Separation of the monopole contribution to the nuclear Hamiltonian

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It is shown that the nuclear Hamiltonian can be separated rigorously as $H = H_m + H_M$. For sufficiently smooth forces the monopole part $H_m$ is entirely responsible for Hartree-Fock selfconsistency and hence saturation properties. The multipole part $H_M$ contains the “residual” terms - pairing, quadrupole, etc. - that will be analyzed in a companion paper. We include a review of the basic results often needed when working with multipole decompositions and average monopole properties.

21.60.Cs, 21.60.Ev, 21.30.xy

It has not been possible, yet, to construct interactions that could satisfy simultaneously three basic conditions:

A) to be realistic, i.e. consistent with the nucleon-nucleon (NN) phase shifts,

B) to ensure good saturation properties, i.e. correct binding energies at the observed radii,

C) to provide good spectroscopy.

As a consequence many forces have been designed for specific contexts or problems: pairing plus quadrupole \[12\], density dependent potentials for mean-field approaches \[34\], Landau-Migdal parametrizations for studies of the giant resonances \[4\], direct fits to two-body matrix elements for shell-model calculations \[5\] and many others. A way out of this unsatisfactory state of affairs would be to exhibit an interaction consistent with conditions A,B,C above. If we only assume that it exists and could be reduced to an effective form smooth enough to do Hartree-Fock (HF) variation, many of its properties can be discovered. The basic tool we shall need is the following separation property:

Given a sufficiently smooth Hamiltonian $H$, it can be separated as $H = H_m + H_M$. Only the monopole field $H_m$ is affected by spherical Hartree-Fock variation. Therefore it is entirely responsible for global saturation properties and single particle behaviour.

As long as we do not find ways of reconciling conditions A and B - a major problem - $H_m$ must be treated phenomenologically. The phenomenological enforcement of good saturation properties is quite feasible in a shell-model context and leads to the pleasing result that the multipole part $H_M$ - which can be extracted rather uniquely from the realistic interactions - has an excellent behaviour \[66\]. Therefore conditions A and C, as well as B and C, are mutually compatible.

An elementary argument explains the situation. The observed nuclear radii, $r \approx 1.2 A^{1/3}$ fm, imply average interparticle distances of some 2.4 fm, and therefore the nucleons “see” predominantly the medium range of the potential. This is a region that is well understood theoretically \[3\] and well described by the realistic forces.

The picture that emerges is quite simple: the short range part of the $NN$ interaction is not well understood, but it is certainly responsible for the repulsive terms necessary for saturation. Therefore, with our present knowledge of the nuclear forces, the phenomenological treatment of $H_m$ is a necessity, but it may perhaps lead to some fresh ideas. The argument is that a Fock representation that can accommodate all possible non-relativistic interactions. Although in principle enormously many matrix elements have to be specified, $H_m$ is described by a small fraction of the total and by some formal manipulations we shall be able to isolate the very few that count (essentially the ones describing the bulk properties of nuclear matter). Therefore it may possible to exhibit some simple $H_m$ that contains few parameters and describes nuclear data satisfactorily. Since something simple in Fock space may be complicated in coordinate space, it may point to ways - so far overlooked - of reconciling conditions A and B.

There are three steps in the task of giving a complete characterization of $H$.

1. The first is technical: we must prove the separation property, and prepare the tools that make it possible to take the greatest advantage of the underlying symmetries: angular momentum, isospin and the unitary symmetries of the monopole world.

2. As mentioned, we know much about $H_M$, but this knowledge comes in huge arrays of matrix elements. It would be far more useful if we could extract from this mass of numbers the truly important ones.

3. It remains to specify the monopole field.

We shall deal with the first point here, and with the second in a companion (next) paper \[10\] where it is shown that $H_M$ is indeed simple and a hint emerges about $H_m$. The hint has been taken up in \[11\], a study of nuclear masses that provides a first approximation to point 3.

Our main purpose now is to prove the separation property of the monopole field. The proof is far more compact if multipole (i.e. angular momentum) techniques are introduced. Furthermore, they are essential for the next paper.
French lectures [12] are the fundamental reference on the subject and - supplemented by a good book on angular momentum [13] - they contain most of the results we may need. In assembling them in the first section and the appendix, we have tried to give a self-contained account, and compress in a few pages many of the things of use in daily shell model practice, putting some emphasis on the connection between coupled and uncoupled representations, and on questions of phase.

The second section deals with separation and HF properties and it aims at making the reader familiar with basic technical and conceptual aspects of the monopole field.

Notations are regrouped at the end of the appendix.

I. BASIC COUPLINGS AND REPRESENTATIONS

This section deals with angular momentum coupling and recoupling of operators following the techniques of French. There is a little here that is not adapted directly from [12]: some changes in notation, the use of scalar product whenever possible, a lesser reluctance to go over the \( m \)-scheme for simple operations and the introduction of the two possible analogues of hermitean conjugation in the coupled schemes.

The reader is reminded that in the Appendix, eqs.(A16-21) contain the elementary formulae that may be needed in handling \( 3j \) and \( 6j \) symbols.

A. Technical preliminaries

The uncoupled representation or \( m \)-scheme

We shall work in spaces containing \( D \) orbits labeled \( i, j, k, \ell \ldots \) each associated with operators \( a_i^+, a_i \) that act on a vacuum \( |0> \)

\[
a_i^+ |0> = |i> , a_i |0> = 0 , \{ a_i, a_j^+ \} = a_i a_j^+ + a_j^+ a_i = \delta_{ij}
\] (1.1)

For a given number of particles \( n \), we can construct \( \binom{D}{n} \) different states, each associated with a state operator \( Z^+ \):

\[
Z^+([i], n)|0> = a_i^+ a_{i_2}^+ \ldots a_{i_n}^+ |0> = |[i], n> , i_1 < i_2 < \ldots < i_n \] (1.2)

Operators of rank \( k \) take the form (we use * for Hermitean - or complex - conjugation so as to reserve the + superscript to creation operators)

\[
R_k = \sum_{i,i'} <[i] k | R_k | [i'] k> > Z^+([i], k) Z([i'], k) , \quad \quad Z([i'], k) = (Z^+([i'], k))^* \] (1.3)

We shall be mainly interested in the \( k = 1 \) case, in the Hamiltonian of rank 1 + 2 (kinetic and potential energies \( K \) and \( V \) respectively):

\[
\mathcal{H} = K + V = \sum_{i,j} K_{ij} a_i^+ a_j - \sum_{k<\ell} V_{ijk\ell} a_i^+ a_j^+ a_k a_{\ell}
\]

\[
K_{ij} = \langle i | K | j > , \quad V_{ijk\ell} = \langle \frac{i j - j i}{\sqrt{2}} | V | \frac{k \ell - \ell k}{\sqrt{2}} > ,
\] (1.4)

and in the special \( k \)-body operator

\[
D_k = \sum_i Z^+([i], k) Z([i], k) = \frac{n(n-1) \ldots (n-k+1)}{k!} = \frac{n^{(k)}}{k!} \equiv \binom{n}{k}
\] (1.5)

whose explicit form is dictated by its rank and the fact that it has eigenvalue 1 when acting on any state of \( k \) particles.

The particle hole (ph) transformations

\[
\begin{align*}
a_i^+ &= \bar{a}_i , & a_i &= \bar{a}_i^+ & \text{if } i \leq i_f \\
a_i^+ &= \bar{a}_i^+, & a_i &= \bar{a}_i & \text{if } i > i_f
\end{align*}
\] (1.6)
consist in interchanging creation and annihilation operators for some orbits, by interchanging vacua

\[ |0 > \rightarrow |0 > = Z^+(s)|0 >, \quad i \leq i_f. \quad (1.7) \]

In its simplest form, \( f = D \) and all operators have to be brought to normal order. In particular

\[
V = - \sum_{i,k} V_{ijkl} a_{ik} a_j a_k a_l = - \sum_i a_i a_j a_i a_j + \delta_{ij}\delta_{jk} \delta_{il} - \delta_{ik}\delta_{lj} - a_i a_j a_i a_j + \delta_{ik}\delta_{lj} + a_i a_j a_i a_l
\]

(1.8)

and

\[
\bar{n} = D - n, \quad \sum Z([i],k)Z^+([i],k) = \sum \bar{Z}^+([i],k)\bar{Z}([i],k) = \left( \bar{n} \right)
\]

(1.9)

\section*{Coupled representations}

To take advantage of the basic symmetries, operators have to be coupled to good angular momentum \( J \) and isospin \( T \). The \( m \)-scheme labels \( i, j \ldots \) will be replaced by pairs \((r, r_z)\), \((s, s_z)\ldots\) where \( r \) specifies the subshell to which the orbit belongs, and \( r_z \) its projection quantum numbers. Following French [12], we shall introduce a product notation in which expressions are independent of the coupling scheme. In \( jt \) formalism, for example, a single tensorial index will represent pairs in spin-isospin (\( JT \)) space

\[
\Gamma \equiv JT, \quad \Gamma_z \equiv MT_z, \quad r \equiv j, \frac{1}{2}, \quad r_z \equiv m_r \tau_z, \quad \text{etc.}
\]

(1.10)

(Note that \( r \) as tensor index has not exactly the same meaning as label of shell \( r \) (i.e. \( r \equiv j, p_r \), where \( p_r \) is the principal quantum number). No confusion can possibly arise from this convention.)

Expressions involving these indices will stand for products as in

\[
(-)^\Gamma_z = (-)^{M + T_z}, \quad (-)^r = (-)^{j + 1/2}, \quad [r] = 2(2j + 1), \quad [\Gamma] = [JT] = (2J + 1)(2T + 1),
\]

(1.11)

more generally:

\[
U(\Gamma \text{ space}) = U(J \text{ space})U(T \text{ space}),
\]

(1.12)

where \( U \) may be some \( 6j \) or Clebsh-Gordan coefficient or similar functions as in

\[
< \gamma \gamma_z \gamma_z' \Gamma \Gamma_z > = < j m j m' j M > < t t_z t_z' | T T_z >
\]

(1.13)

In \( j \) formalism, also called neutron-proton (np), we do not couple explicitly to good \( T \), the tensorial indeces refer to a single space , and the identifications are \( \Gamma = J, r \equiv j, [r] = (2j + 1) \) etc. (Note that when used as label, \( r \) must specify whether the shell is a neutron or a proton one). In \( \ell s \) and \( \ell s t \) formalisms, we have \((L)(S)\) and \((L)(S)(T)\) product spaces respectively, and the necessary identifications are as long as we do not recouple \( L \) and \( S \) explicitly to good \( J \).

Let us introduce the coupled and \textbf{normalized} two body state operators written in terms of \textit{ordered} pairs \((a^+_i a^+_j, i < j)\):

\[
Z^+_{\Gamma_z, rs}(rs) = \zeta^{-1}_{rs} \sum_{(r_z)} < rr_z ss_z | \Gamma \Gamma_z > a^+_{rr_z} a^+_{ss_z}, \quad Z_{\Gamma_z, rs}(rs) = (Z^+_{\Gamma_z, rs}(rs))^*
\]

\[
a^+_{rr_z} a^+_{ss_z} = \zeta^{-1}_{rs} \sum_{(\Gamma)} < rr_z ss_z | \Gamma \Gamma_z > Z^+_{\Gamma_z, rs}(rs), \quad \zeta_{rs} = (1 + \delta_{rs})^{-1/2}
\]

(1.14)

\((r_z)\) means that the sum is restricted to \( r_z < 0 \) if \( \delta_{rs} = 1 \)

\((\Gamma)\) means that \( \Gamma \) must be such that \((-)^{\Gamma + 2r} = -1 \) if \( \delta_{rs} = 1 \)

We can always relax the ordering through

\[
\zeta^{-1}_{rs} \sum_{(r_z)} = \zeta_{rs} \sum_{r_z}
\]

Let us make the following identifications

\[
i \equiv (r, r_z), \quad j \equiv (s, s_z), \quad k \equiv (t, t_z), \quad \ell \equiv (u, u_z)
\]

(1.15)
and take advantage of the ordering in eq.\[(1.14)\] to write directly

\[
V = \sum_{i<j}^{R \cup \ell} V_{ijkl} Z^+(ij)Z(k\ell) = \sum_{r<s \leq u(t)} V^T_{rstu} Z^+_{\Gamma} (rs) \cdot Z_{\Gamma} (tu) \tag{1.16a}
\]

\[
V_{ijkl} = \sum_{(r)\{s\} (t)u} V^T_{rstu} \zeta^{-1} \zeta^{-1} < rrszs_z | \Gamma\Gamma_z > < ttzzu | \Gamma\Gamma_z > \tag{1.16b}
\]

\[
V^\Gamma_{rstu} = V^T_{rstu} = \sum_{(r)\{s\} (t)z} V_{ijkl} \zeta^{-1} \zeta^{-1} < rrszs_z | \Gamma\Gamma_z > < ttzzu | \Gamma\Gamma_z > \tag{1.16c}
\]

\[
V^\Gamma_{rstu} = V^T_{rstu} = -(-)^{r+s-t} V_{srut} = (-)^{r+s+t+u} V^\Gamma_{srut} = -(-)^{t+u-\Gamma} V^T_{srut} \tag{1.16d}
\]

Rotational and isospin invariance are made manifest through the scalar products, which in angular momentum manipulations must be rewritten as zero coupled pairs. There are two natural ways of doing so, depending on the order of the couplings:

\[
Z^+_{\Gamma} \cdot Z_{\Gamma} = \sum_{\Gamma_z} Z^+_{\Gamma_z} Z_{\Gamma_z} = \sum_{\Gamma_z} Z^+_{\Gamma_z} Z_{\Gamma_z-\Gamma_z} (-)^{\Gamma-\Gamma_z} = [\Gamma]^{1/2} (Z^+ Z^-)^0, \text{ or} \tag{1.17a}
\]

\[
Z^{-}_{\Gamma} \cdot Z^+_{\Gamma} = \sum_{\Gamma_z} Z_{\Gamma_z} Z^+_{\Gamma_z} = \sum_{\Gamma_z} Z_{\Gamma_z} Z_{\Gamma_z+\Gamma_z} (-)^{\Gamma+\Gamma_z} = [\Gamma]^{1/2} (Z^- Z^+)^0, \tag{1.17b}
\]

where \( Z \) has been replaced by the most convenient tensor, which is either \( Z^- = \bar{Z}^+ \), the conjugate of \( Z^+ \), or \( Z^- = Z^{-} \), the adjoint of \( Z^+ \). The existence of these operators follows from the commutation rules of \( J \) and \( T \) with any irreducible tensor \( P^\gamma_{\gamma_z} \):

\[
(P^\gamma_{\gamma_z})^* = (-)^{\gamma+\gamma_z} \bar{P}^{-\gamma}_{-\gamma_z} = (-)^{\gamma-\gamma_z} \bar{P}^\gamma_{-\gamma_z}. \tag{1.18}
\]

To stress the unavoidable ambiguity in sign, we have taken the unusual step of introducing two operators, the adjoint \( \bar{P} \) and conjugate \( \bar{P}^\gamma \), that are the inverse of one another:

\[
\bar{P}^\gamma = (-)^{2\gamma} \bar{P}^\gamma, \quad \bar{P}^{-\gamma} = (-)^{2\gamma} P^\gamma, \quad \bar{P} = P\gamma, \tag{1.19}
\]

If \((-)^{2\gamma} = 1\) the two operators are identical. If \((-)^{2\gamma} = -1\), it becomes impossible to ensure - with a single definition - the same sign for scalar product and zero coupling of operators. Which version we choose depends on the problem at hand. For normal ordering as in eq.\[(1.17a)\], and in particular

\[
n_{r} = a^+_{r} \cdot a_{r} = [r]^{1/2} (a^+_{r} a_{r})^0, \tag{1.20}
\]

the natural choice of basic tensors is \( a^+_{r} \) and \( a^+_{r} \), for which we adopt the notation of French:

\[
A_{rrz} = a^+_{r} z_{r} \quad B_{rrz} = a^+_{r} z_{r} = (-)^{r+rz} a_{r-rz}. \tag{1.21}
\]

With antinormal ordering as in eq.\[(1.17b)\], needed in ph transforms we find

\[
Z^+_{\Gamma} \cdot Z^+_{\Gamma} = [\Gamma]^{1/2} (Z^+ Z^+)^0 \equiv [\Gamma]^{1/2} (\bar{Z}^+_{\Gamma} \bar{Z}^-_{\Gamma})^0 \tag{1.22}
\]

which simply amounts to a notation

\[
Z^+_{\Gamma} = \bar{Z}^+_{\Gamma}, \quad Z^+_{\Gamma} = \bar{Z}^-_{\Gamma}, \quad \text{as the bar over the operators means ph transform. But it also means adjoint, and indeed:} \quad \bar{Z}^+_{\Gamma} = Z^+_{\Gamma} \text{ by definition and } \bar{Z}^-_{\Gamma} = Z^+ + Z^+ \text{ by definition of } Z^- \text{ and by the last equality in } \[(1.18)\]. Therefore the ph transform of a state operator is its adjoint. The antitransform of a state operator is its conjugate by exactly the
same argument. Note that for the uncoupled operators ph transform is Hermitean conjugation, better represented by its dual analogues in (1.18) than by the arbitrary choice of one of them. From (1.22) we have the basic transforms:

\[ A_{rrs} = a_{rrs}^- = (-)^{2r} B_{rrs} \]

(1.23)

The coupled operators quadratic in \( A \) and \( B \) are

\[ X^+_{\Gamma \gamma}(rs) = (A_r A_s)_{\Gamma_z}, \quad X^-_{\Gamma \gamma}(rs) = (B_r B_s)_{\Gamma_z}, \quad S^+_{\gamma}(rt) = (A_r B_t)_{\gamma_z} \]

(1.24)

Obviously \( Z^+_{\Gamma \gamma}(rs) = \zeta_{rs} X^+_{\Gamma \gamma}(rs) \). From the easily proved identity

\[ (P^\gamma Q^\gamma)^\Gamma = (-)^{\gamma + \gamma'} \Gamma (Q^\gamma P^\gamma)^\Gamma \]

(1.25)

(equally valid for conjugation), we obtain

\[ \overline{X}^+_{\Gamma}(rs) = -X_{\Gamma}(rs) \quad S^-_{\gamma}(rt) = (-)^{t - r - \gamma} S^+_{\gamma}(rt). \]

(1.26)

For reduced matrix elements we use Racah’s definition

\[ < \alpha \zeta | P^\gamma \beta \zeta > = < \zeta | \gamma > \left( \begin{array}{cc} \alpha & \gamma \\ -\alpha & \beta \end{array} \right) < \alpha \gamma | \beta >. \]

(1.27)

For any operator \( P^\gamma \) it is true that

\[ < \alpha \zeta | P^\gamma \beta \zeta > = < \beta \zeta | (P^\gamma)^* \alpha \zeta >^* \]

(1.28)

and by applying (1.27) to both sides it follows that

\[ < \alpha \gamma | P^\gamma \beta > = (-)^{\alpha - \beta - \gamma} < \beta \gamma | P^\gamma \alpha >, \]

(1.29)

where we have omitted complex conjugation on the rhs because our reduced matrix elements will be real.

The coupled form of a rank 1 operator is deduced from the uncoupled one in (1.3) by using (1.27) and the definition of \( S^\gamma \) in (1.24):

\[ R^\gamma_{\gamma_z}(rt) = \sum_{r,t,r_z} < rr_z | R^\gamma_{\gamma_z} | tt_z > a^+_{rr_z} a_{tt_z} = \sum_{r,t} < r \gamma | R^\gamma | t > (\gamma)^{-1/2} S^\gamma_{\gamma_z}(rt). \]

(1.30)

We can always rewrite an arbitrary \( R^\gamma \) in terms of the symmetric (S) and antisymmetric (A) operators

\[ S^\gamma_{\gamma_z}(rt) = S^\gamma_{\gamma_z}(rt) + (-)^{r-t} S^\gamma_{\gamma_z}(tr), \quad A^\gamma_{\gamma_z}(rt) = S^\gamma_{\gamma_z}(rt) - (-)^{r-t} S^\gamma_{\gamma_z}(tr), \]

(1.31)

Then calling \( R^\gamma_{rt} = < r \gamma | R^\gamma | t > (\gamma)^{-1/2} \), the expansion becomes

\[ R^\gamma_{\gamma_z} = \frac{1}{2} \sum_{r,t} \left[ (R^\gamma_{tr} + (-)^{r-t} R^\gamma_{rt}) S^\gamma_{\gamma_z}(rt) + (R^\gamma_{tr} - (-)^{r-t} R^\gamma_{rt}) A^\gamma_{\gamma_z}(rt) \right]. \]

(1.32)

The \((-)^{r-t}\) phase ensures that the spherical harmonics are symmetric if \( Y_{lm}^* = (-)^m Y_{l-m} \) (Condon and Shortley’s choice), which leads to \( < r \gamma | Y_{l-m} | t > = (-)^{r-t} < t | Y_{l-m} | r > \) and vanishing of the \( A \) term. The phase convention should be changed to \((-)^{r-t} \) if \( Y_{l-m} \rightarrow i^t Y_{l-m} \). Then \( Y_{lm}^* = (-)^{l+m} Y_{l-m} \), and \( < r \gamma | Y_{l-m} | t > = (-)^{r-t} < t | Y_{l-m} | r > \). Note that now \( Y_{l} \cdot Y_{l} = |l|^{1/2}(Y_{l} Y_{l}) \) instead of the usual \( Y_{l} \cdot Y_{l} = (-)^{l} |l|^{1/2}(Y_{l} Y_{l}) \) but \( J \cdot J = -\sqrt{3}(J \cdot J)^{0} \) and \( T \cdot T = -\sqrt{3}(T \cdot T)^{0} \) always, since \( J \rightarrow i J \) makes no sense. Still, positive definite zero coupling for tensors of integer rank could be obtained with the change \( < t \ell m | t > = (-)^{-m} |t|^{-1/2} \) to \((-)^m |t|^{-1/2} \). As we have seen, for half integer rank, no convention will ensure that zero coupling is always definite positive.

Recoupling.

Let us consider a ph transformation (1.0) and using the identification (1.13) assume that orbits \( j = (ssz) \) and \( k = (ttz) \) are transformed while \( i \) and \( \ell \) (i.e. \( r \) and \( u \)) are left untouched. Then we have

\[ V = \sum_{i \leq j \leq k} V_{ijk \ell} a_i^+ a_k a_j^+ a_{\ell} = \sum_{i \leq j \leq k} V_{ijk \ell} a_i^+ a_{\ell} \delta_{jk}. \]

(1.33)
Now use \((1.16)\), relax the ordering for projections in the contraction in \((1.33)\), and introduce the \(A B\) tensors, permuting the middle ones

\[
V = \sum V_{rstu}^\Gamma \left[ |\Gamma \rangle \right]^{1/2} \zeta_{rs} \zeta_{tu} \left\{ \left( A_r \left[ B_t A_s \right] \right)^\Gamma B_u \right\}^\Gamma 0 - \sum_{r<s} <rr_z ss_z|\Gamma \Gamma_z > <ss_z uu_z|\Gamma \Gamma_z > a_{rr_z}^+ a_{uu_z}.
\]

\[
(1.34)
\]

For clarity the internal couplings are indicated by different notations: \(( )^\Gamma\) and \([ | ]^\Gamma\). Recoupling through a normalized \(9j\) symbol yields

\[
\left\{ \left( A_r \left[ B_t A_s \right] \right)^\Gamma B_u \right\}^\Gamma = \sum_{\gamma} [\Gamma \gamma] \left\{ r s \left[ t u \right] \Gamma \right\} \left( S_{rt}^\gamma S_{su}^\gamma \right)^0.
\]

\[
(1.35)
\]

The contraction in \((1.34)\) can be calculated to be

\[
\sum_{r<s} <rr_z ss_z|\Gamma \Gamma_z > <ss_z uu_z|\Gamma \Gamma_z > (-)^{u+s-r} = \delta_{ur} \delta_{u_z r_z} (-)^{u+s-r} \left[ \Gamma \right]_{r}^{1/2},
\]

\[
(1.36)
\]

obtained by reorganizing the couplings with the help of \(3-j\) symbols. All these operations can be summed up by

\[
-(X_{t}^+(rs) X_{r}^+(tu))^0 = \sum_{\gamma} [\Gamma \gamma]^{1/2} (-)^{u+t-\gamma-r} \left\{ r s \left[ t u \right] \Gamma \right\} \left( S_{rt}^\gamma S_{su}^\gamma \right)^0
\]

\[
(1.37)
\]

and its inverse

\[
(S_{rt}^\gamma S_{su}^\gamma)^0 = - \sum_{\Gamma} [\Gamma \gamma]^{1/2} (-)^{u+t-\gamma-r} \left\{ r s \left[ t u \right] \Gamma \right\} (X_{r}^+(rs) X_{r}^+(tu))^0
\]

\[
(1.38)
\]

where the first term can be written by inspection but the contraction needs some care. The hybrid procedure we have adopted - of mixing \(a^+, a\) and \(A, B\) - points to a paradox: French’s notation and techniques simplify many complex coupling problems but complicate a few simple ones. (Hint: try to obtain \((1.37)\) and \((1.38)\) using the - impressive - artillery in Hsu’s appendix to \([12]\); a good exercise).

B. The \(V\) and \(\omega\) representation of \(\mathcal{H}\)

Equations \((1.37)\) and \((1.38)\) were derived to prepare for a ph transform of some orbits. However, beyond the possibility of changing vacua - which may be quite useful occasionally - we are interested in the different representation(s) of the Hamiltonian that these operations entail. Accordingly, we shall always keep the untransformed notation \(a_{r}^+, a_r\) and \(A_r, B_r\) for the orbits.

First we consider the case in which all orbits are transformed. Introducing \(S_{ru} = a_{r}^+ \cdot a_u\), the coupled version of eq.\((1.38)\) becomes,

\[
V = - \sum_{r<s, \Gamma} V_{rstu}^\Gamma Z_{rt}^+(tu) \cdot Z_{r}^+(rs) - \sum_{r<s, \Gamma} |\Gamma \rangle V_{rstu}^\Gamma
\]

\[
(1.39)
\]

\[
- \sum_{r<s, \Gamma} \zeta_{rs} \zeta_{tu} \left[ |\Gamma \rangle \right] V_{rstu}^\Gamma \left( (-)^{u+s-r} \left[ S_{rt}^\gamma \right] \delta_{st} + \left[ S_{rt}^\gamma \right] \delta_{ru} - \left[ S_{rt}^\gamma \right] \delta_{su} - \left[ S_{rt}^\gamma \right] \delta_{ru} \right)
\]

\[
(1.38)
\]

where the first term can be written by inspection but the contraction needs some care. The hybrid procedure we have adopted - of mixing \(a^+, a\) and \(A, B\) - points to a paradox: French’s notation and techniques simplify many complex coupling problems but complicate a few simple ones. (Hint: try to obtain \((1.37)\) and \((1.38)\) using the - impressive - artillery in Hsu’s appendix to \([12]\); a good exercise).
All contractions are calculated using (1.33).
Next we examine the transformations associated to (1.37) and (1.38) in which only the middle operators are interchanged. It is convenient to allow for complete flexibility in the summations and we shall relax the restrictions $r \leq s$, $t \leq u$ by replacing the $\zeta_{rs}$ factors by the $P_{rs}$ convention

$$P_{rs} = (1 + \delta_{rs})^{-1/2} \text{ if } r \leq s, \quad P_{rs} = \frac{1 + \delta_{rs}}{2} \text{ if no restriction} \quad (1.40)$$

so that the sums could be interpreted as restricted or not restricted. We write therefore

$$V = \sum_{r \leq s \leq u} V_{rstu}^\Gamma Z^+_u \cdot Z_{tu} \Gamma = - \sum_{(rstu)\Gamma} P_{rs} P_{tu} [\Gamma]^{1/2} V_{rstu}^\Gamma (X_{rs}^+ X_{tu})^0 \quad (1.41)$$

and from now on we set $Z^+_u (rs) = Z^+_u$, etc.

According to (1.37), $V$ can be transformed into

$$V = \sum_{(rstu)\gamma} P_{rs} P_{tu} [\gamma]^{1/2} \omega_{rstu}^\gamma (S^+_r S^+_u)^0 + \delta_{st} \delta_{ru} [s]^{1/2} \omega^0_{rstu} S^0_u \quad (1.42)$$

where

$$\omega_{rstu}^\gamma = \sum_{(\Gamma)} (-)^{s+t-\gamma-\Gamma} \left\{ r \ s \ \Gamma \atop u \ t \ \gamma \right\} V_{rstu}^\Gamma \quad (1.43a)$$

$$V_{rstu}^\Gamma = \sum_{\gamma} (-)^{s+t-\gamma-\Gamma} \left\{ r \ s \ \Gamma \atop u \ t \ \gamma \right\} \omega_{rstu}^\gamma \quad (1.43b)$$

(Remember that $\sum_{(\Gamma)}$ means that we sum over Pauli allowed $\Gamma$). Eq. (1.38) suggests an alternative to (1.42)

$$V = \sum_{(rstu)\gamma} P_{rs} P_{tu} [\gamma]^{1/2} \omega_{rstu}^\gamma \left[ (S^+_r S^+_u)^0 - (-)^{\gamma+r-s} \frac{\gamma}{r} \right]^{1/2} \delta_{st} \delta_{ru} S^0_u \quad (1.44)$$

where each multipolarity $\gamma$ is associated with a two body operator. The obvious check that (1.44) is indeed (1.42) comes from

$$- \sum_{\gamma} [\gamma]^{1/2} \omega_{rstu}^\gamma (-)^{\gamma+r-s} \left[ \frac{\gamma}{r} \right]^{1/2} = [s]^{1/2} \omega^0_{rstu} \cdot (1.45)$$

The proof is left as an exercise (use (1.43a) and Racah sum rule (A.19)).

In a $Jt$ representation, by introducing explicitly the isospin in $\Gamma = JT$, $\gamma = \lambda \tau$, $r = j_r \frac{1}{2}$ etc and the $6-j$ values

$$\begin{bmatrix} 1/2 & 1/2 & T \end{bmatrix} = \begin{bmatrix} T \tau & 00 & 01 & 10 & 11 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 & 1/6 \end{bmatrix} \quad (1.46)$$

we find

$$\omega_{rstu}^{\lambda 0} = \frac{1}{2} \sum_{(J)} (-)^{j_r + j_s - \lambda - j} \left\{ j_r \ j_s \ j \atop j_r \ j_t \ \lambda \right\} [J] (V_{rstu}^{J0} + 3 V_{rstu}^{J1}) \quad (1.47a)$$

$$\omega_{rstu}^{\lambda 1} = \frac{1}{2} \sum_{(J)} (-)^{j_r + j_s - \lambda - j} \left\{ j_r \ j_s \ j \atop j_u \ j_t \ \lambda \right\} [J] (V_{rstu}^{J0} - V_{rstu}^{J1}) \quad (1.47b)$$

and reciprocally

$$V_{rstu}^{J0} = \frac{1}{2} \sum_{\lambda} (-)^{j_r + j_s - \lambda - j} \left\{ j_r \ j_s \ j \atop j_u \ j_t \ \lambda \right\} [\lambda] (\omega_{rstu}^{\lambda 0} + 3 \omega_{rstu}^{\lambda 1}) \quad (1.48a)$$

$$V_{rstu}^{J1} = \frac{1}{2} \sum_{\lambda} (-)^{j_r + j_s - \lambda - j} \left\{ j_r \ j_s \ j \atop j_u \ j_t \ \lambda \right\} [\lambda] (\omega_{rstu}^{\lambda 0} - \omega_{rstu}^{\lambda 1}) \quad (1.48b)$$

When the Hamiltonian is written as in (1.41) we speak of the normal or $V$-representation while we call the form (1.42) multipole, or $\omega$-representation.
II. THE MONOPOLE FIELD $\mathcal{H}_M$ AND THE SEPARATION $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_M$

Before going into the technical problems, it is worth explaining how the monopole field appears. Several of the statements that follow need a formal proof. It will be supplied in the body of the section.

The separation of the Hamiltonian into an “unperturbed” and a “residual” part, $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}'$, is at the heart of many-body physics, and the idea that $\mathcal{H}_0$, could be represented by some central - i.e. single particle - field is of great heuristic and qualitative value. However, to decide whether it makes sense quantitatively, we must understand how $\mathcal{H}_0$, of rank-2, can be approximated by a rank-1 operator.

What can be done cleanly is to define $\mathcal{H}_0$ as $K$ plus a two body part so that

$$\mathcal{H} = K + \sum_{r \leq s} V_{rs} n_r(n_s - \delta_{rs})/(1 + \delta_{rs}) + \mathcal{H}_0. \quad (2.1)$$

$\mathcal{H}_0$ has two equivalent properties:

- Complete Extraction. When written in a multipole representation $\mathcal{H}'$ contains no number operators. It means that all the $\lambda = 0$ terms that are diagonal in the basis we are using have migrated to $\mathcal{H}_0$.

- Trace Equivalence. The expectation value of $\mathcal{H}_0$ for any basic state is the average energy (i.e. the trace) of the configuration to which it belongs. (A configuration is the set of states with fixed number of particles in each orbit).

To recover and examine the notion of central field we choose as vacuum some determinantal state, i.e. we set $n_r = D_r$, below the Fermi orbit and $n_r = 0$ above. The single - particle and single - hole energies are then calculated from eq.(2.2), by adding or removing a particle from the vacuum. In a HF calculation these energies play an important role and invite the interpretation that the determinantal state generates a mean field in which the particles move.

This interpretation is very good when there is a dominant agent in the Hamiltonian that is responsible for most of the field, as in atoms and planets.

In nuclei it is simply wrong. What the HF calculation produces is a basis of orbitals. When $\mathcal{H}$ is written in that basis, $\mathcal{H}_0$ can be extracted, and it will yield indeed the HF value for the vacuum and single fermion energies. However, there is no reason to linearize $\mathcal{H}_0$ (i.e. to approximate it by a central field) when estimating energies of other configurations: the quadratic effects grow fast. To fix ideas: typically $V_{rs} \approx 300$ keV in the pf shell, not a large number, but it multiplies the $n_r n_s$ operator that may become $O(10)$ or even $O(100)$, and drastically modify the effects of the central field.

There is much empirical evidence that at fixed total number of particles it is a good approximation to keep the same basis for all orbits in the vicinity of the Fermi level, and therefore $\mathcal{H}_0$ is fixed. However we also know that, when we change the number of particles, the orbitals and therefore $\mathcal{H}_0$ must evolve.

Since the evolution of the orbitals is associated to unitary transformations of the basis, the $n_r$ operators become linear combinations of $S^0_{tu}$ ones. Therefore we must generalize the definition of $\mathcal{H}_0$, by extending the notion of “complete extraction” to all $\lambda = 0$ operators, not only the diagonal ones. The resulting object is what we call $\mathcal{H}_m$. As the separation $\mathcal{H} = \mathcal{H}_m + \mathcal{H}_M$ is invariant (representation-independent), $\mathcal{H}_m$ provides the - mathematically and physically - natural definition of unperturbed Hamiltonian.

A. Separation of $\mathcal{H}_m$

Although there is only one $\mathcal{H}_m$, it will have different aspects in $jt$ and $j$ formalisms. We differentiate them by a tag:

$$\mathcal{H}_m = \mathcal{H}_{mT} = \mathcal{H}_{mnp},$$

$\mathcal{H}_{mT}$ is constructed by extracting all the possible $\gamma = 00$ and 01 contributions to eqs.(1.42) or (1.44), while $\mathcal{H}_{mnp}$ contains all the possible $\gamma = 0$ terms. Obviously the kinetic energy is part of $\mathcal{H}_m$, whose rank 2 component must have the form of the $\lambda = 0$ contributions to (1.44) which we call $V_{\lambda=0}$. To give them a transparent aspect we replace $S^0_{rs}$ and $S^0_{rs}$ by

$$S_{rs} = \hat{\delta}_{rs}[r]^{1/2} S^0_{rs}, \quad T_{rs} = \frac{1}{2} \hat{\delta}_{rs}[r]^{1/2} S^0_{rs} \quad (2.2)$$

which, for $\delta_{rs} = 1$, reduce to $S_{rr} = n_r$, $T_{rr} = T_r$. The two body operators

$$S_{rtsu} = \zeta_{rs} \zeta_{tu} (S_{rt} S_{su} - \delta_{st} S_{ru}) \quad (2.3a)$$

are
\[ T_{ruts} = \zeta_{rs} \zeta_{tu}(T_{rt} \cdot T_{su} - \frac{3}{4} \delta_{st} S_{ru}) \]  

(2.3b)

in turn become for \( \delta_{rt} = \delta_{su} = 1 \),

\[ S_{rrss} = n_r (n - \delta_{rs})/(1 + \delta_{rs}), \]

(2.4a)

\[ T_{rrss} = (T_r \cdot T_s - \frac{3}{4} n_r \delta_{rs})/(1 + \delta_{rs}). \]

(2.4b)

and the \( \lambda = 0 \) contribution to \( V \) can be written as

\[ V_{\lambda=0} = \sum_{r < s} [r s]^{-1/2} (\omega_{rtsu}^{00} S_{rtsu} - 4\omega_{rtsu}^{01} T_{rtsu}) \hat{\delta}_{rt} \hat{\delta}_{su}, \]

(2.5)

which follows directly from (2.3a), (2.3b) and (1.44) provided we remember that \( T_{rt} \cdot T_{su} = -\sqrt{3}(T_{rt} T_{su})^0 \).

Although the full rank-2 contribution to \( H_{mT} \) has the form \( V_{\lambda=0} \), the correct values of the \( \omega_{rtsu}^{00} \) and \( \omega_{rtsu}^{01} \) cannot be extracted from eqs.(1.47a) and (1.47b) as

\[ \omega_{rtsu}^{00} = [r s]^{-1/2} \sum_{(J)} [J](V_{rstu}^{00} + 3V_{rstu}^{11}) \hat{\delta}_{rt} \hat{\delta}_{su} \]  

(2.6a)

\[ \omega_{rtsu}^{01} = [r s]^{-1/2} \sum_{(J)} [J](V_{rstu}^{00} - V_{rstu}^{11}) \hat{\delta}_{rt} \hat{\delta}_{su}. \]  

(2.6b)

There are two reasons. One is the Pauli principle, which operates when \( r = s \) or \( t = u \), i.e. when \( \delta(P) = 1 \) with \( \delta(P) \) defined as

\[ \delta(P) = 1 - (1 - \delta_{rs})(1 - \delta_{tu}) \]  

(2.7)

A direct indication of the problem comes if we consider a single shell with a constant interaction, \( V_{rrrr}^T = 1 \). Then from (2.6)

\[ \sum Z_{r\tau \Gamma}^+ \cdot Z_{r\tau \Gamma}^0 = n_r (n_r - 1)/2; \]

but from (2.6a) we obtain \( \omega_{00} = (D_r - 1)/2 \) with \( D_r = |r| \), and inserting in (2.6),

\[ V_{\lambda=0} = ((D_r - 1)/2 D_r) n_r (n_r - 1)/2 \]

, indicating that we miss the correct result by over a factor 2. The remaining half must be found among the \( \omega^{\lambda} \) terms of \( V_{\lambda \neq 0} \), which in the case of \( \delta(P) = 1 \) cannot be linearly independent since they are twice as numerous as the \( V^T \) ones.

The other problem is related to exchange terms that show for \( j_r = j_s = j_t = j_u \) and \( \delta(P) = 0 \), i.e. when \( \delta(e) = 1 \) with

\[ \delta(e) \equiv (\hat{\delta}_{rs} - \delta_{rs})(\hat{\delta}_{tu} - \delta_{tu}), \]

(2.8)

Then, setting \( V_{rstu}^T = 1 \) in (2.6a) and (2.6b) and remembering that \( V_{rstu}^T = -(+)^{j T} V_{rstu}^T \) (since \( (-)^{2r} = 1 \) in \( j t \) formalism)

\[ \omega_{rstu}^{00} = -[r s]^{-1/2} \sum_{(J)} [J](-)^J (V_{rstu}^{00} - 3V_{rstu}^{11}) \]  

(2.9a)

\[ \omega_{rstu}^{01} = -[r s]^{-1/2} \sum_{(J)} [J](-)^J (V_{rstu}^{00} + V_{rstu}^{11}) \]  

(2.9b)

indicating that much \( \lambda = 0 \) monopole strength is hidden in the \( \omega_{rstu}^{\lambda T} \) terms with \( \lambda \neq 0 \).
1. The form of $H_{mT}$

To achieve complete extraction we specify $H_{mT}$ by defect, as suggested at the beginning of the section. It is $H_M$ that will be defined so as to contain no $V_{L=0}$ terms. To illustrate how the prescription works we consider first the $\delta(e) = 0$ case which is simpler.

Since $H_M$ has the form (1.41), let us replace the $V_{rstu}$ matrix elements by

$$ W_{rstu}^{JT} = V_{rstu}^{JT} - \nabla_{rstu}^{T} \hat{\delta}_{rt} \hat{\delta}_{su} \quad \text{if} \quad \delta(e) = 0, \tag{2.10} $$

with the “centroids” $\nabla_{rstu}^{T}$ defined so as to make $\omega_{rstu}^{0}$ vanish:

$$ \hat{\delta}_{rt} \hat{\delta}_{su} \left[ J \right] W_{rstu}^{JT} = 0 \implies \nabla_{rstu}^{T} = \sum_{(J)} \left[ J \right] V_{rstu}^{JT} / \sum_{(J)} \left[ J \right] \tag{2.11} $$

Then $H_{mT}$ is automatically defined in the $V$ representation as

$$ H_{mT}(\delta(e) = 0) = K + \sum_{\substack{r \leq s, \cr t \leq u,T}} \hat{\delta}_{rt} \hat{\delta}_{su} \sum_{(J)} Z_{rstu}^{T} \cdot Z_{tuJT} \tag{2.12} $$

The remaining task is to write the operators in $H_{mT}$ in terms of $S_{rstu}$ and $T_{rstu}$, which we do easily by calling upon eqs.(1.48a) and (1.48b), where ($\hat{\delta}_{rs} \hat{\delta}_{tu} = 1$ of course)

$$ \omega^{\lambda \tau} = \delta_{00} \delta_{\tau0} \implies V^{00}_{rstu} = V^{11}_{rstu} = [rs]^{-1/2} $$

$$ \omega^{\lambda \tau} = \delta_{00} \delta_{\tau1} \implies V^{01}_{rstu} = 3[rs]^{-1/2}, V^{10}_{rstu} = -[rs]^{-1/2} $$

which means

$$ S_{rstu} = \sum_{J} Z_{rsJ0}^{+} \cdot Z_{tuJ0}^{+} = \sum_{J} Z_{rsJ0}^{+} \cdot Z_{tuJ0}^{+} + \sum_{J} Z_{rsJ1}^{+} \cdot Z_{tuJ1} \tag{2.13a} $$

$$ T_{rstu} = \frac{3}{4} \sum_{J} Z_{rsJ0}^{+} \cdot Z_{tuJ0}^{+} - \frac{1}{4} \sum_{J} Z_{rsJ1}^{+} \cdot Z_{tuJ1} \tag{2.13b} $$

and inverting

$$ \sum_{J} Z_{rsJ0}^{+} \cdot Z_{tuJ0}^{+} = \frac{1}{4} (S_{rstu} - 4T_{rstu}), \tag{2.14a} $$

$$ \sum_{J} Z_{rsJ1}^{+} \cdot Z_{tuJ1} = \frac{1}{4} (3S_{rstu} + 4T_{rstu}). \tag{2.14b} $$

The exchange contribution to $H_{mT}$ is a bit trickier to extract, because when $\delta(e) = 1$, two extra operators have to be considered. They can be calculated from (2.14a) and (2.14b) by exchanging $t$ and $u$,

$$ \sum_{J} Z_{rsJ0}^{+} \cdot Z_{utJ0} = -\sum_{J} (-)^{J} Z_{rsJ0}^{+} \cdot Z_{tuJ0} = 4(S_{rstu} - 4T_{rstu}) \tag{2.15a} $$

$$ \sum_{J} Z_{rsJ1}^{+} \cdot Z_{utJ1} = \sum_{J} (-)^{J} Z_{rsJ1}^{+} \cdot Z_{tuJ1} = 4(3S_{rstu} + 4T_{rstu}) \tag{2.15b} $$

and by combining (2.14a), (2.14b), (2.15a) and (2.15b)

$$ \sum_{J} Z_{rsJ0}^{+} \cdot Z_{tuJ0} \frac{(1 \pm (-)^{J})}{2} = \frac{1}{8} ((S_{rstu} - 4T_{rstu}) \mp (S_{rstu} - 4T_{rstu})) \tag{2.16a} $$
Now it is possible to specify completely the matrix elements of $H$.

Through eqs. (2.14a), (2.14b), (2.16a), and (2.16b) we can obtain the form of $T$ and $r_s$ from which it follows that $\delta$ from (2.11) and its generalization to $j$ in the formalism $H_{mT}$ are written in terms of 4 scalars $S$ under another guise: neutron and proton shells are differentiated and the operators $\Omega = V_r T_T V_r T_T \sum \delta_r T_{rs} (1 - \delta(e)) V_r T_T T_{rstu} + \delta(e) V_r T_T T_{rstu}$, $\rho = \text{sign} (-)^j$

$$W_{rstu} = V_r T_T - \delta_r T_{rs} (1 - \delta(e)) V_r T_T + \delta(e) V_r T_T$$

from which it follows that $H_{mT}$ must be

$$H_{mT} = K + \sum_{r \leq s, t \leq u, \rho = \pm} \delta_r T_{rs} \delta_{st} (1 - \delta(e)) V_r T_T T_{rstu} + \delta(e) V_r T_T T_{rstu}$$

$$\Omega_{rstu} = \sum_J Z_{rsJT}^+ \cdot Z_{tuJT} \quad \Omega_{rstu}^\pm = \sum_J Z_{rsJT}^+ \cdot Z_{tuJT} \frac{(1 \pm (-)^j)}{2}$$

Through eqs. (2.14a), (2.14b), (2.16a), and (2.16b) we can obtain the form of $H_{mT}$ in terms of the monopole operators by regrouping the coefficients affecting each of them. To simplify the presentation we adopt the following convention

$$\{ \alpha \equiv rstu \quad r \leq s, t \leq u, \delta_r T_{rs} = 1, \quad \text{BUT} \quad S_\alpha = S_{rstu}, \quad S_\bar{\alpha} = S_{rust}, \quad T_\alpha = T_{rstu}, \quad T_\bar{\alpha} = T_{rstu} \}$$

then

$$H_{mT} = K + \sum_\alpha (1 - \delta(e)) (a_\alpha S_\alpha + b_\alpha T_\alpha) + \delta(e) (a_\alpha^d S_\alpha + b_\alpha^d T_\alpha + a_\alpha^e S_\alpha + b_\alpha^e T_\alpha), \quad \text{with} \quad a_\alpha = \frac{1}{4} (3V_{\alpha} + \bar{V}_{\alpha})$$

$$b_\alpha = \frac{1}{4} (V_{\alpha} - \bar{V}_{\alpha})$$

$$a_\alpha^d = \frac{1}{8} (3V_{\alpha}^+ + 3V_{\alpha}^- + \bar{V}_{\alpha}^0 + \bar{V}_{\alpha}^0)$$

$$a_\alpha^e = \frac{1}{8} (3V_{\alpha}^+ - 3V_{\alpha}^- - \bar{V}_{\alpha}^0 + \bar{V}_{\alpha}^0)$$

$$b_\alpha^d = \frac{1}{2} (3V_{\alpha}^+ + 3V_{\alpha}^- - \bar{V}_{\alpha}^0 - \bar{V}_{\alpha}^0)$$

$$b_\alpha^e = \frac{1}{2} (3V_{\alpha}^+ - \bar{V}_{\alpha}^0 - \bar{V}_{\alpha}^0)$$

From (2.11) and its generalization to $\delta(e) = 1$ we find for the centroids

$$V_{rstu}^T = \sum_{(J)} V_{rstu}^T [J] \cdot \sum_{(J)} [J] \cdot \sum_{(J)} [J] = \frac{1}{4} D_r (D_s + 2\delta(P)(-)^T) / \frac{1}{1 + \delta(P)}$$

$$V_{rstu}^\pm T = \sum_{(J)} V_{rstu}^J [J] (1 \pm (-)^j) / \sum_{(J)} [J] (1 \pm (-)^j)$$

$$\sum_{(J)} [J] (1 \pm (-)^j) = \frac{1}{4} D_r (D_r + 2)$$

$$D_r = [r], \quad \delta_r T_{rs} = 1 \text{ always; } \delta(e) = 1, \text{ for } V_{rstu}^\pm T$$

2. Form of $H_{mnp}$ and $H_m$

In $j$ formalism $H_{mnp}$ is $H_{mT}$ under another guise: neutron and proton shells are differentiated and the operators $T_{rs}$ and $S_{rs}$ are written in terms of 4 scalars $S_{rs,sy} : x, y = n$ or $p$. 

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We may also be interested in remaining in $jt$ formalism and extract only the purely isoscalar contribution to $\mathcal{H}_{mT}$, which we call $\mathcal{H}_{m0}$. The power of French’s product notation becomes particularly evident here, because the form of both terms is identical. It is left as an exercise to find

$$\mathcal{H}_{mnp} \text{ or } \mathcal{H}_{m0} = K + \sum_{\alpha} \left\{ V_\alpha S_\alpha (1 - \delta(e)) + \frac{1}{2}((V_\alpha^+ + V_\alpha^-) S_\alpha + (V_\alpha^- - V_\alpha^+) S_\alpha) \delta(e) \right\} \quad (2.21)$$

with

$$\nabla_{rstu} = \sum_{(\mathcal{I})} \frac{V_{rstu}^T[\mathcal{I}]}{\sum_{(\mathcal{I})} \mathcal{I}} \quad \sum_{(\mathcal{I})} \mathcal{I} = D_r(D_s - \delta(P))/(1 + \delta(P))$$

$$\nabla_{rstu}^\pm = \sum_{\mathcal{I}} V_{rstu}[\mathcal{I}] (1 \pm (-)^{r+2r}) \quad \sum_{\mathcal{I}} \mathcal{I} (1 \pm (-)^{r+2r}) = D_r(D_r \mp (-)^{2r}) \quad (2.22)$$

Of course we must remember that for $\mathcal{H}_{mnp}$ $D_r = 2j_r + 1$, $(-)^{2r} = -1$, $\Gamma \equiv J$, etc., while for $\mathcal{H}_{m0}$

$D_r = 2(2j_r + 1)$, $(-)^{2r} = +1$, $\Gamma \equiv JT$; etc.

It should be noted that $\mathcal{H}_{m0}$ is not obtained by simply discarding the $b$ coefficients in eqs. (2.19), because we can extract some $\gamma = 00$ contribution from the $T_\alpha$ operators. The point will become quite clear when considering the diagonal contributions.

### B. Diagonal forms of $\mathcal{H}_m$

We are going to specialize to the diagonal terms of $\mathcal{H}_{mT}$ and $\mathcal{H}_{mnp}$, which involve only $V_{rsrs}^T$ matrix elements whose centroids will be called simply $V_{rs}$ and $V_{rs}^T$ (the overline in $\nabla_{rstu}^T$ was meant to avoid confusion with possible matrix elements $V_{rstu}^1$ or $V_{rstu}^0$, it can be safely dropped now). Then (2.19) and (2.21) become

$$\mathcal{H}^d_{mT} = K + \sum_{r \leq s} (a_{rs} n_r(n_s - \delta_{rs})/(1 + \delta_{rs}) + b_{rs}(T_r \cdot T_s - \frac{3}{4} n\delta_{rs})/(1 + \delta_{rs})) \quad (2.23a)$$

$$\mathcal{H}^d_{mnp} \text{ or } \mathcal{H}^d_{m0} = K + \sum_{r \leq s} V_{rs} n_r(n_s - \delta_{rs})/(1 + \delta_{rs}) \quad (2.23b)$$

We rewrite the relevant centroids incorporating explicitly the Pauli restrictions

$$V_{rs} = \frac{\sum_{\mathcal{I}} V_{rsrs}^T[\mathcal{I}] (1 - (-)^{r+2r})}{D_r(D_s - \delta_{rs})} \quad V_{rs}^T = \frac{4 \sum_{J} V_{rsrs}^T[J](1 - (-)^{J+2r}\delta_{rs})}{D_r(D_s + 2\delta_{rs}(-)^{2r})} \quad (2.24a)$$

$$a_{rs} = \frac{1}{4}(3V_{rs}^1 + V_{rs}^0) = V_{rs} + \frac{3}{4} \frac{\delta_{rs}}{D_r - 1} b_{rs} \quad b_{rs} = V_{rs}^1 - V_{rs}^0 \quad (2.24b)$$

The relationship between $a_{rs}$ and $V_{rs}^T$ (which should be checked) makes it possible to combine eqs. (2.23a) and (2.23b) in a single form

$$\mathcal{H}^d_m = K + \sum_{r \leq s} V_{rs} n_r(n_s - \delta_{rs})/(1 + \delta_{rs}) + b_{rs} \left( T_r \cdot T_s - \frac{3n_r\bar{n}_r}{4(D_r - 1)} \delta_{rs} \right) / (1 + \delta_{rs}) \quad (2.25)$$

in which now the $b_{rs}$ term can be dropped to obtain $\mathcal{H}^d_{m0}$ or $\mathcal{H}^d_{mnp}$. Note that the diagonal terms depend on the representation:

$\mathcal{H}^d_{mnp} \neq \mathcal{H}^d_{mT}$ in general. It is useful to have at hand the relationship between matrix elements and centroids in the $jt$ and the $j$ schemes. A single shell $r$ in the former becomes a pair $r_n$ and $r_p$ in the latter.
\[ V_{r,s,\bar{t},u} = \frac{(1 + \delta_{rs})^{1/2}(1 + \delta_{tu})^{1/2}}{2} (V_{rstu}^{J1} + V_{rstu}^{J0}) \]
\[ V_{r,s,up,\bar{t},u} = \frac{(1 + \delta_{rs})^{1/2}(1 + \delta_{tu})^{1/2}}{2} (V_{rstu}^{J1} - V_{rstu}^{J0}) \]
\[ V_{r,s,n,t,\bar{u},u} = V_{r,s,\bar{t},u} = V_{r,s}^{J1} \] (2.26)

and for the centroids \((x,y = n \text{ or } p, x \neq y)\)
\[ V_{r,s,x} = V_{r,s}^{1} = \frac{1}{2} \left[ V_{rs}^{1} \left( 1 - \frac{2\delta_{rs}}{D_{r}} \right) + V_{rs}^{0} \left( 1 + \frac{2\delta_{rs}}{D_{r}} \right) \right] \] (2.27)

1. Trace equivalence and propagation

We have already mentioned the important - and known [13] - property of \(H_{\bar{n}}\) that its expectation value for a given state reproduces the trace of \(H\) over configurations \((n_{1} n_{2} \ldots n_{r} \ldots)\) or \(Q = (n_{1} T_{1} n_{2} T_{2} \ldots n_{r} T_{r})\), to which the state belongs. The notation \(Q\) stands for fixed \(n_{r}\) or \(n_{r} T_{r}\) in each subshell. Averages at fixed seniority [15] and for other symmetries are also of much interest [16]. Here we propose a very general proof that relies on Schur’s lemma

\[
D_{\bar{n}} = \sum_{\bar{Y}} Z_{\bar{Y}Q}^{+} Z_{\bar{Y}Q}, \quad |YQ\rangle = Z_{\bar{Y}Q}^{+} |\bar{n}\rangle, \quad |Y\bar{Q}\rangle = Z_{\bar{Y}Q}^{+} |\bar{n}\rangle; \quad (2.28)
\]

The quantum numbers \(\bar{Q}\) are associated to the ph transforms of the operators e.g. \(\bar{Q} = (\bar{n}_{1} T_{1} \bar{n}_{2} T_{2} \ldots)\). \(D_{\bar{Q}}\) is a projector
\[
D_{\bar{Q}} |Y\bar{Q}\rangle = \delta_{QQ} |Y\bar{Q}\rangle \quad \text{if} \quad \bar{n}' \leq \bar{n} \quad (2.29)
\]

From Schur’s lemma it follows that it can be written in terms of the Casimir operators associated with the \(Q\) quantum numbers. It is seen that \(D_{\bar{Q}}\) is a generalization of \(D_{k}\) in [1,3] and for \(Q = (n_{1} n_{2} \ldots n_{r} \ldots)\), it is obvious that
\[
D_{\bar{Q}} = D_{n_{1} n_{2} \ldots n_{r} \ldots} = \prod_{r} \left( \frac{\bar{n}_{r}}{n_{r}} \right); \quad D_{\bar{Q}} = D_{\bar{n}_{1} \bar{n}_{2} \ldots \bar{n}_{r}} \quad (2.30)
\]

where we are forced to distinguish the operator \(\bar{n}_{r}\) and its eigenvalue \(n_{r}\). Note that \(D_{\bar{Q}}\) is not the ph transform of \(D_{Q}\).

We call \(d_{Q} = \sum_{Y} <YQ|YQ>=<\bar{0}|D_{Q}|\bar{0}>\); the dimension of the space over which we take the trace \(<H_{Q}>\); and \(W_{xxq} Z_{xq}^{+} Z_{xq}\) the diagonal contributions to \(H\) (in our case \(x = J, q = rsT\)). Now we can calculate the trace (a step by step explanation follows the equations):
\[
<H_{Q}> = d_{Q}^{-1} \sum_{Y} <YQ| \sum_{xq} W_{xxq} Z_{xq}^{+} Z_{xq}|YQ> = \] (2.31a)
\[
= d_{Q}^{-1} \sum_{xqY} W_{xxq} <\bar{0}|Z_{xq}^{+} Z_{xq}^{+} Z_{xq} Z_{YQ}|\bar{0}> = \] (2.31b)
\[
= d_{Q}^{-1} \sum_{xq} W_{xxq} <\bar{0}|Z_{xq}^{+} D_{Q} Z_{xq}|\bar{0}> = \] (2.31c)
\[
= d_{Q}^{-1} \sum_{q} \left[ <\bar{0}|Z_{xq}^{+} D_{Q} Z_{xq}|\bar{0}> \sum_{x} W_{xxq} \right] = \] (2.31d)
\[ d_Q^{-1} \sum_q \left[ d_q^{-1} \sum_{x'} <QY|Z_{x'q}Z_{x'q}|QY> \sum_x W_{xxq} \right] = \]
\[ = \sum_q D_q d_q^{-1} \sum_x W_{xxq} >Q. \] (2.31f)

Line by line we have

a. definition of trace

b. insert the definition of \(|YQ> = Z_{YQ}|0>\)

c. French’s trick of using \(Z_{YQ}|0>\) instead of \(Z_{x'q}|0>\) for \(|YQ>\) allows to permute operators. Then use eq.(2.28)

d. the crucial step: since the expectation value of \(D_Q\) is independent of \(x\) we take it out of the sum for some arbitrary \(x'\).

e. by the same token we sum over \(x'\) and divide by the number of terms, then commute again the operators in \(q\)

f. rearrange \(e\) to make clear what the *trace equivalent* operator is.

If we specialize this result to our problem, eqs.(2.14a) and (2.14b) for \(r = t, s = u\) provide \(D_q\) and eq.(2.11) provides \(d_q^{-1} \sum W_{xxq}\). Therefore, \(H_{m,n,T}^d\) is indeed the trace equivalent of \(H\). Reference to Schur’s lemma may be avoided for our particular problems. What has to be shown is that we have exactly the number of operators required to calculate \(D_Q\). For \(H_{m,n,p}^d\) the problem is solved in (2.30). For \(H_{m,n,T}^d\) the form of \(D_Q\) is not so simple, but we can easily check that the number of operators available meets exactly our needs. Let us do it for 1 shell (or for total \(n\) and \(T\)), so that \(D_q = D_{n,T}\).

For \(n = 2\) we have from eqs.(2.14a) and (2.14b), \((T^2 = T(T + 1))\),

\[ D_{20} = \frac{1}{8} (n(n + 1) - 4T^2) \quad D_{21} = \frac{1}{8} (3n(n - 1) + 4T^2) \] (2.32)

For \(n = 3\) we need \(D_{3,1}^d\) and \(D_{3,2}^d\) which we construct out of \(n^{(3)}\) and \((n - 2)\) \((T^2 - 3/4n)\), the two available rank 3 operators. For \(n = 4\) we need \(D_{40}, D_{41}, D_{42}\) and we have \(n^4, n^2 T^2 T^4\) from which we can extract rank 4 operators and work out

\[ D_{40} = \frac{1}{12} T^2 - \frac{n(n + 1)}{4} \left( T^2 - \frac{1}{4} n(n + 1) \right) \]
\[ D_{41} = \frac{1}{8} \left( T^2 - \frac{n(n + 1)}{4} \right) \left( T^2 - \frac{3}{4} (n - 2) (n - 4) \right) \]
\[ D_{42} = n^{(4)} \frac{T^2}{24} - D_{40} - D_{41} \] (2.33)

Obviously the construction can be extended to any \(D_{n,T}\).

If we want to deal with averages at fixed intermediate \(T\) values e.g.,

\(Q = (n_1T_1 n_2T_2 n_3T_3 T_{12} T)\), even the counting may become hard; Schur’s lemma is not trivial. Still we can do without it, by direct averaging through standard multipole techniques, as in

\[ \sum_{Y_1Y_2J_1J_2J} <n_1\Gamma_1 Y_1 n_2\Gamma_2 Y_2 \Gamma \| [(A_1 B_1) \gamma (A_2 B_2)] \gamma^0 \| n_1\Gamma_1 Y_1 n_2\Gamma_2 Y_2 \Gamma >= \]
\[ = \sum_{Y_1Y_2J_1J_2J} (-)^{\Gamma_1 + \Gamma_2 - \gamma - \gamma} \left\{ \begin{array}{c} \Gamma_1 \Gamma_2 \Gamma \Gamma_2 \Gamma_1 \end{array} \right\} <\Gamma_1 Y_1 \| S_1^\gamma \| \Gamma_1 Y_1 > <\Gamma_2 Y_2 \| S_2^\gamma \| \Gamma_2 Y_2 > \] (2.34)

where, using the sum rule (A19) for \(J\), only \(\gamma = 01\) and 00 survive, which are absent from \(H_M\), establishing \(H_m\) as trace equivalent for these operators. The only averages that cannot be done in this way are the basic ones involving \(n\) and \(T\) only, but these we have already.

The idea to associate a trace to an equivalent operator and not only to a number, leads to French’s idea of *propagation*. It stresses the fact that by combining the general properties and particular symmetries of a given trace equivalent operator we may *determine* it at some convenient and limited set of \(Q\) values and then *propagate* i.e. apply it to any \(Q\). The task is hard in general but for purely isocalar averages it is often possible to to guess outright the solution, as for the “width” of \(H_M\):
\[ \sigma^2(n) = \langle \mathcal{H}_M^2 \rangle_{n} = \frac{n^{(2)} \mathcal{H}^{(2)}_{n}}{2(D - 2)^{(2)}} \sigma^2(2), \quad \sigma^2(2) = \frac{2}{D^{(2)}} \sum W_{ijkl}^2 \]  

since the trace equivalent operator must be of rank 4, reproduce \( \sigma^2(2) \) at \( n = 2 \), and vanish at \( n = 0, 1, \tilde{n} = 0, 1 \) because the energies of the corresponding states are entirely given by \( \mathcal{H}_m \).

C. The Hartree-Fock property of \( \mathcal{H}_m \). Strict rank operators

Equation (2.35) contains all the possible contractions that may appear in any possible change of vacuum. By construction they all vanish for \( \mathcal{H}_M \) since their coefficients are proportional to centroids that have all migrated to \( \mathcal{H}_m \). As a consequence \( \mathcal{H}_M \) is of rank 2 both for particles and holes (we call this property, strict rank-2) and its expectation value vanishes for any closed shell and any single particle or single hole state, a set we denote by “\( \text{CS} \pm 1 \)”. As a consequence,

\[ < \text{CS} \pm 1 | \mathcal{H} | \text{CS} \pm 1 > = < \text{CS} \pm 1 | \mathcal{H}_m | \text{CS} \pm 1 > \]  

which remains valid under unitary transformations \( a^+_r = \sum U_{rs} b^+_s \) because centroids become linear combinations of centroids. Therefore, if they all vanish in one representation, so they do in any other representation and \( \mathcal{H}_m \) remains a strict rank-2 operator. Of course at any particular basis we can replace \( \mathcal{H}_m \) by \( \mathcal{H}_m^0 \) in (2.36).

From these remarks it follows that

Hartree-Fock variation of \( \mathcal{H} \) is Hartree-Fock variation of \( \mathcal{H}_m \).

Eq. (2.36) can be obtained more directly by arguing that any \( \text{CS} \pm 1 \) state is the only member of the configuration to which it belongs. If we have invoked eq. (2.36) instead, it is to introduce the concept of strict rank-2 operators. It can be certainly generalized and it is quite useful for monopole operators, although it must take a weaker form since for \( \mathcal{H}_M \) the property guarantees vanishing for all \( \text{CS} \pm 1 \) states, which cannot be the case for any diagonal contribution \( \mathcal{H}_m \). For instance: in eq. (2.23) the isoscalar part has been cleanly separated because \( T^2 - 3/4\pi \) of rank 2 in eq. (2.35) has been changed into \( T^2 = \frac{3}{4\pi} \), which is of strict rank 2 for doubly closed shells.

For number operators it is easy to check that

\[ \Gamma_r = \left( \frac{n_r}{D_r} - \frac{n_1}{D_1} \right) D_r = -\left( \frac{\tilde{n}_r}{D_r} - \frac{\tilde{n}_1}{D_1} \right) \]  

is of strict rank 1 for closed shells containing 1 and \( r \), while

\[ \Gamma_{rs} = \left( \frac{2n_r n_s}{D_r D_s} - \frac{n_r^{(2)}}{D_r^{(2)}} - \frac{n_s^{(2)}}{D_s^{(2)}} \right) \frac{D_r D_s}{2} = \bar{\Gamma}_{rs} \equiv \Gamma_{rs}(\tilde{n}_r, \tilde{n}_s) \]  

is of strict rank 2 for closed shells containing \( r \) and \( s \).

It is seen that the set of operators \( n_1, n_2, \ldots, n_r \) can be replaced by the total number \( n \) and \( \nu - 1 \) operators \( \Gamma_r \) (in which the choice of \( n_1 \) is arbitrary). Similarly we have \( n^{(2)}_r, n \Gamma_r \) and \( \Gamma_{rs} \) with \( r \neq 1 \) and \( r \neq s \), replacing the \( n_r n_s \) set.

The form of \( \mathcal{H}_{m0}^d \) obtained with this construction is given in [9], where it made possible an efficient monopole phenomenology (beware of two erratae in this reference: in the definition of \( \epsilon_r \) read \( N_r \) not \( N_{\nu r} \), and in that of \( W_r \) read \( N_{\nu r} \) not \( N_r \). For the form of \( \mathcal{H}_{mnp}^d \) refer to [10], where several applications are given. In particular a reinterpretation of BCS as a number conserving theory which hinges on what is the fundamental point of the construction: That it singles out \( n \) - always a conserved quantity, and the only one of strict rank 0 - from all the other number operators.

To understand the interest of this result in giving a full characterization of \( \mathcal{H}_{m0} \) we note that the energy of a closed shell to which the \( n \Gamma_r \) and \( \Gamma_{rs} \) operators do not contribute, is given by \( W_n(n - 1)/2 \), with

\[ W = \sum V_{rs} D_{rs}/\sum D_{rs} \]  

\[ D_{rs} = D_r(D_s - \delta_{rs})/(1 + \delta_{rs}). \]

It means that a single parameter \( W \) is in principle sufficient to describe the energies of all closed shells. The catch is that all the strict rank properties apply to the particular closed shell being considered and \( W \) is necessarily space-dependent, since defined only for a given set of occupied orbits. In a way we have gained a lot by isolating the strongest contributor but it is of no use to have a parameter that is space dependent if we do not know the dependence. In [10], we shall see how the problem is naturally solved, once we consult the realistic interactions.

The separation property of \( \mathcal{H}_m \) had been anticipated [13], used [9], and very sketchily proved for \( \mathcal{H}_{mnp} \) [14]. The full derivation was long overdue and it might have been given in a briefer form but we hope that the longer one we have chosen will be more useful.
APPENDIX A:

Reduced matrix elements for $\ell$, $\sigma$, $rY_1$ and $q$ operators

Conventions: radial wavefunctions are positive near the origin.
$\hat{j} = \ell + \hat{\sigma}$ (not $\hat{s} + \hat{\ell}$).
Condon and Shortley phases for $Y_{\ell m}$.

A) $\hat{\sigma} = 2\hat{s}$ and $\hat{\ell}$ operators.

\begin{align*}
  j' &= \ell + 1/2 \quad j = \ell - 1/2 \quad f(j) = (j(j + 1)(2j + 1))^{1/2} \\
  \langle j' | \hat{\sigma} | j' \rangle &= f(j')/j' \\
  \langle j' | \hat{\ell} | j' \rangle &= \ell f(j')/j' \\
  \langle j' | \hat{s} | j' \rangle &= \ell f(j')/j' \\
  \langle j | \hat{\sigma} | j \rangle &= -f(j)/(j + 1) \\
  \langle j | \hat{\ell} | j \rangle &= (\ell + 1)f(j)/(j + 1) \\
  \langle j | \hat{s} | j \rangle &= -j\hat{s}|j\rangle = -\langle j | \hat{\ell} | j' \rangle
\end{align*}

(A1)

(A2)

(A3)

B) $r^2 Y_{\lambda}$ operators. $Y_{\lambda} = \left(\frac{2\ell + 1}{4\pi}\right)^{1/2} C_{\lambda}$

\begin{align*}
  \langle p\ell j | r^2 Y_{\lambda} | p'\ell' j' \rangle &= \langle p\ell | r^4 | p'\ell' \rangle (-)^{j' - \lambda - 1/2} [j j']^{1/2} \begin{pmatrix} j & j' & \lambda \\ 1/2 & -1/2 & 0 \end{pmatrix}
\end{align*}

(A4)

B1) $r^1 Y_1$ operators.

\begin{align*}
  \langle p\ell | r | p + 1 \ell + 1 \rangle &= \sqrt{\frac{1}{2}} [p + \ell + 3] \\
  \langle p\ell | r | p + 1 \ell - 1 \rangle &= -\sqrt{\frac{1}{2}} [p - \ell + 2] \\
  \langle p\ell j | rC_1 | p + 1 \ell + 1 j + 1 \rangle &= -\sqrt{\frac{p + \ell + 3}{2}} \left[ \frac{(2j + 1)(2j + 3)}{2(2j + 2)} \right]^{1/2} \\
  \langle p\ell j | rC_1 | p + 1 \ell + 1 j \rangle &= -\sqrt{\frac{p + \ell + 3}{2}} \left[ \frac{2j + 1}{2j(2j + 2)} \right]^{1/2} \\
  \langle p\ell j | rC_1 | p + 1 \ell - 1 j - 1 \rangle &= \sqrt{\frac{p - \ell + 2}{2}} \left[ \frac{2j + 1}{2j(2j + 2)} \right]^{1/2} \\
  \langle p\ell j | rC_1 | p + 1 \ell - 1 j - 1 \rangle &= -\sqrt{\frac{p - \ell + 2}{2}} \left[ \frac{(2j - 1)(2j + 1)}{4j} \right]^{1/2}
\end{align*}

(A5)

(A6)

(A7)

(A8)

(A9)

(A10)

B2) $r^2 Y_2$ operators ($q_{00\omega}$ only)

\begin{align*}
  \langle p\ell | r^2 | p\ell \rangle &= p + 3/2 \\
  \langle p\ell | r^2 | p\ell + 2 \rangle &= -((p - \ell)(p + \ell + 3))^{1/2} \\
  \langle j\ell | r^2 C_2 | j\ell \rangle &= -\frac{p + 3/2}{2} \left[ \frac{(2j + 1)(2j - 1)(2j + 3)}{2j(2j + 2)} \right]^{1/2} \\
  \langle j\ell | r^2 C_2 | j + 1 \ell \rangle &= (p + 3/2) \left[ \frac{3}{2} \frac{(2j + 1)(2j + 3)}{2j(2j + 2)(2j + 4)} \right]^{1/2} \\
  \langle j\ell | r^2 C_2 | j + 1 \ell + 2 \rangle &= -\frac{3}{2} \left[ \frac{(p - \ell)(p + \ell + 3)(2j + 1)(2j + 3)}{2j(2j + 2)(2j + 4)} \right]^{1/2} \\
  \langle j\ell | r^2 C_2 | j + 2 \ell + 2 \rangle &= -\left[ \frac{3}{8} \frac{(p - \ell)(p + \ell + 3)(2j + 5)(2j + 3)(2j + 1)}{(2j + 4)(2j + 2)} \right]^{1/2}
\end{align*}

(A11)

(A12)

(A13)

(A14)

(A15)
Useful formulae involving $3j$ and $6j$ symbols

$$
< jm j' m'|JM > = (-)^{j-j'+M}[J]^{1/2} \left( \begin{array}{ccc}
  j & j' & J \\
  m & m' & -M
\end{array} \right) = \\
(-)^{j+j'-J} < j' m' jm|JM > = (-)^{j+j'-J} < j - m j' - m'|J - M > 
$$

(A16)

$$
\left( \begin{array}{ccc}
  j & j' & J \\
  m & m' & -M
\end{array} \right) = (-)^{j+j'+J} \left( \begin{array}{ccc}
  j' & j & J \\
  m' & m & -M
\end{array} \right) = (-)^{j+j'+J} \left( \begin{array}{ccc}
  j & j' & J \\
  -m & -m' & -M
\end{array} \right) 
$$

(A17)

The $3j$ symbol is invariant under cyclical permutations.

$$
< jm j - m|00 > = [j]^{-1/2}(-)^{j-m} 
$$

(A18)

$$
\sum_j (-)^j[J] \left\{ \begin{array}{ccc}
  j & j' & J \\
  j & k & k
\end{array} \right\} = (-)^{j+j'}[j j']^{1/2} \delta_{k0} 
$$

(A19)

$$
\sum_j [J] \left\{ \begin{array}{ccc}
  j & j' & J \\
  j & j' & k
\end{array} \right\} = 1 
$$

(A20)

$$
\left\{ \begin{array}{ccc}
  j & j' & 0 \\
  j & j' & k
\end{array} \right\} = (-)^{j+j'+k}[j j']^{-1/2} 
$$

(A21)

The $6j$ symbol is invariant under permutations of columns and under interchange of upper and lower indices in two columns.

**NOTATIONS**

We collect here a list of notations used throughout the text, indicating, when useful, the equation or section in which they are introduced or explained. Occasionally some comments are made.

**WARNING:** The same notation may mean different things, but once this is kept in mind it should be very easy to give the correct interpretation in context.

**Three general points**

1) $F^{(n)} = F(F - 1) \cdots s(F - n + 1)$, integer $n$. Boole’s factorial power notation.

2) Hardy’s Order notation $f(x) = O(x)$ means $f(x)/x < \text{constant}$, for $x$ tending to some limit ($x \to \infty$ or $x \to 0$ usually).

3) $T^k_\kappa = T_{\kappa\kappa}$ (never $T^{k\kappa}$) is a tensor operator of rank $k$ and projection $\kappa$. Applies to coupled operators as in $(T^k S^k)^{K\kappa}_K$, in which case the rank $K$ is always a superindex. Whenever possible, projections are omitted. Zero coupling: $(T^k S^k)^0$, when working in product space ($JT$, $LS$ or $LST$), means full scalar. E.g. in $JT$ space $0 = 00$, i.e. scalar, isoscalar.

**Symbols**

- $\Longrightarrow$ implies; $\rightarrow$ transforms, tends to, becomes.

- $[j] = 2j + 1$, but see (1.11) for product spaces. Also $[i] = i_1, i_2, \ldots i_n$, but only in the very first lines of section I.

- $< \phi = \text{trace} \left\{ \phi \right\}$.

- $\cdot$ scalar product $T^k(S^k)^* = \sum_\kappa T^k_\kappa(S^k)^*$

- * Hermitian conjugation for operators, complex conjugation for numbers.
• Creation symbol as in $a^+, Z^+, X^+$ which are creation operators.

• $\text{ph}$ transform (1.6)
  Adjoint (1.18)
  Centroid (2.20)

• $\hat{n}$, $J$, $T$ are used as operators and eigenvalues. When necessary to distinguish them we use $\tilde{n}$, $\tilde{J}$ and $\tilde{T}$ for the operators.

• $\sum (\ )$ restricted sum (2.16a) and (2.16b).

**Acronyms**

CM: center of mass.
CS, CS±1: Closed shell, CS plus 1 particle and 1 hole states.
HF: Hartree Fock.
HO: Harmonic oscillator.
NN: Nucleon-Nucleon.
np or $\nu\pi$: neutron, proton.
ph: particle hole.

rhs and lhs: right hand side, left hand side.

**Operators, variables**

$a^+, a_r$ (1.1), $A_r$, $B_R$ (1.23), creation, annihilation operators

$A$: total number of particles. $A^r$: antisymmetric multipole (1.31)

d: dimension of space.

$D = \sum \rho_r = \sum |\rho_r|$. Degeneracies of space and orbit $r$.

$G$: pairing strength.

$H$: Hamiltonian in full space.

$H$: Hamiltonian in subspace.

$i$, $j$, $k$, $\ell$: individual orbits.

$J$, $j_r$: angular momentum (total or of orbit $r$).

$K$: kinetic energy.

$L$, $\ell_r$: orbital angular momentum (total, or of orbit $r$).

$M$: nucleon mass.

$n$, $n_r$: number of particles (total or for shell $r$), $\bar{n}$, $\bar{n}_r$: number of holes.

$N$: number of neutrons.

$p$, $p_r$: principal quantum number $p_r = 2\nu_r + \ell_r$.

$p$: momentum (see under $r$).

$P_{rs}$ convention (1.4).

$r, s, t, u$: subshells, $r \equiv (j_r p_r)$.

$S^{\gamma}_{rs} = S^{\gamma}_{(rs)} = (A_r B_s)^{\gamma}_{(rs)}$, 1.24, $S^{\gamma}_r$ (1.31).

$S_{rs}$, $T_{rs}$ (2.2): $S_{rstu}$, $T_{rstu}$ (2.3a), (2.3b).

$T$: isospin.

$V$: potential energy. $V$ representation section IB $V_{rstu}$, $V_{ijkl}$, two body matrix elements (1.16).

$V_{rsu}$ (2.20) $V_{rs}$ (2.24a), centroids.

$W$ representation (I.40).

$W_{rstu}^\Gamma$ (II.8).

$X_{\Gamma r s u}$ (rs) = $(A_r A_s)^\Gamma_{\Gamma r s u}$, $X^+_{\Gamma r s u}$ (1.24).

$Z_{\Gamma r s u}$ (rs) or $Z^+_{\Gamma r s u}$ (1.14).

$Z$: number of protons.

$\alpha \equiv rstu$ (i.e. $W_{rstu}^\Gamma = W_{\alpha}^\Gamma$)

$\gamma \equiv \lambda \tau$, $\Gamma = JT$ product tensor indices in the $V$ and omega representations.

$\delta_{rs}$ Kronecker $\delta$.

$\delta_{rs} = \delta_{j_r j_s} \delta_{\pi_r \pi_s}$, $\pi_r = (-)^{p_r}$.
\[ \zeta_{rs} = (1 + \delta_{rs})^{-1/2} \]

\( \lambda, \tau \): angular momentum and isospin in \( \omega \) representation.

\( \nu_r \): nodal quantum number: \( p_r = 2\nu_r + \ell_r \).

\( \nu, \pi \): neutron, proton.

\( \tau_r \): isospin projection for orbit \( r \).

\( \omega_{rstu}^{\gamma} \) \ref{1.42\ref{1.43a}\ref{1.43b}}.

\( \Omega_r = j_r + 1/2 \quad \Omega = \sum_r \Omega_r \)

\( \omega = \text{HO frequency.} \)
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