A novel probe of bubble size statistics and time scales of the epoch of reionization using the contour Minkowski Tensor

Akanksha Kapaha\textsuperscript{1,2}† Pravabati Chingangbam\textsuperscript{1}‡ Stephen Appleby\textsuperscript{3}, and Changbom Park\textsuperscript{3}

\textsuperscript{1} Indian Institute of Astrophysics, Koramangala II Block, Bangalore 560 034, India
\textsuperscript{2} Department of Physics, Indian Institute of Science, C. V. Raman Ave., Bangalore 560 012, India
\textsuperscript{3} Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, South Korea

We employ the rank-2 contour Minkowski Tensor in two dimensions to probe length and time scales of ionized bubbles during the epoch of reionization. We demonstrate that the eigenvalues of this tensor provide excellent probes of the distribution of the sizes of ionized bubbles, and from it the characteristic bubble sizes, at different redshifts. We show that ionized bubbles are not circular, and hence not spherical in three dimensions, as is often assumed for simplified analytic arguments. We quantify their shape anisotropy by using the ratio of the two eigenvalues. The shape parameter provides the characteristic time epochs when bubble mergers begin and end. Our method will be very useful to reconstruct the reionization history using data of the brightness temperature field.

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I. INTRODUCTION

After the epoch of recombination most of the baryonic matter in the Universe must have been in the form of neutral hydrogen. The neutral hydrogen is expected to emit 21-cm emissions and the detection of this emission from the Epoch of Reionization (EoR) \cite{1} is one of the most exciting frontier prospects of observational cosmology. Since the emission would have been redshifted we will observe it in the MHz frequencies. The spatial distribution of the 21-cm emission will contain imprints of reionization physics (see e.g. \cite{2}) and nature and growth of the primordial density perturbations. The EoR commences once ionizing sources get formed and continues as more sources get created and the ionized bubbles grow in size. Two important issues concerning this epoch that we address in this paper are - finding the characteristic size of ionized bubbles \cite{3–7} at different redshifts and reconstructing the reionization history. Currently the best constraint on reionization history is obtained using CMB data from PLANCK \cite{8}.

The detection of 21-cm signal is experimentally extremely challenging in terms of beating instrumental noise and because the signal is dominated by foreground emissions by several orders of magnitude. It is hoped that a thorough understanding of the HI signal from the EoR will help overcome these problems and guide us towards statistical detection of the signal (see e.g. \cite{9}). In comparison to ongoing experiments the upcoming radio interferometer Square Kilometer Array (SKA) will have improved collecting area, angular resolution, and frequency coverage. The SKA might enable direct detection of the 21-cm signal in the redshift range $6 < z < 30$, corresponding to the frequency range $200\text{ MHz} > \nu > 50\text{ MHz}$.

The 21-cm signal, when detected, will help tighten cosmological constraints, complementary to the CMB and large scale structure. To unlock the physical information it is very important to devise efficient statistical tools. The two-point function has been popularly used for this purpose. However, the fields of the EoR are expected to show strong departure from Gaussian nature and hence the two-point function is of limited use. The three-point function has been employed \cite{10,11} to capture some non-Gaussian nature of the fields. Given the expected strong non-Gaussian nature of the fields it is desirable to use statistical methods which can capture all orders of $n$-point functions of the field.

In this paper we use the rank-2 contour Minkowski Tensor (CMT), which is one of the Minkowski Tensors \cite{12,20} that were recently introduced for cosmological random fields in two dimensions \cite{21,23}. Minkowski Tensors are tensor generalizations of the scalar Minkowski Functionals (MFs) \cite{24,30}. In particular, the CMT is the tensor generalization of the second scalar MF, the total contour length. Scalar MFs have been used in many areas of cosmology (see e.g. \cite{31,37}). In the context of 21-cm emissions, the genus, which is one of the scalar MFs, has been used to study different reionization models and history \cite{38,39}. MFs in three dimensions have also been used for similar purpose \cite{41,42}.

The CMT carries information of the anisotropy and relative alignment of structures in two dimensions. We apply it to simulations of fields of the EoR made using 21cmFAST \cite{43,44}. We propose a method using the eigenvalues of the CMT for calculating the probability distribution function of the sizes of ionized bubbles, and from it the characteristic bubble sizes, at different redshifts. We quantify the shape anisotropy of the ionized bubbles by using the ratio of the two eigenvalues. This shape parameter gives the characteristic time epochs when bubble mergers begin and end. We comment on the usefulness of our method for reconstructing the reionization history in the future using observed data.

\* Electronic address: akanksha.kapahtia@iiap.res.in
\‡ Electronic address: prava@iiap.res.in

\*\* Electronic address: prava@iiap.res.in
II. 21-CM BRIGHTNESS TEMPERATURE FIELD

The energy difference between the hyperfine levels of the neutral Hydrogen in ground state corresponds to an excitation temperature of $T_e = 0.068$ K. The emission or absorption for this transition is determined by the spin temperature $T_s$ which is the temperature at which the relative population of the two levels become $n_1/n_0 = 3 \exp(-T_s/T_e)$, if the system is in equilibrium.

Since the redshifted frequency of this transition lies in the radio regime, the intensity of this spectral line is quantified by the brightness temperature $T_b$. The brightness temperature is measured in emission or absorption against a background of CMB. From the equation of radiative transfer along the line of sight, the offset of the 21cm brightness temperature from the CMB temperature is given by:

$$\delta T_b = \frac{T_e - T_s}{1 + z}(1 - e^{-\tau}) \quad (1)$$

Here $\tau$ is the optical depth for the 21cm emission integrated along the line of sight up till the redshift of observation. The expression for the differential brightness temperature $\delta T_b$ for an observed frequency $\nu$ is given by the following expression:

$$\delta T_b(\nu) \approx 27 x_{HI} (1 + \delta_{NL})(1 - \frac{T_s}{T_S}) \frac{\Omega_b h^2}{0.023} \times \sqrt{\frac{1 + z}{10}} \frac{0.15}{\Omega_M h^2} \text{ (in mK)}, \quad (2)$$

where we have dropped a factor containing the peculiar velocity since it is negligible at the redshifts of interest.

In order to study ionization history, we have generated mock 21-cm field in a 200 Mpc box, using the publicly available semi numerical code 21cmFAST [44]. The code generates Gaussian random initial density field in the same way as in N-body simulations and then evolves it using the Zel’dovich approximation. It generates the density $\delta_{NL}(\vec{x})$, spin temperature $T_s(\vec{x})$, peculiar velocity and ionization field $x_{HI}(\vec{x})$ at every grid point $\vec{x}$ and finally calculates the differential brightness temperature $\delta T_b(\vec{x})$ at that point. For our purpose we chose 1024 pixel grid for the initial conditions and 512 pixel grid for the evolved fields. The initial conditions were generated at a redshift of $z = 300$. In order to identify ionized regions the code uses an excursion set approach similar to the Press-Schechter theory of halo mass functions. According to this approach [3] an isolated region of mass $m$ is self ionized if it has sufficient mass in luminous sources. For such a region of size $R(m)$ corresponding to the mass $m$ the number of ionizing photons should be greater than the number of neutral hydrogen atoms. The above criteria boils down to the following condition:

$$f_{coll}(\vec{x}, z, R) \geq \zeta^{-1},$$

where $f_{coll}$ is the fraction of mass residing in collapsed halos inside a sphere of mass $m = 4/3 \pi R^3 \rho [1 + (\delta_{NL})]$. $\zeta$ is a factor that describes the amount of mass a galaxy of mass $m_{gal}$ can ionize: $m_{ion} = \zeta m_{gal}$. $\zeta$ encapsulates information about the efficiency of ionization of the luminous sources during reionization scenario. Therefore different values of $\zeta$ describe different ionization histories. In 21cmFAST a central cell is flagged as ionized if the aforementioned condition is fulfilled at some filter scale while reducing from a maximum value $R_{max}$ to cell size in logarithmic steps. We use $R_{max} = 10$ Mpc. IGM heating is considered throughout the evolution (i.e. $T_e$ is not ignored at lower $z$ values).

We perform our analysis using two different models for the redshift evolution of $x_{HI}$. The two models correspond to two different values of the efficiency factor, $\zeta = 50$ and 10. These values were chosen so as to study a realistic ($\zeta = 50$) and an unrealistically late ($\zeta = 10$) reionization scenario. Reionization ends at $z = 6$ and $z = 2$, respectively, for the two models. In Fig. 1 we

![Fig. 1](image_url)
show the progress of reionization for a two dimensional slice of \( x_{\text{HI}}(x) \) at four different redshifts for the model \( \zeta = 50 \). The bottom plot shows the redshift evolution of the spatial average, \( \bar{x}_{\text{HI}} \), for the two models \( \zeta = 10 \) and 50.

### III. CONTOUR MINKOWSKI TENSOR FOR REIONIZATION FIELDS

Minkowski tensors (MTs) are tensor generalizations of the scalar MFs. For our analysis we restrict attention to the following MT, which we refer to as the ‘contour’ MT, defined for the boundary curve, \( C \), of a structure, as,

\[
W_1 = \frac{1}{4} \int_C \hat{T} \otimes \hat{T} \, ds
\]

where \( \hat{T} \) is the unit tangent vector at every point on the curve, \( \otimes \) denotes the symmetric tensor product, and \( ds \) is the arc length. Our notation follows \[22\]. \( W_1 \) is referred to as \( W_2^{1,1} \) in \[19\] \[21\].

\( W_1 \) is translation invariant, transforms as rank-2 tensor under rotations, and has dimension of length. Since it is symmetric and defined for a closed curve, its two eigenvalues denoted by \( \lambda_1, \lambda_2 \) are real and positive. For convenience of notation let \( \lambda_1 < \lambda_2 \). When the eigenvalues are different they pick out two orthogonal directions and we can effectively approximate the curve as an ellipse whose semi minor axis is aligned with the eigenvector of \( \lambda_1 \) while the semi major is aligned with that of \( \lambda_2 \). Thus, their product is proportional to the effective size of the structure enclosed by the curve. Then the ratio \( \beta = \lambda^1/\lambda^2 \) lies between 0 and 1 and gives a measure of the intrinsic anisotropy of the curve. \( \beta = 1 \) corresponds to isotropic shape or the curve having \( m \)-fold, \( m \geq 3 \), rotational symmetry. Detailed explanations of the anisotropy measure is given in \[19\] \[22\]. The focus of this paper is to exploit \( \lambda_1, \lambda_2 \) and \( \beta \) in order to probe the EoR.

It is worth noting that we could calculate the area of each structure directly and also roughly estimate its shape anisotropy by finding its centre of mass and taking the ratio of the smallest and largest distances of the boundary from the centre. However, the CMT is superior to such simplistic approaches because they are based on very general mathematical framework with well defined transformation properties. This makes them unambiguous and easy to apply to any distribution of structures.

At each given threshold value of a random field the boundaries of the excursion or level set enclose either connected regions or holes. The morphological properties of the excursion sets, and as a consequence that of these boundaries, change systematically as the threshold value is varied. We compute \( W_1 \) for each of the boundary curves at a finite set of threshold values. Our numerical calculation follows the method described in \[22\].

Identifying each closed curve bounding either a connected region or a hole, and then calculating \( W_1 \) for each curve.

For each reionization model we generate the fields \( \delta_{\text{NL}}, T_s, x_{\text{HI}} \) and \( \delta T_b \) at several redshift values. We smooth the fields with Gaussian smoothing kernel using different values of the smoothing scale, \( R_s \). Each of these fields, say \( f \), is redefined as \( f \rightarrow f = (f - \mu)/\sigma \), where \( \mu \) is the mean and standard deviation of \( f \), respectively. The threshold values are then given in units of the standard deviation. We choose 43 threshold values between -7 to 7. This relatively large range was chosen so as to sufficiently sample \( x_{\text{HI}} \) and \( \delta T_b \) which are highly skewed at high and low redshifts. Then, we use two dimensional slices of the fields to compute \( W_1 \), and from it \( \lambda_1, \lambda_2 \) and \( \beta \), for each curve for excursion sets corresponding to different threshold values. \( \lambda_1, \lambda_2 \) are obtained in units of Mpc. For showing our results in the subsequent plots we use 32 slices of thickness 6.4 Mpc each. Our results are robust against reasonable variation (not smaller than smoothing scale) of the slice thickness.

As reionization progresses, at any given redshift we can compare two length scales - the effective distance between ionized bubbles, \( \ell_{\text{eff}} \), and the effective bubble size, \( \ell_{\text{bub}} \) (assuming spherical shape). \( \ell_{\text{eff}} \) will decrease with decreasing redshift at a rate given the rate at which the ionizing sources get formed. Hence it depends on the number density of collapsed fraction of peaks of the matter density field. In contrast, \( \ell_{\text{bub}} \) will grow with decreasing redshift. The rate of growth will be determined by the details of the ionization process. The EoR can then be divided into three time regimes characterized by: (a) \( \ell_{\text{bub}} < \ell_{\text{eff}} \), (b) \( \ell_{\text{bub}} \approx \ell_{\text{eff}} \), and, (c) \( \ell_{\text{bub}} > \ell_{\text{eff}} \). In regime (c) the previously connected neutral region would have fractured into multiple isolated neutral regions (see panels for \( z = 9 \) and 7 of Fig. (1)). Hence \( \ell_{\text{bub}} \) in this case no longer has the meaning of the size of ionized region enclosed by neutral region.

We can expect that the three regimes of the EoR described above will be reflected in the evolution of \( \lambda_1, \lambda_2 \), and \( \beta \) obtained from \( W_1 \). To probe this let us consider the ensemble of all curves obtained from all excursion sets of the entire range of field values for a given field. Then we calculate the probability distribution functions (PDFs) of the corresponding eigenvalues, and their ratio, of \( W_1 \) obtained from these curves. Note that curves associated with different thresholds for a single peak or a trough are correlated. The number of boundaries of connected regions and holes (the so-called Betti numbers) vary with the threshold \[35\] \[36\]. Each Betti number peaks at some threshold and tends to zero for large magnitudes of the threshold. For a Gaussian field, the peak is located at positive threshold value for connected regions and negative threshold value for holes \[35\] \[36\]. For the EoR fields the shape of the Betti numbers will differ from the Gaussian form. The maxima of the PDFs of \( \lambda_1, \lambda_2 \) and \( \beta \) will have the most contribution from curves of the particular excursion set corresponding to
FIG. 2: Redshift evolution of the PDF of ionized bubble sizes encapsulated by $\lambda_1$ (blue filled circles) and $\lambda_2$ (orange triangles). The shift in the peaks of both plots indicates the growth of ionized bubbles as the redshift decreases.

FIG. 3: Redshift evolution of the PDF of the shape parameter $\beta = \lambda_1 / \lambda_2$. The shift of the peak to lower $\beta$ values implies that as the redshift decreases to roughly $z \simeq 9$ the shape of ionized bubbles become more anisotropic. This is due to merger of bubbles becoming numerous. Then, as $z$ decreases further below 9, most mergers are over and the overall bubble sizes grow towards more isotropic shapes again.

the maxima of the Betti numbers. Instead of using all curves for all threshold values we could alternatively use only the curves corresponding to the maxima of the Betti numbers and define the PDF. The purpose of using the excursion sets of the entire field range is to increase the number count of curves and thereby improve the statistics.

In what follows we apply our method to two-dimensional slices of $x_{HI}$ and $\delta T_b$ at each redshift. Before we delve into the results it is important to clarify how we infer the information of the sizes of ionized bubbles from $\lambda_1$ and $\lambda_2$. As can be seen in the first panel of Fig (1), at relatively early redshifts, say $z = 15$, at most threshold values of $x_{HI}$ there will be just one connected region while there will be many holes. The total number of all contours will then be roughly equal to the number of
holes. The contours enclosing holes demarcate ionized bubbles corresponding to the threshold values. Further, as explained in the previous paragraph, the number of contours will be maximum for some particular threshold. Thus, the mean values of $\lambda_1$ and $\lambda_2$ over the entire field range at relatively high redshift gives us information of the mean size of ionized bubbles.

At intermediate redshifts till roughly $z \sim 9$, ionized bubbles become more numerous and also grow in size. As a consequence the single connected region progressively fragments into several parts. The number of contours enclosing connected regions grows and becomes comparable to the number of holes. Hence at around $z \sim 9$ and after, the mean values of $\lambda_1$ and $\lambda_2$ encode information of the sizes of the fragmented neutral regions as well as of ionized bubbles. Further, towards the end of the EoR there are only a few small neutral regions left and these can have ionizing bubbles within them, as can be seen in the panel showing $z = 7$ in Fig (1). The mean values of $\lambda_1$ and $\lambda_2$ should then show a decrease at these redshift values. Since $\delta T_b$ tracks $x_{HI}$ during the redshift values relevant to the EoR, we expect that the PDFs of the two fields will have similar behaviour.

Fig. (2) shows the PDFs of $\lambda_1$ (blue circles) and $\lambda_2$ (orange triangles) at different redshifts. The first and second columns show the PDFs for $x_{HI}$ for two smoothing scales $R_*=2.0$ and 4.5 Mpc. We first observe that $\lambda_1$ and $\lambda_2$ have different PDFs. This is a reflection of the fact that the iso-field contours, and consequently the ionized regions, are not isotropic. Therefore, the typical size of ionized regions must be characterized by at least two length scales. Further, the plots show that at early redshifts the maxima of the PDFs are located at roughly the smoothing scale. The reason for this is that at these redshifts the ionizing sources are small and their effective size is captured by the smoothing scale. As the redshift decreases we find that the PDFs flatten out with the tails getting longer. This indicates that the ionized regions are growing in size. The PDF of the two eigenvalues are not the same at all redshifts. This implies that the bubbles grow with anisotropic shape and remain so at all redshifts. The last two columns of Fig. (2) show the corresponding PDFs for $\delta T_b$. We can see that at earlier redshifts, $z > 14$, the PDF differs from that of $x_{HI}$. This is because at such early redshifts the main contribution to $\delta T_b$ comes from $\delta_{HL}$ and $T_s$. However, at lower redshifts the behaviour of $\delta T_b$ tracks $x_{HI}$.

Let $\lambda_{ch}^i = \int d\lambda_i \lambda_i P(\lambda_i)$, where $i = 1,2$ and $P(\lambda_i)$ is the respective PDF at each $z$ shown in Fig. (2). Then, $\lambda_{ch}^i$ gives the two characteristic length scales. In Fig. (3) we show the growth of $\lambda_{ch}^{i,2}$ as the redshift decreases for $x_{HI}$ and $\delta T_b$. As explained earlier, $\lambda_{ch}^{1,2}$ encodes the ionized bubble size till roughly $z \sim 9$. For $z < 9$ it encodes the typical size of the neutral regions. If we approximate the typical bubble shape to be roughly circular (spherical in three dimensions) the average radius is given by $\lambda_{ch}^i = (\lambda_{ch}^{i,h} + \lambda_{ch}^{i,B})/2$. Then $\lambda_{ch}^i$ gives a measure of $t_{bub}$. We find that the ionized bubbles grow to sizes of order 10 Mpc. We also see that the smoothing scale affects the interpretation of the bubble size. We note that our result is obtained using $d\lambda_i = 2$ MPc and the integral is performed by a simple Riemann sum.

Next we focus on the shape parameter $\beta$. In Fig. (3) we plot its PDF for $x_{HI}$ and $\delta T_b$, for the same redshifts and smoothing scales as in Fig. (2). All the PDFs drop sharply away from $\beta = 1$, implying that exactly spherical bubbles are improbable, as seen also for $\lambda_{1,2}$. In the top panels of the first and second columns showing $z = 18.23$, 14.32 and 9.04, we can see that the peaks of the PDF shift towards lower values of $\beta$. This means that the mean bubble shape becomes more anisotropic from high redshifts towards lower redshifts. This indi-
cates that the EoR is in the regime $\ell_{\text{bub}} \approx \ell_{\text{eff}}$ when ionized bubbles have started merging and hence their shape anisotropy is increasing. In the bottom panels for $x_{\text{HI}}$ showing $z = 9.04$, 7.40 and 6.46, we can see that the PDFs shift back to higher values of $\beta$. This means that the EoR has reached the regime of $\ell_{\text{bub}} > \ell_{\text{eff}}$ during which the bubbles are becoming less anisotropic. The turn over takes place around $z \sim 9$, which corresponds to the mean ionization fraction value $\bar{x}_{\text{HI}} \sim 0.5$, as seen from the plot of $\bar{x}_{\text{HI}}$ in Fig. (1). The third and fourth columns of Fig. (3) show the PDFs of $\delta T_b$. We can see that for higher redshifts $z > 14$ the behaviour is different, while for $z < 14$ the behaviour follows that of $x_{\text{HI}}$.

Let $\langle \beta \rangle \equiv \int d\beta P(\beta)$, where $P(\beta)$ here is the PDF of $\beta$ at each $z$, shown in Fig. (3). In Fig. (5) we show $\langle \beta \rangle$ versus redshift for $x_{\text{HI}}$ and $\delta T_b$. We can see that the regime $\ell_{\text{bub}} < \ell_{\text{eff}}$ corresponds to $z > 14$, $\ell_{\text{bub}} \approx \ell_{\text{eff}}$ corresponds to $14 < z < 9$, and $\ell_{\text{bub}} > \ell_{\text{eff}}$ corresponds to $z < 9$.

We have repeated all calculations for the second model given by $\zeta = 10$. The turnover scale in this case happens at around $z \sim 6$, which corresponds to the mean ionization fraction value $\bar{x}_{\text{HI}} \sim 0.5$, as seen from the plot of $\bar{x}_{\text{HI}}$ in Fig. (1). Apart from this overall shift all other results are similar.

IV. CONCLUSION

We have demonstrated in this paper a proof-of-concept of a new method based on the contour Minkowski Tensor to probe the size statistics of ionized bubbles and important time scales during the EoR. We have shown that the progress of the EoR can be very effectively studied using the eigenvalues, and their ratio, of the CMT. Our first result is that ionized bubbles are not isotropic in shape, as can be expected from visual inspection of some of the ionization fraction field, and our method gives a precise quantification of the anisotropy. The redshift evolution of $\lambda_i$ and $\beta$ that we have shown here will be generic to any reionization model. However, the details of the redshift evolution - the precise shape of $\lambda_{ch}^i$ and $\langle \beta \rangle$ versus $z$ and the location of the transition time scales will depend on the details of the physics that operate during the EoR. Thus, they provide a simple and direct method to constrain reionization history with future data of redshifted 21-cm observations.

For the sake of brevity we have used the set of all curves to interpret the reionization history. It is possible to decipher more details about the reionization models by analyzing the set of contours enclosing connected regions and those enclosing holes separately. We will work with these in upcoming work to discriminate different reionization models. The extension of our work to three dimensional MTs is ongoing. A comparison of the bubble size statistics derived using our method with other methods in the literature will also be carried out in the near future.

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