NUMERICAL RANGE AND THE DYNAMICS OF A RATIONAL FUNCTION

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Abstract. Sometimes we obtain attractive results when associating facts to simple elements. The goal of this work is to introduce a possible alternative in the study of the dynamics of rational maps. In this study we use the family of maps \( f(x) = \frac{x^2 - a}{x^2 - b} \), making some associations with the matrix \( A = \begin{pmatrix} 1 & -a \\ 1 & -b \end{pmatrix} \) of its coefficients.

1. Introduction

The main goal of this article is to present an alternative tool to study the dynamics of a real rational function, using results from the Numerical Range Theory, and has two main parts. The first, comprising Sections 2-3, is concerned to the adequation of the Numerical Range Theory to the data provided from rational maps. As described in Milnor\[8\], each map \( f \), in the space \( \text{Rat}_2 \), can be expressed as a ratio

\[
 f(z) = \frac{p(z)}{q(z)} = \frac{a_0 z^2 + a_1 z + a_2}{b_0 z^2 + b_1 z + b_2},
\]

where \( a_0 \) and \( b_0 \) are not both zero and \( p(z) \), \( q(z) \) have no common root. Milnor\[8\] states that we can obtain a roughly description of the topology of this space \( \text{Rat}_2 \) that can be identified with the Zariski open subset of complex projective 5-space consisting of all points

\[
 (a_0 : a_1 : a_2 : b_0 : b_1 : b_2) \in \mathbb{C}P^5,
\]

for which the resultant

\[
 \text{res}(p, q) = \det \begin{pmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{pmatrix}
\]

is non-zero. Taking \( z = x + i0 \) and \( a_0 = 1, b_0 = 1, a_2 = -a + i0, b_2 = -b + i0, a_1 = 0, b_1 = 0 \), with \( x, a, b \) real numbers, we obtain

\[
 B = \begin{pmatrix} 1 & 0 & -a & 0 \\ 0 & 1 & 0 & -a \\ 1 & 0 & -b & 0 \\ 0 & 1 & 0 & -b \end{pmatrix},
\]

\( \text{res}(p, q) = \det B \). This matrix \( B \) is associated to the real rational map \( f(x) = (x^2 - a) / (x^2 - b) \). The map \( f \) will be the one that we will use in our results associated to the matrix \( B \).
The second part of this article, comprising Sections 4-5, shows how we can apply the Numerical Range Theory to the dynamics of the map \( f \), establishing the relation between some partitions of an ellipse \( \Omega \), and the symbolic space generated by the partition of the domain of \( f \) in real intervals. Moreover, we launch a conjecture that could be a path to generate, in the future, an extension of the usual symbolic space applied to rational maps, allowing us to describe much better the dynamics of this maps.

2. Numerical Range Theory

The classical numerical range of a square matrix \( M_n \), with complex numbers elements, is the set \( W(M_n) = \{u^*M_nu, u \in S(\mathbb{C}^n)\} \), with \( S(\mathbb{C}^n) \) the unit sphere, \( u \) is a vector in \( \mathbb{C}^n \) and \( u^* \) is the transpose conjugate of \( u \).

The numerical range \( W(M_n) \) also can be defined as the image of the Rayleigh quotient \( R_{M_n}(u) = \frac{u^*M_nu}{u^*u}, u \neq 0 \). The set \( W(M_n) \) is closed and limited, and it is also a subset of the Gaussian \( \mathbb{C} \) plane. Toeplitz\([10]\) and Hausdorff\([2]\) proved that \( W(M_n) \) is a convex region. From Kippenhahn\([5]\), the boundary of the numerical range, \( \partial W(M_n) \) is a piecewise algebraic curve.

In the particular case of a square matrix \( M_2 \), with eigenvalues \( \lambda_1, \lambda_2 \), \( W(M_2) \) is a subset limited by an ellipse with foci in \( \lambda_1 \) and \( \lambda_2 \), result known as Elliptical Range Theorem, see Li\([6]\).

**Proposition 2.1.** If \( H_{M_n} = (M_n + M_n^*)/2 \) and \( S_{M_n} = (M_n - M_n^*)/2 \) are the Hermitian and skew-Hermitian parts of \( M_n \), respectively, then \( \text{Re}(W(M_n)) = W(H_{M_n}) \) and \( \text{Im}(W(M_n)) = W(S_{M_n}) \).

**Proof.** The proof can be found in Melo\([7]\). \( \square \)

**Theorem 2.1.** To every complex matrix \( M_n = H_{M_n} + S_{M_n} \) through the equation \( k_{M_n}(\alpha_1, \alpha_2, \alpha_3) \equiv \det(\alpha_1 M_n - i\alpha_2 S_{M_n} + \alpha_3 I_n) = 0 \) is associated a curve of class \( n \) in homogeneous line coordinates in the complex plane. The convex hull of this curve is the numerical range of the matrix \( M_n \).

**Proof.** Adapting the proof in Kippenhahn\([5]\) we have the desired result. \( \square \)

3. Merging \( f(x) \) in \( W(M_n) \)

Hwa-Long Gau\([1]\) states that we can obtain from a \( 4 \times 4 \) matrix an elliptical numerical range, thus we could use \( B \) as defined in section I but we can simplify our results if we use a smaller matrix, \( A_2 \), through a result in linear algebra. The new matrix \( A_2 = \begin{pmatrix} 1 & -a \\ 1 & -b \end{pmatrix} \) will produce equivalent results as the obtained from \( B \).

**Lemma 3.1.** The matrix \( B \) is unitary decomposable in

\[
\begin{pmatrix}
A_2 & \mathbb{O}_2 \\
\mathbb{O}_2 & A_2
\end{pmatrix}
\]

a block diagonal matrix.

**Proof.** In order to prove this result it is sufficient to find an unitary matrix \( E \) such that

\[
E^*BE = \begin{pmatrix} A_2 & \mathbb{O}_2 \\
\mathbb{O}_2 & A_2
\end{pmatrix}.
\]
Proposition 3.2. \( \Psi \)

Proof. By the definition of \( \Psi \) and \( z \)

\[
E = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\( \Box \)

Proposition 3.1. \( W(A_2) = W(B) \)

Proof. By lemma 3.1 \( E^* BE = A_2 \oplus A_2 \) and using the properties of numerical range we have

\[
W(E^* BE) = \text{convex hull}\{W(A_2) \cup W(A_2)\} = \text{convex hull}\{W(A_2)\}.
\]

The convex hull of a convex set is itself, then \( W(E^* BE) = W(A_2) \). But the numerical range of \( B \) is invariant under unitary transformations, see Kippenhahn, so \( W(B) = W(A_2) \).

In our study we use \( f(x) = (x^2 - a)/(x^2 - b) \), with \( a > 0, b > 0 \) and \( a > b \).

Such map takes all real axis with exceptions \( \pm \sqrt{b} \) on \( (-\infty, 1) \cup [\frac{a}{b}, +\infty) \).

With \( \Lambda = \mathbb{R}\{\pm \sqrt{b}\} \times (-\infty, 1) \cup [\frac{a}{b}, +\infty) \) we can define the graphic of \( f \), \( \text{graph}(f) \), as the pair \( (x, f(x)) \in \Lambda \) and \( \theta : \mathbb{R}\{\pm \sqrt{b}\} \longrightarrow \Lambda \).

Definition 3.1. Let \( C = \{v \in \mathbb{C}^2 : v = (x, if(x)) \} \) and

\[
\Psi = \left\{ z \in \mathbb{C} : z = \frac{v^* A_2 v}{v^* v}, v \neq 0 \right\}.
\]

We define \( V : \Lambda \longrightarrow C \) as \( (x, f(x)) \longrightarrow (x, if(x)) \) and \( \Xi : C \longrightarrow \Psi \).

By definition 3.1 the image of \( (x, f(x)) \) is \( z = (v^* A_2 v)/v^* v \) by \( \Xi \circ V \).

Proposition 3.2. \( \Psi \subset W(A_2) \)

Proof. By the definition of \( \Psi \) and \( W(A_2) \), using Rayleigh quotient, the result follows.

From proposition 3.2 we know that \( \Psi \) is a subset of \( W(A_2) \subset \mathbb{C} \), and using definition 3.1 we can calculate the elements \( z \in \Psi \), and as they were defined, they will become a function of \( x \). After some calculations we have

\[
z(x) = \frac{-a^2 b + (2a + b) b x^2 - 3b x^4 + x^6 + i (a + 1) (-a b x + (b + a) x^3 - x^5)}{a^2 + (b^2 - 2a) x^2 + (1 - 2b) x^4 + x^6}.
\]

So, the \( z \in \Psi \) is a function such that \( z(x) = g(x) + i h(x) \), with \( x \in \mathbb{R} \). Some elementary calculus show us that \( g(x) \) and \( h(x) \) are real rational continuous functions in \( \mathbb{R} \), therefore \( z(x) \) is continuous in \( \mathbb{C} \). We call some attention to the fact that \( z(\pm \sqrt{b}) \) and \( z(\infty) \) exists and are well defined in \( \mathbb{C} \).

Observation 3.1. We have \( z(x + 1) = z(x) \) for

\[
x = \frac{1}{2} \left( -1 \pm \sqrt{1 + 6a - 2b \pm 2 \sqrt{4a + 9a^2 - 10ab + b^2}} \right).
\]
Let $x_2$ and $x_{12}$ be the values where $f(x) = 0$. If we calculate $z(0, a/b); z(x_2, 0); z(x_{12}, 0); z(x, -x)$ we obtain four different points of $Ψ$. With some elementary algebra we calculated the ellipse that contain this four points. This ellipse is

$$\Omega = \left\{ \left( \frac{x - \frac{1-b}{2}}{\frac{1+b}{2}} \right)^2 + \frac{y^2}{\left( \frac{1+a}{2} \right)^2} = 1, \ (x, y) \in \mathbb{R}^2, \ a > b, a > 0, b > 0 \right\},$$

with $1 + b$ and $1 + a$ the minor and major axis length of $Ω$, respectively.

Moreover, when we use all points $(x, f(x))$ they will fall in $Ω$ under transformation by $z$.

**Lemma 3.2.** Let $z(x) = g(x) + ih(x)$, then the pair $(\text{Re}(z), \text{Im}(z))$ satisfies $Ω$.

**Proof.** We obtain this result replacing in the equation of $Ω$, $x$ by Re$(z)$ and $y$ by Im$(z)$. □

**Proposition 3.3.** If $S_i \in Ψ$ there are, at least one $x_i$ such that $z(x_i) = S_i$.

**Proof.** Since $z(x) = g(x) + ih(x)$ is a continuous function in $\mathbb{C}$ and by the lemma 3.2 the result follows. □

Then we conclude that $Ψ$ can be represented by the ellipse with equation $Ω$.

Since $Ω$ is constructed in the space $\mathbb{R}^2$ and this space is isomorphic to $\mathbb{C}$, when we refer to an element $z \in Ω$ it can understood has a vector in $\mathbb{R}^2$ or a complex number in the plane $\mathbb{C}$.

There are relations between the functions $f$ and $g$ that we can observe, described in the following lemmas. The proofs are omitted because they result from straight calculus.

**Lemma 3.3.** If $x_0$ is a zero of $f(x)$, then $g(x_0)$ is a relative maximum of $g(x)$.

**Lemma 3.4.** If $x_0$ is a relative minimum of $f(x)$ or $x_0$ is a discontinuity value of $f(x)$, then $g(x_0)$ is a relative minimum of $g(x)$.

There are similar relations between $h(x)$ and $f(x)$.

Follows some results relating $f(x)$ to $Ω$.

**Lemma 3.5.** Let $f(x_0) = \pm x_0$, then $V(x_0, f(x_0))$ is vertex of $Ω$.

**Proof.** If $f(x_0) = x_0$, by $V$ we have $(x_0, ix_0)$. So

$$\Xi((x_0, ix_0)) = \left( \begin{array}{cc} x_0 & -ix_0 \\ \end{array} \right) \left( \begin{array}{cc} 1 & -a \\ 1 & -b \\ \end{array} \right) \left( \begin{array}{c} x_0 \\ ix_0 \\ \end{array} \right) = \frac{1 - b}{2} - i \frac{1 + a}{2} \frac{x_0}{ix_0},$$

And if we look at the equation of $Ω$, we see that $\left( \frac{1-b}{2}, \frac{1+a}{2} \right)$ is a vertex of $Ω$.

If $f(x_0) = -x_0$ we obtain another vertex of $Ω$ in a similar way, which is $\left( \frac{1-b}{2}, \frac{1-a}{2} \right)$. □
Lemma 3.6. The discontinuities of \( f(x) \) and the values where \( f(x) \) has a minimum are transformed by \( \Xi \circ V \circ \theta \) in the vertex \((-b, 0)\) of \( \Omega \), and the roots of \( f(x) \) and the \( \infty \) are transformed by \( \Xi \circ V \circ \theta \) in the vertex \((1, 0)\) of \( \Omega \).

Proof. Since \( z(x) = g(x) + ih(x) = v^*A_2v/v^*v, v = (x, if(x)) \) is a continuous function in \( \mathbb{C} \), we have \( z(\pm \sqrt{b}) = -b \) and \( z(\infty) = 1 \). \( \square \)

4. Partitions of \( \Omega \)

Let \( x_1, x_2, x_5, x_6, x_7, x_8, x_9, x_{12}, x_{13} \) be the solutions of \( g'(x) = 0 \). By lemma 3.2 and lemma 3.3 and considering the order of real axis, we will have \( x_2 \) and \( x_{12} \) as zeros of \( f(x) \); \( x_5 \) and \( x_9 \) as the discontinuities of \( f(x) \) and \( x_7 = 0 \). All this values have image from \( z(x) \), lemma 3.2 including the infinity, being related by \( z(x_{12}) = z(x_9) = z(\infty) \) and equal to vertex \((1, 0)\) in \( \Omega \), see lemma 3.6, \( z(x_5) = z(x_6) = z(0) \) and equal to vertex \((-b, 0)\) in \( \Omega \). Related to the real axis, \( z(x_1) \) is symmetric to \( z(x_{13}) \) and \( z(x_6) \) is symmetric to \( z(x_3) \) in \( \Omega \). Where are the missing \( x_3, x_4, x_{10}, x_{11} \)? They will be the values such that \( z(x_1) = z(x_{11}), z(x_6) = z(x_{10}), z(x_8) = z(x_4) \) and \( z(x_{13}) = z(x_3) \).

Using this special values \( x_i, i = 1,..,13 \), with order \( x_i < x_{i+1} \), we can define a partition function \( pa \), as

\[
pa(x) = \begin{cases} 
I_1, & \text{if } x < x_1 \\
I_i, & \text{if } x_{i-1} < x < x_i \text{ with } 2 \leq i \leq 13 \\
I_{14}, & \text{if } x > x_{13}
\end{cases}
\]

Now we will create partitions in \( \Omega \) using the images \( z(x_i), i = 1,..,13 \) in \( \Omega \). Here, we ask attention for one particular aspect of \( z \), see proposition 3.3. Some intervals \( I_i \) will be transformed in the same arc of \( \Omega \). The only thing that will distinguish them is the orientation and the origin of its end points.

Definition 4.1. Let \( S_i = z(x_i) \), we define \( \text{arc}(S_i, S_{i+1}) \) as the arc of \( \Omega \) starting at \( S_i \) and ending at \( S_{i+1} \), with counterclockwise orientation.

We define \( pa_\Omega \), a partition function, as:

\[
pa_\Omega(w) = \begin{cases} 
J_1, & \text{if } w \in \text{arc}(z(\infty), z(x_1)) \\
J_i, & \text{if } w \in \text{arc}(z(x_{i-1}), z(x_i)) \text{ with } 2 \leq i \leq 13 \\
J_{14}, & \text{if } w \in \text{arc}(z(x_{14}), z(\infty))
\end{cases}
\]

The functions \( pa(x) \) and \( pa_\Omega(w) \) are related by

\[
z(I_i) = \begin{cases} 
J_1 & \text{if } i = 1 \\
-J_i & \text{if } 2 \leq i \leq 6 \\
J_i & \text{if } i = 7 \text{ or } i = 8 \\
-J_i & \text{if } 9 \leq i \leq 13 \\
J_{14} & \text{if } i = 14
\end{cases}
\]

thus we can build a matrix \( T_{14} \) of the transformation \( pa_\Omega(z(pa)) \),

\[
T_{14} = \begin{bmatrix} N_7 & \mathbb{O}_7 \\
\mathbb{O}_7 & N_7 \end{bmatrix},
\]
with

\[
N_7 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

It is easy to see that \( T_{14} T_{14} = I_{14}, \det(T_{14}) = 1, \) and it is an involutary matrix.

5. Dynamics of \( f(x) \)

Now we have a new tool to study the dynamics of \( f \) using a symbolic space. Using \( \Omega \) to study the behavior of \( f \) we will have the same advantages that we would have when studying the behavior of second degree polynomials functions in the unit circle.

If we define a symbolic space using the partitions created by the function \( pa \) in the real axis we will have the problem of dealing with the discontinuities of the function \( f \) and the infinity itself. So, profiting that \( z \) is a continuous complex function in \( \mathbb{C} \cup \{\infty\} \) this problem will vanish.

We can build two distinct symbolic spaces. The first will be the classical association between the intervals produced by \( pa \) in the real axis, see Milnor[8] for further reference, using the domain of the function \( f \), and considering an alphabet \( A \) with designations \( I_i \) for each interval, we will have a symbolic space \( \Sigma_c = A^\mathbb{N} \). The second will be constructed as we consider the alphabet \( B = \{J_1, \ldots, J_{14}\} \), and the set \( \Sigma = B^\mathbb{N} \) of symbolic sequences on the elements of \( B \), introducing the map \( spa : \mathbb{R} \cup \{\infty\} \rightarrow B \).

**Conjecture 5.1.** The symbolic dynamics of \( f \) does not change if we use \( \Sigma \) instead of \( \Sigma_c \).

Both spaces are connected by the transformation matrix \( T_{14} \) and doing some calculus in matrix algebra, since this matrix is an involutary matrix, we could get the result. All computations in our work points in that direction. But we are still working in a suitable proof of this result. Moreover, \( \Sigma \) will work as an extension of \( \Sigma_c \).

It means that we can identify the periodic orbits in the same values of \( a \) and \( b \) as we use both spaces \( \Sigma \) and \( \Sigma_c \). For example for the values \( a = 4.01 \) and \( b = 2.5 \) the critical orbit of \( f \) is periodic in both spaces, such as all the others values of periodicity found in our research. But they are many new sequences in \( \Sigma \) that needs more work to full understood slightly changes caused by the obliteration of \( \infty \). We are in the way.

This work, when finished, will imply a sequential work in the kneading theory and further study of the entropy of rational real maps.

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