PAIRING AND QUANTUM DOUBLE
OF MULTIPLIER HOPF ALGEBRAS

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Abstract. We define and investigate pairings of multiplier Hopf algebras. It is shown that two dually
paired regular multiplier Hopf (∗-)algebras \( A \) and \( B \) yield a quantum double multiplier Hopf (∗-)algebra
which is again regular. Integrals on \( A \) and \( B \) induce an integral on the quantum double. The results
generalize pairing and quantum double construction from ordinary Hopf algebras to multiplier Hopf
algebras.

Introduction

The non-commutative generalization of the abelian \( C^* \)-algebra of continuous complex functions
over a compact group are the so-called compact quantum groups or compact quantum group
algebras \([Wo1,DK]\). The notion of a Hopf algebra enters the construction of such objects. A
multiplier Hopf algebra \( A \) is a not necessarily unital generalization of Hopf algebras where the
image of the comultiplication \( \Delta \) is contained in the multiplier algebra \( M(A \otimes A) \), instead of \( A \otimes A \)
\([VD2]\). If \((A, \Delta)\) has an integral \([VD3,Swe]\) and is regular – i.e. also the co-opposite multiplier
Hopf algebra \((A, \Delta^{op})\) exists– then the dual \((\hat{A}, \hat{\Delta})\) is again a regular multiplier Hopf algebra and
has an invariant integral \([VD3]\). It is also shown in \([VD3]\) that the dual of \((\hat{A}, \hat{\Delta})\) is canonically
isomorphic to \((A, \Delta)\). So, in this case, duality can now be described within the same category.

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For instance the algebra $C_c(G)$ of (continuous) complex functions with compact support on a discrete group $G$ is a multiplier Hopf algebra in a canonical way \cite{VD3,VD4}. Also the discrete quantum groups \cite{ER,VD4} (as well as the compact quantum groups) are multiplier Hopf algebras. And therefore the duality of discrete quantum groups and compact quantum groups \cite{ER,VD4} turns out to be the duality in the category of multiplier Hopf algebras.

There is good hope to extend the notion of (regular) multiplier Hopf algebras (with integral) to a topological version. This could serve as fruitful starting point for a systematic definition of locally compact quantum groups. It seems that the new theory then contains all the special examples existing so far in the literature \cite{MN,PW,Wo2} including the locally compact groups. Also there duality and the existence of a positive integral or Haar measure will play an important rôle.

In the present article we are interested in the more general notion of pairing of (regular) multiplier Hopf algebras. The dual pairing of $A$ and $\hat{A}$ will be seen to be a special case. This has already been announced in \cite{VD3}. We show that two dually paired multiplier Hopf algebras admit the construction of a quantum double object which is again a multiplier Hopf algebra. Regularity and $*$-property as well as the existence of an integral can be proven also for the quantum double. Hence we are able to construct a quantum double multiplier Hopf algebra within the same category. This procedure yields further interesting, non-trivial examples of multiplier Hopf algebras.

The results of this paper generalize the well known properties of Hopf algebra pairing \cite{Ma1,VD1} and the construction of a quantum double out of two dually paired Hopf algebras \cite{Dri,Ma2,VD1}. Although there is an obvious loss of categorical symmetry in the defining equations passing from Hopf algebras to multiplier Hopf algebras many features of the theory of Hopf algebras can be extended to the multiplier Hopf algebra setting. One reason for this is the fact that the defining Hopf relations generalize to the level of the multipliers. However it is not yet clear, for instance, if for the pairing $(A, \hat{A})$ the quantum double multiplier Hopf algebra $D(A)$ can be reconstructed from a category of modules as in the usual Hopf algebra case \cite{JS,Ma3}. This is one of the open questions which are currently under investigation.

In Chapter 1 we repeat the main definitions and results on multipliers and multiplier Hopf algebras and provide several lemmas and propositions which are used in the sequel. Chapter 2 introduces the notion of (pre-)pairings of multiplier Hopf algebras. The definition of so-called multiplier Hopf algebra pre-pairing leads to several equivalent conditions which serve as additional axioms for the definition of multiplier Hopf algebra pairing. The ordinary Hopf algebra pairing is a special case of this construction. Using the results of Chapter 2 we construct in Chapter 3 the quantum double of a dually paired couple of multiplier Hopf algebras $(A, B)$. We will prove that the quantum double is again a regular multiplier Hopf algebra. There exists an integral on the quantum double if those exist on $A$ and $B$. If $A$ and $B$ are multiplier Hopf $*$-algebras we prove that the quantum double has a $*$-structure. In many of our calculations we use a “generalized Sweedler notation” which will be outlined in the Appendix.

1. Preliminaries on Multiplier Hopf Algebras
Henceforth we work with modules over the field $k = \mathbb{C}$ or $k = \mathbb{R}$. By an associative algebra $A$ (over $k$) we mean an algebra which need not contain a unit. Hence this notion is more general than the one for unital algebras. We suppose that all algebras under consideration have a non-degenerate product, i.e. $ab = 0$ for all $a \in A$ implies $b = 0$ and from $ab = 0$ for all $b \in A$ it follows that $a = 0$. With $A$ and $B$ two non-degenerate algebras the tensor algebra $A \otimes B$ is obviously non-degenerate, too.

A multiplier $\rho = (\rho_1, \rho_2)$ of the algebra $A$ is a pair of linear mappings in $\text{End}_k(A)$ such that $\rho_2(a)b = a\rho_1(b)$ for all $a, b \in A$. The set of multipliers of $A$ will be denoted by $\text{M}(A)$. It is a unital algebra which contains $A$ as essential ideal through the embedding $a \mapsto (a, a)$. Hence $\rho \cdot a = (\rho_1(a), \cdot \rho_1(a)) \equiv \rho_1(a)$ and $a \cdot \rho = (\rho_2(a), \cdot \rho_2(a)) \equiv \rho_2(a)$ for all $\rho \in \text{M}(A)$ and $a \in A$. Therefore we will frequently use the identification $a \cdot \rho = \rho_2(a)$ and $\rho \cdot a = \rho_1(a)$. If $A$ is unital then $A = \text{M}(A)$. If $A$ is a $*$-algebra then $\text{M}(A)$ is a $*$-algebra through $\rho^* = (\rho_2^*, \rho_1^*)$ where $\psi^*(a) := \psi(a^*)^*$ for any $a \in A, \psi \in \text{End}_k(A)$. Since the multiplication of $A$ is supposed to be non-degenerate a multiplier $\rho = (\rho_1, \rho_2)$ of $A$ is uniquely determined by its first or second component. For a tensor product of two algebras $A$ and $B$ one obtains the canonical algebra embeddings

\[
A \otimes B \hookrightarrow \text{M}(A) \otimes \text{M}(B) \hookrightarrow \text{M}(A \otimes B) .
\] (1.1)

We often work with extensions of algebra morphisms and module maps without mentioning it explicitly. In the following we will outline this notation. We refer the reader to this exposition whenever she or he suspects to meet extensions in the course of the paper.

Let $A$ and $B$ be algebras, and $\varphi : A \rightarrow \text{M}(B)$ be an algebra morphism. Then $\varphi$ is called non-degenerate algebra morphism if $B = \text{span}\{\varphi(a)b \mid a \in A, b \in B\} = \text{span}\{b\varphi(a) \mid a \in A, b \in B\}$. Analogous conditions hold for non-degenerate $*$-algebra morphisms. We call an $A$-left module $X$ non-degenerate with respect to $A$ if the module map $\mu : A \otimes X \rightarrow X$ is surjective and if $\mu(a \otimes x) = 0$ for all $a \in A$ implies $x = 0$. A similar definition holds for $A$-right modules. The following propositions can now be proved in a similar way as outlined in [VD2].

**Proposition 1.1.** Any non-degenerate algebra morphism has a unique extension to an algebra morphism $\varphi : \text{M}(A) \rightarrow \text{M}(B)$. \qed

**Proposition 1.2.** Let $A$ and $B$ be algebras, and $B$ be a non-degenerate $A$-left module through the module map $\mu : A \otimes B \rightarrow B$ Then there exists a unique extension $\mu : \text{M}(A) \otimes B \rightarrow B$ rendering $B$ an $\text{M}(A)$-left module. \qed

These notions of non-degeneracy are automatic for unital algebras. We will now give the definition of multiplier Hopf algebras as they were introduced in [VD2].

**Definition 1.3.** Let $A$ be an algebra. An algebra morphism $\Delta : A \rightarrow \text{M}(A \otimes A)$ is called a comultiplication on $A$ if for all $a, a' \in A$

\[
\begin{align*}
T_1(a \otimes a') &:= \Delta(a)(1 \otimes a') \\
T_2(a \otimes a') &:= (a \otimes 1)\Delta(a')
\end{align*}
\] (1.2)

and if the linear mappings $T_1, T_2 : A \otimes A \rightarrow A \otimes A$ obey the relation

\[
(T_2 \otimes \text{id}) \circ (\text{id} \otimes T_1) = (\text{id} \otimes T_1) \circ (T_2 \otimes \text{id}) .
\] (1.3)
If \( T_1 \) and \( T_2 \) are bijective then the pair \((A, \Delta)\) is called a multiplier Hopf algebra or shortly MHA. If \( A \) is a \(*\)-algebra we demand \( \Delta \) to be a \(*\)-algebra homomorphism. The multiplier Hopf algebra \((A, \Delta)\) is called regular if in addition \( A_{\text{op}} := (A, \Delta^{\text{op}}) \) is a multiplier Hopf algebra, where \( \Delta^{\text{op}} \) is the opposite comultiplication, \( \Delta^{\text{op}}(a)(b \otimes c) = \tau(\Delta(a)(c \otimes b)) \) for \( a, b, c \in A \), and henceforth \( \tau : A \otimes A \to A \otimes A \) is the usual tensor transposition.

**Remark 1.** Equation (1.3) replaces the coassociativity of the comultiplication of ordinary Hopf algebras. The definition of a multiplier Hopf algebra however guarantees that the comultiplication is coassociative in the sense that

\[
(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta : M(A) \to M(A \otimes A \otimes A). \tag{1.4}
\]

This fact is used in particular in the appendix to define a “generalized Sweedler notation” which will be helpful in the calculations of many proofs in the paper.

From [VD2] it is known that multiplier Hopf algebras automatically possess a unique counit \( \varepsilon \) and an antipode \( S \) such that \( \varepsilon(a) a' = m \circ T_1^{-1}(a \otimes a') \) and \( S(a) a' = (\varepsilon \otimes id) \circ T_1^{-1}(a \otimes a') \). For regular MHA’s the antipode is bijective and \( S_{\text{op}} = S^{-1}, \varepsilon_{\text{op}} = \varepsilon \). In this case the corresponding mappings \( T_{\text{op}1} \) and \( T_{\text{op}2} \) for \( A_{\text{op}} \) can be expressed as follows.

\[
T_{\text{op}1} = (id \otimes S^{-1}) \circ \tau \circ T_{\text{op}2}^{-1} \circ \tau \circ (id \otimes S) \tag{1.5}
\]

\[
T_{\text{op}2} = (S^{-1} \otimes id) \circ \tau \circ T_{\text{op}1}^{-1} \circ \tau \circ (S \otimes id).
\]

For a multiplier Hopf algebra \((A, \Delta)\) and any linear functional \( \omega \in A' \) one can define a multiplier \((id \otimes \omega)\Delta(a) \in M(A)\) for any \( a \in A \) according to \([(id \otimes \omega)\Delta(a)] \cdot a' := (id \otimes \omega)(\Delta(a) \cdot (a' \otimes 1))\) and \( a' \cdot [(id \otimes \omega)\Delta(a)] := (id \otimes \omega)((a' \otimes 1) \cdot \Delta(a))\) for all \( a' \in A \). Analogous results hold for \((\omega \otimes id)\Delta(a)\).

In the same manner the following statements can be proven easily.

**Lemma 1.4.** Let \((A, \Delta)\) be a multiplier Hopf algebra and \( \omega \in A' \) be a linear functional of \( A \). Then \((\omega \otimes id \otimes id)(id \otimes \Delta)\Delta(a)\) is a multiplier in \( M(A \otimes A) \) for all \( a \in A \) and it holds

\[
(\omega \otimes id \otimes id)(id \otimes \Delta)\Delta(a) = \Delta((\omega \otimes id)\Delta(a)). \tag{1.6}
\]

Analogously one obtains the multiplier identity

\[
(id \otimes id \otimes \omega)(\Delta \otimes id)\Delta(a) = \Delta((id \otimes \omega)\Delta(a)). \tag{1.7}
\]

\[ \Box \]

## 2. Pairing of Multiplier Hopf Algebras
In this chapter we consider bilinear functionals between regular multiplier Hopf algebras. We give the definitions of pre-pairing and pairing of multiplier Hopf algebras, and we deduce results which are necessary for the investigation of quantum doubles of MHA$\text{s}$. It will be seen that the pairing of two ordinary Hopf algebras and the pairing of a regular multiplier Hopf algebra with non-trivial integral with its dual $\hat{A}$ [VD3] are special cases of MHA pairings.

**Definition 2.1.** Let $A$ and $B$ be two regular multiplier Hopf algebras, and $\langle \cdot, \cdot \rangle : A \otimes B \to k$ be a linear mapping. Define for all $a \in A$ and $b \in B$ the linear functionals $\omega_a := \langle a, \cdot \rangle \in B'$ and $\omega_b := \langle \cdot, b \rangle \in A'$, and assume that they obey the following properties, where $a' \in A$ and $b' \in B$.

1. $(\omega_a \otimes \text{id})\Delta(b) \in B$ and $(\text{id} \otimes \omega_a)\Delta(b) \in A$,
2. $(\omega_b \otimes \text{id})\Delta(a) \in A$ and $(\text{id} \otimes \omega_b)\Delta(a) \in A$,
3. $\omega(a' \omega \otimes \text{id})\Delta(b) = a' \omega_a(\omega \otimes \text{id})\Delta(b) = a' \omega_a(b)$,
4. $\omega_b(\omega \otimes \text{id})\Delta(a) = \omega(\omega_b \otimes \text{id})\Delta(a) = \omega(b)$.

Then $(A, B, (\cdot, \cdot))$ is called a multiplier Hopf algebra pre-pairing. The pre-pairing $(A, B, (\cdot, \cdot))$ is called non-degenerate if $A$ and $B$ are dual with respect to the bilinear form $(\cdot, \cdot)$.

**Remark 2.** The usual Hopf algebra pairing is a special case of a multiplier Hopf algebra pre-pairing because (1) and (2) of Definition 2.1 are trivially fulfilled, and for instance $\omega_a(\text{id} \otimes a' \omega)\Delta(b) = \langle a \otimes a', \Delta(b) \rangle = a' \omega_a(\omega \otimes \text{id})\Delta(b) = \langle a a', b \rangle = a a' \omega(b)$.

For any multiplier Hopf algebra pre-pairing $(A, B, (\cdot, \cdot))$ we can therefore define the linear mappings

\begin{align*}
\mu^l_{A,B} : & \quad A \otimes B \to B \\
& \quad a \otimes b \mapsto (\text{id} \otimes \omega_a)\Delta(b) \tag{2.1} \\
\mu^r_{A,B} : & \quad B \otimes A \to B \\
& \quad b \otimes a \mapsto (\omega_b \otimes \text{id})\Delta(b) \tag{2.2} \\
\mu^l_{B,A} : & \quad A \otimes A \to A \\
& \quad a \otimes b \mapsto (\text{id} \otimes \omega_b)\Delta(a) \tag{2.3} \\
\mu^r_{B,A} : & \quad A \otimes B \to A \\
& \quad a \otimes b \mapsto (\omega \otimes \text{id})\Delta(a) \tag{2.4}
\end{align*}

In the special case of ordinary Hopf algebra pairings the mappings $\mu^l_{A,B}$, etc. are given by $\mu^l_{A,B}(a \otimes b) = b(1) \langle a, b(2) \rangle$, etc. These mappings are actions [Ma2]. The same is true for multiplier Hopf algebra pre-pairings as it is described in the following proposition.

**Proposition 2.2.** Let $(A, B, (\cdot, \cdot))$ be a multiplier Hopf algebra pre-pairing. Then the maps $\mu^l_{A,B}$ and $\mu^r_{A,B}$ are actions of $A$ on $B$, i.e. $(B, \mu^l_{A,B})$ is an $A$-left module and $(B, \mu^r_{A,B})$ is an $A$-right module respectively. Analogously $\mu^l_{B,A}$ and $\mu^r_{B,A}$ are left and right actions of $B$ on $A$ respectively.

**Proof.** We use Definition 2.1 and in particular $\omega(a' \omega \otimes \text{id})\Delta(b) = a' \omega_a(\omega \otimes \text{id})\Delta(b) = a a' \omega(b)$ for all $a, a' \in A$ and $b, b' \in B$. Then we arrive at

\begin{align*}
\mu^l_{A,B}(a a' \otimes b) \cdot b' &= (\text{id} \otimes a a' \omega)(\Delta(b)(b' \otimes \text{1})) \\
&= (\text{id} \otimes a a' \omega)[(\text{id} \otimes \text{id} \otimes a a' \omega)(\text{id} \otimes \text{\text{Delta}}(b)) \cdot (b' \otimes \text{1})] \\
&= (\text{id} \otimes a a' \omega)\Delta((\text{id} \otimes a a' \omega)\Delta(b)) \cdot b' \\
&= \mu^l_{A,B}(a \otimes \mu^l_{A,B}(a' \otimes b)) \cdot b' \tag{2.5}
\end{align*}
Lemma 2.4. From Lemma 2.3 we immediately get

\[ \mu_{A,B}(a \otimes b) = a \triangleright b \text{ and } \mu_{B,A}(a \otimes b) = a \triangleleft b \] which means “\(a\) acts from the left on \(b\)” and “\(b\) acts from the right on \(a\)”, respectively, according to the direction of the arrows “\(\triangleright\)” and “\(\triangleleft\)

\[ \therefore \]

Henceforth the actions will be denoted by “\(\triangleright\)” and “\(\triangleleft\)” if the meaning is clear. For example \(\mu_{A,B}^l(a \otimes b) = a \triangleright b\) and \(\mu_{B,A}^r(a \otimes b) = a \triangleleft b\) which means “\(a\) acts from the left on \(b\)” and “\(b\) acts from the right on \(a\)” respectively, according to the direction of the arrows “\(\triangleright\)” and “\(\triangleleft\)”.  

Lemma 2.3. Let \((A, B, \langle \cdot, \cdot \rangle)\) be a multiplier Hopf algebra pre-pairing. Then we obtain

\[ \langle b \triangleright a, b' \rangle = \langle a, b' \rangle \]
\[ \langle a \triangleright b, b' \rangle = \langle a, b' \rangle \]
\[ \langle a, a' \triangleright b \rangle = \langle a a', b \rangle \]
\[ \langle a, b \triangleright a' \rangle = \langle a' a, b \rangle \]

for \(a, a' \in A\) and \(b, b' \in B\).  

Proof. The proof is a direct consequence of Definition 2.1. For instance we get for \(a \in A\) and \(b, b' \in B\): \(\langle a, b b' \rangle = \omega_{b'}(a) = \omega_{b'}(\text{id} \otimes \omega_b) \Delta(a) = \langle b \triangleright a, b' \rangle\).  

From Lemma 2.3 we immediately get

Lemma 2.4. If the MHA pre-pairing \((A, B, \langle \cdot, \cdot \rangle)\) is non-degenerate then \((A, \mu_{B,A}^l, \mu_{B,A}^r)\) is a \(B\)-bimodule and \((B, \mu_{A,B}^l, \mu_{A,B}^r)\) is an \(A\)-bimodule.

Proof. Let \(a \in A\) and \(b_1, b_2, b_3 \in B\). Consider \((b_1 \triangleright a) \triangleleft b_2\) and \((b_1 \triangleright a) \triangleleft b_2\) paired with \(b_3\). Using Lemma 2.3 and associativity of \(B\) yields \(\langle (b_1 \triangleright a) \triangleleft b_2, b_3 \rangle = \langle a, b_2 b_3 b_1 \rangle = \langle b_1 \triangleright (a \triangleleft b_2), b_3 \rangle\) and this proves the lemma because of the non-degeneracy of the pre-pairing.

Through the help of Lemma 1.4 the commutation rules of the actions with the comultiplications of a MHA pre-pairing \((A, B, \langle \cdot, \cdot \rangle)\) can be determined. Similarly as in the comments preceding Lemma 1.4 one observes that for all \(a \in A\) and \(b \in B\) the following multipliers can be defined.

\[\begin{align*}
\left(\text{id} \otimes (\cdot) \triangleleft a\right) \Delta_B(b) &\in M(B \otimes B), \\
\left(\text{id} \otimes a \triangleright (\cdot)\right) \Delta_B(b) &\in M(A \otimes A), \\
\left((\cdot) \triangleleft a \otimes \text{id}\right) \Delta_B(b) &\in M(B \otimes B), \\
\left(\text{id} \otimes (\cdot) \triangleright a\right) \Delta_A(a) &\in M(A \otimes A).
\end{align*}\]

For example \([(\text{id} \otimes (\cdot) \triangleleft a) \Delta_B(b)](b' \otimes b'') := (b_1(1) b') \otimes (b_2(2) a) b''\) and \((b' \otimes b'') \cdot [(\text{id} \otimes (\cdot) \triangleleft a) \Delta_B(b)] = (b' b_1(1)) \otimes b''(b_2(2) a)\) for any \(b', b'' \in B\), where the generalized Sweedler notation is used which is explained in the Appendix. Hence

Proposition 2.5. Let \((A, B, \langle \cdot, \cdot \rangle)\) be a MHA pre-pairing. Then for all \(a \in A\) and \(b \in B\) we have

\[\begin{align*}
\Delta_B(a \triangleright b) &= (\text{id} \otimes a \triangleright (\cdot)) \Delta_B(b), \\
\Delta_B(b \triangleleft a) &= ((\cdot) \triangleleft a \otimes \text{id}) \Delta_B(b) \\
\Delta_B^\text{op}(a \triangleright b) &= (a \triangleright (\cdot) \otimes \text{id}) \Delta_B^\text{op}(b), \\
\Delta_B^\text{op}(b \triangleleft a) &= ((\cdot) \triangleleft (\cdot) \triangleleft a) \Delta_B^\text{op}(b).
\end{align*}\]
and analogously for $\Delta_A$.

**Proof.** Let $a, a' \in A$ and $b, b' \in B$. Using Lemma 1.4 and the coassociativity of $\Delta$ yields

\[
\Delta(a \triangleright b) \cdot (a' \otimes b') = \Delta((\text{id} \otimes a_N) \Delta(b))(a' \otimes b')
= (\text{id} \otimes \text{id} \otimes a_N)(\text{id} \otimes \Delta[b(a' \otimes 1)])(1 \otimes b')
= (\text{id} \otimes a_{\triangleright} \cdot) \Delta(b) \cdot (a' \otimes b').
\] (2.9)

Proceeding in an analogous manner completes the proof of the proposition. \qed

**Proposition 2.6.** Let $(A, B, \langle \cdot , \cdot \rangle)$ be a MHA pre-pairing. Then

\[
\langle T_2^A(a \otimes a'), b \otimes b' \rangle = \langle a \otimes a', T_1^B(b \otimes b') \rangle
\] (2.10)

for $a, a' \in A$ and $b, b' \in B$. If in addition $\mu_{B, A}^l$ and $\mu_{A, B}^r$ are surjective, we have

\[
\langle S_A(a), b \rangle = \langle a, S_B(b) \rangle.
\] (2.11)

Analogous results can be derived if $\mu_{B, A}^r$ and $\mu_{A, B}^l$ are supposed to be surjective. Under the surjectivity condition of the proposition also the following identities hold.

\[
S^{\pm 1}(b \triangleright a) = S^{\pm 1}(a) \triangleright S^\pm 1(b) \quad \text{and} \quad S^{\pm 1}(a \triangleright b) = S^{\pm 1}(b) \triangleright S^\pm 1(a).
\] (2.12)

**Proof.** Let $a, a' \in A$ and $b, b' \in B$. Using Lemma 2.3 and the generalized Sweedler notation yields

\[
\langle T_2(a \otimes a'), b \otimes b' \rangle = \langle (a \otimes 1) \Delta(a'), b \otimes b' \rangle
= \langle a a'_{(1)}, b \rangle \langle a'_{(2)}, b' \rangle = \langle a (b \triangleright a'), b \rangle
= \langle b \triangleleft a', b \triangleright a \rangle
\] (2.13)

and

\[
\langle a \otimes a', T_1(b \otimes b') \rangle = \langle a, b_{(1)} \rangle \langle a', b_{(2)} \rangle
= \langle a', (b \triangleright a) b' \rangle = \langle b \triangleright a', b \triangleright a \rangle.
\] (2.14)

which leads to (2.10). Since $T_1$ and $T_2$ are bijective, eqn. (2.10) is also valid for the inverse mappings. With the help of (1.5) one obtains similarly as before

\[
\langle T_2^{-1}(a \otimes a'), b \otimes b' \rangle = \langle (S \otimes \text{id})(\Delta(a)(S^{-1}(a') \otimes 1)), b \otimes b' \rangle
= \langle S((\text{id} \otimes \omega_{b'})\Delta(a)(S^{-1}(a) \otimes 1)), b \rangle
= \langle a S(b \triangleright a'), b \rangle
= \langle S(b \triangleright a'), b \triangleright a \rangle
\]

and on the other hand

\[
= \langle a \otimes a', T_1^{-1}(b \otimes b') \rangle
= \langle a', S((\omega \otimes \text{id})(1 \otimes S^{-1}(b'))\Delta(b)) \rangle
= \langle a', S(b \triangleright a) b' \rangle
= \langle b \triangleright a', S(b \triangleright a) \rangle
\]
which proves (2.11) because \( \mu^l_{B,A} \) and \( \mu^r_{A,B} \) are surjective by assumption. For the verification of (2.12) we are making use of (2.11).

\[
S(b \cdot a) \cdot S(a') = (\text{id} \otimes \omega_b)(S \otimes \text{id})(a' \otimes 1)\Delta(a)
\]
\[
= (\text{id} \otimes \omega_{S^{-1}(b)})(\Delta_{op}(S(a))(S(a') \otimes 1))
\]
\[
= (S(a) \triangleleft S^{-1}(b)) \cdot S(a'). \tag{2.15}
\]

Assuming the surjectivity conditions of Proposition 2.6 we obtain identities which relate \( \mu^r_{A,B} \) and \( \mu^l_{A,B} \) in a multiplier Hopf algebra pre-pairing \( (A, B, \langle \cdot, \cdot \rangle) \).

\[
S_B(S_A(a) \cdot b) S_B(b') = S_B(b' (S_A(a) \cdot b))
\]
\[
= S_B((\text{id} \otimes \omega_{S(a)})(b' \otimes 1)\Delta(b))
\]
\[
= (a, \omega \otimes \text{id})\Delta(S_B(b))(1 \otimes S_B(b'))
\]
\[
= (S_B(b) \trianglelefteq a) S_B(b')
\]

for any \( a \in A \) and \( b, b' \in B \). In a similar manner relations for \( \mu^l_{B,A} \) and \( \mu^r_{B,A} \) can be deduced. Explicitly we have

**Lemma 2.7.** Let \( (A, B, \langle \cdot, \cdot \rangle) \) be a multiplier Hopf algebra pre-pairing and assume that \( \mu^l_{B,A} \) and \( \mu^r_{A,B} \) (or \( \mu^r_{B,A} \) and \( \mu^l_{A,B} \)) are surjective. Then it holds

\[
S^{\pm 1}(b \cdot a) = S^{\pm 1}(a) \triangleleft S^{\pm 1}(b)
\]
\[
S^{\pm 1}(a \cdot b) = S^{\pm 1}(b) \triangleleft S^{\pm 1}(a). \tag{2.16}
\]

for any \( a \in A \) and \( b \in B \). \( \square \)

With the help of the bracket \( \langle \cdot, \cdot \rangle \) of a multiplier Hopf algebra pre-pairing \( (A, B, \langle \cdot, \cdot \rangle) \) we can define multipliers according to

\[
R := (\text{id} \otimes \langle \cdot, \cdot \rangle \otimes \text{id}) \circ (\Delta_A \otimes \Delta_B) : A \otimes B \rightarrow M(A \otimes B)
\]
\[
R(a \otimes b) \cdot (a' \otimes b') := (\text{id} \otimes \langle \cdot, \cdot \rangle \otimes \text{id})((\Delta(a)(a' \otimes 1) \otimes \Delta(b)(1 \otimes b'))
\]
\[
(a' \otimes b') \cdot R(a \otimes b) := (\text{id} \otimes \langle \cdot, \cdot \rangle \otimes \text{id})((a' \otimes 1)\Delta(a) \otimes (1 \otimes b')\Delta(b)) \tag{2.17}
\]

where \( a, a' \in A \) and \( b, b' \in B \). Very analogously we define the mapping

\[
\tilde{R} := (\text{id} \otimes \langle \cdot, \cdot \rangle \circ \tau \otimes \text{id}) \circ (\Delta_B \otimes \Delta_A) : B \otimes A \rightarrow M(B \otimes A). \tag{2.18}
\]

**Proposition 2.8.** Let \( (A, B, \langle \cdot, \cdot \rangle) \) be a multiplier Hopf algebra pre-pairing. Then the following conditions are equivalent.

1. \( R(A \otimes B) = A \otimes B \).
2. \( \mu^l_{B,A} \) is surjective.
3. \( \mu^r_{A,B} \) is surjective.
4. \( \tilde{R}(B \otimes A) = B \otimes A \).
5. \( \mu^l_{A,B} \) is surjective.
6. \( \mu^r_{B,A} \) is surjective.
In this case \( R : A \otimes B \rightarrow A \otimes B \) and \( \tilde{R} : B \otimes A \rightarrow B \otimes A \) are bijective with inverse mappings

\[
R^{-1} := (\text{id} \otimes (\cdot, \cdot) \circ (S^{-1} \otimes \text{id}) \otimes \Delta_A \otimes \Delta_B) : A \otimes B \rightarrow A \otimes B,
\]

\[
\tilde{R}^{-1} := (\text{id} \otimes (\cdot, \cdot) \circ (S^{-1} \otimes \text{id}) \circ \tau \otimes \text{id}) \circ (\Delta_B \otimes \Delta_A) : B \otimes A \rightarrow B \otimes A.
\]

Proof. “(1)\(\Rightarrow\)(2)”: Let \( a \otimes b \in A \otimes B \). Then by assumption there is an \( \sum_i p_i \otimes q_i \in A \otimes B \) s.t. \( R(\sum_i p_i \otimes q_i) = a \otimes b \). Hence applying \( (a' \otimes b') \cdot (\cdot) \) and then \( (\text{id} \otimes \omega) \) to both sides of the equation yields \( a' a \cdot a'' \omega(b' b) = a' \sum_i r_i p_i \) where \( r_i = (\text{id} \otimes \omega)((1 \otimes b') \Delta(q_i)) \). Since \( A \) is non-degenerate algebra this yields the result.

“(2)\(\Rightarrow\)(1)”: Let \( a, a' \in A \) and \( b, b', b'' \in B \). Then

\[
(a' \otimes 1) R(b \cdot a \otimes b')(1 \otimes b'')
\]

\[
= (\text{id} \otimes (\cdot, \cdot) \otimes \text{id})[(a' \otimes 1) \Delta(b \cdot a) \otimes \Delta(b')(1 \otimes b'')]
\]

\[
= (\text{id} \otimes (\cdot, \cdot) \otimes \text{id})[(\text{id} \otimes \omega)(\Delta(a') \otimes b'_1 \otimes (b'_2 b''))]
\]

\[
= (\text{id} \otimes \omega(b_{(1)} b_j))(a' \otimes 1)(a' \otimes 1) \Delta(a) \otimes (b'_2 b'')
\]

\[
= (a' \otimes 1)(b'_{(1)} b \otimes a) \otimes b_{(2)} | 1 \otimes b''
\]

where we used in particular Proposition 2.5 and Definition 2.1. Hence

\[
R \circ (\mu_{B,A}^l \otimes \text{id}) = (\mu_{B,A}^l \otimes \text{id}) \circ (\tau \otimes \text{id}) \circ (\text{id} \otimes \tau T_{op1} \tau) \circ (\tau \otimes \text{id})
\]  

(2.19)

from which the statement follows since \( \mu_{B,A}^l \) is supposed to be surjective.

“(1)\(\Leftrightarrow\)(3)”: The proof of this equivalence works pretty similar to the proofs “(1)\(\Leftrightarrow\)(2)”. We consider \( a' a \otimes b' b = (a' \otimes b') R(\sum_j p'_j \otimes q'_j) \) and then we arrive at the relation \( b' b (a'' \omega) = b' \sum_j q'_j \otimes (\omega(b'') \otimes \text{id}) \Delta(p'_j)) \). On the other side we obtain from \( (a' \otimes 1) R(a \otimes b a')(1 \otimes b') \) the identity

\[
R \circ (\text{id} \otimes \mu_{A,B}^r) = (\text{id} \otimes \mu_{A,B}^r) \circ (\text{id} \otimes \tau) \circ (\tau T_{op2} \tau \otimes \text{id}) \circ (\text{id} \otimes \tau).
\]  

(2.20)

“(4)\(\Leftrightarrow\)(5)\(\Leftrightarrow\)(6)”: In an analogous manner the equivalence of the conditions (4), (5) and (6) can be proved.

“(3)\(\Leftrightarrow\)(5)”: Essentially this proof has been done in (2.16).

To prove that \( \Theta := (\text{id}_A \otimes (\cdot, \cdot) \circ (S^{-1} \otimes \text{id}) \otimes \text{id}_B) \circ (\Delta_A \otimes \Delta_B) : A \otimes B \rightarrow M(A \otimes B) \) is the inverse mapping of \( R \) observe that

\[
\Theta \circ (\mu_{B,A}^l \otimes \text{id}) = (\mu_{B,A}^l \otimes \text{id}) \circ (\tau \otimes \text{id}) \circ [\text{id} \otimes (S^{-1} \otimes \text{id}) T_2 (S \otimes \text{id})] \circ (\tau \otimes \text{id})
\]  

(2.21)

which can be derived similarly as eqn. (2.19). From (1.5) one derives

\[
(\tau \otimes \text{id}) \circ (\text{id} \otimes (S^{-1} \otimes \text{id}) T_2 (S \otimes \text{id})) \circ (\tau \otimes \text{id}) = [(\tau \otimes \text{id}) \circ (\text{id} \otimes \tau T_{op1} \tau) \circ (\tau \otimes \text{id})]^{-1}.
\]

Hence the comparison of (2.19) with (2.21) leads to \( \Theta = R^{-1} \) because \( \mu_{B,A}^l \) is surjective. Similarly the inverse of \( \tilde{R} \) can be determined. \( \square \)

We have provided enough results to define the notion of a pairing of multiplier Hopf algebras.
Definition 2.9. A multiplier Hopf algebra pairing \( (A, B, \langle \cdot, \cdot \rangle) \) is a multiplier Hopf algebra pre-pairing where the conditions of the previous proposition are fulfilled. If \( A \) and \( B \) are multiplier Hopf \( * \)-algebras then we demand additionally \( \langle a^*, b \rangle = \langle a, S(b)^* \rangle \).

Remark 3. (1) In particular an ordinary Hopf algebra pairing \( (H_1, H_2, \langle \cdot, \cdot \rangle)_\text{Hopf} \) [Ma1,VD1] constitutes a multiplier Hopf algebra pairing since, for instance, \( \mu^l_{H_1,H_2} \) is surjective and therefore the conditions of Proposition 2.8 are fulfilled. Conversely a non-degenerate MHA pairing \( (H_1, H_2, \langle \cdot, \cdot \rangle) \) of the Hopf algebras \( H_1 \) and \( H_2 \) is a Hopf algebra pairing. Indeed, the (co-)multiplication and antipode properties of \( \langle \cdot, \cdot \rangle \) are obvious from Definition 2.1 and eq. (2.11). From Definition 2.1.(3) we derive \( \langle 1, 1 \rangle = 1 \) for \( a = a' = 1 \) and \( b = 1 \). If we put \( b = 1 \) and \( a = 1 \) in eq. (2.10) and use the non-degeneracy of \( \langle \cdot, \cdot \rangle \) we arrive at \( \langle a'_{(1)}, 1 \rangle a'_{(2)} = a' \). Applying \( \varepsilon \) on both sides of this equation yields \( \langle a', 1 \rangle = \varepsilon(a') \). Analogously \( \langle 1, b \rangle = \varepsilon(b) \) is shown.

(2) If \( (A, \Delta) \) is a regular multiplier Hopf \( (\ast) \)-algebra with non-trivial (left) integral \( \varphi \), then the dual \( \hat{A} \) of \( A \) is also a regular multiplier Hopf \( (\ast) \)-algebra with integral [VD3]. If we take \( \langle \cdot, \cdot \rangle : A \otimes \hat{A} \to k \) to be the evaluation map, then \( (A, \hat{A}, \langle \cdot, \cdot \rangle) \) is a non-degenerate MH\((\ast)\)A pairing. For the verification of this statement we use the results of [VD3]. Because of [Proposition 3.4, VD3] the algebras \( \hat{A} = \{ \varphi(a) \mid a \in A \} \) and \( A \) are dually paired vector spaces with respect to the bracket \( (a, \varphi(a')) := \varphi(a')a \). It is obvious that \( \omega_{a'(\varphi)} \otimes \text{id} \Delta(a) = (\varphi \otimes \text{id})((a' \otimes 1)\Delta(a)) \in A \), and \( \omega_{a'(\varphi)} \varphi(a') = \varphi(a') \varphi(a') = \omega_{a'(\varphi)} \varphi(a') = \omega_{a'(\varphi)} \varphi(a') \varphi(\text{id} \otimes \omega_{a'(\varphi)}) \Delta = \omega_{a'(\varphi)} \varphi(\text{id} \otimes \omega_{a'(\varphi)}) \Delta \) by [Proposition 4.2, VD3], for any \( a, a' \in A \). Similarly the remaining conditions of Definition 2.1 can be proved. The explicit expression for the action \( \mu^l_{A,\hat{A}} \) is given through \( \varphi(a') \triangleright a = (\text{id} \otimes \varphi)((1 \otimes a')\Delta(a)) \). Hence the action is surjective because of the bijectivity of \( T^0_{(1)} \). Since \( \varphi(a')(a') = \bar{\varphi}(a \cdot S(a'^*) \ast) \) the \( \ast \)-property of the bracket \( \langle \cdot, \cdot \rangle \) is a consequence of [Proposition 4.7, VD3]. Summarizing the results we obtain that \( (A, \hat{A}, \langle \cdot, \cdot \rangle) \) is a non-degenerate MH\((\ast)\)A pairing.

Proposition 2.10. For a non-degenerate multiplier Hopf algebra pairing \( (A, B, \langle \cdot, \cdot \rangle) \) the actions \( \mu^l_{A,B}, \mu^r_{A,B}, \mu^l_{B,A} \) and \( \mu^r_{B,A} \) are non-degenerate in the sense of Proposition 1.2.

Proof. For \( b \in B \) let \( ab = 0 \) for all \( a \in A \). This is equivalent to \( (\text{id} \otimes a\omega)\Delta(b) \cdot b' \) for any \( a \in A \) and \( b' \in B \). Acting with \( \varepsilon_B \) on both sides and using \( (\text{id} \otimes \varepsilon_B)T^0_{(1)}(b \otimes b') = \varepsilon_B(b')b \) (see [VD2]) we arrive at \( \varepsilon_B(b') \langle a, b \rangle = 0 \) for all \( a \in A \) and \( b' \in B \). Since \( \varepsilon \neq 0 \) it follows \( b = 0 \). In the same way the non-degeneracy of all other actions will be proved.

Proposition 2.11. If \( (A, B, \langle \cdot, \cdot \rangle) \) is a multiplier Hopf \( \ast \)-algebra pre-pairing, and for all \( a \in A \) and \( b \in B \) it holds \( \langle a^*, b \rangle = \langle a, S(b)^* \rangle \) and \( \langle a, b^* \rangle = \langle S(a)^*, b \rangle \) then

\[
\begin{align*}
(a \triangleright b)^* &= a^* \triangleleft S(b)^*, \\
(b \triangleright a)^* &= S(b)^* \triangleright a^*, \\
(b \triangleright a)^* &= b^* \triangleleft S(a)^*, \\
(a \triangleright b)^* &= S(a)^* \triangleright b^*.
\end{align*}
\] (2.22)

Proof. Let \( a, a' \in A \) and \( b \in B \). Then we obtain

\[
\begin{align*}
(a b)^* \cdot a' &= [(\omega_b \otimes \text{id})\Delta(a)] \cdot a' \\
&= ((\omega_b \otimes \text{id})(1 \otimes a'^*)\Delta(a))^* \\
&= (\langle a_{(1)}, b \rangle a'^* a_{(2)})^* = \langle (a^*)_{(1)}, S(b)^* a_{(2)}^* \rangle a' \\
&= (a^* \triangleleft S(b)^*) \cdot a'.
\end{align*}
\] (2.23)
where we used the $*$-property of multipliers according to Chapter 1 in the second equation, the generalized Sweedler notation as explained in the Appendix in the third equation, and the $*$-property of the bracket $\langle , \rangle$ according to Definition 2.1 in the fourth equality. Hence the first statement of the proposition is verified. Similarly all other equations in (2.22) can be proven. \hfill \Box

Consider a regular multiplier Hopf ($*$-)algebra $(A, \Delta)$. It is clear form [VD2,VD3] and the results of the previous chapter that the opposite co-opposite object $A_{\text{op}}^{\text{op}} := (A^{\text{op}}, \Delta^{\text{op}})$ is again a regular multiplier Hopf ($*$-)algebra. $A^{\text{op}}$ is the opposite algebra to $A$.

**Proposition 2.12.** Let $(A, B, \langle \cdot, \cdot \rangle)$ be a (non-degenerate) multiplier Hopf ($*$-)algebra pairing. Then $(A_{\text{op}}^{\text{op}}, B_{\text{op}}^{\text{op}}, \langle \cdot, \cdot \rangle)$ is again a (non-degenerate) multiplier Hopf ($*$-)algebra pairing.

**Proof.** For $A_{\text{op}}^{\text{op}}$ and $B_{\text{op}}^{\text{op}}$ we obtain

$$
[(\text{id} \otimes \omega_b)\Delta_{\text{op}}(a)]_{1^{\text{op}}}(a') = (\omega_b \otimes \text{id})(1 \otimes a')\Delta(a) = [(\omega_b \otimes \text{id})\Delta(a)]_{2^{\text{op}}}(a') \tag{2.24}
$$

and analogous results can be found for the second component. Because of Definition 2.1 it follows

$$a\omega(\text{id} \otimes a'\omega)\Delta_{\text{op}}(b) = a\omega((a'\omega \otimes \text{id})\Delta(b) = a\omega(\text{id} \otimes a\omega)\Delta(b) = a\omega
$$

where “$\circ$” is the opposite multiplication. If $(A, B, \langle \cdot, \cdot \rangle)$ is a $*$-pairing then the involution is antimultiplicative w.r.t. the opposite multiplication and $\Delta^{\text{op}}$ is $*$-homomorphism. The $*$-property of $\langle \cdot, \cdot \rangle$ according to Definition 2.1 holds for $(A_{\text{op}}^{\text{op}}, B_{\text{op}}^{\text{op}}, \langle \cdot, \cdot \rangle)$ since $S_{\text{op}}^{\text{op}} = S$. From relation (2.24) one deduces that the action $\mu_{B_{\text{op}}^{\text{op}}, A_{\text{op}}^{\text{op}}}^{\text{op}}$ is surjective because $\mu_{B,A}^{\text{op}}$ is surjective. \hfill \Box

On the tensor product $A_{\text{op}}^{\text{op}} \otimes B_{\text{op}}^{\text{op}}$ we can define a mapping according to eqns. (2.17). We will denote it henceforth by $R_{\text{op}}^{\text{op}} := (\text{id} \otimes \langle \cdot, \cdot \rangle \otimes \text{id}) \circ (\Delta^{\text{op}} \otimes \Delta^{\text{op}})$. If $(A, B, \langle \cdot, \cdot \rangle)$ is multiplier Hopf algebra pairing it follows $R_{\text{op}}^{\text{op}}(A \otimes B) = A \otimes B$ (as sets). From the proof of Proposition 2.8 we obtained particular results which will be important for further calculations and which we would like to collect in a lemma.

**Lemma 2.13.** Let $(A, B, \langle \cdot, \cdot \rangle)$ be a multiplier Hopf algebra pre-pairing then the following identities hold.

$$
R(b \bowtie a \otimes b') = (b'(A_1) b) \bowtie a \otimes b'(A_2)
$$

$$
R^{-1}(b \bowtie a \otimes b') = S^{-1}(S(b) b'(A_1) b) \bowtie a \otimes b'(A_2)
$$

$$
R(a \otimes b \bowtie a') = a(A_1) \bowtie b \bowtie a'(A_2)
$$

$$
R^{-1}(a \otimes b \bowtie a') = a(A_1) \bowtie b \bowtie S^{-1}(a(A_2) b)
$$

And because of the symmetry reasons outlined in Proposition 2.12 it follows immediately

$$
R_{\text{op}}^{\text{op}}(a \bowtie b \otimes b') = a \bowtie (b(A_1) b') \otimes b'(A_1)
$$

$$
R_{\text{op}}^{\text{op}}(a \bowtie b \otimes S^{-1}(a(A_2) b) \otimes b'(A_2)
$$

$$
R_{\text{op}}^{\text{op}}(a \bowtie a' \bowtie b) = a(A_1) \bowtie (a(A_1) a') \bowtie b
$$

$$
R_{\text{op}}^{\text{op}}(a \bowtie a' \bowtie b) = a(A_1) \bowtie S^{-1}(a(A_2) a') \bowtie b
$$

This lemma will be used for the proof of the next proposition.
Proposition 2.14. Let \((A, B, \langle \cdot, \cdot \rangle)\) be a non-degenerate multiplier Hopf algebra pairing. Then the following relations are fulfilled.

\[
R_{\text{op}}^{\text{op}} \circ (S^{\perp 1} \otimes S^{\perp 1}) = (S^{\perp 1} \otimes S^{\perp 1}) \circ R, \\
R \circ R_{\text{op}}^{\text{op}} = R_{\text{op}}^{\text{op}} \circ R.
\] (2.28)

Proof. For the proof one uses the first part of Proposition 2.6 and \(\Delta_{\text{op}} \circ S^{\perp 1} = (S^{\perp 1} \otimes S^{\perp 1}) \circ \Delta\) according to the results of [VD2]. Then

\[
R_{\text{op}}^{\text{op}} \circ (S^{\perp 1} \otimes S^{\perp 1})(a \otimes b) = (\text{id} \otimes \langle \cdot, \cdot \rangle \otimes \text{id})(S^{\perp 1} \otimes (S^{\perp 1} \otimes S^{\perp 1}))(\Delta \otimes \Delta)(a \otimes b) \\
= (S^{\perp 1} \otimes S^{\perp 1}) R(a \otimes b).
\]

The commutativity of \(R\) and \(R_{\text{op}}^{\text{op}}\) will be proved in two steps. At first one verifies without problems that

\[
[R \circ R_{\text{op}}^{\text{op}}(a \otimes b \otimes b')] \cdot (1 \otimes b''') = b_1' \triangleright a \triangleleft (b_3'(b_2') \otimes (b_2')b'''), \\
[R_{\text{op}}^{\text{op}} \circ R(b \otimes a \otimes b')] \cdot (1 \otimes b''') = (b_1')b \triangleright a \triangleleft (b_3'(b_2') \otimes (b_2')b''')
\] (2.29)

where \(a \in A\) and \(b, b', b''' \in B\). Now we operate with \((\langle \cdot, \cdot \rangle c' c) \otimes \text{id})\) on both equations where \(c, c' \in B\). After a little calculation using Lemma 2.3 and keeping the generalized Sweedler notation in mind, we find

\[
(\langle \cdot, \cdot \rangle c' c) \otimes \text{id}) [R \circ R_{\text{op}}^{\text{op}}(a \otimes b \otimes b')] \cdot (1 \otimes b''') = (a \otimes b_3'(c' b')_1'(b_2')b'''), \\
(\langle \cdot, \cdot \rangle c' c) \otimes \text{id}) [R_{\text{op}}^{\text{op}} \circ R(b \otimes a \otimes b')] \cdot (1 \otimes b''') = (b \otimes a_3'(c' b')_1'(b_2')b''').
\] (2.30)

Equations (2.30) prove \(R \circ R_{\text{op}}^{\text{op}} = R_{\text{op}}^{\text{op}} \circ R\) since “\(\triangleright\)” and “\(\triangleleft\)” are non-degenerate because of Proposition 2.10. \(\square\)

Proposition 2.15. For a non-degenerate multiplier Hopf \(\ast\)-algebra pairing \((A, B, \langle \cdot, \cdot \rangle)\) the mapping \(R\) and the involution “\(\ast\)” are related according to

\[
R^{\perp 1} \circ (\ast \otimes \ast) = (\ast \otimes \ast) \circ R^{\perp 1}, \\
(R_{\text{op}}^{\text{op}})^{\perp 1} \circ (\ast \otimes \ast) = (\ast \otimes \ast) \circ (R_{\text{op}}^{\text{op}})^{\perp 1}.
\] (2.31)

Proof. The proof of the proposition is rather straightforward. Let \(a, a' \in A\) and \(b, b' \in B\), then

\[
(a' \otimes b') \cdot R(a'^* \otimes b'^*) = (\text{id} \otimes \langle \cdot, \cdot \rangle \otimes \text{id})(\Delta(a)(a'^* \otimes 1) \otimes \Delta(b)(1 \otimes b'^*))^* \\
= (a_1 a'^*)^* (a_2 S^{-1}(b_1)) \otimes (b_2 b'^*)^* \\
= (\ast \otimes \ast)[R^{-1}(a \otimes b) \cdot (a'^* \otimes b'^*)] \\
= (a' \otimes b') \cdot (\ast \otimes \ast) R^{-1}(a \otimes b)
\] (2.32)

where we used the \(*\)-property of \(\langle \cdot, \cdot \rangle\) and \(\Delta\). The verification of the other cases can be worked out similarly. \(\square\)

The two morphisms \(R\) and \(R_{\text{op}}^{\text{op}}\) are the ingredients for the construction of a twist map which we use in the next chapter for the definition of the multiplication of the quantum double of a multiplier Hopf algebra pairing \((A, B, \langle \cdot, \cdot \rangle)\).
3. The Quantum Double

**Definition 3.1.** The twist map of a non-degenerate MHA pairing \((A, B, \langle \cdot, \cdot \rangle)\) is defined as

\[
T := R \circ (R_{op}^{op})^{-1} \circ \tau : B \otimes A \to A \otimes B.
\]

(3.1)

In the case of Hopf algebra pairings it holds \(T(b \otimes a) = \langle S^{-1}(a_{(1)}), b_{(3)} \rangle \langle a_{(3)}, b_{(1)} \rangle a_{(2)} \otimes b_{(2)}\). This mapping is used in [Ma2,VD1] to construct the multiplication of the quantum double. And indeed, we will see in the following that also for an MHA pairing \((A, B, \langle \cdot, \cdot \rangle)\) the mapping \(T\) has enough properties which enable us to construct a multiplication on the tensor product \(A \otimes B\). Furthermore we can show that even a multiplier Hopf algebra structure on \(A \otimes B\) can be established which generalizes the quantum double construction of usual Hopf algebra pairings to the case of MHA pairings. Before we will prove this fact we have to provide several structural results. Exploiting Lemma 2.13 we arrive at the following proposition.

**Proposition 3.2.** Let \((A, B, \langle \cdot, \cdot \rangle)\) be a non-degenerate MHA pairing. Then the twist map obeys the relations

\[
T(b'' \otimes b \triangleright a \triangleleft b') = (b''(1) a) b \triangleright a S^{-1}(b''(2)) \otimes b'(2),
\]

(3.2)

\[
T(a \triangleright b \triangleleft a' \triangleleft a'') = a''(2) \otimes S^{-1}(S(a) a''(1)) \triangleright b \triangleleft a'(3),
\]

(3.3)

\[
T(a \triangleright b \triangleleft b' \triangleright a') = (b(1) b') a'_{(2)} \otimes S^{-1}(S(a) a'_{(1)}) \triangleright b(2)
\]

(3.4)

\[
T(b \triangleright a \triangleleft a' \triangleleft a'') = a''_{(2)} S(b''(1)) \otimes b(1) \triangleleft a''_{(2)}
\]

(3.5)

\[
(a' \cdot (\cdot \otimes (\cdot) \triangleright a'') T(b \otimes a) = a''_{(2)} \otimes S^{-1}(a_{(1)}) b \triangleleft a(a_{(3)} a''),
\]

(3.6)

\[
(b' \triangleright (\cdot) \otimes (\cdot) \cdot b') T(b \otimes a) = (b' b_{(1)}) b \triangleright a S^{-1}(b_{(3)}) \otimes b(2) b''
\]

(3.7)

for all \(a, a', a'', b, b', b'' \in A\) and \(b, b', b'' \in B\).

**Proof.** We use Lemma 2.13 to verify

\[
T(b'' \otimes b \triangleright a \triangleleft b') = R(b \triangleright a \triangleleft b' S^{-1}(b'')(1)) \otimes S(S^{-1}(b'')(2)).
\]

(3.8)

A short calculation shows that

\[
b' S^{-1}(b''(1)) \otimes \Delta(S(S^{-1}(b'')(2)))(b \otimes 1) = S^{-1}(b''(3) S(b'')) \otimes b''_{(1)} b \otimes b''(2).
\]

(3.9)

Inserting (3.9) into (3.8) leads to

\[
T(b'' \otimes b \triangleright a \triangleleft b') = (b''(1) b) b \triangleright a S^{-1}(b''(2) S(b'')) \otimes b''(2).
\]

which proves (3.2). Analogously identity (3.3) can be verified. Using Lemma 2.13 and Proposition 2.5 according to

\[
T(b \triangleright a \triangleleft a' \triangleleft b') = R(a' \triangleleft a S^{-1}((b \triangleright a)(2) S(b')) \otimes (b \triangleright a)(1))
\]

\[
= R(a' \triangleleft a S^{-1}(b(2) S(b')) \otimes b(1) \triangleleft a)
\]

\[
= [a' \triangleleft S^{-1}(b(2) S(b'))]_{(1)} \otimes b(1) \triangleleft a[a' \triangleleft S^{-1}(b(2) S(b'))]_{(2)}
\]

\[
= a''_{(1)} \triangleleft S^{-1}(b(2) S(b'')) \otimes b(1) \triangleleft a a''_{(2)}
\]
yields (3.5). Similar calculations lead to (3.4). For the proof of (3.6) we consider
\[ (a_1 \cdot (\cdot \otimes (\cdot), a_2) T(a_3 \triangleright b \triangleleft a_4 \otimes a) = a_1 a_2 \otimes S^{-1}(S(a_3) a_4) \triangleright (b \triangleleft a_4) \triangleleft (a_3 \cdot a_2) \]

\[ = a_1 a_2 \otimes S^{-1}(a_1) \triangleright (a_3 \cdot b \triangleleft a_4) \triangleleft (a_3 \cdot a_2) \]

where we used (3.3). Since the actions “⧷” and “◦” are surjective we obtain the result. Similarly (3.7) is shown.

**Proposition 3.3.** The twist map $T$ and the multiplicities $m_A$ and $m_B$ obey the following relations.

\[
T \circ (m_B \otimes \text{id}) = (\text{id} \otimes m_B) \circ (T \otimes \text{id}) \circ (\text{id} \otimes T),
\]

\[
T \circ (\text{id} \otimes m_A) = (m_A \otimes \text{id}) \circ (\text{id} \otimes T) \circ (T \otimes \text{id}).
\]

**Proof.** We prove the first equation. The second one can be derived completely analogous because of the symmetry of the construction. From Proposition 3.2 we obtain

\[
T(b' \otimes a \triangleright b)(1 \otimes b'') = b'(1) \triangleright a \triangleleft S^{-1}(b'(2) S(b)) \otimes (b''(2)b''').
\]

and hence

\[
[(\text{id} \otimes m_B) \circ (T \otimes \text{id}) \circ (\text{id} \otimes T)(b' \otimes b'' \otimes a \triangleright b)](1 \otimes b'')
\]

\[
= (\text{id} \otimes m_B)(T \otimes \text{id})(b' \otimes (b''(1) \triangleright a) \triangleleft S^{-1}(b''(3) S(b)) \otimes b''(2)b''')
\]

\[
= (b'b')(1) \triangleright a \triangleleft S^{-1}((b'b')(2) S(b)) \otimes (b'b')(3)b'''
\]

\[
= T(b'b'' \otimes a \triangleright b) \cdot (1 \otimes b''')
\]

where we used (3.11) two times.

Thus $T$ behaves like a braiding with respect to the multiplication and the identity map. Making use of the properties of $T$ and the associativity of $A$ and $B$, we can therefore define an associative algebra on the tensor product $A \otimes B$ which generalizes the algebra structure of a quantum double of ordinary Hopf algebras \cite{Dri,Ma2,VD1} to multiplier Hopf algebras.

**Definition 3.4.** The quantum double $D(A, B, \langle \cdot, \cdot \rangle)$ of a non-degenerate multiplier Hopf algebra pairing $(A, B, \langle \cdot, \cdot \rangle)$ is the algebra $(A \otimes B, m_D)$ with the multiplication map defined through $m_D := (m_A \otimes m_B) \circ (\text{id} \otimes T \otimes \text{id})$.

**Corollary 3.5.** The multiplication $m_D$ of the quantum double is non-degenerate.

**Proof.** For a fixed $d \in D$ suppose $d \cdot_D d' = 0$ for all $d' \in D$. Then $TT^{-1}(d) \cdot_D d' = 0$ for all $d' \in D$. Because of Proposition 3.3 this is equivalent to $(\text{id} \otimes m_B)(T \otimes \text{id})(\text{id} \otimes m_A \otimes \text{id})(T^{-1}(d) \otimes d) = 0$ for any $d' \in D$. Hence $T(\text{id} \otimes m_A)(T^{-1}(d) \otimes a') = 0$ for any $a' \in A$ since $m_B$ is non-degenerate. Thus it follows $T^{-1}(d) \cdot (1 \otimes a') = 0$ for all $a' \in A$ and therefore $d = 0$. Similarly one proves $d \cdot_D d' = 0 \forall d \in D \Leftrightarrow d' = 0$.

**Proposition 3.6.** Let $(A, B, \langle \cdot, \cdot \rangle)$ be a non-degenerate multiplier Hopf $*$-algebra pairing. Then $\tau_D := T \circ (\text{id} \otimes (\cdot \otimes \cdot) \circ \tau : A \otimes B \rightarrow A \otimes B$ renders $(D, m_D, \tau_D)$ a non-degenerate $*$-algebra.

**Proof.** The antilinearity of $\tau_D$ is clear. From Proposition 2.15 we get

\[
\tau_D^2 = T \circ (\text{id} \otimes (\cdot \otimes \cdot) \circ \tau \circ T \circ (\text{id} \otimes (\cdot \otimes \cdot) \circ \tau
\]

\[
= R \circ R_{\text{op}}^{-1} \circ (\text{id} \otimes (\cdot \otimes \cdot) \circ R_{\text{op}} \circ R^{-1}
\]

\[
= \text{id}.
\]
The antimultiplicativity will be proven as follows. Let \( d, d' \in \mathcal{D} \), then

\[
(d \cdot \mathcal{D} d')^* = \tau_D \circ m_D (d \otimes d')
\]

(3.13)

\[
= T \circ \tau \circ (m_B \otimes m_A) \circ (\tau \otimes \tau) \circ (\ast \otimes \ast \otimes \ast \otimes \ast) \circ (id \otimes T \otimes id)(d \otimes d')
\]

\[
= m_D \circ (id \otimes T (\ast \otimes \ast) \tau T \otimes id) \circ (\tau \otimes \tau) \circ (id \otimes \tau \otimes id) \circ (\tau \otimes \tau) \circ (\ast \otimes id \otimes id \otimes \ast)(d \otimes d')
\]

\[
= m_D \circ (\tau_D \otimes \tau_D)(d \otimes d').
\]

\[
= d^* \cdot \mathcal{D} d^*.
\]

In the second equation of \((3.13)\) the antimultiplicativity of \(\ast\) is used. The third identity is derived with the help of Proposition 3.3 and in the fourth equation we made use of \(\tau_D^2 = \text{id}\). \(\square\)

**Remark 4.** There is no reason why we should prefer \(A \otimes B\) instead of \(B \otimes A\) for the construction of the quantum double. One easily observes that the inverse twist map \(T^{-1} : A \otimes B \rightarrow B \otimes A\) obeys analogous relations like \((3.8)\). If the corresponding quantum double is denoted by \(\overrightarrow{\mathcal{D}} := (B \otimes A, m_{\overrightarrow{\mathcal{D}}})\), it is straightforward to verify that \(T : \overrightarrow{\mathcal{D}} \rightarrow \mathcal{D}\) is a \((\ast)\)-algebra isomorphism.

We are now investigating how multipliers of \(A\) and \(B\), and multipliers of \(A \otimes A\) and \(B \otimes B\) compose to multipliers of \(\mathcal{D}\) and \(\mathcal{D} \otimes \mathcal{D}\) respectively. As usual in this chapter we suppose \((A, B, \langle \cdot, \cdot \rangle)\) to be non-degenerate multiplier Hopf algebra pairing.

**Proposition 3.7.** Let \(m \in M(A)\), \(n \in M(B)\), \(M \in M(A \otimes A)\) and \(N \in M(B \otimes B)\) be multipliers. Then \(\alpha(m \otimes n)\) defined by

\[
\alpha(m \otimes n)_1 := (m_1 \otimes \text{id}) \circ T \circ (n_1 \otimes \text{id}) \circ T^{-1}
\]

\[
\alpha(m \otimes n)_2 := (\text{id} \otimes n_2) \circ T \circ (\text{id} \otimes m_2) \circ T^{-1}
\]

(3.14)

is a multiplier in \(M(\mathcal{D})\), and \(\beta(M \otimes N)\) given through

\[
\beta(M \otimes N)_1 := (M_1)_{13} \circ (T \otimes T) \circ (N_1)_{13} \circ (T^{-1} \otimes T^{-1})
\]

\[
\beta(M \otimes N)_2 := (N_2)_{24} \circ (T \otimes T) \circ (M_2)_{24} \circ (T^{-1} \otimes T^{-1})
\]

(3.15)

is a multiplier in \(M(\mathcal{D} \otimes \mathcal{D})\). \((M_1)_{13}, (N_1)_{13}, (M_2)_{24}, (N_2)_{24} \in \text{End}_k(A \otimes B \otimes A \otimes B)\), for instance \((M_1)_{13}\) operates on the first and third component as \(M_1\).

**Proof.** We give the proof for \(\alpha\). The outlined techniques can be applied in a similar way for the verification of the statement for \(\beta\). Let \(d, d' \in \mathcal{D}\). Then

\[
\alpha(m \otimes n)_2(d) \cdot \mathcal{D} d' = m_D \circ ((\text{id} \otimes n_2) \circ T \circ (\text{id} \otimes m_2) \circ T^{-1} \otimes \text{id} \otimes \text{id})(d \otimes d')
\]

(3.16)

\[
= (m_A \otimes \text{id})(\text{id} \otimes T)(\text{id} \otimes m_B(n_2 \otimes \text{id}) \otimes \text{id})(T (\text{id} \otimes m_2) T^{-1} \otimes T^{-1})
\]

\[
= (\text{id} \otimes m_B)(T \otimes \text{id})(\text{id} \otimes m_A(m_2 \otimes \text{id}) \otimes \text{id})(\text{id} \otimes \text{id} \otimes T(n_1 \otimes \text{id}))(T^{-1} \otimes T^{-1})(d \otimes d')
\]

\[
= d \cdot \mathcal{D} \alpha(m \otimes n)_1(d')
\]
In the second and third equation of (3.16) use has been made of Proposition 3.3. The multiplier property of $m$ and $n$ enters in the third and fourth equality. According to the assertions in Chapter 1 this proves the proposition. 

**Remark 5.** From Proposition 3.7 it is obvious how to proceed for higher tensor products. If $M \in M(A \otimes n)$ and $N \in M(B \otimes n)$ then for example

$$
\gamma(M \otimes N)_2 = (N_2)_{2,4,\ldots,n} \circ (T \otimes \ldots \otimes T) \circ (M_2)_{2,4,\ldots,n} \circ (T^{-1} \otimes \ldots \otimes T^{-1})
$$

is the second component of the multiplier $\gamma(M \otimes N) \in M(D \otimes n)$.

**Corollary 3.8.** The mappings

$$
i_A : \left\{ \begin{array}{ll}
M(A) & \to M(D) \\
m & \mapsto \alpha(m \otimes 1_{M(B)})
\end{array} \right.
$$

and

$$
i_B : \left\{ \begin{array}{ll}
M(B) & \to M(D) \\
n & \mapsto \alpha(1_{M(A)} \otimes n)
\end{array} \right.
$$

are algebra embeddings. If $(A,B,\langle \cdot,\cdot \rangle)$ is a non-degenerate multiplier Hopf $\ast$-algebra pairing then they are $\ast$-algebra morphisms.

**Proof.** We restrict to the proof for $i_A$ because all other cases can be derived similarly. Looking at the first component of $i_A$ it is obvious that it is an algebra embedding. Because of the uniqueness of multipliers coinciding in one of their components it follows that $i_A$ is an algebra embedding. If $(A,B,\langle \cdot,\cdot \rangle)$ is MH($\ast$-)$A$ pairing then we obtain for any $d \in D$

$$
(i_A(m^*))_1(d) = (i_A(m)_2(d^*))^*
= T \tau (\ast \otimes \ast) T (id \otimes m_2) \tau (\ast \otimes \ast)(d)
= (\ast \circ m_2 \circ \ast \otimes id)(d) 
= i_A(m^*)_1(d)
$$

where in the third equation use has been made of $i_B^2 = \text{id}$. Hence $i_A$ is a $\ast$-algebra morphism. 

**Remark 6.** Occasionally we will identify $m$, $n$, $M$ and $N$ with their images under the morphisms of Corollary 3.8. Then it holds for example $M \cdot N = \beta(M \otimes N)$ for $M \in M(A \otimes A)$ and $N \in M(B \otimes B)$.

**Definition 3.9.** The comultiplication $\Delta_D$ of the quantum double $\mathcal{D}$ of a non-degenerate multiplier Hopf algebra pairing $(A,B,\langle \cdot,\cdot \rangle)$ is defined to be the mapping

$$
\Delta_D := \beta \circ (\Delta_A \otimes \Delta_B^\text{op}) : \mathcal{D} \to M(D \otimes D)
$$

where $\beta$ from Proposition 3.7 is used.
Proposition 3.10. The linear mappings $T^D_1$, $T^D_2$, $T^D_{op1}$ and $T^D_{op2} : \mathcal{D} \otimes \mathcal{D} \rightarrow \mathcal{D} \otimes \mathcal{D}$ according to Definition 1.3 are bijective. We obtain the following expressions for $T^D_1$ and $T^D_2$.

\[
\begin{align*}
T^D_1 &= (T^A_1)_{13} \circ (\text{id} \otimes \text{id} \otimes T) \circ (\text{id} \otimes T^B_{op1} \otimes \text{id}) \circ (\text{id} \otimes \text{id} \otimes T^{-1}), \\
T^D_2 &= (T^B_{op2})_{24} \circ (T \otimes \text{id} \otimes \text{id}) \circ (\text{id} \otimes T^A_2 \otimes \text{id}) \circ (T^{-1} \otimes \text{id} \otimes \text{id}).
\end{align*}
\] (3.20)

Proof. We outline the proof for $T^D_2$. All other cases can be worked out in the same fashion. In the proof we use the notation

\[
T(b \otimes a) =: \sum_i a^{(i)} \otimes b^{(i)} \quad \text{and} \quad T^{-1}(a \otimes b) =: \sum_j b^{(j)} \otimes a^{(j)}.
\] (3.21)

Let $a_1, a_2, a_3 \in \mathcal{A}$ and $b_1, b_2, b_3 \in \mathcal{B}$. Then

\[
\begin{align*}
T^D_2(a_1 \otimes b_1 \otimes a_2 \otimes b_2) \cdot (1_D \otimes a_3 \otimes b_3) &= \left(\left([T \otimes T]^\prime \left([T^{-1} \otimes T^{-1}] (a_1 \otimes b_1 \otimes a_3 \otimes b_3) \right) \cdot \Delta_A(a_2)_{24}\right) \cdot \Delta_B^{op}(b_2)_{24}\right) \\
&= \sum T(b_1(i_1) \otimes a_1(i_1)_{(1)} a_2(2)) \otimes (a_3 \otimes b_3) \cdot \Delta(D(a_2(2) \otimes b_2(1)).
\end{align*}
\] (3.22)

Hence

\[
T^D_2(a_1 \otimes b_1 \otimes a_2 \otimes b_2) = \sum T(b_1(i_1) \otimes a_1(i_1)_{(1)} a_2(2)) (1 \otimes b_2(1)) \otimes (a_2(2) \otimes b_2(1))
\]

which yields the result. \hfill \square

Corollary 3.11. $\Delta_D$ is coassociative in the sense of (1.3), i.e.

\[
(T^D_2 \otimes \text{id}) \circ (\text{id} \otimes T^D_1) = (\text{id} \otimes T^D_1) \circ (T^D_2 \otimes \text{id}).
\] (3.23)

Proof. Taking the expressions (3.20) for $T^D_1$ and $T^D_2$ and making use of (1.3) for $(A, \Delta_A)$ and $(B, \Delta_B^{op})$ proves the corollary. \hfill \square

Before we will prove in Proposition 3.15 that $\Delta_D$ is a (\text{-})algebra homomorphism, we need three lemmas. We use the notation (3.21).

Lemma 3.12. Let $(\mathcal{A}, \mathcal{B}, (\cdot, \cdot))$ be a non-degenerate MHA pairing. Then it holds for $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$

\[
\Delta_D(T(b \otimes a)) \cdot (1_D \otimes a' \otimes b')
\]

\[
= \sum (a^{(i)}(1) \otimes (b^{(i)}(2) \otimes ((a^{(i)}(2) \otimes 1) \cdot T((b^{(i)}(1)b^{(j)}(2) \otimes a^{(j)}(1))
\]

and

\[
(\Delta_B^{op}(b) \cdot \Delta_A(a)) \cdot (1_D \otimes a' \otimes b')
\]

\[
= \sum (a(1)(k) \otimes (b(2)(k) \otimes T(b(1)b'(1) \otimes (a(2)a')(1))
\]

\hfill \square
Lemma 3.13. Let $a_1, a_2, a_3, a_4 \in A$ and $b_1, b_2, b_3, b_4, b_5 \in B$. Then
\[
(b_1 \triangleright (\cdot) \otimes b_2 \cdot (\cdot) \otimes a_1 \cdot (\cdot) \otimes \text{id}) [\Delta_D(T(b_3 \triangleleft a_2 \triangleleft a_3 \triangleleft b_4)) \cdot (1_D \otimes a_4 \triangleleft b_5)] \\
= b_1 \triangleright a_3(1) \triangleleft S^{-1}(b_3(3)S(b_4)) \otimes b_2 b_3(2) \otimes a_1 a_3(2) a_4(2) \otimes [S^{-1}(a_4(1)) \triangleright b_3(1) \triangleleft (a_2 a_3(3) a_4(3))] b_5
\]
and
\[
(b_1 \triangleright (\cdot) \otimes b_2 \cdot (\cdot) \otimes a_1 \cdot (\cdot) \otimes \text{id}) [\Delta_D^{op}(b_3 \triangleleft a_2) \cdot \Delta_A(a_3 \triangleleft b_4) \cdot (1_D \otimes a_4 \triangleleft b_5)] \\
= (b_1 b_3(2) \triangleright a_3(1) \triangleleft S^{-1}(b_3(4)S(b_4)) \otimes b_2 b_3(3) \otimes a_1(a_3(2) a_4)(2) \otimes \\
\otimes [S^{-1}((a_3(2) a_4)(1) \triangleright b_3(1) \triangleleft (a_2(a_3(2) a_4)(3))] b_5. \\
\]

\[\square\]

Lemma 3.14. Let $a_1, a_2, a_3, a_4 \in A$ and $b_1, b_2 \in B$. Then the following identity is fulfilled
\[
(b_1 b_2(2) \triangleright a_3(1) \otimes S^{-1}((a_3(2) a_4)(1) \triangleright b_2(1) \otimes a_1(a_3(2) a_4)(2) \otimes a_2(a_3(2) a_4)(3)) \\
= b_1 \triangleright a_3(1) \otimes S^{-1}(a_4(1)) \triangleright b_3 \otimes a_1 a_3(2) a_4(2) \otimes a_2 a_3(3) a_4(3). \\
\tag{3.24}
\]
\[\square\]

If one uses Proposition 2.5 and Proposition 3.2 the proofs of the three lemmas are straightforward, although lengthy calculations are involved. The first part of Lemma 3.12 has already been shown in Proposition 3.10. To prove Lemma 3.13 one uses Lemma 3.12. For the proof of Lemma 3.14 it is convenient to multiply both sides of (3.24) with some $(a_I \cdot (\cdot) \otimes b_I \cdot (\cdot) \otimes (\cdot) \otimes (\cdot) \otimes a_{III})$ and to verify this new equality. The non-degeneracy of the multiplication then yields the identity (3.24). By making use of Lemma 3.13 and Lemma 3.14 we obtain the important proposition.

Proposition 3.15. The comultiplication $\Delta_D$ of the quantum double $D$ of a non-degenerate multiplier Hopf algebra pairing $(A, B, \langle \cdot, \cdot \rangle)$ obeys the identity
\[
\Delta_D \circ T(b \otimes a) = \Delta_D^{op}(b) \cdot \Delta_A(a) \quad \text{for all } a \in A, \ b \in B \tag{3.25}
\]
where the identification has been made according to Remark 6. Hence $\Delta_D : D \to M(D \otimes D)$ is an algebra morphism. It is a $*$-algebra morphism if $(A, B, \langle \cdot, \cdot \rangle)$ is a multiplier Hopf $*$-algebra pairing.

Proof. Lemma 3.13 and Lemma 3.14 immediately lead to equation (3.25). Then it is straightforward to prove that $\Delta_D$ is a ($*$-)algebra homomorphism. We use (3.25), Corollary 3.8 and Remark 6 for the proof.
\[
\Delta_D((a \otimes b) \cdot_D (a' \otimes b')) = \sum \Delta_A(a) \Delta_A(a') \Delta_D^{op}(b) \Delta_D^{op}(b') \\
= \Delta_A(a) \Delta_D^{op}(b) \Delta_A(a') \Delta_D^{op}(b') \\
= \Delta_D(a \otimes b) \cdot_D (a' \otimes b'). \\
\tag{3.26}
\]

Similarly one verifies the $*$-property of $\Delta_D$. \[\square\]

Finally we gather the previous results to prove the main theorem on the construction of a quantum double multiplier Hopf algebra out of a non-degenerate multiplier Hopf algebra pairing $(A, B, \langle \cdot, \cdot \rangle)$. This theorem generalizes the quantum double construction of ordinary Hopf algebra pairings.
Theorem 3.16. Let \((A, B, \langle \cdot, \cdot \rangle)\) be a non-degenerate multiplier Hopf \((\ast, \cdot)\)-algebra pairing. Then \((D, m_D, \Delta_D, \langle \cdot, \cdot \rangle_D)\) is a regular multiplier Hopf \((\ast, \cdot)\)-algebra. Counit and antipode are given through \(\varepsilon_D = \varepsilon_A \otimes \varepsilon_B\) and \(S_D = T \circ \tau \circ (S_A \otimes S_B^{-1})\) respectively. If \(A\) and \(B\) have non-trivial integrals then \(D\) has a non-trivial integral. Explicitely \(\varphi_D = \varphi_A \otimes \psi_B\) is left integral on \(D\), if \(\varphi_A\) is the left integral on \(A\) and \(\psi_B\) is the right integral on \(B\).

Proof. The previous results show that \((D, \Delta_D, \langle \cdot, \cdot \rangle_D)\) is an MH\((\ast, \cdot)\)A. From Proposition 3.10 it follows that \((D, \Delta_D, \langle \cdot, \cdot \rangle_D)\) is regular. Counit and antipode of \(D\) are easily determined through the equations [VD2]

\[
\begin{align*}
  m_D \circ (T_D^D)^{-1} &= \varepsilon_D \otimes \text{id} \\
  m_D \circ (S_D \otimes \text{id}) &= (\varepsilon_D \otimes \text{id}) \circ (T_D^D)^{-1}.
\end{align*}
\]

If \(\varphi_A\) is left integral of \(A\) and \(\psi_B\) is right integral of \(B\), i.e. \((\text{id} \otimes \varphi_A)T_2^A = \text{id} \otimes \varphi_A\) and \((\text{id} \otimes \Psi_B)T_{op}^B = \text{id} \otimes \Psi_B\), then it is not difficult to verify

\[
(id \otimes \varphi_A \otimes \psi_B) \circ T_2^D = (id \otimes \varphi_A \otimes \psi_B).
\]

Hence \(\varphi_A \otimes \psi_B\) is left integral of \(D\). \qed

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Appendix

We present the “generalized Sweedler notation” which is used in the paper. For a regular multiplier Hopf algebra \((A, \Delta)\) the relation (1.4) holds. Since the counit \(\varepsilon\) is (non-degenerate) algebra morphism in the sense of Proposition 1.1 and because of [Theorem 3.6, VD2] one obtains the identity \((\text{id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \text{id}) \circ \Delta = \text{id} : M(A) \to M(A)\) as in the case of ordinary Hopf algebras. Hence we can define \(\Delta^{(n)}\) for any \(n \geq -1\) recursively according to

\[
\begin{align*}
  \Delta^{(-1)} &= \varepsilon \\
  \Delta^{(n)} &= (\text{id} \otimes \Delta^{(n-1)}) \circ \Delta = (\Delta^{(n-1)} \otimes \text{id}) \circ \Delta \quad \text{for all } n \geq 0
\end{align*}
\]

using the fact that \(\Delta\) is coassociative. From the definition of \(\Delta^{(n)}\) it follows immediately that \(\Delta^{(n)} : M(A) \to M(A \otimes A^{n+1})\). We have the following lemma as a direct consequence of this coassociativity, resembling the case of ordinary Hopf algebras.

Lemma A.1.

\[
\Delta^{(n+m+r)} = (\text{id}_{A \otimes A^n} \otimes \Delta^{(m)} \otimes \text{id}_{A \otimes r}) \circ \Delta^{(n+r)} \quad \text{for all } n, m, r \geq 0.
\]

Since the mappings \(T_1, T_2, T_{op}^1\) and \(T_{op}^1\) are linear mappings on \(A \otimes A\), we obtain
Proposition A.2. Let $n,m,r \geq 0$ and $a_i \in A$ for $i \in \{1, \ldots, n\}$ and $a'_j \in A$ for $j \in \{1, \ldots, r\}$. For any $a \in A$, $\epsilon \in \{-1,1\}$ and $p \in \{1, \ldots, n+m+r\}$ denote by $a^{(\epsilon,p)}$ the linear mapping

$$a^{(\epsilon,p)} := \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \lambda_\epsilon(a) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \in \text{End}_k(A^\otimes n+m+r+1)$$

where $\lambda_{-1}$ is the left multiplication and $\lambda_1$ is the right multiplication in $A$. Then it holds

$$\Delta^{(n+m+r)}(a) \cdot a_1^{(\epsilon_1,1)} \cdot \cdots \cdot a_n^{(\epsilon_n,n)} \cdot a_1'(\epsilon_{n+1},n+m+1) \cdot \cdots \cdot a_r'(\epsilon_{n+r},n+m+r) \in A^\otimes n \otimes \Delta^{(m)}(A) \otimes A^\otimes r$$

(A.3)

for $\epsilon_1,\ldots,\epsilon_{n+r} \in \{-1,1\}$. □

This suggests to write symbolically $\Delta^{(n+m+r)}(a) := a_{(1)} \otimes a_{(2)} \otimes \cdots \otimes a_{(n+m+r+1)}$. Then, for instance, we arrive at

$$\Delta^{(n+m+r)}(a) \cdot a_1^{(1,1)} \cdot \cdots \cdot a_n^{(1,n)} \cdot a_1'(1,n+m+1) \cdot \cdots \cdot a_r'(1,n+m+r)$$

(A.4)

and we say that the first $n$ indices and the last $r$ indices are covered and the $m+1$ indices in between are free. Suppose we restrict to multiplications of the type (A.3) which guarantee the proper tensor factorization. Then we can treat covered indices (in a formal sum) as elements of $A$ and the collection of uncovered indices (in the formal sum) as an element in $\Delta^{(m)}(A)$. Therefore we can apply tensor products of morphisms on (A.3) according to this factorization. These rules are obviously compatible, in particular with the successive multiplication with another $\Delta^{(n+m+r)}(\tilde{a})$ and with another $\tilde{a}_1^{(\tilde{\epsilon}_1,1)} \cdot \cdots \cdot \tilde{a}_n^{(\tilde{\epsilon}_n,n)} \cdot \tilde{a}_1'(\tilde{\epsilon}_{n+1},n+m+1) \cdot \cdots \cdot \tilde{a}_r'(\tilde{\epsilon}_{n+r},n+m+r)$, because the multiplier algebra is associative, $\Delta$ is coassociative algebra morphism, and Lemma A.1 and Proposition A.2 hold. Analogous results are true for $\Delta^\text{op}$ since it is also a coassociative algebra morphism. We call the rules figured out in (A.3) and (A.4) the “generalized Sweedler notation” following the common nomenclature for ordinary Hopf algebras [Swe].

References

[DK] M. Dijkhuizen and T.H. Koornwinder, *CQG Algebras: A direct Algebraic Approach to Compact Quantum Groups*, Lett. Math. Phys. 32 (1994), 315-330.

[Dri] V.G. Drinfeld, *Quantum Groups*, Proceedings of the International Congress of Mathematicians, Berkeley, 1986, pp. 798-820.

[ER] E. Effros and Z.-J. Ruan, *Discrete Quantum Groups I. The Haar Measure*, Int. J. Math. 5 (1994), 681-723.

[JS] A. Joyal and R. Street, *Tortile Yang-Baxter Operators in Tensor Categories*, J. Pure Appl. Algebra 71 (1991), 43-51.

[Ma1] S. Majid, *Quasitriangular Hopf algebras and Yang-Baxter equations*, Int. J. Mod. Phys. A 5 (1990), 1-91.

[Ma2] __________, *Doubles of quasitriangular Hopf algebras*, Comm. Algebra 19 (1991), 3061-3073; *Some Remarks on the Quantum Double*, Proc. 3rd Colloq. Quantum Groups, Prague, 1994.

[Ma3] __________, *Representations, duals and quantum doubles of monoidal categories*, Rend. Circ. Mat. Palermo (2) Suppl. 26 (1991), 197-206.

[MN] T. Masuda and Y. Nakagami, *A von Neumann Algebra Framework for the Duality of Quantum Groups*, Publ. RIMS, Kyoto Univ. 30 (1994), 799-850.

[PW] P. Podlés and S.L. Woronowicz, *Quantum Deformation of Lorentz Group*, Commun. Math. Phys 130 (1990), 381-431.
[Swe] E.M. Sweedler, *Hopf Algebras*, Benjamin, New York, 1969.
[VD1] A. Van Daele, *Dual Pairs of Hopf $*$-Algebras*, Bull. London Math. Soc. **25** (1993), 209-230.
[VD2] __________, *Multiplier Hopf algebras*, Trans. AMS **342**, No. **2** (1994), 917-932.
[VD3] __________, *An Algebraic Framework for Group Duality*, Preprint KU Leuven (1996).
[VD4] __________, *Discrete Quantum Groups*, J. Algebra **180** (1996), 431-444.
[Wo1] S.L. Woronowicz, *Compact Matrix Pseudogroups (Quantum Groups)*, Commun. Math. Phys. **111** (1987), 613-665; *Compact Quantum Groups*, Preprint University of Warsaw (1993).
[Wo2] __________, *Unbounded elements affiliated with $C^*$-algebras and non-compact quantum groups*, Commun. Math. Phys. **136** (1991), 399-432.

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