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A Purity Theorem for Abelian Schemes

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1. Introduction

Let $K$ be the field of fractions of a discrete valuation ring $O$. Let $Y$ be a flat $O$-scheme that is regular, and let $U$ be an open subscheme of $Y$ whose complement in $Y$ is of codimension in $Y$ at least 2. We call the pair $(Y, U)$ an extensible pair. Let $q : S \rightarrow \text{Sch}_O$ be a stack over the category $\text{Sch}_O$ of $O$-schemes endowed with the Zariski topology. Let $S_Z$ be the fibre of $q$ over an $O$-scheme $Z$. Answers to the following Question provide information on $S$.

QUESTION 1.1. Is the pull-back functor $S_Y \rightarrow S_U$ surjective on objects?

Question 1.1 has a positive answer in any one of the following three cases:

(i) $S$ is the stack of morphisms into the Néron model over $O$ of an abelian variety over $K$, and $Y$ is smooth over $O$ (see [N]);

(ii) $S$ is the stack of smooth, geometrically connected, projective curves of genus at least 2 (see [M-B]);

(iii) $S$ is the stack of stable curves of locally constant type, and there is a divisor $\text{DIV}$ of $Y$ with normal crossings such that the reduced scheme $Y \backslash U$ is a closed subscheme of $\text{DIV}$ (see [dJO]).

Let $p$ be a prime. If the field $K$ is of characteristic 0, then an example of Raynaud–Gabber–Ogus shows that Question 1.1 does not always have a positive answer if $S$ is the stack of abelian schemes (see [dJO, Sec. 6]). This invalidates [FaC, Chap. IV, Thms. 6.4, 6.4′, 6.8] and leads to the following problem.

PROBLEM 1.2. Classify all those $Y$ with the property that, for any extensible pair $(Y, U)$ with $U$ containing $Y_K$, every abelian scheme (resp., every $p$-divisible group) over $U$ extends to an abelian scheme (resp., to a $p$-divisible group) over $Y$.

We call such $Y$ a healthy (resp., $p$-healthy) regular scheme (cf. [V, 3.2.1(2),(9)]). The counterexample of [FaC, p. 192] and the classical purity theorem of [G, p. 275] indicate that Problem 1.2 is of interest only if $K$ is of characteristic 0 (resp., only if $O$ is a faithfully flat $\mathbb{Z}_{(p)}$-algebra). We shall therefore assume hereafter that $O$ is of mixed characteristic $(0, p)$. Let $e \in \mathbb{N}$ be the index of ramification of $O$. If $e \leq p - 2$, then a result of Faltings states that $Y$ is healthy and $p$-healthy regular, provided it is formally smooth over $O$ (see [Mo, 3.6] and [V, 3.2.2(1) and 3.2.17], a correction to step B of which
is implicitly achieved here by Proposition 4.1). If \( p \geq 5 \), then there are local \( O \)-schemes that are healthy and \( p \)-healthy regular but are not formally smooth over some discrete valuation ring (see [V, 3.2.2(5)]). The goal of this paper is to prove the following theorem.

**Theorem 1.3.** If \( e = 1 \), then any regular, formally smooth \( O \)-scheme is healthy and \( p \)-healthy regular.

The case \( p \geq 3 \) is already known, as remarked previously. The case \( p = 2 \) answers a question of Deligne. In Section 2 we present complements on the crystalline contravariant Dieudonné functor. These complements are needed in Section 3 to prove Lemma 3.1, which pertains to extensions of short exact sequences of finite, flat, commutative group schemes. In Section 4 we use Lemma 3.1 and [FaC] to prove Theorem 1.3.

Milne used an analogue of Question 1.1(i) to define integral canonical models of Shimura varieties (see [Mi, Sec. 2] and [V, 3.2.3, 3.2.6]). Theorem 1.3 implies the uniqueness of such integral canonical models and extends parts of [V] to arbitrary mixed characteristic (see [V, 3.2.3.2, 3.2.4, 3.2.12, etc.]). Also one can use Theorem 1.3 and the integral models of compact, unitary Shimura varieties used in [K] to provide the first concrete examples of Néron models (as defined in [BLR, p. 12]) of projective varieties over \( K \) whose extensions to \( \overline{K} \) are not embeddable into abelian varieties over \( \overline{K} \).

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**2. The Crystalline Dieudonné Functor**

Let \( k \) be a perfect field of characteristic \( p > 0 \). Let \( \sigma_k \) be the Frobenius automorphism of the Witt ring \( W(k) \) of \( k \), and let \( R \) be a regular, formally smooth \( W(k) \)-algebra. Let \( Y := \text{Spec}(R) \). Let \( \Phi_R \) be a Frobenius lift of the \( p \)-adic completion \( R^\wedge \) of \( R \) that is compatible with \( \sigma_k \). Let \( \Omega_R^\wedge \) be the \( p \)-adic completion of the \( R \)-module of relative differentials of \( R \) with respect to \( W(k) \), and let \( d\Phi_{R/p} \) be the differential of \( \Phi_R \) divided by \( p \). For \( n \in \mathbb{N} \), the reduction mod \( p^n \) of \( d\Phi_{R/p} \) is denoted in the same way. If \( Z \) is an arbitrary \( \mathbb{Z}_{(p)} \)-scheme, let

\[
p - FF(Z)
\]

be the category of finite, flat, commutative group schemes of \( p \)-power order over \( Z \).

Let \( \mathcal{MF}^{\nabla}_{[0,1]}(Y) \) be the Faltings–Fontaine category defined as follows. Its objects are quintuples

\[
(M, F, \Phi_0, \Phi_1, \nabla),
\]

where \( M \) is an \( R \)-module, \( F \) is a direct summand of \( M \), both \( \Phi_0 : M \to M \) and \( \Phi_1 : F \to M \) are \( \Phi_R \)-linear maps, and \( \nabla : M \to M \otimes_R \Omega_R^\wedge \) is an integrable, nilpotent mod \( p \) connection on \( M \), such that the following five axioms hold:

1. \( \Phi_0(m) = p\Phi_1(m) \) for all \( m \in F \);
2. \( M \) is \( R \)-generated by \( \Phi_0(M) + \Phi_1(F) \);
3. $\nabla \circ \Phi_0(m) = p(\Phi_0 \otimes d\Phi_{R/p}) \circ \nabla(m)$ for all $m \in M$;
4. $\nabla \circ \Phi_1(m) = (\Phi_0 \otimes d\Phi_{R/p}) \circ \nabla(m)$ for all $m \in F$; and
5. locally in the Zariski topology of $Y$, $M$ is a finite direct sum of $R$-modules of the form $R/p^sR$, where $s \in \mathbb{N} \cup \{0\}$.

A morphism $f : (M, F, \Phi_0, \Phi_1, \nabla) \to (M', F', \Phi_0', \Phi_1', \nabla')$ between two such quintuples is an $R$-linear map $f_0 : M \to M'$ taking $F$ into $F'$ and such that the following three identities hold: $\Phi_0' \circ f_0 = f_0 \circ \Phi_0$, $\Phi_1' \circ f_0 = f_0 \circ \Phi_1$ and $\nabla' \circ f_0 = (f_0 \otimes_R 1_{\Omega_R^\nabla}) \circ \nabla$. We refer to $M$ as the underlying $R$-module of $(M, F, \Phi_0, \Phi_1, \nabla)$. Disregarding the connections (and thus axioms 3 and 4), we obtain the category $\mathcal{MF}_{[0,1]}(Y)$. Categories like $\mathcal{MF}_{[0,1]}(Y)$ and $\mathcal{MF}_{[0,1]}^\nabla(Y)$, in the context of arbitrary smooth $W(k)$-schemes, were first introduced in [Fa] as inspired by [F] and [FL], which worked with the category $\mathcal{MF}_{[0,1]}(\mathrm{Spec}(W(k)))$. In the sequel we will need the following result of Faltings.

**Proposition 2.1.** We assume that $\Omega_R^\nabla$ is a flat $R$-module. Then the category $\mathcal{MF}_{[0,1]}^\nabla(Y)$ is abelian and the functor from it into the category of $R$-modules that takes $f$ into $f_0$ is exact.

**Proof.** This follows from [Fa, pp. 31–33]. Strictly speaking, in [Fa] the result is stated only for smooth $W(k)$-algebras, but the inductive arguments work also for regular, formally smooth $W(k)$-algebras. In fact, we can use Artin’s approximation theorem to reduce Proposition 2.1 to the result in [Fa] as follows.

Let $f$ and $f_0$ be as before. We denote also by $\Phi_0$, $\Phi_1$, $\nabla$ and $\Phi_0'$, $\Phi_1'$, $\nabla'$ the different $\Phi_R$-linear maps and connections obtained from them via restrictions or via natural passage to quotients (for $\nabla$ and $\nabla'$ this makes sense because $\Omega_R^\nabla$ is a flat $R$-module). We need to show that the three quintuples $(\mathrm{Ker}(f_0), F \cap \mathrm{Ker}(f_0), \Phi_0, \Phi_1, \nabla)$, $(f_0(M), f_0(F), \Phi_0', \Phi_1', \nabla')$ and $(M'/f_0(M), F'/f_0(F), \Phi_0', \Phi_1', \nabla')$ are objects of $\mathcal{MF}_{[0,1]}^\nabla(Y)$ and that $f_0(F) = F' \cap f_0(M)$. Since $\Omega_R^\nabla$ is a flat $R$-module, axioms 3 and 4 hold and so from now on we do not mention $\nabla$ and $\nabla'$. Hence we are interested only in the morphism $g : (M, F, \Phi_0, \Phi_1) \to (M', F', \Phi_0', \Phi_1')$ of $\mathcal{MF}_{[0,1]}^\nabla(Y)$ defined by $f_0$. We can assume that $M$ and $M'$ are annihilated by $p^n$ and that $R$ is local. Using devissage as in [Fa, p. 33, ll. 4–11], it is enough to handle the case $n = 1$. So all the $R$-modules involved in the three quintuples listed are in fact $R/pR$-modules. Thus, to check that they are free, we can also assume that $R$ is complete. Based on [Ma, p. 268], there is a $k$-subalgebra $k_1$ of $R/pR$ that is isomorphic to the residue field of $R$. We easily get that $R/pR$ is a $k$-algebra of the form $k_1[[x_1, ..., x_d]]$, where $d \in \mathbb{N} \cup \{0\}$. Because $n = 1$, the choice of $\Phi_R$ plays no role in the study of the three quintuples and so we can also assume that $k_1$ is perfect.

We choose $R/pR$-bases $B$ and $B'$ of $M$ and $M'$ (respectively) such that certain subsets of them are $R/pR$-bases of $F$ and $F'$. With respect to $B$ and $B'$, the functions $f_0$, $\Phi_0$, $\Phi_1$, $\Phi_0'$, and $\Phi_1'$ involve a finite number of coordinates that are elements of $R/pR$. Let $A_0$ be the $k_1$-subalgebra of $R/pR$ generated by all these coordinates, and observe that $A_0$ is of finite type. Hence, from [BLR, p. 91] we derive the existence of an $A_0$-algebra $A_1$ that is smooth over $k_1$ and such that the $k_1$-monomorphism $A_0 \hookrightarrow R/pR$ factors through $A_1$. Localizing $A_1$, we can assume that $A_1$ is the reduction mod $p$ of
a smooth $W(k_1)$-algebra $R_1$. Now fix a Frobenius lift of the $p$-adic completion of $R_1$ that is compatible with $\sigma_k$; hence we can speak about $\mathcal{MF}_{[0,1]}(R_1)$. We get that $g$ is the natural tensorization with $R$ of a morphism $g_1$ of $\mathcal{MF}_{[0,1]}(R_1)$. Applying [Fa, pp. 31–32] to $g_1$ and tensoring with $R$, we deduce that axioms 1, 2, and 5 hold for the three quintuples and that $f_0(F) = F' \cap f_0(M)$. □

**Construction 2.2.** Let $W_n(k) := W(k)/p^nW(k)$. There is a contravariant, $\mathbb{Z}_p$-linear functor

$$\mathbb{D} : p - FF(Y) \to \mathcal{MF}_{[0,1]}^\nabla(Y).$$

Similar functors but with $Y$ replaced by Spec$(W(k))$ (resp., by a smooth $W(k)$-scheme and with $p > 2$) were first considered in [F] (resp. [Fa]). The existence of $\mathbb{D}$ is a modification of a particular case of [BBM, Chap. 3]. We will now include the construction of $\mathbb{D}$ based in essence on [BBM] and [Fa, 7.1]. We will use Berthelot’s crystalline site CRIS$(Y_{W_n(k)}/\text{Spec}(W(k)))$ (see [B, Chap. III, Sec. 4]) and its standard exact sequence

$$0 \to J_{Y_{W_n(k)}}/W(k) \to \mathcal{O}_{Y_{W_n(k)}}/W(k) (\text{see [BBM, p. 12]})$$

Let $G$ be an object of $p - FF(Y)$ that is annihilated by $p^n$. Let $(\tilde{M}, \tilde{\Phi}_0, \tilde{V}_0, \tilde{\nabla})$ be the evaluation of the Dieudonné crystal $\mathbb{D}(G_{Y_k}) = Ext^1_{Y_k=W(k)}(G_{Y_k}, \mathcal{O}_{Y_k}/W(k))$ (see [BBM, p. 116]) at the thickening naturally attached to the closed embedding $Y_k \hookrightarrow Y_{W_n(k)}$. Hence $\tilde{M}$ is an $R$-module, $\tilde{\Phi}_0$ is a $\Phi_R$-linear endomorphism of $\tilde{M}$, $\tilde{V}_0 : \tilde{M} \to \tilde{M} \otimes_R \Phi_R R$ is a Verschiebung map, and $\tilde{\nabla}$ is an integrable and nilpotent mod $p$ connection on $\tilde{M}$. Identifying $\tilde{\Phi}_0$ with an $R$-linear map $\tilde{M} \otimes_R \Phi_R R \to \tilde{M}$, we have

$$\tilde{V}_0 \circ \tilde{\Phi}_0(x) = px \quad \forall x \in \tilde{M} \otimes_R \Phi_R R, \quad \text{and} \quad \tilde{\Phi}_0 \circ \tilde{V}_0(x) = px \quad \forall x \in \tilde{M}.\quad (1)$$

Let $\tilde{F}$ be the direct summand of $\tilde{M}$ that is the Hodge filtration defined by the lift $G_{Y_{W_n(k)}}$ of $G_{Y_k}$. The triple $(\tilde{M}, \tilde{\Phi}_0, \tilde{V}_0, \tilde{\nabla})$ is also the evaluation of $\mathbb{D}(G_{Y_{W_n(k)}}) = Ext^1_{Y_{W_n(k)}/W(k)}(G_{Y_{W_n(k)}}, \mathcal{O}_{Y_{W_n(k)}}/W(k))$ at the trivial thickening of $Y_{W_n(k)}$. So $\tilde{F}$ is the image of the evaluation at this trivial thickening of the functorial homomorphism

$$Ext^1_{Y_{W_n(k)}/W(k)}(G_{Y_{W_n(k)}}, J_{Y_{W_n(k)}}/W(k)) \to Ext^1_{Y_{W_n(k)}/W(k)}(G_{Y_{W_n(k)}}, \mathcal{O}_{Y_{W_n(k)}}/W(k)).$$

To define the map $\tilde{\Phi}_1 : \tilde{F} \to \tilde{M}$ and to check that axioms 1–5 hold for the quintuple $(\tilde{M}, \tilde{F}, \tilde{\Phi}_0, \tilde{\Phi}_1, \tilde{\nabla})$, we can work locally in the Zariski topology of $Y$. Hence we can assume that $G$ is a closed subgroup of an abelian scheme $A'$ over $Y$ (cf. Raynaud’s theorem of [BBM, 3.1.1]). Let $A := A'/G$, and let $i_G : A' \to A$ be the resulting isogeny. We now define $\tilde{\Phi}_1$ using the cokernel of a morphism $f$ of $\mathcal{MF}_{[0,1]}^\nabla(Y)$ associated naturally to $i_G$.

Let $R(n) := R/p^nR$. Let $M := H^1_{\text{crys}}(A_{R(n)}/R(n)) = H^1_{\text{dR}}(A_{R(n)}/R(n))$ as in [BBM, 2.5]. Let $F$ be the direct summand of $M$ that is the reduction mod $p^n$ of the Hodge filtration $F_A$ of

$$H^1_{\text{crys}}(A/R^\wedge) := \text{proj.lim}_{l \in \mathbb{N}} H^1_{\text{crys}}(A_{R(l)}/R(l)) = \text{proj.lim}_{l \in \mathbb{N}} H^1_{\text{dR}}(A_{R(l)}/R(l)).$$

Now let $\Phi_0$ be the reduction mod $p^n$ of the $\Phi_R$-linear endomorphism $\Phi_A$ of $H^1_{\text{crys}}(A/R^\wedge)$, and let $\Phi_1$ be the reduction mod $p^n$ of the $\Phi_R$-linear map $F_A \to H^1_{\text{crys}}(A/R^\wedge)$ taking
\( m \in F_A \) into \( \Phi_A(m)/p \). Let \( \nabla \) be the reduction mod \( p^n \) of the Gauss–Manin connection \( \nabla_A \) of \( A_{R^n} \). That \( C := (M, F, \Phi_0, \Phi_1, \nabla) \) is an object of \( \mathcal{MF}^{[0,1]}(Y) \) is implied by the fact that the quadruple \((H^1_{\text{crys}}(A/R^n), F_A, \Phi_A, \nabla_A)\) is the evaluation at the thickening attached naturally to the closed embedding \( Y_k \hookrightarrow Y^\wedge := \text{Spec}(R^n) \) of a filtered \( F \)-crystal over \( R/pR \) in locally free sheaves (see \([\text{Ka, Sec. 8}]\)). Similarly, starting from \( A' \) we construct \( C' = (M', F', \Phi'_0, \Phi'_1, \nabla') \). Let \( f : C \to C' \) be the morphism of \( \mathcal{MF}^{[0,1]}(Y) \) associated naturally to \( i_G \).

Let \( f_0 : M \to M' \) defining \( f \). Let

\[
\mathbb{D}(G) = (\tilde{M}, \tilde{F}, \tilde{\Phi}_0, \tilde{\Phi}_1, \tilde{\nabla}) := \text{Coker}(f)
\]

(cf. Proposition 2.1). Then \( \tilde{M} := M'/f_0(M), \tilde{F} := F'/f_0(F) \), and so forth. That the quadruple \((\tilde{M}, \tilde{F}, \tilde{\Phi}_0, \tilde{\nabla})\) is as defined previously follows from \([\text{BBM, 3.1.6, 3.2.9, 3.2.10}]\).

The association \( G \to (\tilde{M}, \tilde{F}, \tilde{\Phi}_0, \tilde{\nabla}) \) is functorial. In order to check that \( \tilde{\Phi}_1 \) is well-defined and functorial, we can assume that \( R \) is local. To ease the notations we will check directly that \( \mathbb{D}(G) \) is itself well defined and functorial. So let \( m : G \to H \) be a morphism of \( p - FF(Y) \). If \( H \) is a closed subgroup of an abelian scheme \( B' \) over \( R \), then \( \mathbb{D}(G \times_Y H) \) is computed via the product embedding of \( G \times_Y H \) into \( A' \times_Y B' \). We thus obtain \( \mathbb{D}(G) \oplus \mathbb{D}(H) = \mathbb{D}(G \times_Y H) \). We now define \( \mathbb{D}(m) \). If \( m \) is a closed embedding, then the construction of \( \mathbb{D}(m) \) is obvious because \( i_G \) factors through the isogeny \( i_H : A' \to A'/H \). In general, the homomorphism \((1_G, m) : G \to G \times_Y H \) is a closed embedding. Hence \( \mathbb{D}(m) : \mathbb{D}(H) \to \mathbb{D}(G) \) is defined naturally via the epimorphism \( \mathbb{D}(1_G, m) : \mathbb{D}(G) \oplus \mathbb{D}(H) = \mathbb{D}(G \times_Y H) \to \mathbb{D}(G) \).

One easily checks that \( \mathbb{D}(G) \) and \( \mathbb{D}(m) \) are well-defined; that is, they depend neither on the chosen embeddings into abelian schemes nor on the choice of a power of \( p \) annihilating \( G \) and \( H \). For instance, let \( G \) be a closed subgroup of another abelian scheme \( C'' \) over \( Y \). By embedding \( G \) diagonally into \( A' \times_Y C'' \) and then using the snake lemma in the context of any one of the two projections of \( A' \times_Y C'' \) onto its factors, we get that \( \mathbb{D}(G) \) defined via \( A' \times_Y C'' \) is isomorphic to \( \mathbb{D}(G) \) defined via \( A' \) or \( C'' \). This ends the construction of \( \mathbb{D} \).

**Remarks 2.3.** (1) We have

\[
\tilde{V}_0 \circ \tilde{\Phi}_1(x) = x \quad \forall x \in \tilde{F} \otimes_R \Phi_R R,
\]
as this identity holds in the context of \( A \) and \( A' \). Since \( \tilde{M} \) is \( R \)-generated by the images of \( \tilde{\Phi}_1 \) and \( \tilde{\Phi}_0 \), it follows that \( \tilde{V}_0 \) is uniquely determined by \( \tilde{\Phi}_0 \) and \( \tilde{\Phi}_1 \). We therefore deem it appropriate to denote \((\tilde{M}, \tilde{F}, \tilde{\Phi}_0, \tilde{\Phi}_1, \tilde{\nabla})\) by \( \mathbb{D}(G) \). As \( C \) and \( C' \) depend only on \( A_{Y^{W_{n+1}(k)}} \) and \( A'_{Y^{W_{n+1}(k)}} \) (respectively), \( \mathbb{D}(G) \) also depends only on \( G_{Y^{W_{n+1}(k)}} \).

(2) If \( \tilde{F} \) is neither \( \{0\} \) nor \( \tilde{M} \), then \( \tilde{V}_0 \) has a nontrivial kernel and so \( \tilde{\Phi}_1 \) is not determined by \( \tilde{V}_0 \). The advantage we gain by using \( \tilde{\Phi}_1 \) instead of \( \tilde{V}_0 \) is that we can exploit axiom 5 and the exactness part of Proposition 2.1 (see the proof of Lemma 3.1).
(3) Let \( Y_1 = \text{Spec}(R_1) \) be an affine, regular, formally smooth \( W(k) \)-scheme. We assume that \( R_1 \) is equipped with a Frobenius lift \( \Phi_{R_1} \) compatible with \( \sigma_k \) and that there is a morphism \( l : Y_1 \to Y \) whose \( p \)-adic completion \( l^\wedge \) is compatible with the Frobenius lifts. Let \( l^* : p - FF(Y) \to p - FF(Y_1) \) and \( l^* : \mathcal{M}_{\mathcal{F}^\wedge_{[0,1]}(Y)} \to \mathcal{M}_{\mathcal{F}^\wedge_{[0,1]}(Y_1)} \) be the pull-back functors. Hence \( l^*(G) = G \times_Y Y_1 \) and

\[
l^*(M, F, \Phi_0, \Phi_1, \nabla) = (M \otimes_R R_1, F \otimes_R R_1, \Phi_0 \otimes \Phi_{R_1}, \Phi_1 \otimes \Phi_{R_1}, \nabla_1),
\]

where \( \nabla_1 \) is the natural extension of \( \nabla \) to a connection on \( M \otimes_R R_1 \). These constructions then yield the equality \( \mathbb{D} \circ l^* = l^* \circ \mathbb{D} \) of contravariant, \( \mathbb{Z}_p \)-linear functors from \( p - FF(Y) \) to \( \mathcal{M}_{\mathcal{F}^\wedge_{[0,1]}(Y_1)} \).

(4) As in [Fa, 2.3], we see that the category \( \mathcal{M}_{\mathcal{F}^\wedge_{[0,1]}(Y)} \) does not depend (up to isomorphism) on the choice of the Frobenius lift \( \Phi_R \) of \( R^\wedge \) compatible with \( \sigma_k \). The arguments of [Fa] apply even for \( p = 2 \) because we are dealing with connections that are nilpotent mod \( p \). One can use this to show that remark (3) makes sense even if \( Y \) and \( Y_1 \) are not affine or if no Frobenius lifts are fixed.

(5) If \( R \) is local, complete, and has residue field \( k \), then one can use a theorem of Badra [Ba] on the category \( p - FF(Y) \) to obtain directly that \( \mathbb{D}(G) \) is functorial.

### 3. A Lemma

In this section we prove the following Lemma.

**Lemma 3.1.** Assume that \( e = 1 \). Let \( (Y, U) \) be an extensible pair, with \( Y \) a regular and formally smooth \( O \)-scheme of dimension 2 and with \( U \) containing \( Y_K \). Then any short exact sequence \( 0 \to G_{1U} \to G_{2U} \to G_{3U} \to 0 \) in the category \( p - FF(U) \) extends uniquely to a short exact sequence in the category \( p - FF(Y) \).

**Proof.** Let \( O_X \) be the sheaf of rings on a scheme \( X \). Let \( j : U \hookrightarrow Y \) be the open embedding of \( U \) in \( Y \). For \( i \in \{1, 2, 3\} \), the \( O_Y \)-module \( F_i := j_*(O_{G_{iU}}) \) is locally free (cf. [FaC, Lemma 6.2 of p. 181]). The commutative Hopf algebra structure of the \( O_U \)-module \( O_{G_{iU}} \) extends uniquely to a commutative Hopf algebra structure of \( F_i \). Hence there exists a unique finite, flat, commutative group scheme \( G_i \) over \( Y \) extending \( G_{iU} \). We have to show that the natural complex

\[
0 \to G_1 \to G_2 \to G_3 \to 0
\]

is, in fact, a short exact sequence. This is a local statement for the faithfully flat topology of \( Y \). We may therefore assume that \( Y \) is local and complete and that its residue field \( k \) is separable closed and of characteristic \( p \); we may also assume that \( U \) is the complement in \( Y \) of the maximal point \( y \) of \( Y \). We write \( Y = \text{Spec}(R) \). From Cohen’s coefficient ring theorem (see [Ma, pp. 211, 268]) we have that \( R \) is a \( K(k) \)-algebra, where \( K(k) \) is a Cohen ring of \( k \). Since \( R/pR \) is regular and formally smooth over \( O/pO \) (and thus also over \( k \)), we can identify \( R = K(k)[[x]] \) as \( K(k) \)-algebras. Hence, by replacing \( R \) with the
faithfully, flat $R$-algebra $W(\overline{k})[[x]]$, we can assume that $k = \overline{k}$ and $K(k) = W(k)$ and so can use the notations of Section 2 (e.g. $\Phi_R$, $\Omega^n_R$, ...). Since $\Omega^n_R = dxR$ is a free $R$-module, we can also appeal to Proposition 2.1.

Let $\mathcal{O}$ be the local ring of $Y$, which is a discrete valuation ring that is faithfully flat over $W(k)$. Let $\mathcal{O}_1 := W(k_1)$, where $k_1$ is the algebraic closure of the residue field $k((x))$ of $\mathcal{O}$. We consider a Teichmüller lift $l : \text{Spec}(\mathcal{O}_1) \to \text{Spec}(\overline{R}^\wedge)$ that—at the level of special fibres—induces the inclusion $k[[x]] \hookrightarrow k_1$. Hence, $\mathcal{O}_1$ has a natural structure of an $\mathcal{O}$-algebra. Let

\begin{equation}
(4) \quad 0 \to \mathbb{D}(G_3) \to \mathbb{D}(G_2) \to \mathbb{D}(G_1) \to 0
\end{equation}

be the complex of $\mathcal{M}\mathcal{F}_{[0,1]}(Y)$ corresponding to (3). Let $M_1$, $M_2$ and $M_3$ be the underlying $R$-modules of $\mathbb{D}(G_1)$, $\mathbb{D}(G_2)$, and $\mathbb{D}(G_3)$, respectively. Let

\begin{equation}
(5) \quad 0 \to M_3 \to M_2 \to M_1 \to 0
\end{equation}

be the complex of $R$-modules defined by (4). Let $N_{1,2}$ be the underlying $R$-module of $\text{Coker}(\mathbb{D}(G_2) \to \mathbb{D}(G_1))$. The key point is that $\text{Coker}(\mathbb{D}(G_2) \to \mathbb{D}(G_1))$ exists in the category $\mathcal{M}\mathcal{F}_{[0,1]}(Y)$ and the sequence $M_2 \to M_1 \to N_{1,2} \to 0$ is exact (cf. Proposition 2.1). We show that $N_{1,2} = \{0\}$. Because $N_{1,2}$ is a direct sum of $R$-modules of the form $R/p^sR = W_s(k)[[x]]$ for $s \in \mathbb{N} \cup \{0\}$ (cf. axiom 5), to show that $N_{1,2} = \{0\}$ it is enough to show that $N_{1,2}[\frac{1}{p}] = \{0\}$. It is thus enough to show that the complex

\begin{equation}
(6) \quad 0 \to M_3 \otimes \mathcal{O}_1 \to M_2 \otimes \mathcal{O}_1 \to M_1 \otimes \mathcal{O}_1 \to 0
\end{equation}

obtained from (5) by tensoring with $\mathcal{O}_1$ is a short exact sequence. Note that (6) is the complex obtained by pulling back (3) to $\text{Spec}(\mathcal{O}_1)$, applying $\mathbb{D}$, and then taking underlying $\mathcal{O}_1$-modules (cf. Remark 2.3(4) applied to $l$). But the pull-back of (3) to $\text{Spec}(\mathcal{O}_1)$ is a short exact sequence (since the pull-back of (3) to $U$ is so). Thus (6) is the complex associated via the classical contravariant Dieudonné functor to the short exact sequence $0 \to G_{1k_1} \to G_{2k_1} \to G_{3k_1} \to 0$ (cf. [BBM, pp. 179–180]). From the classical Dieudonné theory we therefore have that (6) is a short exact sequence, cf. [F, p. 128 or p. 153]. So $N_{1,2} = \{0\}$.

Hence the natural $W(k)$-linear map $j_{1,2} : M_2/(x)M_2 \to M_1/(x)M_1$ is an epimorphism. But $j_{1,2}$ is the $W(k)$-linear map associated via the classical contravariant Dieudonné functor to the homomorphism $G_{1k} \to G_{2k}$, so this homomorphism is a closed embedding (cf. the classical Dieudonné theory). It follows by Nakayama’s lemma that $G_1$ is a closed subgroup of $G_2$. Both $G_3$ and $G_2/G_1$ are finite, flat, commutative group schemes extending $G_{3U}$ and so we have $G_3 = G_2/G_1$. Hence (3) is a short exact sequence. This completes the proof. \qed

**Remark 3.2.** For $p > 2$, Lemma 3.1 was proved by Faltings using Raynaud’s theorem [R, 3.3.3] (see [Mo, 3.6] and [V, 3.2.17, Step B]).
4. Proof of Theorem 1.3

Let $O$, $K$, $e$, and $Y$ be as in Section 1. We start with a general Proposition.

**Proposition 4.1.** If $Y$ is $p$-healthy regular then $Y$ is also healthy regular.

*Proof.* Let $(Y,U)$ be an extensible pair with $U$ containing $Y_K$, and let $A_U$ be an abelian scheme over $U$. We need to show that $A_U$ extends to an abelian scheme $A$ over $Y$. Since $Y$ is $p$-healthy regular, the $p$-divisible group $D_U$ of $A_U$ extends to a $p$-divisible group $D$ over $Y$. From now on we forget that $Y$ is $p$-healthy regular and we will use just the existence of $D$ to show that $A$ exists.

Let $N \in \mathbb{N}\setminus\{1,2\}$ be prime to $p$. To show that $A$ exists, we can assume that $Y$ is local, complete, and strictly henselian, that $U$ is the complement of the maximal point $y$ of $Y$, and that $A_U$ has a principal polarization $p_{A_U}$ and a level $N$ structure $l_{U,N}$ (see [FaC, (i)-(iii) of pp. 185, 186]). We write $Y = \text{Spec}(R)$. Let $p_{D_U}$ be the principal quasi-polarization of $D_U$ defined naturally by $p_{A_U}$; it extends to a principal quasi-polarization $p_D$ of $D$ (cf. Tate’s theorem [T, Thm. 4]). Let $g$ be the relative dimension of $A_U$. Let $A_{g,1,N}$ be the moduli scheme over $\text{Spec}(\mathbb{Z}[\frac{1}{N}])$ parameterizing principally polarized abelian schemes over $\text{Spec}(\mathbb{Z}[\frac{1}{N}])$-schemes, of relative dimension $g$ and with level $N$ structure (see [MFK, 7.9, 7.10]). Let $(A, \mathcal{P}_A)$ be the universal principally polarized abelian scheme over $A_{g,1,N}$.

Let $f_U : U \to A_{g,1,N}$ be the morphism defined by $(A_U, p_{A_U}, l_{U,N})$. We show that $f_U$ extends to a morphism $f_Y : Y \to A_{g,1,N}$.

Let $N_0 \in \mathbb{N}$ be prime to $p$. From the classical purity theorem we get that the étale cover $A_U[N_0] \to U$ extends to an étale cover $Y_{N_0} \to Y$. But as $Y$ is strictly henselian, $Y$ no connected étale cover different from $Y$. So each $Y_{N_0}$ is a disjoint union of $N_0^{2g}$-copies of $Y$. Hence $A_U$ has a level $N_0$ structure $l_{U,N_0}$ for any $N_0 \in \mathbb{N}$ prime to $p$.

Let $\overline{A}_{g,1,N}$ be a projective, toroidal compactification of $A_{g,1,N}$ such that (a) the complement of $\overline{A}_{g,1,N}$ in $\overline{A}_{g,1,N}$ has pure codimension 1 in $\overline{A}_{g,1,N}$ and (b) there is a semi-abelian scheme over $\overline{A}_{g,1,N}$ extending $A$ (cf. [FaC, Chap. IV, Thm. 6.7]). Let $\tilde{Y}$ be the normalization of the Zariski closure of $U$ in $Y \times_O \overline{A}_{g,1,N_O}$. It is a projective, normal, integral $Y$-scheme having $U$ as an open subscheme. Let $C$ be the complement of $U$ in $\tilde{Y}$ endowed with the reduced structure; it is a reduced, projective scheme over the residue field $k$ of $y$. The $\mathbb{Z}$-algebras of global functions of $Y$, $U$ and $\tilde{Y}$ are all equal to $R$ (cf. [Ma, Thm. 38] for $U$). So $C$ is a connected $k$-scheme (cf. [H, 11.3, p. 279]).

Let $\overline{A}_{\tilde{Y}}$ be the semi-abelian scheme over $\tilde{Y}$ extending $A_U$. Owing to existence of the $l_{U,N_0}$’s, the Néron–Ogg–Shafarevich criterion (see [BLR, p. 183]) implies that $\overline{A}_{\tilde{Y}}$ is an abelian scheme in codimension at most 1. Therefore, since the complement of $\overline{A}_{g,1,N}$ in $\overline{A}_{g,1,N}$ has pure codimension 1 in $\overline{A}_{g,1,N}$, it follows that $\overline{A}_{\tilde{Y}}$ is an abelian scheme. So $f_U$ extends to a morphism $f_{\tilde{Y}} : \tilde{Y} \to \overline{A}_{g,1,N}$. Let $p_{\overline{A}_{\tilde{Y}}} := f_{\tilde{Y}}^*(\mathcal{P}_A)$. Tate’s theorem implies that the principally quasi-polarized $p$-divisible group of $(\overline{A}_{\tilde{Y}}, p_{\overline{A}_{\tilde{Y}}})$ is the pull-back $(D_{\tilde{Y}}, p_{D_{\tilde{Y}}})$ of $(D, p_D)$ to $\tilde{Y}$. Hence the pull-back $(D_C, p_{D_C})$ of $(D_{\tilde{Y}}, p_{D_{\tilde{Y}}})$ to $C$ is constant; that is, it is the pull-back to $C$ of a principally quasi-polarized $p$-divisible group over $k$. 

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We check that the image \( f_{\tilde{Y}}(C) \) of \( C \) through \( f_{\tilde{Y}} \) is a point \( \{y_0\} \) of \( \mathcal{A}_{g,1,N} \). Since \( C \) is connected, to check this it suffices to show that, if \( \tilde{O}_c \) is the completion of the local ring \( \mathcal{O}_c \) of \( C \) at an arbitrary point \( c \), then the morphism \( \text{Spec}(\tilde{O}_c) \rightarrow \mathcal{A}_{g,1,N} \) defined naturally by \( f_{\tilde{Y}} \) is constant. But as \( (D_C,p_D) \) is constant, this follows from Serre–Tate deformation theory (see [Me, Chaps. 4, 5]). So \( f_{\tilde{Y}}(C) \) is a point \( \{y_0\} \) of \( \mathcal{A}_{g,1,N} \).

Let \( R_0 \) be the local ring of \( \mathcal{A}_{g,1,N} \) at \( y_0 \). Because \( Y \) is local and \( \tilde{Y} \) is a projective \( Y \)-scheme, each point of \( \tilde{Y} \) specializes to a point of \( C \). Hence each point of the image of \( f_{\tilde{Y}} \) specializes to \( y_0 \) and so \( f_{\tilde{Y}} \) factors through the natural morphism \( \text{Spec}(R_0) \rightarrow \mathcal{A}_{g,1,N} \). Since \( R \) is the ring of global functions of \( \tilde{Y} \), the resulting morphism \( \tilde{Y} \rightarrow \text{Spec}(R_0) \) factors through a morphism \( \text{Spec}(R) \rightarrow \text{Spec}(R_0) \). Therefore, \( f_{\tilde{Y}} \) factors through a morphism \( f_Y : Y \rightarrow \mathcal{A}_{g,1,N} \) extending \( f_U \). This ends the argument for the existence of \( f_Y \). We conclude that \( A := f_Y^*(A) \) extends \( A_U \), which completes the proof. \( \Box \)

**Remark 4.2.** In the proof of Proposition 4.1, the use of semi-abelian schemes can be replaced by de Jong’s good reduction criterion [dJ, 2.5] as follows. If we define \( \tilde{Y} \) to be the normalization of the Zariski closure of \( U \) in \( Y \times O \mathcal{A}_{g,1,N} \), then [dJ] implies that the morphism \( \tilde{Y} \rightarrow Y \) of \( O \)-schemes of finite type satisfies the valuative criterion of properness with respect to discrete valuation rings of equal characteristic \( p \). Using (as in the proof of Proposition 4.1) the Néron–Ogg–Shafarevich criterion, one checks that the morphism \( \tilde{Y} \rightarrow Y \) of \( O \)-schemes satisfies the valuative criterion of properness with respect to discrete valuation rings whose fields of fractions have characteristic 0. Hence the morphism \( \tilde{Y} \rightarrow Y \) of \( O \)-schemes is proper. The rest of the argument is entirely the same.

**Conclusion 4.3.** We assume that \( e = 1 \) and that \( Y \) is formally smooth over \( O \). Based on Proposition 4.1, in order to prove Theorem 1.3 it suffices to show that \( Y \) is \( p \)-healthy regular. So let \( (Y,U) \) be an extensible pair with \( U \) containing \( Y_K \). We need to show that any \( p \)-divisible group \( D_U \) over \( U \) extends to a \( p \)-divisible group \( D \) over \( Y \). This is a local statement for the faithfully flat topology, so we can assume that \( Y \) is local, complete, and strictly henselian and that \( U \) is the complement of the maximal point \( y \) of \( Y \) (see [FaC, p. 183]). Write \( Y = \text{Spec}(R) \), and let \( d \in \mathbb{N} \) be the dimension of \( R/pR \). We show the existence of \( D \) by induction on \( d \).

If \( d = 1 \) then, for all \( n, m \in \mathbb{N} \), the short exact sequence \( 0 \rightarrow D_U[p^n] \rightarrow D_U[p^{n+m}] \rightarrow D_U[p^m] \rightarrow 0 \) in the category \( p - \text{FF}(U) \) extends uniquely to a short exact sequence \( 0 \rightarrow D_n \rightarrow D_{n+m} \rightarrow D_m \rightarrow 0 \) in the category \( p - \text{FF}(Y) \) (cf. Lemma 3.1). Hence there is a unique \( p \)-divisible group \( D \) over \( Y \) such that \( D[p^n] = D_n \). Obviously \( D \) extends \( D_U \). For \( d \geq 2 \), the passage from \( d - 1 \) to \( d \) is entirely as in [FaC, pp. 183, 184] applied to \( R \) and any regular parameter \( x \in R \) such that \( R/xR \) is formally smooth over \( O \). This ends the induction and so establishes the existence of \( D \), concluding the proof of Theorem 1.3.

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