Abstract

This paper provides further investigation of the concept of shape $m_{simpl}$-fibrators (previously introduced by the author). The main results identify shape $m_{simpl}$-fibrators among direct products of Hopfian manifolds. First it is established that every closed, orientable manifold homotopically determined by $\pi_1$ with perfectly-Hopfian group (a new class of Hopfian groups that are introduced here) is a shape $m_{simpl}$-fibrator if it is a codimension-2 fibrator (Theorem 6.4). The main result (Theorem 7.1) states that the direct product of two closed, orientable manifolds (of different dimension) homotopically determined by $\pi_1$ and with perfectly-Hopfian fundamental groups (one normally incommensurable with the other one) is a shape $m_{simpl}$-fibrator, if it is a Hopfian manifold and a codimension-2 fibrator.

Keywords: Approximate fibration; Shape $m_{simpl}$-fibrator; Perfectly-Hopfian group; Manifold homotopically determined by $\pi_1$

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1 Introduction

This paper continues an investigation of the proper mappings from $(n + k)$-manifolds onto triangulated manifolds that have closed manifolds as point pre-images, in the PL setting introduced in [34].

The approximate fibrations, introduced by Coram and Duvall [3, 4], are proper mappings that satisfy an approximate version of the homotopy lifting property - the defining property of the more familiar class of fibrations. They form an important class of mappings mostly because of their nice properties. Among them the most useful property is the existence of an exact sequence involving the homotopy groups of domain, target, and shape-theoretical homotopy groups of any point inverse of $p$. Note that, these properties of an approximate fibration reduce to the usual properties of Hurewicz fibration when working with a PL approximate fibration, because the fibers are ANRs, so the $i$th shape homotopy groups are isomorphic to $i$th homotopy groups.

Sometimes a proper map defined on an arbitrary manifold of a specific dimension can be recognized as an approximate fibration due to having point inverses all of a certain homotopy type (or shape). Hence, in order to recognize manifolds that can force a proper map to be an approximate fibrations (when they appear as point pre-images of the map), Daverman introduced the concept of codimension-$k$ (orientable) fibrator [5] and later the concept of PL (orientable) fibrator [7].

In [34] the author introduced the concept of codimension-$k$ shape $m_{\text{simpl}}(o)$-fibrator (and more generally the concept of shape $m_{\text{simpl}}(o)$-fibrator) as PL fibrators in a slightly different PL setting than the one used by Daverman in [7], and provided examples of manifolds that are shape $m_{\text{simpl}}$-fibrators. In addition, in [35] the author provided examples of manifolds that are codimension-$(k + 1)$ shape $m_{\text{simpl}}$-fibrators ($k \geq 2$).

The following is the main question that we address in this paper: Which direct products of Hopfian manifolds are shape $m_{\text{simpl}}$-fibrators?

The question of whether the collection of codimension-$k$ PL (or shape $m_{\text{simpl}}$) fibrators is closed under taking Cartesian product remains unsolved, but seems not likely (because of the examples presented in [10]). Some partial answers to this question for codimension-$k$ PL fibrators (as well as PL fibrators) have been given in [12, 20, 21, 22, 23, 24].

In this paper, we provide examples of shape $m_{\text{simpl}}$-fibrators among direct products of Hopfian manifolds.

Note that analysis of fibrator properties applies mostly to Hopfian manifolds $N$ with Hopfian fundamental groups, hence in search for shape $m_{\text{simpl}}$-fibrators among products of Hopfian manifolds, first we need to look for a particular type of Hopfian groups (the ones that are closed under taking Cartesian products).

Therefore, this paper has two parts. The first part, Sections 3, 4, and 5 introduce and discuss two group properties (perfectly-Hopfian group and normal incommensurability of groups) that are needed to provide closure under taking direct product of Hopfian groups (Theorem 5.5). Section 6 provides examples of perfectly-Hopfian groups among finite and infinite groups, including
the fundamental groups of closed, orientable surfaces with genus \( g > 1 \) (Theorem 3.7). In the next section, Section 4, we list information about normal incommensurability of groups, (e.g., finitely generated groups with \( h \) generators are normally incommensurable with the fundamental group of closed, orientable surfaces with genus \( g > 1 \), if \( h < 2g \) (Theorem 4.3)). Section 5 discusses conditions under which free products (Corollary 5.3) and direct products (Theorem 5.5) of perfectly-Hopfian groups are perfectly-Hopfian.

The second part of the paper, Sections 6 and 7, provide applications to shape fibrators. Namely, Section 6 delivers examples of shape fibrators among codimension-2 fibrators who are closed, orientable manifolds homotopically determined by \( \pi_1 \) with perfectly-Hopfian fundamental groups (Theorem 6.4). Section 7 contains the main results that provide detection of shape fibrators among direct products of Hopfian manifolds (Theorems 7.1 and 7.2).

\[2\] Definitions and notations

Throughout the paper, symbols \( \cong \) and \( \chi \) will denote isomorphism and Euler characteristic respectively, and homology and cohomology groups will be computed with integer coefficients. [31] contains the terminology and definitions used for the material on piecewise-linear topology. Space means topological space and maps are continuous functions. We assume that all spaces are locally compact, ANR. A manifold is assumed to be connected, metric, and boundaryless.

A manifold \( M \) is \textit{aspherical} if \( \pi_i(M) = 0 \) for all \( i > 1 \). If \( M \) is a manifold then \( M^n \) will denote a manifold of dimension represented by the superscript.

A \textit{generalized} \( k \)-\textit{manifold} is a finite dimensional, locally contractible metric space \( X \), such that \( H_*(X, X \setminus \{x\}) \cong H_*(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}) \) for all \( x \in X \). A \textit{simplicial homotopy} \( k \)-\textit{manifold} is a triangulated polyhedron \( K \) in which the link of each \( i \)-simplex has the homotopy type of the \((k-i-1)\)-sphere. Note that simplicial homotopy manifolds are genuine topological manifolds, unlike the polyhedral generalized manifolds, in which vertices possibly fail to have a Euclidean neighborhood.

If \( B \) is a simplicial complex, then \( B^{(j)} \) denotes the \( j \)-skeleton of \( B \) and \( B^j \) denotes the \( j \)-th derived subdivision of \( B \).

A map \( f : N \to N' \) between closed, orientable \( n \)-manifolds is said to have (absolute) \textit{degree} \( d \) if there are choices of generators \( \gamma \in H_n(N) \cong \mathbb{Z}, \gamma' \in H_n(N') \cong \mathbb{Z} \), such that \( f_* (\gamma) = d\gamma' \), where \( d \geq 0 \) is an integer. The \textit{Hopfian manifold} \[9\] is a closed, orientable manifold, such that every degree one self-map which induces a \( \pi_1 \)-isomorphism is a homotopy equivalence. Examples of Hopfian manifolds include: every closed, orientable \( n \)-manifold that (1) is simply connected; or (2) has a finite fundamental group; or (3) has a Hopfian fundamental group and \( n \leq 4 \) [17].

A manifold \( N \) is \textit{homotopically determined by} \( \pi_1 \) [13] if every self map \( f : N \to N \) that induces a \( \pi_1 \)-isomorphism is a homotopy equivalence. Aspherical
manifolds are common examples of manifolds determined by $\pi_1$. No closed $n$-manifold, $n > 1$, with free fundamental group is homotopically determined by $\pi_1$. Additional examples are presented in [13].

A closed manifold $N$ with a non-trivial fundamental group is a special manifold [34] if all self maps with non-trivial, normal images on $\pi_1$-level are homotopy equivalences. All closed, orientable surfaces with negative Euler characteristic are special manifolds [34, Theorem 3.2]. More examples of special manifolds are provided in [34].

A proper, surjective map $p : E \to B$ between locally compact ANR’s is an approximate fibration if $p$ satisfies the following approximate homotopy lifting property: for an arbitrary space $X$, and given a cover $\mathcal{U}$ of $B$ and maps $g : X \to E$ and $H : X \times [0, 1] \to B$, such that $pg = H_0$, there exists a map $\tilde{H} : X \times [0, 1] \to E$, such that $\tilde{H}_0 = g$ and $p\tilde{H}$ and $H$ are $\mathcal{U}$-close (i.e., for each $z \in X \times [0, 1]$, there exists an $U_z \in \mathcal{U}$ such that $\{H(z), p\tilde{H}(z)\} \subset U_z$).

Following P. Hall, we call a group $G$ residually finite if to each non-unit $g$ in $G$, there corresponds a homomorphism giving $G$ onto a finite group and $g$ onto a non-unit element of this image group. In other words, $G$ is a residually finite group if every non-trivial element of $G$ is mapped non-trivially in some finite quotient group of $G$.

Recall that a group $G$ is Hopfian (after Heinz Hopf, 1894-1971) if every epimorphism $\varphi : G \to G$ is an automorphism. In other words, $G$ is Hopfian if it is not isomorphic to a proper factor of itself.

A group $G$ is hyper-Hopfian [8] if every homomorphism $\varphi : G \to G$ with $\varphi(G) \triangleleft G$ and $G/\varphi(G)$ cyclic is necessarily an automorphism.

A group $G$ is ultra-Hopfian [34] if every homomorphism $\varphi : G \to G$ with $\varphi(G)$ a non-trivial, normal subgroup of $G$ is an automorphism.

Recall that a group $G$ is freely indecomposable if, whenever $G$ is expressed as a free product $A \ast B$ either $A$ or $B$ is trivial.

3 Perfectly-Hopfian Groups

In this section we introduce and discuss a new group theoretical property.

A group $G$ is called perfectly-Hopfian if every homomorphism $\varphi : G \to G$ with $\varphi(G)$ a normal subgroup of $G$ and $G/\varphi(G)$ perfect, is an automorphism.

First note that perfectly-Hopfian groups are Hopfian groups by definition and that no perfect group can be perfectly-Hopfian. Also, all ultra-Hopfian groups that are not perfect are perfectly-Hopfian. Furthermore, all non-perfect simple groups are perfectly-Hopfian (since they do not have a proper normal subgroup isomorphic to a factor group of itself). Note that, the simple groups $Z_p$, $p$-prime, are examples of perfectly-Hopfian and ultra-Hopfian groups, that are not hyper-Hopfian groups.

**Theorem 3.1.** All Hopfian, solvable groups are perfectly-Hopfian.

**Proof.** Let $G$ be a Hopfian, solvable group and let $\varphi : G \to G$ be such that $\varphi(G)$ is a normal subgroup of $G$ and $G/\varphi(G)$ is perfect. Since $\varphi(G)$ is solvable
(as a homomorphic image of the solvable group \(G\)), it follows that \(G/\varphi(G)\) is a solvable group too.

No non-trivial solvable group is perfect, hence \(G/\varphi(G)\) must be trivial.

Therefore \(\varphi\) is surjective and the Hopfian property of \(G\) implies that \(\varphi\) is an isomorphisms.

\[\square\]

**Corollary 3.2.** All finitely generated, Abelian groups are perfectly-Hopfian groups.

**Proof.** Follows from Theorem 3.1 since all finitely generated, Abelian groups are Hopfian, solvable groups.

\[\square\]

Theorem 3.1 also implies that all groups of order less than 60 and finite groups of odd order are perfectly-Hopfian, since they are Hopfian, solvable groups.

Recall, that a polycyclic group is a solvable group with finitely generated subgroups.

**Corollary 3.3.** Every polycyclic group is perfectly-Hopfian.

**Proof.** This follows from Theorem 3.1 since polycyclic groups are finitely generated, residually finite groups [25, Theorem 3], hence Hopfian [29].

\[\square\]

**Corollary 3.4.** Every finitely generated, nilpotent group is perfectly-Hopfian.

In particular, all finite \(p\)-groups are perfectly-Hopfian, since they are nilpotent.

**Corollary 3.5.** Every group of order \(p^nq^m\), where \(p, q\) are primes and \(n, m\) are non-negative integers, is perfectly-Hopfian.

**Proof.** This class of groups are solvable (by Burnside’s Theorem) and are Hopfian (since finite), hence they are perfectly-Hopfian by Theorem 3.1.

\[\square\]

Dihedral groups \(D_{2n+1} = \langle x, y \mid x^2 = y^{2n+1} = 1, x^{-1}yx = y^{-1} \rangle\) of order \(2(2n+1)\), where \(2n + 1\) is a prime, are perfectly-Hopfian by Corollary 3.3. They are also hyper-Hopfian [8, Section 4] and ultra-Hopfian [34, Proposition 2.1] groups as well.

Furthermore, \(D_{2n+1} = \langle x, y \mid x^2 = y^{2n+1} = 1, x^{-1}yx = y^{-1} \rangle\) are 2-groups, so perfectly-Hopfian by Corollary 3.3. Note that \(D_{2n+1}\) are not ultra-Hopfian [34, Section 2].

The quaternionic group \(Q = \langle c, d \mid c^2 = (cd)^2 = d^2 \rangle\), of order 8, is a hyper-Hopfian group [8, Section 4] and a perfectly-Hopfian group by Corollary 3.4, which is not ultra-Hopfian [34, Section 2].

On the other hand, the solvable group of order \(p^4\) (\(p\)-prime),

\(\langle x, y \mid x^p = y^p = 1, y^{-1}xy = x^{1+p} \rangle\)

is not hyper-Hopfian [8, Section 4], hence not ultra-Hopfian, but it is perfectly-Hopfian by Theorem 3.1.

\[\square\]
The group of rational numbers, \( \mathbb{Q} \), is a perfectly-Hopfian group since it is Abelian and ultra-Hopfian [33, Section 2].

**Theorem 3.6.** A free group on \( n \) generators, \( 1 \leq n < \infty \), is perfectly-Hopfian.

**Proof.** Let \( F_n \) be a free group of \( n \) generators, \( 1 \leq n < \infty \). Since \( F_1 = \mathbb{Z} \) is perfectly-Hopfian by Corollary [5.2], we restrict our attention to free groups on \( n > 1 \) generators. Let \( f : F_n \to F_n \) be a homomorphism with \( f(F_n) \leq F_n \) and \( F_n / f(F_n) \) perfect. Note that in this case \( f(F_n) \neq 1 \), since \( F_n \) is not a perfect group. Then, \( [F_n, f(F_n)] = k < \infty \) by [28, Corollary 2.10].

Assume \( k \neq 1 \). Then by [28, Corollary 2.10] \( f(F_n) \) is a free group on \( s = k(n - 1) + 1 \) generators. Note that in this case \( s > n \) (since \( n, k > 1 \)) which cannot happen, since the rank of \( f(F_n) \) is less or equal to \( n \).

Therefore, \( [F_n, f(F_n)] = 1 \), i.e., \( f \) is surjective. Since \( F_n \) is Hopfian by [28, Theorem 2.13], it follows that \( f \) is an automorphism.

\[\square\]

In the next few results, we will be using some of the well-known properties of the fundamental group of a closed, orientable surface \( S \) of genus \( g > 1 \), that we list here. Recall that
\[
\pi_1(S) = \langle a_1, b_1, \ldots, a_g, b_g | [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] \rangle,
\]
and \( S \) has a cell structure with one 0-cell, \( 2g \) 1-cells, and one 2-cell. The 1-skeleton is a wedge sum of \( 2g \) circles and the 2-cell is attached along the loop given by the product of the commutators of these generators, \( [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] \).

By [16] Proposition 2.45 \( G \) is torsion-free (since \( S \) is a 2-dimensional CW complex that is a \( K(G, 1) \) space [16, Example 1B.2]. Note that \( S \) has a negative Euler characteristic, \( \chi(S) = 2 - 2g < 0 \). In addition, it is well known that \( G \) is not solvable (hence not Abelian). These groups are residually finite [18] and finitely generated, hence Hopfian [29].

**Theorem 3.7.** Let \( S \) be a closed, orientable surface with genus \( g > 1 \). Then \( G = \pi_1(S) \) is perfectly-Hopfian.

**Proof.** Let \( f : G \to G \) be a homomorphism with \( f(G) \leq G \) and \( G / f(G) \) perfect. Note that \( f(G) \neq 1 \) since \( G \) is not a perfect group.

Consider the following two cases:

**Case 1:** \( [G : f(G)] = k < \infty \), for \( k > 1 \).

This case is impossible, since by [2] Corollary 3.1.9 \( f(G) \) is isomorphic to the fundamental group of a closed, orientable surface of genus \( g_1 = k(g - 1) + 1 > g \), which implies that no epimorphism \( G \to f(G) \) exists as the induced homomorphism on abelianizations would be an epimorphism \( \mathbb{Z}^{2g} \to \mathbb{Z}^{2g_1}, g < g_1 \).

**Case 2:** \( [G : f(G)] = \infty \). Since \( f(G) \) is a non-trivial, finitely generated, normal subgroup of \( G = \pi_1(S) \), by [15, Theorem 6.1] it follows that this case is impossible to occur too.

Hence, \( [G, f(G)] = 1 \), i.e., \( f \) is surjective. Since \( \pi_1(S) \) is Hopfian, it follows that \( f \) is an automorphism. \[\square\]
4 Group Incommensurability

This section investigates another property among groups. A group \( G \) is incommensurable with another group \( H \) if there is no non-trivial homomorphism \( G \to H \). For example, perfect groups are incommensurable with all abelian groups; finite groups are incommensurable with torsion-free groups; and infinite simple groups with finite groups.

In addition, \( \mathbb{Z}_p \) is incommensurable with \( \mathbb{Z}_q \), for distinct primes \( p \) and \( q \).

A group \( G \) is normally incommensurable with another group \( H \) if there is no non-trivial homomorphism \( G \to H \), such that \( f(G) \trianglelefteq K \trianglelefteq H \), for some normal subgroup \( K \) in \( H \).

Note that if a group \( G \) is incommensurable with another group \( H \), then \( G \) is normally incommensurable with \( H \).

**Theorem 4.1.** Finitely generated, solvable groups are normally incommensurable with respect to the fundamental groups of any closed, orientable surface of genus \( g > 1 \).

**Proof.** Let \( f : G \to \pi_1(S) \) be a non-trivial homomorphism from a finitely generated, solvable group \( G \) to the fundamental group of a closed, orientable surface \( S \) of genus \( g > 1 \), such that \( f(G) \trianglelefteq K \trianglelefteq \pi_1(S) \), for some normal subgroup \( K \) in \( \pi_1(S) \). Note that \( f(G) \) is a finitely generated, solvable group.

Consider the following two cases:

**Case 1:** \( [\pi_1(S) : K] = s < \infty \), for \( s \geq 1 \). We show that this case cannot occur. By [2, Corollary 3.1.9], it follows that \( K \) is isomorphic to the fundamental group of a closed, orientable surface, hence without loss of generality we can assume that \( f(G) \trianglelefteq \pi_1(S) \).

Then \( [\pi_1(S) : f(G)] = t < \infty \), for \( t \geq 1 \), by [15, Theorem 6.1]. In this case, \( f(G) \) is isomorphic to the fundamental group of a closed, orientable surface by [2, Corollary 3.1.9], which means it cannot be solvable.

**Case 2:** \( [\pi_1(S) : K] = \infty \). Then by [26, Corollary 1], \( K \) is a free group.

By [23, Theorem 2.10] it follows that \( K \) and \( f(G) \) must be finitely generated, free groups, such that if \( K \) has \( k \) generators, then \( f(G) \) has \( n(k - 1) + 1 \) generators, where \( [K : f(G)] = n < \infty \), \( n \geq 1 \). Hence, \( f(G) \) is a free, solvable group, which means it is isomorphic to \( \mathbb{Z} \). Therefore, \( n(k - 1) + 1 = 1 \), i.e., \( k = 1 \). Then \( K \cong \mathbb{Z} \), which cannot occur by [14, Corollary 4.5].

Hence, such a homomorphism \( f \) does not exist.

**Theorem 4.2.** Finitely generated, virtually Abelian groups are normally incommensurable with respect to the fundamental groups of closed, orientable surfaces with genus \( g > 1 \).

**Proof.** Let \( f : G \to \pi_1(S) \) be a non-trivial homomorphism from a finitely generated, virtually Abelian group \( G \) to the fundamental group of a closed, orientable surface \( S \) with genus \( g > 1 \), such that \( f(G) \trianglelefteq K \trianglelefteq \pi_1(S) \), for some normal subgroup \( K \) in \( \pi_1(S) \).
Since $f(G)$ is a virtually Abelian group, it has an Abelian subgroup of a finite index $h$. In addition, since $f(G)$ is finitely generated, by [24, Theorem 3.5] it has a characteristic subgroup $H$ of finite index. Note that, $H$ is an intersection of all (finitely many) subgroups of $f(G)$ with finite index $h$. Hence, this characteristic subgroup $H$ is a finitely generated, Abelian, normal subgroup of $f(G)$, as well as of $K$.

Then $H \leq K \leq \pi_1(S)$ and $H$ is a finitely generated, solvable group. The two considered cases in the proof of Theorem 4.1 applied in this setting imply that such homomorphism $f$ does not exist.

**Theorem 4.3.** Let $S$ be a closed, orientable surface of genus $g > 1$ and $H$ be a group on $h$ generators, such that $h < 2g$. Then $H$ is normally incommensurable with $\pi_1(S)$.

**Proof.** Suppose $f : H \to \pi_1(S)$ is a non-trivial homomorphisms, such that $f(H) \leq K \leq \pi_1(S)$, for some normal subgroup $K$ in $\pi_1(S)$. Note that $f(H)$ is a non-trivial, finitely generated group on at-most $h$ generators.

Consider the following two cases:

Case 1: $[\pi_1(S) : K] = k < \infty$, for $k \geq 1$. We show that this case cannot occur. By [2, Corollary 3.1.9] it follows that $K$ is isomorphic to the fundamental group of a closed, orientable surface, hence without loss of generality we can assume that $f(H) \leq \pi_1(S)$.

Then, by [15, Theorem 6.1] it follows that $[\pi_1(S) : f(H)] = s < \infty$, for $s \geq 1$. In addition, by [2, Corollary 3.1.9] it follows that $f(H)$ is isomorphic to the fundamental group of a closed, orientable surface of genus $g_1 = [\pi_1(S), f(H)](g-1) + 1$. Therefore, $f(H)$ is a finitely generated group on $2g_1$ generators. Then

$$2g_1 = 2[s(g-1) + 1] \geq 2(g-1) + 2 = 2g > h,$$

which is a contradiction.

Case 2: $[\pi_1(S) : K] = \infty$. This case is also impossible. Namely, in this case $K$ is a free group by [26, Corollary 1].

By [28, Theorem 2.10] it follows that $K$ must be a finitely generated, free group. Hence, $K$ is a non-trivial, finitely generated, normal subgroup of $\pi_1(S)$ of infinite index, which contradicts [15, Theorem 6.1].

Therefore, such a homomorphism $f$ does not exist.

The following two corollaries follow directly from Theorem 4.3.

**Corollary 4.4.** Let $S_1$, $S_2$ be two closed, orientable surfaces with genuses $g_1$, $g_2$ respectively, such that $g_2 > g_1 > 1$. Then $\pi_1(S_1)$ is normally incommensurable with $\pi_1(S_2)$.

**Corollary 4.5.** Let $T$ be an $n$-dimensional torus and $S$ be a closed, orientable surface with genus $g > 1$, such that $n < 2g$. Then $\pi_1(T)$ is normally incommensurable with $\pi_1(S)$. 

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Proof. Since \( \pi_1(T) = \mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z} \) is a finitely generated group on \( n \) generators and \( n < 2g \), the result follows from Theorem 4.3.

**Theorem 4.6.** Let \( F \) be a free group with at least 2 generators, and \( H \) be a finitely generated group with fewer generators than \( F \). Then \( H \) is normally incommensurable with \( F \).

Proof. Suppose \( f : H \to F \) is a non-trivial homomorphism from a group \( H \) on \( h \) generators to the free group \( F \) on \( n \geq 2 \) generators, such that \( f(H) \trianglelefteq K \trianglelefteq F \), for some normal subgroup \( K \) in \( F \). First note that \( f(H) \) is a non-trivial, finitely generated group on \( h \) generators.

Consider the following two cases:

**Case 1:** \([F : K] = k < \infty\), for \( k \geq 1 \). We show that this case cannot occur.

If \( F \) is an infinitely generated, free group, then by [28, Theorem 2.10] it follows that \( K \) is a free group on infinitely many generators, which then implies that \( f(H) \) must be a free group on infinitely many generators. But this is a contradiction with \( f(H) \) being finitely generated.

If \( F \) is a finitely generated free group on \( n > 2 \) generators, then \( K \) is a free group on \( k(n-1)+1 \) generators and \( [K : f(H)] = s < \infty, s \geq 1 \), by [28, Theorem 2.10]. Hence, \( f(H) \) is a finitely generated group on \( sk(n-1)+1 \geq n \) generators. By hypothesis, \( n > h \), hence \( sk(n-1)+1 \geq n > h \geq h_1 \), which is a contradiction.

**Case 2:** \([F : K] = \infty\). This case is also impossible. Namely, in this case, by [28, Theorem 2.10] it follows that \( K \) must be a free group on infinitely many generators, which will imply that \( f(H) \) must be a free group on infinitely many generators as well.

Hence, such a homomorphism \( f \) does not exist.

\( \square \)

5 Products of Perfectly-Hopfian Groups

Next, we investigate when the property of being perfectly-Hopfian (discussed in Section 3) is preserved when taking free products and direct products of finitely generated, perfectly-Hopfian groups.

**Theorem 5.1.** Let \( G_1, G_2 \) be non-trivial, finitely generated, residually finite groups, \( G_2 \neq \mathbb{Z}_2 \), and \( G_1 \ast G_2 \) not perfect. Then \( G_1 \ast G_2 \) is a perfectly-Hopfian group.

Proof. Since \( G_1 \ast G_2 \) is not perfect, and \( G_1 \ast G_2 \) is ultra-Hopfian group by [34, Theorem 2.1], it follows that \( G_1 \ast G_2 \) is perfectly-Hopfian.

\( \square \)

The following Corollaries of Theorem 5.1 follow from [34, Corollaries 2.3, 2.4] respectively.

**Corollary 5.2.** If \( G_1, G_2 \) are non-trivial, finitely generated groups such that \( G_1 \) is non-cyclic, and \( G_1 \ast G_2 \) is Hopfian and not perfect, then \( G_1 \ast G_2 \) is perfectly-Hopfian.
**Corollary 5.3.** If $G_1$, $G_2$ are non-trivial, finitely generated, freely indecomposable, perfectly-Hopfian groups, and $G_1$ is non-cyclic, then $G_1 \ast G_2$ is perfectly-Hopfian.

Corollary 5.3 implies that under some particular conditions, the perfectly-Hopfian property is closed with respect to free products.

Next we focus on direct products of perfectly-Hopfian groups. First, we need the following lemma.

**Lemma 5.4.** Let $\phi : G_1 \times G_2 \rightarrow G_1 \times G_2$ be a homomorphism. In addition, let $i_{G_1} : G_1 \rightarrow G_1 \times G_2$, $i_{G_2} : G_2 \rightarrow G_1 \times G_2$ be inclusions, and $pr_{G_1} : G_1 \times G_2 \rightarrow G_1$, $pr_{G_2} : G_1 \times G_2 \rightarrow G_2$ be projections onto the first and second factor respectively.

1. If $pr_{G_2} \circ \phi \circ i_{G_1}$ is trivial, then $\phi(G_1 \times 1) \subseteq G_1 \times 1$.

2. If $pr_{G_2} \circ \phi \circ i_{G_1}$ is trivial and $pr_{G_2} \circ \phi \circ i_{G_2}$ is an isomorphism, then $\phi((G_1 \times G_2) \cap (G_1 \times 1)) = \phi(G_1 \times 1) = \phi \circ i_{G_1}(G_1)$.

3. If $\phi(G_1 \times G_2)$ is a normal subgroup of $G_1 \times G_2$, then $\phi(G_1 \times G_2) \cap (G_1 \times 1)$ is a normal subgroup of $G_1 \times 1$.

**Proof.**

1. Since $pr_{G_2} \circ \phi \circ i_{G_1}$ is trivial, then $pr_{G_2} \circ \phi \circ i_{G_1}(G_1) = pr_{G_2}(\phi(G_1 \times 1)) = 1$, which implies that $\phi(G_1 \times 1) \subseteq G_1 \times 1$.

2. By part 1 it follows that $\phi(G_1 \times 1) \subseteq \phi(G_1 \times G_2) \cap (G_1 \times 1)$. We only need to prove that $\phi(G_1 \times G_2) \cap (G_1 \times 1) \subseteq \phi(G_1 \times 1)$.

Let $x \in \phi(G_1 \times G_2) \cap (G_1 \times 1)$. Then $x = \phi(g_1, g_2) \in G_1 \times 1$ for some $(g_1, g_2) \in G_1 \times G_2$. Hence

$$x = \phi(g_1, g_2) = \phi((g_1, e_{G_2})(e_{G_1}, g_2)) = \phi(g_1, e_{G_2}) \phi(e_{G_1}, g_2) = (pr_{G_1} \circ \phi \circ i_{G_1}(g_1), pr_{G_2} \circ \phi \circ i_{G_1}(g_1))(pr_{G_1} \circ \phi \circ i_{G_2}(g_2), pr_{G_2} \circ \phi \circ i_{G_2}(g_2)).$$

Since $pr_{G_2} \circ \phi \circ i_{G_1}$ is trivial, it follows that

$$x = (pr_{G_1} \circ \phi \circ i_{G_1}(g_1), e_{G_2})(pr_{G_1} \circ \phi \circ i_{G_2}(g_2), pr_{G_2} \circ \phi \circ i_{G_2}(g_2)) = (pr_{G_1} \circ \phi \circ i_{G_1}(g_1)pr_{G_1} \circ \phi \circ i_{G_2}(g_2), pr_{G_2} \circ \phi \circ i_{G_2}(g_2)) \in G_1 \times 1.$$

Hence $pr_{G_2} \circ \phi \circ i_{G_1}(g_2) = e_{G_2}$, and since $pr_{G_2} \circ \phi \circ i_{G_2}$ is an isomorphism, it follows that $g_2 = e_{G_2}$, i.e., $x = \phi(g_1, e_{G_2}) \in \phi(G_1 \times 1)$.

3. Since $\phi(G_1 \times G_2) \subseteq G_1 \times G_2$, it follows that

$$(g, e_{G_2})(\phi(G_1 \times G_2) \cap (G_1 \times 1))(g^{-1}, e_{G_1}) \subseteq \phi(G_1 \times G_2)$$

for all $g \in G_1$. Moreover,

$$(g, e_{G_2})(a, e_{G_2})(g^{-1}, e_{G_2}) = (gag^{-1}, e_{G_2}) \in G_1 \times 1$$
for all \( a \in \Pr_{G_1}(\phi(G_1 \times G_2) \cap (G_1 \times 1)) \).

Hence,

\[(g, e_{G_2}) (\phi(G_1 \times G_2) \cap (G_1 \times 1)) (g^{-1}, e_{G_2}) \subseteq \phi(G_1 \times G_2) \cap (G_1 \times 1)\]

for all \( g \in G_1 \). Therefore, \( \phi(G_1 \times G_2) \cap (G_1 \times 1) \subseteq G_1 \times 1 \).

\( \square \)

**Theorem 5.5.** Let \( G_1, G_2 \) be perfectly-Hopfian groups and \( G_1 \) be normally incommensurable with \( G_2 \). Then \( G_1 \times G_2 \) is perfectly-Hopfian.

**Proof.** Let \( \phi : G_1 \times G_2 \to G_1 \times G_2 \) be a homomorphism with \( \phi(G_1 \times G_2) \) normal in \( G_1 \times G_2 \) and \( G_1 \times G_2/\phi(G_1 \times G_2) \) perfect. \( G_1 \) and \( G_2 \) being not perfect groups, precludes \( G_1 \times G_2 \) to be perfect, so \( \phi(G_1 \times G_2) \neq 1 \). Examine the following diagram

\[
\begin{array}{ccc}
G_1 & \to & G_1 \\
\downarrow{i_{G_1}} & & \downarrow{\Pr_{G_1}} \\
G_1 \times G_2 & \overset{\phi}{\to} & G_1 \times G_2 \\
\downarrow{i_{G_2}} & & \downarrow{\Pr_{G_2}} \\
G_2 & \to & G_2
\end{array}
\]

where \( i_{G_1}, i_{G_2} \) are inclusions and \( \Pr_{G_1}, \Pr_{G_2} \) are projections.

First, we show that \( \Pr_{G_2} \circ \phi \circ i_{G_2} : G_2 \to G_2 \) is an isomorphism.

Consider the map \( \Pr_{G_2} \circ \phi \circ i_{G_2} : G_1 \to G_2 \). Using that \( \phi(G_1 \times G_2) \subseteq G_1 \times G_2 \)
and \( \Pr_{G_2} \) is onto, it follows that \( \Pr_{G_2} \circ \phi(G_1 \times G_2) \subseteq G_2 \). Note that, \( \phi(G_1 \times 1) \subseteq \phi(G_1 \times G_2) \). Hence, \( \Pr_{G_2} \circ \phi \circ i_{G_1} (G_1) \subseteq \Pr_{G_2} \circ \phi(G_1 \times G_2) \subseteq G_2 \). Since \( G_1 \)
is normally incommensurable with \( G_2 \), it follows that \( \Pr_{G_2} \circ \phi \circ i_{G_1} \) is trivial. Then \( \Pr_{G_2} \circ \phi(G_1 \times G_2) = \Pr_{G_2} \circ \phi \circ i_{G_2}(G_2) \). Hence, \( \Pr_{G_2} \circ \phi \circ i_{G_2}(G_2) = \Pr_{G_2} \circ \phi(G_1 \times G_2) \subseteq G_2 \).

Next, consider the natural epimorphism

\[ f : (G_1 \times G_2) / \phi(G_1 \times G_2) \to G_2 / (\Pr_{G_2} \circ \phi(G_1 \times G_2)) \]

defined with \( f((g_1, g_2) \phi(G_1 \times G_2)) = g_2 \Pr_{G_2} \phi(G_1 \times G_2) \), where \((g_1, g_2) \in G_1 \times G_2 \). Then \( G_2 / (\Pr_{G_2} \circ \phi(G_1 \times G_2)) \) is a perfect group as a homomorphic image of the perfect group \( (G_1 \times G_2) / \phi(G_1 \times G_2) \). Now, the property of \( G_2 \) being perfectly-Hopfian, implies that \( \Pr_{G_2} \circ \phi \circ i_{G_2} : G_2 \to G_2 \) is an isomorphism.

Next we show that \( \Pr_{G_1} \circ \phi \circ i_{G_1} : G_1 \to G_1 \) is an isomorphism.

Using the fact that \( \Pr_{G_1} \circ \phi \circ i_{G_1} \) is trivial, \( \Pr_{G_1} \circ \phi \circ i_{G_1} \) is an isomorphism, and \( \phi(G_1 \times G_2) \) is a normal subgroup of \( G_1 \times G_2 \), by Lemma 5.4 it follows that \( \phi(G_1 \times G_2) \cap (G_1 \times 1) = \phi(G_1 \times 1) = \phi \circ i_{G_1}(G_1) \) and \( \phi(G_1 \times G_2) \cap (G_1 \times 1) \) is a normal subgroup of \( G_1 \times 1 \). Hence, \( \Pr_{G_1} \circ \phi \circ i_{G_1}(G_1) \subseteq G_1 \).

Now consider the following two epimorphisms:

\[ A : (G_1 \times G_2)/\phi(G_1 \times G_2) \to (G_1 \times 1)/\phi \circ i_{G_1}(G_1) \]
defined with \( A ((g_1, g_2) \phi(G_1 \times G_2)) = (g_1, e_{G_2}) \phi \circ i_{G_1}(G_1) \), and
\[
B : \frac{G_1 \times 1/\phi \circ i_{G_1}(G_1)}{G_1/\text{pr}_{G_1} \circ \phi \circ i_{G_1}(G_1)} \rightarrow G_1/\text{pr}_{G_1} \circ \phi \circ i_{G_1}(G_1)
\]
defined with \( B ((g_1, e_{G_2}) \phi \circ i_{G_1}(G_1)) = g_1 \text{pr}_{G_1} \circ \phi \circ i_{G_1}(G_1) \), where \((g_1, g_2) \in G_1 \times G_2\). Then \( G_1/\text{pr}_{G_1} \circ \phi \circ i_{G_1}(G_1) = B \circ A ((G_1 \times G_2)/\phi(G_1 \times G_2)) \) is a perfect group as a homomorphic image of a perfect group. Since \( G_1 \) is perfectly-Hopfian, it follows that \( \text{pr}_{G_1} \circ \phi \circ i_{G_1} \) is an isomorphism.

Now, \( \text{pr}_{G_1} \circ \phi \circ i_{G_1} \) and \( \text{pr}_{G_2} \circ \phi \circ i_{G_2} \) being isomorphisms, force \( \phi \) to be an isomorphism as well.

\[ \square \]

**Corollary 5.6.** Let \( G_1, G_2 \) be perfectly-Hopfian groups and \( G_1 \) be incommensurable with \( G_2 \). Then \( G_1 \times G_2 \) is perfectly-Hopfian.

6 Shape \( m_{\text{simplo}} \)-Fibrators

The following PL setting is used for the rest of this paper: let \( N \) be a fixed closed, PL \( n \)-manifold, \( M \) a (PL) \((n + k)\)-manifold, \( B \) a polyhedron, and \( p : M \rightarrow B \) a proper, surjective (PL) map. The map \( p : M \rightarrow B \) is said to be an \( N \)-shaped (PL) map if each fiber \( p^{-1}(b), b \in B \), has the homotopy type (or more generally the shape \([1, 30]\)) of \( N \).

The closed PL \( n \)-manifold \( N \) is called a codimension-\( k \) shape \( m_{\text{simplo}} \)-fibrator if for every closed, PL \((n + k)\)-manifold \( M \) and \( N \)-shaped PL map \( p : M \rightarrow B \), where \( B \) is a simplicial triangulated manifold, \( p \) is an approximate fibration. Note that the abbreviation \( m_{\text{simplo}} \) points out that the target space is a simplicial triangulated manifold.

Similarly, the manifold \( N \) is a codimension-\( k \) shape orientable \( m_{\text{simplo}} \)-fibrator if for every closed, orientable PL \((n + k)\)-manifold \( M \) and \( N \)-shaped PL map \( p : M \rightarrow B \), where \( B \) is a simplicial triangulated manifold, \( p \) is an approximate fibration. We abbreviate this by writing that \( N \) is a codimension-\( k \) shape \( m_{\text{simplo}} \)-fibrator.

If \( N \) is a codimension-\( k \) shape \( m_{\text{simplo}} \)-fibrator (codimension-\( k \) shape \( m_{\text{simplo}} \)-fibrator) for all \( k \), then \( N \) is called a shape \( m_{\text{simplo}} \)-fibrator (shape \( m_{\text{simplo}} \)-fibrator).

Note that there cannot be much difference between codimension-2 PL fibrators and codimension-2 PL shape \( m_{\text{simpl}} \)-fibrators, since the image spaces \( B \) in codimension-2 are always manifolds \([14\text{, Theorem 3.6]}\). The two classes are precisely the same among Hopfian manifolds with Hopfian fundamental groups.

Let \( p : M \rightarrow B \) be an \( N \)-shaped PL map. The continuity set of \( p, C \), consists of all points \( b \in B \), such that under any retraction \( R : p^{-1}U \rightarrow p^{-1}b \) defined over a neighborhood \( U \subset B \) of \( b \), \( b \) has another neighborhood \( V_b \subset U \), such that for all \( x \in V_b \), \( R | p^{-1}x \rightarrow p^{-1}b \) is a degree one map. Establishing that \( N \)-shaped PL map \( p \) is an approximate fibration, usually requires one to prove that the target space \( B \) equals the continuity set of \( p \), as the next lemma shows. Note that this lemma follows immediately from the definitions and Coram and Duvall's
characterization of approximate fibrations in terms of movability properties [3, Proposition 3.6].

**Lemma 6.1.** Let $N$ be a Hopfian $n$-manifold with a Hopfian fundamental group and $p : M \to B$ be an $N$-shaped PL map, where $M$ is a closed, orientable PL $(n+k)$-manifold, and $B$ is a triangulated manifold. Then the continuity set of $p$, $C$, is equal to $B$ if and only if $p$ is an approximate fibration over $B$.

The next few results listed below are needed for the proof of the main theorem.

**Lemma 6.2.** Let $p : M \to \mathbb{R}^k$, $k \geq 2$, be an $N$-shaped PL map, $M$ an open, orientable $(n+k)$-manifold, and $N$ a Hopfian $n$-manifold. Suppose $T \subset \mathbb{R}^k$ is a closed set, with $\dim T \leq k - 2$. Then $j_1 : \pi_1(p^{-1}(\mathbb{R}^k \setminus T)) \to \pi_1(p^{-1}(\mathbb{R}^k))$ is surjective, where $j : p^{-1}(\mathbb{R}^k \setminus T) \to p^{-1}(\mathbb{R}^k)$ is an inclusion map.

The next result (that we use later) and its proof is the analog to the Fundamental Theorem [34, Theorem 5.5] and its proof.

**Theorem 6.3.** Let $p : M \to \mathbb{R}^k$, $k > 2$, be an $N$-shaped PL map (with respect to a possibly non-standard triangulation of the Euclidian space), $M$ an open, orientable PL $(n+k)$-manifold, and $N$ a closed, orientable PL $n$-manifold, homotopically determined by $\pi_1$ with a perfectly-Hopfian fundamental group. Suppose $T \subset \mathbb{R}^k$ is closed, with $\dim T < k - 2$, and such that $|p|_{p^{-1}(\mathbb{R}^k \setminus T)}$ is an approximate fibration. Then $p$ is an approximate fibration.

**Proof.** Let $T \subset \mathbb{R}^k$ be closed, with $\dim T < k - 2$. The only case that needs to be considered is when $T$ is a minimal closed set such that $|p|_{p^{-1}(\mathbb{R}^k \setminus T)}$ is an approximate fibration. Suppose that $T \neq \emptyset$.

Since $T$ is a closed subset of $B$ and $p$ is an $N$-shaped map, by Daverman and Husch’s work on decompositions and approximate fibrations [11], it follows that there exists an open set $U$, such that $U \cap T \neq \emptyset$ and for all $t \in U \cap T$, a retraction $R : V \to p^{-1}(t)$, (where $V$ is a neighborhood of $p^{-1}(t)$), which restricts to a homotopy equivalence $R| : p^{-1}(t') \to p^{-1}(t)$ for all $t' \in p(V) \cap T$.

Choose $t \in U \cap T$ and let $R : V \to p^{-1}(t)$ be such retraction. Then $p| : V \to p(V)$ is an $N$-shaped PL map. Choose a connected neighborhood $W$ of $t$ in $p(V)$, such that $W \approx \mathbb{R}^k$. Note that $R| : p^{-1}W \to p^{-1}(t)$ is also a retraction, which restricts to a homotopy equivalence $R| : p^{-1}(t') \to p^{-1}(t)$ for all $t' \in W \cap T$. Hence, we only need to prove that $R|_{p^{-1}(x)}$ is a homotopy equivalence for all $x \in W \setminus T$, since then $U \cap T \subset C$. Using Coram and Duvall’s characterization of approximate fibrations in terms of movability properties [4], it will follow that $p|_{p^{-1}(\mathbb{R}^k \setminus (T \setminus U))}$ is an approximate fibration, which will contradict the minimality of $T \cap U$, a closed subset of $\mathbb{R}^k)$. This will imply that $T \setminus U = T$, i.e., $U \cap T = \emptyset$, which will contradict the way $U$ was chosen ($U \cap T \neq \emptyset$). Therefore, $T = \emptyset$ and the result will follow.
To finish the proof we need to prove that $R|_{p^{-1}(x)}$ is a homotopy equivalence for all $x \in W \setminus T$.

Using that $p$ is an approximate fibration over $W \setminus T$, the homotopy exact sequence

$$\pi_1(p^{-1}(x)) \cong \pi_1(N) \xrightarrow{i_1} \pi_1(p^{-1}(W \setminus T)) \xrightarrow{p|_{\pi_1}} \pi_1(W \setminus T) \longrightarrow 1 \cong \pi_0(N)$$

for all $x \in W \setminus T$, gives

$$i_2(\pi_1(N)) = \ker p_2 \cong \pi_1(p^{-1}(W \setminus T)).$$

Hence,

$$\pi_1(p^{-1}(W \setminus T)) / i_2(\pi_1(N)) \cong p_2(p^{-1}(W \setminus T)) = \pi_1(W \setminus T).$$

In addition, by Lemma 6.2 it follows that the map $j_3 : \pi_1(p^{-1}(W \setminus T)) \to \pi_1(p^{-1}W)$ is onto.

Next, look at the long exact homology sequence of the pair $(W, W \setminus T)$:

$$\cdots \to H_2(W) \xrightarrow{\cong} H_2(W, W \setminus T) \xrightarrow{\cong} H_1(W \setminus T) \xrightarrow{\cong} H_1(W) \to \cdots$$

The end groups of the above exact sequence are trivial, hence $H_2(W, W \setminus T) \cong H_1(W \setminus T)$. By Alexander duality [33, p. 342], it follows that $H_2(W, W \setminus T) \cong H^{k-2}_c(T \setminus W)$. Then $H_1(W \setminus T) \cong H^{k-2}_c(T \cap W)$. Since $H^{k-2}_c(T \cap W) \cong 0$ ($\dim(T \cap W) \leq \dim T < k - 2$ and $k > 2$), it follows that $H_1(W \setminus T) \cong 0$, hence $\pi_1(W \setminus T)$ is perfect.

Consider the following diagram:

$$\begin{array}{ccc}
\pi_1(N) & \cong & \pi_1(p^{-1}(x)) \\
\downarrow{i_1} & & \downarrow{i_2} \\
\pi_1(p^{-1}(W \setminus T)) & \xrightarrow{p|_{\pi_1}} & \pi_1(W \setminus T) \\
\downarrow{j_3} & & \downarrow{j_3} \\
\pi_1(p^{-1}W) & \cong & \pi_1(p^{-1}(t)) \\
\downarrow{s} & & \downarrow{R|_{t}} \\
\pi_1(N) & \cong & \pi_1(p^{-1}(t)) \\
\end{array}$$

where $t \in T \setminus U$ and $x \in W \setminus T$.

We need to prove that $i' = (R|_{p^{-1}(x)})_t = R|_{j_3j_2i_2}$ is an isomorphism.

Using (11), and the fact that $j_3$ and $R|_t$ are surjective maps, it follows that $i'(\pi_1(N)) = R|_{j_3j_2i_2}(\pi_1(N)) \cong \pi_1(N)$. Moreover, consider the epimorphism

$$k : \pi_1(p^{-1}(W \setminus T)) / i_2(\pi_1(N)) \to \pi_1(N) / i'(\pi_1(N))$$
defined with \( k(gi_1(\pi_1(N))) = R_{1\sharp j_\sharp}(g)i'(\pi_1(N)) \), where \( g \in \pi_1(p^{-1}(W\setminus T)) \).

Using (2) and the fact that \( \pi_1(W\setminus T) \) is perfect, we conclude that \( \pi_1(N)/i'(\pi_1(N)) \) is perfect.

Then \( \pi_1(N) \) being perfectly-Hopfian implies that \( i' \) (and hence \( (R|_{p^{-1}(x)})_q \)) is an isomorphism.

Now, \( N \) being homotopically determined by \( \pi_1 \) forces \( R|_{p^{-1}(x)} \) to be a homotopy equivalence.

Next we prove the main theorem in this section, that identifies which homotopically determined by \( \pi_1 \) manifolds with perfectly-Hopfian fundamental group are shape fibrators.

**Theorem 6.4.** Let \( N \) be a closed, orientable manifold homotopically determined by \( \pi_1 \) with a perfectly-Hopfian fundamental group. If \( N \) is a codimension-2 shape \( m_{\text{simplo}} \)-fibrator, then \( N \) is a shape \( m_{\text{simplo}} \)-fibrator.

**Proof.** Proceed by induction. Suppose \( N \) is a codimension-(\( k-1 \)) shape \( m_{\text{simplo}} \)-fibrator.

Assume \( p : M^{n+k} \to B \) is an \( N \)-shaped PL map, where \( M \) is a closed, orientable, PL \((n+k)\)-manifold and \( B \) a triangulated manifold. Then by [34] Lemma 8.1 \( p \) is an approximate fibration over \( B\setminus B^{(k-3)} \), and Theorem 6.3 implies that \( p \) is an approximate fibration.

Therefore, \( N \) is a codimension-\( k \) shape \( m_{\text{simplo}} \)-fibrator.

**Remark:** The condition of the manifold \( N \) being a codimension-2 fibrator cannot be omitted. Namely, take an \( n \)-dimensional torus, \( T \). \( T \) is aspherical, so is homotopically determined by \( \pi_1 \). Furthermore, \( \pi_1(T) \) is a finitely generated, Abelian group (so perfectly-Hopfian by Corollary 3.2). But \( T \) is not a codimension-2 fibrator having a factor \( S_1 \) in the product.

**Corollary 6.5.** All closed, orientable surfaces \( S \) with genus \( g > 1 \) are shape \( m_{\text{simplo}} \)-fibrators.

**Proof.** \( S \) is homotopically determined by \( \pi_1 \) (since aspherical), has perfectly-Hopfian fundamental group (Theorem 3.7) and is a codimension-2 shape \( m_{\text{simplo}} \)-fibrator [34] Corollary 6.3]. So by Theorem 6.4 it follows that \( S \) is a shape \( m_{\text{simplo}} \)-fibrators.

### 7 Shape Fibrator’s Properties of Direct Products of Hopfian Manifolds

In this section we discuss the shape \( m_{\text{simplo}} \)-fibrator’s properties of direct product of Hopfian manifolds.

**Theorem 7.1.** Suppose \( N_1, N_2 \) are two closed, orientable manifolds of dimension \( m \) and \( n \), \( m \neq n \), homotopically determined by \( \pi_1 \). Assume also that
$N_1 \times N_2$ is a Hopfian manifold. In addition, $\pi_1(N_1)$ is normally incommensurable with $\pi_1(N_2)$ and $\pi_1(N_1), \pi_1(N_2)$ are perfectly-Hopfian.

If $N_1 \times N_2$ is a codimension-2 shape $m_{\text{simpl}}$-fibrator, then $N_1 \times N_2$ is a shape $m_{\text{simpl}}$-fibrator.

Proof. The proof of [13, Theorem 4.1] implies that $N_1 \times N_2$ is homotopically determined by $\pi_1$. Then the conclusion of the theorem follows from Theorems 5.2 and 6.4.

Theorem 7.2. Suppose $N_1, N_2$ are two closed, orientable, aspherical manifolds. In addition, assume that $\pi_1(N_1)$ is normally incommensurable with $\pi_1(N_2)$ and $\pi_1(N_1), \pi_1(N_2)$ are perfectly-Hopfian.

If $N_1 \times N_2$ is a codimension-2 shape $m_{\text{simpl}}$-fibrator, then $N_1 \times N_2$ is a shape $m_{\text{simpl}}$-fibrator.

Proof. The product $N_1 \times N_2$ is a closed aspherical manifold (as a product of closed, aspherical manifolds). Hence, $N_1 \times N_2$ is homotopically determined by $\pi_1$ with perfectly-Hopfian fundamental group (by Theorem 6.4). Then Theorem 6.4 implies that $N_1 \times N_2$ is a shape $m_{\text{simpl}}$-fibrator.

Since incommensurable implies normally incommensurable, we can state Theorems 7.3 and 7.4 by substituting the condition of normal incommensurability with incommensurability.

Corollary 7.3. Suppose $N_1, N_2$ are two closed, orientable manifolds of dimension $m$ and $n$, $m \neq n$, homotopically determined by $\pi_1$. Assume also that $N_1 \times N_2$ is a Hopfian manifold. In addition, $\pi_1(N_1)$ is incommensurable with $\pi_1(N_2)$ and $\pi_1(N_1), \pi_1(N_2)$ are perfectly-Hopfian.

If $N_1 \times N_2$ is a codimension-2 shape $m_{\text{simpl}}$-fibrator, then $N_1 \times N_2$ is a shape $m_{\text{simpl}}$-fibrator.

Corollary 7.4. Suppose $N_1, N_2$ are two closed, orientable, aspherical manifolds. In addition, assume that $\pi_1(N_1)$ is incommensurable with $\pi_1(N_2)$ and $\pi_1(N_1), \pi_1(N_2)$ are perfectly-Hopfian.

If $N_1 \times N_2$ is a codimension-2 shape $m_{\text{simpl}}$-fibrator, then $N_1 \times N_2$ is a shape $m_{\text{simpl}}$-fibrator.

Remark: Note again the necessity of the requirement for the manifold $N_1 \times N_2$ to be a codimension-2 fibrator. Namely, take an $n$-dimensional torus $T$ and a closed, orientable surface $S$ with genus $g > 1$, such that $n < 2g$. They are both closed, aspherical manifolds with perfectly-Hopfian fundamental groups (Corollary 4.2 and Theorem 6.7). By Theorem 4.6, $\pi_1(T)$ is normally incommensurable with $\pi_1(S)$. But the manifold $T \times S$ is not a codimension-2 fibrator having a factor $S_1$ in the product.

Example 7.5. Let $S_1$ and $S_2$ be two closed, orientable surfaces with genuses $g_1, g_2$ respectively, with $g_2 > g_1 > 1$. Then $S_1, S_2$ are aspherical with perfectly-Hopfian fundamental groups (Theorem 6.7), and $\pi_1(S_1)$ is normally incommensurable with $\pi_1(S_2)$ (Corollary 4.4). [19, Main Theorem p. 9] implies that $S_1 \times S_2$
is a codimension-2 orientable fibrator. Then $S_1 \times S_2$ is a shape $m_{\text{simpl}}$-fibrator by Theorem 7.2.

**Example 7.6.** Let $M^3$ be a closed, orientable 3-manifold with Sol geometry that fibers over $S^1$ [32, Theorem 5.3]. $M^3$ is aspherical, so homotopically determined by $\pi_1$.

Take $S$ to be a closed, orientable surface with genus $g > 1$.

Then $M^3 \times S$ is aspherical as a product of aspherical manifolds, hence homotopically determined by $\pi_1$ and Hopfian.

$\pi_1(M^3)$ is a hyper-Hopfian group [6, Theorem 7.2]. In addition, it is known that $M^3$ has a finitely generated, Hopfian, solvable fundamental group, hence perfectly-Hopfian by Theorem 3.1. Since $\pi_1(M^3)$ is normally incommensurable with $\pi_1(S)$ by Theorem 4.1, the proof of [12, Lemma 5.1] shows that $\pi_1(M^3 \times S)$ is also hyper-Hopfian. Hence, $M^3 \times S$ is a codimension-2 fibrator by [8, Theorem 5.4].

$\pi_1(S)$ is perfectly-Hopfian by Theorem 3.7, hence, by Theorem 7.2 it follows that $M^3 \times S$ is a shape $m_{\text{simpl}}$-fibrator.

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