On Chandrasekhar functional integral inclusion and Chandrasekhar quadratic integral equation via a nonlinear Urysohn–Stieltjes functional integral inclusion

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Abstract
We investigate the existence of solutions for a nonlinear integral inclusion of Urysohn–Stieltjes type. As applications, we give a Chandrasekhar quadratic integral equation and a nonlinear Chandrasekhar integral inclusion.

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1 Introduction
The integral equations of Urysohn–Stieltjes (U-S) type have been studied by some authors; see, for example, [3, 5, 11–15], and [16–22], and reference therein.

The quadratic Chandrasekhar integral equation

\[ x(t) = a(t) + x(t) \int_0^1 \frac{t}{t+s} b_1(s)x(s) \, ds, \quad t \in I = [0, 1] \]

has been studied in some papers; see, for example, [1, 4, 7–10], and [24] and references therein.

Our aim is to study the existence of solutions \( x \in C[0, 1] \) of the U-S nonlinear functional integral inclusion

\[ x(t) - a(t) \in \int_0^1 F \left( t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) \, d_0 g_2(s, \theta) \right) \, d_1 g_1(t, s), \quad t \in I = [0, 1]. \]
As applications, we will prove the existence of solutions $x \in C[0,1]$ of the nonlinear Chandrasekhar functional integral inclusion
\[
x(t) - a(t) \in \int_{0}^{1} \frac{t}{t + s} F \left( b_1(s)x(s), \int_{0}^{1} \frac{s}{s + \theta} b_2(s)x(\theta) d\theta \right) ds, \quad t \in I = [0,1],
\]
and the Chandrasekhar quadratic integral equation
\[
x(t) = a(t) + \int_{0}^{1} \frac{t}{t + s} b_1(s)x(s) \cdot \left( \int_{0}^{1} \frac{s}{s + \theta} b_2(s)x(\theta) d\theta \right) ds, \quad t \in I = [0,1].
\]

The paper is organized as follows. In Sect. 2, we establish the existence and uniqueness results for single-valued nonlinear U-S equations. We also prove the continuous dependence of the unique solution on the $g_i$ ($i = 1, 2$). As an application, we discuss some particular cases by presenting the existence of solutions of nonlinear Chandrasekhar quadratic functional integral equations. In Sect. 3, we add conditions to our problem in order to obtain a new existence result with an application. Our results are generalized in Sect. 4, where we discuss the existence of solutions for set-valued equation (1.1) with continuous dependence on the set $S_F$ and demonstrate a particular case of inclusion by presenting the existence of solutions for set-valued Chandrasekhar nonlinear functional integral equations.

2 Single-valued problem

Here we consider the nonlinear single-valued functional integral equation of U-S type
\[
x(t) = a(t) + \int_{0}^{1} f \left( t, s, x(s), \int_{0}^{1} h(s, \theta, x(\theta)) d_g g_2(s, \theta) \right) d_g g_1(t, s), \quad t \in [0,1]. \tag{2.1}
\]

2.1 Existence of solutions I

Consider the U-S functional integral equation (2.1) under the following assumptions:

(i) $a : [0,1] \to [0,1]$ is a continuous function, with $a = \sup_{t \in [0,1]} |a(t)|$.

(ii) $f : [0,1] \times [0,1] \times R \times R \to R$ is a continuous function, and there exist two continuous functions $m_1, k_1 : [0,1] \times [0,1] \to R$ such that
\[
|f(t,s,x,y)| \leq m_1(t,s) + k_1(t,s)(|x| + |y|).
\]

b) $h : [0,1] \times [0,1] \times R \to R$ is a continuous function, and there exist two continuous functions $m_2, k_2 : [0,1] \times [0,1] \to R$ such that
\[
|h(t,s,x)| \leq m_2(t,s) + k_2(t,s)|x|.
\]

c) $k = \sup \{k_1(t,s) : t,s \in [0,1]\}$, and $m = \sup \{m_1(t,s) : t,s \in [0,1], i = 1,2\}$.

(iii) $g_i : [0,1] \times R \to R$, $i = 1,2$, are continuous functions with
\[
\mu = \max \left\{ \sup \{|g_i(t,1)| + \sup |g_i(t,0)| \mid t \in [0,1]\} \right\}.
\]

(iv) For all $t_1, t_2 \in I$, $t_1 < t_2$, the functions $s \to g_i(t_2, s) - g_i(t_1, s)$ are nondecreasing on $[0,1]$. 

(v) \( g(0, s) = 0 \) for \( s \in [0, 1] \).

(vi) \( k\mu + k^2\mu^2 < 1 \).

Let \( E \) be a Banach space with the norm \( \| \cdot \|_E \), and let \( I = [0, 1] \). Denote by \( C = C(I, E) \) the space of all continuous functions on \( I \) taking values in the space \( E \). This space becomes a Banach space with supnorm

\[
\|x\|_C = \sup_{t \in I} \|x(t)\|_E.
\]

**Remark 2.1** (see [11]) Note that the function \( s \to g(t, s) \) is nondecreasing on the interval \([0, 1] \). Indeed, for \( s_1, s_2 \in [0, 1] \) with \( s_1 < s_2 \), from assumptions (iv) and (v) we obtain

\[
g(t, s_2) - g(t, s_1) = \left[ g(t, s_2) - g(0, s_2) \right] - \left[ g(t, s_1) - g(0, s_1) \right] \geq 0.
\]

**Lemma 2.2** ([11]) Assume that a function \( g \) satisfies assumption (v). Then for arbitrary \( s_1, s_2 \in I \) with \( s_1 < s_2 \), the function \( t \to g(t, s_2) - g(t, s_1) \) is nondecreasing on \( I \).

Indeed, take \( t_1, t_2 \in [0, 1] \) such that \( t_1 < t_2 \). Then by assumption (vi) we get

\[
[g(t_2, s_2) - g(t_2, s_1)] - [g(t_1, s_2) - g(t_1, s_1)] = [g(t_2, s_2) - g(t_1, s_2)] - [g(t_2, s_1) - g(t_1, s_1)] \geq 0.
\]

For the existence of at least one solution of the U-S nonlinear functional integral equation (2.1), we have the following theorem.

**Theorem 2.3** Let the assumptions (i)–(vi) be satisfied. Then the functional integral equation (2.1) has at least one solution \( x \in C[0, 1] \).

**Proof** Define the operator \( A \) by

\[
Ax(t) = a(t) + \int_0^1 f(t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) d_0g_2(s, \theta)) d_1g_1(t, s), \quad t \in I,
\]

and define let the set

\[
Q_r = \{ x \in \mathbb{R} : |x| \leq r \} \subseteq C[0, 1],
\]

where

\[
r = \frac{a + m\mu + km\mu^2}{1 - [k\mu + k^2\mu^2]}.
\]

It is clear that \( Q_r \) is a nonempty, bounded, closed, and convex set.

Let \( x \in Q_r \). Then

\[
|Ax(t)| = \left| a(t) + \int_0^1 f(t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) d_0g_2(s, \theta)) d_1g_1(t, s) \right|
\leq |a(t)| + \int_0^1 \left| f(t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) d_0g_2(s, \theta)) \right| d_1g_1(t, s)
\]
\[
\begin{align*}
\leq a + \int_0^1 \left( m_1(t,s) + k_1(t,s) \left( |x(t)| + \int_0^1 |h(s,\theta,x(\theta))| \, dg_2(s,\theta) \right) \right) \, dg_1(t,s) \\
\leq a + \int_0^1 \left( m_1(t,s) + k_1(t,s) \left( |x(t)| \right. \right. \\
+ \left. \left. \int_0^1 (m_2(s,\theta) + k_2(s,\theta)|x(\theta)| \, dg_2(s,\theta) \right) \right) \, dg_1(t,s) \\
\leq a + \int_0^1 (m_1(t,s) + k_1(t,s)(|x(t)| + (m + kr)\mu) \, dg_1(t,s) \\
\leq a + (m + k(r + (m + kr)\mu))\mu \leq r.
\end{align*}
\]

This proves that the operator \( A : Q_r \rightarrow Q_r \) and the class \( \{Ax\} \) is uniformly bounded on \( Q_r \).

Then, for \( x \in Q_r \), and \( y(s) = \int_0^1 h(s,\theta,x(\theta)) \, dg_2(s,\theta) \), define the set

\[
\theta(\delta) = \sup \{ |f(t_2,s,x,y) - f(t_1,s,x,y)| : t_1, t_2, s \in [0,1], t_1 < t_2, \\
|t_2 - t_1| < \delta, |x| \leq r, |y| \leq r \}.
\]

Then from the uniform continuity of the function \( f : [0,1] \times [0,1] \times Q_r \times Q_r \rightarrow R \) and assumption (ii) we deduce that \( \theta(\delta) \rightarrow 0 \) as \( \delta \rightarrow 0 \), independently of \( x \in Q_r \).

Now let \( t_2, t_1 \in [0,1], |t_2 - t_1| < \delta \). Then we have

\[
\begin{align*}
|Ax(t_2) - Ax(t_1)| &= |a(t_2) + \int_0^1 f(t_2,s,x(s), \int_0^1 h(s,\theta,x(\theta)) \, dg_2(s,\theta)) \, dg_1(t_2,s) \\
&- a(t_1) - \int_0^1 f(t_1,s,x(s), \int_0^1 h(s,\theta,x(\theta)) \, dg_2(s,\theta)) \, dg_1(t_1,s)| \\
\leq |a(t_2) - a(t_1)| + \int_0^1 f(t_2,s,x(s), \int_0^1 h(s,\theta,x(\theta)) \, dg_2(s,\theta)) \, dg_1(t_2,s) \\
- \int_0^1 f(t_1,s,x(s), \int_0^1 h(s,\theta,x(\theta)) \, dg_2(s,\theta)) \, dg_1(t_1,s)| \\
\leq |a(t_2) - a(t_1)| + \int_0^1 f(t_2,s,x(s),y(s)) \, dg_1(t_2,s) \left. \right|_{s} - \int_0^1 f(t_1,s,x(s),y(s)) \, dg_1(t_1,s) \\
+ \int_0^1 f(t_1,s,x(s),y(s)) \, dg_1(t_2,s) \left. \right|_{s} - \int_0^1 f(t_1,s,x(s),y(s)) \, dg_1(t_1,s) \\
\leq |a(t_2) - a(t_1)| + \int_0^1 |f(t_2,s,x(s),y(s)) - f(t_1,s,x(s),y(s))| \, dg_1(t_2,s) \\
+ \int_0^1 |f(t_1,s,x(s),y(s))| \, dg_1(t_2,s) \\
- \int_0^1 f(t_1,s,x(s),y(s)) \, dg_1(t_2,s) \\
\leq |a(t_2) - a(t_1)| + \int_0^1 \theta(\delta) \, dg_1(t_2,s) \\
+ \int_0^1 (m_1(t,s) + k_1(t,s)(|x| + |y|)) \, dg_1(t_2,s) - g_1(t_1,s) \\
\leq |a(t_2) - a(t_1)| + \int_0^1 \theta(\delta) \, dg_1(t_2,s) \\
+ \int_0^1 (m_1(t,s) + k_1(t,s)(|x| + |y|)) \, dg_1(t_2,s) - g_1(t_1,s).
\end{align*}
\]

This inequality means that the class of functions \( \{Ax\} \) is equicontinuous.
To prove the existence of a unique solution of U-S functional integral equation (2.1), let

by condition (ii)

This proves that

and from assumption (ii) (see [23]) we get

This proves that \( Ax_n(t) \to Ax(t) \) and \( A \) is continuous.

Now (see [23]) \( A \) has at least one fixed point \( x \in Q_r \), and (2.1) has at least one solution \( x \in Q_r \subset C[0,1] \).

### 2.2 Uniqueness of the solution

To prove the existence of a unique solution of U-S functional integral equation (2.1), let us replace condition (ii) by

(ii)*

a) the function \( f : I \times I \times R \times R \to R \) is continuous and satisfies the Lipschitz condition

\[
|f(t,s,x_1,y_1) - f(t,s,x_2,y_2)| \leq k_1(|x_1 - x_2| + |y_1 - y_2|).
\]

b) \( h : I \times I \times R \to R \) is continuous and satisfies the Lipschitz condition

\[
|h(t,s,x) - h(t,s,y)| \leq k_2|x - y|.
\]

By condition (ii)*, we have

\[
|f(t,s,x(s),y(s))| - |f(t,s,0,0)| \leq |f(t,s,x(s),y(s)) - f(t,s,0,0)| \leq k_1(|x| + |y|).
\]

Then

\[
|f(t,s,x(s),y(s))| \leq k_1(|x| + |y|) + f_1(t,s,0,0),
\]
and

$$\left| f(t,s,x(s),y(s)) \right| \leq k_1(|x| + |y|) + m_1,$$

where $m_1 = \sup_{t,s \in I} |f(t,s,0,0)|$, and

$$\left| h(t,s,x(s)) \right| - |h(t,s,0)| \leq |h(t,s,x(s)) - h(t,s,0)| \leq k_2|x|.$$

Then

$$|h(t,s,x(s))| \leq k_2|x| + |f_2(t,s,0)|,$$

and

$$|h(t,s,x(s))| \leq k_2|x| + m_2,$$

where $m_2 = \sup_{t,s \in I} |h(t,s,0)|$, $m = \max\{m_1,m_2\}$, and $k = \max\{k_1,k_2\}$.

**Theorem 2.4** Let conditions (i), (ii)*, (iii), and (iv)–(v) be satisfied with $\mu k + k^2\mu^2 \leq 1$. Then the functional integral equation (2.1) has unique solution $x \in C[0,1]$.

**Proof** Let $x_1, x_2$ be solutions of the integral equation (2.1). Then

$$\begin{align*}
|x_1(t) - x_2(t)| &= \left| a(t) + \int_0^1 f(t,s,x_1(s), \int_0^1 h(s,\theta,x_1(\theta)) \, dg_2(s,\theta)) \, dg_1(t,s) ight. \\
&\quad - a(t) + \int_0^1 f(t,s,x_2(s), \int_0^1 h(s,\theta,x_2(\theta)) \, dg_2(s,\theta)) \, dg_1(t,s) \\
&\leq \int_0^1 \left| f(t,s,x_1(s), \int_0^1 h(s,\theta,x_1(\theta)) \, dg_2(s,\theta)) ight. \\
&\quad - f(t,s,x_2(s), \int_0^1 h(s,\theta,x_2(\theta)) \, dg_2(s,\theta)) \right| \, dg_1(t,s) \\
&\leq \int_0^1 k_1 \left( |x_1(s) - x_2(s)| + \int_0^1 |h(s,\theta,x_1(\theta)) - h(s,\theta,x_2(\theta))| \, dg_2(s,\theta) \right) \, dg_1(t,s) \\
&\leq \int_0^1 k_1 \left( |x_1(s) - x_2(s)| + \int_0^1 k_2 \left( |x_1(\theta) - x_2(\theta)| \right) \, dg_2(s,\theta) \right) \, dg_1(t,s) \\
&\leq \int_0^1 k_1 \left( |x_1(s) - x_2(s)| + k_2 \|x_1 - x_2\| \, dg_1(t,s) \\
&\leq k \|x_1 - x_2\| + k^2 \|x_1 - x_2\| \mu^2.
\end{align*}$$

Hence we have

$$\|x_1 - x_2\| \leq (\mu k + k^2\mu^2) \|x_1 - x_2\|.$$
and

\[(1 - (\mu + k^2\mu^2))\|x_1 - x_2\| \leq 0,\]

which implies

\[x_1(t) = x_2(t).\]

### 2.2.1 Continuous dependence of solution on functions \(g_i(t, s)\)

Here we show that the solution of U-S functional integral equation (2.1) continuously depends on the functions \(g_i\).

**Definition 2.5** The solutions of functional integral equation (2.1) continuously depends on the functions \(g_i\) if \(i = 1, 2\), if for every \(\epsilon > 0\), there exists \(\delta > 0\) such that

\[|g_i(t, s) - g_i^*(t, s)| \leq \delta \implies \|x - x^*\| \leq \epsilon.\]

**Theorem 2.6** Let the assumptions of Theorem 2.4 be satisfied. Then the solution of (2.1) depends continuously on functions \(g_i(t, s), i = 1, 2\).

**Proof** Let \(\delta > 0\) be such that \(|g_i(t, s) - g_i^*(t, s)| \leq \delta\) for all \(t \geq 0\). Then

\[
\begin{align*}
|x(t) - x^*(t)| &= \left| a(t) + \int_0^1 f \left( t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) \mathrm{d}_t g_2(s, \theta) \right) \mathrm{d}_t g_1(t, s) \\
&\quad - a(t) + \int_0^1 f \left( t, s, x^*(s), \int_0^1 h(s, \theta, x^*(\theta)) \mathrm{d}_t g_2^*(s, \theta) \right) \mathrm{d}_t g_1^*(t, s) \right| \\
&\leq \left| \int_0^1 f \left( t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) \mathrm{d}_t g_2(s, \theta) \right) \mathrm{d}_t g_1(t, s) \\
&\quad - \int_0^1 f \left( t, s, x^*(s), \int_0^1 h(s, \theta, x^*(\theta)) \mathrm{d}_t g_2(s, \theta) \right) \mathrm{d}_t g_1(t, s) \right| \\
&\quad + \left| \int_0^1 f \left( t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) \mathrm{d}_t g_2^*(s, \theta) \right) \mathrm{d}_t g_1^*(t, s) \right| \\
&\quad - \left| \int_0^1 f \left( t, s, x^*(s), \int_0^1 h(s, \theta, x^*(\theta)) \mathrm{d}_t g_2^*(s, \theta) \right) \mathrm{d}_t g_1^*(t, s) \right| \\
&\leq \int_0^1 f \left( t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) \mathrm{d}_t g_2(s, \theta) \right) \\
&\quad - f \left( t, s, x^*(s), \int_0^1 h(s, \theta, x^*(\theta)) \mathrm{d}_t g_2(s, \theta) \right) \mathrm{d}_t g_1(t, s) \\
&\quad + \left| \int_0^1 f \left( t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) \mathrm{d}_t g_2(s, \theta) \right) \\
&\quad - f \left( t, s, x^*(s), \int_0^1 h(s, \theta, x^*(\theta)) \mathrm{d}_t g_2(s, \theta) \right) \mathrm{d}_t g_1^*(t, s) \right| \\
&\quad - f \left( t, s, x^*(s), \int_0^1 h(s, \theta, x^*(\theta)) \mathrm{d}_t g_2^*(s, \theta) \right) \mathrm{d}_t g_1^*(t, s) \right| \\
&\leq \int_0^1 k_1 \left( |x(s) - x^*(s)| \right)
\end{align*}
\]
\[
\begin{align*}
&+ \int_0^1 \left| h(s, \theta, x(s)) - h(s, \theta, x^*(s)) \right| d_{g_2}(s, \theta) \right) d_{g_1}(t, s) \\
&+ \int_0^1 f \left( t, s, x^*(s), \int_0^1 h(s, \theta, x^*(\theta)) d_{g_2}(s, \theta) \right) d_{g_1}(t, s) \\
&- \int_0^1 f \left( t, s, x^*(s), \int_0^1 h(s, \theta, x^*(\theta)) d_{g_2}^*(s, \theta) \right) d_{g_1}(t, s) \\
&+ \int_0^1 f \left( t, s, x^*(s), \int_0^1 h(s, \theta, x^*(\theta)) d_{g_2}^*(s, \theta) \right) d_{g_1}(t, s) \\
&- \int_0^1 f \left( t, s, x^*(s), \int_0^1 h(s, \theta, x^*(\theta)) d_{g_2}^*(s, \theta) \right) d_{g_1}(t, s) \\
\leq & \int_0^1 k \left( |x(s) - x^*(s)| + \int_0^1 k |x(\theta) - x^*(\theta)| d_{g_2}(s, \theta) \right) d_{g_1}(t, s) \\
&+ \int_0^1 k \left( \int_0^1 h(s, \theta, x^*(\theta)) d_{g_2}(s, \theta) - \int_0^1 h(s, \theta, x^*(\theta)) d_{g_2}^*(s, \theta) \right) d_{g_1}(t, s) \\
&+ \int_0^1 f \left( t, s, x^*(s), \int_0^1 h(s, \theta, x^*(\theta)) d_{g_2}^*(s, \theta) \right) \left[ d_{g_1}(t, s) - d_{g_1}^*(t, s) \right] \\
\leq & \int_0^1 k \left( |x(s) - x^*(s)| + \int_0^1 k |x(\theta) - x^*(\theta)| d_{g_2}(s, \theta) \right) d_{g_1}(t, s) \\
&+ \int_0^1 k \left( \int_0^1 h(s, \theta, x^*(\theta)) d_{g_2}(s, \theta) - \int_0^1 h(s, \theta, x^*(\theta)) d_{g_2}^*(s, \theta) \right) d_{g_1}(t, s) \\
&+ \int_0^1 \left[ m + k \left( |x^*(s)| + \int_0^1 h(s, \theta, x^*(\theta)) d_{g_2}^*(s, \theta) \right) \right] \left[ d_{g_1}(t, s) - d_{g_1}^*(t, s) \right] \\
\leq & \int_0^1 k \left( |x(s) - x^*(s)| + \int_0^1 k |x(\theta) - x^*(\theta)| d_{g_2}(s, \theta) \right) d_{g_1}(t, s) \\
&+ \int_0^1 k \left( \int_0^1 [m + k |x^*(\theta)|] d_{g_2}(s, \theta) - \int_0^1 h(s, \theta, x^*(\theta)) d_{g_2}^*(s, \theta) \right) d_{g_1}(t, s) \\
&+ \int_0^1 \left[ m + k \left( |x^*(s)| + \int_0^1 [m + k |x^*(\theta)|] d_{g_2}^*(s, \theta) \right) \right] \left[ d_{g_1}(t, s) - d_{g_1}^*(t, s) \right] \\
\leq & k\mu \| x - x^* \| + k^2 \mu^2 \| x - x^* \| + k[m + kr] \mu [g_2(s, 1) - g_2^*(s, 1)] \\
&+ \left[ m + k[r + m + kr] \right] \mu [g_1(t, 1) - g_1^*(t, 1)].
\end{align*}
\]

Taking the supremum over \( t \in I \), we get
\[
\| x - x^* \| \leq k\mu \| x - x^* \| + k^2 \mu^2 \| x - x^* \| + [km + kr] \mu \delta + \left[ m + k[r + kr + m] \right] \mu \delta.
\]
Then
\[ \|x - x^*\| \leq \frac{(2km + 2kr + k^2r + m)\mu \delta}{1 - (k\mu + k^2\mu^2)} = \epsilon. \]

Now we get that the solution of (2.1) continuously depends on the functions \( g_i, i = 1, 2 \). □

### 3 Existence of solutions II

Now we replace assumptions (ii) a), (vi) by

(ii') \( a^* \) \( f : [0, 1] \times [0, 1] \times R \times R \rightarrow R \) is a function, and there exist two continuous functions \( m_1, k_1 : [0, 1] \times [0, 1] \rightarrow R \) such that

\[ |f(t, s, x, y)| \leq m_1(t, s) + k_1(t, s)|x| \cdot |y|. \]

(vi') There exists a positive root \( l \) of the algebraic equation

\[ \mu^2k^2l^2 + (k\mu^2m - 1)l + (a + m\mu) = 0. \]

**Theorem 3.1** Let the assumptions of Theorem 2.3 be satisfied with (ii) a) and (vi) replaced by (ii') \( a^* \) and (vi'), respectively. Then equation (2.1) has at least one solution \( x \in C([0, 1]) \).

**Proof** Define the operator \( A^* \) by

\[ A^*x(t) = a(t) + \int_0^1 f \left( t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) \, d_0g_2(s, \theta) \right) \, d_0g_1(t, s), \quad t \in [0, 1], \]

and define the set

\[ Q_l = \{ x \in R : |x| \leq l \} \subseteq C([0, 1]), \]

where \( l \) is a positive root of the algebraic equation

\[ \mu^2k^2l^2 + (k\mu^2m - 1)l + (a + m\mu) = 0. \]

It is clear that \( Q_l \) is a nonempty, bounded, closed, and convex set.

Now let \( x \in Q_l \). Then

\[ |A^*x(t)| \]
\[ = |a(t) + \int_0^1 f \left( t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) \, d_0g_2(s, \theta) \right) \, d_0g_1(t, s)| \]
\[ \leq a + \int_0^1 \left| f \left( t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) \, d_0g_2(s, \theta) \right) \right| \, d_0g_1(t, s) \]
\[ \leq a + \int_0^1 \left( m_1(t, s) + k_1(t, s) \left( |x(t)| \cdot \int_0^1 h(s, \theta, x(\theta)) \, d_0g_2(s, \theta) \right) \right) \, d_0g_1(t, s) \]
\[ \leq a + \int_0^1 \left( m_1(t, s) + k_1(t, s) |x(t)| \cdot \int_0^1 (m_2(s, \theta) + k_2(s, \theta) |x(\theta)|) \, d_0g_2(s, \theta) \right) \, d_0g_1(t, s) \]
\[ a + \int_0^1 (m_1(t,s) + k_1(t,s)(|x(t)| \cdot (m + kl)\mu) \, d_g 1(t,s) \]
\[ \leq a + (m + k(l \cdot (m + kl)\mu))\mu \leq l. \]

This proves that \( A^* : Q_l \rightarrow Q_l \) and the class \( \{A^*x\} \) is uniformly bounded on \( Q_l \).

Now for \( x \in Q_l \) and \( y(s) = \int_0^1 h(s,\theta,x(\theta)) \, d\eta g_2(s,\theta) \), define the set
\[ \theta(\delta) = \sup \{|f(t_2,s,x,y) - f(t_1,s,x,y)| : t_1, t_2, s \in [0,1], t_1 < t_2, \]
\[ |t_2 - t_1| < \delta, |x| \leq l, |y| \leq l \}. \]

Then from the uniform continuity of the function \( f : [0,1] \times [0,1] \times Q_l \times Q_l \rightarrow R \) and assumption (ii*) we deduce that \( \theta(\delta) \rightarrow 0 \) as \( \delta \rightarrow 0 \), independently of \( x \in Q_l \).

Now let \( t_2, t_1 \in [0,1] \) be such that \( |t_2 - t_1| < \delta \). Then we have
\[ |A^*x(t_2) - A^*x(t_1)| \]
\[ = |a(t_2) + \int_0^1 f(t_2,s,x(s), \int_0^1 h(s,\theta,x(\theta)) \, d\eta g_2(s,\theta)) \, d_g 1(t_2,s) \]
\[ - a(t_1) - \int_0^1 f(t_1,s,x(s), \int_0^1 h(s,\theta,x(\theta)) \, d\eta g_2(s,\theta)) \, d_g 1(t_1,s)| \]
\[ \leq |a(t_2) - a(t_1)| + \left| \int_0^1 f(t_2,s,x(s), \int_0^1 h(s,\theta,x(\theta)) \, d\eta g_2(s,\theta)) \, d_g 1(t_2,s) \right| \]
\[ - \left| \int_0^1 f(t_1,s,x(s), \int_0^1 h(s,\theta,x(\theta)) \, d\eta g_2(s,\theta)) \, d_g 1(t_1,s) \right| \]
\[ \leq |a(t_2) - a(t_1)| + \left| \int_0^1 \left( f(t_2,s,x(s),y(s)) - f(t_1,s,x(s),y(s)) \right) \, d_g 1(t_2,s) \right| \]
\[ + \left| \int_0^1 \left( f(t_1,s,x(s),y(s)) \right) \, d_g 1(t_2,s) \right| \]
\[ \leq |a(t_2) - a(t_1)| \]
\[ + \left| \int_0^1 \left( f(t_2,s,x(s),y(s)) - f(t_1,s,x(s),y(s)) \right) \, d_g 1(t_2,s) \right| \]
\[ + \left| \int_0^1 \left( f(t_1,s,x(s),y(s)) \right) \, d_g 1(t_2,s) \right| \]
\[ \leq |a(t_2) - a(t_1)| \]
\[ + \left| \int_0^1 \theta(\delta) \, d_g 1(t_2,s) \right| + \int_0^1 (m_1(t,s) + k_1(t,s)(|x| \cdot |y|)) \, d_g 1(t_2,s) - g_1(t_1,s) | \]
\[ \leq |a(t_2) - a(t_1)| \]

This inequality means that the class of functions \( \{A^*x\} \) is equicontinuous. Therefore \( A^* \) is compact by the Arzelà–Ascoli theorem [25].

Let \( \{x_n\} \subset Q_l \), \( x_n \rightarrow x \). Then
\[ A^*x_n(t) = a(t) + \int_0^1 f(t,s,x_n(s), \int_0^1 h(s,\theta,x_n(\theta)) \, d\eta g_2(s,\theta)) \, d_g 1(t,s), \]
Then the functional integral equation (3.1),

\[ \lim_{n \to \infty} A^* x_n(t) = \lim_{n \to \infty} \left( a(t) + \int_0^1 f \left( t, s, x_n(s), \int_0^1 h(s, \theta, x_n(\theta)) \, dg_2(s, \theta) \right) \, dg_1(t, s) \right), \]

and by assumption (ii*) (see [23]) we get

\[ \lim_{n \to \infty} A^* x_n(t) \]

\[ = a(t) + \int_0^1 \lim_{n \to \infty} f \left( t, s, x_n(s), \int_0^1 h(s, \theta, x_n(\theta)) \, dg_2(s, \theta) \right) \, dg_1(t, s) \]

\[ = a(t) + \int_0^1 f \left( t, s, \lim_{n \to \infty} x_n(s), \int_0^1 h(s, \theta, \lim_{n \to \infty} x_n(\theta)) \, dg_2(s, \theta) \right) \, dg_1(t, s) \]

\[ = a(t) + \int_0^1 f \left( t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) \, dg_2(s, \theta) \right) \, dg_1(t, s) = A^* x(t). \]

This proves that \( A^* x_n(t) \to A^* x(t) \) and \( A^* \) is continuous. So (see [23]) \( A^* \) has at least one fixed point \( x \in Q_r \), and (2.1) has at least one solution \( x \in Q_l \subset C([0,1]) \). □

3.1 Application

Let in equation (2.1), \( h(t, s, x(s)) = b_2(t) x(s) \),

\[ g_1(t, s) = \begin{cases} t \ln \frac{t+s}{t} & \text{for } t \in (0,1], s \in I, \\ 0 & \text{for } t = 0, s \in I, \end{cases} \]

and

\[ g_2(s, \theta) = \begin{cases} s \ln \frac{s+\theta}{s} & \text{for } s \in (0,1], \theta \in I, \\ 0 & \text{for } s = 0, \theta \in I. \end{cases} \]

Then \( g_1, g_2 \) satisfy our assumptions (iii)–(v), and we obtain the nonlinear Chandrasekhar functional integral equation

\[ x(t) = a(t) + \int_0^1 \frac{t}{t+s} f \left( t, s, x(s), \int_0^1 \frac{s}{s+\theta} b_2(s) x(\theta) \, d\theta \right) \, ds. \] (3.1)

Let, in equation (3.1), \( f(t, s, x(s), y(s)) = b_1(s) x(s) \cdot y(s) \), where

\[ y(s) = \int_0^1 \frac{s}{s+\theta} b_2(s) x(\theta) \, d\theta. \]

Then we obtain the Chandrasekhar quadratic functional integral equation of the form

\[ x(t) = a(t) + \int_0^1 \frac{t}{t+s} b_1(s) x(s) \cdot \left( \int_0^1 \frac{s}{s+\theta} b_2(s) x(\theta) \, d\theta \right) \, ds. \] (3.2)

Now, under the assumptions of Theorem 3.1, the Chandrasekhar quadratic functional integral equation (3.2) has at least one solution \( x \in C[0,1] \).
3.2 Example
Consider the following Chandrasekhar quadratic functional integral equation:

\[
x(t) = \frac{e^{-t}}{9 + e^t} + \int_0^1 \frac{t}{t + s} \frac{2 \cos(s)x(s)}{7e^{2s}(1 + \cos^2(s))} \cdot \left( \int_0^1 \frac{s}{s + \theta} \frac{\sin(s)}{4(1 + \sin^2(s))} x(\theta) d\theta \right) ds.
\]  
(3.3)

First, note that equation (3.3) is a particular case of equation (3.2) if we put

\[
a(t) = \frac{e^{-t}}{9 + e^t},
\]
\[
h(t, s, x(s)) = \frac{\sin(t)}{4(1 + \sin^2(t))} x(s),
\]
\[
f(t, s, x(s), y(s)) = \frac{2 \cos(s)x(s)}{7e^{2s}(1 + \cos^2(s))} \cdot y(s),
\]
\[
y(s) = \int_0^1 \frac{s}{s + \theta} \frac{\sin(s)}{4(1 + \sin^2(s))} x(\theta) d\theta,
\]
\[
b_1(s) = \frac{2 \cos(s)}{7e^{2s}(1 + \cos^2(s))}, b_2(s) = \frac{\sin(s)}{4(1 + \sin^2(s))},
\]
with \( k_1 = \frac{2}{7} \) and \( k_2 = \frac{1}{4} \).

Thus conditions (i), (ii∗) and (iii) are satisfied with \( a = \frac{1}{10} \), \( k = \frac{1}{4} \), and \( m = 0 \). By all facts established above, we deduce that condition (vi∗) of the form

\[
\mu^2 k_1^2 \mu + (k \mu^2 - 1) l + (a + m \mu) = 0
\]

has a positive solution \( l \). For example, if \( l \approx 0.1 \) or \( l \approx 33 \), then assumption (vi∗) will be satisfied if we choose one of these values.

As all the conditions of Theorem 3.1 are satisfied, equation (3.3) has at least one solution \( x \in C[0, 1] \).

4 Set-valued problem
Consider the U-S nonlinear functional integral inclusion (1.1),

\[
x(t) \in a(t) + \int_0^1 F\left(t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) d_0 g_2(s, \theta) \right) d_0 g_1(t, s), \quad t \in I,
\]

under the following assumptions:

(i) \( a : [0, 1] \to [0, 1] \) is a continuous function.

(ii)*** \( F : [0, 1] \times [0, 1] \times R \times R \to P(R) \), is a Lipschitzian set-valued map with a nonempty compact convex subset of \( 2^R \), with a Lipschitz constant \( k_1 > 0 \):

\[
\| F(t, s, x_1, y_1) - F(t, s, x_2, y_2) \| \leq k_1 \left( |x_1 - x_2| + |y_1 - y_2| \right).
\]

Remark. From this assumption and Theorem 1 from [2, Sect. 9, Chap. 1] on the existence of Lipschitzian selection we deduce that the set of Lipschitz selections of \( F \) is not empty and there exists \( f \in F \) such that

\[
|f(t, s, x_1, y_1) - f(t, s, x_2, y_2)| \leq k_1 \left( |x_1 - x_2| + |y_1 - y_2| \right).
\]
(b) \( h: [0,1] \times [0,1] \times R \to R \) is a continuous function such that

\[
|h(t,s,x)| \leq m_2(t,s) + k_3(t,s)|x|.
\]

(c) \( k = \sup_{(t,s) \in [0,1] \times [0,1]} k_i(t,s) \) and \( m = \sup_{(t,s) \in [0,1] \times [0,1]} m_i(t,s). \)

(iii) \( g_i: [0,1] \times R \to R, i = 1,2, \) are continuous with

\[
\mu = \max \{ \sup |g_i(t,\varphi(t))| + \sup |g_i(t,0)| \text{ on } [0,1] \}.
\]

(iv) For all \( t_1, t_2 \in [0,1], t_1 < t_2, \) the functions \( s \to g_i(t_2,s) - g_i(t_1,s) \) are nondecreasing on \([0,1].\)

(v) \( g_i(0,s) = 0 \) for any \( s \in [0,1]. \)

(vi) \( k\mu + k^2\mu^2 < 1. \)

4.1 Existence of solution

**Theorem 4.1** Let assumptions (i)–(ii)***, and (iv)–(vi) be satisfied. Then (1.1) has at least one solution \( x \in C[0,1]. \)

**Proof** By assumption (ii)***(a) it is clear that the set of Lipschitz selection of \( F \) is nonempty. So, the solution of the single-valued (2.1) where \( f \in S_F \) is a solution to (1.1).

Note that the Lipschitz selection \( f: [0,1] \times [0,1] \times R \times R \to R \) satisfies

\[
|f(t,s,x_1,y_1) - f(t,s,x_2,y_2)| \leq k_1(|x_1 - x_2| + |y_1 - y_2|).
\]

From this condition with \( m_1 = \sup_{(t,s) \in I} |f(t,s,0,0)| \) we have

\[
|f(t,s,x(s),y(s))| - |f(t,s,0,0)| \leq |f(t,s,x(s),y(s)) - f(t,s,0,0)| \leq k_1(|x| + |y|).
\]

Then

\[
|f(t,s,x(s),y(s))| \leq k_1(|x| + |y|) + |f(t,s,0,0)|,
\]

and

\[
|f(t,s,x(s),y(s))| \leq k_1(|x| + |y|) + m_1,
\]

that is, assumption (ii) of Theorem 2.3 is satisfied. So, all conditions of Theorem 2.3 hold.

Note that if \( x \in C(I,R) \) is a solution of (2.1), then \( x \) is a solution to (1.1). \( \square \)

4.1.1 Continuous dependence on the set of selection \( S_F \)

Here we study the continuous dependence on the set \( S_F \) of all selections of the set-valued function \( F \).

**Definition 4.2** The solution of (1.1) continuously depends on the set \( S_F \) if for all \( \epsilon > 0, \) there exists \( \delta > 0 \) such that if

\[
|f(t,s,x,y) - f^*(t,s,x,y)| < \delta, \quad f,f^* \in S_F, t \in [0,1],
\]

then \( \|x - x^*\| < \epsilon. \)
Now we have the following theorem.

**Theorem 4.3** Let the assumptions of Theorem 4.1 be satisfied with

\[ |h(t, s, x) - h(t, s, y)| \leq k_2|x - y|. \]

Then the solution of (1.1) continuously depends on the set \( S_F \) of all Lipschitzian selections of \( F \).

**Proof** For two solutions \( x(t) \) and \( x^*(t) \) of (1.1) corresponding to two selections \( f, f^* \in S_F \), we have

\[
|x(t) - x^*(t)| \\
= \left| a(t) + \int_0^1 f \left( t, s, x(s), \int_0^1 f (s, \theta, x(\theta)) \, d_g \theta \right) \, d_g (t, s) \\
- \int_0^1 f^* \left( t, s, x^*(s), \int_0^1 h(s, \theta, x^*(\theta)) \, d_g \theta \right) \, d_g (t, s) \right|
\]

\[
\leq \int_0^1 \left| f \left( t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) \, d_g \theta \right) \right| \, d_g (t, s)
\]

\[
+ \int_0^1 \left| f^* \left( t, s, x^*(s), \int_0^1 h(s, \theta, x^*(\theta)) \, d_g \theta \right) \right| \, d_g (t, s)
\]

\[
\leq \int_0^1 \left| f \left( t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) \, d_g \theta \right) \right| \, d_g (t, s) + \int_0^1 \left| f^* \left( t, s, x^*(s), \int_0^1 h(s, \theta, x^*(\theta)) \, d_g \theta \right) \right| \, d_g (t, s)
\]

\[
\leq \int_0^1 \left| f \left( t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) \, d_g \theta \right) \right| \, d_g (t, s) + \int_0^1 \left| f^* \left( t, s, x^*(s), \int_0^1 h(s, \theta, x^*(\theta)) \, d_g \theta \right) \right| \, d_g (t, s)
\]

\[
\leq \int_0^1 \left| f \left( t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) \, d_g \theta \right) \right| \, d_g (t, s) + \int_0^1 \left| f^* \left( t, s, x^*(s), \int_0^1 h(s, \theta, x^*(\theta)) \, d_g \theta \right) \right| \, d_g (t, s)
\]

\[
\leq \int_0^1 \left| f \left( t, s, x(s), \int_0^1 h(s, \theta, x(\theta)) \, d_g \theta \right) \right| \, d_g (t, s) + \int_0^1 \left| f^* \left( t, s, x^*(s), \int_0^1 h(s, \theta, x^*(\theta)) \, d_g \theta \right) \right| \, d_g (t, s)
\]
\[ + \delta \int_0^1 d_1(t, s) \]
\[ \leq \int_0^1 k_1 \left( |x(s) - x^*(s)| + \int_0^1 k_2 |x(\theta) - x^*(\theta)| \, d_2(\theta, \theta) \right) \, d_1(t, s) \]
\[ + \delta \int_0^1 d_1(t, s). \]

Now, taking the supremum over \( t \in I \), we get
\[ \|x - x^*\| \leq k\mu \|x - x^*\| + k^2 \mu^2 \|x - x^*\| + \delta \mu. \]

Hence
\[ \|x - x^*\| \leq \frac{\delta \mu}{1 - (k\mu + k^2 \mu^2)} = \epsilon. \]

Thus from last inequality we get
\[ \|x - x^*\| \leq \epsilon. \]

This proves the continuous dependence of the solution on the set \( S_F \). \( \square \)

### 4.2 Set-valued Chandrasekhar nonlinear quadratic functional integral inclusion

Now, as an application of the nonlinear set-valued functional integral equations of U-S type (1.1), we have the following. Let the functions \( g_i \) be defined by

\[
g_1(t, s) = \begin{cases} 
    t \ln \frac{t+s}{t} & \text{for } t \in (0, 1], s \in I, \\
    0 & \text{for } t = 0, s \in I,
\end{cases}
\]

and

\[
g_2(s, \theta) = \begin{cases} 
    s \ln \frac{t+\theta}{\theta} & \text{for } s \in (0, 1], \theta \in I, \\
    0 & \text{for } s = 0, \theta \in I.
\end{cases}
\]

Let, in (1.1), \( h(t, s, x(s)) = b_2(s)x(s) \) and \( F(t, s, x(s), y(s)) = F(b_1(s)x(s), y(s)) \), where

\[ y(s) = \int_0^s \frac{s}{s+\theta} b_2(s)x(\theta) \, d\theta. \]

Further, since the functions \( g_i \) satisfy assumptions (iii)–(v) (see [6]), we obtain the nonlinear Chandrasekhar functional integral inclusion

\[
x(t) \in a(t) + \int_0^1 \frac{t}{t+s} F\left( b_1(s)x(s), \int_0^1 \frac{s}{s+\theta} b_2(s)x(\theta) \, d\theta \right) \, ds, \quad t \in [0, 1].
\] (4.1)

Now we can state the following existence result for (4.1).

**Theorem 4.4** Under the assumptions of Theorem 4.1, inclusion (4.1) has at least one continuous solution \( x \in C[0, 1] \).
4.3 Example
Consider the following nonlinear Chandrasekhar functional integral inclusion:

\[ x(t) \in te^{-4t} + \int_0^1 \frac{t}{t+s} \frac{\sqrt{\pi} e^{-2\pi} x(s)}{\pi + e^t} \int_0^1 \frac{s}{s+\theta} \sqrt{\pi} e^{x(s)} d\theta \, ds, \quad t \in [0,1]. \]  

(4.2)

Note that this inclusion is a particular case of inclusion (4.1) if we choose \( F : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}_+} \) in (4.2) as follows:

\[
F(b_1(s)x(s), y(s)) = \left[ 0, \frac{s}{s^2 + 1} x(s) \int_0^1 \frac{s}{s+\theta} \sqrt{\pi} e^{x(s)} d\theta \, ds \right].
\]

Further, note that now the terms involved in (4.1) have the form

\[
a(t) = te^{-4t}, \quad y(s) = \int_0^s \frac{1}{s^2 + 1} x(\theta) d\theta, \quad h(t,s,x(s)) = \frac{\sqrt{\pi}}{e^{s+1}} x(\theta),
\]

with \( b_1(s) = \frac{1}{s^2 + 1} \) and \( b_2(s) = \frac{\sqrt{\pi}}{e^{s+1}} \).

Let \( f : [0,1] \times \mathbb{R} \to \mathbb{R} \) be a continuous map. Note that if \( f \in S_F \), then we have

\[
|f(b_1(s)x_1(s), y_1(s)) - f(b_1(s)x_2(s), y_2(s))| \leq \frac{\sqrt{\pi}}{e^{s+1}} |x_1 - x_2|
\]

and

\[
|h(t,s,x_1(t)) - h(t,s,x_2(t))| \leq \frac{1}{e^2} |x_1 - x_2|.
\]

Thus conditions (i) and (ii)\(^*\) are satisfied with \( a = e, k_1 = \frac{\sqrt{\pi}}{e^{s+1}}, \) and \( k_2 = \frac{1}{e^2} \).

Moreover, we have

\[
k \mu + k^2 \mu^2 \approx 0.102607 < 1.
\]

This shows that assumption (vii) is satisfied. So, as all the conditions of Theorem 4.4 are satisfied, inclusion (4.2) has at least one solution \( x \in C[0,1] \).

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