Abstract

We define derived Poincaré–Birkhoff–Witt maps of dg operads or derived PBW maps, for short, which extend the definition of PBW maps between operads of V. Dotsenko and the second author in [11], with the purpose of studying the universal enveloping algebra of dg Lie algebras as a functor on the homotopy category. Our main result shows that the map from the homotopy Lie operad to the homotopy associative operad is derived PBW, which gives us an amenable description of the homology of the universal envelop of an $L_\infty$ algebra in the sense of Lada–Markl. We deduce from this several known results involving universal envelopes of $L_\infty$-algebras of V. Baranovsky and J. Moreno-Fernández, and extend D. Quillen’s classical quasi-isomorphism $\mathcal{C} \rightarrow BU$ from dg Lie algebras to $L_\infty$-algebras; this confirms a conjecture of J. Moreno-Fernández.


**Introduction**

It is a classical result going back to Poincaré, Birkhoff and Witt that, over a field of characteristic zero, the universal enveloping algebra $U(g)$ of a Lie algebra $g$ is isomorphic, as a vector space, to the symmetric algebra $S(g)$ on $g$. One can, in fact, extend without changes the definition of the functor $U$ to the category of dg Lie algebras, and a consequence of the PBW theorem is that for any dg Lie algebra we have a natural isomorphism $UH(g) \rightarrow HU(g)$. This result implies that the underlying homology group of the universal envelope of a dg Lie algebra does not depend on the Lie algebra structure of $g$ but only on the homology group $H(g)$, and that this functor descends to the homotopy category of dg Lie algebras. This category can be modeled through minimal $L_\infty$-algebras and their morphisms up to $L_\infty$-quasi-isomorphism and, similarly, the homotopy category of dga algebras can be modeled through minimal $A_\infty$-algebras and their morphisms up to $A_\infty$-quasi-isomorphism, so it is reasonable to consider the problem of finding a functor at the level of $L_\infty$-algebras representing $U$ on the homotopy category.

This program has been carried out by V. Baranosvky [2] and later by J. Moreno-Fernandez [24], and their results imply that to every minimal $L_\infty$-algebra one can assign a “universal enveloping” minimal $A_\infty$-algebra $\Upsilon(g)$ that enjoys many properties similar to those of the classical universal enveloping algebra functor:

- its underlying vector space is the symmetric algebra $S(g)$, independently of the $L_\infty$-structure of $g$,
- there exists a Quillen type $A_\infty$-quasi-isomorphism $\Omega \mathcal{C}(g) \rightarrow \Upsilon(g)$ (so the algebra $\Upsilon(g)$ does have the correct homotopy type),
- the canonical inclusion $g \rightarrow \Upsilon(g)$ is a strict $L_\infty$-morphism: the antisymmetrized higher multiplication maps on $\Upsilon(g)$ restrict to the higher brackets of $g$.

One of the goals of this paper is to explain, using the methods of V. Dotsenko and second author, explained in [11] and suitably extended to the dg setting, that the results of [2] and [24] both follow from a derived version of the classical Poincaré-Birkhoff-Witt theorem for Lie algebras. To accomplish this, we study, through the theory of operads, the Lada–Markl functor that assigns to an $A_\infty$-algebra the $L_\infty$-algebra obtained by antisymmetrising all the product operations. As in the non-dg case, this functor has a left adjoint, and we prove this functor satisfies a derived Poincaré–Birkhoff–Witt theorem. Our main result implies the following; here $U(g)$ denotes the left adjoint above, while $UH(g)$ is the universal envelope of the Lie algebra $H(g)$:
Theorem. For every minimal $L_\infty$-algebra $\mathfrak{g}$:

1. there is an isomorphism between the symmetric algebra $S(\mathfrak{g})$ and the homology $HU(\mathfrak{g})$ of the $A_\infty$-universal envelope $U(\mathfrak{g})$. This isomorphism is natural with respect to strict $L_\infty$-morphisms.

2. $S(\mathfrak{g})$ can be endowed with a minimal $A_\infty$-algebra structure $A_\infty$-quasi-isomorphic to $U(\mathfrak{g})$ so that the inclusion $\mathfrak{g} \hookrightarrow S(\mathfrak{g})$ is a strict map of $L_\infty$-algebras.

From this theorem, we deduce the three proposed models for universal envelopes of $L_\infty$-algebras are, up to homotopy, one and the same. This means, in particular, that the two existing tentative models for universal envelopes are in fact models. This follows from the following:

Corollary. Let $\mathfrak{g}$ be a minimal $L_\infty$-algebra. Every minimal $A_\infty$-algebra structure on $S(\mathfrak{g})$ for which the restriction of the antisymmetrized structure operations coincide with the operations of $\mathfrak{g}$ is $A_\infty$-quasi-isomorphic to $U(\mathfrak{g})$. Thus, the models of Baranovsky and Moreno-Fernandez are $A_\infty$-isomorphic, and $A_\infty$-quasi-isomorphic to $U(\mathfrak{g})$.

For a final application, consider an $L_\infty$-algebra $\mathfrak{g}$. Then $\mathcal{C}(\mathfrak{g})$ is a commutative dg-coalgebra, where $\mathcal{C}$ is the Quillen construction on $\mathfrak{g}$. Since $U(\mathfrak{g})$ is an $A_\infty$-algebra, it is natural to compare $\mathcal{C}(\mathfrak{g})$ to $BU(\mathfrak{g})$, which is a (non-commutative) dg-coalgebra. This result was conjectured in [24].

Corollary. There is an acyclic cofibration $\mathcal{C}(\mathfrak{g}) \hookrightarrow BU(\mathfrak{g})$ of dg-coalgebras natural with respect to strict $L_\infty$-morphisms. Moreover, for any minimal $L_\infty$-algebra $\mathfrak{g}$, there are acyclic cofibrations $\mathcal{C}(\mathfrak{g}) \hookrightarrow BS(\mathfrak{g})$. In particular, $BU(\mathfrak{g})$ always admits a commutative model.

Structure. The paper is organised as follows. In Section (1) we prove our main theorem relating almost-free and derived PBW morphisms between dg operads. We then use it in Section (2) to show the map from the homotopy Lie operad to the homotopy associative operad is derived PBW and deduce from this several results on universal envelopes of $L_\infty$-algebras, which recover results from Baranovsky and Moreno-Fernandez, and extend the Quillen quasi-isomorphism $\mathcal{C} \to BU$ to $L_\infty$-algebras. We also apply our main theorem to show associative universal envelopes satisfy the derived PBW property as soon as they are PBW. The appendix, written by V. Dotsenko, contains a general result on models of operads obtained by homological perturbation which we use in the particular case of the homotopy associative and the homotopy Poisson operad.
Notation and conventions. We work over a field of characteristic zero, which we write \( k \). We assume the reader is familiar with the theory of algebraic operads, as presented, for example, in [23], with the elements of model theory, as presented, for example, in [19], and with the basic tools of homological algebra. Whenever a new definition is provided, it will appear in **boldfaced italics**.

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1 The derived PBW property

For convenience, we remind the reader of the language of [11]. Let us fix a symmetric monoidal category $C$ that can be either of $\text{Vect}_k$, the category of graded vector spaces, $\text{Ch}_k$, the category of complexes over $k$, $\Sigma\text{Mod}$, the category of $\Sigma$-modules under the Cauchy tensor product or $\text{Ch}_{\Sigma}$, the category of dg $\Sigma$-modules under the same product. We say a map of operads $f : P \to Q$ over $C$ satisfies the \textbf{Poincaré-Birkhoff-Witt property} if there is an endofunctor $T : C \to C$ so that universal enveloping algebra functor $f_!$ is naturally isomorphic to $T$ with respect to $P$-algebra maps. In other words, we demand that $f_!$ depend only on the underlying object of that algebra in $C$, but not on the operations of that particular $P$-algebra. With this at hand, the main result of [11] is the following (we refer the reader to [17] for details on operads and their modules):

\textbf{Theorem.} The map $f : P \to Q$ satisfies the Poincaré-Birkhoff-Witt property if and only if $Q$ is a free right $P$-module. Moreover, in this case, if $T$ is a basis for $Q$ as a right $P$-module, then $f_!$ is isomorphic to $T$, naturally with respect to $P$-algebra maps.  

The corresponding formalism for dg operads is as follows. Fix a dg operad $P$. We say a right dg $P$-module $X$ is \textit{almost-free} if it admits a bounded below exhaustive filtration so that its associated graded module is chain equivalent to a free right $P$-module on a basis of cycles. A morphism of dg operads $f : P \to Q$ is \textit{almost-free} if $Q$ is an almost-free right $P$-module. Note that for every dg module $X$ we have a corresponding right $H\!P$-module $HX$, where $H$ is the homology functor. It follows that we have both a left adjoint $f_!$ to the restriction functor $f^* : Q\text{Alg} \to P\text{Alg}$ and a left adjoint $(Hf)_!$ to the restriction functor $(Hf)^* : HQ\text{Alg} \to H\!P\text{Alg}$. In this way, we obtain two functors

$$H \circ f_! : P\text{Alg} \to HQ\text{Alg}, \quad (Hf)_! \circ H : P\text{Alg} \to HQ\text{Alg}$$

and a natural transformation $F : (Hf)_! \circ H \to H \circ f_!$. The map $f$ is \textit{derived Poincaré-Birkhoff-Witt} if $F$ is a natural isomorphism. Our main result is the following:

\textbf{Theorem 1.1.} Every morphism that is almost-free is derived PBW. Moreover, if $T$ is a basis of cycles for such a morphism, the homology $Hf_!$ of its universal envelope is naturally isomorphic to $TH$ as a functor of algebras on its domain to algebras over the homology of its codomain.
Thus, in the same way that a classical PBW theorem gives an amenable description of universal enveloping algebra dependent only on the underlying object of the input, a derived PBW theorem gives us an amenable description of the homology of the universal enveloping algebra dependent only on the homology of the input.

**Proof.** We split our proof into three steps. Our main tool is a classical spectral sequence argument.

**Step 1.** Suppose that $Q$ is in fact $P$-free on a basis of cycles, so that $Q = T \circ P$ with $dT = 0$. Since $T$ has trivial differential, the Künneth theorem for the circle product gives a natural isomorphism $HQ \rightarrow T \circ HP$, and shows that $HQ$ is a free right $HP$-module. Moreover, for every left $P$-module $X$, we have natural isomorphisms $H(f_i(X)) = H(Q \circ P X) \rightarrow H(T \circ X) \rightarrow T \circ HX$. We also have natural isomorphisms $T \circ HX \rightarrow T \circ HP \circ H P X \rightarrow HQ \circ H P HX = (H f)_!(HX)$, which gives what we wanted: the natural map $F_X : (H f)_!(HX) \rightarrow H(f_i(X))$ is a natural isomorphism, so that $f$ is derived PBW in this case.

**Step 2.** Let us consider now the situation where we have an almost-free filtration $F$ on $Q$, and consider the induced filtration on $f_i(X)$. Linearity on the left of the composite product gives us that $\text{gr}_F(f_i(X))$ is of the form $T \circ X$ where $T$ has trivial differential, so that the domain of the $E^1$-page of the maps of spectral sequences converging to $F_X : (H f)_!(HX) \rightarrow H(f_i(X))$ looks like $T \circ HP$. The arguments above now show that the induced morphism at the $E^1$-page is an isomorphism.

**Step 3.** To conclude, let us suppose that we have a filtration on $Q$ such that $\text{gr}(Q)$ is chain equivalent to a free right $P$-module $Q'$. Arguing as before, we have a map of spectral sequences converging to $F_X : (H f)_!(HX) \rightarrow H(f_i(X))$ with $E^0$ equal to $(\text{gr} Q) \circ P X$, and a map $(\text{gr} Q) \circ P T \rightarrow Q' \circ P X = T \circ X$. Since the circle product is left linear, this map is still a chain equivalence, and thus induces an isomorphism on homology.

The last claim is already part of the content of the main result in [11]. This concludes the proof of the theorem.

We point out that the work of V. Hinich [12, Section 4.6.3] proves that universal envelopes preserve acyclic cofibrations between algebras: a derived PBW theorem extends this to arbitrary weak equivalences.

**Corollary 1.1.** If $f : P \rightarrow Q$ is derived PBW, then $f_i$ preserves weak equivalences.
It is worth pointing out that the PBW theorem of V. Dotsenko and the second author in [11] already implies the following classical result, since there we show that the map of operads $Hf : \text{Lie} \longrightarrow \text{Ass}$ is free and thus PBW in the classical sense.

**Corollary.** Let $\mathfrak{g}$ be a dg Lie algebra. Then the natural map $U\mathcal{H}(\mathfrak{g}) \longrightarrow HU(\mathfrak{g})$ is an isomorphism of algebras, so that a map of dg Lie algebras is a quasi-isomorphism if and only if the map on universal envelopes is one.

**Relation to a theorem of Adams.** It is interesting to point out that in [1] (see also [4, Chapter 9]), the author shows that if $f : \Lambda \longrightarrow \Gamma$ is an inclusion of Hopf algebras and $\Lambda$ is central in $\Gamma$, then $B\Gamma$ admits a filtration whose graded coalgebra is chain equivalent to $(B\Lambda \otimes B\Omega, d \otimes 1)$, so that $Bf$ is almost-free according to our definition. In fact, this result was the main inspiration for our definition of almost-free morphisms. We remark that in the context of differential graded homological algebra, the objects which possess useful homological properties and replace, in a way, the free objects of the classical theory, are sometimes called “semi-free modules”. We chose not to use this terminology to avoid any kind of confusion. It is useful to note that one can just assume that $\Gamma$ is $\Lambda$-free along with the fact that $\Lambda$ is central to deduce this. Hence, Adams’ result states precisely that, under this last extra hypothesis, $B$ preserves almost-free maps. Having appropriate hypotheses and a similar result for $\Omega$, one would obtain results relating classical PBW maps of operads, as defined in [11], to derived PBW maps between cofibrant replacements. We intend to pursue these ideas in the future.

## 2 Applications

### 2.1 The derived PBW property of the morphism $\text{Lie}_\infty \longrightarrow \text{Ass}_\infty$

In this section we prove the derived version of the classical PBW theorem, which then use to obtain results of V. Baranosvky and J. M. Moreno-Fernández and answer in the positive a conjecture of Moreno-Fernández. Observe that the motto of [11] that “the universal envelope of $\mathfrak{g}$ is independent of the Lie algebra structure of $\mathfrak{g}$” is now replaced by “the homology of the universal envelope of $\mathfrak{g}$ is independent of the homotopy Lie algebra structure of $\mathfrak{g}$”.

**Theorem 2.1.** The morphism $\text{Lie}_\infty \longrightarrow \text{Ass}_\infty$ is almost-free, so it is derived PBW.
Proof. We begin with recalling from [23, Prop. 9.1.5] that the Loday–Livernet presentation of the associative operad is given by a commutative non-associative product $x_1 x_2$ and an anti-symmetric Lie bracket $\{x_1, x_2\}$ that is a derivation of the product and satisfies the identity

$$(x_1 x_2) x_3 - x_1 (x_2 x_3) = \{x_2, \{x_1, x_3\}\}.$$ 

Let us consider the weight grading on the space of generators which assigns weight zero to the Lie bracket and weight one to the product. The associated graded relations with respect to this filtration are the relations of the Poisson operad, and the underlying $\Sigma$-modules of Pois and Ass are isomorphic. We are therefore in the situation where result of Appendix applies: there exists a quasi-free resolution of Ass whose differential is obtained from $d_{\text{Pois}_\infty}$ by a perturbation that lowers the weight grading. Moreover, the space of generators of this resolution can be identified with the Koszul dual cooperad of Pois whose underlying $\Sigma$-module is isomorphic to that of the Koszul dual cooperad of Ass; therefore, this resolution has to be minimal and isomorphic to $\text{Ass}_\infty$. The dg suboperad $\text{Lie}_\infty$ is in weight filtration zero, so the weight filtration is a filtration of right $\text{Lie}_\infty$-modules whose associated graded operad is $\text{Pois}_\infty$.

To complete the proof, we will show that there is a chain homotopy equivalence of right $\text{Lie}_\infty$-modules $\pi : \text{Pois}_\infty \rightarrow \text{Com} \circ \text{Lie}_\infty$. For this, we use the language of distributive laws between operads [23, Sec. 8.6.3]. The distributive law $\lambda$ that gives rise to the isomorphism $\text{Pois} = \text{Com} \vee_{\lambda} \text{Lie}$ can be enhanced to a distributive law $\lambda'$ between the operads $\text{Com}$ and $\text{Lie}_\infty$, for which all higher brackets are derivations with respect to the commutative product; there is a surjective quasi-isomorphism $\text{Com} \vee_{\lambda'} \text{Lie}_\infty \rightarrow \text{Com} \vee_{\lambda} \text{Lie} = \text{Pois}$. Hence, we get a surjective quasi-isomorphism

$$\text{Pois}_\infty \rightarrow \text{Com} \circ \text{Lie}_\infty = \text{Com} \vee_{\lambda'} \text{Lie}_\infty.$$ 

Since we are working over a field of characteristic zero, we can produce a map $i : \text{Com} \rightarrow \text{Com}_\infty$ such that $p i = 1$, where $p$ is the projection onto homology. This then gives us a map $j : \text{Com} \circ \text{Lie}_\infty \rightarrow \text{Pois}_\infty$ by composing with the composition of $\text{Pois}_\infty$, and this map is a section of $\pi$. Since $\text{Pois}_\infty$ is free as an operad, and since we’re working over a field of characteristic zero, we can produce an equivariant contracting homotopy $h$ for $j \pi$. This completes the proof that $\text{Lie}_\infty \rightarrow \text{Ass}_\infty$ is almost-free and, by Theorem 1.1, it is derived PBW. □
The algebras over the operad $\text{Com} \lor \Lambda'$ Lie$\infty$ used in the proof are sometimes called *homotopy Poisson algebras* or $P_\infty$-*algebras* in the literature, even though this operad is not cofibrant. These have been considered by A. S. Cattaneo and G. Felder, and independently by T. Voronov, and are related to the theory of Lie and Courant algebroids, and Poisson manifolds, see [7, 13, 15] for example.

### 2.2 Quillen theorem for $L_\infty$-algebras

We recall that if $\mathfrak{g}$ is an $L_\infty$-algebra, the **bar construction on $\mathfrak{g}$** is the commutative dg coalgebra $\mathcal{C}(\mathfrak{g})$ with underlying coalgebra $S^c(s\mathfrak{g})$, the free commutative coalgebra on the suspension of $\mathfrak{g}$, and with differential $d : \mathcal{C}(\mathfrak{g}) \longrightarrow \mathcal{C}(\mathfrak{g})$ induced from the higher brackets of $\mathfrak{g}$; the higher Jacobi identities for these higher brackets are equivalent to the single equation $d^2 = 0$. We write $x_1 \wedge \cdots \wedge x_t$ a generic element from $\mathcal{C}(\mathfrak{g})$, omitting the suspensions signs for ease of notation.

Similarly, if $A$ is an $A_\infty$-algebra, the **bar construction on $A$** is the dg coalgebra $B\mathcal{A}$ with underlying coalgebra $T^c(sA)$, the free coalgebra on the suspension of $A$, and with differential $d : B\mathcal{A} \longrightarrow B\mathcal{A}$ induced from the higher products of $A$; the Stasheff identities for these higher products are equivalent to the single equation $d^2 = 0$. We write $[x_1] \cdot \cdots \cdot [x_t]$ a generic element of $B\mathcal{A}$. Observe that we can apply both constructions, in particular, to dg Lie algebras and dga algebras, of course. Finally, if $C$ is a dga coalgebra, the **cobar construction on $C$** is the dga algebra $\Omega C$ with underlying algebra $T(s^{-1}C)$, the free algebra on the desuspension of $C$, with differential induced from the comultiplication and differential of $C$. Concretely, it is the unique derivation of $\Omega C$ that extends the map $s^{-1}C \longrightarrow \Omega C$ such that $d(s^{-1}c) = s^{-1}dc - s^{-1} \otimes s^{-1}\Delta c$.

Let us recall from [14] that for a dg-Lie algebra $(\mathfrak{g}, d)$ there is a quasi-isomorphism $q : \mathcal{C}(\mathfrak{g}) \longrightarrow B\mathcal{U}(\mathfrak{g})$ where the left hand side is the Quillen construction on $\mathfrak{g}$ (that coincides with the Chevalley-Eilenberg complex of $\mathfrak{g}$) and the right hand side is the associative bar construction on the universal envelope of $\mathfrak{g}$. We remind the reader that $q$ is determined uniquely by a map $\tau : \mathcal{C}(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$, which we call the **twisting cochain associated to $q$**. That this be a twisting cochain is equivalent to the Maurer–Cartan equation $d\tau + \tau \star \tau = 0$. Here, the star product of the convolution dga algebra $A = \text{hom}(\mathcal{C}(\mathfrak{g}), U(\mathfrak{g}))$ is defined by $\star = \mu(- \otimes -)\Delta$.

We also remind the reader that $\tau$ simply sends a generator $s\mathfrak{g} \in s\mathfrak{g}$ to the class of its desuspension in $U(\mathfrak{g})$. The following theorem extends this picture to the case of $L_\infty$-algebras—we will observe below $\tau$ also defines a twisting morphism in this more general setting. In this case, the Maurer–Cartan equation is replaced by a higher ana-
log: since $U(g)$ is now an $A\infty$-algebra, the convolution algebra above is in fact again an $A\infty$-algebra, so that for each $t \in \mathbb{N}$ and each $f_1, \ldots, f_t \in A$, we have higher multiplications defined by $m_t(f_1, \ldots, f_t) = \mu_t(f_1 \otimes \cdots \otimes f_t)\Delta^{(t)}$. The Maurer–Cartan equation incorporates these higher products and now reads:

$$d\tau + \sum_{t \geq 2} \mu_t(\tau, \ldots, \tau) = 0.$$ 

It is useful to remember this conditions simply codifies, with the least amount of information possible, the fact that the corresponding map $\mathcal{C}(g) \to BU(g)$ is one of dg coalgebras, and that this assignment defines a bijection between dg coalgebra maps $\mathcal{C}(g) \to BU(g)$ and twisting cochains $\mathcal{C}(g) \to U(g)$.

Before stating the result, we recall that the category of dg-coalgebras is a model category where the cofibrations are the degree-wise monomorphisms, the weak equivalences are the quasi-isomorphisms and cofibrations satisfy the right lifting property with respect to acyclic fibrations; all objects are cofibrant, and the fibrant dg-coalgebras are the quasi-free ones, see [25]. We remark that the weak equivalences are created by the cobar construction: a map of coalgebras is a weak equivalence if and only if its image under the cobar construction is a quasi-isomorphism of dga algebras. This class is strictly contained in the class of quasi-isomorphisms.

**Theorem 2.2.** For any $L\infty$-algebra $g$ the map $q : \mathcal{C}(g) \to BU(g)$ corresponding to the twisting cochain $\mathcal{C}(g) \to U(g)$ that assigns a generator $sg \in \mathcal{C}(g)$ the class of its desuspension is an acyclic cofibration of dg-coalgebras. If $g$ is minimal, then there are acyclic cofibrations of the form $\mathcal{C}(g) \to BS(g)$.

**Proof.** Let us begin by observing that the Maurer–Cartan equation for $\tau$ is simply a restatement that the higher products in $U(g)$ antisymmetrize to the higher brackets of $g$. Indeed, for a generator $x = x_1 \wedge \cdots \wedge x_t$ in the domain, we have that $\tau d x$ is equal to the higher bracket $[x_1, \cdots, x_t]$, up to signs, while the only non-zero term involving higher products of $\tau$ involves $\Delta^{(t)}(x)$. This is just the signed sum over $\sigma \in S_t$ of $\sigma x$, and then $m_t(\tau, \ldots, \tau)(x)$ is precisely the antisymmetrized higher product of $x_1 \otimes \cdots \otimes x_t$.

To see that $q$ is a weak equivalence one can show that $\Omega \mathcal{C}(g) \to S(g)$ is one. To do this, one can argue as in [2, Theorem 3], or note that there is a morphism of $A\infty$-algebras $BU(g) \to BS(g)$, where the right hand side is Baranovsky’s universal envelope, that is a quasi-isomorphism. Indeed, the map $H(U(g)) \to S(g)$ is an automorphism of the enveloping associative algebra of $(g, \ell_2)$. This implies that $\Omega q$ is a
quasi-isomorphism, since $\varepsilon_{U(g)} : \Omega BU(g) \longrightarrow U(g)$ is one, and $\Omega C(g) \longrightarrow U(g)$ is a quasi-isomorphism that factors as $\varepsilon_{U(g)} \Omega q$.

We write $A^i$ for the homology of $BA$ and, for an $L_\infty$-algebra $g$, we write $g^i$ for the homology of $C(g)$. We also point out that $C$ is the bar construction on Lie algebras, $B$ is that on associative algebras, and $\Omega$ is the cobar construction on associative coalgebras. Note that both of these are, in particular, $A_\infty$-coalgebras. Thus, an immediate corollary of the previous theorem is the following; all claims follow from the general theory of twisting cochains, just like in [2, Theorem 2].

**Corollary 2.1.** Let $g$ be a minimal $L_\infty$-algebra. Then

1. the twisted complex $C(g) \otimes \tau S(g)$ is quasi-isomorphic to $\mathbb{k}[0]$,
2. we have an isomorphism $g^i \longrightarrow S(g)^i$ of minimal $A_\infty$-coalgebras,
3. the categories of $g$-modules and $U(g)$-modules are equivalent and,
4. the functors $\otimes \tau S(g) : D(C(g)) \rightleftarrows D(S(g)) : ? \otimes \tau C(g)$ are mutually inverse derived equivalences.

### 2.3 The universal envelope as a functor on the homotopy category

Suppose that $f : C(g) \longrightarrow C(g')$ is an $L_\infty$-morphism. The dg-coalgebra $BU(g)$ is fibrant and the Quillen map of $g'$ is an acyclic cofibration, so we can produce a lift $U(f) : BU(g) \longrightarrow BU(g')$. This assignment respects composition up to homotopy, meaning that $U(fg) \simeq U(f) \circ U(g)$, and we have the following result.

**Theorem 2.3.** The universal envelope preserves weak equivalences of $L_\infty$-algebras, so it descends to a functor on the homotopy category. Moreover, if for a minimal $L_\infty$-algebra $g$ we identify $S(g)$ with the universal envelope of the Lie algebra $(g, \ell_2)$, then

1. there exists a minimal $A_\infty$-algebra structure on $S(g)$ so that the embedding $g \longrightarrow S(g)$ is a strict map of $L_\infty$-algebras and, moreover,
2. any such $A_\infty$-structure is $A_\infty$-isomorphic to this one, so that the universal envelopes of Baranovsky and Moreno-Fernández are $A_\infty$-isomorphic and $A_\infty$-quasi-isomorphic to $U(g)$.

**Proof.** The claim about weak equivalences is immediate since the Quillen map is a weak equivalence. The first claim follows from the homotopy transfer theorem, and the second one follows from the universal property of the enveloping algebra and our main result. Indeed, suppose that $S(g)$ is endowed with an $A_\infty$-algebra structure as
in the statement of the theorem. The strict map of $L_\infty$-algebras $g \to S(g)$ gives us an $A_\infty$-quasi-isomorphism $BU(g) \to BS(g)$, for the map $U(g) \to S(g)$ induces an isomorphism of algebras: we have shown that $H(U(g))$ is naturally identified with the universal envelope of the Lie algebra $(g, b_2)$, and so does $S(g)$ in Baranosvky’s construction.

Let us recall that one can rectify every $L_\infty$-algebra $g$ to a bona-fide dg Lie algebra. In this way, one can show that the homotopy category of dg Lie algebras $\text{ho}(\text{dgLie})$ and the category of $L_\infty$-algebras up to quasi-isomorphism are equivalent. An analogous result holds for dga algebras. Hence, we can view the universal envelope functor $\text{Lie}_\infty \text{Alg} \to \text{Ass}_\infty \text{Alg}$ as a choice for a representative of the functor that assigns to a dg Lie algebra its universal envelope. A restatement of our results is the following.

**Corollary 2.2.** The constructions of Baranovsky, Lada–Markl and Moreno-Fernández give representatives for the universal envelope functor $\text{ho}(U) : \text{ho}(\text{dgLie}) \to \text{ho}(\text{dgAlg})$. Moreover, if $F : \text{Lie}_\infty \text{Alg} \to \text{Ass}_\infty \text{Alg}$ satisfies the conditions:

1. The $A_\infty$-algebra $F(g)$ is minimal whenever the $L_\infty$ algebra $g$ is minimal,
2. The underlying vector space to $F(g)$ is the symmetric algebra $S(g)$ and,
3. The higher products of $F(g)$ induce the higher brackets on $g \subseteq S(g),$

then $F$ descends to the homotopy category and $\text{ho}(F) = \text{ho}(U)$.

### 2.4 Cohomology groups

Recall that if $P$ is an operad and $A$ is a $P$-algebra, operadic cohomology of $A$ is, by definition, the cohomology of the complex of $P$-derivations $\text{Der}(B, A)$ where $B$ is a cofibrant resolution of $A$ in the model category of $P$-algebras. This complex is quasi-isomorphic to the dg Lie algebra $\text{Der}(B)$ of derivations of $B$ to itself. More generally, the cohomology of $A$ with values in an operadic $A$-module $M$ is, by definition, the cohomology of the complex of $P$-derivations $\text{Der}(B, M)$ where $M$ is given a $B$-module structure through the map $B\to A$. We refer the reader to [17, 23] for details.

Let us fix a minimal $L_\infty$-algebra $g$, a minimal $A_\infty$-algebra $S(g)$ that models $U(g)$, and an acyclic twisting cochain $\mathcal{C}(g) \to S(g)$. Observe that then $\Omega\mathcal{C}(g)$ is a quasi-free model and hence a cofibrant replacement for $S(g)$, so that the operadic cohomology $H^*_\text{Ass}(S(g))$ can be computed through the complex of derivations of the dg-algebra $\Omega\mathcal{C}(g)$ with values in $S(g)$. This receives a map from the complex of derivations...
Der($\Omega C(g), g)$ through the strict map of $L_{\infty}$-algebras $g \longrightarrow S(g)$. We then obtain the following result, which is expected.

**Theorem 2.4.** The maps above induce an isomorphism $H_{\text{Ass}}^*(S(g)) \longrightarrow H_{\text{Lie}}^*(g, S(g))$ and an injection $H_{\text{Lie}}^*(g) \longrightarrow H_{\text{Lie}}^*(g, S(g))$ in operadic cohomology groups.

### 2.5 Derived PBW theorems for associative envelopes

To each symmetric operad $P$ one can assign the universal enveloping functor $U_P$ from the category of $P$-algebras to the category of associative algebras, see [18, 20]. The associative algebra $U_P(V)$ associated with a $P$-algebra $V$ satisfies the universal property that the category of left $U_P(V)$-modules is equivalent to the category of left modules over a $P$-algebra $V$. In other words, the universal enveloping functor $U_P$ is equal to the left adjoint functor $f_!$ assigned to the restriction functor $f : (P, \partial_\kappa) \rightarrow (P, \partial P)$ from the colored operad $(P, \partial_\kappa)$ encoding $P$ and the trivial module $\kappa$ to the colored operad $(P, \partial P)$ encoding $P$ and the pointed right $P$-module $\partial P$. Here $\partial P$, the derivative of the symmetric sequence $P$, is obtained from $P$ by coloring the output and one of the inputs of each operation in $P$.

The first author has shown in [20] that the functor $U_P(?)$ satisfies PBW property whenever $P$ admits a Gröbner basis whose leading monomials are given by left combs. Recall that a shuffle monomial is called a left comb if and only if all its inner vertices belong to the leftmost branch of a shuffle tree, that is, the path connecting the first input and the output. Another criterion was proposed in [20] for a Koszul operad $P$:

- **The functor $U_P$ satisfies the PBW property if and only if the twisted associative algebra $\partial P^!$ is quadratic, Koszul and generated by $\partial X$ as a twisted associative algebra.**

Here $X$ is the generating symmetric sequence of the symmetric operad $P$. Recall that the structure of the twisted associative algebra on $\partial P$ (respectively $\partial P^!$) is given by composition through the new color. We claim that the same conditions are sufficient in the derived setting:

**Theorem 2.5.** If the associative universal envelope $U_P$ satisfies the PBW property then the corresponding derived associative universal envelope $U_{P_{\infty}}$ satisfies the derived PBW property. In particular, if an operad $P$ admits a quadratic Gröbner basis whose leading monomials are given by left combs then the universal enveloping functor $U_{P_{\infty}}$ satisfies derived PBW.
Proof. Let $P^i_\infty := \Omega P^i$ be the minimal cofibrant model of $P$ generated by the Koszul-dual cooperad $P^i$ and denote by $(\partial P^i)^i_{\text{Tw-As}}$ the twisted associative algebra that is Koszul dual to the twisted associative algebra $\partial P^i$. One of the main observations in [20] is the commutation of the coloring procedure $P \mapsto \partial P$ and cobar constructions. So that there is a natural isomorphism $\partial(\Omega P^i) \rightarrow \Omega(\partial P^i)$. Each element of a (colored) cobar construction of a (colored) (co)operad is represented by a (colored) operadic tree $T$. The colored cobar construction of the colored operad $\partial P$ admits the PBW-filtration given by the number of edges connecting the branch colored in a new color and the remaining part of an operadic tree $T$. The associated graded complex is quasi-isomorphic to the composition

$$\Omega_{\text{Tw-As}}(\partial P^i) \circ \Omega P^i \simeq \Omega_{\text{Tw-As}}(\partial P^i) \circ P^i_\infty.$$  

As shown in [20] the Koszulness of the twisted associative algebra $\partial P^i$ is a necessary condition for $U_P$ to satisfy the ordinary PBW criterion. Hence the PBW property for $U_P$ implies the existence of a chain equivalence $s : (\partial P^i)^i_{\text{Tw-As}} \rightarrow \Omega_{\text{Tw-As}}(\partial P^i)$, which shows the map $\partial P^i_\infty$ is almost-free, and finishes our proof of the theorem.

**Corollary 2.3.** Given a $P^i_\infty$-algebra $A$ and a dg $P$-model $B$ of $A$, the corresponding associative universal envelopes are quasi-isomorphic, namely, there is always a quasi-isomorphism $U_P(B) \rightarrow U_{P^i_\infty}(A)$ of dg algebras.

3 Further directions

3.1 Derived universal envelopes

Let us fix a map $f : P \rightarrow Q$ of dg operads over a field $\mathbb{k}$. Although we focused on $P$-algebras—left $P$-modules concentrated in arity 0—, the universal envelope defines a map $f_l : P \text{Mod} \rightarrow \text{Ch}_\mathbb{k}$, which is given explicitly by $f_l(X) = Q \circ_P X$. According to [17], the category $\text{Mod}_P$ of right $P$-modules admits a cofibrantly generated model structure where fibrations and weak-equivalences are defined point-wise. In particular, we can consider a cofibrant replacement $Q^*$ of $Q$, for example, the two sided bar construction $B(Q, P, P)$, and define $\mathbb{L}f_l(X) = Q^* \circ_P X$, which gives us the object $\text{Tor}^P(Q, X)$. Following the procedure of [4, Chapter 7], we can produce a Eilenberg–Moore type spectral sequence which gives a fine tool to study derived PBW phenomena; our arguments essentially consider the situation when there is an immediate collapse of
this sequence due to $Q$ begin almost-free. It would be interesting to consider situations where certain restrictions on $Q$, other than almost-freeness, allow us to obtain derived PBW theorems.

### 3.2 Duflo-type results for higher centres

Although the canonical map $\alpha : S(g) \rightarrow U(g)$ is not a map of algebras (since the source is commutative, but the target is not), we can consider the adjoint action of $g$ on both spaces. It is well known that the map above is then one of $g$-modules and thus induces a map $\alpha^g : S(g)^g \rightarrow U(g)^g$, where $U(g)^g$ is just the center of $U(g)$, a commutative algebra. This map is, however, not an isomorphism of algebras either.

A remarkable result of M. Duflo shows that one can construct from this an isomorphism of algebras $\alpha^g \circ J^{1/2} : S(g)^g \rightarrow U(g)^g$, known as the Duflo isomorphism, through a suitable (and quite involved) modification of this map through an automorphism $J$ of $S(g)$. In fact, M. Pevzner and C. Torossian proved that the Duflo isomorphism is part of an isomorphism of Lie cohomology groups $H^*(g, S(g)) \rightarrow H^*(g, U(g))$ induced from a quasi-isomorphism $C^*(g, S(g)) \rightarrow C^*(g, U(g))$ between the corresponding Chevalley–Eilenberg complexes, following insight of M. Kontsevich. We refer the reader to [6] for details and useful references. In [5] the authors define the $\infty$-centre of a minimal $A_\infty$-algebra, which can be used, for example, to describe the image of the “wrong way” map $H_{*+d}(LX) \rightarrow H_*(\Omega X)$ onto the Pontryagin algebra of a simply connected smooth oriented $d$-manifold $X$: the image of this map is precisely the $\infty$-centre of $H_*(\Omega X)$. In particular we can consider, for any minimal $L_\infty$-algebra, the $\infty$-centre $Z_\infty S(g)$, which is a commutative algebra. It would be interesting to understand this higher centre and explore the possibility of extending the results of Duflo and Pevzner–Torossian to this setting.

### A Models of operads via homological perturbation

This section records an instance general homological perturbation argument which allows one to obtain, in a range of cases, a resolution of a filtered object from the one of its associated graded object. A similar argument for a perturbative construction of a resolution of a shuffle operad with a Gröbner basis is featured in [9, Th. 4.1]. We keep the assumption on the characteristic of the ground field.

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1 by Vladimir Dotsenko
Let us consider a (non-dg) symmetric operad $Q = \mathcal{T}(X)/(R)$ generated by a finite dimensional $\Sigma$-module $X$ concentrated in arities greater than one, subject to a finite dimensional space of relations $R$. Suppose that the $\Sigma$-module $X$ is equipped with a non-negative weight grading, $X = \bigoplus_{n \geq 0} X(n)$. This weight grading gives rise to a weight grading of the free operad $T(X)$, and hence an increasing filtration $F^k \mathcal{T}(X)$ such that $F^k \mathcal{T}(X)$ is spanned by all elements of weight at most $k$. This filtration gives rise to a filtration on each $\Sigma$-submodule of $\mathcal{T}(X)$. In particular, we may consider the operad $P = T(X)/(\text{gr}_F R)$. In general, there is an isomorphism of $\Sigma$-modules between $Q$ and $\text{gr}_F Q = \text{gr}_F T(X)/\text{gr}_F(R)$, and a surjection of $\Sigma$-modules from $P$ onto $Q$.

**Theorem.** Suppose that the underlying $\Sigma$-modules of the operads $P$ and $Q$ are isomorphic. Consider the minimal quasi-free resolution $P_\infty \rightarrow P$ in the category of weight graded operads. There exists a quasi-free resolution $Q_\infty \rightarrow Q$ whose underlying free operad is the same and the differential is obtained from the differential of $P_\infty$ by a perturbation that lowers the weight grading.

**Proof.** Assume that $P_\infty$ is of the form $(\mathcal{T}(W), d)$, where $W$ is an $\Sigma$-module that is bi-graded, by weight and by homological degree. We shall prove that there exists a quasi-free resolution $Q_\infty = (\mathcal{T}(W), d + d')$ of $Q$ so that $d'$ is strictly weight decreasing.

Since the resolution $P_\infty = (\mathcal{T}(W), d)$ is minimal, it follows in particular that $W_0 = X$, $W_1 = k s \otimes \text{gr}_F R$. The operad $\mathcal{T}(W)$ can be mapped to both the operad $P$ and the operad $Q$: one may project it onto its part of homological degree 0, the latter is isomorphic to the free operad $\mathcal{T}(X)$ which admits obvious projection maps to $P$ and to $Q$. Let us choose splittings for those projections; this amounts to exhibiting two idempotent endomorphisms $\bar{\pi}$ and $\pi$ of $\mathcal{T}(W)$ such that both of which annihilate all elements of positive homological degree, and such that the former annihilates the ideal $(\text{gr}_F R) \subset \mathcal{T}(X) = \mathcal{T}(W)_0$ and the latter annihilates the $(R) \subset \mathcal{T}(X) = \mathcal{T}(W)_0$.

Since $P$ is finitely generated and has no generators of arity 1, components of the free operad $\mathcal{T}(W)$ are finite-dimensional, and there exists a weight graded homotopy $h: \mathcal{T}(W) \rightarrow \mathcal{T}(W)$ such that $h^2 = 0$ and $[d, h] = 1 - \bar{\pi}$.

We are going to define a derivation $D: \mathcal{T}(W) \rightarrow \mathcal{T}(W)$ of degree $-1$ and a contracting homotopy $H: \ker(D) \rightarrow \mathcal{T}(W)$ of degree $+1$. Note that a derivation is fully determined by the images of generators, and that $D|_{W_0} = 0$ since $\mathcal{T}(W)$ has no elements of negative homological degree. For each element $x$ of $\mathcal{T}(W)$ of a certain homological degree, we call the “leading term” of $x$ the homogeneous part of $x$ of maximal possible weight grading; we denote it by $\hat{x}$. 


We shall prove by induction on $k$ that one can define the values of $D$ on generators of homological degree $k + 1$ and the values of $H$ on elements of $\ker(D)$ of homological degree $k$ so that the following five conditions hold:

1. for all elements $x \in \mathcal{F}(W)$, the leading term of the difference $D(x) - d(x)$ is of weight lower than that of $x$,
2. we have $D^2 = 0$ on generators of homological degree $k + 1$,
3. the leading term of the difference $H(x) - h(\hat{x})$ is of weight lower than that of $x$,
4. we have $DH = 1 - \pi$ on elements of $\ker(D)$ of homological degree $k$.

As a basis of induction, we shall choose a basis of $W_1 = \mathbb{k}s \otimes \text{gr}_F R$, and set $D(s \otimes r') = r$ where $r$ is some element of $R$ for which $\bar{r} = r'$. We note that $D(sr') - d(sr') = r - r'$ has smaller weight than $r'$, so Condition (1) is satisfied. Condition (2) is satisfied for degree reasons, as there are no elements of negative homological degree. Condition (1) together with the fact that $\bar{\pi} = \pi$ on elements of weight zero implies that the leading term of $D h(\hat{x})$ is

$$dh(\hat{x}) = (1 - \bar{\pi})(\hat{x}) = (1 - \pi)(\hat{x}) = \hat{x} - \pi(\hat{x}),$$

and so we may define $H$ on elements of homological degree zero by induction on weight as follows. On elements $x$ of weight zero, we put $H(x) = h(x)$, and on elements $x$ of positive weight, we put

$$H(x) = h(\hat{x}) + H(x - \pi(x) - Dh(\hat{x})).$$

Both Condition (3) and Condition (4) are proved by induction on weight. For former one, the inductive argument is almost trivial; we shall show how to prove the latter. On elements of weight zero, we have $H(x) = h(x)$ and $\bar{\pi} = \pi$, so Condition (4) is true:

$$DH(x) = dh(x) = [d, h](x) = (1 - \bar{\pi})(x) = (1 - \pi)(x).$$

For elements of positive weight, we have, by induction,

$$DH(x) = Dh(\hat{x}) + DH(x - \pi(x) - Dh(\hat{x})) = Dh(\hat{x}) + (1 - \pi)((1 - \pi)x - Dh(\hat{x})) = (1 - \pi)^2(x) + \pi(Dh(\hat{x})) = (1 - \pi)(x),$$
since $\pi$ vanishes on the image of $D = (R)$ and $1 - \pi$ is a projector. To carry the inductive step, we proceed in a similar way. To define the image under $D$ of a generator of homological degree $k + 1 > 1$, we put
\[ D(x) = d(x) - HDd(x). \]

Condition (1) now easily follows by induction. For Condition (2), we note that
\[
D^2(x) = D(d(x) - HDd(x)) \\
= DDd(x) - DH(Dd(x)) \\
= Dd(x) - (1 - \pi)Dd(x) = \pi(Dd(x)) = 0,
\]
since $Dd(x) \in \ker(D)$, and $\pi$ vanishes on the image of $D$. From that, we see that whenever $x \in \ker(D)$, we have $x - Dh(\tilde{x}) \in \ker(D)$. Using Condition (1) and the fact that $\tilde{\pi}$ vanishes on elements of positive homological degree, we see that the leading term of $Dh(\tilde{x})$ is $d h(\tilde{x}) = (1 - \tilde{\pi})(\tilde{x}) = \tilde{x}$, so the leading term of $x - Dh(\tilde{x})$ is of weight lower than that of $x$. Consequently, we may define $H$ on elements of $\ker(D)$ of homological degree $k > 0$ by the same inductive argument: on elements $x$ of weight zero, we put $H(x) = h(x)$, and on elements $x$ of positive weight, we put
\[ H(x) = h(\tilde{x}) + H(x - Dh(\tilde{x})). \]

Once again, a simple inductive argument shows that Conditions (3) and (4) are satisfied, which completes the construction of $D$ and $H$.

We conclude that $D$ makes $\mathcal{T}(W)$ a dg operad, that the homology of that operad is isomorphic to $Q$, and that differential $D$ is obtained from $d$ by a perturbation $d'$ that lowers the weight grading, as required.

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SCHOOL OF MATHEMATICS, TRINITY COLLEGE DUBLIN, College Green, Dublin 2, Ireland, D02 PN40

E-mail address: pedro@math.tcd.ie

FACULTY OF MATHEMATICS, NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, 20 Myasnitskaya street, Moscow, Russia, 101000 and INSTITUTE FOR THEORETICAL AND EXPERIMENTAL PHYSICS, Bolshaya Cheremushkinskaya 25, Moscow, Russia, 117259

E-mail address: akhoroshkin@hse.ru