FRAME MULTIPLIERS FOR DISCRETE FRAMES ON QUATERNIONIC HILBERT SPACES

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ABSTRACT. In this note, following the complex theory, we examine discrete controlled frames, discrete weighted frames and frame multipliers in a non-commutative setting, namely in a left quaternionic Hilbert space. In particular, we show that the controlled frames are equivalent to usual frames under certain conditions. We also study connection between frame multipliers and weighted frames in the same Hilbert space.

1. Introduction

The notion of frames in Hilbert spaces was introduced by Duffin and Schaeffer in 1952 to address some very deep problems in non-harmonic Fourier series [1]. However the fundamental concept of frames was revived in 1980s by Daubechies, Grossmann and Meyer [2, 3], who showed its significance in signal processing. Frame is a spanning set of vectors which are generally overcomplete (redundant) in a Hilbert space. Therefore a typical frame contains more frame vectors than the dimension of the space and each vector in the space will have infinitely many representations with respect to the frame. This redundancy of frames is the key to their success in applications.

Nowadays frames have broad applications in Mathematics and Engineering in several areas including sampling theory[4], operator theory[5], harmonic analysis[6], wavelet theory[7], wireless communication[8, 9], data transmission with erasures[10, 11], filter banks[12], signal processing[13, 14], image processing[15], geophysics[16] and quantum computing[17]. Hilbert spaces can be defined over the fields $\mathbb{R}$, the set of all real numbers, $\mathbb{C}$, the set of all complex numbers, and $\mathbb{H}$, the set of all quaternions only[18]. The fields $\mathbb{R}$ and $\mathbb{C}$ are associative and commutative but quaternions form a non-commutative associative algebra and this feature highly restricted mathematicians to work out a well-formed theory of functional analysis and harmonic analysis on quaternionic Hilbert spaces. The quaternionic frames have been developed in the mathematical point of view very recently[19, 20]. The applications are yet to be identified and analyzed in terms of these frames.

Frame multiplier is an operator which was introduced by P. Balazs for frames in Hilbert spaces[21]. In these multipliers the analysis coefficients are multiplied by a fixed sequence (called the symbol) before re-synthesis. Fundamental properties of this multiplier were investigated in [21]. Frame multipliers are interesting not only from a theoretical point
of view but also for applications. For example, it is useful in psycho-acoustical modeling \cite{22}, denoising \cite{23}, computational auditory scene analysis \cite{24}, virtual acoustics \cite{25} and seismic data analysis \cite{26}.

The extensions of frames in Hilbert spaces include weighted and controlled frames and these extensions were introduced recently to improve the numerical efficiency of iterative algorithms for inverting the frame operator\cite{27}. In this paper we investigate the connection between the weighted frames and frame multipliers for discrete frames in left quaternionic Hilbert spaces along the lines of the argument of P. Balazs et al. \cite{27}.

This article is organized as follows: In section 2, we collect basic notations and some preliminary results about quaternions and frames as needed for the development of the results obtained in this article. In section 3, we present the concept of controlled frames in quaternionic Hilbert spaces and we will show that controlled frames are equivalent to usual frames under certain conditions. In section 4, we investigate the weighted frames and frame multipliers. We also investigate how the frame multipliers are connected with weighted frames in quaternionic Hilbert spaces. Section 5 ends the article with a conclusion.

2. Mathematical preliminaries

We recall few facts about quaternions, quaternionic Hilbert spaces and quaternion functional analysis which may not be very familiar to the reader.

2.1. Quaternions. Let $\mathbb{H}$ denote the field of quaternions. Its elements are of the form $q = x_0 + x_1i + x_2j + x_3k$, where $x_0, x_1, x_2$ and $x_3$ are real numbers, and $i, j, k$ are imaginary units such that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. The quaternionic conjugate of $q$ is defined to be $\overline{q} = x_0 - x_1i - x_2j - x_3k$. Quaternions do not commute in general. However $q$ and $\overline{q}$ commute, and quaternions commute with real numbers. $|q|^2 = qq = \overline{q}q$ and $\overline{qq} = \overline{q}q$.

2.2. Left Quaternionic Hilbert Space. Let $V^L_{\mathbb{H}}$ is a vector space under left multiplication by quaternionic scalars, where $\mathbb{H}$ stands for the quaternion algebra. For $f, g, h \in V^L_{\mathbb{H}}$ and $q \in \mathbb{H}$, the inner product

$$\langle \cdot | \cdot \rangle : V^L_{\mathbb{H}} \times V^L_{\mathbb{H}} \rightarrow \mathbb{H}$$

satisfies the following properties:

(a) $\langle \overline{f} | g \rangle = \langle g | f \rangle$
(b) $\|f\|^2 = \langle f | f \rangle > 0$ unless $f = 0$, a real norm
(c) $\langle f | g + h \rangle = \langle f | g \rangle + \langle f | h \rangle$
(d) $\langle qf | g \rangle = \langle f | g \rangle q$
(e) $\langle f | qg \rangle = \langle f | g \rangle \overline{q}$.

where $\overline{q}$ stands for the quaternionic conjugate. We assume that the space $V^L_{\mathbb{H}}$ is complete under the norm given above. Then, together with $\langle \cdot | \cdot \rangle$ this defines a right quaternionic Hilbert space, which we shall assume to be separable. Quaternionic Hilbert spaces share most of the standard properties of complex Hilbert spaces. In particular, the Cauchy-Schwartz inequality holds on quaternionic Hilbert spaces as well as the Riesz representation theorem for their duals.
In the left quaternionic Hilbert space scalar multiplication of a vector is always from the left to a vector. Therefore, the Dirac bra-ket notation can be adapted as
\begin{equation}
|q\phi\rangle = |\phi\rangle q, \langle \phi | = |\phi\rangle^\dagger, \langle q\phi | = q\langle \phi |
\end{equation}
for a left quaternionic Hilbert space, with $|\phi\rangle$ denoting the vector $\phi$ and $\langle \phi |$ its dual vector. Let $T$ be an operator on a left quaternionic Hilbert space $V^L_H$ with domain $V^L_H$. The adjoint $T^\dagger$ of $T$ is defined as
\begin{equation}
\langle \psi | T\phi \rangle = \langle T^\dagger \psi | \phi \rangle; \quad \text{for all } \phi, \psi \in V^L_H.
\end{equation}
An operator $T$ is said to be self-adjoint if $T = T^\dagger$.

Let $D(T)$ denotes the domain of $T$. The operator $T$ is said to be left linear if
\begin{align*}
T(\phi + \psi) &= T\phi + T\psi, \\
T(q\phi) &= q(T\phi)
\end{align*}
for all $\phi, \psi \in D(T)$ and $q \in \mathbb{H}$. The set of all left linear operators in $V^L_H$ will be denoted by $\mathcal{L}(V^L_H)$. For a given $T \in \mathcal{L}(V^L_H)$, the range and the kernel will be
\begin{align*}
\text{ran}(T) &= \{ \psi \in V^L_H | T\phi = \psi \text{ for } \phi \in D(T) \} \\
\text{ker}(T) &= \{ \phi \in D(T) | T\phi = 0 \}.
\end{align*}

We call an operator $T \in \mathcal{L}(V^L_H)$ is bounded if
\begin{equation*}
\|T\| = \sup_{\|\phi\| = 1} \|T\phi\| < \infty.
\end{equation*}
or equivalently, there exists $M \geq 0$ such that $\|T\phi\| \leq M\|\phi\|$ for $\phi \in D(T)$. The set of all bounded left linear operators in $V^L_H$ will be denoted by $\mathcal{B}(V^L_H)$.

**Proposition 2.1.** [28] Let $T \in \mathcal{B}(V^L_H)$. Then $T$ is self-adjoint if and only if for each $\phi \in V^L_H$, $\langle T\phi | \phi \rangle \in \mathbb{R}$.

**Definition 2.2.** Let $T_1$ and $T_2$ be self-adjoint operators on $V^L_H$. Then $T_1 \leq T_2$ ($T_1$ less or equal to $T_2$) or equivalently $T_2 \geq T_1$ if $\langle T_1\phi | \phi \rangle \leq \langle T_2\phi | \phi \rangle$ for all $\phi \in V^L_H$. In particular $T_1$ is called positive if $T_1 \geq 0$ or $\langle T_1 f | f \rangle \geq 0$, for all $f \in V^L_H$.

**Theorem 2.3.** [29] Let $A \in \mathcal{B}(V^L_H)$. If $A \geq 0$ then there exists a unique operator in $\mathcal{B}(V^L_H)$, indicated by $\sqrt{A} = A^{1/2}$ such that $\sqrt{A} \geq 0$ and $\sqrt{A} \sqrt{A} = A$.

**Proposition 2.4.** [29] Let $A \in \mathcal{B}(V^L_H)$. If $A \geq 0$, then $A$ is self-adjoint.

**Lemma 2.5.** Let $U^L_H$ and $V^L_H$ be left quaternion Hilbert spaces. Let $T : \mathcal{D}(A) \rightarrow V^L_H$ be a linear operator with domain $\mathcal{D}(T) \subseteq U^L_H$ and $\text{ran}(T) \subseteq V^L_H$, then the inverse $T^{-1} : \text{ran}(T) \rightarrow \mathcal{D}(T)$ exists if and only if $T\phi = 0 \Rightarrow \phi = 0$.

**Proof.** Since the non-commutativity of quaternions does not play a role in the proof, it follows from its complex counterpart. \hfill \Box

**Lemma 2.6.** Let $T \in \mathcal{B}(V^L_H)$ be a self-adjoint operator, then
\begin{equation}
\|T\| = \sup_{\|\phi\| = 1} |\langle \phi | T\phi \rangle|
\end{equation}
Proof. In fact, the non-commutativity of quaternions does not play a role in the proof, the proof follows from its complex counterpart. One may also see [28] where a proof is given. □

We define $\mathcal{GL}(V^L_H)$, the set of all bounded linear operators in $V^L_H$ with bounded inverse.

$$\mathcal{GL}(V^L_H) = \{ T : V^L_H \rightarrow V^L_H : T \text{ bounded and } T^{-1} \text{ bounded} \}.$$ 

Also $\mathcal{GL}^+(V^L_H)$ is the set of positive operators in $\mathcal{GL}(V^L_H)$.

**Proposition 2.7.** Let $T : V^L_H \rightarrow V^L_H$ be a left linear operator. Then the following are equivalent statements:

i. There exist $m > 0$ and $M < \infty$ such that $mI_{V^L_H} \leq T \leq MI_{V^L_H}$;

ii. $T$ is positive and there exist $m > 0$ and $M < \infty$ such that $m \| f \|^2 \leq \| T^\frac{1}{2} f \|^2 \leq M \| f \|^2$;

iii. $T$ is positive and $T^\frac{1}{2} \in \mathcal{GL}(V^L_H)$;

iv. There exists a self-adjoint operator $A \in \mathcal{GL}(V^L_H)$ such that $A^2 = T$;

v. $T \in \mathcal{GL}^+(V^L_H)$.

**Proof.** For (i.)$\Rightarrow$(ii.), suppose that $m > 0$ and $M < \infty$ such that $mI_{V^L_H} \leq T \leq MI_{V^L_H}$.

Let $f \in V^L_H$ then $mI_{V^L_H} f \leq (T f) f \leq M f$, which follows that

$$m \| f \|^2 \leq \langle T f, f \rangle \leq \langle M f, f \rangle.$$ 

Hence $\langle T f, f \rangle \geq m \| f \|^2 \geq 0$, as $m > 0$. Therefore $\langle T f, f \rangle \geq 0$ and $T$ is positive.

Since $T$ is positive, from Theorem 2.3 there exists a unique operator in $\mathcal{B}(V^L_H)$, indicated by $\sqrt{T} = T^{1/2}$ such that $\sqrt{T} \geq 0$ and $\sqrt{T} \sqrt{T} = T$.

Now equation (2.4) becomes

$$m \| f \|^2 \leq \langle T^{1/2}T^{1/2} f, f \rangle \leq M \| f \|^2.$$ 

It follows that

$$m \| f \|^2 \leq \langle T^{1/2} f, (T^{1/2})^{-1} f \rangle \leq M \| f \|^2$$

and

$$m \| f \|^2 \leq \langle T^{1/2} f, T^{1/2} f \rangle \leq M \| f \|^2,$$

as $T^{1/2}$ is positive.

Therefore $m \| f \|^2 \leq \| T^\frac{1}{2} f \|^2 \leq M \| f \|^2$.

For (ii)$\Rightarrow$(i), suppose that $T$ is positive and there exists $m > 0$ and $M < \infty$ such that $m \| f \|^2 \leq \| T^\frac{1}{2} f \|^2 \leq M \| f \|^2$. Then

$$m \| f \|^2 \leq \langle T^{1/2} f, T^{1/2} f \rangle \leq M \| f \|^2.$$ 

It follows that

$$m \| f \|^2 \leq \langle (T^{1/2})^{-1} T^{1/2} f, f \rangle \leq M \| f \|^2$$

and

$$m \| f \|^2 \leq \langle T^{1/2} T^{1/2} f, f \rangle \leq M \| f \|^2,$$

as $T^{1/2}$ is positive.

Hence

$$m \| f \|^2 \leq \langle T f, f \rangle \leq M \| f \|^2.$$
Therefore $m I_{V_{\mathbb{H}}^L} \leq T \leq M I_{V_{\mathbb{H}}^L}$.

For (ii.) $\Rightarrow$ (iii.), suppose that $T$ is positive and there exists $m > 0$ and $M < \infty$ such that $m \|f\|^2 \leq \left\| T^{\frac{1}{2}} f \right\|^2 \leq M \|f\|^2$. Since $T$ is positive, from Theorem 2.3, $T^{\frac{1}{2}}$ is bounded. Let $f \in V_{\mathbb{H}}^L$, assume that $T^{\frac{1}{2}} f = 0$ then $m \|f\|^2 \leq \|0\|^2 \leq M \|f\|^2$. It follows that $f = 0$, and from Lemma 2.5, $(T^{\frac{1}{2}})^{-1} : V_{\mathbb{H}}^L \rightarrow V_{\mathbb{H}}^L$ exists. For $f \in V_{\mathbb{H}}^L$, there exists $g \in V_{\mathbb{H}}^L$ such that $T^{\frac{1}{2}} f = g$. That is $f = (T^{\frac{1}{2}})^{-1} g$. Now $m \|f\|^2 \leq \left\| T^{\frac{1}{2}} f \right\|^2$ implies

$$m \left\| (T^{\frac{1}{2}})^{-1} g \right\|^2 \leq \|g\|^2$$

and

$$\left\| (T^{\frac{1}{2}})^{-1} g \right\| \leq \frac{1}{\sqrt{m}} \|g\|.$$  

It follows that $(T^{\frac{1}{2}})^{-1}$ is bounded and hence $T^{\frac{1}{2}} \in \mathcal{GL}(V_{\mathbb{H}}^L)$. 

For (iii.) $\Rightarrow$ (ii), suppose that $T$ is positive and $T^{\frac{1}{2}} \in \mathcal{GL}(V_{\mathbb{H}}^L)$. Then $T^{\frac{1}{2}}$ is bounded and $(T^{\frac{1}{2}})^{-1}$ is also bounded. Therefore, one may conclude that there exists $m > 0$ and $M < \infty$ such that $m \|f\|^2 \leq \left\| T^{\frac{1}{2}} f \right\|^2 \leq M \|f\|^2$.

For (iii.) $\Rightarrow$ (iv.), assume that $T$ is positive and $T^{\frac{1}{2}} \in \mathcal{GL}(V_{\mathbb{H}}^L)$. Since $T$ is positive, from Theorem 2.3, there exists a unique operator in $\mathcal{B}(V_{\mathbb{H}}^L)$, indicated by $\sqrt{T} = T^{1/2}$ such that $\sqrt{T} \geq 0$ and $(\sqrt{T})^2 = T$. If we take $A = T^{1/2}$ then $A$ is self adjoint as $\sqrt{T} \geq 0$ and $A \in \mathcal{GL}(V_{\mathbb{H}}^L)$. Hence there exists a self-adjoint operator $A \in \mathcal{GL}(V_{\mathbb{H}}^L)$ such that $A^2 = T$.

For (iv.) $\Rightarrow$ (iii.), suppose that there exists a self-adjoint operator $A \in \mathcal{GL}(V_{\mathbb{H}}^L)$ such that $A^2 = T$. If we take $A = T^{1/2}$ then $T^{\frac{1}{2}} \in \mathcal{GL}(V_{\mathbb{H}}^L)$ and $T$ is positive.

For (iv.) $\Rightarrow$ (v.), assume that there exists a self-adjoint operator $A \in \mathcal{GL}(V_{\mathbb{H}}^L)$ such that $A^2 = T$. Then clearly $T \in \mathcal{GL}^+(V_{\mathbb{H}}^L)$.

For (v.) $\Rightarrow$ (iv.), assume that $T \in \mathcal{GL}^+(V_{\mathbb{H}}^L)$. Then $T$ is positive and bounded. From Theorem 2.3, there exists a self-adjoint operator $A(= T^{1/2})$ in $\mathcal{GL}(V_{\mathbb{H}}^L)$ such that $A \geq 0$ and $A A = T$. It follows that there exists a self-adjoint operator $A \in \mathcal{GL}(V_{\mathbb{H}}^L)$ such that $A^2 = T$. 

\section{Frames and Frame operators}

Let $V_{\mathbb{H}}^L$ be a finite dimensional left quaternion Hilbert space. A countable family of elements $\{\phi_k\}_{k \in I}$ in $V_{\mathbb{H}}^L$ is a frame for $V_{\mathbb{H}}^L$ if there exist constants $A, B > 0$ such that

$$A \|\psi\|^2 \leq \sum_{k \in I} |\langle \psi | \phi_k \rangle|^2 \leq B \|\psi\|^2,$$

for all $\psi \in V_{\mathbb{H}}^R$.

The numbers $A$ and $B$ are called frame bounds. They are not unique. The \textit{optimal lower frame bound} is the supremum over all lower frame bounds, and the \textit{optimal upper frame bound} is the infimum over all upper frame bounds. The frame $\{\phi_k\}_{k \in I}$ is said to be normalized if $\|\phi_k\| = 1$, for all $k \in I$.

Let $\{\phi_k\}_{k \in I}$ be a frame on a left quaternionic Hilbert space $V_{\mathbb{H}}^L$ and define a linear mapping

$$T : \mathbb{H}^{|I|} \rightarrow V_{\mathbb{H}}^L, \ T \{q_k\}_{k \in I} = \sum_{k \in I} q_k \phi_k, \ q_k \in \mathbb{H},$$

where $I$ is a finite positive integer set.
where \(|I|\) is the cardinality of \(I\). \(T\) is usually called the pre-frame operator, or the synthesis operator.

The adjoint operator

\[
T^\dagger : \mathbb{H}^{|I|} \rightarrow \mathbb{H}^{|I|}, \text{ by } T^\dagger \psi = \{\langle \psi | \phi_k \rangle \}_{k \in I}
\]

is called the analysis operator.

By composing \(T\) with its adjoint we obtain the frame operator

\[
S : V^L_H \rightarrow V^L_H, \ S \psi = TT^\dagger \psi = \sum_{k \in I} \langle \psi | \phi_k \rangle \phi_k.
\]

Note that in terms of the frame operator, for \(\psi \in V^L_H\)

\[
\langle S\psi | \psi \rangle = \left( \sum_{k=1}^{m} \langle \psi | \phi_k \rangle \phi_k \right) \overline{\psi} = \sum_{k=1}^{m} \langle \psi | \phi_k \rangle \overline{\langle \phi_k | \psi \rangle} = \sum_{k=1}^{m} |\langle \phi_k | \psi \rangle|^2.
\]

**Proposition 2.8.** \([19]\) Let \(\{\phi_k\}_{k \in I}\) be a frame for \(V^L_H\) with frame operator \(S\). Then

i. \(S\) is invertible and self-adjoint.

ii. Every \(\psi \in V^L_H\), can be represented as

\[
\psi = \sum_{k \in I} \langle \psi | S^{-1} \phi_k \rangle \phi_k = \sum_{k \in I} \langle \psi | \phi_k \rangle S^{-1} \phi_k.
\]

### 3. Controlled Frames in Left Quaternion Hilbert Spaces

Controlled frames were first introduced for spherical wavelets in \([30]\) to get a numerically more efficient approximation algorithm. For general frames, it was developed in \([27]\). In this section the concept of controlled frames in quaternionic Hilbert spaces, along the lines of the arguments of \([27]\), is presented.

**Definition 3.1.** Let \(\mathcal{C} \in \mathcal{GL}(V^L_H)\). A countable family of vectors \(\Phi = \{\phi_k \in V^L_H : k \in I\}\) is said to be a frame controlled by the operator \(\mathcal{C}\) or the \(\mathcal{C}\)-controlled frame if there exist constants \(0 < A_{\mathcal{C}S} \leq B_{\mathcal{C}S} < \infty\) such that

\[
A_{\mathcal{C}S} \|\psi\|^2 \leq \sum_{k \in I} \langle \psi | \phi_k \rangle \langle \mathcal{C} \phi_k | \psi \rangle \leq B_{\mathcal{C}S} \|\psi\|^2,
\]

for all \(\psi \in V^L_H\).

Controlled frame operator can be defined as

\[
S_{\mathcal{C}} \psi = \sum_{k \in I} \langle \psi | \phi_k \rangle \mathcal{C} \phi_k.
\]

Now we have the frame operator

\[
S : V^L_H \rightarrow V^L_H, \ S \psi = TT^\dagger \psi = \sum_{k \in I} \langle \psi | \phi_k \rangle \phi_k.
\]
For $\mathcal{C} : V^L H \rightarrow V^L H$, consider

$$
\langle \mathcal{C} \Psi|\Psi \rangle = \mathcal{C} \left( \sum_{k \in I} \langle \Psi|\phi_k \rangle \phi_k \right) |\Psi \rangle
$$

$$
= \sum_{k \in I} \langle \Psi|\phi_k \rangle \mathcal{C} \phi_k |\Psi \rangle
$$

$$
= \sum_{k \in I} \langle \Psi|\phi_k \rangle \langle \mathcal{C} \phi_k|\Psi \rangle .
$$

Now equation (3.1) becomes

$$
(3.4) \quad A_{\mathcal{C}S} \|\Psi\|^2 \leq \langle \mathcal{C} \Psi|\Psi \rangle \leq B_{\mathcal{C}S} \|\Psi\|^2 ,
$$

for all $\psi \in V^L H$. That is, there exist constants $0 < A_{\mathcal{C}S} \leq B_{\mathcal{C}S} < \infty$ such that

$$
A_{\mathcal{C}S} I_{V^L H} \leq \mathcal{C}S \leq B_{\mathcal{C}S} I_{V^L H}.
$$

From Proposition 2.7, $\mathcal{C}S \in GL^+(V^L H)$, and the definition (3.1) is clearly equivalent to $\mathcal{C}S \in GL^+(V^L H)$.

**Proposition 3.2.** Let $\Phi = \{\phi_k \in V^L H : k \in I\}$ be a $\mathcal{C}$- controlled frame in $V^L H$ for $\mathcal{C} \in GL(V^L H)$. Then $\Phi$ is a frame in $V^L H$. Moreover $\mathcal{C}S = \mathcal{S} \mathcal{C}^\dagger$ and

$$
\sum_{k \in I} \langle \Psi|\phi_k \rangle \mathcal{C} \phi_k = \sum_{k \in I} \langle \Psi|\mathcal{C} \phi_k \rangle \phi_k ,
$$

for all $\psi \in V^L H$.

**Proof.** Let $\{\phi_k\}_{k \in I}$ be a $\mathcal{C}$- controlled frame in $V^L H$ for $\mathcal{C} \in GL(V^L H)$. Then there exist constants $0 < A_{\mathcal{C}S} \leq B_{\mathcal{C}S} < \infty$ such that

$$
(3.5) \quad A_{\mathcal{C}S} \|\psi\|^2 \leq \sum_{k \in I} \langle \psi|\phi_k \rangle \langle \mathcal{C} \phi_k|\psi \rangle \leq B_{\mathcal{C}S} \|\psi\|^2 ,
$$

for all $\psi \in V^L H$. It follows that

$$
(3.6) \quad A_{\mathcal{C}S} \langle I_{V^L H} \psi|\psi \rangle \leq \langle S_{\mathcal{C}} \psi|\psi \rangle \leq B_{\mathcal{C}S} \langle I_{V^L H} \psi|\psi \rangle ,
$$

for all $\psi \in V^L H$.

Hence $A_{\mathcal{C}S} I_{V^L H} \leq S_{\mathcal{C}} \leq B_{\mathcal{C}S} I_{V^L H}$. From Proposition 2.7, $S_{\mathcal{C}} \in GL^+(V^L H)$.

Define $\hat{S} = \mathcal{C}^{-1} S_{\mathcal{C}}$. Then $\hat{S} \in GL(V^L H)$ as $\mathcal{C}^{-1}, S_{\mathcal{C}} \in GL(V^L H)$.
Let $\psi \in V_{\mathbb{H}}^L$ then
\[
\hat{S}\psi = \mathcal{C}^{-1}S_{\mathcal{C}}\psi = \mathcal{C}^{-1}\left(\sum_{k \in I} \langle \psi | \phi_k \rangle \mathcal{C}\phi_k\right) = \sum_{k \in I} \langle \psi | \phi_k \rangle \mathcal{C}^{-1}\mathcal{C}\phi_k = \sum_{k \in I} \langle \psi | \phi_k \rangle I_{V_{\mathbb{H}}^L}\phi_k = \sum_{k \in I} \langle \psi | \phi_k \rangle \phi_k = S\psi.
\]
Hence $S$ is everywhere defined and $S \in \mathcal{GL}(V_{\mathbb{H}}^L)$. Thereby $\Phi$ is a frame in $V_{\mathbb{H}}^L$.

Since $S_{\mathcal{C}} \in \mathcal{GL}^+(V_{\mathbb{H}}^L)$, $S_{\mathcal{C}}$ is positive, therefore $S_{\mathcal{C}}$ is self-adjoint.

For $\psi \in V_{\mathbb{H}}^L$,
\[
\mathcal{C}S\psi = \mathcal{C}\left(\sum_{k \in I} \langle \psi | \phi_k \rangle \phi_k\right) = \sum_{k \in I} \langle \psi | \phi_k \rangle \mathcal{C}\phi_k = S_{\mathcal{C}}\psi.
\]
Hence $\mathcal{C} = S_{\mathcal{C}}$.

Now $S_{\mathcal{C}}^\dagger = (\mathcal{C}S)^\dagger = S_{\mathcal{C}}^\dagger = S\mathcal{C}^\dagger$. But $S_{\mathcal{C}}^\dagger = S_{\mathcal{C}}$, as $S_{\mathcal{C}}$ is self-adjoint. Therefore $S_{\mathcal{C}} = S\mathcal{C}^\dagger$.

It follows that $\mathcal{C}S = S\mathcal{C}^\dagger$.

Also for $\psi \in V_{\mathbb{H}}^L$, $\mathcal{C}S\psi = S\mathcal{C}^\dagger\psi$ and $S_{\mathcal{C}}\psi = S\mathcal{C}^\dagger\psi$. Hence
\[
\sum_{k \in I} \langle \psi | \phi_k \rangle \mathcal{C}\phi_k = S_{\mathcal{C}}\psi = S\mathcal{C}^\dagger\psi = \sum_{k \in I} \langle \mathcal{C}^\dagger\psi | \phi_k \rangle \phi_k = \sum_{k \in I} \langle \psi | \mathcal{C}\phi_k \rangle \phi_k.
\]
Thereby $\sum_{k \in I} \langle \psi | \phi_k \rangle \mathcal{C}\phi_k = \sum_{k \in I} \langle \psi | \mathcal{C}\phi_k \rangle \phi_k$. \qed

The above Proposition shows that every controlled frame is a usual frame. But if $\mathcal{C} \in \mathcal{GL}(V_{\mathbb{H}}^L)$ is self-adjoint, we can give a necessary and sufficient condition for a frame to be a controlled frame and vice-versa.

**Proposition 3.3.** Let $\mathcal{C} \in \mathcal{GL}(V_{\mathbb{H}}^L)$ be self-adjoint. The family $\{\phi_k\}_{k \in I}$ is a frame controlled by $\mathcal{C}$ if and only if it is a frame in $V_{\mathbb{H}}^L$ and $\mathcal{C}$ is positive and commutes with the frame operator $S$. 
Proof. Let $\mathcal{C} \in \mathcal{GL}(V_L^H)$ be self-adjoint. Suppose that $\{\phi_k\}_{k \in I}$ is a frame controlled by $\mathcal{C}$. Then from Proposition 3.2 $\{\phi_k\}_{k \in I}$ is a frame in $V_L^H$ and $\mathcal{C} S = S \mathcal{C}^\dagger$. Since $\mathcal{C}$ is self-adjoint, $\mathcal{C} = \mathcal{C}^\dagger$. Hence $\mathcal{C} S = S \mathcal{C}$. Thereby $\mathcal{C}$ commutes with the frame operator $S$. It follows that $\mathcal{C} = S \mathcal{C} S^{-1} = S \mathcal{C}^{-1}$ and $\mathcal{C}$ is positive.

On the other hand suppose that $\{\phi_k\}_{k \in I}$ is a frame in $V_L^H$ with frame operator $S$ and $\mathcal{C}$ is positive and commutes with $S$. Then $S \in \mathcal{GL}^+(V_L^H)$. Therefore $\mathcal{C} S = S \mathcal{C} \in \mathcal{GL}^+(V_L^H)$ and so $S \mathcal{C}$ is positive.

From Proposition 2.7 there exist $A > 0$ and $B < \infty$ such that

\[(3.7) \quad AI_{V_L^H} \leq S \mathcal{C} \leq BI_{V_L^H}.\]

For $\psi \in V_L^H$, (3.7) becomes

\[(3.8) \quad \langle AI_{V_L^H} \psi \mid \psi \rangle \leq \langle S \mathcal{C} \psi \mid \psi \rangle \leq \langle BI_{V_L^H} \psi \mid \psi \rangle.\]

It follows that

\[(3.9) \quad A \|\psi\|^2 \leq \sum_{k \in I} \langle \psi \mid \phi_k \rangle \langle \mathcal{C} \phi_k \mid \psi \rangle \leq B \|\psi\|^2\]

Hence $\{\phi_k\}_{k \in I}$ is a frame controlled by $\mathcal{C}$.

\[\square\]

4. Weighted frames and frame multipliers in $V_L^H$

In this section we present connection between frame multipliers and weighted frames in a left quaternionic Hilbert space.

**Definition 4.1.** Let $\{\phi_k\}_{k \in I}$ be a sequence of elements in $V_L^H$ and $\{\omega_k\}_{k \in I} \subseteq \mathbb{R}^+$ a sequence of positive weights. This pair is called a $\omega-$ frame for $V_L^H$ if there exist constants $A > 0$ and $B < \infty$ such that

\[(4.1) \quad A \|\psi\|^2 \leq \sum_{k \in I} \omega_k \|\langle \psi \mid \phi_k \rangle\|^2 \leq B \|\psi\|^2,\]

for all $\psi \in V_L^H$.

**Definition 4.2.** A sequence $\{\zeta_n\}$ is called semi-normalized if there are bounds $b \geq a > 0$ such that $a \leq |\zeta_n| \leq b$.

**Definition 4.3.** Let $U_L^H$, $V_L^H$ be left quaternionic Hilbert spaces, let $\{\psi_k\}_{k \in I} \subseteq U_L^H$ and $\{\phi_k\}_{k \in I} \subseteq V_L^H$ be frames. Fix $m = \{m_k\} \in \ell^\infty(I)$. Define the operator $M_{m,\{\phi_k\},\{\psi_k\}} : U_L^H \rightarrow V_L^H$ the frame multiplier for the frames $\{\psi_k\}$ and $\{\phi_k\}$, as the operator

\[(4.2) \quad M_{m,\{\phi_k\},\{\psi_k\}}(h) = \sum_{k \in I} m_k \langle h \mid \psi_k \rangle \phi_k; \quad h \in U_L^H.\]

The sequence $m$ is called the symbol of $M$. We will denote $M_{m,\{\phi_k\}} = M_{m,\{\phi_k\},\{\phi_k\}}$

**Proposition 4.4.** Let $\mathcal{C} \in \mathcal{GL}(V_L^H)$ be self-adjoint and diagonal on $\Phi = \{\phi_k\}_{k \in I}$ and assume it generates a controlled frame. Then the sequence $\{\omega_k\}$ which verifies the relation $\mathcal{C}\phi_k = \omega_k \phi_k$ is semi normalized and positive. Furthermore $\mathcal{C} = M_{\omega,\tilde{\phi},\Phi}$, where $\tilde{\Phi} = \{L^{-1}\phi_k\}_{k \in I}$ and $L$ is the frame operator for $\Phi$. 
Proof. Since $\mathcal{C} \in \mathcal{GL}(V_H^L)$ is self-adjoint and $\Phi = \{\phi_k\}$ is a frame controlled by the operator $\mathcal{C}$, from Proposition 3.3, $\mathcal{C}$ is positive. Thereby $\mathcal{C} \in \mathcal{GL}^+(V_H^L)$. Since $\mathcal{C} \in \mathcal{GL}^+(V_H^L)$, from Proposition 2.7, there exists $A > 0$ and $\ell < \infty$ such that

\begin{equation}
A \|\psi\|^2 \leq \left\| \mathcal{C}^{-1} \psi \right\|^2 \leq B \|\psi\|^2,
\end{equation}

for all $\psi \in V_H^L$. Since $\mathcal{C}\phi_k = \omega_k \phi_k$, $\mathcal{C}^\dagger \phi_k = \sqrt{\omega_k} \phi_k$. Now equation 4.3 gives

\begin{equation}
0 < A \leq \omega_k \leq B.
\end{equation}

Hence the sequence $\{\omega_k\}$ is positive and semi-normalized.

Since $\mathcal{C} \in \mathcal{GL}(V_H^L)$ is self-adjoint and $\Phi = \{\phi_k\}$ is a frame in $V_H^L$. Let $\psi \in V_H^L$ then, by Proposition 2.8, $\psi = \sum_{k\in I} \langle \psi | \phi_k \rangle L^{-1} \phi_k$, where $L$ is the frame operator for $\Phi = \{\phi_k\}$.

Now

\[
M_{\omega,\overline{\Phi},\Phi} \psi = \sum_{k\in I} \omega_k \langle \psi | \phi_k \rangle \overline{\phi_k} \\
= \sum_{k\in I} \langle \psi | \phi_k \rangle \omega_k \overline{\phi_k} \text{ as } \omega_k \text{ is real} \\
= \sum_{k\in I} \langle \psi | \phi_k \rangle \mathcal{C} \overline{\phi_k} \\
= \mathcal{C} \left( \sum_{k\in I} \langle \psi | \phi_k \rangle \overline{\phi_k} \right) \\
= \mathcal{C} \left( \sum_{k\in I} \langle \psi | \phi_k \rangle L^{-1} \phi_k \right) \\
= \mathcal{C} \psi.
\]

Hence $\mathcal{C} = M_{\omega,\overline{\Phi},\Phi}$. \hfill \square

**Lemma 4.5.** Let $\{\omega_k\}$ be a semi normalized sequence with bounds $a$ and $b$. If $\{\phi_k\}$ is a frame with bounds $A$ and $B$ then $\{\omega_k \phi_k\}$ is also a frame with bounds $a^2 A$ and $b^2 B$.

**Proof.** Since $\{\omega_k\}$ is semi normalized, there exists $b \geq a > 0$ such that

\begin{equation}
a \leq |\omega_k| \leq b.
\end{equation}

Since $\{\phi_k\}$ is a frame with bounds $A$ and $B$,

\begin{equation}
A \|\psi\|^2 \leq \sum_{k\in I} |\langle \psi | \phi_k \rangle|^2 \leq B \|\psi\|^2,
\end{equation}

for all $\psi \in V_H^L$. Hence

\[
\sum_{k\in I} |\langle \psi | \phi_k \rangle|^2 \leq \frac{B}{A} \sum_{k\in I} |\langle \omega_k \phi_k | \phi_k \rangle|^2 = \frac{B}{A} \sum_{k\in I} |\langle \omega_k \phi_k | \phi_k \rangle|^2 \\
= \frac{B}{A} \sum_{k\in I} |\langle \omega_k \phi_k | \phi_k \rangle|^2.
\]

Now let $\mathcal{C} \in \mathcal{GL}(V_H^L)$ be self-adjoint and $\Phi = \{\phi_k\}$ be a frame controlled by $\mathcal{C}$, then from Proposition 3.3, $\mathcal{C}$ is positive. Thereby $\mathcal{C} \in \mathcal{GL}^+(V_H^L)$. Since $\mathcal{C} \in \mathcal{GL}^+(V_H^L)$, from Proposition 2.7, there exists $A > 0$ and $\ell < \infty$ such that

\begin{equation}
A \|\psi\|^2 \leq \left\| \mathcal{C}^{-1} \psi \right\|^2 \leq B \|\psi\|^2,
\end{equation}

for all $\psi \in V_H^L$. Since $\mathcal{C}\phi_k = \omega_k \phi_k$, $\mathcal{C}^\dagger \phi_k = \sqrt{\omega_k} \phi_k$. Now equation 4.3 gives

\begin{equation}
0 < A \leq \omega_k \leq B.
\end{equation}

Hence the sequence $\{\omega_k\}$ is positive and semi-normalized.

Since $\mathcal{C} \in \mathcal{GL}(V_H^L)$ is self-adjoint and $\Phi = \{\phi_k\}$ is a frame in $V_H^L$. Let $\psi \in V_H^L$ then, by Proposition 2.8, $\psi = \sum_{k\in I} \langle \psi | \phi_k \rangle L^{-1} \phi_k$, where $L$ is the frame operator for $\Phi = \{\phi_k\}$.

Now

\[
M_{\omega,\overline{\Phi},\Phi} \psi = \sum_{k\in I} \omega_k \langle \psi | \phi_k \rangle \overline{\phi_k} \\
= \sum_{k\in I} \langle \psi | \phi_k \rangle \omega_k \overline{\phi_k} \text{ as } \omega_k \text{ is real} \\
= \sum_{k\in I} \langle \psi | \phi_k \rangle \mathcal{C} \overline{\phi_k} \\
= \mathcal{C} \left( \sum_{k\in I} \langle \psi | \phi_k \rangle \overline{\phi_k} \right) \\
= \mathcal{C} \left( \sum_{k\in I} \langle \psi | \phi_k \rangle L^{-1} \phi_k \right) \\
= \mathcal{C} \psi.
\]

Hence $\mathcal{C} = M_{\omega,\overline{\Phi},\Phi}$. \hfill \square

**Lemma 4.5.** Let $\{\omega_k\}$ be a semi normalized sequence with bounds $a$ and $b$. If $\{\phi_k\}$ is a frame with bounds $A$ and $B$ then $\{\omega_k \phi_k\}$ is also a frame with bounds $a^2 A$ and $b^2 B$.

**Proof.** Since $\{\omega_k\}$ is semi normalized, there exists $b \geq a > 0$ such that

\begin{equation}
a \leq |\omega_k| \leq b.
\end{equation}

Since $\{\phi_k\}$ is a frame with bounds $A$ and $B$,

\begin{equation}
A \|\psi\|^2 \leq \sum_{k\in I} |\langle \psi | \phi_k \rangle|^2 \leq B \|\psi\|^2,
\end{equation}

for all $\psi \in V_H^L$. Hence

\[
\sum_{k\in I} |\langle \psi | \phi_k \rangle|^2 \leq \frac{B}{A} \sum_{k\in I} |\langle \omega_k \phi_k | \phi_k \rangle|^2 = \frac{B}{A} \sum_{k\in I} |\langle \omega_k \phi_k | \phi_k \rangle|^2 \\
= \frac{B}{A} \sum_{k\in I} |\langle \omega_k \phi_k | \phi_k \rangle|^2.
\]
for all $\psi \in V^L_H$.
Let $\psi \in V^L_H$ then $|\langle \psi | \omega_k \phi_k \rangle|^2 = |\omega_k|^2 |\langle \psi | \phi_k \rangle|^2$. Thereby

$$
\sum_{k \in I} |\langle \psi | \omega_k \phi_k \rangle|^2 = \sum_{k \in I} |\omega_k|^2 |\langle \psi | \phi_k \rangle|^2 \\
\leq b^2 \sum_{k \in I} |\langle \psi | \phi_k \rangle|^2 \\
\leq b^2 B \|\psi\|^2
$$

Similarly one can prove that $\sum_{k \in I} |\langle \psi | \omega_k \phi_k \rangle|^2 \geq a^2 A \|\psi\|^2$.

Hence $a^2 A \|\psi\|^2 \leq \sum_{k \in I} |\langle \psi | \omega_k \phi_k \rangle|^2 \leq b^2 B \|\psi\|^2$, for all $\psi \in V^L_H$.

It follows that $\{\omega_k \phi_k \}$ is a frame in $V^L_H$ with frame bounds $a^2 A$ and $b^2 B$.

**Lemma 4.6.** Let $\Phi = \{\phi_k\}$ be a frame for $V^L_H$. Let $m = \{m_k\}$ be a positive semi-normalized sequence. Then the multiplier $M_{m, \Phi}$ is the frame operator of the frame $\{\sqrt{m_k} \phi_k\}$ and therefore it is positive, self-adjoint and invertible. If $\{m_k\}$ is negative and semi-normalized then $M_{m, \Phi}$ is negative, self-adjoint and invertible.

**Proof.** We have the frame multiplier for the frame $\Phi = \{\phi_k\}$

\begin{equation}
M_{m, \Phi} \psi = \sum_{k} m_k \langle \psi | \phi_k \rangle \phi_k,
\end{equation}

where $m = \{m_k\}$ is the weight sequence.

Since $\{m_k\}$ is semi-normalized sequence and $\Phi = \{\phi_k\}$ is a frame for $V^L_H$, from Lemma 4.5 $\{\sqrt{m_k} \phi_k\}$ is a frame for $V^L_H$.

Let $\psi \in V^L_H$ then

$$
M_{m, \Phi} \psi = \sum_{k} m_k \langle \psi | \phi_k \rangle \phi_k \\
= \sum_{k} \langle \psi | \sqrt{m_k} m_k \phi_k \rangle \phi_k \\
= \sum_{k} \langle \psi | \sqrt{m_k} \phi_k \rangle \sqrt{m_k} \phi_k \\
= S_{\sqrt{m_k} \phi_k} \psi,
$$

where $S_{\sqrt{m_k} \phi_k}$ is the frame operator for the frame $\sqrt{m_k} \phi_k$.

Hence the frame multiplier $M_{m, \Phi}$ is the frame operator of the frame $\{\sqrt{m_k} \phi_k\}$.

Since the frame operator is always positive, self-adjoint and invertible, $M_{m, \Phi}$ is positive, self-adjoint and invertible.
If \( \{m_k\} \) is negative then \( m_k < 0 \), for all \( k \). So that \( m_k = -\sqrt{|m_k|^2} \). Thereby

\[
M_{m, \Phi} \psi = -\sum_k \sqrt{|m_k|^2} \langle \psi | \phi_k \rangle \phi_k
\]

\[
= -\sum_k \langle \psi | \sqrt{|m_k|^2} \phi_k \rangle \phi_k
\]

\[
= -\sum_k \langle \psi | \sqrt{|m_k|^2} \phi_k \rangle \sqrt{|m_k|^2} \phi_k
\]

\[
= -S \sqrt{|m_k|^2} \phi_k \psi,
\]

Hence \( M_{m, \Phi} \) is negative, self-adjoint and invertible as the frame operator \( S \sqrt{|m_k|^2} \) is always positive, self-adjoint and invertible.

**Theorem 4.7.** Let \( \Phi = \{\phi_k\} \) be a sequence of elements in \( V_{\text{HI}}^L \). Let \( \{\omega_k\} \) be a sequence of positive, semi-normalized weights. Then the following conditions are equivalent:

i. \( \{\phi_k\} \) is a frame.

ii. \( M_{m, \Phi} \) is a positive and invertible operator

iii. There are constants \( A > 0 \) and \( B < \infty \) such that

\[
A \|\psi\|^2 \leq \sum_{k \in I} \omega_k \|\langle \psi | \phi_k \rangle\|^2 \leq B \|\psi\|^2.
\]

iv. \( \{\sqrt{\omega_k} \phi_k\} \) is a frame.

v. \( \{\omega_k \phi_k\} \) is a frame, i.e., the pair \( \{\omega_k\}, \{\phi_k\} \) forms a weighted frame.

**Proof.** For i.\( \Rightarrow \) ii., suppose that \( \{\phi_k\} \) is a frame in \( V_{\text{HI}}^L \). From Lemma 4.6, the multiplier \( M_{m, \Phi} \) is a frame operator of the frame \( \{\sqrt{m_k} \omega_k \phi_k\} \) and therefore it is positive and invertible.

For ii.\( \Leftrightarrow \) iii., suppose that \( M_{m, \Phi} \) is a positive and invertible operator. Then \( M_{m, \Phi} \in \mathcal{GL}^+(V_{\text{HI}}^L) \). From Proposition 2.7, there exists \( A > 0 \) and \( B < \infty \) such that \( AI_{V_{\text{HI}}^L} \leq M_{m, \Phi} \leq BI_{V_{\text{HI}}^L} \). It follows that for \( \psi \in V_{\text{HI}}^L \),

\[
A \|\psi\|^2 \leq \langle M_{m, \Phi} \psi | \psi \rangle \leq B \|\psi\|^2.
\]

If we take \( m = \{\omega_k\} \) then

\[
A \|\psi\|^2 \leq \left\langle \sum \omega_k \langle \psi | \phi_k \rangle \phi_k | \psi \rangle \right\rangle \leq B \|\psi\|^2
\]

and

\[
A \|\psi\|^2 \leq \sum \omega_k \|\langle \psi | \phi_k \rangle\|^2 \leq B \|\psi\|^2.
\]

So that there are constants \( A > 0 \) and \( B < \infty \) such that

\[
A \|\psi\|^2 \leq \sum \omega_k \|\langle \psi | \phi_k \rangle\|^2 \leq B \|\psi\|^2
\]

i.e. the pair \( \{\omega_k\}, \{\phi_k\} \) forms a \( w \)-frame.

On the other hand suppose that there exist constants \( A > 0 \) and \( B < \infty \) such that

\[
A \|\psi\|^2 \leq \sum \omega_k \|\langle \psi | \phi_k \rangle\|^2 \leq B \|\psi\|^2.
\]

It follows that \( A \|\psi\|^2 \leq \langle M_{m, \Phi} \psi | \psi \rangle \leq B \|\psi\|^2 \), for all \( \psi \in V_{\text{HI}}^L \). Hence \( AI_{V_{\text{HI}}^L} \leq M_{m, \Phi} \leq BI_{V_{\text{HI}}^L} \). From Proposition 2.7, \( M_{m, \Phi} \in \mathcal{GL}^+(V_{\text{HI}}^L) \).

Thereby \( M_{m, \Phi} \) is positive and invertible.

For iii.\( \Leftrightarrow \) iv, suppose that there exist constants \( A > 0 \) and \( B < \infty \) such that

\[
A \|\psi\|^2 \leq \sum \omega_k \|\langle f | \phi_k \rangle\|^2 \leq B \|\psi\|^2
\]
for all $\psi \in V^L_H$. Thereby
\begin{equation}
A \|\psi\|^2 \leq \sum |\langle \psi | \sqrt{\omega_k} \phi_k \rangle|^2 \leq B \|\psi\|^2,
\end{equation}
for all $\psi \in V^L_H$, as $\{\omega_k\}$ is the sequence of positive weight. Hence $\{\sqrt{\omega_k} \psi_k\}$ is a frame in $V^L_H$. In similar argument the converse part can be obtained.

For i$\Leftrightarrow$iv, suppose that $\{\phi_k\}$ is a frame and $\{\omega_k\}$ is a sequence of positive, semi-normalized weights. Then there exist constants $A > 0$ and $B < \infty$ such that
\begin{equation}
A \|\psi\|^2 \leq \sum |\langle \psi | \phi_k \rangle|^2 \leq B \|\psi\|^2,
\end{equation}
for all $\psi \in V^L_H$ and there exist constants $b \geq a > 0$ such that
\begin{equation}
a \leq |\omega_k| \leq b.
\end{equation}
Let $\psi \in V^L_H$, from equations 4.11 and 4.12
\[
\sum |\langle \psi | \sqrt{\omega_k} \phi_k \rangle|^2 = \sum |\omega_k| |\langle \psi | \phi_k \rangle|^2 \leq bB \|\psi\|^2.
\]
In similar way one can prove that $\sum |\langle \psi | \sqrt{\omega_k} \phi_k \rangle|^2 \geq aA \|\psi\|^2$. Hence
\begin{equation}
aA \|\psi\|^2 \leq \sum |\langle \psi | \sqrt{\omega_k} \phi_k \rangle|^2 \leq bB \|\psi\|^2,
\end{equation}
for all $\psi \in V^L_H$. Thereby $\{\sqrt{\omega_k} \psi_k\}$ is a frame in $V^L_H$ with bounds $aA$ and $bB$.

On the other hand suppose that $\{\sqrt{\omega_k} \phi_k\}$ is a frame. Then there exist constants $A > 0$ and $B < \infty$ such that
\begin{equation}
A \|\psi\|^2 \leq \sum |\langle \psi | \sqrt{\omega_k} \phi_k \rangle|^2 \leq B \|\psi\|^2,
\end{equation}
for all $\psi \in V^L_H$. It is clear that if $\{w_k\}$ is positive and semi-normalized then $\{w_k^{-1}\}$ is also positive and semi-normalized. Then there are bounds $s \geq r > 0$ such that
\begin{equation}
r \leq |\omega_k^{-1}| \leq s.
\end{equation}
Let $\psi \in V^L_H$, from equations 4.13 and 4.14
\[
\sum \langle \psi | \phi_k \rangle|^2 = \sum |\langle \psi | \sqrt{\omega_k w_k^{-1}} \phi_k \rangle|^2 = \sum |\omega_k^{-1}| \langle \psi | \sqrt{\omega_k} \phi_k \rangle|^2 \leq sB \|\psi\|^2.
\]
Similarly one can get $\sum |\langle \psi | \phi_k \rangle|^2 \geq rA \|\psi\|^2$. It follows that
\begin{equation}
rA \|\psi\|^2 \leq \sum |\langle \psi | \phi_k \rangle|^2 \leq sB \|\psi\|^2,
\end{equation}
for all $\psi \in V^L_H$. Thereby $\{\psi_k\}$ is a frame in $V^L_H$ with frame bounds $rA$ and $sB$.

For $v \Rightarrow i$, suppose that $\{\omega_k \phi_k\}$ is a frame. Then there exists $A > 0$ and $B < \infty$ such that
\begin{equation}
A \|\psi\|^2 \leq \sum |\langle \psi | \omega_k \phi_k \rangle|^2 \leq B \|\psi\|^2,
\end{equation}
for all $\psi \in V^L_H$. We know that $\{\omega_k^2\}$ is positive and semi-normalized as $\{\omega_k\}$ is positive and semi-normalized. Thereby there exist constants $v \geq u > 0$ such that
\begin{equation}
u \leq |\omega_k^2| \leq v.
\end{equation}
Let $\psi \in V_{\mathbb{H}}^L$, from equations 4.16 and 4.17,

$$A \|\psi\|^2 \leq \sum |\langle \psi | \omega_k \phi_k \rangle|^2 = \sum |\omega_k|^2 |\langle \psi | \phi_k \rangle|^2 \leq v \sum |\langle \psi | \phi_k \rangle|^2.$$ 

Hence $A \frac{1}{v} \|\psi\|^2 \leq \sum |\langle \psi | \phi_k \rangle|^2$.

Similarly

$$B \|\psi\|^2 \geq \sum |\langle \psi | \omega_k \phi_k \rangle|^2 = \sum |\omega_k|^2 |\langle \psi | \phi_k \rangle|^2 \geq u \sum |\langle \psi | \phi_k \rangle|^2.$$ 

Hence $\sum |\langle \psi | \phi_k \rangle|^2 \leq \frac{B}{u} \|\psi\|^2$. It follows that

$$\frac{A}{v} \|\psi\|^2 \leq \sum |\langle \psi | \phi_k \rangle|^2 \leq \frac{B}{u} \|\psi\|^2,$$

for all $\psi \in V_{\mathbb{H}}^L$. Therefore $\{ \phi_k \}$ is a frame in $V_{\mathbb{H}}^L$.

i$\Rightarrow$ v follows from Lemma 4.5. This completes the proof. \hfill $\square$

5. Conclusion

We have shown that, in the discrete case, controlled frames, weighted frames and frame multipliers can be defined in a left quaternionic Hilbert space in almost the same way as they have been defined in a complex Hilbert space. The non-commutativity of quaternions does not cause any significant difficulty in the proofs of the results obtained. However, the complex numbers are two dimensional while the quaternions are four dimensional, hence the structure of quaternionic Hilbert spaces are significantly different from their complex counterparts, and therefore in the application point of view the theory developed in this note may provide some advantages or drawbacks. The applications of the discrete case developed here and the corresponding continuous theory in the quaternionic setting are yet to be worked out.

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