The quotient of generating functions of lozenge tilings for certain regions derived from hexagons, obtained with non-intersecting lattice paths

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Abstract

In a recent preprint, Lai showed that the quotient of generating functions of weighted lozenge tilings of two “half hexagons with lateral dents”, which differ only in width, factors nicely, and the same is true for the quotient of generating functions of weighted lozenge tilings of two “quarter hexagons with lateral dents”. Lai achieved this by using “graphical condensation” (i.e., application of a certain Pfaffian identity to the weighted enumeration of matchings).

The purpose of this note is to exhibit how this can be done by the Lindström–Gessel–Viennot method for nonintersecting lattice paths. For the case of “half hexagons”, basically the same observation, but restricted to mere enumeration (i.e., all weights of lozenge tilings are equal to 1), is contained in a recent preprint of Condon.

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1 Lai’s observation for lozenge tilings

In a recent preprint, Lai [4] considers lozenge tilings of “half hexagons with lateral dents” and of “quarter hexagons with lateral dents” and shows that the quotient of the generating functions of such tilings (with a certain weight) for two objects (i.e., two “half” or two “quarter” hexagons) which differ only in width (to be explained below) has a nice factorization. We shall show how these observations can be obtained by the the Lindström–Gessel–Viennot method [5, 3] of non–intersecting lattice paths.

2 Half hexagons

We shall start with the case of “half hexagons”, since it is much simpler.

The literature on tilings enumerations is abundant (see, for instance, [1]); for the experienced reader it certainly suffices to have a look at the left picture in Figure 1: A “half hexagon” is simply the upper half of some hexagon with a horizontal symmetry axis, drawn in the triangular lattice; and “lateral dents” are triangles of this “half hexagon” adjacent to its lateral sides which were removed from the “half hexagon”. All vertical lozenges of a tiling are labelled: This labelling is vertically constant and horizontally increasing by 1 from left to right, such that all vertical lozenges bisected by the vertical symmetry axis of the “half hexagon” have label 0 (see the left picture in Figure 1). Let $T$ be some lozenge tiling whose vertical lozenges are labelled $v_1, v_2, \ldots, v_m$, then the weight of $T$ is defined as

$$w(T) := \prod_{i=1}^{m} \frac{Xq^{v_i} + Yq^{-v_i}}{2}.$$  

Lai observed that if only the width $x$ (i.e., the length of the upper horizontal side) of such “half hexagon with lateral dents” is changed (i.e., the height and the relative positions of the lateral dents are unchanged), then the corresponding generating function of all tilings (weighted as described above) changes by a simple product which does not contain the variables $X$ or $Y$. Lai provided a proof for this fact by “graphical condensation” (i.e., application of a certain Pfaffian identity to the enumeration of matchings).
The left picture shows a “half hexagon” with side lengths 12, 7, 5, 7 in the triangular lattice: The lateral sides have “dents” (i.e., missing triangles; indicated in the picture by black colour), 4 on the left side and 3 on the right side. The triangle “on top” of this “half hexagon” shows the labelling of the vertical lozenges, which is constant vertically and increasing by 1 horizontally (from left to right). The picture also shows a lozenge tiling of this “half hexagon with dents”, where the three possible orientations of lozenges (left–tilted, right–tilted and vertical) are indicated by three different colours: This particular tiling has weight
\[ w_{-7}^2 \cdot w_{-6} \cdot w_{-1} \cdot w_0 \cdot w_3 \cdot w_6, \]
where \( w_i := \frac{Xq^i + Yq^{-i}}{2} \). The non–intersecting lattice paths corresponding to this tiling are indicated by white lines in the left picture; the right picture shows a “reflected, rotated and tilted” version of these paths in the lattice \( \mathbb{Z} \times \mathbb{Z} \), where horizontal edges \((a, b) \rightarrow (a + 1, b)\) are labelled \(a - 2b\) (these labels are shown in the right picture only for the region of interest in our context, i.e., for \(0 \leq y \leq x\)). Clearly, this bijection between lozenge tilings and non–intersecting lattice paths (introduced here “graphically”) is weight–preserving if we define the weight of some family \( P \) of of non–intersecting lattice paths as the product of \( w_i \), where \( i \) runs over the labels of all horizontal edges belonging to paths in \( P \).

Figure 1: Pictures corresponding to Figures 1.2.a and 2.1.a in Lai’s preprint: The length of the upper horizontal side of the “half hexagon” in the left picture is the “width parameter” \( x \) considered by Lai (so \( x = 5 \) in this picture; the “height” by definition is equal to the length of the lateral sides, so it is 7 in this picture).
3 Translation to non–intersecting lattice paths

The literature on the connection between lozenge tilings and non–intersecting lattice paths is abundant (see, for instance, [1, Section 5]); for the experienced reader it certainly suffices to have a look at the pictures in Figure 1: It is easy to see that there is a weight–preserving bijection between lozenge tilings and families of non–intersecting lattice paths in the lattice $\mathbb{Z} \times \mathbb{Z}$ with steps to the right and downwards, where steps to the right from $(a, b)$ to $(a + 1, b)$ are labelled $a - 2b$ and thus have weight

$$\frac{Xq^{a-2b} + Yq^{2a-b}}{2}$$

(and all downward steps have weight 1). As usual, the weight of a lattice path is the product of all the weights of steps it consists of.

It is easy to see that the generating function of all lattice paths from initial point $(a, b)$ to terminal point $(c, d)$ is zero for $a > c$ or $b < d$, otherwise it is equal to:

$$g_f(a, b, c, d) = \prod_{j=1}^{c-a} \left( \frac{Xq^{j-1-2b+a} + Yq^{-j+1+2d-a}}{2} \right) \left( \frac{1 - q^{2(b-d)+2j}}{1 - q^{2j}} \right). \quad (1)$$

This follows immediately by showing that (1) fulfils the recursion for such weighted lattice paths

$$g_f(a, b, a, d) \equiv 1,$$

$$g_f(a, b, c, b) = \prod_{i=a-2b}^{c-2b-1} \frac{Xq^i + Yq^{-i}}{2},$$

$$g_f(a, b, c, d) = \frac{Xq^{a-2b} + Yq^{2b-a}}{2}g_f(a + 1, b, c, d) + g_f(a, b - 1, c, d)$$

for $a \leq c$ and $b \geq d$.

We have to specialize this to our situation, i.e., to initial point $(a, a)$ and terminal point $(c, 0)$ (see the right picture in Figure 1): The generating function $g_f(a, c)$ of all lattice paths from $(a, a)$ to $(c, 0)$ is zero for $c < a$, and for $c \geq a$
it is equal to
\[
\overline{gf}(a, c) = gf(a, a, c, 0) = (2q^{a})^{(a-c)} \prod_{j=1}^{c-a} \frac{(Xq^{j-1} + Yq^{1-j})(1 - q^{2a+2j})}{1 - q^{2j}}. \tag{2}
\]

Note that increasing the width of the “half hexagon with lateral dents” by some \(d \in \mathbb{N}\) corresponds bijectively to shifting all initial and terminal points of the corresponding non–intersecting lattice paths (i.e., \((a, a) \rightarrow (a+d, a+d)\) and \((c, 0) \rightarrow (c+d, 0)\)), and from (2) we immediately obtain
\[
\overline{gf}(a + d, c + d) = \overline{gf}(a, c) \cdot \prod_{j=1}^{c-a} \frac{1 - q^{2a+2d+2j}}{q^{d}(1 - q^{2a+2j})}.
\]

By using the standard \(q\)-Pochhammer notation \((a; q)_{0} := 1\) and
\[
(a; q)_{n} := \prod_{j=0}^{n-1} (1 - a \cdot q^{j}) \quad (3)
\]
\[
(a; q)_{-n} := \frac{1}{(a \cdot q^{-1}; q^{-1})_{n}} \quad (4)
\]
for integer \(n > 0\), we may rewrite this as follows:
\[
\overline{gf}(a + d, c + d) = \overline{gf}(a, c) \cdot \frac{(q^{2d+2}; q^{2})_{c}}{q^{dc} \cdot (q^{2}; q^{2})_{c}} \cdot \frac{q^{da} \cdot (q^{2}; q^{2})_{a}}{(q^{2d+2}; q^{2})_{a}}. \tag{5}
\]

(Note that Lai uses a different notation: In [4, equation (1.1)], \(q\) is replaced by \(-q\).)

By the well–known Lindström–Gessel–Viennot argument [5, 3], the generating function of all families of non–intersecting lattice paths can be written as a determinant, and by the multilinearity of the determinant, we obtain for all \(n \in \mathbb{N}\) and all \(n\)–tuples \((a_1 < a_2 < \cdots < a_n)\) and \((c_1 < c_2 < \cdots < c_n)\) with \(a_i \leq c_i, 1 \leq i \leq n:\)
\[
\frac{\det (\overline{gf}(a_i + d, c_j + d))_{i,j=1}^{n}}{\det (\overline{gf}(a_i, c_j))_{i,j=1}^{n}} = \prod_{m=1}^{n} \left( \frac{(q^{2d+2}; q^{2})_{cm}}{q^{dcm} \cdot (q^{2}; q^{2})_{cm}} \cdot \frac{q^{da_m} \cdot (q^{2}; q^{2})_{a_m}}{(q^{2d+2}; q^{2})_{a_m}} \right). \tag{6}
\]
By the weight–preserving bijection between lozenge tilings and families of non–intersecting lattice paths, this is equivalent to Lai’s observation [4, Theorem 1.1]. (Basically the same simple approach, but restricted to mere enumeration, is contained in a recent preprint of Condon [2].)

4 Quarter hexagons

Lai considered other regions, namely “quarter hexagons” with dents: Again, the idea should become clear quickly when looking at a picture, see Figure 2. By the bijection introduced “graphically” in Figure 1, such tilings correspond to families of non–intersecting lattice paths with initial points \((2b, b)\) and terminal points \((c, 0)\) (see Figure 3, where the same tiling as in Figure 2 is considered, but with labels of vertical lozenges all increased by 1: This “shift of labels” implies starting points \((2b + 1, b)\) for the lattice paths).

For such “quarter hexagons”, Lai considered basically the same weight as for the “half hexagons”, but with two modifications:

- We let \(X = Y = 1\),
- and we let the weight of lozenges with label 0 be \(\frac{1}{2} \left( not \frac{q^0 + q^{-0}}{2} = 1 \right) \).

Observe that for \(X = Y = 1\) we may rewrite (1) as follows (using again notation (3)):

\[
\begin{align*}
\text{gf}(a, b, c, d) \big|_{X=Y=1} & = \\
& \left. 2^{a-c} q^{\frac{1}{2}(a-c)(a+c-1-4d)} \left( -q^{2a-2b-2d}; q^2 \right)_{c-a} \frac{\left( q^{2b-2d+2}; q^2 \right)_{c-a}}{(q^2; q^2)_{c-a}} \right)
\end{align*}
\]

So defining

\[
\begin{align*}
\text{gf}_0(b, c) := \left. \frac{1}{2} \text{gf}(2b + 1, b, c, 0) \right|_{X=Y=1} & + \left. \text{gf}(2b - 1, c, 0) \right|_{X=Y=1} \\
\text{gf}_1(b, c) := \left. \text{gf}(2b + 1, b, c, 0) \right|_{X=Y=1}
\end{align*}
\]
The picture shows a lozenge tiling of the “half of a half hexagon with dents” (width 8 and height 12). The particular tiling shown here is the same as in [4, Figure 1.5.b]. Note that the same picture, but with all labels of vertical lozenges increased by 1, also corresponds to the “half of a half hexagon with dents” of the same height 12, but of width 9: This would give the tiling in [4, Figure 1.5.a].

Figure 2: A lozenge tiling of a “quarter hexagon” (i.e., the “half of a half hexagon”) of width 8 and height 12.
The picture shows the family of non–intersecting lattice paths corresponding bijectively to the tiling in Figure 2 but with all labels increased by 1: The bijection is the same as the one “graphically” introduced in Figure 1. All lattice paths start at some point \((2b + 1, b)\) and end in some point \((c, 0)\), where \(c \geq 2b + 1\). The tiling of Figure 2 (with unchanged labels) would correspond to the same family of lattice paths shifted to the left by 1, i.e., with initial points \((2b, b)\) and terminal points \((c - 1, 0)\).

Figure 3: Non–intersecting lattice paths corresponding to Figure 2.
we obtain from (7):
\[
\tilde{g}_f(b, c) = 2^{2b-c} q^{b(2b-1)-\frac{1}{2}(c-1)c} \left( 1 - q^{2c} \right) \left( q^{4b+4}; q^4 \right)_{c-2b} \quad \text{for } c \geq 2b+1, \quad (8)
\]
\[
\tilde{g}_f(b, c) = 2^{2b+1-c} q^{b(2b+1)-\frac{1}{2}(c-1)c} \left( q^{4b+4}; q^4 \right)_{c-2b-1} \quad \text{for } c \geq 2b+1. \quad (9)
\]

Here, we shall only consider (9): Lais consideration of “increasing” quartered hexagons would correspond to replacing \( b \to b + x \) and \( c \to c + 2x \) for some nonnegative integer \( x \). From (9) we see that the generating function of all lattice paths with initial point \((2x + 1, x)\) and terminal point \((2x + a, 0)\) is zero for \( a < 1 \), otherwise it is
\[
\frac{2^{1-a} q^{-2ax-\frac{1}{2}(a-1)a+2x} \left( q^{4x+4}; q^4 \right)_{a-1}}{\left( q^2; q^2 \right)_{a-1}}.
\]

Hence the generating function of all families of \( m \) non–intersecting lattice paths with

- initial points \((2(x + i - 1) + 1, x + i - 1), i = 1, 2, \ldots m\)
- and terminal points \((2x + a_j, 0), j = 1, 2, \ldots m,\)

where \( \mathbf{a} = (a_j)_{j=1}^m \) is an increasing sequence of nonnegative integers, is given as
\[
2^{m^2 - \sum_{j=1}^m a_j} q^{\frac{1}{2} \sum_{j=1}^m (a_j-a_j^2)} \frac{1}{2} \left( 4m^3 - 3m^2 - m \right) \left( q^{2x} \right)^{m^2 - \sum_{j=1}^m a_j} \times \det \left( [a_j \geq 2i - 1] \cdot \frac{\left( q^{4i+4x}; q^4 \right)_{a_j+1-2i}}{\left( q^2; q^2 \right)_{a_j+1-2i}} \right)_{i,j=1}^m \quad (10)
\]

Here, we used Iverson’s bracket
\[
[some \text{ assertion}] = \begin{cases} 1 & : \text{if the assertion is true,} \\ 0 & : \text{otherwise} \end{cases}
\]
to point out the obvious condition that a lattice path cannot have its end point to the left of its starting point; but note that by (4), we could also omit Iverson’s bracket.
Clearly, the crucial point in (10) is the determinant: Observe that the determinant is zero if and only if \( a_j < 2j - 1 \) for some \( j, 1 \leq j \leq m \).

5 Evaluation of the determinant

**Definition 5.1.** We call a finite sequence of integers \( a = (a_1, a_2, \ldots, a_m) \) admissible if it is strictly increasing and has the additional property

\[
2i - 1 \leq a_i \leq 2m \text{ for all } i, 1 \leq i \leq m.
\]

We call an admissible sequence irreducible if it obeys the stricter condition

\[
2i + 1 \leq a_i \leq 2m \text{ for all } i, 1 \leq i < m, \text{ and } a_m = 2m.
\]

**Remark 5.2.** Admissible sequences are yet another type of objects enumerated by the ubiquitous Catalan–numbers: The number of admissible sequences of length \( m - 1 \) is the Catalan number \( C_m = \frac{1}{m+1} \binom{2m}{m} \). One way to prove this is by giving a simple bijection with Dyck paths of length \( 2m \) (i.e., with lattice paths from \((0,0)\) to \((2m,0)\) which consist of diagonal upwards and downwards steps, but never fall below level 0), which can be easily “seen” by just looking at a picture; see Figure 4.

**Theorem 5.3.** Let \( a = (a_1, a_2, \ldots, a_m) \) be an admissible sequence, and define two \( m \times m \)-matrices \( m(a,x) \) and \( s(a) \) as follows:

\[
m(a,x) := \left( \frac{(q^{2x+2i}; q^2)_{a_j+1-2i}}{(q; q)_{a_j+1-2i}} \right)_{i,j=1}^m \quad \text{and} \quad s(a) := \left( \frac{1}{(q; q)_{a_j+1-2i}} \right)_{i,j=1}^m
\]

(Note that the entries in (11) are zero whenever \( a_j < 2i - 1 \).)

Then we have

\[
\det (m(a,x)) = \left( \prod_{k=1}^{m} (q^{2x+2k}; q)_{a_k-2k+1} \right) \det (s(a)).
\]

(11)
The picture illustrates the bijection for the admissible sequence $a = (3, 6, 7, 8)$ of length 4 and a Dyck path of length $2 \cdot 5 = 10$: For every down–step of the Dyck path except the last, draw a rectangle with one side equal to this down–step and one corner “leaning against the vertical axis”; the heights of these rectangles give an admissible sequence; and any admissible sequence “encodes” a Dyck path.

Figure 4: Illustration of the bijection between Dyck paths of length $2m$ and admissible sequences of length $m - 1$. 
Observe that substituting $q \rightarrow q^2$ in $m(a, x)$ gives precisely the determinant in (10), so by the weight-preserving bijection between lozenge tilings and families of non–intersecting lattice paths, (12) yields Lai’s observation [4, Theorem 1.3].

**Remark 5.4.** The condition $a_m \leq 2m$ in Theorem 5.3 is crucial; otherwise the assertion is wrong, already for $m = 1$.

The rest of this section is devoted to the proof of Theorem 5.3: It consists of several steps, which we indicate by corresponding headlines.

**Reduce the general determinant evaluation to irreducible sequences:**

Observe that if there is some $1 \leq k < m$ such that $a_k \leq 2k$, then the $(i, j)$–entry of $m(a, x)$ is zero for $i > k$ and $j < k$, so the determinant can be written as the product of two minors consisting

- of the first $k$ rows and columns of $m(a, x)$,
- and of the last $m - k$ rows and columns of $m(a, x)$,

respectively. It is easy to see that these minors correspond to the lists $a’ = (a_1, \ldots, a_k)$ and $a'' = (a_{k+1} - 2k, \ldots, a_m - 2k)$, which gives the following factorization:

$$\det (m(a, x)) = \det (m(a', x)) \cdot \det (m(a'', x + k)).$$

Both sequences $a'$ and $a''$ are admissible, and if we manage to prove Theorem 5.3 for the **irreducible case**

$$a_k \geq 2k + 1 \text{ for all } k = 1, \ldots, m - 1,$$

it is easy to see that the factorization above gives the general assertion (12) for all admissible sequences (by induction on the number of “irreducible factors”).

Note that an irreducible sequence $a$ of length $m$ necessarily ends with

$$a = (\ldots, a_{m-1} = 2m - 1, a_m = 2m).$$
Pull out common factors from rows and columns:

We may rewrite the factors of the product in (12) as follows

\[
\prod_{k=1}^{m} (q^{2x+2k}; q)_{a_k-2k+1} = \frac{\prod_{j=1}^{m} (q^{2x}; q)_{a_{j+1}}}{\prod_{i=1}^{m} (q^{2x}; q)_{2i}}
\]

and pull them out from the rows \(i\) and columns \(j\) of \(m(a, x)\). This operation gives a new \(m \times m\)-matrix \(m'(a, x)\) with \((i, j)\)–entry

\[
\frac{(q^{2x}; q)_{2i} (q^{2i+2x}; q^2)_{a_{j+1}+1-2i}}{(q^{2x}; q)_{a_{j+1}} (q; q)_{a_{j+1}+1-2i}} = \frac{(q^{2x+2i}; q^2)_{a_{j+1}+1-2i}}{(q^{2x+2i}; q)_{a_{j+1}} (q; q)_{a_{j+1}+1-2i}}
\]  

and (12) clearly is equivalent to

\[
\det (m'(a, x)) = \det (s(a)).
\]

The modified matrix \(m'(a, x)\) can be triangulating by a lower triangular matrix \(f(a)\) with entries from the field \(\mathbb{C}(q)\) of rational functions in \(q\):

Denote by \(\mathbb{F}\) the field \(\mathbb{C}(q)\) of rational functions in \(q\).

**Lemma 5.5.** Let \(a\) be an irreducible sequence, and consider \(m'(a, x)\) as defined in (13). Then there exists a lower triangular \(m \times m\)–matrix \(f(a)\) whose entries are constants from \(\mathbb{F}\) (i.e., the entries do not depend on \(x\)) with all entries on the main diagonal equal to 1, such that the matrix product

\[
m'(a, x) \cdot f(a)
\]  

is upper triangular, with constants from \(\mathbb{F}\) on its main diagonal.

If we manage to prove Lemma 5.5, then by the multiplicativity of the determinant, we would obtain that

\[
\det (m'(a, x) \cdot f(a)) = \det (m'(a, x)) \cdot \det (f(a)) = \det (m'(a, x)),
\]

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and this determinant is equal to the product of the entries on the main diagonal of (15); so, in particular, it does not depend on \(x\). (This would already be sufficient to deduce Lai's observation concerning the quotient of generating functions [4, Theorem 1.3].) Moreover, for \(q \in \mathbb{C}\) with \(|q| < 1\), (14) would also hold in the limit \(x \to \infty\), so (12) follows since

\[
\lim_{x \to \infty} \frac{(q^{2x+2i}; q^2)_{a_j+1-2i}}{(q^2; q)_{a_j+1-2i}} = \frac{1}{(q; q)_{a_j+1-2i}} \quad \text{for } |q| < 1.
\]

So, the proof of Theorem 5.3 would be complete if we can prove Lemma 5.5.

Finding the triangulating matrix \(f(a)\) amounts to solving \(m - 1\) systems of linear equations:

For some matrix \(n\), let us denote by \([n]_{i \geq s, j \geq t}\) the submatrix of \(n\) consisting

- of all rows \(i \geq s\)
- and all columns \(j \geq t\).

The \(k\)-th column vector in a lower triangular \(m \times m\)-matrix has, by definition, zero entries at all positions \(j < k\), so determining the sub-vector \(v_k\) corresponding to the “interesting” entries of this column vector in the lower triangular \(m \times m\)-matrix \(f(a)\) amounts to finding a solution of the homogeneous system of linear equations

\[
[m'(a, x)]_{i \geq k+1, j \geq k} \cdot v_k = 0,
\]

which has the following additional properties:

1. the entries in \(v_k\) are elements of \(\mathbb{F}\) which do not depend on \(x\): We call such solution \(x\)-invariant,

2. the first entry of \(v_k\) is not zero, whence we can divide \(v_k\) by this first entry (so the lower triangular matrix \(f(a)\) would have entries equal to 1 on its main diagonal, as desired),

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3. the inner product of the first row of \([m'(a, x)]_{i \geq k, j \geq k}\) with \(v_k\) is in \(\mathbb{F}\) and does not depend on \(x\), too.

Clearly, we have to find such solutions for \(k = 1, 2, \ldots, m - 1\).

**Observation:** It suffices to consider a single system of linear equations:

It is easy to see that the submatrix \([m'(a, x)]_{i \geq 2, j \geq 2}\) is precisely the \((m - 1) \times (m - 1)\) -matrix \(m'(\overline{a}, x + 1)\), where

\[\overline{a} = (a_2 - 2, a_3 - 2, \ldots, a_m - 2).\]

Hence it suffices to consider only the first column vector \(v_1\) of the matrix \(f(a)\). This means that we have to show that the system of equations

\[\left[ m'(a, x) \right]_{i \geq 2, j \geq 1} \cdot v_1 = 0 \quad (16)\]

has a solution with the additional properties 1, 2 and 3 from above.

**Express the single system of linear equations more conveniently:**

Observe that we can express property 2 more conveniently: **Prepend** 1 to the irreducible sequence \(a\), i.e., consider the (admissible!) sequence

\[a^* := (1, a_1, \ldots, a_m)\]

(so \(a_1^* = 1\) and \(a_k^* = a_{k-1}\) for \(2 \leq k \leq m + 1\)), and define the \(m \times (m + 1)\) -matrix \(m'(a^*, x)\) with entries as in (13), i.e.:

\[m'(a^*, x)_{i,j} = \frac{(q^{2x+2i}; q^2)_{a_j^*+1-2i}}{(q^{2x+2i}; q)_{a_j^*+1-2i} (q; q)_{a_j^*+1-2i}}. \quad (17)\]

Note that \(m'(a^*, x)_{i,1} = [i = 1]\): Hence an \(x\)-invariant solution of

\[m'(a^*, x) \cdot v_1 = 0 \quad (18)\]
is an $x$–invariant solution of (16) with property 2.

Assume that we can show that (18) has an $x$–invariant solution: Then such solutions would determine a matrix $f(a)$ which might have zeros on the main diagonal; but this cannot happen since we know that $\det(m'(a,x)) \neq 0$ (as it is the generating function of a non–empty set of lattice paths). This means that such solutions “automatically” have property 3: So the proof of Lemma 5.5 boils down to showing that (18) has an $x$–invariant solution.

**Divide all equations by the coefficient corresponding to the last column:**

The system (18) features $m$ equations for $m+1$ variables, so for every fixed $x$ we expect a solution space of dimension $\geq 1$. As pointed out above, element $a_m$ equals $2^m$ in an irreducible sequence of length $m$. By dividing the equation (i.e., $i$–th equation) of the matrix $m'(a^*,x)$ by the (non–zero) coefficient corresponding to $a_{m+1} = a_m = 2m$

$$
\frac{(q^{2x+2i}; q^2)_{2m+1-2i}}{(q^{2x+2i}; q)_{2m+1-2i}(q; q)_{2m+1-2i}}
$$

we do not change the space of solutions, but obtain a modified coefficient matrix $m''(a^*, x)$ with $(i,j)$–entry

$$
\frac{(q^{2x+2i}; q^2)_{aj+1-2i}(q^{2x+2i}; q)_{2m+1-2i}(q; q)_{2m+1-2i}}{(q^{2x+2i}; q^2)_{2m+1-2i}(q^{2x+2i}; q)_{aj+1-2i}(q; q)_{aj+1-2i}} = \frac{(q^{2+a_j-2i}; q)_{2m-a_j}(q^{1+a_j+2x}; q)_{2m-a_j}}{(q^{2+2a_j+2x-2i}; q^2)_{2m-a_j}}.
$$

**Consider a single system of linear equations with more variables:**

It is convenient to view the coefficient matrix $m''(a^*, x)$ as the submatrix corresponding to columns $a_1^*, a_2^*, \ldots, a_{m+1}^*$ of the $m \times 2m$–matrix with $(i,j)$–entry

$$
\frac{(q^{2j-2i}; q)_{2m-j}(q^{1+j+2x}; q)_{2m-j}}{(q^{2+2j+2x-2i}; q^2)_{2m-j}}.
$$
and to reverse the order of columns, i.e., to consider the $m \times 2m$–matrix $\mathbf{m}'''(m, x)$ with $(i, j)$–entry

$$
e(m, i, j, x) := \frac{(q^2(m-i+1)-j; q)_j (q^{2(m+x)-j+1}; q)_j}{(q^{2m+2x-i+1}-2j; q^2)_j},$$

(20)

where $1 \leq i \leq m$ and $0 \leq j \leq 2m - 1$.

Observe that the factor $(q^{2(m-i+1)-j}; q)_j$ in (20) is zero for $j \geq 2 (m - i + 1)$, hence the matrix $\mathbf{m}'''(m, x)$ has a “staircase shape”, with all entries of row $i$ equal to zero for $j \geq 2 (m - i + 1)$. To illustrate this, we present $\mathbf{m}'''(3, x)$ (after expansion of the $q$–Pochhammer symbols involving the parameter $x$ and cancellation; for typesetting reasons, we give the transpose of this matrix):

$$
(\mathbf{m}'''(3, x))^T = \begin{pmatrix}
1 & 1 & 1 \\
(q^4; q)_3 (1-q^{2x+6}) & (q^3; q)_3 (1-q^{2x+6}) & (q^2; q)_3 (1-q^{2x+5}) \\
(q^4q; q)_3 (1-q^{2x+5}(1-q^{2x+6}) & (q^4q; q)_3 (1-q^{2x+5}) & 1 \\
(q^3q; q)_3 (1-q^{2x+5}) & 0 & 1 \\
(q^3q; q)_3 (1-q^{2x+5}) & 0 & 0 \\
(q^2q; q)_3 (1-q^{2x+5}) & 0 & 0 \\
(q^2q; q)_3 (1-q^{2x+5}) & 0 & 0 \\
(q; q)_3 (1-q^{2x+5}) & 0 & 0 \\
(1-q^{2x+5}) & 0 & 0 \\
(1-q^{2x+5}) & 0 & 0 \\
(1-q^{2x+5}) & 0 & 0 \\
(1-q^{2x+5}) & 0 & 0 \\
(1-q^{2x+5}) & 0 & 0 \\
(1-q^{2x+5}) & 0 & 0 \\
(1-q^{2x+5}) & 0 & 0 \\
(1-q^{2x+5}) & 0 & 0 \\
\end{pmatrix}
$$

A useful observation concerning (20):

The attentive reader may have noticed a phenomenon in the example (21): The number of factors in the denominator is not increasing all the way, but apparently becomes constant from $j = m - i$ on. This is no coincidence, but due to certain cancellations: Observe that the factors depending on $x$ in the numerator of (20) end with $(1-q^{2m+2x})$, and from $j = m - i + 1 + t$ on ($t = 0, 1, \ldots$), the “additional factors” $(1-q^{2m+2x-2t})$ appearing in the denominator cancel out with corresponding factors in the numerator, such that in the last two entries of each row only factors with odd $x$–depending $q$–exponents $(1-q^{2x+2t-1})$ survive the cancellation in the numerator; moreover, the last entry is $(1-q)$–times the next–to–last entry. Altogether, this
implies: If we multiply every row \(i\) of \(m''(m, x)\) by the denominator of its last non–zero entry
\[
(q^{2x+2m+2}; q^2)_{m-i},
\]
then the \((i, j)\)–entry of the resulting matrix is equal to \( (q^{2(m-i+1)-j}; q)_j\) times a product of \((m - i)\) factors of the form \((1 - q^{2x+y})\), i.e.,
\[
m''(m, x)_{i,j} \cdot (q^{2x+2m+2}; q^2)_{m-i} = (q^{2(m-i+1)-j}; q)_j \cdot \prod_{k=1}^{m-i} (1 - q^{2x+y_{j,k}}) \tag{22}
\]
for certain integers \(y_{j,k}\).

Claim: The first homogeneous equation of (18) actually has a non–trivial \(x\)–invariant solution:

It might not be clear at first sight that there is a non–trivial \(x\)–invariant solution at all; even for a single equation of (18): In order to show that this is indeed the case, we state and prove a little Lemma.

Recall the definition of the \(q\)–binomial coefficient
\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} \tag{23}
\]
for non–negative integers \(n, k\) with \(0 \leq k \leq n\), and the well–known recursions
\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \left[ \begin{array}{c} n - 1 \\ k - 1 \end{array} \right]_q + q^k \left[ \begin{array}{c} n - 1 \\ k \end{array} \right]_q , \tag{24}
\]
\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_q = q^{n-k} \left[ \begin{array}{c} n - 1 \\ k - 1 \end{array} \right]_q + \left[ \begin{array}{c} n - 1 \\ k \end{array} \right]_q . \tag{25}
\]

An easy consequence of these recursions is the well–known identity
\[
\sum_{k=0}^n (-1)^k q^{k(k+1)/2-kn} \left[ \begin{array}{c} n \\ k \end{array} \right]_q = [n = 0] . \tag{26}
\]

Now define
\[
e(n, k) := \frac{k(k+1)}{2} - kn
\]
and observe that (26) can be generalized as follows:
Lemma 5.6. Let $n, r$ be positive integers, and let $(\gamma_1, \gamma_2, \ldots, \gamma_{n-r})$ be an (arbitrary) sequence of $n - r$ complex numbers. Then we have:

$$\sum_{k=0}^{n} (-1)^k q^{e(n,k)} \left[\begin{array}{c} n \\ k \end{array}\right] \prod_{i=1}^{n-r} (1 - \gamma_i \cdot q^k) = 0. \tag{27}$$

Proof. Let $r$ be arbitrary but fixed, and proceed by induction on $n$: Clearly, for $n \leq r$ the statement is true, since it is equivalent to (26) in this case.

For the inductive step $(n-1) \rightarrow n$, we set $f := \prod_{i=1}^{n-1-r} (1 - \gamma_i \cdot q^k)$ and use the recursions (24) and (25) to rewrite the unsigned $k$-th summand in (27) as

$$q^{e(n,k)} \left(\left[\begin{array}{c} n-1 \\ k \end{array}\right] q^k + \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right] q^{n-k}\right) - \gamma_{n-r} \cdot q^k \left(\left[\begin{array}{c} n-1 \\ k \end{array}\right] q + \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right] q^{n-k}\right) f.$$

Using the obvious relations

$$e(n,k) + k = e(n-1,k) \quad \text{and} \quad e(n,k) + n - 1 = e(n-1,k-1),$$

we may rewrite this as

$$\left(\left[\begin{array}{c} n-1 \\ k \end{array}\right] q^{e(n-1,k)} + q^{1-n} \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right] q^{e(n-1,k-1)}\right) f$$

$$- \gamma_{n-r} \left(\left[\begin{array}{c} n-1 \\ k \end{array}\right] q^{e(n-1,k)} + q \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right] q^{e(n-1,k-1)}\right) f,$$

and the assertion follows by induction. \qed

From this Lemma, we can deduce immediately the following Corollary.

Corollary 5.7. Let $n, r$ be positive integers, and assume that for each for $j = 1, 2, \ldots, n+1$ we are given

- $n - r$ complex numbers $\gamma_{j,1}, \gamma_{j,2}, \ldots, \gamma_{j,n-r}$
- and a rational function $c_j = c_j(q)$ in $\mathbb{F}$. 

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Then each $n+1$ subsequent rows $i = i_0 + 1, i_0 + 2, \ldots, i_0 + n + 1$ (for arbitrary $i_0 \in \mathbb{N}$) of the form

$$
\begin{pmatrix}
c_j \prod_{k=1}^{n-r} \left( 1 - \gamma_{j,k} q^i \right)
\end{pmatrix}_{j=1}^{n+1}
$$

are not linearly independent over $\mathbb{F}$ (by Lemma 5.6), and therefore the same is true for the columns of the matrix

$$
\begin{pmatrix}
c_j \prod_{k=1}^{n-r} \left( 1 - \gamma_{j,k} q^i \right)
\end{pmatrix}_{i=\left(i_0+n+1,1\right)}^{\left(i_0+n+1,n+1\right)}
$$

Hence the homogeneous system of linear equations corresponding to this matrix has a nontrivial solution $b = (b_1, b_2, \ldots, b_{n+1})$ in $\mathbb{F}$.

Since such solution $b$ fulfils the $n$ subsequent equations corresponding to rows $i_0 + 2, i_0 + 3, \ldots, i_0 + n + 1$, it also fulfils the equation corresponding to row $i_0 + n + 2$ (since this row is a linear combination of its $n$ predecessors, by Lemma 5.6) and the equation corresponding to row $i_0 + 0$ (since this is a linear combination of its $n$ successors, again by Lemma 5.6).

In short: Every solution $b$ of the homogeneous system of linear equations corresponding to (29) fulfils the equations corresponding to rows of the form (28) for all $i \in \mathbb{N}$, and in that sense is $i$–invariant. Therefore we have for all complex numbers $q$ with $|q| < 1$

$$
0 = \lim_{i \to \infty} \sum_{j=1}^{n+1} c_j \left( \prod_{k=1}^{n-r} \left( 1 - \gamma_{j,k} q^i \right) \right) b_j = \sum_{j=1}^{n+1} c_j b_j,
$$

so $b$ is a solution to the simpler equation

$$
\sum_{j=1}^{n+1} c_j b_j = 0.
$$

Now observe that Corollary 5.7 applies to the first row of $m''(\mathbf{a}^*, x)$ after multiplication by the denominator of its last non–zero entry (as in (22)): The $j$–th entry in this row is equal to $(q^{2m-j}; q)_j$ times a product of $m - 1$
factors of the form \((1 - q^{2x+y})\), and the length of this row is \(m + 1\); hence we may consider the \(m + 1\) equations for \(x, x + 1/2, \ldots, x + m/2\) and deduce that there exists an \(x\)-invariant solution for the first equation of (18).

Another useful observation concerning (20):

We set \(x = \frac{1}{2} - k\) in \(e(m, i, j, x)\) and observe:

\[
e(m, i, j, \frac{1}{2} - k) = \frac{(q^{2(m-i+1)-j}; q)_j (q^{2(m-k+1)-j}; q)_j}{(q^{2(2m-k+i-1)-2j}; q^2)_j}
\]

\[
= \frac{(q^{2(m-k+1)-j}; q)_j (q^{2((1/2)-i)+1}; q)_j}{(q^{2(2m+(1/2)-k+1)-2j}; q^2)_j}
\]

\[
= e(m, k, j, 1/2 - i).
\]

(30)

Note that this means: Any \(x\)-invariant solution of the equation corresponding to row \(k\) of \(m''(a^*, x)\) is also a solution of all rows \(i \geq k\), but for fixed \(x = \frac{1}{2} - k\).

Put everything together:

For some subsequence \(C \subseteq \{0, 1, \ldots, 2m - 1\}\) of columns of \(m'''(m, x)\), we may consider the homogeneous system of linear equations corresponding to the submatrix of \(m'''(m, x)\) which consists of the columns in \(C\): We call this system the equations corresponding to \(C\), and if we consider only the last \(k\) (\(1 \leq k \leq m\)) of these equations (corresponding to the submatrix of \(m'''(m, x)\) which consists of the rows \(m - k + 1, m - k + 2, \ldots, m\) and the columns in \(C\)), we call this homogeneous system of equations the last \(k\) equations corresponding to \(C\). For arbitrary but fixed \(x\), the set of solutions of the last \(k\) equations corresponding to some \(C\) is a linear subspace of \(F^{[C]}\), which contains the set of \(x\)-invariant solutions, and the latter is a linear subspace of \(F^{[C]}\), too.
Recall that we have to show that the equations corresponding to $\mathbf{a}^\ast$ have a one-dimensional $x$-invariant space of solutions: We shall prove this by working our way up from the last row $m$ to the first.

More precisely, we claim that the last $k$ rows of the equations corresponding to $\mathbf{a}^\ast$

- have an $x$-invariant solutions space whose dimension is equal
  - to the number of elements $j$ in $\mathbf{a}^\ast$ such that $j \leq 2k$
  - minus $k$,
- and that there are no other solutions, i.e.: Every solution (for arbitrary fixed $x$) is, in fact $x$-invariant.

This assertion is immediately clear for $k = 1$, since the (single) last equation corresponding to $\mathbf{a}^\ast$ has a 1–dimensional solution space, which is $x$–invariant since this equation does not contain $x$ at all (see example (21)):

$$1 \cdot c_0 + (1 - q) \cdot c_1 = 0 \iff (c_0, c_1) \in \{ \lambda \cdot (q - 1, 1) : \lambda \in \mathbb{C} \}. \quad (31)$$

So assume we proved our claim for the last $k$ rows

$$m - k + 1, m - k + 2, \ldots, m$$

of the equations corresponding to $\mathbf{a}^\ast$, and assume that for the next row $m - k \geq 1$ there are $l$ columns of $\mathbf{a}^\ast$ which are $\leq 2(k + 1)$ (note that $l \geq k + 1$, since $\mathbf{a}^\ast$ is an admissible sequence). Now recall (22): The entries of row $m - k$ are, after multiplication with the denominator of the entry $\mathbf{m}_m(m, x)_{m-k,2(m-k)-1}$, equal to some element in $\mathbb{F}$ times a product of $k$ factors of the form $(1 - q^{2x+y})$. So by Corollary 5.7, row $m - k$ in the equations corresponding to

- the first $k$ columns of $\mathbf{a}^\ast$,
- plus one of the columns $a_{k+1}, a_{k+2}, \ldots, a_l$ which are less or equal to $2k + 1$
has a non–trivial $x$–invariant solution, and every linear combination of these $(l - k)$ $x$–invariant solutions is again an $x$–invariant solution, so the dimension of the linear space of $x$–invariant solutions is at least $l - k$. Now we combine two observations:

1. Since the first $k$ entries of $a^\star$ are an admissible sequence, the submatrix of $m'''(m, x)$ consisting of
   - the last $k$ rows
   - and the first $k$ columns of $a'$

has determinant $\neq 0$. Hence the $k$ equations are linearly independent, and the dimension of the solution space for arbitrary, but fixed $x$ is $l - k$:

But this implies that all solutions (for any $x$) are, in fact, $x$–invariant.

2. By observation (30), for $i \leq k$ the following rows are identical:
   - row $m - i$ for $x = \frac{1}{2} - (m - k)$,
   - row $m - k$ for $x = \frac{1}{2} - (m - i)$.

This implies that each $x$–invariant solution of the $(m - k)$–th equation corresponding to $a^\star$ is a solution of the last $k + 1$ equations corresponding to $a^\star$ for fixed $x = \frac{1}{2} - (m - k)$, and since we already know that all solutions (for any $x$) of the last $k$ equations are $x$–invariant, the same holds true for the last $k + 1$ equations.

Altogether, this finishes the proof of Lemma 5.5, and thus of Theorem 5.3.

5.1 A simple algorithm for actually finding the solutions which constitute $f(a)$ of Lemma 5.5

For the homogeneous system of the $m$ linearly independent equations in $2m$ variables corresponding to the coefficient matrix $m'''$

$$m''' \cdot c = 0,$$ (32)
we expect a solution space of dimension $m$: We claim that this solution space is $x$–invariant and spanned by a vector

\[ \mathbf{c}_1 = (1, \alpha_1, 0, \alpha_3, \ldots, 0, \alpha_{2m-1}) \]

which equals the $2m$ first elements of a certain infinite sequence in $\mathbb{F}$, which can be constructed recursively and which is independent of $m$, together with $m - 1$ shifts of this vector

\[ \mathbf{c}_2 = (0, 0, 1, \alpha_1, 0, \alpha_3, \ldots, 0, \alpha_{2m-3}), \ldots \mathbf{c}_m = (0, 0, \ldots, 1, \alpha_1). \]

For instance, the solution space for $m = 4$ is the $\mathbb{F}$–span of the columns of the following matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{1-q} & 1 & 0 & 0 \\
0 & \frac{1}{1-q} & 0 & 0 \\
\frac{q}{(1-q)^2(1-q^3)} & 0 & 1 & 0 \\
\frac{q^2(1+q^2)}{(1-q)^3(1-q^5)(1-q^7)} & \frac{q}{(1-q)^3(1-q^3)} & 0 & 1 \\
\frac{q^3(q^8+q^7+3q^6+2q^5+3q^4+2q^3+3q^2+q+1)}{(1-q)^4(1-q^5)^2(1-q^7)} & \frac{q^2(1+q^2)}{(1-q)^3(1-q^5)(1-q^7)} & \frac{q}{(1-q)^2(1-q^5)} & -\frac{1}{1-q}
\end{pmatrix}
\]

Since we already know that all solutions are $x$–invariant, we may work with the simpler matrix $\mathbf{m}''(m, \infty)$ with $(i, j)$–entry

\[ e(m, i, j, \infty) = \lim_{x \to \infty} e(m, i, j, x) = \left( q^{2(m-i+1)-j}; q \right)_j \]

(where we assumed that $|q| < 1$ in taking the limit) and consider

\[ \mathbf{a}^* = (0, 1, 3, 5, \ldots, 2m - 1). \]

Observe, that in each step of “working our way up” from the last equation corresponding to $\mathbf{a}^*$ (as in the proof of Lemma 5.5), precisely one new variable has to be considered. I.e., starting with $\alpha_1 = q - 1$ from the last equation (see (31)), the next–to–last equation becomes a linear equation in a single variable, which, of course can easily be solved: This explains the recursive construction of the solution vector $\mathbf{c}_1$. 24
Now observe that deleting the first two columns and the last row of \( m'''(m, \infty) \), and dividing all remaining rows by their first entry gives \( m'''(m-1, \infty) \), whence we may prepend two zeros to the solution \( c_1 \) (as just described) for \( m - 1 \) and obtain another solution of \( m'''(m, \infty) \): This makes clear that we find the \( m \) solution vectors \( c_1, \ldots, c_m \) (which obviously are linearly independent), as claimed above.

We already know that there exists an \( x \)-invariant solution of the equations corresponding to general \( a^* \) (derived from some admissible sequence \( a \), as in the proof of Lemma 5.5), which may be viewed as a solution vector for (32) where all entries not in \( a^* \) are set to zero: It is easy to see how to construct a solution vector with these “prescribed zeros” as a linear combination of \( c_1, \ldots, c_m \).

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