Modular vector fields attached to Dwork family: \( \mathfrak{sl}_2(\mathbb{C}) \) Lie algebra

YOUNES NIKDELAN

Abstract

We introduce an algebraic group \( G \) that acts from right on the moduli space \( T \) of Calabi-Yau \( n \)-folds arising from Dwork family enhanced with differential forms, and describe its Lie algebra \( \text{Lie}(G) \). We observe that \( \text{Lie}(G) \) together with a modular vector field \( R \) on \( T \) generates another Lie algebra \( \mathfrak{G} \), called AMSY-Lie algebra, such that \( \text{dim}(\mathfrak{G}) = \text{dim}(T) \). We find \( \mathfrak{sl}_2(\mathbb{C}) \) as a Lie subalgebra of \( \mathfrak{G} \) that contains \( R \).

1 Introduction

Modular vector field is a unique vector field on a quasi affine variety \( T \) that satisfies a certain equation involving Gauss-Manin connection. The modular vector field \( R \), in some sense, can be considered as a generalization of the systems of differential equations introduced by G. Darboux [Dar78], G. H. Halphen [Hal81] and S. Ramanujan [Ram16], which have been discussed in [Mov12b], A. Guillot, H. Rebelo and H. Reis in [Gui07], [GR12] and [RR14] treat the semi-completeness and dynamic of Halphen type vector field \( H \), which is a generalization of Darboux-Halphen vector field, in a 3-dimensional complex variety \( W \). They use the action of Lie group \( \text{SL}(2,\mathbb{C}) \) on \( W \) and its Lie algebra \( \mathfrak{sl}_2(\mathbb{C}) \) to reduce the dynamic of \( H \), and then employ a phenomenon of integration which allows to understand the dynamic in details. More specifically, \( H \) together with a constant vector field and the radial vector field constructs \( \mathfrak{sl}_2(\mathbb{C}) \). We are also interested in studying the dynamic of the modular vector field \( R \), for what it will be necessary to know a Lie group action on \( T \) and its Lie algebra, in particular construct \( \mathfrak{sl}_2(\mathbb{C}) \) that contains \( R \). This article deals with these studies.

Since introducing Calabi-Yau manifolds and mirror symmetry, a vast number of works in mathematics and physics has been dedicated to these subjects. A pioneer work on the development of these theories has been done in 1991 by Candelas et al. in [COGP91], where they used mirror symmetry to predict the number of rational curves on quintic 3-folds. Indeed, these numbers come from coefficients of \( q \)-expansion of a function called Yukawa coupling. H. Movasati in [Mov15] reencountered Yukawa coupling by applying an algebraic method, called Gauss-Manin connection in disguise, GMCD for short, which one finds it less complicated, because there is no need to compute the periods and variation of Hodge structure, whereas all theses concepts are hidden in Gauss-Manin connection. H. Movasati started GMCD by applying it to elliptic curves [Mov12b]. This project, so far, contains a sequence of valuable completed works, see for instance [Mov16], [Mov17], [AMSY16], [MN16]. One of the main objects in GMCD is the existence of a unique vector field \( R \) on the moduli space \( T \) of a Calabi-Yau variety enhanced with differential forms.

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2 Universidade do Estado do Rio de Janeiro (UERJ), Instituto de Matemática e Estatística (IME), Departamento de Análise Matemática, Rio de Janeiro, Brazil. e-mail: younes.nikdelan@ime.uerj.br
whose solution has $q$-expansions with integer coefficients, and in particular, in the case of elliptic curves and $K3$ surfaces the solutions can be written in terms of (quasi)-modular forms (see [Mov12b] and the cases $n = 1, 2$ of [MN16]). Since this vector field arises from a moduli space, Movasati called it modular vector field and the $q$-expansion of its solution as Calabi-Yau modular form. One of objectives of the project GMCD is somehow to extend the theory of modular forms, where Calabi-Yau modular forms are candidates of this extension. It is also worth to point out that along this extension we may be able to extend the Rankin-Cohen bracket, see [Zag94], where the derivation can be defined by using of the modular vector field $R$, in the same sense that Zagier in [Zag94] §5 uses the Ramanujan’s system of differential equation to introduce the Rankin-Cohen algebra.

Mirror quintic 3-fold is the main example of [AMS016], where the authors describe a Lie Algebra on the moduli space of Calabi-Yau threefolds enhanced with differential forms and its relation to the Bershadsky-Cecotti-Ooguri-Vafa, BCOV for short, holomorphic anomaly equation (see also [Ali17]). On the other hand, mirror quintic 3-fold is the particular case $n = 3$ of the family of $n$-dimensional Calabi-Yau varieties $X = X_\psi$, $\psi \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ obtained by a quotient and desingularization of the so-called Dwork family:

$$ x_0^{n+2} + x_1^{n+2} + \ldots + x_{n+1}^{n+2} - (n + 2)\psi x_0 x_1 \ldots x_n = 0, $$

where their moduli space $T$ and the relevant modular vector field $R$ have been discussed in [MN16]. Hence, in this manuscript analogously we are going to construct a Lie algebra on the moduli space $T$ arising from Dwork family.

From now on, unless otherwise is stated, by mirror variety $X$ we mean the Calabi-Yau $n$-fold $X = X_\psi$, $\psi \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$, obtained by a quotient and desingularization of Dwork family [MN16] §2], where the quotient refers to the quotient space under a group action (see §2]. By $T = T_n$ we mean the moduli space of the pairs $\{X, [\alpha_1, \ldots, \alpha_n, \alpha_{n+1}]\}$, where $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis of the $n$-th algebraic de Rham cohomology $H^n_{\text{dR}}(X)$ compatible with its Hodge filtration such that its intersection form matrix is constant (see Section 2). We have $\dim(H^n_{\text{dR}}(X)) = n + 1$ and its Hodge numbers $h^{ij}$, $i + j = n$, are all one. In [MN16] we constructed the universal family $\pi : X \to T$ together with global sections $\alpha_i, \ i = 1, \ldots, n + 1$ of the relative algebraic de Rham cohomology $H^n_{\text{dR}}(X/T)$, and we provided $T$ with a complete chart $t = (t_1, t_2, \ldots, t_d)$, where $d = d_n := \dim(T)$ (see (2.21)). Let

$$ \nabla : H^n_{\text{dR}}(X/T) \to \Omega^1_T \otimes_{\mathcal{O}_T} H^n_{\text{dR}}(X/T), $$

be the algebraic Gauss-Manin connection on $H^n_{\text{dR}}(X/T)$, where $\mathcal{O}_T$ is the $\mathbb{C}$-algebra of regular functions on $T$ and $\Omega^1_T$ refers to the $\mathcal{O}_T$-module of differential 1-forms on $T$. The reader who is not familiar with the Gauss-Manin connection can substitute $\nabla$ with $d \int_{\delta_i}$, where $t = (t_1, t_2, \ldots, t_d)$, and $\delta_i$ is an $n$-dimensional homology class in $X_i := \pi^{-1}(t)$. In fact, $\nabla \alpha_i = \sum_{j=1}^{n+1} \zeta_{ij} \otimes \alpha_j$ is equivalent to $d \int_{\delta_i} \alpha_i = \sum_{j=1}^{n+1} \zeta_{ij} \int_{\delta_i} \alpha_j$, where $\zeta_{ij}$ are 1-forms. The notation $\nabla_E$ refers to the Gauss-Manin connection composed with the vector field $E \in \mathfrak{X}(T)$, where $\mathfrak{X}(T)$ is hired for the Lie algebra of vector fields on $T$. We attach an $(n + 1) \times (n + 1)$ matrix $A_E$, called Gauss-Manin connection matrix, to the vector field $E$ defined as follow:

$$ \nabla_E \alpha = A_E \alpha, $$

in which

$$ \alpha := (\alpha_1 \ \alpha_2 \ \ldots \ \alpha_{n+1})^\text{tr}. $$
In the rest of this article the notation \( \alpha \), being clear in the context, either is used for the columnar matrix \((1.3)\), or for the \((n + 1)\)-tuple \((\alpha_1, \alpha_2, \ldots, \alpha_{n+1})\).

Here, before continuing with technical parts and the methodology, we state some explicit examples of \(\mathfrak{sl}_2(\mathbb{C})\) that we find in this work. First, notice that the special linear Lie algebra \(\mathfrak{sl}_2(\mathbb{C})\) is the Lie algebra of \(2 \times 2\) matrices with trace zero. Three matrices

\[
(1.4) \quad e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

form the standard basis of \(\mathfrak{sl}_2(\mathbb{C})\) with commutators:

\[
(1.5) \quad [e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.
\]

From now on \(e, f, h\) refer to the standard basis \((1.4)\) of \(\mathfrak{sl}_2(\mathbb{C})\) with the Lie brackets \((1.5)\). Indeed, if we are going to prove that a determined Lie algebra is isomorphic to \(\mathfrak{sl}_2(\mathbb{C})\), it is enough to introduce the corresponding standard basis. It is a known fact, see for instance \([\text{Mov16}] \, \S 2.12\), that the Ramanujan vector field

\[
R = -(t_1^2 - \frac{1}{12} t_2^2) \frac{\partial}{\partial t_1} - (4 t_1 t_2 - 6 t_3) \frac{\partial}{\partial t_2} - (6 t_1 t_3 - \frac{1}{3} t_2^2) \frac{\partial}{\partial t_3},
\]

together with vector fields

\[
H = 2 t_1 \frac{\partial}{\partial t_1} + 4 t_2 \frac{\partial}{\partial t_2} + 6 t_3 \frac{\partial}{\partial t_3}, \quad F = \frac{\partial}{\partial t_2},
\]

constructs \(\mathfrak{sl}_2(\mathbb{C})\), i.e., \(e = R, \ f = F, \ h = H\). In the point of view of GMCD the vector fields \(R, H, F\) are characterized by

\[
\nabla_R \alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \alpha, \quad \nabla_H \alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \alpha, \quad \nabla_F \alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \alpha,
\]

where \(\alpha = (\alpha_1, \alpha_2)^t\) and \(\nabla\) is the Gauss-Manin connection of the universal family of elliptic curves

\[
y^2 = 4(x - t_1)^3 - t_2(x - t_1) - t_3, \quad \alpha_1 = \left[\frac{dx}{y}\right], \ \alpha_2 = \left[\frac{x dx}{y}\right], \quad \text{with} \ \ 27 t_3^2 - t_2^3 \neq 0.
\]

A similar statement holds for Darboux-Halhen vector field, see \([\text{Mov12a, Gui07}]\). Analogously, on the moduli space \(T = T_n\) arising from Dwork family we find a modular vector field \(R \in \mathfrak{X}(T)\) that together with two other vector fields forms \(\mathfrak{sl}_2(\mathbb{C})\). For example, if \(n = 1\) then we have:

\[
(1.6) \quad R = -(t_1 t_2 - 9(t_1^3 - t_3)) \frac{\partial}{\partial t_1} + (8 t_1 t_2 (t_1^3 - t_3) - t_2^3) \frac{\partial}{\partial t_2} + (-3 t_2 t_3) \frac{\partial}{\partial t_3},
\]

\[
(1.7) \quad H = -t_1 \frac{\partial}{\partial t_1} - 2 t_2 \frac{\partial}{\partial t_2} - 3 t_3 \frac{\partial}{\partial t_3},
\]

\[
(1.8) \quad F = \frac{\partial}{\partial t_2},
\]
and if \( n = 4 \), then we get:

\[
R = (t_3 - t_1 t_2) \frac{\partial}{\partial t_1} + \frac{6 - 2t_3^2 t_4 t_8 - t_1^6 t_2^2 + t_2 t_6}{t_1^3 - t_6} \frac{\partial}{\partial t_2} + \frac{6 - 2t_3^2 t_5 t_8 - 3t_1^6 t_2 t_3 + 3t_2 t_4 t_6}{t_1^4 - t_6} \frac{\partial}{\partial t_3} + \frac{6 - 2t_3^2 t_6 t_8 - t_1^6 t_2 t_4 + t_2 t_4 t_6}{t_1^5 - t_6} \frac{\partial}{\partial t_4} \\
+ \frac{6 - 2t_3^2 t_5 t_8 - 4t_1^6 t_2 t_5 - 2t_1^6 t_3 t_4 + 5t_1^7 t_3 t_8 + 4t_2 t_3 t_6 + 2t_3 t_4 t_6}{2(t_1^6 - t_6)} \frac{\partial}{\partial t_5} + \frac{6 - 2t_2 t_6}{\partial t_6} \frac{\partial}{\partial t_6} + \frac{6 - 2t_3^2 t_5 t_8}{t_1^6 - t_6} \frac{\partial}{\partial t_6} + \frac{3t_1^7 t_3 t_8 + 3t_2 t_6}{t_1^6 - t_6} \frac{\partial}{\partial t_8},
\]

(1.10) \[
H = t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2} + 3t_3 \frac{\partial}{\partial t_3} + t_4 \frac{\partial}{\partial t_4} + 2t_5 \frac{\partial}{\partial t_5} + 6t_6 \frac{\partial}{\partial t_6} + 3t_8 \frac{\partial}{\partial t_8},
\]

(1.11) \[
F = \frac{\partial}{\partial t_2},
\]

where \( t_8^2 = 36(t_1^6 - t_6) \). For details and more examples see Section 5 and Section 6.

Next we state the main theorem of [MN16], and we frequently use its proof and the results given there, hence it is strongly recommended to see its proof in [MN16 §7]. This theorem in a general context is also treated in [Nik15].

**Theorem 1.1.** There is a unique vector field \( R = R_n \in \mathcal{X}(T) \), and there are unique regular functions \( Y_i \), \( 1 \leq i \leq n - 2 \), in \( T \) such that:

\[
\nabla_R \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\vdots \\
\alpha_n \\
\alpha_{n+1}
\end{pmatrix} = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & Y_1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & Y_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & Y_{n-2} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & -1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix} \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\vdots \\
\alpha_n \\
\alpha_{n+1}
\end{pmatrix},
\]

(1.12)\[\quad \text{and } Y\Phi + \Phi Y^{tr} = 0.\]

Using the notation given in (1.2), \( A_R = Y \). In the case \( n = 3 \), \( R_3 \) is explicitly computed in [Mov15] and it is verified that \( Y_1 \) is the Yukawa coupling introduced in [COGP91].

We gave \( R_1, R_2, R_4 \) explicitly in [MN16], where the solutions of \( R_1 \) and \( R_2 \) are in terms of (quasi-)modular forms, and when \( n = 4 \) we found that \( Y_1^2 = Y_2^2 \) is the same as 4-point function presented in [GMP95 Table 1, \( d = 4 \)].

For any positive integer \( n \), let \( G := G_n \) be the following algebraic group:

(1.13) \[
G := \{ g \in \text{GL}(n+1, \mathbb{C}) \mid g \text{ is upper triangular and } g^{tr} \Phi g = \Phi \},
\]

where \( \Phi \) is a constant matrix given in (2.19). Let us fix the notation \( m := \frac{n+1}{2} \) if \( n \) is an odd positive integer, and \( m := \frac{n}{2} \) if \( n \) is an even positive integer. \( G \) acts on \( T \) from right, and it has \( m \) copies of \( G_m \), and \( d - (m + 1) \) copies of \( G_a \) as subgroups, where \( G_m \) and \( G_a \), respectively, refer to the multiplicative group \((\mathbb{C}^*, \cdot)\) and the additive group \((\mathbb{C}, +)\), respectively. We present an interpretation of this group action in the chart \( t = (t_1, t_2, \ldots, t_d) \) of \( T \) and observe that in the case of \( n = 1 \) this action coincides with the action of \( G_m \) and \( G_a \) given in [Mov12b §6.3] for the family of elliptic curves arising from
Weierstrass form of elliptic curves, and in the case \( n = 3 \) this action has been studied in [Mov15] (see Section 3).

The Lie algebra of \( G \) is as follow:

\[
(1.14) \quad \text{Lie}(G) = \{ g \in \text{Mat}(n+1, \mathbb{C}) \mid g \text{ is upper triangular and } g^{tr} \Phi + \Phi g = 0 \}.
\]

\( \text{Lie}(G) \) is a \( d - 1 \) dimensional Lie algebra and we find its canonical basis constructed by \( g_{ab} \)'s, \( 1 \leq a \leq m, \ a \leq b \leq 2m + 1 - a \), given in (4.1) and (4.2).

**Theorem 1.2.** For any \( g \in \text{Lie}(G) \), there exists a unique vector field \( R_g \in \mathfrak{X}(T) \) such that:

\[
(1.15) \quad A_{R_g} = g^{tr},
\]

i.e., \( \nabla_{R_g} \alpha = g^{tr} \alpha \).

The Lie algebra generated by \( R_{g_{ab}} \)'s, \( 1 \leq a \leq m, \ a \leq b \leq 2m + 1 - a \), in \( \mathfrak{X}(T) \) with Lie bracket of vector fields is isomorphic to \( \text{Lie}(G) \) with Lie bracket of matrices. Hence we use \( \text{Lie}(G) \) alternately either as a Lie subalgebra of \( \mathfrak{X}(T) \) or as a Lie subalgebra of \( \text{Mat}(n+1, \mathbb{C}) \).

Define \( \mathfrak{G} \) to be the \( \mathcal{O}_T \)-module generated by \( \text{Lie}(G) \) and the modular vector field \( R \) in \( \mathfrak{X}(T) \). \( \mathfrak{G} \) is a Lie subalgebra of \( \mathfrak{X}(T) \) and we call it \( \text{AMSY-Lie algebra} \). In the following theorem we determine the Lie bracket of \( R \) and the elements of the canonical basis of \( \text{Lie}(G) \). In what follows, \( \delta^k_j \) denotes the Kronecker delta, \( g(n) = 1 \) if \( n \) is an odd integer, and \( g(n) = 0 \) if \( n \) is an even integer, \( Y_j \)'s, \( 1 \leq j \leq n - 2 \), are the functions given in Theorem 1.1, and besides them we let \( Y_0 = -Y_{n-1} = 1 \).

**Theorem 1.3.** Followings hold:

\[
(1.16) \quad [R, R_{g_{11}}] = R, \\
(1.17) \quad [R, R_{g_{22}}] = -R, \\
(1.18) \quad [R, R_{g_{aa}}] = 0, \ 3 \leq a \leq m, \\
(1.19) \quad [R, R_{g_{ab}}] = \Psi_1^{ab}(Y) R_{g_{(a+1)b}} + \Psi_2^{ab}(Y) R_{g_{a(b-1)}}, \ 1 \leq a \leq m, \ a + 1 \leq b \leq 2m + 1 - a,
\]

where

\[
(1.20) \quad \Psi_1^{ab}(Y) := (1 + g(n)\delta_{a+b}^{2m} - \delta_{a+b}^{2m+1}) Y_{a-1}, \\
(1.21) \quad \Psi_2^{ab}(Y) := (1 - 2g(n)\delta_{b}^{n+1}) Y_{n+1-b}.
\]

If \( n = 1, 2 \), then we see that \( \mathfrak{G} \) is isomorphic to \( \mathfrak{sl}_2(\mathbb{C}) \). In general, for \( n \geq 3 \) we have \( \mathfrak{sl}_2(\mathbb{C}) \) as a Lie subalgebra of \( \mathfrak{G} \) and we state it in the following theorem.

**Theorem 1.4.** Let \( H := R_{g_{22}} - R_{g_{11}} \) and \( F := R_{g_{12}} \). The Lie algebra generated by vector fields \( R, H, F \) in \( \mathfrak{X}(T) \) is isomorphic to \( \mathfrak{sl}_2(\mathbb{C}) \); indeed we get:

\[
[R, F] = H, \quad [H, R] = 2R, \quad [H, F] = -2F.
\]

If in \( \mathcal{O}_T \) we attach to any \( t_i \) the weight \( w_i := \text{weight}(t_i) \), where \( w_i \)'s are defined by

\[
H = \sum_{i=1}^{d} w_i t_i \frac{\partial}{\partial t_i},
\]

then \( H \) is the radial vector field and \( F \) is a constant vector field and \( R \) is a quasi-homogeneous vector field in the weighted space \( \mathcal{O}_T \). For example see (1.5), (1.7), (1.8), (1.9), (1.10), (1.11) given above.
The organization of this paper is as follows: Section 2 contains a brief summary of relevant facts and terminologies of [MN16] which are necessary to have a self-contained manuscript. In Section 3 we discuss the action of algebraic group $G$ on $T$; indeed we observe that $\dim(G) = d - 1$ and we present its action in a chart of $T$. Section 4 is devoted to the introducing the Lie($G$) and AMSY-Lie algebra $\mathfrak{G}$, and we give a canonical basis of $\mathfrak{G}$. In this section we prove Theorem 1.2 and Theorem 1.3. In Section 5 we construct the $sl_2(\mathbb{C})$ Lie algebra as a Lie subalgebra of $\mathfrak{G}$ which contains the modular vector field $R$ and we see the proof of Theorem 1.4. Here we provide the elements of $\mathcal{O}_T$ with appropriate weights which turn $R$ to a quasi homogeneous vector field. This section also presents another $sl_2(\mathbb{C})$, where instead of rational vector field $R$ we have a polynomial vector field. The last section, Section 6, gives a brief summary of AMSY-Lie algebra $\mathfrak{G}$ attached to non-rigid compact Calabi-Yau threefolds and shows that there are various copies of $sl_2(\mathbb{C})$ in $\mathfrak{G}$.

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2 Moduli spaces and GMCD

Our recent work [MN16] makes the foundation of this article, hence for the convenience of the reader and also to have a self-contained text, in this section we recall some relevant facts and terminologies discussed in [MN16]. For more details the reader is referred to the same reference.

In (1.1) instead of variable $\psi$, the standard variable which is used in literatures is $z := \psi^{-(n+2)}$. Let $W_z$ be an $n$-dimensional hypersurface in $\mathbb{P}^{n+1}$ given by:

$$f_z(x_0, x_1, \ldots, x_{n+1}) := z x_0^{n+2} + x_1^{n+2} + x_2^{n+2} + \cdots + x_{n+1}^{n+2} - (n + 2)x_0x_1x_2\cdots x_{n+1} = 0.$$ 

$W_z$ represents a family of Calabi-Yau $n$-folds. The group $G := \{ (\zeta_0, \zeta_1, \ldots, \zeta_{n+1}) \mid \zeta_i^{n+2} = 1, \ z_0\zeta_1\cdots\zeta_{n+1} = 1 \}$, acts canonically on $W_z$ as

$$(\zeta_0, \zeta_1, \ldots, \zeta_{n+1}).(x_0, x_1, \ldots, x_{n+1}) = (\zeta_0x_0, \zeta_1x_1, \ldots, \zeta_{n+1}x_{n+1}).$$

The mirror variety $X = X_z$, $z \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$, is obtained by desingularization of the quotient space $W_z/G$ (for more details see [MN16 §2]).

By moduli space of holomorphic $n$-forms $S$ we mean the moduli of the pair $(X, \alpha_1)$, where $X$ is an $n$-dimensional mirror variety and $\alpha_1$ is a holomorphic $n$-form on $X$. We know that the family of mirror varieties $X_z$ is a one parameter family and the $n$-form $\alpha_1$ is unique, up to multiplication by a constant, therefore $\dim(S) = 2$. Analogously to the construction of $X_z$, let $X_{t_1, t_{n+2}}$, $(t_1, t_{n+2}) \in \mathbb{C}^2 \setminus \{(t_1^{n+2} - t_{n+2})t_{n+2} = 0\}$, be the mirror variety obtained by a quotient and desingularization of

$$f_{t_1, t_{n+2}}(x_0, x_1, \ldots, x_{n+1}) := t_{n+2}x_0^{n+2} + x_1^{n+2} + x_2^{n+2} + \cdots + x_{n+1}^{n+2} - (n + 2)t_1x_0x_1x_2\cdots x_{n+1} = 0.$$ 

We fix two $n$-forms $\eta$ and $\omega_1$, respectively, in the family $X_z$ and $X_{t_1, t_{n+2}}$, respectively, such that in the affine space $\{x_0 = 1\}$ are given as follows:

$$\eta := \frac{dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{n+1}}{df_z}, \quad \omega_1 := \frac{dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{n+1}}{df_{t_1, t_{n+2}}}.$$ (2.1)
Any element of $S$ is in the form $(X_z, a\eta)$, where $a$ is a non-zero constant, which can be identified by $(X_{t_1, t_{n+2}}, \omega_1)$ as follow:

\begin{align}
(2.2) \quad (X_z, a\eta) &\mapsto (X_{t_1, t_{n+2}}, \omega_1), \quad (t_1, t_{n+2}) = (a^{-1}, za^{-(n+2)}), \\
(2.3) \quad (X_{t_1, t_{n+2}}, \omega_1) &\mapsto (X_z, t_1^{-1}\eta), \quad z = \frac{t_{n+2}}{t_1}. 
\end{align}

Hence $(t_1, t_{n+2})$ construct a chart for $S$, in the other word

$$S = \text{Spec}(\mathbb{C}[t_1, t_{n+2}, \frac{1}{(t_1^{n+2} - t_{n+2})}]),$$

and the morphism $X \to S$ is the universal family of $(X, \alpha_1)$. The multiplicative group $\mathbb{G}_m := (\mathbb{C}^*, \cdot)$ acts on $S$ from right by $(X, \alpha) \cdot k = (X, k^{-1}\alpha)$, with $k \in \mathbb{G}_m$, $(X, \alpha) \in S$. It can be interpreted in the chart $(t_1, t_{n+2})$ as follow:

\begin{equation}
(2.4) \quad (t_1, t_{n+2}) \cdot k = (kt_1, k^{n+2}t_{n+2}), \quad (t_1, t_{n+2}) \in S, \quad k \in \mathbb{G}_m,
\end{equation}

which follows from isomorphism

\begin{align}
(2.5) \quad & (X_{kt_1, k^{n+2}t_{n+2}}, k\omega_1) \cong (X_{t_1, t_{n+2}}, \omega_1), \\
& (x_1, x_2, \cdots, x_{n+1}) \mapsto (k^{-1}x_1, k^{-1}x_2, \cdots, k^{-1}x_{n+1}).
\end{align}

Let $\nabla : H^n_{\text{dR}}(X/S) \to \Omega^1_S \otimes \Omega^*_S H^n_{\text{dR}}(X/S)$ be the Gauss-Manin connection of the two parameter family of varieties $X/S$. We define $n$-forms $\omega_i$, $i = 1, 2, \ldots, n+1$, as follows

\begin{equation}
(2.6) \quad \omega_i := (\nabla_{\frac{\partial}{\partial t_1}})^i(\omega_1),
\end{equation}

in which $\frac{\partial}{\partial t_1}$ is considered as a vector field on the moduli space $S$. Then $\omega := \{\omega_1, \omega_2, \ldots, \omega_{n+1}\}$ form a basis of $H^n_{\text{dR}}(X)$ which is compatible with its Hodge filtration, i.e.,

\begin{equation}
(2.7) \quad \omega_i \in F^{n+1-i} \setminus F^{n+2-i}, \quad i = 1, 2, \ldots, n+1,
\end{equation}

where $F^i$ is the $i$-th piece of the Hodge filtration of $H^n_{\text{dR}}(X)$. We write the Gauss-Manin connection of $X/S$ in the basis $\omega$ as follow

\begin{equation}
(2.8) \quad \nabla \omega = \tilde{A}\omega, \quad \text{with} \quad \omega = (\omega_1 \quad \omega_2 \quad \ldots \quad \omega_{n+1})^t.
\end{equation}

If we denote by $\tilde{A}[i, j]$ the $(i, j)$-th entry of the Gauss-Manin connection matrix $\tilde{A}$, then we obtain:

\begin{align}
(2.9) \quad & \tilde{A}[i, i] = -\frac{i}{(n+2)t_{n+2}}dt_{n+2}, \quad 1 \leq i \leq n, \\
(2.10) \quad & \tilde{A}[i, i + 1] = dt_1 - \frac{t_1}{(n+2)t_{n+2}}dt_{n+2}, \quad 1 \leq i \leq n, \\
(2.11) \quad & \tilde{A}[n+1, j] = \frac{-S_2(n+2, j)t_1^{n+2}}{t_1^{n+2} - t_{n+2}}dt_1 + \frac{S_2(n+2, j)t_1^{j+1}}{(n+2)t_{n+2}(t_1^{n+2} - t_{n+2})}dt_{n+2}, \quad 1 \leq j \leq n, \\
(2.12) \quad & \tilde{A}[n+1, n+1] = \frac{-S_2(n+2, n+1)t_1^{n+1}}{t_1^{n+2} - t_{n+2}}dt_1 + \frac{n(n+1)t_1^{n+2}}{(n+2)t_{n+2}(t_1^{n+2} - t_{n+2})}dt_{n+2},
\end{align}
where \( S_2(r, s) \) is the Stirling number of the second kind defined by

\[
S_2(r, s) = \frac{1}{s!} \sum_{i=0}^{s} (-1)^i \binom{s}{i} (s-i)^r,
\]

and the rest of the entries of \( \tilde{A} \) are zero. As we mentioned in [MN16], equations (2.11) and (2.12) are checked for \( n = 1, 2, 3, 4 \) and we believe that they are valid for arbitrary \( n \), even though such explicit expressions do not interfere with our proofs. For any \( \xi_1, \xi_2 \in H^n_{dR}(X) \), in the context of de Rham cohomology, the intersection form of \( \xi_1 \) and \( \xi_2 \), denoted by \( \langle \xi_1, \xi_2 \rangle \), is given by

\[
\langle \xi_1, \xi_2 \rangle := \frac{1}{(2\pi i)^n} \int_X \xi_1 \wedge \xi_2.
\]

Intersection form is a non-degenerate \((-1)^n\)-symmetric form. We obtain

\[
\langle \omega_i, \omega_j \rangle = 0, \text{ if } i + j \leq n + 1,
\]

\[
\langle \omega_i, \omega_{n+1} \rangle = (-(n+2))^n \frac{c_n}{t_1^{n+2} - t_{n+2}}, \text{ where } c_n \text{ is a constant},
\]

\[
\langle \omega_j, \omega_{n+2-j} \rangle = (-1)^{j-1} \langle \omega_1, \omega_{n+1} \rangle, \text{ for } j = 1, 2, \ldots, n+1.
\]

from which we can determine all the rest of \( \langle \omega_i, \omega_j \rangle \)'s in a unique way. If we set \( \Omega = \Omega_n := (\langle \omega_i, \omega_j \rangle)_{1 \leq i, j \leq n+1} \), to be the intersection form matrix in the basis \( \omega \), then we have

\[
d\Omega = \tilde{A}\Omega + \Omega \tilde{A}^\text{tr}.
\]

By moduli space of mirror variety \( X \) enhanced with differential forms, denoted by \( T = T_n \), we mean the moduli of pairs \( (X, [\alpha_1, \ldots, \alpha_n, \alpha_{n+1}] \) ), where \( \{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\} \) constructs a basis of \( H^n_{dR}(X) \) with the property

\[
\alpha_i \in F^{n+1-i} \setminus F^{n+2-i}, \quad i = 1, \ldots, n, n+1,
\]

and

\[
[(\alpha_i, \alpha_j)] = \Phi_n.
\]

Here \( \Phi = \Phi_n \) is the following constant \((n+1) \times (n+1)\) matrix:

\[
\Phi_n := \begin{pmatrix}
0_m & J_m \\
-J_m & 0_m
\end{pmatrix}
\]

if \( n \) is odd, and \( \Phi_n := J_{n+1} \) if \( n \) is even,

where by \( 0_k, k \in \mathbb{N} \), we mean a \( k \times k \) block of zeros, and \( J_k \) is the following \( k \times k \) block

\[
J_k := \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]

We find that

\[
d = d_n := \dim(T) = \begin{cases}
\frac{(n+1)(n+3)}{4} + 1, & \text{if } n \text{ is odd} \\
\frac{n(n+2)}{4} + 1, & \text{if } n \text{ is even}
\end{cases}.
\]
Next we are going to present a chart for the moduli space $T$. In order to do this, let $S = (s_{ij})_{1 \leq i,j \leq n+1}$ be a lower triangular matrix, whose entries are indeterminates $s_{ij}$, $i \geq j$ and $s_{11} = 1$. We define

$$
\begin{pmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_{n+1}
\end{pmatrix}^T = S
\begin{pmatrix}
\omega_1 & \omega_2 & \cdots & \omega_{n+1}
\end{pmatrix}^T,
$$

which implies that $\alpha$ forms a basis of $H^n_{dR}(X)$ compatible with its Hodge filtration. We would like that $(X, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}])$ be a member of $T$, hence it has to satisfy

$$(\langle \alpha_i, \alpha_j \rangle)_{1 \leq i,j \leq n+1} = \Phi,$$

from what we get the following equation

$$(2.22) \quad S\Omega S^T = \Phi.$$ 

Using this equation we can express $\frac{(n+2)(n+1)}{2} - d - 2$ numbers of parameters $s_{ij}$'s in terms of other $d - 2$ parameters that we fix them as independent parameters. For simplicity we write the first class of parameters as $t_1, t_2, \ldots, t_d$ and the second class as $t_2, t_3, \ldots, t_{n+1}, t_{n+3}, \ldots, t_d$. We put all these parameters inside $S$ according to the following rule which we write it only for $n = 1, 2, 3, 4, 5$:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
t_2 & t_1 & 0 & 0 & 0 \\
t_4 & t_3 & t_2 & 0 & 0 \\
t_6 & t_5 & t_4 & t_3 & 0 \\
t_8 & t_7 & t_6 & t_5 & t_4 \\
t_{10} & t_9 & t_8 & t_7 & t_6 \\
t_{12} & t_{11} & t_{10} & t_9 & t_8 \\
t_{14} & t_{13} & t_{12} & t_{11} & t_{10} \\
t_{16} & t_{15} & t_{14} & t_{13} & t_{12} \\
t_{18} & t_{17} & t_{16} & t_{15} & t_{14} \\
t_{20} & t_{19} & t_{18} & t_{17} & t_{16} \\
t_{22} & t_{21} & t_{20} & t_{19} & t_{18}
\end{pmatrix}.
$$

Note that we have already used $t_1, t_{n+1}$ as coordinates system of $S$. In particular we find:

$$(2.23) \quad s_{(n+2-i)(n+2-i)} = \frac{(-1)^{n+i+1} t_{i+2} - t_{n+2}}{c_n(n+2)^n s_{ii}}, \quad 1 \leq i \leq m.$$

Hence $t := (t_1, t_2, \ldots, t_d)$ forms a chart for the moduli space $T$, and in fact

$$\begin{align}
(2.24) & \quad T = \text{Spec}(\mathbb{C}[t_1, t_2, \ldots, t_d, 1/t_{n+2}(t_{n+2} - t_1^{n+2})]), \\
(2.25) & \quad \mathcal{O}_T = \mathbb{C}[t_1, t_2, \ldots, t_d, 1/t_{n+2}(t_{n+2} - t_1^{n+2})],
\end{align}$$

where $t$ is a product of $m - 1$ variables among $t_i$'s, $i = 1, 2, \ldots, d$, $i \neq 1, n + 2$. From now on, we alternately use either $s_{ij}$'s, or $t_i$'s and $\tilde{t}_j$'s to refer the entries of $S$. If we denote by $A$ the Gauss-Manin connection matrix of the family $X/T$ written in the basis $\alpha$, i.e., $\nabla \alpha = A \alpha$, then we calculate $A$ as follow:

$$(2.26) \quad A = \left( dS + S \cdot \tilde{A} \right)^{-1}.$$ 

In the following remarks we recall some results deduced from the proof of Theorem 1.2 in [14] in §7.

**Remark 2.1.** We obtain the functions $Y_i$'s given in (1.12) as follows: if $n$ is odd then

$$
(2.27) \quad Y_i = -Y_{n-(i+1)} = \frac{s_{22} s_{(i+1)(i+1)}}{s_{(i+2)(i+2)}}, \quad i = 1, 2, \ldots, n - 3, \\
(2.28) \quad Y_{n-\frac{3}{2}} = (-1)^{\frac{3n+3}{2}} c_n(n+2)^n \frac{s_{22} s_{n+1}^{\frac{1}{2}}}{t_1^{n+2} - t_{n+2}}
$$

and if $n$ is even then

$$
(2.29) \quad Y_i = -Y_{n-(i+1)} = \frac{s_{22} s_{(i+1)(i+1)}}{s_{(i+2)(i+2)}}, \quad i = 1, 2, \ldots, n - 2, \\
(2.30) \quad Y_{n-\frac{4}{2}} = (-1)^{\frac{3n+2}{2}} c_n(n+2)^n \frac{s_{22} s_{n+1}^{\frac{1}{2}}}{t_1^{n+2} - t_{n+2}}.
$$


Remark 2.2. Let $E \in \mathfrak{X}(T)$. If $\nabla_E \alpha = 0$ for any $(X, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}]) \in T$, then $E = 0$.

3 Algebraic group

For any positive integer $n$, let $G := G_n$ be the algebraic group given in (1.13). $G$ acts on $T$ from right as follow:

$$(X, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}]) \bullet g = (X, \alpha^r g),$$

where $\alpha := (\alpha_1 \ \alpha_2 \ \ldots \ \alpha_{n+1})^t$, $g \in G$, and in the right hand side of the above equality $\alpha^r g$ refers to matrix product. It is of interest to interpret this action in the chart $T$. To this end, we first announce some properties of $G$. Before that, note that we defined $m := \frac{n+1}{2}$, if $n$ is an odd integer, and $m := \frac{n}{2}$ if $n$ is an even integer.

The equation $g^r \Phi g = \Phi$ in (1.13) guaranties that $\dim(G) = d - 1$. For $i = 1, 2, \ldots, m$, by defining

$$G_i := \left\{ g = (g_{kl}) \in G \mid g \text{ is a diagonal matrix with } g_{ii} = g_{(n+2-i)(n+2-i)}^{-1} \in \mathbb{C}^*, \right\}$$

we find out that $G_i \cong \mathbb{G}_m$, hence $G$ has $m$ multiplicative subgroups. To describe additive subgroups of $G$, we discuss in the following two cases:

**First case:** If $n$ is odd, then for $1 \leq i \leq \frac{n+1}{2}$, $i + 1 \leq j \leq n + 2 - i$, define $G_{ij}$ as follow:

$$G_{ij} := \left\{ g = (g_{kl}) \in G \mid \begin{array}{l}
\text{g}_{kk} = 1, \text{ for } k = 1, 2, \ldots, n + 1, \\
\text{if } j \leq m, \text{ then } g_{ij} = -g_{(n+2-j)(n+2-i)} \in \mathbb{C}, \\
\text{if } j \geq m + 1, \text{ then } g_{ij} = g_{(n+2-j)(n+2-i)} \in \mathbb{C}, \\
\text{and the rest of the entries are zero.}
\end{array} \right\}$$

One can easily see that $G_{ij} \cong \mathbb{G}_a$ is an additive subgroup of $G$. Thus we find $d - (m + 1)(= \frac{(n+1)^2}{4})$ additive subgroups of $G$.

**Second case:** If $n$ is even, then for $1 \leq i \leq \frac{n}{2}$, $i + 1 \leq j \leq n + 1 - i$, we consider $G_{ij}$ as follow:

$$G_{ij} := \left\{ g = (g_{kl}) \in G \mid \begin{array}{l}
\text{g}_{kk} = 1, \text{ for } k = 1, 2, \ldots, n + 1, \\
\text{g}_{ij} = -g_{(n+2-j)(n+2-i)} \in \mathbb{C}, \\
\text{if } j = \frac{n+2}{2}, \text{ then } g_{i(n+2-i)} = -\frac{1}{2}g_{(n+2-j)(n+2-i)}^2, \\
\text{and the rest of the entries are zero.}
\end{array} \right\}$$

Again it is not difficult to show that $G_{ij} \cong \mathbb{G}_a$ is an additive subgroup of $G$. Therefore, in this case we get $d - (m + 1)(= \frac{n^2}{4})$ additive subgroups of $G$ as well. We give the new order $G_{m+1}, \ldots, G_{d-1}$ to additive subgroups $G_{ij}$'s, in such a way that in this order $G_{i,j}$ appears before $G_{i,j+1}$ provided that $i_1 < i_2$, and in the case that $i_1 = i_2$ then $G_{i,j}$ appears before $G_{i,j+1}$ if $j_1 < j_2$. Any $g_i \in G_i$, $i = 1, 2, \ldots, d - 1$, can be presented by a unique complex number, that we denote it again by $g_i$. For any $g \in G$ there are unique elements $g_i \in G_i$, $i = 1, 2, \ldots, d - 1$, such that $g = g_1g_2 \ldots g_{d-1}$. Therefore, we can represent any $g \in G$ by a $(d - 1)$-tuple $(g_1, g_2, \ldots, g_{d-1})$, where $g_i \in \mathbb{C}^*, i = 1, 2, \ldots, m$, and $g_i \in \mathbb{C}, i = m + 1, m + 2, \ldots, d - 1$; in the other words

$$G \cong \mathbb{G}_m \times \mathbb{G}_m \times \ldots \times \mathbb{G}_m \times \mathbb{G}_a \times \mathbb{G}_a \times \ldots \times \mathbb{G}_a \times G \times \ldots \times G \times \mathbb{G}_a \times \ldots \times G \times \mathbb{G}_a \times \ldots \times G \times \mathbb{G}_a \times \ldots \times G.$$ 

Hence we can summarize the above facts in the following lemma.
Lemma 3.1. $G$ is a $(d - 1)$-dimensional Lie group, and it has $m$ copies of $G_m$ as multiplicative subgroups, and $d - (m + 1)$ copies of $G_n$ as additive subgroups. Moreover, for any $g \in G$, there are unique elements $g_i \in G_i, \ i = 1, 2, \ldots, d - 1$ such that $g = g_1g_2 \cdots g_{d-1}$.

The following example helps to have a better imagination of elements of $G_i$'s.

Example 3.1. If $n=3$, then any $g \in G$ can be written as $g = g_1g_2g_3g_4g_5g_6$, where $g_i \in G_i$ are given as follows:

$$
\begin{align*}
  g_1 &= \begin{pmatrix}
      g_1^{-1} & 0 & 0 & 0 & 0 \\
      0 & 1 & 0 & 0 & 0 \\
      0 & 0 & 1 & 0 & 0 \\
      0 & 0 & 0 & g_1 & 0 \\
      0 & 0 & 0 & 0 & 1 
    \end{pmatrix},
  g_2 &= \begin{pmatrix}
      1 & 0 & 0 & 0 & 0 \\
      0 & g_2^{-1} & 0 & 0 & 0 \\
      0 & 0 & g_2 & 0 & 0 \\
      0 & 0 & 0 & 1 & 0 \\
      0 & 0 & 0 & 0 & 1 
    \end{pmatrix},
  g_3 &= \begin{pmatrix}
      1 & -g_3 & 0 & 0 & 0 \\
      0 & 1 & 0 & 0 & 0 \\
      0 & 0 & 1 & 0 & 0 \\
      0 & 0 & 0 & 0 & 1 \\
      0 & 0 & 0 & 0 & 1 
    \end{pmatrix},
  g_4 &= \begin{pmatrix}
      0 & 1 & 0 & 0 & 0 \\
      0 & 0 & 1 & 0 & 0 \\
      0 & 0 & 0 & 1 & 0 \\
      0 & 0 & 0 & 0 & 1 \\
      0 & 0 & 0 & 0 & 1 
    \end{pmatrix},
  g_5 &= \begin{pmatrix}
      1 & 0 & 0 & g_5 & 0 \\
      0 & 1 & 0 & 0 & g_5 \\
      0 & 0 & 1 & 0 & 0 \\
      0 & 0 & 0 & 1 & 0 \\
      0 & 0 & 0 & 0 & 1 
    \end{pmatrix},
  g_6 &= \begin{pmatrix}
      1 & 0 & 0 & 0 & 0 \\
      0 & 1 & 0 & 0 & 0 \\
      0 & 0 & 1 & 0 & 0 \\
      0 & 0 & 0 & 1 & 0 \\
      0 & 0 & 0 & 0 & 1 
    \end{pmatrix}.
\end{align*}
$$

If $n=4$, then any $g \in G$ can be written as $g = g_1g_2g_3g_4g_5g_6$, in which $g_i \in G_i$ are given below:

$$
\begin{align*}
  g_1 &= \begin{pmatrix}
      g_1^{-1} & 0 & 0 & 0 & 0 & 0 \\
      0 & 1 & 0 & 0 & 0 & 0 \\
      0 & 0 & 1 & 0 & 0 & 0 \\
      0 & 0 & 0 & g_1 & 0 & 0 \\
      0 & 0 & 0 & 0 & 1 & 0 \\
      0 & 0 & 0 & 0 & 0 & 1 
    \end{pmatrix},
  g_2 &= \begin{pmatrix}
      1 & 0 & 0 & 0 & 0 & 0 \\
      0 & g_2^{-1} & 0 & 0 & 0 & 0 \\
      0 & 0 & g_2 & 0 & 0 & 0 \\
      0 & 0 & 0 & g_2 & 0 & 0 \\
      0 & 0 & 0 & 0 & 1 & 0 \\
      0 & 0 & 0 & 0 & 0 & 1 
    \end{pmatrix},
  g_3 &= \begin{pmatrix}
      1 & -g_3 & 0 & 0 & 0 & 0 \\
      0 & 1 & 0 & 0 & 0 & 0 \\
      0 & 0 & 1 & 0 & 0 & 0 \\
      0 & 0 & 0 & 1 & 0 & 0 \\
      0 & 0 & 0 & 0 & 1 & 0 \\
      0 & 0 & 0 & 0 & 0 & 1 
    \end{pmatrix},
  g_4 &= \begin{pmatrix}
      0 & 1 & 0 & 0 & 0 & 0 \\
      0 & 0 & 1 & 0 & 0 & 0 \\
      0 & 0 & 0 & 1 & 0 & 0 \\
      0 & 0 & 0 & 0 & 1 & 0 \\
      0 & 0 & 0 & 0 & 0 & 1 \\
      0 & 0 & 0 & 0 & 0 & 1 
    \end{pmatrix},
  g_5 &= \begin{pmatrix}
      1 & 0 & 0 & -g_5 & 0 & 0 \\
      0 & 1 & 0 & 0 & g_5 & 0 \\
      0 & 0 & 1 & 0 & 0 & g_5 \\
      0 & 0 & 0 & 1 & 0 & 0 \\
      0 & 0 & 0 & 0 & 1 & 0 \\
      0 & 0 & 0 & 0 & 0 & 1 
    \end{pmatrix},
  g_6 &= \begin{pmatrix}
      1 & 0 & 0 & 0 & 0 & 0 \\
      0 & 1 & 0 & 0 & 0 & 0 \\
      0 & 0 & 1 & 0 & 0 & 0 \\
      0 & 0 & 0 & 1 & 0 & 0 \\
      0 & 0 & 0 & 0 & 1 & 0 \\
      0 & 0 & 0 & 0 & 0 & 1 
    \end{pmatrix}.
\end{align*}
$$

From [2,6] we get that for any $k \in C^*$

\[(X_{t_1, t_{n+2}}, S\omega) \cong (X_{kt_1, k^n t_{n+2}}, S \begin{pmatrix}
      k & 0 & \cdots & 0 & 0 \\
      0 & k^2 & \cdots & 0 & 0 \\
      \vdots & \vdots & \ddots & \vdots & \vdots \\
      0 & 0 & \cdots & k^n & 0 \\
      0 & 0 & \cdots & 0 & k^{n+1}
    \end{pmatrix} \omega),\]

in which $\omega = (\omega_1, \omega_2, \ldots, \omega_{n+1})^t$ and $S$ is the basis change matrix $\alpha = S\omega$.

Let $(X_{t_1, t_{n+2}}[\alpha_1, \alpha_2, \ldots, \alpha_{n+1}])$ be a presentation of $t = (t_1, t_2, \ldots, t_d) \in T$. Then for any $g = (g_1, g_2, \ldots, g_{n+1}) \in G$ we have:

\[(X_{t_1, t_{n+2}}, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}]) \circ g = (X_{t_1, t_{n+2}}, (S\omega)^t g)\]

\[(3.1) \quad \cong (X_{g_1t_1, g_1^n t_{n+2}}, S \begin{pmatrix}
      g_1 & 0 & \cdots & 0 & 0 \\
      0 & g_1^2 & \cdots & 0 & 0 \\
      \vdots & \vdots & \ddots & \vdots & \vdots \\
      0 & 0 & \cdots & g_1^n & 0 \\
      0 & 0 & \cdots & 0 & g_1^{n+1}
    \end{pmatrix} \omega) \circ g,\]

and the last pair of the above equations give us the action $(t_1, t_2, \ldots, t_d) \circ g$.

Example 3.2. Following we give the action of $g = (g_1, g_2, \ldots, g_{d-1}) \in G$ on $t = (t_1, t_2, \ldots, t_d) \in T$ for $n = 1, 2, 3, 4$:

$n=1$:

\[(t_1, t_2, t_3) \circ g = (t_1g_1, t_2g_1^2 + g_2, t_3g_1^3).\]

$n=2$:

\[(t_1, t_2, t_4) \circ g = (t_1g_1, t_2g_1 - g_2, t_4g_1^4).\]
n=3:

\[
\begin{align*}
t_1 \cdot g &= t_1g_1, \\
t_2 \cdot g &= (t_2g_1 - g_2g_3)g_2^{-1}, \\
t_3 \cdot g &= t_3g_1^2g_2^{-1}, \\
t_4 \cdot g &= (t_2g_1g_6 + t_4g_1g_2 - g_2g_3g_6 + g_2g_4)g_2^{-1}, \\
t_5 \cdot g &= t_5g_1^5, \\
t_6 \cdot g &= (t_3g_1^2 + t_6g_1^2)g_2^{-1}, \\
t_7 \cdot g &= (t_2g_1g_4 + t_4g_1g_2^2g_3 + t_7g_1g_2 - g_2g_3g_4 + g_2g_5)g_2^{-1},
\end{align*}
\]

n=4:

\[
\begin{align*}
t_1 \cdot g &= t_1g_1, \\
t_2 \cdot g &= (t_2g_1 - g_2g_3)g_2^{-1}, \\
t_3 \cdot g &= t_3g_1^2g_2^{-1}, \\
t_4 \cdot g &= (-t_2g_1g_6 + t_4g_1g_2 + g_2g_3g_6 - g_2g_4)g_2^{-1}, \\
t_5 \cdot g &= (-t_3g_1^2g_6 + t_5g_2g_2^{-1}, \\
t_6 \cdot g &= t_6g_1^6, \\
t_7 \cdot g &= \frac{1}{2}(-t_2g_1g_6^2 + 2t_4g_1g_2g_6 + 2t_7g_1g_2^2 + g_2g_3g_6^2 - 2g_2g_4g_6 - 2g_2g_5)g_2^{-1}, \\
t_8 \cdot g &= t_8g_1^3,
\end{align*}
\]

4 Lie algebra

The Lie algebra Lie(G) of G is given in [14]. Next we give the canonical basis of Lie(G).

If n is odd, then for \(1 \leq a \leq \frac{n+1}{2}\), \(a \leq b \leq n + 2 - a\), define \(g_{ab}\) as follow:

(4.1)

\[
g_{ab} = (g_{kl})_{(n+1) \times (n+1)}, \text{ such that } \begin{cases} 
  \text{if } b \leq m, \text{ then } g_{ab} = 1, & g_{(n+2-b)(n+2-a)} = -1, \\
  \text{if } b = m + 1, \text{ then } g_{ab} = g_{(n+2-b)(n+2-a)} = 1, & \text{and the rest of the entries are zero.}
\end{cases}
\]

And if n is even, then for \(1 \leq a \leq \frac{n}{2}\), \(a \leq b \leq n + 1 - a\), we consider \(g_{ab}\) as follow:

(4.2)

\[
g_{ab} = (g_{kl})_{(n+1) \times (n+1)}, \text{ such that } \begin{cases} 
  \text{if } b = m, \text{ then } g_{ab} = 1, & g_{(n+2-b)(n+2-a)} = -1, \\
  \text{and the rest of the entries are zero.}
\end{cases}
\]

One can easily observe that the set of \(g_{ab}\)'s construct a basis of Lie(G).

Example 4.1. If \(n=3\), then we get:

\[
\begin{align*}
\phi_{11} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \phi_{22} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, & \phi_{12} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
\phi_{13} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \phi_{14} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \phi_{23} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

If \(n=4\), then it follows:

\[
\begin{align*}
\phi_{11} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & \phi_{22} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \phi_{12} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\phi_{13} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \phi_{14} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \phi_{23} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]
4.1 Proof of Theorem 1.2

This proof is somehow analogous to the proof of Theorem 1.1 given in [MN16 §7]. We first construct another moduli space $\tilde{T}$ that contains $T$. In order to do this assume that all the entries $s_{ij}, j \leq i, (i,j) \neq (1,1)$ of $S$ are independent parameters. We denote by $\tilde{T}$ and $\tilde{\alpha}$ the corresponding family of varieties and a basis of differential forms. Indeed $\tilde{T}$ is formed the same as $T$ by removing the condition (2.18).

Let $g \in \text{Lie}(G)$ be arbitrary. In account of (2.20), the existence of a vector field

$$R_g := t_1 \frac{\partial}{\partial t_1} + t_{n+2} \frac{\partial}{\partial t_{n+2}} + \sum_{i=2,j=1}^{n+1,i} s_{ij} \frac{\partial}{\partial s_{ij}},$$

in $\tilde{T}$ with the desired property (1.15) is equivalent to solve the equation

$$(4.3) \quad \dot{S} = A_{R_g} \cdot S - S \cdot \hat{A}(R_g).$$

Note that here $\dot{x} := dx(R_g)$ is the derivation of the function $x$ along the vector field $R_g$ in $\tilde{T}$. The equalities corresponding to the entries $(i,j), j \leq i, (i,j) \neq (1,1)$ serves as the definition of $s_{ij}$. The equality corresponding to $(1,1)$-th and $(1,2)$-th entries give us respectively $t_1$ and $t_{n+2}$. All the rest are trivial equalities $0 = 0$. We conclude the statement of Theorem 1.1 for the moduli space $\tilde{T}$.

To prove the statement for the moduli space $T$, consider the map

$$(4.4) \quad \tilde{T} \to \text{Mat}_{(n+1) \times (n+1)}(\mathbb{C}), \quad (t_1, t_{n+2}, S) \mapsto S \Omega S^\text{tr}.$$

It follows that $T$ is the fiber of this map over the point $\Phi$. We prove that the vector field $R_g$ is tangent to the fiber of the above map over $\Phi$. This follows from

$$\tilde{S} \Omega \tilde{S}^\text{tr} = \dot{S} \Omega \tilde{S}^\text{tr} + S \dot{\Omega} S^\text{tr} + S \tilde{\Omega} \dot{S}^\text{tr}$$

$$= (A_{R_g} S - S \hat{A}(R_g)) \Omega \tilde{S}^\text{tr} + S (\hat{A}(R_g) \Omega + \Omega \hat{A}^\text{tr}(R_g)) S^\text{tr} + S \Omega (S^\text{tr} A_{R_g} - \hat{A}^\text{tr}(R_g) S^\text{tr})$$

$$= A_{R_g} \Phi + \Phi A_{R_g}^\text{tr}$$

$$= g^\text{tr} \Phi + \Phi g$$

$$= 0,$$

where $\dot{x} := dx(R_g)$ is the derivation of the function $x$ along the vector field $R_g$ in $T$. Note that in the above equalities we are using (2.17) and the the fact that $g$ belongs to $\text{Lie}(G)$.

The uniqueness of $R_g$ follows from Remark 2.2.

4.2 Lie(G) as a Lie subalgebra of $\mathfrak{x}(T)$

We know that Gauss-Manin connection $\nabla$ is a flat (or integrable) connection, i.e., for any $E_1,E_2 \in \mathfrak{x}(T)$ we have $\nabla_{[E_1,E_2]} = \nabla_{E_1} \nabla_{E_2} - \nabla_{E_2} \nabla_{E_1}$, where $[E_1,E_2]$ refers to Lie bracket of vector fields. On the other hand, if $\nabla_{E_1} \alpha = A_{E_1} \alpha$ and $\nabla_{E_2} \alpha = A_{E_2} \alpha$, then one obtains

$$\nabla_{E_1} \nabla_{E_2} \alpha = A_{E_2} A_{E_1} \alpha + E_1(A_{E_2}).$$

Hence, if we set $\nabla_{[E_1,E_2]} \alpha = A_{[E_1,E_2]} \alpha$, then we get:

$$(4.5) \quad A_{[E_1,E_2]} = [A_{E_2}, A_{E_1}] + E_1(A_{E_2}) - E_2(A_{E_1}).$$
where on the right hand side by $[A_{E_2}, A_{E_1}]$ we mean the Lie bracket of matrices. Since elements of Lie($G$) are constant matrices, for any $g_1, g_2 \in \text{Lie}(G)$ we obtain:

$$A_{[R_{g_1}, R_{g_2}]} = [A_{R_{g_2}}, A_{R_{g_1}}].$$

Consider the map

$$\varphi : \text{Lie}(G) \to \frakh(T), \quad \varphi(g) = R_g.$$  

For any $g_1, g_2 \in \text{Lie}(G)$, by using (1.15) we have:

$$A_{\varphi([g_1, g_2])} = [g_1, g_2]^\nu = [A_{R_{g_2}}, A_{R_{g_1}}].$$

Therefore in account of (4.6), (4.7) and Remark 2.2 we get $\varphi([g_1, g_2]) = [R_{g_1}, R_{g_2}]$, which implies that the Lie algebra generated by $R_{gab}$'s, $1 \leq a \leq m, a \leq b \leq 2m+1-a$ in $\frakh(T)$ is isomorphic to $\text{Lie}(G)$. Hence we use the same notation $\text{Lie}(G)$ for the Lie algebra generated by $R_{gab}$'s, and by employing (4.6) and Remark 2.2 we can determine the Lie bracket of $\text{Lie}(G) \subset \frakh(T)$ completely.

### 4.3 AMSY-Lie algebra

As we mentioned in Section 1, we define AMSY-Lie algebra $\frak{g}$ as a Lie subalgebra of $\frakh(T)$ generated by $\text{Lie}(G)$ and the modular vector field $R$. Next objective is to determine the Lie bracket of $\frak{g}$, but it is not as easy as the Lie bracket of $\text{Lie}(G)$, since not all members of $\frak{g}$ are constant matrices. In order to do this it is enough to determine $[R, R_{gab}]$, for $1 \leq a \leq m, a \leq b \leq 2m+1-a$, which have been given in Theorem 1.3. Before proving this theorem, note that if we denote by $E = \sum_{i=1}^d \dot{t}_i \frac{\partial}{\partial t_i} \in \frakh(T)$, then we find $\dot{t}_i$'s from

$$\dot{S} = A_E S - S \dot{A}(E),$$

where $\dot{x} := dx(E)$ is the derivation of the function $x$ along the vector field $E$ in $T$. We saw in Section 2 that any $t_i, 1 \leq i \leq d, i \neq 1, n+2$, corresponds to only one $s_{jk}, 1 \leq j, k \leq n+1$.

### 4.4 Proof of Theorem 1.3

Equation (1.5) yields:

$$A_{[R, R_{gab}]} = [A_{R_{gab}}, \frak{Y}] - R_{gab}(\frak{Y}).$$

Hence to determine $[R, R_{gab}]$ we need to compute $R_{gab}(\frak{Y})$. If we apply (4.8) to $R_{g11}$, then the equalities corresponding to (1,1)-th and (1,2)-th entries give us respectively

$$\dot{t}_1 = -t_1, \quad \dot{t}_{n+2} = -(n+2)t_2,$$

from what we obtain the diagonal matrix $\bar{A}(R_{g11}) = \text{diag}(1, 2, \ldots, n+1)$. This implies

$$R_{g11} = \sum_{i=1}^d c_{g11} t_i \frac{\partial}{\partial t_i},$$

where $c_{g11}$'s are constants given as follows:

$$c_{g11} = \begin{cases}  
-1, & \text{if } i = 1, \\
-(n+2), & \text{if } i = n+2, \\
-2, & \text{if } i = d \text{ and } n \text{ is odd}, \\
-1, & \text{if } i = d \text{ and } n \text{ is even}, \\
-k, & \text{if } t_i = s_{jk} \text{ and } i \neq 1, n+2, d. 
\end{cases}$$
For $2 \leq a \leq m$, we employ (4.13) for $R_{gab}$ that yields $i_1 = 0$ and $i_{n+2} = 0$. Hence we find $\hat{A}(R_{gab}) = 0$, which implies

$$R_{gab} = \sum_{i=1}^{d} c_{igaa} t_i \frac{\partial}{\partial t_i},$$

where $c_{igaa}$'s are following constants:

$$c_{igaa} = \begin{cases} 
1, & \text{if } t_i = s_{ak}, \text{ for some } 1 \leq k \leq n + 1, \\
-1, & \text{if } t_i = s_{(n+2-a)k}, \text{ for some } 1 \leq k \leq n + 1, \\
0, & \text{otherwise}.
\end{cases}$$

(4.11)

Analogously for $R_{gab}$, $1 \leq a \leq m$, $a + 1 \leq b \leq 2m + 1 - a$, we obtain $\hat{A}(R_{gab}) = 0$, thus we encounter $R_{gab}$ explicitly using the equation

$$\dot{S} = A_{R_{gab}} S.$$  

(4.12)

The equations (4.10) and (4.11) yield:

$$R_{g_{11}} = -(t_1 \frac{\partial}{\partial t_1} + (n + 2) t_{n+2} \frac{\partial}{\partial t_{n+2}}) - \sum_{k=2}^{m} k s_{kk} \frac{\partial}{\partial s_{kk}} + \hat{R}_{g_{11}},$$

(4.13)

$$R_{g_{aa}} = s_{aa} \frac{\partial}{\partial s_{aa}} + \hat{R}_{g_{aa}}, \quad a = 2, 3, \ldots, m,$$

(4.14)

where in the part $\hat{R}_{g_{aa}}$, $a = 1, 2, 3, \ldots, m$, do not appear the terms including $t_1 \frac{\partial}{\partial t_1}$, $t_{n+2} \frac{\partial}{\partial t_{n+2}}$ and $s_{jj} \frac{\partial}{\partial s_{jj}}$, $j = 2, 3, \ldots, m$. Hence in account of equations (2.27), (2.28) and (2.29) we get:

$$R_{g_{11}}(Y_i) = -Y_i, \quad 1 \leq i \leq m - 1,$$

(4.15)

$$R_{g_{22}}(Y_i) = 2Y_i, \quad R_{g_{22}}(Y_i) = Y_i, \quad 2 \leq i \leq m - 1,$$

(4.16)

$$R_{g_{aa}}(Y_{a-2}) = -Y_{a-2}, \quad R_{g_{aa}}(Y_{a-1}) = Y_{a-1}, \quad 3 \leq a \leq m - 1,$$

(4.17)

$$R_{g_{aa}}(Y_i) = 0, \quad 3 \leq a \leq m - 1, \quad 1 \leq i \leq m - 1 \text{ and } i \neq a - 2, a - 1,$$

(4.18)

if $n$ is odd, then

$$R_{g_{mn}}(Y_{m-2}) = -Y_{m-2}, \quad R_{g_{mn}}(Y_{m-1}) = 2Y_{m-1}, \quad R_{g_{mn}}(Y_i) = 0, \quad 1 \leq i \leq m - 3,$$

(4.19)

and if $n$ is even, then

$$R_{g_{mn}}(Y_{m-2}) = -Y_{m-2}, \quad R_{g_{mn}}(Y_{m-1}) = Y_{m-1}, \quad R_{g_{mn}}(Y_i) = 0, \quad 1 \leq i \leq m - 3.$$

(4.20)

These relations together with (4.19) and Remark 2.2 prove (1.19), (1.17) and (1.18). Using the equation (4.12) we find out that in the expression of $R_{g_{ab}}$, $1 \leq a \leq m$, $a + 1 \leq b \leq 2m + 1 - a$, the terms including $\frac{\partial}{\partial s_{aa}}$ do not appear, hence $R_{g_{ab}}(Y) = 0$, which together with (4.19) and Remark 2.2 finishes the proof of (1.19).

5 \quad \mathfrak{sl}_2(\mathbb{C}) \text{ Lie algebra and the weights}

In the following example we observe that in the cases $n = 1, 2$ the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. 

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Example 5.1. If $n = 1$, then we have:

$$
Y = \left( \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad g_{11}^{tr} = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right), \quad g_{12}^{tr} = \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right),
$$

from what we find $R, H := -R_{g_{11}},$ and $F := R_{g_{12}},$ respectively, given in [1.6], [1.7] and (1.8), respectively. If $n = 2$, then we obtain:

$$
Y = \left( \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right), \quad g_{11}^{tr} = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right), \quad g_{12}^{tr} = \left( \begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right),
$$

$$
R = (t_3 - t_1 t_2) \frac{\partial}{\partial t_1} + (2 t_1 - \frac{1}{2} t_2) \frac{\partial}{\partial t_2} + (-2 t_2 t_3 + 8 t_3^5) \frac{\partial}{\partial t_3} + (-4 t_2 t_4) \frac{\partial}{\partial t_4},
$$

$$
R_{g_{11}} = -t_1 \frac{\partial}{\partial t_1} - t_2 \frac{\partial}{\partial t_2} - 2 t_3 \frac{\partial}{\partial t_3} - 4 t_4 \frac{\partial}{\partial t_4},
$$

$$
R_{g_{12}} = \frac{\partial}{\partial t_2},
$$

where the polynomial equation $t_2^2 = 4(t_1^4 - t_4)$ holds among $t_i$'s. In this case we define $H := -2R_{g_{11}}$ and $F := 2R_{g_{12}}.$ Hence, for $n = 1, 2$ by letting $e = R, f = F, h = H$ we observe that $\mathcal{G}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C}).$

**Proof of Theorem 1.4.** For $n \geq 3$ define $H := R_{g_{22}} - R_{g_{11}}$ and $F := R_{g_{12}}.$ The equation (1.12) yields:

$$
F = \begin{cases} 
\frac{\partial}{\partial t_2}, & \text{if } n \text{ is even}, \\
\frac{\partial}{\partial t_2} - s_{n1} \frac{\partial}{\partial t_3}, & \text{if } n \text{ is odd.}
\end{cases}
$$

If we consider $e = R, f = F, h = H,$ then it is an immediate result of Theorem 1.3 that Lie algebra generated by $R, H, F$ in $\mathfrak{X}(T)$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C}).$ □

In the following example we state $R, H, F$ for $n = 3, 4.$

Example 5.2. If $n = 3,$ then we get:

$$
R = (t_3 - t_1 t_2) \frac{\partial}{\partial t_1} + \frac{t_3^2 t_4 - 5 t_2^2 t_1^5 - t_5}{5 t_5(t_1^2 - t_5)} \frac{\partial}{\partial t_2} + \frac{t_3^2 t_5 - 3 \times 5^4 t_2 t_3(t_1^5 - t_5)}{5 t_5(t_1^2 - t_5)} \frac{\partial}{\partial t_3}
$$

$$
+ (-2 t_2 t_4 - t_7) \frac{\partial}{\partial t_4} + (-5 t_2 t_5) \frac{\partial}{\partial t_5} + (-2 t_3 t_4 + 5 t_3^8) \frac{\partial}{\partial t_6} + (-5 t_1 t_3 - 2 t_2 t_7) \frac{\partial}{\partial t_7},
$$

$$
H = t_1 \frac{\partial}{\partial t_1} + 2 t_2 \frac{\partial}{\partial t_2} + 3 t_3 \frac{\partial}{\partial t_3} + 5 t_5 \frac{\partial}{\partial t_5} + 6 \frac{\partial}{\partial t_6} + 2 t_7 \frac{\partial}{\partial t_7},
$$

$$
F = \frac{\partial}{\partial t_2} - t_4 \frac{\partial}{\partial t_7}.
$$

If $n = 4,$ then we obtain $R, H$ and $F,$ respectively, as given in (1.9), (1.10) and (1.11), respectively.

Remark 5.1. Note that in the cases $n = 2$ and $n = 4,$ respectively, to avoid the second root of $t_1^2$ and $t_3,$ respectively, we added extra variables $t_3 := \tilde{t}_2$ and $t_8 := \tilde{t}_3$ and stated a polynomial equation which follows from (2.23). This happens in all even cases.

Set $H = \sum_{i=0}^{d} w_i t_i \frac{\partial}{\partial t_i},$ $R = \sum_{i=1}^{d} \tilde{t}_i \frac{\partial}{\partial \tilde{t}_i},$ and in $\mathcal{O}_T$ define $weight(t_i) = w_i.$ Hence by above computations, for $n = 1, 2, 3, 4$ we can trivially observe that the rational functions $t_i$'s are quasi-homogeneous functions in $\mathcal{O}_T.$ If we suppose that the derivation of $t_i$'s in
direction of \( R \), namely \( \dot{t}_i \), increases the weight of \( t_i \) by two, then in the expression of \( R \) we observe that \( \text{weight}(\dot{t}_i) = w_i + 2 \). This observation make sense, because in the case \( n = 1, 2 \) we found the solutions of \( R \) in terms of (quasi-)modular forms, and we know that derivation of (quasi-)modular forms increases their weight by two. Thus we believe that this observation should be true for any \( n \).

Moreover, in account of (4.10) and (4.11) we get that the term \( \frac{\partial}{\partial s_n} \) does not appear in the expression of \( H \), hence the weight of \( t_k \) corresponding to \( s_n \) is zero, which implies \( F \) to be a constant vector field and \( H \) to be a radial vector field in the weighted space \( \mathcal{O}_T \). This observation coincides with the definition of Halphen type and semi-simple vector fields given in [Gui07, GR12].

All these facts about weights, motivate us to pursue the studying of dynamic of modular vector field \( R \) in the weighted space \( \mathcal{O}_T \) with assigned weights \( \text{weight}(t_i) = w_i \).

**Remark 5.2.** Let \( f \in \mathcal{O}_T \) be a quasi-homogenous polynomial of degree \( k \). Then \( H(f) = kf \), and hence \( [H, fR] = (k + 2)fR \). In particular if \( f = t_1^{n+2} - t_{n+2} \), then we get

\[
\{fR, F\} = fH, \quad [H, fR] = (n + 4)fR.
\]

Therefore instead of working with \( R \), we may use \( fR \) which is a polynomial vector field and is more convenient to work with.

**Remark 5.3.** Let \( \mathcal{O}_T^k \), \( k \geq 0 \), be the space of quasi-homogenous regular functions of degree \( k \) in \( \mathcal{O}_T \). Then we have:

\[
R : \mathcal{O}_T^k \to \mathcal{O}_T^{k+2}, \\
H : \mathcal{O}_T^k \to \mathcal{O}_T^k, \\
F : \mathcal{O}_T^k \to \mathcal{O}_T^{k-2}.
\]

### 5.1 Another \( \mathfrak{sl}_2(\mathbb{C}) \)

For \( n = 3 \), if we set

\[
E := (t_3 - t_1 t_2) \frac{\partial}{\partial t_1} + (-t_2) \frac{\partial}{\partial t_2} + (-3t_2 t_3) \frac{\partial}{\partial t_3} + (-t_2 t_4 - t_7) \frac{\partial}{\partial t_4} + (-5t_2 t_5) \frac{\partial}{\partial t_5} + (-2t_6 + 5t_1 t_3 + 5^2 t_3) \frac{\partial}{\partial t_6} + (-5^4 t_1 t_3 - t_2 t_7) \frac{\partial}{\partial t_7},
\]

then we find

\[
\mathbf{A}_E = \begin{pmatrix}
0 & 1 & 0 & 0 \\
\frac{t_2 t_4 - t_5}{t_3 (t_1 - t_5)} & \frac{1}{t_3 (t_1 - t_5)} & \frac{t_3}{t_3 (t_1 - t_5)} & 0 \\
-\frac{t_6}{t_3 (t_1 - t_5)} & \frac{t_6}{t_3 (t_1 - t_5)} & \frac{t_6}{t_3 (t_1 - t_5)} & -1 \\
-\frac{t_2 t_3^2 + 2t_2 t_3 t_4 t_6 - t_3^2 t_4}{5^4 (t_1 - t_5)} & \frac{t_2 t_3^2 - t_3 t_4}{5^4 (t_1 - t_5)} & \frac{t_2 t_3^2 - t_3 t_4}{5^4 (t_1 - t_5)} & 0
\end{pmatrix}
\]

\[
= Y - \frac{t_6}{t_3} Y_1 A_{R_{911}} + \frac{t_2 t_6 - t_3 t_4}{t_3} Y_1 A_{R_{912}} + \frac{t_2 t_6^2 - t_3 t_4 t_6}{t_3^2} Y_1 A_{R_{913}} + \frac{t_2 t_6^2 + 2t_2 t_3 t_4 t_6 - t_3^2 t_4}{t_3^2} Y_1 A_{R_{914}} - \frac{t_2^6}{t_3^2} Y_1 A_{R_{921}},
\]
from what we get $E \in \mathcal{G}$. For $n = 4$ let
\[
E := (t_3 - t_1 t_2) \frac{\partial}{\partial t_1} + (-t_2^2) \frac{\partial}{\partial t_2} + (-3 t_2 t_3) \frac{\partial}{\partial t_3} + (-t_2 t_4) \frac{\partial}{\partial t_4}
+ (-2 t_2 t_5 - t_3 t_4) \frac{\partial}{\partial t_5} + (-6 t_2 t_6) \frac{\partial}{\partial t_6} + \frac{6 - 2 t_1^2 - t_1^2}{2 \times 6 - 2} \frac{\partial}{\partial t_7} + (-3 t_2 t_8) \frac{\partial}{\partial t_8}.
\]
In this case we find the $(3, 3)$-th entry of $A_E$ as $-\frac{3 t_1^2 t_5}{t_1^2 - t_6}$, which implies $E \notin \mathcal{G}$. But in both cases $n = 3, 4$ if we let $e = E$, $f = F$, $h = H$, then it follows that the Lie algebra generated by $E$, $H$, $F$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Here $E$ is a polynomial vector field and it follows from $R$ by discarding the non-polynomial terms of the components of $R$. All discussion related with $R$, given in Section 5, about the weights and Remarks 5.2 and 5.3 hold for $E$ as well.

The author believes that for any $n$ we can deduce such a polynomial vector field $E$, which is simpler to work with.

6 $\mathfrak{sl}_2(\mathbb{C})$ attached to Calabi-Yau threefolds

In this section we suppose that $X$ is a non-rigid compact Calabi-Yau threefold on $\mathbb{C}$, and $h := h^{21}$ is the Hodge number of type $(2, 1)$ of $X$. In [AMSY16] the AMSY-Lie algebra $\mathcal{G}$ attached to the Calabi-Yau threefold $X$ is studied and here we first give a brief summary of that, and for more details the reader is referred to the same reference. Then we observe that there are $h$ copies of $\mathfrak{sl}_2(\mathbb{C})$ in $\mathcal{G}$. Note that $\dim(H^3_{dR}(X)) = 2h + 2$ and its Hodge filtration is as follow:

\[
0 = F^1 \subset F^2 \subset F^1 \subset F^0 = H^3_{dR}(X).
\]

The enhanced moduli space $T$ is the moduli of the pairs $(X, [\alpha_1, \alpha_2, \ldots, \alpha_{2h+2}])$, where $X$ is as above and $\{\alpha_i\}_{i=1}^{2h+2}$ is a basis of $H^3_{dR}(X)$ with the properties:

\[
\alpha_1 \in F^3, \quad \alpha_1, \alpha_2, \ldots, \alpha_{h+1} \in F^2, \quad \alpha_1, \alpha_2, \ldots, \alpha_{2h+1} \in F^1, \quad \alpha_1, \alpha_2, \ldots, \alpha_{2h+2} \in F^0,
\]

\[
[(\alpha_i, \alpha_j)] = \Phi,
\]

in which $\Phi$ is the following constant matrix:

\[
\Phi := \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1_{h \times h} & 0 \\
0 & -1_{h \times h} & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]

(6.1)

Here, it is used $(2h + 2) \times (2h + 2)$ block matrices according to the decomposition $2h + 2 = 1 + h + h + 1$ and $1_{h \times h}$ denotes the $h \times h$ identity matrix. The algebraic group

\[
G := \{ g \in \text{GL}(2h + 2, \mathbb{C}) \mid g \text{ is block upper triangular and } g^T \Phi g = \Phi \}
\]

acts from the right on $T$ and its Lie algebra is as follow

\[
\text{Lie}(G) = \{ g \in \text{Mat}(2h + 2, \mathbb{C}) \mid g \text{ is block upper triangular and } g^T \Phi + \Phi g = 0 \}.
\]

Here, by block triangular we mean triangular with respect to the partition $2h + 2 = 1 + h + h + 1$. One finds that

\[
\dim(G) = \frac{3h^2 + 5h + 4}{2}, \quad \dim(T) = h + \dim(G).
\]
In [AMSY16] it is proved that there are unique modular vector fields $R_k$, $k = 1, 2, \ldots, h$ in $T$ and unique $C_{ijk}^{\text{alg}} \in \mathcal{O}_T$, $i, j, k = 1, 2, \ldots, h$ symmetric in $i, j, k$ such that

$$A_{R_k} = \begin{pmatrix} 0 & \delta^j_k & 0 & 0 \\ 0 & 0 & C_{ijk}^{\text{alg}} & 0 \\ 0 & 0 & 0 & \delta^j_i \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

Also, for any $g \in \text{Lie}(G)$ there is a unique vector field $R_g$ in $T$ such that

$$A_{R_g} = g^\text{tr}.$$ 

Here the AMSY-Lie algebra $\mathfrak{g}$ is the $\mathcal{O}_T$-module generated by the vector fields

(6.2) $R_i, R_g, i = 1, 2, \ldots, h, \quad g \in \text{Lie}(G).$

One finds the following canonical basis of $\text{Lie}(G)$:

$$t_{ab} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2}(\delta^j_a \delta^j_b + \delta^j_b \delta^j_a) & 0 & 0 \end{pmatrix}, t_a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta^j_i & 0 & 0 \\ 0 & \delta^j_i & 0 \end{pmatrix}, t := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\xi^a := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\delta^j_i & 0 & 0 & 0 \end{pmatrix}, \xi_b^a := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta^j_i & 0 & 0 & 0 \end{pmatrix}, \xi_0 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

The Lie algebra structure of $\mathfrak{g}$ is given by the following table.

| $R_{g_0}$ | $R_{g_1}$ | $R_{t_{1cd}}$ | $R_{t_1}$ | $R_{t_2}$ | $R_{t_3}$ | $R_c$ |
|-----------|-----------|--------------|-----------|-----------|-----------|-------|
| $R_{g_0}$ | 0         | 0            | $-R_c$    | $-R_c$    | $-R_c$    | $-R_c$ |
| $R_{g_1}$ | 0         | $\delta^i_a R_{t_1}$ | 0         | 0         | 0         | $\frac{1}{2}(\delta^j_b R_{t_2} + \delta^j_b R_{t_1})$ |
| $R_{t_{1cd}}$ | 0 | $\delta^i_a R_{t_{1cd}} + \delta^i_a R_{t_{ac}}$ | 0         | 0         | 0         | $\frac{1}{2}(\delta^j_b R_{t_1} + \delta^j_b R_{t_0})$ |
| $R_{t_1}$ | $R_{t_1}$ | 0            | 0         | 0         | 0         | $2\delta^i_a R_{t_1}$ |
| $R_{t_2}$ | 0         | 0            | 0         | 0         | 0         | $2\delta^i_a R_{t_2}$ |
| $R_{t_3}$ | 0         | 0            | 0         | 0         | 0         | $2\delta^i_a R_{t_3}$ |
| $R_c$ | 0         | 0            | 0         | 0         | 0         | $2\delta^i_a R_{t_0}$ |

Here we find $h$ copies of $\mathfrak{s}(2)(C)$ in $\mathfrak{g}$; indeed if for $k = 1, 2, \ldots, h$ we define

$$H_k := R_{g_0} - R_{g_k}^h, \quad F_k := R_{t_k},$$

then by setting $e = R_k$, $f = F_k$, $h = H_k$ we get that the Lie algebra generated by $R_k, H_k, F_k$ is isomorphic to $\mathfrak{s}(2)(C)$.

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