Novel vortices in a rotating holographic superfluid

Ankur Srivastav*, Sunandan Gangopadhyay†

Department of Theoretical Sciences,
S. N. Bose National Centre for Basic Sciences,
Block-JD, Sector-III, Salt Lake City,
Kolkata 700106, India

We have analytically devised novel vortex solutions in a rotating holographic superfluid. To achieve this result, we have considered a static disc at the AdS boundary and let the superfluid rotate relative to it. This idea has been numerically exploited in [1] where formation of vortices in such a setting was reported. We have found that these vortex solutions are eigenfunctions of angular momentum. We have also shown that vortices with higher quantized rotation of the superfluid.

I. INTRODUCTION

The emergence of gauge/gravity duality in the past decade has been instrumental in our understanding on strongly correlated systems. This connection between a d-dimensional gravity theory and (d − 1)-dimensional QFT has been applied to various physical systems ranging from early universe cosmology to condensed matter systems like high- Tc superconductors and strongly coupled superfluids [2, 3], to name a few. The holographic superconductors and superfluids have been studied in various spacetime settings using numerical as well as analytical methods. Some crucial properties associated with these phenomena have been shown in the past few years [4–13]. In particular, formation of vortex lattice in holographic superconductors near second critical magnetic field has been shown [14–18]. Also, it has been observed that vortices are formed if we rotate a superfluid in a cylindrical container. It is known from various experiments that there are a variety of possible vortices in a superfluid under rotation [19, 20]. Existence of such vortices is of prime interest in a holographic superfluid model. Numerical studies leading to the existence of such vortices in a rotating holographic superfluid have been carried out in [1]. The study made use of the gauge/gravity duality to investigate the dynamics of a strongly coupled superfluid in an uniformly rotating disk at a finite temperature. As the angular velocity of the disk is increased above a critical value, a vortex with quantized vorticity gets excited. With further increase of the angular velocity, higher vortices are generated. In this paper, we have analytically devised novel vortex solutions for a rotating holographic superfluid model proposed in [1]. In our study, we consider that there is a static disc of radius R at the AdS boundary and the superfluid rotates relative to this disc. The superfluid being incompressible, demands no flow along the radial direction and hence it is an equivalent description for the alternate scenario where the superfluid is static in an uniformly rotating disc. The vortex solutions that we have constructed enjoy circular symmetry in the rotating disc of radius R and each of these solutions are eigenfunctions of the angular momentum. To obtain vortices, we have analysed this model very near to the critical value of rotation Ωc, where superfluid vortex state appears. Remarkably, the rotating superfluid also shows the step transitions of the angular velocity observed in [1] leading to the excitation of vortices. Interestingly, we have also discovered a linear relation between the winding number associated with these vortices and the angular velocity of the rotating superfluid.

We have organized this paper in the following way. In section II, we set up the model for holographic superfluid in a static black hole background in AdS3+1 spacetime. In section III, we have constructed vortex solutions, near critical rotation in the rotating disc. Then in the last section (IV) of this paper, we have concluded and made some remarks on our results.

II. THE HOLOGRAPHIC SUPERFLUID

We start by writing down the metric for a static black hole in AdS3+1 spacetime with Eddington-Finkelstein coordinates [1],

\[ ds^2 = \frac{\rho^2}{u^2} [-f(u)dt^2 - 2dtdu + dr^2 + r^2d\theta^2] \] (1)

where the blackening factor is given by,

\[ f(u) = \left(1 - u^3\right). \]

Here l is the AdS radius and u is the bulk direction scaled in such a way that u = 0 is the AdS boundary and u = 1 is the event horizon of the black hole. The coordinates (r, θ) define the 2D flat disc. For convenience, we take unit AdS radius (that is, l = 1) and the cosmological constant \( \Lambda = -3 \). The Hawking temperature associated with the above black hole geometry is given by \( T = \frac{3}{4\pi} \).
We now consider a simple model for holographic superfluid on top of this geometry. The action for the matter section in this model is given by,

$$S = \frac{1}{2\kappa_{4}^{2}e^{2}} \int_{\mathcal{M}} d^{4}x \mathcal{L}_{m}.$$  

(2)

The matter Lagrangian density, $\mathcal{L}_{m}$, consists of a Maxwell field and a complex scalar field minimally coupled to $A_{\mu}$. More precisely $\mathcal{L}_{m}$ is given by following expression,

$$\mathcal{L}_{m} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - |D \Psi|^{2} - m^{2} \Psi^{2}$$  

(3)

where $m$ is the mass of the scalar field while $e$ is its charge. Note that we will be working in the probe limit. In this limit any backreaction of the matter field in the metric is neglected. To achieve this limit, we shall rescale $A_{\mu} \rightarrow A_{\mu}/e$ and $\Psi \rightarrow \Psi/e$, and take the limit $e \rightarrow \infty$. Mathematically, it is equivalent to setting $e = 1$ in the action of our theory.

Now varying the action, $\mathcal{S}$, for $\Psi$ and $A_{\mu}$ we get the following equations of motion for the matter and the gauge fields respectively,

$$\left(D^{2} - m^{2}\right) \Psi = 0$$  

(4)

$$\nabla_{\nu} F^{\mu\nu} = j_{\mu}$$  

(5)

where the bulk current is defined as,

$$j_{\mu} := i \{(D_{\mu} \Psi)^{\dagger} \Psi - \Psi (D_{\mu} \Psi)\}.$$  

(6)

We shall now assume that all the fields are stationary as our interest lies in equilibrium analysis of the rotating superfluid system. Also we would be working with the axial gauge, that is, $A_{r} = 0$, in which case eq.(4) reduces to the following equation,

$$\{\mathcal{D}(u) + \mathcal{D}(r) + \frac{1}{r^{2}} \mathcal{D}(\theta)\} \Psi(u, r, \theta) = 0$$  

(7)

where the segregated derivative operators are given as,

$$\mathcal{D}(u) \equiv u^{2} \partial_{u} \left(\frac{f(u)}{u^{2}} \partial_{u}\right) + iv^{2} \partial_{u} \left(\frac{A_{1}}{u^{2}}\right) + iA_{t} \partial_{u} - \frac{m^{2}}{u^{2}}$$

$$\mathcal{D}(r) \equiv \frac{1}{r} \partial_{r} (r \partial_{r}) - i \partial_{\phi} (r \partial_{r}) - iA_{r} \partial_{r} - A_{r}^{2}$$

$$\mathcal{D}(\theta) \equiv \partial_{\theta}^{2} - i(\partial_{\theta} A_{\phi} + A_{\theta} \partial_{\phi}) - A_{\theta}^{2}.$$  

III. THE VORTEX SOLUTION

Our interest is in the equilibrium state where vortices exist. So we define a deviation parameter, $\epsilon$, from the critical rotation, $\Omega_{c}$, by the following relation,

$$\epsilon := \frac{\Omega - \Omega_{c}}{\Omega_{c}}$$  

(8)

where $\Omega$ is the constant angular velocity of the disc. As argued in [1], one should notice that there is a relative velocity between the superfluid and the disc. Hence, a static superfluid in a rotating disc is justly represented by a rotating superfluid in a static disc. In this analysis, we are visualizing the latter scenario. Now, in order to study this system very near to $\Omega_{c}$, we series expand the matter field $\Psi$, the gauge field $A_{\mu}$ and the bulk current $j_{\mu}$ with respect to $\epsilon$ in the following manner [14],

$$\Psi(u, r, \theta) = \sqrt{\epsilon} (\Psi_{1}(u, r, \theta) + e \Psi_{2}(u, r, \theta) + \ldots)$$  

(9)

$$A_{\mu}(u, r, \theta) = (A_{\mu}^{(0)}(u, r, \theta) + eA_{\mu}^{(1)}(u, r, \theta) + \ldots)$$  

(10)

$$j_{\mu}(u, r, \theta) = e (j_{\mu}^{(0)}(u, r, \theta) + \epsilon j_{\mu}^{(1)}(u, r, \theta) + \ldots)$$  

(11)

A. Zeroth order solutions near AdS boundary

The zeroth order solutions for gauge fields, in axial gauge, that generates the critical rotation field and the chemical potential are given by following relations,

$$A_{t}^{(0)}(u) = \mu (1 - u), \quad A_{r}^{(0)} = 0, \quad A_{\theta}^{(0)}(r) = \Omega_{c} r^{2}.$$  

(12)

Notice that $A_{r}^{(0)} = 0$ restricts any superfluid flow in the radial direction while $A_{\theta}^{(0)}$ allows the superfluid to rotate. Considering these zeroth order solutions for gauge fields near the AdS boundary, we may rewrite eq.(7) for lowest order in $\epsilon$, that is, $O(\sqrt{\epsilon})$, in the following form,

$$\{\mathcal{D}^{(0)}(u) + \mathcal{D}^{(0)}(r) + \frac{1}{r^{2}} \mathcal{D}^{(0)}(\theta)\} \Psi_{1}(u, r, \theta) = 0$$  

(13)

where the derivative operators become,

$$\mathcal{D}^{(0)}(u) \equiv u^{2} \partial_{u} \left(\frac{f(u)}{u^{2}} \partial_{u}\right) + iv^{2} \partial_{u} \left(\frac{A_{1}^{(0)}}{u^{2}}\right) + iA_{t}^{(0)} \partial_{u} - \frac{m^{2}}{u^{2}}$$

$$\mathcal{D}^{(0)}(r) \equiv \frac{1}{r} \partial_{r} (r \partial_{r}) - i \partial_{\phi} (r \partial_{r}) - iA_{r}^{(0)} \partial_{r} - A_{r}^{2}$$

$$\mathcal{D}^{(0)}(\theta) \equiv \partial_{\theta}^{2} - i(\partial_{\theta} A_{\phi}^{(0)} + A_{\theta}^{(0)} \partial_{\phi}) - A_{\theta}^{(0)}.$$  

We now use the method of separation of variables to solve eq.(13) and write $\Psi_{1}(u, r, \theta)$ as a function of $u$ and $(r, \theta)$ separately in the following manner,

$$\Psi_{1}(u, r, \theta) = \Phi(u) \xi(r, \theta).$$  

(14)

With the above separation of matter field, eq.(13) provides the following separated equations,

$$\mathcal{D}^{(0)}(u) \Phi(u) = \lambda \Phi(u)$$  

(15)

$$\{\mathcal{D}^{(0)}(r) + \frac{1}{r^{2}} \mathcal{D}^{(0)}(\theta)\} \xi(r, \theta) = -\lambda \xi(r, \theta)$$  

(16)

where $\lambda$ is an unknown separation constant. Note that both eq.(s)(15, 16) are eigenvalue equations with eigenvalue $\lambda$. In the subsequent discussion we shall proceed to determine $\lambda$. 

B. Solution for vortex in the rotating superfluid

Given the 2D rotational symmetry, we may choose the following ansatz,
\[ \xi(r, \theta) = \eta_p(r)e^{ip\theta} \]  \hspace{1cm} (17)
where \( p \in \mathbb{Z} \) for the single valuedness of the solution. However, one should note that \( \eta_p(r) \) must satisfy certain boundary conditions for regularity at the boundaries. In our case, we would be working with the Neumann boundary conditions at \( r = 0 \) as well as at \( r = R \), that is,
\[ \partial_r \eta_p|_{r=0} = 0 = \partial_r \eta_p|_{r=R} \]  \hspace{1cm} (18)
where \( R \) is the radius of the disc boundary.

Now using the above ansatz in eq.(16), we get the following differential equation to be solved under the boundary conditions defined above,
\[ \partial_r^2 \eta_p(r) + \frac{1}{r} \partial_r \eta_p(r) + \left\{ \lambda - \left( \frac{p}{r} - \Omega r \right)^2 \right\} \eta_p(r) = 0. \]  \hspace{1cm} (19)
To solve for \( \eta_p(r) \), we consider the following ansatz,
\[ \eta_p(r) = F_p(r)e^{-\Omega r^2/2}. \]  \hspace{1cm} (20)
Utilising this form of \( \eta_p(r) \) given by eq.(20), eq.(19) takes the form,
\[ \partial_r^2 F_p(r) + \left( \frac{1}{r} - 2p\Omega \right) \partial_r F_p(r) + \left( \lambda - 2\Omega - \frac{p^2}{r^2} \right) F_p(r) = 0. \]  \hspace{1cm} (21)
We now proceed to solve eq.(21) using the Frobenius series solution method. So we consider that \( F_p(r) \) is given by the following series,
\[ F_p(r) = \sum_{n=0}^{\infty} a_n r^{n+k} \quad , \quad (a_0 \neq 0) \]  \hspace{1cm} (22)
with \( k \) being an integer. The derivatives of the above series solution with respect to \( r \) are given by,
\[ \partial_r F_p(r) = \sum_{n=0}^{\infty} a_n (n+k) r^{n+k-1} \]  \hspace{1cm} (23)
and
\[ \partial_r^2 F_p(r) = \sum_{n=0}^{\infty} a_n (n+k)(n+k-1) r^{n+k-2}. \]  \hspace{1cm} (24)
Using eqs.(22, 23 and 24) in eq.(21), we find the following condition,
\[ \sum_{n=0}^{\infty} a_n \left\{ \left( n+k \right)^2 - p^2 \right\} r^{n+k} + \sum_{n=0}^{\infty} a_n \left\{ \lambda + 2\Omega (p-1-n-k) \right\} r^{n+k+2} = 0. \]  \hspace{1cm} (25)
This implies that coefficient for each order of \( r \) should separately satisfy eq.(25), that is,
\[ r^k : \quad a_0 (k^2 - p^2) = 0 \quad \Rightarrow \quad k = \pm p \]
\[ r^{k+1} : \quad a_1 ((k+1)^2 - p^2) = 0 \quad \Rightarrow \quad (k+1) = \pm p. \]
From the above conditions we consider \( k = p \) for the regularity of the solutions at \( r = 0 \) and this yields \( a_1 = 0 \). The condition \( k = p \) implies that \( p \) is an integer. Similarly setting the coefficient for \( r^{(k+n+2)} \) equal to zero, we get the following recurrence relation,
\[ \frac{a_{n+2}}{a_n} = \frac{(\lambda - 2\Omega (n+1))}{((n+2)^2 + 2p(n+2))} \]  \hspace{1cm} (26)
where we have already used the condition \( k = p \). This recurrence relation connects all the even coefficients with \( a_0 \) and all the odd coefficients with \( a_1 \). Hence, we would get series solution for \( F_p(r) \) with even terms only. Now in order to have normalizable solutions, we must terminate this series at some point, which determines \( \lambda \) in terms of \( \Omega \) and \( n \), that is,
\[ \lambda = 2\Omega (n+1). \]  \hspace{1cm} (27)
The above relation implies that the eigenvalue \( \lambda \) is quantized. With this condition, the above series solution becomes a polynomial of order \( n \). Thus we can write the solution for \( \eta_p(r) \) with an additional index depicting the order of the polynomial as,
\[ \eta_{p,n}(r) = a_0 e^{-\Omega r^2/2} F_{p,n}(r) \]  \hspace{1cm} (28)
where
\[ F_{p,n}(r) = r^p \left( 1 + \frac{a_2}{a_0} r^2 + \frac{a_4}{a_0} r^4 + ... + \frac{a_n}{a_0} r^n \right). \]
Let us now discuss the family of solutions with \( n = 0 \). In this case,
\[ F_{p,0}(r) = r^p \]
and hence,
\[ \eta_{p,0}(r) = a_0 r^p e^{-\Omega r^2/2} ; \quad (\lambda = 2\Omega) \]  \hspace{1cm} (29)
This solution is subjected to the Neumann boundary conditions mentioned earlier. This means the following first derivative of eq.(29) must vanish at the disc boundaries,
\[ \partial_r \eta_{p,0}(r) = a_0 r^{p-1} e^{-\Omega r^2/2} (p - \Omega r^2). \]  \hspace{1cm} (30)
Now the boundary condition at \( r = 0 \) gives the following lower bound for \( p \),
\[ \partial_r \eta_{p,0}(r)|_{r=0} = 0 \quad \Rightarrow \quad p > 1. \]  \hspace{1cm} (31)
Applying the boundary condition at the disc boundary at \( r = R \) gives the following linear relation between \( p \) and \( \Omega \),
\[ \partial_r \eta_{p,0}(r)|_{r=R} = 0 \quad \Rightarrow \quad p = \Omega R^2. \]  \hspace{1cm} (32)
Since \( p \) is an integer, hence the above relation between \( p \) and \( \Omega \) implies a quantization of the angular velocity \( \Omega \) and also a quantization of the angular momenta in the rotating superfluid. Note that the radius \( R \) in the
For this solution, we have,
\[ \eta_{p,2}(r) = a_0 r^p e^{-\Omega r^2/2} \left( 1 - \frac{2\Omega}{(p+2)} r^2 \right); \quad (\lambda = 6\Omega) . \] (33)

For this solution, we have,
\[ \partial_r \eta_{p,2}(r) = a_0 r^{p-1} e^{-\Omega r^2/2} \left( p - 3\Omega r^2 + \frac{2(\Omega r^2)^2}{p+2} \right). \] (34)

In this case, the boundary condition at \( r = 0 \) gives us the same lower bound for \( p \),
\[ \partial_r \eta_{p,2}(r)|_{r=0} = 0 \quad \implies \quad p > 1. \] (35)

However, the boundary condition at \( r = R \) gives us the following condition,
\[ \partial_r \eta_{p,2}(r)|_{r=R} = 0 \quad \implies \quad \left( p - 3\Omega R^2 + \frac{2(\Omega R^2)^2}{p+2} \right) = 0. \] (36)

From this condition, we get,
\[ \Omega R^2 = \frac{3(p+2)}{4} \left( 1 \pm \sqrt{1 - \frac{8p}{9(p+2)}} \right). \] (37)

For \( p >> 2 \), the above result again provides a linear relation between \( p \) and \( \Omega \), that is, \( \Omega R^2 \sim p \).

**IV. CONCLUSION AND REMARKS**

In this work, we have holographically devised vortex solutions with different winding numbers in a rotating superfluid. These solutions may be interpreted as vortices placed at the centre of the disc at \( r = 0 \). Our analysis shows that \( p = \Omega R^2 \) is an exact condition for \( n = 0 \) case while it is true for \( p >> 2 \) for higher order solutions, that is, \( n \neq 0 \). This linear relation between the winding number, \( p \), and the angular velocity, \( \Omega \), seems to be an universal feature of such vortices at least for large \( p \). It is to be noted that due to the Neumann boundary condition at \( r = 0 \), the vortex solution with winding number \( p = 1 \) is absent in this model. However, if one considers the Dirichlet boundary condition, at \( r = 0 \), instead of Neumann boundary condition, then even solutions with winding number \( p = 1 \) are allowed. In Fig.(1), we have shown some vortex solutions with different winding numbers. As a final remark, we would like to emphasize that the results in this work have been obtained analytically making use of the gauge/gravity duality and has similar features to those found numerically in [1].

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