Curvature and Chaos in General Relativity

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Abstract

We clarify some points about the systems considered by Sota, Suzuki and Maeda in Class. Quantum Grav. 13, 1241 (1996). Contrary to the authors’ claim for a non-homoclinic kind of chaos, we show the chaotic cases they considered are homoclinic in origin. The power of local criteria to predict chaos is once more questioned. We find that their local, curvature-based criterion is neither necessary nor sufficient for the occurrence of chaos. In fact, we argue that a merit of their search for local criteria applied to General Relativity is just to stress the weakness of locality itself, free of any pathologies related to the motion in effective Riemannian geometries.

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I. INTRODUCTION

In a recent paper Sota, Suzuki and Maeda [1] found new support to the important issue of chaotic behavior in General Relativity (GR) [2,3]. Based on the sign structure of the eingenvalues of Weyl’s tensor, they also proposed what should be a sufficient, local criterion for the occurrence of chaos in GR (we will refer to this criterion as SSM after the authors). They restrict the analysis only to the case of geodesic motion of test particles in fixed, exact spacetime geometries, more specifically, in static axisymmetric spacetimes in vacuum.

In section 2 firstly we clarify some misunderstandings about the systems considered in [1] related to the SSM statement. Contrary to the authors’ claim, we show that the chaotic cases presented in [1] have a homoclinic origin. We also emphasize that the families of unstable periodic orbits (UPOs) related to the homoclinic tangle are not the same in the static cases considered in [1] and in the time–depending cases studied for example in [4].

In section 3 we find that SSM is neither necessary nor sufficient as a criterion to predict chaos. We address this insufficiency in relation to the deeper question whether local criteria can predict chaos. The central idea and the numerical results of [1] become important to this end. As space–time is intrinsically Riemannian in GR, we argue that a main merit of Sota, Suzuki and Maeda’s search for local criteria applied to GR does not have yet been noted and it is just to exhibit the weakness of locality itself, free of any pathologies related to the motion in effective Riemannian geometries.

II. THE UNDERLYING HOMOCLINIC DYNAMICS

We show in this section that the idea of existing two types of chaos in the systems studied in [1] is wrong. The first type should be associated to the so called homoclinic tangle of the unstable/stable manifolds departing/arriving from/at an UPO. It should rise due to the breaking of the reflection symmetry of the system about the middle plane (as seen in fig. 10 of [1]). The second “new” kind of chaos should rise when that symmetry is preserved
(as seen, e.g., in fig. 4 of [1]) and should be explained by the occurrence of certain locally unstable regions (named LU regions in that paper) rather than a homoclinic tangle. We show in the following that, also in the latter case, the evidences for the homoclinic tangle is a matter of figure scale allied to a careful search of the structures involved in the tangle. It seems that this was what improperly motivated the statement of SSM as a sufficient criterion.

Consider the 2–Curzon system with the same numeric values for energy, angular momentum, etc. of fig. 4. This configuration exhibits chaos yet preserving the reflection symmetry about the middle plane. We found that there exists an UPO confined to the plane $z, p^z$ and characterized by $p^\rho = 0$ and $\rho = \rho_0 \approx 4.4833 R_0$ where $R_0 = GM/c^2$ (this is a local maximum of the effective potential in the middle plane). This UPO is transversal to the middle plane $z = 0$. On the other hand, the geodesic flow has reflection symmetry about that plane in the four–dimensional phase space $(\rho, z, p^\rho, p^z)$. Both facts makes the middle plane ideal, as Poincaré’s section, to unravel the whole homoclinic tangle: the UPO itself will appear in the section as a fixed point and the four (two–dimensional) unstable/stable manifolds emanating from the UPO will intersect the section in four (one–dimensional) X–type branches departing/arriving from/at the fixed point (at the center of the X).

We stress that the (one–dimensional) UPO above plus its associated (two–dimensional) manifolds must not be confused with an unstable equilibrium point plus the (one–dimensional) unstable/stable manifolds emanating from it. In the case with reflection symmetry, the last structure does also exist and lies confined to the middle plane. Although immersed in the same four–dimensional phase space, both structures are in different energy surfaces and hence do not share with the same Poincaré’s section. We also mention that by a suitable periodic perturbation we can break the integrability around the unstable equilibrium point yet maintaining the dynamics confined to the middle plane, as studied, e.g., in [4]. In the latter case we are left with a so called one and a half degree of freedom system in the extended three–dimensional phase space $(\rho, p^\rho, t)$. In this case the Poincaré’s sections are better constructed as discretized maps of the continuos flow in the plane $(\rho, p^\rho)$ at times.
\( t = nT, n = 0, 1, 2, \cdots \), where \( T \) is the period of the perturbation. In the present static situation with reflection symmetry we deal with the whole four-dimensional phase space yet preserving the integrability of the family of orbits lying in the middle plane, to which the unstable equilibrium point together with its homoclinic orbit pertains.

Indeed, all that is shown in our figure 1. Fig. 1–a is basically the same as fig. 4 of [1] except that we add the elements related to the UPO namely, the central fixed point and its X–type branches (only the two right branches are shown in fig. 1–a), and also the boundary of the motion, which is itself one among the whole family of closed orbits confined to the middle plane. If we amplify the left region of fig. 1–a and specialize the study there, we obtain fig. 1–b. We see that among the four branches associated to the central fixed point, only the two on the left are chaotic (at least in the figure scale). For the sake of clarity we show in fig. 1–b only a half of the full homoclinic tangle, i. e., only that associated to the unstable left branch. It was numerically obtained by integrating about 1000 starting points equally spaced on the segment \( \rho/R_0 \in [4.427, 4.477] \), on the straight line given by equation \( p^\rho/\mu c = 0.00461478 \times (\rho/R_0 - 4.477) \). The numerical and hence approximate character of this search is responsible for the filling of the inner region of the half–tangle. The overall figure has about 19000 points. In other words, geodesic equations are a dynamical system prior of being a relativistic one. As such, it must exhibit the universal features — the homoclinic tangle among them — expected within the general context of dynamical systems.

We note that all this remains valid for any other figure of [1]. The question is that finding UPOs in phase space is in general a very difficult task. Moreover, given an UPO, the stable/unstable manifolds emanating from it have a rather complicated cylindrical topology in the full phase space [5]. Then, in generic cases, we need a correspondingly complicated nonplanar Poincaré’s section to follow and suitably intersect those cylindrical manifolds to see the related homoclinic figure. Cases like our fig. 1 are possible only when the symmetries of the motion are strong enough to constrain the cylindrical manifolds emanating from the UPO to remain always longitudinal to a planar section. In this sense, fig. 1 provided by the 2–Curzon system is an amazing exact relativistic example of full homoclinic figure captured
from a differential system.

III. CURVATURE, CHAOS AND GENERAL RELATIVITY

The most celebrated conclusion (and the only really sound until now) about the relation between local, curvature–based properties and chaoticity of the motion is the following: the geodesic flow on a compact manifold with all sectional curvatures negative at every point is chaotic. This very special case has inspired the search for related extensions in both Newtonian theory and GR. In the Newtonian theory, we apply Maupertuis’ principle in the context of Hamiltonian dynamics to provide the motion with an effective Riemannian manifold and search for criteria able to predict chaotic behavior from its curvature properties (see for instance [6,7]). The application of this method to cosmological models [8,9] intends mainly to formulate gauge invariant criteria for the occurrence of chaos in GR.

An alternative to assure gauge invariance to the methods lying on Maupertuis’ principle is proposed in [10] and applied with little changes to the Bianchi IX cosmological model [11]. The method consists also of local analysis of sign structure of eigenvalues, this time based on invariant curvature polynomials constructed after the conformal transformation of the metric (this contrasts, for instance, with Lyapunov’s exponents approach, which is a global yet highly gauge dependent method). The method points to the occurrence of local instability in the Bianchi IX model near the singularity. This adds to a plethora of early analysis and numerics (see for instance the references cited in [12]), which reinforces even more the increasingly accepted idea that Bianchi IX is after all chaotic in some meaningful sense.

This line of search, yet interesting, remains inconclusive to the end of predicting chaos, for several reasons: i) *Locality*. The method deals with the geodesic deviation equation on the effective manifold, however, local instability of the geodesic flow does not imply chaos, even when the curvature is everywhere negative, as exemplified in [13]. ii) *Averaging*. It ever involves some kind of average over the geodesic deviation equation to produce quantitative information about chaos, which may suppress important aspects of the true motion and
even lead to the lack of gauge invariance of the quantities and iii) Pathologies. The effective manifold exhibits boundaries which can trouble the understanding of the true motion in the physical manifold. These problems are pointed out in [8] and discussed in details in [14,14]; we also mention that in a recent paper Szydlowski et. al. [15] propose to circumvent the last difficulty above at the price of abandoning smooth manifolds in favor of lesser stringent differential spaces.

The good idea behind [1] is to work directly in spacetime in GR, since it is intrinsically Riemannian, and to retain a truly local approach of the geodesic deviation equation. This procedure has the merit of testing locality itself, free of any artifacts involving mean motion or boundaries of an effective manifold (in a second part of their paper the authors make averages for some quantitative estimates, which is of no concern here). So, possible relations between curvature and chaos can be studied directly in the physical configuration space, leading eventually to more clean conclusions about them.

SSM proposed in [1] is just an example of such a guessed relation: what is the role of the sign structure of the eingenvalues of the curvature tensor for the onset of chaos? They find correlations that, although interesting, are not enough to state a criterion for the occurrence of chaos in GR. In fact, these correlations associated to the existence of certain locally unstable regions (LU regions), are neither necessary nor sufficient to predict chaos, as it is already clear from the numerical study made by themselves. This can be clearly seen for instance in figures 4 and 10 of their paper. In fig. 4 there exists a region of preserved (small) tori, surrounded by the larger chaotic region all being totally immersed in a LU region, while in fig. 10 the LU region does not meet anywhere the chaotic region. Probably, the authors’ claim for the sufficiency of SSM had its motivation in the subtleties already discussed in section 2. This just exhibits the weakness of locality itself, free of any problems related to the motion in effective Riemannian geometries.

As expected, we can conclude that any local analysis, in effective or even physical spaces, is far from being sufficient to predict a global phenomenon like chaotic motion. Conversely however, it is natural to suppose that chaoticity, if it is there, exerts influence on certain
local properties, e.g., on the sign structure of the eigenvalues of the curvature tensor in that neighborhood. Obviously, this in no way prevent us to find partial methods that may in some sense feel the presence of chaos. This is all we need in many practical situations. However, they are not sufficient for the study of the role played by chaos as a basic phenomenon. In this direction, it is clear that we need more elaborated prescriptions to approach this rather complex subject; as dealt with, for instance, in [13].

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FIGURES

FIG. 1. Poincaré’s sections through the plane $z = 0$ in Weyl coordinates for timelike geodesic orbits in 2-Curzon geometry, with nondimensional parameters $E^2 = 0.913(\mu c^2)^2$ and $L = 6.94G\mu M/c$, where $M$ is the (same) mass of the singularities located at $\pm 2GM/c^2$ on the $z$ axis and $\mu$ is the mass of the test particle. In (a) we show the main global structures of the section namely, the small (on the left) and the great (on the right) regions of preserved tori separated by the X–type unstable/stable branches centered at the fixed point corresponding to the UPO (only the two right branches are exhibited here) and the closed boundary orbit lying in the middle plane. Part (b) is an amplification of the left region of (a) and shows a half of the homoclinic tangle surrounding the small tori, associated to the left (unstable) branch departing from the fixed point, all immersed in a LU region.
