FOURIER-STIELTJES TRANSFORM DEFINED BY INDUCED REPRESENTATION ON LOCALLY COMPACT GROUPS

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Abstract. In this work we extend the Fourier-Stieltjes transform of a vector measure and a continuous function defined on compact groups to locally compact groups. To do so, we consider a representation $L$ of a normal compact subgroup $K$ of a locally compact group $G$, and we use a representation of $G$ induced by that of $L$. Then, we define the Fourier-Stieltjes transform of a vector measure and that of a continuous function with compact support defined on $G$ from the representation of $G$. Then, we extend the Shur orthogonality relation established for compact groups to locally compact groups by using the representations of $G$ induced by the unitary representations of one of its normal compact subgroups. This extension enables us to develop a Fourier-Stieltjes transform in series form that is linear, continuous, and invertible.

1. Introduction. The vector measures generalizing scalar measures attracted a great interest in recent decades due to their numerous applications in functional analysis, control systems, signal analysis, quantum information, quantum theories, and many other domains of applications. For more details on vector measure theory, see for instance, [1, 2] and [3] for some applications on compact groups. Also, Clarkson [4] used theoretical ideas on vector measure to prove that many Banach spaces do not admit equivalent uniformly convex norms. In the same vein, Gel’fand [5] proved that $L_1[0,1]$ is not isomorphic to a dual of a Banach space. Lyapunov [6] showed that the range of a (non-atomic) vector measure is closed and convex. Lyapunov [6]’s work occupies a prominent place in modern mathematics since it lies

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at the intersection of the theory of convex sets and measure theory. The Lyapunov convexity theorem became the starting point of numerous studies in the framework of mathematical analysis as well as in the realm of geometric research into the convex sets that are ranges of non-atomic vector measures [11]. In addition, Bartle [7], Dinculeanu, Klukvánek [1], Dunford and Schwartz [8], and Lindenstrauss and Peččyšk [9] gave many seminal results on vector measure. For instance, Diestel and Uhl Jr [10] provided a comprehensive survey on vector measures. Applications of vector measures were discussed in 1980 in the work by Klukvánek [12]. Fernández and Faranjo [13] studied the Rybakov’s theorem for vector measures in Fréchet spaces. Curbera and Ricker [14] wrote a survey on vector measures, integration, and applications. More information on vector measures can also be found in [10, 14].

To focus on our interest, let $G$ be a locally compact group, $m$ a vector measure on $G$ into a Banach algebra $A$, $\lambda$ a left or right Haar measure on $G$, and $f \in L_1(G, \lambda)$. If the group $G$ is abelian, the Fourier-Stieltjes transform of $m$ is given by the relation

$$\hat{m}(\chi) = \int_G \langle \chi, t \rangle dm(t),$$

while the Fourier transform of $f$ is given by

$$\hat{f}(\chi) = \int_G \langle \chi, t \rangle f(t) d\lambda(t),$$

where $\chi$ denotes a character of $G$. If $G$ is compact and $A = \mathbb{C}$, then the Fourier-Stieltjes transform of $m$ is a family of endomorphisms $(\hat{m}(\sigma))_{\sigma \in \Sigma}$ given by

$$\langle \hat{m}(\sigma) \xi, \eta \rangle = \int_G \langle U_\sigma t \xi, \eta \rangle dm(t),$$

and the associated Fourier transform is provided by the relation

$$\langle \hat{f}(\sigma) \xi, \eta \rangle = \int_G \langle U_\sigma t \xi, \eta \rangle f(t) d\lambda(t),$$

where $U^\sigma$ denotes a unitary representation of the group $G$. If $G$ is compact and $A$ is any Banach algebra, Assiamoua [3] defined a Fourier-Stieltjes transform of a bounded vector measure $m$ on $G$ as a family $(\hat{m}(\sigma))_{\sigma \in \Sigma}$ of sesquilinear mappings of $H_\sigma \times H_\sigma$ with values in $A$ given by the relation,

$$\hat{m}(\sigma)(\xi, \eta) = \int_G \langle U_\sigma t \xi, \eta \rangle dm(t),$$

and the Fourier transform of a function $f \in L_1(G)$ as a family of continuous endomorphisms $(\hat{f}(\sigma))_{\sigma \in \Sigma}$ of sesquilinear applications of $H_\sigma \times H_\sigma$ with value in $A$, given by the relation

$$\hat{f}(\sigma)(\xi, \eta) = \int_G \langle U_\sigma t \xi, \eta \rangle f(t) d\lambda(t).$$

In the continuation of previous investigations by [3], the present work addresses a construction of the Fourier-Stieltjes transform on locally compact groups from a group representation induced by a representation of a compact subgroup. For this purpose, we consider a locally compact group $G$, $K$ a compact normal subgroup of $G$, $\mu$ a $G$-invariant measure on the left coset space $G/K$, and $L^\sigma$ a unitary representation of $K$ into a separable Hilbert space $H_\sigma$. Then, we define the Fourier-Stieltjes transform of a vector measure on the locally compact group $G$ using the representation $U^{L^\sigma}$ of $G$ induced by $L^\sigma$. In this context usually one uses Gel’fand
transform to define Fourier transform. We consider a locally compact group $G$ with Haar measure $dx$. Let $A$ be a unitary commutative Banach algebra and $X(A)$ its spectrum. The Gel’fand transform of $x, x \in A$, is the function $\hat{G}_x : X(A) \to \mathbb{C}$ such that $\hat{G}_x(\chi) = \chi(x)$. The mapping $x \mapsto \hat{G}_x : A \to X(A)$ is called the transformation of Gel’fand associated with $A$. The spherical Fourier transform is the Gel’fand transform associated with $L^1(G)^\#$, the space of integrable, bi-invariant functions by a compact subgroup $K$ of $G$ on $G$. In this case, for $f \in L^1(G)^\#$, $\hat{G}_x$ is denoted $\mathcal{F}f$ or $\hat{f}$ and is defined by

$$\hat{f}(\chi) = \int_G f(x)\chi(x^{-1})dx.$$  \hspace{1cm} (7)

Our method has several advantages over the Gel’fand transform. First, our Fourier transform is injective while the Gel’fand transform is not necessarily injective. Second the Gel’fand transform is limited to spherical functions only. In our case the Fourier transform exists for the whole $L^1(G,A), 1 \leq p < \infty$. Finally, our method is constructive while in Gel’fand transformation the Fourier transformation is obtained by induction.

The paper is organized as follows. In Section 1, we recall the definition of a vector measure and unit representation of a group which are useful in the sequel. In Section 2, we provide the proof of the Shur’s orthogonality property in connection with induced representation and we define the Fourier-Stieltjes transform of a vector measure and the Fourier transform of a function in $L^1(G,A)$. We also report in Section 2 several properties we discovered for our newly developed Fourier-Stieltjes transform.

2. Preliminaries. In this section, for the clarity of the development, we briefly recall useful known main definitions and results, and set our notations. We consider a locally compact space $G$, the Banach spaces $A$ and $\mathcal{F}$ over the field $\mathbb{K}$, ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and denote by $K(G,A)$ the vector space of all continuous functions $f : G \to A$ with a compact support, and by $\mathcal{C}(G,A)$ the space of continuous functions $f : G \to A$.

For simplification, we write $K(G)$ instead of $K(G,\mathbb{R})$ or $K(G,\mathbb{C})$. For each subset $K$ of $G$, denote by $K_K(G,A)$ the space of functions with support contained in $K$. $K_K(G,A)$ is a subspace of $K(G,A)$.

**Definition 2.1.** For every function $f \in K(G,A)$, we define

$$\|f\| := \sup_{t \in G} \|f(t)\|_A.$$  

The mapping $f \mapsto \|f\|$ is a norm on each space $K_K(G,A)$, and defines the topology of uniform convergence on $G$ over $K(G)$.

**Definition 2.2.** On $K(G,A)$, the topology of the compact convergence is the locally convex topology defined by the family of seminorms

$$\|f\|_K = \sup_{t \in K} \|f(t)\|_A,$$

where $K$ takes the elements in the set of compact subsets of $G$.

**Proposition 1.** The space $K(G,A)$ is dense in the space $\mathcal{C}(G,A)$ for the topology of the compact convergence $[1]$.  

Definition 2.3. A vector measure on $G$ with respect to two spaces $A$ and $F$, or an $(A,F)$-measure on $G$, is any linear mapping $m : \mathcal{K}(G,A) \rightarrow F$ having the property that, for each compact set $K \subset G$, the restriction $m$ to the subspace $\mathcal{K}_K(G,A)$ is continuous for the topology of uniform convergence, i.e. for each compact set, there exists a number $a_K > 0$ such that
\[
\|m(f)\| \leq a_K \sup \{\|f(t)\|_A, \ t \in K\}.
\]
The value $m(f)$ of $m$ for a function $f \in \mathcal{K}(G,A)$ is called the integral of $f$ with respect to $m$ also denoted by $\int_G f dm$ or $\int_G f(t) dm(t)$. A vector measure is said to be dominated if there exists a positive measure $\mu$ such that
\[
\int_G |f(t)| d\mu(t), \ f \in \mathcal{K}(G).
\]
If $m$ is dominated, then there exists a smallest positive measure $|m|$, called the modulus or the variation of $m$, that dominates it. A positive measure is said to be bounded if it is continuous in the uniform norm topology of $\mathcal{K}(G)$. A vector measure is said to be bounded if it is dominated by a bounded positive measure. If $m$ is bounded, then $|m|$ is also bounded.

Denoting by $M_1(G,A)$ the Banach algebra of bounded vector measures on $G$, the mapping
\[
m \mapsto \|m\| = \int_G \chi_G d|m|
\]
is a norm on $M_1(G,A)$, where $\chi_G$ represents the characteristic function of $G$.

In the sequel, $K$ will denote a compact subgroup of $G$, $\nu$ and $\lambda$ left Haar measures on $K$ and $G$, respectively.

Definition 2.4. Let $\mu$ be a Radon measure on $G/K$, the homogenous space of left $K$-cosets and $g$ an element of $G$. Define $\mu_g$ by $\mu_g(E) = \mu(ge)$ for Borel subsets $E$ of $G/K$. The measure $\mu$ is called $G$–invariant measure if $\mu_g = \mu$, for $g \in G$. See [15, 16] for more details.

Throughout the paper, $\mu$ will denote the $G$– invariant measure on $G/K$.

Theorem 2.5. For any $f \in \mathcal{K}(G)$, we have [15, 17]:
\[
\int_G f(g) d\lambda(g) = \int_K d\mu(\hat{g}) \int_K f(gk) d\nu(k).
\]
The previous formula (9) extends also to every $f \in L_1(G,\lambda, A)$ [18].

Definition 2.6. A unit representation of $G$ is a homomorphism $L$ from $G$ into the group $U(H)$ of the invertible unitary linear operators on some nonzero Hilbert space $H$, which is continuous with respect to the strong operator topology satisfying for $g_1, g_2 \in G$,
\[
L(g_1 g_2) = L_{g_1} L_{g_2} \quad \text{and} \quad L_1 = Id_H.
\] $H$ is called the representation space of $L$, and its dimension is called the dimension or degree of $L$. 

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Suppose $\mathcal{M}$ is a closed subspace of $H$. $\mathcal{M}$ is called an invariant subspace for $L$ if $L_g\mathcal{M} \subset \mathcal{M}$ for all $g \in G$. If $\mathcal{M}$ is invariant and $\mathcal{M} \neq \{0\}$, then $L^\mathcal{M}$ such that

$$L_g^\mathcal{M} = L_g|_\mathcal{M}$$

defines a representation of $G$ on $\mathcal{M}$, called a subrepresentation of $L$. If $L$ admits an invariant subspace that is nontrivial (i.e. $\neq \{0\}$ or $H$) then $L$ is called reducible, otherwise $L$ is irreducible. If $G$ is compact and $L$ irreducible then the dimension of $L$ is finite.

**Definition 2.7.** Two unit irreducible representations $L$ and $V$ into $H$ and $N$, respectively, are said to be equivalent if there is an isomorphism $T : H \longrightarrow N$ such that, $\forall t \in G$,

$$T \circ L_t = V_t \circ T.$$ 

Consider now the subgroup $K$, $\Sigma$ the coset space (called the dual object of $K$), $\sigma \in \Sigma$, $L^\sigma$ a representation of $\sigma$, $H_\sigma$ a representation space of $L^\sigma$, and $d_\sigma$ its dimension.

**Theorem 2.8.** Let $(L^\sigma_{ij})_{1 \leq i, j \leq d_\sigma}$ be the matrix of $L^\sigma$ in an orthonormal basis $(\xi_i)_{i=1}^{d_\sigma}$ of $H_\sigma$. Then, (see [15, 17, 20, 19]),

$$\int_K L^\sigma_{ij}(t)\overline{L^\sigma_{im}(t)}d\nu(t) = \frac{\delta_{ij}\delta_{im}}{d_\sigma}$$

and

$$\int_K L^\sigma_{ij}(t)\overline{L^\sigma_{im}(t)}d\nu(t) = 0 \text{ if } \sigma \neq \tau. \quad (11)$$

Let $q : G \longrightarrow G/K$ be the canonical quotient map of $G$ into $G/K$ and suppose $H_\sigma$ separable. Denote by $H^{L^\sigma}_0$ the set

$$H^{L^\sigma}_0 = \{u \in \mathcal{C}(G, H_\sigma) : q(\text{Supp}(u)) \text{ is compact and } u(gk) = L^\sigma_k u(g)\}. \quad (12)$$

**Proposition 2.** If $\eta : G \longrightarrow H_\sigma$ is continuous with compact support, then the function $u_\eta$ such that

$$u_\eta(g) = \int_K L^\sigma_k \eta(gk)d\nu(k) \quad (13)$$

belongs to $H^{L^\sigma}_0$, and is uniformly continuous on $G$. Moreover, every element of $H^{L^\sigma}_0$ is of the form $u_\eta$. See [15] for more details.

**Proposition 3.** The mapping:

$$(u, v) \longmapsto \langle u, v \rangle = \int_{G/K} \langle u(g), v(g) \rangle_{H_\sigma} d\mu(g). \quad (14)$$

on $H^{L^\sigma}_0 \times H^{L^\sigma}_0$ is an inner product on $H^{L^\sigma}_0$.

$G$ acts on $H^{L^\sigma}_0$ by left translation, $u \longmapsto L_t u$, so we obtain a unitary representation of $G$ with respect to this inner product on $H^{L^\sigma}_0$. The inner product is preserved by left translations, since $\mu$ is invariant. Hence, if we denote by $H^{L^\sigma}$ the Hilbert space completion of $H^{L^\sigma}_0$, the translation operators $L_t$ extend to unitary operators on $H^{L^\sigma}$. Then the map $t \longmapsto L_t u$ is continuous from $G$ to $H^{L^\sigma}$ for each $u \in H^{L^\sigma}_0$, and then, since the operators $L_t$ are uniformly bounded, they
are strongly continuous on $H^L$. Hence, they define a unitary representation of $G$, called the representation induced by $L^\sigma$, denoted by $U^{L^\sigma}$:

$$U^{L^\sigma}_t u(g) = L_t u(g) = u(t^{-1}g).$$

The representation space is denoted $H^{L^\sigma}$.

**Remark 1.**

- The representations of $G$ induced from $K$ are generally infinite-dimensional unless $G/K$ is a finite set.
- If induced representation $U^{L^\sigma}$ is irreducible then $L^\sigma$ is irreducible \[19\]. The converse of this statement is false.
- $U^{L^\sigma} \in U(H^{L^\sigma})$, where $U(H^{L^\sigma})$ designates the group of invertible unitary linear operators on $H^{L^\sigma}$.

**Lebesgue Spaces**

Let $\lambda$ be a positive measure on the group $G$ into a Banach algebra $A$. For $f : G \rightarrow A$, put

$$N_p(f) = \left( \int_G \|f(t)\|^p d\lambda(t) \right)^{\frac{1}{p}}$$

(15)

$1 \leq p < \infty$, where $\int_G^*$ designates the upper integral \[2\] and

$$N_\infty(f) = \inf \{ \alpha : \|f(t)\|_A \leq \alpha \text{ a.e.} \}.$$ \hspace{1cm} (16)

$L_p(G, \lambda, A)$ denotes the set of all $\lambda$-measurable functions $f : G \rightarrow A$ such $N_p(f) < \infty$, $1 \leq p \leq \infty$. Generally the mapping $f \rightarrow N_p(f)$ is not a norm but a seminorm in $L_p(G, \lambda, A)$, $1 \leq p \leq \infty$.

The relation $R$ defined by $fRg$ if and only if $f - g$ is null almost everywhere is an equivalent relation in $L_p(G, \lambda, A)$. Denoting the equivalence class of $f \in L_p(G, \lambda, A)$ by $[f]$, it follows that $\| [f] \| = N_p(f)$.

**Definition 2.9.** The symbol $L_p(G, \lambda, A)$ will denote the set of equivalence classes $[f]$ of functions $f \in L_p(G, \lambda, A)$.

**Theorem 2.10.** The space $L_p(G, \lambda, A)$ is a normed linear space.

In the sequel the symbol $f$ rather than $[f]$ will be used for an element in $L_p(G, \lambda, A)$.

**Theorem 2.11.** $N_p$ define a norm in $L_p(G, \lambda, A)$ noted $\| \cdot \|_p$.

**Theorem 2.12.** (Hölder inequality) Consider $1 \leq p \leq \infty$, $q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ ( if $p = 1$, $q = \infty$ ) $f \in L_p(G, \lambda, A)$ and $g \in L_q(G, \lambda, A)$ then the function $fg$ is $\lambda$-integrable ie $fg \in L_1(G, \lambda, A)$, and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$ \hspace{1cm} (17)

**Theorem 2.13.** (Minkowski inequality) For $1 \leq p \leq \infty$ and $f, g \in L_p(G, \lambda, A)$, we have

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$ \hspace{1cm} (18)
Remark 2. We deduce by Hölder theorem that for $G$ such that $\lambda(G) < \infty$ and $1 \leq q \leq p \leq \infty$, we have $L_\infty(G, \lambda, \mathcal{A}) \subset L_p(G, \lambda, \mathcal{A}) \subset L_q(G, \lambda, \mathcal{A}) \subset L_1(G, \lambda, \mathcal{A})$ but if $\lambda(G) = \infty$ there are no inclusion between $L_p(G, \lambda, \mathcal{A})$ and $L_q(G, \lambda, \mathcal{A})$ for $q \neq p$.

Proposition 4. $L_p(G, \lambda, \mathcal{A}) \cap L_q(G, \lambda, \mathcal{A})$ is dense in $L_p(G, \lambda, \mathcal{A})$ and in $L_q(G, \lambda, \mathcal{A})$.

Remark 3. If $m$ is a dominated vector measure into a Banach space $F$, then $L_p(G, m, \mathcal{A})$ is by convention the space $L_p(G, |m|, \mathcal{A})$.

Theorem 2.14. (Stone-Weierstrass theorem for locally compact spaces) Let $\Omega$ be a locally compact space, which is non-compact, let $\mathcal{C}_q(\Omega)$ be the algebra of continuous $\mathbb{K}$-valued functions on $\Omega$ that vanish at infinity equipped with the supremum norm and let $\mathcal{P} \subset \mathcal{C}_q^e(\Omega)$ with the following separation properties:

(i) for any two points $\omega_1, \omega_2 \in \Omega$ with $\omega_1 \neq \omega_2$ there exists $f \in \mathcal{P}$ such that $f(\omega_1) \neq f(\omega_2)$

(ii) for any $\omega \in \Omega$ there exists $f \in \mathcal{P}$ with $f(\omega) \neq 0$.

(a) If $\mathbb{K} = \mathbb{R}$ then $\mathcal{P}$ is dense in $\mathcal{C}_q^e(\Omega)$.

(b) If $\mathbb{K} = \mathbb{C}$ and if $\mathcal{P}$ is a $*$-subalgebra (i.e. $f \in \mathcal{P} \Rightarrow \overline{f} \in \mathcal{P}$) then $\mathcal{P}$ is dense in $\mathcal{C}_0^e(\Omega)$.

3. Main results. We have $K$ compact and normal, $H_\sigma$ is finite and separable; then $H^{L_\sigma}$ is also separable. Each of these spaces admits a Hilbertian basis according to Gram-Schmidt process.

In this work, we suppose that $K$ is chosen such that $L^{\sigma}$ and $U^{L_\sigma}$ are both irreducible.

3.1. Schur orthogonality relations.
Let $(\theta_i)_{i=1}^\infty$ be an orthonormal basis of $H^{L^{\sigma}}$ and $(\xi_i)_{i=1}^{d_{\sigma}}$ be an orthonormal basis of $H_\sigma$. We define

$$u^{L^{\sigma}}_{ij}(t) := \left\langle U^t \theta_j, \theta_i \right\rangle_{H^{L^{\sigma}}} = \int_{G/K} \langle \theta_j(t^{-1}g), \theta_i(g) \rangle d\mu(g)$$

(19)

and

$$L^{\sigma}_{ij}(k) := \left\langle L^\sigma_k \xi_j, \xi_i \right\rangle_{H_\sigma}$$

As a result, there is a family of mappings $(\alpha_{is})_{s=1}^{d_{\sigma}}$ of $G$ into $\mathbb{K}$ with $(\mathbb{K} = \mathbb{R}$ or $\mathbb{C})$ such that

$$\theta_i(g) = \sum_{s=1}^{d_{\sigma}} \alpha_{is}(g) \xi_s.$$

We have:

$$\delta_{ij} = \left\langle \theta_j, \theta_i \right\rangle_{H^{L^{\sigma}}} = \int_{G/K} \langle \theta_j(g), \theta_i(g) \rangle_{H_\sigma} d\mu(g) = \sum_{s=1}^{d_{\sigma}} \int_{G/K} \alpha_{js}(g) \overline{\alpha_{is}(g)} d\mu(g).$$

Proposition 5. $\forall \sigma \in \Sigma$ and $\forall i, j \in \{1, 2, \ldots\}$, $u^{L^{\sigma}}_{ij} \in K(G)$. 

Then according to Corollary 1. It follows that $q(supp(u)) \subset q(supp(\eta))$.

Let $A$ be the support of $\eta$. For every $g \in G$ and $k \in K$, $gk \in A \Rightarrow g \in Ak^{-1}$. $AK = \bigcup_{k \in K} Ak^{-1}$ is compact because $A$ and $K$ are compact. Furthermore, $\forall g \notin AK$, $u(g) = 0$ then $supp(u) \subset AK$. Since $supp(u)$ is a closed subset of $AK$ then it is compact.

Now let $B$ and $C$ be $supp(\theta_j)$ and $supp(\theta_i)$, respectively. Assume $D = \{ t \in G : g \in C \Rightarrow t^{-1}g \in B \}$. According to 19, $\langle \theta_j(t^{-1}g), \theta_i(g) \rangle = 0$ if $t \notin D$, and then $u_{ij}^{L_{ij}}(t) = 0$, then $supp(u_{ij}^{L_{ij}}) \subset D$. In addition,

$\forall g \notin AK$, $u(g) = 0$ then $supp(u) \subset AK$. Since $supp(u)$ is a closed subset of $AK$ then it is compact.

Therefore, we have

**Corollary 1.**

$$\int_G \left| u_{ij}^{L_{ij}}(t) \pi_{mij}(t) \right| d\lambda(t) < \infty.$$  

Since $K$ is normal, we have $K \setminus G = G/K$ then

**Lemma 3.1.** $\forall u \in H^{L_{ij}}$, $u(kt) = L_k u(t)$ and $u(tk) = L_{k^{-1}} u(t)$.

**Proof.** There is $\eta \in \mathcal{C}(G, H_{\sigma})$ such that

$$u(t) = \int_K L^\sigma(\xi) \eta(t\xi) d\nu(\xi).$$  \[15\]

$$u(tk) = \int_K L^\sigma(\xi) \eta(tk\xi) d\nu(\xi), \quad k \in K$$

$$= \int_K L^\sigma(k^{-1}\xi) \eta(t\xi) d\nu(\xi)$$

$$= L_{k^{-1}}^\sigma \int_K L^\sigma(\xi) \eta(t\xi) d\nu(\xi)$$

$$= L_{k^{-1}}^\sigma u(t).$$

There exists $\beta \in \mathcal{C}(G, H_{\sigma})$ such that

$$u(t) = \int_K L^\sigma(\xi^{-1}) \beta(\xi) d\nu(\xi).$$  \[19\]

$$u(kt) = \int_K L^\sigma(\xi^{-1}) \beta(kt) d\nu(\xi)$$

$$= \int_K L^\sigma(k\xi^{-1}) \beta(\xi) d\nu(\xi)$$

$$= L_k^\sigma \int_K L^\sigma(\xi^{-1}) \beta(\xi) d\nu(\xi)$$

$$= L_k^\sigma u(t).$$
The following theorem shows the Schur orthogonality relation for the the case of the representation $U^L$ of $G$ induced by the unitary irreducible representation $L^\sigma$ of $K$.

**Theorem 3.2.** We have

\[
\int_G u^L_{ij}(t)\mathfrak{m}^L_{lm}(t)d\lambda(t) = \frac{c_{ijlm}}{d_\sigma} \tag{20}
\]

where

\[
c_{ijlm} := \int_{G/K} d\mu(\hat{t}) \sum_{r,s=1}^{d_\sigma} \alpha_{js}(t^{-1}g)\overline{\alpha}_{ir}(g)d\mu(\hat{g}) \int_{G/K} \alpha_{ms}(t^{-1}h)\overline{\alpha}_{lr}(h)d\mu(\hat{h})
\]

and

\[
\int_G u^L_{ij}(t)\mathfrak{m}^L_{lm}(t)d\lambda(t) = 0 \text{ if } \sigma \neq \tau. \tag{21}
\]

**Proof.** By straightforward computation, we obtain:

\[
\int_{G/K} \langle \theta_{j}(gk), \theta_{i}(g) \rangle_{H_\sigma} d\mu(\hat{g}) = \int_{G/K} \langle L^\sigma(k^{-1}\theta_{j}(g), \theta_{i}(g) \rangle_{H_\sigma} d\mu(\hat{g})
\]

\[
= \sum_{s,r=1}^{d_\sigma} \alpha_{js}(g)\overline{\alpha}_{ir}(g) \langle L^\sigma(k^{-1})\xi_s, \xi_r \rangle_{H_\sigma} d\mu(\hat{g})
\]

\[
= \sum_{s,r=1}^{d_\sigma} L^\sigma_{sr}(k^{-1}) \int_{G/K} \alpha_{js}(g)\overline{\alpha}_{ir}(g) d\mu(\hat{g}).
\]

Besides, we get:

\[
u^L_{ij}(tk) = \int_{G/K} \langle \theta_{j}(k^{-1}t^{-1}g), \theta_{i}(g) \rangle d\mu(\hat{g})
\]

\[
= \int_{G/K} \langle L^\sigma(k^{-1}\theta_{j}(t^{-1}g), \theta_{i}(g) \rangle d\mu(\hat{g})
\]

\[
= \int_{G/K} \langle L^\sigma(k^{-1}\theta_{j}(t^{-1}g), \theta_{i}(g) \rangle d\mu(\hat{g})
\]

\[
= \sum_{s,r=1}^{d_\sigma} L^\sigma_{sr}(k^{-1}) \int_{G/K} \alpha_{js}(t^{-1}g)\overline{\alpha}_{ir}(g) d\mu(\hat{g}).
\]
Hence,
\[
\int_G u_{ij}^\sigma(t)\mu_{lm}^\sigma(t) d\lambda(t) = \int_{G/K} \int_K u_{ij}^\sigma(uk)\mu_{lm}^\sigma(uk) d\nu(k) d\mu(\hat{\iota})
\]
\[
= \int_{G/K} d\mu(\hat{\iota}) \int_K (\sum_{s=1}^{L_s^\sigma} L_{rk}^\sigma(uk^{-1}) \int_{G/K} \alpha_{js}(t^{-1}g)\mu_{sr}(g) d\mu(\hat{\jmath}) \times \sum_{p,q=1} \int_{G/K} \alpha_{mq}(t^{-1}h)\mu_{pr}(h) d\nu(k))
\]
\[
= \int_{G/K} d\mu(\hat{\iota}) \sum_{r,s,p,q=1} \int_K (L_{sk}^\sigma(uk^{-1}) \int_{G/K} \alpha_{js}(t^{-1}g)\mu_{sr}(g) d\mu(\hat{\jmath}) \times \int_{G/K} \alpha_{mq}(t^{-1}h)\mu_{pr}(h) d\nu(k))
\]
\[
= \int_{G/K} \alpha_{mq}(t^{-1}h)\mu_{ps}(h) d\mu(\hat{\iota}) \alpha_{ps}(t^{-1}h)\mu_{tr}(h) d\mu(\hat{\iota})
\]
\[
= \frac{1}{d_\sigma} \sum_{r,s,p,q=1} \int_{(G/K)^3} \alpha_{js}(t^{-1}g)\mu_{sr}(g) \alpha_{ps}(t^{-1}h)\mu_{tr}(h) d\mu(\hat{\iota}) d\nu(k)
\]
\[
= \frac{c_{ijlm}}{d_\sigma}.
\]

If \( \sigma \neq \tau \) then
\[
\int_G u_{ij}^\sigma(t)\mu_{lm}^\tau(t) d\lambda(t) = \int_{G/K} d\mu(\hat{\iota}) \sum_{r,s,p,q=1} \int_K (L_{sk}^\sigma(uk^{-1}) \int_{G/K} \alpha_{js}(t^{-1}g)\mu_{sr}(g) d\mu(\hat{\jmath}) \times \int_{G/K} \alpha_{mq}(t^{-1}h)\mu_{pr}(h) d\nu(k))
\]
\[
= \int_{G/K} \alpha_{mq}(t^{-1}h)\mu_{ps}(h) d\mu(\hat{\iota}) \alpha_{ps}(t^{-1}h)\mu_{tr}(h) d\mu(\hat{\iota}) = 0 \text{ according to } 11
\]

In the case of a particular orthonormal basis, the orthogonality relation reduces to the following:

**Corollary 2.** Choosing an orthonormal basis \( (\xi_i)_{i=1}^{d_\sigma} \) of \( H_\sigma \) such that

\[
\int_{(G/K)^3} \alpha_{js}(t^{-1}g)\mu_{sr}(g) \alpha_{ps}(t^{-1}h)\mu_{tr}(h) d\mu(\hat{\iota}) d\nu(k) = \begin{cases} 
\frac{1}{d_\sigma} & \text{if } j = m \text{ and } i = l \\
0 & \text{if not} 
\end{cases}
\]

(or, also, \( c_{ijlm} = \delta_{il}\delta_{jm} \)) leads to

\[
\int_G u_{ij}^\sigma(t)\mu_{lm}^\sigma(t) d\lambda(t) = \frac{\delta_{il}\delta_{jm}}{d_\sigma}.
\]

(23)
Proof. With the conditions 22, we have

\[
\int_G u_{ij}(t)\overline{u}_{lm}^*(t)d\lambda(t) = \frac{1}{d_\sigma}\sum_{r,s=1}^{d_\sigma} \int_{(G/K)^2} \alpha_j(t^{-1}g)\overline{\alpha}_{ir}(g)\alpha_m(t^{-1}h)\overline{\alpha}_{ir}(h)d\mu(g)d\mu(h)d\mu(t)
\]

\[
= \frac{\delta_{ij}\delta_{jm}}{d_\sigma} \sum_{i,j=1}^{d_\sigma} \frac{1}{d_\sigma^2}
\]

\[
= \frac{\delta_{ij}\delta_{jm}d_\sigma^2}{d_\sigma^2}.
\]

In the sequel, \((\theta_i)_{i=1}^\infty\) will designate an orthonormal basis of \(H^{L^\sigma}\) and \((\xi_i)_{i=1}^{d_\sigma}\) that of \(H^\sigma\), where \((\xi_i)_{i=1}^{d_\sigma}\) is chosen such that

\[
\int_{G/K} d\mu(i) \int_{G/K} \alpha_j(t^{-1}g)\overline{\alpha}_{ir}(g)d\mu(g) \int_{G/K} \alpha_m(t^{-1}h)\overline{\alpha}_{ir}(h)d\mu(h) = \begin{cases} \frac{1}{d_\sigma^2} & \text{if } j = m \text{ and } i = l \\ 0 & \text{if not.} \end{cases}
\]

\[\square\]

3.2. Fourier-Stieltjes transform on a locally compact group with a compact subgroup.

Definition 3.3. Assume \(m \in M_1(G, \mathcal{A})\). We shall define the Fourier-Stieltjes transform of an arbitrary measure \(m\) as the family \((\hat{m}(\sigma))_{\sigma \in \Sigma}\) of sesquilinear mappings of \(H^{L^\sigma} \times H^{L^\sigma}\) into \(\mathcal{A}\), given by the relation

\[
\hat{m}(\sigma)(u, v) = \int_G \langle \overline{U}_{t}^*, u, v \rangle_{H^{L^\sigma}} dm(t) = \int_G \int_{G/K} \langle \overline{m}(t^{-1}g), v(g) \rangle d\mu(g) dm(t). (24)
\]

The Fourier transform of a function \(f \in L_1(G, \mathcal{A}, \lambda)\), where \(\lambda\) denotes a Haar measure on \(G\), is a family \((\hat{f}(\sigma))_{\sigma \in \Sigma}\) of sesquilinear mappings of \(H^{L^\sigma} \times H^{L^\sigma}\) into \(\mathcal{A}\), given by the relation

\[
\hat{f}(\sigma)(u, v) = \int_G \langle \overline{U}_{t}^*, u, v \rangle_{H^{L^\sigma}} f(t)d\lambda(t) = \int_G \int_{G/K} \langle \overline{m}(t^{-1}g), v(g) \rangle f(t)d\mu(g)d\lambda(t). (25)
\]

3.3. Properties of the Fourier-Stieltjes transform.

Let denote by \(S(\Sigma, \mathcal{A}) = \prod_{\sigma \in \Sigma} S(H^{L^\sigma} \times H^{L^\sigma}, \mathcal{A})\), where \(S(H^{L^\sigma} \times H^{L^\sigma}, \mathcal{A})\) is the set of all the sesquilinear mappings of \(H^{L^\sigma} \times H^{L^\sigma}\) into \(\mathcal{A}\). Also, assume \(\prod_{\sigma \in \Sigma} S(H^{L^\sigma} \times H^{L^\sigma}, \mathcal{A})\) is a vector space for addition and multiplication by a scalar of mappings.

Definition 3.4. For \(\Phi \in S(\Sigma, \mathcal{A}) = \prod_{\sigma \in \Sigma} S(H^{L^\sigma} \times H^{L^\sigma}, \mathcal{A})\), note \(\|\Phi\|_\infty\) the quantity is defined by

\[
\|\Phi\|_\infty = \sup_{\sigma \in \Sigma} \{\|\Phi(\sigma)\| : \sigma \in \Sigma\}
\]

(26)

with \(\|\Phi(\sigma)\| = \sup\{\|\Phi(\sigma)(u, v)\|_\mathcal{A} : \|u\|_{H^{L^\sigma}} \leq 1, \|v\|_{H^{L^\sigma}} \leq 1\}\). Let define:

(i) \(S_\infty(\Sigma, \mathcal{A}) = \{\Phi \in S(\Sigma, \mathcal{A}) : \|\Phi\|_\infty < \infty\}\).

(ii) \(S_{00}(\Sigma, \mathcal{A}) = \{\Phi \in S(\Sigma, \mathcal{A}) : \{\sigma \in \Sigma : \Phi(\sigma) \neq 0\} \text{ is finite}\}.\)
Let us consider Proposition 6. There is the matrix \((a_{ij}^\sigma)_{1 \leq i, j \leq \infty}, a_{ij}^\sigma \in A\) such that

\[
\Phi(\sigma) = \sum_{i,j=1}^{\infty} d_\sigma a_{ij}^\sigma \hat{u}_{ij}^\sigma
\]

with \(\hat{u}_{ij}^\sigma\) the Fourier transform of \(u_{ij}^\sigma\) given by

\[
\hat{u}_{ij}^\sigma(\sigma)(u,v) = \int_G \left\langle \mathcal{U}_{ij}^\sigma u, v \right\rangle_{H^\sigma} u_{ij}^\sigma(t) d\lambda(t).
\]

Proof. For all \(u, v \in H^\sigma\) such that \(u = \sum_{j=1}^{\infty} \beta_j \theta_j\) and \(v = \sum_{i=1}^{\infty} \gamma_i \theta_i\),

\[
\Phi(\sigma)(u,v) = \sum_{i,j=1}^{\infty} \beta_j \gamma_i \Phi(\sigma)(\theta_j, \theta_i).
\]

Defining \((\Phi(\sigma)(\theta_j, \theta_i)) = (a_{ij}^\sigma)\), the matrix of \(\Phi(\sigma)\) in the basis \((\theta_i)_{i=1}^{\infty}\) and we have

\[
\hat{u}_{ij}^\sigma(\sigma)(u,v) = \sum_{i,m=1}^{\infty} \beta_m \gamma_i \int_G \mathcal{U}_{im}^\sigma u_{ij}^\sigma(t) d\lambda(t) = \beta_j \gamma_i \frac{d_\sigma}{\delta_{ij}}
\]

which yields

\[
\Phi(\sigma)(u,v) = \sum_{i,j=1}^{\infty} a_{ij}^\sigma \beta_j \gamma_i = \sum_{i,j=1}^{\infty} d_\sigma a_{ij}^\sigma \hat{u}_{ij}^\sigma(u,v)
\]

Corollary 3. For any \(f \in L_1(G, A, \lambda)\), there is a matrix \((a_{ij})_{1 \leq i, j \leq \infty}, a_{ij} \in A\) such that

\[
\hat{f}(\sigma) = \sum_{i,j=1}^{\infty} d_\sigma a_{ij}^\sigma \hat{u}_{ij}^\sigma.
\]

Lemma 3.5. Let us denote by \(\mathcal{L}_\sigma(G)\) the set of finite linear combinations functions \(t \mapsto \left\langle U_{ij}^\sigma u, v \right\rangle_{H^\sigma}\) and \(\mathcal{L}(G) = \bigcup_{\sigma \in \Sigma} \mathcal{L}_\sigma(G)\). \(\mathcal{L}(G)\) is dense in \(C_0(G)\) for the topology of the uniform convergence, where \(C_0(G)\) denotes the space of continuous functions of \(G\) into \(\mathbb{K}\) vanishing at infinity.

Proof. (i) First we have \(\mathcal{L}(G) \subset C_0(G)\) moreover for \(f \in \mathcal{L}(G)\) we have abviously \(\hat{f} \in \mathcal{L}(G)\).

(ii) Let \(t_1\) and \(t_2\) be two elements of \(G\) such that \(t_1 \neq t_2\). Let us suppose by contradiction that for every \(f \in \mathcal{L}(G)\), \(f(t_1) = f(t_2)\). Thus, by chosing \(f = f_k = u_{ik}^\sigma\) we have \(\left\langle U_{t_1}^\sigma \theta_i, \theta_k \right\rangle = \left\langle U_{t_2}^\sigma \theta_i, \theta_k \right\rangle\) for \(i, k \in \{1, 2, \ldots\}\). Let us fix \(i\) and make \(k\) running over the set \(\{1, 2, \ldots\}\). Then, antilinear mappings \(\phi_{U_{t_1}^\sigma \theta_i} = \left\langle U_{t_1}^\sigma \theta_i, \theta_k \right\rangle\) and \(\phi_{U_{t_2}^\sigma \theta_i} = \left\langle U_{t_2}^\sigma \theta_i, \theta_k \right\rangle\) are identically equal in \(H^\sigma\).
Corollary 4. The mapping \( U^L_\sigma \theta_i = U^L_\sigma \theta_i \mapsto \theta_i = U^L_\sigma \theta_i \mapsto U^L_\sigma \theta_i = I_{H^L} \). It means that \( t_2 = t_1 \). There is a contradiction; hence, there is \( f \in \mathcal{L}(G) \) such that \( f(t_1) \neq f(t_2) \).

(iii) Let \( t \) be any element of \( G \); since \( U^L_\sigma \) is invertible, there exist \( i, k \in \{1, 2, ...\} \) such that \( U^L_\sigma \theta_i = \theta_k \). Therefore, we have \( \langle U^L_\sigma \theta_i, \theta_k \rangle = 1 \neq 0 \) i.e \( f(t) \neq 0 \).

According to theorem 2.14 \( \mathcal{L}(G) \) is dense in \( \mathcal{E}_0(G) \).

\[ \square \]

**Theorem 3.6.** The mapping \( m \mapsto \hat{m} \) from \( M_1(G, \mathcal{A}) \) into \( \mathcal{S}_\infty(\Sigma, \mathcal{A}) \) is linear, injective and continuous.

**Proof.** Let \( m, n \in M_1(G, \mathcal{A}) \) such that \( \hat{m} = \hat{n} \). For any \( u, v \in H^L \times H^L \) and any \( \sigma \in \Sigma \), we have:

\[
\hat{m} = \hat{n} \iff \int_G \langle U^L_\sigma u, v \rangle_{H^L} \, dm(t) = \int_G \langle U^L_\sigma u, v \rangle_{H^L} \, dn(t) \\
\iff \int_G \langle U^L_\sigma u, v \rangle_{H^L} \, d(n-m)(t) = 0 \tag{27}
\]

for any \( \sigma \) in \( \Sigma \), and \( u, v \) in \( H^L \).

\( \mathcal{L}(G) \) is dense in \( \mathcal{E}_0(G) \) according to the previous lemma. Then, \( \mathcal{L}(G) \) is dense in \( \mathcal{K}(G) \). Thus \( n - m \) can be viewed as a linear map of \( \mathcal{K}(G) \) which is identically null. Then \( \int_G \langle U^L_\sigma u, v \rangle_{H^L} \, d(n-m)(t) = 0 \implies n - m \equiv 0 \) i.e \( m = n \). The mapping \( m \mapsto \hat{m} \) is therefore injective.

Next, let us now prove the continuity of the mapping \( m \mapsto \hat{m} \).

\[
\|\hat{m}(\sigma)\| = \sup \left\{ \|\hat{m}(\sigma)(u, v)\|_A : \|u\|_{H^L}, \leq 1 \text{ and } \|v\|_{H^L}, \leq 1 \right\} \\
= \sup \left\{ \left\| \int_G \langle U^L_\sigma u, v \rangle_{H^L} \, dm(t) \right\|_A : \|u\|_{H^L}, \leq 1 \text{ and } \|v\|_{H^L}, \leq 1 \right\} \\
\leq \int_G \chi_G |m| \|m\|,
\]

since \( U^L_\sigma \) is unitary. Thus, \( \|\hat{m}(\sigma)\| \leq \|m\|, \sigma \in \Sigma \) and \( \|\hat{m}\|_\infty \leq \|m\| \). As a consequence, \( \hat{m} \in \mathcal{S}_\infty(\Sigma, \mathcal{A}) \) and the mapping is continuous.

**Corollary 4.** The mapping \( f \mapsto \hat{f} \) from \( L_1(G, \lambda, \mathcal{A}) \) into \( \mathcal{S}_\infty(\Sigma, \mathcal{A}) \) is linear, injective, and continuous.

**Proof** Here we consider \( f, g \in L_1(G, \lambda, \mathcal{A}) \) such that \( \hat{f} = \hat{g} \) then

\[
\int_G \langle U^L_\sigma u, v \rangle_{H^L} \, f d\lambda(t) = \int_G \langle U^L_\sigma u, v \rangle_{H^L} \, g d\lambda(t) \implies \int_G \langle U^L_\sigma u, v \rangle_{H^L} \, (f-g) d\lambda(t) = 0.
\]

For the same reasons as the previous proof we have \( f - g \equiv 0 \) then \( f = g \).

\[
\|\hat{f}(\sigma)\| = \sup \left\{ \|f(\sigma)(u, v)\|_A : \|u\|_{H^L}, \leq 1 \text{ and } \|v\|_{H^L}, \leq 1 \right\} \\
= \sup \left\{ \left\| \int_G \langle U^L_\sigma u, v \rangle_{H^L} \, f(t) d\lambda(t) \right\| : \|u\|_{H^L}, \leq 1 \text{ and } \|v\|_{H^L}, \leq 1 \right\} \\
\leq \int_G \|f(t)\| d\lambda(t) = \|f\|_1.
\]
Thus, \( \| \hat{f}(\sigma) \| \leq \| f \|_1, \sigma \in \Sigma \) and \( \| \hat{f} \|_\infty \leq \| f \|_1 \). Hence, \( \hat{f} \in \mathcal{S}_\infty(\Sigma, \mathcal{A}) \) and the mapping is continuous. \( \square \)

### 3.4. Modified Peter-Weyl theorem

The following theorem gives the inverse formula of Fourier transform in the context of our work.

**Theorem 3.7.** For every \( f \in L_2(G, \mathcal{A}) \), there is \( a^\sigma_{ij} \in \mathcal{A} \), \( 1 \leq i, j < \infty, \sigma \in \Sigma \) such that

\[
\hat{f} = \sum_{\sigma \in \Sigma} d_\sigma \sum_{i,j=1}^{\infty} a^\sigma_{ij} u^{L^\sigma}_{ij}.
\]

(28)

**Proof.** With Proposition 6 for \( g \in L_1(G, \lambda, \mathcal{A}) \), there exists \( (a^\sigma_{ij})_{1 \leq i,j \leq \infty} \), \( a^\sigma_{ij} \in \mathcal{A} \) such that

\[
\hat{g}(\sigma) = \sum_{i,j=1}^{\infty} d_\sigma a^\sigma_{ij} u^{L^\sigma}_{ij}
\]

with \( u^{L^\sigma}_{ij}(\sigma)(u, v) = \int_G \left\langle \overline{U_t^{L^\sigma}} u, v \right\rangle_{L^{\infty}(\mathbf{H}^{L^\sigma})} u^{L^\sigma}_{ij}(t) d\lambda(t) \) and \( \hat{g}(\sigma)(\theta_j, \theta_i) = a^\sigma_{ij} \).

Thus \( \hat{g} = \sum_{\sigma \in \Sigma} d_\sigma \sum_{i,j=1}^{\infty} a^\sigma_{ij} u^{L^\sigma}_{ij} \).

It is well known that \( L_1(G, \lambda, \mathcal{A}) \cap L_2(G, \lambda, \mathcal{A}) \) is dense in \( L_2(G, \lambda, \mathcal{A}) \). Then, for every \( f \in L_2(G, \lambda, \mathcal{A}) \) there exists a sequence \((f_n)\) in \( L_1(G, \lambda, \mathcal{A}) \cap L_2(G, \lambda, \mathcal{A}) \) such that \( f_n \xrightarrow{n \to \infty} f \) in \( L_2(G, \lambda, \mathcal{A}) \). \( \hat{f}_n = \sum_{\sigma \in \Sigma} d_\sigma \sum_{i,j=1}^{\infty} \sigma a^\sigma_{ij} \hat{u}^{L^\sigma}_{ij} \) where \( \sigma a^\sigma_{ij} = \hat{f}_n(\sigma)(\theta_j, \theta_i) \).

We have \( \hat{f}_n(\sigma)(\theta_j, \theta_i) \xrightarrow{n \to \infty} \hat{f}(\sigma)(\theta_j, \theta_i) \) i.e. there exists \( a^\sigma_{ij} \in \mathcal{A} \) such that \( \sigma a^\sigma_{ij} \xrightarrow{n \to \infty} a^\sigma_{ij} \).

Finally \( \hat{f} = \sum_{\sigma \in \Sigma} d_\sigma \sum_{i,j=1}^{\infty} a^\sigma_{ij} u^{L^\sigma}_{ij} \).

According to the corollary 4 the mapping \( f \mapsto \hat{f} \) is injective; then \( \hat{f} = \sum_{\sigma \in \Sigma} d_\sigma \sum_{i,j=1}^{\infty} a^\sigma_{ij} \hat{u}^{L^\sigma}_{ij} \)

i.e. \( \hat{f} = \left( \sum_{\sigma \in \Sigma} d_\sigma \sum_{i,j=1}^{\infty} a^\sigma_{ij} u^{L^\sigma}_{ij} \right) \xrightarrow{\sigma} f = \sum_{\sigma \in \Sigma} d_\sigma \sum_{i,j=1}^{\infty} a^\sigma_{ij} u^{L^\sigma}_{ij} \).

(29)

**Corollary 5.** For every \( f \in L_2(G, \mathcal{A}) \), we have

\[
f = \sum_{\sigma \in \Sigma} d_\sigma \sum_{i,j=1}^{\infty} \hat{f}(\theta_j, \theta_i) u^{L^\sigma}_{ij}.
\]

### 4. Concluding remarks

In this work, we have developed a formalism for the Fourier-Stieltjes transform on a locally compact group. More specifically, we have:

- constructed the Fourier-Stieltjes transform of a bounded vector measure on a locally compact group into a Banach algebra,
- derived the Fourier transform of an integrable function, and
- discussed corresponding relevant properties including linearity, injectivity, and continuity.

Also, using the induced representation and its consequences, we established the associated Shur’s orthogonality relation on a locally compact group.
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