SOME HADAMARD LIKE INEQUALITIES VIA CONVEX AND $s$-CONVEX FUNCTIONS AND THEIR APPLICATIONS FOR SPECIAL MEANS

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ABSTRACT. In this study, the author establish some inequalities of Hadamard like based on convex and $s$-convexity in the second sense. Some applications to special means of positive real numbers are also given.

1. PRELIMINARIES

1.1. Definitions.

Definition 1. [15] A function $f : I \to \mathbb{R}$ is said to be convex on $I$ if inequality

\[(1.1) \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)\]

holds for all $x, y \in I$ and $t \in [0,1]$. We say that $f$ is concave if $(-f)$ is convex.

Geometrically, this means that if $P, Q$ and $R$ are three distinct points on the graph of $f$ with $Q$ between $P$ and $R$, then $Q$ is on or below the chord $PR$.

Definition 2. [11] Let $s \in (0,1]$. A function $f : (0, \infty) \to [0, \infty]$ is said to be $s$-convex in the second sense if

\[(1.2) \quad f(tx + (1-t)y) \leq tsf(x) + (1-t)sf(y),\]

for all $x, y \in (0,b]$ and $t \in [0,1]$. This class of $s$-convex functions is usually denoted by $K_s^2$.

Certainly, $s$-convexity means just ordinary convexity when $s = 1$.

1.2. Theorems.

Theorem 1. The Hermite-Hadamard inequality: Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $u, v \in I$ with $u < v$. The following double inequality:

\[(1.3) \quad f \left( \frac{u+v}{2} \right) \leq \frac{1}{v-u} \int_u^v f(x) \, dx \leq \frac{f(u) + f(v)}{2}\]

is known in the literature as Hadamard’s inequality (or Hermite-Hadamard inequality) for convex functions. If $f$ is a positive concave function, then the inequality is reversed.

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Theorem 2. Suppose that \( f : [0, \infty) \to [0, \infty) \) is an \( s \)-convex function in the second sense, where \( s \in (0, 1] \), and let \( a, b \in [0, \infty) \), \( a < b \). If \( f \in L_1 ([0, 1]) \), then the following inequalities hold:

\[
2^{-s} f \left( \frac{u + v}{2} \right) \leq \frac{1}{v - u} \int_u^v f (x) \, dx \leq \frac{f (u) + f (v)}{s + 1}.
\]

The constant \( k = \frac{1}{s + 1} \) is the best possible in the second inequality in (1.4). The above inequalities are sharp. If \( f \) is an \( s \)-concave function in the second sense, then the inequality is reversed.

For recent results and generalizations concerning Hadamard’s inequality and concepts of convexity and \( s \)-convexity see [1]-[20] and the references therein.

Throughout this paper we will use the following notations and conventions. Let \( J = [0, \infty) \subset \mathbb{R} = (-\infty, +\infty) \), and \( u, v \in J \) with \( 0 < u < v \) and \( f' \in L [u, v] \) and

\[
A (u, v) = \frac{u + v}{2}, \quad G (u, v) = \sqrt{uv}, \quad I (u, v) = \frac{1}{e} \left( \frac{b}{a} \right)^{1/e},
\]

\[
L_p (u, v) = \left( \frac{v^{p+1} - u^{p+1}}{(p+1)(v-u)} \right)^{1/p}, \quad u \neq v, \quad p \in \mathbb{R}, \quad p \neq -1, 0
\]

be the arithmetic mean, geometric mean, identric mean, generalized logarithmic mean for \( u, v > 0 \) respectively.

2. SOME NEW HADAMARD LIKE INEQUALITIES

In order to establish our main results, we first establish the following lemma.

Lemma 1. Let \( f : J \to \mathbb{R} \) be a differentiable function on \( J^o \). If \( f' \in L [u, v] \), then

\[
\frac{(v-x)(vf(v) - uf(x)) + (x-u)(vf(x) - uf(u))}{(v-u)^2} = \frac{1}{v-u} \int_u^v f (\mu) \, d\mu - \frac{(v-x)^2}{(v-u)^2} \int_0^1 (tu + (1-t)v) f' (tx + (1-t)v) \, dt
\]

\[
+ \frac{(x-u)^2}{(v-u)^2} \int_0^1 (tv + (1-t)u) f' (tx + (1-t)u) \, dt
\]

for each \( t \in [0, 1] \) and \( x \in [u, v] \).
Proof. Integrating by parts, we get

\[
\begin{align*}
&\frac{(v-x)^2}{(v-u)^2} \int_0^1 (tu + (1-t)v) f'(tx + (1-t)v) \, dt \\
&\quad + \frac{(x-u)^2}{(v-u)^2} \int_0^1 (tv + (1-t)u) f'(tx + (1-t)u) \, dt \\
&= \left( \frac{(v-x)^2}{(v-u)^2} \left[ \frac{(tu + (1-t)v) f'(tx + (1-t)v)}{x-v} \right]_0^1 - \frac{(u-v) f(tx + (1-t)v)}{x-v} \right) \\
&\quad + \left( \frac{(x-u)^2}{(v-u)^2} \left[ \frac{(tv + (1-t)u) f'(tx + (1-t)u)}{x-u} \right]_0^1 - \frac{(v-u) f(tx + (1-t)u)}{x-u} \right) \\
&= \left( \frac{(v-x)^2}{(v-u)^2} \left[ \frac{uf(x) - vf(x)}{x-v} - \frac{v-u}{(x-v)^2} \int_x^v f(\mu) \, d\mu \right] \\
&\quad + \frac{(x-u)^2}{(v-u)^2} \left[ \frac{vf(x) - uf(x)}{x-u} - \frac{v-u}{(x-u)^2} \int_u^x f(\mu) \, d\mu \right] \right) \\
&= \frac{(v-x) (vf(v) - uf(x)) + (x-u) (vf(x) - uf(u))}{(v-u)^2} - \frac{1}{v-u} \int_u^v f(\mu) \, d\mu.
\end{align*}
\]

\[\square\]

**Theorem 3.** Let \( f : J \to \mathbb{R} \) be a differentiable function on \( J \). If \( |f'| \) is convex on \([u,v]\), then

\[
\left| \frac{(v-x) (vf(v) - uf(x)) + (x-u) (vf(x) - uf(u))}{(v-u)^2} - \frac{1}{v-u} \int_u^v f(\mu) \, d\mu \right| 
\leq \frac{(v-x)^2}{6(v-u)^2} \left[ 2u + v \right] f'(x) + \frac{(x-u)^2}{6(v-u)^2} \left[ (u+2v) f'(v) \right] + \frac{(x-u)^2}{6(v-u)^2} \left[ (u+2v) f'(u) \right]
\]

for each \( x \in [u,v] \).
Corollary 1. The proof of Theorem 4 is similar to Theorem 3.

Proof. Using Lemma 1 and from properties of modulus, and since $|f'|$ is convex on $[u, v]$, then we obtain

$$
\left| \frac{(v-x)(vf(v) - uf(x)) + (x-u)(vf(x) - uf(u))}{(v-u)^2} - \frac{1}{v-u} \int_u^v f(\mu) \, d\mu \right|
\leq \frac{(v-x)^2}{(v-u)^2} \int_0^1 (tu + (1-t)v) |f'(tx + (1-t)v)| \, dt
+ \frac{(x-u)^2}{(v-u)^2} \int_0^1 (tv + (1-t)u) |f'(tx + (1-t)u)| \, dt
\leq \frac{(v-x)^2}{6(v-u)^2} \left((2u+v)|f'(x)| + (u+2v)|f'(v)|\right)
+ \frac{(x-u)^2}{6(v-u)^2} \left((u+2v)|f'(x)| + (2u+v)|f'(u)|\right).
$$

The proof is completed. \qed

Theorem 4. Let $f : J \to \mathbb{R}$ be a differentiable function on $J$. If $|f'|$ is $s$-convex on $[u, v]$ for some fixed $s \in (0, 1]$, then

$$
\left| \frac{(v-x)(vf(v) - uf(x)) + (x-u)(vf(x) - uf(u))}{(v-u)^2} - \frac{1}{v-u} \int_u^v f(\mu) \, d\mu \right|
\leq \frac{(v-x)^2}{(v-u)^2} \left[\frac{((s+1)u+v)|f'(x)| + (u+(s+1)v)|f'(v)|}{(s+1)(s+2)}\right]
+ \frac{(x-u)^2}{(v-u)^2} \left[\frac{(u+(s+1)v)|f'(x)| + ((s+1)v+u)|f'(u)|}{(s+1)(s+2)}\right]
$$

for each $x \in [u, v]$.

Proof. The proof of Theorem 4 is similar to Theorem 3. \qed

Remark 1. In Theorem 3 if we take $s = 1$, then Theorem 4 reduces to Theorem 3.

Corollary 1. In Theorem 4 if we choose $x = \frac{u+v}{2}$, we get

$$
\left| \frac{vf(v) - uf(u)}{2(v-u)} + \frac{1}{2} f\left(\frac{u+v}{2}\right) - \frac{1}{v-u} \int_u^v f(\mu) \, d\mu \right|
\leq \frac{1}{4} \left[\frac{((s+1)u+v)|f'(\frac{u+v}{2})| + (u+(s+1)v)|f'(v)|}{(s+1)(s+2)}\right]
+ \frac{1}{4} \left[\frac{(u+(s+1)v)|f'(\frac{u+v}{2})| + ((s+1)v+u)|f'(u)|}{(s+1)(s+2)}\right].
$$
Proposition 1. Let \( u, v \in J^s \), \( 0 < u < v \) and \( s \in (0, 1] \), then

\[
\left| \frac{s+1}{2} L_u^s(u, v) + \frac{1}{2} A^s(u, v) - L_v^s(u, v) \right| \\
\leq \frac{s}{s+1} A(u, v) + \frac{s}{s+2} A(u^s, v^s) + \frac{s^2}{(s+1)(s+2)} A(u^{s^2}, v^{s^2})
\]

(2.1)

Proof. The proof follows from Corollary 1 applied to the \( s \)-convex function \( f(x) = x^s \). Equality in (2.1) holds if and only if \( s = 1 \).

\[\square\]

Corollary 2. In Corollary 1, if we take \( |f'| \leq M \), we get

\[
\left| \frac{vf(v) - uf(u)}{2(v-u)} + \frac{1}{2} f(u+v) - \frac{1}{v-u} \int_u^v f(\mu) d\mu \right| \leq \frac{M}{2} \frac{u+v}{2(s+1)}
\]

Theorem 5. Let \( f : J \to \mathbb{R} \) be a differentiable function on \( J^s \). If \( |f'|^q \) is convex on \( [u, v] \) and \( q > 1 \) with \( 1/p + 1/q = 1 \), then

\[
\left| \frac{(v-x)(vf(v) - uf(x)) + (x-u)(vf(x) - uf(u))}{(v-u)^2} - \frac{1}{v-u} \int_u^v f(\mu) d\mu \right| \leq \frac{(v-x)^2}{(v-u)^2} L_p(u, v) \left( \frac{|f'(v)|^q + |f'(u)|^q}{2} \right)^{\frac{1}{q}} + \frac{(x-u)^2}{(v-u)^2} L_p(u, v) \left( \frac{|f'(v)|^q + |f'(u)|^q}{2} \right)^{\frac{1}{q}}
\]

for each \( x \in [u, v] \).

Proof. Using Lemma 1 and using the well-known Hölder’s inequality and since \( |f''|^{q'} \) is convex on \( [u, v] \), we establish

(2.2)

\[
\left| \frac{(v-x)(vf(v) - uf(x)) + (x-u)(vf(x) - uf(u))}{(v-u)^2} - \frac{1}{v-u} \int_u^v f(\mu) d\mu \right| \leq \frac{(v-x)^2}{(v-u)^2} \int_0^1 (tu + (1-t)v) |f'(tx + (1-t)v)| dt \\
+ \frac{(x-u)^2}{(v-u)^2} \int_0^1 (tv + (1-t)u) |f'(tx + (1-t)u)| dt
\]

\[
\leq \frac{(v-x)^2}{(v-u)^2} \left( \int_0^1 (tu + (1-t)v)^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(tx + (1-t)v)|^q dt \right)^{\frac{1}{q}}
\]

\[
+ \frac{(x-u)^2}{(v-u)^2} \left( \int_0^1 (tv + (1-t)u)^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(tx + (1-t)u)|^q dt \right)^{\frac{1}{q}}.
\]

Since \( |f''| \) is convex on \( [u, v] \), by Hadamard’s inequality, we have

(2.3)

\[
\int_0^1 |f'(tx + (1-t)v)|^q dt \leq |f'(x)|^q + |f'(v)|^q
\]

(2.4)

\[
\int_0^1 |f'(tx + (1-t)u)|^q dt \leq |f'(x)|^q + |f'(u)|^q
\]
It can be easily seen that
\[(2.5)\]
\[
\int_0^1 (tu + (1 - t)v)^p \, dt = \int_0^1 (tv + (1 - t)u)^p \, dt = \frac{v^{p+1} - u^{p+1}}{(v-u)(p+1)} = L_p^p(u,v)
\]
If expressions \((2.3) - (2.5)\) are written in \((2.2)\), we obtain
\[
\left| \frac{(v-x)(vf(v) - uf(x)) + (x-u)(vf(x) - uf(u))}{(v-u)^2} - \frac{1}{v-u} \int_u^v f(\mu) \, d\mu \right| \leq \frac{(v-x)^2 L_p(u,v)}{(v-u)^2} \left( \frac{|f'(x)|q + |f'(v)|q}{s+1} \right)^\frac{1}{q} + \frac{(x-u)^2 L_p(u,v)}{(v-u)^2} \left( \frac{|f'(x)|q + |f'(u)|q}{s+1} \right)^\frac{1}{q}.
\]
The proof is completed. \( \square \)

**Theorem 6.** Let \( f : J \to \mathbb{R} \) be a differentiable function on \( J^o \). If \(|f'|^q\) is \( s\)-convex on \([u,v]\) for some fixed \( s \in (0,1) \) and \( p > 1 \) with \( 1/p + 1/q = 1 \), then
\[
\left| \frac{(v-x)(vf(v) - uf(x)) + (x-u)(vf(x) - uf(u))}{(v-u)^2} - \frac{1}{v-u} \int_u^v f(\mu) \, d\mu \right| \leq \frac{(v-x)^2 L_p(u,v)}{(v-u)^2} \left( \frac{|f'(x)|q + |f'(v)|q}{s+1} \right)^\frac{1}{q} + \frac{(x-u)^2 L_p(u,v)}{(v-u)^2} \left( \frac{|f'(x)|q + |f'(u)|q}{s+1} \right)^\frac{1}{q},
\]
for each \( x \in [u,v] \).

**Proof.** The proof of Theorem 6 is similar to Theorem 5. \( \square \)

**Corollary 3.** In Theorem 6
1. if we take \( x = u \) or \( x = v \), we get
\[(2.6)\]
\[
\frac{|vf(v) - uf(u)|}{v-u} - \frac{1}{v-u} \int_u^v f(\mu) \, d\mu \leq L_p(u,v) \left( \frac{|f'(u)|q + |f'(v)|q}{s+1} \right)^\frac{1}{q}.
\]
2. if we take \( x = \frac{u+v}{2} \) and since \( (\frac{1}{s+1})^\frac{1}{q} \leq 1, s \in (0,1) \), we get
\[(2.7)\]
\[
\frac{L_p(u,v)}{4} \left[ \left( \frac{|f'(\frac{u+v}{2})|^q + |f'(v)|^q}{s+1} \right)^\frac{1}{q} + \left( \frac{|f'(\frac{u+v}{2})|^q + |f'(u)|^q}{s+1} \right)^\frac{1}{q} \right]
\]
\[(2.8)\]
\[
\frac{L_p(u,v)}{4} \left[ \left( \frac{|f'(\frac{u+v}{2})|^q + |f'(v)|^q}{s+1} \right)^\frac{1}{q} + \left( \frac{|f'(\frac{u+v}{2})|^q + |f'(u)|^q}{s+1} \right)^\frac{1}{q} \right]
\]

**Proposition 2.** Let \( u,v \in J^o, \, 0 < u < v \) and \( s \in (0,1) \), then
\[
|sL_s^s(u,v) + A^s(u,v)|
\]
\[
\leq \frac{L_p(u,v)}{2} \left( \frac{s}{s+1} \right)^\frac{1}{q} \left[ \left( \frac{u+v}{2} \right)^{s-1} u^{(s-1)q} + \left( \frac{u+v}{2} \right)^{(s-1)q} + u^{(s-1)q} \right]^\frac{1}{q}
\]

**Proof.** The proof follows from \((2.7)\) applied to the \( s\)-convex function \( f(x) = x^s \). Equality in \((2.1)\) holds if and only if \( s = 1 \) and \( u = v \). \( \square \)
Theorem 7. Let $f : \mathbb{J} \to \mathbb{R}$ be a differentiable function on $\mathbb{J}$. If $|f'|^q$ is $s$-convex on $[u, v]$ for some fixed $s \in (0, 1]$ and $q \geq 1$, then

$$
\left| \frac{(v-x)(vf(v)-uf(x))+(x-u)(vf(x)-uf(u))}{(v-u)^2} - \frac{1}{v-u} \int_u^v f(\mu) \, d\mu \right| 
$$

$$
\leq \frac{(v-x)^2}{(v-u)^2} A^{1-\frac{1}{q}}(u,v) \left( \frac{1}{(s+1)(s+2)} \right)^{\frac{1}{q}} \left( \left( (s+1)u+v \right) |f'(x)|^q + \left( (s+1)v+u \right) |f'(v)|^q \right) dt \right)^{\frac{1}{q}} 
$$

for each $x \in [u,v]$.

Proof. Using Lemma 1 and the well-known power mean inequality and since $|f'|^q$ is $s$-convex on $[u,v]$, we establish

$$
\left| \frac{(v-x)(vf(v)-uf(x))+(x-u)(vf(x)-uf(u))}{(v-u)^2} - \frac{1}{v-u} \int_u^v f(\mu) \, d\mu \right| 
$$

$$
\leq \frac{(v-x)^2}{(v-u)^2} \int_0^1 (tu + (1-t)v) |f'(tx + (1-t)v)| dt + \frac{(x-u)^2}{(v-u)^2} \int_0^1 (tv + (1-t)u) |f'(tx + (1-t)u)| dt 
$$

$$
\leq \frac{(v-x)^2}{(v-u)^2} \left( \int_0^1 (tu + (1-t)v) dt \right)^{1-\frac{1}{q}} \left( \int_0^1 (tu + (1-t)v) |f'(tx + (1-t)v)|^q dt \right)^{\frac{1}{q}} 
$$

$$
+ \frac{(x-u)^2}{(v-u)^2} \left( \int_0^1 (tv + (1-t)u) dt \right)^{1-\frac{1}{q}} \left( \int_0^1 (tv + (1-t)u) |f'(tx + (1-t)u)|^q dt \right)^{\frac{1}{q}} 
$$

$$
\leq \frac{(v-x)^2}{(v-u)^2} \left( \int_0^1 (tu + (1-t)v) dt \right)^{1-\frac{1}{q}} \left( \int_0^1 (tu + (1-t)v) \left( t^s |f'(x)|^q + (1-t)^s |f'(v)|^q \right) dt \right)^{\frac{1}{q}} 
$$

$$
+ \frac{(x-u)^2}{(v-u)^2} \left( \int_0^1 (tv + (1-t)u) dt \right)^{1-\frac{1}{q}} \left( \int_0^1 (tv + (1-t)u) \left( t^s |f'(x)|^q + (1-t)^s |f'(u)|^q \right) dt \right)^{\frac{1}{q}} 
$$

$$
= \frac{(v-x)^2}{(v-u)^2} \left( \frac{u+v}{2} \right)^{1-\frac{1}{q}} \left( \frac{(s+1)u+v}{(s+1)(s+2)} |f'(x)|^q + \frac{(s+1)v+u}{(s+1)(s+2)} |f'(v)|^q \right) \left( \int_0^1 (tu + (1-t)v) \left( t^s |f'(x)|^q + (1-t)^s |f'(v)|^q \right) dt \right)^{\frac{1}{q}} 
$$

$$
+ \frac{(x-u)^2}{(v-u)^2} \left( \frac{u+v}{2} \right)^{1-\frac{1}{q}} \left( \frac{(s+1)v+u}{(s+1)(s+2)} |f'(x)|^q + \frac{(s+1)u+v}{(s+1)(s+2)} |f'(u)|^q \right) \left( \int_0^1 (tv + (1-t)u) \left( t^s |f'(x)|^q + (1-t)^s |f'(u)|^q \right) dt \right)^{\frac{1}{q}} 
$$

The proof is completed. \qed
Theorem 8. Let $f : J \to \mathbb{R}$ be a differentiable function on $J$. If $|f'|^q$ is convex on $[u, v]$ and $q \geq 1$, then

$$
\left| \frac{(v - x)(vf(v) - uf(x)) + (x - u)(vf(x) - uf(u))}{(v - u)^2} - \frac{1}{v - u} \int_u^v f(\mu) \, d\mu \right|
$$

\[ \leq \frac{(v - x)^2}{(v - u)^2} A^{-\frac{q}{q'}} (u, v) \left( \frac{1}{(s + 1)(s + 2)} \right)^\frac{q}{q'} \left( \left( (s + 1) u + v \right) |f'(u)|^q + ((s + 1) v + u) |f'(v)|^q \right) dt \]

for each $x \in [u, v]$.

Proof. In Theorem 8 if we take $s = 1$, then the assertion is proved. \qed

Corollary 4. In Theorem 7

i) if we take $x = u$ or $x = v$, we get

$$
\left| \frac{(v - u)(vf(v) - uf(u))}{(v - u)^2} - \frac{1}{v - u} \int_u^v f(\mu) \, d\mu \right|
$$

\[ \leq A^{-\frac{q}{q'}} (u, v) \left( \frac{1}{(s + 1)(s + 2)} \right)^\frac{q}{q'} \left( \left( (s + 1) u + v \right) |f'(u)|^q + ((s + 1) v + u) |f'(v)|^q \right) dt \]

ii) if we choose $x = u$ or $x = v$ and $s = q = 1$, we get

$$
\left| \frac{(v - u)(vf(v) - uf(u))}{(v - u)^2} - \frac{1}{v - u} \int_u^v f(\mu) \, d\mu \right| \leq \frac{1}{6} ((2u + v) |f'(u)| + (2v + u) |f'(v)|) dt
$$

iii) if we take $x = \frac{u + v}{2}$ and since $(\frac{1}{(s + 1)(s + 2)})^\frac{q}{q'} \leq 1$, $s \in (0, 1]$, we get

$$
\left| \frac{vf(v) - uf(u)}{2(v - u)} + \frac{1}{2} f \left( \frac{u + v}{2} \right) - \frac{1}{v - u} \int_u^v f(\mu) \, d\mu \right|
$$

\[ \leq \frac{1}{4} A^{-\frac{q}{q'}} (u, v) \left( \frac{1}{(s + 1)(s + 2)} \right)^\frac{q}{q'} \left( \left( (s + 1) u + v \right) f' \left( \frac{u + v}{2} \right) \right) \left. \right| \left. \right| + ((s + 1) v + u) |f'(v)|^q dt \]

\[ + \frac{1}{4} A^{-\frac{q}{q'}} (u, v) \left( \frac{1}{(s + 1)(s + 2)} \right)^\frac{q}{q'} \left( \left( (s + 1) v + u \right) f' \left( \frac{u + v}{2} \right) \right) \left. \right| \left. \right| + ((s + 1) u + v) |f'(u)|^q dt \]

\[ \leq \frac{1}{4} A^{-\frac{q}{q'}} (u, v) \left[ \left( (s + 1) u + v \right) f' \left( \frac{u + v}{2} \right) \right] + ((s + 1) v + u) |f'(v)|^q dt \]

\[ + \frac{1}{4} A^{-\frac{q}{q'}} (u, v) \left[ \left( (s + 1) v + u \right) f' \left( \frac{u + v}{2} \right) \right] + ((s + 1) u + v) |f'(u)|^q dt \]
Proposition 3. Let \( u, v \in J^c \), \( 0 < u < v \) and \( s \in (0, 1) \), then
\[
| (s - 1) L_s^x (u, v) + A_s^x (u, v) | \\
\leq \left( \frac{s^q A_q^{-1} (u, v)}{(s + 1)(s + 2)^2} \right)^{\frac{1}{q}} \left( ((s + 1) u + v) L_s^x (u, v) + ((s + 1) v + u) A_s^x (u, v) \right)^{\frac{1}{q}} \\
+ \left( \frac{s^q A_q^{-1} (u, v)}{(s + 1)(s + 2)^2} \right)^{\frac{1}{q}} \left( ((s + 1) v + u) L_s^x (u, v) + ((s + 1) u + v) A_s^x (u, v) \right)^{\frac{1}{q}}
\]

Proof. The proof follows from (2.9) applied to the s-convex function \( f(x) = x^s \). □

Theorem 9. Let \( f : J \to \mathbb{R} \) be a differentiable function on \( J^c \). If \(|f'|^q\) is s-concave on \([u, v]\) for some fixed \( s \in (0, 1) \) and \( q > 1 \) with \( 1/p + 1/q = 1 \), then
\[
\frac{1}{(v - u)^2} \left[ (v - x) \left( vf(v) - uf(x) \right) + (x - u) \left( vf(x) - uf(u) \right) - \frac{1}{v - u} \int_x^v f(\mu) \, d\mu \right] \\
\leq \frac{2^q - 1}{v - u} L_p(u, v) \left[ (v - x)^2 \left| f' \left( \frac{x + v}{2} \right) \right| + (x - u)^2 \left| f' \left( \frac{x + u}{2} \right) \right| \right]
\]
for each \( x \in [u, v] \).

Proof. Using Lemma 1 and using the Hölder inequality and since \(|f'|^q\) is s-concave on \([u, v]\) and using the inequality (1.4), we establish
\[
\frac{1}{(v - u)^2} \left[ (v - x) \left( vf(v) - uf(x) \right) + (x - u) \left( vf(x) - uf(u) \right) - \frac{1}{v - u} \int_x^v f(\mu) \, d\mu \right] \\
\leq \frac{(v - x)^2}{(v - u)^2} \int_0^1 (tu + (1 - t) v) |f'(tx + (1 - t) v)| \, dt + \frac{(x - u)^2}{(v - u)^2} \int_0^1 (tv + (1 - t) u) |f'(tx + (1 - t) u)| \, dt \\
\leq \frac{(v - x)^2}{(v - u)^2} \left( \int_0^1 (tu + (1 - t) v)^p \, dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(tx + (1 - t) v)|^q \, dt \right)^{\frac{1}{q}} \\
+ \frac{(x - u)^2}{(v - u)^2} \left( \int_0^1 (tv + (1 - t) u)^p \, dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(tx + (1 - t) u)|^q \, dt \right)^{\frac{1}{q}} \\
\leq \frac{(v - x)^2}{(v - u)^2} L_p(u, v) \left( 2^{q-1} \left| f' \left( \frac{x + v}{2} \right) \right|^q \right) + \frac{(x - u)^2}{(v - u)^2} L_p(u, v) \left( 2^{q-1} \left| f' \left( \frac{x + u}{2} \right) \right|^q \right)
\]
The proof is completed. □

Theorem 10. Let \( f : J \to \mathbb{R} \) be a differentiable function on \( J^c \). If \(|f'|^q\) is concave on \([u, v]\) and \( q > 1 \) with \( 1/p + 1/q = 1 \), then
\[
\frac{1}{(v - u)^2} \left[ (v - x) \left( vf(v) - uf(x) \right) + (x - u) \left( vf(x) - uf(u) \right) - \frac{1}{v - u} \int_x^v f(\mu) \, d\mu \right] \\
\leq \frac{L_p(u, v)}{(v - u)^2} \left[ (v - x)^2 \left| f' \left( \frac{x + v}{2} \right) \right| + (x - u)^2 \left| f' \left( \frac{x + u}{2} \right) \right| \right]
\]
for each \( x \in [u, v] \).
Proof. Using Lemma 1 and using the Hölder inequality and since \(|f'|^{q}\) is concave on \([u, v]\) and using Hadamard’s inequality for concave functions, we complete the proof. Or, in Theorem 3 if we take \(s = 1\), then the assertion is also proved. \(\square\)

**Corollary 5.** In Theorem 4

i) if we take \(x = u\) or \(x = v\), we get

\[
\left| \frac{(v - u)(vf'(v) - uf'(u))}{(v - u)^2} - \frac{1}{v - u} \int_{u}^{v} f'(\mu) \, d\mu \right| \leq 2\frac{v - 1}{v} L_p(u, v) \left| f'\left(\frac{u + v}{2}\right) \right|
\]

ii) if we take \(x = \frac{u + v}{2}\), we get

\[
(2.11) \quad \left| \frac{vf'(v) - uf'(u)}{2(v - u)} + \frac{1}{2} f\left(\frac{u + v}{2}\right) - \frac{1}{v - u} \int_{u}^{v} f'(\mu) \, d\mu \right| \leq 2\frac{v - 1}{v} L_p(u, v) \left[ \left| f'\left(\frac{u + 3v}{4}\right)\right| + \left| f'\left(\frac{v + 3u}{4}\right)\right| \right]
\]

**Proposition 4.** Let \(u, v \in J^o\), \(0 < u < v\) and \(s = 1\), then

\[
\left| \frac{v \sin v - u \sin u + 2 \cos v - 2 \cos u}{v - u} + A(u, v) \right| \leq L_p(u, v) \left[ \cos \left(\frac{u + 3v}{4}\right) + \cos \left(\frac{v + 3u}{4}\right) \right]
\]

Proof. The proof follows from (2.11) applied to the concave function \(f : [0, \pi] \to [0, 1], f(x) = \sin x\). \(\square\)

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