Some Distributional Properties of Linear Stochastic Differential Equations *

Xue Dong He† and Zhaoli Jiang‡

July 9, 2020

Abstract

In this paper, we prove a sufficient and necessary condition for the transition probability distribution of a general, time-inhomogeneous linear SDE to possess a density function and study the differentiability of the density function and the transition quantile function of the SDE. Moreover, we completely characterize the support of the marginal distribution of this SDE.

Key words: Linear stochastic differential equations, transition probability distribution, quantile, differentiability, support

AMS subject classifications: 60G07

1 Introduction

Stochastic differential equations (SDEs) have wide applications in various fields and linear SDEs are one of the most important classes of SDEs. Examples of applications of linear SDEs include, but are not limited to, the wealth process associated with an affine trading strategy in the BlackScholes market (He et al., 2020), state dynamics in stochastic linear-quadratic control (Yong and Zhou, 1999), and physical systems subject to linear fluctuations (Risken and Eberly, 1985).

*The authors acknowledge financial support from the General Research Fund of the Research Grants Council of Hong Kong SAR (Project No. 14200917).

†Corresponding Author. Room 505, William M.W. Mong Engineering Building, Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong, Telephone: +852-39438336, Email: xdhe@se.cuhk.edu.hk.

‡Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong, Email: zljiang@se.cuhk.edu.hk.
One question that interests researchers in the study of an SDE is the distribution of the SDE at given future time, namely the transition probability distribution of the SDE. For some special linear SDEs, this distribution can be obtained in closed form, examples being the Ornstein-Uhlenbeck process and exponential functionals of Brownian motion with a drift (Yor, 2001). For a general linear SDE, however, there is no closed form for the transition probability distribution.

A general method to study the transition probability distribution of an SDE is the Kolmogorov forward and backward equations. To apply this method, it is crucial to assume certain non-degeneracy conditions so that the transition probability distribution has a density function. As reviewed later on, those conditions do not hold for a general linear SDE, so the method of Kolmogorov equations is not always applicable in this case.

Another question that interests researchers is the support of an SDE and the set of states that the SDE can reach at certain future time. For example, in the study of time-inconsistent stochastic control problems, the set of reachable states of an SDE is an important constituent of the definition of equilibrium strategies in He and Jiang (2019).

In the present paper, we consider a general time-inhomogeneous linear SDE whose coefficients are piece-wise continuous with respect to the time variable. The state space of this SDE is one-dimensional, but the Brownian motion that drives the SDE is multi-dimensional. We prove a sufficient and necessary condition for the transition probability distribution of the SDE to possess a density function and study the joint continuity and differentiability of the density function with respect to the initial data and the terminal state. We then establish a result of regularity of the transition quantile function. On the other hand, by generalizing Strock-Varadhan’s support theorem to handle the linear SDE in our setting and solving an associated deterministic optimal control problem, we derive in closed form the set of reachable states of the SDE at each time in the future. Finally, we apply our result to an SDE that arises from portfolio selection and derive additional distributional properties of the SDE by exploiting certain special structures of the SDE.

The remainder of the paper is organized as follows: In Section 2 we review the literature and compare our results with those in the literature. In Section 3 we present our main results. In Section 4 we apply our results to an SDE arising from portfolio selection. Two technical lemmas and all proofs are presented in the Appendix.
2 Literature Review

2.1 Literature on Regularity of Transition Probability

Hörmander (1967) studies the issue of when a second order differential operator with smooth coefficients on a manifold is so-called hypoelliptic. He proposes a sufficient condition, usually referred to as Hörmander’s hypothesis; see theorem 1.1 in Hörmander (1967), and (5.6), (5.7) in Williams (1981) for the detailed form of Hörmander’s hypothesis. Ichihara and Kunita (1974) apply the result obtained by Hörmander (1967) to study transition probability of a diffusion process, e.g., an SDE, and to this end, one needs to include the differential operator in the time variable, and this leads to a form of Hörmander’s hypothesis applicable to the probability theory; see (5.8) and the discussion following (5.7) in Williams (1981) and also see Condition (H) on page 128 of Nualart (2006). This form of Hörmander’s hypothesis can be applied to time-homogeneous SDEs only. To deal with time-inhomogeneous SDEs, Ichihara and Kunita (1974) consider another form of Hörmander’s hypothesis; see the paragraph preceding (5.8) of Williams (1981) and equation (1.6) of Cattiaux and Mesnager (2002) for details. Following Höpfner et al. (2017), we name this form weak Hörmander’s hypothesis, and for time-homogeneous SDEs, this form is equivalent to the original form of Hörmander’s hypothesis in (5.8) of Williams (1981).

The weak Hörmander’s hypothesis requires the drift and diffusion coefficients of the SDE to be smooth in the time variable. To weaken this requirement, Chaleyat-Maurel and Michel (1984) propose the so-called restricted Hörmander’s hypothesis; see (1.7) of Cattiaux and Mesnager (2002) for the detailed form of this hypothesis. When drift and diffusion coefficients are indeed smooth, the restricted Hörmander’s hypothesis implies the weak Hörmander’s hypothesis. Kusuoka and Stroock (1984) propose a strong ellipticity condition, which implies the restricted Hörmander’s hypothesis.

Florchinger (1990) attempt to prove that Hörmander’s hypothesis, in the form of Condition (H) on page 128 of Nualart (2006) and originally applicable to time-homogeneous SDEs only, can be applied to time-inhomogeneous SDEs as well. Cattiaux and Mesnager (2002), however, point out that there is a flaw in the proof by Florchinger (1990). It is pointed out by Cattiaux and Mesnager (2002) that the dependence on the time variable in diffusions poses nontrivial difficulties compared to the case of time-homogeneous diffusions. On the other hand, Derridj (1971) prove that the weak Hörmander’s hypothesis is almost necessary when the SDE has analytic coefficients.

The linear SDE that we study is a time-inhomogeneous one. The results in Ichihara and
Kunita (1974), Bally (1991), and Höpfner et al. (2017), who assume the weak Hörmander’s hypothesis, cannot apply to this SDE because the the drift and diffusion coefficients in this SDE are not smooth in the time variable and thus do not satisfy the weak Hörmander’s hypothesis; see Section 3.1. On the other hand, Chaleyat-Maurel and Michel (1984) assume restricted Hörmander’s hypothesis globally, namely that the restricted Hörmander’s hypothesis holds for any time and state. The linear SDE that we study, however can be degenerate, namely the diffusion coefficient of the SDE can be zero, at some time and state, so we cannot apply the result in Chaleyat-Maurel and Michel (1984). For the same reason, we cannot apply the result obtained by Kusuoka and Stroock (1984) to our setting either. Cattiaux and Mesnager (2002) assume that the restricted Hörmander’s hypothesis holds locally and that the drift and diffusion coefficients are Hölder continuous in the time variable with Hölder index larger than certain threshold; see Theorem 4.3 therein. For the linear SDE under our study, however, the coefficients are piece-wise continuous in the time variable.

By assuming a local strong ellipticity condition, Stroock (1981) study the differentiability of the transition density function with respect to the initial data and with respect to the terminal state separately. The joint regularity of the transition probability distribution with respect to the initial data and the terminal state of the process is obtained in the literature only when the weak or restricted Hörmander’s hypothesis holds globally; see for instance Theorem 3’ in Ichihara and Kunita (1974), Theorem (38.16) in Rogers and Williams (2000), and the references therein. For the linear SDE under our study, however, the Hörmander’s hypothesis cannot hold globally because the diffusion term of the linear SDE can be degenerate in certain state. Thus, the joint regularity results with respect to the initial data and the terminal state of the linear SDE obtained in the present paper is new.

2.2 Literature on the Support of an SDE

Strock-Varadhan’s support theorem is a crucial tool to study the support of the law, in the space of continuous functions, of the solution to an SDE. The first version of the theorem is proved by Stroock and Varadhan (1972), where the authors assume the diffusion coefficients to be bounded. Gyöngy (1989) study the support theorem for linear SDEs whose diffusion coefficients are not bounded. Gyöngy and Pröhle (1990) and Gyöngy et al. (1995) consider general SDEs under an assumption that is weaker than the one assumed in Stroock and Varadhan (1972). See Ondreját et al. (2018) for a summary of the relevant literature.

With the help of the support theorem, one can represent the support of the marginal distribution of the SDE, namely the support of the distribution of the SDE at a single time
point, by a deterministic optimal control problem. Using the Girsanov transform and the support theorem in Gyöngy and Pröhle (1990), Zak (2014) prove in Lemma 3.4 therein that a particular three-dimensional SDE has support of its marginal distribution to be the whole space $\mathbb{R}^3$. Kunita (1976) show that under the global Hörmander’s hypothesis, the support of the marginal distribution of a time-homogeneous SDE is the whole space; also see the application of this result in Meyn and Tweedie (1993) and Colonius and Kliemann (1999).

To our best knowledge, for a general linear SDE, a complete picture of the support of its marginal distribution has not been derived in the literature. In the present paper, we derive such a complete picture by solving the associated deterministic optimal control problem.

3 Main Results

3.1 Model

We first introduce some notations. For any set $A$ in a metric space, denote its interior as $\text{int}(A)$ and its closure as $\text{cls}(A)$. Fix an interval $[a, b]$. For a metric space $B$, denote by $C([a, b]; B)$ the set of continuous functions from $[a, b]$ to $B$ and denote by $C_{pw}([a, b]; B)$ the set of piece-wise continuous functions from $[a, b]$ to $B$, i.e., the set of functions $\xi$ from $[a, b]$ to $B$ such that $\xi$ is continuous on $[t_{i-1}, t_i)$ with $\lim_{t \uparrow t_i} \xi(t)$ existing, $i = 1, \ldots, N$, for certain partition $a = t_0 < t_1 < \ldots < t_N = b$. Denote by $C^\infty([a, b]; \mathbb{R}^l)$ the set of infinitely differentiable functions from $[a, b]$ to $\mathbb{R}^l$ and by $H([a, b]; \mathbb{R}^l)$ the set of absolutely continuous functions from $[a, b]$ to $\mathbb{R}^l$.

Consider a $d$-dimensional standard Brownian motion $W(t) := (W_1(t), \ldots, W_d(t))^\top$, $t \geq 0$ that lives on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ satisfying the usual condition. Fix $T > 0$. For a $\mathbb{R}^l$-valued diffusion process $X(t)$, $t \geq 0$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, denote by $\mathcal{S}_{X,T}$ the support of $X(t)$, $t \in [0, T]$, conditional of the information at time 0, where $X(t)$, $t \in [0, T]$ is considered to be a random variable taking values in $\mathcal{C}([0, T]; \mathbb{R}^l)$. Denote by $\mathcal{S}_{X(t)}$ the support of the distribution of $X(t)$, conditional on the information at time 0.

We are interested in the following linear SDE:

\[
\begin{cases}
    dX(t) = (c_0(t) + c_1(t)X(t)) \, dt + (c_2(t) + c_3(t)X(t))^\top \, dW(t), & t \in [0, T], \\
    X(0) = x_0 \in \mathbb{R},
\end{cases}
\]  

(3.1)

where $c_0, c_1 \in C_{pw}([0, T]; \mathbb{R})$ and $c_2, c_3 \in C_{pw}([0, T]; \mathbb{R}^d)$. We want to study two distributional properties of this SDE. First, we are concerned about $\mathcal{S}_{X(t)}$.

Second, we want to study the differentiability of conditional probability distribution of
X(T). To this end, for each \( t \in [0, T) \) and \( x \in \mathbb{R} \), denote by \( \tilde{X}(s; t, x), s \in [t, T] \) the solution to (3.1) that starts from time \( t \) and state \( x \), i.e., the solution to the following SDE:

\[
\begin{cases}
  d\tilde{X}(s; t, x) = \left( c_0(s) + c_1(s)\tilde{X}(s; t, x) \right) ds + \left( c_2(s) + c_3(s)\tilde{X}(s; t, x) \right)^\top dW(s), & s \in [t, T], \\
  \tilde{X}(t; t, x) = x.
\end{cases}
\] (3.2)

Define

\[
F(t, x, y) := \mathbb{P}(\tilde{X}(T; t, x) \leq y), \quad y \in \mathbb{R},
\] (3.3)

\[
G(t, x, \alpha) := \sup\{y \in \mathbb{R} : F(t, x, y) \leq \alpha\}, \quad \alpha \in (0, 1)
\] (3.4)

to be respectively the cumulative distribution function (CDF) and the (right-continuous) quantile function of \( \tilde{X}(T; t, x) \).

### 3.2 A Transformation

We can remove the drift of (3.1) by an increasing, affine transformation. More precisely, define

\[
\lambda_0(t) := \int_0^t c_0(z)e^{\int_z^t c_1(s)ds}dz, \quad \lambda_1(t) := e^{\int_0^t c_1(s)ds}, \quad t \in [0, T].
\] (3.5)

Straightforward calculation yields

\[
X(t) = \lambda_0(t) + \lambda_1(t)(x_0 + X^*(t)), \quad t \in [0, T],
\] (3.6)

where

\[
dX^*(t) = \left( c_2^*(t) + c_3(t)X^*(t) \right)^\top dW(t), \quad t \in [0, T], \quad X^*(0) = 0,
\] (3.7)

\[
c_2^*(t) := c_2(t)/\lambda_1(t) + c_3(t)[x_0 + \lambda_0(t)/\lambda_1(t)], \quad t \in [0, T].
\] (3.8)

As a result,

\[
S_{X(t)} = \lambda_0(t) + \lambda_1(t) \left( x_0 + S_{X^*(t)} \right).
\] (3.9)
Similarly, defining
\[
\tilde{\lambda}_0(t) = \int_t^T c_0(z) e^{\int_z^T c_1(s) ds} dz, \quad \tilde{\lambda}_1(t) = e^{\int_t^T c_1(s) ds}, \quad t \in [0, T],
\] (3.10)
we have
\[
\tilde{X}(T; t, x) = \tilde{X}^*(T; t, \tilde{\lambda}_0(t) + \tilde{\lambda}_1(t)x),
\] (3.11)
where
\[
d\tilde{X}^*(s; t, x) = (\tilde{c}_2(s) + c_3(s) \tilde{X}^*(s; t, x))^T dW(s), \quad s \in [t, T], \quad \tilde{X}^*(t; t, x) = x,
\] (3.12)
\[
\tilde{c}_2(s) := c_2(s) \tilde{\lambda}_1(s) - c_3(s) \tilde{\lambda}_0(s), \quad s \in [0, T].
\] (3.13)

Define
\[
F^*(t, x, y) := \mathbb{P}(\tilde{X}^*(T; t, x) \leq y), \quad y \in \mathbb{R},
\] (3.14)
\[
G^*(t, x, \alpha) := \sup \{ y \in \mathbb{R} : F^*(t, x, y) \leq \alpha \}, \quad \alpha \in (0, 1)
\] (3.15)
to be respectively the cumulative distribution function (CDF) and the (right-continuous) quantile function of \(\tilde{X}^*(T; t, x)\). Then we have
\[
F(t, x, y) = F^* \left( t, \tilde{\lambda}_0(t) + \tilde{\lambda}_1(t)x, y \right), \quad G(t, x, \alpha) = G^* \left( t, \tilde{\lambda}_0(t) + \tilde{\lambda}_1(t)x, \alpha \right).
\] (3.16)

The above transformations will be used in the following study of the distributional properties of the SDE (3.1).

### 3.3 Probability Densities

We first present a result of when the transition probability distribution of (3.1) admits a density function.

**Theorem 1** Suppose \(c_0, c_1 \in \mathcal{C}_{pw}([0, T]; \mathbb{R})\) and \(c_2, c_3 \in \mathcal{C}_{pw}([0, T]; \mathbb{R}^d)\). Fix \(t \in [0, T]\).

(i) Suppose that \(\tilde{c}_2(s) = c_3(s) = 0, \forall s \in [t, T]\). Then, \(F(t, x, y) = 1_{y \geq \tilde{\lambda}_0(t) + \tilde{\lambda}_1(t)x}\) for any \(x, y \in \mathbb{R}\).
(ii) Suppose that \( c_3(s) = 0 \) for all \( s \in [t, T) \) and \( \tilde{c}_2(s) \neq 0 \) for some \( s \in [t, T) \). Then, 

\[
F(t, x, y) = \Phi \left( \frac{y - \tilde{\lambda}_0(t) - \tilde{\lambda}_1(t)x}{b_t} \right), \quad y \in \mathbb{R}, x \in \mathbb{R},
\]

where \( b_t := \sqrt{\int_t^T \| \tilde{c}_2(s) \|^2 ds} > 0 \) and \( F(t, x, y) \) is infinitely differentiable in \((x, y)\). Moreover, 

\[
\lim_{|x| \to +\infty} |x|^k |g(t, x, y)| = 0, \quad y \in \mathbb{R}, \quad \lim_{|y| \to +\infty} |y|^k |g(t, x, y)| = 0, \quad x \in \mathbb{R}
\]

holds for any \( k \geq 1 \) and \( g \) to be the partial derivatives of \( F(t, x, y) \) with respect to \( x \) and \( y \) of any order. In addition, \( F(t, x, y) \) and its partial derivatives with respect to \( x \) and \( y \) of any order are bounded in \((x, y) \in \mathbb{R}^2\).

(iii) Suppose that \( \tilde{c}_2(s) + \xi c_3(s) = 0, \forall s \in [t, T) \) for some \( \xi \in \mathbb{R} \) and that \( c_3(s) \neq 0 \) for some \( s \in [t, T) \). Then, 

\[
F(t, x, y) = \begin{cases} 
\Phi \left( \frac{\ln(y - \xi) - \ln(\tilde{\lambda}_0(t)) + \tilde{\lambda}_1(t)(y - \xi) - \bar{a}_t}{b_t} \right) 1_{y > \xi}, & y \in \mathbb{R}, x > \frac{\xi - \tilde{\lambda}_0(t)}{\tilde{\lambda}_1(t)}, \\
1_{y \geq \xi}, & y \in \mathbb{R}, x = \frac{\xi - \tilde{\lambda}_0(t)}{\tilde{\lambda}_1(t)}, \\
1 - \Phi \left( \frac{\ln(x - \xi) - \ln(\tilde{\lambda}_0(t)) - \tilde{\lambda}_1(t)(x) - \bar{a}_t}{b_t} \right) 1_{y < \xi}, & y \in \mathbb{R}, x < \frac{\xi - \tilde{\lambda}_0(t)}{\tilde{\lambda}_1(t)},
\end{cases}
\]

where \( \bar{b}_t := \sqrt{\int_t^T \| c_3(s) \|^2 ds} > 0 \) and \( \bar{a}_t := -\frac{1}{2} \bar{b}_t^2 \), and \( F(t, x, y) \) are infinitely differentiable in \((x, y) \in \mathbb{R}^2 \setminus \{(\xi - \tilde{\lambda}_0(t), \xi)\} \). Moreover, (3.17) holds for any \( k \geq 1 \) and \( g \) to be the partial derivatives of \( F(t, x, y) \) with respect to \( x \) and \( y \) of any order.

(iv) Suppose that for any \( v = (v_1, v_2)^T \in \mathbb{R}^2 \) with \( \|v\| = 1 \), there exists \( s \in [t, T) \) such that \( v_1 c_3(s) + v_2 \tilde{c}_2(s) \neq 0 \). Then, \( F(t, x, y) \) is infinitely differentiable in \((x, y)\). Moreover, \( F(t, x, y) \) and its partial derivatives with respect to \( x \) and \( y \) of any order are bounded in \((x, y) \in \mathbb{R}^2\).

It is straightforward to see that the four cases in Theorem 1 are mutually exclusive and collectively exhaustive. The following corollary, which characterizes when \( \tilde{X}(T; t, x) \) has a density function, is a direct consequence of Theorem 1.

**Corollary 1** Suppose \( c_0, c_1 \in C_{pw}([0, T]; \mathbb{R}) \) and \( c_2, c_3 \in C_{pw}([0, T]; \mathbb{R}^d) \). Fix \( t \in [0, T) \) and \( x \in \mathbb{R} \).
(i) Suppose that $\tilde{c}_2^*(s) + xc_3(s) = 0, \forall s \in [t, T)$. Then, $\tilde{X} \left( T; \frac{x - \tilde{\lambda}_1(t)}{\lambda_1(t)} \right) = \tilde{X}^*(T; t, x) \equiv x$ and thus does not admit a density function.

(ii) Suppose that $\tilde{c}_2^*(s) + xc_3(s) \neq 0$ for some $s \in [t, T)$. Then, $\tilde{X} \left( T; \frac{x - \tilde{\lambda}_1(t)}{\lambda_1(t)} \right) = \tilde{X}^*(T; t, x)$ possesses a smooth density function.

3.4 Differentiability of CDF and Quantile Functions

Define
\[
t^* := \inf \{ t \in [0, T) : \tilde{c}_2^*(s) = c_3(s) = 0, \forall s \in [t, T) \} \tag{3.18}
\]
with the convention that $\inf \emptyset = T$ and
\[
t_* := \inf \{ t \in [0, t^*) : \tilde{c}_2^*(s) + \xi c_3(s) = 0, \forall s \in [t, t^*) \text{ and some } \xi \in \mathbb{R} \} \tag{3.19}
\]
with the convention that $\inf \emptyset = t^*$. Then, by the definition of $t^*$, it is easy to see that there exists unique $\xi \in \mathbb{R}$ such that $\tilde{c}_2^*(s) + \xi c_3(s) = 0, \forall s \in [t^*, t^*)$.

For any interval $[a, b)$ and open set $O$ in $\mathbb{R}^l$, denote by $\mathcal{C}^{0,\infty}([a, b) \times O)$ the set of functions $g(t, z)$ from $[a, b) \times O$ to $\mathbb{R}$ such that its derivatives with respect to $z$ of any order exist and are continuous in $(t, z)$ on $[a, b) \times O$, denote by $\mathcal{C}^{1,\infty}([a, b) \times O)$ the set of functions $g(t, z)$ from $[a, b) \times O$ to $\mathbb{R}$ such that its first-order derivative with respect to $t$ and its derivatives with respect to $z$ of any order exist and are continuous in $(t, z)$ on $[a, b) \times O$. Denote by $\mathcal{C}^{1,\infty}_{pw}([a, b) \times O)$ the set of functions $g(t, z)$ from $[a, b) \times O$ to $\mathbb{R}^d$ such that there exists $a = t_0 < t_1 < \ldots < t_N = b$ with $g \in \mathcal{C}^{1,\infty}([t_{i-1}, t_i) \times O), i = 1, \ldots, N$. Denote by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of positive integers.

For any function $g(t, x, y)$ that is differentiable in $t$ and twice differentiable in $x$, we define
\[
\mathcal{A}g(t, x, y) = g_t(t, x, y) + (c_0(t) + c_1(t)x) g_x(t, x, y) + \frac{1}{2} \| c_2(t) + c_3(t)x \|^2 g_{xx}(t, x, y), \tag{3.20}
\]
where $g_t$, $g_x$, and $g_{xx}$ denote respectively the first-order derivative of $g$ with respect to $t$, first- and second-order derivatives of $g$ with respect to $x$.

The following theorem provides a complete picture of the transition probability distribution $F(t, x, y)$.

**Theorem 2** Consider the SDE (3.2), suppose $c_0, c_1 \in \mathcal{C}_{pw}([0, T); \mathbb{R})$ and $c_2, c_3 \in \mathcal{C}_{pw}([0, T]; \mathbb{R}^d)$, recall $t^*$ and $t_*$ as defined in (3.18) and (3.19), respectively, and recall the unique $\xi \in \mathbb{R}$ such
that $\tilde{c}^*(s) + \xi c_3(s) = 0, \forall s \in [t_*, t^*)$. Consider any partitions $0 = t_0 < t_1 < \cdots < t_m = t_* < t_{m+1} < \cdots < t_n = t^*$ such that $c_0, c_1, c_2,$ and $c_3$ are continuous on $[t_{i-1}, t_i)$ with the left-limits at $t_i$ existent, $i = 1, \ldots, n$. Recall $F$ and $F^*$ as defined in (3.3) and (3.14), respectively, recall $\tilde{\lambda}_0$ and $\tilde{\lambda}_1$ as defined in (3.10), and define $\tilde{\xi}(t) := (\xi - \tilde{\lambda}_0(t))/\tilde{\lambda}_1(t), t \in [0, T]$. Then, $\tilde{\lambda}_0$ and $\tilde{\lambda}_1$ on each $[t_{i-1}, t_i]$ can be extended to $\mathfrak{C}^1([t_{i-1}, t_i]), i = 1, \ldots, n$ and the following hold:

(i) For each $i = m + 1, \ldots, n$, $t \in [t_{i-1}, t_i)$, $F^* \in \mathfrak{C}^{1, \infty}([t_{i-1}, t_i) \times (\mathbb{R}^2 \setminus \{(\xi, \xi)\})), F_t^* \in \mathfrak{C}^{0, \infty}([t_{i-1}, t_i) \times (\mathbb{R}^2 \setminus \{(\xi, \xi)\})), and

$$AF(t, x, y) = 0, \quad \forall (x, y) \in \mathbb{R}^2 \setminus \{(\tilde{\xi}(t), \xi)\}, t \in [t_{i-1}, t_i). \quad (3.21)$$

Moreover, for any $\tau \in [t_*, t^*)$, $F_t(t, x, y)$ is bounded in $(x, y) \in \mathbb{R}^2 \setminus \{(\tilde{\xi}(t), \xi)\}, t \in [t_*, \tau]$ and

$$\sup_{t \in [t_*, \tau], (x, y) \in \mathbb{R}^2 \setminus \{(\tilde{\xi}(t), \xi)\}} \left| \frac{\partial^{j+k} F}{\partial x^j \partial y^k}(t, x, y) \right| < +\infty. \quad (3.22)$$

for any $\delta > 0$, $\ell \in \{0, 1\}$ and $j, k \in \mathbb{N}_0$.

(ii) For each $i = 1, \ldots, m$, $F^*$ and thus $F$ belong to $\mathfrak{C}^{1, \infty}([t_{i-1}, t_i) \times \mathbb{R}^2)$ and

$$AF(t, x, y) = 0, \quad (t, x, y) \in [t_{i-1}, t_i) \times \mathbb{R}^2. \quad (3.23)$$

Moreover, for any $\tau' \in [0, t_*)$ and $i, j \in \mathbb{N}_0$, $|\partial^{j+k} F / \partial x^j \partial y^k(t, x, y)|$ is bounded in $(t, x, y) \in [0, \tau'] \times \mathbb{R}^2$, and $\sup_{t \in [0, \tau'], y \in \mathbb{R}} \left| \frac{\partial^{j+k} F}{\partial x^j \partial y^k}(t, x, y) \right|$ is of polynomial growth in $x$.

(iii) $F$ is continuous on $[0, t_*) \times \mathbb{R}^2 \cup \{(t, x, y) \mid (x, y) \in \mathbb{R}^2 \setminus \{\tilde{\xi}(t), \xi\}, t \in [t_*, t^*)\}$ and for any $j, k \in \mathbb{N}_0$, $\partial^{j+k} F / \partial x^j \partial y^k(t, x, y)$ is continuous on $[0, t_*) \times \mathbb{R}^2 \cup \{t_*\} \times \mathbb{R} \times (\mathbb{R} \setminus \{\xi\}) \cup \{(t, x, y) \mid (x, y) \in \mathbb{R}^2 \setminus \{\tilde{\xi}(t), \xi\}, t \in (t_*, t^*)\}$. For any $\tau \in [t_*, t^*)$, $\delta > 0$, and $j, k \in \mathbb{N}_0$,

$$\sup_{t \in [0, \tau], x \in \mathbb{R}, |y-\xi|>\delta} \left| \frac{\partial^{j+k} F}{\partial x^j \partial y^k}(t, x, y) \right| < +\infty, \quad (3.24)$$

and $\sup_{t \in [0, \tau], |y-\xi|>\delta} \left| \frac{\partial^{j+k} F}{\partial x^j \partial y^k}(t, x, y) \right|$ is of polynomial growth in $x$.

(iv) For any $(t, x, y) \in [0, t_*) \times \mathbb{R}^2 \cup \{(t, x, y) \mid (x, y) \in \mathbb{R}^2 \setminus \{\tilde{\xi}(t), \xi\}, t \in [t_*, t^*)\}$ with $F(t, x, y) \in (0, 1)$, we have $F_x(t, x, y) < 0$. 

10
Corollary 2 Suppose the same conditions as assumed in Theorem 2 hold and denote

\[ \mathcal{D} := \{(t, x) \in [0, t_*) \times \mathbb{R} \cup \{(t, x) \mid x \neq \tilde{\xi}(t), t \in [t_*, t^*)\}. \]

Then, the following are true:

(i) \( G(t, x, \alpha) = \bar{\lambda}_0(t) + \bar{\lambda}_1(t)x \) for all \( t \in [t^*, T] \), \( x \in \mathbb{R} \), and \( \alpha \in (0, 1) \). \( G(t, x, \alpha) = \xi \) for all \( t \in [t_*, t^*) \), \( x = \tilde{\xi}(t) \), and \( \alpha \in (0, 1) \).

(ii) For any \( (t, x) \in \mathcal{D} \) and \( \alpha \in (0, 1) \), \( G(t, x, \alpha) \) is uniquely determined by

\[ F(t, x, G(t, x, \alpha)) = \alpha \]

and continuous in \( (t, x, \alpha) \) in \( \mathcal{D} \times (0, 1) \). Moreover,

\[ G(t, x, \alpha) \neq \xi, \quad \forall x \neq \tilde{\xi}(t), t \in [t_*, t^*), \alpha \in (0, 1), \]

\[ F_y(t, x, G(t, x, \alpha)) > 0, \quad \forall (t, x) \in \mathcal{D}, \alpha \in (0, 1), \]

and \( G(t, x, \alpha) \) is infinitely differentiable in \( (x, \alpha) \) with derivatives to be continuous in \( (t, x, \alpha) \). In particular, we have

\[ G_x(t, x, \alpha) = -\frac{F_x(t, x, G(t, x, \alpha))}{F_y(t, x, G(t, x, \alpha))}, \quad (t, x) \in \mathcal{D}, \alpha \in (0, 1). \]
(iii) \( G \in \mathcal{C}^{1,\infty}([t_{i-1}, t_i] \times \mathbb{R} \times (0, 1)) \) for all \( i = 1, \ldots, m \) and \( G \in \mathcal{C}^{1,\infty}((t, x) \mid x \neq \xi(t), t \in [t_{i-1}, t_i]) \) for all \( i = m + 1, \ldots, n \). In particular,

\[
G_t(t, x, \alpha) = -\frac{F(t, x, G(t, x, \alpha))}{\tilde{F}(t, x, G(t, x, \alpha))}, \quad (t, x) \in \mathcal{D}, \alpha \in (0, 1).
\]

(iv) For any \( x \in \mathbb{R} \) and \( \alpha \in (0, 1) \),

\[
\lim_{t \uparrow t^*, (x', \alpha') \rightarrow (x, \alpha)} G(t, x', x') = G(t^*, x, \alpha) = \tilde{\lambda}_0(t^*) + \tilde{\lambda}_1(t^*)x.
\]

3.5 Support of the SDE

In this section, we focus on the solution to (3.1) and study the support of its solution. We are also interested in the set of states that are reachable by the SDE at a given time point. More precisely, set of reachable states of \( X \) at time \( t \), denoted as \( \mathbb{X}_t \), is defined as follows:

\[
\mathbb{X}_t := \text{int}(\mathbb{S}_{X(t)}) \cup \{ x \in \partial \mathbb{S}_{X(t)} : \mathbb{P}(X(t) \in B_\delta(x) \cap \partial \mathbb{S}_{X(t)}) > 0 \text{ for all } \delta > 0 \},
\]

where \( B_\delta(x) \) denotes the ball with radius \( \delta \) and centered at \( x \) and \( \partial \mathbb{S}_{X(t)} \) is the boundary of \( \mathbb{S}_{X(t)} \). In other words, the \( \mathbb{X}_t \) is the union of \( \text{int}(\mathbb{S}_{X(t)}) \) and \( \partial \mathbb{S}_{X(t)} \) such that \( \mathbb{P}(X(t) \in A) = \mathbb{P}(X(t) \in \partial \mathbb{S}_{X(t)}) \). For an application of the set of reachable sets, see He and Jiang (2019).

By definition, we have \( \mathbb{P}(X(t) \in \mathbb{X}_t) = 1 \). Moreover, we have \( \mathbb{S}_{X(t)} = \text{cls}(\mathbb{X}_t) \) and \( \text{int}(\mathbb{S}_{X(t)}) = \text{int}(\mathbb{X}_t) \). In general, however, \( \mathbb{X}_t \neq \mathbb{S}_{X(t)} \). For example, if \( X(t) \) is a geometric Brownian motion with the starting point \( x_0 > 0 \), then \( \mathbb{X}_t = (0, +\infty) \) and \( \mathbb{S}_{X(t)} = [0, +\infty) \).

The following theorem provides a complete characterization of \( \mathbb{X}_t \).

**Theorem 3** Consider the SDE (3.1) and suppose that \( c_0, c_1 \in \mathcal{C}_{pw}([0, T]; \mathbb{R}) \) and \( c_2, c_3 \in \mathcal{C}_{pw}([0, T]; \mathbb{R}^d) \). Define \( h(s) := -c_2^s(s)\top \frac{c_3(s)}{\|c_3(s)\|^2} 1_{c_3(s) \neq 0}, s \in [0, T], D := \{ s \in [0, T] : c_3(s) \neq 0 \} \), and

\[
\underline{t} := \inf\{ s \in [0, T] : c_2^s(s) \neq 0 \}, \quad (3.26)
\]

\[
\bar{t} := \inf\left\{ s \in [0, T] : \int_0^s \|c_2^z(z) + c_3(z)h(z)\| \, dz > 0 \right\}, \quad (3.27)
\]

with the convention \( \inf\emptyset := T \). Then, \( \underline{t} \leq \bar{t} \). Moreover, for each \( t \in [0, T] \), \( \mathbb{S}_{X(t)} \) is an
interval with the left- and right- ends denoted as $\bar{x}(t)$ and $\bar{x}(t)$, respectively, and $\bar{x}$ and $\bar{x}$ are left-continuous on $[0, T]$. Furthermore, the following hold:

(i) For each $t \in [0, \bar{t}]$, $X_t = \{\lambda_0(t) + \lambda_1(t)x_0\}$. 
(ii) For each $t \in (\bar{t}, T]$, $X(t)$ possesses a density and $X_t = \mathbb{R}$.
(iii) Suppose $\bar{t} < \bar{t}$ and fix any $t \in (t, \bar{t}]$. Define $\tau_t := \sup\{s \in [t, \bar{t}) : c_3(s) \neq 0\}$ with $\sup \emptyset := \bar{t}$. Then, $X(t)$ possesses a density, $\tau_t > \bar{t}$, and the following hold:

(a) Suppose there exist $s_1, s_2 \in [\bar{t}, t)$ such that $h(s_1) < 0$ and $h(s_2) > 0$. Then, $X_t = \mathbb{R}$.
(b) Suppose that $h(s) \leq 0, \forall s \in [\bar{t}, t)$. If $h$ is not decreasing on $[\bar{t}, t) \cap D$, i.e., if there exists $s_1, s_2 \in [\bar{t}, t) \cap D$ with $s_1 < s_2$ and $h(s_1) < h(s_2)$, then $X_t = \mathbb{R}$. Otherwise, $X_t = (\lambda_0(t) + \lambda_1(t)(x_0 + h^*(\tau_t)), +\infty)$, where $h^*(\tau_t) := \lim_{\bar{t} \nearrow \tau_t} h(s)$.
(c) Suppose that $h(s) \geq 0, \forall s \in [\bar{t}, t)$. If $h(s)$ is not increasing on $[\bar{t}, t) \cap D$, i.e., if there exists $s_1, s_2 \in [\bar{t}, t) \cap D$ with $s_1 < s_2$ and $h(s_1) > h(s_2)$, then $X_t = \mathbb{R}$. Otherwise, $X_t = (-\infty, \lambda_0(t) + \lambda_1(t)(x_0 + h^*(\tau_t)))$, where $h^*(\tau_t) := \lim_{\bar{t} \nearrow \tau_t} h(s)$.

4 Linear SDE that Arises from Portfolio Selection

As an application, we consider the following linear SDE

$$
\begin{align*}
\left\{ 
\begin{array}{ll}
    dX(t) &= (\theta_0(t) + \theta_1(t)X(t))^T b(t)dt + (\theta_0(t) + \theta_1(t)X(t))^T \sigma(t)dW(t), & t \in [0, T], \\
    X(0) &= x_0 \in \mathbb{R}
\end{array}
\right.
\end{align*}
$$

(4.1)

that arises from portfolio selection. The parameters satisfy the following assumption:

**Assumption 1** $\theta_0, \theta_1, b \in \mathcal{C}_{pw}([0, T]; \mathbb{R}^n), \sigma \in \mathcal{C}_{pw}([0, T]; \mathbb{R}^{n \times d}),$ and $\sigma(t)\sigma(t)^T$ is positive definite for any $t \in [0, T]$. 

In the SDE (4.1), $b$ and $\sigma$ stand for the mean excess return rate and volatility, respectively, of $n$ stocks in a financial market, $\theta_0$ and $\theta_1$ represent an affine investment strategy, and $X$ stands for the discounted wealth process associated with that strategy. In other words, the discounted dollar amount invested in the stocks in an infinitesimally small period at time $t$ is $\theta_0(t) + \theta_1(t)X(t)$. The positive-definiteness of $\sigma(t)\sigma(t)^T$ is a standard assumption in portfolio selection.
The SDE (4.1) is a special case of (3.1) with
\[ c_0(t) = b(t)\top \theta_0(t), \quad c_1(t) = b(t)\top \theta_1(t), \quad c_2(t) = \sigma(t)\top \theta_0(t), \quad c_3(t) = \sigma(t)\top \theta_1(t). \] (4.2)
As a result, \( c_2^*, \bar{c}_2, \lambda_0, \lambda_1, \lambda_0, \) and \( \bar{\lambda}_1 \) as defined in Section 3.2 become
\[
\lambda_0(t) = \int_0^t e^{\int_s^t b(z)\top \theta_1(z)dz}b(s)\top \theta_0(s)ds, \quad \lambda_1(t) = e^{\int_0^t b(s)\top \theta_1(s)ds},
\] (4.3)
\[
c_2^*(t) = \sigma(t)\top \left[ \theta_0(t)e^{-\int_0^t b(s)\top \theta_1(s)ds} + \theta_1(t) \left( x_0 + \int_0^t b(s)\top \theta_0(s)e^{-\int_0^s b(z)\top \theta_1(z)dz}ds \right) \right],
\] (4.4)
\[
\bar{\lambda}_0(t) = \int_t^T e^{\int_s^T b(z)\top \theta_1(z)dz}b(s)\top \theta_0(s)ds, \quad \bar{\lambda}_1(t) = e^{\int_t^T b(s)\top \theta_1(s)ds},
\] (4.5)
\[
\bar{c}_2^*(t) = \sigma(t)\top \left[ \theta_0(t)e^{\int_t^T b(s)\top \theta_1(s)ds} - \theta_1(t) \int_t^T b(s)\top \theta_0(s)e^{\int_t^s b(z)\top \theta_1(z)dz}ds \right].
\] (4.6)

The following two corollaries are obtained by applying our results in Section 3 to the particular SDE (4.1) and exploiting additional structures of (4.1).

**Corollary 3** Consider the SDE (4.1) and suppose Assumption 1 holds. Then, Theorem 2 and Corollary 2 hold for SDE (4.1) with \( c_i, i = 0, 1, 2, 3, 4, \lambda_i, i = 0, 1, \) and \( c_2^* \) as given in (4.2), (4.3), and (4.4), respectively. Moreover, we have
\[
t^\ast = \inf \{ t \in [0, T) : \theta_0(s) = \theta_1(s) = 0, \forall s \in [t, T) \},
\] (4.7)
\[
t_* = \inf \{ t \in [0, t^\ast) : \text{there exists } \xi \in \mathbb{R} \text{ such that } \theta_0(s) + \xi \theta_1(s) = 0, \forall s \in [t, t^\ast) \},
\] (4.8)
and for any \( t \in [t_*, T], \bar{\xi}(t) = \xi.\)

**Corollary 4** Consider the SDE (4.1) and suppose Assumption 1 holds. Then, Theorem 3 holds for SDE (4.1) with \( c_i, i = 0, 1, 2, 3, 4, \lambda_i, i = 0, 1, \) and \( c_2^* \) as given in (4.2), (4.3), and (4.4), respectively. Moreover,
\[
\bar{t} = \inf \{ s \in [0, T] : \theta_0(s) + x_0 \theta_1(s) \neq 0 \},
\] (4.9)
\[
\bar{t} = \inf \{ s \in [0, T] : \int_0^s \| \theta_0(z) + \bar{h}(z) \theta_1(z) \| dz > 0 \},
\] (4.10)
where
\[
\tilde{h}(t) := -\frac{\theta_0(t)^\top \sigma(t)\sigma(t)^\top \theta_1(t)}{\|\sigma(t)^\top \theta_1(t)\|^2}1_{\theta_1(t) \neq 0}, \quad t \in [0, T], \tag{4.11}
\]
\[
x^*(t) := \int_0^t b(s)^\top \theta_0(s)e^{\int_s^t (\theta_1(z))^\top ds}ds + x_0e^{\int_0^t (\theta_1(s))^\top ds}, \quad t \in [0, T], \tag{4.12}
\]
\[
\mathcal{X}_t = \{x_0\} \text{ and } x^*(t) = x_0 \text{ for any } t \in [0, \bar{t}], \mathcal{X}_t \text{ is increasing in } t \in [0, T], \text{ and}
\]
\[
h(t) = e^{-\int_0^t b(s)^\top \theta_1(s)ds} \left[\tilde{h}(t) - x^*(t)1_{\theta_1(t) \neq 0}\right], \quad t \in [0, T]. \tag{4.13}
\]
Furthermore, with \(\underline{t} < \bar{t}\) and fixing any \(t \in (\underline{t}, \bar{t})\) with \(\theta_1(s) \neq 0, \forall s \in (\underline{t}, t)\), the following are true:

(1) Suppose \(\tilde{h}(s) \leq x^*(s)\) for any \(s \in (\underline{t}, t)\). If \(\tilde{h}\) is not decreasing on \((\underline{t}, t)\), then \(\mathcal{X}_t = \mathbb{R}\). If \(\tilde{h}\) is decreasing in on \((\underline{t}, t)\), then \(\tilde{h}(s) < x^*(s)\) for any \(s \in (\underline{t}, t)\) and \(\mathcal{X}_t = (\tilde{h}(t-), +\infty)\), where \(\tilde{h}(t-):= \lim_{s \uparrow t} \tilde{h}(s)\).

(2) Suppose \(\tilde{h}(s) \geq x^*(s)\) for any \(s \in (\underline{t}, t)\). If \(\tilde{h}\) is not increasing on \((\underline{t}, t)\), then \(\mathcal{X}_t = \mathbb{R}\). If \(\tilde{h}\) is increasing on \((\underline{t}, t)\), then \(\tilde{h}(s) > x^*(s)\) for any \(s \in (\underline{t}, t)\) and \(\mathcal{X}_t = (-\infty, \tilde{h}(t-))\), where \(\tilde{h}(t-):= \lim_{s \uparrow t} \tilde{h}(s)\).

A Two Lemmas

In this section, we provide two technical lemmas that will be used in the proofs of the main results of the present paper. The proofs of these two lemmas are presented at the end of the section.

Lemma 1 Suppose \(c_2^*, c_3 \in \mathcal{C}_{pw}([0, T]; \mathbb{R}^d)\). Fix \(t \in [0, T]\) and define \(\tilde{Z}_1(s; t), s \in [t, T]\) and \(\tilde{Z}_2(s; t), s \in [t, T]\) by
\[
d\tilde{Z}_1(s; t) = \tilde{Z}_1(s; t)c_3(s)^\top dW(s), \quad s \in [t, T], \quad \tilde{Z}_1(t; t) = 1,
\]
\[
d\tilde{Z}_2(s; t) = \left(c_2^*(s) + c_3(s)\tilde{Z}_2(s; t)\right)^\top dW(s),
\]
\[
s \in [t, T], \quad \tilde{Z}_2(t; t) = 0.
\]

Suppose that for any \(v = (v_1, v_2)^\top \in \mathbb{R}^2\) with \(\|v\| = 1\), there exists \(s \in [t, T]\) such that \(v_1c_3(s) + v_2c_2^*(s) \neq 0\). Then, \((\tilde{Z}_1(T; t), \tilde{Z}_2(T; t))\) admits an infinitely differentiable probability
density $g$ on $\mathbb{R}^2$ with

$$
\sup_{z \in \mathbb{R}^2} \|z\|^k \left| \frac{\partial^{i+j} g}{\partial z_1^i \partial z_2^j}(z) \right| < +\infty \tag{A.1}
$$

for any $k \geq 1$, $i \geq 0$, and $j \geq 0$.

**Lemma 2** Consider the following linear SDE

$$
\begin{cases}
    dX(t) = (\beta_0(t) + \beta_1(t)X(t))dt + \sum_{i=1}^d (\beta_{2,i}(t) + \beta_{3,i}(t)X(t))dW_i(t), & t \in [0, T], \\
    X(0) = x_0 \in \mathbb{R}^l,
\end{cases} \tag{A.2}
$$

where $\beta_0, \beta_{2,i} \in \mathcal{C}_{pw}([0, T]; \mathbb{R}^l)$ and $\beta_1, \beta_{3,i} \in \mathcal{C}_{pw}([0, T]; \mathbb{R}^{l \times l})$, $i = 1, \ldots, d$. Denote $\mathcal{U} := \{f_w \in \mathcal{C}([0, T]; \mathbb{R}^l) : w \in \mathcal{S}([0, T]; \mathbb{R}^d)\}$ and $\bar{\mathcal{U}} := \{f_w \in \mathcal{C}([0, T]; \mathbb{R}^l) : w \in \mathcal{C}([0, T]; \mathbb{R}^d)\}$, where $f_w$ is the solution of following ODE:

$$
f_w'(t) = \beta_0(t) - \frac{1}{2} \sum_{i=1}^d \beta_{3,i} \beta_{2,i}(t) + \left[ \beta_1(t) - \frac{1}{2} \sum_{i=1}^d \beta_{3,i}(t) \beta_{3,i}(t) \right] f_w(t)$$

$$+ \sum_{i=1}^d (\beta_{2,i}(t) + \beta_{3,i}(t) f_w(t)) w_i'(t), \quad t \in [0, T], \quad f_w(0) = x_0. \tag{A.3}
$$

Then, the following are true:

(i) $\mathcal{S}_X = \text{cls}(\mathcal{U}) = \text{cls}(\bar{\mathcal{U}})$.

(ii) For each $t \in [0, T]$, $\mathcal{S}_{X(t)} = \text{cls}(\mathcal{U}_t) = \text{cls}(\bar{\mathcal{U}}_t)$, where $\mathcal{U}_t := \{f(t) : f \in \mathcal{U}\}$ and $\bar{\mathcal{U}}_t := \{f(t) : f \in \bar{\mathcal{U}}\}$.

(iii) $\bar{\mathcal{U}}$ and $\mathcal{U}_t$, $t \in [0, T]$, are connected. Consequently, when $l = 1$, $\mathcal{S}_{X(t)}$ is a closed interval for any $t \in [0, T]$.

Lemma 2-(i) is an extension of Theorem 4.1 in Gyöngy (1989) by allowing $\beta_0, \beta_1, \beta_{2,i}$’s and $\beta_{3,i}$’s to be continuous in multiple pieces of $[0, T]$. Lemma 2-(ii) and (iii) are direct consequences of Lemma 2-(i).

**Proof of Lemma 1** In the following, we prove that $(\tilde{Z}_1(T; t), \tilde{Z}_2(T; t))$ is a nondegenerate random vector in the sense of Definition 2.1.1 of Nualart (2006). Then, the lemma is just a consequence of Proposition 2.1.5 of Nualart (2006).

We recall some notations in Malliavin calculus: $(f, g)$ stands for the inner product of $f$ and $g$ in the Hilbert space of square-integrable functions from $[t, T]$ to $\mathbb{R}$. $D$ denotes the
Malliavin derivative operator on the space of square-integrable stochastic processes (see p. 25 of Nualart 2006). Because for each stochastic process $X$, $DX$ is also a stochastic process, so $DX$ can be identified as $D_sX, s \in [t, T]$. $\mathbb{D}^\infty$ denotes the space of certain smooth random variables (see p. 67 of Nualart 2006). The Malliavin matrix of $(\tilde{Z}_1(T; t), \tilde{Z}_2(T; t))$ is defined to be

$$
\Gamma = \begin{pmatrix}
\langle D\tilde{Z}_1(T; t), D\tilde{Z}_1(T; t) \rangle & \langle D\tilde{Z}_1(T; t), D\tilde{Z}_2(T; t) \rangle \\
\langle D\tilde{Z}_2(T; t), D\tilde{Z}_1(T; t) \rangle & \langle D\tilde{Z}_2(T; t), D\tilde{Z}_2(T; t) \rangle
\end{pmatrix}.
$$

Recalling Definition 2.1.1 of Nualart (2006), to prove that $(\tilde{Z}_1(T; t), \tilde{Z}_2(T; t))$ is nondegenerate, we need to prove that (i) $\tilde{Z}_i(T; t) \in \mathbb{D}^\infty, i = 1, 2$ and (ii) $\Gamma$ is invertible almost surely and its determinant, denoted as $\text{det}(\Gamma)$, satisfies $(\text{det}(\Gamma))^{-1} \in \cap_{p\geq 1} L^p(\Omega)$, where $L^p(\Omega)$ denotes the space of random variables $X$ with $\mathbb{E}[|X|^p] < +\infty$. Because $\tilde{c}_2$, and $c_3$ are piece-wise continuous and thus bounded, Theorem 2.2.2 in Nualart (2006) implies $\tilde{Z}_i(T; t) \in \mathbb{D}^\infty, i = 1, 2$.

Thus, we only need to prove (ii) in the following.

Theorem 2.2.1 and equation (2.53) in Nualart (2006) imply that

$$
D_s\tilde{Z}_1(T; t) = c_3(s)^\top \tilde{Z}_1(s; t)\tilde{Z}_0(s), \quad D_s\tilde{Z}_2(T; t) = [\tilde{c}_2(s) + c_3(s)\tilde{Z}_2(s; t)]^\top \tilde{Z}_0(s), \quad s \in [t, T]
$$

where

$$
\tilde{Z}_0(s) = \exp \left\{ \int_s^T -\frac{1}{2} ||c_3(\tau)||^2 d\tau + \int_s^T c_3(\tau)^\top dW(\tau) \right\}.
$$

Then, denoting the component of $\Gamma$ in its $i$-th row and $j$-th column as $\Gamma_{i,j}$, we have

$$
\Gamma_{1,1} = \langle D\tilde{Z}_1(T; t), D\tilde{Z}_1(T; t) \rangle = \int_t^T \|c_3(s)\|^2 \tilde{Z}_1(s; t)^2 \tilde{Z}_0(s)^2 ds,
$$

$$
\Gamma_{2,2} = \langle D\tilde{Z}_2(T; t), D\tilde{Z}_2(T; t) \rangle = \int_t^T \|\tilde{c}_2(s) + c_3(s)\tilde{Z}_2(s; t)\|^2 \tilde{Z}_0(s)^2 ds,
$$

$$
\Gamma_{1,2} = \Gamma_{2,1} = \langle D\tilde{Z}_1(T; t), D\tilde{Z}_2(T; t) \rangle = \int_t^T [\tilde{c}_2(s) + c_3(s)\tilde{Z}_2(s; t)]^\top c_3(s) \tilde{Z}_0(s)^2 ds.
$$

Then, Hölder’s inequality implies that $\text{det}(\Gamma) \geq 0$, so by Lemma 2.3.1 in Nualart (2006), to prove that $\Gamma$ is invertible almost surely and $(\text{det}(\Gamma))^{-1} \in \cap_{p\geq 1} L^p(\Omega)$, we only need to prove...
that for any \( p \geq 2 \), there exists \( \eta(p) > 0 \), such that for all \( \eta \in (0, \eta(p)] \), we have

\[
\sup_{\|v\|=1} \mathbb{P}(v^\top \Gamma v \leq \eta) \leq \eta^p. \tag{A.4}
\]

For all \( v = (v_1, v_2)^\top \in \mathbb{R}^2 \) with \( \|v\| = 1 \), we have

\[
v^\top \Gamma v = \int_t^T \|c_3(s)(v_1 \tilde{Z}_1(s; t) + v_2 \tilde{Z}_2(s; t)) + v_2 c_2^*(s)\|^2 \tilde{Z}_0(s)^2 ds \\
\geq \left( \inf_{s \in [t, T]} \tilde{Z}_0(s)^2 \right) \int_t^T \|c_3(s)(v_1 \tilde{Z}_1(s; t) + v_2 \tilde{Z}_2(s; t)) + v_2 c_2^*(s)\|^2 ds \\
= \left( \inf_{s \in [t, T]} \tilde{Z}_0(s)^2 \right) \int_t^T \|c_3(s)H^v(s) + v_1 c_3(s) + v_2 c_2^*(s)\|^2 ds, \tag{A.5}
\]

where \( H^v(s) := v_1(\tilde{Z}_1(s; t) - 1) + v_2 \tilde{Z}_2(s; t) \) satisfies

\[
dH^v(s) = (H^v(s)c_3(s) + v_1 c_3(s) + v_2 c_2^*(s))^\top dW(s), \quad s \in [t, T], \quad H^v(t) = 0. \tag{A.6}
\]

Consequently, for any \( v \in \mathbb{R}^2 \) with \( \|v\| = 1 \), \( \eta > 0 \), and \( p \geq 2 \),

\[
\mathbb{P}(v^\top \Gamma v \leq \eta) \leq \mathbb{P} \left( \left( \inf_{s \in [t, T]} \tilde{Z}_0(s)^2 \right) \int_t^T \|c_3(s)H^v(s) + v_1 c_3(s) + v_2 c_2^*(s)\|^2 ds \leq \eta \right) \\
= \mathbb{P} \left( \left( \sup_{s \in [t, T]} \tilde{Z}_0(s)^{-2} \right) \int_t^T \|c_3(s)H^v(s) + v_1 c_3(s) + v_2 c_2^*(s)\|^2 ds \geq 1/\eta \right) \\
\leq \mathbb{E} \left[ \left( \sup_{s \in [t, T]} \tilde{Z}_0(s)^{-2} \right)^q \left( \frac{1}{\eta} \right)^{-q} \right] \\
\leq \mathbb{E} \left[ \sup_{s \in [t, T]} \tilde{Z}_0(s)^{-4q} \right]^{1/2} \mathbb{E} \left[ \left( \int_t^T \|c_3(s)H^v(s) + v_1 c_3(s) + v_2 c_2^*(s)\|^2 ds \right)^{2q} \right]^{1/2} \eta^q, \tag{A.7}
\]

where we set \( 1/0 = +\infty \).

Note that \( \tilde{Z}_0(s) = \tilde{Z}_0(t)/M(s) \), where

\[
dM(s) = M(s)c_3(s)^\top dW(s), \quad s \in [t, T], \quad M(t) = 1.
\]
As a result,

\[
E \left[ \sup_{s \in [t,T]} \tilde{Z}_0(s)^{-4q} \right] = E \left[ \tilde{Z}_0(t)^{-4q} \left( \sup_{s \in [t,T]} M(s) \right)^{4q} \right] \\
\leq E \left[ \tilde{Z}_0(t)^{-8q} \right]^{1/2} E \left[ \left( \sup_{s \in [t,T]} M(s) \right)^{8q} \right]^{1/2} < +\infty, \tag{A.8}
\]

where the last inequality is because \( c_3 \) is piece-wise continuous and thus bounded. Thus, recalling (A.7), to prove (A.4), we only need to show for any \( m \geq 2 \),

\[
\sup_{\|v\|=1} E \left[ \frac{1}{\int_t^T \|c_3(s) H^v(s) + v_1 c_3(s) + v_2 \tilde{c}_2(s) \|^2 ds \right]^{m} < +\infty. \tag{A.9}
\]

For each \( v = (v_1, v_2)\top \in \mathbb{R}^2 \) with \( \|v\| = 1 \), because there exists \( s \in [t, T) \), such that \( v_1 c_3(s) + v_2 \tilde{c}_2(s) \neq 0 \) and because \( \tilde{c}_2 \) and \( c_3 \) are right-continuous, we conclude that \( \varphi(v) := \int_t^T \|v_1 c_3(s) + v_2 \tilde{c}_2(s)\|^2 ds > 0 \). It is straightforward to see that \( \varphi(v) \) is continuous in \( v \), so \( L := \inf_{\|v\|=1} \varphi(v) > 0 \). Because \( \tilde{c}_2 \) and \( c_3 \) are in \( \mathcal{C}_{pw}([0, T); \mathbb{R}^d) \), they are bounded on \([0, T)\) by certain constant \( \tilde{C} > 0 \). Then, for any \( v = (v_1, v_2)\top \in \mathbb{R}^2 \) with \( \|v\| = 1 \),

\[
\int_t^T \|c_3(s) H^v(s) + v_1 c_3(s) + v_2 \tilde{c}_2(s)\|^2 ds \\
= \int_t^T \|v_1 c_3(s) + v_2 \tilde{c}_2(s)\|^2 ds + \int_t^T \|c_3(s) H^v(s)\|^2 ds \\
+ 2 \int_t^T H^v(s)c_3(s)\top(v_1 c_3(s) + v_2 \tilde{c}_2(s))ds \\
\geq L - T \sup_{s \in [t,T]} |H^v(s)|^2 \tilde{C}^2 - 4T \sup_{s \in [t,T]} |H^v(s)| \tilde{C}^2. \tag{A.10}
\]

Because \( L - T \tilde{\delta}^2 \tilde{C}^2 - 4T \tilde{\delta} \tilde{C}^2 \geq L/2, \forall \tilde{\delta} \in [0, \delta] \) for certain \( \delta > 0 \), then we conclude from (A.10) that there exists \( \bar{\epsilon} > 0 \) such that for any \( \epsilon \in (0, \bar{\epsilon}) \) and \( v = (v_1, v_2)\top \in \mathbb{R}^2 \) with \( \|v\| = 1 \),

\[
\mathbb{P} \left( \int_t^T \|c_3(s) H^v(s) + v_1 c_3(s) + v_2 \tilde{c}_2(s)\|^2 ds < \epsilon, \sup_{s \in [t,T]} |H^v(s)| < \epsilon^{1/4} \right) = 0. \tag{A.11}
\]
Thus, we have for any $\epsilon \in (0, \bar{\epsilon})$, and any $v = (v_1, v_2) \in \mathbb{R}^2$ with $\|v\| = 1$,

$$
\mathbb{P} \left( \int_t^T \|c_3(s)H^v(s) + v_1c_3(s) + v_2\tilde{c}_2^s(s)\|^2 ds < \epsilon \right)
= \mathbb{P} \left( \int_t^T \|c_3(s)H^v(s) + v_1c_3(s) + v_2\tilde{c}_2^s(s)\|^2 ds < \epsilon, \sup_{s \in [t,T]} |H^v(s)| \geq \epsilon^{1/4} \right)
\leq 2e^{-\frac{1}{2}\epsilon^{-1/2}}, \tag{A.12}
$$

where the equality is the case due to (A.11) and the inequality is the case due to (A.6) and to the inequality (A.5) in Nualart (2006). Sending $\epsilon$ to 0 in the above, we immediately derive that $\int_t^T \|c_3(s)H^v(s) + v_1c_3(s) + v_2\tilde{c}_2^s(s)\|^2 ds > 0$ almost surely. For any $m \geq 2$, denote

$$
Q(y) := \mathbb{P} \left( \left( \int_t^T \|c_3(s)H^v(s) + v_1c_3(s) + v_2\tilde{c}_2^s(s)\|^2 ds \right)^{-m} > y \right), \quad y \geq 0.
$$

Then, we derive from (A.12) that

$$
Q(y) \leq 2e^{-\frac{1}{2}y^{1/(2m)}}, \quad y \geq \bar{\epsilon}^{-m}. \tag{A.13}
$$

As a result,

$$
\mathbb{E} \left[ \left( \int_t^T \|c_3(s)H^v(s) + v_1c_3(s) + v_2\tilde{c}_2^s(s)\|^2 ds \right)^{-m} \right] \leq \bar{\epsilon}^{-m} + \\
+ \mathbb{E} \left[ \left( \int_t^T \|c_3(s)H^v(s) + v_1c_3(s) + v_2\tilde{c}_2^s(s)\|^2 ds \right)^{-m} 1_{f_f^T \|c_3(s)H^v(s) + v_1c_3(s) + v_2\tilde{c}_2^s(s)\|^2 ds < \epsilon} \right]
= \bar{\epsilon}^{-m} + \int_{\epsilon^{-m}}^{\infty} xd(1 - Q(x)) = \bar{\epsilon}^{-m} + \bar{\epsilon}^{-m}Q(\epsilon^{-m}) + \int_{\epsilon^{-m}}^{\infty} Q(x)dx
\leq \bar{\epsilon}^{-m} + 2e^{-\frac{1}{2}\epsilon^{-1/2}} \epsilon^{-m} + \int_{\epsilon^{-m}}^{\infty} 2e^{-\frac{1}{2}x^{1/(2m)}} dx,
$$

where the second inequality is the case due to (A.13). Note that $\int_{\epsilon^{-m}}^{\infty} 2e^{-\frac{1}{2}x^{1/(2m)}} dx < +\infty$, so we immediately derive (A.9). The proof then completes. \hfill \Box

Proof of Lemma 2

(i) For simplicity, we assume $\beta_0, \beta_1, \beta_{2,i},$ and $\beta_{3,i}$, $i = 1, \ldots, d$ are continuous on $[0, t_1)$ with the left-limit at $t_1$ existing and continuous on $[t_1, T]$ for certain $t_1 \in (0, T)$. The case in which $[0, T]$ is divided into multiple pieces and the above functions are continuous
We first prove that \( \mathcal{S}_{\mathcal{X} \mathcal{T}} \subseteq \text{cls}(\mathcal{U}) \). Consider any mollifier \( \eta \), which is a non-negative, infinitely differentiable real function on \( \mathbb{R} \) that is supported on \([0, 1]\) and satisfies \( \int_{\mathbb{R}} \eta(s) \, ds = 1 \). For each \( k \in \mathbb{N} \), define \( W_k(t) = (W_{1,k}(t), ..., W_{d,k}(t)) \) with the convention that \( W_i(t) = 0, t < 0 \). Then, \( W_k \in \mathcal{C}^\infty([0, T], \mathbb{R}^d) \subseteq \mathcal{F}([0, T]; \mathbb{R}^d) \). Because \( \beta_0, \beta_1, \beta_{2,i}, \) and \( \beta_{3,i}, \; i = 1, \ldots, d \) are continuous on \([0, t_1]\), the left-limit at \( t_1 \) existing, Theorem 3.1 of Gyöngy (1989) yields that max_{\( t \in [0,t_1] \)} ||X(t) - f_{W_k}(t)|| converges to 0 in probability. In particular, ||X(t_1) - f_{W_k}(t_1)|| converges to 0 in probability.

Recall that \( X(t) = \tilde{X}(t; X(t_1)), \; t \in [t_1, T] \), where \( \tilde{X}(t, \xi) \) stands for the solution to the SDE (A.2) that starts at time \( t_1 \) with initial value \( \xi \). We also have \( f_{W_k}(t) = f_{W_k, f_{W_k}(t_1)}, \) where \( f_{W_k, f_{W_k}(t_1)} \) stands for the solution to (A.3) that starts from time \( t_1 \) with initial value \( x \). Consider \( \tilde{X}(t; X(t_1)), \; t \in [t_1, T] \) on the filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [t_1, T]}, \mathbb{P})\) and recall \( ||X(t_1) - f_{W_k}(t_1)|| \) converges to 0 in probability. Because \( \beta_0, \beta_1, \beta_{2,i}, \) and \( \beta_{3,i}, \; i = 1, \ldots, d \) are continuous on \([t_1, T]\), Theorem 3.1 of Gyöngy (1989) yields that max_{\( t \in [t_1,T] \)} ||\tilde{X}(t; X(t_1)) - f_{W_k, f_{W_k}(t_1)(t)}|| converges in probability to 0. As a result, max_{\( t \in [t_1,T] \)} ||X(t) - f_{W_k}(t)|| converges in probability to 0 and, consequently, max_{\( t \in [0,T] \)} ||X(t) - f_{W_k}(t)|| converges in probability to 0. In other words, \( f_{W_k} \), viewed as a random variable taking values on \( \mathcal{C}([0, T]; \mathbb{R}^d) \), converges in probability to \( X \), viewed as a random variable on the same space. Then,

\[
\mathbb{P}(X \in \text{cls}(\mathcal{U})) \geq \limsup_{k \uparrow \infty} \mathbb{P}(f_{W_k} \in \text{cls}(\mathcal{U})) = 1,
\]

showing that \( \mathcal{S}_{\mathcal{X} \mathcal{T}} \subseteq \text{cls}(\mathcal{U}) \).

Next, we prove \( \mathcal{S}_{\mathcal{X} \mathcal{T}} \supseteq \text{cls}(\mathcal{U}) \). To this end, we first show that \( \text{cls}(\mathcal{U}) = \text{cls}(\bar{\mathcal{U}}) \), where \( \bar{\mathcal{U}} := \{ f_w \in \mathcal{C}([0, T]; \mathbb{R}^l) : w \in \mathcal{C}^\infty([0, T]; \mathbb{R}^d) \} \). Because \( \beta_0, \beta_1, \beta_{2,i}, \) and \( \beta_{3,i}, \; i = 1, \ldots, d \) are continuous on \([0, t_1]\) with the left-limit at \( t_1 \) existing, the first part of the proof of Theorem 4.1 in Gyöngy (1989) yields that \( \{ f_w \in \mathcal{C}([0, t_1]; \mathbb{R}^l) : w \in \mathcal{C}^\infty([0, t_1]; \mathbb{R}^d) \} \) is dense in \( \{ f_w \in \mathcal{C}([0, t_1]; \mathbb{R}^l) : w \in \mathcal{F}([0, t_1]; \mathbb{R}^d) \} \). Similarly, \( \{ f_w \in \mathcal{C}([t_1, T]; \mathbb{R}^l) : w \in \mathcal{C}^\infty([t_1, T]; \mathbb{R}^d) \} \) is dense in \( \{ f_w \in \mathcal{C}([t_1, T]; \mathbb{R}^l) : w \in \mathcal{F}([t_1, T]; \mathbb{R}^d) \} \). Note that \( f_w \) depends on \( w \) via \( w' \) only. As a result, \( \text{cls}(\mathcal{U}) = \text{cls}(\bar{\mathcal{U}}) \). Therefore, in the following we only need to prove \( \mathcal{S}_{\mathcal{X} \mathcal{T}} \supseteq \text{cls}(\mathcal{U}) \).

Fix any \( w \in \mathcal{C}^\infty([0, T]; \mathbb{R}^d) \) and \( \varepsilon > 0 \). Define \( \tilde{f}_{w,x}(t), \; t \in [t_1, T] \) to be the solution to (A.3) that starts from time \( t_1 \) with initial value \( x \). Then, it is straightforward to show
that there exists $L > 0$ such that

\[
\max_{t \in [t_1, T]} \| \tilde{f}_{w,x}(t) - \tilde{f}_{w,y}(t) \| \leq L \| x - y \|, \quad x, y \in \mathbb{R}^l.
\]  

(A.14)

Because $\beta_0, \beta_1, \beta_{2,i}, \text{ and } \beta_{3,i}, i = 1, \ldots, d$ are continuous on $[0, t_1]$ with the left-limit at $t_1$ existing, Theorem 4.1 of Gyöngy (1989) shows that $S_{X_{t_1}}$ is the closure of $\{f_w \in \mathcal{C}([0, t_1]; \mathbb{R}^l) : w \in \mathcal{S}([0, t_1]; \mathbb{R}^d)\}$, so we have

\[
\mathbb{P}\left( \max_{t \in [0, t_1]} \| X(t) - f_w(t) \| < \frac{\epsilon}{3(L + 1)} \right) > 0.
\]  

(A.15)

Denote by $(\mathbb{P}_{t_1,x})_{x \in \mathbb{R}^l}$ to be the family of probability measures conditional on $X(t_1) = x, x \in \mathbb{R}^l$. Then, because $\beta_0, \beta_1, \beta_{2,i}, \text{ and } \beta_{3,i}, i = 1, \ldots, d$ are continuous on $[t_1, T]$, for each $x \in \mathbb{R}^l$, applying Theorem 4.1 of Gyöngy (1989), we conclude that the support of $\tilde{X}(t; x), t \in [t_1, T]$, which is the solution to the SDE (A.2) that starts at time $t_1$ with initial value $x$, is the same as the closure of $\{\tilde{f}_{w,x} \in \mathcal{C}([t_1, T]; \mathbb{R}^l) : w \in \mathcal{S}([t_1, T]; \mathbb{R}^d)\}$, where $\tilde{f}_{w,x}$ is the solution to the ODE (A.3) that starts from time $t_1$ with initial value $x$. As a result,

\[
\eta(x) := \mathbb{P}_{t_1,x}\left( \max_{t \in [t_1, T]} \| \tilde{X}(t; x) - \tilde{f}_{w,x}(t) \| < \frac{\epsilon}{3} \right) > 0.
\]

Then, denoting by $A$ the event that $\max_{t \in [0, t_1]} \| X(t) - f_w(t) \| < \frac{\epsilon}{3}$ and $\max_{t \in [t_1, T]} \| \tilde{f}_{w,X(t_1)}(t) - \tilde{f}_{w,X(t_1)}(t) \| < \frac{\epsilon}{3}$, we have

\[
\mathbb{P}\left( \max_{t \in [0, T]} \| X(t) - f_w(t) \| < \epsilon \right) \geq \mathbb{P}\left( \max_{t \in [0, t_1]} \| X(t) - f_w(t) \| < \frac{\epsilon}{3}, \max_{t \in [t_1, T]} \| X(t) - f_w(t) \| < \frac{2\epsilon}{3} \right)
\]

\[
= \mathbb{P}\left( \max_{t \in [0, t_1]} \| X(t) - f_w(t) \| < \frac{\epsilon}{3}, \max_{t \in [t_1, T]} \| \tilde{X}(t; X(t_1)) - \tilde{f}_{w,X(t_1)}(t) \| < \frac{2\epsilon}{3} \right)
\]

\[
\geq \mathbb{P}(A \text{ and } \max_{t \in [t_1, T]} \| \tilde{X}(t; X(t_1)) - \tilde{f}_{w,X(t_1)}(t) \| < \frac{\epsilon}{3})
\]

\[
= \mathbb{E}[1_A \mathbb{P}\left( \max_{t \in [t_1, T]} \| \tilde{X}(t; X(t_1)) - \tilde{f}_{w,X(t_1)}(t) \| < \frac{\epsilon}{3} | F_{t_1} \right)]
\]

\[
= \mathbb{E}[1_A \eta(X(t_1))],
\]

where the last equality is the case due to the Markovian property of $X$. By (A.14) and
\(A.15\), we conclude that \(\mathbb{P}(A) > 0\). Because \(\eta(x) > 0\) for all \(x\), we conclude that

\[
\mathbb{P}\left( \max_{t \in [0,T]} ||X(t) - f_w(t)|| < \epsilon \right) \geq \mathbb{E}[1_A \eta(X(t_1))] > 0.
\]

This shows that \(\mathcal{S}_{X_T} \supseteq \text{cls}(\mathcal{U})\).

(ii) We first prove that \(\text{cls}(\mathcal{U}_t) \subseteq \mathcal{S}_{X(t)}\). For any \(y \in \text{cls}(\mathcal{U}_t)\), there exists \(f_n \in \mathcal{U}\) such that \(f_n(t)\) converges to \(y\). Then, for each fixed \(\epsilon > 0\), there exists \(n_\epsilon\) such that \(||f_n(t) - y|| < \epsilon/2\). Consequently,

\[
\mathbb{P}(||X(t) - y|| < \epsilon) \geq \mathbb{P}(||X(t) - f_{n_\epsilon}(t)|| < \epsilon/2) \geq \mathbb{P}(\max_{t \in [0,T]} ||X(t) - f_{n_\epsilon}(t)|| < \epsilon/2) > 0,
\]

where the last inequality is the case because \(f_{n_\epsilon} \in \mathcal{U}\) and \(\mathcal{S}_{X_T} = \text{cls}(\mathcal{U})\).

Next, we prove \(\mathcal{S}_{X(t)} \subseteq \text{cls}(\mathcal{U}_t)\). To this end, consider any \(y \notin \text{cls}(\mathcal{U}_t)\). Then, there exists \(\epsilon_0 > 0\) such that \(||f(t) - y|| \geq \epsilon_0\) for any \(f \in \mathcal{U}\). Because \(\mathcal{S}_{X_T} = \text{cls}(\mathcal{U})\), we have \(||f(t) - y|| \geq \epsilon_0\) for any \(f \in \mathcal{S}_{X_T}\). Consequently, \(A := \{f \in \mathcal{C}([0,T]; \mathbb{R}^d) : ||f(t) - y|| < \epsilon_0\} \subseteq \mathcal{S}_{X_T}^c\), where \(\mathcal{S}_{X_T}^c\) stands for the complement of \(\mathcal{S}_{X_T}\). As a result,

\[
\mathbb{P}(||X(t) - y|| < \epsilon_0) = \mathbb{P}(X \in A) \leq \mathbb{P}(X \in \mathcal{S}_{X_T}^c) = 0,
\]

where the last equality is the case because \(\mathcal{S}_{X_T}\) is the support of \(X\). Thus, \(y \notin \mathcal{S}_{X(t)}\).

In other words, \(\mathcal{S}_{X(t)} \subseteq \text{cls}(\mathcal{U}_t)\). Recall that \(\mathcal{U}_t \subset \mathcal{U}_t\) by their definition. Thus, we have

\(\mathcal{S}_{X(t)} = \text{cls}(\mathcal{U}_t) = \text{cls}(\mathcal{U}_t)\).

(iii) Fix any \(u, v \in \mathcal{C}^\infty([0,T]; \mathbb{R}^d)\) and for any \(\lambda \in [0,1]\), define \(w_\lambda := \lambda u + (1-\lambda)v\). By Gronwall’s inequality, we conclude from equation \((A.3)\) that \(L_1 := \sup_{\lambda \in [0,1]} \max_{t \in [0,T]} f_{w_\lambda}(t) < +\infty\). Moreover, there exists \(L_2 > 0\) such that \(|w_{\lambda_1,i} - w_{\lambda_2,i}| \leq L_2|\lambda_1 - \lambda_2|, \forall \lambda_1, \lambda_2 \in [0,1]\). Applying Gronwall’s inequality again, we conclude from equation \((A.3)\) that \(f_{w_\lambda}\) is continuous in \(\lambda\). Therefore, \(\mathcal{U}\) and \(\mathcal{U}_t, t \in [0,T]\), are connected.

When \(\ell = 1\), because \(\mathcal{U}_t\) is connected, it is an interval. As a result, \(\mathcal{S}_{X(t)} = \text{cls}(\mathcal{U}_t)\) is also an interval. \(\square\)
B Proofs

B.1 Proof of Theorem 1

By (3.16), parts (i)–(iii) are consequences of straightforward calculation, so we only need to prove (iv) in the following.

Recall $\tilde{Z}_1(T; t)$ and $\tilde{Z}_2(T; t)$ as defined in Lemma 1. It is straightforward to see that

$$\tilde{X}^*(T; t, x) = x\tilde{Z}_1(T; t) + \tilde{Z}_2(T; t).$$

Noting that $\tilde{Z}_1(T; t) > 0$, we derive

$$F^*(t, x, y) = \int_0^{+\infty} \int_{-\infty}^{y-z_1 x} g(z_1, z_2; t) dz_2 dz_1,$$

where $g(z_1, z_2; t)$ stands for the density of $(\tilde{Z}_1(T; t), \tilde{Z}_2(T; t))$. By Lemma 1,

$$\sup_{z_1, z_2} (z_1^2 + z_2^2)^{k/2} \left| \frac{\partial^{i+j} g}{\partial z_1^i \partial z_2^j} (z_1, z_2; t) \right| < +\infty$$

for any $k, i, j \geq 0$, so the dominated convergence theorem yields that $F^*(t, x, y)$ is differentiable in $y$ and its derivative is

$$\frac{\partial F^*}{\partial y}(t, x, y) = \int_0^{+\infty} g(z_1, y-z_1 x; t) dz_1,$$  \hspace{1cm} \text{(B.2)}$$

and that $F^*(t, x, y)$ is differentiable in $x$ with derivative

$$\frac{\partial F^*}{\partial x}(t, x, y) = -\int_0^{+\infty} z_1 g(z_1, y-z_1 x; t) dz_1.$$  \hspace{1cm} \text{(B.3)}$$

Moreover, $\frac{\partial F^*}{\partial y}(t, x, y)$ and $\frac{\partial F^*}{\partial x}(t, x, y)$ are continuous in $(x, y)$, so $F^*(t, x, y)$ is differentiable in $(x, y)$. Similar arguments show that $F^*$ is infinitely differentiable in $(x, y)$ with

$$\frac{\partial^{i+j} F^*}{\partial x^i \partial y^j}(t, x, y) = \int_0^{+\infty} \frac{\partial^{i+j} g}{\partial x^i \partial y^j} \left( \int_{-\infty}^{y-z_1 x} g(z_1, z_2; t) dz_2 \right) dz_1.$$  

Moreover, $F^*(t, x, y)$ and its derivatives with respect to $x$ and $y$ of any order are bounded in $(x, y) \in \mathbb{R}^2$.

Finally, recalling (3.16), we complete the proof. \qed
B.2 Proof of Theorem 2

We need to consider \( F^*(t, x, y) \) in following discussion and then recall the transformation (3.16).

We prove (i) first, and we only need to consider the case in which \( t_* < t^* \). Straightforward calculation yields that \( F^* \in C^{1,\infty}(t_{i-1}, t_i) \times (\mathbb{R}^2 \setminus \{(\xi, \xi)\}) \) and (3.21) holds. In particular,

\[
\frac{\partial F^*}{\partial t}(t, x, y) = \begin{cases} 
\phi \left( \frac{\ln(\frac{x-\xi}{y-x})}{b_t} \right) 1_{y>\xi} \left[ g_1(t) \ln \left( \frac{y-\xi}{x-\xi} \right) + g_2(t) \right], & y \in \mathbb{R}, x > \xi, \\
0, & y \neq \xi, x = \xi, \\
-\phi \left( \frac{\ln(\frac{x-\xi}{y-x})}{b_t} \right) 1_{y<\xi} \left[ g_1(t) \ln \left( \frac{y-\xi}{x-\xi} \right) + g_2(t) \right], & y \in \mathbb{R}, x < \xi,
\end{cases}
\]

where \( b_t, \bar{a}_t \) are given in Theorem 1-(iii), and

\[
g_1(t) = \frac{d}{dt}(1/\bar{b}_t), \quad g_2(t) = \frac{d}{dt}(-\bar{a}_t/\bar{b}_t).
\]

By the definition of \( t^* \) and \( t_* \), we have \( b_t > 0 \) for any \( t < t^* \), so for any \( \tau \in [t_*, t^*] \), \( g_1(t) \) and \( g_2(t) \) are bounded in \( t \in [t_*, \tau] \) and \( \phi ((z - \bar{a}_t) / \bar{b}_t) P(z) \) are bounded in \( (t, z) \in [t_*, \tau] \times \mathbb{R} \) for any polynomial function \( P(z) \). Thus,

\[
\sup_{t \in [t_*, \tau], (x,y) \neq (\xi, \xi)} |F^*_1(t, x, y)| < +\infty.
\]

Similar calculation shows that for any \( j, k \in \mathbb{N} \cup \{0\} \), we have

\[
\frac{\partial^{1+j+k} F^*}{\partial t \partial x^j \partial y^k}(t, x, y) = 1_{y>\xi} (x-\xi)^{-j} (y-\xi)^{-k} \times \sum_{n=0}^{j+k} h_n(t, \ln \left( \frac{y-\xi}{x-\xi} \right)) \phi^{(n)} \left( \ln \left( \frac{x-\xi}{y-\xi} \right) \right),
\]

where \( \phi^{(n)} \) stands for the \( n \)-th derivative of \( \phi \) and \( h_n(t, z) \) is certain function of \( (t, z) \) such that \( \sup_{t \in [t_*, \tau]} |h_n(t, z)| < C_n (1 + |z|) \), \( z \in \mathbb{R} \) for certain positive constant \( C_n \). \( \frac{\partial^{1+j+k} F^*}{\partial t \partial x^j \partial y^k}(t, x, y) \) takes a similar form when \( x < \xi \) and is 0 when \( x = \xi \) and \( y \neq \xi \). For each fixed \( \delta > 0 \), \( (x, y) \notin B_2(\xi, \delta) \) implies that

\[
|x-\xi|^{-1} \leq \delta^{-1} \left( 1 + \left( \frac{y-\xi}{x-\xi} \right)^2 \right)^{1/2}, \quad |y-\xi|^{-1} \leq \delta^{-1} \left( 1 + \left( \frac{y-\xi}{x-\xi} \right)^2 \right)^{1/2}.
\]

25
Together with (B.4) and noting that for any \( m \in \mathbb{R} \), \((1 + |\ln z|)(1 + z^m)|\phi^{(m)}((\ln z - \bar{a}_t)/\bar{b}_t)|\) is bounded in \((t, z) \in [t_*, \tau] \times (0, +\infty)\), we immediately conclude (3.22) for \( \ell = 1 \). The case \( \ell = 0 \) can be treated similarly.

Next, we prove (ii), and we only need to consider the case in which \( t_* > 0 \). By the definition of \( t_* \) and \( t^* \), we conclude that for each \( \tau \in (t_{m-1}, t_*) \), it is either the case in which \( c_3(s) = 0, \forall s \in [\tau, T) \) or the case in which for any \( v = (v_1, v_2)^T \in \mathbb{R}^2 \) with \( \|v\| = 1 \), there exists \( s \in [\tau, T) \) such that \( v_1 c_3(s) + v_2 \tilde{c}_2(s) \neq 0 \). Thus, Theorem 1 yields that \( F^*(\tau, x, y) \) is infinitely differentiable in \((x, y) \in \mathbb{R}^2\) with bounded derivatives. Recall that

\[
F^*(t, x, y) = \mathbb{E}[F^*(\tau, \bar{X}^*(\tau; t, x, y))].
\]

Because \( \bar{X}(\tau; t, x) \) is infinitely differentiable in \( x \) pathwisely, and the derivatives of any order have finite arbitrary order moments, the dominated convergence theorem yields that for any \( j, k \in \mathbb{N}_0 \),

\[
\frac{\partial^{j+k}F^*}{\partial x^j \partial y^k}(t, x, y) = \mathbb{E} \left[ \frac{\partial^{j+k}F^*}{\partial x^j \partial y^k}(\tau, \bar{X}^*(\tau; t, x, y)) \right].
\]

Because the derivatives of \( \bar{X}^*(\tau; t, x) \) with respect to \( x \) is continuous in \((t, x)\), the dominated convergence theorem yields that \( \frac{\partial^{j+k}F^*}{\partial x^j \partial y^k}(t, x, y) \) is continuous and bounded in \((t, x, y) \in [t_{m-1}, \tau) \times \mathbb{R}^2\). Moreover, Chapter 5, Theorem 6.1 Friedman (2012) implies that for each fixed \( y \in \mathbb{R} \), \( F^*(t, x, y) \), as a function of \((t, x)\), belongs to \( \mathcal{C}^{1,2}([t_{m-1}, \tau) \times \mathbb{R}) \) and

\[
F^*_t(t, x, y) = -\frac{1}{2} \|\tilde{c}_2(t) + c_3(t)x\|^2 F^*_x(t, x, y), \quad (t, x, y) \in [t_{m-1}, \tau) \times \mathbb{R}^2,
\]

showing that \( F^*_t \) exists and \( F^*_t(t, x, y) \) is continuous in \((t, x, y) \in [t_{m-1}, \tau) \times \mathbb{R}^2 \) and infinitely differentiable in \((x, y) \in \mathbb{R}^2 \), and \( \sup_{t \in [t_{m-1}, \tau), y \in \mathbb{R}} |\frac{\partial^{j+k}F^*}{\partial x^j \partial y^k}(t, x, y)| \) is of polynomial growth in \( x \) for any \( j, k \in \mathbb{N}_0 \). Because \( \tau \) is arbitrarily, we conclude that \( F^* \in \mathcal{C}^{1,\infty}([t_{m-1}, t_*) \times \mathbb{R}^2) \), \( F^*_t \in \mathcal{C}^{0,\infty}([t_{m-1}, t_*) \times \mathbb{R}^2) \), and for any \( \tau' \in [t_{m-1}, t_*) \) and \( j, k \in \mathbb{N}_0 \), \( |\frac{\partial^{j+k}F^*}{\partial x^j \partial y^k}(t, x, y)| \) is bounded in \((t, x, y) \in [t_{m-1}, \tau') \times \mathbb{R}^2 \) and \( \sup_{t \in [t_{m-1}, \tau'), y \in \mathbb{R}} |\frac{\partial^{j+k}F^*}{\partial x^j \partial y^k}(t, x, y)| \) is of polynomial growth in \( x \). Similar arguments show that for any \( i = 1, \ldots, m - 1 \), \( F^* \in \mathcal{C}^{1,\infty}([t_{i-1}, t_i) \times \mathbb{R}^2) \), \( F^*_t \in \mathcal{C}^{0,\infty}([t_{i-1}, t_i) \times \mathbb{R}^2) \), \( |\frac{\partial^{j+k}F^*}{\partial x^j \partial y^k}(t, x, y)| \) is bounded in \((t, x, y) \in [t_{i-1}, t_i) \times \mathbb{R}^2 \) and \( \sup_{t \in [t_{i-1}, t_i), y \in \mathbb{R}} |\frac{\partial^{j+k}F^*}{\partial x^j \partial y^k}(t, x, y)| \) is of polynomial growth in \( x \). Combining (3.13), (3.16), and (B.5), we derive that \( \mathcal{A}F(t, x, y) = 0 \).

Next, we prove (iii), and we only need to consider the case \( t_* \in (0, t^*) \). For any \( t \in [0, t_*) \), we have \( F^*(t, x, y) = \mathbb{E}[F^*(t, \bar{X}^*(t_*; t, x, y))]. \) Fixing \((x_0, y_0) \neq (\xi, \xi)\) and we prove that
$F^*(t, x, y)$ is continuous at $(t_*, x_0, y_0)$ from the left of $t_*$. When $y_0 \neq \xi$, because $F^*(t_*, x', y')$ is continuous in $(x', y') \neq (\xi, \xi)$ and because $\tilde{X}^*(t_*; t, x)$ is continuous in $(t, x)$ pointwisely, the dominated convergence theorem yields that $\lim_{t \uparrow t_*; (x, y) \to (x_0, y_0)} F^*(t, x, y) = F^*(t_*, x_0, y_0)$. When $x_0 \neq \xi$ and $y_0 = \xi$, Corollary 1 shows that $\mathbb{P}(\tilde{X}^*(t_*; t, x) = \xi) = 0$ for any $t$ and $x \neq \xi$, so the dominated convergence theorem again yields that $\lim_{t \uparrow t_*; (x, y) \to (x_0, y_0)} F^*(t, x, y) = F^*(t_*, x_0, y_0)$.

The same calculation as in the proof of part (ii) of the theorem yields that for any $j, k \in \mathbb{N}_0$

$$\sup_{x \in \mathbb{R}} \left| \frac{\partial^{j+k} F^*}{\partial x^j \partial y^k}(t_*, x, y) \right| \leq C_{j,k} |y - \xi|^{-(j+k)}, \quad y \neq \xi$$

(B.6)

for some constant $C_{j,k} > 0$. Because $\tilde{X}(t_*; t, x)$ is infinitely differentiable in $x$ pathwisely, and the derivatives of any order have finite moments of any order, the dominated convergence theorem immediately yields that

$$\frac{\partial^{j+k} F^*}{\partial x^j \partial y^k}(t, x, y) = \mathbb{E} \left[ \frac{\partial^{j+k} F^*}{\partial x^j \partial y^k}(t_*, \tilde{X}^*(t_*; t, x), y) \right], \quad t \in [0, t_*], x \in \mathbb{R}, y \neq \xi$$

and that $\frac{\partial^{j+k} F^*}{\partial x^j \partial y^k}$ is continuous at $(t_*, x, y)$ from the left of $t_*$ for any $x \in \mathbb{R}$ and $y \neq \xi$. Moreover, (B.6) yields that for any $\delta > 0$, $\sup_{t \in [0, t_*], x \in \mathbb{R}, |y - \xi| > \delta} \left| \frac{\partial^{j+k} F^*}{\partial x^j \partial y^k}(t, x, y) \right| < +\infty$, which together with (3.22) yields that (3.24) holds. Recalling (3.23), we have $\sup_{t \in [0, \tau], |y - \xi| > \delta} \left| \frac{\partial^{j+k} F^*}{\partial x^j \partial y^k}(t, x, y) \right|$ is of polynomial growth in $x$ for any $\delta > 0$ and $\tau \in [0, t^*)$.

Now, we prove (iv), and only need to consider the case $t_* > 0$. For each $t \in [0, t_*)$, it is either the case in which $c_3(s) = 0, \forall s \in [t, T)$ or the case in which for any $v = (v_1, v_2)^\top \in \mathbb{R}^2$ with $\|v\| = 1$, there exists $s \in [t, T)$ such that $v_1 c_3(s) + v_2 c_2^2(s) \neq 0$. When $c_3(s) = 0, \forall s \in [t, T)$, the definition of $t_*$ implies that $c_2(s) \neq 0$ for some $s \in [t, t_*)$, and a direct calculation shows that $\frac{\partial F^*}{\partial x}(t, x, y) < 0$, for any $(x, y) \in \mathbb{R}^2$. For the later case and for sake of contradiction, we assume there exists $(x, y) \in \mathbb{R}^2$, such that $F^*(t, x, y) \in (0, 1)$ and $\frac{\partial F^*}{\partial x}(t, x, y) \geq 0$. According to (B.3), we obtain that $\frac{\partial F^*}{\partial y}(t, x, y) = 0$, which together with (B.2) implies that $\frac{\partial F^*}{\partial y}(t, x, y) = 0$. Theorem 1 yields that $\tilde{X}^*(T; t, x)$ has a continuous density and Lemma 2-(iii) then implies that the interior of the support of $\tilde{X}^*(T; t, x)$ is a nonempty interval, so $y$ is in the interior of the support from $F^*(t, x, y) \in (0, 1)$. Proposition 2.1.8 of Nualart (2006) further implies that the density, i.e. $\frac{\partial F^*}{\partial y}(t, x, y)$ is positive when $y$ is in the interior of the support of $\tilde{X}^*(T; t, x)$. Then, we arrive a contradiction.
When \( t_* < t^* \), for any \( t \in [t_*, t^*) \), straightforward calculation yields that

\[
\frac{\partial F^*}{\partial x}(t, x, y) = \begin{cases} 
-\phi \left( \frac{\ln \left( \frac{y - \xi}{t} \right) - \alpha}{b_t} \right) \left( 1_{x > \xi} \right) \frac{1}{b_t(x - \xi)}, & x \in \mathbb{R}, y > \xi \\
0, & x \neq \xi, y = \xi, \\
\phi \left( \frac{\ln \left( \frac{y - \xi}{t} \right) - \alpha}{b_t} \right) \left( 1_{x < \xi} \right) \frac{1}{b_t(x - \xi)}, & x \in \mathbb{R}, y < \xi,
\end{cases}
\]

Because for fixed \( t \in [t_*, t^*) \), \( F^*(t, x, y) \in (0, 1) \) if and only if \( x > \xi, y > \xi \) or \( x < \xi, y < \xi \), then \( \frac{\partial F^*}{\partial x}(t, x, y) < 0 \).

Finally, we prove (v). We consider the case \( x > y \) only, as the case \( x < y \) can be treated similarly. Because \( c_2, c_3 \in \mathcal{C}_{pw}([0, T]; \mathbb{R}^d) \), there exists \( C > 0 \) such that \( \mathbb{E}[|\tilde{X}^*(t^*; t, x') - x'|^2] \leq C|t^* - t| \) for all \( x' \) and \( t \) that is sufficiently close to \( t^* \). By Chebyshev’s inequality, for all \( t \) that is sufficiently close to \( t^* \) and all \( x' > y' \),

\[
F^*(t, x', y') = \mathbb{P}(\tilde{X}^*(t^*; t, x') \leq y') \leq \mathbb{P}(|\tilde{X}^*(t^*; t, x') - x'| \geq x' - y') \\
\leq \mathbb{E}[|\tilde{X}^*(t^*; t, x') - x'|^2]/(x' - y')^2 \leq C|t^* - t|/(x' - y')^2,
\]

which immediately implies \( \lim_{t \uparrow t^*} F^*(t, x', y') = 0 = F^*(t^*, x, y) \).

**B.3 Proof of Corollary 2**

Because of (3.16), we need to consider \( G^*(t, x, y) \) in following discussion. (i) is trivial to prove, so next we assume \( t^* > 0 \) and prove (ii) and (iii).

For any \( t \in [t_*, t^*) \) and \( x \neq \xi \), \( \tilde{X}^*(T; t, x) \) has a continuous density, the support of \( \tilde{X}^*(T; t, x) \) is an interval, and the density is positive in the interior of the support of \( \tilde{X}^*(T; t, x) \). Thus, \( G^*(t, x, \alpha) \) is uniquely determined by \( F^*(t, x, G^*(t, x, \alpha)) = \alpha, \) and \( F_y^*(t, x, G^*(t, x, \alpha)) > 0 \). Suppose \( t_* > 0 \) and consider any \( t \in [0, t_*) \) and \( x \in \mathbb{R} \). Because \( t < t_* \), Theorem 1 yields that \( \tilde{X}^*(T; t, x) \) has a continuous density and Lemma 2-(iii) then implies that the interior of the support of \( \tilde{X}^*(T; t, x) \) is a nonempty interval. Proposition 2.1.8 of Nualart (2006) further implies that the density is positive in the interior of the support of \( \tilde{X}^*(T; t, x) \). Thus, \( G^*(t, x, \alpha) \) is uniquely determined by \( F^*(t, x, G^*(t, x, \alpha)) = \alpha \) and \( F_y^*(t, x, G^*(t, x, \alpha)) > 0 \). Then, (ii) and (iii) of the corollary just follows from the implicit function theorem and Theorem 2.

Finally, we prove (iv). Fix any \( x \in \mathbb{R} \) and \( \alpha \in (0, 1) \). For the sake of contradiction, suppose \( G^*(t, x', \alpha') \) does not converge to \( x \) when \( (t, x', \alpha') \to (t^*, x, \alpha) \). Then, there exists a sequence \( (t_n, x_n, \alpha_n) \) that converges to \( (t^*, x, \alpha) \) and satisfies either \( G^*(t_n, x_n, \alpha_n) \geq x + \delta, n \in \mathbb{N} \).
\( \mathbb{N} \) or \( G^*(t_n, x_n, \alpha_n) \leq x - \delta, n \in \mathbb{N} \) for some \( \delta > 0 \). If \( t_* < t^* \) and \( x_n = \xi \) for infinitely many \( n \), we immediately have \( x = \xi \) and \( G^*(t_n, x_n, \alpha_n) = G^*(t_n, \xi, \alpha_n) = \xi \) for certain \( n \), which is a contradiction. If \( t_* = t^* \) or \( x_n = \xi \) only for finitely many \( n \), we conclude that for sufficiently large \( n \), \( F^*(t_n, x_n, G^*(t_n, x_n, \alpha_n)) = \alpha_n \). In the case in which \( G^*(t_n, x_n, \alpha_n) \geq x + \delta, n \in \mathbb{N} \), we have \( \alpha_n = F^*(t_n, x_n, G^*(t_n, x_n, \alpha_n)) \geq F^*(t_n, x_n, x + \delta) \). Because \( \alpha_n \to \alpha \in (0, 1) \) and \( F^*(t_n, x_n, x + \delta) \to F^*(t^*, x, x + \delta) = 1 \) by Theorem 2-(v), we arrive at contradiction. Similarly, in the case in which \( G^*(t_n, x_n, \alpha_n) \leq x - \delta, n \in \mathbb{N} \), we can also derive contradiction. The proof then completes. \[ \Box \]

**B.4 Proof of Theorem 3**

Because of the transformation (3.9), we only need to consider the support of the distribution of \( X^*(t) \), denoted by \( \mathcal{S}_{X^*(t)} \), and the set of reachable states of \( X^* \) at time \( t \), denoted by \( \mathcal{X}_t^* \). For any \( t \in [0, \bar{t}] \), we have \( c_2^*(t) = 0 \) and thus \( h(t) = 0 \). As a result, \( \bar{t} \leq \bar{t} \). For readability, we divide the remaining proof into several steps.

**B.4.1 Characterize \( \mathcal{S}_{X^*(t)} \) by an optimal control problem**

Lemma 2 shows that \( \mathcal{S}_{X^*(t)} \) is a closed interval with the lower end \( \underline{x}^*(t) \in [-\infty, +\infty) \) and the upper end \( \bar{x}^*(t) \in (-\infty, +\infty] \). Moreover,

\[
\underline{x}^*(t) = \inf_{w \in \mathcal{H}([0, T]; \mathbb{R}^d)} f_w(t) = \inf_{w \in \mathcal{C}([0, T]; \mathbb{R}^d)} f_w(t) = \inf_{w \in \tilde{\mathcal{H}}([0, T]; \mathbb{R}^d)} f_w(t),
\]
\[
\bar{x}^*(t) = \sup_{w \in \mathcal{H}([0, T]; \mathbb{R}^d)} f_w(t) = \sup_{w \in \mathcal{C}([0, T]; \mathbb{R}^d)} f_w(t) = \sup_{w \in \tilde{\mathcal{H}}([0, T]; \mathbb{R}^d)} f_w(t),
\]

where \( \tilde{\mathcal{H}}([0, T]; \mathbb{R}^d) \) denotes the set of absolutely continuous \( w \) with a bounded derivative and for each \( w \in \mathcal{H}([0, T]; \mathbb{R}^d) \), \( f_w \) is the solution to the following ODE:

\[
f'_w(t) = H^w(t, f_w(t)), t \in [0, T], \quad f_w(0) = 0. \tag{B.7}
\]

Here,

\[
H^w(t, x) := (c_2^*(t) + c_3(t)x) \overline{\left(w(t) - \frac{1}{2}c_3(t)\right)} = c_2^*(t) - \frac{1}{2}c_3(t), \quad t \in [0, T], x \in \mathbb{R}. \tag{B.8}
\]
By definition, \( \tau \) respectively, we derive, for any fixed \( \bar{\omega} \),

\[ V_x^w(t, x; \tau) = \int_t^\tau e^\int_s^\tau c_3(z)^\top (-\frac{1}{2} c_3(z) + w'(z))dz \]

Next, straightforward calculation yields

\[ f_w(t) = \int_0^t e^\int_s^t c_3(\tau)^\top (-\frac{1}{2} c_3(\tau) + w'(\tau))d\tau c_3(s)^\top (\frac{1}{2} c_3(s) + w'(s))ds, \quad t \in [0, T]. \]

As a result, for any \( t < s \), by considering a particular \( w \in \mathcal{H}([0, T]; \mathbb{R}^d) \) with \( w'(\tau) = \frac{1}{2} c_3(\tau), \tau \in (t, s] \), it is straightforward to see that

\[ \inf_{w \in \mathcal{H}([0, T]; \mathbb{R}^d)} f_w(s) \leq \inf_{w \in \mathcal{H}([0, T]; \mathbb{R}^d)} f_w(t), \quad \sup_{w \in \mathcal{H}([0, T]; \mathbb{R}^d)} f_w(s) \geq \sup_{w \in \mathcal{H}([0, T]; \mathbb{R}^d)} f_w(t), \]

so \( x^*(s) \leq x^*(t) \) and \( \bar{x}^*(s) \geq \bar{x}^*(t) \), i.e., \( \mathcal{S}_{X^*} \subseteq \mathcal{S}_{X^*} \).

Next, we prove that \( x^* \) and \( \bar{x}^* \) are left-continuous. Fix any \( t \in (0, T] \). Because \( x^* \) is decreasing, we conclude \( x^*(t) \leq \liminf_{s \uparrow t} x^*(s) \). On the other hand, for any \( w \in \mathcal{H}([0, T]; \mathbb{R}^d) \), we have

\[ f_w(t) = \lim_{s \uparrow t} f_w(s) \geq \limsup_{s \downarrow t} f_w(s). \]

As a result, \( x^*(t) = \inf_{w \in \mathcal{H}([0, T]; \mathbb{R}^d)} f_w(t) \leq \limsup_{s \uparrow t} x^*(s) \). Thus, \( x^*(t) = \lim_{s \uparrow t} x^*(s) \). Similarly, \( \bar{x}^*(t) = \lim_{s \uparrow t} \bar{x}^*(s) \).

Because \( \mathcal{S}_{X(t)} = \lambda_0(t) + \lambda_1(t) (x(t) + \mathcal{S}_{X^*}(t)) \), we derive \( \mathcal{S}_{X^*}(t) = \lambda_0(t) + \lambda_1(t) (x(t) + \bar{x}^*(t)) \).

Because \( \lambda_0 \) and \( \lambda_1 \) are continuous, we conclude that \( x \) and \( \bar{x} \) are left-continuous.

Next, we solve \( x^*(t) \) and \( \bar{x}^*(t) \). For any \( w \in \mathcal{H}([0, T]; \mathbb{R}^d) \) and \((t, x) \in [0, T] \times \mathbb{R} \), define \( V^w(t, x; \tau), \tau \in [t, T] \) to be the solution to the following equation

\[ \frac{\partial V^w}{\partial \tau}(t, x; \tau) = H^w(\tau, V^w(t, x; \tau)), \quad \tau \in [t, T], \quad V^w(t, x; t) = x. \]

By definition, \( f_w(s) = V^w(0, 0; s), s \in [0, T] \). Straightforward calculation shows that

\[
V^w(t, x; \tau) := x e^\int_t^\tau c_3(z)^\top (-\frac{1}{2} c_3(z) + w'(z))dz + \int_t^\tau e^\int_s^\tau c_3(z)^\top (-\frac{1}{2} c_3(z) + w'(z))dz c_3(s)^\top (\frac{1}{2} c_3(s) + w'(s))ds.
\]

Consequently, denoting by \( V_t^w \) and \( V_x^w \) the partial derivatives of \( V^w \) with respect to \( t \) and \( x \), respectively, we derive, for any fixed \( \tau \in (0, T] \), that

\[ V_t^w(t, x; \tau) + V_x^w(t, x; \tau) H^w(t, x) = 0, \quad \forall x \in \mathbb{R} \] and almost everywhere \( t \in [0, \tau] \). (B.10)
As a result, for any \( w, \hat{w} \in \mathcal{H}([0, T]; \mathbb{R}^d) \) and \( \tau \in (0, T] \), we have

\[
f_{\hat{w}}(\tau) - f_w(\tau) = V^w(\tau, f_{\hat{w}}(\tau); \tau) - V^w(0, 0; \tau) = \int_0^\tau \frac{\partial V^w}{\partial s}(s, f_{\hat{w}}(s); \tau)ds
\]

\[
= \int_0^\tau \left( V_t^w(s, f_{\hat{w}}(s); \tau) + V_x^w(s, f_{\hat{w}}(s); \tau)H^{\hat{w}}(s, f_{\hat{w}}(s)) \right)ds
\]

\[
= \int_0^\tau V_x^w(s, f_{\hat{w}}(s); \tau)(H^{\hat{w}}(s, f_{\hat{w}}(s)) - H^w(s, f_{\hat{w}}(s)))ds
\]

\[
= \int_0^\tau e^{\int_s^\tau c_3(z)dz}(-\frac{1}{3}c_3(z) + w'(z))d\tau (c_2^*(s) + c_3(s)f_{\hat{w}}(s))^\top (\hat{w}'(s) - w'(s))ds. \tag{B.11}
\]

Because \( c_3 \) is in \( \mathcal{C}_{pw}([0, T]; \mathbb{R}^d) \) and thus bounded on \([0, T]\), there exists a constant \( L > 0 \) such that

\[
||c_2^*(s) + c_3(s)x|| - ||c_2^*(s) + c_3(s)y|| \leq ||c_3(s)x - c_3(s)y|| \leq L|x - y|, \quad \forall x, y \in \mathbb{R}, s \in [0, T].
\]

As a result, for any constant \( k \in \mathbb{R}, \) we can define \( g_k \) to be the unique solution to the following equation:

\[
g_k(t) = \int_0^t k||c_2^*(s) + c_3(s)g_k(s)||ds, \quad t \in [0, T]. \tag{B.12}
\]

Define \( w_k(t) := \int_0^t \frac{c_3(s)}{2}ds + k \int_0^t \frac{c_2^*(s) + c_3(s)g_k(s)}{||c_2^*(s) + c_3(s)g_k(s)||}1_{E_k}(s)ds, \quad t \in [0, T], \) where \( E_k \) is the set of \( s \in [0, T] \) such that \( c_2^*(s) + c_3(s)g_k(s) \neq 0 \). Then, \( w_k \in \mathcal{H}([0, T]; \mathbb{R}^d). \) Straightforward calculation from (B.7) yields

\[
f_{w_k}(t) = \int_0^t k\frac{[c_2^*(s) + c_3(s)f_{w_k}(s)]^\top[c_2^*(s) + c_3(s)g_k(s)]}{||c_2^*(s) + c_3(s)g_k(s)||}1_{E_k}(s)ds, \quad t \in [0, T]. \tag{B.13}
\]

Comparing (B.12) and (B.13), we derive \( f_{w_k} = g_k \). Moreover, by the standard comparison theorem, \( g_k \) is increasing in \( k \).

Now, taking \( \hat{w} = w_k \) in (B.11) and recalling \( g_k = f_{w_k}, \) \( t \in [0, T] \), we derive

\[
g_k(\tau) - f_w(\tau) = \int_0^\tau e^{\int_s^\tau c_3(z)dz}(-\frac{1}{3}c_3(z) + w'(z))d\tau (c_2^*(s) + c_3(s)g_k(s))^\top [w_k'(s) - w'(s)]ds
\]

\[
= k \int_0^\tau e^{\int_s^\tau c_3(z)dz}(-\frac{1}{3}c_3(z) + w'(z))d\tau ||c_2^*(s) + c_3(s)g_k(s)||ds
\]

\[
- \int_0^\tau e^{\int_s^\tau c_3(z)dz}(-\frac{1}{3}c_3(z) + w'(z))d\tau (c_2^*(s) + c_3(s)g_k(s))^\top (w'(s) - \frac{1}{2}c_3(s))ds. \tag{B.14}
\]
Denote $G^w_k(\tau) := \int_0^\tau e^{\int_s^\tau c_3(z)^T (-\frac{1}{2} c_3(z) + w'(z))dz} \|c_2^*(s) + c_3(s)g_k(s)\|ds$. Because $c_3(t)$ and $w'(t)$ are bounded for a.e. $t \in [0, T]$, we conclude from (B.14) that there exists constant $L_w > 0$, such that

\[(k - L_w)G^w_k(\tau) \leq g_k(\tau) - f_w(\tau) \leq (k + L_w)G^w_k(\tau), \quad \forall k \in \mathbb{R}. \quad (B.15)\]

Because (B.15) is true for all $w \in \hat{H}([0, T]; \mathbb{R}^d)$, we immediately derive that

$$\bar{x}^*(t) = \sup_{w \in \hat{H}([0, T]; \mathbb{R}^d)} f_w(t) = \lim_{k \uparrow +\infty} g_k(t), \quad \underline{x}^*(t) = \inf_{w \in \hat{H}([0, T]; \mathbb{R}^d)} f_w(t) = \lim_{k \downarrow -\infty} g_k(t). \quad (B.16)$$

### B.4.2 Proof of Part (i)

By the definition of $\underline{t}$, we conclude $X^*(t) = 0$, $t \in [0, \underline{t}]$ and thus $X(t) = \lambda_0(t) + \lambda_1(t)x_0$, $t \in [0, \underline{t}]$, so part (i) of the theorem follows immediately.

### B.4.3 Proof of Part (ii)

Fix $t \in (\underline{t}, T]$. Note that $X^*(t) = X^*(t; \underline{t}, 0)$. Recalling the definition of $\underline{t}$ and applying Corollary 1-(ii), with $[t, T]$, $x$, and $c_2^*$ therein set to be $[\underline{t}, t]$, 0, and $c_2^*$, respectively, we immediately conclude that $X^*(t) = X^*(t; \underline{t}, 0)$ possesses a probability density function. Then, $X(t) = \lambda_0(t) + \lambda_1(t)(x_0 + X^*(t))$ possesses a probability density function, int$(S_{X(t)})$ is a nonempty open interval and $\mathcal{X}_t = \text{int}(S_{X(t)})$.

Now, fix $t \in (\underline{t}, T]$. For each $s \in [0, T]$, it follows from the definition of $h$ that

$$\min_{x \in \mathbb{R}} \|c_2^*(s) + c_3(s)x\| = \|c_2^*(s) + c_3(s)h(s)\|$$. As a result,

$$|g_k(t)| = |k| \int_0^t \|c_2^*(s) + c_3(s)g_k(s)\|ds \geq |k| \int_0^t \|c_2^*(s) + c_3(s)h(s)\|ds,$$

where the first equality comes from (B.12). Sending $k$ in the above to $+\infty$ and $-\infty$, respectively, and recalling that $\int_0^t \|c_2^*(z) + c_3(z)h(z)\|dz > 0$ because $t \in (\underline{t}, T]$, we immediately conclude that $\bar{x}^*(t) = \lim_{k \uparrow +\infty} g_k(t) = +\infty$ and $\underline{x}^*(t) = \lim_{k \downarrow -\infty} g_k(t) = -\infty$, i.e., $\mathcal{X}_t = \mathcal{X}_t^* = \mathbb{R}$.

### B.4.4 Proof of Part (iii)

Recall that $g_k$ as defined in (B.12). As already showed in Section B.4.1, $g_k$ is increasing in $k$. Moreover, $g_0(s) = 0, \forall s \in [0, T]$, so $g_k(s) \geq 0, s \in [0, T]$ for all $k \geq 0$ and $g_k(s) \leq 0, s \in$
[0, T] for all \( k \leq 0 \). In addition, it is straightforward to see from (B.12) that for \( k \geq 0 \), \( g_k(s) \) is increasing in \( s \in [0, T] \) and for \( k \leq 0 \), \( g_k(s) \) is decreasing in \( s \in [0, T] \).

Assume \( \underline{t} < \bar{t} \) and fix \( t \in (\underline{t}, \bar{t}) \) in the following. We claim that \( \tau_t > \underline{t} \). Otherwise, we have \( c_3(z) = 0, z \in (\underline{t}, \bar{t}) \). By the definition of \( \bar{t} \), we have \( c_2(z) + c_3(z)g(z) = 0 \) for almost everywhere \( z \in [\underline{t}, \bar{t}] \). By the right-continuity of \( c_2 \), we derive \( c_2(z) = 0, \forall z \in [\underline{t}, \bar{t}] \). Then, by the definition of \( \underline{t} \), we have \( \bar{t} > \underline{t} \), which is a contradiction. Thus, we must have \( \tau_t > \underline{t} \).

By the definition of \( \bar{t} \), we have \( c_3^*(z) = -c_3(z)g(z) \) for almost everywhere \( z \in [0, \bar{t}] \). As a result, we conclude from (B.12) that

\[
g_k(s) = k \left( \int_0^s ||c_3(z)||g_k(z) - h(z)dz \right), s \in [0, \bar{t}]. \tag{B.17}
\]

By the definition of \( \underline{t} \), \( c_3^*(z) = 0 \) and thus \( h(z) = 0 \) for all \( z \in [0, \underline{t}] \). As a result, \( g_k(s) = k \left( \int_0^s ||c_3(z)||g_k(z)dz \right), \forall s \in [0, \underline{t}] \), which implies \( g_k(s) = 0, \forall s \in [0, \underline{t}] \), and

\[
g_k(s) = k \int_{\underline{t}}^s ||c_3(z)||g_k(z) - h(z)dz, s \in [\underline{t}, \bar{t}]. \tag{B.18}
\]

Suppose that there exists \( s \in [\underline{t}, \bar{t}) \) such that \( h(s) < 0 \). Then, by the definition of \( h \), we must have \( c_3(s) \neq 0 \). By the right-continuity of \( c_3^* \) and \( c_3 \), there exists \( \epsilon_0 > 0 \) and \( \delta \in (0, t - s) \) such that \( h(z) \leq -\epsilon_0 \) and \( ||c_3(z)|| \geq \epsilon_0 \) for all \( z \in [s, s + \delta] \). As a result, for any \( k \geq 0 \),

\[
\int_{\underline{t}}^t ||c_3(z)||g_k(z) - h(z)dz \geq \int_s^{s+\delta} ||c_3(z)||g_k(z) - h(z)dz \geq \epsilon_0^2 \delta,
\]

where the last inequality is the case because \( g_k(z) \geq 0, z \in [0, T] \) for all \( k \geq 0 \). We then conclude from (B.18) that \( x^*(t) = \lim_{k \uparrow +\infty} g_k(t) = +\infty \).

A similar argument shows that if there exists \( s \in [\underline{t}, t) \) such that \( h(s) > 0 \), then \( x^*(t) = -\infty \). As a result, \( X_{\underline{t}} = X_{\bar{t}}^* = \mathbb{R} \).

Next, we consider the case in which \( h(s) \leq 0 \) for all \( s \in [\underline{t}, t) \) and there exists \( s_1, s_2 \in [\underline{t}, t) \cap D \) with \( s_1 < s_2 \) such that \( h(s_1) < h(s_2) \). Then, because \( h(s_1) < 0 \), as shown in the above, \( x^*(t) = +\infty \). Denote \( g_\infty(s) := \lim_{k \downarrow -\infty} g_k(s), s \in [0, T] \). We claim that \( g_\infty(t) = -\infty \). For the sake of contradiction, suppose it is not the case, i.e., \( g_\infty(t) > -\infty \). Then, because \( g_k(s) \) is decreasing in \( s \in [0, T] \) for each \( k \leq 0 \), we have \( g_\infty(s) > -\infty, \forall s \in [0, t] \). As a result, because \( g_\infty \) is monotone on \([0, t]\), it is continuous almost everywhere on \([0, t]\). Recalling that \( c_2, c_3 \in C_{pw}([0, T]; \mathbb{R}^d) \), that \( s_1, s_2 \in [\underline{t}, t) \cap D \), and that \( h(s_1) < h(s_2) \), we can find \( \bar{s}_1, \bar{s}_2 \in [\underline{t}, \bar{t}) \cap D \) with \( \bar{s}_1 < \bar{s}_2 \) such that \( h(\bar{s}_1) < h(\bar{s}_2) \), that \( h \) is continuous at \( \bar{s}_i, i = 1, 2, \ldots \)
and that \( g_\infty \) is continuous at \( \tilde{s}_i, i = 1, 2 \). Because \( g_\infty(\tilde{s}_1) \geq g_\infty(\tilde{s}_2) \), there exists \( \epsilon_0 > 0 \) such that either \( |g_\infty(\tilde{s}_1) - h(\tilde{s}_1)| > \epsilon_0 \) or \( |g_\infty(\tilde{s}_2) - h(\tilde{s}_2)| > \epsilon_0 \). Without loss of generality, suppose \( |g_\infty(\tilde{s}_1) - h(\tilde{s}_1)| > \epsilon_0 \). Then, if \( g_\infty(\tilde{s}_1) - h(\tilde{s}_1) > \epsilon_0 \), by the continuity of \( g_\infty, c_3, \) and \( h \) at \( \tilde{s}_1 \), we can find \( \delta \in (0, t - \tilde{s}_1) \) such that \( g_\infty(z) - h(z) > \epsilon_0 \) and \( \|c_3(z)\| \geq \frac{1}{2} \|c_3(\tilde{s}_1)\| > 0 \) for all \( z \in [\tilde{s}_1, \tilde{s}_1 + \delta] \), where \( \|c_3(\tilde{s}_1)\| > 0 \) because \( \tilde{s}_1 \in D \). Because \( g_k \) is decreasing in \( k \), we conclude

\[
g_k(z) - h(z) \geq g_\infty(z) - h(z) > \epsilon_0, \quad \forall z \in [\tilde{s}_1, \tilde{s}_1 + \delta], \ k < 0.
\]

As a result, for any \( k < 0 \),

\[
\int_{\tilde{s}_1}^{\tilde{s}_1 + \delta} \|c_3(z)\| \|g_k(z) - h(z)\| dz \geq \int_{\tilde{s}_1}^{\tilde{s}_1 + \delta} \|c_3(z)\| \|g_k(z) - h(z)\| dz \geq \frac{1}{2} \|c_3(\tilde{s}_1)\| \epsilon_0 \delta.
\]

We then conclude from (B.18) by sending \( k \) to \( -\infty \) therein that \( g_\infty(t) = -\infty \), which contradicts the preassumption that \( g_\infty(t) > -\infty \). Thus, we must have \( x^*(t) = g_\infty(t) = -\infty \). Combining with \( x^*(t) = +\infty \), we conclude that \( X_t = X_t^* = \mathbb{R} \).

If \( g_\infty(\tilde{s}_1) - h(\tilde{s}_1) < -\epsilon_0 \), by the continuity of \( g_\infty, c_3, \) and \( h \) at \( \tilde{s}_1 \), there exists \( \delta \in (0, \tilde{s}_1 - t) \) such that \( |g_\infty(z) - g_\infty(\tilde{s}_1)| + |h(z) - h(\tilde{s}_1)| < \frac{1}{3} \epsilon_0 \) and \( \|c_3(z)\| \geq \frac{1}{2} \|c_3(\tilde{s}_1)\| > 0 \) for all \( z \in [\tilde{s}_1 - \delta, \tilde{s}_1] \). Moreover, there exists \( K > 0 \) such that \( |g_k(\tilde{s}_1 - \delta) - g_\infty(\tilde{s}_1 - \delta)| < \frac{1}{K} \epsilon_0 \) for all \( k \leq -K \). As a result, because \( g_k(s) \) is decreasing in \( s \in [0, T] \), we have, for each \( z \in [\tilde{s}_1 - \delta, \tilde{s}_1] \) and \( k \leq -K \), that

\[
g_k(z) - h(z) \leq g_k(\tilde{s}_1 - \delta) - h(z) \leq g_\infty(\tilde{s}_1 - \delta) + \frac{1}{3} \epsilon_0 - h(z) \\
= g_\infty(\tilde{s}_1 - \delta) - g_\infty(\tilde{s}_1) - (h(z) - h(\tilde{s}_1)) + \frac{1}{3} \epsilon_0 + g_\infty(\tilde{s}_1) - h(\tilde{s}_1) < -\frac{1}{3} \epsilon_0.
\]

As a result,

\[
\int_{\tilde{s}_1}^{\tilde{s}_1 + \delta} \|c_3(z)\| \|g_k(z) - h(z)\| dz \geq \int_{\tilde{s}_1 - \delta}^{\tilde{s}_1} \|c_3(z)\| \|g_k(z) - h(z)\| dz \geq \frac{1}{6} \|c_3(\tilde{s}_1)\| \epsilon_0 \delta.
\]

We then conclude from (B.18) by sending \( k \) to \( -\infty \) therein that \( g_\infty(t) = -\infty \), which contradicts the preassumption that \( g_\infty(t) > -\infty \). Thus, we must have \( x^*(t) = g_\infty(t) = -\infty \).

Next, we consider the case in which \( h(s) \leq 0, \forall s \in [t, t) \) and \( h(s) \) is decreasing in
Recall that we already showed that \( g \parallel g \parallel h \) there exists \( r \in k < h \). On the other hand, because \( c \parallel c \parallel \tau \) which implies that \( g \parallel c \parallel h \parallel \tau \). As a result, we conclude from (B.19) that

\[
g_k(s) = \frac{3}{2} \int \frac{f'}{h} \parallel c_3(z) \parallel d\tau = h(s) \left( \frac{3}{2} - \frac{f'}{h} \parallel c_3(z) \parallel d\tau \right) \geq h(s),
\]

which implies that \( \|c_3(s)\|g_k(s) \geq \|c_3(s)\|\bar{h}(s) \). As a result, for any \( s \in \mathbb{L}, t \),

\[
\|c_3(s)\|g_k(s) = h(s)
\]

so by the uniqueness of the solution to (B.18), we derive

\[
g_k(s) = \frac{3}{2} \int \frac{f'}{h} \parallel c_3(z) \parallel d\tau = h(s) \left( \frac{3}{2} - \frac{f'}{h} \parallel c_3(z) \parallel d\tau \right), \quad \forall s \in \mathbb{L}, t.
\]

Fix any \( s \in \mathbb{L}, t \) \( \cap D \). For any \( t \) \( \cap D \), where \( D := \{ s \in (0, T) : [s - \delta, s] \subseteq D \) for some \( \delta > 0 \}. Then, there exists \( r \in \mathbb{L}, s \) such that \( c_3(z) \neq 0 \) for all \( z \in [r, s) \). By (B.20), we have

\[
g_k(s) = e^{\int r_s} \parallel c_3(z) \parallel d\tau g_k(r_s) - k \int r_s e^{\int r_s} \parallel c_3(z) \parallel d\tau \parallel c_3(\tau) \parallel h(\tau) d\tau.
\]

Recall that we already showed that \( g_k(z) = \tilde{g}_k(z) \geq h(z) \) for all \( z \in \mathbb{L}, t \) \( \cap D \) and that \( g_k(z) \leq 0 \) for all \( z \in [0, T] \) and \( k \leq 0 \). Also recall that \( c_3(z) \neq 0, \forall z \in [r, s) \). As a result,

\[
\lim_{k \rightarrow -\infty} e^{\int r_s} \parallel c_3(z) \parallel d\tau g_k(r_s) = 0
\]

On the other hand, because \( h \) is right-continuous and decreasing on \( \mathbb{L}, t \) \( \cap D \), it defines a
measure on \((r_s, s)\), so Fubini’s theorem yields

\[-k \int_{r_s}^s e^k f^*_r \|c_3(z)\|dz \|c_3(\tau)\|h(\tau)d\tau = h(r_s) \left(1 - e^k f^*_r \|c_3(z)\|dz\right) + \int_{(r_s, s)} \left(1 - e^k f^*_r \|c_3(\tau)\|d\tau\right) dh(z).
\]

Because \(\int_{r_s}^s \|c_3(\tau)\|d\tau > 0\) for any \(z \in (r_s, s)\), the dominated convergence theorem yields that the limit of the right-hand side of the above equality, as \(k\) goes to \(-\infty\), is

\[h(r_s) + \int_{(r_s, s)} dh(z) = h(s-):= \lim_{z \uparrow s} h(z).
\]

As a result, \(\underline{x}(s) = \lim_{k \downarrow -\infty} g_k(s) = h(s-).
\]

Now, by the definition of \(\tau_t\) and the right-continuity of \(c_3\), we have \(c_3(z) = 0, \forall z \in [\tau_t, t)\).

Because \(c_2^*(z) + c_3(z)h(z) = 0\) for almost everywhere \(z \in [0, t)\) and because \(c_2^*\) is right-continuous, we derive \(c_2^*(z) = 0, \forall z \in [\tau_t, t)\). As a result, \(X^*(t) = X^*(\tau_t)\) so \(X_t = X^*\).

Thus, \(\underline{x}(s) = x^*(\tau_t) = \lim_{s \uparrow \tau_t} x^*(s)\), where the second equality is the case due to the left-continuity of \(\underline{x}\). By the definition of \(\tau_t\) and recalling that \(c_2^*\) and \(c_3\) are right-continuous, for any \(\epsilon \in (0, \tau_t - \bar{\tau})\), there exists \(s \in (\tau_t - \epsilon, \tau_t)\) such that \(s \in D\) and that \(c_3\) and \(h\) are continuous at \(s\); in particular, \(s \in (\bar{t}, t) \cap \bar{D}\) and thus \(x^*(s) = h(s-) = h(s)\). Also recall that \(h\) is decreasing on \([\bar{t}, t) \cap \bar{D}\). Then, we conclude \(\lim_{s \uparrow \tau_t} x^*(s) = \lim_{[\bar{t}, \tau_t) \cap D \ni \tau \uparrow \tau_t} h(s)\), i.e., \(\underline{x}(t) = \lim_{[\bar{t}, \tau_t) \cap D \ni \tau \uparrow \tau_t} h(s)\). Recalling that \(\mathbb{S}_{X(t)} = \lambda_0(t) + \lambda_1(t) (x_0 + \mathbb{S}_{X^*(t)})\) and that \(\mathbb{X}_t = \text{int}(\mathbb{S}_{X(t)})\), we complete the proof of part (iii-b).

Finally, part (iii-c) can be proved similarly. \(\square\)

### B.5 Proof of Corollary 3

For any \(t \in [0, T]\), because \(\sigma(t)\sigma(t)^T\) is positive definite, so \(c_2^*(t) = c_3(t) = 0\) if and only if \(\theta_0(t) = \theta_1(t) = 0\), so (4.7) holds.

Next, straightforward calculation yields that for each \(\tau \in [0, t^*)\), \(\bar{c}_2^*(t) + \xi c_3(t) = 0, \forall t \in [\tau, t^*)\) if and only if

\[
\theta_0(t) = \theta_1(t) \left[ \int_t^{\tau} b(s)^T \theta_0(s) e^{-\int_t^s b(z)^T \theta_1(z)dz} ds - \xi e^{-\int_t^{\tau} b(s)^T \theta_1(s)ds} \right] = \theta_1(t) \left[ \int_t^{\tau} b(s)^T \theta_0(s) e^{-\int_t^s b(z)^T \theta_1(z)dz} ds - \xi e^{-\int_t^{\tau} b(s)^T \theta_1(s)ds} \right], \quad t \in [\tau, t^*). \tag{B.21}
\]

Because \(\theta_1 \in \mathcal{C}_{pw}([0, T]; \mathbb{R}^d)\) and \(b\) is bounded, (B.21) has a unique solution. In addition, it
is straightforward to verify that

$$\theta_0(t) = -\xi \theta_1(t), \quad t \in [\tau, t^*)$$

solves (B.21). As a result, \(c_2^*(t) + \xi c_3(t) = 0, \forall t \in [\tau, t^*)\) if and only if \(\theta_0(t) + \xi \theta_1(t) = 0, \forall t \in [\tau, t^*)\), so (4.8) holds.

Finally, because \(\theta_0(s) = \theta_1(s) = 0, \forall s \in [t^*, T]\) and \(\theta_0(s) + \xi \theta_1(s) = 0, \forall s \in [t_*, t^*)\), straightforward calculation that for any \(t \in [t_*, t^*)\), \(\tilde{\xi}(t) = \xi. \quad \square\)

### B.6 Proof of Corollary 4

We first prove (4.11), (4.9), and (4.10). By (4.4) and recalling that \(\sigma(t)\sigma(t)\top\) is positive definite for all \(t \in [0, T]\), we conclude that for any \(\tau \in (0, T]\), \(c_2^*(t) = 0, \forall t \in [0, \tau]\) if and only if

$$\theta_0(t) = -\theta_1(t) \left( x_0e^{\int_0^t b(s)\top \theta_1(s)ds} + \int_0^t b(s)\top \theta_0(s) e^{\int_s^t b(z)\top \theta_1(z)dz} ds \right), \quad t \in [0, \tau]. \quad \text{(B.22)}$$

Because \(\theta_1 \in \mathcal{C}_{pw}([0, T]; \mathbb{R}^d)\), the above equation of \(\theta_0\) has a unique solution. Moreover, it is straightforward to verify that \(\theta_0(t) = -x_0 \theta_1(t), t \in [0, \tau]\) solves (B.22). As a result, \(c_2^*(t) = 0, \forall t \in [0, \tau]\) if and only if \(\theta_0(t) = -x_0 \theta_1(t), t \in [0, \tau]\). Consequently, we derive (4.9).

Straightforward calculation yields (4.13) and

$$c_2^*(t) + c_3(t) h(t) = \sigma(t)\top (\theta_0(t) + \theta_1(t)h(t)) e^{-\int_0^t b(s)\top \theta_1(s)ds}, \quad t \in [0, T].$$

Because \(\sigma(t)\sigma(t)\top\) is positive definite for all \(t \in [0, T]\), we immediately derive (4.10).

Next, because \(\theta_0(t) + x_0 \theta_1(t) = 0, \forall t \in [0, t]\), we immediately derive from (4.3) that \(\lambda_0(t) + \lambda_1(t)x_0 = x_0, \forall t \in [0, t]\) and from (4.12) that \(x^*(t) = x_0, \forall t \in [0, t]\). It follows from Theorem 3-(i) that \(\mathbb{X}_t = \{x_0\}\) for all \(t \in [0, t]\).

Next, we prove that \(S_{X(t)}\) is increasing in \(t \in [0, T]\). Lemma 2 shows that \(S_{X(t)}\) is a closed interval with the lower end \(\underline{x}(t) \in [-\infty, +\infty)\) and the upper end \(\bar{x}(t) \in (-\infty, +\infty]\). Moreover,

$$\underline{x}(t) = \inf_{w \in \mathcal{B}([0, T]; \mathbb{R}^d)} f_w(t), \quad \bar{x}(t) = \sup_{w \in \mathcal{B}([0, T]; \mathbb{R}^d)} f_w(t),$$

37
where $f_w$ is given by (A.3) with
\[
\beta_0 = b^\top \theta_0, \quad \beta_1 = b^\top \theta_1, \quad (\beta_{2,1}, \ldots, \beta_{2,d}) = \theta_0^\top \sigma, \quad (\beta_{3,1}, \ldots, \beta_{3,d}) = \theta_1^\top \sigma.
\]

Then, straightforward calculation yields
\[
f_w(t) = x_0 e^{\int_0^t \theta_1(s)^\top \left( b(s) - \frac{1}{2} \sigma(s) \sigma(s)^\top \theta_1(s) + \sigma(s) w'(s) \right) ds + \int_0^t e^{\int_s^t \theta_1(\tau)^\top \left( b(\tau) - \frac{1}{2} \sigma(\tau) \sigma(\tau)^\top \theta_1(\tau) + \sigma(\tau) w'(\tau) \right) d\tau} \times \theta_0(s)^\top \left( b(s) - \frac{1}{2} \sigma(s) \sigma(s)^\top \theta_1(s) + \sigma(s) w'(s) \right) ds, \quad t \in [0, T].
\]

For any $s \in [0, T)$, because $\sigma(s) \sigma(s)^\top$ is invertible, for $u(s) := \sigma(s)^\top \left( \frac{1}{2} \theta_1(s) - (\sigma(s) \sigma(s)^\top)^{-1} b(s) \right)$, we have
\[
b(s) - \frac{1}{2} \sigma(s) \sigma(s)^\top \theta_1(s) + \sigma(s) u(s) = 0.
\]

Then, it is straightforward to see that for any $t < s$,
\[
\inf_{w \in \mathcal{B}([0,T] ; \mathbb{R}^d)} f_w(s) \leq \inf_{w \in \mathcal{B}([0,T] ; \mathbb{R}^d)} f_w(t), \quad \sup_{w \in \mathcal{B}([0,T] ; \mathbb{R}^d)} f_w(s) \geq \sup_{w \in \mathcal{B}([0,T] ; \mathbb{R}^d)} f_w(t),
\]
so $\mathbb{S}_X(t) \subseteq \mathbb{S}_{X(s)}$.

Next, we already showed that $\mathcal{X}_t = \{ x_0 \}$ for all $t \in [0, \underline{t}]$, so $\mathcal{X}_t$ is increasing on $[0, \underline{t}]$. In addition, Theorem 3 shows that for any $t \in (\underline{t}, T)$, $X(t)$ possesses a density function, so $\mathcal{X}_t = \text{int}(\mathbb{S}_{X(t)})$. Then, because $\mathbb{S}_{X(t)}$ is increasing in $t \in [0, T]$, we conclude that $\mathcal{X}_t$ is increasing in $t \in (\underline{t}, T]$. To complete the proof that $\mathcal{X}_t$ is increasing in $t \in [0, T]$, we only need to show that $x_0 \in \mathcal{X}_t$ for any $t \in (\underline{t}, T]$.

Consider any $u$ with $u(s)$ solving (B.25) for any $s \in [0, T]$. For each $K \in \mathbb{R}$, construct $w_K \in \mathcal{U}$ by setting $w_K(t) = u(t) + K \sigma(t)^\top [\theta_0(t) + x_0 \theta_1(t)]$, $t \in [0, T]$. Then, by (B.24) and (B.25), we derive
\[
f_{w_K}(t) = x_0 + K \int_0^t e^{K \int_s^t \sigma(\tau)^\top [\theta_0(\tau) + x_0 \theta_1(\tau)] d\tau} \| \sigma(s)^\top [\theta_0(s) + x_0 \theta_1(s)] \|^2 ds, \quad t \in [0, T].
\]

By the definition of $\underline{t}$ and recalling that $\sigma(s) \sigma(s)^\top$ is positive definite for any $s \in [0, T]$, we derive that $f_{w_K}(t) > x_0$ for any $t \in (\underline{t}, T]$ and $K > 0$ and $f_{w_K}(t) < x_0$ for any $t \in (\underline{t}, T]$ and $K < 0$. Therefore, $\bar{x}(t) < x_0 < \underline{x}(t)$, i.e., $x_0$ is in the interior of $\mathbb{S}_{X(t)}$ and thus in $\mathcal{X}_t$.

Finally, assume $\underline{t} < \tilde{t}$ and fix $t \in (\underline{t}, \tilde{t}]$ such that $\theta_1(s) \neq 0, s \in (\underline{t}, t)$. By the definition of $\tilde{t}$ and recalling (4.10), we derive $\theta_0(s) + \theta_1(s) \tilde{h}(s) = 0$ for almost everywhere $s \in [\underline{t}, t]$ and
thus for all \( s \in (t, t] \) because \( \theta_0, \theta_1 \), and \( \tilde{h} \) are right-continuous on \((t, t)\). Note from (4.13) that \( h \) is of finite variation on \((t, t)\) if and only if \( \tilde{h} \) is of finite variation on \((t, t)\). Thus, we can differentiate both sides of (4.13) provided that one of \( h \) and \( \tilde{h} \) is of finite variation, and the differentiation yields

\[
dh(s) = e^{-\int_0^s b(z)^\top \theta_1(z) dz} d\tilde{h}(s) - b(s)^\top \theta_1(s) e^{-\int_0^s b(z)^\top \theta_1(z) dz} \tilde{h}(s) ds - b(s)^\top \theta_0(s) e^{-\int_0^s b(z)^\top \theta_1(z) dz} ds
\]

\[
= e^{-\int_0^s b(z)^\top \theta_1(z) dz} \tilde{h}(s), \quad s \in (t, t),
\]

where the equality is the case because \( \theta_0(s) + \theta_1(s) \tilde{h}(s) = 0 \), \( s \in (t, t) \). Then, we conclude that \( h \) is decreasing (increasing, respectively) on \((t, t)\) if and only if \( \tilde{h} \) is decreasing (increasing, respectively) on \((t, t)\).

Now, suppose \( \tilde{h}(s) \leq x^*(s), \forall s \in (t, t) \). Then \( h(s) \leq 0, \forall s \in (t, t) \) as \( \theta_1(s) \neq 0, s \in (t, t) \). Because \( h \) is right-continuous at \( t \) if \( \theta_1(t) \neq 0 \) and \( h(t) = 0 \) if \( \theta_1(t) = 0 \), we conclude that \( h(s) \leq 0, \forall s \in [t, t] \). Recall that \( h \) is decreasing on \((t, t)\) if and only if \( \tilde{h} \) is decreasing on \((t, t)\). Then, by Theorem 3-(iii)-(b), we immediately conclude that if \( \tilde{h} \) is not decreasing on \((t, t)\), then \( h \) is not decreasing on \((t, t)\), and thus \( X_t = \mathbb{R} \). Next, we consider the case in which \( \tilde{h} \) is decreasing on \((t, t)\) and thus \( h \) is decreasing on \((t, t)\). Because \( h \) is right-continuous at \( t \) if \( \theta_1(t) \neq 0 \) and \( h(t) = 0 \) if \( \theta_1(t) = 0 \), and because \( h(s) \leq 0, \forall s \in [t, t) \), we conclude that \( h \) is decreasing on \([t, t)\). Then, by Theorem 3-(iii)-(b), we immediately conclude that \( X_t = (\lambda_0(t) + \lambda_1(t)(x_0 + h(t-)), +\infty) \), where \( h(t-) := \lim_{s\uparrow t} h(s) \). By (4.3) and (4.13), straightforward calculation yields \( \lambda_0(t) + \lambda_1(t)(x_0 + h(t-)) = \tilde{h}(t-) \). Thus, \( X_t = (\tilde{h}(t-), +\infty) \).

Moreover, if \( \tilde{h} \) is decreasing on \((t, t)\), we claim that \( \tilde{h}(s) < x^*(s), \forall s \in (t, t) \). For the sake of contradiction, suppose \( \tilde{h}(s_0) = x^*(s_0) \) for some \( s_0 \in (t, t) \). Then, by (4.13), we have \( h(s_0) = 0 \). Recall we have shown that \( h(s) \leq 0, \forall s \in [t, t) \), and \( h \) is decreasing on \((t, t)\) because \( \tilde{h} \) is decreasing on \((t, t)\). As a result, \( h(s) = 0, \forall s \in [t, s_0] \) and, consequently, \( \tilde{h}(s) = x^*(s), \forall s \in [t, s_0] \). It follows from (B.26) that \( dx^*(s) = d\tilde{h}(s) = 0, \forall s \in [t, s_0] \). Because \( x^* \) is continuous on \([0, T] \) and \( x^*(s) = x_0, \forall s \in [0, t] \), we conclude that \( x^*(s) = x_0, \forall s \in [0, s_0] \). As a result, \( \tilde{h}(s) = x^*(s) = x_0, \forall s \in [t, s_0] \). By (4.10), we have \( \theta_0(s) + \tilde{h}(s) \theta_1(s) = 0 \) for almost everywhere \( s \in [0, \tilde{t}] \). Together with the right continuity of \( \theta_0, \theta_1, \) and \( \tilde{h} \) on \((t, t)\), we derive \( \theta_0(s) + x_0 \theta_1(s) = \theta_0(s) + \tilde{h}(s) \theta_1(s) = 0 \) for all \( s \in (t, s_0] \). Because \( \theta_0 \) and \( \theta_1 \) are right-continuous, we derive \( \theta_0(s) + x_0 \theta_1(s) = 0 \) for all \( s \in [t, s_0] \). This contradicts the definition of \( t \).

Finally, the case in which \( \tilde{h}(s) \geq x^*(s), \forall s \in (t, t) \) can be treated similarly. \( \square \)
References

Bally, V. (1991). On the connection between the malliavin covariance matrix and hörmander’s condition, *Journal of Functional Analysis* **96**(2): 219–255.

Cattiaux, P. and Mesnager, L. (2002). Hypoelliptic non-homogeneous diffusions, *Probability Theory and Related Fields* **123**(4): 453–483.

Chaleyat-Maurel, M. and Michel, D. (1984). Hypoellipticity theorems and conditional laws, *Probability Theory and Related Fields* **65**(4): 573–597.

Colonius, F. and Kliemann, W. (1999). Topological, smooth, and control techniques for perturbed systems, *Stochastic Dynamics*, Springer, pp. 181–208.

Derridj, M. (1971). Un problème aux limites pour une classe d’opérateurs du second ordre hypoelliptiques, *Annales de l’institut Fourier*, Vol. 21, pp. 99–148.

Florchinger, P. (1990). Malliavin calculus with time dependent coefficients and application to nonlinear filtering, *Probability Theory and Related Fields* **86**(2): 203–223.

Friedman, A. (2012). *Stochastic differential equations and applications*, Courier Corporation.

Gyöngy, I. (1989). The stability of stochastic partial differential equations and applications. theorems on supports, *Stochastic Partial Differential Equations and Applications II*, Springer, pp. 91–118.

Gyöngy, I., Nualart, D. and Sanz-Sole, M. (1995). Approximation and support theorems in modulus spaces, *Probability Theory and Related Fields* **101**(4): 495–509.

Gyöngy, I. and Pröhle, T. (1990). On the approximation of stochastic differential equation and on stroock-varadhan’s support theorem, *Computers & Mathematics with Applications* **19**(1): 65–70.

He, X. D. and Jiang, Z. (2019). On the equilibrium strategies for time-inconsistent problems in continuous time. SSRN:3308274.

He, X. D., Jiang, Z. and Kou, S. (2020). Portfolio selection under median maximization.

Höpfner, R., Löcherbach, E., Thieullen, M. et al. (2017). Strongly degenerate time inhomogeneous sdes: Densities and support properties. application to hodgkin–huxley type systems, *Bernoulli* **23**(4A): 2587–2616.
Hörmander, L. (1967). Hypoelliptic second order differential equations, *Acta Mathematica* **119**: 147–171.

Ichihara, K. and Kunita, H. (1974). A classification of the second order degenerate elliptic operators and its probabilistic characterization, *Probability Theory and Related Fields* **30**(3): 235–254.

Kunita, H. (1976). The support of diffusion process and controllability problem (partial differential equations and their applications).

Kusuoka, S. and Stroock, D. (1984). Applications of the malliavin calculus, part i, *North-Holland Mathematical Library*, Vol. 32, Elsevier, pp. 271–306.

Meyn, S. P. and Tweedie, R. L. (1993). Stability of markovian processes ii: Continuous-time processes and sampled chains, *Advances in Applied Probability* **25**(3): 487–517.

Nualart, D. (2006). *The Malliavin Calculus and Related Topics*, second edn, Springer.

Ondreját, M., Šimon, P. and Kupsa, M. (2018). Support of solutions of stochastic differential equations in exponential besov–orlicz spaces, *Stochastic Analysis and Applications* **36**(6): 1037–1052.

Risken, H. and Eberly, J. (1985). The fokker-planck equation, methods of solution and applications, *Journal of the Optical Society of America B Optical Physics* **2**: 508.

Rogers, L. C. G. and Williams, D. (2000). *Diffusions, markov processes, and martingales: Volume 1, Itô Calculus*, Vol. 2, 2 edn, Cambridge university press, Cambridge.

Stroock, D. W. (1981). The malliavin calculus and its application to second order parabolic differential equations: Part i, *Mathematical systems theory* **14**(1): 25–65.

Stroock, D. W. and Varadhan, S. R. (1972). On the support of diffusion processes with applications to the strong maximum principle, *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971)*, Vol. 3, pp. 333–359.

Williams, D. (1981). To begin at the beginning:, *Stochastic integrals*, Springer, pp. 1–55.

Yong, J. and Zhou, X. Y. (1999). *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer.
Yor, M. (2001). *Exponential functionals of Brownian motion and related processes*, Springer Science & Business Media.

Zak, F. (2014). Exponential ergodicity of infinite system of interacting diffusions, *arXiv preprint arXiv:1406.1756*. 