A quadratic identity in the shuffle algebra and an alternative proof for de Bruijn’s formula

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March 16, 2021

Abstract

Motivated by a polynomial identity of certain iterated integrals, first observed in [CGM20] in the setting of lattice paths, we prove an intriguing combinatorial identity in the shuffle algebra. It has a close connection to de Bruijn’s formula when interpreted in the framework of signatures of paths.

1 Introduction

A path is a continuous map $X : [0, 1] \to \mathbb{R}^d$. We shall assume that the components of $X$, $X^{(i)}$, for $i = 1, \ldots, d$, are (piecewise) continuously differentiable functions. Describing phenomena parametrized by time, they appear in many branches of science such as mathematics, physics, medicine or finance. For mathematics in particular, see [CGM20] for references.

One way to look at paths is through their iterated integrals

$$\int dX_{t_1}^{(w_1)} \cdots dX_{t_n}^{(w_n)} := \int_{0 < t_1 < \cdots < t_n < 1} \dot{X}_{t_1}^{(w_1)} \cdots \dot{X}_{t_n}^{(w_n)} dt_1 \cdots dt_n,$$

where $n \geq 1$ and $w_1, \ldots, w_n \in \{1, \ldots, d\}$. Here we use the abbreviated notations: $X_{t_i} := X(t_i)$, with $t_i \in [0, 1]$, and $\dot{X}_{t_i}^{(w_i)}$ denotes the derivative of $X^{(w_i)}$ with respect to the variable $t_i$. The first systematic study of these integrals was undertaken by Kuo Tsaï Chen [Che57]. For example, Chen proved that, up to an equivalence relation, iterated integrals uniquely determine a path.

In the field of stochastic analysis paths are usually not differentiable. Nonetheless (stochastic) integrals have played a major role there and this culminated in Terry Lyons’ theory of rough paths [LQ02, FH14]. Owing to their descriptive power of nonlinear phenomena (compare Chen’s uniqueness result above), these objects have recently been successfully applied to statistical learning [LNO14, KSH17, KO19, LZL19].

Iterated integrals possess an interesting Hopf algebraic structure [Reu93]. More recently they have been studied from the viewpoint of (applied) algebraic geometry [AFS19, Gal19, Stu96, PSS19] and representation theory [DR19].

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In [CGM20] an approach of studying signature-path tensors through the lens of toric geometry is presented and the following curious property was discovered.

**Theorem 1.1** ([CGM20, Corollary 3.23]).

\[
\begin{vmatrix}
\int dX^{(1)} dX^{(1)} & \cdots & \int dX^{(1)} dX^{(d)} \\
\vdots & \ddots & \vdots \\
\int dX^{(d)} dX^{(1)} & \cdots & \int dX^{(d)} dX^{(d)}
\end{vmatrix} = \frac{1}{2^d} \left( \sum_{\sigma \in S_d} \text{sign}(\sigma) \int dX^{\sigma(1)} \cdots dX^{\sigma(d)} \right)^2.
\] (1)

Their proof is based on calculations with lattice paths. One consequence of this result is that it gives an inequality for the iterated integrals of order two (i.e. this particular determinant is non-negative). Notice that when working on the Zariski closure of the space of signatures, as in the framework of [AFS19], one can only obtain equalities. See [CGM20, p. 22-23] for more details.

In the current paper we look at this statement from a purely algebraic perspective. (We advise the reader to peek ahead to Section 2 for notation used in the following.) Consider the space of formal linear combinations of words in the alphabet \([d] = \{1, \ldots, d\}\), and denote it by \(T(\mathbb{R}^d)\). Given two words \(w\) and \(v\) in the alphabet \([d]\), the *shuffle product* of them, \(w \shuffle v\), is defined as the sum of all permutations of the concatenated word \(wv\), that keep the respective order of the two words intact. For example,

\[
12 \shuffle 34 = 1234 + 1324 + 1342 + 3124 + 3142 + 3412.
\]

Then, \((T(\mathbb{R}^d), \shuffle)\) is an algebra known as the *shuffle algebra*. In a sense made precise in Section 2 the determinant (in the shuffle algebra) of the matrix

\[
\begin{pmatrix}
11 & \cdots & 1d \\
\vdots & \ddots & \vdots \\
d1 & \cdots & dd
\end{pmatrix}
\]

relates to the determinant (in \(\mathbb{R}\)) appearing in (1). To reformulate the other side of the equality in (1), we recall the basis of \(\text{SL}(\mathbb{R}^d)\)-invariants in \(T(\mathbb{R}^d)\), going back to Weyl [Wey46], and presented, in the language relevant used here, in [DR19]. This basis is indexed by *standard Young tableaux* of shape \((w, \ldots, w)\), for \(w \geq 1\). For us, only the cases \(w = 1, 2\) are relevant and we denote the basis by \(\{\text{inv}(T)\}\), where \(T\) ranges over these standard Young tableaux. The definition of this basis is presented in Section 2.4, and we illustrate with two examples here:

\[
\text{inv} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 12 - 21, \quad \text{and} \quad \text{inv} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1122 - 1221 - 2112 + 2211.
\] (2)

We can now state our main theorem (see Theorem 4.1 in Section 4).
Theorem. For $d \geq 1$ the following two equalities hold:

$$\text{det}_{\mathbb{W}}\left(\begin{array}{ccc}
11 & \ldots & 1d \\
\vdots & \ddots & \vdots \\
d1 & \ldots & dd
\end{array}\right) = \text{inv}\left(\begin{array}{ccc}
1 & 2 \\
3 & 4 \\
\vdots & \vdots \\
d & 2d
\end{array}\right) = \frac{1}{2^d} \text{inv}\left(\begin{array}{c}
1 \\
2 \\
\vdots \\
d
\end{array}\right).$$

In particular, we give a new and purely algebraic proof of the fact that the determinant of the second level of the iterated integrals signature is a square, Theorem 1.1 (see Corollary 4.2). As an application of our approach, we obtain the de Bruijn’s formula for the Pfaffian. We note that de Bruijn’s formula was looked at in the language of shuffle algebras already in [LT02]. Our viewpoint differs in that we approach the topic through invariant theory.

In Section 2, we present the framework of the shuffle algebra and the half-shuffle products, together with their relation with the signatures of paths. Since our proof for the main theorem uses some results from representation theory, we include in Section 2.3 the necessary basic notions. Moreover, in Section 2.4 we relate our setting with the work of [DR19] about SL($\mathbb{R}^d$)-invariants and signed volume. In Section 3 we present already known results that will help us to prove our main results. For instance, we present several identities related to the shuffle product that already appeared in [And83]. Section 4 is dedicated to prove our main theorem. In Section 5, we present the relation of our main theorem with the de Bruijn’s formula for both the even and odd-dimensional cases.

Acknowledgements

The authors express their gratitude to Bernd Sturmfels and Mateusz Michałek for helpful discussions and suggestions, and for bringing the team together. The authors are also grateful to the MPI MiS for giving us the opportunity to work on this project in such a nice environment. We would like to thank Darij Grinberg for making us aware of several inconsistencies in an earlier version. We are also grateful to the anonymous referees, for their careful reading and for their comments that helped us improve the quality of our manuscript.

Funding

Laura Colmenarejo was partially supported by MTM2016-75024-P.

2 Preliminaries

In this section we present several connected frameworks. First, we talk about the shuffle algebra and the half-shuffle operation. A survey on the history of the shuffle product can be found in [FP13] and our presentation here is inspired by [CP18], especially the part concerning the half-shuffle operation. Next, we describe briefly the relation between the setting of signatures of paths and the shuffle algebra. We refer the reader to [DR19, Subsection 2.1] for more details. We also include a brief summary of the
representation theory setting, which can be found in [FH91] with full detail. We finish by presenting a brief overview of the $\text{SL}(\mathbb{R}^d)$-invariants inside of $T(\mathbb{R}^d)$.

2.1 The shuffle algebra and half–shuffle products

Let us consider the alphabet $[d] = \{1, \ldots, d\}$ and denote by $[d]^*$ the set of words of any length in this alphabet, including the empty word $\varepsilon$. Denote by $T(\mathbb{R}^d)$ the space of \textit{finite linear combinations} of these words. Together with the concatenation product, $w \cdot v$, $(T(\mathbb{R}^d), \cdot)$ is an algebra, known as the tensor algebra.

Let us consider another operation on $T(\mathbb{R}^d)$.

\textbf{Definition 2.1.} Consider the words $u$, $v$ and $w$, and the letters $a$ and $b$. The shuffle product of two words is defined recursively by

$$
\varepsilon \shuffle u = u \quad \text{and} \quad (v \cdot a) \shuffle (w \cdot b) = (v \shuffle (w \cdot b)) \cdot a + ((v \cdot a) \shuffle w) \cdot b.
$$

This operation extends bilinearily to a commutative product on all of $T(\mathbb{R}^d)$. The algebra $(T(\mathbb{R}^d), \shuffle)$ is known as the shuffle algebra.

The shuffle product can be seen as the symmetrisation of the right half-shuffle product.

\textbf{Definition 2.2.} The right half-shuffle product is recursively given on words as

$$
w \cdot a = wa, \quad \text{and} \quad w \cdot v = p(w \cdot v) \cdot a = p(w \cdot v) \cdot a + (w \cdot v) \cdot a.
$$

The definition of the half-shuffle goes back to considerations in topology [EML53, Section 18] and in Lie theory [Sch58]. Many variants and generalizations are known, see for example [AL04], [BR10].

Note that if $w, v$ are any non-empty words, then $w \shuffle v = w \cdot v + v \cdot w$. Hence in Definition 2.2 we can replace $w \cdot v = (w \cdot v + v \cdot w) \cdot a$ with the equivalent equality $w \cdot v = (w \cdot v) \cdot a$. In general, for non-empty words, one has

$$
(u \cdot v) \cdot w = (u \cdot v) \cdot w.
$$

In particular, notice that the shuffle product is associative, while the half-shuffle product is not.

2.2 Iterated-integrals signatures of paths

For a (piecewise) smooth path $X : [0, 1] \to \mathbb{R}^d$ the collection of \textit{iterated integrals}

$$
\int dX_{t_1}^{(a_1)} \cdots dX_{t_n}^{(a_n)} := \int_{0 < t_1 < \cdots < t_n < 1} \dot{X}_{t_1}^{(a_1)} \cdots \dot{X}_{t_n}^{(a_n)} dt_1 \cdots dt_n,
$$

4
where \( n \geq 1 \) and \( a_1, \ldots, a_n \in \mathbb{d} \), is conveniently stored in the \textit{iterated-integrals signature}

\[
\sigma(X) := \sum_{n \geq 0, \ w = a_1 \ldots a_n \in [d]^n} \int dX_{t_1}^{(a_1)} \ldots dX_{t_n}^{(a_n)} \ w.
\]

This object is a \textit{formal infinite sum} of words \( w \) in the alphabet \([d]\) whose coefficients are given by the integrals.

Alternatively, \( \sigma(X) \) can be seen as a \textit{linear function} on \( T(\mathbb{R}^d) \) given by

\[
\langle w, \sigma(X) \rangle = \int dX_{t_1}^{(w_1)} \ldots dX_{t_n}^{(w_n)},
\]

for any element \( w \in [d]^n \), and extended linearly to all of \( T(\mathbb{R}^d) \). It turns out that \( \sigma(X) \) is in fact a \textit{multiplicative character} (i.e. an algebra morphism into \( \mathbb{R} \)) on \( (T(\mathbb{R}^d), \cup) \), see [Reu93, Corollary 3.5]. This fact is also called the \textit{shuffle identity} and reads as

\[
\langle w \cup v, \sigma(X) \rangle = \langle w, \sigma(X) \rangle \cdot \langle v, \sigma(X) \rangle, \quad \forall w, v \in T(\mathbb{R}^d).
\]  

### 2.3 Representation theory

This section provides some broad ideas from representation theory that are used in this paper. For more details, we refer the reader to [FH91].

Given a group \( G \) and an \( n \)-dimensional vector space over \( \mathbb{R} \), \( V \), we say that the map \( \rho : G \to \text{GL}(V) \) is a \textit{representation} of \( G \) if \( \rho \) is a group homomorphism, where \( \text{GL}(V) \) is the group of automorphisms of \( V \). In general, \( \rho \) is identified with \( V \), and \( \rho(g)(v) := g \cdot v \), for all \( v \in V \) and \( g \in G \). In this sense, we say that the group acts (on the left) on the vector space. There are different ways to construct representations from other representations; for instance, by constructing the direct sum or the tensor product of representations, or restricting or inducing representations from groups to subgroups, and vice versa.

Another approach to understand representations is by looking at them as modules, which allows us to study representations as vector spaces over the group algebra \( \mathbb{R}[G] \) (i.e. set of all linear combinations of elements in \( G \) with coefficients in \( \mathbb{R} \)).

A (non-zero) representation \( V \) of \( G \) is said to be \textit{irreducible} if the only subspaces of \( V \) invariant under the action of \( G \) are the vector space itself and the trivial subspace. By Schur’s Lemma and Maschke’s Theorem, we know that given any representation of a well-behaved\(^1\) group \( G \) we can decompose it as the direct sum of irreducible representations.

That is \( V = \bigoplus_k I_k \), where \( I_k \cong W_k^{\oplus n_k} \), the \( W_k \) are pairwise non-isomorphic irreducible representations, and \( n_k \) denotes their multiplicity (in \( V \)). The \( I_k \) are called the \textit{isotypic components} of \( V \) and this decomposition is called the \textit{canonical decomposition} of \( V \) or the \textit{decomposition into isotypic components} \( I_k \). Note that the decomposition into the \( I_k \) is unique, while the decomposition of \( I_k \) into the \( n_k \) copies of \( W_k \) is \textit{not}.\(^2\) One of the main general goals in Representation Theory is to characterise the irreducible representations.

\(^1\)\( G \) finite or \( G = \text{SL}(\mathbb{R}^d) \) will do.

\(^2\)The idea for the isotypic decomposition is that there may be several irreducible representations inside of \( V \) that are isomorphic, and they are 'bundled' in \( I_k \).
representations of a group and to give an algorithm to compute the canonical decomposition of an arbitrary representation.

Each representation is also characterised by the trace of the matrix associated to each element of the group. This vector is known as the character of the representation, and it is invariant under conjugation. Character theory is closely related to the theory of symmetric functions.

In this section we focus our attention on the symmetric group and on the general linear group. Let $S_d$ be the group of permutations of $\{1, 2, \ldots, d\}$, for $d \geq 1$. In the case of the symmetric group $S_d$, the conjugacy classes are in bijection with the partitions of $d$, which are weakly decreasing sequences of positive integers that sum up to $d$. This bijection is given by the decomposition of the permutations in cycles. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, we associate to $\lambda$ a Young diagram, which is just an array of boxes with $\lambda_i$ boxes in the $i^{th}$ row. Then, the standard Young tableaux are fillings of Young diagrams with all the numbers in the set $\{1, 2, \ldots, d\}$ that are increasing in columns and rows.

2.4 SL invariants and signed volume

The natural representation of $\text{SL}(\mathbb{R}^d)$ on $\mathbb{R}^d$ induces a representation on $T(\mathbb{R}^d)$. The study of invariants to this action goes back over a century, see [Wey46] for a good starting point of the literature. We recall the presentation used in [DR19].

As mentioned in the introduction, a basis for the invariants is indexed by standard Young tableaux of shape $(w, \ldots, w)$, for $w \geq 1$ arbitrary. Given a Young tableau $T$, the corresponding invariant basis element, denoted $\text{inv}(T)^3$ is obtained as follows: let $n = wd$ be the length of $T$ and consider the word $w = a_1 a_2 \ldots a_n$, where $a_\ell = i$ if and only if $\ell$ is in the $i^{th}$ row of $T$. Then

$$\text{inv}(T) := \sum_\sigma \text{sign}(\sigma) \sigma w,$$

where the sum is over all permutations $\sigma \in S_n$ that leave the values in each column of $T$ unchanged. For example, the tableau $T = \begin{array}{ccc} 1 & 2 \\ 3 & 4 \end{array}$ gives the word $1122$. Therefore,

$$\text{inv} \begin{array}{ccc} 1 & 2 \\ 3 & 4 \end{array} = (\text{id} - (13) - (24) + (13)(24))1122 = 1122 - 2112 - 1221 + 2211.$$

Our study only looks at the following two particular standard Young tableaux for $w = 1, 2$.

$$t_{1,d} := \begin{array}{c} 1 \\ 2 \\ \vdots \\ d \end{array}, \quad t_{2,d} := \begin{array}{ccc} 1 & 2 \\ 3 & 4 \\ \vdots \\ w-1 & 2d \end{array} \quad (5)$$

---

3In [DR19] it is denoted by $\iota(e_T)$.

4The basis element $\text{inv}(t_{1,d})$ has a nice geometric interpretation in the setting of iterated integrals, see [DR19] where it is denoted by $\text{inv}_d$. 

6
In this case, we have that

\[ \text{inv} (t_{1,d}) = \sum_{\sigma \in S_d} \text{sign}(\sigma)\sigma(12 \cdots d) \]  
\[ \text{inv} (t_{2,d}) = \sum_{\sigma \in H_d} \text{sign}(\sigma)\sigma(1122 \cdots dd), \]  

where \( H_d := \langle (1,3), (2,4), (3,5), \ldots, (2d-2,2d) \rangle \subseteq S_{2d}. \)

We also recall the matrix introduced before:

\[ W_d := \begin{pmatrix} 11 & \cdots & 1d \\ \vdots & \ddots & \vdots \\ d1 & \cdots & dd \end{pmatrix}, \]  

which is an element of \( A^{d \times d} \) where \( A = T(\mathbb{R}^d) \), seen as a commutative algebra (over \( \mathbb{R} \)).

3 Auxiliary results

3.1 An identity by Andréief

**Notation 3.1.** Please note that we use the convention of strict left bracketings for the half-shuffle product.\(^5\)

For words \( w_1, \ldots, w_k, w_1 > w_2 > \ldots > w_k := ((w_1 > w_2) > \ldots) > w_k. \)

The following lemma expresses the shuffle product of \( k \) words in terms of half-shuffle products.

**Lemma 3.2.** For any non-empty words \( w_1, \ldots, w_k \), the following equality holds:

\[ w_1 \uplus \cdots \uplus w_k = \sum_{\sigma \in S_k} w_{\sigma(1)} > w_{\sigma(2)} > \ldots > w_{\sigma(k)}. \]  

**Proof.** The identity can be proven by a simple induction on \( k \), using (3). It also follows from a more abstract argument as follows. First, the statement is immediately seen to be true if the words \( w_1, \ldots, w_k \) are in fact single (distinct) letters. Now, \( T(\mathbb{R}^k) \) is the free commutative dendriform algebra over \( k \) letters, [Sch58, p.19], [Lod95, Proposition 1.8]. In particular, for any commutative dendriform algebra \( Z \) and any map \( \phi : \{1,\ldots,k\} \to Z \) there exists a unique morphism \( \Phi : T(\mathbb{R}^k) \to Z \) of commutative dendriform algebras satisfying

\[ \Phi(i) = \phi(i), \quad i = 1, \ldots, k. \]

We specialize to \( Z = T(\mathbb{R}^d) \) and \( \phi(i) = w_i \). Now, in \( T(\mathbb{R}^k) \) we have the following identity

\[ 1 \uplus \cdots \uplus k = \sum_{\sigma \in S_k} \sigma(1) > \ldots > \sigma(k). \]

\(^5\)Recall that the half-shuffle product is non-associative, so the specification of a bracketing is necessary.
Therefore, using that $\Phi$ is a morphism, we have that

$$w_1 \shuffle \ldots \shuffle w_k = \Phi(1) \shuffle \ldots \shuffle \Phi(k) = \Phi \left( \sum_{\sigma \in S_k} \sigma(1) > \ldots > \sigma(k) \right)$$

$$= \sum_{\sigma \in S_k} \Phi(\sigma(1)) > \ldots > \Phi(\sigma(k)) = \sum_{\sigma \in S_k} w_{\sigma(1)} > w_{\sigma(2)} > \ldots > w_{\sigma(k)},$$

as desired. \qed

The following statement seems to first appear in [And83] (see also [dB55, p.1] and [For19]) and is also known as the continuous (or generalized) Cauchy-Binet formula [Joh05, Proposition 2.10].

**Lemma 3.3 ([And83]).** Let $\{1, \ldots, d\}$ and $\{1, \ldots, d\}$ be two alphabets in $d$ letters each. Then

$$\det_{\shuffle} \begin{pmatrix} 11 & 12 & \ldots & 1d \\ 21 & 22 & \ldots & 2d \\ \vdots & \vdots & \ddots & \vdots \\ d1 & d2 & \ldots & dd \end{pmatrix} = \sum_{\sigma, \tau \in S_d} \text{sign}(\sigma) \text{sign}(\tau) [\sigma(1)\tau(1) > \sigma(2)\tau(2) > \ldots > \sigma(d)\tau(d)] \quad (10)$$

**Proof.** For a fixed $\sigma \in S_d$, applying Lemma 3.2 with $w_i := i\sigma(i)$, we get

$$\prod_{i=1}^{d} i\sigma(i) = \sum_{\tau \in S_d} \tau(1)\sigma(\tau(1)) > \tau(2)\sigma(\tau(2)) > \ldots > \tau(d)\sigma(\tau(d)).$$

Then, the left-hand side of (10) is equal to,

$$\sum_{\sigma \in S_d} \text{sign}(\sigma) \prod_{i=1}^{d} i\sigma(i) = \sum_{\sigma \in S_d} \sum_{\tau \in S_d} \text{sign}(\sigma)\tau(1)\sigma(\tau(1)) > \tau(2)\sigma(\tau(2)) > \ldots > \tau(d)\sigma(\tau(d))$$

$$= \sum_{\sigma \in S_d} \sum_{\tau \in S_d} \text{sign}(\sigma \circ \tau) \text{sign}(\tau)\tau(1)\sigma(\tau(1)) > \tau(2)\sigma(\tau(2)) > \ldots > \tau(d)\sigma(\tau(d))$$

$$= \sum_{\rho \in S_d} \sum_{\tau \in S_d} \text{sign}(\rho) \text{sign}(\tau)\tau(1)\rho(1) > \tau(2)\rho(2) > \ldots > \tau(d)\rho(d),$$

as desired. \qed

### 3.2 The shuffle determinant

In this section we present a technical lemma that is crucial for the proof of our main result. This lemma states that, for the determinant $\det_{\shuffle}(W_d)$, the shuffle of letters can be replaced by a “shuffle of blocks of 2 letters”. First, let us see one example.

**Example 3.4.** For $d = 2$, we have that

$$\det_{\shuffle}(W_2) = \det_{\shuffle} \begin{pmatrix} 11 & 12 \\ 21 & 22 \end{pmatrix} = 1122 + 2211 - 1221 - 2112$$

$$= \text{shuffle of the block 11 with the block 22} - \text{shuffle of the block 12 with the block 21}.$$
Further note, that this is equal to $\text{inv} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

The general statement is as follows.

**Proposition 3.5.**

$$\det_{\alpha}(W_d) = \sum_{\sigma \in S_d} \text{sign}(\sigma) \left( \sum_{\tau \in S_d} \prod_{i=1}^{d} \tau(i) \sigma(\tau(i)) \right) = \text{inv} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ \vdots & \vdots \\ 3 & 2d \end{pmatrix},$$

where the product $\prod$ denotes the concatenation product.

Before presenting the proof for Proposition 3.5, we illustrate its idea in the case $d = 3$.

**Example 3.6.** Consider $d = 3$. By Lemma 3.3, we obtain

$$\sum_{\sigma \in S_3} \text{sign}(\sigma) \prod_{i=1}^{3} i \sigma(i) = \sum_{\sigma, \tau \in S_3} \text{sign}(\sigma) \text{sign}(\tau) [\sigma(1) \tau(1) > \sigma(2) \tau(2) > \sigma(3) \tau(3)]$$

$$= (11 > 22) > 33 + (11 > 33) > 22 + (22 > 11) > 33 + (22 > 33) > 11$$

$$+ (33 > 11) > 22 + (33 > 22) > 11 + \cdots - (13 > 22) > 31 - (13 > 31) > 22$$

$$- (22 > 13) > 31 - (22 > 31) > 13 - (31 > 13) > 22 - (31 > 22) > 31 =: \ast.$$

Now observe that the term

$$(11 > 22) > 33 = (11 > 22) \cdot 33 + (11 \updownarrow 2 \updownarrow 3) \cdot 23$$

cancels with the term

$$(11 > 32) > 23 = (11 > 32) \cdot 23 + (11 \updownarrow 3 \updownarrow 2) \cdot 23,$$

to give the term $(11 > 22) \cdot 33 - (11 > 32) \cdot 23$.

Applying this to all terms, we replace the right-most half-shuffle product by a concatenation product. That is,

$$\ast = (11 > 22) \cdot 33 + (11 > 33) \cdot 22 + (22 > 11) \cdot 33 + (22 > 33) \cdot 11 + (33 > 11) \cdot 22 + (33 > 22) \cdot 11 + \cdots$$

$$- (13 > 22) \cdot 31 - (13 > 31) \cdot 22 - (22 > 13) \cdot 31 - (22 > 31) \cdot 13 - (31 > 13) \cdot 22 - (31 > 22) \cdot 31.$$

A similar replacement now needs to be done for the remaining half-shuffle products. As an example of how this works, consider only the terms with $33$ at the end

$$(11 > 22) \cdot 33 + (22 > 11) \cdot 33 - (12 > 21) \cdot 33 - (21 > 12) \cdot 33$$

$$= 11 \cdot 22 \cdot 33 + (1 \downarrow 2) \cdot 12 \cdot 33 + 22 \cdot 11 \cdot 33 + (2 \downarrow 1) \cdot 21 \cdot 33$$

$$- 12 \cdot 21 \cdot 33 - (1 \updownarrow 2) \cdot 21 \cdot 33 - 21 \cdot 12 \cdot 33 - (2 \updownarrow 1) \cdot 12 \cdot 33$$

$$= 11 \cdot 22 \cdot 33 + 22 \cdot 11 \cdot 33 - 12 \cdot 21 \cdot 33 - 21 \cdot 12 \cdot 33,$$
Similarly, applying this procedure for all the other terms, we obtain

\[
\star = +11 \cdot 33 \cdot 22 + 11 \cdot 22 \cdot 33 + 33 \cdot 11 \cdot 22 + 22 \cdot 33 \cdot 11 + 33 \cdot 22 \cdot 11 \\
+ 31 \cdot 23 \cdot 12 + 31 \cdot 12 \cdot 23 + 12 \cdot 31 \cdot 23 + 12 \cdot 23 \cdot 31 + 23 \cdot 31 \cdot 12 + 23 \cdot 12 \cdot 31 \\
+ 13 \cdot 21 \cdot 32 + 13 \cdot 32 \cdot 21 + 21 \cdot 13 \cdot 32 + 21 \cdot 32 \cdot 13 + 32 \cdot 13 \cdot 21 + 32 \cdot 21 \cdot 13 \\
- 11 \cdot 23 \cdot 32 - 11 \cdot 32 \cdot 23 - 23 \cdot 11 \cdot 32 - 32 \cdot 11 \cdot 23 - 11 \cdot 23 \cdot 11 - 32 \cdot 23 \cdot 11 \\
- 12 \cdot 21 \cdot 33 - 12 \cdot 33 \cdot 21 - 21 \cdot 12 \cdot 33 - 21 \cdot 33 \cdot 12 - 33 \cdot 21 \cdot 12 \\
- 13 \cdot 31 \cdot 22 - 13 \cdot 22 \cdot 31 - 13 \cdot 22 \cdot 31 - 31 \cdot 22 \cdot 13 - 22 \cdot 13 \cdot 31 - 22 \cdot 31 \cdot 13,
\]

as desired.

Lemma 3.7. For any (non-commutative) polynomial \( P \) and for any letters \( a, b, c, d \) we have:

\[
(P > [ab]) > [cd] = (P > [ab]) \cdot c \cdot d + (P \uplus a \uplus c) \cdot b \cdot d.
\]  

(11)

Proof. Using (3), we have:

\[
(P > [ab]) > [cd] = (P > [ab]) \uplus c \cdot d = ((P \uplus a) \cdot b) \uplus c \cdot d
\]

\[
= ((P \uplus a) \cdot b) \cdot c + ((P \uplus a) \uplus c) \cdot b \cdot d = (P > [ab]) \cdot c \cdot d + (P \uplus a \uplus c) \cdot b \cdot d.
\]

Proof of Proposition 3.5. The second equality is obtained as follows,

\[
\sum_{\tau \in S_d} \text{sign}(\tau) \left( \sum_{\sigma \in S_d} \prod_{i=1}^{d} \sigma(i) \tau(\sigma(i)) \right) = \sum_{\tau, \sigma \in S_d} \text{sign}(\tau) \prod_{i=1}^{d} \sigma(i) \tau(\sigma(i))
\]

\[
= \sum_{\tau, \sigma \in S_d} \text{sign}(\sigma) \text{sign}(\tau \circ \sigma) \prod_{i=1}^{d} \sigma(i) \tau \circ \sigma(i) = \sum_{\eta, \epsilon \in S_d} \text{sign}(\eta) \text{sign}(\rho) \prod_{i=1}^{d} \eta(i) \rho(i) = \text{inv}(1, 2, 3, 4, \ldots, 2d).
\]

Regarding the first equality, by Lemma 3.3, we have

\[
\det_{\uplus}(W_d) = \sum_{\sigma \in S_d} \text{sign}(\sigma) \prod_{i=1}^{d} i \sigma(i) = \sum_{\sigma, \tau \in S_d} \text{sign}(\sigma) \text{sign}(\tau) [\sigma(1) \tau(1) > \sigma(2) \tau(2) > \ldots > \sigma(d) \tau(d)]
\]

\[
= \sum_{\sigma, \tau \in S_d} \text{sign}(\tau) \text{sign}(\sigma \circ \tau) [\tau(1) \sigma(\tau(1)) > \tau(2) \sigma(\tau(2)) > \ldots > \tau(d) \sigma(\tau(d))],
\]

where the last equality holds by rewriting the summation (and reusing the symbols \( \sigma \) and \( \tau \)). Note that

\[
\text{sign}(\tau) \text{sign}(\sigma \circ \tau) = \text{sign}(\tau) \text{sign}(\sigma) \text{sign}(\tau) = \text{sign}(\sigma),
\]

and so \( \text{sign}(\tau) \) does not appear in our summations anymore.
Then, we denote $d_{-1} := d - 1$ and we split our summation into two, depending on the values of $\tau(d_{-1})$ and $\tau(d)$:

$$
\sum_{\sigma, \tau \in S_d} \text{sign}(\sigma) \left[ \tau(1)\sigma(\tau(1)) > \tau(2)\sigma(\tau(2)) > \ldots > \tau(d)\sigma(\tau(d)) \right]
$$

$$
= \sum_{\sigma, \tau \in S_d} \text{sign}(\sigma) \left[ \tau(1)\sigma(\tau(1)) > \tau(2)\sigma(\tau(2)) > \ldots > \tau(d)\sigma(\tau(d)) \right]
$$

$$
+ \sum_{\sigma, \tau \in S_d} \text{sign}(\sigma) \left[ \tau(1)\sigma(\tau(1)) > \tau(2)\sigma(\tau(2)) > \ldots > \tau(d)\sigma(\tau(d)) \right]
$$

These summations contain terms that cancel each other, and so we want to control those terms. To do so, we first rewrite the summation for $\tau(d_{-1}) > \tau(d)$ in the following way:

$$
\sum_{\sigma, \tau \in S_d} \text{sign}(\sigma) \left[ \tau(1)\sigma(\tau(1)) > \ldots > \tau(d)\sigma(\tau(d)) \right]
$$

$$
= \sum_{\eta, \delta \in S_d} \text{sign}(\eta) \left[ \delta(1)\eta\delta(1) > \ldots > (\delta(d - 2)\eta\delta(d - 2)) > (\delta(d))\eta((\delta(d)) > (\delta(d))\eta((\delta(d))) \right],
$$

where we rewrite the summation by taking $\sigma = \eta$ and $\delta = \tau \circ (d, d - 1)$, so that $\delta(i) = \tau(i)$ for $1 \leq i \leq d - 2$, $\delta(d_{-1}) = \tau(d)$, and $\delta(d) = \tau(d_{-1})$.

Next, we apply Lemma 3.7 to both summations. For the first summation, we have that

$$
\sum_{\sigma, \tau \in S_d} \text{sign}(\sigma) \left[ \tau(1)\sigma(\tau(1)) > \ldots > \tau(d)\sigma(\tau(d)) \right]
$$

$$
= \sum_{\sigma, \tau \in S_d} \text{sign}(\sigma) \left[ \tau(1)\sigma(\tau(1)) > \ldots > \tau(d_{-1})\sigma(\tau(d_{-1})) \right] \tau(d)\sigma(\tau(d))
$$

$$
+ \sum_{\sigma, \tau \in S_d} \text{sign}(\sigma) \left( \tau(1)\sigma(\tau(1)) > \ldots > \tau(d_{-1})\sigma(\tau(d_{-1})) \right) \tau(d_{-1})\sigma(\tau(d_{-1}))\sigma(\tau(d)),
$$

while for the second summation we have that

$$
\sum_{\eta, \delta \in S_d} \text{sign}(\eta) \left[ \delta(1)\eta\delta(1) > \ldots > (\delta(d - 2)\eta\delta(d - 2)) > (\delta(d))\eta((\delta(d)) > (\delta(d))\eta((\delta(d))) \right]
$$

$$
= \sum_{\eta, \delta \in S_d} \text{sign}(\eta) \left[ \delta(1)\eta\delta(1) > \ldots > \delta(d)\eta\delta(d) \right] \delta(d_{-1})\eta\delta(d_{-1})
$$

$$
+ \sum_{\eta, \delta \in S_d} \text{sign}(\eta) \left( \delta(1)\eta\delta(1) > \ldots > \delta(d - 2)\eta\delta(d - 2) \right) \delta(d)\eta\delta(d)\eta\delta(d_{-1}).
$$
Now we are ready to identify the terms that cancel with each other. In fact, we want to see that
\[
\sum_{\sigma, \tau \in S_d} \text{sign}(\sigma) \left( (|\tau(1)\sigma(1)| > \ldots > \tau(d-2)\sigma(\tau(d-2)) \right] \tau(d_{-1}) \tau(d)) \sigma(\tau(d_{-1})) \sigma(\tau(d))
\]
\[
+ \sum_{\eta, \delta \in S_d} \text{sign}(\eta) \left( (|\delta(1)\eta(\delta(1)) > \ldots > \delta(d-2)\eta(\delta(d-2)) \right] \delta(d_{-1}) \delta(d)) \eta(\delta(d_{-1})) = 0.
\]

(12)

In order to prove (12), we identify terms from each summation. Given \( \sigma, \tau \in S_d \), consider the unique permutations \( \eta, \delta \in S_d \) such that \( \delta = \tau \) and \( \eta = \sigma \circ \tau \circ (d, d-1) \circ \tau^{-1} \). Thus, \( \delta(i) \eta(\delta(i)) \), for \( 1 \leq i \leq d-2 \), \( \eta(\delta(d_{-1})) \eta(\delta(d_{-1})) = \sigma(\tau(d_{-1})) \sigma(\tau(d)) \), and \( \text{sign}(\eta) = -\text{sign}(\sigma) \). This implies that the terms in the first summation of (12) cancel with the terms appearing in the second summation. Therefore,
\[
\sum_{\sigma, \tau \in S_d} \text{sign}(\sigma) \left( |\tau(1)\sigma(1)| > \ldots > \tau(d)\sigma(\tau(d)) \right]\]
\[
= \sum_{\sigma, \tau \in S_d} \text{sign}(\sigma) \left( |\tau(1)\sigma(1)| > \ldots > \tau(d_{-1})\sigma(\tau(d_{-1})) \right] \tau(d) \sigma(\tau(d))
\]
\[
+ \sum_{\eta, \delta \in S_d} \text{sign}(\eta) \left( |\delta(1)\eta(\delta(1)) > \ldots > \delta(d)\eta(\delta(d)) \right] \delta(d_{-1}) \eta(\delta(d_{-1})).
\]

Note that if we take \( \eta = \sigma \) and \( \delta = \tau \circ (d, d-1) \), the second summation corresponds to the condition \( \tau(d_{-1}) > \tau(d) \) and the summations merge into one:
\[
\sum_{\sigma, \tau \in S_d} \text{sign}(\sigma) \left( |\tau(1)\sigma(1)| > \ldots > \tau(d)\sigma(\tau(d)) \right]\]
\[
= \sum_{\sigma, \tau \in S_d} \text{sign}(\sigma) \left( |\tau(1)\sigma(1)| > \ldots > \tau(d_{-1})\sigma(\tau(d_{-1})) \right] \tau(d) \sigma(\tau(d))
\]

Our final step is basically using the inductive hypothesis on the remaining half-shuffle corresponding to \( d_{-1} \). However, one should notice that the factor \( \tau(1)\sigma(1) > \ldots > \tau(d_{-1})\sigma(\tau(d_{-1})) \) does not correspond directly to a term in \( S_{d-1} \). However, a permutation can be seen as a comparison between total order relations on a set of objects. In this way, for the inductive step, we can consider the restriction of this total order relation to a subset of \( d-1 \) elements. Thus, replacing step by step all the half-shuffle products by concatenation, we obtain the desired identity
\[
\sum_{\sigma, \tau \in S_d} \text{sign}(\sigma) \prod_{i=1}^{d} \tau(i) \sigma(i) = \sum_{\sigma \in S_d} \text{sign}(\sigma) \left( \sum_{\tau \in S_d} \prod_{i=1}^{d} \tau(i) \sigma(i) \right).
\]

4 Main result

The aim of this section is to prove the following theorem.
Theorem 4.1. For \( d \geq 1 \),
\[
\det_{\omega}(W_d) = \text{inv} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ \vdots & \vdots \\ \omega \cdot 2d & \omega \cdot 2d \end{pmatrix} = \frac{1}{2^d} \text{inv} \begin{pmatrix} 1 & 2 \\ \vdots & \vdots \\ d & d \end{pmatrix}^\omega \, \omega^2,
\]
i.e., using the notation (5), \( \det_{\omega}(W_d) = \text{inv}(t_{2,d}) = \frac{1}{2^d} \text{inv}(t_{1,d})^\omega \).

Before presenting the proof for Theorem 4.1, we show that Theorem 1.1 is a consequence of it.

Corollary 4.2. Equation (1) holds.

Proof.

\[
\begin{aligned}
\det \left( \int dX^{(1)} dX^{(1)} \ldots \int dX^{(d)} dX^{(1)} \ldots \int dX^{(1)} dX^{(d)} \right) &= \det \left( \begin{pmatrix} \langle 11, \sigma(X) \rangle & \ldots & \langle 1d, \sigma(X) \rangle \\ \vdots & \ddots & \vdots \\ \langle d1, \sigma(X) \rangle & \ldots & \langle dd, \sigma(X) \rangle \end{pmatrix} \right) \\
&= \frac{1}{2^d} \text{inv}(t_{1,d})^\omega, \sigma(X) = \frac{1}{2^d} \left( \sum_{\sigma \in S_d} \text{sign}(\sigma) \int dX^{\sigma(1)} \ldots dX^{\sigma(d)} \right)^2.
\end{aligned}
\]

From Proposition 3.5 we already know that the first equality holds. To show the second equality we use the following strategy:

**Step 1** Show that the two terms lie in the same isotypic component of a representation of a certain subgroup of \( S_{2d} \), and that this isotypic component has dimension 1.

**Step 2** Determine the pre-factors, by looking at the factors on a particular word in each side of the equality.

For **Step 1**, we first observe that both \( \text{inv}(t_{2,d}) \) and \( \text{inv}(t_{1,d})^\omega \) are in the irreducible representation for \( S_{2d} \) corresponding to the shape \((2,2,\ldots,2)\). Denote by \( \chi_1 \) the irreducible character for this shape.

Recall that we denote by \( H_d \) the subgroup of \( S_{2d} \) given by
\[
H_d = \left\langle (1,3), (2,4), (3,5), \ldots, (2d-2,2d) \right\rangle \cong S_d \times S_d.
\]

Note that \( \chi_1 \) is also a character for \( H_d \), but it is not irreducible. Let \( \chi_2(h) := \text{sign}(h) \), for all \( h \in H_d \), be the character for the one-dimensional sign-representation restricted from \( S_{2d} \) to \( H_d \). This representation is still irreducible, since it is one-dimensional.
Recall the two standard Young tableaux defined in (5). The following two results show that both (inv(t_{1,d})^\wedge 2) and (inv(t_{2,d})) lie in the isotypic component related of this sign-representation.

**Lemma 4.3.** For all (h \in H_d),

\[ h \cdot \inv_d \begin{pmatrix} 1 \\ 2 \\ \vdots \\ d \end{pmatrix}^\wedge 2 = \sign(h) \cdot \inv_d \begin{pmatrix} 1 \\ 2 \\ \vdots \\ d \end{pmatrix}^\wedge 2. \]

**Proof.** Note that it is enough to show this result for generators of the subgroup (H_d), in particular permutations of the form (h = (i, i + 2), \; i \in \{1, 2, \ldots, 2d - 2\}) are sufficient.

By (6), we have that (\inv(t_{1,d})^\wedge 2 = \sum_{\sigma, \tau \in S_d} \sign(\sigma) \sign(\tau) \left[ \sigma(12 \ldots d) \sqcup \tau(12 \ldots d) \right])

and want to show that

\[ \sum_{\sigma, \tau \in S_d} \sign(\sigma) \sign(\tau) \left[ \sigma(12 \ldots d) \sqcup \tau(12 \ldots d) \right] = \sum_{\sigma, \tau \in S_d} \sign((i, i + 2)) \sign(\sigma) \sign(\tau)(i, i + 2) \left[ \sigma(12 \ldots d) \sqcup \tau(12 \ldots d) \right]. \] (13)

If we expand (\sigma(12 \ldots d) \sqcup \tau(12 \ldots d)) in each side and do not simplify the expression, we obtain on both sides a huge sum of words with coefficients in \{\pm 1\}. To see that both expressions are equal, we show that there is a one-to-one correspondence between the terms in both sums, and that the signs are equal.

We are considering all terms appearing in the sums in Equation (13) and we identify each word \(w\) with a triplet \([\sigma, \tau, P]\), where \(P\) keeps track of the position of the letters coming from \(\sigma(12 \ldots d)\) as an increasing list. To visualize our argument, we color-code the letters \(12 \ldots d\) when they are letters coming from \(\sigma(12 \ldots d)\) and \(12 \ldots d\) when they are letters coming from \(\tau(12 \ldots d)\).

Let \(w = w_1 \ldots w_i w_{i+1} w_{i+2} \ldots w_{2d}\) be a word appearing in \(\sigma(12 \ldots d) \sqcup \tau(12 \ldots d)\). This word corresponds to a triplet \([\sigma, \tau, P]\). Now, we apply \(h = (i, i + 2)\) to \(w\) and we want to describe the triplet \([\sigma', \tau', P']\) corresponding to \(hw\). That is, we want to describe the one-to-one correspondence

\[ w = w_1 \ldots w_i w_{i+1} w_{i+2} \ldots w_{2d} \quad \mapsto \quad hw = w_1 \ldots w_{i+2} w_{i+1} w_i \ldots w_{2d} \]

\[ [\sigma, \tau, P] \quad \mapsto \quad [\sigma', \tau', P'] \]

We describe this map by cases depending on whenever the letters \(w_i, w_{i+1}\) and \(w_{i+2}\) are from \(\sigma(12 \ldots d)\) or from \(\tau(12 \ldots d)\); that is, depending on the colors of the letters \(w_i, w_{i+1},\) and \(w_{i+2}\). Before describing all the possible cases, we need some more notation. Recall that \(\sigma\) and \(\tau\) act on the left, which means that the act on the positions of \(12 \ldots d\). Therefore, \(\sigma(12 \ldots d) = \sigma^{-1}(1)\sigma^{-1}(2) \ldots \sigma^{-1}(d)\) and \(\tau(12 \ldots d) = \tau^{-1}(1)\tau^{-1}(2) \ldots \tau^{-1}(d)\). This implies that the letters appearing in \(w\) corresponds to \(\sigma^{-1}(j)\) or \(\tau^{-1}(j)\), for some \(j \in [d]\).

Now, we are ready to describe all the possible cases. We only look at the relevant part of \(w\), and we include an example using \(\sigma = (1, 3, 4), \; \tau = (1, 2, 3)\) and \(4213 \sqcup 3124\) to illustrate each case.
(C1) If the three letters are coming from $\sigma(12 \ldots d)$, then

\[
\begin{array}{c|c|c|c|c|c}
  w_i & w_{i+1} & w_{i+2} & w_{i+1} & w_i \\
  \sigma^{-1}(j) & \sigma^{-1}(j+1) & \sigma^{-1}(j+2) & \sigma^{-1}(j+1) & \sigma^{-1}(j)
\end{array}
\]

Therefore, $\sigma' = (j, j+2) \sigma$, $\tau' = \tau$, and $P' = P$ since the position of the blue letters does not change.

(C2) If two of the letters are coming from $\sigma(12 \ldots d)$ and the other one is coming from $\tau(12 \ldots d)$, then we have three cases:

(C2.a)

\[
\begin{array}{c|c|c|c|c|c}
  w_i & w_{i+1} & w_{i+2} & w_{i+1} & w_i \\
  \sigma^{-1}(j) & \sigma^{-1}(j+1) & \tau^{-1}(k) & \tau^{-1}(k) & \sigma^{-1}(j)
\end{array}
\]

Therefore, $\sigma' = (j, j+1) \sigma$ and $\tau' = \tau$. Moreover, we know that $P$ is of the form $P = (\ldots, i, i+1, \ldots)$, with no $i+2$ in $P$. Therefore, $P'$ is of the form $P' = (\ldots, i+1, i+2, \ldots)$, with no $i$ in $P'$.

(C2.b)

\[
\begin{array}{c|c|c|c|c|c}
  w_i & w_{i+1} & w_{i+2} & w_{i+1} & w_i \\
  \sigma^{-1}(j) & \tau^{-1}(k) & \sigma^{-1}(j+1) & \tau^{-1}(k) & \sigma^{-1}(j)
\end{array}
\]

Therefore, $\sigma' = (j, j+1) \sigma$, $\tau' = \tau$, and $P' = P$ since the position of the blue letters does not change.

(C2.c)

\[
\begin{array}{c|c|c|c|c|c}
  w_i & w_{i+1} & w_{i+2} & w_{i+1} & w_i \\
  \tau^{-1}(k) & \sigma^{-1}(j) & \sigma^{-1}(j+1) & \sigma^{-1}(j) & \tau^{-1}(k)
\end{array}
\]

Therefore, $\sigma' = (j, j+1) \sigma$ and $\tau' = \tau$. Moreover, we know that $P$ is of the form $P = (\ldots, i+1, i+2, \ldots)$, with no $i$ in $P$. Therefore, $P'$ is of the form $P' = (\ldots, i, i+1, \ldots)$, with no $i+2$ in $P'$.

(C3) If one of the letters is coming from $\sigma(12 \ldots d)$ and the other two are coming from $\tau(12 \ldots d)$, then we have three cases:

(C3.a)

\[
\begin{array}{c|c|c|c|c|c}
  w_i & w_{i+1} & w_{i+2} & w_{i+1} & w_i \\
  \sigma^{-1}(j) & \tau^{-1}(k) & \tau^{-1}(k+1) & \tau^{-1}(k+1) & \sigma^{-1}(j)
\end{array}
\]

15
Therefore, $\sigma' = \sigma$ and $\tau' = (k, k + 1)\tau$. Moreover, we know that $P$ is of the form $P = (\ldots, i, \ldots)$, with no $i + 1, i + 2$ in $P$. Therefore, $P'$ is of the form $P' = (\ldots, i + 2, \ldots)$, with no $i, i + 1$ in $P'$.

(C3.b)

\[\begin{array}{c|c|c|c|c|c}
\tau^{-1}(k) & \sigma^{-1}(j) & \tau^{-1}(k + 1) & \sigma^{-1}(j) & \tau^{-1}(k) \\
\hline
w_i & w_{i+1} & w_{i+2} & w_{i+2} & w_{i+1} & w_i \\
\end{array}\]

Therefore, $\sigma' = \sigma$, $\tau' = (k, k + 1)\tau$, and $P' = P$ since the position of the blue letters does not change.

(C3.c)

\[\begin{array}{c|c|c|c|c|c}
\tau^{-1}(k) & \sigma^{-1}(j) & \tau^{-1}(k + 1) & \sigma^{-1}(j) & \tau^{-1}(k) \\
\hline
w_i & w_{i+1} & w_{i+2} & w_{i+2} & w_{i+1} & w_i \\
\end{array}\]

Therefore, $\sigma' = \sigma$ and $\tau' = (k, k + 1)\tau$. Moreover, we know that $P$ is of the form $P = (\ldots, i + 2, \ldots)$, with no $i, i + 1$ in $P$. Therefore, $P'$ is of the form $P' = (\ldots, i, \ldots)$, with no $i + 1, i + 2$ in $P'$.

(C4) If the three letters are coming from $\tau(12\ldots d)$, we only have one case:

\[\begin{array}{c|c|c|c|c|c}
\tau^{-1}(k) & \tau^{-1}(k + 1) & \tau^{-1}(k + 2) & \tau^{-1}(k + 1) & \tau^{-1}(k) \\
\hline
w_i & w_{i+1} & w_{i+2} & w_{i+2} & w_{i+1} & w_i \\
\end{array}\]

Therefore, $\sigma' = \sigma$, $\tau' = (k, k + 2)\tau$, and $P' = P$ since the position of the blue letters does not change.

Finally, we look at the sign. Notice that in all the cases, $\text{sign}(\sigma)\text{sign}(\tau) = \text{sign}(i, i + 2)\text{sign}(\sigma')\text{sign}(\tau')$ because either $\sigma'$ nor $\tau'$ differ from $\sigma$ and $\tau$, respectively, by a transposition. 

\[\text{Example 4.4.} \quad \text{Consider } \sigma = (1, 3, 4) \text{ and } \tau = (1, 2, 3), \text{ for which } \sigma(1234) = 4213 \text{ and } \tau(1234) = 3124. \quad \text{Then, } w = 43121324 \text{ is a word appearing in } 4213 \sqcup 3124. \quad \text{Let us illustrate the cases in the previous proof with this example with } i = 4: \]

| Case (C1) | Case (C2.a) |
|-----------|--------------|
| $43121324$ | $43131224$ |
| $\sigma(1234) = 4213$ | $\sigma'(1234) = 4312 = (2, 4)\sigma$ |
| $\tau(1234) = 3124$ | $\tau'(1234) = 3124$ |
| $P = [1, 4, 5, 6]$ | $P' = [1, 4, 5, 6]$ |
| $42313124$ | $42313124$ |
| $\sigma(1234) = 4213$ | $\sigma'(1234) = 4231 = (3, 4)\sigma$ |
| $\tau(1234) = 3124$ | $\tau'(1234) = 3124$ |
| $P = [1, 2, 4, 5]$ | $P' = [1, 2, 5, 6]$ |
Case (C2.b)  Case (C2.c)  Case (C3.a)  Case (C3.b)  Case (C3.c)  Case (C3.d)  Case (C4)

|        |        |        |        |        |        |        |
|--------|--------|--------|--------|--------|--------|--------|
| \(\sigma(1234) = 4213\) | \(\sigma'(1234) = 4213\) | \(\tau(1234) = 3124\) | \(\tau'(1234) = 3124\) | \(P = [1, 2, 4, 6]\) | \(P' = [1, 2, 4, 6]\) | \(\sigma(1234) = 4213\) | \(\sigma'(1234) = 4213\) | \(\tau(1234) = 3124\) | \(\tau'(1234) = 3124\) | \(P = [1, 2, 4, 6]\) | \(P' = [1, 2, 4, 6]\) | \(\sigma(1234) = 4213\) | \(\sigma'(1234) = 4213\) | \(\tau(1234) = 3124\) | \(\tau'(1234) = 3124\) | \(P = [1, 2, 4, 6]\) | \(P' = [1, 2, 4, 6]\) |
| \(\sigma(1234) = 4213\) | \(\sigma'(1234) = 4213\) | \(\tau(1234) = 3124\) | \(\tau'(1234) = 3124\) | \(P = [1, 2, 4, 6]\) | \(P' = [1, 2, 4, 6]\) | \(\sigma(1234) = 4213\) | \(\sigma'(1234) = 4213\) | \(\tau(1234) = 3124\) | \(\tau'(1234) = 3124\) | \(P = [1, 2, 4, 6]\) | \(P' = [1, 2, 4, 6]\) | \(\sigma(1234) = 4213\) | \(\sigma'(1234) = 4213\) | \(\tau(1234) = 3124\) | \(\tau'(1234) = 3124\) | \(P = [1, 2, 4, 6]\) | \(P' = [1, 2, 4, 6]\) |

It is immediate that \(\text{inv}(t_{2,d})\) satisfies the analogous statement.

**Lemma 4.5.** For all \(h \in H_d\),

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
\vdots & \vdots \\
\omega & 2d
\end{pmatrix} = \text{sign}(h) \cdot \begin{pmatrix}
1 & 2 \\
3 & 4 \\
\vdots & \vdots \\
\omega & 2d
\end{pmatrix}
\]

Now, we want to see that the dimension of this isotypic component is 1. This follows from the fact that the multiplicity of the irreducible character \(\chi_2\) in \(\chi_1\) is equal to one in the case we are considering.

**Lemma 4.6.** Consider the decomposition (with respect to \(H_d\)) of the representation (with respect to \(S_{2d}\)) corresponding to \(\chi_1\). In this decomposition, the multiplicity of the irreducible sign-representation, i.e. the irreducible representation corresponding to \(\chi_2\), has multiplicity one.

**Proof.** We want to look at the sign representation of \(S_{2d}\), \(\text{sign}\), in terms of representations of \(S_d\) since we have that \(H_d \cong S_d \times S_d\).

Consider \(\lambda = \mu = (1, \ldots, 1)\), so that \(V_\lambda = V_\mu\) is the sign representation of \(S_d\), and \(\nu = (2, \ldots, 2)\).

Denote by \(V_\lambda \boxtimes V_\mu\) the (external) product of representations (see [FH91, Exercise 2.36]). Then, we claim that \(V_\lambda \boxtimes V_\mu \cong \text{sign}|_{H_d}\), where \(\text{sign}|_{H_d}\) denotes the restriction of the representation sign to \(H_d\). By [FH91,
Exercise 2.36] all the irreducible representations of $S_d \times S_d$ are given by the external product of irreducible representations of $S_d$. In our case, there are exactly two one-dimensional irreducible representations of $S_d$, the trivial and the sign representations. Thus, there are exactly 4 different one-dimensional representations of $S_d \times S_d$, obtained by taking the external product of either the trivial or the sign representation with, again, either the trivial or the sign representation. Now, sign $|H_d|$ is one-dimensional, and so it has to be one of these 4 different representations. Therefore, sign $|H_d|$ has to correspond to the one claimed, $V_\lambda \boxtimes V_\mu$.

Hence, the multiplicity (as a $H_d$ representation) of $\chi_2$ inside of $\chi_1$ (as $S_{2d}$ representation) is given by the multiplicity (as $S_d \times S_d$ representation) of $V_\lambda \boxtimes V_\mu$ inside of $V_\nu$. By [FH91, Exercise 4.43, Appendix A.1] the latter is given by the Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$. That is, we are counting the number of semi-standard Young tableaux of skew-shape $\nu/\lambda$ and type $\mu$ such that the reading word is a lattice word (see [Sta99]). In our case, $\nu/\lambda$ is exactly a column and type $\mu$ means that we have to fill out the column exactly with the numbers $1, 2, \ldots, d$ in increasing order. Since there is only one way to do it and the lattice word condition is trivially satisfied, $c^\nu_{\lambda\mu} = 1$.

We are now ready to finish the proof of our main theorem.

**Proof of Theorem 4.1.** Let us start by summarising the situation. By Proposition 3.5 we have that $\det_\omega(W_d) = \text{inv}(t_{2,d})$. Now both $\text{inv}(t_{2,d})$ and $(\text{inv}(t_{1,d}))^{\omega 2}$ live in the irreducible $S_{2d}$-representation corresponding to the shape $(2, 2, \ldots, 2)$. Restricting this representation to the subgroup $H_d$ of $S_{2d}$ and decomposing into irreducible representations of $H_d$, we have that $\det_\omega(W_d)$ and $(\text{inv}(t_{1,d}))^{\omega 2}$ lie in the component corresponding to the sign-representation, see Lemmas 4.3 and 4.5. We also know, by Lemma 4.6, that this component is 1-dimensional.

Our last step is to prove that the prefactors are as stated. Now, the word $11122\ldots d d$ can only be obtained from shuffling $12\ldots d$ with $12\ldots d$, which results in a factor of $2^d$. Using the fact that it reduces to shuffle products of blocks of 2 letters (Proposition 3.5), one sees that in $\det_\omega(W_d)$, the word $11122\ldots d d$ appears with the factor one.

The following example illustrates the computation of the prefactors from Theorem 4.1 for $d = 3$.

**Example 4.7.** The word $112233$ appearing in $(\text{inv}(t_{1,3}))^{\omega 2}$, can only come from the shuffle $123 \shuffle 123$. This can happen in $2^3 = 8$ ways, namely (using the color-code from the proof of Lemma 4.3)

$112233, 112323, 112233, 112323, 112333, 112233, 112333, 112333$.

On the other hand, in $\det_\omega(W_3)$ it only appears once in the term coming from $11 \shuffle 22 \shuffle 33$ (the diagonal of the matrix in the Leibniz formula for the determinant).

## 5 Pfaffians and de Bruijn’s formula

Before stating and proving de Bruijn’s formula, we recall some definitions and present some technical results. Let $\mathcal{A}$ be a commutative algebra over $\mathbb{R}$.

**Lemma 5.1.** Let $A \in \mathcal{A}^{d \times d}$ anti-symmetric, $v \in \mathcal{A}^d$ and $\lambda \in \mathcal{A}$.

\[ \lambda \]
• If $d$ is even, $\det[A + \lambda vv^\top] = \det[A]$.

• If $d$ is odd, $\det[A + \lambda vv^\top] = \sum_{i=1}^d \det[R_i]$, where $R_i$ denotes the matrix $A$, with $i^{th}$ column by that row of $\lambda vv^\top$.

Proof. Let $V := \lambda vv^\top$. For $I \subset [n] := \{1, \ldots, n\}$, let $R_I$ be the matrix $A$ with the rows corresponding to $I$ replaced by the corresponding rows of $V$. Correspondingly, let $C_I$ be the matrix $A$ with the columns corresponding to $I$ replaced by the corresponding columns of $V$. Then

$$\det[A + V] = \sum_{I \subset [n]} \det(R_I) = \det(A) + \sum_{i=1}^n \det(R_{\{i\}})$$

$$\det[A + V] = \sum_{I \subset [n]} \det(C_I) = \det(A) + \sum_{i=1}^n \det(C_{\{i\}}),$$

where we used the fact that $\det(R_I) = \det(C_I) = 0$ if $|I| \geq 2$.

If $d$ is odd, $\det[A] = 0$, since $A$ is anti-symmetric. This yields the statement in this case, with $(R_i := R_{\{i\}})$. If $d$ is even, we add the two expansions to get

$$\det[A + V] = \det[A] + \frac{1}{2} \sum_{i=1}^n (\det(R_{\{i\}}) + \det(C_{\{i\}})),$$

and we get $\det(R_{\{i\}}) = -\det(C_{\{i\}})$, for all $i \in [n]$. This finishes the proof.

For the following statement recall the matrix $W_d$ from (8) with entries in the shuffle algebra $T(\mathbb{R}^d)$.

Lemma 5.2. Write $\text{Sym}[W_d]_4 := \frac{1}{2}(W_d + W_d^\top)$. Then we have

$$\text{Sym}[W_d] = \frac{1}{2} \begin{pmatrix} 1 \\ 2 \\ \vdots \\ d \end{pmatrix} \shuffle \begin{pmatrix} 1 \\ 2 \\ \vdots \\ d \end{pmatrix}^\top.$$

Proof. We have the following computation

$$\begin{pmatrix} 1 \\ 2 \\ \vdots \\ d \end{pmatrix} \shuffle \begin{pmatrix} 1 \\ 2 \\ \vdots \\ d \end{pmatrix} = \begin{pmatrix} 1 \shuffle 1 & 1 \shuffle 2 & \cdots & 1 \shuffle d \\ \vdots & \vdots & \ddots & \vdots \\ d \shuffle 1 & d \shuffle 2 & \cdots & d \shuffle d \end{pmatrix} = \begin{pmatrix} 11 + 11 & 12 + 21 & \cdots & 1d + d1 \\ \vdots & \vdots & \ddots & \vdots \\ d1 + 1d & d2 + 2d & \cdots & dd + dd \end{pmatrix} = W_d + W_d^\top.$$

The following definition is well-known ([Bou07, Chapter 5.2, p.82])

19
Definition 5.3. Let $A = (a_{ij}) \in A^{d \times d}$ be a anti-symmetric matrix, where $d$ is even. Then

$$\text{Pf}_d[A] := \frac{1}{2^{d/2} (d/2)!} \sum_{\pi \in S_d} \text{sign}(\pi) \prod_{i=1}^{d/2} a_{\pi(2i-1)\pi(2i)}$$

is called the Pfaffian of $A$.

For the following statement see for instance [Art57, pages 140-141], [Led93], and references therein.

Lemma 5.4. Let $d$ be even and $A \in A^{d \times d}$ a anti-symmetric matrix. We have the following identity:

$$\text{det}_d[A] = (\text{Pf}_d[A])^2.$$  

5.1 Deducing de Bruijn’s formula from our results

The de Bruijn’s formula has two different formulations depending on the parity of $d$. We compare both formulations with our results.

We recall that any matrix $M$ can be written as the sum of its symmetric part and its anti-symmetric part $M = \text{Sym}[M] + \text{Anti}[M]$, $\text{Sym}[M] := \frac{1}{2}(M + M^\top)$ and $\text{Anti}[M] := \frac{1}{2}(M - M^\top)$.

For $d$ even, de Bruijn’s formula ([dB55], see also [DR19, Remark 3.18]), reads, in modern language, as follows.

Theorem (de Bruijn’s formula - Even case).

$$\text{inv} \begin{pmatrix} 1 & 2 & \vdots & d \end{pmatrix}^{\omega^2} = 2^{d/2} \text{Pf}_{\omega^2}[\text{Anti}[W_d]].$$

Proof. We deduce this from our results in the main text. First, Theorem 4.1 gives

$$\text{inv} \begin{pmatrix} 1 & 2 & \vdots & d \end{pmatrix}^{\omega^2} = 2^d \text{det}_d[W_d].$$

Note that $W_d = \text{Sym}(W_d) + \text{Anti}(W_d)$. Then, by Lemma 5.2, $\text{det}_d[W_d] = \text{det}_d[\text{Anti}[W_d]]$. By properties of the Pfaffian, Lemma 5.4, $\text{det}_d[\text{Anti}[W_d]] = \text{Pf}_d[\text{Anti}[W_d]]^{\omega^2}$. Hence

$$\text{inv} \begin{pmatrix} 1 & 2 & \vdots & d \end{pmatrix}^{\omega^2} = 2^d \text{Pf}_d[\text{Anti}[W_d]]^{\omega^2}.$$
Since the shuffle algebra is commutative and integral, we deduce that:

\[
\text{inv} \begin{pmatrix}
1 \\
2 \\
\vdots \\
d
\end{pmatrix} = \pm 2^{d/2} \text{Pf}_{\omega}[\text{Anti}[W_d]].
\]

By comparing the coefficient of the word 123...d, we deduce that the sign must be positive, and have thus shown de Bruijn’s formula in the even case.

Example 5.5. de Bruijn’s formula in the case \(d = 4\) gives

\[
\text{inv} \begin{pmatrix}
1 \\
2 \\
3 \\
4
\end{pmatrix} = 4 \text{Pf}_{\omega} \begin{pmatrix}
0 & 12 - 21 & 13 - 31 & 14 - 41 \\
21 - 12 & 0 & 23 - 32 & 24 - 42 \\
31 - 13 & 32 - 23 & 0 & 34 - 43 \\
41 - 14 & 42 - 24 & 43 - 34 & 0
\end{pmatrix};
\]

whereas Theorem 4.1 gives

\[
\text{inv} \begin{pmatrix}
1 \\
2 \\
3 \\
4
\end{pmatrix}^{\omega^2} = 16 \det_{\omega} \begin{pmatrix}
11 & 12 & 13 & 14 \\
21 & 22 & 23 & 24 \\
31 & 32 & 33 & 34 \\
41 & 42 & 43 & 44
\end{pmatrix}.
\]

Now, we look at the odd case.

Theorem (de Bruijn’s formula - Odd case). For \(d\) odd, de Bruijn’s formula ([dB55]) reads as

\[
\text{inv} \begin{pmatrix}
1 \\
2 \\
\vdots \\
d
\end{pmatrix} = \text{Pf}_{\omega}[Z_d],
\]

where \(Z_d\) denotes the matrix of the form

\[
Z_d = \left( \begin{array}{cc|c}
2 \text{Anti}[W_d] & 1 \\
2 & 2 \\
\vdots \\
-1 & -2 & \cdots & -d & 0
\end{array} \right).
\]

Proof. We follow a similar strategy as in the even-dimensional case, to show this theorem using our results. As before, we decompose into symmetric and anti-symmetric part, \(W_d = \text{Sym}[W_d] + \text{Anti}[W_d]\).
By Lemma 5.2, Sym[$W_d$] = $\frac{1}{2}vv^T$ for $v^T = [1 \ 2 \ \cdots \ d]$. Then by Lemma 5.1, $\det_w[W_d] = \sum_{i=1}^d \det_w[R_i]$, where $R_i$ is the matrix Anti[$W_d$] with $i^{th}$ row replaced by $\frac{1}{2}v_i v^T$.

Now

$$
\sum_{i=1}^d \det_w[R_i] = 2^{-d} \det_w \begin{pmatrix}
2 \ \text{Anti}[^w W_d] & 1 \\
-1 & -2 & \cdots & -d & 0
\end{pmatrix} = 2^{-d} \det_w(Z_d),
$$

which gives $2^d \det_w[W_d] = \det_w(Z_d)$. Combining with our Theorem 4.1, we get

$$
\begin{pmatrix}
1 \\
2 \\
\vdots \\
d
\end{pmatrix}^{w^2} = 2^d \det_w[W_d] = \det_w(Z_d).
$$

By Lemma 5.4 (note that $Z_d$ is even-dimensional), $\det_w[Z_d] = \text{Pf}_w[Z_d]^{w^2}$. Hence

$$
\begin{pmatrix}
1 \\
2 \\
\vdots \\
d
\end{pmatrix}^{w^2} = \text{Pf}_w[Z_d]^{w^2}.
$$

As for the even case, since the shuffle algebra is commutative and integral, and by comparing the coefficient of the word $123\ldots d$, we deduce de Bruijn’s formula in the odd case.

Example 5.6. de Bruijn’s formula in the case $d = 3$ gives

$$
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}^{w^2} = 8 \det_w \begin{pmatrix}
11 & 12 & 13 \\
21 & 22 & 23 \\
31 & 32 & 33
\end{pmatrix},
$$

whereas Theorem 4.1 gives

$$
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}^{w^2} = \text{Pf}_w \begin{pmatrix}
0 & 12 - 21 & 13 - 31 & 1 \\
21 - 12 & 0 & 23 - 32 & 2 \\
31 - 13 & 32 - 23 & 0 & 3 \\
-1 & -2 & -3 & 0
\end{pmatrix},
$$

which, by the preceding argument, is equal to

$$
\begin{pmatrix}
0 & 12 - 21 & 13 - 31 & 1 \\
21 - 12 & 0 & 23 - 32 & 2 \\
31 - 13 & 32 - 23 & 0 & 3 \\
-1 & -2 & -3 & 0
\end{pmatrix}.$$
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