Shape Optimization of Compliant Pressure Actuated Cellular Structures

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Abstract

Biologically inspired pressure actuated cellular structures can alter their shape through pressure variations. Previous work introduced a computational framework for pressure actuated cellular structures which was limited to two cell rows and central cell corner hinges. This article rigorously extends these results by taking into account an arbitrary number of cell rows, a more complicated cell kinematics that includes hinge eccentricities and varying side lengths as well as rotational and axial cell side springs. The nonlinear effects of arbitrary cell deformations are fully considered. Furthermore, the optimization is considerably improved by using a second-order approach. The presented framework enables the design of compliant pressure actuated cellular structures that can change their form from one shape to another within a set of one-dimensional $C^1$ continuous functions.

Keywords adaptive - biomimetic - cellular - compliant - morphing - structure

1 Introduction

There exists a wide range of technologies that would immensely benefit from robust, strong, lightweight and energy efficient compliant structures that can change their form from one shape to another. For example, currently used aircraft slats are relatively heavy and the gaps between wings and slats increase noise levels, particularly during take off and landing. A comparison of existing actuation principles [4] shows that pressure based actuators have the greatest potential to create such structures (Figure 1). Hence it is not surprising that the pressure driven nastic movement of plants attracted a lot of attention from various scientific communities during the last decade. A considerable research effort in this field was backed up by the Defense Advanced Research Agency, the National Science Foundation and the United States Army [18]. A comprehensive understanding of the nastic movement of plants requires various disciplines that range from biology and chemistry to material science and structural engineering. The focus of this article is on the structural engineering side. Therefore, no attention is given to the functionality of sub-cellular hydration motors [15] or plant cell materials [7]. Instead, it is assumed that cellular structures are made from common engineering materials and that cell pressures are provided by an external source such as a compressor. Furthermore, only prismatic cells are considered. Hence, the problem reduces to the understanding of two-dimensional cell geometries and their interactions.

A brief overview of publications that investigate prismatic pressure actuated cellular structures is subsequently given. A concept based on plane symmetry groups was patented by Dittrich [1]. He combines convex and concave cells to form actuators that can replace double acting cylinders. A similar approach that uses pressurized and void cells was investigated by Luo and Tong [16]. Although not directly related to adaptive structures, Khire et al [8] studied inflatable structures that are made from a large number of uniformly pressurized hexagonal cells. Vos and Barret [17] subsequently patented a similar approach. Further work which investigates pressurized honeycombs can be found in [2, 3, 5]. Numerical tools for the simulation and optimization of two-dimensional cellular structures were, among others, developed by Lv et al [9, 10]. A concept for pressure actuated cellular structures that are made from separately pressurized rows of individually tailored prismatic cells (Figure 2) was patented by Pagitz et al [11, 12]. It was shown in [14] that these structures can be made from arbitrary engineering materials that range from elastomers to steel. Furthermore, it was shown in [13] that cytoskeletons within each cell can reduce the structural weight, while at the same time increasing the overall stiffness. However, the underlying numerical framework of [11] is limited to cellular structures with two cell rows and central cell corner hinges. The aim of this article is to extend the previous work by considering an arbitrary number of cell rows, the presence of hinge eccentricities and rotational as well as axial springs. Furthermore, the optimization is drastically improved by using a second order approach. This allows the design of compliant pressure actuated cellular structures (CPACS) that can change their shape between any given set of one-dimensional $C^1$ continuous functions.

The outline of this article is as follows. Section 2 shows how CPACS can be advantageously abstracted by triangles, pentagons and cell sides. Furthermore, energy terms for these geometric primitives are given. The assembly of energy
Figure 1: (a) Osmotic hydration motors are used to vary cell pressures and thus to alter cell geometries. (b) Power output per unit volume versus strain of various actuation principles (data from Huber et al [4]).

Figure 2: (a) A cantilever that is assembled from a number of identical prismatic cells with pentagonal or hexagonal cross sections deforms after pressurization into a circular arc. The corresponding radius is independent of the cell pressures and solely a function of cell side lengths.

(b) Two cantilevers that are made from either pentagonal or hexagonal cells can be connected if opposite cell sides are of equal length.

(c) Equilibrium shape of a structure that is made from two connected cell rows can be altered by changing the pressure ratio between cell rows.

(d) Individually tailoring the cell side lengths allows the design of cellular structures that can change their shape between any two given one-dimensional $C^1$ continuous functions.
terms for structures with an arbitrary number of cell rows is discussed in Section 3. Section 4 demonstrates the performance of the proposed framework with the help of an example. Conclusions are given in Section 5.

2 Geometric Primitives

Equilibrium shapes of CPACS can be accurately computed by discretizing their cross sectional geometry (Figure 3a) with two-dimensional continuum finite elements. However, optimizing the cross sectional geometry for given target shapes and cell pressures is computationally expensive and not trivial. Hence, a simplified numerical model (Figure 3b) that is based on stiff bars, hinges and rotational/axial springs is required. This simplification is possible due to a concentration of bending strains in regions around cell corners. These local deformations are mainly driven by the large axial cell side forces and the increased central stiffness of sides that are exposed to differential pressures. The numerical model can be broken down into pentagonal and hexagonal cells (Figure 3c) which are bounded by cell sides (Figure 3d). For computational reasons, each hexagonal cell is divided into an independent pentagonal and a triangular subcell. The latter is fully defined by its two neighboring pentagonal cells of the lower row. Cell side geometries are defined by hinge eccentricities, cell corner rotations and the distance between cell corners. Note that the area between a deformed side and the straight line between cell corners is non-zero (Figure 3d).

Figure 3: (a) Compliant pressure actuated cellular structure (CPACS). (b) Numerical model of CPACS with stiff beams, hinges and rotational/axial springs. CPACS can be split into (c) pentagonal, hexagonal cells with central cell corner hinges and (d) cell sides with eccentric hinges and springs. Hexagonal cells can be divided into pentagonal and triangular subcells that are defined by neighboring pentagonal cells of the lower row.
2.1 Pentagonal Cells

The bottom row of CPACS is made from pentagonal cells. Furthermore, pentagonal subcells form the upper part of hexagonal cells. A single pentagonal cell or subcell, shown in Figure 4, has five effective cell side lengths \(a, b_1, b_2, c_1\) and \(c_2\), which are defined as the distance between neighboring cell corners. The dependency between effective cell side lengths and hinge eccentricities, corner rotations is elaborated in Section 2.4. The geometry of the pentagonal cell is further described by the external angles \(\alpha_1\) and \(\alpha_2\) and the internal angles \(\theta_1\) and \(\theta_2\). All effective cell side lengths and angles change with the pressurization of the system and therefore constitute the state of the pentagonal cell. It will prove to be advantageous to split the state variables into the groups

\[
\begin{align*}
\mathbf{u}_0^p &= \begin{bmatrix} a_1 & a_2 & a \end{bmatrix}^T \quad \text{and} \quad \mathbf{v}^p &= \begin{bmatrix} b_1 & b_2 & c_1 & c_2 \end{bmatrix}^T.
\end{align*}
\]

In the unpressurized and undeformed configuration, the vectors have the values \(\mathbf{u}_0^p\) and \(\mathbf{v}^p\). The vector \(\mathbf{u}_0^p\) contains all design variables that are a priori chosen and usually not altered during the optimization process. The vector \(\mathbf{v}^p\) contains all design variables that may be modified during the optimization process. The base side \(a\) is a part of \(\mathbf{u}_0^p\) since it is an abstract term for pentagonal subcells that are a part of hexagonal cells. A superscript “P” is used for pentagonal state variables. The length \(y\) that divides the pentagon into a triangular and quadrilateral part is given by

\[
y = \sqrt{\frac{a^2 + b^1 + b^2}{2}} = \sqrt{\left(a + \sin(\alpha_2) b_2 - \sin(\alpha_1) b_1 \right)^2 + \left(\cos(\alpha_2) b_2 - \cos(\alpha_1) b_1 \right)^2}
\]

and the altitude \(z\) can be expressed as

\[
z = \sqrt{c_1^2 - \frac{1}{4}\left(y^2 + c_2^2 - c_2^2\right)^2}.
\]

The internal angle \(\theta_1\) of a pentagonal cell is completely determined by the state variables through

\[
\theta_1 = \begin{cases}
\alpha_1 + \arcsin\left(\frac{y}{y} / \frac{z}{c_1}\right) + \arcsin\left(\frac{z}{c_1}\right) & c_2^2 < y^2 + c_1^2 \\
\alpha_1 + \arcsin\left(\frac{y}{y} / \frac{z}{c_1}\right) - \arcsin\left(\frac{z}{c_1}\right) + \pi & c_2^2 > y^2 + c_1^2.
\end{cases}
\]

The expressions for the internal angle \(\theta_2\) are derived in a similar manner. The previous equation could be written without a distinction of cases. However, this would result in lengthier expressions. The pressure potential of a pentagonal cell with central cell corner hinges and an internal pressure \(p\) is

\[
\Pi^p = -p A^p = -\frac{p}{2} \left(\cos(\alpha_1) b_1 + \cos(\alpha_2) b_2\right) a + \sin(\alpha_2 - \alpha_1) b_1 b_2 + y z,
\]

where \(A^p\) is the cross-sectional area. The gradients with respect to state variables \(\mathbf{u}_0^p, \mathbf{v}^p\) for the previous expressions can be found in Section 2.4.

2.2 Triangular Cells

A triangular cell is defined by two neighboring pentagonal cells of the lower row. It constitutes together with a pentagonal cell in the upper row a hexagonal cell. Cell side lengths, abstract base lengths as well as state angles and internal angles of a triangular cell are shown in Figure 5. The state variables \(\mathbf{u}_0^t\) and cell sides \(\mathbf{v}^t\) gather the kinematic quantities

\[
\begin{align*}
\mathbf{u}_0^t &= \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_1 & a_2 \end{bmatrix}^T \quad \text{and} \quad \mathbf{v}^t &= \begin{bmatrix} b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & c_4 \end{bmatrix}^T,
\end{align*}
\]
where the superscript “T” is used for state variables of a triangular cell. The abstract base side \( a \) of the adjacent pentagonal cell is given by

\[
a = \sqrt{c_2^2 + c_3^2 + 2c_2c_3 \cos (\theta_1 + \theta_2)} \tag{7}
\]
and the internal angle \( \psi \) of the triangular cell can be expressed as

\[
\psi = \theta_1 - \arccos \left( \frac{a^2 + c_2^2 - c_3^2}{2ac_2} \right). \tag{8}
\]

The pressure potential of the triangular cell without hinge eccentricities is determined by its area \( A^T \) and reads as

\[
\Pi^T = -pA^T = -\frac{p}{2}c_2c_3 \sin (\theta_1 + \theta_2). \tag{9}
\]

The gradients with respect to state variables \( u^T_\alpha, v^T \) for the previous expressions can be found in ??–??.

Figure 5: (a) State variables \( u^T_\alpha \), internal angles and (b) state variables \( v^T \), internal lengths of a triangular cell.

2.3 Cell Sides

The previously published framework \[11\] for pressure actuated cellular structures assumes rigid cell sides that are connected at cell corners via hinges. This assumption is valid as long as cell sides are relatively thin and stiff. In the following, it is outlined how rotational and axial springs as well as hinge eccentricities are taken into account. To simplify matters, it is assumed that undeformed cell sides are straight and that hinge eccentricities are invariant to cell side deformations. The latter assumption is valid since cell corners are usually compact and biaxially stressed. As before, the state variables are split into two parts. The first part

\[
u^S_\kappa = \begin{bmatrix} \kappa_- & \kappa_+ \end{bmatrix}^T \tag{10}
\]
describes the angles between hinge eccentricities and the line that connects both cell corners. The second part

\[
v^S = L \tag{11}
\]
is the distance between both cell corners, i.e. the effective cell side length. Further variables that are required to fully describe a cell side are hinge eccentricities \( d_+ \), rotational springs \( e_\pm \) and an axial spring \( h \) (Figure 6). The distance

\[
L_0 = \sqrt{L_{dx}^2 + L_{dy}^2} = \sqrt{(L - \cos (\kappa_-) d_- - \cos (\kappa_+) d_+)^2 + (\sin (\kappa_-) d_- + \sin (\kappa_+) d_+)^2}. \tag{12}
\]

between both cell side hinges is a function of state variables \( u^S_\kappa, v^S \) and hinge eccentricities \( d_\pm \). It is possible to write the state variables \( u^S_{k,bj} \) for cell sides \( b_1 \) and \( b_2 \) of a single pentagonal cell (Figure 7) as

\[
u^S_{k,bj} = \begin{bmatrix} \kappa_{b,j-} & \kappa_{b,j+} \end{bmatrix}^T = \begin{bmatrix} \kappa_{j-} & \kappa_{j+} \end{bmatrix}^T, \tag{13}
\]

where \( j = 1, 2 \). In contrast, state variables \( u^S_{k,a} \) and \( u^S_{k,c} \) of pentagonal cell sides \( a \) and \( c_1, c_2 \) are a function of cell side \( u^S_\kappa \) and pentagonal \( v^S \) state variables. Furthermore, they depend on an additional global state variable \( \beta \) (Figure 7):

\[
u^S_{k,a1} = \begin{bmatrix} \kappa_1 + \Delta \alpha_1 \\ \kappa_2 + \Delta \alpha_2 \end{bmatrix}, \quad u^S_{k,c1} = \begin{bmatrix} \kappa_1 + \Delta \theta_1 \\ \kappa_3 + \Delta \theta_1 - \Delta \alpha_1 + \Delta \beta \end{bmatrix} \quad \text{and} \quad u^S_{k,c2} = \begin{bmatrix} \kappa_2 + \Delta \theta_2 \\ \kappa_3 - \Delta \theta_2 - \Delta \alpha_2 + \Delta \beta \end{bmatrix}. \tag{14}
\]
Figure 6: (a) State variables $u^S$ and bending angles $\varphi_\pm$. (b) State variable $v^S$ and length $L_h$. (c) Hinge eccentricities $d_\pm$ and rotational $e_\pm$, axial $h$ cell side springs.

For example, $\Delta \alpha = \alpha - \alpha_0$ is the difference between the current (pressurized) and the reference (manufactured) configuration. It is assumed that $\Delta \beta = 0$ for pentagonal cells that are located in the top, boundary cell row. This is due to the fact that global state variables $\beta$ are not required in the top row since there are no further pentagonal cells. Instead, they serve as the basis for variables $\kappa$ and thus can have any value, including zero. Bending angles of a single cell side are

$$\varphi_- = \kappa_- + \arcsin \left( \frac{L_{hy}}{L_h} \right) \quad \text{and} \quad \varphi_+ = \kappa_+ + \arcsin \left( \frac{L_{hy}}{L_h} \right).$$

The pressure potential $\Pi^S_p$ of a cell side is the product of the differential pressure $\Delta p$ and the area between the deformed cell side and the line that connects both cell corners

$$\Pi^S_p = -\frac{\Delta p}{2} \left( \sin (\kappa_+) \cos (\kappa_+) d_+^2 - \sin (\kappa_-) \cos (\kappa_-) d_-^2 + (\sin (\kappa_+) d_+ - \sin (\kappa_-) d_-) L_{ha} \right).$$

The strain potential $\Pi^S_e$ of a cell side consists of the rotational and axial strain energy, i.e.

$$\Pi^S_e = \frac{1}{2} \left( e_- \varphi_-^2 + e_+ \varphi_+^2 + h \Delta L_h^2 \right),$$

where $\Delta L_h = L_h - L_{h0}$. Recall that $\Delta \varphi = \varphi$ since undeformed cell sides are straight. The total energy of a cell side is the sum of the pressure and strain energy

$$\Pi^S = \Pi^S_p + \Pi^S_e.$$  

The gradients of previous expressions with respect to cell side state variables $u^S$, $v^S$ and hinge eccentricities $w^S$ as well as pentagonal state variables $u^P$, $v^P$ can be found in ??–??.

Figure 7: (a) State variables $u^P$, $\beta^P$ and internal angles of a pentagonal cell. (b) State variables $u^S$ for cell corner rotations are defined with respect to cell sides $b_1$ and $b_2$. (c) Derived state variables of cell sides $a$, $c_1$ and $c_2$. 

The gradients of previous expressions with respect to cell side state variables $u^S$, $v^S$ and hinge eccentricities $w^S$ as well as pentagonal state variables $u^P$, $v^P$ can be found in ??–??.
3 Cellular Structure

3.1 Variables

The used notation for state variables, hinge eccentricities and internal angles, lengths of a cellular structure is summarized in Figure 8. The effective cell side lengths \( \mathbf{v} \) between cell corners are

\[
\mathbf{v} = \left[ \mathbf{b}_1^\top \quad \mathbf{c}_1^\top \quad \ldots \quad \mathbf{b}_{n_R}^\top \quad \mathbf{c}_{n_R}^\top \quad \mathbf{a}^\top \right] \in \mathbb{R}^{nv} \tag{19}
\]

where, for example \( \mathbf{b}_i = [b_{i,1} \ldots b_{i,n_P+2-1}]^\top \). Herein, \( n_P \) denotes the number of base pentagons and \( n_R \) is the number of cell rows. It can be seen that \( \mathbf{v} \) incorporates, in contrast to \( \mathbf{v}_P \) and \( \mathbf{v}_T \), the non-abstract pentagonal base sides \( \mathbf{a} \). The total number of cells \( n_C \) and cell sides \( n_v \) of a cellular structure are

\[
n_C = \frac{n_R}{2} (2n_P - n_R + 1) \quad \text{and} \quad n_v = 3n_C + n_P + n_R. \tag{20}\]

It is subsequently assumed that each additional cell row contains one cell less than the previous row. This is not a limitation of the proposed framework since arbitrary topologies at both ends can be modeled with the help of constraints. State variables \( \mathbf{u}_\alpha \) of a cellular structure without hinge eccentricities are

\[
\mathbf{u}_\alpha = \left[ \alpha_1 \quad \ldots \alpha_{2n_P} \quad \beta_{1,1} \quad \ldots \quad \beta_{n_R-1,n_P-n_R+2} \right]^\top \in \mathbb{R}^{n_\alpha} \tag{21}
\]

where the number of state variables \( n_\alpha \) is

\[
n_\alpha = n_C + n_P + n_R - 1. \tag{22}\]

Cellular structures with non-zero hinge eccentricities require additional state variables \( \mathbf{u}_\kappa \) that describe cell corner rotations and are expressed with respect to cell sides \( \mathbf{b} \)

\[
\mathbf{u}_\kappa = \left[ \kappa_1^\top \quad \ldots \quad \kappa_{n_R}^\top \quad \kappa_{n_R+1}^\top \right] \in \mathbb{R}^{n_\kappa} \quad \text{where} \quad \kappa_i = \left[ \begin{array}{c} \kappa_{i,1-} \quad \kappa_{i,1+} \quad \ldots \quad \kappa_{i,n_P-i+2-} \quad \kappa_{i,n_P-i+2+} \end{array} \right]^\top \quad \text{if} \quad i \leq n_R,
\]

\[
\kappa_i = \left[ \begin{array}{c} \kappa_{n_R+1,1-} \quad \ldots \quad \kappa_{n_R+1,n_P-n_R+1-} \end{array} \right]^\top \quad \text{if} \quad i > n_R. \tag{23}\]

The number \( n_\kappa \) of state variables \( \mathbf{u}_\kappa \) equals

\[
n_\kappa = 2n_C + n_P + n_R + 1. \tag{24}\]

Therefore, the state variables of a cellular structure with hinge eccentricities can be written for the reference (manufactured) and current (pressurized) configuration as

\[
\mathbf{u}_0 = \left[ \mathbf{u}_{\alpha 0}^\top \quad \mathbf{v}_0^\top \right]^\top \in \mathbb{R}^{n_\alpha + n_v} \quad \text{and} \quad \mathbf{u} = \left[ \mathbf{u}_\alpha^\top \quad \mathbf{u}_\kappa^\top \quad \mathbf{v}^\top \right]^\top \in \mathbb{R}^{n_\alpha + n_\kappa + n_v}. \tag{25}\]

Cell corner rotations of the reference configuration are \( \mathbf{u}_{\kappa 0} = \mathbf{0} \) since undeformed cell sides are assumed to be straight.
3.2 Transformation Matrices

As illustrated in Figure 9, state variables \( u_{a,i,j}^P \) of the \( j \)-th pentagonal cell in the \((i + 1)\)-th cell row can be expressed in terms of state variables \( u_{a,i,j}^T \) and \( v_{i,j}^T \) of the \( j \)-th triangular cell in the \( i \)-th cell row

\[
\begin{align*}
    u_{a,i,j+1}^P &= \begin{bmatrix} \beta_{i,j}^T \end{bmatrix} \begin{bmatrix} u_{a,i,j}^T \end{bmatrix} + T_{i,j}^{lin} u_{a,i,j}^T + T_{i,j}^{lin} \left( u_{a,i,j}^T, v_{i,j}^T \right) \\
    u_{a,i,j}^P &= \begin{bmatrix} \beta_{i,j}^T \end{bmatrix} \begin{bmatrix} u_{a,i,j}^T \end{bmatrix} + T_{i,j}^{lin} u_{a,i,j}^T + T_{i,j}^{lin} \left( u_{a,i,j}^T, v_{i,j}^T \right)
\end{align*}
\]

(26)

where the linear and nonlinear matrices are

\[
T_{i,j}^P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_{i,j}^{lin} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

and

\[
T_{i,j}^{lin} \left( u_{a,i,j}^T, v_{i,j}^T \right) = \begin{bmatrix} -\psi_{i,j} \left( u_{a,i,j}^T, v_{i,j}^T \right) \\ -\psi_{i,j} \left( u_{a,i,j}^T, v_{i,j}^T \right) \\ \alpha_{i,j} \left( u_{a,i,j}^T, v_{i,j}^T \right) \end{bmatrix}
\]

(27)

The transformation matrix \( T_{i,j}^P \) relates pentagonal state variables \( u_{a,i,j}^P \) to triangular state variables \( u_{a,i,j}^T \). Similarly, the transformation matrix \( T_{i,j}^P \) relates pentagonal state variables \( u_{a,i,j}^P \) to triangular state variables \( v_{i,j}^T \)

\[
T_{i,j}^P = \frac{\partial u_{a,i,j+1}^P}{\partial u_{a,i,j}^P} = T_{i,j}^{lin} + \frac{\partial T_{i,j}^{lin}}{\partial u_{a,i,j}^P} \quad \text{and} \quad T_{i,j}^P = \frac{\partial u_{a,i,j+1}^P}{\partial v_{i,j}^T} = \frac{\partial T_{i,j}^{lin}}{\partial v_{i,j}^T}
\]

(28)

Transformation matrices for reference state variables are derived in a similar manner and denoted as, for example, \( T_{i,j}^{lin} \).

![Figure 9: (a) State variables of pentagonal and hexagonal cells. (b) Internal angles of pentagonal and triangular cells.](image)

3.3 Potential Energy and Equilibrium Configuration

The potential energy of a cellular structure is the sum of the pressure energy of triangular and pentagonal cells as well as the pressure and strain energy of cell sides

\[
\Pi \left( \mathbf{u}_0, \mathbf{u} \right) = \sum_{i=1}^{nR} \left( \sum_{j=1}^{n^{P+2-i}_{a,i,j}} \Pi_{b,i,j}^S \left( \mathbf{u}_0, \mathbf{u} \right) + \sum_{j=1}^{n^{P+2-i}_{a,i,j}} \left( \Pi_{a,i,j}^P \left( \mathbf{u} \right) + \Pi_{a,i,j}^S \left( \mathbf{u}_0, \mathbf{u} \right) + \Pi_{c,i,2j-1}^S \left( \mathbf{u}_0, \mathbf{u} \right) + \Pi_{c,i,2j}^S \left( \mathbf{u}_0, \mathbf{u} \right) \right) \right)
\]

(29)

\[
+ \left( 1 - \delta_i \right) \sum_{j=1}^{n^{P-1}_{a,i,j}} \Pi_{a,i,j}^T \left( \mathbf{u} \right)
\]
where $\delta_{ik}$ is a Kronecker delta. It can be seen that only the energies of cell sides are a function of the reference (manufactured) configuration. The gradient $\Pi^u = \frac{\partial \Pi}{\partial u} \in \mathbb{R}^{nR+nx+nv}$ of the potential energy with respect to the state variables $u$ is computed by adding the contributions of single cell rows. The energy gradient $\Pi^u_i$ incorporates only terms from cells and sides in the $i$-th cell row. It is expressed with respect to state variables $u$ of a cellular structures that solely consists of cell rows. Combinations of cell row pressures are subsequently referred to as pressure sets. A gradient with respect to reference state variables terms such as $\frac{\partial}{\partial q}$ leads to have to satisfy terms such as $\frac{\partial}{\partial q}$ to be small. The change of the current state variables $u$ has to be setting to $\Pi^u = 0$ equilibrium condition.

This nonlinear set of equations for the state variables $u_q$ can be solved by using a Newton based approach. State variables of the $(i+1)$-th Newton iteration are

\[
u_{i+1} = u_q - \left( \Pi^w_q \right)^{-1} \Pi^w_q \]

where $\Pi^w_q$ is the Hessian of the potential energy with respect to state variables $u_q$.

### 3.4 Optimization

The side lengths of a pressure actuated cellular structure with $nR$ cell rows can be optimized such that the outer pentagonal cell corners of an equilibrium configuration are, depending on the pressure set, located on $nR$ different $C^1$ continuous target shapes (Figure [10]). A target shape, indexed by $q$, is approximated by a piecewise linear curve with angles $\Delta \alpha_{q,j}$ between the corner points. Therefore, the state variables $\alpha_q \subset u_{u,q}$ of the $q$-th equilibrium configuration have to satisfy

\[
\Delta \alpha_{q,j} = \alpha_{q,j} - \alpha_{q,j+1} \quad \text{for} \quad j = 1, \ldots, nP - 1.
\]

The deviation between the target shape and an equilibrium configuration at the $j$-th cell corner is given by

\[
r_{q,j} = \Delta \alpha_{q,j} - \alpha_{q,j} + \alpha_{q,j+1}
\]

and gathered in the residual vector $r_q$ for the $q$-th equilibrium configuration

\[
r_q = \left[ r_{q,1} \ldots r_{q,nP-1} \right]^\top.
\]

In turn, the residual vectors of all equilibrium configurations are gathered in

\[
r = \left[ r_1^\top \ldots r_{nP}^\top \right]^\top.
\]

The target angles $\Delta \alpha_{q,j}$ are a function of the base lengths $a_j$ and thus depend on the corresponding axial strains. However, their influence can be neglected since these strains are usually small. The change of the current state variables $u_q$...
Figure 10: (a) Target shapes are (b) approximated by straight lines. (c) Stylized base pentagons with cell sides $a$, $b$ and state variables $\alpha$.

of the $q$-th equilibrium configuration with respect to reference state variables $u_0$ is subsequently derived. Infinitesimally small variations of reference and current state variables need to satisfy

$$\Pi_q^u(u_0 + \Delta u_0, u_q + \Delta u_q) = 0. \quad (38)$$

Neglecting higher order terms leads to

$$\Pi^u_q + \Pi^a_q \Delta u_0 + \Pi^w_q \Delta u_q = 0 \quad (39)$$

where $\Pi^u_q = 0$ from which we can deduce the gradient

$$G_q = \frac{\partial u_q}{\partial u_0} = - (\Pi^w_q)^{-1} \Pi^a_q u_q^\top. \quad (40)$$

The matrix

$$H = \frac{\partial r}{\partial u_0} = \left[ \frac{\partial r_1^\top}{\partial u_0} \ldots \frac{\partial r_{nR}^\top}{\partial u_0} \right]^\top = \left[ G_1^\top B^\top \ldots G_{nR}^\top B^\top \right]^\top \in \mathbb{R}^{nR(nP-1)\times(na+nv)} \quad (41)$$

relates the residual vector $r$ to reference state variables $u_0$ where $B$ is a Boolean matrix. It is not quadratic and therefore not invertible. Hence, there exists a null-space $N = \text{null} (H)$ with $\text{dim} (N) = n_a + n_v - n_R (n_P - 1)$ where changes in state variables $u_0$ do not affect the residual vector $r$. In other words, it is possible to minimize an arbitrary objective function within the null-space [5] where $r = 0$. For example, the objective function $F$ can be chosen as

$$F(u_0) = \frac{1}{2} (u_0 - u_t)^\top (u_0 - u_t) \quad (42)$$

which minimizes the difference between state variables $u_0$ and target values $u_t$. In the following, it is assumed that $u_0 = \{v_0\} \setminus \{a_0\}$. In other words, only reference cell side lengths other than the pentagonal base sides are varied during the optimization. The gradient and Hessian of $F$ are

$$\mathcal{F}^0 = \frac{\partial F}{\partial u_0} = u_0 - u_t \quad \text{and} \quad \mathcal{F}^{00} = \frac{\partial^2 F}{\partial u_0^2} = I, \quad (43)$$

where $I$ is an identity matrix of size $n_v - n_P$. Therefore, the optimization problem can be stated as

minimize $F(u_0)$

subject to $r = 0$

which can be solved with Lagrange multipliers and the Newton method. The Lagrangian

$$\mathcal{L}(u_0, \lambda) = F(u_0) + \lambda^\top r(u_0) \quad (45)$$

is stationary if

$$\frac{\partial \mathcal{L}}{\partial u_0} = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \lambda} = 0. \quad (46)$$
The set of nonlinear equations for $u_0$ and $\lambda$ is iteratively solved with the Newton method which leads to

$$\begin{bmatrix} 1 + Z^i & \mathcal{H}^i \tilde{\mathcal{H}}^T \end{bmatrix} \begin{bmatrix} u_{0}^{i+1} - u_0^i \end{bmatrix} = - \begin{bmatrix} \mathcal{F}^{0,i} + \mathcal{H}^i \tilde{\mathcal{H}}^T \lambda' \end{bmatrix}$$

where $\mathcal{H} = \frac{\partial \mathcal{r}}{\partial u_0}$, $Z_{i,j} = \sum_{k=1}^{nR(ng-1)} \lambda_k \frac{\partial H_{k,i}}{\partial u_{0,j}}$. (47)

Computing the nonlinear contributions of the constraint equation requires third-order derivatives in $Z$ which are computationally expensive. On the other hand, neglecting these terms can slow down convergence. This problem can be overcome by sacrificing the objective function. If the target values are dynamically chosen at each iteration such that

$$u_0^i - u_0^i = 0$$ (48)

then the Newton method reduces to

$$\begin{bmatrix} 1 & \mathcal{H}^i \tilde{\mathcal{H}}^T \end{bmatrix} \begin{bmatrix} u_{0}^{i+1} - u_0^i \end{bmatrix} = - \begin{bmatrix} 0 \\ \mathcal{r}^i \end{bmatrix}$$ (49)

since $\mathcal{F}^{0,i} = 0$ and therefore $\lambda' = 0$.

### 4 Example Structure

An example structure (Figure 11) that consists of two cell rows with 60 pentagonal and 59 hexagonal cells is used to demonstrate the performance of the proposed algorithm. The first target shape is a full circle and the second target shape is a half circle. The presented results are based on a complete structural simulation and optimization so that 359 cell side lengths are optimized. Edge effects cause varying cell side lengths along the structure. Hence it is not possible to solve this problem by investigating only a few cells. The hinge eccentricities $d$, rotational $e$ and axial springs $h$ for a structure with a unit depth are chosen as

$$d = 10 \text{ mm}, \quad e = 1,500 \text{ N} \quad \text{and} \quad h = 3,000 \text{ N/mm}.$$ 

The equilibrium shapes and axial cell side forces of the original and optimized structure are shown in Figure 12. It can be seen that the equilibrium configurations of the optimized structure reassemble a half- and full circle. Furthermore, optimized cell side lengths differ significantly from the initial structure and vary between both ends. Convergence plots for the equilibrium configurations of the initial structure as well as for the optimization (Figure 13) show that the equilibrium configurations are computed in four iterations and the optimized structure is computed in 27 iterations. The presented optimization results are based on the second approach that avoids the computation of third-order derivatives by sacrificing the objective function. This decision is motivated by the fact that, for this example, the results from the constrained optimization are very similar to the results from the second optimization approach.
Figure 12: Equilibrium configurations for the first and second pressure set of the (a) initial and (b) optimized structure.

Figure 13: Convergence for (a) equilibrium configurations of initial structure and (b) optimization.
5 Conclusions

This article presents a novel approach for the simulation and optimization of compliant pressure actuated cellular structures (CPACS). It complements previous work [11] by taking into account an arbitrary number of cell rows, rotational/axial springs and hinge eccentricities at cell corners. The kinematic framework splits naturally into two parts. The first part describes CPACS with central cell corner hinges. The second part adds a rotational degree of freedom at each cell corner to describe the state of nonzero hinge eccentricities. Furthermore, it has been shown that the geometric primitives of CPACS are triangular and pentagonal cells as well as cell sides. This geometric reduction can be used as a basis for an object oriented implementation. The convergence rate of the optimization is, compared to previous work [11], significantly enhanced by using a Newton method. This will enable the computation of the sensitivity of an optimal solution with respect to hinge eccentricities and rotational/axial springs in future work. Hence it is possible to directly use the presented framework for the dimensioning of a cellular structure. This framework will ultimately be the basis for a software tool that can directly send the optimization results to a rapid prototyping machine.

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A Appendix

A.1 Pentagonal Cells

Derivatives of the internal variables and pressure potential of a pentagonal cell with respect to state variables \( u^p, v^p \) are subsequently summarized. Derivatives of length \( y \) are

\[
\frac{\partial y}{\partial u^p_y} = \frac{1}{y} \begin{pmatrix} \sin (\alpha_{12}) b_2 - a \cos (\alpha_{1}) b_1 \\ y \end{pmatrix}^T, \quad \frac{\partial y}{\partial v^p} = \frac{1}{y} \begin{pmatrix} -\sin (\alpha_{1}) a - \cos (\alpha_{12}) b_2 + b_1 \\ \sin (\alpha_{2}) a - \cos (\alpha_{12}) b_1 + b_2 \\ 0 \\ 0 \end{pmatrix}^T \text{ where } \alpha_{12} = \alpha_1 - \alpha_2. \tag{A.1}
\]

Derivatives of altitude \( z \) are

\[
\frac{\partial z}{\partial u^p} = \frac{\partial z}{\partial v^p} = \frac{\partial z}{\partial \alpha^p} = \frac{\partial z}{\partial \psi^p} = \frac{1}{4y^2z} \begin{pmatrix} (c_1^2 - c_2^2)^2 - y^4 \end{pmatrix} \tag{A.2}
\]

where

\[
\frac{\partial z}{\partial \rho^p} = \begin{pmatrix} c_1 \left( y^2 - c_1^2 + c_2^2 \right) \\ c_2 \left( y^2 + c_1^2 - c_2^2 \right) \end{pmatrix}
\]

and the gradients of the pressure potential \( \Pi^p \) are

\[
\Pi^{p, \theta_1} = -\frac{p}{2} \begin{pmatrix} -(\sin (\alpha_1) a + \cos (\alpha_{12}) b_2) b_1 \\ -(\sin (\alpha_2) a - \cos (\alpha_{12}) b_2) b_2 \\ \cos (\alpha_{1}) b_1 + \cos (\alpha_{2}) b_2 \\ 0 \end{pmatrix} + \frac{\partial y}{\partial \alpha^p} \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}
\]

\[
\Pi^{p, \psi} = -\frac{p}{2} \begin{pmatrix} \cos (\alpha_{1}) a^p - \sin (\alpha_{12}) b_2 \\ \cos (\alpha_{2}) a^p - \sin (\alpha_{12}) b_1 \\ 0 \end{pmatrix} + \frac{\partial z}{\partial \psi^p} \begin{pmatrix} 0 & 0 \end{pmatrix} \tag{A.4}
\]

A.2 Triangular Cells

Derivatives of the internal variables and pressure potential of a triangular cell with respect to state variables \( u^t, v^t \) are subsequently summarized. Derivatives of abstract side \( a \) are

\[
\frac{\partial a}{\partial u^t} = -\frac{1}{a} c_2 c_3 \sin (\theta_1 + \theta_2) \frac{\partial (\theta_1 + \theta_2)}{\partial u^t} \tag{A.5}
\]

\[
\frac{\partial a}{\partial v^t} = -\frac{1}{a} \begin{pmatrix} c_2 c_3 \sin (\theta_1 + \theta_2) \frac{\partial (\theta_1 + \theta_2)}{\partial v^t} \end{pmatrix}
\]

Derivatives of angle \( \psi \) are

\[
\frac{\partial \psi}{\partial u^t} = \frac{\partial \theta_1}{\partial u^t} + \frac{\partial \theta_2}{\partial u^t}, \quad \frac{\partial \psi}{\partial v^t} = \frac{\partial \theta_1}{\partial v^t} + \frac{\partial \theta_2}{\partial v^t} + \frac{\partial \psi}{\partial \psi^t} \tag{A.6}
\]

where derivatives with respect to abstract base side \( a \) and cell sides \( e^T = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \end{pmatrix} \) are

\[
\frac{\partial \psi}{\partial a} = \frac{a^2 - c_2^2 + c_3^2}{a^2 \sqrt{(2ac_2)^2 - (a^2 + c_2^2 - c_3^2)^2}}, \quad \frac{\partial \psi}{\partial e^T} = \frac{1}{c_2 \sqrt{(2ac_2)^2 - (a^2 + c_2^2 - c_3^2)^2}} \begin{pmatrix} 0 \\ 2c_2 c_3 \\ 0 \end{pmatrix}. \tag{A.7}
\]

The gradient of the pressure potential is

\[
\Pi^{t, \theta} = -\frac{p}{2} c_2 c_3 \cos (\theta_1 + \theta_2) \frac{\partial (\theta_1 + \theta_2)}{\partial u^t} \tag{A.8}
\]

\[
\Pi^{t, \psi} = -\frac{p}{2} \begin{pmatrix} c_2 c_3 \cos (\theta_1 + \theta_2) \frac{\partial (\theta_1 + \theta_2)}{\partial v^t} + \sin (\theta_1 + \theta_2) \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & c_3^T & c_2^T & 0 \end{pmatrix}
\]
A.3 Cell Sides

Derivatives of the internal variables and pressure, strain potential of a cell side with respect to state variables $u^S$, $v^S$ are subsequently summarized. Derivatives of the length $L_h$ between cell side hinges are

\[
\frac{\partial L_h}{\partial u^S} = \frac{1}{L_h} \left( \frac{\partial L_{h_b}}{\partial u^S} + \frac{L_h}{L_{h_b}} \frac{\partial L_{h_b}}{\partial u^S} \right) = \frac{1}{L_h} \left( L_{h_b} \begin{bmatrix} \sin (\kappa_+) d_+ & \sin (\kappa_+) d_+ \end{bmatrix} + L_{h_b} \begin{bmatrix} \cos (\kappa_+) d_+ & \cos (\kappa_+) d_+ \end{bmatrix} \right) \quad (A.9)
\]

\[
\frac{\partial L_h}{\partial v^S} = \frac{L_h}{L_{h_b}} \left( \begin{bmatrix} 1 & 0 \end{bmatrix} \right)
\]

Derivatives of, for example, the bending angle $\varphi_-$ are

\[
\frac{\partial \varphi_-}{\partial u^S} = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] + \frac{1}{L_{h_b}} \left( \frac{\partial L_{h_b}}{\partial u^S} - \frac{L_{h_b}}{L_h} \frac{\partial L_h}{\partial u^S} \right) \quad \text{and} \quad \frac{\partial \varphi_-}{\partial v^S} = - \frac{L_{h_b}}{L_{h_b} L_h} \frac{\partial L_h}{\partial v^S}.
\]

(A.10)

Gradients of the pressure and strain potential are

\[
\Pi^{S,\kappa}_p = - \frac{\Delta p}{2} \left( \begin{bmatrix} (\sin (\kappa_+))^2 - (\cos (\kappa_+))^2 d_+ \sqrt{2} - (\cos (\kappa_+)) d_+ \sqrt{2} \\ (\cos (\kappa_+))^2 - (\sin (\kappa_+))^2 d_+ \sqrt{2} + (\cos (\kappa_+)) d_+ \sqrt{2} \end{bmatrix} \right) \quad \text{and} \quad \Pi^{S,\kappa}_e = \frac{\Delta \varphi}{2} \frac{\partial \varphi}{\partial u^S} + \Delta \varphi \frac{\partial \varphi}{\partial u^P} + \Delta \varphi \frac{\partial \varphi}{\partial v^S} + h \Delta L_h \frac{\partial L_h}{\partial v^S}.
\]

(A.11)

The gradient of the total cell side energy with respect to pentagonal state variables $u^P = \left[ u^P_\alpha, v^P_\beta \right]^T$ is

\[
\Pi^{S,\alpha,P} = \Pi^{S,\kappa}_p \left( \frac{\partial u^S_\alpha}{\partial u^P_\alpha} \frac{\partial u^P_\alpha}{\partial u^P_\alpha} + \frac{\partial u^S_\alpha}{\partial \beta^P} \frac{\partial \beta^P}{\partial v^P} + \frac{\partial u^S_\alpha}{\partial v^P} \frac{\partial v^P}{\partial u^P} \right). \quad (A.13)
\]