Spatial frequency of unstable eigenfunction of the core-periphery model incorporating differentiated agriculture with transport cost

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Abstract

We extend the core-periphery model with differentiated agriculture with transport cost to those on discrete and continuous multi-regional space and investigate the stability of homogeneous stationary solution, especially for the model on a continuous periodic space. Unstable eigenfunctions can be stable when manufacturing transport cost is sufficiently high, but become unstable when the cost is lower than a certain critical point. As the transport cost decreases further below another critical point, the eigenfunction becomes stable again. It can be observed numerically that the unstable area, which is the range of the transport cost between those two critical points, generally tends to expand with the frequency increases limited to even or odd numbers only.

Keywords new economic geography · continuous racetrack economy · self-organization · spatial patterns · number of cities

JEL classification: R12, R40, C63, C68

1 Introduction

The core-periphery model proposed by [Krugman (1991)] provides a basic framework for modeling economic agglomeration based on the general equilibrium theory. In its most basic form, symmetric equilibrium, where all regions are uniformly populated, is stable when transport cost is sufficiently high, while it becomes unstable when the transport cost is sufficiently low, producing a spatially inhomogeneous structure.

In the most basic formulation of the model, agricultural goods are assumed to be undifferentiated and, in addition, their transport is assumed to

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have no cost. Meanwhile, Fujita et al. (2001, Chapter 7) introduce a model incorporating differentiated agriculture and its transport cost. This model has an interesting property that symmetric equilibrium stabilizes not only when the manufacturing transport cost is sufficiently high but also when the manufacturing transport cost is sufficiently low.

We extend the core-periphery model with differentiated agriculture and its transport cost to finite or infinite multi-regional spaces and consider the stability of stationary solution, especially in the case of one-dimensional periodic continuous space. In the continuous space model, disturbances to the stationary state are decomposed into eigenfunctions, each with its own spatial frequency. The value of the spatial frequency corresponds to the number of cities or economic centers formed when workers and capitals begin to agglomerate.

Depending on the value of various parameters, there would be eigenfunctions which are stable over the entire range of manufacturing transport cost (stable eigenfunctions). On the other hand, there are eigenfunctions that become unstable at certain manufacturing transport cost values (unstable eigenfunctions). We see that unstable eigenfunctions can be stable when manufacturing transport cost is sufficiently high, but become unstable when the cost is lower than a certain threshold value of the transport cost (upper critical point). As the transport cost decreases further below another threshold (lower critical point), the eigenfunction becomes stable again. The behavior of these unstable eigenfunctions is similar to the solution of a linear problem around a homogeneous stationary solution of the original two-regional model by Fujita et al. (2001, Chapter 7).

We obtain expression for the eigenvalue corresponding to the eigenfunction of each frequency and investigate its frequency-dependent behavior. From the numerical observations, it is suggested that as the spatial frequency number of the eigenfunction increases within the even-only or odd-only range, the upper critical point would be non-decreasing (generally increasing) and the lower critical point would be non-increasing (generally decreasing), resulting in expansion of the unstable area where the eigenvalue takes positive value. Meanwhile, it is often observed that the value of lower critical point become larger with increasing frequency that are not limited to even or odd numbers. Nevertheless, if the frequency increases sufficiently, the unstable area generally expands.

1Picard and Zeng (2005) consider a quasi-linear model that takes into account differentiated agriculture and its transport cost. In addition, Zeng and Takatsuka (2016) and Zeng (2021) even treat the case of a footloose entrepreneur model. In the context of international trade theory, models including agriculture have also been devised. For example, Takatsuka and Zeng (2012) consider the transport cost of the homogeneous agricultural sector, and Zeng and Kikuchi (2009) deal with differentiated agriculture with its transport cost.

2Therefore, the symmetric equilibrium becomes unstable when the transport cost is moderate.
The rest of the paper is organized as follows. Section 2 derives the model. Section 3 discusses the homogeneous stationary solution and the eigenvalue of the linearized problem around the stationary solution. Section 4 examines how the eigenvalues depend on various parameters, mainly frequency. Section 5 discusses the results and gives conclusions. Section 6 gives proofs of several propositions omitted in the main part.

2 The model

This section provides an overview of Dixit-Stiglitz framework (Dixit and Stiglitz (1977)) in the context of the differentiated agriculture with transport cost and applies it to derive the multi-regional version of the model which we handle.

2.1 Dixit-Stiglitz framework

The economy is supposed to have a continuously infinite variety of manufactured goods and agricultural goods. Let \( m(i) \geq 0 \) and \( a(j) \geq 0 \) denote demands for the variety \( i \in [0, V_m] \) of the manufactured goods and the variety \( j \in [0, V_a] \) of the agricultural goods. Price of \( i \)-th variety of the manufactured good is denoted by \( p_M(i) \) and that of \( j \)-th variety of agricultural good is denoted by \( p_A(j) \). Given an income \( Y \), and the prices \( p_M(i), p_A(j) \), each consumer faces the following budget constraint.

\[
\int_0^{V_M} p_M(i)m(i)\,di + \int_0^{V_A} p_A(i)a(i)\,di = Y.
\]  

(1)

Facing (1), the consumer is supposed to maximize the following utility function.

\[
M^\mu A^{1-\mu}, \mu \in (0, 1),
\]  

(2)

where

\[
M = \left[ \int_0^{V_M} m(i)^{\frac{\sigma-1}{\sigma}} \,di \right]^{\frac{\sigma}{\sigma-1}}
\]  

(3)

\[
A = \left[ \int_0^{V_A} a(i)^{\frac{\eta-1}{\eta}} \,di \right]^{\frac{\eta}{\eta-1}}.
\]  

(4)

Here, parameters \( \eta > 1 \) and \( \sigma > 1 \) stand for the elasticity of substitution between any two varieties of agricultural goods and that of manufactured goods, respectively. Following Fujita et al. (2001, Chapter 4), we solve this problem in two steps. First, the minimum cost to achieve the partial utility \( M \) and \( A \) is calculated, and then the optimal expenditure on \( M \) and \( A \) under the budget constraint is determined.

\[^3\]For details of the two-stage budgeting, see Deaton and Muellbauer (1980, Chapter 5) or Green (1964, Chapters 2-4)
We obtain a function of consumer demand for the \( j \)-th variety of the agricultural goods as

\[
a(j) = (1 - \mu)Y p_A^j - \eta G^A)^{\eta-1},
\]

where \( Y \) is the income of the consumer, and \( G^A \) is the price index of the agricultural goods defined by

\[
G^A = \left[ \int_0^{V_A} p_A(i)^{1-\eta} \, di \right]^\frac{1}{1-\eta}.
\]

\[ (5) \]

\[ (6) \]

2.2 Multi-regional modeling (Discrete space)

Based on the framework described above, we model a two-sector spatial economy consisting of multiple regions. The modeling of manufacturing sector has already been described in detail in Fujita et al. (2001, Chapters 5 and 6), so we only mention it briefly. On the other hand, the agricultural sector is discussed in more detail here, since Fujita et al. (2001, Chapter 7) only mention the two-regional case.

In this subsection, the economy consists of \( R \) discretely located regions indexed by \( r \) or \( s = 1, 2, \cdots, R \). Following Fujita et al. (2001, Chapter 7), the total population of the mobile and immobile workers are normalized to be \( \mu \) and \( 1 - \mu \), respectively. Then, the mobile and immobile population in region \( r \) is denoted by \( \mu \lambda_r \) and \( (1 - \mu) \phi_r \), respectively. By these, it is easy to see that \( \sum_{r=1}^{R} \lambda_r \equiv 1 \) and \( \sum_{r=1}^{R} \phi_r \equiv 1 \).

Let \( w_M^r \) and \( w_A^r \) denote the nominal wage of the manufacturing and agricultural workers, respectively. Then, the total income of region \( r \) is given by

\[
Y_r = \mu w_M^r \lambda_r + (1 - \mu) w_A^r \phi_r.
\]

\[ (7) \]

The prices of any variety of agricultural goods produced in region \( r \) have equal value \( p_A^r \). Assume that the transportation of the agricultural goods incurs the iceberg transport cost. That is, to transport one unit of agricultural goods from \( r \) to \( s \), \( T^A_{rs} \geq 1 \) times as much agricultural goods must be shipped. Then, the price in region \( s \) of a variety of agricultural good produced in region \( r \) is \( p_{rs}^A = p_A^r T^A_{rs} \).

The number of varieties of agricultural goods produced in region \( r \) denoted by \( n_A^r \) is assumed to be proportional to the agricultural population in region \( r \) as

\[
n_A^r = \phi_r, \quad r = 1, 2, \cdots, R.
\]

\[ (8) \]

This implies that the total number of varieties of agricultural goods in the economy is normalized to 1.

\[ ^4 \text{Of course, similar demand function and price index for the manufactured goods are also derived, but these are already detailed in Fujita et al. (2001, Chapters 4-5) so is not discussed here.} \]
Then from (6) and (8), the price index of the agricultural goods in \( r \) is given by

\[
G^A_r = \left[ \sum_{s=0}^{R-1} \phi_s p^A_{sr}^{1-\eta} \right]^{\frac{1}{1-\eta}}.
\]

By assuming \( p^A_r = w^A_r \), \( (\forall r) \), we obtain the following price index equation

\[
G^A_r = \left[ \sum_{s=1}^{R} \phi_s w^A_s^{1-\eta} T^A_{rs}^{1-\eta} \right]^{\frac{1}{1-\eta}}. \tag{9}
\]

Let us derive the equation for the nominal wage of the agricultural workers. The demand for the \( j \)-th variety of agricultural goods by a consumer with income \( Y \) is given by (5). Thus, the demand from region \( s \) for one variety of agricultural good produced in region \( r \) can be given as

\[
(1 - \mu)Y_s p^A_r^{1-\eta} T^A_{rs}^{1-\eta} G^A_s^{1-\eta}. \tag{10}
\]

We obtain the sales of an agricultural producer in region \( r \) as the following by adding up \( T^A_{rs} \) times the amount in (10) over all regions.

\[
q^A_r = (1 - \mu) p^A_r^{1-\eta} \sum_{s=0}^{N-1} Y_s T^A_{rs}^{1-\eta} G^A_s^{1-\eta}. \tag{11}
\]

Assume that the total supply of agricultural goods in region \( r \) is equal to the population of the agricultural workers there given by \((1 - \mu) \phi_r \). Since the number of varieties of the agricultural goods produced there is \( n^A_r = \phi_r \) according to (8), the supply of one variety of the agricultural goods produced there is given by \((1 - \mu) \phi_r / \phi_r = 1 - \mu \). Thus, the condition that the supply \( 1 - \mu \) and demand (11) for each variety of the agricultural goods are equal, together with the assumption that \( p^A_r = w^A_r \), yields

\[
w^A_r = \left[ \sum_{s=0}^{N-1} Y_s G^A_s^{\sigma-1} T^A_{rs}^{1-\eta} \right]^{\frac{1}{\sigma}}. \tag{12}
\]

For the price index \( G^M_r \) and nominal wage \( w^M_r \) in the manufacturing sector, exactly the same equations are derived as in the one-sector racetrack model (See (5.4) and (5.5) of Fujita et al. (2001, p.63)). That is

\[
G^M_r = \left[ \sum_{s=0}^{N-1} \lambda_s w^M_s^{1-\sigma} e^{-\tau_M (\sigma-1) |r-s|} \right]^{\frac{1}{1-\sigma}} \tag{13}
\]

and

\[
w^M_r = \left[ \sum_{s=0}^{N-1} Y_s G^M_s^{\sigma-1} e^{-\tau_M (\sigma-1) |r-s|} \right]^{\frac{1}{\sigma}}. \tag{14}
\]

5
As mentioned in Fujita et al. (2001, Chapter 7), the real wage of the mobile workers is
\[ \omega_r = w_r^M G_r^M - \mu G_r^\mu - 1. \]  
(15)

We adopt the following ad-hoc dynamics in which the mobile workers flow into regions with above-average real wage, while they flow out of region with below-average real wage. The average real wage is defined by \( \sum_{s=1}^{R} \omega_s \lambda_s \) and the dynamics is described by
\[ \frac{\partial}{\partial t} \lambda_r = \gamma \left[ \omega_r - \sum_{s=1}^{R} \omega_s \lambda_s \right] \lambda_r, \]  
(16)

where \( \gamma > 0 \) stands for adjustment speed.

The equations (7), (9), (12), (13), (14), (15), and (16) constitute the following system of nonlinear algebraic differential equations.

\[
\begin{aligned}
Y_r(t) & = \mu w_r^M(t) \lambda_r(t) + (1 - \mu) w_r^A(t) \phi_r, \\
G_r^A(t) & = \left[ \sum_{s=1}^{R} \phi_s w_s^A(t)^{1 - \eta T_s A^{1 - \eta}} \right]^{\frac{1}{\eta}}, \\
w_r^A(t) & = \left[ \sum_{s=1}^{R} Y_s(t) G_s^A(t)^{\eta - 1} T_s A^{1 - \eta} \right]^{\frac{1}{\eta}}, \\
G_r^M(t) & = \left[ \sum_{s=1}^{R} \lambda_s(t) w_s^M(t)^{1 - \sigma T_s M^{1 - \sigma}} \right]^{\frac{1}{1 - \sigma}}, \\
w_r^M(t) & = \left[ \sum_{s=1}^{R} Y_s(t) G_s^M(t)^{\sigma - 1} T_s M^{1 - \sigma} \right]^{\frac{1}{\sigma}}, \\
\omega_r(t) & = w_r^M(t) G_r^M(t)^{- \mu} G_r^A(t)^{\mu - 1}, \\
\frac{\partial}{\partial t} \lambda_r(t) & = \gamma \left[ \omega_r(t) - \sum_{s=1}^{R} \omega_s(t) \lambda_s(t) \right] \lambda_r(t)
\end{aligned}
\]  
(17)

with an initial condition \( \lambda_r(0) = \lambda_{0,r} \geq 0, \) \( (r = 1, 2, \cdots, R) \) which satisfies \( \sum_{r=1}^{R} \lambda_{0,r} = 1. \)

2.3 Multi-regional modeling (Continuous space)

Let \( \mathcal{N} \) be a \( n(\geq 1) \)-dimensional manifold in a continuous space. It is easy to extend the system (17) to a system on \( \mathcal{N} \). The mobile workers are continuously distributed on \( \mathcal{N} \) and the density at time \( t \in [0, \infty) \), region \( x \in \mathcal{N} \) is denoted by \( \mu \lambda(t, x) \geq 0 \) which satisfies \( \int_{\mathcal{N}} \lambda(t, x) dx = 1 \) for any \( t \). The immobile workers are also distributed on \( \mathcal{N} \) and the density at region \( x \in \mathcal{N} \)
is denoted by \((1 - \mu)\phi(x) \geq 0\) which satisfies \(\int_{N'} \phi(x) dx = 1\). Then, the system \([17]\) naturally becomes the following continuous system\(^5\):

\[
\begin{aligned}
Y(t, x) &= \mu w^M(x) \lambda(t, x) + (1 - \mu) w^A(t, x) \phi(x), \\
G^A(t, x) &= \left[ \int_{N} \phi(y) w^A(t, y) \frac{1}{1-\eta} T^A(x, y)^{1-\eta} dy \right]^{\frac{1}{1-\eta}}, \\
w^A(t, x) &= \left[ \int_{N} Y(t, y) G^A(t, y) \frac{1}{\sigma} T^A(x, y)^{\sigma-1} dy \right]^{\frac{1}{\sigma}}, \\
G^M(t, x) &= \left[ \int_{N} \lambda(t, y) w^M(t, y) \frac{1}{1-\sigma} T^M(x, y)^{1-\sigma} dy \right]^{\frac{1}{1-\sigma}}, \\
w^M(t, x) &= \left[ \int_{N} Y(t, y) G^M(t, y) \frac{1}{\eta} T^M(x, y)^{\eta-1} dy \right]^{\frac{1}{\eta}}, \\
\frac{\partial}{\partial t} \lambda(t, x) &= \gamma \left[ \omega(t, x) - \int_{N} \omega(t, y) \lambda(t, y) \right] \lambda(t, x)
\end{aligned}
\]

with an initial condition \(\lambda(0, x) = \lambda_0(x) \geq 0\), \((\forall x \in N)\) which satisfies \(\int_{N'} \lambda_0(x) dx = 1\).

In the following, we consider in particular a circumference \(S\) of radius \(r > 0\) as a manifold \(N\) and discuss the system \([18]\) on \(S\). The functions \(T^A(x, y) \geq 1\) and \(T^M(x, y) \geq 1\) represent the amount of the agricultural and manufacturing goods that must be shipped to transport one unit of the agricultural and manufacturing goods, respectively, from \(x\) to \(y\). They are also restricted to the forms

\[
\begin{aligned}
T^A(x, y) &= e^{\tau_A |x-y|}, \\
T^M(x, y) &= e^{\tau_M |x-y|},
\end{aligned}
\]

respectively, where \(\tau_A \geq 0\) and \(\tau_M \geq 0\) are cost parameters. Here, the distance function \(|x - y|\) is defined as the shorter distance between \(x\) and \(y\) along the circumference \(S\). Then, the parameters \(\tau_A\), \(\tau_M\), \(\eta\), and \(\sigma\) often appear in the form \(\tau_A(\eta - 1)\) and \(\tau_M(\sigma - 1)\), so it is convenient to define

\[
\begin{aligned}
\alpha &= \tau_A(\eta - 1) \geq 0, \\
\beta &= \tau_M(\sigma - 1) \geq 0.
\end{aligned}
\]

In the following, we identify a function on \(S\) with a corresponding periodic function on the interval \([-\pi, \pi]\), that is, \(Y(t, x), G^A(t, x), w^A(t, x), G^M(t, x), w^M(t, x), \omega(t, x), \) and \(\lambda(t, x)\) of \((t, x) \in [0, \infty) \times S\) are identified with the corresponding functions which are periodic with respect to spatial

\(^5\)To be precise, \(Y(t, x)\) now stands for the “density” of the total income at time \(t\) in region \(x\), not the total income itself.
variables $Y(t, \theta)$, $G^A(t, \theta)$, $w^A(t, \theta)$, $G^M(t, \theta)$, $w^M(t, \theta)$, $\omega(t, \theta)$, and $\lambda(t, \theta)$ of $(t, \theta) \in [0, \infty) \times [-\pi, \pi]$, respectively.

In fact, this identification is possible in the following way. In general, an element $x \in S$ corresponds one-to-one to an element $\theta \in [-\pi, \pi]$, i.e.,

$$x = x(\theta),$$

so for any function $f$ on $S$, there exists a corresponding periodic function $\tilde{f}$ on $[-\pi, \pi]$ such that

$$f(x) = f(x(\theta)) = \tilde{f}(\theta).$$

Then, integral of $f$ on $S$ is calculated by

$$\int_S f(x)dx = \int_{-\pi}^{\pi} \tilde{f}(\theta)r d\theta.$$

The distance between $x = x(\theta)$ and $y = y(\theta')$ is calculated by

$$|x - y| = r \times \min \left\{|\theta - \theta'|_{abs}, 2\pi - |\theta - \theta'|_{abs}\right\},$$

where $|\cdot|_{abs}$ stands for the absolute value.

### 3 Stationary solution

This section considers a homogeneous stationary solution and a linearized system of (18) on $S$ with (19) around the stationary solution.

#### 3.1 Homogeneous stationary solution

We find the homogeneous stationary solution such that all variables $Y$, $w^A$, $G^A$, $w^M$, $G^M$, and $\omega$ are spatially uniform. Then, $\lambda$ and $\phi$ should be

$$\lambda(t, x) \equiv \lambda = \frac{1}{2\pi r},$$

$$\phi(x) \equiv \phi = \frac{1}{2\pi r}.$$

From the first equation of (18), stationary $Y$ becomes

$$\bar{Y} = \mu w^M \lambda + (1 - \mu) w^A \phi. \tag{20}$$

From the second equation of (18), stationary $G^A$

$$\bar{G}^A = \left[\frac{\phi w^A 1 - \eta E_{\alpha}}{(1 - \eta)}\right]$$

In the following, we often refer to “(18) on $S$ with (19)” simply as (18) for brevity when no confusion can arise.
Here, $E_\alpha$ is defined by

$$E_\alpha = \int_S e^{-\alpha|x-y|} dy = \frac{2(1-e^{-\alpha r \pi})}{\alpha}.$$  

From the third equation of (18), stationary $w_A$ becomes

$$\bar{w}^A = \left[YG^A\eta^{-1}E_\alpha\right]^{\frac{1}{\eta}} \quad (22)$$

Then, (22) together with (20) and (21) yield

$$\bar{w}^A = \bar{w}^M (=: \bar{w}).$$

This gives, from (20),

$$\bar{Y} = \frac{\bar{w}}{2\pi r}.$$ 

From the fourth equation of (18), stationary $G^M$ becomes

$$\bar{G}^M = \left[\lambda \bar{w}^{-\sigma} E^\beta\right]^{\frac{1}{1-\sigma}} \quad (23)$$

Here $E_\beta$ is defined by

$$E_\beta = \int_S e^{-\beta|x-y|} dy = \frac{2(1-e^{-\beta r \pi})}{\beta}.$$ 

From the sixth equation of (18) together with (21), (22) and (23), stationary $\omega$ is given by

$$\bar{\omega} = \bar{\lambda}^{\frac{\mu}{\sigma-1}} \phi^{\frac{1-\mu}{\sigma-1}} E_\alpha^{\frac{1-\mu}{\sigma-1}} E_\beta^{\frac{\mu}{\sigma-1}}.$$ 

3.2 Linearization

Let $\Delta Y(t, x), \Delta G^A(t, x), \Delta w^A(t, x), \Delta G^M(t, x), \Delta w^M(t, x), \Delta \omega(t, x),$ and $\Delta \lambda(t, x)$ for $t \in [0, \infty)$ and $x \in S$ be small perturbations. Substituting

$$Y(t, x) = \bar{Y} + \Delta Y(t, x), \quad G^A(t, x) = \bar{G}^A + \Delta G^A(t, x), \quad w^A(t, x) = \bar{w} + \Delta w^A(t, x), \quad G^M(t, x) = \bar{G}^M + \Delta G^M(t, x), \quad w^M(t, x) = \bar{w} + \Delta w^M(t, x), \quad \omega(t, x) = \bar{\omega} + \Delta \omega(t, x), \quad \lambda(t, x) = \bar{\lambda} + \Delta \lambda(t, x)$$

into (18), and ignoring higher order
terms, we obtain the following linearized system

\[
\begin{align*}
\Delta Y(t, x) &= \mu \overline{w} \Delta \lambda + \mu \overline{\lambda} \Delta w^M(t, x) + (1 - \mu) \overline{\phi} \Delta w^A(t, x), \\
\Delta G^A(t, x) &= \frac{w^{-\eta}G^A}{2\pi r} \int_S \Delta w^A(t, y)e^{-\alpha|x-y|}dy, \\
\Delta w^A(t, x) &= \frac{\eta - 1}{\eta} \overline{w}^{-\eta}G^{A^\eta - 2} \int_S \Delta G^A(t, y)e^{-\alpha|x-y|}dy \\
&\quad + \frac{G^{A^\eta - 1}}{\eta} \int_S \Delta Y(t, y)e^{-\alpha|x-y|}dy, \\
\Delta G^M(t, x) &= \overline{G}^M \Delta \lambda \int_S \Delta w^M(t, y)e^{-\beta|x-y|}dy \\
&\quad + \frac{G^M}{1 - \sigma} \int_S \Delta \lambda(t, y)e^{-\beta|x-y|}dy, \\
\Delta w^M(t, x) &= \frac{\sigma - 1}{\sigma} \overline{w}^{-\sigma}G^M \Delta G^M \int_S \Delta \lambda(t, y)e^{-\beta|x-y|}dy \\
&\quad + \int_S \Delta \lambda(t, y)e^{-\beta|x-y|}dy, \\
\Delta \omega(t, x) &= -\mu \overline{w} G^M \int_S \Delta \lambda(t, y)e^{-\beta|x-y|}dy - (1 - \mu) \overline{G} G^{A^\sigma - 1} \Delta G^A(t, x) \\
&\quad + \frac{\overline{G}}{\sigma} \int_S \Delta \lambda(t, y)e^{-\beta|x-y|}dy, \\
\frac{\partial}{\partial t} \Delta \lambda(t, x) &= \gamma \overline{\lambda} \Delta \omega(t, x).
\end{align*}
\]

The Fourier expansion of a periodic function \( \hat{f} \) on \([-\pi, \pi]\), which is identified with the corresponding function \( f \) on \( S \), is given by

\[
\hat{f}(\theta) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \hat{f}_n e^{in\theta}
\]

where \( \hat{f}_n, \ n = 0, \pm 1, \pm 2, \cdots \) are the Fourier coefficients defined by

\[
\hat{f}_n = \int_{-\pi}^{\pi} \hat{f}(\theta)e^{-in\theta'}d\theta'.
\]

Let \( \hat{Y}_n, \hat{G}_n^A, \hat{w}_n^A, \hat{G}_n^M, \hat{w}_n^M, \hat{\omega}_n, \) and \( \hat{\lambda}_n \) be the Fourier coefficients for \( \Delta Y(t, \theta) \), \( \Delta G^A(t, \theta) \), \( \Delta w^A(t, \theta) \), \( \Delta G^M(t, \theta) \), \( \Delta w^M(t, \theta) \), \( \Delta \omega(t, \theta) \), and \( \Delta \lambda(t, \theta) \), respec-
tively. By (24), we obtain the following equation of the Fourier coefficients

\[
\begin{align*}
\hat{Y}_n &= \mu \overline{w}_n + \mu \hat{\lambda}_n \hat{w}_n + (1 - \mu) \overline{w}_n \hat{A}_n, \\
\hat{G}^A_n &= \overline{w}^{-1} G^A_{\eta} H^\alpha_n \hat{A}_n, \\
\hat{w}_n &= \eta - \frac{1}{\eta} \overline{w}^{-1} G^{-1} A^\alpha_n \hat{A}_n + \frac{2\pi r}{\eta} H^{-\alpha}_n \hat{Y}_n, \\
\hat{G}_n &= \frac{G^M}{\overline{w}} H^\beta_n \hat{w}_n + \frac{2\pi r}{1 - \sigma} G^\beta H^\beta_n \hat{\lambda}_n, \\
\hat{w}_n &= \frac{\sigma - 1}{\sigma} \frac{w}{G^M} H^\beta_n \hat{G}_n + \frac{2\pi r}{\sigma} H^\beta \hat{Y}_n, \\
\hat{\omega}_n &= -\mu \overline{w}^{-1} G^{-1} M^\alpha_n - (1 - \mu) \overline{w}^{-1} G^{-1} A^\alpha_n + \overline{w}^{-1} \hat{w}_n \hat{M}_n, \\
\hat{\omega}_n &= \frac{\sigma}{\beta} \frac{w}{G^M} H^\beta_n \hat{G}_n - (1 - \mu) \overline{w}^{-1} G^{-1} A^\alpha_n + \overline{w}^{-1} \hat{w}_n \hat{M}_n,
\end{align*}
\]

for \( n = \pm 1, \pm 2, \cdots \). Solving the first six equations of (25), we get

\[
\hat{\omega}_n = \Omega_n \hat{\lambda}_n.
\]

Here,

\[
\Omega_n = \frac{2\pi r \overline{w}}{(\sigma - 1) D} \left[ \left\{ -\frac{\sigma (\mu^2 + b)}{b} + 1 + B \right\} H^{-\alpha}_n^2 + \frac{\mu (\sigma (b + 1) - 1)}{b} H^{-\beta}_n^2 - B \right],
\]

where

\[
\begin{align*}
b &= 1 - \frac{(1 - \mu) H^\alpha_n}{\eta - (\eta - 1) H^\alpha_n^2}, \\
B &= \frac{\mu \sigma (\sigma - 1)(1 - b) H^\alpha_n}{b}, \\
D &= \sigma - \frac{\mu}{b} H^\beta_n - (\sigma - 1) H^\beta_n^2
\end{align*}
\]

with \( H^\alpha_n \) and \( H^\beta_n \) defined by

\[
\begin{align*}
H^\alpha_n &= \frac{\alpha^2 r^2 (1 - (-1)^{|n|} e^{-\alpha \pi r})}{(n^2 + \alpha^2 r^2)(1 - e^{-\alpha \pi r})}, \\
H^\beta_n &= \frac{\beta^2 r^2 (1 - (-1)^{|n|} e^{-\beta \pi r})}{(n^2 + \beta^2 r^2)(1 - e^{-\beta \pi r})},
\end{align*}
\]

respectively\[7\] By the seventh equation of (24) and (26), we see that the homogeneous stationary solution is unstable (stable) if the eigen value

\[
\gamma \overline{\lambda} \Omega_n
\]

\[7\] Here, (27) and (28) correspond to (7.A.8) in Fujita et al. (2001, p.112) and (7.A.23) in Fujita et al. (2001, p.114).
is positive (negative).

The following lemmas hold as for $b$, $B$, $H^\alpha_n$, and $H^\beta_n$. See Appendix for the proofs. By these lemmas, we can easily see that $D > 0$ in (29).

**Lemma 1.** $H^\alpha_n$ and $H^\beta_n$ are monotonically increasing for $\alpha r \geq 0$ and $\beta r \geq 0$, respectively. In addition,

\[
\lim_{\alpha r \to 0} H^\alpha_n = 0,
\lim_{\alpha r \to \infty} H^\alpha_n = 1,
\lim_{\beta r \to 0} H^\beta_n = 0,
\lim_{\beta r \to \infty} H^\beta_n = 1
\]

hold.

**Lemma 2.** $b$ defined by (27) is monotonically decreasing as a function of $H^\alpha_n$ and satisfies that $\mu < b < 1$ for $\alpha > 0$, and that $b = 1$ for $\alpha = 0$.

**Lemma 3.** $B$ defined by (28) satisfies that $B > 0$ for $\alpha > 0$, and that $B = 0$ for $\alpha = 0$.

Therefore, the sign of the eigenvalue (30) can be determined only by the sign of the content of the square brackets of (26).

**Remark:** By replacing $\zeta$ and $Z$ in Fujita et al. (2001, Chapter 7) with $H^\alpha_n$ and $H^\beta_n$, respectively, we see that Eqs. (7 A.1)-(7 A.3), (7 A.5), (7 A.6), and (7 A.21) in Fujita et al. (2001, pp. 111-112, p. 114) are practically equivalent to the first six equations of (25). Then, (26) corresponds to (7 A.22) in Fujita et al. (2001, p. 114).

### 3.3 Eigenvalue under extreme values of parameters

When so called “No black hole condition” holds, the eigenfunction can be stabilized for sufficiently high manufacturing transport cost (or sufficiently low preference for manufacturing variety). The no black hole condition same as Fujita et al. (2001, p. 59) is given by the following theorem. See Appendix for proof.

**Theorem 1.** If

\[
\frac{\sigma - 1}{\sigma} > \mu
\]

holds, then the eigenvalue (30) satisfies $\lim_{\beta \to \infty} \gamma \lambda \Omega_n < 0$ for $n = \pm 1, \pm 2, \cdots$.

Meanwhile, for sufficiently low manufacturing transport cost (or sufficiently high preference for manufacturing variety), the following theorem hold. See Appendix for proof.
Theorem 2. For $\alpha > 0$, when $\beta \rightarrow 0$, the eigenvalue \((30)\) converges to

\[-\frac{\gamma r \omega}{(\sigma - 1)\sigma} B < 0.\] (32)

In Theorem 2 if $\alpha$ converges to zero, the limit value \((32)\) also converges to zero because $\lim_{\alpha \rightarrow 0} B = 0$ from Lemma 3.

4 Frequency dependence of eigenvalues

This section discusses how the eigenvalue \((30)\) depends on the frequency of the eigenfunction (and the parameters of the model). In this section, the eigenvalues are computed numerically. In doing so, the range of the values of each parameter $X(= \tau_A, \tau_M, \eta, \text{or } \sigma)$ is divided into an appropriate number $N$ of equally spaced discrete points denoted by $X_1, X_2, \ldots, X_N$, where $X_{i+1} - X_i = \Delta X$ for $i = 1, 2, \ldots, N - 1$. Then, the eigenvalues are computed only over these discrete points. As a result, the illustrated curves in the following figures are those complemented by the drawing software.

4.1 Shape of eigenvalue as function of $\tau_M$

Figure 1 plots the eigenvalue \((30)\) for several frequency numbers $n$ with $\tau_M$ on the horizontal axis and eigenvalues themselves on the vertical axis under $\mu = 0.5$, $\eta = 2.0$, $\sigma = 3.0$, and $\tau_A = 2.5$. Here, $\Delta \tau_M = 0.003$ is set. Figure 1(a) shows that for $n = 1$ the eigenvalue is negative for any value of $\tau_M$. On the other hand, Figure 1(b) shows a more general situation for $n = 2$. That is, the eigenvalue is negative at sufficiently high values of $\tau_M$, positive at medium values of $\tau_M$, and negative again at low values of $\tau_M$. In this case, the eigenvalue \((30)\) has two critical points $\tau_M^1 < \tau_M^2$ at which \((30)\) changes its sign.

![Figure 1: Eigenvalues](image.png)
4.2 Critical curve

We investigate how the critical values depend on the parameters. For example, to examine the dependence on $\tau_A$, by plotting the points on the $\tau_M$-$\tau_A$ plane where the value of the eigenvalue (30) is zero, and we can draw a curve that divides the plane into stable and unstable areas. Here, the stable (unstable) area is the area where the homogeneous stationary solution is stable (unstable) for any pair of parameters $(\tau_M, \tau_A)$ in the region.

Figure 2 shows the critical curve in $\tau_M$-$\tau_A$ plane under $\mu = 0.5$, $\eta = 2.0$, and $\sigma = 3.0$, and the frequency number is $n = 3$. Here, $\Delta \tau_A \approx 0.0008$ and $\Delta \tau_M \approx 0.003$ are set. It can be seen from Figure 2 that when $\tau_A$ is sufficiently high, the two critical points $\tau^1_M$ and $\tau^2_M$ approach each other as $\tau_A$ increases, and eventually merge and then disappear.

Figure 2: Critical curve

Figure 3 shows the critical curves plotted for different frequencies $n$ under $\mu = 0.5$, $\eta = 2.0$, and $\sigma = 3.0$. Here, $\Delta \tau_A \approx 0.0008$ and $\Delta \tau_M \approx 0.003$ are set. As can be seen from Figure 3a, the unstable area does not necessarily become larger as $n$ increases. However, if $n$ is limited to even or odd numbers, we would observe that the unstable area expands or at least does not shrink as $n$ increases (Figure 3(b) and 3(c))

In the numerical computation, if the eigenvalue for the discrete point $\tau_{Mi}$ is smaller (resp. greater) than that for $\tau_{Mi+1}$, then we approximate $\tau^{1}_M$ (resp. $\tau^{2}_M$) by $(\tau_{Mi} + \tau_{Mi+1})/2$.

Clearly, the critical curves for $n = 1$ and $n = 2$ intersect.

See Remark below.
Figure 3: Critical curve for $\tau_A$ under different frequencies

For the parameter $\eta > 1$, the critical curves can be drawn in the $\tau_M-\eta$ plane. Figure 4 shows the critical curves on the $\tau_M-\eta$ plane for some $n$ under $\mu = 0.5$, $\sigma = 3.0$, and $\tau_A = 2.5$. Here, $\Delta \eta \approx 0.002$ and $\Delta \tau_M \approx 0.003$ are set. Similar to the previous case in Figure 3, it can be seen from Figure 4 that when the values of $\eta$ are sufficiently large, the two critical points $\tau^*_M$ and $\tau^*_M$ approach each other as $\eta$ value becomes larger, and eventually merge and then disappear. The frequency dependence is also similar to the case of the critical curves on the $\tau_M-\tau_A$ plane.\(^{11}\)

\(^{11}\)See Remark below.
Figure 4. Critical curve for $\eta$ under different frequencies

For the parameter $\sigma > 1$, the critical curves under $\mu = 0.5$, $\eta = 2.0$, and $\tau_A = 2.5$ are shown in Figure 4. Here, $\Delta \sigma \approx 0.03$ and $\Delta \tau_M \approx 0.01$ are set. The shape of the critical curve looks quite different from previous cases in Figure 3 and 4. However, it is similar to the the previous cases qualitatively. Firstly, the two critical points move closer to each other as $\sigma$ increases where the $\sigma$ values are sufficiently large. Secondly, as the frequency number $n$ increases only in the even or odd range, the two critical points move away from each other and the unstable region seems to expand or at least not to shrink. Here, the vertical axis ($\sigma$) is shown on a logarithmic scale. See Remark below.
Remark: Numerical observations alone would not provide absolute confirmation of the claim that the unstable area is truly “monotonically expanding outward as the frequency number increases while remaining even or odd”. This is not only because numerical observations are made only for a finite number of cases, but also because the critical curves corresponding to different frequencies can be so close to each other (especially on $\tau^*_M$ side) that numerically they overlap exactly under some ranges of the parameters. For such cases, it would be impossible to infer the actual positional relationship of the critical curves, taking into account numerical errors.

5 Discussion and conclusion

We have considered a core-periphery model with differentiated agriculture and its transport cost on discrete and continuous multi-regional spaces. The stability of the homogeneous stationary solution of the model on a continuous periodic space has been investigated in detail. We have shown that the unstable eigenfunctions has two critical points at which the eigenvalue changes its sign. It has been observed numerically that the unstable area on a parameter plane between the two critical points generally tends to expand
uniformly outward with the frequency increases only in even or odd numbers. Meanwhile, with the general increases in frequency, it has also observed that the unstable area does not necessarily expands uniformly outward.

Especially, what might be important in our results is the non-monotonic outward expansion of the unstable area with a mere increase in frequency $n$. In particular, the fact that the lower critical point $\tau^*_{M}$ can move to the right with increasing $n$ means that the economy can prefer to agglomerate into a relatively small number of regions rather than decentralize to a relatively large number of regions in a low range of the manufacturing transport cost. This is intuitively incompatible with the fact that the sufficiently low manufacturing transport cost makes the spatial structure more decentralized in the original two-regional model with differentiated agriculture (Fujita et al. (2001, p.110)). This non-trivial result can only be obtained by considering the continuous space (at least multi regional) model, and may reflect a richer spatial complexity than a simple two-regional relationship.

As noted in Remark to the previous section, our results rely on numerical observations. For a rigorous proof, it is necessary to compute analytically how the solution of the quadratic function as for $H^\beta_n$ in (26) behaves with respect to changes in frequency $n$ only within even numbers or only within odd numbers. However, there are difficulties that cannot be attributed to simple differential calculations.

6 Appendix

Proof for Lemma 1

When $n$ is even

In this case, we see

$$H^\beta_n = \frac{\beta^2 r^2}{n^2 + \beta^2 r^2}.$$  

Putting $\beta r = X$, we have

$$H^\beta_n = \frac{X^2}{n^2 + X^2}.$$  

It is easy to see that

$$\lim_{X \to 0} \frac{X^2}{n^2 + X^2} = 0,$$

$$\lim_{X \to \infty} \frac{X^2}{n^2 + X^2} = 1.$$

We also easily see that

$$\frac{d}{dX} H^\beta_n = -\frac{2n^2 X}{(n^2 + X^2)^2} > 0.$$
When \( n \) is odd

In this case, we see

\[
H_n^\beta = \frac{X^2(1 + e^{-X\pi})}{(n^2 + X^2)(1 - e^{-X\pi})}
\]

by putting \( X = \beta r \). By the L’Hôpital’s theorem, we have

\[
\lim_{X \to 0} H_n^\beta = \lim_{X \to 0} \frac{2X(1 + e^{-\pi X}) + X^2(-\pi e^{-\pi X})}{2X(1 - e^{-\pi X}) + (n^2 + X^2)\pi e^{-\pi X}} = 0.
\]

On the other hand, it is obvious that

\[
\lim_{X \to \infty} H_n^\beta = 1.
\]

By differentiating \( H_n^\beta \) by \( X \), we obtain

\[
\frac{d}{dX}H_n^\beta = \frac{2n^2X(1 - e^{-2\pi X}) - 2\pi X^2e^{-\pi X}(n^2 + X^2)}{(n^2 + X^2)^2(1 - e^{-\pi X})^2}
\]

(33)

To show the monotonicity of \( H_n^\beta \) as for \( X \), we need to prove that the numerator of (33) is positive. It is equivalent to

\[
e^{\pi X} - e^{-\pi X} > \frac{\pi X}{n^2}(n^2 + X^2).
\]

(34)

The left hand side of (34) is expanded into a series

\[
2\pi X + 2\left(\frac{\pi X}{3!}\right)^3 + 2\left(\frac{\pi X}{5!}\right)^5 + \cdots.
\]

Therefore, subtracting the right-hand side of (34) from the left-hand side of (34) yields

\[
\pi X + \left\{2\left(\frac{\pi X}{3!}\right)^3 - \frac{\pi}{n^2}X^3\right\} + 2\left(\frac{\pi X}{5!}\right)^5 + \cdots.
\]

(35)

We are left with the task of proving the content of the braces in (35) is positive. It is sufficient to prove for \( n = 1 \), and is obvious that

\[
\pi \left(\frac{\pi^2}{3} - 1\right)X^3 > 0.
\]

Proof for Lemma 2

It is easy to check that

\[
\frac{d}{dH_n^\beta}b = -(1 - \mu)\frac{\eta + (\eta - 1)H_n^{\alpha^2}}{[\eta - (\eta - 1)H_n^{\alpha^2}]^2} < 0.
\]

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It is also immediate that  
\[ b = 1 \]
for \( \alpha = 0 \) because \( \alpha = 0 \iff H_n^\alpha = 0 \), and
\[
\lim_{H_n^\alpha \to 1} b = 1 - \frac{1 - \mu}{\eta - (\eta - 1)} = \mu.
\]
\[ \square \]

**Proof for Lemma 3**

It is evident from (28) and Lemma 2.

\[ \square \]

**Proof for Theorem 1**

When the manufacturing transport cost goes to infinity, i.e., when \( \beta \) goes to infinity, the content in the square brackets of (26) becomes
\[
1 + \frac{-\sigma \mu^2 - \sigma b + \mu \sigma b + \mu \sigma - \mu}{b}.
\]
For there to be no black hole, (36) must be negative, for which it is sufficient that
\[(1 - \sigma + \mu \sigma)(b - \mu) < 0.\]
Since \( b \in (\mu, 1] \) from Lemma 2, it immediately yields (31).

\[ \square \]

**Proof for Theorem 2**

When \( \tau_M \to 0 \) or \( \sigma \to 1 \), i.e., \( \beta \to 0 \), we see that \( H_n^\beta \to 0 \) by Lemma 1. Then, it is easy to see \( D \to \sigma \) by (29). As a result, it immediately follows from (30) that
\[
\Omega_n \to -\frac{2 \pi \sqrt{\omega}}{(\sigma - 1)\sigma} B.
\]
For \( \alpha > 0 \), it is obvious that this limit is negative since \( B > 0 \) from Lemma 3.

\[ \square \]

**References**

**Deaton, A. and J. Muellbauer**, *Economics and Consumer Behavior*, Cambridge University Press, 1980.

**Dixit, A. K. and J. E. Stiglitz**, “Monopolistic competition and optimum product diversity,” *A. E. R.*, 1977, 67 (3), 297–308.

**Fujita, M., P. Krugman, and A. Venables**, *The Spatial Economy: Cities, Regions, and International Trade*, MIT Press, 2001.

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Green, H. A. John, *Aggregation in Economic Analysis*, Princeton University Press, 1964.

Krugman, P., “Increasing returns and economic geography,” *J. Polit. Econ.*, 1991, 99 (3), 483–499.

Picard, P. M. and D.-Z. Zeng, “Agricultural sector and industrial agglomeration,” *J. Dev. Econ.*, 2005, 77 (1), 75–106.

Takatsuka, H. and D.-Z. Zeng, “Trade liberalization and welfare: Differentiated-good versus homogeneous-good markets,” *J. Jpn. Int. Econ.*, 2012, 26 (3), 308–325.

Zeng, D.-Z., “Spatial economics and nonmanufacturing sectors,” *Interdiscip. Inf. Sci.*, 2021, 27 (1), 57–91.

_ and H. Takatsuka, *Spatial Economics (in Japanese)*, TOYO KEIZAI INC., 2016.

_ and T. Kikuchi, “Home market effect and trade costs,” *Jpn. Econ. Rev.*, 2009, 60 (2), 253–270.