SOME THEOREMS ON LEIBNIZ $n$-ALGEBRAS FROM
THE CATEGORY $U_n(Lb)$

M.S. KIM AND R. TURDIBAEV

Abstract. We study the Leibniz $n$-algebra $U_n(L)$, whose multiplication is defined via the
bracket of a Leibniz algebra $L$ as $[x_1, ..., x_n] = [x_1, [\ldots, [x_{n-2}, [x_{n-1}, x_n]] \ldots]]$. We show that
$U_n(L)$ is simple if and only if $L$ is a simple Lie algebra. An analogue of Levi’s theorem for
Leibniz algebras in $U_n(Lb)$ is established and it is proven that the Leibniz $n$-kernel of $U_n(L)$
for any semisimple Leibniz algebra $L$ is the $n$-algebra $U_n(L)$.

Introduction

Leibniz algebras were introduced by A. Bloh [7] in 1960s as algebras satisfying the Leibniz
identity:
$$[[x, y], z] = [[x, z], y] + [x, [y, z]].$$
The Leibniz identity becomes the Jacobi identity if one uses skew-symmetry and in fact, the
category of Lie algebras ($\text{Lie}$) is a full subcategory of the category of Leibniz algebras ($\text{Lb}$).
However, Leibniz algebras did not gain popularity until J.-L. Loday [17] rediscovered them in
1990s while lifting classical Chevalley-Eilenberg boundary map to the tensor module of a Lie
algebra that produces another (Leibniz) chain complex. Leibniz algebras play an important
role in Hochschild homology theory [17] and Nambu mechanics [19, 10].

In 1985 Filippov [12] defined a notion of an $n$-Lie algebra that generalizes Lie algebras by
the arity of the bracket from two to $n$, preserving skew symmetry and satisfying the identity:
$$[x_1, \ldots, x_n], y_1, \ldots, y_{n-1}] = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, [x_i, y_1, \ldots, y_{n-1}], x_{i+1}, \ldots, x_n].$$
For $n = 3$ an example of a 3-Lie algebra appears even earlier in 1973 [19], where the multiplication
for a triple of classical observables on the three-dimensional phase space $\mathbb{R}^3$ is given by
the Jacobian and studied further in [20]. An $n$-Lie algebra is also called a Filippov algebra or
a Nambu algebra in numerous papers and is relevant in string and membrane theory [4, 5].

The generalization of Lie, Leibniz and $n$-Lie algebras is the so called Leibniz $n$-algebras,
introduced in [9]. Leibniz $n$-algebras are defined with the above identity but are not skew-
symmetric with the $n$-ary bracket. As in the case of Leibniz algebras ($n = 2$), the ideal
generated by the $n$-ary brackets, where two elements are the same is called the Leibniz $n$-kernel
and the quotient $n$-algebra by this ideal is an $n$-Lie algebra.

While the structure theory of Leibniz and $n$-Lie algebras has been well-developed, the same
cannot be said about $n$-Leibniz algebras. Although a notion of a solvable radical and some other
classical properties of the radical of Leibniz $n$-algebras are obtained in [13], Levi decomposition
for Leibniz $n$-algebras has not yet been established, while for $n$-Lie and Leibniz algebras they
are proven by Ling [18] and Barnes [6], respectively. Even in small dimensions, there is only a
partial classification of two-dimensional Leibniz $n$-algebras over complex numbers in [11] and [8].

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There are also some instances when results from Leibniz and \( n \)-Lie algebras do not generalize to Leibniz \( n \)-algebras. For example, the Leibniz \( n \)-kernel can coincide with the \( n \)-algebra which is possible for \( n = 2 \) if and only if the Leibniz algebra is abelian. Also, there are examples of Leibniz \( n \)-algebras with invertible right multiplication operators and Cartan subalgebras of different dimensions [2], which are not in accordance with the corresponding results in Lie, Leibniz and \( n \)-Lie algebras.

The motivation of this paper is to continue developing theory on Leibniz \( n \)-algebras and find analogs and variations in relation to results established in \( n \)-Lie algebras. One such direction of study is through a “forgetful” functor \( U_n \) from the category of Leibniz algebras \( \mathbf{Lb} \) to Leibniz \( n \)-algebras \( _n\mathbf{Lb} \). Given a Leibniz algebra \( \mathcal{L} \), authors of [2] introduce a Leibniz \( n \)-algebra structure on the same vector space via \( [x_1, \ldots, x_n] = [x_1, \ldots, [x_{n-2}, [x_{n-1}, x_n]] \ldots] \). We prove that \( U_n(\mathbf{Lb}) \) is not a full subcategory of the category \( _n\mathbf{Lb} \) and our goal is to investigate the structure of Leibniz \( n \)-algebras in the category \( U_n(\mathbf{Lb}) \).

Section 1 is an introduction to Leibniz algebras, Leibniz \( n \)-algebras and the forgetful functor \( U_n(\mathcal{L}) \). Using the recent results of [2] we describe all ideals of a semisimple Leibniz algebra \( \mathcal{L} \) and compare them with the ideals of \( U_n(\mathcal{L}) \) in Section 3.

In Section 2 we discuss the Leibniz \( n \)-kernel of a Leibniz \( n \)-algebra and show that for a Leibniz 3-algebra from \( U_3(\mathbf{Lb}) \), the subspace spanned by the “square” of the elements, \( \text{Span}\{[x, x, y] \mid x, y \in \mathcal{L}\} \) is the Leibniz 3-kernel. For a Leibniz \( n \)-algebra with an invertible right multiplication operator it is known by [2] that the Leibniz \( n \)-kernel is the whole \( n \)-algebra. We present a short proof of this fact and show that the converse does not work for \( U_n(\mathcal{L}) \), where \( \mathcal{L} \) is any semisimple Leibniz algebra.

Section 3 introduces the notions of simplicity and semisimplicity for a Leibniz \( n \)-algebra in a similar way as in \( n \)-Lie algebras. For a non-Lie Leibniz algebras (\( n = 2 \)), both simplicity and semisimplicity are defined differently, taking into account the Leibniz kernel, which is a nontrivial ideal. We establish that a Leibniz \( n \)-algebra \( U_n(\mathcal{L}) \) is simple, if and only if \( \mathcal{L} \) is a simple Lie algebra. Furthermore, we prove an analogue of Levi decomposition and establish that a Leibniz \( n \)-algebra \( U_n(\mathcal{L}) \) is semisimple if and only if \( \mathcal{L} \) is a semisimple Lie algebra.

All vector spaces, modules, algebras and \( n \)-algebras in this work are finite-dimensional over a field \( \mathbb{K} \) of characteristic zero.

1. Preliminaries

1.1. Leibniz algebras. A Leibniz algebra \( \mathcal{L} \) is a vector space over a field \( \mathbb{K} \) with a bilinear bracket \( [-,-] : \mathcal{L} \times \mathcal{L} \to \mathcal{L} \) that satisfies the Leibniz identity:

\[
[[x, y], z] = [[x, z], y] + [x, [y, z]].
\]

If the Leibniz algebra \( \mathcal{L} \) is skew-symmetric i.e. \( [x, x] = 0 \) for all \( x \in \mathcal{L} \), then the Leibniz identity becomes the Jacobi identity. A linear space spanned by the squares of the elements of a Leibniz algebra \( \mathcal{L} \) constitutes a two-sided ideal called the Leibniz kernel and is denoted by \( \text{Leib}(\mathcal{L}) \). The quotient algebra \( \mathcal{L}/\text{Leib}(\mathcal{L}) \) is a Lie algebra, called the liezation of \( \mathcal{L} \). The Leibniz kernel is trivial if and only if the Leibniz algebra is abelian (that is \( [\mathcal{L}, \mathcal{L}] = \{0\} \)) and \( \text{Leib}(\mathcal{L}) = \mathcal{L} \) if and only if \( \mathcal{L} \) is a Lie algebra. Therefore, a non-Lie Leibniz algebra always admits a nontrivial ideal - the Leibniz kernel and we use this definition of simplicity for a non-abelian Leibniz algebra introduced in [3].

Definition 1.1. A Leibniz algebra \( \mathcal{L} \) is called simple if the only ideals are \( \{0\}, \text{Leib}(\mathcal{L}) \) and \( \mathcal{L} \).

Definition 1.2. A subalgebra \( K \) of \( \mathcal{L} \) is solvable if there exists \( m \in \mathbb{N} \) such that \( K^{(m)} = \{0\} \), where \( K^{(n)} = [K^{(n-1)}, K^{(n-1)}] \) for any positive integer \( n \) and \( K^{(0)} = K \).
As in Lie algebra theory, the sum of solvable ideals of a Leibniz algebra is a solvable ideal.

**Definition 1.3.** The maximal solvable ideal of a Leibniz algebra \( \mathfrak{L} \) is called the radical of \( \mathfrak{L} \), denoted \( \text{rad}(\mathfrak{L}) \).

As in the definition of simple Leibniz algebras, the notion of semisimplicity of a Leibniz algebra is different from semisimplicity of Lie algebras.

**Definition 1.4.** A Leibniz algebra \( \mathfrak{L} \) is called semisimple if \( \text{rad}(\mathfrak{L}) = \text{Leib}(\mathfrak{L}) \).

Note that a simple Leibniz algebra is also semisimple. Semisimple Leibniz algebras satisfy \([\mathfrak{L}, \mathfrak{L}] = \mathfrak{L}\), i.e. they are perfect. There is an analog of Levi’s theorem that describes the structure of Leibniz algebra.

**Theorem 1.5.** [21] Let \( \mathfrak{L} \) be a Leibniz algebra. Then \( \mathfrak{L} = \mathfrak{g} \times \text{Leib}(\mathfrak{L}) \), where \( \mathfrak{g} \) is a semisimple Lie subalgebra of \( \mathfrak{L} \).

For a semisimple Leibniz algebra \( \mathfrak{L} \), the semisimple Lie subalgebra \( \mathfrak{g} \) is the liezation \( \mathfrak{g}_L \) of \( \mathfrak{L} \). Since \( \text{Leib}(\mathfrak{L}) \) is a \( \mathfrak{g}_L \)-module, by Weyl’s semisimplicity \( \text{Leib}(\mathfrak{L}) \) decomposes into a direct sum \( \bigoplus_{k=1}^n I_k \) of simple \( \mathfrak{g}_L \)-submodules. Moreover, \( \mathfrak{g}_L \) is a direct sum \( \bigoplus_{i=1}^m \mathfrak{g}_i \) of simple Lie subalgebras and we have the following.

**Corollary 1.6.** Let \( \mathfrak{L} \) be a semisimple Leibniz algebra. Then

\[
\mathfrak{L} = \mathfrak{g}_L \times \text{Leib}(\mathfrak{L}) = (\bigoplus_{i=1}^m \mathfrak{g}_i) \times (\bigoplus_{k=1}^n I_k).
\]

By [21] Theorem 3.5, 3.8] the structure of an indecomposable semisimple Leibniz algebra \( \mathfrak{L} = (\bigoplus_{i=1}^m \mathfrak{g}_i) \times (\bigoplus_{k=1}^n I_k) \) can be determined by a connected bipartite graph with bipartition (\( \{\mathfrak{g}_1, \ldots, \mathfrak{g}_m\}, \{I_1, \ldots, I_n\}\)) and edges between \( \mathfrak{g}_i \)'s and \( I_j \)'s. Let \( \mathfrak{g}_L \) be a subset of \( \mathfrak{g}_L \)-modules, \( I_k \) be a subset of \( I_k \)-modules. Since \( I_k \) for all \( 1 \leq k \leq n \) are simple \( \bigoplus_{i=1}^m \mathfrak{g}_i \)-modules, \( X \) is direct sum of some \( I_k \)'s and we obtain (i).

**Theorem 1.7.** The ideals of an indecomposable semisimple Leibniz algebra \( \mathfrak{L} = (\bigoplus_{i=1}^m \mathfrak{g}_i) \times (\bigoplus_{k=1}^n I_k) \) are the following:

(i) Abelian ideals \( \bigoplus_{k \in A} I_k \), for any subset \( A \subseteq \{1, \ldots, n\} \);

(ii) Non-solvable ideals \( (\bigoplus_{i \in B} \mathfrak{g}_i) \times (\bigoplus_{j \in N(B)} I_j) \oplus (\bigoplus_{k \in C} I_k) \), where \( B \subseteq \{1, \ldots, m\} \) and \( C \subseteq \{1, \ldots, n\} \setminus N(B) \).

**Proof.** Let \( X \) be an ideal of \( \mathfrak{L} \). Since \( \mathfrak{L} \) is a \( \bigoplus_{i=1}^m \mathfrak{g}_i \)-module we obtain that \( X \) is a \( \bigoplus_{i=1}^m \mathfrak{g}_i \)-submodule of \( \mathfrak{L} \).

Assume \( X \subseteq \bigoplus_{k=1}^n I_k \). By Weyl’s semisimplicity \( X \) is a Lie algebra module is a direct sum of simple \( \bigoplus_{i=1}^m \mathfrak{g}_i \)-submodules. Since \( I_k \) for all \( 1 \leq k \leq n \) are simple \( \bigoplus_{i=1}^m \mathfrak{g}_i \)-modules, \( X \) is direct sum of some \( I_k \)'s and we obtain (i).

Now assume that \( X \nsubseteq \bigoplus_{k=1}^n I_k \). By the similar argument we obtain that \( X \) is a Lie algebra module is a direct sum of some \( \mathfrak{g}_i \)'s and \( I_k \)'s. Let \( B \) be a subset of \( \{1, \ldots, m\} \) such that \( \mathfrak{g}_L \in X \) only for \( i \in B \). For any \( I_k \in N(\mathfrak{g}_i) \) by the construction of semisimple Leibniz algebra we have \( I_k = [I_k, \mathfrak{g}_i] \subseteq [I_k, X] \subseteq X \) and therefore, \( \bigoplus_{k \in N(B)} I_k \subseteq X \). Since \([I_k, \mathfrak{g}_i] = 0 \) whenever \( I_k \notin N(\mathfrak{g}_i) \), one can add direct summands \( I_k \)'s to the ideal \( X \) such that \( k \in \{1, \ldots, n\} \setminus N(B) \).

Obviously, ideals of a semisimple Leibniz algebra are all possible summands of the direct sum of the ideals of each indecomposable subideals of the Leibniz algebra.
1.2. Leibniz \( n \)-algebras. A Leibniz \( n \)-algebra \( \mathfrak{g} \) is defined as a \( \mathbb{K} \)-module \( L \) equipped with a linear \( n \)-ary operation \([ -, ..., - ] : L^\otimes n \rightarrow L\) satisfying the identity:

\[
[[x_1, ..., x_n], y_1, ..., y_{n-1}] = \sum_{i=1}^{n} [x_1, ..., x_{i-1}, [x_i, y_1, ..., y_{n-1}], x_{i+1}, ..., x_n].
\]

Note that for \( n = 2 \) this is the definition of Leibniz algebra. Moreover, if the bracket \([ -, ..., - ]\) factors through \( \Lambda^n L \), then \( L \) is an \( n \)-Lie algebra introduced in [12].

Since \( n \)-ary multiplication of Leibniz \( n \)-algebras is not necessarily skew-symmetric, in [2] some variation of the definition of ideal is given.

**Definition 1.8.** A subspace \( I \) of a Leibniz \( n \)-algebra \( L \) is called an \( s \)-ideal of \( L \), if

\[
[L, ..., L, I, L, ..., L] \subseteq I.
\]

If \( I \) is an \( s \)-ideal for all \( 1 \leq s \leq n \), then \( I \) is called an ideal.

Consider the \( n \)-sided ideal

\[
\text{Leib}_n(L) = \text{ideal}([x_1, ..., x_i, ..., x_j, ..., x_n] | \exists i, j : x_i = x_j, x_1, ..., x_n \in L),
\]

which is called the \textit{Leibniz n-kernel} of \( L \). The quotient \( n \)-algebra \( L/\text{Leib}_n(L) \) is clearly an \( n \)-Lie algebra.

**Definition 1.9.** A linear map \( d \) defined on a Leibniz \( n \)-algebra \( L \) is called a derivation if

\[
d([x_1, x_2, ..., x_n]) = \sum_{i=1}^{n} [x_1, ..., d(x_i), ..., x_n].
\]

The space of all derivations of a given Leibniz \( n \)-algebra \( L \) is denoted by \( \text{Der}(L) \) and forms a Lie algebra with respect to the commutator [2]. Given an arbitrary element \( x = (x_2, ..., x_n) \in L^\otimes (n-1) \) consider the operator \( R[x] : L \rightarrow L \) of right multiplication defined for all \( z \in L \) by

\[
R[x](z) = [z, x_2, ..., x_n].
\]

As in \( n \)-Lie algebras, a right multiplication operator is a derivation and the space \( R[L] \) of all right multiplication operators forms a Lie algebra ideal of \( \text{Der}(L) \) [2]. Given a Leibniz \( n \)-algebra \( L \), one can associate the Lie algebras \( R[L] \) or \( \text{Der}(L) \) to \( L \). The following statement from [9] Proposition 3.2] shows how a Leibniz \( n \)-algebra can be constructed from a given Leibniz algebra.

**Proposition 1.10.** Let \( \mathfrak{L} \) be a Leibniz algebra with the product \([ -, - ]\). Then the vector space \( \mathfrak{L} \) can be equipped with the Leibniz \( n \)-algebra structure with the following product:

\[
[x_1, x_2, ..., x_n] := [x_1, ..., [x_{n-2}, [x_{n-1}, x_n]]] ...
\]

It follows that the authors of [9] has built a “forgetful” functor \( U_n : \mathbf{Lb} \rightarrow \mathfrak{nLb} \) from the category of Leibniz algebras to the category of Leibniz \( n \)-algebras. The restriction of \( U_n \) on \( \mathbf{Lie} \) does not necessarily fall into the category of \( n \)-Lie algebras \( \mathfrak{nLie} \). Conversely, given an \( n \)-Lie algebra \( L \), there is a construction by Daletskii and Takhtajan [10] of a Leibniz algebra on \( L^\otimes (n-1) \) with the bracket

\[
[l_1 \otimes \cdots \otimes l_{n-1}, l'_1 \otimes \cdots \otimes l'_{n-1}] = \sum_{1 \leq i \leq n-1} l_1 \otimes \cdots \otimes [l_i, l'_1, ..., l'_{n-1}] \otimes \cdots \otimes l_{n-1}.
\]

This Leibniz algebra is called the \textit{basic Leibniz algebra} of an \( n \)-Lie algebra \( L \) in [10] and comparisons with other Lie algebra constructions from a given \( n \)-Lie algebra are studied in [14]. It is straightforward that Daletskii-Takhtajan’s construction gives rise to a functor
Remarkably, $U_n$ is not a right adjoint of $D_{n-1}$. Indeed, for abelian Leibniz algebra $\mathfrak{L}$ and abelian $n$-algebra $L$ one can check that

$$\text{Hom}_{Lb}(D_{n-1}(L), \mathfrak{L}) = \text{Hom}_K(L^\otimes n-1, \mathfrak{L}),$$
$$\text{Hom}_{Lb}(L, U_n(\mathfrak{L})) = \text{Hom}_K(L, \mathfrak{L}),$$

and they are of different dimensions.

Note that we can define a Leibniz $n$-algebra as an $n$-algebra where every right multiplication operator is a derivation. For Lie and $n$-Lie algebras, an operator of right multiplication is singular. For Leibniz algebras (that is $n = 2$) it is also true [3]. However, for some Leibniz $n$-algebra ($n \geq 3$) an invertible operator of right multiplication exists, as shown in [2, Example 2.3].

**Example 1.11.** An $n$-ary algebra over a field $\mathbb{K}$ with the basis $\{e_1, ..., e_m\}$ with the following multiplication

$$[e_i, e_1, ..., e_{n-1}] = \alpha_i e_i, \quad \alpha \in \mathbb{K}$$

where $\alpha_i \neq 0$ for all $1 \leq i \leq m$, $\sum_{i=1}^{n-1} \alpha_i = 0$ and all other products are zero is a Leibniz $n$-algebra. In this $n$-algebra the operator $R[e]$ is invertible, where $e = (e_1, ..., e_{n-1})$.

Let $H$ be an ideal of a Leibniz $n$-algebra $L$. Put $H^{(1)} = H$ and

$$H^{(m+1)} = \sum_{i_1 + \cdots + i_k = 0}^{n-k} \left[ L, ..., L, H^{(m)}_{i_1}, L, ..., L, H^{(m)}_{i_2}, ..., L, ..., L, H^{(m)}_{i_k}, L, ..., L \right]$$

for all $1 \leq k \leq n$ and $m \geq 1$.

**Definition 1.12.** [3] An ideal $H$ of a Leibniz $n$-algebra $L$ is said to be $k$-solvable with index of $k$-solvability equal to $m$ if there exists $m \in \mathbb{N}$ such that $H^{(m)} = 0$ and $H^{(m-1)} \neq 0$. An ideal $H$ is called solvable if it is $n$-solvable. When $L = H$, $L$ is called a $k$-solvable (correspondingly, solvable) Leibniz $n$-algebra.

Notice that this definition agrees with the definition of $k$-solvability of $n$-Lie algebras given in [16]. Next statement is a straightforward generalization of the similar statement from [18].

**Proposition 1.13.** A $k$-solvable ideal $H$ of a Leibniz $n$-algebra is $k'$-solvable for all $k < k'$. An ideal is called solvable if it is $n$-solvable.

**Definition 1.14.** The maximal $k$-solvable ideal in Leibniz $n$-algebra $L$ is called the $k$-solvable radical of $L$, denoted $\text{Rad}_k(L)$. $\text{Rad}_k(L)$ is called the radical of $L$ if $k = n$ and is denoted $\text{Rad}(L)$.

In [13, Theorem 4.5] the invariance of a $k$-radical under a derivation of a Leibniz $n$-algebra is proven.

**Theorem 1.15.** Let $D$ be a derivation of a Leibniz $n$-algebra $L$. Then $D(\text{Rad}_k(L)) \subseteq \text{Rad}_k(L)$.  

## 2. Leibniz $n$-kernel of a Leibniz $n$-algebra

Consider a Leibniz $n$-algebra $L$ and let us introduce for all $1 \leq i < j \leq n$ the following linear subspaces of $\text{Leib}_n(L)$:

$$I_{ij} = \text{Span}\{[x_1, ..., x_i, ..., x_j, ..., x_n] \mid x_i = x_j, \; x_1, ..., x_n \in L\}.$$ 

Note that for $n = 2$, $I_{12}$ is equal to the Leibniz kernel. In the general case, $\text{Leib}_n(L)$ is generated by $\sum_{1 \leq i < j \leq n} I_{ij}$ and we have the following statement.
Proposition 2.1. The linear space $I_{ij}$ is a 1-ideal of the Leibniz $n$-algebra $L$ for all $1 \leq i < j \leq n$.

Proof. From
\[
[x_1, \ldots, x_i + x_j, \ldots, x_i + x_j, \ldots, x_n] - [x_1, \ldots, x_i, \ldots, x_i, \ldots, x_n] - [x_1, \ldots, x_j, \ldots, x_j, \ldots, x_n] \\
= [x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n] + [x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n]
\]
one obtains
\[
(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n + [x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n] = I_{ij}. \tag{2.1}
\]
Consider the Leibniz $n$-identity
\[
[[x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n], y_2, \ldots, y_n] = \sum_{k \neq i,j} [x_1, \ldots, [x_k, y_2, \ldots, y_n], \ldots, x_n] \\
+ [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_j, \ldots, x_n] + [x_1, \ldots, x_i, \ldots, [x_j, y_2, \ldots, y_n], \ldots, x_n].
\]
If $x_i = x_j$, then the last two terms of the RHS by the property (2.1) belong to $I_{ij}$ and every summand in the sum is in $I_{ij}$. Hence, $I_{ij}$ is a 1–ideal of $L$ for all $1 \leq i < j \leq n$. \hfill \Box

2.1. Results on $L = U_3(\mathfrak{L})$. Since for Leibniz algebras $[a, [b, b]] = 0$, which over the base field of characteristic not equal to 2 is equivalent to $[a, [b, c]] = -[a, [c, b]]$, we obtain
\[
[x, y, y] = 0 \text{ and } [x, y, z] = -[x, z, y]. \tag{2.2}
\]
Hence, the ternary bracket in this case is a linear map $\mathfrak{L} \otimes (\mathfrak{L} \wedge \mathfrak{L}) \to \mathfrak{L}$.

An ideal $I$ of a Leibniz 3-algebra $L$ in the language of the underlying Leibniz algebra $\mathfrak{L}$ is defined by the following inclusions:
\[
[I, [\mathfrak{L}, \mathfrak{L}]] \subseteq I, \quad [\mathfrak{L}, [I, \mathfrak{L}]] \subseteq I, \quad [\mathfrak{L}, [\mathfrak{L}, I]] \subseteq I.
\]
Note that, since $il + li \in \text{Leib}(\mathfrak{L})$ for any $i \in I$, $l \in \mathfrak{L}$ and $[\mathfrak{L}, \text{Leib}(\mathfrak{L})] = 0$ we have $[\mathfrak{L}, [I, \mathfrak{L}]] = [\mathfrak{L}, [\mathfrak{L}, I]]$. Therefore, we have the following statement.

Proposition 2.2. A subspace $V$ of a Leibniz 3-algebra $U_3(\mathfrak{L})$ is an ideal if and only if
\[
[V, [\mathfrak{L}, \mathfrak{L}]] \subseteq V \quad \text{and} \quad [\mathfrak{L}, [V, \mathfrak{L}]] \subseteq V.
\]

Proposition 2.3. The subspace $I_{12}$ of $U_3(\mathfrak{L})$ is an ideal and is equal to $\text{Leib}_3(L)$.

Proof. Using equalities (2.2) we have $I_{23} = \{0\}$ and $I_{13} = I_{12}$. Therefore, the ideal $\text{Leib}_3(L)$ is generated by $I_{12}$. By Proposition 2.1 $I_{12}$ is a 1-ideal. Moreover, one can establish
\[
[x_1, x_2, x_3] - (-1)^{\text{sgn}(\pi)}[x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}] \in I_{12}
\]
for all permutations $\pi \in S_3$, which implies that $I_{12}$ is an ideal and thus, is equal to $\text{Leib}_3(L)$. \hfill \Box

Since a Leibniz 3-algebra is a 3-Lie algebra if and only if $\text{Leib}_3(L) = \{0\}$, we have immediately the following statement.

Corollary 2.4. $U_3(\mathfrak{L})$ is a 3-Lie algebra if and only if the Leibniz algebra $\mathfrak{L}$ satisfies $[x, [x, y]] = 0$ for all $x, y \in \mathfrak{L}$. 

2.2. Right multiplication operator of $U_n(\mathfrak{L})$ and the Leibniz $n$-kernel. Recall, that for a Lie and $n$-Lie algebras, all adjoint (right multiplication) maps are singular due to skew-symmetry. For Leibniz algebras, all right multiplications are also singular $[3]$. However, for Leibniz $n$-algebras, as shown in $[24$ Lemma 2.3$]$ right multiplication operators might be invertible. In this subsection we give a short proof of this fact generalizing the proof of $[15$, Proposition 5$]$ and prove that the converse does not hold.

**Proposition 2.5.** Let $L$ be a finite-dimensional Leibniz $n$-algebra over a field of characteristic zero. If it admits an invertible operator of right multiplication then $\text{Leib}_n(L) = L$.

**Proof.** Let there be an invertible right multiplication operator $R[a] : L \to L$, where $a = (a_2, \ldots, a_n)$. Assume that $\text{Leib}_n(L) \neq L$. If $\text{Leib}_n(L) = \{0\}$, then $L$ is an $n$-Lie algebra which contradicts to invertibility of $R[a]$. Hence, $\text{Leib}_n(L)$ is a non-trivial ideal of $L$ and note that it is an invariant subspace of $R[a]$. The right multiplication map $R[a]$ induces an invertible right multiplication operator in the quotient $n$-Lie algebra $L/\text{Leib}_n(L)$, which is a contradiction. Therefore, $\text{Leib}_n(L) = L$. $\Box$

The converse statement is false, and an example of a Leibniz $n$-algebra with $\text{Leib}_n(L) = L$ and all right multiplications maps being singular is constructed in $[24$ Example 2.4$]$, which is a $U_n$ of a simple Leibniz algebra with $\mathfrak{sl}_2$ as its liezation. Obviously, any right multiplication operator for any Leibniz $n$-algebra in $U_n(\mathfrak{Lb})$ is singular due to a right multiplication being singular in Leibniz algebras $([3], [15])$. Below we prove that any Leibniz $n$-algebra $U_n(\mathfrak{L})$ for a semisimple Leibniz algebra $L$ serves as an example for the converse of Proposition 2.5 to be false.

**Theorem 2.6.** For any semisimple Leibniz algebra $\mathfrak{L}$, the Leibniz $n$-kernel of $U_n(\mathfrak{L})$ coincides with $U_n(\mathfrak{L})$.

**Proof.** Denote by $L = U_n(\mathfrak{L})$ and let us apply the decomposition of semisimple Leibniz algebra from Corollary $[16]$. From Lie theory it is well-known that every simple Lie algebra contains a subalgebra isomorphic to $\mathfrak{sl}_2$. From $[h, e] = 2e$ we obtain $2^{n-1}e = [h, h, \ldots, h, e]$ which belongs to $\text{Leib}_n(L)$. Hence, $\text{Leib}_n(L)$ contains some elements from each simple Lie algebra $\mathfrak{g}_i$ in the decomposition of $\mathfrak{g}_L = \oplus_{i=1}^m \mathfrak{g}_i$. Since $\text{Leib}_n(L)$ is a $\mathfrak{g}_L$-module, in fact $\text{Leib}_n(L)$ contains every $\mathfrak{g}_i$ for all $1 \leq i \leq n$ due to simplicity of the Lie subalgebras.

Next, from $[\text{Leib}(\mathfrak{L}), \mathfrak{g}_L] = \text{Leib}(\mathfrak{L})$ and since $\mathfrak{g}_L$ is perfect, we have

$$\text{Leib}(\mathfrak{L}) = [\text{Leib}(\mathfrak{L}), \mathfrak{g}_L, \ldots, \mathfrak{g}_L] \subseteq [\text{Leib}(\mathfrak{L}), \text{Leib}_n(L), \ldots, \text{Leib}_n(L)] \subseteq \text{Leib}_n(L).$$

Thus $\text{Leib}_n(L) \supseteq \mathfrak{g}_L \oplus \text{Leib}(\mathfrak{L}) = L$ which completes the proof. $\Box$

3. Structure of Leibniz $n$-algebras $U_n(\mathfrak{L})$

It is straightforward that a subalgebra $\mathfrak{A}$ of a Leibniz algebra $\mathfrak{L}$ constitutes a Leibniz $n$-subalgebra $U_n(\mathfrak{A})$ of $U_n(\mathfrak{L})$. Moreover, we have the following statement.

**Proposition 3.1.** $U_n(\mathfrak{L}_1 \oplus \mathfrak{L}_2) = U_n(\mathfrak{L}_1) \oplus U_n(\mathfrak{L}_2)$ for any Leibniz algebras $\mathfrak{L}_1$ and $\mathfrak{L}_2$.

**Proof.** Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \in \mathfrak{L}_1 \oplus \mathfrak{L}_2$. Using the product in $U_n(\mathfrak{L}_1 \oplus \mathfrak{L}_2)$ we have

$$[(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)] = [(x_1, y_1), \ldots, [(x_{n-2}, y_{n-2}), [(x_{n-1}, y_{n-1}), (x_n, y_n)]] \ldots]]$$

$$= [(x_1, y_1), \ldots, [(x_{n-2}, y_{n-2}), [(x_{n-1}, x_n), [y_{n-1}, y_n]] \ldots]]$$

$$= [(x_1, y_1), \ldots, [(x_{n-3}, y_{n-3}), [(x_{n-2}, [x_{n-1}, x_n]), [y_{n-2}, [y_{n-1}, y_n]]] \ldots]]$$

$$= ([x_1, \ldots, x_{n-2}, x_{n-1}, x_n] \ldots [y_1, \ldots, y_{n-2}, y_{n-1}, y_n] \ldots]]$$

$$= ([x_1, x_2, \ldots, x_n], [y_1, y_2, \ldots, y_n])$$
which implies the result. \qed

Next, we introduce of the notion of simplicity as in \cite{[12]}. Note the difference with Definition 1.1 for Leibniz algebras.

**Definition 3.2.** A non-abelian Leibniz $n$-algebra $(n \geq 3)$ is called simple if the only ideals are \{0\} and the $n$-algebra itself.

Recall from \cite{[12]} an important example of an $(n + 1)$-dimensional $n$-Lie algebra which is an analogue of the three-dimensional Lie algebra with the cross product as multiplication.

**Example 3.3.** Let $\mathbb{K}$ be a field and $V_n$ an $(n + 1)$-dimensional $\mathbb{K}$-vector space with a basis $\{e_1, \ldots, e_{n+1}\}$. Then $V_n$, equipped with the skew-symmetric $n$-ary multiplication induced by

$$[e_1, \ldots, e_{i-1}, e_i, e_{i+1}, \ldots, e_{n+1}] = (-1)^{n+1+i}e_i, \quad 1 \leq i \leq n+1,$$

is an $n$-Lie algebra.

This $n$-Lie algebra is a simple $n$-Lie algebra. Remarkably, as shown in \cite{[18]}, over an algebraically closed field $\mathbb{K}$ all simple $n$-Lie algebras are isomorphic to $V_n$.

**Proposition 3.4.** The simple $n$-Lie algebra $V_n$ is not in the category $U_n(Lb)$.

**Proof.** Assume that there exists a Leibniz algebra $\mathfrak{L}$, such that $V_n = U_n(\mathfrak{L})$. Let $\{e_1, \ldots, e_{n+1}\}$ be a basis of $\mathfrak{L}$. Suppose $[e_i, e_j] = \alpha^1_{ij}e_1 + \alpha^2_{ij}e_2 + \ldots + \alpha^{n+1}_{ij}e_{n+1}$ for some $\alpha^k_{ij} \in \mathbb{K}$.

Let us denote by $E$ the right ordered product $[e_{k_1}, \ldots, [e_{k_{n-2}}, e_{k_{n-1}}]] \ldots ]$ of basis elements such that $k_1, \ldots, k_{n-1} \in \{1, \ldots, n+1\}$ and $k_i \neq k_j$ for $i \neq j$. Fix an integer $i \in \{1, \ldots, n+1\}$ and let some $k_j = i$. On one hand,

$$[[e_i, e_i], e_k, e_{k_2}, \ldots, e_{k_{n-2}}, e_{k_{n-1}}] = [[e_i, e_i], E] = [[e_i, E], e_i] + [e_i, [e_i, E]] = 0,$$

since $[e_i, E] = [e_i, e_k, e_{k_2}, \ldots, e_{k_{n-2}}, e_{k_{n-1}}] = 0$. On the other hand,

$$[[e_i, e_i], e_k, e_{k_2}, \ldots, e_{k_{n-2}}, e_{k_{n-1}}] = \sum_{r=1}^{n+1} \alpha^r_{ii}[e_r, e_k, e_{k_2}, \ldots, e_{k_{n-2}}, e_{k_{n-1}}]$$

$$= \alpha^p_{ii}[e_p, e_k, e_{k_2}, \ldots, e_{k_{n-2}}, e_{k_{n-1}}] + \alpha^q_{ii}[e_q, e_k, e_{k_2}, \ldots, e_{k_{n-2}}, e_{k_{n-1}}]$$

$$= \pm \alpha^p_{ii}e_{k} \pm \alpha^q_{ii}e_{k},$$

where $\{p, q\} = \{1, \ldots, n+1\} \setminus \{k_1, \ldots, k_{n-1}\}$. Hence, we obtain $\alpha^p_{ii} = \alpha^q_{ii} = 0$ for all $\{p, q\} = \{1, \ldots, n+1\} \setminus \{k_1, \ldots, k_{n-1}\}$. Choosing all possible subsets of $n - 2$ elements from the set $\{1, \ldots, n+1\} \setminus \{i\}$ for $\{k_1, \ldots, k_{n-1}\} \setminus \{i\}$, similarly we establish that $\alpha^r_{ii} = 0$ for all $r \neq i$. From $0 = [e_i, e_i] = [e_i, [e_i, e_i]] \ldots ] = (\alpha^1_{ii})^{n-1} e_i$ we obtain $\alpha^1_{ii} = 0$ and $[e_i, e_i] = 0$.

Now fix some $1 \leq i \neq j \leq n + 1$ and choose different $k_1, \ldots, k_{n-1}$ from $\{1, \ldots, n+1\}$ so that $i$ and $j$ are also chosen. Similarly, using Leibniz identity and the $n$-Lie bracket one establishes $[[e_i, e_j], E] = 0$. On the other hand, $[[e_i, e_j], E] = \sum_{r=1}^{n+1} \alpha^r_{ij}[e_r, e_j]$ and going through all possible options for $k_1, \ldots, k_{n-1}$ yields $[e_i, e_j] = \alpha^1_{ij}e_i + \alpha^2_{ij}e_j$. From

$$0 = [e_i, \ldots, e_i, e_j] = [e_i, [\ldots, [e_i, e_i, e_j], \ldots]] = \alpha^1_{ij} (\alpha^2_{ij})^{n-2} e_i + (\alpha^2_{ij})^{n-1} e_j$$

we obtain $\alpha^2_{ij} = 0$ and $[e_i, e_j] = \alpha^1_{ij}e_i$. This implies that

$$e_{n+1} = [e_1, e_2, \ldots, e_n] = [e_1, [\ldots, [e_{n-2}, e_{n-1}], e_n], \ldots] \in \text{Span}\{e_1\},$$

which is a contradiction. \qed
Proposition 3.5. Let $I$ be an ideal of a Leibniz algebra $\mathfrak{L}$. Then $I$ is an ideal of the Leibniz $n$-algebra $L = U_n(\mathfrak{L})$.

Proof. Let $s \in \{1, 2, \ldots, n\}$ be given. Then we have $[L, \ldots, L, I, L, \ldots, L] =$ $\mathfrak{L}, \ldots, [I, [\mathfrak{L}, \ldots, \mathfrak{L}, [\mathfrak{L}, \mathfrak{L}], \ldots]]$ $\subseteq \mathfrak{L}, \ldots, [\mathfrak{L}, [\mathfrak{L}, [\mathfrak{L}, I], \ldots]]$ $\subseteq \mathfrak{L}, \ldots, [\mathfrak{L}, [\mathfrak{L}, [\mathfrak{L}, I], \ldots]]$ due to $\mathfrak{L}, \ldots, [\mathfrak{L}, [\mathfrak{L}, I], \ldots] \subseteq \mathfrak{L}$.

By definition we have $[\mathfrak{L}, I] \subseteq I$, and assuming $[\mathfrak{L}, \ldots, [\mathfrak{L}, [\mathfrak{L}, I], \ldots]] \subseteq I$ leads to $[\mathfrak{L}, \ldots, [\mathfrak{L}, [\mathfrak{L}, I], \ldots]] = [\mathfrak{L}, [\mathfrak{L}, \ldots, [\mathfrak{L}, [\mathfrak{L}, I], \ldots]]] \subseteq [\mathfrak{L}, I] \subseteq I$. Hence, by induction we obtain $[L, \ldots, L, I, L, \ldots, L] \subseteq I$ which concludes that $I$ is an ideal of $U_n(\mathfrak{L})$.

The following example shows that the converse of Proposition 3.5 is not true.

Example 3.6. Consider the two-dimensional Leibniz algebra $\mathfrak{L}_2$ with a basis $\{e, f\}$ such that $[e, e] = [e, f] = [f, e] = 0$ and $[f, f] = e$.

$U_n(\mathfrak{L}_2)$ is an abelian $n$-algebra and any subspace is an ideal of $U_n(\mathfrak{L}_2)$. However, Span$\{e\}$ is the only nontrivial ideal of the Leibniz algebra $\mathfrak{L}_2$.

Remark 3.7. The category $U_n(\mathfrak{L}b)$ is not a full subcategory of $n\mathfrak{L}b$. Indeed, consider Leibniz algebra $\mathfrak{L}_2$ from Example 3.6 and note that any linear map from $U_n(\mathfrak{L}_2)$ to $U_n(\mathfrak{L}_2)$ is a Leibniz $n$-algebra homomorphism. However, a linear map $\phi : \mathfrak{L}_2 \to \mathfrak{L}_2$ defined on the basis elements of $\mathfrak{L}_2$ by $\phi(e) = 0, \phi(f) = f$ is not a Leibniz algebra homomorphism due to $0 = \phi(e) = \phi([f, f]) \neq [\phi(f), \phi(f)] = [f, f] = e$.

The following statement describes all simple Leibniz $n$-algebras in $U_n(\mathfrak{L}b)$.

Theorem 3.8. A Leibniz $n$-algebra $U_n(\mathfrak{L})$ is simple if and only if $\mathfrak{L}$ is a simple Lie algebra.

Proof. Let $L = U_n(\mathfrak{L})$ be a simple Leibniz $n$-algebra. Then by Proposition 3.5 the Leibniz algebra $\mathfrak{L}$ does not admit non-trivial ideals which implies that $\mathfrak{L}$ is a simple Lie algebra. Conversely, let $\mathfrak{L}$ be a simple Lie algebra and consider $L = U_n(\mathfrak{L})$. If $I$ is a nontrivial ideal of $L$, then $I \supseteq [I, L, \ldots, L] = [I, [\mathfrak{L}, [\ldots, [\mathfrak{L}, [\mathfrak{L}, \mathfrak{L}], \ldots]]] = [I, \mathfrak{L}]$. Hence, $I$ is a nontrivial ideal of the simple Lie algebra $\mathfrak{L}$, which is a contradiction.

By Proposition 3.5, simple Leibniz $n$-algebras in $U_n(\mathfrak{L}b)$ are not $n$-Lie algebras. One may wonder, what happens if simplicity is defined for Leibniz $n$-algebras as in Leibniz algebras, leaving freedom to admit exactly one nontrivial ideal. The following proposition describes all Leibniz $n$-algebras in $U_n(\mathfrak{L}b)$ that admit exactly one nontrivial ideal. However, that ideal is not the Leibniz $n$-kernel.

Proposition 3.9. A Leibniz $n$-algebra $U_n(\mathfrak{L})$ admits only one nontrivial ideal if and only if $\mathfrak{L}$ is a simple non-Lie Leibniz algebra or a Lie algebra $\mathfrak{g} \ltimes V$, where $\mathfrak{g}$ is a simple Lie algebra and $V$ is a simple $\mathfrak{g}$-module. The nontrivial ideal of $U_n(\mathfrak{L})$ is either $\text{Leib}(\mathfrak{L})$ or $V$, correspondingly.

Proof. Let $I$ be the only ideal of $U_n(\mathfrak{L})$. By Proposition 3.5 it follows that the Leibniz algebra $\mathfrak{L}$ does not admit more than one nontrivial ideal. If $\mathfrak{L}$ is a non-Lie Leibniz algebra, since the Leibniz kernel $\text{Leib}(\mathfrak{L})$ is a non-trivial ideal of $\mathfrak{L}$ it follows that $\mathfrak{L}$ is a simple Leibniz algebra and $I = \text{Leib}(\mathfrak{L})$. If $\mathfrak{L}$ is a Lie algebra, by Theorem 3.8 it is not simple and Proposition 3.5
concludes that it admits exactly one ideal. Therefore, \( \mathfrak{L} \) is a semi-direct product Lie algebra \( \mathfrak{g} \ltimes V \) for some simple Lie algebra \( \mathfrak{g} \) and a simple \( \mathfrak{g} \)-module \( V \), and \( I = V \).

Conversely, let \( \mathfrak{L} \) be a simple non-Lie Leibniz algebra. By Corollary \( 1.6 \) we have \( \mathfrak{L} = \mathfrak{g} \ltimes \text{Leib}(\mathfrak{L}) \) and \( \text{Leib}(\mathfrak{L}) \) is an irreducible \( \mathfrak{g} \)-module and \( \mathfrak{g} \) is a simple Lie algebra. Note that \( \mathfrak{L} \) as a \( \mathfrak{g} \)-module decomposes into the direct sum \( \mathfrak{g} \oplus \text{Leib}(\mathfrak{L}) \) of simple \( \mathfrak{g} \)-modules. Let \( I \) be an ideal of \( U_n(L) \). Then from \( I \supseteq [I, \mathfrak{g}, \ldots, \mathfrak{g}] = \left[ \mathfrak{g}, \ldots, \left[ \mathfrak{g}, \mathfrak{g}, \mathfrak{g} \right] \ldots \right] \)) we obtain that \( I \) is a \( \mathfrak{g} \)-submodule of \( \mathfrak{L} \). \( I \) is a nontrivial module so it must either be \( \text{Leib}(\mathfrak{L}) \) or \( \mathfrak{g} \). However, the latter one is not an ideal of \( U_n(\mathfrak{L}) \) due to \( \text{Leib}(\mathfrak{L}), \mathfrak{g}, \ldots, \mathfrak{g} \rangle = \text{Leib}(\mathfrak{L}), \mathfrak{g} \rangle \). \( \text{Leib}(\mathfrak{L}) \). Therefore, \( I = \text{Leib}(\mathfrak{L}) \) is the only nontrivial ideal of \( L \). Similarly, one obtains \( I = V \) in case \( \mathfrak{L} \) is a Lie algebra \( \mathfrak{g} \ltimes V \).

Note that, from Theorem \( 2.4 \) it follows that the Leibniz \( n \)-kernel of \( U_n(\mathfrak{L}) \) of a Lie or Leibniz algebra \( \mathfrak{L} \) from Proposition \( 3.9 \) is equal to the whole \( n \)-algebra.

**Theorem 3.10.** The ideals of \( U_n(\mathfrak{L}) \) for a semisimple Leibniz algebra \( \mathfrak{L} \) are exactly the ideals of \( \mathfrak{L} \).

**Proof.** The case when \( \mathfrak{L} \) is a Lie algebra is straightforward from Proposition \( 3.11 \). Also, it suffices to consider the case of an indecomposable semisimple Leibniz algebra. Let \( I \) be an ideal of \( U_n(\mathfrak{L}) \). Recall that \( [I, [\mathfrak{L}, \ldots, [\mathfrak{L}, \mathfrak{L}, \mathfrak{L}] \ldots]] \subseteq I \) and \( [\mathfrak{L}, I, \ldots, [\mathfrak{L}, [\mathfrak{L}, \mathfrak{L}]] \ldots] \subseteq I \). Since semisimple Leibniz algebras are perfect, then

\[
(3.1) \quad [I, \mathfrak{L}] \subseteq I \quad \text{and} \quad [\mathfrak{L}, I] \subseteq I.
\]

By Proposition \( 3.3 \) the ideals of \( \mathfrak{L} \) are ideals of \( U_n(\mathfrak{L}) \).

Assume by contradiction that there exists an ideal \( H \) of \( U_n(\mathfrak{L}) \) that is not an ideal of \( \mathfrak{L} \). Then by \( (3.1) \) we have \( [H, \mathfrak{L}] \subseteq H \) which implies \( [H, \mathfrak{g}_L] \subseteq H \), where \( \mathfrak{g}_L \) is a semisimple Lie algebra (liczation of \( \mathfrak{L} \)) and \( H \) is a \( \mathfrak{g}_L \)-module. Using the structure of semisimple Leibniz algebra we conclude that \( H = (\oplus_{i \in I} \mathfrak{g}_i) \oplus (\oplus_{j \in J} \mathfrak{I}_j) \) for some \( I \subseteq \{1, \ldots, m\}, J \subseteq \{1, \ldots, n\} \). Furthermore,

\[
[H, \mathfrak{g}_L] = \left[ \oplus_{i \in I} \mathfrak{g}_i, \oplus_{i=1}^m \mathfrak{g}_i \right] \oplus \left[ \oplus_{j \in I} \mathfrak{I}_j, \oplus_{i=1}^m \mathfrak{g}_i \right] = (\oplus_{i \in I} \mathfrak{g}_i) \oplus (\oplus_{j \in J} \mathfrak{I}_j) = H.
\]

Thus, \( H \supseteq [\mathfrak{L}, H, \mathfrak{L}] = [\mathfrak{L}, [\mathfrak{L}, \mathfrak{L}]] + [\mathfrak{L}, \mathfrak{L}, \mathfrak{L}] \) and \( [H, \mathfrak{L}] \subseteq H \) implies \( H \) is an ideal of \( \mathfrak{L} \) which is a contradiction. □

Following Ling \( 18 \) we extend the following notions from \( n \)-Lie algebras to Leibniz \( n \)-algebras.

**Definition 3.11.** Leibniz \( n \)-algebra \( L \) is called \( k \)-semisimple if \( \text{Rad}_k(L) = \{0\} \). If \( L \) is \( n \)-semisimple it is called semisimle.

Let \( \mathfrak{L} = \mathfrak{g} \ltimes \text{rad}(\mathfrak{L}) \) be Levi decomposition of the Leibniz algebra \( \mathfrak{L} \). We have the following analogue of Levi decomposition for \( n \)-algebras from the category \( U_n(\mathfrak{L}) \).

**Theorem 3.12.** Let \( \mathfrak{L} \) be a Leibniz algebra and \( \mathfrak{g} \ltimes \text{rad}(\mathfrak{L}) \) be Levi decomposition. Let \( \mathfrak{g}_i \) be simple Lie subalgebras from the decomposition \( \mathfrak{g} = \oplus_{i=1}^n \mathfrak{g}_i \). Then in Leibniz \( n \)-algebra \( U_n(\mathfrak{L}) \), the \( k \)-solvable radical coincides with \( \text{rad}(\mathfrak{L}) \) for all \( 2 \leq k \leq n \) and there is the following analogue of the Levi decomposition

\[
U_n(\mathfrak{L}) = (\oplus_{i=1}^n U_n(\mathfrak{g}_i)) + \text{Rad}(U_n(\mathfrak{L})).
\]

**Proof.** Let us prove first that for an ideal \( I \) of \( L \), \( I^{(m)2} \subseteq I^{(m)} \). For \( m = 2 \),

\[
I^{(2)2} = [I, I, L, \ldots] + [I, L, I, L, \ldots] + [L, I, I, L, \ldots] + \cdots + [L, \ldots, I, I] \subseteq [I, [I, \mathfrak{L}]] + [I, [\mathfrak{L}, I]] + [\mathfrak{L}, [I, I]] \subseteq [I, I] = I^{(2)}.
\]

Assume \( I^{(m-1)2} \subseteq I^{(m-1)} \), then \( I^{(m)} \subseteq \left[ I^{(m-1)2}, I^{(m-1)} \right] \) follows similarly and inductively we obtain \( I^{(m)2} \subseteq I^{(m)} \) for all positive integers \( m \).
If \( m \) is the index of solvability of \( \text{rad}(\mathfrak{L}) \), then \( \text{rad}(\mathfrak{L})^{(m)2} \subseteq \text{rad}(\mathfrak{L})^{(m)} = \{0\} \) and by Proposition 1.13 it follows that \( \text{rad}(\mathfrak{L}) \) is a \( k \)-solvable ideal of \( L \) for all \( 2 \leq k \leq n \). Hence, \( \text{rad}(\mathfrak{L}) \subseteq \text{Rad}_k(L) \) for all \( 2 \leq k \leq n \). Let us prove the converse inclusion and that all radicals coincide. Since \( L \) is a \( \mathfrak{g} \)-module and \( \text{Rad}_k(L) \) is an ideal of \( L \) we obtain that \( \text{Rad}_k(L) \) is a \( \mathfrak{g} \)-submodule of \( L \). Let \( \oplus_{i=1}^n \mathfrak{g}_i \) be the decomposition of \( \mathfrak{g} \) into a direct sum of simple Lie subalgebras. Note that if \( \mathfrak{g} \cap \text{Rad}_k(L) \neq \{0\} \) then \( \mathfrak{g}_i \subseteq \text{Rad}_k(L) \) for some \( 1 \leq i \leq n \). However, since \( [\mathfrak{g}_i, \mathfrak{g}_i, \ldots, \mathfrak{g}_i] = \mathfrak{g}_i \) this is a contradiction with solvability of the radical of \( L \). Thus \( \text{Rad}_k(L) = \text{rad}(\mathfrak{L}) \) for all \( 2 \leq k \leq n \).

As a vector space, \( L = \mathfrak{L} = \mathfrak{g} \oplus \text{rad}(\mathfrak{L}) = U_n(\mathfrak{g}) \oplus \text{rad}(L) \). Moreover, by Proposition 3.1 we obtain that \( U_n(\mathfrak{g}) \) is a direct sum of simple Leibniz \( n \)-algebras \( U_n(\mathfrak{g}_i) \)s and we obtain the desired decomposition.

For a semisimple \( U_n(\mathfrak{L}) \) Leibniz \( n \)-algebra, by definition the solvable radical is zero and by the result above, it follows that \( U_n(\mathfrak{L}) \) is a direct sum of simple Leibniz \( n \)-algebras.

**Conjecture 3.13.** A semisimple Leibniz \( n \)-algebra \( (n \geq 3) \) is a direct sum of simple Leibniz \( n \)-algebras.

For \( n \)-Lie algebras this is [18 Proposition 2.7]. The structure of semisimple Leibniz algebra presented in Preliminaries shows why we exclude \( n = 2 \) in this conjecture.

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[MIN SOO KIM] VANDERBILT UNIVERSITY, NASHVILLE, TN 37235
*E-mail address*: min.soo.kim@vanderbilt.edu

[RUSTAM TURDIBAEV] INHA UNIVERSITY IN TASHKENT, 100170 TASHKENT, UZBEKISTAN
*E-mail address*: r.turdibaev@inha.uz