ASYMPTOTIC GROWTH OF SATURATED POWERS AND EPSILON MULTIPLICITY

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1. Introduction

In this paper, we study the growth of saturated powers of modules. In the case of an ideal $I$ in a local ring $(R, m)$, the saturation of $I^k$ in $R$ is

$$(I^k)_{\text{sat}} = I^k :_R m^\infty = \cup_{n=1}^\infty I^k :_R m^n.$$  

There are examples showing that the algebra of saturated powers of $I$, $\bigoplus_{k\geq 0}(I^k)_{\text{sat}}$, is not a finitely generated $R$-algebra; for instance, in many cases the saturated powers are the symbolic powers. As such, it cannot be expected that the “Hilbert function”, giving the length of the $R$-module $(I^k)_{\text{sat}}/I^k$, is very well behaved for large $k$. However, it can be shown that it is bounded above by a polynomial in $k$ of degree $d$, where $d$ is the dimension of $R$. We show that in many cases, there is a reasonable asymptotic behavior of this length.

Suppose that $(R, m)$ is a Noetherian local domain of dimension $d \geq 1$. Let $L$ be the quotient field of $R$. Let $\lambda(M)$ denote the length of an $R$-module $M$. Let $F$ be a finitely generated free $R$-module, and let $E$ be a submodule of $F$ of rank $e$. Let $S = R[F] = \text{Sym}(F) = \bigoplus_{k\geq 0} F^k$ and let $R[E] = \bigoplus_{k\geq 0} E^k$ be the $R$-subalgebra of $S$ generated by $E$. Let

$$E^k :_F m^\infty = \cup_{n=1}^\infty E^k :_F m^n$$

denote the saturation of $E^k$ in $F^k$. We prove the following theorem:

**Theorem 1.1.** Suppose that $(R, m)$ is a local domain of depth $\geq 2$ which is essentially of finite type over a field $K$ of characteristic zero (or over a perfect field $K$ such that $R/m$ is algebraic over $K$). Let $d$ be the dimension of $R$. Suppose that $E$ is a rank $e$ submodule of a finitely generated free $R$-module $F$. Then the limit

$$(1) \quad \lim_{k \to \infty} \frac{\lambda(E^k :_F m^\infty/E^k)}{k^{d+e-1}} \in \mathbb{R}$$

exists.

The conclusions of this theorem follow from Theorem 3.2 and Remark 3.3.

Theorem 1.1 is proven in the case when $E = I$ is a homogeneous ideal and $R$ is a standard graded normal $K$-algebra in our paper [3] with Hà, Srinivasan and Theodorescu. The theorem is proven with the additional assumptions that $R$ is regular, $E = I$ is an ideal in $F = R$, and the singular locus of $\text{Spec}(R/I)$ is $m$ in our paper [4] with Herzog and Srinivasan. Kleiman [13] has proven Theorem 1.1 in the case that $E$ is a direct summand of $F$ locally at every nonmaximal prime of $R$. The theorem is proven for $E$ of low analytic deviation in [4], for the case of ideals, and by Ulrich and Validashti [19] for the case of modules; in the case of low analytic deviation, the limit is always zero. A generalization of

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Partially supported by NSF.
this problem to the case of saturations with respect to non \( m \)-primary ideals is investigated by Herzog, Puthenpurakal and Verma in \[10\]; they show that an appropriate limit exists for monomial ideals.

An example in \[3\] shows that even in the case when \( E \) is an ideal \( I \) in a regular local ring \( R \), the limit may be irrational.

An important technique in the proof of Theorem 1.1 is to use a theorem of Lazarsfeld \[14\] showing that the volume of a line bundle on a complex projective variety can be expressed as a limit of numbers of global sections of powers of the line bundle; Lazarsfeld’s theorem is deduced from an approximation theorem of Fujita \[6\] (generalizations of Fujita’s result to positive characteristic are given in \[17\] and \[15\]).

We can interpret our results in terms of local cohomology. Let \( F_k^L = F_k \otimes_R L \), where \( L \) is the quotient field of \( R \), so that we have natural embeddings \( E_k \subset F_k \subset F_k^L \) for all \( k \). We have identities

\[ H^0_m(F_k/E_k^k) \cong E_k :_{F_k} m^\infty/E_k^k \quad \text{and} \quad H^1_m(E_k^k) \cong E_k :_{F_k^L} m^\infty/E_k^k. \]

Further, these two modules are equal if \( R \) has depth \( \geq 2 \).

We thus obtain the following corollary to Theorem 1.1, which shows that the epsilon multiplicity \( \varepsilon(E) \) of a module, defined as a limsup in \[19\], actually exists as a limit.

**Corollary 1.2.** Suppose that \((R, m)\) is a local domain of depth \( \geq 2 \) which is essentially of finite type over a field \( K \) of characteristic zero (or over a perfect field \( K \) such that \( R/m \) is algebraic over \( K \)). Let \( d \) be the dimension of \( R \). Suppose that \( E \) is a rank \( e \) submodule of a finitely generated free \( R \)-module \( F \). Then the limit

\[ \lim_{k \to \infty} \frac{(d + e - 1)!}{k^{d+e-1}} \lambda(H^0_m(F_k^k/E_k^k)) \in \mathbb{R} \]

exists. Thus the epsilon multiplicity \( \varepsilon(E) \) of \( E \) exists as a limit.

By the above identities of local cohomology, we see that (1) is equivalent to the statement that

\[ \lim_{k \to \infty} \frac{H^0_m(F_k^k/E_k^k)}{k^{d+e-1}} = \lim_{k \to \infty} \frac{H^1_m(E_k^k)}{k^{d+e-1}} \in \mathbb{R} \]

exists when depth\((R) \geq 2\).

In Section 4, we extend our results to domains of dimension \( d \geq 2 \). We prove the following extension of Theorem 1.1, which shows that the second limit of (2),

\[ \lim_{k \to \infty} \frac{H^1_m(E_k^k)}{k^{d+e-1}} \in \mathbb{R} \]

exists when \( R \) is a domain of dimension \( d \geq 2 \).

**Theorem 1.3.** Suppose that \((R, m)\) is a local domain of dimension \( d \geq 2 \) which is essentially of finite type over a field \( K \) of characteristic zero (or over a perfect field \( K \) such that \( R/m \) is algebraic over \( K \)). Suppose that \( E \) is a rank \( e \) submodule of a finitely generated free \( R \)-module \( F \). Then the limit

\[ \lim_{k \to \infty} \frac{\lambda(E_k^k :_{F_k^L} m^\infty/E_k^k)}{k^{d+e-1}} \in \mathbb{R} \]

exists.
Theorem 1.3 follows from Theorem 1.1 and equations (24) and (6). We prove that the first limit of (2),
\[
\lim_{k \to \infty} \frac{H_0^m(F^k/E^k)}{k^{d+e-1}} \in \mathbb{R}
\]
effects when \(R\) is a domain of dimension \(d \geq 2\) and \(E\) is embedded in \(F\) of rank \(< d + e\). I thank Craig Huneke, Bernd Ulrich and Javid Validashti for pointing out this interesting consequence of Theorem 1.3.

**Corollary 1.4.** Suppose that \((R, m)\) is a local domain of dimension \(d \geq 2\) which is essentially of finite type over a field \(K\) of characteristic zero (or over a perfect field \(K\) such that \(R/m\) is algebraic over \(K\)). Suppose that \(E\) is a rank \(e\) submodule of a finitely generated free \(R\)-module \(F\). Suppose that \(\gamma = \text{rank}(F) < d + e\). Then the limits
\[
\lim_{k \to \infty} \frac{\lambda(E^k : F^k m^\infty / E^k)}{k^{d+e-1}} \in \mathbb{R}
\]
and
\[
\lim_{k \to \infty} \frac{(d + e - 1)!}{k^{d+e-1}} \lambda(H_0^m(F^k/E^k)) \in \mathbb{R}
\]
exist. In particular, the epsilon multiplicity \(\varepsilon(E)\) of \(E\) exists as a limit.

In the case when \(e = 1\) and \(F = R\), we get the following statement.

**Corollary 1.5.** Suppose that \((R, m)\) is a local domain of dimension \(d \geq 1\) which is essentially of finite type over a field \(K\) of characteristic zero (or over a perfect field \(K\) such that \(R/m\) is algebraic over \(K\)). Suppose that \(I\) is an ideal in \(R\). Let \((I^k)_{\text{sat}} = I^k : R m^\infty\) be the saturation of \(I^k\). Then the limit
\[
\lim_{k \to \infty} \frac{\lambda((I^k)_{\text{sat}} / I^k)}{k^d} \in \mathbb{R}
\]
exists.

Asymptotic polynomial like behavior of the length of extension functions is studied by Katz and Theodorescu [12], Theodorescu [18] and Crabbe, Katz, Striuli and Theodorescu [2]. By the local duality theorem, we obtain the following corollary to Theorem 1.1.

**Corollary 1.6.** Suppose that \((R, m)\) is a Gorenstein local domain of dimension \(d \geq 2\) which is essentially of finite type over a field \(K\) of characteristic zero (or over a perfect field \(K\) such that \(R/m\) is algebraic over \(K\)). Suppose that \(E\) is a rank \(e\) submodule of a finitely generated free \(R\)-module \(F\). Then the limit
\[
\lim_{k \to \infty} \frac{\lambda(\text{Ext}_R^d(F^k/E^k, R))}{k^{d+e-1}} \in \mathbb{R}
\]
exists.

### 2. Preliminaries

Suppose that \((R, m)\) is a Noetherian local domain of dimension \(d \geq 1\) with quotient field \(L\). Let \(\lambda_R(M)\) denote the length of an \(R\)-module \(M\). When there is no danger of confusion, we will denote \(\lambda_R(M)\) by \(\lambda(M)\).
Let $F$ be a finitely generated free $R$-module of rank $\gamma$, and let $E$ be a submodule of $F$ of rank $e$. Let $S = R[F] = \text{Sym}(F) = \bigoplus_{k \geq 0} F^k$, and let $R[E] = \bigoplus_{k \geq 0} E^k$ be the $R$-subalgebra of $S$ generated by $E$. Let

$$E^k : F^k \ m^\infty = \bigcup_{n=1}^{\infty} E^k : F^k \ m^n$$

denote the saturation of $E^k$ in $F^k$.

Let $F_L^k = F^k \otimes_R L$ (where $L$ is the quotient field of $R$), so that we have natural embeddings $E^k \subset F^k \subset F_L^k$ for all $k$. Let $X = \text{Spec}(R)$, $\widetilde{E}^k$ be the sheafification of $E$ on $X$ and let $u_1, \ldots, u_s$ be generators of the ideal $m$.

There are identities

$$H^0(X \setminus \{m\}, \widetilde{E}^k) = \bigcap_{i=1}^s (E^k)_{u_i} = E^k : F_L^k \ m^\infty. \tag{6}$$

From the exact sequence of cohomology groups

$$0 \to H^0_m(E^k) \to E^k \to H^0_m(X \setminus \{m\}, \widetilde{E}^k) \to H^1_m(E^k) \to 0,$$

we deduce that we have isomorphisms of $R$-modules

$$H^1_m(E^k) \cong E^k : F^k \ m^\infty / E^k \tag{7}$$

for $k \geq 0$. The same calculation for $F^k$ shows that

$$H^1_m(F^k) \cong F^k : F_L^k \ m^\infty / F^k. \tag{8}$$

From the left exact local cohomology sequence

$$0 \to H^0_m(F^k/E^k) \to H^1_m(F^k) \to H^1_m(F^k),$$

we have that

$$H^0_m(F^k/E^k) \cong \left( E^k : F_L^k \ m^\infty \cap F^k \right) / E^k = E^k : F^k \ m^\infty / E^k. \tag{9}$$

From (6), and the fact that $F^k$ is a free $R$-module, we have that $H^0(X \setminus \{m\}, \widetilde{E}^k) = F^k$ and

$$E^k : F_L^k \ m^\infty = E^k : F^k \ m^\infty \text{ if } R \text{ has depth } \geq 2. \tag{10}$$

Let $ES$ be the ideal of $S$ generated by $E$. We compute the degree $n$ part of $(ES)^n$ from the formula

$$[(ES)^n]_n = E^n. \tag{11}$$

Let $R[\mathfrak{m}E] = \bigoplus_{n \geq 0} (\mathfrak{m}E)^n$ be the $R$-subalgebra of $S$ generated by $\mathfrak{m}E$.

Let $X = \text{Spec}(R)$, $Y = \text{Proj}(R[\mathfrak{m}E])$ and $Z = \text{Proj}(R[E])$.

Write $R[E] = R[\overline{x_1}, \ldots, \overline{x_t}]$ as a standard graded $R$-algebra, with deg $\overline{x_i} = 1$ for all $i$. For $1 \leq i \leq t$, let

$$R_i = R[\frac{\overline{x_1}}{\overline{x_i}}, \ldots, \frac{\overline{x_t}}{\overline{x_i}}],$$

and let $V_i = \text{Spec}(R_i)$ for $1 \leq i \leq t$. \{V_i\} is an affine cover of $Z$. Let $u_1, \ldots, u_s$ be generators of the ideal $\mathfrak{m}$. For $1 \leq i \leq s$ and $1 \leq j \leq t$, let

$$R_{i,j} = R\left[ \frac{u_{\alpha} \overline{x_\beta}}{u_i \overline{x_j}} \mid 1 \leq \alpha \leq s, 1 \leq \beta \leq t \right],$$

and $U_{i,j} = \text{Spec}(R_{i,j})$. Then \{U_{i,j}\} is an affine cover of $Y$. Since

$$R_{j} \left[ \frac{u_1}{u_i}, \ldots, \frac{u_s}{u_i} \right] = R_{i,j},$$
we see that $Y$ is the blow up of the ideal sheaf $\mathfrak{m}\mathcal{O}_Z$.

The structure morphism $f : Y \to X$ factors as a sequence of projective morphisms

$$Y \xrightarrow{\varphi} Z \xrightarrow{h} X,$$

where $Y$ is the blow up of the ideal sheaf $\mathfrak{m}\mathcal{O}_Z$. Define line bundles on $Y$ by $\mathcal{L} = g^*\mathcal{O}_Z(1)$ and $\mathcal{M} = \mathfrak{m}\mathcal{O}_Y$. Then $\mathcal{O}_Y(1) \cong \mathcal{M} \otimes \mathcal{L}$.

We have $\mathcal{O}_Z(1)|_{V_j} = \mathfrak{p}_j\mathcal{O}_{V_j}$, $\mathcal{L}|_{U_{i,j}} = \mathfrak{p}_j\mathcal{O}_{U_{i,j}}$ and $\mathcal{M}|_{U_{i,j}} = u_i\mathcal{O}_{U_{i,j}}$.

We give three consequences (Proposition 2.1, Proposition 2.2 and Corollary 2.3) of Serre’s fundamental theorem for projective morphisms which will be useful.

**Proposition 2.1.** $\bigoplus_{k \geq 0} H^i(Y, \mathcal{L}^k)$ are finitely generated $R[E]$-modules for all $i \in \mathbb{N}$.

**Proof.** Let $\tilde{E}^k$ be the sheafication of $E^k$ on $X$. From the natural surjections for $k \geq 0$ of $\mathcal{O}_Z$-modules $g^*(\tilde{E}^k) \to \mathcal{O}_Z(k)$, we obtain surjections $f^*(\tilde{E}^k) \to \mathcal{L}^k$ of $\mathcal{O}_Y$-modules, and a surjection $f^*(\bigoplus_{k \geq 0} \tilde{E}^k) \to \bigoplus_{k \geq 0} \mathcal{L}^k$. Hence $\bigoplus_{k \geq 0} \mathcal{L}^k$ is a finitely generated $f^*(\bigoplus_{k \geq 0} \tilde{E}^k)$-module. By Theorem III.2.4.1 [8], $R^i f_* (\bigoplus_{k \geq 0} \mathcal{L}^k)$ is a finitely generated $\bigoplus_{k \geq 0} \tilde{E}^k$-module for $i \in \mathbb{N}$. Taking global sections on the affine $X$, we obtain the conclusions on the affine $X$, we obtain the conclusions of the proposition. □

**Proposition 2.2.** Suppose that $A$ is a Noetherian ring, and $B = \bigoplus_{k \geq 0} B_k$ is a finitely generated graded $A$-algebra, which is generated by $B_1$ as an $A$-algebra. Let $C = \text{Spec}(A)$ and $D = \text{Proj}(B)$. Let $\alpha : D \to C$ be the structure morphism. Then there exists a positive integer $\overline{k}$ such that $B_k = \Gamma(D, \mathcal{O}_D(k))$ for $k \geq \overline{k}$.

**Proof.** The ring $\bigoplus_{k \geq 0} \Gamma(D, \mathcal{O}_D(k))$ is a finitely generated graded $B$-module by Theorem III.2.4.1 [8]. Hence $(\bigoplus_{k \geq 0} \Gamma(D, \mathcal{O}_D(k)))/B$ is a finitely generated graded $B$-module. Since every element of this module is $B_+ = \bigoplus_{k \geq 0} B_k$ torsion, we have that $B_k/E_k = 0$ for $k \gg 0$. □

Taking the maximum over the $\overline{k}$ obtained from the above proposition applied to a finite affine cover of $W$, we obtain the following generalization of Proposition 2.2.

**Corollary 2.3.** Suppose that $W$ is a Noetherian scheme and $B = \bigoplus_{k \geq 0} B_k$ is a finitely generated graded $\mathcal{O}_W$-algebra, which is locally generated by $B_1$ as a $\mathcal{O}_W$-algebra. Let $W' = \text{Proj}(B)$ and let $\alpha : W' \to W$ be the structure morphism. Then there exists a positive integer $\overline{k}$ such that $B_k = \alpha_* \mathcal{O}_{W'}(k)$ for $k \geq \overline{k}$.

3. Asymptotic Growth

**Proposition 3.1.** Let $(R, \mathfrak{m})$ be a local domain of depth $\geq 2$. Let $d$ be the dimension of $R$. Suppose that $E$ is a rank $e$ $R$-submodule of a finitely generated free $R$-module $F$. Let notation be as above. Then there exist positive integers $k_0$, $k_1$ and $\tau$ such that

1) for $k \geq k_0$, $n \in \mathbb{Z}$ and $p \in X \setminus \{\mathfrak{m}\}$,

$$\Gamma(Y, \mathcal{M}^n \otimes \mathcal{L}^k)_p = (E^k)_p.$$

2) For $k \geq k_1$,

$$E^k \cdot \mathfrak{m}^\infty = \Gamma(Y, \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k).$$

**Proof.** We first establish 1). $U_i = \text{Spec}(R[u_i])$ for $1 \leq i \leq s$ is an affine cover of $X \setminus \{\mathfrak{m}\}$. $g|f^{-1}(U_i)$ is an isomorphism; in fact

$$f^{-1}(U_i) = \text{Proj}(R[\mathfrak{m}E]', u_i) = \text{Proj}(R[E]_{u_i}) = h^{-1}(U_i).$$
By Proposition 2.2 there exist positive integers $a_i$ such that

$$\Gamma(f^{-1}(U_i), \mathcal{M}^{-n} \otimes L^k) = \Gamma(h^{-1}(U_i), \mathcal{O}_Z(k)) = (E^k)_{u_i}$$

for $k \geq a_i$. Let $k_0 = \max\{a_1, \ldots, a_s\}$. Then for $p \in U_i$

$$\Gamma(Y, \mathcal{M}^{-n} \otimes L^k)_p = \Gamma(f^{-1}(U_i), \mathcal{M}^n \otimes L^k)_p = (E^k)_p$$

for $k \geq k_0$, establishing 1).

We now establish 2). Suppose that $n \geq 0$, and $k \geq 0$. Suppose that $\sigma \in E^k :_{F^k} m^n$. Let $i, j$ be such that $1 \leq i \leq s$ and $1 \leq j \leq t$. $\sigma m^n \subset E^k$ implies $u_i^n \sigma \in E^k$ which implies there is an expansion

$$u_i^n \sigma = \sum_{n_1 + \cdots + n_t = k} r_{n_1, \ldots, n_t} x_1^{n_1} \cdots x_t^{n_t}$$

with $r_{n_1, \ldots, n_t} \in R$. Thus

$$u_i^n \sigma = x_j^k \left( \sum_{n_1 + \cdots + n_t = k} r_{n_1, \ldots, n_t} (\frac{x_1}{x_j})^{n_1} \cdots (\frac{x_t}{x_j})^{n_t} \right),$$

so that $\sigma \in u_i^{-n} x_j^k R_{i,j}$. Thus

$$\sigma \in \bigcap_{i,j} u_i^{-n} x_j^k R_{i,j} = \Gamma(Y, \mathcal{M}^{-n} \otimes L^k).$$

We have established that for $k \geq 0$ and $n \geq 0$,

$$E^k :_{F^k} m^n \subset \Gamma(Y, \mathcal{M}^{-n} \otimes L^k).$$

Recall that $S$ is a polynomial ring $S = R[y_1, \ldots, y_r]$ over $R$, where $\gamma$ is the rank of $F$. Let $W = \text{Proj}(S)$, with natural morphism $\alpha : W \to X$. Let $\mathcal{I}$ be the sheafification of the graded ideal $ES$ on $W$. We have expansions

$$\mathfrak{p}_i = \sum_{l=1}^{\gamma} f_{il} y_l$$

with $f_{il} \in R$.

The inclusion $R[E] \subset S$ induces a rational map from $W$ to $Z$.

Let $\beta : W' \to W$ be the blow up of the ideal sheaf $\mathcal{I}$. Let $\mathcal{N} = \mathcal{I} \mathcal{O}_{W'}$ be the induced line bundle. $W'$ has an affine cover $A_{i,j} = \text{Spec}(T_{ij})$ for $1 \leq i \leq t$ and $1 \leq j \leq s$ with

$$T_{ij} = R[y_1, \ldots, y_r|_{\frac{x_1}{x_i}}, \ldots, \frac{x_t}{x_i}].$$

From the inclusions

$$R_i = R[\frac{x_1}{x_i}, \ldots, \frac{x_t}{x_i}] \subset T_{ij}$$

we have induced morphisms $A_{i,j} \to V_i = \text{Spec}(R_i)$ which patch to give a morphism $\varphi : W' \to Z$ which is a resolution of indeterminacy of the rational map from $W$ to $Z$.

We calculate for all $i,j$,

$$\varphi^*(\mathcal{O}_Z(1)) |_{A_{i,j}} = \mathfrak{p}_i \mathcal{O}_{A_{i,j}} = y_j (\sum_{l} f_{il} \frac{y_l}{y_j}) \mathcal{O}_{A_{i,j}} = (\beta^* \mathcal{O}_{W}(1)) |_{A_{i,j}},$$

to see that

$$(\beta^* \mathcal{O}_{W}(1)) \otimes \mathcal{N} \cong \varphi^* \mathcal{O}_Z(1).$$
By Corollary 2.3 there exists a positive integer $k_1 \geq k_0$ such that $\beta_*N^k = T^k$ for $k \geq k_1$. From the natural inclusion $O_Z(k) \subset \varphi_\ast \varphi^\ast O_Z(k)$, we have by the projection formula that for $k \geq k_1$,

$$h_* O_Z(k) \subset h_* \varphi_\ast (\varphi^\ast O_Z(k)) = \alpha_* \beta_\ast (\beta^\ast O_W(k) \otimes N^k)$$

$$= \alpha_* \{ O_W(k) \otimes \beta_\ast N^k \} = \alpha_* \{ O_W(k) \otimes T^k \}$$

$$\subset \alpha_* O_W(k) = \tilde{F}^k,$$

where $\tilde{F}^k$ is the sheafication of the $R$-module $F$ on $X$. Now we have

$$\Gamma(Y, \mathcal{M}^{-n} \otimes \mathcal{L}^k) = \Gamma(X, f_* (\mathcal{M}^{-n} \otimes \mathcal{L}^k))$$

$$\subset \Gamma(X \setminus \{ \mathfrak{m} \}, f_* (\mathcal{M}^{-n} \otimes \mathcal{L}^k)) = \Gamma(X \setminus \{ \mathfrak{m} \}, h_* O_Z(k))$$

$$\subset \Gamma(X \setminus \{ \mathfrak{m} \}, \tilde{F}^k) = F^k$$

since $R$, and hence the free $R$-module $F^k$, have depth $\geq 2$. From (11), we deduce that for $k, n \geq 0$,

$$\left( (ES)^k :_S m^n S \right) \cap F^k = E^k :_{E^k} m^n.$$

By 1.5 [11] or Theorem 1.3 [16], there exists a positive integer $\tau$ such that

$$(ES)^k :_S m^\tau S = (ES)^k :_S (mS)^\infty$$

for all $k \geq 0$. Thus from (15) we have that

$$(ES)^k :_{E^k} m^\tau = E^k :_{E^k} m^\infty$$

for $k \geq 0$. From (16), (12) and (14), we have inclusions

$$E^k :_{E^k} m^\infty \subset \Gamma(Y, \mathcal{M}^{-\tau} \otimes \mathcal{L}^k) \subset F^k$$

for $k \geq k_1$. The conclusions of 2) of the proposition now follow from 1) of the proposition since $E^k :_{E^k} m^\infty$ is the largest $R$-submodule $N$ of $F^k$ which has the property that $N_\mathfrak{p} = (E^k)_\mathfrak{p}$ for $\mathfrak{p} \in X \setminus \{ \mathfrak{m} \}$.

□

**Theorem 3.2.** Suppose that $(R, \mathfrak{m})$ is a local domain of depth $\geq 2$ which is essentially of finite type over a field $K$ of characteristic zero. Let $d$ be the dimension of $R$. Suppose that $E$ is a rank $e$ submodule of a finitely generated free $R$-module $F$. Then the limit

$$\lim_{k \to \infty} \frac{\lambda \left( E^k :_{E^k} m^\infty / E^k \right)}{k^{d+e-1}} \in \mathbb{R}$$

exists.

**Proof.** Let notation be as above.

First consider the short exact sequences

$$0 \to \Gamma(Y, \mathcal{L}^k) / E^k \to E^k :_{E^k} m^\infty / E^k \to E^k :_{E^k} m^\infty / \Gamma(Y, \mathcal{L}^k) \to 0.$$
there exists a constant \( \alpha \) such that \( \lambda(\Gamma(Y, \mathcal{L}^k)/E^k) \leq \alpha k^{d+e-2} \) for all \( k \). From (17), we are now reduced to showing that the limit

\[
\lim_{k \to \infty} \frac{\lambda(E^k : \Gamma(Y, \mathcal{L}^k))^\infty}{k^{d+e-1}}
\]

exists, from which we will have

\[
\lim_{k \to \infty} \frac{\lambda(E^k : \Gamma(Y, \mathcal{L}^k))^\infty}{k^{d+e-1}} = \lim_{k \to \infty} \frac{\lambda(E^k : \Gamma(Y, \mathcal{L}^k))^\infty}{k^{d+e-1}}.
\]

Taking global sections of the short exact sequences

\[
0 \to \mathcal{L}^k \to \mathcal{M}^{-k} \otimes \mathcal{L}^k \to \mathcal{E}^{-k} \otimes \mathcal{L}^k \otimes (\mathcal{O}_Y/m^k \mathcal{O}_Y) \to 0,
\]

we obtain by Proposition 3.1 left exact sequences

\[
0 \to E^k : \mathcal{E}^k \otimes \Gamma(Y, \mathcal{L}^k) \to \Gamma(Y, \mathcal{M}^{-k} \otimes \mathcal{L}^k \otimes (\mathcal{O}_Y/m^k \mathcal{O}_Y)) \to H^1(Y, \mathcal{L}^k)
\]

for \( k \geq k_1 \).

Let \( u_1, \ldots, u_s \) be generators of the ideal \( m \), and set \( U_i = \text{Spec}(R_{u_i}) \), so that \( \{U_1, \ldots, U_s\} \) is an affine cover of \( X \setminus \{m\} \). Then \( \mathcal{L}|f^{-1}(U_i) \) is ample, so there exist positive integers \( b_i \) such that \( R^1 f_*(\mathcal{L}^k) | U_i = 0 \) for \( k \geq b_i \). Let \( k_2 = \max\{b_1, \ldots, b_s\} \). We have that the support of \( H^1(Y, \mathcal{L}^k) \) is contained in \( \{m\} \) for \( k \geq k_2 \).

\[
\bigoplus_{k \geq 0} H^1(Y, \mathcal{L}^k)
\]

is a finitely generated \( R[E] \)-module by Lemma 2.1. Hence the submodule \( M = \bigoplus_{k \geq k_2} H^1(Y, \mathcal{L}^k) \) is a finitely generated graded \( R[E] \)-module. We have that \( \mathcal{m}^r \mathcal{M} = 0 \) for some positive integer \( r \). Since

\[
dim R[E]/\mathcal{m}^r R[E] \leq \dim R + \text{rank } E - 1 = d + e - 1,
\]

and \( R/\mathcal{m}^r \) is an Artin local ring, we have by the Hilbert-Serre theorem that \( \lambda(H^1(Y, \mathcal{L}^k)) \) is a polynomial of degree less than or equal to \( d + e - 2 \) for \( k \gg 0 \). Thus there exists a constant \( c \) such that

\[
\lambda(H^1(Y, \mathcal{L}^k)) \leq c k^{d+e-2}
\]

for all \( k \geq 0 \). By consideration of (18) and (19), we are reduced to proving that the limit

\[
\lim_{k \to \infty} \frac{\lambda(H^0(Y, \mathcal{M}^{-k} \otimes \mathcal{L}^k \otimes \mathcal{O}_Y/m^k \mathcal{O}_Y))}{k^{d+e-1}}
\]

exists.

If \( R/\mathcal{m} \) is algebraic over \( K \), let \( K' = K \). If \( R/\mathcal{m} \) is transcendental over \( K \), let \( t_1, \ldots, t_r \) be a lift of a transcendence basis of \( R/\mathcal{m} \) over \( K \). The rational function field \( K(t_1, \ldots, t_r) \) is contained in \( R \). Let \( K' = K(t_1, \ldots, t_r) \). We have that \( R/\mathcal{m} \) is finite algebraic over \( K' \).

There exists an affine \( K' \)-variety \( X' = \text{Spec}(A) \) such that \( R \) is the local ring of a closed point \( \alpha \) of \( X' \), and \( E \) extends to a submodule \( E' \) of \( A' \), where \( \gamma \) is the rank of the free \( R \)-module \( E \). We then have an inclusion of graded \( A \)-algebras \( A[E'] \subset \text{Sym}(A') \) which extends \( R[E] \). Identify \( \mathcal{m} \) with its extension to a maximal ideal of \( A \). The structure morphism \( Y' = \text{Proj}(A[E']) \to X' \) is projective and its localization at \( \mathcal{m} \) is \( \mathcal{f} : Y' \to X' \). Let \( \overline{X} \) be a projective closure of \( X' \) and let \( \overline{Y} \) be a projective closure of \( Y' \). \( X' \) is an open subset of \( \overline{X} \) and \( Y' \) is an open subset of \( \overline{Y} \). Let \( \overline{f} : \overline{Y} \to \overline{X} \) be the blow up of an ideal sheaf which gives a resolution of indeterminacy of the rational map from \( \overline{Y} \) to \( \overline{X} \). We may assume that the morphism \( \overline{Y} \to \overline{Y} \) is an isomorphism over the locus where the rational map is a morphism, and thus an isomorphism over the subset \( Y' \) of \( \overline{Y} \). Let \( \overline{f} : \overline{Y} \to \overline{X} \) be the resulting morphism. We now establish that \( \overline{f}^{-1}(X') = Y' \). Suppose that \( p \in X' \) and \( q \in \overline{f}^{-1}(p) \). Let \( V \) be a valuation ring of the function field \( L \) of \( \overline{Y} \) (which is also the
function field of $Y'$) which dominates the local ring $O_{Y,q}$. By assumption, $V$ dominates the local ring $O_{X',p}$. $V$ dominates the local ring of a point on $Y'$, by the valuative criterion for properness (Theorem II.4.7 [9]) applied to the proper morphism $Y' \to X'$. Since $V$ dominates the local ring of a unique point on $Y$, we have that $q \in Y'$.

After possibly replacing $Y$ with the blow up of an ideal sheaf on $Y$ whose support is disjoint from $Y'$, we may assume that $L$ extends to a line bundle on $Y$ which we will also denote by $L$. We will identify $m$ with its extension to the ideal sheaf of the point $\alpha$ on $Y$, and identify $M$ with its extension $mO_Y$ to a line bundle on $Y$. Let $A$ be an ample divisor on $Y$. Then there exists $l > 0$ such that $C = f'(A^l) \otimes L$ is generated by global sections and is big.

Set $B = C \otimes M^{-r}$. Tensor the short exact sequences

$$0 \to M^{kr} \to O_Y \to O_Y/m^{kr}O_Y \cong O_Y/m^{kr}O_Y \to 0$$

with $B^k$ to obtain the short exact sequences

$$0 \to C^k \to B^k \to M^{-kr} \otimes L^k \otimes O_Y/m^{kr}O_Y \to 0$$

for $k \geq 0$. Taking global sections, we have exact sequences

(21) \qquad 0 \to H^0(Y, C^k) \to H^0(Y, B^k) \to H^0(Y, M^{-kr} \otimes L^k \otimes O_Y/m^{kr}O_Y) \to H^1(Y, C^k).

For a coherent sheaf $F$ on $Y$, let

$$h^i(Y, F) = \dim_k H^i(Y, F).$$

Since $C$ is semiample (generated by global sections and big) and $Y$ has dimension $d+e-1$, we have that

$$\lim_{k \to \infty} \frac{h^1(Y, C^k)}{k^{d+e-1}} = 0.$$

This follows for instance from [5]. Since $\bigoplus_{k \geq 0} H^0(Y, C^k)$ is a finitely generated $K'$ algebra of dimension $d+e$, as $C$ is generated by global sections and is big (or by the Riemann Roch theorem and the vanishing theorem of [2]) we have that the limit

$$\lim_{k \to \infty} \frac{h^0(Y, C^k)}{k^{d+e-1}} \in \mathbb{Q}$$

exists. Since $B$ is big, by the corollary to [6] given in Example 11.4.7 [14] or [3], we have that the limit

$$\lim_{k \to \infty} \frac{h^0(Y, B^k)}{k^{d+e-1}} \in \mathbb{R}$$

exists. From the sequence (21), we see that

$$\lim_{k \to \infty} \frac{H^0(Y, M^{-kr} \otimes L^k \otimes O_Y/m^{kr}O_Y)}{k^{d+e-1}} \in \mathbb{R}$$

exists. The conclusions of the theorem now follow from (20) and the formula

$$h^0(Y, M^{-kr} \otimes L^k \otimes O_Y/m^{kr}O_Y) = \dim_{K'} H^0(Y, M^{-kr} \otimes L^k \otimes O_Y/m^{kr}O_Y)$$

$$= [R/m : K'] \lambda(H^0(Y, M^{-kr} \otimes L^k \otimes O_Y/m^{kr}O_Y)).$$

□

**Remark 3.3.** The conclusions of Theorem 3.2 are also true if $K$ is a perfect field of positive characteristic and $R/m$ is algebraic over $K$. In this case we have that $K' = K$ in the proof of Theorem 3.2. Let $\overline{K}$ be an algebraic closure of $K$. Since $K$ is perfect, $\overline{Y} \times_K \overline{K}$ is reduced, and to compute the limit, we reduce to computing the sections of the
pullback of \( B^k \) on the disjoint union of the irreducible (integral) components of \( Y \times_K K \). Fujita's approximation theorem is valid on varieties over an algebraically closed field of positive characteristic, as was shown by Takagi [17], from which the existence of the limit now follows.

**Remark 3.4.** Theorem 3.2 is proven for graded ideals in [3]. An example where the limit is an irrational number is given in [3]. The theorem is proven with the additional assumptions that \( R \) is regular, \( E = I \) is an ideal in \( F = R \), and the singular locus of \( \text{Spec}(R/I) \) is \( m \) in [4]. Kleiman [13] has proven Theorem 3.2 in the case that \( E \) is a direct summand of \( F \) locally at every nonmaximal prime of \( R \).

**Corollary 3.5.** Suppose that \((R, m)\) is a local domain of depth \( \geq 2 \) which is essentially of finite type over a field \( K \) of characteristic zero. Let \( d \) be the dimension of \( R \). Suppose that \( E \) is a rank \( e \) submodule of a finitely generated free \( R \)-module \( F \). Then the limit

\[
\lim_{k \to \infty} \frac{(d + e - 1)!}{k^{d+e-1}} \lambda(H^0_m(F^k/E^k)) \in \mathbb{R}
\]

exists. Thus the epsilon multiplicity \( \varepsilon(E) \) of the module \( E \), defined in [19] as a limsup, actually exists as a limit.

The example of [3] shows that \( \varepsilon(E) \) may be an irrational number.

**Proof.** The corollary is immediate from Theorem 3.2 and (9). □

**Remark 3.6.** The conclusions of Corollary 3.5 are valid if \( K \) is a perfect field of positive characteristic and \( R/m \) is algebraic over \( K \), by Remark 3.3.

4. Extension to Domains of Dimension \( \geq 2 \).

In this section, we prove extensions of Theorem 1.1 and Corollary 1.2 to domains of dimension \( \geq 2 \). Let notation be as in Section 2.

Suppose that \( R \) is a domain of dimension \( d \geq 2 \) with a dualizing module. By the Theorem of Finiteness, Theorem VIII.2.1 (and footnote) [7],

\[
\overline{R} = \Gamma(X \setminus \{m\}, \mathcal{O}_X) = \cap_{p \in X \setminus \{m\}} R_p
\]

is a finitely generated \( R \)-module, which lies between \( R \) and its quotient field. Since \( \overline{R}/R \) is \( m \)-torsion,

\[
\lambda_R(\overline{R}/R) < \infty.
\]

Let \( m_1, \ldots, m_\alpha \) be the maximal ideals of \( \overline{R} \) which lie over \( m \). By our construction,

\[
0 = H^1_m(\overline{R}) = H^1_{m_\alpha}(\overline{R}) = \bigoplus_{i=1}^\alpha H^1_{m_i}(\overline{R}),
\]

so

\[
H^1_{m_i}(\overline{R}) = H^1_{m_i}(\overline{R}) \otimes_R \overline{R} = 0
\]

for \( 1 \leq i \leq \alpha \), and thus depth(\( \overline{R}/m_i \)) \( \geq 2 \) for \( 1 \leq i \leq \alpha \).

Let \( F = F \otimes_R \overline{R} \) and \( R[F] = \bigoplus_{k \geq 0} F^k \), so that \( F^k \cong F^k \otimes_R \overline{R} \) for all \( k \). Let \( \overline{F} = \overline{R} E \) be the \( \overline{R} \)-submodule of \( \overline{F} \) generated by \( E \). Let \( \overline{R}[E] = \bigoplus_{k \geq 0} \overline{E}^k \) be the \( \overline{R} \)-subalgebra of \( \overline{R}[F] \) generated by \( \overline{E} \).

Let \( u_1, \ldots, u_s \) be generators of the ideal \( m \). For \( k \in \mathbb{N} \), let \( \overline{E}^k \) be the sheafification of \( E^k \) on \( X = \text{Spec}(R) \).
There are identities
\[ H^0(X \setminus \{m\}, \tilde{E}^k) = \cap_{i=1}^\alpha (E^k)_{u_i} = E^k : E^k \cdot m^\infty. \]
From the exact sequence of cohomology groups
\[ 0 \to H^0_m(E^k) \to E^k \to H^0_m(X \setminus \{m\}, \tilde{E}^k) \to H^1_m(E^k) \to 0, \]
we deduce that we have isomorphisms of \( R \)-modules
\[ H^1_m(E^k) \cong E^k : E^k \cdot m^\infty / E^k \]
for \( k \geq 0 \). The same calculation for \( F^k \) shows that
\[ H^1_m(F^k) \cong F^k : F^k \cdot m^\infty / F^k. \]
From the left exact local cohomology sequence
\[ 0 \to H^0_m(F^k/E^k) \to H^1_m(E^k) \to H^1_m(F^k), \]
we have that
\[ H^0_m(F^k/E^k) \cong \left( E^k : E^k \cdot m^\infty \right) / E^k = E^k : F^k \cdot m^\infty / F^k. \]

**Theorem 4.1.** Suppose that \((R, m)\) is a local domain of dimension \( d \geq 2 \) which is essentially of finite type over a field \( K \) of characteristic zero (or over a perfect field \( K \) such that \( R/m \) is algebraic over \( K \)). Suppose that \( E \) is a rank \( e \) submodule of a finitely generated free \( R \)-module \( F \). Then the limit
\[ \lim_{k \to \infty} \frac{\lambda \left( E^k : F^k \cdot m^\infty / E^k \right)}{k^d + e - 1} \in \mathbb{R} \]
exists.

**Proof.** Since \( E^k : m^\infty / E^k \) are finitely generated \( R \)-torsion \( R \)-modules, we have that
\[ E^k : m^\infty / E^k \cong \bigoplus_{i=1}^\alpha \left( E^k_{m_i} : m_i^\infty / E^k_{m_i} \right). \]
By Theorem 1.1 we have that
\[ \lim_{k \to \infty} \frac{\lambda_{\mathbf{R}_{m_i}} \left( E^k_{m_i} : m_i^\infty / E^k_{m_i} \right)}{k^d + e - 1} \]
exists for \( 1 \leq i \leq \alpha \). Since for any \( \mathbf{R}_{m_i} \) module \( M \) we have that
\[ \lambda_{\mathbf{R}}(M) = [\mathbf{R}/m_i : \mathbf{R}/m] \lambda_{\mathbf{R}_{m_i}}(M), \]
we conclude that
\[ \lim_{k \to \infty} \frac{\lambda_{\mathbf{R}} \left( E^k : m^\infty / E^k \right)}{k^d + e - 1} \]
exists. We have
\[ E^k : m^\infty = \cap_{i=1}^\alpha (E^k)_{u_i} = \cap_{i=1}^\alpha (E^k)_{u_i} = E^k : m^\infty. \]
Consider the short exact sequences
\[ 0 \to E^k / E^k \to E^k : m^\infty / E^k \to E^k : m^\infty / E^k \to 0. \]
Now \( \overline{R[E]}/R[E] \) is a finitely generated \( R[E] \)-module, and the support of the \( R \)-module \( \overline{E}^k/E^k \) is contained in \( \{m\} \) for all \( k \), so there exists a positive integer \( r \) such that \( m^r \) annihilates \( \overline{R[E]}/R[E] \). Thus (as in the argument following equation (17) in the proof of Theorem 3.2), we have that there exists a constant \( \beta \) such that

\[
\lambda_R(\overline{E}^k/E^k) \leq \beta k^{d+e-2}
\]

for all \( k \). The conclusions of the proposition now follow from (29), (31) and (30).

\[ \boxed{} \]

I thank Craig Huneke, Bernd Ulrich and Javid Validashti for pointing out the following consequence of Theorem 4.1.

**Corollary 4.2.** Suppose that \((R, m)\) is a local domain of dimension \( d \geq 2 \) which is essentially of finite type over a field \( K \) of characteristic zero (or over a perfect field \( K \) such that \( R/m \) is algebraic over \( K \)). Suppose that \( E \) is a rank \( e \) submodule of a finitely generated free \( R \)-module \( F \). Suppose that \( \gamma = \text{rank}(F) < d + e \). Then the limits

\[
\lim_{k \to \infty} \frac{\lambda \left( E_k^k : F_k m^\infty / F_k m^\infty \right)}{k^{d+e-1}} \in \mathbb{R}
\]

and

\[
\lim_{k \to \infty} \frac{(d + e - 1)!}{k^{d+e-1}} \lambda(\mathcal{H}_m^0(F_k^k/E_k^k)) \in \mathbb{R}
\]

exist. In particular, the epsilon multiplicity \( \varepsilon(E) \) of \( E \) exists as a limit.

**Proof.** We will establish that the limit (32) exists. We have exact sequences

\[
0 \to E_k^k : F_k m^\infty / F_k^k \to E_k^k : F_k m^\infty / E_k^k \to E_k^k : F_k m^\infty : F_k m^\infty \to 0
\]

and inclusions

\[
E_k^k : F_k m^\infty / E_k^k : F_k m^\infty = E_k^k : F_k m^\infty / \left( (E_k^k : F_k m^\infty) \cap F_k^k \right) \to F_k^k : F_k : m^\infty / F_k
\]

for \( k \geq 0 \).

We have

\[
F_k^k : F_k m^\infty / F_k^k = \mathcal{F}^{k} / F_k^k \cong (R/R)^{(k+\gamma - 1)}. \]

Since \( \gamma = \text{rank}(F) < d + e \), we have

\[
\lim_{k \to \infty} \frac{\lambda_R \left( E_k^k : F_k m^\infty / F_k^k \right)}{k^{d+e-1}} = 0.
\]

The existence of the limit (32) now follows from (34) and Theorem 4.1. The existence of the limit (33) is immediate from (32) and (27).

\[ \boxed{} \]

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