ÉTALE FUNDAMENTAL GROUPS OF KAWAMATA LOG TERMINAL SPACES, FLAT SHEAVES, AND QUOTIENTS OF ABELIAN VARIETIES

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1. INTRODUCTION

Working with a singular complex algebraic variety $X$, one is often interested in comparing the set of finite étale covers of $X$ with that of its smooth locus $X_{\text{reg}}$. More precisely, one may ask the following.
What are the obstructions to extending finite étale covers of $X_{\text{reg}}$ to $X$? How do the étale fundamental groups of $X$ and of its smooth locus differ?

We answer these questions for projective varieties $X$ with Kawamata log terminal (klt) singularities, a class of varieties that is important in the Minimal Model Program. The main result, Theorem 2.1, asserts that in any infinite tower of finite Galois morphisms over a klt variety, where all morphisms are étale in codimension one, almost every morphism is étale. In a certain sense, this result can be seen as saying that the difference between the sets of étale covers of $X$ and of $X_{\text{reg}}$ is small in case $X$ is klt. As an immediate application, we construct in Theorem 1.4 a finite covering $\tilde{X} \to X$, étale in codimension one, such that the étale fundamental groups of $\tilde{X}$ and of its smooth locus $\tilde{X}_{\text{reg}}$ agree. Further direct applications concern global bounds for the index of $\mathbb{Q}$-Cartier divisors on klt spaces, the existence of a quasi-étale “simultaneous index-one cover” where all $\mathbb{Q}$-Cartier divisors are Cartier, and a finiteness result for étale fundamental groups of varieties of weak log Fano-type.

As first major application, we obtain an extension theorem for flat vector bundles on klt varieties: after passing to a finite cover, étale in codimension one, any flat holomorphic bundle, defined on the smooth part of a klt variety extends across the singularities, to a flat bundle that is defined on the whole space. As a consequence, we show that every terminal variety with vanishing first and second Chern class is a finite quotient of an Abelian variety, with a quotient map that is étale in codimension one. In the smooth case, this is a classical result, which follows from the existence of Kähler-Einstein metrics on Ricci-flat compact Kähler manifolds, due to Yau: every Ricci-flat compact Kähler manifold $X$ with $c_2(X) = 0$ is an étale quotient of a compact complex torus. As a further application, we verify a conjecture of Nakayama and Zhang concerning varieties admitting polarised endomorphisms. Building on their results, we obtain a decomposition theorem describing the structure of these varieties.

1.1. Simplified version of the main result. Our main result, Theorem 2.1, is quite general and its formulation therefore somewhat involved. For many applications the following special case suffices.

**Theorem 1.1.** Let $X$ be a normal, complex, quasi-projective variety. Assume that there exists a $\mathbb{Q}$-Weil divisor $\Delta$ such that $(X, \Delta)$ is Kawamata log terminal (klt). Assume we are given a sequence of finite, surjective morphisms that are étale in codimension one,

\[
X = Y_0 \xleftarrow{\gamma_1} Y_1 \xleftarrow{\gamma_2} Y_2 \xleftarrow{\gamma_3} Y_3 \xleftarrow{\gamma_4} \cdots.
\]

If the composed morphisms $\gamma_1 \circ \cdots \circ \gamma_i : Y_i \to X$ are Galois for every $i \in \mathbb{N}^+$, then all but finitely many of the morphisms $\gamma_i$ are étale.

**Remark 1.2** (Galois morphisms). In Theorem 1.1 and throughout this paper, Galois morphisms are assumed to be finite and surjective, but need not be étale, see Definition 3.5 on page 8. The statement of Theorem 1.1 does not continue to hold when one drops the Galois assumption. Section 14.1 discusses an example that illustrates the problems in the non-Galois setting.

**Remark 1.3** (Purity of branch locus). By purity of the branch locus, the assumption that all morphisms $\gamma_i$ of Theorem 1.1 are étale in codimension one can also be formulated in one of the following, equivalent ways.

(1.3.1) All morphisms $\gamma_1 \circ \cdots \circ \gamma_i$ are étale over the smooth locus of $Y_0$.
(1.3.2) Given any $i \in \mathbb{N}^+$, then $\gamma_i$ is étale over the smooth locus of $Y_{i-1}$.
1.2. Direct Applications. Theorems 1.1 and 2.1 have a large number of immediate consequences. As a first direct application, we show that every klt space admits a quasi-étale cover whose étale fundamental group equals that of its smooth locus.

**Theorem 1.4** (Extension of étale covers from the smooth locus of klt spaces). Let \( X \) be a normal, complex, quasi-projective variety. Assume that there exists a \( \mathbb{Q} \)-Weil divisor \( \Delta \) such that \((X, \Delta)\) is klt. Then, there exists a normal variety \( \tilde{X} \) and a finite, surjective Galois morphism \( \gamma: \tilde{X} \to X \), étale in codimension one, such that the following equivalent conditions hold.

1. (1.9.1) Any finite, étale cover of \( \tilde{X} \) extends to a finite, étale cover of \( X \).
2. (1.9.2) The natural map \( \iota_*: \pi_1(\tilde{X}_{\text{reg}}) \to \pi_1(\tilde{X}) \) of étale fundamental groups induced by the inclusion of the smooth locus, \( \iota: \tilde{X}_{\text{reg}} \to \tilde{X} \), is an isomorphism.

In fact, somewhat more general statements are true, cf. Section 10.1.1. To avoid any potential for confusion, we briefly recall the characterisation of the étale fundamental group of a complex variety.

**Fact 1.5** (Étale fundamental group, [Mil80, § 5 and references there]). If \( Y \) is any complex algebraic variety, then the étale fundamental group \( \pi_1(Y) \) is isomorphic to the profinite completion of the topological fundamental group of the associated complex space \( Y^{an} \).

**Remark 1.6** (Reformulation of Theorem 1.4 in terms of étale fundamental groups). Although it might seem natural, Theorem 1.4 does not imply that the kernel of the natural map \( \iota_*: \pi_1(X_{\text{reg}}) \to \pi_1(X) \) is finite. A counterexample is discussed in Section 14.2 on page 34. We do not know whether Theorem 1.4 can be expressed solely in terms of the étale fundamental groups \( \pi_1(X_{\text{reg}}) \) and \( \pi_1(X) \).

**Remark 1.7** (Canonical choice of minimal \( \tilde{X} \)). It is natural to ask whether there exists a canonical choice of \( \tilde{X} \), uniquely determined by a suitable minimality property. This is not the case. We will show in Section 14.3 that in general no ”minimal cover” exists.

The following local variant of Theorem 1.4 considers coverings of neighbourhoods of a given point, rather than coverings of the full space.

**Theorem 1.8** (Local version of Theorem 1.4). Let \( X \) be a normal, complex, quasi-projective variety. Assume that there exists a \( \mathbb{Q} \)-Weil divisor \( \Delta \) such that \((X, \Delta)\) is klt. Let \( p \in X \) be any closed point. Then, there exists a Zariski-open neighbourhood \( X^p \) of \( p \in X \), a normal variety \( \tilde{X} \) and a finite, surjective Galois morphism \( \gamma: \tilde{X} \to X^p \), étale in codimension one, such that the following holds: given any Zariski-open neighbourhood \( U = U(p) \subseteq X^p \) with preimage \( \tilde{U} := \gamma^{-1}(U) \), then \( \pi_1(\tilde{U}_{\text{reg}}) \cong \pi_1(\tilde{U}) \). Equivalently, any finite, étale cover of \( \tilde{U}_{\text{reg}} \) extends to a finite, étale cover of \( \tilde{U} \).

Among its many properties, the covering constructed in Theorem 1.8 can be seen as a simultaneous index-one cover for all divisors on \( X^p \) that are \( \mathbb{Q} \)-Cartier in a neighbourhood of \( p \). In fact, the following much stronger result holds true.

**Theorem 1.9** (Simultaneous index-one cover). In the setting of Theorem 1.8, the following holds for any Zariski-open neighbourhood \( U = U(p) \subseteq X^p \) with preimage \( \tilde{U} = \gamma^{-1}(U) \).

1. (1.9.1) If \( \tilde{D} \) is any \( \mathbb{Q} \)-Cartier divisor on \( \tilde{U} \), then \( \tilde{D} \) is Cartier.
2. (1.9.2) If \( D \) is any \( \mathbb{Q} \)-Cartier divisor on \( U \), then \( \# \text{Gal}(\gamma) \cdot D \) is Cartier.

**Remark 1.10** (Global bound for the index of \( \mathbb{Q} \)-Cartier divisors on klt spaces). Under the assumptions of Theorem 1.8, it follows from (1.9.2) and from quasi-compactness of \( X \) that there exists a number \( N \in \mathbb{N}^+ \) such that \( N \cdot D \) is Cartier, whenever \( D \) is a \( \mathbb{Q} \)-Cartier divisor on \( X \).
Remark 1.11. It seems to be known to experts that a covering space satisfying a weak analogue of (1.9.1) exists for spaces with rational singularities. In contrast to the usual index-one covers discussed in higher-dimensional birational geometry, the Galois group of the morphism γ need not be cyclic.

As a last direct application, we obtain the following generalisation of a recent result by Chenyang Xu, [Xu14, Thm. 2].

Theorem 1.12. Let X be a normal, complex, projective variety. Assume that there exists a Q-Weil divisor Δ such that (X, Δ) is klt, and −(K_X + Δ) is big and nef. Then, the étale fundamental group  \( \hat{\pi}_1(X_{\text{reg}}) \) is finite.

In Section 14.2 we show by way of example that Theorem 1.12 cannot be generalised to rationally connected varieties.

1.3. Extension for flat sheaves on klt base spaces. Consider a normal variety X and a flat, locally free, analytic sheaf \( \mathcal{F}^\circ \), defined on the complex manifold \( X_{\text{an}} \) associated with the smooth locus \( X_{\text{reg}} \) of X. In this setting, a fundamental theorem of Deligne, [Del70, II.5, Cor. 5.8 and Thm. 5.9], asserts that \( \mathcal{F}^\circ \) is algebraic, and thus extends to a coherent, algebraic sheaf \( \mathcal{F} \) on X. If X is klt, we will show that Deligne’s extended sheaf \( \mathcal{F} \) is again locally free and flat, at least after passing to a quasi-étale cover.

Theorem 1.13 (Extension of flat, locally free sheaves). Let X be a normal, complex, quasi-projective variety. Assume that there exists a Q-Weil divisor Δ such that (X, Δ) is klt. Then, there exists a normal variety \( \tilde{X} \) and a finite, surjective Galois morphism \( \gamma : \tilde{X} \to X \), étale in codimension one, such that the following holds. If \( \mathcal{G}^\circ \) is any flat, locally free, analytic sheaf on the complex space \( \tilde{X}_{\text{an}} \), there exists a flat, locally free, algebraic sheaf \( \mathcal{G} \) on \( \tilde{X} \) such that \( \mathcal{G}^\circ \) is isomorphic to the analytification of \( \mathcal{G}|_{\tilde{X}_{\text{reg}}} \).

Theorem 1.13 follows as a consequence of Theorem 1.4. Except for the algebraicity assertion, we do not use Deligne’s result in our proof. In order to avoid confusion, we briefly recall the definition of flat sheaves.

Definition 1.14 (Flat locally free sheaf). If Y is any complex algebraic variety, and \( \mathcal{G} \) is any locally free, analytic sheaf on the underlying complex space \( Y_{\text{an}} \), we call \( \mathcal{G} \) flat if it is defined by a representation of the topological fundamental group \( \pi_1(Y_{\text{an}}) \). A locally free, algebraic sheaf on Y is called flat if and only if the associated analytic sheaf is flat.

Using the partial confirmation of the Lipman-Zariski conjecture shown in [GKKP11], we obtain the following criterion for a klt space to have quotient singularities. We also obtain a first criterion to guarantee that a given projective variety is a quotient of an Abelian variety.

Corollary 1.15 (Criterion for quotient singularities and torus quotients). In the setting of Theorem 1.13, if \( \mathcal{F}_{X_{\text{reg}}} \) is flat, then \( \tilde{X} \) is smooth and X has only quotient singularities. If X is additionally assumed to be projective, then there exists an Abelian variety A and a finite Galois morphism \( A \to \tilde{X} \) that is étale in codimension one.

1.4. Characterisation of torus quotients: varieties with vanishing Chern classes. Consider a Ricci-flat, compact Kähler manifold X whose second Chern class vanishes. As a classical consequence of Yau’s theorem [Yau78] on the existence of a Kähler-Einstein metric, X is then covered by a complex torus, cf. [LB70, Thm. 12.4.3] and [Kob87, Ch. IV, Cor. 4.15]. Building on our main result, we generalise this to the singular case, when X has terminal or a special type of klt singularities.
Theorem 1.16 (Characterisation of torus quotients). Let $X$ be a normal, complex, projective variety of dimension $n$ with at worst klt singularities. Assume that $X$ is smooth in codimension two and that the canonical divisor is numerically trivial, $K_X ≡ 0$. Further, assume that there exist ample divisors $H_1, \ldots, H_{n−2}$ on $X$ and a desingularisation $\pi : \tilde{X} → X$ such that $c_2(\mathcal{T}_{\tilde{X}}) \cdot \pi^∗(H_1) \cdots \pi^∗(H_{n−2}) = 0$.

Then, there exists an Abelian variety $A$ and a finite, surjective, Galois morphism $A → X$ that is étale in codimension two.

In fact, the converse is also true, see Theorem 1.23 for a precise statement. The three-dimensional case has been settled by Shepherd-Barron and Wilson in [SBW94], and our strategy of proof for Theorem 1.16 partly follows their line of reasoning. Apart from our main result, the proof of Theorem 1.16 relies on the semistability of the tangent sheaf of varieties with vanishing first Chern class, on Simpson’s flatness results for semistable sheaves, [Sim92, Cor. 3.10], and on the partial solution of the Lipman-Zariski conjecture mentioned above.

Remark 1.17 (Vanishing of Chern classes). The condition on the intersection numbers posed in Theorem 1.16 is a way of saying “$c_2(X) = 0$” that avoids the technical complications with the definition of Chern classes on singular spaces. Section 4 discusses this in detail.

Remark 1.18 (Terminal varieties, Generalisations). Projective varieties with terminal singularities are smooth in codimension two, [KM98, Cor. 5.18]. Theorem 1.16 therefore applies to this class of varieties. From the point of view of the minimal model program, this seems a very natural setting for our problem.

Using an orbifold version of the second Chern class, Shepherd-Barron and Wilson [SBW94] are able to treat threefolds whose singular set has codimension two. It is conceivable that with sufficient technical work, a similar result could also be obtained in the higher-dimensional setting. We have chosen not to pursue these generalisations here.

As one important step in the proof of Theorem 1.16, we generalise classical flatness results [UY86, Kob87, Sim92, BS94] for semistable vector bundles with vanishing first and second Chern classes to our singular setup, and obtain the following result:

Theorem 1.19 (Flatness of semistable sheaves with vanishing first and second Chern classes). Let $X$ be an $n$-dimensional, normal, complex, projective variety, smooth in codimension two. Assume that there exists a $Q$-Weil divisor $\Delta$ such that $(X, \Delta)$ is klt. Let $H$ be an ample Cartier divisor on $X$, and $\mathcal{E}$ be a reflexive, $H$-semistable sheaf. Assume that the following intersection numbers vanish

\[ c_1(\mathcal{E}) \cdot H^{n−1} = 0, \quad c_1(\mathcal{E})^2 \cdot H^{n−2} = 0, \quad \text{and} \quad c_2(\mathcal{E}) \cdot H^{n−2} = 0. \]

Then, there exists a normal variety $\tilde{X}$ and a finite, surjective Galois morphism $\gamma : \tilde{X} → X$, étale in codimension one, such that $(\gamma^∗\mathcal{E})^{**}$ is locally free and flat, that is, $(\gamma^∗\mathcal{E})^{**}$ is given by a linear representation of $\pi_1(\tilde{X})$.

1.5. Characterisation of torus quotients: varieties admitting polarised endomorphisms. In [NZ10], Nakayama and Zhang study the structure of varieties admitting polarised endomorphisms —the notion of “polarised endomorphism” is recalled in Remark 1.21 below. They conjecture in [NZ10, Conj. 1.2] that any variety of this kind is either uniruled or covered by an Abelian variety, with a covering map that is étale in codimension one. The conjecture has been shown in special cases, for instance in dimensions less than four. As an immediate application of Theorem 1.1, we show that it holds in full generality.
Theorem 1.20 (Varieties with polarised endomorphisms, cf. [NZ10, Conj. 1.2]). Let $X$ be a normal, complex, projective variety admitting a non-trivial polarised endomorphism. Assume that $X$ is not uniruled. Then, there exists an Abelian variety $A$ and finite, surjective morphism $A \to X$ that is étale in codimension one.

Remark 1.21 (Polarised endomorphism). Let $X$ be a normal, complex, projective variety. An endomorphism $f : X \to X$ is called polarised if there exists an ample Cartier divisor $H$ and a positive number $q \in \mathbb{N}^+$ such that $f^*(H) \sim q \cdot H$.

Theorem 1.20 has strong implications for the structure of varieties with endomorphisms. These are discussed in Section 13.2 on page 33.

1.6. Outline of the paper. Section 2 formulates and discusses the main results. Before proving these in Part II, we have gathered in Part I a number of results and facts which will later be used in the proofs. While many of the facts discussed in Sections 3 and 5 are known to experts (though hard to find in the literature), the material of Section 4 is new to the best of our knowledge and might be of independent interest.

The main result is proven in Part II. Starting with a recent result of Chenyang Xu, [Xu14], Sections 6–8 discuss and prove variants of the main result in a number of special cases, with increasing order of complexity. With these preparations at hand, the main results are shown in Section 9. Proofs of the applications are established in Part III. The main results are somewhat delicate and might invite misinterpretation unless care is taken. We have therefore chosen to include a number of examples in Section 14, showing that the assumptions are strictly necessary and that several “obvious” generalisations or reformulations are wrong. The concluding appendices prove uniqueness and equivariance in “Zariski’s Main Theorem in the form of Grothendieck” and invariance of branch loci under Galois closure.

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2. Main results

The introductory section presented a simplified version of our main result. The following more general Theorem 2.1 differs from the simplified version in two important aspects, which we briefly discuss to prepare the reader for the somewhat technical formulation. Figure 2.1 on the next page illustrates the setup schematically.

Aspect 1: Finite vs. quasi-finite morphisms. For simplicity, we have assumed in Theorem 1.1 that all morphisms $\gamma_i$ are finite. However, there are settings where this assumption is too restrictive and where one would like to consider quasi-finite rather than finite morphisms. The proofs of Theorems 1.8 and 1.9 provide examples for this more general setup.

To support this kind of application, Theorem 2.1 allows the morphisms $\gamma_i$ to be quasi-finite, as long as each $Y_i$ is Galois over a suitable open subset $X_i \subseteq X$. 

The figure shows the setup for the main result, Theorem 2.1, schematically. The morphisms $\eta_i$ are Galois covers over a sequence $X \supseteq X_0 \supseteq X_1 \supseteq \cdots$ is increasingly small open subsets of $X$. The morphisms $\gamma_i$ between these covering spaces are étale away from the preimages of $S$. In Theorem 2.1, the set $S$ is of codimension two or more. This aspect is difficult to illustrate and therefore not properly shown in the figure.

**Figure 2.1. Setup for the main result**

For this reason, Theorem 2.1 introduces a descending chain of dense open subsets $X \supseteq X_0 \supseteq X_1 \supseteq \cdots$ and replaces Sequence (1.1.1) by the more complicated Diagram (2.1.1).

**Aspect 2: Specification of the branch locus.** The morphisms $\gamma_i$ of Theorem 1.1 are required to be étale in codimension one, and therefore branch only over the singular locus of the target varieties $Y_i$. However, there are settings where it is advantageous to specify the potential branch locus in a more restrictive manner, requiring that the $\gamma_i$ branch only over a given set $S$.

**Theorem 2.1** (Main result). Let $X$ be a normal, complex, quasi-projective variety of dimension $\dim X \geq 2$. Assume that there exists a $\mathbb{Q}$-Weil divisor $\Delta$ such that $(X, \Delta)$ is klt. Suppose further that we are given a descending chain of dense open subsets $X \supseteq X_0 \supseteq X_1 \supseteq \cdots$, a closed reduced subscheme $S \subset X$ of codimension $\operatorname{codim}_X S \geq 2$, and a commutative diagram of morphisms between normal varieties,

$$
\begin{array}{ccccccc}
Y_0 & \overset{\gamma_1}{\longrightarrow} & Y_1 & \overset{\gamma_2}{\longrightarrow} & Y_2 & \overset{\gamma_3}{\longrightarrow} & \cdots \\
\eta_0 & & \eta_1 & & \eta_2 & & \\
X & \overset{\iota_0}{\longrightarrow} & X_0 & \overset{\iota_1}{\longrightarrow} & X_1 & \overset{\iota_2}{\longrightarrow} & \cdots,
\end{array}
$$

where the following holds for all indices $i \in \mathbb{N}$.

(2.1.2) The morphisms $\iota_i$ are the inclusion maps.
(2.1.3) The morphisms $\gamma_i$ are quasi-finite, dominant and étale away from the reduced preimage set $S_i := \eta_i^{-1}(S)_{\text{red}}$.
(2.1.4) The morphisms $\eta_i$ are finite, surjective, Galois, and étale away from $S_i$.

Then, all but finitely many of the morphisms $\gamma_i$ are étale.

**Remark 2.2** (Interdependence of conditions in Theorem 2.1). Conditions (2.1.3) and (2.1.4) are not independent. The assumption that the morphisms $\gamma_i$ are étale away from $S_i$ follows trivially from Condition (2.1.4) and [Mil80, Cor. 3.6]. We have chosen to include the redundancy for convenience of reference and notation.

**Part I. Preparations**

3. Notation, conventions and facts used in the proof
3.1. **Global conventions.** Throughout this paper, all schemes, varieties and morphisms will be defined over the complex number field. We follow the notation and conventions of Hartshorne’s book [Har77]. In particular, varieties are always assumed to be irreducible. For complex spaces and holomorphic maps, we follow [GR84] whenever possible. For all notations around Mori theory, such as klt spaces and klt pairs, we refer the reader to [KM98]. The empty set has dimension minus infinity, \( \dim \emptyset = -\infty \).

3.2. **Notation.** In the course of the proofs, we frequently need to switch between the Zariski– and the Euclidean topology. We will consistently use the following notation.

**Notation 3.1 (Complex space associated with a variety).** Given a variety or projective scheme \( X \), denote by \( X^{an} \) the associated complex space, equipped with the Euclidean topology. If \( f : X \to Y \) is any morphism of varieties or schemes, denote the induced map of complex spaces by \( f^{an} : X^{an} \to Y^{an} \). If \( F \) is any coherent sheaf of \( \mathcal{O}_X \)-modules, denote the associated coherent analytic sheaf of \( \mathcal{O}_{X^{an}} \)-modules by \( F^{an} \).

**Definition 3.2 (Covers and covering maps, compare with Definition 3.10).** Let \( Y \) be a normal, connected variety or complex space. A cover of \( Y \) is a surjective, finite morphism \( f : X \to Y \), where \( X \) is again normal and connected. The map \( f \) is called covering map. Two covers \( f : X \to Y \) and \( f' : X' \to Y \) are called isomorphic if there exists an isomorphism (resp. biholomorphic map) \( \psi : X \to X' \) such that \( f' \circ \psi = f \). In other words, the following diagram should commute:

\[
\begin{array}{ccc}
X & \xrightarrow{\psi} & X' \\
\downarrow f & \cong & \downarrow f' \\
Y & \xrightarrow{} & Y.
\end{array}
\]

Note that in Definition 3.2 we do not assume \( f \) to be étale.

**Definition 3.3 (Quasi-étale morphisms).** A morphism \( f : X \to Y \) between normal varieties is called quasi-étale if \( f \) is of relative dimension zero and étale in codimension one. In other words, \( f \) is quasi-étale if \( \dim X = \dim Y \) and if there exists a closed, subset \( Z \subseteq X \) of codimension \( \operatorname{codim} X \geq 2 \) such that \( f \mid_{X \setminus Z} : X \setminus Z \to Y \) is étale.

In analogy to the notion of quasi-finite morphism, [Har77, II Ex. 3.5], we make the following definition.

**Definition 3.4 (Quasi-finite holomorphic map).** A holomorphic map \( f : X \to Y \) between complex spaces is called quasi-finite if for any point \( y \in Y \), the preimage \( f^{-1}(y) \) is either empty or a finite set.

3.3. **Galois morphisms.** Galois morphisms appear prominently in the literature, but their precise definition is not consistent. We will use the following definition.

**Definition 3.5 (Galois morphism).** A covering map \( \gamma : X \to Y \) of varieties is called Galois if there exists a finite group \( G \subseteq \operatorname{Aut}(X) \) such that \( \gamma \) is isomorphic to the quotient map.

It is a standard, widely-used fact that any finite, surjective morphism between normal varieties can be enlarged to become Galois.

**Theorem 3.6 (Existence of Galois closure).** Let \( \gamma : X \to Y \) be a covering map of quasi-projective varieties. Then, there exists a normal, quasi-projective variety \( \tilde{X} \) and a finite, surjective morphism \( \tilde{\gamma} : \tilde{X} \to X \) such that the following holds.
(3.6.1) There exist finite groups $H \subseteq G$ such that the morphisms $\Gamma := \gamma \circ \tilde{\gamma}$ and $\tilde{\gamma}$ are Galois with group $G$ and $H$, respectively.

(3.6.2) If $\text{Branch}(\gamma)$ and $\text{Branch}(\Gamma)$ denote the branch loci, with their natural structure as reduced subschemes of $Y$, then $\text{Branch}(\gamma) = \text{Branch}(\Gamma)$.

The morphism $\Gamma : \tilde{X} \to Y$ is often called Galois closure of $\gamma$. A proof of Theorem 3.6 is given in Appendix B.

3.4. Zariski’s Main Theorem. Grothendieck observed in [Gro66] that Zariski’s Main Theorem can be reformulated by saying that any quasi-finite morphism decomposes as a composition of an open immersion and a finite map. If all varieties in question are normal, this decomposition is unique and well-behaved with respect to the action of the relative automorphism group.

Theorem 3.7 (Zariski’s Main Theorem in the equivariant setting). Let $a : V^o \to W$ be any quasi-finite morphism between quasi-projective varieties. Assume that $V^o$ is normal and not the empty set. Then there exists a normal, quasi-projective variety $V$ and a factorisation of $a$ into an open immersion $a : V^o \to V$ and a finite morphism $\beta : V \to W$. The factorisation is unique up to unique isomorphism and satisfies the following additional conditions with respect to group actions.

(3.7.1) If $G$ is any group that acts on $V^o$ and $W$ by algebraic morphisms, and if $a$ is equivariant with respect to these actions, then $G$ also acts on $V$ and the morphisms $a$ and $\beta$ are equivariant with respect to these actions.

(3.7.2) If $W^o := \text{Image}(a)$ is open in $W$ and $a : V^o \to W^o$ is Galois with group $G$, then $\beta : V \to W$ is Galois with group $G$.

Remark 3.8 (Algebraic group actions). If the group $G$ in (3.7.1) is algebraic and acts algebraically on $V^o$, then the induced action on $V$ will clearly also be algebraic.

In the affine setting, parts of Theorem 3.7 are found in [Dré04]. A full proof of Theorem 3.7 is given in Appendix A.

3.5. Topological triviality of algebraic morphisms. The proof of our main theorem relies on the fact that any morphism between complex spaces $f^{an} : X^{an} \to Y^{an}$ that comes from an algebraic morphism $f : X \to Y$ of complex schemes, has the structure of a topologically trivial fibre bundle, at least over a suitable Zariski-open, dense subset of the base. To be more precise, we will use the following result.

Proposition 3.9 (Topological triviality of morphisms between pointed varieties). Let $\phi : X \to B$ be a morphism between normal varieties. Let $S \subseteq X$ be a closed, reduced subscheme such that $\phi|_S : S \to B$ is finite and étale. Then, there exists a Zariski-open, dense subset $U \subseteq B$ with preimage $X_{\phi} \subseteq X$ such that the restriction of $\phi^{an}$ to $X^{an}_{\phi}$ is a topologically locally trivial fibre bundle of pointed spaces with their Euclidean topology.

More precisely, given any closed point $b \in U$ with fibres $X_b \subseteq X$, $S_b \subseteq S$, there exists a neighbourhood $V = V(b) \subseteq U^{an}$, open in the Euclidean topology, with preimage $X^{an}_b : = (\phi^{an})^{-1}(V)$, and a commutative diagram of continuous maps,

\[
\begin{array}{ccc}
X^{an}_b \times V & \xrightarrow{\phi^{an}_V} & X^{an}_V \\
\downarrow \text{projection} & & \downarrow \text{inclusion} \\
V & = & V^c \\
\end{array}
\]

such that $\phi^{an}_V^{-1}(S^{an}_b \cap X^{an}_V) = S^{an}_b \times V$. 

Proposition 3.9 is known to experts, but does not appear in the literature in the exact form given above. For the reader’s convenience, we sketch a brief proof, reducing Proposition 3.9 to a classical result of Verdier.

Sketch of proof. Except for the last equation, the statement of Proposition 3.9 is contained in the work of Verdier, [Ver76, Cor. 5.1], which also holds in the non-normal setting. Proposition 3.9 can easily be deduced by applying Verdier’s result to the non-normal, reducible, reduced complex scheme

$$X' := (X \times \{0\}) \cup (S \times \mathbb{A}^1) \subset X \times \mathbb{A}^1$$

and to the induced map $f' : X' \to B, (x, z) \mapsto f(x)$.

3.6. Algebraic topology. We will need the following elementary criterion for a topological covering space to be homeomorphic to a product. Its standard proof is left to the reader. Since the word cover is used with different meanings in algebraic topology and complex geometry, we clarify our definitions beforehand.

Definition 3.10 (Finite topological covering space, cf. [Hat02, Sect. 1.3]). Given any topological space $Y$, a finite topological covering space is a topological space $X$ together with a surjective, continuous map $\gamma : X \to Y$ such that there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of $Y$ with the property that for each $\alpha \in A$, the preimage $\gamma^{-1}(U_\alpha)$ is a finite disjoint union of open sets in $X$, each of which is mapped homeomorphically onto $U_\alpha$ by $\gamma$.

Remark 3.11 (Comparison with Definition 3.2). In contrast to Definition 3.2, we do not assume that covering spaces are connected. A morphism of varieties that is a cover in the sense of Definition 3.2 is a finite topological covering space if and only if it is étale.

Proposition 3.12 (Covering spaces of products). Consider a product $A \times B$ of two connected, locally path-connected topological spaces, where $B$ is assumed to be contractible. Let $\gamma : X \to A \times B$ be a finite topological covering space. Given any point $b \in B$, write $X_b := \gamma^{-1}(A \times \{b\})$. Then there exists a commutative diagram of continuous maps,

$$
\begin{array}{ccc}
X_b \times B & \cong & X \\
\downarrow \quad \gamma & & \downarrow \\
A \times B & \cong & A \times B.
\end{array}
$$

\[\text{homeomorphic}\]

3.7. Classification of holomorphic covering maps. It is a standard result of topology that covering spaces of a given topological space are classified by conjugacy classes of the fundamental group. Using fundamental results of Grauert, Remmert, and Stein this statement can be generalised to branched holomorphic coverings of a given normal complex space.

Proposition 3.13 (Classification of covering maps with prescribed branch locus). Let $X$ be a connected, normal complex space and $A \subseteq X$ a proper analytic subset. Then the following natural map is bijective:

$$
\begin{align*}
\left\{ \text{Isomorphism classes of holomorphic covering maps } f : Y \to X, \text{ where } f \text{ is locally biholom. away from } f^{-1}(A) \right\} & \to \left\{ \text{Finite-index subgroups of } \pi_1(X \setminus A) \text{ up to conjugation} \right\} \\
& \xrightarrow{f} \left( f|_{Y \setminus f^{-1}(A)} \right)_* \pi_1(Y \setminus f^{-1}(A)).
\end{align*}
$$
Remark 3.14 (Equivalence of categories). The map of Proposition 3.13 induces an equivalence of categories, but we will not need this stronger statement.

Proof of Proposition 3.13. The classification of finite connected topological covering spaces of $X \setminus A$, [Hat02, Thm. 1.38], and the fact that any such cover can be given a uniquely determined complex structure that makes the covering map holomorphic and locally biholomorphic, [DG94, §1.3], together establish a correspondence between isomorphism classes of locally biholomorphic covering maps $f^\circ : Y^\circ \to X \setminus A$ and conjugacy classes of finite-index subgroups of $\pi_1(X \setminus A)$, by sending $f^\circ$ to $(f^\circ)_* \pi_1(Y^\circ)$.

In order to prove our claim, it therefore suffices to show that any locally biholomorphic holomorphic covering map $f^\circ : Y^\circ \to X \setminus A$ can be uniquely extended to a holomorphic covering map $f : Y \to X$, where $Y$ is a connected normal complex space. This however is a special case of [DG94, Thm. 3.4]; for this, observe that the assumption made in [DG94, Thm. 3.4] on the analyticity of $A \cup B$ is automatically fulfilled in our setup, since the critical locus of the covering that we aim to extend is empty. □

Remark 3.15. Theorem [DG94, Thm. 3.4] is a modern version of results obtained by Grauert-Remmert and Stein in their fundamental papers [GR58, Ste56] dating back to the 1950’s.

Corollary 3.16 (Characterisation of biholomorphic maps). Consider a sequence of holomorphic covering maps, $W_2 \to W_1 \to V$. Let $A \subseteq V$ be any a proper analytic subset such that $\eta$ and $\eta \circ \gamma$ are locally biholomorphic away from $A$. Then $\gamma$ is biholomorphic if and only if the following two subgroups of $\pi_1(V \setminus A)$ agree up to conjugation,

$$\eta_* \pi_1(W_1 \setminus \eta^{-1}(A)) \text{ and } (\eta \circ \gamma)_* \pi_1(W_2 \setminus (\eta \circ \gamma)^{-1}(A)).$$

Proof. If $\gamma$ is biholomorphic, then $W_1 \setminus \eta^{-1}(A)$ and $W_2 \setminus (\eta \circ \gamma)^{-1}(A)$ are isomorphic over $V \setminus A$. This implies that the two subgroups $\eta_* \pi_1(W_1 \setminus \eta^{-1}(A))$ and $(\eta \circ \gamma)_* \pi_1(W_2 \setminus (\eta \circ \gamma)^{-1}(A))$ agree up to conjugation.

Now assume that $\eta_* \pi_1(W_1 \setminus \eta^{-1}(A))$ and $(\eta \circ \gamma)_* \pi_1(W_2 \setminus (\eta \circ \gamma)^{-1}(A))$ are conjugate. Then, Proposition 3.13 implies that the covering maps $\eta$ and $\eta \circ \gamma$ are isomorphic. In other words, there exists a biholomorphic map $\mu : W_1 \to W_2$ such that $(\eta \circ \gamma) \circ \mu = \eta$. For any point $p \in V \setminus A$, we therefore have $|\eta \circ \gamma)^{-1}(p)| = |\eta^{-1}(p)|$. Hence, the degree of $\gamma$ is equal to one. As $W_1$ is normal, the analytic version of Zariski’s Main Theorem implies that $\gamma$ is biholomorphic, as claimed. □

4. CHERN CLASSES ON SINGULAR VARIETIES

4.1. Intersection numbers on singular varieties. To prove the characterisation of torus quotients given in Theorem 1.16, we need to discuss intersection numbers of line bundles with Chern classes of reflexive sheaves on varieties with canonical singularities. For our purposes, it is actually not necessary to define Chern classes themselves. The literature discusses several competing notions of Chern classes on singular spaces, all of which are technically challenging, cf. [Mac74, Alu06]. For the reader’s convenience, we have chosen to include a short, self-contained presentation, restricting ourselves to the minimal material required for the proof.

Definition 4.1 (Resolution of a space and a coherent sheaf). Let $X$ be a normal variety and $\mathcal{E}$ a coherent sheaf of $O_X$-modules. A resolution of $(X, \mathcal{E})$ is a proper, birational and surjective morphism $\pi : \tilde{X} \to X$ such that the space $\tilde{X}$ is smooth, and such that the sheaf $\pi^*(\mathcal{E})/\text{tor}$ is locally free. If $\pi$ is isomorphic over the open set where $X$ is smooth and $\mathcal{E}/\text{tor}$ is locally free, we call $\pi$ a strong resolution of $(X, \mathcal{E})$. 

Remark 4.2. In the setup of Definition 4.1, the existence of a resolution of singularities combined with a classical result of Rossi, [Ros68, Thm. 3.5], shows that resolutions and strong resolutions of \((X, \mathcal{E})\) exist.

Definition 4.3 (Intersection of Chern class with Cartier divisors). Let \(X\) be a normal, \(n\)-dimensional, quasi-projective variety and \(\mathcal{E}\) be a coherent sheaf of \(\mathcal{O}_X\)-modules. Assume we are given a number \(i \in \mathbb{N}^+\) such that \(X\) is smooth in codimension \(i\) and such that \(\mathcal{E}\) is locally free in codimension \(i\). Given any resolution morphism \(\pi : \tilde{X} \to X\) of \((X, \mathcal{E})\) and any set of Cartier divisors \(L_1, \ldots, L_{n-i}\) on \(X\), we use the following shorthand notation
\[
c_i(\mathcal{E}) \cdot L_1 \cdots L_{n-i} := c_i(\mathcal{F}) \cdot (\pi^* L_1) \cdots (\pi^* L_{n-i}) \in \mathbb{Z}.
\]
where \(\mathcal{F} := \pi^* \mathcal{E}/\text{tor}\), and where \(c_i(\mathcal{F})\) denotes the classical Chern class of the locally free sheaf \(\mathcal{F}\) on the smooth variety \(\tilde{X}\). More generally, if \(P(y_1, \ldots, y_n)\) is a homogeneous polynomial of degree \(i\) for weighted variables \(y_j\) with \(\deg y_j = j\), we will use the shorthand notation
\[
P(c_1(\mathcal{E}), \ldots, c_n(\mathcal{E})) \cdot L_1 \cdots L_{n-i} := P(c_1(\mathcal{F}), \ldots, c_n(\mathcal{F})) \cdot (\pi^* L_1) \cdots (\pi^* L_{n-i}).
\]

Remark 4.4 (Independence of resolution). In the setting of Definition 4.3, given another smooth variety \(\tilde{X}\) and a diagram
\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\gamma = \pi \circ h} & \tilde{X} \\
\eta, \text{ proper, surjective, birational} & & \pi, \text{ proper, surjective, birational} \\
& & X,
\end{array}
\]
then \((\gamma^* \mathcal{E})/\text{tor} \cong \eta^*(\pi^* \mathcal{E}/\text{tor})\). It follows immediately from the projection formula for Chern classes, [Ful98, Thm. 3.2 on p. 50], that
\[
c_i((\pi^* \mathcal{E})/\text{tor}) \cdot (\pi^* L_1) \cdots (\pi^* L_{n-i}) = c_i((\gamma^* \mathcal{E})/\text{tor}) \cdot (\gamma^* L_1) \cdots (\gamma^* L_{n-i}).
\]
The same holds for the more complicated expressions involving the polynomial \(P\). Since any two resolution maps are dominated by a common third, the numbers defined in Definition 4.3 are independent of the choice of the resolution map.

Remark 4.5 (Alternative computation for semiample divisors). In the setup of Definition 4.3, assume we are given numbers \(m_1, \ldots, m_{n-i} \in \mathbb{N}^+\) and basepoint-free linear systems \(B_j \subseteq |m_j L_j|\), for all \(1 \leq j \leq n-i\). Choose general elements \(\Delta_j \in B_j\) and consider the intersection
\[
S := \Delta_1 \cap \cdots \cap \Delta_{n-i}.
\]
Observe that \(S\) is smooth, entirely contained in the smooth locus \(X_{\text{reg}} \subseteq X\), that \(\mathcal{E}\) is locally free along \(S\), and that
\[
P(c_1(\mathcal{E}), \ldots, c_n(\mathcal{E})) \cdot L_1 \cdots L_{n-i} = \frac{P(c_1(\mathcal{E}|_S), \ldots, c_n(\mathcal{E}|_S))}{m_1 \cdots m_{n-i}} \in \mathbb{Z},
\]
where the left hand side is given as in Definition 4.3 above.

The following proposition is a fairly direct consequence of Remark 4.5 and the projection formula.

Proposition 4.6 (Behaviour under quasi-é etale covers). In the setup of Definition 4.3, if \(\gamma : \tilde{X} \to X\) is any quasi-é etale cover, then \(\tilde{X}\) is smooth in codimension \(i\), the sheaf \(\gamma^* \mathcal{E}\) is locally free in codimension \(i\), and
\[
(\deg \gamma) \cdot P(c_1(\mathcal{E}), \ldots, c_n(\mathcal{E})) \cdot L_1 \cdots L_{n-i} = P(c_1(\gamma^* \mathcal{E}), \ldots, c_n(\gamma^* \mathcal{E})) \cdot (\gamma^* L_1) \cdots (\gamma^* L_{n-i}).
\]
Proof. The statements about smoothness of $\tilde{X}$ and about local freeness of $\gamma^*\mathcal{E}$ follow from purity of the branch locus.

Since any Cartier divisor is linearly equivalent to the difference of two very ample divisors, we can assume without loss of generality that the divisors $L_1, \ldots, L_{n-i}$ are very ample and general in basepoint-free linear systems. The intersection $\mathcal{S} := L_1 \cap \cdots \cap L_{n-i}$ is then smooth, entirely contained in $X_{\text{reg}}$, and the morphism $\gamma$ is étale near $\mathcal{S}$. Writing $\tilde{\mathcal{S}} := \gamma^{-1}(\mathcal{S}) = \gamma^*L_1 \cap \cdots \cap \gamma^*L_{n-i}$, observe that the alternative computation of Remark 4.5 applies to both $\mathcal{S}$ and $\tilde{\mathcal{S}}$, and yields the following.

\[
P(c_1(\gamma^*\mathcal{E}), \ldots, c_n(\gamma^*\mathcal{E})) \cdot (\gamma^*L_1) \cdots (\gamma^*L_{n-i}) = P(c_1(\mathcal{E}|\mathcal{S}), \ldots, c_n(\mathcal{E}|\mathcal{S})) \quad \text{Remark 4.5 for } \tilde{X}
\]

\[
= \deg \gamma \cdot P(c_1(\mathcal{E}|\mathcal{S}), \ldots, c_n(\mathcal{E}|\mathcal{S})) \quad \text{Projection Formula}
\]

\[
= \deg \gamma \cdot P(c_1(\mathcal{E}), \ldots, c_1(\mathcal{E})) \cdot L_1 \cdots L_{n-i} \quad \text{Remark 4.5 for } X.
\]

This finishes the proof of Proposition 4.6. \qed

4.2. Numerically trivial Chern classes. Theorem 1.16 discusses varieties $X$ where the intersection numbers $c_2(\mathcal{F}_X) \cdot H_1 \cdots H_{n-2}$ vanish for certain ample divisors $H_1, \ldots, H_{n-2}$. We will see in Proposition 4.8 that this sometimes implies that all intersection numbers with arbitrary divisors vanish. The following definition is relevant in the discussion.

Definition 4.7 (Numerically trivial Chern class). In the setting of Definition 4.3, we say that $c_i(\mathcal{E})$ is numerically trivial, if $c_i(\mathcal{E}) \cdot L_1 \cdots L_{n-i} = 0$ for all Cartier divisors $L_1, \ldots, L_{n-i}$ on $X$.

Proposition 4.8 (Criterion for numerical triviality of $c_2(\mathcal{F}_X)$). Let $X$ be a normal, $n$-dimensional, projective variety that is smooth in codimension two. Assume that the canonical divisor $K_X$ is $\mathbb{Q}$-Cartier and nef. Then, $c_2(\mathcal{F}_X)$ is numerically trivial if and only if there exist ample Cartier divisors $H_1, \ldots, H_{n-2}$ on $X$ such that

\[(4.8.1) \quad c_2(\mathcal{F}_X) \cdot H_1 \cdots H_{n-2} = 0.
\]

Before proving Proposition 4.8 in Section 4.3 below, we note the following immediate corollary of Propositions 4.6 and 4.8.

Corollary 4.9 (Behaviour of numerically trivial $c_2(\mathcal{F}_X)$ under coverings). Let $X$ be a normal, $n$-dimensional, projective variety that is smooth in codimension two. Assume that the canonical divisor $K_X$ is $\mathbb{Q}$-Cartier and nef. If $\gamma : X' \to X$ is any quasi-étale cover, then $X'$ is smooth in codimension two, and $K_{X'}$ is $\mathbb{Q}$-Cartier and nef. Moreover, $c_2(\mathcal{F}_X)$ is numerically trivial if and only if $c_2(\mathcal{F}_{X'})$ is numerically trivial. \qed

4.3. Proof of Proposition 4.8. The proof of Proposition 4.8 spans the current Section 4.3. For the convenience of the reader, we have subdivided the proof of into several relatively independent steps.

Step 1: Setup of notation. Let $H_1, \ldots, H_{n-2}$ be ample Cartier divisors on $X$ such that (4.8.1) holds. Let further Cartier divisors $L_1, \ldots, L_{n-2}$ be given, and let $\pi : \tilde{X} \to X$ be a strong resolution of singularities. To prove Proposition 4.8, we need to show that

\[(4.9.1) \quad c_2(\mathcal{F}_X) \cdot L_1 \cdots L_{n-2} = 0.
\]

Since each of the $L_i$ can be written as a difference of very ample divisors, it suffices to show the vanishing under the additional assumption that each of the $L_i$ is ample. Replacing the $H_i$ with sufficiently high powers, we can even assume without loss of generality that the following holds.
Step 2: Miyaoka semipositivity. In order to prove (4.9.1) we shall make use of Miyaoka’s Semipositivity Theorem. In our setup, it asserts that the intersection numbers of $c_2(\mathcal{T}_X)$ with nef Cartier divisors on $X$ are never negative.

Claim 4.11 (Miyaoka semipositivity). Assumptions as above. If $A_1, \ldots, A_{n-2}$ are ample Cartier divisors on $X$, then $c_2(\mathcal{T}_X) \cdot A_1 \cdots A_{n-2} \geq 0$.

Proof. Let $\pi : \tilde{X} \to X$ be a strong resolution of singularities and $S \subset X$ a complete intersection surface, as introduced in Remark 4.5. Observe that $S$ avoids the singularities of $X$, and that the strong resolution map $\pi$ is isomorphic along $S$. Since $\pi^*(\mathcal{T}_X)$ and $\mathcal{T}_X$ differ only along the $\pi$-exceptional set, Remark 4.5 shows that

$$c_2(\mathcal{T}_X) \cdot A_1 \cdots A_{n-2} = c_2(\mathcal{T}_{\tilde{X}}) \cdot (\pi^* A_1) \cdots (\pi^* A_{n-1})$$

Using the assumption that the canonical divisor $K_X$ is $\mathbb{Q}$-Cartier and nef, Miyaoka shows in [Miy87, Thm. 6.6] that the right hand side is always non-negative. □

As a first corollary of Claim 4.11 we obtain the following technical result, which shows vanishing of a larger class of intersection numbers.

Consequence 4.12. Assumptions as above. For all numbers $1 \leq k \leq n-2$, we have the following vanishing of numbers,

$$(4.12.1) \quad c_2(\mathcal{T}_X) \cdot (H_1 + L_1) \cdots (H_k + L_k) \cdot H_{k+1} \cdots H_{n-2} = 0.$$

Proof. We prove Consequence 4.12 by induction on $k$. For $k = 0$, Equation (4.12.1) equals Equation (4.8.1), which holds by assumption. For the inductive step, assume that Equation (4.12.1) holds for a given number $k$. We need to show that it holds for $k + 1$. To this end, consider the following two computations,

$$(4.12.2) \quad 0 \leq c_2(\mathcal{T}_X) \cdot (H_1 + L_1) \cdots (H_k + L_k) \cdot (H_{k+1} \pm L_{k+1}) \cdot H_{k+2} \cdots H_{n-2}$$

$$(4.12.3) \quad = \pm c_2(\mathcal{T}_X) \cdot (H_1 + L_1) \cdots (H_k + L_k) \cdot H_{k+1} \cdots H_n \cdot H_{n-2}.$$

The Inequalities (4.12.2) hold by Claim 4.11, because the two bundles $H_{k+1} \pm L_{k+1}$ are both ample by Assumption 4.10. The Equalities (4.12.3) hold by induction hypothesis. Together, the inequalities show that the two numbers computed must both be zero. This finishes the proof of Consequence 4.12. □

Step 3: End of proof. To prove Equation (4.9.1) and thereby finish the proof of Proposition 4.8, apply Consequence 4.12 for $k = n-2$. We obtain the equation

$$(4.12.4) \quad c_2(\mathcal{T}_X) \cdot (H_1 + L_1) \cdots (H_{n-2} + L_{n-2}) = 0.$$

Multiplying out, we write this number as a sum of the form

$$0 = \sum_{j} c_2(\mathcal{T}_X) \cdot A_{1,j} \cdots A_{n-2,j} \quad \text{where } A_{i,j} \in \{H_i, L_i\} \text{ for all indices } i.$$

Recalling that the divisors $H_i$ and $L_i$ are ample, Claim 4.11 thus asserts that none of the summands is negative. Summing up to zero, we obtain that all summands must actually be zero. To finish the proof of Proposition 4.8, observe that $c_2(\mathcal{T}_X) \cdot L_1 \cdots L_{n-2}$ is one of the summands involved. □
5. Bertini-type theorems for sheaves and their moduli

The proofs of our main results use the fact that the restriction of torsion-free or reflexive sheaves to general hyperplanes is again torsion-free or reflexive, respectively, even if the underlying space is badly singular. Moreover, we will use that two reflexive sheaves are isomorphic if and only if their restrictions to a general hyperplane of sufficiently high degree agree. The present section is devoted to the proof of these auxiliary results.

5.1. Restrictions of torsion-free and reflexive sheaves. We begin with a discussion of Bertini-type theorems for torsion-free and reflexive sheaves. In the projective, non-singular setting, similar results appear in [HL10, Sect. 1.1].

Proposition 5.1 (Bertini-type theorem for torsion-freeness). Let \( X \) be a quasi-projective variety of dimension greater than or equal to two and \( \mathcal{F} \) a torsion-free, coherent sheaf of \( \mathcal{O}_X \)-modules. If \( H \subset X \) is a general element of a basepoint-free linear system, then the restriction \( \mathcal{F}|_H = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_H \) is torsion-free as a sheaf of \( \mathcal{O}_H \)-modules.

Proof. As a first step, recall the standard fact that there exists an embedding of \( \mathcal{F} \) into a locally free sheaf \( \mathcal{E} \). For a quick proof, observe that the natural map of torsion-free sheaves \( \mathcal{F} \to \mathcal{F}^{**} \) from \( \mathcal{F} \) into its double dual is generically injective, hence injective. The fact that the reflexive sheaf \( \mathcal{F}^{**} \) can be embedded into a locally free sheaf is shown for example in [Har80, Prop. 1.1].

Next, observe that the statement of Proposition 5.1 is local on \( X \). We can thus assume without loss of generality that \( \mathcal{E} \cong \mathcal{O}_X^{\oplus n} \), that \( X \) is affine, say \( X = \text{Spec} \ R \) for an integral domain \( R \), and that the hyperplane \( H \subset X \) is given as the zero set of a single function \( h \in R \). As \( H \) is general and the linear system is base-point-free, we may apply [Fle77, Satz 5.4 & Satz 5.5] to infer that \( H \) is reduced and irreducible; in other words, \( R/(h) \) is an integral domain. The sheaf \( \mathcal{F} \) is then the sheafification of a submodule \( F \subset R^{\oplus n} \), and the restricted sheaf \( \mathcal{F}|_H \) is the sheafification of the \( R/(h) \)-module \( F_H := F \otimes_R R/(h) \). In order to analyse this module, tensorise the standard exact sequence

\[
0 \to F \to R^{\oplus n} \to R^{\oplus n}/F \to 0
\]

with the \( R/(h) \) and obtain the following exact sequence of \( R/(h) \)-modules,

\[
\ldots \to \text{Tor}_1^R \left( \frac{R}{(h)}, \frac{R^{\oplus n}}{F} \right) \to F \otimes_R \frac{R}{(h)} \to \frac{R^{\oplus n}}{F} \otimes_R \frac{R}{(h)} \to \ldots
\]

We aim to show that \( T = \{0\} \), thus representing the \( R/(h) \)-module \( F_H \) as a submodule of \( (R/(h))^{\oplus n} \), which is torsion-free as \( R/(h) \) is an integral domain. To this end, recall from [Wei94, Sect. 3.1] that

\[
(5.1.1) \quad T = \left\{ f \in \frac{R^{\oplus n}}{F} \mid h \cdot f = 0 \in \frac{R^{\oplus n}}{F} \right\}.
\]

By general choice, \( h \) is not contained in any of the (finitely many) associated primes of \( R^{\oplus n}/F \). Consequently, the right hand side of (5.1.1) is zero. \( \square \)

Proposition 5.2 (Bertini-type theorem for reflexivity). Let \( X \) be a quasi-projective variety of dimension greater than or equal to two and \( \mathcal{F} \) a reflexive, coherent sheaf of \( \mathcal{O}_X \)-modules. If \( H \subset X \) is a general element of a basepoint-free linear system, then the restriction \( \mathcal{F}|_H = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_H \) is reflexive as a sheaf of \( \mathcal{O}_H \)-modules.
Proof. Recall from [Har80, Prop. 1.1] that $\mathcal{F}$ is reflexive if and only if can be included into an exact sequence $0 \to \mathcal{F} \to \mathcal{E} \xrightarrow{h} \mathcal{D} \to 0$ where $\mathcal{E}$ is locally free and $\mathcal{D}$ is torsion-free. Restricting to $H$, we obtain a sequence

$$
(5.2.1) \quad \mathcal{F}|_H \xrightarrow{a|_H} \mathcal{E}|_H \xrightarrow{b|_H} \mathcal{D}|_H \to 0.
$$

The sheaf $\mathcal{E}|_H$ is clearly locally free. Proposition 5.1 asserts that the remaining sheaves $\mathcal{F}|_H$ and $\mathcal{D}|_H$ of Sequence (5.2.1) are at least torsion-free. Using [Har80, Prop. 1.1] again, reflexivity of $\mathcal{F}|_H$ will follow as soon as we show that $a|_H$ is injective. To this end, note that by general choice, the hyperplane $H$ will intersect the open set on $X$ where $\mathcal{F}$ and $\mathcal{D}$ are both locally free. It follows that the torsion-free sheaf ker$(a|_H) \subset \mathcal{F}|_H$ is zero at the general point of $X$, hence zero. □

5.2. Restrictions and isomorphism classes. The following proposition shows that isomorphism of two sheaves can often be checked after restricting to suitable hyperplanes; see [AY08, Thm. 2.2] in case $X$ is a projective space and $H$ a hyperplane. We will later see that Proposition 5.3 can be applied inductively, for bounded families of locally sheaves if the hyperplanes in question are of sufficiently high degrees.

Proposition 5.3 (Bertini-type theorem for isom. classes of reflexive sheaves). Let $X$ be a normal, projective variety of dimension $\dim X \geq 2$ and $\mathcal{E}$, $\mathcal{F}$ two coherent, reflexive sheaves of $O_X$-modules. Let $H \subset X$ be an irreducible, reduced, ample Cartier divisor on $X$ such that the following holds.

(5.3.1) The variety $H$ is normal.

(5.3.2) Setting $\mathcal{H} := \mathcal{Hom}(\mathcal{E}, \mathcal{F})$, the restricted sheaves $\mathcal{E}|_H$, $\mathcal{F}|_H$ and $\mathcal{H}|_H$ are reflexive.

(5.3.3) Let $X' \subseteq X$ be the minimal closed set such that $\mathcal{E}$ and $\mathcal{F}$ are locally free outside of $X'$. Setting $H' := X' \cap H$, we have codim$_H H' \geq 2$.

(5.3.4) The cohomology group $H^1(X, \mathcal{F}_H \otimes \mathcal{H})$ vanishes.

Then, $\mathcal{E}$ is isomorphic to $\mathcal{F}$ if and only if the sheaves $\mathcal{E}|_H$ and $\mathcal{F}|_H$ are isomorphic.

A proof is given in Section 5.3 on the facing page. The following is a consequence.

Proposition 5.4 (Bertini-type theorem for isom. classes of reflexive sheaves). Let $X$ be a normal, projective variety of dimension $\dim X \geq 2$ and $\mathcal{E}$, $\mathcal{F}$ two coherent, reflexive sheaves of $O_X$-modules. If $L \in \text{Pic}(X)$ is ample, then there exists a number $M \in \mathbb{N}$ and for any $m \geq M$ a dense open subset $V_m \subseteq |L|^m$ such that all hyperplanes $H \in V_m$ are reduced and normal, and such that $\mathcal{E} \cong \mathcal{F}$ if and only if $\mathcal{E}|_H \cong \mathcal{F}|_H$.

Proof. Consider the sheaf $\mathcal{H} := \mathcal{Hom}(\mathcal{E}, \mathcal{F})$. Since $\mathcal{F}$ is assumed to be reflexive, so is $\mathcal{H}$. We claim that

$$
(5.4.1) \quad H^1(X, (L^*)^m \otimes \mathcal{H}) = 0 \quad \text{for all sufficiently large } m \gg 0.
$$

If $X$ was smooth and $\mathcal{H}$ was locally free, this could be concluded using Serre duality and Serre vanishing. In our setup, since $X$ is normal and $\mathcal{H}$ is reflexive, $\mathcal{H}$ has depth $\geq 2$ at every point of $X$, [Har80, Prop. 1.3]. Equation (5.4.1) therefore still holds true in our context, as shown in [Gro68, Exp. XII, Prop. 1.5]. Choose one $M \in \mathbb{N}$ such that Equation (5.4.1) holds for all $m \geq M$, and such that the linear systems $|L|^m$ are basepoint-free.

To prepare for the construction of $V_m$, recall from [Har80, Cor. 1.4] that $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{H}$ are locally free outside of a closed subset $X' \subseteq X$ with codim$_X X' \geq 2$. Now, given any number $m \geq M$, let $V_m \subseteq |L|^m$ be the maximal open set such that the following holds for all hyperplanes $H \in V_m$. 


(5.4.2) The hyperplane $H$ is irreducible, reduced and normal.

(5.4.3) The intersection $H' := H \cap X'$ is small, that is, $\text{codim}_H H' \geq 2$.

(5.4.4) The restricted sheaves $\mathcal{E}|_H$, $\mathcal{F}|_H$ and $\mathcal{H}|_H$ are reflexive.

Seidenberg’s theorem, [BS95, Thm. 1.7.1], Bertini’s theorem, and Proposition 5.2 guarantee that none of the open sets $V_m$ is empty. Together with (5.4.1), Proposition 5.3 therefore applies to all $H \in V_m$.

Loosely speaking, Proposition 5.4 asserts that the sheaves $\mathcal{E}$ and $\mathcal{F}$ are isomorphic if and only if their restrictions to general hyperplanes of sufficiently high degrees are isomorphic. Applying this result inductively, we obtain the following.

**Corollary 5.5 (Iterated Bertini-type theorem for isom. classes).** In the setup of Proposition 5.4, assume we are given a positive number $k < \dim X$, a sufficiently increasing sequence $0 \ll m_1 \ll m_2 \ll \cdots \ll m_k$ and general elements $H_i \in |\mathcal{L}^{\otimes m_i}|$. Consider the complete intersection variety $S := H_1 \cap \cdots \cap H_k \subset X$. Then, $\mathcal{E} \cong \mathcal{F}$ if and only if $\mathcal{E}|_S \cong \mathcal{F}|_S$.

Let $X$ be a normal projective variety of dimension $\dim X \geq 2$, let $\mathcal{E}$ be a coherent, reflexive sheaf of $\mathcal{O}_X$-modules, and $F$ be a bounded family of locally free sheaves. If $H$ is any sufficiently ample Cartier divisor, then

$$H^1(X, \mathcal{F}_H \otimes \mathcal{H}(\mathcal{E}, \mathcal{F})) = 0 \quad \text{for all } \mathcal{F} \in F.$$ 

Note that $\mathcal{F}|_H$ is locally free for all $\mathcal{F} \in F$ and for all hyperplanes $H \in |\mathcal{L}^{\otimes m}|$. With the same arguments as above, Proposition 5.3 therefore also yields the following.

**Corollary 5.6 (Iterated Bertini-type theorem for bounded families).** Let $X$ be a normal, projective variety of dimension $\dim X \geq 2$. Let $\mathcal{E}$ be a coherent, reflexive sheaf of $\mathcal{O}_X$-modules, and let $F$ be a bounded family of locally free sheaves. Given an ample line bundle $\mathcal{L} \in \text{Pic}(X)$, a sufficiently increasing sequence $0 \ll m_1 \ll m_2 \ll \cdots \ll m_k$ and general elements $H_i \in |\mathcal{L}^{\otimes m_i}|$ with associated complete intersection variety $S := H_1 \cap \cdots \cap H_k$, then the following holds for all sheaves $\mathcal{F} \in F$. The sheaf $\mathcal{F}$ is isomorphic to $\mathcal{E}$ if and only if $\mathcal{F}|_S$ is isomorphic to $\mathcal{E}|_S$.

**5.3. Proof of Proposition 5.3.** The implication “$\mathcal{E} \cong \mathcal{F} \Rightarrow \mathcal{E}|_H \cong \mathcal{F}|_H$” is clear. For the opposite direction assume for the remainder of the proof that we are given isomorphism $\lambda_H : \mathcal{E}|_H \to \mathcal{F}|_H$. Using the vanishing (5.3.4), we aim to extend $\lambda_H$ to a morphism $\lambda : \mathcal{E} \to \mathcal{F}$, which will turn out to be isomorphic.

**Step 1: Restriction of $\mathcal{H}$.** We compare the restriction $\mathcal{H}|_H$ to the homomorphism sheaf associated with the restrictions, $\mathcal{H}_H := \mathcal{H}(\mathcal{E}|_H, \mathcal{F}|_H)$. Note that the latter has a non-trivial section given by $\lambda_H$. Item (5.3.2) implies that $\mathcal{H}|_H$ and $\mathcal{H}_H$ are reflexive sheaves of $\mathcal{O}_H$-modules. It is also clear that outside of the small set $H' \subset H$ the sheaves $\mathcal{H}|_H$ and $\mathcal{H}_H$ are both locally free and agree. But since two reflexive sheaves are isomorphic if and only if they are isomorphic on the complement of a small set, this immediately yields an isomorphism $\mathcal{H}|_H \cong \mathcal{H}_H$.

**Step 2: Extension of the morphism $\lambda_H$.** We will now show that there exists a morphism $\lambda : \mathcal{E} \to \mathcal{F}$ such that $\lambda_H = \lambda|_H$. To this end, consider the ideal sheaf sequence of the hypersurface $H$, that is, $0 \to \mathcal{F}_H \to \mathcal{O}_X \to \mathcal{E}_H \to 0$. Since $\mathcal{H}$ is torsion-free, this sequence stays exact when tensoring with $\mathcal{H}$, [Wei94, Sect. 3.1]. The associated long exact sequence cohomology sequence then reads

$$\cdots \to H^0(X, \mathcal{H}) \xrightarrow{\partial} H^0(X, \mathcal{H}|_H) \to H^1(X, \mathcal{F}_H \otimes \mathcal{H}) \to \cdots,$$

$=0$ by (5.3.4)
where \( r \) is the natural restriction map. We have seen in Step 1 that the middle term in (5.6.1) is isomorphic to \( H^0(X, \mathcal{K}_H) \), which contains \( \lambda_H \). Surjectivity of \( r \) therefore implies that \( \lambda_H \) can be extended. Choose one extension \( \lambda \) and fix this choice throughout. To finish the proof, we need to show that \( \lambda \) is an isomorphism.

**Step 3: Injectivity of \( \lambda \).** Both kernel and image of the morphism \( \lambda \) are subsheaves of the torsion-free sheaves \( \mathcal{E} \) and \( \mathcal{F} \), respectively, and therefore themselves torsion-free. Let \( X^o \subseteq X \) be the maximal open subset, where \( \ker(\lambda)_* \mathcal{E} \) and \( \im(\lambda) \) are locally free. Recalling from [Har80, Cor. 1.4] that \( \operatorname{codim}_X X \setminus X^o \geq 2 \), the intersection \( H^o := H \cap X^o \) is clearly non-empty, and the restricted sequence

\[
0 \to \ker(\lambda)|_{H^o} \to \mathcal{E}|_{H^o} \xrightarrow{\lambda|_{H^o}} \im(\lambda)|_{H^o} \to 0
\]

remains exact. Since \( \lambda|_{H^o} = \lambda_H|_{H^o} \) is isomorphic, the sheaf \( \ker(\lambda)|_{H^o} \) vanishes, showing that the morphism \( \lambda \) is injective in an open neighbourhood of \( H^o \). Since \( \mathcal{E} \) is reflexive, hence torsion-free, it follows that \( \lambda \) is injective.

**Step 4: Surjectivity of \( \lambda \), end of proof.** Restriction to \( H \) is a functor that is exact on the right. It follows that \( \coker(\lambda)|_{H^o} = \coker(\lambda|_{H^o}) = 0 \). In particular, we see that the support of \( \coker(\lambda) \) does not intersect the hyperplane \( H \). Since \( H \) is ample, it follows that the support of \( \coker(\lambda) \) is finite, and that the injective map \( \lambda \) is isomorphic away from this finite set. Using the standard fact that two reflexive sheaves are isomorphic if and only if they are isomorphic away from a small set, it follows that \( \lambda \) is an isomorphism. This finishes the proof of Proposition 5.4. □

**Part II. Étale covers of klt spaces**

6. **Proof in case where the branch locus is finite**

To prepare for the proof of Theorem 2.1, which is given in Section 9, we consider a number of special cases. This section discusses the case where the set \( S \) is finite. Under this assumption, Theorem 2.1 quickly follows from results of Chenyang Xu.

**Proposition 6.1** (Main theorem in case where the branch locus is finite). In the setting of Theorem 2.1, assume in addition that \( S \) is finite. Then, all but finitely many of the morphisms \( \gamma_i \) are étale.

6.1. **Proof of Proposition 6.1.** We may assume that the set \( S \) contains exactly one point, \( S = \{s\} \) and that \( s \in X_i \) for all \( i \in \mathbb{N} \). Next, choose a descending sequence of analytically open neighbourhoods of \( s \), say \((V_i)_{i \in \mathbb{N}}\), such that \( V_i \subseteq X_i^{an} \), such that the set \( V_0 \) is homeomorphic to the open cone over the link \( \operatorname{Link}(X, s) \), and such that the inclusion maps \( i_j : V_j \to V_0 \) are homotopy equivalences. In summary, we obtain isomorphisms

\[
\hat{\pi}_1(V_i \setminus S) \xrightarrow{(i_j)_*} \hat{\pi}_1 V_0 \setminus S \xrightarrow{\cong} \hat{\pi}_1 \operatorname{Link}(X, s).
\]

We refer the reader to [Dim92, Chapt. I, Sect. 5] for a discussion of links and of the conic structure of analytic sets, and to [Dim92, Thm. 5.1] for the specific result used here. The Local Description of Finite Maps, [GR84, Sect. 2.3.2], allows to assume in addition that the connected components of \( (\eta_i^{an})^{-1}(V_i) \) each contain exactly one point of the fibre \( (\eta_i^{an})^{-1}(s) \).

Choose a sequence of connected components \( W_i \subseteq (\eta_i^{an})^{-1}(V_i) \) such that \( \gamma_i^{an}(W_i) \subseteq W_i^{-1} \), and recall from the Open Mapping Theorem, [GR84, Chapt. 5.4.3], that \( \gamma_i^{an}(W_i) \) is open in \( W_i^{-1} \). Using the isomorphisms \((i_j)_*\) of (6.2.1), we obtain a descending sequence of subgroups,

\[
G_i := (\eta_i^{an})_* \hat{\pi}_1(W_i \setminus S_i) \subseteq \hat{\pi}_1(V_0 \setminus S).
\]
Remark 6.3. The morphisms \( \eta_i \) are Galois by assumption. If \( i \) is any index and \( W'_i \neq W_i \) is any other connected component of \( (\eta_i)^{an}(V_i) \), the groups \( G_i \) and \( G'_i := (\eta_i)^{an} \) are both normal, and in fact equal. The groups \( G_i \) do therefore not depend on the specific choice of the \( W_i \).

The descending sequence \( (G_i) \) stabilises because the algebraic local fundamental group \( \pi^l_{an}(X, s) := \pi_1(\text{Link}(X, s)) \) of the klt base space \( X \) is finite, [Xu14, Thm. 1]. This allows us to choose \( j \in \mathbb{N} \) such that \( G_i = G_j \) for all \( i > j \). Now, given one such \( i \), we aim to apply the characterisation of biholomorphic maps, Corollary 3.16, to the sequence of holomorphic covering maps,

\[
W_i \xrightarrow{\gamma_i^{an}} \gamma_i^{an}(W_i) \xrightarrow{\eta_{i-1}} V_i \subseteq V_0.
\]

These morphisms, as well the open immersion \( \gamma_i^{an}(W_i) \subseteq W_{i-1} \), give rise to a commutative diagram of group morphisms,

\[
\begin{array}{ccc}
\pi_1(W_i) & \xrightarrow{(\eta_i^{an})_*} & \pi_1(\gamma_i^{an}(W_i)) \xrightarrow{(\eta_{i-1}^{an})_*} \pi_1(V_0) \\
& \downarrow{(\text{inclusion})} & \downarrow{(\text{inclusion})} \\
\pi_1(W_{i-1}) & \xrightarrow{(\eta_{i-1}^{an})_*} & \pi_1(V_0).
\end{array}
\]

We obtain that

\[ G_i = (\eta_i^{an})_* \pi_1(\gamma_i^{an}(W_i)) = G_{i-1}, \]

and Corollary 3.16 thus implies that \( \gamma_i^{an}|_{W_i} : W_i \rightarrow \gamma_i^{an}(W_i) \) is biholomorphic. Remark 6.3 implies that the same holds for any other choice of preimage components \((W'_i)_{i \in \mathbb{N}}\). It follows that \( \gamma_i^{an} \) is locally biholomorphic. The morphism \( \gamma_i \) is hence étale, for any \( i > j \). This finishes the proof of Proposition 6.1. \( \square \)

7. Proof in case where the branch locus is relatively finite

7.1. Main result of this section. The following Proposition 7.1 is a major step towards the main result of this paper, Theorem 2.1. It handles the case where \( X \) admits a topologically locally trivial fibration such that \( S \) is étale over the base.

Proposition 7.1 (Main theorem in case where the branch locus is relatively finite). In the setting of Theorem 2.1, assume that there exists a smooth, quasi-projective variety \( B \) and a morphism \( \phi : X \rightarrow B \) that satisfies the following.

(7.1.1) Given any closed point \( b \in B \), the scheme-theoretic fibre \( X_b := \phi^{-1}(b) \) is reduced, normal, and none of its components is contained in the support of \( \Delta \). The cycle \( \Delta_b := \Delta \cap X_b \) is a \( \mathbb{Q} \)-Weil divisor on \( X_b \), and the pair \((X_b, \Delta_b)\) is klt.

(7.1.2) The restriction \( \phi|_S : S \rightarrow B \) is finite and étale. In particular, \( S \) is smooth.

(7.1.3) The induced morphism of complex spaces, \( \phi^{an} : (X^{an}, S^{an}) \rightarrow B^{an} \), is a topologically locally trivial fibre bundle of pointed spaces with their Euclidean topology. In other words, the conclusion of Proposition 3.9 holds without shrinking \( B \).

Then, all but finitely many of the morphisms \( \gamma_i \) are étale.

7.2. Proof of Proposition 7.1. Maintaining notation and assumptions of Theorem 2.1, we aim to apply Proposition 6.1 to a very general fibre of the morphism \( \phi \). Once it is shown that almost all of the morphisms \( \gamma_i \) are étale along preimages this one fibre, we will argue that they are in fact everywhere étale.
Step 1: Setup and notation. We may assume that none of the sets $(S_i)_{i \in \mathbb{N}}$ is empty. Using that finiteness is an open property, [Har77, II Ex. 3.7], and using Seidenberg’s theorem, [BS95, Thm. 1.7.1], we find for each index $i \in \mathbb{N}$ a dense, Zariski-open subset $B_i \subseteq B$ such that the following holds for any closed point $b \in B_i$:

(7.2.1) The scheme-theoretic fibre $Y_{i,b} := (\phi \circ \eta_i)^{-1}(b)$ is reduced, normal, and not empty.

(7.2.2) The restricted morphism $(\phi \circ \eta_i)|_{S_i} : S_i \to B$ is proper, and hence finite, over a neighbourhood of $b$. In particular, the scheme-theoretic fibre $Y_{i,b}$ intersects every irreducible component of the set $S_i$ non-trivially.

Step 2: Choice of a good base point. Since we work over the base field $\mathbb{C}$, the intersection of the countably many dense Zariski-open sets $(B_i)_{i \in \mathbb{N}}$ is not empty. We can therefore find (uncountably many) closed points $b \in B$ such that Properties (7.2.1) and (7.2.2) hold for all indices $i \in \mathbb{N}$. Choose one such $b$ and fix that choice for the remainder of the proof.

Notation 7.3. Given any variety or complex space $X$ that maps to $B$, we denote the associated fibre by $X_b$. If $f : X \to B$ is any morphism over $B$, we denote the associated morphism between fibres by $f_b : X_b \to Y_b$.

Step 3: Induced morphisms between the fibres. Diagram (2.1.1) of Theorem 2.1 induces the following diagram of fibres over $b$,

$$
\begin{array}{ccccccccc}
Y_{0,b} & \xrightarrow{\gamma_{1,b}} & Y_{1,b} & \xrightarrow{\gamma_{2,b}} & Y_{2,b} & \xrightarrow{\gamma_{3,b}} & Y_{3,b} & \xrightarrow{\gamma_{4,b}} & \cdots \\
\eta_{0,b} & & \eta_{1,b} & & \eta_{2,b} & & \eta_{3,b} & & \\
X_b & \xrightarrow{i_{0,b}} & X_{0,b} & \xrightarrow{i_{1,b}} & X_{1,b} & \xrightarrow{i_{2,b}} & X_{2,b} & \xrightarrow{i_{3,b}} & X_{3,b} & \xrightarrow{i_{4,b}} & \cdots \\
\end{array}
$$

(7.4.1)

Recalling from Assumption (2.1.3) of Theorem 2.1 that the morphisms $\gamma_i$ are étale away from the preimage set $S_i$, it follows from stability of étalité under base change, [Mil80, Prop. 3.3], that the restricted morphisms $\gamma_{i,b}$ are étale away from the finite set $S_{i,b}$. Together with Assumption (7.1.1) and Property (7.2.1), this allows to apply Proposition 6.1 to Diagram (7.4.1). We hence obtain that all but finitely many of the morphisms $\gamma_{i,b}$ are étale. Choose a number $M_b \gg 0$ such that the morphisms $\gamma_{i,b}$ are étale for all $i > M_b$.

Claim 7.5. Given any index $i > M_b$ and any point $s_i \in S_i$, the morphism $\gamma_i$ is étale at $s_i$. In particular, $\gamma_i$ is étale.

Claim 7.5 implies Proposition 7.1. To prove it, fix one specific index $i > M_b$ and one point $s_i \in S_i$ for the remainder of the proof. In the next two sections, we will construct a neighbourhood $V$ of $s := \eta_i(s_i)$ that intersects $X_b$ non-trivially and is homeomorphic to a product. We will end the proof with an application of Corollary 3.16, which describes the induced covers over $V$. The setup is sketched in Figure 7.1 on the next page.

Step 4: Preparation for the proof of Claim 7.5, choice of $B^0 \subseteq B^{an}$. Consider the image $\phi(s) \in B$. Since $B$ is smooth, we can find a contractible set $B^0 \subseteq B^{an}$, open in the analytic topology, which contains both $b$ and $\phi(s)$. Since topological fibre bundles are trivial over contractible sets, [Ste51, Cor. 11.6 on p. 53], there exists a homeomorphism

$$
h : B^0 \times X_b^{an} \to (\phi^{an})^{-1}(B^0)
$$

such that the following holds.

(7.5.1) Identifying $X_b^{an}$ and $\{b\} \times X_b^{an}$, we have $h|_{\{b\} \times X_b^{an}} = Id_{X_b^{an}}$. 

The image shows the open set \( V \subseteq X^a_i \) that is constructed in Steps 4 and 5 of the proof. The set \( V \) is relatively compact, contains the point \( s \), intersects the fibre \( X_b \), and is homeomorphic to the product \( B^o \times X^a_b \), where \( B^o \subseteq B \) is contractible.

**Figure 7.1.** Setup constructed in Steps 4 and 5 of the proof

(7.5.2) The map \( h \) is a trivialisation of pointed spaces; i.e., \( h^{-1}(S^a) = B^o \times S^a_b \).

Let \( S^o \subseteq S^a \cap (\phi^a)^{-1}(B^o) \) be the unique connected component that contains the point \( s \). Since \( B^o \) is contractible, \( S^o \) is a section of \( \phi \) over \( B^o \), and hence contains a unique point of intersection with \( X^a_b \), say \( t \in S^b_b \).

To simplify the situation, we will now shrink \( B^o \). To this end, observe that it follows from Assumption (7.2.2) and from the definition of \( s \) that both points \( s \) and \( t \) are contained in \( X^i \). We can therefore choose an injective path \( P : [0,1] \to S^o \cap X^a_i \) that connects \( s \) and \( t \). The composition \( \phi^a \circ P \) is likewise injective and connects \( \phi(s) \) and \( b \). Replacing \( B^o \) by a sufficiently small, contractible neighbourhood of \( \text{Image}(\phi^a \circ P) \), we can thus assume that the following property holds in addition.

**Assumption w.l.o.g. 7.6.** The topological closure \( S^o \subset X^a_i \) is compact and contained in \( X^a_i \).

**Step 5: Preparation for the proof of Claim 7.5,** choice of \( X^o_b \subseteq X^{an}_b \). Choose a small, analytically open neighbourhood of \( t \in X^a_b \), say \( X^o_b \subseteq X^a_b \). Using the homeomorphism \( h \), we obtain an open subset \( V := h(B^o \times X^o_b) \) of \( X^{an}_i \). We also consider the following preimage sets,

\[
W_i \subseteq Y^{an}_i \quad \ldots \quad \text{connected component of } (\eta^{an}_i)^{-1}(V) \text{ that contains } s_i \\
W_{i-1} \subseteq Y^{an}_{i-1} \quad \ldots \quad \text{connected component of } (\eta^{an}_{i-1})^{-1}(V) \text{ that contains } \gamma_i(s_i).
\]

As before, will shrink \( X^o_b \) to simplify the setting. To this end, choose a sufficiently small, relatively compact, contractible neighbourhood \( \Omega \) of \( \gamma_i(s_i) \in Y^{an}_{i-1,b} \). Since the finite, surjective morphism \( \eta^{an}_{i-1} : Y^{an}_{i-1,b} \to X^{an}_{i-1,b} \) is open, \([\text{GR84, Chapt. 5.4.3}]\), the image \( \gamma^{an}_{i-1} \) is an open subset of \( X^{an}_b \). Replacing \( X^o_b \) by this open subset, Assumption 7.6 allows to assume that the following properties hold in addition.

(7.7.1) The topological closure \( \overline{V} \) is contained in \( X^{an}_i \) and therefore also in \( X^{an}_{i-1} \).

(7.7.2) The space \( W_{i-1,b} \) is contractible.
Remark 7.8. Property (7.7.1) implies that the holomorphic map \( \gamma_{i,b}^{an} : W_{i,b} \to W_{i-1,b} \) is finite and surjective. By choice of \( i \), it is locally biholomorphic. Property (7.7.2) thus implies that it is biholomorphic.

Step 7: Proof of Claim 7.5, end of proof. We will now prove Claim 7.5 and thereby end the proof of Proposition 7.1. Once it is shown that the groups

\[(\eta_i^{an}), \pi_1(W_i \setminus S_i^{an}) \quad \text{and} \quad (\eta_i^{an}), \pi_1(W_{i-1} \setminus S_{i-1}^{an})\]

are conjugate in \( \pi_1(V) \), Claim 7.5 follows when we apply Corollary 3.16 to the composition of covering maps \( \gamma_i^{an} : W_i \to W_{i-1} \) and \( \eta_i^{an} : W_{i-1} \to V \). To this end, recall from (7.7.1) and from Proposition 3.12 that the spaces \( W_j \setminus S_j^{an} \) are homeomorphic to \( B^\circ \times (W_{j,b} \setminus S_{j,b}) \), for \( j \in \{i, i-1\} \). Since \( B^\circ \) is contractible, the natural inclusions \( W_{j,b} \setminus S_{j,b} \hookrightarrow W_j \setminus S_j^{an} \) are homotopy equivalences and therefore induce isomorphisms \( \pi_1(W_j \setminus S_j^{an}) \cong \pi_1(W_{j,b} \setminus S_{j,b}) \). To prove Claim 7.5, it will thus suffice to show that the groups

\[(\eta_i^{an}), \pi_1(W_{i,b} \setminus S_i^{an}) \quad \text{and} \quad (\eta_i^{an}), \pi_1(W_{i-1,b} \setminus S_{i-1}^{an})\]

are conjugate in \( \pi_1(X_b^\circ \setminus S_b) \). That, however, follows directly when one applies Remark 7.8 and Corollary 3.16 to the covering maps

\[W_{j,b} \xrightarrow{\gamma_{j,b}^{an}} W_{i-1,b} \xrightarrow{\eta_{i-1}^{an}} X_b^\circ.\]

This finishes the proof of Claim 7.5 and hence of Proposition 7.1. \( \square \)

8. Proof at General Points of the Branch Locus

As a next step in the proof of our main result, we show that the assertion of Theorem 2.1 holds generically, at all general points of the potential branch locus \( S \).

Proposition 8.1. In the setting of Theorem 2.1, if \( T \subseteq S \) is any irreducible component, then there exists a dense, open subscheme \( T^\circ \subseteq T \) such that all but finitely many of the morphisms \( \gamma_i \) are étale near the reduced preimage \( T_i^\circ := \eta_i^{-1}(T^\circ)_{\text{red}} \).

8.1. Proof of Proposition 8.1. If \( S \) is empty, there is nothing to show. If \( T \) is an isolated point of \( S \), the assertion has already been shown in Proposition 6.1. We will therefore make the following assumption throughout the proof.

Assumption w.l.o.g. 8.2. The variety \( T \) is positive-dimensional.

Step 1: Projection to the subvariety \( T \). We will analyse the behaviour of the morphisms \( \gamma_i \) near the generic point of \( T \) using the technique of “projection to a subvariety”, explained in detail in [GKKP11, Sect. 2.G]. More specifically, recall from [GKKP11, Prop. 2.26] that there exists a Zariski-open subset \( X^\circ \subseteq X \) such that \( T^\circ := T \cap X^\circ \) is not empty, that there exist normal, quasi-projective varieties \( \tilde{X} \) and \( B \), and morphisms

\[\begin{array}{ccc}
B & \xrightarrow{\phi} & \tilde{X} \\
\downarrow & & \downarrow \\
\text{finite, étale} & \text{finite, étale} & \text{finite, étale}
\end{array}\]

with the property that the restriction of \( \phi \) to any connected component of \( \tilde{T} := \psi^{-1}(T) \) is an isomorphism. In particular, \( \phi|_{\tilde{T}} : \tilde{T} \to B \) is étale.

Observation 8.3. Since \( \psi \) is finite, there exists a well-defined pull-back divisor on \( \tilde{X} \), say \( \Delta := \psi^*\Delta \). Since \( \psi \) is étale, the pair \( (\tilde{X}, \Delta) \) is klt, [KM98, Prop. 5.20].
Shrinking $B$, $\tilde{X}$ and $X^0$, using generic smoothness of $B$, Seidenberg’s theorem, [BS95, Thm. 1.7.1], the Bertini theorem for klt pairs, [KM98, Lem. 5.17] as well as the topological triviality of morphisms between varieties, Proposition 3.9, we can additionally assume the following.

**Assumption w.l.o.g.** 8.4. The following extra conditions hold.

(8.4.1) The intersection $X^0 \cap S$ is contained in $T$, hence equal to $T^0 \subseteq T$.
(8.4.2) The variety $B$ is smooth.
(8.4.3) For all $b \in B$, the fibre $\tilde{X}_b := \phi^{-1}(b)$ is reduced, normal, and none of its components is contained in the support of $\Delta$. The cycle $\tilde{\Delta}_b := \Delta \cap \tilde{X}_b$ is a $Q$-Weil divisor on $\tilde{X}_b$, and the pair $(\tilde{X}_b, \tilde{\Delta}_b)$ is klt.
(8.4.4) The induced morphism of complex spaces, $\phi^{an} : (\tilde{X}^{an}, \tilde{T}) \to B^{an}$ is a topologically locally trivial fibre bundle of pointed spaces with their Euclidean topology, in the sense of Proposition 3.9.

**Step 2: Fibred products and choice of components.** Taking fibred products with $\tilde{X}$, we obtain the following diagram,

\[
\begin{array}{ccccccccc}
Y_0 \times_X \tilde{X} & \gamma_1 \times \text{Id}_\tilde{X} & Y_1 \times_X \tilde{X} & \gamma_2 \times \text{Id}_\tilde{X} & \cdots \\
\eta_0 \times \text{Id}_\tilde{X} \downarrow & & \eta_1 \times \text{Id}_\tilde{X} \downarrow & & \eta_2 \times \text{Id}_\tilde{X} \downarrow \\
\tilde{X} & \gamma_0 \times \text{Id}_\tilde{X} & \tilde{X} & \gamma_1 \times \text{Id}_\tilde{X} & \cdots \\
\iota_0 \times \text{Id}_\tilde{X} \downarrow & & \iota_1 \times \text{Id}_\tilde{X} \downarrow & & \iota_2 \times \text{Id}_\tilde{X} \downarrow \\
X_0 \times_X \tilde{X} & X_1 \times_X \tilde{X} & X_2 \times_X \tilde{X} & & & & & \\
\end{array}
\]

Its properties are summarised as follows.

(8.5.1) The varieties $\tilde{X}_i := X_i \times_X \tilde{X}$ equal $\psi^{-1}(X_i)$ and thus form a descending chain of dense, open subvarieties of $\tilde{X}$. The morphisms $\tilde{\eta}_i := \iota_i \times_X \text{Id}_\tilde{X}$ are the inclusion maps.
(8.5.2) Since $Y_i \times_X \tilde{X}$ is a base change of the normal variety $Y_i$ under the étale morphism $\psi$, all the schemes $Y_i$, $i \in \mathbb{N}$, are normal. Since quasi-finiteness and étalé are stable under base change, [Gro61, Prop. 6.2.4] and [Mil80, Prop. 3.3(c)], the morphisms $\gamma_i \times_X \text{Id}_\tilde{X}$ are quasi-finite, and they are étale away from the reduced preimage set $(\eta_i \times_X \text{Id}_\tilde{X})^{-1}(\tilde{T})_{\text{red}}$.
(8.5.3) In a similar vein, the morphisms $\eta_i \times_X \text{Id}_\tilde{X}$ are quasi-finite and étale away from the reduced preimage set $(\eta_i \times_X \text{Id}_\tilde{X})^{-1}(\tilde{T})_{\text{red}}$.
(8.5.4) Recall the assumption that the morphisms $\eta_i$ are Galois, say with group $G_i$. The group $G_i$ will then also act on $Y_i \times_X \tilde{X}$, and the morphism $\eta_i \times_X \text{Id}_\tilde{X}$ equals the quotient map for this induced action, [MFK94, Prop. 0.1 on p. 42]. If $\tilde{Y}_i \subseteq Y_i \times_X \tilde{X}$ is any irreducible component, observe that the restricted morphism $\tilde{\eta}_i := (\eta_i \times_X \text{Id}_\tilde{X})|_{\tilde{Y}_i}$ is Galois. In fact, it is the quotient morphism for the action of the subgroup $\text{stab}(\tilde{Y}_i) \subseteq G_i$ that preserves $\tilde{Y}_i$. 

Choose a sequence of irreducible components $\tilde{Y}_i \subseteq Y_i \times X \tilde{X}$ such that for each $i \in \mathbb{N}^+$, the restricted morphism $\tilde{\gamma}_i := (\gamma_i \times X \text{Id}_X)|_{\tilde{Y}_i}$ dominates $\tilde{Y}_{i-1}$. In summary, we obtain a commutative diagram of morphisms between normal, quasi-projective varieties,

\[
\begin{array}{ccccccc}
\tilde{Y}_0 & \overset{\tilde{\gamma}_1}{\longrightarrow} & \tilde{Y}_1 & \overset{\tilde{\gamma}_2}{\longrightarrow} & \tilde{Y}_2 & \overset{\tilde{\gamma}_3}{\longrightarrow} & \tilde{Y}_3 & \overset{\tilde{\gamma}_4}{\longrightarrow} & \cdots \\
\downarrow \quad \eta_0 & & \downarrow \quad \eta_1 & & \downarrow \quad \eta_2 & & \downarrow \quad \eta_3 & & \\
\tilde{X} & \overset{\gamma_0}{\longrightarrow} & \tilde{X}_0 & \overset{\gamma_1}{\longrightarrow} & \tilde{X}_1 & \overset{\gamma_2}{\longrightarrow} & \tilde{X}_2 & \overset{\gamma_3}{\longrightarrow} & \cdots 
\end{array}
\] (8.5.5)

\textbf{Step 3: End of proof.} We aim to apply Proposition 7.1 to Diagram (8.5.5). Observation 8.3 ensures that $\tilde{X}$ is a klt base space, and Properties (8.5.1)-(8.5.4) reproduce Assumptions (2.1.2)-(2.1.4) of Theorem 2.1. The fact that $\phi|_{\tilde{T}} : \tilde{T} \rightarrow B$ is étale and Assumption (8.4.2)-(8.4.4) correspond directly to the assumptions made in Proposition 7.1. We obtain that all but finitely many of the morphisms $\tilde{\gamma}_i$ are étale. It remains to conclude that all but finitely many of the morphisms $\gamma_i$ are étale. To this end, let $i$ be any index where $\tilde{\gamma}_i$ is étale. It follows from the étalité of $\psi$ that

$$
\emptyset = \text{Ramification}(\tilde{\gamma}_i) = (\text{Ramification}(\gamma_i) \times X \tilde{X}) \cap \tilde{Y}_i.
$$

In particular, we see that $\gamma_i$ is étale at all points of $|\gamma_i|^{-1}(X^0)$. Proposition 8.1 thus follows from Assumption (8.4.1). \hfill \Box

\section{Proof of Theorems 2.1 and 1.1}

\subsection{Proof of Theorem 2.1}

We maintain notation and assumptions of Theorem 2.1. We will assume that $S$ is not empty, for otherwise there is nothing to show. The proof proceeds by induction over the dimension of $S$. If $\dim S = 0$, the claim has already been shown in Proposition 6.1. We will therefore assume that $\dim S > 0$ and that Theorem 2.1 has already been shown for all klt pairs $(Y, D)$ and proper subschemes $R \subset Y$ with $\dim R < \dim S$.

Decompose $S$ into irreducible components, $S = S^1 \cup S^2 \cup \cdots \cup S^n$. Applying Proposition 8.1 to each component $S^i$ we obtain open sets $(S^i)^o \subseteq S^i$ and a number $M \in \mathbb{N}$ such that the following holds: if $S^o \subseteq (S^1)^o \cup \cdots \cup (S^n)^o$ is any dense, Zariski-open subset of $S$ and $S':= S \setminus S^o$, then the morphism $\gamma_i$ is étale away from the set-theoretic preimage $|\eta_i|^{-1}(S')_{\text{red}}$, for any $i > M$. Since $\dim S' < \dim S$, we apply the induction hypothesis to the following diagram,

\[
\begin{array}{ccccccc}
Y_M & \overset{\gamma_{M+1}}{\longrightarrow} & Y_{M+1} & \overset{\gamma_{M+2}}{\longrightarrow} & Y_{M+2} & \overset{\gamma_{M+3}}{\longrightarrow} & Y_{M+3} & \overset{\gamma_{M+4}}{\longrightarrow} & \cdots \\
\downarrow \quad \eta_M & & \downarrow \quad \eta_{M+1} & & \downarrow \quad \eta_{M+2} & & \downarrow \quad \eta_{M+3} & & \\
X & \overset{\eta_{M+1}}{\longrightarrow} & Y_{M+1} & \overset{\eta_{M+2}}{\longrightarrow} & X_{M+2} & \overset{\eta_{M+3}}{\longrightarrow} & X_{M+3} & \overset{\eta'_{M+4}}{\longrightarrow} & \cdots 
\end{array}
\]

Theorem 2.1 follows. \hfill \Box

\subsection{Proof of Theorem 1.1}

Under the assumptions of Theorem 1.1, set $S := X_{\text{sing}}$, $X_i := X_i$, $i_i := \text{Id}_X$ and

$$
\eta_i := \begin{cases} 
\text{Id} : Y_0 \rightarrow X_0 & \text{if } i = 0 \\
\gamma_1 \circ \cdots \circ \gamma_1 : Y_i \rightarrow X_i & \text{if } i > 0.
\end{cases}
$$

The spaces and morphisms form a diagram as in (2.1.1) that satisfies all assumptions made in Theorem 2.1. Theorem 1.1 follows. \hfill \Box
Part III. Applications

10. DIRECT APPLICATIONS

We will prove Theorems 1.4, 1.8, 1.9 and 1.12 in this section.

10.1. Proof of Theorem 1.4. Let \( X \) be any variety that satisfies the assumptions of Theorem 1.4. To prove the theorem, we will first show that there exists a quasi-étale Galois cover \( \tilde{X} \to X \) that satisfies Statement (1.4.1). The equivalence between (1.4.1) and (1.4.2) is then shown separately.

Step 1: Proof of Statement (1.4.1). Aiming for a contradiction, we assume that given any normal, quasi-projective variety \( \tilde{X} \) and any quasi-étale, Galois morphism \( \gamma : \tilde{X} \to X \), there exists a normal, quasi-projective variety \( \tilde{X}^o \) and an étale cover \( \psi^o : \tilde{X}^o \to \tilde{X}^{reg} \) that is not the restriction of any étale cover of \( \tilde{X} \).

Given any \( \tilde{X} \) as above, Theorem 3.7 asserts that any étale cover of \( \tilde{X}^{reg} \) extends to a cover of \( \tilde{X} \). The assumption is therefore equivalent to the following.

Assumption 10.1. Given any normal, quasi-projective variety \( \tilde{X} \) and any quasi-étale, Galois morphism \( \gamma : \tilde{X} \to X \), there exists a normal, quasi-projective variety \( \tilde{X} \) and a cover \( \psi : \tilde{X} \to \tilde{X} \) that is étale over \( \tilde{X}^{reg} \), but not étale.

Using Assumption 10.1 repeatedly, and taking Galois closures as in Theorem 3.6, one inductively constructs a sequence of covers,

\[
X = Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow Y_3 \leftarrow \cdots,
\]

where all \( \gamma_i \) are quasi-étale, but not étale and where all composed morphisms \( \gamma_1 \circ \cdots \circ \gamma_i \) are Galois. We obtain a contradiction to Theorem 1.1, showing our initial assumption was absurd. This finishes the proof of Statement (1.4.1).

Step 2: Proof of Implication (1.4.1) \( \Rightarrow \) (1.4.2). If \( \tilde{X} \to X \) is any cover for which Statement (1.4.1) holds, we claim that the push-forward map \( \hat{\tau} : \hat{\pi}_1(\tilde{X}^{reg}) \to \hat{\pi}_1(\tilde{X}) \) of étale fundamental groups is isomorphic. For surjectivity, recall from [FL81, 0.7.B on p. 33] and [Kol95, Prop. 2.10] that the push-forward map \( \hat{\tau} : \pi_1(\tilde{X}^{reg}) \to \pi_1(\tilde{X}^{an}) \) between topological fundamental groups is surjective. Because profinite completion is a right-exact functor, [RZ10, Lem. 3.2.3 and Prop. 3.2.5], it follows that \( \hat{\tau} \) is likewise surjective.

To show that \( \hat{\tau} \) is injective, we use Grothendieck’s equivalence between the category of étale covers and the category of finite sets with transitive action of the étale fundamental group, [Mil80, Sect. 5]. Arguing by contradiction, assume that there exists a non-trivial element \( g \in \ker \hat{\tau} \). Since \( \hat{\pi}_1(\tilde{X}^{reg}) \) is the profinite completion of \( \pi_1(\tilde{X}^{an}) \), it is residually finite. Hence, there exists a finite group \( H \) and a surjective group homomorphism \( \eta : \hat{\pi}_1(\tilde{X}^{reg}) \to H \) such that \( \eta(g) \neq 0 \). By choice of \( g \), the natural action of \( \hat{\pi}_1(\tilde{X}^{reg}) \) on \( H \) is not induced by an action of \( \hat{\pi}_1(\tilde{X}) \). Using Grothendieck’s equivalence, we obtain an associated étale cover of \( \tilde{X}^{reg} \) that is not the restriction of any étale cover of \( \tilde{X} \). This contradiction to Statement (1.4.1) shows that \( \hat{\tau} \) is injective and finishes the proof of Statement (1.4.2).

Step 3: Proof of Implication (1.4.2) \( \Rightarrow \) (1.4.1). This is immediate from Grothendieck’s equivalence. The proof of Theorem 1.4 is thus finished.
10.1.1. **Further remarks.** In the setting of Theorem 1.4, the set \( \tilde{U} := \gamma^{-1}(\overline{X}_{\text{reg}}) \) is a big open subset of the smooth variety \( \overline{X}_{\text{reg}} \). The topological fundamental groups of the complex manifolds \( \tilde{U} \) and \( \overline{X}_{\text{reg}}^{\text{sm}} \) therefore agree, and the inclusions \( \tilde{U} \subseteq \overline{X}_{\text{reg}} \subseteq \overline{X} \) induce a sequence of isomorphisms between étale fundamental groups

\[
\pi_1(\tilde{U}) \xrightarrow{\text{isomorphism}} \pi_1(\overline{X}_{\text{reg}}) \xrightarrow{\text{isomorphism}} \pi_1(\overline{X}).
\]

10.2. **Proof of Theorem 1.8.** The proof of Theorem 1.8 follows the argumentation of Section 10.1 quite closely, using Theorem 2.1 instead of the simpler Theorem 1.1. Maintaining notation and assumptions of Theorem 1.8, we argue by contradiction and assume the following.

**Assumption 10.2.** For any open neighbourhood \( X^0 \) of \( p \) in \( X \) and any quasi-étale, Galois morphism \( \gamma : \overline{X}^0 \to X^0 \), there exists an open neighbourhood \( U = U(p) \subseteq X^0 \) with preimage \( \tilde{U} = \gamma^{-1}(U) \), and coverings

\[
\begin{align*}
\tilde{U} \xrightarrow{\psi} & \tilde{U} \quad \text{étale over } \overline{U}_{\text{reg}, \text{not étale}} \\
\tilde{U} \quad & \text{quasi-étale, Galois} \\
U & \xrightarrow{\gamma|_U} U.
\end{align*}
\]

Using the existence of a Galois closure and the invariance of the branch locus, Theorem 3.6 and (3.6.2), we are free to assume the following in addition.

**Assumption 10.3.** The composed morphism \( \gamma|_U \circ \psi \) is Galois.

Using Assumption 10.2, we will inductively construct an infinite diagram of morphisms as in Theorem 2.1,

\[
\begin{align*}
Y_0 \xrightarrow{\eta_0} & Y_1 \xrightarrow{\gamma_1} Y_2 \xrightarrow{\eta_2} Y_3 \xrightarrow{\gamma_3} \cdots \\
X \xrightarrow{t_0} & X_0 \xrightarrow{t_1} X_1 \xrightarrow{t_2} X_2 \xrightarrow{t_3} X_3 \xrightarrow{t_4} \cdots,
\end{align*}
\]

where all \( \gamma_i \) are quasi-étale, but not étale. This will lead to a contradiction when we apply Theorem 2.1 with \( S = X_{\text{sing}} \).

**Construction 10.5 (Construction up to \( \eta_1 \)).** Applying Assumption 10.2 with \( X^0 := X \) and \( \gamma := \text{Id}_X \), obtain an open neighbourhood \( U = U(p) \subseteq X^0 \) and a covering \( \psi \) as in (10.2.1). Set

\[
\begin{align*}
X_0 & := U \\
Y_0 & := \tilde{U} \\
\eta_0 & := \gamma|_{\tilde{U}} \\
\eta_1 & := \gamma|_{\tilde{U}} \circ \psi \\
\gamma_1 & := \psi \\
\eta_1 & := \text{Id}_U.
\end{align*}
\]

Observe that \( \eta_0 \) and \( \eta_1 \) are quasi-étale. The morphism \( \gamma_1 \) is quasi-étale, but not étale. Assumption 10.3 guarantees that \( \eta_0 \) and \( \eta_1 \) are Galois.

**Construction 10.6 (Construction of \( \eta_{i+1} \)).** Assume that a diagram as in (10.4.1) has been constructed, up to \( \eta_i \). Applying Assumptions 10.2 and 10.3 with \( X^0 := X_i \) and \( \gamma := \eta_i \), we obtain an open neighbourhood \( U = U(p) \subseteq X_i \) and a covering \( \psi \) as in (10.2.1). Set

\[
\begin{align*}
X_{i+1} & := U \\
Y_{i+1} & := \tilde{U} \\
\gamma_{i+1} & := \psi \\
\eta_{i+1} & := \eta_i|_{\tilde{U}} \circ \psi.
\end{align*}
\]

Observe that \( \eta_{i+1} \) is quasi-étale. The morphism \( \gamma_{i+1} \) is quasi-étale, but not étale. Assumption 10.3 guarantees that \( \eta_{i+1} \) is Galois.

In summary, we have obtained a contradiction to Theorem 2.1 showing that Assumption 10.2 was absurd. This finishes the proof of Theorem 1.8. \( \square \)
10.3. **Proof of Theorem 1.9.** Before starting with the proof of Theorem 1.9 we note the following elementary fact which will be used throughout.

**Fact 10.7.** Let $X$ be a quasi-projective variety, $S \subseteq X$ any finite set and $\mathcal{L}$ any invertible sheaf on $X$. Then there exists a Zariski-open set $U \subseteq X$ that contains $S$ and trivialises $\mathcal{L}$, that is, $\mathcal{L}|_U \cong O_U$. □

**Step 1: Proof of Statement (1.9.1).** Maintaining notation and assumptions of Theorem 1.9 and using the assertions of Theorem 1.8, let $U \subseteq X^\circ$ be any Zariski-open neighbourhood of $p$ and $\bar{D}$ be any $Q$-Cartier Weil divisor on $\bar{U}$. Given any point $x \in \bar{U}$, we need to show that $\bar{D}$ is Cartier at $x$.

We claim that there exists a Zariski-open subset $\bar{V} \subseteq \bar{U}$ that contains $p$ and $\gamma(x)$, and a number $n$ such that $\partial_{\bar{V}}(n \cdot D) \cong \partial_{\bar{V}}(\gamma(x))$, where $\bar{V} = \gamma^{-1}(V)$. In order to construct $\bar{V}$, consider the open, Galois-invariant set $S := \gamma^{-1}(p) \cup \gamma^{-1}(\gamma(x))$. By Fact 10.7, there exists an open subset $\bar{W} \subseteq \bar{U}$ that contains $S$ and a number $n$ such that $\partial_{\bar{W}}(n \cdot D) \cong \partial_{\bar{W}}(\gamma(x))$. Let $\bar{V} := \bigcap_{\gamma \in \text{Gal}(\gamma)} \bar{V}^{\gamma}$. This is an open subset of $\bar{W}$ that contains $S$, satisfies $\partial_{\bar{V}}(n \cdot D) \cong \partial_{\bar{V}}(\gamma(x))$ and is invariant under the action of the Galois group. The last point implies that $\bar{V}$ is of the form $\gamma^{-1}(V)$, for an open subset $V \subseteq U$ that contains $p$ and $\gamma(x)$.

Continuing the proof of Statement (1.9.1), choose $n$ minimal, and let $\eta : \bar{V} \to \bar{V}$ be the associated index-one cover. The covering map $\eta$ is quasi-étale and branches exactly over those points of $\bar{V}$ where $\bar{D}$ fails to be Cartier. In particular, its restriction

$$\eta^* := \eta|_{\bar{V}^{\eta \cdot \text{reg}}} : \eta^{-1}(\bar{V}^{\text{reg}}) \to \bar{V}^{\text{reg}}$$

is étale. Recall from Theorem 1.8 that $\eta^*$ admits an étale extension to all of $\bar{V}$. The uniqueness assertion in Zariski’s Main Theorem, Theorem 3.7, therefore implies that this extension equals $\eta$, so that $\eta$ is itself étale. As $x \in \bar{V}$, this finishes the proof of Statement (1.9.1). □

**Step 2: Proof of Statement (1.9.2).** For simplicity of notation, write $G = \text{Gal}(\gamma)$ and let $m$ denote the size of this group. Given any open neighbourhood $U = U(p) \subseteq X^\circ$, any $Q$-Cartier divisor $D$ on $U$ and any point $x \in U$, we need to show that $m \cdot D$ is Cartier at $x$. Consider the pull-back $\bar{D} := \gamma^*D$ and recall from Statement (1.9.1) that $\bar{D}$ is Cartier on $\bar{U}$. Again using Fact 10.7, find an open neighbourhood $V$ of $x$ with preimage $\bar{V} := \gamma^{-1}(V)$ such that $\bar{D}|_{\bar{V}}$ is linearly equivalent to zero. In other words, $\bar{D}|_{\bar{V}} = \text{div}(\bar{f})$, where $\bar{f}$ is a suitable rational function on $\bar{V}$. Taking averages, we obtain a rational function $\bar{f} := \prod_{\gamma \in G} \bar{f} \circ \gamma$, which is Galois invariant, therefore descends to a rational function $F$ on $V$, and defines the divisor $\text{div} F = m \cdot \bar{D}$. To finish the proof of Statement (1.9.2), observe that $\text{div} F = m \cdot D$. This also finishes the proof of Theorem 1.9. □

10.4. **Proof of Theorem 1.12.** We prove Theorem 1.12 as an application of the following, more general Proposition 10.8, which we expect to have further applications, for example in the classification theory of singular varieties with trivial canonical class, cf. [GKP11, Sect. 8.C].

**Proposition 10.8.** Let $\mathcal{R}$ be a set of normal, quasi-projective varieties satisfying the following conditions.

(10.8.1) If $X \in \mathcal{R}$ and if $Y \to X$ is any quasi-étale Galois cover, then $Y \in \mathcal{R}$.
(10.8.2) For each $X \in \mathcal{R}$, there exists a $Q$-Weil divisor $\Delta$ such that $(X, \Delta)$ is klt.
(10.8.3) Each variety $X \in \mathcal{R}$ has finite étale fundamental group.

Then, for each $X \in \mathcal{R}$, the étale fundamental group $\tilde{\pi}_1(X_{\text{reg}})$ of $X_{\text{reg}}$ is finite.
11. Flat sheaves on klt base spaces

We prove Theorem 1.13 and Corollary 1.15 in this section.

11.1. Proof of Theorem 1.13. Recalling the set-up of Theorem 1.13, let $X$ be a normal, complex, quasi-projective variety and assume that there exists a Q-Weil divisor $\Delta$ such that the pair $(X, \Delta)$ is klt. We will prove that there exists a finite, surjective Galois morphism $\gamma: \tilde{X} \to X$, étale in codimension one, such that for any locally free, flat, analytic sheaf $\mathcal{G}^\circ$ on $\tilde{X}_{\text{reg}}$, there exists a locally free, flat, analytic sheaf $\mathcal{G}^{an}$ on $\tilde{X}^{an}$ such that $\mathcal{G}^{an}|_{\tilde{X}^{an}_{\text{reg}}} \cong \mathcal{G}^\circ$. Recalling from [Del70, II.5, Cor. 5.8 and Thm. 5.9], that there exists a coherent, reflexive, algebraic sheaf $\mathcal{G}$ on $\tilde{X}$ whose analytification over $\tilde{X}_{\text{reg}}$ equals $\mathcal{G}^\circ$, the claim will then follow.

Let $\tilde{X} \to X$ be any cover for which the assertions of Theorem 1.4 hold true. To shorten notation, we denote the relevant complex spaces by $Y := \tilde{X}^{an}$ and $Y^\circ := \tilde{X}^{an}_{\text{reg}}$. The inclusion is denoted by $\iota: Y^\circ \to Y$. We have seen in Statement (1.4.2) and Section 10.1.1 that the induced morphism of étale fundamental groups, $\tilde{\pi}_1(Y^\circ) \to \tilde{\pi}_1(Y)$, is isomorphic.

By Definition 1.14, the sheaf $\mathcal{G}^\circ$ corresponds to a representation $\rho^\circ: \pi_1(Y^\circ) \to \text{GL}(\text{rank} \mathcal{F}, \mathbb{C})$. We write $G := \text{img}(\rho^\circ)$. The group $G$ is a quotient of the finitely generated group $\pi_1(Y^\circ)$, hence finitely generated. As a subgroup of the general linear group, $G$ is residually finite by Malcev’s theorem, [Weh73, Thm. 4.2]. Consequently, the profinite completion morphism $a: G \to \hat{G}$ is injective, [RZ10, Sect. 3.2].
To give an extension of \( \rho^\circ \) to a flat sheaf on \( Y \), we need to show that the representation \( \rho^\circ \) is the restriction of a representation \( \rho \) of \( \pi_1(Y) \). More precisely, we need to find a factorisation

\[
\pi_1(Y^\circ) \xrightarrow{\iota_*} \pi_1(Y) \xrightarrow{\rho} G.
\]

To this end, recall from [RZ10, Lem. 3.2.3] that taking profinite completion is functorial. Hence, we obtain a commutative diagram,

\[
\begin{array}{cccc}
\hat{G} & \xrightarrow{\hat{\rho}^\circ} & \tilde{\pi}_1(Y^\circ) & \xrightarrow{\iota_*} & \tilde{\pi}_1(Y) \\
\downarrow & & \downarrow \iota & & \downarrow \\
G & \xrightarrow{\rho^\circ} & \pi_1(Y^\circ) & \xrightarrow{\iota_*} & \pi_1(Y),
\end{array}
\]

where all vertical arrows are the natural profinite completion morphisms. Since \( \iota_* \) is isomorphic by construction, we can set \( \rho := \hat{\rho}^\circ \circ (\iota_\ast)^{-1} \circ \iota \). Recalling from [Kol95, Prop. 2.10] that \( \iota_* \) is surjective, it follows from commutativity that \( \text{img}(\rho) \subseteq \text{img}(a) \). Identifying \( G \) with its image under \( a \), we have thus constructed a factorisation as in (11.0.5). This finishes the proof of Theorem 1.13.

11.2. **Proof of Corollary 1.15.** Let \( \gamma : \tilde{X} \to X \) be any cover for which the assertion of Theorem 1.13 holds true. By assumption, \( \gamma \) is étale in codimension one, hence étale over \( X_{\text{reg}} \). This has two consequences. First, consider the pull-back divisor \( \Delta_{\tilde{X}} := \gamma^*(\Delta) \). The pair \( (\tilde{X}, \Delta_{\tilde{X}}) \) is then klt, [KM98, Prop. 5.20]. Second, it follows that \( \mathcal{F}_X^\circ \cong \gamma^*(\mathcal{F}_{X_{\text{reg}}}^\circ) \) is a locally free, flat sheaf on \( \tilde{X}^\circ := \gamma^{-1}(X_{\text{reg}}) \). Hence, \( \mathcal{F}_X^\circ \) admits an extension to a locally free, flat sheaf \( \mathcal{F} \) on \( \tilde{X} \). Since \( \mathcal{F} \) and \( \mathcal{F}_X^\circ \) agree in codimension one and since \( \mathcal{F}_X^\circ \) is reflexive, both sheaves agree on all of \( \tilde{X} \). It follows that \( \mathcal{F}_X^\circ \) is locally free and flat. In particular, \( K_{\tilde{X}} \) is Cartier, \( \Delta_{\tilde{X}} \) is Q-Cartier, and the pair \( (\tilde{X}, \emptyset) \) is hence klt, [KM98, Cor. 2.35]. The solution of the Lipman-Zariski conjecture for klt spaces, [GKKP11, Thm. 6.1]\(^1\) thus asserts that \( \tilde{X} \) is smooth. Since \( \gamma \) is Galois, hence a quotient map, \( X \) will automatically have quotient singularities. This proves the first statement of Corollary 1.15.

Now assume in addition that \( X \) is projective. Then \( \tilde{X} \) is a smooth, projective variety with flat tangent bundle and therefore the quotient of an Abelian variety \( A \) by a finite group acting freely, [Kob87, Chap. 4, Cor. 4.15]. Taking the Galois closure of the morphism \( A \to X \) as in Theorem 3.6, we find a sequence of covers,

\[
\begin{array}{cccc}
\hat{A} & \xrightarrow{\gamma, \text{Galois, étale in codim. one}} & A & \xrightarrow{\phi, \text{Galois, étale in codim. one}} & \tilde{X} \\
\xrightarrow{\psi, \text{Galois, étale in codim. one}} & \xrightarrow{\gamma, \text{Galois, étale in codim. one}} & & \xrightarrow{\gamma, \text{Galois, étale in codim. one}} & X
\end{array}
\]

Since \( A \) is smooth, it is clear that the morphism \( \phi \), which is a priori étale only in codimension one, is in fact étale. As an étale cover of an Abelian variety, \( \tilde{A} \) is again an Abelian variety.

\[\square\]

12. **Varieties with vanishing Chern classes**

We prove Theorems 1.19 and 1.16 in this section. The proofs rely on a boundedness result for families of flat bundles, which we establish first.

\[\text{See [Gra13, Gra13] for latest results.}\]
12.1. **Boundedness for families of flat bundles.** We will show that the set of flat bundles forms a bounded family, as long as the underlying space has rational singularities. Using the results of Section 3, this will later allow to recover the isomorphism type of a bundle from its restriction to a given hyperplane.

**Proposition 12.1.** Let $X$ be a normal, projective variety with rational singularities and let $r \in \mathbb{N}^+$ be any integer. Then, the set

$$B := \{ \mathcal{F} \mid \mathcal{F} \text{ a locally free, flat, analytic sheaf on } X \text{ with } \text{rank } \mathcal{F} = r \}$$

is a bounded family.

**Proof.** Let $H$ be any ample line bundle on $X$. Since every flat locally free sheaf on $X$ is $H$-semistable, it follows from [HL10, Cor. 3.3.7] that in order to prove the claim, it suffices to show that the Hilbert polynomial of $\mathcal{F}$ with respect to $H$ equals $P$ is the same for any $\mathcal{F} \in B$.

Let $\pi : \tilde{X} \to X$ be a resolution of singularities. Then, for each $\mathcal{F} \in B$, the pull-back $\pi^* \mathcal{F}$ is flat, that is, given by a representation of $\pi_\ast(\tilde{X})$. The Chern classes $c_i(\pi^* \mathcal{F}) \in H^{2i}(\tilde{X}, \mathbb{Z})$ are therefore trivial, for all $i \in \mathbb{N}^+$. Hence, by the Hirzebruch-Riemann-Roch Theorem, we have

$$\chi(\tilde{X}, \pi^* \mathcal{F} \otimes \mathcal{O}_X(mH)) = \text{rank } \mathcal{F} \cdot \chi(\tilde{X}, \pi^* \mathcal{O}_X(mH)) \quad \forall m \in \mathbb{N}.$$  \hspace{1cm} (12.1.1)

Moreover, since $X$ has rational singularities, the Leray spectral sequence and the projection formula imply

$$H^i(\tilde{X}, \pi^* (\mathcal{F} \otimes \mathcal{O}_X(mH))) \cong H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mH)) \quad \forall m \in \mathbb{N}, i \in \mathbb{N}. \hspace{1cm} (12.1.2)$$

Consequently, we have

$$\chi(X, \mathcal{F} \otimes \mathcal{O}_X(mH)) \cong \chi(\tilde{X}, \pi^* (\mathcal{F} \otimes \mathcal{O}_X(mH))) \quad \text{by } (12.1.2)$$

$$\cong \text{rank } \mathcal{F} \cdot \chi(\tilde{X}, \pi^* \mathcal{O}_X(mH)) \quad \text{by } (12.1.1)$$

The Hilbert polynomial of $\mathcal{F}$ with respect to $H$ is thus independent of $\mathcal{F} \in B$, which was to be shown. \hfill \Box

12.2. **Proof of Theorem 1.19.** Let $\gamma : \tilde{X} \to X$ be a quasi-étale, Galois cover enjoying the properties stated in Theorem 1.13. Notice first that $\tilde{X}$ is still smooth in codimension two, since $\gamma$ branches only over the singular set of $X$. Second, it follows from [KM98, Prop. 5.20] that $(\tilde{X}, \Delta)$ is klt, for $\Delta := \gamma^\ast \Delta$. Since $\gamma$ is finite, the Cartier divisor $\tilde{H} := \gamma^\ast (H)$ is ample. It follows from [HL10, Lem. 3.2.2] that $\mathcal{F} := (\gamma^* \mathcal{E})^{\ast}$ is $\tilde{H}$-semistable. Proposition 4.6 and Assumption (1.19.1) guarantee that

$$c_1(\mathcal{F}) \cdot \tilde{H}^{n-1} = 0, \quad c_1(\mathcal{F})^2 \cdot \tilde{H}^{n-2} = 0, \quad \text{and } c_2(\mathcal{F}) \cdot \tilde{H}^{n-2} = 0. \hspace{1cm} (12.2.1)$$

To prove Theorem 1.19, we need to show that $\mathcal{F}$ is locally free and flat.

**Step 1: Construction of a representation.** Recalling from [KM98, Thm. 5.22] that $\tilde{X}$ has only rational singularities, it follows from Proposition 12.1 that the family $B$ of locally free, flat, coherent sheaves on $\tilde{X}$ whose rank equals $r := \text{rank } \mathcal{F}$ is bounded. We may thus apply the Mehta-Ramanathan-Theorem for normal spaces, [Fle84, Thm. 1.2], and Corollary 5.6 to obtain an increasing sequence of numbers, $0 \leq m_1 \leq m_2 \leq \cdots \leq m_{n-2}$, as well as general elements $D_i \in |\mathcal{O}_{\tilde{X}}(m_i \cdot \tilde{H})|$ such that the following holds.

$$\hspace{1cm} (12.2.2) \quad \text{The surface } S := D_1 \cap \cdots \cap D_{n-2} \text{ is smooth and contained in } \tilde{X}_{\text{reg}}.$$

(12.2.3) The restricted sheaf $\mathcal{F}|_S$ is semistable with respect to $\tilde{H}|_S$. 

(12.2.4) Let $\mathbb{B} \in \mathbb{B}$ be any member. Then, $\mathcal{F} \cong \mathbb{B}$ if and only if $\mathcal{F}|_S \cong \mathbb{B}|_S$. Further, it follows from (12.2.1) and Remark 4.5 that
\begin{equation}
(12.2.5) \quad c_1(\mathcal{F}|_S) \cdot (H|_S) = 0 \quad \text{and} \quad \text{ch}_2(\mathcal{F}|_S) = -\frac{1}{2}c_1(\mathcal{F}|_S)^2 - c_2(\mathcal{F}|_S) = 0,
\end{equation}
where $\text{ch}_2$ denotes the second Chern character. With these equalities, it follows from Simpson’s work, [Sim92, Cor. 3.10], that the semistable sheaf $\mathcal{F}|_S$ is flat. In other words, $\mathcal{F}|_S$ is given by a representation $\rho : \pi_1(S^{\text{an}}) \rightarrow \text{GL}(r, \mathbb{C})$. The Lefschetz Theorem for singular spaces, [GM88, Thm. in Sect. II.1.2], asserts that the natural homomorphism $\iota_* : \pi_1(S^{\text{an}}) \rightarrow \pi_1(\tilde{X}^{\text{an}}_{\text{reg}})$, induced by the inclusion $\iota : S^{\text{an}} \hookrightarrow \tilde{X}^{\text{an}}_{\text{reg}}$, is isomorphic. Composing the inverse $(\iota_*)^{-1}$ with $\rho$ we obtain a representation $\tau : \pi_1(\tilde{X}^{\text{an}}_{\text{reg}}) \rightarrow \text{GL}(r, \mathbb{C})$.

**Step 2: End of proof.** The representation $\tau$ defines a flat, locally free, analytic sheaf $\mathcal{F}_\tau$ on $\tilde{X}^{\text{an}}_{\text{reg}}$, which, by choice of $\tilde{X}$, comes from a flat, locally free, algebraic sheaf $\mathcal{F}_\tau|_S$ on $\tilde{X}$. By construction, $\mathcal{F}_\tau|_S$ is isomorphic to $\mathcal{F}|_S$. Applying (12.2.4), we conclude that $\mathcal{F}$ is isomorphic to $\mathcal{F}_\tau$, and therefore flat. This finishes the proof of Theorem 1.19.

12.2.1. **Concluding remarks.** The results of this section clearly remain true if we substitute the polarisation $(H_1, \ldots, H)$ by $H_1, \ldots, H_{n-1}$, where the $H_j$ are not necessarily identical ample divisors.

If $\mathcal{E}$ is polystable, it is possible to avoid the use of [Sim92, Cor. 3.10] in the proof of Theorem 1.19, by using [Don85, Thm. 1] to conclude that $\mathcal{F}|_S$ is Hermite-Einstein. This suffices for the proof of Theorems 12.3 and 1.16 below, since —in the setting of Theorem 1.16— the tangent sheaf is polystable for any polarisation by [GKP11, Cor. 7.3], possibly after a quasi-étale cover of $X$.

12.3. **Proof of Theorem 1.16.** Using the terminology introduced in Section 4 we state the main result of this section. Theorem 1.16 follows directly from this.

**Theorem 12.3 (Characterisation of quotients of Abelian varieties).** Let $X$ be a normal $n$-dimensional projective variety which is klt and smooth in codimension two. Then the following conditions are equivalent:

1. $K_X = 0$, and $c_2(\mathcal{F}_X)$ is numerically trivial in the sense of Definition 4.7.
2. There exists an Abelian variety $A$ and a finite, surjective, Galois morphism $A \rightarrow X$ that is étale in codimension two.

**Remark 12.4 (Comparing the assumptions of Theorem 12.3 and Theorem 1.16).** By Proposition 4.8, the conditions in (12.3.1) are equivalent.

The conditions in (12.3.1) and (12.3.2) are equivalent.

**Remark 12.5.** In dimension three, Theorem 12.3 has been shown without the assumption on the codimension of the singular set by Shepherd-Barron and Wilson, [SBW94], using orbifold techniques. It seems feasible to obtain an analogous result also in higher dimensions with the methods presented here.

Theorem 12.3 is shown in the remainder of this section. The two implications will be shown separately.

**Proof of Theorem 12.3, (12.3.2) ⇒ (12.3.1).** If $\eta : A \rightarrow X$ is any finite map from an Abelian variety, étale in codimension two, then there exists a linear equivalence $\eta^*(K_X) \sim K_A = 0$. In particular, it follows from the projection formula that $K_X$ is numerically trivial. Corollary 4.9 implies that $c_2(\mathcal{F}_X)$ is numerically trivial as well. \(\square\)
Proof of Theorem 12.3, (12.3.1) ⇒ (12.3.2). Assume that (12.3.1) holds. We first reduce to the case of canonical singularities. By the abundance theorem for klt varieties with numerically trivial canonical divisor class, [Amb05, Thm. 4.2], there exists an \( m \in \mathbb{N}^\circ \) such that \( \mathcal{O}_X(mK_X) \cong \mathcal{O}_X \). Let \( m \) be minimal with this property, and let \( v : \bar{X} \to X \) be the associated global index-one cover, which is quasi-étalement over \( X \). Then, \( \mathcal{O}_{\bar{X}}(K_{\bar{X}}) \cong \mathcal{O}_{\bar{X}} \), which together with the fact that \( \bar{X} \) has klt singularities, [KM98, Prop. 5.20], implies that \( \bar{X} \) has canonical singularities. Moreover, applying Corollary 4.9 we see that \( \bar{X} \) is smooth in codimension two, and that \( c_2(\bar{T}_X) \) is numerically trivial. If \( \eta : \tilde{A} \to \bar{X} \) is a finite, surjective, Galois morphism that is étale in codimension two from an abelian variety \( \tilde{A} \) to \( \bar{X} \), then taking the Galois closure of \( \nu \circ \eta \) yields the desired map \( \tilde{A} \to X \). We may therefore make the following simplifying assumption.

Assumption w.l.o.g. 12.6. The variety \( X \) has canonical singularities.

Since \( c_2(X) \) is numerically trivial, we have \( c_2(X) \cdot H^{n-2} = 0 \) for any ample divisor \( H \) on \( X \). Since moreover \( K_X \) is numerically trivial, and since \( X \) has at worst canonical singularities by Assumption 12.6, the tangent sheaf \( \bar{T}_X \) is \( H \)-semistable by [GKP11, Prop. 5.4], and

\[
c_1(\bar{T}_X) \cdot H^{n-1} = c_1(\bar{T}_X)^2 \cdot H^{n-2} = 0 \quad \text{for any ample divisor } H \text{ on } X.
\]

By Theorem 1.19, there hence exists a quasi-étale cover \( \gamma : \bar{X} \to X \) such that \( \bar{T}_X \cong (\gamma^*\bar{T}_X)^\ast \) is a flat, locally free sheaf. Applying Corollary 1.15 to \( \bar{X} \) and possibly taking Galois closure finishes the proof of Theorem 12.3. \( \square \)

13. VARIETIES ADMITTING POLARISED ENDOMORPHISMS

We will prove Theorem 1.20 in Section 13.1. As noted in the introduction, Theorem 1.20 has consequences for the structure theory of varieties with endomorphisms. These are discussed in Section 13.2 below.

13.1. Proof of Theorem 1.20. Theorem 1.20 has been shown in [NZ10, Thm. 3.3] under an additional assumption concerning fundamental groups of smooth loci of Euclidean-open subsets of \( X \). Nakayama and Zhang use this assumption only once, to prove a claim which appears on Page 1004 of their paper. After briefly recalling their setup, we will show that the claim follows directly from our Theorem 1.1, without any additional assumption. Once this is done, the original proof of Nakayama–Zhang applies verbatim.

Under the assumptions of Theorem 1.20, let \( f : X \to X \) be a polarised endomorphism. Nakayama–Zhang start the proof of [NZ10, Thm. 3.3] on page 1004 of their paper by recalling from [NZ10, Thm. 3.2] that \( X \) has at worst canonical singularities, that \( K_X \) is \( \mathbb{Q} \)-linearly trivial, and that \( f \) is étale in codimension one. Given any index \( k \in \mathbb{N}^+ \), they consider the iterated endomorphism \( f^k \) and its Galois closure,

\[
V^k \xrightarrow{\theta_k} X \xrightarrow{\tau_k} X,
\]

where \( \theta_k \) and \( \tau_k \) are Galois and again étale in codimension one, cf. Theorem 3.6. By [NZ10, Lem. 2.5], there exist finite morphisms \( g_k, h_k \), again étale in codimension
one, forming commutative diagrams as follows,

\[
\begin{array}{ccc}
\theta_1 & V_1 & h_1 & V_2 & h_2 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
X & f & X & f & X & \cdots \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\tau_1 & V_1 & g_1 & V_2 & g_2 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
X & f & X & f & X & \cdots \\
\end{array}
\]

Nakayama–Zhang claim in [NZ10, claim on p. 1004] that the morphisms \(h_k\) and \(g_k\) are étale for all sufficiently large \(k\). They prove this claim using their additional assumption on the fundamental groups. Theorem 1.1, however, applies to the associated sequences,

\[
\begin{array}{ccc}
X & \theta_2, \text{Galois} & V_1 & h_1 & V_2 & h_2 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
X & f & X & f & X & f & \cdots \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X & \tau_2, \text{Galois} & V_1 & g_1 & V_2 & g_2 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
X & f & X & f & X & f & \cdots \\
\end{array}
\]

and yields this result without any extra assumption. As pointed out above, the rest of Nakayama–Zhang’s proof applies verbatim. \(\square\)

13.2. **The structure of varieties admitting endomorphisms.** Theorem 1.20 has consequences for the structure theory of varieties with endomorphisms. The following results have been shown in [NZ10, Thm. 1.3], conditional to the assumption that [NZ10, Conj. 1.2] = Theorem 1.20 holds true. The definition \(q^\sharp\) is recalled in Remark 13.2 below.

**Theorem 13.1** (Structure of varieties admitting polarized endomorphisms). Let \(f : X \rightarrow X\) be a non-isomorphic, polarised endomorphism of a normal, complex, projective variety \(X\) of dimension \(n\). Then \(k(X) \leq 0\) and \(q^\sharp(X, f) \leq n\). Furthermore, there exists an Abelian variety \(A\) of dimension \(\dim A = q^\sharp(X, f)\) and a commutative diagram of normal, projective varieties,

\[
\begin{array}{ccc}
A & \omega & Z & \rho & V & \tau & X \\
\downarrow f_A & \downarrow f_Z & \downarrow f_V & \downarrow \tau & \downarrow f & \downarrow \\
A & \text{flat, surjective} & Z & \text{bratl.} & V & \text{finite, surjective, étale in codim. one} & X, \\
\end{array}
\]

where all vertical arrows are polarised endomorphism, and every fibre of \(\omega\) is irreducible, normal and rationally connected. In particular, \(X\) is rationally connected if \(q^\sharp(X, f) = 0\).

Moreover, the fundamental group \(\pi_1(X)\) contains a finitely generated, Abelian subgroup of finite index whose rank is at most \(2 \cdot q^\sharp(X, f)\).

\(\square\)

**Remark 13.2** (Definition of \(q^\sharp\), [NZ10, p. 992f]). In the setting of Theorem 13.1, the number \(q^\sharp(X, f)\) is defined as the supremum of irregularities \(q(\tilde{X'}) = h^1(\tilde{X'}, \mathcal{O}_{\tilde{X}'})\) of a smooth model \(\tilde{X}'\) of \(X'\) for all the finite coverings \(\tau : X' \rightarrow X\) étale in codimension one and admitting an endomorphism \(f' : X' \rightarrow X'\) with \(\tau \circ f' = f \circ \tau\).

14. Examples, counterexamples, and sharpness of results

In this section, we have collected several examples which illustrate to what extend our main results are sharp. In Section 14.1 we construct an infinite sequence of branched non-Galois coverings of the singular Kummer surface, showing that Theorem 1.1 does not to hold without the Galois assumption. Section 14.2 discusses an example of Gurjar–Zhang, showing that Theorem 1.12 is sharp, and that Theorem 1.1 has no simple reformulation in terms of the push-forward between fundamental groups. Finally, Section 14.3 shows by way of an example that there
is generally no canonical, minimal choice for the coverings constructed in Theorem 1.4.

14.1. **Isogenies of Abelian surfaces and the Kummer construction.** We first construct a number of special endomorphisms of Kummer surfaces. For this, fix one Abelian surface $A$ throughout this section.

**Construction 14.1** (Endomorphisms of singular Kummer surfaces). Consider the involutive automorphism $\sigma : A \to A, a \mapsto -a$. Set $X := A/\sigma$ and denote the quotient morphism by $f : A \to X, a \mapsto [a]$. The automorphism $\sigma$ has 16 fixed points. The quotient surface $X$ has 16 rational double points, which are canonical and therefore klt. We call $X$ a “singular Kummer surface”.

Given any number $n \in \mathbb{N}^+$, consider the endomorphism of $A$ obtained by multiplication with $n$, that is, $n_A : A \to A, a \mapsto n \cdot a$. Note that $n_A$ is a finite morphism that commutes with $\sigma$ and therefore induces a finite endomorphism of $X$, which we denote as $n_X : X \to X, [a] \mapsto [n \cdot a]$. By construction, the morphism $n_X$ commutes with $f$.

The main properties of the endomorphisms $n_A$ and $n_X$ are summarised in the following observations.

**Observation 14.2** (Degree and fibres of $n_A$ and $n_X$). The morphism $n_A$ is the quotient for the natural action of the $n$-torsion group $A_n < A$ on $A$. In particular, $n_A$ is finite of degree $4^n$, étale and Galois. Given any point $x \in X$, it follows from commutativity, $n_X \circ f = f \circ n_A$, that the number of points in the fibre is at least $\# n_X^{-1}(x) \geq \frac{1}{2} \cdot 4^n$.

**Observation 14.3** (Endomorphism $n_X$ is quasi-étale, not étale, and not Galois). The morphism $n_X$ is quasi-étale by construction. Let $n > 2$ be any number and $x \in X_{\text{sing}}$ any singular point of $X$. Since $X$ has only 16 singularities, it follows from Observation 14.2 that the fibre $n_X^{-1}(x)$ contains both singular and non-singular points of $X$. This shows that $n_X$ is neither étale nor Galois if $n > 2$.

Construction 14.1 shows that Theorem 1.1 does not hold without the Galois assumption. This is the content of the following proposition.

**Proposition 14.4** (Sharpness of Theorem 1.1). Using the notation of Construction 14.1, choose a number $n > 2$ and consider the tower of finite morphisms

$$
X \xrightarrow{n_X} X \xrightarrow{n_X} X \xrightarrow{n_X} \cdots
$$

Then, all assumptions of Theorem 1.1 are satisfied except that the composed morphisms $(n^k)_X = n_X \circ \cdots \circ n_X$ be Galois, for any $k > 1$. However, none of the finite morphisms $n_X$ is étale. \hfill \Box

14.2. **Comparing the étale fundamental groups of a klt variety and its smooth locus.** In [GZ94, Sect. 1.15], Gurjar and De-Qi Zhang construct a rationally connected, simply-connected, projective surface $X$ with rational double point singularities, admitting a quasi-étale, two-to-one cover $\gamma : \tilde{X} \to X$ where $\tilde{X}$ is smooth and $\pi_1(\tilde{X}^{\text{an}}) \cong \mathbb{Z}^2$.

We claim that $\pi_1(X_{\text{reg}})$ is infinite. In fact, the standard sequence

$$
0 \to \pi_1(\gamma^{-1}(X^{\text{an}}_{\text{reg}})) \xrightarrow{\gamma_*} \pi_1(X^{\text{an}}_{\text{reg}}) \to \mathbb{Z}/2\mathbb{Z} \to 0
$$

shows that the groups $n\mathbb{Z}^2$ are finite-index subgroups of $\pi_1(X^{\text{an}}_{\text{reg}})$, for all $n \in \mathbb{N}^+$. Recalling from the definition that the kernel of the profinite completion map
\[ \pi_1(X_{\text{reg}}) \rightarrow \hat{\pi}_1(X_{\text{reg}}) \] equals the intersection of all finite-index subgroups, the claim follows.

The example of Gurjar–Zhang relates to our results in at least two ways.

**Morphism of fundamental groups in Theorem 1.1.** Given a quasi-projective, klt variety \( X \), we have pointed out in Remark 1.6 that Theorem 1.1 does not assert that the kernel of the natural surjection of \( \text{étale} \) fundamental groups, \( \hat{\iota}_* : \hat{\pi}_1(X_{\text{reg}}) \rightarrow \hat{\pi}_1(X) \), is finite. The example of Gurjar–Zhang proves this point. Note that the singular Kummer surface constructed in Section 14.1 above also exemplifies the same point.

**Sharpness of Theorem 1.12.** It is well-known that rationally chain connected varieties have finite fundamental group, [Kol95, Thm. 4.13]. Hence, it is a natural question whether the condition “\( -(K_X + \Delta) \) is nef and big” of Theorem 1.12 can be weakened to “\( X \) rationally chain connected”. The example of Gurjar–Zhang shows that this is not the case.

### 14.3. Non-existence of a minimal choice in Theorem 1.4.

We will show in this section that, given a klt variety \( X \), there is in general no canonical choice for the cover \( \tilde{X} \) constructed in Theorem 1.4, that is minimal in the sense that any other such cover would factorise over \( \tilde{X} \). For this, we will construct a quasi-projective klt surface \( X \) and two covers \( \gamma_1 : \tilde{X}_1 \rightarrow X, \gamma_2 : \tilde{X}_2 \rightarrow X \) that satisfy the conditions of Theorem 1.4. The geometry of these covers will differ substantially. It will be clear from the construction that the two covers do not dominate a common third. In other words, there is no cover \( \psi : Y \rightarrow X \) that satisfies the conclusions of Theorem 1.4 and fits into a diagram as follows,

\[
\begin{array}{ccc}
\tilde{X}_1 & \xrightarrow{\gamma_1} & Y \\
\downarrow & & \downarrow \psi \\
\tilde{X}_2 & \xrightarrow{\gamma_2} & X
\end{array}
\]

This shows that a canonical, minimal covering cannot exist.

**Construction of the surface \( X \).** Consider the spaces \( A = \mathbb{P}^1 \times \mathbb{P}^1 \) and \( B := \mathbb{P}^1 \). Identifying the group \( \mathbb{Z}_2 \) with the multiplicative group \( \{-1, 1\} \), consider the actions of the Klein four-group \( G := \mathbb{Z}_2 \times \mathbb{Z}_2 \) on \( A \) and \( B \), written in inhomogeneous coordinates as follows

\[
\begin{align*}
(\mathbb{Z}_2 \times \mathbb{Z}_2) \times A & \rightarrow A, \\
((z_1, z_2), (a_1, a_2)) & \mapsto (z_1 \cdot a_1^2, z_1 z_2 \cdot a_2) \\
(\mathbb{Z}_2 \times \mathbb{Z}_2) \times B & \rightarrow B, \\
((z_1, z_2), b) & \mapsto (z_1 \cdot b^2).
\end{align*}
\]

Let \( \pi : A \rightarrow B \) be the projection to the first factor. This map is clearly equivariant and therefore induces a morphism between quotients,

\[
\begin{array}{ccc}
A & \xrightarrow{\pi_* \text{, quotient, four-to-one}} & A/G \\
\downarrow & & \downarrow \rho \\
B & \xrightarrow{\rho_* \text{, quotient, four-to-one}} & B/G \cong \mathbb{P}^1
\end{array}
\]

The quotient curve \( B/G \) is isomorphic to \( \mathbb{P}^1 \). The quotient map \( \rho_* \) has three branch points, \( \{q_1, q_2, q_3\} \subset B/G \), and six ramification points \( \{0, \infty, \pm 1, \pm i\} \subset B \), two over each of the branch points, and each with ramification index two.
The figure shows the singular quotient surface $X$ that is constructed in Section 14.3. The surface contains six $A_1$ quotient singularities which are indicated with the symbol “∗”.

**FIGURE 14.1.** Singular surface with $A_1$ singularities

The quotient surface $A/G$ is singular, with six quotient singularities of type $A_1$, two over each of the branch points $q_i \in B/G$. The quotient $A/G$ is therefore klt. Away from the branch points $q_i$, the morphism $\rho$ has the structure of a $\mathbb{P}^1$-bundle. The fibres $\rho^{-1}(q_i)$ are set-theoretically isomorphic to $\mathbb{P}^1$, but carry a non-reduced, double structure. Choose a fourth point $q_4 \in B/G$ and consider the quasi-projective variety $X := (A/G) \setminus \rho^{-1}(q_4)$. The setup is depicted in Figure 14.1.

**Construction of the covering $\tilde{X}_1$.** Set $\tilde{X}_1 := q_A^{-1}(X)$ and $\gamma_1 := q_A|_{\tilde{X}_1}$. Since $\tilde{X}_1$ is smooth, the condition that $\pi_1(\tilde{X}_1) = \pi_1((\tilde{X}_1)_{\text{reg}})$ is satisfied. The cover $\gamma_1$ is Galois, four-to-one, and branches only over the singularities of $X$.

**Construction of the covering $\tilde{X}_2$.** Consider a double covering of $B/G \cong \mathbb{P}^1$ by an elliptic curve $E$, branched exactly over the points $q_1, \ldots, q_4$. Let $\tilde{X}_2$ be the normalisation of the fibred product $X \times_{B/G} E$ and let $\gamma_2 : \tilde{X}_2 \to X$ be the obvious morphism, which is a Galois, two-to-one cover of $X$, branched exactly over the singularities of $X$. A standard computation shows that $\tilde{X}_2$ is smooth. The condition that $\pi_1(\tilde{X}_2) = \pi_1((\tilde{X}_2)_{\text{reg}})$ is thus again trivially satisfied.

**Non-existence of a minimal cover.** We will now show that a covering $\psi : Y \to X$ forming a diagram as in (14.4.1) cannot exist. Assuming for a moment that $Y$ does exist, recall that $\gamma_2$ is two-to-one. It follows that either $\alpha_2$ or $\psi$ is isomorphic. Neither is possible:

- If $\alpha_2$ was isomorphic, we would obtain a two-to-one covering map $\alpha_2^{-1} \circ \alpha_1 : \tilde{X}_1 \to \tilde{X}_2$. This is impossible, because $\tilde{X}_1$ is rational, while $\tilde{X}_2$ is not.
- If $\psi$ was isomorphic, then $X$ would have to satisfy the condition that $\tilde{\pi}_1(X) = \tilde{\pi}_1((X)_{\text{reg}})$. The existence of covering maps that branch only over the singularities shows that this is not the case.
Appendices

Appendix A. Zariski’s Main Theorem in the equivariant setting

While certainly known to experts, we were not able to find a full reference for Zariski’s Main Theorem in the equivariant setting, Theorem 3.7, in the literature. The following lemma will be used.

Lemma A.1. Consider the composition of a rational map \( a : V \to W_1 \) and a finite morphism \( b : W_1 \to W_2 \), where \( V, W_1 \) and \( W_2 \) are quasi-projective varieties. If \( V \) is normal and if \( b \circ a \) is a morphism, then \( a \) is a morphism.

Proof. Blowing up \( V \) suitably, we obtain a diagram as follows

\[
\begin{array}{c}
\tilde{V} \\
\beta_V, \text{ blow-up} \\
V \\
\beta_W, \text{ morphism} \\
W_1 \xrightarrow{a} W_2.
\end{array}
\]

Since \( V \) is normal, it suffices to show that given any closed point \( v \in V \) with fibre \( F := \beta^{-1}_V(v) \), the image set \( \beta_W(F) \) is a point. To this end, recall Zariski’s main theorem, \([Har77, \text{III Cor. 11.4}]\), which asserts that the fibre \( F \) is connected. Consequently, so is \( \beta_W(F) \). Using the assumption that \( b \circ a \) is a morphism, observe that

\[
b(\beta_W(F)) = (b \circ a)(\beta_V(F)) = (b \circ a)(v) = \text{ a point}.
\]

Since \( b \) is finite, it follows that the connected set \( \beta_W(F) \) is a point, as claimed. \( \square \)

A.1. Proof of Theorem 3.7. Assume that we are given a morphism \( a : V^o \to W \) as in Theorem 3.7. The claims made in the theorem will be shown separately.

Step 1: Existence of a factorisation. Zariski’s Main Theorem in the form of Grothendieck, \([Gro66, \text{Thm. 8.12.6}]\), asserts the existence of a quasi-projective variety \( V' \) and a factorisation of \( a \) into an open immersion \( a_1 : V \to V' \) and a finite morphism \( \beta' : V' \to W \). Let \( \eta : V \to V' \) be the normalisation. Recalling the universal property of the normalisation map, \([Har77, \text{II Ex. 3.8}]\), observe that there exists a morphism \( a' : V^o \to V \) such that \( a' = a \circ \eta \). Finally, set \( \beta := \beta' \circ \eta \).

Step 2: Uniqueness of the factorisation. Assume we are given two factorisations of \( a \) as in Theorem 3.7, say \( a = \beta_1 \circ a_1 = \beta_2 \circ a_2 \). We obtain a commutative diagram of morphisms and rational maps,

\[
\begin{array}{c}
V^o \\
\xrightarrow{a_1, \text{ open immersion}} V_1 \\
\xrightarrow{r := a_2 \circ a_1^{-1}} V_2 \\
V^o \\
\xrightarrow{a_2, \text{ open immersion}} W_1 \\
\xrightarrow{\beta_2, \text{ finite}} W.
\end{array}
\]

Since the composed morphism \( \beta_2 \circ r \) equals \( \beta_1 \), Lemma A.1 asserts that \( r \) is a morphism. The same holds for \( r^{-1} = a_1 \circ a_2^{-1} \), showing that \( r \) is isomorphic.

Step 3: The \( G \)-action on \( V \). To show that \( G \) acts on \( V \), let \( g \in G \) be any element. The action of \( g \) gives rise to a diagram

\[
\begin{array}{c}
V \\
\xrightarrow{g} V^o \\
\beta \\
W \\
\xrightarrow{\beta} W^o.
\end{array}
\]
where all vertical arrows are the natural inclusion maps and where \( \overline{\mathcal{G}} \) is the natural extension of \( g \) to a birational endomorphism of \( V \). The morphism \( a \) is \( G \)-invariant, which means that \( a \circ g = g \circ a \). As a consequence, we see that \( \overline{\beta} \circ \overline{\mathcal{G}} = g \circ \overline{\beta} \). In particular, the rational map \( \overline{\beta} \circ \overline{\mathcal{G}} \) is a morphism. Lemma A.1 thus implies that \( \overline{\mathcal{G}} \) is a morphism, showing that \( G \) acts on \( V' \), as claimed. Equivariance of \( a \) and \( \beta \) is now automatic.

**Step 4: Quotients.** Under the assumptions of (3.7.2), we have seen that \( G \) acts on \( V \). We need to show that \( W \) is the quotient of the \( G \)-action on \( V \) and that \( \beta \) is the quotient map. To this end, consider the following natural diagram of (normal) quotient spaces and quotient maps,

\[
\begin{array}{ccc}
V & \xrightarrow{\beta, \text{finite and } G\text{-invariant}} & W \\
\downarrow{\text{quotient}} & & \downarrow{\text{inclusion}} \\
V^0 & \xrightarrow{\text{quotient}} & W^0, \\
\uparrow{\text{isomorphism}} & & \\
V^0/G & \overset{q}{\longrightarrow} & W^0,
\end{array}
\]

where all vertical arrows are the natural inclusion maps. The map \( q \) is induced by the universal property of the quotient, [Mum08, Thm. 1 on p. 111], using that the map \( \beta \) is \( G \)-invariant. The isomorphism between \( V^0/G \) and \( W^0 \) shows that \( q \) is birational.

Since \( \beta \) is finite, so is \( q \). The standard version of Zariski’s Main Theorem, [Har77, V Thm. 5.2], therefore applies. It asserts that \( V/G \) is isomorphic to \( W \) and that \( \beta \) is the quotient map. This finishes the proof of Theorem 3.7. \( \square \)

### Appendix B. Galois Closure

**B.1. Proof of Theorem 3.6.** We will use the equivalence between the category of connected, étale covers of a variety \( V \) and the category of finite sets \( S \) with transitive action of the étale fundamental group \( \pi_1 (V) \), [Mil80, Sect. 5 and Thm. 5.3].

**Step 1: Construction of \( \tilde{X} \).** Consider the maximal open set \( Y^0 := Y \setminus \text{Branch}(\gamma) \) where \( \gamma \) is étale. Let \( X^0 := \gamma^{-1}(Y^0) \) denote the preimage. Choose a closed point \( y \in Y^0 \) and a closed point \( x \) of the fibre \( X_y := \gamma^{-1}(y) \). The étale map \( \gamma|_{X^0} : X^0 \to Y^0 \) corresponds to a transitive action of the étale fundamental group \( \tilde{\pi} := \pi_1 (Y^0, y) \) on \( X_y \). The group \( \tilde{\pi} \) also acts on

\[ G := \text{img}(\tilde{\pi} \to \text{Permutation group of } X_y) \]

by left multiplication. We obtain a sequence of finite sets with transitive \( \tilde{\pi} \)-action and a corresponding sequence of étale covers,

\[
\begin{array}{c}
G \xrightarrow{g \mapsto \gamma(x)} X_y \\
\text{const.} \{y\} \\
\end{array}
\quad \text{and} \quad \begin{array}{c}
\tilde{X}^0 \xrightarrow{\tilde{\gamma}} X^0 \xrightarrow{\gamma|_{X^0}} Y^0,
\end{array}
\]

where \( G^0 := (\gamma|_{X^0}) \circ \gamma^0 \) is Galois with group \( G \) and \( \gamma^0 \) is Galois with Galois group equal to the isotropy group \( H := \text{Iso}_x \leq G \) of the point \( x \in X_y \), respectively. Applying Theorem 3.7 to the quasi-finite morphism \( \tilde{X}^0 \to X \), we obtain a normal, quasi-projective variety \( \tilde{X} \) and a diagram

\[
\begin{array}{cccc}
\tilde{X}^0 \text{ immersion} & \tilde{X} & \overset{\tilde{\gamma}, \text{Galois with group } H}{\longrightarrow} & X \xrightarrow{\gamma, \text{finite}} Y,
\end{array}
\]
Step 2: Verification of (3.6.1). We need to show that $\Gamma := \gamma \circ \tilde{\gamma}$ is Galois with group $G$. Applying Theorem 3.7 to the quasi-finite morphism $\tilde{X}^o \rightarrow Y$, we obtain a normal, quasi-projective variety $\tilde{X}'$ and morphisms $\tilde{X}^o \rightarrow \tilde{X}' \rightarrow Y$, where $\Gamma'$ is Galois with group $G$. Since $\Gamma$ and $\Gamma'$ are both finite morphisms, the uniqueness statement of Theorem 3.7 shows that $\tilde{X} = \tilde{X}'$ and $\Gamma = \Gamma'$. This establishes (3.6.1).

Step 3: Verification of (3.6.2). As $\Gamma = \gamma \circ \tilde{\gamma}$, it follows from [Gro71, I. Thm. 10.11] that $\text{Branch}(\Gamma) \supseteq \text{Branch}(\gamma)$. On the other hand, the construction immediately shows that $(\gamma|_{\tilde{X}}) \circ \tilde{\gamma}^o$ is étale, so that $\text{Branch}(\Gamma) \subseteq \text{Branch}(\gamma)$. The branch loci of $\gamma$ and $\Gamma$ therefore coincide, as claimed in (3.6.2). This finishes the proof of Theorem 3.6.

\[ \square \]

References

[Alu06] Paolo Aluffi. Classes de Chern des variétés singulières, revisitées. C. R. Math. Acad. Sci. Paris, 342(6):405–410, 2006. ↑ 11

[Amb05] Florin Ambro. The moduli b-divisor of an lc-trivial fibration. Compos. Math., 141(2):385–403, 2005. ↑ 6, 32

[AY08] Takuro Abe and Masahiko Yoshinaga. Splitting criterion for reflexive sheaves. Proc. Amer. Math. Soc., 136(6):1887–1891, 2008. ↑ 16

[BBS94] Shigetoshi Bando and Yum-Tong Siu. Stable sheaves and Einstein-Hermitian metrics. In Geometry and analysis on complex manifolds, pages 39–50. World Sci. Publ., River Edge, NJ, 1994. ↑ 5

[BKLP11] Daniel Greb, Stefan Kebekus, Sándor J. Kovács, and Thomas Peternell. Differential forms on log canonical spaces. Inst. Hautes Études Sci. Publ. Math., 114(1):87–169, November 2011. DOI:10.1007/s10240-011-0036-0 An extended version with additional graphics is available as arXiv:1003.2913. ↑ 4, 22, 29

[GM88] Mark Goresky and Robert D. MacPherson. Stratified Morse theory, volume 14 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1988. ↑ 31

[GR58] Hans Grauert and Reinhold Remmert. Komplexe Räume. Math. Ann., 136:245–318, 1958. ↑ 11
ÉTALÉ FUNDAMENTAL GROUPS, FLAT SHEAVES, AND QUOTIENTS OF ABELIAN VARIETIES  41

and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, second edition, 2010. ↑ 25, 26, 29

[SBW94] Nicholas I. Shepherd-Barron and Pelham M.H. Wilson. Singular threefolds with numerically trivial first and second Chern classes. J. Algebraic Geom., 3(2):265–281, 1994. ↑ 5, 31

[Simp92] Carlos T. Simpson. Higgs bundles and local systems. Inst. Hautes Études Sci. Publ. Math., (75):5–95, 1992. ↑ 5, 31

[Ste51] Norman Steenrod. The Topology of Fibre Bundles. Princeton Mathematical Series, vol. 14. Princeton University Press, Princeton, N. J., 1951. ↑ 20

[Ste56] Karl Stein. Analytische Zerlegungen komplexer Räume. Math. Ann., 132:63–93, 1956. ↑ 11

[Tak80] Shigeharu Takayama. Simple connectedness of weak Fano varieties. J. Algebraic Geom., 9(2):403–407, 2000. ↑ 28

[Uhl86] Karen Uhlenbeck and Shing-Tung Yau. On the existence of Hermitian-Yang-Mills connections in stable vector bundles. Comm. Pure Appl. Math., 39(S, suppl.):S257–S293, 1986. Frontiers of the mathematical sciences: 1985 (New York, 1985). ↑ 5

[Ver76] Jean-Louis Verdier. Stratifications de Whitney et théorème de Bertini-Sard. Invent. Math., 36:295–312, 1976. ↑ 10

[Weh73] Bertram A. F. Wehrfritz. Infinite linear groups. An account of the group-theoretic properties of infinite groups of matrices. Springer-Verlag, New York, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 76. ↑ 28

[Wei94] Charles A. Weibel. An introduction to homological algebra, volume 38 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994. ↑ 15, 17

[Xu14] Chenyang Xu. Finiteness of algebraic fundamental groups. Compositio Math., 150(3):409–414, 2014. ↑ 4, 6, 19

[Yau78] Shing Tung Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. Comm. Pure Appl. Math., 31(3):339–411, 1978. ↑ 4

[Zha06] Qi Zhang. Rational connectedness of log Q-Fano varieties. J. Reine Angew. Math., 590:131–142, 2006. ↑ 28

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