Comment on “Why Einstein, Podolsky and Rosen did not prove that quantum mechanics is 'incomplete’ ”

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Abstract. This comment on the recently published article “Why Einstein, Podolsky and Rosen did not prove that quantum mechanics is 'incomplete’ ” (arXiv:quant-ph/0805.0217) by J.H. Field shows that some conclusions, made in the referred-to article, result from an invalid use of Dirac \(\delta\)-distributions.

Introduction

The purpose of this comment is to point out some mathematically incorrect results in a recently published article \([1]\). Briefly, in this paper we will review the following conclusion by Field, the author of \([1]\):

“The gedanken experiment proposed by EPR cannot be carried out if the usual probabilistic interpretation of Quantum Mechanics is correct, and so no physical conclusions can be drawn from the experiment”

and show that it is derived from an invalid calculation.

The \(\delta\)-“wavefunction”

Using the spatial representation of the well known EPR two-particle-state \(|\Psi\rangle\) in the referred-to paper’s eq.(5), that is \(<x_1,x_2|\Psi\rangle = \Psi(x_1,x_2) = h \, \delta(x_1 - x_2 + x_0)\), where the \(x_i\), \(i = 1,2\), denote the coordinates of particle 1 and 2, respectively, and \(x_0\) their origin, Field gives an expression for the probability \(P(a \leq x_1 \leq b)\) of finding particle 1 in an interval \([a,b]\) with \(a,b \in \mathbb{R}\) and \(a \leq b\) while particle 2 might have any arbitrary position possible. Calculating this probability, usage is made of “the usual probabilistic interpretation of Quantum Mechanics” by writing

\[
P(a \leq x_1 \leq b) = \lim_{L \to \infty} \frac{\int_a^b dx_1 \int_{-\infty}^{\infty} dx_2 |\Psi(x_1,x_2)|^2}{\int_{-L}^{L} dx_1 \int_{-\infty}^{\infty} dx_2 |\Psi(x_1,x_2)|^2} \quad (I)
\]

rendering eq.(6) in Field’s paper. As Field correctly remarks, the above given wavefunctions (in the nominator or denominator) are not square-integrable (taking the limit \(L \to \infty\) for the \(dx_1\)-integration), leading to the referred-to paper’s eq.(6) and (8) with \(P(a \leq x_1 \leq b) \to 0\). This vanishing lets the author of the referred-to paper correctly dismiss the usage of \(\Psi(x_1,x_2) = h \, \delta(x_1 - x_2 + x_0)\) for purposes of expressing eq.(I).

Within the following lines a new aspect has to be investigated - the fact that not eq.(I) vanishes in the limit \(L \to \infty\), as it is stated in the referred-to paper’s eq.(6), but that it is not mathematically well-defined at all. It is the use of \(\delta\)-distributions in the integral’s kernel, \(|\Psi(x_1,x_2)|^2\), that leads to difficulties and violation of the quantum mechanical requirements for proper wavefunctions by the dirac-\(\delta\). As a consequence, not only “no meaningful conclusions can be drawn from any gedanken experiment based upon non square-integrable wave functions” as it is written in the referred-to

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paper, but not even mathematically meaningful
expressions can be gained from such functions.
It will be shown that Field’s correct criticism on
non-square-integrable wavefunctions applies to
methods used in his paper, too. After dealing with
certain preliminaries in this section, in the section
afterwards light is shed on the referred-to paper’s
main proof.

The question why the probability-interpretation
for eq.(I) completely fails, is answered only in
part by the property “non-square-integrability”
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which is not defined, resulting in

again obviously not being defined. Formally one
could write, as it is in perturbative field theoretic
approaches often done, \( \int_a^b dx \) \( \delta^2(x) = \delta(0) \) and then
with \( \delta(0) \) being a “divergent factor”, leading to

Within Colombeau’s theory it is possible to show
the divergence of the distribution \( \delta(x) \) for \( x = 0 \).
This enables us to (roughly) identify the above
mentioned formal factor \( \delta(0) \) with \( \delta(x)|_{x=0} \),
making the formal equation (II) more plausible and
stressing out \( \delta(x)|_{x=0} \neq 1 \) - as it is often wrongly
taken.

1. \( \delta^2 \) hasn’t any counterpart in the distribution
theory of Sobolev and Schwartz (i.e. \( \delta^2 \) isn’t
associated with any \( \iota(\omega) \), \( \omega \in \mathcal{D}' \) where \( \iota \) is the
embedding of \( \mathcal{D}' \) into Colombeau’s associative
algebra) and

2. \( \delta^2 \) in the Colombeau version, applied to any
testfunction, leads to a divergent expression
(of the order 1/\( \epsilon \), with \( \epsilon \to 0 \)). Therefore even
the term \( \int_{-\infty}^{\infty} dx \delta^2(x) \) itself is divergent and
then especially \( \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \delta^2(x_1 - x_2) \).
This leads to the inconvenience of eq.(I) not be-
ing defined, even before taking the limit \( L \to \infty \)
because

We conclude that not “the relative proba-
bility” \( P(a,b) \) of EPR’s Equation (6) also vanishes”
as it is stated in the referred-to paper, but the
relative probability is not well-defined at all if a
\( \delta \)-distribution is taken as “wavefunction”. So
it is invalid using eq.(I) for any considerations
regarding \( P(a,b) \).
Of course one might try to “rescue” eq.(I) by
normalizing \( \Psi(x_1, x_2) \) with a factor of \( 1/\sqrt{\delta(0)} \). But
this formally gives us, using the known rules for
manipulations of the dirac-\( \delta \) :
\( \Psi(x) = \delta(x)/\sqrt{\delta(0)} = \delta(x \cdot \sqrt{\delta(0)}) \), leaving
opportunity for interpretation; additionally, taking
the limit \( L \to \infty \) again would rise a problem.
“Minimally modified” wavefunction

Nearly all results, found in the preceding section, object to Field’s approach of calculating $P(a \leq x_1 \leq b)$ with a ”minimally modified” wavefunction (eq.(9) in his paper), given by

$$
\tilde{\Psi}(x_1, x_2) = \frac{1}{(\sqrt{2\pi}\sigma_x)^{1/2}} \cdot \exp \left( \frac{x_0^2 - 2x_1^2 - 2x_2^2}{16\sigma_x^2} \right) \cdot \delta(x_1 - x_2 + x_0). \tag{III}
$$

Contrary to the claim in the referred-to paper (eq.(10) in that paper), this $\tilde{\Psi}(x_1, x_2)$ again lacks of the wavefunction’s requirements and is not square-integrable - due to the use of $\delta$-distributions. With (III) we get

$$
P(a \leq x_1 \leq b) = \int_a^b \int_\infty^{\infty} dx_1 \int_\infty^{\infty} dx_2 |\tilde{\Psi}(x_1, x_2)|^2
$$

$$
= \int_a^b \int_\infty^{\infty} dx_1 \int_\infty^{\infty} dx_2 \delta^2(x_1 - x_2 + x_0) f^2(x_1, x_2),
$$

where $f(x_1, x_2) = 1/(\sqrt{2\pi}\sigma_x^2) \cdot \exp[(x_0^2 - 2x_1^2 - 2x_2^2)/16\sigma_x^2]$ as it was given in eq.(III). Performing the first integration, this yields (adopting the notion of a “divergent factor” $\delta(0)$), similarly to the case in eq.(III),

$$
\int_a^b \int_\infty^{\infty} dx_1 \int_\infty^{\infty} dx_2 \delta^2(x_1 - x_2 + x_0) f^2(x_1, x_2)
$$

$$
= \delta(0) \int_a^b dx_1 f^2(x_1 = x_2 - x_0)
$$

again showing the formal divergence $\delta(0)$. Canceling the dirac-$\delta$ from eq.(III) would solve the mathematical problems but at the same time jeopardize Field’s further reasoning where the constraint $x_2 = x_1 \pm x_0$ of the mutual particle positions, ensured by the $\delta$, plays a crucial role. Without this constraint the argumentation used in the referred-to paper, namely “measuring $x_1$ in the interval $\delta x_1$ then enables the certain prediction that $x_2$ lies in the interval $\delta x_2$ around $x_2 = x_1 \pm x_0$” doesn’t hold anymore. This is because without a $\delta(x_1 - x_2 \pm x_0)$, the two-particle wavefunction allows for particle positions $x_1$ and $x_2$ with $x_2 \neq x_1 \pm x_0$ and non-vanishing probability. In other words, the vanishing probability of one particle being found in an ever decreasing interval $\delta x_1$ doesn’t affect the other particle’s position-probability.

In fact, the $\tilde{\Psi}$ without the $\delta$, let us call it $\Psi'$ with

$$
\Psi'(x_1, x_2) = \frac{1}{(\sqrt{2\pi}\sigma_x)^{1/2}} \cdot \exp \left( \frac{x_0^2 - 2x_1^2 - 2x_2^2}{16\sigma_x^2} \right),
$$

isn’t anymore a suitable wavefunction for describing a sharply position-entangled EPR-pair of particles.

Discussion

The preceeding sections do not provide a proof to the contrary of the fundamental concerns raised by J.H.Field, but show that the proof of the referred-to paper’s main claim needs some additional work. Another important question, concerning the foundations of a mathematically consistent theory, became obvious throughout the above lines: If the $\delta$, being used as a position-eigenvector in the spatial-representation (i.e. $\langle x|x' \rangle = \delta(x-x')$), is divergent for $x = x'$ - then to what extent should it be used as eigenvector in quantum mechanics? As Field correctly remarks, the dirac-$\delta$ is “a mathematical idealisation never realized in the wavefunction of any actual physical system”, hinting to the emerging problems (see e.g. reference [7] in the referred-to paper). In fact, we saw it isn’t a mathematically valid wavefunction (i.e., loosely speaking, an element of the hilbertspace of the square-integrable functions) at all.

References

[1] J.H.Field, “Why Einstein, Podolsky and Rosen did not prove that quantum mechanics is ‘incomplete’ “, arXiv:quant-ph/0805.0217.

[2] L. Schwartz (1954), Sur l'impossiblité de la multiplications des distributions, C.R.Acad. Sci. Paris 239, pp 847-848.

[3] J.F.Colombeau, New Generalized Functions and Multiplication of Distributions (North Holland, 1984).