Is there a relation between the 2D Causal Set action and the Lorentzian Gauss-Bonnet theorem?

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Abstract. We investigate the relation between the two dimensional Causal Set action, $S$, and the Lorentzian Gauss-Bonnet theorem (LGBT). We give compelling reasons why the answer to the title’s question is no. In support of this point of view we calculate the causal set inspired action of causal intervals in some two dimensional spacetimes: Minkowski, the flat cylinder and the flat trousers.

1. The Gauss-Bonnet theorem

The Gauss-Bonnet (GB) theorem is a remarkable theorem first proved by Gauss, and later by Bonnet in a special case, that relates the geometry of a two-dimensional (2D) Riemannian manifold to its topology. In one of its variations it states that for any 2D compact Riemannian manifold $\mathcal{M}$ with boundary $\partial \mathcal{M}$

$$\frac{1}{2} \int_{\mathcal{M}} R dV + \int_{\partial \mathcal{M}} k_g ds + \sum \theta_{\text{exterior}} = 2\pi \chi(\mathcal{M}).$$

(1.1)

Here $R$ is the scalar curvature of $\mathcal{M}$, $k_g$ is the geodesic curvature of $\partial \mathcal{M}$, $\theta_{\text{exterior}}$ is the exterior angle at non-smooth points of $\partial \mathcal{M}$ and $\chi(\mathcal{M})$ is the Euler character of $\mathcal{M}$.

This theorem was first extended to pseudo-Riemannian 2D manifolds by Avez [1] and Chern [2]. The Lorentzian $^1$ version of the theorem can be viewed as a natural extension of (1.1) where the geometric quantities on the left-hand side are carried over to the Lorentzian context. The essential difference between the Riemannian and Lorentzian theorems then comes from the definition of $\theta_{\text{exterior}}$ which depends on the orientation preserving isometry group of their respective metrics: $SO(2)$ and $SO(1,1)$ for the Euclidean and Lorentzian signatures respectively. There are various versions of the LGBT, here we will be using that due to Law [3] (see also [4, 5])

\textbf{Lorentzian GB Theorem.} Let $D$ be a domain in a 2D Lorentzian manifold $(\mathcal{M}, g)$ with piecewise smooth boundary $\Gamma$ consisting of a finite number of nonnull smooth curves. Then

$$\frac{1}{2} \int_{D} R dV + \int_{\Gamma} k_g ds + \sum \theta_{\text{exterior}, \pm} = \pm 2\pi i \chi(D).$$

(1.2)

$^1$ Note that in 2D the only pseudo-Riemannian metric is Lorentzian
2. The 2D causal set action

Recall that a causal set \([7]\) (causet for short) is a locally finite partial order, \(i.e.\) it is a pair \((\mathcal{C}, \preceq)\) where \(\mathcal{C}\) is a set and \(\preceq\) is a relation on \(\mathcal{C}\) which is reflexive \((x \preceq x)\), acyclic \((y \preceq x \preceq y \Rightarrow y = x)\) and transitive \((z \preceq y \preceq x \Rightarrow z \preceq x)\). Local finiteness is the condition that the cardinality of any order interval is finite, where the (inclusive) order interval between a pair of elements \(y \preceq x\) is defined to be \(I(x, y) := \{z \in \mathcal{C} \mid y \preceq z \preceq x\}\). We call \(x\) the top element and \(y\) the bottom element of \(I(x, y)\). We write \(y < x\) when \(y \preceq x\) and \(y \neq x\). We define \(n(x, y) := |I(x, y)|\) and call a relation \(y < x\) a link if \(n(x, y) = 2\).

We define the 2D action, \(\mathcal{S}\), of a finite causal set \(\mathcal{C}\) to be \([8]\)

\[\mathcal{S}[\mathcal{C}] = N - 2N_1 + 4N_2 - 2N_3\]  

(2.3)

where \(N\) is the cardinality of \(\mathcal{C}\), and \(N_m\) is the number of inclusive order intervals in \(\mathcal{C}\) of cardinality \(m + 1\). \(N_1\) therefore is the number of links in \(\mathcal{C}\), \(N_2\) is the number of order intervals that are 3-chains (3 element chains) and \(N_4\) is the number of order intervals that are 4-chains plus the number that are “diamonds” (with two mutually unrelated elements between the top and bottom elements). The form of \(\mathcal{S}\) as an alternating sum of (weighted) numbers of things is intriguingly reminiscent of certain topological indices.

In order to make a connection with the continuum we need to define the notion of sprinkling: Sprinkling is a random process that produces a causet which is a discretisation of a \(d\)-dimensional, causal, Lorentzian manifold \((\mathcal{M}, g)\). It is a Poisson process of selecting points in \((\mathcal{M}, g)\), independently at random, with density \(\rho\) so that the expected number of points sprinkled in a region of spacetime volume \(V\) is \(\rho V\), where \(\rho = l^{-d}\) and \(l\) is of order the Planck length. This process generates a causet whose elements are (identified with) the sprinkled points and whose order is that induced by the manifold’s causal order restricted to the sprinkled points. We say that a causet \(\mathcal{C}\) is well approximated by a Lorentzian manifold \(\mathcal{M}\) if it could have been generated, with relatively high probability, by sprinkling into \(\mathcal{M}\).

The Poisson distribution gives for the mean of (2.3)

\[\langle \mathcal{S}_\mathcal{M} \rangle = \rho V - 2\rho^2 \int_\mathcal{M} d^d y \sqrt{-g(y)} \int_\mathcal{M} d^d x \sqrt{-g(x)} \left(1 - 2\rho V_{xy} + \frac{1}{2}(\rho V_{xy})^2\right) e^{-\rho V_{xy}} \]  

(2.4)

where \(V_{xy}\) is the volume of the spacetime causal interval, \([x, y] := J^+(y) \cap J^-(x)\), between \(x\) and \(y\) and \(d\) is the dimension of \(\mathcal{M}\).

It can be shown that when \(\mathcal{C}\) is a 2D sprinkling then the mean of \(\mathcal{S}\) tends to something that contains a term \(\frac{1}{4} \int_\mathcal{M} d^2 x \sqrt{-gR}\) plus terms arising from boundary effects. But we know that in 2D gravity the Einstein-Hilbert action plus (the 2D analogue of) the Gibbons-Hawking boundary term are a topological invariant (Eqn. \((1.2)\)), so it is natural to ask to what extent the 2D causet action \(\mathcal{S}\) is a topological invariant? To answer this question we study the action of flat causal intervals in different topologies.
3. Causal intervals in $M^2$

Consider first a causal interval in 2D Minkowski spacetime, $I := [p, q] \subset M^2$. For definiteness consider the interval to have fixed volume (area), $V$. Let us coordinates in which $p$ and $q$ lie on the time axis and $q$ is at the origin. We consider null coordinates $u^x = \frac{1}{\sqrt{2}}(x^0 - x^1)$, $v^x = \frac{1}{\sqrt{2}}(x^0 + x^1)$ and similarly for $u^y$, $v^y$. Then the interval is defined by $u, v \in [0, a]$ for $a = \sqrt{V}$. The action of $I$ is given by

$$\langle S_I \rangle = \rho V - 2 \int_0^a du_x \int_0^a dv_x \int_0^a du_y \int_0^c dv_y \rho^2 p(\rho \Delta u \Delta v) = 1 - e^{-\rho V}$$

(3.5)

where $\Delta u = u_x - u_y$, $\Delta v = v_x - v_y$. As $\rho \to \infty$, $\langle S_I \rangle \to 1$ and we write $\langle S_I \rangle \approx 1$ to denote this. This result is a first hint that $S$ might be topological. Consider now splitting up the interval $I$ into four smaller intervals $I_i$, $i = 1, \ldots, 4$, as shown in Fig. 1a. When computing the expected value of $S_I$ one can split the integral up into the means of the actions of the four subintervals plus the “bilocal” contributions when $x$ and $y$ lie in two different subintervals. More concretely, given any subcausets, $A$ and $B$ of a causal set $C$, we define the bilocal action,

$$S[C; A, B] = N(A, B) - 2N_1(A, B) + 4N_2(A, B) - 2N_3(A, B)$$

(3.6)

where $N(A, B)$ is the number of elements in $A \cap B$ and $N_m(A, B)$ is the number of inclusive order intervals in $C$ of cardinality $m + 1$ with top element in $A$ and bottom element in $B$. The mean of $S[C; A, B]$ we will denote by $\langle S_{C, X, Y} \rangle$ where $X$ and $Y$ are submanifolds of $M$ into which $C$ has been sprinkled, with subcausets $A, B$ sprinkled into $X, Y$ respectively. Note that $S_{M, X, X} = S_X$ if $X$ is a causally convex subset of $M$.

Now, consider $I$ and its subintervals. If we adopt $S_{ij}$ as simplified notation for the bilocal action $S_{I_i, I_j}$, then we have

$$\langle S_I \rangle = \sum_{i=1}^4 \langle S_{I_i} \rangle + \sum_{i,j=1}^4 \langle S_{ij} \rangle$$

(3.7)

$^2$ A causally convex region, $X$, of $M$ is one such that $x, y \in X$ implies that the causal interval in $M$ between $x$ and $y$ is a subset of $X$. 

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**Figure 1:** Splitting up a causal interval in 2D Minkowski to compute the action
The bilocal summands can be computed using the integral in Eq. (3.5) and adjusting the boundaries:

\[
\langle S_{21} \rangle = -2 \int_0^a dv_x \int_b^{b+a} du_x \int_0^a dv_y \int_0^b du_y \rho^2 p(\rho \Delta u \Delta v) \approx -1
\]

and

\[
\langle S_{41} \rangle = -2 \int_a^{a+c} dv_x \int_b^{b+d} du_x \int_0^a dv_y \int_0^b du_y \rho^2 p(\rho \Delta u \Delta v) \approx 1.
\]

The three other bilocal contributions \( \langle S_{ij} \rangle \) can be obtained from \( \langle S_{21} \rangle \) by changing the parameters appropriately. Putting together all parts of Eq. (3.7) one exactly recovers Eq. (3.1).

Now, one can continue this game and split up the interval even further as in Fig. 1b. To compute the mean of the action one must again calculate

\[
\langle S \rangle = \sum_{i=1}^{9} \langle S_{I_i} \rangle + \sum_{i,j=1}^{9} \langle S_{ij} \rangle. \tag{3.8}
\]

We already know the contributions \( \langle S_{I_i} \rangle \approx 1 \) and the bilocal contributions from two intervals that either share an edge or lie above and below a shared vertex (e.g. \( \langle S_{21} \rangle \) and \( \langle S_{51} \rangle \) in Fig. 1b). It remains to compute the bilocal contributions from pairs of intervals such as (4,1), (7,1) and (9,1) in Fig. 1b. It turns out they consist only of exponential terms that are small when intervening intervals are large on the discreteness scale. In the limit of large density, we are left with a contribution of 1 for every subinterval, -1 for every edge and 1 for every vertex. One could write

\[
\langle S \rangle \approx F - E + V \tag{3.9}
\]

where \( F \) denotes the number of faces i.e. intervals, \( E \) the number of edges and \( V \) the number of vertices. This expression is the formula for the Euler character of a polyhedron. This result suggests that \( \langle S \rangle \) might be topological in nature. But if it is related to the LGBT then where is the factor of 2\pi? And i? Also, how does the result depend on the form of the boundary of the region? It can be shown that for any causally convex region, \( R \subset M^2 \), and for any “triangulation” of \( R \) such that no edge is timelike Eqn. (3.5) holds. The first two questions however remain unanswered; unless we take the view that they are proof that there is no connection between \( S \) and the LGBT. Let us not give up yet however and study the mean action of a causal interval on a cylinder.

4. The flat cylinder

Consider a causal interval, \( I_c \), of height \( T \) on a cylinder with circumference \( L \) with \( L \leq T \leq 2L \).

To calculate \( \langle S_{I_c} \rangle \) consider tiling \( I_c \) into subintervals, \( I_i, i = 1, \ldots, 8 \), as shown in Fig. 2. Taking into account the topological identification, we have \( F = 8, E = 12, V = 4 \) thus yielding a predicted high-density expectation value of \( \langle S_{I_c} \rangle \approx 0 \). Indeed one can calculate \( \langle S_{I_c} \rangle \) explicitly by using (2.2) and finds the same result (one needs to be very careful with intervals whose tips have more than one homotopically non-trivial causal curve connecting them, since in these cases the causal volume between them has to be modified appropriately, see [9, 10]). Once again the result is consistent with the Euler character of a cylinder.
5. The flat trousers

As our final example let us look at the action of a causally convex neighbourhood, \( N \), of the flat (1+1)D trousers. The trousers spacetime is a piece of \( M^2 \) with cuts and identifications as shown in Fig. 3. Although the singularity, \( P \), at which the topology changes is by some definitions not strictly in the spacetime since the metric degenerates there, nevertheless the causal order is well defined at the singularity: it is clear what the causal past and causal future of \( P \) are. Therefore we will consider \( P \) as a point of the manifold. Note that in any sprinkling into the trousers almost surely no element will be sprinkled at \( P \). Using the same techniques used in the previous section we can calculate \( \langle S \rangle \) for this neighbourhood and we find

\[
\langle S_N \rangle = 4 \ln(\rho a^2) + 4(\gamma - 1) + O\left(\frac{1}{\rho a^2}\right)
\]  

(5.10)

where \( \gamma \) is Euler’s constant. We see that the expected DA of the neighbourhood of the singularity does not tend to 1 or any constant but grows logarithmically with the density.

Figure 3: The trousers spacetime. \( P \) is the singularity – all three instances of \( P \) are identified – and the shaded region is a neighbourhood of \( P \). There is a vertical cut down from the central copy of \( P \) with the two legs identified as shown.
6. Discussion

The result of the last section, together with the missing factor of $2\pi$, and more crucially $i$, leads us to conclude that the answer to the title’s question is no. But then a few natural questions arise: Why does $S$ display topological properties for some regions of flat spacetimes and how are the boundaries contributing? How do these results carry over to curved spacetimes?

The first question we can answer as follows: Let $D$ be some causally convex region of $M^2$ whose boundary is composed of null segments. We know that in general “$\langle S \rangle \approx \text{bulk} + \text{boundary}”$. So for $D$, $\langle S \rangle \approx \text{boundary}$. But since the geodesic curvature of the boundary is zero we do not expect these boundary contributions to come from the “edges”. When the boundary contains spacelike segments we expect there to be boundary contributions as these segments can have nonzero geodesic curvature. However it is possible to argue that even for such boundaries there will be no contribution to $\langle S \rangle$, see [9]. So the only points where some boundary contribution can come in, is from the points which are close to both past and future boundaries i.e. from the spacelike co-dimension 2 “corners” where the past and future boundaries intersect (There is no reason, from this argument, that timelike boundaries could not contribute and in fact there is evidence that they do, but not in a way which helps establish a connection with the LGBT [9]). Now the GB theorem can hold because as the geometry of the bulk is varied, the extrinsic curvature of the boundary changes in precisely the right way so that their sum remains constant. But if for the causet action the only boundary contribution comes from the $S^0$ corners, then we do not expect these to compensate for a changing bulk term and hence we wouldn’t expect the LGBT to hold. This putative lack of boundary terms could explain why the expected causet action for any causally convex region of $M^2$ is the same. The continuum bulk term is zero. If the mean of $S$ is indeed close to the continuum bulk term plus only a contribution from the $S^0$ corners then that should be the same for all causally convex regions. Presumably, the difference for the neighbourhood of the singularity of the trousers comes from a boundary effect of the non-standard causal structure around the singularity which has a double lobed past and future. These issues all remain to be investigated.

The second question is currently being addressed. As things stand we can only express the hope that similar results will hold in curved 2D spacetimes. Here the bulk term would not be zero in general, so a result of the form “$\langle S \rangle \approx \text{constant}”$ would suggest that the boundary term is getting contributions from the edges as well as the corners. If true, this would possibly revive hope in establishing a connection between the causet action and some modified form of the LGBT — which allows for null boundaries and is real. Of course one would also need the constant appearing on the right hand side to be the same as in flat space for topologically isomorphic causally convex regions, e.g. we would need $\langle S \rangle \approx 1$ for all causally convex domains which are topologically disks. Further work is needed to illuminate these issues.

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