Analysis of data
on low energy $\pi N \rightarrow \pi\pi N$ reaction.
I. Total cross sections. *

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Abstract

This is the first of a series of papers on a consistent model independent analysis of
the complete experimental information on the reaction $\pi N \rightarrow \pi\pi N$ at pion momenta
up to 500 MeV/c. The paper summarizes the theoretical approach and details of
the computational procedure. The complete database on total cross sections in 5
$\pi\pi N$ channels is given together with a critical discussion of their model independent
analysis.

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1 Introduction

The $\pi N \rightarrow \pi \pi N$ processes near threshold is currently a subject of intensive investigations, both theoretically and experimentally [1–25]. There are two main reasons why this process has attracted growing interest: a) the development of experimental techniques made it possible to measure in the nearest neighborhood of threshold not only total cross sections but also angular distributions; b) reliable data on the parameters of low energy $\pi \pi$ interaction are necessary for a precise estimate of higher order corrections in Chiral Perturbation Theory (ChPT), which gives a link between QCD and the world of hadrons (see, for example, [26]). The reaction in question provides an excellent opportunity to get the parameters appearing in the ChPT expansions [27, 28, 29].

In view of the importance of reliable and sufficiently precise results (which has been specially stressed in a recent paper [30], devoted to the problem of estimating of higher order terms of ChPT) it is desirable to perform a complete, model independent analysis based on the full set of available experimental information on $\pi N \rightarrow \pi \pi N$ reaction near threshold. The main goal of such analysis would be to establish the phenomenological form of the reaction amplitude in the low energy domain and to fix (with the help of data) the values of the corresponding parameters appearing in this form.

It is clear that this rather complex task is preferentially solved in various steps, as the systematic experimental and, especially, theoretical investigation of the low energy dynamics of the reaction in question is, in fact, just beginning. The problem of the extraction of low energy $\pi \pi$ – interaction parameters with a desirable accuracy looks, in such a situation, extremely difficult. At the same time, the successive resolution of this problem is important by its own, irrespective of the following application of the results: the pion, which is the lightest hadron, participates — really or virtually — in every strong interaction process (see, for example, [28, 31, 32]). This is a reason why the complete phenomenological analysis of the data on low energy $\pi N \rightarrow \pi \pi N$ reaction is really of actual interest.

In the present paper we make a first step in this direction. It combines two points: it includes the phenomenological ansatz for the amplitude (the generalization of the standard approach used in the isotopic analysis) and the corresponding analysis of the full set of experimental data on total cross sections of 5 channels at $P_{lab} \leq 500 MeV/c$. The goal of this paper is to answer the following questions:

1. Is the modern database on total cross sections internally consistent or are there in a statistical sense “doubtful” points?

2. How strong is the comparative influence of the various mechanisms ($\pi$ – and $\Delta$– exchanges, “background”) on the $\pi N \rightarrow \pi \pi N$ reaction amplitude?

3. What is the capability of data to distinguish between the different contributions?
4. Is it possible to extract the values of the $\pi N \rightarrow \pi \pi N$ amplitude parameters from the data on total cross sections only (of particular interest being the values of the parameters describing the low energy $\pi\pi$–interaction)?

The new precise data on $\pi N \rightarrow \pi \pi N$ total cross sections, which became available in the course of the last years (see Table 1) support the common expectation to contain accurate information on the low energy $\pi\pi$ interaction near threshold. As pointed out above, this information would be of great importance in connection with the problem of a quantitative description of the low energy hadron dynamics from QCD.

To extract the information on $\pi\pi$–scattering from the data on $\pi N \rightarrow \pi \pi N$ reactions, one has to perform a refined analysis, because the detailed mechanism of the latter reaction is not understood in detail and the influence of the $4\pi$–vertex being not necessarily dominant. Its contribution could be (if at all) separated among all others with the help of an accurate analysis of the full set of experimental data including those on differential cross sections.

At this moment, however, the overwhelming majority of recent results on $\pi\pi$–parameters is based on analyses of data on total cross sections only. Most of the analyses having been performed with the help of the Olsson–Turner (OT) model \cite{56} (for a review see \cite{20}). This situation is similar to that of the seventies (see, for example, refs. \cite{27,28}). That time the most popular method of the corresponding data analysis was the Chew–Low extrapolation procedure \cite{57}. This method, however, proved to be not sufficiently reliable, the main uncertainty originating from the phenomenological features of the $\pi N \rightarrow \pi \pi N$ reaction: the contribution of the OPE graph turned out to be small in comparison with those of the other graphs (see, for example, the recent papers \cite{58,59,60}). Due to this reason the results of the extrapolation of experimental data to the unphysical point $\tau = \mu^2$ become extremely sensitive to small variations (errors) of the data used. From the purely mathematical point of view this effect is just a manifestation of J. Hadamard’s general theorem concerning the ambiguities of extrapolation problems (see ref. \cite{33}).

The same situation repeats now again. A good example is given by the comparison of the results on $S$–wave $\pi\pi$–scattering lengths obtained by different groups, where the most interesting conclusion has been made by the OMICRON group in ref. \cite{8}: a single parameter $\xi$ of the OT model is not sufficient for the adequate description of the low energy data on total cross sections of three channels ($\pi^-\pi^+n$, $\pi^-\pi^0p$, $\pi^+\pi^+n$). It should be stressed here that from the purely technical point of view the analysis made by this group is one of the most accurate, because it accounts for all specific features of the OT model. It looks much as if the origin of the difficulty mentioned above is hidden either in these very specific features of the OT model or in the inconsistency of the data.

The original formulae given in ref. \cite{56} are valid only for the threshold values of the amplitudes. Thus, to extract the value of $\xi$ (or, equivalently, $S$–wave scattering lengths) one has to perform the extrapolation of the experimental data on total cross sections to the threshold in all 5 channels of the $\pi N \rightarrow \pi \pi N$ reaction. On this way, however, one meets at least two serious obstacles closely connected to each other. First: to reduce
errors one must use for the extrapolation the datapoints nearest to threshold. Second: one needs to know the form of the extrapolation law. As it is well known the low energy measurements exhibit extremely delicate experimental problems by itself, the difficulties increase approaching to threshold. Thus, those very points which would be the most suitable for the extrapolation, are the most difficult to measure. It must be mentioned that — in addition to purely experimental difficulties — there is the unresolved theoretical problem of data interpretation: in the near threshold domain the radiative corrections become significant. Furthermore, the form of the extrapolation law is more or less known for the case of elastic scattering only (due to general requirements given by quantum mechanics, such as partial wave expansion, unitarity, etc.). In the case under consideration — the $\pi N \rightarrow \pi \pi N$ reaction — these limitations are considerably weaker and the extrapolation law itself is still unknown.

All these obstacles result in large systematic errors of the parameters to be determined. Besides, needless to mention, the OT model — as every other one — has its own limits of applicability.

Out of this situation we started a general analysis of the phenomenological features of the total cross sections of the $5 \pi N \rightarrow \pi \pi N$ channels in the low energy domain ($p_{lab} \leq 500 \text{MeV}/c$). The main goal of this analysis is to get a deeper understanding of the phenomenology of the process and to estimate the informative capacity and selfconsistency of the modern database.

This paper is the first one of a series devoted to the systematic model independent analysis of all available experimental information on the low energy $\pi N \rightarrow \pi \pi N$ reaction. It contains — along with the analysis of data on total cross sections — the detailed description of the theoretical approach and the computational procedure, as both subjects are important for the subsequent analysis of distributions also. That is why the text is divided into 4 sections. The details of the approach are discussed in Sec.2, where we present also the complete database on total cross sections. Sec.3 contains the results of the theoretical analysis of the experimental situation and those of the data fitting. The discussion of the results obtained and the concluding statements are given in Sec.4.

2 Theoretical approach

2.1 Decomposition of the amplitude

The main ingredients of our approach are given in refs. [4, 5, 19]. Here, we just want to recall some important points and give an extension of the formalism.

Our ansatz is model independent in the sense that we do not make a priori suggestions on the form of interactions (other than the well known symmetry requirements), coupling constants (except the $\pi NN$ coupling) or on the off–shell behaviour of the different vertices. It is based on the exact symmetry principles (Lorentz–invariance, isotopic and crossing symmetry, C–, P– and T–invariance and Bose–statistics for pions) and some well
established features of the low energy phenomenology, the basic idea being precisely what is used for the solution of a so called "incorrect" tasks (frequently also called "ill–posed" or "ill–conditioned"; for a detailed review of this subject see ref. [33] and refs. quoted therein). Our procedure can be schematically formulated in the following way:

1. Write down the amplitude $M$ of the process under consideration in the form:

$$M = S + B,$$

where $S$ stands for the “special” contributions with a strong dependence on kinematical variables, which must be separated and where $B$ includes all other contributions (in accordance with the terminology of ref. [19], we call $B$ the ”background”).

2. Take into account all available information on the form of the functions $S$ and $B$. In our case, this step implies the selection of convenient parameterizations for the above mentioned functions in terms of the kinematical variables suitable for our problem. Thus, one obtains the amplitude $M$ of the reaction in question as a function of some set of free parameters. From the decomposition of the invariant amplitude $M$ — as motivated by the underlying dynamics — into $S$, with a sensitive dependence on variables (as will be discussed below, $S$ contains the OPE graph and the $\Delta$–dominated amplitudes), and a smooth background $B$, it is clear that the parameterization of $S$ and $B$ differs in its functional structure.

3. Fit experimental data with the amplitude $M$, obtained in step 2, to determine the influence of the various elements of $M$ on the data and to find, if possible, the values of the parameters or at least some combinations of them. It is clear, that the more detailed information on the form of the functions $S$ and $B$ is involved, the stronger is the model dependence of the results obtained. Hence it is suitable here to fix the meaning of the term ”model independent approach”. To do this, we give here in a shortened form the necessary formulae (detailed explanations are given in ref. [19]).

The amplitude $M_{fl,\mu\nu}^{abc}$ of the reaction

$$\pi_a(k_1) + N_l^\nu(p) \rightarrow \pi_b(k_2) + \pi_c(k_3) + N_f^\mu(q)$$

(2)

$(a, b, c = 1, 2, 3$ and $f, l = 1, 2$ are isotopic indices; $\mu, \nu = 1, 2$ are the nucleon polarizations), can be written in the ”canonical” form:

$$M_{fl,\mu\nu}^{abc} = (\tau^a)_{fl}^{bc} A^{\mu\nu} + (\tau^b)_{fl}^{ac} B^{\mu\nu} + (\tau^c)_{fl}^{ab} C^{\mu\nu} + i\epsilon^{abc} \delta_{fl}^{D^{\mu\nu}},$$

(3)

where $(X = A, B, C, D)$

$$X^{\mu\nu} = \bar{u}^\nu(q)\hat{X} u^\mu(p),$$

(4)

$$\hat{X} = \{S_x \cdot I + \bar{V}_x \cdot \hat{k} + V_x \cdot \hat{\bar{k}} + \frac{i}{2} T_x[\hat{k}, \hat{\bar{k}}](i\gamma_5),$$

(5)

$$k = -k_1 + \epsilon_2 + \epsilon k_3, \quad \bar{k} = -k_1 + \epsilon_2 + \epsilon k_3,$$

(6)

$$\epsilon = exp(i\frac{2\pi}{3}), \quad \bar{\epsilon} = \epsilon^*,$$

(7)
and $\hat{K} = \gamma \mu K^\mu$ with $K = k, \bar{k}$. The 16 invariant form factors $S_x, \bar{V}_x, V_x, T_x$ (with $X = A, B, C, D$) are functions of five independent scalar variables:

$$
\begin{align*}
\tau &= (p - q)^2, \\
\nu &= (p + q) \cdot k, \\
\theta &= (p - q) \cdot k, \\
\bar{\nu} &= (p + q) \cdot \bar{k}, \\
\bar{\theta} &= (p - q) \cdot \bar{k}.
\end{align*}
$$

(8)

The general properties of these functions with respect to the requirements of Bose, crossing and C–symmetry are described in [19] and references therein.

So, up to this point, no suggestions have been made on the form of the amplitude $M$ — only the restrictions due to the exact symmetries of strong interactions are being taken into account.

To proceed further, however, one has to include some additional physical information. First of all, it is necessary to choose the energy domain, which is the most suitable for the solution of our task — the determination of $\pi\pi$–scattering parameters.

In a previous paper [19] we have discussed only general features of our method. The particular form of the parameterization considered there is suitable only for a sufficiently narrow interval of the incident beam momentum $k_1$. In such a case the possible influence of resonances (which were implied as being far from the boundaries of the physical phase space) can be described by slowly varying real functions.

Thus, in a narrow momentum range, physical arguments suggest a representation of $M$ as a sum of two terms: 1. the contribution of the OPE graph (denoted hereafter as $\Pi$); 2. the total contribution of all graphs besides OPE (“background” denoted as $B$). The crucial point is to note that, inside of this narrow kinematical interval, the first term (OPE) is characterized by a strong $\tau$ dependence (due to the pole which is placed close to the boundary of the physical phase space), while the second one (“background”) has no such strong dependence on the variables because of the absence of the corresponding physical mechanisms. Reversely, this means that the background contribution can be approximated by any smooth function of the variables from eq. (8). The choice of the approximating functions is certainly not unique (this weakens the meaning of the term “model independent approach”).

In this note — compared to [19] — our goal is more ambitious; we would like to extend our analysis to the full kinematical region from threshold (or some MeV above it, to avoid problems caused by the rescattering of final particles; see, for example, [27]) up to $k_{1,\text{lab}} \sim 0.5 \text{ GeV}/c$. In this domain the $\pi N \rightarrow \pi\pi N$ reaction amplitude is strongly influenced by the $\Delta$–isobar creation in the initial ($k_{1,\text{lab}} \sim 0.3 \text{ GeV}/c$) and final ($k_{1,\text{lab}} \sim 0.5 \text{ GeV}/c$) $\pi N$ states [19].

In this case the contribution of the graphs with $\Delta$ isobar exchange (simply called $\Delta$ below) shows a sensitive dependence on the scattering energy due to its resonant structure in the initial and final $\pi N$ invariant masses. Therefore, it has to be separated from the background amplitude. This contribution cannot be approximated by linear functions in the kinematical variables with the desirable accuracy. Furthermore, the behavior of the
amplitude near the resonance position is strongly restricted by unitarity. Consequently, our amplitude has to be extended: we must add the Δ–isobar contribution explicitly, for which, in particular, the presence of an imaginary part is necessary.

Keeping in mind all these notes above, we arrive at the conclusion that the appropriate way to construct a suitable approximation for the amplitude \( M \) — in the momentum range investigated — is the decomposition:

\[
M = B + \Pi + \Delta
\]  

(i.e. \( S = \Pi + \Delta \), eq. (9), see also fig. 1), with the following amplitudes:

- \( B \) is the total contribution of all graphs besides those with Δ poles and OPE; It can be approximated by linear polynomials with unknown coefficients. This procedure gives 11 free parameters: \( A_1 - A_{11} \), (see Sec.2.2.1 below).

- \( \Pi \) is the main ingredient of the picture — the OPE graph. It contains four free parameters: \( g_0 - g_3 \), (see Sec.2.2.2 below).

- \( \Delta \) is the Δ–isobar contribution which can be approximated with functions of the Breit–Wigner type (with the terms \( iM_\Delta \Gamma_\Delta \) in the denominator taken into account). The simplest way to write down the unknown polynomials in the corresponding numerators is to use effective Lagrangians, this method guaranteeing the validity of all necessary symmetry conditions (isotopic and discrete symmetries, etc.). According to Sec. 2.2.3 (see below) this introduces 6 extra free parameters.

Taking everything into account, one obtains the amplitude \( M \) of the reaction (1) in the form given by the formulae (3–7), with the functions \( S_x, \bar{V}_x, V_x, T_x \) \( (X = A, B, C, D) \) fixed by 21 free parameters to be determined with the help of experimental data at \( k_{1,lab} \leq 500 \text{ MeV/c} \).

We have to remember that only 10 of these 21 parameters (namely, those, describing the OPE and the Δ contributions) can be interpreted as coupling constants of the corresponding vertices, while the 11 remaining in the background amplitude have no clear relation to underlying physical parameters (e.g. the \( N^*(1440) \) position and width).

Some of 10 coupling constants mentioned above could be fixed (or, at least, constrained) by additional theoretical and experimental information. For example, the \( \pi N \Delta \) coupling could be fixed by the known value of the isobar decay widths (or, equivalently, by the additive quark model) while the \( \Delta \pi \Delta \) one — by the requirements of \( SU(6) \) symmetry. We, however, prefer – for the sake of flexibility of our procedure – to keep them as free parameters. This allows us to avoid the introducing of unnecessary model dependence into our approach.

Of course, it is not to be expected that the restricted data set on total cross sections only is able to fix all 21 parameters in quantitative way. To solve this task one has to analyze the full set of distributions at several values of incident momentum. Our present
goal is rather to answer the questions listed in Sec.1 and to give insight on degree of model dependence of the previously known results.

In closing this section we remark that in order to explore the significance of the various pieces of $M$, both fits with $M = B, \Pi, \Delta$ only and with $M = \Pi + B, \Delta + B, \Pi + \Delta$ were performed in comparison to the fit with the full amplitude $M = B + \Pi + \Delta$.

### 2.2 The amplitude

The explicit inclusion of the $\Delta$ contribution is the main extension of the formulae given in ref. [19]. Therefore we give in the following a detailed description of the formalism; for completeness, we supplement this presentation by the basic ingredients for the parametrization of the OPE and the background terms from ref. [19].

#### 2.2.1 The background contribution

As shown in [19] the aggregate contribution of the background mechanisms can be described as follows:

\begin{align}
S_A &= A_1 + A_2 \tau + A_3 (\theta + \bar{\theta}) \\
V_A &= A_4 + A_5 \tau + A_6 \theta + A_7 \bar{\theta} \\
\bar{V}_A &= A_4 + A_5 \tau + A_6 \bar{\theta} + A_7 \theta \\
T_A &= iA_8 (\nu - \bar{\nu})
\end{align}

\begin{align}
S_B &= A_1 + A_2 \tau + A_3 (\bar{\epsilon} \theta + \epsilon \bar{\theta}) \\
V_B &= A_4 + \epsilon A_5 \tau + A_6 \theta + \bar{\epsilon} A_7 \bar{\theta} \\
\bar{V}_B &= \bar{\epsilon} A_4 + \bar{\epsilon} A_5 \tau + A_6 \bar{\theta} + \epsilon A_7 \theta \\
T_B &= iA_8 (\bar{\epsilon} \nu - \epsilon \bar{\nu})
\end{align}

\begin{align}
S_C &= A_1 + A_2 \tau + A_3 (\epsilon \theta + \bar{\epsilon} \bar{\theta}) \\
V_C &= \bar{\epsilon} A_4 + \epsilon A_5 \tau + A_6 \theta + \bar{\epsilon} A_7 \bar{\theta} \\
\bar{V}_C &= \epsilon A_4 + \epsilon A_5 \tau + A_6 \bar{\theta} + \bar{\epsilon} A_7 \theta \\
T_C &= iA_8 (\epsilon \nu - \bar{\epsilon} \bar{\nu})
\end{align}

\begin{align}
S_D &= 0
\end{align}
\[ V_D = i A_0 \nu \]
\[ \bar{V}_D = -i A_9 \bar{\nu} \]
\[ T_D = A_{10} + A_{11} \tau \]

Here the parameters \( A_1 \div A_{11} \) are real numbers and the kinematical variables \( \tau, \nu, \bar{\nu}, \theta, \bar{\theta} \) are defined by eqs. (8) above.

### 2.2.2 The OPE graph contribution

According to ref. [19] the OPE graph contributes to \( S_A, S_B, S_C \) functions only, the corresponding expressions having the following form.

\begin{align*}
S_A &= \frac{2g}{\tau - \mu^2} (G_0 + G_1(\theta + \bar{\theta}) + G_2(0\bar{\theta}) + G_3(\theta^2 + \bar{\theta}^2)) \\
S_A &= \frac{2g}{\tau - \mu^2} (G_0 + G_1(\bar{\epsilon}\theta + \epsilon\bar{\theta}) + G_2(\epsilon\bar{\theta}) + G_3(\epsilon\theta^2 + \bar{\epsilon}\bar{\theta}^2)) \\
S_A &= \frac{2g}{\tau - \mu^2} (G_0 + G_1(\epsilon\theta + \bar{\epsilon}\bar{\theta}) + G_2(\bar{\epsilon}\bar{\theta}) + G_3(\bar{\epsilon}\theta^2 + \epsilon\bar{\theta}^2))
\end{align*}

(14)

Here \( G_0, \ldots, G_3 \) stand for free real parameters describing the \( \pi\pi \) interaction and the constant \( g \) corresponds to the \( \pi NN \) vertex. It can be expressed via the familiar pseudovector coupling constant \( f \) \((f^2/4\pi \approx 0.08)\) as follows

\[ g = \frac{2m}{\mu} f \]

### 2.2.3 The \( \Delta \) contribution

In accordance with sec.2.1 the propagator of the \( \Delta \)-isobar involves poles of the Breit–Wigner type:

\[ \frac{1}{\omega_{\pi N}^2 - M_\Delta^2 + i M_\Delta \Gamma_\Delta} \]

(15)

Here, \( \omega_{\pi N} \) stands for the invariant mass of the \( \pi N \) pair. There are 6 such pairs:

\begin{align*}
\omega_{\pi i}^2 &\quad (i = 1, 2, 3) : \quad (p + k_1)^2, \quad (p - k_2)^2, \quad (p - k_3)^2, \\
\omega_{qi}^2 &\quad (i = 1, 2, 3) : \quad (q - k_1)^2, \quad (q + k_2)^2, \quad (q + k_3)^2.
\end{align*}

(16) \quad (17)

The numerators of the corresponding terms must have the correct spin–isospin structure \((J_\Delta = I_\Delta = 3/2)\); they have to fulfill also the correct properties with respect to Bose and crossing–symmetry requirements. The simplest way to coordinate all these conditions is to use the method of effective Lagrangians, the corresponding coupling constants being regarded as free parameters.
Though we expect a pronounced energy dependence ($\Delta$-pole) only for the direct graph of fig. 2d (the additional nucleon – and $\Delta$ –poles appearing in graphs of figs. 2a–2d cannot give rise to the strong dependence of the kinematical variables because they are located too far from the physical space) we include in our analysis all graphs of figs. 2a–2d (along with crossing ones) for the sake of consistency.

To calculate the contributions of graphs of figs. 2a–2d we use the following effective Lagrangians:

\begin{align*}
L_{\pi NN} &= g_{\pi NN} \bar{N} \tau^a \gamma_\mu \gamma_5 N \partial^\mu \pi^a, \\
L_{\pi N\Delta} &= g_{\pi N\Delta} \bar{\Delta} \tau^a \gamma_\mu N \partial^\mu \pi^a + h.c. \\
L_{\pi \Delta\Delta} &= g_{\pi \Delta\Delta} \bar{\Delta} \tau^a \gamma_\mu \gamma_5 A^{abc} \Delta^b \mu \pi^c, \\
L_{\pi \pi N\Delta} &= \bar{N} \left( f_0 F_0^{abc} + 3 f_1 F_1^{abc} \right) \gamma_5 \Delta^a \gamma_\mu \gamma_5 \pi^b \partial_\mu \pi^c + h.c. + \bar{N} \left( g_0 F_0^{abc} + 3 g_1 F_1^{abc} \right) \gamma_\mu \gamma_5 \gamma_\nu \gamma_5 \Delta^a \partial_\mu \pi^b \partial_\nu \pi^c + h.c.,
\end{align*}

where

\begin{align*}
A^{abc} &= 2 \delta^{bc} \tau^a + 2 \delta^{ac} \tau^b - 8 \delta^{ab} \tau^c + 5 \varepsilon^{abc}, \\
F_0^{abc} &= i \varepsilon^{abc} + \delta^{ab} \tau^c - \delta^{ac} \tau^b, \\
F_1^{abc} &= \delta^{ab} \tau^c + \delta^{ac} \tau^b - \frac{2}{3} \delta^{bc} \tau^a.
\end{align*}

are the invariant isotopic forms guaranteeing the correct isospin ($I=3/2$) of the isobar field $\Delta^a$:

$$\tau^a \Delta_a = 0 \quad (23)$$

($\tau^a$ are the standard isospin matrices: $[\tau^a, \tau^b] = i \varepsilon^{abc} \tau^c$).

It should be stressed here that the choice (20) of the $\pi\Delta\Delta$ coupling (non–derivative one) has nothing to do with the problem of chiral symmetry breaking, because (due to the equation of motion) the chiral invariant form of the $\pi\Delta\Delta$ coupling

$$L'_{\pi \Delta \Delta} \sim \bar{\Delta} \gamma_\mu \gamma_5 \gamma_\mu \Delta \partial^\mu \pi$$

is equivalent to that of the eq. (21) on the mass shell of the isobar. The off–shell contributions renormalize the constants of the background and the $\pi\pi N\Delta$ vertex. Since the parameters of the background as well as those of the $\Delta$-vertices are considered to be free, the difference between ”chiral” and ”non–chiral” types of $\pi \Delta \Delta$ coupling becomes unimportant; our choice of the $\pi \Delta \Delta$ vertex is dictated by simplicity. The same is true for the $\pi NN$ vertex also. We use the familiar pseudovector type (18).

This note also explains the absence of the terms with the derivative of the nucleon field in (21) — such a term could again only renormalize the parameters already taken into account.
The effective Lagrangians (18–21) describe the ∆ contribution to the \( \pi N \rightarrow \pi \pi N \) amplitude (in the energy region under consideration) in the most general way. They contain 6 real free parameters \( (g_{\pi N \Delta}, g_{\pi \Delta \Delta}, f_0, f_1, g_0, g_1) \), one of them \( g_{\pi N \Delta} \) can be fixed in accordance with the known \( \Delta \rightarrow \pi N \) decay width. We leave it free due to the reasons explained in sect. 2.1.

To complete the description of the formalism we give the form of the ∆ propagator used in our calculations:

\[
D_{\mu\nu}(p) = \frac{p_\mu \gamma^\mu + M_\Delta}{p^2 - M_\Delta^2 + iM_\Delta \Gamma_\Delta} S_{\mu\nu}, \tag{24}
\]

\[
S_{\mu\nu} = -g_{\mu\nu} + \frac{1}{3} \gamma_\mu \gamma_\nu + \frac{2}{3M_\Delta^2} p_\mu p_\nu - \frac{1}{3M_\Delta^2} (p_\mu \gamma_\nu - \gamma_\mu p_\nu), \tag{25}
\]

\( M \) and \( \Gamma \) being the isobar mass and \( \Delta \rightarrow \pi N \) decay widths respectively.

### 2.3 The fitting procedure

To fit experimental data one has at first to calculate the modulo squared \( |M|^2 \) (averaged over initial and summed over final nucleon polarizations) of the theoretical amplitude for every channel of the reaction (1) at any fixed values of the free parameters \( A_i \) (\( i = 1, \ldots, 21 \)).

The second step consists of the integration of \( |M|^2 \) over the appropriate part of the phase space, the total energy \( E \) being fixed. This gives the theoretical expression:

\[
\frac{\partial^{(r)} \sigma}{\partial x_1 \cdots \partial x_r} \big|_{E=const.} = \sum_{m,n=1}^{21} A_m^* Q_{mn}^{(r)}(E, x_1, \ldots, x_r) A_n, \quad (1 \leq r \leq 4). \tag{26}
\]

The distribution \( D \) to be fitted is given by a convolution:

\[
D^{(r)}(E, x_1, \ldots, x_r) = \frac{\partial^{(r)} \sigma}{\partial x_1 \cdots \partial x_r} (E, x_1, \ldots, x_r) \ast \Phi(E, x_1, \ldots, x_r), \tag{27}
\]

where \( \Phi \) is a known apparatus function specifying the type of distribution, experimental conditions, etc.. Expression (20) is a bilinear form in the free parameters \( A_i \) with the coefficient matrix \( Q_{mn}^{(r)} \) being a function of the set of variables \( (E, x_1, \ldots, x_r) \) describing the distribution under consideration.

To perform the fitting procedure, it is necessary to organize the experimental distributions in the form of a finite array of numbers. To do this one has to divide the phase space into a set of \( N \) cells \( C_i \) with ”central points” \( P_i(E, x_1^i, \ldots, x_r^i) \), with \( i = 1, \ldots, N \). Each point \( P_i \) must be supplied with the corresponding experimental number \( W_i \) (integral over the volume of \( C_i \)) and, of course, with the suitable error bars. In other words, one obtains a \( r \)-dimensional histogram representing the experimental data in question.
Precisely the same procedure must be performed for the theoretical distribution on the right hand side of (26). This means that for each central point \(P_i\) of the distribution \(D(r)\) one must prepare the corresponding \(21 \times 21\) matrix \(Q_{mn}(E, x_1, ..., x_r)\), \(N\) matrices in total. When several different experimental distributions are being treated simultaneously, the necessary number of correlation matrices increases proportionally. Finally, to treat data at different energies, one has to prepare the above mentioned full set of correlation matrices for each energy value. Keeping this in mind, one can easily imagine the size of computer memory which is necessary to perform the data treatment. (This also explains why our program [34] could not be run earlier.)

After all preliminary steps have been done, the fitting procedure is performed with the help of the standard program FUMILI [35].

2.4 Integration over the phase space

The question of the integration over the phase space is extremely delicate. The crucial point is that near the threshold the phase space factor strongly depends on the values of particle masses (this sensitivity was pointed out recently by R.E. Cutkosky — see citation in ref. [21]). The dependence of the amplitude modulo squared is weaker (corrections are proportional to mass differences), but, nevertheless, it must be taken into account when one uses the standard version of the Monte Carlo program FOWL [36] (or any other routine) for calculation of the integrals.

Here, we specify the remark by R.E. Cutkosky and show that the amplitude modulo squared contains two different parts, one of which requires special caution with respect to the values of masses used in the course of computation. To explain this statement we must return back to expressions (4) and (5). According to these formulae, the amplitude modulo squared can be written in the form:

\[
\frac{1}{2} \sum_{\mu,\nu=1}^2 |X^{\mu\nu}|^2 = \sum_{i,j=1}^4 X_i^* M_{ij} X_j,
\]

where \(X \equiv (S, \tilde{V}, V, T)\) and the matrix \(M\) is hermitian:

\[
M_{lm} = \frac{1}{2} Tr\{\tilde{\Gamma}_l(\hat{q} + m_f)\Gamma_m(\hat{p} - m_i)\};
\]

\[
\Gamma \equiv \{I, \hat{k}, \frac{i}{2}[\hat{k}, \hat{k}]\}, \quad \tilde{\Gamma} \equiv \{I, \hat{k}, \frac{i}{2}[\hat{k}, \hat{k}]\},
\]

with \(\hat{r} = \gamma_\mu r^\mu\) for \(r = p, q, k, \bar{k}\); \(m_i\) and \(m_f\) are the masses of incoming and outgoing nucleons, respectively.

The matrix \(M\) degenerates at the boundary of the phase space and the quadratic form (28) loses the property of positive definiteness outside of the phase space. The bounds of the phase space domains determined at the physical values of masses \(m_i \neq m_f\) and
at the average ones \( m_i = m_f = \bar{m} \) do not coincide. This means that one must use the physical masses along with momenta to compute both the phase space factor and the matrix elements \( M_{ij} \). Otherwise, incorrect and even negative contributions to the left hand side of (28) are unavoidable. The point is that errors must be compared not with the particle masses, but with the distances from the boundaries of the real physical phase space, the latter being small if the energy is low.

It is easy to understand that all above arguments are valid for the treating the mass differences of the pions also.

Thus, we conclude that one must use the true mass values both for the phase space factor and for the matrix elements (29) to avoid problems with the kinematics near threshold. One can take the averaged values \( \bar{m} \) for the nucleon and \( \bar{\mu} \) for the pion masses for the computation of the amplitude vector \( X \) in (28); in this case the errors will be of the order of the isospin symmetry breaking.

2.5 Description of the database

In our analysis we use the full set of published (up to May 1994) experimental results on total cross sections of 5 channels (namely, \( \pi^-\pi^+n, \pi^-\pi^0p, \pi^0\pi^0n, \pi^+\pi^0p, \pi^+\pi^+n \)) of the \( \pi N \rightarrow \pi\pi N \) reaction at \( P_{lab} \leq 500\,\text{MeV}/c \). These data (105 points in total) along with the corresponding references are listed in the Table 1. For convenience the same data are presented graphically on Figs.3a – 3e in the form of the \( r \)-th (\( r=1,...,5 \)) channel quasi–amplitude \( \tilde{M}_r \) dependence on the incident beam momentum \( p_{lab} \). The quasi–amplitude is defined as follows

\[
\tilde{M}_i = \left\{ \frac{1}{2} \sum_{\mu\nu=1}^2 |M_{i\mu\nu}|^2 d\Gamma_3 \right\}^{1/2} \sim \left\{ \frac{\sigma_{tot}}{\int d\Gamma} \right\}^{1/2}
\]

(see Table 1), where \( d\Gamma_3 \) is the standard element of the 3–particle phase space volume. This method is much more convenient than the one conventionally used (\( \sigma_{tot} \) as a function of \( p_{lab} \)), because the extremely rapid energy dependence of the numerator caused by the phase space volume is cancelled by the denominator. The resulting smooth energy dependence of the quasi amplitude \( \tilde{M}_i \) provides a fairly direct insight in the quality of the various data points. As an immediate consequence the point no. 12, being obviously inconsistent, was excluded from the data set analysis from the very beginning.

2.6 Strategy of the fitting

To answer the questions raised in sec. 1, we choose the following procedure: first, we fit the total cross section data in order to see what we can learn from this information alone. To test the sensitivity of the data to the OPE or \( \Delta \) contribution, we begin to fit with the background amplitude \( B \) only (11 parameters). For comparison with models it is interesting also to fit these data using the OPE amplitude only (4 parameters) and, for completeness and to test the sensitivity, using the \( \Delta \) amplitude only. At this point
it has to be stressed, that the amplitude $\Delta$ alone is not the full amplitude which can be calculated from the diagrams, since its smooth part corresponding to the off-shell $\Delta$ interaction is included into $B$. In contrast, the amplitude $\Pi$ itself has a clear physical meaning because it corresponds to the simplest variant of the OPE model (see ref. [19]).

More physical, however, are the results of a fit with a combination of two of these amplitudes, like $B + \Delta$ or $B + \Pi$. At the end, the fit including the full amplitude with 21 parameters is performed. The full amplitude, according to sec. 2.1, should be general enough to describe every reasonable collection of $N$ data points in the momentum region in question. Any set of data points tolerating a reasonable fit can therefore be considered as being selfconsistent.

The quality of a fit is considered to be good if the value $\bar{\chi}^2 \approx 1$, where

$$\bar{\chi}^2 \equiv \frac{1}{M} \sum_{i=1}^{N} \chi^2_i,$$

with $N$ being the number of datapoints, whereas $M$ being the number of degrees of freedom ($N$ minus number of fit parameters). When there are some data points with individual contributions of $\chi^2_i$ more than 5 (which is of course our own, arbitrary upper limit), these points need a closer look and have been excluded for the next step of the fit.

After the exclusion of some data points the fit with the full amplitude and some fits (leading to a large $\bar{\chi}^2$ before) for a reduced set of parameters are repeated to see the influence of these data points on the results.

3 Analysis of data on total cross sections

As mentioned above our analysis of the $\pi N \rightarrow \pi \pi N$ reactions is based mainly on the results from ref. [13], supplemented by the inclusion of the $\Delta$-isobar. Even in this fairly restricted parametrization the model – as reflected by its dependence on 21 parameters – is highly complex; a finding, which is stressed by the result of the fitting procedure from section 4, that none of the various ingredients from eq. (9) of the model strongly dominates the complex dynamics at low energies. Due to this subtle interplay we expect that the various $\pi N \rightarrow \pi \pi N$ isospin channels exhibit quite different features, such as in their extrapolation to the corresponding threshold or in their angular distributions. To gain deeper insight into our subsequent results and to facilitate their interpretation, we study in appendix A “quasi” $\pi N$ elastic scattering (i.e. equal nucleon and pion mass) as a much simpler system with related dynamics. This consideration will help to understand better the logic of the subsequent analysis of the $\pi N \rightarrow \pi \pi N$ reaction.

Our analysis of the $\pi N \rightarrow \pi N$ reaction at low energies exhibits the following conclusions:

1. Dealing with reactions of particles with spin one has to take account of the existence of several independent scalar amplitudes. In such a case the term "simplest
dynamics” loses its transparency and must be specially defined.

2. The spin structure of the amplitude manifests itself even in the case of unpolarized particles: it causes the ”minimal” (unremovable) dependence of the distributions on kinematical variables (due to the presence of matrix $G$ — see eq. (1.7)). It can be shown that it is impossible to construct a physically correct amplitude which would result in the precise ”phase space-like” behavior of differential cross-sections in all channels (A.3).

3. The explicit form of the ”minimal” kinematical dependence mentioned above is determined by the interplay of the values of parameters describing the independent scalar amplitudes $A_i$. It may be rather complicated even in the ”simplest” cases when all $A_i$ are constants. In what follows we call this phenomenon the ”spin correlation”.

4. The isospin conservation law connects, to some extent, the spin correlations in different channels. Then, crossing symmetry and Bose statistics give the additional limitations on the admissible form of the latter correlations. Altogether, these symmetries result in the existence of spin–isospin – Bose – crossing correlations connecting the kinematical dependence of the amplitudes of different channels. It is easy to understand that the more identical particles are involved in the reaction the stronger are these correlations.

5. The low energy behavior (the form of the extrapolation law) of the total cross section in a given channel depends on the relative magnitudes and signs of the amplitude parameters; it may be different in different channels (see Appendix, Ex.III). To get the form of the extrapolation law one has to analyze the data on distributions at several values of incident momentum (see Appendix, Ex.I–III). Example IV shows that the ”natural” terms (that of order $k^0$) may be absent in low energy expansions of total cross sections of all channels.

Keeping in mind these basic findings from the the “quasi” $\pi N$ elastic scattering we can start now with the analysis of the modern database on total cross sections of the more complicated $\pi N \rightarrow \pi \pi N$ reaction in the low energy domain $P_{\pi} \leq 500 MeV/c$

3.1 General theoretical consideration.

First of all, let us summarize some formulae concerning the $\pi N \rightarrow \pi \pi N$ reaction (hereafter we use the notations of ref. 19). The five experimentally accessible channels of this reaction along with the corresponding amplitudes $X_r$ and statistical factors $f_r$ are listed in Table 2 (in comparison with ref. 13 the numbers of two last channels are changed).
In accordance with the eq. (29) the total cross section of the \( r \)-th channel can be written in the form
\[
\sigma_r = N \int d\omega \cdot f_r \cdot X_r^\dagger MX_r
\] (32)
Here, \( N \) is the normalization factor (see eq.(57) of ref. [19]), \( X_r \) — the amplitude vector of \( r \)-th channel, i.e.
\[
X_r \equiv (S_r, \bar{V}_r, V_r, T_r)
\] (33)
and the hermitian \( 4 \times 4 \) matrix \( M \) (which is the same for all channels, if one neglects the isospin symmetry breaking) has the elements given by eq. (29).

With the help of eq. (32), it is easy to analyze the correspondence of some statements concerning the dynamics of the reaction in question with the experimental data on total cross sections.

I. The OPE — dominance hypothesis.

According to popular wisdom the main contribution to the low energy \( \pi N \rightarrow \pi \pi N \) amplitude originates from the OPE—graph (\( \Pi \); fig.1). It is easy to check that in this case (we assume the leading order of \( 4\pi \)–interaction and the arbitrary form of the nucleon form factor) the amplitude of the reaction (2) takes the form:
\[
S_A = S_B = S_C = S(\tau), \quad S_D = 0 \\
V_X = \bar{V}_X = T_X = 0, \quad (X = A, B, C, D; \text{eq.}(3))
\] (34)
Here, \( S(\tau) \) depends solely on \( \tau \) (see eq. 8). Using the expressions from Table 2 and eq.(32), one obtains the following relations among the total cross sections of five basic channels:
\[
\frac{1}{8} \sigma(\pi^- \pi^+ n) = \sigma(\pi^- \pi^0 p) = \sigma(\pi^0 \pi^0 n) = \sigma(\pi^+ \pi^0 p) = \frac{1}{4} \sigma(\pi^- \pi^+ n)
\] (35)
The quality of this relation is expected to improve with the dominance of the OPE graph. Of course, in the vicinity of threshold \( (p_\pi \leq 300 \text{MeV/c}) \) one must take into account the isospin breaking (see Sec.2.4 above); in this case it would be more correct to consider the corresponding relations for quasi-amplitudes.

Even a short glance at the Table 1 is enough to realize that the relations (35) strongly contradict the experimental data in the area of \( p_\pi \leq 500 \text{MeV/c} \). As a consequence the OPE—dominance hypothesis is questionable and the straightforward application of the Chew-Low extrapolation procedure cannot result in reliable values of \( \pi \pi \) parameters. This statement is now commonly agreed upon.

II. The \( S \)–wave final state.

In accordance with the modern data on distributions [7–9,17,25] at \( p_\pi \leq 380 \text{MeV/c} \) each pair of final particles in reaction (2) is expected to be in a relative \( S \)–wave state. As it was shown in [12], in this case some of relations (35), namely
\[
\sigma(\pi^- \pi^0 p) = \sigma(\pi^+ \pi^0 p) = \frac{1}{4} \sigma(\pi^- \pi^+ n)
\] (36)
are expected to be valid. The comparison of (36) with the corresponding data from Table 1 shows, however, the striking disagreement. Since both parts of the picture (distributions and total cross sections) contain the same dynamics we are forced to conclude that there is either an implicit contradiction between the data on distributions and those on total cross sections or that the assumption on \( S \)-wave final states is too restrictive.

It should be noted here that one of the relations (35, 36):

\[
\sigma(\pi^-\pi^0p) = \sigma(\pi^+\pi^0p)
\] (37)

seems to be supported by data. This relation implies (see Table 2) the smallness of the isotopic amplitude \( D \) in (3) in comparison with \( C \). If this is true, all distributions in both channels must be approximately equal. From the purely theoretical point of view the smallness of \( D \) would imply a delicate cancellation of various contributions. That is why it would be extremely interesting to check this point at momenta \( p_\pi \geq 400\text{MeV/c} \) where the existing data are still scarce.

III. The phase-space like distributions.

Before starting the discussion we would like to stress that our consideration here is only preliminary; as already stated in our introduction, the problems concerning the distributions will be discussed in detail in forthcoming publications.

In accordance with the results of ref. [6, 7, 8, 9, 17, 25] none of the distributions in four channels (except \( \pi^-\pi^+n \)) up to \( p_\pi < 320\text{MeV/c} \) show (within the error bars) any deviation from the purely 3-particle phase space kinematics. What does this fact tell us?

To answer this question it is necessary to use eq. (A.8) along with the explicit expressions for the matrix elements \( M_{ij} \) \((i, j = 1, \ldots, 4)\). A short look at eqs.(56) of ref. [19] shows that almost each element of the matrix \( M \) depends, for example, on \( \tau \). Thus to remove this (“necessary”) \( \tau \)-dependence one has to select the suitable amplitude vectors \( X_r \). The direct calculation, however, shows that this is impossible: no physically reasonable choice of \( X_r \) can ensure the exact independence of \( |M_r|^2 \) of all four kinematical variables. This result becomes more clear from the examples considered in Appendix and the explicit form of the OPE-contribution.

Thus, we are forced to conclude that the experimental observation of the ”phase-space like” behavior of distributions would show that the precision of the data is insufficient to display the \( \tau \)-dependence originating from the OPE-graph. In fact it shows that the experiment cannot distinguish between spin 1/2 and spinless nucleons. Such precision, of course, does not provide the opportunity to extract reliable information on the details of the dynamics of the process under consideration. It can also result in large uncontrollable errors if the total cross section is obtained with the help of integration of distributions over the corresponding variables.

So, even a very general consideration shows that the data set on total cross sections may contain insufficient information for the extraction of some parameters of the amplitude. Moreover, some of the data points (especially, those obtained with the help of
the integration procedure) may be incorrect in addition to other – purely experimental – reasons.

**3.2 Results of the Fit**

In accordance with the general strategy of the data fitting as discussed in Sec. 2.6 we have performed 7 different fits, each one corresponding to the specific set of the amplitude parameters involved: \( B, \pi, \Delta, B + \pi, B + \Delta, \pi + \Delta, B + \pi + \Delta \). Some results of the best fits are given in Table 3. It should be noted here that for each kind of fit there were found several solutions with approximately the same values of \( \chi^2 \); they differ from each other by the parameters values. Thus, in Table 3 we give the results corresponding to only one "typical" solution for each kind of fit.

The results of the primary “filtration” of the data sample (exclusion of “doubtful data points”, DP) show that the latter contains some DP’s. Indeed, 7 points — namely, 6, 64, 73, 75, 88, 91, 93 — appear in all solutions as doubtful ones. The point no.12 omitted from the very beginning must be also added to this list. These 8 points need special verification. Until the latter is not done, they must be excluded from the list of the data used for the subsequent analysis. Excluding them — the database is reduced by this procedure from 105 to 97 points — we can perform the second step of the data filtration to elucidate the status of 4 points (nn. 1, 4, 76, 90) which appear as doubtful in some solutions. The results of this — second — step are given in the Table 4. It is organized in the same manner as the Table 3; the points numbers placed in brackets indicate that the corresponding point gives the individual contribution \( \chi^2_i \) close to the critical value (\( \chi^2_i = 5 \)).

The consideration of Table 4 shows that two points – nn. 1,4 – which appeared in Table 3 as "almost doubtful" can be regarded as being really doubtful. For them we found no solution with \( \chi^2_i \leq 4 \), \( (i = 1, 4) \).

Summarizing, the list of doubtful data points contains:

\[
\begin{align*}
\pi^+\pi^- n & \quad \text{channel} \quad 1, 4, 6, 12; \\
\pi^0\pi^0 n & \quad \text{channel} \quad 64, 73, 75; \\
\pi^+\pi^+ n & \quad \text{channel} \quad 88, 91, 93,
\end{align*}
\]

It should be noted that all DP’s are concentrated in channels with the neutron in the final state; the two remaining channels (those with the final state proton) do not contain DP’s.

The database filtration procedure could be continued further, but we prefer to stop it here, because in the next paper we shall include the additional experimental information (the data of [63] on distributions in \( \pi^-\pi^+ n \) – channel) to get more reliable and well grounded results. It is necessary to note here that we do not regard the list (38) as the final
one. Unfortunately the "TRIUMF–OMICRON problem" in the $\pi^+\pi^+n$-channel (which is clearly visible on fig.3e) is not as trivial as it seems at the first glance. The attentive reader could notice that the point no.90 (TRIUMF) oftenly appears as the doubtful one along with the points nn. 88,91,93 (OMICRON) – at least in the best solutions. Our experience prompts that it is the hint on the impossibility to resolve the above mentioned problem with the help of purely statistical methods on the database in question: it is necessary to use additional independent information. As to our list of DP’s in $\pi^-\pi^+n$- and $\pi^0\pi^0n$-channels, we believe it is correct and the new information can only display some new DP’s in addition to those quoted above.

Our conclusion about the presence of DP’s in the data sample in question contradicts the result derived by Burkhardt and Lowe (BL) in refs. [18, 64] where the authors have stated that the database on total cross sections is internally consistent up to $p_{lab} = 400\text{MeV}/c$ (apart from a momentum region $310 - 370\text{MeV}/c$ in the $\pi^+\pi^+n$-channel). It is interesting to elaborate on this aspect in more detail.

In what follows we discuss only the latest results published in ref. [64], where the new data [25] on total cross sections of the $\pi^0p$-channel are taken into account. In accordance with the prescription of ref. [64] all data points in $\pi^+\pi^+n$-channel corresponding to the momentum interval $p_{lab} = 310 - 370\text{MeV}/c$ (totally 7 points) must be excluded from the fitting procedure. Then, in their work Burkhardt and Lowe use the reduced data set: points nn. 4, 6, 7, 9, 11, 12, 16, 17, 41, 64, 70, 72, 74 are not included in their database. Thus, to compare the results, we have also excluded the points mentioned above and performed the fitting of the remaining database (49 points). Then, using the parameters corresponding to the best fit (or, if several, to one of them) we have computed the individual contributions $\chi^2_i$ for all points in the data set, including those previously omitted. The values of $\chi^2_i$ in the BL fit, which are not listed in ref. [64], have been obtained with the help of the interpolation of the numbers given in the Table 1 of ref. [64]. The corresponding results concerning the DP’s are given in Table 5, where we use the parameters of one of the best solutions with the background amplitude $B$ only (point no.12, as noted above, is omitted from the very beginning).

The comparison of numbers in Table 5 shows that the points in question look inconsistent in both fits. This result confirms our list of DP’s.

It must be noted also that in refs. [18, 64] points nn. 73, 75 have been omitted along with those quoted above, while near threshold points nn.1, 3, 5 (and some others corresponding to higher momenta) were supplied with the beam momentum values considerably different from those quoted in the original papers, this latter difference originating from the additional corrections of purely experimental character which were taken into account in refs. [18, 64] and neglected in our analysis because we use only published re-

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1 The origin of the "TRIUMF–OMICRON problem" has been recently pointed out by Ortner et al. [12]. For the OMICRON collaboration the total cross sections have been obtained from angular distribution data in a restricted angular range assuming a smooth continuation over the full phase space. Angular distributions for the $\pi^-p \rightarrow \pi^+\pi^-n$ channel [14] do not support such a simple extrapolation.
sults. (We are grateful to Prof. J.Lowe for a detailed discussion of this question). In addition, the systematic errors in the $\pi^0\pi^0n$–channel were handled in a different way in refs. [18, 64]. The differences pointed out above between the two data samples give rise to the different corresponding conclusions.

It should be mentioned that if the corrected values of effective beam momentum (in accordance with BL data base) are used in our analysis, only the point no. 1 (corresponding to $p_\pi = 302.5\ MeV/c$ in the BL sample) looks considerably better and must be deleted from the list (B8). All other results remain practically unchanged.

From our findings above a detailed discussion of the parameter dependence of the model would be premature. This can be best illustrated by the mutual comparison of numbers from Table 6 where we give the typical examples of solutions obtained in a course of the pruned database fitting with different groups of parameters. The considerations of this table shows that every statement on the values of $\pi\pi$–parameters based on the analysis of the data on total cross sections only would be strongly model dependent because the present knowledge of the background processes (and, hence, the values of $A_1, \ldots, A_{11}$) is too pure, the latter processes playing a very important role in the $\pi N \rightarrow \pi\pi N$ reaction mechanism.

4 Conclusion

Before the discussion of our results we would like to start with some notes concerning the adequate treatment of the experimental data on $\pi N \rightarrow \pi\pi N$ reaction in the momentum region in question ($p_\pi \leq 500\ MeV/c$).

Even a short glance at Figs. 3a–3e is enough to recognize that the modern database contains some pairs of mutually inconsistent points. Thus, the preliminary selection (or, the same, filtration) of data is unavoidable if one intends to extract with the reasonable accuracy such quantities as $\pi\pi$–scattering parameters. The only question is how to do this selection.

Of course, one can simply exclude the mutually contradictory pairs of points from the database as it was done in refs. [18, 64]. Unfortunately, such a recipe does not guarantee the consistency of the remaining collection of data. Indeed, after removing ”doubtful pairs” one still has 5 independent subsets of data points (the total cross sections of 5 channels). These are related by isospin and Bose symmetries, and it is essential that these symmetries are incorporated in the analysis. The channels are also related by crossing symmetry (which was ignored in the analysis of refs. [18, 64]).

Thus, it is evident that any correct data selection must take into account all symmetry requirements. If this condition is fulfilled one can use a wide range of equivalent approximants for the “data filtration”. Their specific choice (polynomials, spherical harmonics, etc.) can have influence only on the number of free parameters necessary to achieve a good description of the data. From this point of view our choice ($B + \Pi + \Delta$) for the
The transition amplitude \( \pi N \rightarrow \pi \pi N \) seems to be the most economic one. In addition to the symmetry requirements it takes account of rather general features of the process in question: the existence of a smooth background \( B \) and the \( \pi \) and \( \Delta \) poles close to the kinematical regime investigated.

Of course, we could use polynomials of higher degree for the background contribution (this would introduce \( \sim 30 \) new parameters for the second degree polynomials). Our results show, however, that this is not necessary, because a reasonably good description of data is achieved with linear approximants (in our next paper it will be shown that this is true also for the data on distributions).

Summarizing, we conclude that the procedure used here for the primary filtration of data is general enough: in the case when some data points cannot be described with the reasonable accuracy, they need further experimental verification. Until such a verification is not done these points must be excluded from the database.

With this in mind, this conclusion we can make now the following statements.

- The modern collection of the experimental data on total cross sections of 5 channels of \( \pi N \rightarrow \pi \pi N \) reaction at \( p_\pi \leq 500 \text{MeV}/c \) contains some doubtful points (as listed in eq. (38)) which need special verification.

- The experimental data on two channels with a final state proton \((\pi^- \pi^0 p, \pi^+ \pi^0 p)\) are in statistical sense reliable: they do not contain DP’s. Nevertheless, it would be extremely useful to perform new accurate measurements of the \( \pi^+ \pi^0 p \) channel to increase its statistical significance. It would be of special interest to perform such measurements at relatively large values of the incident momentum: \( p_\pi = 400 - 500 \text{MeV}/c \). The corresponding results would give the opportunity to check the approximate equality of cross sections of these two channels.

- The pruned database (i.e. that obtained after the exclusion of DP’s) can be described in terms of our ansatz with a reasonable value of \( \bar{\chi}^2 \) \( (\leq 1.2) \) (see Fig. 3). Unfortunately, there exist at least several quantitatively equivalent solutions corresponding to rather different values of parameters (the correlations among them are extremely strong). Thus, the experimental data in question do not allow the extraction of a unique set of parameters in the \( \pi N \rightarrow \pi \pi N \). Only qualitative conclusions concerning the influence of the various reaction mechanisms are possible from this limited set of data.

- The most important contribution comes from the background term \( B \). This term alone admits solution with \( \bar{\chi}^2 = 1.4 \). The addition of the \( \Pi \) term improves the description significantly: \( \bar{\chi}^2(B + \Pi) = 1.2 \). This fact shows that — in accordance with common expectations — the OPE term plays an essential role, its contribution is definitely visible even on the database in question and, therefore, the corresponding parameters might be extracted if the database is enlarged by the inclusion of the accurate enough data on distributions. The role of \( \Pi \) term, however,
is by no means dominant ($\chi^2(\Pi) = 2.3$), so the well known Chew-Low extrapolation procedure seems questionable. This finding coincides well with the results of refs. [58, 59, 65, 66].

- In contrast to the $\Pi$ term, the $\Delta$ contribution is rather insensitive on the database in question. The only observation which is of some interest can be formulated as follows: the most significant $\Delta$ contribution corresponds to the parameters $g_0$ and $g_3$ appearing in graphs of Fig. 2d. No definite statements concerning the other four parameters associated with the mechanism of the isobar exchanges (graphs of Figs. 2a–2d) can be done. In particular, the values of $g_{\pi N \Delta}$ and $g_{\pi \Delta \Delta}$ cannot be fixed from the data on total cross sections.

To study the $\Delta$ mechanism one needs the more extensive database which must contain the data on distributions at several beam momenta.

- The informative capacity of the modern database on total cross sections is absolutely insufficient for the model independent extraction of the parameters describing the low energy $\pi\pi$ interaction. The widely used statement that these parameters are most sensitive on and, thus, best extracted from the data on total cross sections near threshold, which is based solely on the Olsson–Turner model, is misleading. For the extraction of reliable information on $\pi\pi$ interaction it seems much more reasonable to use the accurate data on distributions at several values of incident momentum in the interval $300 - 500\,\text{MeV}/c$ (see ref. [15]).

Summarizing the present status of our analysis we conclude that information from the total $\pi N \rightarrow \pi\pi N$ cross sections only rather quantitatively fixes the basic features of the dynamics. Evidently more sensitive data, such as angular distributions, are crucial for a quantitative understanding. The analysis of the enlarged database containing data both on total cross sections of 5 channels and on distributions in the $\pi^+\pi^-n$ channel is presently still troubled with serious numerical problems. It will be discussed in a forthcoming paper.

5 Acknowledgements

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A Appendix: “Quasi” $\pi N$ elastic scattering

As an illustrative example on the subtle influence of different dynamical elements on the extrapolation and the angular distributions in different isospin channels, let us consider the familiar process: “quasi” $\pi N$ elastic scattering:

$$\pi^a(k) + N_e^\nu(p) \to \pi^b(k') + N_f^\mu(p') \quad (A.1)$$

The term ”quasi” used above means that — to simplify the reaction kinematics — we consider the hypothetical case of equal pion and nucleon masses $k^2 = p^2 = k'^2 = p'^2 = m^2$. In (A.1) $a, b = 1, 2, 3$ and $e, f = 1, 2$ are the isotopic indices, $\mu, \nu = 1, 2$ the nucleon polarizations.

The amplitude of the process (A.1) can be represented in the following form (for a comprehensive review see ref. [61]):

$$T_{ba,fe}^{\mu\nu} = \delta_{ba} \delta_{fe} T^{\mu\nu(+)} + \epsilon_{bac}(\tau_e)_{fe} T^{\mu\nu(-)} \quad (A.2)$$

Each of the isotopic amplitudes $T^{\mu\nu(\pm)}$ can be written as follows:

$$T^{\mu\nu(\pm)} = \bar{u}_\mu(p') \{ m A^{\mu\nu(\pm)} + \hat{Q} B^{\mu\nu(\pm)} \} \bar{u}_\nu(p) \quad (A.3)$$

here $\hat{Q} = (\hat{k} + \hat{k}')/2$, $A^\pm$ and $B^\pm$ being the functions of scalar kinematical variables: $t$ and $\nu = (s - u)/4$. The coefficient $m$ at $A^\pm$ in (A.3) is introduced for dimensional reasons. The crossing and Bose symmetries imply:

$$A^\pm(\nu,t) = \pm A^\pm(-\nu,t)$$
$$B^\pm(\nu,t) = \mp B^\pm(-\nu,t) \quad (A.4)$$

Formulae (A.2–A.4) determine the so-called ”canonical” form of the amplitude $T_{ba,fe}^{\mu\nu}$ of the process (A.1).

Let us consider now three experimentally accessible channels of the reaction (A.1) and write down the corresponding amplitudes ($T \equiv A, B$):

1. $\pi^+ p \to \pi^+ p \quad T_1 = T^+ - \frac{1}{2} T^-$
2. $\pi^- p \to \pi^- p \quad T_2 = T^+ + \frac{1}{2} T^-$ \quad (A.5)
3. $\pi^- p \to \pi^0 n \quad T_3 = -\frac{\sqrt{2}}{2} T^-$

Evidently, the amplitude $T_r$ of the $r$-th channel can be written in the form of eq. (A.3) with $A_r, B_r$ being constructed in accordance with eqs.(A.5).
Suppose, further, that we analyze the data on three channels (A.5) with unpolarized nucleons. In this case the amplitude modulo squared (summed over the final and averaged over the initial nucleon polarizations) has the form:

\[ |T_r|^2 = \frac{1}{2} Sp\{(\hat{p} + m)(mA^\dagger + \hat{Q}B^\dagger)(\hat{p}' + m)(mA + \hat{Q}B)\} \]  

(A.6)

or, equivalently:

\[ |T_r|^2 = \sum_{k,p=1,2} V_r^\dagger G_{kp} V_k^r \]  

(A.7)

here, \( V_r^r \equiv (A^r, B^r) \) is the "amplitude vector" of the \( r \)-th channel and the real and symmetric matrix \( G \) has the following elements:

\[ G_{11} = m^2(4m^2 - t), \quad G_{12} = G_{21} = 4m^2\nu, \quad G_{22} = 4\nu^2 + \frac{1}{4}t(4m^2 - t) \]  

(A.8)

Thus, the differential cross sections for the \( r \)-th channel may be written as follows:

\[ d\sigma_r = N \cdot V_r^\dagger GV_r^r \cdot dt \]  

(A.9)

where \( N \) contains all "trivial" kinematical and normalization factors.

Let us suppose that it is possible to make use of the "smoothness hypothesis": the amplitudes \( A^\pm \) and \( B^\pm \) in the low energy region area are smooth functions of their variables \( \nu, t \):

\[ A^+(\nu, t) = \alpha_0 + \alpha_1 \cdot t \quad A^-(\nu, t) = 8a\nu \]
\[ B^-(\nu, t) = 2(\beta_0 + \beta_1 \cdot t) \quad B^+(\nu, t) = 4b\nu \]  

(A.10)

The expressions (A.10) are the most general linear forms consistent with the requirements (A.4). These forms are determined by 6 parameters (for the sake of simplicity we consider them to be real) which describe the cross section of each channel (A.5).

Let us consider now 4 instructive examples I,...,IV for the sensitive interplay of the six parameters. In each case we write down explicitly: the values of the parameters (a), the exact expressions for \( |T_r|^2 \) in terms of \( s, t \) (b) and the "extrapolation laws" for the total cross sections \( \sigma_r \sim \int |T_r|^2 dt \), i.e. the leading order terms of the low energy expansion of \( \sigma_r \) in power series of CMS momentum \( k^2 \) (c), together with short comments (d).

Example I

a. \( \alpha_0 = \alpha_1 = a = b = \beta_1 = 0, \quad \beta_0 = d \)

b. \( |T_1|^2 = |T_2|^2 = \frac{1}{2}|T_3|^2 = d^2 \{ (s - 2m^2)^2 + (s - m^2)t \} \)

c. \( \sigma_1 \simeq \sigma_2 \simeq \frac{1}{2}\sigma_3 \simeq k^0(1 + O(k^2)) \)

d. The angular distributions are relatively simple (linear in \( t \)) in all channels, moreover, they are identical due to the presence of the dynamical symmetry (i.e. conditioned solely by the particular values of the parameters). The form of the extrapolation law in each channel is "natural" (\( \sim k^0 \))
Example II
a. \( \alpha_0 = \alpha_1 = b = \beta_0 = \beta_1 = 0, \ a = d/4m^2 \)
b. \( |T_1|^2 = |T_2|^2 = \frac{1}{4}|T_3|^2 = \)
\[ d^2/16m^2 \{ 16m^2(s - 2m^2)^2 - 4(s - 2m^2)(s - 6m^2)t - 4(s - 3m^2)t^2 - t^3 \} \]
c. \( \sigma_1 \simeq \sigma_2 \simeq \frac{1}{2}\sigma_3 \simeq k^0(1 + O(k^2)) \)
d. The same situation as in I with the only difference, that the angular distributions are more complicated.

Example III
a. \( \alpha_1 = a = b = \beta_1 = 0, \ \alpha_0 = \beta_0 = d \)
b. \( |T_1|^2 = d^2\{(s - 4m^2)^2 + (s - 4m^2)t\} \)
\( |T_2|^2 = d^2\{s^2 + st\} \)
\( |T_3|^2 = 2d^2\{(s - 2m^2)^2 + (s - m^2)t\} \)
c. \( \sigma_1 \simeq k^4 \ll \sigma_2 \simeq 2\sigma_3 \simeq 2m^4(1 + O(k^2)) \)
d. Simple angular distributions (linear in \( t \)) in all channels. The "unnatural" form of the extrapolation law (\( \sim k^4 \)) in the first channel, opposite to the natural forms in the third and second channels. In addition here is a dynamical symmetry relation between the total cross sections \( \sigma_2 \) and \( \sigma_3 \).

Example IV
a. \( \alpha_1 = a = b = \beta_1 = d/16m^2, \ \alpha_0 = \beta_0 = -d/4 \)
b. \( |T_1|^2 = d^2/(16m^2)^2 \cdot 4s^2\{(s - 4m^2)^2 + (s - 4m^2)t\} \)
\( |T_2|^2 = d^2/(16m^2)^2 \cdot 4s\{(s - 4m^2)^2 + (s - 4m^2)(3s - 4m^2)t + (3s - 8m^2)t^2 + t^3\} \)
\( |T_3|^2 = d^2/(16m^2)^2 \cdot 2\{-4m^2(s - 4m^2)t + s(s - 8m^2)t^2 - m^2t^3\} \)
c. \( \sigma_1 \simeq 3/2\sigma_2 \simeq 6\sigma_3 \simeq k^4 \)
d. The angular distributions in three channels are quite different, the term \( \sim t^0 \) being absent in the third channel. Unnatural forms (\( \sim k^4 \)) of the extrapolation law show up in all channels.

Table Captions

Table 1:
List of all data for the total cross section from threshold up to \( k_1^{lab} = 500 \) MeV/c for various \( \pi \pi N \) final states.
a) The \( \pi^+ \pi^- n \) – channel
b) The \( \pi^- \pi^0 p \) – channel
c) The \( \pi^0 \pi^0 n \) – channel
d) The \( \pi^+ \pi^0 p \) – channel
e) The \( \pi^+ \pi^+ n \) – channel

Table 2:
Basic channels of the \( \pi N \rightarrow \pi \pi N \) reaction. The isospin amplitudes \( A, B, C, D \) are defined
in eq. (3).

Table 3:
The results of the first step of the data filtration. The different columnes denote:
Column 1: the kind of the fit.
Column 2: the total value of $\chi^2$ corresponding to one of the best solutions.
Column 3: the value of $\bar{\chi}^2$ (from eq. (31))
Column 4: the list of the ”doubtful points” (DP’s); (i.e. points $P_q$ with $\chi^2_q \geq 5$) numbers are given in accordance with Table 1.
Column 5: summed contribution $\chi^2(DP)$ of all DP’s listed in column 4 to the total value of $\chi^2$.

Table 4:
The results of the second step of the data filtration. The notation is as in Table 3, the numbers in bracket refer to points with a $\chi^2$ close to 5.

Table 5:
Comparison of the $\chi^2$ for selected data points from different $\pi\pi N$ channels in the analysis from Burkhardt and Lowe [18, 64] with the results of this analysis.

Table 6:
Examples of solutions obtained in a course of the fitting with different groups of parameters.
Table 1: List of all data for the total cross section from threshold up to $k_{1\text{lab}}^{\text{lab}} = 500 \text{ MeV/c}$ for various $\pi\pi N$ final states.

a) The $\pi^+\pi^- n$ – channel

| channel | $k_{1\text{lab}}$ in $\text{MeV/c}$ | $\sigma_{\text{tot}}$ in $\text{mb}$ | error | $M_i$ | error | reference |
|---------|---------------------------------|---------------------------------|-------|-------|-------|-----------|
| 1       | 295                             | 0.0051                          | 0.0012| 862   | 101   | [6]       |
| 2       | 313                             | 0.0138                          | 0.0015| 685   | 37    | [32]      |
| 3       | 315                             | 0.020                           | 0.0031| 781   | 60    | [6]       |
| 4       | 321                             | 0.015                           | 0.003 | 584   | 58    | [42]      |
| 5       | 334                             | 0.051                           | 0.012 | 837   | 99    | [6]       |
| 6       | 334                             | 0.027                           | 0.005 | 609   | 56    | [42]      |
| 7       | 336                             | 0.031                           | 0.02  | 624   | 208   | [53]      |
| 8       | 342                             | 0.0603                          | 0.0032| 801   | 21    | [32]      |
| 9       | 346                             | 0.053                           | 0.013 | 712   | 87    | [42]      |
| 10      | 354                             | 0.118                           | 0.020 | 959   | 81    | [6]       |
| 11      | 360                             | 0.125                           | 0.028 | 922   | 103   | [42]      |
| 12      | 361                             | 0.060                           | 0.015 | 638   | 80    | [54]      |
| 13      | 369                             | 0.166                           | 0.0061| 965   | 18    | [54]      |
| 14      | 374                             | 0.211                           | 0.036 | 1039  | 89    | [6]       |
| 15      | 374                             | 0.14                            | 0.10  | 843   | 301   | [54]      |
| 16      | 378                             | 0.16                            | 0.06  | 869   | 163   | [42]      |
| 17      | 392                             | 0.4                             | 0.2   | 1229  | 307   | [38]      |
| 18      | 394                             | 0.327                           | 0.041 | 1095  | 69    | [6]       |
| 19      | 395                             | 0.374                           | 0.0146| 1161  | 23    | [42]      |
| 20      | 404                             | 0.38                            | 0.09  | 1098  | 130   | [42]      |
| 21      | 408                             | 0.546                           | 0.036 | 1279  | 36    | [54]      |
| 22      | 413                             | 0.477                           | 0.056 | 1163  | 68    | [6]       |
| 23      | 415                             | 0.57                            | 0.06  | 1253  | 66    | [40]      |
| 24      | 428                             | 0.71                            | 0.17  | 1333  | 88    | [55]      |
| 25      | 432                             | 0.785                           | 0.104 | 1249  | 150   | [6]       |
| 26      | 449                             | 1.160                           | 0.052 | 1483  | 33    | [52]      |
| 27      | 451                             | 1.052                           | 0.125 | 1409  | 84    | [6]       |
| 28      | 456                             | 1.0                             | 0.2   | 1338  | 134   | [42]      |
| 29      | 458                             | 1.39                            | 0.05  | 1562  | 28    | [41]      |
| 30      | 477                             | 1.880                           | 0.077 | 1677  | 34    | [52]      |
| 31      | 485                             | 1.93                            | 0.16  | 1645  | 68    | [19]      |
| 32      | 485                             | 2.4                             | 0.26  | 1834  | 99    | [11]      |
| 33      | 491                             | 1.93                            | 0.37  | 1608  | 154   | [55]      |
| 34      | 495                             | 2.6                             | 0.26  | 1840  | 92    | [41]      |
| 35      | 499                             | 2.40                            | 0.16  | 1740  | 58    | [44]      |
Table 1: b) The $\pi^-\pi^0 p$ channel

| channel \ $\pi^-\pi^0 p$ | $k_{lab}^{\pi^0}$ in MeV/c | $\sigma_{tot}$ in mb | error | $M_i$ | error | reference |
|--------------------------|-----------------------------|----------------------|-------|-------|-------|-----------|
| 36                       | 295                         | 0.00075              | 0.0004| 216   | 58    | [7]       |
| 37                       | 315                         | 0.0022               | 0.007 | 210   | 33    | [4]       |
| 38                       | 334                         | 0.0085               | 0.0016| 301   | 28    | [7]       |
| 39                       | 354                         | 0.020                | 0.005 | 356   | 44    | [7]       |
| 40                       | 375                         | 0.027                | 0.006 | 342   | 38    | [7]       |
| 41                       | 391                         | 0.08                 | 0.13  | 517   | 420   | [7]       |
| 42                       | 394                         | 0.050                | 0.013 | 400   | 52    | [7]       |
| 43                       | 406                         | 0.2                  | 0.1   | 738   | 184   | [7]       |
| 44                       | 413                         | 0.073                | 0.015 | 428   | 44    | [7]       |
| 45                       | 415                         | 0.11                 | 0.05  | 520   | 118   | [17]      |
| 46                       | 415                         | 0.09                 | 0.01  | 469   | 26    | [10]      |
| 47                       | 428                         | 0.14                 | 0.07  | 526   | 142   | [11]      |
| 48                       | 432                         | 0.119                | 0.020 | 493   | 41    | [7]       |
| 49                       | 450                         | 0.157                | 0.037 | 520   | 61    | [7]       |
| 50                       | 456                         | 0.17                 | 0.05  | 527   | 78    | [14]      |
| 51                       | 458                         | 0.17                 | 0.01  | 523   | 15    | [13]      |
| 52                       | 495                         | 0.31                 | 0.07  | 613   | 69    | [11]      |
| 53                       | 499                         | 0.32                 | 0.05  | 613   | 48    | [14]      |
Table 1: c) The $\pi^0\pi^0n$ – channel

| channel | $k_{lab}^{12}$ in MeV/c | $\sigma_{tot}$ in mb | error | $M_i$ | error | reference |
|---------|--------------------------|----------------------|-------|-------|-------|-----------|
| $\pi^0\pi^0n$ |                       |                      |       |       |       |           |
| 54      | 272                      | 0.00038              | 0.00010 | 535   | 70    | [17]      |
| 55      | 276                      | 0.00059              | 0.00014 | 466   | 55    | [17]      |
| 56      | 280                      | 0.00118              | 0.00022 | 465   | 43    | [17]      |
| 57      | 284                      | 0.00206              | 0.00035 | 477   | 40    | [17]      |
| 58      | 286                      | 0.00231              | 0.00065 | 461   | 65    | [17]      |
| 59      | 287                      | 0.00333              | 0.00064 | 523   | 50    | [17]      |
| 60      | 291                      | 0.00381              | 0.00081 | 471   | 50    | [17]      |
| 61      | 293                      | 0.0081               | 0.0013  | 648   | 52    | [17]      |
| 62      | 298                      | 0.0085               | 0.0010  | 563   | 33    | [17]      |
| 63      | 305                      | 0.0171               | 0.0019  | 661   | 37    | [17]      |
| 64      | 310                      | 0.032                | 0.005   | 810   | 63    | [51]      |
| 65      | 314                      | 0.0219               | 0.0020  | 615   | 28    | [17]      |
| 66      | 323                      | 0.0303               | 0.0030  | 620   | 31    | [17]      |
| 67      | 331                      | 0.0598               | 0.0064  | 772   | 41    | [17]      |
| 68      | 339                      | 0.0752               | 0.0073  | 770   | 37    | [17]      |
| 69      | 349                      | 0.0981               | 0.0093  | 785   | 37    | [17]      |
| 70      | 353                      | 0.13                 | 0.02    | 870   | 67    | [51]      |
| 71      | 359                      | 0.118                | 0.011   | 781   | 36    | [17]      |
| 72      | 385                      | 0.32                 | 0.04    | 1040  | 65    | [49]      |
| 73      | 390                      | 0.388                | 0.046   | 1108  | 66    | [17]      |
| 74      | 391                      | 0.27                 | 0.07    | 913   | 118   | [47]      |
| 75      | 400                      | 0.479                | 0.049   | 1151  | 59    | [17]      |
| 76      | 494                      | 1.3                  | 0.1     | 1236  | 48    | [47]      |
Table 1: d) The $\pi^+\pi^0p$– channel

| channel | $k_{lab}^{\pi\pi}$ in $MeV/c$ | $\sigma_{tot}$ in $mb$ | error | $M_i$ | error | reference |
|---------|--------------------------------|------------------------|--------|-------|--------|----------|
| $\pi^+\pi^0p$ | | | | | | |
| 77      | 298                            | 0.001                  | 0.0017 | 223   | 190    | [25]     |
| 78      | 309                            | 0.0027                 | 0.0012 | 267   | 59     | [25]     |
| 79      | 331                            | 0.0068                 | 0.0018 | 278   | 37     | [25]     |
| 80      | 342                            | 0.018                  | 0.012  | 387   | 129    | [18]     |
| 81      | 352                            | 0.0146                 | 0.0026 | 309   | 28     | [25]     |
| 82      | 374                            | 0.026                  | 0.0027 | 336   | 17     | [25]     |
| 83      | 392                            | 0.048                  | 0.034  | 399   | 141    | [18]     |
| 84      | 415                            | 0.12                   | 0.05   | 542   | 113    | [17]     |
| 85      | 418                            | 0.189                  | 0.036  | 669   | 64     | [9]      |
Table 1: e) The $\pi^+\pi^+n$–channel

| channel $\pi^+\pi^+n$ | $k_{lab}^{\text{in}}$ in $MeV/c$ | $\sigma_{\text{tot}}$ in $mb$ | error | $M_i$ | error | reference |
|------------------------|----------------------------------|-------------------------------|-------|-------|-------|-----------|
| 86                     | 288                              | 0.00011                       | 0.0004 | 236   | 32    | [10]      |
| 87                     | 292                              | 0.00028                       | 0.0005 | 249   | 22    | [10]      |
| 88                     | 298                              | 0.0018                        | 0.0004 | 444   | 49    | [8]       |
| 89                     | 299                              | 0.00060                       | 0.00010 | 243  | 20    | [10]      |
| 90                     | 310                              | 0.00146                       | 0.00022 | 246  | 19    | [10]      |
| 91                     | 317                              | 0.0080                        | 0.0018 | 467   | 53    | [8]       |
| 92                     | 335                              | 0.0094                        | 0.0023 | 338   | 41    | [50]      |
| 93                     | 338                              | 0.0217                        | 0.0045 | 513   | 53    | [8]       |
| 94                     | 342                              | 0.030                         | 0.018  | 565   | 170   | [48]      |
| 95                     | 358                              | 0.0274                        | 0.0052 | 440   | 42    | [8]       |
| 96                     | 364                              | 0.0228                        | 0.0048 | 378   | 40    | [54]      |
| 97                     | 378                              | 0.039                         | 0.007  | 432   | 39    | [8]       |
| 98                     | 390                              | 0.026                         | 0.055  | 317   | 336   | [48]      |
| 99                     | 398                              | 0.0451                        | 0.0103 | 395   | 45    | [8]       |
| 100                    | 418                              | 0.065                         | 0.0135 | 415   | 43    | [8]       |
| 101                    | 430                              | 0.0537                        | 0.010  | 353   | 33    | [54]      |
| 102                    | 439                              | 0.074                         | 0.0153 | 394   | 41    | [8]       |
| 103                    | 459                              | 0.083                         | 0.0178 | 380   | 41    | [8]       |
| 104                    | 477                              | 0.12                          | 0.01   | 424   | 18    | [54]      |
| 105                    | 480                              | 0.094                         | 0.0200 | 370   | 39    | [8]       |
Table 2: Basic channels of the $\pi N \rightarrow \pi \pi N$ reaction. The isospin amplitudes $A, B, C, D$ are defined in eq. (3).

| channel                  | $N^0$ | amplitude                      | stat.factor $f$ |
|--------------------------|-------|--------------------------------|-----------------|
| $\pi^- p \rightarrow \pi^- \pi^+ n$ | 1     | $X_1 = \sqrt{2}/2(A + C)$      | 1               |
| $\pi^- p \rightarrow \pi^- \pi^0 p$   | 2     | $X_2 = 1/2 C - D$              | 1               |
| $\pi^- p \rightarrow \pi^0 \pi^0 p$   | 3     | $X_3 = \sqrt{2}/2 A$          | 1/2             |
| $\pi^+ p \rightarrow \pi^+ \pi^0 p$   | 4     | $X_4 = 1/2 C + D$              | 1               |
| $\pi^+ p \rightarrow \pi^+ \pi^+ n$   | 5     | $X_1 = \sqrt{2}/2 (B + C)$     | 1/2             |
Table 3: The results of the first step of the data filtration. The different columns denote:
Column 1: the kind of the fit.
Column 2: the total value of $\chi^2$ corresponding to one of the best solutions.
Column 3: the value of $\bar{\chi}^2$ (from eq. (31))
Column 4: the list of the “doubtful points” (DP’s); (i.e. points $P_q$ with $\chi^2_q \geq 5$); numbers are given in accordance with Table 1
Column 5: summed contribution $\chi^2(DP)$ of all DP’s listed in column 4 to the total value of $\chi^2$.

| Fit     | $\chi^2$ | $\bar{\chi}^2$ | List of DP’s                     | $\chi^2(DP)$ |
|---------|----------|-----------------|---------------------------------|--------------|
| $B$     | 188.6    | 2.03            | 4,6,64,73,75,76,88,91,93,95     | 87.6         |
| $\pi$   | 284.7    | 2.85            | many                           | —            |
| $\Delta$| 1396.0   | 14.2            | many                           | —            |
| $B + \pi$| 149.0   | 1.66            | 1,6,64,73,75,88,90,91,93       | 64.6         |
| $B + \Delta$| 180.8 | 2.08            | 4,6,64,73,75,76,88,91,93       | 78.1         |
| $\pi + \Delta$| 204.4 | 2.17            | 4,6,64,67,72,73,75,76,85 ,88,90,91,93,95 | 109.7      |
| $B + \pi + \Delta$| 148.2 | 1.79            | 1,6,64,73,75,88,90,91,93       | 64.1         |
Table 4: The results of the second step of the data filtration. The notation is as in Table 3, the numbers in bracket refer to points with a $\chi^2$ close to 5.

| Fit       | $\chi^2$ | $\bar{\chi}^2$ | List of DP’s | $\chi^2(DP)$ |
|-----------|-----------|------------------|--------------|--------------|
| $B$       | 123.4     | 1.43             | [1]4,67,72,76,95 | 30.6         |
| $\pi$     | 216.3     | 2.33             | many         | —            |
| $\Delta$  | 1259.0    | 13.8             | many         | —            |
| $B + \pi$ | 98.8      | 1.20             | 1,[4],90     | 11.8         |
| $B + \Delta$ | 119.1    | 1.49             | [1],4,67,72  | 17.3         |
| $\pi + \Delta$ | 139.5   | 1.60             | 4,19,21,67,72,85,95,97 | 52.6         |
| $B + \pi + \Delta$ | 98.2   | 1.29             | 1,[4],85     | 12.0         |

**Figure Captions**

**Figure 1:**
Schematical decomposition of the $\pi N \rightarrow \pi \pi N$ reaction amplitude into the background part ($B$), the $\Delta$-isobar piece ($\Delta$) and the OPE-term ($\Pi$).

**Figure 2:**
Different contributions of the $\Delta$-isobar. (a), (b), (c): $\Delta$-excitation in the first, second and both intermediate states, respectively; (d): nonlinear $\pi \pi N \Delta$ coupling.

**Figure 3:**
Comparison of the model independent fit to the quasi amplitude $\tilde{M}_i$ (eq. (30)) in 5 $\pi \pi N$ channels with the data as a function of $p_\pi (GeV/c)$. Note that “doubtful points”, as discussed in Sect. 3.4, are not included in the data. The experimental data and corresponding references are from Table 1. Data points from Kernel et al. [6, 7, 8, 9] are emphasized by asterisks.

a) The $\pi^+\pi^-n$–channel.
b) The $\pi^-\pi^0p$–channel.
c) The $\pi^0\pi^0n$–channel.
d) The $\pi^+\pi^0p$–channel.
Table 5: Comparison of the $\chi^2$ for selected data points from different $\pi\pi N$ channels in the analysis from Burkhardt and Lowe [18, 64] with the results of this analysis.

| no. point (channel) | Individual contribution $\chi^2_i$ |
|---------------------|----------------------------------|
|                     | BL | Our results |
| 1 ($\pi^-\pi^+n$)  | 5.8 | 5.8 |
| 4 ($\pi^-\pi^+n$)  | 4.2 | 3.9 |
| 6 ($\pi^-\pi^+n$)  | 9.6 | 9.1 |
| 64 ($\pi^0\pi^0n$) | 8.1 | 8.4 |
| 73 ($\pi^0\pi^0n$) | 4.4 | 3.7 |
| 75 ($\pi^0\pi^0n$) | 5.3 | 3.5 |
| 88 ($\pi^+\pi^+n$) | 10.0 | 9.0 |
| 91 ($\pi^+\pi^+n$) | 8.9 | 7.4 |
| 93 ($\pi^+\pi^+n$) | 10.4 | 7.7 |

e) The $\pi^+\pi^+n$ – channel. Compared are the fits with the full data set and with a partial data set (lower and upper line, respectively, 4 points out of 20 are suppressed).
Table 6: Examples of solutions obtained in a course of the fitting with different groups of parameters

| Parameter | $B(\chi^2 = 1.4)$ | $\pi(\chi^2 = 2.3)$ | $B + \pi(\chi^2 = 2.3)$ |
|-----------|-------------------|----------------------|--------------------------|
|           | value $10^3$ | error $10^3$ | correlation factor $10^3$ | value $10^3$ | error $10^4$ | correlation factor $10^4$ | value $10^3$ | error $10^4$ | correlation factor $10^4$ |
| $A_1$     | 0.39            | 1.12               | 4.48                     | 0.0           | fixed          | -                     | -2.70         | 0.18            | 0.20               |
| $A_2$     | -5.00           | 10.9               | 6.79                     | 0.0           | fixed          | -                     | -4.46         | 4.58            | 2.37               |
| $A_3$     | -10.4           | 36.7               | 27.2                     | 0.0           | fixed          | -                     | -12.9         | 7.56            | 2.87               |
| $A_4$     | -0.77           | 0.35               | 2.11                     | 0.0           | fixed          | -                     | -2.10         | 0.34            | 10.6               |
| $A_5$     | 3.77            | 11.1               | 21.2                     | 0.0           | fixed          | -                     | -4.43         | 6.59            | 21.8               |
| $A_6$     | 0.76            | 6.09               | 25.8                     | 0.0           | fixed          | -                     | 5.63          | 0.34            | 0.30               |
| $A_7$     | -6.11           | 7.67               | 4.43                     | 0.0           | fixed          | -                     | -3.16         | 11.6            | 48.9               |
| $A_8$     | 0.85            | 1.86               | 3.66                     | 0.0           | fixed          | -                     | 1.84          | 1.81            | 17.9               |
| $A_9$     | 6.34            | 6.19               | 1.09                     | 0.0           | fixed          | -                     | 7.88          | 1.13            | 0.39               |
| $A_{10}$  | 0.41            | 0.55               | 0.26                     | 0.0           | fixed          | -                     | -0.15         | 0.11            | 0.06               |
| $A_{11}$  | 8.86            | 6.33               | 0.44                     | 0.0           | fixed          | -                     | 0.37          | 2.14            | 0.40               |
| $G_0$     | 0.00            | fixed              | -                        | 9.7           | 3.53           | 4.7                   | -0.18         | 0.03            | 0.57               |
| $G_1$     | 0.00            | fixed              | -                        | -672          | 44.4           | 3.1                   | -2.74         | 0.17            | 0.12               |
| $G_2$     | 0.00            | fixed              | -                        | $1.5 \cdot 10^4$ | 427           | 4.7                   | 20.4          | 12.2            | 3.36               |
| $G_3$     | 0.00            | fixed              | -                        | $2.9 \cdot 10^4$ | 405           | 3.1                   | 18.7          | 11.3            | 2.56               |
Figure 1: Schematical decomposition of the $\pi N \rightarrow \pi \pi N$ reaction amplitude into the background part ($B$), the $\Delta$-isobar piece ($\Delta$) and the OPE-term ($\Pi$).
Figure 2: Different contributions of the $\Delta$-isobar. (a), (b), (c): $\Delta$-excitation in the first, second and both intermediate states, respectively; (d): direct $\pi\pi N\Delta$ coupling (the crossing graphs are implied to be taken into account also).
Figure 3: Comparison of the model independent fit to the quasi amplitude $\tilde{M}_i$ (eq. (30)) in 5 $\pi\pi N$ channels with the data as a function of $p_\pi$ (GeV/c). Note that “doubtful points”, as discussed in Sect. 3.4, are not included in the data. The experimental data and corresponding references are from Table 1. Data points from Kernel et al. [6, 7, 8, 9] are emphasized by asterisks.

a) The $\pi^+\pi^-n$–channel.
Figure 3: b) The $\pi^-\pi^0p$-channel.
Figure 3: c) The $\pi^0\pi^0 n$ channel.
Figure 3: d) The $\pi^+\pi^0 p$ - channel.
Figure 3: e) The $\pi^+\pi^-n$–channel. Compared are the fits with the full data set and with a partial data set (lower and upper line, respectively, 4 points out of 20 are suppressed).