D-Branes on $\mathbb{C}^3_6$ Part I: prepotential and $GW$–invariants

SERGIO CACCIA TORI AND MARCO COMPAGNONI

Dipartimento di Fisica, Università degli Studi dell’Insubria,
Via Valleggio 11, 22100 Como, Italy
Dipartimento di Matematica, Politecnico di Milano,
Via Bonardi 9, 20133 Milano, Italy
sergio.cacciatori@uninsubria.it
marco.compagnoni@polimi.it

Abstract

This is the first of a set of papers having the aim to provide a detailed description of brane configurations on a family of noncompact three-dimensional Calabi-Yau manifolds. The starting point is the singular manifold defined by a given quotient $\mathbb{C}^3 / \mathbb{Z}_6$, which we called simply $\mathbb{C}^3_6$ and which admits five distinct crepant resolutions. Here we apply local mirror symmetry to partially determine the prepotential encoding the $GW$–invariants of the resolved varieties. It results that such prepotential provides all numbers but the ones corresponding to curves having null intersection with the compact divisor. This is realized by means of a conjecture, due to S. Hosono, so that our results provide a check confirming at least in part the conjecture.
1 Introduction

We use (local) mirror symmetry to compute the Gromov-Witten invariants\(^1\) for a family of noncompact Calabi-Yau manifolds obtained resolving an orbifold \(X = \mathbb{C}^3/\mathbb{Z}_6\). The main interest for this example is that it admit five distinct crepant resolutions all birational to \(X\), differing by flops. Going from a resolution to another passing through the singular orbifold realizes a geometrical transition. Geometrically, the transition is obtained by moving the Kähler moduli \(t\) through an orbifold point, where the manifold becomes singular with a curve which shrinks down and reemerges as a flopped curve. As it is well known, in string theory such transition can correspond to smooth physical processes. This can be understood for example by means of a (physically equivalent) dual description using mirror symmetry. Because the orbifold was obtained quotienting by an abelian group, the resulting smooth manifold are indeed toric varieties, so that the powerful toric methods can be employed to work out all details. Mirror symmetry for noncompact CY varieties was developed quite recently in [26]. For toric varieties the mirror manifold result to be defined as the zero locus

\[
Y_x = \{ (\vec{u}, \vec{v}) \in \mathbb{C}^2 \times \mathbb{C}^2^* | F_x(\vec{u}, \vec{v}) = 0 \},
\]

where \(x\) determines a point in the complex structures moduli space of the mirror, corresponding to the point \(t\) specifying the Kähler moduli of the starting manifold \(X_t\), and \(F_x(\vec{u}, \vec{v}) = u_1^2 + u_2^2 + f_x(\vec{v})\) is a certain polynomial fully determined by the toric data describing the starting orbifold. Thus varying the moduli \(t\) corresponds to varying the moduli \(x\) of the mirror manifold. However, whereas \(X_t\) undergoes a flop transition, \(Y_t\) simply changes smoothly its complex moduli.

From the mathematical point of view, the noncompactness of the variety and the particular structure of its cohomology ring introduce some ambiguities in defining the GW invariants and in their interpretation, thus requiring a deeper understanding of the geometrical structures living on a noncompact manifold. From this point of view, a consistent step forward was made by T.-M. Chiang, A. Klemm, S.-T. Yau and E. Zaslow [9], who gave an interpretation to the GW invariants from an enumerative point of view.

From the physical point of view, mirror symmetry looks like a generalization of T-duality equivalence between different perturbative limits of the, supposed to exist, unique M-theory. However, some non perturbative enhancements are provided by adding D-brane configurations. At a semiclassical level, D-branes are described by closed cycles (with bundles) which the branes are supposed to wrap on. One may wonder if any possible cycle is a good candidate as a wrapping locus. Indeed, this is actually a still open question, even if many overcomes have been made in the last decade. A consistence check must be stability, at first place perturbative stability. A first step in favour of perturbative stability is supersymmetry. This gives some

\(^1\)more precisely the lowest genus Gopakumar–Vafa invariants
strong constraint which can depend on the kind of strings to work with. For IIB strings on a CY, supersymmetric configurations are represented by holomorphic (then evendimensional) cycles, whereas for type IIA branes one finds Lagrangian submanifolds (with respect to the Kähler form) as brane representatives, that are halfdimensional subvarieties. In the case in our interest the lasts are three dimensional surfaces. Thus, mirror symmetry must be extended in order to take account of nonperturbative brane configurations. An astounding advance in this direction has been proposed by M. Kontsevich [39] who introduced the concept of homological mirror symmetry. In this case type B branes are described in terms of bounded derived categories of coherent sheaves whereas A branes are substituted by derived Fukaya categories [19]. Such a description reconciles some apparent asymmetry between A an B branes. Indeed, whereas A branes result to be halfdimensional, the dimensions of B branes are heterogeneous so that to any A cycle it can correspond a B cycle of different dimension; on the other hand, lagrangian cycles can be linearly combined to compose new lagrangian cycles (monodromies), which, by mirror symmetry, must match with combinations of holomorphic cycles having different dimensions. In the B side, to monodromies correspond autoequivalences of derived categories. In some sense homological mirror symmetry introduce some democracy since all branes are described in terms of higher dimensional branes in an homogeneous way. From a more topological point of view, brane charges (central charges or masses) are thus described in terms of K-theory groups (even if there are many other indications for this beyond and independently from mirror symmetry, see [25, 45]). From the physical point of view, some new insight in this direction for the case of noncompact CY manifolds was done by X. de la Ossa, B. Florea and H. Skarke [15] who were able to select a distinguished K-theory basis for B-branes configurations adapted to support monodromy correspondence, generalizing (at least at a conjectural level) the corresponding results quite well established in the compact case.

On this side, further progress is due to S. Hosono [30] who found an elegant way to describe local mirror symmetry in terms of cohomology valued hypergeometric series. Mirror symmetry identifies the Kähler moduli of a CY variety with the periods of its mirror, which as functions of the complex structure moduli must satisfy a set of Picard-Fuchs equations, the Gel’fand-Kapranov-Zelevinski system [22]. It results that a particular cohomology valued hypergeometric series $w$ arises naturally providing a basis of solutions for the GKZ system [32], [33], [34], [35]. Hosono was able to recognize such series as a formula identifying the BPS states of the associated physical theory, and proposed an intriguing conjecture, which we dub “the Hosono conjecture”, see conjecture 6.3 in [30], which, beyond identifying the central charge of a brane configuration $F \in K^c(X_t)$ in terms of $w$, interprets the monodromy of the periods via a naturally associated symplectic form on $K^c(X_t)$. Hosono checked very carefully his conjecture for the toric quotients $\mathbb{C}^2/G$, and for the examples $\mathbb{C}^3/\mathbb{Z}_3$, $\mathbb{C}^3/\mathbb{Z}_5$ in three complex dimensions. Among others, a
consequence of the Hosono conjecture is to provide a closed formulation of a prepotential for noncompact quotients also. Indeed, at cohomological level, mirror symmetry provides a map
\[ \text{mir} : K^c(X_t) \rightarrow H_3(Y_t, \mathbb{Z}), \]
transferring the symplectic form on \( K^c(X) \) to a symplectic structure on \( H_3(Y, \mathbb{Z}) \). This is the noncompact analog of the symplectic structure that, combined with Griffiths transversality, ensures the existence of a prepotential in the compact cases. However, due to non compactness, the symplectic structure is generically degenerate. On the \( X \) side, it defines a correspondence between \( H^2(X, \mathbb{Q}) \) and \( H^4(X, \mathbb{Q}) \) which permits a complete determination of the prepotential (and correspondingly of all GW invariants) only when it arises as a vector space isomorphism.

At homological level, mirror symmetry is conjectured to define a map
\[ \text{Mir} : D^\flat \text{Coh}(X) \rightarrow DFuk(Y, \omega) \]
where the symplectic form \( \omega \) is the Kähler form corresponding to the fixed complex structure in \( X \). In this way, monodromies of Lagrangian on \( Y \) correspond to autoequivalences of derived categories on \( X \) described by opportune Mukay transforms which are expected to realize a (quiver) representation of the quotient group by the Mckay correspondence. This has been analyzed for example by Karp [38, 8]. The Hosono conjecture indeed works to this higher level too, and gives some hints to get information on the mirror map \( \text{Mir} \).

In this paper we will work at the lower level, that is at K-theoretical level, postponing the study of the higher (categorical) level mirror map to a future paper. We apply the Hosono conjecture to compute the GW invariants for a family of noncompact toric CY varieties obtained as crepant resolutions of an orbifold quotient \( \mathbb{C}^3/\mathbb{Z}_6 \). We chosen this model because it has quite general properties which make it very interesting to test the conjecture. To begin with, the second and fourth Betti numbers are \( b_2 = 4 \) and \( b_4 = 1 \), so that the symplectic structure result to be highly degenerate. Thus it defines a quite poor correspondence between \( H^2(X, \mathbb{Q}) \) and \( H^4(X, \mathbb{Q}) \). Nevertheless, we will see that the Hosono procedure permits to define a partial prepotential containing a lot of information about local geometry. Indeed, from it we are able to read out almost all GW invariants, leaving out only a three dimensional subcone of the fourdimensional Mori cone.\(^2\) Indeed, it was proposed by B. Forbes and M. Jinzenji [17], [18], a possible way to extend the GKZ system obtaining a complete determination of all GW invariants. To such extension we will devote a future paper. Here, we will only discuss the possible origin for the ambiguity in defining the lacking GW invariants.

A second interesting peculiarity of our model, yet anticipated at the beginning, is that it admits five distinct crepant resolutions, which differ by flops. Thus one expects monodromy to relate different resolution by means of different Fourier

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\(^2\)We are grateful to Professor S. Hosono to give us explanations on this point.
Mukay transforms. This is indeed one of the main targets of this starting studies, but as announced we will not tackle it here. We will limit ourselves to compute the prepotentials and the computable GW invariants for all resolutions, comparing with themselves. Thus in some sense this first paper can be thought as a preparatory one. In this spirit we will try to be as much explicit as possible. In section 2 we include a short overview of the main steps which lead to the introduction of local mirror symmetry to arrive to the Hosono conjecture.

In section 3 we present a detailed analysis of the first resolution, that is the G-Hilbert resolution. We will use the Hosono’s conjecture to construct the cohomological hypergeometric series generating the periods of the mirror manifold. Due to noncompactness, the structure of the cohomology ring does not consent a full definition of the GW–invariants. However, as we will see, the procedure proposed by Hosono permits to equally define a prepotential which generates all GW–invariants associated to the curves in the Mori cone, excluding a codimension one subcone.

In section 4 we repeat the previous analysis for all the other resolutions, deriving a partial determination of the GW–invariants for all of them.

The results will be commented in section 5.

2 Local mirror symmetry and the Hosono conjecture

Here we will recall some main step leading to the conjectures we are testing in this and following papers. The literature on the subject is quite huge, so that we will mainly refer to [27] and references therein.

2.1 Dualities and mirror symmetry

Let us consider a string theory having a toric Calabi-Yau variety $X$ as target space. Thus, there is a nice interpretation of mirror symmetry as a T-duality transformation. Indeed, string theory on $X$ can be described in terms of a two dimensional $U(1)^m$ supersymmetric gauge theory, the so called “gauged linear sigma model” (see [27], sections 7.3, 7.4). It contains a certain number $n > m$ of complex scalar fields $Z = \{Z_\alpha\}_{\alpha=1}^n$ having charges $Q_{a, r}$, $r = 1, 2, \ldots, m$ with respect to the gauge group $U(1)^m$, and with potential energy

$$U(Z) = \frac{1}{2} \sum_{r=1}^m g_r^2 \left( \sum_{\alpha=1}^n Q_{a, r} Z_\alpha \overline{Z_\alpha} - v_r \right)^2.$$
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Here $g_r$ and $r_r$ are the gauge couplings and the Fayet-Iliopoulos terms respectively. Supersymmetric ground states require the vanishing of the potential energy:

$$\sum_{\alpha=1}^{n} Q_{\alpha,r} Z_\alpha \bar{Z}_\alpha = t_r.$$  

For a fixed choice of the $F–I$ parameters, these equations define a toric variety $X$ associated to a fan, in an $(n – m)$–dimensional lattice $N$, generated by an opportune set $\Sigma(1) = v_1, \ldots, v_n$ of vectors in $N$. From this it is possible to conclude that the supersymmetric vacua are identified with the points of a toric variety $X$. Each vector $v_\alpha$ determines an invariant divisor $D_{v_\alpha}$. It is not hard to show (see [27], sec. 7.4) that one can chose a basis $\{C_r\}_{r=1}^{m}$ of irreducible curves of $H_2(X, \mathbb{Z})$ (which indeed result to be $m$–dimensional) such that the charges are given by the intersection numbers $Q_{\alpha,r} = D_{v_\alpha} \cdot C_r$. Also note that the $F–I$ parameters rescale as $|Z|$ so that, if chosen to be positive, they indeed parameterize the points of the Kähler cone. This means that the supersymmetric configurations are completely characterized in geometrical terms.

At this point mirror symmetry can be realized as a T–duality transformation ([27], sec. 20). Indeed, recall that roughly speaking T–duality on a circle transforms a type A string theory on a circle of radius $R$ in a type B string theory on a circle of radius $\alpha' / R$. If $Z_\alpha$ are taking value on a complex variety (indeed the toric variety in the vacuum configuration) then we can T–dualize their phases which define circles in the target manifold. The result ([27], sec. 13) is a Landau-Ginzburg theory with superpotential

$$W(Y, t) = \sum_{\alpha=1}^{n} e^{-Y_\alpha},$$

for a set of chiral superfields related by the set of constraints

$$\sum_{\alpha=1}^{n} Q_{\alpha,r} Y_\alpha = t_r$$

where $t_r$ are the complexified Kähler parameters ($\text{Re}(t_r) = r_r$). In this way, the mirror transformation applied to the twodimensional sigma model gives rise to a Landau–Ginzburg model with superpotential $W(Y, t)$. To take contact with the Batyrev’s geometric construction of mirror manifolds for toric varieties, let us proceed as follows (see [26]) for the cases when the starting linear sigma model describes strings on a crepant resolution of some abelian quotient $\mathbb{C}^3_0 / G$. Being crepant, it will be described by a set of vectors $v_1, \ldots, v_n$ in a three-dimensional lattice such that for some isomorphism $\phi : N \rightarrow \mathbb{Z}^3$ one has $\phi(v_\alpha) = (n_{\alpha,1}, n_{\alpha,2}, 1)$. The solutions of the constraints can thus be written in terms of three independent fields $y_0, y_1, y_2$ as $Y_\alpha = y_0 + n_{\alpha,1} y_1 + n_{\alpha,2} y_2 + c_\alpha$ where $c_\alpha$ are some constant satisfying $\sum_{\alpha=1}^{n} Q_{\alpha,r} Y_\alpha = t_r$. These linear redefinitions do not affect the functional

\(^3\text{t.i. invariant under the toric action}\)
measure, and setting \( w_a = \exp(-y_a), \ a = 0, 1, 2 \) and \( a = \exp(-c_a) \) we get for the superpotential

\[
W(w, a) = w_0 \sum_{\alpha=1}^{n} a_{\alpha} w_1^{\alpha_1} w_2^{\alpha_2}, \quad w_\alpha \in \mathbb{C}_*.
\]

As discussed in [26], we can note that, for what concerns the BPS configurations, this LG model is equivalent to another one, where \( w_0 \in \mathbb{C} \) and with two extra chiral fields \( U, V \in \mathbb{C} \), whose superpotential is

\[
\tilde{W}(U, V; w; a) = W(w, a) - w_0 UV.
\]

Integrating the field \( w_0 \) thus gives a delta function \( \delta(\sum_{\alpha=1}^{n} a_{\alpha} w_1^{\alpha_1} w_2^{\alpha_2} - UV) \) so that the mirror LG model is equivalent to a geometrical theory on a Calabi-Yau manifold

\[
Y_a = \{ (\vec{u}, \vec{w}) \in \mathbb{C}^2 \times \mathbb{C}_*^2 | F_a(\vec{u}, \vec{w}) = 0 \},
\]

where

\[
F_a(\vec{u}, \vec{w}) = u_1^2 + u_2^2 + f_a(\vec{u}, \vec{w}) = u_1^2 + u_2^2 + \sum_{\alpha=1}^{n} a_{\alpha} w_1^{\alpha_1} w_2^{\alpha_2}.
\]

The Kähler parameters \( t \) now parameterize the complex moduli of \( Y \). This is indeed local mirror symmetry as discovered for the first time at physical level in [41], [42].

### 2.2 Branes and homological mirror symmetry

The intuitive picture described above does not takes into account the presence of brane configurations. Because we are looking for supersymmetric vacua, we need to know what kind of brane configurations are admitted on a Calabi-Yau manifold \( X \). In other words, one must search for boundary condition compatible with supersymmetry. This is described for example in [26], sec. 3. The answer depends on the type of string theory one considers. For type A strings, supersymmetric branes are represented (at classical level) by halfdimensional subvarieties \( S, \iota : S \hookrightarrow X \), where the Kähler form \( \omega \) of the C-Y manifold vanishes, \( \iota^* \omega = 0 \), and supporting flat vector bundles. Thus A–branes are Lagrangian submanifolds with respect to the symplectic structure \( \omega \). For type B strings one finds that supersymmetric branes must wrap holomorphic cycles of \( X \) supporting holomorphic vector bundles. In our models it means that type B brane configurations will be described classically by compact divisors, curves of the Mori cone and points.

Thus mirror symmetry should map BPS states of a model into the BPS states of the mirror model, converting A–branes to B–branes and viceversa. However, there is an odd asymmetry between A and B configurations: indeed all A–branes have the same dimensions, whereas this does not happen for B–branes. Now, the point is that in the LG model description branes configurations can change when moduli...
vary. In this picture, BPS states will correspond to critical points of the superpotential. Essentially they determine the points of $Y_t$ around which the supersymmetric three–cycles are defined. Varying $t$, the critical points move on the $W$–plane; when some of these points moves around a branch point, a monodromy transformation can give rise to a new brane configuration ([26]). The boundary states corresponding to the branes are described by the periods of the holomorphic three–form $\Omega$ of $Y_t$ (in the geometric picture). The monodromy thus acts on a basis of cycles recasting them in some linear recombination or equivalently on the periods in the same linear recombination. On the mirror $X$ it should correspond to a recombination of the holomorphic cycles, hard to understand in the naïve geometrical picture where they have different dimensions.

To solve this point a first aid comes from a K-theoretical description, where lower dimensional branes can be described in terms of the top dimensional branes and a tachyon field [45]. K-theory mainly captures topological aspects of the problem, carrying important information on the admissible brane configurations, but it is quite poor from the geometrical point of view. In [16] it was argued that a deeper geometrical understanding of (stable) brane configuration in (topological) type B superstring can be understood in terms of triangulated categories, in particular the derived category of coherent sheaves on the manifold (see also [1], or [4] for a more mathematical point of view). This provided a deep contact between physics and the “homological mirror symmetry” conjectured by Kontsevich [39] who proposed that the usual geometrical mirror symmetry should enhance to homological level as an equivalence between triangulated categories: the derived category of coherent sheaves on a CY manifold $X$ with a fixed complex structure on one side and the derived $\mathcal{A}^\omega$ Fukaya’s category over the mirror manifold $Y$ on the other side, essentially generated by the Lagrangian submanifolds of $(Y, \omega)$, where the symplectic structure $\omega$ is given by the fixed Kähler form on $Y$, dual to the complex form on $X$:

$$\text{Mir} : D^b\text{Coh}(X) \xrightarrow{\sim} DF\text{uk}^\omega(Y, \omega).$$

### 2.3 The Hosono conjecture

As we said, BPS states in the mirror type A string model are described by periods that are integrals of the holomorphic three form $\Omega$ on $Y$ over the Lagrangian cycles. For the noncompact quotients we are describing, the holomorphic three form on the mirror $Y$ is

$$\Omega = \frac{1}{4\pi^3} \text{Res}_{F=0} \left[ \frac{du^1 \wedge du^2 \wedge dw^1 \wedge dw^2}{w^1 w^2 F(u; \bar{u}; a)} \right].$$

Here we have fixed the Kähler form, however $\Omega$ depends explicitly on the complex moduli of $Y$ (as shown by the explicit dependence on $a$ of the polynomial $F$) so

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4for clarity we confine ourselves to the case of Calabi-Yau varieties
that the periods

$$\Pi_{C_i}(a) = \int_{C_i} \Omega,$$

of any set of Lagrangian cycles $C_i$, will be locally holomorphic functions of the moduli. Indeed, they are forced to satisfy a set of hypergeometric differential equations known as the GKZ hypergeometric system, largely studied in [22].

For compact varieties the knowledge of a complete set of solutions for the GKZ system correspond to an exhaustive description of the set of BPS brane configurations on the $A$ side. Furthermore, the special Kähler geometry of the complex structure moduli space of a C-Y manifold can be described in terms of periods [44].

If $x$ parameterizes the structure complex moduli of $Y$ then the Kähler potential of the moduli space can be written as

$$K(x, \bar{x}) = -\log \left(i \sum_{l=0}^{h^{1,2}(Y)} X^l \frac{\partial G}{\partial \bar{x}^l} - \bar{X}^l \frac{\partial G}{\partial X^l}\right),$$

where

$$X^l(x) = \int_{A^I} \Omega(x)$$

are the periods with respect to a canonical symplectic basis $\{A^I, B_I\}$ of $H_3(Y, \mathbb{Z})$.

Finally $G(x)$ is the prepotential

$$G(x) = \frac{1}{2} \sum_{l=0}^{h^{1,2}(Y)} \int_{A^I} \int_{B_I} \Omega.$$ 

Mirror symmetry gives a correspondence between Kähler moduli $t_i$ of $X$ and complex moduli of $Y$ so that

$$t_i = \frac{X^i}{X^0}, \quad i = 1, \ldots, h^{1,2}(Y) = h^{1,1}(X).$$

On the other side, also the Kähler moduli space of $X$ is a special Kähler manifold which can thus be described in terms of a prepotential function $F(t)$. At classical level such geometry is described by the prepotential

$$F^c(t) = \frac{1}{6} d_{ijkl} t^i t^j t^k,$$

where $d_{ijk} = J_i \cdot J_j \cdot J_k$ are the intersection numbers of the Kähler cone generators. Physically they determine the Yukawa couplings of the chiral fields [7]. However these couplings receive quantum corrections which come from worldsheet instantons. At lowest order they corresponds to wrapping of the worldsheet on rational curves in $X$. The energy of such a wrapping is given by the volume of the wrapped cycle as measured by the Kähler metric. Any given (class of) rational curve of degree $\vec{d}$ results to contribute to the prepotential with a term

$$n_{\vec{d}} \text{Li}_3(e^{2\pi i \vec{d} \cdot t}),$$
$n_d$ being the number of classes of curves with the given degree, so that it can be shown that the quantum corrected prepotential takes the form [9]

$$F(t) = \frac{1}{6}d_{ijk}t^it^jt^k - \frac{1}{24}c_2(X) \cdot J \cdot t - \frac{c_3(X)}{16\pi^2} + \sum_{d \in \mathbb{Z}^{h_{1,1}}} n_d Li_3(e^{2\pi i \frac{d}{t}}).$$

More precisely $n_d$ are the Gromov-Witten invariant (in the Gopakumar-Vafa interpretation [23], [24]). See [9] for a mathematical enumerative interpretation. By means of the identification (making use of the Griffith transversality, [9], [44])

$$\left\{ \int_{A^i} \Omega; \int_{B^j} \Omega \right\} \stackrel{h^{2,1}(Y)}{\sim} \left\{ 1, t^i; \partial_t F, 2F - \sum_{j=1}^{h^{1,1}(X)} t^j \partial_t F \right\} \stackrel{h^{1,1}(X)}{\sim}$$

mirror symmetry thus gives a simple way to compute the GW–invariants of $X$.

In a series of papers (see for example [32], [33], [34], [35]) it was provided an efficient strategy to characterize a complete set of the GKZ system for a C-Y hypersurface, which is summarized in [29]. In particular there was introduced a cohomological valued power series whose expansion in the Chow ring $A^*(X) \otimes \mathbb{C}[[x]][\log x]$ gives a basis for the period integrals of the mirror manifold $Y$ in the large complex structure limit (LCSL), (see [29], Claim 5.11). Thus the cohomological series encodes many geometrical information on both the manifolds $X$ and $Y$ so summa-
izing several fundamental aspects of mirror symmetry.

In [31], [30] Hosono extended this picture to local mirror symmetry for noncompact C-Y manifolds, in particular for resolutions of abelian quotients $\mathbb{C}^k/G$, with $k = 2, 3$. For convenience we will state the conjecture in section 3.6. In [30], Hosono verified his conjecture carefully for the case $k = 2$ and reported the analysis for the cases $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^3/\mathbb{Z}_5$, where it was shown the existence of a prepotential for the noncompact cases also.

Here we use Hosono conjecture to analyze the geometry of a quotient $X = \mathbb{C}^3/\mathbb{Z}_6$, which we call for simplicity $\mathbb{C}^3_6$. As stated in the introduction, this singular orbifold admits five distinct crepant resolutions. All these resolutions are related by flop transformations. To noncompactness of the manifold $X$ it corresponds the ambiguity in defining the GW–invariants. In our model this reflects in the fact that the symplectic structure on the half dimensional homology of the mirror $Y$ is degenerate. On the mirror, such structure should determine a pairing between two dimensional and four dimensional cohomology, permitting the reconstruction of the prepotential, but which now becomes degenerate. We determine the LCSL cohomological series for all the resolutions. From each of them, using Hosono’s prescriptions we will able to partially determine a prepotential which codifies all
the $GW$–invariants of the (four dimensional) Mori cone excluding a three dimensional subcone.

3 The tridimensional orbifold $\mathbb{C}^3_6$ and the $G$-Hilb resolution

3.1 Definition of $\mathbb{C}^3_6$

We briefly review the homogeneous coordinates construction of toric varieties [10]. The data of a $d$-dimensional toric variety $X(\Delta)$ can always be specified in terms of a fan $\Delta$ in a lattice $N$ isomorphic to $\mathbb{Z}^d$. Let $\rho_1, \ldots, \rho_r$ be the 1-dimensional cones of $\Delta$ and let $v_i \in \mathbb{Z}^n$ denote the primitive element of $\rho_i$, i.e. the generator of $\rho_i \cap \mathbb{Z}^n$. Then introduce variables $x_i$ for $i = 1, \ldots, r$ in the affine complex space $\mathbb{C}'$. The homogeneous coordinates construction represents $X(\Delta)$ as the quotient

$$X(\Delta) = (\mathbb{C}' \setminus Z)/G$$

for a certain variety $Z$ and some abelian group $G \subset (\mathbb{C}')^r$.

$Z$ is determined as follows. We say that a set of edge generators $I = \{v_1, \ldots, v_s\}$ is primitive if they don’t lie in any cone of $\Delta$ but every proper subset does. Then

$$Z = \bigcup_{I \text{ primitive}} \{x_i = 0, \ldots, x_s = 0\}.$$ 

If $\{e_1, \ldots, e_d\}$ is the standard basis of the dual lattice $M$ and $\langle, \rangle: M \times N \to \mathbb{Z}$ is the natural pairing, the group $G$ is defined as the kernel of the following homomorphism

$$\Phi: (\mathbb{C}')^r \to (\mathbb{C}')^d, \quad (\lambda_1, \ldots, \lambda_r) \mapsto \left(\prod_{i=1}^r \lambda_i^{\langle e_i, v_i \rangle}, \ldots, \prod_{i=1}^r \lambda_i^{\langle e_s, v_i \rangle}\right)$$

and its actions on $\mathbb{C}' \setminus Z$ is by multiplication

$$(\lambda_1, \ldots, \lambda_r) \cdot (x_1, \ldots, x_r) := (\lambda_1 x_1, \ldots, \lambda_r x_r).$$

In this paper we study the three dimensional orbifold $\mathbb{C}^3_6$ defined as the toric variety associated to the fan generated by the vectors

\begin{equation}
(1) \quad v_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},
\end{equation}

in $N \cong \mathbb{Z}^3$. In this case $Z = \emptyset$ and the associated homomorphism is

\begin{equation}
(2) \quad \Phi: (\mathbb{C}^*)^3 \to (\mathbb{C}^*)^3, \quad (\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_1^{-1} \lambda_2^2 \lambda_3^{-1}, \lambda_1^{-1} \lambda_2^{-1} \lambda_3, \lambda_1 \lambda_2 \lambda_3),
\end{equation}

which has kernel

\begin{equation}
(3) \quad G := \ker \Phi = \langle (e, e^2, e^3) \rangle \subset (\mathbb{C}^*)^3, \text{ with } e = e^{2\pi i}.
\end{equation}
Thus $G \simeq \mathbb{Z}_6$ and $\mathbb{C}^3_6 = \mathbb{C}^3 / \mathbb{Z}_6$ where the action on the coordinates is

\begin{equation}
\epsilon \cdot (x_1, x_2, x_3) = (\epsilon_1 x_1, \epsilon_2 x_2, \epsilon_3 x_3).
\end{equation}

$\mathbb{C}^3_6$ is a non compact Calabi-Yau ($K_{\mathbb{C}^3_6}$ is trivial) threefold with an isolated quotient singularity at the origin, because all vectors $v_i$ lie in the plane $z = 1$ (if $(x, y, z)$ are the coordinates on the lattice).\footnote{We refer to section 3.3.1 for an explanation about this CY condition.} In this way, all relevant information is included in the two dimensional intersection of the fan $\Delta$ with the plane $z = 1$. In the figure we have drawn this section for the fan of $\mathbb{C}^3_6$.

\subsection{Crepant resolutions of $\mathbb{C}^3_6$}

A crepant resolution of a variety $X$ is a smooth variety $Y$ together with a proper birational morphism $\tau: Y \to X$ such that $K_Y = \tau^* K_X$. If $X$ is a Calabi-Yau variety this means that $K_Y$ has to be trivial. Any crepant resolution of a toric Calabi-Yau orbifold $X(\Delta) = \mathbb{C}^3 / G$ can be obtained in two simple steps (see [21, 40]). First, add to $\Delta$ all possible edges $\rho_i$ that are generated by the integer vectors $v_i \in \mathbb{N}$ intersecting the fan and lying on the plane determined by $v_1, v_2, v_3$. Next, let one completely triangulate $\Delta$, to obtain the regular fan $\Delta'$ of the toric resolution $X(\Delta')$. If there exist several complete triangulations this means that the orbifold admits multiple crepant resolutions, all related by flops of curves.
Therefore, to obtain the resolutions of the $\mathbb{C}^3_6$ singular variety we add to $\Delta$ the four vectors

\begin{align*}
  v_4 &= \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, & v_5 &= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, & v_6 &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, & v_7 &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\end{align*}

It is easy to show that we have five admissible complete triangulations.

### 3.2.1 Toric $G$-Hilbert resolution

We start considering the $G$-Hilbert resolution, which we call $G-$C$^3_6$. Its general toric construction is given in [13] and we refer to it for a detailed explanation. We can think to $G$-Hilb fan as the “more symmetric” triangulation. We try to illustrate this concept in our case. First we add to $\Delta$ the two dimensional cones generated by $(v_2, v_7)$ and $(v_3, v_7)$, that are necessary to obtain any complete triangulation. Then we extend the line $(v_3, v_7)$ to $v_5$ so obtaining a subdivision of the fan into regular triangles, three of them with edges of length one and the bigger one with edges of length two. Finally we complete the triangulation subdividing this last triangle with a regular tessellation, obtained by drawing all possible internal lines parallel to its edges.

### 3.2.2 $G$-Hilbert resolution as the moduli space of $G$-clusters of $\mathbb{C}^3$

Given an algebraic variety $M$ and a finite group $G$ with an action on $M$, the $G$-Hilb($M$) is defined as the moduli space of $G$-clusters $Z \subset M$. A $G$-cluster is a $G$-invariant zero dimensional subscheme $Z$, with defining ideal $I_Z \subset O_M$ and
structure sheaf $\mathcal{O}_2 = \mathcal{O}_M / \mathcal{I}_2$ isomorphic to the regular representation of $G$, i.e. $H^0(Z, \mathcal{O}_2) \sim R(G)$ with dim $H^0(Z, \mathcal{O}_2) = |G|$. The simplest example of $G$-cluster is a general orbit of $G$ consisting of $N$ distinct points. 

We will study the simple example of $\mathbb{Z}_2$-Hilb$(\mathbb{C}^2)$. Let us consider $\mathbb{C}^2 = \text{Spec } \mathbb{C}[X, Y]$ and the action of $\mathbb{Z}_2$, with generator $\epsilon = -1$, defined on the coordinates as

$$e \cdot (X, Y) = (eX, eY).$$

The orbits of $\mathbb{Z}_2$ are the sets of couple of points

$$(7) \quad \{(p_1, p_2) \in \mathbb{C}^2 \times \mathbb{C}^2 | X(p_1) = -X(p_2), Y(p_1) = -Y(p_2)\}.$$

If $p_1$ has coordinates $(a, b)$, on the open set $X \neq 0$ the $\mathbb{Z}_2$-cluster $Z$ with support over $(p_1, p_2)$ is defined by the equations

$$X^2 = a, \quad Y = b\frac{1}{\alpha}X \implies \mathcal{O}_Z = \frac{\mathbb{C}[XY]}{(X^2-a, Y-\frac{b}{\alpha}X)} \cong \mathbb{C} \oplus \mathbb{C} \cdot X,$$

and on the open set $Y \neq 0$ by

$$Y^2 = b, \quad X = \frac{\alpha}{b}Y \implies \mathcal{O}_Z = \frac{\mathbb{C}[XY]}{(Y^2-b, X-\frac{\alpha}{b}Y)} \cong \mathbb{C} \oplus \mathbb{C} \cdot Y.$$

It is easy to verify all the properties of the $\mathbb{Z}_2$-clusters. Thus we have a bijective relation between generic orbits and $\mathbb{Z}_2$-cluster having support on them. On the set $\{X = 0, Y = 0\}$ we have $\mathbb{Z}_2$-clusters $Z$ of type

$$(10) \quad X^2 = 0, \quad X = \frac{\alpha}{b}Y \implies \mathcal{O}_Z = \frac{\mathbb{C}[XY]}{(X^2X-\frac{\alpha}{b}Y)} \cong \mathbb{C} \oplus \mathbb{C} \cdot X,$$

for any $(\alpha, \beta)$ with $\alpha \neq 0$, or, in alternative, of type

$$(11) \quad Y^2 = 0, \quad Y = \frac{\beta}{\alpha}X \implies \mathcal{O}_Z = \frac{\mathbb{C}[XY]}{(Y^2Y-\frac{\beta}{\alpha}X)} \cong \mathbb{C} \oplus \mathbb{C} \cdot Y,$$

for any $(\alpha, \beta)$ with $\beta \neq 0$. It is evident that the $\mathbb{Z}_2$-Hilb$(\mathbb{C}^2)$ has the structure of the blow-up of $\mathbb{C}^2 / \mathbb{Z}_2$ at the origin and, with the map

$$(12) \quad \tau : \mathbb{Z}_2\text{-Hilb}(\mathbb{C}^2) \to \mathbb{C}^2 / G \quad Z(p_1, p_2) \mapsto (p_1, p_2),$$

it becomes the (crepant) resolution of the orbifold. Let us prove this fact explicitly using toric geometry.

We will follow the construction of toric orbifold given in [21]. Let $L = \mathbb{Z}^2 + \frac{1}{2}(1, 1)$ be the lattice over $\mathbb{Z}^2$; in $L$ the fan of $\mathbb{C}^2 / \mathbb{Z}_2$ is the junior simplex $\Delta_{\text{junior}}$ generated by the standard base $(e_1, e_2)$ of $\mathbb{Z}_2$.

Using the “old construction” of toric variety [21], we have

$$(13) \quad X_{\Delta_{\text{junior}}} = \text{Spec } \mathbb{C}[X^2, XY, Y^2] = \text{Spec } \frac{\mathbb{C}[U, V, W]}{(UW - V^2)}.$$

The toric resolution of $X_{\Delta_{\text{junior}}}$ is obtained adding to $\Delta_{\text{junior}}$ the edge generated by $\frac{1}{2}(e_1 + e_2)$. In the right side of the figure 3 we have drawn the toric fan of the
Figure 3. Fan for $C^2/\mathbb{Z}_2$ and $\mathbb{Z}_2$-$\text{Hilb}(C^2)$ in $L = \mathbb{Z}^2 + \frac{1}{2}(1, 1)$

resolution marked with the coordinates related to the toric curves, expressed as $\mathbb{Z}_2$-invariant ratios of monomials in the orbifold coordinates. Geometrically this is the blow up of $X_{\Delta_{\text{junior}}} = C^2/\mathbb{Z}_2$ in the origin. The two affine open sets are

\begin{align*}
U_{\sigma_a} &= \text{Spec } \mathbb{C}[X^2, Y/X], \\
U_{\sigma_b} &= \text{Spec } \mathbb{C}[Y^2, X/Y].
\end{align*}

Thus $U_{\sigma_a}$, for example, parameterizes equations of the form

\begin{align*}
X^2 &= \xi_a Y, \\
Y &= \eta_a X,
\end{align*}

which define the $\mathbb{Z}_2$-clusters (8). Similar $U_{\sigma_b}$ parameterizes clusters (9) and their intersection $U_{\sigma_a \cap \sigma_b}$ the clusters (10,11). Therefore the crepant toric resolution of $C^2/\mathbb{Z}_2$ is exactly $\mathbb{Z}_2$-$\text{Hilb}(C^2)$.

In a similar way it has been proved in [36, 13] that the toric resolutions of $\mathbb{C}^3/G$ defined in the previous section (for $G \subset SL(3, \mathbb{C})$ abelian) are exactly the $G$-$\text{Hilb}(\mathbb{C}^3)$. In figure 4 we report the $\mathbb{Z}_6$-$\text{Hilb}(\mathbb{C}^3)$ fan marked with the $\mathbb{Z}_6$-invariant ratios associated to the curves of the resolution.

### 3.3 Intersection theory for $G-\mathbb{C}^3_6$

We are interested in finding the Chow ring $A^*(G-\mathbb{C}^3_6)$, the module $A^*_c(G-\mathbb{C}^3_6)$ and the intersection pairing $A^*(G-\mathbb{C}^3_6) \otimes A^*_c(G-\mathbb{C}^3_6) \rightarrow A^*_c(G-\mathbb{C}^3_6)$ (for this section we refer to [20, 21, 40]).

#### 3.3.1 Chow ring $A^*(G-\mathbb{C}^3_6)$

On any variety $X$ the Chow group $A_k(X)$ is defined to be the free abelian group on the $k$-dimensional irreducible closed subvarieties of $X$, modulo the subgroup generated by the cycles of the form $(f)$, where $f$ is a nonzero rational function on
a $k + 1$-dimensional subvariety of $X$. For a toric variety $X = X(\Delta)$, the Chow group $A_k(X)$ is generated by the classes of the closures $V(\sigma) = O(\sigma)$ of orbits of the $n - k$ dimensional cones $\sigma \in \Delta$ under the action of the torus $\mathbb{C}^n\times$. If $\tau$ is a cone of $\Delta$, we define $N_\tau := \mathbb{Z} \cdot \tau$ and $N(\tau) := N/N_\tau$. The relations in $A_k(X)$ are generated by the cycles of the form $(\chi^u) := \sum_i \langle u, v_i \rangle V(\rho_i)$, where $u$ is an element in the dual lattice $M(\tau) = N(\tau)^\circ$, $\rho_i$ are the one dimensional subcones of the projection of $\tau$ in $N(\tau)$ with primitive vectors $v_i$, $\langle , \rangle$ is the natural pairing between $M(\tau)$ and $N(\tau)$, for any cone $\tau \in \Delta$ of dimension $n - k - 1$.

We will study explicitly this construction for $X = G-C_3^6$.

Let us first decorate the fan in figure 5 with labels for the toric invariant subvarieties related to the cones of $\Delta$:

$A_3(X)$ has only one generator, corresponding to the unique zero dimensional cone of $\Delta$, and obviously without relations.

$$A_3(X) = \mathbb{Z} \cdot X \quad X = V(0) \quad (16)$$

$A_2(X)$ has seven generators, related to the seven one dimensional cones of the fan. The relations are generated by the cycles $(\chi^u)$ for $u$ in $M(0) = M$:

$$A_2(X) = \bigoplus_{i=1}^{7} \mathbb{Z} \cdot D_i \quad D_i = V(\rho_i), \quad \rho_i = \mathbb{R}_{\geq 0} \cdot v_i \quad (17)$$

---

6Recall that $(f)$ is the cycle obtained as the sum of the zeros of $f$ minus the poles of $f$, each counted with its multiplicity.
We choose as $u$ the standard basis of the lattice $M$, $e_1^*, e_2^*, e_3^*$, so we obtain these three independent relations

(18) $-D_1 + 2D_2 - D_3 + D_5 - D_6 = 0$, 
(19) $-D_1 - D_2 + D_3 - D_4 - D_5 = 0$, 
(20) $D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 = 0$.

In $A_2(X)$ the divisors $D_5$, $D_6$, $D_7$ can be expressed in terms of the others

(21) $D_5 = -D_1 - D_2 + D_3 - D_4$, 
(22) $D_6 = -2D_1 + D_2 - D_4$, 
(23) $D_7 = 2D_1 - D_2 - 2D_3 + D_4$.

It follows that

(24) $A_2(X) = \bigoplus_{i=1}^{4} \mathbb{Z} \cdot D_i \cong \mathbb{Z}^4$.

Since the variety $G-\mathbb{C}^3_6$ is non singular, we have

(25) $\text{Pic}(G-\mathbb{C}^3_6) \cong A_2(G-\mathbb{C}^3_6) \cong \mathbb{Z}^4$.

Let us make a remark about the canonical divisor. It is a standard fact in toric geometry that the canonical divisor of a variety $X$ is given by $K_X = -\sum_i D_i$ where the sum is over all toric invariant divisors. By relation (20) it then follows that $K_{G-\mathbb{C}^3_6} = 0$ in $\text{Pic}(G-\mathbb{C}^3_6)$ so that $G-\mathbb{C}^3_6$ is a Calabi-Yau variety. This is true for any toric variety with all integer vectors generating the fan lying on the same (hyper)plane.
$A_1(X)$ has twelve generators, the toric invariant curves. The relations are generated by the cycles $(\chi^u)$ for $u$ in $M(\rho_i)$:

$$A_1(X) = \bigoplus \mathbb{Z} \cdot C_{ij} < (\chi^u) > .$$

Therefore, to find the relations we have to study the geometry of any toric invariant divisor. Recall that $D_i = V(\rho_i)$ is a toric variety for any $i$; its fan is called $Star(\rho_i)$ and is obtained by the projection of the cones containing $\rho_i$ into the quotient lattice $N(\rho_i)$. As an example we plot $Star(\rho_7)$ in figure 6.

![Figure 6. Fan for $D_7$ ($Star(\rho_7)$)](image)

$D_7$ has five toric invariant divisors and two relations between them

$$-C_{27} + C_{37} - C_{47} - C_{57} = 0 ,$$

$$2C_{27} - C_{37} + C_{57} - C_{67} = 0 .$$

Doing the same for any divisor $D_i$ we obtain all relations between curves. At the end we find that any two given curves are equivalent:

$$A_1(X) = \mathbb{Z} \cdot C, \quad C = [C_{46}] .$$

Any other invariant curve is related to $C$ by the relations expressed in the decorated fan of figure 7.

$A_0(X)$ is generated by the six toric invariant points of $X$. Every toric variety contains only two kind of toric curves: compact curves isomorphic to $\mathbb{P}_C^1$ and non-compact curves isomorphic to $A^1_C$. Any compact curve gives a rational relation between two invariant points, and in such way we find that any two given points are rationally equivalent. Finally linear equivalence on affine curves says us that
points are rational equivalent to zero. Therefore \(A_0(X)\) is the trivial group (this is true for any noncompact toric variety).

On a nonsingular \(n\)-dimensional variety \(X\), one sets \(A^p(X) := A_{n-p}(X)\). There is an intersection product \(A^p(X) \times A^q(X) \to A^{p+q}(X)\), making \(A^*(X) := \bigoplus A^p(X)\) into a commutative graded ring. For a general toric variety \(X(\Delta)\), if \(\sigma\) and \(\tau\) are cones in \(\Delta\), then

\[
V(\sigma) \cap V(\tau) = \begin{cases} 
V(\gamma) & \text{if } \sigma \text{ and } \tau \text{ span the cone } \gamma, \\
\emptyset & \text{if } \sigma \text{ and } \tau \text{ do not span a cone in } \Delta.
\end{cases}
\]

If \(X(\Delta)\) is nonsingular and the intersection is proper, i.e. each component of the intersection has codimension equal to the sum of the codimension of the two subvarieties, or empty, then \(V(\sigma)\) and \(V(\tau)\) meet transversally in \(V(\gamma)\) (or \(\emptyset\)). In this case we define \([V(\sigma)] \cdot [V(\tau)] = [V(\gamma)]\) (or 0). Otherwise if \(V(\sigma)\) and \(V(\tau)\) do not meet properly, we can always use rational equivalence to replace in \(A^*(X)\) a subvariety (i.e. \(V(\sigma)\)) with another one in the same class and such that it meets \(V(\tau)\) in a proper way.

Again, let us apply these considerations to our example \(X = G-C_3^3\).

First, note that the intersection between \(X\) and any subvarieties \(V(\sigma)\) is obviously equal to \(V(\sigma)\). Therefore \(X\) is the multiplicative identity in \(A^*(X)\).

Any divisor \(D_i\) meets each other properly (or not at all), and so their products give the curves \(D_i \cdot D_j = [C_{ij}]\) (or 0). We have to use the linear equivalences (18) only to find the autointersections \(D_i \cdot D_i\).

Finally, when we intersect divisors and curves we obtain a point (or \(\emptyset\)), but, as we have seen, they are rational equivalent to zero. Any other intersection is always
equivalent to the empty set.

The intersection products in \( A^*(X) \) are summarized in table 1.

| \( X \) | \( D_1 \) | \( D_2 \) | \( D_3 \) | \( D_4 \) | \( C \) |
|---|---|---|---|---|---|
| \( X \) | \( X \) | \( D_1 \) | \( D_2 \) | \( D_3 \) | \( D_4 \) | \( C \) |
| \( D_1 \) | \( D_1 \) | 0 | 0 | 0 | 0 |
| \( D_2 \) | \( D_2 \) | 0 | 0 | \( C \) | 0 |
| \( D_3 \) | \( D_3 \) | 0 | \( C \) | \( C \) | 0 |
| \( D_4 \) | \( D_4 \) | 0 | 0 | 0 | \( -C \) |
| \( C \) | \( C \) | 0 | 0 | 0 | 0 |

Table 1. Intersection product in \( A^*(X) \)

We can see that the product is symmetric and respects the grading. If we call \( R \) the set of relations given by the intersection product we find

\[
A^*(X) = \mathbb{Z}[X, D_1, D_2, D_3, D_4, C]/R.
\]

### 3.3.2 Group \( A^*_c(G - \mathbb{C}^3_0) \) of compactly supported subvarieties

On a noncompact variety \( X \) the group \( A^*_c(X) \) is defined to be the direct limit of the groups \( A^*(Z) \), where \( Z \) are the closed and compact subvarieties of \( X \) ordered by inclusion. This means \( A^*_c(X) = \bigoplus A^*(Z)/R \), where the direct sum is over all compact subvarieties of \( X \) and the relations \( R \) say that two elements \([Z_1]\) and \([Z_2]\) of \( \bigoplus A^*(Z) \) must be identified if exists a compact subvariety \( Z_3 \) that contains them and such that in \( A^*(Z_3) \) they represent the same cycle class. As usual in toric geometry we can restrict our analysis to compact toric invariant subvarieties \( V(\sigma) \); recall that \( X(\Delta) \) is compact in the classical topology if and only if its support \(|\Delta|\) is the whole space \( N_\mathbb{R} \).

Our example has one compact invariant divisor \( D_7 \), six compact curves \((C_{27}, C_{37}, C_{47}, C_{57}, C_{67}, C_{46})\) and six points \( P_i \). It’s easy to see that (as a group)

\[
A^*(P_i) = \mathbb{Z} \cdot P^c_i, \quad A^*(C_{ij}) = \mathbb{Z} \cdot C^c_{ij} \oplus \mathbb{Z} \cdot P^c_{ij},
\]

where \( P^c_{ij} \) represents the point class in the curve \( C^c_{ij} \).

The divisor \( D_7 \) is the toric variety associated to the fan of figure 6, therefore

\[
A^*(D_7) = \mathbb{Z} \cdot D^c_7 \oplus \mathbb{Z} \cdot C^c_{47} \oplus \mathbb{Z} \cdot C^c_{57} \oplus \mathbb{Z} \cdot C^c_{67} \oplus \mathbb{Z} \cdot P^c_7
\]

and the relations with other curves are

\[
C^c_{27} = C^c_{47} + C^c_{67},
\]

\[
C^c_{37} = 2C^c_{47} + C^c_{57} + C^c_{67}.
\]

Now we have to sum all these groups and find relations between different generators. It results that all point classes have to be identified, exactly as the classes of
the same curve. The group of compact subvarieties of $G - \mathbb{C}^3_6$ is then isomorphic to $\mathbb{Z}^6$:

\begin{equation}
A^c_i(X) = \mathbb{Z} \cdot D^c_7 \oplus \mathbb{Z} \cdot C^c_{46} \oplus \mathbb{Z} \cdot C^c_{57} \oplus \mathbb{Z} \cdot C^c_{67} \oplus \mathbb{Z} \cdot C^c_{47} \oplus \mathbb{Z} \cdot P^c.
\end{equation}

### 3.3.3 Intersection pairing

There is a well defined intersection pairing $A^*(X) \otimes A^c_*(X) \to A^c_*(X)$. For any two generators $[Z_1] \in A^*(X)$ and $[Z_2] \in A^c_*(X)$ it is possible to find two representatives which meet properly. Their intersection is a compact subvariety and defines the above pairing $[Z_1][Z_2] = [Z_1 \cap Z_2] \in A^c_*(X)$, which is extendable by linearity to all elements in $A^*(X) \otimes A^c_*(X)$. This product gives the group $A^c_*(X)$ the structure of an $A^*(X)$-module.

For $X = G - \mathbb{C}^3_6$ we obtain the intersection pairing of table 2: 7

|     | $D^c_7$ | $C^c_{46}$ | $C^c_{57}$ | $C^c_{67}$ | $C^c_{47}$ | $P^c$ |
|-----|---------|------------|------------|------------|------------|-------|
| $X$ |         |            |            |            |            |       |
| $D_1$ | 0       | $P^c$      | 0          | 0          | 0          | 0     |
| $D_2$ | $C^c_{27}$ | 0          | $P^c$      | 0          | 0          | 0     |
| $D_3$ | $C^c_{37}$ | 0          | 0          | $P^c$      | 0          | 0     |
| $D_4$ | $C^c_{47}$ | $-P^c$     | $P^c$      | $P^c$      | 0          |       |
| $C$  | $P^c$   | 0          | 0          | 0          | 0          | 0     |

**Table 2.** Intersection pairing $A^*(X) \otimes A^c_*(X) \to A^c_*(X)$

### 3.4 Homology, cohomology, Mori and Kähler cones

For any compact smooth variety $X$ we have two natural homomorphisms

\[ cl_X : A_*(X) \to H_*(X, \mathbb{Z}), \quad cl^X : A^*(X) \to H^*(X, \mathbb{Z}). \]

The map $cl_X$ sends the representant $V$ of an algebraic cycle to the homological cycle $[V]$; it’s well defined because algebraic equivalence implies homological equivalence. The map $cl^X$ is defined by composition of $cl_X$ with Poincaré duality, which associate to an homological $k$-cycle $V$ the $(n - k)$-form $\eta_V$ such that

\[ \int_V \theta = \int_X \theta \wedge \eta_V. \]

7We can quickly obtain the product between divisors and curves which do not intersect properly in this way: suppose $v_1, v_2$ are the minimal lattice points on the edges of $\sigma_C$ and let $v', v''$ be the minimal lattice points of the three-dimensional cones containing $\sigma_C$, then $v' + v'' = a_1v_1 + a_2v_2$ and $D_k \cdot C = -a_kP$. 
In the case of crepant resolutions of toric orbifolds it is possible to prove \cite{[11]} that it exists the following module isomorphism

\[ A^c_*(X) \approx H^c_*(X, \mathbb{Z}), \quad A^*(X) \approx H^*(X, \mathbb{Z}) \]

which respects the intersection product \footnote{The restriction to compact homology is necessary because of the problem in defining integration over non compact cycles.}

\[ H^*(X, \mathbb{Z}) \otimes H^*_c(X, \mathbb{Z}) \to H^*_c(X, \mathbb{Z}). \]

We are interested in determining the Kähler cone of \( X \), which is the set of all forms \( J \) in \( H^2(X, \mathbb{Q}) \) such that

\[ \int_C J \geq 0 \]

for all effective cycles in \( H^2_*(X, \mathbb{Q}) \). We describe the Kähler cone using the module isomorphism of the previous paragraph. We begin defining the Mori cone, i.e. the polyhedral cone in \( A^2(X) \otimes \mathbb{Q} \) generated by effective toric invariant compact curves of \( X \), which are the compact algebraic cycles \( \sum_{i=1}^n a_{ij}C_{ij} \) where all the \( a_{ij} \) are non-negative. Now we can think at the Kähler cone of \( X \) as the dual polyhedral cone in \( A^2(X) \otimes \mathbb{Q} \) of the Mori cone with respect to the intersection pairing.

For \( X = G-\mathbb{C}^3_6 \), in view of the relations (33,34), the Mori cone is generated by:

\[ C_1 := C_{46}^c, \quad C_2 := C_{57}^c, \quad C_3 := C_{67}^c, \quad C_4 := C_{47}^c. \]

Then the Kähler cone has the following dual generators, that satisfied \( T_a \cdot C_b = \delta_{ab} P^c \):

\[ T_1 := D_1, \quad T_2 := D_2, \quad T_3 := D_3, \quad T_4 := -D_1 + D_2 + D_3 - D_4. \]

For completeness we report in table 3 the products between the \( T_i \) in the Chow ring \( A^*(X) \):

|   | \( T_1 \) | \( T_2 \) | \( T_3 \) | \( T_4 \) |
|---|---|---|---|---|
| \( T_1 \) | 0 | 0 | 0 | 0 |
| \( T_2 \) | 0 | 0 | \( C \) | \( C \) |
| \( T_3 \) | 0 | \( C \) | \( C \) | 2\( C \) |
| \( T_4 \) | 0 | \( C \) | 2\( C \) | 2\( C \) |

**Table 3.** Products between the Kähler generators in \( A^*(X) \).
3.5 K-theory

3.5.1 Preliminaries

The K-theory ring of a variety \( X \) is related to the cohomology via the Chern character map. More precisely this is an injective homomorphism of rings from \( K(X) \) to the Chow ring with rational coefficients \( A^*(X) \otimes \mathbb{Q} \). Then composition with \( c_1^X \) gives the homomorphism with \( H^*(X, \mathbb{Q}) \):

\[
\text{ch} : K(X) \to A^*(X) \otimes \mathbb{Q} \cong H^*(X, \mathbb{Q}).
\]

Here we summarize some general properties of Chern map that will be useful in the next sections. The Chern class \( c_i \) is a map from \( K(X) \) to \( A^i(X) \otimes \mathbb{Q} \); the total Chern class is defined as the sum of all Chern classes

\[
c(\mathcal{F}) := c_0(\mathcal{F}) + c_1(\mathcal{F}) + \ldots + c_n(\mathcal{F}),
\]

where \( n \) is the dimension of \( X \). A divisorial \( \mathcal{O}_X(D) \) sheaf has a very simple total Chern class

\[
c(\mathcal{O}_X(D)) = X + D
\]

and, using multiplicative properties of the Chern classes, this implies

\[
\mathcal{F} = \bigoplus_{i=1}^r \mathcal{O}_X(D_i) \implies c(\mathcal{F}) = \prod_{i=1}^r (X + D_i).
\]

The Chern character is defined for such sheaf as

\[
\text{ch}(\mathcal{F}) := \sum_{i=1}^r e^{D_i} = rX + c_1(\mathcal{F}) + \frac{1}{2}(c_1(\mathcal{F})^2 - 2c_2(\mathcal{F})),
\]

where the expansion is stopped to second order in view of the cohomology ring structure of our non compact threefold varieties. In particular for a divisorial sheaf we have

\[
\text{ch}(\mathcal{O}_X(D)) = X + D + \frac{1}{2}D^2.
\]

We recall also the definition of the Todd class:

\[
\text{td}(\mathcal{F}) := \prod_{i=1}^r \frac{D_i}{1 - e^{-D_i}} = X + \frac{1}{2}c_1(\mathcal{F}) + \frac{1}{12}(c_1(\mathcal{F})^2 + c_2(\mathcal{F})).
\]

In particular, we need the Todd class of the tangent bundle \( T_X \); for a toric variety its total Chern class is

\[
c(T_X) = \sum_{\sigma \in \Delta} [V(\sigma)]
\]

and so if \( X \) is a Calabi-Yau non compact toric threefold we have

\[
\text{(39)} \quad c(T_X) = X + \sum [C_{ij}] \quad \Rightarrow \quad \text{td}(X) = \text{td}(T_X) = X + \frac{1}{12} \sum [C_{ij}].
\]
where the sum is over all (compact and non compact) toric curves in $X$.

In the context of non compact varieties we also have to work with the compactly supported K-theory group $K^c(X)$. This group is related to the compactly supported Chow group with rational coefficients $A^*_c(X)$, and therefore to $H^*_c(X, \mathbb{Q})$, via the local Chern character map $\text{ch}^c$:

$$\text{ch}^c : K^c(X) \rightarrow A^*_c(X) \otimes \mathbb{Q} \cong H^*_c(X, \mathbb{Q}).$$

Let us briefly review its definition and properties. Any element $S$ of $K^c(X)$ can be represented by coherent sheaves $S_V$ on a compact subvariety $V$ of $X$. If $i : V \hookrightarrow X$ is the embedding of $V$ in $X$, we can define the local Chern character of $S$ by

$$\text{ch}^c(S) = \text{ch}(i^*S_V).$$

Actually we can compute the local Chern characters with the help of the Grothendieck-Riemann-Roch theorem:

$$i_* (\text{ch}(S_V) \text{td}(V)) = \text{ch}(i_* S_V) \text{td}(X)$$

for any compact subvarieties $V$ of $X$, which implies

$$\text{ch}^c(S) = \text{td}(X)^{-1} i_*(\text{ch}(S_V) \text{td}(V)).$$

The local Chern classes of the divisorial sheaves over the compact subvarieties in a Calabi-Yau non compact toric threefold $X$ are:

$$\text{ch}^c(O_{p^e}) = p^e \quad \text{ch}^c(O_{C^c}(n)) = C^c + (n + 1)p^c$$

$$\text{ch}^c(O_{D^c}(C)) = i_* \left( D^c + \left( C + \frac{1}{2}c_1(D^c) \right) + \frac{1}{2}(C^2 + c_1(D^c)C + \frac{1}{6}(c_1(D^c)^2 + c_2(D^c))) \right) - \frac{1}{12}c_2(X) D^c. \quad (40)$$

In the last character, $C$ is a divisor in $D^c$ and the $c_i(D^c)$ are the Chern classes $c_i(T_{D^c})$, which naturally live in $A^*(D^c)$ and that can be calculated using formula (39). Moreover all the products excepted the last are in $A^*(D^c)$.

### 3.5.2 K-theory generators

Let $G$ be an abelian subgroup of $SL(3, \mathbb{C})$ which acts on the affine space $\mathbb{C}^3$. We write $\pi : \mathbb{C}^3 \rightarrow Y = \mathbb{C}^3/G$ for the quotient, $X = G$-Hilb$(\mathbb{C}^3)$ for the Hilbert scheme with crepant resolution $\tau : X \rightarrow Y$ and the universal scheme $Z = \{(Z(x), x) \in X \times \mathbb{C}^3\}$ where $Z(x)$ is the $G$-cluster over $x$ (see section 3.2.2). Thus we have the commutative diagram

\[
\begin{array}{ccc}
  Z & \xrightarrow{q} & \mathbb{C}^3 \\
 \downarrow^p & & \downarrow^\pi \\
  X & \xrightarrow{\tau} & Y
\end{array}
\]
Let us consider the sheaf $\mathcal{R} := p_*O_Z$ on the resolution $X$. Over any point $x \in X$ the fibre of $\mathcal{R}$ is $H^0(Z(x), O_{Z(x)})$ which supports the regular representation of $G$. In particular the rank of $\mathcal{R}$ is equal to the order of the group $G$. The decomposition of the regular representation into irreducible submodules induces the decomposition

$$\mathcal{R} = \bigoplus_k \mathcal{R}_k \otimes \rho_k$$

into locally free sheaves of rank $\mathcal{R}_i = \dim \rho_i = 1$. We called $\mathcal{R}_k$ the tautological line bundle on $X$ associated to the irreducible representation $\rho_k$ of $G$. At the level of $K$-theory the McKay correspondence states the equivalence of the $G$-equivariant $K$-theory of $\mathbb{C}^n$ and the $K$-theory of the crepant resolutions. In [36] it has been determined the ring isomorphism

$$\varphi : K^G(\mathbb{C}^3) \xrightarrow{\sim} K(X)$$

showing that $\varphi(\rho_i \otimes O_{\mathbb{C}^3}) = \mathcal{R}_i$ and therefore, that the tautological line bundles form a $\mathbb{Z}$-basis of $K(X)$.

In section 3.2.2 we studied the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ and its crepant resolution $\mathbb{Z}_2$-$\text{Hilb} (\mathbb{C}^2)$. We have given a description of $\mathcal{R}$ and its decomposition into line bundles on the two open sets $U_{\sigma_a}$ and $U_{\sigma_b}$. In figure 8 we report the monomial generators of $\mathcal{R}_i$ on the affine pieces.

![Figure 8. Monomial generators of $\mathcal{R}_0$ and $\mathcal{R}_1$ for $\mathbb{Z}_2$-$\text{Hilb} (\mathbb{C}^2)$](image)

With the same procedure we can give the generators of $\mathcal{R}_i$ on $G$-$\text{Hilb} (\mathbb{C}^3)$ for any abelian $G \subset SL(3, \mathbb{C})$. We report in the figure 9 the monomial generators for $\mathbb{Z}_6$-$\text{Hilb} (\mathbb{C}^3)$.

The action of $\mathbb{Z}_6$ on the coordinate ring of $\mathbb{C}^3$ is

$$\epsilon \cdot (X, Y, Z) = (\epsilon X, \epsilon^2 Y, \epsilon^3 Z), \quad \epsilon = e^{2\pi i/6}. \quad (41)$$

Therefore, it is simple to verify that each $\mathcal{R}_i$ supports the irreducible representation $\rho_i$ of $\mathbb{Z}_6$. 


Any line bundle on a smooth algebraic variety is a divisorial bundle. We briefly sketch the standard procedure to find the divisor related to the $R_i$ defined by Reid and proved by Craw, and refer to [11] for a detailed explanation.

The first step consists in decorating the $G$-Hilb fan with the characters of the group. Any curve has to be marked with the character of the monomials in its associated ratio. For any internal vertex $v$ there exists a recipe to associate one or two characters of $G$, depending primarily on the valency of $v$ (i.e. the number of lines meeting at $v$). For a $G$-Hilb fan this is always 3, 4, 5 or 6. There are the following cases:

- A vertex $v$ of valency 3 defines an exceptional $\mathbb{P}^2$. A single character $\chi_k$ marks all three lines meeting at $v$. Mark the vertex $v$ with the character $\chi_m := \chi_k \otimes \chi_k$.

- A vertex $v$ of valency 4 defines an exceptional Hirzebruch surface $\mathbb{F}_r$. There are distinct characters $\chi_k$ and $\chi_l$ each one marking a pair of lines meeting at $v$. Mark the vertex $v$ with the character $\chi_m := \chi_k \otimes \chi_l$.

- A vertex $v$ of valency 5 or 6 (excluding three straight lines meeting at a point) defines an Hirzebruch surface $\mathbb{F}_r$ blown-up in one or two points. There are uniquely determined characters $\chi_k$ and $\chi_l$ each one marking a pair of lines meeting at $v$. Mark the vertex $v$ with $\chi_m := \chi_k \otimes \chi_l$.

- A vertex $v$ at the intersection of three straight lines defines an exceptional Del Pezzo surface of degree six, denoted $\text{dP}_6$. The monomials defining the pair of morphisms $\text{dP}_6 \rightarrow \mathbb{P}^2$ lie in uniquely determined character spaces $\chi_l$ and $\chi_m$ satisfying

$$\chi_l \otimes \chi_m = \chi_i \otimes \chi_j \otimes \chi_k,$$

where $\chi_i$, $\chi_j$ and $\chi_k$ mark the straight lines through the vertex $v$. Mark the vertex $v$ with both $\chi_l$ and $\chi_m$.
Each character of \( G \) appears once on the fan \( \Delta \).

By analyzing of the monomial generators of the tautological line bundles \( R_i \), in [11] the author proved that:

- If \( \chi_k \) marks the line defining the compact curve \( C_k \in H^2(X, \mathbb{Z}) \) on the resolution \( X \), the first Chern class \( c_1(R_k) \) is the dual to \( C_k \) in \( H^2(X, \mathbb{Z}) \):

\[
\int_{C_k} c_1(R_k) = \delta_{lk}.
\]

This means that \( R_k = O_X(T_k) \), where \( T_k \) is the generator of the Kähler cone dual to \( C_k \).

- In \( \text{Pic}(X) \) all relations between tautological line bundles are of the following forms:
  - \( R_m = R_k \otimes R_k \) when \( \chi_m = \chi_k \otimes \chi_k \) marks a vertex \( v \) of valency 3;
  - \( R_m = R_k \otimes R_l \) when \( \chi_m = \chi_k \otimes \chi_l \) marks a vertex \( v \) of valency 4;
  - \( R_m = R_k \otimes R_l \) when \( \chi_m = \chi_k \otimes \chi_l \) marks a vertex \( v \) of valency 5 or 6 (excluding three straight lines meeting at a point);
  - \( R_l \otimes R_m = R_l \otimes R_j \otimes R_k \) when the pair of characters \( \chi_i \) and \( \chi_m \) satisfying \( \chi_i \otimes \chi_m = \chi_i \otimes \chi_j \otimes \chi_i \) marks the intersection point \( v \) of three straight lines.

As usual we apply these considerations to our case \( X = \mathbb{Z}_6\text{-Hilb}(C^3) \) and we summarize them in the decorated fan of figure 10.

![Figure 10. Fan for \( \mathbb{Z}_6\text{-Hilb}(C^3) \) decorated with Reid’s recipe](image)

The resulting tautological line bundles, that give a \( \mathbb{Z} \)-basis of \( K(X) \), are:

\[
R_0 = O_X, \quad R_1 = O_X(D_1), \quad R_2 = O_X(D_2), \quad R_3 = O_X(D_3),
\]
\( R_4 = O_X(-D_1 + D_2 + D_3 - D_4), \quad R_5 = R_2 \otimes R_3 = O_X(D_2 + D_3). \)

### 3.5.3 \( K(X) \) and \( K^c(X) \)

Chosen a base of generators for \( K(X) \) we can find the dual basis for the compact K-theory \( K^c(X) \) as in [36] using the perfect pairing \( (\cdot, \cdot) : K(X) \times K^c(X) \rightarrow \mathbb{Z}, \ (R, S) \mapsto (R|S) = \int_X \text{ch}(R)\text{ch}^c(S)\text{td}(X), \)
so that
\[
(43) \quad (R_i|S_j) = \delta_{ij}.
\]
As usual, the integral is by definition the coefficient of the point class. Using this fact, the standard computations of Chern and Todd characters and the intersection product (42) from condition (43) we find
\[
\begin{align*}
\text{ch}^c(S_0) &= D_7^c - \left(C_{46}^c + C_{57}^c + \frac{3}{2}C_{67}^c + 2C_{47}^c \right) + \frac{7}{6}p^c, \\
\text{ch}^c(S_1) &= C_{46}^c, \\
\text{ch}^c(S_2) &= -D_7^c + \left(C_{57}^c + \frac{1}{2}C_{67}^c + C_{47}^c \right) - \frac{1}{6}p^c, \\
\text{ch}^c(S_3) &= -D_7^c + \left(\frac{3}{2}C_{67}^c + C_{47}^c \right) - \frac{1}{6}p^c, \\
\text{ch}^c(S_4) &= C_{47}^c, \\
\text{ch}^c(S_5) &= D_7^c - \left(\frac{1}{2}C_{67}^c + C_{47}^c \right) + \frac{1}{6}p^c.
\end{align*}
\]

In the spirit of the paper [30] we now express the elements \( S_i \) in terms of a symplectic D-brane basis of \( K^c(X) \). Such basis can be constructed starting from the generators of the compact Chow ring. We choose
\[
(45) \quad B_0 := O_X; \quad B_a := O_{C^\alpha_a}(-T_a); \quad B_5 := O_{D^c_7}(-T_2 - T_3),
\]
with \( a = 1, \ldots, 4 \) and \( O_{C^\alpha_a}(-T_a) := O_{C^\alpha_a} \otimes O_X(-T_a), O_{D^c_7}(-T_2 - T_3) := O_{D^c_7} \otimes O_X(-T_2 - T_3). \) To express the basis \( S_i \) in terms of \( B_j \) we can compare their compact Chern characters. Using (40) and the multiplicative property of Chern character we find
\[
(46) \quad \text{ch}^c(B_0) = p^c \quad \text{ch}^c(B_a) = C^c_a \quad \text{ch}^c(B_5) = D^c_7 - \left(\frac{1}{2}C^c_{67} + C^c_{47} \right) + \frac{1}{6}p^c
\]
and then
\[
\begin{align*}
S_0 &= B_0 - B_1 - B_2 - B_3 - B_4 + B_5, \\
S_1 &= B_1, \\
S_2 &= B_2 - B_3, \\
S_3 &= B_3 - B_5, \\
S_4 &= B_4, \\
S_5 &= B_5.
\end{align*}
\]
Finally we write the $B_i$ basis of $K^c(X)$ in terms of the $S_i$ and its dual basis $\Phi_i$ of $K(X)$ in term of $R_i$:

\begin{align}
B_0 &= S_0 + S_1 + S_2 + S_3 + S_4 + S_5, & \Phi_0 &= R_0, \\
B_1 &= S_1, & \Phi_1 &= -R_0 + R_1, \\
B_2 &= S_2 + S_5, & \Phi_2 &= -R_0 + R_2, \\
B_3 &= S_3 + S_5, & \Phi_3 &= -R_0 + R_3, \\
B_4 &= S_4, & \Phi_4 &= -R_0 + R_4, \\
B_5 &= S_5. & \Phi_5 &= R_0 - R_2 - R_3 + R_5.
\end{align}

(48)

3.6 The Hosono conjecture

We will restate shortly here the Hosono conjecture [30], for convenience. The main point is that the periods for the mirror manifold are solutions of a set of Picard-Fuchs equations, and the general solution can be expressed in terms of an hypergeometric function with value in the cohomology of $X$:

$$w = w(x_1, \ldots, x_4; J_1 \frac{2\pi}{2\pi i}, \ldots, J_4 \frac{2\pi}{2\pi i})$$

Then the conjecture (adapted to our case) states as follows.

3.6.1 Hosono conjecture

Define the basis for $H^*(X, \mathbb{Q})$

$$Q_i := \text{ch}(\Phi_i), \quad i = 0, \ldots, 5$$

and expand the cohomology-valued hypergeometric series $w$ with respect to this basis:

$$w \left( x_1, \ldots, x_4; \frac{J_1}{2\pi i}, \ldots, \frac{J_4}{2\pi i} \right) = \sum_{i=0}^{5} w_i(x_1, \ldots, x_4) Q_i.$$

Thus

1) the coefficient hypergeometric series $w_i(x_1, \ldots, x_4)$ may be identified with the period integrals over the cycles $\text{mir}(B_i)$,

$$w_i(x_1, \ldots, x_4) = \int_{\text{mir}(B_i)} \Omega(Y_{\chi}) ;$$

2) the monodromy of the hypergeometric series is integral and symplectic with respect to the symplectic form defined in $K^c(X)$

$$\chi(B_i, B_j) = \int_X \text{ch}(B_i^\vee) \text{ch}(B_j) \text{td}(X) ;$$
3) the central charge of an element $F \in K^\vee(X)$ is expressed in terms of the cohomology valued hypergeometric $w$ as

$$Z(F) = \int_X \text{ch}(F) w \left( x_1, \ldots, x_4; \frac{J_1}{2\pi i}, \ldots, \frac{J_4}{2\pi i} \right) \text{td}(X).$$

The symplectic form of point 2 can be easily computed with respect to the basis $S_j^i$ following the paper of Ito–Nakajima [36]. Let $Q$ be the 3-dimensional representation given by the inclusion $G \subset SL(3, \mathbb{C})$ and $\{\rho_i\}_{i=0}^r$ be the irreducible representations. The decomposition

$$Q \otimes \rho_j = \bigoplus_k a_{ij} \rho_i$$

is related to the symplectic form by

$$\chi(S_i, S_j) = a_{ji} - a_{ij}.$$

In our example

$$\chi(S_i, S_j) = \begin{pmatrix}
0 & 1 & 1 & 0 & -1 & -1 \\
-1 & 0 & 1 & 1 & 0 & -1 \\
-1 & -1 & 0 & 1 & 1 & 0 \\
0 & -1 & -1 & 0 & 1 & 1 \\
1 & 0 & -1 & -1 & 0 & 1 \\
1 & 1 & 0 & -1 & -1 & 0
\end{pmatrix},$$

and then, for our chosen basis,

$$\chi(B_i, B_j) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & 1 & 1 & 0
\end{pmatrix}.$$

This matrix gives a symplectic correspondence between the space $H^4(X, \mathbb{Q})$ and a one dimensional subspace of $H^2(X, \mathbb{Q})$. There is an obvious ambiguity in such a correspondence, but we will turn back to it later.

### 3.6.2 The cohomological hypergeometric series

The vectors $\ell_a, a = 1, \ldots, 4$ are given by the intersection numbers between the Mori cone generators and the invariant divisors of $X$, so that we find

$$C_1 : \ell_1 = (1, 0, 0, -1, 0, -1, 1),$$

$$C_2 : \ell_2 = (0, 1, 0, 1, -2, 0, 0),$$

$$C_3 : \ell_3 = (0, 0, 1, 1, 0, -1, -1),$$

$$C_4 : \ell_4 = (0, 0, 0, -1, 1, 1, -1).$$

(51)
The hypergeometric series is then

\[
(52) \quad w = \sum_{\vec{m} \in \mathbb{Z}^4_{\geq 0}} \frac{x_1^{m_1+p_1} x_2^{m_2+p_2} x_3^{m_3+p_3} x_4^{m_4+p_4}}{\prod_{i=1}^{7} \Gamma_i(\vec{m} + \vec{\rho})} \bigg|_{j = \frac{\vec{\rho}}{2\pi i}}
\]

\[
\Gamma_1(\vec{m}) = \Gamma(1 + m_1),
\Gamma_2(\vec{m}) = \Gamma(1 + m_2),
\Gamma_3(\vec{m}) = \Gamma(1 + m_3),
\Gamma_4(\vec{m}) = \Gamma(1 - m_1 + m_2 + m_3 - m_4),
\Gamma_5(\vec{m}) = \Gamma(1 - 2m_2 + m_4),
\Gamma_6(\vec{m}) = \Gamma(1 - m_1 - m_3 + m_4),
\Gamma_7(\vec{m}) = \Gamma(1 + m_1 - m_3 - m_4).
\]

We need to expand this function in power series in \( \vec{J} \). Because of the ring relations for \( H^*(X, \mathbb{Q}) \), we see that the expansion stops at order two. The coefficient functions, with respect to the basis \( \{1, \vec{J}, C\} \) of \( H^*(X, \mathbb{Q}) \), are computed in the appendix. However we chosen the basis \( B_i \) in \( K^c(X) \) so that we need to rewrite the expansion in terms of the dual basis \( Q_i = \text{ch}(\Phi_i) \):

\[
(54) \quad Q_0 = 1, \quad Q_1 = J_1, \quad Q_2 = J_2, \quad Q_3 = J_3 + \frac{1}{2}C, \quad Q_4 = J_4 + C, \quad Q_5 = C.
\]

If we make this change of basis and use the mirror symmetry identification

\[
(55) \quad w \left( \frac{\vec{x}}{2\pi i}, \frac{-\vec{J}}{2\pi i} \right) = Q_0 q + \sum_{a=1}^{4} Q_4 t_a + Q_5 g(t_1, \ldots, t_4),
\]

then we find

\[
2\pi i t_1 = \log x_1 + \Psi(x_1 x_3) + \Phi(x_2, x_1 x_4) + N(\vec{x}),
2\pi i t_2 = \log x_2 - \Phi(x_2, x_1 x_4) + 2\Phi(x_1 x_4, x_2),
2\pi i t_3 = \log x_3 - \Phi(x_2, x_1 x_4) + \Psi(x_1 x_3) - N(\vec{x}),
2\pi i t_4 = \log x_4 - \Phi(x_1 x_4, x_2) + \Phi(x_2, x_1 x_4) - \Psi(x_1 x_3) - N(\vec{x}),
\]
and

\[
(2\pi i)^2 g(\vec{J}) = -\frac{\pi^2}{3} - \pi(\log x_3 - \Phi(x_2, x_1 x_4) + \Psi(x_1 x_3) - N(\vec{x}))
-2\pi i(\log x_4 - \Phi(x_1 x_4, x_2) + \Phi(x_2, x_1 x_4) - \Psi(x_1 x_3) - N(\vec{x}))
+7N^{(1)}(\vec{x}) - 3N^{(2)}(\vec{x}) - 2N^{(3)}(\vec{x}) - N^{(4)}(\vec{x}) - N^{(5)}(\vec{x})
-\Psi_1(x_2, x_1 x_4) + \Psi_2(x_2, x_1 x_4) - \Psi_1(x_1 x_4, x_2) + \Psi_2(x_1 x_4, x_2)
-\Psi_1(x_1 x_4, x_2) + \Psi_3(x_1 x_4, x_2) - \Psi_4(x_1 x_3) + \Psi_5(x_1 x_3) + \Psi_6(x_2, x_4)
+\Lambda_1(\vec{x}) - \Lambda_2(\vec{x}) - \Lambda_3(\vec{x})
+\frac{1}{2} \left( \log x_3 \right)^2 + \log x_3 \left[ \Phi(x_1 x_3) - \Phi(x_2, x_1 x_4) - N(\vec{x}) \right]
+[\log x_4]^2 + 2 \log x_4 \left[ \Phi(x_2, x_1 x_4) - \Phi(x_1 x_4, x_2) - \Psi(x_1 x_3) - N(\vec{x}) \right]
+\log x_3 \log x_3 + \log x_2 \left[ \Psi(x_1 x_3) - \Phi(x_2, x_1 x_4) - N(\vec{x}) \right]
+\log x_3 \left[ 2\Phi(x_1 x_4, x_2) - \Phi(x_2, x_1 x_4) \right] + \log x_2 \log x_4
\]
\[ + \log x_3[\Phi(x_2, x_3, x_4) - \Phi(x_1, x_4, x_2) - \Psi(x_1, x_3) - \mathcal{N}(x)] \]
\[ + \log x_4[2\Phi(x_1, x_4, x_2) - \Phi(x_2, x_1, x_4)] + 2 \log x_3 \log x_4 \]
\[ + 2 \log x_5[\Phi(x_2, x_1, x_4) - \Phi(x_1, x_4, x_2) - \Psi(x_1, x_3) - \mathcal{N}(x)] \]
\[ + 2 \log x_4[\Psi(x_1, x_3) - \Phi(x_2, x_1, x_4) - \mathcal{N}(x)]. \]

Using the above expressions we find
\[ g(\vec{i}) = P_2(\vec{i}) + \frac{1}{(2\pi i)^2} \phi(\vec{i}), \]
where \( P_2 \) is the degree two polynomial part
\[ P_2(\vec{i}) = \frac{1}{12} - \frac{1}{2} t_3 - t_4 + \frac{1}{2} t_3^2 + t_2 t_3 + t_2 t_4 + 2 t_3 t_4, \]
and
\[ \phi(\vec{i}) = 7\mathcal{N}^{(1)}(\vec{x}) - 3\mathcal{N}^{(2)}(\vec{x}) - 2\mathcal{N}^{(3)}(\vec{x}) - \mathcal{N}^{(4)}(\vec{x}) - \mathcal{N}^{(5)}(\vec{x}) \]
\[ - \Psi_1(x_2, x_1, x_4) + \Psi_2(x_2, x_1, x_4) - \Psi_1(x_1, x_2, x_4) + \Psi_2(x_1, x_2, x_4) \]
\[ - \Psi_1(x_1, x_3, x_2) + \Psi_2(x_1, x_4, x_3) - \Psi_3(x_1, x_4, x_2) + \Psi_4(x_1, x_3, x_4) + \Psi_5(x_1, x_3, x_4) \]
\[ + \Lambda_1(\vec{x}) - \Lambda_2(\vec{x}) - \Lambda_3(\vec{x}) \]
\[ + \frac{1}{2} \mathcal{S}^2(x_1, x_3) + \frac{1}{2} \mathcal{S}^3(x_1, x_2) + \mathcal{S}^4(x_1, x_2) - \frac{7}{2} \mathcal{S}^2(\vec{x}) \]
\[ - \frac{1}{2} \Phi(x_1, x_3) \Phi(\vec{x}) - \Phi(x_2, x_1, x_4) \Phi(x_1, x_4, x_2), \]
with \( \vec{x} \) expressed as a function of \( \vec{i} \) by inverting system (56), is the part corresponding to instantonic contributions. Following Hosono and using (50) we find

\[ (\partial_{t_1} - \partial_{t_2} - \partial_{t_3}) F(\vec{i}) = g(\vec{i}), \]

where \( F \) is the prepotential. To integrate this equation we must expect for the prepotential to be as usual the sum of a classical term, a cubic polynomial in \( \vec{i} \) and a quantum instantonic contribution. Setting

\[ q_k := e^{2 \pi i t_k}, \]

we then find

\[ F(\vec{i}) = -\frac{t_4}{12} + \frac{t_4^2}{4} + \frac{t_3 t_4}{2} - \frac{t_3^2}{6} - \frac{t_3 t_4}{2} (t_3 + t_4 + 2 t_2) \]
\[ + F_{\text{inst}}(\vec{q}) + P_{\text{class}}(t_2, t_1 + t_3, t_1 + t_3) + Q_{\text{inst}}(q_2, q_1 q_3, q_1 q_4), \]

where \( F_{\text{inst}} \) are the instantonic corrections, obtained integrating \( \phi, P_{\text{class}} \) is an arbitrary cubic polynomial of three variables and \( Q_{\text{inst}} \) an arbitrary function of three variables which we assume analytic in \( (0, 0, 0) \). \( P_{\text{class}} \) and \( Q_{\text{inst}} \) represent the contributions which are undetermined by the equation (61).

As an example, let us compute the Gromov-Witten (GW) invariants up to degree six. Here we use the Gopakumar-Vafa (G-V) reinterpretation, so that by GW–invariants we mean the G-V integral invariants for rational curves, in place of the
original fractional $GW$–invariants. We will use $[d_1, d_2, d_3, d_4]$ to indicate the degree of the curves in the Mori cone, corresponding to the generators $J_1, \ldots, J_4$. Thus we consider the curves with degree $d_1 + d_2 + d_3 + d_4 \leq 6$. The curves with degree in the integer cone generated by $[0, 1, 0, 0], [1, 0, 1, 0], [1, 0, 0, 1]$ must be excluded, because corresponding to the undetermined part of the prepotential. The only nonvanishing invariants in the considered range are

$$GW_{[0,0,0,1]} = GW_{[0,0,1,0]} = GW_{[1,0,0,0]} =$$

$$GW_{[1,0,1,1]} = GW_{[0,1,0,1]} = GW_{[1,1,1,1]} = 1;$$

$$GW_{[0,0,1,1]} = GW_{[0,1,1,1]} = GW_{[1,1,1,2]} = -2;$$

$$GW_{[0,1,1,2]} = GW_{[1,1,2,2]} = 3;$$

$$GW_{[0,1,2,2]} = -4; \quad GW_{[0,1,2,3]} = 5.$$  

(64)

4 The flopped resolutions

In this section we study the remaining four crepant resolutions $X$ of the orbifold $\mathbb{C}^3/\mathbb{Z}_6$.

4.1 Intersection theory

For any resolution we have

$$A^0(X) = \mathbb{Z} \cdot X,$$

$$A^1(X) = \bigoplus_{i=1}^4 \mathbb{Z} \cdot D_i \cong \mathbb{Z}^4,$$

$$A^2(X) = \mathbb{Z} \cdot C,$$

$$A^3(X) = 0.$$  

(65)

In $A_2(X)$ the divisors $D_5, D_6, D_7$ can be expressed in terms of the others

(66) \quad $D_5 = -D_1 - D_2 + D_3 - D_4,$

(67) \quad $D_6 = -2D_1 + D_2 - D_4,$

(68) \quad $D_7 = 2D_1 - D_2 - 2D_3 + D_4.$

The curve $C$ depends on the resolution, as well as the intersection product in $A^\ast(X)$. If we call $R$ the set of relations given by the intersection product, we have

(69) \quad $A^\ast(X) = \mathbb{Z}[X, D_1, D_2, D_3, D_4, C]/R.$

$A^\ast_\zeta(X)$ is an $A^\ast(X)$-module, it is generated as group by the compact divisor $D_\zeta$, the four compact curves depending on the resolution and the point class $P^\zeta$. Finally the intersection pairing $A^\ast(X) \otimes A^\ast_\zeta(X) \to A^\ast_\zeta(X)$ depends on the resolution.

---

See [3] for a nice introduction to $GW$ invariants and their interpretation in physics and in mathematics.
Resolution $X = \mathbb{R}^{2} - C_{6}^{3}$

This resolution differs from the $G$-Hilb by the flop

$$C_{46} \rightarrow C_{17}. $$

We define $C = [C_{14}]$ and we report the relations between any other toric curve and $C$ in the decorated fan of figure 12.

In table 4 we summarize the intersection products, which give the relations $R$ in
the Chow ring \( A^*(X) = \mathbb{Z}[X, D_1, D_2, D_3, D_4, C]/R. \)

\[
A^*(X) = \mathbb{Z} \cdot [X, D_1, D_2, D_3, D_4, C]/R.
\]

| \( X \) | \( D_1 \) | \( D_2 \) | \( D_3 \) | \( D_4 \) | \( C \) |
|---|---|---|---|---|---|
| \( X \) | \( X \) | \( D_1 \) | \( D_2 \) | \( D_3 \) | \( D_4 \) | \( C \) |
| \( D_1 \) | \( D_1 \) | \( C \) | \( 0 \) | \( 0 \) | \(-C\) | \( 0 \) |
| \( D_2 \) | \( D_2 \) | \( 0 \) | \( 0 \) | \(-C\) | \( 0 \) | \( 0 \) |
| \( D_3 \) | \( D_3 \) | \( 0 \) | \(-C\) | \(-C\) | \( 0 \) | \( 0 \) |
| \( D_4 \) | \( D_4 \) | \(-C\) | \( 0 \) | \( 0 \) | \( 2C\) | \( 0 \) |
| \( C \) | \( C \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |

**Table 4.** Intersection product in \( A^*(X) \)

The group of compact sub-varieties of \( X \) is

\[
A^c_*(X) = \mathbb{Z} \cdot D_7 \oplus \mathbb{Z} \cdot C_{17} \oplus \mathbb{Z} \cdot C_{57} \oplus \mathbb{Z} \cdot C_{67} \oplus \mathbb{Z} \cdot C_{47} \oplus \mathbb{Z} \cdot P^c,
\]

with relations to other compact curves

\[
C_{27}^c = 2C_{17} + C_{67} + C_{47},
\]

\[
C_{37}^c = 3C_{17} + C_{57} + C_{67} + 2C_{47}.
\]

In table 5 we summarize the intersection pairing.

| \( X \) | \( D_7^c \) | \( C_{17}^c \) | \( C_{57}^c \) | \( C_{67}^c \) | \( C_{47}^c \) | \( P^c \) |
|---|---|---|---|---|---|---|
| \( D_1 \) | \( C_{17}^c \) | \(-P^c\) | \( 0 \) | \( P^c \) | \( P^c \) | \( 0 \) |
| \( D_2 \) | \( C_{57}^c \) | \( 0 \) | \( P^c \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( D_3 \) | \( C_{67}^c \) | \( 0 \) | \( 0 \) | \( P^c \) | \( 0 \) | \( 0 \) |
| \( D_4 \) | \( C_{47}^c \) | \( P^c \) | \( P^c \) | \( 0 \) | \(-2P^c\) | \( 0 \) |
| \( C \) | \( P^c \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |

**Table 5.** Intersection pairing \( A^*(X) \otimes A^c_*(X) \rightarrow A^c_*(X) \)

The Mori cone generators are \( C_a, a = 1, \ldots, 4 \), with

\[
C_1 = C_{17}, \quad C_2 = C_{57}, \quad C_3 = C_{67}, \quad C_4 = C_{47}.
\]

The Kähler cone is generated by the dual elements \( T_a, a = 1, \ldots, 4 \) with

\[
T_1 = -2D_1 + D_2 + 2D_3 - D_4, \quad T_2 = D_2, \quad T_3 = D_3, \quad T_4 = -D_1 + D_2 + D_3 - D_4.
\]
If we call $J_a$ the Kähler generators in $H^2(X, \mathbb{Q})$ corresponding to the $T_a$ then the cohomology ring is

$$H^*(X, \mathbb{Q}) = \mathbb{Q}[J_1, \ldots, J_4]/\sim,$$

with $\sim$ given by table 6.

**Resolution** $X = \mathbb{R}^3 - \mathbb{C}^3_6$

This resolution differs from the $G$-Hilb by the flop

$$C_{67} \rightarrow C_{34}.$$

We set $C = [C_{34}]$ and we report the relations between any other toric curve and $C$ in the decorated fan of figure 14.
In table 7 we summarize the intersection products, which give the relations $R$ in the Chow ring $A^*(X) = \mathbb{Z}[X, D_1, D_2, D_3, D_4, C]/R$.

|        | $X$ | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $C$ |
|--------|-----|-------|-------|-------|-------|-----|
| $X$    | $X$ | 0     | 0     | 0     | 0     | 0   |
| $D_1$  | $D_1$ | 0     | 0     | 0     | 0     | 0   |
| $D_2$  | $D_2$ | 0     | 0     | $C$   | 0     | 0   |
| $D_3$  | $D_3$ | 0     | $C$   | $2C$  | $C$   | 0   |
| $D_4$  | $D_4$ | 0     | 0     | $C$   | 0     | 0   |
| $C$    | $C$ | 0     | 0     | 0     | 0     | 0   |

Table 7. Intersection product in $A^*(X)$

The group of compact subvarieties of $X$ is

(78) \[ A^c_*(X) = \mathbb{Z} \cdot D^c_1 \oplus \mathbb{Z} \cdot C^c_{46} \oplus \mathbb{Z} \cdot C^c_{57} \oplus \mathbb{Z} \cdot C^c_{67} \oplus \mathbb{Z} \cdot C^c_{47} \oplus \mathbb{Z} \cdot P^c, \]

with relations to other compact curves

(79) \[ C^c_{27} = C^c_{47}, \]

(80) \[ C^c_{37} = C_{37} + 2C^c_{47}. \]

In table 8 we summarize the intersection pairing.
The Mori cone generators are $C_a, a = 1, \ldots, 4$, with
\begin{equation}
C_1 = C_{46}, \quad C_2 = C_{57}, \quad C_3 = C_{34}, \quad C_4 = C_{47}.
\end{equation}
The Kähler cone is generated by the dual elements $T_a, a = 1, \ldots, 4$ with
\begin{equation}
T_1 = D_1, \quad T_2 = D_2, \\
T_3 = D_2 - D_4, \quad T_4 = -D_1 + D_2 + D_3 - D_4.
\end{equation}

| $X$   | $D^c_7$ | $C^c_{46}$ | $C^c_{57}$ | $C^c_{34}$ | $C^c_{47}$ | $P^c$ |
|-------|---------|------------|------------|------------|------------|-------|
| $D_1$ | 0       | $P^c$      | 0          | 0          | 0          | 0     |
| $D_2$ | $C^c_{27}$ | 0         | $P^c$      | 0          | 0          | 0     |
| $D_3$ | $C^c_{37}$ | $P^c$      | 0          | $-P^c$     | $P^c$      | 0     |
| $D_4$ | $C^c_{47}$ | 0         | $P^c$      | $-P^c$     | 0          | 0     |
| $C$   | $P^c$   | 0          | 0          | 0          | 0          | 0     |

\textbf{Table 8.} Intersection pairing $A^\ast(X) \otimes \Delta^\ast(X) \to \Delta^\ast(X)$

If $J_a$ are the Kähler generators in $H^2(X, \mathbb{Q})$ corresponding to the $T_a$ then the cohomology ring is
\begin{equation}
H^\ast(X, \mathbb{Q}) = \mathbb{Q}[J_1, \ldots, J_4]/\sim,
\end{equation}
with $\sim$ given by table 9.
Resolution \( X = R_4 - C_6^3 \)

This resolution differs from the \( G \)-Hilb by the flop

\[ C_{47} \longrightarrow C_{56} \]  

(84)

We set \( C = [C_{56}] \) and we report the relations between any other toric curve and \( C \) in the decorated fan of figure 16.
In table 7 we summarize the intersection products, which give the relations $R$ in the Chow ring $A^*(X) = \mathbb{Z}[X, D_1, D_2, D_3, D_4, C]/R$.

|    | $X$ | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $C$ |
|----|-----|-------|-------|-------|-------|-----|
| $X$ | $X$ | 0     | 0     | 0     | 0     | 0   |
| $D_1$ | 0  | 0     | 0     | 0     | 0     | 0   |
| $D_2$ | 0  | 0     | $C$   | 0     | 0     | 0   |
| $D_3$ | 0  | $C$   | $C$   | 0     | 0     | 0   |
| $D_4$ | 0  | 0     | 0     | 0     | 0     | 0   |
| $C$  | 0  | 0     | 0     | 0     | 0     | 0   |

**Table 10.** Intersection product in $A^*(X)$

The group of compact subvarieties of $X$ is

$$A^c_*(X) = \mathbb{Z} \cdot D_7 \oplus \mathbb{Z} \cdot C^c_{46} \oplus \mathbb{Z} \cdot C^c_{57} \oplus \mathbb{Z} \cdot C^c_{67} \oplus \mathbb{Z} \cdot C^c_{56} \oplus \mathbb{Z} \cdot P^c,$$

with relations to other compact curves

$$C^c_{27} = C^c_{67},$$

$$C^c_{37} = C^c_{57} + C^c_{67}.$$

In table 11 we summarize the intersection pairing.

|    | $D^c_7$ | $C^c_{46}$ | $C^c_{57}$ | $C^c_{67}$ | $C^c_{56}$ | $P^c$ |
|----|---------|------------|------------|------------|------------|-------|
| $X$ | $D^c_7$ | $C^c_{46}$ | $C^c_{57}$ | $C^c_{67}$ | $C^c_{56}$ | $P^c$ |
| $D_1$ | 0       | $P^c$     | 0          | 0          | 0          | 0     |
| $D_2$ | $C^c_{27}$ | 0         | $P^c$     | 0          | 0          | 0     |
| $D_3$ | $C^c_{37}$ | 0         | 0          | $P^c$     | 0          | 0     |
| $D_4$ | 0       | $-2P^c$   | 0          | 0          | $P^c$     | 0     |
| $C$  | $P^c$   | 0          | 0          | 0          | 0          | 0     |

**Table 11.** Intersection pairing $A^*(X) \otimes A^c_*(X) \rightarrow A^c_*(X)$

The Mori cone generators are $C_a$, $a = 1, \ldots, 4$, with

$$C_1 = C_{46}, \quad C_2 = C_{57}, \quad C_3 = C_{67}, \quad C_4 = C_{56}.$$

The Kähler cone is generated by the dual elements $T_a$, $a = 1, \ldots, 4$ with

$$T_1 = D_1, \quad T_2 = D_2, \quad T_3 = D_3, \quad T_4 = 2D_1 + D_4.$$
If $J_a$ are the Kähler generators in $H^2(X, \mathbb{Q})$ corresponding to the $T_a$ then the cohomology ring is
\begin{equation}
H^*(X, \mathbb{Q}) = \mathbb{Q}[J_1, \ldots, J_4]/\sim,
\end{equation}
with $\sim$ given by table 12.

**Resolution** $X = \mathbb{P}^3_{\mathbb{C}}^3$

This resolution differs from the $G$-Hilb by the flops
\begin{equation}
C_{47} \rightarrow C_{56}, \quad C_{57} \rightarrow C_{26}.
\end{equation}
We set $C = [C_{56}]$ and we report the relations between any other toric curve and $C$ in the decorated fan of figure 14.
In table 13 we summarize the intersection products, which give the relations $R$ in the Chow ring $A^\ast(X) = \mathbb{Z}[X, D_1, D_2, D_3, D_4, C]/R$.

![Figure 18](image_url)

**Table 13. Intersection product in $A^\ast(X)$**

|       | $X$ | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $C$ |
|-------|-----|-------|-------|-------|-------|-----|
| $X$   | $X$ | 0     | 0     | 0     | 0     | 0   |
| $D_1$ | $D_1$ | 0     | 0     | 0     | 0     | 0   |
| $D_2$ | $D_2$ | 0     | $C$   | $C$   | 0     | 0   |
| $D_3$ | $D_3$ | 0     | $C$   | $C$   | 0     | 0   |
| $D_4$ | $D_4$ | 0     | 0     | 0     | 0     | 0   |
| $C$   | $C$ | 0     | 0     | 0     | 0     | 0   |

The group of compact subvarieties of $X$ is

$$A_c^\ast(X) = \mathbb{Z} \cdot D_2^c \oplus \mathbb{Z} \cdot C_{26}^c \oplus \mathbb{Z} \cdot C_{67}^c \oplus \mathbb{Z} \cdot C_{56}^c \oplus \mathbb{Z} \cdot P_c^c,$$

with relations to other compact curves

$$C_{27}^c = C_{67}^c,$$

$$C_{37}^c = C_{67}^c.$$

In table 11 we summarize the intersection pairing.
The Mori cone generators are \( C_a, a = 1, \ldots, 4 \), with
\[
C_1 = C_{46}, \quad C_2 = C_{26}, \quad C_3 = C_{67}, \quad C_4 = C_{56}.
\]

The Kähler cone is generated by the dual elements \( T_a, a = 1, \ldots, 4 \) with
\[
T_1 = D_1, \quad T_2 = 2D_1 - D_2 + D_3 + D_4, \quad T_3 = D_3, \quad T_4 = 2D_1 + D_4.
\]

If we call \( J_a \) the Kähler generators in \( H^2(X, \mathbb{Q}) \) corresponding to the \( T_a \) then the cohomology ring is
\[
H^*(X, \mathbb{Q}) = \mathbb{Q}[J_1, \ldots, J_4]/\sim,
\]
with \( \sim \) given by table 15.
4.2 K-theory generators

We have seen that $G$-Hilb is the moduli space of $G$-cluster in $\mathbb{C}^3$. The natural generalization of $G$-cluster is $G$-constellation. For a finite group $G \subset GL(n, \mathbb{C})$, a $G$-constellation is a $G$-equivariant coherent sheaf $F$ on $\mathbb{C}^n$ with global sections $H^0(F)$ isomorphic as a $\mathbb{C}[G]$-module to the regular representation $R$ of $G$. Set

$$ \Theta := \{ \theta \in \text{Hom}_\mathbb{Z}(R(G), \mathbb{Q}) | \theta(R) = 0 \} , $$

where $R(G)$ is the representation ring of $G$. This is an hyperplane in $\mathbb{Q}^r$, where $r$ is the order of $G$. For $\theta \in \Theta$, a $G$-constellation $F$ is said to be $\theta$-stable (or $\theta$-semistable) if every proper $G$-equivariant coherent subsheaf $0 \subset E \subset F$ satisfies $\theta(E) > 0$ (or $\theta(E) \geq 0$). The moduli space $M_\theta$ of $\theta$-stable constellation is constructed using GIT (cf. [43]). The space $\Theta$ is subdivided into polyhedral convex cones $C$ called GIT chamber. Given $\theta$ and $\theta'$ in the same chamber $C$ the moduli spaces $M_\theta$ and $M_{\theta'}$ are isomorphic, so we write $M_C$ in place of $M_\theta$ for any $\theta \in C$.

Ito and Nakajima [36] observed that $G$-Hilb = $M_{C_0}$ for some chamber $C_0 \subset \Theta$ and more generally the method of [5] shows that for any chamber $C \subset \Theta$ there is a crepant resolution $\tau : M_C \to \mathbb{C}^3/G$ and an equivalence of $\Phi_C : D(M_C) \to D^G(\mathbb{C}^3)$ between derived categories of coherent sheaves on $M_C$ and derived categories of $G$-equivariant sheaves on $\mathbb{C}^3$. Craw and Ishii [12] proved that in the Abelian case every crepant resolution may be realized as a moduli space $M_C$ for some chamber. Moreover they uncovered explicit equivalence between the derived categories of moduli $M_\theta$ for parameters lying in adjacent GIT chambers. Therefore starting from $G$-Hilb and by analyzing the chamber structure of $\Theta$, we can define the tautological bundles $R_\rho$ that generate $K(X)$ on every flopped resolution $X$ and that are the Fourier-Mukay transforms of the original tautological bundles on $G$-Hilb.

Here we summarize how calculate the chamber structure of $\Theta$ and the transformation induced by crossing the walls $W$ of the chambers $C$. We refer to [12, 14] for detailed explanations. The derived equivalence $\Phi_C$ induces a $\mathbb{Z}$-linear isomorphism

$$ \varphi_C : K^e(M_C) \to R(G) \quad \sum a_i S_i \mapsto \bigoplus a_i \rho_i $$

where as usual $S_i$ is the element of the basis of $K^e(M_C)$ dual of the tautological bundle $R_i$ and $\rho_i$ is the irreducible representation of character $i$. Let $C \subset \Theta$ be a chamber. Then $\theta \in C$ if and only if

- for every exceptional curve $\ell$, we have $\theta(\varphi_C(O_\ell)) = \sum \theta(\rho_i) \text{deg}(R_{\rho_i}|_\ell) > 0$;\(^{10}\)

\(^{10}\)Recall that if $R_\rho = O_X(D')$ then $\text{deg}(R_{\rho}|_\ell) = D' \cdot \ell$
for every compact reduced divisor \(D\) \(^{11}\) and irreducible representation \(\rho\), we have
\[
\theta(\varphi_C(\mathcal{R}^{-1}_\rho \otimes \omega_D)) < 0 \quad \text{and} \quad \theta(\varphi_C(\mathcal{R}^{-1}_\rho |_D)) > 0,
\]
where \(\omega_D\) is the canonical bundle of \(D\). \(^{12}\)

These inequalities determine the walls of the chamber \(C\), but we have to pay attention that some of them may be redundant.

Let \(\theta \in \Theta\) be a generic parameter, \(C\) the chamber containing it and \(\theta_0\) a parameter on its wall \(W\). The wall is said to be of type 0, I, II or III as follows:

- type 0 if \(\mathcal{M}_{\theta_0}\) isomorphic to \(\mathcal{M}_\theta\),
- type I if \(\mathcal{M}_{\theta_0}\) is obtained from \(\mathcal{M}_\theta\) by the contraction of a curve to a point,
- type II if \(\mathcal{M}_{\theta_0}\) is obtained from \(\mathcal{M}_\theta\) by the contraction of a divisor to a point,
- type III if \(\mathcal{M}_{\theta_0}\) is obtained from \(\mathcal{M}_\theta\) by the contraction of a divisor to a curve.

The inequalities coming from curves determine walls of type I or III, while the others determine walls of type 0. There are no walls of type II.

If \(C'\) is the chamber behind the wall \(W\), the relation between \(\mathcal{M}_{C'}\) and \(\mathcal{M}_C\) and their tautological bundles depends on the type of the wall.

- \(W\) of type 0: \(\mathcal{M}_{\theta'}\) is isomorphic to \(\mathcal{M}_\theta\); the wall \(W \subset \Theta\) is the zero locus of an equation of the form \(R(\theta_0(\rho_1), \ldots, \theta_0(\rho_r)) = a_1\theta_0(\rho_1) + \ldots + a_r\theta_0(\rho_r) = 0\) and, if \(D\) is the divisor defining the wall, the tautological bundles \(\mathcal{R}_i\) and \(\mathcal{R}'_i\) are related as follows:
  - Case +: if \(R(\theta(\rho_1), \ldots, \theta(\rho_r)) > 0\) then
    \[
    \mathcal{R}'_i = \begin{cases} 
    \mathcal{R}_i & \text{if } a_i = 0, \\
    \mathcal{R}_i \otimes \mathcal{O}_{\mathcal{M}_{\theta'}}(D) & \text{if } a_i \neq 0;
    \end{cases}
    \]
  - Case -: if \(R(\theta(\rho_1), \ldots, \theta(\rho_r)) < 0\) then
    \[
    \mathcal{R}'_i = \begin{cases} 
    \mathcal{R}_i & \text{if } a_i = 0, \\
    \mathcal{R}_i \otimes \mathcal{O}_{\mathcal{M}_{\theta'}}(-D) & \text{if } a_i \neq 0.
    \end{cases}
    \]

- \(W\) of type I: \(\mathcal{M}_{\theta'}\) is the variety obtained from \(\mathcal{M}_\theta\) by the flop of the curve \(\ell\) determining the wall; the tautological bundles \(\mathcal{R}_i\) are the proper transform of \(\mathcal{R}_i\).

- \(W\) of type III: \(\mathcal{M}_{\theta'}\) is isomorphic to \(\mathcal{M}_\theta\); if \(D\) is the divisor contracted in \(\mathcal{M}_{\theta_0}\), the tautological bundles \(\mathcal{R}_i\) and \(\mathcal{R}'_i\) are related as follows:

---

\(^{11}\)I.e. \(D = \sum a_i D_i\) where \(D_i\) are compact invariant divisors and the coefficient \(a_i \in \{0, 1, -1\}\).

\(^{12}\)If \(\mathcal{R}_\rho = \mathcal{O}_{\ell}(D')\) and \(\omega_D = \mathcal{O}_{\ell}(K_D)\) then \(\mathcal{R}^{-1}_\rho \otimes \omega_D = \mathcal{O}_{\ell}(-D' \cdot D + K_D)\) and \(\mathcal{R}^{-1}_\rho |_D = \mathcal{O}_{\ell}(-D' \cdot D)\). Then we calculate the inequalities with the help of (40) and (102).
- Case +: if \( \{\text{deg}(R_i|\ell)\} = \{0, 1\} \) then

\[
R'_i = \begin{cases} 
R_i & \text{if deg}(R_i|\ell) = 0, \\
R_i \otimes O_{M'}(D) & \text{if deg}(R_i|\ell) = 1;
\end{cases}
\]

- Case -: if \( \{\text{deg}(R_i|\ell)\} = \{0, -1\} \) then

\[
R'_i = \begin{cases} 
R_i & \text{if deg}(R_i|\ell) = 0, \\
R_i \otimes O_{M'}(-D) & \text{if deg}(R_i|\ell) = -1.
\end{cases}
\]

Thus, crossing walls of type I induces flops, while walls of type 0 and III induce self-equivalence of the derived category of the resolved variety. One can start from the chamber of the \( G \)-Hilb resolution, follow the change of the tautological bundles crossing the walls and reconstruct the chamber structure of \( \Theta \).

In our example the tautological bundles for the \( \mathbb{Z}_6 \)-Hilb are

\[
R_0 = O_X, \quad R_1 = O_X(D_1), \quad R_2 = O_X(D_2), \quad R_3 = O_X(D_3), \\
R_4 = O_X(-D_1 + D_2 + D_3 - D_4), \quad R_5 = R_2 \otimes R_3 = O_X(D_2 + D_3).
\]

We write parameters \( \theta \) as \( (\theta_0, \ldots, \theta_5) \), where \( \theta_i := (\theta(\rho_i)) \). The inequalities defining the \( \mathbb{Z}_6 \)-Hilb chamber are

\[
\begin{align*}
\theta_1 > 0 & \quad \text{wall of type I related to the flop of the curve } C_1; \\
\theta_2 + \theta_5 > 0 & \quad \text{wall of type III+ related to the contraction of the divisor } D_5; \\
\theta_3 + \theta_5 > 0 & \quad \text{wall of type I related to the flop of the curve } C_3; \\
\theta_4 > 0 & \quad \text{wall of type I related to the flop of the curve } C_4; \\
\theta_5 > 0 & \quad \text{wall of type 0+ defined by } \theta(\varphi_C(R_5^{-1}|D_5)) > 0; \\
\theta_2 + \theta_3 + \theta_4 + \theta_5 > 0 & \quad \text{wall of type 0+ defined by } \theta(\varphi_C(R_5^{-1} \otimes \omega_{D_5})) < 0.
\end{align*}
\]

Any other inequality is redundant. As it is proven in section 9 of [12], the flop of any single curve in the \( G \)-Hilb is achieved by crossing a wall of the chamber (generally if we are in a chamber different from the \( G \)-Hilb’s it may be necessary first cross a type 0 wall to realize a flop).
Resolution $R_2-\mathbb{C}^3_6$

Starting from the $G$-Hilb chamber we obtain this resolution by crossing the wall (104). The tautological bundles are again

$$R_0 = O_X, \quad R_1 = O_X(D_1), \quad R_2 = O_X(D_2), \quad R_3 = O_X(D_3), \quad R_4 = O_X(-D_1 + D_2 + D_3 - D_4), \quad R_5 = R_2 \otimes R_3 = O_X(D_2 + D_3).$$

while the pure D-brane basis is

$$B_0 := O_{\rho}; \quad B_a := O_{C_a}(-T_a); \quad B_5 := O_{D_5}(-T_2 - T_3),$$

with $a = 1, \ldots, 4$. In terms of the $R_i$ and their duals $S_i$, the $B_i$-basis of $K(X)$ and its dual $\Phi$-basis of $K^*(X)$ are thus:

$$B_0 = S_0 + S_1 + S_2 + S_3 + S_4 + S_5, \quad \Phi_0 = R_0;$$
$$B_1 = -S_1, \quad \Phi_1 = -R_0 - R_1 + R_3 + R_4;$$
$$B_2 = S_2 + S_5, \quad \Phi_2 = -R_0 + R_2;$$
$$B_3 = S_1 + S_3 + S_5, \quad \Phi_3 = -R_0 + R_3;$$
$$B_4 = S_1 + S_4, \quad \Phi_4 = -R_0 + R_4;$$
$$B_5 = S_5. \quad \Phi_5 = R_0 - R_2 - R_3 + R_5.$$

The symplectic form in the selected basis is

$$\chi(B_i, B_j) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Resolution $R_3-\mathbb{C}^3_6$

Starting from the $G$-Hilb chamber we obtain this resolution by crossing the wall (106). The tautological bundles are again

$$R_0 = O_X, \quad R_1 = O_X(D_1), \quad R_2 = O_X(D_2), \quad R_3 = O_X(D_3), \quad R_4 = O_X(-D_1 + D_2 + D_3 - D_4), \quad R_5 = R_2 \otimes R_3 = O_X(D_2 + D_3).$$

while the pure D-brane basis is

$$B_0 := O_{\rho}; \quad B_a := O_{C_a}(-T_a); \quad B_5 := O_{D_5}(-T_1 - T_2 + T_3 - T_4),$$
with \( a = 1, \ldots, 4 \). In terms of the \( R_i \) and their duals \( S_i \), the \( B_i \)-basis of \( K(X) \) and its dual \( \Phi \)-basis of \( K'(X) \) are thus:

\[
\begin{align*}
B_0 &= S_0 + S_1 + S_2 + S_3 + S_4 + S_5, \\
B_1 &= S_1 + S_3 + S_5, \\
B_2 &= S_2 + S_5, \\
B_3 &= -S_3 - S_5, \\
B_4 &= S_3 + S_4 + S_5, \\
B_5 &= S_5.
\end{align*}
\]

(116)

The symplectic form in the selected basis is

\[
\chi(B_i, B_j) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & -1 & 2 & 0
\end{pmatrix}.
\]

(117)

**Resolution \( R_4 - \mathbb{C}^3_6 \)**

Starting from the \( G \)-Hilb chamber we obtain this resolution by crossing the wall (107). The tautological bundles are again

\[
\begin{align*}
R_0 &= O_X, \\
R_1 &= O_X(D_1), \\
R_2 &= O_X(D_2), \\
R_3 &= O_X(D_3), \\
R_4 &= O_X(-D_1 + D_2 + D_3 - D_4), \\
R_5 &= R_2 \otimes R_3 = O_X(D_2 + D_3).
\end{align*}
\]

(118)

while the pure D-brane basis is

\[
\begin{align*}
B_0 &:= O_p; \\
B_a &:= O_{C_a}(-T_a); \\
B_5 &:= O_{D_5}(-T_2 - T_3),
\end{align*}
\]

with \( a = 1, \ldots, 4 \). In terms of the \( R_i \) and their duals \( S_i \), the \( B_i \)-basis of \( K(X) \) and its dual \( \Phi \)-basis of \( K'(X) \) are thus:

\[
\begin{align*}
B_0 &= S_0 + S_1 + S_2 + S_3 + S_4 + S_5, \\
B_1 &= S_1 + S_4, \\
B_2 &= S_2 + S_4 + S_5, \\
B_3 &= S_3 + S_4 + S_5, \\
B_4 &= -S_4, \\
B_5 &= S_4 + S_5.
\end{align*}
\]

(120)
The symplectic form in the selected basis is

\[
\chi(B_i, B_j) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 2 & -1 & 0
\end{pmatrix}.
\]

Resolution $R_5 = \mathbb{C}^3_6$

Starting from the above chamber of the resolution $R_4 = \mathbb{C}^3_6$, we obtain this resolution by crossing a single wall of type I. The tautological bundles are again

\[
R_0 = O_X, \quad R_1 = O_X(D_1), \quad R_2 = O_X(D_2), \quad R_3 = O_X(D_3),
\]

\[
R_4 = O_X(-D_1 + D_2 + D_3 - D_4), \quad R_5 = R_2 \otimes R_3 = O_X(D_2 + D_3).
\]

while the pure D-brane basis is

\[
B_0 := O_p; \quad B_a := O_{C_a}(-T_a); \quad B_5 := O_{D_5}(T_2 - 2T_3 - T_4),
\]

with $a = 1, \ldots, 4$. In terms of the $R_i$ and their duals $S_i$, the $B_i$-basis of $K(X)$ and its dual $\Phi$-basis of $K^*(X)$ are thus:

\[
B_0 = S_0 + S_1 + S_2 + S_3 + S_4 + S_5, \quad \Phi_0 = R_0, \\
B_1 = S_1 + S_4, \quad \Phi_1 = -R_0 + R_1, \\
B_2 = -S_2 - S_4 - S_5, \quad \Phi_2 = -R_0 - R_1 - R_2 + R_3 - R_4 + R_5, \\
B_3 = S_2 + S_3 + 2S_4 + 2S_5, \quad \Phi_3 = -R_0 + R_3, \\
B_4 = S_2 + S_5, \quad \Phi_4 = -R_0 + R_1 - R_4 + R_5, \\
B_5 = S_4 + S_5. \quad \Phi_5 = R_0 - R_2 - R_3 + R_5.
\]

The symplectic form in the selected basis is

\[
\chi(B_i, B_j) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 3 & 0 & 0
\end{pmatrix}.
\]
4.3 The cohomological hypergeometric series and GW–invariants

The hypergeometric series are specified by the $\ell$ vectors corresponding to large Kähler parameters and the hypergeometric coefficients are determined expanding them with respect to the basis $Q_i = \text{ch}(\Phi_i)$ of $H^*(X, \mathbb{Q})$.

**Invariants for $\mathbb{R}^3_2 - \mathbb{C}^3_6$**

The vectors $\ell_a$, $a = 1, \ldots, 4$ are

\[
\begin{align*}
C_1 & : \quad \ell_1 = (-1, 0, 0, 1, 0, 1, -1), \\
C_2 & : \quad \ell_2 = (0, 1, 0, 1, -2, 0, 0), \\
C_3 & : \quad \ell_3 = (1, 0, 1, 0, 0, -2, 0), \\
C_4 & : \quad \ell_4 = (1, 0, 0, -2, 1, 0, 0).
\end{align*}
\]

(126)

The selected basis of the cohomology is

\[
\begin{align*}
Q_0 = 1, & \quad Q_1 = J_1 - 2C, & \quad Q_2 = J_2, \\
Q_3 = J_3 - \frac{1}{2}C, & \quad Q_4 = J_4 - C, & \quad Q_5 = -C.
\end{align*}
\]

(127)

If we make this change of basis and use the mirror symmetry identification

\[
w\left(\frac{\vec{x}}{2\pi i}, \frac{\vec{f}}{2\pi i}\right) = Q_0 1 + \sum_{a=1}^{4} Q_a t_a + Q_5 g(t_1, \ldots, t_4),
\]

then we find

\[
\begin{align*}
2\pi it_1 &= \log x_1 - \Phi(x_2, x_4) - N(\vec{x}), \\
2\pi it_2 &= \log x_2 + \Phi(x_2, x_4) - 2\Phi(x_4, x_2), \\
2\pi it_3 &= \log x_3 + 2\Phi(x_3), \\
2\pi it_4 &= \log x_4 - 2\Phi(x_2, x_4) + \Phi(x_4, x_2),
\end{align*}
\]

(128)

and

\[
g(\vec{t}) = P_2(\vec{t}) + \frac{1}{(2\pi i)^2} \phi(\vec{t}),
\]

(129)

where $P_2$ is the degree two polynomial part

\[
P_2(\vec{t}) = -2t_1 - \frac{1}{2}t_3 - t_4 + 3t_1^2 + \frac{1}{2}t_3^2 + t_4^2 + 2t_1t_2 + 3t_1t_3 + 4t_1t_4 + t_2t_3 + t_2t_4 + 2t_3t_4,
\]

(130)

and

\[
\phi(\vec{t}) = 6N^{(1)}(\vec{x}) - 3N^{(2)}(\vec{x}) - 2N^{(3)}(\vec{x}) - N^{(6)}(\vec{x}) - \Lambda_4(\vec{x}) - \Lambda_5(\vec{x}) + \Psi_6(x_2, x_4) - 2\Psi_4(x_3) + 2\Psi_5(x_3) + 2\Psi_1(x_2, x_4) + 2\Psi_1(x_4, x_2) - \Psi_2(x_2, x_4) - \Psi_2(x_4, x_2) - \Psi_3(x_2, x_4) - \Psi_3(x_4, x_2) - 3N^2(\vec{x}) + \Phi^2(x_2, x_4) + \Phi^2(x_4, x_2) - \Phi(x_2, x_4)\Phi(x_4, x_2),
\]

(131)
with \( \vec{x} \) expressed as a function of \( \vec{t} \) by inverting system (129), is the part corresponding to instantonic contributions. Following Hosono and using (113) we find
\[
(−\partial_\vec{t})F(\vec{t}) = g(\vec{t}) ,
\]
where \( F \) is the prepotential. Setting
\[
q_k := e^{2\pi i t_k} ,
\]
we then find
\[
F(\vec{t}) = t_1^2 + \frac{1}{2} t_1 t_3 - t_1 t_4 - t_2^3 - \frac{1}{2} t_1 t_2^2 - \frac{1}{2} t_1 t_4^2 - \frac{3}{2} t_2^2 t_3 - 2 t_1 t_4
\]
(135)
\[-t_1 t_2 t_4 - 2 t_1 t_3 t_4 + F_{\text{inst}}(\vec{q}) + P_{\text{class}}(t_2, t_3, t_4) + Q_{\text{inst}}(q_2, q_3, q_4) .
\]
\( P \) and \( Q \) are the undetermined parts. We list the Gromov-Witten invariants for rational curves up to degree six. The curves with degree in the integer cone generated by \([0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]\) must be excluded, because corresponding to the undetermined part of the prepotential. The only nonvanishing invariants in the considered range are
\[
GW_{[1,0,0,0]} = GW_{[1,0,1,0]} = GW_{[1,0,0,1]} =
GW_{[1,1,0,1]} = GW_{[1,1,1,1]} = 1 ;
GW_{[2,0,1,1]} = GW_{[2,1,1,1]} = GW_{[2,1,1,2]} = -2 .
\]
\(136\)

**Invariants for \( \mathbb{R}^3 - \mathbb{C}_6^3 \)**

The vectors \( \ell_a, a = 1, \ldots, 4 \) are
\[
C_1 : \quad \ell_1 = (1, 0, 1, 0, -2, 0) ,
C_2 : \quad \ell_2 = (0, 1, 0, 1, -2, 0) ,
C_3 : \quad \ell_3 = (0, 0, -1, -1, 0, 1) ,
C_4 : \quad \ell_4 = (0, 0, 1, 0, 1, -2) .
\]
(137)

The selected basis of the cohomology is
\[
Q_0 = 1 , \quad Q_1 = J_1 , \quad Q_2 = J_2 , \quad Q_3 = J_3 , \quad Q_4 = J_4 + C , \quad Q_5 = C .
\]
(138)

Making use of the mirror symmetry identification
\[
\Psi : \quad 2\pi i \frac{\vec{x}}{2\pi i} = Q_0 1 + \sum_{a=1}^{4} Q_a t_a + Q_5 g(t_1, \ldots, t_4) ,
\]
then we find
\[
2\pi i t_1 = \log x_1 + 2 \Psi(x_1) ,
2\pi i t_2 = \log x_2 - \Phi(x_2, x_1 x_2 x_4) + 2 \Phi(x_1 x_2^2 x_4, x_2) ,
2\pi i t_3 = \log x_3 - \Psi(x_1) + N(\vec{x}) ,
2\pi i t_4 = \log x_4 - \Phi(x_1 x_2^2 x_4, x_2) - 2 N(\vec{x}) ,
\]
(140)
and
\[ g(\vec{\ell}) = P_2(\vec{\ell}) + \frac{1}{(2\pi i)^2} \phi(\vec{\ell}), \]
where \( P_2 \) is the degree two polynomial part
\[ P_2(\vec{\ell}) = \frac{1}{6} - t_4 + t_4^2 + t_2 t_4, \]
and
\[
\phi(\vec{\ell}) = 8N^{(1)}(\vec{x}) - 4N^{(2)}(\vec{x}) - 2N^{(3)}(\vec{x}) - N^{(4)}(\vec{x}) + \lambda_6(\vec{x}) + \lambda_8(\vec{x}) \\
+ \Psi_3(x_2, 1, x_3^2 x_4) + \Psi_2(x_1, x_2^2, x_3, x_2) + \Psi_3(x_1 x_2^2 x_3, x_2) - 2\Psi_1(x_1, x_3^2 x_4, x_2) \\
- 4N^2(\vec{x}) - 2N(\vec{x})\Phi(x_2, x_1 x_3^2 x_4) - \Phi(x_2, x_1 x_3^2 x_4)\Phi(x_1 x_3^2 x_4, x_2)
\]
(143)

with \( x \) expressed as a function of \( \vec{\ell} \) by inverting system (140), is the part corresponding to instantonic contributions. Using (117) we find
\[ (\partial_{t_3} - 2\partial_{t_4})F(\vec{\ell}) = g(\vec{\ell}), \]
where \( F \) is the prepotential. Setting
\[ q_k := e^{2\pi i \ell_k}, \]
we then find
\[ F(\vec{\ell}) = \frac{1}{6} t_3 + \frac{1}{4} t_4^2 - \frac{1}{6} t_4 - \frac{1}{4} t_2 t_4 \]
\[ + F_{\text{inst}}(\vec{q}) + P_{\text{class}}(t_1, t_2, 2t_3 + t_4) + Q_{\text{inst}}(q_1, q_2, q_3^2 q_4). \]
(146)
P and \( Q \) are the undetermined parts. We list the \( GW \)-invariants up to degree six. The curves with degree in the integer cone generated by \( [1, 0, 0, 0], \ [0, 1, 0, 0], \ [0, 0, 2, 1] \) must be excluded, because corresponding to the undetermined part of the prepotential. The only nonvanishing invariants in the considered range are
\[
\begin{align*}
GW_{[0,0,1,0]} &= GW_{[0,0,1,1]} = GW_{[0,1,1,1]} = 1; \\
GW_{[1,0,1,0]} &= GW_{[1,0,1,1]} = GW_{[1,1,1,1]} = 1; \\
GW_{[0,0,0,1]} &= GW_{[0,1,0,1]} = GW_{[1,0,2,2]} = -2; \\
GW_{[0,1,0,2]} &= -4; \\
GW_{[0,1,1,3]} &= GW_{[1,1,1,3]} = GW_{[0,2,1,3]} = 5; \\
GW_{[0,1,0,3]} &= GW_{[0,2,0,3]} = -6; \\
GW_{[0,0,1,4]} &= 7; \\
GW_{[1,0,0,4]} &= -8; \\
GW_{[0,1,0,5]} &= -10; \\
GW_{[0,2,0,4]} &= -32.
\end{align*}
\]
(147)

**Invariants for \( R_4 = \mathbb{C}^3_6 \)**

The vectors \( \ell_a, a = 1, \ldots, 4 \) are
\[
\begin{align*}
C_1 : \quad \ell_1 &= (1, 0, 0, -2, 1, 0, 0), \\
C_2 : \quad \ell_2 &= (0, 1, 0, -1, 1, -1), \\
C_3 : \quad \ell_3 &= (0, 0, 1, 0, 1, 0, -2),
\end{align*}
\]
The selected basis of the cohomology is
\[ Q_0 = 1, \quad Q_1 = J_1, \quad Q_2 = J_2, \]
\[ Q_3 = J_3 + \frac{1}{2} C, \quad Q_4 = J_4, \quad Q_5 = C. \]

Via the mirror symmetry identification
\[ w\left( \frac{\tilde{x}}{2\pi i} \right) = Q_0 1 + \sum_{a=1}^{4} Q_a t_a + Q_5 g(t_1, \ldots, t_4), \]
we get
\[ 2\pi i t_1 = \log x_1 + 2\Phi(x_1, x_2 x_4) - \Phi(x_2 x_4, x_1), \]
\[ 2\pi i t_2 = \log x_2 + \Phi(x_2 x_4, x_1) - \Psi(x_1, x_3 x_4^2) - N(\tilde{x}), \]
\[ 2\pi i t_3 = \log x_3 - \Phi(x_2 x_4, x_1) - 2N(\tilde{x}), \]
\[ 2\pi i t_4 = \log x_4 + \Psi(x_1, x_3 x_4^2) - \Phi(x_1, x_2 x_4) + \Phi(x_2 x_4, x_1) + N(\tilde{x}), \]
and
\[ g(\tilde{t}) = P_2(\tilde{t}) + \frac{1}{(2\pi i)^2} \phi(\tilde{t}), \]
where \( P_2 \) is the degree two polynomial part
\[ P_2(\tilde{t}) = \frac{1}{6} - \frac{1}{2} t_3 + \frac{1}{2} t_2^3 + t_2 t_3, \]
and
\[ \phi(\tilde{t}) = 8N^{(1)}(\tilde{x}) - 3N^{(2)}(\tilde{x}) - 2N^{(3)}(\tilde{x}) - 2N^{(5)}(\tilde{x}) - N^{(7)}(\tilde{x}) \]
\[ -\Lambda_9(\tilde{x}) + \Lambda_{10}(\tilde{x}) - 2\Lambda_1(\tilde{x}) \]
\[ -\Psi_1(x_2 x_4, x_1) + \Psi_2(x_2 x_4, x_1) + \Psi_5(x_1, x_3 x_4^2) \]
\[ -4N^2(\tilde{x}) - 2N(\tilde{x})\Psi(x_1, x_3 x_4^2) - N(\tilde{x})\Phi(x_2 x_4, x_1) \]
\[ -\Psi(x_1, x_3 x_4^2)\Phi(x_2 x_4, x_1) + \frac{1}{2} \Phi^2(x_2 x_4, x_1) \]
with \( \tilde{x} \) expressed as a function of \( \tilde{t} \) by inverting system (151), is the part corresponding to instantonic contributions. Using (121) we find
\[ (\partial_{t_1} - \partial_{t_2} - 2\partial_{t_3}) F(\tilde{t}) = g(\tilde{t}), \]
where \( F \) is the prepotential. Setting
\[ q_k := e^{2\pi i t_k}, \]
we then find
\[ F(\tilde{t}) = \frac{1}{5} t_4 + \frac{1}{8} t_3^2 - \frac{1}{12} t_3^3 - \frac{1}{2} t_2^2 t_3 + \frac{1}{3} t_2^3 \]
\[ + F_{\text{inst}}(\tilde{q}) + P_{\text{class}}(t_1, t_4 + t_2, 2t_4 + t_3) + Q_{\text{inst}}(q_1, q_4 q_2, q_4^2 q_3), \]
P and \( Q \) being the undetermined parts. We list the GW–invariants up to degree six. The curves with degree in the integer cone generated by \([0, 1, 0, 1], [0, 0, 1, 2], \)
If we make this change of basis and use the mirror symmetry identification

\[
(160)
\]

we get

\[
(161)
\]

and

\[
(162)
\]

and

\[
(163)
\]

where \( P_2 \) is the degree two polynomial part

\[
(164)
\]

and

\[
\phi(\vec{t}) = 9N^{(1)}(\vec{x}) - 3N^{(2)}(\vec{x}) - 3N^{(3)}(\vec{x}) - 3N^{(5)}(\vec{x}) + \Lambda_{12}(\vec{x}) + \Lambda_{13}(\vec{x})
\]
\begin{equation}
+ \Psi_4(x_1 x_2^2 x_4^2) + \Psi_5(x_1 x_2^3 x_3^2) \\
- \frac{1}{2} \Psi^2(x_1 x_2^2 x_3^2) - \frac{9}{2} N^2(\vec{x}) + 3 \Psi(x_1 x_2^2 x_3^2) N(\vec{x})
\end{equation}

with \( \vec{x} \) expressed as a function of \( \vec{t} \) by inverting system (162), is the part corresponding to instantonic contributions. Using (125) we find

\begin{equation}
(\partial_{t_1} - 3 \partial_{t_3}) F(\vec{t}) = g(\vec{t})
\end{equation}

where \( F \) is the prepotential. Setting

\begin{equation}
q_k := e^{2 \pi i \theta_k}
\end{equation}

we then find

\begin{equation}
F(\vec{t}) = \frac{1}{4} t_1 + \frac{1}{12} t_3^2 - \frac{t_3^3}{18} + F_{\text{inst}}(\vec{q}) + P_{\text{class}}(3 t_1 + t_2, t_3) + Q_{\text{inst}}(q_3^2, q_3, q_2, q_4).
\end{equation}

We list the \( GW \)-invariants up to degree six. The curves with degree in the integer cone generated by \( [0, 3, 1, 0], [1, 0, 0, 0], [0, 0, 0, 1] \) must be excluded, because corresponding to the undetermined part of the prepotential. The only nonvanishing invariants in the considered range are

\begin{align*}
GW_{[0,1,0,0]} &= GW_{[0,1,0,1]} = GW_{[1,2,1,1]} = GW_{[1,2,1,2]} = GW_{[1,1,0,1]} = 1; \\
GW_{[0,1,1,0]} &= GW_{[1,1,1,1]} = -2; \\
GW_{[0,2,2,1]} &= GW_{[1,2,2,1]} = -4; \\
GW_{[0,0,2,0]} &= GW_{[0,1,2,0]} = GW_{[0,1,2,1]} = GW_{[1,1,2,1]} = 5; \\
GW_{[0,2,3,0]} &= GW_{[0,2,3,0]} = 7; \\
GW_{[0,1,3,0]} &= GW_{[0,1,4,1]} = GW_{[0,1,4,0]} = -32; \\
GW_{[0,2,4,0]} &= -110; \\
GW_{[0,0,4,0]} &= -192; \\
GW_{[0,1,4,0]} &= GW_{[0,1,4,1]} = 286; \\
GW_{[0,0,5,0]} &= 1695; \\
GW_{[0,1,5,1]} &= 3038; \\
GW_{[0,0,6,0]} &= -17064.
\end{align*}

\section{Conclusions}

As stated in the introduction, this paper is the first one of a short series devoted to a detailed analysis of some aspects of local (homological) mirror symmetry in relation to its physical meaning. As much of such a project results to be quite technical, we preferred to begin with a preparatory article, where we mainly fix our notations and the objects of study. For this reason we tried here to be as elementary as possible and included an introductory section which obviously does not pretend to be neither exhaustive nor self contained. However, we also included the first step of our analysis, that is the application of local mirror symmetry to the construction of a prepotential accounting for the lower genus Gopakumar-Vafa invariants (which we simply called the Gromov-Witten invariants or \( GW \)-invariants). In particular, for this purpose, we have adopted a particularly elegant way introduced by Hosono in [30] and which we dubbed the “Hosono conjecture”. Our results can be thus interpreted also as a positive (partial) check of the Hosono conjecture for the case
of an orbifold with multiple resolutions. Indeed, we applied the Hosono conjecture to an orbifold model admitting five distinct crepant resolutions, showing that, for each resolution, it partially determines a prepotential encoding information about the Gromov-Witten invariants. As we seen, not all \(GW\)-invariant are determined. Indeed it is not even clear how they could be defined as some ambiguity is introduced by noncompactness of the varieties considered. However, we can note that for all resolutions, the only non computable invariants are the one associated to curves having zero intersection with the compact divisor \(D_7\). Curves having negative intersection with \(D_7\) cannot deform out of \(D_7\). When they have non negative intersection with all the other (non compact) divisors, then we expect for the invariant numbers to count the number of deformations in \(D_7\). However, when the intersection with some of the noncompact divisors is negative, then the deformations are constrained on the intersection between the divisors and we expect for the \(G-W\) numbers to vanish.

The invariants predicted by the prepotential agree with the ones computed directly by means of the methods described in \cite{9}. In place of repeating such computations here, we will simply compare some of the invariants of our examples with the ones provided in \cite{9}. Let us start with resolution five. It contains a \(\mathbb{P}^2\) associated the compact divisor \(D_7\) and the curves \(C_27, C_{37}\) and \(C_{67}\), all equivalent. Then, let us fix \(b := C_3 \equiv C_{67}\). It has intersection \(-3\) with \(D_7\) so that it is the null section of the normal bundle of \(D_7\) in \(X\). It also has intersection numbers \(1\) with \(D_3\), \(D_5\), \(D_6\) and \(0\) with the other noncompact divisors. Thus it freely deforms out from all the noncompact divisors, in the sense discussed above, and then one expects that the number of its deformations is just the number of deformations inside \(\mathbb{P}^2\).

From the list (169) we see that, up to degree six, the corresponding numbers are

\[
\begin{align*}
GW[1] &= 3, & GW[2] &= -6, & GW[3] &= 27, \\
GW[4] &= -192, & GW[5] &= 1695, & GW[6] &= -17064,
\end{align*}
\]

which indeed coincide with the \(GW\)-numbers of \(O(-3) \rightarrow \mathbb{P}^2\), see table 1 in \cite{9}.

Let us move to the fourth resolution. It contains the Hirzebruch surface \(\mathbb{F}_1\) associated to \(D_7\) and the curves \(C_{27}, C_{37}, C_{57}, C_{67}\). The independent curves are \(b := C_2 \equiv C_{57}\) and \(f := C_3 \equiv C_{67}\) which define the base and the fiber of the Hirzebruch fibration. Note that \(b\) has intersection \(-1\) with \(D_5\) so that we expect for its eventual deformations in \(\mathbb{F}_1\) to be constrained. This does not happens for \(f\) or for all combinations \([d_B, d_F] \equiv [0, d_B, d_F, 0]\) with \(d_F \geq d_B\) which have negative intersections with \(D_7\) only. Thus we again expect for the \(GW\)-invariants corresponding to \([d_B, d_F]\) to be the same as in \(F_1\). Indeed from table 10 in \cite{9} we see that deformations appears for \(K_{\mathbb{F}_1}\) only for \(d_F \geq d_B\) (apart from the case \([1, 0]\)). We can see that our results, as listed in (158), are in perfect agreement

\[
\begin{align*}
GW[1, 0] &= 1, & GW[0, 1] &= -2, & GW[1, 1] &= 3, & GW[1, 2] &= 5, \\
GW[2, 2] &= -6, & GW[1, 3] &= 7, & GW[1, 4] &= 9, & GW[1, 5] &= 9, \\
GW[3, 3] &= 27, & GW[2, 3] &= -32, & GW[2, 4] &= -110,
\end{align*}
\]
and $GW[i, j] = GW[0, k] = 0$ for $j < i$, $i > 2$, $i + j \leq 6$ and $k = 2, 3, 4, 5$.

A similar comparison can be done for resolution three. In that case, $D_7$ and the curves $C_{27}, C_{37}, C_{47}$ and $C_{57}$ define an Hirzebruch surface $\mathbb{F}_2$ with base $b := C_2 \equiv C_{57}$ and fiber $f := C_4 \equiv C_{47}$. Note that $b$ has intersection 0 with $D_7$ so that curves $[d_B, 0]$ are not countable. These correspond to the first column of table 11 in [9]. Next curves $[d_B, d_F] = [0, d_B, 0, d_F]$ with positive $d_F$ are computable in $D_7$, but only for $d_F > d_B$ their intersections with $D_i$ are negative only when they intersect the compact divisor. We then expect for the curves $[d_B, d_B + 1 + k]$ to determine the same numbers as for $K_{\mathbb{F}_2}$, whereas for $d_F \leq d_B$ they can be constrained by the fact they have negative intersection with $D_5$ also. However, as follows from table 11 in [9] all such numbers vanish (excluding the case $[d_B, d_F] = [1, 1]$) and again our results, collected in (147), agree with the numbers of $K_{\mathbb{F}_2}$

\begin{align}
GW[1, 1] & = -2, \quad GW[1, 2] = -4, \quad GW[1, 3] = GW[2, 3] = -6, \\
GW[1, 4] & = -8, \quad GW[1, 5] = -10, \quad GW[2, 4] = -32,
\end{align}

(172)

and $GW[i, j] = 0$ for all the remaining ones up to degree 6 (and with $j \neq 0$).

All these are only a part of the numbers predicted by means of the Hosono construction. Indeed, these are the ones corresponding to curves having negative intersection number with the compact divisor and thus admitting a representant contained in it. However, as yet remarked, the constructed potential results to determine much more numbers and in particular we note that, for all cases, the only non computable numbers are the ones associated to curves having null intersection with the compact divisor. Actually the true meaning of these facts are not completely clear to us and deserve a deeper analysis, which is left as part of a future work.

One way to proceed in such a direction is to search for an extended GKZ system whose solutions permit to extend the computation of the invariants to all curves, as proposed for example in [17,18]. This also should provide a slight improvement of the Hosono conjecture. Such analysis are actually under investigation. Furthermore, Hosono conjecture goes beyond the determination of the prepotential (or the central charge), involving the monodromy properties of the hypergeometric components and a concrete determination of the mirror map at list at the $K$–theoretical level, and partial information on the homological mirror map $\text{Mir}$. The multiple resolutions of our example, corresponding to a single mirror family, are related by flop transformations and must be related by Fourier–Mukay transforms at the level of derived categories (see section 4.2). In this contest it could be helpful to find the solutions of our GKZ system in the full $B$-moduli space using the approach of [2, 6]. This is the deeper aspect of the conjecture and will be discussed in the third paper.
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Appendix A  The cohomology valued hypergeometric series

Here we compute the coefficient hypergeometric series.

A.1  Computation of the coefficients

First note that $J'_3 = 0$ so that we need the terms up to order two. Also at order zero is survives only the term with $\vec{m} = \vec{0}$, because for non positive integer argument the $\Gamma$ function diverges. Thus, at order zero $w = 1$.

A.1.1  Some properties of the Gamma function

The Euler Gamma function has integral representation

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad Re(0) > 0,$$

and admits analytical continuation to the whole complex plane excluding the non positive integers. Indeed it admit the very useful Weierstrass representation

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}},$$

where

$$\gamma = \lim_{N \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N} - \log(N + 1)\right) \sim 0.5772156649\ldots$$

is the Euler-Mascheroni constant.

From (174) it follows easily the duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma(z + 1/2).$$

Another useful function is the Psi function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$
From (174)

\[ \psi(z) = -\frac{1}{z} + \sum_{n=1}^{\infty} \frac{z^n}{\ln(n+z)} \quad \Rightarrow \quad \psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2} = \zeta(2). \]

Here \( \zeta(a;z) = \zeta(a;z) \) is the usual Hurwitz Zeta function. In particular \( \psi'(1) = \frac{\pi^2}{6} \) and, if \( N \) is a non negative integer, using these relations we have

1) \( \frac{1}{\Gamma(N+1)} = \frac{1}{N!} \), \( \frac{1}{\Gamma(-N)} = 0 \);

2) \( \partial_{\rho} \left( \frac{1}{\Gamma(1+N+\alpha \rho)} \right)_{\rho=0} = \frac{a}{N!} \psi(N+1) = \frac{a}{N!} \left( \gamma - 1 - \frac{1}{2} - \ldots - \frac{1}{N} \right) \)

if \( N \neq 0 \), and

\( \partial_{\rho} \left( \frac{1}{\Gamma(1+\alpha \rho)} \right)_{\rho=0} = a\gamma \);

also

\( \partial_{\rho} \left( \frac{1}{\Gamma(-N+\rho)} \right)_{\rho=0} = (-1)^N N! \).

3) \( \partial_{\rho} \partial_{\sigma} \left( \frac{1}{\Gamma(1+N+\alpha \rho + b \sigma)} \right)_{\rho=0,\sigma=0} = \frac{ab}{N!} (\psi(N+1)^2 - \psi'(N+1)) \);

In particular for \( N = 0 \)

\( \partial_{\rho} \partial_{\sigma} \left( \frac{1}{\Gamma(1+\alpha \rho + b \sigma)} \right)_{\rho=0,\sigma=0} = ab \left( (\gamma^2 - \frac{\pi^2}{6}) \right) \);

4) \( \partial_{\rho} \partial_{\sigma} \left( \frac{1}{\Gamma(-N+\rho + b \sigma)} \right)_{\rho=0,\sigma=0} = -2ab(-1)^N N! \psi(N+1) \).

A.1.2 Order one

To compute the coefficients at order one, we can distinguish three cases:

- the derivative acts on the numerator. This gives a term \( \log x \) only, because the sum contributes only with the term \( \vec{m} = 0 \);
- the derivative acts on a factor of the form \( 1/\Gamma(N+1) \). Then the remaining factors force again \( \vec{m} \) to zero and by (2) we see that it contribute with a factor \( a\gamma \). There is one such factor for any Gamma factor, and all sum up to zero. This is due to the fact that for any fixed \( \ell \) the sum of its components vanishes;
- the main contributions come out when the derivative act on a factor \( 1/\Gamma(-N+1) \). In this case such factor does not contribute to limiting the allowed values for \( \vec{m} \), which is no more constrained to zero.
In conclusion, all the results can be expressed in terms of the following functions

\[ \Psi(x) = \sum_{n=1}^{\infty} \frac{(2n-1)!}{(n!)^2} x^n, \]

\[ \Phi(x, y) = \sum_{(m, k) \in \mathbb{Z}_+} \frac{(2k + 3m - 1)!}{m!(k + 2m)!k!} (-x)^m y^{k+2m}, \]

\[ \mathcal{N}(\vec{x}) = -\sum_{\vec{m} \in \mathbb{Z}_+^4, \vec{m} \neq 0} \frac{(6n_1 + 4n_2 + 2n_3 + 3n_4 - 1)!}{n_1!n_2!n_3!n_4!(2n_1 + n_2 + n_4)!(3n_1 + 2n_2 + n_3 + n_4)!} \cdot (x_1 x_2^2 x_3^3 x_4^4)^{n_1} (x_2 x_3^2 x_4^3)^{n_2} (x_3 x_4)^{n_3} (-x_2 x_3 x_4^2)^{n_4}. \]

### A.1.3 Order two

The second order term is obtained applying the second order operator

\[ O_2 = \frac{1}{2} \partial_{\rho_1}^2 + \partial_{\rho_4}^2 + \partial_{\rho_2 \rho_3}^2 + \partial_{\rho_2 \rho_4}^2 + 2 \partial_{\rho_3}^2 \]

at \( \vec{\rho} = 0 \). In this case there are several contributions\(^{13}\):

- both derivatives act on the numerator in the terms of the series. This gives rise to terms of the form
  \[ (\log x_i)^2, \quad \log x_i \log x_j; \]

- one derivative acts on the numerator and the other one acts on the Gamma factors. This gives terms of the form
  \[ \log x_i w_j^{(1)}; \]

  where \( w_j^{(1)} \) is one of the first order terms computed before;

- both derivatives act on two regular Gamma factors. These can be two distinct factors or the same factor. In both the cases it contributes only the \( \vec{m} = 0 \) term. For two distinct factors we will find a contribution proportional to \( \gamma^2 \) and for the same factor one finds a term proportional to \( \gamma^2 - \pi^2/6 \). A simple argumentation similar to the first order case shows that the terms in \( \gamma^2 \) sum up to zero. Thus we expect only a term proportional to \( \pi^2 \);

- one derivative acts on a regular term and the other one acts on a singular term. This gives rise to a contribution very similar to (179), (180), where the terms of the series are corrected by a multiplicative factor of the form \( \psi(N + 1) \);

\(^{13}\)for simplicity we will call regular the Gamma factors with positive argument, and singular the Gamma factors with negative argument;
both derivatives acts on the same singular term. This gives a contribution very similar to the previous point;
• the derivatives acts on two distinct singular Gamma terms. These give the more complicated series, because there are minimal constrictions for the range of \( \vec{m} \) in the sums.

A.1.4 Second order functions

Here we collect the functions which appear in the second order terms of the cohomological hypergeometric functions. When the ranges \( \vec{m} \in \mathbb{Z}_+^4 \) are intended to be restricted to the subsets where all factorials and psi functions are well defined. The results can then be expressed in terms of the following twenty six functions

\[
\Psi_1(x, y) = \sum_{(m, k) \in \mathbb{Z}_+^2, (m, k) \neq (0, 0)} (2k + 3m - 1)! \frac{\psi(2k + 3m) - \psi(1)}{m!(k + 2m)k!} (-x)^m y^{k + 2m} ;
\]

\[
\Psi_2(x, y) = \sum_{(m, k) \in \mathbb{Z}_+^2, (m, k) \neq (0, 0)} (2k + 3m - 1)! \frac{\psi(1 + k + 2m) - \psi(1)}{m!(k + 2m)k!} (-x)^m y^{k + 2m} ;
\]

\[
\Psi_3(x, y) = \sum_{(m, k) \in \mathbb{Z}_+^2, (m, k) \neq (0, 0)} (2k + 3m - 1)! \frac{\psi(1 + k) - \psi(1)}{m!(k + 2m)k!} (-x)^m y^{k + 2m} ;
\]

\[
\Psi_4(x) = \sum_{m=1}^{\infty} \frac{(2m - 1)!}{(m!)^2} \frac{\psi(2m) - \psi(1)}{(-x)^m} ;
\]

\[
\Psi_5(x) = \sum_{m=1}^{\infty} \frac{(2m - 1)!}{(m!)^2} \frac{\psi(m + 1) - \psi(1)}{(-x)^m} ;
\]

\[
\Psi_6(x, y) = \sum_{(m, n) \in \mathbb{Z}_+^2, 2n - m > 0, 2m - n > 0} \frac{(2m - m - 1)! (2m - n - 1)!}{m! n!} (-x)^m (-y)^n ;
\]

\[
\Psi_7(x) = \sum_{\vec{n} \in \mathbb{Z}_+^4, \vec{n} \neq 0} \frac{(x_1 x_2^2 x_3 x_4^4)^{n_1} (x_2 x_3^2 x_4^3)^{n_2} (x_3 x_4)^{n_3} (-x_2 x_3 x_4)^{n_4}}{m_1! n_2! n_3! n_4! (2n_1 + n_2 + n_4)! (3m_1 + 2n_2 + n_3 + n_4)!} \chi_{\vec{n}}^{(i)} ,
\]

\( i = 1, \ldots, 7 ; \)
\( \chi^{(1)}_{\vec{\kappa}} = (6n_1 + 4n_2 + 2n_3 + 3n_4 - 1)! [\psi(6n_1 + 4n_2 + 2n_3 + 3n_4) - \psi(1)]; \)

\( \chi^{(2)}_{\vec{\kappa}} = (6n_1 + 4n_2 + 2n_3 + 3n_4 - 1)! [\psi(1 + 3n_1 + 2n_2 + n_3 + n_4) - \psi(1)]; \)

\( \chi^{(3)}_{\vec{\kappa}} = (6n_1 + 4n_2 + 2n_3 + 3n_4 - 1)! [\psi(1 + 2n_1 + n_2 + n_4) - \psi(1)]; \)

\( \chi^{(4)}_{\vec{\kappa}} = (6n_1 + 4n_2 + 2n_3 + 3n_4 - 1)! [\psi(1 + n_2) - \psi(1)]; \)

\( \chi^{(5)}_{\vec{\kappa}} = (6n_1 + 4n_2 + 2n_3 + 3n_4 - 1)! [\psi(1 + n_4) - \psi(1)]; \)

\( \chi^{(6)}_{\vec{\kappa}} = (6n_1 + 4n_2 + 2n_3 + 3n_4 - 1)! [\psi(1 + n_1) - \psi(1)]; \)

\( \chi^{(7)}_{\vec{\kappa}} = (6n_1 + 4n_2 + 2n_3 + 3n_4 - 1)! [\psi(1 + n_3) - \psi(1)]; \)

\[
\Lambda_1(\vec{x}) = \sum_{\vec{\kappa} \in \mathbb{Z}^4_{\geq 0}} \frac{n_1!n_2!n_3!n_4!(n_1 + 2n_2 + n_3 + n_4)!}{(n_1 + 3n_2 + 2n_4)!(n_1 + 2n_3)!x_1^{n_1}} \cdot (-x_1^2 x_2 x_4^2)^{n_2}(x_1 x_3)^{n_3}(x_1 x_4)^{n_4};
\]

\[
\Lambda_2(\vec{x}) = \sum_{\vec{n} \in \mathbb{Z}^2_+} \frac{(m_1 - m_2 - m_3 + m_4 - 1)!(m_3 + m_4 - m_1 - 1)!}{m_1!m_2!m_3!(m_4 - 2m_2)!(m_1 - m_3 - m_4)!} \cdot x_1^{m_1}(-x_2)^{m_2}x_3^{m_3}x_4^{m_4};
\]

\[
\Lambda_3(\vec{x}) = \sum_{\vec{n} \in \mathbb{Z}^2_+} \frac{(m_1 + m_3 - m_4 - 1)!(m_3 + m_4 - m_1 - 1)!}{m_1!m_2!m_3!(m_2 + m_3 - m_1 - m_4)!(m_4 - 2m_2)!} \cdot x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4};
\]

\[
\Lambda_4(\vec{x}) = \sum_{\vec{n} \in \mathbb{Z}^2_+} \frac{(m_1 - m_3 - m_4 - 1)!(m_1 - 1)!}{m_1!m_2!(m_1 + m_2 - 2m_4)!(m_4 - 2m_2)!(m_1 - 3m_3)!} \cdot x_1^{m_1}x_2^{m_2}(-x_3)^{m_3}(-x_4)^{m_4};
\]

\[
\Lambda_5(\vec{x}) = \sum_{\vec{n} \in \mathbb{Z}^2_+} \frac{(2m_3 - m_1 - 1)!(2m_2 - 1)!}{m_1!m_2!(m_2 + m_3)!(m_3 - 2m_1)!} \cdot (-x_2)^{m_2}x_3^{m_3}(-x_4)^{m_4};
\]

\[
\Lambda_6(\vec{x}) = \sum_{\vec{n} \in \mathbb{Z}^2_+} \frac{(m_3 - m_4 - m_1 - 1)!(m_3 - m_2 - 1)!}{m_1!m_2!(m_4 - 2m_2)!(m_3 - 2m_2)!(m_3 - 2m_4)!} \cdot (-x_1)^{m_1}(-x_2)^{m_2}x_3^{m_3}(-x_4)^{m_4};
\]
\[ \Lambda_7(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_+^4} \frac{(m_3 - m_2 - 1)! (2m_4 - m_3 - 1)!}{m_1 m_2!(m_1 - m_3 + m_4)!(m_4 - 2m_2)!(m_3 - 2m_2)!} \cdot x_1^{m_1}(-x_2)^{m_2} x_3^{m_3} x_4^{m_4}; \]

\[ \Lambda_8(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_+^4} \frac{(m_3 - m_2 - 1)! (2m_2 - m_4 - 1)!}{m_1 m_2!(m_1 - m_3 + m_4)!(m_3 - 2m_2)!(m_3 - 2m_4)!} \cdot x_1^{m_1}(-x_2)^{m_2} x_3^{m_3} x_4^{m_4}; \]

\[ \Lambda_9(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_+^4} \frac{(m_4 - m_3 + m_2 - m_1 - 1)! (2m_3 + m_2 - m_4 - 1)!}{m_1 m_2 m_3!(m_4 - 2m_1)!(m_3 - 2m_2)!(m_2 - m_4)!} \cdot (-x_1)^{m_1} x_2^{m_2}(-x_3)^{m_3} x_4^{m_4}; \]

\[ \Lambda_{10}(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_+^4} \frac{(m_4 - m_3 + m_2 - m_1 - 1)! (m_4 - m_2 - 1)!}{m_1 m_2 m_3!(m_4 - 2m_1)!(m_4 - 2m_3 - m_2)!} \cdot (-x_1)^{m_1} x_2^{m_2}(-x_3)^{m_3} x_4^{m_4}; \]

\[ \Lambda_{11}(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_+^4} \frac{(m_4 - m_2 - 1)! (2m_3 + m_2 - m_4 - 1)!}{m_1 m_2 m_3!(m_4 - 2m_1)!(m_1 + m_3 - m_2 - m_4)!} \cdot x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}; \]

\[ \Lambda_{12}(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_+^4} \frac{(m_2 - m_3 - m_4 - 1)! (m_2 - m_3 - 1)!}{m_1 m_3!(m_4 - 2m_1)!(m_1 + m_2 - 2m_4)!(m_2 - 3m_3)!} \cdot x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}; \]

\[ \Lambda_{13}(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_+^4} \frac{(m_2 - m_3 - 1)! (3m_3 - m_2 - 1)!}{m_1 m_3 m_4!(m_3 + m_4 - m_2)!(m_4 - 2m_1)!(m_1 + m_2 - 2m_4)!} \cdot x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}. \]
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