Multifractal Analysis of Various Probability Density Functions in Turbulence

Toshihico Arimitsu
Institute of Physics, University of Tsukuba, Ibaraki 305-8571, Japan
Naoko Arimitsu
Graduate School of EIS, Yokohama Nat’l. University, Yokohama 240-8501, Japan

Received 27 January, 2003

Abstract: The probability density functions measured by Lewis and Swinney for turbulent Couette-Taylor flow, observed by Bodenschatz and co-workers in the Lagrangian measurement of particle accelerations and those obtained in the DNS by Gotoh et al. are analyzed in excellent agreement with the theoretical formulae derived with the multifractal analysis, a unified self-consistent approach based on generalized entropy, i.e., the Tsallis or the Renyi entropy. This analysis rests on the invariance of the Navier-Stokes equation under a scale transformation for high Reynolds number, and on the assumption that the distribution of the exponent $\alpha$, introduced in the scale transformation, is multifractal and that its distribution function is given by taking extremum of the generalized entropy with the appropriate constraints. It also provides analytical formula for the scaling exponents of the velocity structure function which explains quite well the measured quantities in experiments and DNS.

Keywords: multifractal analysis, fully developed turbulence, PDF of fluid particle accelerations, Rényi entropy, Tsallis entropy

PACS: 47.27.-i, 47.53.+n, 47.52.+j, 05.90.+m

Mathematics Subject Classification: Mathematical modeling of turbulent processes and determined chaos

1 Introduction

The multifractal analysis of turbulence [1]-[11] is a unified self-consistent approach for the systems with large deviations, which has been constructed based on the Tsallis-type distribution function $\Omega^{\alpha}$ [12, 13] that provides an extremum of the extensive Rényi [14] or the non-extensive Tsallis entropy $\Omega^{\alpha}$ [12, 13, 15] under appropriate constraints. The analysis rests on the scale invariance of the Navier-Stokes equation for high Reynolds number, and on the assumptions that the singularities due to the invariance distribute themselves multifractally in physical space. The multifractal analysis is a generalization of the log-normal model [16]-[18]. It has been shown [5] that the multifractal analysis derives the log-normal model when one starts with the Boltzmann-Gibbs entropy.
In this paper, we derive the formula for various probability density functions (PDF's) in fully developed turbulence by means of the multifractal analysis, and analyze the PDF's observed in three experiments. The first is the PDF of velocity fluctuations at \( R_\lambda = 262 \) of the Taylor microscale Reynolds number measured by Lewis and Swingly [19] in the real experiment for turbulent Couette-Taylor flow in a concentric cylinder system. The second is the PDF of accelerations at \( R_\lambda = 970 \) obtained in the Lagrangian measurement of particle accelerations that was realized by Bodenschatz and co-workers [20, 21] by raising dramatically the spatial and temporal measurement resolutions with the help of the silicon strip detectors. The third is the PDF's of velocity fluctuations, of velocity derivatives and of fluid particle accelerations at \( R_\lambda = 380 \) that was extracted by Gotoh et al. from the DNS of the size \( 1024^3 \) [22].

For high Reynolds number \( Re \gg 1 \), or for the situation where effects of the kinematic viscosity \( \nu \) can be neglected compared with those of the turbulent viscosity, the Navier-Stokes equation, \( \partial \vec{u}/\partial t + (\vec{u} \cdot \nabla) \vec{u} = -\nabla (p/\rho) + \nu \nabla^2 \vec{u} \), of an incompressible fluid is invariant under the scale transformation [23–24] \( \vec{r} \rightarrow \lambda \vec{r} \), \( \vec{u} \rightarrow \lambda^{-\alpha/3} \vec{u} \), \( t \rightarrow \lambda^{1-\alpha/3} t \) and \( (p/\rho) \rightarrow \lambda^{2\alpha/3} (p/\rho) \) where the exponent \( \alpha \) is an arbitrary real quantity. The quantities \( \rho \) and \( p \) represent, respectively, mass density and pressure. The Reynolds number \( Re \) of the system is given by \( Re = \delta u_{in}/\nu = (\ell_{in}/\eta)^{2/3} \) with the Kolmogorov scale \( \eta = (\nu^3/\epsilon)^{1/4} \) [25] where \( \epsilon \) is the energy input rate at the inlet scale \( \ell_{in} \). Here, we introduced \( \delta u_{in} = |u(\bullet + \ell_{n}) - u(\bullet)| \) with the definition of the velocity fluctuation \( \delta u_{n} = |u(\bullet + \ell_{n}) - u(\bullet)| \) where \( u \) is a component of velocity field \( \vec{u} \), and \( \ell_{n} \) is a distance between two points. The pressure (divided by the mass density) difference \( \delta p_{n} = |p(\rho(\bullet + \ell_{n}) - p(\rho(\bullet))| \) between two points separated by the distance \( \ell_{n} \) is another important observable quantity. We are measuring distance by the discrete units \( \ell_{n} = \delta x_{n}^{i} \) with \( \delta x_{n}^{i} = 2^{-n} (n = 0, 1, 2, \ldots) \). The non-negative integer \( n \) can be interpreted as the multifractal depth.\(^4\) However, we will treat it as a positive real number in the analysis of experiments.

Let us consider the quantity \( \delta x_{n} = |x(\bullet + \ell_{n}) - x(\bullet)| \) having the scaling property \( |x_{n}| = |\delta x_{n}/\delta x_{0}| = \delta_{n}^{\alpha/3} \). Its spatial derivative is defined by \( |x'| = \lim \ell_{n}^{-\alpha} \delta x_{n}/\ell_{n} \propto \lim \ell_{n}^{-\alpha} \) which becomes singular for \( \alpha < 3/\kappa \). The values of exponent \( \alpha \) specify the degree of singularity. We see that the scale invariance provides us with \( \delta u_{n}/\delta u_{0} = \delta_{n}^{\alpha/3} \) and \( \delta u_{n}/\delta p_{0} = (\ell_{n}/\ell_{0})^{2\alpha/3} \) giving, respectively, \( \kappa = 1 \) for the velocity fluctuation and \( \kappa = 2 \) for the pressure fluctuation. The velocity derivative and the fluid particle acceleration may be estimated, respectively, by \( |u'| = \lim \ell_{n}^{-\alpha} u_{n}' \) and \( |\vec{a}| = \lim \ell_{n}^{-\alpha} a_{n} \) where we introduced the velocity derivative \( u_{n}' = \delta u_{n}/\ell_{n} \) and the acceleration \( a_{n} = \delta p_{n}/\ell_{n} \) corresponding to the characteristic length \( \ell_{n} \). Note that the acceleration \( \vec{a} \) of a fluid particle is given by the substantive time derivative of the velocity: \( \vec{a} = \partial \vec{u}/\partial t + (\vec{u} \cdot \nabla) \vec{u} \). We see that the velocity derivative and the fluid particle acceleration become singular for \( \alpha < 3 \) and \( \alpha < 1.5 \), respectively, i.e., \( |u'| \propto \lim \ell_{n}^{-\alpha/3} \rightarrow \infty \) and \( |\vec{a}| \propto \lim \ell_{n}^{-\alpha/3} \rightarrow \infty \).

\section{Multifractal analysis}

The multifractal analysis rests on the multifractal distribution of \( \alpha \). The probability \( P^{(n)}(\alpha) d\alpha \) to find, at a point in physical space, a singularity labeled by an exponent in the range \( \alpha \sim \alpha + \Delta \alpha \) is given by \( P^{(n)}(\alpha) = 1 - (\alpha - \alpha_{0}/(\Delta \alpha)^{2})^{n/(1-q)} / Z_{\alpha}^{(n)} \) with an appropriate partition function \( Z_{\alpha}^{(n)} \) and \( (\Delta \alpha)^{2} = 2X/(1-q) \ln 2 \). The range of \( \alpha \) is \( \alpha_{\min} \leq \alpha \leq \alpha_{\max} \) with \( \alpha_{\min} = \alpha_{0} - \Delta \alpha \), \( \alpha_{\max} = \alpha_{0} + \Delta \alpha \). This is consistent with the relation \( P^{(n)}(\alpha) \propto \alpha_{0}^{1-f(\alpha)} \) that is a manifestation of scale invariance and reveals how densely each singularity, labeled by \( \alpha \), fills the multifractal depth \( n \) is related to the number of steps within the energy cascade model. At each step of the cascade, say at the \( n \)th step, eddies break up into two pieces producing an energy cascade with the energy-transfer rate \( \epsilon_{n} \) that represents the rate of transfer of energy per unit mass from eddies of size \( \ell_{n} \) to those of size \( \ell_{n+1} \). The energy dissipation rate becomes singular in the limit \( n \rightarrow \infty \) for \( \alpha < 1 \), i.e., \( \lim n \rightarrow \infty \epsilon_{n}/\epsilon_{0} = \lim n \rightarrow \infty (\ell_{n}/\ell_{0})^{\alpha-1} \rightarrow \infty \).

\(^4\)The multifractal depth \( n \) is related to the number of steps within the energy cascade model. At each step of the cascade, say at the \( n \)th step, eddies break up into two pieces producing an energy cascade with the energy-transfer rate \( \epsilon_{n} \) that represents the rate of transfer of energy per unit mass from eddies of size \( \ell_{n} \) to those of size \( \ell_{n+1} \). The energy dissipation rate becomes singular in the limit \( n \rightarrow \infty \) for \( \alpha < 1 \), i.e., \( \lim n \rightarrow \infty \epsilon_{n}/\epsilon_{0} = \lim n \rightarrow \infty (\ell_{n}/\ell_{0})^{\alpha-1} \rightarrow \infty \).
Physical space. In the present model, the multifractal spectrum \( f(\alpha) \) is given by \[^{2-3} \] \( f(\alpha) = 1 + (1-q)^{-1} \log_2[1 - (\alpha - \alpha_0)^2/\Delta(\alpha)^2] \). In spite of the different characteristics of the entropies, i.e., extensive and non-extensive, the distribution functions \( P^{(n)}(\alpha) \) giving their extremum have the common structure. \[^{5} \]

The dependence of the parameters \( \alpha_0, X \) and \( q \) on the intermittency exponent \( \mu \) is determined, self-consistently, with the help of the three independent equations, i.e., the energy conservation: \( \langle \xi_n \rangle = \epsilon \), the definition of the intermittency exponent \( \mu \): \( \langle \xi_n^2 \rangle = \epsilon^2 \delta_n^{-\mu} \), and the scaling relation \[^{6} \] \( 1/(1-q) = 1/\alpha - 1/\alpha_+ \) with \( \alpha_+ \) satisfying \( f(\alpha_+) = 0 \). The average \( \langle \cdots \rangle \) is taken with \( P^{(n)}(\alpha) \).

It has been shown that the probability \( \Pi^{(n)}(x_n)dx_n \) to find a physical quantity \( x_n \) in the range \( x_n \sim x_n + dx_n \) is given in the form

\[
\Pi^{(n)}(x_n)dx_n = \Pi_S^{(n)}(x_n)dx_n + \Delta \Pi^{(n)}(x_n)dx_n
\]

with the normalization \( \int_{-\infty}^{\infty} dx_n \Pi^{(n)}(x_n) = 1 \). The first term represents the contribution by the singular part of the quantity \( x_n \) stemmed from the multifractal distribution of its singularities in physical space. This is given by \( \Pi_S^{(n)}(\langle x_n \rangle)dx_n \propto P^{(n)}(\alpha) d\alpha \) with the transformation of the variables, \( \langle x_n \rangle = \delta_n^{\alpha / 3} \). Whereas the second term \( \Delta \Pi^{(n)}(x_n)dx_n \) represents the contribution from the dissipative term in the Navier-Stokes equation, and/or the one from the errors in measurements. The dissipative term has been discarded in the above investigation for the distribution of singularities since it violates the invariance under the scale transformation. The contribution of the second term provides a correction to the first one. Note that each term in \[^{1} \] is a multiple of two probability functions, i.e., the one to determine the portion of the contribution among two independent origins, and the other to find \( x_n \) in the range \( x_n \sim x_n + dx_n \). Note also that the values of \( x_n \) originated in the singularity are rather large representing intermittent large deviations, and that those contributing to the correction terms are small in comparison with its deviation.

The \( m \)-th moment of the variable \( \langle x_n \rangle \) is given by \( \langle \langle x_n \rangle^m \rangle \equiv \int_{-\infty}^{\infty} dx_n \langle x_n \rangle^m \Pi^{(n)}(x_n) = 2^{\gamma_0} + (1 - 2\delta_{0,0}^{(m)}) a_{m,0} \xi_{0,0}^{\delta_{0,0}^{(m)}(\alpha - \alpha_0)} \) where \( 2\gamma_0 = \int_{-\infty}^{\infty} dx_n \langle x_n \rangle^m \Delta \Pi^{(n)}(x_n) \), \( a_{q,0} = \{2/[\sqrt{C_q}(1 + \sqrt{C_q})]\}^{1/2} \) with \( C_q = 1 + 2q^2(1-q)X \ln 2 \).

The scaling exponent of the \( m \)th moment of the velocity structure function \( \langle |\delta u_n/\delta u_0|^m \rangle \), i.e., the velocity fluctuations \( \kappa = 1 \). It explains the experimental results, successfully. The formula \[^{2} \] is independent of \( n \), which is a manifestation of the scale invariance.

We now derive the PDF, \( \Pi^{(n)}(\xi_n) \), defined by the relation \( \Pi^{(n)}(\xi_n)dx_n = \Pi^{(n)}(x_n)dx_n \) with the variable \( \xi_n = x_n/\langle x_n^2 \rangle^{1/2} \) scaled by the deviation \( \langle x_n^2 \rangle^{1/2} \). This PDF is to be compared with the observed PDF’s. The variable is related with \( \alpha \) by \( \langle \xi_n \rangle = 2\gamma_0 \delta_{0,0}^{\alpha / 3 - \zeta_{2,0}} \) with \( \xi_n = 2\gamma_0^{\alpha / 3} \delta_{0,0}^{\alpha / 3 - \zeta_{2,0}} + (1 - 2\gamma_0^{(n)})a_{2,0}^{(n)} \) \( \zeta_{2,0} \). It is reasonable to imagine that the origin of intermittent rare events is attributed to the first singular term in \[^{1} \], and that the contribution from the second term is negligible. We then have, for \( \xi_n \leq \xi_n \leq \xi_n^{\max}, \)

\[
\hat{\Pi}^{(n)}(\xi_n)dx_n = \Pi_S^{(n)}(\xi_n)dx_n + \hat{\Pi}_S^{(n)}(\xi_n)dx_n = \hat{\Pi}_S^{(n)}(\xi_n) \left[ 1 - \frac{1-q}{n} \left( \frac{3}{2 \kappa^2 X \ln |\delta_n|} \right)^{n/(1-q)} \right] d\xi_n
\]

with \( \xi_n = 0 = \xi_n^{\alpha_{n,0} / 3 - \zeta_{2,0} / 2}, \xi_n^{\min} = 0, \xi_n^{\max} = \xi_n^{\alpha_{n,0} / 3 - \zeta_{2,0} / 2}, \). On the other hand, for smaller values, the contribution to the PDF comes, mainly, from thermal intermittency.

\[^{5} \] Within the present formulation, the decision cannot be pronounced which of the entropies is underlying the system of turbulence.

\[^{6} \] The scaling relation is a generalization of the one derived first in \[^{20} \text{27} \] to the case where the multifractal spectrum has negative values.
fluctuations or measurement error. It may be described by a Gaussian function, i.e., for $|\xi_n| \leq \xi_n^*$,
\[
\tilde{\hat{H}}^{(n)}(\xi_n) d\xi_n = \left[ H^{(n)}_S(x_n) + \Delta \tilde{\hat{H}}^{(n)}(x_n) \right] dx_n = \tilde{\hat{H}}^{(n)}_S \exp \left\{ -\frac{1}{2} \left[ 1 + \frac{3}{\kappa} f'(\alpha^*) \right] \left[ (\xi_n/\xi_n^*)^2 - 1 \right] \right\} d\xi_n. \tag{4}
\]
This specific form of the Gaussian function is determined by the condition that the two PDF's \[(3)\] and \[(4)\] should have the same value and the same slope at $\xi_n^*$ which is defined by $\xi_n^* = \tilde{\xi}_n \delta_{\alpha}^{m \alpha^*/3 - \zeta_{2 \kappa}^*/2}$ with $\alpha^*$ being the smaller solution of $\zeta_{2 \kappa}/2 - \kappa \alpha/3 + 1 - f(\alpha) = 0$. It is the point at which $\tilde{\hat{H}}^{(n)}(\xi_n^*)$ has the least $n$-dependence for large $n$.

With the help of the second equality in \[(4)\], we obtain $\Delta \tilde{\hat{H}}^{(n)}(x_n)$, and have the formula to evaluate $\gamma_m^{(n)}$ in the form $2\gamma_m^{(n)} = \left( K_m^{(n)} - L_m^{(n)} \right) / \left( 1 + K_0^{(n)} - L_0^{(n)} \right)$ where
\[
K_m^{(n)} = \frac{3}{\kappa \sqrt{2\pi X}} \int_{0}^{1} d z \int_{z_{\min}}^{z_{\max}} z^m \exp \left\{ -\frac{1}{2} \left[ 1 + \frac{3}{\kappa} f'(\alpha^*) \right] \left( z^2 - 1 \right) \right\}, \tag{5}
\]
\[
L_m^{(n)} = \frac{3}{\kappa \sqrt{2\pi X}} \int_{0}^{1} d z \int_{z_{\min}}^{z_{\max}} z^{m-1} \left[ 1 - \frac{1-q}{n} \left( \frac{3 \ln|z/z_0^*|}{2 \kappa^2 X |\ln \delta_n|} \right) \right]^{n/(1-q)}, \tag{6}
\]
with $z_{\min} = \xi_{\min}/\xi_n^* = \delta_{\alpha}^{m (\alpha_{\max} - \alpha^*)/3}$, $z_0^* = \xi_{\alpha 0}/\xi_n^* = \delta_{\alpha}^{m (\alpha_0 - \alpha^*)/3}$. Now, the PDF for the variable scaled by its own deviation, given by \[(3)\] and \[(4)\], is completely determined by the intermittency exponent $\mu$ and the multifractal depth $n$ which gives a length scale $\ell_n$. The PDF’s for velocity fluctuations and for derivatives are given by \[(3)\] and \[(4)\] with $\kappa = 1$, whereas those for pressure fluctuations and for fluid particle accelerations with $\kappa = 2$. The PDF for energy dissipation rates is given with $\kappa = 3$.

3 Analysis of experiments

The PDF’s for velocity fluctuations measured by Lewis and Swinney \[19\] at $R_\lambda = 262$ for turbulent Couette-Taylor flow and those extracted by Gotoh et al. from his DNS data \[22\] at $R_\lambda = 380$ are shown, respectively, in Fig.\(1\) (i) and in Fig.\(1\) (ii). They are analyzed with the derived formulae \[(3)\] and \[(4)\] with $\kappa = 1$ \[6, 7\].

For Fig.\(1\) (i), we adopted the reported value $\mu = 0.28$ \[19\] to calculate the parameters $q = 0.471$, $\alpha_0 = 1.162$ and $X = 0.334$. The dependence of $n$ on $r/\eta$ is extracted as \[6\]
\[
n = -1.019 \times \log_2 r/\eta + 0.901 \times \log_2 Re \tag{7}
\]
with $Re = 540 000$. The Reynolds number is estimated with $\ell_{in} \approx 119.32$ cm and the Kolmogorov scale $\eta \approx 0.006$ cm. The energy input scale $\ell_{in}$ is given by the size of experimental apparatus $2\pi \times 19.00$ cm \[19\]. The definition of the number of steps $\tilde{n}$ within the energy cascade model is given by $\tilde{n} = -\log_2 (r/\ell_{in})$ for the eddies whose diameter is equal to $r$. By putting $r = \ell_n$, this gives us the relation between $\tilde{n}$ and $n$ in the form
\[
\tilde{n} = n - \log_2 (\ell_0/\ell_{in}). \tag{8}
\]

For Fig.\(1\) (ii), we extracted the value $\mu = 0.240$ by analyzing the measured scaling exponents $\zeta_m$ of velocity structure function with the formula \[2\], which gives the values $q = 0.391$, $\alpha_0 = 1.138$ and $X = 0.285$. Through the analyses of the PDF’s for velocity fluctuations, we extracted the formula for the dependence of $n$ on $r/\eta$: \[6, 8\]
\[
n = -1.050 \times \log_2 r/\eta + 16.74 \quad \text{for } \ell_c \leq r, \tag{9}
\]
\[
n = -2.540 \times \log_2 r/\eta + 25.08 \quad \text{for } r < \ell_c. \tag{10}
\]
This shows that the inertial range is divided into two scaling regions separated by the characteristic length \( \ell_c/\eta = 48.26 \) which is close to the Taylor microscale \( \lambda/\eta = 38.33 \) of the system. The equation (9) is consistent with the picture of the energy cascade model in which each eddy breaks up into two pieces at every cascade steps, whereas (10) indicates that, for \( r < \ell_c \), each eddy breaks up, effectively, into 1.33 \[8\] pieces at every cascade steps. This fact may be attributed to a manifestation of structural difference of eddies, which can be checked by visualizing DNS eddies. Actually, one observes that DNS eddies with larger diameters than Taylor microscale \( \lambda \) have rather round shapes, whereas eddies with smaller diameters compared with \( \lambda \) have rather stretched shapes \[28\]. The energy input scale for this DNS is estimated as the longest scale available in the lattice with cyclic boundary condition, i.e., \( \ell_{in}/\eta = \pi/\eta \approx 1220 \) with \( \eta \approx 0.258 \times 10^{-2} \) \[22\]. With this value of \( \ell_{in} \), \[8\] with (9) and (10) gives the number of steps \( \bar{n} \) within the energy cascade model for the DNS.

The analysis of the PDF for velocity derivatives reported by Gotoh et al. \[22\] are performed with \( \mu = 0.240 \). We chose the value \( n = 23.1 (\bar{n} = 17.4) \). The corresponding length \( r = \ell_n \) is calculated by (10) to give \( r/\eta = 1.716 \). This length may give us an estimate for the effective shortest length in processing the DNS data to extract velocity derivatives. Note that it is about the same order of the mesh size \( \Delta r/\eta = 2\pi/(1024 \times \eta) \approx 2.38 \) \[22\] of the DNS lattice.

The PDF’s for longitudinal velocity fluctuations (closed circles; symmetrized by taking averages of the left and the right hand sides data) observed by (i) Lewis and Swinney at \( R_\lambda = 262 \) (Re = 540 000), and by (ii) Gotoh et al. at \( R_\lambda = 381 \). For the experimental data, the distances \( r/\eta = \ell_n/\eta \) are, from top to bottom: (i) 11.6, 23.1, 46.2, 92.5 208, 399, 830, 1440; (ii) 2.38, 4.76, 9.52, 19.0, 38.1, 76.2, 152, 305, 609, 1220. For the theoretical PDF’s (solid line), from top to bottom: (i) \( \mu = 0.28, n (\bar{n}) = 14 (10.7), 13 (9.7), 11 (8.7), 10 (7.7), 9.0 (6.6), 8.0 (5.6), 7.5 (4.6), 7.0 (3.8); \) (ii) \( \mu = 0.240, n (\bar{n}) = 21.5 (15.8), 20.0 (14.3), 16.8 (11.1), 14.0 (8.31), 11.8 (6.11), 10.1 (4.41), 9.30 (3.61), 8.10 (2.41), 7.00 (1.31), 6.00 (0.31). \) For better visibility, each PDF is shifted by −1 unit along the vertical axis.

The PDF’s for fluid particle accelerations measured by Bodenschatz et al. at \( R_\lambda = 970 \) \[20, 21\] and those extracted out from the DNS data by Gotoh et al. at \( R_\lambda = 380 \) \[22\] are shown, respectively, in Fig. 2(i) and in Fig. 2(ii), on log and linear scale \[10, 11\]. They are analyzed with the derived formulae (3) and (4) with \( \kappa = 2 \) \[6, 7\].

For Fig. 2(i), we determined the value \( n = 17.1 \) for this experiment by substituting the reported value 7.1 cm for \( \ell_0 \) and the spatial resolution 0.5 \( \mu \)m of the measurement for \( \ell_n \) into its definition,
$n = \log_2(\ell_0/\ell_0)$. The intermittency exponent $\mu = 0.250$ is extracted by the analysis of the experimental PDF with the derived theoretical formula [10, 11]. Then, we have the values of parameters: $q = 0.413$, $\alpha_0 = 1.144$ and $X = 0.297$. The flatness of the PDF turns out to be $F_3^n \equiv \langle a^4 \rangle / \langle a^2 \rangle^2 = \langle \xi_n^4 \rangle = 61.3$ which is compatible with the value of the flatness $\sim 60$ reported in [20, 21].

For Fig. 2 (ii), with $\mu = 0.240$ for this DNS, we have the values of parameters: $q = 0.391$, $\alpha_0 = 1.138$, $X = 0.285$. The analysis of the PDF obtained by the DNS gives the value $n = 18.3$ ($\bar{n} = 12.6$). Substitution of this value into [10] gives the corresponding characteristic length $r/\eta = 6.36$ [11]. This may be the effective minimum resolution in cooking the DNS data to distill accelerations.

![Figure 2](image)

Figure 2: PDF’s for fluid particle accelerations measured by (i) Bodenschatz et al. at $R_\lambda = 970$ (open circle), and by (ii) Gotoh et al. at $R_\lambda = 380$ (closed circle), plotted on (a) log and (b) linear scale. The experimental data points both on the left- and right-hand sides are plotted altogether. For the theoretical PDF’s (solid line), (i) $\mu = 0.250$ and $n = 17.1$, and (ii) $\mu = 0.240$ and $n = 18.3$ ($\bar{n} = 13.6$) (solid line).

4 Discussions

It was shown that the various experimental PDF’s in turbulence are analyzed successfully with the formulae [3] and [4] derived by the multifractal analysis.

From the above analyses, we see that the contribution of thermal fluctuation and/or measurement error to PDF’s is restricted to smaller values, i.e., $\xi_n \leq \xi_n^*$. In the case of the PDF’s for velocity fluctuations, $\xi_n^* = 1.10 \sim 1.32 \ (\alpha^* = 1.08)$ for Lweis and Swinney [19], and $\xi_n^* = 1.01 \sim 1.39 \ (\alpha^* = 1.07)$ for Gotoh et al. [22]. As for the PDF for velocity derivatives by Gotoh et al. [22], $\xi_n^* = 0.982 \ (\alpha^* = 1.07)$. In the case of the PDF’s for fluid particle accelerations, $\xi_n^* = 0.565 \ (\alpha^* = 1.01)$ for Bodenschatz et al. [20, 21], and $\xi_n^* = 0.551 \ (\alpha^* = 1.005)$ for Gotoh et al. [22]. Within the present approach, the intermittent large deviations $\xi_n \geq \xi_n^*$ are a manifestation of the multifractal distribution of singularities $\alpha$ due to the scale invariance of the Navier-Stokes equation for $Re \gg 1$.

There is no room to incorporate into the present multifractal analysis the energy input scale $\ell_{in}$ and the “system size” $\ell_0$. The former is necessary to determine the number of steps $\bar{n}$ in the energy cascade model. Once $\ell_{in}$ is determined by investigating the structure of experimental apparatuses, the relation between $\bar{n}$ and the multifractal step $n$ is given by [3]. Since main part of
the multifractal analysis rests on the scale invariance, the size of the system under consideration is assumed to be infinite, and therefore, the length \( \ell_0 \) may not have important physical meaning. Actually, the empirical equation (7) extracted from the experimental PDF’s for velocity fluctuations by Lewis and Swinney gives \( \ell_0 \approx 877 \) cm which is large compared with the largest size \( \ell_{\text{in}} \approx 119.32 \) cm of the experimental apparatus. For Gotoh’s DNS, the empirical equation (9) gives \( \ell_0/\eta \approx 63000 \) which is larger than the largest size \( \ell_{\text{in}}/\eta \approx 1220 \) of the DNS lattice. On the other hand, in the analysis of Bodenschatz’s experiment, the assignment of \( \ell_0 \) to the integral length scale 7.1 cm gives us reasonable value \( n \). It is worthwhile to note here that PDF’s derived within the multifractal analysis seem to be sensitive to the characteristic lengths such as the distance of two measuring points, the space resolution in measurement and the mesh size of DNS. How to put the information of characteristic lengths of experimental apparatus into the multifractal analysis is one of the important future problems. It may be resolved when one succeeds to reveal the dynamical foundation underlying the basis of the multifractal analysis starting an investigation by the Navier-Stokes equation with the energy input term.

The authors would like to thank Prof. R.H. Kraichnan and Prof. C. Tsallis for their fruitful and enlightening comments with encouragement, and are grateful to Prof. E. Bodenschatz and Prof. T. Gotoh for the kindness to show their data prior to publication.

References

[1] T. Arimitsu and N. Arimitsu, Analysis of Fully Developed Turbulence in terms of Tsallis Statistics, Phys. Rev. E 61, 3237-3240 (2000).

[2] T. Arimitsu and N. Arimitsu, Tsallis statistics and fully developed turbulence, J. Phys. A: Math. Gen. 33, L235-L241 (2000) [CORRIGENDUM: 34, 673-674 (2001)].

[3] T. Arimitsu and N. Arimitsu, Tsallis statistics and turbulence, Chaos, Solitons and Fractals 13, 479-489 (2002).

[4] T. Arimitsu and N. Arimitsu, Analysis of fully developed turbulence by a generalized statistics, Prog. Theor. Phys. 105, 355-360 (2001).

[5] T. Arimitsu and N. Arimitsu, Analysis of turbulence by statistics based on generalized entropies, Physica A 295, 177-194 (2001).

[6] N. Arimitsu and T. Arimitsu, Multifractal analysis of turbulence by using statistics based on non-extensive Tsallis’ or extensive Rényi’s entropy, J. Korean Phys. Soc. 40, 1032-1036 (2002).

[7] T. Arimitsu and N. Arimitsu, PDF of velocity fluctuation in turbulence by a statistics based on generalized entropy, Physica A 305, 218-226 (2002).

[8] T. Arimitsu and N. Arimitsu, Analysis of velocity fluctuation in turbulence based on generalized statistics, J. Phys.: Condens. Matter 14, 2237-2246 (2002).

[9] N. Arimitsu and T. Arimitsu, Analysis of velocity derivatives in turbulence based on generalized statistics, Europhys. Lett. 60, 60-65 (2002).

[10] T. Arimitsu and N. Arimitsu, Analysis of accelerations in turbulence based on generalized statistics, cond-mat/0203240 (2002).

[11] T. Arimitsu and N. Arimitsu, Multifractal analysis of fluid particle accelerations in turbulence, cond-mat/0210274 (2002).
[12] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, *J. Stat. Phys.* **52**, 479-487 (1988).

[13] C. Tsallis, Nonextensive statistics: Theoretical, experimental and computational evidences and connections, *Braz. J. Phys.* **29**, 1-35 (1999); On the related recent progresses see at [http://tsallis.cat.cbpf.br/biblio.htm](http://tsallis.cat.cbpf.br/biblio.htm).

[14] A. Rényi, On measures of entropy and information, *Proc. 4th Berkeley Symp. Maths. Stat. Prob.* **1**, 547 (1961).

[15] J.H. Havrda and F. Charvat, Quantification methods of classification processes: Concepts of structural $\alpha$ entropy, *Kybernatica* **3**, 30-35 (1967).

[16] A.M. Oboukhov, Some specific features of atmospheric turbulence, *J. Fluid Mech.* **13**, 77-81 (1962).

[17] A.N. Kolmogorov, A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number, *J. Fluid Mech.* **13**, 82-85 (1962).

[18] A.M. Yaglom, The influence of fluctuations in energy dissipation on the shape of turbulent characteristics in the inertial interval, *Sov. Phys. Dokl.* **11**, 26-29 (1966).

[19] G.S. Lewis and H.L. Swinney, Velocity structure functions, scaling, and transitions in high-Reynolds-number Couette-Taylor flow, *Phys. Rev. E* **59**, 5457-5467 (1999).

[20] A. La Porta, G. A. Voth, A. M. Crawford, J. Alexander and E. Bodenschatz, Fluid particle accelerations in fully developed turbulence, *Nature* **409**, 1017-1019 (2001).

[21] G. A. Voth, A. La Porta, A. M. Crawford, J. Alexander and E. Bodenschatz, Measurement of particle accelerations in fully developed turbulence, *J. Fluid Mech.* **469**, 121 (2002).

[22] T. Gotoh, D. Fukayama and T. Nakano, Velocity field statistics in homogeneous steady turbulence obtained using a high resolution DNS, *Phys. Fluids* **14**, 1065 (2002).

[23] U. Frisch and G. Parisi, On the singularity structure of fully developed turbulence, Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, ed. by M. Ghil, R. Benzi and G. Parisi (North-Holland, New York, 1985) 84.

[24] C. Meneveau and K. R. Sreenivasan, The multifractal spectrum of the dissipation field in turbulent flows, *Nucl. Phys. B (Proc. Suppl.)* **2**, 49-76 (1987).

[25] A.N. Kolmogorov, The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers, *Dokl. Akad. Nauk SSSR* **30**, 301-305 (1941). Dissipation of energy in locally isotropic turbulence, *Dokl. Akad. Nauk SSSR* **31**, 538-540 (1941).

[26] U.M.S. Costa, M.L. Lyra, A.R. Plastino and C. Tsallis, Power-law sensitivity to initial conditions within a logisticlike family of maps: Fractality and nonextensivity, *Phys. Rev. E* **56**, 245-250 (1997).

[27] M.L. Lyra and C. Tsallis, Nonextensivity and multifractality in low-dimensional dissipative systems, *Phys. Rev. Lett.* **80**, 53-56 (1998).

[28] Private communication with Prof. M. Tanahashi at Tokyo Institute of Technology.