Black Strings in Gauss-Bonnet Theory are Unstable

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Resumen

We report the existence of unstable, s-wave modes, for black strings in Gauss-Bonnet theory (which is quadratic in the curvature) in seven dimensions. This theory admits analytic uniform black strings that in the transverse section are black holes of the same Gauss-Bonnet theory in six dimensions. All the components of the perturbation can be written in terms of a single one and its derivatives. For this latter component we find a master equation which admits bounded solutions provided the characteristic time of the exponential growth of the perturbation is related with the wave number along the extra direction, as it occurs in General-Relativity. It is known that these configurations suffer from a thermal instability, and therefore the results presented here provide evidence for the Gubser-Mitra conjecture in the context of Gauss-Bonnet theory. Due to the non-triviality of the curvature of the background, all the components of the metric perturbation appear in the linearized equations. As it occurs for spherical black holes, these black strings should be obtained as the short distance limit $r \ll \alpha^{1/2}$ of the black string solution of Einstein-Gauss-Bonnet theory, which is not know analytically, where $\alpha$ is the Gauss-Bonnet coupling.\footnote{alexgiacomini@uach.cl, julio.oliva@uach.cl, aldovera@udec.cl}
I. INTRODUCTION

Gravity in higher dimensions has been an important scenario to test how generic are the notions we have gained from four-dimensional gravity. Also motivated by String Theory and supergravity, many results have been learned in the last decades concerning gravity in dimensions higher than four, as for example the existence of the asymptotically flat black rings and all its extensions (for a review see [1] and [2]). These objects were conjectured to be unstable for large angular momentum, as they inherit the Gregory-Laflamme instability [5] of non-extremal black strings and black p-branes [3]-[4]. Indeed the black ring instability has been confirmed in [6], [7], [8]. The Gregory-Laflamme instability can be guessed from thermodynamical arguments since, as a function of the mass, the entropy of the black hole and the black string cross at a given critical mass \( M_c \). This can be seen from the fact that the entropy of the black hole grows as \( S_{BH} \sim M^{D-2} \), while the entropy of the black string grows as \( S_{BS} \sim M^{D-4} \). For masses below \( M_c \), the black hole is thermally favoured and above \( M_c \) is the black string solution the one with greater entropy and therefore the most favoured. This relation between thermal and perturbative instabilities led Gubser and Mitra to conjecture that both kinds of instabilities always appear together for black hole configurations with extended directions [9], and is was recently proved in [10] for General Relativity in vacuum. To understand the complete evolution of the unstable mode a non-linear analysis is required. Recent outstanding numerical results seem to indicate that the black string evolves toward a non-homogenous configuration with section for which the size of the string eventually shrinks to zero generating a null singularity and providing a counterexample of the cosmic censorship conjecture [12] (for a historical review on this problem see Chapter 2 of [2]).

An interesting problem is whether higher curvature corrections may modify this scenario. In the particular case of higher curvature Lovelock theories [13] it is difficult to construct analytic, homogenous, black strings due to the fact that the new dimensionful coupling constants introduce a length scale that induces the existence of a cosmological constant. Numerical and approximate results in this context have been reported in [14]-[18]. The situation in theories that have a single Lovelock term is much more like the one in General Relativity, since as shown in [19] homogeneous black strings and black p-branes can be constructed analytically. These solutions are also important since they should be obtained as the short distance configuration \( r << \alpha^{1/2} \) of the black string solution of Einstein-
Gauss-Bonnet theory, which is not known analytically. Here $\alpha$ is the Gauss-Bonnet coupling. This is what occurs for example with the “good branch” of the spherically symmetric black hole in Einstein-Gauss-Bonnet gravity, which is defined by the following action:

$$ I_{EGB} [g] = \frac{1}{16\pi G} \int d^D x \sqrt{-g} \left[ R + \alpha \left( R^2 - 4 R_{\mu \nu} R^{\mu \nu} + R_{(\alpha \beta \gamma \delta)} R^{(\alpha \beta \gamma \delta)} \right) \right]. $$

(1)

This theory admits the following black hole solution [20]:

$$ ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2, $$

(2)

where

$$ f(r) = 1 + \frac{r^2}{(D-3)(D-4)\alpha} \left[ 1 - \sqrt{1 + \frac{64\pi G (D-3)(D-4)\alpha M}{(D-2)V(S^{D-2})r^{D-1}}} \right]. $$

(3)

where the integration constant $M$, is the mass. Here $\alpha$ has dimensions of length square and we can analyze the behavior of this metric function for $r >> \sqrt{\alpha}$ and $r << \sqrt{\alpha}$ which respectively read:

$$ f(r) \approx \frac{32\pi G M}{(D-2)V(S^{D-2})r^{D-3}} + \ldots $$

(4)

$$ f(r) \approx \frac{64\pi GM}{(D-3)(D-4)(D-2)\alpha V(S^{D-2})} \frac{1}{r^{D-5}} + \ldots. $$

(5)

In the former case the solution reduces to the Schwarzschild-Tangherlini black hole, while in the latter it reduces to the solution found in [21]. Therefore, we have that for large distances, the effects of the quadratic curvature correction are sub-leading, whereas for short distance (as compared with $\sqrt{\alpha}$) the quadratic terms dominate and one recovers a solution of Gauss-Bonnet theory.

As shown in [19], the asymptotically flat black holes constructed in [21] can be oxidated to construct homogenous black string and black p-brane solutions. These spacetimes are solutions of the theory that contains only the $k-th$ order term in the Lovelock theory, being the case $k = 1$ the one of General Relativity. For simplicity let’s consider only the quadratic Gauss-Bonnet term in seven dimension:

$$ I_{EGB} [g] = \frac{\alpha}{16\pi G} \int d^7 x \sqrt{-g} \left[ R^2 - 4 R_{\mu \nu} R^{\mu \nu} + R_{(\alpha \beta \gamma \delta)} R^{(\alpha \beta \gamma \delta)} \right]. $$

(6)

This theory has the following two solution

$$ ds^2 = -\left( 1 - \frac{\mu}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{\mu}{r}} + r^2 d\Omega_5^2, $$

(7)
and
\[ ds^2 = - \left( 1 - \frac{m}{r^{1/2}} \right) dt^2 + \frac{dr^2}{1 - \frac{m}{r^{1/2}}} + r^2 d\Omega_4^2 + dz^2 , \tag{8} \]
which correspond to an spherically symmetric black hole and a black string, respectively. The constants \( m \) and \( \mu \) determine the masses of the configurations while \( d\Omega_n \) stands for the line element of the \( n \)–sphere, \( S^n \). By the experience gained from the spherically symmetric black hole, one can expect that these black strings should be obtained as the short distance limit of the, not known analytically, black string of Einstein-Gauss-Bonnet theory in seven dimensions.

The black strings and black p-branes constructed in this way where proved to be thermally unstable \[19\] exactly in the same manner than the black strings in General Relativity, since the entropies, as a function of the mass for \( (7) \) and \( (8) \) read \( S_{BH}^{GB} \sim M^{3/2} \) and \( S_{BS}^{GB} \sim M^2 \), respectively and they do cross at a critical mass \( M_{c}^{GB} \).

A natural question is whether such thermal instability has a perturbative counterpart. In this paper we show this is indeed the case. In the next section we show that the black strings of Gauss-Bonnet theory \( (8) \) are unstable under the s-wave mode and that such instability disappears for compactified black strings that are short enough.

II. THE PERTURBATIVE INSTABILITY

Here we will be concerned with gravitational perturbations in the context of Gauss-Bonnet theory. The field equations are therefore given by
\[ E_{\mu\nu} := 2RR_{\mu\nu} - 4R_{\mu\rho\sigma\tau}R^{\rho\sigma\tau\nu} - 4R_{\mu\rho\tau}R_{\nu}^{\rho\sigma\tau} - 4R_{\mu\rho}R_{\nu}^{\rho} - \frac{1}{2}g_{\mu\nu} \left( R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \right) = 0 . \tag{9} \]
For simplicity we will focus in the seven dimensional case. As mentioned above, this theory admits the homogenous black string solution \( (8) \). The radius of the horizon reads \( r_+ = m^2 \).

In order to work with a finite range of parameters, let’s consider the change in the radial coordinate given by
\[ r = \left( \frac{m}{1 - x} \right)^2 , \tag{10} \]
that maps the region outside the event horizon \( r \in [m^2, +\infty] \) to \( x \in [0, 1] \). In this new coordinated the metric (8) reads
\[
d s^2_{BS7} = -x dt^2 + \frac{4m^4 dx^2}{x(1-x)^6} + \left( \frac{m}{1-x} \right)^2 d\Omega^2 + dz^2 .
\] (11)
The s-wave perturbation on the background, black string metric (11) reads
\[
h_{\mu\nu}(t, x, z) = e^{i\Omega t} e^{ikz} \begin{pmatrix}
H_{tt}(x) & H_{tx}(x) & 0 & 0 \\
H_{tx}(x) & H_{xx}(x) & 0 & 0 \\
0 & 0 & H(x) & \sigma_{S^4} \\
0 & 0 & 0 & 0
\end{pmatrix},
\]
where \( \sigma_{S^4} \) is the metric of the four sphere and \( k \) is the wave number along the \( z \) direction. An unstable mode is defined as a bounded solution of the linearized Gauss-Bonnet equations (9) with positive \( \Omega \). It easy to show that the linearized field equations imply that the components of the perturbation can be written in terms of \( H_{tx}(x) \) in the following manner
\[
H_{tt}(x) = \frac{(1-x)^6 x^2}{4m^4\Omega^4} \frac{d^2}{dx^2} H_{tx} + \frac{x(1-x)^6}{4m^4\Omega} H_{tx}
\] (12)
\[
H_{xx}(x) = -\frac{x}{\Omega} H_{tx}'' + \frac{2(1-4x)}{(1-x)\Omega} H_{tx}' + \left( \frac{(3k^2 x + 4\Omega^2) m^4}{x(1-x)^6 \Omega} + \frac{6}{\Omega(1-p)} \right) H_{tx}
\] (13)
\[
H(x) = \frac{x^2(1-x)^2}{6\Omega} H_{tx}'' + \frac{(1-3x)(1-x)x}{2\Omega} H_{tx}' + \left( \frac{(1-x)(1-7x)}{6\Omega} - \frac{m^4(3k^2 x + 4\Omega^2)}{6(1-x)^4 \Omega} \right) H_{tx},
\] (14)
where the prime (‘) denotes differentiation with respect to \( x \). The component \( H_{tx}(x) \) fulfils the following linear, second order, master equation
\[
A(x) H_{tx}'' + B(x) H_{tx}' + C(x) H_{tx} = 0 ,
\] (15)
with
\[
A(x) = (1-x)^6 x^2 \left( (1-x)^6 - (12k^2 x + 16\Omega^2)m^4 \right) ,
\] (16)
\[
B(x) = 3x(1-x)^5 \left( (32k^2 x^2 + 48x\Omega^2 - 8k^2 x - 16\Omega^2)m^4 + (1-x)^7 \right) ,
\] (17)
\[
C(x) = 4 \left( 4\Omega^2 + 3k^2 x \right)^2 m^8 + (1-x)^6(45k^2 x^2 + 164x\Omega^2 + 3k^2 x - 20\Omega^2)m^4 + (1-x)^{12} .
\] (18)
Then, one can see that all the linearized field equations are solved provided (12)-(14) and (15), hold. Note that the master equation is invariant under
\[
m \to \alpha m, \Omega \to \alpha^{-2}\Omega, k \to \alpha^{-2}k ,
\] (19)
therefore it is enough to study the existence of unstable modes for a fixed value of the horizon radius \( r_+ = m^2 \), since the other can be obtained by applying the scaling symmetry (19).

We are then left with finding a well-behaved solution of the master equation (15). This equation implies that the solution \( H_{tx}(x) \) admits the following asymptotic behaviors at the horizon \( (x \to 0) \) and at infinity \( (x \to +1) \), respectively:

\[
H_{tx} \to_{x \to 0} C_\pm x^{-1 \pm 2m^2\Omega} (1 + \mathcal{O}(x)), \\
H_{tx} \to_{x \to 1} E_\pm (1-x)^{\alpha_\pm} e^{\mp m^2\sqrt{\Omega^2+4\Omega^2} + (\sqrt{3k^2+4\Omega^2})m^2\sqrt{3k^2+4\Omega^2}} (1 + \mathcal{O}(1-x)),
\]

with

\[
\alpha_\pm = \frac{1}{8} \frac{-12(3k^2 + 4\Omega^2)^2 \pm m^2(144k^2\Omega^2 + 128\Omega^4 + 27k^4)\sqrt{3k^2 + 4\Omega^2}}{(3k^2 + 4\Omega^2)^2}.
\]

Since we are looking for unstable modes, we need to find a numerical solution that interpolates between the plus sign in (20) and the minus sign in (21). It’s natural to think that in order to have a well posed behavior at the horizon we need to impose \( \Omega > \Omega_{c,GB} := \frac{1}{2m^2} \), as it was originally considered in [22] where it was proved that in the five dimensional black string in General Relativity there is no non-singular, single, unstable mode in this family (in G.R. in five dimensions \( \Omega_{c,G.R.} = \frac{1}{r_+} \)). Nevertheless in General Relativity, in the range \( 0 < \Omega < \Omega_{c,GR} \) one can construct a perturbation that is a composition of single divergent modes at the horizon in such a manner that the divergences cancel, as it occurs for the instabilities in some colored black holes [23] and originally observed by Vishveshwara in [24]. This can be seen also considering the fact that a \( t = const \) surface intersects the bifurcation surface rather than the future horizon. It it therefore necessary consider Kruskal-like coordinates, where the \( T = const \) surfaces do indeed intersect the future event horizon. Then, by going to Kruskal coordinates, it is easy to see that the unstable modes we find below are regular at the future horizon provided we choose the branch with the plus sign in (20).

In order to find whether the master equation (15) admits a bounded solution for some positive values of \( \Omega \), we will follow the approach developed in [25] for quasinormal modes. Briefly, the method consists in proposing a power series solution around the horizon, then selecting the well behaved branch and finally truncate the power series to some order \( N \). Then use the fact that, due to the pole structure of the equation such a power series has a convergence radius that at least includes \( x = 1 \) and therefore we can request for the truncated series to vanish at infinity \( (x = 1) \). Such an equation provides for the spectrum of unstable modes. For details we refer to the original work [25].
The results of the previous analysis are depicted in Figure 1:

![Graph showing Ω vs k for the homogeneous black string in Gauss-Bonnet in D = 7. The parameter m in the solution has been fixed to 1, and any value for the mass can be obtained by applying the scaling transformation in (19). The numerical precision is such that all the digits in Ωs are stable (the continuous curve has been included to facilitate the visualization).](image)

From these results we see that there is a minimum wavelength $\lambda_{\text{min}}$ above which instability occurs. This also implies the existence of a critical length for the string, above which the instability takes place.

As it occurs for the black string in General Relativity [4], as far as $k \neq 0$, one can show that the perturbation cannot be gauged away. Another straightforward method to check that these perturbations are physical and cannot be gauged away is to consider the following scalar invariant

$$A = 881 R_{abcd} R^{e}{}_{ef} R^{f}{}_{ab} + 2428 R^{ab} R^{cd} R^{ce} R^{df} R^{df} R^{ae},$$

which vanishes identically on the unperturbed metric, but it is non-vanishing for the perturbed black string.

We have then found a set of physical s-wave modes on the black string in Gauss-Bonnet theory [8], which drive the instability of the background, and therefore black strings in Gauss-Bonnet theory are unstable.

**III. CONCLUSIONS**

In this paper we have shown that the black strings in Gauss-Bonnet theory are unstable under gravitational perturbations. Following the arguments in [4], one can prove that the
instability we have found cannot be gauged away, and therefore it represents a truly physical
instability. Since the field equations are quadratic in the curvature, the linearization around
the maximally symmetric Minkowski vacua does not provide any equation at all\textsuperscript{1}, therefore in
order to study the perturbative properties of the solutions of Gauss-Bonnet gravity one needs
to perturb around solutions that have a non-trivial Riemann tensor as it is the case of the
black string. As mentioned above the linearized equations around such a background are non-
degenerate since all the perturbed metric components appear in the linearized equations. In
order the black strings to be unstable, the wavelength of the perturbation along the extended
direction has to be above some minimum critical value $\lambda_c$. This critical value tends to zero
in the large $D$ limit in General Relativity \textsuperscript{27}-\textsuperscript{28}, and since the large $D$ behavior of G.R. is
qualitatively similar to the one in gravity theories with a single Lovelock term \textsuperscript{29}, one may
also expect $\lambda_c \to 0$ as $D$ grows for the black strings and black p-branes constructed in \textsuperscript{19}. Given the results presented in this work it is natural to expect that the black string solution
of the full Einstein-Gauss-Bonnet theory will suffer from the Gregory-Laflamme instability,
which will induce an instability for large angular momentum in the rotating version of the
static black string constructed numerically in \textsuperscript{30}.

For Einstein-Gauss-Bonnet gravity different stability analysis of black holes have been
performed in \textsuperscript{31}-\textsuperscript{35} and it would be interesting to extend such analysis to the black holes
of \textsuperscript{21} that are in the transverse section of the black strings we have constructed here.\textsuperscript{2}

It’s worth also to explore whether the results presented here can be extended to all the
black strings and black p-branes obtained in \textsuperscript{19}, and even to the compactifications with
Einstein manifold in four dimensions that were obtained in \textsuperscript{41} for Lovelock theories. Work
along these lines is in progress.

Acknowledgments

The authors are grateful to Marco Astorino, Fabrizio Canfora, Gustavo Dotti and Sourya
Ray for useful discussions. The authors also thank Gaston Giribet for enlightening comments.
This work has been supported by FONDECYT Regular grants 1141073 and 1150246. This
project was also partially funded by Proyectos CONICYT, Research Council UK (RCUK)

\textsuperscript{1} As an example for how to deal with the phase space structure of such degenerate systems see e.g. \textsuperscript{26}.
\textsuperscript{2} It would be also interesting to extend these results to cases with more general asymptotic behaviours as
the ones e.g. \textsuperscript{36}-\textsuperscript{40}.
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