Research Article

Pullback Attractors for Nonautonomous Degenerate Kirchhoff Equations with Strong Damping

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Omega$. We consider the following Kirchhoff wave model with strong damping:

\[
\begin{align*}
\rho \ddot{u} - \Delta u - \phi \left( \|\nabla u\| \right) \Delta u + f(u) &= h(x, t), \quad \text{in } \Omega \times (t, \infty), \\
\left. u \right|_{\partial \Omega} &= 0, \quad u(x, t) = u^0_t(x), \quad \left. u_t \right|_{\partial \Omega} = u^1_t(x), \quad x \in \Omega, \quad t \in \mathbb{R},
\end{align*}
\]

where $h(x, t)$ is a time-dependent external force term, $u^0_t$ and $u^1_t$ are initial data, and $\phi$ and $f$ are nonlinear functions specified later.

To describe small vibrations of an elastic stretched string, Kirchhoff [1] introduced the equation

\[
\rho \ddot{u} + \frac{E h}{2L} \int_0^L \left( \frac{\partial u}{\partial x} \right)^2 \, dx \, \partial^2 u + g,
\]

where $u = u(x, t)$ is the lateral deflection, $0 < x < L$ the space coordinate, $t \geq 0$ the time, $E$ the Young’s modulus, $\rho$ the mass density, $h$ the cross-section area, $L$ the length, $\rho_0$ the initial axial tension, and $g$ the external force. It has been called the Kirchhoff equation since then. In general, we call the Kirchhoff equation nondegenerate if the stiffness $\phi$ satisfies the strict hyperbolicity condition $\phi(s) \geq c > 0$ and degenerate if $\phi(s) \geq 0$ on $\mathbb{R}^+$. Obviously, the degenerate stiffness coefficient $\phi(s)$ in (1) corresponds to the case that the initial axial tension equals zero.

From the mathematical point of view, global existence of the model like (2) has been proven in a multitude of special situations in $\Omega \subset \mathbb{R}^n$. We refer to [2–5] for the analytic data, [6–9] for the dispersive equations and small data, and [10–15] for the weak damped equations.

Introducing the strong damping term $-\Delta u$, provides an additional a priori estimate. Certainly, from the physical point of view, the dissipative plays an important spreading role for the energy gathered arising from the nonlinearity in a real process. Concerning Kirchhoff equations with strong dissipation, the first result on the well-posedness we are aware of was obtained by Nishihara [16]. He proved the global existence of the solution for the model $\rho \ddot{u} - \Delta u - m(\|\nabla u\|) \Delta u = 0$. In recent years, many mathematicians and physicists paid their attentions to this type of problem and obtained the well-posedness under different types of hypotheses, such as the absent source term [17] and the subcritical source term [18–23]. In general, the exponent $p^* = n + 2/(n - 2)$ is called to be critical when someone studies the problem in $H_0^1(\Omega) \times L^2(\Omega)$. Assuming the stiffness factor is nondegenerate ($\phi(s) \geq \phi_0 > 0$), References [18–24] also proved the existence of the attractor. In the case of possible degeneration of the stiffness coefficient and the case of
supercritical source term \( p^* < p < (n + 4)/(n - 4) \), the first result on the well-posedness we are aware of is given by Chueshov [25]. However, when he proved the existence of a global attractor for problem (1) in the natural energy space \( L^2(\Omega) \times L^4(\Omega) \) endowed with a partially strong topology (in the sense, if \((u^n, u^n) \to (u_0, u_1)\) with a partially strong topology, then \((u^n, u^n) \to (u_0, u_1)\) strongly in \( H^1_0 \times L^2 \) and \( u^n \rightharpoonup u_0 \) weakly in \( L^{p^*-1} \)), he assumed that
\[
\phi(s) > 0, \quad \forall s \geq 0, \; \phi \in C^1(\mathbb{R}^+).
\] (3)

Under this condition, one can conclude that \( \phi \left( \|V(t)\|^2 \right) \geq c_0 > 0 \) if \( \|V(t)\| \) is bounded for \( t \in \mathbb{R}^+ \). Recently, Ma et al. [26] proved the existence of the global attractor in the case of degeneration for the autonomous Kirchhoff system.

The pullback attractor is a basic concept to study the longtime dynamics of nonautonomous evolution equations (see [27–32] and references therein). It is worth mentioning that there are only a few recent results devoted to the pullback attractor for nonautonomous systems like (1). In 2013, Wang and Zhong [33] investigated the upper semicontinuity of pullback attractors for problem (1) with \( \phi(s) = 1 + es \) (\( e > 0 \) and \( |f'(u)| \leq C(\|u\|^{2/(n-4)} + 1) \)). Recently, Li and Yang [34] studied the robustness of pullback attractors with \( \phi(s) \geq 0, \phi(0) = \phi_0 > 0 \). We notice that all these publications assume that the stiffness factor is nondegenerate, or more precisely, \( \phi(0) > 0 \) and \( \phi \) is nondecreasing.

In this paper, we consider the problem (1) under the degenerate hyperbolicity condition \( \phi(s) \geq 0 \). We do not assume that \( \phi \) is monotone and allow \( \phi(0) = 0 \), such as \( \phi(s) = b_0^s \) (degenerate and monotone) or \( \phi(s) = (1 + \sin^2 s)^{s^\gamma} \) (degenerate and nonmonotone) with \( \gamma \geq 1 \). Based on the result in [25, 26], we prove the existence of pullback attractors in \( H^1_0(\Omega) \times L^2(\Omega) \) if \( \phi \) is really degenerate. To overcome the difficulties caused by the degeneracy, we first established a method (condition \( D\text{-PC} \)) via “the measure of noncompactness” (some ideas come from [35, 36]) to prove that the process is pullback \( D\text{-asymptotically} \) compact.

The paper is organized as follows. In Section 2, we introduce some preliminaries and establish a necessary abstract result (see Theorem 5). In Section 3, we discuss the existence of pullback attractors for the equation (1) (see Theorem 12).

### 2. Preliminaries

In this section, we will give some notations and results. As usual, we denote by \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) the norm and the inner product in \( L^2(\Omega) \), respectively. Let \( \mathfrak{H} = H^1_0(\Omega) \times L^2(\Omega) \). We define the norms in \( \mathfrak{H} \) by
\[
\|u_0, u_1\|_{\mathfrak{H}}^2 = \|V_{u_0}\|^2 + \|u_1\|^2.
\]

Let \( X \) be a Banach space and \( U(t, \tau) \) be a process acting on \( X \). In the following, we recall some definitions and results related to the pullback attractors; more details can be found in [27, 29, 33].

**Definition 1.** A family of compact sets \( \mathcal{A} = \{A(t)\}_{t \in \mathbb{R}} \) is said to be a pullback attractor for process \( U(\cdot, \cdot) \) if
\begin{itemize}
  \item[(i)] \( \mathcal{A} \) is invariant, that is, \( U(t, \tau)A(\tau) = A(t) \), for all \( t \geq \tau \)
  \item[(ii)] \( \mathcal{A} \) is pullback attracting, i.e., \( d(U(t, \tau), \mathcal{A}) \to 0 \) as \( \tau \to +\infty \), for all bounded subset \( B \) of \( X \), where \( d(B, \mathcal{A}) \) is the Hausdorff semidistance
\end{itemize}

**Definition 2.** A family of sets \( \Phi = \{D(t)\}_{t \in \mathbb{R}} \) is said to be a pullback absorbing family for process \( U(\cdot, \cdot) \), if for all \( t \in \mathbb{R} \) and all bounded \( B \subset X \), there exists \( T = T(B) > 0 \), such that \( U(t, \tau)B \subset D(t) \), for all \( \tau \geq T \). In addition, the family \( \Phi \) is said to be pullback \( \Phi \text{-absorbing} \), if for any \( t \in \mathbb{R} \), there exists \( T > 0 \) such that \( U(t, \tau)B \subset D(t) \) for \( \tau \geq T \).

**Lemma 4** (see [29]). Let the family \( \Phi = \{D(t)\}_{t \in \mathbb{R}} \) be pullback absorbing and \( U(\cdot, \cdot) \) be continuous and pullback \( \Phi\text{-asymptotically} \) compact in \( X \). Then, the family \( \mathcal{A} = \{A(t)\}_{t \in \mathbb{R}} \) defined by
\[
A(t) = \bigcap_{s \leq 0} \bigcup_{t \in \mathbb{R}} U(t, s)D(t),
\]

is a pullback attractor for \( U(\cdot, \cdot) \).

To verify the pullback \( \Phi\text{-asymptotically} \) compact property in \( X \), it suffices to check the following condition.

**Theorem 5.** Let the family \( \Phi = \{D(t)\}_{t \in \mathbb{R}} \) be a pullback \( \Phi\text{-asymptotically} \) compact family of the process \( U(t, \tau) \). If the \( \Phi \text{-pullback condition} \ (\Phi\text{-PC}) \) holds, then \( U(\cdot, \cdot) \) is pullback \( \Phi\text{-asymptotically} \) compact in \( X \).

**Proof.** By Definition 3, the result will be proven if we can show that for any \( t \in \mathbb{R} \), any sequences \( \tau_n \to +\infty \) and \( x_n \in D(t, -\tau_n) \), \( \{U(t, \tau_n)x_n\}_{n \in \mathbb{N}} \) is relatively compact in \( X \).

For every \( \delta > 0 \), condition \( (\Phi\text{-PC}) \) implies that there exist \( \tau_0 \) and \( X_1 \) such that (5) holds. Then, we have
\[
\gamma(U(t, \tau_0)D(t, -\tau_0)) \leq \gamma([P(U(t, \tau_0)D(t, -\tau_0)]) + \gamma(((1 - P)(U(t, \tau_0)D(t, -\tau_0))) \leq \gamma(N(0, \delta)) \leq 2\delta,
\]

(6)
where \( \gamma \) is the measure of noncompactness defined as
\[
\gamma(B) = \inf \{ \delta > 0 | B \text{ admits a finite cover by sets whose diameter } \leq \delta \}.
\] (7)

On the other hand, the properties of \( \mathcal{D} \) give that there exists \( t_{1-\tau_0} > 0 \), such that for \( t \geq t_{1-\tau_0} \), \( U(t - \tau_0, t - \tau_0 - \tau) \subset D(t - \tau_0) \) and

\[
U(t, t - \tau_0 - \tau) D(t - \tau_0 - \tau) = U(t, t - \tau_0) U(t - \tau_0, t - \tau_0 - \tau) D(t - \tau_0 - \tau)
\] (8)
\[
\cdot \subset U(t, t - \tau_0) D(t - \tau_0).
\]

Then, we can find \( \mathcal{N}_0 \) such that \( \gamma(\bigcup_{n \in \mathbb{N}} U(t, t - \tau_n) x_n) \leq 2\delta_0 \), which means that \( \{ U(t, t - \tau_n) x_n \}_{n \in \mathbb{N}} \) has a finite \( 4\delta_0 \) -net for any \( \delta_0 > 0 \). The proof is complete. \( \square \)

### 3. Existence of Pullback Attractors

In this section, we will prove the existence of the pullback attractor when \( \phi(s) \) is really degenerate and \( f(u) \) is subcritical. We assume that \( f, \phi \), and \( h \) satisfy the following conditions.

**Assumption 6.** The function \( \phi \in C^1(\mathbb{R}^+) \), \( \phi(s) \geq \min \{ L_2 s^s, L_2 \} \) for \( s \in \mathbb{R}^+ \), and some constants \( a \geq 0 \), \( L_1, L_2 > 0 \). Moreover, there exists \( \delta_0 > 0 \) such that

\[
\liminf_{s \to +\infty} \frac{\phi(s) - \delta_0 \Phi(\phi(s))}{s} > -\infty,
\] (9)

where \( \Phi(s) = \int_0^s \phi(t) dt \).

**Assumption 7.** \( f(u) \) is a \( C^1 \) function, \( f(0) = 0 \), \( f'(s) \geq -c_1 \), and \( s \in \mathbb{R} \),

\[
\mu_j = \lim_{|s| \to +\infty} \frac{f(s)}{s} > -\lambda_1 \Phi(\phi) \quad \text{with} \quad \Phi(\phi) = \liminf_{s \to +\infty} \frac{\phi(s)}{s},
\] (10)

and the following properties hold:

(i) if \( n = 1 \), then \( f \) is arbitrary

(ii) if \( n = 2 \), then

\[
|f'(u)| \leq C(1 + |u|^{p-1}) \quad \text{with} \quad 1 \leq p < +\infty,
\] (11)

(iii) if \( n \geq 3 \), then

\[
|f'(u)| \leq C(1 + |u|^{p-1}) \quad \text{with} \quad 1 \leq p < p_* = \frac{n + 2}{n - 2},
\] (12)

where \( c_1 \) and \( C \) are positive constants and \( \lambda_1 \) is the first eigenvalue of \(-\Delta\).

**Assumption 8.** \( h, \partial_r h \in L^2_{\text{loc}}(\mathbb{R}, L^2(\Omega)) \) and

\[
\int_{-\infty}^t \|h(\cdot, s)\|^2 ds < +\infty, \quad \forall t \in \mathbb{R}.
\] (13)

**Remark 9.** (1) \( \phi(s) = L_1 s^s \) or \( \phi(s) = (1 + \sin^2 s)^{s} (\alpha \geq 1) \) satisfies Assumption 6. It indicates that we include into the consideration the case of possibly degenerate \( \phi \) since \( \phi(0) = 0 \). Moreover, because \( \phi_{\infty} = +\infty \) in this case, \( \mu_j > -\lambda_1 \Phi(\phi) \) becomes \( \mu_j > -\infty \). If \( \alpha = 0 \), then \( \phi(s) \) is a constant, and equation (1) is the nonlinear wave equation with strong damping.

(2) Assumptions 6 and 7 imply that there exist constants \( c_0 > 0 \), \( \theta_1 > 0 \) with \( 0 < \phi_1 < \phi_{\infty}, 0 < \phi_1, \lambda_1 - \theta_1 < 1 \) such that

\[
\Phi(s) \geq \phi_1 s - c_0 \phi_1 \quad \forall s \in \mathbb{R}^+,
\] (14)

\[
F(s) \geq -\frac{\theta_1}{2} s^2 - c_2, f(s) s \geq -\theta_1 s^2 - c_2, f(s) s - F(s)
\] (15)

\[
\geq -c_1 s^2, \quad \forall s \in \mathbb{R},
\]

where \( F(s) = \int_0^s f(t) dt \).

The well-posedness of the problem

\[
\partial_t u - \sigma (\|u(t)\|^2) \Delta u - \phi(\|u(t)\|^2) \Delta u + f(u) = h(x), \quad \text{in} \; \Omega \times (0, \infty),
\]

\[
u|_{\partial \Omega} = 0, u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \quad x \in \Omega,
\] (16)

has been established by Chueshov [25] in the autonomous case. Noticing that the conditions of \( \phi, f \) are more general than the above Assumptions 6–8, we can obtain the following Proposition 10 by a similar argument as in [25], except for the treatment of \( h(x, t) \). The reader is referred to the Appendix for a detailed proof of these facts.

**Proposition 10.** Let Assumptions 6–8 be in force. Then, for \( \tau, T \in \mathbb{R}(\tau < T) \) and \( (u_0, u_1) \in \mathcal{H} \), problem (1) has a unique weak solution \( u \) with \( u, u_t \in C([\tau, T]; \mathcal{H}) \) and

(1) for every \( t \in [\tau, T] \), there exists \( C = C_{R, \tau, T} > 0 \) such that

\[
\|u_t(t)\|^2 + \|\nabla u(t)\|^2 + \int_\tau^t \|\nabla u_t(s)\|^2 ds \leq C,
\] (17)

\[
E(u(t), u_t(t)) + 2 \int_{\Omega} (\|\nabla u_t\|^2 - (h, u_t)) dt
\]

\[
= E(u(s), u_t(s)), \quad t > \tau,
\] (18)

where \( E(u_0, u_1) = \|u_0\|^2 + \Phi(\|u_0\|^2) + 2 \int_\Omega F(u_0) dx, \)

\[
\|u_0, u_1\|_{\mathcal{H}} \leq R
\]

(2) for every \( t \in [\tau, T] \), there exists \( K = K_{R, \tau, T} > 0 \) such that

\[
\|u_t(t)\|^2 + \|\nabla u_t(t)\|^2 \leq K \left( 1 + \frac{1}{(t - \tau)^2} \right) \left( 1 + \int_\tau^t \|h(\cdot, s)\|^2 + \|h_t(\cdot, s)\|^2 ds \right).
\] (19)
(3) the Lipschitz stability
\[ \|(\zeta(t), z_1(t))\|_\mathcal{X}^2 \leq K\|(\zeta(t), z_1(t))\|_\mathcal{X}^2, \]
(20)
holds for \( z(t) = u^i(t) - u^j(t) \), where \( u^i, u^j \) are two weak solutions of problem (1) with initial data \( (u^i_0, u^j_1) \), \( \|(u^i_0, u^j_1)\|_\mathcal{X} \leq R, i = 1, 2. \)

We define the solution operator \( U(t, r) : \mathcal{X} \rightarrow \mathcal{X} \) associated to problem (1) as
\[ U(t, r)(u^0_i, u^1_i) = (u(t), u_r(t)), \quad \forall t \geq r, r \in \mathbb{R}, \]
(21)
where \( u \) is the weak solution of problem (1) corresponding to initial data \( (u^0_i, u^1_i) \in \mathcal{X} \). Then, we know from Proposition 10 that \( U(t, r) : \mathcal{X} \rightarrow \mathcal{X} \) is a continuous evolution process. For convenience, we denote by \( \xi_n(t) = (u(t), u_r(t)) \) for any function \( u \). As \( (u(t), u_r(t)) = (u^0_i, u^1_i) \), we also denote \( (u^0_i, u^1_i) \) by \( \xi_n(t) \).

**Lemma 11.** Let Assumptions 6–8 be valid. Then, the process \( U(\cdot, \cdot) \) defined in (21) has a pullback \( \mathcal{D} \)-absorbing family \( \mathcal{D} = \{ D(t) \}_{t \in \mathbb{R}} \). Moreover, \( D(t) \) is bounded in \( \mathcal{X} = H_0^1(\Omega) \times H_0^1(\Omega) \) for every \( t \in \mathbb{R} \).

**Proof.** As usual, the argument below can be justified by considering Galerkin approximations. Using the multiplier \( u_i + \eta u \) in Equation (1), we have that
\[ \frac{d}{dt} W^n(\xi_n(t)) + K(\xi_n(t)) = 0, \quad t \geq r, \]
(22)
where
\[ W^n(\xi_n(t)) = ||u||^2 + \Phi(||\nabla u||^2) + 2(F(u), 1) + \eta ||||\nabla u||^2 + 2||u||^2 \geq (1 - \eta)||u||^2 + \Phi(||\nabla u||^2) + 2(F(u), 1), \]
\[ + \phi_1 \cdot ||\nabla u||^2 - c_0 \phi_1 \Phi(||\nabla u||^2) - 2c_2 \cdot \text{mes} \Omega + \eta ||\nabla u||^2 - \eta ||u||^2 \geq \kappa ||\xi_n(t)||_\mathcal{X}^2 - C_3, \]
(23)
and
\[ K(\xi_n(t)) = 2||\nabla u||^2 - 2\eta ||u||^2 \geq 2\eta \big[ \Phi(||\nabla u||^2) ||\nabla u||^2 + (F(u), u) \big] - 2(h, u + \eta u), \]
(24)
for \( \eta > 0 \) which is small enough, \( \kappa > 0 \) is a positive constant, and \( \kappa, C_3 \) are independent of \( \xi_n(t) \).

Since Assumption 6 implies that there exists \( L_3 > 0 \) such that
\[ \delta_0 \Phi(||\nabla u||^2) \leq \Phi(||\nabla u||^2) ||\nabla u||^2 + L_3, \]
(25)
combining with (15), we have that
\[ W^n(\xi_n(t)) \leq (1 + \eta) ||u||^2 + \Phi(||\nabla u||^2) + 2(F(u), u) + c_1 ||\nabla u||^2 + \eta ||\nabla u||^2 \leq \kappa ||\xi_n(t)||_\mathcal{X}^2 + \Phi(||\nabla u||^2) + 2(F(u), u) \leq \kappa ||\xi_n(t)||_\mathcal{X}^2 + \Phi(||\nabla u||^2) + c_3 ||u||^{p+1} + C_4, \]
(26)
\[ K(\xi_n(t)) \geq ||\nabla u||^2 + (\lambda_1 - 2\eta) ||u||^2 + (2\eta - \epsilon) \Phi(||\nabla u||^2) ||\nabla u||^2 + \epsilon(\delta_0 \Phi(||\nabla u||^2) - L_3) + 2(\eta u, u) \geq \delta ||u||^2 - \delta_0 ||u||^2 - \delta_0 \geq \frac{2}{\delta} ||h(\cdot, t)||^2. \]
(27)
Then, we can find \( \eta > 0, \epsilon > 0, \delta > 0 \) small enough such that
\[ W^n(\xi_n(t)) - \delta W^n(\xi_n(t)) \geq ||\nabla u||^2 + (\lambda_1 - 2\eta) ||u||^2 + (2\eta - \epsilon) \Phi(||\nabla u||^2) ||\nabla u||^2 + \epsilon(\delta_0 \Phi(||\nabla u||^2) - L_3) + 2(\eta u, u) \geq \delta ||u||^2 - \delta_0 ||u||^2 - \delta_0 \geq \frac{2}{\delta} ||h(\cdot, t)||^2. \]
(28)
By (22) and (28), we get that
\[ \frac{d}{dt} W^n(\xi_n(t)) + \delta W^n(\xi_n(t)) + ||\nabla u||^2 \leq C(1 + ||h(\cdot, t)||^2). \]
(29)
According to the Gronwall inequality, we have
\[ W^n(\xi_n(t)) \leq W^n(\xi_n(r)) e^{-\delta(t-r)} + C \left( 1 + \int_r^t ||h(\cdot, s)||^2 \, ds \right). \]
(30)
Then, (23), (26), and \( H_0^1(\Omega)^{L^{p+1}(\Omega)} \) yield that
\[ ||\xi_n(t)||_\mathcal{X}^2 \leq \frac{1}{K} \left( W^n(\xi_n(r)) e^{-\delta(t-r)} + C \left( 1 + \int_r^t ||h(\cdot, s)||^2 \, ds \right) \right) \]
\[ \leq \frac{\kappa}{K} ||\xi_n(r)||_\mathcal{X}^2 + \frac{1}{K} \Phi(||\nabla u||^2) + \frac{C_3}{K} ||u||^{p+1} + \frac{C_4}{K} e^{-\delta(t-r)} \]
\[ + C \left( 1 + \int_r^t ||h(\cdot, s)||^2 \, ds \right) \]
\[ \leq C \left( \xi_n(r) \right) + \Phi \left( ||\xi_n(r)||_\mathcal{X}^2 \right) + ||\xi_n(\xi_n(t))||_\mathcal{X}^{(p+1)/2} e^{-\delta(t-r)} \]
\[ + C \left( 1 + \int_r^t ||h(\cdot, s)||^2 \, ds \right) \]
\[ + C \left( 1 + \int_r^t ||h(\cdot, s)||^2 \, ds \right). \]
(31)
where $Q(x) = C(x + \Phi(x) + x^{\beta+1}/2) > 0$ is a monotone positive function on $\mathbb{R}^+$. Let

$$D_0(t) = \{ \xi \in \mathcal{H} \mid \|\xi\|_{\mathcal{H}} \leq R(t) \}, \text{ with } R^2(t) = 2C\left(1 + \|h\|_{L_2(-\infty,L^2)}^2 \right), \quad t \in \mathbb{R}. \quad (32)$$

Obviously, $\mathcal{D}_0 = \{ D_0(t) \}_{t \in \mathbb{R}}$ is a pullback absorbing family of the process $U(t, \tau)$ in $\mathcal{H}$. Moreover, for every $t \in \mathbb{R}$, there exists a $T_t > 0$ such that

$$U(t, t - \tau)D_0(t - \tau) \subset D_0(t),$$

$$U(t - 1, t - \tau)D_0(t - (t - 1)) \subset D_0(t - 1), \quad \text{for } \tau \geq T_t. \quad (33)$$

Let $D(t) = \bigcup_{t \in \mathbb{R}} U(t, t \tau)D_0(t \tau)$. By a standard procedure (see, e.g., Theorem 3.1 of [34]), we know that $\mathcal{D} = \{ D(t) \}_{t \in \mathbb{R}}$ is a pullback absorbing family. Moreover, $D(t)$ is bounded in $\mathcal{H}$, for every $t \in \mathbb{R}$, and there exists a $T_t > 0$ such that

$$U(t, t - \tau)D(t - \tau) \subset D(t) \text{ for } \tau \geq T_t. \quad \square$$

For simplicity, we assume that $\alpha > 0$ and $L_1 = L_2 = 1$ in the following.

**Theorem 12.** Let Assumptions 6–8 be in force. Then, the process $U(t, \cdot)$ possesses a pullback attractor $\mathcal{A} = \{ A(t) \}_{t \in \mathbb{R}}$ as shown in (4). Moreover, $A(t)$ is bounded in $\mathcal{H}$ for every $t \in \mathbb{R}$.

**Proof.** According to Lemma 4, Theorem 5, Lemma 11, and the continuity of $U(t, \tau) : \mathcal{H} \rightarrow \mathcal{H}$, it suffices to show that $U(t, \tau)$ satisfies the condition $(\mathcal{D}$-$\mathcal{P}$-$\mathcal{C})$. Let $\{ e_j \}_{j \in \mathbb{N}}$ be an orthonormal basis and $\{ \lambda_j \}_{j \in \mathbb{N}}$ be the corresponding eigenvalues of $L^2(\Omega)$ which consists of eigenvectors of $-\Delta_j$, i.e., $-\Delta_j e_j = \lambda_j e_j$, $j \in \mathbb{N}$. Let $V_m \times W_m = \text{span} \{ e_1, \ldots, e_m \} \times \text{span} \{ e_1, \ldots, e_m \}$ in $\mathcal{H}$ and $P_m = (P_m^1, P_m^2) : \mathcal{H} \rightarrow V_m \times W_m$ be an orthogonal projector. Denote $Q_m = I - P_m$, $u = P_m u + Q_m u = U + U^2$, and $\xi_u^\tau(t) = (u(t), u(t)) = U(t, \tau)/(u_0^\tau, u_1^\tau)$ with $u_0^\tau, u_1^\tau \in D(t \tau), \tau \geq t$.

Let $\varepsilon > 0$ and $t_0 \in \mathbb{R}$ be given. Without loss of generality, we assume $\varepsilon < 1/4$.

For every $\tau \geq 1$ and every $(u_0^{t - t_0}, u_1^{t - t_0}) \in D(t_0 - \tau), t \lesssim \rho^2$,

$$(u, u_1)(t) = \xi_u^\tau(t) = U(t, t_0 - \tau)\left( u_0^{t_0 - t}, u_1^{t_0 - t}\right) \in U(t, t_0 - \tau) \cdot D(t_0 - \tau) \subset U(t, t_0 - \tau)D_0(t_0 - \tau). \quad (34)$$

Denote $Z(t) = (1/2)(\|u_t\|^2 + \|u\|^2 + \|\nabla u\|^2)$. It is easy to see that

$$Z(t_0 - \tau + 1) \leq \frac{1}{2} \left(1 + \frac{1}{R(t)}\right) \|\xi_u^\tau(t_0 - \tau + 1)\|^2_{\mathcal{H}}$$

$$\leq \frac{1}{2} \left(1 + \frac{1}{R(t)}\right) Q\left(\|\xi_u^\tau(t_0 - \tau)\|^2_{\mathcal{H}} e^{-\varepsilon}\right) + C \left(1 + \int_{-\infty}^{t_0} \|h(s, s^2)\|^2_{\mathcal{H}} ds\right). \quad (35)$$

Since $\xi_u^\tau(t_0 - \tau) = (u_0^{t_0 - t}, u_1^{t_0 - t}) \in D(t_0 - \tau) \subset D_0(t_0 - \tau)$, we find

$$\|\xi_u^\tau(t_0 - \tau)\|^2_{\mathcal{H}} \leq R(t_0 - \tau) \leq R(t_0), \quad \forall \tau \geq 0. \quad (36)$$

Thus,

$$Z(t_0 - \tau + 1) \leq \frac{1}{2} \left(1 + \frac{1}{R(t_0)}\right) \left(\|\xi_u^\tau(t_0 - \tau)\|^2_{\mathcal{H}} e^{-\varepsilon}\right) + C \left(1 + \int_{-\infty}^{t_0} \|h(s, s^2)\|^2_{\mathcal{H}} ds\right)$$

$$\leq C(t_0), \quad \forall \tau \geq 0. \quad (37)$$

where $C_{1}(t_0)$ is independent of $\tau$. Then, there exists $t_0 > 1$ such that

$$Z(t_0 - (t_0 - 1)) e^{-2\varepsilon(t_0 - 1)} < \frac{\varepsilon^2}{2}. \quad (38)$$

On the other hand, for every $(u_0^{t_0 - t_0}, u_1^{t_0 - t_0}) \in D(t_0 - \tau_0)$, using (16) and (18), we get that

$$\|\xi_u^\tau(t_0 - \tau_0)\|^2_{\mathcal{H}} \leq K_0, \quad \text{for } t \in [t_0 - \tau_0 + 1, t_0]. \quad (39)$$

where $(u(t), u_1(t)) = U(t, t_0 - \tau_0)(u_0^{t_0 - t_0}, u_1^{t_0 - t_0})$. Using $H^1_0(\Omega)^* L^q(\Omega) (2 \leq q \leq p^* = 2n/(n - 2))$, one can find $M \geq 1$, $L_0 < t_0$ (without loss of generality, we assume $L_0 < 0$), such that for every $t \in [t_0 - \tau_0 + 1, l_0]$,

$$\|f(u(t))\|_{L^{q}(\Omega)} + \left(\int_{t_0}^{t_0} \|u_t(s)\|^2_{\mathcal{H}} ds\right)^{1/2} + \left(\int_{-\infty}^{t_0} \|h(s, s)\|^2_{\mathcal{H}} ds\right)^{1/2}$$

$$< M \left(\int_{-\infty}^{t_0} \|h(s, s)\|^2_{\mathcal{H}} ds\right)^{1/2}.$$ \quad (40)

By the Sobolev embedding theorem, we know that the embedding $H^1_0(\Omega)^* L^q(\Omega) \subset L^2(\Omega)$ is compact. Then, the boundedness of $\{u(t), u_1(t)\}_{t \in [t_0 - \tau_0, t_0]}$ in $H^1_0(\Omega)^* L^q(\Omega)$ implies that $\{u(t), u_1(t)\}_{t \in [t_0 - \tau_0 + 1, t_0]}$ is compact in $L^2(\Omega) \times L^2(\Omega)$. Therefore, for $\epsilon_1 = \varepsilon^2/2M(1 + \sqrt{t_0 - L_0})$, there exists $m_0 \in \mathbb{Z}_+$, such that for every $t \in [t_0 - \tau_0 + 1, t_0]$,

$$\|u(t, u_1(t))\|_{L_t^2}^2 \leq \epsilon_1, \quad (41)$$

$$\|u^2\|_{L_t^\infty} \leq \|u_t^2\|_{L_t^\infty} \leq C \|u^2\|_{L_t^\infty} \leq C_0 \|u^2\|_{L_t^\infty} < \frac{\varepsilon^2}{M}. \quad (42)$$
where \( u = P_{m_0} u + (I - P_{m_0}) u \equiv u^1 + u^2, \ u_t = P_{m_0} u_t + (I - P_{m_0}) u_t \equiv u^1_t + u^2_t, \) and \( 1/(p + 1) = \theta/2 + (1 - \theta)/p^*. \)

Now, we will consider two situations. Without loss of generality, we assume \( 0 < \epsilon < 1/3. \)

Case 1. For every \((u^0_{t_0 - \tau_0}, u^1_{t_0 - \tau_0}) \in D(t_0 - \tau_0),\) the inequality
\[
\|\nabla u(t)\| > \epsilon,
\tag{43}
\]
holds for any \( t \in [t_0 - \tau_0 + 1, t_0], \) where \((u, u_t)(t) = \xi^u_{t_0 - \tau_0}(t).\)

Multiplying (1) by \( u^2, \) we have that
\[
\frac{d}{dt} \left( u^2_t + \frac{1}{2} \| \nabla u^2 \|^2 \right) + \phi \left( \| \nabla u(t) \|^2 \right) \| \nabla u^2 \|^2 \\
\leq \| u^2_t \|^2 + \left( f(u), u^2 \right) + \left( h(t, u), u^2 \right) \\
+ \| f(u) \|_{L^p(V_{p+1})} \| \nabla u \|_{L^2} \cdot \| h(t, u) \|_{L^1} \cdot \| u^2 \| = \mathcal{W}^2(t). \tag{44}
\]

By Gronwall’s inequality, we obtain that
\[
Y(t) \leq Y(t - (\tau - 1)) e^{-2\epsilon^2(t-1)} + e^{-2\epsilon^2} \int_{t-(\tau-1)}^t e^{2\epsilon^2 \mathcal{W}^2(s)} ds. \tag{46}
\]

Since
\[
Y(t_0 - \tau + 1) \leq Z(t_0 - \tau + 1) \leq C_3(t_0), \tag{47}
\]
(37) yields that
\[
Y(t_0 - (\tau_0 - 1)) e^{-2\epsilon^2(\tau_0 - 1)} < \frac{\epsilon^2}{2}. \tag{48}
\]
Combining (45), we have
\[
Y(t_0) \leq Y(t_0 - \tau_0 + 1) e^{-2\epsilon^2(\tau_0 - 1)} + e^{-2\epsilon^2} \int_{t_0 - \tau_0 + 1}^{t_0} e^{2\epsilon^2 \mathcal{W}^2(s)} ds \\
\leq \frac{\epsilon^2}{2} + e^{-2\epsilon^2} \int_{t_0 - \tau_0 + 1}^{t_0} e^{2\epsilon^2 \mathcal{W}^2(s)} ds \\
+ e^{-2\epsilon^2} \int_{t_0 - \tau_0 + 1}^{t_0} e^{2\epsilon^2 \mathcal{W}^2(s)} \| h(s) \| \| u^2(s) \| ds \\
\leq \frac{\epsilon^2}{2} + \left( 1 + \frac{1}{\epsilon^2} \right) \epsilon^2 e^{-2\epsilon^2} \int_{t_0 - \tau_0 + 1}^{t_0} e^{2\epsilon^2 \mathcal{W}^2(s)} \| h(s) \| \| u^2(s) \| ds. \tag{49}
\]

If \( L_0 \leq t_0 - \tau_0 + 1, \) by the Hölder inequality, we have that
\[
I_1 = e^{-2\epsilon^2 t_0} \int_{t_0 - \tau_0 + 1}^{t_0} e^{2\epsilon^2 \mathcal{W}^2(s)} \| h(s) \| \| u^2(s) \| ds \\
\leq e^{-2\epsilon^2 t_0} \epsilon \epsilon^2 \epsilon \cdot \sqrt{t_0 - L_0} M < \frac{\epsilon^2}{4}. \tag{50}
\]

On the other hand, if \( L_0 > t_0 - \tau_0 + 1, \) we get that
\[
I_1 \leq e^{-2\epsilon^2 t_0} \int_{t_0}^{t_0} e^{2\epsilon^2 \mathcal{W}^2} \| h(s) \| \| u^2(s) \| ds + \frac{1}{2} \int_{t_0}^{t_0} e^{2\epsilon^2 \mathcal{W}^2} \| u^2(s) \|^2 + \| h(s) \|^2 ds \\
\leq \epsilon^2 \cdot \sqrt{t_0 - L_0} M + \epsilon^2 \frac{e^2}{4} + \frac{\epsilon^2}{2} < \frac{\epsilon^2}{2}. \tag{51}
\]

The above inequalities guarantee that \( Y(t_0) < 9\epsilon^2/8. \) And because
\[
Y(t_0) = (u^0(t_0), u^2(t_0)) + \frac{1}{2} \| u^2(t_0) \|^2 \geq \frac{1}{2} \| u^2(t_0) \|^2 \\
- \| u^2(t_0) \| \| u^2(t_0) \| \geq \frac{1}{2} \| u^2(t_0) \|^2 - \epsilon^2, \tag{52}
\]
we get that
\[
\| u^2(t_0) \|^2 \leq 2 \left( Y(t_0) + \epsilon^2 \right) < 2 \left( \frac{9\epsilon^2}{8} + \frac{\epsilon^2}{8} \right) < 4\epsilon^2, \tag{53}
\]
i.e., \( \| u^2(t_0) \| < 2\epsilon. \)

Case 2. There exist \((u^0_{t_0 - \tau_0}, u^1_{t_0 - \tau_0}) \in D(t_0 - \tau_0) \) and \( t_1 \in [t_0 - \tau_0 + 1, t_0] \) such that
\[
\| \nabla u(t_1) \| \leq \epsilon \text{ with } (u, u_t)(t) = \xi^u_{t_0 - \tau_0}(t). \tag{54}
\]

In this case, we claim that the following inequality is true, i.e., for every \( t_1 \leq t \leq t_0, \)
\[
\| \nabla u^2(t) \| < 2\epsilon, \quad \text{for } u^2 = Q^1_{u_0} u. \tag{55}
\]

In fact, if this claim is not true, the continuity of \( \| \nabla u^2(t) \| \) gives that
\[
E = \{ t \mid t \in [t_1, t_0], \| \nabla u^2(t) \| = 2\epsilon \}, \tag{56}
\]
is not an empty set. Let \( t_3 = \inf E. \) It is easy to prove that \( \| \nabla u^2(t_3) \| = 2\epsilon. \) Moreover, by the definition of \( t_3, \) we have that
\[
\| \nabla u^2(t) \| < 2\epsilon, \quad \forall t \in [t_1, t_3]. \tag{57}
\]

According to the intermediate value theorem, we know that the set
\[ E_1 = \left\{ t \mid t \in (t_1, t_2), \| \nabla u^2(t) \| = \frac{3}{2} \varepsilon \right\}. \]  

(58)

is not empty. Denoting \( t_2 = \sup E_1 \), we can conclude from the definition of supremum that

\[ \| \nabla u^2(t_2) \| = \frac{3}{2} \varepsilon, \]  

(59)

Thus,

\[ \frac{3}{2} \varepsilon < \| \nabla u^2(t) \| < 3 \varepsilon, \forall t \in (t_2, t_3), \| \nabla u^2(t_3) \| = \frac{3}{2} \varepsilon. \]  

(60)

Notice that \( \| \nabla u \| \geq \| \nabla u^2 \| \) and \( \| \nabla u^2 \| \leq 1 \) for \( t \in [t_2, t_3] \); we have that \( \phi(\| \nabla u^2 \|) \geq \| \nabla u^2 \|^{2\alpha} \). Then, integrating (43) on \( (t_2, t_3) \), we have that

\[ \left( (u^2(t_3), u^2(t_3) + \frac{1}{2} \| \nabla u^2(t_3) \| + \frac{1}{2} \| \nabla u^2(t_3) \| ) \right) + \int_{t_2}^{t_3} \phi(\| \nabla u^2(s) \|)\| \nabla u^2(s) \|^2 ds \leq \int_{t_2}^{t_3} \left( (u^2(s))^2 + (f(u(s)), u^2(s)) + (h(s), u^2(s)) \right) ds. \]  

(61)

It implies that

\[ \| \nabla u^2(t_3) \| + 2 \int_{t_2}^{t_3} \left| (u^2(t_3))^2 + 2 (u^2(t_3), u^2(t_3)) + \frac{1}{2} \| \nabla u^2(t_3) \| \right| \| \nabla u^2(s) \|^2 ds \leq (\| \nabla u^2(t_2) \|)^2 \]  

(62)

Combing (40), (41), and (59), we get

\[ \| \nabla u^2(t_2) \| + 2 \int_{t_2}^{t_3} \left| (u^2(t_2))^2 + 2 (u^2(t_2), u^2(t_2)) + \frac{1}{2} \| \nabla u^2(t_2) \| \right| \| \nabla u^2(s) \|^2 ds \leq \sqrt{\| \nabla u^2(t_2) \|} \]  

(63)

Thus, \( \| \nabla u^2(t_2) \|^2 < (11/4)\varepsilon^2 \), which is in contradiction with (59), and condition (\( \mathcal{G} \)-PC) holds. This completes the proof.

**Appendix**

**A. Proof of Proposition 10**

We prove the well-posedness of Problem (1) using the same method as in [25].

**Step 1.** We start with the case when \( u^0 \in H^2(\Omega) \cap H_0^1(\Omega) \) and assume that \( \| (u^0, u_0) \|_{\mathcal{F}} \leq R \) with some \( R > 0 \). We seek for the approximate solutions of the form

\[ u^N(t) = \sum_{k=1}^{N} \Phi_k(t) \phi_k, \quad N = 1, 2, \ldots, \]  

(A1)

satisfying the finite-dimensional projections of (1). Moreover, we have that

\[ \| (u^N(t), u_0^N(t)) \|_{\mathcal{F}} \leq C_R, \]  

(\( A2 \)) \[ \| (u^N(t) - u^0, u_0^N(t) - u_0) \|_{\mathcal{F}} \rightarrow 0, \text{ as } N \rightarrow \infty. \]  

We omit the superscript \( N \) in the sequel. Now, we use the multiplier \( u^N(t) \) and get that

\[ \frac{d}{dt} \left[ \frac{1}{2} \| u^2(t) \|^2 + \int_0^t \int_{\Omega} \left( \frac{1}{2} \| \nabla u^2(t) \|^2 + \Phi(t) \| \nabla u^2(t) \|^2 - f(u(t)) \| u^2(t) \|^2 + \| \nabla u^2(t) \|^2 - (h, u) \right) dt \right] = 0. \]  

(A3)

Similarly, multiplying (1) by \( u \), we have that

\[ \frac{d}{dt} \left[ \frac{1}{2} \| u^2(t) \|^2 + \int_0^t \int_{\Omega} \left( \frac{1}{2} \| \nabla u^2(t) \|^2 + \Phi(t) \| \nabla u^2(t) \|^2 - f(u(t)) \| u^2(t) \|^2 + \| \nabla u^2(t) \|^2 - (h, u) \right) dt \right] = 0. \]  

(A4)

Let

\[ E(u_0, u_1) = \frac{1}{2} \| u_1 \|^2 + \Phi(\| \nabla u_0 \|^2) + F(u_0), \]  

(\( A5 \)) \[ W^N(u_0, u_1) = E(u_0, u_1) + \eta \left[ (u_0, u_1) + \frac{1}{2} \| \nabla u_0 \|^2 \right]. \]  

From (A3) and (A4),

\[ \frac{d}{dt} W^N(u(t), u(t)) + \| \nabla u^2(t) \|^2 - (h, u(t)) \]  

(\( A6 \)) \[ = \eta \left[ \| u_1 \|^2 + \Phi(\| \nabla u_1 \|^2) \| u^2 \|^2 - (f(u), u) + (h, u) \right]. \]  

Using (14), (24), and \( |(h, u(t))| \leq \lambda_1/2 \| u_1 \|^2 + 1/2 \lambda_1 \| h(\cdot, t) \|^2 \), we find that
\[
\frac{d}{dt} W^n(u(t), u_1(t)) + \frac{1}{2} \|\nabla u_1\|^2 \leq \eta \|u_1\|^2 - \eta \delta_0 \Phi(\|\nabla u\|^2) + \eta \phi_1 \|u_1\|^2 + c_1 \|u(\cdot, t)\|^2 + c_2. \tag{A7}
\]

where \(c_1, c_2\) is independent of \(t\). Obviously,
\[
W^n(u_0, u_1) \leq \|u_1\|^2 + \frac{1}{2} \Phi(\|\nabla u_0\|^2) + \tilde{\varepsilon}_0 \|\nabla u_0\|^2. \tag{A8}
\]

By (13) and (14), there exists \(\eta_0 > \delta_1 > 0\), for any \(\eta \in (0, \eta_0)\),
\[
W^n(u_0, u_1) \geq \left( \frac{1}{2} - \eta \right) \|u_1\|^2 + \frac{1}{2} \phi_1 \cdot \|\nabla u_0\|^2 - \frac{\theta_1}{2} \|u_0\|^2 - \frac{\eta}{2} \|\nabla u_0\|^2 - \tilde{\varepsilon}_1 \geq \frac{1}{4} \|u_1\|^2 + \delta_1 \|\nabla u_0\|^2 - \tilde{\varepsilon}_2. \tag{A9}
\]

Combining (A8) and the above inequalities, we have that
\[
\frac{d}{dt} W^n(u(t), u_1(t)) + \frac{1}{2} \|\nabla u_1\|^2 \leq C_1 W^n(u(t), u_1(t)) + \frac{1}{2} \|u\|^2 + C_3. \tag{A10}
\]

Therefore, using Gronwall’s inequality, we obtain
\[
W^n(u(t), u_1(t)) \leq \tilde{C}_{R,T} + C_2 e^{\tilde{C}_{R,T} T} \int_t^{\infty} \|h(s, \cdot)\|^2 \|u_0\|^2 \leq \tilde{C}_{R,T}, \quad \forall t \in [\tau, T], \tag{A11}
\]

which means that
\[
\|u(t), u_1(t)\|_{\mathcal{X}} \leq C_{R,T}, \quad \forall t \in [\tau, T],
\]
\[
\int_\tau^T \|\nabla u(t)\|^2 dt \leq C_{R,T}, \tag{A12}
\]

Now, multiplying (1) by \(-\Delta u\), we have
\[
\frac{d}{dt} \left[ -(u, \Delta u) + \frac{1}{2} \|\Delta u\|^2 \right] + \phi(\|\nabla u(t)\|^2) \|\Delta u\|^2 + \left( f'(u), |\nabla u|^2 \right) \leq \|\nabla u(t)\|^2 + \frac{1}{2} \|h(\cdot, t)\|^2 + \frac{1}{2} \|\Delta u\|^2. \tag{A13}
\]

Since \(H_0^1(\Omega)^* L^{p+1}(\Omega)\) when \(n \geq 3\) and \(H_0^1(\Omega)^* L^q(\Omega)\) for any \(q \geq 1\) when \(n = 2, H_0^1(\Omega)^* L^{\infty}(\Omega)\) when \(n = 1\), we easily obtain that
\[
\left( f'(u), |\nabla u|^2 \right) \leq C \int_\Omega (1 + |u|^{p-1}) |\nabla u|^2 dx \leq C |\nabla u|^2 + C \|u\|_{L^{p+1}}^p \|\nabla u\|^2 + C \|u\|_{L^{p+1}}^{p-1} \|\Delta u\|^2 \leq C(1 + \|\Delta u\|^2). \tag{A14}
\]

It follows that
\[
\frac{d}{dt} \left[ -(u, \Delta u) + \frac{1}{2} \|\Delta u\|^2 \right] \leq \|\nabla u(t)\|^2 + \frac{1}{2} \|h(\cdot, t)\|^2 + C_1 \|\Delta u\|^2 + C_2, \tag{A15}
\]

for every \(t \in [\tau, T]\). Let
\[
\Psi(t) = E(u(t), u_1(t)) + \varepsilon \left[ -(u_1, \Delta u) + \frac{1}{2} \|\Delta u\|^2 \right], \quad \varepsilon > 0. \tag{A16}
\]

We can choose \(\varepsilon_0 > 0\), such that
\[
\Psi(t) \geq C_{R,T} \left( \|u_1\|^2 + \|\Delta u\|^2 \right) - C, \quad \forall 0 < \varepsilon < \varepsilon_0. \tag{A17}
\]

Thus, combing (A3), (A12), and (A15), we have that
\[
\frac{d}{dt} \Psi(t) \leq C_1 \|\Delta u\|^2 + C_2 \|h(\cdot, t)\|^2 + C_3 \leq C_1 \Psi(t) + C_2 \|h(\cdot, t)\|^2 + C_3. \tag{A18}
\]

This implies that
\[
\|\Delta u(t)\|^2 \leq C_{R,T} \left( 1 + \|\Delta u(t)\|^2 \right), \quad t \in [\tau, T]. \tag{A19}
\]

The above a priori estimates show that \((u^N, u_1^N)\) is bounded in
\[
L^{\infty}(\tau, T; H^2(\Omega) \cap H_0^1(\Omega)) \times L^{\infty}(0, T; L^2(\Omega)) \cap L^2([\tau, T]; H_0^1(\Omega)), \tag{A20}
\]

for every \(T > \tau\). Moreover, using the equation for \(u^N(t)\), we can show \(\int_\tau^T \|u_1^N\|^2 m dt \leq C_{R,T}\) for some \(m \geq \max \{1, n/2\}\). Thus, there exists a subsequence, still denoted \(u^N\), and \(u\), such that
\[
\begin{align*}
&u^N \rightharpoonup u, \quad \text{in } C([\tau, T]; H_0^1(\Omega)), \\
&u^N \rightharpoonup u, \quad \text{in } L^{\infty}(\tau, T; H^2(\Omega)) \text{ weak-star}, \\
&u_1^N \rightharpoonup u_1, \quad \text{in } L^2(\tau, T; H^1(\Omega)) \cap C(\tau, T; H^{-1}(\Omega)), \\
&u_1^N \rightharpoonup u_1, \quad \text{in } L^2(\tau, T; H_0^1(\Omega)) \text{ weak}, \tag{A21}
\end{align*}
\]

as \(N \to \infty\). Moreover, by the Lions lemma (see Lemma 1.3 in [37]) we have that
\[ f(u^N(x, t)) \rightarrow f(u(x, t)), \text{ in } L^2([r, T] \times \Omega), \quad (A22) \]

as \( N \rightarrow \infty \). Then, making a limit transition in the non-linear term, we prove the existence of a weak solution under the additional condition \( u^0 \in H^2(\Omega) \cap H^1_0(\Omega) \). One can see that this solution \( u \) satisfies (14) and (15).

**Step 2.** Now, let \( u^1(t) \) and \( u^2(t) \) be weak solutions to (1) with different initial data \( (u^0_0, u^0_1) \in \mathcal{S} \) such that

\[
\left\| (u'(t), u^0_1(t)) \right\|_\mathcal{S} + \int_r^T \| \nabla u_1(t) \|^2 \, dt \leq C_R, \quad \forall t \in [r, T],
\]

for some \( R > 0 \). Notice that we do not assume \( u^0_1 \in H^2(\Omega) \) here. Since \( \phi \in C^1 \), we conclude from (60) that

\[
|\phi(|\nabla u_1|)|, |\phi'(||\nabla u_1||)| \leq M, t \in [r, T]. \quad (A24)
\]

We can see that \( z(t) = u^1(t) - u^2(t) \) solves the equation

\[
z_{tt} - \Delta z_t = -\frac{1}{2} \phi_{12}(t) \Delta z - \frac{1}{2} \left( \phi_1(t) - \phi_2(t) \right) (\Delta u^1 + \Delta u^2) + f'(u^1) - f'(u^2), \quad (A25)
\]

where \( \phi_{12}(t) = \phi_1(t) + \phi_2(t) \) with \( \phi_1(t) = \phi(|\nabla u_1|) \).

By the definition of a weak solution, we can multiply (A25) by \( z \) in \( L^2(\Omega) \) and reduce that

\[
\frac{d}{dt} \left[ (z, z_t) + \frac{1}{2} \| \nabla z_t \|^2 - \frac{1}{2} \phi_{12}(t) \| \nabla z \|^2 \right] + (f'(u^1) - f'(u^2), z) + \frac{1}{2} \left( \phi_1(t) - \phi_2(t) \right) (\nabla u^1 + \nabla u^2, \nabla z) = 0.
\]

(A26)

Using \( H^2(\Omega) \rightarrow L^q(\Omega) \) for every \( 1 \leq q < +\infty \) when \( n = 1, 2 \) and \( H^2(\Omega) \rightarrow L^{2q/(n-2)}(\Omega) \) when \( n \geq 3 \), we have that

\[
|f'(u^1) - f'(u^2)| \leq C \int_\Omega \left( 1 + |u^1|^{p-1} + |u^2|^{p-1} \right) |z|^2 \, dx \leq C_R \| \nabla z \|^2.
\]

(A27)

Therefore, combining with

\[
|\phi_1(t) - \phi_2(t)| = \int_0^1 \phi'(\lambda) \left( |\nabla u_1(t)|^2 + (1 - \lambda) |\nabla u_2(t)|^2 \right) \, d\lambda \cdot (\nabla u^1 + \nabla u^2, \nabla z) \leq C \| \nabla z \|,
\]

we can conclude that

\[
\frac{d}{dt} \left[ (z, z_t) + \frac{1}{2} \| \nabla z_t \|^2 \right] \leq |z|^2 + C_R \| \nabla z \|^2. \quad (A29)
\]

Now consider the multiplier \( z_1 \). Since \( z \in L^\infty([r, T]; H^1_0(\Omega)), z_1 \in L^2([r, T]; H^0_0(\Omega)), \) and \( z_{1r} \in L^2([r, T]; H^{-1}(\Omega)) \), we can multiply (A26) by \( z_1 \) and obtain

\[
\frac{1}{2} \frac{d}{dt} \| z_1 \|^2 + \| \nabla z_1 \|^2 + \frac{1}{2} \phi_{12}(t) (\nabla z_1, \nabla z_2) + (f(u^1) - f(u^2), z_1) - \frac{1}{2} \left( \phi_1(t) - \phi_2(t) \right) (\Delta u^1 + \Delta u^2, z_1) = 0.
\]

(A30)

Similar to (A29), we can get

\[
\frac{d}{dt} \| z_1 \|^2 + 2 \| \nabla z_1 \|^2 \leq C \| \nabla z \| \cdot \| \nabla z_1 \|. \quad (A31)
\]

Similar to (A27), we have

\[
|f(u^1) - f(u^2), z_1| \leq C_R \| \nabla z \| \cdot \| \nabla z_1 \|. \quad (A32)
\]

Therefore, we can conclude from Young’s inequality that

\[
\frac{d}{dt} \| z_1 \|^2 + \| \nabla z_1 \|^2 \leq C \| \nabla z \|^2 + \| \nabla z_1 \|^2 \]. \quad (A33)
\]

Let

\[
\Gamma(t) = \| z_1 \|^2 + \varepsilon \left[ (z, z_t) + \frac{1}{2} \| \nabla z \|^2 \right], \quad (A34)
\]

for \( \varepsilon > 0 \) small enough. Then, there exists a positive constants \( C_1 \) such that

\[
C_1 \| z_1 \|^2 + \| \nabla z \|^2 \leq \Gamma(t) \leq C_2 \| z_1 \|^2 + \| \nabla z \|^2. \quad (A35)
\]

From (A29) and (A33), we have the estimation

\[
\frac{d}{dt} \Gamma(t) + \| \nabla z_1(t) \|^2 \leq C_{t,R} \Gamma(t). \quad (A36)
\]

Using Gronwall’s inequality, we get that

\[
\| z_1(t) \|^2 + \| \nabla z(t) \|^2 + \int_0^t || \nabla z(s) ||^2 \, ds \leq C_{R,T} \left( \| z_1(t) \|^2 + \| \nabla z(t) \|^2 \right), \quad (A37)
\]

for all \( t \in [r, T] \), which implies the desired conclusion in (20). By this inequality, we can prove the existence of weak solutions for initial data \( (u^0_0, u^0_1) \in \mathcal{S} \). Indeed, we can choose a sequence \( \{ u^{0, n}_r, u^{1, n}_r \} \in (H^2(\Omega) \cap H^1_0(\Omega)) \times L^2(\Omega) \) such that \( (u^{0, n}_r, u^{1, n}_r) \rightarrow (u^0_r, u^1_r) \) in \( \mathcal{S} \). Owing to
(20), the corresponding solutions \((u^n(t), u^n_r(t))\) converge to functions \((u, u_r)\) in \(L^\infty(T, T; \mathcal{H})\). From the boundedness for \(\{u^n_r\}\) in \(L^2(T, H^2_0(\Omega))\) we also have weak convergence of \(\{u^n_r\}\) to \(u_r\) in the space \(L^2(T, H^2_0(\Omega))\). This implies that \(u(t)\) is a weak solution of Problem (1). By (19), this solution is unique.

**Step 3.** For the proof of smoothness properties stated in (18), we use the same method as [18, 38]. As usual, the argument below can be justified by considering Galerkin approximations. Set \(v = u_r\) and differentiate (1) with respect to time. This yields

\[
v_{tt} - \Delta v_t - \phi(\|\nabla u\|^2)\Delta v - 2\phi'(\|\nabla u\|^2)\Delta u(\nabla u\nabla u_t) + f'(u)v = h_t.
\]

(A38)

Multiplying the above equation by \(v\), we obtain that

\[
\frac{d}{dt}(v, v) + \frac{1}{2}\|\nabla v\|^2 + \phi(\|\nabla u\|^2)\|\nabla v\|^2 + (f'(u)v, v) \\
\leq \|v_t\|^2 + C_R(\|\nabla u\|\|v\|)^2 + (h_t, v).
\]

(A39)

This implies that

\[
\frac{d}{dt}(v, v) + \frac{1}{2}\|\nabla v\|^2 \leq \|v_t\|^2 + C_R(\|\nabla u\|\|v\|^2 + \|h_t(t)\|^2).
\]

(A40)

Multiplying the above equation by \(\mathcal{A}^{-1}v_t\) with \(\mathcal{A} = -\Delta\) and using Young’s inequality, we obtain that

\[
\frac{d}{dt}(\|\mathcal{A}^{-\frac{1}{2}}v_t\|^2 + \|v_t\|^2) \leq C_R(\|\nabla u\|^2 + \|h_t(t)\|^2).
\]

(A41)

Denote

\[
Y(t) = \|\mathcal{A}^{-\frac{1}{2}}v_t\|^2 + \|v_t\|^2 + \varepsilon(\min(v, v_t) + \frac{1}{2}\|\nabla v\|^2),
\]

then we have that

\[
a_1(\|\mathcal{A}^{-\frac{1}{2}}v_t\|^2 + \|v_t\|^2) \leq Y(t) \leq a_2(\|\mathcal{A}^{-\frac{1}{2}}v_t\|^2 + \|v_t\|^2).
\]

(A43)

for some positive constants \(a_1, a_2\) depending on \(\varepsilon\). Due to (A40) and (A41), it is apparent that

\[
\frac{dY(t)}{dt} + \frac{1}{2}\|v_t\|^2 \leq C_0\|\nabla v\|^2 + C_\varepsilon\|h_t(t)\|^2 \leq \tilde{C}_3 Y(t) + C_4\|h_t(t)\|^2.
\]

(A44)

Multiplying (A44) by \((s - \tau)^2\), we get that

\[
\frac{d}{ds}((s - \tau)^2 Y(s)) + \frac{(s - \tau)^2}{2}\|v_t(s)\|^2 \leq 2(s - \tau) Y(s) + \tilde{C}_3(s - \tau)^2 Y(s) + C_4(s - \tau)^2\|h_t(s)\|^2.
\]

(A45)

It is easy to know

\[
2(s - \tau) Y(s) \leq 1 + (s - \tau)^2 Y^2(s) \leq 1 + (s - \tau)^2
\]

\[
\cdot a_2(\|\mathcal{A}^{-\frac{1}{2}}v_t\|^2 + \|\nabla v\|^2) Y(s).
\]

(A46)

Since

\[
\mathcal{A}^{-1}u_t = -u_t - \phi(\|\nabla u\|^2)u - \mathcal{A}^{-1}(f(u) - h),
\]

(A47)

one can see that \(\|\mathcal{A}^{-\frac{1}{2}}v_t\| \leq C_R(1 + \|h(s, t)\|).\) Using \(\|\mathcal{A}^{-\frac{1}{2}}v_t\|^2 \leq C\|\mathcal{A}^{-\frac{1}{2}}v_t\| \cdot \|v_t\| \) and Young’s inequality, we get

\[
\frac{d}{ds}((s - \tau)^2 Y(s)) + \frac{(s - \tau)^2}{4}\|v_t(s)\|^2 \leq C_5(s - \tau)^2
\]

\[
\cdot (1 + \|\nabla u\|^2) Y(s) + C_6(s - \tau)^2(\|h(s, t)\|^2 + \|h_t(s)\|^2).
\]

(A48)

By Gronwall’s inequality and (16), one can find

\[
(t - \tau)^2 Y(t) \leq C_{R,t}(1 + \|h(s, t)\|^2 + \|h_t(s)\|^2).
\]

(A49)

This implies (18). The proof is completed.

**Data Availability**

All data used to support the findings of this study are included within the article.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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**References**

[1] G. Kirchhoff, *Vorlesungen über Mechanik*, Teubner, Stuttgart, 1883.

[2] A. Arosio and S. Spagnolo, “Global solutions to the Cauchy problem for a nonlinear hyperbolic equation, nonlinear partial differential equations and their applications,” in *Coll ege de Fracnes seminar*, vol. VI (Paris, 1982/1983), pp. 1–26, Res. Notes in Math.,109 Pitman, Boston, MA, 1984.

[3] S. Berstein, “Sur une classe de é quations fonctionnelles aux de rive’es,” *Izvestiya Akademii Nauk SSSR. Seriya Matematiches-
kaya*, vol. 4, pp. 17–26, 1940.
Y. Yamada, “On an abstract weakly hyperbolic equation modelling the nonlinear vibrating string,” in \textit{Developments in Partial Differential Equations and Applications to Mathematical Physics} (Ferrara, 1991), pp. 27–32, Plenum, New York, 1992.

P. D’Ancona and S. Spagnolo, “Global solvability for the degenerate Kirchhoff equation with real analytic data,” \textit{Inventiones Mathematicae}, vol. 108, pp. 247–262, 1992.

P. D’Ancona and S. Spagnolo, “A class of nonlinear hyperbolic problems with global solutions,” \textit{Archive for Rational Mechanics and Analysis}, vol. 124, no. 3, pp. 201–219, 1993.

J. M. Greenberg and S. C. Hu, “The initial value problem for a stretched string,” \textit{Quarterly of Applied Mathematics}, vol. 38, no. 3, pp. 289–311, 1980.

T. Matsuyama and M. Ruzhansky, “Global well-posedness of Kirchhoff systems,” \textit{Journal de Mathématiques Pures et Appliquées}, vol. 100, no. 2, pp. 220–240, 2013.

T. Yamazaki, “Global solvability for the Kirchhoff equations in exterior domains of dimension three,” \textit{Journal of Differential Equations}, vol. 210, no. 2, pp. 290–316, 2005.

E. H. De Brito and J. Hale, “The damped elastic stretched string equation generalized: existence, uniqueness, regularity and stability,” \textit{Applicable Analysis}, vol. 13, no. 3, pp. 219–233, 1982.

M. Ghisi and M. Gobbino, “Global existence and asymptotic behavior for a mildly degenerate dissipative hyperbolic equation of Kirchhoff type,” \textit{Asymptotic Analysis}, vol. 40, no. 1, pp. 25–36, 2004.

M. Ghisi, “Global solutions for dissipative Kirchhoff strings with non-Lipschitz nonlinear term,” \textit{Journal of Differential Equations}, vol. 230, no. 1, pp. 128–139, 2006.

K. Nishihara and Y. Yamada, “On global solutions of some degenerate quasilinear hyperbolic equations with dissipative terms,” \textit{Fako de l’Funkcialaj Ekvacioj Japana Matematika Societo}, vol. 33, pp. 151–159, 1990.

K. Ono, “On global existence, asymptotic stability and blowing up of solutions for some degenerate nonlinear wave equations of Kirchhoff type with a strong dissipation,” \textit{Mathematical Methods in the Applied Sciences}, vol. 20, pp. 151–177, 1997.

Y. Yamada, “On some quasilinear wave equations with dissipative terms,” \textit{Nagoya Mathematical Journal}, vol. 87, pp. 17–39, 1982.

K. Nishihara, “Degenerate quasilinear hyperbolic equation with strong damping,” \textit{Fako de l’Funkcialaj Ekvacioj Japana Matematika Societo}, vol. 27, pp. 125–145, 1984.

M. M. Cavalcanti, V. N. D. Cavalcanti, J. S. P. Filho, and J. A. Soriano, “Existence and exponential decay for a Kirchhoff-Carrier model with viscosity,” \textit{Journal of Mathematical Analysis and Applications}, vol. 226, no. 1, pp. 40–60, 1998.

V. Kalantarov and S. Zelik, “Finite-dimensional attractors for the quasi-linear strongly-damped wave equation,” \textit{Journal of Differential Equations}, vol. 247, no. 4, pp. 1120–1155, 2009.

M. Nakao, “An attractor for a nonlinear dissipative wave equation of Kirchhoff type,” \textit{Journal of Mathematical Analysis and Applications}, vol. 353, pp. 652–659, 2009.

M. Nakao and Z. Yang, “Global attractors for some quasilinear wave equations with a strong dissipation,” \textit{Advances in Mathematical Sciences and Applications}, vol. 17, pp. 89–105, 2007.

Z. J. Yang, P. Y. Ding, and Z. M. Liu, “Global attractor for the Kirchhoff type equations with strong nonlinear damping and supercritical nonlinearity,” \textit{Applied Mathematics Letters}, vol. 33, pp. 12–17, 2014.

Z. J. Yang and Y. Wang, “Global attractor for the Kirchhoff equation with a strong dissipation,” \textit{Journal of Differential Equations}, vol. 249, pp. 3258–3278, 2010.

Z. J. Yang and X. Li, “Finite-dimensional attractors for the Kirchhoff equation with a strong dissipation,” \textit{Journal of Mathematical Analysis and Applications}, vol. 375, no. 2, pp. 579–593, 2011.

H. L. Ma, J. Zhang, and C. K. Zhong, “Global existence and asymptotic behavior of global smooth solutions to the Kirchhoff equations with strong nonlinear damping,” \textit{Discrete and Continuous Dynamical Systems - Series B}, vol. 24, no. 9, pp. 4721–4737, 2019.

I. Chueshov, “Long-time dynamics of Kirchhoff wave models with strong nonlinear damping,” \textit{Journal of Differential Equations}, vol. 252, no. 2, pp. 1229–1262, 2012.

H. L. Ma, J. Zhang, and C. K. Zhong, “Attractors for the degenerate Kirchhoff wave model with strong damping: existence and the fractal dimension,” \textit{Journal of Mathematical Analysis and Applications}, vol. 484, no. 1, article 123670, 2020.

T. Caraballo, A. N. Carvalho, J. A. Langa, and F. Rivero, “Existence of pullback attractors for pullback asymptotically compact processes,” \textit{Nonlinear Analysis}, vol. 72, no. 3–4, pp. 1967–1976, 2010.

T. Caraballo, P. E. Kloeden, and J. Real, “Pullback and forward attractors for a damped wave equation with delays,” \textit{Stochastics and Dynamics}, vol. 4, no. 3, pp. 405–423, 2004.

T. Caraballo, G. Lukaszewicz, and J. Real, “Pullback attractors for asymptotically compact non-autonomous dynamical systems,” \textit{Nonlinear Analysis: Theory, Methods & Applications}, vol. 64, no. 3, pp. 484–498, 2006.

A. N. Carvalho, J. A. Langa, J. C. Robinson, and A. Suárez, “Characterization of non-autonomous attractors of a perturbed infinite-dimensional gradient system,” \textit{Journal of Differential Equations}, vol. 236, no. 2, pp. 570–603, 2007.

P. E. Kloeden, “Pullback attractors of non-autonomous semidynamical systems,” \textit{Stochastics and Dynamics}, vol. 3, no. 1, pp. 101–112, 2003.

C. Y. Sun, D. M. Cao, and J. Q. Duan, “Non-autonomous dynamics of wave equations with nonlinear damping and critical nonlinearity,” \textit{Nonlinearity}, vol. 19, pp. 2645–2665, 2006.

Y. H. Wang and C. K. Zhong, “Upper semicontinuity of pullback attractors for nonautonomous Kirchhoff wave models,” \textit{Discrete and Continuous Dynamical Systems}, vol. 7, pp. 3199–3209, 2013.

Y. N. Li and Z. J. Yang, “Robustness of attractors for non-autonomous Kirchhoff wave models with strong nonlinear damping,” \textit{Applied Mathematics & Optimization}, vol. 84, no. 1, pp. 245–272, 2021.

Q. F. Ma, S. H. Wang, and C. K. Zhong, “Necessary and sufficient conditions for the existence of global attractors for semigroups and applications,” \textit{Indiana University Mathematics Journal}, vol. 51, pp. 1541–1559, 2002.

Y. J. Wang, C. K. Zhong, and S. F. Zhou, “Pullback attractors of nonautonomous dynamical systems,” \textit{Discrete and Continuous Dynamical Systems}, vol. 16, pp. 587–614, 2006.

J. Lions, \textit{Quelques Méthodes de Résolution des Problèmes Aux Limites Non Linéaires}, Dunod, Paris, 1969.

A. V. Babin and M. I. Vishik, \textit{Attractors of Evolution Equations}, North-Holland, Amsterdam, 1992.