Research Article

Existence and Uniqueness of Solutions for Fractional Boundary Value Problems under Mild Lipschitz Condition

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This paper deals with the following boundary value problem

\[ \begin{cases} D^\alpha u(t) = f(t, u(t)), & t \in (0, 1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \]

where \( 3 < \alpha \leq 4, D^\alpha \) is the standard Riemann–Liouville fractional derivative, and the nonlinearity \( f \), which could be singular at both \( t = 0 \) and \( t = 1 \), is required to be continuous on \( (0, 1) \times \mathbb{R} \) satisfying a mild Lipschitz assumption. Based on the Banach fixed point theorem on an appropriate space, we prove that this problem possesses a unique continuous solution \( u \) satisfying \( |u(t)| \leq c\omega(t) \), for \( t \in [0, 1] \) and \( c > 0 \), where \( \omega(t) = t^{\alpha-2}(1-t)^2 \).

1. Introduction

Higher order fractional differential equations subject to two-point boundary value problems occur naturally when modeling various phenomena in the applied sciences (see, for example, [1–4] and references therein). The study of existence, uniqueness, and qualitative properties of the solutions of such problems subject to various type of boundary conditions become an active area of research (see, for instance, [5–13] and references therein).

In [10], the authors considered the following problem

\[ \begin{cases} D^\alpha u(t) = f(t, u(t)), & t \in (0, 1), \\ u(0) = u'(0) = u(1) = 0, \end{cases} \]

where \( 3 < \alpha \leq 4, D^\alpha \) is the standard Riemann–Liouville fractional derivative.

By reducing problem (1) to an equivalent Fredholm integral equation and using some fixed-point theorems, they have proved the existence, multiplicity, and uniqueness of positive solutions. In their approach, properties of the corresponding Green’s function are used.

In [6], the authors proved the existence and uniqueness of positive solutions of the problem

\[ \begin{cases} D^\alpha u(t) = p(t)u'(t), & t \in (0, 1), \\ u(0) = u'(0) = u(1) = 0, \end{cases} \]

where \( \sigma \in (-1, 1), 3 < \alpha \leq 4, D^\alpha \) is the standard Riemann–Liouville fractional derivative. Their approach relies on properties of Karamata regular variation functions and the Schauder fixed point theorem.

Recently, in [13], Zou and He, by using the Banach fixed point theorem on an appropriate space, they have proved the existence and uniqueness of a solution to the following problem

\[ \begin{cases} D^\alpha u(t) + f(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = u'(0) = u(1) = 0, \end{cases} \]
where $2 < \alpha \leq 3$, $D^\alpha$ denotes the standard Riemann–Liouville fractional derivative and $f$ satisfying:

(C1) \( f \in C((0, 1) \times \mathbb{R}, \mathbb{R}) \) and $f(s, 0) \in L^1((0, 1))$.

(C2) There exists $p \in C((0, 1), [0, \infty))$ such that

\[ |f(t, u) - f(t, v)| \leq p(t)|u - v|, \forall t \in (0, 1), u, v \in \mathbb{R} \]

\[ 0 < \int_0^1 p(s)ds < \infty. \tag{4} \]

In this paper, following the approach used in [13], we will address the existence and uniqueness of a solution to the following fractional problem involving fractional boundary derivatives

\[ \begin{cases} D^\alpha u(t) = f(t, u(t)), t \in (0, 1), \\ u(0) = u(1) = D^{\alpha-1}u(0) = u'(1) = 0, \end{cases} \tag{5} \]

where $3 < \alpha \leq 4$ and $D^\alpha$ denotes the standard Riemann–Liouville fractional derivative. The nonlinear term $f$ is required to satisfy:

(H1) $f \in C((0, 1) \times \mathbb{R}, \mathbb{R})$ and \( \int_0^1 s(1-s)^{\alpha-1}|f(s, 0)|ds < \infty. \)

(H2) There exists $p \in C((0, 1), [0, \infty))$ such that

\[ |f(t, u) - f(t, v)| \leq p(t)|u - v|, \forall t \in (0, 1), u, v \in \mathbb{R} \]

\[ 0 < L_{p,\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}p(s)ds < \infty. \tag{6} \]

Remark 1. It is worth mentioning that problem (5) is different form problem (3) and that conditions (H1) and (H2) are weaker than (C1) and (C2). Indeed, for example, the function \( f(t, u) = (1-t)^{-\alpha/3} \cos u \) with $\alpha \in (3, 4]$ satisfies (H1) and (H2) (see Example 8) but not (C1) and (C2).

To simplify our statement, for $\alpha \in (3, 4]$, we fix the following notation:

(i) $G_\alpha(t, s)$ denotes the Green’s function of the operator $u \rightarrow D^\alpha u$ with boundary conditions $u(0) = u(1) = D^{\alpha-1}u(0) = u'(1) = 0$, which is given (see [6], Lemma 5) for $t, s \in [0, 1]$ by

\[ G_\alpha(t, s) = \frac{1}{\Gamma(\alpha)} \left( t^{\alpha-1}(s-t)^{\alpha-2}[s-t + (\alpha - 2)s(1-t)] + (\max (t-s, 0))^{\alpha-1}. \right) \tag{7} \]

(ii) Let $L$ be the minimum positive constant such that

\[ \int_0^1 G_\alpha(t, s)\omega(t)p(s)ds \leq L\omega(t), \tag{8} \]

where

\[ \omega(t) = t^{\alpha-2}(1-t)^{2}, \text{ for } t \in [0, 1]. \tag{9} \]

We will prove that $L$ satisfies

\[ \alpha(\alpha-2)L_{p,\alpha+1} \leq L \leq \gamma L_{p,\alpha}, \tag{10} \]

where $\gamma = \max ((\alpha - 2)^2, \alpha - 1)$.

(iii) $C_\omega([0, 1]): = \{ u \in C([0, 1]) \text{ there is } \sigma > 0 \text{ such that } |u(t)| \leq \sigma \omega(t), t \in [0, 1] \}$

Note that $C_\omega([0, 1])$ is a Banach space equipped with the $\| \cdot \|_\omega$-norm

\[ \|u\|_\omega = \inf \{ \sigma > 0 : |u(t)| \leq \sigma \omega(t), t \in [0, 1] \} = \sup_{t \in [0, 1]} \frac{|u(t)|}{\omega(t)}. \tag{11} \]

Our main result is the following.

Theorem 2. Assume that (H1) and (H2) hold and $L < 1$. Then, problem (5) has a unique solution $u \in C_\omega([0, 1])$.

In addition, for any $u_0 \in C_\omega([0, 1])$, the iterative sequence $u_k(t) = \int_0^t G_\alpha(t, s)f(s, u_{k-1}(s))ds$ converges to $u$ with respect to the $\| \cdot \|_\omega$-norm, and we have

\[ \|u_k - u\|_\omega \leq \frac{L^k}{1-L}\|u_1 - u_0\|_\omega. \tag{12} \]

Remark 3. Assume that (H1) and (H2) hold. If $\gamma L_{p,\alpha} < 1$, then from (10), it follows that $L < 1$.

Note that the condition $\gamma L_{p,\alpha} < 1$ is satisfied by a large class of functions $p$ including singular ones. As examples, we have

(i) If $p$ is a positive continuous function on $(0, 1)$ with $\sup_{t \in (0, 1)} |p(t)| \leq 1$, then

\[ \gamma L_{p,\alpha} < 1. \tag{13} \]

(ii) If $p(s) \equiv (1-s)^{-\alpha/3}$, then

\[ \gamma L_{p,\alpha} = \frac{\gamma}{\Gamma(\alpha)} \left( \alpha \left( \frac{2\alpha}{3} \right) \right) < 1. \tag{14} \]
Lemma 5. Let $\gamma L_{p,\alpha} = \frac{\gamma}{\Gamma(\alpha)} \left( \frac{2\alpha}{3}, \frac{2\alpha}{3} \right) < 1$, \hspace{1cm} (15)

where $\Gamma$ denotes the Beta function.

The paper is organized as follows. In Section 2, we recall basic properties of the Green’s function $G_\alpha(t, s)$, and we prove that $L$ satisfies the range estimates (10). In Section 3, we prove our main result. To illustrate our existence result, some examples and approximations are given.

2. Preliminaries

For the convenience of the reader, we recall the following definition.

Definition 4 ([2, 14]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a measurable function $g : (0, \infty) \rightarrow \mathbb{R}$ is defined as

$$D^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t (t-s)^{n-1} g(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$, where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of $\alpha$.

The next key lemma is useful. For the proof, we refer the reader to [6].

Lemma 5. Let $G_\alpha(t, s)$ be the Green’s function given by (7). Then

(i) $G_\alpha(t, s)$ is a nonnegative continuous function on $[0, 1] \times [0, 1]$.

(ii) For all $t, s \in [0, 1]$, we have

$$(\alpha - 2)H_\alpha(t, s) \leq G_\alpha(t, s) \leq \gamma H_\alpha(t, s),$$

where $H_\alpha(t, s) := (1/\Gamma(\alpha))(t^{\alpha-3}(1-t)s^{\alpha-3} \min(t, s)(1 - \max(t, s))$.

(iii) Let $g$ be a function such that the map $t \rightarrow t^2 (1-t)^{\alpha-2} g(t)$ is continuous and integrable on $(0, 1)$. Then, $Vg(t) := \int_0^1 G_\alpha(t, s)g(s) ds$ is the unique solution in $C([0, 1])$ of the boundary value problem

$$\begin{cases}
D^\alpha u(t) = g(t), t \in (0, 1), \\
u(0) = u(1) = D^{\alpha-3}u(0) = u'(1) = 0.
\end{cases}$$

Lemma 6. Let $p \in C((0, 1), [0, \infty))$ such that $0 < L_{p,\alpha} < \infty$. Then

$$\alpha(\alpha - 2)L_{p,\alpha+1} \leq L \leq \gamma L_{p,\alpha},$$

where $\gamma = \max ((\alpha - 2)^2, \alpha - 1)$, and $L$ is the constant defined in (8).

Proof. Consider the set

$$E = \left\{ a > 0 : \int_0^1 G_\alpha(t, s)\omega(s) p(s) ds \leq a\omega(t), t \in [0, 1] \right\},$$

where $\omega(t) = t^\alpha - (1-t)^\alpha, t \in [0, 1]$.

From the upper inequality in (17), we obtain

$$\int_0^1 G_\alpha(t, s)\omega(s) p(s) ds \leq \gamma \int_0^1 H_\alpha(t, s)\omega(s) p(s) ds \leq \gamma L_{p,\alpha}\omega(t).$$

(21)

It follows that $E \neq \emptyset$ and $L \leq \gamma L_{p,\alpha}$.

On the other hand, from the lower inequality in (17), we deduce for any $a \in E$ that

$$a\omega(t) \geq \frac{(\alpha - 2)}{\Gamma(\alpha)} t^{\alpha-3}(1-t)^{\alpha-1} \min(t, s) \cdot (1 - \max(t, s)) \omega(t) ds
\geq \frac{(\alpha - 2)}{\Gamma(\alpha)} t^{\alpha-3}(1-t)^{\alpha-1} t \omega(t) (1-t) (1-s) p(s) ds$$

$$= a(\alpha - 2)\omega(t) L_{p,\alpha+1}.$$

(22)

Hence, for each $a \in E$,

$$a \geq a(\alpha - 2) L_{p,\alpha+1}.$$

(23)

Therefore, $L \geq a(\alpha - 2) L_{p,\alpha+1}$. That is $L \in [a(\alpha - 2) L_{p,\alpha+1}, \gamma L_{p,\alpha}]$.

3. Main Results

Assume that (H1) and (H2) hold and $L < 1$. We prove that problem (5) has a unique solution $u$ in $C_\omega([0, 1])$. In addition, for any $u_0 \in C_\omega([0, 1])$, the iterative sequence $u_k(t) = \int_0^1 G_\alpha(t, s)f(s, u_{k-1}(s)) ds$ converges to $u$ with respect to the $\|\cdot\|_\omega$-norm, and we have

$$\|u_k - u\|_\omega \leq \frac{L^k}{1 - L} \|u_1 - u_0\|_\omega.$$

(24)

Consider the operator $T$ defined by

$$Tu(t) := \int_0^1 G_\alpha(t, s)f(s, u(s)) ds, t \in [0, 1], u \in C_\omega([0, 1]).$$

(25)

We claim that $T$ is a contraction operator from $(C_\omega([0, 1]), \|\cdot\|_\omega)$ into itself.
Indeed, let $u \in C_{\alpha}(0, 1]$ and $\sigma > 0$ such that $|u(t)| \leq \sigma \omega(t)$, for all $t \in [0, 1]$.

Since by Lemma 5 (ii), $0 \leq G_{\alpha}(t, s) \leq (\gamma / \Gamma(\alpha))(1-s)^{\alpha-1}$, it follows from (H2) that

$$\begin{align*}
|G_{\alpha}(t, s)f(s, u(s))| &\leq \frac{\gamma}{\Gamma(\alpha)}(1-s)^{\alpha-3}(|f(s, u(s)) - f(s, 0)| + |f(s, 0)|) \\
&\leq \frac{\gamma}{\Gamma(\alpha)}(1-s)^{\alpha-3}(p(s)|u(s)| + |f(s, 0)|) \\
&\leq \frac{\gamma}{\Gamma(\alpha)}(\sigma^{\alpha-1}(1-s)^{\alpha-3}p(s) + s(1-s)^{\alpha-3}|f(s, 0)|).
\end{align*}$$

(26)

Using the fact that $G_{\alpha}(t, s)$ is continuous on $[0, 1] \times [0, 1]$, we deduce from (H1), (H2), and the dominated convergence theorem that $Tu \in C([0, 1])$.

On the other hand, from Lemma 6 (ii), we have

$$0 \leq G_{\alpha}(t, s) \leq \frac{\gamma}{\Gamma(\alpha)}(1-s)^{\alpha-3} \omega(t).$$

(27)

By using (27) and similar arguments as above, we obtain

$$|Tu(t)| \leq \gamma \left[ \frac{\sigma L_{p,\alpha}}{\Gamma(\alpha)} \int_0^1 (s-1-s)^{\alpha-3}|f(s, 0)|ds \right] \omega(t).$$

(28)

So, $T(C_{\alpha}([0, 1])) \subset C_{\alpha}([0, 1])$.

Let $u, v \in C_{\alpha}([0, 1])$. By using (H2) and (8), we obtain for $t \in [0, 1]$,

$$\begin{align*}
|Tu(t) - Tv(t)| &\leq \int_0^1 G_{\alpha}(t, s)|f(s, u(s)) - f(s, v(s))|ds \\
&\leq \int_0^1 G_{\alpha}(t, s)p(s)|u(s) - v(s)|ds \\
&\leq \|u - v\|_\omega \int_0^1 G_{\alpha}(t, s)p(s)\omega(s)ds \leq L\|u - v\|_\omega \omega(t).
\end{align*}$$

Hence

$$\|Tu - Tv\|_\omega \leq L\|u - v\|_\omega.$$  

(30)

Since $L < 1$, we deduce that $T$ is a contraction operator on $C_{\alpha}([0, 1])$. Hence, there exists a unique $u \in C_{\alpha}([0, 1])$ satisfying

$$u(t) = \int_0^1 G_{\alpha}(t, s)f(s, u(s))ds, t \in [0, 1].$$

(31)

We need to prove that $u$ is a solution of problem (5). Indeed, it is clear that $s \rightarrow s^2(1-s)^{\alpha-2}f(s, u(s))$ is continuous on $(0, 1)$.

On the other hand, by writing

$$|f(s, u(s))| \leq |f(s, u(s)) - f(s, 0)| + |f(s, 0)| \leq p(s)|u(s)| + |f(s, 0)|,$$

(32)

we deduce from (H1), (H2), and $u \in C_{\alpha}([0, 1])$ that the function $s \rightarrow s^2(1-s)^{\alpha-2}f(s, u(s))$ is integrable on $(0, 1)$.

Hence, from (31) and Lemma 5 (iii), we conclude that $u$ is the unique solution of problem (5).

Finally, it is well known that for any $u_0 \in C_{\alpha}([0, 1])$, the iterative sequence $u_k := T(u_{k-1})$ converges to $u$, and we have

$$\|u_k - u\|_\omega \leq \frac{L^k}{1 - L}\|u_1 - u_0\|_\omega.$$  

(33)

**Corollary 7.** Let $3 < \alpha \leq 4$ and $f$ be a function satisfying (H1) and (H3) for all $t \in (0, 1)$ and $u, v \in \mathbb{R}$

$$|f(t, u) - f(t, v)| \leq L_d|u - v|,$$

(34)

where $0 < L_d < \Gamma(2\alpha)/\max((5\alpha - 2)^2, \alpha - 1)\Gamma(\alpha)$.

Then, problem (5) has a unique solution $u$ in $C_{\alpha}([0, 1])$.

**Proof.** The conclusion follows from Lemma 6 and Theorem 2.

**Example 8.** Let $3 < \alpha \leq 4, \gamma = \max((5\alpha - 2)^2, \alpha - 1)$ and consider the following singular fractional problem:

$$\begin{align*}
D^{\alpha}u(t) &= (1 - t)^{-\alpha/3} \cos u, \\
u(0) &= u(1) = D^{\alpha-3}u(0) = u'(1) = 0.
\end{align*}$$

(35)

To verify that hypotheses (H1) and (H2) are satisfied, set $f(t, u) := (1 - t)^{-\alpha/3} \cos u$, for all $t \in (0, 1)$ and $u \in \mathbb{R}$.

We have $f \in C((0, 1) \times \mathbb{R})$ and $\int_0^1 s^{1-1}(1-s)^{\alpha-3}ds < \infty$. So condition (H1) is satisfied.

On the other hand, we have for all $t \in (0, 1)$ and $u, v \in \mathbb{R}$

$$|f(t, u) - f(t, v)| \leq p(t)|u - v|,$$

(36)

where $p(t) = (1 - t)^{-\alpha/3}$ and

$$0 < L_{p, \alpha} = \frac{1}{\Gamma(\alpha)} \int_0^1 s^{\alpha-1}(1-s)^{2 \alpha-3}ds < \infty.$$  

(37)

Furthermore, since by Lemma 6 and Remark 3 (ii), $L \leq \gamma L_{p, \alpha} < 1$, we deduce from Theorem 2 that problem (35) has a unique solution $u \in C_{\alpha}([0, 1])$.

In particular, for $\alpha = 7/2$, this solution is approximated (see Figure 1) by the iterative sequence $u_k(t) = \int_0^1 G_{7/2}(t, s)(1-s)^{-7/2} \cos(u_{k-1}(s))ds$ with $u_0(t) = \gamma^{1/2}(1-t)^2, t \in [0, 1]$.

The sequences of functions $u_0(t), u_1(t), u_2(t), u_3(t)$, and $u_4(t)$ are illustrated in Figure 1.
Example 9. Consider the following singular fractional problem:

\[
\begin{aligned}
D^{10/3} u(t) &= t^{-10/9} (1 - t)^{-10/9} (1 + u(t)), \\ u(0) &= u(1) = D^{1/3} u(0) = u'(1) = 0.
\end{aligned}
\] (38)

By using similar arguments as in the previous example, we verify that conditions \((H1)\) and \((H2)\) are fulfilled.

Hence, by applying Theorem 2, problem (38) has a unique solution \(u \in C_\omega([0, 1])\).

Furthermore, the iterative sequence defined by \(u_k(t) = \int_0^t G_{10/3}(t, s) s^{-10/9} (1 - s)^{-10/9} (1 + u_{k-1}(s)) ds\) and \(u_0(t) = t^{4/3} (1 - t)^2\) converges to \(u\). Some iterations are depicted in Figure 2.

**Data Availability**

No data were used to support this study.
Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors’ Contributions

All authors contributed equally to writing of this paper. All authors read and approved the final manuscript.

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