SUPERSYMMETRY AND THE 
ATIYAH-SINGER INDEX THEOREM I: 
Peierls Brackets, Green’s Functions, and a 
Supersymmetric Proof of the Index 
Theorem 

Ali Mostafazadeh 

Center for Relativity 
The University of Texas at Austin 
Austin, Texas 78712, USA 

March 28, 2022 

Abstract 

The Peierls bracket quantization scheme is applied to the supersymmetric system corresponding to the twisted spin index theorem. A detailed study of the quantum system is presented, and the Feynman propagator is exactly computed. The Green’s function methods provide a direct derivation of the index formula.
1 Introduction

The Peierls bracket quantization is explained and applied to many interesting examples in [1]. In particular, it is used to quantize a supersymmetric system described by the Lagrangian:

\[ L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} g_{\mu\nu} \dot{\psi}_\alpha^\mu D_{\mu} \psi_\alpha^\nu + \frac{1}{8} R_{\mu\nu\sigma\tau} \psi_\alpha^\mu \psi_\beta^\nu \psi_\gamma^\sigma \psi_\delta^\tau \]  

(1)

\[ \alpha, \beta = 1, 2 \quad \mu, \nu, \sigma, \tau = 1, ..., m. \]

The configuration space of (1) is an \((m, 2m)\)-dimensional supermanifold [1] with \(x\) and \(\psi\) denoting its bosonic (commuting) and fermionic (anticommuting) coordinates, respectively. The quantization of (1) gives rise to a rigorous supersymmetric proof of the Gauss-Bonnet(-Chern-Avez) theorem, [2, Part I, p. 395].

The Gauss-Bonnet theorem is the best known example of an index theorem. It is also called the index theorem for the deRham complex [3]. The idea of using supersymmetric quantum mechanics to give proofs of index theorems was originally suggested by Witten who discovered the relation between the two subjects in his study of supersymmetry breaking, [4]. Windey [5] and Alvarez-Gaume [6, 7, 8] provided the first supersymmetric proofs of the index theorem. Few years later, WKB methods were applied to give an alternative approach to the path integral evaluation of the index by Mañes and Zumino, [9]. The same was achieved in a concise paper by Goodman [10] who used the mode expansion techniques. Among other remarkable works on this subject are a difficult paper by Getzler [11] and a superspace approach by Friedan and Windey [12].

The present paper uses the basic Green’s function methods discussed in [1] to arrive at a proof of the index theorem. The strategy is the same as the earlier supersymmetric proofs, but the methods differ appreciably. In particular, the Feynman propagators are computed exactly for the first time.
In Section 2, the Atiyah-Singer index theorem and its relation to supersymmetric quantum mechanics are reviewed. Section 3 starts with a brief description of the Peierls bracket quantization. The superclassical system corresponding to the twisted spin complex is then introduced and its quantization is performed. The quantum mechanical Hamiltonian involves a scalar curvature factor. Section 4 includes a discussion of the quantum systems associated with the spin and twisted spin complexes. In particular, it is shown that the quantization of the supersymmetric charge leads to its identification with the corresponding Dirac operator. In Section 5, the path integral representation of the index is presented and the Green’s function methods are discussed. Sections 6 and 7 are devoted to the proofs of the spin and twisted spin index theorems, respectively. Section 8 includes the final remarks.

Throughout this paper \( \hbar \) will be set to 1, except in Section 3.

2 The Atiyah-Singer Index Theorem and Supersymmetric Quantum Mechanics

The Atiyah-Singer index theorem is one of the most substantial achievements of modern mathematics. It has been a developing subject since its original proof by Atiyah and Singer, [13]. A clear exposition of the index theorem is given in the classic paper of Atiyah, Bott, and Patodi, [14]. For the historical origins of the index theory, see [15]. Several mathematical proofs of the index theorem and its variations and generalizations are available in the literature, [16, 17, 18, 19, 20]. A common feature of all these proofs is the use of K-theory, [21]. In particular, the original cobordism proof [13], the celebrated heat kernel proof [14, 18], and the supersymmetric proofs, [3, 8], of the index theorem are based on a result of K-theory which reduces the proof of the “general” index theorem to the special case of twisted signature or alternatively twisted spin index theorems, [14, 17, 18, 19].
In “general”, one has an elliptic differential operator,

\[ D : C^\infty(V_+) \longrightarrow C^\infty(V_-) \]  

(2)

between the spaces of sections of two hermitian vector bundles \( V_+ \) and \( V_- \) over a closed \(^2\) Riemannian manifold, \( M \). Alternatively one can speak of the short elliptic complex:

\[ 0 \longrightarrow C^\infty(V_+) \xrightarrow{D} C^\infty(V_-) \longrightarrow 0. \]

The index theorem provides a closed formula for the analytic index of \( D \). The latter is defined by

\[ \text{index}(D) := \dim[\ker(D)] - \dim[\text{coker}(D)]. \]

(3)

The symbols \( \dim, \ker, \) and \( \text{coker} \) are abbreviations of \( \text{dimension}, \ker, \) and \( \text{cokernel} \), respectively. Indices of elliptic operators are of great interest in mathematics and physics because they are topological invariants.

The “general” index formula computes the index as an integral involving some characteristic classes associated with the vector bundles \( V_\pm \), the base manifold \( M \), and the operator \( D \). The twisted (or generalized) spin index theorem is a special case.

Let \( M \) be a closed \( m = 2l \)-dimensional spin manifold, \(^3\). Let \( S = S^+ \oplus S^- \) denote the spin bundle of \( M \). The sections of \( S^\pm \) are called the \( \pm \) chirality spinors. Let \( D : C^\infty(S) \to C^\infty(S) \) be the Dirac operator \(^{17,18,3,22}\), and \( \partial := D |_{C^\infty(S^+)} \). The short complex

\[ 0 \longrightarrow C^\infty(S^+) \xrightarrow{\partial} C^\infty(S^-) \longrightarrow 0 \]  

(4)

is called the spin complex.

---

1A hermitian vector bundle is a complex vector bundle which is endowed with a hermitian metric and a compatible connection \(^{14}\).

2compact, without boundary
Theorem 1 Let $\mathcal{D}$ be as in (4), and define the so-called $\hat{A}$-genus density:

$$
\hat{A}(M) := \prod_{i=1}^{l} \left[ \frac{\Omega_i}{\sinh(\frac{\Omega_i}{4\pi})} \right].
$$

(5)

where, $\Omega_i$ are the 2-forms defined by block-diagonalizing the curvature 2-form $\Omega$:

$$
\Omega := \left( \frac{1}{2}R_{\mu\nu\gamma\lambda}dx^\gamma \wedge dx^\lambda \right) =: \text{diag} \left( \begin{bmatrix} 0 & \Omega_i \\ -\Omega_i & 0 \end{bmatrix} : i = 1 \cdots l \right).
$$

(6)

Then,

$$
\text{index}(\mathcal{D}) = \int_M \left[ \hat{A}(M) \right]_{\text{top}}.
$$

(7)

In (7), “top” means that the highest rank form in the power series expansion of (5) is integrated.

Let $V$ be a hermitian vector bundle with fiber dimension $n$, base manifold $M$, and connection 1-form $A$. Then, the operator:

$$
\mathcal{D}_V : C^\infty(S^+ \otimes V) \to C^\infty(S^+ \otimes V)
$$

defined by

$$
\mathcal{D}_V(\Psi \otimes v) := \mathcal{D}(\Psi) \otimes v + (-1)^\Psi \Psi \otimes D_A(v),
$$

(8)

is an elliptic operator, called the twisted Dirac operator. In (8), $\Psi \in C^\infty(S^+) \Leftrightarrow (-1)^\Psi = \pm 1$, $v \in C^\infty(V)$, and $D_A$ is the covariant derivative operator defined by $A$.\cite{[18]} Also define

$$
\mathcal{D}_V := \mathcal{D}_V \big|_{C^\infty(S^+) \otimes C^\infty(V)}.
$$

(9)

The twisted spin complex is the following short complex:

$$
0 \to C^\infty(S^+ \otimes V) \xrightarrow{\mathcal{D}_V} C^\infty(S^- \otimes V) \to 0.
$$

(10)

\text{The block-diagonalization is always possible since } \Omega \text{ is antisymmetric.}
Theorem 2 Let $\partial V$ be as in (9), and

$$ch(V) := \text{tr} \left[ \exp \left( \frac{iF}{2\pi} \right) \right]$$

be the Chern character of $V$. Then,

$$\text{index} (\partial V) = \int_M \left[ ch(V) \hat{A}(M) \right]_{\text{top}}.$$  \hspace{1cm} (12)

In (11), $F$ is the curvature 2-form of the connection 1-form $A$ (written in a basis of the structure Lie algebra of $V$):

$$F := \left( \frac{1}{2} F^{ab}_{\chi\gamma} dx^\chi \wedge dx^\gamma \right).$$  \hspace{1cm} (13)

Throughout this paper, the Greek indices refer to the coordinates of $M$, they run through $1, \ldots, m = \text{dim}(M)$, and the indices from the beginning of the Latin alphabet refer to the fibre coordinates of $V$ and, hence, run through $1, \ldots, n$.

The relevance of the index theory to supersymmetry has been discussed in almost every article written in this subject in the past ten years. The idea is to realize the parallelism between the constructions involved in the index theory, (2), and the supersymmetric quantum mechanics. In the latter, the Hilbert (Fock) space is the direct sum of the spaces of the bosonic and the fermionic state vectors. These correspond to the spaces of sections of $V_\pm$ in (8). Moreover, the supersymmetric charge $Q$ plays the role of the elliptic operator $D$, and the Hamiltonian $H$ is the analog of the Laplacian $\Delta$ of $D$. One has:

$$\Delta := \{ D, D^\dagger \},$$

and

$$H = \frac{1}{2} \{ Q, Q^\dagger \}. \hspace{1cm} (14)$$

Here, “$\dagger$” denotes the adjoint of the corresponding operator. (14) is also known as the superalgebra condition. One can also define selfadjoint super-
symmetric charges $\hat{Q}$, $Q_\alpha$, in terms of which (14) becomes:

$$H = Q_\alpha^2, \forall \alpha.$$  \hspace{1cm} (15)

The next step is to recall

$$\text{coker}(D) = \ker(D^\dagger),$$

$$\ker(\Delta) = \ker(D) \oplus \ker(D^\dagger)$$

and use (3), and (11) to define:

$$\text{index}_W := n_{b,0} - n_{f,0}.$$  

Here “$W$” refers to Witten [4] and $n_{b,0}$ and $n_{f,0}$ denote the number of the zero-energy bosonic and fermionic eigenstates of $H$. Realizing that due to supersymmetry any excited energy eigenstate has a superpartner, one has the following set of equalities:

$$\text{index}_W = n_b - n_f = tr[(-1)^f]$$

$$= tr[(-1)^f e^{-i\beta H}]$$

$$=: \text{str}[e^{-i\beta H}].$$  \hspace{1cm} (16)

In (16), $n_b$ and $n_f$ denote the number of the bosonic and fermionic energy eigenstates, and $f$ is the fermion number operator. See [4, 23, 8] for a more detailed discussion of (16).

The appearance of $e^{-i\beta H}$ in (16) is quite interesting. It had, however, been noticed long before supersymmetry was introduced in physics. The heat kernel proof of the index theorem is essentially based on (14), [14, 19, 18].

---

4 $\alpha = 1, \ldots, 2N$, where $N$ is the number of nonselfadjoint charges (type N-SUSY).

5 In the heat kernel proof, one computes the index using the formula:

$$\text{index} = tr[e^{-\beta \Delta_+}] - tr[e^{-\beta \Delta_-}],$$

where $\Delta_+ := D^\dagger D$ and $\Delta_- := DD^\dagger.$
The major advantage of the supersymmetric proofs is that one can compute $\text{index}_W$ using its path integral representation. In particular, since $\text{index}_W$ is independent of $\beta$, the WKB approximation, i.e. the first term in the loop expansion, yields the index immediately.

3 The Superclassical System and Its Quantization

Consider a superclassical system described by the action functional $S = S[\Phi]$, with $\Phi = (\Phi^i)$ denoting the coordinate (field) variables. The dynamical equations are given by $\delta S = 0$, i.e.

$$S_{,i} := S[\Phi] \frac{\delta}{\delta \Phi^i(t)} = 0.$$  

(17)

Throughout this paper the condensed notation of [1] is used where appropriate. The following example demonstrates most of the conventions:

$$i_j S_{,ji} G_{jk} = \int dt' \left( \frac{\delta}{\delta \Phi^i(t)} S[\Phi] \frac{\delta}{\delta \Phi^j(t')} \right) G^{ij}(t', t).$$

In general, the second functional derivatives of $S$ “are” second order differential operators [2], e.g. see (30) below. Let $G_{\pm i'k''}$ denote the corresponding advanced and retarded Green’s functions:

$$i_j S_{,j} G_{\pm i'k''} = -\delta(t - t'') \delta^k_i.$$  

(18)

where $\delta(t - t'')$ and $\delta^k_i$ are the Dirac and Kronecker delta functions, respectively. Furthermore, define:

$$\tilde{G}^{jk'} := G^{jk'} - G^{-jk'}.$$  

(19)

These are called the Jacobi operators.

8
Then, the Peierls bracket \([24, 1]\) of any two scalar fields, \(A\) and \(B\), of \(\Phi\) is defined by:

\[
(A, B) := A, \tilde{G}^{ij}j', B.
\]

(20)

In particular, one has

\[
(\Phi^i, \Phi^{j'}) := \tilde{G}^{ij}.
\]

(21)

The Green’s functions \(G^{+ij'}\) satisfy the following reciprocity relation \([1]\),

\[
G^{-ij'} = (-1)^{ij'} G^{+j'i}.
\]

(22)

In (22), the indices \(i\) and \(j'\) in \((-1)^{ij'}\) are either 0 or 1 depending on whether \(\Phi^i(\Phi^{j'})\) is a bosonic or fermionic variable, respectively.

The quantization is performed by promoting the superclassical quantities to the operators acting on a Hilbert space and forming a superalgebra defined by the following supercommutator:

\[
[A, B]_{\text{super}} := i\hbar (A, B),
\]

(23)

where the right hand side is defined up to factor ordering.

The superclassical system of interest is represented by the Lagrangian \([1, 1]\):

\[
L = \left[ \frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + \frac{i}{2} g_{\lambda\gamma}(x) \psi^\lambda \frac{D}{dt} \psi^\gamma \right]_1 + \kappa \left[ i \eta^a \left( \eta^a + \dot{x}^\sigma A^{ab}(x) \eta^b \right) + \frac{1}{2} F_{ab} \psi^\lambda \psi^\gamma \eta^a \eta^b \right]_2 + \left[ \frac{a}{2} \eta^a \eta^a \right]_3.
\]

(24)

In (24), \(x^\mu\) are the bosonic variables corresponding to the coordinates of \(M\). \(g_{\mu\nu}\) are components of the metric tensor on \(M\). \(\psi^\lambda\) and \(\eta^a\) are fermionic real and complex variables associated with the bundles \(S\) and \(V\), respectively. \(A^{ab}\) and \(F_{ab}^{\lambda\gamma}\) are the components of the connection 1-form and the curvature 2-form on \(V\) (written in an orthonormal basis of the structure Lie algebra). One has the following well-known relation in the Lie algebra:

\[
F_{\mu\nu} = A_{\nu, \mu} - A_{\mu, \nu} + [A_\mu, A_\nu].
\]

(25)
Both $A_\mu$ and $F_{\mu\nu}$ in (25) are antihermitian matrices. This makes $L$ real up to total time derivatives. In (24), $\kappa = 0,1$ correspond to switching off and on of the twisting, respectively. $\alpha$ is a scalar parameter whose utility will be discussed in Section 4. $\beta$ is the time parameter: $t \in [0, \beta]$. Finally, “dot” means ordinary time derivative $\frac{d}{dt}$, and $\overline{\frac{D}{dt}}$ denotes the covariant time derivative defined by the Levi Civita connection, e.g.

$$D_{\mu} \psi^\gamma := \dot{\psi}^\gamma + \dot{x}^\mu \Gamma^\gamma_{\mu\theta} \psi^\theta.$$

The following set of infinitesimal supersymmetric transformations,

$$\delta x^\mu = i \psi^\mu \delta \xi,$$
$$\delta \psi^\gamma = \dot{\psi}^\gamma \delta \xi,$$
$$\delta \eta^a = i A^a_{\gamma} \psi^\gamma \eta^b \delta \xi,$$
$$\delta \eta^a* = -i A^b_{\gamma} \psi^\gamma \eta^b* \delta \xi,$$

$\delta \xi := \text{an infinitesimal fermionic variable}$

leaves the action

$$S := \int_0^\beta L dt$$

invariant. Hence, the system is supersymmetric. It is easy to see that the first two equations in (23) leave $L_1 := [...]_1$ in (24) invariant. Thus, for $\kappa = \alpha = 0$, one has a supersymmetric subsystem. This subsystem will be used to compute the index of spin complex (the Dirac $\hat{A}$ genus) in Section 5.

$L_1$ can also be obtained from (1) by reducing the system of (1) by setting $\psi_1^\mu = \psi_2^\mu$, for all $\mu$.

The Dynamical equations (17) are:

$$S_{\tau} = -g_{\mu\tau} \frac{D}{dt} \dot{x}^\mu + i \Gamma^\gamma_{\epsilon\tau} \psi^\gamma \frac{D}{dt} \psi^\epsilon + \frac{i}{2} R_{\epsilon\gamma\tau\sigma} \dot{x}^\sigma \psi^\gamma \psi^\epsilon +$$
$$+ \kappa \left[ -i A^a_{\tau} (\dot{\eta}^a* \eta^b + \eta^a* \dot{\eta}^b) + i \dot{\eta}^\sigma (A^a_{\sigma,\tau} - A^a_{\tau,\sigma}) \eta^a* \eta^b +$$
$$+ \frac{1}{2} F^a_{\epsilon\gamma,\tau} \psi^\epsilon \psi^\gamma \eta^a* \eta^b \right] = 0$$

(28)
\[ S_{\epsilon} = g_{\epsilon\gamma} \frac{D}{dt} \psi^\gamma + \kappa [iF_{\gamma\epsilon}^a \psi^\gamma \eta^a \eta^b] = 0 \]

\[ S_{\alpha} = \kappa [i\eta^a + i\dot{x}^\sigma A_{\sigma}^a \eta^b + \frac{1}{2}F_{\gamma\epsilon}^a \psi^\gamma \eta^b] + \frac{\alpha}{\beta} \eta^a = 0 \]

\[ S_{b^*} = \kappa [-i\dot{\eta}^{b*} + i\dot{x}^\sigma A_{\sigma}^{a*} \eta^{b*} + \frac{1}{2}F_{\gamma\epsilon}^a \psi^\gamma \psi^a \eta^{b*}] + \frac{\alpha}{\beta} \eta^{b*} = 0 \]

where the indices from the beginning of the Greek alphabet (\(\gamma, \delta, \epsilon, \theta, \eta\)) label \(\psi\)'s and those of the middle of the Greek alphabet (\(\kappa, \cdots\)) label \(x\)'s, e.g.

\[ S_{,\tau} := \frac{\delta}{\delta x^\tau} \quad \text{and} \quad S_{,\epsilon} := \frac{\delta}{\delta \psi^\epsilon}. \]

Similarly,

\[ S_{,\alpha} := \frac{\delta}{\delta \eta^a} \quad \text{and} \quad S_{,a^*} := \frac{\delta}{\delta \eta^{a*}}. \]

The supersymmetric charge \(Q\) corresponding to (26) is given by

\[ Q \propto g_{\gamma\mu} \psi^\gamma \dot{x}^\mu. \]  \hspace{1cm} (29)

The second functional derivatives of the action are listed in the following:

\[ \tau, S_{,\tau'} = \left[ -g_{\tau\tau} \frac{\partial^2}{\partial t^2} + \left( -2\Gamma_{\tau\mu\nu} \dot{x}^\nu + i\frac{2}{\kappa} R_{\epsilon\gamma\tau\tau} \psi^\gamma \psi^\epsilon + i\kappa \left( A_{\alpha}^{ab} - A_{\alpha}^{ab} \right) \eta^a \eta^b + i\Gamma_{\gamma\tau\Gamma} \psi^\gamma \psi^\epsilon \right) \frac{\partial}{\partial t} \right] \delta(t - t') \]

\[ \tau, S_{,\epsilon'} = \left[ i\Gamma_{\gamma\tau\epsilon} \psi^\epsilon \frac{\partial}{\partial t} + C_{\epsilon\gamma} \right] \delta(t - t') \]

\[ \tau, S_{,\alpha'} = \kappa \left[ -iA_{\epsilon}^{a*} \eta^a \frac{\partial}{\partial t} + C_{\epsilon\alpha} \right] \delta(t - t') \]

\[ \epsilon, S_{,\tau'} = \left[ i\Gamma_{\epsilon\mu\tau} \psi^\mu \frac{\partial}{\partial t} + C_{\epsilon\tau} \right] \delta(t - t') \]

\[ \epsilon, S_{,\epsilon'} = \left[ ig_{\epsilon\gamma} \frac{\partial}{\partial t} + C_{\epsilon\gamma} \right] \delta(t - t') \]

\[ \epsilon, S_{,\alpha'} = \kappa \left[ iA_{\epsilon}^{a*} \eta^a \frac{\partial}{\partial t} + C_{\epsilon\alpha} \right] \delta(t - t') \]
\begin{equation}
\begin{aligned}
\epsilon, S_{a'} &= \kappa [C_{\epsilon a}] \delta(t - t') \\
\epsilon, S_{a''} &= \kappa [C_{\epsilon a''}] \delta(t - t') \\
b, S_{a', \gamma} &= \kappa \left[-i A_{\pi}^{cb} \eta^{c} \frac{\partial}{\partial t} + C_{b \pi}\right] \delta(t - t') \\
b, S_{a, \gamma} &= \kappa [C_{b \gamma}] \delta(t - t') \\
b, S_{a'} &= \kappa [C_{ba}] \delta(t - t') \\
b, S_{a''} &= \kappa \left[i \delta_{ba} \frac{\partial}{\partial t} + C_{ba''}\right] \delta(t - t') \\
b, S_{a', \sigma} &= \kappa \left[i A_{\pi}^{bc} \eta^{c} \frac{\partial}{\partial t} + C_{b \pi}\right] \delta(t - t') \\
b, S_{a, \sigma} &= \kappa [C_{b \sigma}] \delta(t - t') \\
b, S_{a'} &= \kappa \left[i \delta_{ba} \frac{\partial}{\partial t} + C_{ba'}\right] \delta(t - t') \\
b, S_{a''} &= \kappa [C_{b a''}] \delta(t - t').
\end{aligned}
\end{equation}

In (30) \[C_{ij}\]'s \[\Gamma\] are terms which do not involve any time derivative. These terms do not actually contribute to the equal-time commutation relations of interest, (38). However, they will contribute in part to \(s\text{det}[G^{ij}]\) in Section 5. One has:

\[
\begin{align*}
C_{\tau \pi} &= -g_{\mu \nu, \pi} \dot{x}^\mu - \Gamma_{\mu \nu, \pi} \dot{x}^\nu \dot{x}^\mu + i \Gamma_{\epsilon \gamma, \pi} \dot{\psi}^\epsilon \dot{\psi}^\gamma + i (\Gamma_{\epsilon \gamma, \pi} \Gamma_{\delta, \pi}) \dot{x}^\sigma \dot{\psi}^\gamma \dot{\psi}^\delta + \\
&\quad + \frac{i}{2} R_{\epsilon \gamma, \pi} \dot{x}^\sigma \dot{\psi}^\epsilon \dot{\psi}^\gamma + \kappa \left[-i A_{\epsilon, \pi}^{cd} (\eta^{c \epsilon} \eta^{d} + \eta^{c} \eta^{d \epsilon}) + \\
&\quad + i (A_{\epsilon, \pi}^{cd} - A_{\epsilon, \pi}^{cd}) \dot{x}^\sigma \eta^{c \epsilon} \eta^{d} + \frac{1}{2} F_{\epsilon \gamma, \pi} \dot{\psi}^\epsilon \dot{\psi}^\gamma \eta^{c \epsilon} \eta^{d}\right] \\
C_{\tau \gamma} &= -i \Gamma_{\epsilon \gamma, \pi} \dot{\psi}^\epsilon + i (\Gamma_{\epsilon \delta} \Gamma_{\gamma}^{\delta} - \Gamma_{\epsilon \gamma} \Gamma_{\delta, \pi}) \dot{x}^\sigma \dot{\psi}^\delta + i R_{\epsilon, \delta} \dot{x}^\sigma \dot{\psi}^\epsilon + \\
&\quad + \kappa \left[F_{\epsilon \gamma, \pi} \dot{\psi}^\epsilon \eta^{c \epsilon} \eta^{d}\right] \\
C_{\tau a} &= -i A_{\epsilon, \pi}^{ca} \dot{\eta}^c + i (A_{\epsilon, \pi}^{ca} - A_{\epsilon, \pi}^{ca}) \dot{x}^\sigma \eta^{c \epsilon} + \frac{1}{2} F_{\epsilon \pi}^{ca} \dot{\psi}^\epsilon \dot{\psi}^\gamma \eta^{c \epsilon} \\
C_{\tau a^*} &= i A_{\epsilon, \pi}^{ac} \dot{\eta}^c - i (A_{\epsilon, \pi}^{ac} - A_{\epsilon, \pi}^{ac}) \dot{x}^\sigma \eta^{c \epsilon} + \frac{1}{2} F_{\epsilon \pi}^{ac} \dot{\psi}^\epsilon \dot{\psi}^\gamma \eta^{c \epsilon} \\
C_{\epsilon \pi} &= i \Gamma_{\epsilon, \delta} \dot{x}^\mu \dot{\psi}^\delta + i g_{\epsilon, \delta} \dot{\psi}^\delta - \kappa \left[F_{\epsilon, \pi}^{\delta c} \dot{\psi}^\delta \eta^{c \epsilon} \eta^{d}\right]
\end{align*}
\]

\(\Phi^j \in (x^\mu, \dot{x}^\mu, \psi^\gamma, \eta^\epsilon, \eta^\sigma, \eta^{d \epsilon})\) are generic variables.
The advanced Green’s functions $G^{+ij'}$ are calculated using (18) and (30). The results are listed below:

\[
C_{e\gamma} := i\Gamma_{e\gamma\mu'}\xi^\mu + \kappa F_{e\gamma}^{ab}\eta^a\eta^b \\
C_{ea} := -F_{dx}^{ca}\psi^d\eta^a \\
C_{ea^*} := F_{d}\psi^d\eta^c \\
C_{b\pi} := -iA_{a^b}^{cb}x^\sigma\eta^c - \frac{1}{2}F_{\gamma\delta}\psi^\gamma\psi^\delta\eta^c \\
C_{b\gamma} := F_{\delta\gamma}\psi^\delta\eta^c \\
C_{ba} := 0 \\
C_{ba^*} := -iA_{a^b}^{cb}x^\sigma - \frac{1}{2}F_{\delta\theta}\psi^\theta - \frac{\alpha}{n\beta}F_{ab} \\
C_{b^*\pi} := iA_{a^b}^{bc}x^\sigma\eta^c + \frac{1}{2}F_{\gamma\pi}\psi^\gamma\psi^\pi\eta^c \\
C_{b^*\gamma} := -F_{\delta\gamma}\psi^\delta\eta^c \\
C_{b^*a} := iA_{a^b}^{cb}x^\sigma + \frac{1}{2}F_{\delta\theta}\psi^\theta + \frac{\alpha}{n\beta}F_{ba} \\
C_{b^*a^*} := 0.
\]

The advanced Green’s functions $G^{+ij'}$ are calculated using (18) and (30). The results are listed below:

\[
G^{+\xi\pi'} = \theta(t' - t) \left[ -g^{\xi\pi'}(t - t') + \frac{1}{2}g^{\xi\pi'} \left( 2\Gamma_{\tau\nu\mu'}\xi^\mu' - \frac{i}{2}R_{\tau\nu\gamma\epsilon'}\psi^\gamma\psi^\epsilon' \right) \\
+ i\kappa F_{\tau'\nu'}^{\alpha\epsilon'}\eta^{\alpha'}\eta^{\epsilon'} \right] g^{\nu'\pi'}(t - t')^2 + O(t - t')^3 \right] \\
G^{+\xi\gamma'} = \theta(t' - t) \left[ g^{\xi\gamma'} \Gamma_{\tau\nu\gamma\epsilon'}(t - t') + O(t - t')^2 \right] \\
G^{+\xi a'} = \theta(t' - t) \left[ g^{\xi\gamma'} A_{\tau\gamma\epsilon'}^{a\epsilon'}\eta^{\epsilon'}(t - t') + O(t - t')^2 \right] \\
G^{+\xi a^*} = \theta(t' - t) \left[ g^{\xi\gamma'} A_{\tau\gamma\epsilon'}^{a\epsilon'}\eta^{\epsilon'}(t - t') + O(t - t')^2 \right] \\
G^{+\epsilon\pi'} = \theta(t' - t) \left[ g^{\pi\epsilon'} \Gamma_{\tau\epsilon\gamma'}(t - t') + O(t - t')^2 \right] \\
G^{+\epsilon\gamma'} = \theta(t' - t) \left[ -ig^{\epsilon\gamma'} + O(t - t') \right] \\
G^{+\epsilon a'} = \theta(t' - t) \left[ O(t - t') \right] \\
G^{+\epsilon a^*} = \theta(t' - t) \left[ O(t - t') \right] \\
G^{+b\pi'} = \theta(t' - t) \left[ g^{\pi\epsilon'} A_{\tau\epsilon\gamma'}^{b\epsilon'}\eta^{\epsilon'}(t - t') + O(t - t') \right] \\
G^{+b\gamma'} = \theta(t' - t) \left[ O(t - t') \right] \\
G^{+b a'} = \theta(t' - t) \left[ O(t - t') \right] \\
G^{+b a^*} = \theta(t' - t) \left[ O(t - t') \right].
\]
The interesting equal-time Peierls brackets are the following:

\[ G^{ba'c'} = \theta(t' - t) \left[ -\frac{i}{\hbar}\delta^{ba'} + O(t - t') \right] \]
\[ G^{b'c'a} = \theta(t' - t) \left[ -g^{a'c'} A^{ba}_c(t - t') + O(t - t')^2 \right] \]
\[ G^{b'c'a} = \theta(t' - t) \left[ -\frac{i}{\hbar}\delta^{c'a'} + O(t - t') \right] \]
\[ G^{b'c'a} = \theta(t' - t) \left[ -\frac{i}{\hbar}\delta^{a'c'} + O(t - t') \right] \]

The Green's functions \( G^{-ij'} \) and \( \tilde{G}^{ij'} \) are then obtained using (22) and (19), respectively. Substituting the latter in (21) leads to the following Peierls brackets:

\[ (x^\xi, x^{\pi'}) = -g^{\xi\pi'}(t - t') + \frac{1}{2}g^{\xi\pi'} \left( 2\Gamma_{\gamma'\nu'\nu'}\dot{x}^{\nu'} - \frac{i}{2}R_{\nu'\gamma'\nu'} \psi^{\gamma'} \psi^{\nu'} + \right. \]
\[ \left. -i\kappa F_{\nu'\nu}^a \eta^{\nu'} \eta^{\nu'} \right) g^{\nu'\pi'}(t - t')^2 + O(t - t')^3 \]
\[ (x^\xi, \psi^{\gamma'}) = g^{\xi\gamma'} \Gamma_{\gamma'\beta'} \dot{\psi}^{\beta'} + O(t - t')^2 \]
\[ (x^\xi, \eta^{a'}) = g^{\xi\gamma'} A^{a'c'} \eta^{\gamma'}(t - t') + O(t - t')^2 \]  \( (33) \)
\[ (x^\xi, \eta^{a^*}) = -g^{\xi\gamma'} A^{a^*c'} \eta^{\gamma'}(t - t') + O(t - t')^2 \]
\[ (\psi^{\gamma}, \psi^{\gamma'}) = -ig^{\gamma'\gamma} + O(t - t') \]
\[ (\eta^b, \eta^{a^*}) = -\frac{i}{\hbar}\delta^{b'a'} + O(t - t') \].

Other possible Peierls brackets are all of the order \((t - t')\) or higher. Differentiating the necessary expressions in (33) with respect to \( t \) and \( t' \), one arrives at the Peierls brackets among the coordinates \((x, \psi, \eta, \eta^*)\) and their time derivatives. The interesting equal-time Peierls brackets are the following:

\[ (x^\xi, x^{\pi'}) = (x^\xi, \psi^{\gamma'}) = (x^\xi, \eta^{a'}) = (x^\xi, \eta^{a^*}) = 0 \]
\[ (\psi^{\gamma}, \eta^{a'}) = (\psi^{\gamma}, \eta^{a^*}) = (\eta^b, \eta^{a'}) = (\eta^b, \eta^{a^*}) = 0 \]
\[ (\psi^{\gamma}, \psi^{\gamma'}) = -ig^{\gamma'\gamma} \]
\[ (\eta^a, \eta^{b^*}) = -\frac{i}{\hbar}\delta^{ab} \]
\[ (\dot{x}^\xi, x^{\pi'}) = -g^{\xi\pi} \]

14
\[
(\dot{x}^\xi, \dot{x}^\pi) = g^{\xi\pi} \left[ (\Gamma_{\nu\tau\mu} - \Gamma_{\tau\nu\mu}) \dot{x}^\mu \right. 
+ \left. \frac{i}{2} R_{\tau\nu\epsilon\gamma} \psi^\epsilon \psi^\gamma + i \kappa F^{ab}_{\tau\nu} \eta^{a*} \eta^b \right] g^{\nu\pi}
\]
\[
(\dot{x}^\xi, \psi^\gamma) = g^{\xi\gamma} \Gamma^\tau_{\tau\epsilon} \psi^\epsilon
\]
\[
(\dot{x}^\xi, \eta^a) = g^{\xi a} A^c_{\tau} \eta^c
\]
\[
(\dot{x}^\xi, \eta^{a*}) = -g^{\xi a} A^{ca}_{\tau} \eta^{c*}.
\]

The next step is to define the appropriate momenta conjugate to \(x^\nu\). The canonical momenta are given by:
\[
p^{\text{(canonical)}}_{\nu} := \frac{\partial}{\partial \dot{x}^\nu} = g_{\nu\mu} \dot{x}^\mu + \frac{i}{2} \Gamma_{\epsilon\gamma\nu} \psi^\epsilon \psi^\gamma + i \kappa A^{ab}_{\nu} \eta^{a*} \eta^b.
\]

A more practical choice is provided by:
\[
p_{\nu} := g_{\nu\mu} \dot{x}^\mu.
\]

This choice together with the use of (20) and (34) lead to:
\[
(p_{\nu}, x^\mu) = -\delta_{\nu}^\mu
\]
\[
(p_{\nu}, \psi^\gamma) = \Gamma^\gamma_{\nu\epsilon} \psi^\epsilon
\]
\[
(p_{\nu}, \eta^a) = A^c_{\nu} \eta^c
\]
\[
(p_{\nu}, \eta^{a*}) = -A^{ca}_{\nu} \eta^{c*}
\]
\[
(p_{\nu}, p_{\mu}) = \frac{i}{2} R_{\nu\epsilon\gamma\mu} \psi^\epsilon \psi^\gamma + \kappa \left[ i F^{ab}_{\nu\mu} \eta^{a*} \eta^b \right].
\]

The quantization is performed via (23). Enforcing (23) and using (34) and (37), one has the following supercommutation relations. For convenience, the commutators, [.,.], and the anticommutators, {.,.}, are distinguished.
\[
[x^\mu, x^\nu] = [x^\mu, \psi^\gamma] = [x^\mu, \eta^a] = [x^\mu, \eta^{a*}] = 0
\]
\[
\{\psi^\epsilon, \eta^a\} = \{\psi^\epsilon, \eta^{a*}\} = \{\eta^a, \eta^b\} = \{\eta^{a*}, \eta^{b*}\} = 0
\]
\[
\{\psi^\epsilon, \psi^\gamma\} = \hbar \delta^\gamma_{\epsilon}
\]
\[
\{\eta^a, \eta^{b*}\} = \frac{\hbar}{\kappa} \delta^{ab}
\]
\[
[x^\mu, p_{\nu}] = i\hbar \delta^\mu_{\nu}.
\]
\[
[\psi^\gamma, p_\nu] = -i \hbar \gamma^\gamma_{\nu\rho} \psi^\rho \\
[\eta^a, p_\nu] = -i \hbar A^a_\nu \eta^c \\
[\eta^{a*}, p_\nu] = i \hbar A^{a*}_\nu \eta^{c*} \\
[p_\mu, p_\nu] = -\frac{\hbar}{2} R_{\mu\nu\rho\gamma} \psi^\rho \psi^\gamma + \kappa \left[ -\hbar F^{ab}_{\mu\nu} \eta^{a*} \eta^b \right].
\]

One must note that in general there may be factor ordering ambiguities in the right hand side of (23). Indeed, for the example considered in this paper there are three inequivalent choices for the last equation in (38). These correspond to the following choices of ordering \( \eta^{a*} \eta^b \) in (37):

\( \eta^{a*} \eta^b, \quad -\eta^b \eta^{a*}, \quad \) and \( \frac{1}{2} (\eta^{a*} \eta^b - \eta^b \eta^{a*}) \).

(39)

The first choice is selected in (38) because, as will be seen in Section 4, it leads to the identification of the supersymmetric charge with the twisted Dirac operator. The quantum mechanical supersymmetric charge corresponding to (29), which is also hermitian, is given by:

\[
Q = \frac{1}{\sqrt{\hbar}} \psi^\nu g^{\frac{1}{4}} p_\nu g^{-\frac{1}{4}}.
\]

(40)

Hermiticity is ensured in view of the identity:

\[
\psi^\nu g^{\frac{1}{4}} p_\nu g^{-\frac{1}{4}} = g^{-\frac{1}{2}} p_\nu g^{\frac{1}{2}} \psi^\nu.
\]

Here, \( g \) is the determinant of \( (g_{\mu\nu}) \) and the proportionality constant, \( 1/\sqrt{\hbar} \), is fixed by comparing the reduced form of (41) to the case: \( \psi = \eta = 0 \) and \( M = \mathbb{R}^n \). Equation (40) together with (15) yield the Hamiltonian:

\[
H = Q^2 = \frac{1}{4} g^{-\frac{1}{2}} p_\nu g^{\frac{1}{2}} g^{\mu\nu} p_\rho g^{-\frac{1}{2}} + \frac{\hbar^2}{8} R + \kappa \left[ -\frac{1}{2} F^{ab}_{\gamma\rho} \psi^\rho \psi^\gamma \eta^{a*} \eta^b \right].
\]

(41)

The derivation of (41) involves repeated use of (38). In particular, the appearance of the scalar curvature term, \( \frac{\hbar^2}{8} R \), is a consequence of the third and the last equations in (38) and the symmetries of the Riemann curvature tensor. Reducing the system of (24) to a purely bosonic one, i.e. setting
ψ = η = 0, leads to the problem of the dynamics of a free particle moving on a Riemannian manifold. Equation (41) is in complete agreement with the analysis of the latter problem by Bryce DeWitt [1]. One has to emphasize, however, that here the Hamiltonian is obtained as a result of the superalgebra condition (13). One can also check that (41) reduces to the classical Hamiltonian, i.e., $L_{\alpha} = \frac{\partial}{\partial \psi^i} \phi^j - L$, as $\hbar \to 0$, for the Lagrangian (24) with $\alpha = 0$.

It turns out that this is precisely what one needs for the proof of the twisted spin index theorem. See Section 4, for a more detailed discussion of this point.

## 4 The Quantum System

In the rest of this paper $\hbar$ will be set to 1.

### 4.1 The Case of Spin Complex ($\kappa = \alpha = 0$)

Let $\{e^i_\mu \, dx^\mu\}$ be a local orthonormal frame for the cotangent bundle, $TM^*$, i.e., $e^i_\mu e^j_\nu \delta_{ij} = g_{\mu \nu}$ and $\{e^i_\mu \partial / \partial x^\mu\}$ be its dual in $TM$. Consider,

\[ \gamma^i := i\sqrt{2}e^i_\mu \psi^\mu. \] (42)

Then, (38) implies:

\[ \{\gamma^i, \gamma^j\} = -2\delta_{ij}. \] (43)

In the mathematical language one says that $\gamma^i$’s are the generators of the Clifford algebra $\mathcal{C}(TM^*_x) \otimes \mathbb{C}$, [2, Part II, p. 6]. Furthermore, for $i = 1, \ldots, l := m/2$ define [3, 4]:

\[ \xi^i := \frac{1}{2}(\gamma^{2i-1} + i\gamma^{2i}) \]
\[ \xi^{i\dagger} := -\frac{1}{2}(\gamma^{2i-1} - i\gamma^{2i}). \] (44)
Equations (43) and (44) yield the following anticommutation relations:

\[
\{ \xi^i, \xi^j \} = \delta^{ij}
\]
\[
\{ \xi^i, \xi^j \} = \{ \xi^{i\dagger}, \xi^{j\dagger} \} = 0.
\] (45)

Equation (45) indicates that \( \xi^i \) and \( \xi^{i\dagger} \) behave as fermionic annihilation and creation operators. The basic kets of the corresponding Fock space are of the form:

\[
| i_r, \ldots, i_1, x, t \rangle := \xi^{i_r\dagger} \ldots \xi^{i_1\dagger} | x, t \rangle.
\] (46)

The wavefunctions are given by:

\[
\Psi_{i_1, \ldots, i_r}(x, t) = \langle x, t, i_1, \ldots, i_r | \Psi \rangle
\] (47)

where

\[
\langle x, t, i_1, \ldots, i_r | := | i_r, \ldots, i_1, x, t \rangle^\dagger.
\]

The chirality operator \((-1)^f\) of (16) is defined by

\[
\gamma^{m+1} := i^l \gamma^1 \ldots \gamma^m = \prod_{i=1}^l (1 - 2 \xi^{i\dagger} \xi^i).
\] (48)

Equation (48) is a clear indication of the relevance of the system to the spin complex. In fact, in terms of \( \gamma \)'s the supersymmetric charge, (40), is written as

\[
Q = -i \sqrt{2} g^{1/2} \gamma^\mu p_\mu g^{1/2},
\] (49)

where

\[
\gamma^\mu := e^\mu_i \gamma^i.
\] (50)

It is not difficult to see that indeed \( Q \) is represented by the Dirac operator \( \slashed{D} \) in the coordinate representation, i.e.

\[
\langle x, t, i_1, \ldots, i_r | p_\mu = -i \slashed{D}_\mu \langle x, t, i_1, \ldots, i_r | \]

with

\[
\slashed{D}_\mu := \frac{\partial}{\partial x^\mu} - \frac{1}{8} \omega_\mu.
\] (52)
The following commutation relations can be easily computed:

\[
\begin{align*}
[x^\mu, -i \partial_\nu] &= i \delta^\mu_\nu, \\
[\gamma^i, -i \partial_\nu] &= -i \omega^i_{\mu \nu} \gamma^j, \\
[-i \partial_\mu, -i \partial_\nu] &= \frac{1}{4} R_{\mu \nu} \epsilon^\delta_{\epsilon \gamma} \gamma^\epsilon \gamma^\delta.
\end{align*}
\] (53, 54, 55)

In (52) and (54) \(\omega_\mu\) and \(\omega^i_{\mu \nu}\) refer to the spin connection:

\[
\omega^i_{\mu \nu} : = \Gamma^i_{\nu \sigma} \epsilon^\mu_{\mu} \epsilon^\sigma_{\mu} - \epsilon^i_{\mu \nu} \epsilon^\mu_{\mu} =: \omega_{\nu ij}.
\] (56)

\[
\omega_\mu : = \omega_{\mu ij} [\gamma^i, \gamma^j] = 2 \omega_{\mu ij} \gamma^i \gamma^j.
\] (57)

In the derivation of (55) one uses the symmetries of \(\omega_{\nu ij}\), especially the identity:

\[
\omega_{\nu ij} = -\omega_{\nu ji}.
\]

Comparing (53), (54), (55) with the last three equations in (38) justifies (51) and the claim preceding it.

Following the analysis of [1, §6.7], the coherent state representation can be used to give a path integral representation of the supertrace of any operator, \(\hat{O}\). The following relations summarize this procedure. The coherent states are defined by

\[
\begin{align*}
| x, \xi; t \rangle : &= e^{\frac{1}{2} \xi^\dagger \xi} e^\dagger(t) | x, t \rangle \\
\langle x, \xi^*; t \mid : &= | x, \xi; t \rangle^\dagger,
\end{align*}
\] (58)

The "\(\dagger\)" is used to distinguish the operators from the scalars where necessary. Equation (58) leads to

\[
\text{str}(\hat{O}) = \frac{1}{(2\pi i)^l} \int \langle x, \xi^*; t \mid \hat{O} \mid x, \xi; t \rangle d^m x \ d^d \xi^* \ d^d \xi.
\] (59)

In particular, one has

\[
\text{str}(e^{-i\beta H}) = \frac{1}{(2\pi i)^l} \int \langle x, \xi^*; t + \beta \mid x, \xi; t \rangle d^m x \ d^d \xi^* \ d^d \xi.
\] (60)
The following notation is occasionally used:

\[ K(x, \xi; \beta) := \langle x, \xi^*; \beta | x, \xi; 0 \rangle \tag{61} \]

(61) has a well-known path integral representation. One can change the variables \( \xi \)'s to \( \psi \)'s in (60) and (61) to compute the index. This will be pursued in Section 6.

### 4.2 The Case of the Twisted Spin Complex (\( \kappa = 1 \))

The commutation relations between \( \eta \)'s and \( \eta^* \)'s in (38), with \( \hbar = 1 \) and \( \eta^\dagger := \eta^* \), read

\[
\{ \eta^a, \eta^b\dagger \} = \delta^{ab} \\
\{ \eta^a, \eta^b \} = \{ \eta^{a\dagger}, \eta^{b\dagger} \} = 0 . \tag{62}
\]

Thus, \( \eta \) and \( \eta^\dagger \) can be viewed as the annihilation and creation operators for “\( \eta \)-fermions”. The total Fock space \( \mathcal{F}_{tot} \) is the tensor product of the Fock space \( \mathcal{F}_0 \) of the \( \kappa = 0 \) case and the one constructed by the action of \( \eta^\dagger \)'s on the vacuum. The basic kets are:

\[
| a_p, ..., a_1, i_r, ..., i_1, x, t \rangle := | a_p, ..., a_1, x, t \rangle \otimes | i_r, ..., i_1, x, t \rangle,
\]

where

\[
| a_p, ..., a_1, x, t \rangle := \eta^{a_p\dagger} ... \eta^{a_1\dagger} | x, t \rangle.
\]

The relevant Fock space for the twisted spin complex, however, is the subspace \( \mathcal{F}_V \) of \( \mathcal{F}_{tot} \), spanned by the 1-\( \eta \)-particle state vectors. These are represented by the following basic kets:

\[
| a, i_r, ..., i_1, x, t \rangle. \tag{63}
\]

In the coordinate representation one has:

\[
\langle x, t, i_1, ..., i_r, a | p_\mu = -i(\bar{\theta}_\mu + A_\mu)\langle x, t, i_1, ..., i_r, a |, \tag{64}
\]

20
where
\[ \mathcal{A}_\mu := A^{ab}_\mu \eta^a \eta^b. \]

This is justified by computing the following commutation relations and comparing them with the last five relations in (38):
\[
\begin{align*}
[x^\mu, -i(\vartheta_\nu + \mathcal{A}_\nu)] &= i\delta^\mu_\nu \\
[\gamma^i, -i(\vartheta_\nu + \mathcal{A}_\nu)] &= -i\omega^i_\nu \gamma^j \\
[\eta^a, -i(\vartheta_\nu + \mathcal{A}_\nu)] &= -iA^{ab}_\nu \eta^b \\
[\eta^{a*}, -i(\vartheta_\nu + \mathcal{A}_\nu)] &= iA^{ca}_\nu \eta^{*c} \\
[-i(\vartheta_\mu + \mathcal{A}_\mu), -i(\vartheta_\nu + \mathcal{A}_\nu)] &= \frac{1}{4} R^{\mu\nu\delta\gamma} \gamma^\gamma - F^{ab}_{\mu\nu} \eta^{a*} \eta^b.
\end{align*}
\]

Again, the supersymmetric charge \( Q \) of (40) is identified with the twisted Dirac operator, \( \vartheta_V \), in the coordinate representation. In view of (40), (42), and (64), one has
\[
\langle x, t, i_1, \ldots, i_r, a | Q = \frac{-1}{\sqrt{2}} g^{\frac{1}{4}} [\gamma^\mu(\vartheta_\mu + \mathcal{A}_\mu)] g^{-\frac{1}{4}} \langle x, t, i_1, \ldots, i_r, a |.
\]

Once more, \( \gamma^{m+1} \) of (48) serves as the chirality operator. In particular, \( Q \) switches the \( \pm 1 \)-eigenspaces of \( \gamma^{m+1} \), i.e. \( \{ \gamma^{m+1}, Q \} = 0 \).

Coherent states are defined by
\[
|x, \xi, \eta; t\rangle := e^{-\frac{1}{2} \eta^{a*} \eta^a + \eta^{a*} \eta^a} |x, \xi; t\rangle.
\]

The supertrace formula, the analog of (59), is given by:
\[
str(\hat{O}) = \frac{1}{(2\pi i)^{l+n}} \int \langle x, \xi, \eta^*; t | \hat{O} | x, \xi, \eta; t \rangle d^m x d^l \xi d^n \eta.
\]

The application of the latter equation to the time evolution operator leads to
\[
str(e^{-i\beta H}) = \frac{1}{(2\pi i)^{l+n}} \int \langle x, \xi, \eta^*; t + \beta | x, \xi, \eta; t \rangle d^m x d^l \xi d^n \eta.
\]
Equation (66) does not, however, provide the index. This is because in (66) the supertrace is taken over $F_{tot}$, rather than $F_V$. This is remedied by including a term of the form $e^{i\alpha \hat{n}^a \hat{n}^a}$, in (66), and considering
\[
\text{str} \left[ e^{-i\beta H} e^{i\alpha \hat{n}^a \hat{n}^a} \right].
\]
(67)
The linear term in $\epsilon := e^{i\alpha}$ in (67) is precisely the index of $\partial V$. The term $[...]_3$ in the original Lagrangian (24) is added to fulfill this objective, [5, 9]. In Section 7,
\[
H_{\text{eff}} := H - \frac{\alpha}{\beta} \hat{n}^a \hat{n}^a
\]
(68)
will be used in the path integral evaluation of the kernel:
\[
K(x, \xi, \eta; \beta) := \langle x, \xi^*, \eta^*; \beta | x, \xi, \eta; 0 \rangle.
\]
(69)
The index of $\partial V$ is then given by
\[
\text{index}(\partial V) = \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \text{str}(e^{-i\beta H_{\text{eff}}})
\]
(70)

5 The Path Integral Evaluation of the Kernel, the Loop Expansion and the Green’s Function Methods

The path integral evaluation of the kernel, (71) below, is discussed in [4, §5]. In general, for a quadratic Lagrangian the following relation holds:
\[
K(\Phi'', t'' | \Phi', t') := \langle \Phi'', t'' | \Phi', t' \rangle = Z \int_{\langle \Phi', t' \rangle}^{(\Phi'', t'')} e^{iS[\Phi]} \left( s\text{det}G^+ \right)^{-\frac{1}{2}} D\Phi.
\]
(71)

\footnote{Note that (67) is a polynomial in $\epsilon$}
Here, $Z$ is a (possibly infinite) constant of functional integration. In the loop expansion of (71), one expands the field (coordinate) variables around the (classical) solutions of the dynamical equations, $\Phi_0(t)$:

$$\Phi^i(t) = \Phi^i_0(t) + \phi^i(t). \quad (72)$$

Substituting (72) in (71) and expanding in power series around $\Phi_0$, one has:

$$K(\Phi', t' \mid \Phi', t') = Z(s\text{det}G_0^{1/2})^{-1/2} e^{i S_0} \int e^{i \phi^j i \phi^j} \{1 + \cdots\} D\phi. \quad (73)$$

Here, “···” denotes the higher order terms starting with the 2-loop terms. The subscript “0” means that the corresponding quantity is evaluated at the classical solution $\Phi_0$, e.g.

$$\phi^i := \phi^i_{\Phi_0}.$$

The lowest order approximation of (71), which is explicitly shown in (73), is the well-known WKB approximation. This term can be further simplified if the surviving Gaussian functional integral in (73) is evaluated. Generalizing the ordinary Gaussian integral formula, one has:

$$\int e^{\frac{1}{2} i \phi^i i \phi^j} D\phi = c[\text{sdet}(G)]^{1/2}. \quad (74)$$

The Green’s function $G = (G^{ij})$, which appears in (74), is the celebrated Feynman propagator. It is the inverse of $(\phi_{\Phi_0})$:

$$i \phi^i i \phi^j G^{jk} = -\delta^j_k \delta(t - t''), \quad (75)$$

defined by the boundary conditions which fix the end points:

$$\Phi(t') = \Phi' \quad \text{and} \quad \Phi(t'') = \Phi''. \quad (76)$$

\footnote{There are additional complications if $M$ has nontrivial first homology group, \cite{1}. However, this is not relevant to the computation of the index. See Section 8 for a further discussion of this problem.}

\footnote{The 2-loop terms will be analyzed in \cite{25}. They are of order $\beta$ or higher.}

\footnote{In (74), $c$ is a constant of functional integration, it may be identified with 1 if the action is appropriately rescaled. However, this does not play any role in the application of (74) and thus is not pursued here.}
An important property of $G^{ij}$ is the following \[1\]:

$$G'^{ij} = (-1)^{ij'} G^{i'j}.$$  

(77)

The emergence of $s\text{det}(G^+)$ in (71) is quite important and must not be underestimated. An explicit computation of $s\text{det}(G^+)$ for (24) is in order. The main tool is the definition of the $s\text{det}$ as the solution of the variational equation \[1, \S 1\]:

$$\delta \ln [s\text{det}(G^+)] := \text{str}[(G^+)^{-1} \delta G^+] .$$  

(78)

Using the definition of $G^+$ (13):

$$(G^{+ij})^{-1} := -(i, S_j)$$  

(79)

one has

$$\delta \ln \left[ s\text{det} \left( G^+ \right|_{\Phi^{+''}, t^{''}} \right) \right] = \int_{t^{''}}^{t'''} d\tau \int_{t'}^{t''} d\tau' (-1)^{i} i \delta S_{,j} G^{+j'i}. $$  

(80)

In (80), the variation in the action is with respect to the functional variation of the metric and the connection fields, i.e. $\delta g_{\mu \nu}$ and $\delta A_{ab}^\mu$, respectively. In view of (30) and (32), equation (80) becomes:

$$\delta \ln \left[ s\text{det} \left( G^+ \right|_{\Phi^{+''}, t^{''}} \right) \right] = \int_{t^{''}}^{t'''} dt \left[ \frac{d}{dt} (\delta \ln g) - i G^{+\gamma \epsilon} \delta g_{\epsilon \gamma} + i \tilde{\epsilon}\tilde{\alpha} (\delta C_{\beta a} + \delta C_{\epsilon a}) + i g^{\epsilon \gamma} \delta C_{\gamma} \right] \theta(0). $$  

(81)

In (81), $G_1^{+\gamma \epsilon}$ are the coefficients of the linear terms in the expansion of $G^{+\gamma \epsilon}$ in (32), i.e.

$$G^{+\gamma \epsilon} =: \theta(t - t') \left[ -ig^{\gamma \epsilon} + G_1^{+\gamma \epsilon} (t - t') + O(t - t')^2 \right],$$  

(82)

and $\delta C$'s are the variations of the corresponding terms in (31). It is quite remarkable that although other higher order terms in (32) originally enter in (80), their contributions cancel and one is finally left with (81). The fortunate
cancellations seem to be primarily due to supersymmetry. Incidentally, the calculation of \( G_1^{\gamma \epsilon} \) is quite straightforward. The 16 coupled equations, \((32)\), which give the next order terms in \((22)\), decouple miraculously to yield:

\[
G_1^{\gamma \epsilon} = g^{\delta \gamma} g^{\delta \epsilon} C \delta \theta - g^{\pi \tau} \Gamma^{\gamma \epsilon \pi \delta \epsilon \tau} \psi \delta \psi \theta.
\]  

Substituting \((83)\) in \((81)\), the second term in \((83)\) drops after contracting with \( \delta g_{\epsilon \gamma} \). The surviving term combines with the last term in \((81)\) to produce a factor of

\[
i \delta (g^{\pi \epsilon} C_{\epsilon \gamma}).
\]

Finally, using \((31)\) and the identity

\[
\Gamma^{\gamma \mu} = \frac{1}{2} \partial_{\mu} (\ln g)
\]

one obtains:

\[
\delta \ln \det (G^{\pi \epsilon} \Phi_{\epsilon \tau}^{\mu}, \Phi_{\tau \mu}^{\epsilon}) = \int_{t'}^{t''} \delta \left[ \frac{1}{2} \frac{d}{dt} (\ln g) \right] \theta (0) dt.
\]  

Adapting \( \theta (0) = 1/2 \), [1, §6.4], and integrating \((84)\), one has:

\[
\det (G^{\pi \epsilon} \Phi_{\epsilon \tau}^{\mu}, \Phi_{\tau \mu}^{\epsilon}) = \text{const.} g^{\frac{1}{2}} (x'') g^{-\frac{1}{2}} (x').
\]  

Specializing to the Euclidean case, i.e. \( g_{\mu \nu} = \delta_{\mu \nu} \), the “const.” is identified with “1”. Furthermore, for the periodic boundary conditions \((89)\), equation \((85)\) reduces to

\[
\det (G^{\pi}) = 1 \quad \text{(for: } x'' = x').
\]

Equation \((86)\) is a direct consequence of supersymmetry. It results in a great deal of simplifications in \((73)\), particularly in the higher-loop calculations, \([25]\).
6 The Derivation of the Index of Dirac Operator (A Proof of Theorem 1)

Combining (16), (50), (74), (75), (76) and (86), one obtains the index of $\theta$ in the form:

$$\text{index} (\hat{\partial}) = \frac{1}{(2\pi i)^4} \int K(x, \psi; \beta \to 0) \Theta d^n x d^n \psi,$$

(87)

where

$$K(x, \psi; \beta \to 0) := \langle x, \psi; \beta \to 0 | x, \psi; 0 \rangle^\text{WKB} \equiv Z ce^{iS_0} [sdet(G)]^\frac{1}{2},$$

(88)

and $\Theta$ is the superjacobian associated with the change of variables of integration from $\xi, \xi^*$'s to $\psi$'s.

In (88), the periodic boundary conditions must be adapted, i.e.

$$x(0) = x(\beta) =: x_0$$

$$\psi(0) = \psi(\beta) =: \psi_0.$$  

(89)

This is consistent with supersymmetry (26), [26, 8, 10].

Requiring (89) and considering $\beta \to 0$, the only solution of the classical dynamical equations (28), with $\kappa = 0$, is the constant configuration:

$$x_0(t) = x_0$$

$$\psi_0(t) = \psi_0.$$  

(90)

Substituting (90) in (24), one has [12]:

$$S_0 = 0.$$  

(91)

Thus, the factor $e^{iS_0}$ in (88) drops. Another important consequence of (90) is that unlike $i, S_j$, (30), $i, S_{0j}$ are tensorial quantities. This allows one to work with normal coordinates, [2], centered at $x_0$, in which

$$g_{0\mu\nu} := g_{\mu\nu}(x_0) = \delta_{\mu\nu}$$

(92)

$$g_{\mu\nu,\sigma}(x_0) = \Gamma_{\mu\nu}^\sigma(x_0) = 0.$$

(93)

[12] One must take note of the end point contribution to $S_0$. For details see the Appendix.
In the rest of this section, all the fields are evaluated in such a coordinate system. The final results hold true for arbitrary coordinates since the quantities of interest are tensorial.

Substituting (90) in (30), using (93), and noting that all the C’s vanish and one obtains:

\[ \tau, S_{0, \pi'} = \left[ -g_{0 \tau \pi} \frac{\partial^2}{\partial t^2} + R_{\tau \pi} \frac{\partial}{\partial t} \right] \delta(t - t') \]

\[ \tau, S_{0, \gamma'} = \epsilon, S_{0, \pi'} = 0 \] (94)

\[ \epsilon, S_{0, \gamma'} = \left[ i g_{0 \epsilon \gamma} \frac{\partial}{\partial t} \right] \delta(t - t'). \]

Here, \( g_{0 \tau \pi} = \delta_{\tau \pi} \) are retained for convenience, and

\[ R_{\tau \pi} := i \frac{1}{2} R_{\delta \theta \tau \pi}(x_0) \psi_0^\beta \psi_0^\theta. \] (95)

The Green’s functions (75) are also tensorial quantities. They can be explicitly computed:

\[ G^{\pi \xi'} = g_0^{\pi \mu} \left[ \theta(t - t') \frac{(e^{R(t - \beta)} - 1)(1 - e^{-Rt'})}{R(e^{R\beta} - 1)} \right]_{\mu}^{\xi} \] (96)

\[ G^{\pi \delta'} = 0 \] (97)

\[ G^{\gamma \xi'} = 0 \] (98)

\[ G^{\gamma \delta'} = i \frac{1}{2} g_0^{\gamma \delta} \left[ \theta(t - t') - \theta(t' - t) \right]. \] (99)

In (96), the expression inside the bracket is to be interpreted as a power series in:

\[ R := \left( R_\tau^\mu \right) := (g_0^{\mu \nu} R_{\tau \nu}). \] (100)

This is a finite series due to the presence of \( \psi_0 \)'s in (93). Equation (96) is

\[ \text{One can use all the properties of the “exp” and other analytic functions with arguments such as } R. \text{ The only rule is that the power series expansion must be postponed until all other operations are performed.} \]
obtained starting from the following ansatz:

\[ G^{\pi\xi'} = \left[ \theta(t - t')(\frac{t}{\beta} - 1)t' g^{\pi\mu} X_{\mu}(t, t') + \theta(t' - t)(\frac{t'}{\beta} - 1)t g^{\pi\mu} X_{\mu}(t, t') \right]. \]  

(101)

(101) satisfies the boundary conditions (76), i.e.

\[ G^{\pi\xi'} = 0 \]  

if \( t \) or \( t' \) = 0 or \( \beta \).

Moreover, the reduction to \( \psi = 0 \) case is equivalent to choosing \( X_{\pm} = 1_{m \times m} \).

Imposing (75) on (101) and using (94), one obtaines the following equations:

\[ t > t' : (t - \beta) \left[ \frac{\partial^2}{\partial t^2} X_+ - R \frac{\partial}{\partial t} X_+ \right] + 2 \left[ \frac{\partial}{\partial t} X_+ - \frac{1}{\beta} RX_+ \right] = 0 \]  

(103)

\[ t < t' : t \left[ \frac{\partial^2}{\partial t^2} X_- - R \frac{\partial}{\partial t} X_- \right] + 2 \left[ \frac{\partial}{\partial t} X_- - \frac{1}{\beta} RX_- \right] = 0 \]  

(104)

\[ t = t' : \left[ \frac{t}{\beta} X_+ - \left( \frac{t}{\beta} - 1 \right) X_- \right] + t(t - 1) \left( \frac{\partial}{\partial t} X_+ - \frac{\partial}{\partial t} X_- - R X_+ + R X_- \right) \bigg|_{t = t'} = 1_{m \times m}. \]  

(105)

(103) and (104) are easily solved by the power series method. The final result, (96), follows using (77) and (105). The derivation of (97), (98), and (99) is straightforward.

The next step is to compute \( sdet(G) \). This is accomplished by considering a functional variation in the metric tensor, \( \delta g_{\mu\nu} \), and using the definition of \( sdet \), (78). After considerable amount of algebra and repeated use of the symmetries of \( g_{\mu\nu} \), and \( R_{\mu}^\nu \), one arrives at the following expression:

\[ \delta \ln[sdet(G)] = \theta(0)tr \left[ 2\delta \ln(\beta R) + \beta \delta R \left( \frac{1 + e^{\beta R}}{1 - e^{\beta R}} \right) \right]. \]  

(106)

Setting \( \theta(0) = 1/2 \) and integrating the right hand side of (106), one has:

\[ \delta \ln[sdet(G)] = \delta tr \left( \ln \left[ \frac{-\beta R}{2 \sinh(\beta R)} \right] \right). \]  

(107)
Since $\mathcal{R}$ is antisymmetric in its indices, it can be put in the following block-diagonal form:

$$
\mathcal{R} = \text{diag} \left( \begin{array}{cc}
0 & \mathcal{R}_i \\
-\mathcal{R}_i & 0 \\
\end{array} \right) : i = 1, \ldots, l
$$

(108)

Another important observation is that $\frac{\partial \mathcal{R}}{\partial \mathcal{K}}$ and hence $\ln\left[-\frac{\partial \mathcal{R}}{\partial \mathcal{K}}\right]$ are polynomials in $(\frac{\partial \mathcal{R}}{2})^2$. In view of (108), it is easy to see that $\mathcal{R}^2$ is diagonal, namely

$$
\mathcal{R}^2 = \text{diag}(-\mathcal{R}^2_1, -\mathcal{R}^2_1, \ldots, -\mathcal{R}^2_l, -\mathcal{R}^2_l).
$$

(109)

Implementing (109) in (107), one has:

$$
\delta \ln[\text{sdet}(G)] = 2\delta \ln \prod_{j=1}^{l} \left[ -\frac{\beta \mathcal{R}_j}{2} \sinh\left(\frac{\beta \mathcal{R}_j}{2}\right) \right],
$$

and finally

$$
\text{sdet}(G)^{\frac{1}{2}} = \tilde{c} \prod_{j=1}^{l} \left[ \frac{\beta \mathcal{R}_j}{2} \sinh\left(\frac{\beta \mathcal{R}_j}{2}\right) \right],
$$

(110)

where $\tilde{c}$ is a constant of functional integration. Substituting (110) and (90) in (88), one obtains the kernel in the form:

$$
K(x, \psi; \beta \to 0) = Zc \tilde{c} \prod_{j=1}^{l} \left[ \frac{\beta \mathcal{R}_j}{2} \sinh\left(\frac{\beta \mathcal{R}_j}{2}\right) \right].
$$

(111)

To write $\mathcal{R}_j$ as functions of $x_0$ and $\psi_0$, one needs to also block-diagonalize:

$$
\left( \frac{1}{2} R_{\mu\nu\delta\theta}(x_0) \psi_0^\delta \psi_0^\theta \right) = \text{diag} \left( \begin{array}{cc}
0 & \Omega_{j\delta\theta} \psi_0^\delta \psi_0^\theta \\
-\Omega_{j\delta\theta} \psi_0^\delta \psi_0^\theta & 0 \\
\end{array} \right) : j = 1, \ldots, l
$$

(112)

Combining (95), (100), (108), (111) and (112), one is led to

$$
K(x_0, \psi_0; \beta \to 0) = Zc \tilde{c} \prod_{j=1}^{l} \left[ \frac{\beta \mathcal{R}_j}{2} \sinh\left(\frac{\beta \mathcal{R}_j}{2}\right) \right].
$$

(113)
The proportionality constant \( Zc \bar{c} =: \tilde{Z} \) is determined by specializing to the \( M = \mathbb{R}^m \) case. Using the results of \([4, \S 5, \S 6]\), one has

\[
\tilde{Z} = (2\pi i dt)^{-\frac{\delta m}{2\pi}} \times (2\pi i)^{-\frac{\delta l}{2\pi}}. \tag{114}
\]

The limit \( \beta \to 0 \) is taken by setting

\[
\beta = dt. \tag{115}
\]

Finally, combining (87,113) and (113), and realizing that \( e^\mu(x_0) = \delta^\mu_l \) so that \( \Theta = i^{-l} \), one has

\[
\text{index}(\varnothing) = \frac{1}{(2\pi)^l} \int M \prod_{j=1}^{l} \left[ \frac{\Omega_j}{\sinh(\frac{\Omega_j}{2})} \right] \frac{d^m \psi_0 d^m x_0}{(2\pi i)^{\beta} l}. \tag{116}
\]

The \( \psi_0 \)-integration rules, \([4]\):

\[
\begin{aligned}
\int d\psi_0 &= 0 \\
\int \psi_0 d\psi_0 &= \sqrt{2\pi i}
\end{aligned} \tag{117}
\]

allow only the highest degree term in the integrand to survive. This implies the cancellation of \( \beta \)'s. Performing the \( \psi_0 \)-integrations yields an expression for (116) which is identical with the following:

\[
\text{index}(\varnothing) = \frac{1}{(2\pi)^l} \int M \prod_{j=1}^{l} \left[ \frac{\Omega_j}{\sinh(\frac{\Omega_j}{2})} \right]_{\text{top}}
\]

\[
= \int M \prod_{j=1}^{l} \left[ \frac{\Omega_j}{\sinh(\frac{\Omega_j}{2})} \right]_{\text{top}}. \tag{118}
\]

In (118), \( \Omega_j \) are defined by (4) and the identity

\( \Omega_j = \Omega_{j\delta\theta} dx^\delta \wedge dx^\theta. \)

(118) is precisely the statement of Theorem 1, i.e. (\( j \)). The fact that the integrand in (118) is an even polynomial in \( \frac{\Omega_j}{4\pi} \) implies that only for \( l = 2k \), i.e. \( m = 4k \), is the index nonvanishing.
The Derivation of the Index of the Twisted Dirac Operator (A Proof of Theorem 2)

Equations (66), (69), (70), (71), (73), (74), and (86) provide the following formula for the index

\[ \text{index}(\partial V) = \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \left[ \frac{1}{(2\pi i)^{l+n}} \int K(x, \psi, \eta; \beta \to 0) \Theta d^m x d^n \psi d^n \eta^* d^n \eta \right] \]  

(119)

where

\[ K(x, \psi, \eta; \beta \to 0) := \langle x, \psi, \eta^*; \beta \to 0 | x, \psi, \eta; 0 \rangle \overset{\text{WKB}}{=} Z \psi e^{is_0} [sdet(G)]^{\frac{1}{2}}. \]  

(120)

The first step in the calculation of the kernel is to obtain the classical solution of the dynamical equations, (28), in the \( \beta \to 0 \) limit. As one can see from the analysis of Section 6, absorbing a factor of \( \sqrt{\beta} \) in \( \psi \)'s can take care of the limiting process automatically. Furthermore, following [9], define the parameter:

\[ s := \frac{t}{\beta}, \]

and expand the coordinate variables in powers of \( \beta \):

\[ x(t) = \bar{x}_0(s) + \bar{x}_1(s) \beta + O(\beta^2) \]

\[ \psi(t) = \frac{1}{\sqrt{\beta}} \left[ \bar{\psi}_0(s) + \bar{\psi}_1(s) \beta + O(\beta^2) \right] \]

\[ \eta(t) = \bar{\eta}_0(s) + \bar{\eta}_1(s) \beta + O(\beta^2). \]

\[ \eta^*(t) = \bar{\eta}_0(s) + \bar{\eta}_1^*(s) \beta + O(\beta^2). \]

(121)

The appropriate boundary conditions for (120) are the periodic boundary conditions [4], i.e., (89) and:

\[ \eta = \eta' := \eta(t = 0) = \eta(\beta) =: \eta'' \]

\[ \eta^* = \eta^*'. \]

\[ ^{14} \text{In fact, one can adopt antiperiodic boundary conditions for } \eta \text{'s and } \eta^* \text{'s. [3]. Since the relevant trace is taken over the 1-} \eta \text{-particle state vectors, this would introduce an extra minus sign.} \]
\[ \eta' := \eta^*(t = 0) = \eta^*(t = \beta) =: \eta'' = \eta^*. \] (122)

It follows from (89), (122), and (121) that the dynamical equations are solved by:

\[
\begin{align*}
x_0(t) & = x_0 + O(\beta) \\
\psi_0(t) & = \frac{1}{\sqrt{\beta}} \left[ \tilde{\psi}_0 + \tilde{\psi}_1(s)\beta + O(\beta^2) \right] \quad (123) \\
\eta_0(t) & = \tilde{\eta}_0(s) + O(\beta) \\
\eta_0^*(t) & = \tilde{\eta}_0^*(s) + O(\beta),
\end{align*}
\]

where \( x_0 \) and \( \psi_0 =: \frac{1}{\sqrt{\beta}} \tilde{\psi}_0 \) are constants. Adopting a normal coordinate frame centered at \( x_0 \) and using (92), (93) and \( A_{\mu}^{ab}(x_0) = 0 \), one has:

\[
\begin{align*}
\frac{d}{ds} \tilde{\psi}_0^\gamma - ig_\gamma^\epsilon \bar{F}_{\epsilon\delta}^{ab}(x_0) \tilde{\psi}_0^\delta \tilde{\eta}_0^a \tilde{\eta}_0^b & = 0 \\
\frac{d}{ds} \tilde{\eta}_0^a - i(\bar{\mathcal{F}}^{ab} + \alpha \delta^{ab}) \tilde{\eta}_0^b & = 0 \\
\frac{d}{ds} \tilde{\eta}_0^* + i(\bar{\mathcal{F}}^{ba} + \alpha \delta^{ba}) \tilde{\eta}_0^* & = 0. 
\end{align*}
\] (124)

Here \( \bar{\mathcal{F}} \) is defined by

\[ \bar{\mathcal{F}} = (\bar{\mathcal{F}}_{ab}) := \left( \frac{1}{2} F_{\gamma\epsilon}^{ab}(x_0) \tilde{\psi}_0^\gamma \tilde{\psi}_0^\epsilon \right). \] (125)

The quantity:

\[ \mathcal{F} = (\mathcal{F}_{ab}) := \left( \frac{1}{2} F_{\gamma\epsilon}^{ab}(x_0) \psi_0^\gamma \psi_0^\epsilon \right) = \frac{1}{\beta} \bar{\mathcal{F}} \] (126)

will also be used.

The next step is to compute \( S_0 \). This is done in the Appendix. The final result is:

\[ S_0 = -i \eta^a \left[ e^{i(\beta \mathcal{F} + \alpha 1_{n \times n})} - 1_{n \times n} \right]^{ab} \eta^b. \] (127)
Next, one needs to calculate the Feynman propagator $G$. This is done in two steps. First, consider the special case of

$$
x_0(t) = x_0
$$
$$
\psi_0(t) = \psi_0 = \frac{1}{\sqrt{\beta}} \tilde{\psi}_0
$$
$$
\eta_0(t) = 0
$$
$$
\eta_0^*(t) = 0.
$$

It is clear that (128) satisfies the dynamical equations, (28). Substituting (128) in (30) and (31), one recovers (94). The other nonvanishing $i, S_{0,j}$'s are:

$$
b, S_{0,a} = \left[ i \delta_{ba} \frac{\partial}{\partial t} - (F_{ab} + \frac{\alpha}{\beta} \delta_{ba}) \right] \delta(t - t')
$$
$$
b^*, S_{0,a} = \left[ i \delta_{ba} \frac{\partial}{\partial t} + (F_{ba} + \frac{\alpha}{\beta} \delta_{ba}) \right] \delta(t - t').
$$

In other words, one has:

$$
(i, S_{0,j'}) = \begin{bmatrix} \tau, S_{0,\pi'} & 0 & 0 & 0 \\
0 & c, S_{0,\gamma'} & 0 & 0 \\
0 & 0 & 0 & b, S_{0,a'} \\
0 & 0 & b^*, S_{0,a'} & 0 \end{bmatrix}.
$$

(130) suggests:

$$
(G^{ij'}) = \begin{bmatrix} G^{\pi\xi'} & 0 & 0 & 0 \\
0 & G^{\gamma\eta'} & 0 & 0 \\
0 & 0 & 0 & G^{ac'} \\
0 & 0 & G^{a'c'} & 0 \end{bmatrix}.
$$

In view of (72), it is clear that $G^{\pi\xi'}$ and $G^{\gamma\eta'}$ are given by equations (96) and (99), respectively. Defining

$$
G_1(t, t') = (G^{ac}_{1}(t, t')) := \left( G^{a'c'} \right)
$$
$$
G_2(t, t') = (G^{ac}_{2}(t, t')) := \left( G^{ac'} \right),
$$
and using (131),(94),(129),(96),(99), and (75), one obtains:

\[
\begin{align*}
\left[i \frac{\partial}{\partial t} - (F^* + \alpha \beta)\right] G_1(t, t') &= -\delta(t - t') \\
\left[i \frac{\partial}{\partial t} + (F + \alpha \beta)\right] G_2(t, t') &= -\delta(t - t') .
\end{align*}
\]

In (132), use has been made of the fact that $F$ is hermitian and hence

\[F^{\text{transpose}} = F^*.\]

To compute $G_1$ and $G_2$, consider the ansatz:

\[G_k(t, t') = \theta(t - t') e^{iX_k(t, t')} - \theta(t' - t) e^{iY_k(t, t')} \quad (k = 1, 2).\]  

(133)

Substituting (133) in (132), one obtains:

\[
\begin{align*}
t > t' & : \frac{\partial}{\partial t} X_1 = -(F^* + \alpha \beta), \quad \frac{\partial}{\partial t} X_2 = F + \alpha \beta \\
t < t' & : \frac{\partial}{\partial t} Y_1 = -(F^* + \alpha \beta), \quad \frac{\partial}{\partial t} Y_2 = F + \alpha \beta \\
t = t' & : [e^{iX_k} + e^{iY_k}]_{t=t'} = i \quad k = 1, 2.
\end{align*}
\]

Finally, using (77):

\[G_1(t, t') = -G_2^{\text{transpose}}(t', t),\]

one arrives at the following expressions:

\[
\begin{align*}
G_1(t, t') &= (G^a e^c) = \frac{i}{2} e^{-i(F^* + \frac{\alpha}{\beta})(t - t')} [\theta(t - t') - \theta(t' - t)] \\
G_2(t', t) &= (G^{ac} e^c) = \frac{i}{2} e^{i(F + \frac{\alpha}{\beta})(t - t')} [\theta(t - t') - \theta(t' - t)].
\end{align*}
\]

(134)

Next step is to observe that, in the limit: $\beta \to 0$, (131) with (96),(99), and (134) actually satisfies equation (75) even in the general case of (123). To see this, it is sufficient to examine:

\[
\begin{align*}
\tau, S_0, \pi' &= \beta^{-3} \left[-g_{0\tau \pi} \frac{\partial^2}{\partial s^2} + i \frac{1}{2} R_{c\gamma \pi \tau}(x_0) \tilde{\psi}_\gamma \tilde{\psi}_\pi \frac{\partial}{\partial s} + O(\beta)\right] \delta(s - s') \quad (135) \\
\epsilon, S_0, \gamma' &= \beta^{-2} \left[ ig_{0\gamma} \frac{\partial}{\partial s} + O(\beta)\right] \delta(s - s') \quad (136)
\end{align*}
\]

34
and
\[
\begin{align*}
\delta S_{0,a^*} &= \beta^{-2} \left[ i\delta_{ba} \frac{\partial}{\partial s} - (\tilde{F}_{ab} + \alpha \delta_{ab}) + O(\beta) \right] \delta(s - s') \\
\delta S_{0,a^*} &= \beta^{-2} \left[ i\delta_{ba} \frac{\partial}{\partial s} + (\tilde{F}_{ba} + \alpha \delta_{ba}) + O(\beta) \right] \delta(s - s').
\end{align*}
\]  

(137)

The next step is to compute \(s\text{det}(G)\). Since \(G\) is block-diagonal, one has:
\[
\text{sdet}(G) = \text{sdet}(G_0).s\text{det}(G_\eta)
\]

(138)

where
\[
G_0 := \begin{pmatrix} G^{\alpha\xi} & 0 \\ 0 & G^{\alpha'\beta'} \end{pmatrix} \quad \text{and} \quad G_\eta := \begin{pmatrix} 0 & G^{ac\beta'} \\ G^{a^*c} & 0 \end{pmatrix}.
\]

Clearly, \(s\text{det}(G_0)\) is given by (110). \(s\text{det}(G_\eta)\) is computed following the procedure of Section 6. Applying (78) and (80) to \(G_\eta\) and writing only the nonvanishing terms, one has:
\[
\delta \ln s\text{det}(G_\eta) = \int_0^\beta dt \int_0^\beta dt' \left[ -c, \delta S_{0,a^*}G^{a^*c} - c, \delta S_{0,a^*}G^{a'c'} \right].
\]

(139)

Let us define:
\[
Z(t,t'') := \int_0^\beta dt' \left[ -c, \delta S_{0,a^*}G^{a^*c''} - c, \delta S_{0,a^*}G^{a'c''} \right].
\]

(140)

Taking the functional variation of (137) with respect to \(\delta F\) and substituting the result in (140), one has:
\[
\begin{align*}
Z(t,t'') &= -\delta F_{ac}G^{a^*c''} + \delta F_{ca}G^{ac''} \\
&= \text{tr} \left[ -\delta F^* G_1(t,t'') + \delta F G_2(t,t'') \right] \\
&= \text{tr} \left[ \delta e^{-i(F^* + \theta)(t-t'')} + \delta e^{i(F + \theta)(t-t'')} \right] \left[ \frac{\theta(t-t'') - \theta(t''-t)}{i(t''-t)} \right] \\
&= \text{tr} \left[ \delta \left( \cos \left[ (F + \frac{\alpha}{\beta})(t-t'') \right] \right) \left[ \frac{\theta(t-t'') - \theta(t''-t)}{i(t''-t)} \right] \right] \\
&= \delta \text{tr} \left[ \frac{1}{2} (F + \frac{\alpha}{\beta})^2 (t''-t) + O(t-t'')^3 \right] \left[ \theta(t-t'') - \theta(t''-t) \right].
\end{align*}
\]
Combining equations (139), (140), and (141), one obtains:

$$\delta \ln s\text{det}(G_{\eta}) = \int_0^\beta dt \mathcal{Z}(t, t) = 0,$$

and hence:

$$s\text{det}(G_{\eta}) = \text{const.}$$

Equations (110), (138), and (142) yeild:

$$[s\text{det}(G)]^{1/2} = \tilde{c}' \prod_{j=1}^l \left[ \frac{\beta R_j}{2 \sinh(\beta R_j/2)} \right].$$

Substituting (127) and (143) in (120), one finds:

$$K(x_0, \psi_0, \eta; \beta \to 0) = Z'c'\tilde{c}' \prod_{j=1}^l \left[ \frac{\beta R_j}{2 \sinh(\beta R_j/2)} \right] \exp \left( \eta^{a*} \left[ e^{i(\beta \mathcal{F} + \alpha 1_{n\times n})} - 1_{n\times n} \right]^{ab} \eta^b \right).$$

The constant $\tilde{Z}' := \tilde{Z}' = (2\pi i \beta)^{-l}$.

Substituting (144) in (119) and performing the $\eta^*$ and $\eta$ integrations, one has:

$$\text{index}(\phi_V) = \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon = 0} \left( \frac{1}{(2\pi)^l} \prod_{j=1}^l \left[ \frac{\beta R_j}{2 \sinh(\beta R_j/2)} \right] \text{det} \left[ e^{i(\beta \mathcal{F} + \alpha 1_{n\times n})} - 1_{n\times n} \right] \right).$$

At this stage, one can take the $\epsilon$-derivative. Since $\mathcal{F}$ is hermitian it can be diagonalized, i.e.

$$\mathcal{F} =: \text{diag}(\mathcal{F}_1, \cdots, \mathcal{F}_n).$$
Then,
\[
\left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \det \left[ e^{i(\beta F + \alpha)1_{n \times n}} - 1_{n \times n} \right] = \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \left[ \prod_{a=1}^{n} (\epsilon e^{i\beta F_a} - 1) \right]
\]
\[
= (-1)^n \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \ln \left[ \prod_{a=1}^{n} (\epsilon e^{i\beta F_a} - 1) \right]
\]
\[
= \sum_{a=1}^{n} e^{i\beta F_a}
\]
\[
= \text{tr} (e^{i\beta F}).
\] (146)

Using (146) and performing \(\psi_0\)-integrations, one finally arrives at:

\[
\text{index}(\phi_V) = \frac{1}{(2\pi)^l} \int_{M} \left[ \prod_{j=1}^{l} \left( \frac{\Omega_j}{\sinh(\frac{\Omega_j}{2\pi})} \right) \text{tr} (e^{iF}) \right]_{\text{top.}}
\]
\[
= \int_{M} \left[ \prod_{j=1}^{l} \left( \frac{\Omega_j}{\sinh(\frac{\Omega_j}{2\pi})} \right) \text{tr} (e^{i\frac{F}{2\pi}}) \right]_{\text{top.}}. \] (147)

In (147), \(\Omega_j\) and \(F\) are defined by (4) and (13), respectively. This completes the proof of Theorem 2.

## 8 Remarks and Discussion

The following is a list of remarks concerning some of the aspects of the present work:

1. The reality condition on the Lagrangian (24), requires \(F_{\mu\nu}\) to be antihermitian matrices. This is equivalent to requiring the structure group of the bundle \(V\) to be (a subgroup) of \(U(n)\). This is consistent with the existence of a hermitian structure on \(V\). For, any hermitian vector bundle can be reduced to one with \(U(n)\) as its structure group.
2. The term $i\eta^a\dot{\eta}^a$ in the Lagrangian (24) can be replaced with

$$\frac{i}{2}(\eta^a\dot{\eta}^a - \dot{a}^a\eta^a)$$

with no consequences for the action. This is because the boundary conditions on $\eta$’s and $\eta^*$’s are periodic.

3. In the Peierls quantization scheme, the momenta, “$p_\mu$”, are not a priori fixed. They may be chosen in a way that facilitates the analysis of the problem. In particular, the choice: (36) has obvious advantages. The factor ordering problem may arise in general. In the case of the index problem, however, the requirement that the supersymmetry generator be identified with the elliptic operator in question, enables one to choose the appropriate ordering. It is also remarkable that in the special case that the structure group of $V$ can be chosen $SU(n)$, the factor ordering ambiguities, (39), disappear. It seems that the obstruction for having a unique quantum system is the first Chern class.

4. The problem of nonuniqueness of the momentum operators was not addressed in Section 4. In general,

$$\mathbf{p} := p_\mu dx^\mu$$

is unique up to closed forms, $\omega \in H^1(M, \mathbb{R})$, [1]. This is reflected in the path integral formulation as the necessity of considering different homotopy meshes of paths. This is, however, not relevant to the index problem. In general, the index of an elliptic operator is a map:

$$\text{index} : K(TM) \to \mathbb{Z},$$

where $K(TM) \cong K(M) \cong \bigoplus_i H^{2i}(M, \mathbb{Z})$ is the Grothendieck’s K-group (ring), [17]. Thus the index does not detect the first cohomology group of $M$. 

38
5. The procedure used in the evaluation of the index in this paper are essentially based on a single basic assumption. This is the Gaussian functional integral formula, (74). The other important ingredient is the definition of the superdeterminant, (78), which is assumed to hold even for infinite dimensional matrices. The latter can be easily shown to be a consequence of (74).

6. The factor $[sdet \, G^+]^{-\frac{1}{2}}$, in (71), is usually constant for the case of flat spaces. However, it contributes in an essential way in the case of curved spaces. For the system considered in this paper, it was explicitly calculated and shown to be 1. It is not difficult to see that the amazing cancellations are due to supersymmetry. Considering the complexity of the system, (24), (30), and (31), this might be a general pattern for a large class of supersymmetric systems.

7. The appearance of $\frac{\hbar^2}{8} R$ in the Hamiltonian, (11), can be verified independently by comparing the heat kernel and loop expansions of the path integral, (71). This term contributes to the linear term in the heat kernel expansion, [1]. On the other hand, the presence of $\hbar^2$ is reminiscent of the contribution to the path integral in the 2-loop order. The complete 2-loop analysis of this problem is the subject of [25].

9 Conclusion

The Peierls quantization scheme provides a systematic procedure for quantizing superclassical systems. Supersymmetry results in remarkable simplifications in the path integral evaluation of the kernel. It also leads to a direct proof of the twisted spin index theorem. The supersymmetric proof of the Atiyah-Singer index theorem is based on the assumption that the ordinary
Gaussian integral formula holds even in the functional integral case. The index theorem is one of the best established mathematical results. Thus, the existence of its supersymmetric proof can be viewed as another sign of the validity and power of the path integral techniques. In particular, it is remarkable to observe that even the normalization constants agree with those chosen by mathematicians.

Acknowledgements

This project was suggested to the author by Prof. Bryce DeWitt. He would like to thank Prof. DeWitt for invaluable discussions and his guidance and support. He would also like to acknowledge Prof.’s Luis Boya and Cecile Dewitt-Morette for reading the first draft and for their most helpful comments and suggestions.

Appendix: Coherent State Path Integral and Calculation of $S_0$

In the coherent state path integral formula for the kernel, one must also include the end point contributions to the action,\cite{27, 28}. This can be done implicitly by including step functions at the end points of the paths. An example of this is provided in \cite{1}.

Consider a fermionic quantum system and let $\hat{\zeta^i}$ and $\hat{\zeta^j}$ be the creation and annihilation operators:

\[
\{\hat{\zeta^i}, \hat{\zeta^j}\} = \{\hat{\zeta^i}, \hat{\zeta^j}\} = 0.
\]

\[
\{\hat{\zeta^i}, \hat{\zeta^j}\} = \delta^{ij}.
\]

\footnote{In fact, the action function can be shown to include the end point contributions automatically. This was pointed out to the author by Cecile DeWitt.}
The coherent states are defined by:

\[ |\zeta\rangle := e^{\frac{i}{2}\zeta^*\hat{\zeta} + \hat{\zeta}^i\zeta^i} |0\rangle \]

\[ \langle\zeta^*| := |\zeta\rangle^\dagger, \]

where \(|0\rangle\) is the vacuum and \(\zeta \in \mathbb{C}\). Then one has:

\[ \hat{\zeta}^i |\zeta\rangle = \zeta^i |\zeta\rangle \quad \text{and} \quad \langle\zeta^*| \hat{\zeta}^i = \zeta^i \langle\zeta^*| . \]

In the path integral formula for the kernel:

\[ K(\zeta''; t'' | \zeta', t') := \langle\zeta''; t'' | \zeta'; t'\rangle = \int e^{iS(sdet G^+)^{-\frac{1}{2}}}D\zeta^* D\zeta, \quad (148) \]

one must note that \(\zeta'' \not= (\zeta')^*\). In fact a priori only

\[ \zeta(t') = \zeta' \quad \text{and} \quad \zeta^*(t'') = \zeta'' \quad (149) \]

are fixed. The boundary conditions on

\[ \zeta(t'') = \zeta'' \quad \text{and} \quad \zeta^*(t') = \zeta' \]

depend on the specifics of the problem. In (148), the action is given by:

\[ S = \int_{\zeta', t'}^{\zeta'', t''} dt \left[ \frac{1}{2i}(\zeta^i \hat{\zeta}^i - \hat{\zeta}^i \zeta^i) - h(\zeta^*, \zeta, t) \right] \quad (150) \]

where \(h(\zeta^*, \zeta, t)\) is the normal symbol of the Hamiltonian, \[ [27]. \] The paths are given by:

\[ \zeta^*(t) = \lim_{\epsilon \to 0} \left[ \theta(t - t' - \epsilon)\zeta^*_c(t) + \theta(t' - t + \epsilon)\zeta^* \right] \]
\[ \zeta(t) = \lim_{\epsilon \to 0} \left[ \theta(t'' - t - \epsilon)\zeta_c(t) + \theta(t - t'' + \epsilon)\zeta'' \right]. \quad (151) \]

In (151), \(\zeta^*_c(t)\) and \(\zeta_c(t)\) are paths connected to the end points, i.e. :

\[ \zeta^*_c(t'') = \zeta'' \quad \text{and} \quad \zeta_c(t') = \zeta' \quad (152) \]

In general,

\[ \zeta^*_c(t') \not= \zeta^* \quad \text{and} \quad \zeta_c(t'') \not= \zeta''. \]
To compute $S_0$, which appears in (120), one needs to specialize to the classical paths. In the limit $\beta \to 0$, these are given by (123) and (124). Equations (124) are easily solved to give:

\begin{align*}
\tilde{\eta}_0^a(s) &= e^{i(\tilde{F} + \alpha_1 n \times n) s} \eta_a' \quad (153) \\
\tilde{\eta}_0^a(s) &= e^{\eta_{0a}} \left[ e^{i(\tilde{F} + \alpha_1 n \times n)(1-s)} \right]^{ba} \quad (154) \\
\tilde{\psi}_1^\gamma(s) &= ig_0^\gamma e_{\alpha}^b(x_0) \tilde{\psi}_0 E^{ab}(s) + C^\gamma, \quad (155)
\end{align*}

where

$$E^{ab}(s) := \int_0^1 ds \tilde{\eta}_0^a(s) \tilde{\eta}_0^b(s).$$

In particular, one has:

$$\frac{i}{2} g_{0\gamma} \tilde{\psi}_0^\gamma \tilde{\psi}_1^\gamma(s) = i \eta_{0\gamma}' e^{i(\tilde{F} + \alpha_1 n \times n) s} \eta_a' s + C. \quad (156)$$

In (153) and (156) $C^\gamma$ and $C$ are constants to be determined by the boundary conditions on $\psi$'s.

In view of equations (123) and (24), one has

$$S_0 = \int_0^1 ds \left[ i g_{0\gamma} \psi_0^\gamma \tilde{\psi}_1^\gamma(s) + \frac{i}{2} \left( \tilde{\eta}_0^a(s) \tilde{\eta}_0^a(s) - \tilde{\eta}_0^a(s) \tilde{\eta}_0^a(s) \right) + \tilde{\cal F}^{ab}(s) \tilde{\eta}_0^a(s) \tilde{\eta}_0^b(s) \right] + O(\beta). \quad (157)$$

Defining

$$\tilde{\xi}^i := \sqrt{\beta} \xi^i,$$

and using (12) and (44), one may express the first term on the right hand side of (157) in the form:

$$\int_0^1 ds \left[ i \frac{1}{2} g_{0\gamma} \tilde{\psi}_0^\gamma \tilde{\psi}_1^\gamma(s) \right] = \int_0^\beta dt \left[ i \frac{1}{2} g_{0\gamma} \psi_0^\gamma(t) \tilde{\psi}_0^\gamma(t) \right] + O(\beta) \quad (158)$$

$$= \int_0^\beta dt \left[ \frac{i}{2} \left( \xi_0^i(t) \tilde{\xi}_0^i(t) - \frac{\xi_0^i(t) \tilde{\xi}_0^i(t)}{\beta} \right) \right] + O(\beta)$$

$$= \int_0^1 ds \left[ \frac{i}{2} \left( \xi_0^i(s) \tilde{\xi}_0^i(s) - \frac{\xi_0^i(s) \tilde{\xi}_0^i(s)}{\beta} \right) \right] + O(\beta).$$
In view of (158), it is clear that (157) is already in the form demanded by (150). Next step is to evaluate (157). Let us define:

\[ I_1 := \int_0^1 ds \left[ \frac{1}{2} g_{0c} \tilde{\gamma} \tilde{\psi}_0 \tilde{\psi}_1(s) \right] \]

\[ I_2 := \int_0^1 ds \left[ \frac{i}{2} \left( \tilde{\xi}_0(s) \tilde{\xi}_0(s) - \tilde{\xi}_0(s) \tilde{\xi}_0(s) \right) \right] \]

so that

\[ S_0 = I_1 + I_2. \]

Replacing \( \zeta \)'s by \( \tilde{\xi}_0 \)'s and setting \( t' = 0 \) and \( t'' = 1 \) in (151), it is a matter of simple algebra to show that:

\[ I_1 = I_1c + \partial \]

\[ I_1c := \int_0^1 ds \left[ \frac{i}{2} \left( \tilde{\xi}_{0c}(s) \tilde{\xi}_{0c}(s) - \tilde{\xi}_{0c}(s) \tilde{\xi}_{0c}(s) \right) \right] \]

\[ \partial := -\frac{i}{2} \left[ \tilde{\xi}_{0c} \left( \tilde{\xi}_{0c}(1) - \tilde{\xi}_{0c}(0) \right) + \left( \tilde{\xi}_{0c} - \tilde{\xi}_{0c} \right) \tilde{\xi}_{0c} ' \right]. \]

Imposing periodic boundary conditions, (140), (142):

\[ \tilde{\xi}_{0c} = \tilde{\xi}_{0c} =: \tilde{\xi}_0 \]

\[ \tilde{\xi}_{0c}' = \tilde{\xi}_{0c}' =: \tilde{\xi}_0' \]

\[ \tilde{\eta} = \eta =: \eta \]

\[ \tilde{\eta}' = \eta' =: \eta' \]

\[ \tilde{\eta}'' = \eta'' =: \eta'' \]

and using (156) and (158), one has:

\[ I_{1c} = \int_0^1 ds \left[ \frac{1}{2} g_{0c} \tilde{\gamma} \tilde{\psi}_0 \tilde{\psi}_{c1}(s) \right] = -\eta_{0c} \left[ e^{i(\tilde{F} + \alpha)} 1_{n \times n} \right] \tilde{F} \eta_{0c} \]

\[ \partial = -\frac{i}{2} \left[ \tilde{\xi}_{0c} \left( \tilde{\xi}_{0c}(1) - \tilde{\xi}_{0c}(0) \right) + \left( \tilde{\xi}_{0c} - \tilde{\xi}_{0c} \right) \tilde{\xi}_{0c} ' \right]. \]

Next one needs to use equations (42), (44), and (157) to compute \( \tilde{\xi}_{0c}(1) \) and \( \tilde{\xi}_{0c}^*(0) \). In (153), the boundary conditions must be chosen appropriately. The correct choice is the following:

choose: \( \psi(0) = \psi_0 = \frac{\tilde{\psi}_0}{\sqrt{\beta}} \Rightarrow \tilde{\psi}_{c1}(0) = 0 \) to compute: \( \tilde{\xi}_{0c}(1) \)

choose: \( \psi(\beta) = \psi_0 = \frac{\tilde{\psi}_0}{\sqrt{\beta}} \Rightarrow \tilde{\psi}_{c1}(1) = 0 \) to compute: \( \tilde{\xi}_{0c}^*(0) \).

43
This leads to:

\[ \partial = \eta^a [e^{i(\tilde{F} + \alpha n \times n)} \tilde{F}]^{ab} \eta^b, \]

and hence

\[ I_1 = 0 \quad (163) \]

The computation of \( I_2 \) is straightforward. One replaces \( \zeta \)'s by \( \eta_0 \)'s in (151), and sets \( t' = 0 \) and \( t'' = 1 \). The final result is obtained using (153), (154), and (162):

\[ I_2 = -i \eta^a [e^{i(\tilde{F} + \alpha n \times n)} - 1_{n \times n}]^{ab} \eta^b. \quad (164) \]

Combining (161), (163), and (164), one has:

\[ S_0 = -i \eta^a [e^{i(\tilde{F} + \alpha n \times n)} - 1_{n \times n}]^{ab} \eta^b. \]

This is used in Section 7, (127). Taking \( \tilde{F} = \eta = \eta^* = 0 \), the situation reduces to the case of \( \kappa = 0 \). In this case, one has \( S_0 = 0 \).

References

[1] B. S. DeWitt, *Supermanifolds*, Cambridge Univ. Press, Cambridge (1992)

[2] Y. Choquet-Bruhat, C. DeWitt-Morette, with M. Dillard-Bleick, *Analysis, Manifolds and Physics Parts I*, North-Holland, Amsterdam (1989); Y. Choquet-Bruhat and C. DeWitt-Morette, *Analysis, Manifolds and Physics Part II*, North-Holland, Amsterdam (1989)

[3] T. Eguchi, P. B. Gilkey and A. J. Hanson: “Gravitation, gauge theories and differential geometry,” Phys. Rep. 66 (1980) 213

[4] E. Witten: “Constraints on supersymmetry breaking,” Nucl. Phys. B202 (1982) 253
[5] P. Windey: “Supersymmetric quantum mechanics and the Atiyah-Singer index theorem,” Acta Physica Polonica B15 No:5 (1984) 435

[6] L. Alvarez-Gaumé: “Supersymmetry and the Atiyah-Singer index theorem,” Commun. Math. Phys. 90 (1983) 161

[7] L. Alvarez-Gaumé: “A note on the Atiyah-Singer index theorem,” J. Phys. A16 no:5 (1983) 4177

[8] L. Alvarez-Gaumé, “Supersymmetry and Index Theorem” in Supersymmetry, Proceedings of the 1984 NATO School at Bonn, Eds. K. Dietz, R. Flume, G. V. Gehlen and V. Rittenberg, (1984)

[9] J. Mañes and B. Zumino: “WKB method, SUSY quantum mechanics and the index theorem,” Nucl. Phys. B270 (1986) 651

[10] M. W. Goodman: “Proof of character-valued index theorems,” Commun. Math. Phys. 107, (1986) 391

[11] E. Getzler: “Pseudodifferential operators on supermanifolds and the Atiyah-Singer index theorem,” Commun. Math. Phys. 92 (1983) 163

[12] D. Friedan and P. Windey: “Supersymmetric derivation of the Atiyah-Singer index theorem and the chiral anomaly,” Nucl. Phys. B235 (1984) 395

[13] M. F. Atiyah and I. M. Singer, Bull. Am. Math. Soc. 69 (1963) 422; M. F. Atiyah and I. M. Singer, Ann. Math. 87 (1968) 484; ibid., 546

[14] M. F. Atiyah, R. Bott and V. K. Patodi: “On the heat equation and the index theorem,” Inv. Math. 19 (1973) 279

[15] M. F. Atiyah, The Index of Elliptic Operators, Colloquium Lectures, Amer. Math. Soc., Dallas (1973), in Collected works vol. III, Oxford Uni. Press (1988)
[16] R. S. Palais, *Seminar on the Atiyah-Singer Index Theorem*, Ann. of Math. Study, vol. 57, Princeton Univ. Press (1965)

[17] P. Shanahan, *The Atiyah-Singer Index Theorem*, Lect. Notes in Math., vol. 638, Springer (1978)

[18] P. B. Gilkey, *Invariance Theory, the Heat Equation and the Atiyah-Singer Index Theorem*, Math. Lecture series vol. 11, Publish or Perish (1984)

[19] B. Booss and D. D. Bleecker, *Topology and Analysis*, Springer (1985)

[20] N. Berline, E. Getzler, M. Vergne, *Heat Kernels and Dirac Operators*, Springer (1991)

[21] M. F. Atiyah, *K Theory*, W. A. Benjamin, Inc. (1967)

[22] C. Nash, *Differential Topology and Quantum Field Theory*, Academic Press (1991)

[23] L. E. Gendenshtein and I. V. Krive: “Supersymmetry in quantum mechanics,” Sov. Phys. Usp. 28(8) (1986) 645

[24] R. E. Peierls: “The commutation laws of relativistic field theory,” Proc. Roy. Soc. (London) A214 (1952) 143

[25] A. Mostafazadeh: “Supersymmetry and the Atiyah-Singer Index Theorem II: The Scalar Curvature Factor in the Schrödinger Equation” (1993)

[26] S. Cecotti and L. Girardello: “Functional measure, topology and dynamical supersymmetry breaking,” Phys. Lett. 110B, no:1 (1982) 39

[27] C. Itzykson and J. B. Zuber, “Quantum Field Theory”, McGraw-Hills (1985)
[28] L. D. Faddeev, “Introduction to Functional Methods” in Les Houches Lectures XXVIII (1975), “Methods in Field Theory”, Ed.’s: R. Balian and J. Zinn-Justin, North-Holland.