Simultaneous Sparse Recovery and Blind Demodulation

Youye Xie, Michael B. Wakin, and Gongguo Tang
Department of Electrical Engineering, Colorado School of Mines, USA

Abstract—The task of finding a sparse signal decomposition in an overcomplete dictionary is made more complicated when the signal undergoes an unknown modulation (or convolution in the complementary Fourier domain). Such simultaneous sparse recovery and blind demodulation problems appear in many applications including medical imaging, super resolution, self-calibration, etc. In this paper, we consider a more general sparse recovery and blind demodulation problem in which each atom comprising the signal undergoes a distinct modulation process. Under the assumption that the modulating waveforms live in a known common subspace, we employ the lifting technique and recast this problem as the recovery of a column-wise sparse matrix from structured linear measurements. In this framework, we recover the sparse coefficient vector $c$ and all of the modulating waveforms simultaneously by minimizing the induced atomic norm $\| \cdot \|_\Omega$, which in this problem corresponds to $\ell_{2,1}$ norm minimization.

B. Setup and Notation

To better illustrate our main contributions and compare to related work, we first define our signal model and the corresponding atomic norm minimization problem.

Throughout this paper, we use bold uppercase, $X$, bold lowercase, $x$, and non-bold letters, $x$, to represent matrices, vectors, and scalars. We denote the complex conjugate of a scalar, vector, or matrix with an overline $\overline{\cdot}$. The symbol $C$ denotes a constant. $X_T (x_T, \text{resp.})$ is a matrix (vector, resp.) that zeros out the columns (entries, resp.) not in $T$. We call $T$ the support of the matrix $X$ (and vector $x$), and we use $X_T$ to denote the sub-matrix after removing the zero rows or columns in $X$. $\text{sign}(x) = x/||x||_2$ when $||x||_2 \neq 0$ and $0$ otherwise. $\text{sign}(X) = [\text{sign}(x_1), \cdots, \text{sign}(x_M)]$. We use $||\cdot||$ to indicate the spectral norm, which returns the maximum singular value of a matrix. The $\ell_{2,1}$ norm of a matrix $X = [x_1 \cdots x_M]$, denoted by $||X||_{2,1}$, is defined to be $\sum_{j=1}^{M}||x_j||_2$.

C. Problem Formulation

In this paper, we study a generalized sparse recovery and blind demodulation problem in which the coefficient vector is unknown and each atom (column) of the dictionary undergoes an unknown modulation process. Specifically, we assume the system receives a composite signal

$$y = \sum_{j=1}^{M} c_j D_j a_j \in \mathbb{C}^N$$  \hspace{1cm} (I.1)

where $c_j \in \mathbb{C}$ is an unknown scalar, $D_j \in \mathbb{C}^{N \times N}$ is an unknown diagonal modulation matrix, and $a_j \in \mathbb{C}^N$ is the $j$-th atom from a known dictionary $A = [a_1 \ a_2 \ \cdots \ a_M] \in \mathbb{C}^{N \times M}$ with $N < M$. Our goal is to recover both $c_j$ and $D_j$ for all $j$ from the observation $y$.

To make this problem well-posed, among the $M$ overcomplete atoms, we assume only $J < M$ of them contribute...
to the observed signal; that is, at most $J$ coefficients $c_j$ are nonzero. We furthermore assume that each modulation matrix obeys a subspace constraint:

\[
\mathbf{D}_j = \text{diag}(\mathbf{Bh}_j),
\]

where $\mathbf{B} \in \mathbb{C}^{N \times K}$ ($N > K$) is a known basis for the $K$-dimensional subspace of possible modulating waveforms, and $\mathbf{h}_j \in \mathbb{C}^K$ is an unknown coefficient vector. Similar subspace assumptions have been made in deconvolution and demixing papers [12], [13]. With this assumption, recovering $c_j$ and $\mathbf{D}_j$ equals recovering $c_j$ and $\mathbf{h}_j$. Since $c_j\mathbf{D}_j\mathbf{a}_j = c_j \text{diag}(\mathbf{Bh}_j)\mathbf{a}_j = (kc_j)\text{diag}(\mathbf{B}(\mathbf{h}_j))\mathbf{a}_j$ for any $k \neq 0$, without loss of generality, we assume $\mathbf{h}_j$ has unit norm and $c_j \geq 0$ with its complex phase and sign absorbed by $\mathbf{h}_j$.

Define $\mathbf{B}^H = [b_1^T, b_2^T, \cdots, b_N^T] \in \mathbb{C}^{K \times N}$ and note that the $n$-th entry of the observed signal can be expressed as

\[
y(n) = \sum_{j=1}^{M} c_j b_n^H \mathbf{h}_j = \text{Tr} \left( e_n b_n^H \sum_{j=1}^{M} c_j \mathbf{h}_j \overline{a}_j^H \right) = \sum_{j=1}^{M} c_j \mathbf{h}_j \overline{a}_j^H, \tag{1.3}
\]

where $\mathbf{G} = \sum_{j=1}^{M} c_j \mathbf{h}_j \overline{a}_j^H$, and $e_n$ is the $n$-th column of the $N \times N$ identity matrix. From (1.3), we see that the measurement vector $\mathbf{y}$ depends linearly on the matrix $\mathbf{G}$ which encodes all of the unknown parameters of interest. We denote this linear sensing process as $y = \mathcal{L}'(\mathbf{G})$ and recast the recovery problem as that of recovering $\mathbf{G}$ (and its components) from the linear measurements.

The unknown matrix $\mathbf{G}$ can be viewed as a linear combination of $J$ rank-1 matrices from the atomic set $\mathcal{A} := \{\mathbf{h}\mathbf{a}^H : \mathbf{a} \in \{a_1, \ldots, a_M\}, \|\mathbf{h}\|_2 = 1\}$ and thus we propose to recover $\mathbf{G}$ using the corresponding atomic norm minimization:

\[
\underset{\mathbf{G} \in \mathbb{C}^{N \times K}}{	ext{minimize}} \|\mathbf{G}\|_A \quad \text{subject to} \quad \mathbf{y} = \mathcal{L}'(\mathbf{G}). \tag{1.4}
\]

The atomic norm appearing in (1.4) is defined as $\|\mathbf{G}\|_A := \inf \{\sum_k \|\mathbf{g}_k\|_2 : \mathbf{G} = \sum_k \mathbf{g}_k \mathbf{a}_k, \mathbf{a}_k \in \mathcal{A}\}$. Moreover, the following result establishes its equivalence with the $\ell_{2,1}$ norm.

**Proposition 1.** The atomic norm optimization problem (1.4) can be equivalently expressed as the following $\ell_{2,1}$ norm optimization problem

\[
\underset{\mathbf{X} \in \mathbb{C}^{N \times M}}{	ext{minimize}} \|\mathbf{X}\|_{2,1} \quad \text{subject to} \quad \mathbf{y} = \mathcal{L}(\mathbf{X}) \tag{1.5}
\]

where $\mathbf{X} = [c_1 \mathbf{h}_1, c_2 \mathbf{h}_2, \cdots, c_M \mathbf{h}_M] \in \mathbb{C}^{K \times M}$ and $\mathcal{L}$ represents the following linear sensing process

\[
y(n) = \langle \mathbf{X}, b_n^H \mathbf{A} \rangle = b_n^H \mathbf{X} a_n', \tag{1.6}
\]

in which $b_n'$ and $a_n'$ are the $n$-th column of $\mathbf{B}^H$ and $\mathbf{A}^T$.

**Proof.** We first note that the atomic norm can be equivalently expressed as $\|\mathbf{G}\|_A = \inf \{\sum_{j=1}^{M} |c_j| : \mathbf{G} = \sum_{j=1}^{M} c_j \mathbf{h}_j \overline{a}_j^H, \|\mathbf{h}_j\|_2 = 1\}$. To see this, consider any decomposition of $\mathbf{G}$ of the form $\mathbf{G} = \sum_k \mathbf{g}_k$ with $g_k \in \mathcal{A}$. Define $N_j = \{k : g_k = \mathbf{h}_k \overline{a}_j^H\}$ and write $\mathbf{G} = \sum_{j=1}^{M} \sum_{k \in N_j} c_k \mathbf{h}_k \overline{a}_j^H$. This is equivalent to writing $\mathbf{G} = \sum_{j=1}^{M} c_j \mathbf{h}_j \overline{a}_j^H$ where $c_j = \sum_{k \in N_j} c_k \mathbf{h}_k^H$ and $c_j = \|\sum_{k \in N_j} c_k \mathbf{h}_k\|_2$. Finally, note that $|c_j| \leq \sum_{k \in N_j} |c_k|$. Next, to establish the equivalence with the $\ell_{2,1}$ norm, for any $c_j$ and $\mathbf{h}_j$ with $\|\mathbf{h}_j\|_2 = 1$, define $x_j = c_j \mathbf{h}_j$ and $\mathbf{x} = [x_1, x_2, \cdots, x_M]$. Then

\[
\|\mathbf{G}\|_A = \inf \left\{ \sum_{j=1}^{M} |c_j| : \mathbf{G} = \sum_{j=1}^{M} c_j \mathbf{h}_j \overline{a}_j^H, \|\mathbf{h}_j\|_2 = 1 \right\} = \inf \left\{ \sum_{j=1}^{M} \|x_j\|_2 : \mathbf{G} = \sum_{j=1}^{M} x_j \overline{a}_j^H \right\} = \inf \{\|\mathbf{X}\|_{2,1} : \mathbf{G} = \mathbf{X} \mathbf{A}^H\}.
\]

Finally, to establish the equivalence of the linear sensing process, (1.3) indicates that for $\mathbf{G} = \mathbf{X} \mathbf{A}^H$,

\[
y(n) = \langle \mathbf{G}, b_n' e_n^H \rangle = \langle \mathbf{X}, b_n' e_n^H \mathbf{A} \rangle = b_n'^H \mathbf{X} a_n'.
\]

The above optimization focuses on recovering the structured matrix $\mathbf{X}$ from linear measurements. Once the optimization is solved, the unknown parameters can be easily extracted from the solution $\hat{\mathbf{X}}$ as follows:

\[
c_j = \|\hat{x}_j\|_2, \quad \mathbf{h}_j = \frac{\hat{x}_j}{\|\hat{x}_j\|_2}, \quad \mathbf{D}_j = \text{diag}(\mathbf{Bh}_j). \tag{1.7}
\]

for $\hat{x}_j \neq 0$ and $1 \leq j \leq M$. The linear operator $\mathcal{L}$ also has a matrix-vector multiplication form. Note that $\mathcal{L}(\mathbf{X}) = \mathbf{F} \cdot \text{vec}(\mathbf{X})$, where $\mathbf{F} \in \mathbb{C}^{N \times KM}$ is

\[
\mathbf{F} = \begin{bmatrix} \phi_{1,1} & \cdots & \phi_{K,1} & \cdots & \phi_{1,M} & \cdots & \phi_{K,M} \end{bmatrix} \tag{1.8}
\]

in which $\phi_{i,j} = \text{diag}(b_i) a_j \in \mathbb{C}^{N \times 1}$ and $b_i$ is the $i$-th column of $\mathbf{B}$. Furthermore,

\[
\mathbf{F}^H = [\phi_1' \phi_2' \cdots \phi_N'] \in \mathbb{C}^{KM \times N}. \tag{1.9}
\]

where $\phi_i' = \overline{a}_i' \overline{b}_i' \in \mathbb{C}$. Finally, we note that the observed signal could be contaminated with noise. In this case, our observation model becomes

\[
y = \sum_{j=1}^{M} c_j \mathbf{D}_j a_j + n \tag{1.10}
\]

for some unknown noise vector $n \in \mathbb{C}^{N \times 1}$ which we suppose satisfies $\|n\|_2 \leq \eta$. In this case, we can write $\mathbf{y} = \mathcal{L}(\mathbf{X}_0) + n$, where $\mathbf{X}_0$ is the ground truth solution. As an alternative to equality-constrained $\ell_{2,1}$ norm minimization (1.5), we then consider the following relaxation:

\[
\underset{\mathbf{X} \in \mathbb{C}^{N \times M}}{	ext{minimize}} \|\mathbf{X}\|_{2,1} \quad \text{subject to} \quad \|\mathbf{y} - \mathcal{L}(\mathbf{X})\|_2 \leq \eta. \tag{1.11}
\]
D. Main Contributions

Our contributions are twofold. First, we employ $\ell_{2,1}$ norm minimization to achieve sparse recovery and blind demodulation simultaneously given the generalized signal model from equation (I.1). Second, for perfect recovery of all parameters in the noiseless case, we derive near optimal sample complexity bounds for the cases where $A$ is a random Gaussian and a random subsampled Fourier dictionary. Both of bounds require the number of observations $N$ to be proportional to the number of degrees of freedom, $O(JK)$, up to log factors. We also provide bounds on recovering the column-wise sparse matrix in the noisy case; these bounds show that the recovery error scales linearly with respect to the strength of the noise.

E. Related Work

The $\ell_{2,1}$ norm has been widely used to promote sparse recovery in multiple measurement vector (MMV) problems [14], [15]. The MMV problem involves a collection of sparse signal vectors that are stacked as the rows of a matrix $X$. These signals have a common sparsity pattern, which results in a column-wise sparse structure for $X$. As in our setup, the $\ell_{2,1}$ norm is used to recover $X$ from linear measurements of the form $y = \Phi_{MMV} \cdot \text{vec}(X^T)$. However, $\Phi_{MMV}$ has a block diagonal structure where all diagonal sub-matrices are the same which is the dictionary matrix. This is different from the structure of the linear measurements in our problem; see for example [18].

Our work is also closely related to certain recent works in model-based deconvolution, self-calibration, and demixing. When all $D_j$ in (I.1) are the same, our signal model coincides with the self-calibration problem in [9], although that work employs $\ell_1$ norm minimization rather than $\ell_{2,1}$ norm minimization to recover $X$. A more recent paper [10] does apply the $\ell_{2,1}$ norm for the self-calibration problem but again assumes a common modulation matrix $D$. The paper [12] generalizes the work of [9] and considers a blind deconvolution and demixing problem which can be interpreted as the self-calibration scenario with multiple sensors whose calibration parameters might be different. However, the signal model in that paper is not directly comparable to our model, and the recovery approach studied in that paper involves nuclear norm minimization and requires knowledge of the number of sensors. A blind sparse spike deconvolution is studied in [13], wherein the dictionary consists of sampled complex sinusoids over a continuous frequency range and all atoms undergo the same modulation. Inspired by [13], [8] generalizes the model to the case of different modulating waveforms. Like [13], however, [8] also considers a sampled sinusoid dictionary over a continuous frequency range, and it employs a random sign assumption on the coefficient vectors $h_j$ which makes it difficult to derive recovery guarantees with noisy measurements. More works considering a common modulation process can be found in [7], [17], [18].

Our work can be viewed as a generalization of the self-calibration [9] and blind deconvolution problems [7]. Moreover, our analysis is quite different from the works considering the continuous sinusoid dictionary [13], [8], since the tools in those papers are specialized to the continuous sinusoids dictionary and we consider discrete Gaussian and random Fourier dictionaries in both noiseless and noisy settings.

The rest of the paper is organized as follows. In Section II we present our main theorems regarding perfect parameter recovery in the noiseless setting and matrix denoising in the noisy setting. Sections III and IV contain the detailed proofs of the main theorems. Several numerical simulations are provided in Section V to illustrate the critical scaling relationships, and we conclude in Section VI.

II. Main Results

We present our main theorems in this section. In each of the noiseless and noisy cases, we consider two models for the dictionary matrix $A$. In the first model, $A \in \mathbb{R}^{N \times M}$ is a real-valued random Gaussian matrix, with each entry sampled independently from the standard normal distribution. In the second model, $A \in \mathbb{C}^{N \times M}$ is a complex-valued random Fourier matrix, with each of its $N(< M)$ rows chosen uniformly with replacement from the $M \times M$ discrete Fourier transform matrix $F$ where $F^H F = M I_M$. Our first theorem concerns perfect parameter recovery in the noiseless setting.

Theorem II.1. (Noiseless case) Consider the observation model in equation (I.7), assume that at most $J(< M)$ coefficients $c_j$ are nonzero, and furthermore assume that the nonzero coefficients $c_j$ are real-valued and positive. Suppose that each modulation matrix $D_j$ satisfies the subspace constraint (II.1), where $B^H B = I_K$ and each $h_j$ has unit norm.

Then the solution $\hat{X}$ to problem (I.5) is the ground truth solution $X_0$—which means that $c_j$, $h_j$, and $D_j$ can all be successfully recovered for each $j$ using (I.7)—with probability at least $1 - O(N^{-\alpha+1})$

- if $A \in \mathbb{R}^{N \times M}$ is a random Gaussian matrix and

$$
\frac{N}{\log^* N} \geq C_A \mu_{\max}^2 K J (\log(M - J) + \log(N)).
$$

- if $A \in \mathbb{C}^{N \times M}$ is a random Fourier matrix and

$$
N \geq C_A \mu_{\max}^2 K J \log(4 \sqrt{2} J \gamma).
$$

$$
(\log(M - J) + \log(K + 1) + \log(N))
$$

where $\gamma = \sqrt{2M \log(2KM) + 2M + 1}$.

In both cases, $C_A$ is a constant defined for $\alpha > 1$ and the coherence parameter

$$
\mu_{\max} = \max_{i,j} \sqrt{N} |B_{ij}|.
$$

We note that both of the sample complexity bounds in Theorem II.1 require the number of observations $N$ to be proportional to the number of degrees of freedom, $O(KJ)$, up to log factors. We also note that the sample complexity bounds scale with the square of the coherence parameter $\mu_{\max} = \max_{i,j} \sqrt{N} |B_{ij}|$. This term is minimized when the
energy of each column of $B$ is not concentrated on a few entries but spread across the whole column. The assumption $B^H B = I_K$ requires the columns of $B$ to be orthonormal.

Our second theorem provides bounds on recovering the column-wise sparse matrix in the noisy case; these bounds show that the recovery error scales linearly with respect to the strength of the noise.

**Theorem II.2. (Noisy case)** Consider the observation model in equation (I.10), assume that at most $J (\leq M)$ coefficients $c_j$ are nonzero, and furthermore assume that the norm of the noise is bounded, $\|n\|_2 \leq \eta$. Suppose also that each modulation matrix $D_j$ satisfies the subspace constraint (I.2), where $B^H B = I_K$.

Then with probability at least $1 - O(N^{-\alpha + 1})$, the solution $\hat{X}$ to problem (I.1) satisfies

- if $A \in \mathbb{R}^{N \times M}$ is a random Gaussian matrix,
  \[
  \|\hat{X} - X_0\|_F \leq (C_1 + C_2 \sqrt{J}) \eta \tag{II.3}
  \]
  when
  \[
  \frac{N}{\log^2 N} \geq C_0 \mu_{\max}^2 K J \left( \log(C \mu_{\max} \sqrt{JK}) + 1 \right) \left( \log(M - J) + \log(M K) + \log(N) \right)
  \]
  where $C$ is a constant.

- if $A \in \mathbb{C}^{N \times M}$ is a random Fourier matrix,
  \[
  \|\hat{X} - X_0\|_F \leq (C_1 + C_2 \sqrt{P J}) \eta \tag{II.5}
  \]
  when
  \[
  N \geq C_0 \mu_{\max}^2 K J \log(4 \sqrt{2 J} \gamma) \left( \log(M - J) + \log(M K) + \log(N) \right)
  \]
  where $\gamma = \sqrt{2M \log(2KM) + 2M + 1}$ and $P \geq \log(4 \sqrt{2 J} \gamma) / \log 2$.

In both cases, $C_0$ is defined for $\alpha > 1$. $C_1$ and $C_2$ are constant.

Although Theorem II.2 focuses exclusively on bounding the recovery error of the matrix $X_0$, one can also attempt to estimate the parameters $c_j, h_j,$ and $D_j$ from $\hat{X}$ using (III.3). We do not bound the error of these individual parameter estimates here. However, as results on structured matrix recovery from (possibly noisy) linear measurements, we believe that Theorems II.1 and II.2 may be of independent interest outside of the sparse recovery and blind demodulation problem.

**III. PROOF OF THEOREM II.1**

To begin our proof of the main theorem in the noiseless case, we first derive sufficient conditions for exact recovery.

**A. Sufficient conditions for exact recovery**

Sufficient conditions for exact recovery are the null space property and an alternative sufficient condition derived from the null space property.

**Proposition 2. (The null space property)** The matrix $X_0 = [c_1 h_1, \ldots, c_M h_M] \in \mathbb{C}^{K \times M}$ with support $T$ is the unique solution to the inverse problem (I.5) if

\[ -\|H_T, \text{sign}(X_0)\|_F + \|H_{T^c}\|_{2,1} > 0 \]

for any $H \neq 0$ in the nullspace of $L$.

**Proof.** Let $\tilde{X} = X_0 + H$ be a solution to problem (I.5), with $L(H) = 0$. To prove $X_0$ is the unique solution, it is sufficient to show that $\|X_0\|_2 > \|X_0\|_2$ if $H \neq 0$. Start by observing that

\[ \|X_0 + H\|_2 \leq \|X_0, T + H_T\|_2 + 1 + \|H_{T^c}\|_{2,1} \leq \|X_0, T + H_T, \text{sign}(X_0, T)\|_2 + \|H_{T^c}\|_{2,1} \]

where $\|X_0, T\|_2 \geq \|X_0, T + H_T, \text{sign}(X_0, T)\|_2 - \|H_{T^c}\|_{2,1}$ where $\text{sign}(X_0, T) = \text{sign}(X_0)$ and the first inequality comes from the fact that

\[ \|H_T, \text{sign}(X_0)\|_F + \|H_{T^c}\|_{2,1} > 0 \]

for any $H \neq 0$ in the nullspace of $L$, $X_0$ is the unique solution.

**Proposition 3.** The matrix $X_0 \in \mathbb{C}^{K \times M}$ with support $T$ is the unique solution to the inverse problem (I.5) if there exists $\gamma > 0$ and a matrix $Y$ in the range space of $L^*$ such that

\[ \|Y_T - \text{sign}(X_0)\|_F \leq \frac{1}{4\sqrt{2}\gamma} \quad \text{and} \quad \|Y_{T^c}\|_{2,\infty} \leq \frac{1}{2} \]

and the operator $L$ satisfies $\{L_T(X), \{b_n^H X_n,_{n=1}\}^N\}$

\[ \|L_T^* L_T - I_T\| \leq \frac{1}{2} \quad \text{and} \quad \|L\| \leq \gamma. \tag{III.2} \]

**Proof.** Proposition 2 shows that to establish uniqueness, it is sufficient to prove that $-\|H_T, \text{sign}(X_0)\| + \|H_{T^c}\|_{2,1} > 0$ for any $H \neq 0$ in the nullspace of $L$. Note that

\[ -\|H_T, \text{sign}(X_0)\| + \|H_{T^c}\|_{2,1} = -\|H_T, \text{sign}(X_0) - Y_T\| + \|H_T, Y_T\| + \|H_{T^c}\|_{2,1} \]

\[ \geq -\|H_T, \text{sign}(X_0) - Y_T\| - \|H_{T^c}, Y_{T^c}\| + \|H_{T^c}\|_{2,1} \]

since $(H_T, Y_T) = (H_{T^c}, Y_{T^c})$. By applying the Hölder inequality, we get a stronger condition

\[ -\|\text{sign}(X_0) - Y_T\|_F \|H_T\|_F + (1 - \|Y_{T^c}\|_{2,\infty}) \|H_{T^c}\|_{2,1} > 0 \]

Since $\|L_T^* L_T - I_T\| \leq \frac{1}{2}$ and $\|L\| \leq \gamma$, we have $\|L(H_T)\|_F \geq \frac{1}{\gamma^2} \|H_T\|_F$, $\|L(H_{T^c})\|_F \leq \gamma \|H_{T^c}\|_F$ and

\[ \|L_T^* L_T - I_T\| \leq \|L(H_T)\|_F \leq \|L(H_{T^c})\|_F \leq \|H_{T^c}\|_{2,1}. \tag{III.3} \]
Plugging (I.3) into the stronger condition above yields
\[
1 - ||Y_{TC}||_{2,\infty} - \sqrt{2\gamma}||\text{sign}(X_0) - Y_{TC}||_F||H_{TC}||_{2,1} > 0.
\]
Therefore, if \( ||Y_T - \text{sign}(X_0,T)||_F \leq \frac{1}{\sqrt{2\gamma}}, \) \( ||Y_{TC}||_{2,\infty} \leq \frac{1}{2}, \) and \( H_{TC} \neq 0, \) the left hand side is positive. On the other hand, if \( H_{TC} = 0 \), from (I.3), \( H_T = 0 \) and \( H = 0. \)

**Theorem III.1.** If \( ||\Phi_T^H \Phi_T - I_T|| \leq \frac{1}{2}, \) there exists \( Y \) in the range space of \( \mathcal{L}^* \) such that
\[
Y_T = \text{sign}(X_0,T) \quad \text{and} \quad ||Y_{TC}||_{2,\infty} \leq \frac{1}{2}
\]
with probability at least \( 1 - (M - J)e^{-\alpha} \) when \( N \geq 40\alpha \mu_{\text{max}}^2 K J \) for \( \alpha \geq \log(M - J). \)

**Proof.** To simplify the notation, without loss of generality, we assume the support of \( X_0 \) is the first \( J \) columns. Let \( \mathbf{Y} \) be the dual certificate matrix defined in (I.3). After removing the columns of \( Y \) on support \( T, \) we obtain \( \text{vec}(Y_{TC}) \in \mathbb{C}^{K(M - J)} \) which takes the form
\[
\text{vec}(Y_{TC}) = \Phi_{T}^H p
\]
\[
= [\phi_{1,j+1}^H, \ldots, \phi_{K,j+1}^H, \phi_{1,j+2}^H, \ldots, \phi_{K,M}^H]^T
\]
\[
= [a_{j+1}^H \text{diag}(\tilde{b}_1), \ldots, a_{j+1}^H \text{diag}(\tilde{b}_K)] p,
\]
\[
= a_{j+1}^H \text{vec}(\tilde{b}_K) p.
\]

The columns of \( \Phi_{T}^H \) are independent of \( p \) since \( p \) is constructed with \( \alpha_j \) \((j \in T)\). Equivalently,
\[
\tilde{Y}_{TC} = \begin{bmatrix}
    a_{j+1}^H \text{diag}(\tilde{b}_1) & \cdots & a_{j+1}^H \text{diag}(\tilde{b}_K) \\
    a_{j+2}^H \text{diag}(\tilde{b}_1) & \cdots & a_{j+2}^H \text{diag}(\tilde{b}_K) \\
    \vdots & \ddots & \vdots \\
    a_{j+N}^H \text{diag}(\tilde{b}_K) & \cdots & a_{j+N}^H \text{diag}(\tilde{b}_K)
\end{bmatrix}
\]

Thus \( ||Y_{TC}||_2 = ||P a_j||_2 \) \((j > J)\) where \( a_j \) is real and
\[
P = \begin{bmatrix}
    p^T \text{diag}(\tilde{b}_1) \\
    p^T \text{diag}(\tilde{b}_2) \\
    \vdots \\
    p^T \text{diag}(\tilde{b}_K)
\end{bmatrix} \in \mathbb{C}^{K \times N}.
\]

We set \( \Sigma = PHP \in \mathbb{C}^{N \times N} \) and have
\[
\text{Tr} (\Sigma) = ||P||_F^2 \leq \frac{2\mu_{\text{max}}^2 K J}{N}
\]
since each row of \( P \) can be bounded by
\[
||P^T \text{diag}(\tilde{b}_i)||_2^2 \leq \frac{\mu_{\text{max}}^2}{N} ||\tilde{b}_i||_2^2
\]
\[
= \frac{\mu_{\text{max}}^2}{N} \text{vec}(\text{sign}(\tilde{X}_{0,T}))^H (\Phi_T^H \Phi_T)^{-1} \text{vec}(\text{sign}(\tilde{X}_{0,T}))
\]
\[
\leq \frac{2\mu_{\text{max}}^2}{N} ||\text{sign}(\tilde{X}_{0,T})||_F^2 = \frac{2\mu_{\text{max}}^2}{N}
\]
since we assume \( ||\Phi_T^H \Phi_T - I_T|| \leq \frac{1}{2} \) which implies \( ||(\Phi_T^H \Phi_T)^{-1}|| \leq 2. \) By generalizing Proposition 1 in (20) to our case, we have
\[
\text{Pr} \left( ||P a_j||_2^2 > \text{Tr} (\Sigma) + 2\sqrt{\text{Tr} (\Sigma^2) \alpha + 2||\Sigma||\alpha} \right) \leq e^{-\alpha}
\]
In addition, because \( \Sigma \) is positive semi-definite and all its eigenvalues are non-negative, \( \text{Tr} (\Sigma^2) = \sum_{i=1}^N \lambda_i^2 \leq (\sum_{i=1}^N \lambda_i)^2 = \text{Tr} (\Sigma^2) \) where \( \lambda_i \) is the \( i \)-th eigenvalue of
According to Section 4.2.1 in [9], there exists a partition of the $N$ observations into $P$ disjoint subsets such that each subset, $\Gamma_p$, contains $Q$ elements and

$$
\max_{1 \leq p \leq P} ||\mathbf{P}_p - \frac{Q}{N} \mathbb{I}_K|| < \frac{Q}{4N},
$$

where $\mathbf{P}_p = \sum_{l \in \Gamma_p} b_l^H b_l$ and $Q > C\mu_{\max} K \log(N)$. So

$$
\max_{1 \leq p \leq P} ||\mathbf{B}_p|| \leq \frac{5Q}{4N}.
$$

Define $L_p(X) = \{b_l^H X a_l^H\}_{l \in \Gamma_p}$ and 0 on entries $l \notin \Gamma_p$. $L_p^*(x) = \sum_{l \in \Gamma_p} x_l b_l^H a_l^H$. The golfing scheme iterates through

$$
\mathbf{Y}_p = \mathbf{Y}_{p-1} - \frac{N}{Q} L_p^* L_p(\mathbf{Y}_{p-1} - T - \text{sign}(\mathbf{X}_0, T)), \quad \mathbf{Y}_0 = 0.
$$

(III.7)

**Theorem III.2.** If $\mathbf{X}_0$ is the ground truth solution to problem (I.3), there exists a matrix $\mathbf{Y} \in \mathbb{L}^*$ such that

$$
||\mathbf{Y}_{T} - \text{sign}(\mathbf{X}_0, T)||_F \leq \frac{1}{4\sqrt{\gamma}} \text{ and } ||\mathbf{Y}_{T^c}||_2 \leq \frac{1}{2}
$$

with probability at least $1 - 2N^{-\alpha + 1}$ for $\alpha > 1$ when

$$
N = PQ, \quad P \geq \frac{\log(4(2J\gamma))}{2}\log 2
$$

and

$$
Q \geq C\mu_{\max} K \log(M - J + \log(K + 1) + \log(N))
$$

where $C\alpha$ is a constant determined by $\alpha$.

**Proof.** If we define $\mathbf{W}_p = \mathbf{Y}_{p,T} - \text{sign}(\mathbf{X}_0,T)$, (III.7) gives

$$
\mathbf{W}_p = \frac{N}{Q} \left( \frac{Q}{N} \mathbb{L}_p^* \mathbb{L}_p \mathbf{W}_p \right) - \mathbb{L}_p \mathbf{W}_p, \quad \mathbf{W}_0 = \mathbb{L}_p \mathbf{W}_0
$$

(III.8)

where $\mathbb{L}_p(\mathbf{X}) = \{b_l^H X a_l^H\}_{l \in \Gamma_p}$ with 0 on entries $l \notin \Gamma_p$ and $\mathbb{L}_p^*(x) = \sum_{l \in \Gamma_p} x_l b_l^H a_l^H$, which are used to generate the sequence $\mathbf{Y}_{p,T}$. And we can obtain

$$
||\mathbf{W}_p||_F \leq \frac{N}{Q} \left( \frac{Q}{N} - \mathbb{L}_p^* \mathbb{L}_p \mathbf{W}_p \right) ||\mathbf{W}_p - \mathbf{W}_0||_F \leq \frac{1}{2} ||\mathbf{W}_0||_F
$$

(III.9)

with probability at least $1 - N^{-\alpha + 1}$ when $Q \geq C\alpha \mu_{\max} K \log(N)$ with $\alpha > 1$ applying Lemma 4.6 in [9]. Therefore,

$$
||\mathbf{W}_p||_F \leq 2^{-P} ||\mathbf{W}_0||_F = 2^{-P} \text{sign}(\mathbf{X}_0,T) \text{ for } 2^{-P} \sqrt{J}.
$$

(III.10)

To ensure that $||\mathbf{W}_p||_F = ||\mathbf{Y}_{T,T} - \text{sign}(\mathbf{X}_0,T)||_F \leq \frac{1}{4\sqrt{\gamma}}$, where $\mathbf{Y}_p = \mathbf{Y}$ is the final constructed dual certificate after $P$ iterations, we need

$$
P \geq \frac{\log(4\sqrt{2J\gamma})}{\log 2}.
$$

(III.11)

We now turn to find the conditions such that $||\mathbf{Y}_{T^c}||_2 \leq \frac{1}{2}$. Note that substituting $\mathbf{W}_p$ into equation (III.7) yields

$$
\mathbf{Y} = -\frac{N}{Q} \sum_{p=1}^P \mathbb{L}_p^* \mathbb{L}_p(\mathbf{W}_{p-1}).
$$
It is sufficient to show \( \| \Pi_{T^C} (L_p^* L_p W_{p-1}) \|_2 \leq 2^{-p-1} \frac{N}{Q} \), where \( \Pi_{T^C} \) is the projector on \( T^C \). Furthermore, we have \( \Pi_{T^C} \) is the projector on \( T^C \), so that \( \Pi_{T^C} (L_p^* L_p W_{p-1}) = 0 \) because support \( T \) and \( 0 \) on \( T^C \). Therefore, for \( i \in T^C \), \( E(z_{i,i}) = 0 \). Moreover,

\[
\| z_{i,i} \|_2 \leq \sqrt{K \left( \frac{\mu_{\max} \sqrt{KJ}}{N} \| w \|_2 \right)^2} \leq \frac{\mu_{\max} K \sqrt{J} \| w \|_2}{N}.
\]

Because each entry of \( z_{i,i} \) can be bounded by

\[
| \langle \phi_i^H \phi_j^H w, e_{K(i-1)+j} \rangle | = | e_{K(i-1)+j}^H \phi_i \phi_j w | = | e_{K(i-1)+j}^H \phi_i |^2 \| (\hat{a}_i^H b_j^H) w \|_2 \leq \frac{\mu_{\max} \sqrt{KJ}}{N} \| w \|_2 \leq \frac{\mu_{\max} \sqrt{KJ}}{N} \| w \|_2^2 \leq \frac{\mu_{\max} \sqrt{KJ}}{N} \| w \|_2 \leq \frac{\mu_{\max} \sqrt{KJ}}{N} \| w \|_2^2.
\]

where the third equality holds because \( w = \text{vec}(W) \) and \( W \) has support \( T \). The variance of \( z_{i,i} \) is also bounded:

\[
\max \left\{ \left\| \sum_{i \in \Gamma_p} E(z_{i,i} z_{i,i}^T) \right\|, \left\| \sum_{i \in \Gamma_p} E(z_{i,i}^T z_{i,i}) \right\| \right\} \leq \sum_{i \in \Gamma_p} E(\| z_{i,i} \|_2^2) \leq \frac{5\mu_{\max} K Q \| w \|_2}{4N^2}.
\]

because for each element of \( \| z_{i,i} \|_2 \), we have

\[
E(\| \phi_i^H \phi_j^H w, e_{K(i-1)+j} \|_2^2) = E(\| e_{K(i-1)+j}^H \phi_i \phi_j w \|_2^2) \leq \frac{\mu_{\max} \sqrt{KJ}}{N} \| w \|_2 \leq \frac{\mu_{\max} \sqrt{KJ}}{N} \| w \|_2^2.
\]

and therefore

\[
E(\| z_{i,i} \|_2^2) \leq \frac{2 \mu_{\max} K}{N} w^H (I_M \otimes b_i^H b_i^H) w.
\]

Furthermore,

\[
\sum_{i \in \Gamma_p} E(\| z_{i,i} \|_2^2) \leq \sum_{i \in \Gamma_p} \frac{\mu_{\max} K}{N} w^H (I_M \otimes b_i^H b_i^H) w \leq \frac{5\mu_{\max} K Q \| w \|_2}{4N^2}.
\]

The second inequality in (III.13) applies the inequality (III.6) and \( |I_M \otimes B_p| = |I_M| = |B_p| \). We then apply the matrix Bernstein inequality from Theorem 1.6 in [23]. If we set \( u = \text{vec}(W_{p-1}) \) and we know from (III.10) that \( \| w \|_2 = \| W_{p-1} \|_F \leq 2^{-p+1} \sqrt{J} \), we obtain

\[
\Pr \left( \left\| \sum_{i \in \Gamma_p} z_{i,i} \right\|_2 \geq t \right) \leq (K + 1) \exp \left( -\frac{3t^2}{30 \mu_{\max}^2 K Q \| w \|_2^2} + 2 \mu_{\max} K \sqrt{J} \| w \|_2 \right)
\]

\[
\leq (K + 1) \exp \left( -\frac{3Q}{128 \mu_{\max} K J} \right).
\]
where $t = 2^{-p-1} \frac{Q}{N}$, for a particular $i \in T^C$ and $p$. We then take the union over all $i \in T^C$ and get
\[
\Pr \left( \| \Pi_{T^C} (L_p^* L_p(W_{p-1})) \|_{2, \infty} \geq 2^{-p-1} \frac{Q}{N} \right) \\
\leq \left( M - J \right) \left( K + 1 \right) \exp \left( \frac{-3Q}{128 \mu_{\max} K J} \right).
\]

To ensure $\| \Pi_{T^C} (L_p^* L_p(W_{p-1})) \|_{2, \infty} \leq 2^{-p-1} \frac{Q}{N}$ for all $p$, we obtain
\[
\Pr \left( \| \Pi_{T^C} (L_p^* L_p(W_{p-1})) \|_{2, \infty} \leq 2^{-p-1} \frac{Q}{N}, \forall 1 \leq p \leq P \right) \\
\geq 1 - P \left( M - J \right) \left( K + 1 \right) \exp \left( \frac{-3Q}{128 \mu_{\max} K J} \right) \\
\geq 1 - P N^{-\alpha} \geq 1 - N^{-\alpha+1}
\]
when $Q \geq \frac{128 \mu_{\max} K J \alpha}{4 \log (M - J) + \log (K + 1) + \log (N)}$ using the same $\alpha$ as in deriving equation (III.9). Setting $C_\alpha = \max \{C, C_{\alpha,1}, C_{\alpha,2} \}$, where $C$ is a constant comes from equation (III.6), gives us Theorem III.2.

**F. Proof of Theorem III.1 for random Fourier dictionary**

We now complete the proof of Theorem III.1 in the case when $A$ is a random Fourier matrix. First, combining the conditions and probabilities from Lemma III.1 and III.2, we know that the operator $L$ satisfies the inequalities $\| L \cdot T - I_T \| \leq \frac{1}{2}$ and $\| L \| \leq \gamma = \sqrt{2M \log(2KM) + 2M + 1}$ with probability at least $1 - \left( N + 1 \right) \left( N^{-\alpha} \geq 1 - N^{-\alpha+1} \right)$ when $N \geq C_{\alpha,2} \mu_{\max} K J \log(N)$ for some constant, $C_{\alpha,2}$, that grows linearly with $\alpha > 1$.

Applying the same $\alpha$ in Theorem III.2, the desired dual matrix exists with probability at least $1 - 2 N^{-\alpha+1}$ when $N \geq C_{\alpha,2} \mu_{\max} K J \log(4 \sqrt{2} \gamma)(\log(M - J) + \log(K + 1) + \log(N))$. Merging the requirement on $N$ by setting $C_\alpha = \max \{C_{\alpha,1}, C_{\alpha,2} \}$ and combining the probabilities, we complete the proof by applying Proposition 3.

**IV. PROOF OF THEOREM III.2**

To derive our recovery guarantee in the presence of measurement noise, the main ingredient of the proof is Theorem IV.1 which is a variation of the Theorem 4.33 in [19] from the infinity norm optimization to $\ell_2, 1$ norm optimization problem.

**Theorem IV.1.** Define $\Phi \in \mathbb{C}^{N \times KM}$ and $\Phi \cdot \text{vec}(X) = \mathcal{L}(X)$. Suppose the ground truth $X_0$ to (II.1) has $J$ non-zero columns with support $T$ and the observation vector $y = \mathcal{L}(X_0) + \nu$ with $\| \nu \|_2 \leq \eta$. For $\delta, \beta, \theta, \gamma, \tau > 0$ and $\delta < 1$, assume that
\[
\max_{i \in T^C} \| \Phi_H^* \Phi_{T^C} \| \leq \beta,
\]
\[
\| \Phi_H^* \Phi_T - I_T \| \leq \delta
\]
and that there exists a matrix $Y_T = \mathcal{L}^*(p) \in \mathbb{C}^{K \times M}$ such that
\[
\| Y_T - \text{sign}(X_0,T) \|_F \leq \frac{\beta}{\sqrt{2} \gamma}, \quad \| Y_T \|_{2,\infty} \leq \theta,
\]
and $\| p \|_2 \leq \tau \sqrt{J}$.

If $\rho := \theta + \frac{\delta}{4 \sqrt{2} \gamma (1 - \delta)} < 1$, then the minimizer, $\hat{X}$, to (II.1) satisfies
\[
\| \hat{X} - X_0 \|_F \leq (C_1 + C_2 \sqrt{J}) \eta
\]
where $C_1$ and $C_2$ are two constants depending on $\delta, \beta, \theta, \gamma, \tau$.

**Proof.** Due to our assumption on the noise, $X_0$ is a feasible solution. Assume the final minimizer to (II.1) is $\hat{X} = X_0 + H$, which implies
\[
\| X_0 \|_{2,1} \geq \| X_0 + H \|_{2,1} = \| X_0, T \|_{2,1} + \| H_{T^C} \|_{2,1}
\]
\[
\geq (\| X_0, T \| + H_T, \text{sign}(X_0,T)) + \| H_{T^C} \|_{2,1}
\]
\[
\geq \| X_0 \|_{2,1} - (| H_T, \text{sign}(X_0,T) |) + \| H_{T^C} \|_{2,1}
\]
where the second inequality comes from equation (III.1). Thus
\[
\| H_{T^C} \|_{2,1} \leq (| H_T, \text{sign}(X_0,T) |)
\]
\[
\leq | (H_T, \text{sign}(X_0,T) - Y_T) + (H_T, Y_T) |
\]
\[
\leq \frac{1}{4 \sqrt{2} \gamma} \| H_T \|_F + \| (H, Y) \| + \| H_{T^C}, Y_{T^C} \|_F
\]
\[
\leq \frac{1}{4 \sqrt{2} \gamma} \| H_T \|_F + 2 \eta \sqrt{J} + \theta \| H_{T^C} \|_{2,1}
\]

The last inequality comes from the Hölder inequality and our assumption $\| \nu \| \leq \eta$, which tells us
\[
\| \mathcal{L}(H) \|_2 = \| \mathcal{L}(\hat{X} - X_0) \|_2 = \| \mathcal{L}(\hat{X}) - \mathcal{L}(X_0) \|_2
\]
\[
\leq \| \mathcal{L}(\hat{X}) - y \|_2 + \| y - \mathcal{L}(X_0) \|_2 \leq 2 \eta
\]
and
\[
|\langle H, Y \rangle | = |\langle H, \mathcal{L}^*(p) \rangle | = |\langle \mathcal{L}(H), p \rangle | \leq \tau \sqrt{J} \| \mathcal{L}(H) \|_2
\]
\[
\leq 2 \tau \sqrt{J}.
\]
Moreover, $\| H_{T^C} \|_F$ can also be bounded as follows.
\[
\| H_{T^C} \|_F = \| \Phi_{T^C} \Phi_{T^C}^{-1} \Phi_{T} \Phi_{T} \cdot \text{vec}(H_{T^C}) \|_2
\]
\[
\leq \| \Phi_{T} \Phi_{T} \cdot \text{vec}(H_{T}) \|_2 = \frac{1}{1 - \delta} \| \Phi_{T} \Phi_{T} \cdot \text{vec}(H_{T}) \|_2
\]
\[
= \frac{1}{1 - \delta} \| \Phi_{T} \Phi_{T} \cdot \text{vec}(H - T) \cdot \text{vec}(H_{T^C}) \|_2
\]
\[
\leq \frac{1}{1 - \delta} \| \Phi_{T} \Phi_{T} \cdot \text{vec}(H) \|_2 + \frac{1}{1 - \delta} \| \Phi_{T} \Phi_{T} \cdot \text{vec}(H_{T^C}) \|_2
\]
\[
\leq \frac{1}{1 - \delta} \| \Phi_{T} \Phi_{T} \cdot \text{vec}(H) \|_2 + \frac{1}{1 - \delta} \| \Phi_{T} \Phi_{T} \cdot \text{vec}(H_{T^C}) \|_2
\]
\[
\leq 2 \eta \sqrt{1 + \delta} + \beta \| H_{T^C} \|_{2,1}
\]

because $\| \Phi_{T} \Phi_{T} \cdot \text{vec}(H) \|_2 \leq \delta$ ensures that $\| (\Phi_{T} \Phi_{T}^{-1} \Phi_{T}) \|_F \leq \frac{1}{1 - \delta}$ and $\| \Phi_{T} \Phi_{T} \|_F \leq \sqrt{1 + \delta}$ according to Lemma A.12 and Proposition A.15 in [19] respectively. Furthermore,
\[
\| \Phi_{T} \Phi_{T^C} \cdot \text{vec}(H_{T^C}) \|_2
\]
\[
= \| \sum_{i \in T^C} \Phi_{T} \Phi_{T} \cdot \text{vec}(H_{K(i-1)+1} \cdots K(i-1)+K) \|_2
\]
\[
\leq \sum_{i \in T^C} \| \Phi_{T} \Phi_{T} \cdot \text{vec}(H_{K(i-1)+1} \cdots K(i-1)+K) \|_2 \| h_i \|_2
\]
\[
\leq \sum_{i \in T^C} \beta \| h_i \|_2 = \beta \| H_{T^C} \|_{2,1}
\]

\[
(IV.1)
\]

\[
(IV.2)
\]
in which $h_i$ is the $i$-th column of $H$. By setting $\rho = \theta + \frac{\beta}{2\sqrt{2\gamma(1-\rho)}}$, $\mu = \frac{\eta \mu}{2\sqrt{2\gamma(1-\rho)}}$, and substituting the inequality \[(IV.2)\] into \[(IV.1)\], we obtain

$$
\|H_{T^C}\|_{2,1} \leq \frac{\eta \mu}{2\sqrt{2\gamma(1-\rho)}} + \frac{2\eta \gamma \sqrt{J}}{1-\rho}. \tag{IV.3}
$$

Substituting inequality \[(IV.3)\] into \[(IV.2)\] yields

$$
\|H_T\|_F \leq 2\eta \mu + \frac{\beta \mu}{1-\delta} \left( \frac{\eta \mu}{2\sqrt{2\gamma(1-\rho)}} + \frac{2\eta \gamma \sqrt{J}}{1-\rho} \right). 
$$

Combining the above two inequalities, we obtain

$$
\|H\|_F \leq \|H_T\|_F + \|H_{T^C}\|_F \leq \|H_T\|_F + \|H_{T^C}\|_{2,1} \leq \left(2\mu + \frac{\eta \mu}{2\sqrt{2\gamma(1-\rho)}} + \frac{\beta \mu}{2\sqrt{2\gamma(1-\delta)(1-\rho)}} \right) \eta + \left(2\tau + \frac{2\beta \tau}{(1-\delta)(1-\rho)} \right) \sqrt{J} \eta
$$

$$
= (C_1 + C_2 \sqrt{J}) \eta. \tag{IV.4}
$$

Next, we specify the values of the variables $\theta$, $\tau$, $\delta$, and $\beta$ when $A$ is a random Gaussian and Fourier matrix. The Orlicz-1 norm $\|\|$ and associated matrix Bernstein inequality are needed for determining the value of $\beta$ when $A$ is Gaussian. Specifically, the Orlicz-1 norm is defined as [1]

$$
\|Z\|_{\psi_1} = \inf_{u \geq 0} \{|E[\exp(|Z|/u)]| \leq 2\}. \tag{IV.5}
$$

Its associated matrix Bernstein inequality is provided in Proposition 3 in [1] which can be rewritten as

\[\text{Proposition 4. Let } Z_1, \ldots, Z_N \text{ be independent } M \times M \text{ random matrices with } E(Z_j) = 0. \text{ Suppose}
\]

$$
\max_{1 \leq j \leq N} \|Z_j\|_{\psi_1} \leq R
$$

and define

$$
\sigma^2 = \max \left\{ \|E(Z_jZ_j^H)\|_1, \|E(Z_j^H Z_j)\|_1 \right\}.
$$

Then there exists a constant $C$ such that for $t > 0$

$$
Pr \left( \| \sum_{j=1}^{N} Z_j \| > t \right) \leq 2M \exp \left( -C \sigma^2 \log \left( \frac{\sqrt{N}Rt}{\sigma} \right) \right).
$$

The following theorem utilizes the Proposition [4] and depicts the conditions under which $\beta = 1$.

\[\text{Theorem IV.2. For } \Phi \text{ defined in } \Phi \text{ and } \mathcal{L}(X) = \Phi \text{vec}(X), \]

$$
\max_{i \in T} \|\Phi_T^H[\Phi_{K(i-1)+1} \cdots \Phi_{K(i-1)+K}]\| \leq 1
$$

with probability at least $1 - N^{-\alpha + 1}$.

- if $A$ is a random Gaussian matrix and

$$
N \geq C_\alpha \mu_{\max}^2 KJ \left( (\log(\mu_{\max} \sqrt{KJ}) + 1) \cdot (\log(KM) + \log(M - J) + \log(N)) \right),
$$

- if $A$ is a random Fourier matrix and

$$
N \geq C_\alpha \mu_{\max}^2 KJ (\log(KM) + \log(M - J) + \log(N)),
$$

where $C_\alpha$ is a constant that grows linearly with $\alpha > 1$ and $C$ is a constant.

\[\text{Proof. We first prove the Gaussian case; the Fourier case is very similar. Note that for an arbitrary } i \in T^C
\]

$$
\left\| \Phi_T^H \left[ \Phi_{K(i-1)+1} \cdots \Phi_{K(i-1)+K} \right] \right\|_1
$$

$$
= \left\| \Phi_T^H \Phi_i \right\|_1 = \left\| \sum_{j=1}^{N} (\bar{a}_{j,i}^T \otimes b_j) \cdot (\bar{a}_{j,i}^H \otimes b_j^H) \right\|_1
$$

$$
= \left\| \sum_{j=1}^{N} (\bar{a}_{j,i}^T \bar{a}_{j,i}^H) \otimes (b_j^H b_j^H) \right\|_1 = \left\| \sum_{j=1}^{N} Z_j \right\|_1
$$

where $\Phi_i \in C^{N \times KM}$ is $\Phi$ but only contains values in the $(K(i-1) + 1)$-th to $(K(i-1) + K)$-th columns and is zero otherwise. $\Phi_i$ can also be viewed as an extension of $\Phi_{T^C, K(i-1)+1} \cdots \Phi_{T^C, K(i-1)+K}$ by padding zero columns. Moreover, $\bar{a}_{j,i}$ is the conjugate of the $j$-th column of $A^T$ who has only one non-zero value in the $i$-th entry. In addition, $E(Z_j) = E(\bar{a}_{j,i}^H \otimes b_j^H) = E(\bar{a}_{j,i}^H \bar{a}_{j,i}^H) \otimes b_j^H b_j^H = 0$ for $i \in T^C$. By applying the property of the Kronecker product, we estimate the spectral norm of $Z_j$ which can be used to determine its Orlicz-1 norm:

$$
\|Z_j\|_1 = \|\bar{a}_{j,i}^H \otimes b_j^H\|_1 = \|b_j^H b_j^H\|_1 \cdot \|\bar{a}_{j,i}^H \bar{a}_{j,i}^H\|_1
$$

$$
= \|b_j^H b_j^H\|_1 \cdot \|\bar{a}_{j,i}^H \bar{a}_{j,i}^H\|_1 \leq \frac{\mu_{\max}^2 K}{N} \|\bar{a}_{j,i}^H \bar{a}_{j,i}^H\|_1
$$

$$
= \frac{\mu_{\max}^2 K}{N} \|\bar{a}_{j,i}^H \|_2 \|\bar{a}_{j,i}^H\|_2
$$

$$
\leq \frac{\mu_{\max}^2 K}{N} \cdot \|\bar{a}_{j,i}^H\|_2^2 + \|\bar{a}_{j,i}^H\|_2^2 = \frac{\mu_{\max}^2 K}{2N} \|\bar{a}_{j,(T,i)}^H\|_2^2
$$

in which $\bar{a}_{j,(T,i)}$ contains non-zero values in the entries indexed by $\{T,i\}$. Therefore, $\|\bar{a}_{j,(T,i)}\|_2^2$ follows the Chi-squared distribution with $J + 1$ degrees of freedom which implies that $\|Z_j\|_{\psi_1} \leq \frac{C_\alpha \mu_{\max} K(J+1)}{2N} \leq \frac{C_\alpha \mu_{\max} K2J}{2N} = \frac{C_\alpha \mu_{\max} KJ}{R}$ for some constant $C$ according to the proof of Lemma 4.7 in [9] and the definition of Orlicz-1 norm.
Moreover,
\[
\left\| \sum_{j=1}^{N} E(Z_j^H Z_j) \right\| = \left\| \sum_{j=1}^{N} E \left[ (\tilde{a}_j^H, \tilde{a}_j^H) \otimes (b_j b_j^H) \cdot (\tilde{a}_j^H, \tilde{a}_j^H, \tilde{a}_j^H) \otimes (b_j b_j^H) \right] \right\| \\
= \left\| \sum_{j=1}^{N} E \left( \tilde{a}_j, \tilde{a}_j^H, \tilde{a}_j^H \right) \otimes (b_j b_j^H, b_j b_j^H) \right\| \\
= \left\| J_{I, M} \otimes \left( \sum_{j=1}^{N} \left\| b_j^H \right\| \right) \right\| \\
\leq \mu_{\text{max}}^2 K J N \left\| J_{I, M} \right\| \cdot \left\| \sum_{j=1}^{N} b_j b_j^H \right\| = \frac{\mu_{\text{max}}^2 K J N}{N},
\]
following from the fact that \( E(\tilde{a}_j, \tilde{a}_j^H, \tilde{a}_j^H) \otimes (b_j b_j^H, b_j b_j^H) = J_{I, M, i} \) for all \( j \) and \( \sum_{j=1}^{N} b_j b_j^H = I_{K} \) from the assumption. On the other hand,
\[
\left\| \sum_{j=1}^{N} E(Z_j^H Z_j^H) \right\| = \left\| \sum_{j=1}^{N} E \left( \tilde{a}_j^H \tilde{a}_j^H, \tilde{a}_j^H, \tilde{a}_j^H \right) \otimes b_j^H b_j^H b_j^H \right\| \\
= \left\| J_{I, M} \otimes \left( \sum_{j=1}^{N} ||b_j^H|| \right) \right\| \\
\leq \frac{\mu_{\text{max}}^2 K J N}{N} \left\| J_{I, M} \right\| \cdot \left\| \sum_{j=1}^{N} b_j b_j^H \right\| = \frac{\mu_{\text{max}}^2 K J N}{N}.
\]
Therefore, \( \max \{ \left\| \sum_{j=1}^{N} E(Z_j^H Z_j^H) \right\|, \left\| \sum_{j=1}^{N} E(Z_j^H Z_j) \right\| \} = \frac{\mu_{\text{max}}^2 K J N}{N} \) and substituting the variables \( R \) and \( \sigma^2 \) into Proposition 4 and taking the union bound over all \( i \in T^C \) results in
\[
\Pr \left( \max_{i \in T^C} \left\| \Phi_{T^C}^H \Phi_{T^C, K(i-1)+1} \cdots \Phi_{T^C, K(i-1)+K} \right\| \leq 1 \right) \\
\leq 2(M - J) K M \exp \left( - \frac{1}{C_0} \frac{N \mu_{\text{max}}^2 K J + \log \left( C \mu_{\text{max}} \sqrt{K J} \right) C \mu_{\text{max}}^2 K J}{N} \right).
\]
Define a variable \( \alpha > 1 \) and set
\[
N \geq C_0 \mu_{\text{max}}^2 K J \left( \log(C \mu_{\text{max}} \sqrt{K J}) C + 1 \right). \\
\left( \log(K M) + \log(M - J) + \log(N) \right) \\
\geq C_0 \mu_{\text{max}}^2 K J \left( \log(C \mu_{\text{max}} \sqrt{K J}) C + 1 \right). \\
\left( \log(K M) + \log(M - J) + \alpha \log(N) \right),
\]
where \( C_0 = C_0 \alpha \). Simplifying the probability term gives
\[
\Pr \left( \max_{i \in T^C} \left\| \Phi_{T^C}^H \Phi_{T^C, K(i-1)+1} \cdots \Phi_{T^C, K(i-1)+K} \right\| \leq 1 \right) \\
> 1 - 2N^{-\alpha} \geq 1 - N \cdot N^{-\alpha} = 1 - N^{-\alpha+1}.
\]
Following the same procedures, when \( A \) is a random Fourier matrix and for any \( i \in T^C \), we have \( E(Z_j^i) = E(\tilde{a}_j^H T \tilde{a}_j^H) \otimes b_j b_j^H = 0 \) and \( \left\| Z_j \right\| = \frac{2 \mu_{\text{max}} K}{N} \left\| \tilde{a}_j \right\| \leq \frac{2 \mu_{\text{max}} K J}{N} = R \) and \( \sigma^2 = \frac{\mu_{\text{max}}^2 K J}{N} \). The matrix Bernstein inequality implies
\[
\Pr \left( \max_{i \in T^C} \left\| \Phi_{T^C}^H \Phi_{T^C, K(i-1)+1} \cdots \Phi_{T^C, K(i-1)+K} \right\| \geq 1 \right) \\
\leq 2(M - J) K M \exp \left( - \frac{N \mu_{\text{max}}^2 K J}{2 \mu_{\text{max}}^2 K J + 2 \frac{3}{2} \mu_{\text{max}}^2 K J} \right).
\]
Similarly, if we define a variable \( \gamma > 1 \) and let
\[
N \geq C_0 \mu_{\text{max}}^2 K J \log(K M) + \log(M - J) + \log(N)
\]
\[
\geq \left( \frac{2 \mu_{\text{max}}^2 K J}{N} \right) \left( \frac{2 \mu_{\text{max}}^2 K J}{N} \right) \left( \frac{2 \mu_{\text{max}}^2 K J}{N} \right) \log(K M) + \log(M - J) + \alpha \log(N),
\]
by setting \( C_0 = \frac{2}{3} \alpha \), simplifying the probability gives us
\[
\Pr \left( \max_{i \in T^C} \left\| \Phi_{T^C}^H \Phi_{T^C, K(i-1)+1} \cdots \Phi_{T^C, K(i-1)+K} \right\| \leq 1 \right) \\
> 1 - 2N^{-\alpha} \geq 1 - N \cdot N^{-\alpha} = 1 - N^{-\alpha+1}.
\]
\[\square\]

A. Proof of Theorem 7.2 for random Gaussian dictionary

According to Section III-D, \( \left\| \Phi_{T^C}^H \Phi_{T^C} - I_T \right\| \leq \frac{1}{2} = \delta \), \( \left\| Y_{T^C} \right\| \geq \frac{1}{2} = \theta \) and \( \gamma = \sqrt{M \log(M N/2) + \alpha \log(N)} \) with probability at least \( 1 - 3N^{-\alpha+1} \) when \( \frac{N \log N}{N} \geq \frac{C_0 \mu_{\text{max}}^2 K J \log(N)}{C_0 \mu_{\text{max}}^2 K J + \log(M - J) + \log(N)} \). Moreover, in Theorem 7.1 where we construct the dual certificate matrix when \( A \) is a random Gaussian matrix, we define \( p = \Phi_{T^C}^H \Phi_{T^C} - \Phi_{T^C} \Phi_{T^C} \Phi_{T^C} \Phi_{T^C} \left( \text{sign}(\tilde{X}_{0, T_i}) \right) \in C_{N \times 1} \text{ and } \Phi_{T^C}^H \Phi_{T^C} - I_T \right\| \leq \frac{1}{2} \text{ leads to } \left\| \Phi_{T^C}^H \Phi_{T^C} - I_T \right\| \leq 2 \). So
\[
\left\| p \right\|_2 = \sqrt{\text{vec}(\text{sign}(\tilde{X}_{0, T_i}))^H (\Phi_{T^C}^H \Phi_{T^C} - \Phi_{T^C} \Phi_{T^C}) \Phi_{T^C} \Phi_{T^C} \Phi_{T^C} \left( \text{sign}(\tilde{X}_{0, T_i}) \right)} \\
\leq \sqrt{2 \left\| \text{vec}(\text{sign}(\tilde{X}_{0, T_i})) \right\|_2^2 = \sqrt{2} J}
\]
which implies \( \delta = \sqrt{2} \). If we use the same \( \alpha \) in Theorem 7.2, we have \( \beta = 1 \) with probability at least \( 1 - N^{-\alpha+1} \) when
\[
N \geq C_0 \mu_{\text{max}}^2 K J \left( \log(C \mu_{\text{max}} \sqrt{K J}) C + 1 \right). \\
\left( \log(K M) + \log(M - J) + \log(N) \right).
\]
Combining the requirement on \( N \) and setting \( C_\alpha = \max \{ C_{\alpha, 1}, C_{\alpha, 2} \} \) yield
\[
N \geq C_\alpha \mu_{\text{max}}^2 K J \left( \log(C \mu_{\text{max}} \sqrt{K J}) C + 1 \right). \\
\left( \log(K M) + \log(M - J) + \log(N) \right)\]
\[\text{(IV.6)}\]
Therefore, the conditions in Theorem 7.1 are satisfied with probability at least \( 1 - 4N^{-\alpha+1} \) when \( N \) is as defined in
equation (IV.6). In addition, after substituting the parameters \( \rho = \theta + \frac{2}{\sqrt{2} - 1} > 1 \) and \( \mu = \frac{1}{\sqrt{2} - 1} = \sqrt{6} \) into (IV.4), \( 2\mu + \frac{2}{\sqrt{2} - 1} \mu > \frac{2}{\sqrt{2} - 1} (1 - \rho) + \frac{2}{\sqrt{2} - 1} (1 - \rho) = 2\sqrt{6} + \frac{3\sqrt{6}}{\sqrt{2} - 1} \leq 5\sqrt{6} = C_1 \) and \( \frac{1}{\rho - 1} + \frac{2}{\sqrt{2} - 1} (1 - \rho) = \frac{24\sqrt{6}}{\sqrt{2} - 1} \leq 24 = C_2 \).

### B. Proof of Theorem II.2 for random Fourier dictionary

In the proof of Theorem II.2, we have derived \( Y = -\frac{N}{Q} \sum_{p=1}^{P} \mathcal{L}_p(W_{p-1}) \). Since the sets \( \Gamma_p \) are disjoint, the indices of non-zero entries of \( \mathcal{L}_p(W_{p-1}) \) for different \( p \) are disjoint and \( Y = L'(-\frac{N}{Q} \sum_{p=1}^{P} \mathcal{L}_p(W_{p-1})) = L'(p) \). Moreover, \( W_{p-1} \) has support \( T \) from its definition in (III.8) which gives us

\[
||p||_2^2 \leq \frac{N^2}{Q^2} \sum_{p=1}^{P} ||\mathcal{L}_p(W_{p-1})||_2^2 = \frac{N^2}{Q^2} \sum_{p=1}^{P} ||\mathcal{L}_p(T(W_{p-1}))||_2^2
\]

\[
= \frac{N^2}{Q^2} \sum_{p=1}^{P} \text{vec}(W_{p-1})^H \Phi_{p,T}^H \Phi_{p,T} \text{vec}(W_{p-1})
\]

\[
\leq \frac{N^2}{Q^2} \sum_{p=1}^{P} ||\Phi_{p,T}^H \Phi_{p,T}||_2 ||W_{p-1}||_F^2 \leq \frac{N^2}{Q^2} \sum_{p=1}^{P} \frac{3Q}{2N} 4^{-p+1} J
\]

\[
\leq \frac{2NJ}{Q} = 2PJ
\]

because \( ||\Phi_{p,T}^H \Phi_{p,T}||_2 \leq \frac{3Q}{2N} \) and \( ||W_{p-1}||_F^2 < 4^{-p+1} J \) following from Lemma 4.6 in (2) and equation (III.10) respectively. \( \Phi_{p,T} \) is \( \Phi \) constructed with \( A_T \) and only rows indexed by \( \Gamma_p \) are non-zero. Therefore, \( ||p||_2 \leq 2\sqrt{FJ} \) and \( \tau = \sqrt{2P} \) with \( P \geq \log(4\sqrt{2}\gamma) / \log 2 \) defined in equation (III.11). In addition, from Section III.B and Theorem II.1 we have \( \delta = \frac{1}{2}, \theta = \frac{1}{2} \) and \( \gamma = \sqrt{2M \log(2KM) + 2M + 1} \) with probability at least \( 1 - 4N^{-\alpha+1} \) when

\[
N \geq C_{\alpha,1} \mu_{\max}^2 KJ \log(4\sqrt{2}\gamma).
\]

Applying the same \( \alpha \) to Theorem IV.2 \( \beta = 1 \) with probability at least \( 1 - N^{-\alpha+1} \) when \( N \geq C_{\alpha,2} \mu_{\max} KJ \log(4\sqrt{2}\gamma) + \log(M - J) + \log(K + 1) + \log(N) \). One can easily examine that \( \rho = \theta + \frac{2}{\sqrt{2} - 1} > 1 \).

If we set \( C_{\alpha} = \max(C_{\alpha,1}, C_{\alpha,2}) \) and merge the requirements on \( N \), we obtain

\[
N \geq C_{\alpha} \mu_{\max}^2 KJ \log(4\sqrt{2}\gamma):
\]

\[
(\log(M - J) + \log(MK) + \log(N)).
\]

Thus, the conditions in Theorem IV.1 are satisfied with probability at least \( 1 - 5N^{-\alpha+1} \) when \( N \) satisfies (IV.7). Moreover, since \( \mu = \frac{1}{\sqrt{2} - 1} > 2/\sqrt{2} - 1 \), \( 2\mu + \frac{2}{\sqrt{2} - 1} \mu > \frac{2}{\sqrt{2} - 1} (1 - \rho) + \frac{2}{\sqrt{2} - 1} (1 - \rho) = 2\sqrt{6} + \frac{3\sqrt{6}}{\sqrt{2} - 1} \leq 5\sqrt{6} = C_1 \) and \( \frac{1}{\rho - 1} + \frac{2}{\sqrt{2} - 1} (1 - \rho) = \frac{24\sqrt{6}}{\sqrt{2} - 1} \leq 24\sqrt{P} = C_2 \sqrt{P} \) with \( P \geq \log(4\sqrt{2}\gamma) / \log 2 \).

### V. Numerical Simulations

Here we present numerical simulations that illustrate and support our theoretical results. We set \( B \in \mathbb{C}^{M \times K} \) to be the first \( K \) columns of the normalized DFT matrix \( \sqrt{M} \mathbf{F} \in \mathbb{C}^{M \times M} \). The ground truth parameters \( c_j \) and \( h_j \) are generated by sampling independently from the standard normal distribution, and \( J \) non-zero columns of the ground truth solution \( X_0 = [c_j h_j \cdots c_j h_M] \) are selected uniformly. 40 simulations are run for each setting, based on which we compute the percentage of successful recovery. Both the dictionary, \( A \), and the ground truth solution, \( X_0 \), including the support and its content, are sampled independently for each simulation. We solve problems (1.5) and (1.11) via CVX (24), and in the noiseless case if the relative error between the solution \( X \) and the ground truth \( X_0 \) is smaller than \( 10^{-5} \), \( \frac{||X - X_0||_F}{||X_0||_F} \leq 10^{-5} \), we count it as a successful recovery.

In the first noiseless simulation, we examine the recovery rate with respect to the parameters \( K \) and \( J \). We fix \( M = 200 \) and \( N = 100 \) and let \( K \) and \( J \) range from 1 to 20. The results are summarized in the phase transition plots of Fig. 1 for the random Gaussian dictionary and Fig. 2 for the random Fourier dictionary. The results for the two dictionaries are similar. The reciprocal nature of the phase transition boundary supports the linear scaling with \( KJ \) in equations (1.11) and (1.2). Roughly when \( KJ \leq 60 \), the recovery success rate is satisfactory.

To further illustrate the linear scaling of the required number
The same simulation but switching the roles of $K$ for the random Gaussian and Fourier dictionaries, respectively. We call $20 \log |\mathbf{A}|$ sides by sides.

We test the noisy case, we set $M = 200$, $K = J = 5$, and $N = 100$, and we let $N$ and $J$ range from 30 to 100 and 1 to 20, respectively. The results are recorded in Figs. 3 and 4 for the random Gaussian and Fourier dictionaries, respectively. The same simulation but switching the roles of $K$ and $J$ is also implemented, and the results are shown in Figs. 5 and 6. These results support the linear scaling of Theorem II.1.

To test the noisy case, we set $M = 200$, $K = J = 5$, and $N = 100$, and we let $y = \mathcal{L}(X_0) + n$ with $\|n\|_2 \leq \eta$. Theorem II.2 gives a recovery guarantee of the form $\|X - X_0\|_F \leq C \cdot \eta$ for a constant $C$. Therefore, after dividing both sides by $\|X_0\|_F$, setting $\|n\|_2 = \eta$ and changing the units to decibels (dB), we obtain

$$20 \log_{10} \left( \frac{\|X - X_0\|_F}{\|X_0\|_F} \right) \leq 20 \log_{10} \left( \frac{\|n\|_2}{\|X_0\|_F} \right) + 20 \log_{10}(C).$$

We call $20 \log_{10} \left( \frac{\|X - X_0\|_F}{\|X_0\|_F} \right)$ the relative error in dB and $20 \log_{10} \left( \frac{\|n\|_2}{\|X_0\|_F} \right)$ the noise-to-signal ratio in dB. To examine the linear relation between the relative error and the noise-to-signal ratio in equation (V.1), we sample the real and complex components of the noise vector $n$ independently from a standard Gaussian distribution and scale $\|n\|_2$ to attain different noise-to-signal ratios. Similar to the previous plots, 40 independent simulations are run for each noise-to-signal ratio and the range of the standard deviation and mean (computed before transforming to dB) of the relative error in dB are recorded in Figs. 7 and 8. The dashed lines show the theoretical error bound from Theorem II.2 by substituting the parameters into equations (II.3) and (II.5), and the slope of each dashed line is 1. We observe that when noise-to-signal ratio is smaller than 0 dB, the relative error scales linearly with respect to the noise-to-signal ratio with slope 1 for both random Gaussian and Fourier dictionaries. This confirms that $\|X - X_0\|_F$ grows linearly with respect to $\eta$ in Theorem II.2. Moreover, if the noise dominates the observed signal, solving the problem (II.11) results in $X = 0$ and the relative error becomes 0 dB.

**VI. CONCLUSION**

In this paper, we introduce the generalized sparse recovery and blind demodulation model and achieve sparse recovery.
and blind demodulation simultaneously. Under the assumption that the modulating waveforms live in a known common subspace, we employ the lifting technique and recast this problem as the recovery of a column-wise sparse matrix from structured linear measurements. In this framework, we accomplish sparse recovery and blind demodulation simultaneously by minimizing the induced atomic norm, which in this problem corresponds to \( \ell_{2,1} \) norm minimization. In the noiseless case, we derive near optimal sampling complexity that is proportional to the number of degrees of freedom, and in the noisy case we bound the recovery error of the structured matrix. Numerical simulations support our theoretical results. In addition to extending the class of dictionaries we have considered, an interesting future direction would be to relax the constraint that each \( D_j \) is diagonal while preserving the low-dimensional subspace assumption.

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REFERENCES

[1] Y. Xie, M. B. Wakin, and G. Tang, “Sparse recovery and non-stationary blind demodulation,” in 2019 IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP), IEEE, 2019.
[2] M. Lustig, D. L. Donoho, J. M. Santos, and J. M. Pauly, “Compressed sensing MRI,” IEEE Signal Processing Magazine, vol. 25, no. 2, pp. 72–82, 2008.
[3] F. J. Herrmann and G. Hennenfent, “Non-parametric seismic data recovery with curvelet frames,” Geophysical Journal International, vol. 173, no. 1, pp. 233–248, 2008.
[4] S. Pudlewski and T. Melodia, “On the performance of compressive video streaming for wireless multimedia sensor networks,” in 2010 IEEE International Conference on Communications (ICC), pp. 1–5, IEEE, 2010.
[5] Y. Zhang, M. Roughan, W. Willinger, and L. Qiu, “Spatio-temporal compressive sensing and internet traffic matrices,” in ACM SIGCOMM Computer Communication Review, vol. 39, pp. 267–278, ACM, 2009.
[6] A. Goldsmith, Wireless communications. Cambridge university press, 2005.
[7] A. Ahmed, B. Recht, and J. Romberg, “Blind deconvolution using convex programming,” IEEE Transactions on Information Theory, vol. 60, no. 3, pp. 1711–1732, 2014.
[8] D. Yang, G. Tang, and M. B. Wakin, “Super-resolution of complex exponentials from modulations with unknown waveforms,” IEEE Transactions on Information Theory, vol. 62, no. 10, pp. 5809–5830, 2016.
[9] S. Ling and T. Strohmer, “Self-calibration and biconvex compressive sensing,” Inverse Problems, vol. 31, no. 11, p. 115002, 2015.
[10] V. Chandrasekaran, B. Recht, P. A. Parrilo, and A. S. Willsky, “The convex geometry of linear inverse problems,” Foundations of Computational Mathematics, vol. 12, no. 6, pp. 805–849, 2012.
[11] Y. Xie, S. Li, G. Tang, and M. B. Wakin, “Radar signal demixing via convex optimization,” in 2017 22nd International Conference on Digital Signal Processing (DSP), pp. 1–5, IEEE, 2017.
[12] S. Ling and T. Strohmer, “Blind deconvolution meets blind demixing: Algorithms and performance bounds,” IEEE Transactions on Information Theory, vol. 63, no. 7, pp. 4497–4520, 2017.
[13] Y. Chi, “Guaranteed blind sparse deconvolution via lifting and convex optimization,” J. Sel. Topics Signal Processing, vol. 10, no. 4, pp. 782–794, 2016.
[14] S. F. Cotter, B. D. Rao, K. Engan, and K. Kreutz-Delgado, “Sparse solutions to linear inverse problems with multiple measurement vectors,” IEEE Transactions on Signal Processing, vol. 53, no. 7, pp. 2477–2488, 2005.
[15] J. Chen and X. Huo, “Sparse representations for multiple measurement vectors (mmv) in an over-complete dictionary,” in 2005 IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP), vol. 4, pp. iv–257, IEEE, 2005.
[16] C.-Y. Hung and M. Kaveh, “Low rank matrix recovery for joint array self-calibration and sparse model doa estimation,” arXiv preprint arXiv:1712.05890, 2017.
[17] Y. C. Eldar, W. Liao, and S. Tang, “Sensor calibration for off-the-gridspectral estimation,” arXiv preprint arXiv:1707.03378, 2017.
[18] A. Flinth, “Sparse blind deconvolution and demixing through 1, 2-minimization,” Advances in Computational Mathematics, vol. 44, no. 1, pp. 1–21, 2018.
[19] S. Foscarini and H. Rauhut, A mathematical introduction to compressive sensing. Birkhäuser Basel, 2013.
[20] D. Hsu, S. Kakade, T. Zhang, et al., “A tail inequality for quadratic forms of subgaussian random vectors,” Electronic Communications in Probability, vol. 17, 2012.
[21] D. Gross, “Recovering low-rank matrices from few coefficients in any basis,” IEEE Transactions on Information Theory, vol. 57, no. 3, pp. 1548–1566, 2011.
[22] E. J. Candès and Y. Plan, “A probabilistic and ripless theory of compressed sensing,” IEEE Transactions on Information Theory, vol. 57, no. 11, pp. 7235–7254, 2011.
[23] J. A. Tropp, “User-friendly tail bounds for sums of random matrices,” Foundations of Computational Mathematics, vol. 12, no. 4, pp. 389–434, 2012.
[24] M. Grant, S. Boyd, and Y. Ye, “Cvx: Matlab software for disciplined convex programming,” 2008.