Thermodynamics of AdS/QCD within the 5D dilaton-gravity model

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Abstract

We calculate the pressure, entropy density, trace anomaly and speed of sound of the gluon plasma using the dilaton potential of Ref. \[1\] in the dilaton-gravity theory of AdS/QCD. The finite temperature observables are calculated from the Black Hole solutions of the Einstein equations, and using the Bekenstein-Hawking equality of the entropy with the area of the horizon. Renormalization is well defined, because the $T = 0$ theory has asymptotic freedom. Comparison with lattice simulations is made.

Keywords: QCD thermodynamics, Gauge-gravity correspondence, Black Holes

1. Introduction

The duality of string theory with 5-dimensional gravity is nowadays a powerful tool to study the strong coupling properties of gauges theories and in particular of QCD, either at zero or finite temperature. In conformal $AdS_5$ the metric is well known. It has a horizon in the bulk space at $r_H \approx \ell$, where $\beta = 1/T$ and $\ell$ is the size of the AdS-space. Conformal solutions for entropy scale like $s \propto T^3$, since the 3-dimensional area of the horizon is given as $A \propto 2\pi r_H^2$. Promising solutions of this conformal theory have been proposed to the problem of viscosity $\eta$ with a small constant value for $\eta/s$. Top to bottom approaches based on the conformal SYM with fermions have been investigating the chiral phase transition and problems at finite density \[3\, 4\].

To extend this duality to SU($N_c$) Yang-Mills theory, one of the first tasks is to control the breaking of conformal invariance. In this work we will studied a 5D dilaton-gravity model introduced in \[5\], with the action

\[ S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-G} \left( R - \frac{4}{3} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right) - \frac{1}{8\pi G_5} \int d^5x \sqrt{-H} K. \] (1)

The last term is the Gibbons-Hawking term, with $K$ being the extrinsic curvature and $H$ the induced metric on the boundary $\partial M$. The dilaton potential $V(\phi)$ is related to the $\beta$-function in a one-to-one relation. While the $\beta$-function is well known at high energies, there is no general consensus about its IR behavior. We assume the following parameterization

\[ \beta(\alpha) = -2b_2 \alpha + \left[ b_2 a + \left( \frac{b_2}{\tilde{a}} - \beta_0 \right) \alpha^2 \right] \] (2)

which was proposed in Ref. \[1\] for the computation of the heavy $Q\bar{Q}$ potential at zero temperature within this dilaton-gravity model. This parameterization has the standard behavior in the UV region limit $\beta(\alpha) \approx -b_2 a^2 - \beta_1 \alpha^3 + \ldots$ for $\alpha \ll \tilde{a}$, and generates confinement in the IR region, where it behaves as $\beta(\alpha) \approx -b_2 a$ for $\alpha \gg \tilde{a}$. The optimum values to reproduce the $Q\bar{Q}$ potential at $T = 0$ are \[1\]

\[ b_2 = 2.3, \quad \tilde{a} = 0.45, \quad \ell = 4.389\text{GeV}^{-1}. \] (3)

This parameterization also allows to obtain the running QCD-coupling $\alpha(E)$ in the $\overline{\text{MS}}$-scheme from the running of the dilaton $\phi(z)$ in the bulk by mapping the bulk coordinate $z$ to the energy scale in Ref. \[1\]. Small values of the bulk coordinate $z$ correspond to large energies and large values of $z$ describe the infrared physics.

2. Equations of motion

The dilaton-gravity model of Eq. \[1\] has two different types of solutions for the metric. The thermal gas solution corresponds to the confined phase, and the black
hole solution characterizes the deconfined phase and it has a horizon localized in the bulk coordinate at \( z = z_h \) similar to the situation in 4-dim gravity [5]. We have studied in Ref. [8] the heavy \( Q\bar{Q} \) free energy in the confined phase of QCD by using a thermal gas metric. The black hole metric takes the form

\[
d\ell^2 = b^2(z) \left( f(z) dt^2 + dx_i dx^i + \frac{dz^2}{f(z)} \right),
\]

where \( z = \ell^2/r \) and \( f(z) \) has the properties

\[
f(z_h) = 0, \quad f'(z_h) = -4\pi T.
\]

The second expression in Eq. (5) follows from regularity of the metric at the horizon. This solution only exists at high enough temperatures [5]. The classical equations of motion corresponding to the BH solution read [5, 7, 8]

\[
W' = \frac{16}{9} b^2 W^2 - \frac{1}{f} \left( Wf' - \frac{3}{4} bV \right),
\]

\[
b' = -\frac{4}{9} b^2 W',
\]

\[
\alpha' = \alpha \sqrt{bW'},
\]

\[
f'' = \frac{4}{3} f' b W',
\]

where the superpotential is defined as \( W(z) = -9b'(z)/(4b^2(z)) \).

### 3. Thermodynamics

The phase transition from the confined glueball gas to the deconfined gluon plasma can be qualitatively understood as follows. At low temperatures the infrared physics is determined by the thermal gas metric with \( f = 1 \). At a minimal temperature the black hole solution appears with a horizon at \( z_h \) and \( f \neq 1 \) and becomes the preferred solution at the phase transition which is first order. The phase transition temperature is fully fixed by the scales of the theory at zero temperature. The position of the black hole horizon intervenes between the confinement region in bulk space and determines from them on the physics. Let us describe our new results starting with the highest temperatures, where expansions in the thermal coupling \( \alpha_h = \alpha(z_h) \) are possible. Starting from the ultraviolet expansion of the \( \beta \)-function, one can solve analytically the equations of motion in the UV. In doing that, one arrives at the following expansion for the metric factor up to \( \mathcal{O}(\alpha^2) \)[8]:

\[
b(\alpha) = \frac{\ell^2}{4} \left[ 1 - \frac{4}{9} \beta_0 \alpha + \frac{2}{81} (22\beta_0^2 - 9\beta_1) \alpha^2 \right].
\]

\[\text{(10)}\]

\[\text{1}\text{ Analytical expressions up to } \mathcal{O}(\alpha^3) \text{ will be presented in Ref. [8].}\]
constant qcd perturbation theory (pQCD). This is easy to see, because the coefficient of $O(a_s)$ in the expansion of the pressure in the holographic model ($p_v^{\text{vis}} = \frac{5}{4}T^3$), Eq. (14), is a factor two larger than the corresponding one in pQCD ($p_v^{\text{QCD}} = \frac{\pi^3}{240}T^4$) [12]. As a consequence, the value of $G_5^c$ given by Eq. (15) leads to lower values of the thermodynamic quantities for all temperatures in comparison with lattice data, also in the regime close to $T_c$. To reproduce lattice data in the regime $T_c < T < 5T_c$ one must use a value of $G_5$ which is a factor 1.46 smaller than $G_5^\infty$, i.e.

$$\frac{1}{G_5} = 1.46 \times \frac{1}{G_5^\infty},$$  \hspace{1cm} (16)

spoiling the Stefan-Boltzmann limit at high temperatures. Note, however, that lattice data for pressure and entropy density taken from Refs. [9] and [10] are not consistent each other at high temperatures, and it is hard to believe that both computations fulfill the Stefan-Boltzmann’s law. This discrepancy introduces an error in $G_5$ of the order of 10%, which in either case is not enough to explain the factor in Eq. (16). The gravity model seems to be hardly consistent to reproduce at the same time lattice data at very high temperatures, and close to the phase transition. In this sense, there exists the possibility that the ideal gas limit doesn’t correspond to the limit of the black hole gravity theory at high temperatures. A natural question arises: is it possible that the gravity theory allows more degrees of freedom at high temperatures expected from pQCD. Neat comparison with lattice data, also in the regime close to $T_c$, of motion have been obtained [8]. We show in Figs. 1, 2 and 3 the pressure, entropy density and trace anomaly as a function of $T$ (in units of $T_c$). See Fig. 1 for details.

To reach this expression one has to evaluate Eq. (10) at $T \to T_c$. Using

$$T \frac{dp}{dT} = -\beta_0 \alpha_s^2 - \beta_1 \alpha_s^3 + O(\alpha_s^4),$$  \hspace{1cm} (13)

it is easy to derive the pressure from Eq. (12), as it is related to the entropy density by $s = dp/dT$. One gets

$$p(T) = \frac{\pi^3 T^3}{16G_5} \left[ 1 - \frac{4}{3} \beta_0 \alpha_s - \frac{2}{9} \left( 4 \beta_0^2 - 3 \beta_1 \right) \alpha_s^2 \right].$$  \hspace{1cm} (14)

From Eqs. (12) and (14) one can immediately compute the energy density $\epsilon = TS - p$, and the trace anomaly $(\epsilon - 3p)/T^4 = s/T^3 - 4p/T^4$.

In order to obtain the thermodynamic functions for all temperatures, numerical solutions of the equations of motion have been obtained [8]. We show in Figs. 1 and 2 the pressure, entropy density and trace anomaly for pure gluodynamics with $N_c = 3$. The lattice data are taken from Refs. [9][10]. We also plot the analytical results of Eqs. (12) and (14). It is noteworthy and visible in the figures that the expansion in terms of $\alpha_s$ converge quite rapidly. Even at temperatures $T \approx 1.5T_c$, orders of $\alpha_s$ give a sufficient approximation, quite opposite to the $a(T)$ expansion in conventional high temperature QCD perturbation theory (pQCD).

The question arises: which value of the gravitational constant $G_5$ should we choose in the 5-dim gravity action? In principle, $G_5$ can be chosen to reproduce the Stefan-Boltzmann limit at high temperatures, i.e.

$$\frac{\pi^3 T^3}{16G_5} = (N_c^2 - 1) \frac{\pi^2 T_c^4}{45},$$  \hspace{1cm} (15)

If one believes that the high temperature region $T \sim (100T_c - 1000T_c)$ is a perturbative regime which can be described by the perturbative $\beta$-function, then this model reaches the Stefan-Boltzmann limit much slower than what lattice data suggest [11].

Figure 3: Trace anomaly density $(\epsilon - 3p)/T^4$ as a function of $T$ (in units of $T_c$).
In this model from the $Q\bar{Q}$-potential. The mapping, however, between the $a$ obtained from the gravity theory and $\alpha_{\infty}$ has not been achieved. The thermal couplings $\alpha(z_0)$ in the gravity model of Ref. [13] are very much smaller than the $\alpha(T)$ in the corresponding QCD-lattice simulations, thereby making the calculation compatible with the asymptotic Stefan-Boltzmann pressure. The swift change from a perturbative to a nonperturbative $\beta$-function facilitates the steep rise of the thermodynamic functions at low temperatures.

The description of Ref. [11] does well for all quantities which are calculated in the string framework, like the free energy of a $Q\bar{Q}$-pair. In a forthcoming work [8] we will demonstrate the agreement between the computation of the free energy from the Bekenstein-Hawking entropy formula presented here, and the method followed in Ref. [5] based on the regularization of the Einstein-Hilbert action of Eq. (1). We will also extend the computation to other thermodynamic observables, like the Polyakov loop and the spatial string tension.

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