Minor summation formula and a proof of Stanley’s open problem

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Abstract

In the open problem session of the FPSAC’03, R.P. Stanley gave an open problem about a certain sum of the Schur functions (See [20]). The purpose of this paper is to give a proof of this open problem. The proof consists of three steps. At the first step we express the sum by a Pfaffian as an application of our minor summation formula ([8]). In the second step we prove a Pfaffian analogue of Cauchy type identity which generalize [23]. Then we give a proof of Stanley’s open problem in Section 4. At the end of this paper we present certain corollaries obtained from this identity involving the Big Schur functions and some polynomials arising from the Macdonald polynomials, which generalize Stanley’s open problem.

Keywords. Schur functions, determinants, Pfaffians, minor summation formula of Pfaffians.

1 Introduction

In the open problem session of the 15th Anniversary International Conference on Formal Power Series and Algebraic Combinatorics (Vadstena, Sweden, 25 June 2003), R.P. Stanley gave an open problem on a sum of Schur functions with a weight including four parameters, i.e. Theorem 1.1 (See [20]). The purpose of this paper is to give a proof of this open problem. In the process of our proof, we obtain a Pfaffian identity, i.e. Theorem 3.1 which generalize the Pfaffian identities in [23]. Note that certain determinant and Pfaffian identities of this type first appeared in [19], and applied to solve some alternating sign matrices enumerations under certain symmetries stated in [16]. Certain conjectures which intensively generalize the determinant and Pfaffian identities of this type were stated in [18], and a proof of the conjectured determinant and Pfaffian identities was given in [7]. Now we know that various methods may be adopted to prove this type of identity. We can prove it algebraically using Dodgson’s formula or the usual expansion formula of Pfaffians. Here we state an analytic proof since this proof is due to the author and is not stated in other places. Our proof proceeds by three steps. In the first step we utilize the minor summation formula ([8]) to express the sum of Schur functions as a Pfaffian. In the second step we express the Pfaffian
by a determinant using a Cauchy type Pfaffian formula (also see [17], [18]
and [7]), and try to simplify it as much as possible. In the process of
this step, it is conceivable that the determinants we treat may be closely
related to characters of representations of $\text{Sp}_{2n}$ and $\text{SO}_m$ (See [3], [9]
and [13]). In the final step we complete our proof using a key proposition, i.e.
Proposition 4.1 (See [19] and [22]). At the end of this paper we state
some corollaries which generalize Stanley’s open problem to the big Schur
functions, and to certain polynomials arising from the Macdonald poly-
momials. Furthermore, in the forthcoming paper [10], we study a finite
version of Boulet’s theorem and present certain relations with orthogo-
nal polynomials and the basic hypergeometric series. In the paper we
find more applications of the Pfaffain expression of Stanley’s weight
$\omega(\lambda)$ obtained in this paper, and also study a certain summation of Schur’s
$Q$-functions weighted by $\omega(\lambda)$.

We follow the notation in [15] concerning symmetric functions. In
this paper we use a symmetric function $f$ in $n$ variables $(x_1, \ldots, x_n)$,
which is usually written as $f(x_1, \ldots, x_n)$, and also a symmetric function
$f$ in countably many variables $x = (x_1, x_2, \ldots)$, which is written as $f(x)$
(for detailed description of the ring of symmetric functions in countably
many variables, see [15], I, sec.2). To simplify this notation we express
the $n$-tuple $(x_1, \ldots, x_n)$ by $X_n$, and sometimes simply write $f(X_n)$ for
$f(x_1, \ldots, x_n)$. When the number of variables is finite and there is no fear of
confusion what this number is, we simply write $X$ for $X_n$ in abbreviation.

Thus $f(x)$ is in countably many variables, but $f(X)$ is in finitely many
variables and the number of variables is clear from the assumption.

Given a partition $\lambda$, define $\omega(\lambda)$ by

$$\omega(\lambda) = a\sum_{i \geq 1}[^{\lambda_2i - 1}/2] b\sum_{i \geq 1}[^{\lambda_2i - 1}/2] c\sum_{i \geq 1}[^{\lambda_2i}/2] d\sum_{i \geq 1}[^{\lambda_2i}/2],$$

where $a$, $b$, $c$ and $d$ are indeterminates, and $[x]$ (resp. $\lfloor x \rfloor$) stands for the
smallest (resp. largest) integer greater (resp. less) than or equal to $x$
for a given real number $x$. For example, if $\lambda = (5, 4, 4, 1)$ then $\omega(\lambda)$ is
the product of the entries in the following diagram for $\lambda$.

\begin{center}
\begin{tabular}{cccc}
  a & b & a & b \\
  c & d & c & d \\
  a & b & a & b \\
  c & & & \\
\end{tabular}
\end{center}

Let $s_{\lambda}(x)$ denote the Schur function corresponding to a partition $\lambda$. R. P. Stan-
ley gave the following conjecture in the open problem session of FPSAC’03.

**Theorem 1.1.** Let

$$z = \sum_\lambda \omega(\lambda)s_{\lambda}.$$ 

Here the sum runs over all partitions $\lambda$. Then we have

$$\log z = \sum_{n \geq 1} \frac{1}{2n} a^n (b^n - c^n) p_{2n} = \sum_{n \geq 1} \frac{1}{4n} a^n b^n c^n d^n p_{2n}^2 \in \mathbb{Q}[ [p_1, p_3, p_5, \ldots] ]. \quad (1.1)$$

Here $p_r = \sum_{i \geq 1} x_i^r$ denote the $r$th power sum symmetric function.
As a special case of this open problem, if we put \( b = c = a^{-1} \) and \( d = a \), and check the constant term of the both sides, then we obtain the following simple case:

**Corollary 1.2.** Let
\[
y = \sum_{\lambda, \lambda' \text{ even}} s_\lambda(x).
\]
Here the sum runs over all partitions \( \lambda \) such that \( \lambda \) and \( \lambda' \) are even partitions (i.e. with all parts even). Then we have
\[
\log y - \sum_{n \geq 1} \frac{1}{4n} p_{2n}^2 \in \mathbb{Q}[[p_1, p_3, p_5, \ldots]].
\]

In the rest of this section we briefly recall the definition of Pfaffians. For a detailed explanation of Pfaffians, the reader can consult [11] and [21]. Let \( n \) be a non-negative integer and assume we are given a \( 2n \times 2n \) skew-symmetric matrix \( A = (a_{ij})_{1 \leq i, j \leq 2n} \), (i.e. \( a_{ji} = -a_{ij} \)), whose entries \( a_{ij} \) are in a commutative ring. The Pfaffian of \( A \) is, by definition,
\[
\text{Pf}(A) = \sum \varepsilon(\sigma_1, \sigma_2, \ldots, \sigma_{2n-1}, \sigma_{2n}) a_{\sigma_1 \sigma_2} \cdots a_{\sigma_{2n-1} \sigma_{2n}}.
\]
where the summation is over all partitions \( \{\{\sigma_1, \sigma_2\} <, \ldots, \{\sigma_{2n-1}, \sigma_{2n}\} \} \) of \([2n]\) into 2-elements blocks, and where \( \varepsilon(\sigma_1, \sigma_2, \ldots, \sigma_{2n-1}, \sigma_{2n}) \) denotes the sign of the permutation
\[
\begin{pmatrix}
1 & 2 & \cdots & 2n \\
\sigma_1 & \sigma_2 & \cdots & \sigma_{2n}
\end{pmatrix}.
\]
We call a partition \( \{\{\sigma_1, \sigma_2\} <, \ldots, \{\sigma_{2n-1}, \sigma_{2n}\} \} \) of \([2n]\) into 2-elements blocks a matching or 1-factor of \([2n]\).

## 2 Minor Summation Formula

First we restrict our attention to the finite variables case. Let \( n \) be a non-negative integer. We put
\[
y_n = y_n(x_{2n}) = \sum_{\lambda, \lambda' \text{ even}} s_\lambda(X_{2n}) = \sum_{\lambda, \lambda' \text{ even}} s_\lambda(x_1, \ldots, x_{2n}).
\]
where \( s_\lambda(X_{2n}) \) is the Schur function corresponding to a partition \( \lambda \) in \( 2n \) variables \( x_1, \ldots, x_{2n} \). Then there is a known formula which is originally due to Littlewood as follows. (See [23]).
\[
y_n(x_{2n}) = \prod_{1 \leq i < j \leq 2n}(x_i - x_j)^{-1} \text{Pf} \left[ \begin{array}{c}
x_i - x_j \\
1 - x_i x_j
\end{array} \right]_{1 \leq i < j \leq 2n}.
\]
The aim of this section is to prove the following theorem which generalize this identity.

**Theorem 2.1.** Let \( n \) be a positive integer and let \( \omega(\lambda) \) be as defined in Section 1. Let
\[
z_n = z_n(x_{2n}) = \sum_{\ell(\lambda) \leq 2n} \omega(\lambda) s_\lambda(X_{2n}) = \sum_{\ell(\lambda) \leq 2n} \omega(\lambda) s_\lambda(x_1, \ldots, x_{2n})
\]
(2.3)
be the sum restricted to $2n$ variables. Then we have
\begin{equation}
  z_n(X_{2n}) = \prod_{1 \leq i < j \leq 2n} \frac{(abcd)^{\binom{n}{2}}}{(I_n + x_i - x_j)} \text{Pf} (p_{ij})_{1 \leq i < j \leq 2n},
\end{equation}

where $p_{ij}$ is defined by
\begin{equation}
  p_{ij} = \begin{vmatrix}
    x_i + ax^2_i & 1 - a(b + c)x_i - abx^2_i \\
    x_j + ax^2_j & 1 - a(b + c)x_j - abx^2_j \\
  \end{vmatrix}
  (1 - abx^3_i)(1 - abx^3_j)(1 - abdx^3_j).
\end{equation}

Let $m$, $n$, and $r$ be integers such that $r \leq m, n$ and let $T$ be an $m \times n$ matrix. For any index sets $I = \{i_1, \ldots, i_r\} \subseteq [m]$ and $J = \{j_1, \ldots, j_r\} \subseteq [n]$, let $\Delta_{IJ}(A)$ denote the submatrix obtained by selecting the rows indexed by $I$ and the columns indexed by $J$. If $r = m$ and $I = [m]$, we simply write $\Delta_I(A)$ for $\Delta_{[m]}^m(A)$. Similarly, if $r = n$ and $J = [n]$, we write $\Delta^J(A)$ for $\Delta_{[n]}^n(A)$. For any finite set $S$ and a non-negative integer $r$, let $\binom{S}{r}$ denote the set of all $r$-element subsets of $S$. We cite a theorem from [S] which we call a minor summation formula:

**Theorem 2.2.** Let $n$ and $N$ be non-negative integers such that $2n \leq N$. Let $T = (t_{ij})_{1 \leq i \leq 2n, 1 \leq j \leq N}$ be a $2n \times N$ rectangular matrix, and let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a skew-symmetric matrix of size $N$. Then
\begin{equation}
  \sum_{I \in \binom{[N]}{\lfloor \frac{n}{2} \rfloor}} \text{Pf} \left( \Delta^I(T) \right) \det \left( \Delta_I(T) \right) = \text{Pf} \left( TA^T \right).
\end{equation}

If we put $Q = (Q_{ij})_{1 \leq i, j \leq 2n} = TA^T$, then its entries are given by
\begin{equation}
  Q_{ij} = \sum_{1 \leq k < \leq N} a_{ik} \text{det} \left( \Delta^J_{kl}(T) \right), \quad (1 \leq i, j \leq 2n).
\end{equation}

Here we write $\Delta^J_{kl}(T)$ for $\Delta^J_{\{k\}\{l\}}(T) = \begin{vmatrix} t_{ik} & t_{il} \\ t_{jk} & t_{jl} \end{vmatrix}$. \hfill \Box

Before we proceed to the proof of Theorem 2.1 we cite a lemma from [S]. The proof is not difficult, but we omit the proof and the reader can consult [S], Section 4, Lemma 7.

**Lemma 2.3.** Let $x_i$ and $y_j$ be indeterminates, and let $n$ be a non-negative integer. Then
\begin{equation}
  \text{Pf} [x_{ij}]_{1 \leq i < j \leq 2n} = \prod_{i=1}^{n} x_{2i-1} \prod_{i=1}^{n} y_{2i}. \hfill \Box
\end{equation}

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ satisfying $\ell(\lambda) \leq m$, we associate a decreasing sequence $\lambda + \delta_m$ which is usually denoted by $l = (l_1, \ldots, l_m)$, where $\delta_m = (m-1, m-2, \ldots, 0)$.

**Lemma 2.4.** Let $n$ be a non-negative integer. Let $\lambda = (\lambda_1, \ldots, \lambda_{2n})$ be a partition such that $\ell(\lambda) \leq 2n$, and put $l = (l_1, \ldots, l_{2n}) = \lambda + \delta_{2n}$. Define a $2n$ by $2n$ skew-symmetric matrix $A = (a_{ij})_{1 \leq i, j \leq 2n}$ by
\begin{equation}
  a_{ij} = a^{[l_i-1/2]} b^{[l_i-1/2]} c^{[l_j/2]} d^{[l_j/2]}
\end{equation}
for $i < j$, and as $a_{ij} = -a_{ij}$ holds for any $1 \leq i, j \leq 2n$. Then we have
\begin{equation}
  \text{Pf} [A]_{1 \leq i, j \leq 2n} = (abcd)^{\binom{n}{2}} \omega(\lambda).
\end{equation}
Proof. By Lemma 2.2, we have
\[ \text{Pf}[A] = \prod_{i=1}^{n} a^{i(k_{2i-1} - 1/2) b_{i(k_{2i-1} - 1/2)} c_{i(k_{2i-1} - 1/2)} d_{i(k_{2i-1} - 1/2)}.} \]

Since \( k_{2i-1} - 1 = \lambda_{2i-1} + 2(n - i) \) and \( l_{2j} = \lambda_{2j} + 2(n - j) \), this Pfaffian becomes
\[ \prod_{i=1}^{n} a^{i(k_{2i-1} - 1/2) + n-i} b_{i(k_{2i-1} - 1/2) + n-i} c_{i(k_{2i-1} - 1/2) + n-j} d_{i(k_{2i-1} - 1/2) + n-j}, \]

which is easily seen to be \((abcd)^{G2}_{\omega}(\lambda). \)

Now we are in the position to give a proof of Theorem 2.3.

**Proof of Theorem 2.3.** By Theorem 2.2 it is enough to compute
\[ \beta_{ij} = \sum_{k \geq 0}^{a(k-1/2)} b_{(k-1/2)} c_{(k-1/2)} d_{(k-1/2)} \left| \begin{array}{cc} a^k & x^i_j \\ b^j_k & x^j_i \end{array} \right|. \]

Let \( f_{kl}^{ij} = a^{i(k-1/2)} b_{l(k-1/2)} c_{d(l-1/2)} x^k_i x^j_l \), then, this sum can be divided into four cases, i.e.
\[ \beta_{ij} = \sum_{k=2r+1, l=2s} f_{kl}^{ij} + \sum_{k=2r+1, l=2s} f_{kl}^{ij} + \sum_{k=2r+1, l=2s} f_{kl}^{ij} + \sum_{k=2r+1, l=2s} f_{kl}^{ij}. \]

We compute each case:

(i) If \( k = 2r+1 \) and \( l = 2s \) for \( r \geq s \geq 0 \), then
\[ \sum_{k=2r+1, l=2s} f_{kl}^{ij} = \sum_{r \geq s \geq 0} a^{r} b^{r} c^{s} d^{s} \left| \begin{array}{cc} x^{2r+1}_{j} & x^{2s}_{j} \\ x^{r+1}_{j} & x^{s}_{j} \end{array} \right| = \sum_{r \geq s \geq 0} c^{s} d^{s} \frac{a^{r} b^{r} x^{2s+1}_{j}}{1-abx^2_j} x^{4s}_{j} = \frac{(x_i - x_j)(1+abx_i x_j)}{(1-abx^2_j)(1-abx^2_j)(1-abcdx^2_i x_j)}. \]

In the same way we obtain the followings by straight forward computations.

(ii) If \( k = 2r \) and \( l = 2s \) for \( r \geq s \geq 0 \), then
\[ \sum_{k=2r, l=2s} f_{kl}^{ij} = \frac{a(x_i - x_j^2)}{(1-abx^2_j)(1-abx^2_j)(1-abcdx^2_i x_j)}. \]

(iii) If \( k = 2r+1 \) and \( l = 2s+1 \) for \( r \geq s \geq 0 \), then
\[ \sum_{k=2r+1, l=2s+1} f_{kl}^{ij} = \frac{abx_i x_j (x_i^2 - x_j^2)}{(1-abx^2_j)(1-abx^2_j)(1-abcdx^2_i x_j)}. \]

(iv) If \( k = 2r+2 \) and \( l = 2s+1 \) for \( r \geq s \geq 0 \), then
\[ \sum_{k=2r+2, l=2s+1} f_{kl}^{ij} = \frac{acx_i x_j (x_i - x_j)(1+abx_i x_j)}{(1-abx^2_j)(1-abx^2_j)(1-abcdx^2_i x_j)}. \]
Summing up these four identities, we obtain

\[ \beta_{ij} = \frac{(x_i - x_j)(1 + abx_i x_j + ax_i + x_j + abcx_i x_j + x_i x_j + acx_i x_j(1 + abx_i x_j))}{(1 - abx_i^2)(1 - abx_j^2)(1 - abcdx_i x_j^2)}. \]

It is easy to see the numerator is written by the determinant, and this completes the proof. □

3 Cauchy Type Pfaffian Formulas

The aim of this section is to derive (3.4). In the next section we will use this identity to prove Stanley’s open problem. First we prove a fundamental Pfaffian identity, i.e. Theorem 3.1 and deduce all the identities in this section from this theorem. In the latter half of this section we also show that we can derive Sundquist’s Pfaffian identities obtained in [24] from our theorem although these identities have no direct relation to Stanley’s open problem. In that sense our theorem can be regarded as a generalization of Sundquist’s Pfaffian identities (See [24]). An intensive generalization was conjectured in [13] and proved in [7]. There must be several ways to prove this type of identity. In [24] Sundquist gave a combinatorial proof of his Pfaffian identities. In [7] the authors adopted an algebraic method to prove identities of this type. Here we give an analytic proof of our theorem, in which we regard both sides of this identity as meromorphic functions and check the Laurent series expansion at each isolated pole in the Riemann sphere. The idea to use complex analysis to prove various determinant and Pfaffian identities is first hinted by Prof. H. Kawamoko to the author, and the author recognized this can be a powerful tool to prove various identities including determinants and Pfaffians. This idea was also used to prove a Pfaffian-Hafnian analogue of Borchardt’s identity in [5]. In this section, we first state our theorems and later give proofs of them.

First we fix notation. Let \( n \) be an non-negative integer. Let \( X = (x_1, \ldots, x_{2n}), Y = (y_1, \ldots, y_{2n}), A = (a_1, \ldots, a_{2n}) \) and \( B = (b_1, \ldots, b_{2n}) \) be \( 2n \)-tuples of variables. Set \( V^n_{ij}(X, Y; A, B) \) to be

\[
\begin{cases}
  a_i x_i^{n-j} y_j^{1-j} & \text{if } 1 \leq j \leq n, \\
  b_i x_i^{n-j} y_j^{1-j} & \text{if } n+1 \leq j \leq 2n,
\end{cases}
\]

for \( 1 \leq i \leq 2n \), and define \( V^n(X, Y; A, B) \) by

\[ V^n(X, Y; A, B) = \det (V^n_{ij}(X, Y; A, B))_{1 \leq i \leq j \leq 2n}. \]

For example, if \( n = 1 \), then we have \( V^1(X, Y; A, B) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \) and if \( n = 2 \), then \( V^2(X, Y; A, B) \) looks as follows:

\[
V^2(X, Y; A, B) = \begin{vmatrix}
  a_1 x_1 & a_1 y_1 & b_1 x_1 & b_1 y_1 \\
  a_2 x_2 & a_2 y_2 & b_2 x_2 & b_2 y_2 \\
  a_3 x_3 & a_3 y_3 & b_3 x_3 & b_3 y_3 \\
  a_4 x_4 & a_4 y_4 & b_4 x_4 & b_4 y_4
\end{vmatrix}.
\]

The main result of this section is the following theorem.
Theorem 3.1. Let \( n \) be a positive integer. Let \( X = (x_1, \ldots, x_{2n}) \), \( Y = (y_1, \ldots, y_{2n}) \), \( A = (a_1, \ldots, a_{2n}) \), \( B = (b_1, \ldots, b_{2n}) \), \( C = (c_1, \ldots, c_{2n}) \) and \( D = (d_1, \ldots, d_{2n}) \) be \( 2n \)-tuples of variables. Then

\[
Pf \begin{bmatrix}
  a_i & b_i & c_i & d_i \\
  a_j & b_j & c_j & d_j \\
  x_i & y_i & x_j & y_j
\end{bmatrix}_{1 \leq i < j \leq 2n} = \frac{V^n(X, Y; A, B)V^n(X, Y; C, D)}{\prod_{1 \leq i < j \leq 2n} (x_i - x_j)(1 - tx_i x_j)}.
\]

(3.1)

The following proposition is obtained easily by elementary transformations of the matrices and we will prove it later.

**Proposition 3.2.** Let \( n \) be a positive integer. Let \( X = (x_1, \ldots, x_{2n}) \) be a \( 2n \)-tuple of variables and let \( t \) be an indeterminate. Then

\[
V^n(X, 1 + tX^2; X, 1) = (-1)^{\binom{n}{3}} \prod_{1 \leq i < j \leq 2n} (x_i - x_j),
\]

(3.2)

where \( 1 \) denotes the \( 2n \)-tuple \((1, \ldots, 1)\), and \( 1 + tX^2 \) denotes the \( 2n \)-tuple \((1 + tx_1^2, \ldots, 1 + tx_{2n}^2)\).

Let \( t \) be an arbitrary indeterminate. If we set \( y_i = 1 + tx_i^2 \) in (3.1), then

\[
\begin{bmatrix} x_i & 1 + tx_i^2 \\ x_j & 1 + tx_j^2 \end{bmatrix} = (x_i - x_j)(1 - tx_i x_j)
\]

and (3.2) immediately implies the following corollary.

**Corollary 3.3.** Let \( n \) be a non-negative integer. Let \( X = (x_1, \ldots, x_{2n}) \), \( A = (a_1, \ldots, a_{2n}) \), \( B = (b_1, \ldots, b_{2n}) \), \( C = (c_1, \ldots, c_{2n}) \) and \( D = (d_1, \ldots, d_{2n}) \) be \( 2n \)-tuples of variables. Then

\[
Pf \begin{bmatrix}
  (a_i b_j - a_j b_i)(c_i d_j - c_j d_i) \\
  (x_i - x_j)(1 - tx_i x_j)
\end{bmatrix}_{1 \leq i < j \leq 2n} = \frac{V^n(X, 1 + tX^2; A, B)V^n(X, 1 + tX^2; C, D)}{\prod_{1 \leq i < j \leq 2n} (x_i - x_j)(1 - tx_i x_j)}.
\]

(3.3)

In particular, we have

\[
Pf \begin{bmatrix}
  a_i b_j - a_j b_i \\
  1 - tx_i x_j
\end{bmatrix}_{1 \leq i < j \leq 2n} = (-1)^{\binom{n}{3}} t^{\binom{n}{2}} \prod_{1 \leq i < j \leq 2n} (1 - tx_i x_j).
\]

(3.4)

In the latter half of this section we show that we can derive Sundquist’s Pfaffian identities from ours. Let \( X = (x_1, \ldots, x_{2n}) \) and \( A = (a_1, \ldots, a_{2n}) \) be \( 2n \)-tuples of variables and let \( S_{2n} \) act on each by permuting indices. For compositions \( \alpha \) and \( \beta \) of length \( n \), we write

\[
a_{\alpha, \beta}(X; A) = \sum_{\sigma \in S_{2n}} \epsilon(\sigma) \sigma \left( a_1 x_1^{\alpha_1} \cdots a_{n-1} x_{n-1}^{\alpha_{n-1}} x_n^{\beta_1} \cdots x_{2n}^{\beta_n} \right).
\]

This is to say

\[
a_{\alpha, \beta}(X; A) = \begin{vmatrix}
  a_1 x_1^{\alpha_1} & \cdots & a_1 x_1^{\alpha_n} & x_1^{\beta_1} & \cdots & x_1^{\beta_n} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  a_{2n} x_{2n}^{\alpha_1} & \cdots & a_{2n} x_{2n}^{\alpha_n} & x_{2n}^{\beta_1} & \cdots & x_{2n}^{\beta_n}
\end{vmatrix}.
\]
Let $n$ be a non-negative integer. Let $\mathcal{P}_n$ denote the set of all partitions of the form $\lambda = (\alpha_1, \ldots, \alpha_r | \alpha_1 + 1, \ldots, \alpha_r + 1)$ in Frobenius notation with $\alpha_r \leq n - 1$. For example,
\[ \mathcal{P}_4 = \{ \emptyset, 1^2, 2^1, 2^3, 3^1, 3^2 1, 3^2 2^1, 3^3 \}. \]

We put
\[ U^n(X; A) = \sum_{\lambda \in \mathcal{P}_n} \sum_{\mu \in \mathcal{P}_n} a_{\lambda, \mu}(X; A). \]

**Theorem 3.4.** Let $n$ be a non-negative integer. Let $X = (x_1, \ldots, x_{2n})$ and $A = (a_1, \ldots, a_{2n})$ be $2n$-tuples of variables. Then
\[ U^n(X; A) = V^n(X, 1 - X^2; A, 1), \tag{3.5} \]
where $1 - X^2 = (1 - x_1^2, \ldots, 1 - x_{2n}^2)$ and $1 = (1, \ldots, 1)$.

From Theorem 3.1, Corollary 3.3 and Theorem 3.4 we obtain the following Pfaffian identities which are obtained in [24] (See [23], Theorem 2.1).

**Corollary 3.5.** (Sundquist)
\[
Pf \begin{bmatrix} a_i - a_j \\ x_i + x_j \end{bmatrix}_{1 \leq i < j \leq 2n} = (-1)^{\binom{2n}{2}} a_{2i, 2j}(X; A) \prod_{1 \leq i < j \leq 2n} (x_i + x_j) \tag{3.6} \]
\[
Pf \begin{bmatrix} a_i - a_j \\ 1 + x_i x_j \end{bmatrix}_{1 \leq i < j \leq 2n} = \frac{U^n(X; A)}{\prod_{1 \leq i < j \leq 2n} (1 + x_i x_j)}. \tag{3.7} \]

Now we state the proofs of our theorems. Before we prove Theorem 3.4 we need two lemmas. Let $n$ and $r$ be integers such that $2n \geq r \geq 0$. Let $X = (x_1, \ldots, x_{2n})$ be a $2n$-tuple of variables and let $1 \leq k_1 < \cdots < k_r \leq 2n$ be a sequence of integers. Let $X^{(k_1, \ldots, k_r)}$ denote the $(2n - r)$-tuple of variables obtained by removing the variables $x_{k_1}, \ldots, x_{k_r}$ from $X_{2n}$.

**Lemma 3.6.** Let $n$ be a positive integer. Let $X = (x_1, \ldots, x_{2n})$ and $A = (a_1, \ldots, a_{2n})$ be $2n$-tuples of variables. Let $k, l$ be any integers such that $1 \leq k < l \leq 2n$. Then
\[ V^n(X, 1; A, 1)_{x_j = x_k} = (-1)^{k+l+n}(a_k - a_l) \]
\[ \times \prod_{i \neq k, l}^{2n} (x_i - x_k) \cdot V^{n-1}(X^{(k,l)}, 1^{(k,l)}; A^{(k,l)}, 1^{(k,l)}), \]
where $1$ denotes the $2n$-tuple $(1, \ldots, 1).

**Proof.** Without loss of generality we may assume that $k = 2n - 1$ and $l = 2n$. From the definition, $V^n(X, 1; A, 1)$ is in the form of
\[ \det \begin{bmatrix} a_{i}x_i^{n-j} & \text{if } 1 \leq j \leq n, \\ x_i^{2n-j} & \text{if } n + 1 \leq j \leq 2n. \end{bmatrix}_{1 \leq i, j \leq 2n}. \]
For example, when $n = 3$, if we substitute $x_6 = x_5$ into this determinant, we obtain
\[ V^n(X, 1; A, 1)_{x_6 = x_5} = \begin{bmatrix} a_1 x_1^2 & a_1 x_1 & a_1 & x_1 & 1 \\ a_2 x_2^3 & a_2 x_2 & a_2 & x_2 & 1 \\ a_3 x_3^4 & a_3 x_3 & a_3 & x_3 & 1 \\ a_4 x_4^5 & a_4 x_4 & a_4 & x_4 & 1 \\ a_5 x_5^6 & a_5 x_5 & a_5 & x_5 & 1 \\ a_6 x_6^7 & a_6 x_6 & a_6 & x_6 & 1 \end{bmatrix}. \]
First subtract the last row from the second last row, and next factor out $(a_{2n-1} - a_{2n})$ from the second last row. Then, subtract $a_{2n}$ times the second last row from the last row. Thus we obtain

\[ V^n(X, 1; A, 1)|_{x_{2n} = x_{2n-1}} = (a_{2n-1} - a_{2n}) \]

\[
\times \det \begin{pmatrix}
 x_{i,j}^{n-j} & \text{if } 1 \leq i \leq 2n-2 \text{ and } 1 \leq j \leq n, \\
 x_{i,j}^{2n-j} & \text{if } 1 \leq i \leq 2n-2 \text{ and } n + 1 \leq j \leq 2n, \\
 x_{i,j}^{n-j} & \text{if } i = 2n-1 \text{ and } 1 \leq j \leq n, \\
 0 & \text{if } i = 2n-1 \text{ and } n+1 \leq j \leq 2n, \\
 0 & \text{if } i = 2n \text{ and } 1 \leq j \leq n, \\
 x_{i,j}^{2n-j} & \text{if } i = 2n \text{ and } n+1 \leq j \leq 2n.
\end{pmatrix}_{1 \leq i,j \leq 2n}
\]

If we use our example, then the matrix after these transformations looks like

\[
V^3(X, 1; A, 1)|_{x_6 = x_5} = (a_5 - a_6)
\]

\[
\begin{array}{ccccccccc}
a_1x_1^2 & a_1x_1 & a_1 & x_1^2 & x_1 & 1 \\
a_2x_2^2 & a_2x_2 & a_2 & x_2^2 & x_2 & 1 \\
a_3x_3^2 & a_3x_3 & a_3 & x_3^2 & x_3 & 1 \\
a_4x_4^2 & a_4x_4 & a_4 & x_4^2 & x_4 & 1 \\
x_5^2 & x_5 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & x_5^2 & x_5 & 1
\end{array}
\]

Now subtract $x_{2n-1}$ times the second column from the first column, then subtract $x_{2n-1}$ times the third column from the second column, and so on, until we subtract $x_{2n-1}$ times the $n$th column from the $(n-1)$th column. Next subtract $x_{2n-1}$ times the $(n+1)$th column from the $(n+1)$th column, then subtract $x_{2n-1}$ times the $(n+2)$th column from the $(n+2)$th column, and so on, until we subtract $x_{2n-1}$ times the $2n$th column from the $(2n-1)$th column. If we expand the resulting determinant along the last two rows and factor out $(x_i - x_{2n})$ from the ith row for $1 \leq i \leq 2n-2$, then we obtain

\[
V^n(X, 1; A, 1)|_{x_{2n} = x_{2n-1}} = (-1)^{n+1} (a_{2n-1} - a_{2n}) \prod_{i=1}^{2n-2} (x_i - x_{2n-1})
\]

\[
\times V^{n-1}(X, (2n-1, 2n), 1, (2n-1, 2n), A, (2n-1, 2n), 1, (2n-1, 2n)),
\]

and this proves our lemma. \(\square\)

**Lemma 3.7.** Let $n$ be a positive integer. Let $X = (x_1, \ldots, x_{2n})$, $A = (a_1, \ldots, a_{2n})$ and $C = (c_1, \ldots, c_{2n})$ be $2n$-tuples of variables. Then the following identity holds.

\[
\sum_{k=1}^{2n-1} \frac{(x_k - x_{2n})}{x_k - x_{2n}} (a_k - a_{2n}) (c_k - c_{2n})
\]

\[
\times V^{n-1}(X, (2n-1, 2n), 1, (2n-1, 2n)) V^{n-1}(X, (2n-1, 2n), 1, (2n-1, 2n), C, (2n, 2n))
\]

\[
= \frac{V^n(X, 1; A, 1)V^n(X, 1; C, 1)}{\prod_{i=1}^{2n-1} (x_i - x_{2n})}.
\]

Here $1$ denotes the $2n$-tuples $(1, \ldots, 1)$.

**Proof.** It is enough to prove this lemma as an identity for a rational function in the complex variable $x_{2n}$. Further we may assume $x_1, \ldots, x_{2n-1}$ are distinct complex numbers. Denote by $F(x_{2n})$ the left-hand
The residue of the function \( F \) reside at \( G \).

On the other hand, the residue of the function claim is

\[
\lim_{x \to x_k} (x - x_k)F(x) = \text{Res}_{x = x_k} F(x).
\]

Thus it is shown that the principal part of \( \lim_{x \to x_k} (x - x_k)F(x) \) is up to \( (x - x_k) \), and deduce the general case to this special case. Thus our first step is to prove the first theorem in this section.

First we prove (3.1) when

\[
\text{Proof of Theorem 3.1.}
\]

Now we are in the position to prove the first theorem in this section.

**Proof of Theorem 3.1.** First we prove \( \text{Claim} \) when \( Y = B = D = 1 = (1, \ldots, 1) \), and deduce the general case to this special case. Thus our first claim is

\[
\text{Pf} \left[ \frac{(a_i - a_j)(c_i - c_j)}{x_i - x_j} \right]_{1 \leq i < j \leq 2n} = \frac{V^n(X; 1; A, 1)}{\prod_{1 \leq i < j \leq 2n} (x_i - x_j)} V^n(X; 1; C, 1) \tag{3.8}
\]

We proceed by induction on \( n \). When \( n = 1 \), it is trivial since \( V^1(X; 1; A, 1) = a_1 - a_2 \) and \( V^1(X; 1; C, 1) = c_1 - c_2 \). Assume \( n \geq 2 \) and the identity holds up to \( (n - 1) \). Expanding the Pfaffian along the last row/column and using the induction hypothesis, we obtain

\[
\text{Pf} \left[ \frac{(a_i - a_j)(c_i - c_j)}{x_i - x_j} \right]_{1 \leq i < j \leq 2n} = \prod_{k=1}^{2n-1} \left( -1 \right)^{k-1} \frac{(a_k - a_{2n})(c_k - c_{2n})}{x_k - x_{2n}}
\]

\[
\times \prod_{1 \leq i < j \leq 2n} (x_i - x_j)
\]

Substituting \( \prod_{1 \leq i \leq 2n-1} (x_i - x_j) \equiv \left( -1 \right)^{k-1} \prod_{1 \leq i \leq 2n-1} (x_i - x_j) \) into this formula.
identity, we have
\[
Pf \left[ \frac{(a_i - a_j)(c_i - c_j)}{x_i - x_j} \right]_{1 \leq i < j \leq 2n} = \prod_{1 \leq i < j \leq 2n} \frac{\prod_{k=1}^{2n-1} (a_k - a_{2n})(c_k - c_{2n}) \cdot \prod_{i \neq k}^{2n-1} (x_k - x_i)}{x_k - x_{2n}} \\
\times V_{n-1}(X(k, 2n), 1(k, 2n), A(k, 2n), C(k, 2n), 1(k, 2n)).
\]

Thus, by Lemma 5.8, this equals
\[
\frac{V^n(X, 1; A, 1)V^n(X, 1; C, 1)}{\prod_{1 \leq i < j \leq 2n} (x_i - x_j)},
\]
and this proves 5.8. The general identity 5.8 is an easy consequence of 5.5 by substituting \(\frac{x_i}{y_i} \) into \(x_i\), \(\frac{x_i}{y_i} \) into \(b_i\) and \(\frac{x_i}{y_i} \) into \(c_i\) for \(1 \leq i \leq 2n\) in the both sides of 5.5. In fact, if we write \(X/Y\) for \((x_1/y_1, \ldots, x_{2n}/y_{2n})\), \(A/B\) for \((a_1/b_1, \ldots, a_{2n}/b_{2n})\), and \(C/D\) for \((c_1/d_1, \ldots, c_{2n}/d_{2n})\), then we have
\[
\begin{align*}
V^n(X/Y, 1; A/B, 1) &= \prod_{i=1}^{2n} b_i^{-1} \prod_{i=1}^{2n} y_i^{-n+1} \cdot V^n(X, Y; A, B) \\
V^n(X/Y, 1; C/D, 1) &= \prod_{i=1}^{2n} d_i^{-1} \prod_{i=1}^{2n} y_i^{-n+1} \cdot V^n(X, Y; C, D).
\end{align*}
\]

Since the denominator of the right-hand side of 5.8 is
\[
\prod_{1 \leq i < j \leq 2n} (x_i/y_i - x_j/y_j) = \prod_{i=1}^{2n} y_i^{-2n+1} \prod_{1 \leq i < j \leq 2n} \left|\frac{x_i}{x_j} - \frac{y_i}{y_j}\right|,
\]
the right-hand side of 5.8 becomes
\[
\frac{V^n(X/Y, 1; A/B, 1)V^n(X/Y, 1; C/D, 1)}{\prod_{1 \leq i < j \leq 2n} (x_i/y_i - x_j/y_j)} = \prod_{i=1}^{2n} b_i^{-1} \prod_{i=1}^{2n} d_i^{-1} \prod_{i=1}^{2n} y_i^{-n+1} \cdot \frac{V^n(X, Y; A, B)V^n(X, Y; C, D)}{\prod_{1 \leq i < j \leq 2n} \left|\frac{x_i}{x_j} - \frac{y_i}{y_j}\right|}
\]
\]
On the other hand, the left-hand side of 5.8 becomes
\[
Pf \left[ \frac{(a_i - a_j)(c_i - c_j)}{y_i - y_j} \right]_{1 \leq i < j \leq 2n} = \prod_{i=1}^{n} (b_i^{-1}d_i^{-1}y_i) \cdot Pf \left[ \begin{array}{c|c|c|c}
\frac{a_i}{a_j} & b_i & c_i & d_i \\
\frac{b_i}{y_i} & x_i & y_i & y_j \\
\frac{c_j}{d_j} & x_j & y_j & y_j
\end{array} \right]_{1 \leq i, j \leq 2n}
\]
and this proves 5.1. □

Next we give a proof of Proposition 5.2.

**Proof of Proposition 5.2.** From definition, \(V^n(X, 1+tX^2; X, 1)\) is equal to
\[
\begin{pmatrix}
\left(\frac{x_i^{n-j+1}}{x_i} + (1 + tx_i^2)^{j-1}\right) & \text{if } 1 \leq j \leq n, \\
\left(\frac{x_i^{2n-j}}{x_i} + (1 + tx_i^2)^{j-n-1}\right) & \text{if } n + 1 \leq j \leq 2n
\end{pmatrix}
\]
\]
\]
\]
For example, if $n = 3$, then $V^3(X, 1 + tX^2; X, 1)$ is equal to
\[
\begin{vmatrix}
 x_1^3 & x_1^2(1 + tx_1^2) & x_1(1 + tx_1^2) & (1 + tx_1^2)^2 \\
x_2^3 & x_2^2(1 + tx_2^2) & x_2(1 + tx_2^2) & (1 + tx_2^2)^2 \\
x_3^3 & x_3^2(1 + tx_3^2) & x_3(1 + tx_3^2) & (1 + tx_3^2)^2 \\
x_4^3 & x_4^2(1 + tx_4^2) & x_4(1 + tx_4^2) & (1 + tx_4^2)^2 \\
x_5^3 & x_5^2(1 + tx_5^2) & x_5(1 + tx_5^2) & (1 + tx_5^2)^2 \\
x_6^3 & x_6^2(1 + tx_6^2) & x_6(1 + tx_6^2) & (1 + tx_6^2)^2
\end{vmatrix}
\]

By expanding $(1 + tx_i^2)^{j-1}$ and $(1 + tx_i^2)^{j-n-1}$, and performing appropriate elementary column transformations, this determinant becomes
\[
\det \begin{pmatrix}
x_i^n & x_i^{n-1} & x_i^{n-2} & \ldots & x_i & 1 \\
x_i^{n-1} & x_i^{n-2} & x_i^{n-3} & \ldots & x_i & 1 \\
x_i^{n-2} & x_i^{n-3} & x_i^{n-4} & \ldots & x_i & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x_i & 1 & & & & \\
x_i & 1 & & & & \\
x_i & 1 & & & &
\end{pmatrix}
\]

For example, if $n = 3$, then this determinant equals
\[
\begin{vmatrix}
x_1^3 & x_1^2 & x_1 & 1 \\
x_2^3 & x_2^2 & x_2 & 1 \\
x_3^3 & x_3^2 & x_3 & 1 \\
x_4^3 & x_4^2 & x_4 & 1 \\
x_5^3 & x_5^2 & x_5 & 1 \\
x_6^3 & x_6^2 & x_6 & 1
\end{vmatrix}
\]

We subtract $t$ times the first column from the $(n + 2)$th column, and subtract the $(n+1)$th column from the second column. Then we subtract $t$ times the second column from the $(n+3)$th column, and subtract the $(n+2)$th column from the third column. We continue this transformation until we subtract $t$ times the $(n-1)$th column from the $2n$th column, and subtract the $(2n-1)$th column from the $r$th column. Thus we obtain
\[
V^n(X, 1 + tX^2; X, 1) = \begin{pmatrix}
(t^{j-1}x_i^{j+n-1}) & \text{if } 1 \leq j \leq n, \\
x_i^{2n-j} & \text{if } n+1 \leq j \leq 2n
\end{pmatrix}_{1 \leq i, j \leq 2n}
\]

If we illustrate by the above example, then this determinant looks like
\[
\begin{vmatrix}
x_1^3 & tx_1^2 & t^2x_1 & x_1 & 1 \\
x_2^3 & tx_2^2 & t^2x_2 & x_2 & 1 \\
x_3^3 & tx_3^2 & t^2x_3 & x_3 & 1 \\
x_4^3 & tx_4^2 & t^2x_4 & x_4 & 1 \\
x_5^3 & tx_5^2 & t^2x_5 & x_5 & 1 \\
x_6^3 & tx_6^2 & t^2x_6 & x_6 & 1
\end{vmatrix}
\]

By the Vandermonde determinant, we can easily conclude that
\[
V^n(X, 1 + tX^2; X, 1) = (-1)^\binom{n+1}{2} t^{0+1+2+\cdots+(n-1)} \prod_{1 \leq i < j \leq 2n} (x_i - x_j).
\]

This completes the proof. \(\square\)

Corollary \ref{cor3.3} is an immediate consequence of Theorem \ref{thm5.4} and Proposition \ref{prop5.1}. So, in the rest of this section we derive Corollary \ref{cor3.3} from Theorem \ref{thm5.4}. The following proposition is an immediate consequence of the Laplace expansion formula and the Littlewood formula:
\[
\sum_{\lambda \in \mathbb{P}_n} s_\lambda(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (1 + x_ix_j), \quad (3.9)
\]
Thus, by Proposition 3.8, we complete our proof.

Proof. We write \( \Delta(X_I) = \prod_{i < j} (x_i - x_j) \). By the Laplace expansion formula, we have

\[
a_{\lambda,\mu}(X; A) = \sum_{I \in \binom{[2n]}{n}} (-1)^{|I| + (\frac{n+1}{2})} a_I \prod_{i,j \in I, i < j} (x_i - x_j)(1 + x_i x_j),
\]

where \( X_I \) stands for the \( n \)-tuple of variables with index set \( I \) in \([2n]\) and \( a_I = \prod_{i \in I} a_i \) for any subset \( I \subseteq [2n] \).

Proof of Theorem 3.4. As before we apply the Laplace expansion formula to \( V^n(X, 1 - X^2; A, 1) \) to obtain

\[
V^n(X, 1 - X^2; A, 1) = \sum_{I \in \binom{[2n]}{n}} (-1)^{|I| + (\frac{n+1}{2})} a_I \det \Delta^I(M) \det \Delta^{I^c}(M),
\]

where \( M = (x_i^{n-j}(1 - x_i^j)^{j-1})_{1 \leq i \leq 2n, 1 \leq j \leq n} \). By the Vandermonde determinant, we obtain

\[
\det \Delta^I(M) = \prod_{i, j \in I, i < j} \{x_i(1 - x_j^2) - x_j(1 - x_i^2)\}
= \prod_{i, j \in I, i < j} (x_i - x_j)(1 + x_i x_j).
\]

Thus we conclude that

\[
V^n(X, 1 - X^2; A, 1) = \sum_{I \in \binom{[2n]}{n}} (-1)^{|I| + (\frac{n+1}{2})} a_I \prod_{i,j \in I, i < j} (x_i - x_j)(1 + x_i x_j)
\]
\[
\quad \times \prod_{i,j \in I^c, i < j} (x_i - x_j)(1 + x_i x_j).
\]

Thus, by Proposition 3.8, we complete our proof. \( \square \)

Proof of Corollary 3.5. If we substitute \( x_i^2 \) into \( x_i \) and \( 1 \) into \( y_i, b_i \) and \( d_i \) for \( 1 \leq i \leq 2n \), then we obtain

\[
Pf \left[ \frac{a_i - a_j}{x_i + x_j} \right]_{1 \leq i < j \leq 2n} = \frac{V^n(X^2, 1; A, 1)V^n(X^2, 1; X, 1)}{\prod_{1 \leq i < j \leq 2n} (x_i^2 - x_j^2)},
\]
where $X^2 = (x_1^2, \ldots, x_{2n}^2)$. By Proposition 4.2,

$$V^n(X^2, 1, X, 1) = (-1)^{\frac{n}{2}} \prod_{1 \leq i < j \leq 2n} (x_i - x_j)$$

and this proves (4.3).

Next we state a proof of another identity. Substituting $t = -1$ and $b_i = 1$ for $1 \leq i \leq 2n$ in (4.3), we obtain

$$\text{Pr} \left[ \frac{a_i - a_j}{1 + x_i x_j} \right]_{1 \leq i < j \leq 2n} = \frac{V^n(X, 1 - X^2; A, 1)}{\prod_{1 \leq i < j \leq 2n} (1 + x_i x_j)}.$$

Thus, by Theorem 4.1, we immediately obtain the desired identity. This completes the proof. \(\square\)

### 4 A Proof of Stanley’s Open Problem

The key idea of our proof is the following proposition, which the reader can find in [19], Exercise 7.7, or [22], Section 3.

**Proposition 4.1.** Let $f(x_1, x_2, \ldots)$ be a symmetric function with infinite variables. Then $f \in \mathbb{Q}[p_\lambda]$ : all parts $\lambda_i > 0$ are odd if and only if

$$f(t, -t, x_1, x_2, \ldots) = f(x_1, x_2, \ldots).$$

Our strategy is simple. If we set $v_n(X_{2n})$ to be

$$\log z_n(X_{2n}) = \sum_{k \geq 1} \frac{1}{2k} a^k (b^k - c^k)p_{2k}(X_{2n}) - \sum_{k \geq 1} \frac{1}{4k} a^k b^k c^k d^k p_{2k}(X_{2n})^2$$

then we claim it satisfies

$$v_{n+1}(t, -t, X_{2n}) = v_n(X_{2n}).$$

(4.2)

This will eventually prove Theorem 1.1. As an immediate consequence of (4.2), (4.4), and (4.3), we obtain the following theorem:

**Theorem 4.2.** Let $X = (x_1, \ldots, x_{2n})$ be a 2n-tuple of variables. Then

$$z_n(X_{2n}) = (-1)^{\frac{n}{2}} V^n(X^2, 1 + abcdX^4; X + aX^2, 1 - a(b + c)X^2 - abcX^3) \prod_{i=1}^{2n} (1 - abx_i^2) \prod_{1 \leq i < j \leq 2n} (x_i - x_j) (1 - abcdx_i^2 x_j^2)$$

where $X^2 = (x_1^2, \ldots, x_{2n}^2)$, $1 + abcdX^4 = (1 + abdx_1^2, \ldots, 1 + abdx_{2n}^2)$, $X + aX^2 = (x_1 + ax_1^2, \ldots, x_{2n} + ax_{2n}^2)$ and $1 - a(b + c)X^2 - abcX^3 = (1 - a(b + c)x_1^2 - abcx_1^4, \ldots, 1 - a(b + c)x_{2n}^2 - abcx_{2n}^4)$. \(\square\)

The (4.3) is key expression to prove that $v_n(X_{2n})$ satisfies (4.2). Once one knows (4.3), then it is straightforward computation to prove Stanley’s open problem. The following proposition is the first step.

**Proposition 4.3.** Let $X = (x_1, \ldots, x_{2n})$ be a 2n-tuple of variables. Put

$$f_n(X_{2n}) = V^n(X^2, 1 + abcdX^4; X + aX^2, 1 - a(b + c)X^2 - abcX^3).$$

Then $f_n(X_{2n})$ satisfies

$$f_{n+1}(t, -t, X_{2n}) = (-1)^{n} 2t (1 - abt^2)(1 - act^2) \prod_{i=1}^{2n} (t^2 - x_i^2) \prod_{i=1}^{2n} (1 - abcdt^2 x_i^2) \cdot f_n(X_{2n}).$$

(4.4)
Proof. First, we put \( \xi_i = x_i^2, \eta_i = 1 + abdx_i, \alpha_i = x_i + ax_i^2, \beta_i = 1 - a(b + c)x_i^2 + abcx_i^2 \) and \( \zeta_i = \xi_i^{-1} \eta_i = x_i^{-2} + abdx_i^2 \) for \( 1 \leq i \leq 2n \).

Then

\[
\begin{align*}
f_{n+1}(X_{2n+2}) &= \det \left( \begin{bmatrix} \alpha_i \xi_i^{n+1-j} \eta_i^{j-1} & \beta_i \xi_i^{n+2-j} \eta_i^{j-2} \end{bmatrix} \right)_{1 \leq i, j \leq 2n+2} \\
&= \lim_{n \to \infty} \prod_{i=1}^{2n} \xi_i^n \cdot \det \left( \begin{bmatrix} \alpha_i \xi_i^{j-1} & \beta_i \xi_i^{j-n-2} \end{bmatrix} \right)_{1 \leq i, j \leq 2n+2}.
\end{align*}
\]

For example, if \( n = 2 \) then \( f_3(X_6) \) looks as follows:

\[
\begin{align*}
\prod_{i=1}^{2n} \xi_i^2 &= \begin{vmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\
\alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_1 \\
\alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_1 & \alpha_2 \\
\alpha_4 & \alpha_5 & \alpha_6 & \alpha_1 & \alpha_2 & \alpha_3 \\
\alpha_5 & \alpha_6 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\alpha_6 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5
\end{vmatrix}.
\end{align*}
\]

Now we subtract \( \zeta_1 \) times the \( n \)th column from the \((n+1)\)th column, then subtract \( \zeta_1 \) times the \((n-1)\)th column from the \( n \)th column, and so on, until we subtract \( \zeta_1 \) times the first column from the second column. Next we subtract \( \zeta_1 \) times the \((2n)\)th column from the \((2n+1)\)th column, then subtract \( \zeta_1 \) times the \((2n)\)th column from the \((2n+1)\)th column, and so on, until we subtract \( \zeta_1 \) times the \((n+2)\)th column from the \((n+3)\)th column. Thus we obtain \( f_{n+1}(X_{2n+2}) \) is equal to

\[
\begin{align*}
\lim_{n \to \infty} \prod_{i=1}^{2n} \xi_i^n \cdot \det \left( \begin{bmatrix} \alpha_i & \beta_i \xi_i^{-n-3}(\xi_i - \zeta_1) \end{bmatrix} \right)_{1 \leq i, j \leq 2n+2}.
\end{align*}
\]

If we illustrate in the above example, then this determinant looks

\[
\begin{align*}
\prod_{i=1}^{2n} \xi_i^2 &= \begin{vmatrix}
\alpha_1 & 0 & 0 & \beta_1 \\
\alpha_2 & \alpha_2 \xi_2 - \xi_1 & \beta_2 & \beta_2 \xi_2 - \zeta_1 \\
\alpha_3 & \alpha_3 \xi_3 - \xi_2 & \beta_3 & \beta_3 \xi_3 - \zeta_1 \\
\alpha_4 & \alpha_4 \xi_4 - \xi_3 & \beta_4 & \beta_4 \xi_4 - \zeta_1 \\
\alpha_5 & \alpha_5 \xi_5 - \xi_4 & \beta_5 & \beta_5 \xi_5 - \zeta_1 \\
\alpha_6 & \alpha_6 \xi_6 - \xi_5 & \beta_6 & \beta_6 \xi_6 - \zeta_1
\end{vmatrix}.
\end{align*}
\]

Here, if we assume \( \xi_1 = \xi_2 \) and \( \xi_1 = \xi_2 \) hold, then \( f_{n+1}(X_{2n+2}) \) is equal to

\[
(-1)^n \begin{vmatrix}
\alpha_1 & \beta_1 \\
\alpha_2 & \beta_2
\end{vmatrix} \prod_{i=3}^{2n+2} \xi_i \eta_i \cdot f_n(x_3, \ldots, x_{2n+2}).
\]

Now we substitute \( X_{2n+2} = (t, -t, X_{2n}) \) into this identity, then, since we have \( \xi_1 = \xi_2 = t^2, \xi_1 = \xi_2 = t^2 + abct^2 \) and

\[
\begin{align*}
\left| \begin{array}{cc}
\xi_1 & \eta_1 \\
\xi_2 & \eta_2
\end{array} \right| &= (t^2 - x_1^2)(1 - abdx_1), \\
\left| \begin{array}{cc}
\alpha_1 & \beta_1 \\
\alpha_2 & \beta_2
\end{array} \right| &= 2t(1 - abt^2)(1 - acd^2),
\end{align*}
\]

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thus we obtain
\[
f_{n+1}(t, -t, X_{2n}) = (-1)^n \cdot 2t(1 - abt^2)(1 - ac^2) \prod_{i=1}^{2n}(t^2 - x_i^2)(1 - abcdt^2 x_i^2) \cdot f_n(X_{2n}).
\]
This proves our proposition. \(\square\)

**Proposition 4.4.** Let \(X = (x_1, \ldots, x_{2n})\) be a 2n-tuple of variables. Then
\[
z_{n+1}(t, -t, X_{2n}) = \frac{1 - act^2}{(1 - abt^2)(1 - abcdt^4)} \prod_{i=1}^{2n}(1 - abcdt^2 x_i^2) \cdot z_n(X_{2n}).
\]
(4.5)

**Proof.** By Theorem \(\ref{thm:proof} \) we have
\[
z_n(X_{2n}) = (-1)(^2) \frac{f_n(X_{2n})}{\prod_{i=1}^{2n}(1 - abx_i^2) \prod_{1 \leq i < j \leq 2n} (x_i - x_j) (1 - abcdt^2 x_i^2)}.
\]
This implies
\[
z_{n+1}(t, -t, X_{2n}) = (-1)(^2+1) \frac{2t(1 - abt^2)^2(1 - abcdt^4) \prod_{i=1}^{2n}(t^2 - x_i^2)(1 - abcdt^2 x_i^2)^2}{(1 - abt^2)(1 - abcdt^4) \prod_{i=1}^{2n}(1 - abcdt^2 x_i^2)} \cdot f_{n+1}(t, -t, X_{2n}) \prod_{1 \leq i < j \leq 2n} (x_i - x_j) (1 - abcdt^2 x_i^2).
\]
Thus, substituting (4.4), we obtain the desired identity. \(\square\)

Now we are in the position to complete our proof of Stanley’s open problem.

**Proof of Theorem \(\ref{thm:main} \).** Immediately implies
\[
\log z_{n+1}(t, -t, X_{2n}) = \log z_n(X_{2n}) + \log \frac{1}{1 - abt^2} - \log \frac{1}{1 - act^2} + \log \frac{1}{1 - abcdt^4} + \sum_{i=1}^{2n} \log \frac{1}{1 - abcdt^2 x_i^2}.
\]
(4.6)

On the other hand, \(p_{2k}(t, -t, X_{2n}) = 2t^{2k} + \sum_{i=1}^{2n} x_i^{2k}\) implies
\[
\sum_{k \geq 1} \frac{a^n (b^n - c^n)}{2k} p_{2k}(t, -t, X_{2n}) = \sum_{k \geq 1} \frac{a^n (b^n - c^n)}{2k} p_{2k}(X_{2n}) + \log \frac{1}{1 - abt^2} - \log \frac{1}{1 - act^2},
\]
\[
\sum_{k \geq 1} \frac{a^n b^n c^n d^n}{4k} p_{2k}(t, -t, X_{2n})^2 = \sum_{k \geq 1} \frac{a^n b^n c^n d^n}{4k} p_{2k}(X_{2n})^2 + \log \frac{1}{1 - abcdt^4} + \sum_{i=1}^{2n} \log \frac{1}{1 - abcdt^2 x_i^2}.
\]
Thus, putting \(v_n(X_{2n})\) as in (4.4), we easily find \(v_n(X_{2n})\) satisfies (4.2) from (4.6). This completes our proof of Theorem \(\ref{thm:main} \) \(\square\)
5 Corollaries

The author tried to find an analogous formula when the sum runs over all distinct partitions by computer experiments using Stembridge’s SF package (cf. [2] and [3]). But the author could not find any conceivable formula when the sum runs over all distinct partitions, and, instead, found the following formula involving the big Schur functions and certain symmetric functions arising from the Macdonald polynomials as byproducts. But, later, Prof. R. Stanley and Prof. A. Lascoux independently pointed out these are derived from Theorem 1.1 as corollaries.

Let
\[ S_{\lambda}(x; t) = \det(q_{\lambda,-i+j}(x; t))_{1 \leq i,j \leq \ell(\lambda)} \]
be the big Schur function corresponding to the partition \( \lambda \), where
\[ q_r(x; t) = Q_r(x; t) \]
denotes the Hall-Littlewood function (See [15], III, sec.2).

Corollary 5.1. Let
\[ Z(x; t) = \sum_{\lambda} \omega(\lambda) S_{\lambda}(x; t), \]
Here the sum runs over all partitions \( \lambda \). Then we have
\[
\log Z(x; t) = \sum_{n \geq 1} \frac{1}{2n} a^n b^n (1-t^n) p_{2n} = \sum_{n \geq 1} \frac{1}{4n} a^n b^n c^n d^n (1-t^{2n})^2 p_{2n} \\
\in \mathbb{Q}[p_1, p_2, p_3, \ldots].
\]
(5.1)

Proof. This proof was originally suggested by Prof. R. Stanley. Let \( \Lambda_x \) denote the ring of symmetric functions in countably many variables \( x_1, x_2, \ldots \). (For details see [15], I, sec.2). Let \( \theta_x \) be the ring homomorphism \( \Lambda_x \rightarrow \Lambda_x[t] \) taking \( h_n(x) \) to \( q_n(x; t) \). By the Jacobi-Trudi identity we have
\[ s_{\lambda}(x) = \det(h_{\lambda,-i+j}(x)). \]
(5.2)
(cf. [15], I, sec.2 (3.4).) Applying \( \theta_x \) to both sides, and using the definition of the big Schur \( S_{\lambda}(x; t) = \det(q_{\lambda,-i+j}(x; t)) \) (I, III, sec.4, (4.5)), we obtain
\[ \theta_x(s_{\lambda}(x)) = S_{\lambda}(x; t). \]
By taking logarithms of
\[ \sum_{\lambda} S_{\lambda}(x; t)s_{\lambda}(y) = \prod_{i,j \geq 1} \frac{1-tx_i y_j}{1-x_i y_j}, \]
(I, III, sec.4, (4.7)), the product on the right-hand side is
\[ \exp \sum_{n \geq 1} \frac{1}{n} (1-t^n)p_n(x)p_n(y). \]
Similarly, by taking logarithms of the right-hand side of
\[ \sum_{\lambda} s_{\lambda}(x)s_{\lambda}(y) = \prod_{i,j \geq 1} (1-x_i y_j)^{-1} \]
(1, sec.4, (4.3)), we have
\[ \sum_{\lambda} s_{\lambda}(x)s_{\lambda}(y) = \exp \sum_{n \geq 1} \frac{1}{n} p_n(x)p_n(y), \]
and it follows that \( \theta(p_n) = (1-t^n)p_n \). The identity (5.1) now follows by applying \( \theta \) to equation (1.1). ✷
This corollary is generalized to the two parameter polynomials defined by I. G. Macdonald. Define
\[
T_\lambda(x; q, t) = \det \left(Q_{(\lambda, -1+i)}(x; q, t)\right)_{1 \leq i, j \leq \ell(\lambda)}
\]
where \(Q_\lambda(x; q, t)\) stands for the Macdonald polynomial corresponding to the partition \(\lambda\), and \(Q_{(r)}(x; q, t)\) is the one corresponding to the one row partition \((r)\) (See [15], IV, sec.4).

**Corollary 5.2.** Let
\[
Z(x; q, t) = \sum_\lambda \omega(\lambda) T_\lambda(x; q, t),
\]
Here the sum runs over all partitions \(\lambda\). Then we have
\[
\log Z(x; q, t) = \sum_{n \geq 1} \frac{1}{2n} \frac{a^n(b^n - c^n)}{1 - q^{2n}p_{2n}} - \sum_{n \geq 1} \frac{1}{4n} \frac{a^n b^n c^n d^n (1 - t^{2n})^2}{(1 - q^{2n})^2 p_{2n}}
\]
\[\in \mathbb{Q}[[p_1, p_3, p_5, \ldots]].\] (5.4)

**Proof.** The proof proceeds almost parallel to that of Corollary 5.1 except that we define the ring homomorphism \(\theta : \Lambda \to \Lambda(t, q)\) by \(\theta(h_n) = g_n(x; q, t)\). Here we write \(g_n(x; q, t) = Q_{(n)}(x; q, t)\), following the notation in [15]. Since
\[
\sum_{n \geq 0} g_n(x; q, t) y^n = \prod_{i \geq 1} \frac{(tx_i y; q)_\infty}{(x_i y; q)_\infty},
\]
then we have \(g_r(x; t) = h_r(\xi)\), and therefore, by Jacobi-Trudi identity (5.2), this implies \(T_\lambda(x; t) = s_\lambda(\xi)\). By (5.5) we obtain
\[
\sum_\lambda T_\lambda(x; q, t) s_\lambda(x) = \prod_{i, j \geq 1} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty}.
\]
The rest of the arguments is almost the same as in the proof of Corollary 5.1. 

**Remark 5.3.** Prof. A. Lascoux said that Corollary 5.1 and Corollary 5.2 are obtained as corollaries of Theorem 1.1 by \(\Lambda\)-ring arguments. We cite his comment here. What we need to do is to modify the argument of the symmetric functions. For Corollary 5.1 we pass the argument \(X \to X(1 - t) = X - tX\). For Corollary 5.2 we use \(X(1 - t)/(1 - q)\). This defines two transformations on the complete symmetric functions, and therefore transformations on all the other functions. In particular, for the power sums, it transforms \(p_k \to (1 - t^k)p_k\) or \(p_k \to (1 - t^k)p_k/(1 - q^k)\). Any identity on symmetric functions which is valid for an infinite alphabet \(X\) remains valid for \((1 - t)\) and \((1 - t)/(1 - q)\) and thus Theorem 1.1 implies Corollary 5.1 and Corollary 5.2. About the \(\Lambda\)-rings, the reader can consult [14].

We also checked the Hall-Littlewood functions case, and could not find a formula for the general case, but found some nice formulas if we substitute \(-1\) for \(t\).
Conjecture 5.4. Let

\[ w(x; t) = \sum_{\lambda} \omega(\lambda) P_\lambda(x; t), \]

where \( P_\lambda(x; t) \) denote the Hall-Littlewood function corresponding to the partition \( \lambda \), and the sum runs over all partitions \( \lambda \). Then

\[
\log w(x; -1) + \sum_{n \geq 1 \text{ odd}} \frac{1}{2n} a^n c^n p_{2n} + \sum_{n \geq 2 \text{ even}} \frac{1}{2n} a^n c^n (a^n c^n - 2b^n d^n) p_{2n} \in \mathbb{Q}[p_1, p_3, p_5, \ldots]
\]

would hold. \( \square \)

We might replace the Hall-Littlewood functions \( P_\lambda(x; t) \) by the Macdonald polynomials \( P_\lambda(x; q, t) \) in this conjecture. Let \( P_\lambda(x; q, t) \) denote the Macdonald polynomial corresponding to the partition \( \lambda \) (Sec [15], IV, sec.4).

Conjecture 5.5. Let

\[ w(x; q, t) = \sum_{\lambda} \omega(\lambda) P_\lambda(x; q, t). \]

Here the sum runs over all partitions \( \lambda \). Then

\[
\log w(x; q, -1) + \sum_{n \geq 1 \text{ odd}} \frac{1}{2n} a^n c^n p_{2n} + \sum_{n \geq 2 \text{ even}} \frac{1}{2n} a^n c^n (a^n c^n - 2b^n d^n) p_{2n} \in \mathbb{Q}[p_1, p_3, p_5, \ldots]
\]

would hold. \( \square \)

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References

[1] G.E. Andrews, “On a Partition Function of Richard Stanley”, Electron. J. Combin. 11 (2) (2004), R1.

[2] C.E. Boulet, “A four-parameter partition identity”, \texttt{arXiv:math.CO/0308012} to appear in Ramanujan J.

[3] C.E. Boulet, “A four-parameter partition identity”, Formal Power Series and Algebraic Combinatorics (Vancouver 2004), extended abstract.

[4] W. Fulton and J. Harris, Representation theory, a first course, Springer-Verlag (1991).

[5] M. Ishikawa, H. Kawamuko and S. Okada, “A Pfaffian-Hafnian analogue of Borchardt’s identity”, Electron. J. Combin. 12(1), (2005) N9. \texttt{arXiv:math.CO/0408364}

[6] M. Ishikawa, S. Okada and M. Wakayama, “Applications of minor summation formulas I, Littlewood’s formulas”, J. Alg. 208 (1998), 480–525.

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M. Ishikawa, H. Tagawa, S. Okada and J. Zeng, “Generalizations of Cauchy’s determinant and Schur’s Pfaffian”, arXiv:math.CO/0411280 to appear in Adv. in Appl. Math.

M. Ishikawa and M. Wakayama, “Minor summation formula of Pfaffians”, Linear and Multilinear Algebra 39 (1995), 285-305.

M. Ishikawa and M. Wakayama, “Applications of minor summation formula III, Plücker relations, lattice paths and Pfaffian identities”, arXiv:math.CO/0312358 to appear in J. Combin. Theory Ser. A.

M. Ishikawa and J. Zeng, “The Andrews-Stanley Partition functions and Al-Salam-Chihara polynomials”, arXiv:math.CO/0506128.

D.E. Knuth, “Overlapping pfaffians”, Electronic J. of Combi. 3, 151–163.

C. Krattenthaler, “Advanced determinant calculus’, Seminaire Lotharingien Combin. 42 (“The Andrews Festschrift”) (1999), Article B42q, 67.

G. Kuperberg, “Symmetry classes of alternating-sign matrices under one roof”, Ann. of Math. (2) 156 (2002), 835-866.

A. Lascoux, Symmetric functions & Combinatorial operators on polynomials, CBMS/AMS Lectures Notes 99, (2003).

I.G. Macdonald, Symmetric functions and Hall polynomials, 2nd Edition, Oxford University Press, (1995).

S. Okada, Applications of minor-summation formulas to rectangular-shaped representations of classical groups, J. Alg. 205 (1998), 337–367.

S. Okada, “Enumeration of symmetry classes of alternating sign matrices and characters of classical groups”, arXiv:math.CO/0308234.

S. Okada, Determinant and Pfaffian formulae of Cauchy type and their applications, in “Aspects of Combinatorial Representation Theory”, Sūrikaisekikenkyūsho Kōkyūroku 1382 (2004), 198–215 (in Japanese).

R.P. Stanley, Enumerative combinatorics, Volume II, Cambridge University Press, (1999).

R.P. Stanley, “Open problem”, International Conference on Formal Power Series and Algebraic Combinatorics (Vadstena 2003), June 23 - 27, 2003, available from http://www-math.mit.edu/~rstan/trans.html.

J. Stembridge, “Nonintersecting paths, Pfaffians and plane partitions”, Adv. Math. 83 (1990), 96–131.

J. Stembridge, “Enriched P-partitions”, Trans. Amer. Math. Soc. 349 (1997), 763–788.

T. Sundquist “Two variable Pfaffian identities and symmetric functions” J. Alg. Combin. 5 (1996), 135–148.

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