Comparing and Combining Methods for Automatic Query Expansion

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\textbf{Abstract.} Query expansion is a well known method to improve the performance of information retrieval systems. In this work we have tested different approaches to extract the candidate query terms from the top ranked documents returned by the first-pass retrieval. One of them is the cooccurrence approach, based on measures of cooccurrence of the candidate and the query terms in the retrieved documents. The other one, the probabilistic approach, is based on the probability distribution of terms in the collection and in the top ranked set. We compare the retrieval improvement achieved by expanding the query with terms obtained with different methods belonging to both approaches. Besides, we have developed a na"ıve combination of both kinds of method, with which we have obtained results that improve those obtained with any of them separately. This result confirms that the information provided by each approach is of a different nature and, therefore, can be used in a combined manner.

1 Introduction

The reformulation of the user queries is a common technique in information retrieval to cover the gap between the original user query and his necessity of information. The most used technique for query reformulation is query expansion, where the original user query is expanded with new terms extracted from different sources. Queries submitted for users are usually very short and query expansion can complete the information need of the users.

A very complete review on the classical techniques of query expansion was done by Efthimiadis [?]. The main problem of query expansion is that in some cases the expansion process worsen the query performance. Improving the robustness of query expansion has been the goal of many researchers for the last years, and most proposed approaches use external collections [], such as the Web documents, to extract candidate terms for the expansion. There are other methods that extract the candidate terms from the same collection where the search is performed []. Some of these methods are based on global analysis where the list of candidate terms is generated from the whole collection, but the are computationally very expensive and its effectiveness is not better than the one of methods based on local analysis. We also use the same collection where the
search is performed, but applying local query expansion, also known as pseudo-feedback or blind feedback, which does not use the global collection or external sources for the expansion. This approach was first proposed by Xu and Croft and extracts the expansion terms from the documents retrieved for the original user query in a first pass retrieval.

In this work we have tested different approaches to extract the candidate terms from the top ranked documents returned by the first-pass retrieval. There exist two main approaches to rank the terms extracted from the retrieval documents. One of them is the cooccurrence approach, based on measures of cooccurrence of the candidate and the query terms in the retrieved documents. The other one is the probabilistic approach and is based on the differences of the probability distribution of terms in the collection and in the top ranked set. In this paper we are interested in evaluating the different techniques existing to generate the candidate term list. Our thesis is that the information obtained with the cooccurrence methods is different from the information obtained with probabilistic methods and these two kinds of information can be combined to improve the performance of the query expansion process. Accordingly, our goal has been to compare the performance of the cooccurrence approach and the probabilistic techniques and to study the way of combining them to improve the query expansion process.

After the term extraction step, the query expansion process requires a further step that is to re-compute the weights of the query terms that will be used in the search process. We present the results of combining different methods for the term extraction and the reweighting steps.

Two important parameters have to be adjusted for the described process. One of them is the number of documents retrieved in the first pass to be used for the term extraction. The other one is the number of candidate terms that are finally used to expand the original user query. We have performed experiments to set both of them to its optimal value in each considered method.

The rest of the paper proceeds as follows: sections 2 and 3 describe the cooccurrence and probabilistic approaches, respectively; section 4 presents our proposal of combining both approaches; section 5 describes the different reweighting methods considered to assigned new weight to the query terms after the expansion process; section 6 is devoted to show the experiments performed to evaluate the different expansion techniques separately and combined and section 7 summarizes the main conclusions of this work.

2 Cooccurrence Methods

The methods based on term cooccurrence have been used since the 70’s to identify some of the semantic relationships that exist among terms. In the first works of K. Van Rijsbergen and K. Sparck Jones we find the idea of using cooccurrence statistics to detect some kind of semantic similarity between terms and using it to expand the user’s queries. In fact, this idea is based on the Association Hypothesis:
If an index term is good at discriminating relevant from non-relevant documents then any closely associated index term is likely to be good at this.

The main problem with the cooccurrence approach was mentioned by Peat and Willet that claim that similar terms identified by cooccurrence tend to occur also very frequently in the collection and therefore these terms are not good elements to discriminate between relevant and non-relevant documents. This is true when the cooccurrence analysis is done on the whole collection but if we apply cooccurrence analysis only on the top ranked documents the problem exposed by Peat and Willet is smoothed.

For our experiments we have used the well-know Cosine, Dice and Tanimoto coefficients:

\[
Tanimoto(t_i, t_j) = \frac{c_{ij}}{c_i + c_j - c_{ij}} \quad (1)
\]

\[
Dice(t_i, t_j) = \frac{2 * c_{ij}}{c_i + c_j} \quad (2)
\]

\[
Cosine(t_i, t_j) = \frac{c_{ij}}{\sqrt{c_i * c_j}} \quad (3)
\]

where \(c_i\) and \(c_j\) are the number of documents in which terms \(t_i\) and \(t_j\) occur, respectively, and \(c_{i,j}\) is the number of documents in which \(t_i\) and \(t_j\) cooccur.

We apply these coefficients to measure the similarity between terms represented by the vectors. The result is a ranking of candidate terms where the most useful terms for expansion are in the top.

In the selection method the most likely terms are selected using the next equation:

\[
rel(q, t_e) = \sum_{t_i \in q} q_i * ASS(t_i, t_e) \quad (4)
\]

where \(ASS\) is one of the cooccurrence coefficients: Tanimoto, Dice, or Cosine. The equation ?? boosted the terms related with more terms of the original query.

The results obtained with each of these measures, presented in section ??, show that Tanimoto performs better.

3 Distribution Analysis Approaches

One of the main approaches to query expansion is based on studying the difference of term distribution between the whole collection and the subsets of documents that can be relevant for the query. It is expected that terms with little informative content have a similar distribution in any document of the collection. On the contrary terms closely related to those of the original query are expected to be more frequent in the top ranked set of documents retrieved with the original query than in other subsets of the collection.
3.1 Information-theoretic approach

One of the most interesting approaches based on term distribution analysis has been proposed by C. Carpineto et. al. [?], and uses the concept the Kullback-Liebler Divergence to compute the divergence between the probability distributions of terms in the whole collection and in the top ranked documents obtained for a first pass retrieval using the original user query. The most likely terms to expand the query are those with a high probability in the top ranked set and low probability in the whole collection. This divergence is computed as:

$$KLD_{(PR,PC)} = P_{R}(t) * \log \frac{P_{R}(t)}{P_{C}(t)}$$

where $P_{R}(t)$ is the probability of the term $t$ in the top ranked documents, and $P_{C}(t)$ is the probability of the term $t$ in the whole collection.

3.2 Divergence From Randomness term weighting model

The Divergence From Randomness (DFR) [?] term weighting model infers the informativeness of a term by the divergence between its distribution in the top-ranked documents and a random distribution. The most effective DFR term weighting model is the Bo1 model that uses the Bose-Einstein statistics [?,?]:

$$w(t) = tf_{x} * \log_{2} \left( \frac{1 + P_{n}}{P_{n}} \right) + \log(1 + P_{n})$$

where $tf_{x}$ is the frequency of the query term in the top-ranked documents and $P_{n}$ is given by $\frac{F}{N}$, where $F$ is the frequency of the query term in the collection and $N$ is the number of documents in the collection.

4 Combined query expansion method

The two approaches tested in this work can complement each other because they rely on different information. The performance of the cooccurrence approach is reduced by words, which are not stop-words, but are very frequent in the collection [?]. Those words, which represent a kind of noise, can reach a high position in the term index, thus worsen the expansion process. However, precisely because their high probability in any set of the document collection, these words tend to have a low score in KLD or Bo1. Accordingly, combining the cooccurrence measures with others based on the informative content of the terms, such as KLD or Bo1, helps to eliminate the noisy terms, thus improving the retrieved information with the query expansion process.

Our combined model amounts to applying both, a cooccurrence method and a distributional method and then obtaining the list of candidate terms by intersecting the lists provided by each method separately. Finally, the terms of the resulting list are assigned a new weight by one of the reweighting method considered.
In the combined approach the number of selected terms depends on the overlapping between the term sets proposed by both approaches. To increase the intersection area and obtain enough candidate terms in the combined list it is necessary to increase the number of selected terms for the non-combined approaches. This issue has been studied in the experiments.

5 Methods for Reweighting the Expanded Query Terms

After the list of candidate terms has been generated by one of the methods described above, the selected terms which will be added to the query must be re-weighted. Different schemas have been proposed for this task. We have compared these schemas and tested which is the most appropriate for each expansion method and for our combined query expansion method.

The classical approach to term re-weighting is the Rocchio algorithm. In this work we have used Rocchio’s beta formula, which requires only the $\beta$ parameter, and computes the new weight $qtw$ of the term in the query as:

$$qtw = \frac{qtf}{qtf_{max}} + \beta \cdot \frac{w(t)}{w_{max}(t)}$$

(7)

where $w(t)$ is the old weight of term $t$, $w_{max}(t)$ is the maximum $w(t)$ of the expanded query terms, $\beta$ is a parameter, $qtf$ is the frequency of the term $t$ in the query and $qtf_{max}$ is the maximum term frequency in the query $q$. In all our experiments, $\beta$ is set to 0.1.

We have also tested other reweighting schemes, each of which directly comes from one of the proposed methods for the candidate term selection. These schemes use the ranking values obtained applying the function defined by each method. Each of them can only be applied to reweight terms selected with the method it derives from. This is due to these methods require data, collected during the selection process, which are specific of each of them.

For the case of the reweighting scheme derived from KLD, the new weight is directly obtained applying KLD to the candidate terms. Terms belonging to the original query maintain their value.

For the scheme deriving from the co-occurrence method, that we called SumASS, the weights of the candidate terms are computed by:

$$qtw = \frac{rel(q, t_e)}{\sum_{t_i \in q} q_i}$$

(8)

where $\sum_{t_i \in q} q_i$ is the sum of the weights of the original terms.

Finally, for the reweighting scheme deriving from the Bose-Einstein statistics, a normalization of Bo1, that we call BoNorm, we have defined a simple function based in the normalization of the values obtained by Bose-Einstein computation:

$$qtw = \frac{Bo_1}{\sum Bo_{t \in cl}}$$

(9)

where $\sum Bo_{t \in cl}$ is the sum of the Bose-Einstein values for all terms included in the candidate list obtained applying Bose-Einstein statistics.
6 Experiments

Lucene Vector Space Model implementation has been used to build our information retrieval system. Stemming and stopword removing has been applied in indexing and expansion process. Evaluation is carried out on the Spanish EFE94 corpus which is part of the CLEF collection [?] (approximately 215K documents of 330 average word length and 352K unique index terms) and the 2001 Spanish topic set, with 100 topics corresponding to 2001 and 2002 years, of which we only used the title (of 3.3 average word length).

We have used different measures to evaluate each method. Each of them provides a different estimation of the precision of the retrieved documents, which is the main parameter to optimize when doing query expansion, since recall is always improved by the query expansion process. The measures considered have been:

- MAP (Mean Average Precision), which is the average of the precision (percent of retrieved documents that are relevant) value obtained for the top set documents existing after each relevant document is retrieved. In this way MAP measures precision at all recall levels and provides a view of both aspects.
- GMAP, a variant of MAP, that uses a geometric mean rather than an arithmetic mean to average individual topic results.
- Precision@X, which is precision after X documents (whether relevant or non-relevant) have been retrieved. If X documents were not retrieved for a query, then all missing documents are assumed to be non-relevant.
- R-Precision, which measures precision after R documents have been retrieved, where R is the total number of relevant documents for a query. If R is greater than the number of documents retrieved for a query, then the non-retrieved documents are all assumed to be non-relevant.

First of all we have tested the different cooccurrence methods described above. Table ?? shows the results obtained for the different measures considered in this work. We can observe that Tanimoto provides the best results for all the measures, except for P@10, but in this case the difference with the result of Dice, which is the best, is very small. According to the results we have selected the Tanimoto similarity function as cooccurrence method for the rest of the work.

6.1 Selecting the Reweighting Method

The next set of experiments have had the goal of determining the most appropriate reweighting method for each candidate term selection method. Table ?? shows the results of different reweighting methods (Rocchio and SumASS) applied after selecting the candidate terms by the cooccurrence method. We can observe that the results are quite similar for both reweighting methods, though Rocchio is slightly better.
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Abstract. Query expansion is a well known method to improve the performance of the information retrieval systems. In this work we have tested different approaches to extract the candidate query terms from the top ranked documents returned by the first-pass retrieval. One of them is the co-occurrence approach, based on measures of co-occurrence of the candidate and the query terms in the retrieved documents. The other one, the probabilistic approach, is based on the probability distribution of terms in the collection and in the top ranked set. We compare the retrieval improvement obtained expanding the query with terms obtained with different methods belonging to both approaches. Besides, we have developed a naïve combination of both approaches, with which we have obtained results that improve the obtained with any of them separately. This result confirms that the information provided by each approach has a different nature and, therefore can be used in a combined manner.

1 Introduction

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A very complete review on the classical techniques of query expansion was done by Efthimiadis [7]. Different methods of query expansion have been used to improve the retrieval performance in the different tracks of Text Retrieval Conference TREC especially in Robust and HARD tracks but also in Web and Terabyte tracks.

The main problem for the query expansion methods is that in some cases the expansion process worsen the query performance. Improving the robustness of query expansion has been the goal of many researchers for the last years, and most proposed approaches use external collections, such as the Web documents, to extract candidate terms for the expansion. There are other methods that extract the candidate terms from the same collection where the search is performed. Some of these methods are based on global analysis where the list
of candidate terms is generated from the whole collection, but the are computationally very expensive and its effectiveness is not better than the one of methods based on local analysis. We follow also use the same collection where the search is performed, but applying local query expansion, also known as pseudo-feedback or blind feedback, which does not uses the global collection or external sources to expansion. This approach was first proposed by Xu and Croft and uses the documents retrieved for the original user query in a first pass for the term extraction.

In this work we have tested different approaches to extract the candidate terms from the top ranked documents returned by the first-pass retrieval. There exist two main approaches to rank the terms extracted from the retrieval documents. One of them is the co-occurrence approach, based on measures of co-occurrence of the candidate and the query terms in the retrieved documents. The other one, the probabilistic approach, is based on the probability distribution of terms in the collection and in the top ranked set. In this paper we are interested in testing the different techniques existing to generate the candidate term list. Our thesis is that the information obtained with the co-occurrence methods is different to the information obtained with probabilistic methods and these two kinds of information can be combined to improve the performance of the query expansion process. Our main goal is to compare the performance of the co-occurrence approach and the probabilistic techniques and to study the way of combining them to improve the query expansion process.

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The rest of the paper proceeds as follows: sections 2 and 3 describes the co-occurrence and the probabilistic approaches, respectively; section 4 presents the different reweighting methods considered to assigned new weight to the query terms after the expansion process; section 5 is devoted to show the experiments performed to evaluate the different expansion techniques separately and combined and section 6 summarizes the main conclusions of this work.

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where \(c_i\) and \(c_j\) are the number of documents in which terms \(t_i\) and \(t_j\) occur, respectively, and \(c_{ij}\) is the number of documents in which \(t_i\) and \(t_j\) co-occur. The results obtained with each of these measures, shown in section ??, show that Tanimoto performs better.

We apply these coefficients to measure the similarity between terms represented by the vectors. The result is a ranking of candidate terms where the most useful terms for expansion are in the top.

In the selection method the most likely terms are selected using the next equation:

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One of the main approaches to query expansion is based on studying the difference of term distribution between the whole collection and the subsets of documents that can be relevant for the query. It is expected that terms with little informative content have a similar distribution in any document of the collection. On the contrary terms closely related to those of the original query are expected to be more frequent in the top ranked set of documents retrieved with the original query than in other subsets of the collection.
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One of the most interesting approaches based on term distribution analysis has been proposed by C. Carpineto et. al.[?], and uses the concept the Kullback-Liebler Divergence to compute the divergence between to probability distributions of terms in the whole collection and in the top ranked documents obtained for a first pass retrieval using the original query user. The most likely terms to expand the query are those with a high probability in the top ranked set and low probability in the whole collection. This divergence is computed as:

\[ KLD_{PR,PC} = P_R(t) \log \frac{P_R(t)}{P_C(t)} \]  

where \( P_R(t) \) is the probability of the term \( t \) in the top ranked documents, and \( P_C(t) \) is the probability of the term \( t \) in the whole collection.

3.2 Divergence From Randomness term weighting model

The Divergence From Randomness (DFR)[?] term weighting model infers the informativeness of a term by the divergence between its distribution in the top-ranked documents and a random distribution. The most effective DFR term weighting model is the Bo1 model that uses the Bose-Einstein statistics[?]:

\[ w(t) = tf_x \log_2 \left( \frac{1 + P_n}{P_n} \right) + \log(1 + P_n) \]  

where \( tf_x \) is the frequency of the query term in the top-ranked documents and \( P_n \) is given by \( F/overN \), where \( F \) is the frequency of the query term in the collection and \( N \) is the number of documents in the collection.

4 Combined query expansion method

The two approaches tested in this work can complement each other because they rely on different information. Specifically, the the performance of the cooccurrence approach can be reduced by those words, which are not stop-words, but are very frequent in the collection. Those words, which represent a kind of noise, can reach probability a high position in the term index, thus worsen the expansion process. However, this kind of word well, in general, have a low score in KLD or Bo1, precisely because their high probability in any set of the document collection. Accordingly, combining the cooccurrence measures with others based on the informative content of the terms, such as KLD or Bo1, helps to eliminate the noise terms, thus improving the query expansion process, and thus the retrieved information.

Our combined model consists in retrieving In the combined approaches the number of selected terms depends of the overlapping between the terms proposed by both approaches.
5 Methods for Reweighting the Expanded Query Terms

After candidate list has been generated by the methods showed above, the selected terms that will be added to the query must be re-weighted. Different schemas have been proposed for this task. We have compared these schemas and experimented which is the most appropriate for each expansion method and for our combined query expansion method.

The classical approach to term re-weighting is the Rocchio algorithm \[?]\. In this work we have used Rocchio’s beta formula, which requires only the $\beta$ parameter, and computes the new weight $qtw$ of the term in the query as:

$$ qtw = \frac{qtf}{qtf_{max}} + \beta \cdot \frac{w(t)}{w_{max}(t)} $$ \hspace{1cm} (7)

where $w(t)$ is the old weight of term $t$, $w_{max}(t)$ is the maximum $w(t)$ of the expanded query terms, $\beta$ is a parameter, $qtf$ is the frequency of the term $t$ in the query and $qtf_{max}$ is the maximum term frequency in the query $q$. In all our experiments, $\beta$ is set to 0.1.

We have also tested other reweighting schemes, each of which directly comes from the proposed methods for the candidate term selection. These schemes use the ranking values obtained applying the functions defined by each method. Each of them can be only be applied to reweight terms selected with the method they derive from. It is due to these methods require data collected during the selection process, which are specific of each method.

For the case of the reweighting scheme derived from KLD, the new weight is obtained directly applying KLD to the candidate terms. Terms belonging to the original query maintain their value \[?\].

For scheme deriving from the co-occurrence method, that we called SumASS, the weights of the candidate terms are computed by:

$$ qtw = \frac{rel(q,t_e)}{\sum_{i \in q} q_i} $$ \hspace{1cm} (8)

where $\sum_{i \in q} q_i$ is the sum of the weight of the original terms\[?\].

Finally, for the reweighting scheme deriving from the Bose-Einstein statistics, a normalization of Bo1, that we call BoNorm, we have defined a simple function based in the normalization of the values obtained by Bose-Einstein computation:

$$ qtw = \frac{Bo_i}{\sum Bo_{i \in cl}} $$ \hspace{1cm} (9)

where $\sum Bo_i \in cl$ is the sum of the Bose-Einstein values for all terms included in the candidate list obtained applying Bose-Einstein statistics.

6 Experiments

Lucene Vector Space Model implementation has been used to build our information retrieval system. Stemming and stopword removing has been applied in
indexing and expansion process. Evaluation is carried out on the Spanish EFE94 corpus which is part of the CLEF collection [?] (approximately 215K documents of 330 average word length and 352K unique index terms) and the 2001 Spanish topic set, with 100 topics corresponding to 2001 and 2002 years, of which we only used the title (of 3.3 average word length).

We have used different measures to evaluate each method. Each of them provides a different estimation of the precision of the retrieved documents, which is the main parameter to optimize when doing query expansion, since recall is always improved by the query expansion process. The measures considered have been:

- **MAP** (Mean Average Precision), which is the average of the precision (percent of retrieved documents that are relevant) value obtained for the top set of documents existing after each relevant document is retrieved. In this way MAP measures precision at all recall levels and thus provides a view of both aspects.
- **GMAP**, a variant of MAP, that uses a geometric mean rather than an arithmetic mean to average individual topic results.
- **Precision@X**, precision after X documents (whether relevant or non-relevant) have been retrieved. Values averaged over all queries. If X docs were not retrieved for a query, then all missing docs are assumed to be non-relevant.
- **R-Precision**, which measures precision after R docs have been retrieved, where R is the total number of relevant docs for a query. If R is greater than the number of docs retrieved for a query, then the non-retrieved docs are all assumed to be non-relevant.

First of all we have tested the different cooccurrence methods described above. Table ?? shows the results obtained for the different measures considered in this work. We can observe that Tanimoto provides the best results all the measures, except for P@10, but in this case the difference the result obtained with Dice, which is the best, is very small. According to which we have selected the Tanimoto similarity function for the rest of the work.

| Method    | MAP   | GMAP  | R-PREC | P@5    | P@10   |
|-----------|-------|-------|--------|--------|--------|
| Baseline  | 0.4006| 0.1941| 0.4044 | 0.5340 | 0.4670 |
| Cosine    | 0.4698| 0.2375| 0.4530 | 0.6020 | 0.5510 |
| Tanimoto  | 0.4831| 0.2464| 0.4623 | 0.6060 | 0.5520 |
| Dice      | 0.4772| 0.2447| 0.4583 | 0.6020 | 0.5530 |

**Table 1.** Comparing different cooccurrence methods. The Baseline row corresponds to the results of the query without expansion. P@5 stands for precision after the first five documents retrieved, P@10 after the first ten, and R-PREC stands for R-precision.
6.1 Selecting the Reweighting Method

The next set of experiments have had the goal of determining the most appropriate reweighting method for each candidate term selection method. Table ?? shows the results of different reweighting methods (Rocchio and SumASS) applied after selecting the candidate terms by the cooccurrence method. We can observe that the results are quite similar for both reweighting methods.

|                | MAP  | GMAP | R-PREC | P@5  | P@10 |
|----------------|------|------|--------|------|------|
| Baseline       | 0.4006 | 0.1941 | 0.4044 | 0.5340 | 0.4670 |
| CooRocchio     | **0.4831** | **0.2464** | 0.4623 | **0.6060** | **0.5520** |
| CooSumASS      | 0.4798 | 0.2386 | **0.4628** | **0.6080** | 0.5490 |

Table 2. Comparing different reweighting methods for cooccurrence. CooRocchio corresponds to using cooccurrence as selection terms method and Rocchio as reweighting method. CooSumASS corresponds to using cooccurrence as selection terms method and SumASS as reweighting method. Best results appear in boldface.

Table ?? shows the results of different reweighting methods (Rocchio and kld) applied after selecting the candidate terms with KLD. The best results are obtained using kld as reweighting method.

|                | MAP  | GMAP | R-PREC | P@5  | P@10 |
|----------------|------|------|--------|------|------|
| Baseline       | 0.4006 | 0.1941 | 0.4044 | 0.5340 | 0.4670 |
| KLDRocchio     | 0.4788 | 0.2370 | 0.4450 | 0.5960 | 0.5480 |
| KLDkld         | **0.4801** | **0.2376** | **0.4526** | **0.6080** | **0.5510** |

Table 3. Comparing different reweighting methods for KLD. KLDRocchio corresponds to using KLD as selection terms method and Rocchio as reweighting method. KLDkld corresponds to using KLD as selection terms method and kld as reweighting method. Best results appear in boldface.

Table ?? shows the results of different reweighting methods (Rocchio and BoNorm) applied after selecting the candidate terms with Bo1. In this case, the best results are obtained using BoNorm as reweighting method.

The results of this section show that the best reweighting method after selected term by cooccurrence is Rocchio, while for the distribution analysis methods used as selection method the best reweighting is obtained with the their derived method, though Rocchio also provides results very close in all cases.

6.2 Parameter Study

We have studied two parameters that are fundamental in query expansion, the number of candidate terms to expand the query and the number of documents
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Abstract. Query expansion is a well known method to improve the performance of the information retrieval systems. In this work we have tested different approaches to extract the candidate query terms from the top ranked documents returned by the first-pass retrieval. One of them is the co-occurrence approach, based on measures of co-occurrence of the candidate and the query terms in the retrieved documents. The other one, the probabilistic approach, is based on the probability distribution of terms in the collection and in the top ranked set. We compare the retrieval improvement obtained expanding the query with terms obtained with different methods belonging to both approaches. Besides, we have developed a naïve combination of both approaches, with which we have obtained results that improve the obtained with any of them separately. This result confirms that the information provided by each approach has a different nature and, therefore can be used in a combined manner.

1 Introduction

The reformulation of the user queries is a common technique in information retrieval to cover the gap between the original user query and his necessity of information. The most used technique for query reformulation is query expansion, where the original query user is expanded with new terms extracted from different sources. Queries submitted for users are usually very short and query expansion can complete the information need of the users.

A very complete review on the classical techniques of query expansion was done by Efthimiadis [?]. Different methods of query expansion have been used to improve the retrieval performance in the different tracks of Text Retrieval Conference TREC especially in Robust and HARD tracks but also in Web and Terabyte tracks.

The main problem for the query expansion methods is that in some cases the expansion process worsen the query performance. Improving the robustness of query expansion has been the goal of many researchers for the last years, and most proposed approaches use external collections, such as the Web documents, to extract candidate terms for the expansion. There are other methods that extract the candidate terms from the same collection where the search is performed. Some of these methods are based on global analysis where the list
of candidate terms is generated from the whole collection, but the are computationally very expensive and its effectiveness is not better than the one of methods based on local analysis. We follow also use the same collection where the search is performed, but applying local query expansion, also known as pseudo-feedback or blind feedback, which does not uses the global collection or external sources to expansion. This approach was first proposed by Xu and Croft and uses the documents retrieved for the original user query in a first pass for the term extraction.

In this work we have tested different approaches to extract the candidate terms from the top ranked documents returned by the first-pass retrieval. There exist two main approaches to rank the terms extracted from the retrieval documents. One of them is the co-occurrence approach, based on measures of co-occurrence of the candidate and the query terms in the retrieved documents. The other one, the probabilistic approach, is based on the probability distribution of terms in the collection and in the top ranked set. In this paper we are interested in testing the different techniques existing to generate the candidate term list. Our thesis is that the information obtained with the co-occurrence methods is different to the information obtained with probabilistic methods and these two kinds of information can be combined to improve the performance of the query expansion process. Our main goal is to compare the performance of the co-occurrence approach and the probabilistic techniques and to study the way of combining them to improve the query expansion process.

After the term extraction step, the query expansion process requires a further step that is to re-compute the weights of the query terms that will be used in the search process. We present the results of combining different methods for the term extraction and the reweighting steps.

Two important parameters have to be adjusted for the described process. One of them is the number of documents retrieved in the first pass to be used for the term extraction. The other one is the number of candidate terms that are finally used to expand the original query user. We have performed experiments to set both of them to its optimal value in each considered method.

The rest of the paper proceeds as follows: sections 2 and 3 describes the co-occurrence and the probabilistic approaches, respectively; section 4 presents the different reweighting methods considered to assigned new weight to the query terms after the expansion process; section 5 is devoted to show the experiments performed to evaluate the different expansion techniques separately and combined and section 6 summarizes the main conclusions of this work.

2 Cooccurrence Methods

The methods based on term co-occurrence have been used since the 70’s to identify some of the semantic relationships that exit among terms. In the first works of K. Van Rijsbergen and K. Sparck Jones we find the idea of using co-occurrence statistics to detect some kind of semantic similarity between terms.
and using it to expand the user’s queries. In fact, this idea is based on the Association Hypothesis:

*If an index term is good at discriminating relevant from non-relevant documents then any closely associated index term is likely to be good at this.*

The main problem with the co-occurrence approach was mentioned by Peat and Willet that claim that similar terms identified by co-occurrence tend to occur also very frequently in the collection and therefore these terms are not good elements to discriminate between relevant and non-relevant documents. This is true when the co-occurrence analysis is done on the whole collection but if we apply co-occurrence analysis only on the top ranked documents the problem exposed by Peat and Willet is smoothed.

For our experiments we have used the well-know Cosine, Dice and Tanimoto coefficients:

\[
Tanimoto(t_i, t_j) = \frac{c_{ij}}{c_i + c_j - c_{ij}}
\]

\[
Dice(t_i, t_j) = \frac{2 * c_{ij}}{c_i + c_j}
\]

\[
Cosine(t_i, t_j) = \frac{c_{ij}}{\sqrt{c_i * c_j}}
\]

where \(c_i\) and \(c_j\) are the number of documents in which terms \(t_i\) and \(t_j\) occur, respectively, and \(c_{i,j}\) is the number of documents in which \(t_i\) and \(t_j\) co-occur. The results obtained with each of these measures, shown in section ??, show that Tanimoto performs better.

We apply these coefficients to measure the similarity between terms represented by the vectors. The result is a ranking of candidate terms where the most useful terms for expansion are in the top.

In the selection method the most likely terms are selected using the next equation:

\[
rel(q, t_e) = \sum_{t_i \in q} q_i * ASS(t_i, t_e)
\]

where \(ASS\) is one of the cooccurrence coefficients: Tanimoto, Dice, or Cosine. The equation ?? boosted the terms related with more terms of the original query.

### 3 Distribution Analysis Approaches

One of the main approaches to query expansion is based on studying the difference of term distribution between the whole collection and the subsets of documents that can be relevant for the query. It is expected that terms with little informative content have a similar distribution in any document of the collection. On the contrary terms closely related to those of the original query are expected to be more frequent in the top ranked set of documents retrieved with the original query than in other subsets of the collection.


3.1 Information-theoretic approach

One of the most interesting approaches based on term distribution analysis has been proposed by C. Carpineto et. al.\cite{Carpineto}, and uses the concept the Kullback-Liebler Divergence to compute the divergence between to probability distributions of terms in the whole collection and in the top ranked documents obtained for a first pass retrieval using the original query user. The most likely terms to expand the query are those with a high probability in the top ranked set and low probability in the whole collection. This divergence is computed as:

\[ KLD_{(PR,PC)} = P_R(t) \times \frac{\log \frac{P_R(t)}{P_C(t)}}{ } \] (5)

where \(P_R(t)\) is the probability of the term \(t\) in the top ranked documents, and \(P_C(t)\) is the probability of the term \(t\) in the whole collection.

3.2 Divergence From Randomness term weighting model

The Divergence From Randomness (DFR)\cite{DFR} term weighting model infers the informativeness of a term by the divergence between its distribution in the top-ranked documents and a random distribution. The most effective DFR term weighting model is the Bo1 model that uses the Bose-Einstein statistics\cite{Bose-Einstein}:

\[ w(t) = tf_x \times \log_2 \left( \frac{1 + P_n}{P_n} + \log(1 + P_n) \right) \] (6)

where \(tf_x\) is the frequency of the query term in the top-ranked documents and \(P_n\) is given by \(F/\overline{N}\), where \(F\) is the frequency of the query term in the collection and \(N\) is the number of documents in the collection.

4 Combined query expansion method

The two approaches tested in this work can complement each other because they rely on different information. Specifically, the performance of the cooccurrence approach can be reduced by those words, which are not stop-words, but are very frequent in the collection. Those words, which represent a kind of noise, can reach probability a high position in the term index, thus worsen the expansion process. However, this kind of word well, in general, have a low score in KLD or Bo1, precisely because their high probability in any set of the document collection. Accordingly, combining the cooccurrence measures with others based on the informative content of the terms, such as KLD or Bo1, helps to eliminate the noise terms, thus improving the query expansion process, and thus the retrieved information.

Our combined model consists in retrieving In the combined approaches the number of selected terms depends of the overlapping between the terms proposed by both approaches.
5 Methods for Reweighting the Expanded Query Terms

After candidate list has been generated by the methods showed above, the selected terms that will be added to the query must be re-weighted. Different schemas have been proposed for this task. We have compared these schemas and experimented which is the most appropriate for each expansion method and for our combined query expansion method.

The classical approach to term re-weighting is the Rocchio algorithm \[\text{Rocchio}\]\. In this work we have used Rocchio’s beta formula, which requires only the $\beta$ parameter, and computes the new weight $qtw$ of the term in the query as:

$$qtw = \frac{qtf}{qtf_{max}} + \beta \frac{w(t)}{w_{max}(t)}$$  \hspace{1cm} (7)

where $w(t)$ is the old weight of term $t$, $w_{max}(t)$ is the maximum $w(t)$ of the expanded query terms, $\beta$ is a parameter, $qtf$ is the frequency of the term $t$ in the query and $qtf_{max}$ is the maximum term frequency in the query $q$. In all our experiments, $\beta$ is set to 0.1.

We have also tested other reweighting schemes, each of which directly comes from the proposed methods for the candidate term selection. These schemes use the ranking values obtained applying the functions defined by each method. Each of them can be only be applied to reweight terms selected with the method they derive from. It is due to these methods require data collected during the selection process, which are specific of each method.

For the case of the reweighting scheme derived from KLD, the new weight is obtained directly applying KLD to the candidate terms. Terms belonging to the original query maintain their value\[\text{KLD}\].

For scheme deriving from the co-ocurrence method, that we called SumASS, the weights of the candidate terms are computed by:

$$qtw = \frac{rel(q, t_e)}{\sum_{t_i \in q} q_i}$$  \hspace{1cm} (8)

where $\sum_{t_i \in q} q_i$ is the sum of the weight of the original terms\[\text{SumASS}\].

Finally, for the reweighting scheme deriving from the Bose-Einstein statistics, a normalization of Bo1, that we call $BoNorm$, we have defined a simple function based in the normalization of the values obtained by Bose-Einstein computation:

$$qtw = \frac{Bo_1}{\sum Bo_{t \in cl}}$$  \hspace{1cm} (9)

where $\sum Bo_t \in cl$ is the sum of the Bose-Einstein values for all terms included in the candidate list obtained applying Bose-Einstein statistics.

6 Experiments

Lucene Vector Space Model implementation has been used to build our information retrieval system. Stemming and stopword removing has been applied in
indexing and expansion process. Evaluation is carried out on the Spanish EFE94 corpus which is part of the CLEF collection [?] (approximately 215K documents of 330 average word length and 352K unique index terms) and the 2001 Spanish topic set, with 100 topics corresponding to 2001 and 2002 years, of which we only used the title (of 3.3 average word length).

We have used different measures to evaluate each method. Each of them provides a different estimation of the precision of the retrieved documents, which is the main parameter to optimize when doing query expansion, since recall is always improved by the query expansion process. The measures considered have been:

- MAP (Mean Average Precision), which is the average of the precision (percent of retrieved documents that are relevant) value obtained for the top set of documents existing after each relevant document is retrieved. In this way MAP measures precision at all recall levels and thus provides a view of both aspects.
- GMAP, a variant of MAP, that uses a geometric mean rather than an arithmetic mean to average individual topic results.
- Precision@X, precision after X documents (whether relevant or non-relevant) have been retrieved. Values averaged over all queries. If X docs were not retrieved for a query, then all missing docs are assumed to be non-relevant.
- R-Precision, which measures precision after R docs have been retrieved, where R is the total number of relevant docs for a query. If R is greater than the number of docs retrieved for a query, then the non-retrieved docs are all assumed to be non-relevant.

First of all we have tested the different cooccurrence methods described above. Table ?? shows the results obtained for the different measures considered in this work. We can observe that Tanimoto provides the best results all the measures, except for P@10, but in this case the difference the result obtained with Dice, which is the best, is very small. According to which we have selected the Tanimoto similarity function for the rest of the work.

|             | MAP   | GMAP  | R-PREC | P@5  | P@10 |
|-------------|-------|-------|--------|------|------|
| Baseline    | 0.4006| 0.1941| 0.4044 | 0.5340 | 0.4670 |
| Cosine      | 0.4698| 0.2375| 0.4530 | 0.6020 | 0.5510 |
| Tanimoto    | 0.4831| 0.2464| 0.4623 | 0.6060 | 0.5520 |
| Dice        | 0.4772| 0.2447| 0.4583 | 0.6020 | 0.5530 |

Table 1. Comparing different cooccurrence methods. The Baseline row corresponds to the results of the query without expansion. P@5 stands for precision after the first five documents retrieved, P@10 after the first ten, and R-PREC stands for R-precision.
6.1 Selecting the Reweighting Method

The next set of experiments have had the goal of determining the most appropriate reweighting method for each candidate term selection method. Table 2 shows the results of different reweighting methods (Rocchio and SumASS) applied after selecting the candidate terms by the cooccurrence method. We can observe that the results are quite similar for both reweighting methods.

|                  | MAP  | GMAP | R-PREC | P@5  | P@10 |
|------------------|------|------|--------|------|------|
| Baseline         | 0.4006 | 0.1941 | 0.4044 | 0.5340 | 0.4670 |
| CooRocchio       | 0.4831 | 0.2464 | 0.4623 | 0.6060 | 0.5520 |
| CooSumASS        | 0.4798 | 0.2386 | 0.4628 | 0.6080 | 0.5490 |

Table 2. Comparing different re-weighting methods for Co-occurrence. CooRocchio corresponds to using cooccurrence as selection terms method and Rocchio as reweighting method. CooSumASS corresponds to using cooccurrence as selection terms method and SumASS as reweighting method. Best results appear in boldface.

Table 3 shows the results of different reweighting methods (Rocchio and kld) applied after selecting the candidate terms with KLD. The best results are obtained using kld as reweighting method.

|                  | MAP  | GMAP | R-PREC | P@5  | P@10 |
|------------------|------|------|--------|------|------|
| Baseline         | 0.4006 | 0.1941 | 0.4044 | 0.5340 | 0.4670 |
| KLDRocchio       | 0.4788 | 0.2370 | 0.4450 | 0.5960 | 0.5480 |
| KLDkld           | 0.4801 | 0.2376 | 0.4526 | 0.6080 | 0.5510 |

Table 3. Comparing different re-weighting methods for KLD. KLDRocchio corresponds to using KLD as selection terms method and Rocchio as reweighting method. KLDkld corresponds to using KLD as selection terms method and kld as reweighting method. Best results appear in boldface.

Table 4 shows the results of different reweighting methods (Rocchio and BoNorm) applied after selecting the candidate terms with Bo1. In this case, the best results are obtained using BoNorm as reweighting method.

The results of this section show that the best reweighting method after selected term by cooccurrence is Rocchio, while for the distribution analysis methods used as selection method the best reweighting is obtained with the their derived method, though Rocchio also provides results very close in all cases.

6.2 Parameter Study

We have studied two parameters that are fundamental in query expansion, the number of candidate terms to expand the query and the number of documents
Preface

This textbook is intended for use by students of physics, physical chemistry, and theoretical chemistry. The reader is presumed to have a basic knowledge of atomic and quantum physics at the level provided, for example, by the first few chapters in our book *The Physics of Atoms and Quanta*. The student of physics will find here material which should be included in the basic education of every physicist. This book should furthermore allow students to acquire an appreciation of the breadth and variety within the field of molecular physics and its future as a fascinating area of research.

For the student of chemistry, the concepts introduced in this book will provide a theoretical framework for that entire field of study. With the help of these concepts, it is at least in principle possible to reduce the enormous body of empirical chemical knowledge to a few basic principles: those of quantum mechanics. In addition, modern physical methods whose fundamentals are introduced here are becoming increasingly important in chemistry and now represent indispensable tools for the chemist. As examples, we might mention the structural analysis of complex organic compounds, spectroscopic investigation of very rapid reaction processes or, as a practical application, the remote detection of pollutants in the air.

April 1995

Walter Olthoff
Program Chair
ECOOP’95
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Hamiltonian Mechanics unter besonderer Berücksichtigung der höheren Lehranstalten

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\textbf{Abstract.} The abstract should summarize the contents of the paper using at least 70 and at most 150 words. It will be set in 9-point font size and be inset 1.0 cm from the right and left margins. There will be two blank lines before and after the Abstract.

\section{1 Fixed-Period Problems: The Sublinear Case}

With this chapter, the preliminaries are over, and we begin the search for periodic solutions to Hamiltonian systems. All this will be done in the convex case; that is, we shall study the boundary-value problem

\[
\dot{x} = JH'(t,x) \\
x(0) = x(T)
\]

with \(H(t,\cdot)\) a convex function of \(x\), going to \(+\infty\) when \(\|x\|\to\infty\).

\subsection{1.1 Autonomous Systems}

In this section, we will consider the case when the Hamiltonian \(H(x)\) is autonomous. For the sake of simplicity, we shall also assume that it is \(C^1\).

We shall first consider the question of nontriviality, within the general framework of \((A_\infty, B_\infty-)\)-subquadratic Hamiltonians. In the second subsection, we shall look into the special case when \(H\) is \((0,b_\infty-)\)-subquadratic, and we shall try to derive additional information.

\textbf{The General Case: Nontriviality.} We assume that \(H\) is \((A_\infty, B_\infty-)\)-subquadratic at infinity, for some constant symmetric matrices \(A_\infty\) and \(B_\infty\), with \(B_\infty - A_\infty\) positive definite. Set:

\[
\gamma := \text{smallest eigenvalue of } B_\infty - A_\infty \tag{1}
\]

\[
\lambda := \text{largest negative eigenvalue of } J \frac{d}{dt} + A_\infty . \tag{2}
\]
Theorem 1 tells us that if $\lambda + \gamma < 0$, the boundary-value problem:

\[
\begin{align*}
\dot{x} &= JH'(x) \\
x(0) &= x(T)
\end{align*}
\]

has at least one solution $\pi$, which is found by minimizing the dual action functional:

\[
\psi(u) = \int_0^T \left[ \frac{1}{2} (A_0^{-1}u, u) + N^*(-u) \right] dt
\]

(4)

on the range of $A$, which is a subspace $R(A)_L^2$ with finite codimension. Here

\[
N(x) := H(x) - \frac{1}{2} (A_\infty x, x)
\]

(5)

is a convex function, and

\[
N(x) \leq \frac{1}{2} ((B_\infty - A_\infty) x, x) + c \quad \forall x.
\]

(6)

**Proposition 1.** Assume $H'(0) = 0$ and $H(0) = 0$. Set:

\[
\delta := \liminf_{x \to 0} 2N(x) \|x\|^{-2}.
\]

(7)

If $\gamma < -\lambda < \delta$, the solution $\pi$ is non-zero:

\[
\pi(t) \neq 0 \quad \forall t.
\]

(8)

**Proof.** Condition (7) means that, for every $\delta' > \delta$, there is some $\varepsilon > 0$ such that

\[
\|x\| \leq \varepsilon \Rightarrow N(x) \leq \frac{\delta'}{2} \|x\|^2.
\]

(9)

It is an exercise in convex analysis, into which we shall not go, to show that this implies that there is an $\eta > 0$ such that

\[
f \|x\| \leq \eta \Rightarrow N^*(y) \leq \frac{1}{2\delta'} \|y\|^2.
\]

(10)

---

**Fig. 1.** This is the caption of the figure displaying a white eagle and a white horse on a snow field
Since $u_1$ is a smooth function, we will have $\|hu_1\|_\infty \leq \eta$ for $h$ small enough, and inequality (10) will hold, yielding thereby:

$$\psi(hu_1) \leq \frac{h^2}{2} \frac{1}{\lambda} \|u_1\|^2 + \frac{h^2}{2} \frac{1}{\delta} \|u_1\|^2 .$$  \hfill (11)

If we choose $\delta'$ close enough to $\delta$, the quantity $\left(\frac{1}{\lambda} + \frac{1}{\delta'}\right)$ will be negative, and we end up with

$$\psi(hu_1) < 0 \quad \text{for } h \neq 0 \text{ small}. \hfill (12)$$

On the other hand, we check directly that $\psi(0) = 0$. This shows that 0 cannot be a minimizer of $\psi$, not even a local one. So $\pi \neq 0$ and $\pi \neq A_o^{-1}(0) = 0$. \hfill $\Box$

**Corollary 1.** Assume $H$ is $C^2$ and $(a_\infty, b_\infty)$-subquadratic at infinity. Let $\xi_1, \ldots, \xi_N$ be the equilibria, that is, the solutions of $H'(\xi) = 0$. Denote by $\omega_k$ the smallest eigenvalue of $H''(\xi_k)$, and set:

$$\omega := \text{Min} \{\omega_1, \ldots, \omega_k\} .$$  \hfill (13)

If:

$$\frac{T}{2\pi} b_\infty < -E \left[ -\frac{T}{2\pi} a_\infty \right] < \frac{T}{2\pi} \omega$$  \hfill (14)

then minimization of $\psi$ yields a non-constant $T$-periodic solution $\pi$.\hfill $\Box$

We recall once more that by the integer part $E[\alpha]$ of $\alpha \in \mathbb{R}$, we mean the $a \in \mathbb{Z}$ such that $a < \alpha \leq a + 1$. For instance, if we take $a_\infty = 0$, Corollary 2 tells us that $\pi$ exists and is non-constant provided that:

$$\frac{T}{2\pi} b_\infty < 1 < \frac{T}{2\pi}$$  \hfill (15)

or

$$T \in \left(\frac{2\pi}{\omega}, \frac{2\pi}{b_\infty}\right) .$$  \hfill (16)

**Proof.** The spectrum of $A$ is $\frac{2\pi}{T} \mathbb{Z} + a_\infty$. The largest negative eigenvalue $\lambda$ is given by $\frac{2\pi}{T} k_\alpha + a_\infty$, where

$$\frac{2\pi}{T} k_\alpha + a_\infty < 0 \leq \frac{2\pi}{T}(k_\alpha + 1) + a_\infty .$$  \hfill (17)

Hence:

$$k_\alpha = E \left[ -\frac{T}{2\pi} a_\infty \right] .$$  \hfill (18)

The condition $\gamma < -\lambda < \delta$ now becomes:

$$b_\infty - a_\infty < -\frac{2\pi}{T} k_\alpha - a_\infty < \omega - a_\infty$$  \hfill (19)

which is precisely condition (14). \hfill $\Box$
Lemma 1. Assume that $H$ is $C^2$ on $\mathbb{R}^{2n}\setminus\{0\}$ and that $H''(x)$ is non-degenerate for any $x \neq 0$. Then any local minimizer $\bar{x}$ of $\psi$ has minimal period $T$.

Proof. We know that $\bar{x}$, or $\bar{x} + \xi$ for some constant $\xi \in \mathbb{R}^{2n}$, is a $T$-periodic solution of the Hamiltonian system:

$$\dot{x} = JH'(x).$$

(20)

There is no loss of generality in taking $\xi = 0$. So $\psi(x) \geq \psi(\bar{x})$ for all $\bar{x}$ in some neighbourhood of $x$ in $W^{1,2}(\mathbb{R}/TZ; \mathbb{R}^{2n})$.

But this index is precisely the index $i_T(\bar{x})$ of the $T$-periodic solution $\bar{x}$ over the interval $(0, T)$, as defined in Sect. 2.6. So

$$i_T(\bar{x}) = 0.$$  

(21)

Now if $\bar{x}$ has a lower period, $T/k$ say, we would have, by Corollary 31:

$$i_T(\bar{x}) = i_{kT/k}(\bar{x}) \geq ki_{T/k}(\bar{x}) + k - 1 \geq k - 1 \geq 1.$$  

(22)

This would contradict (21), and thus cannot happen.  

Notes and Comments. The results in this section are a refined version of [1]; the minimality result of Proposition 14 was the first of its kind.

To understand the nontriviality conditions, such as the one in formula (16), one may think of a one-parameter family $x_T$, $T \in (2\pi\omega^{-1}, 2\pi b^{-1})$ of periodic solutions, $x_T(0) = x_T(T)$, with $x_T$ going away to infinity when $T \to 2\pi\omega^{-1}$, which is the period of the linearized system at 0.

Table 1. This is the example table taken out of The T_{Ex}book, p. 246

| Year        | World population |
|-------------|------------------|
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Theorem 1 (Ghoussoub-Preiss). Assume $H(t, x)$ is $(0, \varepsilon)$-subquadratic at infinity for all $\varepsilon > 0$, and $T$-periodic in $t$

$$H(t, \cdot) \quad \text{is convex } \forall t$$  

(23)

$$H(\cdot, x) \quad \text{is } T-\text{periodic } \forall x$$  

(24)

$$H(t, x) \geq n(\|x\|) \quad \text{with } n(s)s^{-1} \to \infty \quad \text{as } s \to \infty$$  

(25)
∀ε > 0 , ∃c : H(t, x) ≤ \frac{ε}{2} \|x\|^2 + c . \quad (26)

Assume also that H is C^2, and H''(t, x) is positive definite everywhere. Then there is a sequence \( x_k, k \in \mathbb{N}, \) of kT-periodic solutions of the system

\[ \dot{x} = JH'(t, x) \quad (27) \]

such that, for every \( k \in \mathbb{N}, \) there is some \( p_o \in \mathbb{N} \) with:

\[ p \geq p_o \Rightarrow x_{pk} \neq x_k . \quad (28) \]

□

Example 1 (External forcing). Consider the system:

\[ \dot{x} = JH'(x) + f(t) \quad (29) \]

where the Hamiltonian \( H \) is \((0, b_\infty)-\)subquadratic, and the forcing term is a distribution on the circle:

\[ f = \frac{d}{dt} F + f_o \quad \text{with} \quad F \in L^2 \left( \mathbb{R}/T \mathbb{Z}; \mathbb{R}^{2n} \right) , \quad (30) \]

where \( f_o := T^{-1} \int_0^T f(t)dt. \) For instance,

\[ f(t) = \sum_{k \in \mathbb{N}} \delta_k \xi , \quad (31) \]

where \( \delta_k \) is the Dirac mass at \( t = k \) and \( \xi \in \mathbb{R}^{2n} \) is a constant, fits the prescription. This means that the system \( \dot{x} = JH'(x) \) is being excited by a series of identical shocks at interval \( T. \)

Definition 1. Let \( A_\infty(t) \) and \( B_\infty(t) \) be symmetric operators in \( \mathbb{R}^{2n}, \) depending continuously on \( t \in [0, T], \) such that \( A_\infty(t) \leq B_\infty(t) \) for all \( t. \)

A Borelian function \( H : [0, T] \times \mathbb{R}^{2n} \to \mathbb{R} \) is called \((A_\infty, B_\infty)-\)subquadratic at infinity if there exists a function \( N(t, x) \) such that:

\[ H(t, x) = \frac{1}{2} (A_\infty(t)x, x) + N(t, x) \quad (32) \]

\[ \forall t , \quad N(t, x) \quad \text{is convex with respect to} \quad x \quad (33) \]

\[ N(t, x) \geq n(\|x\|) \quad \text{with} \quad n(s)s^{-1} \to +\infty \quad \text{as} \quad s \to +\infty \quad (34) \]

\[ \exists c \in \mathbb{R} : \quad H(t, x) \leq \frac{1}{2} (B_\infty(t)x, x) + c \quad \forall x . \quad (35) \]

If \( A_\infty(t) = a_\infty I \) and \( B_\infty(t) = b_\infty I, \) with \( a_\infty \leq b_\infty \in \mathbb{R}, \) we shall say that \( H \) is \((a_\infty, b_\infty)-\)subquadratic at infinity. As an example, the function \( \|x\|^{\alpha}, \) with \( 1 \leq \alpha < 2, \) is \((0, \varepsilon)-\)subquadratic at infinity for every \( \varepsilon > 0. \) Similarly, the Hamiltonian

\[ H(t, x) = \frac{1}{2} k \|k\|^2 + \|x\|^{\alpha} \quad (36) \]

is \((k, k + \varepsilon)-\)subquadratic for every \( \varepsilon > 0. \) Note that, if \( k < 0, \) it is not convex.
Notes and Comments. The first results on subharmonics were obtained by Rabinowitz in [5], who showed the existence of infinitely many subharmonics both in the subquadratic and superquadratic case, with suitable growth conditions on $H'$. Again the duality approach enabled Clarke and Ekeland in [2] to treat the same problem in the convex-subquadratic case, with growth conditions on $H$ only.

Recently, Michalek and Tarantello (see [3] and [4]) have obtained lower bound on the number of subharmonics of period $kT$, based on symmetry considerations and on pinching estimates, as in Sect. 5.2 of this article.

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Hamiltonian Mechanics

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Abstract. The abstract should summarize the contents of the paper using at least 70 and at most 150 words. It will be set in 9-point font size and be inset 1.0 cm from the right and left margins. There will be two blank lines before and after the Abstract.

1 Fixed-Period Problems: The Sublinear Case

With this chapter, the preliminaries are over, and we begin the search for periodic solutions to Hamiltonian systems. All this will be done in the convex case; that is, we shall study the boundary-value problem

\[ \dot{x} = JH'(t, x) \]
\[ x(0) = x(T) \]

with \(H(t, \cdot)\) a convex function of \(x\), going to \(+\infty\) when \(|x| \to \infty\).

1.1 Autonomous Systems

In this section, we will consider the case when the Hamiltonian \(H(x)\) is autonomous. For the sake of simplicity, we shall also assume that it is \(C^1\).

We shall first consider the question of nontriviality, within the general framework of \((A_\infty, B_\infty)\)-subquadratic Hamiltonians. In the second subsection, we shall look into the special case when \(H\) is \((0, b_\infty)\)-subquadratic, and we shall try to derive additional information.

The General Case: Nontriviality. We assume that \(H\) is \((A_\infty, B_\infty)\)-subquadratic at infinity, for some constant symmetric matrices \(A_\infty\) and \(B_\infty\), with \(B_\infty - A_\infty\) positive definite. Set:

\[ \gamma := \text{smallest eigenvalue of } B_\infty - A_\infty \]
\[ \lambda := \text{largest negative eigenvalue of } J \frac{d}{dt} A_\infty \]

Theorem 21 tells us that if \(\lambda + \gamma < 0\), the boundary-value problem:

\[ \dot{x} = JH'(x) \]
\[ x(0) = x(T) \]
has at least one solution $\pi$, which is found by minimizing the dual action functional:

$$\psi(u) = \int_0^T \left[ \frac{1}{2} (A_o^{-1} u, u) + N^*(-u) \right] dt$$

(4)

on the range of $\Lambda$, which is a subspace $R(A)^2_L$ with finite codimension. Here

$$N(x) := H(x) - \frac{1}{2} (A_{\infty} x, x)$$

(5)

is a convex function, and

$$N(x) \leq \frac{1}{2} ((B_{\infty} - A_{\infty}) x, x) + c \quad \forall x.$$  

(6)

**Proposition 1.** Assume $H'(0) = 0$ and $H(0) = 0$. Set:

$$\delta := \liminf_{x \to 0} 2N(x) \|x\|^{-2}.$$  

(7)

If $\gamma < -\lambda < \delta$, the solution $u$ is non-zero:

$$\pi(t) \neq 0 \quad \forall t.$$  

(8)

Proof. Condition (7) means that, for every $\delta' > \delta$, there is some $\varepsilon > 0$ such that

$$\|x\| \leq \varepsilon \Rightarrow N(x) \leq \frac{\delta'}{2} \|x\|^2.$$  

(9)

It is an exercise in convex analysis, into which we shall not go, to show that this implies that there is an $\eta > 0$ such that

$$f \|x\| \leq \eta \Rightarrow N^*(y) \leq \frac{1}{2\delta'} \|y\|^2.$$  

(10)

---

**Fig. 1.** This is the caption of the figure displaying a white eagle and a white horse on a snow field

Since $u_1$ is a smooth function, we will have $\|hu_1\|_\infty \leq \eta$ for $h$ small enough, and inequality (10) will hold, yielding thereby:

$$\psi(hu_1) \leq \frac{h^2}{2} \frac{1}{\lambda} \|u_1\|^2 + \frac{h^2}{2} \frac{1}{\delta'} \|u_1\|^2.$$  

(11)
If we choose $\delta'$ close enough to $\delta$, the quantity $\left(\frac{1}{\lambda} + \frac{1}{\nu} \right)$ will be negative, and we end up with

$$\psi(hu_1) < 0 \quad \text{for } h \neq 0 \text{ small}. \quad (12)$$

On the other hand, we check directly that $\psi(0) = 0$. This shows that 0 cannot be a minimizer of $\psi$, not even a local one. So $\varpi \neq 0$ and $\varpi \neq \Lambda^{-1}(0) = 0$. \hfill \Box

**Corollary 1.** Assume $H$ is $C^2$ and $(a_{\infty}, b_{\infty})$-subquadratic at infinity. Let $\xi_1, \ldots, \xi_N$ be the equilibria, that is, the solutions of $H'(\xi) = 0$. Denote by $\omega_k$ the smallest eigenvalue of $H''(\xi_k)$, and set:

$$\omega := \text{Min} \{\omega_1, \ldots, \omega_k\}. \quad (13)$$

If:

$$\frac{T}{2\pi} b_{\infty} < -E \left[ -\frac{T}{2\pi} a_{\infty} \right] < \frac{T}{2\pi} \omega \quad (14)$$

then minimization of $\psi$ yields a non-constant $T$-periodic solution $\varpi$.

We recall once more that by the integer part $E[\alpha]$ of $\alpha \in \mathbb{R}$, we mean the $a \in \mathbb{Z}$ such that $a < \alpha \leq a + 1$. For instance, if we take $a_{\infty} = 0$, Corollary 2 tells us that $\varpi$ exists and is non-constant provided that:

$$\frac{T}{2\pi} b_{\infty} < 1 < \frac{T}{2\pi} \quad (15)$$

or

$$T \in \left(\frac{2\pi}{\omega}, \frac{2\pi}{b_{\infty}}\right). \quad (16)$$

**Proof.** The spectrum of $\Lambda$ is $\frac{2\pi}{T}\mathbb{Z} + a_{\infty}$. The largest negative eigenvalue $\lambda$ is given by $\frac{2\pi}{T} k_0 + a_{\infty}$, where

$$\frac{2\pi}{T} k_0 + a_{\infty} < 0 \leq \frac{2\pi}{T} (k_0 + 1) + a_{\infty}. \quad (17)$$

Hence:

$$k_0 = E \left[ -\frac{T}{2\pi} a_{\infty} \right]. \quad (18)$$

The condition $\gamma < -\lambda < \delta$ now becomes:

$$b_{\infty} - a_{\infty} < -\frac{2\pi}{T} k_0 - a_{\infty} < \omega - a_{\infty} \quad (19)$$

which is precisely condition (14). \hfill \Box

**Lemma 1.** Assume that $H$ is $C^2$ on $\mathbb{R}^{2n} \setminus \{0\}$ and that $H''(x)$ is non-degenerate for any $x \neq 0$. Then any local minimizer $\tilde{x}$ of $\psi$ has minimal period $T$. 

Proof. We know that $\tilde{x}$, or $\tilde{x} + \xi$ for some constant $\xi \in \mathbb{R}^{2n}$, is a $T$-periodic solution of the Hamiltonian system:

$$\dot{x} = JH'(x). \quad (20)$$

There is no loss of generality in taking $\xi = 0$. So $\psi(x) \geq \psi(\tilde{x})$ for all $\tilde{x}$ in some neighbourhood of $x$ in $W^{1,2}(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n})$.

But this index is precisely the index $i_T(\tilde{x})$ of the $T$-periodic solution $\tilde{x}$ over the interval $(0,T)$, as defined in Sect. 2.6. So

$$i_T(\tilde{x}) = 0. \quad (21)$$

Now if $\tilde{x}$ has a lower period, $T/k$ say, we would have, by Corollary 31:

$$i_T(\tilde{x}) = i_{kT/k}(\tilde{x}) \geq ki_{T/k}(\tilde{x}) + k - 1 \geq k - 1 \geq 1. \quad (22)$$

This would contradict (21), and thus cannot happen. $\square$

Notes and Comments. The results in this section are a refined version of 1980; the minimality result of Proposition 14 was the first of its kind.

To understand the nontriviality conditions, such as the one in formula (16), one may think of a one-parameter family $x_T$, $T \in (2\pi\omega^{-1}, 2\pi b^{-1})$ of periodic solutions, $x_T(0) = x_T(T)$, with $x_T$ going away to infinity when $T \to 2\pi\omega^{-1}$, which is the period of the linearized system at 0.

Table 1. This is the example table taken out of The TeXbook, p. 246

| Year      | World population |
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