Correlation functions for open XXZ spin 1/2 quantum chains with unparallel boundary magnetic fields

G Niccoli and V Terras

1 Univ Lyon, Ens de Lyon, Univ Claude Bernard, CNRS, Laboratoire de Physique, F-69342 Lyon, France
2 Université Paris-Saclay, CNRS, LPTMS, 91405, Orsay, France

E-mail: veronique.terras@universite-paris-saclay.fr

Received 17 April 2022, revised 17 June 2022
Accepted for publication 28 June 2022
Published 23 September 2022

Abstract

In this paper we continue our derivation of the correlation functions of open quantum spin 1/2 chains with unparallel magnetic fields on the edges; this time for the more involved case of the XXZ spin 1/2 chains. We develop our study in the framework of the quantum separation of variables, which gives us both the complete spectrum characterization and simple scalar product formulae for separate states, including transfer matrix eigenstates. Here, we leave the boundary magnetic field in the first site of the chain completely arbitrary, and we fix the boundary field in the last site \( N \) of the chain to be a specific value along the \( z \)-direction. This is a natural first choice for the unparallel boundary magnetic fields. We prove that under these special boundary conditions, on the one side, we have a simple enough complete spectrum description in terms of homogeneous Baxter like \( TQ \)-equation. On the other side, we prove a simple enough description of the action of a basis of local operators on transfer matrix eigenstates as linear combinations of separate states. Thanks to these results, we achieve our main goal to derive correlation functions for a set of local operators both for the finite and half-infinite chains, with multiple integral formulae in this last case.

Keywords: quantum integrable systems, separation of variables, correlation functions

Contents

1. Introduction 2
2. The open spin-1/2 XXZ quantum chain 6

*Author to whom any correspondence should be addressed.
1. Introduction

In [1], the correlation functions of an open spin 1/2 XXX quantum chain with unparallel boundary magnetic fields were derived. Here, we continue this program by considering the more complicated case of an open XXZ chain with unparallel boundary magnetic fields. More precisely, we explain how to compute correlation functions in the case of an open spin 1/2 XXZ quantum chain coupled with a completely arbitrary magnetic field on the first site of the chain and with a specific fixed longitudinal magnetic field on the last site of the chain.

We develop our analysis in the framework of the quantum separation of variables (SoV) [2–39]. This approach, first pioneered by Sklyanin [2–7] in the framework of the quantum inverse scattering method (QISM) [40–49], has been more recently reformulated in [50] on the pure basis of the integrable structure of the model, hence extending its range of application even to higher rank cases³ in [50, 53–59]. On the one hand, and contrary to Bethe ansatz [60] and its variants, the SoV approach does not rely on an ansatz, which means that the completeness of the SoV description of the eigenvalues and eigenstates of the considered models follows by construction. On the other hand, the SoV approach has the advantage to produce naturally and quite universally determinant formulae for the scalar products of the so-called separate states [22, 24–30, 33, 35, 37–39],⁴ a class of states which notably contains the eigenstates of the model. In particular, in [39] we derived scalar products formulae in determinant form for the separate states of the open spin 1/2 XXZ quantum chains under quite general boundary conditions; this is the first required step to compute correlation functions.

In the last decades, there have been impressive progresses concerning the exact determination of correlation functions of quantum integrable models. However, the state of the art remains quite unsatisfactory with respect to the type of boundary conditions for which such results are available. In fact, most of the results that have been obtained so far concern the quantum non-linear Schrödinger model or the XXX/XXZ spin 1/2 quantum chains with periodic boundary conditions or, for the latter models, with open boundary conditions with longitudinal boundary magnetic fields, i.e. along the $\mathbf{z}$-direction. More precisely, correlation functions of the XXX/XXZ spin 1/2 chains were first computed in infinite volume and at zero temperature in the framework of the $q$-vertex operator approach in [64–66]. These results were confirmed and generalized to the case of a non-zero global magnetic field by means of the study of the periodic chain in finite volume by algebraic Bethe ansatz (ABA) in [67–73], and to the temperature cases in [74–79]. See also [80–90] for relevant results on correlation functions by the use of the hidden Grassmann structure. Still in this periodic background, analytical study in the thermodynamic limit of long distances two-point and multi-point correlation functions have been developed in [91–101] and numerical study of the dynamical structure factors in [102–104] have made accessible comparison with experimental settings as neutron scattering [105]. Correlation functions in the case of an open spin chain were considered in [106–109], but only for longitudinal boundary fields, i.e. along the $\mathbf{z}$-direction. In fact, for different kind of boundary conditions, usual Bethe ansatz techniques can in principle no longer be used to compute correlations. In the case of the XXX spin chain, these limitations on the computation of correlation functions have been overcome only recently by the consideration of the model in the SoV framework: in [110], correlation functions were computed for all types of quasi-periodic (including anti-periodic) boundary conditions, and in [1], correlation functions were computed in the open case with unparallel boundary magnetic fields. As previously announced,

³See also [7, 10, 51, 52] for previous developments.

⁴This is the case for the rank one models while in [61] it has been shown how these type of formulae extend to the $gl(3)$ higher rank case, see also the interesting and recent papers [62, 63].
we will extend these results here to the case of an open XXZ spin 1/2 quantum chains with unparallel boundary magnetic fields.

Open quantum spin chains have been recently at the center of a large research activity, see e.g. [16, 21, 24, 29–31, 106–109, 111–131] and related references, with application to the studies of classical stochastic models, as asymmetric simple exclusion models [132], and also to modelling numerous applications in quantum condensed matter physics, such as out-of-equilibrium and transport properties in the spin chains [133]. Exact results on the spectrum for parallel magnetic fields have been accessible since the seminal works in coordinate Bethe ansatz [111] and ABA [112]. Instead, for general unparallel boundary magnetic fields, the description of the spectrum was only addressed recently, and is still not completely well understood, in particular concerning the accurate description of the Bethe roots for the ground state in the thermodynamic limit. In fact, by using the reflection equation first introduced by Cherednik [134], Sklyanin [112] has extended the QISM to open quantum chains. Whereas this algebraic QISM framework applies \textit{a priori} to both parallel and unparallel boundary magnetic fields, its ABA implementation as proposed in [112] holds only for diagonal boundary matrices, i.e. for longitudinal parallel boundary magnetic fields. Other integrable techniques have been developed to handle the non-diagonal case, i.e. generally speaking the unparallel cases. In [115], under a special constrain relating the parameters of the two boundary matrices, a first description of the spin chain spectrum with unparallel boundary fields has been obtained, by using the fusion procedure [135], in terms of polynomial solutions of some $TQ$-equation of Baxter’s type [136], for the roots of unity points and later in [116] for general values of the anisotropy parameter. Under such a constraint, generalized Bethe eigenstates have also been constructed [114, 117, 118] by using Baxter’s vertex-IRF transformations [137] so as to simplify the boundary matrices. Such a constraint appears also in other integrable approaches leading to the spectrum description in terms of polynomial solutions of ordinary $TQ$-equations, such as by coordinate Bethe ansatz with elements of matrix product ansatz [121, 122], $q$-Onsager algebra [119, 120] etc. This constraint has been overcome only more recently. In [124] a description of the spectrum of these open chains was proposed on the basis of analytic properties and functional relations satisfied by the transfer matrices. The proposed spectrum description has been there presented by polynomial solutions of inhomogeneous $TQ$-equations, i.e. equations admitting some extra term, see also [138]. Such a description of the spectrum in terms of an inhomogeneous $TQ$-equation also appears in the framework of the modified ABA [125–127, 129] dealing with unconstrained boundary conditions. The SoV method has been used, in particular, for open spin chains with the most general unconstrained non-diagonal boundary matrices [16, 21, 24, 29, 30], and in [31] SoV approach was used to prove that the complete spectrum characterization of these open chains, indeed, can be formulated in terms of polynomial solutions to functional $TQ$-equations of Baxter type which, for the most general unconstrained boundary matrices, have the aforesaid inhomogeneous extra term.

Our aim is here to derive exact expressions for the correlation functions of local/quasi-local operators in the vicinity of the first site of the chain. It is natural to expect such correlation functions to be strongly influenced by the magnetic field on this first site; in fact, we expect an explicit dependence of the result on the boundary field at site 1. Instead, for long enough chains, we expect the influence of the boundary field at the other side of the chain to be only indirect (i.e. mainly encoded in the structure of the ground state and not in the explicit expression of the correlation functions themselves, see [131]). This type of physical argument motivates us to initiate our study by considering some special choice of the boundary magnetic field at site

\footnote{The boundary matrices allow to parametrize the magnetic fields at the boundaries of the quantum spin chain.}
The idea is to adjust the magnetic field at site $N$ in such a way that we can gauge out some of the difficulties due to the consideration of non-diagonal boundaries while keeping unparallel boundary magnetic fields. In particular, we fix the magnetic field on site $N$ to a fixed value along the $z$-direction, whereas the magnetic field on site 1 is completely arbitrary. On the one hand, this enables us to have an easy description of the thermodynamic ground state distribution of the Bethe roots, which, as already mentioned, is not known in the most general case: indeed, in this special case of unparallel boundary fields that we consider here, the transfer matrix happens to be isospectral to the transfer matrix of a chain with parallel longitudinal boundary magnetic fields. On the other hand, this special choice of unparallel boundary magnetic fields is done here so as to simplify the computation of the action of the local/quasi local operators on the transfer matrix eigenstates. Let us recall that a reconstruction formula expressing the local operators in terms of the generators of the reflection algebra is so far missing in the literature, and that the alternative method proposed in [108] consists in using the known reconstruction in terms of the bulk Yang–Baxter algebra, and in decomposing the boundary eigenstates in terms of bulk Bethe states (boundary-bulk decomposition) so as to be able to act with the local operators on these states. This is possible due to the fact that the reflection algebra generators are quadratic functions of the bulk ones, and that the eigenstates are expressed in the form of Bethe states in terms of these generators. Here, we extend these constructions to the current unparallel boundary conditions. This requires first the introduction of a gauge deformation of the Yang–Baxter algebra and the generalization to it of the reconstruction formulae for the local operators. Then, we have to compute the boundary-bulk decomposition of the separate states in terms of some gauged version of bulk Bethe states. Finally, we have to compute the action of local operators first on the gauged bulk Bethe states and then on the boundary separate states by using the boundary-bulk decomposition. Here, the special choice of unparallel boundary magnetic fields is done so as to simplify the form of the boundary-bulk decomposition and to have an easier access to the formulae for the action of local operators. In this paper we develop all these technical steps which, together with the known formulae for the scalar products in the SoV framework\(^6\), give us access to the computation of the correlation functions here presented.

The paper is organized as follows.

Section 2 provides a technical introduction to the model we consider, the XXZ spin 1/2 chain with general boundary conditions. There, we introduce the reflection algebras leading to the integrable description of the spin chains by using two kinds of symmetries which reverse the role of the plus/minus reflection algebras, and which happen to be convenient in that they enable us to make a direct use of some former results [39]. We also introduce the gauge-transformed version of this algebra by Baxter’s vertex-IRF transformation, which enables one to simplify the form of the boundary matrices.

In section 3, we recall the complete characterization of the transfer matrix spectrum that was obtained for this model in the SoV framework. There, we particularize the SoV spectrum construction to the special boundary conditions associated to one arbitrary magnetic field in the site 1 and a fixed magnetic field oriented along the $z$-direction in the site $N$. In such a case, the transfer matrix spectrum is completely characterized by the polynomial solutions of a homogeneous $TQ$-equation of Baxter type, and the separate eigenstates can be constructed in the form of Bethe states. We also prove the isospectrality of this case with the case of parallel boundary magnetic fields along the $z$-direction.

\(^6\) It is worth mentioning that, the recent and interesting results on scalar products for open chains with general boundary conditions [139], see also [123, 130, 140–143], may put a basis for the computation of correlation functions in a generalized/modified Bethe ansatz framework.
In section 4, we compute the action of a basis of quasi-local operators on Bethe-type boundary states. More precisely, we define a set of quasi-local operators on the first \( m \) sites of the chain, which can be written as monomials in terms of the generator of the gauged Yang–Baxter algebra. We prove that for any fixed finite \( m \), this operator set is an operator basis of the corresponding \( m \)-sites quasi-local operators and we compute the action of its elements on gauged bulk Bethe-type states. By means of the boundary-bulk decomposition of the boundary states, we rewrite this action as an action on general boundary gauged Bethe-type states.

Finally, in section 5, this result concerning the action of local operators on transfer matrix eigenstates, as well as the known scalar product formulae (previously derived in the SoV framework \([35, 39]\)) allow us to compute the correlation functions. Here, we explicitly present the expressions for the correlation functions of a particular class of elements of the aforementioned local operator basis, both on the finite chain and on the half-infinite chain. In the latter case, we obtain multiple integral representations that are quite similar, in their form, to the one previously obtained in \([108]\) for the case of a chain with diagonal boundary conditions, with however some differences related to the different choice of boundary field at site 1 with respect to \([108]\).

The paper contains also three appendices with some technical details. In appendix A, we gather some useful properties of the bulk gauge Yang–Baxter algebra. We show notably that the latter can in fact be described in terms of a single operator family with three continuous parameters, written as linear combinations of the ungauged Yang–Baxter generators: one parameter being the spectral parameter and the other two—the gauge parameters—defining the coefficients of the linear combination. In particular, we derive the commutation relations of this operator family and its action on gauged reference states and gauged Bethe-type states. Appendix B is devoted to the boundary-bulk decomposition: we notably explain how to decompose the transfer matrix eigenstates, and more generally the SoV separate states as linear combinations of generalized gauged Bethe states; i.e. a gauged version of the so-called boundary-bulk decomposition \([108, 109]\). Finally, in appendix C, we show how, taking the limit in which the boundary at site 1 becomes diagonal, we can obtain from our final result the result of \([108]\).

### 2. The open spin-1/2 XXZ quantum chain

The Hamiltonian of the integrable quantum spin-1/2 XXZ chain with open boundary conditions and general boundary fields can be written as:

\[
H = \sum_{n=1}^{N-1} \left[ \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \cosh \eta \sigma_n^z \sigma_{n+1}^z \right] \\
+ \frac{\sinh \eta}{\sinh \varsigma_-} \left[ \sigma_n^z \cosh \varsigma_- + 2\kappa_- \left( \sigma_n^x \cosh \tau_- + i\sigma_n^y \sinh \tau_- \right) \right] \\
+ \frac{\sinh \eta}{\sinh \varsigma_+} \left[ \sigma_n^z \cosh \varsigma_+ + 2\kappa_+ \left( \sigma_n^x \cosh \tau_+ + i\sigma_n^y \sinh \tau_+ \right) \right].
\] (2.1)

The operators \( \sigma_n^\alpha, \alpha \in \{x, y, z\} \), act as the corresponding Pauli matrices on the local quantum spin space \( \mathcal{H}_n \cong \mathbb{C}^2 \) at site \( n \), so that \( H \) is an operator acting on the \( 2^N \)-dimensional quantum space \( \mathcal{H} = \bigotimes_{n=1}^N \mathcal{H}_n \). The anisotropy parameter is \( \Delta = \cosh \eta \). The boundary fields are parametrized by the six boundary parameters \( \varsigma_\pm, \kappa_\pm, \tau_\pm \). We shall also use the following re-parametrization \( \varphi_\pm, \psi_\pm \) of the boundary parameters \( \varsigma_\pm, \kappa_\pm \):
\[
\sinh \varphi \pm \cosh \psi \pm = \sinh \varsigma \pm 2 \kappa \pm, \\
\cosh \varphi \pm \sinh \psi \pm = \cosh \varsigma \pm 2 \kappa \pm.
\] (2.2)

The Hamiltonian (2.1) corresponds to the most general case of open boundary conditions with non-diagonal integrable boundary interactions. It is worth noting that this Hamiltonian is manifestly invariant under the following simultaneous change of sign:

\[
\{\eta, \varsigma\pm\} \rightarrow \{-\eta, -\varsigma\pm\},
\] (2.3)

while the others parameters \(\kappa\pm\) and \(\tau\pm\) remain unchanged. It is also invariant under a relabelling of each site \(n\) by \(N - n + 1\) together with the interchange of the boundary parameters + and −:

\[
\begin{align*}
\{n\} &\rightarrow \{N - n + 1\}, & 1 \leq n \leq N, \\
\{\varsigma, \kappa, \tau\} &\rightarrow \{\varsigma, \kappa, \tau\}.
\end{align*}
\] (2.4)

2.1. Reflection algebra and symmetries

The open chain with Hamiltonian (2.1) can be studied in the formalism introduced in [112], that we recall here.

Let us introduce the six-vertex trigonometric solution of the Yang–Baxter equation,

\[
R_{12}(\lambda) = \\
\begin{pmatrix}
\sinh(\lambda + \eta) & 0 & 0 & 0 \\
0 & \sinh \lambda & \sinh \eta & 0 \\
0 & \sinh \eta & \sinh \lambda & 0 \\
0 & 0 & 0 & \sinh(\lambda + \eta)
\end{pmatrix} \in \text{End} (\mathbb{C}^2 \otimes \mathbb{C}^2),
\] (2.5)

and let us consider, on \(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathcal{H}\), the reflection equation for the so-called boundary monodromy matrix \(\mathcal{U}_-(\lambda) \in \text{End} (\mathbb{C}^2 \otimes \mathcal{H})\):

\[
R_{21}(\lambda - \mu)\mathcal{U}_-(\lambda)R_{12}(\lambda + \mu - \eta)\mathcal{U}_-(\mu) = \mathcal{U}_-(\mu)R_{21}(\lambda + \mu - \eta)\mathcal{U}_-(\lambda)R_{12}(\lambda - \mu).
\] (2.6)

Here the subscripts parametrize the subspaces of \(\mathbb{C}^2 \otimes \mathbb{C}^2\) on which the corresponding operators act non-trivially, and \(R_{21}(\lambda)\) is obtained from \(R_{12}(\lambda)\) (2.5) as \(R_{21}(\lambda) = P_{12}R_{12}(\lambda)P_{12}\), where \(P_{12}\) is the permutation operator on \(\mathbb{C}^2 \otimes \mathbb{C}^2\).

For convenience, we shall define here two different realizations of the reflection algebra, i.e. construct two different solutions \(\mathcal{U}_-(\lambda)\) of the reflection equation (2.6). These realizations slightly differs from the standard one introduced by Sklyanin [112] and used in our previous work [39] in that they rely on the two types of symmetries (2.4) and (2.3) of the model. They can easily be related to each other as described below.

Let us first introduce the following monodromy matrix \(T_0(\lambda) \in \text{End} (\mathcal{H}_0 \otimes \mathcal{H})\), where \(\mathcal{H}_0 = \mathbb{C}^2\) is the so-called auxiliary space:

\[
T_0(\lambda) = R_{01}(\lambda - \xi_1 - \eta/2) \cdots R_{0N}(\lambda - \xi_N - \eta/2).
\] (2.7)

Note that here we have intentionally reversed the order in the quantum sites w.r.t. the one that is often used in the literature, and in particular in our previous series of papers. \(T_0(\lambda)\) is a solution of the Yang–Baxter equation on \(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathcal{H}\):
In other words, we have here used the symmetry (2.4) of the model with respect to our previous paper.

Let us define from (2.7) the following boundary monodromy matrix:

$$\mathcal{U}_{-\beta}(\lambda) = T_0(\lambda)K_{-\beta}(\lambda)\tilde{T}_0(\lambda) = \begin{pmatrix} A_-(\lambda) & B_-(\lambda) \\ C_-(\lambda) & D_-(\lambda) \end{pmatrix},$$

solution of the reflection equation (2.6), where

$$\tilde{T}_0(\lambda) = (-1)^N \sigma_0^- T_0^\eta (-\lambda) \sigma_0^+$$

$$= R_{0\eta}(\lambda + \xi_N - \eta/2) \cdots R_{0\omega}(\lambda + \xi_1 - \eta/2),$$

and

$$K_-(\lambda) = K(\lambda; \zeta_+, \kappa_+, \tau_+), \quad K_+(\lambda) = K(\lambda + \eta; \zeta_-, \kappa_-, \tau_-).$$

We have here introduced the boundary parameters plus in the minus boundary $K$-matrix and vice versa, while the definition of the $K$-matrix is unchanged with respect to our previous paper [39]:

$$K(\lambda; \zeta, \kappa, \tau) = \frac{1}{\sinh \zeta} \begin{pmatrix} \sinh(\lambda - \eta/2 + \zeta) & \kappa e^{-\tau} \sinh(2\lambda - \eta) \\ \kappa e^{\tau} \sinh(2\zeta - \lambda + \eta/2) & \sinh(\zeta - \lambda + \eta/2) \end{pmatrix}. \tag{2.12}$$

In other words, we have here used the symmetry (2.4) of the model with respect to our previous works. We recall that (2.12) is the most general scalar solution [113, 144, 145] of the reflection equation (2.6) for general values of the parameters $\zeta, \kappa$ and $\tau$.

It follows [112] that the transfer matrices,

$$T(\lambda) = \text{tr}_0 \{ K_+\beta(\lambda)T_0(\lambda)K_-\beta(\lambda)\tilde{T}_0(\lambda) \},$$

form a one-parameter family of commuting operators on $\mathcal{H}$. The Hamiltonian (2.1) of the spin-1/2 open chain can be obtained in the homogeneous limit $\xi_m = 0, m = 1, \ldots, N$, as the following derivative of the transfer matrix (2.13):

$$H = \frac{2(\sinh \eta)^{1-2N}}{\text{tr} \{ K_+(\eta/2) \} \text{tr} \{ K_-(\eta/2) \}} \frac{d}{d\lambda} T(\lambda)_{\lambda=\eta/2} + \text{constant}. \tag{2.13}$$

We also recall that the quantum determinant of $\mathcal{U}_-(\lambda)$,

$$\det_q \mathcal{U}_-(\lambda) = \sinh(2\lambda - 2\eta)[A_-(\eta/2 \pm \lambda)A_-(\eta/2 \mp \lambda) + B_-(\eta/2 \pm \lambda)C_-(\eta/2 \mp \lambda)]$$

$$= \sinh(2\lambda - 2\eta)[D_-(\eta/2 \pm \lambda)D_-(\eta/2 \mp \lambda) + C_-(\eta/2 \pm \lambda)B_-(\eta/2 \mp \lambda)]$$

$$= \det_q T(\lambda)\det_q T(-\lambda)\det_q K_-(\lambda), \tag{2.15}$$

is a central element of the reflection algebra: $[\det_q \mathcal{U}_-(\lambda), \mathcal{U}_-(\lambda)] = 0$. In (2.15), $\det_q T(\lambda)$ stands for the bulk quantum determinant which can be expressed as

$$\det_q T(\lambda) = a(\lambda + \eta/2)d(\lambda - \eta/2), \tag{2.16}$$

where

$$R_{1,2}(\lambda - \mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R_{1,2}(\lambda - \mu). \tag{2.8}$$
where

\[ a(\lambda) = \prod_{n=1}^{N} \sinh(\lambda - \xi_n + \eta/2), \quad d(\lambda) = \prod_{n=1}^{N} \sinh(\lambda - \xi_n - \eta/2), \] (2.17)

and \( \det_q K_+ (\lambda) \) stands for the quantum determinant of the scalar boundary matrix \( K_+ (\lambda) \). The quantum determinant of the scalar boundary matrices \( K_\pm (\lambda) \) (2.11) can be expressed as

\[
\frac{\det_q K_\pm (\lambda)}{\sinh(2\lambda \mp 2\eta)} = \mp \left( \frac{\sinh^2 \lambda - \sinh^2 \varphi_\pm}{\sinh^2 \varphi_\pm \cosh^2 \psi_\pm} \right) \left( \sinh^2 \lambda + \cosh^2 \psi_\pm \right). (2.18)
\]

The boundary quantum monodromy matrix \( U_- (\lambda) \) (2.9) satisfies the quantum inversion relation

\[
U_- (\lambda + \eta/2) U_- (-\lambda + \eta/2) = \frac{\det_q U_- (\lambda)}{\sinh(2\lambda - 2\eta)}, (2.19)
\]

and its elements satisfy the following relations:

\[
A_- (\lambda) = \sinh \eta \sinh(2\lambda) D_- (\lambda) + \sinh(2\lambda - \eta) D_- (-\lambda), \quad B_- (\lambda) = -\sinh(2\lambda + \eta) B_- (-\lambda), \quad C_- (\lambda) = -\sinh(2\lambda + \eta) C_- (-\lambda), \quad D_- (\lambda) = \sinh(2\lambda - \eta) D_- (-\lambda). (2.20, 2.21, 2.22)
\]

Let us now introduce a second realization relying on the symmetry (2.3). We define

\[
\bar{\eta} = -\eta, \quad \bar{\xi}_\pm = -\xi_\pm, \quad \bar{\kappa}_\pm = \kappa_\pm, \quad \bar{\tau}_\pm = \tau_\pm, (2.23)
\]

Let us consider the \( R \)-matrix

\[
\bar{R}_{12}(\lambda) = \begin{pmatrix}
\sinh(\lambda - \eta) & 0 & 0 & 0 \\
0 & \sinh \lambda & -\sinh \eta & 0 \\
0 & -\sinh \eta & \sinh \lambda & 0 \\
0 & 0 & 0 & \sinh(\lambda - \eta)
\end{pmatrix} = -R_{12}(-\lambda), (2.24)
\]

which corresponds to a change of parameter \( \eta \to \bar{\eta} = -\eta \) with respect to (2.5), and let us introduce the bulk monodromy matrix

\[
M(\lambda) = \bar{R}_{0N}(\lambda - \xi_N + \eta/2) \ldots \bar{R}_{01}(\lambda - \xi_1 + \eta/2) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, (2.25)
\]

which satisfies the Yang–Baxter equation w.r.t. \( \bar{R}_{12}(\lambda) \). From it we also define the matrix

\[
\tilde{M}(\lambda) = (-1)^N \sigma_0^T M_0(-\lambda) \sigma_0^T \\
= \bar{R}_{01}(\lambda + \xi_1 + \eta/2) \ldots \bar{R}_{0N}(\lambda + \xi_N + \eta/2). (2.26)
\]
Note that the bulk monodromy matrices (2.25) and (2.7) are related by
\[ M(-\lambda) = (-1)^N \hat{T}(\lambda), \quad \hat{M}(-\lambda) = (-1)^N T(\lambda). \] (2.27)

Similarly, we define the analogue of (2.12) with a change of parameter \( \eta \rightarrow \bar{\eta} = -\eta \):
\[ \tilde{K}(\lambda; \zeta, \kappa, \tau) = \frac{1}{\sinh \zeta} \begin{pmatrix} \sinh(\lambda + \eta/2 + \zeta) & \kappa e^{-\tau} \sinh(2\lambda + \eta) \\ \kappa e^{\tau} \sinh(2\lambda - \zeta - \eta/2) & \sinh(\zeta - \lambda - \eta/2) \end{pmatrix}, \] (2.28)
and we introduce
\[ \tilde{K}_-(\lambda) = \tilde{K}(\lambda; \zeta-, \kappa-, \tau-), \quad \tilde{K}_+(\lambda) = \tilde{K}(\lambda - \eta; \zeta+, \kappa+, \tau+), \] (2.29)
in terms of the parameters (2.23). From (2.25)–(2.29), we define the boundary monodromy matrices
\[ \tilde{U}_-(\lambda) \equiv \tilde{U}_-(\lambda|\eta, \zeta-, \kappa-, \tau-) = M(\lambda)\tilde{K}_-(\lambda)\hat{M}(\lambda), \] (2.30)
\[ \tilde{U}^0_+(\lambda) \equiv \tilde{U}^0_+(\lambda|\eta, \zeta+, \kappa+, \tau+) = M^0(\lambda)\tilde{K}^0_+(\lambda)\hat{M}^0(\lambda). \] (2.31)

These boundary matrices can easily be expressed in terms of the previous ones. More precisely, the following identities hold, which can be shown by direct computation:

**Proposition 2.1.** The boundary monodromy matrix (2.9) can be expressed in terms of the bulk monodromy matrix (2.25) as
\[ \tilde{U}_-(\lambda) = \hat{M}(-\lambda)\tilde{K}_-(\lambda)M(-\lambda), \] (2.32)
whereas the boundary matrices (2.11) and (2.29) are related by:
\[ \tilde{K}_-(\lambda) = \sigma_0^y \tilde{K}^0_-\sigma_0^y, \quad \tilde{K}_+(\lambda) = \sigma_0^x \tilde{K}^0_+\sigma_0^x. \] (2.33)

As a consequence, the boundary monodromy matrix (2.9) can be expressed in terms of the boundary matrix (2.31) as
\[ \tilde{U}_-(\lambda) = \sigma_0^x \tilde{U}^0_-\sigma_0^x. \] (2.35)

In the following, for technical reasons, it will be convenient to use the representation (2.32) of the boundary monodromy matrix \( \tilde{U}_-(\lambda) \) in terms of the bulk monodromy matrix \( M(\lambda) \).

### 2.2. Gauge transformation of the reflection algebra

The quantum version of the separation of variable (SoV) approach [2, 3, 6], see also [50, 57] for a more general version of the SoV approach, has been used to solve the spectral model of the boundary transfer matrices (2.13). Even if this is not required in the latter framework, we recall here the gauge transformations introduced in [30, 117] and used in [39] to transform the model into an effective one in which at least one of the boundary matrices becomes triangular [24].
2.2.1. Vertex-IRF transformation. Let us recall the definition of the trigonometric solid-on-solid (SOS) (or dynamical) R-matrix:

\[
R_{SOS}(\lambda|\beta) = \begin{pmatrix}
\sinh(\lambda + \eta) & 0 & 0 \\
0 & \sinh(\eta(\beta + 1)) & \sinh(\lambda + \eta\beta) \\
0 & \sinh(\eta(\beta - 1)) & \sinh(\lambda + \eta)
\end{pmatrix},
\]

(2.36)

which also depends on the so-called parameter \(\beta\). This R-matrix can be related to the six-vertex R-matrix (2.5) by the so-called vertex-IRF transformation as

\[
R_{12}(\lambda - \mu)S_1(\lambda|\alpha, \beta)S_2(\mu|\alpha, \beta + \sigma_2^1) = S_2(\mu|\alpha, \beta)S_1(\lambda|\alpha, \beta + \sigma_2^1)R_{12}^{SOS}(\lambda - \mu|\beta),
\]

(2.37)

or equivalently as

\[
R_{12}(\lambda - \mu)S_2(-\mu|\alpha, \beta)S_1(-\lambda|\alpha, \beta + \sigma_2^1) = S_1(-\lambda|\alpha, \beta)S_2(-\mu|\alpha, \beta + \sigma_2^1)R_{12}^{SOS}(\lambda - \mu|\beta).
\]

(2.38)

The relations (2.37) and (2.38) involve the vertex-IRF transformation matrix

\[
S(\lambda|\alpha, \beta) = \begin{pmatrix}
\exp(\lambda - \eta(\beta + \alpha)) & \exp(\lambda + \eta(\beta - \alpha)) \\
1 & 1
\end{pmatrix},
\]

(2.39)

which depends on the spectral parameter \(\lambda\), on the dynamical parameter \(\beta\), and on an arbitrary shift \(\alpha\) of the spectral parameter.

2.2.2. Gauge transformation of the boundary monodromy matrices. Let us introduce the following gauged transformed boundary monodromy matrix \(U_- (\lambda)\):

\[
U_- (\lambda|\alpha, \beta) = S_0^{-1}(\eta/2 - \lambda|\alpha, \beta)U_- (\lambda)S_0(\lambda - \eta/2|\alpha, \beta) \\
= \begin{pmatrix}
A_- (\lambda|\alpha, \beta) & B_- (\lambda|\alpha, \beta) \\
C_- (\lambda|\alpha, \beta) & D_- (\lambda|\alpha, \beta)
\end{pmatrix}.
\]

(2.40)

It is worth remarking that, by definition,

\[
B_- (\lambda|\alpha, \beta) = C_- (\lambda|\alpha, -\beta)
\]

\[
= \frac{\exp\left[\exp\left(\lambda - \eta(\beta + \alpha)\right)C_- (\lambda) + D_- (\lambda) - \exp(2\lambda - \eta)A_- (\lambda) - \exp(-2\lambda - \eta)B_- (\lambda)\right]}{2 \sinh(\eta\beta)},
\]

(2.41)

so that \(\sinh(\eta\beta)e^{-\eta\beta}B_- (\lambda|\alpha, \beta)\) and \(\sinh(\eta\beta)e^{\eta\beta}C_- (\lambda|\alpha, \beta)\) depend on the external gauge parameters only through the combinations \(\alpha - \beta\) and \(\alpha + \beta\), respectively. We will denote:

\[
\tilde{B}_- (\lambda|\alpha - \beta) = \sinh(\eta\beta)e^{-\eta\beta}e^{-(\lambda - \eta/2)}B_- (\lambda|\alpha, \beta),
\]

(2.42)

\[
\tilde{C}_- (\lambda|\alpha + \beta) = \sinh(\eta\beta)e^{\eta\beta}e^{-(\lambda - \eta/2)}C_- (\lambda|\alpha, \beta).
\]

(2.43)
Moreover, the gauged monodromy matrix satisfies the dynamical reflection equation:

\[
R^{\text{SOS}}_{21}(\lambda - \mu | \beta) U_{-1}(\lambda | \alpha, \beta + \sigma_1^+ \beta) R^{\text{SOS}}_{12}(\lambda + \mu - \eta | \beta) U_{-2}(\mu | \alpha, \beta + \sigma_1^+ \beta) = U_{-2}(\mu | \alpha, \beta + \sigma_1^+ \beta) R^{\text{SOS}}_{21}(\lambda + \mu - \eta | \beta) U_{-1}(\lambda | \alpha, \beta + \sigma_1^+ \beta) R^{\text{SOS}}_{12}(\lambda - \mu | \beta).
\]  

(2.44)

From the inversion relation (2.19) of \( U_{-}(\lambda) \), we derive the inversion relation for the matrix \( U_{-}(\lambda | \alpha, \beta) \):

\[
U_{-}(\lambda + \eta/2 | \alpha, \beta) U_{-}(- \lambda + \eta/2 | \alpha, \beta) = \frac{\det_q U_{-}(\lambda)}{\sinh(2\lambda - 2\eta)}.
\]

(2.45)

where \( \det_q U_{-}(\lambda) \) is the quantum determinant (2.15), so that

\[
\frac{\det_q U_{-}(\lambda)}{\sinh(2\lambda - 2\eta)} = A_{-}(\eta/2 + \epsilon \lambda | \alpha, \beta) A_{-}(\eta/2 - \epsilon \lambda | \alpha, \beta) + B_{-}(\eta/2 + \epsilon \lambda | \alpha, \beta) C_{-}(\eta/2 - \epsilon \lambda | \alpha, \beta) + D_{-}(\eta/2 + \epsilon \lambda | \alpha, \beta) D_{-}(\eta/2 - \epsilon \lambda | \alpha, \beta)
\]

(2.46)

for any \( \epsilon \in \{+, -\} \). Let us also recall the relations between the elements of (2.40), which can easily be deduced from the relations (2.20)–(2.22):

\[
D_{-}(\lambda | \alpha, \beta + 1) = \frac{\sinh \eta \sinh(2\lambda + \eta \beta)}{\sinh(\eta(\beta + 1)) \sinh(2\lambda)} A_{-}(\lambda | \alpha, \beta - 1) + e^{2\lambda} \frac{\sinh(\eta \beta) \sinh(2\lambda - \eta)}{\sinh(\eta(\beta - 1)) \sinh(2\lambda)} A_{-}(\lambda | \alpha, \beta - 1),
\]

(2.47)

\[
A_{-}(\lambda | \alpha, \beta - 1) = -\frac{\sinh \eta \sinh(2\lambda - \eta \beta)}{\sinh(\eta(\beta - 1)) \sinh(2\lambda)} A_{-}(\lambda | \alpha, \beta + 1) + e^{2\lambda} \frac{\sinh(\eta \beta) \sinh(2\lambda + \eta)}{\sinh(\eta(\beta - 1)) \sinh(2\lambda)} D_{-}(\lambda | \alpha, \beta + 1),
\]

(2.48)

\[
B_{-}(\lambda | \alpha, \beta) = -e^{2\lambda} \frac{\sinh(2\lambda + \eta)}{\sinh(2\lambda - \eta)} B_{-}(\lambda | \alpha, \beta),
\]

(2.49)

\[
C_{-}(\lambda | \alpha, \beta) = -e^{2\lambda} \frac{\sinh(2\lambda + \eta)}{\sinh(2\lambda - \eta)} C_{-}(\lambda | \alpha, \beta),
\]

(2.50)

It will be convenient in the following to use the boundary-bulk decomposition (2.32) of the boundary monodromy matrix \( U_{-}(\lambda) \) (2.9) in terms of the bulk monodromy matrix \( M(\lambda) \) (2.25). The latter can be rewritten at the level of the gauge-transformed boundary monodromy matrix (2.40) as

\[
U_{-}(\lambda | \alpha, \beta) = \hat{M}(-\lambda | (\gamma, \delta), (\alpha, \beta)) K_{-}(\lambda | (\gamma, \delta), (\gamma', \delta')) M(-\lambda | (\gamma', \delta'), (\alpha, \beta)),
\]

(2.51)

in which we have defined
We will also use the notation:

\[
M(\lambda|\alpha, \beta, (\gamma, \delta)) = S^{-1}(-\eta/2 - \lambda|\alpha, \beta)M(\lambda)S(-\eta/2 - \lambda|\gamma, \delta)
\]

\[
= \begin{pmatrix}
A(\lambda|\alpha, \beta, (\gamma, \delta)) & B(\lambda|\alpha, \beta, (\gamma, \delta)) \\
C(\lambda|\alpha, \beta, (\gamma, \delta)) & D(\lambda|\alpha, \beta, (\gamma, \delta))
\end{pmatrix},
\]

(2.52)

\[
\tilde{M}(\lambda|\alpha, \beta, (\gamma, \delta)) = S^{-1}(\lambda + \eta/2|\gamma, \delta)\tilde{M}(\lambda|\gamma/2|\alpha, \beta)
\]

(2.53)

and

\[
K_-(\lambda|\gamma, \delta, (\gamma', \delta')) = S^{-1}(\eta/2 - \lambda|\gamma, \delta)K_-(\lambda|\gamma - \eta/2|\gamma', \delta').
\]

(2.54)

It is easy to see that, up to a global normalization factor, the gauged operators \( D_\pm \) can be expressed in terms of the elements of the gauged monodromy matrix as follows:

\[
\lambda = \bar{\alpha}, \beta, \gamma, \delta
\]

\[
\lambda = \bar{\alpha}, \beta, \gamma, \delta
\]

Proposition 2.2. Under the following choice of the gauge parameters:

\[
\eta \alpha = -\tau_+ + \frac{\epsilon^\prime}{2} (\varphi_+ - \psi_+) - \frac{\epsilon + \epsilon^\prime}{4}i\pi \mod i\pi,
\]

(2.55)

\[
\eta \beta = \frac{\epsilon + \epsilon^\prime}{2} (\varphi_+ - \psi_+) + \frac{2 + \epsilon - \epsilon^\prime}{4}i\pi \mod i\pi,
\]

(2.56)

for \( \epsilon, \epsilon^\prime \in \{1, -1\} \), the transfer matrix can be expressed in terms of the elements of the gauged monodromy matrix as follows:

\[
T(\lambda) = \tilde{a}_+(\lambda) \frac{\sinh(2\lambda + \eta)}{\sinh 2\lambda} \tilde{A}_-(\lambda|\alpha, \beta - 1) + \tilde{a}_-(\lambda) \frac{\sinh(2\lambda - \eta)}{\sinh 2\lambda} \tilde{A}_-(\lambda|\alpha, \beta - 1),
\]

(2.57)

\[
= \tilde{d}_+(\lambda) \frac{\sinh(2\lambda + \eta)}{\sinh 2\lambda} \tilde{D}_-(\lambda|\alpha, \beta + 1) + \tilde{d}_-(\lambda) \frac{\sinh(2\lambda - \eta)}{\sinh 2\lambda} \tilde{D}_-(\lambda|\alpha, \beta + 1),
\]

(2.58)
where
\begin{align*}
\bar{a}_+(\lambda) &= \epsilon_+ e^{-\lambda + \frac{\eta}{2}} \frac{\sinh(\lambda - \frac{\eta}{2} + \epsilon_+ \phi_-) \cosh(\lambda - \frac{\eta}{2} - \epsilon_- \psi_-)}{\sinh \phi_- \cosh \psi_-}, \\
\bar{d}_+(\lambda) &= -\epsilon_- e^{-\lambda + \frac{\eta}{2}} \frac{\sinh(\lambda - \frac{\eta}{2} - \epsilon_+ \phi_-) \cosh(\lambda - \frac{\eta}{2} + \epsilon_- \psi_-)}{\sinh \phi_- \cosh \psi_-}.
\end{align*}

(2.62)

(2.63)

\textbf{Proof.} This is just a rewriting of our known results, see for example [39]. \hfill \Box

3. Transfer matrix spectrum and eigenstates by SoV

The transfer matrix $T(\lambda)$ is a polynomial in $\sinh^2 \lambda$ of degree $N + 2$ which satisfies the following centrality conditions:
\begin{align*}
T(\lambda) &\sim \kappa_+ \kappa_- \cosh(\tau_+ - \tau_-) e^{\pm 2(N+1)\lambda} \frac{\sinh \zeta_+ \sinh \zeta_-}{2^{2N+1}}, \quad \lambda \to \pm \infty, \quad \zeta_n = \frac{\tau_n}{2} + \frac{\zeta}{2}, \quad n = 1, \ldots, N, \quad \zeta = \frac{\eta}{2}. \tag{3.1}
\end{align*}
and
\begin{align*}
T(\eta/2) &= 2(-1)^N \cosh \eta \det_q T(0), \tag{3.2} \\
T(\eta/2 + i\pi/2) &= -2 \cosh \eta \coth \zeta_+ \coth \zeta_- \det_q T(i\pi/2), \tag{3.3}
\end{align*}
plus the quantum determinant identity:
\begin{align*}
T(\xi_n + \eta/2)T(\xi_n - \eta/2) &= -\frac{\det_q K_+ (\xi_n) \det_q U_- (\xi_n)}{\sinh(2\xi_n + \eta) \sinh(2\xi_n - \eta)}, \quad \forall n \in \{1, \ldots, N\}. \tag{3.4}
\end{align*}

A basis of the space of states (the SoV basis) which separates the variables for the $T(\lambda)$-spectral problem at particular values of the spectral parameter $\lambda$ (related to the inhomogeneity parameters of the model) was constructed in [24, 30] in the framework of a generalization of the Sklyanin’s SoV approach. There, the complete spectrum characterization of the transfer matrix was derived. This characterization was rewritten in [31] in terms of polynomial solutions of some functional $TQ$-equation: when the boundary parameters satisfy a particular constraint (see (3.46) below), the $TQ$-equation is a usual one, otherwise the $TQ$-equation proposed in [31] contains an additional inhomogeneous term which modifies accordingly the resulting Bethe equations. Let us also mention that, in [57], the construction of [24, 30] was generalized beyond the pseudo-diagonalizability of the gauged $B$-operator, i.e. beyond the applicability of the generalized Sklyanin’s SoV approach.

Here, we briefly recall the main results concerning the characterization of the transfer matrix spectrum in this framework. We more particularly focus on the case with a constraint, for which the SoV description of the spectrum of [24, 30] can be reformulated in terms of solutions of a usual (homogeneous) $TQ$-equation and the eigenstates as generalized Bethe states [31].

3.1. On the SoV basis

In this subsection, we recall briefly the SoV basis both in the generalized Sklyanin’s approach [24, 30] and in the new SoV schema [54], which are at the basis of the SoV characterization of the transfer matrix spectrum presented in the next subsections.
Let us introduce some notations. We define\(^7\)
\[
\mathbf{b}_-(\lambda|\alpha, \beta) = \mathbf{c}_-(\lambda|\alpha, -\beta) = \frac{e^{\lambda+\eta\beta-\eta/2}}{\sinh(\eta\beta)} \mathbf{b}_- (\alpha - \beta),
\]
where
\[
\mathbf{b}_-(x) = -\frac{1}{2} \sinh\left(\kappa_+ x\right) \frac{\sinh(x\eta + \tau_+) + e^{x+}}{\sinh(x\eta + \tau_-)}
\]
and from them the following product along the chain:
\[
N_.(\alpha, \beta, \{\xi\}) \equiv \prod_{j=1}^{N} \left[ \frac{\mathbf{b}_-(\frac{x}{2} - \xi_j|\alpha, \beta + 1 + N - 2j)}{\mathbf{g}_-(\frac{x}{2} - \xi_j)} \frac{\sinh(\eta(\beta + 1 + N - 2j))^{1/2}}{\sinh(\eta(j - \beta + N))} \right].
\]
Here, \(\mathbf{g}_-(\lambda)\) is a function such that
\[
\mathbf{g}_-(\lambda + \eta/2)\mathbf{g}_-(-\lambda + \eta/2) = \frac{\det_q K_-(\lambda)}{\sinh(2\lambda - 2\eta)}.
\]
Then, we also introduce the coefficients:
\[
\mathbf{a}_\varepsilon(\lambda) = (-1)^\varepsilon \frac{\sinh(2\lambda + \eta)}{\sinh(2\lambda)} \mathbf{a}_\varepsilon(\lambda) d(-\lambda),
\]
where
\[
\mathbf{a}_\varepsilon(\lambda) = \frac{\sinh(\lambda - \frac{\eta}{2} + \varepsilon_+ \varphi_+)}{\sinh(\varepsilon_+ \varphi_+)} \frac{\cosh(\lambda - \frac{\eta}{2} + \varepsilon_+ \psi_+)}{\cosh(\psi_+)}
\times \frac{\sinh(\lambda - \frac{\eta}{2} - \varepsilon_- \varphi_-)}{\sinh(\varepsilon_- \varphi_-)} \frac{\cosh(\lambda - \frac{\eta}{2} - \varepsilon_- \psi_-)}{\cosh(\psi_-)},
\]
for any choice of \(\varepsilon \equiv (\varepsilon_+, \varepsilon_-, \varepsilon_+, \varepsilon_-) \in \{-1, 1\}^4\) such that \(\varepsilon_+ \varepsilon_- \varepsilon_+ \varepsilon_- = 1\), which satisfy the quantum determinant condition:
\[
\mathbf{a}_\varepsilon(\lambda + \eta/2)\mathbf{a}_\varepsilon(-\lambda + \eta/2) = -\frac{\det_q K_+(\lambda)\det_q K_-(-\lambda)}{\sinh(2\lambda + \eta)\sinh(2\lambda - \eta)}.
\]
Finally, we define, for any \(N\)-tuple of variables \((x_1, \ldots, x_N)\), the generalized Vandermonde determinant \(\tilde{V}(x_1, \ldots, x_N)\) as
\[
\tilde{V}(x_1, \ldots, x_N) = \det_{1<j<k<N} \left[ \sinh(2j-1)x_j \right] = \prod_{j<k}(\sinh^2 x_k - \sinh^2 x_j),
\]
and by them the following function of the inhomogeneities
\[
N(\{\xi\}) = \frac{\tilde{V}(\xi_1^{(0)}, \ldots, \xi_N^{(0)})}{\tilde{V}(\xi_1^{(1)}, \ldots, \xi_N^{(1)})}.
\]
\(^7\) Note these are the coefficients that enters in the triangularization of the \(K_.(\lambda)\) matrix by gauge transformation.
where
\[ \xi_n^{(b)} = \xi_n + \eta/2 - h\eta, \quad 1 \leq n \leq N, \quad h \in \{0, 1\}. \] (3.15)

### 3.1.1. Generalized Sklyanin’s SoV basis.

Following standard notations, let us define
\[ \langle 0 \rangle = \otimes_{n=1}^{N} (1, 0)_n, \quad \langle 0 \rangle = \otimes_{n=1}^{N} (0, 1)_n, \] (3.16)
\[ |0\rangle = \otimes_{n=1}^{N} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |0\rangle = \otimes_{n=1}^{N} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \] (3.17)

For each choice of \( e \in \{-1, 1\}^4 \) and each N-tuple \( h \equiv (h_1, \ldots, h_N) \in \{0, 1\}^N \), we define the following states\(^8\)
\[ |h, \alpha, \beta + 1\rangle_{Sk} = \frac{1}{N_{\langle \alpha, \beta, \{\xi\} \rangle}} \prod_{j=1}^{N} \left( \frac{D_{\langle j, \alpha, \beta, \{\xi\} \rangle}}{k_j A_{\langle j, \alpha, \beta, \{\xi\} \rangle}} \right)^{h_j} S_{1,N}(\{\xi\}|\alpha, \beta) |Q\rangle, \] (3.18)
\[ \sk\langle \alpha, \beta - 1, h \rangle = \frac{1}{N_{\langle \alpha, \beta, \{\xi\} \rangle}} (0) |1\rangle S_{1,N}(\{\xi\}|\alpha, \beta) \prod_{j=1}^{N} \left( \frac{A_{\langle j, \alpha, \beta, \{\xi\} \rangle}}{A_{\langle j, \alpha, \beta, \{\xi\} \rangle}} \right)^{1-h_j}, \] (3.19)

in which \( h_j, j \in \{1, \ldots, N\} \), denotes the \( j \)th component of the N-tuple \( h \). Here we have defined the following product of local gauge matrices (2.39) on \( H = \otimes_{n=1}^{N} H_n \):
\[ S_{1,N}(\{\xi\}|\alpha, \beta) = S_1(-\xi|\alpha, \beta) S_2(-2|\alpha, \beta + \sigma_1^+) \cdots S_N(-\xi|\alpha, \beta + \cdots + \sigma_{N-1}^+) \]
\[ = \prod_{n=1}^{N} S_n \left( -\xi_n|\alpha, \beta + \sum_{j=1}^{n-1} \sigma_j^+ \right), \] (3.20)

where the arrow indicates in which order we have to consider the product of the non-commuting operators. The normalization coefficients \( k_j \) and \( A_{\langle \lambda \rangle} \) are chosen as
\[ k_j = \frac{\sinh(2\xi_j + \eta)}{\sinh(2\xi_j - \eta)}, \quad A_{\langle \lambda \rangle} = g_{\langle \lambda \rangle} a(\lambda)d(-\lambda), \] (3.21)

with here \( g_{\langle \lambda \rangle} \) given in terms of \( e \) and of the boundary parameters \( \varphi_\pm, \psi_\pm \) as
\[ g_{\langle \lambda \rangle} = e_{\pm} e_{\varphi_\pm} (-1)^N \frac{\sinh(\lambda + e_{\varphi_\pm}) \cosh(\lambda + e_\psi_+) \cosh(e_{\psi_+})}{\sinh(e_{\varphi_+}) \cosh(\lambda - e_{\psi_-}) \sinh(\lambda + e_{\psi_-})}, \] (3.22)

for some fixed \( e_{\pm} \in \{-1, 1\} \). Then, the following proposition holds:

---

\(^8\) Up to a different normalization and the use of the symmetry (2.4), the states (3.18) correspond to the states defined in equation (4.6) of [39] multiplied on the right by \( S_{1,N}(\{\xi\}|\alpha, \beta) \), whereas the states (3.19) correspond to the states defined in equation (4.7) of [39] multiplied on the left by \( S_{1,N}(\{\xi\}|\alpha, \beta)^{-1} \).
Proposition 3.1. Let us suppose that the inhomogeneity parameters are generic, i.e.
\[
\xi_j, \xi_j \pm \xi_k \not\in \{0, -\eta, \eta\} \mod(i\pi), \quad \forall j, k \in \{1, \ldots, N\}, \ j \neq k,
\]
and that the following nonzero conditions,
\[
\prod_{j=1}^N b_{\alpha - (\beta + 1 + N - 2j)} \neq 0,
\]
are satisfied. Then, \(B_{-}(\lambda|\alpha, \beta)\) is right and left pseudo-diagonalizable with right and left pseudo-eigenstates (3.18) and (3.19), and its action on these states is given as
\[
B_{-}(\lambda|\alpha, \beta)_{sk} = (-1)^N a_h(\lambda) a_h(-\lambda) \times b_{-}(\lambda|\alpha, \beta - N - 1) \frac{\sinh(\eta\beta)}{\sinh(\eta(\beta + N))} \langle \alpha, \beta - 1, h | B_{-}(\lambda|\alpha, \beta + 1)_{sk}, (3.25)
\]
where
\[
a_h(\lambda) = \prod_{n=1}^N \sinh(\lambda - \xi_n + \eta/2 + h_n\eta). (3.27)
\]
Moreover, the two left and right basis satisfy the following orthogonal conditions:
\[
\langle \alpha, \beta - 1, h | k, \alpha, \beta + 1 \rangle_{sk} = \delta_{hk} \frac{N(\{\xi\}) e^{2\sum_{n}^{N} h_n \xi_n}}{V(\xi_n^{(h)} \cdots, \xi_N^{(h)})}. (3.28)
\]
In [24, 30], it has been shown that such basis are SoV basis of generalized Sklyanin’s type for the spectral problem of the open chain transfer matrix under the choice (2.58) and (2.59) of the gauge parameters. The main reason for these basis to be SoV basis is that under the choice (2.58) and (2.59) the transfer matrix become diagonal in \(\mathcal{A}_{-}(\lambda|\alpha, \beta - 1)\) and \(\mathcal{D}_{-}(\lambda|\alpha, \beta + 1)\) (see proposition 2.2) and these gauged operators act as shift operators on the pseudospectrum of \(B_{-}(\lambda|\alpha, \beta \pm 1)\), respectively, on the left and right basis elements.

3.1.2. The new general SoV basis. One of the interesting achievements of the quantum SoV approach, in particular of the new SoV approach developed in [54], is its ability to describe in an universal way the fundamental objects of different models. This is for example the case for SoV basis, for which mainly the same characterization derived in proposition 3.1 of [1] for the XXX spin chain holds true also for the XXZ chain:

Proposition 3.2. For a given co-vector \(S\), let us define the following co-vectors
\[
\langle h | = \langle S | \prod_{n=1}^N \left( \frac{T(\xi_n - \eta/2)}{A_\epsilon(\eta/2 - \xi_n)} \right)^{1-k_n} = \langle h_1, \ldots, h_N \rangle \in \{0, 1\}^N, (3.29)
\]
and vectors

\[ |h\rangle = \prod_{n=1}^{N} \left( \frac{T(\zeta_n + \eta/2)}{t_n A_2(\eta/2 - \xi_n)} \right)^{h_n} |R\rangle, \quad h \in \{0, 1\}^{N}, \]  

(3.30)

where

\[ t_n = e^{-2\zeta_n} \frac{\sinh 2(\zeta_n + \eta)}{\sinh 2(\zeta_n - \eta)}, \]  

(3.31)

and where \(|R\rangle\) is such that

\[ \langle h | R \rangle = \delta_{h, h'}, \]  

(3.32)

Let us moreover assume that the boundary matrices \(K_-(\lambda)\) and \(K_+(\lambda)\) are not both proportional to the identity. Then, for almost any choice of the co-vector \(|S\rangle\) and of the inhomogeneity parameters satisfying (3.23), the vector \(|R\rangle\) is uniquely defined by (3.32) and the families of co-vectors (3.29) and of vectors (3.30) are basis of \(H^*\) and of \(H\) respectively, which moreover satisfy the following orthogonality conditions:

\[ \langle h | h' \rangle = \delta_{h, h'} \frac{N_{\{\xi\}))(\{\xi'\})}{V(\xi_1, \ldots, \xi_N)} \]  

(3.33)

The fact that these basis are SoV basis is here mainly a clear consequence of their very definitions. Indeed, they imply the factorized form of the transfer matrix wave functions in terms of the transfer matrix eigenvalues. In theorem 3.2 of [54], it was shown that the complete characterization of the spectrum follows by imposing the quantum determinant condition together with the known polynomial form in \(\cosh 2\lambda\) to the transfer matrix eigenvalues.

3.2. The transfer matrix spectrum and eigenstates

From the previous SoV basis follows a characterization of the transfer matrix spectrum in terms of discrete equations [24, 30, 39, 54]. Here, following [31, 39, 54], we present instead its equivalent characterization in terms of \(TQ\)-functional equations, and we more particularly insist on the special case of interest for the present paper, in which one constrain is imposed between the boundary parameters of the sites 1 and \(N\), so that the corresponding \(TQ\)-equation is a homogeneous equation with polynomial \(Q\)-solutions.

3.2.1. SoV characterization of the transfer matrix spectrum and eigenstates in terms of solutions of a functional \(TQ\)-equation. Let us start by recalling some further notations:

\[ u_n = \frac{\sinh(2\zeta_n - \eta)}{\sinh(2\zeta_n + \eta)} \frac{a(\zeta_n + \eta/2)d(-\xi_n - \eta/2)}{a(-\zeta_n + \eta/2)d(\zeta_n - \eta/2)} \]  

\[ = -\prod_{j \neq n} \frac{\sinh(\zeta_n - \zeta_j + \eta) \sinh(\zeta_n + \zeta_j + \eta)}{\sinh(\zeta_n + \xi_j - \eta) \sinh(\zeta_n - \xi_j - \eta)}, \]  

(3.34)

and

\[ v_{n, \epsilon} = \frac{\epsilon(\zeta_n + \frac{\eta}{2})}{\epsilon(-\zeta_n + \frac{\eta}{2})} = \frac{\epsilon(\xi_n + \frac{\eta}{2})}{\epsilon(-\xi_n + \frac{\eta}{2})}, \]  

(3.35)
so that
\[
\frac{\sinh(2\xi_n - 2\eta)}{\sinh(2\xi_n + 2\eta)} \frac{A_{\xi}(\xi_n + \eta)}{A_{-\xi}(\xi_n - \eta)} = u_n V_{n\varepsilon}.
\] (3.36)

Moreover, we denote by \(\Sigma^M_\varepsilon\) the set of polynomials in \(\cosh(2\lambda)\) of degree \(M\) of the form
\[
Q(\lambda) = \prod_{j=1}^{M} \frac{\cosh(2\lambda) - \cosh(2\lambda_j)}{2} = \prod_{j=1}^{M} \left(\sinh^2 \lambda - \sinh^2 \lambda_j\right),
\] (3.37)
with
\[
\cosh(2\lambda) \neq \cosh(2\xi^{(h)}), \quad \forall (j, n, h) \in \{1, \ldots, M\} \times \{1, \ldots, N\} \times \{0, 1\}.
\] (3.38)

Finally, for \(r \in \mathbb{N}\) and \(\varepsilon \equiv \{\epsilon_{\varphi_+}, \epsilon_{\varphi_-}, \epsilon_{\psi_+}, \epsilon_{\psi_-}\} \in \{-1, 1\}^4\), we define
\[
f_\varepsilon^{(r)} = \sum_{n=0}^{N} \frac{2\kappa_+ \kappa_-}{\sinh \xi \sinh \xi} \theta^{(r)}(\tau_+, \tau_-, \varphi_+, \varphi_-, \psi_+, \psi_-),
\] (3.39)
with
\[
\theta^{(r)}(\tau_+, \tau_-, \varphi_+, \varphi_-, \psi_+, \psi_-) = \cosh(\tau_+ - \tau_-) - \epsilon_{\varphi_+} \epsilon_{\varphi_-} \cosh(\epsilon_{\varphi_+} \varphi_+ + \epsilon_{\varphi_-} \varphi_- + \epsilon_{\psi_+} \psi_+ - \epsilon_{\psi_-} \psi_- + (N - 1 - 2r)\eta).
\] (3.40)

**Theorem 3.1 ([31, 39]).** Let the two boundary matrices be not both proportional to the identity matrix, the inhomogeneity parameters be generic, and the following identity be satisfied:
\[
\forall r \in \{0, \ldots, N - 1\}, \quad f_\varepsilon^{(r)}(\tau_+, \tau_-, \varphi_+, \varphi_-, \psi_+, \psi_-) \neq 0.
\] (3.41)

with \(\varepsilon \equiv \{\epsilon_{\varphi_+}, \epsilon_{\varphi_-}, \epsilon_{\psi_+}, \epsilon_{\psi_-}\} \in \{-1, 1\}^4\) and \(\epsilon_{\varphi_+} \epsilon_{\varphi_-} \epsilon_{\psi_+} \epsilon_{\psi_-} = 1\). Then, the transfer matrix \(T(\lambda)\) is diagonalizable with simple eigenvalues, and the set \(\Sigma_T\) of its eigenvalues is given by the set of entire functions \(\tau(\lambda)\) such that there exists a polynomial \(Q(\lambda) \in \Sigma_0\) satisfying with \(\tau(\lambda)\) the TQ-equation
\[
\tau(\lambda)Q(\lambda) = A_{\varepsilon}(\lambda)Q(\lambda - \eta) + A_{-\varepsilon}(\lambda)Q(\lambda + \eta) + F(\varepsilon),
\] (3.42)
with inhomogeneous term
\[
F(\varepsilon) = \int_{e}^{(\eta)} a(\lambda) d(-\lambda) d(-\lambda) [\cosh^2(2\lambda) - \cosh^2 \eta].
\] (3.43)

Moreover, in that case, the corresponding \(Q(\lambda) \in \Sigma_0\) satisfying (3.42) with \(\tau(\lambda)\) is unique, and the unique (up overall constants) left and right \(T(\lambda)\) eigenstates can be expressed as
\[
|Q\rangle = \sum_{h \in \{0, 1\}^N} \prod_{n=1}^{N} \frac{Q(\xi_{h^n})}{Q(\xi_{h^n}^{(0)})} e^{-\sum_{j=1}^{N} h_j \xi_j} V(\xi_1^{(h_1)}, \ldots, \xi_N^{(h_N)}) |h\rangle,
\] (3.44)
\[
\langle Q| = \sum_{h \in \{0, 1\}^N} \prod_{n=1}^{N} \left( u_{h, n \varepsilon} V(\xi_1^{(h_1)}, \ldots, \xi_N^{(h_N)}) |h\rangle\right) e^{-\sum_{j=1}^{N} h_j \xi_j} V(\xi_1^{(h_1)}, \ldots, \xi_N^{(h_N)}) |h\rangle.
\] (3.45)
Note that the characterization of the transfer matrix spectrum and eigenstates of theorem 3.1 is complete. However, as announced above, we are more particularly interested here in the special case in which the following type of constraint on the boundary parameters is satisfied:

\[ f(\tau, \varphi, \psi) = 0, \quad (3.46) \]

for some \( r \in \{0, \ldots, N\} \), for which the transfer matrix spectrum and eigenstates can be (at least partially) characterized in terms of solutions of some usual (homogeneous) functional \( TQ \)-equation of Baxter type. More precisely, we can state the following proposition:

**Proposition 3.3 ([31, 39])**. Let us suppose that the inhomogeneity parameters are generic, and that the two boundary matrices are not both proportional to the identity matrix. We moreover suppose that the condition (3.46) is satisfied for a given \( r = M \in \{0, \ldots, N\} \) and a given choice of \( \epsilon \equiv \left(\epsilon_\varphi, \epsilon_\psi\right) \in \{-1, 1\}^4 \) such that \( \epsilon_\varphi \epsilon_\psi = 1 \). Then, the transfer matrix \( T(\lambda) \) is diagonalizable with simple spectrum, and any entire function \( \tau(\lambda) \) such that there exists \( Q(\lambda) \in \Sigma^M_T \) satisfying the following homogeneous \( TQ \)-equation,

\[ \tau(\lambda)Q(\lambda) = A_\epsilon(\lambda)Q(\lambda - \eta) + A_\epsilon(-\lambda)Q(\lambda + \eta), \quad (3.47) \]

is an eigenvalue of \( T(\lambda) \) (we write \( \tau(\lambda) \in \Sigma_T \)). Moreover, in that case, the corresponding \( Q(\lambda) \in \Sigma^M_T \) satisfying (3.47) with \( \tau(\lambda) \) is unique, and the unique (up overall constants) left and right \( T(\lambda) \) eigenstates can be expressed in terms of \( Q \) as in (3.44) and (3.45).

Contrary to theorem 3.1, the above proposition does not a priori provide a complete characterization of the spectrum: as explained in [31, 39], a part of the spectrum is associated to solutions of the inhomogeneous \( TQ \)-equation (3.42) if the condition (3.46) is satisfied only for a fixed \( r = M < N \), that is if (3.46) is satisfied for \( r = M < N \) being

\[ \theta^{(M)}(\tau, \varphi, \psi) = 0, \quad (3.48) \]

while holding

\[ \frac{\kappa_+ \kappa_-}{\sinh \varsigma_+ \sinh \varsigma_-} \neq 0. \quad (3.49) \]

This gives instead a complete characterization of the spectrum in terms of only homogeneous \( TQ \)-equations when we have either

\[ \theta^{(M)}(\tau, \varphi, \psi) = 0, \quad (3.50) \]

or

\[ \frac{\kappa_+ \kappa_-}{\sinh \varsigma_+ \sinh \varsigma_-} = 0. \quad (3.51) \]

More precisely, we have in that case the following result:

**Theorem 3.2 ([31, 39])**. Let the two boundary matrices be not both proportional to the identity matrix, the inhomogeneity parameters be generic, and one of the two conditions (3.50)
and (3.51) be satisfied. Then, for almost any choice of the boundary parameters, the transfer matrix $T(\lambda)$ is diagonalizable with simple spectrum. Moreover, defined

$$
\Sigma_Q = \begin{cases} 
\Sigma_Q^N & \text{if } \frac{\kappa_+ \kappa_-}{\sinh \varsigma_+ \sinh \varsigma_-} \neq 0, \\
\cup_{n=0}^{N} \Sigma_Q^n & \text{if } \frac{\kappa_+ \kappa_-}{\sinh \varsigma_+ \sinh \varsigma_-} = 0,
\end{cases}
$$  

(3.52)

the set $\Sigma_T$ of the transfer matrix eigenvalues is given by the set of entire functions $\tau(\lambda)$ such that there exists a polynomial $Q(\lambda) \in \Sigma_Q$ satisfying with $\tau(\lambda)$ the homogeneous TQ-equation (3.47). For $\tau(\lambda) \in \Sigma_T$, the corresponding $Q(\lambda) \in \Sigma_Q$ solution of (3.47) is unique, and the corresponding unique (up overall constants) left and right eigenstates of $T(\lambda)$ are respectively given by (3.45) and (3.44).

3.2.2. On the ABA rewriting of separate states: the Sklyanin’s case. Some remarks are in order here. In the previous results, we have used the new SoV basis to provide the SoV characterization of the transfer matrix eigenstates. Indeed, this SoV characterization holds for a larger set of boundary conditions than the generalized Sklyanin’s one. However, we should comment that, when the generalized Sklyanin’s SoV approach works, the transfer matrix eigenstates can alternatively be expressed by the same formulas as (3.44) and (3.45) by using the basis (3.18) and (3.19) instead of (3.29) and (3.30), i.e. as

$$
|Q\rangle_{SK} = \sum_{\mathbf{h} \in \{0,1\}^N} \prod_{n=1}^{N} \frac{Q(\xi^{(h_n)})}{Q(\xi_n^{(0)})} e^{-\sum_{j} h_j \varphi_{j}(\alpha \xi^{(h_1)}, \ldots, \xi^{(h_N)}_{N})} | h, \alpha, \beta + 1 \rangle_{SK},
$$  

(3.53)

$$
\langle Q |_{SK} = \sum_{\mathbf{h} \in \{0,1\}^N} \prod_{n=1}^{N} \left[ (u_{n\alpha})^{h_n} \frac{Q(\xi^{(h_n)})}{Q(\xi_n^{(0)})} \right] e^{-\sum_{j} h_j \varphi_{j}(\alpha \xi^{(h_1)}, \ldots, \xi^{(h_N)}_{N})} \langle h, \alpha, \beta - 1 |_{SK},
$$  

(3.54)

once the gauge parameters are fixed by (2.58) and (2.59). Moreover, due to the simplicity of the transfer matrix spectrum, these two SoV representations (Sklyanin and non-Sklyanin ones) of the eigenstates must coincide up to nonzero normalization, i.e. it must holds:\footnote{That is, the above proportionality holds independently from the precise relation between the two SoV basis.}

$$
|Q\rangle_{SK} = c^S_Q |Q\rangle, \quad \text{sk} \langle Q | = \langle Q | / c^S_Q,
$$  

(3.55)

for some normalization coefficient $c^S_Q \neq 0$, and this is always the case for any choice of the origin states $|S\rangle$ and $|R\rangle$, used to construct the SoV bases (3.29) and (3.30).

Moreover, in the range of validity of the generalized Sklyanin’s approach, separate states of the form (3.53) and (3.54) for any $Q \in \Sigma^M_Q$ admit an ABA form. More precisely, defining

$$
|\Omega_{\alpha,\beta+1} \rangle = \frac{1}{N(\xi)} \sum_{\mathbf{h} \in \{0,1\}^N} e^{-\sum_{j} h_j \varphi_{j}(\alpha \xi^{(h_1)}, \ldots, \xi^{(h_N)}_{N})} | h, \alpha, \beta + 1 \rangle_{SK},
$$  

(3.56)

$$
\langle \Omega_{\alpha,\beta-1} | = \frac{1}{N(\xi)} \sum_{\mathbf{h} \in \{0,1\}^N} \prod_{n=1}^{N} (u_{n\alpha})^{h_n} e^{-\sum_{j} h_j \varphi_{j}(\alpha \xi^{(h_1)}, \ldots, \xi^{(h_N)}_{N})} \langle h, \alpha, \beta - 1 |_{SK},
$$  

(3.57)
and using the following shortcut notation for a product of gauged boundary operators (2.42):
\[
\hat{\mathcal{R}}_{-M}^a(\{\lambda_j\}_{j=1}^M|\alpha - \beta + 1) = \hat{\mathcal{R}}_{-}^a(\lambda_1|\alpha - \beta + 1) \ldots \hat{\mathcal{R}}_{-}^a(\lambda_M|\alpha - \beta + 2M - 1) = \prod_{j=1}^M \hat{\mathcal{R}}_{-}^a(\lambda_j|\alpha - \beta + 2j - 1),
\] (3.58)
we can easily show from (3.25) and (3.26) that, under the condition (3.24), any separate state of the form (3.53) and (3.54) for \(Q \in \Sigma_Q^M\) (not necessarily solution of a \(TQ\)-equation) can be rewritten as
\[
|Q\rangle_{Sk} = c_{Q,ABA}^{(R)} \hat{\mathcal{R}}_{-M}^a(\{\lambda_j\}_{j=1}^M|\alpha - \beta + 1|\Omega_{\alpha,\beta+1-2M}),
\]
(3.59)
\[
\langle Q|_{Sk} = c_{Q,ABA}^{(L)} \Omega_{\alpha,\beta+1+2M} \hat{\mathcal{R}}_{-M}^a(\{\lambda_j\}_{j=1}^M|\alpha - \beta + 1 - 2M),
\]
(3.60)
with normalization coefficients
\[
c_{Q,ABA}^{(R)} = \frac{N(\{\xi\})(-1)^{NM}e^{NM\eta}}{\prod_{j=1}^M \sigma_0(\lambda_j)\sigma_0(-\lambda_j)\sinh(2\lambda_j - \eta)\mathcal{B}_a(\alpha - \beta + 2j - N - 1)},
\]
(3.61)
\[
c_{Q,ABA}^{(L)} = \frac{N(\{\xi\})(-1)^{NM}e^{-NM\eta}}{\prod_{j=1}^M \sigma_0(\lambda_j)\sigma_0(-\lambda_j)\sinh(2\lambda_j - \eta)\mathcal{B}_a(\alpha - \beta - 2j + N + 1) \prod_{j=0}^{2M-1} \frac{\sinh(\eta(\beta + N + j))}{\sinh(\eta(\beta + j))}}.
\]
(3.62)
Here \(\lambda_1, \ldots, \lambda_M\) label the roots of \(Q\) similarly as in (3.37). We underline once again that (3.59) and (3.60) hold for arbitrary \(Q\) of the form (3.37), i.e. not necessarily solution of a \(TQ\)-equation.

This ABA representation of the separate states enables us to reformulate theorem 3.2 in the Sklyanin’s case as follows:

**Corollary 3.1.** *Let us suppose that the generalized Sklyanin’s approach is applicable and that one of the two conditions (3.50) and (3.51) is satisfied. Then the transfer matrix \(T(\lambda)\) is diagonalizable with simple spectrum and the set \(\Sigma_T\) of its eigenvalues is given by the set of entire functions \(\tau(\lambda)\) such that there exists a polynomial \(Q(\lambda) \in \Sigma_Q\) satisfying with \(\tau(\lambda)\) the homogeneous \(TQ\)-equation (3.47). For \(\tau(\lambda) \in \Sigma_T\), the corresponding \(Q(\lambda) \in \Sigma_Q\) solution of (3.47) is unique, and the corresponding unique (up overall constants) left and right eigenstates of \(T(\lambda)\) admit the following ABA representations:
\[
\hat{\mathcal{R}}_{-N_{K_+}}(\{\lambda_j\}_{j=1}^{N_{K_+}}|\alpha - \beta + 1|\Omega_{\alpha,\beta+1-2N_{K_+}}),
\]
(3.63)
and
\[
(\Omega_{\alpha,\beta+1+2N_{K_+}}^\dagger \hat{\mathcal{R}}_{-N_{K_+}}(\{\lambda_j\}_{j=1}^{N_{K_+}}|\alpha - \beta + 1 - 2N_{K_+})).
\]
(3.64)
Here \(N_{K_+}\) denotes the degree of the polynomial \(Q \in \Sigma_Q\), \(\lambda_1, \ldots, \lambda_{N_{K_+}}\) label its roots similarly as in (3.37), and the gauge parameters are fixed in terms of the boundary parameters by (2.58) and (2.59).*

\[10\] That is, let us suppose that the inhomogeneity parameters are generic and that the nonzero condition (3.24) is satisfied imposing (2.58) and (2.59).
Let us comment that the left and right reference states $\langle \Omega_\alpha^{\beta-1}, $ and $| \Omega_\alpha^{\beta+1} \rangle$, with slightly different notations and normalizations, have been presented in our previous paper [39]. Here, we are interested in reformulating these left and right reference states by using, under special conditions, states introduced in [42, 117] in the context of ABA. In order to do so, let us first introduce some notations that will be used throughout the rest of the paper:

\[
\langle x, \eta \rangle \equiv \otimes_{n=1}^{N} (-1, e^{-(N-n+x)\eta - \xi_n})_n, \quad | \eta, x \rangle \equiv \otimes_{n=1}^{N} \left( e^{-(n-N+x)\eta - \xi_n} \right)_n.
\]

It is easy to verify that such states result from the action of the vertex-IRF transformation (2.39) or of its inverse on the states (3.16) and (3.17):

\[
\langle \alpha - \beta, \eta \rangle = e^{-\sum_{n}^{N} \xi_n} N_{\alpha,\beta,\eta}(0) \prod_{n=1}^{N} S_{n}(-\xi_n|\alpha, \beta - N + n), \quad (3.66)
\]

\[
| \eta, \alpha + \beta \rangle = \prod_{n=1}^{N} S_{n}(-\xi_n|\alpha, \beta + n - N) 0 \rangle, \quad (3.67)
\]

with

\[
N_{\alpha,\beta,\eta} = 2^{N} e^{-\alpha N \eta} \prod_{n=1}^{N} \sinh(\eta(n - N + \beta)). \quad (3.68)
\]

By using these states, we can present the announced closed form of the left and right reference states.

**Proposition 3.4.** Let us suppose that the generalized Sklyanin’s approach is applicable, with $\alpha$ and $\beta$ fixed by the gauge conditions (2.58) and (2.59) and

\[
\epsilon_{\varphi^{+}} = \epsilon_{\psi^{+}}, \quad \epsilon_{\varphi^{-}} = \epsilon_{\psi^{-}} = \epsilon_{-} = \epsilon_{-}^{\prime}.
\]

Then, under the condition

\[
\eta(\alpha + \beta + N - 1 - 2M) + \tau^{+} = -\epsilon_{\varphi^{+}}(\varphi^{+} + \psi^{+}) + \frac{1 - \epsilon_{\varphi^{+}}}{2} i \pi \mod 2 \pi i,
\]

for $M \leq N$, the following identity holds:

\[
| \eta, \alpha + \beta + N - 1 - 2M \rangle = \mathcal{R}_{M, \text{ref}}^{(B)} | \Omega_{\alpha,\beta-2M+1} \rangle,
\]

where $\mathcal{R}_{M, \text{ref}}^{(B)}$ is a nonzero scalar factor which only depends on $M$.

Let us suppose that the generalized Sklyanin’s approach is applicable, with $\alpha$ and $\beta$ fixed by the gauge conditions (2.58) and (2.59) and

\[
\epsilon_{\varphi^{+}} = \epsilon_{\psi^{+}}, \quad \epsilon_{\varphi^{-}} = \epsilon_{\psi^{-}} = -\epsilon_{-} = -\epsilon_{-}^{\prime}.
\]

(3.72)
Then, under the condition
\[ \eta(\alpha + \beta - N + 1 + 2M) + \pi_+ = \epsilon_{\varphi_+}(\varphi_+ + \psi_+) + \frac{1 + \epsilon_{\varphi_+}}{2}\pi \mod 2\pi, \]
for \( M \leq N \), the following identity holds:
\[ \langle \alpha + \beta - N + 2M + 1, \eta | = c_{M,\text{ref}}^{(L)}(\Omega_{\alpha,\beta + 2M - 1}), \]
where \( c_{M,\text{ref}}^{(L)} \) is a nonzero scalar factor which only depends on \( M \).

**Proof.** It follows from (3.70) that the condition (3.46) is satisfied for \( M \leq N \) when we impose the condition (3.69). Let \( Q(\lambda) \in \Sigma^M_Q \) solution of the homogeneous \( TQ \)-equation (3.47) for some entire function \( \tau(\lambda) \), i.e. such that the parameters \( \lambda_1, \ldots, \lambda_M \) labelling the roots of \( Q \) satisfy the corresponding Bethe equations. Then, the associated right transfer matrix eigenstate can be written from corollary 3.1 as
\[ \hat{B}_{-M}^{(L)}(\lambda)^M_{j=1}|(\alpha - \beta + 1)|\Omega_{\alpha,\beta + 2M - 1}. \] (3.75)

On the other hand, it follows from (3.70) and from lemma B.1 that the state \( |\eta, \alpha + \beta + N - 1 - 2M \rangle \) is an eigenstate of the operator \( A_-|\lambda|\alpha, \beta - 1\rangle\) as (3.75). Due to the simplicity of the transfer matrix spectrum, the two states (3.75) and (3.77) are therefore collinear:
\[ \hat{B}_{-M}^{(L)}(\lambda)^M_{j=1}|(\alpha - \beta + 1)|\eta, \alpha + \beta + N - 1 - 2M \rangle \] (3.77)
\[ = c_{Q,\text{ref}}\hat{B}_{-M}^{(L)}(\lambda)^M_{j=1}|(\alpha - \beta + 1)|\Omega_{\alpha,\beta + 2M - 1}. \] (3.78)
for some constant \( c_{Q,\text{ref}} \) which depends *a priori* on the particular choice of \( Q \), i.e. of the set of Bethe roots \( \{\lambda_j\}^M_{j=1} \). From the fact that the operators \( \hat{B}_+^{(L)}(\lambda_j|\alpha - \beta + j - 1 \rangle \) are invertible when evaluated at any admissible Bethe root \( \lambda_j \), i.e. satisfying (3.38), we get that
\[ |\eta, \alpha + \beta + N - 1 - 2M \rangle = c_{Q,\text{ref}}|\Omega_{\alpha,\beta + 2M - 1}. \] (3.79)
This relation should be true for any \( Q \in \Sigma^M_Q \) solution of the \( TQ \)-equation, so that the proportionality constant \( c_{Q,\text{ref}} \) should in fact coincide for all such \( Q \in \Sigma^M_Q \).

The second assertion can be proven similarly. \( \square \)
Remark 1. The two conditions (3.70) and (3.73) are in general not compatible, and we made different choices for the gauge parameters in (3.71) and in (3.74), so that we cannot in principle use the ABA form of the right separate states with reference state (3.71) and of the left separate states with reference state (3.74) simultaneously. In the following, in our study of correlation functions, we shall in fact only use the ABA form of the right separate state with condition (3.70).

3.2.3. Spectrum and eigenstates for the case of a generic magnetic field on site 1 and a special 3.2.3. Spectrum and eigenstates for the case of a generic magnetic field on site 1 and a special
condition. Here, we write explicitly the form that takes the spectrum and eigenstates of the transfer matrix under the particular case considered in this paper: when the most general boundary conditions are considered on the site 1, i.e. general values of the boundary parameters \( \varsigma_{-}, \kappa_{-} \) and \( \tau_{-} \), whereas the following very special choice of the boundary conditions is made on the site 1:

\[
K_{-0}(\lambda) = K_{-0}(\lambda; \varsigma_{+} = -\infty, \kappa_{+}, \tau_{+})
\]

\[
= \begin{pmatrix}
\cosh(\lambda N/2) & 0 \\
0 & \cosh(\lambda N/2)
\end{pmatrix}
\]

Then the transfer matrix spectrum characterization reads:

**Proposition 3.5.** Let us fix the boundary condition at site N by imposing (3.80), and let us suppose that the inhomogeneity parameters are generic, i.e. satisfy (3.23). Let us moreover define

\[
\Sigma_{Q} = \cup_{n=0}^{N} \Sigma_{Q}^n.
\]

Then, for almost any choice of the boundary parameters \( \varsigma_{-}, \kappa_{-} \) and \( \tau_{-} \) at site 1, the transfer matrix \( T(\lambda) \) is diagonalizable with simple spectrum. Moreover, the set \( \Sigma_{T} \) of its eigenvalues is given by the set of entire functions \( \tau(\lambda) \) such that there exists a polynomial \( Q(\lambda) \in \Sigma_{Q} \) satisfying with \( \tau(\lambda) \) the homogeneous \( TQ \)-equation (3.47) with coefficient \( A_{\epsilon}(\lambda) \) defined as in (3.10) in terms of

\[
a_{\epsilon}(\lambda) = \frac{\sinh(\lambda - \frac{\pi}{2} + \epsilon_{\phi} \psi_{-}) \cosh(\lambda - \frac{\pi}{2} - \epsilon_{\phi} \psi_{-})}{\sin(\epsilon_{\phi} \psi_{-}) \cosh(\epsilon_{\phi} \psi_{-})}.
\]

for \( \epsilon_{\phi} = \epsilon_{\psi} \in \{+1, -1\} \). For \( \tau(\lambda) \in \Sigma_{T} \), the corresponding \( Q(\lambda) \in \Sigma_{Q} \) solution of (3.47) is unique, and the corresponding unique (up overall constants) left and right eigenstates of \( T(\lambda) \) are respectively given by the SoV states \( |Q\rangle \) (3.44) and \( \langle Q| \) (3.45).

Moreover, if \( \lambda_{1}, \ldots, \lambda_{M} \) label the roots of \( Q(\lambda) \) similarly as in (3.37), then the Bethe states

\[
\mathcal{B}_{-M}(\{\lambda_{i}\}_{i=1}^{M}|\alpha - \beta + 1| \eta, \alpha + \beta + N - 2M - 1),
\]

for \( \alpha \) and \( \beta \) fixed in terms of the boundary parameters by (2.58) and (2.59) with \( \epsilon_{-} = \epsilon'_{-} = \epsilon_{\phi} = \epsilon_{\psi} \), and

\[
(\alpha + \beta + 2M - N + 1, \eta |\mathcal{B}_{-M}(\{\lambda_{i}\}_{i=1}^{M}|\alpha - \beta + 1 - 2M),
\]

for \( \alpha \) and \( \beta \) fixed in terms of the boundary parameters by (2.58) and (2.59) with \( \epsilon_{-} = \epsilon'_{-} = -\epsilon_{\phi} = -\epsilon_{\psi} \), are respectively collinear to \( |Q\rangle \) and \( \langle Q| \).
Finally, the transfer matrix is isospectral to the transfer matrix of an open spin chain with diagonal boundary conditions with boundary parameters $\varsigma^{(D)}_{\pm}$ given as

$$
\varsigma^{(D)}_\varphi = \epsilon_\varphi, \quad \varsigma^{(D)}_\psi = -\epsilon_\varphi + i\pi/2,
$$

for $\epsilon_\varphi = 1$ or $-1$.

**Proof.** The condition (3.80) implies that $K_{-\vartheta}(\lambda)$ is non-proportional to the identity so that the requirements of theorem 3.2 are satisfied. Therefore the above characterization is a corollary of theorem 3.2 once we remark that the coefficients (3.83) are compatible with the quantum determinant condition:

$$
\det_q K_{-\vartheta}(\lambda) = 1, \quad (3.87)
$$

and that they can be obtained as the limit $\varsigma_+ \rightarrow -\infty$ of the general coefficients:

$$
\lim_{\varsigma_+ \rightarrow -\infty} \frac{\sinh(\lambda - \frac{\vartheta}{2} + \epsilon_\varphi \varphi_+)}{\sinh(\epsilon_\varphi \varphi_+)} \frac{\cosh(\lambda - \frac{\vartheta}{2} + \epsilon_\psi \psi_+)}{\cosh(\epsilon_\psi \psi_+)} = 1, \quad (3.88)
$$

for the choice $\epsilon_\varphi = \epsilon_\psi$. Indeed, the limit $\varsigma_+ \rightarrow -\infty$ can be achieved by taking

$$
\varphi_+ \rightarrow -\infty, \quad \psi_+ \rightarrow +\infty \quad \text{with } \varphi_+ - \psi_+ - \varsigma_+ \text{ finite}. \quad (3.89)
$$

In the limit $\varsigma_+ \rightarrow -\infty$, both conditions (B.8) and (B.11) of lemma B.1 are satisfied, so that one can show similarly as in proposition 3.5 by direct ABA computations that the states (3.84) and (3.85) are eigenstates of the transfer matrix with the same eigenvalue as $\langle Q \rangle$ and $\langle Q \rangle$, so that they should be collinear to the latter due to the simplicity of the transfer matrix spectrum.

The second part of the proposition, about isospectrality, follows once we observe that under the identification of the boundary parameters given in (3.83) the corresponding diagonal transfer matrix and the original non-diagonal one have their eigenvalues completely characterized by the same homogeneous $TQ$-equation.

In the following, for the computation of correlation functions for the chain with such special boundary conditions, we will use the fact that boundary Bethe states of the form (3.84) can be decomposed into bulk Bethe states, similarly as what was done in [108], as explained in appendix B.2. This will enable us to compute the action of local operators on such states as done in [108]. The computation of such an action is the purpose of the next section.

4. **Action of local operators on boundary states**

In this section, we compute the action of a basis of local operators on boundary states of ABA form (3.84). The strategy is similar to the one used in the diagonal case [108], i.e.

(a) We decompose the boundary Bethe states of ABA form in terms of bulk Bethe states;
(b) We act with local operators on the bulk Bethe states by means of the solution of the bulk inverse problem;
(c) We reconstruct the resulting states in terms of boundary states.

However, here, all this process should be done in terms of the gauged boundary and bulk operators.
The boundary-bulk decomposition of arbitrary states of the form (3.84) in terms of gauged bulk Bethe states is done in appendix B.2 for the particular boundary conditions that we consider in this paper, i.e. with the boundary matrix (3.80). From proposition B.1, it takes the form

$$\tilde{B}_{\sigma}(\{\lambda_i\}_{i=1}^M|\alpha - \beta + 1) = h_M(x;\alpha,\beta) \sum_{\sigma_1,\ldots,\sigma_M = \pm} H_{\sigma_1,\ldots,\sigma_M}(\{\lambda_i\}_{i=1}^M) \times B_M(\{\sigma_i\lambda_i\}_{i=1}^M|x - 1,\alpha - \beta|\eta,\alpha + \beta + N - M - 1),$$

(4.1)

where the coefficient $H_{\sigma_1,\ldots,\sigma_M}$ is given by (B.32), that we recall here:

$$H_{\sigma_1,\ldots,\sigma_M}(\{\lambda_i\}_{i=1}^M) = \prod_{j=1}^M \left[ \sigma_j \left( -\lambda_j^{(\sigma)} \right) \frac{\sinh(2\lambda_j - \eta)}{\sinh(2\lambda_j)} \right] \prod_{1 \leqslant i < j \leqslant M} \frac{\sinh(\lambda_i^{(\sigma)} + \lambda_j^{(\sigma)} + \eta)}{\sinh(\lambda_i^{(\sigma)} + \lambda_j^{(\sigma)})},$$

(4.2)

and where $h_M(x;\alpha,\beta) \equiv h_M(x,\alpha - \beta,\alpha + \beta + N - 2M - 1)$ is an overall non-zero constant depending exclusively on the gauge parameters $x,\alpha,\beta$ and on the number $M$ of $\tilde{B}$ operators (see (B.33)):

$$h_M(x;\alpha,\beta) = (-1)^{M(M+1)/2} \prod_{j=1}^M \frac{\sinh(\eta(\beta - M - j))}{\sinh(\eta - 2\alpha + 2\beta + 2N - 2M - 1 - 2j)}.$$  

(4.3)

In (4.2) and in the following, we use the shortcut notations $\lambda_j^{(\sigma)} \equiv \sigma_j \lambda_j, 1 \leqslant j \leqslant M$, and

$$\tilde{B}_M(\{\mu_i\}_{i=1}^M|x - 1,\alpha - \beta|\eta,\alpha + \beta + N - M - 1) = \prod_{j=1}^M B(\mu_j|x - j,\alpha - \beta + j - 1|\eta,\alpha + \beta + N - M - 1).$$

(4.4)

To compute correlation functions, we therefore need to act with local operators on the bulk gauged states of the form (4.4). In subsection 4.1, we explain how to reconstruct local operators in terms of the elements of the gauged bulk monodromy matrix (2.52). In subsection 4.2, we identify a basis of local operators whose action on states of the form (4.4) is relatively simple, and we explicitly compute this action. Finally, in subsection 4.3, we compute the resulting action on the boundary states (4.1).

4.1. Reconstruction of local operators in terms of bulk gauged Yang–Baxter generators

Let us start remarking that, using the ordinary bulk inverse relation,

$$\tilde{M}(\pm \lambda - \eta/2) = (-1)^{N-1}(\mp \lambda - \eta/2)\det M(\mp \lambda),$$

(4.5)

we obtain the following inversion relation for the gauge bulk monodromy matrix (2.52):

$$\tilde{M}(\pm \lambda - \eta/2)(\alpha,\beta,\gamma,\delta)) = (-1)^{N-1}(\mp \lambda - \eta/2)(\alpha,\beta,\gamma,\delta)\det M(\mp \lambda).$$

(4.6)
Let us now define, for $i, j \in \{1, 2\}$, the following local operators at the site $n$ of the chain:

$$E_{n}^{ij}(\lambda|\alpha, \beta), (\gamma, \delta)) = S_{n}(\lambda|\gamma, \delta)E_{n}^{ij}S_{n}^{-1}(\lambda|\alpha, \beta),$$

(4.7)

where $E_{n}^{ij} \in \text{End}(\mathbb{C}^2)$ is such that $(E_{n}^{ij})_{j,k} = \delta_{j,k} \delta_{i,l}$. Then, by the ungauged results [67, 68, 108], one can show the following reconstruction:

**Proposition 4.1.** The local operators (4.7) can be reconstructed in terms of the elements of the bulk monodromy matrix (2.52) as

$$E_{n}^{ij}(\xi_n|\alpha, \beta), (\gamma, \delta)) = \prod_{k=1}^{n-1} [t(\xi_k - \eta/2)]^{-1} M_{ij}(\xi_n|\alpha, \beta), (\gamma, \delta)] \prod_{k=1}^{n} [t(\xi_k - \eta/2)]^{-1},$$

(4.8)

and

$$E_{n}^{ij}(\xi_n|\gamma, \delta), (\alpha, \beta)) = (-1)^n \prod_{k=1}^{n} [t(\xi_k - \eta/2)]^{-1}$$

$$\times \frac{M_{ij}(-\eta/2 - \xi_n|\alpha, \beta), (\gamma, \delta))}{\det_q M(\xi_n)} \prod_{k=1}^{n-1} [t(\xi_k - \eta/2)]^{-1}$$

$$= (-1)^n \frac{\det S(-\xi_n|\alpha, \beta)}{\det S(-\xi_n|\gamma, \delta)} \prod_{k=1}^{n} [t(\xi_k - \eta/2)]^{-1}$$

$$\times \frac{M_{3-13-\beta}(\xi_n + \eta/2, (\alpha - 1, \beta), (\gamma - 1, \delta))}{\det_q M(\xi_n)} \prod_{k=1}^{n-1} [t(\xi_k - \eta/2)]^{-1},$$

(4.9)

where $t(\lambda)$ is the bulk transfer matrix:

$$t(\lambda) = \text{tr}_0[M_0(\lambda)],$$

(4.10)

and

$$M_{ij}(\lambda|\alpha, \beta), (\gamma, \delta)) = \text{tr}_0\left[ E_{0}^{ij}M_0(\lambda|\alpha, \beta), (\gamma, \delta)) \right],$$

(4.11)

$$\hat{M}_{ij}(\lambda|\alpha, \beta), (\gamma, \delta)) = \text{tr}_0\left[ E_{0}^{ij}\hat{M}_0(\lambda|\alpha, \beta), (\gamma, \delta)) \right].$$

(4.12)

**Remark 2.** It will be convenient, for further computations, to express the product of inverse transfer matrices $[t(\xi_k - \eta/2)]^{-1}$ in terms of the transfer matrices $t(\xi_k + \eta/2)$ thanks to the quantum determinant condition:

$$[t(\xi_k - \eta/2)]^{-1} = \frac{t(\xi_k + \eta/2)}{\det_q M(\xi_k)},$$

(4.13)

**Proof.** We use the known reconstruction [67, 68, 108] for the local operators $E_{n}^{ij}(\xi_n|\alpha, \beta), (\gamma, \delta))$:
\[ E_0^{ij}(\xi_0| (\alpha, \beta), (\gamma, \delta)) \]
\[ = \prod_{k=1}^{n-1} \left[ \tau(\xi_k - \eta/2) \text{tr}_0 \left[ E_0^{ij}(\xi_0| (\alpha, \beta), (\gamma, \delta)) M_0(\xi_0 - \eta/2) \right] \tau(\xi_k - \eta/2) \right]^{-1} \]
\[ = \prod_{k=1}^{n} \tau(\xi_k - \eta/2) \frac{\text{tr}_0 \left[ E_0^{ij}(\xi_0| (\alpha, \beta), (\gamma, \delta)) \sigma_0^\gamma M_0(\xi_0 + \eta/2) \sigma_0^\beta \right]}{\det_\eta M(\xi_0)} \prod_{k=1}^{n-1} \tau(\xi_k - \eta/2) \right]^{-1}, \]

(4.14)

and remark that
\[ \text{tr}_0 \left[ E_0^{ij}(\xi_0| (\alpha, \beta), (\gamma, \delta)) M_0(\xi_0 - \eta/2) \right] = \text{tr}_0 \left[ E_0^{ij} M_0(\xi_0 - \eta/2| (\alpha, \beta), (\gamma, \delta)) \right], \]

(4.15)

and
\[ \text{tr}_0 \left[ E_0^{ij}(\xi_0| (\gamma, \delta), (\alpha, \beta)) \sigma_0^\alpha M_0(\xi_0 + \eta/2) \sigma_0^\gamma \right] = (-1)^{\epsilon_2} \text{tr}_0 \left[ E_0^{ij} M_0(\xi_0 - \eta/2| (\alpha, \beta), (\gamma, \delta)) \right] \]
\[ = \frac{\det \mathcal{S}(\lambda + \eta/2| (\alpha, \beta))}{\det \mathcal{S}(\lambda + \eta/2| (\gamma, \delta))} \text{tr}_0 \left[ E_0^{ij} \sigma_0^\alpha M_0(\xi_0 + \eta/2| (\alpha - 1, \beta), (\gamma - 1, \delta)) \right], \]

(4.16)

in which we have used (A.1).

As a consequence, one can show, similarly as for the ungauged case, the following identities for the gauged monodromy matrix elements:

**Corollary 4.1.** For any \( \epsilon, \epsilon', \epsilon'' \in \{1, 2\} \), and any choice of parameters \( \alpha, \beta, \gamma, \delta, \gamma', \delta' \) such that \( \sinh(\eta\beta) \neq 0, \sinh(\eta\delta) \neq 0 \) and \( \sinh(\eta\delta') \neq 0 \), the gauged Yang–Baxter algebra generators satisfy the following annihilation relations:

\[ M_{\epsilon,\epsilon'}(\xi_0 - \eta/2| (\alpha, \beta), (\gamma, \delta)) M_{\epsilon',\epsilon''}(\xi_0 + \eta/2| (\alpha - 1, \beta), (\gamma' - 1, \delta')) = 0, \]

(4.17)
\[ M_{\epsilon,\epsilon'}(\xi_0 + \eta/2| (\gamma - 1, \delta), (\alpha - 1, \beta)) M_{\epsilon',\epsilon''}(\xi_0 - \eta/2| (\gamma', \delta'), (\alpha, \beta)) = 0. \]

(4.18)

One also has, for any \( \epsilon, \epsilon', \epsilon'' \in \{1, 2\} \), and any choice of parameters \( \alpha, \beta, \alpha', \beta', \gamma, \delta, \gamma', \delta' \) such that \( \sinh(\eta\beta) \neq 0, \sinh(\eta\beta') \neq 0, \sinh(\eta\delta) \neq 0 \) and \( \sinh(\eta\delta') \neq 0 \),

\[ M_{\epsilon',\epsilon''}(\xi_0 - \eta/2| (\alpha, \beta), (\gamma, \delta)) M_{\epsilon,\epsilon'-\epsilon''}(\xi_0 + \eta/2| (\alpha - 1, \beta), (\gamma' - 1, \delta')) \]
\[ = (-1)^{\epsilon'' - \epsilon'} \frac{e^{\eta\beta'} \sinh(\eta\beta')}{e^{\eta\beta} \sinh(\eta\beta)} M_{\epsilon',\epsilon''}(\xi_0 - \eta/2| (\alpha', \beta'), (\gamma, \delta)) M_{\epsilon,\epsilon'-\epsilon''}(\xi_0 + \eta/2| (\alpha', \beta'), (\gamma' - 1, \delta')) \]
\[ \times (\xi_0 + \eta/2| (\alpha', \beta'), (\gamma' - 1, \delta')), \]

(4.19)
\[ M_{\epsilon',\epsilon''}(\xi_0 + \eta/2| (\gamma - 1, \delta), (\alpha - 1, \beta)) M_{\epsilon,\epsilon'-\epsilon''}(\xi_0 - \eta/2| (\gamma', \delta'), (\alpha, \beta)) \]
\[ = (-1)^{\epsilon'' - \epsilon'} \frac{e^{\eta\beta'} \sinh(\eta\beta')}{e^{\eta\beta} \sinh(\eta\beta')} M_{\epsilon',\epsilon''}(\xi_0 + \eta/2| (\gamma - 1, \delta), (\alpha - 1, \beta')) \]
\[ \times M_{\epsilon,\epsilon'-\epsilon''}(\xi_0 - \eta/2| (\gamma', \delta'), (\alpha', \beta')). \]

(4.20)
Proof. The proof works as for the ungauged case. From the definition (4.7) and the reconstruction formulas (4.8) and (4.9), we have:

\[
S_n(-\xi_n|\gamma, \delta)E_n^\epsilon E_n^\epsilon S_n^{-1}(-\xi_n|\gamma', \delta') \\
= E_n^\epsilon(-\xi_n|\alpha, \beta, (\gamma, \delta)) \cdot E_n^\epsilon(-\xi_n|\gamma', \delta', (\alpha, \beta)) \\
= (-1)^{k-\ell} \frac{\det S(-\xi_n|\alpha, \beta)}{\det S(-\xi_n|\gamma', \delta') \det_q M(\xi_n)} \prod_{a=1}^{n-1} \delta(\xi_n - \eta/2) \\
\times M_{j,\xi_n}(\xi_n - \eta/2|\alpha, \beta, (\gamma, \delta)) \cdot M_{j,\xi_n}(\xi_n + \eta/2|\alpha - 1, \beta), \\
\times (\gamma' - 1, \delta')) \prod_{a=1}^{n-1} \delta(\xi_n - \eta/2)^{-1}, \\
= (-1)^{k-\ell} \frac{\det S(-\xi_n|\gamma, \delta)}{\det S(-\xi_n|\alpha, \beta) \det_q M(\xi_n)} \prod_{a=1}^{n} \delta(\xi_n - \eta/2) \\
\times M_{3-k,\xi_n}(\xi_n + \eta/2|\alpha - 1, \beta), \cdot M_{j,\xi_n}(\xi_n - \eta/2|\gamma', \delta'), \\
\times (\alpha, \beta) \prod_{a=1}^{n} \delta(\xi_n - \eta/2)^{-1}. \tag{4.21}
\]

To prove (4.17) and (4.18), it is then enough to notice that (4.21) vanishes when \( k \neq j \), i.e. when \( j = 3 - k \), which implies the cancellation of the corresponding product of monodromy matrix elements.

(4.19) and (4.20) follow from the fact that

\[
E_n^\epsilon E_n^\epsilon = E_n^\epsilon E_n^\epsilon , \tag{4.22}
\]

which implies

\[
E_n^\epsilon(-\xi_n|\alpha, \beta, (\gamma, \delta)) \cdot E_n^\epsilon(-\xi_n|\gamma', \delta', (\alpha, \beta)) \\
= E_n^\epsilon(-\xi_n|\alpha', \beta', (\gamma, \delta)) \cdot E_n^\epsilon(-\xi_n|\gamma', \delta', (\alpha', \beta')) , \tag{4.23}
\]

and one uses the first identity of (4.21), respectively the second identity of (4.21).

\[\square\]

As a particular case of (4.19) for \( \epsilon = \epsilon' = 2, \bar{\epsilon} = \bar{\epsilon} = 1 \), we notably have

\[
C(\xi_n - \eta/2|\alpha, \beta, (\gamma, \delta))B(\xi_n + \eta/2|\alpha - 1, \beta, (\gamma' - 1, \delta')) \\
= -\frac{e^{\alpha_\epsilon \delta}}{e^{\alpha' \bar{\epsilon}}} A(\xi_n - \eta/2|\alpha', \beta', (\gamma, \delta))D(\xi_n + \eta/2|\alpha' - 1, \beta', (\gamma' - 1, \delta')). \tag{4.24}
\]

Note also that (4.17) and (4.18) can be extended to the case in which the two operators belong to a larger product of operators:

**Corollary 4.2.** Let \( n \leq m \) and let \( \alpha = (\alpha_1, \ldots, \alpha_m), \beta = (\beta_1, \ldots, \beta_m) \), \( \alpha' = (\alpha'_1, \ldots, \alpha'_m) \), \( \beta' = (\beta'_1, \ldots, \beta'_m) \), \( \gamma = (\gamma_1, \ldots, \gamma_m) \) and \( \delta = (\delta_1, \ldots, \delta_m) \) be arbitrary \((m + n - 1)\)-tuples of
parameters such that all operators
\[
\mathcal{E}_{n,m}^{\omega'}(\omega, \omega') = \prod_{k=1}^m S_k(-\xi_k|\omega_k, \omega'_k),
\] (4.25)

for \((\omega, \omega') \in \{(\alpha, \beta), (\alpha', \beta'), (\gamma, \delta)\}\) are invertible. Then, the product of operators
\[
\prod_{k=n-m}^m M_{k,\epsilon_k}(\epsilon_k - \eta/2)(\gamma_k, \delta_k), (\alpha_k, \beta_k)) \prod_{k=n-m}^m M_{k,\epsilon'_k}(\epsilon'_k + \eta/2)(\gamma_k - 1, \delta_k, (\alpha'_k - 1, \beta'_k))
\] (4.26)
vanishes as soon as there exists some \(k \in \{n, \ldots, m\}\) such that \(\epsilon_k = \epsilon'_k\).

**Proof.** This is once again a direct consequence of the reconstruction formulas (4.8) and (4.9). We have
\[
\prod_{k=n-m}^m E_{k,\epsilon_k}(\epsilon_k - \eta)(\gamma_k, \delta_k), (\alpha_k, \beta_k)) \prod_{k=n-m}^m E_{k,\epsilon'_k}(\epsilon'_k + \eta)(\gamma_k - 1, \delta_k, (\alpha'_k - 1, \beta'_k))
\] (4.27)
\[
= \mathcal{E}_{n,m}^{\alpha,\beta}(\alpha, \beta) \prod_{k=n}^m E_{k,\epsilon_k}(\epsilon_k - \eta)(\gamma_k, \delta_k), (\alpha_k, \beta_k)) \prod_{k=n}^m E_{k,\epsilon'_k}(\epsilon'_k + \eta)(\gamma_k - 1, \delta_k, (\alpha'_k - 1, \beta'_k))
\] (4.28)
in which we have used (4.8) to reconstruct the first product of operators and (4.9) to reconstruct the second product. (4.27) obviously vanishes if \(\epsilon_k = \epsilon'_k\) for some \(k\), which achieves the proof. \(\square\)

We can now use the above results to produce the reconstruction of a basis of quasi-local operators:

**Proposition 4.2.** Let us consider the following monomials of local operators:
\[
\prod_{n=1}^m E_{n,\epsilon_n}^{\alpha_n,\beta_n}(\xi_n|\alpha_n, \beta_n) = \mathcal{E}_{n,m}^{\alpha,\beta}(\alpha, \beta) \prod_{n=1}^m E_{n,\epsilon_n}^{\alpha_n,\beta_n}(\xi_n|\alpha_n, \beta_n),
\] (4.29)

obtained from the elementary basis of local operators \(\prod_{n=1}^m E_{n,\epsilon_n}^{\alpha_n,\beta_n}\), in which we have denoted \(\epsilon = (\epsilon_1, \ldots, \epsilon_n), \epsilon' = (\epsilon'_1, \ldots, \epsilon'_n)\), by means of the tensor product like operators \(\mathcal{E}_{n,m}^{\alpha,\beta}\) and \(\mathcal{E}_{n,m}^{\alpha',\beta'}\) defined as in (4.25). Here \(\alpha \equiv (\alpha_1, \ldots, \alpha_m), \beta \equiv (\beta_1, \ldots, \beta_m), \alpha' \equiv (\alpha'_1, \ldots, \alpha'_m)\) and \(\beta' \equiv (\beta'_1, \ldots, \beta'_m)\) are arbitrary \(m\)-tuples of parameters such that the operators \(\mathcal{E}_{n,m}^{\alpha,\beta}\) and \(\mathcal{E}_{n,m}^{\alpha',\beta'}\) are invertible. Then, the elements (4.29) admit the following reconstruction in
terms of the gauge transformed Yang–Baxter generators:

\[
\prod_{i=1}^{m} E_{i}^{\epsilon_{i}^{\prime},\epsilon_{i}}(\xi_{i}|(\alpha_{i}', \beta_{i}'), (\alpha_{i}, \beta_{i})) = \prod_{i=1}^{m} \frac{\det S(-\xi_{i}|\gamma_{i}, \delta_{i})}{\det S(-\xi_{i}|\alpha_{i}', \beta_{i}')} \prod_{i=1}^{m} M_{i}^{\epsilon_{i}^{\prime},\epsilon_{i}}(\xi_{i} - \eta/2(\gamma_{i}, \delta_{i}), (\alpha_{i}, \beta_{i})) \\
\times \prod_{i=m+1}^{m} \frac{M_{i}^{\epsilon_{i}^{\prime},\epsilon_{i}}(\xi_{i} + \eta/2(\gamma_{i} - 1, \delta_{i}), (\alpha_{i}' - 1, \beta_{i}'))}{\det M(\xi_{i})},
\]

(4.30)

for any \(m\)-tuples of arbitrary parameters \(\gamma \equiv (\gamma_{1}, \ldots, \gamma_{m})\) and \(\delta \equiv (\delta_{1}, \ldots, \delta_{m})\) such that the operator \(S_{(\gamma, \delta)}^{l,m}\) defined similarly as in (4.25), is invertible.

Proof. This reconstruction follows from the identities

\[
E_{n}^{\epsilon_{n},\epsilon_{n}} E_{n}^{\epsilon_{n},\delta_{n}} = E_{n}^{\epsilon_{n},\alpha_{n}},
\]

(4.31)

for any \(\epsilon_{n}, \epsilon_{n}' \in \{1, 2\}\), which enable us to write

\[
\prod_{n=1}^{m} E_{n}^{\epsilon_{n},\epsilon_{n}}(\xi_{n}|(\alpha_{n}', \beta_{n}'), (\alpha_{n}, \beta_{n})) = \mathcal{E}_{(\alpha, \beta)}^{1,m} \prod_{n=1}^{m} E_{n}^{\epsilon_{n},\alpha_{n}}(\xi_{n}|(\alpha_{n}', \beta_{n}'), (\alpha_{n}, \beta_{n}))^{-1} \mathcal{E}_{(\gamma, \delta)}^{1,m} \prod_{n=1}^{m} E_{n}^{\epsilon_{n},\delta_{n}}(\xi_{n}|(\alpha_{n}', \beta_{n}'), (\gamma_{n}, \delta_{n})).
\]

(4.32)

Then we use (4.8) to reconstruct the first product of operators:

\[
\prod_{i=1}^{m} E_{i}^{\epsilon_{i},\epsilon_{i}}(\xi_{i}|(\gamma_{i}, \delta_{i}), (\alpha_{i}, \beta_{i})) = \prod_{i=1}^{m} M_{i}^{\epsilon_{i},\epsilon_{i}}(\xi_{i} - \eta/2(\gamma_{i}, \delta_{i}), (\alpha_{i}, \beta_{i})) \prod_{i=1}^{m} [n(\xi_{i} - \eta/2)]^{-1},
\]

(4.33)

and (4.9) to reconstruct the second product of operators:

\[
\prod_{i=1}^{m} E_{i}^{\epsilon_{i},\epsilon_{i}}(\xi_{i}|(\alpha_{i}', \beta_{i}'), (\gamma_{i}, \delta_{i})) = \prod_{i=1}^{m} \frac{\det S(-\xi_{i}|\gamma_{i}, \delta_{i})}{\det S(-\xi_{i}|\alpha_{i}', \beta_{i}')} \prod_{i=1}^{m} n(\xi_{i} - \eta/2) \\
\times \prod_{i=m+1}^{m} \frac{M_{i}^{\epsilon_{i},\epsilon_{i}}(\xi_{i} + \eta/2(\gamma_{i} - 1, \delta_{i}), (\alpha_{i}' - 1, \beta_{i}'))}{\det M(\xi_{i})},
\]

(4.34)

to obtain the reconstruction (4.30).
4.2. Action of a basis of local operators on gauged bulk states

Here, we identify a basis of local operators and we compute the action of its elements on bulk gauged Bethe like states.

**Proposition 4.3.** For two given gauge parameters $\alpha$ and $\beta$, the set of monomial of local operators

$$\mathcal{E}_m(\alpha, \beta) = \left\{ \prod_{n=1}^{m} E_n^{\varepsilon_n, \varepsilon'_n}(\zeta_n(a_n, b_n), (\tilde{a}_n, \tilde{b}_n)) \mid \varepsilon, \varepsilon' \in \{1, 2\}^m \right\},$$

where the gauge parameters $a_n, \bar{a}_n, b_n, \bar{b}_n$, $1 \leq n \leq m$, are fixed in terms of $\alpha, \beta$ and of the $m$-tuples $\varepsilon \equiv (\epsilon_1, \ldots, \epsilon_m)$ and $\varepsilon' \equiv (\epsilon'_1, \ldots, \epsilon'_m)$ as

$$a_n = \alpha + 1, \quad b_n = \beta - \sum_{r=1}^{n} (-1)^{\epsilon_r},$$

$$\bar{a}_n = \alpha - 1, \quad \bar{b}_n = \beta + \sum_{r=n+1}^{m} (-1)^{\epsilon'_r} - \sum_{r=1}^{m} (-1)^{\epsilon'_r} = b_n + 2\bar{m}_{n+1},$$

with

$$\bar{m}_{n} = \sum_{r=n}^{m} (\epsilon'_r - \epsilon_r) = \sum_{r=n}^{m} \frac{(-1)^{\epsilon'_r} - (-1)^{\epsilon_r}}{2}$$

defines a basis of $\mathcal{End}(\otimes_{n=1}^{m} \mathcal{H}_n)$, except for a finite numbers of values of $\beta \mod 2\pi/\eta$.

In other words, this means that any local operator acting on the first $m$ sites of the chain can be expressed as a linear combination of elements of (4.35).

**Proof.** Thanks to the tensor product form of the elements of (4.35), we have just to prove that, for any fixed $n \in \{1, \ldots, m\}$ the four operators $E_n^{\varepsilon, \varepsilon'_n}(\zeta_n(a_n, b_n), (\tilde{a}_n, \tilde{b}_n))$ associated with the four possible choices of $(\epsilon_n, \epsilon'_n) \in \{1, 2\}^2$, form a basis independently from the choice of the other $\varepsilon_j, \varepsilon'_j$ for $j \neq n$. Here, the main point is that the gauge parameters $b_n$ and $\bar{b}_n$, defined in (4.36) and (4.37), depend not only on $(\epsilon_n, \epsilon'_n) \in \{1, 2\}^2$ but also on the choice of the other $\varepsilon_j, \varepsilon'_j$ for $j \neq n$. We evidence this dependence as follows:

$$b_n = b_n - (-1)^{\epsilon_n}, \quad \text{with} \quad b_n = b - \sum_{r=1}^{n-1} (-1)^{\epsilon_r},$$

$$\bar{b}_n = \bar{b}_n - (-1)^{\epsilon'_n}, \quad \text{with} \quad \bar{b}_n = \bar{b}_n + 2\bar{m}_{n+1},$$

so that $\bar{b}_n$ and $\tilde{\bar{b}}_n$ do not depend on $\epsilon_n, \epsilon'_n$, but only on the other $\varepsilon_j, \varepsilon'_j$, $j \neq n$. Let us define the coefficients $E_n^{\epsilon_i, \epsilon'_n}(\zeta_n(a_n, b_n), (\tilde{a}_n, \tilde{b}_n))$, $1 \leq i, j \leq 2$, by the decomposition

$$E_n^{\epsilon_i, \epsilon'_n}(\zeta_n(a_n, b_n), (\tilde{a}_n, \tilde{b}_n)) = \sum_{j=1}^{2} \sum_{j=1}^{2} E_n^{\epsilon_i, \epsilon'_n}(\zeta_n(a_n, b_n), (\tilde{a}_n, \tilde{b}_n)) E_n^{j'},$$

where $E_n^{\epsilon_i, \epsilon'_n}(\zeta_n(a_n, b_n), (\tilde{a}_n, \tilde{b}_n)) = \sum_{j=1}^{2} \sum_{j=1}^{2} E_n^{\epsilon_i, \epsilon'_n}(\zeta_n(a_n, b_n), (\tilde{a}_n, \tilde{b}_n)) E_n^{j'}$.
in the natural basis $E^{ij}_n$, $1 \leq i, j \leq 2$, of local operators on the site $n$. The determinant

$$\det_{(\epsilon_j, \epsilon'_j) \in \{1,2\}^2} \left[ E^{\epsilon_j\epsilon'_j}_{n,\alpha_j\beta_j} (\zeta_n(a_n, b_n), (\tilde{a}_n, \tilde{b}_n)) \right]$$

(4.42)

can easily be computed, and one can remark that it is a rational function of $e^{-2\beta}$ with coefficients being integers powers of $e^{-2\beta}$. Hence this determinant is always nonzero, up to a finite number of values of $\beta$ (modulo the periodicity $i\pi/\eta$), so that the local operators $E^{\epsilon_j\epsilon'_j}_{n,\alpha_j\beta_j} (\zeta_n(a_n, b_n), (\tilde{a}_n, \tilde{b}_n))$ form a basis for any choice of $\epsilon_j, \epsilon'_j, j \neq n$. 

The choice of this basis is due to the fact that the action of its elements on the gauged Bethe-type bulk states (4.4) has a relatively simple form, as shown in the remaining part of this section. To compute this action, we use the following reconstruction of the local operators elements of (4.35) (see proposition 4.2):

$$\prod_{n=1}^{m} E^{\epsilon_j\epsilon'_j}_{n,\alpha_j\beta_j} (\zeta_n(a_n, b_n), (\tilde{a}_n, \tilde{b}_n)) = \prod_{n=1}^{m} \frac{\det S(-\zeta_n, c_n, d_n)}{\det S(-\zeta_n, a_n, b_n)} \prod_{n=1}^{m} M_{\epsilon_j\epsilon'_j\alpha_j\beta_j}(\zeta_n - \eta/2)(c_n, d_n, (a_n, \tilde{b}_n))$$

$$\times \prod_{n=m+1}^{N} M_{\epsilon_j\epsilon'_j\alpha_j\beta_j}(\zeta_n + \eta/2)(c_n - 1, d_n, (a_n - 1, b_n))$$

$$\det M(\zeta_n),$$

(4.43)

with the following choice for the internal gauge parameters:

$$c_n = \frac{x + \alpha + \beta + N - 1}{2} - M,$$

(4.44)

$$d_n = \frac{-x + \alpha + \beta + N - 1}{2} - M - \sum_{r=1}^{n} (-1)^r,$$

(4.45)

for any arbitrary choice of $x$ which leaves $\mathcal{S}_{(\epsilon, \epsilon')}^{(1a)}$ invertible.

In the remaining part of this subsection, we derive the action of (4.43) on the Bethe-type states (4.4) in full details. In order to do so we introduce the following notations.

Let us associate, to two given $m$-tuples $\epsilon \equiv (\epsilon_1, \ldots, \epsilon_m)$, $\epsilon' \equiv (\epsilon'_1, \ldots, \epsilon'_m) \in \{1,2\}^m$, the following sets of integers, for any $n \in \{1, \ldots, m\}$:

1. $\{i^{(n)}_h\}_{p \in \{1, \ldots, s(n)\}}$, with $i^{(n)}_k < i^{(n)}_h$ for $0 < k < h \leq s(n)$, (4.46)
2. $\{i^{(n)}_h\}_{p \in \{s(n) + 1, \ldots, s(n) + s'_n\}}$, with $i^{(n)}_k > i^{(n)}_h$ for $s(n) < k < h \leq s(n) + s'_n$, (4.47)

defined by the conditions

$$j \in \{i^{(n)}_p\}_{p \in \{1, \ldots, s(n)\}} \quad \text{iff} \quad n \leq j \leq m \quad \text{and} \quad \epsilon_j = 2,$$

(4.48)

$$j \in \{i^{(n)}_p\}_{p \in \{s(n) + 1, \ldots, s(n) + s'_n\}} \quad \text{iff} \quad n \leq j \leq m \quad \text{and} \quad \epsilon'_j = 1.$$ (4.49)

Note that we have

$$s(n) = \sum_{j=n}^{m} (\epsilon_j - 1), \quad s'_n = \sum_{j=n}^{m} (2 - \epsilon'_j), \quad s(n) + s'_n = m - n + 1 + \sum_{j=n}^{m} (\epsilon_j - \epsilon'_j).$$ (4.50)
We use the following simplified notations for the case \( n = 1 \),
\[
s = s_{(n=1)}, \quad s' = s'_{(n=1)} \quad \text{and} \quad i_p = i_p^{(n=1)} \quad \forall \ p \in \{1, \ldots, s + s'\}. \tag{4.51}
\]

We can relate these sets for two different values of \( n \) by
\[
i_1^{(n-1)} = (n - 1)\delta_{n-1,2} + (1 - \delta_{n-1,2}) i_1^{(n)},
\tag{4.52}
\]
\[
i_p^{(n-1)} = \delta_{n-1,1} i_p^{(n)} + \delta_{n-1,2} i_{p+1}^{(n)}, \quad \forall \ p \in \{1, \ldots, s_{(n)}\},
\tag{4.53}
\]
\[
i_{(n-1)}^{(n-1)} = n\delta_{n-1,1} + (1 - \delta_{n-1,1}) i_{(n)} + i_{(n-1)}',
\tag{4.54}
\]
\[
i_{(n-1)}^{(n-1)} = s_{(n-1)} + s_{(n)}',
\tag{4.55}
\]

with
\[
s_{(n-1)} = \delta_{n-1,2} + s_{(n)}, \quad s'_{(n-1)} = \delta_{n-1,1} + s_{(n)}'.
\tag{4.56}
\]

Then, we can derive the following result:

**Proposition 4.4.** For any given \( m \)-tuples \( \epsilon \equiv (\epsilon_1, \ldots, \epsilon_m), \epsilon' \equiv (\epsilon'_1, \ldots, \epsilon'_m) \in \{1, 2\}^m \) and gauge parameters \( \alpha, \beta, x \), let us consider, for each \( n \in \{1, \ldots, m\} \), the following monomials of elements of the bulk gauged monodromy matrix:
\[
M_{(n,m)}^{(\alpha, \beta, x)} = \prod_{k=n-m}^{n} M_{\epsilon_k \epsilon'_k}^{(\alpha, \beta, x)}(\xi_k - \eta/2; (c_k, d_k), (\bar{a}_k, \bar{b}_k))
\times \prod_{k=m-n}^{m} \frac{M_{3-\epsilon_k-\epsilon'_k}^{(\alpha, \beta, x)}(\xi_k + \eta/2; (c_k - 1, d_k), (a_k - 1, b_k))}{\det_{q} M(\xi_k)},
\tag{4.57}
\]

where we have defined
\[
a_k = \alpha + 1, \quad b_k = \beta + \sum_{r=1}^{k} (-1)^{r'},
\tag{4.58}
\]
\[
\bar{a}_k = \alpha - 1, \quad \bar{b}_k = \beta + \sum_{r=k+1}^{m} (-1)^{r'} - \sum_{r=1}^{m} (-1)^{r} = b_k + 2\bar{m}_{k+1},
\tag{4.59}
\]
\[
c_k = \frac{x + \alpha + \beta + N - 1}{2} - M, \quad d_k = \frac{-x + \alpha + \beta + N - 1}{2} - M - \sum_{r=1}^{k} (-1)^{r'}
\tag{4.60}
\]

with
\[
\bar{m}_k = \sum_{r=k}^{m} (\epsilon'_r - \epsilon_r) = \sum_{r=k}^{m} (-1)^{r'} - (-1)^{r'}.
\tag{4.61}
\]
Then, the action of $M^{(n,m)}_{\epsilon,\epsilon'}(\alpha, \beta, x)$ on the following gauged Bethe-like bulk states

\[
B_M(\{\mu_i\}^M_{i=1}|x_n-1, z_n) \eta, y_n \}
\]

where

\[
x_n = x + \sum_{r=1}^{n-1} (-1)^r = c_n - d_n - (-1)^s, \quad (4.63)
\]

\[
z_n = \alpha - \beta + \sum_{r=1}^{n-1} (-1)^r = a_n - b_n - (-1)^s - 1, \quad (4.64)
\]

\[
y_n = \alpha + \beta + N - M - 1 - \sum_{r=1}^{n-1} (-1)^r = a_n + b_n + (-1)^s + N - M - 2 = c_n + d_n + M - (-1)^s, \quad (4.65)
\]

is

\[
M^{(n,m)}_{\epsilon,\epsilon'}(\alpha, \beta, x) \ B_M(\{\mu_i\}^M_{i=1}|x_n-1, z_n) \eta, y_n \}
\]

\[
= \sum_{\mathcal{B}_{\epsilon,\epsilon'}} F_{\mathcal{B}_{\epsilon,\epsilon'}}(\{\mu_j\}^M_{j=1}, \{\xi_j^{(1)}\}^m_{j=1}|\alpha, \beta, x)
\]

\[
\times \ B_M+y_n(\{\mu_j\}_{j \in \mathcal{A}_{\epsilon,\epsilon'}}|x_n-1, z_n-2\tilde{m}_n) \eta, y_n + \tilde{m}_n), \quad (4.66)
\]

where we have defined $\mu_{M+j} \equiv c_{m+1-j}$ for $j \in \{1, \ldots, m+1-n\}$. In (4.66), the sum runs over all the possible sets of integers $\mathcal{B}_{\epsilon,\epsilon'} = \{\mathcal{B}_{1}^{(n)}, \ldots, \mathcal{B}_{M+m}^{(n)}\}$ whose elements satisfy the conditions

\[
\begin{align*}
\{\mathcal{B}_{p}^{(n)} \in \{1, \ldots, M\} \setminus \{\mathcal{B}_{1}^{(n)}, \ldots, \mathcal{B}_{p-1}^{(n)}\} & \quad \text{for } 0 < p \leq s_{(n)}, \\
\{\mathcal{B}_{p}^{(n)} \in \{1, \ldots, M + m + 1 - t_{(n)}\} \setminus \{\mathcal{B}_{1}^{(n)}, \ldots, \mathcal{B}_{p-1}^{(n)}\} & \quad \text{for } s < p \leq s_{(n)} + s'_{(n)}.
\end{align*}
\]

whereas

\[
\mathcal{A}_{\epsilon,\epsilon'}^{(n)} \equiv \{\mathcal{A}_1^{(n)}, \ldots, \mathcal{A}_{M+m}^{(n)}\} = \{1, \ldots, M + m + 1 - n\} \setminus \mathcal{B}_{\epsilon,\epsilon'}^{(n)}. \quad (4.68)
\]

Finally
\begin{align*}
  \mathcal{F}_{\epsilon_{\alpha,i}}^{(n,m)}(\mu_1, \ldots, \mu_n; \xi_1, \ldots, \xi_m) &= \prod_{k=1}^{n+m-1} e^{(\xi_{\alpha,k} - \mu_{\alpha,k+1} - M)} \\
  &= \frac{2 \sinh(\eta(d_{n+1} + k - M + 1))}{2 \sinh(\eta(d_{n+1} + k + M))} \quad \text{if } \bar{m}_n > 0, \\
  &= \frac{2 \sinh(\eta(d_{n+1} - k - M + 1))}{2 \sinh(\eta(d_{n+1} - k + M))} \quad \text{if } \bar{m}_n < 0.
\end{align*}

Proof. Let us prove this result by induction. We can easily prove that it holds for \( n = m \) using the actions (A.78) and (A.79) and the identity (4.42) according to the required case.

So let us assume that the stated form of the action holds for \( 2 \leq n + 1 \leq m \) and let us prove it for \( n \). We distinguish the four possible choices of \((\epsilon_\alpha, \epsilon_\beta) \in \{1, 2\}^2\).

(A) Let us first consider the case \((\epsilon_\alpha, \epsilon_\beta) = (1, 1)\). We have to act on the state (4.62) with the monomial

\begin{align*}
  \frac{e^{2\mu_n}}{4 \sinh^2(\eta d_{n+1})} A^{(1)}(c_n + d_n + b_n + \tilde{b}_n)_{M_{\epsilon_\alpha, \epsilon_\beta}^{(n+1,m)}}(\alpha, \beta, x) \times \frac{D(c_n + d_n - 1, a_n - b_n - 1)}{\det(M(z))}.
\end{align*}

in which we have used that

\begin{align*}
  a_n &= b_n - 1 = z_n + (-1)^\epsilon, \\
  c_n + d_n &= y_n - M = (-1)^\epsilon - 1, \\
  c_n - d_n &= x_n = x_{n+1}.
\end{align*}
\[ \bar{a}_n + \bar{b}_n = y_{n+1} - N + M + 2 \bar{m}_{n+1} = \bar{y}_{n+1} - N + (M + \bar{m}_{n+1}), \]  

(4.75)

with the shortcut notations

\[ \bar{z}_n = z_n - 2\bar{m}_n, \quad \bar{y}_n = y_n + \bar{m}_n. \]  

(4.76)

We can use (A.79) to act with the last factor of (4.71) on this state. It follows from corollary 4.2 that only the direct term of the action of the D-operator can finally lead to a non-zero contribution, so that the result of this action that we have to take into account is

\[
\frac{d_{\alpha}^{(R)}}{d(\xi^{(1)}_1)} \prod_{b=1}^{M} \frac{\sinh(\mu_b - \xi^{(1)}_b)\mathcal{B}(\{\mu_j\}_{j=1}^{M}|x_n - 2, z_n)|\eta, y_n + 1)}{\sinh(\mu_b - \xi^{(0)}_b\mathcal{B}(\{\mu_j\}_{j=1}^{M}|x_n - 1, z_{n+1} + 1)} \times |\eta, y_{n+1}\rangle,
\]

(4.77)

with

\[
\frac{d_{\alpha}^{(R)}}{d(\xi^{(1)}_1)} = \frac{e^{-(a_1-M-1)\eta} - e^{-(a_1-N)\eta}}{e^{\eta/2}} \frac{\sinh(\eta y_{n+1} - M + 2)}{\sinh(\eta y_n - a_1 + 2M + 2)},
\]

\[
= \frac{\sinh(\eta d_n)}{\sinh(\eta(d_n + M))},
\]

(4.78)

We now use the induction formula to compute the action of \( M_{\alpha,\beta}^{(n+1,m)}(\alpha, \beta, x) \) on this state, which gives

\[
\frac{d_{\alpha}^{(R)}}{d(\xi^{(1)}_1)} \prod_{b=1}^{M} \frac{\sinh(\mu_b - \xi^{(1)}_b)\sum_{\mathcal{F}(\alpha, \beta, x)} \mathcal{B}(\{\mu_j\}_{j=1}^{M}|x_n - 1, \bar{z}_{n+1})|\eta, \bar{y}_{n+1}\rangle}{\sinh(\mu_b - \xi^{(0)}_b\mathcal{B}(\{\mu_j\}_{j=1}^{M}|x_n - 1, z_{n+1} + 1))} \times \mathcal{B}(\{\mu_j\}_{j=1}^{M}|x_n - 1, \bar{z}_{n+1} + 2)|\eta, \bar{y}_{n+1} - 1\rangle.
\]

(4.79)

We finally act with the last A-operator on this state using (A.78). We have

\[
\mathcal{A}(\xi^{(1)}_1|x_{n+1}, \bar{y}_{n+1} - N + M + \bar{m}_{n+1}) \mathcal{B}(\{\mu_j\}_{j=1}^{M}|x_n - 1, \bar{z}_{n+1} + 2)|\eta, \bar{y}_{n+1} - 1\rangle
\]

\[
= a_{\alpha}^{(L)} \sum_{\nu \in \mathcal{A}(\xi^{(1)}_1, |(M+m+1-n))} d(\nu) \sinh(\nu_{\zeta^{(1)}} - \mu_a + \eta(\bar{b}_n - 1)) \times \prod_{j=1}^{\mathcal{A}(\xi^{(1)}_1, |(M+m+1-n))} \sinh(\nu_{\zeta^{(1)}} - \mu_a)
\]

(4.80)
in which we have used that
\[ \frac{\tilde{y}_{n+1} - N - \tilde{z}_{n+1} - 1 + (M + \tilde{m}_{n+1})}{2} = b_n + 2\tilde{m}_{n+1} - 1 = \tilde{b}_n - 1, \quad (4.81) \]
and where we have defined
\[
a^{(L)}_n = \frac{e^{-(\gamma_{n+1} - M - \tilde{m}_{n+1})/2} - e^{-\tilde{\gamma}_{n+1}/2}}{\sinh(\eta(\tilde{b}_n - 1 - M - \tilde{m}_n))} \]
\[
\times e^{-\gamma_{n+1} - M - \tilde{m}_{n+1}} \sinh(\eta(\tilde{b}_n - 1 - M - \tilde{m}_n)) \]
\[
eq \frac{2 \sinh(\eta(\tilde{b}_n - 1 - M - \tilde{m}_n))}{e^{\gamma_{n+1}/2} \sinh(\eta(\tilde{b}_n - 1 - M - \tilde{m}_n))} \quad (4.82) \]

Note that here
\[
\tilde{m}_{n+1} = \tilde{m}_n, \quad x_{n+1} = x_n - 1, \quad \tilde{z}_{n+1} = \tilde{z}_n - 1, \quad \tilde{y}_{n+1} = \tilde{y}_n + 1. \quad (4.83) \]

Note also that, with our notations,
\[
s_{(n)} = s_{(n+1)}, \quad s'_{(n)} = s'_{(n+1)} + 1, \quad (4.84) \]
\[
B_{(n')}^{(n+1)} = B_{(n')}^{(n+1)} \cup \{ s_{(n)} + s'_{(n)} \}, \quad (4.85) \]
with
\[
b_{(l)}^{(n)} = n_{(l)}^{(n+1)} \quad \forall l \in \{ 1, \ldots, s_{(n)} + s'_{(n)} - 1 \}, \quad (4.86) \]
\[
b_{(s_{(n)} + s'_{(n)})} \in A_{(n')}^{(n+1)} \cup \{ M + m + 1 - n \} \]
\[
= \{ 1, \ldots, M + m + 1 - n \} \backslash \{ n_{(1)}^{(n)}, \ldots, n_{(s_{(n)} + s'_{(n)}) - 1}^{(n+1)} \}, \quad (4.87) \]

being \( l_{(n)}^{(n+1)} = l_{(n)}^{(n+1)}(1 \leq p \leq s_{(n)} + s'_{(n)} - 1) \) and \( l_{(n)}^{(n+1)} + s'_{(n)} = n \). Hence, gathering all contributions, we can rewrite the action of (4.71) on (4.62) as
\[
e^{\gamma_{n+1} - M - \tilde{m}_{n+1}} \prod_{p=1}^{M} \sinh(\mu_{(p)} - \zeta_{(1)}^{(1)}) \prod_{p=1}^{M} \sinh(\mu_{(p)} - \zeta_{(0)}^{(0)}) \sum_{B_{(n')}^{(n+1)}} \mathcal{F}_{B_{(n')}^{(n+1)}} \times d(\mu_{(n)}) \sinh(\zeta_{(n)}^{(1)} - \mu_{(n)}) - \eta(1 - \tilde{b}_n) \]
\[
\times \prod_{j \in A_{n'}^{(n+1)}} \sinh(\mu_{(n)}) \prod_{j \in A_{n'}^{(n+1)}} \sinh(\mu_{(n)}) \]
\[
\times \mathcal{B}_{M + \tilde{m}_n} \left\{ \left( \mu_j \right) \big| x_n = 1, \tilde{z}_n, \eta, \tilde{y}_n \right\}, \quad (4.88) \]
with

\[ a_n^{(\text{tot})} = \frac{e^{(\xi_n + \eta_n + 1)\Delta_1}a_n^{(L)}}{4 \sinh(\eta \Delta_1)} \]

\[ = \frac{\sinh(\eta(d_n + M + \tilde{m}_n)) \sinh(\eta(b_n - M - 1))}{\sinh(\eta(d_n + M)) \sinh(\eta(b_n - M - 1 + \tilde{m}_n))} \]

\[ = \frac{\sinh(\eta(d_{n-1} + M + \tilde{m}_n + 1)) \sinh(\eta(b_{n-1} - M))}{\sinh(\eta(d_{n-1} + M + 1)) \sinh(\eta(b_{n-1} - M + \tilde{m}_n))}. \]  \tag{4.89}

We therefore obtain the following recursion relation:

\[
\mathcal{F}_{\mathbf{\epsilon}, \mathbf{\alpha}, \mathbf{\beta}}^{(n)}(\{\mu_j\}_{j=1}^{M}, \{\xi_j^{(1)}\}_{j=n+1}^{n} | \alpha, \beta, x) = \frac{e^{(\xi_n + \eta_n + 1)\Delta_1}a_n^{(L)}}{4 \sinh(\eta \Delta_1)} \prod_{b=1}^{M} \frac{\sinh(\mu_b - \zeta_{n+1}^{(1)})}{\sinh(\mu_b - \zeta_n^{(1)})} \sinh(\xi_j^{(1)} - \mu_{b_n}^{(n)} - \eta(1 - \bar{b}_n)) \\
\times \prod_{j\in\mathcal{A}_{n+1}^{(1)}} \sinh(\mu_j - \mu_{b_n}^{(n)} - \eta) \mathcal{F}_{\mathbf{\epsilon}, \mathbf{\alpha}, \mathbf{\beta}}^{(n+1)}(\{\mu_j\}_{j=1}^{M}, \{\xi_j^{(1)}\}_{j=n+1}^{n+1} | \alpha, \beta, x), \tag{4.90}
\]

which, using that

\[
\prod_{j\in\mathcal{A}_{n+1}^{(1)}} \sinh(\mu_j - \mu_{b_n}^{(n)} - \eta) \]

\[ = \prod_{j=1}^{M} \sinh(\mu_j - \mu_{b_n}^{(n)} - \eta) \prod_{j=n+1}^{M} \sinh(\xi_j^{(1)} - \mu_{b_n}^{(n)} - \eta) \\
= \frac{\prod_{j=1}^{M} \sinh(\mu_j - \mu_{b_n}^{(n)} - \eta)}{\prod_{j=1}^{M} \sinh(\xi_j^{(1)} - \mu_{b_n}^{(n)} - \eta)}. \]

\[
\prod_{j\in\mathcal{A}_{n+1}^{(1)}} \sinh(\mu_j - \mu_{b_n}^{(n)} - \eta) \]

\[ = \prod_{j=1}^{M} \sinh(\mu_j - \mu_{b_n}^{(n)} - \eta) \prod_{j\in\mathcal{A}_{n+1}^{(1)}} \sinh(\xi_j^{(1)} - \mu_{b_n}^{(n)} - \eta) \\
= \frac{\prod_{j=1}^{M} \sinh(\mu_j - \mu_{b_n}^{(n)} - \eta)}{\prod_{j=1}^{M} \sinh(\xi_j^{(1)} - \mu_{b_n}^{(n)} - \eta)}. \]

can be rewritten as
\[\mathcal{F}_{x_{\nu}, \xi}^{(0)}(\mu_j^M)_{j=1}^M, \{\xi_j^m\}_{j=m}^M | (\alpha, \beta, x) = \frac{e^{i(\eta_n - \theta_n + 1)}a_n^{(tot)}}{\sinh(\beta)} \sum_{\nu=1}^M \frac{d(\mu_n^{(\nu)})}{\sinh(\mu_n^{(\nu)} + \epsilon_n^{(\nu)})} \frac{\prod_{j=1}^M \sinh(\mu_j^{(\nu)} - \mu_j^{(\nu)} - \eta)}{\prod_{j=1}^M \sinh(\mu_j^{(\nu)} + \epsilon_n^{(\nu)} + \eta)} \times \sinh(\eta_n - \theta_n - \eta(1 - \tilde{b}_n)) \times \prod_{j=1}^M \frac{\sinh(\eta_n - \theta_n - \eta)(\mu_j^{(\nu)} - \mu_j^{(\nu)} - \eta)}{\sinh(\mu_j^{(\nu)} - \mu_j^{(\nu)} + \eta)} \times \prod_{j=1}^M \frac{\sinh(\mu_j^{(\nu)} - \mu_j^{(\nu)} - \eta)}{\sinh(\mu_j^{(\nu)} + \epsilon_n^{(\nu)} + \eta)} \mathcal{F}_{x_{\nu}, \xi}^{(0)}(\mu_j^M)_{j=1}^M, \{\xi_j^m\}_{j=m}^M | (\alpha, \beta, x), (4.91)\]

and gives the result once we notice that

\[f_{x, \xi}^{(\alpha, \beta)}(\alpha, \beta, x) = a_n^{(tot)(\alpha+1,m)}(\alpha, \beta, x). (4.92)\]

(B) Let us now consider the case \((\epsilon_n, \epsilon'_n) = (1,2)\), which means that we have to act on (4.62) with the monomial

\[\frac{e^{2\eta_n}}{4 \sinh(\eta_n) d_n} B_{\eta_n}(\eta_n | c_n - d_n, \tilde{a}_n - \tilde{b}_n) M_{x_{\nu}, \xi}^{(n+1,m)}(\alpha, \beta, x) \frac{D(\xi_j^{(0)})_{j=n+1} - 1, a_n - b_n + 1)}{\det(M(\xi_j))} (4.93)\]

in which we have used (4.72)–(4.74) and

\[a_n - \tilde{b}_n = \tilde{a}_n + 1 = 2\tilde{a}_n + 1 = \tilde{z}_n + 1 = 1. (4.94)\]

As in the previous case, only the direct term (4.77) from the action of the \(D\)-operator contributes to the final result, and the action of \(M_{x_{\nu}, \xi}^{(n+1,m)}(\alpha, \beta, x)\) on this contribution gives (4.79). Note that here

\[\tilde{m}_n + 1 = \tilde{m}_n - 1, \quad \tilde{x}_n + 1 = \tilde{x}_n - 1, \quad \tilde{z}_n + 1 = \tilde{z}_n - 1, \quad \tilde{y}_n + 1 = \tilde{y}_n. (4.95)\]

Note also that, with our notations,

\[s_{(n)} = s_{(n+1)}, \quad s'_{(n)} = s'_{(n+1)}, (4.96)\]

\[B_{x_{\nu}, \xi}^{(n)} = B_{x_{\nu}, \xi}^{(n+1)}, \quad A_{x_{\nu}, \xi}^{(n)} = A_{x_{\nu}, \xi}^{(n+1)} \cup \{M + m + 1 - n\}. (4.97)\]

with \(\tilde{p}_n = \tilde{p}_{n+1}, 1 \leq p \leq s_{(n)} + s'_{(n)}\). Gathering all contributions, we can therefore rewrite the action of (4.93) on (4.62) as

\[\frac{e^{2\eta_n}}{4 \sinh(\eta_n) d_n} \prod_{b=1}^M \frac{\sinh(\mu_b^{(\nu)} - \xi_j^{(0)}_{j=1} \mathcal{F}_{x_{\nu}, \xi}^{(0)}(\mu_j^M)_{j=1}^M, \{\xi_j^m\}_{j=m}^M | (\alpha, \beta, x))}{\sinh(\mu_b^{(\nu)} - \xi_j^{(0)}_{j=1} \mathcal{F}_{x_{\nu}, \xi}^{(0)}(\mu_j^M)_{j=1}^M, \{\xi_j^m\}_{j=m}^M | (\alpha, \beta, x))} \times B_{s_{(n)} - 1, \tilde{z}_n - 2} M_{n, \tilde{a}_n - 1} | (\mu_j^M)_{j=1}^M, \{\xi_j^m\}_{j=m}^M | (\alpha, \beta, x). (4.98)\]
We therefore obtain the following recursion relation:

\[
F_{\xi,\nu}(\{\mu_j\}_{j=1}^M, \{\xi_j\}_{j=n+1}^M | \alpha, \beta, x) = e^{i(\alpha - \eta + \beta)} \frac{b^{(n)}_n}{\sinh(\eta d_n)} \prod_{b=1}^{M} \sinh(\eta d_b - \xi^{(1)}_b) \mathcal{F}_{\xi,\nu}(\{\mu_j\}_{j=1}^M, \{\xi_j\}_{j=n+1}^M | \alpha, \beta, x),
\]

with

\[
b^{(n)}_n = \frac{e^{i(\alpha + n) - n/2} \sinh(\eta b_n - M - 1)}{2 \sinh(\eta (d_n + M))} = \frac{2 \sinh(\eta (d_n + M + 1))}{2 \sinh(\eta (d_n + M + 1))},
\]

which gives the result once we notice that

\[
f_{\xi,\nu}(\alpha, \beta, x) = b^{(n+1,0)}_n(\alpha, \beta, x).
\]

(C) Let us now consider the case \((\epsilon_n, \epsilon'_n) = (2, 2)\). We act on (4.62) with the monomial:

\[
\frac{e^{2\gamma_n}}{4 \sinh^2(\eta d_n)} \mathcal{D}(\xi^{(1)}_n | c_n + d_n, \tilde{b}_n - b_n) M^{(n+1,0)}(\alpha, \beta, x) \frac{A(\xi^{(0)}_n | c_n - d_n - 1, a_n + b_n - 1)}{\det M(\xi_n)} = \frac{e^{2\gamma_n}}{4 \sinh^2(\eta d_n)} \mathcal{D}(\xi^{(1)}_n | y_{n+1} - M, \tilde{z}_{n+1} - 1) M^{(n+1,0)}(\alpha, \beta, x) \frac{A(\xi^{(0)}_n | x_n, y_n - N + M)}{\det(\xi^{(1)}_n)}.
\]

(4.102)

From (A.78), and since \(d(\xi^{(0)}_n) = 0\), the only non-zero contributions from the action of the \(A\)-operator on the state (4.62) come from indirect actions, so that it produces

\[
A(\xi^{(0)}_n | x_n, y_n - N + M) \mathcal{D}(\{\mu_j\}_{j=1}^M | x_n - 1, \tilde{z}_n | \eta, y_n) = a^{(R)}_n \sum_{\mu_n} A^{(R)}_n \prod_{\mu_m} \frac{\sinh(\mu_j - \mu_n - \eta)}{\sinh(\xi^{(0)}_n - \mu_n)} \mathcal{D}(\{\mu_j\}_{j=1}^M \cup \{\xi^{(0)}_n\} | x_n, \tilde{z}_n + 1 | \eta, y_n - 1),
\]

in which we have used that

\[
y_n - N - \tilde{z}_n - 1 + M = b_n,
\]

(4.104)

and defined

\[
a^{(R)}_n = \frac{e^{-(\epsilon_n - M)\eta} - e^{-\gamma_n}}{\sinh(\eta)} = \frac{e^{-(\epsilon_n - d_n - (-1)^n - M)\eta} - e^{-(\epsilon_n + d_n + M + (-1)^n)\eta}}{\sinh(\eta)},
\]

\[
= \frac{2 \sinh(\eta (d_n + 1 + M))}{2 \sinh(\eta (d_n + 1 + M))},
\]

(4.105)
Noticing that
\[ x_n = x_{n+1} - 1, \quad z_n = z_{n+1} - 1, \quad y_n = y_{n+1} - 1, \quad \tilde{m}_n + 1 = \tilde{m}_{n+1} \] (4.106)
we can use the induction hypothesis so as to compute the action of \( M_{(n+1),m}^{(\alpha, \beta, \xi)}(\alpha, \beta, \xi) \) on the above state. We remark that, since \( dt(\xi^{(0)}) = 0 \), the action of the monomial \( M_{(n+1),m}^{(\alpha, \beta, \xi)}(\alpha, \beta, \xi) \) cannot result into a replacement of the argument \( \xi^{(0)} \) in the product of \( B \)-operators, so that we can make the following identification:
\[ B_j^{(m)} = B_{j+1}^{(m+1)} \quad \forall j \in \{1, \ldots, s(n) + s'(n) - 1\}, \quad \text{and} \quad B'_1^{(m)} = B, \] (4.107)
with here
\[ s(n) = s(n+1) + 1, \quad s'(n) = s'(n+1), \] (4.108)
\[ B_{\alpha, \beta}^{(m)} = B_{\alpha, \beta}^{(m+1)} \cup \{ n' \}, \quad A_{\alpha, \beta}^{(m)} = A_{\alpha, \beta}^{(m+1)} \cup \{ M + m + 1 - n \} \setminus \{ n' \}. \] (4.109)

Hence it gives
\[
M_{(n+1),m}^{(\alpha, \beta, \xi)}(\alpha, \beta, \xi)A(\xi^{(0)})|x_n, y_n = N + MB_{\alpha, \beta}^{(m)}(\mu_j)_{j=1}^M |x_n - 1, z_n, \eta, \gamma_n \rangle
\]
\[ = A_{\alpha, \beta}^{(m)} \sum_{B_{\alpha, \beta}^{(m)}, A_{\alpha, \beta}^{(m)}} \sinh(c_{\alpha, \beta}^{(1)} - \mu_{\alpha, \beta}^{(m)} + \eta(b_{\alpha, \beta} + 1)) \prod_{j=1}^M \sinh(c_{\alpha, \beta}^{(1)} - \mu_{\alpha, \beta}^{(m)} + \eta) \prod_{j \neq \mu_{\alpha, \beta}^{(m)}}^M \sinh(c_{\alpha, \beta}^{(1)} - \mu_{\alpha, \beta}^{(m)} - \eta) \]
\[ \times F_{\alpha, \beta}^{(m+1)}(\mu_j)_{j=1}^M |x_n - 1, z_n, \eta, \gamma_n \rangle 
\times B_{\alpha, \beta}^{(m+1)}(\mu_j)_{j=1}^M \cup \{ \xi^{(0)} \} |x_n + 1, z_n + 1, \eta, \gamma_n \rangle. \] (4.110)

Note that we have
\[ \{ \mu_j \}_{j=1}^{M+M-n} \cup \{ \xi^{(0)} \} = \{ \mu_j \}_{j \in A_{\alpha, \beta}^{(m)}} \setminus \{ c_{\alpha, \beta}^{(1)} \} \cup \{ \xi^{(0)} \}. \] (4.111)
It remains to act with the last \( D \)-operator on this state. From (4.17), the only non-zero contribution of this action comes from the indirect term replacing \( \xi^{(0)} \) by \( \xi^{(1)} \) in the product of \( B \):
\[
D(\xi^{(1)}) |y_{n+1} = M, z_n = 1, \eta, \gamma_n \rangle
\]
\[ = -\sum_{j=1}^M \sinh(\xi^{(0)} - \mu_j - \eta) \sinh(\xi^{(0)} - \mu_j) B_{M+\tilde{m} + 1} \sum_{j \in A_{\alpha, \beta}^{(m)}} (\mu_j)_{j \in A_{\alpha, \beta}^{(m)}} |x_n - 1, z_n, \eta, \gamma_n \rangle. \] (4.112)
Hence, gathering all contributions, we obtain that the action of \((4.102)\) on \((4.62)\) results in

\[
d^{(L)}_n = \frac{e^{-G_n+1-1+M+\hat{m}_n\eta} - e^{-G_n+1-N\eta}}{e^{\eta/2}} \sinh \left( -\eta + \frac{\hat{b}_n + \hat{m}_n + \frac{1}{2}}{2} \right) \sinh \left( \frac{\eta (\hat{b}_n + \hat{m}_n + 1 + \frac{1}{2})}{2} \right)
\]

\[
= \frac{2 \sinh (\eta(b_n - M + \hat{m}_n)) \sinh (\eta d_n)}{e^{\eta/2} e^{\eta m}} \sinh (\eta(d_n + M + \hat{m}_n + 1)) \tag{4.113}
\]

Hence, gathering all contributions, we obtain that the action of \((4.102)\) on \((4.62)\) results in

\[
- \frac{e^{2\eta} a_n^{(L)} d^{(L)}_n}{4 \sinh^2 (\eta d_n) d^{(L)}_n} \sum_{\nu_{\alpha_i}^{(n)}} d(\nu_{\alpha_i}^{(n)} \sinh(\xi_{\alpha_i}^{(j)}) - \mu_{\nu_{\alpha_i}^{(n)} + \eta} (b_n + 1) \prod_{j=1}^M \sinh(\mu_j - \mu_{\nu_{\alpha_i}^{(n)} + \eta} \prod_{j=1}^M \sinh(\mu_j - \mu_{\nu_{\alpha_i}^{(n)}}) + \eta) F_{\nu_{\alpha_i}^{(n)}}^{(L)} (\{\mu_j\}_{j=1}^M \backslash \{\mu_{\nu_{\alpha_i}^{(n)}}\} \cup \{\xi_{\alpha_i}^{(j)}\} \times \{\xi_{\alpha_i}^{(j+1)}\}_{j=n+1}^m | x_1, x_2). \tag{4.114}
\]

We therefore obtain the following recursion relation:

\[
F_{\nu_{\alpha_i}^{(n)}}^{(L)} (\{\mu_j\}_{j=1}^M, \{\xi_{\alpha_i}^{(j)}\}_{j=n+1}^m | \alpha, \beta, x) = - \frac{e^{2\eta} a_n^{(L)} d^{(L)}_n}{4 \sinh^2 (\eta d_n) d^{(L)}_n} \prod_{j=1}^M \sinh(\mu_j - \mu_{\nu_{\alpha_i}^{(n)}}) \prod_{j=1}^M \sinh(\mu_j - \mu_{\nu_{\alpha_i}^{(n)} + \eta}) F_{\nu_{\alpha_i}^{(n)}}^{(L)} (\{\mu_j\}_{j=1}^M \backslash \{\mu_{\nu_{\alpha_i}^{(n)}}\} \cup \{\xi_{\alpha_i}^{(j)}\} \times \{\xi_{\alpha_i}^{(j+1)}\}_{j=n+1}^m | \alpha, \beta, x). \tag{4.115}
\]

Using moreover that

\[
F_{\nu_{\alpha_i}^{(n)}}^{(L)} (\{\mu_j\}_{j=1}^M, \{\xi_{\alpha_i}^{(j)}\}_{j=n+1}^m | \alpha, \beta, x) = \prod_{\mu_{\nu_{\alpha_i}^{(n)}}} \sinh(\xi_{\alpha_i}^{(j)} - \mu_{\nu_{\alpha_i}^{(n)}}) \sinh(\mu_{\nu_{\alpha_i}^{(n)}} - \mu_{\nu_{\alpha_i}^{(n)}}) \sinh(\mu_{\nu_{\alpha_i}^{(n)}} - \mu_{\nu_{\alpha_i}^{(n)} + \eta}) \prod_{\mu_{\nu_{\alpha_i}^{(n)}}} \sinh(\xi_{\alpha_i}^{(j)} - \mu_{\nu_{\alpha_i}^{(n)}}) \sinh(\mu_{\nu_{\alpha_i}^{(n)}} - \mu_{\nu_{\alpha_i}^{(n)}}) \sinh(\mu_{\nu_{\alpha_i}^{(n)}} - \mu_{\nu_{\alpha_i}^{(n)} + \eta}) \tag{4.116}
\]

and that

\[
\prod_{\mu_{\nu_{\alpha_i}^{(n)}}} \sinh(\xi_{\alpha_i}^{(j)} - \mu_{\nu_{\alpha_i}^{(n)}}) \sinh(\mu_{\nu_{\alpha_i}^{(n)}} - \mu_{\nu_{\alpha_i}^{(n)} + \eta}) \prod_{\mu_{\nu_{\alpha_i}^{(n)}}} \sinh(\mu_{\nu_{\alpha_i}^{(n)}} - \mu_{\nu_{\alpha_i}^{(n)} + \eta}) \prod_{\mu_{\nu_{\alpha_i}^{(n)}}} \sinh(\mu_{\nu_{\alpha_i}^{(n)}} - \mu_{\nu_{\alpha_i}^{(n)} + \eta}) \tag{4.117}
\]
the relation (4.115) becomes

\[
\mathcal{F}_{\epsilon',\epsilon}^{(\alpha)}(\{\mu_j\}^M_{j=1}, \{\xi_j\}^m_{j=\alpha}(\alpha, \beta, x) = - \frac{\mathcal{F}_{\epsilon',\epsilon}^{(\alpha)}(\{\mu_j\}^M_{j=1}, \{\xi_j\}^m_{j=\alpha}(\alpha, \beta, x)}{\sinh(\eta d_{\alpha})} \frac{d(\mu_{n,0}^{(\alpha)})}{d(\xi_j^{(1)})} \prod_{i=1}^M \sinh(\mu_{n,0}^{(\alpha)} - \xi_j^{(1)}) \sinh(\eta - \xi_j^{(1)}) \times \frac{\sinh(\mu_{n,0}^{(\alpha)} - \xi_j^{(1)}) - \eta(b_n + 1))}{\sinh(\mu_{n,0}^{(\alpha)} - \eta)} \prod_{j=n+1}^m \sinh(\mu_{n,0}^{(\alpha)} - \xi_j^{(1)}) \sinh(\eta - \xi_j^{(1)}) \times \prod_{j=2}^{n(\alpha)+1} \frac{\sinh(\mu_{n,0}^{(\alpha)} - \xi_j^{(1)}) - \eta}{\sinh(\mu_{n,0}^{(\alpha)} - \xi_j^{(1)}) - \eta} \mathcal{F}_{\epsilon',\epsilon}^{(\alpha+1)}(\{\mu_j\}^M_{j=1}, \{\xi_j\}^m_{j=\alpha}(\alpha, \beta, x),
\]

in which

\[
d_n^{(\alpha)} = \frac{\mathcal{F}_{\epsilon',\epsilon}^{(\alpha)}(\{\mu_j\}^M_{j=1}, \{\xi_j\}^m_{j=\alpha}(\alpha, \beta, x)}{4 \sinh(\eta d_{\alpha})} \frac{d(\mu_{n,0}^{(\alpha)})}{d(\xi_j^{(1)})} \prod_{i=1}^M \sinh(\mu_{n,0}^{(\alpha)} - \xi_j^{(1)}) \sinh(\eta - \xi_j^{(1)}) \times \frac{\sinh(\mu_{n,0}^{(\alpha)} - \xi_j^{(1)}) - \eta(b_n + 1))}{\sinh(\mu_{n,0}^{(\alpha)} - \eta)} \prod_{j=n+1}^m \sinh(\mu_{n,0}^{(\alpha)} - \xi_j^{(1)}) \sinh(\eta - \xi_j^{(1)}) \times \prod_{j=2}^{n(\alpha)+1} \frac{\sinh(\mu_{n,0}^{(\alpha)} - \xi_j^{(1)}) - \eta}{\sinh(\mu_{n,0}^{(\alpha)} - \xi_j^{(1)}) - \eta} \mathcal{F}_{\epsilon',\epsilon}^{(\alpha+1)}(\{\mu_j\}^M_{j=1}, \{\xi_j\}^m_{j=\alpha}(\alpha, \beta, x),
\]

which gives the result once we notice that \(i_p^{(n)} = i_p^{(n+1)} (1 \leq p \leq s_n(\alpha) + s_n'(\alpha) - 1)\) and \(i_1^{(n)} = n\) and that

\[
f_{\epsilon',\epsilon}^{(0,0)}(\alpha, \beta, x) = d_{\alpha}^{(\alpha)} f_{\epsilon',\epsilon}^{(\alpha+1,0)}(\alpha, \beta, x).
\]

(D) Let us now consider the case \((\epsilon_n, \epsilon_n') = (2, 1)\). Then we have to act on the state (4.62) with the monomial

\[
\frac{\mathcal{F}_{\epsilon,\epsilon}^{(2)}(\{\mu_j\}^M_{j=1}, \{\xi_j\}^m_{j=\alpha}(\alpha, \beta, x)}{4 \sinh(\eta d_{\alpha})} \prod_{i=1}^M \sinh(\mu_{n,0}^{(\alpha)} - \xi_j^{(1)}) \sinh(\eta - \xi_j^{(1)}) \times \frac{\sinh(\mu_{n,0}^{(\alpha)} - \xi_j^{(1)}) - \eta(b_n + 1))}{\sinh(\mu_{n,0}^{(\alpha)} - \eta)} \prod_{j=n+1}^m \sinh(\mu_{n,0}^{(\alpha)} - \xi_j^{(1)}) \sinh(\eta - \xi_j^{(1)}) \times \prod_{j=2}^{n(\alpha)+1} \frac{\sinh(\mu_{n,0}^{(\alpha)} - \xi_j^{(1)}) - \eta}{\sinh(\mu_{n,0}^{(\alpha)} - \xi_j^{(1)}) - \eta} \mathcal{F}_{\epsilon,\epsilon}^{(\alpha+1,0)}(\{\mu_j\}^M_{j=1}, \{\xi_j\}^m_{j=\alpha}(\alpha, \beta, x),
\]

The action (4.110) is similar as in the previous case, except that here we have to make the following identification:

\[
B_j^{(n)} = B_j^{(n+1)} \quad \forall j \in \{1, \ldots, s_n(\alpha) + s_n'(\alpha) = s_n(\alpha) + s_n'(\alpha) - 2\}, \quad \text{and} \quad B_1^{(n)} = n,
\]

45
Using (4.24), which can be rewritten as

$$B_n^{(n+1)} = B_n^{(n+1)} \cup \{b_1^{(n)}, b_0^{(n)} + \epsilon_n\},$$

$$A_{x, \epsilon}^{(n)} = A_{x, \epsilon}^{(n+1)} \cup \{m + m + 1 - n\} \setminus \{b_1^{(n)}, b_0^{(n)} + \epsilon_n\}. $$

Setting also

$$\hat{B}_e^{(n)} = B_{e, \epsilon}^{(n+1)} \cup \{b_1^{(n)}\} = \{b_e^{(n)}\}_{j=1}^{b_0^{(n)} + \epsilon_n - 1},$$

we therefore can write

$$M_{x, \epsilon}^{(n+1, m)}(x, \beta, \eta)A(n_0) |x_n - N + M, B_n^{(n)} \rho| \{\mu_j^{(n)}\}_{j=1}^{b_0^{(n)}} |x_n = 1, \eta, y_n\} = a_n^{(n)} \sum_{e, \epsilon} d(\mu_{e, \epsilon}^{(n)}) \sinh(\xi_n^{(n)} - \mu_{\epsilon, \epsilon}^{(n)} + \eta) \prod_{j=1}^{b_0^{(n)}} \sinh(\mu_j - \mu_{n_j}^{(n)} - \eta)$$

$$\times F_{e, \epsilon}^{(n+1)} \{[\mu_j^{(n)}]_{j=1}^{b_0^{(n)}} \cup \{\xi_n^{(n)}\}, \{\epsilon_j^{(n+1)}\}_{j=1}^{b_0^{(n)} + \epsilon_n - 1} |x_n = 1, \eta, y_n + 1\}.$$

Using (4.24), which can be rewritten as

$$C(\xi_n - \eta/2|y, w)B(\xi_n + \eta/2|x - 1, z)$$

$$= \frac{\sinh(\eta/2 - x')}{\sinh(\eta/2 - x)} C(\xi_n - \eta/2|y', w)D(\xi_n + \eta/2|y' - 1, z),$$

we obtain that

$$C(\xi_n^{(1)}|y_{n+1} - 2, \tilde{y}_{n+1} - N + M, \tilde{m}_{n+1})$$

$$= C(\xi_n^{(1)}|y_{n+1} - M, \tilde{y}_{n+1} - N + (M + \tilde{m}_{n+1}))B(\xi_n^{(0)}|x_{n+1} - 2, \tilde{y}_{n+1} - 1)$$

$$= C(\xi_n^{(1)}|y_{n+1} - 2, \tilde{y}_{n+1} - N + (M + \tilde{m}_{n+1}))B(\xi_n^{(0)}|x_{n+1} - 2, \tilde{y}_{n+1} - 1)$$

$$= - \frac{\sinh(\eta/2 - y_{n+1} - 2)}{\sinh(\eta/2 - y_{n+1} - 2 + \tilde{y}_{n+1} - 1)} A(\xi_n^{(1)}|x_{n+1} - 2, \tilde{y}_{n+1} - N + M + \tilde{m}_{n+1})$$

$$\times B(\xi_n^{(0)}|x_{n+1} - 2, \tilde{y}_{n+1} - 1, \tilde{y}_{n+1} - 1, \tilde{y}_{n+1} - 1)$$

$$\times (\{\mu_j^{(1)}\}_{j=1}^{b_0^{(n)} + \epsilon_n - 1} |x_n = 1, \eta, y_n + 1\}.$$
Here we have defined

\[ D(\xi_n) | x_n - 1, y_n = N + M + \tilde{m}_n + 1 \]

Then, the action of the last \( A \)-operator computed in \( \xi_n \) can be both direct and indirect and it generates a further sum over the index

\[ n^{(a)}(\tilde{a}^{(a)} + 1) \in \{ 1, \ldots, M + m + 1 - n \} \setminus B^{(a)}_{\tilde{a}^{(a)}}, \]

so that we get

\[ C(\xi_n) | x_{n+1} - 1, y_{n+1} = N + (M + \tilde{m}_{n+1}) \]

and that, thanks to (4.17), the only non-zero contribution comes from the direct action of \( D \). Here we have defined

\[ \hat{d}_n^{(a)} \frac{\sinh(\eta d_n)}{\sinh(\eta (d_n + 2))} \]

Then, the action of the last \( A \)-operator computed in \( \xi_n \) can be both direct and indirect and it generates a further sum over the index

\[ n^{(a)}(\tilde{a}^{(a)} + 1) \in \{ 1, \ldots, M + m + 1 - n \} \setminus B^{(a)}_{\tilde{a}^{(a)}}, \]

so that we get

\[ C(\xi_n) | x_{n+1} - 1, y_{n+1} = N + (M + \tilde{m}_{n+1}) \]

and that, thanks to (4.17), the only non-zero contribution comes from the direct action of \( D \). Here we have defined

\[ \hat{d}_n^{(a)} \frac{\sinh(\eta d_n)}{\sinh(\eta (d_n + 2))} \]

Then, the action of the last \( A \)-operator computed in \( \xi_n \) can be both direct and indirect and it generates a further sum over the index

\[ n^{(a)}(\tilde{a}^{(a)} + 1) \in \{ 1, \ldots, M + m + 1 - n \} \setminus B^{(a)}_{\tilde{a}^{(a)}}, \]

so that we get

\[ C(\xi_n) | x_{n+1} - 1, y_{n+1} = N + (M + \tilde{m}_{n+1}) \]

and that, thanks to (4.17), the only non-zero contribution comes from the direct action of \( D \). Here we have defined

\[ \hat{d}_n^{(a)} \frac{\sinh(\eta d_n)}{\sinh(\eta (d_n + 2))} \]

Then, the action of the last \( A \)-operator computed in \( \xi_n \) can be both direct and indirect and it generates a further sum over the index

\[ n^{(a)}(\tilde{a}^{(a)} + 1) \in \{ 1, \ldots, M + m + 1 - n \} \setminus B^{(a)}_{\tilde{a}^{(a)}}, \]

so that we get

\[ C(\xi_n) | x_{n+1} - 1, y_{n+1} = N + (M + \tilde{m}_{n+1}) \]

and that, thanks to (4.17), the only non-zero contribution comes from the direct action of \( D \). Here we have defined

\[ \hat{d}_n^{(a)} \frac{\sinh(\eta d_n)}{\sinh(\eta (d_n + 2))} \]
\[
\prod_{j=1}^{M+n} \frac{\sinh(\mu_j - \mu_k^{(0)})}{\sinh(\mu_j - \mu_k^{(1)})} \times \prod_{j=1}^{M+n-1} \frac{\sinh(\mu_j - \mu_k^{(0)})}{\sinh(\mu_j - \mu_k^{(1)})}
\]
\[
\times B_{M+n} \left( \left\{ \mu_j \right\}_{j=1}^{M+n-1} | x_n - 1, \bar{z}_n \right) \eta, \bar{\eta}, \right),
\]
with
\[
\hat{a}^{(L)}_n = \frac{e^{-(x_n - 1 - M + \bar{m}_n)\eta} - e^{-(z_n + 1 - M - \bar{m}_n)\eta}}{e^{\eta/2} \sinh((\eta - \bar{m}_n) + (1 - M - \bar{m}_n))}
\]
\[
= 2 \frac{\sinh(\eta(b_n + 1 + \bar{m}_n) - M)}{\sinh(\eta(b_n + 1 - M + \bar{m}_n))}
\]
Setting
\[
\hat{c}^{(L)}_n = \frac{\sinh(\eta d_n) \hat{a}^{(L)}_n}{\sinh(\eta(d_n + 2))} \hat{c}^{(L)}_n
\]
\[
= 4 \frac{\sinh(b_n + 1 + \bar{m}_n - M)}{\sinh(\eta(b_n + 1 - M - \bar{m}_n))}
\]
and gathering all contributions, we obtain that the action of the monomial (4.121) on the state
\[
(4.62)
\] results into
\[
\frac{e^{2\eta} \hat{g}^{(R)} \hat{c}^{(L)}_n}{4 \sinh^2(\eta d_n)} \sum_{B^{(m + n), \alpha, \beta}} d(\mu_k^{(0)}) \times \sinh(\xi^{(1)}_n - \mu_k^{(0)}) \prod_{j=1}^{M+n} \sinh(\mu_j - \mu_k^{(0)} - \eta) \prod_{j=1}^{M+n} \sinh(\mu_j - \mu_k^{(0)})
\]
\[
\times \mathcal{F}_{B^{(m + n), \alpha, \beta}} \left( \left\{ \mu_j \right\}_{j=1}^{M+n} \setminus \left\{ \mu_k^{(0)} \right\} \cup \left\{ \xi^{(1)}_j \right\}_{j=n+1}^{M+n} | \alpha, \beta, x \right)
\]
\[
\times \sinh(\xi^{(1)}_n - \mu_k^{(0)} - \eta(1 - \bar{b}_n)) \prod_{j=1}^{M+n} \sinh(\mu_j - \mu_k^{(0)} + \eta(1 - \bar{b}_n))
\]
\[
\times \left. \prod_{j=1}^{M+n} \sinh(\mu_j - \mu_k^{(0)} + \eta(1 - \bar{b}_n)) \times \left. \prod_{j=1}^{M+n} \sinh(\mu_j - \mu_k^{(0)} + \eta(1 - \bar{b}_n)) \right) \times \mathcal{F}_{B^{(m + n), \alpha, \beta}} \left( \left\{ \mu_j \right\}_{j=1}^{M+n} \setminus \left\{ \mu_k^{(0)} \right\} \cup \left\{ \xi^{(1)}_j \right\}_{j=n+1}^{M+n} | \alpha, \beta, x \right)
\]
\[
\times \left. \prod_{j=1}^{M+n} \sinh(\mu_j - \mu_k^{(0)} + \eta(1 - \bar{b}_n)) \times \left. \prod_{j=1}^{M+n} \sinh(\mu_j - \mu_k^{(0)} + \eta(1 - \bar{b}_n)) \right) \times \mathcal{F}_{B^{(m + n), \alpha, \beta}} \left( \left\{ \mu_j \right\}_{j=1}^{M+n} \setminus \left\{ \mu_k^{(0)} \right\} \cup \left\{ \xi^{(1)}_j \right\}_{j=n+1}^{M+n} | \alpha, \beta, x \right)
\]
\[
\times B_{M+n} \left( \left\{ \mu_j \right\}_{j=1}^{M+n-1} | x_n - 1, \bar{z}_n \right) \eta, \bar{\eta}, \right),
\]
(4.135)
We therefore obtain the following recursion relation

\[
\mathcal{F}_{\psi,s}^{(\mu)} (\{\mu_j\}_{j=1}^M, \{\xi_j\}_{j=1}^M | \alpha, \beta, x) = \frac{c^{2\nu \rho} \rho_0^{(e,\nu)} \mathcal{C}_n^{(L)}}{4 \sinh(\eta \rho_0)} \\
\times \frac{d(\mu_{s,0}) \sinh(\xi_{s,0} - \mu_{s,0}) + \eta(b_n + 1) }{d(\xi_{s,0} - \mu_{s,0})} \prod_{j=1}^M \frac{\sinh(\mu_j - \mu_{s,0}) - \eta}{\sinh(\xi_{s,0} - \mu_{s,0})} \prod_{j=1}^M \sinh(\mu_j - \mu_{s,0}) \\
\times \left( \prod_{j=1}^{M+M-n} \frac{\sinh(\mu_j - \mu_{s,0}) - \eta}{\sinh(\xi_{s,0} - \mu_{s,0})} \sinh(\mu_j - \xi_{s,0}) \right) \mathcal{F}_{\psi,s}^{(\mu)} (\{\mu_j\}_{j=1}^M \setminus \{\mu_{s,0}\}) \\
\times \bigcup \left( \{\xi_{s,0}^{(0)}\}, \{\xi_j^{(1)}\}_{j=1}^M | \alpha, \beta, x \right),
\]

(4.136)

which can equivalently be rewritten, using (4.116), as

\[
\mathcal{F}_{\psi,s}^{(\mu)} (\{\mu_j\}_{j=1}^M, \{\xi_j^{(1)}\}_{j=1}^M | \alpha, \beta, x) = \frac{c^{2\nu \rho} \rho_0^{(e,\nu)} \mathcal{C}_n^{(L)}}{4 \sinh(\eta \rho_0)} \\
\times \left( \prod_{j=1}^{M+M-n} \frac{\sinh(\mu_j - \mu_{s,0}) - \eta}{\sinh(\xi_{s,0} - \mu_{s,0})} \sinh(\mu_j - \xi_{s,0}) \right) \mathcal{F}_{\psi,s}^{(\mu)} (\{\mu_j\}_{j=1}^M \setminus \{\mu_{s,0}\}) \\
\times \left( \prod_{j=1}^{M} \frac{\sinh(\mu_j - \mu_{s,0}) - \eta}{\sinh(\xi_{s,0} - \mu_{s,0})} \sinh(\mu_j - \xi_{s,0}) \right) \mathcal{F}_{\psi,s}^{(\mu)} (\{\mu_j\}_{j=1}^M \setminus \{\mu_{s,0}\}) \\
\times \left( \prod_{j=1}^{M+M-n} \frac{\sinh(\mu_j - \mu_{s,0}) - \eta}{\sinh(\xi_{s,0} - \mu_{s,0})} \sinh(\mu_j - \xi_{s,0}) \right) \mathcal{F}_{\psi,s}^{(\mu)} (\{\mu_j\}_{j=1}^M \setminus \{\mu_{s,0}\}) \\
\times \bigcup \left( \{\xi_{s,0}^{(0)}\}, \{\xi_j^{(1)}\}_{j=1}^M | \alpha, \beta, x \right),
\]

(4.137)
with

\[ c_n^{(\text{tot})} = \frac{e^{i(\alpha+\alpha_n-1)a_n(E_n^\ell \ell_d)}}{4 \sinh(\eta d_n)} = \frac{2 \sinh(\eta(d_n + 1 + M))}{\sinh(\eta(b_n - M))} = [b_n^{(\text{tot})}]_{n+1}^{-1} \]

\[ = \frac{2 \sinh(\eta(d_{n-1} + M))}{\sinh(\eta(b_{n-1} - M - 1))} = [b_n^{(\text{tot})}]_{n-1}^{-1}. \tag{4.138} \]

Noticing that \( \hat{\hat{t}}(n) n, \hat{\hat{t}}(0) n, 1 \) for \( 1 \leq p \leq s(n) + s'(n) - 2 \) and \( \hat{\hat{t}}(0) = n \), and that

\[ f^{(\epsilon, \alpha)}_{\epsilon, \alpha}(\alpha, \beta, x) = c_n^{(\text{tot})} f^{(\alpha + 1, \mu)}_{\epsilon, \alpha}(\alpha, \beta, x), \tag{4.139} \]

we get the result.

The previous result implies the following one for the action of the generic element of the basis (4.35) on the gauged Bethe-like bulk states (4.4).

**Theorem 4.1.** For any given \( m \)-tuples \( \epsilon \equiv (\epsilon_1, \ldots, \epsilon_m), \epsilon' \equiv (\epsilon'_1, \ldots, \epsilon'_m) \in \{1, 2\}^m \) and gauge parameters \( \alpha, \beta, x \), the action of the generic element \( \prod_{n=1}^m E_{\epsilon_n}^{\epsilon_n^*}(\xi_n(a_n, b_n)), (\bar{a}_n, \bar{b}_n)) \) of the basis (4.35), with

\[ a_n = \alpha + 1, \quad b_n = \beta - \sum_{r=1}^n (-1)^r, \tag{4.140} \]

\[ \bar{a}_n = \alpha - 1, \quad \bar{b}_n = \beta + \sum_{r=1}^m (-1)^r - \sum_{r=1}^m (-1)^r, \tag{4.141} \]

on the gauged Bethe-like bulk states (4.4) is

\[ \prod_{n=1}^m E_{\epsilon_n}^{\epsilon_n^*}(\xi_n(a_n, b_n), (\bar{a}_n, \bar{b}_n)) B_M:\{\mu\}_{j=1}^M | x - 1, \alpha - \beta \]

\[ \times | \eta, \alpha + \beta + N - M - 1 \rangle = \sum_{\mathcal{B}_{\epsilon, \epsilon'}} F_{\mathcal{B}_{\epsilon, \epsilon'}}(\{\mu\}_{j=1}^M, \{\xi_{n=1}^{(1)}\}_{j=1}^m | \alpha, \beta, x) \]

\[ \times B_M : \tilde{\tilde{a}}_{\epsilon, \epsilon'} :| \langle \eta, \alpha + \beta + N - M + 1 + \tilde{m}_{\epsilon, \epsilon'} \rangle \]

\[ \tag{4.142} \]

in which we have defined \( \mu_{M+j} = \xi_{n=1-j}^{(1)} \) for \( j \in \{1, \ldots, m\} \) and

\[ \tilde{m}_{\epsilon, \epsilon'} = \sum_{r=1}^m (\epsilon'_r - \epsilon_r) = m - s + s'. \tag{4.143} \]

In (4.142), the sum runs over all possible sets of integers \( \mathcal{B}_{\epsilon, \epsilon'} = \{B_1, \ldots, B_{s+s'}\} \) whose elements satisfy the conditions

\[ \{B_p \in \{1, \ldots, M\} \setminus \{B_1, \ldots, B_{p-1}\} \quad \text{for } 0 < p \leq s, \]

\[ \{B_p \in \{1, \ldots, M + m - 1 + i_p\} \setminus \{B_1, \ldots, B_{p-1}\} \quad \text{for } s < p \leq s + s', \tag{4.144} \]

50
whereas
\[ A_{\epsilon,e'} \equiv \{ A_{\alpha}, \ldots, A_{M+\tilde{a}_{\epsilon,e'}} \} = \{ 1, \ldots, M + m \} \setminus B_{\epsilon,e'}. \]  

Finally,
\[
\mathcal{F}_{B_{\epsilon,e'}}(\{ \xi_j \}_{j=1}^{M}, \{ \zeta_j^{(1)} \}_{j=1}^{m} | \alpha, \beta, x) = \prod_{n=1}^{m} \frac{e^{\xi_j^{(1)}}}{\sinh(\eta \beta_j)} f_{\epsilon,e'}^{(1,m)}(\alpha, \beta, x)
\]
\[
\times \left[ \prod_{p=1}^{\alpha} \prod_{k=p+1}^{\beta} \frac{\sinh(\xi_k^{(1)} - \mu_{p}) - \eta(1 + \beta_p)) \sinh(\mu_{p})}{\sinh(\xi_k^{(1)} - \mu_{p})} \right] \prod_{p=1}^{\alpha} \prod_{k=p+1}^{\beta} \frac{\sinh(\xi_k^{(1)} - \mu_{p}) - \eta(1 - \beta_p)) \sinh(\mu_{p})}{\sinh(\xi_k^{(1)} - \mu_{p})} \right].
\]

where
\[
f_{\epsilon,e'}^{(1,m)}(\alpha, \beta, x) = \begin{cases} 
\prod_{k=1}^{\tilde{a}_{\epsilon,e'}} \left( \frac{e^{\xi_k^{(1)} - \mu_{p}}}{2 \sinh(\eta \beta + k - M - 1)} \right) & \text{if } \tilde{a}_{\epsilon,e'} > 0, \\
\prod_{k=1}^{\tilde{a}_{\epsilon,e'}} \left( \frac{e^{\xi_k^{(1)} - \mu_{p}}}{2 \sinh(\eta \beta - k + M)} \right) & \text{if } \tilde{a}_{\epsilon,e'} < 0,
\end{cases}
\]

In all these expressions we have used the notations (4.51).

**Proof.** This is a direct consequence of the reconstruction (4.43) and of proposition 4.4. 

Note that the expression of this action appears as the direct gauge generalization of (5.11)–(5.13) of [108], once we take into account the change \( \eta \) into \( -\eta \) and our slightly different definition of the inhomogeneity parameters with respect to [108] which result here in a dependence of the formula into \( \zeta_j^{(1)} \), \( 1 \leq n \leq m \), instead of \( \xi_j \), \( 1 \leq n \leq m \), in (5.13) of [108].

4.3. Action of local operators on gauged boundary states

We now express, for the very specific boundary conditions that we consider in the framework of this paper, corresponding to the choice (3.80) of the boundary matrix, the action of the generic
element of the basis of local operators (4.35) on a generic gauged boundary Bethe-type state of the form

\[ \hat{R}_{-M}^{(\alpha)}(\mu_1^{M} | \alpha - \beta + 1) \eta, \alpha + \beta + N - 1 - 2M \]. \quad (4.148)\]

The result of this action is a direct consequence of theorem 4.1 and of the boundary-bulk decomposition (B.31):

**Theorem 4.2.** Let \( \prod_{n=1}^{m} E_n^{(\alpha)}(\xi_n | (a_n, b_n), (\tilde{a}_{n}, \tilde{b}_{n})) \) be the generic element of the basis (4.35) of local operators on the first \( m \) sites of the chain, where we have defined

\[ a_n = \alpha + 1, \quad b_n = \beta - \sum_{r=1}^{n} (-1)^{r}, \quad (4.149) \]
\[ \tilde{a}_n = \alpha - 1, \quad \tilde{b}_n = \beta + \sum_{r=1}^{n} (-1)^{r} - \sum_{r=1}^{m} (-1)^{r}. \quad (4.150) \]

Then, its action on the boundary separate states (4.148) reads

\[ \prod_{n=1}^{m} E_n^{(\alpha)}(\xi_n | (a_n, b_n), (\tilde{a}_{n}, \tilde{b}_{n})) \hat{R}_{-M}^{(\beta)}(\mu_1^{M} | \alpha - \beta + 1) \]
\[ \times | \eta, \alpha + \beta + N - 1 - 2M \rangle = \sum_{B_{\epsilon,s'}} F_{B_{\epsilon,s'}}(\mu_1^{M}, (\xi_j^{(1)})_j^{m} | \alpha, \beta) \]
\[ \times \hat{R}_{-M+\tilde{b}_{\epsilon,s'}}^{(\alpha)}(\mu_1^{M+m} | \alpha - \beta + 1 - 2\tilde{m}_{\epsilon,s'}) | \eta, \alpha + \beta + N - 1 - 2M \rangle, \quad (4.151) \]

where we have defined \( m_{\epsilon,s'} = \xi^{(1)}_{m+1-j} \) for \( j \in \{1, \ldots, m\} \), and

\[ \tilde{m}_{\epsilon,s'} = \sum_{r=1}^{m} (\epsilon'_r - \epsilon_r) = m - (s + s'). \quad (4.152) \]

The sum runs over all possible sets of integers \( B_{\epsilon,s'} = \{ B_1, \ldots, B_{s'} \} \) whose elements satisfy the conditions

\[ \left\{ \begin{array}{l} B_p \in \{ 1, \ldots, M \} \backslash \{ B_1, \ldots, B_{p-1} \} \quad \text{for } 0 < p \leq s, \\ B_p \in \{ 1, \ldots, M + m + 1 - i_p \} \backslash \{ B_1, \ldots, B_{p-1} \} \quad \text{for } s < p \leq s + s', \end{array} \right. \quad (4.153) \]

and

\[ 52 \]
Here, the sum is performed over all $\sigma_j \in \{+,-\}$ for $j \in \alpha_+$, we have defined $\mu_{\alpha} = \sigma_i \mu_i$ for $i \in B_{\epsilon, \rho}$, with $\sigma_i = 1$ if $i > M$, and

$$\alpha_+ = B_{\epsilon, \rho} \cap \{1, \ldots, M\}, \quad \alpha_- = \{1, \ldots, M\} \setminus \alpha_+, \quad (5.155)$$

$$\gamma_- = \{M + m + 1 - \bar{j}\} \cup B_{\epsilon, \rho} \cap \{N+1, \ldots, N+m\}, \quad \gamma_+ = \{1, \ldots, m\} \setminus \gamma_. \quad (5.156)$$

The function $H^{\alpha_+}_N(\lambda)$ is the coefficient appearing in the boundary-bulk decomposition.

Once again, we can compare the expression of this action to the one obtained in the ungauged diagonal case (see proposition 5.2 of [108]): up to the change $\eta$ in $-\eta$, and considering our slightly different definition of the inhomogeneity parameters with respect to [108], (4.151)–(4.154) appear as a direct generalization of (5.14)–(5.15) of [108].

5. Correlation functions

Let us now derive the exact expressions for the correlation functions associated with the local operators of the type

$$\prod_{n=1}^m E^{\alpha, \epsilon}_{\epsilon_\rho}(\xi_n|a_n, b_n, (\bar{a}_n, \bar{b}_n)), \quad (5.1)$$

with $a_n, b_n, \bar{a}_n, \bar{b}_n$ given by (4.36) and (4.37) in terms of

$$\eta \alpha = -\tau_- + i \frac{\pi}{2}, \quad \eta \beta = \epsilon_\varphi (-\varphi_- - \psi_-) + i \frac{\pi}{2}. \quad (5.2)$$

For simplicity, we shall consider here only the local operators satisfying the following constrain:

$$s + s' = m \quad \text{or equivalently} \quad \bar{\eta}_{\epsilon, \rho} = \sum_{\epsilon_i = 1}^m (\epsilon'_i - \epsilon_i) = 0. \quad (5.3)$$
5.1. General strategy for the computation of matrix elements of local operators in the limit
\( \zeta_+ = -\infty \)

Let us here explain how to compute the mean value of some quasi-local operator \( O_{1 \rightarrow m} \in \text{End}(\otimes_{n=1}^m \mathcal{H}_n) \),

\[
\langle O_{1 \rightarrow m} \rangle = \frac{\langle Q | O_{1 \rightarrow m} | Q \rangle}{\langle Q | Q \rangle},
\]

in some eigenstate \( |Q \rangle \) of the transfer matrix under the boundary conditions (3.80) which are out of the range of validity of Sklyanin’s SoV framework. Here \( |Q \rangle \) and \( \langle Q | \) denote the right and left eigenstates of \( T_\lambda \) represented as the SoV states (3.44) and (3.45) (written in the new general SoV basis) for some \( Q \) of the form (3.37) with roots labelled by \( \lambda_1, \ldots, \lambda_M \). The latter are admissible solutions of the Bethe equations issued from the homogeneous \( TQ \)-equation (3.47).

Since the SoV state \( |Q \rangle \) and the boundary Bethe state \( \hat{Q}^{-M} \{ \{ \lambda_j \}_{j=1}^M | \alpha - \beta + 1 \} \eta, \alpha + \beta + N - 2M - 1 \} \) (with \( \alpha \) and \( \beta \) given in terms of the ‘+’ boundary parameters by (2.58) and (2.59)) are eigenvectors of the transfer matrix with the same eigenvalue, they are collinear due to the simplicity of the transfer matrix spectrum. Hence we can write:

\[
\langle O_{1 \rightarrow m} \rangle = \frac{\langle Q | O_{1 \rightarrow m} | \hat{Q}^{-M} \{ \{ \lambda_j \}_{j=1}^M | \alpha - \beta + 1 \} \eta, \alpha + \beta + N - 2M - 1 \} \rangle}{\langle Q | \hat{Q}^{-M} \{ \{ \lambda_j \}_{j=1}^M | \alpha - \beta + 1 \} \eta, \alpha + \beta + N - 2M - 1 \} | Q \rangle}.
\]

Expressing the quasi-local operator \( O_{1 \rightarrow m} \) on the basis (4.35),

\[
O_{1 \rightarrow m} = \sum_{\epsilon, \epsilon' \in \{1, 2\}} f_{O, \epsilon, \epsilon'} \prod_{n=1}^m E_n^{\epsilon, \epsilon'}(\zeta_n)(a_n, b_n, (\bar{a}_n, \bar{b}_n)),
\]

and using the action (4.151) of these basis elements on the boundary Bethe state \( \hat{Q}^{-M} \{ \{ \lambda_j \}_{j=1}^M | \alpha - \beta + 1 \} \eta, \alpha + \beta + N - 2M - 1 \} \), we can write

\[
\langle O_{1 \rightarrow m} \rangle = \sum_{\epsilon, \epsilon' \in \{1, 2\}} f_{O, \epsilon, \epsilon'} \sum_{\bar{B}_{\epsilon, \epsilon'}} \hat{B}_{\epsilon, \epsilon'}(\{ \lambda_j \}_{j=1}^M, \{ \zeta_j \}_{j=1}^m | \beta) \text{SP}_Q(\{ \lambda_i \}_{i=\epsilon, \epsilon'}),
\]

where the set \( \{ \lambda_i \}_{i=\epsilon, \epsilon'} \) is a subset of the set \( \{ \lambda_j \}_{j=1}^M \cup \{ \zeta_n \}_{n=1}^m \) (see (4.145)), and where \( \text{SP}_Q(\{ \lambda_i \}_{i=\epsilon, \epsilon'}) \) is the following ratio of scalar products:

\[
\text{SP}_Q(\{ \lambda_i \}_{i=\epsilon, \epsilon'}) = \frac{\langle \hat{Q}^{-M+\tilde{m}} \{ \lambda_i \}_{i=\epsilon, \epsilon'} | \alpha - \beta + 1 - 2\tilde{m}, \epsilon' \} \eta, \alpha + \beta + N - 1 - 2M \rangle}{\langle \hat{Q}^{-M} \{ \lambda_j \}_{j=1}^M | \alpha - \beta + 1 \} \eta, \alpha + \beta + N - 2M \rangle}.
\]

The whole problem is now to compute the ratio of scalar products (5.8) in a simple way.

To this aim, let us remark that the boundary conditions (3.80) can be reached as the \( \zeta_+ \rightarrow -\infty \) limit of some continuous trajectories in the three-dimensional space of the ‘+’ boundary parameters \( \zeta_+, \kappa_+, r_+ \) on which Sklyanin’s SoV approach can be used. We can choose a particular trajectory on which the condition (3.70) holds all along the trajectory for
fixed $\alpha$ and $\beta$ given in terms of the (fixed) ‘$-$’ boundary parameters by (2.58) and (2.59). Along such a trajectory and for finite $\varsigma_+$, it follows from proposition 3.4 that

$$|\eta, \alpha + \beta + N - 1 - 2M, \varsigma_+\rangle = \Theta_{M,\text{ref}}^{(R_{K+L})} \Omega_{\alpha,\beta-2M+1}^{(\varsigma_+)}.$$  (5.9)

Note that both the Sklyanin’s reference state (3.56) and the scalar factor in (5.9) depend in a continuous way on the ‘$+\,$’ boundary parameters, and therefore on the position on the trajectory, and we underline this dependance by a superscript $(\varsigma_+)$. In the same way, the boundary operators $\hat{B}_-$ depend continuously on the ‘$+' boundary parameters, and we denote them along the trajectory by $\hat{B}_+^{(\varsigma_+)}$.

On the other hand, all along the trajectory and in the limit $\varsigma_+ \to -\infty$, the transfer matrix is diagonalizable with simple spectrum, and its eigenvalues depend continuously on the ‘$+$’ boundary parameters. Hence we have no crossing of levels along the trajectory and we can define in an unambiguous way the SoV eigenstate $\langle Q^{(\varsigma_+)} |$ (still written in the new general SoV basis) that converges toward $|Q\rangle$. Once again, the state $\langle Q^{(\varsigma_+)} |$ depends in a continuous way of the ‘$+$’ boundary parameter along the trajectory, and we have from (3.55) that it is collinear to the corresponding eigenstate $s_k | Q^{(\varsigma_+)} \rangle$ defined in the Sklyanin’s SoV framework as in (3.54).

Hence we can write

$$\text{SP}\langle \{\hat{\lambda}_i\} \in A_{\varsigma'} | = \lim_{\varsigma_+ \to -\infty} \frac{\Theta_{M,\text{ref}}^{(R_{K+L})} \Omega_{\alpha,\beta-2M+1}^{(\varsigma_+)}}{\Theta_{M,\text{ref}}^{(R_{K+L})} \Omega_{\alpha,\beta-2M+1}^{(\varsigma_+)}} \Theta_{M,\text{ref}}^{(R_{K+L})} \Omega_{\alpha,\beta-2M+1}^{(\varsigma_+)}.}$$  (5.11)

Note here that we keep the roots $\lambda_i$ in the Bethe vectors entering the ratio of scalar products in the right-hand side of (5.11) fixed along the trajectory, so that these Bethe vectors are a priori no longer eigenstates at finite $\varsigma_+$.

The ratio of scalar products that appear in the right-hand side of (5.11) can now be computed in the framework of Sklyanin’s SoV, at least for $m_{\epsilon,\epsilon'} = 0$ for which we can use the collinearity relations (3.59). From now on, we therefore restrict our study to operators $O_{1-m_n}$ in the sector $m_{\epsilon,\epsilon'} = 0$, i.e. which can be obtained as a linear combinations of quasi-local operators of the form (5.1) with condition (5.3). Let us denote by $\bar{Q}_{A_{\varsigma'}}$, the polynomial of the form (3.37) with

\begin{itemize}
  \item Note that, in the limit $\varsigma_+ \to -\infty$, the degree of the polynomial $Q^{(\varsigma_+)}$ written as in (3.37) may not be conserved. In other words, it may happen that some roots of $Q^{(\varsigma_+)}$ tend to infinity when $\varsigma_+ \to -\infty$. Note also that $Q^{(\varsigma_+)}$ may no longer be solution of a homogenous functional $TQ$-equation, but instead on the inhomogeneous one (3.42). However, the state (3.45) itself is perfectly well defined and continuous all along the trajectory and in the limit $\varsigma_+ \to -\infty$ by the condition

$$\langle Q^{(\varsigma_+)} | \langle Q^{(\varsigma_+)} | \frac{\langle Q^{(\varsigma_+)} |}{\langle Q^{(\varsigma_+)} |}.$$

being both the transfer matrix eigenvalue $\tau^{(\varsigma_+)}$ and the function $A^{(\varsigma_+)}$ (3.10) continuous with respect to the ‘$+$’ boundary parameters and well defined in the $\varsigma_+ \to -\infty$ limit.

\end{itemize}
$M + \tilde{m}_{c',c} = M$ roots labelled by $\tilde{\lambda}_i, i \in A_{c',c}$. Recall that we denote by $Q$ the polynomial (3.37) with roots labelled by $\lambda_1, \ldots, \lambda_M$. From (3.59), we can again transform (5.11) as

$$\text{SP}_Q(\{\tilde{\lambda}_i\}_{i \in A_{c',c}}) = \lim_{\zeta_+ \to -\infty} \frac{C^{(R)}_{QABA}}{C^{(R)}_{Q_{A_{c',c}}ABA}} \frac{\text{Sk}(Q^{(c+)}A_{c',c})\text{Sk}}{\text{Sk}(Q^{(c+)}\text{Sk})}.$$  \hfill (5.12)

with

$$\frac{C^{(R)}_{QABA}}{C^{(R)}_{Q_{A_{c',c}}ABA}} = \prod_{j=1}^M \frac{d_0(\tilde{\lambda}_j)d_0(-\tilde{\lambda}_j) \sinh(2\tilde{\lambda}_j - \eta)}{d_0(\lambda_j)d_0(-\lambda_j) \sinh(2\lambda_j - \eta)}.$$  \hfill (5.13)

The ratio of scalar products of separate states in the right-hand side of (5.12) can, as usual, be expressed as a ratio of determinants:

$$\frac{\text{Sk}(Q^{(c+)}A_{c',c})\text{Sk}}{\text{Sk}(Q^{(c+)}\text{Sk})} = \frac{\det_{1 \leq i,j \leq N} \left[ \sum_{h=0}^{\lambda_1} \left( -\frac{\eta_0}{a_{\lambda_1}} (Q^{(c+)}Q_{A_{c',c}})_{\lambda_1}^{(c+)} \right) \cosh(2\xi_i^{(c+)} - \xi_i^{(c)}) \right]^{h-1}}{\det_{1 \leq i,j \leq N} \left[ \sum_{h=0}^{\lambda_1} \left( -\frac{\eta_0}{a_{\lambda_1}} (Q^{(c+)}Q_{A_{c',c}})_{\lambda_1}^{(c+)} \right) \cosh(2\xi_i^{(c+)} - \xi_i^{(c)}) \right]^{h-1}}.$$  \hfill (5.14)

It is now easy to take the $\zeta_+ \to -\infty$ in the above expression, and we obtain

$$\text{SP}_Q(\{\tilde{\lambda}_i\}_{i \in A_{c',c}}) = \prod_{j=1}^M \frac{d(\tilde{\lambda}_j)d(-\tilde{\lambda}_j) \sinh(2\tilde{\lambda}_j - \eta)}{d(\lambda_j)d(-\lambda_j) \sinh(2\lambda_j - \eta)} \frac{\det_{1 \leq i,j \leq N} \left[ \sum_{h=0}^{\lambda_1} \left( -\frac{\eta_0}{a_{\lambda_1}} (Q^{(c+)}Q_{A_{c',c}})_{\lambda_1}^{(c+)} \right) \cosh(2\xi_i^{(c+)} - \xi_i^{(c)}) \right]^{h-1}}{\det_{1 \leq i,j \leq N} \left[ \sum_{h=0}^{\lambda_1} \left( -\frac{\eta_0}{a_{\lambda_1}} (Q^{(c+)}Q_{A_{c',c}})_{\lambda_1}^{(c+)} \right) \cosh(2\xi_i^{(c+)} - \xi_i^{(c)}) \right]^{h-1}}.$$  \hfill (5.15)

Such a formula can be transformed similarly as in [39], and we finally obtain

$$\text{SP}_Q(\{\tilde{\lambda}_i\}_{i \in A_{c',c}}) = \prod_{j=1}^M \frac{\sinh(2\lambda_j + \eta)}{\sinh(2\lambda_j + \eta)} \frac{\det_{1 \leq i,j \leq N} [S(\lambda, \lambda)]}{\text{det}_{1 \leq i,j \leq N} [S(\lambda, \lambda)]}.$$  \hfill (5.16)

in which $S(\mu, \lambda)$, for $\mu = (\mu_1, \ldots, \mu_M)$ and $\lambda = (\lambda_1, \ldots, \lambda_M)$, is the $M \times M$ matrix with elements

$$[S(\mu, \lambda)]_{jk} = Q(\mu_j) \frac{\partial Q(\mu)}{\partial \lambda_k} = A_{\tilde{\mu}}(\mu_j)Q(\mu_j - \eta)[\tau(\mu_j + \lambda_k - \eta/2) - \tau(\mu_j - \lambda_k - \eta/2)]$$

$$- A_{\tilde{\mu}}(-\mu_j)Q(\mu_j + \eta)[\tau(\mu_j + \lambda_k + \eta/2) - \tau(\mu_j - \lambda_k + \eta/2)],$$  \hfill (5.17)

where $\tau_Q$ is the transfer matrix eigenvalue associated with the Bethe roots $\lambda_1, \ldots, \lambda_M$ labelling the roots of the polynomial $Q$, and

56
\[ n(\lambda) = \frac{\sinh \eta}{\sinh(\lambda - \eta/2) \sinh(\lambda + \eta/2)} = \coth(\lambda - \eta/2) - \coth(\lambda + \eta/2). \] (5.18)

Note that (5.17) is regular for \( \mu_j = \lambda_j \) due to the fact that \( \lambda_1, \ldots, \lambda_M \) satisfy the Bethe equations, and can be rewritten as

\[ [S(\mu, \lambda)]_{jk}|_{\mu_j=\lambda_j} = -A_{\bar{\varepsilon}}(-\lambda_j)Q(\lambda_j + \eta) \times \{2N\delta_{j,k} \Xi_{\varepsilon,Q}(\lambda_j) + 2\pi[iK(\lambda_j - \lambda_k) - K(\lambda_j + \lambda_k)]\}. \] (5.19)

in which we have defined

\[ \Xi_{\varepsilon,Q}(\mu) = \frac{i}{2N} \frac{\partial}{\partial \mu} \left( \log \frac{A_{\bar{\varepsilon}}(-\mu)Q(\mu + \eta)}{A_{\varepsilon}(\mu)Q(\mu - \eta)} \right). \] (5.20)

\[ K(\lambda) = \frac{i \sinh(2\eta)}{2\pi \sinh(\lambda + \eta) \sinh(\lambda - \eta)} = \frac{i}{2\pi} \left[ n(\lambda + \eta/2) + n(\lambda - \eta/2) \right]. \] (5.21)

In particular, if \( \{\lambda_j\}_M^{j=1} = \{\lambda_a\}_{a<0} \cup \{\lambda_b\}_{b>0} \) and \( \{\bar{\lambda}_j\}_M^{j=1} = \{\lambda_a\}_{a<0} \cup \{\xi^{(1)}_{b}\}_{b>0} \), then

\[ \text{SP}_G(\{\lambda\}_M^{j=1}) = \prod_{b<\alpha+} \frac{\sinh(2\lambda_b + \eta)A_{\varepsilon}(\lambda_b - \xi^{(1)}_{b} + \eta)}{\sinh(2\xi^{(1)}_{b} + \eta)A_{\varepsilon}(\lambda_b + \eta)} \times \frac{\sinh^2 \lambda_a - \sinh^2 \lambda_b}{\sinh^2 \xi^{(1)}_{a} - \sinh^2 \xi^{(1)}_{b}} \prod_{a,b<0} \frac{\sinh^2 \lambda_a - \sinh^2 \lambda_b}{\sinh^2 \xi^{(1)}_{a} - \sinh^2 \xi^{(1)}_{b}} \det_M M(\bar{\lambda}, \lambda) \det_M N(\lambda), \] (5.22)

in which

\[ [\mathcal{N}(\lambda)]_{jk} = 2N\delta_{jk} \Xi_{\varepsilon,Q}(\lambda_j) + 2\pi[iK(\lambda_j - \lambda_k) - K(\lambda_j + \lambda_k)], \] (5.23)

\[ [\mathcal{M}(\lambda, \lambda)]_{jk} = \begin{cases} [\mathcal{N}(\lambda)]_{jk} & \text{if } k \in \alpha-, \\ [i\tau(\xi_{\alpha} - \lambda_j) - \tau(\xi_{\alpha} + \lambda_j)] & \text{if } k \in \alpha+. \end{cases} \] (5.24)

Note that, in that case, the ratio of the determinants of \( \mathcal{M} \) and \( \mathcal{N} \) in (5.22) reduces to the determinant of a matrix of size \( |\alpha+| \);

\[ \frac{\det_M M(\bar{\lambda}, \lambda)}{\det_M N(\lambda)} = \frac{\det_M \mathcal{R}_{\alpha,\beta}}{\det_M \mathcal{R}_{\alpha,\beta}} \] (5.25)

5.2. Expression of the correlation functions in the finite chain

We can now give the exact expression of the mean value, in the eigenstate \( |Q\rangle \) with Bethe roots \( \lambda_1, \ldots, \lambda_M \), of the quasi-local operator (5.1) with condition (5.3) and under the special boundary conditions (3.80) corresponding to \( \zeta_+ = -\infty \) on the site \( N \). For \( |Q\rangle \) being the ground state of the chain, this gives the zero-temperature correlation functions of all combinations of local operators in the sector (5.3).

From the previous study, we therefore obtain the following multiple sum representation of these quantities:
Theorem 5.1. Under the boundary conditions (3.80), the correlation functions of the quasi-local operators (5.1) satisfying the condition (5.3) can be written as

\[
\langle \prod_{\alpha=1}^{m} E_{n_{\alpha}}^{\sigma_{\alpha}}(\xi_{\alpha}|(a_{\alpha}, b_{\alpha}), (\bar{a}_{\alpha}, \bar{b}_{\alpha})) \rangle = \sum_{n_{1}=1}^{M} \cdots \sum_{n_{m+1}=1}^{M} \sum_{n_{m+2}=1}^{M+1} \cdots \times \sum_{n_{m+1}=1}^{M+1} \prod_{1 \leq \ell < k \leq m} \sinh(\xi_{\ell} - \xi_{k}) \prod_{1 \leq \ell < q \leq m} \sinh(\xi_{\ell} + \xi_{q}),
\]

(5.26)

in which

\[
H_{[s_{j}]}(\{\lambda\}|\beta) = \sum_{n=1}^{m} \frac{e^{\beta n}}{\sinh(\beta a_{n})} \sum_{\sigma_{n}} (-1)^{s_{j}} \prod_{i=1}^{m} \prod_{j=1}^{m} \sinh(\lambda_{n_{i}}^{\sigma_{n_{i}}} + \xi_{i}/2) \sinh(\lambda_{n_{j}}^{\sigma_{n_{j}}} - \xi_{j} - \eta) \sinh(\lambda_{n_{j}}^{\sigma_{n_{j}}} + \xi_{j} + \eta) \times \prod_{p=1}^{s_{j}} \left\{ \sinh(\lambda_{n_{p}}^{\sigma_{n_{p}}} - \xi_{p}^{(1)} - \eta(1 + \bar{b}_{p})) \prod_{k=1}^{p-1} \sinh(\lambda_{n_{p}}^{\sigma_{n_{p}}} - \xi_{k}^{(1)}) \prod_{k=p+1}^{m} \sinh(\lambda_{n_{p}}^{\sigma_{n_{p}}} - \xi_{k}^{(1)} + \eta) \right\} \times \prod_{p=m+1}^{m} \left\{ \sinh(\lambda_{n_{p}}^{\sigma_{n_{p}}} - \xi_{p}^{(1)} + \eta(1 - \bar{b}_{p})) \prod_{k=1}^{p-1} \sinh(\lambda_{n_{p}}^{\sigma_{n_{p}}} - \xi_{k}^{(1)}) \prod_{k=p+1}^{m} \sinh(\lambda_{n_{p}}^{\sigma_{n_{p}}} - \xi_{k}^{(1)} + \eta) \right\} \times \prod_{k=1}^{m} \frac{\sinh(\xi_{k}^{(1)} - \xi_{k}^{(D)}) \sinh(\xi_{k}^{(1)} + \xi_{k}^{(D)})}{\sinh(\lambda_{n_{p}}^{\sigma_{n_{p}}} - \xi_{p}^{(D)} + \eta/2) \sinh(\lambda_{n_{p}}^{\sigma_{n_{p}}} - \xi_{p}^{(D)} - \eta/2)} \det \Omega,
\]

(5.27)

where the parameters \(\xi_{s_{j}}^{(D)}\) are given in terms of the boundary parameters at site 1 by (3.86). Here the sum is performed over all \(\sigma_{n_{i}} \in \{+, -\}\) for \(B_{1} \leq M\), and \(\sigma_{n_{i}} = 1\) for \(B_{1} > M\), and the \(m \times m\) matrix \(\Omega\) reads

\[
\Omega_{k} = -\delta_{M+1-k}, \quad \text{for } B_{1} > M,
\]

(5.28)

\[
\Omega_{k} = R_{B_{1}-k}, \quad \text{for } B_{1} \leq M,
\]

(5.29)

in terms of the matrix \(R\) (5.25).

5.3. Expression of the correlation functions in the half-infinite chain

Let us now consider the thermodynamic limit \(N \to \infty\) of these correlation functions, which can be obtained quite similarly as in [108]. We recall here that the spectrum of the spin chain with boundary conditions (3.80) at site \(N\) is isospectral to that of a chain with diagonal boundary conditions and boundary parameters \(\xi_{s_{j}}^{(D)}\) (3.86), so that we can use the known description of the ground state in the diagonal case to derive the thermodynamic limit of the expression (5.26).

The configuration for the Bethe roots for the ground state of the open spin chain with diagonal boundary conditions in the thermodynamic limit \(N \to \infty\) has been studied in [146, 147], see also [131] for a more accurate description of the ground state in the massive regime.

In the thermodynamic limit \(N \to \infty\), nearly all Bethe roots for the ground state of the spin chain condensate on an interval \((0, \Lambda)\) of the real axis (in the regime \(|\Delta| < 1\), for which \(\Lambda =\)
+∞) or of the imaginary axis (in the regime Δ > 1, for which Λ = −iπ/2), with some density function ρ(λ) solution of the following integral equation:

\[ \rho(\lambda) + \int_0^\Lambda [K(\lambda - \mu) + K(\lambda + \mu)] \rho(\mu) d\mu = \frac{p'(\lambda)}{\pi}, \]

which can be extended by parity on the whole interval (−Δ, Δ) as

\[ \rho(\lambda) + \int_{-\Lambda}^\Lambda K(\lambda - \mu) \rho(\mu) d\mu = \frac{p'(\lambda)}{\pi}. \]

Here K is given by (5.21) and p′ is given in terms of t (5.18) as

\[ p'(\lambda) = \frac{i \sinh \eta}{\sinh(\lambda + \eta/2) \sinh(\lambda - \eta/2)} = it(\lambda). \]

Explicitly, we have

\[ \rho(\lambda) = \begin{cases} 
\frac{1}{\zeta \cosh(\pi \lambda/\zeta)} & \text{with } \zeta = i\eta > 0 \text{ if } |\Delta| < 1, \\
\frac{i}{\pi} \phi_q(0, q) \phi_q(1, \lambda, q) & \text{with } q = e^i(\eta < 0) \text{ if } \Delta > 1.
\end{cases} \]

Furthermore, the set of Bethe roots for the ground state may contain some extra isolated complex roots, that we call boundary roots, and that may tend either to η/2 − ς(D+), or to η/2 − ς(D−), with exponentially small corrections in the large N limit. We will denote such a boundary root by \( \lambda_+ \) in the former case and by \( \lambda_- \) in the latter case. The presence of such a boundary root within the set of roots for the ground state depends on both boundary parameters \( \varsigma(D+), \varsigma(D−) \) and on the parity of the number of sites N of the chain (see [131]). We refer to [131] for a detailed study of the set of Bethe roots describing the ground state, and in particular of the cases in which it contains such a boundary root, in the regime \( \Delta > 1 \).

In the expression of the correlation functions (5.26), the sum over real Bethe roots, with indices \( B_j \) running from 1 to M, become integrals over the density functions in the thermodynamic limit \( N \to \infty \) according to the following rule:

\[ \frac{1}{N} \sum_{B_j = 1}^M \sum_{\sigma_j = \pm} \sigma_{\lambda_j} f(\lambda_j^0) \int_0^\Lambda d\lambda_j \rho(\lambda_j) \sum_{\sigma_j = \pm} \sigma_{\lambda_j} f(\lambda_j^0) = \int_{-\Lambda}^\Lambda d\lambda_j f(\lambda_j) \rho(\lambda_j), \]

while the sum over the \( B_j > M \) can be written as contour integrals thanks to the identity

\[ 2\pi \text{Res } \rho(\lambda - \zeta), \lambda = \xi + \eta/2 = -2. \]

The ratio of determinants can also be computed in the thermodynamic limit similarly as in [108]. If \( \lambda_j \) corresponds to a real root, we have

\[ [\Lambda(\Lambda)]_{j,k} = 2\pi N \delta_{jk} \left[ \rho(\lambda_j) + O \left( \frac{1}{N} \right) \right] + 2\pi [K(\lambda_j - \lambda_k) - K(\lambda_j + \lambda_k)], \]

Here and in the following, the terms ‘real root’ are used to designate a Bethe root \( \lambda \) which belongs to the interval \( (0, \Lambda) \), i.e. which is indeed real in the regime \( |\Delta| < 1 \), but which is instead purely imaginary in the regime \( \Delta > 1 \); in the latter case, we have to make an appropriate change of variable \( \alpha = i\lambda \) to recover a real root.
and, due to (5.31),
\[ \sum_{p=1}^{M} \mathcal{N}_{jp} \frac{\rho(\lambda_p - \xi_k) - \rho(\lambda_p + \xi_k)}{2N\rho(\lambda_p)} \xrightarrow{N \to \infty} i[t(\xi_k - \lambda_j) - t(\xi_k + \lambda_j)]. \] (5.37)

If instead \( \lambda_j \) is a boundary root of the form \( \lambda_j = \bar{\lambda}_\sigma = \eta/2 - \zeta^{(D)}_\sigma + \bar{\epsilon}_\sigma \) with \( \bar{\epsilon}_\sigma \) being an exponentially small factor is compensated by the prefactor \( \frac{1}{\sinh(\bar{\epsilon}_\sigma)} \), and due to (5.31),
\[ [\mathcal{N}(\lambda)]_{jk} = -\frac{i}{\epsilon_\sigma} \left[ \delta_{jk} \left( 1 + \delta_{\zeta^{(D)},\zeta^{(D)}} \right) + O(\epsilon_\sigma) \right]. \] (5.38)

Hence, in the thermodynamic limit, the elements of the matrix (5.25) are given by
\[ \mathcal{R}_{n,h} \sim \begin{cases} \frac{i\pi \epsilon_\sigma \left[ \rho(\lambda_a - \xi_k) - \rho(\lambda_a + \xi_k) \right]}{1 + \delta_{\zeta^{(D)_{\zeta^{(D)}}}}} & \text{if } \lambda_a = \bar{\lambda}_\sigma, \\ \frac{\rho(\lambda_a - \xi_k) - \rho(\lambda_a + \xi_k)}{2N\rho(\lambda_a)} & \text{if } \lambda_a \in (0, \Lambda). \end{cases} \] (5.39)

Note that, if the boundary root \( \bar{\lambda}_\sigma \) belongs to the set of roots \( \{ \lambda_j \}_{j=1}^{M} \cap \{ \lambda_n \}_{n=1}^{M} \), the corresponding row in \( \mathcal{R} \) is proportional to \( \eta/\epsilon_\sigma \), which is exponentially small in \( N \). However, this exponentially small factor is compensated by the prefactor
\[ \frac{1}{\sinh(-\lambda - \zeta^{(D)} + \eta/2)} \sim \frac{1}{\epsilon_\sigma}, \] (5.40)
so that the final contribution is of order 1. This contribution can be written as a contour integral around the point \( \zeta^{(D)} - \eta/2 \).

Therefore, the following result holds:

**Theorem 5.2.** Under the boundary conditions (3.80), the correlation functions of the quasi-local operators (5.1) satisfying the condition (5.3) can be written in the thermodynamic limit as
\[ \langle \prod_{n=1}^{m} E^{\mu_{\alpha_n}}_{\sigma_n}(\xi_n)(a_n, b_n), (a_n, b_n) \rangle = \prod_{n=1}^{m} \frac{\delta^{\eta_{\mu}}_{n}}{\sinh(\eta b_n)} \prod_{j=1}^{M} \frac{(-1)^{j}}{\sinh(\xi_j - \xi) \prod_{i \neq j} \sinh(\xi_i + \xi_j)} \times \prod_{j=1}^{M} d\lambda_j \int_{\xi_j}^{\xi_{j+1}} d\lambda_j H_{n}^{\lambda_j} \left\{ \{ \lambda_j \}_{j=1}^{M} \right\} \left\{ \xi_k \right\}_{k=1}^{m} \det [\Phi(\lambda_j, \xi_k)], \] (5.41)

where we have denoted
\[ \Phi(\lambda_j, \xi_k) = \frac{1}{2} \left[ \rho(\lambda_j - \xi_k) - \rho(\lambda_j + \xi_k) \right], \] (5.42)

and
\[ \text{The factor } 2 \text{ for } \zeta^{(D)} = \zeta^{(D)} \text{ is due to the fact that, in that case, the boundary root approaches a double pole of the Bethe equation, which corresponds to a doubling of the same term in (5.38), see [131].} \]
\[ H_m(\{\lambda_j\}_{j=1}^m; \{\xi_k\}_{k=1}^m) \]
\[ = \prod_{j=1}^m \prod_{k=1}^m \frac{\sinh(\lambda_j + \xi_k + \eta/2)}{\sinh(\lambda_j + \eta)} \times \prod_{j=1}^s \left( \sinh(\lambda_j - \xi_j^{(1)} + \eta(1 + b_j)) \prod_{k=1}^{i_j-1} \sinh(\lambda_j - \xi_j^{(1)}) \prod_{k=i_j+1}^m \sinh(\lambda_j - \xi_j^{(1)} - \eta) \right) \]
\[ \times \prod_{p=s+1}^m \left( \sinh(\lambda_p - \xi_p^{(1)} + \eta(1 - b_p)) \prod_{k=1}^{i_p-1} \sinh(\lambda_p - \xi_p^{(1)}) \prod_{k=i_p+1}^m \sinh(\lambda_p - \xi_p^{(1)} + \eta) \right) \]
\[ \times \prod_{k=1}^m \frac{\sinh(\xi_k - \xi_k^{(D)}) \sinh(\xi_k - \xi_k^{(D)})}{\sinh(\lambda_k - \xi_k^{(D)} + \eta/2) \sinh(\lambda_k - \xi_k^{(D)} + \eta/2)}, \] (5.43)

in which we have used the identification (3.86). The contours \( C \) is defined as
\[ C = [-\Lambda, \Lambda] \] (5.44)
if the set of Bethe roots for the ground state do not contain any boundary root, and as
\[ C = [-\Lambda, \Lambda] \cup \Gamma(\xi^{(D)}_n - \eta/2) \] (5.45)
if the set of Bethe roots for the ground state contains the boundary root \( \tilde{\lambda}_n \). The contour \( C_\xi \) is defined as
\[ C_\xi = C \cup \Gamma(\{\xi_j^{(1)}\}_{k=1}^m). \] (5.46)

Here \( \Gamma(\xi^{(D)}_n - \eta/2) \) (respectively \( \Gamma(\{\xi_j^{(1)}\}_{k=1}^m) \)) surrounds the point \( \xi^{(D)}_n - \eta/2 \) (respectively the points \( \xi_j^{(1)}, \ldots, \xi_m^{(1)} \)) with index 1, all other poles being outside.

We recall that a full description of the configuration of Bethe roots for the ground states in terms of the boundary parameters \( \xi_j^{(D)} \) has been done in [131] in the regime \( \Delta > 1 \).

It is interesting to compare the above result to the corresponding formulas (6.8)–(6.10) of [108] that were obtained in the diagonal case. To this aim, we can rewrite (5.41)–(5.43) in a closer form as (6.8)–(6.10) of [108] by changing the sign of the integration variables \( \lambda_j \), \( 1 \leq j \leq m \), and by redefining new inhomogeneity parameters as
\[ \tilde{\xi}_j = -\xi_j + \eta/2, \quad 1 \leq j \leq m. \] (5.47)

It gives
\[ \langle \prod_{n=1}^m E_n^{(\bar{a}, \bar{b})}(\xi_n|\{a_n, b_n\}, (\bar{a}_n, \bar{b}_n)) \rangle \]
\[ = \prod_{n=1}^m \frac{e^{\eta}}{\sinh(\eta b_n)} \int \prod_{j=1}^s d\lambda_j \int \prod_{1 \leq j \leq m} d\lambda_j \tilde{H}_m(\{\lambda_j\}_{j=1}^m; \{\xi_k\}_{k=1}^m) \det_{1 \leq j, k \leq m} [\tilde{\Phi}(\lambda_j, \tilde{\xi}_k)], \] (5.48)
where we have denoted
\[ \hat{\Phi}(\lambda_j, \xi_k) = \frac{1}{2} \left[ \tilde{\rho}(\lambda_j - \xi_k) - \tilde{\rho}(\lambda_j - \eta + \xi_k) \right], \]  
with
\[ \tilde{\rho}(\lambda) = \rho(\lambda + \eta/2) = \begin{cases} \frac{i}{\zeta \sinh(\pi \lambda/\zeta)} & \text{with } \zeta = i\eta > 0 \text{ if } |\Delta| < 1, \\ \frac{i}{\vartheta_1(0, q) \vartheta_2(0, q)} & \text{with } q = e^{\eta}(\eta < 0) \text{ if } \Delta > 1. \end{cases} \] 
and
\[ \hat{H}_m(\{\lambda_j\}_{j=1}^M, \{\xi_k\}_{k=1}^m) = \prod_{1 \leq j \leq m} \sinh(\lambda_j - \lambda_j + \eta) \sinh(\lambda_i + \lambda_j - \eta) \]
\[ \times \prod_{p=1}^s \left\{ \sinh(\lambda_{ip} - \xi_{ip} + \eta(1 + \tilde{b}_{ip})) \prod_{k=1}^{ip-1} \sinh(\lambda_{ip} - \xi_k + \eta) \prod_{k=ip+1}^m \sinh(\lambda_{ip} - \xi_k + \eta) \right\} \]
\[ \times \prod_{p=s+1}^m \left\{ \sinh(\lambda_{ip} - \xi_{ip} - \eta(1 - \tilde{b}_{ip})) \prod_{k=1}^{ip-1} \sinh(\lambda_{ip} - \xi_k - \eta) \prod_{k=ip+1}^m \sinh(\lambda_{ip} - \xi_k - \eta) \right\} \]
\[ \times \prod_{k=1}^m \frac{\sinh(\xi_k + \zeta^{(D)}_+ - \eta/2) \sinh(\xi_k + \zeta^{(D)}_- - \eta/2)}{\sinh(\lambda_k + \zeta^{(D)}_+ - \eta/2) \sinh(\lambda_k + \zeta^{(D)}_- - \eta/2)}. \]  

Here the contour \( \tilde{C} \) is defined as (5.44) if the set of Bethe roots for the ground state does not contain any boundary root, and as
\[ \tilde{C} = [-\Lambda, \Lambda] \cup \Gamma(\eta/2 - \zeta^{(D)}) \]  
if the set of Bethe roots for the ground state contains the boundary root \( \tilde{\lambda}_\sigma \), and the contour \( \tilde{C}_\xi \) is defined as
\[ \tilde{C}_\xi = \tilde{C} \cup \Gamma(\xi_k)_{k=1}^m. \]  

With this rewriting and this redefinition of the inhomogeneity parameters, we see a clear correspondence with the results (6.9) and (6.10) of [108] for the diagonal case. The only difference comes from the boundary factor (last line of (5.51)) which here contains two factors and involves both parameters \( \zeta^{(D)}_+ \) and \( \zeta^{(D)}_- \), the related fact that the residues at both boundary roots \( \tilde{\lambda}_+ \) and \( \tilde{\lambda}_- \) may be involved through the definition of the integration contour (5.52), and an additional dependence in terms of the gauge parameters \( b_\sigma \) and \( b_\tau \), which are themselves defined in terms of the boundary parameters through the gauge condition (2.59). This strong similarity of the two results (the one of [108] and the present one) is not surprising since, as we have shown, the chain that we consider here is isospectral to the one considered in [108], and the computation of the local operator actions and of the resulting scalar products are very similar in the two cases, except for the contribution of the gauge parameters involved in the definition of the eigenstates and in the local operators. On the other hand, the fact that the two
boundary parameters $\bar{\varsigma}^{(D)}$ and $\bar{\varsigma}^{-}$ and their corresponding boundary roots $\bar{\lambda}^{+}$ and $\bar{\lambda}^{-}$ appear explicitly in the result\textsuperscript{15} is also not surprising since, contrary to what happens in [108], these two boundary parameters both label the boundary field at site 1 through (3.86).

**Remark 3.** It is interesting to consider the limit
\[ \psi_{-} \to +\infty \quad \text{with} \quad \varphi_{-} \sim \varsigma_{-} \quad \text{finite}, \] (5.54)
in which one recovers diagonal boundary conditions on both ends (in that case $\kappa_{-} \sim e^{-\psi_{-}} \to 0$). In appendix C, we explicitly show how the result of [108] for the correlation functions can be inferred from (5.48)–(5.51).

Finally, let us mention that, from such a formula, the homogeneous limit can be taken in both regimes of the chain completely similarly as in [108] (see formulas (6.15) and (6.18) of that paper).

### 6. Conclusion

Here we have shown how to compute correlation functions of a class of local operators for an open chain with unparallel boundary fields and with particular diagonal boundary conditions at the last site. The boundary field at the first site of the chain is here kept completely general.

Let us briefly summarize our method and results. The boundary matrix at site 1 can be made diagonal by means of the use of a vertex-IRF gauge transformation, and the spectrum and eigenstates of the corresponding transfer matrix can be described within the new SoV approach introduced in [54]. These eigenstates can also be rewritten in the form of generalized Bethe states constructed by means of the gauge transformed boundary Yang–Baxter algebra. On the one hand we can use, as in [108], a boundary-bulk decomposition of the generalized Bethe states, as well as the solution of the bulk inverse problem, to compute the action of a basis of local operators on these generalized Bethe states, the main difference being that all this should be done in terms of the gauged transformed boundary and bulk Yang–Baxter operators. On the other hand, since our special boundary conditions at site $N$ can be obtained as some limit of some more general non-diagonal ones for which the generalized Sklyanin’s SoV approach can be used, we can compute the resulting scalar products as some limit of the determinant representations obtained in [39] for the scalar products of separate states. As in [108], we hence obtain correlation functions as multiple sums of determinants (5.26) and, in the half-infinite chain, as multiple integrals (5.41). Let us also mention that, due to the fact the open chain that we consider is isospectral to a chain with diagonal boundary condition, the description of the ground state and computation of the scalar products have the same form than in [108]. Hence, our final results are very similar to the ones of [108], except that our explicit final formula involves the description of the boundary field at site 1 through two boundary parameters (instead of only one in [108]), and possibly through the contribution of two different isolated complex roots (boundary roots) converging towards these two boundary parameters. Finally, one should mention that the dependence into the boundary parameters at site 1 also appears in a slightly more intricate way via the gauge parameters (2.59).

Let us underline once again that we have here only computed the correlation of a particular class of local operators, the ones which satisfy the condition (5.3), i.e. which do not change the

\textsuperscript{15} And not only indirectly through the definition of the ground state, see [131].
number of gauged $B$-operators when acting on bulk (or boundary) generalized gauged Bethe vectors, see (4.142) and (4.151). The computation of completely general correlation functions indeed leads to more complicated types of scalar products, for which at the moment there exists no known simple determinant representation. We intend to consider this interesting but more delicate problem in a further study.

Finally, we expect that our study of correlation function can be adapted to the consideration of some other type of boundary conditions at site $N$, for instance non-diagonal ones with the constraint (3.46). We also intend to consider such problems in a further publication.

Acknowledgments
GN is supported by CNRS and Laboratoire de Physique, ENS-Lyon. VT is supported by CNRS.

Data availability statement
All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. On the bulk gauge Yang–Baxter algebra
In this appendix, we list some useful properties satisfied by (2.52).

A.1. A useful identity
From the definitions (2.52) and (2.53), one can show the following useful identity:

\[
\hat{M}(\lambda|(\alpha, \beta), (\gamma, \delta)) = \frac{\sigma_0^y [S^0(\lambda + \eta/2|\gamma, \delta)M^0(-\lambda)\sigma_0^yS(\lambda + \eta/2|\alpha, \beta)]^0}{(-1)^N \det S(\lambda + \eta/2|\gamma, \delta)} \\
= \frac{\sigma_0^y [\sigma_0^yS^0(\lambda + \eta/2|\alpha, \beta)\sigma_0^yM(-\lambda)S(\lambda + \eta/2|\gamma, \delta)]^0}{(-1)^N \det S(\lambda + \eta/2|\gamma, \delta)} \\
= \frac{\sigma_0^y[S^{-1}(\lambda - \eta/2|\alpha - 1, \beta)M(-\lambda)S(\lambda - \eta/2|\gamma - 1, \delta)]^0}{(-1)^N \det S(\lambda + \eta/2|\alpha, \beta) S(\lambda - \eta/2|\gamma - 1, \delta)} \\
= (-1)^N \det S(\lambda + \eta/2|\alpha, \beta) S(\lambda - \eta/2|\gamma, \delta) \sigma_0^yM^0(-\lambda)(\alpha - 1, \beta), (\gamma - 1, \delta))\sigma_0^y. \tag{A.1}
\]

or in components:

\[
\hat{M}(\lambda|(\alpha, \beta), (\gamma, \delta)) = (-1)^N e^{-\eta(\alpha - \gamma, \beta)} \sinh(\eta\beta) \sinh(\eta\delta) \\
\times \begin{pmatrix}
D(-\lambda|(\alpha - 1, \beta), (\gamma - 1, \delta)) & -B(-\lambda|(\alpha - 1, \beta), (\gamma - 1, \delta)) \\
-C(-\lambda|(\alpha - 1, \beta), (\gamma - 1, \delta)) & A(-\lambda|(\alpha - 1, \beta), (\gamma - 1, \delta))
\end{pmatrix}. \tag{A.2}
\]
A.2. Commutation relations

The bulk monodromy matrix (2.25) satisfies the following Yang–Baxter relation with the R-matrix (2.24):

\[ R_{12}(\lambda - \mu) M_1(\lambda) M_2(\mu) = M_2(\mu) M_1(\lambda) R_{12}(\lambda - \mu), \]  

(A.3)

Let us define

\[ S(\lambda |\alpha, \beta) = \begin{pmatrix} e^{\lambda + \eta(\beta + \alpha)} & e^{\lambda - \eta(\beta - \alpha)} \\ 1 & 1 \end{pmatrix} = S(\lambda | -\alpha, -\beta) \]  

(A.4)

and

\[ R^{SOS}(\lambda |\beta) = \begin{pmatrix} \sinh(\lambda - \eta) & 0 & 0 & 0 \\ 0 & \sinh(\eta(\beta + 1)) & \sinh(\lambda - \eta(\beta)) & \sinh(\eta) \\ 0 & \sinh(\eta(\beta - 1)) & \sinh(\eta(\beta + 1)) & \sinh(\lambda - \eta) \\ 0 & 0 & 0 & \sinh(\lambda - \eta) \end{pmatrix}. \]  

(A.5)

the analogue of (2.39) and (2.36) with \( \eta \) replaced by \( \bar{\eta} = -\eta \). We have

\[ R_{12}(\lambda - \mu) \bar{S}_2(-\mu|\alpha, \beta) \bar{S}_1(-\lambda|\alpha, \beta + \sigma_2^\gamma) = \bar{S}_1(-\lambda|\alpha, \beta) \bar{S}_2(-\mu|\alpha, \beta + \sigma_1^\beta) R^{SOS}_{21}(\lambda - \mu|\beta), \]  

(A.6)

which is the analogue of (2.38) with \( \eta \) replaced by \( \bar{\eta} = -\eta \). The latter relation implies that

\[ R^{SOS}_{21}(\lambda - \mu| -\delta) = S_2(-\mu - \eta/2|\gamma, \delta - \sigma_1^\gamma)^{-1} S_1(-\lambda - \eta/2|\gamma, \delta)^{-1} \times R_{12}(\lambda - \mu) S_2(-\mu - \eta/2|\gamma, \delta) S_1(-\lambda - \eta/2|\gamma, \delta - \sigma_2^\gamma), \]  

(A.7)

so that, multiplying (A.3) on the left by \( S_2(-\mu - \eta/2|\alpha, \beta - \sigma_1^\gamma)^{-1} S_1(-\lambda - \eta/2|\gamma, \delta)^{-1} \) and on the right by \( S_2(-\mu - \eta/2|\alpha, \beta - \sigma_2^\gamma)^{-1} S_1(-\lambda - \eta/2|\gamma, \delta)^{-1} \), we obtain:

\[ R^{SOS}_{21}(\lambda - \mu| -\delta) S_1(-\lambda - \eta/2|\alpha, \beta - \sigma_2^\gamma)^{-1} S_2(-\mu - \eta/2|\alpha, \beta - \sigma_2^\gamma)^{-1} M_1(\lambda) M_2(\mu) \times S_2(-\mu - \eta/2|\gamma, \delta) S_1(-\lambda - \eta/2|\gamma, \delta - \sigma_2^\gamma) = S_2(-\mu - \eta/2|\alpha, \beta - \sigma_1^\gamma)^{-1} S_1(-\lambda - \eta/2|\gamma, \delta)^{-1} \times M_2(\mu) M_1(\lambda) S_1(-\lambda - \eta/2|\gamma, \delta) S_2(-\mu - \eta/2|\gamma, \delta - \sigma_1^\gamma) R^{SOS}_{12}(\lambda - \mu| -\delta). \]  

(A.8)

Notice that

\[ R^{SOS}_{21}(\lambda| -\beta) = R^{SOS}_{12}(\lambda|\beta). \]  

(A.9)
The commutation relations of the matrix elements of (2.52) can be deduced from (A.8). In particular, we have

\[ A(\lambda)(\alpha, \beta - 1, \gamma, \delta - 1)A(\mu)(\alpha, \beta, \gamma, \delta) \]
\[ = A(\mu)(\alpha, \beta - 1, \gamma, \delta - 1)A(\lambda)(\alpha, \beta, \gamma, \delta), \quad (A.10) \]

\[ B(\lambda)(\alpha, \beta - 1, \gamma, \delta + 1)B(\mu)(\alpha, \beta, \gamma, \delta) \]
\[ = B(\mu)(\alpha, \beta - 1, \gamma, \delta + 1)B(\lambda)(\alpha, \beta, \gamma, \delta), \quad (A.11) \]

\[ C(\lambda)(\alpha, \beta + 1, \gamma, \delta - 1)C(\mu)(\alpha, \beta, \gamma, \delta) \]
\[ = C(\mu)(\alpha, \beta + 1, \gamma, \delta - 1)C(\lambda)(\alpha, \beta, \gamma, \delta), \quad (A.12) \]

\[ D(\lambda)(\alpha, \beta + 1, \gamma, \delta + 1)D(\mu)(\alpha, \beta, \gamma, \delta) \]
\[ = D(\mu)(\alpha, \beta + 1, \gamma, \delta + 1)D(\lambda)(\alpha, \beta, \gamma, \delta). \quad (A.13) \]

We also have

\[ A(\mu)(\alpha, \beta - 1, \gamma, \delta + 1)B(\lambda)(\alpha, \beta, \gamma, \delta) \]
\[ = \frac{\sinh(\lambda - \mu - \eta) \sinh(\eta \beta)}{\sinh(\lambda - \mu) \sinh(\eta \delta - 1)}B(\lambda)(\alpha, \beta - 1, \gamma, \delta - 1)A(\mu)(\alpha, \beta, \gamma, \delta) \]
\[ - \frac{\sinh \eta \sinh(\lambda - \mu - \eta \beta)}{\sinh(\eta \delta - 1) \sinh(\lambda - \mu)}B(\mu)(\alpha, \beta - 1, \gamma, \delta - 1)A(\lambda)(\alpha, \beta, \gamma, \delta), \quad (A.14) \]

and

\[ D(\lambda)(\alpha, \beta - 1, \gamma, \delta + 1)B(\mu)(\alpha, \beta, \gamma, \delta) \]
\[ = \frac{\sinh(\lambda - \mu - \eta) \sinh(\eta \beta)}{\sinh(\lambda - \mu) \sinh(\eta \delta - 1)}B(\mu)(\alpha, \beta + 1, \gamma, \delta + 1)D(\lambda)(\alpha, \beta, \gamma, \delta) \]
\[ + \frac{\sinh \eta \sinh(\lambda - \mu + \eta \beta)}{\sinh(\eta \beta - 1) \sinh(\lambda - \mu)}B(\lambda)(\alpha, \beta + 1, \gamma, \delta + 1)D(\mu)(\alpha, \beta, \gamma, \delta). \quad (A.15) \]

Using the notation (2.55), these commutation relations can be rewritten as

\[ A(\lambda)(\alpha - \beta + 1, \gamma + \delta - 1)A(\mu)(\alpha - \beta, \gamma + \delta) \]
\[ = A(\mu)(\alpha - \beta + 1, \gamma + \delta - 1)A(\lambda)(\alpha - \beta, \gamma + \delta), \quad (A.16) \]

\[ B(\lambda)(\alpha - \beta + 1, \gamma - \delta - 1)B(\mu)(\alpha - \beta, \gamma - \delta) \]
\[ = B(\mu)(\alpha - \beta + 1, \gamma - \delta - 1)B(\lambda)(\alpha - \beta, \gamma - \delta), \quad (A.17) \]

\[ C(\lambda)(\alpha + \beta + 1, \gamma + \delta - 1)C(\mu)(\alpha + \beta, \gamma + \delta) \]
\[ = C(\mu)(\alpha + \beta + 1, \gamma + \delta - 1)C(\lambda)(\alpha + \beta, \gamma + \delta), \quad (A.18) \]

\[ D(\lambda)(\alpha + \beta + 1, \gamma - \delta - 1)D(\mu)(\alpha + \beta, \gamma - \delta) \]
\[ = D(\mu)(\alpha + \beta + 1, \gamma - \delta - 1)D(\lambda)(\alpha + \beta, \gamma - \delta). \quad (A.19) \]
The vertex-IRF (A.7) can be easily extended to a transformation between bulk monodromy matrices:

\[
\tilde{M}^{\text{SOS}}(\lambda|\beta) = \tilde{S}_{1-N}(\{\xi\}|\alpha, \beta - \sigma_0) \tilde{S}_0(-\lambda - \eta/2|\alpha, \beta) M(\lambda) \times \tilde{S}_{1-N}(\{\xi\}|\alpha, \beta) S_0(-\lambda - \eta/2|\alpha, \beta - S'),
\]

(A.28)
in which
\[
\tilde{S}_{1,N}(\xi | \alpha, \beta) = \prod_{n=N+1} S_n \left( -\xi_n \bigg| \alpha, \beta - \sum_{j=n+1}^{N} \sigma_j^2 \right),
\]
(A.29)

and
\[
\tilde{M}^{\text{SOS}}(\lambda | \beta) = \prod_{n=N+1} \tilde{R}_{n0}^{\text{SOS}} \left( \lambda - \xi_n/2 \bigg| \beta + \sum_{j=n+1}^{N} \sigma_j^2 \right),
\]
\[
= \left( \tilde{A}^{\text{SOS}}(\lambda | \beta) \quad \tilde{B}^{\text{SOS}}(\lambda | \beta) \right).
\]
(A.30)
The relation (A.28) can also be written in terms of the matrix (2.52) as
\[
\tilde{M}^{\text{SOS}}(\lambda | \beta) = \tilde{S}_{1,n}^{-1}(\xi | \alpha, \beta - \sigma_0^2) S_0(\alpha, \beta | \alpha', \beta')
\times M(\lambda(\alpha', \beta'), (\gamma, \delta)) \tilde{S}_{1,n}(\xi | \alpha, \beta) S_0(\gamma, \delta | \alpha, \beta - \tilde{S}),
\]
(A.31)
in which
\[
S(\alpha, \beta | \alpha', \beta') = S^{-1}(\lambda | \alpha, \beta) S(\lambda | \alpha', \beta')
\]
\[
= \frac{e^{\eta \beta}}{\sinh(\eta \beta)} \begin{pmatrix}
 e^{\eta(\beta - \alpha)} & -e^{-\eta(\alpha + \beta)} & e^{\eta(\beta - \alpha')} & -e^{-\eta(\alpha' + \beta')}
\end{pmatrix}.
\]
(A.32)

Note that the matrix (A.32) does not depend on the spectral parameter \( \lambda \).

From (A.31) and from the fact that
\[
\tilde{A}^{\text{SOS}}(\lambda | \beta | 0) = d(\lambda) | 0 \rangle,
\]
\[
\tilde{B}^{\text{SOS}}(\lambda | \beta | 0) = \frac{\sinh(\eta(\beta - N))}{\sinh(\eta \beta)} a(\lambda) | 0 \rangle,
\]
\[
\tilde{C}^{\text{SOS}}(\lambda | \beta | 0) = 0,
\]
(A.33)\(\)\(\)\(\)\(\)

which follows from (A.30), one obtains that
\[
C(\lambda(\alpha, \beta), (\alpha, \beta - N)) \eta, \alpha + \beta \rangle = 0,
\]
(A.36)

\[
A(\lambda(\alpha, \beta), (\alpha, \beta - N)) \eta, \alpha + \beta \rangle = d(\lambda) | \eta, \alpha + \beta - 1 \rangle,
\]
(A.37)

\[
D(\lambda(\alpha, \beta), (\alpha, \beta - N)) \eta, \alpha + \beta \rangle = \frac{\sinh(\eta(\beta - N))}{\sinh(\eta \beta)} a(\lambda) | \eta, \alpha + \beta + 1 \rangle,
\]
(A.38)

and more generally, by using the expression (A.32), that
\[
A(\lambda(\alpha', \beta'), (\alpha, \beta - N)) \eta, \alpha + \beta \rangle = \frac{e^{\eta \beta} (e^{\eta(\beta - \alpha')} - e^{-\eta(\alpha + \beta)})}{2 \sinh(\eta \beta')} d(\lambda) \eta, \alpha + \beta - 1 \rangle,
\]
(A.39)

\[
D(\lambda(\alpha, \beta), (\gamma, \delta)) \eta, \alpha + \beta \rangle = \frac{e^{\eta \beta} (e^{\eta(\beta - \gamma)} - e^{-\eta(\alpha + \beta - N)})}{2 \sinh(\eta \beta)} a(\lambda) \eta, \alpha + \beta + 1 \rangle,
\]
(A.40)
in which

\[ |\eta, \alpha + \beta \rangle = \tilde{S}_{1,...,N}(\{\xi\}|\alpha, \beta)|0\rangle \]  \hspace{1cm} (A.41)

coincides with the definition (3.67). Similarly, from (A.31) and from the fact that

\[ \langle 0 | A^{SOS}(\lambda) - \beta \rangle = d(\lambda)(0|, \]  \hspace{1cm} (A.42)

\[ \langle 0 | D^{SOS}(\lambda) - \beta \rangle = \frac{\sinh(\eta(\beta - N))}{\sinh(\eta\beta)} a(\lambda)(0|, \]  \hspace{1cm} (A.43)

\[ \langle 0 | B^{SOS}(\lambda) - \beta \rangle = 0, \]  \hspace{1cm} (A.44)

one obtains that

\[ \langle \alpha - \beta + 1, \eta | B(\lambda|\alpha, \beta), (\alpha, \beta - N) \rangle = 0, \]  \hspace{1cm} (A.45)

\[ \langle \alpha - \beta + 1, \eta | A(\lambda|\alpha, \beta), (\gamma, \delta) \rangle = \frac{e^{\eta(\beta - N - \alpha)} - e^{-\eta(\alpha + \beta)}}{2 \sinh(\eta\beta)} d(\lambda)(\alpha - \beta, \eta|, \]  \hspace{1cm} (A.46)

\[ \langle \alpha - \beta + 1, \eta | D(\lambda|\alpha', \beta'), (\alpha, \beta - N) \rangle = \frac{e^{\eta(\beta - \alpha')} - e^{-\eta(\alpha + \beta')}}{2 \sinh(\eta\beta')} a(\lambda)(\alpha - \beta + 2, \eta|, \]  \hspace{1cm} (A.47)

in which

\[ \langle \alpha - \beta, \eta \rangle = e^{-\sum_{e_{\nu}} N_{\alpha,\beta,\eta} (0| \tilde{S}_{1,...,N}^{-1}(\{\xi\}|\alpha, \beta) \]  \hspace{1cm} (A.48)

coincides with the definition (3.66). With the normalization (2.55) and (A.36)–(A.40) can be rewritten as

\[ C(\lambda|\alpha + \beta, \alpha + \beta - N)|\eta, \alpha + \beta \rangle = 0, \]  \hspace{1cm} (A.49)

\[ A(\lambda|\alpha' - \beta', \alpha + \beta - N)|\eta, \alpha + \beta \rangle = \frac{e^{\eta(\beta' - \alpha') - \eta(\alpha + \beta)}}{e^{\eta/2}} d(\lambda)|\eta, \alpha + \beta - 1\rangle, \]  \hspace{1cm} (A.50)

\[ D(\lambda|\alpha + \beta, \gamma - \delta)|\eta, \alpha + \beta \rangle = \frac{e^{\eta(\gamma - \delta)} - e^{-\eta(\alpha + \beta - N)}}{e^{\eta/2}} a(\lambda)|\eta, \alpha + \beta + 1\rangle, \]  \hspace{1cm} (A.51)

and (A.45)–(A.47) can be rewritten as

\[ \langle \alpha - \beta + 1, \eta | B(\lambda|\alpha - \beta, \alpha - \beta + N) = 0, \]  \hspace{1cm} (A.52)

\[ \langle \alpha - \beta + 1, \eta | A(\lambda|\alpha - \beta, \gamma + \delta) \rangle = \frac{e^{\eta(\beta - \alpha) - \eta(\gamma + \delta)}}{e^{\eta/2}} d(\lambda)(\alpha - \beta, \eta|, \]  \hspace{1cm} (A.53)

\[ \langle \alpha - \beta + 1, \eta | D(\lambda|\alpha' + \beta', \alpha - \beta + N) \rangle = \frac{e^{\eta(\beta' - \alpha') - \eta(\alpha + \beta')}}{e^{\eta/2}} a(\lambda)(\alpha - \beta + 2, \eta|. \]  \hspace{1cm} (A.54)
A.4. Expression of the gauged operators in terms of a single operator

From the definition (2.52), all the elements of the gauged bulk monodromy matrix $M(\lambda \mid \alpha, \beta, \gamma, \delta)$ are linear combinations of the elements of $M(\lambda)$. More precisely, we have

$$
\begin{pmatrix}
A(\lambda \mid \alpha, \beta, \gamma, \delta) & B(\lambda \mid \alpha, \beta, \gamma, \delta) \\
C(\lambda \mid \alpha, \beta, \gamma, \delta) & D(\lambda \mid \alpha, \beta, \gamma, \delta)
\end{pmatrix} = \frac{(-\Theta(\lambda | \alpha - \beta, \gamma + \delta) - \Theta(\lambda | \alpha - \beta, \gamma - \delta))}{2 e^{-\lambda - \eta(\alpha + 1) / 2} \sinh \eta \beta},
$$

(A.55)

in which the operator $\Theta(\lambda \mid x, y)$ stands for the following $(x, y)$-dependant linear combination of the operators entries $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $D(\lambda)$ of $M(\lambda)$:

$$
\Theta(\lambda \mid x, y) = B(\lambda) + A(\lambda) e^{-\lambda - \eta / 2 - y} - C(\lambda) e^{-2\lambda - \eta - \eta x + 1} - D(\lambda) e^{-\lambda - \eta / 2 - y}.
$$

(A.56)

It is indeed an interesting point that all these gauged generators can be written in terms of a single operator, function of three parameters $\lambda, x, y$, and computed in different values of its parameters. It follows in particular that:

$$
B(\lambda \mid \alpha, \beta, \gamma, \delta) = A(\lambda \mid \alpha, \beta, \gamma, -\delta),
$$

(A.57)

$$
C(\lambda \mid \alpha, \beta, \gamma, \delta) = A(\lambda \mid \alpha, -\beta, \gamma, \delta),
$$

(A.58)

$$
D(\lambda \mid \alpha, \beta, \gamma, \delta) = A(\lambda \mid \alpha, -\beta, \gamma, -\delta).
$$

(A.59)

So that we can claim the following

Lemma A.1. There is just one (three parameters) generator $\Theta(\lambda \mid x, y)$ of the gauged Yang–Baxter algebra. The latter satisfies in particular the following two-shifts commutation relations:

$$
\Theta(\lambda \mid x + 1, y - 1) \Theta(\mu \mid x, y) = \Theta(\mu \mid x + 1, y - 1) \Theta(\lambda \mid x, y),
$$

(A.60)

and

$$
\Theta(\lambda \mid x + 1, y + 1) \Theta(\mu \mid x, z) = \frac{\sinh(\eta \frac{1 - \lambda}{2}) \sinh(\lambda - \mu + \eta)}{\sinh(\lambda - \mu) \sinh(\eta \frac{1}{2} + 1)} \Theta(\mu \mid x + 1, z + 1) \Theta(\lambda \mid x, y)
$$

$$
- \frac{\sinh \eta \sinh(\lambda - \mu + \frac{1 - \lambda}{2} \eta)}{\sinh(\lambda - \mu) \sinh(\eta \frac{1}{2} + 1)} \Theta(\lambda \mid x + 1, z + 1) \Theta(\mu \mid x, y),
$$

(A.61)

and

$$
\Theta(\lambda \mid x - 1, y - 1) \Theta(\mu \mid w, y)
$$

$$
= \frac{\sinh(\eta \frac{1 - \lambda}{2}) \sinh(\lambda - \mu - \eta)}{\sinh(\lambda - \mu) \sinh(\eta \frac{1}{2} + 1)} \Theta(\mu \mid w - 1, y - 1) \Theta(\lambda \mid x, y)
$$

$$
+ \frac{\sinh \eta \sinh(\lambda - \mu + \frac{1 - \lambda}{2} \eta)}{\sinh(\lambda - \mu) \sinh(\eta \frac{1}{2} + 1)} \Theta(\lambda \mid w - 1, y - 1) \Theta(\mu \mid x, y).
$$

(A.62)

We have also the following lemma on the action of the gauged Yang–Baxter generator on the reference states:
Lemma A.2. The following identities hold:

\[
\langle x+1, \eta | \Theta(\lambda| x, x+N) = 0, \quad (A.63)
\]

\[
\langle x+1, \eta | \Theta(\lambda| x, y) = \frac{e^{-\eta y} - e^{-(x+N)\eta}}{e^{x+\eta/2}} d(\lambda) \langle x, \eta |, \quad (A.64)
\]

\[
\langle x+1, \eta | \Theta(\lambda| y, x+N) = \frac{e^{-\eta y} - e^{-y\eta}}{e^{y+\eta/2}} d(\lambda) \langle x+2, \eta |, \quad (A.65)
\]

and similarly,

\[
\Theta(\lambda| x, x-N| \eta, x = 0, \quad (A.66)
\]

\[
\Theta(\lambda| y, x-N| \eta, y = \frac{e^{-\eta y} - e^{-y\eta}}{e^{x+\eta/2}} d(\lambda) \langle \eta, x = 1), \quad (A.67)
\]

\[
\Theta(\lambda| x, y| \eta, x = \frac{e^{-\eta y} - e^{-\eta y-x-N\eta}}{e^{x+\eta/2}} d(\lambda) \langle \eta, x + 1). \quad (A.68)
\]

By using these identities, we can write the action of the gauge transformed Yang–Baxter generator \(\Theta(\lambda| x, y)\) on a Bethe-like vector of the following form:

\[
\Theta_M^M(\lambda_j|_{j=1}^M| \alpha - \beta - 1, \gamma - \delta + 1 - 2M| \eta, \gamma + \delta + N), \quad (A.69)
\]

in which we have defined

\[
\Theta_M^M(\lambda_j|_{j=1}^M| x - 1, y + 1) = \prod_{j=1}^{\alpha=\delta} \Theta(\lambda_j| x - j, y + j)
\]

\[
= \Theta(\lambda_1| x - 1, y + 1) \Theta(\lambda_2| x - 2, y + 2) \ldots
\]

\[
\ldots \Theta(\lambda_M| x - M + 1, y + M - 1) \Theta(\lambda_M| x - M, y + M).
\]

(A.70)

Note that, due to the pseudo-commutativity (A.60) of the gauged \(\Theta\)-operator, the above operator product is independent w.r.t. the order of the \(\lambda_j\)’s, i.e.

\[
\prod_{j=1}^{\alpha=\delta} \Theta(\lambda_j| x - j, y + j) = \prod_{j=1}^{\alpha=\delta} \Theta(\lambda_{\pi j}| x - j, y + j), \quad (A.71)
\]

where \(\pi\) is any permutation of the set \(\{1, \ldots, M\}\), so that the notation (A.70) is indeed coherent. Then, the following lemma holds:

Lemma A.3. The actions of the gauged Yang–Baxter generator on a Bethe state of the form (A.69) reads:

\[
\Theta(\lambda_M| x - \beta - \gamma + M) \Theta_M^M(\lambda_j|_{j=1}^M| x - \beta - 1, \gamma - \delta + 1 - 2M| \eta, \gamma + \delta + N)
\]

\[
= \frac{e^{-\gamma y} - e^{-(x+M\gamma)\eta}}{e^{x+y/2}} \sum_{a=1}^{M+1} \theta(\lambda_a| x - \lambda_a + \eta| \gamma - \delta + 1 + \frac{N}{2})
\]

\[
\times \prod_{j=1}^{M+1} \sinh(\lambda_j - \lambda_j + \eta) \Theta_M^M(\lambda_j|_{j=1}^M| x - \beta, \gamma - \delta + 2 - 2M| \eta, \gamma + \delta + N - 1), \quad (A.72)
\]
and
\[
\Theta(\alpha + \beta, \gamma + \delta + M) B_{\tilde{M}}(\{\lambda_j\}_{j=1}^{M} | \alpha - \beta - 1, \gamma - \delta + 1 - 2M) \eta, \gamma + \delta + N)
\]
\[
= \frac{e^{-\gamma M \eta} - e^{-(\gamma + M \eta)}}{e^{\eta / 2}} \frac{\sinh(\beta \eta)}{\sinh(\gamma M \eta)} \prod_{j=1}^{M} \frac{\sinh(\lambda_j + \eta(\delta - 1 + \frac{3M}{2}))}{\sinh(\lambda_j + \frac{\eta}{2})} \prod_{j \neq n} \frac{\sinh(\lambda_n - \lambda_j + \eta)}{\sinh(\lambda_n - \lambda_j)}
\]
\[
\times \frac{\sinh(\lambda_n - \lambda_n + \eta(\delta - 1 + \frac{3M}{2}))}{\sinh(\lambda_n - \lambda_n)} \prod_{j \neq n} \frac{\sinh(\lambda_n - \lambda_j + \eta)}{\sinh(\lambda_n - \lambda_j)}
\]
\[
\times \frac{\sinh(\lambda_n - \lambda_n + \eta(\delta - 1 + \frac{3M}{2}))}{\sinh(\lambda_n - \lambda_n)} \prod_{j \neq n} \frac{\sinh(\lambda_n - \lambda_j + \eta)}{\sinh(\lambda_n - \lambda_j)}
\]
\[
\times \frac{\sinh(\lambda_n - \lambda_n + \eta(\delta - 1 + \frac{3M}{2}))}{\sinh(\lambda_n - \lambda_n)} \prod_{j \neq n} \frac{\sinh(\lambda_n - \lambda_j + \eta)}{\sinh(\lambda_n - \lambda_j)}
\]
\[
\times \frac{\sinh(\lambda_n - \lambda_n + \eta(\delta - 1 + \frac{3M}{2}))}{\sinh(\lambda_n - \lambda_n)} \prod_{j \neq n} \frac{\sinh(\lambda_n - \lambda_j + \eta)}{\sinh(\lambda_n - \lambda_j)}
\]
\[
\times \frac{\sinh(\lambda_n - \lambda_n + \eta(\delta - 1 + \frac{3M}{2}))}{\sinh(\lambda_n - \lambda_n)} \prod_{j \neq n} \frac{\sinh(\lambda_n - \lambda_j + \eta)}{\sinh(\lambda_n - \lambda_j)}
\]
(\ref{A.75})

which gives (\ref{A.72}), (\ref{A.73}) can be shown similarly by means of (\ref{A.62}) and (\ref{A.68}).

Lemma A.3 can be rewritten in terms of the more usual notations A, B, C, D (2.55), and provides the action of the gauged operators A and D on a bulk gauge Bethe state:

**Lemma A.4.** The actions of the diagonal gauge transformed Yang–Baxter generators on a gauged bulk Bethe state read:

\[
A(\alpha + \beta, \gamma + \delta + M) B_{\tilde{M}}(\{\lambda_j\}_{j=1}^{M} | \alpha - \beta - 1, \gamma - \delta + 1 - 2M) \eta, \gamma + \delta + N)
\]
\[
= \frac{e^{-(\gamma + M \eta)}}{e^{\eta / 2}} \sum_{a=1}^{M+1} \frac{\sinh(\lambda_a + \eta(\delta - 1 + \frac{3M}{2}))}{\sinh(\lambda_a + \frac{\eta}{2})} \prod_{j \neq n} \frac{\sinh(\lambda_n - \lambda_j + \eta)}{\sinh(\lambda_n - \lambda_j)}
\]
\[
\times \frac{\sinh(\lambda_n - \lambda_n + \eta(\delta - 1 + \frac{3M}{2}))}{\sinh(\lambda_n - \lambda_n)} \prod_{j \neq n} \frac{\sinh(\lambda_n - \lambda_j + \eta)}{\sinh(\lambda_n - \lambda_j)}
\]
\[
\times \frac{\sinh(\lambda_n - \lambda_n + \eta(\delta - 1 + \frac{3M}{2}))}{\sinh(\lambda_n - \lambda_n)} \prod_{j \neq n} \frac{\sinh(\lambda_n - \lambda_j + \eta)}{\sinh(\lambda_n - \lambda_j)}
\]
\[
\times \frac{\sinh(\lambda_n - \lambda_n + \eta(\delta - 1 + \frac{3M}{2}))}{\sinh(\lambda_n - \lambda_n)} \prod_{j \neq n} \frac{\sinh(\lambda_n - \lambda_j + \eta)}{\sinh(\lambda_n - \lambda_j)}
\]
\[
\times \frac{\sinh(\lambda_n - \lambda_n + \eta(\delta - 1 + \frac{3M}{2}))}{\sinh(\lambda_n - \lambda_n)} \prod_{j \neq n} \frac{\sinh(\lambda_n - \lambda_j + \eta)}{\sinh(\lambda_n - \lambda_j)}
\]
\[
\times \frac{\sinh(\lambda_n - \lambda_n + \eta(\delta - 1 + \frac{3M}{2}))}{\sinh(\lambda_n - \lambda_n)} \prod_{j \neq n} \frac{\sinh(\lambda_n - \lambda_j + \eta)}{\sinh(\lambda_n - \lambda_j)}
\]
\[
\times \frac{\sinh(\lambda_n - \lambda_n + \eta(\delta - 1 + \frac{3M}{2}))}{\sinh(\lambda_n - \lambda_n)} \prod_{j \neq n} \frac{\sinh(\lambda_n - \lambda_j + \eta)}{\sinh(\lambda_n - \lambda_j)}
\]
These actions can be rewritten as

\[
A(\lambda_{M+1}|x,y + M)B_M(\{\lambda\}_{j=1}^M|x - 1, z + 1|\eta, y + N) = e^{-i\zeta - M}\rho - e^{-i\zeta + N}\rho \sum_{a=1}^{M+1} d(\lambda_a) \frac{\sinh(\lambda_{M+1} - \lambda_a + \frac{\eta y - z + 2M}{2})}{\sinh(\frac{\eta y - z + 2M}{2})} \prod_{k=1}^{M} \sinh(\lambda_a - \lambda_k + \eta) \prod_{j=1}^{M+1} \sinh(\lambda_a - \lambda_j - \eta) \times B_M(\{\lambda\}_{j=1}^M|x, z + 2|\eta, y + N - 1),
\]

and

\[
D(\lambda_{M+1}|y - M, z) B_M(\{\lambda\}_{j=1}^M|x - 1, z + 1|\eta, y) = e^{-i\zeta - M}\rho - e^{-i\zeta + N}\rho \sum_{a=1}^{M+1} d(\lambda_a) \frac{\sinh(\lambda_{M+1} - \lambda_a + \frac{\eta y - z + 2M}{2})}{\sinh(\frac{\eta y - z + 2M}{2})} \prod_{k=1}^{M} \sinh(\lambda_a - \lambda_k + \eta) \prod_{j=1}^{M+1} \sinh(\lambda_a - \lambda_j - \eta) \times B_M(\{\lambda\}_{j=1}^M|x, z + 2|\eta, y + N - 1),
\]

\[
(A.76)
\]

\[
(A.77)
\]

\[
(A.78)
\]
Here, we construct an $A - B$. 1. Construction of an $U - U$ with the notation

we recall that the elements of the gauge boundary monodromy matrix Appendix B. On the boundary-bulk decomposition

w.r.t. In that case, in terms of the gauged boundary matrix (2.54). In this appendix, we prove several consequences of this boundary bulk decomposition. Let us remark that the matrix $K_-(\lambda(\gamma, \delta), (\gamma', \delta'))$ can be made upper triangular identically w.r.t. $\lambda$ if

$\gamma + \delta = \gamma' + \delta'$.  

In that case,

$[K_-(\lambda(\gamma, \delta), (\gamma', \delta'))]_{21} = 0$  

is equivalent to

$\frac{K_+}{\sinh \zeta_+} \left( \sinh(\eta(\gamma + \delta) + \tau_+) + \sinh(\varphi_+ + \psi_+) \right) = 0$.  

Similarly, the matrix $K_-(\lambda(\gamma, \delta), (\gamma', \delta'))$ can be made lower triangular identically w.r.t. $\lambda$ if

$\gamma - \delta = \gamma' - \delta'$.  

In that case,

$[K_-(\lambda(\gamma, \delta), (\gamma', \delta'))]_{12} = 0$  

is equivalent to

$\frac{K_+}{\sinh \zeta_+} \left( \sinh(\eta(\gamma - \delta) + \tau_+) + \sinh(\varphi_+ + \psi_+) \right) = 0$.  

B.1. Construction of an $A_-(\lambda|\alpha, \beta - 1)$ and $D_-(\lambda|\alpha, \beta + 1)$ eigenvector

Here, we construct an $A_-(\lambda|\alpha, \beta - 1)$ right eigenvector and a $D_-(\lambda|\alpha, \beta + 1)$ left eigenvector, for a special choice of the gauge parameters $(\alpha, \beta)$.

Lemma B.1. Under the condition

$\frac{K_+}{\sinh \zeta_+} \left[ \sinh(\eta(\alpha + \beta + N - 1) + \tau_+) + \sinh(\varphi_+ + \psi_+) \right] = 0$.  

(A.80)

Appendix B. On the boundary-bulk decomposition

We recall that the elements of the gauge boundary monodromy matrix $\mathcal{U}_-(\lambda|\alpha, \beta)$ can be expressed in terms of the elements of the gauge bulk monodromy matrix by formula (2.51), which can also be written by components as

$\mathcal{U}_-(\lambda|\alpha, \beta) = \frac{(-1)^N}{4 \sinh \eta \sinh \eta'} \begin{pmatrix}
D(\lambda|\delta + 1, \alpha - \beta - 1) & -B(\lambda|\gamma - \delta - 1, \alpha - \beta - 1) \\
-C(\lambda|\gamma - \delta + 1, \alpha + \beta - 1) & A(\lambda|\gamma - \delta - 1, \alpha + \beta - 1)
\end{pmatrix}

\times K_-(\lambda|\gamma - \delta, (\gamma', \delta')) \begin{pmatrix}
A(-\lambda|\gamma' - \delta', \alpha + \beta) \\
B(-\lambda|\gamma' - \delta', \alpha - \beta)
\end{pmatrix}

\times \begin{pmatrix}
C(-\lambda|\gamma' + \delta', \alpha + \beta) \\
D(-\lambda|\gamma' + \delta', \alpha - \beta)
\end{pmatrix}$.  

in terms of the gauged boundary matrix (2.54). In this appendix, we prove several consequences of this boundary bulk decomposition.

74
the vector $|\eta, \alpha + \beta + N - 1\rangle$ is an $A_- (\lambda |\alpha, \beta - 1\rangle$ eigenvector:

$$A_- (\lambda |\alpha, \beta - 1\rangle |\eta, \alpha + \beta + N - 1\rangle) = (-1)^N a(\lambda) d(-\lambda) [K_- (\lambda |\alpha, \beta + N - 1\rangle, (\alpha, \beta + N - 1)]]_{11} |\eta, \alpha + \beta + N - 1\rangle,$$  \hspace{1cm} (B.9)

and is annihilated by $C_- (\lambda |\alpha, \beta - 1\rangle$:

$$C_- (\lambda |\alpha, \beta - 1\rangle |\eta, \alpha + \beta + N - 1\rangle = 0.$$  \hspace{1cm} (B.10)

Under the condition

$$\frac{K_+}{\sinh \zeta_+} [\sinh(\eta (\alpha + \beta - N + 1) + \tau_+) + \sinh(\varphi_+ + \psi_+)] = 0,$$  \hspace{1cm} (B.11)

the covector $\langle \alpha + \beta - N + 1, \eta |D_- (\lambda |\alpha, \beta + 1\rangle$ is an $D_- (\lambda |\alpha, \beta + 1\rangle$ eigencovector:

$$\langle \alpha + \beta - N + 1, \eta |D_- (\lambda |\alpha, \beta + 1\rangle = (-1)^N a(\lambda) d(-\lambda) [K_- (\lambda |\alpha, \beta - N + 1\rangle, (\alpha, \beta - N + 1)]_{22} |\eta, \alpha + \beta - N + 1, \eta \rangle.$$  \hspace{1cm} (B.12)

**Proof.** To prove (B.9) and (B.10), let us use the boundary bulk decomposition of the gauged monodromy given in (B.1), where we have fixed the internal parameters by

$$\gamma = \gamma' = \alpha, \quad \delta = \delta' = \beta + N - 1.$$  \hspace{1cm} (B.13)

Then under the conditions (B.8), the matrix $K_- (\lambda |\alpha, \beta + N - 1\rangle, (\alpha, \beta + N - 1)\rangle$ is upper triangular. Noticing moreover that

$$C(-\lambda |\alpha + \beta + N - 1, \alpha + \beta - 1\rangle |\eta, \alpha + \beta + N - 1\rangle = 0,$$  \hspace{1cm} (B.14)

we obtain that

$$A_- (\lambda |\alpha, \beta - 1\rangle |\eta, \alpha + \beta + N - 1\rangle = \frac{(-1)^N e^{2\eta} e^{2\eta} [K_- (\lambda |\alpha, \beta + N - 1\rangle, (\alpha, \beta + N - 1)]_{11}}{4 \sinh(\eta (\beta - 1)) \sinh(\eta (\beta + N - 1))} \times D(\lambda |\alpha + \beta + N - 2, \alpha - \beta) A(-\lambda |\alpha - \beta + N + 1, \alpha + \beta - 1\rangle \times |\eta, \alpha + \beta + N - 1\rangle.$$  \hspace{1cm} (B.15)

By using the following identities:

$$A(-\lambda |\alpha - (\beta + N - 1), \alpha + \beta - 1\rangle |\eta, \alpha + \beta + N - 1\rangle = \frac{2 \sinh \eta (\beta + N - 1)}{e^{\eta (\alpha + 1/2)}} d(-\lambda |\eta, \alpha + \beta + N - 2\rangle,$$  \hspace{1cm} (B.16)

and

$$D(\lambda |\alpha + \beta + N - 2, \alpha - \beta) |\eta, \alpha + \beta + N - 2\rangle = \frac{2 \sinh \eta (\beta - 1)}{e^{\eta (\alpha - 1/2)}} a(\lambda |\eta, \alpha + \beta + N - 1\rangle,$$  \hspace{1cm} (B.17)
we can compute the rhs of (B.15). (B.10) is a consequence of the identity:
\[
C(\lambda|\alpha + \beta + N - 2, \alpha + \beta - 2)A(-\lambda|\alpha - \beta - N + 1, \alpha + \beta - 1)\eta, \alpha + \beta + N - 1) \\
\propto C(\lambda|\alpha + \beta + N - 2, \alpha + \beta - 2)\eta, \alpha + \beta + N - 2) = 0. \tag{B.18}
\]

One can prove similarly, using (B.1) in the case (B.6), that, under the condition
\[
\frac{\kappa_+}{\sinh \zeta_+} \left[ \sinh(\eta(\alpha - \beta + N + 1 + \tau_+) + \sinh(\varphi_+ + \psi_+) \right] = 0, \tag{B.19}
\]
the covector \( \langle \alpha - \beta + N + 1, \eta \rangle \) is an \( \mathcal{A}_-(\lambda|\alpha + \beta - 1) \) eigenvector:
\[
\langle \alpha - \beta - N + 1, \eta \mathcal{A}_-(\lambda|\alpha, \beta - 1) = (-1)^N a(\lambda)d(-\lambda)[K_-(\lambda|\alpha, \beta + N - 1), \rangle \tag{B.20}
\]
Since, for any gauge parameter \( \beta' \), one has the identification
\[
\mathcal{A}_-(\lambda|\alpha, \beta') = \mathcal{D}_-(\lambda|\alpha, -\beta'), \tag{B.21}
\]
\[
[K_-(\lambda|\alpha, \beta'), (\alpha, \beta')])_{11} = [K_-(\lambda|\alpha, -\beta'), (\alpha, -\beta')])_{22}, \tag{B.22}
\]
this proves the second part of the lemma.

Note that these vectors have been first constructed as generalized reference states for the gauge transformed boundary monodromy matrix in the framework of a generalization of ABA for the open chain [117]. Here, they enter in our SoV description of the transfer matrix spectrum, where no ansatz on the eigenstate construction is done and where the complete spectrum description is a built-in feature (see proposition 3.4).

### B.2. Boundary-bulk decomposition of boundary gauge Bethe states

Here, we present the boundary-bulk decomposition of ABA states of the form
\[
\hat{B}_-(\lambda|z + 1|z + 2j - 1|\eta, y) \equiv \prod_{j=1}^{\mathcal{M}} \hat{B}_-(\lambda|z + 2j - 1|\eta, y) \tag{B.23}
\]
for the very specific boundary conditions that we consider in the framework of this paper, i.e. the one associated to the choice of the boundary matrix (3.80). Then under the gauge transformation for any choice \( (\gamma, \delta) \), we have
\[
K_-(\lambda, \zeta_+ = -\infty, \kappa_+, \tau_+|\gamma, \delta) = e^{(\lambda-\eta)/2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{B.24}
\]

**Lemma B.2.** The boundary operator \( \hat{B}_-(\lambda|\alpha - \beta) \) can be expressed in terms of the gauge bulk operators elements of (2.55) as
\[
\hat{B}_-(\lambda|\alpha - \beta) = \frac{(-1)^N e^{\delta(\gamma + \alpha - \beta)}}{4 \sinh \gamma (\delta + 1)} \sinh(2\lambda - \eta) \times \left[ B(\lambda|\gamma - \delta - 1, \alpha - \beta - 1)D(\lambda|\gamma + \delta, \alpha - \beta) \\
- B(\lambda|\gamma - \delta - 1, \alpha - \beta - 1)D(-\lambda|\gamma + \delta, \alpha - \beta) \right], \tag{B.25}
\]

76
for any gauge parameters $\alpha, \beta, \gamma, \delta$, which can be equivalently rewritten as

$$\hat{B}_-(\lambda|z) = \frac{(-1)^N e^{\frac{i}{4} \pi N z}}{4 \sinh(\eta)} \left[ \sinh(2\lambda - \eta) \sum_{\sigma = \pm} \sigma \beta \sigma \geq 1, z - 1 \right] \left[ \sinh(2\lambda) \right]$$

for any gauge parameter $z, x, y$.

**Proof.** By definition, it holds

$$\hat{B}_-(\lambda|\alpha - \beta) = \frac{(-1)^N e^{\frac{i}{4} \pi N z}}{4 \sinh(\eta)} \left[ \sinh(2\lambda - \eta) \sum_{\sigma = \pm} \sigma \beta \sigma \geq 1, \alpha - \beta - 1 \right] \left[ \sinh(2\lambda) \right]$$

and the following identity:

$$\sinh(\eta \sinh(2\lambda + \delta \eta)) \sinh(2\lambda) \sinh(\eta(\delta + 1) = -1 = \frac{\sinh(\eta \sinh(2\lambda - \eta)) \sinh(2\lambda) \sinh(\eta(\delta + 1))}{\sinh(2\lambda)}.$$  \[ \text{(B.29)} \]

It follows from this result that boundary gauge Bethe states of the form \( \hat{B}_-(\lambda|\alpha - \beta) \), for any gauge parameters $z$ and $y$, can be expressed as linear combinations of bulk gauge Bethe states of the form

$$\mathcal{B}_M(\{\sigma_i \lambda_i\}_{i=1}^M| x - 1, z) \eta, y + M \geq 1 \right] \left[ \sinh(2\lambda) \right]$$

for any choice of the gauge parameter $x$, where $\sigma_i \in \{+, -\}, i = 1, \ldots, M$. More precisely, we can formulate the following result:

**Proposition B.1.** Boundary gauge Bethe states of the form \( \hat{B}_-(\lambda|\alpha - \beta) \), for any arbitrary set of spectral parameters \( \{\lambda_i, \ldots, \lambda_M\} \) and any choice of gauge parameters $z$ and $y$, can be expressed as the following linear combination of bulk gauge Bethe states:

$$\hat{B}_- \left( \{\lambda_i\}_{i=1}^M | z + 1 \right) \eta, y \geq 1 \right] \left[ \sinh(2\lambda) \right]$$

with

$$H_{\sigma_1, \ldots, \sigma_M}(\{\lambda_i\}_{i=1}^M) = \prod_{j=1}^M \left[ \sinh(\lambda_j - \eta) \sinh(2\lambda_j) \right]$$

and

$$\left( \sinh(\lambda^\sigma_j + \lambda_j + \eta) \right) \sinh(\lambda^\sigma_j + \lambda_j + \eta).$$  \[ \text{(B.32)} \]
in which we have used the shortcut notation \( \lambda_j^{(g)} \equiv \sigma_j \lambda_j \), \( 1 \leq j \leq M \), and where \( h_M(x, z, y) \) is an overall non-zero constant depending exclusively on the gauge parameters \( x, z, y \) and on the number \( M \) of \( \hat{B} \) operators. More precisely we have

\[
h_M(x, z, y) = (-1)^M e^{\frac{\pi M}{2}} \frac{\sinh \left( \frac{\pi (y + M - 2)}{2} \right)}{\sinh \left( \frac{\pi (y - M - 2)}{2} \right)}. \tag{B.33}
\]

**Proof.** The proof can be done by induction on \( M \) following the same lines as in the ungauged case \([108]\).

For \( M = 1 \), (B.31) is a direct consequence of (B.26) and of the action (A.51) of the gauge operator \( D \) on the gauge reference state. Let us now assume (B.31) for a given \( M \) and let us prove it for \( M + 1 \). Then, from (B.26) and the recursion hypothesis, we can write

\[
\hat{B}_{-M+1}(\{\lambda_j\}_{j=1}^{M+1}|z+1, \eta, y) = \hat{B}_{-M}(\{\lambda_j\}_{j=1}^{M}|z+1, \eta, y)
\]

\[
= h_M(x, z + 2, y) \sum_{\sigma_1, \ldots, \sigma_M = \pm} H_{\sigma_1, \ldots, \sigma_M}(\{\lambda_j\}_{j=1}^{M}) \hat{B}_{-M}(\{\lambda_j\}_{j=1}^{M}|z+3, \eta, y + M)
\]

\[
= h_M(x, z + 2, y) (-1)^M e^{\frac{\pi}{2}} \frac{\sinh(2\lambda_{M+1} - \eta)}{\sinh(2\lambda_{M+1})}
\]

\[
\times \sum_{\sigma_1, \ldots, \sigma_M = \pm} H_{\sigma_1, \ldots, \sigma_M}(\{\lambda_j\}_{j=1}^{M}) \sinh(2\lambda_{M+1} - \eta)
\]

\[
\times \sum_{\sigma} \sigma B(\sigma \lambda_{M+1}|y, z+1) B_{M}(\{\lambda_j\}_{j=1}^{M}|x - 1, z + 2, \eta, y + M)
\]

\[
\times (\{\sigma_j \lambda_{j+1}\}_{j=1}^{M}|x - 1, z + 2, \eta, y + M). \tag{B.34}
\]

From (A.79), it is easy to verify that the direct action of \( D(\sigma \lambda_{M+1}|y, z+1) \) leads to the contribution

\[
h_{M+1}(x, z, y) = h_M(x, z + 2, y) (-1)^M e^{\frac{\pi}{2}} \frac{\sinh(2\lambda_{M+1} - \eta)}{\sinh(2\lambda_{M+1})} e^{-\frac{\pi}{2}}
\]

\[
= (-1)^M e^{\frac{\pi}{2}} \frac{\sinh(\eta y + M - 1)}{\sinh(\eta y - M - 1)} h_M(x, z + 2, y), \tag{B.36}
\]

\[
H_{\sigma_1, \ldots, \sigma_{M+1}}(\{\lambda_j\}_{j=1}^{M+1}) = \sigma_{M+1} a(\sigma_{M+1}^{(g)}) \sinh(2\lambda_{M+1} - \eta) \prod_{j=1}^{M} \sinh(\eta - \lambda_j^{(g)}) \sinh(\eta - \lambda_j^{(g)})
\]

\[
\times H_{\sigma_1, \ldots, \sigma_M}(\{\lambda_j\}_{j=1}^{M}), \tag{B.37}
\]
which gives the left-hand side of (B.31) for $M + 1$. Hence, it remains to prove that the contributions due to the indirect terms in the action of $D(\sigma \lambda_{M+1}|y, z + 1)$ vanish. These terms lead to bulk gauge Bethe vectors of the form

$$B_{M+1}(\{\lambda_i^{(\sigma)}\}_{i=1}^M \cup \{\lambda_{M+1}^{(\sigma)}, -\lambda_{M+1}\} \\setminus \{\lambda_{a_i}^{(\sigma)}\} | x - 1, z | \eta, y + M + 1) \quad (B.38)$$

for some $a \in \{1, \ldots, M\}$, and it is easy to see that the coefficient of such a vector contains a factor of the form

$$\frac{\sigma_a \sigma_{M+1} \sinh(\lambda_{a,M+1}^{(\sigma)} + \lambda_{M+1}^{(\sigma)} + \kappa)}{\sinh(\lambda_{a,M+1}^{(\sigma)} + \lambda_{M+1}^{(\sigma)})}$$

This expression vanishes due to the fact that

$$\frac{\sinh(\lambda_{a,M+1} + \kappa)}{\sinh \lambda_{a,M+1}} = \frac{\sinh(\lambda_{a,M+1} - \kappa)}{\sinh \lambda_{a,M+1}} = \frac{\sinh(\lambda_{a,M+1} + \kappa)}{\sinh \lambda_{a,M+1}}$$

(B.39)

where

$$\kappa = \eta \frac{x - y - 2}{2}. \quad (B.40)$$

This expression vanishes due to the fact that

$$\frac{\sinh(\lambda + \kappa)}{\sinh \lambda} + \frac{\sinh(\lambda - \kappa)}{\sinh \lambda} = 2 \cosh \kappa, \quad (B.41)$$

which completes the proof.

\[\square\]

**Appendix C. The diagonal limit**

In this appendix, we explicitly show how the expressions (6.8)–(6.10) of [108] for the correlation function in the diagonal case can be inferred by taking the limit

$$\psi_- \to +\infty \quad \text{with} \quad \varphi_- \sim \zeta_- \quad \text{finite}, \quad (C.1)$$

in our result (5.48)–(5.51). In this limit, $\kappa_- \sim e^{-\psi_-} \to 0$, so that we indeed have diagonal boundary conditions at both ends of the chain.

Let us fix for instance $\epsilon_{\varphi_-} = \epsilon_{\psi_-} = 1$. From (5.2), it follows that, in the limit (C.1),

$$\eta \beta \sim \eta b_n \sim \eta \tilde{b}_n \sim -\psi_- \to -\infty. \quad (C.2)$$

Hence we have

$$-2 \sinh(\eta b_n) e^{-\eta} E_n^{1,1}\langle \zeta_n \rangle(a_n, b_n, (a_n, \tilde{b}_n)) \sim e^{-\eta b_n} E_n^{1,1}, \quad (C.3)$$

$$-2 \sinh(\eta b_n) e^{-\eta} E_n^{1,2}\langle \zeta_n \rangle(a_n, b_n, (a_n, \tilde{b}_n)) \sim e^{-\eta b_n + \hat{a}_n + \hat{e}_n - \zeta_n} E_n^{1,2}, \quad (C.4)$$

$$-2 \sinh(\eta b_n) e^{-\eta} E_n^{2,1}\langle \zeta_n \rangle(a_n, b_n, (a_n, \tilde{b}_n)) \sim e^{\hat{a}_n + \hat{e}_n} E_n^{2,1}, \quad (C.5)$$

$$-2 \sinh(\eta b_n) e^{-\eta} E_n^{2,2}\langle \zeta_n \rangle(a_n, b_n, (a_n, \tilde{b}_n)) \sim e^{-\eta b_n + 1} E_n^{2,2}. \quad (C.6)$$
so that
\[
\prod_{n=1}^{m} E_n^{s,s_n} \sim (-2)^m \prod_{n=1}^{m} \left[ e^{-\eta \sinh(\eta b_n)} \right] \prod_{p=1}^{s} e^{-\xi_p - \eta (b_{p-1})} \times \prod_{p=s+1}^{m} E_n^{s,s_n} (\xi_n | (a_n, b_n), (\tilde{a}_n, \tilde{b}_n))
\]
(C.7)
in which we have used that \(s + s' = m\). On the other hand, from (5.51), we get in the limit
\[
\tilde{H}_m^{\text{diag}}(\{\lambda_j\}_{j=1}^{M}; \{\tilde{\xi}_k\}_{k=1}^{m}) \sim \prod_{1 \leq i < j \leq m} \sinh(\lambda_i + \tilde{\xi}_j - \eta) \sinh(\lambda_j + \lambda_i - \eta) \\
\times \prod_{p=1}^{s} \left\{ \frac{e^{-\xi_p + \tilde{\xi}_p - \eta(1+b_{p-1})}}{2} \prod_{k=1}^{p-1} \sinh(\lambda_p - \tilde{\xi}_k) \prod_{k=p+1}^{m} \sinh(\lambda_p - \tilde{\xi}_k + \eta) \right\} \\
\times \prod_{k=1}^{m} e^{-\tilde{\xi}_k + \lambda_i} \prod_{k=1}^{m} \sinh(\tilde{\xi}_k + \zeta - \eta/2) \frac{1}{\sinh(\lambda_k + \zeta - \eta/2)},
\]
(i.e.
\[
\tilde{H}_m^{\text{diag}}(\{\lambda_j\}_{j=1}^{M}; \{\tilde{\xi}_k\}_{k=1}^{m}) \sim (-2)^{-m} \prod_{p=1}^{s} e^{-\xi_p + \tilde{\xi}_p - \eta(1+b_{p-1})} \prod_{p=s+1}^{m} e^{-\xi_p + \tilde{\xi}_p - \eta b_{p-1}} \\
\times \tilde{H}_m^{\text{diag}}(\{\lambda_j\}_{j=1}^{M}; \{\tilde{\xi}_k\}_{k=1}^{m}),
\]
(C.9)
in which
\[
\tilde{H}_m^{\text{diag}}(\{\lambda_j\}_{j=1}^{M}; \{\tilde{\xi}_k\}_{k=1}^{m}) \sim \prod_{1 \leq i < j \leq m} \sinh(\lambda_i + \tilde{\xi}_j - \eta) \sinh(\lambda_j + \lambda_i - \eta) \\
\times \prod_{p=1}^{s} \left\{ \frac{\sinh(\lambda_p - \tilde{\xi}_k)}{\sinh(\lambda_p - \tilde{\xi}_k + \eta)} \right\} \\
\times \prod_{k=1}^{m} \frac{1}{\sinh(\lambda_k + \zeta - \eta/2)},
\]
(C.10)
From (C.7) and (C.9) and using (5.47), we therefore obtain in the limit
\[
\langle m \prod_{n=1}^{m} E_{\varepsilon_{n}}^{a_{n}} \rangle = \frac{(-1)^{m-s}}{\prod_{i<j} \sinh(\zeta_{i} - \zeta_{j}) \prod_{i \leq j} \sinh(\zeta_{i} + \zeta_{j} - \eta)}
\times \int_{\tilde{C}} \prod_{j=1}^{s} d\lambda_{j} \prod_{j=1}^{m} d\lambda \tilde{H}_{m}^{\text{diag}}(\{\lambda_{j}\}_{j=1}^{M}; \{\tilde{\zeta}_{k}\}_{k=1}^{m}) \det_{1 \leq i,j,k \leq m} \left[ \tilde{\Phi}(\lambda_{j}, \tilde{\zeta}_{k}) \right],
\]
(C.11)
in which \(\tilde{C}\) is defined by (5.44) if the set of roots for the ground state does not contain the boundary root \(\tilde{\lambda}_{-}\) converging towards \(\eta/2 - \zeta_{-}\), and by
\[
\tilde{C} = [-\Lambda, \Lambda] \cup \Gamma(\eta/2 - \zeta_{-})
\]
(C.12)
if the set of Bethe roots for the ground state contains the boundary root \(\tilde{\lambda}_{-}\), whereas \(\tilde{C}_{\xi}\) is defined as in (5.53). This result coincides with (6.8)–(6.10) of [108].

ORCID iDs

V Terras © https://orcid.org/0000-0002-1643-4319

References

[1] Niccoli G 2021 Correlation functions for open XXX spin 1/2 quantum chains with unparallel boundary magnetic fields (arXiv:2105.07992)

[2] Sklyanin E K 1985 The quantum Toda chain Non-Linear Equations in Classical and Quantum Field Theory ed N Sanchez (Berlin: Springer) pp 196–233

[3] Sklyanin E K 1985 Goryachev–Chaplygin top and the inverse scattering method J. Math. Sci. 31 3417

[4] Sklyanin E K 1990 Functional Bethe ansatz Integrable and Superintegrable Systems ed B Kupershmidt (Singapore: World Scientific) pp 8–33

[5] Sklyanin E K 1992 Quantum inverse scattering method. Selected topics Quantum Group and Quantum Integrable Systems (Nankai Lectures in Mathematical Physics) ed M-L Ge (Singapore: World Scientific) pp 63–97 (arXiv:hep-th/9211111)

[6] Sklyanin E K 1995 Separation of variables Prog. Theor. Phys. Suppl. 118 35

[7] Sklyanin E K 1996 Separation of variables in the quantum integrable models related to the Yangian \(Y(s[3])\) J. Math. Sci. 80 1861

[8] Babelon O, Bernard D and Smirnov F A 1996 Quantization of solitons and the restricted sine-Gordon model Commun. Math. Phys. 182 319

[9] Smirnov F A 1998 Structure of matrix elements in the quantum Toda chain J. Phys. A: Math. Gen. 31 8953

[10] Smirnov F A 2002 Separation of variables for quantum integrable models related to \(U_{q}(\widehat{sl}_{n})\) Math-Phys Odyssey 2001 (Progress in Mathematical Physics vol 23) ed M Kashiwara and T Miwa (Basel: Birkhäuser) pp 455–65

[11] Derkachov S È, Korchemsky G P and Manashov A N 2001 Noncompact Heisenberg spin magnets from high-energy QCD Nucl. Phys. B 617 375

[12] Derkachov S È, Korchemsky G P and Manashov A N 2003 Separation of variables for the quantum \(SL(2, \mathbb{R})\) spin chain J. High Energy Phys. JHEP07(2003)047

[13] Derkachov S È, Korchemsky G P and Manashov A N 2003 Baxter \(Q\)-operator and separation of variables for the open \(SL(2, \mathbb{R})\) spin chain J. High Energy Phys. JHEP10(2003)053

[14] Bytsko A G and Teschner J 2006 Quantization of models with non-compact quantum group symmetry: modular XXZ magnet and lattice sinh-Gordon model J. Phys. A: Math. Gen. 39 12927
[15] von Gehlen G, Jorgov N, Pakuliak S and Shadura V 2006 The Baxter–Bazhanov–Stroganov model: separation of variables and the Baxter equation J. Phys. A: Math. Gen. 39 7257
[16] Frahm H, Seel A and Wirth T 2008 Separation of variables for integrable spin-boson models Nucl. Phys. B 839 604
[17] Amico L, Frahm H, Osterloh A and Wirth T 2010 Separation of variables for integrable spin-boson models Nucl. Phys. B 835 263
[18] Frahm H, Seel A and Wirth T 2010 Separation of variables for integrable spin-boson models Nucl. Phys. B 839 604
[19] Niccoli G and Teschner J 2010 The sine-Gordon model revisited: I J. Stat. Mech. P09014
[20] Niccoli G 2010 Reconstruction of Baxter $Q$-operator from Sklyanin SOV for cyclic representations of integrable quantum models Nucl. Phys. B 835 263
[21] Niccoli G 2010 Completeness of Bethe ansatz by Sklyanin SOV for cyclic representations of integrable quantum models J. High Energy Phys. JHEP03(2011)123
[22] Frahm H, Grelik J H, Seel A and Wirth T 2011 Functional Bethe ansatz methods for the open XXX chain J. Phys. A: Math. Theor. 44 015001
[23] Grosjean N, Maillet J M and Niccoli G 2012 On the form factors of local operators in the lattice sine-Gordon model J. Stat. Mech. P09014
[24] Niccoli G 2012 Non-diagonal open spin-1/2 XXZ quantum chains by separation of variables: complete spectrum and matrix elements of some quasi-local operators J. Stat. Mech. P10025
[25] Niccoli G 2013 Antiperiodic spin-1/2 XXZ quantum chains by separation of variables: complete spectrum and form factors Nucl. Phys. B 870 397
[26] Niccoli G 2013 An antiperiodic dynamical six-vertex model: I. Complete spectrum by SOV, matrix elements of the identity on separate states and connections to the periodic eight-vertex model J. Phys. A: Math. Theor. 46 075003
[27] Niccoli G 2013 Form factors and complete spectrum of XXX antiperiodic higher spin chains by quantum separation of variables J. Math. Phys. 54 053516
[28] Grosjean N, Maillet J-M and Niccoli G 2015 On the form factors of local operators in the Bazhanov–Stroganov and chiral Potts models Ann. Henri Poincaré 16 1103
[29] Faldella S and Niccoli G 2014 SOV approach for integrable quantum models associated with general representations on spin-1/2 chains of the eight-vertex reflection algebra J. Phys. A: Math. Theor. 47 115202
[30] Faldella S, Kitanine N and Niccoli G 2014 The complete spectrum and scalar products for the open spin-1/2 XXZ quantum chains with non-diagonal boundary terms J. Stat. Mech. P01011
[31] Kitanine N, Maillet J M and Niccoli G 2014 Open spin chains with generic integrable boundaries: Baxter equation and Bethe ansatz completeness from separation of variables J. Stat. Mech. P05015
[32] Niccoli G and Terras V 2015 Antiperiodic XXZ chains with arbitrary spins: complete eigenstate construction by functional equations in separation of variables Lett. Math. Phys. 105 989
[33] Levy-Bencheton D, Niccoli G and Terras V 2016 Antiperiodic dynamical six-vertex model by separation of variables: II. Functional equations and form factors J. Stat. Mech. 033110
[34] Niccoli G and Terras V 2016 The eight-vertex model with quasi-periodic boundary conditions J. Phys. A: Math. Theor. 49 044001
[35] Kitanine N, Maillet J M, Niccoli G and Terras V 2016 On determinant representations of scalar products and form factors in the SoV approach: the XXX case J. Phys. A: Math. Theor. 49 104002
[36] Jiang Y, Komatsu S, Kostov I and Serban D 2016 The hexagon in the mirror: the three-point function in the SoV representation J. Phys. A: Math. Theor. 49 174007
[37] Kitanine N, Maillet J M, Niccoli G and Terras V 2017 The open XXX spin chain in the SoV framework: scalar product of separate states J. Phys. A: Math. Theor. 50 224001
[38] Maillet J M, Niccoli G and Pezelier B 2017 Transfer matrix spectrum for cyclic representations of the six-vertex reflection algebra I Scipost Phys. 2 009
[39] Kitanine N, Maillet J M, Niccoli G and Terras V 2018 The open XXZ spin chain in the SoV framework: scalar product of separate states J. Phys. A: Math. Theor. 51 485201
[40] Faddeev L D and Sklyanin E K 1978 Quantum-mechanical approach to completely integrable field theory models Sov. Phys. Dokl. 23 902
[41] Sklyanin E K, Takhtadzhyan L A and Faddeev L D 1979 Quantum inverse problem method I Theor. Math. Phys. 40 688
Faddeev L D, Sklyanin E K and Takhtajan L A 1979 Teor. Mat. Fiz. 40 194–220 (translated)
[42] Takhtadzhan L A and Faddeev L D 1979 The quantum method of the inverse problem and the Heisenberg XYZ model *Russ. Math. Surv.* **34** 11
[43] Sklyanin E K 1979 Method of the inverse scattering problem and the non-linear quantum Schrödinger equation *Sov. Phys. Dokl.* **24** 107
[44] Sklyanin E K 1979 On complete integrability of the Landau–Lifshitz equation *LOMI E-79-3*
[45] Faddeev L D and Takhtajan L A 1981 Quantum inverse scattering method *Sov. Sci. Rev. Math. C* **1** 107
[46] Sklyanin E K 1982 Quantum version of the method of inverse scattering problem *J.Math.Sci.* **19** 1546
[47] Faddeev LD1984 Integrable models in (1 + 1)-dimensional quantum field theory *Les Houches 1982 (Recent Advances in Field Theory and Statistical Mechanics)* ed J B Zuber and R Stora (Amsterdam: North Holland) pp 561–608
[48] Korepin V E, Bogoliubov N M and Izergin A G 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge Monographs on Mathematical Physics) (Cambridge: Cambridge University Press)
[49] Maillet J M and Niccoli G 2018 On quantum separation of variables *J. Math. Phys.* **59** 091417
[50] Martin D and Smirnov F 2016 Problems with using separated variables for computing expectation values for higher ranks *Lett. Math. Phys.* **106** 469
[51] Gromov N, Levkovich-Maslyuk F and Sizov G 2017 New construction of eigenstates and separation of variables for SU(N) quantum spin chains *J. High Energy Phys.* JHEP09(2017)111
[52] Ryan P and Volin D 2019 Separated variables and wave functions for rational gl(n) spin chains in the companion twist frame *J. Math. Phys.* **60** 032701
[53] Maillet J M and Niccoli G 2019 On separation of variables for reflection algebras *J. Stat. Mech.* **094020**
[54] Maillet J M, Niccoli G and Vignoli L 2020 On scalar products in higher rank quantum separation of variables (arXiv:1903.06281)
[55] Cavaglià A, Gromov N and Levkovich-Maslyuk F 2019 Separation of variables and scalar products at any rank *J. High Energy Phys.* JHEP09(2019)052
[56] Jimbo M and Miwa T 1995 *Algebraic Analysis of Solvable Lattice Models* (CBMS Regional Conference Series in Mathematics vol 85) (Providence, RI: American Mathematical Society)
[57] Jimbo M, Miki K, Miwa T and Nakayashiki A 1992 Correlation functions of the XXZ model in the gapless regime *J. Phys. A: Math. Gen.* **29** 2923
[58] Kitanine N, Maillet J M and Terras V 1999 Form factors of the XXZ Heisenberg finite chain *Nucl. Phys. B* **554** 647
[59] Maillet J M and Terras V 2000 On the quantum inverse scattering problem *Nucl. Phys. B* **575** 627
[60] Kitanine N, Maillet J M and Terras V 2000 Correlation functions of the XXZ Heisenberg spin-chain in a magnetic field *Nucl. Phys. B* **567** 554
[61] Kitanine N, Maillet J M, Slavnov N A and Terras V 2002 Spin–spin correlation functions of the XXZ-Heisenberg chain in a magnetic field *Nucl. Phys. B* **641** 487
[62] Kitanine N, Maillet J M, Slavnov N A and Terras V 2005 Master equation for spin–spin correlation functions of the chain *Nucl. Phys. B* **712** 600
[72] Kitanine N, Maillet J M, Slavnov N A and Terras V 2005 Dynamical correlation functions of the spin-chain Nucl. Phys. B 729 558
[73] Kitanine N, Kozlowski K, Maillet J M, Slavnov N A and Terras V 2007 On correlation functions of integrable models associated with the six-vertex R-matrix J. Stat. Mech. P01022
[74] Göhmann F, Klümper A and Seel A 2004 Integral representations for correlation functions of the XXZ chain at finite temperature J. Phys. A: Math. Gen. 37 7625
[75] Göhmann F, Klümper A and Seel A 2005 Integral representation of the density matrix of the XXZ chain at finite temperatures J. Phys. A: Math. Gen. 38 1833
[76] Boos H E, Göhmann F, Klümper A and Suzuki J 2007 Factorization of the finite temperature correlation functions of the XXZ chain in a magnetic field J. Phys. A: Math. Theor. 40 10699
[77] Göhmann F and Suzuki J 2010 Quantum spin chains at finite temperatures New Trends in Quantum Integrable Systems (Singapore: World Scientific) pp 81–100
[78] Dugave M, Göhmann F, Kozlowski K K and Suzuki J 2015 Low-temperature spectrum of correlation lengths of the XXZ chain in the antiferromagnetic massive regime J. Phys. A: Math. Theor. 48 334001
[79] Göhmann F, Karbach M, Klümper A, Kozlowski K K and Suzuki J 2017 Thermal form-factor approach to dynamical correlation functions of integrable lattice models J. Stat. Mech. 113106
[80] Boos H, Jimbo M, Miwa T, Smirnov F and Takeyama Y 2005 Traces on the Sklyanin algebra and correlation functions of the eight-vertex model J. Phys. A: Math. Gen. 38 7629
[81] Boos H, Jimbo M, Miwa T, Smirnov F and Takeyama Y 2006 Reduced qKZ equation and correlation functions of the XXZ model Commun. Math. Phys. 261 245–76
[82] Boos H, Jimbo M, Miwa T, Smirnov F and Takeyama Y 2006 Density matrix of a finite sub-chain of the Heisenberg anti-ferromagnet Lett. Math. Phys. 75 201
[83] Boos H, Jimbo M, Miwa T, Smirnov F and Takeyama Y 2006 Algebraic representation of correlation functions in integrable spin chains Ann. Henri Poincaré 7 1395
[84] Boos H, Jimbo M, Miwa T, Smirnov F and Takeyama Y 2006 A recursion formula for the correlation functions of an inhomogeneous XXX model St. Petersburg Math. J. 17 85
[85] Boos H, Jimbo M, Miwa T, Smirnov F and Takeyama Y 2007 Hidden Grassmann structure in the XXZ model Commun. Math. Phys. 272 263
[86] Boos H, Jimbo M, Miwa T, Smirnov F and Takeyama Y 2009 Hidden Grassmann structure in the XXZ model: II. Creation operators Commun. Math. Phys. 286 875
[87] Jimbo M, Miwa T and Smirnov F 2009 Hidden Grassmann structure in the XXZ model: III. Introducing the Matsubara direction J. Phys. A: Math. Theor. 42 304018
[88] Jimbo M, Miwa T and Smirnov F 2011 Hidden Grassmann structure in the XXZ model: V. Sine-Gordon model Lett. Math. Phys. 96 325
[89] Mestyán M and Pozsgay B 2014 Short distance correlators in the XXZ spin chain for arbitrary string distributions J. Stat. Mech. P09020
[90] Pozsgay B 2017 Excited state correlations of the finite Heisenberg chain J. Phys. A: Math. Theor. 50 074006
[91] Kitanine N, Kozlowski K K, Maillet J M, Slavnov N A and Terras V 2009 Riemann–Hilbert approach to a generalised sine Kernel and applications Commun. Math. Phys. 291 691
[92] Kitanine N, Kozlowski K K, Maillet J M, Slavnov N A and Terras V 2009 Algebraic Bethe ansatz approach to the asymptotic behavior of correlation functions J. Stat. Mech. P04003
[93] Kitanine N, Kozlowski K K, Maillet J M, Slavnov N A and Terras V 2009 On the thermodynamic limit of form factors in the massless XXZ Heisenberg chain J. Math. Phys. 50 095209
[94] Kozlowski K K, Maillet J M and Slavnov N A 2011 Long-distance behavior of temperature correlation functions in the one-dimensional Bose gas J. Stat. Mech. P03018
[95] Kozlowski K K, Maillet J M and Slavnov N A 2011 Correlation functions for one-dimensional bosons at low temperature J. Stat. Mech. P03019
[96] Kozlowski K K and Terras V 2011 Long-time and large-distance asymptotic behavior of the current–current correlators in the non-linear Schrödinger model J. Stat. Mech. P04013
[97] Kitanine N, Kozlowski K K, Maillet J M, Slavnov N A and Terras V 2011 The thermodynamic limit of particle–hole form factors in the massless XXX Heisenberg chain J. Stat. Mech. P05028
[98] Kitanine N, Kozlowski K K, Maillet J M, Slavnov N A and Terras V 2011 A form factor approach to the asymptotic behavior of correlation functions in critical models J. Stat. Mech. P12010
[99] Kitanine N, Kozlowski K K, Maillet J M, Slavnov N A and Terras V 2012 Form factor approach to dynamical correlation functions in critical models J. Stat. Mech. P09001

84
[100] Dugave M, Göhmann F and Kozlowski K K 2013 Thermal form factors of the XXZ chain and the large-distance asymptotics of its temperature dependent correlation functions J. Stat. Mech. P07010

[101] Kitanine N, Kozlowski K K, Maillet J M and Terras V 2014 Large-distance asymptotic behaviour of multi-point correlation functions in massless quantum models J. Stat. Mech. P05011

[102] Caux J-S, Hagemans R and Maillet J M 2005 Computation of dynamical correlation functions of Heisenberg chains: the gapless anisotropic regime J. Stat. Mech. P09003

[103] Caux J-S and Maillet J M 2005 Computation of dynamical correlation functions of Heisenberg chains in a magnetic field Phys. Rev. Lett. 95 077201

[104] Pereira R G, Sirker J, Caux J-S, Hagemans R, Maillet J M, White S R and Affleck I 2006 Dynamical spin structure factor for the anisotropic spin-1/2 Heisenberg chain Phys. Rev. Lett. 96 257202

[105] Kenzelmann M, Coldea R, Tennant D A, Visser D, Hofmann M, Smeibidl P and Tylczynski Z 2002 Order-to-disorder transition in the XY-like quantum magnet Cs2CoCl4 induced by noncommuting applied fields Phys. Rev. B 65 144432

[106] Jimbo M, Kedem R, Kojima T, Konno H and Miwa T 1995 XXZ chain with a boundary Nucl. Phys. B 441 437

[107] Jimbo M, Kedem R, Konno H, Miwa T and Weston R 1995 Difference equations in spin chains with a boundary Nucl. Phys. B 448 429

[108] Kitanine N, Kozlowski K K, Maillet J M, Niccoli G, Slavnov N A and Terras V 2007 Correlation functions of the open XXZ chain: I J. Stat. Mech. P10009

[109] Kitanine N, Kozlowski K K, Maillet J M, Niccoli G, Slavnov N A and Terras V 2008 Correlation functions of the open XXZ chain: II J. Stat. Mech. P07010

[110] Niccoli G, Pei H and Terras V 2021 Correlation functions by separation of variables: the XXX spin chain SciPost Phys. 10 006

[111] Alcaraz F C, Barber M N, Batchelor M T, Baxter R J and Quispel G R W 1987 Surface exponents of the quantum XXZ, Ashkin–Teller and Potts models J. Phys. A: Math. Gen. 20 6397

[112] Sklyanin E K 1988 Boundary conditions for integrable quantum systems J. Phys. A: Math. Gen. 21 2375

[113] Ghoshal S and Zamolodchikov A 1994 Boundary S matrix and boundary state in two-dimensional integrable quantum field theory Int. J. Mod. Phys. A 09 3841

[114] Fan H, Hou B-Y, Shi K-J and Yang Z-X 1996 Algebraic Bethe ansatz for the eight-vertex model with general open boundary conditions Nucl. Phys. B 478 723

[115] Nepomechie R I 2002 Solving the open XXZ spin chain with nondiagonal boundary terms at roots of unity Nucl. Phys. B 622 615

[116] Nepomechie R I 2004 Bethe ansatz solution of the open XXZ chain with nondiagonal boundary terms J. Phys. A: Math. Gen. 37 433

[117] Cao J, Lin H-Q, Shi K-J and Wang Y 2003 Exact solution of XXZ spin chain with unparallel boundary fields Nucl. Phys. B 663 487

[118] Yang W-L and Zhang Y-Z 2007 On the second reference state and complete eigenstates of the open XXZ chain J. High Energy Phys. JHEP04(2007)044

[119] Basilevskii P 2006 The q-deformed analogue of the Onsager algebra: beyond the Bethe ansatz Nucl. Phys. B 754 309

[120] Basilevskii P and Koizumi K 2007 Exact spectrum of the XXZ open spin chain from the q-Onsager algebra representation theory J. Stat. Mech. P09006

[121] Crampé N, Ragoucy E and Simon D 2010 Eigenvectors of open XXZ and ASEP models for a class of non-diagonal boundary conditions J. Stat. Mech. P11038

[122] Crampé N, Ragoucy E and Simon D 2011 Matrix coordinate Bethe ansatz: applications to XXZ and ASEP models J. Phys. A: Math. Theor. 44 405003

[123] Filali G and Kitanine N 2011 Spin chains with non-diagonal boundaries and trigonometric SOS model with reflecting end Symmetry, Integrability Geometry Methods Appl. 7 012

[124] Cao J, Yang W-L, Shi K and Wang Y 2013 Off-diagonal Bethe ansatz solutions of the anisotropic spin-chains with arbitrary boundary fields Nucl. Phys. B 877 152

[125] Belliard S and Crampé N 2013 Heisenberg XXX model with general boundaries: eigenvectors from algebraic Bethe ansatz Symmetry, Integrability Geometry Methods Appl. 9 072

[126] Belliard S 2015 Modified algebraic Bethe ansatz for XXZ chain on the segment: I. Triangular cases Nucl. Phys. B 892 1

[127] Belliard S and Pimenta R A 2015 Modified algebraic Bethe ansatz for XXZ chain on the segment: II. General cases Nucl. Phys. B 894 527
[128] Belliard S and Pimenta R A 2015 Slavnov and Gaudin–Korepin formulas for models without $U(1)$ symmetry: the twisted XXX chain Symmetry, Integrability Geometry Methods Appl. 11 099
[129] Avan J, Belliard S, Grosjean N and Pimenta R A 2015 Modified algebraic Bethe ansatz for XXZ chain on the segment: III. Proof Nucl. Phys. B 899 229
[130] Belliard S and Pimenta R A 2016 Slavnov and Gaudin-Korepin formulas for models without $U(1)$ symmetry: the XXX chain on the segment J. Phys. A: Math. Theor. 49 17LT01
[131] Grijalva S, De Nardis J and Terras V 2019 Open XXZ chain and boundary modes at zero temperature SciPost Phys. 7 23
[132] de Gier J and Essler F H L 2005 Bethe ansatz solution of the asymmetric exclusion process with open boundaries Phys. Rev. Lett. 95 240601
[133] Prosen T 2011 Open XXZ spin chain: nonequilibrium steady state and a strict bound on ballistic transport Phys. Rev. Lett. 106 217206
[134] Cherednik I V 1984 Factorizing particles on a half-line and root systems Theor. Math. Phys. 61 977
[135] Kulish P P, Reshetikhin N Y and Sklyanin E K 1981 Yang–Baxter equation and representation theory: I Lett. Math. Phys. 5 393
[136] Baxter R J 1982 Exactly Solved Models in Statistical Mechanics (London: Academic) https://physics.anu.edu.au/theophys/_files/Exactly.pdf
[137] Baxter R 1973 Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain: II. Equivalence to a generalized ice-type lattice model Ann. Phys., NY 76 25
[138] Nepomechie R I 2013 An inhomogeneous T–Q equation for the open XXX chain with general boundary terms: completeness and arbitrary spin J. Phys. A: Math. Theor. 46 442002
[139] Belliard S, Pimenta R A and Slavnov N A 2021 Scalar product for the XXZ spin chain with general integrable boundaries (arXiv:2103.12501)
[140] Belliard S and Slavnov N A 2019 Scalar products in twisted XXX spin chain. Determinant representation Symmetry, Integrability Geometry Methods Appl. 15 066
[141] Slavnov N, Zabrodin A and Zotov A 2020 Scalar products of Bethe vectors in the eight-vertex model J. High Energy Phys. JHEP06(2020)123
[142] Yang W-L, Chen X, Feng J, Hao K, Hou B-Y, Shi K-J and Zhang Y-Z 2011 Determinant representations of scalar products for the open XXZ chain with non-diagonal boundary terms J. High Energy Phys. JHEP01(2011)006
[143] Duval A and Pasquier V 2015 Pieri rules, vertex operators and Baxter $Q$-matrix (arXiv:1510.08709)
[144] de Vega H J and Ruiz A G 1993 Boundary $K$-matrices for the six vertex and the $n(2n − 1)A_{n−1}$ vertex models J. Phys. A: Math. Gen. 26 L519
[145] de Vega H J and Gonzalez-Ruiz A 1994 Boundary $K$-matrices for the XYZ, XXZ and XXX spin chains J. Phys. A: Math. Gen. 27 6129
[146] Skorik S and Saleur H 1995 Boundary bound states and boundary bootstrap in the sine-Gordon model with Dirichlet boundary conditions J. Phys. A: Math. Gen. 28 6605
[147] Kapustin A and Skorik S 1996 Surface excitations and surface energy of the antiferromagnetic XXZ chain by the Bethe ansatz approach J. Phys. A: Math. Gen. 29 1629