Random Polynomials and the Friendly Landscape

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Abstract

In hep-th/0501082 a field theoretic "toy model" for the Landscape was proposed. We show that the considerations of that paper carry through to realistic effective Lagrangians, such as those that emerge out of string theory. Extracting the physics of the large number of metastable vacua that ensue requires somewhat more sophisticated algebro-geometric techniques, which we review.
1. Introduction and Summary

One of the striking observations of KKLT [1] was that the existence of a large number of long-lived metastable vacua in certain compactifications of string theory (dubbed the “landscape” [2]) provides a concrete instance in which the anthropic principle [3,4,5,6,7,8,9] might be realized in Nature. Unfortunately, many discussions of these ideas get bogged down because it is hard to disentangle the intricacies of the string theoretic constructions from the “anthropic” questions one would like to address.

The paper of Arkani-Hamed, Dimopoulos and Kachru [10] was, therefore, very useful in clearing away the string-theoretic underbrush and presenting a simple, tractable field-theoretic model with a large number of vacua in which anthropic questions could be addressed (see also [11]).

One of the concepts to emerge from their investigation was the notion of a “friendly landscape.” In general, all of the couplings, $c_a$ of the theory will vary between the different vacua. However, there is a qualitative difference between those couplings which “scan” (those whose standard deviation is much larger than their mean value) and those couplings which “don’t scan” (those which are sharply-peaked about their mean value). In making anthropic arguments, one usually considers the situation in which one coupling is allowed to vary, while the others are held fixed. If all the couplings vary appreciably, the anthropic bounds are much weaker, or go away entirely. So it was very useful for the authors of [10] to identify a mechanism by which the couplings one would like to “tune” anthropically scan, whereas the remaining couplings are sharply peaked.

Their model had three basic features:

1) A large number, $N$, of scalar fields, $\phi_i$.

2) A decoupled form for the scalar potential, $V(\phi) = \sum_i V_i(\phi_i)$.

3) A decoupled form for the $\phi$-dependence of the observable couplings of the model, $c_a(\phi) = \sum_i c_{ai}(\phi_i)$.

The first has a fairly natural realization in string theory. Many compactification of string theory have hundreds of moduli which, when the physics which lifts the vacuum degeneracy is included [12,13,14,15,16,17,18,19,20,21,22], form natural candidates for the $\phi_i$. We can, quite plausibly, take “$N$” to be the number of (complex) moduli. However, 2,3) are rather unnatural from this point of view and, indeed, it’s hard to imagine that such a decoupled structure might emerge from string theory.

In the present work, we would like to overcome this drawback and present what we hope is a realistic version of the scenario of [10]. Our approach will be to start with, essentially, the most general $N = 1$ supersymmetric effective field theory with a large number of “moduli” chiral multiplets, $\Phi^i$. We will then see what conditions must be imposed in order to realize a friendly landscape.

In §2, we will discuss the theory of $N$ chiral multiplets coupled to $N = 1$ supergravity. Whereas any such theory must be cut off at a scale $M_c < M_p$, we will see that, at large-$N$, we will need to impose the stronger condition $M_c < M_p/\sqrt{N}$, in order to have a sensible effective field theory. Moreover, we will find, in §2.1, that the couplings among the
moduli chiral multiplets must be suitably small; we will summarize the condition that we must impose on these couplings by saying that they are \textit{generically small}. We then turn to a polynomial truncation of the superpotential of the model. While not indispensable, such a truncation makes the analysis of the vacuum structure amenable to perturbation theory. As usual, for this to be a \textit{reliable} guide to the vacuum structure, the super-renormalizable terms in the superpotential must have coefficients governed by a mass scale \( M_r \ll M_c \) (\S\ 2.2). In \S\ 2.3 we will see how these considerations mesh with the most popular arena for landscape considerations — F-theory vacua with fluxes.

Next (\S\ 2.4), we impose the discrete R-symmetry found by \cite{10} to lead to a friendly landscape for the cosmological constant, and use the \( GL(N, \mathbb{C}) \) symmetry of field redefinitions to simplify our problem (\S\ 2.5).

In \S\ 3 we lay out the algebro-geometric techniques used to determine the vacuum structure and extract information about the distribution of values for various physical quantities among the \( 2^N \) vacua of the theory. In \S\ 4 we use these techniques to compute certain “holomorphic moments” which characterize the distribution of values of the cosmological constant. In \S\ 5 we discuss the physics that ensues when one assumes that the couplings are chosen from some (unspecified) probability distribution, generalizing the considerations of \cite{10}. We apply our analysis both to the superpotential and to other holomorphic couplings.

Finally, in \S\ 6 we discuss some generalizations of our techniques and future directions.

2. General Features of SUSY Landscape Sectors

As a field theoretic model for the landscape sector, we could start with an arbitrary SUSY field theory of \( N \) chiral superfields and \( N' \) vector superfields, all coupled to \( \mathcal{N} = 1 \) SUGRA. For simplicity, we will restrict ourselves to the case \( N' = 0 \) and only briefly touch on generalizations to gauged hidden sectors in this work. Thus, the vacuum structure of the model can be determined using the two-derivative effective action of the SUSY non-linear \( \sigma \)-model describing the \( N \) chiral fields at energies below a cutoff scale, \( M_c \).

Clearly, for any sort of effective field theory to be valid, we must take \( M_c < M_p \). However, as noted in \cite{10}, when one has a large number, \( N \gg 1 \), of fields, radiative stability of Newton’s constant requires

\[
\frac{M_c^2}{M_p^2} < \frac{1}{N} \tag{2.1}
\]

To see this, note that the action for \( \mathcal{N} = 1 \) supergravity interacting with \( N \) chiral fields is, in superspace notation (we use \( M_p \) for the reduced Planck mass and the conventions of \cite{23}),

\[
S = -3M_p^2 \int d^8z E^{-1} e^{-\frac{1}{3M_p^2}K(\Phi, \bar{\Phi})} + \int d^6z \phi^3 W(\Phi) + \text{h.c.} \tag{2.2}
\]

where \( E^{-1} \) is the superdeterminant of the vielbein, \( z \) is a superspace coordinate, and \( \phi \) is a compensator superfield. The Einstein-Hilbert term comes from the leading (\( \Phi \)-independent) piece of the first term in (2.2). At one-loop, this receives a quadratically-divergent contribution,

\[
\Delta K_W^{(1)} \bigg|_{\Phi = \bar{\Phi} = 0} \sim \left( \frac{M_c^2}{16\pi^2} \right) g^3 \partial_i \bar{\partial}_j K \sim N \left( \frac{M_c^2}{16\pi^2} \right) \tag{2.3}
\]
where \( g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K \) is the Kähler metric of the \( \sigma \)-model. The enhancement which comes from having \( N \) fields running around the loop requires us to set the cutoff, \( M_c \) to be parametrically smaller than \( M_p \).

In \( \mathcal{N} = 1 \) supergravity, the chiral multiplets parameterize a Kähler manifold, whose Kähler form, \( \omega = \frac{i}{2} \partial \bar{\partial} K \). The superpotential, \( W(\Phi) \) transforms as a section of a line bundle, \( \mathcal{L} \), whose first Chern class,

\[
c_1(\mathcal{L}) = \frac{1}{\pi M_p^2} \omega = \frac{1}{2\pi i M_p^2} \partial \bar{\partial} K.
\]

(2.4)

The fiber metric on \( \mathcal{L} \) is

\[
h(\Phi, \bar{\Phi}) = e^{K(\Phi, \bar{\Phi})/M_p^2}
\]

(2.5)

and the connection is of type (1,0), \( D_i = \partial_i + \partial_i K/M_p^2 \). Under Kähler transformations, \( W(\Phi) \to e^{f(\Phi)/M_p^2} W(\Phi) \),

\[
K(\Phi^i, \bar{\Phi}^{\bar{i}}) \to K(\Phi^i, \bar{\Phi}^{\bar{i}}) + f(\Phi^i) + \bar{f}(\bar{\Phi}^{\bar{i}}),
\]

(2.6)

The supersymmetric vacua of this model are the critical points of the superpotential with respect to the Chern connection, i.e. the points at which \( D_i W \in \Gamma(T^* X \otimes \mathcal{L}) \) intersect the zero section of \( T^* X \otimes \mathcal{L} \),

\[
D_i W = \partial_i W + (\partial_i K) W/M_p^2 = 0.
\]

(2.7)

More generally, all the vacua of this model, supersymmetric or not, are critical points of its scalar potential,

\[
V = e^{K/M_p^2} \left( g^{i\bar{j}} D_i W D_j \bar{W} - 3|W|^2/M_p^2 \right).
\]

(2.8)

Since we will be interested in vacua in which \( |W| \sim M_c^3 < M_p^3 \ll M_p^3 \), we will always be safe in neglecting the connection term in (2.7) and hunting for ordinary critical points, \( \partial_i W = 0 \).

In fact, to justify a perturbative analysis of the effective Lagrangian (2.2), we require that \( W(\Phi) \) have a convergent Taylor expansion in a polydisk, \( |\Phi_i| < M_c \). Just as the radiative stability of Newton’s constant imposed constraints on the cutoff scale, \( M_c \) of our effective Lagrangian, radiative corrections to the Kähler metric imposes, at large-\( N \), further constraints on the coefficients.

### 2.1. Quantum Corrections to the Kähler Potential and Large \( N \) Scaling

Before embarking upon a study of the vacua of these models, we should study the radiative stability of these models in the limit of large \( N \). As our theory is an effective field theory, we expect that the characteristic radius of curvature for \( X \) is given by the cutoff scale, \( M_c \). Thus, in a small enough neighborhood of a smooth point \( p \in X \), we can choose local coordinates \( \Phi^i \) such that \( \Phi^i(p) = 0 \) and the Kähler potential takes the form (modulo Kähler transformations),

\[
K(\Phi, \bar{\Phi}) = g_{i\bar{j}} \Phi^i \bar{\Phi}^{\bar{j}} + \text{Re} \sum_{I_n, \bar{I}_n} \frac{1}{n! \bar{n}! M_c^{n+\bar{n}-2}} K_{I_n, \bar{I}_n} \Phi^{I_n} \ldots \Phi^{i} \bar{\Phi}^{\bar{i}} \ldots \bar{\Phi}^{\bar{I}_n}
\]

(2.9)
where the $K_{I_n I_n}$ are dimensionless and symmetric in the multi-indices $I_n$ and $\bar{I}_n$ and $g_{ij}$ is the Kähler metric at $p$. Further, as the superpotential $W$ is holomorphic, it can be locally expanded as a power series in the holomorphic coordinates $\Phi^i$ about the origin $\Phi^i(p) = 0$,

$$W(\Phi^i) = A_0 + \sum_{n I_n} \frac{A_{I_n}}{n!} \Phi^{i_1} \cdots \Phi^{i_n} = M_c^2 W_0 + \sum_{n I_n} \frac{W_{I_n}}{n M_c^{n-3}} \Phi^{i_1} \cdots \Phi^{i_n},$$  

(2.10)

where the $A_{I_n}$ are symmetric in the multi-indices $I_n$ and have mass dimension $3 - n$ while the $W_{I_n} = M_c^{n-3} A_{I_n}$ are dimensionless.

While the superpotential is holomorphic and not renormalized, the Kähler potential is renormalized. It is important to check the radiative stability of its assumed form given the form of $W$ in the large $N$ limit. More precisely, we compute the effective Kähler potential (in the Wilsonian sense) at a scale $M_c$ lower than $M_c$ by integrating out the modes in a shell of momenta between $M_c$ and $M_c'$, and require that the corrections are not parametrically larger in $N$ than the bare values. As we will see, this requirement will restrict the asymptotic growth of both the $A_{I_n}$ and $K_{I_n I_n}$ parametrically in $N$.

Let us first consider the one-loop corrections to the Kähler potential coming from the superpotential $W$,

$$\Delta K_W^{(1)} \sim \left( \frac{1}{8\pi^2} \log \frac{M_c^2}{M_{c'}^2} \right) g^{i j} g^{j k} \partial_i \Phi^k \bar{W} \partial_i W$$

$$\sim \left( \frac{1}{8\pi^2} \log \frac{M_c^2}{M_{c'}^2} \right) \sum_{I_n \bar{I}_n} (n + 1)(\bar{n} + 1) g^{i j} g^{j k} W_{i n} \bar{W}_{j \bar{n}} \Phi^{i_1} \cdots \Phi^{i_n} \Phi^{j_1} \cdots \Phi^{j_{\bar{n}}},$$

(2.11)

where $g^{i j}$ denotes the uncorrected inverse Kähler metric at the point $p$, the origin in field space. For $n = \bar{n} = 1$, we get a one loop correction to the Kähler metric at the origin $p$ arising from the dimensionless superpotential couplings $A_{i j k}$,

$$\Delta g_{i j}^{(1)} \sim \left( \frac{1}{2\pi^2} \log \frac{M_c^2}{M_{c'}^2} \right) g^{i k} g^{j l} A_{i k l} \bar{A}_{j k l}.$$  

(2.12)

Radiative stability requires that $\Delta g_{i j}^{(1)}$ is “small” compared to $g_{i j}$, that the coordinate-invariant norms of tangent vectors to $X$ at $p$ under $\Delta g_{i j}^{(1)}$ are smaller that those under $g_{i j}$, or roughly,

$$g^{i j} \Delta g_{i j}^{(1)} \sim \left( \frac{1}{2\pi^2} \log \frac{M_c^2}{M_{c'}^2} \right) g^{i j} g^{k k} g^{l l} A_{i k l} \bar{A}_{j k l} \lesssim g^{i j} g_{i j} = \delta_i^j.$$  

(2.13)

We can rephrase this requirement in precise, coordinate invariant terms by taking traces and determinants of both sides,

$$\det \left[ g^{i j} \Delta g_{i j}^{(1)} \right] \lesssim 1, \quad \text{Tr} \left[ g^{i j} \Delta g_{i j}^{(1)} \right] \lesssim N.$$  

(2.14)

As promised, this gives us parametric bounds on the growth of the dimensionless superpotential couplings $A_{i j k}$,

$$g^{i i} g^{j j} g^{k k} A_{i j k} \bar{A}_{i j k} \sim \mathcal{O}(N),$$  

$$\det_{i j} \left[ g^{i j} g^{k k} g^{l l} A_{i k l} \bar{A}_{j k l} \right] \sim \mathcal{O}(1).$$  

(2.15)

(2.16)
Since $A_{ijk}$ is a tensor, the interpretation of these bounds on the size of its components is, of course, a coordinate dependent question. Now, in a Kähler manifold, $d\omega = 0$ implies that it is always possible (see [24] p.107) to choose local holomorphic coordinates about any smooth point $p$ such that $\Phi^i(p) = 0$ and
\begin{equation}
\partial_i \bar{\partial}_j K(\Phi, \bar{\Phi}) = g_{ij}(\Phi, \bar{\Phi}) = \delta_{ij} + \text{terms of order } \geq 2 \text{ in the } \Phi, \bar{\Phi}.
\end{equation}
(2.17)

In these coordinates the above bounds simplify to,
\begin{equation}
\sum_{i,j,k} |A_{ijk}|^2 \sim O(N), \quad \det_{ij} \left[ \sum_{k,l} A_{ijkl} \bar{A}_{jkl} \right] \sim O(1).
\end{equation}
(2.18)

Thus, in these special coordinates, we see that in order to satisfy these bounds, the generic components of $A_{ijk}$ must have parametrically suppressed magnitudes at large $N$,
\begin{equation}
|A_{ijk}| \sim O(N^{-1})
\end{equation}
(2.19)

though $O(N)$ of them can still be as large as $O(1)$ in that limit. Therefore, we will refer to a tensor $A_{ijk}$ obeying (2.15) as being generically small.

We can similarly obtain bounds on the coefficients $A_{I_1}$ and $K_{I_1, I_2}$ by considering the leading loop correction to the quadratic term in the Kähler potential coming from the corresponding higher order terms. One finds a slew of conditions similar in form to (2.18). For instance,
\begin{equation}
M_c^{2n-2} g^{i_1 i_2} \cdots g^{i_n+1 i_n+2} A_{i_1 \cdots i_n+2} \bar{A}_{i_1 \cdots i_n+2} \sim O(N),
\end{equation}
(2.20a)
\begin{equation}
\det_{ij} \left[ M_c^{2n-2} g^{j_1 j_2} g^{i_1 i_2} \cdots g^{i_n+1 i_n+2} A_{i_1 \cdots i_n+2} \bar{A}_{i_1 \cdots i_n+2} \right] \sim O(1),
\end{equation}
(2.20b)
\begin{equation}
g^{i_1 i_2} \cdots g^{i_n i_1} K_{i_1 \cdots i_n+1} \sim O(N),
\end{equation}
(2.20c)
\begin{equation}
\det_{ij} \left[ g^{j_1 j_2} g^{i_1 i_2} \cdots g^{i_n+1 i_n+2} K_{i_1 \cdots i_n+1} \right] \sim O(1).
\end{equation}
(2.20d)

In particular, this implies that $O(N)$ of the coefficients $A_{I_1}$ could be $O(1)$, while the generic coefficient must be small,
\begin{equation}
|A_{i_1 \cdots i_n}| \sim M_c^{-(n-3)} O(N^{-(n-1)/2}). \quad (n \geq 3)
\end{equation}
(2.21)

Thus, just as with $A_{ijk}$, we will refer to any tensor obeying bounds of the form (2.20) as generically small. In general, these conditions, as well as a very large number of others corresponding to loop diagrams involving multiple vertices, constrain all the coefficients of the higher order terms in the Kähler potential and superpotential to be generically small at large $N$.

\footnote{We note that there is an additional constraint on the size of the coefficients coming from the requirement that the power series we have been writing down actually absolutely converge in a region of size $\sim M_c$. These constraints, while stronger than those coming from radiative stability, only constrain the asymptotics of the $A_{I_1}$ for large $n$, rather than constraining the terms at some particular order in $n$.}
2.2. SUSY Vacua of the Renormalizable Wess-Zumino Model at Large $N$

We now wish to study the vacuum structure of this effective field theory. As discussed in the previous subsection, this amounts to studying the critical points of the superpotential, $\partial_i W = 0$. For large $N$, we expect that the number of critical points of $W$ within a polydisk of radius $M_c$ to scale exponentially with $N$.

Since we assume that the power series expansion for $W$ is absolutely convergent in the polydisk, a field theorist might reasonably take the approach of truncating the power series at some finite order and looking for the critical points of the resulting polynomial. Of course, we cannot hope that a truncation to any finite order polynomial can be an accurate guide to the critical points in the entire polydisk of radius $M_c$. At best, we will hope to obtain the critical points inside some much smaller polydisk, of radius $M_r \ll M_c$. Generically, however, we would not expect to find any critical points inside this smaller polydisk. The criterion for finding (trustworthy) critical points within this smaller polydisk for small $N$ is well-known: we require that the coefficients of the linear and quadratic terms in \[ (2.10) \] be $A_{I_1} \sim O(M_r^2)$ and $A_{I_2} \sim O(M_r)$, respectively, where

$$\frac{M_r}{M_c} = \epsilon \ll 1$$

(2.22)

As long as the quartic and higher terms in the superpotential are not anomalously large, they give negligible corrections to the critical points determined by truncating $W$ to cubic order\(^2\).

At large $N$, this is not quite sufficient. Even assuming that the higher $A_{I_n}$ are generically small, in the sense of the previous subsection, we still need to require

$$\epsilon < \frac{1}{\sqrt{N}}$$

(2.23)

in order for these higher-order corrections to be negligible. To see this, we can look at the invariant quantity,

$$\left( M_c^{-4} g^{ij} \partial_i W \partial_j W \right)_{\Phi = \Phi_*},$$

(2.24)

where $\Phi_*$ is the critical point derived from the cubic approximation of $W$. Using the generic smallness of the $|A_{I_n}| \sim M_c^{-(n-3)} N^{-(n-1)/2}$ and $|\Phi_i| \sim M_r = \epsilon M_c$, we see that the corrections grow parametrically with $N$, unless (2.23) is satisfied. In what follows, we will keep $\epsilon$ as a free parameter, cognizant of the fact that it must be sufficiently small for the story to work.

In non-supersymmetric theories, ensuring that under radiative corrections the coefficients of the super-renormalizable terms in the potential remain much smaller than the cutoff is called the hierarchy problem. In a supersymmetric theory, there are no perturbative corrections to the superpotential. Nonetheless, the fact that the coefficients of the super-renormalizable terms in the superpotential are of order $M_r$, rather than $M_c$, means that the point about which we are expanding is, in some sense, “special.” That will be further brought home in \[ \text{2.4.} \] where we will assume that the point $\Phi = 0$ will be a point where there is an unbroken $\mathbb{Z}_4$ R-symmetry.

\(^2\)As long as we are away from the discriminant locus to be discussed below.
2.3. Distributions of Models and Flux Compactifications of IIB

Since F-theory compactifications with flux are the prime motivating example of “landscape” models with a large number of vacua, let us pause to consider how such models fit in with our general considerations, as developed so far. Any given set of fluxes satisfying the requisite tadpole cancellation conditions in an F-theory or IIB Orientifold compactification gives rise to a Gukov-Vafa-Witten \[25\] superpotential for the complex structure moduli. Since there are, in general, many solutions to the tadpole cancellation conditions, we have an ensemble of theories with different superpotentials, labeled by the possible fluxes.

In the IIB case, the elements of the ensemble are labeled by all choices of integer fluxes through the \(2b^2,1 + 2\) three cycles \(\Sigma_a\) of the Calabi-Yau 3-fold \(M\),

\[
\frac{1}{(2\pi)^2\alpha'} \int_{\Sigma_a} F = N_a \in \mathbb{Z}, \quad \frac{1}{(2\pi)^2\alpha'} \int_{\Sigma_a} H = M_a \in \mathbb{Z}.
\]

(2.25)

compatible with the tadpole cancellation condition for induced D3-brane charge on \(M\),

\[
Q_3 = \frac{1}{(2\pi)^4\alpha'^2} \int_{M} F \wedge H.
\]

(2.26)

where \(Q_3\) includes the contribution of space-filling, mobile D3-branes, D7-branes, and O3-planes. Now, if \(Q_3 \sim b_{2,1} \sim \mathcal{O}(N)\) then tadpole cancellation requires that each of the integer fluxes can then be at most \(N_a \sim M_a \sim \mathcal{O}(N^{1/2})\). Each choice of fluxes gives rise to a Gukov-Vafa-Witten superpotential for the IIB axio-dilaton \(\tau\) and the complex structure moduli \(z_i = \int_{A_i} \Omega\) of \(M\),

\[
W = \int_{M} G \wedge \Omega.
\]

(2.27)

where \(G = F - \tau H\) and the \(A_i\) and \(B^i\) form a symplectic basis of 3-cycles \(\{\Sigma_a\} = \{A_i, B^i\}\) in \(M\) with respect to the intersection pairing on \(H_3(M,\mathbb{Z})\). In particular, we see that the tadpole cancellation condition places bounds on the growth of \(W\) on \(N\),

\[
W \succ \sum_i \int_{B^i} G \int_{A_i} \Omega \sim \sum_i (N^i - \tau M^i) z_i \sim \mathcal{O}(N).
\]

(2.28)

which is certainly consistent with the condition that the couplings of the moduli are indeed generically small. However, it is not at all clear that there is any point in the moduli space about which the cubic truncation gives a good approximation to the locations of its critical points – that is, we do not expect the renormalizable approximation to be a useful guide to the vacuum structure.

Further, notice that the superpotential is odd under \(G \to -G\), while the tadpole cancellation condition is even, so if \(G\) satisfies the tadpole constraint, then so does \(-G\). This means that if we find a supersymmetric vacuum \(z_i^*\), a critical point of \(W\) with a given flux \(G\), and a corresponding value \(W^*\) of the superpotential, then \(z_i^*\) also corresponds to a vacuum of the model with flux \(-G\) and superpotential \(-W^*\). Thus, even though the average value of \(W\) at the supersymmetric vacua for any fixed choice of the fluxes may not vanish, the ensemble average of \(W\) certainly does vanish.
Motivated by this example, we will be interested in situations where we are actually given a distribution or ensemble of models. That is, we will assume that physics at high energies \( \sim M_p \) can be understood as providing a distribution of coefficients \( A_{I_n} \) for the effective low-energy \( \lesssim M_c \) models described above. Indeed, in the IIB flux vacua, the physics which determines the values of the quantized fluxes is not even field theoretic in nature - it likely involves high energy string/brane dynamics, topology change, etc. In particular, the scanning of the cosmological constant in these models is explained by high energy physics. Certainly, we could accept such a high energy explanation and restrict our consideration to distributions of coefficients which are symmetric under \( A_{I_n} \rightarrow -A_{I_n} \) and share this property of flux vacua. However, we will focus on situations where there may be a low energy explanation for this scanning.\(^3\) One way to achieve this \([10]\) is through the imposition of an R-symmetry, which we discuss presently.

2.4. R-Symmetry, The Renormalizable Wess-Zumino Model, and \( \Lambda \)

The general renormalizable Wess-Zumino model of \( N \) interacting chiral superfields \( \Phi^i \) is described by a quadratic Kähler potential,

\[
K(\Phi, \bar{\Phi}) = g_{ij} \Phi^i \bar{\Phi}^j, \tag{2.29}
\]

and a cubic superpotential,

\[
W(\Phi^i) = A_i + A_{ij} \Phi^j + \frac{1}{2} A_{ijk} \Phi^j \Phi^k + \frac{1}{3} A_{ijk} \Phi^j \Phi^k \Phi^k. \tag{2.30}
\]

Radiative stability of the scaling of the Kähler potential at large \( N \) restrict us to the case that generically \( |A_{ijk}| \sim N^{-1} \) with \( \mathcal{O}(N) \) terms which are \( \mathcal{O}(1) \). The supersymmetric vacua for this theory are points where,

\[
\partial_i W = A_i + A_{ij} \Phi^j + A_{ijk} \Phi^j \Phi^k = 0, \tag{2.31}
\]

a set of \( N \) complex algebraic equations in the \( \Phi^i \). In particular, these polynomials determine an ideal\(^4\) in the ring of polynomials in the \( \Phi^i \), \( \langle \partial_i W \rangle \subset \mathbb{C}[\Phi^1, \ldots, \Phi^N] \). Bezout’s theorem (see Chapter 3, Theorem 5.5 of [26]) guarantees that for a generic choice of the coefficients

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\(^3\) Some high energy input, however, may be inevitable. For example, one might worry about the origin of the small parameter \( \epsilon = M_r / M_c \) that we required in order to make the cubic approximation. This appears to be a tuning of \( \mathcal{O}(N) \) relevant couplings in the model. Without some further explanation, one might worry that, together, these represents a fine-tuning of order \( \epsilon N \), which would wipe out whatever “advantage” we gained in having \( 2^N \) vacua. This need not be the case if the smallness of each of the \( \mathcal{O}(N) \) couplings has a common explanation. Perhaps there is an approximate symmetry, broken weakly by the effects that generate the superpotential, which guarantees that these couplings are small. Indeed, such a symmetry would be reflected in symmetries of the resulting probability distribution for the couplings. For example, if the coefficients of the relevant operators were all selected from the same distribution, we would indeed only require a single fine tuning of the distribution.

\(^4\) See section [8] for a basic review of the commutative algebra language used here, and [20,27] for a more detailed introduction.

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8
of $W$, these $N$ simultaneous quadratic equations have $2^N$ roots. From an algebraic point of view, this translates into the fact that the quotient ring,

$$\mathbb{C}[\Phi^1, \ldots, \Phi^N]/\langle \partial_i W \rangle \cong \mathbb{C}^{2^N}$$

(2.32)

is a $2^N$-dimensional vector space over $\mathbb{C}$ (see Chapter 3, Theorem 6.2 of [26]) generated by the images of the monomials $\Phi^{i_1} \cdots \Phi^{i_r}$ with $i_1 < \cdots < i_r \leq N$. Explicitly, the quotient ring should be understood as the “polynomial functions” on the algebraic variety cut out by the equations generating the ideal, which in this case are the functions on a set of $2^N$ points, i.e. a $2^N$ dimensional vector space.

Further, we saw that the critical points are generically contained in a neighborhood $U$ of radius $\sim M_r$ about the origin. Note that if we assume that the constant term $A$ in $W$ is also of order $\lesssim M_r^3$, the superpotential at each of the critical points in $U$ is roughly of the order of

$$W \sim M_r^3 \times \mathcal{O}(N).$$

(2.33)

This gives a supersymmetric contribution to the vacuum energy

$$\Lambda = -3 \frac{|W|^2}{M_p^2} \left(1 + \mathcal{O}(M_r^2/M_p^2)\right) \sim -N \epsilon^2 M_r^4,$$

(2.34)

where $\epsilon = M_r/M_c \lesssim 1/\sqrt{N}$ and $M_c/M_p \lesssim 1/\sqrt{N}$. Note that all corrections coming from the inclusion of the Chern connection terms and higher order terms in the Kähler potential and superpotential are parametrically suppressed at large $N$.

When supersymmetry is broken, we get a positive contribution, $\Lambda_S$ to the vacuum energy. The hope is that the distribution of values for $W$ among the $2^N$ vacua will be such that (2.34) can very nearly cancel $\Lambda_S$.

As the authors of [10] noted, this is easy to arrange. If the superpotential is odd under $\Phi^i \rightarrow -\Phi^i$, then supersymmetric vacua come in pairs $\Phi^i \ast$ and $-\Phi^i \ast$ with opposite values of $W$ and we would have $\langle W \rangle = 0$. We can enforce this by imposing a $\mathbb{Z}_4$ R-symmetry

$$\Phi^i(y, \theta) \rightarrow -\Phi^i(y, i\theta),$$

under which the superpotential must have charge 2 and therefore is an odd polynomial,

$$W(\Phi^i) = \sum_{i=1}^{N} A_i \Phi^i + \frac{1}{3} \sum_{i,j,k=1}^{N} A_{ijk} \Phi^i \Phi^j \Phi^k.$$  

(2.35)

Such an R-symmetry is clearly non-generic. In particular, the origin in these coordinates must be a special point for such a symmetry to hold, as an expansion of the superpotential at

5By generic, we mean that a certain polynomial in the coefficients known as a resultant is non-vanishing – see [28].

6As a test, we note that the prototypical example of a landscape superpotential, the Gukov-Vafa-Witten superpotential for the complex structure moduli of a IIB orientifold model indeed scales this way - see [22]. Again, in a appropriate basis, the nonzero coefficients in the Gukov-Vafa-Witten superpotential can be $\sim \mathcal{O}(1)$, but, they are, indeed, very sparse.

7See [28, 29, 30, 31, 32, 33, 34, 35] for discussions regarding the scale of supersymmetry breaking in landscape models.
any nearby point in field space certainly would not exhibit the same R-symmetry. That is, the R-symmetry is spontaneously broken by a vev for the $\Phi^i$. In fact, given an arbitrary SUSY non-linear sigma model, there is no reason to believe that its superpotential will generically ever have a point in the target space where $W$ has such a symmetry. Thus, the imposition of an R-symmetry means that we are restricting our consideration to very special SUSY non-linear sigma models expanded locally about a special point in their target spaces. This is the price that one must pay for a low-energy explanation for the scanning of the vacuum energy. For further discussion regarding this issue, see [36,37,38]. Of course, R-symmetry is phenomenologically desirable for many other reasons as well (see the discussion in [10]).

The algebraic consequences of the R-symmetry and their geometric interpretations will be of use in our discussion of the statistics of the vacua of this model. Note that the conditions for unbroken SUSY are actually equations for the scalars $\phi^i$ in the chiral superfields $\Phi^i$, and the R-symmetry acts as a $\mathbb{Z}_2$ parity on these scalars, $\phi^i \to -\phi^i$. In particular, the SUSY vacua of this model are determined by $N$ quadratic equations,

$$\partial_i W = A_i + \sum_{j,k=1}^N A_{ijk} \phi^j \phi^k = 0 \quad (2.36)$$

which are invariant under the $\mathbb{Z}_2$ parity. As a result, these quadratics actually determine an ideal in the ring of $\mathbb{Z}_2$ invariants constructed from polynomials in the $\phi^i$,

$$\langle \partial_i W \rangle = \langle A_i + A_{ijk} \phi^j \phi^k \rangle \in \mathbb{C}[\phi^1, \ldots, \phi^N]^{\mathbb{Z}_2}. \quad (2.37)$$

We can interpret the ring of invariants $\mathbb{C}[\phi^1, \ldots, \phi^N]^{\mathbb{Z}_2}$ as the “polynomial functions” on the orbifold $\mathbb{C}^N/\mathbb{Z}_2$. Then, in analogy with the general case, we can consider the quotient ring,

$$\mathbb{C}[\phi^1, \ldots, \phi^N]^{\mathbb{Z}_2}/\langle \partial_i W \rangle \cong \mathbb{C}^{2^{N-1}}, \quad (2.38)$$

which is a $2^{N-1}$-dimensional vector space over $\mathbb{C}$ generated by the images of the even monomials $\Phi^{i_1} \cdots \Phi^{i_{2r}}$ with $i_1 < \cdots < i_{2r} \leq N$. Geometrically, this quotient ring can be interpreted as the polynomial functions on the variety cut out by the $\partial_i W$ in the orbifold, which consists of the $2^{N-1}$ images of the $2^N$ critical points of $W$ in $\mathbb{C}^N/\mathbb{Z}_2$.

### 2.5. Fermat Form of the Cubic

We are interested in the properties of distributions of models with superpotentials respecting the R-symmetry [2.36] and quadratic Kähler potentials [2.29]. At first glance, one might expect to describe such a distribution of models as an arbitrary probability distribution on the space of all superpotential and Kähler couplings, $f(A_i, A_{ijk}, g_{ij})$. However, we should not distinguish between models which differ by field redefinitions. Of course, arbitrary field redefinitions will not preserve the form of the Wess-Zumino models we are considering. However, it is easy to see that the field redefinitions which do are of the form $\phi^i \to G^i_m \phi^m$, where $G^i_m \in GL_N(\mathbb{C})$. In particular, they act on the space of couplings as,

$$A_i \to A_m G^m_i, \quad A_{ijk} \to A_{mnp} G^m_i G^n_j G^p_k, \quad g_{ij} \to g_{mn} G^m_i G^n_j. \quad (2.39)$$
If we posit some probability distribution on the space of couplings, this distribution gets averaged over $GL_N(\mathbb{C})$ orbits to produce a probability distribution on the space of theories. That is awkward to deal with. If possible, it is much better to fix the redundancy by choosing a gauge for the $GL_N(\mathbb{C})$ field redefinitions. Since $GL_N(\mathbb{C})$ is $N^2$ dimensional, such a gauge condition should involve precisely $N^2$ independent, complex algebraic conditions. Now, in working with algebraic equations, it is often convenient to redefine our variables in order to make the coefficients of the terms of highest degree as simple as possible. If $W$ is a generic polynomial of degree $d$, we can use the $GL_N(\mathbb{C})$ symmetry to enforce $N^2$ conditions on $A_{IJ}$,

$$A_{i\cdots i} = 1, \quad A_{ij\cdots j} = A_{jij\cdots j} = \cdots = A_{ji\cdots j} = 0, \quad \text{for } i \neq j. \quad (2.40)$$

We will refer to a polynomial satisfying these conditions as being in Fermat form,

$$W = \frac{1}{d} \left( (\phi^1)^d + \cdots + (\phi^N)^d \right) + \text{ (terms of degree } \leq d-1 \text{ in each } \phi^i). \quad (2.41)$$

Note that if $W$ is odd (so it has a $\mathbb{Z}_2$ R-symmetry as above), then the Fermat form is somewhat stronger, as the additional terms in $(2.41)$ actually have degree at most $(d - 2)$ in any $\phi^i$. In particular, for the case of interest here, $d = 3$, with the R-symmetry above, the superpotential takes the form,

$$W(\phi^i) = \sum_{i=1}^{N} \left( \frac{1}{3} (\phi^i)^3 - a_i \phi^i \right) - \sum_{i<j<k} b_{ijk} \phi^i \phi^j \phi^k, \quad (2.42)$$

where $a_i$ has mass dimension 2 and is generically $M_r^2 \times \mathcal{O}(1)$ and the $b_{ijk}$ are symmetric, traceless (so $b_{iii} = b_{ijj} = b_{iji} = b_{jji} = 0$), dimensionless and generically $|b_{ijk}| \sim \mathcal{O}(N^{-1})$ with at most $\mathcal{O}(N)$ of them $\mathcal{O}(1)$. In going to Fermat form, it is important to note that we cannot also simultaneously set $g_{ij} = \delta_{ij}$. Further, the genericity of $W$ is important here — not all polynomials can be put into Fermat form. For example, the following model with spontaneous breaking of supersymmetry via the O’Raifeartaigh Mechanism,

$$W = Z(X^2 - a) + YX^2 \quad (2.43)$$

cannot be put into this form. In fact, it is easy to show by analytic computations using eigenvalue methods (see §3) that for $N = 3$ and $d = 3$ no polynomial in Fermat form exhibits the O’Raifeartaigh Mechanism.

We believe that this may be true much more generally. More precisely, we conjecture that all cubic, R-symmetric superpotentials of the Fermat form $(2.42)$ have at least a pair of supersymmetric vacua. While a proof of this fact is far beyond the scope of this work, we hope to return to this question in the future.

### 2.6. Symmetries of the Fermat Form

Just as is often the case in gauge fixing, it is important to note that our gauge condition $(2.40)$ does not completely fix the $GL_N(\mathbb{C})$ redundancy. For any generic $A_{IJ}$, there is a finite subgroup $H$ of $GL_N(\mathbb{C})$ transformations which transforms the coefficients in such a way as
to keep $A_{\mathbf{4}}$ in the Fermat form (2.41). Namely, $h_i^m \in H$ are the solutions to $N^2$ algebraic equations of degree $d$ in the $N^2$ complex components $h_i^m$:

$$A_{m_1 \cdots m_d} h_i^{m_1} \cdots h_i^{m_d} = 1, \quad A_{m_1 \cdots m_d} h_i^{m_1} h_j^{m_2} \cdots h_j^{m_d} = 0, \quad \text{for } i \neq j.$$  

(2.44)

If these were generic degree $d$ equations, Bezout’s theorem would tell us that they have $d^{N^2}$ solutions. However, it turns out that this is not the case for generic $A_{\mathbf{4}}$, and $d^{N^2}$ is actually a strict upper bound on the order of $H$. We will show this by exhibiting a large subgroup of $H$ which is independent of $A_{\mathbf{4}}$ and whose order does not divide $d^{N^2}$. First, note that permutations of the $\phi^i$ certainly won’t take us out of Fermat form, so we expect that the symmetric group on $N$ variables $S_N \subset H$. Next, note that multiplying any $\phi^i$ by a $d^{th}$ root of unity also won’t ruin the special form, so we get a subgroup of $H$ isomorphic to $(\mathbb{Z}_d)^N$ given by,

$$h_i^j = \delta_i^j \zeta^{n_i}, \quad \zeta = e^{2\pi i/d}, \quad 1 \leq n_i \leq d. \quad (2.45)$$

Thus, we see that $S_N \times (\mathbb{Z}_d)^N$, which has order $N! \times d^N$, must be a subgroup of $H$. Since $N!$ generally does not divide $d^{N^2}$, we see that,

$$N! \times d^N \leq |H| < d^{N^2} \quad (2.46)$$

In the simplest non-trivial case, $N = 3$ and $d = 3$, we numerically solved (2.44) using Mathematica for various values of the coefficients and found that $H$ is generically of order $648 = 2^3 \times 3^4 = 3! \times 3^3 \times 4$. In particular, there are indeed elements of $H$ which depend on $A_{\mathbf{4}}$, which suggests that $|H|$ may generally be strictly larger than $S_N \times (\mathbb{Z}_d)^N$. This is not unexpected. Rather generally, the space of common solutions of these polynomial equations is an algebraic variety. Different values for the coefficients of the polynomials will nonetheless lead to isomorphic algebraic varieties. $H$ is the modular group, and we will not attempt to give a full characterization of $H$ beyond its “obvious” $S_N \times (\mathbb{Z}_d)^N$ subgroup. We will content ourselves with exploiting the constraints that stem from this latter subgroup.

To summarize, after fixing gauge and imposing R-symmetry, each model is uniquely described up to a discrete symmetry $H$ containing $S_N \times (\mathbb{Z}_3)^N$ by a cubic superpotential in Fermat form (2.42) and a quadratic Kähler potential (2.29). Thus, a distribution of field theory landscape sectors is a probability distribution on the space of couplings,

$$f(a_i, b_{ijk}, g_{ij}) da_i db_{ijk} dg_{ij} \quad (2.47)$$

which is invariant under $H \supset S_N \times (\mathbb{Z}_3)^N$,

$$a_i \to a_m h_i^m, \quad b_{ijk} \to b_{mnk} h_i^m h_j^n h_k^p, \quad g_{ij} \to g_{mn} h_i^m h_j^n \quad (2.48)$$

We assume that this distribution is fixed by high energy physics. Lacking any clear physical input which distinguishes among the different vacua of a given low-energy model, we simply treat these vacua democratically. In the following sections, we use algebraic methods to compute some statistical properties of various physical quantities averaged over the $2^N$ vacua of each model as a function of the couplings. Of course, these model averages can then be further averaged over the ensemble using the high energy distribution function.
3. Eigenvalue Methods for Solving Algebraic Equations

The solution of \( N \) simultaneous linear equations in \( N \) variables is something we all learn to do from an early age using several different methods. For example, we may decide to do it via Gaussian elimination or via matrix methods - using determinants and Cramer’s rule to invert the matrix of coefficients. A natural question to ask is if there exist analogues of these methods which could be used to solve simultaneous equations of higher degree. Indeed, this is the case. The method of Gaussian elimination has a vast generalization in the study of algebraic elimination theory and Gröbner bases methods (see [26,27]). These methods provide algorithms which allow one to systematically eliminate variables one by one in any given system of equations at the cost of increasing the degrees of the resulting system of equations in fewer variables and, with some further work, find (approximate) solutions. Unfortunately, these are not directly useful for us as we will be interested in the statistical properties (such as the average and variance of the superpotential and other observables) of such solutions as a function of the coefficients - i.e. in the properties of solutions of families of such equations. It turns out that the generalization of matrix methods to the case of simultaneous equations of higher degree does end up being quite useful for these purposes.

3.1. Rings, Ideals and a Toy Model - Quadratic \( W \)

We will introduce these methods by applying them to the simple case of a generic quadratic superpotential,

\[
W = W_0 - A_i z_i + \frac{1}{2} C_{ij} z_i z_j = W_0 - A \cdot z + \frac{1}{2} z \cdot C \cdot z. \tag{3.1}
\]

Of course, the “statistics of vacua” in this case may seem to be, well, vacuous - the critical point of this polynomial is given by the unique solution of \( N \) linear equations,

\[
\partial_i W = -A_i + C_{ij} z_j = 0 \Rightarrow C \cdot z = A, \tag{3.2}
\]

which is, of course,

\[
z = C^{-1} \cdot A, \tag{3.3}
\]

where the genericity condition is that \( \det C \neq 0 \), or \( C \in GL_N(\mathbb{C}) \). However, if we are given a distribution of coefficients for \( W \), we might be interested in the corresponding distribution of the values of \( W \) at the critical point. To attack such problems more generally, it is essential (though, in this case, akin to trying to kill an ant with a machine gun) to recast these questions in the algebraic language of polynomial rings and ideals.

To begin with, let us quickly review what we will need about ideals, rings, and varieties. Roughly, a ring is a set \( R \) whose algebraic properties mimic many of those of the integers \( \mathbb{Z} \). That is, we may add, subtract, and multiply elements of \( R \) and obtain new elements in \( R \), though the same may not be true for division. Recall that an ideal \( I \) contained in a ring \( R \) is defined by the property that if \( f, g \in I \) and \( r, s \in R \) then \( rf + sg \in I \). In particular, this implies that the quotient ring of \( R \) by \( I \), \( Q = R/I \) where we identify all elements of \( R \) differing by an element in \( I \), is well-defined. Note that all the rings that we consider are
Noetherian rings, which means that all their ideals are finitely generated, so

\[ I = \langle f_1, \ldots, f_m \rangle = \left\{ \sum r_i f_i \mid r_i \in R \right\} = \langle f_i \rangle. \]  \hspace{1cm} (3.4)

The rings we will be most interested are rings of polynomial functions in \( N \) variables, \( R = \mathbb{C}[z_1, \ldots, z_N] \), and their quotients by ideals \( I \) generated by some polynomials \( f_i(z_1, \ldots, z_N) \in R \). Thus, \( I \) is just the set of all polynomials which vanish on the zero locus in \( \mathbb{C}^N \) of the simultaneous set of polynomial equations \( f_i(z_j) = 0 \), \( i = 1, \ldots, m \). This zero-locus is called the algebraic variety in \( \mathbb{C}^N \) corresponding to the ideal \( I \). Thus, non-vanishing elements of the quotient ring \( Q = R/I \) are precisely the non-trivial residues of polynomial functions on \( \mathbb{C}^N \) restricted to the variety - they define a notion of “polynomial functions” on the variety.

As we have already mentioned above, the polynomials \( \partial W \) which determine the critical point generate an ideal in the ring of polynomials in \( N \) variables,

\[ I = \langle \partial_i W \rangle = \langle -A_i + C_{ij} z_j \rangle = \langle -A + C \cdot z \rangle. \] \hspace{1cm} (3.5)

Clearly, any linear combination of the generators \( \partial_i W \) with complex coefficients is also in the ideal. In particular, if \( \det C \neq 0 \) then \( C \) is invertible and we can consider the \( N \) elements of \( I \), \( z - C^{-1} \cdot A \), obtained by multiplying the \( N \) generators \( \partial_i W \) by the matrix \( C^{-1} \). In fact, as \( C \) is invertible, these are also generators of the ideal \( I \). Thus, in the quotient ring \( Q = \mathbb{C}[z_1, \ldots, z_N]/I \), we can use the \( N \) relations \( z_i - C^{-1} A_j = 0 \) to systematically eliminate the \( z_i \) and rewrite any element of \( Q \) in terms of multiples of the identity - constants. So, we see that

\[ Q = \mathbb{C}[z_1, \ldots, z_N]/I \cong \mathbb{C}, \] \hspace{1cm} (3.6)

a one dimensional complex vector space, generated by constants. In particular, the residue of any polynomial \( f(z_1, \ldots, z_N) \in \mathbb{C}[z_1, \ldots, z_N] \) in \( Q \) is obtained by setting to zero all elements of the ideal \( I \), that is, by substituting \( z = C^{-1} \cdot A \in \mathbb{C}^N \) into \( f \). Thus, we see that the residue of \( f \) in \( Q \) is precisely its value at the critical point of \( W \), and that the variety corresponding to \( I \) is the single critical point \( z = C^{-1} \cdot A \in \mathbb{C}^N \). Indeed, as the only functions on a point are constant functions - the value at the point - this is consistent with our intuition that the quotient ring \( Q = \mathbb{C}[z_1, \ldots, z_N]/I \) should be interpreted as the “polynomial functions” on a point. Further, note that as \( Q \) is isomorphic as a ring to \( \mathbb{C} \), the product structure of the ring of polynomials is preserved in the quotient. That is, we can think of the residue of \( f \) in \( Q \) as an operator on other elements of \( Q \) which is multiplication by its value at the critical point. In particular, using the fact that \( \partial W, z - C^{-1} \cdot A \in I \), we can compute the residue of \( W \) in \( Q \) by,

\[ W = W_0 - \frac{1}{2} A \cdot z + \frac{1}{2} z \cdot \partial W \sim W_0 - \frac{1}{2} A \cdot z \sim W_0 - \frac{1}{2} A \cdot C^{-1} \cdot A \in Q. \] \hspace{1cm} (3.7)

Of course, we could have just plugged in our solution, but it turns out that this kind of computation generalizes to the case of higher degree in a way that is useful for computing statistical properties.
3.1.1. Non-Generic Quadratic Superpotentials

It is also interesting to ask what happens if $C$ is not generic, if $\det C = 0$. Then, $C$ has a non-trivial null space which is the space solutions of the homogeneous linear equations we get in the limit that we take $A_i \to 0$. We can describe this limit more precisely by lifting the equations to projective space, to $\mathbb{C}P^N$. That is, we add a new coordinate $z_0$ (which we may think of physically as a chiral superfield corresponding to a (complexified) scale factor) and then consider the homogeneous system of $N$ linear equations $C_{ij}z_j - A_i z_0 = 0$ up to (complex) scaling in $\mathbb{C}P^N$. In the open set $U_0 = \{z_i \sim \lambda z_i \in \mathbb{C}^{N+1} - \{0\}|z_0 \neq 0\} \subset \mathbb{C}P^N$, the $u_i = \frac{z_i}{z_0}$ are good coordinates and the above equation reduces to the inhomogeneous equation with $z_i \to u_i$. Thus, if $\det C \neq 0$, the unique solution of the inhomogeneous equation in $U_0$ gives us the unique solution in $\mathbb{C}P^N$ as well. However, the advantage of projectivizing the problem is that we have added points corresponding to solutions even in the non-generic case. If $\det C = 0$ then we have at least one solution in $\mathbb{C}P^N$ which has $z_0 = 0$ given by a one-dimensional subspace of the null space of $C$. As the $u_i \to \infty$ as we take $z_0 \to 0$, we can think of such a solution as “a solution at infinity” from the perspective of $U_0$. Thus, we see that non-generic points in the space of couplings of the quadratic superpotential correspond to situations in which the supersymmetric vacuum has “run off to infinity” in field space. Now, since all of our field theory computations are approximations only valid locally in field space, these non-generic points should more properly be understood physically as points in the space of couplings in which those approximations break down.

3.2. The General Case

Now, let’s consider the generalization of the above to the case that $W$ is of degree $d > 2$,

$$W(z_1, \ldots, z_N) = \sum_{n=1}^{d} \frac{A_{1n}}{n} z_{i_1} \cdots z_{i_n}. \quad (3.8)$$

The critical points of $W$ are then given by the simultaneous solutions of $N$ degree $d - 1$ polynomials, $\partial_i W = 0$, and, as we have already mentioned above, the polynomials $\partial_i W$ generate an ideal in the ring of polynomials in $N$ variables,

$$I = \langle \partial_i W \rangle = \langle \sum_{n=1}^{d} A_{1n-1} z_{i_1} \cdots z_{i_{n-1}} \rangle. \quad (3.9)$$

Just as before, we may now try to simplify the generators by taking $\mathbb{C}$-linear combinations of them. In particular, it would be nice to simplify the form of the terms of highest degree as we did in obtaining the Fermat form. So, define the matrix $C$ by,

$$A_{i \cdots n} = C_{ii}, \quad A_{ij \cdots n} = A_{jij \cdots n} = \cdots = A_{j \cdots nji} = C_{ij}, \quad \text{for } i \neq j. \quad (3.10)$$

Now, just as in the case of the quadratic superpotential, if $\det C \neq 0 \Rightarrow C \in GL_N(\mathbb{C})$, we can multiply the $\partial_i W$ by the $C^{-1}$ and obtain a simpler set of generators,

$$I = \langle \sum_{n=1}^{d} \sum_{j=1}^{N} C_{ij}^{-1} A_{j1n-1} z_{i_1} \cdots z_{i_{n-1}} \rangle = \langle z_i^{d-1} + (\text{terms of degree } \leq d - 2 \text{ in any } z_j) \rangle. \quad (3.11)$$
Of course, if \( W \) is in Fermat form, then \( C_{ij} = \delta_{ij} \) and so the generators \( \partial_i W \) themselves take precisely this form.\(^8\) As we have argued earlier, the Fermat Form is convenient for other reasons as well, so we will assume henceforth that \( W \) is given in Fermat form. Just as before, the above form suggests that in the quotient ring \( Q = \mathbb{C}[z_1, \ldots, z_N]/I \) we should be able to use the generators in (3.11) to systematically eliminate all powers of \( z_i \) greater than \((d-2)\) and write any element of \( Q \) only in terms of monomials with powers of \( z_i \) less than or equal to \((d-2)\). This is in fact the case as long as \( W \) is a \textit{generic} polynomial of degree \( d \) (for a rigorous proof of this fact, see Chapter 3, Theorem 6.2 of [26]). It is not difficult to show this explicitly in the case of interest \( d = 3 \), but the result is not particularly enlightening and will not be presented here. We only note here that the computation requires that several large matrices in the coefficients be invertible, which we believe is a computational manifestation of the assumption of genericity. For the following, we will proceed assuming that \( W \) is generic, and discuss what we mean by this more precisely at the end of the discussion.

Thus, for \( W \) generic, the monomials \( z_1^{d_1} \cdots z_N^{d_N} \) with \( 0 \leq d_i \leq d-2 \) form a basis for \( Q \) as a vector space, and the relations will allow us to express the residue of any polynomial \( f(z_1, \ldots, z_N) \in \mathbb{C}[z_1, \ldots, z_N] \) in \( Q \) in terms of this basis. This is particularly simple in the case of the superpotential itself, as we have

\[
W = \frac{1}{d} \sum_i z_i \partial_i W + \sum_{n=0}^{d-1} \frac{d-n}{d} A_{1n} z_1 \cdots z_{i_n} \to \sum_{n=0}^{d-1} \frac{d-n}{d} A_{1n} z_1 \cdots z_{i_n} \in Q. \tag{3.12}
\]

Thus, we see that the quotient ring,

\[
\mathbb{C}[z_1, \ldots, z_N]/(\partial_i W) \cong \mathbb{C}^{(d-1)^N},
\]

is a \((d-1)^N\)-dimensional vector space over \( \mathbb{C} \) generated by the images of the monomials \( z_1^{d_1} \cdots z_N^{d_N} \) with \( 0 \leq d_i < d-1 \). Further, as we mentioned earlier, this quotient ring should be understood as the “polynomial functions” on the variety cut out by the generators of the ideal, which in this case are the functions on a set of \((d-1)^N\) points. In particular, in analogy with with the quadratic case, we expect that the residue of any polynomial \( f(z_1, \ldots, z_N) \in \mathbb{C}[z_1, \ldots, z_N] \) in \( Q \) should be related to the values of \( f \) at the critical points of \( W \). However, the fundamental difference between the case \( d > 2 \) and the simple case of a quadratic superpotential is that here, \textit{the residue of \( f \) in \( Q \) is a vector}, and it is not immediately obvious how the components of that vector in any given basis are related to its values at the critical points. However, as \( Q \) is a ring, \( f \) also acts by multiplication on \( Q \). Since \( Q \) is a vector space, and as multiplication by \( f \) is \( \mathbb{C} \)-linear, \( f \) is also a linear operator on \( Q \), which we can represent as a matrix \( f \) in our basis of monomials in an obvious way. In particular, we could consider \( f = z_i \) as an operator or matrix \( z_i \) in this sense. Clearly, the action of \( z_i \) on most basis vectors (for \( d_i < d-2 \)) is totally obvious,

\[
z_i : z_1^{d_1} \cdots z_i^{d_i} \cdots z_N^{d_N} \to z_1^{d_1} \cdots z_i^{d_i+1} \cdots z_N^{d_N}, \quad d_i < d-2. \tag{3.14}
\]

\(^8\)Now, while it is obvious that a superpotential in Fermat form has \( \det C \neq 0 \), one might wonder if this condition is in general necessary and sufficient for the existence of a \( GL_N(\mathbb{C}) \) coordinate transformation taking any given superpotential to Fermat form. However, this is not at all clear, as \( C \) involves only nearly diagonal components of \( A_{1i} \) while the transformation to Fermat form involves all the components of \( A_{1i} \). We will not need to delve into this issue further for our purposes.
However, for \( d_i = d - 2 \) we must use the relations in the ideal to re-express the resulting monomial as a linear combination of the basis elements. Assuming that this is done for each \( z_i \), we obtain \( N \) matrices \( z_i \), each representing multiplication by one of the \( z_i \). We can consider the characteristic equation for each of these matrices,

\[
0 = \det (z_i - \lambda \mathbb{I}) = P_i(\lambda),
\]

which is a polynomial equation of degree \( (d - 1)^N \) in \( \lambda \). Generically, this equation will have \( (d - 1)^N \) distinct roots, corresponding to the \( (d - 1)^N \) eigenvalues of the matrix \( z_i \). In particular, the matrix \( z_i \) must be diagonal in the corresponding basis of eigenvectors. Since \( Q \) is a commutative ring, all matrices corresponding to multiplication by functions must mutually commute, and this must in particularly be true for the \( N \) matrices \( z_i \). Thus, all the matrices corresponding to multiplication by functions \( f \in \mathbb{C}[z_1, \ldots, z_N] \) can be simultaneously diagonalized in the basis of eigenvectors of the \( z_i \). In particular, in this eigenbasis, \( Q \) splits as a ring into the direct sum of \( (d - 1)^N \) rings isomorphic \( \mathbb{C} \),

\[
Q = \mathbb{C}[z_1, \ldots, z_N]/\langle \partial_i W \rangle \cong \mathbb{C} \oplus \cdots \oplus \mathbb{C}.
\]

each of which one may naturally associate with the ring of functions on one of the critical points of \( W \). Thus, generalizing our result from the toy model, the eigenvalue of \( f \) associated with each eigenvector corresponds to the value of \( f \) at the corresponding critical point. If we take \( f = z_i \), then the eigenvalues of \( z_i \) are precisely the \( z_i \) coordinates of the critical points. Further, this means that the trace of \( f \) in any basis is the sum of the values of \( f \) at the critical points of \( W \). In particular, we can do this trace explicitly in the monomial basis. Now, as powers of \( f \) correspond to powers of the corresponding matrix \( f \), we can take their traces to compute the sums of powers of the values of \( f \) at the critical points as well. Thus, we see that by computing traces in the monomial basis, we can compute all holomorphic moments of \( f \) at the critical points of a generic superpotential \( W \). In other words, we can effectively compute all (holomorphic) statistical properties of the values of any polynomial \( f \) at the critical points of \( W \).

### 3.2.1. Resultants and Genericity

There are two basic ways in which the generic situation of \( (d - 1)^N \) vacua can break down. Roots of the system of polynomials can coincide, or roots can run off to infinity. The former is something that already was implicit in the setup of \( [10] \), where the \( N \) chiral fields were assumed to be decoupled. When roots run together, the tunneling between the respective vacua is no longer suppressed, and one should not really count them as independent vacua. It is not difficult to modify the following considerations to deal with this situation, but we will not discuss this further here (see \( [26] \) for details). Having roots “run off to infinity” is a phenomenon that can occur only when interactions between the fields are turned on, and our analysis breaks down in this case.

It turns out \( [26] \) that both the collision and expulsion of roots can be captured by a single genericity condition.\(^9\) As it is only the latter non-genericity which is fatal, it is useful to have

\(^9\)It is a certain integral polynomial in the coefficients \( A_{Ia} \) of \( W \) known as the u-resultant.
a precise criterion when it obtains. In the case of a quadratic superpotential, we saw that the critical point runs off to infinity precisely when \( \det \mathbf{C} = 0 \). Just as before, we can describe this more precisely by considering the homogenization of the polynomial equations \( \partial_i W = 0 \) and lifting them to to equations in \( \mathbb{C}P^N \). It turns out (see \cite{26} Chapter 3) that there is an analogous integral polynomial in the coefficients of the monomials in \( W \) of highest degree \( A_{1d} \) known as a multi-polynomial resultant which plays the same role as \( \det \mathbf{C} \) in degree \( > 2 \). The vanishing of this particular resultant indicates that some solutions have “run off to infinity”, or equivalently, the presence of solutions in the compliment of the open set \( U_0 \). We will refer to the vanishing locus of the resultant as the discriminant locus. While we will not describe this resultant in general, we will be able to compute it exactly in an example \( d = N = 3 \) and see the advertised behavior explicitly in the next section. Finally, note that the cubic polynomial superpotentials we are interested in are approximations valid only within some small polydisk about the origin. Indeed, in the real situation, we don’t literally have roots running off to infinity. Once they leave the polydisk we are considering, our approximation of truncating to the renormalizable terms in the Lagrangian breaks down. We simply no longer trust those solutions which have wandered too far from the origin.

4. Holomorphic Moments and the Statistics of SUSY Vacua

We will now restrict our consideration to the physically relevant case of a renormalizable, R-symmetric, cubic superpotential for \( N \) chiral fields in Fermat form,

\[
W(z_i) = \sum_{i=1}^{N} \left( \frac{1}{3} z_i^3 - a_i z_i \right) - \sum_{i<j<k} b_{ijk} z_i z_j z_k, \tag{4.1}
\]

where we recall that \( b_{ijk} \) is symmetric with vanishing diagonals, i.e. \( b_{iii} = b_{iji} = b_{jii} = 0 \). Dimensional analysis as well as the \( S_N \times (\mathbb{Z}_3)^N \) symmetry will be important tools for the analysis to follow. As the superpotential \( W \) has mass dimension 3, the \( z_i \) must have mass dimension 1 while the \( a_i \) have mass dimension 2 and the \( b_{ijk} \) are dimensionless. Further, \( \zeta \in (\mathbb{Z}_3)^N \) acts as,

\[
z_i \rightarrow z_i \zeta_i, \quad a_i \rightarrow a_i \zeta_i^{-1}, \quad b_{ijk} \rightarrow b_{ijk} \zeta_i^{-1} \zeta_j^{-1} \zeta_k^{-1}, \quad \zeta = \{ \zeta_i \}, \tag{4.2}
\]

where \( \zeta_i \) is in the \( i^{th} \) \( \mathbb{Z}_3 \), under which \( z_i \) has charge +1 while the \( a_i \) and \( b_{ijk} \) have charge -1. Note that the equations for the critical points of the cubic in Fermat form are,

\[
z_i^2 - a_i - \sum_{j<k} b_{ijk} z_j \zeta_k = 0, \tag{4.3}
\]

and a basis for the quotient ring \( Q \) is given by the \( 2^N \) monomials,

\[
\{1, z_i, z_i z_j, \ldots, z_1 \cdots z_N\}, \tag{4.4}
\]

where at most one power of each \( z_i \) appears in each monomial. The residue of \( W \) in \( Q \) written in this basis is,

\[
W = -\frac{2}{3} \sum_{i=1}^{N} a_i z_i. \tag{4.5}
\]
Note that tr (W) and all other holomorphic moments of odd powers of W or  \( z_i \) vanish due to the R-symmetry. Explicitly, since the relations \( \mathbb{Z}_3 \) are even, any odd power of the operator \( z_i \) maps monomials of even degree to those of odd degree and vice-versa and therefore never has any non-zero diagonal components.

We will be interested in determining the distribution of values of \( W \) among the \( 2^N \) vacua which appear for generic values of the couplings. These are the eigenvalues of \( W \). By the R-symmetry, the values of \( W \) at the critical points occur in pairs, \( \pm \lambda \), and are the roots of the characteristic equation,\(^\dagger\)

\[
0 = \det\left( \frac{3}{2} W - \lambda \mathbb{I} \right) \equiv P(\lambda^2) = (\lambda^2)^{2N-1} - 2^{N-1} \sum_{k=1}^{2N-1} F_k(a, b)(\lambda^2)^{2N-1-k}. \tag{4.6}
\]

We can express the \( F_k(a, b) \) in terms of the holomorphic moments of \( W \) by noting that,

\[
\det\left( \frac{3}{2} W - \lambda \mathbb{I} \right) = \exp \text{tr} \left( \log\left( \frac{3}{2} W - \lambda \mathbb{I} \right) \right) = \lambda^{2N} \exp \text{tr} \left( - \sum_{k=1}^{2N-1} \frac{1}{2k} \left( \frac{3}{2} W \right)^{2k} \right), \tag{4.7}
\]

where the last equality is only meaningful for positive powers of \( \lambda \) and encodes the form of the \( F_k(a, b) \). Dimensional analysis and invariance under the \( S_N \times (\mathbb{Z}_3)^N \) symmetry can be used to restrict the form of the \( F_k(a, b) \). They must be homogeneous polynomials in the \( a_i \), of degree \( 3k \), whose coefficients are rational functions of the \( b_{ijk} \). Of particular importance to us is \( F_1 \), which is proportional to the “holomorphic variance” of \( W \). Using the symmetries, we may parameterize this variance as,

\[
\langle \frac{3}{4} W^2 \rangle \equiv 2^{-N} \text{tr} \left( \frac{3}{4} W^2 \right) = F_1(a, b) = f(b) \sum_i a_i^3 + \sum_{i<j<k} g^{ijk}(b) a_i a_j a_k + \sum_{i\neq j} h^{ij}(b) a_i^2 a_j, \tag{4.8}
\]

where the normalization factor converts the trace into the average value of \( \frac{3}{4} W^2 \) among the \( 2^N \) critical points. Now, as \( F_1 \) must be invariant under \( S_N \times (\mathbb{Z}_3)^N \), we see that \( f(b) \) must also be invariant, while \( g^{ijk}(b) \) is symmetric with vanishing diagonals and must have charge +1 under the \( i^{th}, j^{th}, \) and \( k^{th} \) \( \mathbb{Z}_3 \), and \( h^{ij}(b) \) has vanishing diagonal and must have charge −1 under the \( i^{th} \) and +1 under the \( j^{th} \) \( \mathbb{Z}_3 \). While we will not be able to compute the coefficient functions \( f(b), g^{ijk}(b), \) and \( h^{ij}(b) \) in closed form for arbitrary large \( N \), we can certainly do so for any small fixed \( N \). We will now turn to the simple example of the case \( N = 3 \).

### 4.1. Example I - W odd, \( d = 3 \), \( N = 3 \)

Consider the case of the generic, odd, cubic superpotential in three variables in Fermat form,

\[
W = \frac{1}{3} \left( z_1^3 + z_2^3 + z_3^3 \right) - (a_1 z_1 + a_2 z_2 + a_3 z_3 + b z_1 z_2 z_3). \tag{4.9}
\]

\(^\dagger\)Note that in general, the number of roots that run off to infinity as one approaches the discriminant locus is controlled by the order of the pole of \( F_{2N-1}(a, b) \). For a pole of order \( m \), \( 2m \) roots run off to infinity. If the order of the pole is less than the maximal \( (2^{N-1}) \), then some of the roots remain finite, even in the limit.
The equations for its critical points are,
\[ z_1^2 - a_1 - b_2 z_2 z_3 = 0, \quad z_2^2 - a_2 - b_3 z_1 z_3 = 0, \quad z_3^2 - a_3 - b_1 z_1 z_2 = 0. \] \hfill (4.10)

A basis for the quotient ring \( Q \) is given by the \( 2^3 \) monomials,
\[ \{1, z_1, z_2, z_3, z_1 z_2, z_1 z_3, z_2 z_3, z_1 z_2 z_3\}. \] \hfill (4.11)

In this basis, the residue of \( W \) in \( Q \) is,
\[ W = -\frac{2}{3} (a_1 z_1 + a_2 z_2 + a_3 z_3). \] \hfill (4.12)

It is an easy exercise to determine the form of the matrix \( W \),
\[
W = -\frac{2}{3} \begin{pmatrix}
0 & a_1^2 & a_2^2 & a_3^2 & 0 & 0 & 0 & \frac{3a_1a_2a_3b}{1-b^3} \\
 a_1 & 0 & 0 & 0 & a_2^2 + a_3a_1b^2 & 0 & \frac{2a_2a_3b}{1-b^3} & 0 \\
 a_2 & 0 & 0 & 0 & 0 & a_3^2 + a_1a_2b^2 & 0 & 0 \\
 a_3 & 0 & 0 & 0 & 0 & 0 & a_2^2 + a_1a_3b^2 & 0 \\
 0 & a_2 & a_1 & a_3b & 0 & 0 & 0 & a_2^2 + 2a_1a_3b^2 \\
 0 & a_3 & a_2b & a_1 & 0 & 0 & 0 & a_1^2 + 2a_2a_3b^2 \\
 0 & a_1b & a_3 & a_2 & 0 & 0 & 0 & a_1^2 + 2a_2a_3b^2 \\
 0 & 0 & 0 & 0 & a_3 & a_2 & a_1 & 0
\end{pmatrix}. \] \hfill (4.13)

Now, it is clear that \( \text{tr}(W) = 0 \) as expected from the R-symmetry. More interesting is the fact that we can just as easily compute,
\[ \langle \frac{9}{4} W^2 \rangle = 2^{-3} \text{tr} \left( \frac{9}{4} W^2 \right) = \left( \frac{a_1^3 + a_2^3 + a_3^3}{3} \left( 1 - \frac{1}{4} b^3 \right) + \frac{9}{2} a_1a_2a_3b^2 \right) \left( 1 - b^3 \right), \] \hfill (4.14)

as well as all higher moments of \( W \). In particular, we can explicitly read off the coefficient functions of \( \mathbf{1.58} \) for the \( N = 3 \) case,
\[
f(b) = \frac{1}{4} \left( 4 - b^3 \right), \quad g(b) = \frac{3b^2}{2(1 - b^3)}, \quad h(b) = 0. \] \hfill (4.15)

Further, if we wish to explicitly compute the values of \( W \) at the critical points, we can use these moments to compute the characteristic polynomial for \( W \) and numerically solve it. Indeed, this is easy to do using Mathematica, and one can verify that numerical solution of the above equations through other means agree with the eigenvalue methods. Further, using similar techniques, we can also compute the moments of any polynomial over these vacua.

Finally, we note that \( \mathbf{1.15} \) are singular in the limit that \( b \) approaches a third root of unity. It is easy to check numerically that six of the eight critical points run off to infinity in this limit, and that these three points indeed comprise the discriminant locus of the \( N = 3 \) case. Thus, we see explicitly in this case that the multi-polynomial resultant we discussed earlier must be proportional to \( (1 - b^3) \) for \( N = 3 \).
4.2. Example II - The Fermat Cubic for Large $N$ and small $b_{ijk}$

Since the dimension of the vector space $Q$ grows exponentially with the number of variables, the matrix methods introduced above become quickly intractable, even numerically, for $N \gtrsim 10$. In particular, explicit computations require that we can compute holomorphic moments like

$$\mathcal{W}^2 = 2^{-N} \text{tr} \left( \frac{2}{4} W^2 \right) = \sum_{i,j} a_i a_j (2^{-N} \text{tr} (z_i z_j)) = \sum_{i,j} a_i a_j \mathbb{E}[z_i z_j],$$

(4.16)

which in turn require that we can compute the moments of products of the $z_i$'s. This could be done if we could easily compute the matrix elements of $z_i$ in this basis. However, as we mentioned earlier, this is very computationally intensive, and involves the inversion of exponentially large matrices.

Thus, it is useful to understand if there exists a limit in which the above methods become more tractable at large $N$, at least perturbatively. Now, note that the decoupled limit with $b_{ijk} = 0$ considered in [10] is certainly such a simple case, so the $b_{ijk}$ seem to be natural candidates for small parameters. Indeed, radiative stability of the Kähler potential requires that generically $|A_{ijk}| \sim O(N^{-1})$ while at most $O(N)$ of them may be $O(1)$. That is, the $b_{ijk}$'s are generically small. There are, of course, a large number of them, so this large-$N$ suppression of the magnitudes of the $b_{ijk}$ is compensated by the sum. But say that we make the further assumption that perturbation theory is good — that is, that the magnitude of the $b_{ijk}$ is further suppressed by some small parameter $\delta$. Then it makes sense to expand our expressions for $\mathbb{E}[f(z)]$ as a (formal) power series in the $b_{ijk}$.

4.2.1. Computations of Holomorphic Moments for Small $b_{ijk}$

To begin with, we can use dimensional analysis as well as the $S_N \ltimes (\mathbb{Z}_3)^N$ symmetry to constrain the form of successive terms in the power series expansions for $\mathbb{E}[f(z)]$. Let us consider $\mathbb{E}[z_i z_j]$ as an example. First note that if we have $i = j$,

$$\mathbb{E}[z_i^2] = \mathbb{E}[a_i \mathbb{1} + \sum_{j < k} b_{ijk} z_j z_k] = a_i + \sum_{j < k} b_{ijk} \mathbb{E}[z_j z_k],$$

(4.17)

so we only need focus on the case $i \neq j$. Using the symmetries and dimensional analysis, it is easy to check that the first few terms in the (formal) power series expansion of $\mathbb{E}[z_i z_j]$ for $i \neq j$ must take the form,

$$\mathbb{E}[z_i z_j] = \alpha_1^{(2)} b_{ijk}^2 a_k + \sum_{k < l} b_{ikl} b_{jkl} \left[ \alpha_1^{(3)} (b_{ijkl} a_i + b_{ikl} a_j) + \alpha_2^{(3)} (b_{ijkl} a_i + b_{iik} a_k) \right] + O(b^4),$$

(4.18)

where the $\alpha_i^{(n)} \in \mathbb{Q}$ are rational numbers which we will see are independent of $N$ for any fixed $\alpha_i^{(n)}$ and $N$ sufficiently large. The $\alpha_i^{(n)}$ can be found by computing the diagonal matrix elements of the operator $z_i z_j$ in the monomial basis, summing them, and dividing by $2^N$. Of course, the only way one obtains a diagonal element is if the relations (4.13) are used, so such diagonal elements only arise from the action of $z_i z_j$ on basis elements containing $z_i$.
or $z_j$. Each use of the relation results either in the introduction of one factor of $a_i$ or $b_{ijk}$, so the $\alpha_i^{(n)}$ can be computed using the relations precisely $(n + 1)$ times. For example, the contribution of the monomial $z_i$ to $\alpha_1^{(n)}$ is found by,

\[
z_i z_j \cdot z_i = a_i z_j + \sum_{k<i} b_{ikl} z_k z_l z_j + \sum_{l \neq (i,j)} b_{ijl} z_i z_l + z_j
\]

\[
= \cdots + \sum_{l \neq (i,j)} b_{ijl} \left( a_j z_l + \sum_{m<n \neq (l,j)} b_{jmn} z_m z_n z_l + \sum_{m \neq (j,l)} b_{jlm} z_i z_m \cdot z_l \right)
\]

\[
= \cdots + \sum_{l \neq (i,j)} \sum_{m \neq (l,j)} b_{ijl} b_{jlm} a_k z_m + \mathcal{O}(b^3)
\]

\[
= \cdots + \sum_k b_{ijk}^2 a_k z_i + \cdots .
\]

In fact, we can greatly simplify this computation by noting that the $S_N$ symmetry implies that we only need to compute how many times any single term of a sum over dummy indices appears in the diagonal matrix elements. For example, to find $\alpha_1^{(2)}$, we only need to ask how many times the single term $b_{ijk}^2 a_k$ with $i,j$, and $k$ all fixed appears on the diagonal of $z_i z_j$. This is very easy to do. First of all, this term only involves $a_k$ and one particular coefficient $b_{ijk}$. As these two coefficients arise in the relations \[\ref{4.3}\] only in terms involving $z_i$, $z_j$, and $z_k$, we can completely ignore all the other variables, which just go along for the ride.\footnote{Formally, this is equivalent to working in $Q$ modulo the ideal $\langle z_i \rangle$, $l \neq i, j, k$.} Thus, to calculate $\alpha_1^{(2)}$, we only need to compute the coefficient of the term $b_{ijk}^2 a_k$ in the trace of $z_i z_j$ restricted to the $2^3$ monomials of the form $z_i^{a_1} z_j^{a_2} z_k^{a_3}$, $a_i = 0, 1$ in our basis \[\ref{4.4}\], ignoring all terms involving any other variables (and divide the result by $2^3$). That is, we have shown that this coefficient can be computed by considering just the case $N = 3$. For instance, we can expand \[\ref{4.14}\] in a power series in $b$ and find $\alpha_1^{(2)} = \frac{3}{4}$.

We can more easily compute this using a diagrammatic technique. To compute the contribution of $z_i z_j \cdot z_i^{a_1} z_j^{a_2} z_k^{a_3}$ to $\alpha_1^{(2)}$, first draw a labeled external line for $z_i$ and $z_j$ as well as all the $z_i$’s present in the basis element. Each $b_{ijk}$ corresponds to a vertex which takes in two identically labeled lines corresponding to one of its indices and outputs two lines corresponding to the other two indices, while $a_k$ is an external sink for a pair of $z_k$ lines. Construct all possible graphs with two $b_{ijk}$ vertices, and one $a_k$ external sink from the given lines, noting that graphs which differ by a choice of which lines of the same index are contracted at a vertex or sink are equivalent. Count the number of distinct graphs you drew and divide by $2^3$ - this is the contribution. It turns out that there is precisely one possible graph for each basis element with either $z_i$ or $z_j$ in it, so we again find $\alpha_1^{(2)} = \frac{3}{4}$.

The above method, of course, generalizes in the obvious way to the computation of any $\alpha_i^{(n)}$ by restricting ourselves to just the relevant $N = n + 1$ variables. For example, to compute the coefficient $\alpha_1^{(3)}$ of $b_{ikl} b_{jkl}^2 a_i$, we consider $N = 4$ and graphs with one $b_{ikl}$ vertex and two $b_{jkl}$ vertices as well as a single sink $a_i$. Note that it is useful to start with monomials of lowest total degree and proceed to those of higher degree as any graph associated with a given basis monomial also contributes to all monomials it divides. Thus, without much
trouble, one can compute the terms cubic in \( b_{ijk} \),

\[
\langle z_i z_j \rangle = \frac{3}{4} \left[ \sum_k b_{ijk}^2 a_k + \sum_{k<l} b_{ikl} b_{jkl} \left[ (b_{jkl} a_i + b_{ikl} a_j) + 2(b_{ijk} a_l + b_{ijl} a_k) \right] + O(b^4) \right],
\]

so \( \alpha_1^{(3)} = \frac{3}{4}, \alpha_2^{(3)} = \frac{3}{2} \). Using this result as well as (4.14) we can compute,

\[
\langle \frac{9}{4} W^2 \rangle = \left( \sum_i a_i^2 \langle z_i^2 \rangle + 2 \sum_{i<j} a_j a_j \langle z_i z_j \rangle \right) = \sum_i a_i^3 + \sum_{i<j} \left[ 2a_i a_j + \sum_l a_{ijl}^2 \right] \langle z_i z_j \rangle
\]

\[
= \left( 1 + \frac{3}{4} \sum_{j<k<l} b_{jkl}^3 \right) \sum_i a_i^3 + \sum_{i<j} \frac{9}{2} \left( b_{ij}^2 + 2 \sum_{l} b_{ijkl} b_{jkl} \right) a_i a_j a_k + \sum_{i \neq j} \left( \frac{3}{4} \sum_{k<l} b_{ikl} b_{jkl}^2 \right) a_i^2 a_j + O(b^4).
\]

Thus, we can read off the coefficients functions of \( \langle 4.14 \rangle \) up to terms of order \( b^4 \),

\[
f(b) = 1 + \frac{3}{4} \sum_{j<k<l} b_{jkl}^3 + \ldots,
\]

\[
g^{ijk}(b) = \frac{9}{2} \left( b_{ij}^2 + 2 \sum_{l} b_{ijkl} b_{jkl} \right) + \ldots,
\]

\[
h^{ij}(b) = \frac{3}{4} \sum_{k<l} b_{ikl} b_{jkl}^2 + \ldots.
\]

As noted, each additional power of \( b \) brings an additional sum over a dummy index, which has \( O(N) \) terms. If \( |b_{ijk}| \sim N^{-1} \delta \), then the term of order \( b^k \) goes like \( N \delta^k \), so this is a systematic expansion in powers of \( \delta \).

We can continue and compute higher holomorphic moments in an expansion in powers of \( b_{ijk} \) - for instance, the following holomorphic moments will be useful (with \( i \neq j \neq k \neq l \),

\[
\langle z_i^4 \rangle - \langle z_i^2 \rangle \langle z_i^2 \rangle = \sum_{j<k} b_{ijk}^2 a_j a_k + O(b^3)
\]

\[
\langle z_i^3 z_j \rangle - \langle z_i^2 \rangle \langle z_i z_j \rangle = \sum_k b_{ijk}^2 a_i a_k + O(b^3)
\]

\[
\langle z_i^2 z_j^2 \rangle - \langle z_i \rangle \langle z_j \rangle \langle z_j \rangle = \sum_{k<l} b_{ikl} b_{jkl} a_k a_l + O(b^3)
\]

\[
\langle z_i^2 z_j z_k \rangle - \langle z_i z_j \rangle \langle z_i z_k \rangle = a_i a_j + \sum_{k<l} b_{ikl} b_{jkl} a_k a_l + O(b^3)
\]

\[
\langle z_i z_j z_k z_l \rangle - \langle z_i z_j \rangle \langle z_i z_k \rangle \langle z_j z_l \rangle = b_{ijk} a_j a_k + O(b^3)
\]

\[
\langle z_i^2 z_j z_k \rangle - \langle z_i z_j \rangle \langle z_i z_k \rangle = b_{ijk} a_j a_k + \frac{3}{4} \sum_{l} b_{jkl}^2 a_i a_l + O(b^3)
\]

\[
\langle z_i z_j z_k z_l \rangle - \langle z_i z_j \rangle \langle z_i z_k \rangle \langle z_k z_l \rangle = O(b^3).
\]
Therefore, we see that it is possible to effectively compute holomorphic moments and any desired holomorphic statistical property in the limit of large \( N \) as a systematic expansion in small \( b_{ijk} \).

For any fixed \( N \), we can, with more effort, crank out some explicit formulæ, capturing the dependence on the discriminant locus. For \( N = 3 \), we see explicitly, that each of the moments in \( (4.23) \) has a double pole at the discriminant locus.

\[
\langle \langle z_1^4 \rangle \rangle - \langle \langle z_1^2 \rangle \rangle^2 = \frac{b^2(8(2 + b^3)a_2a_3 + 3b^4a_1^2)}{16(1 - b^3)^2}
\]

\[
\langle \langle z_1^3z_2 \rangle \rangle - \langle \langle z_1^2 \rangle \rangle \langle \langle z_1z_2 \rangle \rangle = \frac{b^2((16 - b^3)a_1a_3 + 12ba_2^2)}{16(1 - b^3)^2}
\]

\[
\langle \langle z_1^2z_2^2 \rangle \rangle - \langle \langle z_1^2 \rangle \rangle \langle \langle z_2^2 \rangle \rangle = \frac{b^3((16 - b^3)a_1a_2 + 12ba_3^2)}{16(1 - b^3)^2}
\]

\[
\langle \langle z_1^2z_2z_3 \rangle \rangle - \langle \langle z_1^2 \rangle \rangle \langle \langle z_2z_3 \rangle \rangle = \frac{8(2 + b^3)a_1a_2 + 3b^4a_3^2}{16(1 - b^3)^2}
\]

\[
\langle \langle z_1^2z_2z_3 \rangle \rangle - \langle \langle z_1z_2 \rangle \rangle \langle \langle z_1z_3 \rangle \rangle = \frac{b(16 - b^3)a_2a_3 + 12ba_1^2}{16(1 - b^3)^2}
\]

\[
\langle \langle z_1^2z_2z_3 \rangle \rangle - \langle \langle z_1z_2 \rangle \rangle \langle \langle z_1z_3 \rangle \rangle = \frac{b((16 - b^3)a_1a_3 + 12ba_2^2)}{16(1 - b^3)^2}
\]

\[
\langle \langle z_1^2z_2z_3 \rangle \rangle - \langle \langle z_1z_2 \rangle \rangle \langle \langle z_1z_3 \rangle \rangle = \frac{b((16 - b^3)a_2a_3 + 12ba_1^2)}{16(1 - b^3)^2}
\]

5. Holomorphic Moments and Scanning

Now, let us consider the relationship between the holomorphic moments we have computed and the real-valued statistical properties of the vacua. \( W = W_1 + iW_2 \) is complex. Averaged over the \( 2^N \) vacua, the mean value is, of course, zero.

\[
\langle \langle W_1 \rangle \rangle = \langle \langle W_2 \rangle \rangle = 0
\]

The variances involve the sums of squares of the eigenvalues, and so are encoded in \( \frac{9}{2}F_1 = \langle \langle W^2 \rangle \rangle \). The variances we are interested in are \( \delta W_1^2 \), \( \delta W_2^2 \) and the covariance, \( \langle \langle W_1W_2 \rangle \rangle \). Two linear combinations of these are “holomorphic” in the eigenvalues,

\[
\delta W_1^2 - \delta W_2^2 = \text{Re} \langle \langle W^2 \rangle \rangle \quad (5.1a)
\]

\[
\langle \langle W_1W_2 \rangle \rangle = \text{Im} \langle \langle W^2 \rangle \rangle \quad (5.1b)
\]

The remaining linear combination requires more detailed knowledge, though it is easy to find a lower bound from the Cauchy-Schwarz inequality,

\[
\delta W_1^2 + \delta W_2^2 \geq |\langle \langle W^2 \rangle \rangle| \quad (5.1c)
\]

In the following, we will turn to the implications of these formulæ when one averages over some ensemble of couplings. A lower bound, like \( (5.1c) \) will be quite sufficient to prove that a given coupling “scans”. But one requires something of an upper bound in order to prove that a coupling doesn’t scan. These methods generalize straightforwardly to other coupling which depend holomorphically on the \( \phi_i \) (superpotential couplings for the Standard model fields, holomorphic gauge couplings, etc.).
5.1. Ensembles of Theories and the Scanning of \( \Lambda \)

Let us return to (4.8),

\[
\langle \langle W^2 \rangle \rangle = \frac{4}{9} \left[ f(b) \sum_i a_i^3 + \sum_{i<j<k} g^{ijk}(b)a_i a_j a_k + \sum_{i\neq j} h^{ij}(b)a_i^2 a_j \right], \tag{5.2}
\]

which encodes the variance of \( W \) among the \( 2^N \) vacua for fixed values of the couplings. What we would like to do now is discuss what happens if we have some ensemble of theories. For example, rather than studying F-theory vacua for fixed flux, we might wish to study the ensemble of all possible fluxes.

No one, currently, has a compelling proposal for what probability measure to choose for that ensemble. Neither do we. What we hope to do in this section is to explore the implications one can extract from such a choice, given the results of the previous sections.

The authors of [10] worked with \( b_{ijk} \equiv 0 \), and assumed that the \( a_i \) were \( N \) independent random variables, chosen from some common distribution. In their limit, the second two terms of (5.2) vanish, and we have

\[
\langle \langle W^2 \rangle \rangle = \frac{4}{9} \sum_{i=1}^N a_i^3
\]

Since there are \( N \) terms, and each is an independent random variable, the sum grows like \( O(N) \). At first glance, for nonzero \( b_{ijk} \), the situation appears to change dramatically. The second term in (5.2) contains \( O(N^3) \), and the third term contains \( O(N^2) \) independent random variables. At least, that would be the case if the \( b_{ijk} \) coefficients were all generically of \( O(1) \).

However, as we have argued, radiative stability requires that the \( b_{ijk} \) be generically small. So, despite appearances, each of the terms in (5.2) is \( O(N) \). The cosmological constant scans (and, in fact, the variance grows like \( O(N) \)) in this more general context, just as it did in the model of [10] where \( b_{ijk} \equiv 0 \).

5.2. Other Couplings

Other holomorphic couplings which depend on the \( \phi_i \) can be treated similarly. Consider some such coupling, \( c(\phi) \), which might be a holomorphic gauge coupling, or a coupling in the superpotential for the Standard Model fields. It is reasonable to assume that \( c(\phi) \) has definite parity under the \( \mathbb{Z}_4 \) R-symmetry that constrained the form of \( W(\phi) \). The crucial distinction in [10] is between those cases where \( c(\phi) \) is odd and hence \( \langle \langle c(\phi) \rangle \rangle = 0 \) (like the superpotential, \( W(\phi) \)) versus those for which \( c(\phi) \) is even. The former couplings “scan,” whereas the latter do not: the standard deviation of \( c(\phi) \) among the \( 2^N \) vacua is much smaller than its mean value.

As with the superpotential, we assume that \( c(\phi) \) has a Taylor expansion, convergent in a polydisk of radius \( M_c \). For simplicity, we will take it to be dimensionless; the case of the \( \mu \) parameter in the Standard Model is an easy generalization. So we have,

\[
c_{\text{ev}}(\phi) = c_0 + \sum_{i \leq j} c_{ij}\phi_i\phi_j/M_c^2 + \ldots
\]

\[
c_{\text{odd}}(\phi) = \sum_i c_i\phi_i/M_c + \ldots \tag{5.3}
\]
depending on the parity of \( c(\phi) \). As in our previous discussion, radiative stability constrains the form of the coefficients in this expansion for large-\( N \). The \( c_{ijk...} \) must be generically small. Specifically, we have constraints of the form
\[
g^{\eta \rho} g^{\rho \tau} c_{ij} \sim O(N)
\]
and so forth for the higher coefficients. Thus, the mean value of the \((2k)^{th}\) term in the series \( c_0 + \epsilon^2 f(b) + \epsilon^4 h^{ijkl}(b)c_{ijkl} + \cdots \) (the mean value of the odd terms vanish), rather than going like \( N^{2k}(M_r/M_c)^{2k} \) actually goes like \( N(M_r/M_c)^{2k} \). So, we have a systematic expansion in powers of \( \epsilon^2 = (M_r/M_c)^2 \). For sufficiently small \( \epsilon \), the leading, \( \epsilon^2 \) term can compensate for the overall factor of \( N \), while the subleading terms are negligible. More explicitly, if we let \( a_i = M^2_t \tilde{a}_i \), we have,
\[
\langle \langle c_{\text{odd}} \rangle \rangle = 0
\]
\[
\langle \langle c_{\text{ev}} \rangle \rangle = c_0 + \epsilon^2 \left[ f(b) \sum_i c_{ii} \tilde{a}_i + \sum_{i \neq j} h^{ij}(b) c_{ij} \tilde{a}_j + \frac{1}{2} \sum_{i \neq j \neq k} g^{ijk}(b) c_{ijk} \tilde{a}_k \right] + O(\epsilon^4)
\]
where the \( O(\epsilon^4) \) terms represent the contributions of quartic and higher terms in \( c(\phi) \). Note that the rational functions of \( b \): \( f(b), g^{ijk}(b) \) and \( h^{ij}(b) \) are the same ones from \( 4.8 \) that appeared in the computation of the variances of the superpotential and were computed up to terms of \( O(b^4) \) in \( 4.22 \). It is easy to check that, as announced, the leading term behaves as \( N\epsilon^2 \), with the subleading terms suppressed by higher powers of \( \epsilon^2 \).

The variance is calculated similarly. In the odd case,
\[
\langle \langle \delta^2 \rangle \rangle = \epsilon^2 \left[ f(b) \sum_i c_i^2 \tilde{a}_i + \sum_{i \neq j} h^{ij}(b) c_i^2 \tilde{a}_j + \frac{1}{2} \sum_{i \neq j \neq k} g^{ijk}(b) c_i c_j \tilde{a}_k \right] + O(\epsilon^4).
\]

Thus, given a distribution for the couplings we can definitively show that this coupling scans if this variance is non-vanishing.

In the even case, let \( \hat{c} = c - \langle \langle c \rangle \rangle \). The variance is then given by,
\[
\langle \langle \hat{c}^2 \rangle \rangle = \sum_{i,j,k,l} \frac{C_{ijkl}}{M_c^4} \left[ \langle \langle Z_i Z_j Z_k Z_l \rangle \rangle - \langle \langle Z_i \rangle \langle \langle Z_j \rangle \langle \langle Z_k \rangle \rangle \langle \langle Z_l \rangle \rangle \rangle \right] + O(\epsilon^6).
\]

As in \( 4.22 \) we can easily compute \( 5.7 \) in an expansion in powers of \( \epsilon^2 \) using \( 4.23 \). For finite \( N \), we can do better – for \( N = 3 \), we have \( 4.24 \). As with the superpotential, we can write \( \hat{c} = c_1 + ic_2 \), and we have
\[
\delta c_1^2 - \delta c_2^2 = \text{Re} \langle \langle \hat{c}^2 \rangle \rangle ,
\]
\[
\langle \langle c_1 \hat{c}_2 \rangle \rangle = \text{Im} \langle \langle \hat{c}^2 \rangle \rangle ,
\]
\[
\delta c_1^2 + \delta c_2^2 \geq |\langle \langle \hat{c}^2 \rangle \rangle |.
\]

The right-hand-side of \( 5.5b \) goes like \( N\epsilon^2 \), whereas the right-hand-side of \( 5.5a \) goes like \( N\epsilon^4 \). We would like to conclude that the standard deviations, \( \delta c_1/c_1 \) and \( \delta c_2/c_2 \), behave as
\[
\frac{\sqrt{N\epsilon^2}}{N\epsilon^2} = \frac{1}{\sqrt{N}}.
\]
Unfortunately, (5.8c) is only a lower bound, so more detailed information is necessary to really show that this coupling does not scan. Indeed, this is a potentially serious drawback to the whole notion of a “friendly landscape.” If the lower bound in (5.8c) drastically underestimates the true variance, then these couplings will vary appreciably over this ensemble of vacua. And anthropic arguments, based on holding them fixed while varying other coupling, like the cosmological constant, are incorrect.

6. Generalizations

The “space” of vacua discussed here is the complex affine algebraic variety, $\mathbb{C}[z_1, \ldots, z_N]/(\partial_i W)$. As such, it was amenable to the techniques of complex algebraic geometry. If we were studying $N$ real scalar fields, $\phi_i$, with potential, $V(\phi)$, we would be faced with a problem in real algebraic geometry. This problem is harder, both because real algebraic geometry is harder, and less well-developed than the complex case and because the characterization of the desired space of vacua is more subtle. We are not interested in $\mathbb{R}[x_1, \ldots, x_N]/(\partial_i V)$. We are only interested in minima of $V$, as opposed to all critical points. At large $N$, “most” critical points of $V$ are actually saddle points, and it’s algebraically a little awkward to pick out just the minima.

A more interesting generalization is the case in which some of our complex chiral multiplets are charged under a $U(1)^k$ gauge symmetry. Physically, this leaves open the possibility of supersymmetry-breaking, in this “moduli” sector, via the inclusion of Fayet-Iliopoulos terms. Mathematically, this gauging moves us from the realm of complex affine algebraic geometry to that of toric geometry. The toric version of our problem is nearly as well-developed mathematically as the affine case we have discussed. It would be very interesting to generalize our considerations to that case.

Finally, we have deliberately eschewed discussion of the microphysics that determines the (ensemble of) couplings in our low-energy effective Lagrangian. Douglas and collaborators [39, 40, 41, 42, 43, 28, 44, 45, 46, 47], for instance, have pursued the idea of treating all possible choices of fluxes in an F-theory compactification “democratically,” assigning equal weight to the low-energy theory that arises from each choice of flux. Further developments along this vein appear in [48, 30, 19, 30, 51, 52, 53]. It has been argued [22] that in the context of type IIA flux compactifications this may not be a reasonable choice, as there are instances where the number of possible choices of flux is infinite. Independent of this more subtle question, we believe that our methods will prove useful in analyzing the vacuum structure of any given theory whenever one has a large number, $N$, of light chiral multiplets.

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References

[1] S. Kachru, R. Kallosh, A. Linde, and S. P. Trivedi, “De Sitter vacua in string theory,” *Phys. Rev.* **D68** (2003) 046005, [hep-th/0301240](http://link.springer.com/article/10.1103/PhysRevD.68.046005).

[2] L. Susskind, “The anthropic landscape of string theory,” [hep-th/0302219](http://arxiv.org/abs/hep-th/0302219).

[3] S. Weinberg, “Anthropic bound on the cosmological constant,” *Phys. Rev. Lett.* **59** (1987) 2607.

[4] J. D. Brown and C. Teitelboim, “Dynamic neutralization of the cosmological constant,” *Phys. Lett.* **B195** (1987) 177–182.

[5] S. Weinberg, “The cosmological constant problem,” *Rev. Mod. Phys.* **61** (1989) 1–23.

[6] H. Martel, P. R. Shapiro, and S. Weinberg, “Likely values of the cosmological constant,” *Astrophys. J.* **492** (1998) 29, [astro-ph/9701099](http://arxiv.org/abs/astro-ph/9701099).

[7] J. Garriga and A. Vilenkin, “On likely values of the cosmological constant,” *Phys. Rev.* **D61** (2000) 083502, [astro-ph/9908115](http://arxiv.org/abs/astro-ph/9908115).

[8] S. Weinberg, “A priori probability distribution of the cosmological constant,” *Phys. Rev.* **D61** (2000) 103505, [astro-ph/0002387](http://arxiv.org/abs/astro-ph/0002387).

[9] R. Bousso and J. Polchinski, “Quantization of four-form fluxes and dynamical neutralization of the cosmological constant,” *JHEP* **06** (2000) 006, [hep-th/0004134](http://arxiv.org/abs/hep-th/0004134).

[10] N. Arkani-Hamed, S. Dimopoulos, and S. Kachru, “Predictive landscapes and new physics at a TeV,” [hep-th/0501082](http://arxiv.org/abs/hep-th/0501082).

[11] K. R. Dienes, E. Dudas, and T. Gherghetta, “A calculable toy model of the landscape,” [hep-th/0412185](http://arxiv.org/abs/hep-th/0412185).

[12] S. B. Giddings, S. Kachru, and J. Polchinski, “Hierarchies from fluxes in string compactifications,” *Phys. Rev.* **D66** (2002) 106006, [hep-th/0105097](http://arxiv.org/abs/hep-th/0105097).

[13] S. Kachru, M. B. Schulz, and S. Trivedi, “Moduli stabilization from fluxes in a simple iib orientifold,” *JHEP* **10** (2003) 007, [hep-th/0201028](http://arxiv.org/abs/hep-th/0201028).

[14] B. S. Acharya, “A moduli fixing mechanism in M theory,” [hep-th/0212294](http://arxiv.org/abs/hep-th/0212294).

[15] B. S. Acharya, “Compactification with flux and Yukawa hierarchies,” [hep-th/0303234](http://arxiv.org/abs/hep-th/0303234).

[16] A. Giryavets, S. Kachru, P. K. Tripathy, and S. P. Trivedi, “Flux compactifications on Calabi-Yau threefolds,” *JHEP* **04** (2004) 003, [hep-th/0312104](http://arxiv.org/abs/hep-th/0312104).
[17] F. Denef, M. R. Douglas, and B. Florea, “Building a better racetrack,” *JHEP* **06** (2004) 034, hep-th/0404257.

[18] F. Denef, M. R. Douglas, B. Florea, A. Grassi, and S. Kachru, “Fixing all moduli in a simple F-theory compactification,” hep-th/0503124.

[19] V. Balasubramanian and P. Berglund, “Stringy corrections to Kähler potentials, SUSY breaking, and the cosmological constant problem,” *JHEP* **11** (2004) 085, hep-th/0408054.

[20] V. Balasubramanian, P. Berglund, J. P. Conlon, and F. Quevedo, “Systematics of moduli stabilisation in Calabi-Yau flux compactifications,” *JHEP* **03** (2005) 007, hep-th/0502058.

[21] J. P. Conlon, F. Quevedo, and K. Suruliz, “Large-volume flux compactifications: Moduli spectrum and D3/D7 soft supersymmetry breaking,” hep-th/0505076.

[22] O. DeWolfe, A. Giryavets, S. Kachru, and W. Taylor, “Type IIA moduli stabilization,” hep-th/0505160.

[23] S. J. Gates, M. T. Grisaru, M. Rocek, and W. Siegel, *Superspace, or one thousand and one lessons in supersymmetry*, vol. 58 of *Frontiers of Physics*. 1983. hep-th/0108200.

[24] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*. Wiley, 1978.

[25] S. Gukov, C. Vafa, and E. Witten, “CFT’s from Calabi-Yau four-folds,” *Nucl. Phys.* **B584** (2000) 69–108, hep-th/9906070.

[26] D. Cox, J. Little, and D. O’Shea, *Using Algebraic Geometry*. Springer, 1998.

[27] D. Cox, J. Little, and D. O’Shea, *Ideals, Varieties, and Algorithms*. Springer, second ed., 1996.

[28] M. R. Douglas, “Statistical analysis of the supersymmetry breaking scale,” hep-th/0405279.

[29] N. Arkani-Hamed and S. Dimopoulos, “Supersymmetric unification without low energy supersymmetry and signatures for fine-tuning at the LHC,” hep-th/0405159.

[30] N. Arkani-Hamed, S. Dimopoulos, G. F. Giudice, and A. Romanino, “Aspects of split supersymmetry,” *Nucl. Phys.* **B709** (2005) 3–46, hep-ph/0409232.

[31] L. Susskind, “Supersymmetry breaking in the anthropic landscape,” hep-th/0405189.

[32] R. Kallosh and A. Linde, “Landscape, the scale of susy breaking, and inflation,” *JHEP* **12** (2004) 004, hep-th/0411011.

[33] M. Dine, E. Gorbatov, and S. Thomas, “Low energy supersymmetry from the landscape,” hep-th/0407043.
[34] M. Dine, “Supersymmetry, naturalness and the landscape,” hep-th/0410201
[35] M. Dine, “The intermediate scale branch of the landscape,” hep-th/0505202
[36] O. DeWolfe, A. Giryavets, S. Kachru, and W. Taylor, “Enumerating flux vacua with enhanced symmetries,” JHEP 02 (2005) 037, hep-th/0411061
[37] M. Dine and M. Graesser, “CPT and other symmetries in string / M theory,” JHEP 01 (2005) 038, hep-th/0409209
[38] M. Dine and Z. Sun, “R symmetries in the landscape,” hep-th/0506246
[39] M. R. Douglas, “The statistics of string / M theory vacua,” JHEP 05 (2003) 046, hep-th/0303194
[40] S. Ashok and M. R. Douglas, “Counting flux vacua,” JHEP 01 (2004) 060, hep-th/0307049
[41] M. R. Douglas, “Statistics of string vacua,” hep-ph/0401004
[42] M. R. Douglas, B. Shiffman, and S. Zelditch, “Critical points and supersymmetric vacua,” Commun. Math. Phys. 252 (2004) 325–358, math.cv/0402326
[43] F. Denef and M. R. Douglas, “Distributions of flux vacua,” JHEP 05 (2004) 072, hep-th/0404116
[44] M. R. Douglas, B. Shiffman, and S. Zelditch, “Critical points and supersymmetric vacua, II: Asymptotics and extremal metrics,” math.cv/0406089
[45] M. R. Douglas, “Basic results in vacuum statistics,” Comptes Rendus Physique 5 (2004) 965–977, hep-th/0409207
[46] F. Denef and M. R. Douglas, “Distributions of nonsupersymmetric flux vacua,” JHEP 03 (2005) 061, hep-th/0411183
[47] M. R. Douglas, B. Shiffman, and S. Zelditch, “Critical points and supersymmetric vacua. III: String/M models,” math-ph/0506015
[48] A. Giryavets, S. Kachru, and P. K. Tripathy, “On the taxonomy of flux vacua,” JHEP 08 (2004) 002, hep-th/0404243
[49] R. Blumenhagen, F. Gmeiner, G. Honecker, D. Lust, and T. Weigand, “The statistics of supersymmetric D-brane models,” Nucl. Phys. B713 (2005) 83–135, hep-th/0411173
[50] A. Misra and A. Nanda, “Flux vacua statistics for two-parameter Calabi-Yau’s,” Fortsch. Phys. 53 (2005) 246–259, hep-th/0407252
[51] J. P. Conlon and F. Quevedo, “On the explicit construction and statistics of Calabi-Yau flux vacua,” JHEP 10 (2004) 039, hep-th/0409215
[52] J. Kumar and J. D. Wells, “Landscape cartography: A coarse survey of gauge group rank and stabilization of the proton,” Phys. Rev. D71 (2005) 026009, hep-th/0409218.

[53] B. S. Acharya, F. Denef, and R. Valandro, “Statistics of M theory vacua,” hep-th/0502060.