Primitive contractions of Calabi–Yau threefolds II

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Abstract

We construct several new examples of Calabi–Yau threefolds with Picard group of rank 1. Each of these examples is obtained by smoothing the image of a primitive contraction with exceptional divisor being a del Pezzo surface of degree 4, 5, 6, or 7, or $\mathbb{P}^1 \times \mathbb{P}^1$.

1. Introduction

The lack of understanding of Calabi–Yau threefolds is the main gap in the classification of 3-dimensional algebraic varieties (see [10, 19, 21, 30–32]). For various reasons, manifolds with Picard group of rank 1 play a special role among all Calabi–Yau threefolds (see [3, 5, 10, 14, 17, 18, 23]). By the ‘Reid fantasy’ conjecture they are expected to lead, via sequences of conifold transitions, to all other Calabi–Yau threefolds. The already known examples of such threefolds are obtained as complete intersections in homogeneous varieties, by the method of Kawamata and Namikawa (see [14, 17]) based on smoothing normal crossing varieties, or obtained by covering singular Fano threefolds (see [18, 28]). The aim of this paper is to apply a method inspired by the ‘Reid fantasy’ to construct new families of Calabi–Yau threefolds with Picard group of rank 1 that are not of any of the standard types. Four of them were predicted by van Enckevort and van Straten in [5].

The method of construction can be described as follows. We first find a singular Calabi–Yau threefold $X'$ that contains a del Pezzo surface $D'$. Then consider the following schematic diagram.

Here the morphism $X \to X'$ is an appropriate resolution of singularities such that $X$ is a smooth Calabi–Yau threefold, and the strict transform $D$ of $D'$ is isomorphic to $D'$. The morphism $X \to Y$ is a primitive contraction of type II (see [31]) having $D$ as an exceptional divisor. The arrow $\mathcal{M} \to X'$ (respectively, $Y \leftarrow Y$) means that $X'$ can be smoothed inside the moduli space of smooth Calabi–Yau threefolds, $\mathcal{M}$ (respectively, $Y$ inside $\mathcal{Y}$). The result of the construction is a generic Calabi–Yau threefold from the family $\mathcal{Y}$.

Our constructions of Calabi–Yau threefolds $X'$ are analogous to those in [13, Section 5]. We first embed the del Pezzo surface $D'$ by the anti-canonical embedding into a linear subspace of a projective space (or a homogeneous space). Then, choosing an appropriate hypersurface from the ideal of the del Pezzo surface, we construct a singular Calabi–Yau threefold $X'$ containing $D'$. Next, using the results of [4], we prove that the resulting Calabi–Yau varieties have only

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of Y are new. The other Calabi–Yau threefolds are discussed in Remarks 2.5–2.8. The values of the rank of the Picard group of the exceptional locus of a primitive contraction of type II. This will be done if we prove that the strict transform $D$ appropriate generic complete intersection has only ODP singularities. The second problem is ordinary double points (ODP) singularities. In fact, Theorem 2.1 shows in general that an appropriate generic complete intersection has only ODP singularities. The second problem is to prove that the Picard group of $X$ is 2. The last statement is proved in Theorem 2.2 using a generalization of the Grothendieck–Lefschetz theorem given in [22].

After smoothing the images of the constructed primitive contractions, we obtain several examples of Calabi–Yau threefolds. From the results of [13] we compute the Hodge numbers of a (generic element of $Y$) and compare them with the Hodge numbers of known families. We also find other important invariants of the threefolds obtained: the degree of the second Chern class, that is, $c_2 \cdot H$, and the degree of the generator of the Picard group, that is, $H^3$, where $H$ is the generator of the Picard group of $Y$. For each of the first thirteen Calabi–Yau threefolds $Y_i$ presented in Table 1, the linear system $|H|$ is very ample. In each case we have $h^{1,1}(Y_i) = 1$.

The Calabi–Yau threefolds 2, 8, 10, and 15 were predicted in [5] and 3, 4, 5, 9, 12, 13, and 17 are new. The other Calabi–Yau threefolds are discussed in Remarks 2.5–2.8. The values of $h^0(H)$ in Table 1 were found with the help of the computer algebra system SINGULAR [8].

The second aim of this paper is to determine which del Pezzo surfaces can be the exceptional locus of a primitive contraction of type II. From [9] we know that the exceptional locus of such a contraction is either a normal Gorenstein rational del Pezzo surface, or a non-normal Gorenstein surface (that is, such that $\omega_E^{-1}$ is ample) of degree 7, or a cone over an elliptic curve of degree at most 3. We show in Theorem 2.5 that each of the above surfaces, except $\mathbb{P}^2$ blown up in one point, can be the exceptional locus of a primitive contraction of type II.

### 2. Nodal threefolds

In this section we show a method that permits us to check if a given complete intersection is nodal.
THEOREM 2.1. Let $D \subset \mathbb{P}^{s+2}$ be a smooth surface of codimension $s$, that is, the scheme-theoretic base locus of a linear system of hypersurfaces of degree $d$. Then a generic complete intersection of $s - 1$ hypersurfaces of degree $d$ containing $D$ is a nodal threefold.

Proof. First observe that a generic complete intersection $G$ of $s - 2$ hypersurfaces $g_1, \ldots, g_{s-2}$ of degree $d$ is smooth. Indeed, it follows from the Bertini theorem that it is smooth away from $D$. Singular points on $D$ appear when the rank of the Jacobian matrix of $g_1, \ldots, g_{s-2}$ is smaller than $s - 2$. As the differentials $dg_1, \ldots, dg_{s-2}$ correspond to global sections of the globally generated vector bundle $I_S/I_S^2(d)$, we conclude that the singular points appear on $c_3(I_D/I_D^2(d))$, but this is zero for dimensional reasons (see [7, Example 14.4.3]).

Denote by $\pi : \hat{G} \rightarrow G$ the blow-up of $G$ along $D$, and by $E$ its exceptional divisor. Now $D \subset G$ is a scheme-theoretic base locus of the linear system $\Lambda$ on $G$ induced by the hypersurfaces of degree $d$ in $\mathbb{P}^{s+2}$ containing $D$. It follows that the linear system $\pi^*\Lambda - E$ is base-point-free. Therefore the strict transform of a general element of $\Lambda$ is non-singular.

We claim that the singularities of a generic element $L$ of $\Lambda$ are exactly the points above which the strict transform of $\Lambda$ on $\hat{G}$ contains a fibre of $\pi|_E$. Indeed, it is enough to prove the above fact in local analytic coordinates around a point of $D$. To do this we follow [4, Claim 2.2].

We claim that a generic element of $\pi^*\Lambda - E$ cuts $E$ along a finite number of fibres of $\pi|_E$. Consider the morphism

$$\varphi : \hat{G} \rightarrow \mathbb{P}(H^0(\mathcal{O}(\pi^*\Lambda - E))) =: \mathbb{P}^N$$

given by $[\pi^*\Lambda - E]$. It follows from the assumptions that the images of the fibres of $\pi|_E$ are lines in $\mathbb{P}^N$, so the codimension of the set of divisors of the system $\Lambda$ that have a singularity at a given point of $D$ is 2. We deduce that a generic element of $\Lambda$ cannot have singularities along a curve on $D$ (see [4, p. 75]).

To complete the proof, observe that the fibres of $\pi$ over $D$ are isomorphic to $\mathbb{P}^1$. It follows that a generic element of $\Lambda$ has a small resolution with $\mathbb{P}^1$ as exceptional divisor. Now, using the Bertini theorem on $E$ for the linear system $(\pi^*\Lambda - E)|_E$, observe that a generic element of $\pi^*\Lambda - E$ cuts $E$ along a non-singular surface. It follows that the normal bundle to an exceptional curve $C$ of the small resolution has normal bundle with sub-bundle $\mathcal{O}_C(-1)$ (corresponding to the smooth strict transform of $D$).

From [16, Theorem 4.4] we know that a generic element $L$ has only singularities of $cA$ type that are normal (hypersurface singularities that are regular in codimension 1). In small resolutions of such singularities there are only rational curves with normal bundles $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ or $\mathcal{O} \oplus \mathcal{O}(-2)$ (see [6, Remark 1.7]). Thus the normal bundle of an exceptional curve is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ in our cases, so the singularities of a generic $\Lambda$ are ordinary double points (see [27]).

J. Wiśniewski remarked that we can directly compute the normal bundle of a contracted curve $C$. Indeed, the normal bundle $N_{E|\hat{G}}$ is $\mathcal{O}_E(-1)$ (since $\pi$ is the blow-up along $D$), and, as observed above, $C$ is contained in a smooth surface that is contained in $E$, which gives the other summand.

\[ \square \]

REMARK 2.1. To compute the number of nodes on $L$ we proceed as follows. From the Bertini theorem the double points of $L$ lie on $D$. More precisely, since $L$ is a complete intersection of $s - 1$ hypersurfaces, the corresponding sections of $I_D/I_D^2(d)$ are linearly dependent precisely at the singular points of $L$. This means that the number of nodes is equal to $c_2(I_D/I_D^2(d))$. 

Remark 2.2. It is interesting to compare the above theorem with the Namikawa theorem, namely, that a Calabi–Yau threefold with terminal singularities can be deformed to a Calabi–Yau threefold with ordinary double points (see [21]).

2.1. Embedding in a complete intersection of four quadrics

Let \( D_i' \subset \mathbb{P}^i \) for \( i = 4, 5, 6, 7 \) be smooth del Pezzo surfaces of degree \( i \) embedded by the anticanonical system. We choose in \( \mathbb{P}^7 \) a linear subspace \( L_i \) of dimension \( i \) such that \( D_i' \subset L_i \subset \mathbb{P}^7 \).

Let \( X'_i \) denote the intersection of four generic quadrics from the ideal defining \( D_i' \) in \( \mathbb{P}^7 \). From Theorem 2.1 the threefold \( X'_i \) is a nodal Calabi–Yau threefold.

Blowing up \( D_i' \) we resolve the singularities of \( X'_i \), and flopping the exceptional divisors we obtain a smooth Calabi–Yau threefold \( X_i \) such that the strict transform \( \tilde{D}_i \) of \( D_i' \) is isomorphic to \( D_i' \). Denote by \( H^* \) the pull-back to \( X_i \) of the hyperplane section \( H \) of \( \mathbb{P}^7 \).

Theorem 2.2. The rank of the Picard group of \( X_i \) is 2. Moreover, this group is generated by \( D_i \) and \( H^* \).

Proof. Observe first that \( \text{Pic}(X_i) \) is isomorphic in a natural way to the Picard group of the blowing up \( \tilde{X}'_i \) of \( X'_i \) along \( D'_i \). Let \( R_i \) be a generic complete intersection of three quadrics containing \( D'_i \). From the proof of Theorem 2.1 it follows that \( R_i \) is smooth. Let \( \pi_i : \tilde{\mathbb{P}}^7 \rightarrow \mathbb{P}^7 \) be the blow-up along \( D'_i \). The strict transforms of quadrics containing \( D'_i \) define a base-point-free linear system \( \Lambda_i \) on the strict transform \( \tilde{R}_i \) of \( R_i \).

Observe that \( \Lambda_i \) is also big. Indeed, in the cases \( i = 4, 5, 6 \) this is clear since linear forms already separate points on \( \mathbb{P}^7 - L_i \).

Claim. The system \( |2H - D'_i| \) on \( R_i - D'_i \) of quadrics containing \( D'_i \) is big and does not contract any divisor to a curve.

Recall first that the ideal of \( D_i \subset \mathbb{P}^7 \) is scheme-theoretically defined by the \( 2 \times 2 \) minors of a \( 3 \times 4 \) matrix

\[
\begin{pmatrix}
t & x & y & s \\
x & l & z & r \\
y & z & u & v
\end{pmatrix}
\]

obtained by deleting the last row from a symmetric matrix with generic linear forms in \( \mathbb{P}^7 \). Since the first syzygies of this set of quadrics are generated by linear forms, we can use a theorem of Room (see [1, Proposition 3.1]). Since \( (R_i \cap \text{Sec}(D'_i)) \neq R_i \subset \mathbb{P}^7 \) (where \( \text{Sec}(\cdot) \) denotes the secant variety), we find that \( |2H - D'_i| \) gives a birational morphism. Moreover, the closure of an at least 2-dimensional fibre of \( |2H - D'_i| \) is a linear space \( W \) that cuts \( D'_i \) along a curve of degree 2. It follows that the dimension of \( W \) is 2 and that \( W \) cuts \( D'_i \) along a curve from the system \( |L - E_1| \) (\( D'_i \) is a blow-up of \( \mathbb{P}^2 \) in two points, \( L \subset \mathbb{P}^2 \) is a generic line, and \( E_1 \) and \( E_2 \) are exceptional divisors). Suppose now that \( |2H - D'_i| \) contracts a divisor on \( R_i \) to a curve. We obtain a contradiction since \( R_i \) does not contain the 3-dimensional scrolls determined by \( |L - E_1| \) and \( |L - E_2| \).

We conclude that the linear system \( \Lambda_i \) is big and defines a morphism that contracts no divisor on \( \tilde{R}_i \) to a point or to a curve. From [22, Theorem 6] one has an exact sequence

\[0 \rightarrow K \rightarrow \text{Pic}(\tilde{R}_i) \rightarrow \text{Pic}(\tilde{X}'_i) \rightarrow Q \rightarrow 0,\]

where \( K \) is the subgroup generated by the divisors in \( \tilde{R}_i \) that map to points under the map given by \( \Lambda \), and \( Q \) is the group generated by irreducible components of the traces on \( \tilde{X}'_i \) of those divisors on \( \tilde{R}_i \) that map to a curve under this map. We obtain \( \text{Pic}(\tilde{R}_i) \cong \text{Pic}(\tilde{X}'_i) \). Now, arguing as before, we conclude that \( \text{Pic}(\tilde{R}_i) \) is isomorphic to the Picard group of the strict transform in
\(\mathbb{P}^7\) of an intersection of two quadrics containing \(D'_i\). Therefore it is also isomorphic to \(\text{Pic}(\mathbb{P}^7)\), which is generated by the exceptional divisor and the pull-back of a hyperplane.

**Remark 2.3.** We can also compute the number \(h^{1,2}(X_i)\) in SINGULAR [8] using the method described in [11, Remark 4.1]. From the Euler characteristic of \(X_i\), we can also deduce that \(h^{1,1}(X_i) = 2\).

Since \(D_i|_{D_i} = K_{D_i}\) and \(H^*|_{D_i} = K_{D_i}\), using [13, Lemma 2.5] we conclude that \(D_i\) is the exceptional divisor of a primitive contraction of type II.

**Theorem 2.3.** The morphism \(\varphi_{|H^* + D_i|} : X_i \to Y_i\) associated to the linear system \(|H^* + D_i|\) is a primitive contraction of type II with exceptional divisor \(D_i\). Moreover, \(Y_i\) is smoothable and if \(i = 6\) there are two different smoothings.

**Proof.** We can show using [13, Lemma 2.5] that some multiple \(n(H^* + D_i)\) gives a primitive contraction. To prove that it is enough to take \(n = 1\), we shall show that \(|H^* + D_i|\) is basepoint-free. Indeed, it is enough to find a divisor \(G_i \in |H^* + D_i|\) such that \(G_i \cap D_i = \emptyset\). We first construct a surface \(G'_i \subset X'_i\). A generic quadric (or linear form) that contains \(D_i\) cuts \(X_i\) along a divisor \(D'_i + S_i\), where \(S_i\) is an irreducible surface passing through all of the singularities of \(X'_i\) (this follows from the fact that \(D'_i\) is smooth and there is no smooth Cartier divisor passing through a singular point). With SINGULAR [8], we compute that there exists a cubic (or a quadric) that contains \(S_i\) but does not contain \(D'_i\). This cubic cuts \(X'_i\) along the divisor \(S_i + G'_i\). Using SINGULAR, we show that \(G'_i \cap D'_i\) is a finite set of points (these will be all of the singular points of \(X'_i\)). Moreover, we find that the ideal \(I_{D'_i} + IG'_i\) is radical, so \(G'_i\) and \(D'_i\) intersect transversally at these points. It follows that their strict transforms \(G_i\) and \(D_i\) are disjoint.

We claim that \(|H^* + D_i|\) gives a birational morphism that contracts exactly \(D_i\). Indeed, the system \(|H^*| + D_i\) separates points from \(X_i - D_i\) and \(H^* + D_i|_{D_i} = 0\). It remains to prove the normality of the image \(Y_i\).

Since \(|H^*|\) separates points (also infinitely near to) outside of \(D_i\), it is enough to prove that the image is normal at the point \(\varphi_{|G_i|}(D_i)\). We choose a generic hyperplane \(K\) passing through this point. It is enough to show that this hyperplane cuts the image along a normal surface. Observe that the strict transform of \(K\) in \(X_i\) is a smooth element \(H'_i \in |H^*|\). The divisor \(H'_i\) is the strict transform of a hyperplane section \(H_1 \subset X'_i\). Since \(D'_i \cap H_1 \subset H_1\) is a linearly normal elliptic curve, the morphism

\[
H^0(\mathcal{O}_{H_1 \cap X'_i}(H)) \longrightarrow H^0(\mathcal{O}_{D'_i \cap H_1}(H))
\]

is a surjection. This implies that

\[
H^0(\mathcal{O}_{X_H}(G_i) \otimes \mathcal{O}_{X_H}(-D_i)) \longrightarrow H^0(\mathcal{O}_Z(G_i) \otimes \mathcal{O}_Z(-D_i)),
\]

where \(Z = D_i \cap H_1\) and \(X_H = X_i \cap H^*_1\), is a surjection.

To prove that \(\varphi_{|Z|}(X_H)\) is normal, we can now argue as in [12, p. 416] using the fact that \(H^0(\mathcal{O}_Z(-D_i)) = H^0(\mathcal{O}_Z(-Z)) = m_{P}\)/\(m^2_{P}\) (see [25, Theorem 4.23]), where \(P\) is the elliptic singularity obtained after the contraction of \(Z\).

To prove that \(Y_i\) can be smoothed, we use [9, Theorem 5.8]. To prove that in the case \(i = 6\) there are two different smoothings, we use [20, Theorem 10] or look carefully at the proof of Theorem 4.3 in [9]. Indeed, the versal Kuranishi space of a cone over a del Pezzo surface of degree 6 has two components with different dimensions. It is enough to observe that
since \( H^1_E(X, \Theta_X) = 0 \) both of these components lead to smoothings of \( Y_i \) (we use the local cohomology sequence as in the proof of Theorem 3.3 in [13]).

**Remark 2.4.** The above proof gives an algorithm to compute the number of nodes on \( X'_i \); this number is equal to the degree of \( (G'_i \cap D'_i) \), provided that the latter is finite.

Using the results of [13] we can compute the Hodge numbers of smooth Calabi–Yau threefolds \( Y^1_i \) from smoothing families of \( Y_i \). We obtain \( h^{1,1}(Y^1_i) = 1 \) and \( h^{1,2}(Y^1_i) = h^{1,1}(Y^1_i) - \frac{1}{2} \chi(Y^1_i) \).

Denote by \( T \) the generator of the Picard group of \( Y^1_i \). We compute the degree \( T^3 \) (see Table 2) using the fact that \( (H^* + D_i)^3 \) is not divisible by a cube and that the image of \( H^* + D_i \) in \( Y^i_i \) is a Cartier divisor (see the proof of Proposition 3.1 in [13]).

**Remark 2.5.** In the first case the obtained Calabi–Yau threefold is the intersection of the Grassmannian \( G(2, 5) \subset \mathbb{P}^9 \) with a linear space and two quadrics. In the second case we obtain the Calabi–Yau threefold of degree 21 predicted in [5].

It is also natural to find \( \dim([T]) \) or, equivalently, \( T \cdot c_2/12 = \chi(O(T)) - T^3/6 \) in the examples above. From the following two lemmas we deduce that \( \dim([T]) = 9 \) in each case.

**Lemma 2.1.** The dimension of the linear system \( |G_i| \) is 9.

**Proof.** We have \( G_i \vert_{G_i} = K_{G_i} \) and the image of \( G_i \) in \( Y_i \) is a hyperplane section of \( Y_i \) (since \( h^1(O_X(G)) = 0 \)). It follows that it is enough to prove that \( G'_i \subset \mathbb{P}^7 \) (with the notation from the proof above) is an embedding given by the canonical divisor. Since, by the previous computer calculations, \( G_i \in |H^* + D_i| \), we have that \( K_{G_i} = (H^* + D_i)|_{G_i} = H^* \vert_{G_i} \), so \( H \vert_{G'_i} = K_{G'_i} \). Observe now that the system of cubics that contains \( S'_i \) and does not contain \( D'_i \) is base-point-free on \( X'_i \) outside the singularities of \( X'_i \). Indeed, we can add to such a cubic a cubic obtained by multiplying a linear form with the generator of the ideal of \( D'_i \subset \mathbb{P}^7 \) that defines \( S'_i \). It follows that a generic surface \( G'_i \subset \mathbb{P}^7 \) constructed above is smooth outside the singularities of \( X'_i \). Since the ideal \( I_D'_i + I_{G'_i} \) is radical, \( G'_i \) is smooth everywhere. Finally, since the divisor \( H^* - G \) is not effective, the surface \( G'_i \subset \mathbb{P}^7 \) is not contained in a hyperplane. It remains to show that \( G'_i \subset \mathbb{P}^7 \) is embedded by a complete linear system. First, the surface \( D'_i \subset \mathbb{P}^7 \) is arithmetically Cohen–Macaulay. Since the last property is preserved via Gorenstein linkage (the general properties of linkages are described in [15]), we obtain \( h^1(I_{G'_i}(1)) = 0 \). Hence \( G'_i \subset \mathbb{P}^7 \) is linearly normal. 

Thus \( Y_i \subset \mathbb{P}^8 \) is linearly normal, so \( h^1(I_{Y_i}(1)) = 0 \). It remains to use the semi-continuity theorem [12, III 12.8] and the following lemma.

**Table 2. Illustration.**

| \( i \) | Number of nodes on \( X'_i \) | \( \chi(X'_i) \) | \( \chi(Y^1_i) \) | \( T^3 = \deg(Y^1_i) \) |
|---|---|---|---|---|
| 4 | 12 | −104 | −120 | 20 |
| 5 | 18 | −92 | −102 | 21 |
| 6 | 24 | −80 | −84 and −86 | 22 |
| 7 | 30 | −64 | −68 | 23 |
Lemma 2.2. Let $i: Y \hookrightarrow \mathbb{P}^n$ be a smoothable (normal Gorenstein) Calabi–Yau threefold. Then $Y$ can be smoothed inside $\mathbb{P}^n$.

Proof. Let $\text{Def}(Y|\mathbb{P}^n)$ be the functor from the category of germs of complex spaces to the category of sets, such that the image of a germ $(S, x)$ is the set of isomorphism classes of deformations

$$
\begin{array}{ccc}
Y & \xrightarrow{i} & \mathbb{P}^{n+1} \\
\downarrow & & \downarrow \\
\mathcal{Y} & \rightarrow & \mathbb{P}^{n+1} \times S
\end{array}
$$

of the embedding $Y \hookrightarrow \mathbb{P}^n$ (here $\mathcal{Y} \rightarrow S$ is flat). We need to show that for each germ $S$ the natural map

$$\text{Def}(Y|\mathbb{P}^n)(S) \rightarrow \text{Def}(Y)(S)$$

is surjective. The latter follows from [29, Theorem 1.7(ii)] if we show that $H^1(\Theta_Y|_{\mathbb{P}^n}) = 0$.

Indeed, since $\dim(\text{sing}(Y)) = 0$ and $Y$ is normal Gorenstein, it follows from [2, Proposition 1.1] that $H^1(\mathcal{O}_Y(1)) = 0$. The claim follows from the long exact cohomology sequence associated to

$$0 \rightarrow \mathcal{O}_Y \rightarrow (\mathcal{O}_Y(1))^{n+1} \rightarrow \Theta_{\mathbb{P}^n}|_Y \rightarrow 0.$$ 

Example 2.1. Let us show how to compute the number of nodes in the case $i = 7$. From Remark 2.1 we need to find $c_2(\mathcal{I}_{D_7}/\mathcal{I}_{D_7}^2(2))$. We use the exact sequence

$$0 \rightarrow \mathcal{T}_{D_7} \rightarrow \mathcal{T}_{\mathbb{P}^7}|_{D_7} \rightarrow \mathcal{N}_{\mathbb{P}^7/D_7} \rightarrow 0.$$ 

Observe that $c_t(D_7) = 1 - ht + \chi_{\text{top}}(D_7)Pt^2 = 1 - ht + \frac{5}{7}h^2t^2$ and $c_t(\mathcal{T}_{\mathbb{P}^7}|_{D_7}) = (1 + ht)^8$, where $h$ is a hyperplane section in $\mathbb{P}^7$ and $P \in D_7$ is a point. Consequently, $c_t(\mathcal{T}_{\mathbb{P}^7}|_{D_7})c_t^{-1}(D_7) = c_t(\mathcal{N}_{\mathbb{P}^7/D_7})$, and it remains to apply [7, Remark 3.2.3].

Example 2.2. In the same way we can embed del Pezzo surfaces $D'$ of degree 4, 5, and 6 (see Table 3) into nodal complete intersections of a cubic and two quadrics (we first embed into a smooth complete intersection of two quadrics and then use Theorem 2.1). The Calabi–Yau threefolds $\mathcal{Y}_i$ from the resulting smoothing families have $h^{1,1}(\mathcal{Y}_i) = 1$.

Remark 2.6. In the case when $\deg(D') = 4$, the invariants are similar to the invariants of a complete intersection of four quadrics, and when $\deg(D') = 5$ to one constructed in [18]. The existence of a Calabi–Yau threefold of degree 18 with Euler characteristic $-88$ was predicted in [5].

| $\deg(D')$ | Number of nodes on $X'$ | $\chi(X)$ | $\chi(\mathcal{Y}_i)$ | $\deg(\mathcal{Y}_i)$ |
|------------|------------------------|-----------|----------------------|---------------------|
| 4          | 16                     | -112      | -128                 | 16 or 2             |
| 5          | 23                     | -98       | -108                 | 17                  |
| 6          | 30                     | -84       | -88 and -90          | 18                  |
2.2. Embedding in complete intersections in Grassmannians

We can use a similar construction to find primitive embeddings of del Pezzo surfaces into Calabi–Yau threefolds that are complete intersections in Grassmannians.

Let $D'_1 \subset \mathbb{P}^9$ be a del Pezzo surface of degree 5 anti-canonically embedded in a codimension 4 linear section. It is known that $D'_1$ can be seen as a section of the Grassmannian $G(2, 5)$ embedded via the Plücker embedding.

Let $X'_1$ be the intersection of $G(2, 5)$ with a hyperplane $L$ and two quadrics containing $D'_1$.

**Lemma 2.3.** The threefold $X'_1$ is a nodal Calabi–Yau threefold that can be resolved by blowing-up $D'_1$. Flopping the exceptional curves of this blow-up, we obtain a small resolution $X_1 \to X'_1$. The rank of the Picard group of $X_1$ is 2.

**Proof.** To prove that $X'_1 \subset G(2, 5) \cap L$ is nodal, we follow the proof of Theorem 2.1. Since the system on $G(2, 5) \cap L$ of quadrics that contain $D'_1$ separates points on $(G(2, 5) \cap L) - D'_1$, we can argue as in the proof of Theorem 2.2 to obtain the second part. In particular, we construct the surface $G'_1$.

We compute that $X'$ has fifteen ordinary double points (the radical of the ideal $I_{D'_1} + I_{G'_1}$ is 0-dimensional and has degree 15). The strict transform of $D'_1$ in $X$ is an exceptional locus of a primitive contraction of type II.

In the same way, we can embed a del Pezzo surface $D'_2 \subset \mathbb{P}^9$ of degree 5 into a nodal Calabi–Yau threefold $X'_2$ with twenty nodes that is a complete intersection of $G(2, 5)$ with two hyperplanes and a cubic. Let $X_2 \to X'_2$ be the appropriate small resolution, and let $D_2$ be the strict transform of $D'_2$. We obtain, as before, a primitive contraction given by $H^* + D$, with $D_2$ as exceptional locus.

The del Pezzo surface $D'_3 \subset \mathbb{P}^6$ of degree 6 can be seen as a special linear section of the Grassmannian $G(2, 6) \subset \mathbb{P}^{14}$. More precisely, it can be described by the $4 \times 4$ Pfaffians of a $6 \times 6$ extra-symmetric matrix (that is, skew-symmetric and symmetric with respect to the other diagonal see [26]). We embed $D'_3$ into a nodal Calabi–Yau threefold $X'_3 \subset \mathbb{P}^{10}$ that is a complete intersection of $G(2, 6)$ with a quadric and four linear forms. We compute that $X'_3$ has twelve nodes (see Table 4).

Denote by $Y_i$ the images of the above primitive contractions and by $Y^i_t$ their smoothing families. In each case, we have $h^{1,1}(Y^i_t) = 1$.

**Remark 2.7.** These computations suggest that the smooth elements $Y_i$ from the smoothing family of $Y_2$ are generic complete intersections of $G(2, 5)$ with a hyperplane and two quadrics. However, calculations in SINGULAR show that the surface $G'$ (see the proof of Theorem 2.2), which is isomorphic to a hyperplane section of $Y_2$, is not generated by quadrics. The elements from the smoothing family of $Y_1$ have the same invariants as a Calabi–Yau threefold constructed in [18] (see also [5]).

**Table 4. Illustration.**

| $i$ | Number of nodes on $X'_i$ | $\chi(X'_i)$ | $\chi(Y^i_t)$ | $\deg(Y^i_t)$ |
|-----|--------------------------|-------------|--------------|---------------|
| 1   | 15                       | −105        | −100         | 25            |
| 2   | 20                       | −130        | −120         | 20            |
| 3   | 12                       | −104        | −96 and −98  | 34            |
2.3. Embeddings of quadrics

We shall embed as before a quadric \( D' \subset \mathbb{P}^3 \) (this embedding is given by half of the canonical divisor on \( D' \)) into a nodal quintic in \( \mathbb{P}^4 \), a nodal complete intersection of two cubics in \( \mathbb{P}^5 \), a complete intersection of a cubic and two quadrics, and a complete intersection of four quadrics.

Let \( X' \subset \mathbb{P}^5 \) be a generic complete intersection of two cubics containing \( D' \subset L \subset \mathbb{P}^5 \), where \( L \) is a codimension 2 linear section. We check that the assumptions of Theorem 2.1 hold, so that \( X' \) is a nodal Calabi–Yau threefold. The blow-up of \( D' \subset X' \) is a small resolution. Denote by \( X \) the flop of the exceptional curves of this resolution and by \( D \subset X \) the strict transform of \( D' \). We prove, as in Theorem 2.2, that the rank of the Picard group of \( X \) is 2. It follows that \( D \) is the exceptional locus of a primitive contraction of type II.

We can perform the same construction for \( X' \subset \mathbb{P}^N \) being a nodal quintic (respectively, a complete intersection of four quadrics and a complete intersection of a cubic and two quadrics). If \( Y \) is the image of the constructed contraction and \( Y_i \) is the smoothing family of \( Y \), then we obtain as before \( h^1(Y_i) = 1 \). In these cases, however, the primitive contractions are given by the linear systems \( |2H^* + D| = |G| \) (because \( 2H^*|_D = -K_D \)). To compute the dimension \( h^0(\mathcal{O}_X(G)) \) we observe that \( G|_G = K_G \simeq 2H|_{G'} \). Arguing as in the proof of Lemma 2.1, we obtain \( H^1(\mathcal{I}_{G'}(2)) = 0 \). Now, the elements of degree 2 in the ideal \( \mathcal{I}_{G'} \) are exactly the quadrics containing \( X' \) (use SINGULAR). Since \( h^0(\mathcal{I}_{G'}(2)) \) is the dimension of the kernel of the map \( H^0(\mathcal{O}_{\mathbb{P}^n}(2H)) \to H^0(\mathcal{O}_{G'}(2H)) \), we are ready to compute \( h^0(\mathcal{O}_X(G)) \) in Table 5.

**Remark 2.8.** The Calabi–Yau threefold \( Y_i \) obtained in the first row of Table 5 has similar invariants to the complete intersection \( Y_{3,4} \subset \mathbb{P}(1,1,1,1,1,2) \). In the second row we probably obtain the Calabi–Yau threefold of degree 10 predicted in [5], and in the third row the Calabi–Yau threefold defined by the \( 4 \times 4 \) Pfaffians of a \( 5 \times 5 \) skew-symmetric matrix that has one row and one column with quadric entries.

2.4. Embeddings in weighted projective spaces

Let us show one more application of Theorems 2.1 and 2.2. Let \( D' \subset \mathbb{P}(1,1,1,1,2,3) \) (with coordinates \( x, y, z, t, u, v \)) be a del Pezzo surface of degree 1 defined by \( x = y = s = 0 \), where \( s \) is a generic sextic in \( \mathbb{P}(1,1,1,1,2,3) \). Let \( q = xf + yg \), where \( f \) and \( g \) are generic quadrics. Denote by \( X' = X_{3,6}^4 \) the variety defined in \( \mathbb{P}(1,1,1,1,1,2,3) \) by the equations \( s = q = 0 \).

**Lemma 2.4.** The threefold \( X' \) is a nodal Calabi–Yau threefold with four nodes.

**Proof.** Consider the sixfold ramified covering

\[
c : \mathbb{P}^5 \ni (x, y, z, t, u_1, v_1) \longrightarrow (x, y, z, t, u_1^2, v_1^3) \subset \mathbb{P}(1,1,1,1,2,3).
\]

| Table 5. Illustration. |
|------------------------|
| \( \deg(X') \) | ODP on \( X' \) | \( \chi(X') \) | \( \chi(Y_i) \) | \( (2H^* + D)^3 \) | \( h^0(2H^* + D) \) |
|------------------------|
| 5  | 24 | -200 | -156 | 48 | 16 |
| 9  | 16 | -144 | -116 | 80 | 22 |
| 12 | 14 | -144 | -120 | 104 | 27 |
| 16 | 13 | -128 | -106 | 136 | 33 |
Since a generic sextic omits the singularities of \( \mathbb{P}(1, 1, 1, 1, 2, 3) \), it follows from the Bertini theorem that the singularities of \( X' \) can occur only when \( x = y = 0 \). To prove that these are ordinary double points, it is enough to show that \( c^{-1}(X') \subset \mathbb{P}^5 \) has only nodes. The threefold \( c^{-1}(X') \) is defined by the sextic \( s_1 = s(x, y, z, t, u_1^2, v_1^3) \) and a cubic of the form
\[
xq_1(x, y, z, t) + xu_1^2 + yq_2(x, y, z, t) + yu_1^2.
\]

Since the system of such cubics defines the surface \( c^{-1}(D') \) scheme-theoretically on the fourfold \( s_1 = 0 \), we can conclude as in the proof of Theorem 2.1.

To compute the number of nodes, we find the degree of the Jacobian ideal of \( X' \) (using \textsc{Singular}). We deduce that \( c^{-1}(X') \) has 24 nodes at general points of \( D' \).

Let \( X' \) be, as defined before, the small resolution of \( X' \) such that the strict transform \( D \) of \( D' \) is isomorphic to \( D' \). Since the system of cubics of the form
\[
xq_1(x, y, z, t) + xu + yq_2(x, y, z, t) + yu
\]
induces a system that separates points on \( W - D' \), we follow the proof of Theorem 2.2 to obtain \( \rho(X) = 2 \). Hence we can find, as before, a primitive contraction with exceptional locus \( D \).

We find analogous constructions for a del Pezzo surface of degree 2 embedded in a natural way into the nodal Calabi–Yau threefolds \( X'_{3,4} \subset \mathbb{P}(1, 1, 1, 1, 2) \) and \( X'_6 \subset \mathbb{P}(1, 1, 1, 1, 2) \). Assuming that \( \rho(X') = 2 \), we deduce, with the same notation as in Section 2.3, that \( h^1(X_1) = 1 \).

Denote by \( G' \) the smooth surface in \( X' \) defined by \( f = g = 0 \) and by \( G \) its strict transform in \( X \). The calculations in Table 6 suggest the following theorem.

**Theorem 2.4.** The linear system \( |6G| \) gives a primitive contraction with \( D \) as exceptional locus into a Calabi–Yau threefold that is isomorphic to a complete intersection \( X_{2,6} \subset \mathbb{P}(1, 1, 1, 1, 3) \).

**Proof.** Observe that \( G' \) and \( D' \) intersect transversally at 24 general points defined by
\[
g(x, y, z, t, u_1^2) = f(x, y, z, t, u_1^2) = s_1 = 0.
\]

Moreover, \( G' \in |H' + D'| \) (where \( H' \) is an element from the system \( \mathcal{O}_{\mathbb{P}(1,1,1,1,2,3)}(1) \)). It follows that \( G \cap D = \emptyset \). Since the system \( |6H'| \) is very ample, we can argue as in [13, Proposition 4.3] to show that \( |6G| \) gives a birational morphism \( \varphi_{|6G|} \). Observe that \( G' \) is a surface of general type naturally isomorphic to \( X_{2,6} \subset \mathbb{P}(1, 1, 1, 1, 3) \). It follows that \( 6G_{|G} \) gives a morphism into a projectively normal surface. We deduce, as in the proof of Proposition 4.3 in [13], that the image of \( \varphi_{|6G|} \) is projectively normal with one singular point. Now, as in the proof of Theorem 4.5 in [13], we can compute that the Hilbert series of \( \bigoplus_{n=0}^\infty H^n(\mathcal{O}_X(nG)) \) is equal to
\[
P(t) = \frac{(1 - t^2)(1 - t^6)}{(1 - t)^5(1 - t^3)}.
\]

We conclude that the image of \( \varphi_{|6G|} \) is isomorphic to a normal complete intersection \( X_{2,6} \subset \mathbb{P}(1,1,1,1,1,3) \) with an isolated rational Gorenstein singularity.

**Table 6. Illustration.**

| \( X' \) | The number of nodes on \( X' \) | \( \chi(X') \) | \( \chi(\mathcal{Y}_t) \) | deg(\( \mathcal{Y}_t \)) |
|---|---|---|---|---|
| \( X'_{3,6} \) | 4 | -200 | -256 | 4 |
| \( X'_{3,4} \) | 8 | -148 | -176 | 8 or 1 |
| \( X'_6 \) | 20 | -184 | -200 | 5 |
Remark 2.9. An analogous theorem holds for del Pezzo surfaces of degree 2 naturally embedded into $X'_{4,4} \subset \mathbb{P}(1,1,1,1,1,2)$ or $X'_{5} \subset \mathbb{P}(1,1,1,1,2)$. The image of the primitive contraction is then isomorphic to $X'_{2,4} \subset \mathbb{P}^3$ or to $X'_{5} \subset \mathbb{P}^4$, respectively.

Remark 2.10. Note that we cannot give an analogous construction for the del Pezzo surface $D' \subset X'_{4,6} \subset \mathbb{P}(1,1,1,2,2,3)$ of degree 1. In fact, $X'_{4,6}$ will have a curve of singularities.

2.5. Primitive contractions of singular del Pezzo surfaces

Let $D' \subset \mathbb{P}^3$ be a cubic surface with du Val singularities (rational Gorenstein del Pezzo surface of degree 3).

Let $X'$ be the quintic defined in $\mathbb{P}^4$ by the equation $r = cq + lp$, where $c$ is a cubic, $l$ is a linear form such that $c = l = 0$ defines $D' \subset \mathbb{P}^3$, $p$ is a generic quartic, and $q$ is a generic quadric.

Lemma 2.5. The threefold $X'$ is a nodal Calabi–Yau threefold with 24 nodes at points lying outside the singularities of $D'$.

Proof. The partial derivatives of $r$ give a system of equations

\[
q \frac{dc}{dx_1} + c \frac{dq}{dx_1} + l \frac{dp}{dx_1} + p \frac{dl}{dx_1} = 0
\]

\[
q \frac{dc}{dx_2} + c \frac{dq}{dx_2} + l \frac{dp}{dx_2} + p \frac{dl}{dx_2} = 0.
\]

From the Bertini theorem the singularities of $X'$ lie on $D'$. For a generic choice of $p$ and $q$ the solution of this system is the set of 24 points where $p = q = l = c = 0$. Indeed, we can assume that $l = x_1$; if we put $c = l = 0$ then the first equation takes the form $p + q\frac{dc}{dx_1} = 0$. We infer that the singularities of $X'$ lie outside those of $D'$. Furthermore, at points $P \in D'$, where $D'$ is non-singular, one of the partial derivatives of the equation defining $D'$, say $\frac{dc}{dx_2}$, is non-zero. Taking the second equation we obtain $q(P) = 0$, and thus $p(P) = 0$.

Let $X$ be the Calabi–Yau threefold obtained by blowing up $X'$ along the smooth surface defined by $l = q = 0$. It follows from Theorem 2.2 that the rank of the Picard group of $X$ is 2. The strict transform $D \simeq D'$ of $D'$ in $X$ is the exceptional locus of a primitive contraction of type II.

Using the constructions from Sections 2.1, 2.3, and 2.4, we can find, arguing as in Section 2.5, primitive contractions with exceptional loci being rational Gorenstein del Pezzo surfaces of degree 1 to 7, and a (possibly singular) quadric. The exceptional divisor of a primitive contraction of type II is either a normal Gorenstein del Pezzo surface with rational double points, or a non-normal del Pezzo surface of degree 7 whose normalization is a non-singular rational ruled surface, or a Gorenstein del Pezzo surface of degree at most 3 that is a cone over an elliptic curve (see [32, Lemma 3.2]). The converse also holds.

Theorem 2.5. If $E$ is a del Pezzo surface as above that is not isomorphic to the blow-up of $\mathbb{P}^2$ in one point, then we can find a primitive contraction of type II with exceptional locus isomorphic to $E$. 

Proof. It suffices to consider the case when $E$ is non-normal. Then, from [24] and [32], $E$ is isomorphic to one of the following:

1. $\tilde{F}_{5,1}$, the projection from a point that lies in the plane spanned by the line $l$ and a fibre of the rational normal scroll that is a join in $\mathbb{P}^8$ of $l$ and a rational normal curve of degree 6; or

2. $\tilde{F}_{3,2}$, the projection from a point on the plane spanned by the conic of the scroll that is the join in $\mathbb{P}^8$ of a conic and a rational normal curve of degree 5.

We find that $\tilde{F}_{3,2} \subset \mathbb{P}^7$ is defined by fourteen quadrics. After a suitable change of coordinates $x, z, t, u, v, w, p$, and $q$, they can be written as $p^2 - wq, wp - vq, vp - uq, up - tq, w^2 - uq, vw - tq, uw - tp, zw - xq, v^2 - tp, uw - tw, zw - xp, u^2 - tw, zu - xw$, and $zt - xv$. These quadrics define a big system that does not contract any divisor to a curve on $R_i - \tilde{F}_{3,2}$ ($R_i$ is a smooth intersection of three such quadrics), so we can argue as before. We first choose four generic quadrics from the ideal of $\tilde{F}_{3,2} \subset \mathbb{P}^7$, and we denote by $X'$ their intersection. We prove as in Lemma 2.5 that the singularities of $X'$ are outside the singularities of $\tilde{F}_{3,2}$. Then, arguing as in Theorem 2.1, we prove that there are 30 ordinary double points. Let $X''$ be the blow-up of $X'$ along $\tilde{F}_{3,2}$. We deduce as in the proof of Theorem 2.2 that $\rho(X'') = 2$.

Analogously, we find that, after a suitable change of coordinates, the ideal defining $\tilde{F}_{5,1} \subset \mathbb{P}^7$ is defined by $p^2 - wq, wp - vq, vp - uq, up - tq, w^2 - uq, vw - tq, tw - 2xp + 2zp - tp - 22q, v^2 - tp, uw - tw, tw - 2xw + 2w^2 - 2xp - tp - 22q, u^2 - tw, tu - 2xv + 2v^2 - 2xw - 2yp - tp - 22q, and i^2 - 2xu + 2zu - 2xv - 2xw - 2xp - tp - 22q$ (it can be shown that $\tilde{F}_{5,1}$ is a degeneration of $\tilde{F}_{3,2}$; see [32]).

The following natural question arises.

Problem 2.1. Is there a primitive contraction of type II with exceptional divisor isomorphic to $\mathbb{P}^2$ blown up in one point?

The answer is probably ‘yes’, but an example is difficult to find because of the high codimension of the anti-canonical image of this surface.

Remark 2.11. Note that in all of the above cases the singular del Pezzo surface $D' \subset X'$ can be smoothed by a deformation of $X'$. We recall that Wilson asked in [32] whether this is always possible for primitive contractions.

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