Hidden correlations in quantum optics and quantum information

To cite this article: Margarita A Man’ko 2018 J. Phys.: Conf. Ser. 1071 012015

View the article online for updates and enhancements.
Hidden correlations in quantum optics and quantum information

Margarita A Man’ko
Lebedev Physical Institute, Leninskii Prospect 53, Moscow 119991, Russia
E-mail: mmanko@sci.lebedev.ru

Abstract. We review the notion of quantum correlations in composite systems and show that noncomposite systems have similar correlations (which we call “hidden” correlations); these quantum correlations are analogous to the ones available in composite (multipartite) systems. We demonstrate new entropic-information inequalities characterizing the hidden correlations in noncomposite systems and present the examples of some qudit states with such correlations.

Keywords: qubit, probability distributions, quantum correlations, entropy, entanglement.

1. Introduction
The aim of this work is to review recent studies of the knew notion of hidden correlations in quantum systems, which have no subsystems and present the entropic inequalities characterizing such correlations. Before discussing quantum systems, we recall a crucial role of the notion of probability in quantum mechanics and quantum optics. This notion is related to classical systems with fluctuating random variables. The probability distributions are associated with states of classical systems; they are described by nonnegative functions satisfying the normalization conditions.

All the statistical properties of classical systems like, e.g., correlations of different degrees of freedom in systems containing two (or more) subsystems are coded by the joint probability distributions. Historical aspects and foundations of the classical probability theory can be found in the book by Kolmogorov [1]. An important notion determined by the probability distribution is entropy, which gets the name of Shannon entropy due to milestone work by Shannon [2].

In quantum mechanics and quantum optics, the states are associated with the wave function (pure states) [3] and the density matrices (mixed states) [4, 5]. The notion of quantum entropy was introduced by von Neumann, and this entropy, called von Neumann entropy, is expressed in terms of the density matrix. The quantum von Neumann entropy also characterizes the correlations in the systems containing the subsystems (e.g., two spin-1/2 systems, two-level atoms, or qubits).

Recently, the new representation of quantum states was introduced [6–8]. This representation, called the probability representation of quantum states, is based on the results where the tomographic probability distributions were measured in quantum optics experiments [9] to reconstruct the Wigner function of photon states.
In the probability representation of quantum states, fair probability distributions\textsuperscript{1} are associated with quantum-system states. Thus, the known properties of classical probability distributions can be used to study quantum systems.

The clarification of the classical–quantum entropic relations and obtaining some new relations characterizing the quantum-system properties is one of the goals of this work based on the publications \([10–18]\). The procedure to obtain the entropic-information relations for the density matrix \(\rho\) of a single qudit (single spin \(j\) or \(N\)-level atom) is based on a simple tool of applying the invertible map of the matrix indices \(s\) and \(s'\) onto the pairs of indices \(s \leftrightarrow (jk)\) and \(s' \leftrightarrow (j'k')\) or triples of indices \(s \leftrightarrow (jk\ell)\) and \(s' \leftrightarrow (j'k'\ell')\), etc.

Following this way, we reconsider the density matrix \(\rho_{ss'}\) of a single qudit (single spin \(j\) or \(N\)-level atom) state as the density matrix \(\rho_{jk,j'k'}\) (or \(\rho_{jk\ell,j'k'\ell'}\)) of artificial composite bipartite (or tripartite) system state. In view of the applied invertible map of the matrix indices, the entropic relations obvious for composite systems are transformed to the relations for the density matrices of the single qudit (single spin \(j\)) state. It is just a basis of our idea to show the entropic-information relations, which are new for the noncomposite systems (single qudits).

For composite systems, the entropic-information relations correspond to quantum correlations between the subsystem degrees of freedom. These correlations are known to be a resource for quantum technologies, e.g., correlations, which are responsible for the phenomenon of entanglement \([19, 20]\) in multipartite quantum systems. The existence of analogous entropic-information relations (inequalities and equalities) in single qudit states (single spin-\(j\) states) reflects the existence of quantum correlations (analogous to entanglement) in noncomposite systems, which do not contain any subsystems. In principle, these correlations can provide an additional resource for quantum technologies based on manipulations with single qudit states analogous to processing the multipartite system states.

For diagonal density matrices \(\rho_{ss'}\), the presented map of the matrix indices provides the possibility to map any classical probability distribution \(P(s) \equiv \rho_{ss}\) onto the joint probability distribution of bipartite system \(p(j, k) \equiv \rho_{jk,jk}\) or tripartite systems \(p(j, k, \ell) \equiv \rho_{jk\ell,jk\ell}\). This tool can be used to describe hidden correlations in classical systems with one random variable.

This paper is organized as follows.

In section 2, we study classical hidden correlations. In section 3, we consider entropic-information inequalities for the classical coin and compass. In section 4, we present quantum subadditivity and strong subadditivity conditions and discuss hidden correlations in bipartite quantum systems in section 5. In section 6, we derive new entropic inequalities for the five-level atom and study qubit–qutrit systems in section 7 and a single qudit with spin \(j = 2\) in section 8. Finally, we give our conclusions and prospectives in section 9.

2. Classical hidden correlations

We formulate the idea of our approach recalling the known relations for probability distributions of bipartite and tripartite classical systems.

If for bipartite system \((1, 2)\), the joint probability distribution is given by the nonnegative function \(p(j, k), j = 1, 2, \ldots, n\) and \(k = 1, 2, \ldots, m\), the known entropic inequality, called the entropic subadditivity condition for Shannon entropies of system \(H(1, 2)\) and its two subsystems\textsuperscript{1}.

\textsuperscript{1}The tomographic probability distribution is a fair (actual) probability distribution; this terminology is often used in the literature as canonical one to stress that the tomographic probability distribution is, in fact, the standard notion for probability distributions used in classical probability theory. In the quantum case, it is fair (actual) one in contrast to quasidistributions like the Wigner function, the Husimi–Kano function, and the Glauber–Sudarshan \(P\)-distribution.
\( H(1) \) and \( H(2) \), holds

\[
- \sum_{j=1}^{n} \sum_{k=1}^{m} p(j, k) \ln p(j, k) \leq - \sum_{k=1}^{m} \left[ \left( \sum_{j=1}^{n} p(j, k) \right) \ln \left( \sum_{j=1}^{n} p(j, k) \right) \right] \\
- \sum_{j=1}^{n} \left[ \sum_{k=1}^{m} p(j, k) \right] \ln \left( \sum_{j=1}^{n} p(j, k) \right).
\]

(1)

Here, the left-hand side is equal to Shannon entropy \( H(1, 2) \), and the right-hand side is equal to the sum of Shannon entropies \( H(1) \) and \( H(2) \), respectively.

For tripartite system \( (1, 2, 3) \) described by the joint probability distribution \( p(j, k, \ell) \), \( j = 1, 2, \ldots, n \), \( k = 1, 2, \ldots, m \), and \( \ell = 1, 2, \ldots, s \), the entropic inequality holds

\[
- \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{\ell=1}^{s} p(j, k, \ell) \ln p(j, k, \ell) - \sum_{k=1}^{m} \sum_{\ell=1}^{s} \left( \sum_{j=1}^{n} p(j, k, \ell) \right) \ln \left( \sum_{j=1}^{n} p(j, k, \ell) \right) \\
\leq - \sum_{j=1}^{n} \sum_{k=1}^{m} \left( \sum_{\ell=1}^{s} p(j, k, \ell) \right) \ln \left( \sum_{\ell=1}^{s} p(j, k, \ell) \right) - \sum_{k=1}^{m} \sum_{\ell=1}^{s} \left( \sum_{j=1}^{n} p(j, k, \ell) \right) \ln \left( \sum_{j=1}^{n} p(j, k, \ell) \right).
\]

(2)

Here, the left-hand side is equal to the sum of Shannon entropy \( H(1, 2, 3) \) of tripartite system \( (1, 2, 3) \) and Shannon entropy \( H(2) \) of its second subsystem, and the right-hand side is equal to the sum of Shannon entropies \( H(1, 2) \) and \( H(2, 3) \) of subsystems \( (1, 2) \) and \( (2, 3) \), respectively. Inequality (2) is called the strong subadditivity condition.

Below we show that such inequalities exist also for systems with one random variable. For this aim, we use the map of indices described above. To introduce the approach, we start with a simplest example of the probability distribution of one classical coin, then consider two coins.

3. Entropy of a classical coin and entropic inequalities for two coins and compass

We consider an example of the classical bit, taking into account that it has a quantum counterpart like the two-level atom and one qubit (or spin \( j = 1/2 \) system). By nonnegative number \( p_1 \leq 1 \), we describe the probability for the coin to be in state “UP,” and by nonnegative number \( p_2 \leq 1 \) we describe the probability for the coin to be in state “DOWN” (Fig. 1). The two numbers satisfy the normalization condition \( p_1 + p_2 = 1 \leftrightarrow \sum_{n=1}^{2} p_n = 1 \).

Shannon entropy [2] \( H = -p_1 \ln p_1 - p_2 \ln p_2 = -\sum_{n=1}^{2} p_n \ln p_n \) satisfies the inequality \( \ln 2 \geq H \geq 0 \). Shannon entropy \( H = 0 \) for “complete order,” i.e., either \( p_1 = 1 \) and \( p_2 = 0 \) or \( p_1 = 0 \) and \( p_2 = 1 \), and \( H = \ln 2 \) for \( p_1 = p_2 = 1/2 \).

There exists another entropy called Tsallis entropy [21] or \( q \)-entropy

\[
T(q) = \frac{p_1^q + p_2^q - 1}{1 - q} = -\sum_{n=1}^{2} p_n^q \frac{1-p_n}{1-q}.
\]

(3)

In the limit \( q \to 1 \), \( T(1) = H \), Tsallis entropy converts to Shannon entropy. Both entropies are nonnegative. For Tsallis entropy, also there exist inequalities which can be used within the framework of our approach.

For two coins, there are four possibilities: “UP” “UP”, “UP” “DOWN”, “DOWN” “UP”, “DOWN” “DOWN”.
Let the probabilities for the two coins to have these positions to be described by nonnegative numbers $1 \geq p_1, p_2, p_3, p_4 \geq 0$, such that $p_1 + p_2 + p_3 + p_4 = 1$ or $\sum_{n=1}^{4} p_n = 1$. On the other hand, the physical meaning of these four numbers is reflected by the notation $p(jk)$, such that one has

$$p(11) = p_1, \quad p(12) = p_2, \quad p(21) = p_3, \quad p(22) = p_4,$$

where $p(jk)$ is the joint probability distribution that the first coin is in the $j$th position (“UP” or “DOWN”) and the second coin, in the $k$th position (“UP” or “DOWN”). This means that the position “UP” “UP” for two coins is described by the probability $p(11) = p_1$, and analogously for the other three positions of these coins. We see that, in fact, we use the map of the integers $1 \leftrightarrow 1, 2 \leftrightarrow 1 2, \ 3 \leftrightarrow 2 1, \ 4 \leftrightarrow 2 2$ to label the four probabilities. We can use either four numbers $p_1, p_2, p_3, p_4$ or the same four numbers labeled as $p(11), p(12), p(21), p(22)$.

We can formulate this observation as follows.

Given four nonnegative numbers $p_1, p_2, p_3, p_4$, which can be considered as components of the probability vector $\vec{p} = (p_1, p_2, p_3, p_4)$. These four numbers can be identified either with a joint probability distribution described by the table $p(jk)$ or as a probability distribution $p(s)$, $s = 1, 2, 3, 4$ for four events, which have four different possibilities, e.g., four different orientations of compass “North”, “South”, “East”, “West” (Fig. 2).

From the viewpoint of mathematical relations between these four numbers, the relations like equalities or inequalities do not depend on the interpretation of these numbers connected either with the positions of two coins or with the compass orientation. But if we identify these four numbers with the joint probability distribution $p(jk)$, i.e., $1 \geq p(jk) \geq 0$, $\sum_{j=1}^{2} \sum_{k=1}^{2} p(jk) = 1$, we elaborate the approach to study entropies associated with the probabilities $p(jk)$.

For example, the probability distribution describing the positions “UP” or “DOWN” only for the first coin reads

$$p(1)_j = \sum_{k=1}^{2} p(jk) = \begin{cases} p(11) + p(12) & \equiv p_1 + p_2, \\ p(21) + p(22) & \equiv p_3 + p_4, \end{cases} \quad (4)$$

and analogously the probability distribution describing the positions “UP” or “DOWN” only for the second coin is

$$p(2)_k = \sum_{j=1}^{2} p(jk) = \begin{cases} p(11) + p(21) & \equiv p_1 + p_3, \\ p(12) + p(22) & \equiv p_2 + p_4. \end{cases} \quad (5)$$

Shannon entropies associated with two probability distributions $p(1)_j$, $p(2)_k$ and the entropy associated with joint probability distribution $p(jk)$ are known to satisfy the inequality (subadditivity condition) $H(1) + H(2) \geq H(12)$, where

$$H(1) = -(p(1)_1 \ln p(1)_1 - (p(1)_2 \ln p(1)_2),$$
$$H(2) = -(p(2)_1 \ln p(2)_1 - (p(2)_2 \ln p(2)_2),$$
$$H(1, 2) = -p(11) \ln p(11) - p(12) \ln p(12) - p(21) \ln p(21) - p(22) \ln p(22). \quad (6)$$

The physical meaning of this inequality consists in the fact that the degree of “disorder” in composite system (two coins) is smaller than the sum of degrees of “disorder” in the behavior of two subsystems (first coin and second coin).
The difference $I = H(1) + H(2) - H(12) \geq 0$ is called the mutual Shannon information. If the behaviors of both coins are completely independent (there is no correlations between the subsystems – coins), information $I = 0$. If there is a strong correlation in the behaviors of two coins (i.e., the position of the first coin determines the position of the second coin), information is maximum, i.e., $I = \ln 2$.

The absence of correlations between two coins means that the probability vector $\vec{p} = (p_1, p_2, p_3, p_4)$ has the product form $\vec{p} = (x y, x(1 - y), (1 - x)y, (1 - x)(1 - y))$, where the numbers $x$ and $y$ are the probabilities to have the position of the first coin “UP” and of the second coin also “UP.” In this case, the system entropy is equal the sum of entropies of subsystems, $H(1, 2) = H(1) + H(2) \to I = 0$. A complete correlation means that the position of the first coin determines the position of the second coin.

If the probability distributions are such that the coin positions are completely correlated, e.g., the probability vector $\vec{p} = (p_1, p_2, p_3, p_4) = (x, y, y, x)$ has the form $(x, 1/2 - x, 1/2 - x, x)$, and it just describes this situation. In fact, $(p(1))_1 = 1/2, (p(1))_2 = 1/2, (p(2))_1 = 1/2, (p(2))_2 = 1/2$; this means that the Shannon information $I = 2 \ln 2 + 2x \ln x + 2(1/2 - x) \ln(1/2 - x)$. For $x = 1/2$ (or for $x = 0$), the Shannon information is maximum, $I = \ln 2$.

Thus, one can see that for an arbitrary probability vector $\vec{p} = (p_1, p_2, p_3, p_4)$, the vector of the form $(x, 1/2 - x, 1/2 - x, x)$ at $x = 0$ and $x = 1/2$ provides a maximum of the function

$$I = (p_1 \ln p_1 + p_2 \ln p_2 + p_3 \ln p_3 + p_4 \ln p_4)$$

$$- (p_1 + p_2) \ln(p_1 + p_2) - (p_3 + p_4) \ln(p_3 + p_4)$$

$$- (p_1 + p_3) \ln(p_1 + p_3) - (p_2 + p_4) \ln(p_2 + p_4). \quad (7)$$

For two coins, complete correlations of the coins described by the probability vector $(x, 1/2 - x, 1/2 - x, x)$ means that, if the first coin is in the position ”UP” the second coin is also definitely in the position “UP” (for $x = 1/2$) or, if the first coin is in the position ”UP” the second coin is definitely in the position ”DOWN” (for $x = 0$). This means that such distribution corresponds to the “entanglement” of classical coins.

The function $I$ can be interpreted as Shannon information for two “artificial coins” associated with a compass. The correlations in the compass orientations analogous to the correlations in positions of two coins provides the possibility to extend the notion of mutual information known for bipartite system to the case of a single (noncomposite) system.

The systems under consideration – “two coins” and “compass” are quite different systems. The former one is a composite system containing two subsystems. The latter one is noncomposite system which does not contain subsystems. But from the viewpoint of mathematical properties, the fair probability distributions describing the results of measuring the states of these systems are identical. This means that all notions based on the properties like entropies, information, equalities and inequalities for entropies and information can be introduced for both composite (“two coins”) systems and noncomposite (“compass”) system. The difference in the introduced notions is in the interpretation of entropic inequalities, since the same numerical inequalities reflect the presence or absence of correlations in the systems, but the physical nature of the correlations is different.

In composite (“two coins”) systems, correlations providing the nonnegativity of the mutual information are correlations between the states of the subsystems (two different coins). In noncomposite (“compass”) system, “mutual information” introduced for this system describes the correlations between the states of the same system, i.e., between the “compass” orientations. Specific properties of the correlations are related to the events which correlate. In the case of compass, a pair of compass orientations is one event but, the pairs, in principle, are not independent. Formally, namely, the dependence of pairs of the compass-orientation events is mimicking correlations between the events in experiments with two coins.
We make another comment. Different noncomposite systems can exist; for example, in addition to the compass already considered, there exists “casino roulette” with four outputs (Fig. 4). It is different system but the probability description of the casino roulette is identical to the statistics of the compass behavior. The correlations in the casino roulette processing are described by the same entropic-information formula existing for the compass correlations and two-coin correlations.

From this discussion, one can draw the conclusion:

Entropic relations obvious for composite (“two coins”) system are found for noncomposite (nondivisible) system, which has no subsystems. For example, if there is no correlations in the behavior of two coins, the entropy of this composite system is the sum of entropies of both subsystems. It is worth noting that completely analogous properties exist for noncomposite system like casino roulette.

In the case of maximum correlations in the behavior of two coins, the position of the first coin determines the position of the second coin, and the mutual information takes a maximum possible value. An analogous property for compass or casino roulette is not obvious but it also exists.

Applying the map between the system states, we can get and clarify the meaning of the inequalities, which are “new” entropic inequalities for noncomposite systems. This idea is the general idea for the approach we are presenting in this work and not only for classical systems but quantum (qudit) systems as well.

4. Quantum subadditivity and strong subadditivity conditions

In previous sections, we discussed classical probability distributions of composite and noncomposite systems and known inequalities. Now we present the relations known for quantum systems and called the subadditivity and strong subadditivity conditions. A basic concept for quantum states, e.g., the qudit state is the density matrix $\rho$ with matrix elements $\rho_{ss'}$, where $s, s' = 1, 2, \ldots, N$. The von Neumann entropy of the state is $S = -\text{Tr}(\rho \ln \rho)$.

If the system is composite with two subsystems 1 and 2, for example, two qudits, the density matrix $\rho(1, 2)$ has matrix elements $\rho_{jk,j'k'}(1, 2)$, with $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$.

The density matrix $\rho(1)$ of the first subsystem is given by the so-called partial tracing tool that provides the matrix elements of this matrix

$$
\rho_{jj'}(1) = \sum_{k=1}^{n} \rho_{jk,j'k}(1, 2), \quad j, j' = 1, 2, \ldots, m.
$$

For the density matrix $\rho(2)$ of the second subsystem, we have the matrix elements, in view of the partial tracing with respect to indices connected with the first subsystem, i.e.,

$$
\rho_{kk'}(2) = \sum_{j=1}^{m} \rho_{jk,jk'}(1, 2).
$$

The known subadditivity condition for three entropies of composite system and its two subsystems

$$
S(1) = -\text{Tr}(\rho(1) \ln \rho(1)), \quad S(2) = -\text{Tr}(\rho(2) \ln \rho(2)) \quad S(1, 2) = -\text{Tr}(\rho(1, 2) \ln \rho(1, 2))
$$

(10)
reads
\[ S(1) + S(2) \geq S(1, 2), \] (11)

which means nonnegativity of the mutual quantum information
\[ I_q = S(1) + S(2) - S(1, 2) \geq 0. \]

For tripartite system with three subsystems 1,2,3, for example, three qudits, the density matrix \( \rho(1, 2, 3) \) has matrix elements \( \rho_{jkl,j'k'l'}(1, 2, 3) \), with \( j, j' = 1, 2, \ldots, m \), \( k, k' = 1, 2, \ldots, n \), and \( l, l' = 1, 2, \ldots, s \). The density matrices of the subsystems have matrix elements obtained from these ones by the partial tracing tool.

The density matrix \( \rho(1, 2) \) of subsystems 1 and 2 reads \( \rho_{jk,j'k'}(1, 2) = \sum_{l=1}^{s} \rho_{jk, j'k'l}(1, 2, 3) \), and \( \rho(2, 3) \) of subsystems 2 and 3 is \( \rho_{k,l,k'l'}(2, 3) = \sum_{j=1}^{m} \rho_{jkl,j'k'l'}(1, 2, 3) \); for the density matrix \( \rho(2) \) of the second subsystem, we have the matrix elements \( \rho_{kk'}(2) = \sum_{l=1}^{s} \rho_{kk'l}(2, 3) \).

The quantum strong subadditivity condition is known as the entropic inequality
\[ S(1, 2) + S(2, 3) \geq S(1, 2, 3) + S(2), \]

where the von Neumann entropies are
\[ S(1, 2) = -\text{Tr} (\rho(1, 2) \ln \rho(1, 2)), \quad S(2, 3) = -\text{Tr} (\rho(2, 3) \ln \rho(2, 3)), \]
\[ S(1, 2, 3) = -\text{Tr} (\rho(1, 2, 3) \ln \rho(1, 2, 3)). \] (12)

The nonnegative number \( I_{\text{cond}} = S(1, 2) + S(2, 3) - S(1, 2, 3) - S(2) \) is called the conditional information.

Our approach is to employ the same map of indices, which we used for classical probability distributions in previous sections, but now we apply the map to the density matrix \( \rho_{ss'} \) of a single qudit (noncomposite system) and obtain the corresponding entropic inequalities. This means that we interpret the matrix \( \rho_{ss'} \) either as the density matrix of bipartite system, \( \rho_{ss'} \leftrightarrow \rho_{jkl,j'k'l'} \), or as the matrix of tripartite system, \( \rho_{ss'} \leftrightarrow \rho_{jkl,j'k'l'} \).

In the following sections, we give particular examples of the approach.

5. Hidden correlations in bipartite systems

In previous sections, we discussed hidden correlations in systems with one random variable. We can associate the tool of partitions, which provides the possibility to map the set of indices (integers) \( s = 1, 2, \ldots, N \), \( N = n_1 n_2 \) onto pairs of indices (integers) \( (j, k) \), \( j = 1, 2, \ldots, n_1 \), \( k = 1, 2, \ldots, n_2 \), and the tool can be used to associate a bipartite system, say, of two qudits with a tripartite system (with three artificial subsystems). Then we introduce the notion of hidden correlations not for a single qudit but for a system of two qudits. The correlations existing in these bipartite systems are characterized by strong subadditivity conditions known for composite tripartite systems.

Such approach can be developed if the integer number \( N \) can be presented in two forms — one form is the mentioned product form \( N = n_1 n_2 \), and the other form is \( N = m_1 m_2 m_3 = n_1 n_2 \), where \( m_1, m_2, \) and \( m_3 \) are integers. This equality permits to consider the density matrix \( \rho_{jkl,j'k'l'} \) of two-qudit state as the density matrix \( \rho_{mnl,m'n'l'} \) of three-qudit state, where three integer indices \( mnl, m'n'l' \) are taken from domains \( 1 \leq m, m' \leq m_1, 1 \leq n, n' \leq m_2, \) and \( 1 \leq l, l' \leq m_3 \). We can construct this map since there exists the bijective correspondence of the matrix elements \( \rho_{ss'} \leftrightarrow \rho_{jkl,j'k'l'} \) and \( \rho_{ss'} \leftrightarrow \rho_{mnl,m'n'l'}, \) where \( 1 \leq s, s' \leq N, 1 \leq j, j' \leq n_1, \) and \( 1 \leq k, k' \leq n_2 \).

In view of this fact, if one has the density matrix of bipartite system \( \rho_{jkl,j'k'l'} \), this density matrix can be interpreted first as the density matrix of noncomposite system \( \rho_{ss'}, \) which is the
same numerical $N \times N$-matrix. Then, this density matrix (again the same numerical matrix) can be interpreted as the density matrix of tripartite quantum state. For tripartite quantum states, we have the strong subadditivity condition for von Neumann entropies of the system and its three subsystems. The entropic inequality for this composite system with three artificial qudits (the initial density matrix of bipartite system is the matrix corresponding to two qudits) provides the relation (inequality) for the matrix elements of the density matrix of this two-qudit state. This inequality characterizes hidden three-partite-like correlations in bipartite quantum systems.

We give the example of such inequality for the system containing the spin-1/2 subsystem and the spin-3/2 subsystem; the density $8 \times 8$-matrix of this system has matrix elements $\rho_{m_1m_2,m'_1m'_2}$, where $m_1, m'_1 = -1/2, +1/2$ and $m_2, m'_2 = -3/2, -1/2, +1/2, +3/2$. We employ the map of indices $ss' \leftrightarrow mm'$ of the form

\begin{align*}
-1/2 - 3/2 &\leftrightarrow 1, & -1/2 - 1/2 &\leftrightarrow 2, & -1/2 + 1/2 &\leftrightarrow 3, & -1/2 + 3/2 &\leftrightarrow 4, \\
+1/2 - 3/2 &\leftrightarrow 5, & +1/2 - 1/2 &\leftrightarrow 6, & +1/2 + 1/2 &\leftrightarrow 7, & +1/2 + 3/2 &\leftrightarrow 8,
\end{align*}

and interpret the matrix $\rho_{m_1m_2,m'_1m'_2}$ as the density matrix of noncomposite system $\rho_{ss'}$ corresponding to the qudit system with $j = 7/2$. Then we apply another map of indices, namely, $\rho_{ss'} \leftrightarrow \rho_{M_1M_2M_3,M'_1M'_2M'_3}$, where $M_1, M_2, M_3, M'_1, M'_2$, and $M'_3$ take values $\pm 1/2$, i.e.,

\begin{align*}
1 &\leftrightarrow +1/2 + 1/2 + 1/2, & 2 &\leftrightarrow +1/2 + 1/2 - 1/2, & 3 &\leftrightarrow +1/2 - 1/2 + 1/2, \\
4 &\leftrightarrow +1/2 - 1/2 - 1/2, & 5 &\leftrightarrow -1/2 + 1/2 + 1/2, & 6 &\leftrightarrow -1/2 + 1/2 - 1/2, \\
7 &\leftrightarrow -1/2 - 1/2 + 1/2, & 8 &\leftrightarrow -1/2 - 1/2 - 1/2.
\end{align*}

Producing the partial trace procedure with the density matrix of three artificial qubits, we arrive at the density matrices $\rho_{M_1M_2M_3,M'_1M'_2M'_3}(1,2)$, $\rho_{M_2M_3,M'_2M'_3}(2,3)$, and $\rho_{M_2M'_3}(2)$ of the qubit subsystems; in terms of matrix elements $\rho_{ss'}$, the matrices read

\begin{align*}
\rho(1,2) &= \left( \begin{array}{cccccccc}
\rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} & \rho_{15} + \rho_{26} & \rho_{17} + \rho_{28} \\
\rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} & \rho_{35} + \rho_{46} & \rho_{37} + \rho_{48} \\
\rho_{51} + \rho_{62} & \rho_{52} + \rho_{64} & \rho_{55} + \rho_{66} & \rho_{57} + \rho_{68} \\
\rho_{71} + \rho_{82} & \rho_{72} + \rho_{84} & \rho_{75} + \rho_{86} & \rho_{77} + \rho_{88}
\end{array} \right), \\
\rho(2,3) &= \left( \begin{array}{cccccccc}
\rho_{11} + \rho_{55} & \rho_{12} + \rho_{56} & \rho_{13} + \rho_{57} & \rho_{14} + \rho_{58} \\
\rho_{21} + \rho_{65} & \rho_{22} + \rho_{66} & \rho_{23} + \rho_{67} & \rho_{24} + \rho_{68} \\
\rho_{31} + \rho_{75} & \rho_{32} + \rho_{76} & \rho_{33} + \rho_{77} & \rho_{34} + \rho_{78} \\
\rho_{41} + \rho_{85} & \rho_{42} + \rho_{86} & \rho_{43} + \rho_{87} & \rho_{44} + \rho_{88}
\end{array} \right), \\
\rho(2) &= \left( \begin{array}{cccc}
\rho_{11} + \rho_{22} + \rho_{55} + \rho_{66} & \rho_{13} + \rho_{24} + \rho_{57} + \rho_{68} \\
\rho_{11} + \rho_{42} + \rho_{75} + \rho_{86} & \rho_{13} + \rho_{44} + \rho_{77} + \rho_{88}
\end{array} \right),
\end{align*}

The hidden correlations in the bipartite system with spin $-1/2$ and spin $-3/2$ subsystems satisfy the strong subadditivity condition

\begin{equation}
-Tr \rho \ln \rho - Tr \rho(2) \ln \rho(2) \leq -Tr \rho(1,2) \ln \rho(1,2) - Tr \rho(2,3) \ln \rho(2,3);
\end{equation}

thus, we obtain the new entropic inequality for the bipartite system containing the two-level atom and the four-level atom as subsystems. The hidden correlations in this bipartite system correspond to correlations of artificial qubits. If the considered density matrix $\rho$ describes completely chaotic states, the off-diagonal elements in matrices $\rho(1,2)$, $\rho(2,3)$, and $\rho(2)$ are equal to zero, and the inequality obtained corresponds to the inequality for classical probability
distribution \( P(s) = \rho_{ss} \) given explicitly as follows:

\[
-\sum_{s=1}^{8} \rho_{ss} \ln \rho_{ss} - (P(1) + P(2) + P(5) + P(6)) \ln (P(1) + P(2) + P(5) + P(6)) \\
-(P(3) + P(4) + P(7) + P(8)) \ln (P(3) + P(4) + P(7) + P(8)) \\
\leq -(P(1) + P(2)) \ln (P(1) + P(2)) - (P(3) + P(4)) \ln (P(3) + P(4)) \\
-(P(5) + P(6)) \ln (P(5) + P(6)) - (P(7) + P(8)) \ln (P(7) + P(8)) \\
-(P(1) + P(5)) \ln (P(1) + P(5)) - (P(2) + P(6)) \ln (P(2) + P(6)) \\
-(P(3) + P(7)) \ln (P(3) + P(7)) - (P(4) + P(8)) \ln (P(4) + P(8)).
\]

All inequalities obtained from the above one by arbitrary permutations of the integers 1, 2, 3, 4, 5, 6, 7, 8 are also valid.

Thus, we derived some entropic inequalities valid for an arbitrary Hermitian 8×8-matrix \( \rho = \rho^\dagger \), \( \text{Tr} \rho = 1 \), and \( \rho \geq 0 \), but the physical meaning of the inequalities depends on the interpretation of this matrix. For the case of three-qubit states, the entropic inequality is the standard strong subadditivity condition characterizing the degree of quantum correlations of the three qubits. In the classical case, say, if one has the system – a coin plus the compass, the obtained inequality characterizes the degree of correlations of three artificial classical coins for which their joint probability distribution of three random variables is mapped onto the joint probability distribution of two random variables (one coin and one compass). The entropic inequalities obtained can be checked in experiments with superconducting circuits discussed in [22–32].

6. New entropic inequality for the five-level atom (spin \( j = 2 \))

First, we formulate, without derivation, new entropic-information inequality for qudit (spin \( j = 2 \)) or the five-level atom. Then we show how to obtain these inequalities.

Qudit with \( j = 2 \) has five states. These states can be modeled by states of a five-level atom. The atomic states can be realized by superconducting circuit states based on the Josephson junction. On the other hand, these states can be considered as states of qubit–qudit having six states but one of these six states is not occupied.

In Fig. 4, the atomic levels are labeled either by numbers 1, 2, 3, 4, 5 or by pairs of numbers (2, 2), (2, 1), (2, 0), (2, −1), (2, −2) corresponding to labeling the states of spin \( j = 2 \) with spin projections \( m = 2, 1, 0, -1, -2 \).

Thus, we use the invertible map of the labels

\[ 1 \leftrightarrow (2, 2), \quad 2 \leftrightarrow (2, 1), \quad 3 \leftrightarrow (2, 0), \quad 4 \leftrightarrow (2, -1), \quad 5 \leftrightarrow (2, -2). \]

For the density matrix of the five-level atom, we can use both possibilities to label the matrix elements.

The qudit (spin \( j = 2 \)) density matrix \( R_{mm'} \) \( (m, m' = -2, -1, 0, 1, 2) \) realized as states of the five-level atom reads

\[
R_{mm'} = \begin{pmatrix}
R_{22} & R_{21} & R_{20} & R_{2-1} & R_{2-2} \\
R_{12} & R_{11} & R_{10} & R_{1-1} & R_{1-2} \\
R_{02} & R_{01} & R_{00} & R_{0-1} & R_{0-2} \\
R_{-22} & R_{-21} & R_{-20} & R_{-2-1} & R_{-2-2} \\
R_{-12} & R_{-11} & R_{-10} & R_{-1-1} & R_{-1-2}
\end{pmatrix}
\equiv \begin{pmatrix}
\tilde{R}_{11} & \tilde{R}_{12} & \tilde{R}_{13} & \tilde{R}_{14} & \tilde{R}_{15} \\
\tilde{R}_{21} & \tilde{R}_{22} & \tilde{R}_{23} & \tilde{R}_{24} & \tilde{R}_{25} \\
\tilde{R}_{31} & \tilde{R}_{32} & \tilde{R}_{33} & \tilde{R}_{34} & \tilde{R}_{35} \\
\tilde{R}_{41} & \tilde{R}_{42} & \tilde{R}_{43} & \tilde{R}_{44} & \tilde{R}_{45} \\
\tilde{R}_{51} & \tilde{R}_{52} & \tilde{R}_{53} & \tilde{R}_{54} & \tilde{R}_{55}
\end{pmatrix}.
\]

\[ \text{Figure 4. Qubit–qudit model of the five-level atom} \quad | n \rangle \equiv | j, m \rangle, \quad j = 2. \]
Using the other notation for the matrix $R$ and its indices, i.e., $R_{mm'} \leftrightarrow \tilde{R}_{jk}$, the matrix $\tilde{R}$ can be interpreted as the density matrix of the five-level atom.

One can construct two density matrices of artificial “atoms” (qubit and qutrit). In terms of matrix elements $R_{mm'}$, the density matrix of the qubit (two-level “atom”) state is

$$r_{MM'} = \begin{pmatrix} R_{22} + R_{00} + R_{-2-2} & R_{21} + R_{1-1} \\ R_{12} + R_{-11} & R_{11} + R_{-1-1} \end{pmatrix}, \quad M, M' = 1/2, -1/2.$$ 

The density matrix of the qutrit (“three-level” atom) state reads

$$\rho_{\mu\mu'} = \begin{pmatrix} R_{22} + R_{11} & R_{20} + R_{1-1} & R_{2-2} \\ R_{02} + R_{-11} & R_{00} + R_{-1-1} & R_{0-2} \\ R_{-22} & R_{-20} & R_{-2-2} \end{pmatrix}, \quad \mu, \mu' = 1, 0, -1.$$

Now we are in the position to write some entropic inequalities for these three density matrices, namely, $R_{mm'}$: $m, m' = -2, -1, 0, 1, 2$ of qudit (spin $j = 2$) state realized by the state of superconducting circuit based on Josephson junction, $r_{MM'}$: $M, M' = 1/2, -1/21$ of the qubit (two-level “atom”) state, and $\rho_{\mu\mu'}$: $\mu, \mu' = 1, 0, -1$ of the qutrit (three-level “atom”) state.

The subadditivity condition for von Neumann entropies determined by the three density matrices reads

$$-\text{Tr} (R \ln R) \leq -\text{Tr} (r \ln r) - \text{Tr} (\rho \ln \rho).$$

An analog of the Araki–Lieb inequality known for bipartite quantum systems is

$$-\text{Tr} (R \ln R) \geq | -\text{Tr} (r \ln r) + \text{Tr} (\rho \ln \rho) |.$$

For the mutual information of the five-level atomic system, we obtain

$$I = \text{Tr} (R \ln R) - \text{Tr} (r \ln r) - \text{Tr} (\rho \ln \rho) \geq 0.$$ 

The inequalities can be checked for arbitrary density 5×5-matrices $\tilde{R}_{jk}$: $j, k = 1, 2, 3, 4, 5$, for example, if it is obtained in experiments with superconducting qudits. The approach with such ideas was employed in [30–32].

Now we present another inequality, which is an analog of the conditional von Neumann information nonnegativity known for tripartite quantum systems; we write it for entropies of the following density matrices:

The density 4×4-matrix $\tilde{R}_{\mu\mu'}$: $(\mu, \mu' = 3/2, 1/2, -1/2, -3/2)$ of an artificial qudit (spin $j = 3/2$) expressed in terms of matrix elements $R_{mm'}$ reads

$$\tilde{R}_{\mu\mu'} = \begin{pmatrix} R_{22} + R_{-2-2} & R_{21} & R_{20} & R_{2-1} \\ R_{12} & R_{11} & R_{10} & R_{1-1} \\ R_{02} & R_{01} & R_{00} & R_{0-1} \\ R_{-22} & R_{-20} & R_{-2-2} & R_{-2-1} \end{pmatrix}.$$ 

The density matrix $\tilde{\rho}_{mm'}$: $m, m' = 1, 0, -1$ of an artificial qutrit (spin $j = 1$) expressed in terms of matrix elements $R_{mm'}$ is

$$\tilde{\rho}_{mm'} = \begin{pmatrix} R_{22} + R_{11} & R_{20} + R_{1-1} & R_{2-2} \\ R_{02} + R_{-11} & R_{00} + R_{-1-1} & R_{0-2} \\ R_{-22} & R_{-20} & R_{-2-2} \end{pmatrix}.$$ 

The density matrix $\tilde{\rho}_{mm'}$: $(m, m' = \pm 1/2)$ of an artificial qubit is

$$\tilde{\rho}_{mm'} = \begin{pmatrix} R_{11} + R_{22} + R_{-2-2} & R_{20} + R_{1-1} \\ R_{02} + R_{-11} & R_{00} + R_{-1-1} \end{pmatrix}.$$
The new conditional information nonnegativity (an analog of the strong subadditivity condition) for the five-level atomic state reads

$$I_{\text{con}} = -\text{Tr} \tilde{\rho} \ln \tilde{\rho} - \text{Tr} \tilde{R} \ln \tilde{R} + \text{Tr} R \ln R + \text{Tr} \tilde{\rho} \ln \tilde{\rho} \geq 0;$$  \hspace{1cm} (16)

it can be checked experimentally, if one measures the density matrix of five-level “atom” or qudit with \( j = 2 \) in experiments with superconducting circuits.

For completely decoherent state with diagonal matrix \( R \), relation (16) yields the inequality for the population of the five-level atom,

$$I_{\text{con}} = -(R_{22} + R_{11}) \ln(R_{22} + R_{11}) - (R_{22} + R_{2-2}) \ln(R_{22} + R_{2-2}) + R_{22} \ln R_{22} + (R_{22} + R_{11} + R_{2-2}) \ln(R_{22} + R_{11} + R_{2-2}) \geq 0. \hspace{1cm} (17)$$

One can see that, in this case, the conditional information depends only on the three parameters \( R_{11}, R_{22}, \text{ and } R_{2-2} \). It is a new matrix inequality.

The inequalities obtained for the density matrix of the five-level atom can be formulated as mathematical inequalities for an arbitrary 5x5-matrix \( \rho \) such that \( \rho^1 = \rho \), \( \text{Tr} \rho = 1 \), and \( \rho \geq 0 \), i.e., eigenvalues of the matrix are nonnegative.

For example, for such 5x5-matrix \( \rho \), the mathematical subadditivity condition reads

$$-\text{Tr} \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} & \rho_{25} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} & \rho_{35} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} & \rho_{45} \\ \rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & \rho_{55} \end{pmatrix} \ln \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} & \rho_{25} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} & \rho_{35} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} & \rho_{45} \\ \rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & \rho_{55} \end{pmatrix} \leq -\text{Tr} \begin{pmatrix} \rho_{11} + \rho_{33} + \rho_{55} & \rho_{12} + \rho_{34} \\ \rho_{21} + \rho_{43} & \rho_{22} + \rho_{44} \end{pmatrix} \ln \begin{pmatrix} \rho_{11} + \rho_{33} + \rho_{55} & \rho_{12} + \rho_{34} \\ \rho_{21} + \rho_{43} & \rho_{22} + \rho_{44} \end{pmatrix}$$

The mathematical strong subadditivity condition for such 5x5-matrix \( \rho \) is

$$-\text{Tr} \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} & \rho_{25} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} & \rho_{35} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} & \rho_{45} \\ \rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & \rho_{55} \end{pmatrix} \ln \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} & \rho_{25} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} & \rho_{35} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} & \rho_{45} \\ \rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & \rho_{55} \end{pmatrix} \leq -\text{Tr} \begin{pmatrix} \rho_{11} + \rho_{22} + \rho_{55} & \rho_{13} + \rho_{24} \\ \rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} \end{pmatrix} \ln \begin{pmatrix} \rho_{11} + \rho_{22} + \rho_{55} & \rho_{13} + \rho_{24} \\ \rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} \end{pmatrix}$$

Inequalities (18) and (19) are new matrix inequalities.
7. Qubit–qutrit state

Now we consider the qubit–qutrit system and explain how the results for qudit with \( j = 2 \) are obtained. The pure states \( |m M\rangle \) of the system consisted of qubit (spin \( s_1 = 1/2 \)) and qutrit (spin \( s_2 = 1 \)), where spin projections are \( m = \pm 1/2 \) and \( M = 1, 0, -1 \), can be described also by the density 6×6-matrix \( \rho(1, 2) \) with matrix elements \( \rho_{mM, m'M'}(1, 2) \) of the form

\[
\rho_{mM, m'M'}(1, 2) = \\
\begin{pmatrix}
\rho_{1, 1}^{1, 1} & \rho_{1, 1}^{1, 0} & \rho_{1, 1}^{1, -1} & \rho_{1, 1}^{1, -0} & \rho_{1, 1}^{1, -1} & \rho_{1, 1}^{1, -2} \\
\rho_{1, 1}^{0, -1} & \rho_{1, 1}^{0, 0} & \rho_{1, 1}^{0, -1} & \rho_{1, 1}^{0, -2} & \rho_{1, 1}^{0, -3} & \rho_{1, 1}^{0, -4} \\
\rho_{1, 1}^{1, -1} & \rho_{1, 1}^{1, -0} & \rho_{1, 1}^{1, -1} & \rho_{1, 1}^{1, -2} & \rho_{1, 1}^{1, -3} & \rho_{1, 1}^{1, -4} \\
\rho_{1, 1}^{0, -2} & \rho_{1, 1}^{0, -1} & \rho_{1, 1}^{0, 0} & \rho_{1, 1}^{0, 1} & \rho_{1, 1}^{0, 2} & \rho_{1, 1}^{0, 3} \\
\rho_{1, 1}^{1, -2} & \rho_{1, 1}^{1, -1} & \rho_{1, 1}^{1, 0} & \rho_{1, 1}^{1, 1} & \rho_{1, 1}^{1, 2} & \rho_{1, 1}^{1, 3} \\
\rho_{1, 1}^{0, -3} & \rho_{1, 1}^{0, -2} & \rho_{1, 1}^{0, -1} & \rho_{1, 1}^{0, 0} & \rho_{1, 1}^{0, 1} & \rho_{1, 1}^{0, 2} \\
\end{pmatrix}
\]

The density 2×2-matrix of qubit \( \rho_{mm'}(1) \) is obtained by the partial trace of the matrix \( \rho_{mM, m'M'}(1, 2) \), i.e., \( \rho_{mm'}(1) = \sum_{M=-1}^{1} \rho_{mM, m'M}(1, 2) \). The density 3×3-matrix of qutrit \( \rho_{MM'}(2) \) is obtained by the partial trace of the matrix \( \rho_{mM, m'M'}(1, 2) \), i.e., \( \rho_{MM'}(2) = \sum_{m=-1/2}^{1/2} \rho_{mM, m'M'}(1, 2) \). These matrices read

\[
\rho_{mm'}(1) = \\
\begin{pmatrix}
\rho_{1, 1}^{1, 1} + \rho_{2, 0}^{1, 0} + \rho_{2, -1}^{1, -1} & \rho_{1, 1}^{1, -1} + \rho_{2, 0}^{1, -0} + \rho_{2, -1}^{1, -1} & \rho_{1, 1}^{1, -1} + \rho_{2, 0}^{1, -0} + \rho_{2, -1}^{1, -1} & \rho_{1, 1}^{1, -1} + \rho_{2, 0}^{1, -0} + \rho_{2, -1}^{1, -1} \\
\rho_{1, 1}^{0, -1} + \rho_{2, 0}^{1, 0} + \rho_{2, -1}^{1, -1} & \rho_{1, 1}^{0, -1} + \rho_{2, 0}^{1, 0} + \rho_{2, -1}^{1, -1} & \rho_{1, 1}^{0, -1} + \rho_{2, 0}^{1, 0} + \rho_{2, -1}^{1, -1} & \rho_{1, 1}^{0, -1} + \rho_{2, 0}^{1, 0} + \rho_{2, -1}^{1, -1} \\
\rho_{1, 1}^{1, 0} + \rho_{2, 0}^{1, 0} + \rho_{2, -1}^{1, -1} & \rho_{1, 1}^{1, 0} + \rho_{2, 0}^{1, 0} + \rho_{2, -1}^{1, -1} & \rho_{1, 1}^{1, 0} + \rho_{2, 0}^{1, 0} + \rho_{2, -1}^{1, -1} & \rho_{1, 1}^{1, 0} + \rho_{2, 0}^{1, 0} + \rho_{2, -1}^{1, -1} \\
\rho_{1, 1}^{0, 0} + \rho_{2, 0}^{1, 0} + \rho_{2, -1}^{1, -1} & \rho_{1, 1}^{0, 0} + \rho_{2, 0}^{1, 0} + \rho_{2, -1}^{1, -1} & \rho_{1, 1}^{0, 0} + \rho_{2, 0}^{1, 0} + \rho_{2, -1}^{1, -1} & \rho_{1, 1}^{0, 0} + \rho_{2, 0}^{1, 0} + \rho_{2, -1}^{1, -1} \\
\end{pmatrix}
\]

\[
\rho_{MM'}(2) = \\
\begin{pmatrix}
\rho_{1, 1}^{1, 1} & \rho_{1, 1}^{1, 0} & \rho_{1, 1}^{1, -1} & \rho_{1, 1}^{1, -0} & \rho_{1, 1}^{1, -1} & \rho_{1, 1}^{1, -2} \\
\rho_{1, 1}^{0, -1} & \rho_{1, 1}^{0, 0} & \rho_{1, 1}^{0, -1} & \rho_{1, 1}^{0, -2} & \rho_{1, 1}^{0, -3} & \rho_{1, 1}^{0, -4} \\
\rho_{1, 1}^{1, -1} & \rho_{1, 1}^{1, -0} & \rho_{1, 1}^{1, -1} & \rho_{1, 1}^{1, -2} & \rho_{1, 1}^{1, -3} & \rho_{1, 1}^{1, -4} \\
\rho_{1, 1}^{0, -2} & \rho_{1, 1}^{0, -1} & \rho_{1, 1}^{0, 0} & \rho_{1, 1}^{0, 1} & \rho_{1, 1}^{0, 2} & \rho_{1, 1}^{0, 3} \\
\rho_{1, 1}^{1, -2} & \rho_{1, 1}^{1, -1} & \rho_{1, 1}^{1, 0} & \rho_{1, 1}^{1, 1} & \rho_{1, 1}^{1, 2} & \rho_{1, 1}^{1, 3} \\
\rho_{1, 1}^{0, -3} & \rho_{1, 1}^{0, -2} & \rho_{1, 1}^{0, -1} & \rho_{1, 1}^{0, 0} & \rho_{1, 1}^{0, 1} & \rho_{1, 1}^{0, 2} \\
\end{pmatrix}
\]

The matrices \( \rho(1, 2), \rho(1), \) and \( \rho(2) \) are known to satisfy the subadditivity condition, which means the nonnegativity of the von Neumann mutual information,

\[
I_q = \text{Tr} (\rho(1, 2) \ln \rho(1, 2)) - \text{Tr} (\rho(1) \ln \rho(1)) - \text{Tr} (\rho(2) \ln \rho(2)) \geq 0.
\]

If there is no correlations, i.e., \( \rho(1, 2) = \rho(1) \otimes \rho(2) \), the mutual information is equal to zero.

8. Single qudit with \( j = 2 \)

Now we consider the density 5×5-matrix \( \rho_{\mu\mu'} \) of the single-qudit state with \( j = 2 \), where \( \mu, \mu' = 2, 1, 0, -1, -2 \), and explain how we obtained the relations of section 6. To reduce the problem to the consideration of the 6×6-matrix, we construct the 6×6 matrix \( \rho_{mM, m'M'}(1, 2) \) by adding a zero column and a zero row, i.e.,

\[
\rho_{mM, m'M'}(1, 2) = \\
\begin{pmatrix}
\rho_{22} & \rho_{21} & \rho_{20} & \rho_{2-1} & \rho_{2-2} & 0 \\
\rho_{12} & \rho_{11} & \rho_{10} & \rho_{1-1} & \rho_{1-2} & 0 \\
\rho_{02} & \rho_{01} & \rho_{00} & \rho_{0-1} & \rho_{0-2} & 0 \\
\rho_{-22} & \rho_{-21} & \rho_{-20} & \rho_{-2-1} & \rho_{-2-2} & 0 \\
\end{pmatrix}
\]
After this, we construct matrices $\rho_{m_1 m_2 m_3 m_1' m_2' m_3'}(1, 2, 3)$ in the same way, as we did for composite system, and obtain the matrices which numerically are the same as we already considered. This means that matrices $\rho(1, 2) \equiv \tilde{R}$, $\rho(1) \equiv \tilde{r}$, and $\rho(2) \equiv \tilde{\rho}$ of “subsystems” by partial tracing. Numerically they are the same matrices as were obtained for $j = 2$ qudit; this means that they satisfy the strong subadditivity condition, which we have already described and presented in the previous section as the nonnegativity of conditional mutual information $I_{\text{con}}$ for the five-level atom.

As a particular result, the strong subadditivity condition applied to diagonal density matrix of the five-level atom yields the inequality for any three-level populations $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ of the atom of the form

$$I_{\text{con}} = (\mathcal{P}_1 + \mathcal{P}_2) \ln(\mathcal{P}_1 + \mathcal{P}_2) + (\mathcal{P}_1 + \mathcal{P}_3) \ln(\mathcal{P}_1 + \mathcal{P}_3) - \mathcal{P}_1 \ln \mathcal{P}_1 - (\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3) \ln(\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3) \geq 0.$$ 

Formally this inequality means the nonnegativity of the conditional mutual information calculated for the three qubit state with a specific diagonal density $8 \times 8$-matrix, and it can be checked experimentally.

The inequality is valid for any three nonnegative numbers $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$, satisfying the inequality $\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 \leq 1$ independently of the interpretation of these numbers.

9. Conclusions

To conclude, we list the main results of our work.

First of all, it is worth pointing out that in this work we did not derive new inequalities but did use the well-known for bipartite and multipartite systems entropic-information relations to attract attention to the fact that, by employing the tool of a map of indices, these inequalities can also be applied to the system which does not contain subsystems, e.g., for a single qudit. For noncomposite systems, the relations discussed here, like the subadditivity condition, were not discussed in the literature.

The motivation of writing such entropic relations in explicit matrix form is connected with recent discussions of experimental studies of qudit density matrices, where the entropic-information matrix formulas can be checked using the data characterizing the states of superconducting quantum circuits realized by Josephson junctions [22–32].

We reviewed the approach to the set of nonnegative numbers and Hermitian nonnegative matrices with unit trace, in view of the interpretation of the numbers and the matrices as the probability distributions and the density matrices, respectively.
Symmetries in Science XVII

IOP Conf. Series: Journal of Physics: Conf. Series 1071 (2018) 012015
doi:10.1088/1742-6596/1071/1/012015

We showed that the known entropic inequalities, which are applied to composite systems, both classical and quantum, can be also applied to the systems without subsystems. We obtained new entropic inequalities, in view of the interpretation of the density matrix of the system state as the density matrix of artificial bipartite or tripartite system.

In fact, the approach presented provides the possibility to extend all known entropic-information relations for classical and quantum composite systems to the case of systems without subsystems, and the relations reflect the presence of correlations either classical or quantum ones of the system degrees of freedom. The quantum correlations of the single qudit states, which we call hidden correlations [33, 34] can be used in quantum technologies, analogously to the employment of entanglement as a quantum resource.

To clarify the physical meaning of hidden correlations in noncomposite (nondivisible) systems, we define the “events” of these systems as “pairs” or “triples” of the “initial events,” statistics of which we study employing either the probability distribution $p(s)$ in classical domain or the density matrix $\rho_{ss'}$ in quantum domain.

Acknowledgments
The author is grateful to the Organizers of the 2017 Symposium “Symmetries in Science” (Bregenz, Austria, July 30 – August 4, 2017) and especially to Prof. Dr. Dieter Schuch for invitation and kind hospitality.

References
[1] Kolmogorov A N 1956 Foundations of the Theory of Probability, 2nd ed., Chelsea Publishing Company, New York, USA
[2] Shannon C E 1948 Bell. Syst. Tech. J. 27 379; 623
[3] Schrödinger E 1926 Ann. Phys. 79, 361; 81, 109
[4] Landau L D 1927 Z. Phys. 45 430
[5] von Neumann J 1927 Göttingenische Nachrichten 11 245
[6] Mancini S, Man’ko V I and Tombesi P 1996 Phys. Lett. A 213 1
[7] Dodonov V V and Man’ko V I 1997 Phys. Lett. A 229 335
[8] Man’ko V I and Man’ko O V 1997 J. Exp. Theor. Phys. 85 430
[9] Smitey T D, Beck M, Raymer M G and Faridani A 1993 Phys. Rev. Lett. 70 1244
[10] Man’ko M A and Man’ko V I 2011 Found. Phys. 41 330
[11] Man’ko M A and Man’ko V I 2011 “Dynamic symmetries and entropic inequalities in the probability representation of quantum mechanics,” Latin-American School of Physics XI ELAF: Symmetries in Physics, AIP Conf. Ser., Vol. 1334, pp. 217–248
[12] Man’ko M A and Man’ko V I 2012 “Tomographic entropic inequalities in the probability representation of quantum mechanics,” in: R. Bijker, (Ed.), Beauty in Physics: Theory and Experiment: in Honor of Francesco Iachello on the Occasion of His 70th Birthday, Hacienda Cocoyoc, Mexico, 14-18 May 2012, AIP Conference Proceedings, Vol. 1488, pp. 110–121
[13] Man’ko M A 2013 Phys. Scr. 87 038113
[14] Man’ko M A and Man’ko V I 2014 Phys. Scr. T160, 014030
[15] Man’ko M A, Man’ko V I, Marmo G, Simoni A and Ventriglia F 2013 Nuovo Cim. C 36 163
[16] Man’ko M A and Man’ko V I 2014 Int. J. Quantum Inf. 12 156006.
[17] Man’ko M A and Man’ko V I 2016 J. Phys.: Conf. Ser. 698 012004
[18] Man’ko M A and Man’ko V I 2015 Entropy 17 2876
[19] Schrödinger E 1935 Naturwissenschaften 23 807
[20] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information, Cambridge University Press, Cambridge, UK
[21] Tsallis C 2001 “Nonextensive statistical mechanics and thermodynamics: historical background and present status,” in: S. Abe and Y. Okamoto (Eds.), Nonextensive Statistical Mechanics and Its Applications, Lecture Notes in Physics, Springer, Berlin, Vol. 560, pp. 3–98
[22] Dodonov V V, Man’ko V I and Man’ko O V 1989 J. Sov. Laser Res. 10 413
[23] Man’ko V I 1991 J. Sov. Laser Res. 12 383
[24] Man’ko O V 1994 J. Korean Phys. Soc. 27 1
[25] Pashkin Yu A, Yamamoto T, Astafiev O, Nakamura Y, Averin D V and Tsai J S 2003 Nature 421 823
[26] Devoret M H, Wallraff A and Martinis J M 2004 “Superconducting qubits: A short review” arXiv:cond-mat/0411174v1
[27] Devoret M H and Schoelkopf R J 2013 Science 339 1169
[28] Shalibo Y, Resh R, Fogel O, Shwa D, Bialczak R, Martinis J M and Katz N 2013 Phys. Rev. Lett. 110 100404
[29] Braumüller J, Cramer J, Schlör S, Rotzinger H, Radtke L, Lukashenko A, Yang P, Marthaler M, Guo L, Ustinov A V and Weides M 2015 Phys. Rev. B 91 054523
[30] Fedorov A K, Kiktenko E O, Man’ko O V and Man’ko V I 2015 Phys. Rev. A 91 042312
[31] Kiktenko E O, Fedorov A K, Strakhov A A and Man’ko V I 2015 Phys. Lett. A 379 1409
[32] Glushkov E, Glushkova A and Man’ko V I 2015 J. Russ. Laser Res. 36 448
[33] Man’ko M A and Man’ko V I 2015 J. Russ. Laser Res. 36 301
[34] Man’ko M A 2017 J. Russ. Laser Res. 38 211