Properties of chromatic polynomials of hypergraphs not held for chromatic polynomials of graphs

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Abstract

In this paper, we present some properties on chromatic polynomials of hypergraphs which do not hold for chromatic polynomials of graphs. We first show that chromatic polynomials of hypergraphs have all integers as their zeros and contain dense real zeros in the set of real numbers. We then prove that for any multigraph \( G = (V, E) \), the number of totally cyclic orientations of \( G \) is equal to the value of \( |P(H_G, -1)| \), where \( P(H_G, \lambda) \) is the chromatic polynomial of a hypergraph \( H_G \) which is constructed from \( G \). Finally we show that the multiplicity of root “0” of \( P(H, \lambda) \) may be at least 2 for some connected hypergraphs \( H \), and the multiplicity of root “1” of \( P(H, \lambda) \) may be 1 for some connected and separable hypergraphs \( H \) and may be 2 for some connected and non-separable hypergraphs \( H \).

1 Main results

For any graph \( G = (V, E) \), the chromatic polynomial of \( G \) is the function \( P(G, \lambda) \) such that for any positive integer \( \lambda \), \( P(G, \lambda) \) is the number of proper \( \lambda \)-colourings of \( G \), where a proper \( \lambda \)-colouring of \( G \) is a mapping \( \phi : V \to \{1, 2, \cdots, \lambda\} \) such that \( \phi(u) \neq \phi(v) \) holds for each pair of adjacent vertices \( u \) and \( v \) in \( G \). This graph-function \( P(G, \lambda) \) was originally introduced by Birkhoff [5] in 1912 in the hope of proving the four-colour theorem (i.e., \( P(G, 4) > 0 \) holds for any loopless planar graph \( G \)).

A hypergraph \( \mathcal{H} \) consists of an order pair of vertex set \( \mathcal{V} \) and edge set \( \mathcal{E} \), where \( \mathcal{E} \) is a subset of \( \{e \subseteq \mathcal{V} : |e| \geq 1\} \). If \( |e| \leq 2 \) for all \( e \in \mathcal{E} \), then \( \mathcal{H} \) is a graph.

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For any integer $\lambda \geq 1$, a weak proper $\lambda$-colouring of a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is a mapping $\phi : \mathcal{V} \rightarrow \{1, 2, \ldots, \lambda\}$ such that $|\{\phi(v) : v \in e\}| > 1$ holds for each $e \in \mathcal{E}$ (see [24, 25, 40]). Thus $\mathcal{H}$ does not have any weak proper $\lambda$-colouring if $|e| = 1$ for some edge $e \in \mathcal{E}$. A strong proper $\lambda$-colouring of $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is a mapping $\phi : \mathcal{V} \rightarrow \{1, 2, \ldots, \lambda\}$ such that $|\{\phi(v) : v \in e\}| = |e|$ holds for each $e \in \mathcal{E}$ (see [24, 25, 40] also).

Note that the number of distinct strong proper $\lambda$-colourings in $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is equal to $P(G, \lambda)$, where $G$ is the graph with vertex set $\mathcal{V}$ in which any two vertices $u, v \in \mathcal{V}$ are adjacent if and only if they are contained in an edge $e \in \mathcal{E}$ in $\mathcal{H}$ (i.e., $G$ is obtained from $\mathcal{H}$ by changing each edge in $\mathcal{H}$ to a clique in $G$). Thus the function counting the number of strong proper $\lambda$-colourings in $\mathcal{H}$ is not different from the chromatic polynomial of a graph. It is probably for this reason that most articles on chromatic polynomials of hypergraphs in the past two decades focused on the function counting the number of weak proper $\lambda$-colourings in a hypergraph $\mathcal{H}$ (see [1, 2, 3, 7, 12, 35, 36, 37, 38, 40, 41]).

Let $P(\mathcal{H}, \lambda)$ be the number of weak proper $\lambda$-colourings of $\mathcal{H}$. It is obvious that this graph-function $P(\mathcal{H}, \lambda)$ is an extension of the chromatic polynomial of a graph. In this paper, $P(\mathcal{H}, \lambda)$ is called the chromatic polynomial of $\mathcal{H}$, and it is indeed a polynomial in $\lambda$ of degree $|\mathcal{V}|$. This graph-function $P(\mathcal{H}, \lambda)$ appeared in the work of Helgason [19] in 1972, and it is unknown if it had been introduced earlier. It may have been noticed to be a polynomial by Chvátal [10]. It has been studied extensively in the past twenty years by many researchers, such as Allagan [1, 2, 3], Borowiecki and Lazuka [7], Dohmen [12], Tomescu [35, 36, 37, 38], Voloshin [40] and Walter [41]. They extended many properties of chromatic polynomials of graphs on computations, expressions, factorizations, etc, to chromatic polynomials of hypergraphs.

The following are some known properties on chromatic polynomials of graphs which also hold for chromatic polynomials of hypergraphs:

(A.1). [24] Deletion/Contraction property, cited as Theorem 5;

(A.2). (by definition) multiplicativity with respect to disjoint unions, cited as Proposition 2;

(A.3). [7, 24, 46] the factorization formula for the chromatic polynomial of a graph with a cut-set which induced a clique, cited as Theorem 6;

(A.4). [31] the roots of chromatic polynomials of graphs are dense in the complex plane;

(A.5). [22, 34] (as the family of hypergraphs include all graphs) the real roots of chromatic polynomials of graphs are dense in the interval $[32/27, \infty)$;

(A.6). [12, 44] Whitney’s Broken-cycle Theorem, extended by Dohmen to the set of hypergraphs which do not contain any edge with an odd size and contain edges
of size 2 in every cycle;

(A.7) [26] for fixed $\lambda$, computing $P(H, \lambda)$ is polynomial time computable for classes of graphs (hypergraphs) of bounded tree width. The proof also works for hypergraphs;

(A.8) [6, 13, 12, 28, 29, 35, 44] the chromatic polynomial has integer coefficients and leading coefficient 1. Its degree is equal to the number of vertices in the graph;

(A.9) [6, 13, 12, 28, 29, 35, 44] 0 is a root of every chromatic polynomial and all positive integers are roots of chromatic polynomials.

But chromatic polynomials of graphs also have the following properties on its coefficients not held for chromatic polynomials of hypergraphs:

(B.1) [6, 13, 28, 29, 35, 44] for a graph $G$ of order $n$ and component number $c$, if $P(G, \lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$, then $a_i \neq 0$ if and only if $c \leq i \leq n$;

(B.2) [6, 13, 28, 29, 35, 44] the coefficients $1, a_{n-1}, \ldots, a_c$ in the expansion of $P(G, \lambda)$ alternate in signs;

(B.3) [20, 21, 28, 29, 35] log-concavity of the sequence $1, |a_{n-1}|, \ldots, |a_c|$.

In this article, we will present some properties on chromatic polynomials of hypergraphs which are different from the following properties on chromatic polynomials of graphs:

(i) [22, 34] $(-\infty, 0), (0, 1)$ and $(1, 32/27]$ are zero-free intervals for chromatic polynomials of graphs;

(ii) [32] $|P(G, -1)|$ counts the number of acyclic orientations (i.e., orientations without any directed cycle) of a graph $G$;

(iii) [15, 28] $P(G, \lambda)$ has no factor $\lambda^2$ whenever $G$ is connected;

(iv) [28, 43, 45] $P(G, \lambda)$ has a factor $(\lambda - 1)^2$ whenever $G$ is connected and separable.

For any simple graph $G = (V, E)$ (i.e., it has no loops nor parallel edges), let $H \bullet G$ be the hypergraph with vertex set $V = V \cup \{w\}$ and edge set $E = \{\{u, v, w\} : uv \in E\}$, i.e., $H \bullet G$ is obtained from $G$ by adding a new vertex $w$ and changing each edge $\{u, v\}$ in $G$ to an edge $\{u, v, w\}$ in $H \bullet G$.

The independence polynomial of a graph $G$ is defined to be $I(G, x) = \sum_A x^{|A|}$, where the sum runs over all independent sets $A$ of $G$. One of the main purposes in this article is to establish a relation between $P(H \bullet G, \lambda)$ and $I(G, x)$ in the following theorem.
Theorem 1. For any simple graph $G$ of order $n$,

$$P(H_{\star G}, \lambda) = \lambda(\lambda - 1)^n I(G, 1/(\lambda - 1)).$$  

(1)

Theorem 1 implies that the multiplicity of root “1” in $P(H_{\star G}, \lambda)$ is $n - \alpha(G)$, where $\alpha(G)$ is the independence number of $G$, and whenever $z$ is a zero of $I(G, x)$, $1 + 1/z$ is a zero of $P(H_{\star G}, \lambda)$. If $G$ is the complete graph $K_n$, then $I(G, x) = 1 + nx$ and by Theorem 1, $1 - n$ is a zero of $P(H_{\star G}, \lambda)$.

Brown, Hickman and Nowakowski [8] showed that real roots of independence polynomials are dense in $(-\infty, 0]$ while the complex roots of these polynomials are dense in $\mathbb{C}$ (i.e. the whole complex plane). Chudnovsky and Seymour [9] proved that if $G$ is clawfree, then all the roots of its independence polynomial are real. By Theorem 1 and results in [8, 9], we have the following conclusions immediately except Corollary 1 (b) whose proof will be given in Section 3.

Corollary 1. (a) The complex roots of $P(H_{\star G}, \lambda)$ for all graphs $G$ are dense in the whole complex plane;

(b) The real roots of $P(H_{\star G}, \lambda)$ for all graphs $G$ are dense in the set of real numbers;

(c) Every negative integer is a root of $P(H_{\star G}, \lambda)$ for some graph $G$;

(d) If $G$ is clawfree, then all roots of $P(H_{\star G}, \lambda)$ are real.

For a given multigraph $G = (V, E)$, $H_G = (V, \mathcal{E})$ is another hypergraph constructed from $G$, where $\mathcal{V} = V \cup \{w_e : e \in E\}$ and $\mathcal{E} = \{\{u_e, v_e, w_e\} : e \in E\}$, where $u_e$ and $v_e$ are the two ends of $e$, i.e., $H_G$ is obtained from $G$ by adding $|E|$ new vertices $\{w_e : e \in E\}$ and changing each edge $e$ in $G$ to an edge $\{u_e, v_e, w_e\}$ in $H_G$. We will express $P(H_G, \lambda)$ in terms of the Tutte polynomial $T_G(x, y)$ for $G = (V, E)$, where $T_G(x, y)$ is defined below:

$$T_G(x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)}(y - 1)^{|A| - r(A)},$$

(2)

where $r(A) = |V| - c(A)$ and $c(A)$ is the number of components of the spanning subgraph $(V, A)$ of $G$ for any $A \subseteq E$.

Theorem 2. For any multigraph $G = (V, E)$ with order $n$ and size $m$,

$$P(H_G, \lambda) = \lambda^{m-n+2c(G)} \cdot (-1)^{n+c(G)} \cdot T_G(1 - \lambda^2, (\lambda - 1)/\lambda),$$

(3)

where $c(G)$ is the number of components of $G$, i.e., $c(G) = c(E)$.

Stanley [32] showed that for any multigraph $G$, $|P(G, -1)| = T_G(2, 0)$ counts the number of acyclic orientations of $G$, where an acyclic orientation of $G$ is an orientation of all edges in $G$ such that the digraph obtained does not have any directed cycle. By Theorem 2, the number of totally cyclic orientations of $G$ (i.e., orientations on which each arc is in some cycle) can be determined by the value of $|P(H_G, -1)|$. 
Corollary 2. For any multigraph \( G \), \( |P(H_G, -1)| = T_G(0, 2) \) counts the number of totally cyclic orientations of \( G \).

It is well known that for any connected graph \( G \), \( P(G, \lambda) \) has a factor \( \lambda \) but no factor \( \lambda^2 \). However, \( P(H, \lambda) \) may have a factor \( \lambda^2 \) for a connected hypergraph \( H \).

A hypergraph \( H = (V, E) \) is said to be connected if for any two vertices \( v_1, v_2 \) in \( H \), there exists a sequence of edges \( e_0, e_1, \ldots, e_k \) in \( H \) such that \( v_1 \in e_0 \), \( v_2 \in e_k \) and \( e_i \cap e_{i+1} \neq \emptyset \) holds for all \( i = 0, 1, \ldots, k - 1 \). Now assume that \( H \) is connected. An edge \( e \) in \( H \) is called as a bridge of \( H \) if \( H - e \) (i.e., the hypergraph obtained from \( H \) by removing \( e \)) is disconnected. Let \( B(H) \) be the set of bridges of \( H \). If \( H \) is a graph (i.e., \( |e| = 2 \) for all \( e \in E \)), then the spanning subgraph \( (V, B(H)) \) is connected if and only if \( B(H) = E \) and \( H \) is a tree. However, if \( H \) has some edge \( e \) with \( |e| \geq 3 \), it is possible that \( (V, B(H)) \) is connected while \( B(H) \) is a proper subset of \( E \). Such an example is given in Figure 1 (a).

![Figure 1: Two hypergraphs](image)

Theorem 3. Let \( H = (V, E) \) be any connected hypergraph. If \( B(H) \) is a proper subset of \( E \) and the sub-hypergraph \( (V, B(H)) \) is connected, then \( \lambda^2 \) is a factor of \( P(H, \lambda) \).

It is well known that for a connected graph \( G \), if \( G \) is separable (i.e., \( G \) has a cut-vertex), then \( (\lambda - 1)^2 \) is a factor of \( P(G, \lambda) \). Can this property be extended to hypergraphs?

We first need to make it clear what are separable hypergraphs. A connected hypergraph \( H = (V, E) \) is said to be separable at a vertex \( w \) if the hypergraph \( H - w \) obtained from \( H \) by removing \( w \) and all edges containing \( w \) is disconnected. This definition is a natural extension of the one for separable graphs. Observe that \( H = (V, E) \) is separable at \( w \) if and only if \( V \) has two subsets \( V_1 \) and \( V_2 \) such that \( V_1 \cup V_2 = V \), \( V_1 \cap V_2 = \{w\} \) and for each \( e \in E \), either \( w \in e \) or \( e \subseteq V_i \) for some \( i \in \{1, 2\} \). Note that if \( H \) is a graph, then \( w \in e \) implies that \( e \subseteq V_i \) for some \( i \). But if \( H \) is not a graph, it is possible that \( |e \cap V_i| \geq 2 \) for both \( i \in \{1, 2\} \).
The hypergraph in Figure 1 (b) is connected and separable at \( w \), but its chromatic polynomial is \( \lambda(\lambda-1)(\lambda^2+\lambda-1) \) which does not contain a factor \( (\lambda-1)^2 \). Actually this hypergraph is contained in a family of connected and separable hypergraphs whose chromatic polynomials have no factor \( (\lambda - 1)^2 \). For any \( H = (V, E) \), let \( F(H) \) be the set of vertices \( w \in V \) such that \( w \in e \) for every \( e \in E \). Observe that \( w \in F(H) \) if and only if \( e \not\subseteq V \setminus \{ w \} \) for every \( e \in E \). If \( H \) does not have parallel edges (i.e., edges \( e_1, e_2 \) with \( e_1 = e_2 \)), then \( F(H) = V \) if and only if \( H \) is connected and \( |E| = 1 \). It is trivial that if \( |E| = 1 \), then \( P(H, \lambda) \) does not have a factor \( (\lambda - 1)^2 \). We will show that if \( |E| \geq 2 \) and \( F(H) \neq \emptyset \), then \( P(H, \lambda) \) has a factor \( (\lambda - 1)^2 \) if and only if \( |F(H)| = 1 \).

For a hypergraph \( H = (V, E) \) and any \( V_0 \subseteq V \), let \( H \cdot V_0 \) denote the hypergraph obtained from \( H \) by identifying all vertices in \( V_0 \) as one vertex and let \( H[V_0] \) be the hypergraph with vertex set \( V_0 \) and edge set \( \{ e \in E : e \subseteq V_0 \} \). We call \( H[V_0] \) the sub-hypergraph of \( H \) induced by \( V_0 \). Let \( H - V_0 \) be the induced sub-hypergraph \( H[V - V_0] \). A hypergraph is said to be empty if it contains no edges. Let \( I(H) \) be the set of those subsets \( V_0 \) of \( V \) such that \( H[V_0] \) is an empty graph. A hypergraph \( H \) is said to be Sperner if \( e_1 \not\subseteq e_2 \) for each pair of edges \( e_1, e_2 \) in \( H \).

For the case that \( F(H) = \emptyset \) and \( H \) is separable, we also give an equivalent statement for \( P(H, \lambda) \) to have a factor \( (\lambda - 1)^2 \).

**Theorem 4.** Let \( H = (V, E) \) be any connected and Sperner hypergraph with \( |E| \geq 2 \).

(i) If \( F(H) \neq \emptyset \), then \( P(H, \lambda) \) has a factor \( (\lambda - 1)^2 \) if and only if \( |F(H)| = 1 \);

(ii) If \( F(H) = \emptyset \) and \( H \) has a vertex \( w \) and two proper subsets \( V_1 \) and \( V_2 \) of \( V \) such that \( V_1 \cup V_2 = V \), \( V_1 \cap V_2 = \{ w \} \) and for each \( e \in E \), either \( w \in e \) or \( e \subseteq V_i \) for some \( i \), then \( P(H, \lambda) \) has a factor \( (\lambda - 1)^2 \) if and only if one of the following conditions is satisfied:

(a) \( V_i \notin I(H) \) for both \( i = 1, 2 \);

(b) for some \( i \in \{1, 2\} \), \( V_i \in I(H) \), \( V_{3-i} \notin I(H) \) and \( P(H \cdot V_i, \lambda) \) has a factor \( (\lambda - 1)^2 \).

We will prove Theorems 1-4 in Sections 3-5 after some fundamental results are introduced in Section 2. We will also propose some open problems regarding multiplicities of roots “0” and “1” of \( P(H, \lambda) \) for a hypergraph \( H \).

## 2 Preliminary

In this section, we present several known results on chromatic polynomials of hypergraphs, which will be applied later. The first one follows directly from the definition of weak proper colourings of a hypergraph.
Proposition 1. Let $e_1, e_2$ be any two edges in a hypergraph $H$. If $e_1 \subseteq e_2$, then

$$P(H, \lambda) = P(H - e_2, \lambda),$$

where $H - e_2$ is the hypergraph obtained from $H$ by removing $e_2$.

By Proposition 1, we need only to consider Sperner hypergraphs in the function $P(H, \lambda)$.

For a hypergraph $H = (V, E)$, a component of $H$ is an induced and connected subgraph $H[V_0]$ such that $H[V_0 \cup \{v\}]$ is disconnected for any $v \in V - V_0$. By the definition of $P(H, \lambda)$, we have the following expression for $P(H, \lambda)$ when $H$ is disconnected.

Proposition 2. Assume that $H_1, \cdots, H_k$ are components of $H$. Then

$$P(H, \lambda) = \prod_{1 \leq i \leq k} P(H_i, \lambda).$$

For any hypergraph $H = (V, E)$ and $V_0 \subset V$, recall that $H \cdot V_0$ is obtained from $H$ by identifying all vertices in $V_0$ as one, i.e., $H \cdot V_0$ is the hypergraph with vertex set $(V - V_0) \cup \{w\}$ and edge set

$$\{e' \in E : e' \cap V_0 = \emptyset\} \cup \{(e' - V_0) \cup \{w\} : e' \cap V_0 \neq \emptyset\},$$

where $w \notin V$. For an edge $e$ in $H$, let $H/e$ be the hypergraph $(H - e) \cdot e$. This hypergraph $H/e$ is said to be obtained from $H$ by contracting $e$.

The deletion-contraction formula for chromatic polynomials of graphs is very important for the study of this polynomial. It was extended to chromatic polynomials of hypergraphs by Jones [24].

Theorem 5 ([24]). Let $H = (V, E)$ be a hypergraph. For any $e \in E$,

$$P(H, \lambda) = P(H - e, \lambda) - P(H/e, \lambda).$$

(4)

Note that Theorem 5 can be equivalently stated below: for any subset $e$ of $V$,

$$P(H, \lambda) = P(H + e, \lambda) + P(H \cdot e, \lambda),$$

(5)

where $H + e$ is the hypergraph obtained from $H$ by adding a new edge $e$.

A hypergraph $H = (V, E)$ is written as $H_1 \cup H_2$, where $H_i = (V_i, E_i)$ is a hypergraph for $i = 1, 2$, if $V = V_1 \cup V_2$, $E = E_1 \cup E_2$ and for any $e \subseteq V_1 \cap V_2$, $e \in E_1$ if and only if $e \in E_2$. If $\{u, v\} \in E_1 \cap E_2$ for each pair $\{u, v\} \subseteq V_1 \cap V_2$, then write $H_1 \cap H_2 = K_p$, where $p = |E_1 \cap E_2|$. Borowiecki and Lazuka [7] extended Zyklov’s result [46] on the chromatic polynomial of the graph $G_1 \cup G_2$ with $G_1 \cap G_2 \cong K_p$.

Theorem 6 ([7]). If $H = H_1 \cup H_2$ and $H_1 \cap H_2 = K_p$, then

$$P(H, \lambda) = \frac{P(H_1, \lambda)P(H_2, \lambda)}{P(K_p, \lambda)}.$$  

(6)
3 Proof of Theorem 1 and Corollary 1 (b)

Let $G = (V, E)$ be a simple graph, i.e., $G$ has neither parallel edges nor loops. For $v \in V$, let $N_G(v)$ (or simply $N(v)$) be the set $\{u \in V : uv \in E\}$, and let $N[v] = N(v) \cup \{v\}$. The degree of $v$ in $G$, denoted by $d(v)$, is the size $|N(v)|$ of $N(v)$.

Proposition 3. For any vertex $v \in V$,

$$P(H \ast G, \lambda) = (\lambda - 1) \cdot P(H \ast G - \{v\}, \lambda) + (\lambda - 1)^{d(v)} \cdot P(H \ast G - N[v], \lambda).$$

(7)

Proof. Let $w$ be the new vertex in $H \ast G$ when it is produced from $G$. By (5),

$$P(H \ast G, \lambda) = P(H \ast G + e, \lambda) + P(H \ast G \cdot e, \lambda),$$

(8)

where $e = \{w, v\}$.

Observe that $e \subset \{w, v, v_i\}$ for all $v_i \in N(v)$. By Proposition 1,

$$P(H \ast G + e, \lambda) = P(H \ast G + e - \mathcal{E}(v), \lambda),$$

where $\mathcal{E}(v) = \{\{w, v, v_i\} : v_i \in N(v)\}$. By Theorem 6,

$$P(H \ast G + e - \mathcal{E}(v), \lambda) = (\lambda - 1) \cdot P(H \ast G - \{v\}, \lambda) = (\lambda - 1) \cdot P(H \ast G - \{v\}, \lambda).$$

(9)

Note that the edges $\{w, v, v_i\}$ in $H \ast G$, where $v_i \in N(v)$, are changed to $\{w, v_i\}$ in $H \ast G \cdot e$, and thus all edges $\{w, v_i, u\}$ in $H \ast G$, where $u \in N(v_i) - \{v\}$, can be removed by Proposition 1. By Theorem 6 again,

$$P(H \ast G \cdot e, \lambda) = (\lambda - 1)^{d(v)} \cdot P(H \ast G - N[v], \lambda).$$

(10)

Hence the result follows from (8), (9) and (10).

The following property on the independence polynomial of a graph is needed for proving Theorem 1.

Proposition 4 ([9]). Let $G = (V, E)$ be any simple graph and $v \in V$. Then $I(G, x) = I(G - \{v\}, x) + xI(G - N[v], x)$.

Now we are ready to prove Theorem 1 by applying Propositions 3 and 4.

Proof of Theorem 1: Suppose that the result fails. Assume that $G = (V, E)$ is a simple graph for which the result fails and $|V| + |E|$ has the minimum value among all those graphs for which the result fails. We shall complete the proof by showing the following claims.
Claim 1: \( n = |V| \geq 2 \).

Assume that \( n = |V| = 1 \). Then \( E = \emptyset \) as \( G \) is simple. Thus \( \mathcal{H}_G \) is the hypergraph with two vertices and no edges, implying that \( P(\mathcal{H}_G, \lambda) = \lambda^2 \). Observe that the right-hand side of (1) is

\[
\lambda(\lambda - 1)(1 + 1/(\lambda - 1)) = \lambda^2,
\]

implying that Theorem 1 holds for this graph, a contradiction.

Claim 2: \( G \) does not exist.

Let \( v \) be any vertex of \( G \), by Proposition 3, we have

\[
P(\mathcal{H}_G, \lambda) = (\lambda - 1) \cdot P(\mathcal{H}_{(G \setminus \{v\})}, \lambda) + (\lambda - 1)^{d(v)} \cdot P(\mathcal{H}_{(G \setminus N[v])}, \lambda). \tag{11}
\]

By the assumption on \( G \), Theorem 1 holds for both \( G \setminus \{v\} \) and \( G \setminus N[v] \). Thus

\[
P(\mathcal{H}_{(G \setminus \{v\})}, \lambda) = \lambda \cdot (\lambda - 1)^{n-1} \cdot I(G \setminus v, \frac{1}{\lambda - 1}) \tag{12}
\]

and

\[
P(\mathcal{H}_{(G \setminus N[v])}, \lambda) = \lambda \cdot (\lambda - 1)^{n-d(v)-1} \cdot I(G \setminus N[v], \frac{1}{\lambda - 1}). \tag{13}
\]

From Proposition 4 and equalities (11), (12) and (13), we obtain

\[
P(\mathcal{H}_G, \lambda) = \lambda(\lambda - 1)^n \cdot I(G \setminus \{v\}, \frac{1}{\lambda - 1}) + \lambda \cdot (\lambda - 1)^{n-1} \cdot I(G \setminus N[v], \frac{1}{\lambda - 1}) \]

\[
= \lambda \cdot (\lambda - 1)^n \cdot I(G, \frac{1}{\lambda - 1}).
\]

Thus equality (1) holds for \( G \), a contradiction. Therefore Claim 2 is proved and Theorem 1 holds. \( \square \)

We end this section by providing a proof of Corollary 1 (b).

Proof of Corollary 1 (b). It has been shown in [8] that the real roots of independence polynomials are dense in the interval \((-\infty, 0]\). Then Theorem 1 implies that the real roots of chromatic polynomials of hypergraphs \( \mathcal{H}_G \) for all graphs \( G \) are dense in the interval \((-\infty, 1]\). By Corollary 4, which follows from Proposition 8 in Section 5 directly, the real roots of the chromatic polynomials of hypergraphs \( \mathcal{H}_G + K_1 \) for all graphs \( G \) are dense in the interval \((-\infty, 2]\), where \( \mathcal{H}_G + K_1 \) is the hypergraph obtained from \( \mathcal{H}_G \) by adding a new vertex \( u \) and adding new edges \( \{u, v\} \) for all vertices \( v \) in \( \mathcal{H}_G \). Repeating this process or applying the fact that the real roots of chromatic polynomials of graphs are dense in \([2, \infty)\), the result of Corollary 1 (b) holds. \( \square \)
4 Proof of Theorem 2

Let $G = (V, E)$ be a graph of order $n$ and size $m$. Assume that $G$ may have loops or parallel edges. We first establish the following recursive formula for $P(\mathcal{H}_G, \lambda)$.

**Proposition 5.** For any $e \in E(G)$,

$$P(\mathcal{H}_G, \lambda) = \lambda P(\mathcal{H}_{G-e}, \lambda) - P(\mathcal{H}_{G/e}, \lambda).$$

(14)

**Proof.** Assume that $u$ and $v$ are the two ends of $e$ in $G$. It is possible that $u = v$, as $e$ may be a loop. Let $e' = \{u, v, e\}$. So $e'$ is the edge in $\mathcal{H}_G$ corresponding to $e$. By Theorem 5, we have

$$P(\mathcal{H}_G, \lambda) = P(\mathcal{H}_{G-e'}, \lambda) - P(\mathcal{H}_{G/e'}, \lambda)$$

(15)

Note that $\mathcal{H}_{G-e'}$ consists of an isolated vertex and the hypergraph $\mathcal{H}_{G-e}$, and $\mathcal{H}_{G/e'}$ is actually the hypergraph $\mathcal{H}_{G/e}$. Thus the result holds.

Listed below are some other properties of $P(\mathcal{H}_G, \lambda)$ which can be proved easily by applying Theorem 6 and Proposition 5.

**Proposition 6.** Let $G$ be a multigraph of order $n$.

(a) If $G$ is an empty graph, then $P(\mathcal{H}_G, \lambda) = \lambda^n$;

(b) If $e$ is a loop of $G$, then $P(\mathcal{H}_G, \lambda) = (\lambda - 1)P(\mathcal{H}_{G-e}, \lambda)$;

(c) If $e$ is a bridge of $G$, then $P(\mathcal{H}_G, \lambda) = (\lambda^2 - 1)P(\mathcal{H}_{G/e}, \lambda)$.

Some fundamental properties on Tutte polynomials $T_G(x, y)$ are needed for proving Theorem 2.

**Proposition 7 ([16, 42]).** Let $G$ be a multigraph.

(a) If $G$ is an empty graph, then $T_G(x, y) = 1$;

(b) If $e$ is a loop of $G$, then $T_G(x, y) = y \cdot T_{G-e}(x, y)$;

(c) If $e$ is a bridge of $G$, then $T_G(x, y) = x \cdot T_{G/e}(x, y)$;

(d) For any $e \in E(G)$, if $e$ is neither a loop nor a bridge, then $T_G(x, y) = T_{G-e}(x, y) + T_{G/e}(x, y)$. 

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We are now ready to prove Theorem 2.

**Proof of Theorem 2:** We will prove the result by induction on the size $m$ of $G$.

If $m = 0$, Theorem 2 holds for $G$ by Propositions 6 (a) and 7 (a).

Assume that Theorem 2 holds for any graph of size less than $m$, where $m > 0$. Now we assume that $G = (V, E)$ is a graph of size $m$. Let $e$ be any edge in $G$.

**Case 1:** $e$ is a loop.

By Propositions 6 (b) and 7 (b),

$$ P(H_G, \lambda) = (\lambda - 1) P(H_{G-e}, \lambda), \quad T_{G}(x, y) = yT_{G-e}(x, y). \quad (16) $$

By the inductive assumption, Theorem 2 holds for $G - e$, i.e.,

$$ P(H_{G-e}, \lambda) = \lambda^{m-1-n+2c(G)} \cdot (-1)^{n+c(G)} \cdot T_{G-e}(1 - \lambda^2, (\lambda - 1)/\lambda). \quad (17) $$

Thus Theorem 2 holds for $G$ by equalities in (16) and (17).

**Case 2:** $e$ is a bridge.

By Propositions 6 (c) and 7 (c),

$$ P(H_G, \lambda) = (\lambda^2 - 1) P(H_{G/e}, \lambda), \quad T_{G}(x, y) = xT_{G/e}(x, y). \quad (18) $$

By the inductive assumption, Theorem 2 holds for $G/e$, i.e.,

$$ P(H_{G/e}, \lambda) = \lambda^{m-n+2c(G)} \cdot (-1)^{n-1+c(G)} \cdot T_{G/e}(1 - \lambda^2, (\lambda - 1)/\lambda). \quad (19) $$

Thus Theorem 2 holds for $G$ by equalities in (18) and (19).

**Case 3:** $e$ is neither a bridge nor a loop.

Then $c(G-e) = c(G/e) = c(G)$. By inductive assumption, Theorem 2 holds for $G - e$ and $G/e$, i.e.,

$$ P(H_{G-e}, \lambda) = \lambda^{m-1-n+2c(G)} \cdot (-1)^{n+c(G)} \cdot T_{G-e}(1 - \lambda^2, (\lambda - 1)/\lambda) \quad (20) $$

and

$$ P(H_{G/e}, \lambda) = \lambda^{m-n+2c(G)} \cdot (-1)^{n-1+c(G)} \cdot T_{G/e}(1 - \lambda^2, (\lambda - 1)/\lambda). \quad (21) $$

By (20), (21) and Proposition 7 (d), it can be verified that Theorem 2 holds for $G$. $\Box$
5 Proofs of Theorems 3 and 4

In this section, we will complete the proofs of Theorems 3 and 4.

Proof of Theorem 3: By a result due to Tomescu [35], the coefficient of $\lambda$ in $P(H, \lambda)$ is equal to

$$a_1 = \sum_j (-1)^j N_j$$

where $N_j$ is the number of connected and spanning sub-hypergraphs of $H$ with exactly $j$ edges. By the given conditions, any sub-hypergraph $(V, E')$ of $H$ is connected if and only if $B(H) \subseteq E'$. Assume that $r = |B(H)|$ and $k = |E| - |B(H)| > 0$. Then

$$N_j = \binom{k}{j-r}$$

and

$$a_1 = \sum_{j=r}^{r+k} (-1)^j \binom{k}{j-r} = 0.$$ 

Thus the result holds.

Some results are needed for proving Theorem 4. Let $\Phi(H)$ be the set of those partitions $\{V_1, \ldots, V_k\}$ of $V$ such that each $V_i$ is an non-empty member in $I(H)$. By the definition of $P(H, \lambda)$,

$$P(H, \lambda) = \sum_{\{V_1, \ldots, V_k\} \in \Phi(H)} (\lambda)_k, \tag{22}$$

where $(x)_0 = 1$ and $(x)_k = x(x-1) \cdots (x-k+1)$ for any number $x$ and positive integer $k$. By (22), we can deduce the following result.

**Proposition 8.** Let $w$ be a fixed vertex in $H = (V, E)$. Then

$$P(H, \lambda) = \lambda \sum_{w \in V_0 \in I(H)} P(H - V_0, \lambda - 1). \tag{23}$$

**Proof.** By (22), we can assume that

$$P(H, \lambda) = \sum_{\{V_0, V_1, \ldots, V_k\} \in \Phi(H)} (\lambda)_{k+1}. \tag{24}$$

For any $V_0 \in I(H)$ with $w \in V_0$, $\{V_0, V_1, \ldots, V_k\} \in \Phi(H)$ if and only if $\{V_1, \ldots, V_k\} \in \Phi(H - V_0)$. Thus

$$P(H, \lambda) = \lambda \sum_{w \in V_0 \in I(H)} \sum_{\{V_1, \ldots, V_k\} \in \Phi(H - V_0)} (\lambda - 1)_k. \tag{25}$$

By (22) again, the result holds.
By Proposition 8, \( \lambda \) is a factor of \( P(\mathcal{H}, \lambda) \) for any hypergraph \( \mathcal{H} \). Actually \( \lambda - 1 \) is also a factor of \( P(\mathcal{H}, \lambda) \) whenever \( \mathcal{H} \) is not an empty graph.

**Corollary 3.** For any hypergraph \( \mathcal{H} = (V, E) \), \( \lambda^{c(\mathcal{H})} \) and \( (\lambda - 1)^{c'(\mathcal{H})} \) are factors of \( P(\mathcal{H}, \lambda) \), where \( c(\mathcal{H}) \) is the number of components of \( \mathcal{H} \) and \( c'(\mathcal{H}) \) is the number of those components of \( \mathcal{H} \) which contain edges.

**Proof.** We just prove that \( (\lambda - 1)^{c'(\mathcal{H})} \) is a factor of \( P(\mathcal{H}, \lambda) \). It suffices to show that if \( \mathcal{H} \) is connected and non-empty, then \( \lambda - 1 \) is a factor of \( P(\mathcal{H}, \lambda) \). As \( \mathcal{H} \) is not empty, \( V \notin I(\mathcal{H}) \). Thus \( P(\mathcal{H} - V_0, \lambda) \) has a factor \( \lambda \) for every \( V_0 \in I(\mathcal{H}) \). By Proposition 8, \( \lambda - 1 \) is a factor of \( P(\mathcal{H}, \lambda) \).

Recall that for a hypergraph \( \mathcal{H} \), \( \mathcal{H} + K_1 \) is the hypergraph obtained from \( \mathcal{H} \) by adding a new vertex \( u \) and adding new edges \( \{u, v\} \) for all vertices \( v \) in \( \mathcal{H} \). By Proposition 8, the following result is obtained.

**Corollary 4.** For any hypergraph \( \mathcal{H} \), \( P(\mathcal{H} + K_1, \lambda) = \lambda P(\mathcal{H}, \lambda - 1) \).

By Corollary 4, \( P(\mathcal{H} + K_1, \lambda) \) has a factor \( (\lambda - 1)^2 \) if and only if \( P(\mathcal{H}, \lambda) \) has a factor \( \lambda^2 \). If \( \mathcal{H} \) is connected, then \( \mathcal{H} + K_1 \) is not separable. By Theorem 3, there exist non-separable hypergraphs whose chromatic polynomials have a factor \( (\lambda - 1)^2 \).

Now we establish two important results for proving Theorem 4.

**Proposition 9.** Let \( \mathcal{H} = (V, E) \) be any connected hypergraph with a vertex \( w \) and two proper subsets \( V_1 \) and \( V_2 \) of \( V \) such that \( V_1 \cup V_2 = V \), \( V_1 \cap V_2 = \{w\} \) and for each \( e \in E \), either \( w \in e \) or \( e \subseteq V_i \) for some \( i \). Then \( (\lambda - 1)^2 \) is a factor of \( P(\mathcal{H}, \lambda) \) if and only if \( \lambda^2 \) is a factor of the following polynomial:

\[
\sum_{V_0 \in I(V_1, V_2)(\mathcal{H})} P(\mathcal{H} - V_0, \lambda), \tag{26}
\]

where \( I(V_1, V_2)(\mathcal{H}) \) is the set of those \( V_0 \in I(\mathcal{H}) \) with \( V_i \subseteq V_0 \) for some \( i \in \{1, 2\} \).

**Proof.** By Proposition 8,

\[
P(\mathcal{H}, \lambda) = \lambda \sum_{V_0 \in I(V_1, V_2)} P(\mathcal{H} - V_0, \lambda - 1) + \lambda \sum_{w \in V_0 \in I(\mathcal{H}) \setminus I(V_1, V_2)} P(\mathcal{H} - V_0, \lambda - 1). \tag{27}
\]

For any \( V_0 \in I(\mathcal{H}) \) with \( w \in V_0 \), if \( V_0 \notin I(V_1, V_2)(\mathcal{H}) \), then \( \mathcal{H} - V_0 \) is disconnected and Corollary 3 implies that \( \lambda^2 \) is a factor of \( P(\mathcal{H} - V_0, \lambda) \) and thus \( (\lambda - 1)^2 \) is a factor of \( P(\mathcal{H} - V_0, \lambda - 1) \). Hence the result follows from (27).  \( \square \)
Let \( \mathcal{I}_{(V_1, V_2)}(H) \) be the set of those members \( V_0 \) in \( \mathcal{I}_{(V_1, V_2)}(H) \) such that \( H - V_0 \) is connected. For each \( V_0 \in \mathcal{I}_{(V_1, V_2)}(H) - \mathcal{I}_{(V_1, V_2)}(H), \) \( H - V_0 \) is disconnected and thus \( P(H - V_0, \lambda) \) has a factor \( \lambda^2 \) by Corollary 3. By Proposition 9, we get the following result.

**Corollary 5.** Let \( H = (V, E) \) be any connected hypergraph with a vertex \( w \) and two proper subsets \( V_1 \) and \( V_2 \) of \( V \) such that \( V_1 \cup V_2 = V, \) \( V_1 \cap V_2 = \{w\} \) and for each \( e \in E, \) either \( w \in e \) or \( e \subseteq V_i \) for some \( i. \) Then \( (\lambda - 1)^2 \) is a factor of \( P(H, \lambda) \) if and only if \( \lambda^2 \) is a factor of \( P(H - V_0, \lambda) \).

\[
\sum_{V_0 \in \mathcal{I}_{(V_1, V_2)}(H)} P(H - V_0, \lambda). \quad (28)
\]

We are now going to prove Theorem 4.

**Proof of Theorem 4:** (i) First assume that \( F(H) = \{w\}. \) As \( |E| \geq 2 \) and \( H \) is Sperner, \( |V| \geq 3. \) Then \( H \) is separable at \( w. \) Assume that \( V_1 \) and \( V_2 \) are proper subsets of \( V \) such that \( V_1 \cap V_2 = \{w\} \) and \( V_1 \cup V_2 = V. \) As \( w \) is the only member in \( F(H), \) \( V - \{u\} \notin \mathcal{I}(H) \) for every \( u \in V - \{w\}. \) Thus, for each \( V_0 \in \mathcal{I}(H) \) with \( w \in V_0, \) we have \( |V_0| \leq |V| - 2 \) and so \( H - V_0 \) is an empty graph of order at least 2, implying that \( \lambda^2 \) is a factor of \( P(H - V_0, \lambda). \) By Proposition 8 or Proposition 9, \( (\lambda - 1)^2 \) is a factor of \( P(H, \lambda). \)

Now consider the case that \( k = |F(H)| \geq 2. \) By (5), we have

\[
P(H, \lambda) = P(H + F(H), \lambda) + P(H \cdot F(H), \lambda). \quad (29)
\]

As \( F(H) \subseteq e \) for each \( e \in E, \) Proposition 1 implies that

\[
P(H + F(H), \lambda) = P(H_0, \lambda) = \lambda^{|V| - k} (\lambda^k - \lambda), \quad (30)
\]

where \( H_0 \) is the hypergraph with vertex set \( V \) and edge set \( \{F(H)\}. \) By (29) and (30),

\[
P(H, \lambda) = \lambda^{|V| - k} (\lambda^k - \lambda) + P(H \cdot F(H), \lambda). \quad (31)
\]

Observe that \( H \cdot F(H) \) is Sperner, connected and has as many edges as \( H. \) As \( H \cdot F(H) \) has at least two edges and \( |F(H) \cdot F(H)| = 1, \) \( P(H \cdot F(H), \lambda) \) has a factor \( (\lambda - 1)^2 \) by the result proved above. Since \( \lambda^{|V| - k} (\lambda^k - \lambda) \) does not have a factor \( (\lambda - 1)^2, \) (31) implies that \( P(H, \lambda) \) does not have a factor \( (\lambda - 1)^2. \)

(ii) As \( F(H) = \emptyset, \) it is impossible that \( V_i \in \mathcal{I}(H) \) for both \( i = 1, 2. \) By Proposition 9, if \( V_i \notin \mathcal{I}(H) \) for both \( i = 1, 2, \) then \( (\lambda - 1)^2 \) is a factor of \( P(H, \lambda). \)

Now we assume that \( V_1 \in \mathcal{I}(H) \) but \( V_2 \notin \mathcal{I}(H). \) By Proposition 9, \( (\lambda - 1)^2 \) is a factor of \( P(H, \lambda) \) if and only if \( \lambda^2 \) is a factor of the following polynomial:

\[
\sum_{V_0 \in \mathcal{I}_{V_1}(H)} P(H - V_0, \lambda). \quad (32)
\]
where $\mathcal{I}_V(\mathcal{H})$ is the set of those $V_0 \in \mathcal{I}(\mathcal{H})$ with $V_1 \subset V_0$. Observe that

$$\sum_{V_0 \in \mathcal{I}_V(\mathcal{H})} P(\mathcal{H} - V_0, \lambda) = \sum_{V_1 \cup V' \in \mathcal{I}(\mathcal{H}) \setminus V' \subseteq V_2 - \{w\}} P(\mathcal{H} - (V_1 \cup V'), \lambda).$$

Let $\mathcal{H}_0$ denote the hypergraph $\mathcal{H} \cdot V_1$ and let $w_0$ denote the vertex in $\mathcal{H}_0$ which is produced after identifying all vertices $V_1$ as one. Thus the vertex set of $\mathcal{H}_0$ is $(V_2 - \{w\}) \cup \{w_0\}$. Observe that for any $V' \subseteq V_2 - \{w\}$, $V_1 \cup V' \in \mathcal{I}(\mathcal{H})$ if and only if $\{w_0\} \cup V' \in \mathcal{I}(\mathcal{H}_0)$, and $\mathcal{H} - (V_1 \cup V')$ is exactly the hypergraph $\mathcal{H}_0 - (\{w_0\} \cup V')$. Thus, by (33),

$$\sum_{V_0 \in \mathcal{I}_V(\mathcal{H})} P(\mathcal{H} - V_0, \lambda) = \sum_{V' \cup \{w_0\} \in \mathcal{I}(\mathcal{H}_0)} P(\mathcal{H}_0 - (V' \cup \{w_0\}), \lambda).$$

By Proposition 8, the right-hand side of (34) has a factor of $\lambda^2$ if and only if $P(\mathcal{H}_0, \lambda)$ has a factor $(\lambda - 1)^2$.

Hence (ii) holds. \qed

For any graph $G$, if $G$ has two edges $e_1, e_2$ which have no any common end, then $\mathcal{H}_{\bullet G}$ is separable at vertex $w$ which is the vertex not in $G$ and $\mathcal{I}(V_1, V_2)(\mathcal{H}_{\bullet G}) = \emptyset$ for suitable $V_1, V_2$ with $e_i \subseteq V_i$ for $i = 1, 2$, implying that $P(\mathcal{H}_{\bullet G}, \lambda)$ has a factor $(\lambda - 1)^2$ by Proposition 9.

By Theorem 4, we can easily get examples of separable hypergraphs $\mathcal{H}$ whose chromatic polynomials don’t have a factor $(\lambda - 1)^2$. Let $\mathcal{H}$ be a hypergraph with vertex set $\{w\} \cup \{x_i : 1 \leq i \leq s\} \cup \{y_j : 1 \leq j \leq t\}$ and edge set

$$\{\{y_j : 1 \leq j \leq t\} \cup \{w, x_1, x_2, \ldots, x_s, y_j\} : 1 \leq j \leq t\},$$

where $s \geq 1$ and $t \geq 1$. Then $V_1 = \{w\} \cup \{x_i : 1 \leq i \leq s\}$ is a member of $\mathcal{I}(\mathcal{H})$ while $V_2 = \{w\} \cup \{y_j : 1 \leq j \leq t\}$ is not. By Theorem 4, $P(\mathcal{H}, \lambda)$ has a factor $(\lambda - 1)^2$ if and only if $P(\mathcal{H} \cdot V_1, \lambda)$ has a factor $(\lambda - 1)^2$. Observe that $\mathcal{H} \cdot V_1$ is a hypergraph with vertex set $\{w\} \cup \{y_j : 1 \leq j \leq t\}$ and edge set

$$\{\{y_j : 1 \leq j \leq t\} \cup \{w, y_j\} : 1 \leq j \leq t\}.$$  

By Corollary 4,

$$P(\mathcal{H} \cdot V_1, \lambda) = \lambda P(\mathcal{H} - V_1, \lambda - 1) = \lambda((\lambda - 1)^t - (\lambda - 1)),$$

which does not have a factor $(\lambda - 1)^2$.

Theorem 4 actually provides a method of producing a separable hypergraph $\mathcal{H}$ from a given hypergraph $\mathcal{H}'$ such that $P(\mathcal{H}, \lambda)$ has a factor $(\lambda - 1)^2$ if and only if $P(\mathcal{H}', \lambda)$ has a factor $(\lambda - 1)^2$. For any connected and Sperner hypergraph $\mathcal{H}'$ with at least
two edges and any vertex \( w \) in \( H' \), let \( H \) be a connected hypergraph with vertex set \( V(H') \cup S \), where \( S \) is a non-empty set, and edge set

\[
(E(H') - E') \cup \{e \cup S_e : e \in E', S_e \subseteq S\},
\]

where \( E' \) is a set of some edges \( e \in E(H') \) with \( w \in e \) and \( S_e \) is any subset of \( S \). As \( H \) must be connected, \( \bigcup_{e \in E'} S_e = S \). By Theorem 4, \( P(H, \lambda) \) has a factor \((\lambda - 1)^2\) if and only if \( P(H', \lambda) \) has a factor \((\lambda - 1)^2\).

6 Conclusions and further research

It is well known that for any simple graph \( G \), the multiplicity of root “0” of \( P(G, \lambda) \) is equal to the number of components of \( G \) ([15, 28]). However, Theorem 3 shows that for a connected hypergraph \( H \), \( P(H, \lambda) \) may have a factor \( \lambda^2 \), implying that the multiplicity of root “0” of the chromatic polynomial of a hypergraph can be as large as twice the number of components of this hypergraph. But we don’t know what is the largest possible multiplicity of root “0” of \( P(H, \lambda) \) and the relation between the multiplicity of root “0” of \( P(H, \lambda) \) and the structure of \( H \). Thus we propose the following problem regarding the multiplicity of root “0” of \( P(H, \lambda) \) for a hypergraph \( H \).

**Problem 1.** Let \( H \) be a connected hypergraph.

(a) What is a necessary and sufficient condition for \( P(H, \lambda) \) to have a factor \( \lambda^2 \)?

(b) Is it possible that the multiplicity of root “0” of \( P(H, \lambda) \) is larger than 2 for some non-separable hypergraph \( H \)?

(c) What is the relation between the multiplicity of root “0” of \( P(H, \lambda) \) and the structure of \( H \)?

Note that for a bridge \( e \) in a connected hypergraph \( H \), by Theorem 5, \( P(H, \lambda) \) has a factor \( \lambda^2 \) if and only if \( P(H/e, \lambda) \) has a factor \( \lambda^2 \). Thus the study of Problem 1(a) can be focused on those connected hypergraphs without bridges.

It is also well known that for a connected graph \( G \), \( (\lambda - 1)^2 \) is a factor of \( P(G, \lambda) \) if and only if \( G \) is separable (i.e., \( G \) has a cut-vertex), and the multiplicity of root “1” of \( P(G, \lambda) \) is equal to the number of blocks of \( G \) ([43, 45]). However, such result does not hold for hypergraphs. Theorem 4 shows that \( (\lambda - 1)^2 \) is a factor of \( P(H, \lambda) \) for some but not all connected and separable hypergraphs \( H \). For connected but non-separable hypergraphs, Theorem 3 and Corollary 4 also imply their chromatic polynomials may have a factor \((\lambda - 1)^2\). Regarding the multiplicity of root “1” of \( P(H, \lambda) \) for a connected hypergraph \( H \), we propose the following problem.
Problem 2. Let $\mathcal{H}$ be a connected hypergraph.

(a) What is a necessary and sufficient condition for $P(\mathcal{H}, \lambda)$ to have a factor $(\lambda - 1)^2$?

(b) What is the relation between the multiplicity of root “1” of $P(\mathcal{H}, \lambda)$ and the structure of $\mathcal{H}$?

Properties (B.1) and (B.2) in Section 1 imply that $P(G, \lambda)$ does not have negative real roots for every graph $G$. This property was only extended to a set of hypergraphs by Dohmen [12] who showed that $P(\mathcal{H}, \lambda)$ have no negative real roots if each edge in a hypergraph $\mathcal{H}$ has an even size and each cycle in $\mathcal{H}$ has an edge of size 2, although Corollary 1 tells that chromatic polynomials of hypergraphs have dense roots in the interval $(-\infty, 0)$. It is possible that Dohmen’s result in [12] can be extended to a larger family of hypergraphs. Now we propose the following problem on negative roots of chromatic polynomials of hypergraphs.

Problem 3. Let $\mathcal{H}$ be a connected hypergraph of order $n$.

(a) What is a necessary and sufficient condition for $P(\mathcal{H}, \lambda)$ to have no negative roots?

(b) Is it true that if $\mathcal{H}$ contains even-size edges only, then $P(\mathcal{H}, \lambda)$ has no negative roots?

(c) Determine a function $f(n)$ such that $P(\mathcal{H}, \lambda)$ has no roots in $(-\infty, f(n))$.

The study of chromatic polynomials of planar graphs is very important in the topic of chromatic polynomials of graphs. We will end this paper with some problems on the chromatic polynomials of planar hypergraphs. The planarity of hypergraphs has been studied by some researchers such as Heise, Panagiotou, Pikhurko and Taraz [18], Johnson and Pollak [23], Verroust and Viaud [39], Zykov [47], etc. Zykov [47] and Johnson and Pollak [23] gave different definitions for planar hypergraphs. Here we take Zykov’s planarity which seems more natural. His definition associates edges in a hypergraph with faces in a plane graph.

Definition 1 ([47]). Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph. If there is a plane graph $G$ with vertex set $\mathcal{V}$ such that for each $e \in \mathcal{E}$, $e$ is the set of vertices in some face of $G$, then $\mathcal{H}$ is said to be Zykov-planar.

For example, if $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is a hypergraph with $\mathcal{V} = \{a, b, c, d, f, g, x, y\}$ and $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5\}$, where $e_1 = \{a, b, c\}$, $e_2 = \{c, d, y\}$, $e_3 = \{b, c, f, g\}$, $e_4 = \{f, g, y\}$ and $e_5 = \{x, y, h\}$, then $\mathcal{H}$ is Zykov-planar, as its edges are sets of vertices of some faces in the plane graph shown in Figure 2.
Johnson and Pollak [23] showed that a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is Zykov-planar if and only if the bipartite graph with vertex set $\mathcal{V} \cup \mathcal{E}$ and edge set $\{ve : v \in e \in \mathcal{E}\}$ is planar.

For any planar graph $G$ without loops, Birkhoff and Lewis [6] showed that $P(G, \lambda) > 0$ holds for all real $\lambda \geq 5$, implying that $P(G, \lambda)$ has no real roots in the interval $[5, \infty)$. They also conjectured in the same paper that $P(G, \lambda) > 0$ holds for all real $\lambda$ with $4 \leq \lambda < 5$. By the Four-color Theorem [4, 30] and Johnson and Pollak’s characterization on Zykov-planar hypergraphs in [23], every Zykov-planar hypergraph has a weak proper colouring with 4 colours, i.e., $P(\mathcal{H}, 4) > 0$ holds for every Zykov-planar hypergraph $\mathcal{H}$. Then it is natural to extend Birkhoff and Lewis’s result and conjecture [6] on chromatic polynomials of planar graphs to Zykov-planar hypergraphs.

**Conjecture 1.** For any Zykov-planar hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with $|e| \geq 2$ for all $e \in \mathcal{E}$, $P(\mathcal{H}, \lambda) > 0$ holds for all real numbers $\lambda$ with $\lambda > 4$.

Thomassen [34] proved that the roots of chromatic polynomials of planar graphs are dense in the interval $(32/27, 3)$. In the same paper, he conjectured that the roots of chromatic polynomials of planar graphs are dense in the interval $[3, 4)$. Recently Perret and Thomassen [27] proved that this conjecture holds except for a small interval $(t_1, t_2)$ around the number $\frac{5 + \sqrt{5}}{2} \approx 3.618033$, where $t_1 \approx 3.618032$ and $t_2 \approx 3.618356$. Now we end this section with the following conjecture.

**Conjecture 2.** The roots of chromatic polynomials of Zykov-planar hypergraphs $\mathcal{H}$ with $|e| \geq 3$ for some edge $e$ in $\mathcal{H}$ are dense in the interval $(32/27, 4),$

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