A renormalization group method by harmonic extensions and the classical dipole gas

Hao Shen
Princeton University

November 12, 2013

Abstract

In this paper we develop a new renormalization group method, which is based on conditional expectations and harmonic extensions, to study functional integrals related with small perturbations of Gaussian fields. In this new method one integrates Gaussian fields inside domains at all scales conditioning on the fields outside these domains, and by variation principle solves local elliptic problems. It does not rely on an a priori decomposition of the Gaussian covariance. We apply this method to the model of classical dipole gas on the lattice, and show that the scaling limit of the generating function with smooth test functions is the generating function of the renormalized Gaussian free field.

1 Introduction

In this paper we develop a renormalization group (RG) method to estimate functional integrals, based on ideas of conditional expectations and harmonic extensions. We demonstrate this method with the model of classical dipole gas, which has always been considered as a simple model to start with for this type of problems. For the classical dipole model, earlier important works are [FP78, FS81c]. The renormalization group approach to this model originated from the works by Gawedzki and Kupiainen [GK80, GK83], based on Kadanoff spin blockings. A different method by Brydges, Yau, Slade and so on uses the idea of decomposition of the covariance of the Gaussian field, which was initiated from [BY90], and was simplified and pedagogically presented in the lecture notes [Bry09], see also [Dim09]. The latter method has achieved several important applications in other problems such as the two-dimensional Coulomb gas model [DH00, Fal12], \( \phi^4 \) field theories [BDH95, BDH98, BMS03] and self-avoiding walks [BIS09, BS10, BBST12].

Our method is different from the above two methods, and may be as well regarded as a variation of the method by Brydges et al. Their decomposition of covariance scheme, which was also used by other people such as [Gal85], could be implemented by Fourier analysis. In [BGM04], a decomposition of Gaussian covariance with every piece of covariance having finite range was constructed using elliptic partial differential equation techniques, which also depends to some extent on Fourier analysis, and this decomposition is the foundation of
the simplified version of their RG method (see also [BT06, Bau12, AKM13] for alternative constructions of such decompositions). We don’t perform such a decomposition of covariance. Instead we directly take harmonic extensions as our basic scheme and use the Poisson kernel to smooth the Gaussian field. We don’t need Fourier analysis; instead, real space decay rates of Poisson kernels and (derivatives of) Green’s functions are essential. Some complexities in [BGM04] such as proof of elliptic regularity theorem on lattice are avoided. Many elements of this method such as the polymer expansions and so on are very close to the method by Brydges et al, especially to [Bry09], while we also have some new features, such as simpler norms and regulators. We keep notations as close as possible to [Bry09] for convenience of the readers who are familiar with [Bry09].

Very roughly speaking, our method is aimed to study functional integrals of the form

\[ Z = \mathbb{E} \left[ e^{V(\phi)} \right] \]

where \( \phi \) is a Gaussian field and \( \mathbb{E} \) is an expectation with respect to a Gaussian measure. Similarly with [Bry09] we will rewrite the integrand into a local expansion over subsets \( X \) of an explicit part and an implicit remainder, for instance in the model considered in this paper

\[ Z \approx \mathbb{E} \left[ \sum_X e^{\sigma \sum_{x \in X} (\partial \phi(x))^2} K(X, \phi) \right] \]

where \( K(X, \phi) \) depends only on \( \{ \phi(x) : x \in \tilde{X} \} \) and \( \tilde{X} \) is a subset slightly enlarger than \( X \). We will take a family of conditional expectations at a sequence of scales parametrized by integer \( j \):

\[ Z \approx e^{\sigma j} \mathbb{E} \left[ \sum_Y e^{\sigma \sum_{x \in Y} (\partial \phi(x))^2} e^{[\partial \phi(x)]^2} \mathbb{E} \left[ K(Y, \phi) | Y^c \right] \right] \]

where \( \mathbb{E} [F(\phi)|X^c] \) for a function of the field \( F(\phi) \) means integrating all the variables \( \{ \phi(x) : x \in X \} \) with \( \{ \phi(x) : x \in X^c \} \) fixed, \( \sigma_j \) is the most important dynamical parameter (which corresponds to renormalization of the dielectric constant in the dipole model), \( B_x \) is a block containing \( x \). This idea of conditional expectation is close to Frohlich and Spencer’s work on Kosterlitz-Thouless transition [FS81a, FS81b] where the authors take inside an expectation conditional integrations, each over all variables \( \{ \phi(x) : x \in \Omega \} \) where \( \Omega \) is a bounded region around a charge density \( \rho \) with diameter \( \sim 2^j \). They didn’t take dynamical system viewpoint very explicitly.

Such conditional expectations can be carried out by minimizing the quadratic form in the Gaussian measure with conditioning variables fixed. Since the Gaussian is associated to a Laplacian these minimizers are harmonic extensions of \( \phi \) from \( X^c \) into \( X \). These harmonic extensions result in smoother dependence of the integrand of the expectation on the field. Some elliptic PDE methods along with random walk estimates will be used. We remark that this variational viewpoint also shows up in Balaban’s RG method (see for instance [Bal83] or Section 2.2 - 2.3 of [Dim11]). Hopefully our approach would to some extent help in understanding those works.

ACKNOWLEDGEMENT: I would like to thank my advisor Weinan E who has been supporting my work on renormalization group methods over years and giving me many good
suggestions. I am very grateful for the kind hospitality of David Brydges during my visits to University of British Columbia, as well as a lot of encouragement and helpful conversations by him. I also appreciate numerous discussions with Stefan Adams, Arnulf Jentzen, and especially Roland Bauerschmidt.

2 Outline of the paper

2.1 Settings, notations and conventions

Let $\mathbb{Z}^d$ be the $d$ dimensional lattice with $d \geq 2$. Denote the sets of lattice directions as $\mathcal{E}_+ = \{e_1, \ldots, e_d\}$ and $\mathcal{E}_- = \{-e_1, \ldots, -e_d\}$ where $e_k = (0, \cdots, 1, \cdots, 0)$ with only the $k$-th element being $1$. Let $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_-$. For $e \in \mathcal{E}$, $\partial_e f(x) = f(x + e) - f(x)$ is the lattice derivative. For $x, y \in \mathbb{Z}^d$, we say that $(x, y)$ is a nearest neighbor pair and write $x \sim y$ if there exists an $e \in \mathcal{E}$ such that $x = y + e$. Denote $E(\mathbb{Z}^d)$ to be the set of all nearest neighbor pairs of $\mathbb{Z}^d$. For $X \subset \mathbb{Z}^d$, we define $E(X) := \{(x, y) \in E(\mathbb{Z}^d) : x, y \in X\}$.

Let $L$ be a positive odd integer, and $N \in \mathbb{N}$. Let

$$\Lambda = [-L^N/2, L^N/2]^d \cap \mathbb{Z}^d$$

and we will consider functions on $\Lambda$ with periodic boundary condition. In other words we view $\Lambda$ as a torus by identifying the boundary points of $\Lambda$ in the usual way.

For $x, y \in \Lambda$, define $d(x, y)$ to be the length of the shortest path of nearest neighbor sites in the torus $\Lambda$ connecting $x$ and $y$. Also define $\partial X$ to be the “outer boundary”: $\partial X = \{x \in \mathbb{Z}^d : d(x, X) = 1\}$. Write $X^c$ to be the complement of $X$.

For a function $\phi$ on $\mathbb{Z}^d$, when it doesn’t cause confusions, we write for short

$$\sum_x (\partial \phi)^2 = \sum_{x \in \mathcal{X}} (\partial \phi(x))^2 := \frac{1}{2} \sum_x \sum_{e \in \mathcal{E}} (\partial_e \phi(x))^2$$

and similarly for other such type of summations. If $E$ is the expectation over $\phi$, we will use a short-hand notation for conditional expectation

$$\mathbb{E}[-|X] := \mathbb{E}[-|\{\phi(x)|x \in X\}]$$

namely, the expectation with $\phi|_X$ fixed.

Unless we specify otherwise, Poisson kernels and Green’s functions will be associated with operator $-\Delta + m^2$ where $m$ is a small mass regularization. For any set $X$, $P_X$ or $P_X(x, y)$ ($x \in X$, $y \in \partial X$) is the Poisson kernel for $X$. If $x \notin X$ then $P_X f(x) = f(x)$ is always understood. In other words, $P_X f$ is the harmonic extension of $f$ from $X^c$ into $X$ with $f|_{X^c}$ unchanged.

2.2 The dipole gas model and the scaling limit

Let $\mu$ be the Gaussian measure on the space of functions $\{\phi(x) : x \in \Lambda\}$ with mean zero and covariance $\mathcal{C}_m = (-\Delta + m^2)^{-1}$ where $m > 0$. In other words, $\phi$ is the Gaussian free field on
the $\Lambda$ with covariance $C_m$, and $\mathbb{E}$ be the expectation over $\phi$. Then the classical dipole gas model is defined by the following measure:

$$\nu(\phi) = e^{zW(\phi)} \mu(\phi)$$

where

$$W(\phi) := \sum_{x \in \Lambda} \sum_{e \in E} \cos \left( \sqrt{\beta} \partial_e \phi(x) \right)$$

Such a measure can be also obtained by a definition of the model via the great canonical ensemble followed by a Sine-Gordon transformation, for instance, see [BY90].

We would like to study the problem of scaling limit. More precisely, let $\tilde{\Lambda} := [-\frac{1}{2}, \frac{1}{2}]^d \subset \mathbb{R}^d$. Given a mean zero function $\tilde{f} \in C^\infty(\tilde{\Lambda})$, $\tilde{\int}_\Lambda f = 0$ with periodic boundary condition, we study the (real) generating function

$$Z_N(f) := \lim_{m \to 0} \mathbb{E} \left[ e^{\sum_{x \in \Lambda, e \in E} f(x) \phi(x) e^{zW(\phi)}} \right]$$

where

$$f(x) = f_N(x) := L^{-(d+2)/2} \tilde{f}(L^{-N}x)$$

The main question is the scaling limit of $Z_N(f)$ as $N \to \infty$.

### 2.3 Some preparative steps before RG

As the start of our strategy to study this problem, we perform an a priori tuning of the Gaussian measure, which we describe now. Define

$$V(\phi) := \frac{1}{4} \sum_{x \in \Lambda, e \in E} (\partial_e \phi(x))^2$$

The tuning is to split part of the quadratic form of the Gaussian measure into the integrand, so that the resulting Gaussian field has covariance $[\epsilon(-\Delta + m^2)]^{-1}$, with the associated expectation called $\mathbb{E}^\epsilon$:

$$Z_N(\epsilon) = \lim_{m \to 0} \frac{\mathbb{E}^\epsilon \left[ e^{\sum_{x \in \Lambda, e \in E} f(x) \phi(x) e^{(\epsilon - 1) V(\Lambda, \phi) + zW(\Lambda, \phi)}} \right]}{\mathbb{E}^\epsilon \left[ e^{(\epsilon - 1) V(\Lambda, \phi) + zW(\Lambda, \phi)}} \right]$$

Note that normalization factors caused by re-definition of Gaussian:

$$\mathbb{E}^\epsilon \left[ \exp ((\epsilon - 1) V(\Lambda, \phi)) \right]$$

appear in both numerator and denominator and are thus cancelled.

We would like to make the expectation (and thus the RG maps which we will define later) independent of $\epsilon$. So we rescale $\phi \to \phi / \sqrt{\epsilon}$ and let $\sigma = \epsilon^{-1} - 1$ and obtain

$$Z_N(\epsilon) = \lim_{m \to 0} \frac{\mathbb{E}^\epsilon \left[ e^{\sum_{x \in \Lambda, e \in E} f(x) \sqrt{\epsilon} \phi(x) / \sqrt{\epsilon} e^{-\sigma V(\phi) + zW(\sqrt{1 + \sigma} \phi)}} \right]}{\mathbb{E}^\epsilon \left[ e^{-\sigma V(\phi) + zW(\sqrt{1 + \sigma} \phi)}} \right]}$$

4
We also shift the Gaussian field to get rid of the linear term. Write \(-\Delta_m = -\Delta + m^2\) and make a translation \(\phi \to \phi + \xi\) where \(\xi = (-\sqrt{\epsilon}\Delta_m)^{-1}f\) in the numerator in (2.4). Then

\[
Z_N(f) = \lim_{m \to 0} e^{\frac{1}{2} \sum_{x \in X} f(x)(-\epsilon \Delta_m)^{-1}f(x)} Z'_N(\xi)/Z'_N(0)
\]  

(2.5)

where

\[
Z'_N(\xi) = \mathbb{E} \left[ e^{-\sigma V(\Lambda, \phi + \xi) + \frac{h}{\sqrt{\epsilon}} \xi} \right]
\]

(2.6)

Let \(-\tilde{\Delta}_m = -\tilde{\Delta} + m^2\), where \(\tilde{\Delta}\) is the Laplacian in continuum, \(\tilde{\mathcal{C}}_m := (-\tilde{\Delta}_m)^{-1}\) and \(\tilde{\xi} := (-\sqrt{\epsilon}\tilde{\Delta}_m)^{-1}\tilde{f}\). We can verify that \(L^{-2N}C_{L^{-N}}(L^N x) = \tilde{C}_m(x)\) and \(L^{-\frac{d+2}{2}N}\tilde{\xi}(L^N x) = \tilde{\xi}(x)\). Let \(q < \frac{d}{d-1}\) and

\[
R = \sup_{m>0} \max(||\tilde{C}_m||_{L^p}, ||\partial \tilde{C}_m||_{L^p}) < \infty
\]

We will assume that \(\|	ilde{f}\|_{L^p} \leq h/R\) (\(p > d\)), for a constant \(h\) to be specified later, so that for \(\alpha = 0, 1\)

\[
\||\partial^\alpha \tilde{\xi}||_{L^p} \leq hL^{-\left(\frac{d+2}{2} + \alpha\right)N}
\]

(2.7)

by Young’s inequality.

Before the RG steps, we write both \(Z'_N(\xi)\) and \(Z'_N(0)\) into a form of “polymer expansion”. For any set \(X \subseteq \Lambda\), write

\[
W(X, \phi) := \sum_{x \in X} \sum_{e \subseteq \phi} \cos \left( \sqrt{\beta} \partial_x \phi(x) \right)
\]

**Proposition 1.** With \(Z'_N(\xi)\) given by (2.6), we have

\[
Z'_N(\xi) = \mathbb{E} \left[ \sum_{X \subseteq \Lambda} I(\Lambda \setminus X, \phi + \xi)K(X, \phi + \xi) \right]
\]

(2.8)

where \(I(X) = \prod_{x \in X} I(\{x\})\) and

\[
I(\{x\}, \phi + \xi) = e^{-\frac{1}{2} \sigma \sum_{e \subseteq \phi} (\partial_x \phi(x) + \partial_x \xi(x))^2}
\]

(2.9)

\[
K(X, \phi) = \prod_{x \in X} e^{-\frac{1}{2} \sigma \sum_{e \subseteq \phi} (\partial_x \phi(x) + \partial_x \xi(x))^2} \left( e^{W(\{x\}, \phi + \xi)/\sqrt{\epsilon}) - 1 \right)
\]

(2.10)

The quantity \(Z'_N(0)\) has the same form of expansion with \(\xi = 0\). The proof is given in Appendix B.

For \(Z'_N(0)\) we perform (2.8)-(2.10) with \(\xi = 0\).

### 2.4 Outline of main ideas

Our renormalization group method is based on the idea of rewriting the expectation into an expression involving many conditional expectations. We will carry out a multiscale analysis; an RG map will be iterated from one scale to the next one, during which we will re-arrange the conditional expectations. A basic algebraic structure and analytical bound will be propagated to every scale. In order to describe these structures and bounds, we first give some definitions.
2.4.1 Basics of polymers

1. We call blocks of size $L^j$ j-blocks which are translations of $\{x \in \mathbb{Z}^d : |x| < \frac{1}{2}(L^j - 1)\}$ by vectors in $(L^j \mathbb{Z})^d$. In particular a 0-block is a single site in $\mathbb{Z}^d$. A j-polymer $X$ is a union of j-blocks. In particular the empty set is also a j-block. The number of lattice sites in $X \subset \mathbb{Z}^d$ is denoted by $|X|$. The number of j-blocks in a j-polymer $X$ is denoted by $|X|_j$.

2. $X \subset \mathbb{Z}^d$ is said to be connected if for any two points $x, y \in X$ there exists a path $(x_i : i = 0, \ldots, n)$ with $|x_{i+1} - x_i|_\infty = 1$ connecting $x$ and $y$. Here, $|x|_\infty$ is the maximum of all coordinates of $x$; note that for instance $\{(0, 0), (1, 1)\}$ is connected if $d = 2$. Connected sets are not empty. Two sets $X, Y$ are said to be strictly disjoint if there is no path from $x$ to $y$ when $x \in X$ and $y \in Y$; otherwise we say that they touch.

3. For a j-polymer $X$ we have the following notations. $\mathcal{B}_j(X)$ is the set of all j-blocks in $X$. $\mathcal{P}_j(X)$ is the set of all j-polymers in $X$. $\mathcal{P}_{j,c}(X)$ is the set of all connected j-polymers in $X$. We sometimes just write $\mathcal{B}_j, \mathcal{P}_j, \mathcal{P}_{j,c}$ and so on when $X = \Lambda$.

4. Let $X \in \mathcal{P}_j$. Define for $j \geq 1$

\[ \hat{X} := \bigcup \{ B \in \mathcal{B}_j : B \text{ touches } X \} \]

\[ X^+ = \bigcup \{ x \in \Lambda : d(x, X) \leq \frac{1}{3}L^j \} \]

\[ \hat{X} = \bigcup \{ x \in \Lambda : d(x, X) \leq \frac{1}{6}L^j \} \]

\[ X = \bigcup \{ x \in \Lambda : d(x, X) \leq \frac{1}{12}L^j \} \]

Note that we have $X \subset \hat{X} \subset \hat{X} \subset X^+ \subset \hat{X}$. Only $X, \hat{X}$ belong to $\mathcal{P}_j$.

5. When $j = 0$ and $X \in \mathcal{P}_0$, we define $\hat{X} = \hat{X} = X^+ = \hat{X} = X$, and the Poisson kernel at scale 0 is understood as $P_{X^+} := \text{id}$.

We also have the following notations for functions of the fields.

1. Define $\mathcal{N}$ to be the set of functions of $\phi$. Define $\mathcal{N}(X) \subseteq \mathcal{N}$ to be the set of functions of $\{\phi(x) | x \in X\}$. $\mathcal{N}(\mathcal{P}_j)$ is the set of maps $K : \mathcal{P}_j \to \mathcal{N}$ such that $K(X) \in \mathcal{N}(\hat{X})$. We define $\mathcal{N}(\mathcal{P}_j), \mathcal{N}(\mathcal{P}_{j,c})$ similarly.

2. For $I \in \mathcal{N}(\mathcal{P}_j)$ we write

\[ I(X) = I^X := \prod_{B \in \mathcal{P}_j(X)} I(B) \quad \text{for } X \in \mathcal{P}_j \]

For $K \in \mathcal{N}(\mathcal{P}_j)$ we say that $K$ factorizes over connected components and write $K \in \mathcal{N}(\mathcal{P}_{j,c})$ if

\[ K(X) = \prod_{Y \in \mathcal{C}(X)} K(Y) \quad (2.11) \]
The basic structure that we want to propagate to every scale of the RG iterations is, for $j \geq 0$

$$Z'_N(\xi) = e^{E_j} \mathbb{E} \left[ \sum_{X \in \mathcal{P}_j(\Lambda)} I_j(\Lambda \setminus \hat{X}, \phi, \xi) K_j(X, \phi, \xi) \right] \tag{2.12}$$

Here, $e^{E_j}$ is a $\phi, \xi$ independent constant factor. This constant will be shown to be the same for $Z'_N(\xi)$ and $Z'_N(0)$ and thus cancels. $K_j(X, \phi, \xi)$ only depends on the values of $\phi, \xi$ in a small neighborhood of $X$. Note that there is a “corridor” between each $X$ and $\Lambda \setminus \hat{X}$. One can also have the viewpoint that there is a factor $I(\hat{X} \setminus X)$. These “corridors” will be important in our conditional expectation method.

Furthermore, $I_j$ will have a local form in the sense that it factorizes over $j$-blocks

$$I_j(B, \phi, \xi) = e^{-\frac{1}{4}\sigma_j \sum_{x \in B} \varphi(x) + \varphi(\xi)} \tag{2.13}$$

$I_j(B)$ is essentially determined by the dynamical parameter $\sigma_j$. On the other hand, $K_j$ will only factorize over “connected components of polymer”.

The basic bounds that hold on every scale about $K_j$ whose form will not be explicit is

$$\sum_{n=0}^{4} \frac{1}{n!} \left\| K^{(n)}_j(X, \phi, \xi) \right\| \leq \|K\|_j A^{-|X|} G(X, X^+) \tag{2.14}$$

and for $X \subset Y$, $G(X, Y)$ is a normalized conditional expectation called “regulator”

$$G(X, Y) = \mathbb{E} \left[ e^{\frac{1}{2} \sum \varphi(x)} \mid \phi_Y \right] / N(X, Y) \tag{2.15}$$

and the normalization factor is

$$N(X, Y) = \mathbb{E} \left[ e^{\frac{1}{2} \sum \varphi(x)} \mid \phi_Y = 0 \right] \tag{2.16}$$

This form of regulator is different from the one defined in [Bry99]; in particular it is itself a conditional expectation. It will be shown to have some interesting properties.

Now we outline the steps to go from scale $j$ to scale $j+1$ while the structure (2.12) is preserved.

1) Extraction and reblocking.

Reblocking is a procedure which rewrites (2.12) into an expansion over “$j+1$ scale polymers”; and we extract the components that grow too fast under this reblocking.

**Lemma 2.** Suppose that $L$ is sufficiently large. If at the scale $j$ one has

$$Z'_N(\xi) = e^{E_j} \mathbb{E} \left[ \sum_{X \in \mathcal{P}_j} I_j^{\Lambda \setminus \hat{X}}(\phi, \xi) K_j(X, \phi, \xi) \right] \tag{2.17}$$
with $I_j \in \mathcal{N}(\mathcal{B})$ given by (2.13), then there exist $E_{j+1}, I_{j+1} \in \mathcal{N}(\mathcal{B}_{j+1})$ and $K_j \in \mathcal{N}(\mathcal{B}_{j+1}^c)$, so that the following expansion at the scale $j+1$ holds

$$Z_N'(\xi) = e^{E_{j+1}} \mathbb{E} \left[ \sum_{U \in \mathcal{P}_{j+1}} I_{j+1}^U (\phi, \xi) K_j^U (U, \phi, \xi) \right]$$

where $\mathcal{E}'$ is a constant independent of $\phi, \xi$, and for every $D \in \mathcal{B}_{j+1}$,

$$I_{j+1}(D) = e^{-\sigma_{j+1} \sum_{x \in \partial_D^U \phi(x) + \partial_D^U \xi(x)}}$$

for some constant $\sigma_{j+1}$.

We will prove this Lemma in Section 3.

2) Conditional expectation.

This step is the main difference between this new method and [Bry09]. As above, we will first make a corridor around $K_j$ by writing $I_j = (I_j - 1) + 1$ and glue some "$j+1$ blocks" onto $U$; these "$j+1$ blocks" are the ones where $I_j - 1$ live on while all the rest "$j+1$ blocks" are neither touching them nor touching $U$. Then we will have a form

$$\mathbb{E} \left[ \sum_{U \in \mathcal{P}_{j+1}} I_{j+1}^U (\phi, \xi) K_j^U (U, \phi, \xi) \right]$$

where $\hat{U} \setminus U$ is the corridor we just made, of width $L^{j+1}$. We then take conditional expectation

$$\mathbb{E} \left[ \sum_{U \in \mathcal{P}_{j+1}} I_{j+1}^U (\phi, \xi) \mathbb{E} \left[ K_j^U (U, \phi, \xi) | (U^+)^c \right] \right]$$

where $U \subset U^+ \subset \hat{U}$. For notation conventions, see subsection 2.1. This conditional expectation followed by factoring out $\phi, \xi$ independent constant gives $K_j$ and we’re back to the form (2.12) with all $j$ replaced by $j+1$. In case $U = \Lambda$, we just integrate (unconditionally): $\mathbb{E} \left[ K_j^\Lambda (\Lambda, \phi) \right]$, but to streamline expressions we still write (2.20) keeping in mind the special treatment for the $U = \Lambda$ term.

Remark 3. The reason that we have to create corridors before conditional expectation is, obviously, to make $I_{j+1}$ intact, while the conditioning can be a bit away from $U$, that is $(U^+)^c$. As we will see, an important ingredient that makes the method work is the $O(L^{j+1})$ distance between $U$ and $(U^+)^c$.

We point out two important facts about the conditional expectation step. The first one is that we can write the Gaussian field $\phi$ into a sum of two decoupled parts. Let $P_U$ be the Poisson kernel for $U$ and recall our convention that $P_U \phi(x) = \phi(x)$ for $x \notin U$ as in subsection 2.1.
Proposition 4. Let $U \subset V$ be finite graphs. Define $\zeta$ via $\phi(x) = P_U \phi(x) + \zeta(x)$. Then the quadratic form

$$- \sum_{x \in V} \phi(x) \Delta \phi(x) = - \sum_{x \in V} \zeta(x) \Delta_{U,m}^D \zeta(x) - \sum_{x \in V} P_U \phi(x) \Delta m P_U \phi(x)$$

(2.21)

where $-\Delta_{U,m}^D = -\Delta_U + m^2$ and $\Delta_U^D$ is the Dirichlet Laplacian for $U$, $m \geq 0$.

Notice that $x \in U$ don’t contribute to the last summation since $\Delta m P_U \phi(x) = 0$ in $U$. By this proposition, taking expectation of a function $K(\phi)$ conditioned on $\{\phi(x) | x \in U^c\}$ is simply integrating out a Gaussian field $\zeta$:

$$\mathbb{E} \left[ K(\phi, \xi) | U^c \right] = \mathbb{E}_\xi \left[ K(P_U \phi + \zeta, \xi) \right]$$

(2.22)

where the covariance of $\zeta$ is the $C_U^D -$ the Dirichlet Green’s function for $U$. In particular, we observe that $I_j$ defined in (2.13) has an alternative representation

$$I_j(B, \phi, \xi) = e^{-\frac{1}{2} \sigma_j \sum_{e \in B \setminus e} \mathbb{E} \left[ \partial \phi(x) + \partial \xi(x) \right]^{(B^+)^\nu}}$$

(2.23)

It’s conceptually helpful to keep in mind that we’re just re-arranging the following structure (comparing with (2.8)-(2.9))

$$\mathbb{E} \left[ \sum_{x \in \mathcal{D}_j} e^{-\frac{1}{2} \sigma_j \sum_{e \in x \setminus e} \mathbb{E} \left[ \partial \phi(x) + \partial \xi(x) \right]^{(B^+)^\nu}} \mathbb{E} \left[ \cdots \left| (X^+)^c \right| \right] \right]$$

(2.24)

namely an outmost (unconditional) expectation of a simple combination of many conditional expectations.

Remark 5. In the paper, $P_U \phi$ will always be well-defined: by Prop 1.11 of [Kum10], if the probability that the random walk starting from any point in $U$ exits $U$ in finite time is 1, then the harmonic extension exists and is unique. Domains $U \subset \Lambda$ will always satisfy this condition because the random walk hits any point in $\Lambda$ in finite time with probability one.

The next fact is as follows:

Proposition 6. Let $d \geq 2$, $x \in X \subset U \subset \Lambda$. If $d(x, \partial X) \geq cL^j$, then

$$| (\partial_x P_x \xi^D) (\partial_x P_x) \xi^D(x, x) | \leq O(1) L^{-d}$$

(2.25)

where $O(1)$ depends on $c$, and $C_U^D$ is the Dirichlet Green’s function for $U$.

See Lemma[12]. This result gives the expected scaling for the covariance of $\partial P_x \xi$ where $P_x$ is a Poisson kernel obtained from the previous RG step. We take a heuristic test to see the necessity of this proposition: setting $\xi = 0$, for $X \subset U$, if we perform an expectation conditioned on $\{\phi(x) | x \in X^c\}$, followed by another expectation conditioned on $\{\phi(x) | x \in U^c\}$, by (2.22)

$$\mathbb{E}_{\zeta_U, \zeta_U} \left[ K(P_x (P_U \phi + \zeta_U) + \zeta_X) \right] = \mathbb{E}_{\zeta_U, \zeta_U} \left[ K(P_U \phi + P_X \zeta_U + \zeta_X) \right]$$

(2.26)

then we need this proposition to deal with $P_X \zeta_U$ when integrating over $\zeta_U$.

Proofs of the above two results are in the following sections.
Linearization and stable manifold theorem

We have just outlined a single RG map

\[(\sigma_j, \sigma_{j+1}, E_{j+1}, K_j) \rightarrow K_{j+1}\]

We will show smoothness of this map in Section 5. Note that two issues haven’t been discussed: 1) choice of \(\sigma_{j+1}, E_{j+1}\), which should be a function of \((\sigma_j, K_j)\), so that the RG map becomes \((\sigma, K) \rightarrow (\sigma_{j+1}, K_{j+1})\) (notice that we won’t regard \(E_{j+1}\) as dynamical parameter and we’ll factorize it out); 2) choice of \(\sigma\) in the a priori tuning step. We will outline how to treat these two issues now.

Clearly \((\sigma, K) = (0, 0)\) is a fixed point of the RG map. In Section 6 we show that the linearization of the map \((\sigma_j, \sigma_{j+1}, E_{j+1}, K_j) \rightarrow K_{j+1}\) around \((0, 0, 0, 0)\) has a form \(\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3\) where \(\mathcal{L}_1\) captures the “large polymers” contributions to \(K_{j+1}\), and \(\mathcal{L}_2\) involves the remainder of second order Taylor expansion of conditionally expected \(K_j\) on “small polymers”, both of which will be shown contractive with arbitrarily small norm by suitable choices of constants \(L\) and \(A\) introduced above. Furthermore, \(\mathcal{L}_3\) will roughly have a form

\[\mathcal{L}_3(D) \approx L^d E_{j+1} + \sigma_{j+1} \sum_{x \in D} (\partial P_D \cdot \phi(x))^2 - \sigma_j (\sum_{x \in D} (\partial P_D \cdot \phi(x))^2 + \delta E_j) + \text{Tay} \tag{2.27}\]

where \(\text{Tay}\) is the second order Taylor expansion of conditionally expected \(K_j\) on small polymers, which consists of constant and quadratic terms, and \(D\) is a \(j+1\) block. Now it’s easy to see that there is a way to choose \(E_{j+1}\) and \(\sigma_{j+1}\) so that \(\mathcal{L}_3\) is almost 0, up to a localization procedure for “\(\text{Tay}\)”. For proofs see Section 6.

Once we have shown a way to choose the constants \(\sigma_{j+1}, E_{j+1}\) to ensure contractivity of the above linear map, a stable manifold theorem can be applied to prove that there exists a suitable tuning of \(\sigma\) so that

\[|\sigma_j| \lesssim 2^{-j} \quad ||K_j||_j \lesssim 2^{-j} \tag{2.28}\]

Main result: the scaling limit

**Theorem 7.** For any \(p > d\) there exists constants \(M > 0\) and \(z_0 > 0\) so that: for all \(||\tilde{f}||_{L^p} \leq M\) and all \(|z| \leq z_0\) there exists a constant \(\epsilon\) depending on \(z\) and

\[\lim_{N \to \infty} Z_N(f) = \exp \left( \frac{1}{2} \int_{\hat{\Lambda}} \tilde{f}(x)(-\hat{\Delta})^{-1} \tilde{f}(x)d^d x \right) \tag{2.29}\]

where \(\hat{\Lambda}\) is the Laplacian in continuum.

The main ingredient of the proof is that at scale \(N - 1\) (we don’t want to continue all the way to the last step since it would be a bit awkward to define \(I_{N-1}\) and \(I_N\)), by eq. (2.12)

\[Z_N'(\xi) \approx \lim_{m \to 0} e^{\delta_{N-1}} \sum_{x \in \mathcal{O}_{N-1}} (1 + 2^{-N})\hat{\Lambda} \cdot \hat{\Lambda} 2^{-N} \tag{2.30}\]

Bounding the number of terms by \(2^d\) we see that it is almost \(e^{\delta_{N-1}}\) as \(N\) becomes large. The constant \(e^{\delta_{N-1}}\) will be the same for \(Z_N'(\xi)\) and \(Z_N''(0)\). So only the exponential factor
in equation (2.5) survives in the $N \to \infty$ limit and it goes to the right hand side of (2.29). Details are in Section 7. We remark that the assumption on $\tilde{f}$, which makes $f$ smooth at the scale $N$ is for simplicity of the demonstration of the method.

3 The renormalization group steps

3.1 Some additional definitions

1. A j-polymer $X$ is called a small set or small polymer if it is connected and $|X| \leq 2^d$. Otherwise it’s called large. We write by $\mathcal{I}_j(X)$ the set of all small j-polymers in $X$.

2. Define $\hat{\mathcal{S}}_j$ to be the set of pairs $(B,X)$ so that $X \in \mathcal{S}_j$ and $B \in B_j(X)$.

3. We let $\mathcal{C}(X)$ be the set of connected components of $X$.

4. We also introduce a notation $Y \in_X \mathcal{P}_j$ which means $Y \in \mathcal{P}_j$ and that if $X = \emptyset$ then $Y = \emptyset$.

5. Let $X \in \mathcal{P}_j$. Define its closure $\bar{X} \in \mathcal{P}_{j+1}$ to be the smallest $(j+1)$-polymer that contains $X$.

6. We define a notation $\chi_{\mathcal{A}}$ where $\mathcal{A}$ is a set of polymers: $\chi_{\mathcal{A}} = 1$ if any two polymers in $\mathcal{A}$ are strictly disjoint as $j$-polymers and $\chi_{\mathcal{A}} = 0$ otherwise. Also, if $\mathcal{A}$ is a set of polymers, let’s write $X_\mathcal{A}$ to be the union of all elements of $\mathcal{A}$.

3.2 Renormalization group steps

Now we focus on a single RG map from scale $j$ to $j + 1$. For simpler notations we omit the subscript $j$ and objects at scale $j + 1$ will be labelled by a prime, e.g. $K', \mathcal{P}'$. The guidance principle will be that for all kinds of $I$’s below, $I - 1$ and their difference $\delta I$ and $K$ will be small, so their products will be higher order small quantities. These remarks will make more sense after we discuss the linearization of the smooth RG map.

Extraction and Reblocking

We start to prove Lemma 2. Notice that according to the conclusion of Lemma 2 one has to construct a corridor of width $L^{j+1}$ between the set where $I'$ lives on (i.e. $\Lambda \setminus \hat{U}$) and the set where $K^2$ lives on (i.e. $U$). This will be important for taking the conditional expectation below. The way to construct $I', K^2$ is not unique. Our construction below has the foresight that certain components in $K^2$ will be well separated, see Remark 8.

Proof. (of Lemma 2) Define $\bar{I} \in \mathcal{M}(\mathcal{P})$ as

$$\bar{I}(B) = e^{E' - \frac{4}{\sigma'} \sum_{e \in \mathcal{E}} \left( \partial_e p(e) \cdot \phi(x) + \partial_e \xi(x) \right)^2} \tag{3.1}$$

where $E'$ and $\sigma'$ will be chosen later. Denote

$$\langle X \rangle := \cup \{ B \in \mathcal{P}_j ; (B)^+ \cap \bar{X} \neq \emptyset \}$$
where the + operation is on the scale \( j + 1 \) and the hat is on the scale \( j \). Then we let

\[
\begin{align*}
1(B) &= (1 - e^{E'}) + e^{E'} & \text{if } B \subseteq \hat{X} \setminus X \\
1(B) &= (1(B) - e^{E'}) + e^{E'} & \text{if } B \subseteq \langle X \rangle \setminus \hat{X} \\
1(B) &= \delta I(B) + \bar{I}(B) & \text{if } B \subseteq \langle X \rangle^c \\
K(X) &= \sum_{B \in \mathcal{B}(X)} \frac{1}{|X|} K(B, X) & \text{if } X \in \mathscr{I}
\end{align*}
\]

where \( \delta I \) is defined implicitly, and \( K(B, X) := K(X) \). Insert these summations into the product factors in (2.17), and expand, we obtain

\[
Z_N' (\xi) = e^{\mathcal{E}} \left[ \sum_X f(X) \prod_{Y \in \mathcal{E}(X) \setminus \mathcal{I}} K(Y) \prod_{Y \in \mathcal{E}(X) \setminus \mathcal{I}} K(Y) \right]
= e^{\mathcal{E}} \left[ \sum_{\mathcal{J} \cup \mathcal{Q}} \chi_{\mathcal{J} \cup \mathcal{Q}} \sum_{P \in \mathcal{Q}, Z} (1 - e^{E'})^P (1 - e^{E'})^Q (1 - e^{E'})^{\langle X \rangle \setminus (P \cup Q)} \delta I \bar{I}^{(X)^c \setminus Z} \\
\cdot \prod_{Y \in \mathcal{I}} K(Y) \prod_{(B, Y) \in \mathcal{Q}} \frac{1}{|Y|} K(B, Y) \right] \tag{3.2}
\]

where the first summation is over \( \mathcal{J} \) which is a family of connected large polymers, and \( \mathcal{Y} \) which is a family of elements in \( \mathcal{I} \) i.e. \( \mathcal{Y} = \{(B_i, Y_i) \in \mathcal{I}) \}^{1 \leq i \leq n} \), and \( n \geq 0 \), and we have defined \( \mathcal{Y} := \{Y_i\}^{1 \leq i \leq n} \). In the above equation and in the sequel of this proof,

\[
X := X_{\mathcal{J} \cup \mathcal{Q}}
\]

and the second summation above is over \( P \in \mathcal{P}(\hat{X} \setminus X), Q \in \mathcal{P}(\langle X \rangle \setminus \hat{X}), \) and \( Z \in \mathcal{P}(\langle X \rangle^c) \).

Now observe that one can re-arrange the above summations in the following way:

\[
\sum_{\mathcal{J} \cup \mathcal{Q}} \chi_{\mathcal{J} \cup \mathcal{Q}} \sum_{P \in \mathcal{Q}, Z} = \sum_{V \in \mathcal{P}(\hat{X} \setminus X) \cup \mathcal{Q}, Z} \sum_{V} \sum_{(P \cup Q, Z)} 1_{P \cup Q, Z} (\langle \Im, 1 \rangle) \cup X = V \tag{3.3}
\]

where the second summation on the right hand side means

\[
\sum_{(P \cup Q, Z)} := \sum_{\mathcal{J} \cup \mathcal{Q}} \sum_{P \in \mathcal{Q}, Z} \sum_{V} \sum_{(P \cup Q, Z)} 1_{P \cup Q, Z} (\langle \Im, 1 \rangle) \cup X = V
\]

We would like to write the factors \( \bar{I} \) and \( e^{E'} \) into parts in \( V \) and outside \( V \):

\[
\bar{I}^{(X)^c \setminus Z} = P^{V \cap (X)^c \setminus Z} P^{V \cap (X)^c \setminus Z} \tag{3.4}
\]

\[
(e^{E'})^{\langle X \rangle \setminus (P \cup Q)} = (e^{E'})^{V \cap (X)^c \setminus (P \cup Q)} (e^{E'})^{V \cap (X)^c \setminus (P \cup Q)} (3.4)
\]

Note that \( V^c \cap \langle X \rangle^c \) (where some \( \bar{I} \) live on) could possibly touch \( V \), so our next step is to make a corridor so that such touchings will be avoided. Write \( I = (\bar{I} - e^{E'}) + e^{E'} \) and expand,

\[
I^{V \cap (X)^c} = \sum_{W \in \mathcal{P}(\mathcal{V})} (I - e^{E'})^{W \cap (X)^c} (e^{E'})^{(V \setminus W) \cap (X)^c}
\]

12
For each $V$ and $W$, define $U$ to be the smallest union of connected components of $V \cup W$ that contains $V$:

$$U = U_{W,V} := \cap \{ T \mid T \in \mathcal{W}^c(V \cup W), T \supseteq V \} \in \mathcal{P}$$

where $\mathcal{W}^c(V \cup W)$ is the set of unions of $(j+1 \text{ scale})$ connected components of $V \cup W$.

Observe that if $L$ is sufficiently large, one has $\langle X \rangle \subseteq \hat{\mathcal{V}} \subseteq \hat{\mathcal{U}}$. So

$$\mathcal{I}^{V^c \cap (X)^c} = \sum_{W \in \mathcal{P}(V^c)} (\hat{I} - e^{E})^{W\setminus \hat{\mathcal{U}}} (\hat{I} - e^{E})^{W \cap \langle X \rangle^c} (e^{E})^{(V^c \cap W) \setminus \hat{\mathcal{U}} \cap \langle X \rangle^c}$$

Let $R := W \setminus U = W \setminus \hat{\mathcal{U}}$. Note that one has the following identities for the sets appeared in the above equation: $W \cap U = U \cap V$ and

$$(V^c \setminus W) \setminus \hat{\mathcal{U}} = \hat{\mathcal{U}} \setminus R$$

$$\langle V^c \setminus W \rangle \setminus \hat{\mathcal{U}} = \hat{\mathcal{U}} \setminus U$$

The summation over $W$ amounts to a summation over $U$ and $R$:

$$\mathcal{I}^{V^c \cap (X)^c} = \sum_{U \in \mathcal{P}(U \cup V)} \sum_{R \in \mathcal{P}(X \setminus U)} (\hat{I} - e^{E})^{R}(\hat{I} - e^{E})^{U \cap \langle X \rangle^c} (e^{E})^{(\hat{\mathcal{U}} \setminus U) \cap \langle X \rangle^c}$$

The factor $(e^{E})^{V^c \cap \langle X \rangle^c}$ appeared in (3.4) is treated as follows. Since $\langle X \rangle \subseteq \hat{\mathcal{U}}$

$$(e^{E})^{V^c \cap \langle X \rangle^c} = (e^{E})^{V^c \cap \langle X \rangle} (e^{-E})^{V^c \cap \langle X \rangle}$$

$$(e^{E})^{(\hat{\mathcal{U}} \setminus U) \cap \langle X \rangle} (e^{-E})^{V^c \cap \langle X \rangle \cap U} (e^{-E})^{V^c \cap \langle X \rangle}$$

Combine (3.2), (3.3), (3.4), (3.5), (3.6),

$$Z_{\mathcal{X}}(\xi) = e^{\mathcal{E}} \mathbb{E} \left[ \sum_{U \in \mathcal{P}(U \cup V)} \mathcal{I}^{V^c \cap \langle X \rangle^c} \cdot (e^{E})^{\hat{\mathcal{U}}} K^R(U) \right]$$

where for $U \neq \emptyset$

$$K^R(U) := \sum_{V \subseteq U, V \neq \emptyset} \sum_{(P,Q,Z,R)_{(X)}} (1 - e^{E})^P (I - e^{E})^Q \delta I^Z \prod_{Y \in \mathcal{X}} K(Y) \prod_{(B,Y) \in \mathcal{X}} \frac{1}{|Y|} K(B,Y)$$

$$\cdot (\hat{I} - e^{E})^{U \cap \langle X \rangle^c} (e^{E})^{\langle X \rangle \cap U \setminus (P \cup Q)} (e^{-E})^{U \cup \langle X \rangle \setminus \hat{\mathcal{U}} \cap \langle X \rangle^c}$$

Factorizing the constant $e^{E}$ by letting

$$e^{\mathcal{E}}' = e^{\mathcal{E}} + E'[\Lambda]$$

$$I'(D) = e^{-L'\mathcal{E}'} \prod_{B \in \mathcal{B}(D)} I(B) = e^{-\frac{1}{2} \sigma_{ij+1} \sum_{d \in D, e \in c} \partial_d P_{d+} \phi(d) + \partial_d \xi(x)}$$
for $D \in \mathcal{B}'$, we obtain

$$Z_N(\xi) = e^{\mathcal{E}^*} \mathbb{E} \left[ \sum_{U \in \mathcal{C}} (I')^{\mathcal{N} \setminus U} K^#(U) \right]$$

Remark 8. In Remark We have to create corridors as well before extraction and rebloking because: $K^#(U)$ is a complicated product of $K, l - 1, \delta I, -e^E$; each $K(X)$ has an $L^j$ corridor created in the previous $(j - 1$'th RG step) and depends on $\phi$ in an $L'/3$ neighborhood of $X$, but $\delta I, l - 1$ both depend on $\phi$ in an $O(L^{j+1})$ neighborhood that would intrude into the $O(L^j)$ corridor of $K(X)$ which would be bad for the estimates. Gluing some $l - 1$ onto $K$ is unharmful because $l - 1$ only depends on $\phi$ in an $L'/3$ neighborhood, which can’t penetrate the $L^j$ corridor of $K(X)$. In the other words, our arrangement is such that the factors $\tilde{I} - e^E$ and $l$ always live outside $X = \langle X_{\mathcal{P}} \cup \mathcal{B} \rangle$. Therefore, although in the definition of $l$ eq. (3.1) the Poisson kernel $P_{(B)^+}$ is a quite long range one, the set $(B)^+$ actually doesn’t intersect with the domains where the regulators for $K$’s are defined on; this will be important when we prove the smoothness of the RG map later.

Conditional expectation

Lemma 9. $K^#$ factorizes over $j + 1$ scale connected components, namely

$$K^#(U) = \prod_{V \in \mathcal{C}_{j+1}(U)} K^#(V) \quad (3.8)$$

where $\mathcal{C}_{j+1}(U)$ is the set of connected components of $U$ as a $j + 1$ polymer.

Proof. Let $V_1, \ldots, V_{|\mathcal{C}(U)|}$ be all the connected components of $U$. For any $E$ which may stand for $U, Z, P, Q$, elements of $\mathcal{P} \cup \mathcal{N}$, one of the $B_i$, or $X = X_{\mathcal{P}} \cup \mathcal{B}$, let $E(p) = E \setminus \cup_{q \neq p} V_q$. It’s easy to check that for $i \neq j, E^{(i)}$ and $E^{(j)}$ are strictly disjoint on scale $j$. Then the lemma is proved by the factorization property of $I, K$ on scale $j$. 

We are now ready to take the expectation of $K^#(V)$ conditioned on $\phi$ outside $V^+$ for each $V \in \mathcal{C}(U) \setminus \{\Lambda\}$, because $\Lambda \setminus \mathcal{N}$ and $V^+$ don’t touch. In the case $V = \Lambda$, we just take expectation of $K^#(V)$ without conditioning, but write $\mathbb{E} \left[ K^#(\Lambda) \left| (\Lambda^+)^c \right. \right] := \mathbb{E} \left[ K^#(\Lambda) \right]$ to shorten the notations.

$$Z_N(\xi) = e^{\mathcal{E}_{j+1}} \mathbb{E} \left[ \sum_{U \in \mathcal{C}_{j+1}} \prod_{V \in \mathcal{C}(U)} \mathbb{E} \left[ K^#(V) \left| (V^+)^c \right. \right] \right] \quad (3.9)$$

Now we come back to the basic structure (2.12) with $j$ replaced by $j + 1$. Obviously, $K_{j+1}(U) \in \mathcal{P}_{j+1,c}$. In Section 4 we give precise definitions for norms and spaces of the $K_j$ above, and in section 5 we prove smoothness of the above map $(\sigma_j, E_{j+1}, \sigma_{j+1}, K_j) \rightarrow K_{j+1}$.
3.3 Properties about conditional expectation

The variation principle

One of our main ideas is to write the Gaussian field $\phi$ into a sum of two decoupled parts. This is important for the conditional expectation.

**Fact.** Given any positive definite quadratic form $Q(v)$ for vector $v$, if $v = (x,y)$, one can write $Q(v) = Q_1(x) + L(x,y) + Q_2(y)$ where $Q_{1,2}$ are positive definite quadratic forms and $L(x,y)$ is the crossing term. Let $\tilde{x}(y)$ be the minimizer of $Q(v) = Q(x,y)$ with $y$ fixed. Then, one can cancel $L(x,y)$ by shifting $x$ by $\tilde{x}$:

$$Q(v) = Q_1(x - \tilde{x}) + Q((\tilde{x},y)) \quad (3.10)$$

Before introducing the next proposition, let’s recall our convention that $P_U \phi(x) = \phi(x)$ for $x \notin U$ as in subsection 2.1.

**Proposition 10.** Let $U \subset V \subset \mathbb{Z}^d$ be two finite sets. Let $\phi_U$ and $\phi_{U^c}$ be the restriction of $\phi$ to $U$ and $U^c$. Let $P_U$ be the Poisson kernel for $U$ and write $\phi(x) = P_U \phi(x) + \zeta(x)$. Then,

$$-\sum_{x \in V} \phi(x) \Delta \phi(x) = -\sum_{x \in U} \zeta(x) \Delta^D_P \zeta(x) - \sum_{x \in V} P_U \phi(x) \Delta m P_U \phi(x) \quad (3.11)$$

where $\Delta^D_P$ is the Dirichlet Laplacian for $U$.

**Proof.** We can apply the Fact (3.10) for $\phi = (\phi_U, \phi_{U^c})$, and

$$Q(\phi) = -\sum_{x \in V} \phi(x) \Delta \phi(x)$$

$$= -\sum_{x \in U} \phi_U(x) \Delta^D_P \phi_U(x) + L(\phi_U, \phi_{U^c}) - \sum_{x \in U^c} \phi_{U^c}(x) \Delta^D_{U^c} \phi_{U^c}(x)$$

where $L$ is the crossing term, and $\Delta^D_{U^c}$ is the Dirichlet Laplacian for $U^c$. Since the minimizer of $Q(\phi)$ with $\phi_{U^c}$ fixed is $P_U \phi$,

$$-\sum_{x \in V} \phi(x) \Delta \phi(x) = -\sum_{x \in U} (\phi_U - P_U \phi)(x) \Delta^D_P (\phi_U - P_U \phi)(x) - Q((P_U \phi, \phi_{U^c}))$$

$$= -\sum_{x \in U} \zeta(x) \Delta^D_P \zeta(x) - \sum_{x \in V} P_U \phi(x) \Delta m P_U \phi(x)$$

By this proposition, taking expectation of a function $K(\phi)$ conditioned on \{$(\phi(x) | x \in U^c)$\} is equivalent to simply integrating out $\zeta$:

$$E[K(\phi) | U^c] = E_{\zeta} [K(P_U \phi + \zeta)] \quad (3.12)$$

where the covariance of $\zeta$ is the Dirichlet Green’s function for $U$. 

15
The important scaling

We first prove some general results about harmonic functions on the lattice, such as averaging properties and that the derivative of a harmonic function is bounded by itself with a factor of dimension \([1/\text{length}]\).

For \(R > 0\) we introduce a cube of size \(R\) centered at \(a\) if
\[
K_R := \left\{ y \in \mathbb{Z}^d | |y - a|_{\infty} \leq R \right\}
\]
for some \(a \in \mathbb{Z}^d\).

**Lemma 11.** Let \(K_R\) and \(K_{R/2}\) be cubes of sizes \(R, \frac{R}{2}\) respectively centered at the same point. Assume that \(u\) is harmonic in a cube \(K_R\). Let \(X = \mathcal{K}_R \setminus \mathcal{K}_{R/2}, x \in \mathcal{K}_{R/2}\) and \(d(x, \partial \mathcal{K}_{R/2}) > R/6\). Then
\[
u(x) = O(R^{-d}) \sum_{y \in X} u(y) \quad (3.14)
\]
and for \(e \in E\),
\[
|\partial_e u(x)| \leq O(R^{-1}) \sup_{y \in X} |u(y)| \quad (3.16)
\]

**Proof.** For any integer \( \frac{R}{2} < b < R\), let \(K_b\) be cubes of sizes \(b\) centered with \(K_R\). Let \(w\) be the random walk starting from \(x\) and \(\tau_b := \inf \{ t > 0 | w_t \in \partial K_b \}\). By Lemma [38] there exists a constant \(c\) so that
\[
\mathbb{P}^x(w_{\tau_b} = y) \leq c b^{-(d-1)}
\]
for all \(y \in \partial K_b\). Then since \(u\) is harmonic,
\[
u(x) = \mathbb{E}^x \left[ u(w_{\tau_b}) \right] \leq c b^{-(d-1)} \sum_{y \in \partial K_b} u(y)
\]
Multiply both sides by \(b^{d-1}\) and sum over \( \frac{R}{2} < b < R\), we have
\[
R^d u(x) \leq c' \sum_{y \in X} u(y) \quad (3.17)
\]
which proves (3.14). By Cauchy-Schwartz inequality,
\[
u(x) \leq O(R^{-d}) \left( \sum_{y \in X} u(y)^2 \right)^{1/2} |X|^{1/2}
\]
which proves (3.15). Let \(X^o\) be the interior of \(X\), namely \(X^o \cup \partial X^o = X\). In (3.17) replace \(u\) by \(\partial_e u\), which is harmonic in \(X^o\), and apply summation by parts along each line parallel to \(e\),
\[
R^d |\partial_e u(x)| \leq c' \left| \sum_{y \in X, y+e \in X^o} u(y+e) + \sum_{y \in X, y-e \in X^o} u(y-e) \right| \leq O(R^{d-1}) \sup_{y \in X} |u(y)|
\]
which proves (3.16).
The next Lemma plays an important role in controlling the fundamental scaling.

**Lemma 12.** Let $x \in X \subset U \subset \Lambda$. If $d(x, \partial X) \geq cL^j$, then

$$\sum_{y_1, y_2 \in \partial X} (\partial_{x,e} P_x)(x, y_1) C_U(y_1, y_2) (\partial_{x,e} P_x)(x, y_2) \leq O(1) L^{-d/j}$$

(3.18)

for all $e \in E, m > 0$ where the constant $O(1)$ depends on $c$. Here $\partial_{x,e}$ is the discrete derivative w.r.t. the argument $x$ to the direction $e$.

**Proof.** Notice that $C_U \leq C_A$ as quadratic forms, so it’s enough to prove the statement with $C_U$ replaced by $C_A$. Since $y_2 \in \partial X$ and $C_A(x - y_2)$ is $-\Delta_m$-harmonic in $x \in X$.

$$\sum_{y_1 \in \partial X} P_x(x, y_1) C_A(y_1, y_2) = C_A(x, y_2)$$

Taking derivative w.r.t. $x$ on the above equation we obtain that the left hand side of eq. (3.18) equals

$$\sum_{y_2 \in \partial X} \partial_{x,e} C_A(x, y_2) (\partial_{x,e} P_x)(x, y_2)$$

(3.19)

Now let $R = cL^j/3$ and define a cube $K_R$ centered on $x$. Apply lemma [2]

$$|\partial_{x,e} P_x(x, y_2)| \leq O(L^{-j}) |P_x(x^*_j, y_2)|$$

where $x^*_j \in K_R$. By Corollary [3] and the assumption $d(x, \partial X) \geq cL^j$, 

$$|\partial_{x,e} C_A(x, y_2)| \leq O(L^{-(d-1)j})$$

so (3.19) is bounded by $O(L^{-d/j})$ since $\sum_{y_2 \in \partial X} P_x(x^*_j, y_2) \leq 1$ for all $m > 0$. 

**Remark 13.** One may find that our method also resembles Gawedzki and Kupiainen’s approach [4, 5] because the Poisson kernel here plays a similar role as their spin blocking operator. However, there’re many differences. For example, our fluctuation fields $\xi$ have finite range covariances; the integrands at different scales don’t have to be in Gibbsian forms; and our polymer arrangements are closer to Brydges [6].

**4 Norms**

**4.1 Definitions of norms**

Define $h_j = hL^{-d-2}/2$ for some constant $h > 0$. We first define the norm for the fields. Let’s recall that $\xi$ is the field introduced in Section [2]. For $j > 0$ and $X \subset Y$ define

$$\|f, \lambda \xi\|_{\Phi_j(X,Y)} := h_j^{-1} \sup_{x \in X, e} |L^j \partial_e (P_y f(x) + \lambda \xi(x))|$$

(4.1)

$\|f\|_{\Phi_j(X,Y)}$ will be understood as $\|(f, 0)\|_{\Phi_j(X,Y)}$. As a special case, if $X \in \mathcal{P}_j$ then we write

$$\|(f, \lambda \xi)\|_{\Phi_j(X)} := \|(f, \lambda \xi)\|_{\Phi_j(X, X^+)}$$

(4.2)
We then define differentials for functions of the fields, and their norm. For test functions \((f, \lambda)^{\times n} := (f_1, \lambda_1 \xi, \ldots, f_n, \lambda_n \xi)\), the \(n\)-th differential of \(K(X, \phi, \xi)\) is

\[
K^{(n)}(X, \phi, \xi; (f, \lambda)^{\times n}) := \left. \frac{\partial^n}{\partial t_1 \ldots \partial t_n} K(X, \phi + \sum_{i=1}^n t_i f_i, \xi + \sum_{i=1}^n t_i \lambda_i \xi) \right|_{t_i=0}
\]

It is normed with a space of test functions \(\Phi\) by

\[
\|K^{(n)}(X, \phi, \xi)\|_{T^n_\phi(\Phi)} := \sup_{\|(f, \lambda, \xi)\|_{\Phi} \leq 1} |K^{(n)}(X, \phi, \xi; (f, \lambda)^{\times n})|
\]

In most of our discussions \(\Phi\) above will be chosen to be \(\Phi_j(X)\). We then measure the amplitude of \(K(X, \phi, \xi)\) at a fixed function \(\phi\) by incorporating all its derivatives at \(\phi\) that we want to control:

\[
\|K(X, \phi, \xi)\|_{T^n_\phi(\Phi)} := \sum_{n=0}^4 \frac{1}{n!} \|K^{(n)}(X, \phi, \xi)\|_{T^n_\phi(\Phi)}
\]

Define “regulators”:

\[
G(X, Y) := \mathbb{E} \left[ e^{\sum_{x \in X, \phi \in \phi} (\partial_x \phi(x))^2} \right] / N(X, Y)
\]

for \(X \subset Y\) where the normalization factor is defined by

\[
N(X, Y) := \mathbb{E} \left[ e^{\sum_{x \in X, \phi \in \phi} (\partial_x \phi(x))^2} \big| \phi_Y = 0 \right]
\]

Define

\[
\|K(X)\|_j := \sup_{\phi} \|K(X, \phi, \xi)\|_{T^n_\phi(\Phi_j(X))} G(X, X^+) \]^{-1}
\]

Finally, for \(A > 0\),

\[
\|K\|_j := \sup_{X \in \mathcal{P}_j} \|K(X)\|_j A^{|X|}
\]

For the case \(j = 0\): \((4.1)-(4.3)\) are still defined for \(j = 0\) with \(P_X = id\) and \(X = Y\) (recall these conventions made in Section \(2\)). \((4.5)\) is defined with \(G\) replaced by

\[
G_j(X) := e^{\sum_{x \in X, \phi \in \phi} (\partial_x \phi(x))^2}
\]

4.2 Properties

**Lemma 14.** Let \(F\) be function of \(\phi, X \subset Y \subset U\). We have the following property for the \(T^n_\phi(\Phi)\) norms:

\[
\|F^{(n)}(\phi)\|_{T^n_\phi(\Phi_j(Y, U))} \leq \|F^{(n)}(\phi)\|_{T^n_\phi(\Phi_j(X, U))}
\]

which also holds without \(n\).

**Proof.** The proof is immediate because \(\|f\|_{\Phi_j(Y, U)} \geq \|f\|_{\Phi_j(X, U)}\).
For further properties we first exploit a kind of functions $K(X, \phi, \xi)$ with an “special structure”: it depends on $\phi, \xi$ via $P_X \phi + \xi$; in other words there exists a function $\tilde{K}(X, \psi)$ so that
\begin{equation}
K(X, \phi, \xi) = \tilde{K}(X, P_X \phi + \xi)
\end{equation}
In view of this special structure we define new function spaces $\tilde{\Phi}_j(X, Y)$ for all $X \subset Y$
\begin{equation}
\tilde{\Phi}_j(X, Y) := \{ \text{functions harmonic on } Y \} \oplus \mathbb{R} \xi
\end{equation}
(this is really a direct sum since $\xi$ is either zero or non-harmonic) equipped with norm
\begin{equation}
\| g \oplus \lambda \xi \|_{\tilde{\Phi}_j(X, Y)} := h_j^{-1} \sup_{x \in X, e} | L^i \partial_e (g(x) + \lambda \xi(x)) |
\end{equation}
The following result roughly says that conditional expectation of a product followed by taking norm is bounded by the other way around, with norms taken on each factor.

**Lemma 15.** Let $X_k \subset Y_k \subset U$, $k = 1, 2, \ldots, m$, suppose that $K_k(\phi, \xi) = \tilde{K}_k(P_Y \phi + \xi)$. Then $E \left[ \prod_k K_k(\phi, \xi) \mid U^c \right]$ depends on $\phi, \xi$ via $P_U \phi + \xi$.

Furthermore, let $\psi = P_Y \phi + \xi$ and $\tilde{F}(\psi) = E_\xi \left[ \prod_k \tilde{K}_k(\psi + P_Y \xi) \right]$ where $\xi$ is the Gaussian field with Dirichlet Green’s function on $U$ as covariance, then
\begin{equation}
\| \tilde{F}(\psi) \|_{\bar{T}_e(\tilde{\Phi}_j(X_1 \cup X_2, U))} \leq E \left[ \prod_k \| K_k(\phi, \xi) \|_{\bar{T}_e(\tilde{\Phi}_j(X_k, Y_k))} \right] | U^c |
\end{equation}

**Proof.** The first statement holds because
\begin{equation}
E \left[ \prod_k K_k(\phi, \xi) \mid U^c \right] = E_\xi \left[ \prod_k \tilde{K}_k(P_Y \phi + \xi + \xi) \right]
\end{equation}
and $P_Y P_U = P_U$. For the second statement, without of generality let $m = 2$. By the definition of $\tilde{\Phi}_j$ norm, one has
\begin{equation}
\| \tilde{F}(\psi) \|_{\bar{T}_e(\tilde{\Phi}_j(X_1 \cup X_2, U))} \leq \sup_{\| g \oplus \lambda \xi \xi \|_{\tilde{\Phi}_j(X_1 \cup X_2, U)} \leq 1} \left| \partial_n^i \xi_{i=0}^{n} E_\xi \left[ \prod_k \tilde{K}_k(P_U \phi + P_Y \xi + \sum_{i=1}^{n} t_i g_i + \xi + \sum_{i=1}^{n} t_i \lambda_i \xi) \right] \right|
\end{equation}
\begin{equation}
\leq E_\xi \left[ \sup_{\| g \oplus \lambda \xi \xi \|_{\tilde{\Phi}_j(X_1 \cup X_2, U)} \leq 1} \left| \partial_n^i \xi_{i=0}^{n} \prod_k K_k(P_U \phi + \xi + \sum_{i=1}^{n} t_i g_i, \xi + \sum_{i=1}^{n} t_i \lambda_i \xi) \right| \right]
\end{equation}
where we used the harmonicity of $P_U \phi$ and $g_i$. By the product rule of derivatives, the fact that
harmonic functions on $U$ are harmonic on $Y_k$, and Lemma \[14\]

\[
\sum_{n=0}^{3} \frac{1}{n!} \| \tilde{F}^{(n)}(\psi) \|_{T_0^\phi(\Phi,(X_1 \cup X_2, U))} \\
\leq \sum_{n=0}^{3} \frac{1}{n!} \mathbb{E} \left[ \sum_{r=0}^{n} \binom{n}{r} \sup_{t_i=1, \ldots, r=1} \left| \partial_{t_i}^{n-r} \right|_{t_i=0} K_1(\phi + \sum_{i=1}^{n} t_i g_i, \xi + \sum_{i=1}^{n} t_i \lambda_i \xi) \right]
\leq \mathbb{E} \left[ \prod_k \left| K_k(\phi, \xi) \right|_{T_0^\phi(\Phi,(X_k, Y_k))} \right]
\]

where in the last step we used

\[
K_k(\phi + \sum_{i=1}^{n} t_i g_i, \xi + \sum_{i=1}^{n} t_i \lambda_i \xi) = K_k(P_k \phi + \sum_{i=1}^{n} t_i g_i + \xi + \sum_{i=1}^{n} t_i \lambda_i \xi)
\]

so we’re effectively deforming $(\phi, \xi)$ using test functions in $\Phi_j(X_k, Y_k)$ to come back to the $T_0^\phi(\Phi_j(X_k, Y_k))$ norm.

Before the next lemma we introduce a short notation

\[
(\partial_m f)^2 := (\partial f)^2 + m^2 f^2
\]

**Lemma 16.** We have the following properties for the regulator:

1. $G(X, Y, \phi = 0) = 1$
2. If $X_1 \subset Y_1, X_2 \subset Y_2$, and $Y_1 \cup \partial Y_1, Y_2 \cup \partial Y_2$ are disjoint, then

\[
G(X_1, Y_1)G(X_2, Y_2) = G(X_1 \cup X_2, Y_1 \cup Y_2)
\]

3. We have an alternative representation of $G(X, Y)$

\[
G(X, Y) = \exp \left( \frac{\kappa}{2} \sum_X (\partial \psi_1)^2 - \frac{1}{2} \sum_Y (\partial_m \psi_1)^2 + \frac{1}{2} \sum_Y (\partial_m \psi_2)^2 \right)
\]

where $\psi_1$ is the minimizer of $\sum_X (\partial_m \phi)^2 - \kappa \sum_X (\partial \phi)^2$ with $\phi_{Y^c}$ fixed, and $\psi_2$ is the minimizer of $\sum_Y (\partial_m \phi)^2$ with $\phi_{Y^c}$ fixed.

4. Fixing $Y$, $G(X, Y)$ is monotonically increasing in $X$ for all $X \subset Y$.

5. With $\psi_{1,2}$ defined in \[3\],

\[
\exp \left( \frac{\kappa}{2} \sum_X (\partial \psi_2)^2 \right) \leq G(X, Y) \leq \exp \left( \frac{\kappa}{2} \sum_X (\partial \psi_1)^2 \right)
\]
Lemma 18.

Proof. \( \text{(1)(2)} \) hold by definition and the fact that \( G(X,Y) \) is a function of \( \phi \) on \( \partial Y \). For \( \text{(3)}, \)

\[
G(X,Y) = \frac{\int e^{\frac{1}{2} \sum_{x} (\partial \phi)^2 - \frac{1}{2} \sum_{L} (\partial_{m} \phi)^2} \, dY \phi}{\int e^{-\frac{1}{2} \sum_{x} (\partial \phi)^2 - \frac{1}{2} \sum_{L} (\partial_{m} \phi)^2} \, dY \phi}
\]

(4.22)

where \( dY \phi \) is the Lebesgue measure on \( \{ \phi(x) : x \in Y \} \cong \mathbb{R}^Y \), \( \partial D \) takes Dirichlet boundary condition on \( \partial Y \). Using Fact (3.10) for both quadratic forms \( \frac{1}{2} \sum_{x} (\partial \phi)^2 + \frac{1}{2} \sum_{L} (\partial_{m} \phi)^2 \) and \( \frac{1}{2} \sum_{L} (\partial_{m} \phi)^2 \), we obtain \( \text{(3)} \), where the quantity \( \int e^{\frac{1}{2} \sum_{x} (\partial \phi)^2 - \frac{1}{2} \sum_{L} (\partial_{m} \phi)^2} \, dY \phi \) appears in both numerator and denominator and thus cancels, and so does the quantity \( \int e^{-\frac{1}{2} \sum_{x} (\partial \phi)^2 - \frac{1}{2} \sum_{L} (\partial_{m} \phi)^2} \, dY \phi \). \( \text{(4)} \) holds because of \( \text{(3)} \) and that

\[
\inf_{\phi} \left\{ \sum_{Y} (\partial_{m} \phi)^2 - \kappa \sum_{X} (\partial \phi)^2 | \Sigma_{X} \right\} \quad \text{(4.23)}
\]

is monotonically decreasing in \( X \). The two inequalities in \( \text{(5)} \) hold by replacing \( \psi_1 \) by \( \psi_2 \) or replacing \( \psi_2 \) by \( \psi_1 \), and using definitions of \( \psi_1, \psi_2 \). \( \square \)

Remark 17. The regulator in \([\text{Bry09}]\) has the form \( e^{\kappa \sum (\partial \phi)^2 + \text{other terms}} \), since the smoothed field \( \phi' \) there is analogous to our \( \psi \), the last property above implies that our regulator has about the same amplitude as the one in \([\text{Bry09}]\), except that we are not concerned about what the “other terms” are.

Before proving a furthur property we recall a formula. If \( U \) is a finite set and \( \psi = \{ \psi(x) : x \in U \} \) is a family of centered Gaussian random variables with covariance identity, and \( T : l^2(U) \rightarrow l^2(U) \) satisfies \( \|T\| < 1 \) then

\[
\mathbb{E} \left[ \exp \left( \frac{1}{2} \langle \psi, T \psi \rangle_{l^2(U)} \right) \right] = \det (1 - T)^{-1/2} = \exp \left( \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} Tr(T^n) \right)
\]

(4.24)

The next lemma shows that the conditional expectations almost automatically do the work when one wants to see how the regulators undergo integrations, except that we need to manually control a ratio of normalizations.

Lemma 18. For \( X \subset Y \subset U \), and \( d(X,Y^c) = c_0 |L| \), one has the bound

\[
\mathbb{E} \left[ G(X,Y)|U^c \right] \leq c^{L-d |X|} |G(X,U)|
\]

(4.25)

if \( U \neq \Lambda \), for some constant \( c \) only depending on \( c_0 \). One also has, as the superficial case, the short-hand notation and bound

\[
\mathbb{E} \left[ G(X,Y)|\Lambda^+ \right] := \mathbb{E} [G(X,Y)] \leq c^{L-d |X|}
\]

In particular if \( X = \tilde{X}_0 \) for some \( X_0 \in \mathcal{P} \), then the factor \( c^{L-d |X|} \) can be written as \( c^{|X_0|} \). Furthurmore, \( G_0 \) also satisfies the same bound.
Taking logarithm, we need to compute \( \psi \) has covariance identity, where \( C_Y \) is the Dirichlet Green’s function for \( Y \). Then define \( T_Y = \frac{1}{2} \sum_{e \in E}(\partial_e C_Y^{1/2})^* \left( \partial_e C_Y^{1/2} \right) \) as an operator on \( l^2 = l^2(\Lambda) \). We define similar operators \( C_U, T_U \) in the same way for \( U \). Let \( \partial_e^D, -\Delta_Y \) take Dirichlet boundary condition on \( \partial Y \). Because \( C_Y \) is the inverse of \( -\Delta_Y \)

\[
(f, T_Y f)_\mathbb{L} = \frac{1}{2} \sum_{x \in \mathcal{X}, e \in \mathcal{E}} (\partial_e C_Y^{1/2} f(x))^2 \leq \frac{1}{2} \sum_{x \in \mathcal{Y}, e \in \mathcal{E}} (\partial_e^D C_Y^{1/2} f(x))^2
\]

\[
\leq \sum_{x \in \mathcal{Y}} C_Y^{1/2} f(x)(-\Delta_Y C_Y^{1/2} f(x))
\]

(4.26)

So \( \|T_Y\| \leq 1 \). Similarly \( \|T_U\| \leq 1 \). By (4.24)

\[
\frac{N(X, U)}{N(X, Y)} = \frac{\mathbb{E} \left[ e^{\frac{\lambda}{2}(\psi, T_Y \psi)} \right]}{\mathbb{E} \left[ e^{\frac{\lambda}{2}(\psi, T_Y \psi)} \right]} = \left( \frac{\det(1 - \kappa T_U)}{\det(1 - \kappa T_Y)} \right)^{-1/2}
\]

(4.27)

Taking logarithm, we need to compute

\[
Tr(\log(1 - \kappa T_U) - \log(1 - \kappa T_Y)) \leq O(1) Tr(\kappa T_U - \kappa T_Y)
\]

where we have used \( \|T_Y\| \leq 1, \|T_U\| \leq 1, \kappa \) is small, and \( \log(1 - x) \) is Lipschitz on \( x \in [-\frac{1}{2}, \frac{1}{2}] \). Since \( C_U - C_Y = P_Y C_U \),

\[
Tr(T_U - T_Y) = \frac{1}{2} \sum_{e \in \mathcal{E}, x \in \mathcal{X}} \partial_e (C_U - C_Y) \partial_e^* (x, x)
\]

\[
= \frac{1}{2} \sum_{e \in \mathcal{E}, x \in \mathcal{X}, y \in \partial Y} \partial_e P_Y (x, y) \partial_e C_U (y, x)
\]

(4.28)

By Lemma 11 and proceed similarly as eq. (3.19) in proof of Lemma 12, making use of the \( O(L^d) \) distance between \( x \) and \( y \), the above expression is bounded by \( O(L^{-d}) |X| \) which concludes the proof.

The bound on \( \mathbb{E} \left[ G(X, Y) \right] \left( \Lambda^+ \right)^c \) is similar. The only modification is to replace \( C_U \) by \( C_\Lambda \) which satisfies periodic instead of Dirichlet boundary condition. For \( G_0 \), we can directly bound \( \mathbb{E} \left[ e^{\frac{\lambda}{2} \sum_{e \in \mathcal{E} \setminus \mathcal{E}^c} \partial_e \phi(x)} \right| U^c \) by \( c|X| \).

5 Smoothness of RG

In this section we prove that the RG map constructed in Section 3 is smooth. The proof is a bit tedious but straightforward. First of all, we need some geometric results from [Bry09].
Lemma 19. (Brydges [Bry99]) There exists an \( \eta = \eta(d) > 1 \) such that for all \( L \geq 2^d + 1 \) and for all large connected sets \( X \in \mathcal{P}_j \), \( |X| \geq \eta |X|_{j+1} \). In addition, for all \( X \in \mathcal{P}_j \), \( |X| \geq |X|_{j+1} \), and

\[
|X| \geq \frac{1}{2} (1 + \eta) |\bar{X}|_{j+1} - \frac{1}{2} (1 + \eta) 2^d + 1 |c(X)|
\]  

(5.1)

Proof. The lemma is the same with [Bry99] (Lemma 6.15 and 6.16), so we omit the proof.

\( \square \)

Proposition 20. Let \( B'(N_{j+1}^{\mathcal{P}_j+1}) \) be a ball centered on the origin in \( N_{j+1}^{\mathcal{P}_j+1} \). There exists \( A(d, L, B') \) and \( A'(d, A) \) such that for \( A > A(d, L, B') \) and \( A' > A'(d, A) \), the map \((\sigma_j, E_j, K_j) \mapsto K_{j+1} \) defined above is smooth from \((-A^{-1}, A^{-1})^3 \times B_{A^{-1}}(N_{j+1}^{\mathcal{P}_j+1}) \) to \( B'(N_{j+1}^{\mathcal{P}_j+1}) \) where \( B_{A^{-1}}(N_{j+1}^{\mathcal{P}_j+1}) \) is a ball centered on the origin in \( N_{j+1}^{\mathcal{P}_j+1} \) with radius \( A^{-1} \).

Proof. We omit subscript \( j \) for objects at scale \( j \) and write a prime for objects at scale \( j+1 \), as in Section [5] Let

\[
A^{*-1} \ll \kappa
\]

(5.2)

For \( U \in \mathcal{P}_j \), \( U \neq \emptyset \), by definition of \( K^\emptyset \),

\[
K'(U) = \sum_{V \subseteq U, V \neq \emptyset} \sum_{P, Q, Z, \mathcal{X}, \mathcal{Y}} (1 - e^{-E'})^p (e^{-E'})^q (X \setminus U \setminus \emptyset) (c_{W'})^{U+} \leq \mathcal{O}(A^*/2)^{P} \mathcal{O}(A^{*-1})\mathcal{O}(A^{*-1}) (A^{*-1}) \mathcal{O}(A^{*-1}) (A^{*-1})
\]

(5.3)

where, with \( \prod K := \prod_{Y \in \mathcal{X}} K(Y) \prod_{(B, Y) \in \mathcal{Y}} \frac{1}{|Y|} K(B, Y) \) as a short-hand notation,

\[
E^{U+} := \mathbb{E}\left[(1 - e^{-E'})^{(U \setminus V) \cap (X \setminus Z)} \delta I^2 (1 - e^{-E'}) Q \prod K | (U^+)^c \right] \\
\equiv \mathbb{E}\left[(1 - e^{-E'})^{(U \setminus V) \cap (X \setminus Z)} \delta I^2 (1 - e^{-E'}) Q | (W^+)^c \right] \prod K | (U^+)^c
\]

(5.4)

where \( W = U \setminus \hat{X} \) (recall that \( X := X_{\mathcal{X} \cup \mathcal{Y}} \)) and the last step used the corridors around \( K(Y) \) in order to make sense of the \((W^+)^c \) conditional expectation. In the above \( W^+ \) is a \( + \) operation at scale \( j \) and \( U^+ \) is a \( + \) operation at scale \( j+1 \).

We first control \( E^{W^+} \). With \( \phi = P_W, \phi + \zeta \) and the inequality \((a + b)^2 \leq 2a^2 + 2b^2 \), and using assumption \((5.2), \) Lemma \([39] \) we list the estimates for each factors.

\[
\| (1 - e^{-E'}) (B) \|_{T^\emptyset (\phi_j (B))} \leq 5 (\kappa A^* - 1) e^{\frac{\delta}{2} (\phi \setminus \mathcal{Z})} \mathcal{O} (\mathcal{P}_W + \phi)^2 + \frac{\delta}{2} \mathcal{O} (\mathcal{P}_W + \zeta)^2
\]

for all \( B \in Q \), where \( B^+ \subseteq W^+ \) since \( Q \subseteq (X) \setminus \hat{X} \); and,

\[
\| (1 - e^{-E'}) (B) \|_{T^\emptyset (\phi_j (B))} \leq 5 (\kappa A^* - 1) e^{\frac{\delta}{2} (\phi \setminus \mathcal{Z})} \mathcal{O} (\mathcal{P}_W + \phi)^2 + \frac{\delta}{2} \mathcal{O} (\mathcal{P}_W + \zeta)^2
\]

for all \( B \in Q \), where \( B^+ \subseteq W^+ \) since \( Q \subseteq (X) \setminus \hat{X} \); and,
for all $B \in \mathcal{B}_j((U \setminus V) \cap \langle X \rangle^c)$, where $(\bar{B})^+ \subseteq W^+$ since $\langle X \rangle$ is designed to ensure that; and

$$\|I(B)\|_{T_\phi(\Phi_j(B))} \leq 2e^{\frac{5}{2} \sum_\delta (\partial P_{w+} \phi)^2 + \frac{5}{2} \sum_\delta (\partial P_{\bar{b}} + \xi)^2}$$

for all $B \in \mathcal{B}_j(V \cap \langle X \rangle^c \setminus Z)$, where $(\bar{B})^+ \subseteq W^+$ since $B \subseteq \langle X \rangle^c$; and

$$\|\delta I(B)\|_{T_\phi(\Phi_j(B))} \leq \|I(B) - 1\|_{T_\phi(\Phi_j(B))} + \|I(B) - 1\|_{T_\phi(\Phi_j(B))}$$

$$\leq 8(\kappa A^*)^{-1} e^{\frac{5}{2} \sum_\delta (\partial P_{w+} \phi)^2 + \frac{5}{2} \sum_\delta (\partial P_{\bar{b}} + \xi)^2}$$

by $e^a + e^b \leq 2e^{a+b} (a, b > 0)$ for all $B \in \mathcal{B}_j(Z)$, where $(\bar{B})^+ \subseteq W^+$ since $Z \subseteq \langle X \rangle^c$. Combining all above estimates, together with Lemma 15, we have

$$\|\mathbf{E}^{W^+}\|_{T_\phi(\Phi_j(W))} \leq (\kappa A^*/8)^{-|Q,\mathcal{Y}^c(U \setminus V)\setminus \langle X \rangle|} e^{\frac{5}{2} \sum_\delta (\partial P_{w+} \phi)^2} \mathcal{M}$$

(5.5)

where

$$\mathcal{M} \leq \mathbb{E}_\xi^F \left[ e^{\frac{5}{2} \sum_\delta B(\bar{B}) \sum_\delta (\partial P_{w+} \phi)^2 + \frac{5}{2} \sum_\delta (\partial P_{\bar{b}} + \xi)^2} e^{\frac{5}{2} \sum_\delta (\partial P_{\bar{b}} + \xi)^2} \right]$$

(5.6)

In the next Lemma we show that $\mathcal{M} \leq c|U|^j$.

Now we proceed to control $\mathbf{E}^{U^+}$. Instead of $(a + b)^2 \leq 2a^2 + 2b^2$ we use properties of the regulator established in Section 4. Since for all $X \in \mathcal{P}_{j,c}$

$$\|K_j(X)\|_{T_\phi(\Phi_j(X))} \leq A^{-1} G(\bar{X}, X^+) A^{-|X|}$$

By Lemma 16 (2)(4)(5) and Lemma 18

$$\|\mathbf{E}^{U^+}\|_{T_\phi(\Phi_j(W))} \leq c|U|^j \cdot (\kappa A^*/8)^{-|Z,\mathcal{Y}^c(U \setminus V)\setminus \langle X \rangle|} \mathbb{E}_\xi^F \left[ e^{\frac{5}{2} \sum_\delta (\partial P_{w+} \phi)^2} \right]$$

$$\times \prod_{Y \in \mathcal{Y}} G(\bar{Y}_k, Y_k^+) \prod_{Y \in \mathcal{Y}} G(\bar{Y}_i, Y_i^+) \|U^+\|_{\langle X \rangle} A^{-|X|} \mathbb{E}_\xi^F \left[ e^{\frac{5}{2} \sum_\delta (\partial P_{w+} \phi)^2} \right]$$

(5.7)

$$\leq c|U|^j \cdot (\kappa A^*/8)^{-|Z,\mathcal{Y}^c(U \setminus V)\setminus \langle X \rangle|} \mathbb{E}_\xi^F \left[ e^{\frac{5}{2} \sum_\delta (\partial P_{w+} \phi)^2} \right]$$

$$\times \prod_{Y \in \mathcal{Y}} (1 + \eta)$$

We can bound the number of terms in the summation in (5.3) by $k|U|^j$ with $k = 2^j$, because every $j$-block in $U$ either belongs to $V$ or $V^c$, and the same statement is true if $V$ is replaced by $P, Q, Z, X_{\mathcal{Y}}, Y_{\mathcal{Y}}$, and if it’s in $Y \in \mathcal{Y}$ it’s either the $B$ of $(B, Y) \in \mathcal{G}$ or not. By Lemma 19 for $a = \frac{1}{2} (1 + \eta)$, with $\mathcal{Y} = \{X_k\}$, $\mathcal{G} = \{(B_i, Y_i)\}$

$$a|U|^j \leq a|Z|^j + a|\cup \bar{B}i|_{j+1} + a|\cup \bar{X}_k|_{j+1} + a|\bar{Q}^+|_{j+1} + a|(U \setminus V) \cap \langle X \rangle^c |_{j+1}$$

$$\leq (|Z| + a2^{d+1} |\mathcal{C}(Z)|) + a|\mathcal{D}| + (\sum_k |X_k|_{j+1} + a2^{d+1} |\mathcal{Y}|)$$

$$+ (|Q| + a2^{d+1} |\mathcal{C}(Q)|) + aL^j |(U \setminus V) \cap \langle X \rangle^c |_{j+1}$$

$$\leq (1 + a2^{d+1} )(|Z| + |Q|) + a|\mathcal{D}| + (|X_{\mathcal{Y}}| + a2^{d+1} |\mathcal{Y}|) + aL^j |(U \setminus V) \cap \langle X \rangle^c |_{j+1}$$

Then we can easily check that with $A, A^*$ sufficiently large as assumed in the proposition

$$\|K \|_{j+1} = \sup_{U \in \mathcal{P}_{j^c}} \|K'(U)\|_{j+1} A^{a|U|^j} A^{(1-a)|U|^j+1} < r$$

24
where \( r \) is the radius of \( B'(\mathcal{N}_j') \), because \( A^{[X]}_j \) is cancelled by its inverse in \((5.7)\), and

\[
\lim_{A \to \infty} A^{(1-\alpha)[U]_j+A^{-[X]}_j} k^{[U]_j} c^{[U]_j} e^{[W]_j+[X]_j} \frac{2}{2} \frac{1}{2} (X) (X') \frac{1}{2} (P(U)Q) \frac{1}{2} (U) = 0 \quad (5.8)
\]

\[
\lim_{A \to \infty} \left( kA^*/8 \right)^{-(Q \cup Z \cup ([U]V \setminus (X)) - [X]) - [\mathcal{Y}]}
\]

\[
\cdot A^{(1+a2^{d+1})[Q][Z]} e^{[\mathcal{Y}] + al/2(U'V) \cap (X')_j} = 0 \quad (5.9)
\]

The derivatives of the map \((\sigma_j, E_{j+1}, \sigma_{j+1}, K_j) \mapsto K_{j+1}\) are bounded similarly. \(\square\)

**Lemma 21.** Let \( \mathcal{M} \) be the quantity introduced in the proof of Proposition \((20)\). There exists a constant \( c \) independent of \( L, A, A^* \) such that

\[
\mathcal{M} \leq c^{[U]_j} \quad (5.10)
\]

**Proof.** Defining \( \zeta = c^{1/2}_W \psi \) where \( c_W \) is the Dirichlet Green function for \( W^+ \) and \( \psi \in L^2(W^+) \), \( \mathcal{M} \) is bounded by

\[
\mathbb{E}_\psi \exp \left\{ 4 k \sum_{x \in W} \psi(x) T \psi(x) \right\} \quad (5.11)
\]

where \( \psi \) has identity covariance and

\[
T = \frac{1}{4} \sum_{B \in \mathscr{B}_j(W), e \in \mathcal{E}} \left( C^{1/2}_U P^P \partial_x C_{U^+} \partial_x + C^{1/2}_U P^P \partial_x C_{U^+} \partial_x + C^{1/2}_U P^P \partial_x C_{U^+} \partial_x + C^{1/2}_U P^P \partial_x C_{U^+} \partial_x \right) \quad (5.12)
\]

is a linear map from \( L^2(W^+) \) to itself. \( T_1, T_2 \) are defined to be the two terms respectively. We have by Lemma \((12)\)

\[
Tr(T) = \frac{1}{4} \sum_{B \in \mathscr{B}_j(W), e \in \mathcal{E}} \left( \sum_{x \in B} \partial_x P^P \partial_x C_{U^+} \partial_x C_{U^+} \partial_x \right) (x, x) + \sum_{x \in B} \partial_x P^P \partial_x C_{U^+} \partial_x C_{U^+} \partial_x \right) (x, x) \quad (5.13)
\]

\[
\leq O(1)(L^{-d+j} + L^{-d(j+1)} |W|) \leq O(1)|W|_j
\]

For the next step we bound \( ||T|| \). In fact,

\[
(f, T_1 f)_2 = \frac{1}{4} \sum_{B \in \mathscr{B}_j(W), x \in B, e} \left( \partial_x C_{U^+}^2 f(x) \right)^2 \leq \frac{1}{4} \sum_{B \in \mathscr{B}_j(W), x \in B, e} \left( \partial_x C_{U^+}^2 f(x) \right)^2 \quad (5.14)
\]

\[
\leq c_d \sum_{x \in W, e} \left( \partial_x C_{U^+}^2 f(x) \right)^2
\]

25
where we used the fact that the harmonic extension minimizes the Dirichlet form to get rid of the Poisson kernels. The constant \( c_{ij} \) comes from overlapping of \( B^+ \)'s. Then we can proceed as (4.26) to bound the above expression by \( c_{ij}(f,f)_{d^2} \). \( T_2 \) is bounded in the same way. Now by \( |Tr(T^n)| \leq |Tr(T)||T|^{n-1} \), and formula (4.24) the proof of the lemma is completed.

6 Linearized RG

Having established smoothness, in this section we study the linearization of the RG map.

In view of Lemma 15, we can show, by induction along all the RG steps, that \( K_j(X) \) depends on \( \phi,\xi \) via \( P_X,\phi + \xi \) (at scale 0, \( I_0, K_0 \) depend on \( \phi,\xi \) via \( \phi + \xi \)). We write

\[
\text{TayE} [K_j(X)|(U^+)^c]
\]
to be the second order Taylor expansion of \( E[K_j(X)|(U^+)^c] \) in \( P_{U^+},\phi + \xi \).

**Proposition 22.** The linearization of the map \( (\sigma_j, E_{j+1}, \sigma_{j+1}, K_j) \to K_{j+1} \) around \( (0,0,0,0) \) is \( \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 \) where

\[
\mathcal{L}_1 K_j(U) = \sum_{X \in \mathcal{P}_{j+1}, X \in U} \mathbb{E}[K_j(X)|(U^+)^c]
\]

\[
\mathcal{L}_2 K_j(U) = \sum_{B \in \mathcal{P}_{j+1}, B = U} \sum_{X \in \mathcal{P}_j, X \geq B} \frac{1}{|X|} (1 - \text{Tay}) \mathbb{E}[K_j(X)|(U^+)^c]
\]

\[
\mathcal{L}_3 (\sigma_j, E_{j+1}, \sigma_{j+1}, K_j)(U) = \sum_{B \in \mathcal{P}_{j+1}, B = U} \left( E_{j+1}(B) + \frac{\sigma_{j+1}}{4} \sum_{x \in B,e} \left( \partial_{P_B} \phi(x) + \xi(x) \right)^2 \right) + \sum_{X \in \mathcal{P}_j, X \geq B} \frac{1}{|X|} \text{TayE} [K_j(X)|(U^+)^c]
\]

**Proof.** In Proposition 20 we proved that the map \( (\sigma_j, E_{j+1}, \sigma_{j+1}, K_j) \to K_{j+1} \) is smooth around \( (0,0,0,0) \) so that we can linearize the map. In (3.7) since \( V \neq 0 \), \( \hat{I} - e^{E_{j+1}} \) factor doesn’t contribute to the linear order. Also if \( X = \emptyset \) then \( \hat{X} = \langle X \rangle = \emptyset \), so \( 1 - e^{E_{j+1}} \) and \( I_j - e^{E_{j+1}} \) don’t contribute to the linear order either. The terms that contribute to the linear order correspond to \( (Z, |\mathcal{P}|, |\mathcal{P}|) \) equal to \( (\emptyset, 0, 1) \) or \( (\emptyset, 1, 0) \) or \( (B, 0, 0) \) where \( B \in \mathcal{P}_j \). Grouping these terms into large sets part and small sets part with Taylor leading terms and remainder we obtain the above linear operators.

6.1 Large sets

**Lemma 23.** Let \( L \) be sufficiently large and \( A \) be sufficiently large depending on \( L \). Then \( \mathcal{L}_1 \) in Proposition 22 is a contraction. Moreover, \( \lim_{L \to \infty} \lim_{A \to \infty} \| \mathcal{L}_1 \| = 0 \).
Lemma 24. For $F$ a function of $\mathcal{K}$, therefore by Lemma 19,

\[ \|L^1 K_j(U)\|_{j+1} \leq \sum_{X \in \mathcal{P}_{j+1}} \|K_j\| \|\varphi\|^{|X|}A^{-|X|} \]  

(6.4)

therefore by Lemma 19

\[ \|L^1 K_j\|_{j+1} = \sup_{U \in \mathcal{P}_{j+1}} \|L^1 K_j(U)\|_{j+1} A^{|U|_{j+1}} \]
\[ \leq \left[ \sup_{U \in \mathcal{P}_{j+1}} A^{|U|_{j+1}} \sum_{X \in \mathcal{P}_{j+1} \setminus \mathcal{P}_{j}, X \subseteq U} c^{|X|}A^{-|X|} \right] \|K_j\|_j \]
\[ \leq \left[ \sup_{U \in \mathcal{P}_{j+1}} A^{|U|_{j+1}} 2^{|U|_{j+1}} (A / \eta)^{|U|_{j+1}} \right] \|K_j\|_j \]

(6.5)

where $\eta > 1$ is introduced in Lemma 19. The bracketed expression goes to zero as $A \to \infty$. \qed

6.2 Taylor remainder

We prepare to show contractivity of $L_2$. We first show that the Taylor remainder after the second derivative is bounded by the third derivative. It’s a general result about the $T_\phi(\Phi)$ norm with no need to specify the test function space $\Phi$.

Lemma 24. For $F$ a function of $\phi$ let $\text{Tay}_n$ be its $n$-th order Taylor expansion about $\phi = 0$, and $\Phi$ be a space of test functions, then

\[ \|(1 - \text{Tay})F(\phi)\|_{T_\phi(\Phi)} \leq (1 + \|\phi\|_\phi)^3 \sup_{t \in [0,1]} \|F^{(k)}(\phi)\|_{T_\phi(\Phi)} \]

(6.6)

Proof. By Taylor remainder theorem,

\[ \|(1 - \text{Tay}_2)F(\phi)\|_{T_\phi(\Phi)} = \sum_{n=0}^4 \frac{1}{n!} \sup_{\{f_1, \ldots, f_n: \|f_i\|_\phi \leq 1\}} \left| (F^{(n)} - \text{Tay}_2 F^{(n)}) (\phi; f^{\times n}) \right| \]

\[ = \sum_{n=0}^4 \frac{1}{n!} \sup_{\{f_1, \ldots, f_n: \|f_i\|_\phi \leq 1\}} \left| (F^{(n)} - \text{Tay}_2 F^{(n)}) (\phi; f^{\times n}) \right| \]

\[ = \sum_{n=0}^4 \frac{1}{n!} \sup_{\{f_1, \ldots, f_n: \|f_i\|_\phi \leq 1\}} \left| \left( \int_0^1 \frac{(1-t)^{2-n}}{(2-n)!} \partial^3 t F^{(n)}(\phi; f^{\times n}) + 1_{\{n \geq 3\}} F^{(n)}(\phi; f^{\times n}) \right) \right| \]

\[ = \sum_{n=0}^4 \frac{1}{n!} \sup_{\{f_1, \ldots, f_n: \|f_i\|_\phi \leq 1\}} \left| \left( \int_0^1 \frac{(1-t)^{2-n}}{(2-n)!} F^{(3)}(\phi; (3-n) f^{\times n}) + 1_{\{n \geq 3\}} F^{(n)}(\phi; f^{\times n}) \right) \right| \]

(6.7)

27
where \( \phi \times (3-n) \) means \( 3 - n \) test functions \( \phi \). Calculating the time integrals,

\[
\|(1 - T\partial_t^2)F(\phi)\|_{T^1(\Phi)} \leq \sum_{n=0}^{\frac{3}{n!}} 1 \sup_{t \in [0,1]} \left| \frac{1}{(3-n)!} \right| \sup_{t \in [0,1]} |F^{(3)}(t; \phi \times (3-n), f \times n)| + \|F^{(4)}(\phi)\|_{T^1(\Phi)}
\]

\[
\leq \left(1 + \|\phi\|_\Phi\right)^3 \sup_{t \in (0,1), k=3, 4} \|F^{(k)}(t; \phi)\|_{T^1(\Phi)}
\]

(6.8)

where in the last step binomial theorem is applied.

\[\square\]

**Lemma 25.** Let \((B, X) \in \mathcal{B}, B = U\), if \( \kappa \) is small enough depending on \( L \), and \( h \) is large enough depending on \( \kappa \) and \( L \), then

\[
(2 + \|\phi\|_{\Phi^{j+1}(X, U^+)} )^3 G(X, U^+) \leq qG(U, U^+)
\]

(6.9)

for a constant \( q \), where the dot(s) operations on \( X \) are at scale \( j \), and \( + \) operation on \( U \) is at scale \( j + 1 \).

**Proof.** Since \( X, U \) are \( j \) and \( j + 1 \) scale small sets, we may take for convenience the mass regularization \( m = 0 \) here which obviously doesn’t change the value of \( \Theta \). For the first step, let \( \psi_2 = P_{U^+} \phi \). For each \( e \in \mathcal{E}, \partial_e \psi_2 \) is harmonic in \( U^+ \cap (U^+ - e) \). By assumptions we can find a \( Y \subset U \) s.t. \( d(X, Y) \geq O(L^j), d(Y, \partial U) = O(L^{j+1}), |Y| = O(L^{d-j}) \), and finally \( Y \) satisfies the assumptions of Lemma 38 so that (A.9) holds. Then,

\[
\sup_{e \in \mathcal{E}, x \in X} |\partial_e \psi_2(x)|^2 \leq O(L^{-d-j}) \sum_{e \in E(Y)} (\partial_e \psi_2(y))^2
\]

(6.10)

namely,

\[
\|\phi\|_{\Phi^{j+1}(X, U^+)}^2 \leq h^{-2} \sum_{e \in E(Y)} (\partial_e \psi_2(y))^2
\]

(6.11)

So there exists \( q \) so that

\[
(2 + \|\phi\|_{\Phi^{j+1}(X, U^+)} )^3 \leq q \exp \left( \frac{h^{-2}}{2} \sum_{e \in E(Y)} (\partial_e \psi_2(y))^2 \right)
\]

(6.12)

Therefore the left hand side of (6.9) is bounded by

\[
q \exp \left\{ \frac{\kappa}{2} \sum_{e \in E(U^+)} (a_e \partial_e \psi_1)^2 + \frac{h^{-2}}{2} \sum_{e \in E(Y)} (\partial_e \psi_2(y))^2 - \frac{1}{2} \sum_{U^+} (\partial \psi_1)^2 + \frac{1}{2} \sum_{U^+} (\partial \psi_2)^2 \right\}
\]

where the function \( a_e = 1 \) if \( e \in E(\tilde{X}) \) and decays to zero in a neighborhood of \( \tilde{X} \), and the support of \( a_e \), more precisely \( \tilde{X} := \{x : \exists e \in \mathcal{E} \text{ s.t. } a_{x+e} \neq 0\} \), is still separated from \( Y \) by an \( O(L^j) \) distance and from \( \partial U \) by an \( O(L^{d-j}) \) distance, and \( |\nabla^k a_e| \leq O(L^{-k-j}) \), and

\[
\psi_1 \text{ maximizes } \kappa \sum_{e \in E(U^+)} (a_e \partial_e \phi)^2 - \sum_{U^+} (\partial \phi)^2 \text{ fixing } \phi \big|_{|U^+|} \]

(6.13)
Notice that we “enlarged” the set $\hat{X}$ in $G(\hat{X}, U^+)$ by smoothing out the coefficient $a_e$, followed by a replacement with the maximizer $\psi_1$ solving the new elliptic problem. In the following we show that by choosing $h$ large enough

$$\frac{h^2}{2} \sum_{e \in E(Y)} (\partial_e \psi_2(y))^2 \leq \frac{K}{2} \sum_{e \in E(Y)} (\partial_e \psi_1(y))^2$$

(6.14)

such that the left hand side of (6.9) is bounded by

$$q \exp \left\{ \frac{K}{2} \sum_{e \in E(U)} (\partial_e \psi_1)^2 - \frac{1}{2} \sum_{U^+} (\partial \psi_1)^2 + \frac{1}{2} \sum_{U^+} (\partial \psi_2)^2 \right\} \leq qG(\hat{U}, U^+)$$

which holds by a replacement of $\psi_1$ with the maximizer of $\frac{K}{2} \sum_{e \in E(U)} (\partial_e \psi_1)^2 - \frac{1}{2} \sum_{U^+} (\partial \psi_1)^2$.

To show (6.14), let $P^{(0)}_{U^+}$ be the Poisson kernel for $-\Delta$ on $U^+$, and $P^{(1)}_{U^+}$ be the Poisson kernel for the non-constant coefficient Laplacian associated with (6.13). Let $\tilde{a} = 1 + \frac{\kappa}{2} \cdot a$. Since

$$\nabla \cdot (\tilde{a} \nabla P^{(1)}_{U^+}) - \Delta P^{(1)}_{U^+} = 0$$

we have

$$\nabla \cdot (\tilde{a}(P^{(1)}_{U^+} - P^{(0)}_{U^+})) = \nabla_i((1 \delta^i_k - \tilde{a}^i) \nabla_k P^{(0)}_{U^+})$$

$$P^{(1)}_{U^+} - P^{(0)}_{U^+} = 0 \quad \text{on } \partial U^+$$

Our arrangement in the following is based on the knowledge on the decay rate of the Green’s function (but not its derivative) for the non-constant coefficient Laplacian [Dei99], and the trick to bound derivatives of harmonic functions (w.r.t. the constant coefficient Laplacian) by scaling factors times these harmonic functions themselves. In fact,

$$\partial (P^{(1)}_{U^+} - P^{(0)}_{U^+})(x) = \int_{\hat{X}} \partial G(x, y) \nabla_i((1 \delta^i_j - \tilde{a}^i) \nabla_j P^{(0)}_{U^+})(y) dy$$

$$= - \int_{\hat{X}} G(x, y) \partial \left[ \nabla_i((1 \delta^i_j - \tilde{a}^i) \nabla_j P^{(0)}_{U^+})(y) \right] dy$$

$$+ \int_{\partial \hat{X}} G(x, y) \nabla_i((1 \delta^i_j - \tilde{a}^i) \nabla_j P^{(0)}_{U^+})(y) dy$$

(6.15)

where $G$ is the Green’s function for the non-constant coefficient Laplacian. We apply the chain rule and estimate each term separately. For instance

$$\left| \int_{\hat{X}} G(x, y) (\partial \nabla_i \tilde{a}^i)(\nabla_j P^{(0)}_{U^+})(y) dy \right| \leq \kappa O(L^{d^j}O(L^{-(d-2)^j})O(L^{-2 j})|\nabla P^{(0)}_{U^+}(x)|$$

$$\leq \kappa O(1)|\nabla P^{(0)}_{U^+}(x)|$$

Other terms in (6.15) are bounded similarly. Shifting $\phi \rightarrow \phi + c$ so that $\phi > 0$ on $\partial U^+$, and if $\kappa$ is small enough,

$$|\nabla P^{(0)}_{U^+} \phi(x)| \leq \frac{1}{1 - \kappa O(1)} |\nabla P^{(1)}_{U^+} \phi(x)|$$

and thus (6.14) is proved. \qed
Before the next Lemma we define

\[ F_X(U, \phi, \xi) := \mathbb{E} \left[ K_j(X, \phi, \xi) \right] \ (U^+) \]  

and we know that it depends on \( \phi, \xi \) via \( \psi := P_U \phi + \xi \), i.e. there exists a function \( \tilde{F}_X \) such that \( F_X(U, \phi, \xi) = \tilde{F}_X(U, \psi) \).

**Lemma 26.** Let \( L \) be sufficiently large. Then \( \mathcal{L}_2 \) in Proposition 22 is a contraction.

**Proof.** By Lemma 14 and Lemma 24 with test function space \( \Phi = \tilde{\Phi}_j(\tilde{X}, U^+) \) we have

\[
\left\| (1 - \text{Tay}) \tilde{F}_X(U, \phi, \xi) \right\|_{T_p(\Phi_j,(U^+))} \leq \left\| (1 - \text{Tay}) F_X(U, \phi, \xi) \right\|_{T_p(\Phi_j,(X,U^+))} \]

\[
= \left\| (1 - \text{Tay}) \tilde{F}_X(U, \psi) \right\|_{T_p(\Phi_j,(X,U^+))} \]

\[
\leq \left( 1 + \left\| \psi \right\|_{\Phi_j,(X,U^+)} \right) \sup_{k=3,4} \left\| \tilde{F}_X^{(k)}(U, \psi) \right\|_{T_p(\Phi_j,(X,U^+))} \]  

(6.17)

where \( \text{Tay} \) always means second order Taylor expansion in \( \psi = P_U \phi + \xi \) so that the equality above holds. Now by linearity of \( \tilde{F}_X^{(k)} \) in test functions and Lemma 15, we have

\[
\left\| \tilde{F}_X^{(3)}(U, \psi) \right\|_{T_p(\Phi_j,(X,U^+))} \leq L^{-\frac{1}{2}d} \left\| \tilde{F}_X^{(3)}(U, \psi) \right\|_{T_p(\Phi_j,(X,U^+))} \]

\[
\leq L^{-\frac{1}{2}d} \mathbb{E} \left[ \sup_{\left\| f \right\|_{\Phi_j,(X,U^+)} \leq 1} \left| \frac{\partial^3 f}{\partial \xi^3} \right| \right] \left( \sum_{i=1}^3 t_i f_i, \xi + \sum_{i=1}^3 t_i \lambda_i \xi \right) \left( (U^+)^c \right) \]  

(6.18)

\[
\leq L^{-\frac{1}{2}d} 3! \mathbb{E} \left[ \left| K_j(X, \phi, \xi) \right| \left| (U^+)^c \right| \right] \]

\[
\leq O(L^{-\frac{1}{2}d}) K_j(X) \lambda c^{(X)} |G(\tilde{X}, U^+) | \]

where in the last step Lemma 18 is applied. Also, \( \tilde{F}_X^{(3)} \) satisfies an even smaller bound. Next we estimate

\[
\left\| \psi \right\|_{\Phi_j,(X,U^+)} \leq h_j^{-1} \sup_{x \in \mathcal{X}, e} \left| L^j \partial_e P_U \phi(x) \right| + h_j^{-1} \sup_{x \in \mathcal{X}, e} \left| L^j \partial_e \xi(x) \right| \]

\[
\leq \left\| \psi \right\|_{\Phi_j,(X,U^+)} + 1 \]

by (2.7). By (6.17), (6.18) and Lemma 25 and (4) of Lemma 16

\[
\left\| (1 - \text{Tay}) F_X(U) \right\|_{j+1} \leq O(L^{-3d/2}) c^{(X)} j K_j(X) \left\| j \leq O(L^{-3d/2}) \left( \frac{A}{c} \right)^{-|X|} \right\| \| K \| \]  

(6.19)

thus by Lemma 19

\[
\left\| \mathcal{L}_2 K \right\|_{j+1} = O(L^{-3d/2}) \left[ \sup_{U \in \mathcal{B}_j} A^{(U)} j+1 \sum_{B \in \mathcal{B}, B = U \times \mathcal{X} \times \emptyset } \frac{1}{|X|} \left( \frac{A}{c} \right)^{-|X|} \right] \| K \| \]

\[
\leq O(L^{-3d/2}) \left[ \sup_{U \in \mathcal{B}_j} A^{(U)} j+1 O(L^d) A^{-|U|} c^{2d} \right] \| K \| \]  

(6.20)

\[
= O(L^{-d/2}) \| K \| \]

\( \square \)
6.3 $L_3$ and determination of coupling constants

We now localize the last term in $L_3$, which is the second order Taylor expansion of $\tilde{F}_X(U, \psi)$ in $\psi$ (recall that $F_x(U, \psi)$ and $\psi$ are introduced before Lemma 26). To do this we fix a point $z \in B$, and replace $\psi(x)$ by $x \cdot \partial \psi(z)$ (which according to our convention means $\frac{1}{2} \sum_{e \in E} x_e \partial_e \psi(z)$), and then average over $z \in B$. We will show that the error of this replacement is irrelevant. Then

$$\frac{1}{2} \tilde{F}_X^{(2)}(U, 0; \psi, \psi) = \text{Loc}K_j(B, X, U) + (1 - \text{Loc})K_j(B, X, U)$$

where we have defined

$$\text{Loc}K_j(B, X, U) := \frac{1}{8|B|} \sum_{z \in B, \mu, \nu \in \mathcal{E}} \partial_{t_2}^2 \left| \frac{\partial}{\partial t_1} \right| \frac{1}{t_1} \mathbb{E}_\zeta \left[ K_j(X, t_1 x_\mu + t_2 x_\nu + \zeta) \right] \partial_\mu \psi(z) \partial_\nu \psi(z)$$

and

$$(1 - \text{Loc})K_j(B, X, U) := \frac{1}{2|B|} \sum_{z \in B} \left( \partial_{t_2}^2 \left| \frac{\partial}{\partial t_1} \right| \frac{1}{t_1} \mathbb{E}_\zeta \left[ K_j(X, t_1 x \cdot \partial \psi(z) + t_2 x \cdot \partial \psi(z) + \zeta) \right] - \partial_{t_2}^2 \left| \frac{\partial}{\partial t_1} \right| \frac{1}{t_1} \mathbb{E}_\zeta \left[ K_j(X, t_1 x \cdot \partial \psi(z) + t_2 x \cdot \partial \psi(z) + \zeta) \right] \right)$$

$$= \frac{1}{2|B|} \sum_{z \in B} \left( \tilde{F}_X^{(2)}(U, 0; \psi - x \cdot \partial \psi(z), \psi) + \tilde{F}_X^{(2)}(U, 0; \psi - x \cdot \partial \psi(z), x \cdot \partial \psi(z)) \right)$$

We show that $\psi - x \cdot \partial \psi(z)$ gives additional contractive factors as going to the next scale:

**Lemma 27.** If $\psi = P_U \phi + \xi \in \Phi_j(\hat{X}, U^+)$,

$$\|\psi - x \cdot \partial \psi(z)\|_{\Phi_j(\hat{X}, U^+)} \leq O(L^{-\frac{j}{2} - 1}) \left( \|\phi\|_{\Phi_j+1(U)} + 1 \right)$$  \hspace{1cm} (6.22)

**Proof.** Since $P_U x = x$,

$$\|\psi - x \cdot \partial \psi(z)\|_{\Phi_j(\hat{X}, U^+)} = h_j^{-1} \sup_{x \in X, e} \int |\partial_e P_U \phi(x) + \partial_e \xi(x) - \partial_e P_U \phi(z) - \partial_e \xi(z)|$$

$$= h_j^{-1} \sup_{x \in X, e} \int \left( |\partial_e P_U \phi(x) - \partial_e P_U \phi(z)| + |\partial_e \xi(x) - \partial_e \xi(z)| \right)$$

For the first term we apply Newton-Leibniz formula along a curve connecting $x, z$, and then apply (3.16) with $R = O(L^{j+1})$ using the distance $O(L^{j+1})$ between $X$ and $\partial U$,

$$h_j^{-1} \sup_{x \in X, e} \int |\partial_e P_U \phi(x) - \partial_e P_U \phi(z)|$$

$$\leq h_j^{-1} \sup_{x \in \hat{U}} L \|\phi\|_{\Phi_j+1(U)}$$  \hspace{1cm} (6.24)
where \( \text{diam}(X) = O(L^d) \) since \( X \) is small. The second term in (6.23) can be bounded by

\[
h_j^{-1} \sup_{x \in X, e} L^j |\partial_x \xi (x) - \partial_x \tilde{\xi} (z)| \leq O(L^{-\frac{d}{2} (N-j)}) \leq O(L^{-\frac{d}{2}}) \]  
(6.25)

as long as \( j + 1 < N \), and by \( d \geq 2 \) and (2.7). Therefore

\[
\|\psi - x \cdot \partial \psi(z)\|_{\Phi_j(X, U^+)} \leq O(L^{-\frac{d}{2}-1}) \left( \|\Phi\|_{\Phi_j(U)} + 1 \right) \]  
(6.26)

**Lemma 28.** If \( L \) be sufficiently large and define

\[
\mathcal{L}_j^l K_j(U) = \sum_{B=U} \sum_{X \subseteq B} (1 - \text{Loc}) K_j(B, X, U) \]  
(6.27)

then \( \mathcal{L}_j^l \) is contractive with arbitrarily small norm; namely, \( \|\mathcal{L}_j^l\| \to 0 \) as \( L \to \infty \).

**Proof.** Recall that \( \psi = P_{U^+} \phi + \tilde{\xi} \) and let

\[
H_{z, X}(U, \phi, \xi) = F_X^{(2)}(U, 0; \psi - x \cdot \partial \psi(z), \psi) \]  
(6.28)

then with \( \tilde{f} := P_{U^+} f + \lambda \xi \)

\[
H_{z, X}^{(1)}(U, \phi, \xi; (f, \lambda \xi)) = F_X^{(2)}(U, 0; \psi - x \cdot \partial \psi(z), \tilde{f} + F_X^{(2)}(U, 0; \tilde{f} - x \cdot \partial \tilde{f}(z), \psi) ;
\]

\[
H_{z, X}^{(2)}(U, \phi, \xi; (f_1, \lambda_1 \xi), (f_2, \lambda_2 \xi)) = F_X^{(2)}(U, 0; \tilde{f}_1 - x \cdot \partial \tilde{f}_1(z), \tilde{f}_2 + F_X^{(2)}(U, 0; \tilde{f}_2 - x \cdot \partial \tilde{f}_2(z), \tilde{f}_1) \]  
(6.29)

and \( H_{z, X}^{(3)} = 0 \). In the calculations here, though \( z \) is fixed, \( P_{U^+} \phi(z) \) should also participate in the differentiations: \( P_{U^+} \phi(z) \to P_{U^+} (\phi + tf)(z) \).

Similarly with the previous lemma,

\[
\|P_{U^+} f - x \cdot \partial P_{U^+} f(z)\|_{\Phi_j(X, U^+)} \leq O(L^{-\frac{d+2}{2}}) \|f\|_{\Phi_j(U)} \]  
(6.30)

Since \( \|\Phi_{j, (X, U^+)}\| \leq \|\Phi_{j, (X, U^+)}\| \leq L^{-d/2} \|\Phi_{j, (U)}\| \) we also have estimates

\[
\|\psi\|_{\Phi_j(X, U^+)} \leq O(L^{-d/2}) \left( \|\Phi\|_{\Phi_j(U)} + 1 \right) ;
\]

\[
\|P_{U^+} f\|_{\Phi_j(X, U^+)} \leq O(L^{-d/2}) \|f\|_{\Phi_j(U)} \]  
(6.31)

Combine (6.29), (6.30), (6.31) we have for \( n = 0, 1, 2 \)

\[
\left| H_{z, X}^{(n)}(U, \phi, \xi; (f, \lambda \xi)^n) \right| \leq O(L^{-d-1}) \left| \tilde{F}_X^{(2)}(U, 0) \right|_{T_f(\Phi_j(X, U^+))} \cdot \left( \|\Phi\|_{\Phi_j(U)} + 1 \right) 2^{-n} \prod_{i=1}^n \|f_i, \lambda_i \xi\|_{\Phi_j(U)} \]  
(6.32)
So by the same arguments as (6.12) and Lemma 16(5),

\[
\|H_{t,X}(U, \phi, \xi)\|_{T_\phi(U)} \leq O(L^{-d-1}) \|F_X^{(2)}(U, 0)\|_{T_\phi(U)} (1 + \|\phi\|_{U})^2
\]

\[
\leq O(L^{-d-1}) \|F_X^{(2)}(U, 0)\|_{T_\phi(U)} G(U, U^+)
\]

By Lemma 15, Lemma 18, Lemma 16(11) and \(X \in \mathcal{J}_j\)

\[
\|F_X^{(2)}(U, 0)\|_{T_\phi(U)} \leq \mathbb{E} \left[ \left. \|K_j(X, \phi, \xi = 0)\|_{T_\phi(U)} \|\phi(U)\| \right| \phi(U) = 0 \right]
\]

\[
\leq \mathbb{E} \left[ \left. \|K_j(X)\| G(X, X^+) \|\phi(U)\| \right| \phi(U) = 0 \right] \leq \|K_j(X)\|_{X_j}
\]

\[
\leq O(1) A^{-1} \|K_j\|_j
\]

Combining the above inequalities, we obtain

\[
\|H_{t,X}(U)\|_{j+1} \leq O(L^{-d-1}) A^{-1} \|K\|_j
\]

The other term in (6.21) is estimated similarly.

Therefore

\[
\|L_j^d K(U)\|_{j+1} \leq \sum B = U \sum \sum B \leq \frac{1}{|B|} O(L^{-d-1}) A^{-1} \|K\|_j \leq O(L^{-d-1}) A^{-1} \|K\|_j
\]

Since \(L_j^d K_j(U) = 0\) unless \(U\) is a block, \(\|L_j^d K_j\|_{j+1} \leq O(L^{-1}) \|K\|_j\).

Now we turn to \(LocK_j\). We observe that the coefficient of \(\partial_\mu \psi(z) \partial_\nu \psi(z)\) is \(4\) times

\[
\alpha_{\mu \nu}(B) := \frac{1}{2|B|} \sum_{X \in j, X \supset B} \frac{\partial^2}{\partial \xi_z} \mathbb{E}_\xi \left[ K_j(X, t_1 x_\mu + t_2 x_\nu + \xi) \right]
\]

Noticing \(\|x_\mu\|_{D_j} \leq h^{-1} d_j/2\) we have \(\|\alpha_{\mu \nu}(B)\| \leq O(1) h^{-2} \|K_j\|_j A^{-1}\).

Note that for a fixed \(D \in \mathcal{B}_{j+1}\), and for all \(B \in \mathcal{B}_{j}\), \(\alpha_{\mu \nu}(B)\) depends on the position of \(B\) in \(D\) because \(\xi\) is not translation invariant. This problem wasn’t present in the method [Bry09]. We cure this problem by the following lemma.

**Lemma 29.** Let \(D \in \mathcal{B}_{j+1}\), and let \(B_{ct} \in \mathcal{B}_j\) be the \(j\)-block at the center of \(D\). Then with definition (6.36),

\[
\left| \alpha_{\mu \nu}(B) - \alpha_{\mu \nu}(B_{ct}) \right| \leq O(L^{-d}) h^{-2} \|K_j\|_j A^{-1}
\]

for all \(B \in \mathcal{B}_j\) such that \(B = D\).

**Proof.** Let \(T\) be a translation so that \(TB = B_{ct}\), and \(\xi_{D^+, \xi_{TD^+}}\) be Gaussian fields on \(D^+, TD^+\) with Dirichlet Green’s functions \(C_{D^+}, C_{TD^+}\) as covariances respectively.

\[
\left| \alpha_{\mu \nu}(B) - \alpha_{\mu \nu}(B_{ct}) \right|
\]

\[
\leq \frac{1}{8|B_{ct}|} \sum_{X \in \mathcal{J}_j, X \supset B_{ct}} \left| \frac{\partial^2}{\partial \xi_z} \right|_{t_1 = 0} \left[ \mathbb{E}_\xi \left[ K_j(X, t_1 x_\mu + t_2 x_\nu + \xi_{TD^+}) \right] - \mathbb{E}_{\xi_D} \left[ K_j(X, t_1 x_\mu + t_2 x_\nu + \xi_{D^+}) \right] \right]
\]

33
To estimate the difference of the two expectations, define
\[ C(t) := tC_D^+ + (1-t)C_{TD}^+ \]
and recall that \( K_j \) depends on \( \zeta \) via \( \nabla \zeta \), let
\[ \mathcal{K}(\nabla \zeta) := K_j(X, t_1x_\mu + t_2x_\nu + \zeta) \]
Then, one has the formula
\[
\mathbb{E}_{\nabla^2 C(1)} \mathcal{K} - \mathbb{E}_{\nabla^2 C(0)} \mathcal{K} = \int_0^1 \frac{d}{dt} \mathbb{E}_{\nabla^2 C(t)} \mathcal{K} dt = \frac{1}{2} \int_0^1 \mathbb{E}_{\nabla^2 C(t)} \left[ \Delta \nabla^2 C(t) \mathcal{K} \right] dt
\]
where for any covariance \( C \) the Laplacian is defined as
\[
\Delta C := \sum_{x,y} C(x,y) \frac{\delta}{\delta \phi(x)} \frac{\delta}{\delta \phi(y)}
\]
In order to bound \( \nabla^2 \dot{C}(t) = \nabla^2 C_D^+ - \nabla^2 C_{TD}^+ \), let \( E = TD^+ \cup D^+ \). Clearly for any \( X \in \mathcal{R}, X \supset B_{\delta}, \) we have \( X^+ \subset TD^+ \cap D^+ \) and \( d(\partial E, \partial X^+) = O(L^{j+1}) \). Then, for \( x,y \in \partial X^+ \),
\[
|\nabla^2 C_{TD}^+ (x,y) - \nabla^2 C_D^+ (x,y)| \\
\leq \max \left\{ |\nabla^2 C_E (x,y) - \nabla^2 C_D^+ (x,y)|, |\nabla^2 C_E (x,y) - \nabla^2 C_{TD}^+ (x,y)| \right\} \\
\leq \max \left\{ |\nabla^2 P_{D^-} C_E (x,y), \nabla^2 P_{TD^-} C_E (x,y)| \right\}
\]
Since \( x,y \) have distance of \( O(L^{j+1}) \) from \( \partial D^+ \) and \( \partial TD^+ \), and \( C_E \) can be bounded by \( C_{2d}^j \), we can proceed as the arguments following (3.19) in proof of Lemma 11 or the arguments following (4.28) in proof of Lemma 15, the above expression is bounded by \( O(L^{-d(j+1)}) \). Note that one has \( \|x_\mu\|_\Phi \leq h^{-1} L^{d/2} \) and \( \|\delta_x\|_\Phi \leq h^{-1} L^{-d/2} \) and \( |X| = O(L^{d/2}) \), therefore
\[
|\alpha_{\mu \nu} (B) - \alpha_{\mu \nu} (B_{D_k})| \leq O(L^{-d}) h^{-4} \|K_j\|_J A^{-1}
\]
which proves the Lemma.

Let \( D \in \mathcal{B}_{j+1} \). Define \( \alpha_{\mu \nu} := \alpha_{\mu \nu} (B_{D_k}) \) where \( B_{D_k} \in \mathcal{B}_j \) is at the center of \( D \). Clearly it’s well defined (independent of \( D \)). By reflection and rotation symmetries, there exists an \( \alpha \) so that \( \alpha_{\mu \nu} = \frac{1}{2} \alpha (\tilde{\delta}_{\mu \nu} + \delta_{\mu,-\nu}) \).

**Lemma 30.** With \( \psi := P_U \phi + \bar{\xi} \)
\[
\mathcal{L}^\alpha := \frac{1}{4} \sum_{B \in D} \left( \sum_{x \in B, x \notin B_{\delta}} \alpha (\partial_x \psi(x))^2 - \sum_{x \in B, x \notin B_{\delta}} \alpha_{\mu \nu} (\partial_x \psi(x))^2 \right)
\]
is contractive.
Proof. This is essentially Lemma 10 of [Dim09], so the proof is omitted. \qed

**Proposition 31.** We can choose \( E_{j+1} \) and \( \sigma_{j+1} \) so that if \( L \) be sufficiently large then \( \mathcal{L}_3 \) in Proposition 22 is contractive.

**Proof.** As the first step with \( D = \mathcal{E}_j \in \mathcal{D}_1(\Lambda), \phi = P_D, \phi + \zeta \) we compute

\[
\mathbb{E} \left[ \sum_{x \in B, \rho \in \mathcal{E}} (\partial_x P_B \phi(x) + \partial_x \xi(x))^2 | (D^+)^c \right] = \sum_{x \in B, \rho \in \mathcal{E}} (\partial_x P_B \phi(x) + \partial_x \xi(x))^2 + \delta E_j \tag{6.41}
\]

where \( \delta E_j = \sum_{x \in B, \rho \in \mathcal{E}} \mathbb{E} \left[ (\partial_x P_B \xi)^2 \right] = O(1) \) by Lemma 12.

Let \( \psi = P_D, \phi + \zeta \). By Lemma 28, Lemma 29 and Lemma 30 it remains to show the contractivity of

\[
\mathcal{L}_3 = \sum_{B = U} \left[ E_{j+1}(B) + \frac{\sigma_{j+1}}{4} \sum_{x \in B, \rho \in \mathcal{E}} (\partial_x \psi(x))^2 - \frac{\sigma_j}{4} \left( \sum_{x \in B, \rho \in \mathcal{E}} (\partial_x \psi(x))^2 + \delta E_j \right) \right.
\]

\[
+ \mathbb{E} \left[ K_j(X, \zeta) \right] + \frac{\alpha}{4} \sum_{x \in B, \rho \in \mathcal{E}} (\partial_x \psi(x))^2 \right] \tag{6.42}
\]

Choose

\[
\sigma_{j+1} = \sigma_j - \alpha \\
E_{j+1} = \sigma_j \delta E_j - \mathbb{E} \left[ K_j(X, \zeta) \right] \tag{6.43}
\]

then we actually have \( \mathcal{L}_3 = 0 \). \qed

By the above choice of \( E_{j+1} \) we can easily see that it’s the same number for \( Z_N(\xi) \) and \( Z_N(0) \). Therefore \( e^{\xi} \) is the same for \( Z_N(\xi) \) and \( Z_N(0) \), for all \( j \).

**7 Proof of scaling limit of the generating function**

**Proposition 32.** Let \( L \) be sufficiently large; \( A \) sufficiently large depending on \( L; \) \( \kappa \) sufficiently small depending on \( L, A \); \( h \) sufficiently large depending on \( L, A, \kappa \); and \( r \) sufficiently small depending on \( L, A, \kappa, h \). Then for \( |z| < r \) there exists a constant \( \sigma \) depending on \( z \) so that the dynamic system

\[
\sigma_{j+1} = \sigma_j + \alpha(K_j) \\
K_{j+1} = \mathcal{L} K_j + f(\sigma_j, K_j) \tag{7.1}
\]

satisfies

\[
|\sigma_j| \leq 2^{-j} \\
\|K_j\| \leq r 2^{-j} \tag{7.2}
\]

**Proof.** By contractivity of \( \mathcal{L} \) we apply Theorem 2.16 in [Bry09] (i.e. the stable manifold theorem) to obtain a smooth function \( \sigma = h(K_0) \) so that (7.2) hold. Since \( K_0 \) depends on \( z \) and \( \sigma \), we solve \( \sigma \) from equation \( \sigma - h(K_0(z, \sigma)) = 0 \), using Lemma 42. Noting that this equation holds with \( (\sigma, z) = 0 \), and that \( K_0(z = 0, \sigma) = 0 \), the derivative of left hand side w.r.t. \( \sigma \) is 1. So by implicit function theorem there exists a \( \sigma \) depending on \( z \) so that \( \sigma = h(K_0(z, \sigma)) \). Therefore the proposition is proved. \qed
With the generating function $Z_N(f)$ defined in (2.1), we have

**Theorem 33.** For any $p > d$ there exists constants $M > 0$ and $z_0 > 0$ so that for all $\| \tilde{f} \|_{L^p} \leq M$, and all $|z| \leq z_0$ there exists a constant $\varepsilon$ depending on $z$ so that

$$\lim_{N \to \infty} Z_N(f) = \exp\left(-\frac{1}{2} \int_{\tilde{\Lambda}} \tilde{f}(x) (-\varepsilon \tilde{\Lambda})^{-1} \tilde{f}(x) d^d x \right)$$

where $\tilde{\Lambda}$ is the Laplacian in continuum.

**Proof.** By (2.5),

$$Z_N(f) = \lim_{m \to 0} e^{\frac{1}{2} \sum_{\xi \in \Lambda} f(x)(-\varepsilon \Delta_m)^{-1} f(x)} Z'_N(\xi)/Z'_N(0)$$

(7.3)

In fact, since $\int_{\tilde{\Lambda}} \tilde{f} = 0$

$$e^{\frac{1}{2} \sum_{\xi \in \Lambda} f(x)(-\varepsilon \Delta_m)^{-1} f(x)} \to e^{-\frac{1}{2} \int_{\tilde{\Lambda}} \tilde{f}(x) (-\varepsilon \tilde{\Lambda})^{-1} \tilde{f}(x) d^d x}$$

(7.4)

as $m \to 0$ followed by $N \to \infty$.

At scale $N - 1$ (we don’t want to continue all the way to the last step since it would be a bit awkward to define $I_{N-1}$ and $I_N$), by Prop. 32 and Lemma 18

$$|Z'_N(\xi) - e^{\delta N - 1}| = e^{\delta N - 1} |E[I_{N-1} \delta K_{N-1}] - 1|$$

$$\leq e^{\delta N - 1} \left[ \sum_{\theta \neq X \in \mathbb{P}_{N-1}} (1 + 2^{-N+1} |X|_{N-1} 2^{-N+1} E|G(X, X^+) - 1| + |I_{N-1}^A - 1| \right]$$

$$\leq e^{\delta N - 1} \left( \sum_{\theta \neq X \in \mathbb{P}_{N-1}} (1 + 2^{-N+1} |X|_{N-1} 2^{-N+1} c |X|_{N-1} + 2^{-N+1}) \right)$$

(7.5)

Since the constant $e^{\delta N - 1}$ is identical for $Z'_N(\xi)$ and $Z'_N(0)$, and $Z'_N(0)$ satisfies the same bound above, so $Z'_N(\xi)/Z'_N(0) \to 1$. Therefore the theorem is proved. \qed

## A Decay of Green’s functions and Poisson kernels

The decay rates of Green’s functions and their derivatives are essential in our method. First of all consider the Green’s function of $-\Delta_m = -\Delta + m^2$ on $\mathbb{Z}^d$. If $d \geq 3$, let $G_m = (-\Delta_m)^{-1}$. If $d = 2$ let $G_m(x) = (-\Delta_m)^{-1}(x) - (-\Delta_m)^{-1}(0)$ for $m > 0$ and from [Law91] we know that $\lim_{m \to 0} G_m(x)$ exists. Write $\tilde{G} = G_{m=0}$.

**Lemma 34.** Let $\tilde{G}(x) = a_d |x|^{-d}$ if $d \geq 3$ or $\tilde{G}(x) = a_d \log |x|$ if $d = 2$ where $a_d$ only depends on $d$. Let $k = \frac{2 \gamma - \log 8}{\pi}$ if $d = 2$ where $\gamma$ is Euler’s constant and $k = 0$ if $d \geq 3$. As $|x| \to \infty$

$$G(x) = \tilde{G}(x) + k + O(|x|^{-d})$$

(A.1)

Furthermore, for all $e \in \mathcal{E}$

$$\partial_e G(x) = \partial_e \tilde{G}(x) + O(|x|^{-(d+1)})$$

(A.2)

where $\partial_e \tilde{G}(x)$ is also discrete derivative.
Proof. See [LL10] Theorem 4.3.1, 4.4.4, Corollary 4.3.3, 4.4.5. The only difference here is a sharper estimate of the error term for $\nabla G$, which is remarked after those corollaries and thus the proof is omitted.

Lemma 35. Let $d \geq 2$. For all $e \in \mathcal{E}$, $x \in \Lambda$ where $\Lambda$ is the torus defined in subsection 2.2 and $m \geq 0$,

$$\left| \sum_{y \in \mathbb{Z}^d \setminus \{0\}} \partial_e G_m(x + L^N y) \right| \leq c_d L^{-(d-1)N} \quad (A.3)$$

where $c_d$ only depends on $d$.

Remark 36. Note that the left hand side is not absolutely summable uniformly in $m \geq 0$.

Proof. It’s enough to show the proof for $m = 0$. Denote $D_{\mu}$ to be the smooth derivative. Without loss of generality assume $e = e_1$. The term $O(|x|^{-(d+1)})$ in (A.2) is summable:

$$\left| \sum_{y \in \mathbb{Z}^d \setminus \{0\}} O(|x + L^N y|^{-(d+1)}) \right| = O(L^{-(d+1)N}) \quad (A.4)$$

Up to this term, $\partial_{e_1} G(x + L^N y)$ is equal to

$$\tilde{G}(x + e_1 + yL^N) - \tilde{G}(x + yL^N)$$

$$= \left( \tilde{G}(yL^N) + (x + e_1) \cdot D\tilde{G}(yL^N) + \frac{1}{2} (x + e_1)^2 \cdot D^2 \tilde{G}(yL^N) + O(L^{2N} \sup |D^3 \tilde{G}|) \right)$$

$$- \left( \tilde{G}(yL^N) + x \cdot D\tilde{G}(yL^N) + \frac{1}{2} x^2 \cdot D^2 \tilde{G}(yL^N) + O(L^{2N} \sup |D^3 \tilde{G}|) \right)$$

$$= D_1 \tilde{G}(yL^N) + (x \cdot DD_1 \tilde{G}(yL^N) + \frac{1}{2} D_1^2 \tilde{G}(yL^N)) + O(L^{2N} \sup |D^3 \tilde{G}|)$$

where the last term comes from Taylor remainder theorem. It’s a straightforward calculation to see that the summation over $y \neq 0$ of the first three terms is zero due to cancellations. The summation over $y \neq 0$ of the last term gives $O(L^{-(d-1)N})$.

Corollary 37. Let $d \geq 2$ and $C_m$ be the Green’s function of $-\Delta + m^2$ on the torus $\Lambda$. For all $e \in \mathcal{E}$, $x \in \Lambda$ and $m \geq 0$,

$$|\partial_e C_m(x)| \leq c_d |x|^{-(d-1)} \quad (A.6)$$

where $c_d$ only depends on $d$.

Proof. The statement is immediately shown by

$$\partial_e C_m(x) = \sum_{y \in \mathbb{Z}^d} \partial_e G_m(x + L^N y) \quad (A.7)$$

and Lemma 34, 35. 

37
Lemma 38. Let $\mathcal{K}_R = \{ x \in \mathbb{Z}^d : x_k \in [1, R-1], k = 1, \ldots, d \}$ be a cube of size $R$. If $d(x, \partial \mathcal{K}_R) > R/3$, $y \in \partial \mathcal{K}_R$ then

$$P_{\mathcal{K}_R}(x, y) \leq \frac{O(1)}{R^{d-1}}$$

Furthermore, there exists constants $\gamma_d < \bar{\gamma}_d \in (0, \frac{1}{2})$ such that if $d(x, \partial \mathcal{K}_R) \in (\gamma_d R, \bar{\gamma}_d R)$, $y \in \partial \mathcal{K}_R$ then

$$|\nabla P_{\mathcal{K}_R}(x, y)| \geq \frac{O(1)}{R^d}$$

Proof. It’s enough to prove it for large $R$. Without loss of generality we assume that $y \in \partial_1 \mathcal{K}_R = \{ x \in \partial \mathcal{K}_R : x_1 = R \}$. The Poisson kernel $P_{\mathcal{K}_R}(x, y)$ associated to the standard Laplacian is, by Prop 8.1.3 of [LL10], equal to

$$\left( \frac{2}{R} \right)^{d-1} \sum_{z \in \partial_1 \mathcal{K}_R} \frac{\sinh(\alpha z_1 \pi / R)}{\sinh(\alpha \pi / R)} \prod_{i=2}^d \sin \left( \frac{z_i \pi}{R} \right) \prod_{i=2}^d \sin \left( \frac{z_i \pi}{R} \right)$$

where $\alpha_c$ is the unique nonnegative number satisfying

$$\cosh\left( \frac{\alpha_c \pi}{R} \right) + \sum_{i=2}^d \cos \left( \frac{z_i \pi}{R} \right) = d$$

Then the bounds can be obtained from the above formula by straightforward checking, noticing that $\sinh(cz)/\sinh(z)$ decays exponentially for $c < 1$, and $\alpha_c$ grows linearly in $z$. \qed

B The initial expansion

Consider equation (2.6), following Mayer expansion,

$$Z'_N(\xi) = E \left[ e^{iW(\Lambda) - \sigma V(\Lambda)} \right] = E \left[ \prod_{x \in \Lambda} \left( e^{-\sigma V(\{x\})} + \left( e^{iW(\{x\})} - 1 \right) e^{-\sigma V(\{x\})} \right) \right]$$

$$= E \left[ \sum_{X \in \mathcal{P}_0} I_0^X K_0(X) \right] = E \left[ (I_0 \circ K_0) (\Lambda) \right]$$

where we have defined $I_0 \in \mathcal{N} \mathcal{P}_0$ and $K_0 \in \mathcal{N} \mathcal{P}_{0,c}$ as

$$I_0(\{x\}) = e^{-\sigma V(\{x\})} = \exp \left( -\frac{\sigma}{4} \sum_{c \in \mathcal{E}} (\partial_c \phi(x) + \partial_c \xi(x))^2 \right)$$

$$K_0(X) = \prod_{x \in X} \left( e^{iW(\{x\})} - 1 \right) e^{-\sigma V(\{x\})}$$

This proves the statement (2.8).
C Estimates

In Section 4 we defined norms for functions of the fields. In the Appendix we give estimates in terms of these norms of some functions of interest.

**Lemma 39.** There exists a constant $c > 0$ so that if $\sigma / \kappa < c$ and $h^2 \sigma < c$, $B \in \mathcal{B}_j$, $j < N - 1$,

$$\|e^{-\frac{\sigma}{2} \sum_{x \in B, \varepsilon}(\partial_x P_{B^+} \phi(x) + \partial_x \xi(x))^2}\|_{T_{\mathcal{B}}(\Phi_j(B))} \leq 2e^{\frac{\sigma}{2} \sum_{\partial P_{B^+} \phi}^2} \tag{C.1}$$

$\|e^{-\frac{\sigma}{2} \sum_{x \in B, \varepsilon}(\partial_x P_{B^+} \phi(x) + \partial_x \xi(x))^2}\|_{T_{\mathcal{B}}(\Phi_j(\hat{B}, \hat{B}^+))} \leq 2e^{\frac{\sigma}{2} \sum_{\partial P_{B^+} \phi}^2} \tag{C.2}$

And

$$\|e^{-\frac{\sigma}{2} \sum_{x \in B, \varepsilon}(\partial_x P_{B^+} \phi(x) + \partial_x \xi(x))^2} - 1\|_{T_{\mathcal{B}}(\Phi_j(B))} \leq 4e^{-1}h^2 \|e^{\frac{\sigma}{2} \sum_{\partial P_{B^+} \phi}^2}\| \tag{C.3}$$

$$\|e^{-\frac{\sigma}{2} \sum_{x \in B, \varepsilon}(\partial_x P_{B^+} \phi(x) + \partial_x \xi(x))^2} - 1\|_{T_{\mathcal{B}}(\Phi_j(\hat{B}, \hat{B}^+))} \leq 4e^{-1}h^2 e^{\frac{\sigma}{2} \sum_{\partial P_{B^+} \phi}^2} \tag{C.4}$$

**Remark 40.** Note that the prefactors on the exponentials on the right hand sides are always $\frac{\sigma}{2}$, whereas the prefactor in our regulator defined in Section 4 is $\frac{\xi}{2}$.

**Proof.** To prove (C.1), let

$$V = -\frac{1}{2} \sum_{x \in B, \varepsilon} (\partial_x P_{B^+} \phi(x) + \partial_x \xi(x))^2$$

and let $\|(f, \lambda \xi)^{\times n}\|_{\Phi_j(B)} \leq 1$. By $|\partial_x \xi|^2 \leq h^2 L^{-d} N$ it’s straightforward to check that if $\sigma / \kappa$ is sufficiently small, for $n = 0, 1, 2$,

$$\left| (\sigma V)^{(n)}(\phi, \xi; (f, \lambda \xi)^{\times n}) \right| \leq \frac{\kappa}{2^{n+1}} \sum_{x \in B, \varepsilon} (\partial_x P_{B^+} \phi(x))^2 + 2\sigma h^2 \tag{C.5}$$

and for $n \geq 3$, $V^{(n)} = 0$. Therefore for $n = 0, \ldots, 4$

$$\frac{1}{n!} \left| (e^{\sigma V})^{(n)}(\phi, \xi; (f, \lambda \xi)^{\times n}) \right| \leq e^{\sigma V} e^{(\sigma V)^{(1)} + (\sigma V)^{(2)}}$$

$$\leq e^{\frac{\sigma}{2} \sum_{x \in B, \varepsilon}(\partial_x P_{B^+} \phi(x))^2 + 8\sigma h^2}$$

$$\leq 2e^{\frac{\sigma}{2} \sum_{x \in B, \varepsilon}(\partial_x P_{B^+} \phi(x))^2}$$

if $h^2 \sigma$ is sufficiently small, where we bounded the polynomials in $\sigma V^{(n)}$ by $e^{|\sigma V^{(1)}| + |\sigma V^{(2)}}$. So (C.1) is proved. (C.2) is proved in the same way.

To prove (C.3), note that similarly as above one can show that $e^{\sigma V}$ is analytic in $\sigma$, so

$$\|e^{\sigma V} - 1\|_{T_{\mathcal{B}}(\Phi_j(B))} = \left| \frac{1}{2\pi i} \int_{|z| = c h^2} \frac{\sigma e^{zV}}{z - \sigma} dz \right|_{T_{\mathcal{B}}(\Phi_j(B))} \leq 4e^{-h^2} \|e^{\frac{\sigma}{2} \sum_{\partial P_{B^+} \phi}^2}\|$$

and (C.4) is proved in the same way. \hfill $\square$
Another example is the estimate of the initial interaction. At step $j = 0$ a block $B$ is a single lattice point $x$. Define

$$
\hat{W}(\{x\}, \phi, u) = \frac{1}{2} \sum_{e \in \mathcal{E}, x + e \in \Lambda} \cos(u \partial_e \phi(x))
$$

We also write $W(\{x\}, \phi) = \hat{W}(\{x\}, \phi, \sqrt{B(1 + \sigma)})$.

**Lemma 41.** If $\kappa \geq h^{-1}$, then

1) $\hat{W}(\{x\}, \phi, u)$ satisfies

$$
\sum_{n=0}^{3} \frac{1}{n!} \sup_{|\partial f(x)| \leq h} \left| \partial_{e_1 \ldots e_n} \right|_{t=0} \left| \partial_u^m W(\{x\}, \phi + \sum_{i=1}^{n} t_i f_i) \right| \leq (2h)^m e^{\frac{h}{2}} \sum_{e \in \mathcal{E}} (\partial_e \phi(x))^2
$$

(C.6)

for $m = 0, 1, 2, \ldots$. 

2) Let $|| - ||_0$ be the $|| - ||_0$ norm with $G = 1$. For $|z|$ sufficiently small,

$$
||e^{zW(B)}||_0 \leq 2
$$

(C.7)

**Proof.** 1) The case $m = 0$ holds even without $e^{\frac{h}{2}} \sum_{e \in \mathcal{E}} (\partial_e \phi(x))^2$ by straightforward computations and thus is omitted. For $m > 0$,

$$
\partial_u^m W = \pm \frac{1}{2} \sum_{e \in \mathcal{E}, x + e \in \Lambda} \sin(\cos(u \partial_e \phi(x))) (\partial_e \phi(x))^m
$$

(C.8)

and we bound

$$
\sum_{n=0}^{4} \left| \partial_{e_1 \ldots e_n} \right|_{t=0} \left( \partial_u \phi(x) + \sum_{i=1}^{n} t_i f_i(x) \right)^m \leq (2h)^m e^{\frac{h}{2}} \sum_{e \in \mathcal{E}} (\partial_e \phi(x))^2
$$

(C.9)

The bound for $\partial_u^m W$ follows by product rule of differentiations and the case $m = 0$.

2) For $|z|$ sufficiently small,

$$
||e^{zW(B)}||_0 \leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!} ||W(B)||_0 \leq \exp \left( 4d |z|^h \right) \leq 2
$$

(C.10)

**Lemma 42.** Given $r > 0$, if $|z|$ and $|\sigma|$ are sufficiently small, then $||K_0|| < r$. Furthermore, $K_0$ is smooth in $z$ and $\sigma$.

**Proof.** As in the proof of (C.7), one has

$$
||e^{zW(B)} - 1||_0 \leq \exp \left( 4d |z|^h \right) - 1 \leq c |z|
$$

for some constant $c$. Write $V_0(B) = -\frac{1}{2} \sum_u (\partial_u \phi(x) + \partial_u \xi(x))^2$. By Lemma 39

$$
||(e^{zW(B)} - 1)e^{\sigma V_0(B)}||_0 \leq 2c |z|
$$

40
therefore
\[ \| K_0 \|_0 = \sup_{X \in \mathcal{F}_{0,0}} \| K_0(X) \|_0 A^{X|0} \leq \sup_{X \in \mathcal{F}_{0,0}} (2c|z|A)^{X|0} < r \]

The derivative of \( \prod_{B \in \mathcal{F}_{0,0}} (e^{zW(B)} - 1) \) w.r.t. \( \sigma \) is equal to
\[ \sum_{B_0 \subseteq X} zW'(B_0) \frac{1}{2\sqrt{1 + \sigma}} \prod_{B \subseteq X \setminus B_0} (e^{zW(B)} - 1) \]
therefore its \( \| - \|_0 \) norm is bounded by \( c'A|z| \) for some constant \( c' \). The derivative of \( e^{\sigma V_0(B)} \) and higher derivatives can be bounded similarly. The derivative of \( \prod_{B \in \mathcal{F}_{0,0}} (e^{zW(B)} - 1) \) w.r.t. \( z \) is equal to
\[ \sum_{B_0 \subseteq X} W(B_0) \prod_{B \subseteq X \setminus B_0} (e^{zW(B)} - 1) \]
which can be bounded in the same way. \( \square \)

References

[AKM13] Stefan Adams, Roman Kotecký, and Stefan Müller. Finite range decomposition for families of gradient Gaussian measures. J. Funct. Anal., 264(1):169–206, 2013.

[Bał83] Tadeusz Balaban. Ultraviolet stability in field theory. The \( \phi^4 \) model. In Scaling and self-similarity in physics (Bures-sur-Yvette, 1981/1982), volume 7 of Progr. Phys., pages 297–319. Birkhäuser Boston, Boston, MA, 1983.

[Bau12] Roland Bauerschmidt. A simple method for finite range decomposition of quadratic forms and gaussian fields. arXiv preprint arXiv:1206.2212, 2012.

[BBS12] R. Bauerschmidt, D.C. Brydges, and G. Slade. Structural stability of a dynamical system near a non-hyperbolic fixed point. arXiv preprint arXiv:1211.2477, 2012.

[BDH95] D. Brydges, J. Dimock, and T. R. Hurd. The short distance behavior of \( (\phi^4)_3 \). Comm. Math. Phys., 172(1):143–186, 1995.

[BDH98] D. Brydges, J. Dimock, and T. R. Hurd. A non-Gaussian fixed point for \( \phi^4 \) in \( 4 - \epsilon \) dimensions. Comm. Math. Phys., 198(1):111–156, 1998.

[BGM04] David C. Brydges, G. Guadagni, and P. K. Mitter. Finite range decomposition of Gaussian processes. J. Statist. Phys., 115(1-2):415–449, 2004.

[BIS09] David C. Brydges, John Z. Imbrie, and Gordon Slade. Functional integral representations for self-avoiding walk. Probab. Surv., 6:34–61, 2009.

[BMS03] D. C. Brydges, P. K. Mitter, and B. Scoppola. Critical \( (\Phi^4)_{3,\epsilon} \). Comm. Math. Phys., 240(1-2):281–327, 2003.
[Bry09] David C. Brydges. Lectures on the renormalisation group. In *Statistical mechanics*, volume 16 of *IAS/Park City Math. Ser.*, pages 7–93. Amer. Math. Soc., Providence, RI, 2009.

[BS10] David Brydges and Gordon Slade. Renormalisation group analysis of weakly self-avoiding walk in dimensions four and higher. In Proceedings of the International Congress of Mathematicians. Volume IV, pages 2232–2257, New Delhi, 2010. Hindustan Book Agency.

[BT06] David Brydges and Anna Talarczyk. Finite range decompositions of positive-definite functions. *J. Funct. Anal.*, 236(2):682–711, 2006.

[BY90] David Brydges and Horng-Tzer Yau. Grad φ perturbations of massless Gaussian fields. *Comm. Math. Phys.*, 129(2):351–392, 1990.

[Del99] Thierry Delmotte. Parabolic Harnack inequality and estimates of Markov chains on graphs. *Rev. Mat. Iberoamericana*, 15(1):181–232, 1999.

[DH00] J. Dimock and T. R. Hurd. Sine-Gordon revisited. *Ann. Henri Poincaré*, 1(3):499–541, 2000.

[Dim09] J. Dimock. Infinite volume limit for the dipole gas. *J. Stat. Phys.*, 135(3):393–427, 2009.

[Dim11] Jonathan Dimock. The renormalization group according to balaban - i. small fields. *arXiv preprint arXiv:1108.1335*, 2011.

[Fal12] Pierluigi Falco. Kosterlitz-Thouless transition line for the two dimensional Coulomb gas. *Comm. Math. Phys.*, 312(2):559–609, 2012.

[FP78] Jürg Fröhlich and Yong Moon Park. Correlation inequalities and the thermodynamic limit for classical and quantum continuous systems. *Comm. Math. Phys.*, 59(3):235–266, 1978.

[FS81a] Jürg Fröhlich and Thomas Spencer. Kosterlitz-Thouless transition in the two-dimensional plane rotator and Coulomb gas. *Phys. Rev. Lett.*, 46(15):1006–1009, 1981.

[FS81b] Jürg Fröhlich and Thomas Spencer. The Kosterlitz-Thouless transition in two-dimensional abelian spin systems and the Coulomb gas. *Comm. Math. Phys.*, 81(4):527–602, 1981.

[FS81c] Jürg Fröhlich and Thomas Spencer. On the statistical mechanics of classical Coulomb and dipole gases. *J. Statist. Phys.*, 24(4):617–701, 1981.

[Gal85] Giovanni Gallavotti. Renormalization theory and ultraviolet stability for scalar fields via renormalization group methods. *Rev. Modern Phys.*, 57(2):471–562, 1985.
[GK80] K. Gawędzki and A. Kupiainen. A rigorous block spin approach to massless lattice theories. *Comm. Math. Phys.*, 77(1):31–64, 1980.

[GK83] K. Gawędzki and A. Kupiainen. Block spin renormalization group for dipole gas and $(\nabla \varphi)^4$. *Ann. Physics*, 147(1):198–243, 1983.

[Kum10] T. Kumagai. Random walks on disordered media and their scaling limits. *Notes of St. Flour lectures, available at http://www.kurims.kyoto-u.ac.jp/ kumagai/*, 2010.

[Law91] Gregory F. Lawler. *Intersections of random walks*. Probability and its Applications. Birkhäuser Boston Inc., Boston, MA, 1991.

[LL10] Gregory F. Lawler and Vlada Limic. *Random walk: a modern introduction*, volume 123 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.