}) + a \|x_n - \bar{x}\|^\alpha) \\
= \lim_{n \to +\infty} \left[ f(\bar{x}) + \|x_n - \bar{x}\|^\alpha \left( v \left( \frac{x_n - \bar{x}}{\|x_n - \bar{x}\|} \right) \|x_n - \bar{x}\|^{1-\alpha} + a \right) \right] \\
= +\infty.
\]

(the last equality follows since a continuous linear functional is bounded). To prove that \( g(x) \) is lower bounded observe that for every \( M \in \mathbb{R} \), there exists \( k > 0 \) such that \( z \setminus \{0\} \) since \( z \). We endow \( x \in X : \|x\| < k \) with the distance defined by \( d(x, y) \) compatible with the topology of uniform convergence on bounded sets.

**Corollary 5.2.** Let \( f : X \to Y \) be a \( C \)-convex, \( C \)-lsc function and for \( \xi \in C^* \setminus \{0\} \) and \( a > 0, \alpha \geq 1 \) set \( g_\xi(x) = \langle \xi, f(x) \rangle + a \|x - \bar{x}\|^\alpha \). Then, \( \lim_{\|x\| \to +\infty} g_\xi(x) = +\infty \) and \( g \) is lower bounded.

**Proof.** The proof follows from Proposition 5.1 since \( f \) \( C \)-convex and \( C \)-lsc implies \( g \) convex and lsc for every \( \xi \in C^* \setminus \{0\} \). \( \square \)

Let \( \mathcal{F} \) be the set of \( C \)-convex and \( C \)-lsc functions \( f : X \to Y \). We endow \( \mathcal{F} \) with the distance defined by \( (1.3) \), compatible with the topology of uniform convergence on bounded sets.

**Theorem 5.3.** Let \( \mathcal{F} \) be the set of \( C \)-convex and \( C \)-lsc functions endowed with the topology of uniform convergence on bounded sets and let \( \bar{\mathcal{F}} \) be the set of functions \( f \in \mathcal{F} \) such that \( \text{Eff}(X, f) \neq \emptyset \) and problem \( (X, f) \) is DH-wp at some point \( \bar{x} \in \text{Eff}(X, f) \). Then \( \bar{\mathcal{F}} \) contains a dense \( G_\delta \) set.

**Proof.** The initial argument of the proof is inspired to that of Theorem 2.1 in [23]. If we fix \( \bar{x} \in \text{int} C \), we can find \( \xi \in C^* \) such that \( \langle \xi, \bar{x} \rangle = 1 \). Consider the set

\[
\mathcal{Z} = \{ z : X \to \mathbb{R} \text{ such that } z(x) = \langle \xi, f(x) \rangle, f \in \mathcal{F} \}
\]

Since \( f \) is \( C \)-lsc, \( z \) is lsc. Endow \( \mathcal{Z} \) with the topology of uniform convergence on bounded sets and let \( S : \mathcal{F} \to \mathcal{Z} \) be the map \( S(f) = z \), with \( z \) defined as before. Then \( S \) is a continuous map. Let

\[
(5.1) \quad \mathcal{A}_n = \{ z \in \mathcal{Z} : \exists a > \inf_{x \in X} z, \text{ diam } L_z(a) < \frac{1}{n} \}
\]

where \( L_z(a) = \{ x \in X : z(x) \leq a \} \). Observe that \( L_z(a) \) are closed convex sets since \( z \) is convex and lsc. Then, it is known (see e.g. [22]) that if \( z_n \to z \) in the uniform convergence, then \( \text{diam } L_{z_n}(a) \to \text{diam } L_z(a) \), which gives continuity of the diam function. Hence \( \mathcal{A}_n \) is an open set for all \( n \) and this implies \( S^{-1}(\mathcal{A}_n) \) is an open set for all \( n \). We claim that the set \( \mathcal{W} \) of
those functions \( h \in \mathcal{F} \) such that problem \((X, S(h))\) is T-wp is dense in \( \mathcal{F} \). Since

\[
\mathcal{W} = \bigcap_{n=1}^{+\infty} S^{-1}(A_n)
\]

then it is a \( G_\delta \) set i.e. the countable intersection of open sets. Let \( f \in \mathcal{F}, \sigma > 0 \) and take \( j \) so large that setting \( g(x) = f(x) + \frac{1}{j} \|x - \theta\|k^0 \) it holds \( d(f, g) < \frac{\sigma}{2} \). Setting \( g_\xi(x) = \langle \xi, g(x) \rangle \) we have \( \lim_{\|x\| \to +\infty} g_\xi(x) = +\infty \) and \( g_\xi \) is lower bounded by Corollary 5.2. The proof now follows along the lines of Theorem 4.4. We can find \( M > 0 \) such that

\[
\{ x \in X : g(x) \leq \inf_{x \in X} g(x) + 1 \} \subseteq B(\theta, M)
\]

Let \( h : X \to Y \) be defined as

\[
h(x) = g(x) + \varepsilon \|x - \bar{x}\|k^0
\]

and let \( s = \sum_{k=0}^{+\infty} \frac{1}{2^k} \|k + M\|k^0\| \). Apply Theorem 3.7 with \( \varepsilon = \frac{s}{2^j} \) and arbitrary \( r \) to find a point \( \bar{x} = \bar{x}_\sigma \in X \) such that \( \|\bar{x} - \theta\| \leq M \), \( \bar{x} \) is the unique minimizer of

\[
S(h)(x) = \langle \xi, g(x) \rangle + \varepsilon \|x - \bar{x}\|
\]

and problem \((X, S(h))\) is T-wp at \( \bar{x} \) and hence \( h \in \mathcal{W} \). This implies that problem \((X, h)\) is DH-wp at \( \bar{x} \) by Lemma 4.3. Now observe that

\[
d(h, g) \leq \varepsilon s = \frac{\sigma}{2}
\]

and then \( d(f, h) < \sigma \). Hence \( \mathcal{F} \) contains a dense \( G_\delta \) set, which concludes the proof.

\[ \square \]

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(M. Rocca) DEPARTMENT OF ECONOMICS, UNIVERSITÀ DEGLI STUDI DELL’INSUBRIA, VIA MONTE GENEROSO 71, 21100 VARESE, ITALY

Email address: matteo.rocca@uninsubria.it