Negative-Weight Single-Source Shortest Paths in Almost-linear Time
(Preliminary Version)

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Abstract

We present a randomized algorithm that computes single-source shortest paths (SSSP) in $m^{1+o(1)} \log W$ time when edge weights are integral and can be negative. This essentially resolves classic negative-weight SSSP problem. The previous bounds are $\tilde{O}(m^{1 + n^{1.5}} \log W)$ [BLNPSSW FOCS’20] and $m^{4/3+o(1)} \log W$ [AMV FOCS’20]. In contrast to all recent developments that rely on sophisticated continuous optimization methods and dynamic algorithms, our algorithm is based on a simple graph decomposition and elementary combinatorial tools. In fact, ours is the first combinatorial algorithm for negative-weight SSSP to break through the classic $\tilde{O}(m \sqrt{n} \log W)$ bound from over three decades ago [Gabow and Tarjan SICOMP’89]. Beside being combinatorial, an important feature of our algorithm is in its simplicity: treating our graph decomposition as a black-box, we believe that the reader can reconstruct our algorithm and analysis from our 6-page overview.

Independent result. Independently from our result, the recent major breakthrough by Chen, Kyng, Liu, Peng, Gutenberg, and Sachdeva [CKL+22] achieve an almost-linear time bound for min-cost flow, implying the same bound for our problem. We discuss this result at the end of the introduction.

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1Throughout, $n$ and $m$ denote the number of vertices and edges, respectively, and $W \geq 2$ is such that every edge weight is at least $-W$. $\tilde{O}$ hides polylogarithmic factors.
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1 Introduction

We consider the single-source shortest paths (SSSP) problem with (possibly negative) integer weights. Given an $m$-edge $n$-vertex directed weighted graph $G = (V,E,w)$ with integral edge weight $w(e)$ for every edge $e \in E$ and a source vertex $s \in V$, we want to compute the distance from $s$ to $v$, denoted by $\text{dist}_G(s,v)$, for every vertex in $v$. It can be assumed without lost of generality that $s$ can reach all vertices.

Two textbook algorithms for SSSP are Bellman-Ford and Dijkstra’s algorithm. Dijkstra’s algorithm is near-linear time ($O(m + n \log n)$ time), but restricted to nonnegative edge weights. With negative weights, we can use the Bellman-Ford algorithm, which only requires that there is no negative-weight cycle in $G$; in particular, the algorithm either returns $\text{dist}_G(s,v) \neq -\infty$ for every vertex $v$ or reports that there is a cycle whose total weight is negative. Unfortunately, the runtime of Bellman-Ford is $O(mn)$.

Designing faster algorithms for SSSP with negative edge weights (negative-weight SSSP thereafter) is one of the most fundamental and long-standing problems in graph algorithms, and has witnessed waves of exciting improvements every few decades since the 50s. Early works in the 50s, due to Shimbel [Sh55], Ford [For56], Bellman [Bel58], and Moore [Moo59] resulted in the $O(mn)$ runtime. In the 80s and 90s, the scaling technique led to a wave of improvements (Gabow [Gab85], Gabow and Tarjan [GT89], and Goldberg [Gol95]), resulting in runtime $O(\sqrt{nm} \log W)$, where $W \geq 2$ is the minimum integer such that $w(e) \geq -W$ for all $e \in E$. \footnote{The case when $n$ is big and $W$ is small can be improved by the $O(n^2 W)$-time algorithms of Sankowski [San05], and Yuster and Zwick [YZ05]. The special case of planar directed graphs can be solved in $O(n \log^2 n/ \log \log n)$-time [FR06, MW10, KMW10, LRT79, HKRS97].}

In the last few years, advances in continuous optimization and dynamic algorithms have led to a new wave of improvements, which achieve faster algorithms for the more general problems of transshipment and min-cost flow, and thus imply the same bounds for negative-weight SSSP (Cohen, Madry, Sankowski, Vladu [CMSV17]; Axiotis, Madry, Vladu [AMV20]; BLNPSSSW [BLN+20, BLL+21, BLSS20]). This line of work resulted in an almost-linear runtime ($\tilde{O}((m + n^{1.5}) \log W)$ time) on moderately dense graphs [BLN+20] and $n^{4/3+o(1)} \log W$ runtime on sparse graphs [AMV20]. \footnote{$\tilde{O}$-notation hides polylogarithmic factors. The dependencies on $W$ stated in [AMV20, BLN+20] are slightly higher than what we state here. These dependencies can be reduced by standard techniques (weight scaling, adding dummy source, and eliminating high-weight edges).}

These results motivate two natural questions:

1. Can we get almost-linear runtime for all graphs?

2. Can we achieve efficient algorithms without complex machinery?

For the second question, note that currently all state-of-the-art results for negative-weight SSSP are based on min-cost flow algorithms, and hence rely on sophisticated continuous optimization methods and a number of complex dynamic algebraic and graph algorithms (e.g. [SW19, NSW17, CGL+20, BBG+20, NS17, Wu17]). It would be useful to develop simple efficient algorithms that are specifically tailored to negative-weight SSSP, and thus circumvent the complexity currently inherent in flow algorithms; the best known bound of this kind is still the classic $O(m\sqrt{n} \log(W))$ from over three decades ago [GT89, Gol95]. A related question is whether it is possible to achieve efficient algorithms for the problem using combinatorial tools, or whether there are fundamental barriers that make continuous optimization necessary.

Our Contribution. In this paper we resolve both of the above questions for negative-weight SSSP: we present a simple combinatorial algorithm that reduces the running time all the way down to almost-linear.

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Theorem 1.1. There exists a randomized algorithm that takes \(O(m^{1+o(1)} \log(W))\) time for an \(m\)-edge input graph \(G\) and source \(s\) and behaves as follows:

- if \(G\) contains a negative-weight cycle, then the algorithm always returns an error message, and
- if \(G\) contains no negative-weight cycle, then the algorithm returns correct distances from \(s\) to every node with probability at least \(1 - 1/n\) and otherwise returns an error message.

Remark: Our algorithm can be modified into a Las Vegas algorithm that returns a negative-weight cycle when \(G\) contains one. In this preliminary version, we focus on the Monte Carlo algorithm in Theorem 1.1.

Our algorithm relies only on basic combinatorial tools; the presentation is self-contained and only uses standard black-boxes such as Dijkstra’s and Bellman-Ford algorithms. In particular, it is a scaling algorithm enhanced by a simple graph decomposition algorithm called Low Diameter Decomposition, which can be obtained in a manner similar to some known algorithms [LS93, Bar96, BGK+14, MPX13, PRS+18, FG19, CZ20, BGW20, FGdV21] (see Section 3 for a more detailed discussion). Our main technical contribution is showing how low-diameter decomposition—which works only on graphs with non-negative weights—can be used to develop a recursive scaling algorithm for SSSP with negative weights. As far as we know, all previous applications of this decomposition were used for parallel/distributed/dynamic settings for problems that do not involve negative weights, and our algorithm is the first to take advantage of it in the classical sequential setting; we also show that in this setting, there is a simple and efficient algorithm to compute it.

We present an overview of our techniques in Section 4. We have purposefully made the overview quite detailed, as the algorithm is simple enough that the reader can reconstruct the algorithm and most of the analysis from the 6-page overview alone.

Perspective on Other Problems: While our result is specific to negative-weight SSSP, we note that question (2) above in fact applies to a much wider range of problems. The current landscape of graph algorithms is that for many of the most fundamental problems, including ones taught in undergraduate courses and used regularly in practice, the state-of-the-art solution is a complex algorithm for the more general min-cost flow problem: some examples include negative-weight SSSP, bipartite matching, the assignment problem, edge/vertex-disjoint paths, s-t cut, densest subgraph, max flow, transshipment, and vertex connectivity. This suggests a research agenda of designing simple algorithms for these fundamental problems, and perhaps eventually their generalizations such as min-cost flow. We view our result on negative-weight SSSP as a first step in this direction.

Independent Result [CKL+22]. Independently from our result, the recent major breakthrough by Chen, Kyng, Liu, Peng, Gutenberg, and Sachdeva [CKL+22] culminates the line of works based on continuous optimization (e.g. [Mad13, LS14, Mad16, CMSV17, CLS19, Bra20, LS20, AMV20, BLSS20, BLN+20, BLL+21]) and achieves an almost-linear time bound\(^4\) for min-cost flow. The authors thus achieve the same bound as us for negative-weight SSSP as a special case of their result. The two results are entirely different, and as far as we know there is no overlap in techniques.

The above landmark result essentially resolves the complexity for a wide range of fundamental graph problems. We believe that this makes it a natural time to pursue question (2) for these problems, outlined above.

\[^4\ddot{O}(m^{1+o(1)} \log^2 U)\) time when vertex demands, edge costs, and upper/lower edge capacities are all integral and bounded by \(U\) in absolute value.


2 Preliminaries

Throughout, $O$ hides $\log(n)$ factors. For any weighted graph $G = (V, E, w)$, define $V(G) = V, E(G) = E$, and $E^{\text{neg}}(G) := \{ e \in E \mid w(e) < 0 \}$. Define $W_G := \max\{0, -\min_{e \in E}\{w(e)\}\}$; that is, $W_G$ is the most negative edge weight in the graph. Given any set of edges $S \subseteq E$ we define $\eta(S) = \sum_{e \in S} w(e)$. We say that a cycle $C$ in $G$ is a negative-weight cycle if $w(C) < 0$.

Consider graph $G = (V, E, w)$ and consider subsets $V' \subseteq G$ and $E' \subseteq E$. For subgraph $H = (V', E')$, we define $E^{\text{neg}}(H) := E^{\text{neg}}(G) \cap E'$; we drop the subscript when the context is clear. We slightly abuse notation and write $H = (V', E', w)$ to denote the subgraph where the weight function $w$ is restricted to edges in $E'$. We define $G \setminus E'$ to be the graph $G' = (V, E \setminus E', w)$. We say that a subgraph $H$ of $G$ has weak diameter $D$ if for any $u, v \in V(H)$ we have that $\text{dist}_G(u, v) \leq D$.

Throughout the paper, we rely on the following assumption.

**Assumption 2.1.** Throughout, we can assume that in our input graph $G$ we have $W_G = O(1)$. This is justified because the scaling framework of Goldberg [Gol95] shows that an algorithm for this case implies an algorithm for general $G$ at the expense of an extra $\log(W_G)$ factor.\(^5\)

Most of our algorithms will compute distances from a dummy source with particular properties. To this end we introduce the following definitions

**Definition 2.2** (Universal and Canonical Source). Consider a graph $G = (V, E, w)$ and a vertex $s \in V$. We say that $s$ is a universal source if for every other $v \in V$ we have $(s, v) \in E$ and $(v, s) \notin E$. We say that $s$ is a canonical source if $s$ is universal, and if additionally, for every $v \in V$, we have $w(s, v) \geq 0$ and $\text{dist}_G(s, v) \leq 0$. Note that changing the weight function cannot affect whether $s$ is universal, but may affect whether it is canonical.

The definition below will allow us to quantify the number of negative edges on the shortest path from a source $s$ to a vertex $v$.

**Definition 2.3** ($\eta(v; G, s), P(v; G, s), (s, k)$-negative and $\Delta$-negative). Consider graph $G = (V, E, w)$ and let $s \in V$ be some fixed source. Define

$$\eta(v; G, s) := \begin{cases} \infty & \text{if } \text{dist}_G(s, v) = -\infty \\ \min\{|E^{\text{neg}}(G) \cap P| : P \text{ is a shortest } sv\text{-path}| & \text{otherwise.} \end{cases}$$

When $\text{dist}_G(s, v) \neq -\infty$, let $P(v; G, s)$ be a shortest $sv$-path such that $|E^{\text{neg}}(P(v))| = \eta(v; G, s)$. When the context is clear, we write $\eta(v)$ and $P(v)$ instead of $\eta(v; G, s)$ and $P(v; G, s)$. For any vertex $v \in V$ and any non-negative $k$, we say that $v$ is $(s, k)$-negative if $\eta(v; G, s) \leq k$. Finally, for any real $\Delta \geq 0$, we say that $(G, s)$ is $\Delta$-negative if

$$\sum_{v \in V} \eta(v) \leq \Delta.$$ 

Observe that if $G$ contains no negative cycle, then $(G, s)$ is $n^2$-negative; otherwise, it is not $\Delta$-negative for any finite $\Delta$.

\(^5\)Quoting [Gol95]: “Note that the basic problem solved at each iteration of the bit scaling method is a special version of the shortest paths problem where the arc lengths are integers greater or equal to $-1$.”
2.1 Price Functions and Equivalence

Our algorithm heavily relies on price functions, originally introduced by Johnson [Joh77].

**Definition 2.4** (Price function and equivalence). Consider a graph \( G = (V, E, w) \) and let \( \phi \) be any function: \( V \rightarrow \mathbb{Z} \). Then, we define \( w_\phi \) to be the weight function \( w_\phi(u, v) = w(u, v) + \phi(u) - \phi(v) \) and we define \( G_\phi = (V, E, w_\phi) \). We will refer to \( \phi \) as a price function on \( V \). Note that \( (G_\phi)_\psi = (G_\phi + \psi) \).

Finally, given two graphs \( G = (V, E, w) \) and \( G' = (V, E, w') \), we say that \( G \) and \( G' \) are equivalent (in short, \( G \equiv G' \)) if \( G' = G_\phi \) for some price function \( \phi \).

**Lemma 2.5** ([Joh77]). Consider any graph \( G = (V, E, w) \) and price function \( \phi \). For any pair \( u, v \in V \) we have \( \text{dist}_{G_\phi}(u, v) = \text{dist}_G(u, v) + \phi(u) - \phi(v) \). This implies that any shortest path in \( G \) is also a shortest path in \( G' \) and vice versa. Similarly if one of the graphs has a negative cycle, then so does the other.

The overall goal of our algorithm will be to compute a price function \( \phi \) such that all edge weights in \( G_\phi \) are non-negative (or return a negative-weight cycle); we can then run Dijkstra to \( G_\phi \). The lemma below, originally used by Johnson, will be one of the tools we use.

**Lemma 2.6** ([Joh77]). Let \( G = (V, E) \) be a directed graph with no negative-weight cycle and consider any \( s \in V \) such that all vertices in \( V \) are reachable from \( s \). Let \( \phi(v) = \text{dist}_G(s, v) \) for all \( v \in V \). Then, all edge weights in \( G_\phi \) are non-negative and \( \text{dist}_{G_\phi}(s, v) = 0 \) for all \( v \in V \). Note in particular that if \( s \) is universal in \( G \) then \( s \) is canonical in \( G_\phi \).

2.2 Simple Subroutines

Our algorithm relies on several simple shortest path subroutines.

**Lemma 2.7.** There exists an algorithm Dijkstra\((G, s)\) that takes as input a graph \( G \) with non-negative edge weights and a vertex \( s \in V \) and outputs a shortest path tree from \( s \) in \( G \). The running time is \( O(m + n \log(n)) = \tilde{O}(m) \).

The next subroutine shows that in a graph with negative weights, it is easy to compute a shortest path from \( s \) to \( v \) if that path contains few negative edges. This algorithm is obtained via a simple combination of Dijkstra and Bellman-Ford.

**Lemma 2.8.** There exists an algorithm SPWithFewNegEdges\((G, s, k)\) that takes as input a graph \( G = (V, E, w) \), any canonical source \( s \in V \), and a non-negative integer \( k \). The algorithm outputs, for every \( v \in V \), a distance estimate \( \hat{\text{dist}}(s, v) \) such that:

- \( \hat{\text{dist}}(s, v) = \text{dist}(s, v) \) if \( v \) is \((s, k)\)-negative in \( G \).
- We always have \( 0 \geq \hat{\text{dist}}(s, v) \geq \text{dist}(s, v) \) (note that \( \text{dist}(s, v) \leq 0 \) because \( s \) is canonical).

The running time of the algorithm is \( O(mk + nk \log(n)) = \tilde{O}(mk) \).

**proof sketch.** The proof follows easily by alternating between executing Dijkstra and one iteration of Bellman-Ford for a total of \( k + 1 \) times, exploiting that we only need exact distance estimates for vertices whose shortest paths from \( s \) contain at most \( k \) negative edges; the path relaxation property of shortest paths then easily implies the lemma. A full proof can be found in the appendix.
Recall that our goal is to compute a price function $\phi$ such that $w_\phi \geq 0$. It is easy to see that if $G$ is a DAG (Directed Acyclic Graph), such a $\phi$ can be computed by looping over the topological order $v_1, \ldots, v_n$, and setting $\phi(v_i)$ so that all incoming edges have non-negative weight. The lemma below generalizes this approach to graphs where only the “DAG part” has negative edges.

**Lemma 2.9.** There exists an algorithm $\text{FixDAGEdges}(G, \mathcal{P})$ that takes as input a graph $G$ and a partition $\mathcal{P} := \{V_1, V_2, \ldots \}$ of vertices of $G$ such that

1. for every $i$, the induced subgraph $G[V_i]$ contains no negative-weight edges, and
2. when we contract every $V_i$ into a node, the resulting graph is a DAG (i.e. contains no cycle).

The algorithm outputs a price function $\phi : V \to \mathbb{Z}$ such that $w_\phi(u, v) \geq 0$ for all $(u, v) \in E$. The running time is $O(m + n)$.

**Proof.** See Algorithm 1 for pseudocode. Let us first consider the running time. Note that computing each $\mu_j$ requires time proportional to $O(1) + \text{[the number of edges entering in } E \text{ entering } V_j \text{]}$; since the $V_j$ are disjoint, the total time to compute all of the $\mu_j$ is $O(m + n)$. Similarly, it is easy to check that the for loop in Line 4 only considers each vertex once, so the total runtime of the loop is $O(n + q) = O(n)$.

To prove correctness, we need to show that $w_\phi(u, v) \geq 0$ for all $(u, v) \in E$. Say that $u \in V_i$ and $v \in V_j$ and note that because the algorithm labels the sets in topological order, we must have $i < j$. Moreover, by definition of $\mu_j$ we have $\mu_j < w(u, v)$. Thus, we have

$$w_\phi(u, v) = w(u, v) + \phi(u) - \phi(v) = w(u, v) + M_i - M_j = w(u, v) - \sum_{k=i+1}^{j} \mu_k \geq w(u, v) - \mu_j \geq 0.$$

\[ \square \]

**Algorithm 1:** Algorithm for $\text{FixDAGEdges}(G = (V, E, w), \mathcal{P} = \{V_1, V_2, \ldots \})$

1. Relabel the sets $V_1, V_2, \ldots$ so that they are in topological order in $G$. That is, after relabeling, if $(u, v) \in E$, with $u \in V_i$ and $v \in V_j$, then $i \leq j$.
2. Define $\mu_j = \min \{w(u, v) \mid (u, v) \in E^{\text{neg}}(G), u \notin V_j, v \in V_j\}$; here, let $\min \{\emptyset\} = 0$.
   // $\mu_j$ is min negative edge weight entering $V_j$, or 0 if no such edge exists.
3. Define $M_1 \leftarrow \mu_1$.
4. for $j = 2$ to $q$ do // make edges into each $V_2, \ldots, V_q$ non-negative
5.   $M_j \leftarrow M_{j-1} + \mu_j$ ; // Note: $M_j = \sum_{k \leq j} \mu_k$
6.   Define $\phi(v) \leftarrow M_j$ for every $v \in V_j$
7. return $\phi$

### 3 Low Diameter Decomposition

One of our key subroutines is an algorithm that decomposes any directed graph with non-negative edge weights into strongly-connected components (SCCs) of small diameter. In particular, the algorithm computes a small set of edges $E^{\text{sep}}$ such that all SCCs in the graph $G \setminus E^{\text{sep}}$ have small weak diameter. Although the lemma below only applies to non-negative weights, we show in the high-level overview that it is in fact extremely useful for our problem.
Lemma 3.1. There is an algorithm $\text{LowDiamDecomposition}(G, D)$ with the following guarantees:

- **INPUT**: an $m$-edge, $n$-vertex graph $G = (V, E, w)$ with non-negative integer edge weight function $w$ and a positive integer $D$.
- **OUTPUT**: A set of edges $E_{\text{sep}}$ with the following guarantees:
  - each SCC of $G - E_{\text{sep}}$ has weak diameter at most $D$; that is, if $u, v$ are in the same SCC, then $\text{dist}_G(u, v) \leq D$ and $\text{dist}_G(v, u) \leq D$.
  - For every $e \in E$, $\Pr[e \in E_{\text{sep}}] = O(w(e) \cdot (\log n)^2 / D + n^{-10})$. These probabilities are not guaranteed to be independent.
- **RUNNING TIME**: The algorithm has running time $\tilde{O}(m + n)$

The decomposition above is similar to other low-diameter decompositions used in both undirected and directed graphs [LS93, Bar96, MPX13, PRS+18, FG19, CZ20, BGW20]. The closest similarity is to the algorithm $\text{Partition}$ of Bernstein, Probst-Gutenberg, and Wulff-Nilsen [BGW20]. The main difference is that the algorithm of [BGW20] needed to work in the more general dynamic setting, and as a result their algorithm is too slow for our purposes. Our decomposition algorithm follows the general framework of [BGW20], but with several key differences to ensure faster running time; our algorithm is also simpler, since it only applies to the static setting. For the reader’s convenience, we present the entire algorithm from scratch in Section 7.

4 Overview

We now give a detailed overview of our algorithm.

**Two Simplifying Assumptions**: Throughout most of this section, we rely on the two following simplifying assumptions. We discuss at the end of the section how to do without them.

The first assumption is that we have a deterministic version of Lemma 3.1; this will allow us to postpone bounding error probabilities. In particular, note that the second output property of Lemma 3.1 tells that for any path $P$ in $G$, we have $\mathbb{E}[|P \cap E_{\text{sep}}|] = \tilde{O}(w(P)/D)$. We will assume that this bound holds deterministically.

**Assumption 4.1** (temporary simplifying assumption). We assume that whenever we run $\text{LowDiamDecomposition}(G, D)$, the following always holds for any path $P$:

$$|P \cap E_{\text{sep}}| = \tilde{O}(w(P)/D)$$

For simplicity, the overview focuses on the case where $G$ contains no negative-weight cycle.

**Assumption 4.2.** We assume that the input graph has no negative-weight cycles.

**Input Source is Canonical**: The first step of the algorithm will be to create a new source $s$ with edges of weight 0 to every $v \in V$; we thus assume for the rest of the overview that source $s$ is canonical in $G$. This assumption is justified because if we can compute all $\text{dist}_G(s, v)$ for a canonical $s$, then we can use price functions to remove all negative-edge weights (Lemma 2.6), at which point we can easily compute shortest paths from any source vertex using Dijkstra.

**Roadmap**: We will build up to the final algorithm in several steps. First we will show how our framework can be used to achieve a running time of $\tilde{O}(m \sqrt{n})$. Next we will outline how to improve this to $m^{1+o(1)}$. Finally, we will discuss how to remove the two simplifying assumptions above.
Our framework features two main algorithms: SPmain and ScaleDown. The first computes the actual shortest distances; the second computes a price function which makes edge-weights less negative. We will start with a basic version of these algorithms in our $\tilde{O}(m \sqrt{n})$-section, and then refine them as we proceed through the overview. In the final refinement, these algorithms will recursively call each other, and will have probabilistic guarantees.

### 4.1 Warm Up: An $\tilde{O}(m \sqrt{n})$ Algorithm.

We start by describing in detail an $\tilde{O}(m \sqrt{n})$ algorithm for computing all $\text{dist}(s,v)$. Although the run-time is slow, this algorithm contains all of the main ideas of our final algorithm. For this $\tilde{O}(m \sqrt{n})$ algorithm, we only need a basic (non-recursive) version of the algorithm ScaleDown.

**Definition 4.3** (ScaleDown($G,s,B$): basic version). The algorithm ScaleDown($G,s,B$) takes as input a graph $G = (V,E,w)$, a source $s$, and a positive number $B$. It has the following input and output guarantees

- **Input:** the source $s$ must be canonical and $w(e) \geq -B$ for all $e \in E$
- **Output:** a price function $\phi$ such that (1) $w_\phi(e) \geq -B/2$ for all $e \in E$, (2) $s$ is canonical in $G_\phi$, and (3) $E^{neg}(G_\phi) \subseteq E^{neg}(G)$

The main challenge of the warm up is to implement ScaleDown in $\tilde{O}(m \sqrt{n})$ time. Note that if we only required the first two output properties, we could use the earlier scaling algorithm of Goldberg [Gol95]. But the scaling approach of Goldberg can introduce many new negative edge-weights to the graph, and so violates property (3). This property will be crucial once we introduce recursion, so we develop an entirely novel scaling approach.

Before presenting the algorithm for ScaleDown, we show how ScaleDown can be used to compute all distances $\text{dist}(s,v)$.

#### 4.1.1 Using ScaleDown to compute distances

To compute all $\text{dist}_G(s,v)$, the algorithm simply calls ScaleDown($\tilde{G},s,\cdot$) repeatedly to bring down weights; after $\sim \log(n)$ iterations this will produce a price function $\phi$ such that $w_\phi \geq -1/n$. By Lemma 2.5, shortest paths in $G_\phi$ are the same as those in $G$. But note that distances in $G$ are integral, so intuitively, a weight of $-1/n$ cannot affect the shortest path structure. In particular, it is not hard to show that if we set $w^*(e) = w_\phi(e) + 1/n$, then any shortest path in $G^* = (V,E,w^*)$ is still shortest in $G_\phi$ and in $G$. Running Dijkstra($G^*,s$) then completes the algorithm.

To avoid dealing with non-integer weights/prices, our algorithm actually starts by scaling up all edge weights by a factor of $2n$ and then executing ScaleDown until all weights are $\geq -1$. Also, we note that property (2) of ScaleDown is crucial (Definition 4.3), because to bring down weights, each iteration of ScaleDown must use the price function computed by the previous iteration, so $s$ needs to remain canonical under all of these price functions. See Algorithm 3 in the main body for details.

#### 4.1.2 Implementing ScaleDown (basic version) in $\tilde{O}(m \sqrt{n})$ time.

See algorithm 2 for complete pseudocode. The algorithm first creates a new graph $\tilde{G} = (V,E,\tilde{w})$, where $\tilde{w} = w(e) + B/2$ for $e \in E^{neg}$ and $\tilde{w}(e) = w(e)$ for $e \notin E^{neg}$ (Line 1). Note that $\tilde{G}$ can still have negative edge-weights, but as will become clear, it is much easier to work with than $G$. In particular, we will show how to compute a price function $\phi$ such that all edge weights in $G_\phi$ are
Algorithm 2: Algorithm ScaleDown\(G = (V, E, w), s, B\): Simplified \(\tilde{O}(m\sqrt{n})\) version.

// INPUT: \(w(e) \geq -B\) for all \(e \in E\), \(s\) canonical in \(G\).
// See also Assumptions 4.1 and 4.2
1 Define \(\hat{G} = (V, E, \hat{w})\), where \(\hat{w}(e) = w(e) + \lceil B/2 \rceil\) for all \(e \in E^{neg}(G)\) and \(\hat{w}(e) = w(e)\) for \(e \in E \setminus E^{neg}(G)\).
2 \(E^{sep} \leftarrow \text{LowDiamDecomposition}(\hat{G}_{\geq 0} \setminus s, B\sqrt{n})\), where \(\hat{G}_{\geq 0} = (V, E, \hat{w}_{\geq 0})\) and \(\hat{w}_{\geq 0}(e) = \max\{0, w(e)\}\) for all \(e \in E\).
3 Compute strongly connected components (SCCs) of \(\hat{G} \setminus E^{sep}\), denoted by \(V_1, V_2, \ldots\)

// Phase 1: make edges inside SCCs of \(\hat{G}[V_1], \hat{G}[V_2], \ldots\) non-negative
4 foreach \(V_i\) do
5 \(\text{Add a dummy source } s^*_{i} \text{ to } \hat{G}[V_i] \text{ and an edge } (s^*_{i}, v) \text{ of weight 0 for every } v \in V_i\)
6 \(\hat{\text{dist}}(s^*_{i}, v) \leftarrow \text{SPWithFewNegEdges}(\hat{G}[V_i], s^*_{i}, 2\sqrt{n})\) (Lemma 2.8)
7 \(\phi_1(v) \leftarrow \hat{\text{dist}}(s^*_{i}, v)\) for every \(v \in V_i\)

// ASSERT: \(\hat{w}_{\phi_1}(u, v) \geq 0\) for all \((u, v) \in E\) (requires Assumption 4.2).

// Phase 2: make all edges in \(\hat{G} \setminus E^{sep}\) non-negative
8 \(\psi \leftarrow \text{FixDAGEdges}((\hat{G} \setminus E^{sep})_{\phi_1}, \{V_1, V_2, \ldots\})\) (Lemma 2.9)

// Above line makes all edge weights in \((\hat{G} \setminus E^{sep})_{\phi_1 + \psi}\) non-negative
9 \(\psi' \leftarrow \text{Dijkstra}((\hat{G} \setminus E^{sep})_{\phi_1 + \psi}, s)\) // Make s canonical
10 \(\phi_2 \leftarrow \phi_1 + \psi + \psi'\); // ASSERT: \(\hat{w}_{\phi_2}(e) \geq 0\) \(\forall e \notin E^{sep}\) and \(s\) canonical in \(\hat{G}_{\phi_2}\)

// Phase 3: make all edges in \(\hat{G}\) non-negative
11 \(\text{dist}(s, v) \leftarrow \text{SPWithFewNegEdges}(\hat{G}_{\phi_2}, s, \hat{O}(\sqrt{n}))\)
12 \(\phi_3(v) \leftarrow \text{dist}(s, v) - \phi_2(s) + \phi_2(v)\) for all \(v \in V\)
13 return \(\phi_3\); // ASSERT: \(\hat{w}_{\phi_3}(v) \geq 0\) and hence \(w_{\phi_3}(v) \geq -B/2\) for all \(v \in V\)

non-negative and \(s\) is canonical in \(\hat{G}_{\phi}\). It is easy to check that \(\phi\) then satisfies output properties (1), (2), and (3) of ScaleDown (Definition 4.3).

The hard part is computing \(\phi\) such that \(\hat{w}_{\phi}\) is non-negative. The key idea is to use LowDiamDecomposition (Lemma 3.1). We cannot apply the decomposition to \(\hat{G}\) directly because it has negative weights. We thus round edge weights up to 0; define \(\hat{G}_{\geq 0} = (V, E, \hat{w}_{\geq 0})\), where \(\hat{w}_{\geq 0} = \max\{0, w(e)\}\). We then compute \(E^{sep} \leftarrow \text{LowDiamDecomposition}(G_{\geq 0}, D := B\sqrt{n})\). Let \(V_1, V_2, \ldots\) be the SCCs of \(\hat{G}_{\geq 0} \setminus E^{sep}\), which by Lemma 3.1 have small weak diameter. We start by relabeling these in topological order, so that if \((u, v) \in E\), with \(u \in V_i\) and \(v \in V_j\), then \(i \leq j\) (Line 3).

The Three Phases: Although we compute \(E^{sep}\) and \(V_1, \ldots\) using weights \(\hat{w}_{\geq 0}\), we now turn back to the unrounded weight function \(\hat{w}\). Since \(\hat{G}\) and \(\hat{G}_{\geq 0}\) differ only by weights, the sets \(V_1, V_2, \ldots\) are also SCCs of \(\hat{G} \setminus E^{sep}\). Observe that all edges in \(\hat{G}\) are in one of three categories (1) The edges inside a single SCC \(V_i\), i.e. edges in \(\hat{G}[V_i]\), (2) Edges between two SCCs in \(\hat{G} \setminus E^{sep}\), and (3) the edges of \(E^{sep}\). (Some edges in \(E^{sep}\) may be in Category (1) as well.) Our algorithm consists of three phases, which correspond to the above edge classes. Phase 1 will compute a price function \(\phi_1\) such that (1)-edges have non-negative weight in \(\hat{G}_{\phi_1}\). Phase 2 will compute \(\phi_2\) such that (1)- and (2)-edges have non-negative weight in \(\hat{G}_{\phi_2}\). Finally phase 3 will compute the final \(\phi_3\) such that all edges have non-negative weight in \(\hat{G}_{\phi_3}\).
Phase 1: Fixing Edges Inside Each SCC $V_i$ (Lines 4-7) Here the goal is to make all the edges of $\hat{G}[V_i]$ non-negative. We focus on some some fixed $V_i$. We start by adding a dummy source $s_i^*$ and an edge $(s_i^*, v)$ of weight 0 to every $v \in V_i$ (Line 5); note that $s_i^*$ is canonical in $\hat{G}[V_i]$. The key claim of this phase is that shortest paths from $s_i^*$ contain few negative edges.

Claim 4.4. Every $v \in V_i$ is $(s_i^*, 2\sqrt{n})$-negative in $\hat{G}[V_i]$. (See Definition 2.3.)

Before proving the claim, let us show why it completes Phase 1. By the claim, we can use $\text{SPWithFewNegEdges}(\hat{G}[V_i], 2\sqrt{n})$—which is a simple combination of Dijkstra’s and Bellman-Ford algorithms—to compute $\text{dist}(s_i^*, v)$ for every $v \in V_i$ (Line 6). Setting $\phi_1(v) = \text{dist}(s_i^*, v)$ then gives the desired result by Lemma 2.6.

Proof of Claim 4.4. The main idea is that if there are more than $2\sqrt{n}$ negative-weight edges between a shortest $uv$-path in $\hat{G}[V_i]$ for some $u, v \in V_i$, then such path must have weight less than $-B\sqrt{n}$ in $G$. This creates a negative-weight cycle due to the weak diameter of $D = B\sqrt{n}$ of $G[V_i]$.

In more detail, note that by construction of $s_i^*$ we have $\text{dist}(s_i^*, v) \leq 0$ in $\hat{G}[V_i]$. Now say, for contradiction, that there were a shortest $s_i^* - v$ path $P$ in $\hat{G}[V_i]$ such that $|P \cap E^{\text{neg}}(G)| > 2\sqrt{n}$. Let $u$ be the second vertex on $P$, i.e., the vertex right after $s_i^*$, let $P_{uv}$ be the suffix of $P$ from $u$ to $v$, and note that $\hat{w}(P_{uv}) \leq 0$. Recall $\hat{G}$ and $G$ from Line 1 and note that $E^{\text{neg}}(\hat{G}) \subseteq E^{\text{neg}}(G)$. We have:

$$w(P_{uv}) = \hat{w}(P_{uv}) - \frac{B}{2}|P_{uv} \cap E^{\text{neg}}(G)| \leq -\frac{B}{2}|P_{uv} \cap E^{\text{neg}}(G)|$$

$$\leq \frac{B}{2}|P_{uv} \cap E^{\text{neg}}(G)| < -\frac{B}{2} \cdot 2\sqrt{n} = -B\sqrt{n}. \quad (1)$$

On the other hand, since $u, v \in V_i$, we know by the low-diameter guarantee of Line 2 that $\text{dist}_G(v, u) \leq \text{dist}_{\hat{G}_{\geq 0}}(v, u) \leq D = B \cdot \sqrt{n}$. But combining this with the above equation yields a $u$-$v$ cycle whose weight in $G$ is $w(P_{uv}) + \text{dist}_G(v, u) < B\sqrt{n} - B\sqrt{n} < 0$, contradicting Assumption 4.2 that $G$ has no negative-weight cycle.

Phase 2: Fixing Edges Between SCCs in $E \setminus E^{\text{sep}}$ (Lines 8-10). In phase 2, our goal is to compute $\phi_2$ such that $\hat{w}_{\phi_2}(e) \geq 0$ for all $e \in E \setminus E^{\text{sep}}$. We exploit the fact that the $(2)$-edges—are the edges in $E \setminus E^{\text{sep}}$ that are between the SCCs $V_1, V_2, \ldots$—form a DAG, and making all negative-weight edges in a DAG positive is easy. In particular, we achieve this by simply calling $\psi \leftarrow \text{FixDAGEdges}$ in Line 8. Temporarily ignoring $\psi'$ in Line 9, consider $\phi_1 + \psi$. The output guarantee of $\text{FixDAGEdges}$ (Lemma 2.9) ensures that $\hat{w}_{\phi_1 + \psi}(e) = (\hat{w}_{\phi_1})_{\psi}(e) \geq 0$ for each $e \in E \setminus E^{\text{sep}}$, as desired. It is also easy to check that the input guarantees of $\text{FixDAGEdges}$ are satisfied; the first by the guarantees of $\phi_1$ at end of phase 1 and the second because $V_1, \ldots, V_q$ are SCCs of $G \setminus E^{\text{sep}}$.

The reason we add $\psi'$ in Line 9 is to additionally ensure that $s$ is canonical in $\hat{G}_{\phi_2}$; this follows directly from Lemma 2.6.

Phase 3: Edges in $E^{\text{sep}}$ (Lines 11-13). In phase 3, our goal is to compute $\phi_3$ such that $\hat{w}_{\phi_3}(e) \geq 0$ for all $e \in E$. We will start with $\hat{G}_{\phi_2}$ from phase 2; note that $E^{\text{neg}}(\hat{G}_{\phi_2}) \subseteq E^{\text{sep}}$. The key claim of Phase 3 is to show that shortest $s - v$ paths in $\hat{G}_{\phi_2}$ use $\hat{O}(\sqrt{n})$ negative edges.

Claim 4.5. Every vertex $v$ is $(s, \hat{O}(\sqrt{n}))$-negative in $\hat{G}_{\phi_2}$.

Before proving the claim, let us see how we use it to compute $\phi_3$. By the claim, we can compute all $\text{dist}_{\hat{G}_{\phi_2}}(s, v)$ by running $\text{SPWithFewNegEdges}(\hat{G}_{\phi_2}, 2\sqrt{n})$ (Line 11). We then set $\phi_3(v) \leftarrow \text{dist}_{\hat{G}_{\phi_2}}(s, v) - \phi_2(s) + \phi_2(v) = \text{dist}_G(s, v)$, where the last equality follows from Lemma 2.5.
By Lemma 2.6, using \( \phi_3(v) = \text{dist}_G(s, v) \) ensures that all edge-weights in \( \hat{G}_{\phi_3} \) are non-negative and that \( s \) is canonical, as desired.

**Proof of Claim 4.5.** The key idea is that only edges in \( E^{sep} \) can be negative in \( \hat{G}_{\phi_2} \) and every \( s \) – \( v \) shortest path in \( \hat{G} \) contains \( \tilde{O}(\sqrt{n}) \) such edges due to Assumption 4.1.

In more detail, let \( P \) be any \( s \) – \( v \) shortest path in \( \hat{G} \), and note that by Lemma 2.5, \( P \) is also a shortest path in \( \hat{G}_{\phi_2} \). We now show that \( |P \cap E^{sep}| = \tilde{O}(\sqrt{n}) \); this will complete the claim because as noted above \( E^{neg}(\hat{G}_{\phi_2}) \subseteq E^{sep} \).

By Assumption 4.1, \( |P \cap E^{sep}| = \tilde{O}(\tilde{w}_{\geq 0}(P)/D) \), so we need to bound \( \tilde{w}_{\geq 0}(P) \). By the input guarantees to ScaleDown, we know that \( w(P) \leq 0 \) (because \( s \) is canonical), and so \( \tilde{w}(P) \leq nB/2 \). We also know that \( w(e) \geq -B \) and \( \tilde{w}(e) \geq -B/2 \) for all \( e \in E \). We thus have:

\[
\tilde{w}_{\geq 0}(P) \leq \sum_{e \in P \cap \not\in E^{neg}(\hat{G})} \tilde{w}(e) - \sum_{e \in E^{neg}(\hat{G}) \cap P} \tilde{w}(e) \leq nB - \sum_{e \in E^{neg}(\hat{G}) \cap P} \tilde{w}(e) \leq 3nB/2. \tag{2}
\]

Thus, by Assumption 4.1 we have \( |P \cap E^{sep}| = \tilde{O}(\tilde{w}_{\geq 0}(P)/D) = \tilde{O}(nB/(\sqrt{n}B)) = \tilde{O}(\sqrt{n}) \).

### 4.2 Improving the runtime to \( m^{1+o(1)} \)

The \( \tilde{O}(m\sqrt{n}) \) algorithm in Section 4.1 contains all of the main ideas of our algorithm; improving the runtime of \( m^{1+o(1)} \) is just a matter of applying the framework recursively. Full pseudocode for the faster algorithm appears in Algorithm 4 of the main body; one can see at a glance that it is extremely similar to the \( \tilde{O}(m\sqrt{n}) \)-algorithm (Algorithm 2).

The \( \tilde{O}(m\sqrt{n}) \) algorithm started with an arbitrary graph where every vertex is \((s, n)\)-negative, and then in Phases 1 and 3 it computed shortest paths in a graph where every vertex is \((s, \tilde{O}(\sqrt{n}))\)-negative. In the \( m^{1+o(1)} \) algorithm, we use multiple layers: we show that given a graph where all vertices are \((s, \Delta)\)-negative, we can use the same three phases above with different parameters to ensure that vertices in Phase 1 are \((s, n^{o(1)})\)-negative and vertices in Phase 3 are \((s, \Delta/n^{o(1)})\)-negative; to solve Phase 1 we can just use SPWithFewNegEdges as before, whereas Phase 3 is handled recursively.

We now define our two main algorithms SPmain and ScaleDown with this additional parameter \( \Delta \) in mind. These are simplified versions of the algorithms defined in Section 5; the key difference is that because of Assumption 4.1, we here assume deterministic input and output guarantees.

**Definition 4.6 (SPmain and ScaleDown – simplified version).** The algorithms SPmain\((G, s, \Delta, B)\) and ScaleDown\((G, s, \Delta, B)\) both take as input a graph \( G \) with the following properties: \( s \) is canonical in \( G \), every \( v \in V \) is \((s, \Delta)\)-negative, and \( w(e) \geq -B \) for all \( e \in E \). The output of SPmain is \( \text{dist}_G(s, v) \) for every \( v \in V \). The output of ScaleDown\((G, s, \Delta, B)\) is the same as in Definition 4.3.

**Algorithm SPmain\((G, s, \Delta, B)\).** The algorithm is the same as the one outlined in Section 4.1.1: we scale all weights up by \( 2n \) and then run ScaleDown\((G, s, \Delta, \cdot)\) for \( \log(n) \) iterations until all weights are \( \geq -1 \); we then increase all weights by \( 1 \) to achieve non-negative weights and run Dijkstra on the resulting graph.

Note that Property (3) of Definition 4.3 is crucial for this recursive version of ScaleDown. The reason is that if \( G \) is \( \Delta \)-negative and ScaleDown\((G, s, \Delta, \cdot)\) returns \( \phi \), we can only proceed to ScaleDown\((G_\phi, s, \Delta, B/2)\) if \( G_\phi \) is also \( \Delta \)-negative. Property (3) trivially guarantees that this is the case.

**Algorithm ScaleDown\((G, s, \Delta, B)\).** The algorithm is extremely similar to the \( \tilde{O}(m\sqrt{n}) \)-algorithm, so we just detail the changes we make to Algorithm 2.
The setup (Lines 1 - 3) is the same except that in Line 2 we use a different parameter: we set $E^{\text{sep}} \leftarrow \text{LowDiamDecomposition}(\hat{G}_{\geq 0} \setminus s, dB)$, where $d = n^{o(1)}$.

Phase 1 is also the same, except for a corresponding change of parameter in Line 6: we set $\text{dist}(s^*_v, v) \leftarrow \text{SPWithFewNegEdges}(\hat{G}[\Delta_v], s^*_v, 4d)$. To prove correctness, we show that every $v \in \hat{G}[\Delta]$ is $(s^*_v, 4d)$-negative; the proof is identical to that of Claim 4.4, just with different parameters.

Phase 2 is completely the same; no changes are required.

Phase 3: in line 11, instead of using SPWithFewNegEdges we recursively call SPmain with smaller $\Delta$ parameter. This recursive call is the entire reason for the speedup from $O(m\sqrt{n})$ to $m^{1+o(1)}$. In particular, for all $v \in V$, we set $\text{dist}(s, v) \leftarrow \text{SPmain}(\hat{G}^2, s, \Delta/n^{o(1)}, O(nB))$. To show that this returns correct distances we need to prove that the input guarantees of SPmain are satisfied. First we need to show that every vertex $v$ is $(s, \Delta/n^{o(1)})$-negative in $\hat{G}^2$: the proof strategy is entirely identical to that of Claim 4.5, just with the parameters altered. We also need to show that $\hat{w}_{\phi^2}(e) \geq -3nB$ for every $e \in E$: this follows from the fact that $\hat{w}(e) \geq -B$ (by the input assumption to ScaleDown), and that, as can be easily verified, all the price functions $\hat{\phi}$ computed during the course of the algorithm have $|\phi(v)| = O(nB)$ (see Lemma 6.13).

### 4.3 Removing Assumption 4.1: A Probabilistic Analysis.

We now remove Assumption 4.1 and instead show how to make our algorithm work with the probabilistic guarantees of Lemma 3.1. The algorithm itself is essentially the same, but the analysis is somewhat more technical.

The first crucial change is the interpretation of the parameter $\Delta$. In Definition 4.6, we assumed that every vertex $v$ is $(s, \Delta)$-negative. The problem is that whenever we make some recursive call SPmain($G', s', \Delta', B'$) in Phase 3, although we can argue that any particular vertex $v$ in $G'$ will be $(s', \Delta')$-negative in expectation, this is very unlikely to hold for all vertices $v$ simultaneously.

To overcome this issue, we instead let $\Delta$ correspond to a single graph-wide aggregate: we say that $(G, s)$ is $\Delta$-negative if the total number of negative edges on all $n$ different $s - v$ shortest paths is at most $\Delta$; see Definition 2.3 for details. The idea is that because we only care about a single quantity, there is a relatively high probability that it will behave according to expectation.

**Bounding Success Probability and the Number of Recursive Calls:** Unlike in Definition 4.6, the final probabilistic algorithm does not strictly assume that in any call to SPmain($G, s, \Delta, B$) or ScaleDown($G, s, \Delta, B$), we have that $G$ is $(\Delta, s)$-negative; instead the goal is to show that this holds with high-enough probability.

Consider a single call to ScaleDown($G, s, \Delta, B$) made during the recursive algorithm and say that $(G, s)$ is $\Delta$-negative. ScaleDown makes a recursive call to SPmain($\hat{G}^2, s, \Delta/n^{o(1)}, 3nB$). Just like in Section 4.2, we can use the expectation guarantee of LowDiamDecomposition (Lemma 3.1), combined with the argument of Claim 4.5, to argue that in expectation $(\hat{G}^2, s)$ is $\Delta/n^{o(1)}$-negative. We say that the call succeeds if this statement in fact holds; note that whether a call succeeds depends only its immediate child in the recursion tree, not on the rest of its descendants. Setting slack parameters appropriately, and using Markov’s inequality, we can show that $\Pr[\text{call succeeds}] \geq 1 - 1/\gamma$, for some $\gamma = n^{o(1)}$.

We now take a union bound to show that with probability $1/2$, all recursive frames succeed. The fan-out of the recursion tree is $O(\log(n))$ (because SPmain makes $O(\log(n))$ calls to ScaleDown), and the number of layers is sub-logarithmic (because $\Delta$ decreases by $n^{o(1)}$ between layers); setting the $n^{o(1)}$ parameters appropriately, we end up with a total of $\tau = n^{o(1)}$ recursive calls, with $\tau < \gamma/2$. A union bound thus ensures that $\Pr[\text{all frames succeed}] \geq 1/2$, so w.h.p the overall algorithm will only need to make $O(\log(n))$ attempts before success. (We show in the main body that the algorithm can easily differentiate between success and failure.)
Putting it all together, the algorithm starts with initial call SPmain\((G, s, \Delta := n^2, B := W_G)\); here we clearly have that \((G, s)\) is \(\Delta\)-negative, because all graphs are \(n^2\)-negative (assuming no negative-weight cycle). If we are in the (likely) case that all frames succeed, we can prove via induction down the tree that every recursive call \((G', \Delta')\) will also have the property that \(G'\) is \(\Delta'\)-negative; that is, we are likely to satisfy all the same guarantees as the deterministic algorithm from Section 4.2 that relied on Assumption 4.1.

5 Main Algorithms: SPmain and ScaleDown

This section develops our two main algorithms, SPmain (Algorithm 3) and ScaleDown (Algorithm 4), which recursively call each other.

Input. Both procedures take \((G, s, \Delta, B)\) as inputs, where \(G = (V, E, w)\) is a graph, \(s\) is a source node in \(G\), and \(\Delta\) and \(B\) are non-negative integer integers.

Output. Both algorithms return either error messages or,
- for SPmain, distance \(\hat{\text{dist}}(s, v) \in \mathbb{Z}\) for all \(v \in V\), and
- for ScaleDown, price function \(\phi : V \to \mathbb{Z}\).

5.1 Proper Inputs and Outputs

Although the formal output requirements of our algorithm are minimal, a key step in our analysis will be to show that both algorithms either return an error message or return an output that satisfies certain properties, denoted proper output

Definition 5.1 (Proper output). For SPmain\((G, s, \Delta, B)\), the proper output is the distances

\[\hat{\text{dist}}(s, v) = \text{dist}_G(s, v) \neq -\infty\]

for every \(v \in V\). (Note that if \(G\) contains a negative-weight cycle there is no proper output, because \(\text{dist}_G(s, v) = -\infty\) for some vertex \(v\).)

For ScaleDown\((G, s, \Delta, B)\), the proper output is a price function \(\phi : V \to \mathbb{Z}\) that satisfies the following properties:

1. \(w_\phi(e) \geq -B/2\) for all \(e \in E\).
2. \(s\) is a canonical source in \(G_\phi\).
3. \(E_{\text{neg}}(G_\phi) \subseteq E_{\text{neg}}(G)\).

Lemma 5.2 (Proper outputs; proof in Section 6.2). Any call SPmain\((G, s, \Delta, B)\) (Algorithm 3) or ScaleDown\((G, s, \Delta, B)\) (Algorithm 4) always returns either an error message or a proper output.

Note that Lemma 5.2 says nothing about the probability that our algorithms return error messages. But it ensures that if the algorithms do not return error messages, then they are guaranteed by Lemma 5.2 to return proper outputs. For example, this means that SPmain\((G, s, \Delta, B)\) never returns \(\hat{\text{dist}}(s, v) \neq \text{dist}_G(s, v)\) for any vertex \(v\), and ScaleDown\((G, s, \Delta, B)\) never returns \(\phi\) with \(w_\phi(e) < -B/2\) for some edge \(e\). Moreover, the lemma implies that if \(G\) has a negative-weight cycle, SPmain\((G, \ldots)\) always returns error.
Proper Inputs: We also introduce the notion of a proper input. Unlike with proper outputs, we cannot guarantee that all calls to ScaleDown and SPmain will have proper inputs. But in our analysis, we will show that whenever the algorithm has a proper input, it succeeds (i.e. doesn’t return error) with relatively high probability; if the algorithm has a non-proper input, we allow it to return error with any probability.

Definition 5.3 (Proper input). Any input \((G, s, \Delta, B)\), where \(G = (V, E, w)\) is a graph, \(s\) is a source node in \(G\), and \(\Delta\) and \(B\) are non-negative integers, is called proper if it satisfies the following conditions: (i) \(s\) is a canonical source (Definition 2.2), (ii) \(\Delta \leq n^2\), (iii) \(B \geq 2\) is such that \(w(e) \geq -B\) for all \(e \in E\), and (iv) \((G, s)\) is \(\Delta\)-negative. Note that if \(G\) contains a negative-weight cycle then the input is not proper because \((G, s)\) cannot be \(\Delta\)-negative.

Error probabilities: We have not yet discussed the error probability of our algorithms. This will be done in Section 6. We will show that with high-enough probability, if the graph contains no negative-weight cycle, then all the recursive calls made by our algorithm have proper inputs and proper outputs; it might help to read through our algorithms with these guarantees in mind.

5.2 Top-Level Recursive Call

The main theorem we need to prove is that the top-level call to SPmain always has small running time and that if the graph has no negative-weight cycle then it returns a proper output (rather than an error) with probability at least 1/2. We now state the theorem formally and then show that it easily implies our main result in Theorem 1.1.

Theorem 5.4 (Proved in Section 6). For any weighted graph \(G_{in} = (V, E, w)\) such that \(W_{G_{in}} = \text{poly}(n)\), \(s\) is a canonical source, and any parameter \(1/\sqrt{\log n} \leq \epsilon < 1\), \(\text{SPmain}(G_{in}, s, \Delta := n^2, B := W_{G_{in}})\) (Algorithm 3) has time complexity \(O((mn^2)\tau(n, W_{G_{in}}, \epsilon))\), where \(\tau(n, W_{G_{in}}, \epsilon) = (\epsilon^{-1}\log(n) + \log(W_{G_{in}}))^{O(1/\epsilon)}\). Moreover,

- if \(G_{in}\) contains a negative-weight cycle, then the algorithm always returns an error, and
- if \(G_{in}\) contains no negative-weight cycle, then the algorithm returns correct distances from \(s\) to every node with probability at least 1/2, and otherwise returns an error message.

Technical Note: The assumptions \(W_{G_{in}} = \text{poly}(n)\) and \(\epsilon > 1/\sqrt{\log n}\) in the theorem statement are used to simplify our calculations. They can be removed if necessary. Even though the term \(W_{G_{in}}\) is subsumed by other terms, we keep this term so that we can keep track of its role in the algorithms and analysis.

Before proving Theorem 5.4 we show how it implies the main result of this paper (Theorem 1.1).

Proving Theorem 1.1 Let \(G, s\) be the input graph and source to the main result (Theorem 1.1). We cannot apply Theorem 5.4 directly because \(s\) might not be canonical in \(G\). Thus the algorithm starts by adding a new universal source \(s^*\) to \(G\). Define \(V^* = V \cup \{s^*\}\), \(E^* = E \cup \{(s^*, v) \mid v \in V\}\), define \(w(s^*, v) = 0\) for all \(v \in V\), and let \(G^* = (V^*, E^*, w)\). Note that \(s^*\) is a canonical source of \(G^*\), that \(G^*\) contains a negative-weight cycle if an only if \(G^*\) does, and that that by assumption 2.1 we have \(W_{G^*} = W_G = O(1)\).

The algorithm then executes \(\text{SPmain}(G^*, s^*, \Delta := n^2, B := W_{G^*})\) for \(\log(n)\) iterations or until \(\text{SPmain}\) returns distances (rather than error). If \(\text{SPmain}\) returns error every time then our algorithm also outputs error. By Theorem 5.4, if \(G\) contains a negative-weight cycle, then our algorithm always outputs error, but if \(G\) does not contain a negative-weight cycle then our algorithm returns error with probability \(\leq 1/n\).
Now let us consider the case that come execution of $\text{SPmain}(G^*, s^*, \Delta := n^2, B := W_{G^*})$ returns correct distances $\text{dist}_{G^*}(s^*, v)$ for all $v \in V$. Define $\phi(v) = \text{dist}_{G^*}(s^*, v)$ for every $v \in V$. By Lemma 2.6 we know that $w_\phi(u, v) \geq 0$ for all $(u, v) \in E$. The algorithm can thus now run Dijkstra$(G_\phi, s)$ and output $\text{dist}_{G^*}(s, v) = \text{dist}_{G_\phi}(s, v) - \phi(s) + \phi(v)$.

We now bound the running time. It is clear that the overall running time is upper bounded by $\log(n) \cdot \text{[Running time of SPmain].}$ Setting $\epsilon = 1/\sqrt{\log(n)}$, Theorem 5.4 implies that $\text{SPmain}$ has a runtime of $m^{1+o(1)}$.

**Global Variables:** Our main goal in the analysis is to prove Theorem 5.4. In particular, all of our algorithms and analysis will refer to the top-level call $\text{SPmain}(G_{in}, s, \Delta := n^2, B := W_{G_{in}})$. We define several global variables that are used in the lower-level recursive calls $\text{SPmain}$ and $\text{ScaleDown}$

**Definition 5.5** (global variables $\epsilon, \gamma, d$). We use the following variables throughout our algorithm and analysis:

- We consider $\epsilon$ as in Theorem 5.4.
- Define $\gamma = (\epsilon^{-1} \log(n) + \log(W_{G_{in}}))^{c/\epsilon}$ for some big enough constant $c$, where $G_{in}$ is the input graph to Theorem 5.4.
- Define $d = 2 \log^2(n) \gamma n^\epsilon$

To keep all variables (except $\epsilon$) integral, we round $\gamma$ and $d$ up so they are integral. We omit the rounding in the paper to keep calculations simple.

### 5.3 SPmain Algorithm

We now describe the simpler of our two main algorithms, SPmain. See Algorithm 3 for pseudocode. Recall, for the sake of intuition, that if the input is not proper then SPmain is allowed to return error with any probability. So our goal is an algorithm such that: (1) no matter what the input, the algorithm either outputs error or a proper output and (2) if the input is proper then it outputs a proper output with relatively high probability.

The base case is $\Delta = 0$. If the input is proper then $G$ cannot contain any negative-weight edges, because otherwise there would be a shortest path with at least one negative edge (here we use that $s$ is canonical.) Thus, if $G$ does contain a negative-weight edge, the algorithm can safely return “Error” (Line 1). Otherwise, if all edges are positive, the algorithm simply runs Dijkstra$(G, s)$.

If $\Delta > 0$, the basic idea is to first scale up weights by $2n$ (Lines 4-5) so that everything remains integral, and then repeatedly call ScaleDown until we have a price function $\phi_t$ such that $w_{\phi_t}(e) \geq -1$. The algorithm then defines a graph $G^* = (V, E, w^*)$ with $w^*(e) = w_{\phi_t}(e) + 1$ (Line 9). In the analysis, we will argue that because we are dealing with a scaled-up graph (Line 4), the additive $+1$ is insignificant and does not affect the shortest path structure (Lemma 6.6), so running Dijkstra on $G^*$ will return correct shortest paths in $G$ (Lines 10 and 11).

### 5.4 ScaleDown Algorithm

We now describe algorithm ScaleDown. This algorithm is discussed in detail in the overview (Section 4), and Algorithm 2 of the overview contains pseudocode for a simplified (and slower) algorithm which captures all the main ideas. We thus omit a more detailed discussion here, and point the reader directly to the pseudocode (Algorithm 4).

We highlight one minor technical point. In Line 3, we call LowDiamDecomposition on the graph $G_{\geq 0} \backslash s$: the reason we remove the vertex $s$ is to ensure that no edge incident to $s$ ends up in $E_{\text{sep}}$. 

14
Algorithm 3: Algorithm for $\text{SPmain}(G = (V, E, w), s, \Delta, B)$

// Whenever a subroutine returns an error message, so does this algorithm
1 if $\Delta = 0$ then
2   If $w(u, v) < 0$ for some $(u, v) \in E$ return "Error"
3   Otherwise use Dijkstra$(G, s)$ to return $\hat{\text{dist}}(s, v) = \text{dist}_G(s, v)$ for all $v \in V$.
4   $\bar{w}(e) \leftarrow w(e) \cdot 2n$ for all $e \in E$, $\bar{G} \leftarrow (V, E, \bar{w})$, $B \leftarrow B \cdot 2n$, $\phi_0 = 0$ // scale $B$ up
5   Round $B$ up to nearest power of 2 // still have $\bar{w}(e) \geq -B$ for all $e \in E$
6   for $i = 1$ to $t := \log_2(B)$ do
7     $\psi_i \leftarrow \text{ScaleDown}(G_{\phi_{i-1}}, s, \Delta, B/2^{i-1})$
8     $\phi_i \leftarrow \phi_{i-1} + \psi_i$ // (Invariant 6.11) $w_{\phi_i}(e) \geq -B/2^i$ for all $e \in E$ and $(G_{\phi_i}, s)$ is $\Delta$-negative if $(G, s)$ is
9   $G^* \leftarrow (V, E, w^*)$ where $w^*(e) \leftarrow \bar{w}_{\phi_t}(e) + 1$ for all $e \in E$.
   // Observe: $G^*$ in above line has non-negative weights
10  Compute a shortest path tree $T$ from $s$ using Dijkstra$(G^*, s)$ (Lemma 2.7)
   // (Lemma 6.6) Will Show: any shortest path in $G^*$ is also shortest in $G$
11  Set $\text{dist}(s, v)$ to be the length of the $sv$-path in $T$ using the original (unscaled) weight
   function $w$ of the input graph $G$.
12 return $\text{dist}(s, v)$ for all $v \in V$

6 Analysis

This section is devoted to proving Theorem 5.4 above. To simplify notations, let $W := W_{G_{in}}$. When we call $\text{SPmain}(G_{in}, s, \Delta = n^2, B = W)$ for some input graph $G_{in}$ and canonical source node $s$, the algorithm makes several recursive calls to $\text{SPmain}(G, s, \Delta, B)$ and $\text{ScaleDown}(G, s, \Delta, B)$.

Lemma 6.1 below bounds the number of recursive calls.

Lemma 6.1 (Number of calls; proof in Section 6.1). For any $n$-node graph $G_{in}$, source node $s$, and parameter $\epsilon > 0$ (defined in Algorithm 4), after we call $\text{SPmain}(G_{in}, s, \Delta = n^2, B = W)$, the total number of calls to $\text{SPmain}$ and $\text{ScaleDown}$ in Algorithms 3 and 4 is at most $\tau(n, W, \epsilon)$, where

$$\tau(n, W, \epsilon) = (\epsilon^{-1} \log(n) + \log(W))^{O(1/\epsilon)}.$$

Moreover, all recursive calls of $\text{SPmain}(G, s, \Delta, B)$ and $\text{ScaleDown}(G, s, \Delta, B)$, are such that

$$B = n^{O(1/\epsilon)}W.$$

The bounds in Lemma 6.1 always hold regardless of the randomness. The proof of Lemma 6.1 follows from a standard recurrence analysis; see Section 6.1. The runtime of our algorithm follows almost immediately from Lemma 6.1 (see the proof of theorem 5.4 below). Lemma 6.1 will also be useful when we later analyze the error probability (Lemma 6.3).

Bounding the error probability. Note that in Theorem 5.4 we only need to prove a probability bound when $G_{in}$ does not contain a negative-weight cycle. More generally, we will only prove probability bounds for calls to $\text{SPmain}$ and $\text{ScaleDown}$ that have proper inputs (Definition 5.3). Note in particular that if $G_{in}$ does not have a negative-weight cycle then the input in Theorem 5.4 is proper.

Consider some recursive call to $\text{ScaleDown}(G, s, \Delta, B)$ and say that it has proper input. It is crucial for our analysis to argue that the shortest $sv$-path in $\bar{G}$ has a small intersection with $E^{sep}$,
Algorithm 4: Algorithm for ScaleDown($G = (V, E, w), s, \Delta, B$)

// Whenever a subroutine returns an error message, so does this algorithm
1 Recall variable $d$ from Definition 5.5
2 Define $\hat{G} = (V, E, \hat{w})$, where $\hat{w}(e) = w(e) + \lfloor B/2 \rfloor$ for all $e \in E^{neg}(G)$ and $\hat{w}(e) = w(e)$ for $e \in E \setminus E^{neg}(G)$;
3 $E^{sep} \leftarrow \text{LowDiamDecomposition}(\hat{G}_{\geq 0} \setminus s,dB)$, where $\hat{G}_{\geq 0} = (V, E, \hat{w}_{\geq 0})$ and $\hat{w}_{\geq 0}(e) = \max\{0, w(e)\}$ for all $e \in E$; // Decompose the graph (Lemma 3.1) after rounding negative weights up to 0 and deleting $s$.
4 Compute strongly connected components (SCCs) of $\hat{G} \setminus E^{sep}$, denoted by $V_1, V_2, \ldots$

// Phase 1: make edges inside SCCs of $\hat{G}[V_1], \hat{G}[V_2], \ldots$ non-negative
5 foreach $V_i$ do
6 Add a dummy source $s^*_i$ to $\hat{G}[V_i]$ and an edge $(s^*_i, v)$ of weight 0 for every $v \in V$;
7 $\hat{\text{dist}}(s^*_i, v) \leftarrow \text{SPWithFewNegEdges}(\hat{G}[V_i], s^*_i, 4d)$ (Lemma 2.8)
8 $\hat{\phi}_1(v) \leftarrow \hat{\text{dist}}(s^*_i, v)$ for every $v \in V_i$;
9 If $\hat{w}_{\phi_1}(u, v) < 0$ for some $u, v \in V_i$ for some $i$, terminate the entire recursive algorithm and return “Error”; // If $\phi_1$ does not make edges inside SCCs non-negative, the input graph must contain a negative cycle (consequence of Lemma 6.12)

// Phase 2: make all edges in $\hat{G} \setminus E^{sep}$ non-negative
10 $\psi \leftarrow \text{FixDAGEdges}((\hat{G} \setminus E^{sep})_{\phi_1}, \{V_1, V_2, \ldots\})$ (Lemma 2.9)
11 $\psi' \leftarrow \text{Dijkstra}((\hat{G} \setminus E^{sep})_{\phi_1+\psi}, s)$ (Lemma 2.7); // Make $s$ canonical
12 $\phi_2 \leftarrow \phi_1 + \psi + \psi'$; // (Lemma 6.13) $s$ is canonical in $\hat{G}_{\phi_2}$, $\hat{w}_{\phi_2}(e) \geq -3nB$ for all $e \in E$ and $\hat{w}_{\phi_2}(e) \geq 0$ for all $e \notin E^{sep}$

// Phase 3: make edges in $\hat{G}$ non-negative
13 $\text{dist}(s, v) \leftarrow \text{SPmain}(\hat{G}_{\phi_2}, s, [\Delta/n], 3nB)$ // Recursively compute distances from $s$ to every $v \in V$ in $\hat{G}_{\phi_2}$ and store in $\text{dist}(s, v)$. (Edges from $E^{sep}$ are now included.) See Lemma 6.13 for correctness.
14 $\phi_3(v) \leftarrow \text{dist}(s, v) - \phi_2(s) + \phi_2(v)$ for all $v \in V$
15 return $\phi_3$; // ASSERT: $\hat{w}_{\phi_3}(v) \geq 0$ and hence $w_{\phi_3}(v) \geq -B/2$ for all $v \in V$
where $\hat{G}$ is the graph defined in Line 2, and $E^{\text{sep}}$ is the output of LowDiamDecomposition in Line 4 of Algorithm 4; see overview sections 4.1.2 and 4.2 for intuition. The probabilistic guarantee of LowDiamDecomposition (Lemma 3.1) will allow us to prove that this holds with relatively high probability. We now want to lower bound the probability that this desired event always holds for all recursive calls.

Let $r$ be the random string that the algorithm uses to make its random choices. Consider the top-level call $\text{SPmain}(G_{in}, s, \Delta = n^2, B = W)$ in Theorem 5.4, and note that for a fixed random string $r$, the algorithm behaves deterministically. We define $r$ to be good random string if all recursive calls to ScaleDown made during the course of the top-level $\text{SPmain}(G_{in}, \ldots)$ satisfy the guarantee of the above paragraph. Formally:

**Definition 6.2 (Good random string).** Consider the top-level call $\text{SPmain}(G_{in}, s, \Delta = n^2, B = W)$ made in Theorem 5.4 with random string $r$. We say that $r$ is good with respect to $(G_{in}, s)$ if for all recursive calls $\text{ScaleDown}(G, s, \Delta, B)$ (Algorithm 4) such that $(G, s, \Delta, B)$ is a proper input, we have

$$\sum_{v \in V} |P(v; \hat{G}, s) \cap E^{\text{sep}}| \leq \frac{\Delta}{n^\epsilon} \tag{4}$$

where $P(v; \hat{G}, s)$ is the shortest $sv$-path defined in Definition 2.3.

We prove that at least half of the random strings are good:

**Lemma 6.3 (Proof in Section 6.3).** For any input $(G_{in}, s)$ where $G_{in}$ contains no negative cycle and $s$ is a canonical super source,

$$P_r[r \text{ is good w.r.t. } (G_{in}, s)] \geq 1/2,$$

where the probability is over all random strings $r$.

Then, we show that $\text{SPmain}(G_{in}, s, \Delta = n^2, B = W)$ gives the correct distances when $r$ is good:

**Lemma 6.4 (Proof in Section 6.4).** If the random string $r$ used by $\text{SPmain}(G_{in}, s, \Delta = n^2, B = W)$ is good, and $G_{in}$ contains no negative-weight cycle, then the algorithm returns a proper output, i.e. $\text{dist}_G(s, v) \neq -\infty$ for all $v \in V$.

**Proof of Theorem 5.4** We now conclude the proof Theorem 5.4. The runtime of our algorithm follows easily from Lemma 6.1: among all steps of Algorithms 3 and 4, the most expensive non-recursive step is Line 7 of Algorithm 4. Recalling the global variables from Definition 5.5, the time complexity of this step is

$$O(md) = \tilde{O}\left(mn^\epsilon (\epsilon^{-1} \log(n) + \log(B))^{O(1/\epsilon)}\right) = \tilde{O}\left(mn^\epsilon (\epsilon^{-1} \log(n) + \log(W))^{O(1/\epsilon)}\right)$$

where the equality is because of (3) in Lemma 6.1. (That is, even though the parameter $B$ in lower-level recursive calls can be larger than $W$, it is bounded by (3).) Multiplying this by the bound on the total number of recursive calls in Lemma 6.1 yields the desired result.

If $G_{in}$ contains a negative-weight cycle, then by Lemma 5.2 the algorithm returns an error message, as there does not exist a proper output.

If $G_{in}$ contains no negative-weight cycle and the random string $r$ used by the algorithm is good, then the algorithm output correct distances $\text{dist}_{G_{in}}(s, v) \neq -\infty$ due to Lemma 6.4. By Lemma 6.3, this happens with probability at least 1/2. With the remaining probability, the algorithm returns either an error messages or the correct distances (by Lemma 5.2). This completes the proof of Theorem 5.4.
6.1 Proof of Lemma 6.1 (Number of Recursive Calls)

Fix a random string \( r \) used by the algorithm throughout this subsection. Let \( f_n(\Delta, B) \) be the upper bound of the total number of calls to SPmain and ScaleDown when we execute SPmain\((H, s, \Delta, B)\) in Algorithm 3 for any \( n \)-node graph \( H \), source \( s \), and parameters \( \Delta \) and \( B \). Define \( g_n(\Delta, B) \) similarly for executing ScaleDown\((H, s, \Delta, B)\) in Algorithm 4. For simplicity, we ignore the subscripts in the rest of this subsection. We can derive the following recurrence relations.

\[
    f(\Delta, B) = \begin{cases} 
    1 & \text{when } \Delta = 0 \text{ (by Line 2 of Algorithm 3)} \\
    \log_2(2nB) + \sum_{i=1}^{\log_2(2nB)} g(\Delta, (2nB)/2^{i-1}) & \text{otherwise (by Line 7 of Algorithm 3)}. 
    \end{cases}
\]

\[
    g(\Delta, B) = f(\lfloor \Delta/n^\epsilon \rfloor, 3nB) \quad \text{(by Line 13 of Algorithm 4)}
\]

Note that the terms \( 4nB \) above are because we multiply the input \( B \) by \( n \) and round this up to the nearest power of two (Lines 4-5 of Algorithm 3). It follows from the above that

\[
    f(\Delta, B) \leq \log_2(4nB) + \log_2(4nB) \cdot f(\lfloor \Delta/n^\epsilon \rfloor, 4n \cdot 3nB) \quad \text{when } \Delta > 0.
\]

For \( f(\Delta = n^2, B = W) \), the recursion depth is at most \( \log_{n^\epsilon}(n^2) \leq \lceil 2/\epsilon \rceil \leq 3/\epsilon \) (for \( \epsilon < 1 \)). Thus, the largest value of input \( B \) (at the leaves for the recursion tree) is at most \( (4n \cdot 3n)^{3/\epsilon}W = n^{O(1/\epsilon)}W \). This proves Equation 3 in Lemma 6.1.

We thus have that in all recursive calls \( \log(B) = O(e^{-1}\log(n)+\log(W)) \). Applying the recurrence formula above it follows that

\[
    f(n^2, W) \leq \sum_{i=1}^{3/\epsilon} \left( \log_2 \left( 4n \cdot n^{O(1/\epsilon)}W \right) \right)^i \leq \frac{3}{\epsilon} \left( \log_2 \left( n^{O(1/\epsilon)}W \right) \right)^{3/\epsilon} = (e^{-1}\log(n)+\log(W))^{O(1/\epsilon)} \quad \text{(because } 3/\epsilon \leq 2^{O(1/\epsilon)}).\]

6.2 Proof of Lemma 5.2 (Proper Outputs)

In short, the key to proving Lemma 5.2 is to inductively show that (i) the shortest path tree \( T \) of \( G^* \) computed in Line 10 of Algorithm 3 is also a shortest path tree of \( G \) (see Lemmas 6.5 and 6.6) and (ii) \( \phi_3 \) computed in Algorithm 4 is such that \( \phi_3(v) = \text{dist}_G(s, v) \) for every vertex \( v \) (Equation (7)).

In more detail, we prove the following statements inductively on the value of \( \Delta \).

- **A(\Delta):** For any integer \( \Delta \geq 0 \), let \( A(\Delta) \) denote the statement that, for any \( G, s, \) and \( B \), SPmain\((G, s, \Delta, B)\) returns either an error message or proper output, i.e. the distances \( \text{dist}_G(s, v) \neq -\infty \) for every \( v \in V \).

- **B(\Delta):** For any integer \( \Delta \geq 0 \), let \( B(\Delta) \) denote the statement that, for any \( G, s, \) and \( B \), ScaleDown\((G, s, \Delta, B)\) returns either an error message or the proper output, i.e. the price function \( \phi : V \rightarrow \mathbb{Z} \) that satisfies Items 1 to 3 in Definition 5.3.

Note that if we prove the above statements for all \( \Delta \), Lemma 5.2 immediately follows. The base case of our induction will be \( A(0) \), which holds because of line 1 of Algorithm 3. Below, we divide our inductive proof into two parts. Recall that \( \epsilon \) is a global variable defined in definition 5.5.

- Part 1: We prove that for any \( \Delta > 0 \), if \( B(\Delta) \) holds, then \( A(\Delta) \) also holds.
- Part 2: We prove that for any \( \Delta \geq 0 \), if \( A(\lfloor \Delta/n^\epsilon \rfloor) \) holds, then \( B(\Delta) \) also holds.
By a simple induction, we can argue that $A(\Delta)$ and $B(\Delta)$ hold for any $\Delta \geq 0$, proving Lemma 5.2. (Note that $B(0)$ holds by Part 2, since $\lfloor 0/n^\epsilon \rfloor = 0$, and our base case is that $A(0)$ holds.)

**Part 1: For any $\Delta > 0$, if $B(\Delta)$ holds, then $A(\Delta)$ also holds.**

Fix any $\Delta > 0$. Consider the calls of $\text{SPmain}(G, s, \Delta, B)$ (over all possible $(G, s, B)$ and random string $r$) that do not output an error message. We will be done if we can show that all these calls return proper outputs.

Consider the execution of any such call until we complete the for loop in Line 6 of Algorithm 3. We know that none of the calls to $\text{ScaleDown}$ in the for loop returns an error (since otherwise our current execution would return error), so $B(\Delta)$ implies that all of them return proper outputs, and in particular that we end with a price function $\phi_t$ that satisfies output properties in Definition 5.1; that is,

$$\bar{w}_{\phi_t}(e) \geq -1 \quad \forall e \in E \quad (5)$$

by Item 1 in Definition 5.1. Equation (5) implies the following.

**Lemma 6.5.** $\bar{G}_{\phi_t}$ contains no negative-weight cycle.

*Proof.* Assume for a contradiction that such cycle $C$ exists in $\bar{G}_{\phi_t}$. Observe that $\bar{w}(C) = \bar{w}_{\phi_t}(C)$ (price functions do not change the weight of cycles). By (5), $\bar{w}_{\phi_t}(C) \geq -n$. However, since edge weights in $\bar{G}$ are multiples of $n$, $\bar{w}_{\phi_t}(C) = \bar{w}(C) \leq -2n$, a contradiction. \qed

Moreover, Equation (5) implies that $G^*$ from Line 9 of Algorithm 3 contains no negative-weight edges, so calling $\text{Dijkstra}(G^*, s)$ in Line 10 gives us a shortest path tree $T$ in $G^*$ rooted at $s$. The claim below implies $A(\Delta)$ and concludes Part 1.

**Lemma 6.6.** $\widetilde{\text{dist}}(s, v) = \text{dist}_G(s, v)$ for all $v \in V$; i.e., $\text{SPmain}$ (Algorithm 3) correctly returns distances from $s$ in $G$.

*Proof.* Fix any vertex $v$. Let $P_v = (u_0 = s, u_1, u_2, \ldots, u_k = v)$ be a shortest $sv$-path in $G^*$. Recall the definitions of $\bar{G}, \bar{w}$ from Line 4 and $G^*, w^*$ from Line 9. To prove the lemma, we need to show that $P_v$ is also a shortest path in $G$.

Now define $G' = (V, E, w')$ where

$$w'(e) = \bar{w}(e) + 1$$

for all $e \in E$. Observe that $G^*$ is equivalent to $G'$ (because $w^*(e) = w'_{\phi_t}(e)$ for every edge $e$); thus $P_v$ is a shortest $sv$-path in $G'$.

We next claim that

$$P_v \text{ is a shortest } sv \text{-path in } \bar{G}. \quad (6)$$

To see this, assume for contradiction that there exists an $sv$-path $Q_v$ in $\bar{G}$ such that $\bar{w}(Q_v) < \bar{w}(P_v)$. Since every edge weight $\bar{w}(e)$ is a multiple of $2n$,

$$\bar{w}(Q_v) + 2n \leq \bar{w}(P_v).$$

Thus, $w'(Q_v) < \bar{w}(Q_v) + 2n \leq \bar{w}(P_v) \leq w'(P_v)$, contradicting that fact that $P_v$ is the shortest path in $G'$. This proves (6).

Finally, $P_v$ is a shortest $sv$-path in $G$ since multiplying edge weights by a positive number preserves shortest paths. This implies that $\widetilde{\text{dist}}(s, v) = \text{dist}_G(s, v)$ as desired. \qed
For part 2 of the inductive proof, we will need the following easy observation:

**Observation 6.7.** Consider the top-level call $\text{SPmain}(G_{in}, s, \Delta, B)$, where $s$ is a universal source. For all the recursive calls $\text{SPmain}(G, s, \ldots)$ and $\text{ScaleDown}(G, s, \ldots)$, the source $s$ is universal in $G$. (See Definition 2.2.) This follows from the fact that these algorithms never change the topology of $G$, only the edge weights.

**Part 2:** For any $\Delta \geq 0$, if $\mathcal{A}(\lfloor \Delta/n^\epsilon \rfloor)$ holds, then $\mathcal{B}(\Delta)$ also holds. Fix any $\Delta \geq 0$. Like Part 1, consider the calls of $\text{ScaleDown}(G, s, \Delta, B)$ (over all possible $(G, s, B)$ and random string $r$) that do not output an error message. We will be done if we can show that such calls always return proper outputs.

Consider executing any such call until we reach Line 13 of Algorithm 4, where we recursively call $\text{SPmain}(\hat{G}_{\phi_3}, s, \lfloor \Delta/n^\epsilon \rfloor, Bn^4)$. By assumption our call does not return an error, so neither do the subcall to $\text{SPmain}$, so $\mathcal{A}(\lfloor \Delta/n^\epsilon \rfloor)$ implies that $\hat{G}_{\phi_3}$ contains no negative-weight cycle and

$$\forall v \in V : \ \hat{\text{dist}}(s, v) = \text{dist}_{\hat{G}_{\phi_3}}(s, v) = \text{dist}_{\hat{G}}(s, v) + \phi_3(s) - \phi_2(v).$$

Thus,

$$\phi_3(v) = \text{dist}_{\hat{G}}(s, v). \quad (7)$$

This implies that $\hat{w}_{\phi_3}(e) \geq 0$ for every $e \in E$; thus, $w_{\phi_3}(e) \geq -B/2$ for every $e \in E$. By Lemma 2.6 and (7), $\text{dist}_{\hat{G}_{\phi_3}}(s, v) = 0$ for all $v \in V$; consequently, $\text{dist}_{G_{\phi_3}}(s, v) \leq 0$ and so $s$ is canonical in $G_{\phi_3}$ (we can always apply Lemma 2.6 to source $s$ because of Observation 6.7). Finally, since $\hat{G}_{\phi_3}$ contains no negative-weight edge, $E_{neg}(G_{\phi_3}) \subseteq E_{neg}(G)$. So, $\phi_3$ is a proper output (Definition 5.1) and $\mathcal{B}(\Delta)$ holds. This concludes Part 2.

### 6.3 Proof of Lemma 6.3 (Good Random String Probability)

Throughout this section, probabilities and expectations are over the random strings used by our algorithms. To prove Lemma 6.3, the key is to analyze equation (4) for a single call of $\text{ScaleDown}(G, s, \Delta, B)$ (Algorithm 4); see Lemma 6.8. To ease notations, define $P_v := P(v; G, s)$, $\hat{P}_v := P(v; \hat{G}, s)$, and $\eta(v) := \eta(v; G, s)$ (see Definition 2.3); thus, (4) can be rewritten as $\sum_{v \in V} |\hat{P}_v \cap E^{sep}| \leq \frac{\Delta}{n^\epsilon}$. Recall also the global variables from 5.5

**Lemma 6.8.** For any proper input $(G, s, \Delta, B)$, $\text{ScaleDown}(G, s, \Delta, B)$ (Algorithm 4) gives $E^{sep}$ such that $Pr[(4) \text{ holds}] \geq 1 - \gamma^{-1}$, i.e.

$$Pr \left[ \sum_{v \in V} |\hat{P}_v \cap E^{sep}| \leq \frac{\Delta}{n^\epsilon} \right] \geq 1 - \gamma^{-1}.$$ 

To prove Lemma 6.8, note the following.

**Claim 6.9.** For every $v \in V$, $\hat{w}_{\geq 0}(\hat{P}_v) \leq \eta(v)B$.

**Proof.** Fix $v \in V$ and recall $\hat{G}$ from Line 2. Observe that\(^6\)

$$|E_{G}^{neg}(\hat{P}_v)| \leq |E_{G}^{neg}(P_v)| = \eta(v) \quad (8)$$

\(^6\)Recall from Section 2 that $E_{G}^{neg}(\hat{P}_v) = P_v \cap E_{neg}(G)$ and $E_{G}^{neg}(P_v) = P_v \cap E_{neg}(G)$.
because \( \hat{w}(\hat{P}_v) \leq \hat{w}(P_v), w(\hat{P}_v) \geq w(P_v) \), \( \hat{w}(\hat{P}_v) = w(\hat{P}_v) + |E_G^{neg}(\hat{P}_v)|B/2 \) and \( \hat{w}(P_v) = w(P_v) + |E_G^{neg}(P_v)|B/2 \). Also observe that

\[
\hat{w}(\hat{P}_v) \leq \hat{w}(P_v) = w(P_v) + |E_G^{neg}(P_v)|B/2 \leq \eta(v)B/2.
\]  

(9)

where the first inequality is because \( \hat{P}_v \) is the shortest \( sv \)-path in \( \hat{G} \) and the last inequality is because \( w(P_v) \leq 0 \) (by definition of proper input, \( s \) is a canonical source in \( G \) (Definition 2.2)). So, we have

\[
\hat{w}_{\geq 0}(\hat{P}_v) \leq \hat{w}(\hat{P}_v) + |E_G^{neg}(\hat{P}_v)|B/2 \quad \text{since} \quad \hat{w}(e) \geq -B/2 \quad \text{for all} \ e \in E
\]

\[
\leq \hat{w}(\hat{P}_v) + |E_G^{neg}(\hat{P}_v)|B/2 \quad \text{since} \quad E^{neg}(\hat{G}) \subseteq E^{neg}(G)
\]

\[
\leq \hat{w}(\hat{P}_v) + \eta(v)B/2 \quad \text{by (8)}
\]

\[
\leq \eta(v)B \quad \text{by (9)}
\]

Proof of Lemma 6.8. By Claim 6.9 and lemma 3.1 with \( D = dB = [2 \log^2(n)\gamma n^\epsilon]B \) (see Line 3 of Algorithm 4), we have that for every \( v \in V \):

\[
E[\hat{P}_v \cap E^{sep}] \leq \frac{\log^2(n)\eta(v)B}{2 \log^2(n)\gamma n^\epsilon} + n^{-9} \leq \frac{\eta(v)}{2\gamma n^\epsilon} + n^{-9},
\]

where the expectation is over all random strings \( r \). If \( G \) is \( \Delta \)-negative, i.e. \( \sum_{v \in V} \eta(v) \leq \Delta \), we have

\[
E \left[ \sum_{v \in V} |\hat{P}_v \cap E^{sep}| \right] \leq \sum_{v \in V} \left( \frac{\eta(v)}{2\gamma n^\epsilon} + n^{-9} \right) \leq \frac{\Delta}{2\gamma n^\epsilon} + n^{-8} \leq \frac{\Delta}{\gamma n^\epsilon}
\]

where the last inequality is because \( 2\gamma n^\epsilon \leq n^8 \) for big enough \( n \), because \( \epsilon < 1 \) and \( \gamma \leq n \). (The fact that \( \gamma \leq n \) follows from the fact that we assume in Theorem 5.4 that \( W = \text{poly}(n) \) and \( \epsilon \geq 1/\sqrt{\log n} \).) By Markov’s inequality,

\[
Pr \left[ \sum_{v \in V} |\hat{P}_v \cap E^{sep}| \leq \frac{\Delta}{n^\epsilon} \right] \geq 1 - \gamma^{-1}.
\]

Proof of Lemma 6.3. This follows easily from Lemma 6.8. In short, Lemma 6.8 allows us to use union bound to show that the probability (over all \( r \)) that (4) holds for all recursive calls of ScaleDown is at least \( 1 - \gamma \tau(n,W,\epsilon) \) where \( \tau(n,W,\epsilon) \) is the upper bound of the number of recursive calls over all random strings (Lemma 6.1). For a large enough constant \( c \) in the definition of \( \gamma \) (Algorithm 4) this probability is at least \( 1/2 \).

In more detail, consider generating the random string \( r \) while executing the algorithm. For any \( i \geq 0 \), consider the \( i^{th} \) recursive call of ScaleDown\((G,s,\Delta,B)\) where \((G,s,\Delta,B)\) is a proper input and \( G \) is \( \Delta \)-negative. Assume that the random string used by previous calls are already fixed. Since the \( i^{th} \) call uses new random bits that are independent from those used by the previous calls, we can apply Lemma 6.8 to conclude that for the \( i^{th} \) call, \( Pr[\text{(4)} \text{ does not hold}] \leq 1/\gamma \). Since the recursion tree contains at most \( \tau(n,W,\epsilon) = (\epsilon^{-1}\log(n) + \log(W))^{O(1/\epsilon)} \) recursive calls (Lemma 6.1), the probability that (4) does not hold for any of these calls is at most \( \gamma \cdot \tau(n,W,\epsilon) \). Recall from Algorithm 4 that \( \gamma = (\epsilon^{-1}\log(n) + \log(W))^{c/\epsilon} \). So, for a large enough constant \( c \), we have \( \gamma \cdot \tau(n,W,\epsilon) \leq 1/2 \). Thus, the probability that (4) holds for all recursive calls, i.e. the random string is good with respect to \((G_{(n)},s)\), is at least \( 1/2 \).

\footnote{the first inequality is not equality because edges incident to \( s \) are excluded when computing \( E^{sep} \). (Line 3
6.4 Proof of Lemma 6.4 (Correctness under Good Random Strings)

Throughout this section, we consider $G_{in}$ containing no negative-weight cycle and assume that the random string $r$ used by the algorithm is good\(^8\). Similar to Section 6.2, we prove Lemma 6.4 by inductively proving the following statements.

- **$\mathcal{A}'(\Delta)$**: For any fixed $\Delta \geq 0$, let $\mathcal{A}'(\Delta)$ denote the statement that all recursive calls of SPmain($G, s, \Delta, B$) (Algorithm 3) such that $(G, s, \Delta, B)$ is a proper input return the proper output, i.e. $\text{dist}_G(s, v) \neq -\infty$ for all $v \in V$.

- **$\mathcal{B}'(\Delta)$**: For any fixed $\Delta \geq 0$, let $\mathcal{B}'(\Delta)$ denote the statement that all recursive calls of ScaleDown($G, s, \Delta, B$) (Algorithm 3) such that $(G, s, \Delta, B)$ is a proper input return the proper output, i.e. a price function $\phi$ satisfying properties in Definition 5.1.

Observe that Lemma 6.4 follows from $\mathcal{A}'(n^2)$ since $(G_{in}, s, \Delta = n^2, B = W_{G_{in}})$ is a proper output and $(G_{in}, s)$ is $n^2$-negative when $G_{in}$ contains no negative-weight cycle. For our induction, the base case is $\mathcal{A}'(0)$: SPmain obviously returns a proper output when $G$ is 0-negative by calling Dijkstra on Line 1 of Algorithm 3. Below, we divide our proof into two parts:

- Part 1: We prove that for any $\Delta > 0$, if $\mathcal{B}'(\Delta)$ holds, then $\mathcal{A}'(\Delta)$ also holds.

- Part 2: We prove that for any $\Delta \geq 0$, if $\mathcal{A}'(\lceil \Delta / n^2 \rceil)$ holds, then $\mathcal{B}'(\Delta)$ also holds.

By a simple induction, we can argue that $\mathcal{A}'(\Delta)$ and $\mathcal{B}'(\Delta)$ hold for any $\Delta \geq 0$, proving Lemma 6.4.

Let us start with the following simple observation.

**Observation 6.10.** If the input graph $G_{in}$ to the top-level recursive call contains no negative-weight cycle, then all input graphs in all lower-level calls (Line 7 of Algorithm 3 and Line 13 of Algorithm 4) also contain no negative-weight cycle.

**Proof.** The recursive calls made by SPmain($G, \ldots$) have as input a graph $G_\phi$ for some price function $\phi$, and by Lemma 2.5, if $G$ does not contain a negative-weight cycle then neither does $G_\phi$. Similarly, the recursive call made by ScaleDown($G, \ldots$) is on an input graph $G_\phi$, where edge-weights in $G$ are larger than in $G$, so again if $G$ does not contain a negative-weight cycle then neither does $G_\phi$. \[\square\]

So, in the rest of this subsection, we assume that all graphs $G$ given to all calls of SPmain and ScaleDown contain no negative cycle.

**Part 1: For any $\Delta > 0$, if $\mathcal{B}'(\Delta)$ holds, then $\mathcal{A}'(\Delta)$ also holds.** Throughout this part, fix any $\Delta > 0$ and assume that $\mathcal{B}'(\Delta)$ holds. Consider any call of SPmain($G, s, \Delta, B$) with proper input $(G, s, \Delta, B)$. Our main task is to show that all calls of ScaleDown($G_{\phi_{i-1}}, s, \Delta, B/2^{i-1}$) on Line 7 of Algorithm 3 do not lead to errors, which can be brute-force checked in the proof below.

**Invariant 6.11.** For every $1 \leq i \leq \log_2 B$, calling ScaleDown($G_{\phi_{i-1}}, s, \Delta, B/2^{i-1}$) on Line 7 of Algorithm 3 results in $\psi_i$ and $\phi_i = \phi_{i-1} + \psi_i$ such that (a) $w_{\phi_i}(e) \geq -B/2^i$ for all $e \in E$, (b) $s$ is a canonical source in $G_{\phi_i}$, and (c) $(G_{\phi_i}, s)$ is $\Delta$-negative.

**Proof sketch.** (a)-(c) hold for $i = 0$ by the property of $(G, s, \Delta, B)$. (Note that if $(G, s)$ is $\Delta$-negative, then so is $(G, s)$.) For any $1 \leq i \leq \log_2 (B)$, assume that (a)-(c) hold for $i - 1$, i.e. calling ScaleDown($G_{\phi_{i-2}}, s, \Delta, B/2^{i-2}$) gives $\psi_{i-1}$ and $\phi_{i-1}$ such that

\(^8\)The good random string assumption is required only in Lemma 6.14.
(a') \( w_{\phi_{i-1}}(e) \geq -B/2^{i-1} \) for all \( e \in E \), (b') \( s \) is a canonical source in \( G_{\phi_{i-1}} \), and (c') 
\( (G_{\phi_{i-1}}, s) \) is \( \Delta \)-negative.

Because of (a') and (b') (and since \( B/2^{i-1} \geq 2 \) and \( \Delta \leq n^2 \)), \( \text{ScaleDown}(G_{\phi_{i-1}}, s, \Delta, B/2^{i-1}) \) called on Line 7 of Algorithm 3 gets a proper input. This together with (c') allow us to apply \( B'(\Delta) \) to conclude that \( \text{ScaleDown}(G_{\phi_{i-1}}, s, \Delta, B/2^{i-1}) \) returns an output that satisfies Items 1 to 3 in Definition 5.1. Items 1, 2 and 3 in Definition 5.1 imply (a), (b), and (c) respectively (note that \( (G_{\phi_{i-1}}, \psi_i) = G_{\phi_{i-1}+\psi_i} = G_{\phi_i} \)). Thus (a)-(c) hold as well. The invariant follows by induction. \( \square \)

The invariant implies that calling \( \text{ScaleDown} \) on Line 7 always results in proper outputs and not errors. In other words, the for-loop in Algorithm 3 does not lead to an error. This implies that Algorithm 3 does not return an error. By Lemma 5.2, Algorithm 3 returns a proper output. Thus, \( \mathcal{A}'(\Delta) \) holds.

**Part 2:** For any \( \Delta \geq 0 \), if \( \mathcal{A}'([\Delta/n^\epsilon]) \) holds, then \( \mathcal{B}'(\Delta) \) also holds. Throughout this part, fix any \( \Delta \geq 0 \) and assume that \( \mathcal{A}'([\Delta/n^\epsilon]) \) holds. First, note that when \( \text{LowDiamDecomposition} \) is called on Line 3, its input conditions (Lemma 3.1) are satisfied since edge weights in \( G_{\geq 0} \) are all non-negative. Now we analyze each phase of Algorithm 4.

**Phase 1:** Recall that in Phase 1, we make edges inside SCCs of \( \hat{G}[V_1], \hat{G}[V_2], \ldots \) non-negative. We show that Phase 1 terminates with errors only when \( G \) contains a negative-weight cycle. Note that \( \text{SPWithFewNegEdges} \) called on Line 7 only requires that \( s_i \) is canonical, which is the case by the construction in the previous line. Now we prove one of our key claims:

**Lemma 6.12.** If \( G \) contains no negative-weight cycle, then \( \text{SPWithFewNegEdges}(\hat{G}[V_i], s_i^*, 4d) \) always return \( \hat{\text{dist}}(s_i^*, v) = \hat{\text{dist}}_{\hat{G}[V_i]}(s_i^*, v) \) for all \( i \).

**Proof.** Assume that for some \( i \), \( \hat{\text{dist}}(s_i^*, v) \neq \hat{\text{dist}}_{\hat{G}[V_i]}(s_i^*, v) \) for some \( v \in V \). This means that \( v \) is not \( (s, 4d) \)-negative in \( \hat{G}[V_i] \) (Definition 2.3), i.e. the shortest \( s^*v \)-path \( P = (u_0 = s_i^*, u_1, u_2, \ldots, u_\ell = v) \) contains more than \( 4d \) negative-weight edges in \( \hat{G}[V_i] \). We have

\[
\text{dist}_{\hat{G}}(u_1, v) < \hat{w}(P) - (4d)[B/2] \leq -((4d)[B/2] \leq -2d(B - 1) \leq -dB \tag{10}
\]

where the first inequality is because we obtain \( \hat{G} \) by adding \( [B/2] \) to negative-weight edges in \( G \) (Line 2) and edge \((s_i^*, u_1)\) has weight zero in \( \hat{G}[V_i] \), the second inequality is because \( \hat{\text{dist}}_{\hat{G}[V_i]}(s_i^*, v) \leq 0 \) (because \((s_i^*, u_1)\) has weight zero), and the last inequality holds when \( B \geq 2 \). We also have

\[
\text{dist}_{\hat{G}}(v, u_1) \leq \text{dist}_{\hat{G}}(u, u_1) \leq \text{dist}_{\hat{G}_{\geq 0}} \leq dB \tag{11}
\]

where the last inequality is because \( \text{LowDiamDecomposition}(\hat{G}_{\geq 0}, dB) \) (called on Line 3; Lemma 3.1) always guarantee that \( \text{dist}_{\hat{G}_{\geq 0}}(v, u_1) \leq dB \) (since \( u_1 \) and \( v \) are in the same \( V_i \)). Equations (10) and (11) imply that \( G \) contains a negative-weight cycle, a contradiction. This proves Lemma 6.12. \( \square \)

Lemma 6.12 implies that \( \hat{w}_{\phi_i}(u, v) \geq 0 \) for all \( u, v \in V_i \) for all \( i \) (by Lemma 2.6).\(^9\) Thus, Phase 1 terminates with errors only when \( G \) contains a negative-weight cycle.

**Phase 2:** Recall that in Phase 2, we aim to make all edges in \( \hat{G} \setminus E^{sep} \) non-negative. For this phase, we prove the following.

\(^9\)Note that we can apply Lemma 2.6 because \( s \) can reach all vertices in \( G \).
**Lemma 6.13.** After Phase 2, we get a price function \( \phi_2 \) such that (i) \( s \) is canonical in \( \hat{G}_{\phi_2} \), (ii) \( \hat{w}_{\phi_2}(e) \geq 0 \) for all \( e \notin E^{sep} \), and (iii) \( \hat{w}_{\phi_2}(e) \geq -3nB \) for all \( e \in E \).

**Proof.** We follow the algorithm on Lines 10 and 11 and tediously prove (i) and (ii). (The proof of (iii) is slightly more interesting.)

Line 10: When FixDAGEdges (Lemma 2.9) is called on Line 10, the first input condition in Lemma 2.9 holds because otherwise the algorithm would terminate on the previous line. The second condition holds because the partition \( \{V_1, V_2, \ldots \} \) was obtained by the SCC decomposition of \( \hat{G}_{\geq 0} \) (Line 3), and \( \hat{G}_{\geq 0} \) and \( \hat{G} \) share the same underlying unweighted graph (when weights are ignored). Thus, by Lemma 2.9, we get a price function \( \psi \) such that all edge weights in the SCCs of \( (\hat{G} \setminus E^{sep})_{\phi_1} = (G \setminus E^{sep})_{\phi_1 + \psi} \) are non-negative.

Line 11: Define \( H = (\hat{G} \setminus E^{sep})_{\phi_1 + \psi} \). Since all edge weights in \( H \) are non-negative, Dijkstra(\( H, s \)) on Line 11 returns a price function \( \psi' \) such that \( \psi'(v) = \text{dist}_H(s, v) \) for all \( v \in V \). It follows that (a) \( \text{dist}_{H_{\phi_2}}(s, v) = 0 \) for all \( v \in V \) and (b) all edge weights are non-negative in \( H_{\psi'} = (\hat{G} \setminus E^{sep})_{\phi_2} \) (by Lemma 2.6). Item (a) implies (i) in the lemma statement since adding edges only makes the distance smaller. Item (b) implies (ii).

It remains to show (iii). For every \( v \in V \), since
\[
\psi'(v) = \text{dist}_H(s, v) = \text{dist}_{\hat{G}}(s, v) + \phi_1(s) + \psi(s) - \phi_1(v) - \psi(v),
\]
we have
\[
\phi_2(v) = \phi_1(v) + \psi(v) + \psi'(v) = \text{dist}_{\hat{G}}(s, v) + \phi_1(s) + \psi(s).
\]
So,
\[
w(u, v) = \hat{w}(u, v) + \phi_2(u) - \phi_2(v) = \hat{w}(u, v) + \text{dist}_{\hat{G}}(s, u) - \text{dist}_{\hat{G}}(s, v) \geq -3nB
\]
where the last inequality is because \( \hat{w}(u, v) \geq -B \), \( \text{dist}_{\hat{G}}(s, u) \geq -nB \) and \( \text{dist}_{\hat{G}}(s, v) \leq \text{dist}_{\hat{G}}(s, v) + Bn/2 \leq Bn/2 \) (because \( s \) is canonical in \( G \), i.e. \( \text{dist}_{\hat{G}}(s, v) \leq 0 \)). This concludes (iii). \( \square \)

Phase 3: Recall that in Phase 3, we aim to make edges in \( \hat{G} \) non-negative. Lemma 6.14 below is the key that makes Phase 3 and the whole algorithm return the correct output. Note that this is the only place we require a good random string \( r \).

**Lemma 6.14.** Assuming that the random string \( r \) is good and \( (G, s) \) is \( \Delta \)-negative, \( (\hat{G}_{\phi_2}, s) \) is \( \lfloor \Delta/n^{\epsilon} \rfloor \)-negative.

**Proof.** Let \( P(v; \hat{G}, s) \) be a shortest \( sv \)-path in \( \hat{G} \) as defined in Definition 2.3. Since \( (G, s) \) is \( \Delta \)-negative, we have
\[
\sum_{v \in V} \left| E_{\hat{G}_{\phi_2}}^{\text{neg}}(P(v; \hat{G}, s)) \right| \leq \sum_{v \in V} \left| P(v; \hat{G}, s) \cap E^{\text{sep}} \right| \leq \frac{\Delta}{n^{\epsilon}}
\]
where the first inequality is because of Lemma 6.13 (ii) (i.e. only edges in \( E^{\text{sep}} \) can be of negative weight in \( G_{\phi_2} \)), and the second inequality is exactly (4), which holds because \( r \) is good and \( (G, s, \Delta, B) \) is a proper input. Lemma 6.14 follows. \( \square \)

Lemmas 6.13 and 6.14 together imply that, when \( (G, s) \) is \( \Delta \)-negative, SPmain(\( \hat{G}_{\phi_2}, s, \lfloor \Delta/n^{\epsilon} \rfloor, 3nB \)) is called on Line 13 with a proper input (Definition 5.3). Consequently, we can apply \( \mathcal{A}'([\Delta/n^{\epsilon}]) \) to conclude that SPmain(\( \hat{G}_{\phi_2}, s, \lfloor \Delta/n^{\epsilon} \rfloor, 3nB \)) returns a proper output. In other words, ScaleDown(\( G, s, \Delta, B \)) does not return an error message. Thus, by Lemma 5.2, it returns a proper output. So, \( \mathcal{B}(\Delta) \) holds, concluding Part 2.
7 Algorithm for Low-Diameter Decomposition

In this section, we prove Lemma 3.1 which we restate here for convenience:

Lemma 3.1. There is an algorithm LowDiamDecomposition(G, D) with the following guarantees:

- **INPUT:** an m-edge, n-vertex graph \( G = (V, E, w) \) with non-negative integer edge weight function \( w \) and a positive integer \( D \).
- **OUTPUT:** A set of edges \( E^{sep} \) with the following guarantees:
  - Each SCC of \( G - E^{sep} \) has weak diameter at most \( D \); that is, if \( u, v \) are in the same SCC, then \( \text{dist}_G(u, v) \leq D \) and \( \text{dist}_G(v, u) \leq D \).
  - For every \( e \in E \), \( \Pr[e \in E^{sep}] = O(w(e) \cdot (\log n)/D + n^{-10}) \). These probabilities are not guaranteed to be independent.
- **RUNNING TIME:** The algorithm has running time \( \tilde{O}(m + n) \)

7.1 Notation

For a directed graph \( H = (V_H, E_H, w_H) \) with non-negative integer edge weight function \( w_H \), we let \( H_{rev} \) denote \( H \) with all edge directions reversed but with edge weights kept. We will identify vertices in \( H \) with their copies in \( H_{rev} \) and edges in \( H \) with their reverse counterparts in \( H_{rev} \). Given any subset \( V' \subseteq V \) we define \( G[V'] \) to be the induced graph on \( V' \). We define \( G - V' \) to be \( G[V - V'] \).

Denote by \( \deg_H(v) \) the in- plus out-degree in \( H \) of a vertex \( v \in V_H \). For a vertex \( s \in V_H \) and an integer \( r \geq 0 \), we introduce the following notation:

- \( V_H(s, r) = \{v \in V_H | d_H(s, v) \leq r\} \),
- \( B_H(s, r) = H[V_H(s, r)] \),
- \( n_H(s, r) = |V_H(s, r)| \),
- \( b_H(s, r) = 1 + \sum_{v \in V_H(s, r)} \deg_H(v) \),
- \( L_H(s, r) = \{(u, v) \in E_H | u \in V_H(s, r) \land v \notin V_H(s, r)\} \).

In words, \( B_H(s, r) \) is the ball in \( H \) centered at \( s \) and having radius \( r \) and we refer to \( L_H(s, r) \) as a layer. Note that if \( R \) is the largest finite distance from \( s \) to a vertex of \( V_H \), the above definitions are still valid for \( r > R \); for instance \( L_H(s, r) = \emptyset \) for such \( r \).

Note that Dijkstra’s algorithm can compute \( B_H(s, r) \) and \( L_H(s, r) \) in \( \tilde{O}(b_H(s, r)) \) time; the added 1 in the definition of \( b_H(s, r) \) deals with the special case where \( u \) has no incident edges, in which case the algorithm still needs to spend constant time.

Define a canonical range to be a half-open interval of the form \( [2^j, 2^{j+1}) \) where \( j \) is a non-negative integer.

Consider a coin whose probability of heads is \( p \in (0, 1] \). The geometric distribution \( \text{Geo}(p) \) is the probability distribution of the number \( X \) of independent coin tosses until obtaining the first heads. We have \( \Pr[X = k] = p(1 - p)^{k-1} \) for every \( k \in \mathbb{N} \).

LowDiamDecomposition will randomly sample BFS layers from a geometric distribution which can be thought of as the discrete version of an exponential distribution that previous low-diameter decomposition algorithms rely on. We find the geometric distribution to be more natural to use and it has the advantage of being easy to sample from. In fact, the most basic sampling algorithm will
Lemma 7.1. We obtain by repeatedly sampling layers randomly from a geometric distribution and then recursing.

The generic algorithm is an efficient implementation of a generic algorithm which we present in the following lemma. The generic algorithm is more of a method than an algorithm as it does not specify the exact order of steps nor how they are implemented. Instead, it specifies conditions that need to hold when executing each step. These conditions are essentially that $E^{\text{sep}}$ is obtained by repeatedly sampling layers randomly from a geometric distribution and then recursing on the two subgraphs induced by each such layer:

**Lemma 7.2.** Let $r$ and $c$ be positive integers. The generic algorithm takes as input an $n$-vertex graph $G = (V, E, w)$ with non-negative integer edge weight function $w$, and it outputs a subset $E^{\text{sep}} \subseteq E$. Initially, $E^{\text{sep}} = \emptyset$ and the generic algorithm then grows $E^{\text{sep}}$ by any number of executions of the following sequence of steps:

1. pick a vertex $s$ and an integer $i_{\max} \geq r$ such that either $n_G(s, i_{\max}) \leq \frac{2n}{3}$ or $n_{G_{\text{rev}}}(s, i_{\max}) \leq \frac{2n}{3}$; let $i_{\min} = i_{\max} - r \geq 0$.
2. pick an integer $i_{\text{rand}} = i_{\min} + \min\{i, r\}$ where $\tilde{i} \sim \text{Geo}(\lambda)$ and $\lambda = 2c(\ln n)/r$.
3. execute exactly one of the following (note that at least one of them is applicable):
   - **Case 1:** if $n_G(s, i_{\max}) \leq \frac{2n}{3}$ then add $L_G(s, i_{\text{rand}})$ to $E^{\text{sep}}$ and recurse on $B_G(s, i_{\text{rand}})$ and on $G - V_G(s, i_{\text{rand}})$.
   - **Case 2:** if $n_{G_{\text{rev}}}(s, i_{\max}) \leq \frac{2n}{3}$ then add $L_{G_{\text{rev}}}(s, i_{\text{rand}})$ to $E^{\text{sep}}$ and recurse on $B_{G_{\text{rev}}}(s, i_{\text{rand}})$ and on $G - V_{G_{\text{rev}}}(s, i_{\text{rand}})$.

Then at termination, $\Pr[e \in E^{\text{sep}}] \leq O(cw(e)(\log n)^2/r + n^{-c})$ for every $e \in E$.

To prove this lemma, we need the following result which bounds the probability of including an edge in a random layer chosen by the generic algorithm in a particular iteration. Note that if $w(e) > r/2 - 1$ or $\lambda > \frac{1}{2}$ then the probability bound of the above lemma automatically holds, so in the lemma below, we focus on the case where $w(e) \leq r/2 - 1$ and $\lambda \leq \frac{1}{2}$.

**Lemma 7.2.** Let $G$ be an $m$-edge, $n$-vertex graph with non-negative integer edge weight function $w$, let $s \in V$, let $r$ and $i_{\min}$ be positive integers, let $c > 0$, and assume that $\lambda = 2c(\ln n)/r \leq \frac{1}{2}$. Let $\tilde{i} \sim \text{Geo}(\lambda)$ and let $i_{\text{rand}} = i_{\min} + \min\{i, r\}$. Then for every $e = (u, v) \in E$ of weight $w(e) \leq r/2 - 1$,

$$u \in V_G(s, i_{\min} + r - w(e)) \Rightarrow \Pr[v \notin V_G(s, i_{\text{rand}}) | u \in V_G(s, i_{\text{rand}})] \leq \frac{cw(e) \ln n}{r}$$

$$u \notin V_G(s, i_{\min} + r - w(e)) \Rightarrow \Pr[u \in V_G(s, i_{\text{rand}})] \leq n^{-c}.$$
High-level idea for Lemma 7.2 Before proving the lemma, let us describe in words what it expresses. The random number \( i_{\text{rnd}} \) corresponds to the layer \( L_G(s,i_{\text{rnd}}) \) chosen by the generic algorithm, which is exactly the set of edges added to \( E^{\text{sep}} \). The bad event that an edge \( e = (u,v) \) belongs to this layer is exactly the event that \( u \in V_G(s,i_{\text{rnd}}) \) and \( v \notin V_G(s,i_{\text{rnd}}) \). Since the probabilities on the right-hand sides of the two implications are both upper bounds on the probability of this event, the lemma thus bounds the probability that \( e \in L_G(s,i_{\text{rnd}}) \).

Note that the upper bound on the conditional probability is much weaker than the \( n^{-c} \) bound on the second probability. As we show later, the type of conditional event \( u \in V_G(s,i_{\text{rnd}}) \) can only happen a logarithmic number of times during the course of the generic algorithm and this offsets the weaker upper bound of the first implication of the lemma.

Let us sketch why the first implication holds for the case \( w(e) = 1 \). Conditioning on \( u \in V_G(s,i_{\text{rnd}}) \), the event \( v \notin V_G(s,i_{\text{rnd}}) \) can only happen if \( d_G(s,u) \leq i_{\text{rnd}} \) and \( d_G(s,v) > i_{\text{rnd}} \) and hence \( d_G(s,u) = i_{\text{rnd}} \) and \( d_G(s,v) = i_{\text{rnd}} + 1 \) (since \( w(e) = 1 \Rightarrow d_G(s,v) \leq d_G(s,u) + 1 \)). Suppose we had defined \( i_{\text{rnd}} \) slightly differently as \( i_{\text{rnd}} = \min \{i, r\} \) with \( \tilde{r} \sim \text{Geo}(\lambda) \). Then we can use the memorylessness property of \( \text{Geo}(\lambda) \) which implies that the probability of a random variable with this distribution having value \( i \) conditioned on having value at least \( \tilde{r} \) is \( \lambda \). In particular, with \( i = d_G(s,u) - i_{\text{min}} \), the conditional probability of the lemma is at most

\[
\Pr[i_{\text{rnd}} = d_G(s,u) | i_{\text{rnd}} \geq d_G(s,u)] = \Pr[\tilde{i} = i | \tilde{i} \geq \tilde{i}] = \lambda = (2c \ln n)/r,
\]
as desired.

Unfortunately, this analysis fails when using the actual definition \( i_{\text{rnd}} = i_{\text{min}} + \min \{\tilde{i}, r\} \) of the lemma which gives rise to a "truncated" geometric distribution with no probability mass for values above \( r \). Suppose that \( d_G(s,u) = i_{\text{max}} = i_{\text{min}} + r \). Then the event \( u \in V_G(s,i_{\text{rnd}}) \) implies \( i_{\text{max}} \leq i_{\text{min}} + r \) and hence \( i_{\text{rnd}} = i_{\text{max}} \); in this case, the choice of \( i_{\text{rnd}} \) is deterministic and the conditional probability is 1 if \( d_G(s,v) = i_{\text{max}} + 1 \). The inequality on the right-hand side of the implication would then be false for sufficiently large \( r \). However, this is where we rely on our assumption \( u \in V_G(s,i_{\text{min}} + r - w(e)) \) which is equivalent to \( u \in V_G(s,i_{\text{max}} - w(e)) \). This effectively allows us to ignore the truncation and apply the analysis sketched above for the modified definition of \( i_{\text{rnd}} \).

For the second implication of the lemma, the assumption \( u \notin V_G(s,i_{\text{min}} + r - w(e)) \) implies that \( u \notin V_G(s,i_{\text{min}} + r/2) \); here we use that Lemma 7.2 assumes \( w(e) \leq r/2 - 1 \). In other words, \( u \) is far away from \( s \) and a tail bound on \( \text{Geo}(\lambda) \) then gives a small probability that \( \tilde{i} \) is large enough for the event \( u \in V_G(s,i_{\text{rnd}}) \) to occur.

We now give the formal proof of the lemma.

**Proof of Lemma 7.2.** Let \( e = (u,v) \in E \). Pick \( \ell_u \) and \( \ell_v \) such that \( d_G(s,u) = i_{\text{min}} + \ell_u \) and \( d_G(s,v) = i_{\text{min}} + \ell_v \). Note that \( \ell_v \leq \ell_u + w(e) \).

We first focus on the first property of the lemma and assume \( u \in V_G(s,i_{\text{min}} + r - w(e)) \). Then \( i_{\text{min}} + \ell_u \leq i_{\text{min}} + r - w(e) \) and hence \( \ell_u + w(e) \leq r \). We have

\[
v \in V_G(s,i_{\text{rnd}}) \iff i_{\text{min}} + \ell_v \leq i_{\text{rnd}} = i_{\text{min}} + \min \{\tilde{i}, r\} \iff \ell_v \leq \min \{\tilde{i}, r\} \iff \ell_v \leq \tilde{i},
\]

where the last step follows from \( \ell_v \leq \ell_u + w(e) \leq r \). Since \( w(e) \geq 0 \), we have \( \ell_u \leq r - w(e) \leq r \) and hence

\[
u \in V_G(s,i_{\text{rnd}}) \iff i_{\text{min}} + \ell_u \leq i_{\text{min}} + \min \{\tilde{i}, r\} \iff \ell_u \leq \min \{\tilde{i}, r\} \iff \ell_u \leq i_{\text{min}}.
\]

Since \( \ell_v - \ell_u \leq w(e) \), the memorylessness property of the geometric distribution now gives

\[
\Pr[v \in V_G(s,i_{\text{rnd}}) | u \in V_G(s,i_{\text{rnd}})] = \Pr[\tilde{i} \geq \ell_v | \tilde{i} \geq \ell_u] = \Pr[\tilde{i} \geq \ell_v - \ell_u] \geq \Pr[\tilde{i} \geq w(e)] \geq (1 - \lambda)^{w(e)}.
\]
Since \( \lambda < \frac{1}{2} \), we have \( 1 - \lambda \geq e^{-\lambda/(1-\lambda)} > e^{-\lambda/2} \) and hence \( (1 - \lambda)^w(e) > e^{-w(e)\lambda/2} \).

Using \( 1 + x \leq e^x \) for all \( x \in \mathbb{R} \) gives

\[
\Pr[v \notin V_G(s, i_{\text{rnd}})|u \in V_G(s, i_{\text{rnd}})] < 1 - e^{-w(e)\lambda/2} \leq w(e)\lambda/2 = \frac{cw(e)\ln n}{r},
\]

as desired.

It remains to consider the case \( u \notin V_G(s, i_{\text{min}} + r - w(e)) \). This is equivalent to \( i_{\text{min}} + \ell_u > i_{\text{min}} + r - w(e) \) which again is equivalent to \( \ell_u > r - w(e) \). Since \( (w(e) + 1)/r \leq \frac{1}{2} \),

\[
\Pr[u \in V_G(s, i_{\text{rnd}})] = \Pr[i_{\text{min}} + \ell_u \leq i_{\text{rnd}}] = \Pr[\ell_u \leq \min \{i, r\}] \leq \Pr[\ell_u \leq i] < \Pr[r - w(e) \leq i] = (1 - \lambda)^{(r - (w(e) + 1))} \leq e^{-(r - (w(e) + 1))\lambda} = e^{(2c\ln n)(1 - (w(e) + 1)/r)} \leq n^{-c},
\]

as desired. \( \square \)

We are now ready to prove Lemma 7.1.

Since \( c \) is arbitrary, it suffices to show a probability bound of \( O(cw(e)(\log n)^2/r + n^{1-c}) \) for each edge since replacing \( c \) by \( c+1 \) will then give the desired bound of \( O((c+1)w(e)(\log n)^2/r + n^{1-(c+1)}) = O(cw(e)(\log n)^2/r + n^{-c}) \).

Since \( E^\text{sep} \) is monotonically growing over time, any vertex \( x \) will belong to SCCs of \( G - E^\text{sep} \) of monotonically decreasing vertex size during the course of the algorithm. Define the level \( \ell(x) \in \{0, 1, \ldots, \log_{3/2} n\} \) of \( x \) to be the value such that \( x \) belongs to an SCC of \( G - E^\text{sep} \) of vertex size in \((3/2)^{\ell(x)}, (3/2)^{\ell(x)+1}\). Thus, \( \ell(x) \) is monotonically decreasing over time.

To show the lemma, let \( e = (u, v) \in E \) be given. Note that the probability bound for \( e \) is trivial if \( w(e) > r/2 \) so assume \( w(e) \leq r/2 \) in the following. This will allow us to apply Lemma 7.2.

Let us first fix our attention to a single execution of the sequence of steps in the generic algorithm where \( n_G(s, i_{\text{max}}) \leq \frac{2n}{3} \), i.e., where Case 1 is executed. Then the event that \( e \) is added to \( E^\text{sep} \) occurs exactly when \( u \in V_G(s, i_{\text{rnd}}) \) and \( v \notin V_G(s, i_{\text{rnd}}) \). Since \( n_G(s, i_{\text{rnd}}) \leq n_G(s, i_{\text{max}}) \leq \frac{2n}{3} \), the event that \( u \in V_G(s, i_{\text{rnd}}) \) implies the event that \( \ell(u) \) decreases by at least 1.

It follows that during all executions of Case 1 by the algorithm, the event that \( e \) belongs to a set of the form \( V_G(s, i_{\text{rnd}}) \) occurs no more than \( \log_{3/2} n \) times. Furthermore, the total number of executions of Case 1 is at most \( n - 1 \); this is because every execution of Case 1 splits the vertex set into two disjoint non-empty sets, so it increases the total number of sets by 1, and at the end there are at most \( n \) sets. Since \( w(e) \leq r/2 - 1 \) and \( \lambda \leq \frac{1}{2} \), Lemma 7.2 and a union bound then imply that the probability that \( e \) is added to \( E^\text{sep} \) during all executions of Case 1 is at most

\[
\frac{2cw(e)\ln n}{r} \log_{3/2} n + (n - 1) \cdot n^{-c} = O(cw(e)(\log n)^2/r + n^{-c+1}).
\]

By replacing \( u \) with \( v \) in the above analysis, we get the same bound on the probability that \( e \) is added to \( E^\text{sep} \) over all executions of Case 2. A union bound over the two cases now completes the proof of Lemma 7.1.
7.3 Growing balls in parallel

As mentioned earlier, LowDiamDecomposition will be an efficient implementation of the generic algorithm. A major challenge is that the two subgraphs that the generic algorithm recurses on (see Cases 1 and 2 in Lemma 7.1) do not necessarily constitute a balanced split of \( G \). To get a near-linear time bound, LowDiamDecomposition thus needs to identify the smaller of the two subgraphs in time roughly proportional to its size. Let us consider this for Case 1 of the generic algorithm. Since \( n_G(s, i_{\text{rnd}}) \leq n_G(s, i_{\text{max}}) \leq \frac{2n}{3} \), the time spent on identifying \( B_G(s, i_{\text{rnd}}) \) should be \( \Theta(b_G(s, i_{\text{rnd}})) \). However, as will become clear later, obtaining \( B_G(s, i_{\text{max}}) \) and \( n_G(s, i_{\text{max}}) \). We will deal with this by ensuring that \( b_G(s, i_{\text{rnd}}) = \Theta(b_G(s, i_{\text{max}})) \) so that the generic algorithm can afford to compute \( B_G(s, i_{\text{max}}) \) and \( n_G(s, i_{\text{max}}) \). Case 2 is similar if we replace \( G \) by \( G_{\text{rev}} \).

In both cases, LowDiamDecomposition can use Dijkstra’s algorithm to grow the ball centered at \( s \) up to any chosen radius \( i_{\text{max}} \) in time near-linear in the size of the ball. The issue is that LowDiamDecomposition needs to know which of the two cases in the generic algorithm is applicable before growing a ball of too large size for a case that does not apply. The solution is essentially to grow balls from \( s \) in \( G \) and in \( G_{\text{rev}} \) in parallel and then stop as soon as Case 1 or Case 2 applies.

Growing balls in parallel will be done by a deterministic algorithm that we describe in Corollary 7.4 below. We first need the following lemma. It either certifies \( f \) or a center vertex \( s \) of large vertex size and small radius or that there is a large range such that all balls centered at \( s \) with radii in this range have roughly the same \( b_G \)-value. The latter is useful since we need \( b_G(s, i_{\text{rnd}}) = \Theta(b_G(s, i_{\text{max}})) \) as described above.

Lemma 7.3. There is a deterministic algorithm \( \text{LayerRange}(G, s, D) \) with the following guarantees

- **INPUT:** an \( n \)-vertex graph \( G = (V, E, w) \) with non-negative integer weights, a vertex \( s \in V \), and an integer \( D \geq 15 \lg n \)
- **OUTPUT:** a pair \((\text{Condition}, i)\) where \( \text{Condition} \in \{1, 2\} \) and \( i \leq D \) is a non-negative integer such that
  - if \( \text{Condition} = 1 \) then \( n_G(s, i) > \frac{2n}{3} \),
  - if \( \text{Condition} = 2 \) then \( n_G(s, i) \leq \frac{2n}{3} \) and \( i \geq D/(3\lg n) \); moreover, \( b_G(s, i) \) and \( b_G(s, i - \lceil D/(3\lg n) \rceil) \) are in the same canonical range.
- **RUNNING TIME:** The algorithm has running time \( \tilde{O}(b_G(s, i)) \) which is \( \tilde{O}(b_G(s, i - \lceil D/(3\lg n) \rceil)) \) if \( \text{Condition} = 2 \).

Proof. \( \text{LayerRange} \) grows a shortest path tree from \( s \) using Dijkstra’s algorithm. Let \( i \) be the first integer found satisfying one of the following conditions:

- \( n_G(s, i) > \frac{2n}{3} \),
- \( n_G(s, i) \leq \frac{2n}{3} \), \( i \geq D/(3\lg n) \), and \( b_G(s, i - \lceil D/(3\lg n) \rceil) \) and \( b_G(s, i) \) are in the same canonical range.

In the first case, \( \text{LayerRange} \) returns \((1, i)\), in the second case, it returns \((2, i)\).

For the correctness proof, it suffices to consider the case \( n > 144 \) and to show that the algorithm terminates with \( i \leq D \). Assume for contradiction that this does not happen. Then for any integer
\[ x \in [1, \frac{5}{2} \log n - 1], \quad b_G(s, (x-1)[D/(3 \log n)]) \text{ and } b_G(s, x[D/(3 \log n)]) \text{ are in distinct canonical ranges; here we used that} \]
\[ 1 \leq \frac{D}{15 \log n} \Rightarrow x[D/(3 \log n)] \leq \left( \frac{5}{2} \log n - 1 \right) \left( \frac{D}{3 \log n} + 1 \right) \leq \left( \frac{5}{2} \log n - 1 \right) \frac{2D}{5 \log n} < D. \]

It follows that \( b_G(s, x[D/(3 \log n)]) > 2^x \) for \( x = 1, 2, \frac{5}{2} \log n - 1 \). The biggest of these \( b_G \)-values thus exceeds \( 2^{\frac{5}{2} \log n - 1} \geq \frac{1}{4} n^{5/2} \), contradicting that no ball has \( b_G \)-value greater than \( 1 + 2|E| \leq 2n^2 \) which is less than \( \frac{1}{4} n^{5/2} \) since \( n > 144 \Rightarrow \frac{1 + 2n^2}{4n^2} < \frac{3n^2}{4n^2} = 12 < \sqrt{n} \Rightarrow 1 + 2n^2 < \frac{1}{4} n^{5/2} \).

We conclude that the algorithm terminates with \( i \leq D \), as desired.

The time bound is clear if \( b_G(s, i) > \frac{2n}{3} \). In the other case, the time bound follows since \( b_G(s, i - [D/(3 \log n)]) \) and \( b_G(s, i) \) are in the same canonical range. \( \square \)

The following corollary is now easily obtained by running \( \text{LayerRange} \) in parallel from \( s \) in \( G \) and in \( G_{rev} \).

**Corollary 7.4.** There exists a deterministic algorithm \( \text{CoreOrLayerRange}(G, s, D) \) with the following guarantees:

- **INPUT:** an \( n \)-vertex graph \( G = (V, E, w) \) with non-negative integer weights, a vertex \( s \in V \), and an integer \( D \geq 15 \log n \)
- **OUTPUT:** a pair \((\text{Condition}, i)\) where \( \text{Condition} \in \{1, 2, 3\} \) and \( i \leq D \) is a non-negative integer such that
  - if \( \text{Condition} = 1 \) then \( n_G(s, i) > \frac{2n}{3} \) and \( n_{G_{rev}}(s, i) > \frac{2n}{3} \),
  - if \( \text{Condition} = 2 \) then \( n_G(s, i) \leq \frac{2n}{3} \) and \( b_G(s, i) \) and \( b_G(s, i - [D/(3 \log n)]) \) are in the same canonical range,
  - if \( \text{Condition} = 3 \) then \( n_{G_{rev}}(s, i) \leq \frac{2n}{3} \) and \( b_{G_{rev}}(s, i) \) and \( b_{G_{rev}}(s, i - [D/(3 \log n)]) \) are in the same canonical range.

If \( \text{Condition} \in \{2, 3\} \) then \( i \geq D/(3 \log n) \).

- **RUNNING TIME:** If \( \text{Condition} = 1 \), the algorithm runs in time \( \tilde{O}(m + n) \), if \( \text{Condition} = 2 \), it runs in time \( \tilde{O}(b_G(s, i)) = \tilde{O}(b_G(s, i - [D/(3 \log n)])) \), and if \( \text{Condition} = 3 \), it runs in time \( \tilde{O}(b_{G_{rev}}(s, i)) = \tilde{O}(b_{G_{rev}}(s, i - [D/(3 \log n)])) \).

**Proof.** Let \((C, j) = \text{LayerRange}(G, s, D)\) and let \((C_{rev}, j_{rev}) = \text{LayerRange}(G_{rev}, s, D)\).

\( \text{CoreOrLayerRange}(G, s, D) \) runs \( \text{LayerRange}(G, s, D) \) and \( \text{LayerRange}(G_{rev}, s, D) \) in parallel until one of them terminates. If \( \text{LayerRange}(G, s, D) \) terminated with \( C = 1 \), \( \text{CoreOrLayerRange} \) finishes executing \( \text{LayerRange}(G_{rev}, s, D) \). Similarly, if \( \text{LayerRange}(G_{rev}, s, D) \) terminated with \( C_{rev} = 1 \), \( \text{CoreOrLayerRange} \) finishes executing \( \text{LayerRange}(G, s, D) \).

If both \( \text{LayerRange}(G, s, D) \) and \( \text{LayerRange}(G_{rev}, s, D) \) were executed to completion and if \( C = C_{rev} = 1 \), \( \text{CoreOrLayerRange} \) outputs \((\text{Condition}, i) = (1, \max\{j, j_{rev}\})\).

Otherwise, at least one of the following two cases must hold:

- \( \text{LayerRange}(G, s, D) \) was executed to completion and \( C = 2 \),
• LayerRange($G_{rev}, s, D$) was executed to completion and $C_{rev} = 2$.

In the first case, CoreOrLayerRange returns (Condition, $i$) = (2, $j$) and in the second case, it returns (Condition, $i$) = (3, $j_{rev}$).

Correctness and the time bounds follow immediately from Lemma 7.3.

For a given source vertex $s$, LowDiamDecomposition needs to efficiently identify one of the two cases in a step of the generic algorithm. Comparing Lemma 7.1 and Corollary 7.4, we see that CoreOrLayerRange can be applied by LowDiamDecomposition, provided that Condition $\in \{2, 3\}$. The case where Condition $= 1$ is more tricky to deal with since neither case in the generic algorithm is satisfied. In this case, LowDiamDecomposition calls another algorithm, RandomTrim, which picks other source vertices for which CoreOrLayerRange is guaranteed to identify one of the two cases of the generic algorithm. This is described in more detail in the following subsection.

### 7.4 The LowDiamDecomposition algorithm

We are ready to present our algorithm LowDiamDecomposition. Algorithm 5 shows the pseudocode. We first give a high-level description. As we will show, this algorithm follows the steps of the generic algorithm. This can be seen from the inspection of the pseudocode when Condition $\in \{2, 3\}$.

When Condition $= 1$, neither of the two cases of the generic algorithm applies since $n_{G}(s, i_{\max}) > \frac{2n}{3}$ and $n_{rev}(s, i_{\max}) > \frac{2n}{3}$. In this case, a call is made to RandomTrim whose pseudocode can be seen in Algorithm 6. This algorithm exploits the fact that $n_{G}(s, i_{\max}) > \frac{2n}{3}$ and $n_{rev}(s, i_{\max}) > \frac{2n}{3}$ and hence that the size of the "core" $V_{G}(s, i_{\max}) \cap V_{rev}(s, i_{\max})$ is greater than $\frac{n}{3}$. RandomTrim starts by picking a vertex $v$ whose distance to $s$ or from $s$ is large. It then picks a random layer $i_{\text{rnd}}$ so that the ball centered at $v$ and with radius $i_{\text{rnd}}$ does not intersect the core which guarantees that the ball contains at most $\frac{4n}{3}$ vertices. Cutting this random layer thus executes a step of the generic algorithm. The "trimmed" ball is then partitioned with a recursive call to LowDiamDecomposition. The growing vertex set $M$ indicates the set of vertices trimmed so far and RandomTrim proceeds on the subgraph of $G$ induced by $V - M$. When trimming is no longer possible, it implies that all remaining vertices (i.e., vertices of $V - M$) have small distance to and from $s$ in $G$. The untrimmed part thus forms an SCC of low weak diameter and RandomTrim terminates since it has found a desired decomposition.

We now formally analyze LowDiamDecomposition in order to prove Lemma 3.1. In order to apply Corollary 7.4, we need to assume $D \geq 15\lg n$ in the following. Note that the case where $D < 15\lg n$ is trivial for Lemma 3.1 since we can simply let $E^{sep}$ be the set of all edges of $E$ of weight at least 1 in that case.

To make the pseudo-code and analysis simpler, we will only require LowDiamDecomposition to give a weak diameter upper bound of at most $4D$; to get the desired upper bound of $D$ in Lemma 3.1, simply call LowDiamDecomposition($G, \frac{D}{4}$).

We prove Lemma 3.1 (with weak diameter upper bound $4D$) by induction on $n$. Hence, for a graph of vertex size strictly smaller than $n$, we inductively assume in all of the following that LowDiamDecomposition satisfies the guarantees of Lemma 3.1.

We start by analyzing the auxiliary algorithm RandomTrim.

**Lemma 7.5.** There is an algorithm RandomTrim($G, s, D$) with the following guarantees

• **INPUT:** an $m$-edge, $n$-vertex graph $G = (V, E, w)$ with non-negative integer weights, a vertex $s \in V$, and a positive integer $D$ such that $n_{G}(s, D) > \frac{2n}{3}$ and $n_{rev}(s, D) > \frac{2n}{3}$.

• **OUTPUT:** a set of edges $E^{sep}$ with the following guarantees
every SCC of $G - E^{sep}$ has weak diameter at most $4D$,

- **RUNNING TIME:** The algorithm has running time $\tilde{O}(m + n)$,
- **RandomTrim** is an implementation of the generic algorithm.

**Proof.**

**Implementation of the generic algorithm:** We first show that RandomTrim is an implementation of the generic algorithm. From Lemma 7.1 and inspection of the pseudo-code of RandomTrim, it suffices to show the following properties:

1. if $d_{T_{in}}(s, v) > 2D$ in line 11 then $n_{G[V-M]}(v, i_{\text{max}}) \leq \frac{2|V-M|}{3}$,

2. if $d_{T_{in}}(s, v) \leq 2D$ in line 11 then $n_{G[V-M]rev}(v, i_{\text{max}}) \leq \frac{2|V-M|}{3}$ in line 14.

Note that if $d_{T_{in}}(s, v) \leq 2D$ then since $v \in V_{\text{far}}$ we have $d_{T_{out}}(s, v) > 2D$ in line 14. By symmetry, it thus suffices to show the first property. Assume therefore $d_{T_{in}}(s, v) > 2D$.

Consider the “core” $C = \{u \in V | \max(d_{T_{out}}(s, u), T_{in}(s, u)) \leq D\}$. In the next paragraph, we show that $|C| \geq \frac{n}{3}$ and $V_{G[V-M]}(v, i_{\text{max}}) \cap C = \emptyset$. This implies $n_{G[V-M]}(v, i_{\text{max}}) \leq |V-M| - |C| \leq |V-M| - \frac{n}{3} \leq \frac{2|V-M|}{3}$, showing the desired property.

The input assumptions $n_{G}(s, D) > \frac{2n}{3}$ and $n_{Grev}(s, D) > \frac{2n}{3}$ imply that $|C| = |V_{G}(s, D) \cap V_{Grev}(s, D)| \geq \frac{n}{3}$, as desired. To show $V_{G[V-M]}(v, i_{\text{max}}) \cap C = \emptyset$, observe that since $d_{T_{in}}(s, v) > 2D$ and $i_{\text{max}} = D$, we have for each $u \in V_{G[V-M]}(v, i_{\text{max}})$,

$$d_{T_{in}}(s, u) \geq d_{T_{in}}(s, v) - d_{Grev}(u, v) > 2D - d_{G}(v, u) \geq 2D - d_{G[V-M]}(v, u) \geq 2D - i_{\text{max}} = D,$$

so $V_{G[V-M]}(v, i_{\text{max}}) \cap C = \emptyset$.

**Weak diameter bound:** For the output set $E^{sep}$, let $K \subseteq V$ be an SCC of $G - E^{sep}$ and let $u, v \in K$. We need to show that $d_{G}(u, v) \leq 4D$.

Assume first that $u, v \in V - M$. Since $V_{\text{far}} - M = \emptyset$ and hence $V_{\text{far}} \subseteq M$, neither $u$ nor $v$ belongs to $V_{\text{far}}$ so $d_{G}(u, v) \leq d_{G}(u, s) + d_{G}(s, v) \leq 2D + 2D = 4D$.

Next, assume that exactly one of $u, v$ belongs to $M$ and the other to $V - M$. Then by inspection of the pseudo-code, we see that either there is no path from $u$ to $v$ in $G - E^{sep}$ or there is no path from $v$ to $u$ in $G - E^{sep}$. This contradicts that $u$ and $v$ belong to the same SCC $K$ of $G - E^{sep}$.

It remains to consider the case where $u, v \in M$. Some call to LowDiamDecomposition in the while loop must have been to a subgraph containing both $u$ and $v$ since otherwise we get the same contradiction as above. By our induction hypothesis for LowDiamDecomposition, this algorithm correctly splits the subgraph into SCCs of weak diameter at most $4D$. Thus, $d_{G}(u, v) \leq 4D$.

**Running time analysis:** The time spent outside the while-loop is dominated by the $O(m + n \log n)$ time to compute SSSP trees using Dijkstra’s algorithm.

Consider one execution of the while-loop and assume that $d_{T_{in}}(s, v) > 2D$; the case where $d_{T_{out}}(s, v)$ is symmetric. The time spent is dominated by the time to grow $B = B_{G[V-M]}(v, i_{\text{rnd}})$ and to sample $i_{\text{rnd}}$.

We will show how to sample $i_{\text{rnd}}$ within the time to grow $B$. We start by growing the first $i_{\text{min}}$ layers of $B$. For each remaining layer, a coin is flipped whose probability of heads is $2c(\ln n)/r$. The process stops once heads is obtained or once $r$ tails have been obtained since then layer $i_{\text{rnd}}$ has been found. The process also stops if an empty layer is encountered since then all later layers
are empty as well and $B$ has thus been obtained. Since every sampled layer is non-empty, it follows that the time spent is dominated by the size of $B$ and hence by the time to grow $B$.

We pay for the time cost of growing $B$ by charging each of its vertices $v$ a cost of $O(1 + \deg_G(v))$.

We have already argued above that when recursing on $B_G[V - M](v, i_{\text{rnd}})$, we have $\nu_G[V - M](v, i_{\text{rnd}}) \leq 3n_{G[V - M]}(v, i_{\max}) \leq 2n$ and when recursing on $B_G[V - M_{\text{rev}}](v, i_{\text{rnd}})$, we have $\nu_G[V - M_{\text{rev}}](v, i_{\text{rnd}}) \leq n_{G[V - M_{\text{rev}}]}(v, i_{\max}) \leq 2n$. The near-linear time bound now follows since each vertex $v$ is charged a time cost of $O(1 + \deg_G(v))$ no more than $\log_3/2 n$ times.

We are now ready to prove Lemma 3.1.

**Probability bounds:** We start by showing the probability bounds. Algorithm 5 applies CoreOrLayerRange and proceeds depending on which of the conditions of Corollary 7.4 holds. The case where Condition $= 1$ is handled by RandomTrim which we analyzed above. The case where Condition $\in \{2, 3\}$ is handled by directly following the steps of the generic algorithm. By Lemma 7.5, the entire execution of LowDiamDecomposition (including calls to RandomTrim) is thus an implementation of the generic algorithm. The probability bounds of Lemma 3.1 now follow from Lemma 7.1.

**Weak diameter bound:** Note that when LowDiamDecomposition does not call RandomTrim, it recurses on two subgraphs whose vertex sets form a partition of $V$. Additionally, since either $L_G(s, i_{\text{rnd}})$ or $L_{G_{\text{rev}}}(s, i_{\text{rnd}})$ is added to the returned set $E^{\text{sep}}$, every SCC of $G - E^{\text{sep}}$ must be fully contained in one of the two subgraphs. At the leaves of the recursion, RandomTrim is called; by Lemma 7.5, this algorithm ensures that every SCC in $G - E^{\text{sep}}$ has weak diameter at most $4D$.

**Running time:** It remains to show that LowDiamDecomposition runs in near-linear time. But this follows using the same analysis that we did for RandomTrim and observing that the time cost of CoreOrLayerRange is $O(b_G(s, i_{\text{rnd}}))$ when Condition $= 2$ and $O(b_{G_{\text{rev}}}(s, i_{\text{rnd}}))$ when Condition $= 3$.

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**Algorithm 5:** Algorithm LowDiamDecomposition($G = (V, E, w), D$)

1. if $V = \emptyset$ then return;
2. Pick arbitrary $s \in V$
3. $(\text{Condition}, i_{\max}) \leftarrow \text{CoreOrLayerRange}(G, s, D)$
4. if Condition = 1 then
   5. return RandomTrim($G, s, D$)
6. $r \leftarrow \left\lceil D/(3 \lg n) \right\rceil$
7. $i_{\min} \leftarrow i_{\max} - r$
8. Let $i_{\text{rnd}} = i_{\min} + \min\{\tilde{r}, r\}$ where $\tilde{r} \sim \text{Geo}(2c(\ln n)/r)$
9. if Condition = 2 then
   10. return $L_G(s, i_{\text{rnd}}) \cup \text{LowDiamDecomposition}(B_G(s, i_{\text{rnd}}), D) \cup$ $\text{LowDiamDecomposition}(G - V_G(s, i_{\text{rnd}}), D)$
      // Condition = 3:
11. return $L_{G_{\text{rev}}}(s, i_{\text{rnd}}) \cup \text{LowDiamDecomposition}(B_{G_{\text{rev}}}(s, i_{\text{rnd}}), D) \cup$ $\text{LowDiamDecomposition}(G - V_{G_{\text{rev}}}(s, i_{\text{rnd}}), D)$

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Algorithm 6: Algorithm RandomTrim($G = (V, E, w), s, D$)

1. $E_{sep} \leftarrow \emptyset$
2. Compute SSSP tree $T_{out}$ from $s$ in $G$ and SSSP tree $T_{in}$ from $s$ in $G_{rev}$
3. $V_{far} \leftarrow \{v \in V | \max\{d_{T_{out}}(s, v), d_{T_{in}}(s, v)\} > 2D\}$
   
   // $M$ is a growing set of marked vertices, indicating the trimmed part:
4. $M \leftarrow \emptyset$
5. $i_{max} \leftarrow D$
6. $r \leftarrow \lceil D/(3 \log n) \rceil$
7. $i_{min} \leftarrow i_{max} - r$
8. while $V_{far} - M \neq \emptyset$ do
9.     Pick a vertex $v \in V_{far} - M$
10.    Let $i_{rnd} = i_{min} + \min\{\tilde{i}, r\}$ where $\tilde{i} \sim \text{Geo}(2c(\ln n)/r)$
11.    if $d_{T_{in}}(s, v) > 2D$ then
12.        $E_{sep} \leftarrow E_{sep} \cup L_{G[V - M]}(v, i_{rnd}) \cup \text{LowDiamDecomposition}(B_{G[V - M]}(v, i_{rnd}), D)$
13.        $M \leftarrow M \cup V_{G[V - M]}(v, i_{rnd})$
14.    else
15.        $E_{sep} \leftarrow E_{sep} \cup L_{G[V - M]}(v, i_{rnd}) \cup \text{LowDiamDecomposition}(B_{G[V - M]}(v, i_{rnd}), D)$
16.        $M \leftarrow M \cup V_{G[V - M]}(v, i_{rnd})$
17. return $E_{sep}$

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Recall that Dijkstra’s algorithm executes an initialization step where all distance estimates are set to $\infty$ except the source whose estimate is set to $0$. Our proof will rely on a slight variant of Dijkstra whose initial priority queue contains all vertices with arbitrary (possibly negative) initial distance estimates $\hat{\text{dist}}(s,u)$ as keys for each $u \in V$. The algorithm updates these distance estimates as edges are relaxed. Note that this variant does not necessarily extract $s$ as the first vertex from the priority queue.

Let $G_+$ denote the graph $G$ with $E^{\text{neg}}(G)$ removed. SPWithFewNegEdges($G, s, k$) initializes $\hat{\text{dist}}(s, u) \leftarrow \infty$ for each $u \in V \setminus \{s\}$ and $\hat{\text{dist}}(s, s) \leftarrow 0$ (this corresponds to the standard initialization step of Dijkstra). It then executes $k + 1$ iterations where each iteration consists of the following
two steps: first, run the variant of Dijkstra’s algorithm above in $G_+$ (initialized with the current estimates $\hat{d}(s, u)$); then relax all edges in $E^\text{neg}(G)$, updating estimates $\hat{d}(s, u)$ further.

Clearly, $\text{SPWithFewNegEdges}(G, s, k)$ has the desired running time. Since edge relaxations are the only operations that update distance estimates and since for each $v \in V$, $d(s, v) \leq 0$, we have $0 \geq \hat{d}(s, v) \geq d(s, v)$.

It remains to show that for each $v$ that is $(s, k)$-negative in $G$, we have $\hat{d}(s, v) = d(s, v)$. The shortest path $P = P(v; G, s)$ contains at most $k$ negative edges of $E^\text{neg}(G)$. Partition this path into maximal subpaths each of which is either contained in $G_+$ or consists of a single negative edge; let $P_1, e_1, P_2, e_2, \ldots, e_{r-1}, P_r$ be these subpaths in order along $P$, where each $P_i$ is contained in $G_+$ and each $e_i \in E^\text{neg}(G)$ (possibly with some $P_i$-paths consisting of a single vertex). We have $r \leq k + 1$.

Denote by $s_i$ and $t_i$ the first and last vertex of $P_i$, respectively, for each $i$. We will show by induction on $i \in \{1, 2, \ldots, r\}$ that after the $i$th iteration of $\text{SPWithFewNegEdges}(G, s, k)$, $\hat{d}(s, u) = d(s, u)$ for each vertex $u$ of $P_i$ as well as for the vertex $s_{i+1}$ if it exists (i.e., if $i < r$).

We first consider the base case $i = 1$. After Dijkstra has been executed, $\hat{d}(s, u) = d(s, u)$ for each vertex $u$ of $P_1$; this is immediate from the correctness of Dijkstra. Since $e_1$ is subsequently relaxed, we have $\hat{d}(s, s_2) = d(s, s_2)$ at the end of the first iteration. This shows the base case.

Now assume $i > 1$ and that the claim holds for smaller values. In particular, $\hat{d}(s, s_i) = d(s, s_i)$. Note that Dijkstra’s algorithm is executed on $G_+$. A simple modification of the correctness proof of Dijkstra then shows that our variant of Dijkstra relaxes the edges of $P_i$ in order. Hence, afterwards, $\hat{d}(s, u) = d(s, u)$ for each vertex $u$ of $P_i$. If $i < r$, the subsequent relaxation of $e_i$ ensures that $\hat{d}(s, s_{i+1}) = d(s, s_{i+1})$. This completes the inductive step.

It follows that at termination, $\hat{d}(s, v) = d(s, v)$ for the last vertex $v$ of $P$, as desired.