Chern-Simons Supergravities with Off-Shell Local Superalgebras *

Ricardo Troncoso and Jorge Zanelli†
Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago 9, Chile
and
Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile.

A new family of supergravity theories in odd dimensions is presented. The Lagrangian densities are Chern-Simons forms for the connection of a supersymmetric extension of the anti-de Sitter algebra. The superalgebras are the supersymmetric extensions of the AdS algebra for each dimension, thus completing the analysis of van Holten and Van Proeyen, which was valid for \( N = 1 \) and for \( D = 2, 3, 4, \mod 8 \). The Chern-Simons form of the Lagrangian ensures invariance under the gauge supergroup by construction and, in particular, under local supersymmetry. Thus, unlike standard supergravity, the local supersymmetry algebra closes off-shell and without requiring auxiliary fields. The Lagrangian is explicitly given for \( D = 5, 7 \) and 11. In all cases the dynamical field content includes the vielbein \( (e^a_{\mu}) \), the spin connection \( (\omega_{\mu}^{ab}) \), \( N \) gravitini \( (\phi_{\mu}) \), and some extra bosonic “matter” fields which vary from one dimension to another. The superalgebras fall into three families: \( \mathfrak{osp}(m|N) \) for \( D = 2, 3, 4, \mod 8 \), \( \mathfrak{osp}(N|m) \) for \( D = 6, 7, 8, \mod 8 \), and \( \mathfrak{su}(m-2,2|N) \) for \( D = 5 \mod 4 \), with \( m = 2^{D/2} \). The possible connection between the \( D = 11 \) case and M-Theory is also discussed.

Abstract

I. INTRODUCTION

A good part of the results presented in this lecture were also discussed in \( \Box \) and also presented at the January ’98 meeting in Bariloche \( \Box \) —where the detailed construction of the superalgebra can be found—, but it was at the meeting covered by these proceedings where these results were first presented.

Three of the four fundamental forces of nature are consistently described by Yang-Mills (YM) quantum theories. Gravity, the fourth fundamental interaction, resists quantization in spite of several decades of intensive research in this direction. This is intriguing in view of the fact that General Relativity (GR) and YM theories have a deep geometrical foundation: the gauge principle. How come two theories constructed on almost the same mathematical basis produce such radically different physical behaviors? What is the obstruction for the application of the methods of YM quantum field theory to gravity?

The final answer to these questions is beyond the scope of this paper, however one can note a difference between YM and GR which might turn out to be an important clue: YM theory is defined on a fiber bundle, with the connection as the dynamical object, whereas the dynamical fields of GR cannot be interpreted as components of a connection. Therefore, gravitation does not lend itself naturally for a fiber bundle interpretation.

The closest one could get to a connection formulation for GR is the Palatini formalism, with the Hilbert action

\[
I[\omega, e] = \int \varepsilon^{abc} R^{ab} \wedge e^c + \Lambda^a \wedge \lambda^b,
\]

where \( R^{ab} = d\omega^{ab} + \omega^c_e \wedge \omega_e^{bc} \) is the curvature two-form, and \( e^a \) is a local orthonormal frame. This action is sometimes claimed to describe a gauge theory for local translations. However, in our view this is a mistake. If \( \omega \) and \( e \) were the components of the Poincaré connection, under local translations they should transform as

\[
\delta \omega^{ab} = 0, \quad \delta e^a = D\lambda^a = d\lambda^a + \omega_b^a \wedge \lambda^b.
\]

Invariance of \( I[\Box] \) under \( \Box \) would require the torsion-free condition,

\[
T^a = de^a + \omega_b^a \wedge e^b = 0.
\]

This condition is an equation of motion for the action \( I[\Box] \). This means that the invariance of the action \( I[\Box] \) under \( \Box \) could not result from the transformation properties of the fields alone, but it would be a property of their dynamics as well. The torsion-free condition, being one of the field equations, implies that local translational invariance is at best an on-shell symmetry, which would probably not survive quantization.

The contradiction stems from the identification between local translations in the base manifold (diffeomorphisms)

\[
x^\mu \rightarrow x'^\mu = x^\mu + \zeta^\mu(x),
\]

—which is a genuine invariance of the action \( I[\Box] \)–, and local translations in the tangent space \( \Box \).

Since the invariance of the Hilbert action under general coordinate transformations \( \Box \) is reflected in the closure of the first-class hamiltonian constraints in the Dirac formalism, one could try to push the analogy between the Hamiltonian constraints \( \mathcal{H}_\mu \) and the generators of a gauge algebra. However, the fact that the constraint algebra requires structure functions, which depend on the

---

*Talk presented at the Sixth Meeting on Quantum Mechanics of Fundamental Systems: Black Holes and the Structure of the Universe, Santiago, August 1997.
†John Simon Guggenheim fellow
dynamical fields, is another indication that the generators of diffeomorphism invariance of the theory do not form a Lie algebra but an open algebra (see, e. g., [1]).

More precisely, the subalgebra of spatial diffeomorphisms is a genuine Lie algebra in the sense that its structure constants are independent of the dynamical fields of gravitation,

\[ [H_i, H'_j] \sim H'_j \delta_{ij} - H_i \delta_{ij}. \] (5)

In contrast, the generators of timelike diffeomorphisms form an open algebra,

\[ [H_{\perp}, H'_{\perp}] \sim g^{ij} H'_j \delta_{i\perp}. \] (6)

This comment is particularly relevant in a Chern-Simons theory, where spatial diffeomorphisms are always part of the true gauge symmetries of the theory. The generators of timelike displacements \( (H_{\perp}) \), on the other hand, are combinations of the internal gauge generators and the generators of spatial diffeomorphism, and therefore do not generate independent symmetries [3].

**Higher D** The minimal requirements for a consistent theory which includes gravity in any dimension are: general covariance and second order field equations for the metric. For \( D > 4 \) the most general action for gravity satisfying this criterion is a polynomial of degree \( [D/2] \) in the curvature, first discussed by Lanczos for \( D = 5 \) [4] and, in general, by Lovelock [5].

**First order theory**

If the theory contains spinors that couple to gravity, it is necessary to decouple the affine and metric properties of spacetime. A metric formulation is sufficient for spinless point particles and fields because they only couple to the symmetric part of the affine connection, while a spinning particle can “feel” the torsion of spacetime. Thus, it is reasonable to look for a formulation of gravity in which the spin connection \( (\omega_{ab}^\mu) \) and the vielbein \( (e^a_\mu) \) are dynamically independent fields, with curvature and torsion standing on a similar footing. Thus, the most general gravitational Lagrangian would be of the general form \( L = L(\omega, e) \) [6].

Allowing an independent spin connection in four dimensions does not modify the standard picture in practice because any occurrence of torsion in the action leaves the classical dynamics essentially intact. In higher dimensions, however, theories that include torsion can be dynamically quite different from their torsion-free counterparts.

As we shall see below, the dynamical independence of \( \omega_{ab} \) and \( e^a \) also allows defining these gravitation theories in \( 2n + 1 \) dimensions on a fiber bundle structure as a Yang-Mills theory, a feature that is not shared by General Relativity except in three dimensions.

**II. SUPERGRAVITY**

For some time it was hoped that the nonrenormalizability of GR could be cured by supersymmetry. However, the initial glamour of supergravity (SUGRA) as a mechanism for taming the wild ultraviolet divergences of pure gravity, was eventually spoiled by the realization that it too would lead to a nonrenormalizable answer [6]. Again, one can see that SUGRA is not a gauge theory either in the sense of a fiber bundle, and that the local symmetry algebra closes naturally only on shell. The algebra can be made to close off shell at the cost of introducing auxiliary fields, but they are not guaranteed to exist for all \( D \) and \( N \) [1].

Whether the lack of fiber bundle structure is the ultimate reason for the nonrenormalizability of gravity remains to be proven. However, it is certainly true that if GR could be formulated as a gauge theory, the chances for proving its renormalizability would clearly grow.

In three spacetime dimensions both GR and SUGRA define renormalizable quantum theories. It is strongly suggestive that precisely in \( 2+1 \) dimensions both theories can also be formulated as gauge theories on a fiber bundle [12]. It might seem that the exact solvability miracle was due to the absence of propagating degrees of freedom in three-dimensional gravity, but the power counting renormalizability argument rests on the fiber bundle structure of the Chern-Simons form of those systems.

There are other known examples of gravitation theories in odd dimensions which are genuine (off-shell) gauge theories for the anti-de Sitter (AdS) or Poincaré groups [15, 16]. These theories, as well as their supersymmetric extensions have propagating degrees of freedom [2] and are CS systems for the corresponding groups as shown in [17].

**A. From Rigid Supersymmetry to Supergravity**

Rigid SUSY can be understood as an extension of the Poincaré algebra by including supercharges which are the “square roots” of the generators of rigid translations, \( \{ \bar{Q}, Q \} \sim \Gamma \cdot P \). The basic strategy to generalize this idea to local SUSY was to substitute the momentum \( P_{\mu} = i\partial_{\mu} \) by the generators of diffeomorphisms, \( \mathcal{H} \), and relate them to the supercharges by \( \{ \bar{Q}, Q \} \sim \Gamma \cdot H \). The resulting theory has on-shell local supersymmetry algebra [18].

An alternative point of view—which is the one we advocate here—would be to construct the supersymmetry on the tangent space and not on the base manifold. This approach is more natural if one recalls that spinors provide a basis of irreducible representations for \( SO(N) \), and not for \( GL(N) \). Thus, spinors are naturally defined relative to a local frame on the tangent space rather than in the coordinate basis. The basic point is to reproduce the 2+1 “miracle” in higher dimensions. This idea has been successfully applied by Chamseddine in five dimensions.
and by us for pure gravity \cite{13,16} and in supergravity \cite{14}. The SUGRA construction has been carried out for spacetimes whose tangent space has AdS symmetry \cite{2}, and for its Poincaré contraction in \cite{17}. In \cite{14}, a family of theories in odd dimensions, invariant under the supertranslation algebra whose bosonic sector contains the Poincaré generators was presented. The anticommutator of the supersymmetry generators gives a translation plus a tensor “central” extension,

\[ \{ Q^\alpha, \bar{Q}_\beta \} = -i (\Gamma^\alpha)_{\beta}^\gamma P_\gamma - i (\Gamma a b c d e f)_{\alpha}^\beta Z a b c d e f, \] \hspace{1cm} (7)

The commutators of \( Q, \bar{Q} \) and \( Z \) with the Lorentz generators can be read off from their tensorial character. All the remaining commutators vanish. This algebra is the continuation to odd-dimensional spacetimes of the \( \mathcal{D} = 10 \) superalgebra of van Holten and Van Proeyen \cite{19}, and yields supersymmetric theories with off-shell Poincaré superalgebra. The existence of these theories suggests that there should be similar supergravities based on the AdS algebra.

### B. Assumptions of Standard Supergravity

Three implicit assumptions are usually made in the construction of standard SUGRA:

(i) The fermionic and bosonic fields in the Lagrangian should come in combinations such that their propagating degrees of freedom are equal in number. This is usually achieved by adding to the graviton and the gravitini a number of lower spin fields (\( s < 3/2 \)) \cite{8}. This matching, however, is not necessarily true in AdS space, nor in Minkowski space if a different representation of the Poincaré group (e.g., the adjoint representation) is used \cite{24}.

The other two assumptions concern the purely gravitational sector. They are as old as General Relativity itself and are dictated by economy: (ii) gravitons are described by the Hilbert action (plus a possible cosmological constant), and, (iii) the spin connection and the vielbein are not independent fields but are related through the torsion equation. The fact that the supergravity generators do not form a closed off-shell algebra can be traced back to these assumptions.

The procedure behind (i) is tightly linked to the idea that the fields should be in a \textit{vector} representation of the Poincaré group \cite{21}, and that the kinetic terms and couplings are such that the counting of degrees of freedom works like in a minimally coupled gauge theory. This assumption comes from the interpretation of supersymmetric states as represented by the in- and out-plane waves in an asymptotically free, weakly interacting theory in a minkowskian background. These conditions are not necessarily met by a CS theory in an asymptotically AdS background. Apart from the difference in background, which requires a careful treatment of the unitary irreducible representations of the asymptotic symmetries \cite{21}, the counting of degrees of freedom in CS theories is completely different from the one for the same connection one-forms in a YM theory.

### III. LANZOS–LOVELOCK GRAVITY

#### A. Lagrangian

For \( D > 4 \), assumption (ii) is an unnecessary restriction on the available theories of gravitation. In fact, as mentioned above, the most general action for gravity – generally covariant and with second order field equations for the metric – is the Lanczos-Lovelock Lagrangian (LL). The LL Lagrangian in a \( D \)-dimensional Riemannian manifold can be defined in at least four ways:

(a) As the most general invariant constructed from the metric and curvature leading to second order field equations for the metric \( g_{\alpha \beta} \).

(b) As the most general \( D \)-form invariant under local Lorentz transformations, constructed with the vielbein, the spin connection, and their exterior derivatives, without using the Hodge dual \( * \) \cite{22}.

(c) As a linear combination of the dimensional continuation of all the Euler classes of dimension \( 2p < D \). \cite{8,2}

(d) As the most general low energy effective gravitational theory that can be obtained from string theory \cite{24}.

Definition (a) was historically the first. It is appropriate for the metric formulation and assumes vanishing torsion. Definition (b) is slightly more general than the first and allows for a coordinate-independent first-order formulation, and even allows torsion-dependent terms in the action \cite{8}. As a consequence of (b), the field configurations that extremize the action obey first order equations for \( \omega \) and \( e \). Assertion (c) gives directly the Lanczos–Lovelock solution as a polynomial of degree \( [D/2] \) in the curvature of the form

\[ I_G = \int \sum_{p=0}^{[D/2]} \alpha_p L^p, \] \hspace{1cm} (8)

where \( \alpha_p \) are arbitrary constants and

\[ L^p_G = \varepsilon_{a_1 \ldots a_D} R^{a_1 a_2} \ldots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \ldots e^{a_D}, \] \hspace{1cm} (9)

where wedge product of forms is understood throughout.

Statement (d) reflects the empirical observation that the vanishing of the superstring \( \beta \)-function in \( D = 10 \) gives rise to an effective Lagrangian of the form \( \Box \) \cite{8,24}.

\[ ^1 \text{For even and odd dimensions the same expression} \] \cite{8} \text{can be used, but for odd} \( D \), Chern-Simons forms for the Lorentz connection could also be included (this point is discussed below).
In even dimensions, the last term in the sum is the Euler character, which does not contribute to the equations of motion. However, in the quantum theory, this term in the partition function would assign different weights to nonhomeomorphic geometries.

The large number of dimensionful constants $\alpha_p$ in the LL theory contrasts with the two constants of the EH action ($G$ and $\Lambda$) [23,24,15]. This feature could be seen as an indication that renormalizability would be even more remote for the LL theory than in ordinary gravity. However, as already mentioned, the torsion-free postulate is at best a good description of the classical dynamics only. Thus, an off-shell treatment of gravity should allow for dynamical torsion even in four dimensions. In the first order formulation, the theory has second class constraints due to the presence of a large number of “coordinates” which are actually “momenta” [20], thus complicating the dynamical analysis of the theory.

On the other hand, if torsion is assumed to vanish, $\omega$ could be solved as a function of $e^{-1}$ and its first derivatives, but this would restrict the validity of the approach to nonsingular configurations for which $\det(e_a^\mu) \neq 0$. In this framework, the theory has no second class constraints and the number of degrees of freedom is the same as in the Einstein-Hilbert theory, namely $\frac{D(D-3)}{2}$ [23].

### B. Equations

Consider the Lovelock action [8], viewed as a functional of the spin connection and the vielbein,

$$I_{LL} = I_{LL} [\omega^{ab}, e^a].$$

Varying with respect to the vielbein, the generalized Einstein equations are obtained,

$$\sum_{p=0}^{n-1} \alpha_p (D - 2p) \varepsilon_{a_1 \ldots a_D} R^{a_1 \ldots a_2} \ldots R^{a_{2p-1} a_{2p}} \times e^{a_{2p+1}} \ldots e^{a_{D-1}} = 0. \quad (11)$$

Varying with respect to the spin connection, the torsion equations are found,

$$\sum_{p=0}^{n-1} \alpha_p (D - 2p) \varepsilon_{a b a_3 \ldots a_D} R^{a_3 a_4} \ldots R^{a_{2p-1} a_{2p}} \times e^{a_{2p+1}} \ldots e^{a_{D-1}} T^{a_{D}} = 0. \quad (12)$$

The presence of the arbitrary coefficients $\alpha_p$ in the action implies that static, spherically symmetric Schwarzschild-like solutions possess a large number of horizons [2], and time-dependent solutions have an unpredictable evolution [23,28]. However, as shown below, for a particular choice of the constants $\alpha_p$ the dynamics is significantly better behaved.

Additional terms containing torsion explicitly can be included in the action. It can be shown, however, that the presence of torsional terms in the Lagrangian does not change the degrees of freedom of gravity in four dimensions. Indeed, the matter-free theory with torsion terms is indistinguishable (at least classically) from GR, [23]. However, in higher dimensions, the situation is completely different [1].

### C. The vanishing of Classical Torsion

Obviously $T^a = 0$ solves [2]. However, for $D > 4$ this equation does not imply vanishing torsion in general. In fact, there are choices of the coefficients $\alpha_p$ and configurations of $\omega^{ab}, e^a$ such that $T^a$ is completely arbitrary. On the other hand, as already mentioned, the torsion-free postulate is at best a good description of the classical dynamics only. Thus, an off-shell treatment of gravity should allow for dynamical torsion even in four dimensions. In the first order formulation, the theory has second class constraints due to the presence of a large number of “coordinates” which are actually “momenta” [20], thus complicating the dynamical analysis of the theory.

On the other hand, if torsion is assumed to vanish, $\omega$ could be solved as a function of $e^{-1}$ and its first derivatives, but this would restrict the validity of the approach to nonsingular configurations for which $\det(e_a^\mu) \neq 0$. In this framework, the theory has no second class constraints and the number of degrees of freedom is the same as in the Einstein-Hilbert theory, namely $\frac{D(D-3)}{2}$ [23].

### D. Dynamics and Degrees of Freedom

Imposing $T^a = 0$ from the start, the action is $I = I_{LL} [e^a, \omega(e)]$ and varying respect to $e$, the “1.5 order formalism” [18] is obtained,

$$\delta I = \frac{\delta I_{LL}}{\delta e^a} \delta e^a + \frac{\delta I_{LL}}{\delta \omega^{bc}} \delta \omega^{bc} \frac{\delta}{\delta e^a} \delta e^a. \quad (13)$$

Assuming $\frac{\delta I_{LL}}{\delta e^a} = 0$ the equations of motion consist of the Einstein equations (11), defined on a restricted configuration space.

For $D \leq 4$, $T^a = 0$ is the unique solution of eqn.(12). In those dimensions, the different variational principles (first-, second- and 1.5-th order) are classically equivalent in the absence of sources. On the contrary, for $D > 4$, $T^a = 0$ is not logically necessary and is therefore unjustified.

The LL–Lagrangians [4] include the Einstein-Hilbert (EH) theory as a particular case, but they are dynamically very different in general. The classical solutions of the LL theory are not perturbatively related to those of the Einstein theory. For instance, it was observed that the time evolution of the classical solutions in the LL theory starting from a generic initial state can be unpredictable, whereas the EH theory defines a well-posed Cauchy problem.

It can also be seen that even for some simple minisuperspace models, the dynamics could become quite messy because the equations of motion are not deterministic in the classical sense, due to the vanishing of some eigenvalues of the Hessian matrix on critical surfaces in phase space [23,28].
E. Choice of Coefficients

At least for some simple minisuperspace geometries the indeterminate classical evolution can be avoided if the coefficients are chosen so that the Lagrangian is based on the connection for the AdS group,

\[ \alpha_p l^{D-2p} = \begin{cases} (D-2p)^{-1} \left( \frac{n-1}{p} \right), & D = 2n-1 \\ (n/p), & D = 2n. \end{cases} \]

(14)

This corresponds to the Born-Infeld theory in even dimensions [20], and to the AdS Chern-Simons theory in odd dimensions [15,31].

1. \( D = 2n \): Born-Infeld Gravity

In even dimensions the choice (14) gives rise to a Lagrangian of the form

\[ L = \kappa \epsilon_{a_1 \ldots a_D} \left( R^{a_1 a_2} + \frac{e^{a_1} e^{a_2}}{l^2} \right) \cdots \left( R^{a_D-1 a_D} + \frac{e^{a_D-1} e^{a_D}}{l^2} \right). \]

(15)

This is the Pfaffian of the two–form \( R^{ab} + \frac{1}{l^2} e^a e^b \), and, in this sense it can be written in the Born-Infeld-like form,

\[ L = \kappa \sqrt{\det (R^{ab} + \frac{1}{l^2} e^a e^b)}. \]

(16)

The combinations \( R^{ab} + \frac{1}{l^2} e^a e^b \) are the components of the AdS curvature (c.f. (19) below). This seems to suggest that the system might be naturally described in terms of an AdS connection [21]. However, this is not the case: In even dimensions, the Lagrangian (15) is invariant under local Lorentz transformations and not under the entire AdS group. As will be shown below, it is possible, in odd dimensions, to construct gauge invariant theories of gravity under the full AdS group.

2. \( D = 2n - 1 \): AdS Gauge Gravity

The odd-dimensional case was discussed in [15,14], and later also in [15]. Consider the action (8) with the choice given by (14) for \( D = 2n - 1 \). The constant parameter \( l \) has dimensions of length and its purpose is to render the action dimensionless. This also allows the interpretation of \( \omega \) and \( e \) as components of the AdS connection [22], \( A = \frac{1}{2} \omega^{ab} J_{ab} + e^a J_{aD+1} = \frac{1}{2} W^{ABC} J_{ABC} \), where

\[ W^{ABC} = \begin{bmatrix} \omega^{ab} & e^a/l & 0 \\ -e^a/l & 0 & 0 \end{bmatrix}, \quad A, B = 1, \ldots, D + 1. \]

(17)

The resulting Lagrangian is the Euler-CS form. Its exterior derivative is the Euler form in 2n dimensions,

\[ dL^\text{AdS}_{2n-1} = \kappa \epsilon_{A_1 \ldots A_{2n}} R^{A_1 A_2} \cdots R^{A_{2n-1} A_{2n}} = \kappa E_{2n}, \]

(18)

where \( R^{AB} = dW^{AB} + W^{A}_{\;C} W^{CB} \) is the AdS curvature, which contains the Riemann and torsion tensors,

\[ R^{AB} = \begin{bmatrix} R^{ab} + \frac{1}{l^2} e^a e^b & T^a/l \\ -T^b/l & 0 \end{bmatrix}. \]

(19)

The constant \( \kappa \) is quantized [13] (in the following we will set \( \kappa = l = 1 \)).

In general, a Chern-Simons Lagrangian in \( 2n - 1 \) dimensions is defined by the condition that its exterior derivative be an invariant homogeneous polynomial of degree \( n \) in the curvature, that is, a characteristic class. In the case above, (17) defines the CS form for the Euler class \( 2n \)-form.

A generic CS Lagrangian in \( 2n - 1 \) dimensions for a Lie algebra \( g \) can be defined by

\[ dL^g_{2n-1} = \langle F^n \rangle, \]

(20)

where \( \langle \rangle \) stands for a multilinear function in the Lie algebra \( g \), invariant under cyclic permutations such as Tr, for an ordinary Lie algebra, or STr, in the case of a superalgebra. In the case above, the only nonvanishing brackets in the algebra are

\[ \langle J_{A_1 A_2}, \ldots, J_{A_{2n-1} A_{2n}} \rangle = \epsilon_{A_1 \ldots A_{2n}}. \]

(21)

3. \( D = 2n - 1 \): Poincaré Gauge Gravity

Starting from the AdS theory (17) in odd dimensions, a Wigner- Inönü contraction deforms the AdS algebra into the Poincaré one. The same result is also obtained choosing \( \alpha_p = \delta_p^a \). Then, the Lagrangian (8) becomes:

\[ L^P = \epsilon_{a_1 \ldots a_D} R^{a_1 a_2} \cdots R^{a_{2n-2} a_{2n-1}} e^{a_D}. \]

(22)

In this way the local symmetry group of (8) is extended from Lorentz (\( \text{SO}(D-1,1) \)) to Poincaré (\( \text{ISO}(D-1,1) \)). Analogously to the anti-de Sitter case, one can see that the action depends on the Poincaré connection: \( A = e^a P_a + \frac{1}{2} \omega^{ab} J_{ab} \). It is straightforward to verify the invariance of the action under local translations,

\[ \delta e^a = D \lambda^a, \quad \delta \omega^{ab} = 0. \]

(23)

Here \( D \) stands for covariant derivative in the Lorentz connection. If \( \lambda \) is the Lie algebra-valued zero-form, \( \lambda = \lambda^a P_a \), the transformations (23) are read from the general gauge transformation for the connection, \( \delta A = \nabla \lambda \), where \( \nabla \) is the covariant derivative in the Poincaré connection.
Moreover, the Lagrangian \( \mathcal{L} \) is a Chern-Simons form. Indeed, with the curvature for the Poincaré algebra, \( F = dA + A \cdot A = \frac{1}{2} R^{ab} J_{ab} + T^a P_a \), \( L^P \_G \) satisfies
\[ d L^P _G = (F_{n+1}), \] (24)
where the only nonvanishing components in the bracket are
\[ \langle J_{a_1 a_2} \cdots J_{a_{D-2} a_{D-1}}, P_{a_2} \rangle = \epsilon_{a_1 \cdots a_D}. \] (25)

Thus, the Chern character for the Poincaré group is written in terms of the Riemann curvature and the torsion as
\[ \langle F^3 \rangle = \epsilon_{a_1 \cdots a_D} R^{a_1 a_2} \cdots R^{a_{D-2} a_{D-1}} T^{a_D}. \] (26)

The simplest example of this is ordinary gravity in 2+1 dimensions, where the Einstein-Hilbert action with cosmological constant is a genuine gauge theory of the AdS group, while for zero cosmological constant it is invariant under local Poincaré transformations. Although this gauge invariance of 2+1 gravity is not always emphasized, it lies at the heart of the proof of integrability of the theory \( \mathcal{L} \).

IV. ADS GAUGE GRAVITY

As shown above, the LL action assumes spacetime to be a Riemannian, torsion-free, manifold. That assumption is justified a posteriori by the observation that \( T^a = 0 \) is always a solution of the classical equations, and means that \( e \) and \( \omega \) are not dynamically independent. This is the essence of the second order or metric approach to GR, in which distance and parallel transport are not independent notions, but are related through the Christoffel symbol. There is no fundamental justification for this assumption and this was the issue of the historic discussion between Einstein and Cartan \( \mathcal{L} \).

In four dimensions, the equation \( T^a = 0 \) is algebraic and could in principle be solved for \( \omega \) in terms of the remaining fields. However, for \( D > 5 \), CS gravity has more degrees of freedom than those encountered in the corresponding second order formulation \( \mathcal{L} \). This means that the CS gravity action has propagating degrees of freedom for the spin connection. This is a compelling argument to seriously consider the possibility of introducing torsion terms in the Lagrangian from the start.

Another consequence of imposing a dynamical dependence between \( \omega \) and \( e \) through the torsion-free condition is that it spoils the possibility of interpreting the local translational invariance as a gauge symmetry of the action. Consider the action of the Poincaré group on the fields as given by \( \mathcal{L} \); taking \( T^a = 0 \) implies
\[ \delta \omega^{ab} = \frac{\partial \omega^{ab}}{\partial e^c} \delta e^c \neq 0, \] (27)
which would be inconsistent with the transformation of the fields under local translations \( \mathcal{L} \). Thus, the spin connection and the vielbein—the soldering between the base manifold and the tangent space—cannot be identified as the compensating fields for local Lorentz rotations and translations, respectively.

In our construction \( \omega \) and \( e \) are assumed to be dynamically independent and thus torsion necessarily contains propagating degrees of freedom, represented by the torsion tensor \( k_{\mu} := \omega_{\mu} - \omega_{\mu} (e, \ldots) \), where \( \tilde{\omega} \) is the spin connection which solves the (algebraic) torsion equation in terms of the remaining fields.

The generalization of the Lovelock theory to include torsion explicitly can be obtained assuming definition (b). This is a cumbersome problem due to the lack of a simple algorithm to classify all possible invariants constructed from \( e^a, R^{ab} \) and \( T^a \). In Ref. \( \mathcal{L} \) a useful “recipe” to generate all those invariants is given.

A. The Two Families of AdS Theories

Similarly to the theory discussed in section III, the torsional additions to the Lagrangian bring in a number of arbitrary dimensionful coefficients \( \beta_k \), analogous to the \( \alpha_p \)'s. Also in this case, one can try choosing the \( \beta \)'s in such a way as to enlarge the local Lorentz invariance into an AdS gauge symmetry. If no additional structure (e.g., inverse metric, Hodge-* etc.) is assumed, AdS invariants can only be produced in dimensions \( 4k \) and \( 4k - 1 \).

The proof of this claim is as follows: invariance under AdS requires that the D-form be at least Lorentz invariant. Then, in order for these scalars to be invariant under AdS as well, it is necessary and sufficient that they be expressible in terms of the AdS connection \( \mathcal{L} \). As is well-known (see, e.g., \( \mathcal{L} \)), in even dimensions, the only D-form invariant under \( SO(N) \) constructed according to the recipe mentioned above are the Euler character (for \( N = D \), and the Chern characters (for any \( N \)). Thus, the only AdS invariant D-forms are the Euler class, and linear combinations of products of the type
\[ P_{r_1 \cdots r_s} = c_{r_1} \cdots c_{r_s}, \] (28)
with \( 2(r_1 + r_2 + \cdots + r_s) = D \), where
\[ c_r = \text{Tr}(F^r), \] (29)
defines the \( r \)-th Chern character of \( SO(N) \). Now, since the curvature two-form \( F \) in the vectorial representation

\(^2\)For simplicity we will not always distinguish between different signatures. Thus, if no confusion can occur, the AdS group in \( D \) dimensions will also be denoted as \( SO(D+1) \). The de Sitter case can be obtained replacing \( \alpha_p \) by \((-1)^p \alpha_p \) in \( \mathcal{L} \).
is antisymmetric in its indices, the exponents \( \{ r_i \} \) are necessarily even, and therefore \( \{ r_i \} \) vanishes unless \( D \) is a multiple of four. Thus, one arrives at the following lemmas:

**Lemma:** 1 For \( D = 4k \), the only D-forms built from \( e^a, R^{ab} \) and \( T^a \), invariant under the AdS group, are the Chern characters and the torsional terms are parity violating.

**Lemma:** 2 For \( D = 4k+2 \), there are no AdS-invariant D-forms constructed from \( e^a, R^{ab} \) and \( T^a \).

In view of this, it is clear why attempts to construct gravitation theories with local AdS invariance in even dimensions have been unsuccessful [31,34].

Since the forms \( P_{r_1 \ldots r_s} \) are closed, they are at best boundary terms in \( 4k \) dimensions—which do not contribute to the classical equations, but could assign different weights to configurations with nontrivial torsion in the quantum theory. In other words, they can be locally expressed as

\[
P_{r_1 \ldots r_s} = d L^{AdS}_{\{r\}4k-1}(W). \tag{30}
\]

Thus, for each collection \( \{r\} \), the \((4k-1)\)-form \( L^{AdS}_{\{r\}4k-1} \) defines a Lagrangian for the AdS group in \( 4k-1 \) dimensions. It takes direct computation to see that these Lagrangians involve torsion explicitly. These results are summarized in the following

**Theorem:** There are two families of gravitational first order Lagrangians for \( e \) and \( \omega \), invariant under local AdS transformations:

a: Euler-Chern-Simons form in \( D = 2n - 1 \), whose exterior derivative is the Euler character in dimension \( 2n \), which do not involve torsion explicitly, and

b: Pontryagin-Chern-Simons forms in \( D = 4k - 1 \), whose exterior derivatives are the Chern characters in \( 4k \) dimensions, which involves torsion explicitly.

It must be stressed that locally AdS-invariant gravity theories only exist in odd dimensions. They are genuine gauge systems, whose action comes from topological invariants in one dimension above. These topological invariants can be written as the trace of a homogeneous polynomial of degree \( n \) in the AdS curvature. Obviously, for dimensions \( 4k - 1 \) both \( a \)- and \( b \)-families exist. The most general Lagrangian of this sort is a linear combination of the two families.

An important difference between these two families is that under a parity transformation the first is even while the second is odd [1]. The parity invariant family has been extensively studied in [23,25,26]. In what follows we concentrate on the construction of the pure gravity sector as a gauge theory which is parity-odd. This construction was discussed in [35], and also briefly in [27]

---

*Parity is understood here as an inversion of one coordinate, both in the tangent space and in the base manifold. Thus, for instance the Euler character is invariant under parity, while the Lorentz Chern characters and the torsional terms are parity violating.*

---

**B. Even dimensions**

In \( D = 4 \), the the only local Lorentz-invariant 4-forms constructed with the recipe just described are [8]:

\[
\begin{align*}
\mathcal{E}_4 &= \epsilon_{abcd} R^{ab} R^{cd} \\
L_{EH} &= \epsilon_{abcd} e^a e^b e^c e^d \\
L_C &= \epsilon_{abcd} e^a e^b e^c e^d \\
C_2 &= R^{ab} R_{ab} \\
L_{T_1} &= R^{ab} e_a e_b \\
L_{T_2} &= T^a T_a.
\end{align*}
\]

The first three terms are even under parity and the rest are odd. Of these, \( \mathcal{E}_4 \) and \( C_2 \) are topological invariant densities (closed forms): the Euler character and the rest are odd. Of these, \( \mathcal{E}_4 \) and \( C_2 \) are topological invariant densities (closed forms): the Euler character and the second Chern character for \( SO(4) \), respectively. The remaining four terms define the most general gravity action in four dimensions,

\[
I = \int_{M_4} [\alpha L_{EH} + \beta L_C + \gamma L_{T_1} + \rho L_{T_2}]. \tag{31}
\]

It can also be seen, that by choosing \( \gamma = -\rho \), the last two terms are combined into a topological invariant density (the Nieh-Yan form). Thus, with this choice the odd part of the action becomes a boundary term. Furthermore, \( C_2, L_{T_1} \) and \( L_{T_2} \) can be combined into the second Chern character of the AdS group,

\[
R_b^a R^b_a + 2(T^a T_a - 2 R^{ab} e_a e_b) = R^A B R^B_A. \tag{32}
\]

This is the only AdS invariant constructed with \( e^a, \omega^{ab} \) and their exterior derivatives alone, confirming that there are no locally AdS invariant gravities in four dimensions.

In general, the only AdS-invariant functionals in higher dimensions can be written in terms of the AdS curvature as [8]

\[
\hat{I}_{r_1 \ldots r_s} = \int_M C_{r_1} \cdots C_{r_s}, \tag{33}
\]

or linear combinations thereof, where \( C_r = Tr[(R^A_B)^r] \) is the \( r \)-th Chern character for the AdS group. For example, in \( D = 8 \) the Chern characters for the AdS group are

\[
Tr[(R^A_B)^4] = C_4, \tag{34}
\]

Similar Chern classes are also found for \( D = 4k \). (As already mentioned, \( \hat{I}_{r_1 \ldots r_s} \) vanishes if one of the \( r \)'s is odd, which is the case in \( 4k + 2 \) dimensions.)

Thus, there are no AdS-invariant gauge theories in even dimensions.
C. Odd dimensions

The simplest example is found in three spacetime dimensions where there are two locally AdS-invariant Lagrangians, namely, the Einstein-Hilbert with cosmological constant,

\[ L^{\text{AdS}}_{G_3} = \epsilon_{abc}[R^{ab}e^c + \frac{1}{3!} \epsilon^{a}e_{b}e_{c}], \quad (35) \]

and the “exotic” Lagrangian

\[ L^{\text{AdS}}_{T_3} = L^*_{3}(\omega) + 2e_a T^a, \quad (36) \]

where

\[ L^*_{3} \equiv \omega^a_b du^b_a + \frac{2}{3} \omega^a_b \omega_b^c \omega^c_a, \quad (37) \]

is the Lorentz Chern-Simons form. Note that in (36), the SO\(3\) grows as the partitions of \(k\), in correspondence with the number of possible exotic forms grows as the partitions of \(k\), in correspondence with the number of composite Chern invariants of the form \(P(r) = \prod_j C_r^j\). The most general action for gravitation in \(D = 3\), which is invariant under \(SO(4)\) is therefore a linear combination \(\alpha L^{\text{AdS}}_{G_3} + \beta L^{\text{AdS}}_{T_3}\).

For \(D = 4k - 1\), the number of possible exotic forms grows as the partitions of \(k\), in correspondence with the number of composite Chern invariants of the form \(P(r) = \prod_j C_r^j\). The most general Lagrangian in \(4k - 1\) dimensions takes the form \(\alpha L^{\text{AdS}}_{G_4k-1} + \beta \{ \} \), where \(dL^{\text{AdS}}_{T_4k-1} = P(r)\), with \(\sum_j r_j = 4k\). These Lagrangians have proper dynamics, and, unlike the even dimensional cases, they are not boundary terms. For example, in seven dimensions one finds \([33,34]\)

\[ L^{\text{AdS}}_{T_7} = \beta_{2,2}[R^{a}_{b}R^{b}_{a} + 2(T^a T_a - R^{ab}e_a e_b)]L^{\text{AdS}}_{T_3} + \beta_4[L^*_{7}(\omega) + 2(T^a T_a + R^{ab}e_a e_b)T^a e_a + 4T^a R^a b R^b c e c], \]

where \(L^{\text{AdS}}_{2n-1}\) is the Lorentz-CS \((2n-1)\)-form,

\[ dL^{\text{AdS}}_{2n-1}(\omega) = Tr[(R_{b}^{a})^n]. \quad (38) \]

Summarizing: The requirement of local AdS symmetry is rather strong and has the following consequences:

- Locally AdS invariant theories of gravity exist in odd dimensions only.
- For \(D = 4k - 1\) there are two families: one involving only the curvature and the vielbein (Euler Chern-Simons form), and the other involving torsion explicitly in the Lagrangian. These families are even and odd under space reflections, respectively.
- For \(D = 4k + 1\) only the Euler-Chern-Simons forms exist. These ar parity even and don’t involve torsion explicitly.

V. EXACT SOLUTIONS

As stressed here, the local symmetry of odd-dimensional gravity can be extended from Lorentz to AdS by an appropriate choice of the free coefficients in the action. The resulting Lagrangians (with or without torsion terms), are Chern-Simons \(D\)-forms defined in terms of the AdS connection \(A\), whose components include the vielbein and the spin connection [see eqn. (17)]. This implies that the field equations \(\{11,13\}\) obtained by varying the vielbein and the spin connection respectively, can be written in an AdS-covariant form

\[ < F^{a-1} J_{A B} > = 0, \quad (39) \]

where \(F = \frac{1}{2} R^{A B} J_{A B}\) is the AdS curvature with \(R^{A B}\) given by \([19]\) and \(J_{A B}\) are the AdS generators.

It is easily checked that any locally AdS spacetime is a solution of \([19]\). Apart from anti-de Sitter space itself, some interesting spacetimes with this feature are the topological black holes of Ref. \([27]\), and some “black branes” with constant curvature worldsheet \([28]\). For any \(D\), there is also a unique static, spherically symmetric, asymptotically AdS black hole solution \([15]\), as well as their topological extensions which have nontrivial event horizons \([23]\).

Exact solutions of the form \(\text{AdS}_4 \times S^{D-4}\) have also been found \([10,11]\) as well as alternative four-dimensional cosmological models.

All of the above geometries can be extended into solutions of the gravitational Born-Infeld theory \([10]\) in even dimensions. Friedmann-Robertson-Walker like cosmologies have been shown to exist in even dimensions \([26]\), and it could be expected that similar solutions exist in odd dimensions as well.

VI. CHERN-SIMONS SUPERGRAVITIES

We now consider the supersymmetric extensions of the locally AdS theories defined above. The idea is to enlarge the AdS algebra incorporating SUSY generators. The closure of the algebra (Jacobi identity) forces the addition of further bosonic generators as well \([15]\). In order to accommodate spinors in a natural way, it is useful to cast the AdS generators in the spinor representation of \(SO(D + 1)\). In particular, one can write

\[ dL^{\text{AdS}}_{4k-1} = -\frac{1}{2^{4k}} Tr[(R^{A B} \Gamma_{A B})^{2k}]. \quad (40) \]

The de-Sitter case (\(\Lambda > 0\)) was discussed in \([41]\) for the torsion-free theory. Changing the sign in the cosmological constant has deep consequences. In fact, the solutions are radically different, and locally supersymmetric extensions for positive cosmological constant don’t exist in general.
A. Superalgebra and Connection

The smallest superalgebra containing the AdS algebra in the bosonic sector is found following the same approach as in [19], but lifting the restriction of \( N = 1 \) [33]. The result, for odd \( D > 3 \) is (see [3] for details)

\[
\{ Q \bar{Q} \} \sim \Gamma^a P_a + \Gamma^{ab} Z_{ab} + \Gamma^{abcdef} Z_{abcdef}. \tag{41}
\]

B. D=5 Supergravity

In this case, as in every dimension \( D = 4k + 1 \), there is no torsional Lagrangians \( L_T \) due to the vanishing of the Pontrjagin \( 4k + 2 \)-forms for the Riemann curvature. This fact implies that the local supersymmetric extension will be of the form \( L = L_G + \cdots \).

As shown in the previous table, the appropriate AdS superalgebra in five dimensions is \( su(2, \mathbb{C}|2, \mathbb{C}) \), whose generators are \( K, J_{ab}, Q_\alpha, Q_\beta, M^{ij} \), with \( a, b = 1, \ldots, 5 \) and \( i, j = 1, \ldots, N \). The connection is

\[
A = b k + e^a J_a + \frac{i}{2} \omega^{ab} J_{ab} + \bar{\psi}\psi Q_i - \bar{Q}\psi j, \tag{42}
\]

where \( D \) is the covariant derivative on the bosonic connection, \( D\epsilon_j = (d + \frac{1}{4} [e^a \Gamma_a + \frac{1}{2} \omega^{ab} \Gamma_{ab} + \frac{1}{4} \delta^r i \Gamma_{[r]}]) \epsilon_j - \omega_{ij} \epsilon_i \).

In each of these cases, \( m = 2^{[D/2]} \) and the connection takes the form

\[
A = \frac{1}{2} \omega^{ab} J_{ab} + e^a J_a + \frac{1}{4} \delta^r i \Gamma_{[r]} \zeta \psi_j + \frac{1}{2} \bar{\psi}\psi_M M^{ij}, \tag{43}
\]

with \( \omega^{ab} \) as in [19], but lifting the restriction of \( N = 1 \) [33]. The result, for odd \( D > 3 \) is (see [3] for details)

\[
\{ Q \bar{Q} \} \sim \Gamma^a P_a + \Gamma^{ab} Z_{ab} + \Gamma^{abcdef} Z_{abcdef}. \tag{41}
\]

\[
\text{Conjugation Matrix} \quad \text{Internal Metric}
\]

\begin{align*}
8k - 1 & \quad \text{osp}(N|m) & C^i = C & u^i = -u
\end{align*}

\begin{align*}
8k + 3 & \quad \text{osp}(m|N) & C^i = -C & u^i = u
\end{align*}

\begin{align*}
4k + 1 & \quad \text{su}(m|N) & C^i = C & u^i = u
\end{align*}

The generators \( J_{ab}, J_a \) span the AdS algebra and the \( Q_\alpha \)'s generate (extended) supersymmetry transformations. The \( Q \)'s transform in a vector representation under the action of \( M_{ij} \) and as spinors under the Lorentz group. Finally, the \( Z \)'s complete the extension of AdS into the larger algebras \( so(m), sp(m) \) or \( su(m) \), and \( [r] \) denotes a set of \( r \) antisymmetrized Lorentz indices.
\[ F = \left[ \begin{array}{cc} \bar{R}^\alpha_{\beta} & D\psi^a_eta \\ -D\bar{\psi}^\beta_a & F^a_j \end{array} \right] \] (45)

where
\[
\begin{align*}
D\psi^\alpha_j &= d\psi^\alpha_j + \bar{\Omega}^\alpha_j \psi^\beta - A^j_i \psi_i^\alpha, \\
\bar{R}^\alpha_{\beta} &= R^\alpha_{\beta} - \psi^\alpha_i \bar{\psi}^\beta_i, \\
F^a_j &= F^a_j - \bar{\psi}^\alpha_j \psi^\beta_a.
\end{align*}
\]

Here \( F^j_i = dA^i_j + A^i_k A^k_j + \frac{1}{4} db \delta^j_i \) is the \( su(N) \) curvature, and \( \bar{R}^\alpha_{\beta} = d\bar{\psi}^\alpha_i \bar{\psi}^\beta_i + \frac{1}{4} \bar{\psi}^\alpha \psi^\beta \) is the \( u(2,2) \) curvature. In terms of the standard \((2n-1)\)-dimensional fields, \( R^\alpha_{\beta} \) can be written as
\[
R^\alpha_{\beta} = i \frac{d}{d\psi^\alpha_j} + \frac{1}{2} [T^a \Gamma_a + (R^{ab} + e^a e^b) \Gamma_{ab}]^\alpha_{\beta}.
\] (47)

In six dimensions the only invariant form is
\[
P = i \text{Str} \left[ F^3 \right],
\] (48)

which in this case reads
\[
P = Tr [R^3] - Tr [F^3] + 3 \left[ D\bar{\psi}(R + F)D\psi - \bar{\psi}(R^2 - F^2 + [R - F](\psi^2)\psi) \right],
\] (49)

where \( (\psi^2)^2 = \bar{\psi} \psi \). The resulting five-dimensional C-S density can be decomposed as a sum of a gravitational part, a \( b \)-dependent piece, a \( su(N) \) gauge part, and a fermionic term,
\[
L = L_G^{\text{AdS}} + L_b + L_{su(N)} + L_F,
\] (50)

with
\[
L_G^{\text{AdS}} = \frac{1}{4} e_{abcd}(R^{ab} R^{cd} e^e + \frac{2}{3} R^{ab} e^d e^e + \frac{2}{5} e^a e^b e^c e^d e^e)
\]
\[
L_b = -(\frac{1}{12} - \frac{1}{4})(db)^2 + \frac{1}{2} (T^a \Gamma_a - R^{ab} e_a e_b - \frac{1}{4} R^{ab} R_{ab}) b + \frac{1}{12} \bar{R}^\alpha_{\beta} \Gamma_a f^a_j f^j_i
\]
\[
L_{su(N)} = -(a^j_i a^j_i + \frac{3}{2} a^j_i a^k_i a^k_i a^l_i + \frac{3}{4} a^j_i a^k_i a^l_i a^m_i)\]
\[
L_F = \frac{3}{2} \left[ \bar{\psi}(\bar{R} + \bar{F}) D\psi - \frac{1}{2} (\psi^2) (\bar{\psi} D\psi) \right].
\] (51)

The action is invariant under local gauge transformations, which contain the local SUSY transformations
\[
\begin{align*}
\delta e^a &= -\frac{1}{2} (\bar{\psi} \Gamma^a \psi_i - \bar{\psi} \Gamma^a \psi_i) \\
\delta \omega_{ab} &= \frac{1}{4} (\bar{\psi} \Gamma^{ab} \psi_i - \bar{\psi} \Gamma^{ab} \psi_i) \\
\delta b &= i (\bar{\psi} \psi_i - \bar{\psi} \psi_i) \\
\delta \psi_i &= D\epsilon_i \\
\delta \bar{\psi}^\alpha &= \bar{D} \epsilon^\alpha \\
\delta a^j_i &= i (\bar{\psi} \psi_j - \bar{\psi} \psi_j).
\end{align*}
\] (52)

As in \( 2 + 1 \) dimensions, the Poincaré supergravity theory is recovered contracting the super \( AdS \) group. Consider the following rescaling of the fields
\[
\begin{align*}
e^a &\to \frac{1}{\kappa} e^a \\
\omega_{ab} &\to \frac{1}{\kappa} \omega_{ab} \\
b &\to \frac{1}{\kappa} b \\
\psi_i &\to \frac{1}{\kappa} \psi_i \\
a_i^j &\to \frac{1}{\kappa} a_i^j.
\end{align*}
\] (53)

Then, if the gravitational constant is also rescaled as \( \kappa \to \alpha \kappa \), in the limit \( \alpha \to \infty \) the action becomes that in \([7]\), plus a \( su(N) \) CS form,
\[
I = \frac{1}{8} \int \epsilon_{abcd} R^{ab} R^{cd} e^e - R^{ab} R_{ab} b - 2 R^{ab} (\bar{\psi} \Gamma_{ab} D\psi + D\bar{\psi} \Gamma_{ab} \psi_i) + L_{su(N)}).
\] (54)

The rescaling \([53]\) induces a contraction of the super \( AdS \) algebra \( su(m)N \) into \( su(2) \), where the second factor is an automorphism.

C. D=7 Supergravity

The smallest AdS superalgebra in seven dimensions is \( osp(2|8) \). The connection \([12]\) is \( A = \frac{1}{2} \epsilon^{ab} J_{ab} + e^a J_a + \frac{1}{2} \bar{e} M^i ) \), where \( M^i \) are the generators of \( sp(2) \). In the representation given above, the bracket \( \{ \) is the supertrace and, in terms of the component fields appearing in the connection, the CS form is
\[
L_7^{osp(2|8)}(A) = 2^{-4} L_G^{AdS}(\omega, e) - \frac{1}{2} L_G^{AdS}(\psi, e)
\]
\[
- L_7^{sp(2)}(a) + L_F(\psi, \omega, e, a).
\] (55)

Here the fermionic Lagrangian is
\[
L_F = 4 \bar{\psi}^j (R^2 \delta^j_i + R f_i^j + (f^2)^i_j) D\psi_i
\]
\[
+ 4 (\bar{\psi}^j \psi_i) [(\bar{\psi}^j \psi_i) (\bar{\psi} D\psi_i) - \bar{\psi} D\psi_i] - R \delta^j_k + f_i^k) \psi_i + D\bar{\psi} D\psi_i,
\]
\[
-2 (\bar{\psi} D\psi_i) |\bar{\psi} (\bar{R}^2 + f_i^k) \psi_i + D\bar{\psi} D\psi_i,
\]
\[
\text{where} \ f_i^j = da^j_i + a^j_i a^k_i + R = \frac{1}{4} (R^{ab} + e^a e^b) \Gamma_{ab} + \frac{1}{2} T^a \Gamma_a \text{ are the sp(2) and so(8) curvatures, respectively. The supersymmetry transformations }([13]) \text{ read}
\]
\[
\begin{align*}
\delta e^a &= \frac{1}{4} \bar{\epsilon} \Gamma^a \psi_i \\
\delta \omega_{ab} &= - \frac{1}{4} \bar{\epsilon} \Gamma^{ab} \psi_i \\
\delta b &= D\epsilon_i \\
\delta \psi_i &= D\epsilon_i \\
\delta a_i^j &= \bar{\epsilon} \psi_j - \bar{\epsilon} \psi_j.
\end{align*}
\]

Standard seven-dimensional supergravity is an \( N = 2 \) theory (its maximal extension is \( N=4 \)), whose gravitational sector is given by the Einstein-Hilbert action with cosmological constant and with an \( osp(2|8) \) invariant background \([17][18]\). In the case presented here, the extension to larger \( N \) is straightforward: the index \( i \) is allowed to run from 2 to 2s, and the Lagrangian is a CS form for \( osp(2s|8) \).
D. D=11 Supergravity

In this case, the smallest AdS superalgebra is $osp(32|1)$ and the connection is $A = \frac{1}{8} e^{ab} J_{ab} + e^a J_a + \frac{1}{49} h^{abde} J_{abde} + Q \psi$, where $b$ is a totally antisymmetric fifth-rank Lorentz tensor one-form. Now, in terms of the elementary bosonic and fermionic fields, the CS form in (32) reads

$$L^{osp(32|1)}(A) = L^{sp(32)}(\Omega) + L_F(\Omega, \psi),$$

where $\Omega = \frac{1}{2} (e^a \Gamma_a + \frac{1}{2} \omega^{ab} \Gamma_{ab} + \frac{1}{49} h^{abde} \Gamma_{abde})$ is an $sp(32)$ connection. The bosonic part of (32) can be written as

$$L^{sp(32)}(\Omega) = 2^{-6} L^{AdS}_G(\omega, e) - \frac{1}{2} L^{AdS}_T(\omega, e) + L_{11}(b, \omega, e).$$

The fermionic Lagrangian is

$$L_F = 6(\bar{\psi} R^2 \psi) - 3 \left( (\bar{D} \psi \bar{D} \psi) + (\bar{\psi} R^2 \psi) \right) + 3 \left( (\bar{D} \psi \bar{D} \psi) - (\bar{\psi} R^2 \psi) \right) + 2 \left( (\bar{D} \psi \bar{D} \psi) + (\bar{\psi} R^2 \psi) \right) \left( (\bar{D} \psi + D \psi) \right),$$

where $R = d\Omega + \Omega^2$ is the $sp(32)$ curvature. The supersymmetry transformations (43) read

$$\delta e^a = \frac{1}{8} \Gamma^a \psi \quad \delta \omega^{ab} = -\frac{1}{8} \epsilon^{ab} \psi \quad \delta \psi = D \epsilon \quad \delta h^{abde} = \frac{1}{8} \epsilon^{abde} \psi. \quad \delta \alpha = \frac{1}{8} \omega^{ab} \psi \alpha \quad \delta \beta = \frac{1}{8} \omega^{ab} \psi \beta \quad \delta \phi = \frac{1}{8} \omega^{ab} \psi \delta \psi \quad \delta \delta \psi = \frac{1}{8} \epsilon^{abde} \psi.$$

Standard eleven-dimensional supergravity [42] is an $N=1$ supersymmetric extension of Einstein-Hilbert gravity that cannot accommodate a cosmological constant [44]. An $N > 1$ extension of this theory is not known. In our case, the cosmological constant is necessarily nonzero by construction and the extension simply requires including an internal $so(N)$ gauge field coupled to the fermions, and the resulting Lagrangian is an $osp(32|N)$ CS form [35].

VII. DISCUSSION

The supergravities presented here have two distinctive features: The fundamental field is always the connection $A$ and, in their simplest form, they are pure CS systems (matter couplings are discussed below). As a result, these theories possess a larger gravitational sector, including propagating spin connection. Contrary to what one could expect, the geometrical interpretation is quite clear, the field structure is simple and, in contrast with the standard cases, the supersymmetry transformations close off shell without auxiliary fields.

A. Torsion. It can be observed that the torsion Lagrangians ($L_T$) are odd while the torsion-free terms ($L_G$) are even under spacetime reflections. The minimal supersymmetric extension of the AdS group in $4k - 1$ dimensions requires using chiral spinors of $SO(4k)$ [21]. This in turn implies that the gravitational action has no definite parity, but requires the combination of $L_T$ and $L_G$ as described above. In $D = 4k + 1$ this issue doesn’t arise due to the vanishing of the torsion invariants, allowing constructing a supergravity theory based on $L_G$ only, as in [4]. If one tries to exclude torsion terms in $4k - 1$ dimensions, one is forced to allow both chiralities for $SO(4k)$ duplicating the field content, and the resulting theory has two copies of the same system [53].

B. Field content and extensions with $N>1$. The field content compares with that of the standard supergravities in $D = 5, 7, 11$ as follows:

| $D$ | Standard supergravity | CS supergravity |
|-----|-----------------------|-----------------|
| 5   | $e^a \psi_i \psi_{ij}$ | $e^a \bar{\omega}^{ab} \psi_i \psi_{ij}$ |
| 7   | $e^a \psi_i \psi_{ij}$ | $e^a \bar{\omega}^{ab} \psi_i \psi_{ij}$ |
| 11  | $e^a \psi_i \psi_{ij}$ | $e^a \bar{\omega}^{ab} \psi_i \psi_{ij}$ |

Standard supergravity in five dimensions.... The theory obtained with our scheme is the same one discussed by Chamseddine in [14].

Standard seven-dimensional supergravity is an $N = 2$ theory (its maximal extension is $N = 4$), whose gravitational sector is given by Einstein-Hilbert gravity with cosmological constant and with a background invariant under $OSp(2|8)$ [17,18]. Standard eleven-dimensional supergravity [12] is an $N=1$ supersymmetric extension of Einstein-Hilbert gravity that cannot accomodate a cosmological constant [13,50]. An $N > 1$ extension of this theory is not known.

In the case presented here, the extensions to larger $N$ are straightforward in any dimension. In $D = 7$, the index $i$ is allowed to run from 2 to 2$s$, and the Lagrangian is a CS form for $osp(2|8)$. In $D = 11$, one must include an internal $so(N)$ field and the Lagrangian is an $osp(32|N)$ CS form [2]. The cosmological constant is necessarily nonzero in all cases.

C. Spectrum. The stability and positivity of the energy for the solutions of these theories is a highly nontrivial problem. As shown in Ref. [2], the number of degrees of freedom of bosonic CS systems for $D > 5$ is not constant throughout phase space and different regions can have radically different dynamical content. However, in a region where the rank of the symplectic form is maximal the theory behaves as a normal gauge system, and this condition is stable under perturbations. As it is shown in [23], there exists a nontrivial extension of the AdS superalgebra with one abelian generator for which anti-de Sitter space without matter fields is a background of maximal rank, and the gauge superalgebra is realized in the Dirac brackets. For example, for $D = 11$ and $N = 32$, the only nonvanishing anticommutator reads

$$\{ Q^i, \bar{Q}^j \} = \frac{1}{8} \delta^{ij} \left[ \Gamma^a J_a + \Gamma^{ab} J_{ab} + \Gamma^{abde} Z_{abde} \right]_{\alpha\beta}$$

11
where $M^{12}$ are the generators of $SO(32)$ internal group. On this background the $D = 11$ theory has $2^{12}$ fermionic and $2^{12} - 1$ bosonic degrees of freedom. The (super)charges obey the same algebra with a central extension. This fact ensures a lower bound for the mass as a function of the other bosonic charges. 

D. Classical solutions. The field equations for these theories in terms of the Lorentz components $(\omega, e, b, a, \psi)$ are spread-out expressions for $<F^{\mu\nu}G_{(a)}>=0$, where $G_{(a)}$ are the generators of the superalgebra. It is rather easy to verify that in all these theories the anti-de Sitter space is a classical solution, and that for $\psi = b = a = 0$ there exist spherically symmetric, asymptotically $AdS$ standard $[26]$, as well as topological $[37]$ black holes. In the extreme case these black holes can be shown to be BPS states.

E. Matter couplings. It is possible to introduce a minimal couplings to matter of the form $A\cdot J$. For $D = 11$, the matter content is that of a theory with (super-) 0, 2, and 5-branes, whose respective worldhistories couple to the spin connection and the $b$ fields.

F. Standard SUGRA. Some sector of these theories might be related to the standard supergravities if one identifies the totally antisymmetric part of the contorsion tensor in a coordinate basis, $k_{\mu\nu\lambda}$, with the abelian 3-form, $A_{[3]}$. In 11 dimensions one could also identify the antisymmetric part of $b$ with an abelian 6-form $A_{[6]}$, whose exterior derivative, $dA_{[6]}$, is the dual of $F_{[4]} = dA_{[3]}$. Hence, in $D = 11$ the CS theory possibly contains the standard supergravity as well as some kind of dual version of it.

ACKNOWLEDGEMENTS

The authors are grateful to R. Aros, M. Bañados, O. Chandía, M. Contreras, A. Dabholkar, S. Deser, G. Gibbons, A. Gomberoff, M. Günyaydin, M. Henneaux, C. Martínez, F. Méndez, S. Mukhi, R. Olea, C. Teitelboim and E. Witten for many enlightening discussions and helpful comments. This work was supported in part by grants 1960229, 1970151, 1980788 and 3960009 from FONDECYT (Chile), and 27-953/ZI-DICYT (USACH). Institutional support to CECS from Fuerza Aérea de Chile and a group of Chilean private companies (Business Design Associates, CGE, CODELCO, COPEC, Empresas CMPC, Minera Collahuasi, Minera Escondida, NOVAGAS and XEROX-Chile) is also acknowledged.

[1] R. Troncoso and J. Zanelli, Phys. Rev. D 58 R101703, (1998).

[2] R. Troncoso and J. Zanelli, Gauge Supergravities for all Odd Dimensions, lecture presented at the Third Meeting Quantum Gravity in the Southern Cone, Bariloche, January 1998. [hep-th/9807029]

[3] R. Troncoso and J. Zanelli, Gauge Supergravities for all Odd Dimensions, lecture presented at the Third Meeting Quantum Gravity in the Southern Cone, Bariloche, January 1998.

[4] M. Henneaux, Phys. Rep. 126 (1985) 1.

[5] M. Bañados, L. J. Garay and M. Henneaux, Phys. Rev. D53 R593 (1996); Nucl. Phys. B476 611 (1996).

[6] C. Lanczos, Ann. Math. 39 (1938) 842.

[7] J. Mardones and J. Zanelli, Nucl. Phys. B291 (1988) 257.

[8] J. Mardones and J. Zanelli, Phys. Lett. B268 (1991) 27.

[9] E. Witten, Nucl. Phys. B121 (1983) 38.

[10] J. C. Taylor and V. O. Rivelles, Phys. Lett. B104 (1981) 133; B121 (1983) 38.

[11] J. W. van Holten and A. Van Proeyen, J. Phys. A 15 (1982) 3763.

[12] M. Henneaux, Phys. Rep. 28 (1985) 39.

[13] A. Chamessian, Phys. Lett. B233 (1989) 291.

[14] A. Chamessian, Nucl. Phys. B346 (1990) 213.

[15] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. D49 (1994) 975.

[16] J. Zanelli, Phys. Rev. D51 (1995) 490.

[17] M. Bañados, R. Troncoso and J. Zanelli, Phys. Rev. D54 (1996) 2605.

[18] P. van Nieuwenhuizen, Phys. Rep. 68 (1981) 1.

[19] J. W. van Holten and A. Van Proeyen, J. Phys. A 15 (1982) 3763.

[20] M. S. Sohnius, Phys. Rep. 28 (1985) 39.

[21] M. Günyaydin and C. Sacioglu, Comm. Math. Phys. 87 (1982) 159.

[22] A. Chamessian, Phys. Lett. 137 (1986) 31.

[23] C. Teitelboim and J. Zanelli, Class. and Quantum Grav. 4 (1987) L125.

[24] B. Zwiebach, Phys. Lett. 156B (1985) 315.

[25] D. G. Boulware and S. Deser, Phys. Rev. Lett. 55 (1985) 2656.

[26] M. Bañados, C. Teitelboim and J. Zanelli, Lovelock-Born-Infeld Theory of Gravity in J. J. Giambiagi Festschrift, H. Falomir, E. Gamboa-Saravi, P. Leal, and F. Schaposnik (eds.), World Scientific, Singapore, 1991.

[27] J.T.Wheeler, Nucl. Phys. B268 (1986) 737; B273 (1986) 732. B.Witt, Phys. Rev. D38 (1988) 3001. R. C. Myers and J. Simon, Phys. Rev. D38 (1988) 2434. D.L.Wiltshire, ibid., 38 (1988) 2445.

[28] M. Henneaux, C. Teitelboim and J. Zanelli, Gravity in Higher Dimensions, in SILARG V, M. Novello, (ed.), World Scientific, Singapore, 1987; Phys. Rev. A 36 (1987) 4417.

[29] R. Hojman, C. Mukku and W.A. Sayed, Phys.Rev. D22 (1980) 1915.

[30] M. Contreras and J. Zanelli, A note on the spin connection formulation of gravity (to appear).

[31] P. Freund, Introduction to Supergravity Cambridge University Press, Cambridge, U.K., 1989.

[32] P. Debever, (ed.), Elie Cartan – Albert Einstein, Lettres sur le Parallelisme Absolu, 1929-1932 Académie Royal de Belgique & Princeton University Press (1979).

[33] M. Nakahara, Geometry, Topology and Physics Adam Hilger, New York, 1990.
