Limit Cycle Analysis of 3-D Nonlinear systems

Souma Mazumdar¹, Premashis Kumar² and Gautam Gangopadhyay³

¹Department of Theoretical Sciences
²³Department of Chemical, Biological and Macro-molecular Sciences
S. N. Bose National Centre for Basic Sciences
Block - JD, Sector - III, Salt Lake City, Kolkata - 700 106

Abstract

Considering Limit Cycles as one of the limits of Lienard equation, an analysis analogous to centre manifold analysis has been done for a 3 − D nonlinear system exhibiting Limit Cycle. A rigorous study on radius of the Limit Cycle orbit has been done by considering $\lambda - \omega$ equations for the particular system and subsequently converting the system equations from cartesian to polar form. It has been shown through an analysis analogous to Centre Manifold Analysis and reduction of the system dynamics on a lower dimensional space, the Limit cycle radius undergoes an increment change. One example is provided to support the theoretical predictions.

Keywords: Nonlinear Dynamics, Lienard Equation, Centre Manifold Analysis, Reduction of dimensions, Limit Cycles, Limit Cycle Manifold

1 Introduction

Limit cycles [25, 3, 20] are interesting features of stability that often arise in Nonlinear Dynamical Systems [23, 20]. For 2 − D nonlinear dynamical systems it is often observed in the phase portraits the systems move in closed orbits around a fixed point. Thus Limit Cycles give valuable information about the stability and the asymptotic behaviour of a system. Limit cycles can be of both types: Attracting and Repelling depending on the behaviour of the system for $t$ approaching positive or negative $\infty$ respectively. Numerous examples of nonlinear systems with Limit Cycle behaviour are abound in nature like the Nonlinear Van Der Pol Oscillators [6, 22, 1]. So studying systems which have Limit Cycles are subjects of general interest in Nonlinear Dynamics. In this paper we have made an attempt to do so.

The purpose of the paper is a rigorous study on the radius of the Limit Cycles, the overall asymptotic behaviour of the system when with the standard two degrees of freedom we couple a third degree of freedom and following an analysis similar to Centre manifold Analysis [24, 8, 12], a reduction of the dynamics on a reduced space is obtained. Then following standard approach of Limit cycle analysis we calculate the effect on radius on the Limit Cycle on reduced space due the additional coupled direction.

With the beginning of discussions Limit Cycles are shown to be one of the two limits of Lienard equation [23, 4, 2], the other limit being a Centre. When the middle term of the Lienard equation is equated to 0 evaluated at the fixed point $(0,0)$ Limit Cycle comes out as one of the limits when the evaluated value is less than 0 and the Centre comes out as another limit when the evaluated value is 0. This fine difference between a Limit cycle and a Centre in terms of their mathematical definitions as per the value of the middle term at the fixed point $(0,0)$ is brought out nicely through our work. Adding a third degree of freedom to the two dimensional Lienard equation and doing an analysis similar to the centre manifold analysis [24] we show mathematically the two criterion for obtaining a Limit cycle and Centre and thereby establish the true motivation behind our work.

The second motivation behind our work comes from analysing the effect on the Limit Cycle radius on addition of a coupled third direction. It is known that Limit Cycle radius undergoes change on coupling with an oscillator which results in change in the Limit Cycle area. Further the oscillations can be brought down to zero by changing the coupling strengths where the coupling parameters can be described as controls. We in our
work show that by addition of a stable third direction the Limit Cycle radius undergoes an increment change where the eigenvalue of the stable direction acts as the control parameter. The equations established points towards the fact that by changing the eigenvalue of the stable direction we can change the Limit Cycle radius by a desired amount.

Although the main focus and motivation of our work involve finding theoretical aspects of generating and controlling the limit cycle oscillation by carrying out a new analytical method, it is capable of satisfying different requirements of various well-known and promising technological applications due to its general applicability.

This study can be implemented in designing a controller for manipulating the amplitude of the limit cycle oscillation in various mechanical systems. The requirement of desired periodic motions in the manufacturing processes can be met at low energy dissipation by adding control to the marginally stable limit cycle of a conservative system. The nonlinear behaviors of uncontrolled aeroelastic systems are well established. Our finding in this report can be crucial in developing a reliable control technique of limit cycle oscillations in such aeroelastic systems. For example, our methodology can find direct application in a two-dimensional aeroelastic system with nonlinear stiffness and revealing the role of stiffness on limit cycle oscillation suppression of aeroelastic system by estimating the amplitude of oscillation more accurately under different conditions.

The amplitude of limit cycle oscillations in nonlinear aeroelastic systems can be minimized by exploiting this method. The amplitude suppression of limit cycle oscillation can be achieved by a suitable choice of control gains and a time delay state feedback. Here we have demonstrated the possibility of having such phenomena in generic nonlinear systems by adding a coupled direction. Effective controllers of the amplitude of the limit cycle in a general feedback scheme can be realized through our method of considering the third degree of freedom coupled with the other two. Limit-cycle control and fuel control are popular for their usage in the automotive applications and keeping the limit cycle oscillations' upper and lower amplitude at a recommended value is at the heart of these controllers. Therefore we believe that our study of controlling limit cycle oscillation can help in improving the performance of these controllers.

Besides these applications, many important biological processes like glycolysis, circadian rhythm follow limit cycle oscillation. Therefore studying peculiarities in limit cycle oscillation would help to get more insights into those biological oscillations and understand the effect of different external and internal parameters on those processes. The general nature of our methodology makes it worthwhile enough to study chemical oscillations from a different perspective. In fact, the amplitude of the limit cycle plays a guiding role in the concentration dynamics of the system and thus dictating the dissipation and energetics of the system. The general coupling term described in the report can be regarded as different coupling and controls that commonly exist in a model chemical system. Thus many amplitude-mediated emerging and counterintuitive phenomena of chemical oscillatory systems in the presence of different coupling can be understood at a fundamental level by implementing our approach.

The paper is organised as follows. In the next section we bring up the Lienard equation and write it in a two dimensional form. Then adding a third direction we do an analysis similar to centre manifold analysis and show the two criterion mathematically for obtaining the Limit Cycle and a Centre. Then in subsequent sections we take up our analysis for a general 3 – D system and on reduction on a two dimensional space was found to possess a Limit Cycle. Then bringing in the λ – ω form for the reduced 2 – D system and converting the equations from cartesian to polar form we perform a deep investigation on the peculiarities of the radius of the Limit Cycle. As Limit Cycle and Centre are two limits of the Lienard Equation our motivation prompts us to do an analysis similar to centre manifold analysis which we term as ”Limit Cycle Manifold Analysis”. It is shown that the radius of the Limit Cycle orbit changes with an increment if the third added coupled direction is considered to be a stable direction. One example is provided to support the Theoretical predictions. Finally we conclude by adding some remarks.

2 Lienard Equation and its Limits

The starting point of our discussion is the Lienard Equation. The Lienard Equation is a general equation which covers all the second order equations which have nonlinear damping terms.

The standard form of Lienard equation is given by the following,

\[ \ddot{x} + F(x, \dot{x})\dot{x} + G(x) = 0 \]  

(1)

There are some prescribed conditions for a Lienard System to exhibit Limit Cycle. These are,

- \( xG(x) > 0 \) for \( |x| > 0 \)
- \( \int_{0}^{\infty} G(x)dx = \int_{0}^{-\infty} G(x)dx = \infty \)
- \( F(0,0) < 0 \)
Among these we take the third condition that is
Most of the second order nonlinear systems have $G(x) = x$. So for simplicity we have the value $x$ for $G(x)$. Writing as two first order systems Lienard Equation can be written as
\[ \dot{x} = y \\
\dot{y} = -F(x, y) - x \] \hspace{1cm} (2)

Now in the nonlinear term $F(x, y)$ we segregate the nonlinear part containing only $x, y$ and the constant part. We write $F(x, y) \to F(x, y) - k, (k > 0)$. Then the Lienard Equation takes the form,
\[ \dot{x} = y \\
\dot{y} = -[F(x, y) - k]y - x \] \hspace{1cm} (3)

On fixed point $(0, 0)$ consideration $F(0, 0) = 0$. Further if $k = 0$ we get a Centre and if $k > 0$ we get a Limit Cycle.

### 2.1 Limit Cycle Manifold

Now we add a third direction coupled with the other two. The third direction is considered to be a stable direction. Then the system equations take the form,
\[ \dot{x} = y \\
\dot{y} = k(y - F(x, y)y - x) \\
\dot{z} = -\lambda z + f_3(x, y, z) \] \hspace{1cm} (4)

where $-\lambda(\lambda > 0)$ is the eigenvalue of the stable direction and $f_3(x, y, z)$ is the nonlinear term for the third direction. Now we add nonlinear couplings for the first and second direction and write the equations in matrix form as follows.
\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
-1 & k & 0 \\
0 & 0 & -\lambda
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} +
\begin{bmatrix}
f_1(x, y, z) \\
f_2(x, y, z) - F(x, y) \\
f_3(x, y, z)
\end{bmatrix}
\] \hspace{1cm} (5)

where $f_1$ and $f_2$ are the nonlinear couplings for the first and second direction respectively. Note that the above system is in the block diagonal form for the linear part. Following an analysis similar to Centre Manifold analysis we consider $z$ as a polynomial of $x, y$. We write,
\[ z = h(x, y) = a_0x^2 + a_1xy + a_2y^2 + O(3) \] \hspace{1cm} (6)

Differentiating with respect to $t$ we have the above equation as,
\[ \dot{z} - \frac{\partial h}{\partial x} \dot{x} - \frac{\partial h}{\partial y} \dot{y} = 0 \] \hspace{1cm} (7)

Substituting the values of $\dot{x}, \dot{y}, \dot{z}$ from the matrix equations we have,
\[ -\lambda z + f_3(x, y, z) - (2a_0x + a_1y)(y + f_1(x, y, z)) - (2a_2y + a_1x)(-x + ky - F(x, y) + f_2(x, y, z)) = 0 \] \hspace{1cm} (8)

Now for fixed point $(0, 0)$ consideration we put $F(x, y) = 0$ as it contains terms only in $x, y$. Then the above equation takes the form,
\[ -\lambda z + f_3(x, y, z) - (2a_0x + a_1y)(y + f_1(x, y, z)) - (2a_2y + a_1x)(-x + ky + f_2(x, y, z)) = 0 \] \hspace{1cm} (9)

Now we consider polynomial forms for $f_1, f_2, f_3$. We consider,
\[
\begin{align*}
f_1(x, y, z) &= c_0x^2 + c_1y^2 + c_2z^2 + c_3xy + c_4yz + c_5xz \\
f_2(x, y, z) &= d_0x^2 + d_1y^2 + d_2z^2 + d_3xy + d_4yz + d_5xz \\
f_3(x, y, z) &= e_0x^2 + e_1y^2 + e_2z^2 + e_3xy + e_4yz + e_5xz
\end{align*}
\] \hspace{1cm} (10)
Substituting the values of \( f_1, f_2, f_3 \) in equation (9) we have,

\[
- \lambda z + (e_0 x^2 + e_1 y^2 + e_2 z^2 + e_3 x y + e_4 y z + e_5 z x)
- (2a_0 x + a_1 y)(y + c_0 x^2 + c_1 y^2 + c_2 z^2 + c_3 x y + c_4 y z + c_5 z x)
- (2a_2 y + a_1 x)(-x + ky + d_0 x^2 + d_1 y^2 + d_2 z^2 + d_3 x y + d_4 y z + d_5 z x)
\]

(11)

Substituting the value of \( z \) from (6) we have,

\[
- \lambda (a_0 x^2 + a_1 x y + a_2 y^2) + (e_0 x^2 + e_1 y^2 + e_2(a_0 x^2 + a_1 x y + a_2 y^2) + e_3 x y + e_4 y(a_0 x^2 + a_1 x y + a_2 y^2)
+ e_5 x(a_0 x^2 + a_1 x y + a_2 y^2)) - (2a_0 x + a_1 y)(y + c_0 x^2 + c_1 y^2 + c_2(a_0 x^2 + a_1 x y + a_2 y^2) + c_3 x y
+ e_4 y(a_0 x^2 + a_1 x y + a_2 y^2) + e_5 x(a_0 x^2 + a_1 x y + a_2 y^2)) - (2a_2 y + a_1 x)(-x + ky + d_0 x^2 + d_1 y^2
+ d_2(a_0 x^2 + a_1 x y + a_2 y^2) + d_3 x y + d_4 y(a_0 x^2 + a_1 x y + a_2 y^2) + d_5 x(a_0 x^2 + a_1 x y + a_2 y^2)) = 0
\]

(12)

Equating the coefficients of \( x^2, y^2, x y \) to 0 we from the above equation we have,

\[
- \lambda a_0 + e_0 + a_1 = 0
- \lambda a_2 + e_1 - a_1 - 2a_2 k = 0
- \lambda a_1 + e_3 - 2a_0 + 2a_2 - a_1 k = 0
\]

(13)

Solving for \( a_0, a_1, a_2 \) we have,

\[
a_0 = \frac{\lambda e_0 + e_3 + \frac{2(e_0 + e_1)}{\lambda + 2k} + e_4 k}{\lambda^2 + 2 + \frac{2\lambda}{\lambda + 2k} + \lambda k}
\]

\[
a_1 = \frac{\lambda e_3 + \frac{2\lambda e_1}{\lambda + 2k} - 2e_0}{\lambda^2 + 2 + \frac{2\lambda}{\lambda + 2k} + \lambda k}
\]

\[
a_2 = \frac{\lambda^2 e_1 + 2(e_0 + e_1) + \lambda k e_1 - \lambda e_3}{(\lambda + 2k)(\lambda^2 + 2 + \frac{2\lambda}{\lambda + 2k} + \lambda k)}
\]

(14)

Choosing \( e_0 = e_1 = 0, e_3 = 1 \) we have,

\[
a_0 = \frac{1}{\lambda^2 + 2 + \frac{2\lambda}{\lambda + 2k} + \lambda k}
\]

\[
a_1 = \frac{\lambda}{\lambda^2 + 2 + \frac{2\lambda}{\lambda + 2k} + \lambda k}
\]

\[
a_2 = \frac{-\lambda}{(\lambda + 2k)(\lambda^2 + 2 + \frac{2\lambda}{\lambda + 2k} + \lambda k)}
\]

(15)

From equation (6) we have,

\[
z = a_0 x^2 + a_1 x y + a_2 y^2
= \frac{x^2}{\lambda^2 + 2 + \frac{2\lambda}{\lambda + 2k} + \lambda k} + \frac{\lambda x y}{\lambda^2 + 2 + \frac{2\lambda}{\lambda + 2k} + \lambda k} - \frac{\lambda y^2}{(\lambda + 2k)(\lambda^2 + 2 + \frac{2\lambda}{\lambda + 2k} + \lambda k)}
\]

(16)

Puting \( k = 0 \) in the above equation we get the Equation of Centre Manifold. Therefore the Centre Manifold Equation is,

\[
z = \frac{x^2}{\lambda^2 + 4} + \frac{\lambda x y}{\lambda^2 + 4} - \frac{\lambda y^2}{\lambda(\lambda^2 + 4)}
\]

(17)

Following the same spirit as in the case of Centre Manifold if we put \( k > 0 \) in equation in (16) we get the Equation which we term as ”Limit Cycle Manifold Equation”. This forms the basis of our motivation to carry out Limit Cycle Analysis along the line of Centre Manifold Analysis.

### 3 Limit Cycle Manifold Analysis

As the purpose of our paper is to do the Limit Cycle Analysis of Dissipative systems we consider general systems which exhibit Limit Cycles which takes into account Limit Cycle behaviour of the Dissipative systems which
Substituting the values of $\dot{\gamma}$ from equations [10, 11].

General systems which exhibit Limit Cycle behaviour can be cast into the following form:

$$\dot{x} = \lambda(r)x - w(r)y$$

$$\dot{y} = w(r)x + \lambda(r)y$$

(18)

where $\lambda(r), w(r)$ are functions of $r$ satisfying the conditions $\lambda(r) > 0$ when $r < r_0$ and $\lambda(r) < 0$ when $r > r_0$, $r_0$ being the Limit Cycle radius and $w(r) > 0$. It is usually chosen as,

$$\lambda(r) = \gamma - r^2$$

$$w(r) = 1$$

(19)

Let us try to find out the Limit cycle radius when the system is moving on the two dimensional plane. We have,

$$\dot{x} = (\gamma - r^2)x - y$$

$$\dot{y} = x + (\gamma - r^2)y$$

(20)

Therefore,

$$x\dot{x} = (\gamma - r^2)x^2 - xy$$

$$y\dot{y} = xy + (\gamma - r^2)y^2$$

(21)

Adding the above two equations,

$$r\dot{r} = (\gamma - r^2)r^2 \implies \dot{r} = (\gamma - r^2)r$$

(22)

Now the Limit Cycle condition is given by $\dot{r} = 0$ which gives $r = \sqrt{\gamma}$. This is the Limit Cycle radius which we denote by $R_0$.

Now adding a third direction which we considered as a stable direction as shown in the preceding section for Limit Cycle Manifold we have the system equations as,

$$\dot{x} = (\gamma - (x^2 + y^2))x - y + f_1(x, y, z)$$

$$\dot{y} = x + (\gamma - (x^2 + y^2))y + f_2(x, y, z)$$

$$\dot{z} = -\lambda z + f_3(x, y, z)$$

(23)

where $f_1, f_2, f_3$, are the nonlinear parts of the above equations, $-\lambda(\lambda > 0)$ is the eigenvalue of the stable direction and $r^2 = x^2 + y^2$. Writing in matrix form the above equations take the form,

$$\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{pmatrix} =
\begin{bmatrix}
\gamma & -1 & 0 \\
1 & \gamma & 0 \\
0 & 0 & -\lambda
\end{bmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} +
\begin{bmatrix}
f_1(x, y, z) - x(x^2 + y^2) \\
f_2(x, y, z) - y(x^2 + y^2) \\
f_3(x, y, z)
\end{bmatrix}$$

(24)

Again linear part of the above system is in block diagonal form. The Linear part of the Limit Cycle equations(first two equations) is decoupled from the stable direction(third equation). Considering a "Limit Cycle Manifold" we write $z$ as a polynomial function of $x, y$.

$$z = h(x, y) = a_0x^2 + a_1xy + a_2y^2 + O(3)$$

(25)

Differentiating the above equation with respect to $t$ we have,

$$\ddot{z} - \frac{\partial h}{\partial x} \dot{x} - \frac{\partial h}{\partial y} \dot{y} = 0$$

(26)

Substituting the values of $\dot{x}, \dot{y}, \dot{z}$ in the above equation we get,

$$-\lambda z + f_3(x, y, z) - (2a_0x + a_1y)[(\gamma - (x^2 + y^2))x - y + f_1(x, y, z)]$$

$$-(a_1x + 2a_2y)[x + (\gamma - (x^2 + y^2))y + f_2(x, y, z)] = 0$$

(27)

Substituting the value of $z$ from equation (25) we have,

$$-\lambda(a_0x^2 + a_1xy + a_2y^2) + f_3(x, y, z) - (2a_0x + a_1y)[(\gamma - (x^2 + y^2))x - y + f_1(x, y, z)]$$

$$-(a_1x + 2a_2y)[x + (\gamma - (x^2 + y^2))y + f_2(x, y, z)] = 0$$

(28)
Now we assume polynomial forms for \( f_1, f_2, f_3 \). We write,
\[
\begin{align*}
  f_1(x, y, z) &= c_0 x^2 + c_1 y^2 + c_2 z^2 = c_3 x y + c_4 y z + c_5 x z \\
  f_2(x, y, z) &= d_0 x^2 + d_1 y + d_2 z^2 + d_3 x y + d_4 y z + d_5 x z \\
  f_3(x, y, z) &= e_0 x^2 + e_1 y^2 + e_2 z^2 + e_3 x y + e_4 y z + e_5 x z
\end{align*}
\]
(29)

Substituting the values of \( f_1, f_2, f_3 \) from equation (28) we have the above equation as,
\[
\begin{align*}
  -\lambda(a_0 x^2 + a_1 x y + a_2 y^2) + e_0 x^2 + e_1 y^2 + e_2(a_0 x^2 + a_1 x y + a_2 y^2)^2 + e_3 x y + e_4 y(a_0 x^2 + a_1 x y + a_2 y^2) + e_5 a_0^2 x^2 + e_5 a_1 x y + e_5 a_2 y^2 \\
  + e_5 x(a_0 x^2 + a_1 x y + a_2 y^2) - (2a_0 + a_1)(\gamma - (x^2 + y^2)) - y + c_0 x^2 + c_1 y^2 + c_2(a_0 x^2 + a_1 x y + a_2 y^2)^2 \\
  + c_3 x y + c_4 y(a_0 x^2 + a_1 x y + a_2 y^2) + c_5 x(a_0 x^2 + a_1 x y + a_2 y^2)^2 - (a_1 x + 2a_2 y)(x + (\gamma - (x^2 + y^2))) y + d_0 x^2 \\
  + d_1 y^2 + d_1 a_0 x^2 + a_1 x y + a_2 y^2 + d_3 x y + d_4 y(a_0 x^2 + a_1 x y + a_2 y^2) + d_5 x(a_0 x^2 + a_1 x y + a_2 y^2)^2 = 0
\end{align*}
\]
(30)

Equating the coefficients of the \( x^2, y^2, x y \) to 0 from the above equation we have,
\[
\begin{align*}
  -\lambda a_0 + e_0 - 2a_0 \gamma - a_1 &= 0 \\
  -\lambda a_2 + e_1 + a_1 - 2a_2 \gamma &= 0 \\
  -\lambda a_1 + e_3 + 2a_0 - 2a_1 \gamma - 2a_2 &= 0
\end{align*}
\]
(31)

Solving for \( a_0, a_1, a_2 \) we have,
\[
\begin{align*}
  a_0 &= \frac{e_0(\lambda + 2\gamma)}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} \\
  a_1 &= \frac{e_0(\lambda + 2\gamma)^2 + 2(e_0 + e_1) - e_3(\lambda + 2\gamma)}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} \\
  a_2 &= \frac{e_0 + e_1}{\lambda + 2\gamma} \frac{e_0(\lambda + 2\gamma)^2 + 2(e_0 + e_1) - e_3(\lambda + 2\gamma)}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4}
\end{align*}
\]
(32)

Choosing \( e_0 = e_1 = 0, e_3 = 1 \) we have,
\[
\begin{align*}
  a_0 &= -\frac{1}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} \\
  a_1 &= \frac{\lambda}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} \\
  a_2 &= \frac{1}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4}
\end{align*}
\]
(33)

Then the "Limit Cycle Manifold" Equation is given by,
\[
\begin{align*}
  z &= \frac{x^2}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} + \frac{(\lambda + 2\gamma)x y}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} + \frac{y^2}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4}
\end{align*}
\]
(34)

The reduced equations on the "Limit Cycle Manifold" is given by
\[
\begin{align*}
  \dot{x} &= (\gamma - (x^2 + y^2))x - y + y[\frac{x^2}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} + \frac{(\lambda + 2\gamma)x y}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4}] \\
  \dot{y} &= x + (\gamma - (x^2 + y^2))y + x[\frac{x^2}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} + \frac{(\lambda + 2\gamma)x y}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4}]
\end{align*}
\]
(35)

on substituting the value \( z \) from equation (31).

Writing \( x = R \cos \theta, y = R \sin \theta \) we have \( \dot{x} = \dot{R} \cos \theta - R \sin \theta \dot{\theta}, \dot{y} = \dot{R} \sin \theta + R \cos \theta \dot{\theta} \). Putting the values of \( \dot{x} \) and \( \dot{y} \) in the above two equations we have,
\[
\begin{align*}
  \dot{R} \cos \theta - R \sin \theta \dot{\theta} &= (\gamma - R^2)R \cos \theta - R \sin \theta \sin \theta \\
  + R \sin \theta \left[ -\frac{R^2 \cos^2 \theta}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} + \frac{(\lambda + 2\gamma)R^2 \cos \theta \sin \theta}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} \right. \\
  + R \cos \theta \left. \left[ -\frac{R^2 \cos^2 \theta}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} + \frac{(\lambda + 2\gamma)R^2 \sin \theta \cos \theta}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} \right] \right]
\end{align*}
\]
(36)
Solving for \( \dot{R} \) and \( \dot{\theta} \) from the above equations we have

\[
\dot{R} = (\gamma - R^2)R + R \sin 2\theta \left[ -\frac{R^2 \cos 2\theta + (\lambda + 2\gamma) R^2 \sin 2\theta}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} \right]
\]

\[= (\gamma - R^2)R + R^3 \left[ -\frac{\sin 4\theta - (\lambda + 2\gamma) \cos 4\theta + \lambda + 2\gamma}{2[\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4]} \right] \quad \text{(37)}
\]

\[
\dot{\theta} = 1 + R^2 \cos 2\theta \left[ -\frac{\cos^2 \theta + \sin \theta \cos(\lambda + 2\gamma) + \sin^2 \theta}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} \right]
\]

\[= 1 + R^2 \cos 2\theta \left[ -\frac{\cos 2\theta + \sin 2\theta(\lambda + 2\gamma)}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} \right] \quad \text{(38)}
\]

We plot \( R - t \) as shown in the Figure 1. The \( R - t \) plot shows an oscillatory motion about a mean radius which

![Figure 1: Limit Cycle Radius](image)

happens to be the new Limit cycle radius \( R' \). But the velocity is maximum on the mean radius\( (R'_0) \). Thus we can write,

\[
\dot{R}_{max} = \gamma R'_0 - R'_0^3 + R'_0^3 \left[ -\frac{\sin 4\theta - (\lambda + 2\gamma) \cos 4\theta + \lambda + 2\gamma}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} \right] \quad \text{(39)}
\]

Now we make an observation. We find that if we put \( \theta + \frac{\pi}{4} \) in place of \( \theta \), the motion is repeated. Therefore when \( \theta \) equals \( \theta + \frac{\pi}{4} \) the velocity should be maximum but in the opposite direction which is true for any oscillatory motion. So we can write,

\[
-\dot{R}_{max} = \gamma R'_0 - R'_0^3 + R'_0^3 \left[ -\frac{\sin 4(\theta + \frac{\pi}{4}) - (\lambda + 2\gamma) \cos 4(\theta + \frac{\pi}{4}) + \lambda + 2\gamma}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} \right] \quad \text{(40)}
\]
The system moves in a mean circular path with phase increasing with time with a mean angular velocity

\[ \omega = \frac{\lambda + 2\gamma}{4(\lambda + 2\gamma) + 2(\lambda + 2\gamma) + 4} \]  \hspace{1cm} (41)

which gives the mean radius or the Limit Cycle radius on addition of a stable third direction. Without the third direction Limit Cycle radius is given by \[ R_0 = \sqrt{\pi}. \] Thus we find that the Limit Cycle radius has undergone an increment change on addition of the stable direction. As a further check we see from equation (34) when the \( z \) direction is absent that is \( z = 0 \) which implies \( \lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4 = \infty \) we have \( R_0' = R_0. \) The increment of orbital radius on addition of a stable third direction as calculated above is a highly nontrivial result which our study reveals.

### 3.1 Asymptotic Amplitude and Angular Velocity

#### Asymptotic Amplitude:

As the system settles in a limit cycle of mean radius \( R_0' = \sqrt{\frac{3\pi}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4}} \) the Asymptotic Amplitude is given by \( R_0' \).

#### Asymptotic Angular Velocity:

The system moves in a mean circular path with phase increasing with time with a mean angular velocity which is the Asymptotic Angular Velocity. From equation (18) we have,

\[ \dot{\theta} = 1 + R^2 \cos 2\theta \left[ -\cos 2\theta + \frac{\sin 2\theta}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} \right] \]

Now the above equation is periodic with respect to \( \theta \) with period \( 2\pi \). To get the mean angular velocity we have to take average on the L.H.S. and R.H.S. of the above equation (19). Taking average on both sides of the above equation we have,

\[ \bar{\dot{\theta}} = \frac{1}{2\pi} \left[ \int_0^{2\pi} \dot{\theta} d\theta - \frac{\dot{\theta}}{\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4} \right] \]

\[ = \frac{1}{2\pi} \left[ \int_0^{2\pi} \dot{\theta} d\theta - \frac{\dot{R}^2}{2(\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4)} \right] \]

\[ = 1 - \frac{\dot{R}^2}{2(\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4)} \]

Now \( \dot{R}^2 = R_0'. \) Puting the value of \( R_0'^2 \) from equation (42) in place of \( \dot{R}^2 \) we have,

\[ \bar{\dot{\theta}} = 1 - \frac{1}{2(\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4)} \]

\[ = 1 - \frac{2\gamma}{4(\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4) - (\lambda + 2\gamma)} \]

**Time period for 1 complete rotation around the Limit Cycle:**

The time period for 1 complete rotation around the Limit Cycle is given by,

\[ T = \frac{2\pi}{\bar{\dot{\theta}}} \]

\[ = \frac{2\pi}{1 - \frac{2\gamma}{4(\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4) - (\lambda + 2\gamma)}} \]

\[ = \frac{2\pi}{1 - \frac{2\gamma}{4\lambda(\lambda + 2\gamma) + 2\gamma(\lambda + 2\gamma) + 4 - (\lambda + 2\gamma)}} \]
3.2 Total number of oscillations in one full cycle

We have shown velocity reverses to its maximum at phase separation of angle $\frac{\pi}{4}$. Therefore velocities are in the same phase when the phase separation is $\frac{\pi}{2}$. At this phase separation velocities are maximum in the same direction while crossing the mean radius. So we can say one oscillation is equivalent to phase difference $\frac{\pi}{2}$.

Therefore the number of oscillations in 1 complete cycle is given by,

$$\text{Number Of Oscillations in a full cycle} = \frac{2\pi}{\frac{\pi}{2}} = 4$$  \hspace{1cm} (46)

3.3 Examples

We show two examples of Nonlinear System occurring in nature which exhibit Limit Cycles. By the above analysis they will undergo a increase in Limit cycle radius on addition of a stable third direction.

We cite the example of Van der Pol system which frequently occurs in nature. The 2 − $D$ model of this system is the following.

Van der Pol:

\begin{align*}
\dot{x} & = y \\
\dot{y} & = -\mu(x^2 - 1)y - x
\end{align*}

(47)

The O.D.E s in case of Van der Pol is given in polar form as,

\begin{align*}
\dot{r} \cos \theta - r \sin \theta \dot{\theta} & = r \sin \theta \\
\dot{r} \sin \theta + r \cos \theta \dot{\theta} & = -\mu(r^2 \cos^2 \theta - 1)r \sin \theta - r \cos \theta
\end{align*}

(48)

Solving for $\dot{r}$ we get,

$$\dot{r} = -\mu r^3 \cos^2 \theta \sin^2 \theta + \mu r \sin^2 \theta$$

(49)

Now the above equation is periodic with respect to $\theta$ with period $2\pi$. To get the mean radius we average over the R.H.S and L.H.S of the above equation. Taking the average on both sides of equation (49) we get,

$$\dot{\bar{r}} = \frac{1}{2\pi} \left[ \frac{-\mu r^3}{8} \int_{0}^{2\pi} (1 - \cos 4\theta) d\theta + \mu \bar{r} \int_{0}^{2\pi} (1 - \cos 2\theta) d\theta \right]$$

(50)

Therefore $\dot{\bar{r}} = 0$ when $\bar{r} = 2$. Therefore the mean radius of the Limit Cycle radius is given by $R_0 = 2$. Therefore $\gamma = 4$. On addition of a stable third direction the Limit Cycle radius is given by

$$R_0' = \sqrt{\frac{4}{1 - \frac{8}{4\lambda + 8 + 8(\lambda + 8) + 4}}}$$

(51)

Asymptotic Amplitude and Angular Velocity:

The Asymptotic Amplitude is given by the above $R_0'$. The Asymptotic Angular Velocity is given by

$$\dot{\theta} = 1 - \frac{8}{4\lambda + 8 + 8(\lambda + 8) + 4(\lambda + 8)}$$

(52)

We plot the $R - t$ plot for the Van der Pol systems as shown in Figure 2.

The figure shows the $R - t$ plots for Van der Pol system for different values of the parameter $\gamma$. The parameter values are chosen as $\gamma = 4, 9, 16, 25$. As $R_0 = \sqrt{\gamma}$ the mean radius($R_0$) = 2, 3, 4, 5. On addition of a stable third direction, the mean radius($R_0'$) comes nearly equal to $R_0$ with difference in the second place of decimals. So the oscillations as shown in the plots is close to $R_0$.

To see the dependence of modified radius with respect to parameter $\lambda$ we plot the modified mean radius $R_0'$ with respect to $\lambda$ as shown in figure 3. Here the value of the parameter $\gamma$ is kept fixed at $\gamma = 4$.
4 Conclusion

We try to conclude the article with a short conclusion adding few points and remarks on our work. The first thing we like to point out is we have identified the Limit Cycle as one of the limits of Lienard Equation, the other limit being a Centre. This initiated the motivation behind carrying out an analysis following the centre manifold analysis which we have appropriately termed as "Limit Cycle Manifold" analysis. We have added a third direction with our original two degrees of freedom and following an analysis analogous to centre manifold analysis reduced the system on a lower dimensional space. Through the initial discussion we successfully established the motivation behind our work. As our second motivation was to study the peculiarities of Limit Cycle radius on addition of a coupled direction, we brought in the $\lambda - \omega$ system which is an inherent form for writing the differential equations for $2-D$ nonlinear systems exhibiting Limit Cycle. We did a detailed analysis on radius of the orbit of a Limit Cycle following the standard limit cycle analysis by converting the equations from cartesian to polar form. As we in our equation have added a third degree of freedom coupled with the other two we examined rigorously its effect on the Limit Cycle radius. What we have found that on addition of a stable direction the Limit Cycle radius changes with an increment which in our view is a completely new and interesting result that we have obtained theoretically. To tally with our theoretical prediction we have included three graphs which agree with the prediction.

References

[1] Barbosa, R. S., Machado, J. T., Vinagre, B. M., and Calderon, A. J. (2007). Analysis of the van der pol oscillator containing derivatives of fractional order. *Journal of Vibration and Control*, 13(9-10):1291–1301.

[2] Burton, T. (1965). The generalized lienard equation. *Journal of the Society for Industrial and Applied Mathematics, Series A: Control*, 3(2):223–230.

[3] Christopher, C. and Li, C. (2007). *Limit cycles of differential equations*. Springer Science & Business Media.

[4] Feng, Z. (2002). On explicit exact solutions for the lienard equation and its applications. *Physics Letters A*, 293(1-2):50–56.

[5] Graef, J. R. (1972). On the generalized liénard equation with negative damping. *Journal of Differential Equations*, 12(1):34–62.

[6] Kanamaru, T. (2007). Van der pol oscillator. *Scholarpedia*, 2(1):2202.

[7] Leloup, J.-C., Gonze, D., and Goldbeter, A. (1999). Limit cycle models for circadian rhythms based on transcriptional regulation in drosophila and neurospora. *Journal of Biological Rhythms*, 14(6):433–448.

[8] Liu, L., Wong, Y., and Lee, B. (2000). Application of the centre manifold theory in non-linear aeroelasticity. *Journal of Sound and Vibration*, 234(4):641–659.

[9] Maccari, A. (2003). Vibration control for the primary resonance of the van der pol oscillator by a time delay state feedback. *International Journal of Non-Linear Mechanics*, 38(1):123–131.

[10] Murray, J. (2001). *Mathematical biology II: spatial models and biomedical applications*, volume 3. Springer-Verlag.

[11] Murray, J. D. (2007). *Mathematical biology: I. An introduction*, volume 17. Springer Science & Business Media.

[12] Namachchivaya, N. S. and Van Roessel, H. (2003). A centre-manifold analysis of variable speed machining. *Dynamical systems*, 18(3):245–270.

[13] NAYFEH, A. H., HARB, A. M., and CHIN, C.-M. (1996). Bifurcations in a power system model. *International Journal of Bifurcation and Chaos*, 06(03):497–512.

[14] Ribbens, W. B. (2013). Chapter 1 - the systems approach to control and instrumentation. In Ribbens, W. B., editor, *Understanding Automotive Electronics (Seventh Edition)*, pages 1–49. Butterworth-Heinemann, Oxford, seventh edition edition.

[15] Sanders, J. A., Verhulst, F., and Murdock, J. (2007). *Averaging methods in nonlinear dynamical systems*, volume 59. Springer.

[16] Schiehlen, W. and Guse, N. (2005). Control of limit cycle oscillations. In *IUTAM Symposium on Chaotic Dynamics and Control of Systems and Processes in Mechanics*, pages 429–439. Springer.
[17] SEL’KOV, E. E. (1968). Self-oscillations in glycolysis 1. a simple kinetic model. European Journal of Biochemistry, 4(1):79–86.

[18] Shukla, H. and Patil, M. J. (2017). Controlling limit cycle oscillation amplitudes in nonlinear aeroelastic systems. Journal of Aircraft, 54(5):1921–1932.

[19] Strganac, T. W., Ko, J., Thompson, D. E., and Kurdila, A. J. (2000). Identification and control of limit cycle oscillations in aeroelastic systems. Journal of Guidance, Control, and Dynamics, 23(6):1127–1133.

[20] Strogatz, S. H. (2018). Nonlinear dynamics and chaos with student solutions manual: With applications to physics, biology, chemistry, and engineering. CRC press.

[21] TANG, J. and CHEN, Z. (2006). Amplitude control of limit cycle in van der pol system. International Journal of Bifurcation and Chaos, 16(02):487–495.

[22] Tavazoei, M. S., Haeri, M., Attari, M., Bolouki, S., and Siami, M. (2009). More details on analysis of fractional-order van der pol oscillator. Journal of Vibration and Control, 15(6):803–819.

[23] Villari, G. (1987). On the qualitative behaviour of solutions of lienard equation. Journal of differential equations, 67(2):269–277.

[24] Wiggins, S. (2003). Introduction to applied nonlinear dynamical systems and chaos, volume 2. Springer Science & Business Media.

[25] Ye, Y.-Q. and Lo, C. Y. (1986). Theory of limit cycles, volume 66. American Mathematical Soc.
Figure 2: Limit Cycle Radius: Van der Pol
Figure 3: Changing Limit Cycle radius with respect to parameters: Van der Pol