On the Diophantine Approximations of logarithms in cyclotomic fields.

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Abstract. I present here the proofs of results, which are obtained in my papers "On the linear forms, whose coefficients are linear combinations with algebraic coefficients of logarithms of algebraic numbers," VINITI, 1996, 1617-B96, pp. 1 – 23 (in Russian), and "On the systems of linear forms, whose coefficients are linear combinations with algebraic coefficients of logarithms of algebraic numbers," VINITI, 1996, 2663-B96, pp. 1 – 18 (in Russian).

Let \( T \) is a real number, \( \Delta, m \) and \( n \), are positive integers, \( \Delta \) is greater or equal than 2, \( K_m = \mathbb{Q}[\exp(2\pi i/m)] \) is a cyclotomic field, \( \mathbb{Z}_{K_m} \) is the ring of all the integers of the field \( K_m \), \( \Lambda(n) \) is the Mangold’s function, \( \epsilon^2 = \epsilon \).

Let \( \Lambda_0(m) = 0 \), if \( m \) is odd and \( \Lambda_0(m) = \Lambda(m/2) \), if \( m \) is even. Let further \( \omega_1(m) = (m - 1)/2 \), if \( m \) is odd, \( \omega_1(m) = m/2 - 2 \), if \( m \equiv 2 \pmod{4} \) and \( \omega_1(m) = m/2 - 1 \), if \( m \equiv 0 \pmod{4} \).

(1) \[
\begin{align*}
\Delta(T) &= \sqrt{\Delta^3(3 - T^2) + 1} + 16\Delta^2T^2 + \Delta^2(3 - T^2) + 1
\end{align*}
\]
and the values \( V^*_\Delta, V_\Delta(m), \ell_\Delta(\epsilon, T), g_{\Delta,k}(m) \) and \( h_{\Delta,k}(m) \) are defined by the equalities

(2) \[
\begin{align*}
V^*_\Delta &= (\Delta + 1) + \log((\Delta - 1)^{(\Delta-1)/2}(\Delta + 1)^{(\Delta+1)/2})
\end{align*}
\]
\[\frac{\pi}{2} \sum_{\mu=0}^{1}(1 - 2\mu) \sum_{\kappa=1}^{[(d-1)/2]+\mu} \cot \left( \frac{\pi \kappa}{d - 1 + 2\mu} \right),
\]

(3) \[
V_\Delta(m) = V^*_\Delta + (\Delta + 1)\Lambda_0(m)/\phi(m),
\]

(4) \[
\begin{align*}
\ell_\Delta(\epsilon, T) &= -\log \left( 4(\Delta + 1)^{\Delta+1}(1 - 1/\Delta)(\Delta - 1) \right) + \\
\frac{1}{2} \log \left( (2\Delta + (-1)^\epsilon w_\Delta(T) + (\Delta + 1))^2 + T^2\Delta^2 \left( 1 + \frac{(-1)^\epsilon 2\Delta}{w_\Delta(T)} \right)^2 \right) + \\
\frac{1}{2} \log \left( (2\Delta + (-1)^\epsilon w_\Delta(T) - (\Delta + 1))^2 + T^2\Delta^2 \left( 1 + \frac{(-1)^\epsilon 2\Delta}{w_\Delta(T)} \right)^2 \right) +
\end{align*}
\]
\[
\frac{(\Delta - 1)}{2} \log \left( (2\Delta + (-1)^{w_\Delta(T)})^2 + T^2 \Delta^2 \left( 1 + \frac{(-1)^{2\Delta}}{w_\Delta(T)} \right)^2 \right),
\]

(5) \[g_{\Delta, \epsilon}(m) = (-1)^{\epsilon}(l_\Delta(\epsilon, \tan(\pi \omega_1(m)/m) + V_\Delta(m))),\]

(6) \[h_\Delta(m) = -V_\Delta(m) - l_\Delta(1, \tan(\pi/m)),\]

where \(m \neq 2, k = 0, 1\). Let, finally, the values \(\beta(d, m)\) and \(\alpha(\Delta, m)\) are defined by the equalities

\[\beta(\Delta, m) = g_{\Delta, 0}(m)/h_\Delta(m), \quad \alpha(\Delta, m) = \beta(\Delta, m) - 1 + g_{\Delta, 1}(m)/h_\Delta(m).\]

**Theorem.** Let \(m\) is a positive integer different from one, two and six, \(\Delta \in \{5, 7\}\).

Then

(7) \[h_\Delta(m) > 0\]

and for each \(\epsilon > 0\) there exists \(C_{\Delta, m}(\epsilon) > 0\) such that

(8) \[\max_{\sigma \in \text{Gal}(K/Q)} (|q^\sigma \log((2 + \exp(2\pi i/m))^\sigma) - p^\sigma|) \geq C_{\Delta, m}(\epsilon)(\max_{\sigma \in \text{Gal}(K_m/Q)} (|q^\sigma|)^{-\alpha(\Delta, m) - \epsilon},\]

where \(p \in \mathbb{Z}_{K_m}\) and \(q \in \mathbb{Z}_{K_m} \setminus \{0_{K_m}\}\); moreover, for any \(q \in \mathbb{Z}_{K_m} \setminus \{0_{K_m}\}\) and any \(\epsilon > 0\) there exists \(C^*_{\Delta, m}(q, \epsilon) > 0\) such that

(9) \[b^{\beta(\Delta, m) + \epsilon} \max_{\sigma \in \text{Gal}(K/Q)} (|q^\sigma b \log((2 + \exp(2\pi i/m))^\sigma) - p^\sigma|) \geq C^*_{\Delta, m}(q, \epsilon),\]

where \(p \in \mathbb{Z}_{K_m}\), \(b \in \mathbb{N}\).

For the proof I use the same method, as in [41] - [71]. I must work on the Riemann surface \(\mathcal{F}\) of the function \(\log(z)\) and identify it with the direct product of the multiplicative group \(\mathbb{R}_+^* = \{r \in \mathbb{R}: r > 0\}\) of all the positive real numbers with the operation \(\times\), not to be written down explicitly as usual, and the additive group \(\mathbb{R}\) of all the real numbers, so that

\[z_1 z_2 = (r_1 r_2, \phi_1 + \phi_2)\]

for any two points \(z_1 = (r_1, \phi_1)\) and \(z_2 = (r_2, \phi_2)\) on \(\mathcal{F}\). I will illustrate the appearing situations on the half plain \((\phi, r)\), where \(r > 0\).

For each \(z = (r, \phi) \in \mathcal{F}\), let

\[\theta_0(z) = r \exp i\phi, \quad \log(z) = \ln(r) + i\phi, \quad \eta^*_\alpha(z) = (r, \phi - \alpha),\]
where \( \alpha \in \mathbb{R} \). Clearly, \( \log(z_1 z_2) = \log(z_1) + \log(z_2) \) for any \( z_1 \in \mathbb{R}, z_2 \in \mathbb{R} \). Let \( \rho(z_1, z_2) = |\log(z_1) - \log(z_2)| \), where \( z_1 \in \mathbb{R} \) and \( z_2 \in \mathbb{R} \); clearly, \((\mathbb{R}, \rho)\) is a metric space. Clearly, \( \rho(z_1, z_2) = \rho(z_1, z_2) \) for any \( z_1, z_2 \) in \( \mathbb{R} \). Clearly, \( \theta_0(z) = \exp(\log(z)) \) for any \( z \in \mathbb{R} \). Clearly, for any \( \alpha \in \mathbb{R} \) the map \( z \to \eta_0^{\alpha}(z) \) is the bijection of \( \mathbb{R} \) onto \( \mathbb{R} \) and

\[
\theta_0(\eta_0^{\alpha}(z)) = \exp(-ima)\theta_0(z)
\]

for each \( z = (r, \phi) \in \mathbb{R}, \alpha \in \mathbb{R} \) and \( m \in \mathbb{Z} \). Clearly, the group \( \mathbb{R} \) may be considered as \( \mathbb{C} \)-linear space, if for any \( z \in \mathbb{R} \) and any \( s \in \mathbb{C} \) we let

\[
z^s = (|\exp(s\log(z))|, \Im(s\log(z))).
\]

Let us fix a domain \( D \) in \( \mathbb{R} \). Let \( f(z) = f^\wedge(r, \phi) \) for a complex-valued function \( f(z) \) on \( D \). It is well known that \( f(z) \) is holomorphic in \( D \) if the complex-valued function \( f^\wedge(r, \phi) \) of two real variables \( r \) and \( \phi \) has continuous partial derivatives in \( D \), and the Cauchy-Riemann conditions

\[
r((\partial/\partial r)f^\wedge)(r, \phi) = -i((\partial/\partial \phi)f^\wedge)(r, \phi) :=
\]

\[
(\delta f)(z) := \theta_0(z)((\partial/\partial z)f)(z)
\]

are satisfied for every point \( z = (r, \phi) \in D \). The equalities (10) determine a differentiations \( \frac{\partial}{\partial z} \) and \( \delta = \theta_0(z)\frac{\partial}{\partial s} \) on the ring of all the holomorphic in the domain \( D \) functions. In particular, the function \( \log(z) \) is holomorphic on \( \mathbb{R} \) and we have the equalities

\[
((\partial/\partial z)\log(z)) = \theta_0(z^{-1}), \quad (\delta\log)(z) = 1.
\]

For the proof I use the functions of C.S.Mejer. Let \( \Delta \in \mathbb{N} + 1, \delta_0 = 1/\Delta \),

\[
\gamma_1 = (1 - \delta_0)/(1 + \delta_0), \quad d_1 = \Delta + (-1)^l, \quad l = 1, 2.
\]

To introduce the first of my auxiliary function \( f_1(z, \nu) \), I use the auxiliary set

\[
\Omega_0 = \{ z \in \mathbb{R} : |z| \leq 1 \}.
\]

I prove that, for each \( \nu \in \mathbb{N} \), the function \( f_1(z, \nu) \) belongs to the ring \( \mathbb{Q}[\theta_0(z)] \); therefore using the principle of analytic continuation we may regard it as being defined in \( \mathbb{R} \). For \( \nu \in \mathbb{N} \), let

\[
f_1(z, \nu) = -(-1)^\nu(\Delta + 1)G_{1, 2}^{(1, 1)} \left( z \left| \begin{array}{cc}
-\nu d_1, & 1 + \nu d_2 \\
0, & \nu
\end{array} \right. \right)
\]

\[
= -(-1)^\nu(\Delta + 1) \frac{1}{2\pi i} \int_{L_1} g_{2, 2}^{(1, 1)}(s) ds,
\]

where

\[
g_{2, 2}^{(1, 1)}(s) = \theta_0(z^s)\Gamma(-s)\Gamma(1 + d_1 \nu + s)/(\Gamma(-\nu + s)\Gamma(1 + d_2 \nu - s))
\]

and the curve \( L_1 \) passes from \( +\infty \) to \( +\infty \) encircling the set \( \mathbb{N} - 1 \) in the negative direction, but not including any point of the set \( -\mathbb{N} \). So, for the parameters of the Meyer’s functions we have

\[
p = q = 2, \quad m = n = 1, \quad a_1 = -\nu d_1, \quad a_2 = 1 + \nu d_2, \quad b_1 = 0, \quad b_2 = \nu,
\]
\[ \Delta^* = \left( \sum_{k=1}^{q} b_k \right) - \sum_{j=1}^{p} a_j = -\nu - 1 < -1, \]

and, since we take \(|z| \leq 1\), convergence conditions of the integral in (11) hold. To compute the function \(f_1(z, \nu)\), we use the following formula

\[
G = (-1)^k \sum_{s \in S_k} \text{Res}(g; s),
\tag{12}
\]

where \(k = 1\), \(G\) denotes the integral (11) with \(L = L_k\), \(g\) denotes the integrand of the integral (11), \(S_k\) denotes the set of all the unremovable singularities of \(g\) encircled by \(L_k\), and \(\text{Res}(g; s)\) denotes the residue of the function \(g\) at the point \(s\). Then we obtain the equation

\[
f_1(z, \nu) = \frac{(\nu d_1)!/(\nu \Delta)!}{(-1)^\nu} |z|^{\nu} (-1)^\nu \Delta \sum_{k=0}^{\nu \Delta} (-\theta_0(z))^k \binom{\nu \Delta}{k} \left( \frac{\nu \Delta + k}{\nu d_1} \right).
\]

Therefore, as it has been already remarked, using the principle of analytic continuation we may regard it as being defined in \(\mathfrak{F}\). Let

\[ \Omega_1 = \{ z \in \mathfrak{F} : |z| \geq 1 \}. \]

Now, let me introduce my second auxiliary function defined for \(z \in \Omega_1\). For \(\nu \in \mathbb{N}\), let

\[
f_2(z, \nu) = \frac{(-1)^{(\nu+1)}}{2\pi i} \int_{L_2} g_{2,2}^{(2,1)}(s) ds,
\tag{13}
\]

where

\[
g_{2,2}^{(2,1)}(s) = \theta_0((\eta_2(z))^s) \Gamma(-s) \Gamma(\nu - s) \Gamma(1 + d_1 \nu + s) / \Gamma(1 + d_2 \nu - s).
\]

and the curve \(L_2\) passes from \(-\infty\) to \(-\infty\) encircling the set \(-\mathbb{N}\) in the positive direction, but not including any point of the set \(\mathbb{N} - 1\). So, for the parameters of the Meyer’s functions we have

\[
p = q = m = 2, n = 1, a_1 = -\nu d_1, a_2 = 1 + \nu d_2, b_1 = 0, b_2 = \nu,
\]

\[ \Delta^* = \left( \sum_{k=1}^{q} b_k \right) - \sum_{j=1}^{p} a_j = -\nu - 1 < -1, \]

and, since we take \(|z| \geq 1\), convergence conditions of the integral in (13) hold. To compute the function \(f_2(z, \nu)\), we use the formula (12) where \(k = 2\), \(G\) denotes the integral in (11) with \(L = L_k\), \(g\) denotes the integrand of the integral in (13), \(S_k\) denotes the set of all the unremovable singularities of \(g\).
encircled by \( L_k \), and \( \text{Res}(g; s) \) denotes the residue of the function \( g \) at the point \( s \). Then we obtain the equality

\[
(14) \quad f_2(z, \nu)(\nu \Delta)!/(\nu d_1)! = (-1)^\nu \sum_{t=\nu+1}^{\infty} R_0(t; \nu) \theta_0(z^{-t+\nu}),
\]

where

\[
R_0(t; \nu) = (\nu \Delta)!/(\nu d_1)! \left( \prod_{\kappa=\nu+1}^{\nu \Delta} (t - \kappa) \right) \prod_{\kappa=0}^{\nu \Delta} (t + \kappa)^{-1}.
\]

Let further

\[
(15) \quad f^*_k(z, \nu) = f_k(z, \nu)(\nu \Delta)!/(\nu d_1)!,
\]

where \( k = 1, 2 \). Expanding the function \( R_0(t; \nu) \) into partial fractions, we obtain the equality

\[
R_0(t; \nu) = \sum_{k=0}^{\nu \Delta} \alpha^*_{\nu, k}/(t + k)
\]

with

\[
(16) \quad \alpha^*_{\nu, k} = (-1)^{\nu+\nu \Delta+k} \left( \frac{\nu \Delta}{k} \right) \left( \frac{\nu \Delta + k}{\nu \Delta - \nu} \right),
\]

where \( k = 0, \ldots, \nu \Delta \). It follows from (13), (14), (15) and (16) that

\[
(17) \quad f^*_2(z, \nu) = (-\theta_0(z)\nu) \sum_{t=1+\nu}^{+\infty} (\theta_0(z))^{-t} R_0(t; \nu) = \sum_{t=1+\nu}^{+\infty} (\theta_0(z))^{-t+\nu} \sum_{k=0}^{\nu \Delta} \alpha^*_{\nu, k}/(t + k)
\]

\[
= (-\theta_0(z))\nu \sum_{t=1+\nu}^{+\infty} ((\theta_0(z))^{-t+k}/(t + k)) \sum_{k=0}^{\nu \Delta} \alpha^*_{\nu, k}(\theta_0(z))^k = (-\theta_0(z))\nu \sum_{k=0}^{\nu \Delta} \alpha^*_{\nu, k}(\theta_0(z))^k \sum_{\tau=1+\nu+1}^{+\infty} ((\theta_0(z))^{-\tau}/\tau) = \alpha^*(z; \nu)(-\log(1 - 1/\theta_0(z))) - \phi^*(z; \nu),
\]

where \( \log(\zeta) \) is a branch of \( \text{Log}(\zeta) \) with \( |\text{arg}(\zeta)| < \pi \),

\[
(18) \quad \alpha^*(z; \nu) = (-\theta_0(z))\nu \sum_{k=0}^{\nu \Delta} \alpha^*_{\nu, k}(\theta_0(z))^k = f^*_1(z; \nu),
\]

\[
(19) \quad \phi^*(z; \nu) = (-\theta_0(z))\nu \sum_{k=0}^{\nu \Delta} \alpha^*_{\nu, k}(\theta_0(z))^k \sum_{\tau=1}^{\nu + k} ((\theta_0(z))^{-\tau}/\tau) = (-\theta_0(z))\nu \sum_{\tau=1}^{\nu} ((\theta_0(z))^{-\tau} \alpha^*(z; \nu)/\tau +
\]

The change of order of summation by passage to (17) is possible, because the series in the second sum in (17) is convergent, if \( |z| \geq 1 \) and \( \theta_0(z) \neq 1 \). Since

\[
(-\theta_0(z))^{r_{\Delta}} \sum_{k=0}^{\nu_{\Delta}} \alpha^*_{\nu,k}(\theta_0(z))^k \sum_{\tau=1+\nu}^{\nu+k} ((\theta_0(z))^{-\tau}/\tau).
\]

it follows that

\[
\alpha^*(1;\nu) = \text{Res}(R_0(t;\nu); t = \infty) = 0
\]

So in the domain \( D_0 = \{ z \in \mathfrak{F} : |z| > 1 \} \) the function \( f^*_2(z,\nu) \) coincides with the function

\[
f^*_0(z,\nu) = \alpha^*(z;\nu)(-\log(1 - 1/\theta_0(z))) - \phi^*(z;\nu),
\]

The form (20) may be used for various applications. Especially it is pleasant, when both \( 1/\theta_0(z) \) and \( \alpha^*(z;\nu) \) for some \( z \) is an integer algebraic number. The following Lemma corresponds to this remark.

**Lemma 1.** Let \( m \in \mathbb{N}, m > 2 \) \( m \neq 2p^\alpha \), where \( p \) run over the all the prime numbers and \( \alpha \) run over \( \mathbb{N} \). Then \( 1 + \exp(2\pi i/m) \) belongs to the group of the units of the field \( K_m \). If \( m = 2p^\alpha \), where \( p \) is a prime number and \( \alpha \in \mathbb{N} \), then the ideal \( \mathfrak{I} = (1 + \exp(2\pi i/m)) \) is a prime ideal in the field \( K_m \), and \( \varphi(m) = (p) \).

**Proof.** Let polynomial \( \Phi_m(z) \) is irreducible over \( \mathbb{Q} \), has the leading coefficient equal to one and \( \Phi_m(\exp(2\pi i/m)) = 0 \). Let \( \Lambda(n) \), as usual, denotes the Mangold’s function. Since (see, for example, [31], end of the chapter 3)

\[
\Phi_m(z) = \prod_{d|m} (z^{m/d} - 1)^{\mu(d)},
\]

it follows that

\[
\Phi_m(-1) = (-2)^\left( \sum_{d|m} \mu(d) \right) = 1,
\]

if \( m \in 1 + 2\mathbb{N} \),

\[
\Phi_m(z) = \prod_{d|(m/2)} (((z)^{m/(2d)} - 1)^{\mu(2)((-z)^{m/d} - 1)/((-z) - 1)})^{\mu(d)},
\]

\[
\Phi_m(-1) = \lim_{z \to -1} \prod_{d|(m/2)} (((-z)^{m/d} - 1)/((-z) - 1))^\mu(d) \times \n \times \n
\left( -2 \right)^{\mu(2)\left( \sum_{d|(m/2)} \mu(d) \right)} = \exp\left( \sum_{d|(m/2)} \ln(m/(2d))\mu(2d) \right) = \exp(\Lambda(m/2)),
\]

if \( m \in 2(1 + 2\mathbb{N}) \),

\[
\Phi_m(z) = \prod_{d|(m/2)} (((-z)^{m/d} - 1)/((-z) - 1))^\mu(d),
\]
and

$$\Phi_m(-1) = \lim_{z \to 1} \prod_{d|m/2} \left( ((-z)^{m/d} - 1)/((-z) - 1))^{\mu(d)} = \right.$$

$$\exp \left( \sum_{d|m/2} \ln(m/(2d))\mu(d) \right) = \exp(\Lambda(m/2)),$$

if \( m \in 4\mathbb{N} \). If \( m = 2p^\alpha \) with \( \alpha \in \mathbb{N} \), then \( \Phi_m(-1) = \exp(\Lambda(m/2)) = p \), and ideals \( \mathfrak{i}_k = (1 + \exp(2\pi ik/m)) \), where \( (k, m) = 1 \), divide each other and in the standard equality \( efg = n \) (see, [31], chapter 3, section 10) we have

$$e = n = \phi(m), \ f = g = 1. \ ■.$$ 

In connection with the above remark and with the Lemma 1, the following case is interesting for us:

$$\theta_0(z) = (-\rho)(1 + \exp(-i\beta)) = -(\rho \exp(i\beta/2))/(2\cos(\beta/2)) =$$

$$-(\rho \exp(i\psi))(2\cos(\psi)) = -(1 + i \tan(\psi))/2$$

with \( \rho > 2/3, |\beta| < \pi \) and \(-\pi/2 < \psi = \beta/2 < \pi/2 \); then

$$\Re(1 - 1/\theta_0(z)) = \Re(2 + \exp(i\beta)/\rho) > 1/2,$$

and we have no problems with \( \log(1 - 1/\theta_0(z)) \). Of course, according to the Lemma 1, the case \( \rho = 1 \) is interesting especially. So, we will take further

$$z = (\rho/(2 \cos(\psi)), \psi - \pi) = (\rho/(-2\cos(\theta), \theta),$$

where \( \rho > 2/3, |\psi| < \pi/2 \) and \(-3\pi/2 < \theta = \psi - \pi < -\pi/2 \); clearly, the function (20) is analytic in the domain

$$D_1 = \{ z = (\rho(2 \cos(\psi))^{-1}, \psi - \pi) \colon \rho > 2/3, -\pi/2 < \psi < \pi/2 \} =$$

$$\{ z = ((-2\rho \cos(\theta))^{-1}, \theta) \colon \rho > 2/3, -3\pi/2 < \theta < -\pi/2 \}.$$ 

Let

$$D_2(\delta_0) = \{ z \in \mathfrak{F} \colon |z| > 1 + \delta_0/2 \}, \ D_3 = D_2(\delta_0) \cup D_1.$$ 

So, the function \( f_2^*(z, \nu) \) coincides with the function (20) in \( D_2(\delta_0) \subset D_0 \). Since \( D_2(\delta_0) \cap D_1 \neq \emptyset \), it follows that the join \( D_3 = D_2(\delta_0) \cup D_1 \) of the domains \( D_2(\delta_0) \) and \( D_1 \) is a domain in \( \mathfrak{F} \) and the function (20) is analytic in this domain.

The conditions, which imply the equality

$$(-1)^{m+p-n} \exp(-i\alpha) \theta_0(z) \times$$

$$\left( \prod_{j=1}^p (\delta + 1 - a_j) \right) (G \circ \eta_\alpha^*) (z) = \left( \prod_{k=1}^q (\delta - b_k) \right) (G \circ \eta^*_\alpha) (z)$$
hold in our case for the Mejer’s function
\[ G = G^{(m,n)}_{p,q}(z \begin{bmatrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{bmatrix}). \]

We have \( p = q = 2, m = n = 1, \alpha = 0 \) for the function \( f_1(z, \nu) \) and the equation \( (24) \) takes the form
\[ \theta_0(z)((\delta + 1 + d_1 \nu)(\delta - d_2 \nu)f_1)(z, \nu) = (\delta(\delta - \nu)f_1)(z, \nu) \]

We have \( p = q = m = 2, n = 1, \alpha = \pi \) for the function \( f_2(z, \nu) \) and the equation \( (24) \) takes the form
\[ \theta_0(z)((\delta + 1 + d_1 \nu)(\delta - d_2 \nu)f_2)(z, \nu) = (\delta(\delta - \nu)f_2)(z, \nu). \]

We see that both the functions \( f(z, \nu) = f_k^*(z, \nu) \), where \( k = 1, 2 \) satisfy to the same differential equation
\[ \theta_0(z)((\delta + 1 + d_1 \nu)(\delta - d_2 \nu)f(z, \nu) = (\delta(\delta - \nu)f)(z, \nu). \]

in the domain \( D_0 \). According to the general properties of the Mejer’s functions we have the equality
\[ \left( \prod_{\kappa=1}^{\Delta-1} (\nu(\Delta - 1) + \kappa) \right) \prod_{\kappa=1}^{d_2} (\delta - d_2 \nu - \kappa) f_k^*(z, \nu + 1) = \]
\[ \left( \prod_{\kappa=1}^{\Delta} (\nu \Delta + \kappa) \right) (\delta - \nu) \prod_{\kappa=1}^{d_1} (\delta + d_1 \nu + \kappa) f_k^*(z, \nu), \]

where \( k = 1, 2 \) and \( z \in D_0 \). Since \( f_k^*(z, \nu) \) and polynomial \( f_1^*(z, \nu) \) are analytic in the domain \( D_0 \cup D_1 \), and \( f_0^*(z, \nu) \) coincides with \( f_2^*(z, \nu) \), it follows that the equations \( (25) \) and \( (26) \) hold in \( D_0 \cup D_1 \) for \( k = 0, 1 \).

Let
\[ D^\nu(w, \eta) = (\eta + 1)(\eta + \gamma_1) - 2(1 + \gamma_1)w \eta, \]
\[ D^\lambda(z, \eta) = D^\nu(\theta_0(z), \eta), \]

where, in view of \( (21) \),
\[ w = \theta_0(z) = -r \exp(i\psi), r = 1/(2\cos(\psi)), |\psi| < \pi/2. \]

In view of \( (29) \), the polynomial \( (27) \) coincides with the polynomial \( (1) \) in [63]. Let
\[ h^\sim(\eta) = (\eta - 1)(1 - \delta_0)^{-d_1}(\eta + 1)2^{-2} \eta^{d_1}. \]

As in [55], we consider \( \nu^{-1} \) as an independent variable taking its values in the field \( \mathbb{C} \) including 0. Let \( F \) be a bounded closed subset of \( \mathfrak{F} \) (in particular, this compact \( F \) may be an one-point set). Let \( \mathfrak{S}_0(F) \) be the subring of all those functions in \( \mathbb{Q}(w) \), which are well defined for every \( w \in \theta_0(F) \). For \( \varepsilon \in (0, 1), \)
let $\mathcal{H}(F, \varepsilon)$ be the subring of all those functions in $\mathbb{Q}(w, \nu^{-1})$, which are well defined for every $(w, \nu^{-1})$ with $w \in \theta_0(F)$, $|\nu^{-1}| \leq \varepsilon_0$.

**Lemma 2.** Let $F$ be a closed bounded subset of $D_0 \cup D_1$ (in particular, $F$ may be an one-point set). Let further for any $z \in F$ the polynomial (28) has only simple roots and on the set of all the roots $\eta$ of the polynomial $D^\wedge(z, \eta)$ the map

$$\eta \to h^\sim(\eta)$$

is injective. Then there is $\varepsilon \in (0, 1)$ such that, for any $z \in F, \nu \in \mathbb{N} + [1 / \varepsilon]$, the functions $f_0^\ast(z, \nu), f_1^\ast(z, \nu) = \alpha^\ast(z; \nu)$ and $\phi^\ast(z; \nu)$ are solutions of the difference equation

$$x(z, \nu + 2) + \sum_{j=0}^{1} q_j^\ast(z, \nu^{-1})x(z, \nu + j) = 0,$$

moreover,

$$q_j^\ast(z, \nu^{-1}) \in \mathcal{H}(F, \varepsilon)$$

for $j = 0, 1$, and trinomial

$$w^2 + \sum_{j=0}^{1} q_j^\ast(z, 0)w^j$$

coincides with

$$\prod_{k=0}^{1}(w - h(\eta_k)),$$

if

$$\prod_{k=0}^{1}(w - \eta_k),$$

coincides with $D^\vee(w, \eta)$ from (27).

**Proof.** Proof may be found in [55].

This Lemma shows the importance of the properties of the roots of the polynomial (27). In correspondence with (22) and with notations in [63], let

$$\rho > 2/3, r = \rho/(2 \cos(\psi)), t = \cos(\psi), |\psi| < \pi/2.$$  

Let $u = r^2, \delta_0 \leq 1/2 < 2/3 < \rho$. Then

$$2\delta_0 \leq 2/5 < 2/3 < \rho < 2\sqrt{u} = 2r.$$  

Clearly,

$$(\partial/\partial \psi)r = (\rho \sin(\psi))/(2 \cos^2(\psi)) = -2\rho(\sin(\psi) - 1) - 2\rho/(\sin(\psi) + 1),$$

$$(\partial/\partial \psi)^2r = (2\rho \cos(\psi))/(\sin(\psi) - 1)^2 + (2\rho \cos(\psi))/(\sin(\psi) + 1)^2 > 0,$$

if $|\psi| < \pi/2$ In view of (3.1.10) in [56],

$$|D_0(r, \psi, \delta_0)|^2 = r^4 + r^2 + (\delta_0/2)^4 +$$
In view of (3.1.68) in [56], (3.1.70) – (3.1.71) in [56] and (39),

\[ 2r^2(\delta_0/2)^2(2t^2 - 1) + 2r(r^2 + (\delta_0/2)^2)t = u^2 + u + (\delta_0/2)^4 + (\delta_0/2)^2(r^2 - 2u) + \rho(u + (\delta_0/2)^2) = u^2 + u(\rho + 1 - (\delta_0)^2/2) + (\delta_0/2)^2(r^2 + \rho + (\delta_0/2)^2), \]

(39) \[ |R_0(r, \psi, \delta_0)|^2 = |D_0(r, \psi, \delta_0)| = \sqrt{u^2 + u(\rho + 1 - (\delta_0)^2/2) + (\delta_0/2)^2(r^2 + \rho + (\delta_0/2)^2)}. \]

In view of (3.1.41) - (3.1.43) in [56] and (39),

(40) \[ p_1 = 8(|R_0^*(r, \psi, \delta_0)|^2 + |R_0(r, \psi, \delta_0)|^2)/(1 + \delta_0)^2 = 8(r^2 + rt + 1/4 + |D_0(r, \psi, \delta_0)|)/(1 + \delta_0)^2 = 8(1 + \delta_0)^{-2} \times \left( u + \rho/2 + 1/4 + \sqrt{u^2 + u(\rho + 1 - (\delta_0)^2/2) + (\delta_0/2)^2(r^2 + \rho + (\delta_0/2)^2)} \right), \]

(41) \[ p_2 = (8(|R_0^*(r, \psi, \delta_0)|^2 + |R_0(r, \psi, \delta_0)|^2))/(1 + \delta_0)^2 = 8(r^2 - r\delta_0 + (\delta_0)^2/4 + |D_0(r, \psi, \delta_0)|)/(1 + \delta_0)^2 = 8(u - \delta_0\rho/2 + (\delta_0)^2/4)/(1 + \delta_0)^2 + 8(1 + \delta_0)^{-2}\sqrt{u^2 + u(\rho + 1 - (\delta_0)^2/2) + (\delta_0/2)^2(r^2 + \rho + (\delta_0/2)^2)} = 8(1 + \delta_0)^{-2}u(2 + (\rho + 1 - \delta_0\rho)/(2u) + O(1/u^2)), \]

(42) \[ q_1(r, \psi, \delta_0) = ((1 - \delta_0)/(1 + \delta_0))^2, q_2(r, \psi, \delta_0) = (4r/(1 + \delta_0))^2 = (16u)/(1 + \delta_0)^2. \]

In view of (91) in [63], (36) and (37),

(43) \[ s = s_0(r, \psi) = |r \exp(i\psi)| + 1/2 = \sqrt{(r^2 + 1 + 2r\cos(\psi))}/4 = \sqrt{(u + 1 + \rho)/4} \in (\max(|r - 1|/2, \delta_0/4), (r + 1)/2] \]

and

\[ t = \cos(\psi) = (4s^2 - r^2 - 1)/(2r). \]

In view of (3.1.68) in [56], (3.1.70) - (3.1.71) in [56] and (39),

\[ |R_{-1}^*(r, \psi, \delta_0)|^2 = r^2 + (2 + \delta_0)^2/4 + r(2 + \delta_0)\cos(\psi) = u + (2 + \delta_0)^2/4 + \rho(2 + \delta_0)/2, \]

(44) \[ p_0 = 8(|R_{-1}^*(r, \psi, \delta_0)|^2 + |R_0(r, \psi, \delta_0)|^2)/(1 + \delta_0)^2 = 8(u + (2 + \delta_0)^2/4 + \rho(2 + \delta_0)/2)/(1 + \delta_0)^2 + 8(1 + \delta_0)^{-2}\sqrt{u^2 + u(\rho + 1 - (\delta_0)^2/2) + (\delta_0/2)^2(r^2 + \rho + (\delta_0/2)^2)}, \]

(45) \[ q_0(r, \psi, \delta_0)(1 + \delta_0)^2/16 = (r^2 + 1 + 2r\cos(\psi)) = (u + 1 + \rho). \]
According to Lemma 4.4 in [63], (23) and (37),

\[ |\eta_1^\wedge(r, \psi, \delta_0) + \epsilon| < |\eta_0^\wedge(r, \psi, \delta_0) + \epsilon|, \]

if \( \epsilon^2 = \epsilon \) and \( z \in D_3 \). Therefore, according to (40), (42) and (46),

\[ (-1)^k(\partial/\partial u)|\eta_k^\wedge(r, \psi, \delta_0)| > 0, \]

where \( \frac{1}{3} < \rho/2 < \sqrt{u} = r, k^2 = k \). According to a) and c) of the Lemma 4.6 in [63], and in view of (23) and (43),

\[ |\eta_1^\wedge(r, \psi, \delta_0) - 1| < |\eta_0^\wedge(r, \psi, \delta_0) - 1|, \]

if \( z \in D_3 \). In view of (38),

\[ |D_0(r, \psi, \delta_0)|^2 = \]

\[ u^2 + u(\rho + 1 - (\delta_0)^2/2) + (\delta_0/2)^2(\rho^2 + \rho + (\delta_0/2)^2) = \]

\[ (u + (\rho + 1)/2 - (\delta_0)^2/4)^2 + (\delta_0/2)^2(\rho^2 + \rho + (\delta_0/2)^2) = \]

\[ (((\rho + 1)/2)^2 - (\rho + 1)(\delta_0)^2/4 + (\delta_0/2)^4) = \]

\[ (u + (\rho + 1)/2 - (\delta_0)^2/4)^2 + (\delta_0/2)^2(\rho^2 + 2\rho + 1) - (\rho + 1)^2/4 = \]

\[ (u + (\rho + 1)/2 - (\delta_0)^2/4)^2 - (\rho + 1)^2(1 - (\delta_0)^2)/4. \]

Consequently,

\[ |D_0(r, \psi, \delta_0)| = u + \frac{\rho + 1}{2} - \frac{(\delta_0)^2}{4} + O(1/u), \]

where \( u \geq 1/4 \). Since \( u \geq 1/4 > (\delta_0)^2/4 \), it follows that

\[ u + (\rho + 1)/2 - (\delta_0)^2/4 > \sqrt{1 - (\delta_0)^2}(\rho + 1)/2. \]

If \( \rho = 1, u = 1/4 \) then in view of (49),

\[ |D_0(r, \psi, \delta_0)|^2 = (5/4 - (\delta_0)^2/4)^2 - (1 - (\delta_0)^2) = \]

\[ (\tau - 5/4)^2 + 4\tau - 1, \]

where \( 0 < \tau = \frac{(\delta_0)^2}{4} < \frac{1}{100} \); moreover, in this case

\[ (\partial/\partial \tau)|D_0(r, \psi, \delta_0)|^2 = 2\tau - 5/2 + 4 > 0; \]

therefore if \( \delta_0 \leq 1/5 \), then

\[ |D_0(r, \psi, \delta_0)|^2 \bigg|_{u=1/4, \rho=1} \leq (1, 24)^2 - 0, 96 = 0, 5776 \]

and

\[ |D_0(r, \psi, \delta_0)|^2 \bigg|_{u=1/4, \rho=1} \leq 0, 76. \]

In view of (49),

\[ 1 < (\partial/\partial u)|D_0(r, \psi, \delta_0)| = \]
\[
\sqrt{\frac{(u + (r + 1)/2 - (\delta_0)^2/4)^2}{(u + (r + 1)/2 - (\delta_0)^2/4)^2 - (r + 1)^2(1 - (\delta_0)^2)/4}} = 1 + O(1/u^2),
\]
in view of (40), (41) and (44),

\[\eqref{eq:51}\]
\[
(\partial/\partial u)p_\epsilon = 8(2 + O(1/u^2))/(1 + \delta_0)^2,
\]
where \(\epsilon^3 = \epsilon\), and \((\partial/\partial u)|D_\theta(r, \psi, \delta_0)|\) decreases with increasing \(u\); consequently,

\[(\partial/\partial u)^2|D_\theta(r, \psi, \delta_0)| < 0,
\]
if \(u \geq 1/4\). In view of (40), (41) and (44),

\[\eqref{eq:52}\]
\[
(\partial/\partial u)^2p_\epsilon = (\partial/\partial u)^2|D_\theta(r, \psi, \delta_0)| < 0,
\]
where \(u \geq 1/4, 0 < \delta_0 < 2/3 < \rho\), \(\epsilon^3 = \epsilon\). In view of (41), (42), (49) and (50), if \(\rho = 1\), \(u > 1/4\), \(0 < \delta_0 \leq 1/5\), then

\[\eqref{eq:53}\]
\[
q_2((\partial/\partial u)p_2)/((\partial/\partial u)q_2 - p_2/2 =
\]
\[
8u(1 + (u + 1 - (\delta_0)^2/4))/|D_\theta(r, \psi, \delta_0)|/(1 + \delta_0)^2-
\]
\[
4(u - \delta_0/2 + (\delta_0)^2/4 + |D_\theta(r, \psi, \delta_0)|)/(1 + \delta_0)^2 -
\]
\[
4(u + \delta_0/2 - (\delta_0)^2/4)/(1 + \delta_0)^2+
\]
\[
4((1 + \delta_0)^2|D_\theta(r, \psi, \delta_0)|^{-1}(2u^2 + u(2 - (\delta_0)^2/2) -
\]
\[
4((1 + \delta_0)^2|D_\theta(r, \psi, \delta_0)|^{-1}(u^2 + u(2 - (\delta_0)^2/2) + (\delta_0/2)^2(2 + (\delta_0/2)^2)) =
\]
\[
4(u + \delta_0/2 - (\delta_0)^2/4)/(1 + \delta_0)^2+
\]
\[
4(1 + \delta_0)^2|D_\theta(r, \psi, \delta_0)|^{-1}(u^2 - (\delta_0/2)^2(2 + (\delta_0/2)^2)) > 0,
\]
\[
q_2((\partial/\partial u)p_2)/((\partial/\partial u)q_2 - p_2 = \frac{8}{u}(1 + (u + 1 - (\delta_0)^2/4))/|D_\theta(r, \psi, \delta_0)|/(1 + \delta_0)^2-
\]
\[
8(u - \delta_0/2 + (\delta_0)^2/4 + |D_\theta(r, \psi, \delta_0)|)/(1 + \delta_0)^2 =
\]
\[
8u(2 + O(1/u^2))/(1 + \delta_0)^2-
\]
\[
8(u - \delta_0/2 + (\delta_0)^2/4 + u + 1 - (\delta_0)^2/4 + O(1/u))/(1 + \delta_0)^2 =
\]
\[
-8(1 - \frac{\delta_0}{2} + O(1/u))/(1 + \delta_0)^2.
\]
In view of (44), (45), (53), (49), (51), (50), if \(\rho = 1\), \(u > 1/4\), \(0 < \delta_0 \leq 1/5\), then

\[\eqref{eq:54}\]
\[
q_0((\partial/\partial u)p_0)/((\partial/\partial u)q_0 - p_0/2 = (u + 2)(\partial/\partial u)p_0 - p_0/2 >
\]
\[\eqref{eq:55}\]
\[
(u + 1)(\partial/\partial u)p_0 - p_0/2 > 0,
\]
Therefore, according to (55), (51) and (45),

\begin{equation}
q_0(\partial/\partial u)p_0)/(\partial/\partial u)q_0 - p_0 = 8(u + 2)(2 + O(1/u^2))/(1 + \delta_0)^2 - 8(u + (2 + \delta_0)^2/4 + (2 + \delta_0)/2 + u + 1 - (\delta_0)^2/4)/(1 + \delta_0)^2 = 8(4 + O(1/u))/(1 + \delta_0)^2 - (2 + \delta_0)^2/4 - (2 + \delta_0)/2 - 1 + (\delta_0)^2/4 + O(1/u) = 8(1 - (3/2)\delta_0 + O(1/u))/(1 + \delta_0)^2,
\end{equation}

where \( u > 1/4 \). In view of (45), (54) and (52),

\begin{equation}
(\partial/\partial u)((q_0(\partial/\partial u)p_0)/(\partial/\partial u)q_0 - p_0)(\partial/\partial u)p_0 + (\partial/\partial u)q_0) = (\partial/\partial u)((u + 2)(\partial/\partial u)p_0 - p_0)(\partial/\partial u)p_0 = ((\partial/\partial u)p_0)^2 + ((u + 2)(\partial/\partial u)^2 p_0 - (\partial/\partial u)p_0)(\partial/\partial u)p_0 + ((u + 2)(\partial/\partial u)p_0 - p_0)(\partial/\partial u)^2 p_0 = (2(u + 2)(\partial/\partial u)p_0 - p_0)(\partial/\partial u)^2 p_0 < 0.
\end{equation}

Therefore, according to (55), (51) and (45),

\begin{equation}
\inf\{(u + 2)(\partial/\partial u)p_0 - p_0)(\partial/\partial u)p_0 + (\partial/\partial u)q_0 : u \geq 1/4\} = \\
\lim_{u \to +\infty} ((u(\partial/\partial u)p_0 - p_0)(\partial/\partial u)p_0 + (\partial/\partial u)q_0) = \\
128(1 - (3/2)\delta_0)/(1 + \delta_0)^4 + 16/(1 + \delta_0)^2 > 0.
\end{equation}

According to the Lemma 4.17 in [63] and in view of (53), (54), (56),

\begin{equation}
(\partial/\partial u)|\eta_0(r, \psi, \delta_0) + \epsilon|^2 > 0,
\end{equation}

where \( \epsilon^2 = 1 \), \( u > 1/4 \),

\begin{equation}
(\partial/\partial u)|\eta_1(r, \psi, \delta_0) - 1|^2 < 0,
\end{equation}

where \( u > 1/4 \). The following Lemma describes the behavior of the value \( h^\sim(\eta_k(r, \psi, \delta_0)) \) with \( k^2 = k \) and \( h^* \) in (30).

**Lemma 3.** If \( \Delta \geq 5 \), then

\begin{equation}
(\partial/\partial u)||h^\sim(\eta_0(r, \psi, \delta_0))|| > 0, \\
(\partial/\partial u)||h^\sim(\eta_1(r, \psi, \delta_0))|| < 0,
\end{equation}

where \( u \in (1/4, +\infty) \).

**Proof.** The inequality (59) directly follows from (46), (57) and (30). So, we must prove the inequality (30) Clearly, if \( \beta < 1 \), \( u > 1/4 \) then

\begin{equation}
(\partial/\partial u)(u^{3/4} + (3/4)\beta u^{-1/4}) > 0.
\end{equation}

We take

\begin{equation}
\beta = (4/3)(\delta_0/2)^2(2 + (\delta_0)^2)/4/(2 - (\delta_0)^2/2).
\end{equation}
Then, clearly, \( \beta < (\delta_0)^2 = 1/(\Delta)^2 < 1 \). Therefore, in view of (40) and (49), if \( \rho = 1 \), then

\[
p_1 u^{-1/4} = (u^{3/4} + (3/4)u^{-1/4} + \sqrt{u^{3/2} + u^{1/4}(\rho + 1 - (\delta_0)^2/2)(u^{3/4} + (3/4)\beta u^{-1/4})}) \]

increases together with increasing \( u \in (1/4, +\infty) \), and, in view of (42),

\[
|\eta_0(r, \psi, \delta_0)|^2 u^{-1/4} = p_1 u^{-1/4}/2 + \sqrt{(p_1 u^{-1/4}/2 - q_1 u^{-1/2})^2 - q_1 u^{-1/2}}
\]

increases together with increasing \( u \in (1/4, +\infty) \).

In view of (47), (42), (60), (57) and (58), if \( \Delta \geq 5 \), then

\[
|\eta_1(r, \psi, \delta_0)|^{2(\Delta-1)}|\eta_1(r, \psi, \delta_0)|^2 - 1| = \frac{(q_1)^4}{(|\eta_0(r, \psi, \delta_0)|^2 u^{-1/4})^2} \times \frac{16}{(1 + \delta_0)^2} |\eta_0(r, \psi, \delta_0) + 1|^{-2}|\eta_1(r, \psi, \delta_0) - 1|^2
\]

decreases together with increasing \( u \in (1/4, +\infty) \).

Let \( D \) is bounded domain in \( \mathbb{C} \) or \( \mathfrak{F} \), and \( D^* \) is closure of \( D \). Let

\[
a_0^\sim(z), \ldots, a_n^\sim(z)
\]

are the functions continuous on \( D^* \) and analytic in \( D \). Let \( a_n^\sim(z) = 1 \) for any \( z \in D^* \). Let

\[
T(z, \lambda) = \sum_{i=0}^{n} a_i^\sim(z) \lambda^i.
\]

Let \( s \in \mathbb{N} \), \( n_i \in \mathbb{N} - 1 \), where \( i = 1, \ldots, s \) and \( \sum_{i=1}^{s} n_i = n \). We say that polynomial \( T(z, \lambda) \) has \( (n_1, \ldots, n_s) \)-disjoint system of roots on \( D^* \), if for any \( z \in D^* \) the set of all the roots \( \lambda \) of the polynomial \( T(z, \lambda) \) splits in \( s \) classes \( \mathfrak{K}_1(z), \ldots, \mathfrak{K}_s(z) \) with following properties:

a) the sum of the multiplicities of the roots of the klass \( \mathfrak{K}_i \) is equal to \( n_i \) for \( i = 1, \ldots, s \);

b) if \( i \in [1, s] \cap \mathbb{N} \), \( j \in (i, s] \cap \mathbb{N} \) and \( n_i n_j \neq 0 \), then the absolute value of each roots of the klass \( \mathfrak{K}_i(z) \) is greater than absolute value of the each roots of the klass \( \mathfrak{K}_j(z) \).

If the polynomial (62) has \( (n_1, \ldots, n_s) \)-disjoint system of roots on \( D^* \), then for each \( i = 1, \ldots, s \) we denote by \( \rho^*_i(z) \) and \( \rho^*_{i,1}(z) \) respectively the maximal and minimal absolute value of the roots of the klass \( \mathfrak{K}_i(z) \).

Let \( D \) is bounded domain in \( \mathfrak{F} \) such that \( D^* \in D_3 \). Let

\[
F^\wedge(z, \eta) = \prod_{i=1}^{2} (\theta_0(z) - h(\eta_{i-1}(r, \psi, \delta_0))),
\]

\[
n = s = 2, n_1 = n_2 = 1, \mathfrak{K}_1(z) = \{h(\eta_{i-1}(r, \psi, \delta_0))\}, \rho_{1,0} = \rho_{1,1} = |h(\eta_{i-1}(r, \psi, \delta_0))|,
\]
L.A. Gutnik, On the Diophantine Approximations of logarithms in cyclotomic fields.

where \( i = 1, 2 \).

**Lemma 4.** The polynomial \( F^*(z, \eta) \) in (63) has \((1, 1)\)-disjoint system

of roots on \( D^* \).

**Proof.** The assertion of the Lemma follows from (46) and (48). □

**Corollary.** The map (31) is injective for every \( z \in D^* \); all the conditions

of the Lemma 2 are fulfilled for the functions \( f_i(z, \nu) \) from (20), \( \alpha^*(z, \nu) \)
from (18) and \( \phi^*(z, \nu) \) from (19) in every \( z \in D^* \); therefore for every \( z \in D^* \)
these functions are solutions of the difference equation of Poincaré type (32),

and the polynomial (35) coincides with characteristic polynomial of this equation.

Let for each \( \nu \in \mathbb{N} - 1 \) are given continuous on \( D^* \) functions

\[
(64) \quad a_0(z; \nu), \ldots, a_n(z; \nu),
\]

which are analytic in \( D \).

Let \( a_n(z; \nu) = 1 \) for any \( z \in D^* \) and any \( \nu \in \mathbb{N} - 1 \). We suppose that for
any \( i = 1, \ldots, n - 1 \) the sequence of functions \( a_i(z; \nu) \) converges to \( a_i^*(z) \)
uniformly on \( D^* \), when \( \nu \to \infty \). Let us consider now the difference equation

\[
(65) \quad a_0(z; \nu) y(\nu + 0) + \ldots + a_n(z; \nu) y(\nu + n) = 0,
\]

i.e. we consider a difference equation of the Poincaré type, coefficients (64)
of this equation are continuous on \( D^* \) and analytic in \( D \), and they uniformly

converge to limit functions (61), when \( \nu \to \infty \).

**Lemma 5.** Let polynomial (62) has \((n_1, \ldots, n_s)\)-disjoint system of roots
on \( D^* \). Let \( y(z, \nu) \) is a solution of the equation (65), and this solution is
continuous on \( D^* \) and analytic on \( D \). Let further \( i \in [1, s] \cap \mathbb{Z} \). Let us consider
the set of all the \( z \in D \), for which the following inequality holds

\[
(66) \quad \limsup_{\nu \in \mathbb{N}, \nu \to \infty} |y(z, \nu)|^{1/\nu} < \rho_{i, 1}(z);
\]

if this set has a limit point in \( D \), then the inequality (66) holds in \( D^* \).

**Proof.** The proof may be found in [35] (Theorem 1 and its Corollary). □

**Lemma 6.** Let \( D \) is bounded domain in \( \mathfrak{X} \) such that \( D^* \in D_3 \). Then

\[
(67) \quad \limsup_{\nu \in \mathbb{N}, \nu \to \infty} |f_0^*(z, \nu)|^{1/\nu} < \rho_{1, 1}(z) = |h^*(\eta_0(r, \psi, \delta_0))|\]

for any \( z \in D^* \).

**Proof.** In view of (23), expanding the domain \( D \), if necessary, we can
suppose that \( \{(r, \phi): r \in [2, 3], \phi = 0\} \in D \). Making use the same arguments,
as in [59], Lemma 4.2.1, we see that the inequality (67) holds for
any point \( z = (r, \phi) \in \{(r \in [2, 3], \phi = 0\} \). According to the Lemma 5, the
inequality (67) holds for any \( z \in D^* \).

For each prime \( p \in \mathbb{N} \) let \( v_p \) denotes the \( p \)-adic valuation on \( \mathbb{Q} \).

**Lemma 7.** Let \( p \in \mathbb{N} + 2 \) is a prime number,

\[
d \in \mathbb{N} - 1, r \in \mathbb{N} - 1, r < p.
\]

Then

\[
v_p((dp + r)!/((-p)^d! d!) - 1) \geq 1.
\]
Lemma 8. Let \( p \in \mathbb{N} + 2 \) be a prime number, \( d \in \mathbb{N} - 1, d_1 \in \mathbb{N} - 1, \)
\[
(68) \quad r \in [0, p - 1] \cap \mathbb{N}, r_1 \in [0, p - 1] \cap \mathbb{N}, d_1 p + r_1 \leq dp + r.
\]
Then
\[
(69) \quad v_p \left( \left( \frac{dp + r}{d_1 p + r_1} \right) \right) = v_p \left( \left( \frac{d}{d_1} \right) \right),
\]
if \( r_1 \leq r, \)
\[
(70) \quad v_p \left( \left( \frac{dp + r}{d_1 p + r_1} \right) \left( \left( \frac{d}{d_1} \right) \left( \frac{r}{r_1} \right)^{-1} - 1 \right) \right) \geq 1,
\]
if \( r_1 \leq r, \)
\[
(71) \quad v_p \left( \left( \frac{dp + r}{d_1 p + r_1} \right) \right) = 1 + v_p \left( \left( d - d_1 \right) \left( \frac{d}{d_1} \right) \right),
\]
if \( r < r_1, \)
\[
(72) \quad v_p \left( (-1)^{r_1 - r - 1} \frac{dp + r}{d_1 p + r_1} \left( \frac{r_1}{r} \right) (r_1 - r) \left( p \left( \frac{d}{d_1} \right) (d - d_1) \right)^{-1} - 1 \right) \geq 1.
\]

**Proof.** Clearly, \( d_1 \leq d. \) If \( r_1 \leq r, \) then let \( r_2 = r - r_1, d_2 = d - d_1. \) On the other hand, if \( r_1 > r, \) then, in view of (68), \( d \geq d_1 + 1; \) therefore in this case we let
\[
(73) \quad r_2 = p + r - r_1, d_2 = d - d_1.
\]
Then \( d = d_1 + d_2, r = r_1 + r_2, \)
\[
\left( \frac{dp + r}{d_1 p + r_1} \right) = (dp + r)!(d_1 p + r_1)!(d_2 p + r_2)! \cdot (d_1 p + r_1)! \cdot (d_2 p + r_2)!
\]
According to the Lemma 7,
\[
(74) \quad v_p \left( \left( \frac{dp + r}{d_1 p + r_1} \right) (-p)^{-d + d_1 + d_2} d_1! d_2! r_1! r_2! / (d! r!) - 1 \right) \geq 1,
\]
\[
(75) \quad v_p \left( \left( \frac{dp + r}{d_1 p + r_1} \right) \right) = d - d_1 - d_2 + \]
\[
v_p(d! r! / (d_1! r_1! d_2! r_2!)).
\]
The equality (69) and the inequality (13) directly follow from (74) and (75). If
the inequality $r < r_1$ holds, then in view of (73) – (75),
\[ r_2! \prod_{j=1}^{r_1 - r - 1} (p + r - r_1 + j) = (p-1)!, \]
\[ v_p(r_2!(r_1 - r - 1)!(-1)^{r_1 - r} - 1) \geq 1, \]
and (72) holds.

**Corollary 1.** Let $p \in \mathbb{N}$ is a prime number,
\[ d \in \mathbb{N} - 1, r \in \mathbb{N} - 1, d_1 \in \mathbb{N} - 1, d_2 \in \mathbb{N} - 1, r_1 \in \mathbb{N} - 1, r_2 \in \mathbb{N} - 1, \]
\[ \max(r_1, r_2) < p. \]
Then
\[ p^{-d}(dp + r)! \in (-1)^d d!r! + p\mathbb{Z}, \]
\[ \left( \frac{d_1 + d_2 + r_1 + r_2}{d_1 p + r_1} \right) \in \left( \frac{d_1 + d_2}{d_1} \right) \left( \frac{r_1 + r_2}{r_1} \right) + p\mathbb{Z}. \]

**Proof.** This is a direct corollary of the Lemma 7 and Lemma 8. See also Lemma 9 in [58].

**Corollary 2.** Let $p \in \mathbb{N} + 2$ is a prime number,
\[ d \in \mathbb{N}, r_1 \in \mathbb{N}, r_1 < p, d_1 \in \mathbb{N} - 1, d < d. \]
Then
\[ v_p \left( \left( \frac{dp}{d_1 p + r_1} \right) \left( \frac{d - 1}{d_1} \right) \left( \frac{p}{r_1} \right) \right) + 1 \geq 1 \]
\[ (76) \]
\[ v_p \left( \left( \frac{dp}{d_1 p + r_1} \right) \left( \frac{d - 1}{d_1} \right) \left( \frac{p}{r_1} \right) \right) + 1 \geq 1 \]
\[ \text{Proof.} \quad \text{Since,} \]
\[ d \left( \frac{d - 1}{d_1} \right) = \left( d - d_1 \right) \left( \frac{d}{d_1} \right), \quad v_p \left( \left( \frac{p}{r_1} \right) \right) \geq 1, \]
the equality (76) directly follows from (72).

**Corollary 3.** Let $p \in \mathbb{N} + 2$ is a prime number,
\[ d \in \mathbb{N}, r_1 \in \mathbb{N}, r_1 < p, d^\sim \in \mathbb{N} - 1, d^\sim \geq d. \]
Then
\[ \left( \frac{dp}{d_1 p + r_1} \right) \in d \left( \frac{d - 1}{d^\sim} \right) \left( \frac{p}{r_1} \right) + p^2 \mathbb{Z}. \]

**Proof.** This is a corollary of the Corollary 2. See also Lemma 10 in [58].

Let let $p$ be prime in $(2, +\infty)$, let $K$ be a finite extension of $\mathbb{Q}$ let $p$ be a prime ideal in $\mathbb{Z}_K$ and $p \in p$, let $f$ be the degree of $p$, let $(p) = p^\beta \mathfrak{b}$, with entire ideal $\mathfrak{b}$ not contained in $p$, let $v_p$ be additive $p$-valuation, which prolongs $v_p$; so, if $\pi$ is a $p$-prime number, then $v_p(\pi) = 1/e$. If $f$ is the degree of the ideal $p$ then
\[ (77) \]
\[ v_p \left( w^{v_p} - w \right) \geq 1, \]
where $\beta \in \mathbb{N} f, w \in K$ and
\[ v_p(w) \geq 0. \]
In view of (77), (18), and (16),
\[ v_p(\alpha^*(z; p^\beta l) - \alpha^*(z; l)) > 1/e, \]
if \( \beta \in \mathbb{N} \), \( \theta_0(z) \in K \) and \( v_p(\theta_0(z)) \geq 0 \). In view of (19),
\[
\phi^*(z; \nu) = (-\theta_0(z))^{\nu} \sum_{k=0}^{\nu \Delta} \alpha^*_{\nu,k}(\theta_0(z))^{k} \sum_{\tau=1}^{\nu+k} ((\theta_0(z))^{-\tau}/\tau) = \\
(-\theta_0(z))^{\nu} \sum_{\tau=1}^{\nu} ((\theta_0(z))^{-\tau} \alpha^*(z; \nu)/\tau + \\
(-\theta_0(z))^{\nu} \sum_{k=0}^{\nu \Delta} \alpha^*_{\nu,k}(\theta_0(z))^{k} \sum_{\tau=1+\nu}^{\nu+k} ((\theta_0(z))^{-\tau}/\tau)) = \\
(-1)^{\nu} \sum_{\tau=1}^{\nu(\Delta+1)} \frac{1}{\tau} \sum_{k=\max(0,\tau-\nu)}^{\nu \Delta} \alpha^*_{\nu,k}(\theta_0(z))^{\nu-\tau+k};
\]
therefore, if \( \nu = p^\beta l, f = 1, \beta \in \mathbb{N} \), \( p > l(\Delta+1) \), \( \theta_0(z) \in K \) and \( v_p(\theta_0(z)) \geq 0 \), then, according to the Lemma 2,
\[
1 - \beta \leq \\
v_p \left( \phi^*(z; \nu) - \sum_{\eta \in [1, \Delta+1]} \frac{(-1)^{pl}}{p^\beta \eta} (\theta_0(z))^{p^\beta(l-\eta)+k} \alpha^*_{\nu,p,k} \right),
\]
(79)
\[
1/e - \beta \leq \\
v_p \left( \phi^*(z; \nu) - \sum_{\eta \in [1, \Delta+1]} \frac{(-1)^{pl}}{p^\beta \eta} (\theta_0(z))^{p^\beta(l-\eta)+k} \alpha^*_{\nu,p,k} \right).
\]
(80)
We make the pass (79) \( \rightarrow \) (80) \( \beta \) times and obtain the inequality
\[
1/e - \beta \leq \\
v_p \left( \phi^*(z; \nu) - p^{-\beta} \phi^*(z; l) \right),
\]
(81)
where
\[ \{l, \beta\} \subset \mathbb{N}, p > l(\Delta+1), p \in \mathfrak{p} \]
and \( \mathfrak{p} \) is ideal of the first degree.

**Lemma 9.** If \( m \in \mathbb{N} + 1, K = \mathbb{Q}[\exp(2\pi i/m)], \)
\[ \alpha^*(z; l_1) \phi^*(z; l_2) \neq 0 \]
for some \( z \in K \setminus \{0\}, l_1 \in \mathbb{N}, l_2 \in \mathbb{N} \), then for any \( l \in \mathbb{N} \) the sequences

\[
(82) \quad \alpha^*(z; \nu), \phi^*(z; \nu),
\]

where \( \nu \in l + \mathbb{N} \) form a linear independent system over \( K \).

**Proof.** There exists \( d^* \in \mathbb{N} \) such that

\[
d^*z \in \mathbb{Z}_K, d^*z\alpha^*(z; l_1) \in \mathbb{Z}_K, d^*z\phi^*(z; l_2) \in \mathbb{Z}_K.
\]

Let a prime \( p \in \mathbb{N}m + 1 \) satisfies to the inequality

\[
p > |Nm_{K/\mathbb{Q}}(d^*z\alpha^*(z; \nu))| + |Nm_{K/\mathbb{Q}}(d^*z\phi^*(z; \nu)) + |Nm_{K/\mathbb{Q}}(d^*) + (\Delta + 1)(l_1 + l_2).
\]

Let \( p \) is a prime ideal containing \( p \). Then

\[
v_p(\alpha^*(z; l_1)) = v_p(\phi^*(z; l_2)) = 0,
\]

and, in view of (81),

\[
v_p(\phi^*(z; p^\beta l_2)) = -\beta,
\]

but

\[
v_p(\alpha^*(z; p^\beta l_1)) = 0.
\]

with \( \beta \in \mathbb{N} \).

Let \( m \in \mathbb{N}, k \in \mathbb{Z}, 2 \leq 2|k| < m \), and let \( m \) and \( k \) have no common divisor with expection \( \pm 1 \). Let further \( K_m = \mathbb{Q}[\exp(2\pi i/m)] \) is a cyclotomic field, \( \mathbb{Z}_{K_m} \) is the ring of all the integers of the field \( K_m \).

**Lemma 10.** Let \( \Delta \in \{5, 7\} \). In correspondence with (21), (22) and (23), let \( z = (1/(2\cos(k\pi i/m)), k\pi i/m - \pi) \), where \( |k| < m/2, (|k|, m) = 1 \).

Then for each \( l \in \mathbb{N} \) the two sequences (82) form a linear independent system over \( \mathbb{C} \).

**Proof.** We check the fulfilment of the conditions of the Lemma 9.

Let \( \mathfrak{M} = \mathbb{N} \setminus \{1, 2, 6\} \) and \( \mathfrak{M}_0 = \{m \in \mathfrak{M} : \Lambda_0(m) = 0\} \). According to the condition of the Lemma, \( \theta_0(z) = -1/(1 + \exp(2i\pi/m)) \) with \( m \in \mathfrak{M} \). If \( m \in \mathfrak{M} \) and \( \phi(m) > \Delta \), then, in view of (18) and (16), \( \alpha^*(z; 1) \neq 0 \), because the numbers \( (1 + \exp(2i\pi/m))^k \), where \( k = 0, \ldots, \phi(m) - 1 \), form a basis of the field \( K_m \). Let \( \Delta = p \in 2\mathbb{N} + 1 \), where \( p \) is a prime, \( \mathfrak{p} \) is a prime ideal containing \( p \), and, as before, let \( (p) = \mathfrak{b} \mathfrak{p}^e \), \( 1_{K_m} \in \mathfrak{b} + \mathfrak{p} \). Then

\[
(83) \quad \left(\begin{array}{c} 2p - 1 \\ p \\ p - 1 \end{array}\right) \equiv p \mod p^2, v_p\left(\left(\begin{array}{c} p + k \\ 1 + k \\ k \end{array}\right)\right) = 2,
\]

where \( k = 1, \ldots, p - 2 \),

\[
(84) \quad \left(\begin{array}{c} p \\ 1 \\ 0 \end{array}\right) = p, \left(\begin{array}{c} 2p \\ p + 1 \\ p \end{array}\right) \equiv 2p \mod p^2.
\]

If \( m \in \mathfrak{M} \) and \((m, p) = 1 \), or, if \( m \in \mathfrak{M}_0 \), then, according to the Lemma 1,

\[
(85) \quad (1 + \exp(2i\pi/m), p) = (1)
\]
and, according to the Lemmata 7 and 8,

\[ \alpha^*(z; 1)/(p\theta_0(z)) \equiv 1 + (\theta_0(z))^{-1} - 2(\theta_0(z))^p \equiv 1 + (\exp(2i\pi/m) + 3)/(1 + \exp(2i\pi/m)) \equiv \exp(2ip\pi/m) + \exp(2i\pi/m) + 4 \not\equiv 0 \mod p. \]

If \( m = q^a \) with \( \alpha \in \mathbb{N} \) and prime \( q \) and there exists \( l \) in \( \{0, \ldots, \phi(m) - 1\} \) such that \( p \equiv l \mod (m) \), then

\[ \exp(2i\pi/m) + \exp(2i\pi/m) + 4 \not\equiv 0 \mod p. \]

If \( m = 2q^a \) with odd prime \( q \) and \( \alpha \in \mathbb{N} \), and there exist \( l \) in \( \{0, \ldots, \phi(m/2) - 1\} \) such that \( p \equiv 2l \mod (m/2) \), then \( (87) \) holds.

If \( p = 5 \), then \( \{3, 4, 5, 7, 8, 10, 12\} = \{m \in \mathfrak{M}_1: \phi(m) \leq p\} \).

If \( m = 3, 4, 5, 8, 10 \) then, clearly, \( (87) \) holds.

If \( m = 12 \), then \( 1, \exp(i\pi/2), \exp(2i\pi/3), \exp(i\pi/6) \), form a entire basis of \( K_{12} \), \( \exp(5i\pi/6) = \exp(i\pi/2) - \exp(i\pi/6) \), and \( (87) \) holds.

If \( p = 7 \) then \( \{3, 4, 5, 7, 8, 9, 10, 12, 14, 18\} = \{m \in \mathfrak{M}_1: \phi(m) \leq p\} \).

If \( m = 3, 4, 5, 7, 9, 14 \) then, clearly, \( (87) \) holds.

If \( m = 8 \), then \( \exp(7i\pi/4) = -\exp(3i\pi/4) \) and \( (87) \) holds.

If \( m = 12 \), then \( 1, \exp(i\pi/2), \exp(2i\pi/3), \exp(i\pi/6) \), form a entire basis of \( K_{12} \), \( \exp(7i\pi/6) = -\exp(i\pi/6) \), and \( (87) \) holds.

If \( m = 18 \), then

\[ \exp(7i\pi/9) = -\exp(-2i\pi/9) = \exp(4i\pi/9) + \exp(10i\pi/9), \]

and \( (87) \) holds.

The coefficient at \( \phi^*(z; \nu) \) in the expression \( (19) \) of \( \phi^*(z; \nu) \) is equal to

\[ \sum_{k=0}^{\nu \Delta} (-1)^k \alpha^*_{\nu,k}/(\nu + k) \]

and, if \( \Delta = p, \nu = 1 \), then in view \( (83) \) -- \( (84) \), the value of \( v_p \) on this coefficient is equal to 0. Therefore, if \( m \in \mathfrak{M}_1 \) and \( \phi(m) > p = \Delta \), then \( \phi^*(z; 1) \not\equiv 0 \).

If \( m \in \mathfrak{M}_1 \setminus \mathfrak{M}_0 \), and \( m \equiv 0 \mod p \) then \( m = 2p^a \), where \( \alpha \in \mathbb{N} \). According to the Lemma 1, \( p = (1 + \exp(2i\pi/m)) \) is a prime ideal in \( K_m \), and, furthermore, \( p^{\phi(m)} = (p) \). Let \( v_p \) be the \( p \)-adic valuation, which prolongs the valuation \( v_p \). Clearly, \( v_p(1 + \exp(2i\pi/m) = 1/\phi(m), v_p(\theta_0(z)) = -1/\phi(m) \) In view of \( (19) \) with \( \nu = 1 \), for the summands of the sum

\[ \sum_{k=1}^{\nu \Delta} \alpha^*_{\nu,k}(\theta_0(z))^{1+k} \sum_{\tau=2}^{1+k} ((\theta_0(z))^{-\tau}/\tau) \]

we have the inequality

\[ v_p((\theta_0(z))^{\Delta+k} - \theta_{v_{\nu,k}}^{\nu}) \geq -(k - 1)/\phi(m) + 2 \quad \text{if } k = 1, \ldots, p - 2, \text{ because in this case } \tau \in [2, p - 1], \]

\[ v_p((\theta_0(z))^{\Delta+k} - \theta_{v_{\nu,k}}^{\nu}) \geq -(k)/\phi(m) + 1 \quad \text{if } k = 1, \ldots, p - 2, \text{ because in this case } \tau \in [2, p - 1], \]

\[ v_p((\theta_0(z))^{\Delta+k} - \theta_{v_{\nu,k}}^{\nu}) \geq -(k)/\phi(m) + 1 \quad \text{if } k = 1, \ldots, p - 2, \text{ because in this case } \tau \in [2, p - 1], \]
where \( k \in \{q - 1, q\} \), and the equality reaches only for \( k = \tau = p \); on the other hand, \( \nu_p(\alpha^*(z; 1)) \geq 1 - (p + 1)/\phi(m) \geq -2/(p - 1) \geq -2/\phi(m) \). So, if \( p \geq 5 \), then \( \nu_p(\phi^*(z; 1)) = -(p - 1)/\phi(m) \). If \( m \in \mathfrak{M} \setminus \mathfrak{M}_0 \), then \( m = 2q^a \), with prime \( q \), according to the Lemma 1, \( I = (1 + \exp(2i\pi/m)) \) is a prime ideal in \( K_m \), and \( \phi(m) = (q) \). Therefore in this case \( \nu_p(\theta_0(z)) = 0 \) If \( m \in \mathfrak{M}_0 \), then, according to the Lemma 1, \( \nu_p(\theta_0(z)) = 0 \). According to (19), in both last cases,

\[
\nu_p(\phi^*(z; 1)) + \alpha_{\nu,p-1}/p + \theta_0(z)\alpha_{\nu,p}/p \geq 1.
\]

In view of (83), (84),

\[
\nu_p(\alpha_{\nu,p-1}/p + \theta_0(z)\alpha_{\nu,p}/p) = \\
\nu_p(\exp(2i\pi/m) - 1)/(\exp(2i\pi/m) + 1)).
\]

If \( p = 5 \) and \( m \in \{3, 4, 5, 7, 8, 9, 10\} \) then, clearly,

\[
(88) \quad \nu_p((\exp(2i\pi/m) - 1)) \leq 1/4.
\]

If \( p = 5 \) and \( m = 12 \), then \( Nm_{K_{12}}((\exp(i\pi/6) - 1)) = 3 \) and (88) holds.

If \( p = 7 \), and \( m \in \{3, 4, 5, 7, 8, 9, 10, 12, 14, 18\} \) then

\[
\nu_p((\exp(2i\pi/m) - 1)) \leq 1/6.
\]

Lemma 11. Let are fulfilled all the conditions of the Lemma 10. Then

\[
(89) \quad \limsup_{\nu \in \mathbb{N}, \nu \to \infty} \left| f_0^*(z, \nu) \right|^{1/\nu} = \rho_{2,1}^*(z) = \begin{cases} \\
\rho_{2,1}(z) = -1/(1+\exp(2ik\pi/m)) \quad \text{if } \theta_0(z) = -1/(1+\exp(2ik\pi/m)), \\
h^n(\eta_1(1/(2\cos(k\pi/m)), k\pi/m, \delta_0)) \quad \text{if } \theta_0(z) = -1/(1+\exp(2ik\pi/m)) \end{cases}
\]

where \( h^n(\eta) \) is defined in (30).

Proof. According to the Lemma 2, (20) and Lemma 10, \( f_0^*(z, \nu) \) is a nonzero solution of the Poincaré type difference equation (32). According to the Perron’s theorem and Lemma 5, the equality (89) holds. 

Let \( K/\mathbb{Q} \) be the finite extension of the field \( \mathbb{Q} \),

\[
[K: \mathbb{Q}] = d.
\]

Let the field \( K \) has \( r_1 \) real places and \( r_2 \) complex places. Each such place is the monomorphism of the field \( K \) in the field \( \mathbb{R} \), if a place is real, or in the field \( \mathbb{C} \), if a place is not real; we will denote these monomorphisms respectively by \( \sigma_1, \ldots, \sigma_{r_1+r_2} \). Then \( d = r_1 + 2r_2 \). Let \( \mathfrak{B} \) be the fixed integer basis

\[
\omega_1, \ldots, \omega_d
\]

of the field \( K \) over \( \mathbb{Q} \). Clearly, \( K \) is an algebra over \( \mathbb{Q} \). With extension of the ground field from \( \mathbb{Q} \) to \( \mathbb{R} \) appears an isomorphism of the algebra \( K = K \otimes \mathbb{R} \) onto direct sum

\[
\mathbb{R} \oplus \cdots \oplus \mathbb{R} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}
\]

\( r_1 \) times \( r_2 \) times
of $r_1$ copies of the field $\mathbb{R}$ and $r_2$ copies of the field $\mathbb{C}$. We identify by means of this isomorphism the algebra $\mathcal{R}$ with the specified direct sum. We denote below by $\pi_j$, where $j = 1, \ldots, r_1 + r_2$, the projection of $\mathcal{R}$ onto its $j$-th direct summand and also the extension of this projection onto all kinds of matrices which have all the elements in $\mathbb{R}$ and $\pi_j(\mathcal{R}) = \mathbb{C}$ for $j = r_1 + 1, \ldots, r_1 + r_2$. Further by $i_{\mathcal{R}}$ we denote the embedding of $\mathbb{R}$ in $\mathcal{R}$ in diagonal way and also the extension of this embedding onto all kinds of the real matrices. So, $\mathbb{R}$ is imbedded by means $i_{\mathcal{R}}$ in $\mathcal{R}$ in diagonal way. Each element $Z \in \mathcal{R}$ has a unique representation in the form:

$$Z = \begin{pmatrix} z_1 \\ \vdots \\ z_{r_1+r_2} \\ \sigma_{r_1+1} \\ \vdots \\ \sigma_{r_1+r_2} \end{pmatrix},$$

with $z_j = \pi_j(Z) \in \mathbb{R}$ for any $j = 1, \ldots, r_1$ and with $z_j = \pi_j(Z) \in \mathbb{C}$ for any $j = r_1 + 1, \ldots, r_1 + r_2$. Further by $Tr_{\mathcal{R}}(Z)$ we denote the sum

$$\sum_{j=1}^{r_1} z_j + \sum_{j=r_1+1}^{r_1+r_2} 2\Re(z_j) =$$

$$\sum_{j=1}^{r_1} \pi_j(Z) + \sum_{j=r_1+1}^{r_1+r_2} 2\Re(\pi_j(Z)),$$

and by $q^{(\mathcal{R})}(Z)$ we denote the value

$$\max(|z_1|, \ldots, |z_{r_1+r_2}|) = \max(|\pi_1(Z)|, \ldots, |\pi_{r_1+r_2}(Z)|).$$

Clearly,

$$q^{(\mathcal{R})}(Z_1Z_2) \leq q^{(\mathcal{R})}(Z_1)q^{(\mathcal{R})}(Z_2),$$

$$q^{(\mathcal{R})}(Z_1 + Z_2) \leq q^{(\mathcal{R})}(Z_1) + q^{(\mathcal{R})}(Z_2),$$

$$q^{(\mathcal{R})}(i_{\mathcal{R}}(\lambda)Z) = |\lambda|q^{(\mathcal{R})}(Z)$$

for any $Z_1 \in \mathcal{R}, Z_2 \in \mathcal{R}, Z \in \mathcal{R}$ and $\lambda \in \mathbb{R}$. The natural extension of the norm $q^{(\mathcal{R})}$ on the set of all the matrices, which have all the elements in $\mathcal{R}$ (i.e. the maximum of the norm $q^{(\mathcal{R})}$ of all the elements of the matrix) also will be denoted by $q^{(\mathcal{R})}$. If

$$Z = \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} \in K,$$

then

$$z_j = \sigma_j(Z),$$
where \( j = 1, \ldots, r_1 + r_2 \),
\[
z_{r_1+r_2+j} = \sigma_{r_1+j}(Z),
\]
where \( j = 1, \ldots, r_2 \). In particular,
\[
\omega_k = \begin{pmatrix}
\sigma_1(\omega_k) \\
\vdots \\
\sigma_{r_1+r_2}(\omega_k) \\
\sigma_{r_1+1}(\omega_k) \\
\vdots \\
\sigma_{r_1+r_2}(\omega_k)
\end{pmatrix},
\]
As usually, the ring of all the integer elements of the field \( K \) will be denoted by \( \mathbb{Z}_K \). The ring \( \mathbb{Z}_K \) is embedded in the ring \( \mathfrak{r} \) as discrete lattice. Moreover, if \( Z \in \mathbb{Z}_K \setminus \{0\} \), then
\[
\left( \prod_{i=1}^{r_1} |\sigma_j(Z)| \right) \left( \prod_{i=1}^{r_2} |\sigma_{r_1+i}(Z)| \right)^2 = |Nm_{K/Q}(Z)| \in \mathbb{N}
\]
and therefore \( q_{\infty}^{(\mathfrak{r}_i)}(Z) \geq 1 \), for any \( Z \in \mathbb{Z}_K \setminus \{0\} \). The elements of \( \mathbb{Z}_K \) we name below by \( K \)-integers. For each \( Z \in \mathfrak{r} \) let
\[
\|Z\|_K = \inf_{W \in \mathbb{Z}_K} \{ q_{\infty}^{(\mathfrak{r}_i)}(Z - W) \}.
\]
Let \( \{m, n\} \subset \mathbb{N} \),
\[a_{i,k} \in \mathfrak{r}\]
for \( i = 1, \ldots, m, \ k = 1, \ldots, n \),
\[\alpha_j^\wedge(\nu) \in \mathbb{Z}_K\]
where \( j = 1, \ldots, m + n \) and \( \nu \in \mathbb{N} \). Let there are \( \gamma_0, r_1^\wedge \geq 1, \ldots, r_m^\wedge \geq 1 \) such that
\[
q_{\infty}^{(\mathfrak{r}_i)}(\alpha_i(\nu)) < \gamma_0 (r_i^\wedge)^\nu
\]
where \( i = 1, \ldots, m \) and \( \nu \in \mathbb{N} \). Let
\[
y_k(\nu) = -\alpha_{m+k}^\wedge(\nu) + \sum_{i=1}^{m} a_{i,k} \alpha_i^\wedge(\nu)
\]
where \( k = 1, \ldots, n \) and \( \nu \in \mathbb{N} \). If \( X = \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} \in \mathfrak{r}^n \), then let
\[
y^\wedge(X) = y^\wedge(X, \nu) = \sum_{k=1}^{n} y_k^\wedge(\nu)Z_k
\]
for \( \nu \in \mathbb{N} \), let

\[
\phi_i(X) = \sum_{k=1}^{n} a_{i,k} Z_k
\]

for \( i = 1, \ldots, m \), and let

\[
\alpha_0^\wedge(X, \nu) = \sum_{k=1}^{n} \alpha_{m+k}(\nu) Z_k
\]

for \( \nu \in \mathbb{N} \). Clearly,

\[
y^\wedge(X, \nu) = -\alpha_0^\wedge(X, \nu) + \sum_{i=1}^{m} \alpha_i^\wedge(\nu) \phi_i(X)
\]

for \( X \in \mathfrak{R}^n \) and \( \nu \in \mathbb{N} \),

\[
\alpha_0^\wedge(X, \nu) \in \mathbb{Z}_K
\]

for \( X \in (\mathbb{Z}_K)^n \) and \( \nu \in \mathbb{N} \).

**Lemma 12.** Let \( \{l, n\} \subset \mathbb{N} \), \( \gamma_1 > 0 \), \( \gamma_2 > \frac{1}{2}, R_1 \geq R_2 > 1, \)

\[
\alpha_i = (\log(r_i^\wedge R_1/R_2))/\log(R_2),
\]

where \( i = 1, \ldots, m \), let \( X \in (\mathbb{Z}_K)^n \setminus \{(0)\}, \)

\[
\gamma_3 = \gamma_1(R_1)^{-\log(2\gamma_2 R_2)/\log(R_2)}, \quad \gamma_4 = \gamma_3 \left( \sum_{i=1}^{m} \gamma_0(r_i^\wedge)^{(\log(2\gamma_2))/\log(R_2)+l} \right)^{-1}
\]

and let for each \( \nu \in \mathbb{N} - 1 \) hold the inequalities

\[
\gamma_1(R_1)^{-\nu} q_\infty^{(\mathfrak{g})}(X) \leq \sup\{q_\infty^{(\mathfrak{g})}(y^\wedge(X, \kappa)) : \kappa = \nu, \ldots, \nu + l - 1\},
\]

\[
q_\infty^{(\mathfrak{g})}(y^\wedge(X, \nu)) \leq \gamma_2(R_2)^{-\nu} q_\infty^{(\mathfrak{g})}(X)
\]

Then

\[
\sup\{\|\phi_i(X)\|_{K(q_\infty^{(\mathfrak{g})}(X))^{\alpha_i}} : i = 1, \ldots, m\} \geq \gamma_4.
\]

**Proof.** Proof may be found in [60], Theorem 2.3.1. ■

**Corollary.** Let \( a \in \mathfrak{R} \),

\[
(90) \quad \alpha_1^\wedge(\nu) \in \mathbb{Z}_K, \quad \alpha_2^\wedge(\nu) \in \mathbb{Z}_K, y(\nu) = -\alpha_2^\wedge(\nu) + a\alpha_1^\wedge(\nu)
\]

where \( \nu \in \mathbb{N} \). Let there are \( \gamma_0, r_1^\wedge \geq 1 \) such that

\[
q_\infty^{(\mathfrak{g})}(\alpha_1(\nu)) < \gamma_0(r_1^\wedge)^{\nu},
\]

where \( \nu \in \mathbb{N} \). Let \( l \in \mathbb{N} \), \( \gamma_1 > 0, \gamma_2 > \frac{1}{2}, R_1 \geq R_2 > 1, \)

\[
\alpha_1 = (\log(r_1^\wedge R_1/R_2))/\log(R_2), \quad \gamma_3 = \gamma_1(R_1)^{-\log(2\gamma_2 R_2)/\log(R_2)},
\]

\[
\gamma_4 = \gamma_3 \left( \gamma_0(r_1^\wedge)^{(\log(2\gamma_2))/\log(R_2)+l} \right)^{-1},
\]

\( X \in \mathbb{Z}_K \) and let for each \( \nu \in \mathbb{N} - 1 \) hold the inequalities

\[
\gamma_1(R_1)^{-\nu} q_\infty^{(\mathfrak{g})}(X) \leq \sup\{q_\infty^{(\mathfrak{g})}(y_1(\kappa)X) : \kappa = \nu, \ldots, \nu + l - 1\}\},
\]
\( q_{\infty}^{(g)}(y(\nu)X) \leq \gamma_2(R_2)^{-\nu}q_{\infty}^{(g)}(X) \)

Then

\[(91) \quad \|aX\|_K(q_{\infty}^{(g)}(X))^\alpha \geq \gamma_4.\]

**Proof.** This Corrolary is the Lemma 12 for \( m = n = 1 \). ■

Let \( B \in \mathbb{N} \), \( D^*(B) = \inf\{q \in \mathbb{N}: d/\kappa \in \mathbb{N}, \kappa \in \mathbb{N}, \kappa \leq B\} \). It is known that

\( D^*(B) = \exp(B + O(B/\log(B))). \)

Let \( d_0^*(\Delta, \nu) = D^*(\nu(\Delta + 1)) \). Then

\[(92) \quad d_0^*(\Delta, \nu) = \exp(\nu(\Delta + 1) + O(\nu/\log(\nu))),\]

when \( \nu \to \infty \).

Probably G.V. Chudnovsky was the first man, who discovered, that the numbers (16) have a great common divisor; Hata ([17]) in details studied this effect. Therefore I name the mentioned common divisor by Chudnovsky-Hata’s multiplier and denote it by \( d_1^*(\Delta, \nu) \). According to the Hata’s results,

\[(93) \quad \log(d_1^*(\Delta, \nu)) = (1 + o(1))\nu \times \]

\[ \sum_{\mu=0}^{1} \left( \frac{\Delta + (-1)^{\mu}}{2} \log \left( \frac{\Delta}{\Delta + (-1)^{\mu}} \right) + (-1)^{\mu} \frac{\pi}{2} \sum_{\kappa=1}^{\left[ \frac{\Delta + (-1)^{\mu}}{2} \right]} \cot \left( \frac{\pi \kappa}{\Delta + (-1)^{\mu}} \right) \right). \]

In view of (92),

\[(94) \quad d_0^*(5, \nu) = \exp(6\nu(\Delta + 1) + O(\nu/\log(\nu))), d_0^*(7, \nu) = \exp(8\nu(8) + O(\nu/\log(\nu))).\]

In view of (94)

\[(95) \quad \log(d_1^*(5, \nu)) = (1 + o(1))\nu \times \]

\[ (-3 \log(1.2) + 2 \log(0.8) + (\pi/2)(\cot(\pi/6) + \cot(\pi/3) + \cot(\pi/4))) = (1 + o(1))\nu \times 1.956124..., \]

\[(96) \quad \log(d_1^*(7, \nu)) = (1 + o(1))\nu \times \]

\[ (4 \log(7/8) + 3 \log(7/6)) + (1 + o(1))(\pi/2)\nu \times \]

\[ (-\cot(\pi/6) - \cot(\pi/3) + \cot(\pi/8) + \cot(3\pi/8) + \cot(\pi/4)) = (1 + o(1))\nu(4 \log(7/8) + 3 \log(7/6) + \pi(-2/\sqrt{3} + 2/\sqrt{2} + 1/2) = (1 + o(1))\nu \times 2.314407... \],

when \( \nu \to \infty \).

In view of (18) and (19),

\[ \alpha^*(z; \nu)d_0^*(\nu)/d_1^*(\nu) \in \mathbb{Z}[z], \]

\[ \phi^*(z; \nu)d_0^*(\nu)/d_1^*(\nu) \in \mathbb{Z}[z]. \]
Let

\begin{equation}
U_\Delta(m, \nu) = d_0^*(\nu)/d_1^*(\nu), \Lambda_0(m) = 0,
\end{equation}

if \( m \neq 2p^\alpha \), where \( p \) run over the all the prime numbers and \( \alpha \) run over \( \mathbb{N} \) and let

\begin{equation}
U_\Delta(m, \nu) = \frac{d_0^*(\nu)}{d_1^*(\nu)} p^{[(\Delta+1)\nu/\phi(m)]+1}, \Lambda_0(m) = \Lambda(m/2),
\end{equation}

if \( m = 2p^\alpha \), where \( p \) is a prime number and \( \alpha \in \mathbb{N} \). In view of the (18), (19) and Lemma 1,

\begin{equation}
\alpha^*(z; \nu) \bigg|_{z = \left( \frac{1}{2 \cos \left( \frac{\nu \pi}{m} \right)}, \frac{k \pi - \pi}{m} \right)} U_\Delta(m, \nu) \in \mathbb{Z}_{\mathbb{Q}[\exp(2\pi i/m)]},
\end{equation}

\begin{equation}
\phi^*(z; \nu) \bigg|_{z = \left( \frac{1}{2 \cos \left( \frac{\nu \pi}{m} \right)}, \frac{k \pi - \pi}{m} \right)} U_\Delta(m, \nu) \in \mathbb{Z}_{\mathbb{Q}[\exp(2\pi i/m)]},
\end{equation}

where \((k, m) = 1\). In view of (98), (97), (93), (92), (2) and (3)

\begin{equation}
\frac{d_0^*(\nu)}{d_1^*(\nu)} = \nu(1 + o(1))V_\Delta \log(U_\Delta(m, \nu)) = \nu(1 + o(1))V_\Delta(m),
\end{equation}

when \( \nu \to \infty \).

The polynomial (28) take the form

\[ D^\Delta(z, \eta) = (\eta + 1) \left( \eta + \frac{\Delta - 1}{\Delta + 1} \right) + \frac{2\Delta \exp(i\psi)\eta}{(\Delta + 1) \cos(\psi)} = ((\Delta + 1)\eta^2 + 2\Delta(2 + iT)\eta + (\Delta - 1))/(\Delta + 1), \]

where \( \psi \in (-\pi/2, \pi/2) \) and \( T = \tan(\psi) \); its roots are equal to

\[ -\left( 2\Delta + \Delta iT + R \right)/(\Delta + 1), \]

where \( R^2 = \Delta^2(3 - T^2) + 1 + 4\Delta^2iT \). In view of (1), Then

\[ R \in \{ \pm \left( w_\Delta(T) + i2\Delta^2iT/w_\Delta(T) \right) \} . \]

In view of (102) and (46),

\[ \eta_j^\Delta(r, \psi, \delta_0) = \frac{-2\Delta + \Delta iT + (-1)^j (w_\Delta(T) + i2\Delta^2iT/w_\Delta(T))}{\Delta + 1} = \frac{-2\Delta + (-1)^j w_\Delta(T) + iT\Delta (1 + (-1)^j 2\Delta/w_\Delta(T))}{\Delta + 1} , \]

where \( j = 0, 1, \)

\[ |\eta_j^\Delta(r, \psi, \delta_0) + k|^2 = \]
\[
\frac{(2\Delta + (-1)^j w_\Delta(T) - k(\Delta + 1))^2 + T^2 \Delta^2 \left(1 + (-1)^j 2\Delta/w_\Delta(T)\right)^2}{(\Delta + 1)^2},
\]
where \( j = 0, 1; \) \( k = 0, 1, -1. \) Therefore, in view of (30) and (4)

\[
\ln |h^{\sim}(\eta_j^0(r, \psi, \delta_0))| =
\]

\[
(\eta_j(r, \psi, \delta_0) - 1)(1 - \delta_0)^{-d_1}(\eta_j(r, \psi, \delta_0) + 1)^{-2} \eta_j(r, \psi, \delta_0)d_1 =
\]

\[
- \log (4(\Delta + 1)^{\Delta+1}(1 - 1/\Delta)(\Delta - 1)) +
\]

\[
\frac{1}{2} \log \left( (2\Delta + (-1)^j w_\Delta(T) + (\Delta + 1))^2 + T^2 \Delta^2 \left(1 + \frac{(-1)^j 2\Delta}{w_\Delta(T)}\right)^2 \right) +
\]

\[
\frac{1}{2} \log \left( (2\Delta + (-1)^j w_\Delta(T) - (\Delta + 1))^2 + T^2 \Delta^2 \left(1 + \frac{(-1)^j 2\Delta}{w_\Delta(T)}\right)^2 \right) +
\]

\[
\frac{\Delta - 1}{2} \log \left( (2\Delta + (-1)^j w_\Delta(T))^2 + T^2 \Delta^2 \left(1 + \frac{(-1)^j 2\Delta}{w_\Delta(T)}\right)^2 \right) =
\]

\[
l_\Delta(j, T),
\]
where \( j = 0, 1. \) Clearly,

\[
w_\Delta(0) = \sqrt{3\Delta^2 + 1},
\]

\[
\eta_j^0(1/2, 0, \delta_0) = - \frac{2\Delta + (-1)^j \sqrt{3\Delta^2 + 1}}{\Delta + 1},
\]
where \( j = 0, 1,

\[
|\eta_j^0(1/2, 0, \delta_0) + k| = \left| \frac{2\Delta + (-1)^j \sqrt{3\Delta^2 + 1} - k(\Delta + 1)}{\Delta + 1} \right|,
\]
where \( j = 0, 1; k = 0, 1, -1. \) Therefore

\[
l_\Delta(\epsilon, 0) = \langle \log |h^{\sim}(\eta_j^0(1/2, 0, \delta_0))| \rangle =
\]

\[
\log \left( |\eta_\epsilon(1/2, 0, \delta_0) - 1|(1 - \delta_0)^{-d_1}(\eta_\epsilon(1/2, 0, \delta_0) + 1)^{-2} \eta_\epsilon(1/2, 0, \delta_0)d_1 | \right) =
\]

\[
- \log (4(\Delta + 1)^{\Delta+1}(1 - 1/\Delta)(\Delta - 1)) +
\]

\[
\log \left( |2\Delta + (-1)^\epsilon \sqrt{3\Delta^2 + 1} - (\Delta + 1)| \right) +
\]

\[
\log \left( |2\Delta + (-1)^\epsilon \sqrt{3\Delta^2 + 1} + (\Delta + 1)| \right) +
\]

\[
(\Delta - 1) \log \left( |2\Delta + (-1)^\epsilon \sqrt{3\Delta^2 + 1}| \right).
\]

Consequently

\[
l_\epsilon(1, 0) = - \log(4) - 6 \log 6 - 4 \log(0.8) +
\]

\[
\log(\sqrt{76} - 4) + \log(16 - \sqrt{76}) + 4 \log(10 - \sqrt{76})
\]

I made computations below ”by hands” using calculator of the firm ”CASIO.”

\[
\log 4 = 1.386294361...; \ 6 \log(6) = 10.7505682...;
\]
\[
4 \log(0.8) = -0.892574205...;
\]
\[
\sqrt{76} = 8, 717797887...; \sqrt{76} - 4 = 4, 717797887...;
\]
\[
16 - \sqrt{76} = 7, 282202113...; \ 10 - \sqrt{76} = 1, 282202113...;
\]
\[
\log \left( \sqrt{76} - 4 \right) = 1.551342141...; \ \log \left( 16 - \sqrt{76} \right) = 1.985433305...;
\]
\[
\log \left( 10 - \sqrt{76} \right) = 0.248579...; \ 4 \log \left( 10 - \sqrt{76} \right) = 0, 994316001...;
\]
\[
16 - \sqrt{76} = 7, 282202113...; \ 10 - \sqrt{76} = 1, 282202113...;
\]
\[
\log \left( \sqrt{76} - 4 \right) = 1.551342141...; \ \log \left( 16 - \sqrt{76} \right) = 1.985433305...;
\]
\[
\log \left( 10 - \sqrt{76} \right) = 0.248579...; \ 4 \log \left( 10 - \sqrt{76} \right) = 0, 994316001...;
\]

(105) \[ l_5(1, 0) = -6.713196909...; \]

\[
l_7(1, 0) = -\log(4) - 8 \log(8) - 6 \log(6) + 6 \log(7) + \log \left( \sqrt{148} - 6 \right) + \log \left( 22 - \sqrt{148} \right) + 6 \log \left( 14 - \sqrt{148} \right); \]

\[
8 \log 8 = 16, 63553233...; \ 6 \log 6 = 10, 75055682...; \ 6 \log 7 = 11, 67546089...;
\]
\[
\sqrt{148} = 12, 16552506...; \sqrt{148} - 6 = 6, 16552506...;
\]
\[
22 - \sqrt{148} = 9, 83474939...; \ 14 - \sqrt{148} = 1, 83474939...;
\]
\[
\log(\sqrt{148} - 6) = 1, 818973301; \ \log(22 - \sqrt{148}) = 2, 285894063...;
\]
\[
\log(14 - \sqrt{148}) = 0, 606758304...; \ 6 \log(14 - \sqrt{148}) = 3, 640549824...;
\]

(106) \[ l_7(1, 0) = -9, 35150543... . \]

In view of (2), (92), (93), (95), (96) and (101),

(107) \[ V_5^* = 6 - 1.956124... = 4, 04387...; \ V_7^* = 8 - 2.314407 = 5, 685593. \]

In view (105) – (107),

(108) \[-V_5^* - l_5(1, 0) > 0, \ -V_7^* - l_7(1, 0) > 0. \]

So, the key inequalities (108) are checked ”by hands”. I view of (103), (108) and Lemma 3,

\[-V_5^* - l_5(1, \tan(\pi/m)) > 0, \ -V_7^* - l_7(1, \tan(\pi/m)) > 0, \]

where \( m > 2 \). Since \( (\log(p))/(p^{\alpha-1}(p - 1)) \) decreases together with increasing of \( p \in (3, +\infty) \) with fixed \( \alpha \geq 1 \), or increasing of \( \alpha \in (1, +\infty) \) with fixed \( p \geq 2 \) (or, of course, increasing both \( \alpha \in (1, +\infty) \) and \( p \in (3, +\infty) \)), and

\[ \lim_{p \to \infty} ((\log(p))/(p^{\alpha-1}(p - 1))) = 0, \]

where \( \alpha \geq 1, \)

\[ \lim_{\alpha \to \infty} ((\log(p))/(p^{\alpha-1}(p - 1))) = 0, \]

where \( p \geq 2 \), it follows that the inequality (7) holds for all the sufficient big integers \( m \). Computations on computer of class ”Pentium” show that the inequality (7) holds for \( m = 3, m = 4, m = 5 \) and \( m = 2 \times 5; \) therefore
inequality (7) holds for all the $m > 2 \times 3$. Let $\varepsilon_0 = h_\Delta(m)/2$, with $h_\Delta(m)$ defined in (6). In view of (7), $\varepsilon_0 > 0$. We take now $K = K_m = \mathbb{Q}[\exp(2\pi i/m)]$. Let further $\{\sigma_1, \ldots, \sigma_{\phi(m)}\} = \text{Gal}(K/\mathbb{Q})$. For each $j = 1, \ldots, \phi(m)$ there exists $k_j \in (-m/2, m/2) \cap \mathbb{Z}$ such that

$$\left(\lfloor k_j \rfloor, m\right) = 1, \sigma_j\left(\exp\left(\frac{2\pi i}{m}\right)\right) = \exp\left(\frac{2\pi ik_j}{m}\right).$$

Let $a$ be the element of $\mathfrak{R}$, such that

$$\pi_j(a) = \log(2 + \sigma_j(\exp(2\pi i/m))) = \log(2 + \exp(2\pi ik_j/m)),$$

where $j = 1, \ldots, \phi(m)$; we suppose that $k_1 = 1$. In view of (99) and (100), let $\alpha^\vee_1(\nu), \alpha^\vee_2(\nu), \alpha^\wedge_1(\nu), \alpha^\wedge_2(\nu)$, are elements in $\mathfrak{R}$ such that

$$\pi_j(\alpha^\vee_1(\nu)) = \alpha^\ast(z; \nu)\left|_{z = \left(\frac{1}{2 \cos(\frac{\nu \pi i}{m})}, \frac{k_j \pi i}{m} - \pi\right)}\right.,$$

$$\pi_j(\alpha^\vee_2(\nu)) = \phi^\ast(z; \nu)\left|_{z = \left(\frac{1}{2 \cos(\frac{\nu \pi i}{m})}, \frac{k_j \pi i}{m} - \pi\right)}\right.,$$

$$\pi_j(\alpha^\wedge_1(\nu)) = \alpha^\ast(z; \nu)\left|_{z = \left(\frac{1}{2 \cos(\frac{\nu \pi i}{m})}, \frac{k_j \pi i}{m} - \pi\right)}\right. U_\Delta(m, \nu),$$

$$\pi_j(\alpha^\wedge_2(\nu)) = \phi^\ast(z; \nu)\left|_{z = \left(\frac{1}{2 \cos(\frac{\nu \pi i}{m})}, \frac{k_j \pi i}{m} - \pi\right)}\right. U_\Delta(m, \nu),$$

where $j = 1, \ldots, \phi(m)$. Then $\alpha^\wedge_k(\nu) \in \mathbb{Z}_K$ for $k = 1, 2$.

$$y^\vee(\nu) = -\alpha^\vee_2(\nu) + a\alpha^\vee_1(\nu),$$

and let $y(\nu)$ is defined by means the equality (90). According to the Corollary of the Lemma 4, to the Theorem 4 in [62] (or Theorem 7 in [70]), to the Lemma 8, to (103), there exist $m^*_1 \in \mathbb{N}$ having the following property:

for any $\varepsilon \in (0, \varepsilon_0)$ there exist $\gamma_0(\varepsilon) > 0$, $\gamma_1(\varepsilon) > 0$, and $\gamma_2(\varepsilon) > 0$ such that

$$\left|\pi_j(\alpha^\wedge_k(\nu))\right| \leq \gamma_0(\varepsilon) \exp\left((l_\Delta(\tan((k_j \pi i)/m), 0) + \varepsilon/3)\nu\right),$$

where $k = 1, 2$, $j = 1, \ldots, \phi(m)$ and $\nu \in \mathbb{N} - 1 + m^*_1$,

$$\gamma_1(\varepsilon) \exp((l_\Delta(\tan((k_j \pi i)/m), 1) - \varepsilon/3)\nu) \leq \max(|\pi_j(y^\vee(\nu))|, |\pi_j(y^\vee(\nu + 1))| \leq \gamma_2(\varepsilon) \exp((l_\Delta(\tan((k_j \pi i)/m), 1) + \varepsilon/3)\nu),$$

where $j = 1, \ldots, \phi(m)$ and $\nu \in \mathbb{N} - 1 + m^*_1$. 

L.A.Gutnik, On the Diophantine Approximations of logarithms in cyclotomic fields. 29
Let $\omega_1(m) = (m - 1)/2$, if $m$ is odd, $\omega_1(m) = m/2 - 2$, if $m \equiv 2 \pmod{4}$, and $\omega(m) = m/2 - 1$, if $m \equiv 0 \pmod{4}$. Then

$$\omega_1(m) = \sup \{k \in \mathbb{N}: k_j < m/2, (k, m) = 1\}.$$ 

According to the Lemma 3 and (103),

$$l_\Delta(\tan((k_j \pi i)/m), 0) \leq l_\Delta(\tan((\omega_1(m) \pi i)/m), 0),$$

(114)

$$l_\Delta(\tan((\omega_1(m) \pi i)/m), 1) \leq l_\Delta(\tan((k_j \pi i)/m), 1) \leq l_\Delta(\tan((\pi i)/m), 1)$$

where $j = 1, \ldots, \phi(m)$. In view of (112) – (115),

$$|\pi_j(\alpha_k^*(\nu))| \leq \gamma_0(\varepsilon) \exp((l_\Delta(\tan((\omega_1(\nu) \pi i)/m), 0) + \varepsilon/3)\nu),$$

where $k = 1, 2, j = 1, \ldots, \phi(m)$ and $\nu \in \mathbb{N} - 1 + m_1^*$. In view of (101), there exists $m_2^* \in \mathbb{N} - 1 + m_1^*$, such that

$$\exp(V_\Delta(m) - \varepsilon/3)\nu \leq U_\Delta(m, \nu) \leq \exp(V_\Delta(m) - \varepsilon/3)\nu$$

where $\nu \in \mathbb{N} - 1 + m_2^*$.

In view of (115) – (118), (109) – (111), (6), (5),

$$|\pi_j(\alpha_k(\nu))| \leq \gamma_0(\varepsilon) \exp((g_{\Delta,0}(m) + 2\varepsilon/3)\nu),$$

where $k = 1, 2, j = 1, \ldots, \phi(m)$ and $\nu \in \mathbb{N} - 1 + m_2^*$.

$$\gamma_1(\varepsilon) \exp((-g_{\Delta,1}(m) - 2\varepsilon/3)\nu) \leq \gamma_2(\varepsilon) \exp((-h_\Delta(m) + 2\varepsilon/3)\nu),$$

where $j = 1, \ldots, \phi(m)$ and $\nu \in \mathbb{N} - 1 + m_2^*$.

Let $X \in \mathbb{Z}_{K_m} \setminus \{0\}$. Then, in view of (119) and (120),

$$|\pi_j(X\alpha_k(\nu))| \leq \gamma_0(\varepsilon) \exp((g_{\Delta,0}(m) + 2\varepsilon/3)\nu)|\pi_j(X)| \leq \gamma_0(\varepsilon) \exp((g_{\Delta,0}(m) + 2\varepsilon/3)\nu)g_\infty^{(g_0)}(X),$$

where $k = 1, 2, j = 1, \ldots, \phi(m)$ and $\nu \in \mathbb{N} - 1 + m_2^*$.

$$\gamma_1(\varepsilon) \exp((-g_{\Delta,1}(m) - 2\varepsilon/3)\nu)|\pi_j(X)| \leq \gamma_1(\varepsilon) \exp((-g_{\Delta,1}(m) - 2\varepsilon/3)\nu)|\pi_j(X)| \leq \gamma_1(\varepsilon) \exp((-g_{\Delta,1}(m) - 2\varepsilon/3)\nu)\gamma_2(\varepsilon) \exp((-h_\Delta(m) + 2\varepsilon/3)\nu)f(X).$$
max(|\pi_j(Xy^\nu(\nu))|, |\pi_j(Xy^\nu(\nu + 1))| \leq
max(q_{\infty}^{(g)}(Xy^\nu(\nu)), q_{\infty}^{(g)}(Xy^\nu(\nu + 1)),$
where \( j = 1, \ldots, \phi(m) \) and \( \nu \in \mathbb{N} - 1 + m^*_2 \),

\[
(123) \quad \max(|\pi_j(Xy^\nu(\nu))|, |\pi_j(Xy^\nu(\nu + 1))| \leq
\gamma_2(\varepsilon) \exp((-h_\Delta(m) + 2\varepsilon/3)\nu)|\pi_j(X)| \leq
\gamma_2(\varepsilon) \exp((-h_\Delta(m) + 2\varepsilon/3)\nu)q_{\infty}^{(g)}(X),
\]
where \( j = 1, \ldots, \phi(m) \) and \( \nu \in \mathbb{N} - 1 + m^*_2 \).

In view of (121)

\[
(124) \quad q_{\infty}^{(g)}(X\alpha_k(\nu)) \leq
\gamma_0(\varepsilon) \exp((g_{\Delta,0}(m) + 2\varepsilon/3)\nu)q_{\infty}^{(g)}(X),
\]
where \( k = 1, 2 \), and \( \nu \in \mathbb{N} - 1 + m^*_2 \). In view of (123),

\[
(125) \quad \max(q_{\infty}^{(g)}(Xy^\nu(\nu)), q_{\infty}^{(g)}((Xy^\nu(\nu + 1)) =
\sup(|\pi_j(Xy^\nu(\nu + \varepsilon))|, : \varepsilon \in \{0, 1\}, j = 1, \ldots, \phi(m))) \leq
\gamma_2(\varepsilon) \exp((-h_\Delta(m) + 2\varepsilon/3)\nu)q_{\infty}^{(g)}(X),
\]
where \( \nu \in \mathbb{N} - 1 + m^*_2 \).

Taking in account (124), (125) and (122), we see that all the conditions of the Corollary of the Lemma 12 are fulfilled for

\[
\varepsilon \in (0, \varepsilon_0), \gamma_0(\varepsilon), \gamma_1(\varepsilon), \gamma_2(\varepsilon), y = y(\nu), \alpha_1(\nu), \alpha_2(\nu),
\]

\[
r_1 = r_1(\varepsilon) = \exp(g_{\Delta,0}(m) + 2\varepsilon/3),
R_1 = R_1(\varepsilon) = \exp(g_{\Delta,1}(m) + 2\varepsilon/3),
R_2 = R_2(\varepsilon) = \exp(h_\Delta(m) - 2\varepsilon/3),
\]
and this proves the part of our Theorem connected with the inequality (8).

Let again \( X \in \mathbb{Z}_{K_m} \setminus \{0\} \) and let

\[
q_{\min}^{(g)}(X) = \inf(|\{\pi_j(X)\} : j = 1, \ldots, \phi(m))
\]
Clearly, \( q_{\min}^{(g)}(X) > 0 \) According to the Theorem 4 in [62], or to the Theorem 7 in [70], there exist \( m^*_1 \in \mathbb{N} \) having the following property: for any \( \varepsilon \in (0, \varepsilon_0) \) there exist \( \gamma_0^*(X, \varepsilon) > 0, \gamma_1^*(X, \varepsilon) > 0, \) and \( \gamma_2^*(X, \varepsilon) > 0 \) such that

\[
|\pi_j(\alpha_k^\nu(\nu))| \leq \gamma_0^*(\varepsilon) \exp((l_\Delta(\tan((\omega_m\pi)\nu)) / m) + \varepsilon/3)\nu,
\]
where \( k = 1, 2, j = 1, \ldots, \phi(m) \) and \( \nu \in \mathbb{N} - 1 + m^*_1 \),

\[
\gamma_1^*(X\varepsilon) \exp((l_\Delta(\tan((\pi)\nu) / m), 1 - \varepsilon/3)\nu) \leq
\max(|\pi_j(y^\nu(\nu))|, |\pi_j(y^\nu(\nu + 1))| \leq
\gamma_2(\varepsilon) \exp((l_\Delta(\tan((\pi)\nu) / m), 1 + \varepsilon/3)\nu),
\]
where \( j = 1, \ldots, \phi(m) \) and \( \nu \in \mathbb{N} - 1 + m_1^* \). Repeating the previous considerations, we see that all the conditions of the Corollary of the Lemma 12 are fulfilled for \( \varepsilon \in (0, \varepsilon_0) \),

\[
\gamma_0 = \gamma_0^*(X, \varepsilon), \quad \gamma_1 = \gamma_1^*(X, \varepsilon), \quad \gamma_2 = \gamma_2^*(X, \varepsilon),
\]

and\( \gamma \)\( \gamma \)

\[
y = y(\nu), \quad \alpha_1(\nu), \alpha_2(\nu), \quad r_1 = r_1(\varepsilon) = \exp(g_{\Delta, 0}(m) + 2\varepsilon/3),
\]

and

\[
R_1 = R_2 = R_2(\varepsilon) \exp(h_{\Delta}(m) - 2\varepsilon/3),
\]

and this proves the part of our Theorem connected with the inequality (9).

Below are values of \( \beta \) and \( \alpha \) computed for \( \Delta \in \{5, 7\} \) and some \( m \in \mathbb{N} \).

\[
(m; \Delta; \beta; \alpha) = (3; 5; 3, 111228...; 3, 111228...),
\]

\[
(m; \Delta; \beta; \alpha) = (3; 7; 3, 073525...; 3, 073525...),
\]

\[
(m; \Delta; \beta; \alpha) = (4; 5; 11, 458947...; 11, 458947...),
\]

\[
(m; \Delta; \beta; \alpha) = (4; 7; 10, 551730...; 10, 551730...),
\]

\[
(m; \Delta; \beta; \alpha) = (5; 4, 826751...; 5, 607961...),
\]

\[
(m; \Delta; \beta; \alpha) = (5; 7, 4, 837858...; 5, 684622...),
\]

\[
(m; \Delta; \beta; \alpha) = (7; 5; 5, 701485...; 6, 977258...),
\]

\[
(m; \Delta; \beta; \alpha) = (7; 7; 5, 724804...; 7, 114963...),
\]

\[
(m; \Delta; \beta; \alpha) = (8; 5; 9, 436901...),
\]

\[
(m; \Delta; \beta; \alpha) = (8; 7; 8, 932056...; 7, 960502...),
\]

\[
(m; \Delta; \beta; \alpha) = (9; 5; 6, 932056...; 7, 960502...),
\]

\[
(m; \Delta; \beta; \alpha) = (9; 7; 6, 335274...; 8, 134962...),
\]

\[
(m; \Delta; \beta; \alpha) = (10; 5; 43, 546644...; 46, 230614...),
\]

\[
(m; \Delta; \beta; \alpha) = (10; 7; 35, 648681...; 38, 043440...),
\]

\[
(m; \Delta; \beta; \alpha) = (11; 5; 6, 786990...; 8, 735234...),
\]

\[
(m; \Delta; \beta; \alpha) = (11; 7; 6, 806087...; 8, 934922...),
\]

\[
(m; \Delta; \beta; \alpha) = (12; 5; 5, 638541...; 6, 813222...),
\]

\[
(m; \Delta; \beta; \alpha) = (12; 7; 5, 696732...; 6, 983870...),
\]

\[
(m; \Delta; \beta; \alpha) = (13; 5; 9, 376030...),
\]

\[
(m; \Delta; \beta; \alpha) = (13; 7; 7, 190814...; 9, 94580...),
\]

\[
(m; \Delta; \beta; \alpha) = (14; 5; 19, 659885...; 21, 835056...),
\]

\[
(m; \Delta; \beta; \alpha) = (14; 7; 18, 447228...; 20, 668254...),
\]

\[
(m; \Delta; \beta; \alpha) = (15; 5; 7, 508714...; 9, 922761...),
\]

\[
(m; \Delta; \beta; \alpha) = (15; 7; 7, 516606...; 10, 156245...),
\]

\[
(m; \Delta; \beta; \alpha) = (16; 5; 7, 951153...; 9, 876454...),
\]
(m; Δ; β; α) = (16; 7; 7; 945763...; 10, 039605...),
(m; Δ; β; α) = (17; 5; 7; 797153...; 10, 399610...),
(m; Δ; β; α) = (17; 7; 7; 799343...; 10, 645404...),
(m; Δ; β; α) = (18; 5; 9; 486110...; 10, 955534...),
(m; Δ; β; α) = (18; 7; 9; 406368...; 10, 989150...),
(m; Δ; β; α) = (19; 5; 8; 052478...; 10, 822446...),
(m; Δ; β; α) = (19; 7; 8; 049182...; 11, 078690...),
(m; Δ; β; α) = (20; 5; 6; 696241...; 8, 559091...),
(m; Δ; β; α) = (20; 7; 6; 733979...; 8, 774063...),
(m; Δ; β; α) = (21; 5; 8; 281548...; 11, 202268...),
(m; Δ; β; α) = (21; 7; 8; 273039...; 11, 467583...),
(m; Δ; β; α) = (22; 5; 13; 134623...; 15, 504916...),
(m; Δ; β; α) = (22; 7; 12; 815391...; 15, 331975...),
(m; Δ; β; α) = (23; 5; 8; 489281...; 11, 547024...),
(m; Δ; β; α) = (23; 7; 8; 475843...; 11, 820351...),
(m; Δ; β; α) = (24; 5; 7; 088338...; 9, 210037...),
(m; Δ; β; α) = (24; 7; 7; 116679...; 8, 439782...),
(m; Δ; β; α) = (25; 5; 8; 679328...; 11, 862643...),
(m; Δ; β; α) = (25; 7; 8; 661235...; 12, 143143...),
(m; Δ; β; α) = (26; 5; 12; 172520...; 14, 674949...),
(m; Δ; β; α) = (26; 7; 11; 944943...; 14, 618461...),

...(m; Δ; β; α) = (32; 5; 8; 654733...; 11, 466214...),
(m; Δ; β; α) = (32; 7; 8; 637697...; 11, 705492...),
(m; Δ; β; α) = (33; 5; 9; 310125...; 12, 911341...),
(m; Δ; β; α) = (33; 7; 9; 275806...; 13, 214792...),

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