Abstract

The main purpose of this paper is to introduce the random tensor with normal distribution, which promotes the matrix normal distribution to a higher order case. Some basic knowledge on tensors are introduced before we focus on the random tensors whose entries follow normal distribution. The random tensor with standard normal distribution (SND) is introduced as an extension of random normal matrices. As a random multi-array deduced from an affine transformation on a SND tensor, the general normal random tensor is initialised in the paper. We then investigate some equivalent definitions of a normal tensor and present the description of the density function, characteristic function, moments, and some other functions related to a random matrix. A general form of an even-order multi-variance tensor is also introduced to tackle a random tensor. Finally some equivalent definitions for the tensor normal distribution are described.

keywords: Tensor; mixed effect tensor model; parameter estimation; Normal distribution; Characteristic function.

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1 Introduction

The multivariate statistics have been used in many areas including the medical imaging. The classical treatment of a normal random matrix $X$ is to use the traditional multivariate normal distribution after the vectorization of $X$. Basser has studied the high-order diffusion tensor (DT) of water through the magnetic resonance image (MRI), called the DT-MRI, since 1999 [1, 2, 3]. As Basser pointed out in [2], the traditional approach to tackle normal random matrices does not satisfy the requirement of the DT-MRI for the
distribution and moments of both its components and its eigenvalues and eigenvectors. However Basser and his cooperators stay in the matrix style to describe the properties of those tensors, which somehow impeded further investigation of DT-MRI.

The systematic treatment of multivariate statistics through matrix theory has been developed since 1970s[8, 28]. In multivariate statistics, the \( k \)-moment of a random vector \( \mathbf{x} \in \mathbb{R}^n \) is conventionally described by a matrix for any positive integer \( k > 1 \). The multivariate normal distribution, also called the joint normal distribution, usually deals with random vectors with normal distributions. The study of random matrices, motivated by quantum mechanics and wireless communications [12, 13, 21, 22] etc. in the past 70 years, mainly focuses on the spectrum properties [22] and can be used in many areas such as the classical analysis and number theory using enumerative combinatorics [22], Fredholm determinants [20], diffusion processes [4], integrable systems [5], the Riemann Hilbert problem [15] and the free probability theory in the 1990s. The theory of random matrices in multivariate statistics basically focuses on the distribution of the eigenvalues of the matrices [27].

Recently we found some need to use the high order tensors has manifested in many areas more than half a century ago, and the recent growing development of multivariate distribution theory poses new challenge for finding some novel tools to describe classical statistical concepts e.g. moment, characteristic function and covariance etc. This in turn has facilitated the development of higher order tensor theory for multilinear regression model [10] and the higher order derivatives of distribution functions. Meanwhile, the description of an implicit multi-relationship among a family of random variables pose a challenge to modern statisticians. The applications of the high order tensors in statistics was initialized by Cook etc. [10, 11] when the envelope models were established.

In this paper, we first use tensors to express the high order derivatives, which in turn leads to the simplification of the high-order moments and the covariances of a random matrix. We also introduce the normal distributions of a random matrix as well as that of a random tensor. The Gaussian tensors are investigated to extend the random matrix theory.

By a random vector \( \mathbf{x} := (x_1, \ldots, x_n)^\top \in \mathbb{R}^n \) we mean that each component \( x_i \) is a random variable (r.v.). Here we usually do not distinguish a row and a column vector unless specifically mentioned. There are several equivalent definitions for a random vector to be Gaussian. Given a constant vector \( \mu \in \mathbb{R}^n \) and a positive semidefinite matrix \( \Sigma \in \mathbb{R}^{n \times n} \). A random vector \( \mathbf{x} \in \mathbb{R}^n \) is called a Gaussian or normal vector with parameter \((\mu, \Sigma)\) if it is normally distributed with \( E[\mathbf{x}] = \mu \) and \( \text{Var}(\mathbf{x}) = \Sigma, \alpha \in \mathbb{R}^n \). This is equivalent to a single variable normal distribution of \( \alpha^\top \mathbf{x} \) for all \( \alpha \in \mathbb{R}^n \). It is obvious from this fact that each component of a normal vector is normal. The converse is however not true. A random tensor \( \mathcal{A} = (A_{i_1i_2\ldots i_m}) \) is an
m-order tensor whose entries are random variables. As a special case, a random matrix is a matrix whose entries are random variables.

The covariance matrix of a random vector \( \mathbf{x} \) restores the variances and covariances of its coordinates and plays a very important role in the statistical analysis. However, it cannot demonstrate the multi-variances of a group of variables. On the other hand, the tensor form of derivative is that we can locate and label easily any entry of \( \frac{\partial^m f}{\partial \mathbf{x}^m} \).

We denote \( [n] \) for the set \( \{1, 2, \ldots, n\} \) for any positive integer \( n \) and \( \mathbb{R} \) the field of real numbers. Throughout we follow the convention to use the lowercase italic letters \( a, b, \ldots \) for scalars, uppercase italic letters \( A, B, \ldots \) for matrices, lowercase boldface upright letters (e.g. \( \mathbf{x}, \mathbf{y}, \ldots \)) for random vectors while the lowercase italic letter such as \( x, y, \ldots \), for their values, sample points or observations, the uppercase script letters \( \mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{Y}, \ldots \) to denote tensors. We reserve the letters \( X, Y, Z \) (in either case or style) strictly for random objects and all other letters for deterministic objects. Finally a moment of a random vector is denoted by \( \mathbf{m} \) which could be either a scalar, a vector, a matrix or a tensor.

Given a matrix \( \mathbf{A} \in \mathbb{R}^{m \times n} \), we use \( \text{vec}(\mathbf{A}) \) to denote the \( mn \)-dimensional column vector \( \mathbf{v} = (v_1, v_2, \ldots, v_{mn})^\top \) formed by stacking the columns of \( \mathbf{A} \), i.e., \( \mathbf{v}^\top = (\beta_1^\top, \beta_2^\top, \ldots, \beta_n^\top) \) where \( \mathbf{A} = [\beta_1, \beta_2, \ldots, \beta_n] \). A tensor \( \mathcal{A} \) of size \( d := d_1 \times d_2 \times \ldots \times d_m \) is an \( m \)-way matrix or an \( m \)-order tensor. \( \mathcal{A} \) is called an \( m \)-th order \( n \)-dimensional real tensor if \( d_1 = \ldots = d_m = [n] \). Let \( \mathcal{T}(d) \) be the set of all the \( m \)-order tensors indexed by \( d \), \( \mathcal{T}_{m;n} \) be the set of all \( m \)-th order \( n \)-dimensional real tensors, and \( \mathcal{T}_m \) be the set of all \( m \)-th order tensors. Thus a scalar, a vector and a matrix is respectively a tensor of order zero, one and two. For a tensor \( \mathcal{A} \), we usually denote by \( A_{i_1 \ldots i_p} \) or \( A_{\sigma} \), the component of \( \mathcal{A} \) associated with the indices \( \sigma := (i_1, i_2, \ldots, i_p) \). A tensor \( \mathcal{A} \in \mathcal{T}_{m;n} \) is said to be a symmetric tensor if each entry of \( \mathcal{A} \) is invariant for any permutation of its indices. An \( m \)-th order \( n \)-dimensional real tensor \( \mathcal{A} \in \mathcal{T}_{m;n} \) is associated with an \( m \)-order \( n \)-variate homogeneous polynomial in the form

\[
f_{\mathcal{A}}(\mathbf{x}) = \mathbf{A} \mathbf{x}^m = \sum_{i_1, i_2, \ldots, i_m} A_{i_1 i_2 \ldots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}
\]

(1.1)

where the summation is taken over the index set \( S(m, n) := \{(i_1, i_2, \ldots, i_m) : i_k \in [n], \forall k \in [m]\} \)

We denote by \( \mathcal{ST}_{m;n} \) the set of all \( m \)-th order \( n \)-dimensional symmetric tensors. A symmetric tensor \( \mathcal{A} \in \mathcal{ST}_{m;n} \) is said to be positive semidefinite
(PSD) if \( f_A(x) := Ax^m > 0 \) (\( \geq 0 \)) for all \( 0 \neq x \in \mathbb{R}^n \), and \( A \) is called a copositive tensor if \( f_A(x) \geq 0 \) for all nonnegative vector \( x \). For the study of the symmetric tensors, PSD tensors, the copositive tensors, including their spectrum, decompositions and other properties, we refer the reader to \cite{7, 9, 23, 24} and \cite{9}.

Let \( I := I_1 \times \ldots \times I_m \) where each \( I_k \) represents an index set (usually a set in form \( \{1, 2, \ldots, n_k\} \)) and \( A \in \mathcal{T}(I) \). For any \( k \in [m] \) and \( j \in I_k \), \( A \)'s \( j \)-slice along the \( k \)-mode (denoted by \( A^{(k)}[j] \)) is defined as an \((m-1)\)-order tensor \( (A_{i_1 \ldots i_{k-1} j_{k+1} \ldots i_m}) \) where the \( k \)th subscript of each entry of \( A \) is fixed to be \( j \in [I_k] \). An \( m \)-order tensor can be sliced into a series of \((m-1)\)-order tensors along any of the \( m \) modes. A high order tensor can be flattened or unfolded into a matrix by slicing iteratively. For example, an \( m \times n \times p \) tensor \( A \) can be unfolded to be a matrix \( A_{ij1} \in \mathbb{R}^{m \times np} \) along the first mode and \( A_{ij} \in \mathbb{R}^{n \times pm} \) if along the second mode. There are ten options to flatten a 4-order tensor \( m \times n \times p \times q \) into a matrix: four to reserve one mode and stack the other three and six to group two modes together to form a matrix.

The product of tensors can be defined in many different ways. Given any tensors \( A, B \) of appropriate size, the \( k \)-mode contractive product of \( A \) and \( B \) w.r.t. the chosen mode(s). This can be regarded as a generalisation of the matrix product. For more detail, we refer the reader to \cite{26} and \cite{6}.

Given a random vector \( x \in \mathbb{R}^n \). The characteristic function (CF) of \( x \), is defined by

\[
\phi_x(t) = E[exp(it'x)], \forall t \in \mathbb{R}^n
\]

For any positive integer \( k \), the \( k \)-moment of a random vector \( x \) is defined by

\[
m_k(x) = \frac{1}{i^k} \left. \frac{d^k}{dt^k} \phi_x(t) \right|_{t=0} \quad (1.2)
\]

Note that \( m_1(x) = \frac{d}{dt} \phi_x(t) \in \mathbb{R}^n \) is a vector and \( m_k(x) = \frac{d^k}{dt^k} \phi_x(t) \in \mathbb{R}^{n \times n^{k-1}} \) is a matrix for each \( k > 1 \) by the conventional definition.

To simplify the definition of the \( k \)-moments of a random vector, we present a tensor form of the high order multivariate function. Let \( y = f(x) \) be a mapping from \( \mathcal{C}^n \) to \( \mathcal{C}^m \), i.e., \( y = (y_1, \ldots, y_m)^T \in \mathcal{C}^m \) with each components \( y_i = f_i(x) \) sufficiently differentiable. Then we define \( H(y, x) = (h_{ij}) \) as the Jacobi matrix of \( y \) w.r.t. \( x \) defined by \( h_{ij} := \frac{\partial y_i}{\partial x_j} \) for all \( i \in [m], j \in [n] \).

Thus \( H(y, x) \in \mathbb{R}^{m \times n} \). Now we define the \( k \)-order differentiation tensor by

\[
H^k(y, x) = (h_{ij_1j_2\ldots j_k}) \quad (1.3)
\]

which is an \((k+1)\)-order tensor of size \( m \times n \times \ldots \times n \) where

\[
h_{ij_1j_2\ldots j_k} = \frac{\partial^k y_i}{\partial x_{j_1} \partial x_{j_2} \ldots \partial x_{j_k}} \quad (1.4)
\]
for any \( i \in [m], j_1, \ldots, j_k \in [n] \). Recall that the conventional form for the \( k \)-order differentiation of a mapping \( y = f(x) \) produces an \( m \times n^k \) matrix \( H^k(y, x) = (h_{ij}) \) where \( h_{ij} = \frac{\partial^k y_i}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_k}} \) with \( j = \sum_{s=1}^{k} j_s n^{k-s} \) \((j = 1, 2, \ldots, n^k)\), making the location of each entry ambiguous.

In this paper, we use tensor form to simplify high order differentiations. The \( k \)-moment of a random vector \( x \in \mathbb{R}^n \) is defined by

\[
\mathbf{m}_k[x] = E[x^k] = E\left[\prod_{j=1}^{k} x \right] \quad (1.5)
\]

This is a natural extension of the \( k \)-moment in the univariate case since

\[
\mathbf{m}_1[x] = E[x] \in \mathbb{R}^n, \quad \mathbf{m}_2[x] = E[xx^\top] \in \mathbb{R}^{n \times n}, \ldots, \quad \mathbf{m}_k[x] = E[x^k] \in \mathcal{T}_{m \times n}.
\]

The definition is identical to the one through characteristic function in the tensor form, as in the following.

**Lemma 1.1.** Let \( k \) be any positive integer and \( x \in \mathbb{R}^n \) be a random vector with characteristic function \( \phi_x(t) \) (with \( t \in \mathbb{R}^n \)). Then

\[
E[x^k] = \frac{1}{i^k} \left. \frac{d^k}{dt^k} \phi_x(t) \right|_{t=0} \quad (1.6)
\]

where \( 0 \) is a zero vector in \( \mathbb{R}^n \).

**Proof.** It is easy to see from the definition of the characteristic function \( \phi_x(t) \) and (1.3) and (1.4) that

\[
\frac{d^k}{dt^k} \phi_x(t) = i^k E[\exp\{it^\top x\} x^k] \quad (1.7)
\]

Thus (1.6) holds. \( \square \)

Similarly we can also simplify the definition of the \( k \)-central moment \( \mathbf{\tilde{m}}_k[x] \) by \( \mathbf{\tilde{m}}_k[x] = E[(x - E[x])^k] \). Note that our definition is consistent with the traditional one for \( k \leq 2 \). In the following section, we will extend the \( k \)-moment, the characteristic function, and the related terminology to the case for the random matrices.

## 2 The tensor forms of derivatives of matrices

Let \( \theta := \{\theta_k : k = 1, 2, \ldots, p\} \) be a 2-partition of \([2p]\) \((|\theta_k| = 2 \text{ for each } k)\), and let \( A^{(k)} \in \mathbb{R}^{m_k \times n_k}, k \in [p] \). We may assume w.l.g. that \( \theta_k := \{a_k, b_k\} \) with \( a_k < b_k \) for each \( k \). The outer product of the matrix sequence \( \{A^{(k)}\} \)
along partition $\theta$ is the $2p$-order tensor $A$, denoted $A = A^{(1)} \times_{\theta_1} A^{(2)} \times_{\theta_2} \ldots \times_{\theta_{p-1}} A^{(p)},$ with

$$A_{i_1j_1i_2j_2\ldots i pj_p} = A^{(1)}_{i_1a_1b_1} A^{(2)}_{i_2a_2b_2} \ldots A^{(p)}_{i pb_p}.$$ 

For $p = 2$, we let $\theta_1 := \{s,t\} \subset [4]$ with $s < t$ and $\theta_2 := \{p,q\}$ the complement of $\theta_1$ with $p < q$. The outer product $A \times B$ is the 4-order tensor with the $(s,t)$-modes attributed to $A$ and $(p,q)$-modes attributed to $B$. Thus

$$(A \times_{(1,2)} B)_{i_1i_2i_3i_4} = A_{i_1i_2} B_{i_3i_4}, \quad (A \times_{(1,3)} B)_{i_1i_2i_3i_4} = A_{i_1i_3} B_{i_2i_4}. $$

For $\{(s,t)\} = \{(1,2)\}$, we simply write $A \times B$ instead of $A \times_{(1,2)} B$. For a tensor $A \in \mathbb{R}^{m \times n \times p \times q}$ and a matrix $B \in \mathbb{R}^{p \times q}$, the product $C = AB$ refers to the contractive product, which is defined as

$$C_{ij} = \sum_{i'j'} A_{ij'i'j'} B_{i'j'}. $$

Note that the product $A \times 4 B$ is still a 4-order tensor since

$$(A \times 4 B)_{i_1i_2i_3i_4} = \sum_k A_{i_1i_2i_3k} B_{ki_4}. $$

We denote $A^{[2]} := A \times A$ and $A^{(2)} = A \times_{(2,4)} A$ for a matrix $A \in \mathbb{R}^{m \times n}$, i.e.,

$$A^{[2]}_{i_1i_2i_3i_4} = A_{i_1i_2} A_{i_3i_4}, \quad A^{(2)}_{i_1i_2i_3i_4} = A_{i_1i_3} A_{i_2i_4}. $$

It is obvious that $A^{[2]}$ has size $m \times n \times m \times n$ while $A^{(2)}$ has size $m \times m \times n \times n$. 

**Proposition 2.1.** Let $A, B, C, D$ be any matrices of appropriate sizes and $\{s,t\} \subset [4]$ and $\{p,q\} = \{s,t\}^c$. Then

1. $(A \times_{(s,t)} B) \times_{(s,t)} C = (B,C)A$ where $B,C$ are of same size and $(X,Y)$ stands for the inner product of $X,Y$.
2. $(A \times_{(s,t)} B) \times_{(p,q)} C = A \times_{(s,t)} (B \times_{(p,q)} C)$.
3. $(A \times B) \times 4 C = A \times (BC)$.
4. $(A \times_{(s,t)} B)(C \times_{(s,t)} D) = (AC) \times_{(s,t)} (B \times_{(s,t)} D)$.

**Proof.** (1). We may assume that $A \in \mathbb{R}^{m \times n}, B, C \in \mathbb{R}^{p \times q}$ (the equation is valid only if $B, C$ are of the same size). For simplicity, we let $\{s,t\} = \{1,2\}$.
and denote \( D = (A \times_{(1,2)} B) \times_{(1,2)} C \). Then for any pair \((i, j)\), we have by definition
\[
D_{ij} = \sum_{i'j'}(A \times_{(1,2)} B)_{i'j'}C_{i'j'}
\]
\[
= \sum A_{i'j'}B_{i'j'}C_{i'j'} = (B, C)A_{ij}
\]
which immediately implies (1) for \((s, t) = (1, 2)\). Similarly we can also show its validity for other cases. The second and the third item can also be checked using the same technique. To show the last item, we consider the case \((s, t) = (1, 3)\) and rewrite it in form
\[
(A_1 \times A_2)(B_1 \times B_2) = (A_1 B_1) \times (A_2 B_2)
\]
(2.1)
where \( \times := \times_{(1,3)} \), \( A_i \in \mathbb{R}^{n_i \times n_i} \), \( B_i \in \mathbb{R}^{n_i \times n_i} \) for \( i = 1, 2 \). Denote the tensor of the left hand side and the right hand side resp. by \( L \) and \( R \). Then we have
\[
R_{i_1i_2i_3i_4} = (A_1 B_1)_{i_1i_3}(A_2 B_2)_{i_2i_4}
\]
\[
= (\sum_{j=1}^{n_1} a_{i_1j}^{(1)} b_{j i_3}^{(1)})(\sum_{k=1}^{n_2} a_{i_2k}^{(2)} b_{k i_4}^{(2)})
\]
\[
= \sum_{j,k} a_{i_1j}^{(1)} a_{i_2k}^{(2)} b_{j i_3}^{(1)} b_{k i_4}^{(2)}
\]
\[
= \sum_{j,k} (A_1 \times A_2)_{i_1i_2j}k(B_1 \times B_2)_{jk i_3i_4} = L_{i_1i_2i_3i_4}
\]
for all possible \( i_1, i_2, i_3, i_4 \). Thus (2.1) holds. This argument can also be extended to other cases. \( \square \)

The outer product of matrices can be extended to the case for any number of any order. Let \( A_1, A_2, \ldots, A_s \) be tensors of orders \( p_1, p_2, \ldots, p_s \) respectively and let \( S_1, S_2, \ldots, S_s \) be a partition of set \([p]\), \( p = p_1 + p_2 + \cdots + p_k \) where \( |S_j| = p_j \), \( \forall j \in [s] \). Denote this partition by \( \gamma := \{S_1, S_2, \ldots, S_s\} \) for our convenience. The outer product of \( A_1, A_2, \ldots, A_s \) under partition \( \gamma \) as
\[
(A_1 \times A_2 \times \cdots \times A_s)_{i_1i_2\ldots i_p} = (A_1)_{i_11\ldots i_{p_1}}(A_2)_{i_{p_1+1}i_{p_1+2}\ldots i_{p_2}} \cdots (A_s)_{i_{p_s+1}\ldots i_{p_s}}
\]
(2.2)
outer product where \( S_j := \{t_{j1}, t_{j2}, \ldots, t_{jp_j}\} \) with entries in increasing order. Here we allow some \( p_j \)s to be zero, i.e., \( A_j \) is a scalar, which corresponds to an empty underlying set \( S_j \). We denote this outer product of \( A_1, A_2, \ldots, A_s \) under (index) partition \( \gamma \) by \([A_1, A_2, \ldots, A_s]_{\gamma}\). Note that the assumption of the increasing order of the index can be dropped when \( A_j \) is symmetric. We note that when two tensors \( A_k, A_l \) agree, the order of the sets corresponding
to these two tensors in the partition has no influence on the product. In particular, the outer power of a tensor $A$ along an unordered partition $\gamma = \{S_1, S_2, \cdots, S_k\}$ is defined as

$$A^\gamma := [A, A, \ldots, A]. \quad (2.3)$$

We now define the transposition of tensors. Given a tensor $A$ of order $p$ and a permutation $\pi$ on set $[p]$. We define $(\text{Trans}_\pi A)_i := (A)_{i^\pi}$, or more specifically

$$(\text{Trans}_\pi A)_{i_1i_2\cdots i_p} := A_{i_{\pi(1)}i_{\pi(2)}\cdots i_{\pi(p)}}. \quad (2.4)$$

Note that this is a left action on the symmetric group of the index set, i.e., $\text{Trans}_\pi \circ \text{Trans}_\rho A = \text{Trans}_\pi \text{Trans}_\rho A$. Obviously that a tensor is symmetric if it is preserved by all transpositions.

The outer product of tensors along a partition can be expressed in terms of the transposition of a canonical outer product, i.e.,

$$A_1 \times S_1 A_2 \times S_2 \cdots \times S_{k-1} A_k = \text{Trans}_\pi A_1 \times A_2 \times \cdots \times A_k, \quad (2.5)$$

where $\pi$ is a permutation of $[p]$ mapping the block

$$\left\{ \sum_{k=1}^{j-1} p_k + 1, \sum_{k=1}^{j-1} p_k + 2, \cdots, \sum_{k=1}^{j} p_k \right\},$$

increasingly onto $S_j$ for all $j \in [k]$. The requirement of the increasing order on the blocks can be removed if all the $A_j$’s are symmetric.

The introduction of transposition allows us to formulate the beautiful fact that the outer products of symmetric tensors $A_1, A_2, \cdots, A_k$ along all suitable partitions exhaust all transposes of the canonical outer product $A_1 \times A_2 \times \cdots \times A_k$. For the simple case when all $A_j$’s are the same, say $A := A_1 = A_2 = \cdots = A_k$ with order $m$ ($p = mk$), we denote $A^k := [A, A, \ldots, A]$ for canonical $k$-power of $A$.

Now let us go back to the case when we are given two matrices, say $A \in \mathbb{R}^{m_1 \times n_1}, B \in \mathbb{R}^{m_2 \times n_2}$. The cross (outer) product of $A$ and $B$, written as $A \times_c B$, is the 4-order tensor with $(A \times_c B)_{i_1i_2i_3i_4} = A_{i_1i_3}B_{i_2i_4}$, that is, $A \times_c B = [A, B]_{\gamma}$ with $\gamma = \{\{1, 3\}, \{2, 4\}\}$ as a partition of $[4]$. Note that the canonical outer product $A \times B = [A, B]_{\eta}$ where $\eta := \{\{1, 2\}, \{3, 4\}\}$. On the other hand, the contractive product of a 4-order tensor $A \in \mathbb{R}^{m \times n \times m \times n}$ with a matrix $P \in \mathbb{R}^{m \times n}$ is defined as

$$(AP)_{ij} = \sum_{i', j'} A_{ij'i'} P_{i'j'}, \quad (PA)_{ij} = \sum_{i', j'} P_{i'j'} A_{i'j'ij}$$

The following results can be verified easily (thus proof omitted).
Corollary 2.2. (1) \((A \times_{(s,t)} I_n) \times_{(s,t)} I_n = nA\) for any 2-subset \(\{s, t\}\) of \([4]\).

(2) \((I_m \times I_n)A = \text{Tr}(A)I_m\) for any \(A \in \mathbb{R}^{n \times n}\).

(3) \(A \times (I_m \times_c I_n) = (I_m \times_c I_n)A = A\) for any \(A \in \mathbb{R}^{m \times n}\).

(4) \(A^\top (I_n \times_{(2,3)} I_m) = A, (I_m \times_{(2,3)} I_n)A^\top = A\) for any \(A \in \mathbb{R}^{m \times n}\).

Note also that tensor \(I_m \times_c I_n\) can be regarded as the identity tensor in the space \(\mathbb{R}^{m \times n \times m \times n}\) due to (3) of Corollary 2.2.

Recall that a commutation matrix \(K_{p,q} = (K_{ij})\) is a \(p \times q\) block matrix where each block \(B_{ij} \in \mathbb{R}^{q \times p}\) has a unique nonzero entry 1 at position \((j, i)\). Thus \(K_{2,3}\) is an 6 \(\times\) 6 matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

In \([29]\), we define the commutation tensor \(K_{n,m} = (K_{ijkl})\) as an \(n \times m \times m \times n\) tensor that transforms any matrix \(A \in \mathbb{R}^{m \times n}\) into its transpose, i.e., \(K_{n,m}A = A^\top\). We also show that \(K_{m,n} = I_m \times_{(2,3)} I_n\).

Let \(X = (x_{ij}) \in \mathbb{R}^{m \times n}\) be a random matrix whose entries are independent variables. Let \(Y = (y_{ij}) \in \mathbb{R}^{p \times q}\) be a matrix each of whose entries \(y_{ij}\) is a function of \(X\). The derivative \(\frac{dY}{dX}\) is interpreted as the 4-order tensor \(A = (A_{i_1i_2i_3i_4})\) of size \(m \times n \times p \times q\) whose entries are defined by

\[A_{i_1i_2i_3i_4} = \frac{dY_{i_3i_4}}{dX_{i_1i_2}}\]

In order to simplify the expressions of high order moments of random matrices, we now use tensors to describe the derivatives of matrices. In the following, we will present some known results in tensor forms other than in the conventional matrix versions. The following lemma is the derivative chain rule in the matrix version.

Lemma 2.3. Let \(X \in \mathbb{R}^{m_1 \times n_1}, Y \in \mathbb{R}^{m_2 \times n_2}, Z \in \mathbb{R}^{m_3 \times n_3},\) and \(Z = Z(Y), Y = Y(X)\). Then we have

\[
\frac{dZ}{dX} = \frac{dY}{dX} \times \frac{dZ}{dY}
\]

(2.6)

Proof. We denote \(A = \frac{dY}{dX}, B = \frac{dZ}{dY}\) and \(C = \frac{dZ}{dX}\). By definition we have

\[A \in \mathbb{R}^{m_1 \times n_1 \times m_2 \times n_2}, B \in \mathbb{R}^{m_2 \times n_2 \times m_3 \times n_3}, C \in \mathbb{R}^{m_1 \times m_1 \times m_3 \times n_3}.
\]
Then for any given \((i_1, i_2, i_3, i_4) \in [m_1] \times [n_1] \times [m_3] \times [n_3]\), we have

\[
A_{i_1i_2i_3i_4} = \frac{dZ}{dX}_{i_1i_2i_3i_4} = \frac{dz_{i_3i_4}}{dx_{i_1i_2}} = \sum_{j_1, j_2} \frac{dy_{j_1j_2}}{dx_{i_1i_2}} \frac{dz_{i_3i_4}}{dy_{j_1j_2}} = \frac{dY}{dx} \times \frac{dZ}{dY}_{i_1i_2i_3i_4}
\]

Thus (2.6) holds.

Lemma 2.3 can be extended to a more general case:

**Lemma 2.4.**

1. Let \(Z = Z(Y_1, Y_2, \ldots, Y_n)\) be the matrix-valued function of \(Y_1, Y_2, \ldots, Y_n\) where \(Y_k = Y_k(X)\) for all \(k \in [n]\). Then

\[
\frac{dZ}{dX} = \sum_{k=1}^{n} \frac{dY_k}{dX} \times \frac{dZ}{dY_k}
\]

(2.7)

2. Let \(X, Y, Z, U\) be matrix forms of variables and \(U = U(Z), Z = Z(Y), Y = Y(X)\). Then we have the chain

\[
dU = dX \times (dY/dX) \times (dZ/dY) \times (dU/dZ)
\]

(2.8)

where \(dX = (dx_{ij})\).

It is easy to verify the results in Lemma 2.4 so that we omit its proof here. The following results on the matrix derivatives are useful and will be used in the next section.

**Theorem 2.5.** Let \(X = (X_{ij}) \in \mathbb{R}^{m \times n}\) be a matrix whose elements are independent variables. Then

1. \(\frac{dX}{dX} = I_m \times_c I_n\).
2. \(\frac{dX^T}{dX} = I_m \times_{(2,3)} I_n = K_{m,n}\).
3. \(\frac{d(YZ)}{dX} = \frac{dY}{dX} \times_4 Z + \frac{dZ}{dX} \times_3 Y^\top\).
4. \(\frac{dX^2}{dX} = I_n \times_c X + X^\top \times_c I_n\) when \(m = n\).
5. \(\frac{dX^k}{dX} = \sum_{p=0}^{k-1} [(X^\top)^p \times_c X^{k-1-p}]\) where \(X \in \mathbb{R}^{n \times n}\).
Proof. Let $A = \frac{dX}{dX}$. Then we have by definition and the independency of the elements of $X$ that

$$A_{i_1i_2i_3i_4} = \frac{dx_{i_4}}{dx_{i_1i_2}} = \delta_{i_1i_3} \delta_{i_2i_4} = (I_m \times_c I_n)_{i_1i_2i_3i_4}$$

Thus (1) is proved. Similarly we can prove (2) by noticing that

$$\begin{bmatrix} dx_i^\top \end{bmatrix}_{i_1i_2i_3i_4} = \delta_{i_1i_3} \delta_{i_2i_4}$$

which implies

$$\frac{dX^\top}{dX} = I_m \times_{(2,3)} I_n = K_{m,n}.$$

To prove (3), we let $Y \in \mathbb{R}^{p \times r}$, $Z \in \mathbb{R}^{r \times q}$. Then $\frac{d(YZ)}{dX} \in \mathbb{R}^{m \times n \times p \times q}$ whose elements are

$$
\left( \frac{d(YZ)}{dX} \right)_{i_1i_2i_3i_4} = \frac{d\left[ \sum_k y_{i_3k} z_{k,i_4} \right]}{dx_{i_1i_2}} = \sum_k \frac{d(y_{i_3k} z_{k,i_4})}{dx_{i_1i_2}} \\
= \sum_k \left[ \frac{d(y_{i_3k})}{dx_{i_1i_2}} z_{k,i_4} + y_{i_3k} \frac{d(z_{k,i_4})}{dx_{i_1i_2}} \right] \\
= \frac{dY}{dX} \times_4 Z_{i_1i_2i_3i_4} + \left( \frac{dZ}{dX} \times_3 Y^\top \right)_{i_1i_2i_3i_4}
$$

To prove (4), we let $X \in \mathbb{R}^{n \times n}$ and take $Y = Z = X$. By (3) and (1), we have

$$\frac{dX^2}{dX} = \frac{dX}{dX} \times_4 X \times_3 \frac{dX}{dX} = (I_n \times_c X) \times_4 X \times_3 (I_n \times_c X) = I_n \times_c X \times_3 X^\top \times_c I_n$$

To prove (5), we use the induction method to $k$. For $k = 1$, the result is immediate since both sides of (5) are identical to $I_n \times_c I_n$ by (1). The result is also valid for $k = 2$ by (4). Now suppose it is valid for a positive integer $k > 2$. We come to show its validity for $k + 1$. By (3) we have

$$\frac{dX^{k+1}}{dX} = \frac{dX^k}{dX} \times_4 X + (X^\top)^k \times_3 \frac{dX}{dX}$$

$$= \sum_{p=0}^{k-1} \left( (X^\top)^p \times_c X^{k-p} \right) + (I_n \times_c I_n) \times_3 (X^k)^\top$$

$$= \sum_{p=0}^{k-1} \left( (X^\top)^p \times_c X^{k-p} \right) + (X^\top)^k \times I_n$$

$$= \sum_{p=0}^{k} \left( (X^\top)^p \times X^{k-p} \right)$$
Thus we complete the proof of (5).

\[ \square \]

**Corollary 2.6.** Let \( X = (X_{ij}) \in \mathbb{R}^{m \times n} \) be a matrix whose elements are independent, and \( A \in \mathbb{R}^{p \times m}, B \in \mathbb{R}^{n \times q} \) be the constant matrices. Then

1. \[ \frac{d(AXB)}{dX} = A^\top \times cB. \]
2. \[ \frac{d(\det(X))}{dX} = \det(X)X^{-\top}. \]
3. \[ \frac{d(\text{Tr}(X))}{dX} = I_n \text{ for } X \in \mathbb{R}^{n \times n}. \]
4. \[ \frac{dX^{-1}}{dX} = -X^{-\top} \times X^{-1} \text{ when } X \in \mathbb{R}^{n \times n} \text{ is invertible.} \]

**Proof.** To prove (1), we take \( Y = A, Z = XB \). By (4) of Theorem 2.5, we get

\[ \frac{d(AXB)}{dX} = \frac{d(XB)}{dX} \times_3 A^\top = A^\top \times_3 \left( \frac{d(X)}{dX} \times_4 B \right) = A^\top \times_3 \left( [I_m \times I_n] \times_4 B \right) = A^\top \times B \tag{2.9} \]

To prove (2), we denote \( A = \frac{d(\det(X))}{dX} = (A_{ij}) \). Then for any given pair \((i, j) \in [m] \times [n]\), we have by the expansion of the determinant

\[ A_{ij} = \frac{d(\det(X))}{X_{ij}} = \frac{d}{dX_{ij}} \left( \sum_{k=1}^{n} (-1)^{i+k}X_{ik} \det(X(i|k)) \right) \]

\[ = (-1)^{i+j} \det(X(i|j)) = [\det(X)X^{-1}]_{ji} \]

where \( X(i|j) \) represents the submatrix of \( X \) obtained by the removal of the \( i \)th row and the \( j \)th column of \( X \). Thus we have \( \frac{d(\det(X))}{dX} = \det(X)X^{-\top}. \)

Now (3) can be verified by noticing the fact that for all \((i, j)\)

\[ \left[ \frac{d(\text{Tr}(X))}{dX} \right]_{ij} = \sum_{k=1}^{n} \frac{d(X_{kk})}{dX_{ij}} = \sum_{k=1}^{n} \delta_{ik}\delta_{jk} = (e_i, e_j) = \delta_{ij} \]

where \( e_i \in \mathbb{R}^n \) is the \( i \)th row of the identity matrix \( I_n \). Now we prove (4). Using (4) of Theorem 2.5 on the equation \( XX^{-1} = I_n \) (here \( X = (X_{ij}) \in \mathbb{R}^{n \times n} \)), we have

\[ \left( \frac{d(X)}{dX} \times_4 X^{-1} + X \times_2 \frac{dX^{-1}}{dX} \right) \times_4 X^{-1} + X \times_2 \frac{dX^{-1}}{dX} \]

It follows that

\[ \frac{dX^{-1}}{dX} = -X^{-1} \times_2 \left( \frac{d(X)}{dX} \times_4 X^{-1} \right) \]

\[ = -X^{-1} \times_2 \left( [I_n \times I_n] \times_4 X^{-1} \right) \]

\[ = -X^{-1} \times_2 \left( [I_n \times X^{-1}] \right) \]

\[ = -X^{-\top} \times X^{-1} \]
Given any two 4-order tensors, say,
\[ A = (A_{i_1i_2j_1j_2}) \in \mathbb{R}^{m_1 \times m_2 \times n_1 \times n_2}, B = (B_{i_1i_2j_1j_2}) \in \mathbb{R}^{n_1 \times n_2 \times q_1 \times q_2}. \]
The product of \( A, B \), denoted by \( AB \), is referred to as the 4-order tensor of size \( m_1 \times m_2 \times q_1 \times q_2 \) whose entries are defined by
\[
(AB)_{i_1i_2j_1j_2} = \sum_{k_1,k_2} A_{i_1i_2k_1k_2}B_{k_1j_2j_1j_2}
\]
This definition can also be carried over along other pair of directions. We will not go into detail at this point in this paper, and want to point out that all the results concerning the tensor forms of the derivatives can be transformed into the conventional matrix forms, which can be achieved by Kronecker product.

3 On Gaussian matrices

In this section, we introduce and study the random matrices with Gaussian distributions and investigate the tensor products of such matrices. We denote \( \| \cdot \| \) for the Euclidean norm and \( S^{k-1} := \{ s \in \mathbb{R}^k : \| s \| = 1 \} \) for the unit sphere in \( \mathbb{R}^k \) for any positive integer \( k > 1 \). Let \( m, n > 1 \) be two positive integers. Then \( \alpha \in S^{m-1}, \beta \in S^{n-1} \) implies \( \alpha \otimes \beta \in S^{mn-1} \) since \( \| \alpha \otimes \beta \| = \| \alpha \| \cdot \| \beta \| = 1 \).

Let \( X = (x_{ij}) \in \mathbb{R}^{m \times n} \) be a random matrix. The characteristic function (CF) of \( X \) is defined by
\[
\phi_X(T) = E[\exp(i\text{Tr}(T'X))], \quad \forall T \in \mathbb{R}^{m \times n}
\]
Note that \( \phi_X(T) = \phi_X(t) \) where \( x = \text{vec}(X) \) and \( t = \text{vec}(T) \) are respectively the vectorization of \( X \) and \( T \). While vectorization allows us to treat all derivative (and thus the high order moments) of random matrices, it also pose a big challenge for identifying the \( k \)-moment corresponding to each coordinate of \( X \). We introduce the tensor expression for all these basic terminology thereafter. For our convenience, we denote by \( X_{i} \) (\( X_{j} \)) the \( i \)th row (resp. \( j \)th column) of a random matrix \( X \). \( X \) is called a standard normally distributed (SND) or a SND matrix if

1. \( X_{i} \)'s i.i.d. with \( X_{i} \sim N_n(0, I_n) \);
2. \( X_{j} \)'s i.i.d. with \( X_{j} \sim N_m(0, I_m) \).

that is, all the rows (and columns) of \( X \) are i.i.d. with standard normal distribution. This is denoted by \( X \sim N_{m,n}(0, I_m, I_n) \).

The following lemma concerning a necessary and sufficient condition for a SND random vector will be frequently used in the paper.
Lemma 3.1. Let \( x \in \mathbb{R}^n \) be a random vector. Then \( x \sim N_n(0, I_n) \) if and only if \( \alpha^T x \sim N(0, 1) \) for all unit vectors \( \alpha \in S^{n-1} \).

Lemma 3.2. Let \( X = (x_{ij}) \in \mathbb{R}^{m \times n} \) be a random matrix. The following conditions are equivalent:

1. \( X \sim N_{m,n}(0, I_m, I_n) \).
2. \( \text{vec}(X) \sim N_{mn}(0, I_{mn}) \).
3. All \( x_{ij} \) are i.i.d. with \( x_{ij} \sim N(0,1) \).
4. \( \alpha^T X \beta \sim N(0,1), \ \forall \alpha \in S^{m-1}, \beta \in \mathbb{R}^{n-1} \).

Proof. The equivalence of (2) and (3) is obvious. We now show (1) \( \iff \) (2) \( \iff \) (4). To show (2) \( \implies \) (1), we denote \( x := \text{vec}(X) \in \mathbb{R}^{mn} \) and suppose that \( x \sim N_{mn}(0, I_{mn}) \). Then \( \text{cov}(X_i, X_{.j}) = \Sigma_{ij} = 0 \) for all distinct \( i, j \in [n] \). So the columns of \( X \) are independent. Furthermore, we have by Lemma 3 that

\[
\alpha^T x \sim N(0,1), \ \forall \alpha \in S^{mn-1} \tag{3.1}
\]

Now set \( \alpha = e_j \otimes \beta \in \mathbb{R}^{mn} \) (\( \forall j \in [n] \)) where \( e_j \in \mathbb{R}^n \) is the \( j \)th coordinate vector of \( \mathbb{R}^n \) and \( \beta \in S^{m-1} \). Then \( \alpha \in S^{mn-1} \) and by (3.1) we have

\[
\beta^T X e_j = \beta^T X e_j = (e_j^T \otimes \beta^T) x = (e_j \otimes \beta)^T x = \alpha^T x \sim N(0,1)
\]

It follows by Lemma 3 that \( X_{.j} \sim N_m(0, I_m) \) for all \( j \in [n] \). Consequently (2) implies (1) by definition.

(2) \( \implies \) (4): Denote \( \gamma := \beta \otimes \alpha \) for any given \( \alpha \in S^{m-1}, \beta \in S^{n-1} \). Then \( \gamma \in S^{mn-1} \). Since \( \text{vec}(X) \sim N_{mn}(0, I_{mn}) \), we have, by Lemma 3, that \( \alpha^T X \beta = (\beta^T \otimes \alpha^T) \text{vec}(X) = \gamma^T \text{vec}(X) \sim N(0,1) \), which proves (4).

To show (4) \( \implies \) (2), we let \( \gamma \in S^{mn-1} \). Then there is a unique matrix \( A = (a_{ij}) \in \mathbb{R}^{m \times n} \) such that \( \gamma = \text{vec}(A) \) and \( \|A\|_F^2 = \sum_{i,j} a_{ij}^2 = \|\alpha\|^2 = 1 \) (\( \|A\|_F \) denotes the Frobenius norm of matrix \( A \)). Since \( \|\text{vec}(A)\|_2 = \|A\|_F = 1 \), \( \text{vec}(A) \in S^{mn-1} \). By (4), we have \( \gamma^T \text{vec}(X) = \text{vec}(A)^T \text{vec}(X) \sim N(0,1) \). Consequently we get \( X \sim N_{mn}(0, I_{mn}) \) by Lemma 3.

The density function and the characteristic function of a SND random matrix \([9, 17]\) can also be obtained by Lemma 3.2.

Proposition 3.3. Let \( X \sim N_{m,n}(0, I_m, I_n) \). Then
\[
f_X(T) = (2\pi)^{-mn/2} \exp \left\{ -\frac{1}{2} \text{Tr}(T^\top T) \right\} \text{ where } T \in \mathbb{R}^{m \times n}.
\]

\[
\phi_X(T) = \exp \left\{ -\frac{1}{2} \text{Tr}(T^\top T) \right\} \text{ where } T \in \mathbb{R}^{m \times n}.
\]

Let \( M = (m_{ij}) \in \mathbb{R}^{n_1 \times n_2} \), and \( \Sigma_k = (\sigma^{(k)}_{ij}) \in \mathbb{R}^{n_k \times n_k} \) be positive definite for \( k = 1, 2 \). A random matrix \( X = (X_{ij}) \in \mathbb{R}^{n_1 \times n_2} \) is called a Gaussian matrix with parameters \((M, \Sigma_1, \Sigma_2)\), written as \( X \sim \mathcal{N}_{m,n}(M, \Sigma_1, \Sigma_2) \), if

(a) Each row \( X_i \) follows a Gaussian distribution with

\[
X_i \sim \mathcal{N}_{n_1}(M_i, \sigma^{(1)}_{ii} \Sigma_2), \quad \forall i \in [m],
\]

(b) Each column vector \( X_j \) follows a Gaussian distribution with

\[
X_j \sim \mathcal{N}_{n_2}(M_j, \sigma^{(2)}_{jj} \Sigma_1), \quad \forall j \in [n]
\]

Such a random matrix \( X \) is called a Gaussian matrix. It follows that the vectorization of a Gaussian matrix \( X \) is a Gaussian vector, i.e.,

\[
\text{vec}(X) \sim \mathcal{N}_{mn}(\text{vec}(M), \Sigma_2 \otimes \Sigma_1)
\]

A Gaussian vector cannot be shaped into a Gaussian matrix if its covariance matrix possesses no Kronecker decomposition of two PSD matrices. For any two random matrices (vectors) \( X, Y \) of the same size, we denote \( X = Y \) if their distributions are identical. The following statement, which can be found in [17], shows that an affine transformation preserves the Gaussian distribution.

**Lemma 3.4.** Let \( X \sim \mathcal{N}_{n_1,n_2}(\mu, \Sigma_1, \Sigma_2) \) and \( Y = B_1 X B_2^\top + C \) with \( B_i \in \mathbb{R}^{n_i \times n_i} \) (\( i = 1, 2 \)) being constant matrices. Then

\[
Y \sim \mathcal{N}_{m_1,m_2}(C + B_1 \mu B_2^\top, B_1 \Sigma_1 B_1^\top, B_2 \Sigma_2 B_2^\top)
\]

The following statement can be regarded as an alternative definition of a Gaussian matrix.

**Lemma 3.5.** Let \( X = (x_{ij}) \in \mathbb{R}^{n_1 \times n_2} \) be a random matrix and \( \Sigma_i = A_i A_i^\top \) with each \( A_i \in \mathbb{R}^{n_i \times n_i} \) nonsingular (\( i = 1, 2 \)). Then \( X \sim \mathcal{N}_{n_1,n_2}(M, \Sigma_1, \Sigma_2) \) if and only if there exist a SND random matrix \( Z \in \mathbb{R}^{p \times q} \) such that

\[
X = A_1 Z A_2^\top + M
\]

where \( M \in \mathbb{R}^{m \times n} \) is a constant matrix.

---

\(^1\) We do not use the term normal matrix since it is referred to a matrix satisfying \( XX^\top = X^\top X \).
The proof of Lemma 3.5 can be found in [5]. Now we let \( X \sim \mathcal{N}_{n_1,n_2}(M, \Sigma_1, \Sigma_2) \) be a Gaussian matrix where \( M = (m_{ij}) \in \mathbb{R}^{n_1 \times n_2} \) and \( \Sigma_k = (\sigma_{ij}^{(k)}) \in \mathbb{R}^{n_k \times n_k} \) \((k = 1, 2)\) being positive definite. Write
\[
\omega_T = \text{Tr}[(T - M)\Sigma_1^{-1}(T - M)\Sigma_2^{-1}]
\]
(3.6)
where \( T \in \mathbb{R}^{n_1 \times n_2} \) is arbitrary. The characteristic function (CF) of \( X \) is defined as
\[
\phi_X(T) := E[\exp(i \langle X, T \rangle)], \quad T \in \mathbb{R}^{n_1 \times n_2}
\]
We have

**Corollary 3.6.** Let \( X \in \mathbb{R}^{n_1 \times n_2} \) be a random matrix. Then the density and the characteristic function of \( X \) are respectively
\[
f_X(T) = (2\pi)^{-n_1 n_2 / 2} (\det(\Sigma_1))^{-n_2 / 2} (\det(\Sigma_2))^{-n_1 / 2} \exp\left\{-\frac{1}{2} \omega_T\right\}
\]
and
\[
\phi_X(T) = \exp\left\{i \text{Tr}(T\Sigma_1 - \frac{1}{2} \text{Tr}(T \Sigma_1 T \Sigma_2))\right\}
\]
(3.7)
where \( T \) takes values in \( \mathbb{R}^{n_1 \times n_2} \).

Lemma 3.4 can be used to justify the definition of Gaussian matrices if we take Lemma 3.5 as the original one. Let \( X \sim \mathcal{N}_{n_1,n_2}(\mu, \Sigma_1, \Sigma_2), \ A = I_m \) and \( B = e_j \ (\forall j \in [n]) \) is the \( j \)th coordinate vector of \( \mathbb{R}^n \). Then we have
\[
AXB = X, j, AMB = \mu, j, A\Sigma_1 A^\top = \Sigma_1, B\Sigma_2 B^\top = \sigma_{jj}^2,
\]
Thus \( x_j \sim \mathcal{N}_{m_1,1}(\mu, j, \Sigma_1, \sigma_{jj}^2) \) by Lemma 3.4 which is equivalent to (3.3). This argument also applies to prove (3.2). Furthermore, \( x_{ij} \sim \mathcal{N}(\mu_{ij}, (\sigma_{ij}^{(1)})^2) \) for all \( i \in [n_1], j \in [n_2] \).

For any matrix \( A \in \mathbb{R}^{m \times n} \), we use \( A[S_1|S_2] \) to denote the submatrix of \( A \) whose entries \( a_{ij} \)'s are confined in \( i \in S_1, j \in S_2 \) where \( \emptyset \neq S_1 \subset [m], \emptyset \neq S_2 \subset [n] \). This is denoted by \( A[S] \) when \( S_1 = S_2 = S \). It follows from Lemma 3.4 that any submatrix of a Gaussian matrix is also Gaussian.

**Corollary 3.7.** Let \( X \sim \mathcal{N}_{n_1,n_2}(\mu, \Sigma_1, \Sigma_2) \) and \( \neq S_i \subset [n] \) with cardinality \( |S_i| = r_i \) for \( i = 1, 2 \). Then
\[
X[S_1|S_2] \sim \mathcal{N}_{r_1,r_2}(\mu[S_1|S_2], \Sigma_1[S_1], \Sigma_2[S_2])
\]
(3.10)

**Proof.** We may assume that \( S_1 = \{i_1 < i_2 < \ldots < i_{r_1}\}, S_2 = \{j_1 < j_2 < \ldots < j_{r_2}\} \), and for \( i = 1, 2 \), choose matrix
\[
P^\top_1 = [e_{i_1}, e_{i_2}, \ldots, e_{i_{r_1}}], \quad P^\top_2 = [f_{j_1}, f_{j_2}, \ldots, f_{j_{r_2}}]
\]
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where \( c_k \in \mathbb{R}^{n_1} \) is the \( k \)th coordinate (column) vector in \( \mathbb{R}^{n_1} \), and \( f_k \in \mathbb{R}^{n_2} \) is the \( k \)th coordinate (column) vector in \( \mathbb{R}^{n_2} \), thus we have \( P_i \in \mathbb{R}^{n_1 \times n_{1i}} \) for \( i = 1, 2 \). Since \( X[S_1|S_2] = P_i X P_i^T \), we have

\[
X[S_1|S_2] \sim \mathcal{N}_{r_1,r_2}(P_1 \mu P_2^T, P_1 \Sigma_1 P_1^T, P_2 \Sigma_2 P_2^T)
\]

Then (3.10) follows by noticing that \( \mu[S_1|S_2] = P_1 \mu P_2^T \) and \( \Sigma_i[S_i] = P_1 \Sigma_i P_1^T \) (\( i = 1, 2 \)).

For a random vector \( x \sim \mathcal{N}(\mu, \Sigma) \), we have \( x = \mu + Ay \) with \( \mu \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n} \) satisfying \( AA^\top = \Sigma \) and \( y \sim \mathcal{N}_n(0, I_n) \). It follows that \( m_1[x] = \mu \) and

\[
m_2[x] = E[xx^\top] = E[(\mu + Ay)(\mu + Ay)^\top] = \mu \mu^\top + AE[yy^\top]A^\top = \mu \mu^\top + AA^\top = \mu \mu^\top + \Sigma
\]

since \( E[yy^\top] = m_2[y] = I_n \). The \( k \)-moment of a random matrix \( X \in \mathbb{R}^{m \times n} \) is defined as the \( 2k \)-order tensor \( E[X^{(k)}] \) which is of size \( m^k \times n^k \). Write \( m_k[X] = (\mu_{i_1,i_2,\ldots,i_k})_{i_1,i_2,\ldots,i_k} \). By definition

\[
\mu_{i_1,i_2,\ldots,i_k} = E[x_{i_1,i_2,\ldots,i_k}]
\]

Lemma 3.8. Let \( Y \in \mathbb{R}^{m \times n} \) be a standard normal matrix (SNM), i.e., \( Y \sim \mathcal{N}_{m,n}(0, I_m, I_n) \). Then

\[
m_2[Y] = E[Y \times Y] = I_m \times_c I_n
\]

Proof. Denote \( y := \text{vec}(Y) \). Then \( y \sim \mathcal{N}_{mn}(0, I_{mn}) \) by the hypothesis. It follows that \( m_2[y] = I_{mn} \) by Lemma 3.2. Now let \( Z = Y \times Y \) and \( M^{(2)} = m_2[Y] \). Then

\[
M^{(2)}_{i_1,i_2,i_3,i_4} = E[y_{i_1,i_2}y_{i_3,i_4}] = \text{cov}(y_{i_1,i_2}, y_{i_3,i_4}) = \delta_{i_1,i_3}\delta_{i_2,i_4} = (I_m \times_c I_n)_{i_1,i_2,i_3,i_4}
\]

(3.12)

for any \( (i_1, i_2, i_3, i_4) \). So (3.11) holds. The third equality in (3.12) is due to Lemma 3.2.

Let \( X \) be a random matrix following MND \( X \sim \mathcal{N}_{n_1,n_2}(\mu, \Sigma_1, \Sigma_2) \) with \( \mu \in \mathbb{R}^{n_1 \times n_2} \) and \( \Sigma_i \in \mathbb{R}^{n_i \times n_i} \), \( i = 1, 2 \), \( \Sigma_i = A_i A_i^\top \) with \( A_i \in \mathbb{R}^{n_i \times n_i} \) being nonsingular \( (i = 1, 2) \). Denote \( Z = A_i^{-1}(X - \mu) A_i^\top \). Then \( Z \in \mathbb{R}^{n_1 \times n_2} \) is a SND matrix and we have

\[
X = \mu + A_1 Z A_2^\top
\]

(3.13)

It follows that \( m_1[X] = E[X] = \mu \in \mathbb{R}^{n_1 \times n_2} \) and \( m_2[X] = E[X \times X] \) whose entries are defined by

\[
m^{(2)}_{i_1,i_2,i_3,i_4} = E[x_{i_1,i_2}x_{i_3,i_4}] = \text{cov}(X_{i_1,i_2}, X_{i_3,i_4})
\]

(3.14)
Thus \( m_2[X] \in \mathbb{R}^{n_1 \times n_2 \times n_1 \times n_2} \). An \( k \)-moment of \( X \) is defined as an \( 2k \)-order tensor \( M_k[X] := E[X^{[k]}] \) (with size \( (n_1 \times n_2)^{[k]} \)). Then each entry of \( M_k[X] \) can be described as

\[
(M_k[X])_{i_1i_2...i_kj_1j_2...j_h} = E[X_{i_1j_1}X_{i_2j_2}...X_{i_kj_k}]
\]

For any matrices \( A, B, C, D \) and non-overlapped subset \( \{s_i, t_i\} \subset [8] \) with \( s_i < t_i \). The tensor

\[
T = (T_{i_1i_2...i_k}) \equiv A \times(s_2,t_2) B \times(s_3,t_3) C \times(s_4,t_4) D
\]
yields an 8-order tensor whose entries are defined by

\[
T_{i_1i_2...i_k} = A_{i_1i_2}B_{i_2i_3}C_{i_3i_4}D_{i_4i_5}
\]

where \( \{s_i, t_i\} = (\cup_{k=2}^{4} \{s_k, t_k\})^c \).

Let \( A, B \) be tensors of order \( p \) and \( q \) respectively. The tensor product \( C := A \times B \) is an \( (p + q) \)-order tensor whose components are defined by

\[
C_{i_1i_2...i_pj_1j_2...j_q} = A_{i_1i_2...i_p}B_{j_1j_2...j_q}
\]

For \( p = q \), we can also define the cross tensor product of \( A, B \), as the \( 2p \)-order tensor \( D = A \times_c B \) defined by

\[
D_{i_1i_2...i_pj_1j_2...j_p} = A_{i_1i_2...i_p}B_{j_1j_2...j_p}
\]

Note that when \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{m \times n} \), we have

\[
A \times B = A \times_{\{3,4\}} B \in \mathbb{R}^{m \times n \times m \times n}, \quad A \times_c B = A \times_{\{2,4\}} B \in \mathbb{R}^{m \times m \times n \times n}.
\]

In the following when we write \( A \times B \) we usually mean \( A \times_{\{3,4\}} B \), the tensor product of \( A \) and \( B \), if not mentioned the other way.

**Lemma 3.9.** Let \( A(T) := \eta \mu - \Sigma_1 T \Sigma_2 \). Then we have

(1) \[
\frac{dA}{dT} = -\Sigma_1 \times \Sigma_2.
\]

(2) \[
\frac{d(A \times_c A)}{dT} = \frac{dA}{dT} \times_{\{3,6\}} A + \frac{dA}{dT} \times_c A.
\]

**Proof.** We write \( B = (B_{i_1i_2j_1j_2}) = \frac{dA(T)}{dT} \) i.e., the derivative of \( A(T) \) w.r.t. \( T \), which, according to the definition, is of size \( n_1 \times n_1 \times n_2 \times n_2 \). Then

\[
B_{i_1i_2j_1j_2} = \frac{dA_{i_2j_2}}{dT_{i_1j_1}} = -\frac{d}{dT_{i_1j_1}} \sum_{k,l} \sigma^{(1)}_{i_2k} \sigma^{(2)}_{i_2l} T_{kl} = -\sum_{k,l} \sigma^{(1)}_{i_2k} \sigma^{(2)}_{i_2l} \delta_{i_1k} \delta_{j_1l} = -\sigma^{(1)}_{i_2i_1} \sigma^{(2)}_{j_1j_2}
\]

It follows that

\[
B = \frac{dA}{dT} = -\Sigma_1 \times \Sigma_2.
\]

To prove (2), we first note that \( A \times_c A \in \mathbb{R}^{n_1 \times n_2 \times n_2 \times n_2} \). Thus \( \frac{d(A \times_c A)}{dT} \)
is an 6-order tensor. Denote \( C = (C_{i_1i_2i_3j_1j_2j_3}) = \frac{d(A \times_v A)}{dT}. \) Then \( C \) is of size \( n_1^{[3]} \times n_2^{[3]} \equiv n_1 \times n_1 \times n_2 \times n_2 \times n_2 \times n_2, \) and

\[
C_{i_1i_2i_3j_1j_2j_3} = \frac{d[A_{i_2i_3j_1j_2j_3}]}{dT_{i_1j_1}} = (\frac{dA}{dT})_{i_1i_2j_1j_2}A_{i_3j_3} + (\frac{dA}{dT})_{i_1i_3j_1j_3}A_{i_2j_2} \quad (3.16)
\]

for all possible \( i_1, j_1, \) which completes the proof of (2).

Let \( X \sim \mathcal{N}_{n_1,n_2}(\mu, \Sigma_1, \Sigma_2) \) and let \( \phi := \phi_X(T) \) be its characteristic function. For our convenience, we denote \( \phi' \) (\( A' \)) for the first order derivative of \( \phi \) (\( A(T) \)) w.r.t. \( T \), \( \phi'' \) (\( A'' \)) for the second order derivative of \( \phi \) (\( A(T) \)) w.r.t. \( T \), and \( \phi^{(k)} \) (\( A^{(k)} \)) for the \( k \)-order derivative of \( \phi \) (\( A(T) \)) w.r.t. \( T \). By Lemma 3.9 we can characterize the derivatives of the characteristic function of a Gaussian matrix \( X \), as in the following:

**Theorem 3.10.** Let \( X \sim \mathcal{N}_{n_1,n_2}(\mu, \Sigma_1, \Sigma_2) \), \( \phi := \phi_X(T) \) be its characteristic function and \( A = A(T) \) be defined as above. Then

1. \( \phi' = \phi'(\mu - \Sigma_1T\Sigma_2) = \phi A \).
2. \( \phi'' = -\phi \left[ \mu^{[2]} + \Sigma_1 \times \Sigma_2 - \Sigma_1^{[2]}T^{[2]}\Sigma_2^{[2]} + \iota(I_{n_1} \times \Sigma_1)(\mu \times T + T \times \mu)(I_{n_2} \times \Sigma_2) \right] \)
3. \( \phi^{(3)} = A \times_{(3,6)} \phi'' + \phi(A \times A)' \).
4. \( \phi^{(k+1)} = \sum_{i+j=k} A^{(i)} \times_{(3,6)} \phi^{(j)} + \sum_{i+j=k-1} \phi^{(i)} \times_{(2,5)} A^{(j)} \)

**Proof.** In order to prove (1), we denote \( f = \text{Tr}(\Sigma_1 T \Sigma_2 T') \). Then \( \frac{df}{dT} \in \mathbb{R}^{n_1 \times n_2} \). Since

\[
\text{Tr}(\Sigma_1 T \Sigma_2 T') = (\Sigma_1 T, (T \Sigma_2)') = \sum_{i,j,k,l} \sigma_{ij}^{(1)} \sigma_{lk}^{(2)} t_{jk} t_{il},
\]

It follows that for any pair \( (u, v) \) where \( u \in [n_1], v \in [n_2] \), we have

\[
(df/dT)_{uv} = df/dt_{uv} = \sum_{i,j,k,l} \sigma_{ij}^{(1)} \sigma_{lk}^{(2)} (\delta_{ju} \delta_{kv} t_{il} + \delta_{iu} \delta_{lv} t_{jk}) \]

\[
= \sum_{i,l} \sigma_{iu}^{(1)} \sigma_{il}^{(2)} t_{il} + \sum_{j,k} \sigma_{ij}^{(1)} \sigma_{jk}^{(2)} t_{jk} \]

\[
= 2(\Sigma_1 T \Sigma_2)_{uv}.
\]

Thus we have

\[
\frac{d(\text{Tr}(\Sigma_1 T \Sigma_2 T'))}{dT} = 2 \Sigma_1 T \Sigma_2 \quad (3.17)
\]

It follows that \( \phi'(T) := \frac{d\phi}{dT} = \phi(\mu - \Sigma_1 T \Sigma_2) \) due to (3.17). Thus (1) is proved.
To prove (2), we denote $A := A(T) = \mu - \Sigma_1 T \Sigma_2$ as in Lemma 3.9. Then again by Lemma 3.9 we get

$$\frac{d^2 \phi}{dT^2} = \frac{d}{dT}(\phi') = \frac{d}{dT}(\phi A) = \frac{d\phi}{dT} \times (2,4) A + \phi' \frac{dA}{dT}$$

$$= -\phi [\mu \times (2,4) \mu + \Sigma_1 \times (2,4) \Sigma_2 - U \times (2,4) U + \iota(\mu \times (2,4) U + U \times (2,4) \mu)]$$

where $U = \Sigma_1 T \Sigma_2$. By (4) of Lemma 2.1, $U \times U = (\Sigma_1 \times \Sigma_1)(T \times T)(\Sigma_1 \times \Sigma_1)$ and thus (2) holds.

Now (3) can be verified by using Lemma 3.9, and (4) is also immediate if we use the induction approach to take care of it.

**Corollary 3.11.** Let $X \sim N_{n_1, n_2}(0, \Sigma_1, \Sigma_2)$. Then

1. $m_2[X] = \Sigma_1 \times \Sigma_2$.
2. $m_k[X] = 0$ for all odd $k$.
3. $m_4[X] = \Sigma_1 \times (2,4) \Sigma_2 \times (5,7) \Sigma_1 \times (6,8) \Sigma_2 + \Sigma_1 \times (2,4) \Sigma_2 \times (3,7) \Sigma_1 \times (6,8) \Sigma_2 + \Sigma_1 \times (2,6) \Sigma_2 \times (5,7) \Sigma_1 \times (4,8) \Sigma_2$

**Proof.** (1). By definition we have

$$m_2[X] = \frac{1}{\iota^2} \phi''(T)_{T=0} = -\phi''(T)_{T=0}.$$

The result is followed by (2) of Theorem 3.10.

(2). It is obvious that $m_1[X] = 0$. By the hypothesis, we have $A(0) = 0$, $A'(0) = -\Sigma_1 \times \Sigma_2$. Thus we have

$$(A \times A)'|_{T=0} = A'(0) \times (3,6) A(0) + A'(0) \times A(0) = 0,$$

We now use the induction to $k$ to prove (2). By Theorem 3.10 we have

$$m_3[X] = \frac{1}{\iota^3} \phi'(3)(T)_{T=0}$$

$$= -\iota \left(A(0) \times (3,6) \phi''(0) + \phi(A \times A)'\right) |_{T=0}$$

$$= -\iota A(0) \times (3,6) \phi''(0) = 0$$

Now we assume the result holds for an odd number $k$. Then by (4) of Theorem 3.10 we have

$$m_{k+2} = \frac{1}{\iota^{k+2}} \phi^{(k+2)}|_{T=0}$$

$$= \frac{1}{\iota^{k+2}} \left(A \times (3,6) \phi^{(k+1)} + A' \times (3,6) \phi^{(k)} + A \times (3,6) \phi'^{(k+1)} \right) |_{T=0}$$

$$= \frac{1}{\iota^{k+2}} \left(A(0) \times (3,6) \phi^{(k+1)}(0) + A'(0) \times (3,6) \phi^{(k)}(0) + A(0) \times (3,6) \phi'^{(k+1)}(0) \right)$$

$$= 0$$

since $A(0) = 0$ and $\phi^{(k)}(0)$ by the hypothesis. Thus the result is proved.

(3). This can be shown by using (4) of Theorem 3.10. But we can also prove it by comparing the item (iv) in Theorem 2.2.7 (Page 203) in [17].

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4 Random tensors with Gaussian distributions

We start this section by considering a 3-order random tensor \( Z \in \mathbb{R}^{n_1 \times n_2 \times n_3} \). The unfolding of \( Z \) along mode-\( k \) for any \( k \in [3] \) is a random matrix \( Z[k] \in \mathbb{R}^{n_k \times n_{m_k}} \), where \( \{i,j,k\} = [3] \). A Gaussian tensor is a random tensor with a Gaussian distribution. For our convenience, we use \( Z(:,j,k) \) to denote the \((j,k)\)-th fibre of \( Z \) (notation borrowed from MATLAB) given \( j \in [n_2], k \in [n_3] \), and use \( A(i,:) \) and \( A(i,j,:) \) for the similar cases. Now we state an equivalent definition for a 3-order SND tensor.

**Definition 4.1.** A random tensor \( Z \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) is said to follow a standard normal distribution (SND), denoted \( Z \sim \mathcal{N}(0, I_{n_1}, I_{n_2}, I_{n_3}) \), if the following three conditions hold:

1. For all \( j \in [n_2], k \in [n_3] \), \( Z(:,j,k) \in \mathbb{R}^{n_1} \)'s are i.i.d. with \( \mathcal{N}(0_1, I_{n_1}) \).
2. For all \( i \in [n_1], k \in [n_3] \), \( Z(i,:,k) \in \mathbb{R}^{n_2} \)'s are i.i.d. with \( \mathcal{N}(0_2, I_{n_2}) \).
3. For all \( i \in [n_1], j \in [n_2] \), \( Z(i,j,:) \in \mathbb{R}^{n_3} \)'s are i.i.d. with \( \mathcal{N}(0_3, I_{n_3}) \).

Similar to the random matrix case, we have

**Theorem 4.2.** Let \( Z = (Z_{ijk}) \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) be a random tensor, \( n = n_1 n_2 n_3 \) and \( m_l = n/m_l \) for \( l = 1, 2, 3 \). The following items are equivalent:

1. \( Z \sim \mathcal{N}(0, I_{n_1}, I_{n_2}, I_{n_3}) \).
2. \( Z[k] \sim \mathcal{N}(0, I_{n_k}, I_{m_k}) \) for all \( k = 1, 2, 3 \).
3. \( \text{vec}(Z) \sim \mathcal{N}_n(0, I_n) \).
4. All the \( Z_{ijk} \)'s are i.i.d. with \( Z_{ijk} \sim \mathcal{N}(0, 1) \).
5. \( Z \times_1 \alpha^{(1)} \times_2 \alpha^{(2)} \times_3 \alpha^{(3)} \sim \mathcal{N}(0, 1/l) \), \( \forall \alpha^{(l)} \in S^{m_l-1}, l = 1, 2, 3 \).

**Proof.** We first show that (1) \( \implies \) (2). It suffices to show that \( Z[1] \sim \mathcal{N}_{n_1,m_1}(0, I_{n_1}, I_{m_1}) \) since \( X[1] \in \mathbb{R}^{n_1 \times m_1} \). By definition of a SND tensor, we know that \( Z(:,j,k) \sim \mathcal{N}_{n_1}(0, I_{n_1}) \) for all \( j, k \). On the other hand, for any \( i \in [n_1] \), we have \( \text{vec}(Z(i,:,:)) \sim \mathcal{N}_{m_1}(0, I_{m_1}) \) by \( Z(i,:,:) \sim \mathcal{N}_{n_2,n_3}(0, I_{n_2}, I_{n_3}) \) and Lemma 3.2. Thus \( Z[1](i,:) = (\text{vec}(Z(i,:,:)))^\top \sim \mathcal{N}_{m_1}(0, I_{m_1}) \). The implication (2) \( \implies \) (1) is obvious.

The equivalence (2) \( \iff \) (3) is directly from Lemma 3.2 and (3) \( \iff \) (4) is obvious. We now prove (3) \( \implies \) (5). Let \( \alpha^{(l)} \in S^{m_l-1} \) for \( l = 1, 2, 3 \) and let \( \beta = \alpha^{(2)} \otimes \alpha^{(3)} \). Then we have \( \beta \in S^{m_1-1} \). Furthermore, we have

\[
Z \times_1 \alpha^{(1)} \times_2 \alpha^{(2)} \times_3 \alpha^{(3)} = (\alpha^{(1)})^\top Z[1] \beta
\]

which is SND by Lemma 3.2. Thus we have \( Z \times_1 \alpha^{(1)} \times_2 \alpha^{(2)} \times_3 \alpha^{(3)} \sim \mathcal{N}(0, 1) \) for all \( \alpha^{(l)} \in S^{m_l-1} \) \( (l = 1, 2, 3) \). Thus (5) holds.
Conversely, we let (5) hold and want to show (2). For any given \( \alpha \in S^{n_1-1} \), we denote \( A(\alpha) = Z \times_1 \alpha \). Then \( A(\alpha) \in \mathbb{R}^{n_2 \times n_3} \). Furthermore, for any \( \alpha^{(2)} \in S^{n_2-1}, \alpha^{(3)} \in S^{n_3-1} \), we have

\[
Z \times_1 \alpha^{(1)} \times_2 \alpha^{(2)} \times_3 \alpha^{(3)} = (\alpha^{(2)})^\top A(\alpha) \alpha^{(3)} \in \mathcal{N}(0, 1)
\]

It follows by Lemma 3.2 that \( A(\alpha) \in \mathcal{N}_{n_2, n_3}(0, I_{n_2}, I_{n_3}) \) for every \( \alpha \in S^{n_1-1} \). Specifically, if we take \( \alpha = e_i \in \mathbb{R}^{n_1} \) to be the \( i \)th coordinate vector in \( \mathbb{R}^{n_1} \), then we have \( A(\alpha) = Z(i, :, :) \) (\( i \in [n_1] \)). Hence we have

\[
Z(i, :, :) \in \mathcal{N}_{n_2, n_3}(0, I_{n_2}, I_{n_3}), \forall i \in [n_1].
\]

This shows that all the slices of \( Z \) along the mode-1 is a SND matrix. We can also show that all the slices (along the other two directions) are SND matrices by employing the same technique. This complete the proof that all the five items are equivalent.

From Definition 4.1 we can see that a hypercubic random tensor \( Z \in \mathcal{T}_{3,n} \) (\( \mathcal{T}_{3,n} := \mathbb{R}^{n \times n \times n} \)) is SND if \( Zx^3 \sim \mathcal{N}(0, 1) \) for any unit vector \( x \in \mathbb{R}^n \). It is easy to see from Definition 4.1 that

**Corollary 4.3.** Let \( Z = (Z_{ijk}) \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) be a random tensor. Then \( Z \) is a SND tensor if and only if all \( Z_{ijk} \)'s are iid with \( Z_{ijk} \sim \mathcal{N}(0, 1) \).

Similar to the matrix case, we can also define a general Gaussian tensor. Denote \( N := n_1 n_2 \ldots n_m, m_k := N/n_k \). Let \( \mathcal{M} \in \mathbb{R}^{n_1 \times \ldots \times n_m} \) be a constant tensor and \( \Sigma_k \in \mathbb{R}^{n_k \times n_k} \) be a positive semidefinite matrix for each \( k \in [m] \). For any random tensor \( Z \in \mathbb{R}^{n_1 \times \ldots \times n_m} \), we denote \( Z^{(k)}(:, j) \) for the \( j \)th column (fiber) vector of \( Z^{(k)} \) where \( Z[k] \) is the flattened matrix of \( Z \) along mode \( k \). \( Z \) is called a Gaussian tensor with parameters \((M, \Sigma_1, \ldots, \Sigma_m)\) if \( Z^{(k)}(:, 1), Z^{(k)}(:, 2), \ldots, Z^{(k)}(:, m_k) \) are independent with

\[
Z^{(k)}(:, j) \sim \mathcal{N}_{n_k, m_k}(M^{(k)}(:, j), \lambda_k \Sigma_k), \quad \forall k \in [m]. \tag{4.1}
\]

We note that the above definition for a general Gaussian tensor reduces to a multivariate Gaussian distribution when \( m = 1 \) and to a Gaussian matrix when \( m = 2 \). Obviously a tensor \( Z \in \mathcal{T}_{m,n} \) is a Gaussian tensor if \( Zx^m \) follows a Gaussian distribution for every nonzero vector \( x \in \mathbb{R}^n \).

The following theorem tells that each flattened matrix of an 3-order SND tensor along any direction is a SND matrix.

**Theorem 4.4.** Let \( Z \in \mathbb{R}^{n_1 \times \ldots \times n_m} \) be a Gaussian tensor with \( I := n_1 \times \ldots \times n_m \). Then \( Z[k] \in \mathbb{R}^{n_k \times N_k} \) is a Gaussian matrix for each \( k \in [m] \), where \( Z[k] \) is the flattened matrix of \( Z \) along the \( k \)-mode.

Theorem 4.4 is directly from Definition 4.1.

Now we consider an \( m \)-order tensor \( \mathcal{A} \in \mathcal{T}(I) \) of size \( I := d_1 \times d_2 \times \ldots \times d_m \).
and denote \( a[k,j] \) the \( j \)th fibre of \( A \) along the \( k \)-mode where \( k \in [m] \) and \( j \) ranges from 1 to \( N_k := d_1d_2\ldots d_m/d_k \). We call \( A \) a standard Gaussian tensor if 
\[
a[k,j] \sim N_{d_k}(0, I_{d_k}) \quad \text{for each } k, j,
\]
and denote \( A \sim \mathcal{N}_I(0, \Sigma) \). A random tensor \( A \in T(I) \) is said to follow a Gaussian (or normal) distribution if 
\[
a[k,j] \sim N_{n_k}(M[k,j], \Sigma_k) \quad \text{for each } k, j.
\]
The following result also applies to a general case.

**Theorem 4.5.** Let \( Y = (Y_{ijk}) \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) be a 3-order random tensor, \( M = (M_{ijk}) \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) be a constant tensor and \( U_k \in \mathbb{R}^{n_k \times n_k} \) be invertible matrices. If
\[
Y = M + X \times_1 U_1 \times_2 U_2 \times_3 U_3 \quad \text{with } X \in \mathbb{R}^{n_1 \times n_2 \times n_3}
\]
where \( X \) is a standard normal tensor. Then \( A \) follows a Gaussian distribution with parameters \((M, \Sigma_{1\cdot\cdot\cdot m})\) where
\[
\Sigma_k = U_k U_k^T.
\]

**Conjecture 4.6.** A random tensor \( Y \in T_{m,n} \) follows a normal distribution \( Y \sim \mathcal{N}_{m,n}(M, \Sigma_1, \ldots, \Sigma_m) \) iff there exist some matrices \( U_k \) \((k \in [m])\) such that 
\[
Y = X \times_1 U_1 \times_2 U_2 \ldots \times_m U_m \quad \text{and} \quad X \text{ obeys a standard Gaussian distribution.}
\]

**Theorem 4.7.** A random tensor \( Y \in T_1 \) follows a normal distribution \( Y \sim \mathcal{N}_1(\Sigma_1, \ldots, \Sigma_m) \) iff
\[
\mathcal{Y}[k] \sim \mathcal{N}_{n_k,m_k}(M_k, \Sigma_k, \Omega_k)
\]
where \( \mathcal{Y}[k] \) is the unfolding of \( \mathcal{Y} \) along mode-\( k \) and \( \Omega_k := \Sigma_m \otimes \ldots \otimes \Sigma_{k+1} \otimes \Sigma_{k-1} \ldots \otimes \Sigma_1 \) for each \( k \in [m] \).

**Proof.** This is true for \( m = 1, 2 \) by the result on random vector and random matrix cases. Using the unfolding of tensor \( \mathcal{Y} \) and induction on \( m \), we easily get the result. \( \square \)

Let \( X \sim \mathcal{N}_1(0, I_{d_1}, \ldots, I_{d_m}) \) be a random following a standard normal distribution(SND). The density function of \( X \) is defined by
\[
f_X(T) = (2\pi)^{-d/2} \exp\left(-\frac{1}{2} \langle T, T \rangle \right)
\]
where \( d = d_1d_2\ldots d_m \).

**Theorem 4.8.** Let \( X \sim \mathcal{N}_1(0, I_{d_1}, \ldots, I_{d_m}) \) be a random following a SND. Then the CF of \( X \) is
\[
\phi_X(T) = \exp\left(-\frac{1}{2} \langle T, T \rangle \right)
\]
where \( T = (T_{i_1\ldots i_m}) \in \mathbb{R}^{d_1 \times d_2 \times \ldots \times d_m} \).
Proof. By definition of CF, we have
\[
\phi_{X}(T) = E[\exp(\langle T, X \rangle)] \\
= E[\exp(\sum_{i_1,\ldots,i_m} T_{i_1\ldots i_m} X_{i_1\ldots i_m})] \\
= E[\prod_{i_1,\ldots,i_m} \exp(\langle i(T_{i_1\ldots i_m} X_{i_1\ldots i_m}) \rangle)] \\
= \prod_{i_1,\ldots,i_m} E[\exp(\langle i(T_{i_1\ldots i_m} X_{i_1\ldots i_m}) \rangle)] \\
= \prod_{i_1,\ldots,i_m} \exp(-\frac{1}{2}T_{i_1\ldots i_m}^2) \\
= \exp[-\frac{1}{2} \sum_{i_1,\ldots,i_m} T_{i_1\ldots i_m}^2] \\
= \exp[-\frac{1}{2} \langle T, T \rangle]
\]

\[
\square
\]

Compliance with ethical standards

Conflicts of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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