Inflationary Models Driven by Adiabatic Matter Creation

L. R. W. Abramo\textsuperscript{1,a}, J. A. S. Lima\textsuperscript{1,2,b}

\textsuperscript{1} Physics Department, Brown University, Providence, RI 02912, USA.
\textsuperscript{2} Departamento de Física Teórica e Experimental, Universidade Federal do Rio Grande do Norte, 59072 - 970, Natal, RN, Brazil.

Abstract

The flat inflationary dust universe with matter creation proposed by Prigogine and coworkers is generalized and its dynamical properties are reexamined. It is shown that the starting point of these models depends critically on a dimensionless parameter $\Sigma$, closely related to the matter creation rate $\psi$. For $\Sigma$ bigger or smaller than unity flat universes can emerge, respectively, either like a Big-Bang FRW singularity or as a Minkowski space-time at $t = -\infty$. The case $\Sigma = 1$ corresponds to a de Sitter-type solution, a fixed point in the phase diagram of the system, supported by the matter creation process. The curvature effects have also been investigated. The inflating de Sitter is a universal attractor for all expanding solutions regardless of the initial conditions as well as of the curvature parameter.

\textsuperscript{a}e-mail: abramo@het.brown.edu
\textsuperscript{b}e-mail: limajas@het.brown.edu
1 Introduction

Much effort has been spent to understand the effects of the matter creation process on the universe evolution[1-7]. In the framework of the Friedman-Robertson-Walker (FRW) geometries, a self-consistent phenomenological description for matter creation has recently been proposed by Prigogine and coworkers[8, 9]. The leitmotiv of this approach is that the matter creation process, at the expense of the gravitational field, can happen only as an irreversible process constrained by the usual requirements of nonequilibrium thermodynamics. The crucial ingredient, however, is the explicit use of a balance equation to the number density of created particles in addition to the Einstein field equations (EFE). When properly combined with the second law of thermodynamics, such an equation leads naturally to a reinterpretation of the stress tensor, corresponding to an additional pressure term, which in turn, depends on the matter creation rate.

An extended manifestly covariant version of such a formulation has also appeared in the literature[10, 11]. The nonequivalence between the matter creation process and the mechanism of bulk viscosity, which has been widely used in the literature as a phenomenological description to the former, has been recently clarified[12, 13]. In connection to this, we remark that the general thermodynamic properties of the formulation by Prigogine and coworkers has been more carefully investigated than its dynamic counterpart, which was also presented in the mentioned papers as a new example of nontraditional cosmology.
In this article we focus our attention on the cosmological scenario proposed in Refs.\cite{8,9}. As will be seen, our study will provide some general results often useful to bring forward certain subtleties in the cosmological solutions which apparently were not perceived at first by the mentioned authors. To be more specific, in the scenario proposed in the quoted papers, the spacetime starts from a Minkowski phase at $t = 0$, with a particle number density $n_o$ describing the initial fluctuation (Cf., for instance, \cite{8}, pg. 773). As will be shown here, however, there is no initial fluctuation since the Minkowski spacetime starts at $t = -\infty$ with a number of particles precisely equal to zero, as it should be. Subsequently, these models evolve smoothly to a de Sitter phase, in such a way that at $t = 0$ all solutions describe an expanding (inflationary) flat FRW-type universe. Another unnoticed aspect is related to the existence of a new large class of solutions starting from a big-bang FRW singularity and also approaching de Sitter spacetime for late cosmological times. For all values of the parameter of the equation of state $p = (\gamma - 1)\rho$, the starting point of the flat universes depend critically on a dimensionless parameter closely related to the matter creation rate. It can also be shown that all expanding solutions converge to a flat inflating de Sitter solution. In other words, the latter is an attractor independent of initial conditions.

Using phase space portrait techniques, the above analysis is extended to curved spacetimes. Due to degeneracy in flat space, the class of models proposed by Prigogine et al. is shown to split into two distinct classes. In
particular, it is shown that only de Sitter and a space-like singularity (Big Crunch) are attracting nodes (stable points) in the phase space of solutions. For the first class, the former attractor is independent of the initial conditions, that is, all expanding solutions end up like a inflating de Sitter spacetime regardless of the values of the curvature parameter.

This paper is organized as follows: in Section 2, the basic equations describing a FRW-type Cosmology with matter creation are presented. In Section 3 we discuss the flat case for a matter content satisfying the $\gamma$-law equation of state. The flat dust model proposed by Prigogine et al. is discussed in detail and a second set of exact FRW-type solutions not foreseen by those authors are established. Finally, in Section 4 we develop a qualitative analysis in order to investigate the curvature effects. The matter creation ansatz of Refs.\[8, 9\] is generalized and its dynamic consequences are discussed.

2 Basic Equations

Consider now the homogeneous and isotropic FRW line element

$$ds^2 = dt^2 - R^2(t)(\frac{dr^2}{1 - kr^2} + r^2d\theta^2 + r^2\sin^2(\theta)d\phi^2) ,$$

(1)

where $R$ is the scale factor and $k = 0, \pm 1$ is the curvature parameter. Throughout we use units such that $c = 1$.

In that background, the basic dynamic equations for a self-gravitating perfect fluid endowed with matter creation reduce to

$$\Box \phi = \rho,$$

$$\dot{\rho} + 3H(\rho + \rho_\text{m}) - \lambda \rho^2,$$

where $\lambda$ is the matter creation velocity parameter.
i) Einstein’s field equations (EFE):

\[ \chi \rho = 3 \frac{\dot{R}^2}{R^2} + 3 \frac{k}{R^2}, \tag{2} \]

\[ \chi (p + p_c) = -2 \frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{k}{R^2}, \tag{3} \]

whereas the energy conservation law,

\[ \dot{\rho} + 3(\rho + p + p_c)H = 0, \tag{4} \]

is the usual consistency condition contained in the independent EFE (2) and (3). In the above equations, \( \chi = 8\pi G \), \( \rho \), \( p \) and \( p_c = -\frac{\alpha \psi}{3H} \) are, respectively, the energy density, equilibrium and creation pressures. An overdot means time derivative and \( H = \dot{R}/R \) is the Hubble parameter.

ii) The balance equation for the particle number density:

\[ \dot{n} + 3nH = \psi, \tag{5} \]

where \( \psi \) is the particle source (\( \psi > 0 \)) or sink (\( \psi < 0 \)).

The phenomenological parameter \( \alpha \) appearing in the definition of the creation pressure was originally introduced in Refs. [10, 11]. As shown there, the Prigogine et al. formulation is recovered in the “adiabatic limit”, namely when \( \psi \) is different from zero but the specific entropy (per particle) is constant. From now on, we consider just the “adiabatic case” for which the \( \alpha \) parameter is given by (see Refs. [11, 12])
\[ \alpha = \frac{\rho + p}{n} , \]  

with the creation pressure assuming the following form

\[ p_c = -\frac{\rho + p}{3nH}\psi. \]  

In order to obtain a definite cosmological scenario with matter creation, one still needs to provide two additional relations: the equation of state and the matter creation rate. In the cosmological domain, the former is usually expressed in terms of the “gamma-law” equation of state

\[ p = (\gamma - 1)\rho , \quad 0 \leq \gamma \leq 2 , \]  

where \( \gamma \) is the “adiabatic index”.

Note that, from (2), (3), (7) and (8), the evolution equation for the scale factor \( R \) can be written as

\[ R\ddot{R} + \left( \frac{3\gamma - 2}{2} - \frac{\gamma\psi}{2nH} \right)\dot{R}^2 + \left( \frac{3\gamma - 2}{2} - \frac{\gamma\psi}{2nH} \right)k = 0 , \]  

which depends only on the matter creation rate \( \psi \) and the number density of particles \( n \).

Furthermore, using (4), (5) and (7), it is easy to show that \( n \) and \( \rho \) satisfy the general relation (see Ref. [15])

\[ n = n_o\left( \frac{\rho}{\rho_o} \right)^{\frac{1}{\gamma}} , \]  

where \( \psi \) is the “adiabatic index”.

Note that, from (2), (3), (7) and (8), the evolution equation for the scale factor \( R \) can be written as
so that, using the EFE (2) one can obtain \( n(R, H^2) \), with Eq.(9) now depending only on \( \psi \).

In what follows, we will first discuss the simple and exactly integrable flat case \((k = 0)\). This class includes Prigogine et al.’s dust model [8, 9]. The contributions of curvature terms will be analyze in section 4, but for now it suffices to say that flat solutions are not necessarily stable with respect to curvature perturbations. In the language of dynamical systems, FRW cosmology without matter creation is “structurally stable” with respect to curvature perturbations, while models with matter creation are not.

3 Flat Case: Two Classes of Models

Let us now discuss the flat case. In terms of the Hubble parameter the evolution equation (9) for the scale factor reduces to:

\[
\dot{H} + \frac{3\gamma}{2} H^2 = \frac{\gamma \psi}{2n} H ,
\]

(11)

while from (2) and (10) the particle number density \( n \) assume the form below,

\[
n = n_o \left( \frac{H}{H_o} \right)^{\frac{\gamma}{2}} .
\]

(12)

Now, in order to properly generalize the dust universe of Refs. [8, 9] we consider a class of models endowed with the same matter creation rate

\[
\psi = 3\beta H^2 .
\]

(13)
In the above expression $\beta$ is a new phenomenological parameter ($\frac{\alpha}{3}$ in the notation of Refs. [8, 9]). Parenthetically, since we are dealing with a spatially flat spacetime, the above equation implies that $\psi$ is proportional to the energy density, more precisely, $\psi = \chi \beta \rho$. Of course, these relations are not equivalent in curved spacetimes, and so doing we are effectively working with two different creation rates (see section 4).

Inserting Eqs. (12) and (13) into (11) one obtains
\[ \dot{H} + \frac{3\gamma}{2} H^2 \left[ 1 - \left( \frac{H}{H_d} \right)^{\frac{\gamma - 2}{\gamma}} \right] = 0 \quad (\gamma \neq 2), \tag{14} \]
and
\[ \dot{H} + 3 \left( 1 - \frac{\beta H_o}{n_o} \right) H^2 = 0 \quad (\gamma = 2). \tag{15} \]

In Eqs. (12)-(15), $n_o$ is the value of $n$ for $H = H_o$ and $H_d$ is given by
\[ H_d = H_o \left( \frac{n_o}{\beta H_o} \right)^{\frac{\gamma - 2}{\gamma}} \quad (\gamma \neq 2). \tag{16} \]
The case $\gamma = 2$ is the simplest one to be analyzed. However, since it does not exhibit the peculiarities present in the class of solutions of (14), which contains the model of Refs. [8, 9], we do not consider it in what follows. Apart from some interpretation problems, Prigogine's et al. model must be recovered by taking $\gamma = 1$ in the general solution of (14). A straightforward integration of this later equation yields
\[ H = H_d (1 + AR^{\frac{3(\gamma - 2)}{2}})^{\frac{\gamma}{2 - \gamma}}, \tag{17} \]
where $A$ is a dimensional integration constant.

Now, in order to introduce in our approach a useful dimensionless parameter, we compute the value of $A$ as a function of $H_d$, $H_o$ and $R_o$, where $R_o$ is the value of $R$ when $H = H_o$. From (17) one reads

$$A = (\Sigma - 1)R_o^{\frac{3}{2} \gamma (2-\gamma)} ,$$  \hspace{1cm} (18)

where

$$\Sigma = \frac{n_o}{\beta H_o} .$$ \hspace{1cm} (19)

In terms of $\Sigma$ the Hubble parameter takes the form

$$H = H_d \left[ 1 + (\Sigma - 1)\left(\frac{R}{R_o}\right)^{\frac{3}{2} \gamma (2-\gamma)} \right]^{\frac{2}{2-\gamma}} ,$$ \hspace{1cm} (20)

and from (12) and (17)-(19), the particle number density is given by

$$n = n_o \Sigma^{\frac{2}{2-\gamma}} \left[ 1 + (\Sigma - 1)\left(\frac{R}{R_o}\right)^{\frac{3}{2} \gamma (2-\gamma)} \right]^{\frac{2}{2-\gamma}} ,$$ \hspace{1cm} (21)

with the time-dependent net number of particles, $N = nR^3$, assuming the following form:

$$N(t) = n_o R^3 \Sigma^{\frac{2}{2-\gamma}} \left[ 1 - (\Sigma - 1)\left(\frac{R}{R_o}\right)^{\frac{3}{2} \gamma (2-\gamma)} \right]^{\frac{2}{2-\gamma}} .$$ \hspace{1cm} (22)

For these models, the qualitative behavior at early and late times can be easily determined from Eqs. (20) and (21) or (22). If $\Sigma = 1$, the Hubble parameter is constant, $H = H_d$, with the solutions reducing to a flat de
Sitter-type universe regardless of the value of $\gamma$. In this way, we may have, for instance, a de Sitter universe supported by the adiabatic creation of photons ($\gamma = 4/3$) or dust ($\gamma = 1$). Such a fact had already been observed in Ref. [8]. Indeed, $H = H_d$ is a special solution, a singular point in the phase diagram of the system $\{\dot{H}(\rho, H), \dot{\rho}(\rho, H)\}$ (see Eqs. (34) and (35) in section 4). It should be noticed that for large values of $R$, that is, $(R/R_o)^{-3(2-\gamma)/2} \ll 1$, the models evolve to this inflating solution.

How these flat universes emerge depends only on the sign of $A$ or equivalently, if the dimensionless parameter $\Sigma$ is bigger, smaller or equal unity. For $\Sigma > 1$, the second term on the rhs of (20) is dominant for small values of $R$, that is, $(R/R_o)^{-2-\gamma} \gg 1$, thereby leading to the usual FRW singularity, $R \sim t^{2/3\gamma}$. Since $H \to \infty$ in this limit, it follows from (11) and (12) that $\rho$ and $n$ are infinite as $t$ goes to zero. These results characterize a class of singular solutions not previously perceived in Refs. [8, 9].

On the other hand, if $\Sigma < 1$ it follows from (21) that there is a minimal value of $R$, namely:

$$R_{\text{min}} = R_o(1 - \Sigma)^{\frac{2}{3(2-\gamma)}},$$

for which $H = 0$. In addition, Eqs. (11), (12) and (21), yield $\rho = n = N = 0$ at $R = R_{\text{min}}$, making explicit that such models start, for all values of $\gamma$, as a Minkowski vacuum. We remark that, from (7) and (12), the creation pressure $p_c$ scales with $H^{3\gamma-2}$. Thus, according to the above results, in the beginning of the universe the creation pressure was either zero or infinite if,
respectively, $\Sigma > 1$ or $\Sigma < 1$. Only in the de Sitter case ($\Sigma = 1$) the creation pressure has a finite (and constant) value.

The above results are rather important in what follows. For $R = R_o$ one obtains $H = H_o$, $n = n_o$ and $N = n_o R_o^3$ for all values of $\gamma$ and $\Sigma \neq 1$. For $\Sigma = 1$, (20) and (21) reduce to the constant values $H = H_d$ and $n = n_o$ as required by the symmetries of the de Sitter spacetime. The latter case is different from the de Sitter phase attained in the course of the evolution for $\Sigma \neq 1$. In fact, for $R \gg R_o$, (21) yields

$$n_d = n_o \Sigma^{-2/3},$$

which may also be obtained from (12) taking $H = H_d$. The important point to keep in mind here is that models starting as Minkowski ($\Sigma < 1$) have $n = 0$ when $R = R_{min}$, that is, $n_o$ cannot be the number density at the beginning of the universe.

Therefore, since for $\gamma = 1$ the energy density is $\rho = nM$ (where $M$ is the mass of the created dust particles) we can say that the authors of Refs. [8, 9] studied only the case $\Sigma < 1$, that is, $M > \frac{3n_o}{\chi \beta^2}$. Note that $n_o$ and $\beta$ are coupled to give the natural mass scale of the models so that for $\gamma = 1$ the condition $\Sigma \leq 1$ ($\Sigma \geq 1$) assumes the interesting form $M \geq M_c$ ($M \leq M_c$), where the critical mass is $M_c = \frac{3n_o}{\chi^3 \beta^2}$, just the mass of the particles created in the de Sitter spacetime. For this steady state scenario, we see from (5) and (13) that $\beta = n_d H_d^{-1}$, and using the present data it is readily obtained $M_c \approx 1 GeV$ i.e., the proton mass. Naturally, the above condition may also be
translated as a constraint on $\beta$, the free parameter of the models. In terms of the mass $M$, the critical value of this parameter is given by $\beta_c = \frac{3H_d}{M^2}(\frac{M^2_{pl}}{M})$, where $M_{pl}$ is the Planck mass. In contrast with the approach developed here, the mass of the created particles in Refs.[8, 9] has been estimated by assuming that the de Sitter phase is unstable and evolves continuously (up to first derivatives) to the FRW radiation phase.

3.1 Prigogine et al.’s Model ($\Sigma < 1$, $\gamma = 1$)

Let us now analyze with more detail the case $\gamma = 1$. As remarked above, in this case $\chi \rho_o = \chi n_o M = 3H_o^2$, so that (16) and (24) reduce to

$$H_d = \frac{\chi \beta M}{3},$$

and

$$n_d = \frac{\chi \beta^2 M}{3}.$$  

The aforementioned results are presented, respectively, in Eqs. (20) and (4) of Refs.[8, 9]. Now, in order to relate (23) and (26) with the characteristic time scale of the de Sitter phase we insert (25) into (20) and integrate it with $\gamma = 1$ and $\Sigma < 1$ to obtain

$$R(t) = R_o[1 + \Sigma(e^{\frac{\beta M t}{2}} - 1)]^{2/3},$$

which is the same expression for the scale function presented in Refs.[8, 9]. Therefore, after a characteristic time $\tau_c = 2H_d^{-1}$, the universe reaches
the de Sitter phase expanding as

\[ R(t) \simeq R_o \left( \frac{n_o}{n_d} \right)^{1/2} e^{H_d t}, \quad (28) \]

where \( n_d \) and \( H_d \) were defined by (25) and (26). On the other hand, it follows from (23) that the minimal value of \( R \) for \( \gamma = 1 \) is given by

\[ R_{\text{min}} = R_o [1 - \Sigma]^{2/3}, \quad (29) \]

which depends on \( n_o \) through \( \Sigma \). Hence, if we consider \( R_o = 1 \) as in Refs. [8, 9], one has \( R_{\text{min}} < 1 \), with (29) making explicit the constraint \( \Sigma < 1 \) as noticed earlier. Of course, the natural choice is to take not \( R_o \) but \( R_{\text{min}} = 1 \).

However, the important step is to determine at what time the scale factor assumes its minimal value \( R_{\text{min}} \). From (27) we see that \( R = R_{\text{min}} \) only if \( t = -\infty \), which is coherent with our qualitative analysis. Note that for \( t = 0 \) (27) yields \( R = R_o \) and, (21), \( n = n_o \). In this context we remark that the authors of Refs. [8, 9] truncated arbitrarily the time coordinate at \( t = 0 \), and since the arbitrary value of \( R_o \) has been fixed equal to unity, they obtained \( R(0) = 1 \); in their words: “the universe emerges without singularity at \( t = 0 \), with a particle number density \( n_o \) describing the initial Minkowskian fluctuation” (see Ref. [8], pg.773). Indeed, for \( t = 0 \), in addition to \( R = R_o \) and \( n = n_o \), one obtains from (12) or equivalently from (16), (19) and (20) that the Hubble parameter itself is \( H = H_o \), making it clear that the spacetime is not Minkowski. It thus follows that at \( t = 0 \) the solution (27) describes an expanding FRW-type universe driven by the matter creation
process. Indeed, such a result is valid for all values of \( \gamma \) in the considered interval and \( \Sigma \neq 1 \). As a matter of fact, the extension of the time coordinate for \( t = -\infty \) may also be justified looking for the expression of the total number of particles given by (22). By taking \( \gamma = 1 \) in the later equation and replacing the expression of \( R \) given in (27) one obtains

\[
N(t) = N_o e^{3Mt},
\]

(30)

which is the same expression derived in Refs. [8, 9] using a different approach. Notice that the Minkowskian limit, \( N \to 0 \), is recovered only if \( t \to -\infty \) in accordance with our previous comments, while for \( t = 0 \) or equivalently \( R = R_o \) one has \( N = N_o \), as it should be. Despite the fact that this “initial fluctuation” \( n_o \) does not exist, it will be shown that there is a structural instability related with these types of models, which may be discussed using the dynamical systems technique (see section 4).

Summarizing, this class of spacetimes (\( \Sigma < 1 \)), which includes the dust case, emerges at \( t = -\infty \) as a true Minkowski spacetime, that is, \( \rho = n = H = 0 \) and \( R = R_{\text{min}} \), with a general expression given by (23). There is no initial fluctuation. Due to the matter creation process, the universe evolves smoothly from Minkowski to a de Sitter spacetime. This is an interesting example of spacetimes which are unbounded in time, that is, their evolution ranges the time interval \( (-\infty, \infty) \) and are also free of physical singularities. Naturally, the characteristic quantities of the de Sitter phase may, for certain special values of \( \gamma \), be independent of \( n_o \); a particular value of \( n \) arbitrarily
fixed at $t = 0$. This is not remarkable and happens just for $\gamma = 1$ as a consequence of the relation $\rho = nM$ (see Eqs. (25), (26) and the definition of $\tau_c$).

### 3.2 A New Model ($\Sigma > 1$, $\gamma = 1$)

Having in mind that the case $\Sigma = 1$ is trivial we now analyze the last possibility, that is, $\Sigma > 1$ or $\chi\beta^2 M < 3n_o$. In this case, as expected, Eq. (20) is not modified, but on the other hand there is no $R_{min}$, so that $R \to 0$ and $H \to \infty$, characterizing an initial FRW-type singularity. Taking into account such remarks in the integration process of (20), it is easily found, for $\gamma = 1$,

$$R(t) = R_o[(\Sigma - 1)(e^{\frac{3}{2}H_o t} - 1)]^{2/3}. \tag{31}$$

This expression is consistent with our earlier qualitative analysis. For times $t > \tau_c = \frac{2}{3}H_o^{-1}$, the model approaches the de Sitter regime. If $t \ll \tau_c$ then (31) yields

$$R(t) \approx R_o[(\Sigma - 1)(1 + \frac{3}{2}H_o t + \cdots - 1)]^{2/3}, \tag{32}$$

so that

$$R(t) \approx R_o[\frac{3}{2}H_o(\Sigma - 1)t]^{2/3}. \tag{33}$$

This was expected, since for large values of $\Sigma = H_o/H_d$, the equation above approaches $R \sim R_o(\frac{2}{3}H_o t)^{2/3}$, which is the standard form of the FRW dust model\cite{14}. 

}\end{verbatim}
4 General Case: Qualitative Analysis

The flat case have already showed striking richness, and a natural extension to curved spacetimes is already more than justified. However, since is rather difficult to analytically derive the complete set of solutions, we will employ a different approach provided by the qualitative analysis of dynamical systems \cite{16, 17}. The basic idea is to reduce the field equations to a bidimensional autonomous system and perform the qualitative analysis of all solutions. As we know, the first step is set up a convenient set of dynamic variables, which we choose as being the energy density $\rho$ and the Hubble parameter $H$. Using the definition $H = \dot{R}/R$ and the constraint (2) we rewrite (9) as

$$\dot{H} = \gamma \left( \frac{\psi}{3nH} - 1 \right) \chi \rho + \frac{1}{3} \left( \chi \rho - 3H^2 \right).$$

(34)

The second dynamical equation is just the energy conservation law (4). Inserting (7) and (8) it takes the following form

$$\dot{\rho} = 3\gamma \left( \frac{\psi}{3nH} - 1 \right) \rho H.$$

(35)

Since the ratio $\Psi/n$ is a function of the energy density or Hubble parameter (or some combination of them) our dynamical system is fully defined by Eqs. (34) and (35). In what follows we consider only a positive-definite $\gamma$. First of all, we remark that the flat solutions corresponding to $\chi \rho = 3H^2$ are reconstructed in the phase space irrespectively of the matter creation rate. In fact, taking the ratio of (34) to (35) we see that the parabola $\chi \rho = 3H^2$ corresponds to the solutions for the flat case. Therefore, such a curve is a
separatrix for the general solutions even in presence of matter creation. This is just a reminder that (2) constrains open and closed spacetimes in such a way that they cannot evolve into one another. In general, there are at least two singular points $\dot{H} = \dot{\rho} = 0$ in the phase plane:

\[
[H, \rho] = [0, 0] \quad \text{(36)}
\]

\[
[H, \rho] = [H_d, \rho_d] \quad \text{(37)}
\]

which correspond to Minkowski and de Sitter spacetimes. As will be shown ahead, the former plays a completely different role in this new context, while the later does not exist in the absence of matter creation.

A phase diagram (or portrait) is a plot of the solutions to the dynamical system in the plane of the variables, with a flow corresponding to the arrow of time. As we know, this kind of graph can be compactified by changing the variables in order to bring infinity to a boundary in the phase portrait of the transformed dynamical system. A particularly useful new set of variables may be defined as the conformal mapping\[18\]

\[
H = H_d \frac{r}{1 - r} \sin \phi \quad \text{(38)}
\]

and

\[
\chi \rho = 3H_d^2 \left( \frac{r}{1 - r} \cos \phi \right)^2 \quad \text{(39)}
\]

where $H_d$ is an arbitrary scale that will be set equal to the previously defined Hubble parameter of the de Sitter spacetime. The inverse transformation is
easily computed as

\[
    r = \left[ 1 + \frac{H_d}{(\chi \rho + H^2)^{\frac{3}{2}}} \right]^{-1},
\]

(40)

\[
    \phi = \arctan \frac{3H^2}{\chi \rho}.
\]

(41)

It is readily seen that infinite values of \(\rho\) and \(H\) are brought onto the circle of radius \(r = 1\). In terms of the new variables \(\{r, \phi\}\), we can recast the dynamical system (34)-(35) as

\[
    \dot{r} = rH_d \left\{ \sin \phi \cos 2\phi - \frac{3\gamma}{2} \cos \phi \sin 2\phi \left[ 1 - \frac{(1 - r)}{r} \frac{\psi}{3nH_d \sin \phi} \right] \right\}
\]

(42)

and

\[
    \dot{\phi} = \frac{r}{1 - r} H_d \cos \phi \cos 2\phi \left\{ 1 - \frac{3\gamma}{2} \left[ 1 - \frac{(1 - r)}{r} \frac{\psi}{3nH_d \sin \phi} \right] \right\}.
\]

(43)

The phase portrait of usual FRW models (\(\psi = 0\); see, e.g., Ref. [4]) is conformally transformed into the diagram of Fig.1, which have been shown for further comparison. The parabola \(\chi \rho = 3H^2\) in the original phase space is mapped onto the radiuses at \(\phi = \pm \pi/4\), and the Big Bang (Big Crunch) singularities, respectively \((\rho, H) = (\infty, \pm \infty)\), are tamed to the points \(r = 1\), \(\phi = \pm \pi/4\). The shape of solutions is kept intact near the origin of the transformed graph. As expected, for these simple FRW models we are able to find exact expressions to the integral curves shown in Fig.1, since we know
these systems to be integrable (see, e.g., Ref. [19]). By taking the ratio of (42) to (43) with $\psi = 0$ we immediately solve the equation $dr/d\phi$ to find the solution

$$r(\phi) = 1 - (1 - r_o)\left[|\cos \phi| |\cos 2\phi|^{-\frac{3\gamma}{2-3\gamma}}\right]^{\frac{2}{2-3\gamma}},$$

(44)

where $r_o$ and $r$ are constrained to the interval $[0, 1]$. Note that this formula is valid even for the unphysical region $\rho < 0$ ($\cos \phi < 0$) which is also depicted for completeness. This will later prove helpful in order to analyze the nature of equilibrium points such as the Minkowski spacetime at the origin $r = 0$ ($\rho = H = 0$), and the physical singularities at $r = 1$ and $\phi = \pm \pi/4$ (Big Bang and Big Crunch respectively).

At this point we should recall that although in flat spacetimes the energy density and Hubble parameter are indistinguishable, in curved spacetimes this is not so. In this way, the ansatz proposed in Refs. [8, 9], $\psi_{\text{flat}} = 3\beta H^2$, must be somewhat generalized to include the presence of curvature. In principle, if we mimic the bulk viscosity mechanism, a more reasonable phenomenological law for $\Psi$ should be a power-law dependence $\Psi = \eta \rho^\nu$, where $\eta$ is a dimensional constant and $0 \leq \nu \leq 1$ (see, for instance, Ref.[4]). In this case, regardless of the curvature parameter, the Prigogine et al. ansatz corresponds to $\nu = 1$ and $\eta = \beta \chi$. However, this kind of phenomenological law does not include the case $\Psi = 3\beta nH$ ($\beta$ constant), discussed in Ref.[15]. In this connection, we recall that the matter creation rate is a degree of freedom introduced in the theory through the balance equation for the particle.
number density. In this way, we assume here a rather general expression which include both cases, namely:

\[ \psi = 3\beta n \left( \frac{\chi \rho}{3} \right)^{\mu/2} H^{\nu+1}. \]  

(45)

where \( \beta \) is a constant.

As noted above, a general feature of models with matter creation is the presence of a de Sitter solution. It is easy to see how these solutions come up: consider the curve \( C(\rho, H) \) corresponding to the equality \( \psi(\rho, H) = 3nH \) which, from Eqs. (34)-(35) always crosses the parabola \( \chi \rho = 3H^2 \) (or, equivalently, the straight line \( \phi = \pi/4 \) of Fig.1). At that precise point \([H_d, \rho_d]\) (hereafter indicated as \( \mathcal{I} \)), it is readily seen that both \( \dot{\rho} \) and \( \dot{H} \) are zero, meaning an exponentially expanding universe with constant density, i.e., an inflating universe. The question of whether models evolve towards it or outwards from it is related to the nature of the singular point \( \mathcal{I} \). This is done by finding the Liapunov coefficients of the linearized dynamical system at that point \([16]\). For the matter creation rate of the form assumed above, after some simple algebra we find

\[ \lambda_1 = -2 \]

and

\[ \lambda_2 = \frac{3\gamma}{2}(\mu + \nu) \]

This means that the fixed point \( \mathcal{I} \) is either a saddle point, an attracting
Jordan node or a simple attracting node (attractor), if $\lambda_2 > 0$, $\lambda_2 = 0$ or $\lambda_2 < 0$ respectively. The absence of complex Liapunov coefficients for any $\gamma$ or $\beta$ guarantees no closed orbits around $I$, which is obvious given the constraint (3) and the discussion after Eq.(35). The reader may verify for himself that $\lambda_2$ is negative if we consider $\Psi = \beta \rho$, since in this case $\nu = -1$ and $\mu = 2(\gamma - 1)/\gamma \leq 1$. It thus follows that the inflating universe is an attractor for the class of models considered in Refs.[8, 9]. The phase portrait is shown in Fig.2. As explained, the point $I$ is an attractor, and all expanding solutions converge there. Since $H = 0$ ($H$ in Fig.2) is a line of singular points, expanding and contracting models are never in contact. Such an effect is probably due to the creation pressure $p_c$ which diverges at that line (see Eq.(7)). Actually, it is easy to see that all along line $H$ the rate $\dot{H}$ (see Eq.(34)) goes to infinity.

In this model, the expanding universe seems to have appeared either from a FRW-type singularity at $B$ ($r = 1, \phi = \pi/4$), or at any point in the singular line $H$. In other words, all expanding spacetimes evolved from a singularity of some kind. Even Minkowski (denoted $M$), which is a nonsingular saddle point in FRW models (see fig.1), is a multiple equilibrium singular point here. Note also that closed universes show no turning points, either expanding or contracting forever.

Contracting universes in this scheme can begin either from Minkowski or from the same singular line as the expanding ones, $H$. All solutions now evolve to a crunch, exactly as in the FRW case. In the region of negative
energy (left half of Fig.2), contracting universes emerge either from the singular line $\mathcal{H}$ or from $\mathcal{M}$, and finish with infinite negative energy at $(r = 1, \phi = -\pi)$. Expanding universes with negative energy all start from $\mathcal{H}$, and end up as flat empty universes at $\mathcal{M}$. In sum, if the universe described by this model started from a fluctuation with both $H > 0$ and $\rho > 0$, it would evolve towards the inflating universe at $\mathcal{I}$. On the other hand, if it started from a curvature fluctuation ($H \neq 0, \rho = 0$), it would certainly end up either in a crunch or in an empty Minkowski spacetime. The case of a matter fluctuation ($H = 0, \rho \neq 0$) is somewhat ill-defined, since $p_c$ diverges on the line $\mathcal{H}$.

5 Conclusion

In this paper, by allowing both pressure and curvature, we have generalized the flat pre-inflationary stage with matter creation proposed by Prigogine and coworkers. All expanding solutions evolve to the flat de Sitter spacetime in the infinite cosmic time of their evolution. Physically, open and closed models are always singular, while the flat solutions split into two distinct subclasses which depend on the matter creation rate. The beginning of the universe in this case may be either a Minkowskian vacuum or a FRW type singularity. The avoidance of a singularity depends on the strength of the matter creation process, which has been translated here as a condition on the dimensionless $\Sigma$ parameter. It thus appear that a certain degree of fine tuning is needed in order to escape from a physical singularity in these models.
6 Acknowledgments

It is a pleasure to thank R. Brandenberger, A. Sornborger and N. Lemos for valuable suggestions and a critical reading of the manuscript. JASL is particularly grateful for A. Germano for many discussions in the beginning of this work. One of us (JASL) is grateful for the hospitality of the Physics Department of Brown University. This work was partially supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq (Brazilian Research Agency), and by the US Department Of Energy under grant DE-FG02-91ER40688, Task A.

References

[1] L. Parker, Phys. Rev. Lett. 21, 562 (1968); Phys. Rev. 183, 1057 (1969); S. A. Fulling, L. Parker and B. L. Hu, Phys. Rev. 10, 3905, (1974); B. L. Hu and L. Parker, Phys. Rev. 17, 933, (1978); N. J. Papastamation and L. Parker, Phys. Rev. 19, 2283, (1979).

[2] Ya. B. Zel’dovich, JETP Lett. 12, 307 (1970).

[3] B. L. Hu, Phys. Lett. A90, 375 (1982); Adv. Astrop. 1, 23 (1983).

[4] V. A. Belinski and I. M. Khalatnikov, Sov. Phys. JETP 45, 1 (1977).

[5] M. S. Turner, Phys. Rev. D28, 1243 (1983).

[6] J. D. Barrow, Phys. Lett B180, 335 (1986); Nucl. Phys. B310, 743 (1988).
[7] J. V. Narlikar, “Nonstandard Cosmologies”, in Vth Brazilian School of Cosmology and Gravitation (ed. M. Novello), World Scientific (1987).

[8] I. Prigogine, J. Geheniau, E. Gunzig, P. Nardone, Gen. Rel. Grav. 21, 767 (1989).

[9] E. Gunzig, P. Nardone, Int. J. Theor. Phys. 28, 943 (1989); I. Prigogine, Int. J. Theor. Phys. 28, 927 (1989). See also I. Prigogine and I. Stengers, “Entre le Temps et l’Éternité”, Fayard, Paris (1988).

[10] J. A. S. Lima, M. O. Calvão, I. Waga, “Cosmology, Thermodynamics and Matter Creation”, in :Frontier Physics, Essays in Honor of Jayme Tiomno, Eds. S. Macdowell, N. Nussensweig and A. Tropper, World Scientific, Singapore (1991).

[11] M. O. Calvão, J. A. S. Lima, I. Waga, Phys. Lett. A162, 223 (1992).

[12] J. A. S. Lima, A. S. M. Germano, Phys. Lett. A170, 373 (1992).

[13] J. Gabriel, G. Le Denmat, Phys. Lett A200, 11 (1995).

[14] S. Weinberg, Gravitation and Cosmology, John Wiley Sons, New York, (1972).

[15] J. A. S. Lima, A. S. M. Germano and L. R. W. Abramo, “FRW Type Cosmologies with Adiabatic Matter Creation”, Phys. Rev. D53, 4287 (1996).
[16] A. A. Andronov, “Qualitative theory of second-order dynamic systems”, J. Wiley (1973).

[17] O. Bogoyavlensky, “Methods in the Qualitative Theory of Dynamical Systems in Astrophysics and Gas Dynamics”, Spring Verlag, Berlin (1985).

[18] H. P. de Oliveira, *J. Math. Phys.* 36 (6), 2988 (1995).

[19] M. J. D. Assad and J. A. S. Lima, *GRG* 20, 527 (1988).
Captions for Figures

Fig. 1

Fig. 1 - Phase diagram of the standard FRW models mapped onto a disk. Flat universes are represented by the straight lines at $\phi = \pm \pi/4$. The origin $\mathcal{M}$ is an attracting node. Point $\mathcal{B}$ corresponds to the Big Bang singularity, while $\mathcal{C}$ is the Big Crunch at the end of the evolution of closed universes.

Fig. 2

Fig. 2 - Phase diagram of a model with $\psi = \beta \chi \rho$. Again flat universes corresponds to straight lines at $\phi = \pm \pi/4$. However, the point $\mathcal{I}$ is an attractor for all expanding solutions. $\mathcal{H}$ (corresponding to $H = 0$, or $\phi = 0$) is a line of singular points and Minkowski ($\mathcal{M}$) now is a Saddle point.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9606064v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9606064v1