2-SPINORS VIA LINEAR ALGEBRA

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Abstract. We give a streamlined account of 2-spinors, up to and including the Dirac equation, using little more than the resources of linear algebra. We prove that the Dirac bundle is isomorphic to the associated bundles $SL_2(\mathbb{C}) \times SU_2 S$ and $SL_2(\mathbb{C}) \times SU_2 \overline{S}$. A solution of the Dirac equation determines a pair of conjugate 2-spinor fields over the mass shell $X_m$.

Contents

1. Introduction 1
2. 2-spinors 2
2.1. 4-spinors as a Clifford module 6
2.2. Momentum space 7
3. The Dirac Equation 9
3.1. The Dirac Bundle 9
3.2. The Dirac bundle as an associated vector bundle 11
3.3. Dirac equation 12
3.4. A pair of conjugate 2-spinor fields 13
3.5. Comments 13
4. The transition to spinor fields on space-time 13
References 14

1. Introduction

The classic work on spinors and space-time is by Penrose and Rindler [PR]. In this magisterial work, the authors demonstrate that 2-spinors are woven into the very fabric of space-time. This point is further made in the books by O’Donnell [O] and Wald [Chapter 13] [W].

It is our aim in this Note to show that the basic theory of 2-spinors, up to and including the Dirac equation, can be formulated using little more than the resources of linear algebra.

Throughout this Note, we define the vector space of 4-spinors as

$$\mathcal{S} = S \oplus \overline{S}$$
where $S$ is the vector space of 2-spinors and $\overline{S}$ is the conjugate vector space of $S$

Our approach is canonical except at the moment in §2.2 when one has to identify a
real finite-dimensional vector space $V$ with its space of characters.

The Dirac bundle is defined and studied in some detail in the books by Simms
[Sim] and Varadarajan [Var]. Our new result is that the Dirac bundle as an $SL_2(\mathbb{C})$-
bundle is isomorphic to the associated bundles

(1) \[ SL_2(\mathbb{C}) \times_{SU_2} S \]

and

(2) \[ SL_2(\mathbb{C}) \times_{SU_2} \overline{S} \]

where $SU_2$ acts on $S$ and $\overline{S}$ as the spin 1/2 representation. As a consequence, a
solution of the Dirac equation determines a pair of conjugate 2-spinor fields over the
mass shell $X_m$.

In section 2, we present the basic theory of 2-spinors via linear algebra. In section 3,
we define the Dirac bundle and prove that it is isomorphic to the associated vector
bundles (1) and (2). In section 4, we briefly discuss the transition from the vector
space of 2-spinors to spinor fields on space-time.

2. 2-spinors

Let $W$ be a complex vector space. Then $W$ comprises an abelian group $A$ and a
scalar-multiplication map

(3) \[ m : (\mathbb{C}, A) \to A, \quad (\lambda, v) \mapsto \lambda v \]

Each complex vector space $W$ has a companion vector space, the \textit{conjugate vector
space} denoted $\overline{W}$. Now $W$ and $\overline{W}$ share the same underlying abelian group $A$, but
the scalar-multiplication map for $\overline{W}$ is given by

(4) \[ \overline{m} : (\mathbb{C}, A) \to A, \quad (\lambda, v) \mapsto \overline{\lambda} v \]

Let $v \in W$. This means that $v \in A$ equipped with the scalar-multiplication (3).

The vector spaces $W$ and $\overline{W}$ enjoy a perfect duality, in the sense that each is the
conjugate of the other.

If we wish to refer to $v \in A$ equipped with the scalar-multiplication (4) we shall
write $\overline{v}$ instead of $v$. From a logical point of view, we have $v = \overline{v} \in A$ but $v$ and $\overline{v}$ lie in the distinct vector spaces $W$ and $\overline{W}$. From a practical point of view, it is
good to adopt the following convention: we shall always write

\[ v \in W \quad \text{and} \quad \overline{v} \in \overline{W} \]

In that case, in accordance with (4), we have

\[ \overline{\overline{v}} = \overline{\lambda v} \]

If $W$ is finite-dimensional with basis $\{e_1, \ldots, e_n\}$ then $\overline{W}$ admits the basis$\{\overline{e_1}, \ldots, \overline{e_n}\}$
so that $W$ and $\overline{W}$ have the same dimension.

Given $A \in GL(W)$ define the \textit{conjugate} of $A$ as follows:

\[ \overline{A} := \overline{A} \]
Then we have
\[
\overline{A}(\lambda v + \mu w) = A(\overline{\lambda}v + \overline{\mu}w) \\
= \overline{\lambda}Av + \overline{\mu}Aw \\
= \lambda Av + \mu Aw \\
= \lambda A\overline{v} + \mu A\overline{w}
\]
so that \(\overline{A}\) is a linear map \(W \to \overline{W}\). We have \(\overline{A} \in \text{GL}(\overline{W})\). The matrix of the conjugate of \(A\) is the conjugate of the matrix of \(A\).

Now let \(S\) denote a complex vector space of dimension 2. Consider the tensor product over \(\mathbb{C}\)

\[
S \otimes \overline{S}
\]
This is a complex vector space of dimension 4. Note the standard tensor product rule:

\[
\lambda(x \otimes \overline{y}) = (\lambda x) \otimes \overline{y} = x \otimes \overline{\lambda y}
\]
This vector space admits a canonical involution \(J\) defined on elementary tensors as follows:

\[
J : S \otimes \overline{S} \to S \otimes \overline{S}, \quad x \otimes \overline{y} \mapsto y \otimes \overline{x}
\]
We have \(J^2 = I\) the identity map on \(S \otimes \overline{S}\). The map \(J\) admits two eigenvalues, namely \(+1\) and \(-1\).

The \(+1\)-eigenspace of \(J\) will be denoted \(V\). We define

\[
V := \{v \in S \otimes \overline{S} : Jv = v\}
\]
Then \(V\) is a canonical subspace of \(S \otimes \overline{S}\). We will view \(V\) as a real vector space of dimension 4.

Let \(\wedge^2 S\) denote the exterior square of \(S\). Choose a basis \(\{e_1, e_2\}\) for \(S\). Then the 2-vector \(e_1 \wedge e_2\) is a basis for the 1-dimensional vector space \(\wedge^2 S\). Define an isomorphism \(\varphi : \wedge^2 S \simeq \mathbb{C}\) as follows:

\[
\varphi(\lambda e_1 \wedge e_2) = \lambda
\]
Next, define

\[
\varepsilon(x, y) := \varphi(x \wedge y)
\]
Then we have

\[
\varepsilon(x, y) = \varphi(x \wedge y) \\
= \varphi(-y \wedge x) \\
= -\varphi(y \wedge x) \\
= -\varepsilon(x, y)
\]
and also

\[
\varepsilon(e_1, e_2) = \varphi(e_1 \wedge e_2) = 1
\]
so that \(\varepsilon\) is a symplectic form on \(S\) and \(\{e_1, e_2\}\) is a symplectic basis, a dyad.

We emphasize that the symplectic form \(\varepsilon\) arises from, and is determined by, a non-canonical choice of isomorphism

\[
\varphi : \wedge^2 S \simeq \mathbb{C}
\]
Define \[ \text{SL}(S) := \{ A \in \text{GL}(S) : \det(A) = 1 \} \].

Once a basis in \( S \) has been chosen, we have
\[ \text{SL}(S) \cong \text{SL}_2(\mathbb{C}) \]

Let \( A \in \text{SL}(S) \). If \( Ae_1 = \alpha_{11}e_1 + \alpha_{12}e_2 \) and \( Ae_2 = \alpha_{21}e_1 + \alpha_{22}e_2 \) then
\[
\varepsilon(Ae_1, Ae_2) = \varphi(Ae_1 \wedge Ae_2) = \varphi(\det(A)e_1 \wedge e_2) = \varphi(e_1 \wedge e_2) = \varepsilon(e_1, e_2) = 1
\]
so that \( \text{SL}(S) \) acts freely and transitively on the set of all symplectic bases (dyads).

The basis \( \{ e_1, e_2 \} \) determines an isomorphism \( \varphi : \Lambda^2 \overline{S} \cong \mathbb{C} \) as follows:
\[ \varphi(\lambda e_1 \wedge e_2) = \lambda \]
and allows us to define
\[ \varepsilon(x, y) := \varphi(x \wedge y) \]

as a symplectic form on \( \overline{S} \).

We have
\[ \Lambda^2 \overline{S} = \overline{\Lambda^2 S} \]
and so
\[ \varepsilon(x, y) = \overline{\varepsilon(x, y)} \]

We have
\[
\overline{Ae_1} = \overline{Ae_1} = \overline{a_{11}e_1 + a_{12}e_2} = a_{11} \overline{e_1} + a_{12} \overline{e_2}
\]
and
\[
\overline{Ae_2} = \overline{Ae_2} = \overline{a_{21}e_1 + a_{22}e_2} = a_{21} \overline{e_1} + a_{22} \overline{e_2}
\]
so that
\[ \det(A) = \det(\overline{A}) = 1 \]
and
\[ \overline{A} \in \text{SL}(\overline{S}) \]

Then
\[ \varepsilon(\overline{Ae_1}, \overline{Ae_2}) = 1 \]
as above and \( \text{SL}(\overline{S}) \) acts freely and transitively on the set of dyads for \( \overline{S} \).

**Definition 2.1.** We now define
\[ h(a \otimes \overline{b}, c \otimes \overline{d}) : = \varepsilon(a, c) \cdot \overline{\varepsilon(b, d)} \]
This determines a bilinear form on $S \otimes \overline{S}$, thanks to the bilinearity of $\varepsilon$ and $\overline{\varepsilon}$. Furthermore, this is a symmetric bilinear form, for $h$ is invariant under the simultaneous interchanges $a \to c$ and $b \to \overline{d}$.

We are especially interested in the restriction of $h$ to $V$. We denote this restriction by $g$:

$$g = h|_V$$

This will be a symmetric bilinear form on $V$, and so will have an associated quadratic form $Q$. We next determine the rank and signature of this quadratic form.

Conceptually, $h$ is given by the following map:

$$S \otimes \overline{S} \otimes S \otimes \overline{S} \cong S \otimes S \otimes \overline{S} \otimes S \to \bigwedge^2 S \otimes \bigwedge^2 \overline{S} \cong \mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$$

Note that this map depends only on $\varepsilon$.

**Theorem 2.2.** Let $x \in S \otimes \overline{S}$. If $Jx = x$ then $g$, defined as

$$g(x) := h(x, x)$$

is a quadratic form of rank 4 and signature 2, i.e. a Lorentz quadratic form.

**Proof.** We will do this via an explicit diagonalization of $g$. Let $\{e_1, e_2\}$ be a basis of $S$. Then $e_i \otimes \overline{e}_j$ is a basis of $S \otimes \overline{S}$ with $1 \leq i, j \leq 2$. We will write

$$e_i \cdot \overline{e}_j := e_i \otimes \overline{e}_j$$

Following [PR 3.1.20], we define

$$u_0 = (e_1 \cdot \overline{e}_1 + e_2 \cdot \overline{e}_2)/\sqrt{2}$$

$$u_1 = e_1 \cdot \overline{e}_2 + e_2 \cdot \overline{e}_1/\sqrt{2}$$

$$u_2 = i(e_1 \cdot \overline{e}_2 - e_2 \cdot \overline{e}_1)/\sqrt{2}$$

$$u_3 = e_1 \cdot \overline{e}_1 - e_2 \cdot \overline{e}_2/\sqrt{2}$$

Then we have

$$Ju_j = u_j$$

for all $j$ and so $u_j \in V$ for all $j$. We also have

$$g(u_0, u_0) = g(e_1 \cdot \overline{e}_1 + e_2 \cdot \overline{e}_2, e_1 \cdot \overline{e}_1 + e_2 \cdot \overline{e}_2)$$

$$= \frac{\varepsilon(e_1, e_2)\varepsilon(\overline{e}_1, \overline{e}_2)}{2} + \frac{\varepsilon(e_2, e_1)\varepsilon(\overline{e}_2, \overline{e}_1)}{2}$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1$$

Similarly, we have

$$g(u_j, u_j) = -1 \quad 1 \leq j \leq 3$$

$$g(u_i, u_j) = 0 \quad i \neq j, \ 0 \leq i, j \leq 3$$

so that $g$ determines a quadratic form $Q$ of rank 4 and signature 2, which we will denote as follows:

$$+ --$$
This is the Lorentz quadratic form.

It is a truly remarkable fact that the Lorentz quadratic form emerges from an apparently symmetrical situation.

The space $S \oplus \overline{S}$ is a complex vector space of dimension 4. It is called the space of 4-spinors. We will write

$$\mathfrak{S} := S \oplus \overline{S}.$$  

2.1. 4-spinors as a Clifford module. Thanks to Paul Robinson for help with this section.

Let $a, b, c \in S$. Since $S$ has dimension 2, there must be a linear relation among $a, b, c$. We make this precise.

**Lemma 2.3.** [PR 1.6.19]. Let $a, b, c \in S$. Then we have the cyclic identity

$$\varepsilon(b, c)a + \varepsilon(c, a)b + \varepsilon(a, b)c = 0 \quad (7)$$

**Proof.** Fix $s \in S$ and define

$$f(a, b, c) := \varepsilon(s, \varepsilon(b, c)a + \varepsilon(c, a)b + \varepsilon(a, b)c).$$

Then $f$ is a skew 3-form on a 2-space, hence vanishes for all $s \in S$. Now invoke the non-singularity of $\varepsilon$. \hfill \Box

Define a map

$$\phi : S \otimes \overline{S} \to \text{End}(\mathfrak{S})$$

as follows:

$$\phi(p \otimes \overline{q})(a \oplus \overline{b}) := \sqrt{2}[\varepsilon(\overline{b}, \overline{q})p \oplus \varepsilon(p, a)q]$$

**Lemma 2.4.** [PW §2.3]. For all $X, Y \in S \otimes \overline{S}$, we have

$$\phi(X)\phi(Y) + \phi(Y)\phi(X) = h(X, Y)I_{\mathfrak{S}}$$

**Proof.** We have

$$\frac{1}{2}\phi(r \otimes \overline{s})\phi(p \otimes \overline{q})(a \oplus \overline{b}) = \frac{1}{\sqrt{2}}\phi(r \otimes \overline{s})[\varepsilon(\overline{b}, \overline{q})p \oplus \varepsilon(p, a)q]$$

$$= \varepsilon(\varepsilon(p, a)\overline{q}, \overline{s})r \oplus \varepsilon(r, \varepsilon(\overline{b}, \overline{q})p)\overline{s}$$

$$= \varepsilon(p, a)\varepsilon(\overline{q}, \overline{s})r \oplus \varepsilon(r, p)\varepsilon(\overline{b}, \overline{q})\overline{s}$$

Next, we symmetrize and add:

$$\frac{1}{2}\{\phi(p \otimes \overline{q})\phi(r \otimes \overline{s}) + \phi(r \otimes \overline{s})\phi(p \otimes \overline{q})\}(a \oplus \overline{b})$$

$$= \varepsilon(p, a)\varepsilon(\overline{q}, \overline{s})r \oplus \varepsilon(r, p)\varepsilon(\overline{b}, \overline{q})\overline{s} + \varepsilon(r, a)\varepsilon(\overline{s}, \overline{q})p \oplus \varepsilon(p, r)\varepsilon(\overline{b}, \overline{s})\overline{q}$$

$$= \varepsilon(\varepsilon(p, a)r - \varepsilon(r, a)p) \oplus \varepsilon(p, r)[\varepsilon(\overline{b}, \overline{s})\overline{q} - \varepsilon(\overline{q}, \overline{b})\overline{s}]$$

Applying the cyclic identity \(\overline{[q]}\), we obtain

$$\varepsilon(p, a)r - \varepsilon(r, a)p = \varepsilon(p, r)a$$

$$\varepsilon(\overline{b}, \overline{s})\overline{q} - \varepsilon(\overline{q}, \overline{b})\overline{s} = \varepsilon(\overline{q}, \overline{s})\overline{b}$$
Our conclusion is that
\[
\frac{1}{2} \{ \phi(p \otimes \overline{q}) \phi(r \otimes \overline{s}) \} \phi(p \otimes \overline{q}) \phi(r \otimes \overline{s}) (a \oplus \overline{b}) = \varepsilon(p, r) \overline{\varepsilon(q, s)} (a \oplus \overline{b})
\]
as required. \(\square\)

In view of Lemma 2.4, the map \(\phi\) will lift to a morphism of \(C\)-algebras
\[
\mathcal{Cl}(V, Q) \otimes_{\mathbb{R}} C \to \text{End}(\mathcal{G})
\]
where \(\mathcal{Cl}(V, Q)\) is the Clifford algebra of \(V\) with respect to the quadratic form \(Q\). Therefore, \(\mathcal{G}\) slots in as a pointwise irreducible \(\mathbb{Z}/2\mathbb{Z}\)-graded Clifford module as in [ABS].

2.2. Momentum space. Let \(x \in V\). The following equation defines a character of \(V\):
\[
\hat{x}(y) = \exp(ig(x, y))
\]
and the map
\[
V \to \hat{V}, \quad x \mapsto \hat{x}
\]
(8) secures a (non-canonical) isomorphism of the real vector space \(V\) onto its Pontryagin dual \(\hat{V}\). The vector space \(\hat{V}\) admits an \(SL_2(\mathbb{C})\)-action as follows:
\[
(A \cdot \hat{x})(y) = \hat{x}(A^{-1} y)
\]
The isomorphism \(V \to \hat{V}\) commutes with the action of \(SL_2(\mathbb{C})\).

We recall the basis \(\{u_0, u_1, u_2, u_3\}\) determined by the symplectic basis \(\{e_1, e_2\}\), see [6]. A basis of \(\hat{V}\) now presents itself, namely
\[
v_j := \hat{u}_j
\]
As explicit characters, we have
\[
v_0(x) = \exp(ix_0), \quad v_1(x) = \exp(-ix_1), \quad v_2(x) = \exp(-ix_2), \quad v_3(x) = \exp(-ix_3)
\]
The basis \(\{u_0, u_1, u_2, u_3\}\) determines an isomorphism
\[
V \to \mathbb{R}^4, \quad x \mapsto (x_0, x_1, x_2, x_3)
\]
and the basis \(\{v_0, v_1, v_2, v_3\}\) determines an isomorphism
\[
\hat{V} \to \mathbb{R}^4, \quad p \mapsto (p_0, p_1, p_2, p_3).
\]
We will write
\[
\mathbb{P}^4 = \hat{V}.
\]
Now \(V\) equipped with its Lorentz quadratic form is isomorphic to \(\mathbb{P}^4\) with its quadratic form
\[
Q(p) = p_0^2 - p_1^2 - p_2^2 - p_3^2
\]
We record this in the following

**Lemma 2.5.** Thanks to the map (8), we have
\[
(V, Q) \cong (\mathbb{P}^4, Q)
\]
and this map commutes with the action of \(SL_2(\mathbb{C})\).
Lemma 2.5 allows us, by transport of structure, to obtain the following

**Corollary 2.6.** For all $x, y \in \mathbb{P}^4$ we have a morphism

$$\phi : \mathcal{C}(\mathbb{P}^4, Q) \to \text{End}(\mathcal{G})$$

for which

$$\phi(x)\phi(y) + \phi(y)\phi(x) = 2g(x, y)$$

In particular, we have

$$\phi(p)^2 = Q(p)$$

for all $p \in \mathbb{P}^4$.

**Definition 2.7.** Let $A \in \text{SL}_2(\mathbb{C})$. The canonical representation $\pi$ of $\text{SL}_2(\mathbb{C})$ on $S \otimes \overline{S}$ is defined, on elementary tensors, by the equation

$$\pi(A) := A \otimes \overline{A}, \quad (x \otimes \overline{y}) \mapsto Ax \otimes \overline{A}y$$

The canonical representation $\tau$ of $\text{SL}_2(\mathbb{C})$ on $\mathcal{G} = S \oplus \overline{S}$ is defined, on elementary tensors, by the equation

$$\tau(A) := A \oplus \overline{A}, \quad (x \oplus \overline{y}) \mapsto Ax \oplus \overline{A}y$$

The representations $\pi$ and $\tau$ are intimately related to the Clifford module map $\phi$ in the following way.

**Theorem 2.8.** For all $A \in \text{SL}_2(\mathbb{C})$, $X \in V$, we have

$$\phi(\pi(A)(X)) = \tau(A)\phi(X)\tau(A^{-1})$$

**Proof.** Let $X = p \otimes \overline{q}$. Then the LHS is

$$\phi(\pi(A)(X))(a \oplus \overline{b}) = \phi(Ap \otimes \overline{A}\overline{q})(a \oplus \overline{b}) = \overline{\varepsilon(b, \overline{A}\overline{q})}Ap \oplus \varepsilon(Ap, a)\overline{A}\overline{q}$$

We also have

$$\phi(p \otimes \overline{q})\tau(A^{-1})(a \oplus \overline{b}) = \phi(p \otimes \overline{q})(A^{-1}a \oplus \overline{A^{-1}}b) = \overline{\varepsilon(A^{-1}b, \overline{q})}p \oplus \varepsilon(A^{-1}a, \overline{q})$$

so that the RHS is

$$\tau(A)\phi(p \otimes \overline{q})\tau(A^{-1})(a \oplus \overline{b}) = \overline{\varepsilon(A^{-1}b, \overline{q})}Ap \oplus \varepsilon(Ap, a)\overline{A}\overline{q}$$

which is the LHS. Note that we always have

$$\varepsilon(Ax, Ay) = \det(A) \cdot \varepsilon(x, y) = \varepsilon(x, y)$$

for all $A \in \text{SL}_2(\mathbb{C})$. \qed

**Lemma 2.9.** The restriction of $\pi$ to $V$ preserves the Lorentz quadratic form $Q$. 

Proof. The map $\pi(A)$ commutes with $J$:

$$\pi(A)J(x \otimes \bar{y}) = \pi(A)(y \otimes \bar{x})$$
$$= Ay \otimes \bar{A}x$$
$$= \pi(A)(y \otimes \bar{x})$$
$$= \pi(A)J(x \otimes \bar{y})$$

for all $A \in \text{SL}_2(\mathbb{C})$ and all elementary tensors $x \otimes \bar{y}$. The map $\pi(A)|_V$ is well-defined because

$$Jv = v \implies \pi(A)v = \pi(A)Jv = J\pi(A)v$$

for all $A \in \text{SL}_2(\mathbb{C})$ and all $v \in V$. Finally, we have

$$h(\pi(X), \pi(Y)) = h(Ap \otimes \bar{A}q, Ar \otimes \bar{A}s)$$
$$= \varepsilon(Ap, Ar)\varepsilon(\bar{A}q, \bar{A}s)$$
$$= \varepsilon(p, r)\varepsilon(\bar{q}, \bar{s})$$
$$= h(X, Y).$$

□

We have a commutative diagram

$$\begin{array}{ccc}
V & \longrightarrow & S \otimes \bar{S} \\
\pi(A)|_V \downarrow & & \downarrow \pi(A) \\
V & \longrightarrow & S \otimes \bar{S}
\end{array}$$

in which the horizontal maps are canonical inclusions and the left vertical map is a Lorentz transformation.

3. The Dirac Equation

3.1. The Dirac Bundle. We consider one of the orbits of $\text{SL}_2(\mathbb{C})$ acting on $\mathbb{P}^4$. Let $m > 0$ and let $v_0 = (1, 0, 0, 0) \in \mathbb{P}^4$. Explicitly, $v_0$ is the character of $\mathbb{R}^4$ given by $x \mapsto \exp(ix_0)$. Define

$$p_0 = mv_0 \in \mathbb{P}^4$$

and define $X_m$ to be the orbit of $p_0$:

$$X_m := \text{SL}_2(\mathbb{C}) \cdot p_0 \subset \mathbb{P}^4$$

This orbit is the mass shell associated with the positive mass $m$.

We recall that

$$\mathfrak{S} = S \oplus \bar{S}.$$

We begin with the following trivial vector bundle over $X_m$:

$$X_m \times \mathfrak{S} \to X_m, \quad (p, \Psi) \mapsto p$$

With $A \in \text{SL}_2(\mathbb{C})$, we define

$$A \cdot p = \pi(A)p$$
$$A \cdot \Psi = \tau(A)\Psi$$
$$A \cdot (p, \Psi) = (A \cdot p, A \cdot \Psi)$$
We obtain a trivial \( SL_2(\mathbb{C}) \)-bundle of rank 4, i.e. the total space and the base space admit an action of \( SL_2(\mathbb{C}) \) which commutes with the projection.

We now construct the Dirac bundle as a sub-bundle. The total space is \( \mathcal{B}_m := \{(p, \Psi) : p \in X_m, \Psi \in \mathcal{G}, \phi(p)\Psi = m\Psi\} \) the base space is \( X_m \), and the projection is \( \mathcal{B}_m \to X_m, (p, \Psi) \mapsto p \).

The fibre at \( p \in X_m \) is the linear subspace of \( \mathcal{G} \) given by \( \{(p, \Psi) : \phi(p)\cdot \Psi = m\Psi\} \)

Now we have
\[
\phi(p)\Psi = m\Psi
\]
\[
\implies \tau(A)\phi(p)\Psi = m\tau(A)\Psi
\]
\[
\implies \tau(A)\phi(p)\tau(A^{-1})\tau(A)\Psi = m\tau(A)\Psi
\]
\[
\implies \phi(Ap) (\tau(A)\Psi) = m\tau(A)\Psi
\]

by Theorem (2.8). That is to say,
\[
\phi(p)\Psi = m\Psi \implies \phi(Ap)(A\cdot \Psi) = m(A\cdot \Psi)
\]

This shows that \( SL_2(\mathbb{C}) \) sends, for each \( A \in SL_2(\mathbb{C}) \), the fibre at \( p \) to the fibre at \( A\cdot p \). Therefore, the Dirac bundle \( \mathcal{B}_m \to X_m \) is an \( SL_2(\mathbb{C}) \)-bundle.

We have \( \phi(p_0) = m\phi(v_0) = m\gamma_0 \) and so the fibre at \( p_0 \) is given by
\[
\{(\Psi) : \gamma_0\Psi = \Psi\}
\]

This is the eigenspace of \( \gamma_0 \) with eigenvalue 1. Now \( \gamma_0^2 = 1 \), so \( \gamma_0 \) has two eigenspaces \( \mathcal{G}_+, \mathcal{G}_- \) of dimension 2 with eigenvalues \(+1, -1\).

Therefore, the Dirac bundle is an \( SL_2(\mathbb{C}) \)-bundle of rank 2.

The isotropy subgroup of \( SL_2(\mathbb{C}) \) at \( p_0 \) is the compact Lie group \( SU_2 \). By Lemma 2.5 and Theorem 2.8 we have
\[
\tau(A)\gamma_0 = \gamma_0\tau(A)
\]

for all \( A \in SU_2 \). We have
\[
\Psi \in \mathcal{G}_+ \implies \gamma_0\Psi = \Psi \implies \tau(A)\gamma_0\Psi = \tau(A)\Psi \implies \gamma_0\tau(A)\Psi = \tau(A)\Psi \implies \tau(A)\Psi \in \mathcal{G}_+
\]

so that \( \tau|_{SU_2} \) leaves \( \mathcal{G}_+ \) invariant.

The vector space \( \mathcal{G} \) admits a natural basis, namely \( \{e_1, e_2, \bar{e}_1, \bar{e}_2\} \). We calculate that
\[
\phi(v_0)(e_2 + \bar{e}_1) = e_2 + \bar{e}_1
\]
\[
\phi(v_0)(e_1 - \bar{e}_2) = e_1 - \bar{e}_2
\]
\[
\phi(v_0)(e_1 + \bar{e}_2) = -(e_1 + \bar{e}_2)
\]
\[
\phi(v_0)(e_2 - \bar{e}_1) = -(e_2 - \bar{e}_1)
\]

The \(+1\)-eigenspace of \( \phi(v_0) \) is therefore given by
\[
\mathcal{G}_+ = \text{span of } \{e_1 - \bar{e}_2, e_2 + \bar{e}_1\}
Let
\[ A = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \]
Then we have
\[ \tau(A)e_1 = e^{it}e_1, \quad \tau(A)e_2 = e^{-it}e_2, \quad \tau(A)e_1^\dagger = e^{-it}e_1^\dagger, \quad \tau(A)e_2^\dagger = e^{it}e_2^\dagger \]
\[ \tau(A)(e_1 - e_2) = e^{it}(e_1 - e_2) \]
\[ \tau(A)(e_2 + e_1^\dagger) = e^{-it}(e_2 + e_1^\dagger) \]
The character of \( \tau(A) \) is therefore \( e^{it} + e^{-it} \) and so \( \tau|_{SU_2} \) is the spin \( 1/2 \) representation of \( SU_2 \) on \( \mathcal{S}_+ \).

### 3.2. The Dirac bundle as an associated vector bundle.

Let \( p_0 = (m, 0, 0, 0) \) and note that
\[ SL_2(\mathbb{C})/SU_2 \cong X_m, \quad A \mapsto Ap_0. \]
We construct the associated vector bundle
\[ SL_2(\mathbb{C}) \times_{SU_2} \mathcal{S}_+ \]
where \( SU_2 \) acts on \( \mathcal{S}_+ \) via the spin \( 1/2 \) representation \( \tau|_{SU_2} \). The elements of the associated vector bundle are equivalence classes \([A, \Psi]\) with \( A \in SL_2(\mathbb{C}) \) and \( \Psi \in \mathcal{S}_+ \), defined as
\[ [A, \Psi] := \{(AT, \tau(T)^{-1}\Psi) : T \in SU_2\} \]

**Theorem 3.1.** There is an isomorphism of \( SL_2(\mathbb{C}) \)-bundles as follows:
\[ \beta : SL_2(\mathbb{C}) \times_{SU_2} \mathcal{S}_+ \cong \mathcal{B}_m \]
\[ [A, \Psi] \mapsto (Ap_0, \tau(A)\Psi) \]

**Proof.** Note that \( \mathcal{S}_+ \subset \mathcal{S} \) and \( \tau(A)\Psi \in \mathcal{S} \). The map \( \beta \) is well-defined: replacing \( A \) by \( AT \) and \( \Psi \) by \( \tau(T^{-1})\Psi \), we have
\[ (AT, \tau(T)^{-1}\Psi) \mapsto (ATp_0, \tau(AT)\tau(T)^{-1}\Psi) = (Ap_0, \tau(A)\Psi). \]
We have
\[ \phi(Ap_0)\tau(A)\Psi = \tau(A)\phi(p_0)\tau(A^{-1})\tau(A)\Psi \quad \text{by Theorem 2.8} \]
\[ = \tau(A)\phi(p_0)\Psi \]
\[ = \tau(A)m\gamma_0\Psi \]
\[ = \tau(A)m\Psi \quad \text{since } \Psi \in \mathcal{S}_+ \]
\[ = m\tau(A)\Psi \]
so that \((Ap_0, \tau(A)\Psi) \in \mathcal{B}_m\) as required.

Conversely, given \((p, \Psi) \in \mathcal{B}_m\). Choose \( A \) such that \( p = Ap_0 \) and set \( \Phi = \tau(A)^{-1}\Psi \). Then we have
\[ \phi(p)\Psi = m\Psi \]
\[ \implies \tau(A)\phi(p_0)\tau(A)^{-1}\tau(A)\Phi = m\tau(A)\Phi \]
\[ \implies \tau(A)\phi(p_0)\Phi = m\tau(A)\Phi \]
\[ \implies \gamma_0\Phi = \Phi \]
\[ \implies \Phi \in \mathcal{S}_+ \]
Therefore, we have

\[ \beta : (A, \Phi) \mapsto (Ap_0, \tau(A)\Phi) = (p, \Psi) \]

as required.

Now \( A \) is determined mod SU\(_2\). If we replace \( A \) by \( AT \) with \( T \in SU_2 \) then we must replace \((A, \Phi)\) by \((AT, y)\) with \( y \) defined by

\[ y = \tau(AT)^{-1}\Psi = \tau(T)^{-1}\tau(A)^{-1}\Psi = \tau(T)^{-1}\Phi \in \mathcal{G}_+ \]

This leads to the pair

\[ (AT, \tau(T)^{-1}\Phi) \]

which is in the same equivalence class as \((A, \Phi)\). Therefore, the inverse of the map \( \beta \) is well-defined. \( \square \)

**Corollary 3.2.** The Dirac bundle is isomorphic as an SL\(_2(\mathbb{C})\)-bundle to the associated vector bundles

\[ SL_2(\mathbb{C}) \times_{SU_2} S \]

and

\[ SL_2(\mathbb{C}) \times_{SU_2} \overline{S} \]

**Proof.** There is a natural action of SL\(_2(\mathbb{C})\) on \( S \). The action on the conjugate vector space is defined as

\[ A \cdot v = \overline{Av} \]

with \( A \in SL_2(\mathbb{C}) \) and \( v \in S \). Now use the fact that \( \mathcal{G}_+, S \) and \( \overline{S} \) are isomorphic as SU\(_2\)-modules. \( \square \)

**3.3. Dirac equation.** The condition

\[ \phi(p)\Psi = m\Psi \]

with \( \Psi \in \mathcal{G} \) is, written in full,

\[ (p_0\gamma_0 + p_1\gamma_1 + p_2\gamma_2 + p_3\gamma_3)\Psi = m\Psi \]

In the slashed notation [F] p.118, this equation appears as

(9)

\[ \not{\phi}\Psi = m\Psi \]

This is the Dirac equation. After Fourier transform of the variables \( p_0, p_1, p_2, p_3 \), the equation appears in the more familiar form

\[ \sum_{r=0}^{3} i\gamma_r \frac{\partial}{\partial x_r} \Psi = m\Psi \]

where \( \gamma_\mu := \phi(v_\mu) \) with \( \mu = 0, 1, 2, 3 \), are \( \gamma \)-matrices satisfying the conditions

\[ \gamma_0^2 = 1 \]

\[ \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1 \]

\[ \gamma_\mu \gamma_\nu = -\gamma_\nu \gamma_\mu \]

for all \( \mu \neq \nu \).
3.4. **A pair of conjugate 2-spinor fields.** Let \( \Psi \) be a solution of the Dirac equation (9), i.e. \( \phi(p)\Psi = m\Psi \). To mark the dependence on \( p \), we will write this equation as

\[
\phi(p)\Psi_p = m\Psi_p
\]

Then \( \{\Psi_p : p \in X_m\} \) is a section of the Dirac bundle \( \mathcal{B}_m \). By Corollary 3.2, \( \mathcal{B}_m \) is isomorphic to the associated bundle

\[
(10) \quad \text{SL}_2(\mathbb{C}) \times_{\text{SU}_2} \mathcal{S}
\]

The section \( \{\Psi_p : p \in X_m\} \) therefore determines a section of (10), which is a 2-spinor field over the manifold \( X_m \). Now \( \mathcal{B}_m \) is also isomorphic to the conjugate bundle

\[
(11) \quad \text{SL}_2(\mathbb{C}) \times_{\text{SU}_2} \overline{\mathcal{S}}
\]

and so the section \( \{\Psi_p : p \in X_m\} \) determines a section of (11), which is the conjugate 2-spinor field. In this way, a solution of the Dirac equation determines a pair of conjugate 2-spinor fields over the manifold \( X_m \).

3.5. **Comments.** It is worth noting that the Clifford module structure of \( \mathcal{S} \) survives up to and including the Dirac equation, for \( \phi(p) \) belongs to the Clifford algebra \( \mathcal{C}(\mathbb{P}^4, Q) \).

In the literature, in the mathematical accounts of the Dirac equation [Var, Eqn.(136)], [Sim, p.72], [Jos, Eqn.1.3.52] one finds the map

\[
A \mapsto A \oplus (A^*)^{-1}
\]

instead of our map

\[
\tau(A) = A \oplus \overline{A}
\]

From a conceptual point of view, the adjoint \( A^* \) of \( A \in \text{SL}(\mathcal{S}) \) is not defined, because \( \mathcal{S} \) is not equipped with a sesquilinear inner product. The space \( \mathcal{S} \) is endowed only with the symplectic form \((x, y) \mapsto \varepsilon(x, y)\).

4. **The Transition to Spinor Fields on Space-Time**

So far, we have dealt only with 2-spinors. In order to apply this theory to spinor fields on space-time, we proceed as follows.

Let \( \mathcal{M} \) be a smooth real 4-manifold. The first observation is that we can view \( \mathcal{S} \) as a (smooth) complex 2-plane bundle over the base space \( \mathcal{M} \). The second observation is that the operations of conjugation, direct sum and tensor product are as easy to apply to vector bundles as to vector spaces. Then \( \overline{\mathcal{S}} \) is the conjugate of \( \mathcal{S} \), \( \varepsilon \) is a non-degenerate symplectic form on \( \mathcal{S} \), \( V \) is the real part of \( \mathcal{S} \otimes \overline{\mathcal{S}} \), and \( g \) is the Lorentz metric with signature \(+\,-\,-\,-\) given by

\[
g = \varepsilon \otimes \overline{\varepsilon}.
\]

Then \( V \) is a real 4-plane bundle on \( \mathcal{M} \), equipped with the Lorentz metric tensor. At this point, we have to identify \( V \) with the tangent bundle \( T\mathcal{M} \) of \( \mathcal{M} \):

\[
T\mathcal{M} = V.
\]

In the terminology of [PR, p.211], we must identify the world-vectors in \( V \) with the tangent vectors to the manifold \( \mathcal{M} \). In that case, we can view \( \mathcal{M} \) as a model of
curved space-time in general relativity. The smooth sections of the vector bundle $S$ are called 2-spinor fields; the smooth sections of $\mathcal{G}$ are called 4-spinor fields.

The whole of §2 carries over, \textit{at no extra cost}, to the theory of spinor fields on manifolds. In this formulation, the curved space-time $\mathcal{M}$ of general relativity is \textit{already equipped} with spinor fields.

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