ON COMPLETE INTERSECTIONS CONTAINING A LINEAR SUBSPACE

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Abstract. Consider the Fano scheme $F_k(Y)$ parameterizing $k$-dimensional linear subspaces contained in a complete intersection $Y \subset \mathbb{P}^m$ of multi-degree $\underline{d} = (d_1, \ldots, d_s)$, with $1 \leq s \leq m - 2$. We will assume that $Y$ is neither a linear subspace nor a quadric, cases to be considered as trivial. Thus we will constantly assume that $\Pi_{i=1}^s d_i > 2$.

Let $S_{\underline{d}} := \bigoplus_{i=1}^s H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d_i))$, and consider its Zariski open subset $S_{\underline{d}}^* := \bigoplus_{i=1}^s \left( H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d_i)) \setminus \{0\} \right)$. For any $u := (g_1, \ldots, g_s) \in S_{\underline{d}}^*$, let $Y_u := V(g_1, \ldots, g_s) \subset \mathbb{P}^m$ denote the closed subscheme defined by the vanishing of the polynomials $g_1, \ldots, g_s$. When $u \in S_{\underline{d}}^*$ is general, $Y_u$ is a smooth, irreducible variety of dimension $m - s \geq 2$. For any integer $k \geq 1$, we define the locus

$W_{\underline{d},k} := \{ u \in S_{\underline{d}}^* \mid F_k(Y_u) \neq \emptyset \}$

and set

$t(m, k, \underline{d}) := \sum_{i=1}^s \left( \frac{d_i + k}{k} \right) - (k + 1)(m - k)$.

If no confusion arises, we will simply denote $t(m, k, \underline{d})$ by $t$.

First of all, consider the case $t \leq 0$. This is the most studied case in the literature, and it is now well understood (cf. e.g. [2, 3, 9, 7]). In particular, the following holds.

Result 1. Let $m, k, s$ and $\underline{d} = (d_1, \ldots, d_s)$ be such that $\Pi_{i=1}^s d_i > 2$ and $t \leq 0$. Then:

(a) $W_{\underline{d},k} = S_{\underline{d}}^*$;
(b) for general $u \in S_{\underline{d}}^*$, $F_k(Y_u)$ is smooth, of dimension $\dim(F_k(Y_u)) = -t$ and it is irreducible when $\dim(F_k(Y_u)) \geq 1$.

The proof of this result can be found e.g. in [2 Prop.2.1, Cor.2.2, Thm. 4.1], for the complex case, and in [8 Thm. 2.1, (b) & (c)], for any algebraically closed field. In addition, in [8 Thm. 4.3] the authors compute $\deg(F_k(Y_u))$ under the Plücker embedding $F_k(Y_u) \subset \mathbb{G}(k, m) \hookrightarrow \mathbb{P}^N$, with $N = \binom{m+1}{k+1} - 1$. Their formulas extend to any $k \geq 1$ enumerative formulas by Libgober in [4], who computed $\deg(F_1(Y_u))$ when $t(m, 1, \underline{d}) = 0$.

On the other hand, we are interested in the case $t > 0$, where the known results can be summarized as follows.

Result 2. Let $m, k, s$ and $\underline{d} = (d_1, \ldots, d_s)$ be such that $\Pi_{i=1}^s d_i > 2$ and $t > 0$. Then:

(a) $W_{\underline{d},k} \subsetneq S_{\underline{d}}^*$.
(b) $W_{\underline{d},k}$ contains points $u$ for which $Y_u \subset \mathbb{P}^m$ is a smooth complete intersection of dimension $m - s$ if and only if $s \leq m - 2k$.
(c) For $s \leq m - 2k$, set $H_{\underline{d},k} := \{ u \in W_{\underline{d},k} \mid Y_u \subset \mathbb{P}^m \text{ is smooth, of dimension } m - s \}$. If $d_i \geq 2$ for any $1 \leq i \leq s$, then $H_{\underline{d},k}$ is irreducible, unirational and $\text{codim}_{S_{\underline{d}}}(H_{\underline{d},k}) = t$. Moreover, for general $u \in H_{\underline{d},k}$, $F_k(Y_u)$ is a zero-dimensional scheme.

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The proof of Result 2 (a) is contained in [3] Thm. 2.1 (a), whereas that of assertions (b) and (c) is contained in [3] Cor. 1.2, Rem. 3.4; both proofs therein hold for any algebraically closed field.

The main result of this paper, which improves on Result 2, is the following.

**Theorem 1.1.** Let \( m, k, s \) and \( d = (d_1, \ldots, d_s) \) be such that \( \Pi^0 = \Pi^1 \cdots = \Pi^s = 0 \). Then \( W_{d,k} \subset S^*_d \) is non-empty, irreducible and rational, with \( \text{codim}_{S^*_d}(W_{d,k}) = t \). Furthermore, for a general point \( u \in W_{d,k} \), the variety \( Y_u \subset \mathbb{P}^m \) is a complete intersection of dimension \( m-s \) whose Fano scheme \( F_k(Y_u) \) is a zero-dimensional scheme of length one. Moreover, \( Y_u \) has singular locus of dimension \( \max\{-1, 2k + s - m - 1\} \) along its unique \( k \)-dimensional linear subspace (in particular, \( Y_u \) is smooth if and only if \( m-s \geq 2k \)).

The proof of this theorem is contained in Section 2 and it extends [1] Prop. 2.3 to arbitrary \( k \geq 1 \). Theorem 1.1 improves, via different and easier methods, Miyazaki’s results in [5] Cor. 1.2, showing that for general \( u \in W_{d,k} \) one has \( \deg(F_k(Y_u)) = 1 \), which implies the rationality of \( W_{d,k} \). Moreover, we also get rid of Miyazaki’s hypothesis \( m-s \geq 2k \).

2. The proof

This section is devoted to the proof of Theorem 1.1.

**Proof of Theorem 2.** Let \( G := G(k, m) \) be the Grassmannian of \( k \)-linear subspaces in \( \mathbb{P}^m \) and consider the incidence correspondence

\[
J := \left\{ ([\Pi], u) \in G \times S^*_d | \Pi \subset Y_u \right\} \subset G \times S^*_d
\]

with the two projections

\[
G \xrightarrow{\pi_1} J \xrightarrow{\pi_2} S^*_d.
\]

The map \( \pi_1: J \rightarrow G \) is surjective and, for any \( [\Pi] \in G \), one has \( \pi_1^{-1}([\Pi]) = \bigoplus_{i=1}^s \left( H^0(\mathcal{I}_{\Pi}/\mathbb{P}^m(d_i)) \setminus \{0\} \right) \), where \( \mathcal{I}_{\Pi}/\mathbb{P}^m \) denotes the ideal sheaf of \( \Pi \) in \( \mathbb{P}^m \).

Thus \( J \) is irreducible with \( \text{dim}(J) = \text{dim}(G) + \text{dim}(\pi_1^{-1}([\Pi])) = (k+1)(m-k) + \sum_{i=1}^s h^0(\mathcal{I}_{\Pi}/\mathbb{P}^m(d_i)) \). From the exact sequence

\[
0 \rightarrow \bigoplus_{i=1}^s \mathcal{I}_{\Pi}/\mathbb{P}^m(d_i) \rightarrow \bigoplus_{i=1}^s \mathcal{O}_{\mathbb{P}^m}(d_i) \rightarrow \bigoplus_{i=1}^s \mathcal{O}_{\Pi}(d_i) \cong \bigoplus_{i=1}^s \mathcal{O}_{\mathbb{P}^m}(d_i) \rightarrow 0,
\]

one gets

\[
\text{dim}(J) = (k+1)(m-k) + \sum_{i=1}^s \left( d_i + m \right) - \sum_{i=1}^s \left( d_i + k \right) = \text{dim}(S^*_d) - t. \tag{2.1}
\]

The next step recovers [3] Cor. 1.2 via different and easier methods, and we also get rid of the hypothesis \( m-s \geq 2k \) present there. We essentially adapt the argument in [2] Proof of Prop. 2.1], used for the case \( t \leq 0 \).

**Step 1.** The map \( \pi_2: J \rightarrow S^*_d \) is generically finite onto its image \( W_{d,k} \), which is therefore irreducible and unirational. Moreover \( \text{codim}_{S^*_d}(W_{d,k}) = t \).

For general \( u \in W_{d,k} \), \( F_k(Y_u) \) is a zero-dimensional scheme and \( Y_u \) has singular locus of dimension \( \max\{-1, 2k + s - m - 1\} \) along any of the \( k \)-dimensional linear subspaces in \( F_k(Y_u) \).

**Proof of Step 1.** One has \( W_{d,k} = \pi_2(J) \), hence \( W_{d,k} \subset S^*_d \) is irreducible and unirational, because \( J \) is rational, being an open dense subset of a vector bundle over \( G \). Once one shows that \( \pi_2: J \rightarrow W_{d,k} \) is generically finite, one deduces that \( \text{codim}_{S^*_d}(W_{d,k}) = t \) from (2.1). Therefore, we focus on proving that \( \pi_2 \) is generically finite, i.e. that if \( u \in W_{d,k} \) is a general point, then \( \text{dim}(\pi_2^{-1}(u)) = 0 \).

Let \( [\Pi] \in G \) and choose \([y_0, y_1, \ldots, y_m]\) homogeneous coordinates in \( \mathbb{P}^m \) such that the ideal of \( \Pi \) is \( I_{\Pi} := (y_{k+1}, \ldots, y_m) \). For general \( ([\Pi], u) \in \pi_2^{-1}([\Pi]) \subset J \), with \( u = (g_1, \ldots, g_s) \in W_{d,k} \), we can write

\[
g_i = \sum_{h=k+1}^m y_h p_i^{(h)} + r_i, \quad 1 \leq i \leq s,
\]

with

\[
r_i \in (I_{\Pi}^2)^{d_k}, \quad \text{whence} \quad p_i^{(h)} = \sum_{|\mu|=d_k-1} c^{(h)}_{i,\mu} y^{\mu} \in \mathbb{C}[y_0, y_1, \ldots, y_k]_{d_k-1}, \quad 1 \leq i \leq s, \quad k+1 \leq h \leq m, \quad (2.3)
\]

where \((I_{\Pi}^2)_{d_k}\) is the homogeneous component of degree \( d_k \) of the ideal \( I_{\Pi}^2 \), \( \mu := (\mu_0, \ldots, \mu_k) \in \mathbb{Z}_{\geq 0}^{k+1} \), \(|\mu| := \sum_{r=0}^k \mu_r \), \( y^{\mu} := y_0^{\mu_0} y_1^{\mu_1} \cdots y_k^{\mu_k} \). By the generality assumption on \( u \), the polynomials \( p_i^{(h)} \) and \( r_i \) are general.
The Jacobian matrix \( \begin{pmatrix} \frac{\partial y_i}{\partial y_j} \end{pmatrix}_{1 \leq i \leq s, 0 \leq j \leq m} \) computed along \( \Pi \) takes the block form

\[
M = \begin{pmatrix} 0 & P \end{pmatrix}
\]

where the 0–block has size \( s \times (k+1) \) and \( P \) has size \( s \times (m-k) \), where \( m-k \geq s \) because of course \( \dim(Y_u) = m-s \geq k \).

By the generality of the polynomials \( p_i^{(k)} \), the locus of \( \Pi \) where \( \ker(M) < s \), which coincides with the singular locus of \( Y_u \) along \( \Pi \), has dimension max\( \{-1, 2k + s - m - 1\} \) and, by Bertini’s theorem, it coincides with the singular locus of \( Y_u \).

Next we consider the following exact sequence of normal sheaves

\[
0 \to N_{\Pi/Y_u} \to N_{\Pi/P^m} \cong \mathcal{O}_{P^m}(1)^{\oplus (m-k)} \to N_{Y_u/P^m}|_{\Pi} \cong \bigoplus_{i=1}^{s} \mathcal{O}_{P^m}(d_i) \tag{2.4}
\]

(see [8 Lemma 68.5.6]). Any \( \xi \in H^0(\Pi, N_{\Pi/P^m}) \) can be identified with a collection of \( m-k \) linear forms on \( \Pi \cong \mathbb{P}^k \)

\[
\varphi_{h}^{\xi}(y) := a_{h,0}y_0 + a_{h,1}y_1 + \cdots + a_{h,k}y_k, \; k+1 \leq h \leq m,
\]

whose coefficients fill up the \((m-k) \times (k+1)\) matrix

\[
A_{\xi} := (a_{h,j}), \; \kappa+1 \leq h \leq m, \; 0 \leq j \leq k;
\]

by abusing notation, one may identify \( \xi \) with \( A_{\xi} \).

Thus the map \( H^0(\Pi, N_{\Pi/P^m}) \overset{\sigma}{\to} H^0(\Pi, N_{Y_u/P^m}|_{\Pi}) \), arising from (2.4) is given by (cf. e.g. [2 formula (4)])

\[
A_{\xi} \overset{\sigma}{\to} \left( \sum_{0 \leq j < k \leq m} a_{h,j}y_j p_i^{(h)} \right)_{1 \leq i \leq s}. \tag{2.5}
\]

Notice that the assumption \( t > 0 \) reads as

\[
(k+1)(m-k) = h^0(\Pi, N_{\Pi/P^m}) < h^0(\Pi, N_{Y_u/P^m}|_{\Pi}) = \sum_{i=1}^{s} \binom{d_i+k}{k}.
\]

Claim 2.1. The map \( H^0(\Pi, N_{\Pi/P^m}) \overset{\sigma}{\to} H^0(\Pi, N_{Y_u/P^m}|_{\Pi}) \) is injective, equivalently \( h^0(\Pi, N_{Y_u/P^m}) = 0 \). In particular, for a general point \( u \in W_{d,k} \), the Fano scheme \( F_k(Y_u) \) contains \(|\Pi|\) as a zero–dimensional integral component.

Proof of Claim 2.1. Using (2.3), the polynomials on the right–hand–side of (2.5) read as

\[
\sum_{h=k+1}^{m} \sum_{j=0}^{k} a_{h,j}y_j \left( \sum_{|\mu|=d_i-1} c_{i,h}^{(h)} y_\mu \right), \; 1 \leq i \leq s.
\]

Ordering the previous polynomial expressions via the standard lexicographical monomial order on the canonical basis \( \{y_\mu\} \) of \( \mathbb{C}[y_0, y_1, \ldots, y_k] \) one has \( \sigma^i = H^0(\mathcal{O}_{P^m}(d_i)) \), \( 1 \leq i \leq s \), the injectivity of the map \( \sigma \) is equivalent for the homogeneous linear system

\[
\sum_{0 \leq j < k \leq m} c_{i,h}^{(h)} a_{h,j} = 0, \; 1 \leq i \leq s, \tag{2.6}
\]

to have only the trivial solution, where \( c_{i,h}^{(h)} := (\nu_0, \nu_1, \ldots, \nu_k) \in \mathbb{Z}_{\geq 0}^{k+1} \) is such that \( |\nu| = d_i \), \( \xi_{i,h} \) is the \((j+1)\)–th vertex of the standard \((k+1)\)–simplex in \( \mathbb{Z}_{\geq 0}^{k+1} \) \( \setminus \{0\} \), and \( c_{i,h}^{(h)} = 0 \) when \( \nu - \xi_{i,h} \not\in \mathbb{Z}_{\geq 0}^{k+1} \) (this last condition stands for “\( \nu - \xi_{i,h} \) improper” as formulated in [2 p. 29]). The linear system (2.6) consists of \( \sum_{i=1}^{s} \binom{d_i+k}{k} \) equations in the \((k+1)(m-k)\) indeterminates \( a_{h,j} \), with coefficients \( c_{i,h}^{(h)}, 0 \leq j < k \leq h \leq m \).

Let \( C := (c_{i,h}^{(h)}) \) be the coefficient matrix of (2.6); one is reduced to show that, for general choices of the entries \( c_{i,h}^{(h)} \), the matrix \( C \) has maximal rank \((k+1)(m-k)\). This can be done arguing as in [2 p. 29]. Namely, row–indices of \( C \) are determined by the standard lexicographical monomial order on the canonical basis of \( \bigoplus_{i=1}^{s} \mathbb{C}[y_0, y_1, \ldots, y_k] \), whereas column–indices of \( C \) are determined by the standard lexicographic order on the set of indices \((h,j)\). If one considers the square sub–matrix \( \bar{C} \) of \( C \) formed by the first \((k+1)(m-k)\) rows and by all the columns of \( C \), then \( \det(\bar{C}) \) is a non–zero polynomial in the indeterminates \( c_{i,h}^{(h)} \). Indeed, take the lexicographic order on the set of indices

\[
(h, i, \mu), \; \text{where} \; k+1 \leq h \leq m, \; |\mu| = d_i - 1, \; 1 \leq i \leq s,
\]
and order the monomials appearing in the expression of \( \det(\hat{C}) \) according to the following rule: the monomials \( m_1 \) and \( m_2 \) are such that \( m_1 > m_2 \) if, considering the smallest index \((h, i, \mu)\) for which \( c_{i,\mu}^{(h)} \) occurs in the monomial \( m_1 \) with exponent \( p_1 \) and in the monomial \( m_2 \) with exponent \( p_2 \neq p_1 \), one has \( p_1 > p_2 \). The greatest monomial (in the monomial ordering described above) appearing in \( \det(\hat{C}) \) has coefficient \( \pm 1 \), since in each column the choice of the \( c_{i,\mu}^{(h)} \) entering in this monomial is uniquely determined. By maximality of such monomial, it follows that \( \det(\hat{C}) \neq 0 \). Therefore, the condition \( \delta(m, k, d) \) described above) appearing in \( \det(\hat{C}) \) is zero-dimensional and reduced, i.e. the map \( \sigma \) is injective.

The injectivity of \( \sigma \) and \( \delta(m, k, d) \) yield \( h^0(N_{k/k}) = 0 \). Since \( R^0(N_{k/k}) \) is the tangent space to \( F_k(Y_u) \) at its point \([\Pi]\), one deduces that \( \{[\Pi]\} \) is a zero-dimensional, reduced component of \( F_k(Y_u) \), as claimed.

Finally, by monodromy arguments, the irreducibility of \( J \) and Claim \( \delta \) ensure that for general \( u \in W_{d,k} \), the Fano scheme \( F_k(Y_u) \) is zero-dimensional and reduced, i.e. \( \pi_2: J \to W_{d,k} \) is generically finite, and that \( Y_u \) has a singular locus of dimension \( \max\{-1, 2k + s - m - 1\} \) along any of the \( k \)-dimensional linear subspaces in \( F_k(Y_u) \). This completes the proof of Step \( \Pi \).

To conclude the proof of Theorem \( \Pi \), we need the following numerical result.

\textbf{Step 2.} For \( 0 \leq h \leq k - 1 \) integers, consider the integer
\[
\delta_h(m, k, d) := \sum_{i=1}^{s} \left( d_i + \frac{k + i}{h} \right) - \sum_{i=1}^{s} \left( d_i + \frac{h}{h} \right) = (k - h)(m + h + 1 - k).
\]

If \( \delta_h(m, k, d) \leq 0 \), then
\[
t(m, k, d) \leq 0.
\]

\textbf{Proof of Step \( \Pi \).} In order to ease notation, we set \( \delta_h := \delta_h(m, k, d) \). Therefore, the condition \( \delta_h \leq 0 \) implies \( m \geq 1 - \delta_h \). Plugging the previous inequality in the expression of \( t \), one has
\[
t \leq - \sum_{i=1}^{s} \left( h + \frac{k + i}{k} \right) \left( d_i + \frac{k}{k} \right) - \sum_{i=1}^{s} \left( h + \frac{k + i}{k} \right) \left( d_i + \frac{h}{h} \right) = (k + 1)(h + 1).
\]

Set \( D(x) := \frac{h + 1}{k - h} \left( \sum_{i=1}^{s} \left( d_i + \frac{k + i}{k} \right) \right) - \frac{h + 1}{k - h} \left( \sum_{i=1}^{s} \left( d_i + \frac{h}{h} \right) \right) \). Thus, \( \delta \) reads
\[
t \leq - \sum_{i=1}^{s} D(d_i) + (k + 1)(h + 1).
\]

The assumption \( 0 \leq h \leq k - 1 \) gives
\[
D(d_i) = \frac{(h + 1)(d_i + 1) \cdots (d_i + d_i)}{k! (k - h)} \left( (d_i + 1) \cdots (d_i + k) - (k + 1)k \cdots (h + 2) \right), 1 \leq i \leq s.
\]

The polynomial \( D(x) \) vanishes for \( x = 1 \), which is its only positive root. Notice that
\[
D(1) = \frac{h + 1}{k - h} \left( \sum_{i=1}^{s} \left( d_i + \frac{k + i}{k} \right) \right) - \frac{h + 1}{k - h} \left( \sum_{i=1}^{s} \left( d_i + \frac{h}{h} \right) \right) = \frac{(h + 1)(k + 1)}{2} > 0.
\]

In particular, \( D(x) \) is increasing and positive for \( x > 1 \), so from \( \delta \) it follows that
\[
t \leq - \sum_{i=1}^{s} D(d_i) + (k + 1)(h + 1) = - s D(2) + (k + 1)(h + 1) = (k + 1)(h + 1) \left( 1 - \frac{s}{2} \right).
\]

Therefore, when \( s \geq 2 \), we have \( t \leq 0 \) and we are done in this case.

If \( s = 1 \), set \( d := d_1 \). In this case \( \delta \) is \( t \leq - D(d) + (k + 1)(h + 1) \), where again \( D(d) \) is increasing and positive for \( d > 1 \). When \( s = 1 \), we have \( d \geq 3 \) by assumption. Thus, one computes
\[
D(1) = (k + 1)(h + 1) \frac{k + h + 5}{6}
\]

and so, for any \( d \geq 3 \), one has
\[
t \leq - D(d) + (k + 1)(h + 1) \leq - D(3) + (k + 1)(h + 1) = (k + 1)(h + 1) \frac{1 - k - h}{6}.
\]

Thus, \( t \leq 0 \), completing the proof of Step \( \Pi \).
Step 3. For general \( u \in W_{\mathbb{A}^k} \), the zero-dimensional Fano scheme \( F_k(Y_u) \) has length one. In particular, the map \( \pi_2 : J \to W_{\mathbb{A}^k} \) is birational and \( W_{\mathbb{A}^k} \) is rational.

Proof of Step 3. Let us consider the (locally closed) incidence correspondence

\[
I := \left\{ ([\Pi_1], [\Pi_2], u) \in G \times G \times S^*_{\mathbb{A}^k} | \Pi_1 \neq \Pi_2, \Pi_i \subset Y_u, 1 \leq i \leq 2 \right\} \subset G \times G \times S^*_{\mathbb{A}^k}.
\]

If \( I \) is not empty, let \( \varphi : I \to J \) be the map defined by

\[
\varphi (([\Pi_1], [\Pi_2], u)) = ([\Pi_1], u).
\]

We need to prove that \( \varphi \) is not dominant. To do this, consider the (locally closed) subset

\[
I_h := \left\{ ([\Pi_1], [\Pi_2], u) \in I | \Pi_1 \cap \Pi_2 \cong \mathbb{P}^h \right\}, \quad \text{where} \quad -1 \leq h \leq k - 1
\]

(we set \( \mathbb{P}^{-1} = \emptyset \), i.e. the case \( h = -1 \) occurs when \( \Pi_1 \) and \( \Pi_2 \) are skew). Clearly, one has \( I = \bigcup_{h=-1}^{k-1} I_h \). Setting \( \varphi_h := \varphi|_{I_h} \), it is sufficient to prove that \( \varphi_h \) is not dominant, for any \(-1 \leq h \leq k - 1\).

So, let \( h \) be such that \( I_h \) is not empty, and let \( T_h \) be an irreducible component of \( I_h \). Of course, if \( \dim(T_h) \prec \dim(J) \), the restriction \( \varphi_h|_{T_h} : T_h \to J \) is not dominant. On the other hand, suppose that \( \dim(T_h) \prec \dim(J) \). For any such a component, the map \( \varphi_h|_{T_h} \) cannot be dominant, otherwise the composition \( T_h \overset{\varphi_h}{\to} J \overset{\pi_2}{\to} W_{\mathbb{A}^k} \) would be dominant, as \( \pi_2 \) is, which would imply that the general fiber of \( \pi_2 \) is positive dimensional, contradicting Step 1.

Therefore, it remains to investigate the case \( \dim(T_h) = \dim(J) \). We estimate the dimension of \( T_h \) as follows. Consider

\[
G^2_h := \left\{ ([\Pi_1], [\Pi_2]) \in G \times G | \Pi_1 \cap \Pi_2 \cong \mathbb{P}^h \right\} \subset G \times G,
\]

which is locally closed in \( G \times G \). The projection

\[
\bar{\pi}_1 : G^2_h \to G, \quad ([\Pi_1], [\Pi_2]) \mapsto [\Pi_1]
\]

is surjective onto \( G \) and any \( \bar{\pi}_1 \)-fiber is irreducible, of dimension equal to \( \dim \left( G \times (k-h-1, m-h-1) \right) = (h+1)(k-h) + (k-h)(m-k) \). Thus

\[
\dim G^2_h = (k+1)(m-k) + (h+1)(k-h) + (k-h)(m-k).
\]

One has the projection

\[
\psi_h : T_h \overset{\varphi_h}{\to} G^2_h, \quad ([\Pi_1], [\Pi_2], u) \mapsto ([\Pi_1], [\Pi_1]),
\]

which is surjective, because the projective group acts transitively on \( G^2_h \). Hence \( \dim(T_h) = \dim(G^2_h) + \dim(\mathfrak{F}_h) \), where \( \mathfrak{F}_h := \bigoplus_{i=1}^{s_2} \left( H^0 \left( \mathcal{I}_{\Pi_1 \cup \Pi_2/\mathbb{P}^n}(d_i) \right) \setminus \{0\} \right) \) is the general fiber of \( \psi_h|_{T_h} \) and where \( \mathcal{I}_{\Pi_1 \cup \Pi_2/\mathbb{P}^n} \) denotes the ideal sheaf of \( \Pi_1 \cup \Pi_2 \) in \( \mathbb{P}^m \).

Claim 2.2. For every positive integer \( d \) one has

\[
h^0(\mathcal{I}_{\Pi_1 \cup \Pi_2/\mathbb{P}^n}(d)) = \dim(S_d) - 2 \binom{d + k}{k} + \binom{d + h}{h}.
\]

Proof of Claim 2.2. We have

\[
h^0(\mathcal{I}_{\Pi_1/\mathbb{P}^n}(d)) = \dim(S_d) - \binom{d + k}{k}.
\]

Consider the linear system \( \Sigma \) cut out on \( \Pi_2 \) by \( |\mathcal{I}_{\Pi_1/\mathbb{P}^n}(d)| \). We claim that \( \Sigma \) is the complete linear system of hypersurfaces of degree \( d \) of \( \Pi_2 \) containing \( \Pi := \Pi_1 \cap \Pi_2 \). Indeed \( \Sigma \) contains all hypersurfaces consisting of a hyperplane through \( \Pi \) plus a hypersurface of degree \( d-1 \) of \( \Pi_2 \), which proves our claim. In the light of this fact, and arguing as in 2.1 and 2.2, we deduce that

\[
h^0(\mathcal{I}_{\Pi_1 \cup \Pi_2/\mathbb{P}^n}(d)) = h^0(\mathcal{I}_{\Pi_1/\mathbb{P}^n}(d)) - (\dim(\Sigma) + 1) = h^0(\mathcal{I}_{\Pi_1/\mathbb{P}^n}(d)) - \binom{d + k}{k} - \binom{d + h}{h},
\]

which, by 2.1, yields the assertion. \( \square \)

By Claim 2.2 we have

\[
\dim(\mathfrak{F}_h) = \dim(S^*_{\mathbb{A}^k}) - 2 \sum_{i=1}^{s_2} \binom{d_i + k}{k} + \sum_{i=1}^{s} \binom{d_i + h}{h}.
\]
Hence
\[
\dim(T_h) = \dim(S_h) + \dim(G^2_h) = \\
= \dim(S^*_{2h}) - 2 \sum_{i=1}^{s} \left( \frac{d_i + k}{k} \right) + \sum_{i=1}^{s} \left( \frac{d_i + h}{h} \right) + \\
+ (k + 1)(m - k) + (h + 1)(k - h) + (k - h)(m - k) = \\
= \dim(J) - \sum_{i=1}^{s} \left( \frac{d_i + k}{k} \right) + \sum_{i=1}^{s} \left( \frac{d_i + h}{h} \right) + (k - h)(m + h + 1 - k) = \\
= \dim(J) - \delta_h.
\] (2.10)

Since \( \dim(T_h) = \dim(J) \), (2.10) implies \( \delta_h = 0 \). When \( 0 \leq h \leq k - 1 \), Step 2 gives \( t \leq 0 \), contrary to our assumption. When \( h = -1 \), one has \( 0 = \delta_{-1} = t \), again against our assumptions.

Since no component \( T_h \subset I_h \) can dominate \( J \), the map \( \varphi: I \to J \) is not dominant. We conclude therefore that the map \( \pi_2 : J \to W_{d,k} \) is birational, completing the proof of Step 3.

Steps 1–3 prove Theorem 1.1.

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References

[1] F. Bastianelli, C. Ciliberto, F. Flamini, P. Supino, Gonality of curves on general hypersurfaces, arxiv:1707.07252, 23 Jul 2017, 1–23.
[2] C. Borcea, Deforming varieties of \( k \)-planes of projective complete intersections, Pacific J. Math. 143 (1990), 25–36.
[3] O. Debarre, L. Manivel, Sur la variété des espaces linéaires contenus dans une intersection complète, Math. Ann. 312 (1998), 549–574.
[4] A. S. Libgober, Numerical characteristics of systems of straight lines on complete intersections, Math. Notes 13 (1973), 51–56, English translation of the original paper in Mat. Zametki 13 (1973), 87–96.
[5] C. Miyazaki, Remarks on \( r \)-planes in complete intersections, Tokyo J. Math. 39 (2016), 459–467.
[6] U. Morin, Sull’insieme degli spazi lineari contenuti in una ipersuperficie algebrica, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. 24 (1936), 188–190.
[7] A. Predonzan, Intorno agli \( S_k \) giacenti sulla varietà intersezione completa di più forme, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. 5 (1948), 238–242.
[8] The Stacks Project Authors, The stack project, (2018), https://stacks.math.columbia.edu

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