Exact Non-Markovian Evolution with Several Reservoirs

A. E. Teretenkov\textsuperscript{a, b, *}

\textsuperscript{a}Steklov Mathematical Institute, Russian Academy of Sciences, Moscow, 119991 Russia
\textsuperscript{b}Moscow State University, Moscow, 119991 Russia
* e-mail: taemsu@mail.ru

Received December 20, 2019; revised January 16, 2020; accepted January 29, 2020

Abstract—The model of a multilevel system interacting with several reservoirs is considered. The exact reduced evolution of a system’s density matrix can be obtained for this model without using the Markov approximation. Namely, this evolution is completely defined by the finite set of linear differential equations. The results obtained earlier for one Lorentz peak in the spectral density are generalized to the case of an arbitrary number of such peaks. The contribution of Ohmic spectral density is also considered.

DOI: 10.1134/S1063779620040711

1. INTRODUCTION

A rigorous derivation of the equations for the reduced evolution of system’s density matrix in the Markov approximation originates in the paper of Krylov and Bogolyubov [1]. The methods described there were developed in the context of the theory of stochastic limit, a modern exposition of which can be found in [2]. In this case, the kinetic equations of the density matrix of the system are written as the Gorini–Kossakowski–Sudarshan–Lindblad equation (GKSL equation) [3, 4].

However, the problems of non-Markovian evolution have attracted increasing interest in recent times [5–12]. In particular, it is natural to ask when the non-Markovian evolution of the system can be dilated to Markovian evolution for a finite-dimensional density matrix. In this paper, we consider a model of a multilevel system interacting with reservoirs at zero temperature, and the evolution of the density matrix of the system can be obtained for this model exactly in terms of the finite-dimensional Schrodinger equations with a non-Hermitian Hamiltonian. We analyze cases when the evolution of the reduced density matrix can be dilated to Markovian evolution with a larger but finite dimension.

This work develops and generalizes the results obtained in [13] and [14]. In section Model, we describe the model under consideration and present the results of [14], which are necessary for this article. We refer the reader to [13] and [14] for a detailed discussion of how this model is related to other models known in the literature. We only mention that the model is closely related to the Friedrichs model [15], and the method we use in Propositions 1 and 3 is a development of the pseudo-mode method proposed in [16–18]. In section Model, we also consider the case when the spectral density of the reservoir is a combination of Lorentz peaks, and we take into consideration the Ohmic contribution to the spectral density. Finally, in the Conclusions section, we summarize and outline possible directions for further research.

2. MODEL

We consider evolution in a Hilbert space

$$\mathcal{H} = (\mathbb{C} \oplus \mathbb{C}^N) \otimes \bigotimes_{i=1}^N \mathcal{F}_i(\mathbb{R}).$$

Here, $\mathbb{C} \oplus \mathbb{C}^N$ is an $(N + 1)$-dimensional Hilbert space with a distinguished one-dimensional linear subspace. The degrees of freedom $\mathbb{C}^N$ describe the excited states of the system, and the distinguished subspace corresponds to the ground state. Let $|i\rangle, i = 0, 1, \ldots, N$ be the orthonormal basis of the space $\mathbb{C} \oplus \mathbb{C}^N$, where $|0\rangle$ corresponds to the distinguished space. $\mathcal{F}_i(\mathbb{R})$ are the bosonic Fock spaces corresponding to reservoirs. Denote by $|\Omega\rangle$ the vacuum state of these reservoirs. We also introduce the creation operator $a_i$ satisfying the canonical commutation relations $[b_i(k), b_i^\dagger(k')] = \delta(k-k')$, $[b_i(k), b_j^\dagger(k')] = 0$, $b_i(k)|\Omega\rangle = 0$.

We consider the Hamiltonian, which vanishes in the ground state; otherwise, it has a general form. Namely,

$$\hat{H}_S = \sum_i \varepsilon_i |i\rangle \langle i| + \sum_{i,j} J_{ij} |i\rangle \langle j| = \mathbb{0} \oplus \hat{H}_S,$$

$$i, j = 1, \ldots, N,$$
where we do not assume that $\hat{H}_S$ is diagonalized in the basis $|i\rangle$. From a physics perspective, $|i\rangle$ plays the role of a local basis [19].

The Hamiltonian of the reservoir is the sum of the identical Hamiltonians of free bosonic fields (with the same dispersion relations $\alpha(k)$)

$$\hat{H}_B = \sum_{i=1}^{N} \alpha(k) b_i^\dagger(k) b_i(k) dk. \tag{2}$$

The interaction is described by the following Hamiltonian

$$\hat{H}_I = \sum_{i} \left( g^*(k) |0\rangle \langle i| \otimes b_i^\dagger(k) + g(k) |i\rangle \langle 0| \otimes b_i(k) \right) dk. \tag{3}$$

Note that this expression considers interactions of each level with its own reservoir, and the functions $g(k)$ (they are sometimes called form factors [21]) are the same for all reservoirs. From a physics perspective, such a Hamiltonian implies that the dipole approximation is performed (since only terms which are linear in the creation operator and annihilation operator are included), and the rotating wave approximations are performed too (this is expressed in the fact that there are no terms $|i\rangle \langle 0| \otimes b_i^\dagger(k)$ in the Hamiltonian).

We consider the Schrödinger equation

$$d \frac{d}{dt} |\Psi(t)\rangle = -i\hat{H} |\Psi(t)\rangle, \tag{4}$$

with the Hamiltonian $\hat{H} = \hat{H}_S \otimes I + I \otimes \hat{H}_B + \hat{H}_I$, and the initial condition

$$|\Psi(0)\rangle = (|\psi(0)\rangle + \psi_b(0) |0\rangle) \otimes |\Omega\rangle, \langle 0| \psi(0)\rangle = 0, \tag{5}$$

that is, the initial condition is fully factorized and assumes a zero temperature of the reservoir.

We will be interested in the evolution of the reduced density matrix

$$\rho_\Sigma(t) \equiv \text{tr}_R |\Psi(t)\rangle \langle \Psi(t)|,$$

where $\text{tr}_R$ is a partial trace with respect to the reservoir, that is, with respect to the spaces $\bigotimes_{j=1}^{N} L_2(\mathbb{R})$. In [14], we proved the following theorem, which allows one to describe the evolution of the reduced density matrix in terms of a finite-dimensional solution of the integro-differential equation.

**Theorem 1.** Let the integral converge (the reservoir correlation function)

$$G(t) = \int g(k) e^{-i\alpha(k)t} \, dk \tag{6}$$

for an arbitrary instant of time $t \in \mathbb{R}_+$ and defines a continuous function, then

$$\rho_\Sigma(t) = \begin{pmatrix}
1 - ||\psi(t)||^2 & \psi^*_0(0)\langle \psi(t) | \\
\psi^*_0(0) \langle \psi(t) | & \langle \psi(t) | \langle \psi(t) |
\end{pmatrix}$$

where $|\psi(t)\rangle$ is the solution of the integro-differential equation

$$\frac{d}{dt} |\psi(t)\rangle = -iH_S |\psi(t)\rangle - \int_0^t ds G(t-s) |\psi(s)\rangle \tag{7}$$

with the initial condition $|\psi(0)\rangle = |\psi(0)\rangle$.

In the physics literature, the spectral density $\hat{f}(\omega)$, but not the function $g(k)$ or $G(t)$, is often considered, that is, the Fourier transform of the function $G(t)$

$$G(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \hat{f}(\omega), \quad \hat{f}(\omega) = \int_{-\infty}^{+\infty} G(t) e^{i\omega t} \, dt.$$ 

In [14], we considered the case of spectral densities with one Lorentz peak. In the next sections, we generalize these results to the case of an arbitrary finite number of Lorentz peaks and consider the contribution of the Ohmic spectral density.

### 3. COMBINATION OF LORENTZ PEAKS

In this section, we reduce the evolution with spectral density in the form of a positive combination of Lorentz peaks

$$\hat{f}_L(\omega) = \sum_{j=1}^{K} \frac{g_j^2}{\omega_j^2 + (\omega - \varepsilon_j)^2}, \tag{8}$$

$$g_j > 0, \quad \gamma_j > 0, \quad K \in \mathbb{N}$$

to a system of linear equations. The finite set of numbers $g_j$ and the function $g(k)$ are denoted by the same letter, since they have close physical meaning. It should be remembered that $g(k)$ are complex-valued functions, and $g_j$ are strictly positive real numbers. In the case of spectral density (8), the reservoir correlation function takes the form

$$G_{\text{Lorentz}}(t) = \sum_{j=1}^{K} g_j^2 e^{-\frac{\gamma_j t}{2}} e^{-i\varepsilon_j t}.$$ 

Consequently, the following proposition holds.

**Proposition 1.** Let $|\psi(t)\rangle$ be a solution to Eq. (7) with the initial condition $|\psi(0)\rangle = |\psi(0)\rangle$ in the case when $G(t) = G_{\text{Lorentz}}(t)$ is defined by formula (9), then a $(K+1)N$-dimensional vector $|\tilde{\psi}(t)\rangle \equiv |\psi(t)\rangle \otimes \bigoplus_{j=1}^{K} \mid \varphi_j(t) \rangle \in \bigoplus_{j=1}^{K+1} C^N$, where

$$|\varphi_j(t)\rangle \equiv -ig_j \int_0^t ds e^{-\frac{\gamma_j}{2} (t-s)} |\psi(s)\rangle, \tag{10}$$
satisfies the Schrödinger equation with a non-Hermitian Hamiltonian

\[
\frac{d}{dt} |\psi(t)\rangle = -iH\text{eff} |\psi(t)\rangle,
\]

with the initial condition of \( |\psi(0)\rangle = |\psi(0)\rangle \oplus 0 \).

**Proof.** Substituting (9) into (7) and considering (10), we obtain

\[
\frac{d}{dt} |\psi(t)\rangle = -iH_S |\psi(t)\rangle - i\sum_{j=1}^{K} g_j |\varphi_j(t)\rangle,
\]

and differentiating (10) with respect to time \( t \), we obtain \( K \) more differential equations

\[
\frac{d}{dt} |\varphi_j(t)\rangle = -i\gamma_j |\varphi_j(t)\rangle - \left( \frac{\gamma_j}{2} + i\epsilon_j \right) |\varphi_j(t)\rangle.
\]

Combining these differential equations into one equation for the vector \( |\psi(t)\rangle = |\psi(t)\rangle \oplus |\varphi_1(t)\rangle \oplus ... \oplus |\varphi_K(t)\rangle = |\psi(t)\rangle \oplus \oplus_{j=1}^{K} |\varphi_j(t)\rangle \), we obtain (11). In addition, definition (10) leads to \( |\varphi_j(0)\rangle = 0 \), i.e., to the initial condition \( |\psi(0)\rangle = |\psi(0)\rangle \oplus 0 \).

It is shown in [13] that in the case when the matrix \( V = \frac{i}{2} (H\text{eff} - H\text{eff}) \) is nonnegative definite, then such a non-Hermitian Hamiltonian can be used to construct the (one-particle) GKSL equation so that \( \rho_j(t) \) is obtained by taking the trace with respect to \( KN \) auxiliary degrees of freedom that arise in equation (11) compared with (7). Following [17], we call such degrees of freedom pseudo-modes, and the matrix \( V \), following [20], will be called the optical potential. Without dwelling on this issue in detail, we note that

\[
V = 0 \oplus \frac{\gamma_1}{2} I_N \oplus ... \oplus \frac{\gamma_K}{2} I_N
\]

and the nonnegative definiteness of the matrix \( V \) follows from the positivity of \( \gamma_j \). Thus, in the case when the spectral density is a positive combination of the Lorentz peaks, in accordance with [13], we found that reduced evolution can always be dilated to Markovian (GKSL) evolution of higher dimension.

4. OHMIC SPECTRAL DENSITY

In this section, we consider the influence of the contribution of Ohmic spectral density

\[
\tilde{\mathcal{J}}_{\text{Ohmic}}(\omega) = \eta \omega, \quad \eta > 0.
\]

Since the Fourier transform of such a function exists only in the context of generalized functions, Theorem 1 is not directly applicable. Therefore, we consider a family of spectral densities with exponential cutoff

\[
\tilde{\mathcal{J}}_{\text{Ohmic}}(\omega) = \eta \omega e^{-\frac{\omega}{\Xi}}
\]

parameterized by the cutoff frequency \( \Omega \), which we will tend to \( +\infty \). The corresponding reservoir correlation function is

\[
G_{\text{Ohmic}}(t) = -i\eta \frac{2\eta \Omega^3}{\pi(1 + (\Omega t)^2)}.
\]

We also need a family of Hamiltonians instead of \( H_S \)

\[
H_S(\Omega) = H_S^c + \frac{\eta \Omega}{\pi}.
\]

Further, we will see that \( \frac{\eta \Omega}{\pi} \) plays the role of a counterterm [22]. Since we will tend \( \Omega \to +\infty \), from a physics perspective this corresponds to the fact that the energies of transitions from the ground state to the excited state are much larger than the transition energies between excited states. Our studies were motivated by transfer models in biological systems, where such a condition is satisfied [23–26]. Moreover, it is this condition that makes it legitimate to neglect the multielectron states in our model described above.

**Proposition 2.** Let \( |\psi_{\text{Ohmic}}(t)\rangle \) be a solution to (7) with the initial condition of \( |\psi_{\text{Ohmic}}(0)\rangle = |\psi(0)\rangle \), where \( G(t) = G(\Omega) + G_c(t) \); \( G(\Omega) \) is defined by formula (12), and \( G_c(t) \) is a continuous function; \( H_S = H_S(\Omega) \) is defined by formula (13). Let the limit \( \lim_{\Omega \to +\infty} |\psi_{\text{Ohmic}}(t)\rangle = |\psi_{\text{Markovian}}(t)\rangle \) exist for \( t > 0 \) and define
the infinitely differentiable function for $t \in (0, +\infty)$, then $\psi^{(r)}(t)$ is a solution of the equation

$$\frac{d}{dt}|\psi^{(r)}(t)\rangle = -\frac{1}{1 + i \frac{\Omega}{2}} H_S^{(r)}|\psi^{(r)}(t)\rangle$$

$$- \int_0^t ds \frac{1}{1 + i \frac{\eta}{2}} G_c(t-s)|\psi^{(r)}(s)\rangle$$

with the initial condition of $|\psi^{(r)}(0)\rangle = \frac{1}{1 + i \frac{\eta}{2}}|\psi(0)\rangle$.

Proof. We give only a brief sketch of the proof. Note that

$$G_{\Omega}(t) = \frac{e}{\pi} \frac{f_{\Omega}(t)}{1 + \frac{\Omega t}{2}},$$

where $f_{\Omega}(t)$ is a symmetric $\delta$-shaped family. Then,

$$-\int_0^t ds G_{\Omega}(t-s)|\psi_{\Omega}(s)\rangle = -\frac{e}{\pi} \int_0^t ds f_{\Omega}(t-s)|\psi_{\Omega}(s)\rangle$$

$$= \frac{e}{\pi} \int_0^t ds \frac{d}{ds} f_{\Omega}(t-s)|\psi_{\Omega}(s)\rangle$$

$$= \frac{e}{\pi} \int_0^t ds f_{\Omega}(t-s)|\psi(0)\rangle$$

$$- \frac{e}{\pi} \int_0^t ds f_{\Omega}(t-s) \frac{d}{ds}|\psi_{\Omega}(s)\rangle.$$ 

Substituting this expression into the integral form of Eq. (7), we have

$$|\psi_{\Omega}(t)\rangle - |\psi(0)\rangle = -iH_S^{(r)} |\psi_{\Omega}(t)\rangle$$

$$- \int_0^t d\tau \int_0^\tau ds G_c(\tau-s)|\psi_{\Omega}(s)\rangle$$

$$- \int_0^t d\tau \int_0^\tau ds f_{\Omega}(\tau-s) \frac{d}{ds}|\psi_{\Omega}(s)\rangle.$$ 

(The terms of the form $\int_0^t d\tau \frac{e}{\pi} \Omega|\psi_{\Omega}(\tau)\rangle$ are cancelled). For $t > 0$, $\Omega \rightarrow +\infty$, we have

$$\left[\psi^{(r)}(t)\rangle - |\psi(0)\rangle = -iH_S^{(r)} \int_0^t d\tau |\psi^{(r)}(\tau)\rangle - \int_0^t d\tau \int_0^\tau ds G_c(\tau-s)|\psi^{(r)}(s)\rangle - i \frac{\eta}{2} |\psi(0)\rangle - i \frac{\eta}{2} \left(\psi^{(r)}(t)\rangle - |\psi(0)\rangle\right).$$

Differentiating and cancelling $1 + \frac{\eta}{2}$, we obtain (14). In addition, setting $t = 0$ in this equation, we obtain the required initial condition.

Note that if we restrict ourselves to the Ohmic term and set $G_c(t) = 0$, then Eq. (14) takes the form of the Schrodinger equation (11) with the non-Hermitian Hamiltonian $H_{\text{eff}} = \frac{1}{1 + i \frac{\eta}{2}} H_S^{(r)}$. The corresponding optical potential is $V = \frac{\eta}{2} H_S^{(r)}$. Thus, in order to construct the Markovian dilation, it is necessary that the Hamiltonian $H_S^{(r)}$ be positive definite

$$H_S^{(r)} \geq 0.$$ 

Note that from a physics perspective, this means that the transition to excited levels of the Hamiltonian of the system $\hat{H}_S$ from the ground state corresponds to higher energies than $\frac{\eta \Omega}{\pi}$, which is determined by the cutoff frequency.

By analogy with Proposition 1, we can obtain the following proposition.

**Proposition 3.** Let $|\psi^{(r)}(t)\rangle$ be a solution to (14) with the initial condition of $|\psi(t)\rangle|_{t=0} = |\psi(0)\rangle$ in the case when $G_c(t) = G_{\text{Lorentz}}(t)$ is defined by formula (9), then a $(K+1)N$-dimensional vector $|\bar{\psi}(t)\rangle = |\psi(t)\rangle \oplus \oplus_{j=1}^K |\phi_j(t)\rangle \in \oplus^{K+1} C^N$, where $|\phi_j(t)\rangle$ is defined similarly to (10), satisfies the Schrodinger equation in the form (11) with non-Hermitian Hamiltonian $H_{\text{eff}} = \frac{1}{1 + i \frac{\eta}{2}} H_S^{(r)}$.
In this case, to verify the nonnegative definiteness of the optical potential, we note that if we pass to the global basis (eigenbasis $H_S$) with respect to each subspace $\mathbb{C}^N$ of the space $\otimes^{k=1} \mathbb{C}^N$, then $H_{\text{eff}}$ will be decomposed into the direct sum of blocks in the form ($E_\alpha$ are eigenvalues of $H_S$)

$$H_{\text{eff,}\alpha} = \left(\begin{array}{cc}
\frac{1}{2}E_\alpha & -\frac{1}{2}g^T
\end{array}\right)
\left(\begin{array}{c}
1 + i\frac{n}{2}
g
\end{array}\right),$$

$$g^T = (g_1, \ldots, g_K), \quad \varepsilon = \text{diag}(\varepsilon_1, \ldots, \varepsilon_K),$$

$$\gamma = \text{diag}(\gamma_1, \ldots, \gamma_K).$$

Then, the optical potential will be decomposed into the direct sum of blocks in the form

$$V_\alpha = \frac{i}{2} (H_{\text{eff,}\alpha} - H_{\text{eff,}\alpha}^\dagger)$$

$$= \left[ \frac{n}{2}E_\alpha - \frac{1}{2}g^T \right]
\left(\begin{array}{cc}
1 + \left(\frac{n}{2}\right)^2
2 \left(1 + i\frac{n}{2}\right)
\end{array}\right)
\left(\begin{array}{c}
\frac{n}{2}g
\frac{1}{2} \gamma
\end{array}\right).$$

Since the submatrix $\frac{1}{2} \gamma$ is positive definite, for the positive definiteness of $V_\alpha$, it suffices to verify the positivity of the determinant

$$\det V_\alpha = \left[ \frac{n}{2}E_\alpha - \frac{1}{2}g^T \gamma^{-1} \right] \det \left(\frac{1}{2} \gamma\right).$$

Obviously, $\det V_\alpha > 0$ is equivalent to

$$E_\alpha > \frac{n}{4} g^T \gamma^{-1} g = \sum_{j=1}^K \frac{g_j^2}{\gamma_j}.$$  

To verify nonnegative definiteness, generally speaking [27], one needs to check the nonnegativity of all diagonal minors. However, all the minors containing the element $\frac{n}{2}E_\alpha$, will have a form similar to $V_\alpha$, but instead of $g$, they will contain vectors containing all possible subsets of the elements of vector $g$. Therefore, if the inequality

$$E_\alpha > \frac{n}{4} \sum_{j=1}^K \frac{g_j^2}{\gamma_j}$$

satisfies, then the inequalities for the remaining minors are automatically satisfied due to the positiveness of $\frac{g_j^2}{\gamma_j}$. This expression (criterion for the possibility of Markovian dilation) can be rewritten in matrix form

$$H_S^{(\nu)} > \sum_{j=1}^K \frac{g_j^2}{\gamma_j} I_N,$$

which is a generalization of the condition for the presence of Markovian dilation (15), obtained earlier in the case of $G_\nu(t) = 0$. Physically, this corresponds to the fact that the energies of transition to excited levels should be greater than the sum of the counterterm, which is proportional to the cutoff frequency, and the term proportional to the sum of the maxima of the Lorentz peaks.

5. CONCLUSIONS

The exact evolution of the reduced density matrix expressed in terms of solving the equation with a non-Hermitian Hamiltonian is obtained for the model described above. We considered cases when the spectral density has the form of a combination of the Lorentz peaks, and also when there is an Ohmic contribution to the spectral density. Conditions when the reduced evolution can be dilated to Markovian evolution are obtained.

As a possible direction for further research, we are interested in trying to obtain the exact non-Markovian evolution described in this article as the limit of the approximate evolution of a more general form, similar to how the Markov equations appear in the Bogolyubov–Van Hove limit [2].

ACKNOWLEDGMENTS

I am very grateful to I.V. Volovich, S.V. Kozyrev, A.I. Mikhailov, and A.S. Trushechkin for their participation in discussions of the issues considered in this article.

FUNDING

This work was supported by the Russian Science Foundation, project no. 17-71-20154.

REFERENCES

1. N. M. Krylov and N. N. Bogolyubov, On the Fokker–Planck Equations Obtained in Perturbation Theory Using an Approach Based on the Spectral Properties of the Perturbation Hamiltonian, Vol. 2: N. N. Bogolyubov. Select-
1. L. Accardi, Y. G. Lu, and I. Volovich, *Quantum Theory and Its Stochastic Limit* (Springer, Berlin, 2002).

2. V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, “Completely positive dynamical semigroups of N-level systems,” J. Math. Phys. 17, 821–825 (1976).

3. G. Lindblad, “On the generators of quantum dynamical semigroups,” Commun. Math. Phys. 48, 119–130 (1976).

4. H. P. Breuer, B. Kappler, and F. Petruccione, “Stochastic wave-function method for non-Markovian quantum master equations,” Phys. Rev. A 59, 1633 (1999).

5. H. P. Breuer, “Non-Markovian generalization of the Lindblad theory of open quantum systems,” Phys. Rev. A 75, 022103 (2007).

6. A. Kossakowski and R. Rebolledo, “On non-Markovian time evolution in open quantum systems,” Open Syst. Inf. Dyn. 14, 265–274 (2007).

7. D. Chruscinski and A. Kossakowski, “Non-Markovian quantum dynamics: Local versus nonlocal,” Phys. Rev. Lett. 104, 070406 (2010).

8. N. Singh and P. Brumer, “Efficient computational approach to the non-Markovian second order quantum master equation: Electronic energy transfer in model photosynthetic systems,” Mol. Phys. 110, 1815–1828 (2012).

9. N. Tang, T.-T. Xu, and H.-S. Zeng, “Comparison between non-Markovian dynamics with and without rotating wave approximation,” Chin. Phys. B 22, 030304 (2013).

10. I. A. Luchnikov, S. V. Vintskevich, H. Ouerdane, and S. N. Filippov, “Simulation complexity of open quantum dynamics: Connection with tensor networks,” Phys. Rev. Lett. 122, 160401 (2019).

11. A. Strathearn P. Kirton, D. Kildal, J. Keeling, and B. W. Lovett, “Efficient non-Markovian quantum dynamics using time-evolving matrix product operators,” Nat. Commun. 9, 3322 (2018).

12. A. E. Teretenkov, “Pseudomode approach and vibronic non-Markovian phenomena in light-harvesting complexes,” Proc. Steklov Inst. Math. 306, 258–272 (2019).

13. A. E. Teretenkov, “Non-Markovian evolution of multilevel system interacting with several reservoirs. Exact and approximate,” Lobachevskii J. Math. 40, 1587–1605 (2019).

14. A. E. Teretenkov, “Non-Markovian evolution of multilevel system interacting with several reservoirs. Exact and approximate,” Lobachevskii J. Math. 40, 1587–1605 (2019).

15. K. O. Friedrichs, “On the perturbation of continuous spectra,” Comm. Pure Appl. Math. 1, 361–406 (1948).

16. B. M. Garraway and P. L. Knight, “Cavity modified quantum beats,” Phys. Rev. A 54, 3592 (1996).

17. B. M. Garraway, “Nonperturbative decay of an atomic system in a cavity,” Phys. Rev. A 55, 2290 (1997).

18. A. S. Trushechkin and I. V. Volovich, “Perturbative treatment of inter-site couplings in the local description of open quantum networks,” EPL. 113, 30005 (2016).

19. D. Chruscinski and S. Pascazio, “A brief history of the GKLS equation,” Open Syst. Inf. Dyn. 24, 1740001 (2017).

20. S. V. Kozyrev, A. A. Mironov, A. E. Teretenkov, and I. V. Volovich, “Flows in non-equilibrium quantum systems and quantum photosynthesis,” Infinit. Dimens. Anal. Quantum Probab. Relat. Top. 20, 1750021 (2017).

21. M. Mohseni, P. Rebentrost, S. Lloyd, and A. Aspuru-Guzik, “Environment-assisted quantum walks in photosynthetic energy transfer,” J. Chem. Phys. 129, 11B603 (2008).

22. F. R. Gantmakher, *The Theory of Matrices*, 5th ed. (Fizmatlit, Moscow, 2004) [in Russian].