On the $L^2$–Poincaré duality for incomplete riemannian manifolds: a general construction with applications

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Abstract

Let $(M, g)$ be an open, oriented and incomplete riemannian manifold of dimension $m$. Under some general conditions we show that it is possible to build a Hilbert complex $(L^2\Omega^i(M, g), d_{M,i})$ such that its cohomology groups, labeled with $H^i_{L^2}(M, g)$, satisfy the following properties:

- $H^i_{L^2}(M, g) = \ker(d_{\text{max},i})/\text{im}(d_{\text{min},i})$
- $H^i_{L^2}(M, g) \cong H^{m-i}_{L^2}(M, g)$ (Poincaré duality holds)

Finally in the rest of the paper we study some properties of this complex with particular attention to the sufficient conditions which make it a Fredholm complex.

Introduction

Poincaré duality is one of the best known and most important property of the de Rham cohomology on a closed and oriented differential manifold $M$. Putting a riemannian metric $g$ on $M$ and using the results coming from Hodge theory, Poincaré duality says that there exists an isomorphism between $H^i_{dR}(M)$ and $H^{m-i}_{dR}(M)$ for all $i = 0, \ldots, m$, where $m$ is the dimension of $M$. In particular, among all its applications, Poincaré duality reduces the study of the de Rham cohomology of $M$ to the study of the first $m^2/m+1$ de Rham cohomology groups respectively when $m$ is even or odd. As is it well known this is not longer true when $M$ is not compact.

When $(M, g)$ is an open and oriented riemannian manifold two natural and important variations of the de Rham cohomology are provided by the $L^2$–de Rham cohomology and by the reduced $L^2$–de Rham cohomology. In the non compact setting they are an important argument and indeed they have been the subject of many researches during the last decades. In this case, as it is well known, the completeness of $(M, g)$ plays a fundamental role. When $(M, g)$ is complete, the Laplacian $\Delta_i$, with domain given by the smooth and compactly supported forms $\Omega^i_c(M)$, is an essentially self-adjoint operator on $L^2\Omega^i(M, g)$. In particular it follows that Poincaré duality holds for the reduced $L^2$–cohomology of $(M, g)$. Therefore, when the $L^2$–cohomology is finite dimensional, it coincides with the reduced $L^2$–cohomology and so it satisfies Poincaré duality. All this properties in general do not hold when $(M, g)$ is incomplete. Generally in this case the differential $d_i$ acting on smooth $i$–forms with compact support admits several different closed extensions when we look at it as an unbounded operator between $L^2\Omega^i(M, g)$ and $L^2\Omega^{i+1}(M, g)$. Therefore, depending on the closed extensions fixed, we will get different kinds of $L^2$–cohomology groups and $L^2$–reduced cohomology groups for which, in general, Poincaré duality does not hold. Anyway incomplete riemannian manifolds appears naturally in the context of riemannian geometry and in that of global analysis, in particular when we deal with space with "singularities" such as stratified pseudomanifolds or algebraic subvariety. Therefore it is an interesting question looking for some general construction for the $L^2$–cohomology of $(M, g)$, when $g$ is incomplete, such that Poincaré duality holds. In the literature other papers have dealt with this question: for example we mention \[1\], \[3\] and \[5\]. This paper is organized in the following way: the beginning of first section contains some back
ground material about Hilbert complexes while its final part contains some news results about Poincaré duality for abstract Hilbert complexes.

The second section is divided in three parts. In the first one, after recalled the notion of $L^2$-de Rham cohomology, we prove, using the results about abstract Poincaré duality proved in the previous section, two of the main results of the paper. We can summarize them in the following way:

**Theorem 1.** Let $(M, g)$ be an open, oriented and incomplete riemannian manifold of dimension $m$. Then:

$$\ker(d_{\max,i})/\text{im}(d_{\min,i,-})$$

is a finite sequence of vector spaces with Poincaré duality.

Assume now that for each $i = 0, \ldots, m$, $\text{ran}(d_{\min,i})$ is closed in $L^2\Omega^{i+1}(M, g)$. Then there exists a Hilbert complex $(L^2\Omega^{i}(M, g), d_{\min,i})$ such that for each $i = 0, \ldots, m$

- $\mathcal{D}(d_{\min,i}) \subset \mathcal{D}(d_{\min,i}) \subset \mathcal{D}(d_{\max,i})$, that is $d_{\max,i}$ is an extension of $d_{\min,i}$ which is an extension of $d_{\min,i}$,
- $\text{ran}(d_{\min,i})$ is closed in $L^2\Omega^{i+1}(M, g)$
- $\mathcal{H}^{2,\Omega}_{i}(M, g) = \frac{\ker(d_{\max,i})}{\text{ran}(d_{\min,i})}$ where $\mathcal{H}^{2,\Omega}_{i}(M, g)$ is the cohomology of the Hilbert complex $(L^2\Omega^{i}(M, g), d_{\min,i})$.

In particular $(L^2\Omega^{i}(M, g), d_{\min,i})$ is a Hilbert complex with Poincaré duality.

The second part of the second section is devoted to the study of some necessary and/or sufficient conditions in order to get $(L^2\Omega^{i}(M, g), d_{\min,i})$ a Fredholm complex. In particular we will show that this property is strictly connected to the behavior of the Gauss-Bonnet operator

$$d + \delta : L^2\Omega^{i}(M, g) \rightarrow L^2\Omega^{i}(M, g).$$

More precisely we will show that:

**Theorem 2.** Let $(M, g)$ be an open, oriented and incomplete riemannian manifold which satisfies the assumptions of Theorem 1. Then the following properties are equivalent:

1. $(L^2\Omega^{i}(M, g), d_{\min,i})$ is a Fredholm complex.
2. The complex $(\Omega^{i}(M), d_{i})$ has only a finite number of closed extensions and they are all Fredholm. Moreover, for all $i = 0, \ldots, m$, we have that

$$\mathcal{D}(d_{\max,i})/\mathcal{D}(d_{\min,i})$$

is a finite dimensional vector space whose dimension equals the number of closed extensions of $d_{i} : \Omega^{i}(M) \rightarrow \Omega^{i+1}(M)$.
3. $(d + \delta)_{\max} : \mathcal{D}(d + \delta)_{\max} \rightarrow L^2\Omega^{i}(M, g)$ is a Fredholm operator on its domain endowed with the graph norm.
4. Every closed extension of $d + \delta : \Omega^{i}(M) \rightarrow \Omega^{i}(M)$ is a Fredholm operator. Therefore there are only a finite number of closed extensions and this number is given by

$$\mathcal{D}(d + \delta)_{\max}/\mathcal{D}(d + \delta)_{\min}.$$ 

The rest of the second part contains some corollaries and some applications of this result. In particular, in the third part of the second section, we define, using the complex $(L^2\Omega^{i}(M, g), d_{\min,i})$, an $L^2$-Euler characteristic $\chi_{2,\Omega}(M, g)$ and an $L^2$-signature $\sigma_{2,\Omega}(M, g)$ for $(M, g)$ and then we show that they are the index of some suitable Fredholm operators. More precisely:

**Theorem 3.** Let $(M, g)$ be an open, oriented and incomplete riemannian manifold such that $(L^2\Omega^{i}(M, g), d_{\min,i})$ is a Fredholm complex. Then $(d_{\min} + d_{\min})^{ev}$ and $(d + \delta)^{ev}_{\max}$ are Fredholm operators on their respective domains endowed with the graph norm and we have:

$$\text{ind}((d_{\min} + d_{\min})^{ev}) = \chi_{2,\Omega}(M, g) = \dim(\ker(((d + \delta)^{ev})_{\max})) - \dim(\ker(((d + \delta)^{odd})_{\max})) \quad (1)$$

If $m = 4l$ then we have also

$$\sigma_{2,\Omega}(M, g) = \dim(\ker((D^{sign,+})_{\max})) - \dim(\ker((D^{sign,-})_{\max})) = \text{ind}((d + \delta)_{M}^{+}) \quad (2)$$
where \((d_{2\Omega} + d_{2\Omega}^*)^n\) and \((d + \delta)^n_M\) are suitable extensions respectively of the Gauss-Bonnet operator and of the signature operator that we will define in the same subsection of Theorem 3.

In the third part of the paper we will provide some examples of incomplete Riemannian manifolds \((M, g)\) for which the complex \((L^2\Omega(M, g), d_{2\Omega})\) is a Fredholm complex. Therefore all the results proved in the paper apply to those examples.

Finally we conclude mentioning that in a subsequent paper we plan to come back again on this subject investigating some topological properties of the vector spaces \(H^2_{2\Omega}(M, g)\) with particular attention to the cases when they are finite dimensional.

Acknowledgments. I wish to thank Pierre Albin, Jochen Brünning, Erich Leichtnam, Rafe Mazzeo and Paolo Piazza for interesting comments and emails.

1 Hilbert Complexes

We start the section recalling the notion of Hilbert complex and its main properties. For a complete development of the subject we refer to [5].

**Definition 1.** A Hilbert complex is a complex, \((H_*, D_*)\) of the form:

\[
0 \rightarrow H_0 \xrightarrow{D_0} H_1 \xrightarrow{D_1} H_2 \xrightarrow{D_2} \ldots \xrightarrow{D_{n-1}} H_n \rightarrow 0,
\]

(3)

where each \(H_i\) is a separable Hilbert space and each map \(D_i\) is a closed operator called the differential such that:

1. \(D(D_i)\), the domain of \(D_i\), is dense in \(H_i\).
2. \(\text{im}(D_i) \subset D(D_{i+1})\).
3. \(D_{i+1} \circ D_i = 0\) for all \(i\).

The cohomology groups of the complex are \(H^i(H_*, D_*) := \text{Ker}(D_i)/\text{im}(D_{i-1})\). If the groups \(H^i(H_*, D_*)\) are all finite dimensional we say that it is a Fredholm complex.

Given a Hilbert complex there is a dual Hilbert complex

\[
0 \leftarrow H_0 \xleftarrow{D_0^*} H_1 \xleftarrow{D_1^*} H_2 \xleftarrow{D_2^*} \ldots \xleftarrow{D_{n-1}^*} H_n \leftarrow 0,
\]

(4)

defined using \(D_i^* : H_{i+1} \rightarrow H_i\), the Hilbert space adjoints of the differentials \(D_i : H_i \rightarrow H_{i+1}\). The cohomology groups of \((H^*_j, (D_j)^*)\), the dual Hilbert complex, are

\[
H^i(H^*_j, (D_j)^*) := \text{Ker}(D^*_{n-i-1})/\text{im}(D^*_n).
\]

For all \(i\) there is also a laplacian \(\Delta_i = D_i^* D_i + D_{i-1} D_{i-1}^*\) which is a self-adjoint operator on \(H_i\) with domain

\[
\mathcal{D}(\Delta_i) = \{ v \in \mathcal{D}(D_i) \cap \mathcal{D}(D_{i-1}^*) : D_i v \in \mathcal{D}(D_i^*), D_{i-1}^* v \in \mathcal{D}(D_{i-1}) \}
\]

(5)

and nullspace:

\[
\mathcal{H}^i(H_*, D_*) := \ker(\Delta_i) = \text{Ker}(D_i) \cap \text{Ker}(D_{i-1}^*).
\]

(6)

Another important self-adjoint operator associated to (3) is the following: let us label \(H := \bigoplus_{i=0}^n H_i\) and let

\[
D + D^* : H \rightarrow H
\]

(7)

be the self-adjoint operator with domain

\[
\mathcal{D}(D + D^*) = \bigoplus_{i=0}^n (\mathcal{D}(D_i) \cap \mathcal{D}(D_{i-1}^*))
\]
and defined as
\[ D + D^* := \bigoplus_{i=0}^{n} (D_i + D_i^*). \]

The following propositions that now we are going to recall are standard results for this topic. The first result is a weak Kodaira decomposition:

**Proposition 1.** [5, Lemma 2.1] Let \((H_i, D_i)\) be a Hilbert complex and \((H_i, (D_i)^*)\) its dual complex, then:
\[ H_i = \mathcal{H}^i \oplus \overline{\text{im}(D_{i-1})} \oplus \text{im}(D_i^*). \] (8)

The reduced cohomology groups of the complex are:
\[ \overline{\mathcal{H}}^i(H_*, D_*) := \text{Ker}(D_i)/\overline{\text{im}(D_{i-1})}. \]

By the above proposition there is a pair of weak de Rham isomorphism theorems:
\[
\begin{align*}
\mathcal{H}^i(H_*, D_*) & \cong \overline{\mathcal{H}}^i(H_*, D_*) \\
\mathcal{H}^i(H_*, D_*) & \cong \overline{\mathcal{H}}^{n-i}(H_*, (D_*)^*)
\end{align*}
\] (9)

where in the second case we mean the cohomology of the dual Hilbert complex.

The complex \((H_*, D_*)\) is said weak Fredholm if \(\mathcal{H}_i(H_*, D_*)\) is finite dimensional for each \(i\). By the next propositions it follows immediately that each Fredholm complex is a weak Fredholm complex.

**Proposition 2.** [5, corollary 2.5] If the cohomology of a Hilbert complex \((H_*, D_*)\) is finite dimensional then, for all \(i\), \(\text{im}(D_{i-1})\) is closed and \(H^i(H_*, D_*) \cong \mathcal{H}^i(H_*, D_*)\).

**Proposition 3.** The following properties are equivalent:
1. \([5]\) is a Fredholm complex
2. for all \(i = 0, \ldots, n\), \(\Delta_i : D(\Delta_i) \to H_i\) is a Fredholm operator on its domain endowed with the graph norm
3. the operator defined in \([7]\) is a Fredholm operator on its domain endowed with the graph norm.

*Proof.* See [5] Theorem 2.4

**Proposition 4.** [5, corollary 2.6] A Hilbert complex \((H_j, D_j), j = 0, \ldots, n\) is a Fredholm complex (weak Fredholm) if and only if its dual complex, \((H_j, D_j^*)\), is Fredholm (weak Fredholm). If it is Fredholm then
\[ \mathcal{H}_i(H_j, D_j) \cong H_i(H_j, D_j) \cong H_{n-i}(H_j, (D_j)^*) \cong \mathcal{H}_{n-i}(H_j, (D_j)^*). \] (10)

Analogously in the weak Fredholm case we have:
\[ \mathcal{H}_i(H_j, D_j) \cong \overline{\mathcal{H}}_i(H_j, D_j) \cong \overline{\mathcal{H}}_{n-i}(H_j, (D_j)^*) \cong \mathcal{H}_{n-i}(H_j, (D_j)^*). \] (11)

Now we recall some definitions from [5]. We refer to the same paper for more properties and comments.

Given a pair of Hilbert complexes \((H_j, D_j)\) and \((H_j, D'_j)\) we will write \((H_j, D_j) \subseteq (H_j, D'_j)\) if for each \(j\) one of the two following properties is satisfied:
1. \(D'_j : H_i \to H_{j+1}\) is equal to \(D_j : H_j \to H_{j+1}\)
2. \(D'_j : H_j \to H_{j+1}\) is a proper closed extension of \(D_j : H_i \to H_{j+1}\)

We will write \((H_j, D_j) \subset (H_j, D'_j)\) when the second of the above properties is satisfied.

Consider again a pair of Hilbert complexes \((H_i, D_i)\) and \((H_i, L_i)\) with \(i = 0, \ldots, n\).

**Definition 2.** The pair \((H_i, D_i)\) and \((H_i, L_i)\) is said to be complementary if the following property is satisfied
• for each \( i \) there exist an isometry \( \phi_i : H_i \to H_{n-i} \) such that \( \phi_i(D(D_i)) = D(L_{n-i-1}^\ast) \) and \( L_{n-i-1}^\ast \circ \phi_i = C_i(\phi_{i+1} \circ D_i) \) on \( D(D_i) \) where \( L_{n-i-1}^\ast : H_{n-i} \to H_{n-i-1} \) is the adjoint of \( L_{n-i} : H_{n-i-1} \to H_{n-i} \) and \( C_i \neq 0 \) is a constant which depends only on \( i \).

Furthermore we call the maps \( \phi_i \) duality maps.

We have the following propositions:

**Proposition 5.** Let \( (H_i, D_i) \) and \( (H_i, L_i) \) be complementary Hilbert complexes. Then

1. Also \( (H_i, L_i) \) and \( (H_i, D_i) \) are complementary Hilbert complexes. Moreover if \( \{ \phi_i \} \) are the duality maps which make \( (H_i, D_i) \) and \( (H_i, L_i) \) complementary then \( \{ \phi_i^\ast \} \), the family obtained taking the adjoint maps, are the duality maps which make \( (H_i, L_i) \) and \( (H_i, D_i) \) complementary.

2. Each \( \phi_i \) induces an isomorphism between \( \mathcal{H}'(H_i, D_i) \) and \( \mathcal{H}^{n-i}(H_i, L_i) \).

3. The complexes \( (H_i, D_i) \) and \( (H_i, L_i) \) have isomorphic cohomology groups and isomorphic reduced cohomology groups. In the same way the complexes \( (H_i, L_i) \) and \( (H_i, D_i^\ast) \) have isomorphic cohomology groups and isomorphic reduced cohomology groups.

4. The following isomorphism hold: \( \overline{\mathcal{H}'}(H_i, D_i) \cong \overline{\mathcal{H}}^{n-i}(H_i, L_i) \).

**Proof.** See \([3]\) Prop. 5

Now we recall the following definition:

**Definition 3.** Let \( V_0, V_1, ..., V_n \) be a finite sequence of vector spaces. We will say that it is a finite sequence of vector spaces with Poincaré duality if for each \( i \):

\[ V_i \cong V_{n-i} \]

that is \( V_i \) and \( V_{n-i} \) are isomorphic vector spaces.

Finally we are in position to state the main results of the section:

**Theorem 4.** Let \( (H_j, D_j) \subseteq (H_j, L_j) \) be a pair of complementary Hilbert complexes. Then:

\[ \text{Ker}(L_j)/\text{im}(D_{j-1})) \]

is a finite sequence of vector spaces with Poincaré duality.

**Proof.** Consider, for each \( j = 0, ..., n \) the following complex:

\[
0 \to H_0 \xrightarrow{D_0} H_1 \xrightarrow{D_1} ... \xrightarrow{D_{j-1}} H_j \xrightarrow{L_j} ... \xrightarrow{L_{n-1}} H_n \to 0,
\]

The dual Hilbert complex is clearly:

\[
0 \leftarrow H_0 \xleftarrow{D_0^\ast} ... \xleftarrow{D_{j-1}^\ast} H_j \xleftarrow{L_j^\ast} ... \xleftarrow{L_{n-1}^\ast} H_n \leftarrow 0,
\]

Therefore, by \([4]\) we know that

\[ \text{Ker}(L_j)/\text{im}(D_{j-1})) \cong \text{Ker}(L_j) \cap \text{Ker}(D_{j-1}^\ast). \]

By Prop. \([3]\) we know that \( \phi_{n-j} \) induces an isomorphism between \( \text{Ker}(L_j) \) and \( \text{Ker}(D_{n-j-1}^\ast) \) and between \( \text{Ker}(L_{n-j}) \) and \( \text{Ker}(D_{n-j-1}^\ast) \). Therefore it induces an isomorphism between \( \text{Ker}(L_j) \cap \text{Ker}(D_{n-j}^\ast) \) and \( \text{Ker}(D_{n-j-1}) \cap \text{Ker}(L_{n-j}) \) and so by \([15]\) we get the conclusion.

**Theorem 5.** Let \( (H_j, D_j) \subseteq (H_j, L_j) \), \( j = 0, ..., n \), be a pair of Hilbert complexes. Suppose that for each \( j \) \( \text{im}(D_j) \) is closed in \( H_{j+1} \). Then there exists a third Hilbert complex \( (H_j, P_j) \) such that

1. \( (H_j, D_j) \subseteq (H_j, P_j) \subseteq (H_j, L_j) \) and the image of each \( P_j \) is closed for each \( j \).
2. \(H_j(H_*, P_*) = \text{Ker}(L_j)/\text{im}(D_{j-1})\).

3. If \((H_j, D_j) \subseteq (H_j, L_j)\) are complementary then \((H_j, P_j)\) is a Hilbert complex with Poincaré duality.

4. If \((H_j, P_j)\) is a Fredholm complex then both \((H_j, D_j)\) and \((H_j, L_j)\) are Fredholm complexes.

Proof. To prove the first part of the proposition we have to exhibit a Hilbert complex which satisfies the assertions of the statement. To do this consider the following Hilbert space

\[\mathcal{D}(L_j), <, >_g\]

which is by definition the domain of \(L_j\) endowed with the graph scalar product, that is for each pair of elements \(u, v \in \mathcal{D}(L_j)\) we have

\[<u, v>_g := <u, v>_{H_j} + <L_ju, L_jv>_{H_{j+1}}.

During the rest of the proof we will work with this Hilbert space and therefore all the direct sum that will appear and all the assertions of topological type are referred to this Hilbert space \((\mathcal{D}(L_j), <, >_g)\). We can decompose \((\mathcal{D}(L_j), <, >_g)\) in the following way:

\[\mathcal{D}(L_j), <, >_g = \text{Ker}(L_j) \oplus V_j\]

where \(V_j = \{ \alpha \in \mathcal{D}(L_j) \cap \text{im}(L_j^*) \}\) and it is immediate to check that these subspaces are both closed in \((\mathcal{D}(L_j), <, >_g)\).

Consider now \((\mathcal{D}(D_j), <, >_g)\); it is a closed subspace of \((\mathcal{D}(L_j), <, >_g)\) and we can decompose it as

\[\mathcal{D}(D_j), <, >_g = \text{Ker}(D_j) \oplus A_j\]

By the assumption that the range of \(D_j\) is closed it follows that also the range of \(D_j^*\) is closed. So, analogously to the previous case, \(A_j = \{ \alpha \in \mathcal{D}(D_j) \cap \text{im}(D_j^*) \}\) and obviously these subspaces are both closed in \((\mathcal{D}(L_j), <, >_g)\). Obviously if \(\text{Ker}(D_j) = \text{Ker}(L_j)\) then the Hilbert complex \((H_j, D_j)\) satisfies the first part of the proposition. So we can suppose that \(\text{Ker}(D_j)\) is properly contained in \(\text{Ker}(L_j)\). Let \(\pi_1\) be the orthogonal projection of \(A_j\) onto \(\text{Ker}(L_j)\) and analogously let \(\pi_2\) be the orthogonal projection of \(A_j\) onto \(V_j\). We have the following properties:

1. \(\pi_2\) is injective
2. \(\text{im}(\pi_2)\) is closed.

The first property follows from the fact that \(\text{Ker}(\pi_2) = A_j \cap \text{Ker}(L_j)\). But \(L_j\) is an extension of \(D_j\); therefore if an element \(\alpha\) lies in \(A_j \cap \text{Ker}(L_j)\) then it lies also in \(\text{Ker}(D_j)\) and so \(\alpha = 0\). For the second property consider a sequence \(\{\gamma_m\}_{m \in \mathbb{N}} \subseteq A_j\) such that \(\pi_2(\gamma_m)\) converges to \(\gamma \in V_j\). Then

\[\lim_{m \to \infty} D_j(\gamma_m) = \lim_{m \to \infty} L_j(\gamma_m) = \lim_{m \to \infty} L_j(\pi_2(\gamma_m)) = L_j(\gamma)\]

This implies that

\[\lim_{m \to \infty} D_j(\gamma_m) = L_j(\gamma)\]

and therefore the limit exists. So by the assumptions about the range of \(D_j\) it follows that there exists an element \(\eta \in A_j\) such that

\[\lim_{m \to \infty} D_j(\gamma_m) = D_j(\eta)\]

Moreover \(L_j(\gamma) = D_j(\eta) = L_j(\eta) = L_j(\pi_2(\eta))\). This implies that \(L_j(\pi_2(\eta) - \gamma) = 0\) and therefore \(\pi_2(\eta) = \gamma\) because \(\pi_2(\eta), \gamma \in V_j\) and \(L_j\) is injective on \(V_j\). In this way we showed that \(\text{im}(\pi_2)\) is closed.

Now define \(N_j\) as the range of \(\pi_2\). Finally define \(W_j\) as the vector space generated by the sum
of $\text{Ker}(L_j)$ and $N_j$. By the fact that $\text{Ker}(L_j)$ and $N_j$ are orthogonal to each other we have $W_j = \text{Ker}(L_j) \oplus N_j$ and therefore $W_j$ is closed in $\mathcal{D}(L_j)$. Finally define $P_j$ as

$$P_j := L_j|_{W_j}$$

By the fact that $W_j$ is closed in $\mathcal{D}(L_j)$ and that $\pi_1(A_j), \pi_2(A_j) \subseteq W_j$ it follows that $P_j$ is a closed extension of $D_j$ which is in turn extended by $L_j$. Moreover, by the construction, it is clear that $\text{Ker}(P_j) = \text{Ker}(L_j)$. Finally, again by the definition of $P_j$ and its domain, it is follows that $\text{im}(P_j) = L_j(\pi_2(A_j)) = \text{im}(D_j)$. Therefore we got that $\text{im}(P_j)$ is closed and that

$$\frac{\text{Ker}(P_j)}{\text{im}(P_j)} = \frac{\text{Ker}(L_j)}{\text{im}(D_j)}.$$ 

This complete the proof of the first two statements.

The third statement follows combining the second statement of this Theorem with Theorem [\ref{lem:closed}]. Finally if $(H_j, P_j)$ is Fredholm then we know that $\frac{\text{Ker}(L_j)}{\text{im}(D^1)}$ is finite dimensional. This clearly implies in turn that also $\frac{\text{Ker}(L_j)}{\text{im}(D^1)}$ and $\frac{\text{Ker}(D_j)}{\text{im}(D^1)}$ are finite dimensional for each $j$, that is $(H_j, L_j)$ and $(H_j, D_j)$ are Fredholm complexes. This completes the proof. 

\section{Poincaré duality for $L^2$–cohomology}

\subsection{General results}

Now we want to show that riemannian geometry is a context in which pairs of complementary Hilbert complexes appear in a natural way. Let $(M, g)$ be an open, oriented and incomplete riemannian manifold. Consider the de Rham complex $(\mathcal{C}_0^\infty \Omega^t(M), d_*)$ where each form $\omega \in \mathcal{C}_0^\infty \Omega^t(M)$ is a $i$–form with compact support. Using the riemannian metric $g$ and the associated volume form $\text{vol}_g$ we can construct for each $i$ the Hilbert space $L^2 \Omega^t(M, g)$. To turn the previous complex into a Hilbert complex we must specify a closed extension of $d$. With the two following propositions we will recall the two canonical closed extensions of $d$

\begin{definition}
The maximal extension $d_{\max}$; this is the operator acting on the domain:

$$\mathcal{D}(d_{\max, i}) = \{ \omega \in L^2 \Omega^t(M, g) : \exists \eta \in L^2 \Omega^{t+1}(M, g) \}$$

s.t. $\langle \omega, \delta_i \zeta \rangle_{L^2(M, g)} = \langle \eta, \zeta \rangle_{L^2(M, g)} \forall \zeta \in \mathcal{C}_0^\infty \Omega^{t+1}(M)$

In this case $d_{\max, i} \omega = \eta$. In other words $\mathcal{D}(d_{\max, i})$ is the largest set of forms $\omega \in L^2 \Omega^t(M, g)$ such that $d_i \omega$, computed distributionally, is also in $L^2 \Omega^{t+1}(M, g)$.

\end{definition}

\begin{definition}
The minimal extension $d_{\min, i}$; this is given by the graph closure of $d_i$ on $\mathcal{C}_0^\infty \Omega^t(M)$ respect to the norm of $L^2 \Omega^t(M, g)$, that is,

$$\mathcal{D}(d_{\min, i}) = \{ \omega \in L^2 \Omega^t(M, g) : \exists \{ \omega_j \}_{j \in J} \subset \mathcal{C}_0^\infty \Omega^t(M, g), \omega_j \rightarrow \omega, \ d_i \omega_j \rightarrow \eta \in L^2 \Omega^{t+1}(M, g) \}$$

and in this case $d_{\min, i} \omega = \eta$

Obviously $\mathcal{D}(d_{\min, i}) \subset \mathcal{D}(d_{\max, i})$. Furthermore, from these definitions, it follows immediately that

$$d_{\min, i}(\mathcal{D}(d_{\min, i})) \subset \mathcal{D}(d_{\min, i+1}), \ d_{\min, i+1} \circ d_{\min, i} = 0$$

and that

$$d_{\max, i}(\mathcal{D}(d_{\max, i})) \subset \mathcal{D}(d_{\max, i+1}), \ d_{\max, i+1} \circ d_{\max, i} = 0.$$ 

Therefore $(L^2 \Omega^*(M, g), d_{\max/min, *})$ are both Hilbert complexes and their cohomology groups are denoted by $H^*_\text{max/min}(M, g)$.

Another straightforward but important fact is that the Hilbert complex adjoint of $(L^2 \Omega^*(M, g), d_{\max/min, *})$ is $(L^2 \Omega^*(M, g), \delta_{\min/max, *})$ with $\delta_*$ the formal adjoint of $d_*$, that is

$$(d_{\max, i})^* = \delta_{\min, i}, \ (d_{\min, i})^* = \delta_{\max, i}. \tag{21}$$
Moreover are a pair of complementary Hilbert complexes. Let
\[ L^2\Omega^i(M,g) = \mathcal{H}_{abs/rel}^i \oplus \text{ran}(d_{\max/min,i-1}) \oplus \text{ran}(\delta_{\min/max,i}) \]
with summands mutually orthogonal in each case. The first summand in the right, called the absolute or relative Hodge cohomology, respectively, is defined as the orthogonal complement of the other two summands. Since \( \text{ran}(d_{\max,i-1}) \) and \( \text{ran}(d_{\min,i-1}) \), we see that
\[ H_{abs/rel}^i = \text{Ker}(d_{\max/min,i}) \cap \text{Ker}(\delta_{\max/min,i}). \]

Now consider the following operators:
\[ \Delta_{abs,i} = \delta_{min,i}d_{max,i} + d_{max,i-1}\delta_{min,i-1}, \Delta_{rel,i} = \delta_{max,i}d_{min,i} + d_{min,i-1}\delta_{max,i-1} \]
These are selfadjoint and satisfy:
\[ H_{abs}^i(M,g) = \text{Ker}(\Delta_{abs,i}), \quad H_{rel}^i(M,g) = \text{Ker}(\Delta_{rel,i}) \]
and
\[ \text{ran}(\Delta_{abs,i}) = \text{ran}(d_{max,i-1}) \oplus \text{ran}(\delta_{min,i}), \quad \text{ran}(\Delta_{rel,i}) = \text{ran}(d_{min,i-1}) \oplus \text{ran}(\delta_{max,i}). \]

Furthermore if \( H^i_{\max/min}(M,g) \) is finite dimensional then the range of \( d_{max/min,i-1} \) is closed and \( H_{abs/rel}^i(M,g) \cong H^i_{\max/min}(M,g) \). On \( L^2\Omega^i(M,g) \) we have also a third weak Kodaira decomposition.
\[ L^2\Omega^i(M,g) = \mathcal{H}_{max}^i \oplus \text{ran}(d_{min,i-1}) \oplus \text{ran}(\delta_{min,i}) \]
where \( H_{max}^i(M,g) \) is \( \text{Ker}(d_{max,i}) \cap \text{Ker}(\delta_{max,i-1}) \) and it is called the \( i-th \) maximal Hodge cohomology group.

Now we are in position to state the following results:

**Proposition 6.** Let \((M,g)\) be an open, oriented and incomplete riemannian manifold of dimension \(m\). Then the complexes
\[(L^2\Omega^i(M,g), d_{max,i})\] and \[(L^2\Omega^i(M,g), d_{min,i})\]
are a pair of complementary Hilbert complexes.
Moreover
\[ \text{Ker}(d_{max,i})/\text{im}(d_{min,i-1}) = (L^2\Omega^i(M,g), d_{max,i-1}) \]
is a finite sequence of vector spaces with Poincaré duality.

**Proof.** See [3] Theorem 11 for the first part of the theorem. The second part follows from Theorem 4 \( \square \)

**Theorem 6.** Let \((M,g)\) be an open, oriented and incomplete riemannian manifold of dimension \(m\). Suppose that for each \(i = 0, \ldots, m\) \(\text{ran}(d_{min,i})\) is closed in \(L^2\Omega^{i+1}(M,g)\). Then there exists a Hilbert complex \((L^2\Omega^i(M,g), d_{2\mathbb{R},i})\) such that for each \(i = 0, \ldots, m\)
\[ \mathcal{D}(d_{min,i}) \subset \mathcal{D}(d_{2\mathbb{R},i}) \subset \mathcal{D}(d_{max,i}), \]
that is \(d_{max,i}\) is an extension of \(d_{2\mathbb{R},i}\), which is an extension of \(d_{min,i}\), \(\text{ran}(d_{2\mathbb{R},i})\) is closed in \(L^2\Omega^{i+1}(M,g)\) and
\[ H_{2,\mathbb{R}}^i(M,g) = \frac{\text{Ker}(d_{max,i})}{\text{ran}(d_{min,i})} \]
where \(H_{2,\mathbb{R}}^i(M,g)\) is the cohomology of the Hilbert complex \((L^2\Omega^i(M,g), d_{2\mathbb{R},i})\). Moreover \((L^2\Omega^i(M,g), d_{2\mathbb{R},i})\) is a Hilbert complex with Poincaré duality.

**Proof.** The proof is an application of Theorem 4 combined with the above proposition. \( \square \)

We conclude the section with the following **Hodge theorem** for the \(L^2\)-cohomology groups \(H_{2,\mathbb{R}}^2(M,g)\).
Theorem 7. In the same assumptions of Theorem[6] Let \( \Delta_i : \Omega^i_c(M) \to \Omega^i_c(M) \) be the Laplacian acting on the space of smooth compactly supported \( i \)-forms. Then there exists a self-adjoint extension \( \Delta_{2\mathcal{O}} : L^2\Omega^i(M) \to L^2\Omega^i(M,g) \) with closed range such that

\[
\text{Ker}(\Delta_{2\mathcal{O}_i}) \cong \text{Ker}(d_{\max,i}) / \text{ran}(d_{\min,i}).
\]

Proof. Consider the Hilbert complex \((L^2\Omega^i(M,g),d_{2\mathcal{O}_i})\). For each \( i = 0, \ldots, n \) define

\[
\Delta_{2\mathcal{O}_i} := d_{2\mathcal{O}_i}^* \circ d_{2\mathcal{O}_i} + d_{2\mathcal{O}_i-1} \circ d_{2\mathcal{O}_i-1}
\]

with domain given by

\[
\mathcal{D}(\Delta_{2\mathcal{O}_i}) = \{ \omega \in \mathcal{D}(d_{2\mathcal{O}_i}) \cap \mathcal{D}(d_{2\mathcal{O}_i-1}) : d_{2\mathcal{O}_i}(\omega) \in \mathcal{D}(d_{2\mathcal{O}_i-1}) \text{ and } d_{2\mathcal{O}_i-1}(\omega) \in \mathcal{D}(d_{2\mathcal{O}_i-1}) \}.
\]

In other words, for each \( i = 0, \ldots, n \), \( \Delta_{2\mathcal{O}_i} \) is the \( i \)-th Laplacian associated to the Hilbert complex \((L^2\Omega^i(M,g),d_{2\mathcal{O}_i})\). So, as recalled in the first section, it follows that \( \Delta_{2\mathcal{O}_i} \) is a self-adjoint operator. Moreover, by Theorem[6] we know that \( d_{2\mathcal{O}_i} \) has closed range for each \( i \). This implies that also \( d_{2\mathcal{O}_i}^* \) has closed range for each \( i \). This means that for the Hilbert complex \((L^2\Omega^i(M,g),d_{2\mathcal{O}_i})\) the \( L^2 \)-cohomology and the reduced \( L^2 \)-cohomology are exactly the same and so we can apply[9] to get the first conclusion. Moreover, by the assumptions, it follows that \( \text{ran}(\Delta_{2\mathcal{O}_i}) = \text{ran}(d_{2\mathcal{O}_i-1} \oplus d_{2\mathcal{O}_i}^*) \). The reason of the previous equality is the following: clearly we have always \( \text{ran}(\Delta_{2\mathcal{O}_i}) \subseteq \text{ran}(d_{2\mathcal{O}_i-1} \oplus d_{2\mathcal{O}_i}) \). Now let \( \omega \in \text{ran}(d_{2\mathcal{O}_i-1} \oplus d_{2\mathcal{O}_i}) \). Applying repeatedly the decomposition recalled in Prop.[9] and keeping in mind that \( d_{2\mathcal{O}_i} \) and \( d_{2\mathcal{O}_i}^* \) have closed range in all degree, we get that

\[
\omega = d_{2\mathcal{O}_i-1}(d_{2\mathcal{O}_i}^*(d_{2\mathcal{O}_i-1}(\eta_1))) + d_{2\mathcal{O}_i}^*(d_{2\mathcal{O}_i}(d_{2\mathcal{O}_i-1}(\eta_2)))
\]

for some \( \eta_1 \in \mathcal{D}(d_{2\mathcal{O}_i-1}) \) and \( \eta_2 \in \mathcal{D}(d_{2\mathcal{O}_i}^*) \). Clearly, by the construction of \( \eta_1 \) and \( \eta_2 \), it follows that

\[
d_{2\mathcal{O}_i-1}(\eta_1) + d_{2\mathcal{O}_i}^*(\eta_2) \in \mathcal{D}(\Delta_{2\mathcal{O}_i})
\]

and

\[
d_{2\mathcal{O}_i-1}(d_{2\mathcal{O}_i}^*(d_{2\mathcal{O}_i-1}(\eta_1))) + d_{2\mathcal{O}_i}^*(d_{2\mathcal{O}_i}(d_{2\mathcal{O}_i-1}(\eta_2))) = \Delta_{2\mathcal{O}_i}(d_{2\mathcal{O}_i-1}(\eta_1) + d_{2\mathcal{O}_i}(\eta_2)).
\]

Therefore we got \( \text{ran}(\Delta_{2\mathcal{O}_i}) \supset \text{ran}(d_{2\mathcal{O}_i-1} \oplus d_{2\mathcal{O}_i}^*) \) and in this way we can conclude that \( \Delta_{2\mathcal{O}_i} \) is an operator with closed range. This completes the proof.

According to[9] we have \( \text{ker}(\Delta_{2\mathcal{O}_i}) = \text{ker}(d_{2\mathcal{O}_i}) \cap \text{ker}(d_{2\mathcal{O}_i}^*) \). We will label these spaces as \( \mathcal{H}_{2\mathcal{O}_i}^{\text{d}}(M,g) \).

2.2 When \( H_{2,\mathcal{O}_i}^*(M,g) \) is finite dimensional?

In this subsection we explore some necessary and sufficient conditions in order to get \( H_{2,\mathcal{O}_i}^*(M,g) \) finite dimensional for all \( i \), or equivalently, such that the Hilbert complex \((L^2\Omega^i(M,g),d_{2\mathcal{O}_i})\) is a Fredholm complex.

First of all we recall some definitions. Consider again the complex \((\Omega^*_c(M),d_c)\) any Hilbert complex \((L^2\Omega^i(M,g),D_i)\) where \( D_i : L^2\Omega^i(M,g) \to L^2\Omega^{i+1}(M,g) \) is a densely defined, closed operator which extends \( d_i : \Omega^*_c(M,g) \to \Omega^*_{c+1}(M,g) \) and such that the action of \( D_i \) on \( \mathcal{D}(D_i) \), its domain, coincides with the action of \( d_i \) on \( \mathcal{D}(D_i) \) in a distributional way. Obviously for every closed extension of \((\Omega^*_c(M),d_c)\) we have \( (L^2\Omega^i(M,g),d_{\min,i}) \subseteq (L^2\Omega^i(M,g),D_i) \subseteq (L^2\Omega^i(M,g),d_{\max,i}) \). We will label with \( H_{2,D_i}^*(M,g) \) and with \( \overline{H}_{2,D_i}^*(M,g) \) respectively the cohomology groups and the reduced cohomology group of \((L^2\Omega^i(M,g),D_i)\) and with \( \mathcal{H}_{2}^{\text{d}}(M,g) \) its Hodge cohomology groups.

Proposition 7. Let \((M,g)\) be an open, oriented and incomplete riemannian manifold which satisfies the assumptions of Theorem[6] Then we have the following properties:

1. Consider the natural inclusion of complexes \((L^2\Omega^i(M,g),d_{\min,i}) \subset (L^2\Omega^i(M,g),d_{2\mathcal{O}_i})\).

Then the natural map induced between cohomology groups is injective for all \( i = 0, \ldots, m \).
2. Let \( (L^2\Omega(M,g), D) \) be a closed extension of \((\Omega^i(M), d)\). Then there exists an injective map \( \phi_i : \mathcal{T}^i_{2,D}(M,g) \to H^i_{2,\mathcal{R}}(M,g) \).

**Proof.** The first property follows immediately by the fact that

\[ H^i_{2,\mathcal{R}}(M,g) = \text{Ker}(d_{max,i})/\text{ran}(d_{min,i-1}). \]

For the second property, by Prop. 4 we have \( H^i_{2,D}(M,g) \cong \mathcal{H}^i_{D_1}(M,g) \). So applying \( \mathcal{T} \) we get \( \mathcal{T}^i_{2,D}(M,g) \cong \text{Ker}(D_i) \cap \text{Ker}(D^*_{i-1}) \). Applying the same statements to the complex \((L^2\Omega(M,g), d_{\mathcal{R}}, i)\) we get \( H^i_{2,\mathcal{R}}(M,g) \cong \text{Ker}(d_{\mathcal{R},i}) \cap \text{Ker}(d^*_{\mathcal{R},i-1}) \). But \( \text{Ker}(d_{\mathcal{R},i-1}) \) is the orthogonal complement of \( \text{ran}(d_{\mathcal{R},i-1}) = \text{ran}(d_{min,i-1}) \) and the orthogonal complement of this last space is \( \text{Ker}(\delta_{max,i-1}) \). Therefore we got that \( H^i_{2,\mathcal{R}}(M,g) \cong \text{Ker}(d_{max,i}) \cap \text{Ker}(\delta_{max,i-1}) \) and clearly \( \text{Ker}(D_i) \cap \text{Ker}(D^*_{i-1}) \subset \text{Ker}(d_{max,1}) \cap \text{Ker}(\delta_{max,i-1}) \). Summarizing we proved that

\[ \mathcal{T}^i_{2,D}(M,g) \cong \text{Ker}(D_i) \cap \text{Ker}(D^*_{i-1}) \subset \text{Ker}(d_{max,i}) \cap \text{Ker}(\delta_{max,i-1}) \cong H^i_{2,\mathcal{R}}(M,g) \]

and this complete the proof. \( \square \)

**Corollary 1.** Let \((M,g)\) be as in the above proposition. Consider the following properties:

1. \((L^2\Omega(M,g), d_{max/min,i})\) are Fredholm complexes.

2. Every closed extension \((L^2\Omega(M,g), D_1)\) of \((\Omega^i(M), d_1)\) is a weak Fredholm complex.

Then the above conditions are all necessary conditions in order to have \( H^i_{2,\mathcal{R}}(M,g) \) finite dimensional for all \( i \).

From now on the goal of this subsection is to produce some sufficient or equivalent conditions in order to get \( H^i_{2,\mathcal{R}}(M,g) \) finite dimensional. First of all we need to recall some standard tools.

Consider again an open oriented and incomplete riemannian manifold \((M,g)\). Let us label with \( d \) the differential acting on the whole space of smooth forms with compact support, that is

\[ d : \bigoplus_{i=0}^{m} \Omega^i(M) \to \bigoplus_{i=0}^{m} \Omega^i(M). \]

Let \( \delta \) be the formal adjoint of \( d \) and let \( d_{max/min}, \delta_{max/min} \) be the maximal and the minimal extension of \( d \) and \( \delta \) respectively. Finally consider the operator

\[ d + \delta : \bigoplus_{i=0}^{m} \Omega^i(M) \to \bigoplus_{i=0}^{m} \Omega^i(M) \]

and let \( (d + \delta)_{max/min} \) be its maximal and minimal extension respectively. For the sake of simplicity let us label \( \bigoplus_{i=0}^{m} \Omega^i(M) \) with \( \Omega^i(M) \) and analogously \( \bigoplus_{i=0}^{m} L^2\Omega^i(M,g) \) with \( L^2\Omega^i(M,g) \). We have the following proposition which is of independent interest because it provides a description of the null space of \( (d + \delta)_{max} \) in term of weakly closed and weakly cocolsed \( L^2 \)-forms.

**Proposition 8.** Let \((M,g)\) be an open, oriented and incomplete riemannian manifold. Then we have:

\[ \text{Ker}((d + \delta)_{max}) = \text{Ker}(d_{max}) \cap \text{Ker}(\delta_{max}). \]

**Proof.** We start proving the inclusion \( \supset \). Let \( \omega \in \text{Ker}(d_{max}) \cap \text{Ker}(\delta_{max}) \). Then for all \( \phi \in \Omega^*(M) \) we have \( \langle \omega, d\phi \rangle_{L^2\Omega^*(M,g)} = 0 = \langle \omega, \delta \phi \rangle_{L^2\Omega^*(M,g)} \). Therefore, for all \( \phi \in \Omega^*(M) \) we have \( \langle \omega, (d + \delta) \phi \rangle_{L^2\Omega^*(M,g)} = 0 \) and this means that \( \omega \in \text{Ker}((d + \delta)_{max}) \).

Now we prove the opposite inclusion. First of all consider the following operator:

\[ (d + \delta)_{min} \circ (d + \delta)_{max} : \mathcal{D}((d + \delta)_{min} \circ (d + \delta)_{max}) \to L^2\Omega^*(M,g) \]

\[ (d + \delta)_{min} \circ (d + \delta)_{max} : \mathcal{D}((d + \delta)_{min} \circ (d + \delta)_{max}) \to L^2\Omega^*(M,g) \] (31)
where $D((d+\delta)_{\min} \circ (d+\delta)_{\max}) := \{ \omega \in D((d+\delta)_{\max}) \text{ such that } (d+\delta)_{\max} \omega \in D((d+\delta)_{\min}) \}$. Let us call this operator with $\Delta_N$. It is clearly a self-adjoint extension of $\Delta : \Omega^2(M) \rightarrow \Omega^2(M)$. Now define

$$\Delta_{N,i} := \Delta_N|_{L^2\Omega^i(M,g)}$$

with domain $D(\Delta_{N,i}) := D(\Delta_N) \cap L^2\Omega^i(M,g)$. Also in this case it is easy to show that $\Delta_{N,i}$ is a self-adjoint extension of $\Delta : \Omega^2_c(M) \rightarrow \Omega^2_c(M)$. Now if we consider

$$\Delta_{N,i} : L^2\Omega^i(M,g) \rightarrow L^2\Omega^i(M,g)$$

and

$$\bigoplus_{i=0}^m \Delta_{N,i} : \bigoplus_{i=0}^m L^2\Omega^i(M,g) \rightarrow \bigoplus_{i=0}^m L^2\Omega^i(M,g)$$

with domain given by $\bigoplus_{i=0}^m D(\Delta_{N,i})$ we have that they are both self-adjoint extension of $\Delta : \Omega^2_c(M) \rightarrow \Omega^2_c(M)$. Moreover it is clear that $\Delta_N$ is an extension of $\bigoplus_{i=0}^m \Delta_{N,i}$. Therefore we can conclude that they are the same operator. Now if we consider $\eta \in Ker((d+\delta)_{\max})$ we have $\eta \in Ker(\Delta_N)$, because $Ker(\Delta_N) = Ker((d+\delta)_{\max})$, and therefore if

$$\eta = \eta_0 + \eta_1 + \ldots + \eta_m \in L^2\Omega^0(M,g) \oplus \ldots \oplus L^2\Omega^m(M,g)$$

we can conclude that $\eta_i \in Ker(\Delta_{N,i})$ for all $i = 0, \ldots, m$. So we proved that $\eta_i \in Ker(\Delta_N)$ and this implies that $\eta_i \in Ker((d+\delta)_{\max})$. Therefore for all $\phi \in \Omega^2_c(M)$ we have $\langle \eta_i, (d+\delta)\phi \rangle_{L^2\Omega^i(M,g)} = 0$. But $\eta_i$ is an $i$-form and therefore the last equality is equivalent to say that for all $\phi \in \Omega^i_c(M)$ and for all $\psi \in \Omega^{i+1}_c(M)$ we have $\langle \eta_i, d_{i-1}\phi + \delta\psi \rangle_{L^2\Omega^{i+1}(M,g)} = 0$ and this means that, for all $i = 0, \ldots, m$, $\eta_i \in Ker(d_{\max,i}) \cap Ker(\delta_{\max,i-1})$. By the fact that it is immediate to check that

$$\bigoplus_{i=0}^m (Ker(d_{\max,i}) \cap Ker(\delta_{\max,i-1})) = Ker(d_{\max}) \cap Ker(\delta_{\max})$$

we can conclude that $\eta \in Ker(d_{\max}) \cap Ker(\delta_{\max})$ and this complete the proof.

Using Prop. 8 we are now in position to prove the main result of the section. In particular, involving the operator $(d+\delta)_{\max}$, we will exhibit some conditions which are equivalent to say that the complex $(L^2\Omega^i(M,g), d_{\max,i})$ is Fredholm. From the theorem it will be clear that the behavior of the complex $(L^2\Omega^i(M,g), d_{\max,i})$ is strictly connected to that of the operator $(d+\delta)_{\max}$.

**Theorem 8.** Let $(M,g)$ be an open, oriented and incomplete riemannian manifold which satisfies the assumptions of Theorem 7. Then the following properties are equivalent:

1. $(L^2\Omega^i(M,g), d_{\max,i})$ is a Fredholm complex.
2. The complex $(\Omega^i_c(M), d_i)$ has only a finite number of closed extensions and they are all Fredholm. Moreover, for all $i = 0, \ldots, m$, we have that

$$D(d_{\max,i})/D(d_{\min,i})$$

is a finite dimensional vector space whose dimension equals the number of closed extensions of $d_i : \Omega^i_c(M) \rightarrow \Omega^{i+1}_c(M)$.
3. $(d+\delta)_{\max} : D((d+\delta)_{\max}) \rightarrow L^2\Omega^i(M,g)$ is a Fredholm operator on its domain endowed with the graph norm.
4. Every closed extension of $d+\delta : \Omega^2_c(M) \rightarrow \Omega^2_c(M)$ is a Fredholm operator. Therefore there are only a finite number of closed extensions and this number is given by

$$D((d+\delta)_{\max})/D((d+\delta)_{\min})$$
Proof. The equivalence between 3) and 4) is proved in [3] Lemma 1.3.15. We recall the proof for the sake of completeness. Assume that \((d + \delta)_{\text{max}}\) is Fredholm operator on its domain endowed with the graph norm. Then \(\text{ran}(d + \delta)_{\text{max}}\) is closed in \(L^2\Omega^*(M, g)\). Therefore also \(\text{ran}(d + \delta)_{\text{min}}\) is closed in \(L^2\Omega^*(M, g)\) because \(((d + \delta)_{\text{max}})^* = (d + \delta)_{\text{min}}\). From this it follows that \(L^2\Omega^*(M, g)/\text{ran}(d + \delta)_{\text{min}}\) is finite dimensional because \(L^2\Omega^*(M, g)/\text{ran}(d + \delta)_{\text{min}}\) \(\cong \text{Ker}(d + \delta)_{\text{max}}\) which is finite dimensional. Moreover we have clearly \(\text{Ker}(d + \delta)_{\text{min}} \subset \text{Ker}(d + \delta)_{\text{max}}\) which is finite dimensional. So we can conclude that \((d + \delta)_{\text{min}}\) is a Fredholm operator on its domain endowed with the graph norm. By the fact that \((d + \delta)_{\text{max/min}}\) are both Fredholm operators on their domains endowed with the graph norm it follows immediately that every other closed extension of \(d + \delta : \Omega^*_1(M) \rightarrow \Omega^*_1(M)\) is a Fredholm operator on its domain endowed with the graph norm. Finally, if we consider the natural inclusion \(i : \mathcal{D}(d + \delta)_{\text{min}} \rightarrow \mathcal{D}(d + \delta)_{\text{max}}\) we have \((d + \delta)_{\text{min}} = (d + \delta)_{\text{max}} \circ i\). Therefore also \(i : \mathcal{D}((d + \delta)_{\text{min}}) \rightarrow \mathcal{D}((d + \delta)_{\text{max}})\) is a Fredholm operator with index given by \(-\dim(\mathcal{D}(d + \delta)_{\text{max}})/\mathcal{D}((d + \delta)_{\text{min}}))\) which therefore is a finite dimensional vector space. It follows immediately that \(d + \delta : \Omega^*_1(M) \rightarrow \Omega^*_1(M)\) admits only a finite number of closed extensions and this number is exactly \(\dim(\mathcal{D}((d + \delta)_{\text{max}})/\mathcal{D}((d + \delta)_{\text{min}}))\).

Now from 4) it follows immediately that \((\Omega^*_1(M), d_i)\) admits only a finite number of closed extensions and all these extensions are Fredholm complexes. The reason is given by the fact that to a closed extension \((L^2\Omega^1(M, g), D_i)\) of \((\Omega^*_1(M), d_i)\) we can associate a closed extension of \(d + \delta : \Omega^*_1(M) \rightarrow \Omega^*_1(M)\) taking

\[
D + D^* : L^2\Omega^*(M, g) \rightarrow L^2\Omega^*(M, g)
\]

with domain given by

\[
\mathcal{D}(D + D^*) = \bigoplus_{i=0}^m (\mathcal{D}(D_i) \cap \mathcal{D}(D_{i-1}^*))
\]

and defining \(D + D^*\) as

\[
D + D^* := \bigoplus_{i=0}^m (D_{i-1}^* + D_i).
\]

As proved in [3], Lemma 2.3, this construction is injective and therefore \((\Omega^*_1(M), d_i)\) admits only a finite number of closed extensions. Moreover, from Prop. 3 it follows that they are all Fredholm extensions. This in turn implies clearly that for all \(i = 0, \ldots, m\) \(\mathcal{D}(d_{\text{max},i})/\mathcal{D}(d_{\text{min},i})\) is a finite dimensional vector space. In this way we proved that properties 4) implies properties 2). Clearly 2) implies 1). Finally consider the property 1). If \((L^2\Omega^1(M, g), d_{\text{min},i})\) is Fredholm then, by definition, we know that \(H^2_{\text{min}}(M, g)\) is finite dimensional for all \(i\). This implies immediately that both \((L^2\Omega^1(M, g), d_{\text{max/min},i})\) are Fredholm complexes. Moreover, as showed in proposition 7 we know that \(\text{Ker}(d_{\text{min},i}) \cap \text{Ker}(d_{\text{max},i-1}) = \text{Ker}(d_{\text{max},i}) \cap \text{Ker}(d_{\text{min},i-1})\). So, using Prop. 6 we know that \(\text{Ker}((d + \delta)_{\text{max}})\) is finite dimensional. Now, by the fact that \((L^2\Omega^1(M, g), d_{\text{max},i})\) is a Fredholm complex, it follows that

\[
d_{\text{max}} + \delta_{\text{min}} : L^2\Omega^*(M, g) \rightarrow L^2\Omega^*(M, g)
\]

is Fredholm operator on its domain endowed with the graph norm as recalled in Prop. 4. This means that \(L^2\Omega^*(M, g)/\text{ran}(d_{\text{max}} + \delta_{\text{min}})\) is finite dimensional. But clearly we have a natural and surjective map

\[
(L^2\Omega^*(M, g)/\text{ran}(d_{\text{max}} + \delta_{\text{min}})) \rightarrow L^2\Omega^*(M, g)/\text{ran}(d + \delta)_{\text{max}}.
\]

In this way we can conclude that also \(L^2\Omega^*(M, g)/\text{ran}(d + \delta)_{\text{max}}\) is finite dimensional and therefore that \(\text{ran}(d + \delta)_{\text{max}}\) is closed in \(L^2\Omega^*(M, g)\). In this way we got that \((d + \delta)_{\text{max}}\) is a Fredholm operator on its domain endowed with the graph norm and this proves that 1) implies 3). This complete the proof.

From theorem 1 we get the following corollaries:

**Corollary 2.** Let \((M, g)\) be an open, oriented and incomplete riemannian manifold which satisfies the assumptions of Theorem 1. Then the following properties are equivalent:

1. Every closed extension of \(d + \delta : \Omega^*_1(M) \rightarrow \Omega^*_1(M)\) is a Fredholm operator on its domain endowed with the graph norm.
2. Every self-adjoint extension of \( d + \delta : \Omega_c^\ast(M) \to \Omega_c^\ast(M) \) is a Fredholm operator on its domain endowed with the graph norm.

**Proof.** Obviously the first property implies the second one. Assume now that every self-adjoint extension of \( d + \delta : \Omega_c^\ast(M) \to \Omega_c^\ast(M) \) is a Fredholm operator on its domain endowed with the graph norm. In particular

\[
d_{2\mathbb{R}} + d_{2\mathbb{R}}^* : \mathcal{D}(d_{2\mathbb{R}} + d_{2\mathbb{R}}^*) \to L^2\Omega^\ast(M, g)
\]

is a Fredholm operator and therefore, according to Prop. 7, \( (L^2\Omega^\ast(M, g), d_{2\mathbb{R}, i}) \) is a Fredholm complex. Finally applying Theorem \( 8 \) we get the conclusion. \( \square \)

**Corollary 3.** Let \((M, g)\) be an open oriented and incomplete riemannian manifold of dimension \( m \). Assume that \((L^2\Omega^\ast(M, g), d_{\max, i})\) is a Fredholm complex and that \( d_{\max, i} = d_{\min, i} \) for all \( i = 0, ..., m \). Then

\[
(d + \delta)_{\max} : L^2\Omega^\ast(M, g) \to L^2\Omega^\ast(M, g)
\]

is a Fredholm operator on its domain endowed with the graph norm. Therefore

\[
d + \delta : \Omega_c^\ast(M) \to \Omega_c^\ast(M)
\]

admits only a finite number of closed extensions and they are all Fredholm operators on their respective domains endowed with the graph norm.

**Proof.** It follows immediately applying the second statement of Theorem \( 5 \). \( \square \)

Before to state the next corollary we recall that two riemannian metrics \( g \) and \( h \) over the same manifold \( M \) are said quasi isometric if there exists a positive real number \( c \) such that \( c^{-1} g \leq h \leq c g \).

**Corollary 4.** Let \( M \) be an open and oriented manifold and let \( g \) and \( h \) be two incomplete metrics over \( M \). Let us label with \( \delta_g \) the formal adjoint of \( d \) with respect to \( g \) and analogously \( \delta_h \) the formal adjoint of \( d \) with respect to \( h \). If \( g \) and \( h \) are quasi isometric then

\[
(d + \delta_g)_{\max} : \mathcal{D}((d + \delta_g)_{\max}) \to L^2\Omega^\ast(M, g)
\]

is a Fredholm operator on its domain endowed with the graph norm if and only if

\[
(d + \delta_h)_{\max} : \mathcal{D}((d + \delta_h)_{\max}) \to L^2\Omega^\ast(M, h)
\]

it is a Fredholm operator on its domain endowed with the graph norm.

**Proof.** If \( g \) and \( h \) are quasi isometric then \( L^2\Omega^\ast(M, g) = L^2\Omega^\ast(M, h) \), \( \mathcal{D}(d_{\max/min, i}) \) (respect to \( g \)) = \( \mathcal{D}(d_{\max/min, i}) \) (respect to \( h \)), \( \ker(d_{\max/min, i}) \) (respect to \( g \)) = \( \ker(d_{\max/min, i}) \) (respect to \( h \)) and \( \text{im}(d_{\max/min, i}) \) (respect to \( g \)) = \( \text{im}(d_{\max/min, i}) \) (respect to \( h \)). This in particular implies that \( H^1_{\mathbb{R}^2}(M, g) = H^1_{\mathbb{R}^2}(M, h) \). Therefore, applying Theorem \( 7 \), \( \delta_g \) is Fredholm if and only if \( L^2\Omega^\ast(M, g), d_{2\mathbb{R}, i} \) is a Fredholm complex if and only if \( L^2\Omega^\ast(M, h), d_{2\mathbb{R}, i} \) is Fredholm and if and only if \( (d + \delta_h)_{\max} \) is Fredholm. \( \square \)

**Corollary 5.** Let \((M, g)\) be an open oriented and incomplete riemannian manifold of dimension \( m \) such that \((L^2\Omega^\ast(M, g), d_{2\mathbb{R}, i})\) is a Fredholm complex. Then the following two properties are equivalent:

1. The inclusion \( \mathcal{D}(d_{\min, i}) \subset L^2\Omega^\ast(M, g) \) is compact for every \( i = 0, ..., m \).
2. The inclusion \( \mathcal{D}(d_{\max, i}) \subset L^2\Omega^\ast(M, g) \) is compact for every \( i = 0, ..., m \).

Moreover they implies that the inclusion

\[
\mathcal{D}((d + \delta)_{\max}) \subset L^2\Omega^\ast(M, g)
\]

is compact.
Proof. Clearly 2) implies 1) . Assume now 1). Let \( \{ \eta_n \} \) a bounded sequence in \( D(d_{\text{max}},i) \). Consider \( D(d_{\text{max}},i) \) as a Hilbert space endowed with the graph product. Then \( D(d_{\text{min}},i) \) is a closed subspace. Let \( V \) be the orthogonal complement of \( D(d_{\text{min}},i) \) in \( D(d_{\text{max}},i) \) with respect to the graph product. By Theorem 5 we know that \( V \) is finite dimensional. Now let \( \pi_j, j = 1, 2 \) be the orthogonal projections on \( D(d_{\text{min}},i) \) and \( V \) respectively. Clearly \( \{ \pi_1(\eta_n) \} \) and \( \{ \pi_2(\eta_n) \} \) are both bounded in the graph norm of \( D(d_{\text{max}},i) \). So, having assumed 1), we know that there exists a subsequence \( \{ \eta'_{n} \} \) of \( \{ \eta_n \} \) such that \( \pi_1(\eta'_n) \) converges in \( L^2\Omega(M,g) \). Now from \( \{ \eta'_n \} \) we can extract a subsequence \( \{ \eta''_n \} \) such that \( \pi_2(\eta''_n) \) converges in \( L^2\Omega(M,g) \). In this way we can finally conclude that \( \{ \eta''_n \} \) converges in \( L^2\Omega(M,g) \).

Now consider the following operator:

\[
d_{\text{max}} + \delta_{\text{min}} : L^2\Omega^*(M,g) \rightarrow L^2\Omega^*(M,g)
\]

where the domain is described in (7). Clearly the inclusions \( D((d + \delta)_{\text{min}}) \subset D(d_{\text{max}} + \delta_{\text{min}}) = D(d_{\text{max}}) \cap D(\delta_{\text{min}}) \) are continuous with the respective graph norms. Moreover it is clear that

\[
D(d_{\text{max}}) \cap D(\delta_{\text{min}}) = \bigoplus_{i=0}^{m} D(d_{\text{max},i}) \cap D(\delta_{\text{min},i-1}) \subset \bigoplus_{i=0}^{m} D(d_{\text{max},i}) \subset L^2\Omega^*(M,g).
\]

By 2) we know that the last inclusion is compact and so we can conclude that the inclusion

\[
D((d + \delta)_{\text{min}}) \subset L^2\Omega^*(M,g)
\]

is compact. Furthermore we know that \( (d + \delta)_{\text{min}} \) is a Fredholm operator and that \( W \), the orthogonal complement of \( D((d + \delta)_{\text{min}}) \) in \( D((d + \delta)_{\text{max}}) \) with respect the graph product, is finite dimensional. So we can use the same argument we used to prove 2) implies 1) in order to prove that the inclusion \( D((d + \delta)_{\text{max}}) \subset L^2\Omega^*(M,g) \) is compact. This completes the proof.

Corollary 6. Let \( (M,g) \) be an open oriented and incomplete riemannian manifold of dimension \( m \) such that \( (L^2\Omega(M,g), d_{\Omega^i}) \) is a Fredholm complex. Assume that the first property of corollary 5 holds for all \( i = 0, \ldots, m \). Then for every closed extension \( D \) of

\[
d + \delta : \Omega^*_\varepsilon(M) \rightarrow \Omega^*_\varepsilon(M)
\]

\( D^*D \) is a discrete operator with compact resolvent.

Proof. In order to prove the statement, as it is showed for example in [11] Cor. 4.2.3, it is enough to prove that the inclusion \( D(D^*) \subset L^2\Omega^*(M,g) \) is compact. By Theorem 5 and Corollary 5 we know that if we consider \( D(D^*) \) endowed with its graph norm then the inclusion \( D(D^*) \subset L^2\Omega^*(M,g) \) is compact. Clearly the inclusion \( D(D^*) \subset D(D) \), where each domain is endowed with its graph norm, is continuous. The reason is given by the fact that if \( \eta \in D(D^*) \) then:

\[
\| D\eta \|_{L^2\Omega^*(M,g)}^2 = \langle \eta, D^* D\eta \rangle_{L^2\Omega^*(M,g)} \leq \frac{1}{2} \| \eta \|_{L^2\Omega^*(M,g)}^2 + \| D^* D\eta \|_{L^2\Omega^*(M,g)}^2
\]

and from this previous inequality we get:

\[
\| \eta \|_{L^2\Omega^*(M,g)}^2 + \| D\eta \|_{L^2\Omega^*(M,g)}^2 \leq \frac{3}{2} \| \eta \|_{L^2\Omega^*(M,g)}^2 + \| D^* D\eta \|_{L^2\Omega^*(M,g)}^2.
\]

Therefore if we consider \( D(D^*) \) endowed with its graph norm then we can conclude that the inclusion \( D(D^*) \subset L^2\Omega^*(M,g) \) is compact.

Finally we conclude this section with the following proposition:

Proposition 9. Let \( (M,g) \) be an open and oriented manifold. The the following properties are equivalent:

1. \( D(d_{\text{min},i}) = D(d_{\text{max},i}) \) for all \( i = 0, \ldots, m \).
2. \( \text{im}(d_{\text{min},i}) = \text{im}(d_{\text{max},i}) \) for all \( i = 0, \ldots, n \).
Moreover if \((L^2Ω^i(M,g),\delta_{\max/min,i})\) is a Fredholm complex then we have the following list of equivalent properties:

1. \(\mathcal{D}(d_{\min,i}) = \mathcal{D}(d_{\max,i})\) for all \(i = 0,...,m\).
2. \(\text{im}(d_{\min,i}) = \text{im}(d_{\max,i})\) for all \(i = 0,...,m\).
3. \(\text{Ker}(d_{\min,i}) = \text{Ker}(d_{\max,i})\) for all \(i = 0,...,m\).
4. \(H_{2,\max}^2(M,g) \cong H_{2,\min}^2(M,g)\) for all \(i = 0,...,m\).
5. \(H_{2,\min}^i(M,g) \cong H_{2,\max}^i(M,g)\) for all \(i = 0,...,m\).

**Proof.** Start proving the equivalence of the first pair of statements. Clearly 1) implies 2). Assume now that 2) holds. Then we know that also \(\text{im}(d_{\min,i}) = \text{im}(d_{\max,i})\) for all \(i = 0,...,m\). Therefore we get \(\text{Ker}(d_{\min,i}) = \text{Ker}(d_{\max,i})\) and finally, using the Hodge star operator we get \(\text{Ker}(d_{\min,i}) = \text{Ker}(d_{\max,i})\) for all \(i = 0,...,m\). Now let \(\eta \in \mathcal{D}(d_{\max,i})\). Then there exists \(\omega \in \mathcal{D}(d_{\max,i})\) such that \(d_{\max,i} \eta = d_{\min,i} \omega\). This means that \(\eta - \omega \in \text{Ker}(d_{\max,i})\) and therefore there exist \(\psi \in \text{Ker}(d_{\min,i})\) such that \(\eta - \omega = \psi\). Summarizing we got \(\eta = \omega + \psi \in \mathcal{D}(d_{\min,i})\) and this concludes the proof of the first part.

Now we prove the second part of the proposition. First of all we observe that using the Hodge star operator, it follows easily that \((L^2Ω^i(M,g),\delta_{\max,i})\) is Fredholm if and only if \((L^2Ω^i(M,g),d_{\min,i})\) is Fredholm. Now from the first part we know that the first two assertions are equivalent and they imply the remaining statements. Assume now that 3) holds. Then applying the Hodge star operator we know that also \(\text{Ker}(\delta_{\min,i}) = \text{Ker}(\delta_{\max,i})\) and therefore \(\text{im}(d_{\min,i}) = \text{im}(d_{\max,i})\) for all \(i = 0,...,m\) because, being \((L^2Ω^i(M,g),\delta_{\max/min,i})\) a Fredholm complexes, it follows that \(\text{im}(d_{\max/min,i})\) is closed. So we can apply the first part of the proposition to get the conclusion.

Now assume that 4) holds. Then \(H_{2,\min}^2(M,g)\) is finite dimensional. We already know that \(H_{2,\max}^2(M,g) \cong \text{Ker}(\delta_{\min,i-1}) \cap \text{Ker}(d_{\min,i}) \subset \text{Ker}(\delta_{\max,i-1}) \cap \text{Ker}(d_{\max,i}) \cong H_{2,\max}^2(M,g)\). Combining with 4) we get

\[\text{Ker}(\delta_{\min,i-1}) \cap \text{Ker}(d_{\max,i}) = \text{Ker}(\delta_{\max,i-1}) \cap \text{Ker}(d_{\max,i})\]

and therefore using the weak Kodaira decompositions \((22)\) and \((27)\) we have:

\[\text{im}(d_{\max,\min,i}) = \text{im}(d_{\min,\max,i}) \oplus \text{im}(\delta_{\min,\max,i})\]

In this way we get: \(\text{im}(d_{\min,\max,i}) = \text{im}(d_{\min,\max,i})\) for each \(i\). So we are in position to apply the first part of the proposition and therefore we proved that 4) \(\Rightarrow\) 1). In the same way, with the obvious modifications, we can prove that 5) \(\Rightarrow\) 1). The Proposition is thus proved.

### 2.3 \(L^2\)-Euler characteristic and \(L^2\)-signature

The aim of this subsection is to give some geometric applications using the results previously proved. In particular we will show that when \((L^2Ω^i(M,g),\delta_{\min,i})\) is a Fredholm complex then we can define a \(L^2\)-Euler characteristic and a \(L^2\)-signature for \((M,g)\). Moreover we will prove that they correspond respectively to the index of some suitable Fredholm operators.

**Definition 6.** Let \((M,g)\) be an open and oriented manifold such that \((L^2Ω^i(M,g),\delta_{\min,i})\) is a Fredholm complex. Then the \(L^2\)-Euler characteristic of \((M,g)\) associated to \((L^2Ω^i(M,g),\delta_{\min,i})\) is defined as

\[
\chi_{2,\min}^i(M,g) := \sum_{i=0}^m (-1)^i b_{2,i,\min}(M,g)
\]

where \(b_{2,i,\min}(M,g) := \text{dim}(H_{2,\min}^i(M,g))\).

We have the following immediate corollary:

**Corollary 7.** Let \((M,g)\) be as in the previous definition. If \(m\) is odd then \(\chi_{2,\min}^i(M,g) = 0\).
If $m$ is even

$$\chi_{2,2m}(M, g) = b_{2,0,2m}(M, g) + b_{2,m,2m}(M, g)$$

if $\frac{m}{2}$ is still even while

$$\chi_{2,2m}(M, g) = b_{2,0,2m}(M, g) - b_{2,m,2m}(M, g)$$

if $\frac{m}{2}$ is odd.

Proof. It follows immediately by the fact that $H^i_{2,2m}(M, g) \cong H^{m-i}_{2m}(M, g)$. \qed

Now consider the operator

$$d_{2m} + d_{2m}^*: L^2\Omega^*(M, g) \to L^2\Omega^*(M, g)$$

defined according to (7). Let us label $L_{2m}$ if $m$ is even, and $L_{2m}$ if $m$ is odd.

Furthermore consider the following operator

$$d_{2m} : L^2\Omega^c(M, g) \to L^2\Omega^c(M, g)$$

defined as the maximal extension of $d_{2m}$.

Let $\Omega_{2m}^c(M, g) := \bigoplus_{i=0}^m L^2\Omega^{2i}(M, g)$ and analogously $L^2\Omega^{2i+1}(M, g)$.

Define

$$(d_{2m} + d_{2m}^*)^{ev/odd} : L^2\Omega^{ev/odd}(M, g) \to L^2\Omega^{ev/odd}(M, g)$$

as the restriction of $d_{2m} + d_{2m}^*$ to $L^2\Omega^{ev/odd}(M, g)$ with domain given by

$$\mathcal{D}(d_{2m} + d_{2m}^*)^{ev/odd} := \mathcal{D}(d_{2m} + d_{2m}^*) \cap L^2\Omega^{ev}(M, g).$$

Furthermore consider the following operator

$$(d + \delta)^{ev/odd} : L^2\Omega^{ev/odd}(M, g) \to L^2\Omega^{ev/odd}(M, g)$$

defined as the maximal extension of

$$d + \delta : \Omega_{2m}^{ev/odd}(M) \to \Omega_{2m}^{ev/odd}(M)$$

where $\Omega_{2m}^{ev}(M) = \bigoplus_{i=0}^m \Omega_{2i}^c(M)$ and $\Omega_{2m}^{odd}(M) = \bigoplus_{i=0}^m \Omega_{2i+1}^c(M)$. It is immediate to check that, if we consider

$$(d + \delta)^{max} : L^2\Omega^{*}(M, g) \to L^2\Omega^{*}(M, g)$$

then we have

$$\mathcal{D}((d + \delta)^{max}) \cap L^2\Omega^{ev/odd}(M, g) = \mathcal{D}(((d + \delta)^{ev/odd})^{max}).$$

Finally we are ready for the next theorem.

**Theorem 9.** Let $(M, g)$ be an open, oriented and incomplete riemannian manifold such that $(L^2\Omega^c(M, g), d_{2m})$ is a Fredholm complex. Then $(d_{2m} + d_{2m}^*)^{ev}$ and $((d + \delta)^{ev})^{max}$ are Fredholm operators on their respective domains endowed with the graph norm and we have:

$$\text{ind}((d_{2m} + d_{2m}^*)^{ev}) = \chi_{2,2m}(M, g) = \text{dim}(\ker((d + \delta)^{ev})^{max})) - \text{dim}(\ker(((d + \delta)^{odd})^{max})$$

Proof. By Theorem 3 we know that $d_{2m} + d_{2m}^*$ is a Fredholm operator on its domain endowed with the graph norm. Clearly we have

$$\ker((d_{2m} + d_{2m}^*)^{ev}) = \ker(d_{2m} + d_{2m}^*) \cap L^2\Omega^{ev}(M, g)$$

and

$$\text{im}((d_{2m} + d_{2m}^*)^{ev}) = \text{im}(d_{2m} + d_{2m}^*) \cap L^2\Omega^{odd}(M, g).$$

It follows immediately that $\ker((d_{2m} + d_{2m}^*)^{ev})$ is finite dimensional and that $\text{im}((d_{2m} + d_{2m}^*)^{ev})$ is closed with finite dimensional orthogonal complement. So we got that $(d_{2m} + d_{2m}^*)^{ev}$ is a Fredholm operator on its domain endowed with the graph norm. Analogously, using again theorem 3 we know that $((d + \delta)^{ev})^{max}$ is Fredholm. By (33) we know that

$$\ker(((d + \delta)^{ev})^{max}) = \ker((d + \delta)^{max}) \cap L^2\Omega^{ev}(M, g)$$
and therefore it is finite dimensional. Moreover we have a natural and surjective map

$$\pi_{2,20}^\text{odd} \rightarrow \pi_{2,20}^\text{odd}$$

and from this it follows that $\text{im}((d + \delta)^{\text{ev}})_{\text{max}}$ is closed with finite dimensional orthogonal complement. Therefore we can conclude that $((d + \delta)^{\text{ev}})_{\text{max}}$ is a Fredholm operator on its domain endowed with the graph norm.

Now using (0), (8), (9) and the fact that $\ker((d_{20} + d_{20})^{\text{ev}}) = \sum_{i=0}^n \ker(\Delta_{20,2i})$ and that $(\text{im}((d_{20} + d_{20})^{\text{odd}}))^{-1} = \ker((d_{20} + d_{20})^{\text{odd}}) = \sum_{i=0}^n \ker(\Delta_{20,2i+1})$ we get the first equality of (34).

For the second equality by (33) we know that $\ker(((d + \delta)^{\text{ev/odd}})_{\text{max}}) = \ker((d + \delta)_{\text{max}}) \cap L^2\Omega^{\text{ev/odd}}(M, g)$ and combining this with Prop. (3) we get

$$\ker(((d + \delta)^{\text{ev}})_{\text{max}}) = \bigoplus_{i=0}^m (\ker(d_{\text{max,2i}}) \cap \ker(\delta_{\text{max,2i-1}}))$$

and analogously

$$\ker(((d + \delta)^{\text{odd}})_{\text{max}}) = \bigoplus_{i=0}^m (\ker(d_{\text{max,2i+1}}) \cap \ker(\delta_{\text{max,2i}})).$$

As we already know (35) and (36) are isomorphic respectively to

$$\bigoplus_{i=0}^m H^2_{2,20}(M, g)$$

and to

$$\bigoplus_{i=0}^m H^{2+1}_{2,20}(M, g).$$

In this way also the second equality is established.

In the rest of this subsection we will describe how to define a $L^2-$signature for $(M, g)$ using $H^2_{2,20}(M, g)$. To this aim, first of all, let us label $\pi_{2,20}^i(M, g)$ the vector spaces defined as

$$\pi_{2,20}^i(M, g) := \ker(d_{\text{max,2i}})/\text{im}(d_{\text{min,2i-1}}).$$

The first step is to show that using the wedge product we can construct a well defined and non degenerate pairing between $\pi_{2,20}^i(M, g)$ and $\pi_{2,20}^{n-i}(M, g)$ where $m = \text{dim} M$.

We define:

$$\pi_{2,20}^i(M, g) \times \pi_{2,20}^{n-i}(M, g) \rightarrow \mathbb{R}, ([\eta], [\omega]) \mapsto \int_M \eta \wedge \omega$$

(37)

where $\omega$ and $\eta$ are any representative of $[\eta]$ and $[\omega]$ respectively.

**Proposition 10.** Let $(M, g)$ be an open, oriented and incomplete riemannian manifold of dimension $m$. Then (37) is a well defined and non degenerate pairing.

**Proof.** The first step is to show that (37) is well defined. Let $\eta', \omega'$ other two forms such that $[\eta] = [\eta']$ in $\pi_{2,20}^i(M, g)$, $[\omega] = [\omega']$ in $\pi_{2,20}^{n-i}(M, g)$. Then there exist $\alpha \in \text{im}(d_{\text{min,2i-1}})$ and $\beta \in \text{im}(d_{\text{min,2n-2i+1}})$ such that $\eta = \eta' + \alpha$ and $\omega = \omega' + \beta$. Therefore

$$\int_M \eta' \wedge \omega = \int_M (\eta' + \alpha) \wedge (\omega' + \beta) = \int_M \eta' \wedge \omega' + \int_M \eta' \wedge \beta + \int_M \alpha \wedge \omega + \int_M \alpha \wedge \beta$$

Now

$$\int_M \eta' \wedge \beta = \pm \int_M \langle \eta', *\beta \rangle \text{dvol}_g = \langle \eta', *\beta \rangle_{L^2\Omega^i(M, g)} = 0$$

because $*\beta \in \text{im}(\delta_{\text{min,2i}})$ and $\alpha \in \ker(d_{\text{max,2i}})$. In the same way:

$$\int_M \alpha \wedge \beta = \pm \int_M \langle \alpha, *\beta \rangle \text{dvol}_g = \langle \alpha, *\beta \rangle_{L^2\Omega^i(M, g)} = 0$$
because $\alpha \in \text{im}(d_{\min,i-1})$ and $*\beta \in \text{im}(\delta_{\min,i})$. Finally
\[
\int_M \alpha \wedge \omega' = \pm \int_M \langle \alpha, *\omega' \rangle d\text{vol}_g = \langle \alpha, *\omega' \rangle_{L^2Q^i(M,g)} = 0
\]
because $\alpha \in \text{im}(d_{\min,i-1})$ and $\omega' \in \ker(\delta_{\max,i-1})$. So we can conclude that (37) is well defined. Now fix $[\eta] \in \overline{\Omega}_{2,2R}(M,g)$ and suppose that for each $[\omega] \in \overline{\Omega}_{2,2R}(M,g)$ the pairing (37) vanishes. Then this means that for each $\omega \in \ker(d_{\min,m-1})$ we have $\int_M \eta \wedge \omega = 0$. We also know that $\int_M \eta \wedge \omega = \langle \eta, *\omega \rangle_{L^2Q^i(M,g)}$ and that $\ker(d_{\max,m-1}) = \ker(\delta_{\max,m-1})$. So by the fact that $(\ker(\delta_{\max,i-1}))^\perp = \text{ran}(d_{\min,i-1})$ we obtain that $[\eta] = 0$. In the same way if $[\omega] \in \overline{\Omega}_{2,2R}(M,g)$ is such that for each $[\eta] \in \overline{\Omega}_{2,2R}(M,g)$ the pairing (37) vanishes then we know that for each $\eta \in \ker(d_{\max,i})$ we have $\int_M \eta \wedge \omega = 0$. But we know that $\int_M \eta \wedge \omega = \langle \eta, *\omega \rangle_{L^2Q^i(M,g)}$. By the fact that $(\text{ran}(d_{\min,m-1})) = \ker(\delta_{\min,m-1})$ we obtain that $[\omega] = 0$. So we can conclude that the pairing (37) is well defined and non degenerate and this establish the proposition.

We have the following immediate corollary:

**Corollary 8.** Let $(M,g)$ be an open, oriented and incomplete riemannian manifold of dimension $m = 4n$. Then on $\overline{\Omega}_{2,2R}^n(M,g)$ the pairing (37) is a symmetric bilinear form.

We can now state the following definition:

**Definition 7.** Let $(M,g)$ be an open, oriented and incomplete riemannian manifold of dimension $m = 4n$ such that, for $i = 2n$, $\overline{\Omega}_{2,2R}^n(M,g)$ is finite dimensional. Then we define the $L^2$–signature of $(M,g)$ associated to $\overline{\Omega}_{2,2R}^n(M,g)$ and we label it $\sigma_{2,2R}(M,g)$ as the signature of the pairing (37) applied on $\overline{\Omega}_{2,2R}^n(M,g)$.

Before to conclude the subsection with the next theorems we need to introduce some notations. Let $(M,g)$ be an open oriented and incomplete riemannian manifolds of dimension $m = 2l$. Consider the complexified cotangent bundle $T^*_C M \cong T^*M \otimes \mathbb{C}$. Then the metric $g$ admits a natural extension as a positive definite hermitian metric on $T^*M \otimes \mathbb{C}$ and therefore, in complete analogy to the real case, we can build $L^2\Omega^*_C(M,g) \cong L^2\Omega^*(M,g) \otimes \mathbb{C}$, $d_{\max/min,i} : L^2\Omega^*_C(M,g) \rightarrow L^2\Omega_i^{1+i}(M,g)$, $(d + \delta)_{\max/min} : L^2\Omega^*_C(M,g) \rightarrow L^2\Omega^*_C(M,g)$ etc etc. Consider now the endomorphism $\epsilon : \Lambda^*_C(T^*M) \rightarrow \Lambda^*_C(T^*M)$ defined by $\epsilon(\eta) = (\sqrt{-1})^{l(l-1)+1}\eta$ where $\eta \in \Omega^1(M,\mathbb{C})$. This is the well known endomorphism of the classical signature theorem. In fact we have $e^2 = 1d$ and therefore we get the well known $\mathbb{Z}_2$ graduation of the signature theorem given by the autospace of $\epsilon : \Lambda^*_C(M) \cong (\Lambda^*_C(M))^+ \oplus (\Lambda^*_C(M))^-$, $\Omega^*(M,\mathbb{C}) \cong (\Omega^*(M,\mathbb{C}))^+ \oplus (\Omega^*(M,\mathbb{C}))^-$. Clearly we can extend this $\mathbb{Z}_2$ graduation also in the $L^2$ setting getting $L^2\Omega^*_C(M,g) \cong (L^2\Omega^*_C(M,g))^+ \oplus (L^2\Omega^*_C(M,g))^-$. Another well known property is that $d + \delta$ is odd with respect to $\epsilon$. So we can recall the definition of the signature operator as the operator acting in the following way:

$$d + \delta : (\Omega^*_C(M,\mathbb{C}))^+ \rightarrow (\Omega^*_C(M,\mathbb{C}))^-.$$  

We will label it $D_{\text{sign},+}$. Clearly $D_{\text{sign},-}$, that is $d + \delta : (\Omega^*_C(M,\mathbb{C}))^- \rightarrow (\Omega^*_C(M,\mathbb{C}))^+$ is the formal adjoint of $D_{\text{sign},+}$. Finally we will label with $\Delta^+: := D_{\text{sign},-} \circ D_{\text{sign},+}$, $\Delta^- : (\Omega^*_C(M,\mathbb{C}))^- \rightarrow (\Omega^*_C(M,\mathbb{C}))^+$. Consider now the operator $(d + \delta)_{\max/min} : L^2\Omega_C^i(M,g) \rightarrow L^2\Omega_C^i(M,g)$. Clearly if $\eta \in \mathcal{D}((d + \delta)_{\max/min})$ then also $*\eta \in \mathcal{D}((d + \delta)_{\max/min})$. In other words $\mathcal{D}((d + \delta)_{\max/min})$ is invariant under the action of the Hodge star operator and therefore the same conclusion applies also to $\epsilon$. So we have:

$$\mathcal{D}((d + \delta)_{\max/min}) = (\mathcal{D}((d + \delta)_{\max/min}))^+ \oplus (\mathcal{D}((d + \delta)_{\max/min}))^-$$  

(38)

\[\text{In (38) we introduced a different } L^2\text{–signature for } (M,g) \text{ using another kind of } L^2\text{–cohomology groups. So when } (M,g) \text{ is incomplete we may have different kinds of } L^2\text{–signatures and so we have to specify which } L^2\text{–signature we are referring.}\]
Moreover a simple check shows that
\[ (\mathcal{D}((d + \delta)_{\text{max}}))^{+/-} = \mathcal{D}((D^{\text{sign},+/-}_{\text{max}})) \]  
and that
\[ (d + \delta)_{\text{max}}|_{(\mathcal{D}(d+\delta))^{+/-}} = (D^{\text{sign},+/-}_{\text{max}}) \]  
Finally we observe that if we consider \( \Delta_N : L^2\Omega^2_c(M,g) \to L^2\Omega^2_c(M,g) \), see (31) for the definition, then we have that also \( \mathcal{D}(\Delta_N) \) is invariant under the action of \( \epsilon \). Therefore, also in this case, we get:
\[ \mathcal{D}(\Delta_N) = (\mathcal{D}(\Delta_N))^+ \oplus (\mathcal{D}(\Delta_N))^- \]  
Moreover, if we define \( \Delta_N^+ := (D^{\text{sign},-}_{\text{min}})^{\circ}(D^{\text{sign},+}_{\text{max}}) \) and \( \Delta_N^- := (D^{\text{sign},+}_{\text{min}})^{\circ}(D^{\text{sign},-}_{\text{max}}) \) then we have:
\[ (\mathcal{D}(\Delta_N))^{+/-} = (\mathcal{D}(\Delta_N^+)^{-}) \]  
and that
\[ \Delta_N|_{(\mathcal{D}(\Delta_N)^{+/-})} = \Delta_N^{+/-} \]  

**Theorem 10.** Let \((M,g)\) be an open, oriented and incomplete riemannian manifold of dimension \( m = 4l \) which satisfies the assumptions of theorem [4] and such that \((L^2\Omega^l(M,g),d_{2R} \cdot \cdot \cdot)\) is a Fredholm complex. Then we have
\[ \sigma_{2,2R}(M,g) = \dim(\ker((D^{\text{sign},+}_{\text{max}}))) - \dim(\ker((D^{\text{sign},-}_{\text{max}}))). \]  
*Proof.* The crucial observations for the proof are given by (38)–(45). The rest of the proof follows the classic proof of the signature Theorem, see for example [4]. First of all, for the benefit of the reader, we recall that
\[ \ker(d_{\text{max}}) \cap \ker(\delta_{\text{max}}) = \ker(d_{2R}) \cap \ker(d_{2R}^\ast), \quad \text{that } \ker((d + \delta)_{\text{max}}) = \ker(\Delta_N) \]  
and that according to Prop. [5]
\[ \ker((d + \delta)_{\text{max}}) = \ker(d_{\text{max}}) \cap \ker(\delta_{\text{max}}). \]  

Clearly we have
\[ \dim(\ker((D^{\text{sign},+}_{\text{max}}))) - \dim(\ker((D^{\text{sign},-}_{\text{max}}))) = \dim(\ker(\Delta_N^+)) - \dim(\ker(\Delta_N^-)) \]  
and
\[ \ker(\Delta_N^{+/-}) = \bigoplus_{k=0}^{2l-1} (\ker(\Delta_N^{+/-}) \cap (L^2\Omega^k_c(M,g) \oplus L^2\Omega^{m-k}_c(M,g))) \oplus (\ker(\Delta_N^{+/-}) \cap L^2\Omega^2_c(M,g)). \]  
Now if \( \omega \in \ker(\Delta_N^+) \cap (L^2\Omega^k_c(M,g) \oplus L^2\Omega^{m-k}_c(M,g)) \) with \( k \leq 2l - 1 \) then \( \omega = \eta + \epsilon(\eta) \) with \( \eta \in H^k_{\text{max}}(M,g) \). On the other hand if \( \eta \in H^k_{\text{max}}(M,g) \) then \( \eta + \epsilon(\eta) \in \ker(\Delta_N^+) \cap (L^2\Omega^k_c(M,g) \oplus L^2\Omega^{m-k}_c(M,g)) \). Therefore we can conclude that \( \ker(\Delta_N^+) \cap (L^2\Omega^k_c(M,g) \oplus L^2\Omega^{m-k}_c(M,g)) = \{ \eta + \epsilon(\eta), \eta \in H^k_{\text{max}}(M,g) \}. \) The same observations lead to the conclusion that \( \ker(\Delta_N^-) \cap (L^2\Omega^k_c(M,g) \oplus L^2\Omega^{m-k}_c(M,g)) = \{ \eta - \epsilon(\eta), \eta \in H^k_{\text{max}}(M,g) \}. \) In this way it follows immediately that
\[ \bigoplus_{k=0}^{2l-1} (\ker(\Delta_N^+) \cap (L^2\Omega^k_c(M,g) \oplus L^2\Omega^{m-k}_c(M,g))) \]
is isomorphic to
\[ \bigoplus_{k=0}^{2l-1} (\ker(\Delta_N^k) \cap (L^2 \Omega^k(M, g) \oplus L^2 \Omega^{m-k}(M, g))). \]

So we got that
\[ \dim(\ker((D^{\text{sign, +}})_{\text{max}})) - \dim(\ker((D^{\text{sign, -}})_{\text{max}})) = \dim(\ker(\Delta_N^k) \cap L^2 \Omega^k(M, g)) - \dim(\ker(\Delta_N^k) \cap L^2 \Omega^{m-k}(M, g)). \]

Now if \( \eta \in \ker(\Delta_N^k) \cap L^2 \Omega^k(M, g) \) this means that \( \eta \in \mathcal{H}^{2l}_{\text{max}}(M, g) \) and that \( \epsilon(\eta) = \eta \) that is \( *\eta = \eta. \) Analogously if \( \eta \in \ker(\Delta_N^k) \cap L^2 \Omega^{m-k}(M, g) \) this means that \( \eta \in \mathcal{H}^{2l}_{\text{max}}(M, g) \) and that \( \epsilon(\eta) = -\eta \) that is \( *\eta = -\eta. \) Therefore we proved that
\[ \dim(\ker(\Delta_N^k) \cap L^2 \Omega^k(M, g)) - \dim(\ker(\Delta_N^k) \cap L^2 \Omega^{m-k}(M, g)) = \dim(\mathcal{H}^i_{\text{max}}(M, g)) = \dim(\mathcal{H}^i_{\text{min}}(M, g)) = \sigma_{2,2\mathfrak{g}}(M, g) \]
and this completes the proof.

\[ \square \]

**Remark 1.** In formula \([10]\) the signature \( \sigma_{2,2\mathfrak{g}}(M, g) \) is expressed as the difference of the dimensions of two nullspaces. What we want to remark here is that it is not an index formula in the sense that
\[ \dim(\ker((D^{\text{sign, +}})_{\text{max}})) - \dim(\ker((D^{\text{sign, -}})_{\text{max}})) \neq \text{ind}((D^{\text{sign, +}})_{\text{max}}) \]
because
\[ \text{ind}((D^{\text{sign, +}})_{\text{max}}) = \dim(\ker((D^{\text{sign, +}})_{\text{max}})) - \dim(\ker((D^{\text{sign, -}})_{\text{min}})). \]

Moreover it would be natural to search for a connection between \( \sigma_{2,2\mathfrak{g}}(M, g) \) and \( d_{2\mathfrak{g}} + d^*_{2\mathfrak{g}} \) but a fundamental obstruction in order to do this is given by the fact that the domain of \( d_{2\mathfrak{g}} + d^*_{2\mathfrak{g}} \) is not in general invariant under the action of the Hodge star operator \( * \) and therefore it is not in general invariant under the action of \( \epsilon. \)

So to describe \( \sigma_{2,2\mathfrak{g}}(M, g) \) as an index we need to consider a different extension of \( D^{\text{sign, +}}. \) Consider again the elliptic complex \((\Omega^k(M), d_k)\) and consider the following Hilbert complex:
\[ 0 \to L^2 \Omega^0(M, g) \xrightarrow{d_{M,0}} L^2 \Omega^1(M, g) \xrightarrow{d_{M,1}} L^2 \Omega^2(M, g) \xrightarrow{d_{M,2}} \cdots \xrightarrow{d_{M,m-1}} L^2 \Omega^m(M, g) \to 0 \quad (47) \]
where
\[ d_{M,i} = \begin{cases} d_{\min,i} & i < 2l \\ d_{\max,i} & i \geq 2l \end{cases} \quad (48) \]

According to \([7]\) define
\[ (d + \delta)_M : L^2 \Omega^*(M, g) \to L^2 \Omega^*(M, g) \quad (49) \]
as the self-adjoint operator arising from the Hilbert complex \([17].\) Finally define \( \Delta_M := (d + \delta)_M \circ (d + \delta)_M. \) A simple check shows that both \( D((d+\delta)_M) \) and \( D(\Delta_M) \) are invariant under the action of \( \epsilon \) and that \( (d + \delta)_M \) is odd with respect to \( \epsilon. \) In particular we get
\[ D((d + \delta)_M) = (D((d + \delta)_M))^+ \oplus (D((d + \delta)_M))^- \quad (50) \]
and analogously
\[ D(\Delta_M) = (D(\Delta_M))^+ \oplus (D(\Delta_M))^- \quad (51) \]

Define now \( (d + \delta)^{+/−}_M \) as the restriction of \( (d + \delta)_M \) to \( D((d + \delta)_M)^{+/−}. \) Clearly \( (d + \delta)^{+/−}_M \) is the adjoint of \( (d + \delta)_M \) and they are both Fredholm operators on their domains endowed with the graph norm when \( (d + \delta)_M : L^2 \Omega^*(M, g) \to L^2 \Omega^*(M, g) \) it is Fredholm. In the same way define \( \Delta^{+/−}_M \) as \( \Delta^{+/−}_M := (d + \delta)^{−/+}_M \circ (d + \delta)^{−/+}_M \) and analogously \( \Delta^{−/+}_M := (d + \delta)^{−/+}_M \circ (d + \delta)^{−/+}_M. \) Finally we are in position to prove the last theorem of this subsection:
Let $(M,g)$ be as in Theorem 10. Then we have

$$\sigma_{2,\mathbb{R}}(M,g) = \text{ind}(d + \delta)^+_{M}$$

Proof. We have

$$\text{ind}((d + \delta)^+_{M}) = \dim(Ker(\Delta^+_M)) - \dim(Ker(\Delta^-_M))$$

and

$$\ker(\Delta^+_M) = \bigoplus_{k=0}^{2l-1} (\ker(\Delta^+_M) \cap (L^2\Omega^k(M,g) \oplus L^2\Omega^{m-k}(M,g))) \oplus (\ker(\Delta^-_M) \cap L^2\Omega^m_{\mathbb{C}}(M,g)).$$

Now if $\omega \in \ker(\Delta^+_M) \cap (L^2\Omega^k(M,g) \oplus L^2\Omega^{m-k}(M,g))$ with $k \leq 2l - 1$ then $\omega = \eta + \epsilon(\eta)$ with $\eta \in \mathcal{H}_{\text{rel}}^k(M,g)$. On the other hand if $\eta \in \mathcal{H}_{\text{rel}}^k(M,g)$ then $\eta + \epsilon(\eta) \in \ker(\Delta^+_M) \cap (L^2\Omega^k(M,g) \oplus L^2\Omega^{m-k}(M,g))$. Therefore we can conclude that $\ker(\Delta^+_M) \cap (L^2\Omega^k(M,g) \oplus L^2\Omega^{m-k}(M,g)) = \{\eta + \epsilon(\eta), \eta \in \mathcal{H}_{\text{rel}}^k(M,g)\}$. The same observations lead to the conclusion that $\ker(\Delta^-_M) \cap (L^2\Omega^k(M,g) \oplus L^2\Omega^{m-k}(M,g)) = \{\eta - \epsilon(\eta), \eta \in \mathcal{H}_{\text{rel}}^k(M,g)\}$. In this way we get that

$$\bigoplus_{k=0}^{2l-1} (\ker(\Delta^+_M) \cap (L^2\Omega^k(M,g) \oplus L^2\Omega^{m-k}(M,g)))$$

is isomorphic to

$$\bigoplus_{k=0}^{2l-1} (\ker(\Delta^-_M) \cap (L^2\Omega^k(M,g) \oplus L^2\Omega^{m-k}(M,g))).$$

So we get that

$$\text{ind}((d + \delta)^+_{M}) = \dim(\ker(\Delta^+_M) \cap L^2\Omega^m_{\mathbb{C}}(M,g)) - \dim(\ker(\Delta^-_M) \cap L^2\Omega^m_{\mathbb{C}}(M,g)).$$

But $\ker(\Delta_M) \cap L^2\Omega^m_{\mathbb{C}}(M,g) = \mathcal{H}^m_{\text{max}}(M,g)$ and this implies that $\ker(\Delta^+_M)^+ \cap L^2\Omega^m_{\mathbb{C}}(M,g) = (\mathcal{H}^m_{\text{max}}(M,g))^+$. From now on the proof coincides with that of Theorem 10. □

## 3 Some examples

It is not difficult to show examples of open, oriented and incomplete riemannian manifolds $(M,g)$ such that $\text{ran}(d_{\text{min},i})$ is closed in $L^2\Omega^{i+1}(M,g)$ for all $i = 0, \ldots, m$. We can use for example to a compact and orientable manifold with boundary endowed with a smooth metric up to the boundary such as in [5], admissible riemannian pseudomanifold as in [10] or in [15], compact stratified pseudomanifold endowed with a quasi edge metric with weights as in [2] or to the Weil-Petersen metric on the regular part of the moduli space of curves as in [10]. Therefore, in all these cases, we can always build the complex $(L^2\Omega(M,g), d_{2\mathbb{R},i})$. What is much more complicated is to find examples of open, oriented and incomplete riemannian manifolds $(M,g)$ such that $(L^2\Omega(M,g), d_{2\mathbb{R},i})$ is a Fredholm complex. This last section is devoted to this task.

First of all we recall that two riemannian metrics $g$ and $h$ are said quasi isometric if there exists a positive real number $c$ such that $\frac{1}{c}h \leq g \leq ch$. Now we describe the first example; we start with the following definition from [6].

Let $M$ a compact manifold with boundary $N := \partial M$. Let us label its interior with $M$. Let $U \cong [0,1) \times N$ a collar neighborhood for $N$. Let $g$ be a riemannian metric over $M$ such that $g$ restricted to $U$ is isometric to $h(x)(dx^2 + x^2g_N(x))$ where $g_N(x)$ is a family of metric on $N$ depending on $x$ which varies smoothly in $[0,1)$ and continuously $[0,1]$ and $h \in C^\infty((0,1) \times N)$ satisfies:

$$\sup_{p \in N}|(x\partial_x^j x^{-c}h(x,p) - 1)| = O(x^\delta) \quad \text{as } x \to 0, \quad j = 0, 1$$

$$\sup_{p \in N}|h(x,p)^{-1}d_N h(x,p)||T^p N, g_N(x)) = O(x^\delta) \quad \text{as } x \to 0$$

and

$$\sup_{p \in N}|(g^i - g^0)|_N(x,p) + x|\omega^0 - \omega^1|_N(x,p)) = O(x^\delta) \quad \text{as } x \to 0$$

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for some $\delta > 0$ and $c > -1$ and where $g^0 := dx^2 + x^2 g_N(0)$, $g_1 = dx^2 + x^2 g_N(x)$ and $\omega^0, \omega^1$ are the connection forms of the Levi-Civita connection $\nabla^0, \nabla^1$ of $g^0$ and $g^1$ respectively. The metric $g$ is called a conformally conic metric. As it is showed in $[7, 8]$ and $[9]$ if we consider a complex projective curve $V \subset \mathbb{P}^n(\mathbb{C})$ and $g$ is the riemannian metric induced by the Fubini-Study metric of $\mathbb{P}^n(\mathbb{C})$ on the regular part of $V$ then $g$ is a conformally conic metric. According to $[5]$ if $(M, g)$ is a conformally conic riemannian manifold then every closed extension of $(\Omega^*_c(M), d_i)$ is a Fredholm extension. Therefore we have the following corollary:

**Corollary 9.** Let $(M, g)$ be an open and oriented riemannian manifold where $M$ is the interior of a compact manifold with boundary and $g$ is a riemannian metric on $M$ quasi isometric to a conformally conic metric. Then Theorems $6, 7, 8, 9, 10$ and $11$ and their relative corollaries hold for $(M, g)$. In particular they hold when $M$ is the regular part of a complex projective curve $V \subset \mathbb{P}^n(\mathbb{C})$ and $g$ is any riemannian metric on $M$ quasi isometric to the metric induced by the Fubini-Study metric of $\mathbb{P}^n(\mathbb{C})$.

Another example is the following: consider again a compact and oriented riemannian manifold with boundary and finally with $U \cong [0, 1)$ a collar neighborhood of $N$. Let $g$ a riemannian metric on $M$ such that, over $U$, it takes the form $dx^2 + x^{2\beta} h$ where $\beta > 1$ and $h$ is a riemannian metric on $N$. A metric like this is called metric horn. In $[14]$ the authors prove that the Gauss-Bonnet operator

$$d + \delta : L^2 \Omega^*(M, g) \to L^2 \Omega^*(M, g)$$

with domain given by $\Omega^*_c(M)$, admits only a finite number of closed extensions and all these extensions are Fredholm operators on their domains endowed with the graph norm. So we have the following corollary:

**Corollary 10.** Let $(M, g)$ be an open and oriented riemannian manifold where $M$ is the interior of a compact manifold with boundary and $g$ is a metric horn. Then Theorems $6, 7, 8, 9, 10$ and $11$ and their relative corollaries hold for $(M, g)$.

Finally we mention that recently, in his PhD thesis $[12]$, Frank Lapp generalised the result of Lesch and Peyerimhoff to the following case: consider again a compact and oriented manifold with boundary $\overline{M}$ such that the boundary, that we still label with $N$, is diffeomorphic to a product of closed manifolds: $N \cong N_1 \times \ldots \times N_q$. Let $U$ be a collar neighborhood of $N$ let $g$ be riemannian metric over $M$ such that over $U \cong [0, 1) \times N_1 \times \ldots \times N_q$ it takes the form

$$dx^2 + h_1^2(x) g_1 + \ldots + h_q^2(x) g_q$$

(52)

where $h_i(x) \in C^\infty((0, 1), (0, \infty))$. A metric of this shape is called a multiply warped product metric. In his thesis, see $[12]$, Lapp proved that if for some constant $K > 0$

$$\max_{j=1,\ldots,q} h_j(x) \leq K r^\beta \; x \in (0, 1), \beta > 1$$

and for every $j = 1, \ldots, q$ there exists a real number $c_j$ such that

$$\int_0^1 x |\log x| \frac{h'_j(x)}{h_j(x)} - \frac{c_j}{x} \leq x < \infty$$

then the Gauss-Bonnet operator

$$d + \delta : L^2 \Omega^*(M, g) \to L^2 \Omega^* (M, g)$$

with domain given by $\Omega^*_c(M)$, admits only a finite number of closed extensions and all these extensions are Fredholm operators on their domains endowed with the graph norm. In particular this is true when $g$ is a multiply metric horns that is in $[52]$ all the warping functions satisfy the following requirement:

$$h_i(x) = x^{\beta_i}, \; \beta_i > 1, \; i = 1, \ldots, q.$$ 

Therefore, in this case as well, we have the following corollary:
Corollary 11. Let \((M, g)\) be an open and oriented riemannian manifold where \(M\) is the interior of a compact manifold with boundary \(\overline{M}\). Suppose that the boundary is diffeomorphic to a product \(\partial M \cong N_1 \times \cdots \times N_q\). Let \(g\) be a riemannian metric on \(M\) quasi isometric to a multiply warped product metric. Then Theorems 6, 7, 8, 9, 10 and 11 and their relative corollaries hold for \((M, g)\).

Finally we remark that in this last context, unlike the previous two examples, the Gauss-Bonnet operator is not generally essentially self-adjoint when the dimension of the manifold is even. This means that in general the \(L^2\)-signature \(\sigma_{2, M}(M, g)\) that we introduced in the previous section is different from that we defined in \([3]\) and that analogously the \(L^2\)-Euler characteristic \(\chi_{2, M}(M, g)\) is different from that we introduced in \([3]\). For what concerns the odd dimensional case we recall that \(d + \delta\) is not generally essentially self-adjoint in all the examples we recalled. Therefore we can conclude again that, in general, \(\chi_{2, M}(M, g)\) is different from the \(L^2\)-Euler characteristic we introduced in \([3]\).

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