Idempotent Fourier multipliers acting contractively on $H^p$ spaces

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Our problem

- Goal: Describe idempotent Fourier multipliers that act contractively on $H^p(\mathbb{T}^d)$ for $1 \leq p \leq \infty$, $p \neq 2$, and $d \geq 1$.

First an easier problem:

- Describe idempotent Fourier multipliers that act contractively on $L^p(\mathbb{T}^d)$.

Some notation and terminology: We represent functions $f$ in $L^p(\mathbb{T}^d)$ by their Fourier series $f(z) \sim \sum_{\alpha \in \mathbb{Z}^d} \hat{f}(\alpha) z^\alpha$, where

$$
\hat{f}(\alpha) := \int_{\mathbb{T}^d} f(z) \overline{z^\alpha} \, dm_d(z)
$$

and $m_d$ denotes the Haar measure of the $d$-dimensional torus $\mathbb{T}^d$. For $\Lambda$ a non-empty subset of $\mathbb{Z}^d$,

$$
P_\Lambda f(z) := \sum_{\alpha \in \Lambda} \hat{f}(\alpha) z^\alpha.
$$

We say that $\Lambda$ is a **contractive projection set for** $L^p(\mathbb{T}^d)$ when $P_\Lambda$ extends to a contraction on $L^p(\mathbb{T}^d)$. A subset $\Lambda$ of $\mathbb{Z}^d$ is a **coset** in $\mathbb{Z}^d$ if $\Lambda$ is equal to the coset of a subgroup of $(\mathbb{Z}^d, +)$.
Theorem (After Andô (1965) and Rudin (1962))

Let $d$ be a non-negative integer and fix $1 \leq p \leq \infty$, $p \neq 2$. A subset $\Lambda$ of $\mathbb{Z}^d$ is a contractive projection set for $L^p(\mathbb{T}^d)$ if and only if $\Lambda$ is a coset in $\mathbb{Z}^d$. 
The proof of the $L^p$ theorem—necessity

**Lemma (Linear reflection)**

Fix $1 \leq p \leq \infty$, $p \neq 2$, and set $c_p := 2/p - 1$. Then

$$\|c_p \varepsilon \overline{z} + 1 + \varepsilon z\|_p < \|1 + \varepsilon z\|_p$$

for every sufficiently small $\varepsilon > 0$.

**Lemma (Triangular reflection)**

Fix $1 \leq p < \infty$, $p \neq 2$, and set $c_p := 1 - p/2$.

$$\|1 + \varepsilon (z_1 + z_2) + c_p \varepsilon^2 z_1 z_2\|_p < \|1 + \varepsilon (z_1 + z_2)\|_p$$

for every sufficiently small $\varepsilon > 0$.

Think of the $z$ above as $z^{\alpha - \beta}$ with $\alpha, \beta$ in $\Gamma$. Then all linear and triangular reflections constitute the 1-extension of $\Gamma$. 
Figure: The points $\lambda$ obtained by linear and triangular reflection starting from the set $\Gamma = \{(3, 0, 0), (0, 3, 0), (1, 1, 1)\}$, in the plane $z = 3 - x - y$. The shaded triangle represents the intersection of this plane and the narrow cone.
Frequently encountered examples

- F. Wiener’s inequality, appearing already in classical work of Bohr’s (1914): The case $d = 1$ of the above theorem.

$$P f(z) = \sum_{k \in \mathbb{Z}} \hat{f}(kn) z^{kn} = \frac{1}{n} \sum_{j=0}^{n-1} f(zw^j)$$

where $w$ is primitive $n$’th root of unity.

- The restriction to the $m$-homogeneous terms of a power series in $d$ variables.

$$P f(z) = \int_{\mathbb{T}} f(z_1 \zeta, z_2 \zeta, \ldots, z_d \zeta) \zeta^m \, dm_1(\zeta)$$
Contractive projection sets for $H^p(\mathbb{T}^d)$

- $H^p(\mathbb{T}^d)$ is the subspace of $L^p(\mathbb{T}^d)$ comprised of functions $f$ with $\hat{f}(\alpha) = 0$ for every $\alpha$ in $\mathbb{Z}^d \setminus \mathbb{N}_0^d$, where $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

- A subset $\Lambda$ of $\mathbb{N}_0^d$ is a **contractive projection set for $H^p(\mathbb{T}^d)$** if $P_\Lambda$ extends to a contraction on $H^p(\mathbb{T}^d)$.

If $\Lambda$ is a coset in $\mathbb{Z}^d$, then $\Lambda \cap \mathbb{N}_0^d$ is a contractive projection set for $H^p(\mathbb{T}^d)$.

**Question:** Are there other contractive projection sets for $H^p(\mathbb{T}^d)$?

The dimension of the affine span of $\Lambda$, $\dim(\Lambda)$ plays a nontrivial role in this problem.

**Definition**

Suppose that $1 \leq k \leq d$. We say that $H^p(\mathbb{T}^d)$ enjoys the **contractive restriction property of dimension $k$** if every $k$-dimensional contractive projection set for $H^p(\mathbb{T}^d)$ is of the form $\Lambda \cap \mathbb{N}_0^d$ with $\Lambda$ a coset in $\mathbb{Z}^d$. 
Main theorem (from low to high dimensions)

**Theorem**

Suppose that $1 \leq p \leq \infty$.

(a) If $d = 2$ or $k = 1$, then $H^p(\mathbb{T}^d)$ enjoys the contractive restriction property of dimension $k$ if and only if $p \neq 2$.

(b) If either $d = k = 3$ or $d \geq 3$ and $k = 2$, then $H^p(\mathbb{T}^d)$ enjoys the contractive restriction property of dimension $k$ if and only if $p \neq 2, 4$.

(c) If $d \geq 4$ and $k \geq 3$, then $H^p(\mathbb{T}^d)$ enjoys the contractive restriction property of dimension $k$ if and only if $p$ is not an even integer.

The hardest part of the theorem is item (b) which can be thought of as representing the two cases of intermediate dimension, namely $d = k = 3$ and $d \geq 3, k = 2$. These two cases require completely different methods ...
The geometry of the case $p = 2n$

Let $\Gamma$ be a non-empty subset of $\mathbb{N}_0^d$ and suppose that $\lambda$ is in $\Lambda(\Gamma)$, which is the coset generated by $\Gamma$. The distance from $\Gamma$ to $\lambda$ is

$$d(\Gamma, \lambda) := \inf \max \left( \sum_{m_{\gamma,\alpha} > 0} m_{\gamma,\alpha}, - \sum_{m_{\gamma,\alpha} < 0} m_{\gamma,\alpha} \right)$$

where the infimum is taken over all possible representations

$$\lambda = \gamma + \sum_{\substack{\alpha \in \Gamma \\ \alpha \neq \gamma}} m_{\gamma,\alpha} (\alpha - \gamma), \quad \gamma \in \Gamma.$$ 

For a non-negative integer $n$, the $n$-extension of $\Gamma$ is

$$E_n(\Gamma) := \left\{ \lambda \in \Lambda(\Gamma) \cap \mathbb{N}_0^d : d(\Gamma, \lambda) \leq n \right\}.$$
Let $n \in \mathbb{N}$. A set $\Gamma$ in $\mathbb{N}_0^d$ is a contractive projection set for $H^{2(n+1)}(\mathbb{T}^d)$ if and only if $E_n(\Gamma) = \Gamma$. 

**Theorem**
Example: $E_1(\Gamma) = \Gamma$ and $E_2(\Gamma) \neq \Gamma$

Figure: Points $\lambda$ which satisfy $d(\Gamma, \lambda) = 1$ and $d(\Gamma, \lambda) = 2$ for
$\Gamma = \{(3, 0, 0), (0, 3, 0), (1, 1, 1)\}$, in the plane $z = 3 - x - y$. The shaded triangle
represents the intersection of this plane and the narrow cone $\mathbb{N}_0^d$. 
Duality formulation for $1 \leq p < \infty$

Lemma

Fix $1 \leq p < \infty$ and $d \geq 1$. A set of frequencies $\Gamma$ in $\mathbb{N}^d_0$ is a contractive projection set for $H^p(\mathbb{T}^d)$ if and only if

$$\int_{\mathbb{T}^d} |f(z)|^{p-2} f(z) \overline{z^\lambda} \, dm_d(z) = 0$$

for every $f$ in $H^p(\mathbb{T}^d)$ of the form $f(z) = \sum_{\gamma \in \Gamma} a_\gamma z^\gamma$ and every $\lambda$ in $(\Lambda(\Gamma) \cap \mathbb{N}^d_0) \setminus \Gamma$.

- The fact that $\Gamma$ in $\mathbb{N}^d_0$ is a contractive projection set for $H^{2(n+1)}(\mathbb{T}^d)$ if and only if $E_n(\Gamma) = \Gamma$ is a geometric reformulation of this result.
- The case $1 \leq p < \infty$, $p \neq 2n$, follows almost immediately (next slide).
The case $1 \leq p < \infty$, $p$ not even

If $\Gamma$ is not the restriction of a coset in $\mathbb{Z}^d$ to $\mathbb{N}_0^d$, then there is some $\lambda$ in $(\Lambda(\Gamma) \cap \mathbb{N}_0^d) \setminus \Gamma$. There is an affinely independent subset $\{\gamma_0, \gamma_1, \ldots, \gamma_n\}$ of $\Gamma$ which generates $\Lambda(\Gamma)$, where $n = \dim(\Lambda(\Gamma))$. Hence

$$\lambda = \gamma_0 + \sum_{j=1}^{n} m_j (\gamma_j - \gamma_0).$$

Set

$$f(z) := z^\gamma_0 + \varepsilon \sum_{j=1}^{n} z^\gamma_j$$

for $0 < \varepsilon < 1/n$. We use the binomial series to express

$$\int_{\mathbb{T}^d} |f(z)|^{p-2} f(z) \overline{z^\lambda} \, dm_d(z)$$

as a non-trivial power series in $\varepsilon$, which is in conflict with the preceding lemma.
Key lemmas for the sufficiency part, $d \geq 3$ and $k = 2$

**Lemma (Main lemma)**

Fix $d \geq 3$ and let $T$ be a set in $\mathbb{N}_0^d$ with $\dim(T) = 2$. Then the 2-completion of $T$ is $\Lambda(T) \cap \mathbb{N}_0^d$.

To prove this result, we start from the special case of three points:

**Lemma**

Let $T$ be a set of three affinely independent points in $\mathbb{N}_0^d$ for $d \geq 3$. Then the 2-completion of $T$ is $\Lambda(T) \cap \mathbb{N}_0^d$.

The proofs are quite arithmetic. We prove the main lemma by a kind of Euclidean algorithm, starting from the three point lemma.
The case $d = k$

The extension problem is of a rather different nature when $d = k$. Indeed, our job is then mainly to reach $\infty$ inside the narrow cone. This is reflected in the following basic result.

**Lemma**

Let $T$ be a subset of $\mathbb{N}_0^d$. If there are points $\alpha$ and $\beta$ in $E^\infty_n(T)$ such that $\beta - \alpha$ is in $\mathbb{N}^d$, then

$$E^\infty_n(T) = E^\infty_1(T \cup \{\alpha, \beta\}) = \Lambda(T) \cap \mathbb{N}_0^d.$$ 

Here

$$E^\infty_n(T) := \bigcup_{k=1}^{\infty} E^k_n(T),$$

i.e., $E^\infty_n(T)$ is the smallest subset $\Gamma$ of $\mathbb{N}_0^d$ such that $T \subset \Gamma$ and $E_n(\Gamma) = \Gamma$. 

Key lemmas for the sufficiency part, $d = k = 2$ and $d = k = 3$

In view of the preceding lemma, all we need are the following two results.

**Lemma**

Let $T$ be a set of three affinely independent points in $\mathbb{N}^2_0$. Then for every $\alpha$ in $T$ there exists a point $\beta$ in $E_1^\infty(T) \setminus \{\alpha\}$ such that $\beta - \alpha$ is in $\mathbb{N}^2$.

**Lemma**

Let $T$ be a set of four affinely independent points in $\mathbb{N}^3_0$. Then for every $\alpha$ in $T$ there exists a point $\beta$ in $E_2^\infty(T) \setminus \{\alpha\}$ such that $\beta - \alpha$ is in $\mathbb{N}^3$.

The proof of both lemmas rely on a simple idea (next slide), but the proof of the latter is quite hard and requires a somewhat involved combinatorial argument.
Increasing iteratively the “negativity index”

Definition (Negativity index)

Given a set $U$ of $d$ linearly independent vectors $u = (u_1, \ldots, u_d)$ in $\mathbb{Z}^d$, we define the negativity index of $U$ as

$$\text{ind}(U) := \sum_{j=1}^{d} \min \left( 0, \min_{u \in U} u_j \right).$$

Proof idea: Successively change $U$ by making 1- or 2-extensions of $\alpha + U$ to get to new vectors with a larger negativity index. For this to work, it is crucial that linear independence of the vectors of $U$ be preserved during the course of the iteration!
The necessity part of our main theorem are proved by finding suitable example sets.

- The example $\Gamma := \{(3, 0, 0), (0, 3, 0), (1, 1, 1)\}$ settles the necessity of the condition for $d \geq 3$ and $k = 2$.

- The example $\Gamma := \{(4, 0, 0), (0, 4, 0), (0, 0, 4), (1, 1, 1)\}$ settles the necessity of the condition for $d = k = 3$. 
The example \((n \geq 3)\)

\[
(n, 1, 0, 1), (n + 1, 0, 1, 0), (0, 0, n + 1, 0), (0, 0, 0, n + 1), (0, n + 1, 0, 0)
\]

settles the necessity of the condition for \(d \geq 4\) and \(4 \leq k \leq d\) in part (c). (The set equals its \(n\)-extension.)

The \(n + 1\)-extension of the set is the full coset intersected with the narrow cone.
Using the above example sets, it is easy to cook up an explicit linear operator to arrive at the following result.

**Theorem**

*Fix an integer $n \geq 1$. There is a linear operator $T_n$ that is densely defined on $H^p(\mathbb{T}^\infty)$ for every $1 \leq p \leq \infty$, and that extends to a bounded operator on $H^p(\mathbb{T}^\infty)$ if and only if $p = 2, 4, \ldots, 2(n + 1)$.*

This result exemplifies quite strikingly the impossibility of interpolating between Hardy spaces on the infinite-dimensional torus, as studied in depth in a recent paper of Bayart and Mastylo (2019).
Is there an even more exotic linear operator on $H^p(\mathbb{T}^\infty)$?

**Question**

Is there a linear operator $T_\infty$ that is densely defined on $H^p(\mathbb{T}^\infty)$ for every $1 \leq p \leq \infty$, and that extends to a bounded operator on $H^p(\mathbb{T}^\infty)$ if and only if $p = 2n$, $n = 1, 2, \ldots$?

This question is related to an old problem in the theory of Hardy spaces of Dirichlet series: For which $p$ is there an absolute constant $C_p$ such that

$$
\int_0^1 |F(1/2 + it)|^p dt \leq C_p \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T |F(it)|^p dt
$$

for all Dirichlet polynomials $F(s) = a_1 + a_2 2^{-s} \cdots a_n N^{-s}$? This is true when $p = 2n$ (easy) and is known to fail when $0 < p < 2$ by a recent theorem of Harper (2020) (a deep result).

Failure of (1) for $p \neq 2n$, $p > 2$, would, via the Bohr lift, yield such an exotic $T_\infty$. 