Generalized Harmonic Progression

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Abstract

This paper presents formulae for the sum of the terms of a harmonic progression of order $k$ with integer parameters, $HP_k(n)$, and for the partial sums of its two associated Fourier series, $C_{k}^m(a, b, n)$ and $S_{k}^m(a, b, n)$. These new formulae are a generalization of the formulae created in a previous paper and were achieved using a slightly modified version of the reasoning employed before.

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1 Introduction

Building upon the results of [3], Generalized Harmonic Numbers, this new paper demonstrates how to obtain exact formulae for the sum of the first $n$ terms of a harmonic progression
of order $k$ with integer parameters, $a$ and $b$:

$$ HP_k(n) = \sum_{j=1}^{n} \frac{1}{(aj + b)^k} $$

Even though formulae for $HP_k(n)$ can probably be created using the digamma function, $\psi$, the ones derived here are arguably more interesting, and they must be the first such formulae other than $\psi$.

We also create formulae for the partial sums of two Fourier series associated with $HP_k(n)$:

$$ C_m^k(a, b, n) = \sum_{j=1}^{n} \frac{1}{(aj + b)^k} \cos \frac{2\pi(aj + b)}{m} $$

and

$$ S_m^k(a, b, n) = \sum_{j=1}^{n} \frac{1}{(aj + b)^k} \sin \frac{2\pi(aj + b)}{m} $$

My first manuscript\(^3\) received some criticism for including too many details, so in this one we omit unnecessary details, which can be easily understood from a reading of [3], for the interested reader.

This new manuscript is concise and focuses solely on closed-forms for $HP_k(n)$, $C_m^k(a, b, n)$ and $S_m^k(a, b, n)$\(^1\). Please refer to [4] and [5] for the demonstrations on how to obtain the limits of these expressions as $n$ approaches infinity, and to [6] for how to obtain their analytic continuations.

We make use of Faulhaber’s formula\(^2\) for the sum of the $i$-th powers of the first $n$ positive integers:

$$ \sum_{k=1}^{n} k^i = \sum_{j=0}^{i} \frac{(-1)^j i! B_j n^{i+1-j}}{(i+1-j)! j!}, $$

where $B_j$ are the Bernoulli numbers.

Since odd Bernoulli numbers are always 0, except for $B_1$, the above formula can be simplified for even and odd powers as follows:

$$ \sum_{k=1}^{n} k^{2i} = \frac{n^{2i}}{2} + \sum_{j=0}^{i} \frac{(2i)! B_{2j} n^{2i+1-2j}}{(2j)!(2i+1-2j)!} \quad (1) $$

$$ \sum_{k=1}^{n} k^{2i+1} = \frac{n^{2i+1}}{2} + \sum_{j=0}^{i} \frac{(2i+1)! B_{2j} n^{2i+2-2j}}{(2j)!(2i+2-2j)!} \quad (2) $$

2 Lagrange’s Identities

In [3] we introduced an indicator function, $k$ divides $n$ ($\mathbb{1}_{k|n}$), and its analog as key elements of the method used to solve the harmonic numbers. For the harmonic progression, we need to

\(^1\)In a stricter sense, closed-forms don’t include integrals.
modify those functions and obtain their Taylor series, though unlike in the first paper here we focus on only one of them, as the other one has an analogous behavior.

The below is the function we need for the odd case. We can write this sum as a closed-form using Lagrange’s trigonometric identities:

\[
\sum_{j=1}^{k} \sin \frac{2\pi n(a_j + b)}{k} = -\frac{1}{2} \sin \frac{2\pi bn}{k} + \frac{1}{2} \sin 2\pi n \left( a + \frac{b}{k} \right) + \sin \pi n \left( a + \frac{2b}{k} \right) \sin \pi an \cot \frac{\pi an}{k}
\]

Now, we can derive a power series for the left-hand side of the above equation with the employment of (2), and through comparison we can deduce the following power series for each function on the right-hand side (they hold for all real \( n, a \) and \( b \) and for all integer \( k \geq 1 \)):

\[
\sum_{i=0}^{\infty} (-1)^i \left( \frac{2\pi bn}{k} \right)^{2i+1} \sum_{j=0}^{i} \left( \frac{(ak/b)^{2j}}{(2j)!(2i+1-2j)!} + \frac{(ak/b)^{2j+1}}{(2j+1)!(2i+2-2j)!} \right) = \sin 2\pi n \left( a + \frac{b}{k} \right) \quad (3)
\]

\[
\sum_{i=0}^{\infty} (-1)^i \left( \frac{2\pi bn}{k} \right)^{2i+1} \sum_{j=0}^{i} \left( \frac{(a/b)^{2j}}{(2i+1-2j)!} \sum_{p=0}^{j} \frac{B_{2p}k^{2j+1-2p}}{(2j+1-2p)!(2p)!} + \frac{(a/b)^{2j+1}}{(2j+1-2p)!} \sum_{p=0}^{j} \frac{B_{2p}k^{2j+2-2p}}{(2j+2-2p)!(2p)!} \right) = \sin \pi n \left( a + \frac{2b}{k} \right) \sin \pi an \cot \frac{\pi an}{k} \quad (4)
\]

For the harmonic progressions of even order, the analogous function is:

\[
\sum_{j=1}^{k} \cos \frac{2\pi n(a_j + b)}{k} = -\frac{1}{2} \cos \frac{2\pi bn}{k} + \frac{1}{2} \cos 2\pi n \left( a + \frac{b}{k} \right) + \cos \pi n \left( a + \frac{2b}{k} \right) \sin \pi an \cot \frac{\pi an}{k}
\]

### 3 Approach based on sine

The rationale to build a formula for \( HP_k(n) \) is to use the Taylor series expansion of \( \sin 2\pi (a_j + b) \),\(^1\) and exploit the fact that it’s 0 for all integer \( a_j + b \) (hence the need for \( a \) and \( b \) to be integers):

\[
\sin 2\pi (a_j + b) = 0 \Rightarrow 2\pi (a_j + b) = -\sum_{i=1}^{\infty} \frac{(-1)^i}{(2i+1)!} (2\pi (a_j + b))^{2i+1} \quad (5)
\]

If one divides both sides of (5) by \( 2\pi (a_j + b)^2 \), one obtains a power series for \( 1/(a_j + b) \) that only holds for integer \( a_j + b \).
Besides, on the right-hand side of the resulting equation the exponents of \( a j + b \) are positive integers, allowing us to apply the Faulhaber’s formula mentioned in the introduction. By doing so we end up with a very convoluted power series that can be turned into integrals by means of equations (3) and (4), which in turn were derived using Lagrange’s identities. That about sums up the logic.

### 3.1 Harmonic Progression

We start by dividing both sides of (5) by \( 2\pi(a j + b)^2 \):

\[
\frac{1}{a j + b} = \sum_{i=0}^{\infty} \frac{(-1)^i(2\pi)^{2i+2}(a j + b)^{2i+1}}{(2i + 3)!}
\]  

(6)

Now, we sum \( 1/(aj+b) \) over \( j \) and expand \( (aj+b)^{2i+1} \) using the binomial theorem and apply the respective Faulhaber’s formulae, ending up with the following power series after all the calculations are done:

\[
\sum_{j=1}^{n} \frac{1}{aj+b} = -\frac{1}{2b} + \frac{1}{2} \sum_{i=0}^{\infty} (-1)^i \frac{(2\pi b)^{2i+2}(2i + 1)!}{b(2i + 3)!} \sum_{j=0}^{i} \frac{(an/b)^{2j}}{(2j)!}(2j+1-2j)!(2j+1-2j)! + \frac{(an/b)^{2j+1}}{(2j+1)!}(2j+1-2j)! + \frac{(an/b)^{2j+1}}{(2j+1)!}(2j+1-2j)! + \frac{B_{2p}n^{2j+2-2p}}{(2j+2-2p)!}(2p)!
\]

The above sums can be manipulated conveniently and then obtained from (3) and (4), the two equations we derived previously, by means of decompositions into linear combinations followed by integrations. Though we have omitted the details here, the reader can refer to the precursor paper for a more detailed description of the steps involved.

After all the appropriate calculations are performed, one ends up with:

\[
\frac{1}{2b} \sum_{i=0}^{\infty} (-1)^i \frac{(2\pi b)^{2i+2}(2i + 1)!}{(2i + 3)!} \sum_{j=0}^{i} \frac{(an/b)^{2j}}{(2j)!}(2j+1-2j)! + \frac{(an/b)^{2j+1}}{(2j+1)!}(2j+1-2j)! = \frac{2\pi(an+b) - \sin 2\pi(an+b)}{4\pi(an+b)^2}
\]

\[
\sum_{i=0}^{\infty} (-1)^i \frac{(2\pi b)^{2i+2}(2i + 1)!}{b(2i + 3)!} \sum_{j=0}^{i} \frac{(a/b)^{2j}}{(2j+1-2j)!}(2j+1-2j)! + \frac{(a/b)^{2j+1}}{(2j+1)!}(2j+1-2j)! + \frac{B_{2p}n^{2j+2-2p}}{(2j+2-2p)!}(2p)! = 2\pi \int_{0}^{1} (1-u) \sin \pi anu \sin \pi(an+2b)u \cot \pi au \, du
\]

Now, summing up all the results (disregarding the sine of multiples of \( \pi \)), we arrive at a formula for \( HP(n) \):

\[
\sum_{j=1}^{n} \frac{1}{aj+b} = -\frac{1}{2b} + \frac{1}{2(an+b)} + 2\pi \int_{0}^{1} (1-u) \sin \pi(an+2b)u \sin \pi anu \cot \pi au \, du
\]

In the next sections we state a brief generalization of this result.
3.2 General Formula

If we keep dividing (6) by \(aj + b\), we obtain recursions similar to the ones we obtained for the generalized harmonic numbers. All the results presented next follow from reasonings analogous to those from reference [3].

3.3 Harmonic Progression of Order \(2k\)

We have the following recursion for \(HP_{2k}(n)\):

\[
HP_{2k}(n) = -\frac{1}{2b^{2k}} \sum_{j=0}^{k} \frac{(-1)^j(2\pi b)^{2j}}{(2j+1)!} + \frac{1}{2(an+b)^{2k}} \sum_{j=0}^{k} \frac{(-1)^j(2\pi(an+b))^{2j}}{(2j+1)!}
\]

\[
-\sum_{j=1}^{k-1} \frac{(-1)^{k-j}(2\pi)^{2k-2j}}{(2k+1-2j)!} HP_{2j}(n) - \frac{(-1)^{k}(2\pi)^{2k}}{(2k)!} \int_{0}^{1} (1-u)^{2k} \cos \pi(an+2b)u \sin \pi an u \cot \pi au du
\]

Note that \(HP_{0}(n) = 0\) for all positive integer \(n\) (just like \(H_{0}(n) = 0\), previously). Therefore, from the recursion, we conclude that for all integer \(a \neq 0, b \neq 0\) and \(k \geq 1\):

\[
\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k}} = -\frac{1}{2b^{2k}} + \frac{1}{2(an+b)^{2k}} + \frac{(-1)^{k}(2\pi)^{2k}}{2} \int_{0}^{1} \sum_{j=0}^{k} \frac{B_{2j}(2-2^{2j})(1-u)^{2k-2j}}{(2j)!(2k-2j)!} \cos \pi(an+2b)u \sin \pi an u \cot \pi au du
\]

And if we turn the product of cosine and sine into a sum of sines, and replace \(1 - u\) with \(u\) (this doesn’t change the integral or each of its individual parts), we derive another, perhaps more useful, way to express this formula:

\[
\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k}} = -\frac{1}{2b^{2k}} + \frac{1}{2(an+b)^{2k}}
\]

\[
-\frac{(-1)^{k}(2\pi)^{2k}}{2} \int_{0}^{1} \sum_{j=0}^{k} \frac{B_{2j}(2-2^{2j})u^{2k-2j}}{(2j)!(2k-2j)!} \sin 2\pi(an+b)u - \sin 2\pi bu \cot \pi au du
\]

This formula also holds for the generalized harmonic numbers \((a = 1, b = 0)\), if the term \(-1/(2b^{2k})\) is disregarded. In fact, if we disregard any term of the equation that has a null denominator, the equation still holds.
3.4 Harmonic Progression of Order $2k + 1$

We have the following recursion for $HP_{2k+1}(n)$:

$$HP_{2k+1}(n) = -\frac{1}{2b^{2k+1}} \sum_{j=0}^{k} (-1)^j (2\pi b)^{2j} + \frac{1}{2(an+b)^{2k+1}} \sum_{j=0}^{k} (-1)^j (2\pi(an+b))^{2j} \frac{k}{(2j+1)!}$$

$$-\sum_{j=0}^{k-1} \frac{(-1)^{k-j}(2\pi)^{2k-2j}}{(2k+1-2j)!} HP_{2j+1}(n) + \frac{(-1)^{k}(2\pi)^{2k+1}}{(2k+1)!} \int_0^1 (1-u)^{2k+1} \sin(\pi(an+2b)u \sin \pi an \cot \pi au du$$

Therefore, for all integer $a \neq 0$, $b \neq 0$ and $k \geq 0$:

$$\sum_{j=1}^{n} \frac{1}{(aj + b)^{2k+1}} = -\frac{1}{2b^{2k+1}} + \frac{1}{2(an+b)^{2k+1}}$$

$$+ \frac{(-1)^{k}(2\pi)^{2k+1}}{2} \int_0^1 \sum_{j=0}^{k} \frac{B_{2j} (2 - 2^{2j}) (1-u)^{2k+1-2j}}{(2j)! (2k+1-2j)!} \sin(\pi(an+2b)u \sin \pi an \cot \pi au du$$

Again, if we turn the product of sines into a sum of cosines, and replace $1-u$ with $u$ (this only flips the sign of the integral, ditto for each of its individual parts), we obtain:

$$\sum_{j=1}^{n} \frac{1}{(aj + b)^{2k+1}} = -\frac{1}{2b^{2k+1}} + \frac{1}{2(an+b)^{2k+1}}$$

$$+ \frac{(-1)^{k}(2\pi)^{2k+1}}{2} \int_0^1 \sum_{j=0}^{k} \frac{B_{2j} (2 - 2^{2j}) u^{2k+1-2j}}{(2j)! (2k+1-2j)!} (\cos 2\pi(an+b)u - \cos 2\pi bu) \cot \pi au du$$

3.5 Generating functions

As seen in [3], the polynomials in $u$ within the integrals are generated by the functions:

$$f(x) = \frac{x \cos x(1-u)}{\sin x} \Rightarrow f^{(2k)}(0) = \frac{(-1)^k}{(2k)!} \sum_{j=0}^{k} B_{2j} (2 - 2^{2j}) (1-u)^{2k-2j}$$

$$= \frac{(-1)^k}{(2k)!} \sum_{j=0}^{k} B_{2j} (2 - 2^{2j}) (1-u)^{2k+1-2j}$$

4 Approach based on exponential

This approach consists in using equation $e^{2\pi i(a+b)} = 1$, which combines the cosine and sine equations into one. It has a possible advantage over the sine-based approach, since it yields a single formula for both odd and even powers. For this one we skip the step-by-step demonstration and go straight to the final result.
4.1 General formula

For all integer $a$, $b$ and $k \geq 1$:

$$
\sum_{j=1}^{n} \frac{1}{(aj+b)^k} = -\frac{1}{2b^k} + \frac{1}{2(an+b)^k} + \frac{i(2\pi i)^k}{2} \int_{0}^{1} \sum_{j=0}^{k} B_j (1-u)^{k-j} j! (k-j)! \left( e^{2\pi i(an+b)u} - e^{2\pi i bu} \right) \cot \pi au \, du
$$

We can introduce a constant $c$ in the formula such that, provided that $ca$ and $cb$ are both integers, the following modified formula still holds:

$$
\sum_{j=1}^{n} \frac{1}{(aj+b)^k} = -\frac{1}{2b^k} + \frac{1}{2(an+b)^k} + \frac{i(-2\pi c)^k}{2} \int_{0}^{1} \sum_{j=0}^{k} B_j u^{k-j} j! (k-j)! \left( e^{2\pi i c(an+b)u} - e^{2\pi i c bu} \right) \cot \pi cu \, du
$$

Note the negative sign is a consequence of replacing $1-u$ with $u$, it has nothing to do with the constant. The constant can be useful. For example, if we have $a = 1$ and $b = 1/3$, setting $c = 3$ will make the formula right.

4.2 Decoupling

It’s possible to remove the complex numbers out of the picture, in which case the formula becomes:

$$
\sum_{j=1}^{n} \frac{1}{(aj+b)^k} = -\frac{1}{2b^k} + \frac{1}{2(an+b)^k} - (2\pi)^k \int_{0}^{1} \sum_{j=0}^{k} B_j (1-u)^{k-j} j! (k-j)! \cos \left( (an + 2b) \frac{k\pi}{2} \right) \sin \pi au \cot \pi au \, du
$$

which in turn can be transformed into forms pretty similar to the originals from (3.2).

For the even powers we have:

$$
\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k}} = -\frac{1}{2b^{2k}} + \frac{1}{2(an+b)^{2k}} - (-1)^k (2\pi)^{2k} \int_{0}^{1} \left( -\frac{u^{2k-1}}{2(2k-1)!} + \sum_{j=0}^{k} \frac{B_{2j} u^{2k-2j}}{(2j)! (2k-2j)!} \right) \left( \sin 2\pi (an + b)u - \sin 2\pi bu \right) \cot \pi au \, du
$$
and for the odd powers:

\[
\begin{align*}
\sum_{j=1}^{n} \frac{1}{(aj + b)^{2k+1}} &= -\frac{1}{2b^{2k+1}} + \frac{1}{2(an + b)^{2k+1}} \\
&+ \frac{(-1)^k(2\pi)^{2k+1}}{2} \int_0^1 \left( -\frac{u^{2k}}{2(2k)!} + \sum_{j=0}^{k} \frac{B_{2j}u^{2k+1-2j}}{(2j)!(2k+1-2j)!} \right) \left( \cos 2\pi(an + b)u - \cos 2\pi bu \right) \cot \pi au \, du
\end{align*}
\]

### 4.3 Generating function

The generating function of the polynomial in \( u \) within the integral is:

\[
f(x) = \frac{xe^{x(1-u)}}{e^x - 1} \Rightarrow f^{(k)}(0) = \sum_{j=0}^{k} \frac{B_j(1-u)^{k-j}}{j!(k-j)!}
\]

### 5 The partial Fourier series

Let’s briefly recall the formulae we found for \( C_m^n(n) \) and \( S_m^n(n) \), the partial sums of the Fourier series associated with the generalized harmonic numbers, \( H_k(n) \), from reference [3]. Next to each one we show their harmonic progression analogs, \( C_m^m(a,b,n) \) and \( S_m^m(a,b,n) \), which are based entirely on analogy.

The following expressions hold for all complex \( m, a \) and \( b \) (unlike the \( HP_k(n) \) formulae), and for all integer \( n \geq 1 \). As mentioned before, we can disregard any term that has zero in the denominator, and the equation still holds (technically, we take the limit as the parameter tends to 0). See [5] and [6], for examples.

By definition \( H_0(n) = 0 \) and \( HP_0(n) = 0 \) for all positive integer \( n \), so they have no effect in the sums and to avoid confusion we skip them in the formulae.

#### 5.1 \( C_m^m(n) \) and \( C_m^m(a,b,n) \)

For all integer \( k \geq 1 \):

\[
\begin{align*}
\sum_{j=1}^{n} \frac{1}{j^{2k}} \cos \frac{2\pi j}{m} &= \frac{1}{2\pi^2} \left( \cos \frac{2\pi n}{m} - \sum_{j=0}^{k} \frac{(-1)^j(\frac{2\pi}{m})^{2j}}{(2j)!} \right) + \sum_{j=1}^{k} \frac{(-1)^{k-j}(\frac{2\pi}{m})^{2k-2j}}{(2k-2j)!} H_{2j}(n) \\
&+ \frac{(-1)^k(\frac{2\pi}{m})^{2k}}{2(2k-1)!} \int_0^1 (1-u)^{2k-1} \sin \frac{2\pi nu}{m} \cot \frac{\pi u}{m} \, du
\end{align*}
\]
\[
\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k}} \cos \frac{2\pi j}{m} = -\frac{1}{2b^{2k+1}} \left( \cos \frac{2\pi b}{m} - \sum_{j=0}^{k-1} (-1)^{j} \left(\frac{2\pi b}{m}\right)^{2j} \frac{1}{(2j)!} \right) \\
+ \frac{1}{2(2an+b)^{2k}} \left( \cos \frac{2\pi (an+b)}{m} - \sum_{j=0}^{k-1} (-1)^{j} \left(\frac{2\pi (an+b)}{m}\right)^{2j} \frac{1}{(2j)!} \right) + \frac{1}{2(2k-1)!} \int_{0}^{1} (1-u)^{2k-1} \left( \sin \frac{2\pi (an+b)u}{m} - \sin \frac{2\pi bu}{m} \right) \cot \frac{\pi au}{m} \, du
\]

5.2 \( S_{2k+1}^{m}(n) \) and \( S_{2k+1}^{m}(a, b, n) \)

For all integer \( k \geq 0 \):

\[
\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k+1}} \sin \frac{2\pi j}{m} = \frac{1}{2n^{2k+1}} \left( \sin \frac{2\pi n}{m} - \sum_{j=0}^{k} (-1)^{j} \left(\frac{2\pi n}{m}\right)^{2j+1} \frac{1}{(2j+1)!} \right) \\
+ \frac{1}{2(2an+b)^{2k+1}} \left( \sin \frac{2\pi (an+b)}{m} - \sum_{j=0}^{k} (-1)^{j} \left(\frac{2\pi (an+b)}{m}\right)^{2j+1} \frac{1}{(2j+1)!} \right) + \frac{1}{2(2k-1)!} \int_{0}^{1} (1-u)^{2k} \left( \sin \frac{2\pi (an+b)u}{m} - \sin \frac{2\pi bu}{m} \right) \cot \frac{\pi au}{m} \, du
\]

5.3 \( C_{2k+1}^{m}(n) \) and \( C_{2k+1}^{m}(a, b, n) \)

For all integer \( k \geq 0 \):

\[
\sum_{j=1}^{n} \frac{1}{j^{2k+1}} \cos \frac{2\pi j}{m} = \frac{1}{2n^{2k+1}} \left( \cos \frac{2\pi n}{m} - \sum_{j=0}^{k-1} (-1)^{j} \left(\frac{2\pi n}{m}\right)^{2j} \frac{1}{(2j)!} \right) \\
+ \frac{1}{2(2an+b)^{2k+1}} \left( \cos \frac{2\pi (an+b)}{m} - \sum_{j=0}^{k-1} (-1)^{j} \left(\frac{2\pi (an+b)}{m}\right)^{2j} \frac{1}{(2j)!} \right) + \frac{1}{2(2k-1)!} \int_{0}^{1} (1-u)^{2k} \left( \cos \frac{2\pi (an+b)u}{m} - \cos \frac{2\pi bu}{m} \right) \cot \frac{\pi au}{m} \, du
\]
5.4 $S_{2k}^m(n)$ and $S_{2k}^m(a, b, n)$

For all integer $k \geq 1$:

\[
\sum_{j=1}^{n} \frac{1}{j^{2k}} \sin \frac{2\pi j}{m} = \frac{1}{2n^{2k}} \left( \sin \frac{2\pi n}{m} - \sum_{j=0}^{k-1} (-1)^j \left( \frac{2\pi n}{m} \right)^{2j+1} \frac{1}{(2j+1)!} \right) - \sum_{j=0}^{k-1} (-1)^{k-j} \left( \frac{2\pi}{m} \right)^{2k-1-2j} \frac{1}{(2k-1-2j)!} H_{2j+1}(n)
\]

\[
- \frac{(-1)^k \left( \frac{2\pi}{m} \right)^{2k}}{2(2k-1)!} \int_0^1 (1-u)^{2k-1} \left( \cos \frac{2\pi nu}{m} - 1 \right) \cot \frac{\pi u}{m} du
\]

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