Equivalence of Weighted DT-Moduli of (Co)convex Functions

Malik Saad Al-Muhja\textsuperscript{1,2}, Habibulla Akhadkulov\textsuperscript{1},
Nazihah Ahmad\textsuperscript{1}

\textsuperscript{1}Department of Mathematics and statistics
Universiti Utara Malaysia
06010 Sintok, Kedah, Malaysia

\textsuperscript{2}Department of Mathematics and Computer Application
University of Al-Muthanna
Samawa 66001, Iraq

email: dr.al-muhja@hotmail.com; malik@mu.edu.iq

(Received April 10, 2020, Accepted July 15, 2020)

Abstract

This paper presents new definitions for weighted DT moduli. Similarly, a general result in an equivalence of moduli of smoothness is obtained. It is known that, for any $r \in \mathbb{N}_0$, $0 < p \leq \infty$, $1 \leq \eta \leq r$ and $\phi(x) = \sqrt{1 - x^2}$, the equivalences

$$\omega_{i+1,r}^\phi(f^{(r)}; \|\theta_N\|_{w_{\alpha,\beta,p}}) \sim \omega_{i,r+1}^\phi(f^{(r+1)}; \|\theta_N\|_{w_{\alpha,\beta,p}})$$

and

$$\omega_{i+\eta}^\phi(f; \|\theta_N\|_{\alpha,\beta,p}) \sim \|\theta_N\|^{-\eta} \omega_{i,2\eta}^\phi(f^{(2\eta)}; \|\theta_N\|_{\alpha+\eta,\beta+\eta,p})$$

are valid.

1 Introduction

Hierarchy foundations of the moduli of smoothness began modern with the work of Ditzian and Totik (1987), (see [6]), and Kopotun (2006-2019), (see [8, 9, 10, 11, 12, 14, 15, 16, 18]). Ditzian and Totik established better continuous moduli of the function in a normed space. Then Kopotun contributed

\underline{Key words and phrases:} Moduli of Smoothness, Jacobi Weights, Weighted Approximation, Lebesgue Stieltjes integral.

\underline{AMS (MOS) Subject Classifications:} 41A10, 41A25, 26A42.

ISSN 1814-0432, 2021, http://ijmcs.future-in-tech.net
to properties of various moduli of smoothness like univariate piecewise polynomial functions (splines) [16]. He has a significant impact on the hierarchy between moduli of smoothness for the past 14 years including his effect on the $k$th symmetric difference (see, [9, proof of Lemma 4.1]). Let $\Delta^k_h(f, x)$ be the $k$th symmetric difference of $f$ [6] given by

$$\Delta^k_h(f, x) = \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x + \frac{2i-k}{2} h) ; & x \pm \frac{kh}{2} \in [-1, 1] , \\ 0 ; & \text{otherwise} \end{cases}$$

The space $L_p([-1, 1])$, $0 < p < \infty$, denotes the space of all measurable functions $f$ on $[-1, 1]$, [15] such that

$$\|f\|_{L_p[-1,1]} = \begin{cases} \left( \int_{-1}^{1} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty ; & \text{if } 0 < p < \infty , \\ \text{esssup}_{x \in [-1,1]} |f(x)| ; & \text{if } p = \infty . \end{cases}$$

Let $\|\cdot\|_p = \|\cdot\|_{L_p[-1,1]}$, $0 < p \leq \infty$ and $\phi(x) = \sqrt{1 - x^2}$. The Ditzian-Totik modulus of smoothness (DTMS) of a function $f \in L_p[-1, 1]$, is defined [5] by

$$\omega^\phi_{k,r}(f, t) = \sup_{0 < h \leq t} \|\phi^r \Delta^k_{h\phi}(f, x)\|_p, \ k, r \in \mathbb{N}_0.$$ 

Also, the $k$th modulus of smoothness of $f \in L_p[-1, 1]$ is defined [6] by

$$\omega_k(f, \delta, [-1, 1])_p = \sup_{0 < h \leq \delta} \|\Delta^k_h(f, x)\|_p, \ \delta > 0, p \leq \infty.$$ 

Denote by $AC_{loc}(-1, 1)$ and $AC[-1, 1]$ the sets of functions which are locally absolutely continuous on $(-1, 1)$ and absolutely continuous on $[-1, 1]$ respectively. We accept the following:

**Definition 1.1.** [18] Let $w_{\alpha, \beta}(x) = (1 + x)^\alpha (1 - x)^\beta$ be the (classical) Jacobi weight, and let

$$\alpha, \beta \in J_p = \begin{cases} (-1/p, \infty), & \text{if } p < \infty , \\ [0, \infty), & \text{if } p = \infty , \end{cases}$$

Define

$$\mathbb{L}^\alpha = \{ f : [-1, 1] \rightarrow \mathbb{R} : \|w_{\alpha, \beta}f\|_p < \infty , \ \text{and} \ 0 < p < \infty \},$$

$$\mathbb{L}^\alpha_{p,r} = \{ f : [-1, 1] \rightarrow \mathbb{R} : f^{(r-1)} \in AC_{loc}(-1, 1), 1 \leq p \leq \infty, \|w_{\alpha, \beta}f^{(r)}\|_p < \infty \},$$

and for convenience use $\mathbb{L}^\alpha_{p,0} = \mathbb{L}^\alpha_p$. 

Let \( f \in L_{p,r}^{\alpha,\beta} \), we write \( \| \cdot \|_{w_{\alpha,\beta,p}} \). If \( r = 0 \), then we use \( \| \cdot \|_{\alpha,\beta,p} \).

**Definition 1.2.** [20] Let \( X \) be a subset of \( \mathbb{R}^n \). A function \( f \) is called convex on \( X \) if \( f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y) \), for all \( x, y \in X \) and \( \lambda \in [0, 1] \).

**Definition 1.3.** [7] Let \( Y_s = \{ y_i \}_{i=1}^s \), \( s \in \mathbb{N} \) be a partition of \([-1, 1]\); that is, a collection of \( s \) fixed points \( y_i \) such that \( y_{s+1} = -1 < y_s < \cdots < y_1 < 1 = y_0 \).

Let \( \Delta^{(2)}(Y_s) \) be the set of continuous functions on \([-1, 1]\) that are convex downwards on the segment \([y_{i+1}, y_i]\) if \( i \) is even and convex upwards on the same segment if \( i \) is odd. The functions from \( \Delta^{(2)}(Y_s) \) are called coconvex.

**Definition 1.4.** [21] The partition \( \hat{T}_{\eta} = \{ t_j \}_{j=0}^\eta \), where

\[
t_j = t_{j,\eta} = \begin{cases} \cos(\frac{\pi j}{\eta}); & \text{if } 0 \leq j \leq \eta, \\ -1; & \text{if } j < 0, \end{cases}
\]

and \( t_j \)'s as the knots of a Chebyshev partition.

**Definition 1.5.** [17] A function \( f \) is said to be \( k \)-monotone, \( k \geq 1 \) on \([a, b]\), if and only if for all choices of \( k + 1 \) distinct points \( x_0, \ldots, x_k \) in \([a, b]\) the inequality \( f[x_{0}, \ldots, x_{k}] \geq 0 \), holds, where

\[
f[x_0, \ldots, x_k] = \sum_{j=0}^{k} f(x_j) \theta'(x_j), \quad \theta(x_j) = \prod_{j=0}^{k} (x - x_j)
\]

denotes the \( k \)th divided difference of \( f \) at \( x_0, \ldots, x_k \).

Now, we present the most important Kopotun’s methods and some further developments of his contribution to the \( k \)th symmetric difference. He stated that [16] ”\( A \) is equivalent to \( B, A \sim B \), if \( c^{-1}A \leq B \leq cA \) such that \( c \) is a positive constant”. Let us recall:

First, for a piecewise polynomial \( s \) on a Chebyshev partition of \([-1, 1]\), we have [12]:

\[
\omega_{k+\eta}^\phi(s, t)_p \leq ct^\eta \omega_{k,\eta}^\phi(s^\eta, t)_p, \quad 0 < p < 1, \quad t > 0,
\]

and

\[
\omega_{k-\eta,\eta}^\phi(s^\eta, n^{-1})_p \sim \omega_k^\phi(s^\eta, n^{-1})_p.
\]
Secondly, in 2007, Kopotun dedicated his attention on the computation of several results on the equivalence of moduli of smoothness (see [16], for example):
\[ n^\eta \omega_{k-\eta}^\phi(s^{(n)}, n^{-1})_p \sim \omega_k(s, n^{-1})_p, \quad 1 \leq p \leq \infty, \quad 1 \leq \eta \leq \min\{k, m + 1\}. \tag{1.1} \]

Thirdly, in 2009, Kopotun examined the equivalence [14]:
\[ \omega_k(f, \delta)_p \leq Ac_\delta(k, q, p) \|f\|_p, \]
where \( f \) is satisfied (1.1), \( q < p \) and
\[
c_\delta(k, q, p) = \begin{cases} 
\frac{\delta^{\frac{2}{q}}}{\delta}, & \text{if } k \geq 2, \\
\frac{\delta^{\frac{2}{2q}}}{\delta}, & \text{if } k = 1 \text{ and } p < 2q, \\
\frac{1}{\delta^{\frac{1}{p}}}, & \text{if } k = 1 \text{ and } p = 2q, \\
\frac{\delta^{\frac{1}{p}}}{\delta}, & \text{if } k = 1 \text{ and } p > 2q.
\end{cases}
\]

The first to deal with development moduli of smoothness were Kopotun, Leviatan and Shevshuk [8]. They were interested in discussing various properties of the new modulus of smoothness
\[ \omega_{k,r}^\phi(f(r), t)_p = \sup_{0 < h \leq t} \|W_{kh}^r(\cdot)\Delta_k^{\phi}(f(r), \cdot)\|_p, \tag{1.2} \]
where
\[ W_{kh}^r(x) = \begin{cases} 
((1 - x - \delta^{\phi(x)}_2)(1 + x - \delta^{\phi(x)}_2))^\frac{1}{2}; & \text{if } 1 \pm x - \delta^{\phi(x)}_2 \in [-1, 1], \\
0; & \text{otherwise.}
\end{cases} \]

However, they contributed to the \( k \)th symmetric difference of modulus by \( K \)-functional [9, proof of Lemma 4.1].

The following result was proven by a different method of modulus of smoothness [11]:

**Theorem 1.6.** Let \( k, n \in \mathbb{N}, r \in \mathbb{N}_0, A > 0, 0 < p \leq \infty, \alpha + \frac{r}{2}, \beta + \frac{r}{2} \in J_p. \)

Let \( 0 < t \leq \frac{a n}{r^2} \), where \( a \) is some positive constant that depends only on \( \alpha, \beta, k \) and \( q \). Then, for any \( p_n \in \pi_n \),
\[ \omega_{k,r}^\phi(p_n^{(r)}(t), A, t)_{\alpha,\beta,p} \sim \Omega_{k,r}(p_n^{(r)}(t), A, t)_{\alpha,\beta,p} \sim t^k \|w_{\alpha,\beta}\delta^r p_n^{k+r}\|_p, \]

where
\[ \Psi_{k,r}(p_n^{(r)}(t), A, t)_{\alpha,\beta,p} = \sup_{0 \leq h \leq t} \|w_{\alpha,\beta}\delta^r \Delta_k^{\phi}(p_n^{(r)}(t), x)\|_p, \]
\[ \Omega_{k,r}^\phi(p_n^r(\cdot), A; t)_{\alpha,\beta,p} = \sup_{0 \leq h \leq t} \| w^{\alpha,\beta}_r \phi^r \Delta^k_{h\phi}(p_n^r(x); T_{A,h}) \|_{L_p(T_{A,h})}, \]

and the equivalence constants depend only on \( k, r, \alpha, \beta, A \) and \( q \).

**Definition 1.7.** [10] For \( r \in \mathbb{N}_0 \) and \( 0 < p \leq \infty \), denote \( \mathbb{B}_p^\alpha(\alpha, \beta) = \mathbb{L}_p^\alpha,\beta \) and
\[
\mathbb{B}_p^\alpha(\alpha, \beta) = \{ f : f^{(r-1)} \in AC_{loc}(-1, 1), \phi^r f^{(r)} \in \mathbb{L}_p^\alpha,\beta \}, r \geq 1.
\]

In 2018, Kopotun et al. ([10, Lemmas 2.2, 2.3]) proposed a function \( f \in \mathbb{B}_p^\alpha(\alpha, \beta) \) and \( \alpha + \frac{r}{2} \geq 0, \beta + \frac{r}{2} \geq 0 \). Then,
\[
\omega_{k,r}^\phi(f^{(r)}, t)_{\alpha,\beta,p} \leq c \| w^{\alpha,\beta}_r \phi^r f^{(r)} \|_p, \quad t > 0,
\]
and
\[
\lim_{t \to 0^+} \omega_{k,r}^\phi(f^{(r)}, t)_{\alpha,\beta,p} = 0.
\]

## 2 Notations and Further Results

In this section, we will present the linear space for functions of Lebesgue Stieltjes integrable-i. First, recall the definition of the Lebesgue Stieltjes integrable-i [3]:

**Definition 2.1.** Let \( \mathbb{D} \) be a measurable set, \( f : \mathbb{D} \to \mathbb{R} \) be a bounded function, and \( \mathcal{L}_i : \mathbb{D} \to \mathbb{R} \) be nondecreasing function for \( i \in \Lambda \). For a Lebesgue partition \( \mathbb{P} \) of \( \mathbb{D} \), put \( \mathcal{L}(f, \mathbb{P}, \mathcal{L}) = \sum_{j=1}^{n} \prod_{i \in \Lambda} \mathcal{L}_i(\mu(\mathcal{D}_j)) \) and \( \mathcal{L}(f, \mathbb{P}, \mathcal{L}) = \sum_{j=1}^{n} \prod_{i \in \Lambda} M_j \mathcal{L}_i(\mu(\mathcal{D}_j)) \) where \( \mu \) is a measure function of \( \mathbb{D} \), \( m_j = \inf \{ f(x) : x \in \mathcal{D}_j \} \), \( M_j = \sup \{ f(x) : x \in \mathcal{D}_j \} \), and \( \mathcal{L} = \mathcal{L}_1, \mathcal{L}_2, \cdots \). Also, \( \mathcal{L}_i(x_j) - \mathcal{L}_i(x_{j-1}) > 0, \mathcal{L}(f, \mathbb{P}, \mathcal{L}) \leq \mathcal{L}(f, \mathbb{P}, \mathcal{L}), \)
\[
\prod_{i \in \Lambda} \int_{\mathbb{D}_i} f d\mathcal{L} = \sup \{ \mathcal{L}(f, \mathcal{L}) \} \text{ and } \prod_{i \in \Lambda} \int_{\mathbb{D}_i} f d\mathcal{L} = \inf \{ \mathcal{L}(f, \mathcal{L}) \}, \]
where \( \mathcal{L}(f, \mathcal{L}) = \{ \mathcal{L}(f, \mathbb{P}, \mathcal{L}) : \mathbb{P} \text{ part of set } \mathbb{D} \} \) and \( \mathcal{L}(f, \mathcal{L}) = \{ \mathcal{L}(f, \mathbb{P}, \mathcal{L}) : \mathbb{P} \text{ part of set } \mathbb{D} \} \). If \( \prod_{i \in \Lambda} \int_{\mathbb{D}_i} f d\mathcal{L} = \prod_{i \in \Lambda} \int_{\mathbb{D}_i} f d\mathcal{L} \) where \( d\mathcal{L} = d\mathcal{L}_1 \times d\mathcal{L}_2 \times \cdots \), then \( f \) is integral \( \int_{\mathbb{D}_i} f d\mathcal{L} \) according to \( \mathcal{L}_i \) for \( i \in \Lambda \).

**Lemma 2.2.** [2] If \( f \) is a function of Lebesgue Stieltjes integral-i, then \( vf \) is a function of Lebesgue Stieltjes integral-i, where \( v > 0 \) is real number, and
\[
\prod_{i \in \Lambda} \int_{\mathbb{D}_i} v f d\mathcal{L} = v \prod_{i \in \Lambda} \int_{\mathbb{D}_i} f d\mathcal{L},
\]
holds.
Lemma 2.3. [2] If the functions \( f_1, f_2 \) are integrable on the set \( D \) according to \( L_i \), for \( i \in \Lambda \), then \( f_1 + f_2 \) is the function of integrable according to \( L_i \), for \( i \in \Lambda \), such that

\[
\prod_{i \in \Lambda} \int_{D} (f_1 + f_2) \, dL_i = \prod_{i \in \Lambda} \int_{D} f_1 \, dL_i + \prod_{i \in \Lambda} \int_{D} f_2 \, dL_i.
\]

Definition 2.4. [4] A domain \( D \) of convex polynomial \( p_n \) of \( \Delta^{(2)} \) is a subset of \( X \subseteq \mathbb{R} \), satisfying the following properties:

1. \( D \in K^N \), where
   \[
   K^N = \{ D : D \text{ is a compact subset of } X \}
   \]
   is the class of all domains of convex polynomials,

2. there is \( t \in X/D \) such that
   \[
   |p_n(t)| > \sup\{ |p_n(x)| : x \in D \}, \text{ and}
   \]

3. there is the function \( f \) of \( \Delta^{(2)} \), such that
   \[
   \|f - p_n\| \leq \frac{c}{n^\gamma} \omega_{k,2}^\phi(f'', \frac{1}{n}).
   \]

Definition 2.5. [4] A domain \( D \) of coconvex polynomial \( p_n \) of \( \Delta^{(2)}(Y_s) \) is a subset of \( X \) and \( X \subseteq \mathbb{R} \), satisfying the following properties:

1. \( D \in K^N(Y_s) \), where
   \[
   K^N(Y_s) = \{ D : D \text{ is a compact subset of } X, \text{ and } p_n \text{ changes convexity at } D \}
   \]
   is the class of all domains of coconvex polynomials,

2. \( y_i \)'s are inflection points such that
   \[
   |p_n(y_i)| \leq \frac{1}{2}, \ i = 1, ..., s, \text{ and}
   \]

3. there is the function \( f \) of \( \Delta^{(2)}(Y_s) \) such that
   \[
   \|f - p_n\| \leq \frac{c}{n^\gamma} \omega_{k,2}^\phi(f'', \frac{1}{n}).
   \]

From Definitions 2.1, 2.4 and 2.5, if the function \( f \) is convex, then \( D \) is the domain of \((co)\)convex functions of \( f \).

Remark 2.6. [2] Let \( I_f \) be the class of all functions of integrable \( f \) satisfying Definition 2.1; i.e.,

\[
I_f = \{ f : f \text{ is an integrable function according to } L_i, \ i \in \Lambda \}
= \{ f : \prod_{i \in \Lambda} \int_{D} f \, dL = \prod_{i \in \Lambda} \int_{D} f \, dL_i \}.
\]
Remark 2.7. [13] Let \( x_i \in \left[ \frac{x_i + x_{i+1}^\#}{2}, \frac{x_i + x_{i-1}^\#}{2} \right] \subseteq \theta_N \). Let

\[
x^\# = x_{j(i)+1}, \quad x_\# = x_{j(i)-2},
\]

where \( \theta_N = \theta_N[-1, 1] = \{x_i\}_{i=0}^N = \{-1 = x_0 \leq \cdots \leq x_{N-1} \leq x_N = 1\} \) and \( \|\theta_N\| = \max_{0 \leq i \leq N-1} \{x_{i+1} - x_i\} \) is the length of the largest interval in that partition.

Definition 2.8. For \( r \in \mathbb{N}_0 \), the weighted DTMS in \( L^p_{\alpha, \beta} \cap I_f \), we define

\[
\Delta^i_h(f, x) = \begin{cases} 
\prod_{i \in \Lambda} J^i f \, dL & \text{if } f \in I_f, \\
0 & \text{otherwise.}
\end{cases}
\]

Equivalence of Weighted DT-Moduli of (Co)convex Functions

For \( \Phi^r \) and \( \omega_{i, r}^\phi (f^{(r)}, \|\theta_N\|)[-1, 1]\), we denote

\[
\Phi^p (w_{\alpha, \beta}) = \{ f : f \in L^\alpha_{p, r} \cap I_f \text{ and } \omega_{i, r}^\phi (f^{(r)}, \|\theta_N\|)[-1, 1] < \infty \},
\]

and \( \Phi^{p, 0} (w_{\alpha, \beta}) = \Phi^p (w_{\alpha, \beta}) \).

We focus on the applications of results that were obtained in [2, Theorems 3.1, 3.3] and [1, Theorem 2.11].

A set of all piecewise polynomial approximation \( S(\hat{T}_n, r+2) \) of order \( r+2 \), with the knots of a Chebyshev partition \( \hat{T}_n \).

Theorem 2.10. [2] For \( r \in \mathbb{N}_0, \alpha, \beta \in J_p \), there is a constant \( c = c(r, \alpha, \beta, p) \) such that if \( f \in \Delta(2) \cap L^p_{\alpha, \beta}, \) there is a number \( N = N(f, \omega^\delta_{i, r}, I\alpha, \|\theta_N\|, I\beta) \) for \( n \geq N \) and \( S \in S(\hat{T}_n, r+2) \cap \Delta(2) \cap L^p_{\alpha, \beta} \) such that

\[
\|f^{(r)} - S^{(r)}\|_{w_{\alpha, \beta}} \leq c_{r, \alpha, \beta, p, \omega_{i, r}^\phi} \min\{\omega^\phi_{i, r} (f^{(r)}, \|\theta_N\|, I\alpha)_{w_{\alpha, \beta}}, \omega^\phi_{i, r} (f^{(r)}, \|\theta_N\|, I\beta)_{w_{\alpha, \beta}}\},
\]

where

\[
\Delta^i_h \phi (f^{(r)}, x) = \int_1^D \int_2^D \cdots \int_{i-1}^D f^{(r)} \, dL_{1t, \alpha} dL_{2t, \alpha} \cdots dL_{it, \alpha} \cdots = \prod_{i \in \Lambda} \int_i^D f^{(r)} \, dL_{t\phi, \alpha},
\]

(2.2)
\[ \Delta_{\phi, \beta}^i(f^{(r)}, x) = \int_1^D \int_2^D \cdots \int_i^D f^{(r)} \, d\mathcal{L}_{1\phi, \beta} \, d\mathcal{L}_{2\phi, \beta} \cdots d\mathcal{L}_{i\phi, \beta} = \prod_{i \in \Lambda} \int_i^D f^{(r)} \, d\mathcal{L}_{\phi, \beta}. \] (2.3)

Moreover, if \( r, \alpha, \beta = 0 \), then
\[ \|f - S\|_p \leq c(\omega_1^\phi) \omega_i^\phi(f, \|\theta_N\|, I)_{p}. \]

In particular,
\[ \|f^{(r)} - S^{(r)}\|_{w_{\alpha, \beta}, p} \leq c_r \omega_i^\phi(f^{(r)}, \|\theta_N\|, I)_{w_{\alpha, \beta}, p}. \]

**Theorem 2.11.** [2] Let \( \Delta^k \) be the space of all \( k \)-monotone functions. If \( f \in \Delta^k \cap \mathbb{L}_{p,r}^{\alpha, \beta} \) is such that \( f^{(r)}(x) = p_n^{(r)}(x) \), where \( p_n \in \pi_n \cap \Delta^k, N \geq k \geq 2 \) and \( s \in \mathbb{S}(\bar{T}_n, r + 2) \cap \Delta^k \cap \mathbb{L}_{p,r}^{\alpha, \beta} \), then
\[ \|f - s\|_{w_{\alpha, \beta}, p} \leq c(f, p, k, \alpha, \beta, x, x^\#) \omega_i^\phi(f, \|\theta_N\|, I)_{w_{\alpha, \beta}, p}. \]

In particular, if \( f \) is a convex function and \( p_n \) is a convex polynomial or a piecewise convex polynomial, then
\[ \|f - s\|_{w_{\alpha, \beta}, p} \leq c_k \omega_i^\phi(f, \|\theta_N\|, I)_{w_{\alpha, \beta}, p}. \]

**Definition 2.12.** [1] For \( \alpha, \beta \in J_p \) and \( f \in I_f \), we set
\[ E_n(f, w_{\alpha, \beta})_{\alpha, \beta, p} = \mathbb{E}_n(f)_{\alpha, \beta, p} = \inf \{ \|f - p_n\|_{\alpha, \beta, p} : p_n \in \pi_n \cap I_f, f \in \Delta^{(2)}(Y_s) \cap \Phi^p(w_{\alpha, \beta}) \} \]
and
\[ E_n^{(2)}(f, w_{\alpha, \beta}, Y_s)_{p} = \inf \{ \|f - p_n\|_{\alpha, \beta, p} : p_n \in \pi_n \cap \Delta^{(2)}(Y_s) \cap I_f, f \in \Delta^{(2)}(Y_s) \cap \Phi^p(w_{\alpha, \beta}) \} \]
respectively which denote the degree of best unconstrained and (co)convex polynomial approximation of \( f \).

**Theorem 2.13.** [1] Let \( \sigma, m, n \in \mathbb{N}, \sigma \neq 4, s \in \mathbb{N}_0 \) and \( \alpha, \beta \in J_p \). If \( f \in \Delta^{(2)}(Y_s) \cap \Phi^p(w_{\alpha, \beta}) \), then
\[ \sup \{ n^\sigma E_n^{(2)}(f, w_{\alpha, \beta}, Y_s)_{p} : n \geq m \} \leq c \sup \{ n^\sigma E_n(f)_{\alpha, \beta, p} : n \in \mathbb{N} \}. \] (2.4)

In particular, suppose that \( Y_s \in \mathbb{Y}_s \) and \( s \geq 1 \). Then
\[ E_n^{(2)}(f, w_{\alpha, \beta}, Y_s)_{p} \leq c n^{-\sigma} \omega_i^\phi(f^{(r)}, \|\theta_N\|, I)_{w_{\alpha, \beta}, p}, \quad n \geq \|\theta_N\|. \]
Remark 2.14. If \( f \) in \( I_f \) is a function of Lebesgue Stieltjes integral \(-i\) and \( f \) is a differentiable function, then

\[
f' = \frac{df}{dx} = \frac{d}{dx}\left( \int_0^x \frac{df(u)}{dl_{1,\mu,D_0}} \times dl_{1,\mu,D_0} \right)
\]

\[
= \frac{d}{dx}\left( \int_0^x \frac{d^2 f(u)}{dl_{1,\mu,D_0} \times dl_{2,\mu,D_0}} \times dl_{1,\mu,D_0} \times dl_{2,\mu,D_0} \times \cdots \times dl_{i,\mu,D_0} \cdots \right)
\]

\[
= \frac{d}{dx}\left( \prod_{i \in A} \int_{I_x} f^{(i)}(u) \left( dl_{\mu,D_0} \right) \right), \quad x \in I_f = [0, x] \subseteq \mathbb{D}_0, \quad u \in \mathbb{D}_0, \quad \text{and} \quad \ell_{\mu,D_0} = \ell(\mu(\mathbb{D}_0))
\]

Lemma 2.15. We have

\[
\Phi^{p,r+1}(w_{\alpha,\beta}) = \Phi^{p,r}(w_{a+\frac{1}{p},\beta+\frac{1}{p}}).
\]

Proof. First, suppose \( 1 \leq p < \infty \), and \( w_{\alpha,\beta}(x) = (1 + x)^{\alpha}(1 - x)^{\beta} \). Let \( f \in \Phi^{p,r+1}(w_{\alpha,\beta}) \) and assume \( f \) satisfies Definition 2.8. Next,

\[
\|w_{\alpha,\beta} \phi^{r+1} \Delta_{h \phi}^i (f^{(r+1)}, x)\|_p = \left( \int_{-1}^{1} |w_{\alpha,\beta} \phi^{r+1} \Delta_{h \phi}^i (f^{(r+1)}, x)|^p dx \right)^\frac{1}{p}, \quad 0 < h \leq \|\theta_N\|
\]

\[
= \left( \int_{-1}^{1} |w_{\alpha,\beta} \phi^{r+1} \prod_{i \in A} \int_{I_x} f^{(r+1)} d\mathcal{L}_\phi| d\mathcal{L}_\phi \right)^\frac{1}{p}.
\]

Next, from [2, proof of Lemma 3.2], [18] and Remark 2.14, we have

\[
\|w_{\alpha,\beta} \phi^{r+1} \Delta_{h \phi}^i (f^{(r+1)}, x)\|_p = \left( \int_{-1}^{1} |w_{\alpha+\frac{1}{p},\beta+\frac{1}{p}} \phi^{r} \prod_{i \in A} \int_{I_x} f^{(r)} d\mathcal{L}_\phi| d\mathcal{L}_\phi \right)^\frac{1}{p}
\]

\[
= \|w_{\alpha+\frac{1}{p},\beta+\frac{1}{p}} \phi^{r} \Delta_{h \phi}^{i+1} (f^{(r)}, x)\|_p, \quad 0 < h \leq \|\theta_N\|.
\]

Remark 2.16. By virtue of Lemma 2.15, we immediately get

\[
\omega_{i,r+1} (f^{(r+1)}, \|\theta_N\|)_{w_{\alpha,\beta},p} = \omega_{i+1,r} (f^{(r)}, \|\theta_N\|)_{w_{\alpha+\frac{1}{p},\beta+\frac{1}{p}},p}.
\]
3 Main Results for Weighted DT Moduli

**Theorem 3.1.** Let $s, r \in \mathbb{N}_0$, $0 < p \leq \infty$ and $f \in \Delta^{(2)}(Y_s) \cap \Phi^{\rho,r}(w_{\alpha,\beta})$. Let $D$ be defined in Definition 2.1 such that $|D| \leq \delta_\circ$, for some $\delta_\circ \in \mathbb{R}^+$. Then

$$\omega^{\phi}_{i+1,r} (f^{(r)}, \|\theta_N\|)_{w_{\alpha,\beta},p} \leq c(\delta_\circ) \omega^{\phi}_{i,r+1} (f^{(r+1)}, \|\theta_N\|)_{w_{\alpha,\beta},p},$$

(3.1)

where the constant $c$ depends on $\delta_\circ$.

**Proof.** Suppose that $f \in \Delta^{(2)}(Y_s) \cap \Phi^{\rho,r}(w_{\alpha,\beta})$,

$$\phi^D_{\text{d} \cap \text{d}^j} = \prod_{i \in \Lambda} \int_i^{D \cap \text{d}^j} f \, dL_{\phi}$$

and

$$\phi^D_{\text{d} \cap \text{d}^j} = \prod_{i \in \Lambda} \int_i^{D \cap \text{d}^j} f \, dL_{\phi}.$$

In addition, assume that $D_j \subset D$ such that

$$f(x) = \begin{cases} |D|, & \text{if } |D| \leq \delta_\circ, \\ (\phi^D_{\text{d} \cap \text{d}^j}) \to (\phi^D_{\text{d} \cap \text{d}^j}), & \text{if } D_k, D_j \text{ are Lebesgue measurable sets,} \\ 0, & \text{otherwise.} \end{cases}$$

(3.2)

Then

$$\|w_{\alpha,\beta} \phi^r \Delta^{i+1}_{h_\circ} (f^{(r)}, x)\|_p = \left( \int_{-1}^{1} |w_{\alpha,\beta} \phi^r \prod_{i \in \Lambda} \int_i^{D} f^{(r)} \, dL_\phi|^p \, dx \right)^{\frac{1}{p}}$$

$$= \begin{cases} \left( \int_{-1}^{1} |w_{\alpha,\beta} \phi^r \prod_{i \in \Lambda} \int_i^{D} |D| \, dL_\phi|^p \, dx \right)^{\frac{1}{p}} = I_0(x), & \text{if } |D| \leq \delta_\circ, \delta_\circ \in \mathbb{R}^+ \\ \left( \int_{-1}^{1} |w_{\alpha,\beta} \phi^r \lim_{k \to \infty} \prod_{i \in \Lambda} \int_i^{D_k \cap \text{d}^j} f^{(r)} \, dL_\phi|^p \, dx \right)^{\frac{1}{p}} = I_1(x), \text{if } D_k, D_j \text{ are Lebesgue measurable} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, (3.2) implies that

$$I_0(x) = \left( \int_{-1}^{1} |w_{\alpha,\beta} \phi^r \prod_{i \in \Lambda} \int_i^{D} |D| \, dL_\phi|^p \, dx \right)^{\frac{1}{p}} \leq \delta_\circ,$$

for some $\delta_\circ \in \mathbb{R}^+$, while

$$I_1(x) = \left( \int_{-1}^{1} |w_{\alpha,\beta} \phi^r \lim_{k \to \infty} \prod_{i \in \Lambda} \int_i^{D_k \cap \text{d}^j} f^{(r)} \, dL_\phi|^p \, dx \right)^{\frac{1}{p}}$$
By Remark 2.14, we have

\[ I_1(x) \leq c \left( \int_{-1}^{1} |w_{\alpha,\beta} \phi^r \prod_{i \in A} \int_{D \cap D_j} f^{(r)} \, dL_\phi |^p \, dx \right)^{\frac{1}{p}}. \]

Taking supremum, we obtain (3.1).

The following corollary is clear.

**Corollary 3.2.** Let \( s, r \in \mathbb{N}_0 \), \( 0 < p \leq \infty \) and \( f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta}) \). Let \( D \) and \( \delta_0 \) be defined in Theorem 3.1. Then

\[ \omega_{i+1,r}(f^{(r)}, \|\theta_N\|)_{w_{\alpha,\beta},p} \leq c(\delta_0) \omega_{i+1,r}(f^{(r)}, \|\theta_N\|)_{w_{\alpha+\frac{1}{2},\beta+\frac{1}{2}},p}, \]

where the constant \( c \) depends on \( \delta_0 \).

Figure 1: Graph of partitions of the coconvex function on the interval \([-1, 2]\).
Theorem 3.3. Let \( s, r \in \mathbb{N}_0 \), \( \alpha, \beta \in J_p \) and \( 0 < p \leq \infty \). Let \( P \) be a Lebesgue partition of \( \mathbb{D} \) and \( \hat{T}_\eta \) be a Chebyshev partition with \( P \cap \hat{T}_\eta \neq \emptyset \), \( 1 \leq \eta \leq r \). If \( f \in \Delta^{(2)}(Y_s) \cap \Phi^p_r(w_{\alpha, \beta}) \), then there is a constant \( c \) depending on \( \eta \) and \( J_{j, \eta} \) such that

\[
\omega_{i, \eta}^\phi (f, \|\theta_N\|)_{w_{\alpha, \beta}; p} \leq c \|\theta_N\|^{-\eta} \omega_{i, 2\eta}^\phi (f^{(2\eta)}, \|\theta_N\|)_{w_{\alpha+\eta, \beta+\eta}; p}. \tag{3.3}
\]

Proof. Recall that \( P \) is a Lebesgue partition of \( \mathbb{D} \) and \( \hat{T}_\eta \) is a Chebyshev partition. Since \( P \cap \hat{T}_\eta \neq \emptyset \), by virtue of [2, proof of Lemma 2.3], for \( \varepsilon > 0 \), there is a Lebesgue partition \( P_\varepsilon \) of \( \mathbb{D} \) such that \( \hat{T}_\eta \cap P_\varepsilon = P \).

We can construct \( J_{j, \eta} = [u_{j-(\eta+1)}, u_{j-(\eta+1)+1}] \) for some \( y_i \in \cup_{j=0}^{\eta} J_{j, \eta} \) and \( y_i \) is an inflection point of \( Y_s \), \( s \in \mathbb{N}_0 \), (see Figure 1). Next, if \( f \in \Delta^{(2)}(Y_s) \cap \Phi^p_r(w_{\alpha, \beta}) \), then

\[
\omega_{i, \eta}^\phi (f, \|\theta_N\|)_{w_{\alpha, \beta}; p} = \sup \{ \|w_{\alpha, \beta} \phi^r \Delta_{h_\phi}^{i+n}(f, x)\|_{L^p(J_{j, \eta}, dx)} \}.
\]

By virtue of [19] and Theorem 2.13 or (see [1, proof of Theorem 2.11]), we have

\[
\|\theta_N\|^\eta \omega_{i, \eta}^\phi (f, \|\theta_N\|)_{w_{\alpha, \beta}; p} \leq c \sum_{j=0}^{\eta} \left( \int_{-1}^{1} \left| \prod_{i \in \Lambda} \int_{j+\eta}^{J_{j, \eta}} (f - f^{(n)} + f^{(n)}) d\mathcal{L}_\phi \right| d\mathcal{L}_\phi \right) dx
\]

\[
\leq c \sum_{j=0}^{\eta} \left( \int_{-1}^{1} \left| \prod_{i \in \Lambda} \int_{j+\eta}^{J_{j, \eta}} (f - f^{(n)}) d\mathcal{L}_\phi \right| d\mathcal{L}_\phi + \int_{-1}^{1} \left| \prod_{i \in \Lambda} \int_{j+\eta}^{J_{j, \eta}} f^{(n)} d\mathcal{L}_\phi \right| d\mathcal{L}_\phi \right)
\]

\[
\leq c \sum_{j=0}^{\eta} \left( \|w_{\alpha, \beta} \phi^r \Delta_{h_\phi}^{i+n}(f - f^{(n)}, x)\|_{L^p(J_{j, \eta}, dx)} + \|w_{\alpha, \beta} \phi^r \Delta_{h_\phi}^{i+n}(f^{(n)}, x)\|_{L^p(J_{j, \eta}, dx)} \right)
\]

\[
\leq c \sum_{j=0}^{\eta} \left( \|w_{\alpha, \beta} \phi^r \Delta_{h_\phi}^{i+n}(f - f^{(n)}, x)\|_{L^p(J_{j, \eta}, dx)} \right)
\]

\[
+ \sup_{j=0}^{\eta} \left( \|w_{\alpha, \beta} \phi^r \Delta_{h_\phi}^{i+n}(f^{(n)}, x)\|_{L^p(J_{j, \eta}, dx)} \right)
\]

\[
\leq c(\eta, J_{j, \eta}) \times \sup_{j=0}^{\eta} \left( \|w_{\alpha, \beta} \phi^r \Delta_{h_\phi}^{i+n}(f^{(n)}, x)\|_{L^p(J_{j, \eta}, dx)} \right)
\]

\[
0 < h \leq \|\theta_N\|.
\]
Let \( P \) be a Lebesgue partition of \( D \), and \( \hat{T}_\eta \) be a Chebyshev partition with \( P \cap \hat{T}_\eta \neq \emptyset \), \( 1 \leq \eta \leq r \). We have

\[
\|w_{\alpha,\beta} \phi^n f^{(n)}\|_p \geq c(\eta, J_{\eta}) \left\{ \begin{array}{ll}
\omega_{i+\eta,\eta}(f^{(i+\eta)}, \|\theta_N\|)_{w_{\alpha,\beta},p} & \text{if } |D| \leq c(\eta, J_{\eta}), \\
\omega_{i+2\eta}(f^{(i+2\eta)}, \|\theta_N\|)_{w_{\alpha+\beta,\frac{p+2}{2}},p} & \text{if } |D| > c(\eta, J_{\eta}).
\end{array} \right.
\]

Finally, by virtue of Theorems 3.1 and 3.3, we have

\[
\|w_{\alpha,\beta} \phi^n f^{(n)}\|_p \geq c(\eta, J_{\eta}) \left\{ \begin{array}{ll}
\omega_{i+\eta,\eta}(f^{(i+\eta)}, \|\theta_N\|)_{w_{\alpha,\beta},p} & \text{if } |D| \leq c(\eta, J_{\eta}), \\
\omega_{i+2\eta}(f^{(i+2\eta)}, \|\theta_N\|)_{w_{\alpha+\beta,\frac{p+2}{2}},p} & \text{if } |D| > c(\eta, J_{\eta}).
\end{array} \right.
\]
4 Conclusions and Direct Estimates

Theorem 4.1. Assume that \( s, r \in \mathbb{N}_0 \), \( \alpha, \beta \in J_p \), \( 0 < p \leq \infty \) and \( f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta}) \). If \( \mathcal{P} \) is a Lebesgue partition of \( \mathbb{D} \), and \( \mathcal{T}_\eta \) is a Chebyshev partition with \( \mathcal{P} \cap \mathcal{T}_\eta \neq \emptyset \). Then, for any constant \( c \) depending on \( \eta, J_{j,n} \), and \( |\mathcal{D}| \leq \delta_e \), we have

\[
\omega_{i+1,r}^{(r)} \left( f^{(r)}, \|\theta_N\| \right)_{w_{\alpha,\beta},p} \sim c(\delta_e)\omega_{r+1,1}^{(r+1)} \left( f^{(r+1)}, \|\theta_N\| \right)_{w_{\alpha,\beta},p} \sim \\
c(\delta_e) \times \omega_{i+1,r}^{(r)} \left( f^{(r)}, \|\theta_N\| \right)_{w_{\alpha,\beta},p} \sim \|w_{\alpha,\beta}\phi_n f^{(n)}\|_p \\
c(\eta, J_{j,n}) \{\omega_{i+2\eta,1}^{(i+\eta)} \left( f^{(i+\eta)}, \|\theta_N\| \right)_{w_{\alpha,\beta},p} : |\mathcal{D}| \leq c(\eta, J_{j,n}) \}
\]

and

\[
\|\theta_N\| \times \omega_{i+1}^{(r)} \left( f, \|\theta_N\| \right)_{w_{\alpha,\beta},p} \sim c(\eta, J_{j,n})\omega_{i,2\eta}^{(2\eta)} \left( f^{(2\eta)}, \|\theta_N\| \right)_{w_{\alpha,\beta},p} \sim \\
\|w_{\alpha,\beta}\phi_n f^{(n)}\|_p \sim c(\eta, J_{j,n}) \{\omega_{i+2\eta,1}^{(i+2\eta)} \left( f^{(i+2\eta)}, \|\theta_N\| \right)_{w_{\alpha,\beta},p} : |\mathcal{D}| > c(\eta, J_{j,n}) \}.
\]

Corollary 4.2. \((s = 0)\) For \( r \in \mathbb{N}_0 \) and \( \alpha, \beta \in J_p \), there is a constant \( c \) depending on \( r, \alpha, \beta, p, \omega_{i,r}^{(r)} \) and \( r, \alpha, \beta, p, \omega_{i,r}^{(2r)}, \eta \) and \( J_{j,n} \) such that \( f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta}) \), \( J_{j,n} = [u_{j-(\eta+i)}, u_{j-(\eta+i)+1}] \) and \( 1 \leq \eta \leq r \). Then

\[
\mathcal{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_p) \leq c(\|\theta_N\|^n) \omega_{i+1}^{(r)} \left( f^{(r)}, \|\theta_N\| \right)_{w_{\alpha,\beta},p}
\]

and

\[
\mathcal{E}^{(2)}_n(f^{(n)}, w_{\alpha,\beta}, Y_p) \leq c(\|\theta_N\|^n) \omega_{i+1}^{(2n)} \left( f^{(2n)}, \|\theta_N\| \right)_{w_{\alpha,\beta},p}.
\]

Corollary 4.3. \((s \geq 1)\) Suppose that \( Y_s \in \mathbb{Y}_s \), \( \sigma, s, n \in \mathbb{N} \) and \( \sigma \neq 4 \). If \( f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta}) \), then

\[
\mathcal{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_p) \leq c(\delta_e) \times n^{-\sigma} \omega_{i+1,r}^{(r)} \left( f^{(r)}, \|\theta_N\| \right)_{w_{\alpha,\beta},p}
\]

and

\[
\mathcal{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_p) \leq c(\eta, J_{j,n}) \times n^{-\sigma} \omega_{i+2\eta,i+\eta}^{(i+\eta)} \left( f^{(i+\eta)}, \|\theta_N\| \right)_{w_{\alpha,\beta},p}.
\]

Acknowledgements. The first author is indebted to Prof. Eman Samir Bhaya (University of Babylon) for useful discussions of the subject. The first author is supported by University of Al-Muthanna while studying for his Ph.D. We would like to thank Universiti Utara Malaysia (UUM) for the financial support. The corresponding author is Nazihah Ahmad.
Equivalence of Weighted DT-Moduli of (Co)convex Functions

References

[1] M. Al-Muhja, H. Akhadkulov, N. Ahmad, Derivation of new Degrees for Best (Co)convex and Unconstrained Polynomial Approximation in $L^p_{\alpha,\beta}$ space: I, Int. J. Psychosocial Rehabilitation, 24, (2020), 2866–2886.

[2] M. Al-Muhja, H. Akhadkulov, N. Ahmad, Estimates for constrained approximation in $L^p_{\alpha,\beta}$ space: Piecewise polynomial, Int. J. Math. Comput. Sci., 16, no. 1, (2021), 389–406.

[3] M. Al-Muhja, A Korovkin type approximation theorem and its applications, Abstract and Applied Analysis, 2014, DI 859696, 6 pages.

[4] M. Al-Muhja, A. Almyaly, On extending the domain of (co)convex polynomial, (to appear), https://arxiv.org/abs/2001.07545

[5] R. DeVore, G. Lorentz, Constructive approximation: A Series of Comprehensive Studies in Mathematics, 303, (1993), Springer-Verlag, New York, NY.

[6] Z. Ditzian, V. Totik, Moduli of smoothness: Springer Series in Computational Mathematics, 9, Springer-Verlag, New York, NY, 1987.

[7] H. Dzyubenko, V. Zalizko, Pointwise estimates for the coconvex approximation of differentiable functions, Ukrainian Mathematical Journal, 57, (2005), 52–69.

[8] K. Kopotun, D. Leviatan, I. Shevchuk, New moduli of smoothness. Publications de l’Institut Mathématique (Beograd), 96, (2014), 169–180.

[9] K. Kopotun, D. Leviatan, I. Shevchuk, New moduli of Smoothness: Weighted DT Moduli Revisited and Applied, Constructive Approximation, 42, (2015), 129–159.

[10] K. Kopotun, D. Leviatan, I. Shevchuk, On moduli of smoothness with Jacobi weights, Ukrainian Mathematical Journal, 70, (2018), 379–403.

[11] K. Kopotun, D. Leviatan, I. Shevchuk, On some properties of moduli of smoothness with Jacobi weights, Topics in Classical and Modern Analysis, Applied and Numerical Harmonic Analysis, Springer, (2019), 19–31.
[12] K. Kopotun, D. Leviatan, I. Shevchuk, On equivalence of moduli of smoothness of splines in $L_p$, $0 < p < 1$, J. Approximation Theory, 143, (2006), 36–43.

[13] K. Kopotun, On $k$-monotone interpolation, In: Advances in Constructive Approximation, Proc. Int. Conf. (M. Neamtu, ed.), Nashville, TN, USA, 2003, Brentwood, TN: Nashboro Press, Modern Methods in Mathematics, (2004), 265–275.

[14] K. Kopotun, On moduli of smoothness of $k$-monotone functions and applications, Math. Proc. Cambridge Philos. Soc., 146, (2009), 213–223.

[15] K. Kopotun, B. Popo, Moduli of smoothness of splines and applications in constrained approximation, Jean Journal on Approximation, 2, (2010), 79–91.

[16] K. Kopotun, Univariate splines: Equivalence of moduli of smoothness and applications. Mathematics of Computation, 76, (2007), 931–945.

[17] K. Kopotun, Whitney theorem of interpolatory type for $k$-monotone functions, Constructive approximation, 17, (2001), 307–317.

[18] K. Kopotun, Weighted moduli of smoothness of $k$-monotone functions and applications, J. Approximation Theory, 192, (2015), 102–131.

[19] D. Leviatan, I. Shevchuk, Some positive results and counterexamples in comonotone approximation, II, J. Approximation Theory, 100, (1999), 113–143.

[20] B. Mordukhovich, N. Nam, An easy path to convex analysis and applications, Morgan and Claypool Publishers, 2014.

[21] H. Salzer, Lagrangian interpolation at the Chebyshev points $x_{n,v} \equiv \cos(\frac{\pi v}{n})$, $v = o(1)n$; some unnoted advantages, Comput. J., 15, (1972), 156–159.