Determination of the spin parameter and the inclination angle from the primary and secondary images caused by gravitational lensing

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We study the primary images (PIs) and secondary images (SIs) caused by strong gravitational lensing around a Kerr black hole shadow, which carry some essential signatures related to the black hole space-time. We define a new celestial coordinate, whose origin is the center of the black hole shadow, to locate the PIs and SIs of luminous celestial objects. Based on the dragging effect caused by the rotating black hole and the inclination angle of the observer, the relative positions between the PIs and SIs are different for different values of the Kerr spin parameter $a$ and the observer’s inclination angle $i$; hence, it can be used to determine the values of $a$ and $i$. We propose a specific approach to measure $a$ and $i$ using the PIs and SIs. The time delays between the PIs and SIs are different for different values of $a$ and $i$. The time delays, in conjunction with the relative positions between the PIs and SIs, can enable us to measure $a$ and $i$ more precisely. These PIs and SIs around the black hole shadow act as unique fingerprints for the black hole space-time, using which we can further determine other parameters of different types of compact objects and verify various theories of gravity. Our results provide a new method to implement parameter estimation in the study of black hole physics and astrophysics.

black hole shadow, gravitational lensing, the primary images, the secondary images

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1 Introduction

The Event Horizon Telescope (EHT) Collaboration has recently revealed the first images of supermassive black holes at the center of the giant elliptical galaxy M87 [1-6] and the Milky Way galaxy [7-12]. The EHT collaboration marks the new beginning of a new era in the fields of astrophysics and black hole physics and attracts an increasing number of researchers to study black hole images. The black hole shadow, which is a dark silhouette [13-16], is an extremely important element in a black hole image. This dark shadow appears in the observer’s sky because the light rays close to a black hole are captured by the black hole. Because a black hole shadow can carry considerable information regarding the space-time geometry around a compact object, the shadow plays a vital role in the study of black holes (constraining black hole...
parameters) [17-23] and enables investigations into the fundamental issues related to physics, including dark matter [24-28] and the verification of various theories of gravity [29-32]. Black hole shadows have also been researched in refs. [33-105]. The information imprinted in black hole shadows may be captured during future astronomical observations using the upgraded EHT.

A black hole at the center of a galaxy possesses spin. A neutral rotating black hole is well described by the Kerr metric having two parameters: mass $M$ and spin $a$. The Kerr black hole shadow gradually becomes a D-shaped silhouette with increasing spin parameter [14]. Furthermore, it also depends on the inclination angle between the observer’s line of sight and the spin axis of the black hole. Hioki and Maeda [17] calculated the observable radius $R$, and the deviation parameter $δ_r$ of a Kerr black hole shadow to determine the spin parameter $a$ and the observer’s inclination angle $i$. These observables can also be used to determine other parameters of Kerr-like black holes [18-22] and to test alternative theories of gravity [31].

Some images of luminous celestial objects caused by gravitational lensing are visible around the black hole shadow when observing a black hole [106]. Two infinite sets of images of one light source are present on either side of the black hole shadow. However, only two of these images can be easily detected: the primary image (PI) and the secondary image (SI). Other images that are concentrated on the shadow boundary are called relativistic images [107]. The black hole shadow and the PIs & SIs carry some important signatures related to the black hole space-time. The relative positions between the PIs and SIs around a Kerr black hole shadow are influenced by the dragging effect and inclination angle. These PIs and SIs around the black hole shadow act as unique fingerprints of the black hole space-time, using which we can determine parameters of all types of compact objects, such as Kerr-like black holes, and verify various theories of gravity. Our results could provide a new method to implement parameter estimation in the study of black hole physics and astrophysics.

The paper is organized as follows. In sect. 2, we briefly introduce the celestial coordinates used to locate the coordinates of the image points in the observer’s sky and the observables of a Kerr black hole shadow to determine the values of $a$ and $i$. In sect. 3, we calculate four couples of PI and SI around the black hole shadow for different $a$ and $i$ values and determine the values of $a$ and $i$ based on the relative positions between the PIs and SIs. In sect. 4, we calculate the time delays between the PIs and SIs, which can help to determine the values of $a$ and $i$ with high precision. Finally, we present the conclusion. In this paper, we employ the geometric units $G = c = M = 1$.

## 2 Determination of the Kerr spin parameter and the inclination angle with respect to the Kerr black hole shadow

The Kerr metric describes a neutral rotating black hole specified by two parameters: mass $M$ and spin $a$. According to the Boyer-Lindquist coordinates, the Kerr metric is described as follows:

\[
d s^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2d\theta^2 + \sin^2\theta \left(\frac{\rho^2}{\mu} + 2Mr^2\sin^2\theta\right)d\phi^2
\]

\[
- \frac{4Mr\sin^2\theta}{\mu^2}d\rho dt d\phi,
\]

where

\[
\Delta = a^2 + r^2 - 2Mr, \quad \rho^2 = r^2 + a^2 \cos^2\theta.
\]

The event horizon of a Kerr black hole is at $r_h = M \pm \sqrt{M^2 - a^2}$ and only exists for $|a| \leq M$. Light rays entering the event horizon are absorbed by the black hole, casting the black hole shadow. However, the boundaries of the black hole shadow are determined by the photon sphere. The photon sphere is composed of unstable photon spherical orbits that satisfy

\[
\begin{align*}
& r = 0, \quad \text{and} \quad \tilde{r} = 0. \\
& \text{Light rays that enter the photon sphere are captured by the black hole, light rays that pass by the photon sphere can reach infinity, and light rays that spiral asymptotically toward the photon sphere constitute the black hole shadow boundary.}
\end{align*}
\]

In the observer’s sky, celestial coordinates can be used to define the coordinates of the image points in black hole images. In refs. [32, 46-48, 53, 54, 57-59], we calculate the celestial coordinates in axially symmetric space-times as:

\[
x = -\frac{p^\phi}{p^r|_{(r, \theta)}} , \quad y = \frac{p^\phi}{p^\theta|_{(r, \theta)}} ,
\]

where $p^\phi$ denotes the four-momentum of photons measured locally by the observer at $(r_\theta, \theta)$. The locally measured four-momentum $p^\phi$ can be expanded using the four-momentum $p^\mu$ of a photon as follows [14, 32, 46-52]:

\[
\begin{align*}
p^\phi &= \sqrt{\frac{g_{\phi \phi}}{g_{rr} - g_{\phi \phi}} E - \frac{g_{\phi \theta}}{g_{\phi \phi}} \sqrt{\frac{g_{\phi \phi}}{g_{rr} - g_{\phi \phi}} L_z}}, \\
p^\phi &= \frac{1}{\sqrt{g_{rr}}} p^r , \quad p^\phi = \frac{1}{\sqrt{g_{\phi \phi}}} p^\theta , \quad p^\phi = \frac{1}{\sqrt{g_{rr}}} L_z ,
\end{align*}
\]

where $E$ and $L_z$ are the two conserved quantities in photon motion: energy and the $z$-component of angular momentum, which are expressed as:

\[
E = -p_t = -g_{tt} \hat{t} - g_{t \phi} \hat{\phi} , \quad L_z = p_\phi = g_{\phi \phi} \hat{\phi} + g_{\theta \phi} \hat{\theta} .
\]
The image points in the boundary of the black hole shadow correspond to the light rays that spiral asymptotically toward the photon sphere. Because the observer is far away from the black hole, \( r_o \) takes the limit \( r_o \rightarrow \infty \). The analytic expressions for the boundary of the Kerr black hole shadow in the celestial coordinates are [14, 32, 46-55]

\[
x = -\frac{\eta}{\sin \theta_o},
\]

\[
y = \pm \sqrt{\sigma + a^2 \cos^2 \theta_o - \eta^2 \cos^2 \theta_o},
\]

where the impact parameter \( \eta \) and \( \sigma \) are the conserved quantities in photon motion,

\[
\eta = \frac{L}{E} = \frac{r^2(r - 3M) + a^2(r + M)}{a(r - M)},
\]

\[
\sigma = \frac{r^3(4a^2M - r - 3M^2)}{a^2(r - M)^2}.
\]

A Kerr black hole shadow depends on the spin parameter \( a \) and the observer’s inclination angle \( i \) (i.e., \( \theta_o \)) in ref. [14]. Thus, \( a \) and \( i \) can be determined by calculating the observable radius of the shadow \( R \) and the deviation parameter \( \delta \) that describes the deviation of the shape of the shadow from a circle [17]. We calculated several additional observables for the Kerr black hole shadow that might be used to quantify \( a \) and \( i \) more easily. In a Kerr black hole shadow with \( a = 0.998M, i = 90^\circ \), Figure 1, four characteristic points: the leftmost point \((x_l, y_l)\), the rightmost point \((x_r, y_r)\), the topmost point \((x_t, y_t)\), and the bottommost point \((x_b, y_b)\), of the shadow can be easily determined. Tables 1-3 exhibit the coordinates of \((x_l, y_l)\), \((x_r, y_r)\), and \((x_t, y_t)\), respectively, for a Kerr black hole shadow under different values of \( a \) and \( i \) using the celestial coordinates reported in a previous study (7). For the bottommost point, \((x_b, y_b) = (x_t, -y_t)\). In addition, we define the shadow center as \((x_c, y_c) = \left( \frac{x_t + x_b}{2}, \frac{y_t + y_b}{2} \right) \) and the coordinates \((x_c, y_c)\) under the different \( a \) and \( i \) are listed in Table 4. \( x_c \) and \( x_{(b)} \) increase with \( a \) and \( i \) under the influence of the dragging effect. A distortion observed in the Kerr black hole shadow is expressed as \( x_c \neq x_{(b)} \). Using the above mentioned point coordinates, we define four black hole shadow observables: the width \( W = x_r - x_l \), the height \( H = y_t - y_b \), the oblateness \( K = W/H \), and the distortion \( \delta = x_c - x_{(b)} \). Figure 2 shows the contour maps of the width \( W \), height \( H \), oblateness \( K \), and distortion \( \delta \) of the Kerr black hole shadow in terms of the spin parameter \( a \) and inclination angle \( i \). When \( a \) increases, the width \( W \) decreases owing to the dragging effect, as shown in Figure 2(a). The width \( W \) is almost independent of the inclination angle \( i \) when \( a < 0.9M \) and decreases as \( i \) increases when \( a > 0.9M \). This phenomenon occurs because a D-shaped silhouette shadow is cast when \( a > 0.9M \) and \( i \) gradually converge to \( 90^\circ \). Figure 2(b) shows that the height \( H \) of shadow decreases as \( a \) increases and increases as \( i \) increases. In particular, the width \( H \) is independent of \( a \) when \( i = 90^\circ \) but decreases as \( a \) increases when \( i = 0^\circ \). Figure 2(c) shows that regardless of the values of \( a \) and \( i \) (except \( a = i = 0 \)), the oblateness \( K \) is always less than 1 and decreases as \( a \) and \( i \) increase. Figure 2(d) shows that the distortion \( \delta \) of the shadow increases as \( a \) and \( i \) increase, i.e., \( a \) and \( i \) can manifest the distortion of the Kerr black hole shadow. For a given \( a \) and \( i \), only a set of values of \( W, H, K, \) and \( \delta \) can be found in Figure 2. After locating the four characteristic points of the black hole shadow, the width \( W \), height \( H \), oblateness \( K \), and distortion \( \delta \) can be measured, and then \( a \) and \( i \) can be determined using the one-to-one correspondence between \((W, H, K, \) and \( \delta \) and \((a \) and \( i \), as shown in Figure 2.

For the case of the spin parameter \( a < 0.4M \), changes in Kerr black hole shadows with \( a \) and \( i \) are minor; hence, neither the observables \( W, H, K, \) and \( \delta \) of the shadow we proposed nor the observables \( R, \) and \( \delta \) proposed in a previous study [17] can determine \( a \) and \( i \) with high accuracy. Furthermore, measuring the distortion of a black hole shadow requires an astronomical telescope of an accuracy level exceeding that of the current EHT. Therefore, better measurements or more visible observables than currently available are needed to determine the \( a \) and \( i \).

3 Determination of the Kerr spin parameter and the inclination angle from the PIs and SIs

When observing a black hole, images of luminous celestial objects generated owing to gravitational lensing can be observed in addition to the black hole shadow [106]. For a
luminous celestial object, two infinite sets of images exist on either side of the black hole shadow. However, only two of these images can be easily detected: the PI and SI. Because the photons forming PI and SI travel less than a loop around the black hole (deflection angle \( \alpha < 2\pi \)), they are the outermost images. Other images are concentrated near the shadow boundary because the photons forming them travel complete loops (deflection angle \( \alpha \geq 2\pi \)) that are near the photon sphere; these images are called the relativistic images \[107\]. The light traveling \( n \) loops around the black hole correspond to the \( n \)-th relativistic images. Figure 3(b) exhibits a Kerr black hole image with a background light source. The background light source is shown in Figure 3(a), which is a photo of the center of the Milky Way galaxy taken by the Cerro Tololo Inter-American Observatory \[108\]. A vast number of luminous celestial objects can be seen in the photo. Herein, we set the background light source as a sphere with a radius of 200M, with the Kerr black hole located at its center. Figure 3(a) shows the image of the spherical light source after spreading. The Kerr black hole image is obtained using the backward ray-tracing method \[32, 46-55\]. In the backward ray-tracing method, the light rays are assumed to evolve
backward in time from the observer, and an image is obtained by numerically solving the null geodesic equations. In Kerr black hole space-time, the null geodesic equations are expressed as:

\[
\begin{align*}
\dot{t} &= \frac{E g_{\phi \phi} + L_z g_{t \phi}}{g_{t \phi}^2 - g_{t t} g_{\phi \phi}}, \\
\ddot{r} &= \frac{1}{2 g_{rr}} \left( \frac{\partial}{\partial r} g_{tt} \dot{t}^2 + 2 \frac{\partial}{\partial r} g_{rt} \dot{t} \phi + \frac{\partial}{\partial r} g_{\phi \phi} \dot{\phi}^2 - \frac{\partial}{\partial r} g_{rr} \dot{r}^2 \right) \\
\ddot{\theta} &= \frac{1}{2 g_{\theta \theta}} \left( \frac{\partial}{\partial \theta} g_{tt} \dot{t}^2 + 2 \frac{\partial}{\partial \theta} g_{rt} \dot{t} \phi + \frac{\partial}{\partial \theta} g_{\phi \phi} \dot{\phi}^2 + \frac{\partial}{\partial \theta} g_{\theta \theta} \dot{\theta}^2 \right) \\
\ddot{\phi} &= \frac{E g_{\phi t} + L_z g_{t \phi}}{g_{\phi \phi} g_{t t} - g_{\phi t}^2},
\end{align*}
\]

where the dot represents the derivative with respect to the affine parameter \( \lambda \). Because the light rays start from the observer, we set the initial condition as \( t(0) = 0, r(0) = r_0 = 200 M, \theta(0) = \theta_0 = \pi/2, \phi(0) = 0 \). Next, we evolve the light rays in all directions by solving the null geodesic equations (9), where the direction can be expressed using the four-momentum \( \{ E, p_r(0), p_\theta(0), L_z \} \) corresponding to the initial value \( \{ t(0), r(0), \theta(0), \phi(0) \} \). The light rays falling into the event horizon \( r = r_h \) correspond to the black hole shadow. With the directions of these lights at the observer, their celestial coordinates \( (x, y) \) can be calculated using eqs. (4) and (5) and should be marked as black (the black region in Figure 3(b)). The light ray returns to \( r = 200 M \), i.e., it arrives at the spherical light source. Using \( \theta \) and \( \phi \) at this point, the position of this light source can be determined in Figure 3(a). The observer will see the image of this light source at the celestial coordinate \( (x, y) \), as shown in the color region in Figure 3(b). In the background light source, a luminous celestial object is visible inside the green circle shown in Figure 3(a). The observer can see the PI and SI of the object around the black hole shadow generated owing to gravitational lensing, and the images are marked inside the green circles in Fig-

Figure 2  (Color online) (a) The contour map of the width \( W \) of Kerr black hole shadows in terms of the spin parameter \( a \) and inclination angle \( i \); (b) the contour map of the height \( H \) of Kerr black hole shadows in terms of \( a \) and \( i \); (c) the contour map of the oblateness \( K \) of Kerr black hole shadows in terms of \( a \) and \( i \); (d) the contour map of the distortion parameter \( \delta \) of Kerr black hole shadows in terms of \( a \) and \( i \).
Figure 3 (Color online) (a) The background light source with a vast number of luminous celestial objects can be seen in the photo of the center of the Milky Way galaxy taken by the Cerro Tololo Inter-American Observatory [108]; (b) the numerically simulated image of a Kerr black hole with the spin parameter $a = 0.998M$ and the inclination angle $i = 90^\circ$.

Figure 4 (Color online) (a) The diagram of gravitational lensing; (b) the PIs, SIs, and relativistic images of luminous celestial objects around the black hole shadow.
Figure 5 (Color online) The four pairs of PI and SI (arrows) around the Kerr black hole shadows (black) with the spin parameter \( a = 0.998M \). (a) The case with the inclination angle \( i = 0^\circ \); (b) the case with \( i = 45^\circ \); (c) the case with \( i = 90^\circ \). Here \( r_o = 200M \).
amplified with the increasing spin parameter $a$. To illustrate the positional deviations clearly, the PIs corresponding to the SIs (0, 10), (0, −10), (−10, 0), and (10, 0) under the different $a$ and $i$ are shown in Figure 7(a)-(d). The positions of the PIs corresponding to each SI are different with different $i$ except for $a = 0$. The positional deviation of PIs increases with increasing $a$. Thus, the relative positions between the PIs and SIs could be used to determine the $a$ and $i$.

The relative positions between the PIs and SIs can be expressed as:

\[ \Delta x' = x'_i' - x'_p, \quad \Delta y' = y'_i' - y'_p, \]

where $(x'_p, y'_p)$ and $(x'_i, y'_i)$ are the coordinates of the PI and SI, respectively, in the $(x', y')$ coordinate system. We exhibit the contour maps of $\Delta x'$ and $\Delta y'$ between the SI (0, 10) and the corresponding PIs in terms of $a$ and $i$ in Figure 8(a) and (b). We observe that $\Delta x' = 0$ when $a = 0$; $\Delta x'$ increases and $\Delta y'$ decreases with increasing $a$. With increasing $i$, $\Delta x'$ first increases and then decreases and $\Delta y'$ first decreases and then increases. The contour lines of these $\Delta x'$ and $\Delta y'$ shown in Figure 8(c) reveal that some couples of $(\Delta x', \Delta y')$ cannot uniquely determine $a$ and $i$. Figure 9(a) and (b) show the contour maps of $\Delta x'$ and $\Delta y'$ between the SI (0, −10) and the corresponding PIs in terms of $a$ and $i$. A line corresponding to $\Delta x' = 0$ (the red dashed line) can be observed in Figure 9(a). Above this line, $\Delta x'$ increases as $a$ increases; below this line, $\Delta x'$ decreases as $a$ increases. Moreover, $\Delta x'$ decreases as $i$ increases. As shown in Figure 9(b), $\Delta y'$ increases with $a$ and first decreases and then increases with increasing $i$. The contour lines of $\Delta x'$ and $\Delta y'$ are shown in Figure 9(c), which illustrate the one-to-one correspondence between $(\Delta x', \Delta y')$ and $(a, i)$. Thus, $\Delta x'$ and $\Delta y'$ between the SI (0, −10) and the corresponding PI can determine the $a$ and $i$ as per these contour maps. Furthermore, the span of $\Delta x'$ is $|9.2 - (-3.68)| = 12.88 > 2R_s \approx 10.4$, which is larger than the diameter of the Kerr black hole shadow and can be distinguished more easily. However, the span of $\Delta y'$ is only $|-66.3 - (-67.49)| = 1.19$. Figure 10 shows the contour maps of $\Delta x'$ and $\Delta y'$ between the SI (−10, 0) and the corresponding PIs in terms of $a$ and $i$. It shows that $\Delta x'$ decreases as $a$ and $i$ increase; $\Delta y'$ increases as $a$ increases and decreases as $i$ increases. Figure 10(c) shows the one-to-one correspondence between $(\Delta x', \Delta y')$ and $(a, i)$ that can be used to determine $a$ and $i$. The spin of $\Delta x'$ is $|-67.07 - (-79.63)| = 12.56$, and the span of $\Delta y'$ is $[4.23 - 0] = 4.23$. Figure 11 shows the contour maps of $\Delta x'$ and $\Delta y'$ between the SI (10, 0) and the
corresponding PIs in terms of $a$ and $i$. It shows $\Delta x'$ decreases as $a$ and $i$ increase; $\Delta y'$ decreases as $a$ increases and increases as $i$ increases. $\Delta x'$ and $\Delta y'$ between SI (10, 0) and the corresponding PI can also be used to determine $a$ and $i$, as shown in Figure 11(c). The spin of $\Delta x'$ is $[67.49 - 59.19] = 8.3$, and the span of $\Delta y'$ is $|0 - (-3.68)| = 3.68$. The spans of $\Delta x'$ and $\Delta y'$ between the PIs and SIs are sufficiently large to aid in determining more accurate values of $a$ and $i$. In astronomical observations, one could even use the PIs and SIs of multiple light sources to jointly determine the $a$ and $i$.

For a given Kerr black hole image with the PIs and SIs of luminous celestial objects, the specific approach to measure the $a$ and $i$ is expressed as follows:

1) First, the distances of the black hole and the light sources from the observer must be measured, i.e., $r_\text{p}$ and $r_\text{s}$. Next, the center of the black hole shadow as the origin of the new celestial coordinate system $(x', y')$ must be determined. Finally, the coordinates $(x'_p, y'_p)$ of PIs and $(x'_s, y'_s)$ of SIs in the black hole image, where the scale of the coordinate system is completely flexible, must be determined.

2) The coordinates $(x_{p(a,i)}, y_{p(a,i)})$ of PIs and $(x_{s(a,i)}, y_{s(a,i)})$ of SIs in the celestial coordinates system $(x, y)$ with different spin parameter $a$ and inclination angle $i$ can be determined as follows:

$$x_{(a,i)} = \frac{(x_r - x_t)|_{(a,i)} x' + x_t|_{(a,i)}}{x'_r - x'_t}, \quad y_{(a,i)} = \frac{|y_t|_{(a,i)} y'}{y'_t},$$

where $x_r, x_t, x_s, y_t$ are the functions of $(a, i)$ as seen in eq. (7) and Tables 1-4, and $x'_r, x'_s, y'_t$ can be measured in the black hole shadow.

3) According to the coordinates $(x_{p(a,i)}, y_{p(a,i)})$ of the PIs and $(x_{s(a,i)}, y_{s(a,i)})$ of the SIs, the locations of their respective light sources are calculated using the backward ray-tracing method. The values of $a$ and $i$, when PI and SI share a common light source, are equal to the $a$ and $i$ of the black hole.

4 Time delays between the PIs and SIs

Figure 4 shows that the photons forming the SIs have longer paths than those forming the PIs. Moreover, the photons forming the SI are subject to a greater gravitational field owing to their greater proximity to the black hole. Hence, the
photons forming the SIs take more time to travel to the observer. Thus, a time delay exists in the appearance of intrinsic variability between the PI and the SI. This time delay is substantial in gravitational lensing observations. Measuring the time delays is useful in determining the mass and length scale of a gravitational lensing system. In a cosmological context, time delay has been used as a probe of the Hubble parameter \[109-111\].

Under different spin parameters \(a\) and inclination angle \(i\), the different paths between the PIs and SIs result in different time delays. The light starts from the observer, where \(t(0) = 0\). We evolve the light rays that form the PI and SI backward in time by numerically solving the null geodesic equations (9) until they reach the light source \((r = r_s)\) at times \(t_s = -t_p\) and \(-t_s\), respectively. Thus, for the same signal starting from the light source, the observer will receive the signal at time \(t_p\) in the PI and at time \(t_s\) in the SI.

This time delay between the PI and the SI is expressed as

Figure 8  (Color online) (a) The contour map of \(\Delta x'\) between the SI \((0, 10)\) and corresponding PIs in terms of \(a\) and \(i\); (b) the contour map of \(\Delta y'\) between the SI \((0, 10)\) and corresponding PIs in terms of \(a\) and \(i\); (c) the contour lines of \(\Delta x'\) and \(\Delta y'\) between the SI \((0, 10)\) and corresponding PIs in terms of \(a\) and \(i\).

Figure 9  (Color online) (a) The contour map of \(\Delta x'\) between the SI \((0, -10)\) and corresponding PIs in terms of \(a\) and \(i\); (b) the contour map of \(\Delta y'\) between the SI \((0, -10)\) and corresponding PIs in terms of \(a\) and \(i\); (c) the contour lines of \(\Delta x'\) and \(\Delta y'\) between the SI \((0, -10)\) and corresponding PIs in terms of \(a\) and \(i\).

Figure 10  (Color online) (a) The contour map of \(\Delta x'\) between the SI \((-10, 0)\) and corresponding PIs in terms of \(a\) and \(i\); (b) the contour map of \(\Delta y'\) between the SI \((-10, 0)\) and corresponding PIs in terms of \(a\) and \(i\); (c) the contour lines of \(\Delta x'\) and \(\Delta y'\) between the SI \((-10, 0)\) and corresponding PIs in terms of \(a\) and \(i\).
\( \Delta t = (t_s - t_p) \). Figure 12 shows the time delays \( \Delta t \) between the PIs and the SIs under different values of \( a \) and \( i \), where the PIs and the SIs are the same as those in Figures 6 and 7. Figures 12(a) and (b) show the time delays \( \Delta t \) between SI \((0, 10)\) and the corresponding PI and between SI \((0, -10)\) and the corresponding PI, respectively. Because these PIs and SIs are almost on the \( y' \)-axis, the time delays \( \Delta t \) between the PIs and SIs remain nearly constant corresponding to the changes in \( i \) for a fixed \( a \); however, for \( a = 0.998M \), the time delay decreases as \( i \) increases. For fixed \( i \), the time delays \( \Delta t \) decrease as \( a \) increases. Figures 12(c) and (d) show the time delays \( \Delta t \) between SI \((-10, 0)\) and the corresponding PI and between SI \((10, 0)\) and the corresponding PI. Owing to the dragging effect, the time delay \( \Delta t \) between SI \((-10, 0)\) and the corresponding PI increases with \( a \). Because the PIs and SIs are almost on the \( x' \)-axis, the increase in \( i \) indicates that they are getting closer to the black hole equator. This proximity leads to an increase in the time delay \( \Delta t \) owing to the increasing dragging effect. However, the time delay \( \Delta t \) between SI \((10, 0)\) and the corresponding PI decreases as \( a \) increases.

**Figure 11** (Color online) (a) The contour map of \( \Delta x' \) between the SI \((10, 0)\) and corresponding PIs in terms of \( a \) and \( i \); (b) the contour map of \( \Delta y' \) between the SI \((10, 0)\) and corresponding PIs in terms of \( a \) and \( i \); (c) the contour lines of \( \Delta x' \) and \( \Delta y' \) between the SI \((10, 0)\) and corresponding PIs in terms of \( a \) and \( i \).

**Figure 12** (Color online) (a) The time delays \( \Delta t = (t_s - t_p) \) between SI \((0, 10)\) and the corresponding PI for different spin parameter \( a \) and inclination angle \( i \); (b) \( \Delta t \) between SI \((0, -10)\) and the corresponding PI; (c) \( \Delta t \) between SI \((-10, 0)\) and the corresponding PI; (d) \( \Delta t \) between SI \((10, 0)\) and the corresponding PI. These PIs and SIs are the same as those in Figures 6 and 7.
and $i$ increase. The time delays $\Delta t$ between the PIs and SIs in conjunction with their relative positions could allow us to measure the $a$ and $i$ with high precision.

5 Conclusion

We study the primary and secondary images generated owing to strong gravitational lensing around a Kerr black hole shadow. These images carry some essential signatures regarding the black hole space-time. To locate the PIs and SIs of luminous celestial objects, we define a new celestial coordinate whose origin is the center of the black hole shadow. Under the influence of the dragging effect caused by the rotating black hole and based on the inclination angle of the observer, the relative positions between the PIs and SIs are different under different values of $a$ and $i$. We mark four SIs (above, below, on the left, and on the right of the shadow center) and observe that the positions of the PIs are different for different values of $a$ and $i$. In astronomical observations, the relative positions between the PIs and SIs around the black hole shadow can be used to determine $a$ and $i$. Furthermore, we provide a specific approach to measure $a$ and $i$ based on PIs and SIs. The time delays between the PIs and SIs are different for different values of $a$ and $i$. The time delays, in conjunction with the relative positions between the PIs and SIs, allow us to measure $a$ and $i$ with increased precision. These PIs and SIs act as unique fingerprints for black hole space-time, using which other parameters of all types of compact objects, such as Kerr-like black holes, can be determined, and various theories of gravity can be verified. Our results provide a new method to implement parameter estimation in the study of black hole physics and astrophysics.

The black hole pictures captured by the EHT Collaboration are the image of the accretion disk around the black hole shadow. The accretion disk is the light source close to the black hole. Further research is needed to determine whether the parameters of black holes and the inclination angle can be extracted using the accretion disk image. In addition, the questions of what an image of a naked singularity would look like and whether the parameters of the naked singularity could be extracted from the images remain unanswered. The study of naked singularity images could help us further test the cosmic censorship hypothesis. The study of the PIs and SIs of luminous celestial objects in astronomical images could provide a crucial validation of the gravitational theory.

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Conflict of interest The authors declare that they have no conflict of interest.
