Nonsingular bouncing cosmology in general relativity: physical analysis of the spacetime defect

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Abstract
In this paper, we describe physical effects occurring in the regularized Robertson–Walker spacetime which can reveal the presence of the defect. Our analysis is based on two main physical quantities: the compressive forces acting on (human) observers and the energy possessed by massive particles and photons during their dynamical evolution. In section 3, we claim that with a characteristic length scale of the order of the Planck length compressive forces become so intense near the defect that no (human) observer is able to cross it. In section 4, we show that the energy exhibits an unusual character over a small time interval around the bounce contrasting with the behaviour in the standard cosmology picture. We conclude the paper with some considerations and open problems related to our results.

Keywords: general relativity, big bang theory, mathematical and relativistic aspects of cosmology

(Some figures may appear in colour only in the online journal)

1. Introduction

Spacetime defect has been recently proposed in the literature as a tool to tame the big-bang singularity [1–4]. Such an object can be described by a degenerate metric with a vanishing determinant on a three-dimensional submanifold of the spacetime and a nonzero length scale $b$, which acts as a ‘regulator’ of the Friedmann singularity. This new metric has been called the
regularized-big-bang metric. It gives rise to a nonsingular spatially flat Friedmann-type solution of the Einstein gravitational field equation which allows for a ‘pre-big-bang’ phase with a bounce-type behaviour of the cosmic scale factor. Cosmological observables occurring in this geometry such as past particle horizon and modified Hubble diagrams have been investigated in reference [3] and the effective violation of the null energy condition (NEC) in the vicinity of the defect was first pointed out in reference [2] (for details see reference [4]). A gravitational model for the nonsingular bounce solution involving a Brans–Dicke-type scalar field having, in the potential action, a ‘wrong-sign’ kinetic component and a quartic interaction term has been presented in reference [5], where it is shown that the bounce behaviour appears if the boundary conditions provide for a kink-type solution of the Brans–Dicke-type scalar field.

The physical investigation of the defect is a subtle issue, since classical physics may not be valid at \( t = 0 \). Possible connections between the characteristic length scale \( b \) and the Planck length suggested by loop quantum cosmology [6] and string cosmology [7] have been explored in appendix B of reference [2] (see also appendix C of reference [4]). Furthermore, a new explanation about the origin of \( b \) has been set forth recently in reference [8] (see also references [9, 10]), where it has been shown that the classical regularized-big-bang metric can, in principle, emerge from the IIB matrix model (i.e. a nonperturbative formulation of type-IIB superstring theory [11]). This means that physics of the spacetime defect might require knowledge of the underlying model and quantities at \( t = 0 \) might originate from the underlying theory. The situation might be similar to that of an atomic crystal, where classical physics is a reliable source of information everywhere except at the atomic defect, whose details necessitate quantum mechanics. In the same way, according to the picture of reference [8], the classical spacetime can be emergent for any value of the \( t \) variable and the ‘point’ \( t = 0 \) is only defined via a limit procedure.

Motivated by the fact that the description of physics at \( t = 0 \) is a delicate point, in this paper we will examine physical effects pointing out the presence of the defect in the regularized Robertson–Walker (RW) spacetime. We will see that if the length scale \( b \) had a quantum nature, then the defect would be shaped as an object allowing no human observer to go across it. Furthermore, we will explain how massive particles and photons energy display, in the proximity of the defect, a behaviour deviating from the expectations of standard RW cosmology.

The plan of the paper is as follows. In section 2, we recall some basic concepts of modified RW spacetime and define two different freely falling observers which will fulfil a crucial role in our analysis: the Eulerian observer and the non-comoving observer, which we will call the ‘traveller’. In section 3, we will investigate how compressive forces affect these observers. For this purpose, we will suppose to deal with human observers, i.e. observers made of atoms. In section 4, we propose a definition of energy suitable for our model and analyze its features. Eventually, concluding remarks are made in section 5.

Throughout the paper, we use metric signature \((- + + +)\) and natural units with \( c = 1 \) and \( \hbar = 1 \).

2. Freely falling observers in the regularized Robertson–Walker geometry

In references [1, 2] it has been shown that the big-bang singularity underlying the standard Friedmann cosmology can be regularized by employing the following ansatz for the modified spatially flat RW metric:

\[
 ds^2 = -\frac{\rho^2}{\rho^2 + b^2} \, dt^2 + \alpha(t)^2 \delta_{ij} \, dx^i \, dx^j, \quad (i, j = 1, 2, 3), \tag{1a}
\]
\[ b^2 > 0, \quad (1b) \]
\[ a(t) \in \mathbb{R}, \quad (1c) \]
\[ t \in (-\infty, +\infty), \quad (1d) \]
\[ x^t \in (-\infty, +\infty), \quad (1e) \]

where \( t \) denotes the cosmic time coordinate, \{ \( x^1, x^2, x^3 \) \} the comoving spatial Cartesian coordinates, \( a(t) \) the cosmic scale factor and \( b > 0 \) corresponds to the characteristic length scale of the spacetime defect localized at \( t = 0 \). By employing the metric (1) and the energy-momentum tensor of a homogeneous perfect fluid having energy density \( \rho \) and pressure \( P \), the Einstein equation with a vanishing cosmological constant gives [1, 2]

\[
\left[ 1 + \frac{b^2}{r^2} \right] \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho, \quad (2a)
\]

\[
\left[ 1 + \frac{b^2}{r^2} \right] \left[ \frac{\ddot{a}}{a} + \frac{1}{2} \left( \frac{\dot{a}}{a} \right)^2 \right] - \frac{b^2 \dot{a}}{r^2 a} = -4\pi GP, \quad (2b)
\]

\[
\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + P) = 0, \quad (2c)
\]

\[
P = P(\rho), \quad (2d)
\]

where the dot signifies a differentiation with respect to the \( t \) variable. Equations (2a) and (2b) are the modified first-order and second-order spatially flat Friedmann equations, respectively, equation (2c) the energy-conservation equation of the matter, and equation (2d) the equation of state. Two remarks are in order. First, since the inverse metric from (1a) has the \( g^{00} \) component which diverges at \( t = 0 \), the reduced field equations at \( t = 0 \) must be obtained carefully from the limit \( t \to 0 \) (see section 3.3.1 of reference [12] for further details). Second, the new \( b^2 \) terms in the modified Friedmann equations (2a) and (2b) are a manifestation of the different differential structure of (1a) compared to the differential structure of the standard spatially flat RW metric [13, 14] which gives the standard Friedmann equations (see references [1–3] for details).

From equation (2), we obtain for the function \( a(t) \) the following solutions (with normalization \( a(t_0) = 1 \) at \( t_0 > 0 \)) [1, 2]:

\[
a(t) = \begin{cases} \left( \frac{b^2 + t^2}{b^2 + t_0^2} \right)^{1/3}, & \text{nonrelativistic matter,} \\ \left( \frac{b^2 + t^2}{b^2 + t_0^2} \right)^{1/4}, & \text{relativistic matter,} \end{cases} \quad (3)
\]

depending on whether a relativistic-matter or a nonrelativistic-matter equation of state for the perfect fluid is used. From equation (3), the bouncing behaviour of the positive scale factor is evident: \( a(t) \) decreases (resp. increases) for negative (resp. positive) \( t \) and the bounce occurs at \( t = 0 \) where \( a(t) \) has a vanishing time derivative.

For later use, we also give the affine connection component

\[
\Gamma^t_{tt} = \frac{b^2}{t(b^2 + t^2)}, \quad (4)
\]
One of the main advantages of our model consists in the fact that it depends on only one free parameter, i.e. the defect length scale $b$. In this paper, we will examine the effects produced by the possible range of values of $b$. In that regard, we recall that in reference [3] a Gedankenexperiment which makes use of modified Hubble diagrams has been proposed in order to evaluate $b$ numerically. Furthermore, it should be stressed that the study of cosmological perturbations is a precious mean to work out observable constraints on $b$. Indeed, this analysis makes it possible to determine several observables like the spectral index of the primordial curvature perturbations, the tensor-to-scalar ratio, and the running of the spectral index, permitting thus a comparison with the latest Planck 2018 data [15] (for a thorough investigation of cosmological perturbations framed in extended theories of gravity we refer the reader to [16–18] and references therein). Scalar metric perturbations for the modified RW geometry (1) have been tackled in reference [4], where it has been proved the stability of the bounce under small perturbations of the metric and the matter. Likewise modified Friedmann equation (2), which represent singular differential equations having nonsingular solutions [1, 2], the metric perturbations exhibit nonsingular solutions, although they are described by singular differential equations (the singularity appears at $t = 0$) [4]. However, it is worth mentioning that since the pre-bounce contracting phase is unstable to the growth of anisotropies, our model is plagued by the Belinskii–Khalatnikov–Lifshitz (BKL) instability [19].

In the study of the generation era of the primordial perturbation modes from Bunch–Davies vacuum state [20, 21], a key parameter is represented by the comoving Hubble radius, which is defined as

$$R_H(t) = \frac{1}{\alpha(t)H(t)}, \quad H(t) = \frac{\dot{a}(t)}{a(t)}$$

(5)

$H(t)$ being the Hubble rate. The scale factor (3) leads immediately to the following expressions:

$$R_H(t) = \begin{cases} \frac{2t}{2(b^2 + t^2)^{3/4} (b^2 + t_0^2)^{1/4}}, & \text{relativistic matter}, \\ \frac{3}{2} \left( b^2 + t^2 \right)^{2/3} (b^2 + t_0^2)^{1/3}, & \text{nonrelativistic matter} \end{cases}$$

(6)

The plot of (the absolute value of) $R_H(t)$ is displayed in figures 1 and 2. It is clear that the comoving Hubble radius diverges both at $t = 0$ (which is a peculiar feature of all bouncing cosmologies) and, most importantly, at great distances from the defect. Therefore, the asymptotic behaviour of the comoving Hubble radius (6) is the same as the one predicted by those bouncing models characterized by primordial perturbation modes generated at very large negative cosmic times (see e.g. [16]). The other viable picture occurs when $R_H(t)$ vanishes asymptotically, resulting in perturbation modes produced near the bouncing epoch (see reference [18] for further details).

In view of our forthcoming analysis, it will be crucial to define two different types of observers. First of all, we have the freely falling comoving observer known as the Eulerian observer. It is easy to show that for this observer, whose proper time will be indicated with $\tau$, the unit timelike four-velocity vector can be written as [14]

$$\frac{dx^a}{d\tau} \equiv n^a = \frac{\sqrt{t^2 + b^2}}{|t|} (1, 0, 0, 0),$$

(7)

1 Vector and tensor perturbations are also briefly analyzed in appendix B of reference [4].
Figure 1. The absolute value of the comoving Hubble radius (6) for the nonrelativistic-matter solution and with $b = 1$ and $t_0 = 4\sqrt{5}$. The suffix ‘NR’ stands for ‘nonrelativistic’ cosmological matter content.

Figure 2. The absolute value of the comoving Hubble radius (6) for the relativistic-matter solution and with $b = 1$ and $t_0 = 4\sqrt{5}$. The suffix ‘R’ stands for ‘relativistic’ cosmological matter content.

where we have introduced the usual notation $x^\mu = (t, x^i)$. It will soon be clear that the motion of the Eulerian observer is characterized by a vanishing conserved momentum II (see equation (10) below). We also note that equation (7) is not defined at $t = 0$ (cf equation (1d)). However, we will see that this fact will not prevent us from computing quantities we are interested in.

The second observer, which throughout the paper will be referred to as the ‘traveller’, is a generic freely falling non-comoving observer having a unit timelike four-velocity vector given by

$$\frac{dx^\alpha}{d\tau'} \equiv \upsilon^\alpha,$$

where
where $\tau'$ denotes the traveller’s proper time. Due to the spatial maximal symmetry of (1a) and the associated conserved angular momentum, we will suppose that, without loss of generality, the traveller moves in the $(t, x)$-plane. Therefore, from the condition $v \cdot v = -1$ we obtain
\[
\left( \frac{dt}{d\tau'} \right)^2 = \left( \frac{t^2 + b^2}{t^2} \right) \left[ 1 + a(t)^2 \left( \frac{dx}{d\tau'} \right)^2 \right].
\] (9)

Furthermore, we can define the conserved momentum $\Pi$ along the traveller’s timelike geodesic as
\[
\Pi = V^{(x)}_\alpha \frac{dx^\alpha}{d\tau'} = a(t)^2 \frac{dx}{d\tau'},
\] (10)

$V^{(x)}$ being the spacelike $x$-translational Killing vector. Therefore, the nonvanishing components of the traveller’s four velocity (8) read as
\[
v^t = \begin{pmatrix} t^2 + b^2 \end{pmatrix} \left( 1 + \frac{\Pi^2}{a(t)^2} \right),
\] (11a)
\[
v^x = \frac{\Pi}{a(t)^2}.
\] (11b)

Bearing in mind the above equations, if we consider a setup where the traveller moves along the increasing direction of the $x$-axis (i.e. $\Pi > 0$ and $dx/dt > 0$), then we can write the corresponding geodesic equation as
\[
\frac{dx(t)}{dt} = \frac{|t|}{\sqrt{t^2 + b^2}} \frac{1}{a(t)^2 \left( 1 + a(t)^2 / \Pi^2 \right)}.
\] (12)

We have solved equation (12) and have seen that the traveller’s timelike geodesics turn out to be well-behaved at $t = 0$. This agrees with the analysis of null geodesics performed in section 3.1 of reference [3]. However, it should be stressed that, since Levi-Civita connection of degenerate metrics is not unique, some ambiguities in the study of geodesic equation can arise (see reference [12] for further details).

3. Compressive forces acting on observers

In this section, we will evaluate the compressive forces felt by the Eulerian observer and the traveller as they travel towards the defect. In that regard, we will suppose to deal with human observers, meaning that we will assume that they are made of atoms.

In order to evaluate the abovementioned compressive forces, we will employ two different instantaneous rest frames: the proper reference frame $e_\alpha = (e_t, e_x, e_y, e_z)$ of the Eulerian observer and the proper reference frame $e_{\alpha'} = (e_{t'}, e_{x'}, e_{y'}, e_{z'})$ of the traveller. Such frames will be constructed in equations (14)–(16) and (27) and (28), respectively. We are aware of the fact that in our model (1) the equivalence principle does not hold at $t = 0$ and for this reason we will employ the limit $t \to 0$ to evaluate at $t = 0$ the quantities we are interested in (details can be found in reference [12]).

The compressive forces felt by a (human) observer are measured via the components of the Riemann curvature tensor occurring in the geodesic deviation equation evaluated in his/her
local orthonormal frame. We recall that the geodesic deviation equation can be written as [14]

\[ \Delta a^\alpha = - F_{\text{compressive}}^\alpha, \] (13a)

\[ F_{\text{compressive}}^\alpha = R^\alpha_{\beta\gamma\delta} u^\beta \xi^\gamma u^\delta, \] (13b)

where \( \Delta a = \nabla_u \nabla_u \xi \) is the relative acceleration of two freely falling test particles (i.e. two nearby geodesics) having separation vector \( \xi \) and four-velocity \( u \).

With the above premises, we are ready to construct the proper reference frame \( \{ \hat{e}_\alpha \} \) of the Eulerian observer and to calculate compressive forces acting on him/her. We will see that we can get the traveller’s proper reference frame \( \{ \hat{e}_\alpha' \} \) by simply applying a Lorentz boost to Eulerian observer’s frame. Since both the Eulerian observers and the traveller are freely falling observers, their proper reference frames turn out to be local Lorentz frames all along their geodesics worldline (with the exclusion of \( t = 0 \), i.e. the defect). The employed coordinates (which are Riemann normal coordinates with axes marked by gyroscopes, see section 13.6 of reference [14] and also references [22, 23] for details) are known as Fermi normal coordinates (we have explicitly checked that the conditions \( \nabla_n \hat{e}_\alpha = 0 \) and \( \nabla_v \hat{e}_\alpha' = 0 \) hold). By invoking the usual tetrad formalism [24] we obtain

\[ g = \eta_{\mu\nu} \theta^\mu \otimes \theta^\nu, \] (14a)

\[ \theta^\tau = \sqrt{\frac{t^2}{t^2 + b^2}} \, dt, \] (14b)

\[ \theta^{\xi'} = a(t) \, d\xi'. \] (14c)

From the relations

\[ \theta^\tau = e^\tau_{\mu} \, dx^\mu, \] (15a)

\[ e^\mu_{\alpha} e^\nu_{\beta} = \delta^\mu_{\beta}, \] (15b)

\[ e_{\tau} = e^\tau_{\mu} e_{\mu}. \] (15c)

we get (see equation (7))

\[ e_\tau = \sqrt{\frac{t^2 + b^2}{t^2}} \, e_\tau = n, \] (16a)

\[ e^{\xi'} = \frac{1}{a(t)} \, e^{\xi'}. \] (16b)

The components \( R_{\hat{\gamma}\hat{\beta}\hat{\delta}} \) of the Riemann tensor in the proper reference frame \( (e_\tau, e_\xi, e_\eta, e_\zeta) \) can be obtained from those computed in the frame \( e_\mu = (e_\tau, e_\xi, e_\eta, e_\zeta) \) through

\[ R_{\hat{\gamma}\hat{\beta}\hat{\delta}} = e^\gamma_{\mu} e^\beta_{\nu} e^\delta_{\rho} R_{\mu\nu\rho\delta}, \] (17a)

\[ R_{\hat{\beta}\hat{\gamma}\hat{\delta}} = \eta_{\hat{\gamma}\hat{\delta}} R_{\hat{\beta}\hat{\delta}}. \] (17b)

At this stage, we can analyze the compressive forces acting on the Eulerian human observer by evaluating the geodesic deviation equation in his/her proper reference. If we set \( u = n \) in equation (13) and exploit equations (16a) and (17) along with the condition \( \xi \cdot n = -\xi^\tau = 0, \)
we obtain (no sum over \( j \)):

\[ \Delta a^j = -R^j_{\hat{\tau} \hat{\tau} \hat{\tau}} \xi^j, \]  
\[ \Delta a^j = \frac{d^2}{d\tau^2} \xi^j, \]  
\[ -R^j_{\hat{\tau} \hat{\tau} \hat{\tau}} = \left( \frac{b^2 + t^2}{t^2} \right) \frac{\ddot{a}}{a} - \frac{b^2}{t^3} \frac{\dot{a}}{a} \right) = \begin{cases} \frac{-2}{9 (b^2 + t^2)}, & \text{nonrelativistic matter}, \\ \frac{-1}{4 (b^2 + t^2)}, & \text{relativistic matter}. \end{cases} \]

(18a)  
(18b)  
(18c)

The fact that equation (18) depends on both first order and second order derivatives of \( a(t) \) marks a difference with standard RW cosmology, which predicts compressive forces depending only on the ratio \( \dot{a}/a \). On the other hand, the minus sign occurring in front of the quantities contained inside the curly bracket in equation (18c) shows that we are dealing with compressive forces (cf equation (13)), in strict analogy with standard RW cosmology [13], where Friedmann equations foretell a scale factor having \( \ddot{a} < 0 \) as long as \( \rho > 0 \) and \( P \geq 0 \) (\( \rho \) and \( P \) being, as pointed out in section 2, the matter energy density and the matter pressure, respectively).

Equation (18c) implies that compressive forces acting at \( t = 0 \) on the Eulerian human observer are given by

\[ \lim_{t \to 0} (-R^\hat{\tau}_{\hat{\tau} \hat{\tau}}) = \lim_{t \to 0} \left( -R^\hat{\tau}_{\hat{\tau} \hat{\tau}} \right) = \lim_{t \to 0} \left( -R^\hat{\tau}_{\hat{\tau} \hat{\tau}} \right) = \begin{cases} -2/(9b^2), & \text{nonrelativistic matter}, \\ -1/(4b^2), & \text{relativistic matter}, \end{cases} \]

(19)

meaning that such observer is subjected to forces inversely proportional to the square of the length scale \( b \) when he/she gets at the defect.

Let us analyze separately the two terms occurring in the square brackets of equation (18c). First of all, we have

\[ \frac{\ddot{a}(t)}{a(t)} = \begin{cases} 2 \left( \frac{3b^2 - t^2}{9 (b^2 + t^2)} \right), & \text{nonrelativistic matter}, \\ \left( \frac{2b^2 - t^2}{4 (b^2 + t^2)} \right), & \text{relativistic matter}, \end{cases} \]

(20)

which means that in a finite time interval around \( t = 0 \) (i.e. near the defect) we have\(^2\)

\[ \frac{\ddot{a}(t)}{a(t)} > 0 \Leftrightarrow \begin{cases} -\sqrt{3} b < t < \sqrt{3} b, & \text{nonrelativistic matter}, \\ -\sqrt{2} b < t < \sqrt{2} b, & \text{relativistic matter}. \end{cases} \]

(21)

\(^2\)In equation (18c) the factor \( \ddot{a}/a \) is multiplied by the term \((b^2 + t^2)/t^2\) which is always positive except at \( t = 0 \), where it is not defined.
Equations (20) and (21) imply that the factor \( \ddot{a}/a \) occurring in equation (18c) produces stretching forces in the neighbourhood of the defect and compressive forces far from it. In particular, if we define

\[
t^* \equiv \sqrt{n} b, \tag{22a}
\]

\[
n = \begin{cases} 
3, & \text{nonrelativistic matter}, \\
2, & \text{relativistic matter}, 
\end{cases} \tag{22b}
\]

we can say that if \( t < -t^* \) (contracting-universe phase) or \( t > t^* \) (expanding-universe phase) the term \( \ddot{a}/a \) generates a compressive force, whereas if \(-t^* < t < 0 \) (contracting epoch) or \( 0 < t < t^* \) (expanding epoch) \( \ddot{a}/a \) leads to stretching forces. Here, we can appreciate the anti-gravitational action of the defect, which produces an accelerated contraction or expansion rate of the universe for \( t \in (-t^*, t^*) \).

The second term appearing in the square brackets of equation (18c) always causes compressive forces, since we have

\[
-\frac{\ddot{a}(t)}{a(t)} \frac{b^2}{t^3} = \begin{cases} 
\frac{2b^2}{3 \left( b^2 + t^2 \right)}, & \text{nonrelativistic matter}, \\
-\frac{b^2}{2 \left( b^2 + t^2 \right)}, & \text{relativistic matter}.
\end{cases} \tag{23}
\]

However, we have already seen that the sum of the two contributions of equation (18c) amounts to a compressive force. This means that near the defect, where the terms \( \ddot{a}/a \) and \( \left[-(\ddot{a}/a)(b^2/t^3)\right] \) give opposite contributions, the modulus of \( \left[-(\ddot{a}/a)(b^2/t^3)\right] \) "wins" against \( \ddot{a}/a \) (which is positive for \(-t^* < t < 0 \), cf equation (21)) so that their sum gives a compressive force.

In principle, it could be possible to build a scenario where both compressive and stretching forces act on the Eulerian human observer in the vicinity of the defect. However, this would require, in the spacetime (1), a scale factor for which (at least) third-order time derivatives are nonvanishing at \( t = 0 \). In other words, such a model would entail an odd \( a(t) \) function.

For future purposes, it is important to express the components \( R^{ij}_{\hat{j}\hat{p}} \) (no sum over \( \hat{j} \)) of the Riemann tensor appearing in equation (18) in an equivalent way. Indeed, the Riemann tensor written in terms of the Christoffel symbols \( \Gamma^{\gamma}_{\hat{a}\hat{b}\hat{c}} \) only reads as [24]

\[
R^{\hat{a}}_{\hat{j}\hat{b}\hat{c}} = \delta^{\hat{a}}_{\hat{p}} \left[ \Gamma^{\hat{a}}_{\hat{j}\hat{b}\hat{c}} - \delta^{\hat{a}}_{\hat{c}} \left( \Gamma^{\hat{a}}_{\hat{j}\hat{b}} + \Gamma^{\hat{a}}_{\hat{b}\hat{c}} + \Gamma^{\hat{a}}_{\hat{c}\hat{b}} \right) \right], \tag{24}
\]

and the relation between the connection coefficients \( \Gamma^{\gamma}_{\hat{a}\hat{b}\hat{c}} \) in the basis \( \{e_{\hat{a}}\} \) and \( \Gamma^{\nu}_{\hat{p}\hat{\mu}\hat{\lambda}} \) in the basis \( \{e_{\hat{p}}\} \) is given by [24]

\[
\Gamma^{\gamma}_{\hat{a}\hat{b}} = e^{\hat{\nu}}_{\hat{p}} e^{\hat{\mu}}_{\hat{a}} \left( \partial_{\hat{\nu}} e_{\hat{b}} + e^{\hat{\mu}}_{\hat{b}} \Gamma^{\nu}_{\hat{p}\hat{\mu}\hat{\lambda}} \right). \tag{25}
\]

Therefore, from equations (24) and (25) we find

\[
-R^{\hat{i}}_{\hat{j}\hat{p}} = \delta^{\hat{i}}_{\hat{a}} \left[ \Gamma^{\hat{i}}_{\hat{j}\hat{p}} - \left( \Gamma^{\hat{i}}_{\hat{j}\hat{p}} \right)^2 \right] = e^{\hat{\nu}}_{\hat{i}} e^{\hat{\mu}}_{\hat{a}} \left( \partial_{\hat{\nu}} e_{\hat{p}} + e^{\hat{\mu}}_{\hat{p}} \Gamma^{\nu}_{\hat{p}\hat{a}\hat{c}} \right), \tag{26}
\]
which, after having exploited equations (14)--(16), leads to the same result as equation (18c).

The most important aspect of the above calculation is that equation (26) has no contribution from $\Gamma^t_\tau \tau$ (i.e. the only Christoffel symbol of (1) which does not depend on $a(t)$ and its derivatives, see equation (4)). Indeed, this observation will be crucial in section 4, where we will claim that compressive forces (18) show no substantial differences with respect to the corresponding standard cosmology case because the Eulerian observer cannot measure contributions related to $\Gamma^t_\tau \tau$ in his/her proper reference frame (see the discussion below equation (62)).

At this stage, we are ready to move to the proper reference frame $(\hat{e}_\tau', \hat{e}_\tau', \hat{e}_\tau', \hat{e}_\tau')$ of the traveller by means of a boost along the $\hat{e}_\tau$ direction. We can calculate the ordinary boost velocity $V^i$ as

$$V^i = \frac{\text{proper distance along } \hat{e}_\tau \text{ as seen by the Eulerian observer}}{\text{proper lapse time as seen by the Eulerian observer}}$$

$$= \sqrt{g_{\tau\tau}} \frac{dx}{\sqrt{-g_{tt}} \frac{dt}{dt}} = \frac{1}{\sqrt{1 + a(t)^2/\Pi^2}} \equiv V,$$

(27)

where we have exploited equation (11). In terms of the orthonormal bases $\{ \hat{e}_\alpha \}$ of the Eulerian observer and $\{ \hat{e}_\alpha' \}$ of the traveller, the boost is expressed by

$$\begin{align*}
\hat{e}_\tau &= e_\tau + \gamma V e_\tau,
\hat{e}_\tau' &= e_\tau + \gamma V e_\tau,
\hat{e}_\tau' &= e_\tau,
\hat{e}_\tau' &= v,
\gamma &= \frac{1}{\sqrt{1 - V^2}} = \sqrt{1 + \Pi^2/a(t)^2}.
\end{align*}$$

(28a) (28b)

The components $R_{\hat{e}_\alpha' \hat{e}_\alpha' \hat{e}_\alpha' \hat{e}_\alpha'}$ of the Riemann tensor in the traveller’s proper frame can be obtained by applying the usual Lorentz transformation to the components $R_{\hat{e}_\alpha \hat{e}_\alpha \hat{e}_\alpha \hat{e}_\alpha}$ measured by the Eulerian observer in his/her proper frame. Moreover, since the separation vector $\xi_T$ is purely spatial in the traveller’s frame, the geodesic deviation equation (13) can be written as

$$\begin{align*}
\Delta a^\tau = \Delta a_\tau &= -R_{\hat{e}_\alpha' \hat{e}_\alpha' \hat{e}_\alpha' \hat{e}_\alpha'} \xi_\tau^T,
\Delta a^\tau &= \frac{d^2}{d\tau^2} \xi_\tau^T.
\end{align*}$$

(29a) (29b)

Bearing in mind the above equation, the compressive forces felt by the human traveller as he/she passes through the defect are given by

$$\lim_{t \to 0} (-R_{\xi_T' \xi_T' \xi_T' \xi_T'}) = \begin{cases} -2/(9b^2), & \text{nonrelativistic matter,} \\
-1/(4b^2), & \text{relativistic matter,}
\end{cases}$$

(30a)
\[
\lim_{t \to 0} (-R_{\tau'\tau'\tau'}) = \lim_{t \to 0} (-R_{\tau'\tau'\tau'}) \\
= \begin{cases} 
-\left[2/(9b^2)\right](1 + 3\Pi^2/a(0)^2), & \text{nonrelativistic matter,} \\
-\left[1/(4b^2)\right](1 + 2\Pi^2/a(0)^2), & \text{relativistic matter.}
\end{cases}
\]

At this stage, if we take into account the hypothesis developed in reference [2] according to which the defect length scale \( b \) can be of order of the Planck length \( \ell_P \) and we also suppose that

\[
t_0 \gg b, \\
\frac{|\Pi|t_0}{b} \gg 1,
\]

we can write (30b) approximately as

\[
\lim_{t \to 0} (-R_{\tau'\tau'\tau'}) = \lim_{t \to 0} (-R_{\tau'\tau'\tau'}) \\
\approx \begin{cases} 
-\left[2\Pi^2(t_0)^{1/3}\right]/\left[3b^2(b)^{4/3}\right], & \text{nonrelativistic matter,} \\
-\left[\Pi^2 |t_0|\right]/\left[2b^4\right], & \text{relativistic matter.}
\end{cases}
\]

Some comments on the results obtained so far are in order. First of all, equations (19) and (30) show that compressive forces are finite at \( t = 0 \) provided that \( b \neq 0 \). This may be seen as an equivalent proof of the fact that in the RW geometry (1) the big-bang singularity can be regularized via the introduction of the defect [1]. Furthermore, the analysis of compressive forces acting on both the Eulerian observer and the traveller reveals that the spacetime defect can be identified as the three-dimensional spacelike hypersurface of the regularized RW spacetime where compressive forces become as intense as possible. This is clear from figure 3, where we have chosen to plot compressive forces experienced by the Eulerian observer in the case of nonrelativistic-matter solution (cf equation (3)). Compressive forces on the Eulerian observer for the relativistic-matter solution and on the traveller (for both the matter-dominated and the radiation-dominated universe) display the same behaviour as the one in figure 3. Moreover, equations (19), (30) and (32) show that if \( b \sim \ell_P \) then both the Eulerian human observer and the human traveller are subjected to very large compressive forces proportional to \( 1/b \) as they pass the defect, which therefore amounts to a gravitational obstacle between two different universes, the one with \( t < 0 \) and that with \( t > 0 \). To have an idea, a rough calculation reveals that if we assume that a human body cannot withstand an acceleration gradient of ten times the Earth’s gravity (i.e. \( 10g_\oplus \approx 98 \text{ m s}^{-2} \)) per metre, then equation (19) implies that the Eulerian human observer can survive the compressive forces generated at \( t = 0 \) only if \( b \gtrsim 10^7 \text{ m} \). This suggests that the results of this section can concur with the analysis performed in references [2, 4, 8] (see also reference [10]) where a quantum origin of the defect length scale is proposed. Indeed, as spelled out in reference [10], our model (1) could yield two possible scenarios, depending on the values assumed by the defect length scale \( b \). The first pattern leads to a nonsingular-bouncing-cosmology model, whereas the second provides a new physics phase at \( t = 0 \) which pair-produces a ‘universe’ for \( t > 0 \) and an ‘anti-universe’ for \( t < 0 \). The first framework is built on classical Einstein theory and hence may be possible if \( b \gg \ell_P \), whereas the second might apply if \( b \sim \ell_P \). Our investigation shares some similar features with this last
situation since we have shown that if $b \sim \ell_P$ then compressive forces become so large at $t = 0$ that the universe having $t < 0$ and the one with $t > 0$ can be viewed as separated.

It should be stressed that equations (18) and (29) (likewise those derivable thereof) will not hold exactly, since they do not account for the forces between the atoms comprising the observers. Nevertheless, when gravitational compressive forces become very strong such interatomic forces can be neglected and hence the equations derived above will be valid to good approximation. Therefore, we can conclude that when compressive forces tear the Eulerian human observer and the human traveller apart, the very atoms of which they are composed must ultimately undergo the same fate.

4. The energy of massive particles and photons

We have examined, in the previous section, compressive forces acting on the Eulerian human observer and the human traveller. In order to complete our physical description of the spacetime defect underlying the modified RW cosmology (1), we will now analyze the energy behaviour of both the Eulerian observer and the traveller during their motion. Furthermore, our investigation will involve the energy of photons. We also note that for these considerations we have no need to think of the Eulerian observer and the traveller as human observers.

First of all, we need to define the concept of massive particle or photon energy in a proper way. Before we set out this theme, let us describe two simple situations where we can safely define this physical quantity. As a first example, consider the Schwarzschild solution. In this case, it is possible to define the total energy of a massive particle or a photon due to the presence of a timelike Killing vector field related to the time-translation invariance of Schwarzschild geometry; moreover, in the case of timelike geodesics, this conserved quantity reduces, at large distances from the centre of attraction, to the usual special relativistic formula for the total energy per unit mass of the particle [13]. Another example is furnished by the energy of a massive particle or a photon with four-momentum $p$ which is (locally) measured by a generic observer having four-velocity $u_{obs}$ in his/her proper reference frame (whose orthonormal basis
four-vectors are such that $e_0 = u_{\text{obs}}$. In this case, we find

$$E_{\text{measured}} = -p \cdot u_{\text{obs}} = p^\hat{0} = (\hat{\theta}^0, p),$$

(33)

where $\{\theta^i\}$ is the basis one-forms of the observer’s proper reference frame (cf equation (14a)) and $\langle \cdot, \cdot \rangle$ is the usual inner product. However, in this case there is a caveat: expression (33) only represents an *intrinsic* energy from motion and inertia, *not* the total energy of the massive particle or the photon. In that regard, we will see that in our model equation (33) yields, in agreement with the equivalence principle (which holds for all times except at $t = 0$), an expression for the energy measured by the Eulerian observer where there is no room for the term $\Gamma_t^{\nu\sigma}$, which, as we have pointed out in section 3, fulfills an important role in our analysis. Indeed, we will interpret this $\Gamma_t^{\nu\sigma}$ term as due to the (anti-)gravitational action of the defect (see comments above equation (54)).

To formulate an energy definition suitable for our model, we need to briefly recall some topics. It is well-known that the Hamiltonian formulation of general relativity as well as the analysis of the Cauchy (initial value) problem can be performed by employing the $3 + 1$ formalism, which relies on a theorem stating that any globally hyperbolic spacetime can be foliated by a constant-$t$ family $(\Sigma_t)_{t \in \mathbb{R}}$ of spacelike (Cauchy) hypersurfaces [13, 25–27]. Within this framework, the spacetime metric can be written as

$$g_{\mu\nu} = \begin{bmatrix} N N^i - N^2 & N_i \\ N_i & h_{ik} \end{bmatrix},$$

(34)

where $N, N_i$ are the lapse function and the shift vector, respectively, while $h_{ik}$ is the (induced) three-metric on the generic hypersurface $\Sigma_t$ belonging to the set $(\Sigma_t)_{t \in \mathbb{R}}$. An inspection of equation (1a), reveals that in our model we have

$$N = \sqrt{t^2 + b^2},$$

(35a)

$$N_i = 0,$$  

(35b)

$$h_{ik} = a(t)^2 \delta_{ik}.$$  

(35c)

Furthermore, in the context of the geometry of foliation the Eulerian observer four-velocity can be written in general as $n^\alpha = (\frac{1}{N}, \frac{-N^i}{N})$ and it represents the timelike and future-oriented unit four-vector normal to the generic hypersurface $\Sigma_t$. In our modified RW setup, it follows from equations (35a) and (35b) that equation (7) can be written equivalently as

$$n^\alpha = \frac{1}{N}(1, 0, 0, 0).$$

(36)

An important object of the $3 + 1$ approach is the so-called normal evolution (four-)vector [25], defined as the timelike vector field

$$m \equiv N n,$$

(37)

which carries (or Lie drags) the hypersurface $\Sigma_t$ (defined, as pointed out before, by the condition $t = \text{const}$) to the neighbouring hypersurface $\Sigma_{t+\delta t}$. In other words, the hypersurface $\Sigma_{t+\delta t}$

---

3 We have labelled the orthonormal basis vector as $e_0$ and not as $e_\tau$ or $e_{\tau'}$ because the measurement process (33) can be performed by any observer, not only those characterized by a free-fall motion.
can be obtained from the neighbouring $\Sigma_t$ by a small displacement $m \, dt$ of each point of $\Sigma_t$. From equation (36) we easily find that our regularized RW geometry is characterized by the following normal evolution (four-)vector:

$$m = e_t. \tag{38}$$

Bearing in mind the above results, we propose for model (1) the following definition of the total energy. We identify the total energy $E_{\text{tot}}$ of a massive particle or a photon having four-momentum $p$ with the projection of $p$ along the normal evolution (four-)vector, i.e.

$$E_{\text{tot}} = -p \cdot m. \tag{39}$$

This is justified by the fact that, as explained before, the four-vector $m$ regulates the temporal evolution of the spacetime (note however that this evolution is described in terms of the coordinate time variable $t$, see below). Therefore, the procedure underlying equation (39) turns out to be similar to the method adopted by an observer who measures the (intrinsic) energy of a massive particle/photon in his/her proper reference frame by simply projecting the massive particle/photon four-momentum $p$ along $e^0 = u^0_{\text{obs}}$, i.e. along the time direction defined by the observer’s clock (cf equation (33)). In a similar way, we propose through equation (39) to define the total energy by projecting $p$ along the four-vector $m$ ruling the evolution of the spacetime. We will justify later in which sense (39) defines a total energy (see comments below equation (54)). However, in our definition (39) there is one important aspect which must be taken into account: while $e^0$ governs the flow of the observer’s proper time, $m$ can be seen as the vector controlling the flow of the coordinate time.

We can analyze the consequences of our definition (39) by considering the form assumed by $E_{\text{tot}}$ in the most general situations. Consider for instance the generic four-dimensional space-time metric written as in equation (34). According to our definition (39), the total energy $\mathcal{E}$ of the Eulerian observer is given by (for the sake of simplicity, the rest masses of all observers are set to one)

$$\mathcal{E} = -n \cdot m = N = -n_t. \tag{40}$$

Now consider a generic particle having four-momentum $p$. According to our prescription (39), the total energy will be

$$E_{\text{particle}} = -p \cdot m = N^2 p^t. \tag{41}$$

In this case, $E_{\text{particle}} \neq -p_t$ since

$$-p_t = N^2 p^t - N_t p^t - N^t p^t, \tag{42}$$

and hence

$$E_{\text{particle}} = -p_t \iff N^t = 0. \tag{43}$$

Equations (41)–(43) reveal the advantages of our definition (39). Indeed, equation (41) depends only on the gauge function $N$ (reflecting thus the fact that the energy is not a scalar and hence its expression changes according to the coordinates adopted), whereas (42) depends on both $N$ and $N^t$. However, this last circumstance would lead to an ill-defined concept of energy since the contributions due to $N^t$ can always be gauged away once comoving spatial coordinates are invoked. On the contrary, by adopting the definition (41), the energy does not depend on terms which can be set to zero by an appropriate coordinate transformation.
In model (1), the normal evolution (four-)vector is represented by equation (38) and hence we can define the total energy of the traveller and the photon as, respectively,

$$E = -v \cdot m = -v_\tau, \quad (44a)$$

$$E_{\text{ph}} = -k \cdot m = -k_\tau. \quad (44b)$$

$k$ being the photon four momentum. In addition, the total energy of the Eulerian observer will be represented by (40), which by means of equation (35a) reads as

$$E = \sqrt{\frac{t^2}{t^2 + b^2}}, \quad (45)$$

Let $E_{\text{local}}$ represent the traveller energy as measured by the Eulerian observer in his/her proper reference frame $(e^\tau, e_x, e_y, e_z)$. Then we have from equation (33)

$$E_{\text{local}} = -v \cdot n = -v_\tau = v^\tau = \langle \theta^\tau, v \rangle = \langle \theta^\tau, v^\nu e_\nu \rangle = \sqrt{-g_{\tau\tau}} \frac{dt}{d\tau} = \sqrt{1 + \Pi^2 / a(t)^2} = \gamma, \quad (46)$$

where we have exploited the well-known relation $\langle dx^\mu, e_\nu \rangle = \delta_\nu^\mu$ and equations (11a), (14b) and (28c). Similarly, the photon energy $E_{\text{ph,local}}$ measured by the Eulerian observer in his/her proper reference frame is given by

$$E_{\text{ph,local}} = -k \cdot n = -k_\tau = k^\tau = \sqrt{-g_{\tau\tau}} \frac{ds}{ds} = \frac{|\Pi_{\text{ph}}|}{a(t)}, \quad (47)$$

$s$ being the affine parameter along the photon’s null geodesics and $\Pi_{\text{ph}} \equiv a(t)^2 \frac{dx}{ds}$ the conserved momentum. Therefore, the total energy of the traveller and the photon reads as, respectively,

$$E = -v_\tau = \sqrt{-g_{\tau\tau}} E_{\text{local}} = \sqrt{\frac{t^2}{t^2 + b^2}} \left( \sqrt{1 + \Pi^2 / a(t)^2} \right), \quad (48a)$$

$$E_{\text{ph}} = -k_\tau = \sqrt{-g_{\tau\tau}} E_{\text{ph,local}} = \sqrt{\frac{t^2}{t^2 + b^2}} \left( \frac{|\Pi_{\text{ph}}|}{a(t)} \right). \quad (48b)$$

Equations (46)–(48) show that even if we reject our proposal of defining the total energy according to equation (39), we can at least say that the total energy $E$ of the traveller and the total energy $E_{\text{ph}}$ of the photon make sense at great distances from the defect (i.e. if $|t| \gg b$), where they reduce to $E_{\text{local}}$ and $E_{\text{ph,local}}$, respectively. This is due to the fact that for large values of $t$ the lapse function (35a) is such that $N \to 1$, meaning that there is no difference between the Eulerian velocity $n$ and the normal evolution (four-)vector $m$ if $|t| \gg b$ (cf equations (36)–(38)). Therefore, we can write

$$E = -v \cdot m \xrightarrow{|t|\gg b} -v \cdot n = E_{\text{local}}, \quad (49a)$$

$$E_{\text{ph}} = -k \cdot m \xrightarrow{|t|\gg b} -k \cdot n = E_{\text{ph,local}}. \quad (49b)$$
Figure 4. The traveller energy (48a) for the nonrelativistic-matter solution (3) and with \(\Pi = 1\), \(b = 1\), \(t_0 = 4\sqrt{5}\). The suffix ‘NR’ stands for ‘nonrelativistic’ cosmological matter content.

The geodesic equation provides us with the equations governing the dynamical evolution of \(E\) and \(E_{ph}\). After an easy calculation, we obtain

\[
\dot{E} + \frac{\dot{a}}{a}E - \frac{\dot{a}}{a} \left( \frac{t^2}{b^2 + t^2} \right) = \Gamma^t_{tt}E, \tag{50a}
\]

\[
\dot{E}_{ph} + \frac{\dot{a}}{a}E_{ph} = \Gamma^t_{tt}E_{ph}. \tag{50b}
\]

A comparison with standard RW cosmology allows us to derive the equations ruling the dynamical behaviour of \(E_{local}\) and \(E_{ph,local}\), i.e.

\[
\dot{E}_{local} + \frac{\dot{a}}{a}E_{local} - \frac{\dot{a}}{a} \frac{1}{E_{local}} = 0, \tag{51a}
\]

\[
\dot{E}_{ph,local} + \frac{\dot{a}}{a}E_{ph,local} = 0. \tag{51b}
\]

The main difference between equations (50) and (51) stems from the presence on the right-hand side of the former of the connection coefficient \(\Gamma^t_{tt}\). As anticipated before, we will interpret this term as due to the (anti-)gravitational action generated by the defect. Furthermore, we note in equations (50a) and (51a) a contribution proportional to the inverse of the energy originating from the nonvanishing magnitude of traveller’s four-velocity \(v\) via the condition \(v \cdot v = -1\).

The plots of the traveller energy \(E\) and the photon energy \(E_{ph}\) are shown in figures 4–7.

From figures 4 and 5, it is clear that: in the negative-\(t\) branch, the total energy of the traveller (48a) starts decreasing after a time interval during which it has increased; for \(t > 0\) we have the mirrored behaviour with respect to \(t < 0\); for \(t = 0\) the energy vanishes and \(\lim_{t \to \pm \infty} E = 1\) (corresponding to the traveller’s rest mass). Moreover, apart from \(t = 0\), the energy (48a) is always positive. The traveller energy has thus a usual (i.e. standard) behaviour only away from the defect, where it increases (resp. decreases) while the universe contracts (resp. expands); on the contrary, near the defect the energy diminishes (resp. grows) as the universe undergoes a contracting (resp. expanding) era.
Something similar happens for the photon energy (48b), as figures 6 and 7 witness. In this case, $E_{\text{ph}}$ is always positive except for $t = 0$ and for large values of $|t|$, where it vanishes. In addition, the simple form assumed by equation (48b) permits to evaluate readily the sign of the time derivative $\dot{E}_{\text{ph}}$. Indeed we find that (see equation (22))

$$\dot{E}_{\text{ph}} > 0 \iff t < -\bar{t} \cup 0 < t < \bar{t},$$

$$\bar{t} \equiv \begin{cases} \frac{r^*}{\sqrt{2}}, & \text{nonrelativistic matter,} \\ r^*, & \text{relativistic matter.} \end{cases}$$

**Figure 5.** The traveller energy (48a) for the relativistic-matter solution (3) and with $\Pi = 1, b = 1, t_0 = 4\sqrt{5}$. The suffix ‘R’ stands for ‘relativistic’ cosmological matter content.

**Figure 6.** The photon energy (48b) for the nonrelativistic-matter solution (3) and with $\Pi_{\text{ph}} = 1, b = 1, t_0 = 4\sqrt{5}$. The suffix ‘NR’ stands for ‘nonrelativistic’ cosmological matter content.
Following equation (52), we see that the photon energy \((48b)\) increases when \(t < -\tilde{t}\) or \(0 < t < \tilde{t}\) and decreases if \(-\tilde{t} < t < 0\) or \(t > \tilde{t}\). Thus, as for the traveller energy, \(E_{\text{ph}}\) has the usual behaviour only for \(|t| \gg b\).

From the above analysis it is thus clear that the unusual character of the total energies \(E\) and \(E_{\text{ph}}\) is due to the (anti-)gravitational action exerted by the defect through the term \(\Gamma_{\nu_{\mu}}\), as equation (50) shows. Furthermore, the same nonstandard behaviour affects also the Eulerian observer energy \(E\) being, as it is clear from equation (45), monotonically decreasing for \(t < 0\) and monotonically increasing for \(t > 0\) (indeed the shape of the energy function \(E\) resembles the traveller energy displayed in figures 4 and 5).

We also note that we can interpret as a discontinuity in the (instantaneous) power the discontinuity of first kind that the total energy of the Eulerian observer, the traveller and the photon, equations (45) and (48), shows at \(t = 0\) (see figures 4–7).

At this stage, we can consider the traveller energy \(E_{\text{local}}\) and the photon energy \(E_{\text{ph,local}}\) as measured by the Eulerian observer in his/her proper reference frame \((\hat{e}, \hat{e}, \hat{e}, \hat{e})\) (see equations (46) and (47)). Their plots are shown in figures 8–11, respectively. It is thus clear that the energies (46) and (47) have always the usual character since, as we will show below, no contribution from \(\Gamma_{\nu_{\mu}}\) can be measured by the Eulerian observer in his/her proper reference frame (however, a first clear evidence is given by equation (51)).

By employing hypotheses (31), we have that

\[
\lim_{t \to 0} E_{\text{local}} = \sqrt{1 + \Pi^2/a(0)^2} \approx \begin{cases} \left|\Pi(t_0^2)^{1/3}/(b^2)^{1/3}\right|, & \text{nonrelativistic matter,} \\ \left|\Pi(t_0^2)^{1/4}/(b^2)^{1/4}\right|, & \text{relativistic matter,} \end{cases}
\]

and hence it is clear that, at \(t = 0\), \(E_{\text{local}}\) is proportional to a fractional power of \(1/b^2\) (see figures 8 and 9). This signifies an enormous amount of energy, if we suppose that \(b\) is proportional to the Planck length \(\ell_P\). The same conclusions as those expressed by equation (53) hold also for the photon energy \(E_{\text{ph,local}}\) if we enforce (31a).

We have already seen (cf equation (49)) that for large values of \(t\) the total energies \(E\) and \(E_{\text{ph}}\) reduce to the corresponding local intrinsic quantities \(E_{\text{local}}\) and \(E_{\text{ph,local}}\), respectively. At

**Figure 7.** The photon energy (48b) for the relativistic-matter solution (3) and with \(\Pi_{\text{ph}} = 1, b = 1, t_0 = 4\sqrt{5}\). The suffix ‘R’ stands for ‘relativistic’ cosmological matter content.
Figure 8. The traveller energy (46) for the nonrelativistic-matter solution (3) and with $\Pi = 1$, $b = 1$, $t_0 = 4\sqrt{5}$. It is clear that the maximum is reached at $t = 0$. The suffix ‘NR’ stands for ‘nonrelativistic’ cosmological matter content.

Figure 9. The traveller energy (46) for the relativistic-matter solution (3) and with $\Pi = 1$, $b = 1$, $t_0 = 4\sqrt{5}$. It is clear that the maximum is attained when $t = 0$. The suffix ‘R’ stands for ‘relativistic’ cosmological matter content.

At this stage, we can provide a further explanation of this point. If we first consider the case of photon energy, we can see that if $|t| \gg b$ then, from equation (4), $\Pi'$ tends to zero and hence equation (50b) reduces to (51b), whose solution is represented by (47). The same holds also for the traveller energy, where in addition the term occurring in equation (50a), i.e.

$$\frac{t^2}{t^2 + b^2},$$

tends to one for large times and hence we can recover (51a), which in turn yields the expression (46).
Figure 10. The photon energy (47) for the nonrelativistic-matter solution (3) and with \( \Pi_{ph} = 1, b = 1, t_0 = 4\sqrt{5} \). It is clear that the maximum is reached at \( t = 0 \). The suffix ‘NR’ stands for ‘nonrelativistic’ cosmological matter content.

Figure 11. The photon energy (47) for the relativistic-matter solution (3) and with \( \Pi_{ph} = 1, b = 1, t_0 = 4\sqrt{5} \). It is clear that the maximum is reached at \( t = 0 \). The suffix ‘R’ stands for ‘relativistic’ cosmological matter content.

As pointed out before, we can interpret \( \Gamma_{tt} \) as the term embodying the (anti-)gravitational action exerted by the defect. At this stage, we can account for this assumption. First of all, we have already explained that \( \Gamma_{tt} \) fulfils an important role in the equations governing the dynamical evolution of the total energy of the traveller and the photon (cf equation (50)). Moreover, the fact that \( \Gamma_{tt} \) gets closer to zero if \( |t| \gg b \) reflects that the action of the defect phases out as a particle travels away from it. Furthermore, \( \Gamma_{tt} \) diverges if \( t \to 0 \): the closer a particle gets to the defect, the stronger is the ‘force’ it experiences. Moreover, it is possible to write \( \Gamma_{tt} \) as (cf
equation (4))
\[ \Gamma^\alpha_{\mu\nu} = -\frac{1}{t} \frac{\rho_{\text{defect}}}{\rho}, \]
where (see equation (2.4c) in reference [4])
\[ \rho_{\text{defect}} \equiv -\frac{b^2 \rho}{b^2 + t^2}. \]
This means that the (anti-)gravitational ‘force’ produced by the defect depends on the ratio between the effective energy density \( \rho_{\text{defect}} \) of the defect and the energy density \( \rho \) of matter (or equivalently the ratio between their masses) as well as the inverse time separation from the defect.

The above analysis indicates also that \( E \) and \( E_{\text{ph}} \) can be regarded as total energies since they take into account also the contribution due to \( \Gamma_{tt} \), unlike the intrinsic quantities \( E_{\text{local}} \) and \( E_{\text{ph,local}} \). Indeed, once \( \Gamma_{tt} \) vanishes, the energy will not contain the contributions coming from the gravitational action of the defect but it will include only those terms connected to the kinetic energy and the inertia. As a consequence, when \(|t| \gg b\) the total energies reduce to the corresponding intrinsic expressions, as we have just demonstrated.

We have already seen that the proper reference frame of the Eulerian observer is a freely falling frame whose coordinates amount to be Fermi normal coordinates. In such a frame, the (Fermi normal) coordinates \( \hat{x}^{\mu} \) of a generic point \( \mathcal{P} \) located near the Eulerian observer’s worldline are given by
\[ \hat{x}^{\mu} (\mathcal{P}) = \left( \tau, x^j \right) = \left( \tau, s r^j \right), \]
where \( r = r^j e_j \) is the tangent vector to the (spacelike) geodesic originating from the Eulerian observer’s worldline at the specific (Eulerian observer’s) proper time \( \tau \) and \( s \) the proper length along such geodesic. If we employ the notation \( f(x^\tau = \tau, x^j = 0) \equiv f|_G \) to indicate that a quantity is evaluated along the Eulerian observer’s geodesic, we know from reference [23] that Christoffel symbols satisfy the following relations:
\[ \partial_k \Gamma^{\gamma}_{\mu\nu}|_G = 0, \]
\[ \partial_k \Gamma^{\gamma}_{\mu\nu}|_G = R^{\gamma}_{\mu\nu}|_G, \]
\[ \partial_k \Gamma^{\gamma}_{ij}|_G = -\frac{1}{3} \left( R^{\gamma}_{ijk}|_G + R^{\gamma}_{ikj}|_G \right), \]
from which we derive the following expansion for the Christoffel symbols:
\[ \Gamma^{\gamma}_{\mu\nu} \left( \tau, x^j \right) = \partial_k \Gamma^{\gamma}_{\mu\nu}|_G \hat{x}^k + O \left( |x|^2 \right). \]
From the above equation, we obtain the following relation, valid for our modified RW model:
\[ \Gamma^{\gamma}_{\tau\tau} \left( \tau, x^j \right) = O \left( |x|^2 \right). \]
This means that the connection coefficient \( \Gamma^{\gamma}_{\tau\tau} \) vanishes not only along the geodesic \( \mathcal{G} \) of the Eulerian observer but also in the neighbourhood of \( \mathcal{G} \) (within the precision of \( O(|x|^2) \)). Accordingly, the energies \( E_{\text{local}} \) and \( E_{\text{ph,local}} \) measured by the Eulerian observer in his/her proper
reference frame will not include the effects coming from $\Gamma^\tau_{\gamma\gamma}$. This offers another explanation of the fact that, once the underlying computations are performed in the coordinate system $(t, x, y, z)$, the dynamical evolution of $E_{\text{local}}$ and $E_{\text{ph,local}}$ are ruled by equation (51), where no contribution from $\Gamma^\tau_{\nu}$ appears. In other words, the Eulerian observer will not take into account the gravitational action of the defect (represented, as we said before, by $\Gamma^\nu_{\mu}$) when he/she measures the energy of the photon or the massive particle.

We can also explain this point with a more direct approach. We know that, in Fermi normal coordinates, the geodesic equation can be written as

$$\frac{d^2x^\hat{i}}{d\lambda^2} + \Gamma^\hat{i}_{\hat{\nu}\hat{\mu}}(\tau, x^j) \frac{dx^\hat{\nu}}{d\lambda} \frac{dx^\hat{\mu}}{d\lambda} = 0,$$

(60)

where $\lambda$ is the affine parameter along the geodesic (which in the case of timelike geodesic can be identified with the particle’s proper time). Therefore, the energy $p^\hat{i}$ of a massive particle or a photon having four-momentum $p^\hat{i} = \frac{dx^\hat{i}}{d\lambda}$ as measured by the Eulerian observer will obey the equation (we simply write $\Gamma^\tau_{\nu\mu}(\tau, x^j) \equiv \Gamma^\tau_{\nu\mu}$ in order to ease the notation)

$$\frac{dp^\hat{i}}{d\lambda} + \Gamma^\tau_{\beta\gamma}P^\beta P^\gamma = 0.$$

(61)

Bearing in mind equations (14)–(16) and (25), we find that the coefficients $\Gamma^\tau_{\alpha\beta}$ occurring in equation (61) are given by

$$\Gamma^\tau_{\gamma\gamma} = \delta e^\gamma + e^\gamma \Gamma^\gamma_{\mu} = 0,$$

(62a)

$$\Gamma^\tau_{\gamma j} = e^\gamma \Gamma^\gamma_{ij} = 0,$$

(62b)

$$\Gamma^\tau_{ij} = e^i e^j \left( \delta e^i + e^i \Gamma^\gamma_{\mu} \right) = 0,$$

(62c)

$$\Gamma^\tau_{ij} = e^i e^j \Gamma^\gamma_{ij} = \sqrt{\frac{b^2 + t^2 \dot{a}(t)}{\dot{a}(t)}} \delta^i_j.$$

(62d)

By means of equation (62), we can show that equation (61) reduces to equation (51) once all calculations are performed in $(t, x, y, z)$ coordinates. However, the most important point of this computation is that equation (62) clearly shows that $\Gamma^\tau_{\gamma\gamma}$ is the only connection coefficient occurring in equation (61) which depends on $\Gamma^\tau_{\nu}$. Since $\Gamma^\tau_{\gamma\gamma}$ vanishes, no contribution from $\Gamma^\tau_{\nu}$ appears in (51). As a result, the Eulerian observer will not measure the effects related to $\Gamma^\tau_{\nu}$, i.e. those contributions we are interpreting as due to the (anti-)gravitational action of the defect.

At this stage, let us stress another point. We have seen in section 3 that compressive forces felt by the Eulerian (human) observer show no deviations from the expectations of standard cosmology. It is now clear that this result is due to the fact that the effects introduced by $\Gamma^\tau_{\nu}$ cannot be measured in the freely falling frame $(e_\gamma, e_\gamma, e_\gamma, e_\gamma)$, see equation (26) and comments below. Bearing in mind the previous analysis regarding the local energies (46) and (47), this means that physical quantities measured by the Eulerian observer do not display an unusual behaviour because in the proper reference frame $(e_\gamma, e_\gamma, e_\gamma, e_\gamma)$ no contribution from $\Gamma^\tau_{\nu}$ can appear.

In our analysis a final question must be answered. For the sake of clarity, in the following calculations we will keep the observers’ rest mass explicit. Accordingly, let $m_{\text{E}}$ denote the Eulerian observer’s rest mass. We know that comoving coordinates $(t, x, y, z)$ introduced in equations (1d) and (1e) allow for the description of the modified model of universe from the
point of view of the Eulerian observer. Recalling that such observer is always at rest in these coordinates, we are thus led to wonder about the reasons for which our definition (39) of total energy leads to equation (40) instead of an expression like $E = \mu E$. The answer is that (39) defines a total energy and hence it takes into account also the defect’s gravitational action. This means that the total energy of the Eulerian observer (40) (or equivalently (45)) receives a contribution from the defect such that $E \neq \mu E$. Indeed, we simply have

$$E = -\mu E (n \cdot m) = \mu E \sqrt{\frac{t^2}{t^2 + b^2}} = \sqrt{\frac{\rho^{\text{defect}}}{\rho}}.$$  

(63)

In other words, $E$ differs from $\mu E$ by a term involving the ratio $\rho^{\text{defect}}/\rho$, i.e. the same term occurring in the expression of $\Gamma_{tt}$ (see equation (54)). Therefore, the presence of the defect makes $E$ differ from the Eulerian observer’s rest mass $\mu E$. Similarly, the total energy of the traveller (48a) can be written as

$$E = -\mu T (v \cdot m) = \sqrt{\frac{2}{\mu_T^2} + \frac{\rho^{\text{defect}}}{\rho}} \sqrt{1 + \frac{\Pi^2}{a(t)^2}}.$$  

(64)

$\mu_T$ being the traveller’s rest mass. In this expression we can recognize both the defect’s gravitational action, represented by the term $\mu_T \rho^{\text{defect}}/\rho$ and the ‘kinetic’ term proportional to $\Pi^2/a(t)^2$. Finally, the photon energy (48b) reads as

$$E_{\text{ph}} = \sqrt{1 + \frac{\rho^{\text{defect}}}{\rho} \left| \frac{\Pi_{\text{ph}}}{a(t)} \right|}.$$  

(65)

In this case, we can interpret the ‘correction’ term $\rho^{\text{defect}}/\rho$ as a photon’s effective mass induced by the defect.

Equations (63)–(65) support, once again, our proposal of interpreting (39) as the total energy of a massive particle/photon with four-momentum $p$. On the other hand, (33) denotes an intrinsic energy which does not take into account the defect’s gravitational force, represented by the term $\Gamma_{tt}$. Indeed, in equations (63)–(65) the term $\rho^{\text{defect}}/\rho$, originating from $\Gamma_{tt}$, appears explicitly. On the contrary, in equations (46) and (47) no contribution coming from $\Gamma_{tt}$ can occur, as we have explained before (cf equation (51) and comments following equations (59) and (62)).

5. Conclusions and open problems

The main purpose of this paper consists in seeking physical observables which can point out the presence of the spacetime defect characterizing the regularized RW geometry (1), where the big-bang singularity has been tamed by a nonzero length parameter $b$. Our description relies on the analysis of two physical quantities: the compressive forces acting on (human) observers and the energy of massive particles and photons crossing it. The first topic has been explored in section 3. We have devised a reasonable criterion to single out the defect, which can be defined as the three-dimensional hypersurface where the modulus of compressive forces attain their maximum value (see figure 3). Furthermore, we have found that if we take the proposal $b \sim \ell_P$ made in appendix B of reference [2] seriously, then the defect can be modelled as a gravitational obstacle with compressive forces proportional to $1/b$ (see equations (19), (30) and (32)). A rough calculation reveals that a Eulerian human observer can withstand compressive forces generated at $t = 0$ if $b \gtrsim 10^7$ m. This result can lead to interesting implications due to the possibility of having a quantum-inspired defect length scale. In particular, we have explained
how our investigation seems to agree with the scenario drawn in reference [10] suggesting that a new physics phase at \( t = 0 \) could create a pair of separated universes.

In section 4, we have provided a definition of a total energy suitable for our model (see comments accompanying equations (39) and (49)) which differentiates it from the (local) intrinsic energy measured by the Eulerian observer (see comments below equations (54) and (65)). We have seen that both the total energy of the generic freely falling observer and of the photon, defined according to (39) and given in equation (48), exhibit an unusual character over a finite time interval around \( t = 0 \): they grow (resp. diminish) as the universe expands (resp. contracts); see figures 4–7. As an inspection of equation (50) reveals, this scenario is due to the effects introduced by the Christoffel symbol \( \Gamma^\alpha_{\mu\nu} \) (cf equation (4)), which we propose to interpret as the term embodying the (anti-)gravitational action exerted by the defect (see comments below figure 11). Furthermore, the same nonstandard behaviour affects also the Eulerian observer energy (45). On the other hand, the intrinsic energy of the generic freely falling traveller and of the photon, given in equations (46) and (47), respectively, displays no unusual property (see figures 8–11 and comments therein). This absence of discrepancy with respect to standard cosmology predictions stems from the fact that the Eulerian observer cannot measure contributions related to the Christoffel symbol \( \Gamma^\alpha_{\mu\nu} \) in his/her proper reference frame (see comments following equations (59) and (62)).

A possible explanation of the nonstandard behaviour of the energy can come from the following hypothesis involving gravitationally repulsive negative masses. A negative mass is an exotic matter which would violate one or more energy conditions. It can be implemented in general relativity theory [28], where the equivalence principle implies that the inertial mass equals the passive gravitational mass\(^4\). Consider the situation depicted in figure 12, where, for simplicity, the gravitational positive-negative mass interaction is explored by means of Newtonian theory and the negative mass is supposed to be fixed. The gravitational repulsive force \( \vec{F}_g \) experienced by the moving body and the resulting acceleration \( \vec{a} \) point along the same direction. In panel (a), the positive mass approaches with velocity \( \vec{v} \) the negative mass. Since the work done by \( \vec{F}_g \) is negative, the positive mass slows down. In panel (b), the positive mass departs from the fixed body. In this case, it is clear that the kinetic energy of the moving body increases. Furthermore, the mechanical energy is conserved in both situations. This classical example can give us some insight into the nature of the defect. First of all, recall that in a generic spacetime there will not be a well-defined notion of gravitational potential energy (although in special cases it exists). In our relativistic model, if we conceive the defect as an object having negative (active gravitational) mass, then it is possible to account for the behaviour of the energy as shown in figures 4–7. Indeed, over a small time interval around the bounce, where antigravitational phenomena reveal an increasing importance, the energy of both massive particles and photons decreases for negative values of the time variable \( t \) and grows when \( t \) becomes positive. This reflects the behaviour of the kinetic energy in the classical example of figure 12. Moreover, at great distances from the defect, the energy displays its standard behaviour, meaning that antigravitational effects are negligible (as witnessed by the fact that \( \Gamma^\alpha_{\mu\nu} \) goes to zero if \(|t| \gg b\), see equation (4)). Furthermore, this scenario allows for the fact that the energy drops to zero at \( t = 0 \): the defect drains the energy of particles until, at the bounce, it vanishes; after that, the defect gives to particles the required energy to carry on with their motion. Therefore, we can conclude that the effective NEC violation featuring the defect and the related antigravity effects represent the source of the nonstandard behaviour of the total energy shown in figures 4–7. Incidentally, the possibility of modelling the defect as an object having negative

\(^4\) Active and passive gravitational masses are identical due to the law of conservation of momentum.
Figure 12. Gravitational repulsion experienced by a positive (yellow) mass due to the presence of a body with negative (red) mass, which is supposed to be fixed. The interaction is described through Newtonian theory. (a) The positive mass goes towards the fixed body with velocity $\vec{v}$; (b) the positive mass moves away from the fixed body with velocity $\vec{v}$. In both cases, the gravitational repulsive force $\vec{F}_g$ and the resulting acceleration $\vec{a}$ have the same direction and point away from the negative mass.

mass has been discussed also in reference [29]. This hypothesis can open interesting perspectives. As an example, recently in reference [30] gravitationally repulsive negative masses have been proposed as natural candidates for the description of both dark matter and dark energy. Thus, we might wonder if also the defect can represent such candidate.

The analysis of section 4 has shown that the nonstandard properties of massive particles and photons energy are not confined to the single point $t = 0$, but they concern a finite time interval around the defect’s location which is of the order of the characteristic length scale $b$. This pattern is in accordance with the aforementioned proposed interpretation of the connection coefficient $\Gamma^t_{tt}$ since, as we have explained in the comments above equation (54), this function attains large values in a region around $t = 0$ (except for $t = 0$, where it is not defined) and approaches zero as $|t| \gg b$. This means that the unusual energy’s behaviour configures as a sort of ‘spreading effect’ able to encompass regions near the defect. This is not the first example of a ‘spreading effect’ occurring in the modified RW geometry (1), the other one being represented by the effective NEC violation which, as reported in references [2, 4], can be extended to a finite interval around $t = 0$. Therefore, these results lead quite naturally to one important (open) question: are there other ‘spreading phenomena’ in nonsingular-bouncing-cosmology settings?

Another fascinating issue to be addressed, especially in light of the outcome of this paper, regards the origin of the defect length scale $b$. Indeed, if its quantum origin were proven, then an intriguing task would consists in trying to reconcile this fact with the arguments spelled out in our paper. On the other hand, a broader investigation is performed in reference [8], where it is argued that $b$ could be a remnant of a new (not necessarily quantum) physics phase replacing Einstein gravity (see also reference [10]). It would be interesting to determine if there exists a
connection between the physical effects discussed in this paper and the new phase mentioned in reference [8]. Finally, the rich mathematical structure underlying regularized RW geometry (1) (likewise degenerate metrics in general, see reference [12]) still deserves further investigation. Indeed, this can represent, on the one hand, a way to find other physical phenomena associated with the defect, and, on the other, we can expect to obtain equivalent explanations for the physical effects described in this paper.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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