Sudden death and birth of entanglement beyond the Markovian approximation

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Abstract

We investigate the entanglement dynamics of two initially entangled qubits interacting independently with two uncorrelated reservoirs beyond the Markovian approximation. Quite different from the Markovian reservoirs [C. E. López et al., Phys. Rev. Lett. 101 (2008) 080503], we find that entanglement sudden birth (ESB) of the two reservoirs occurs without certain symmetry with respect to the entanglement sudden death (ESD) of the two qubits. A phenomenological interpretation of entanglement revival is also given.

Key words: Entanglement dynamics, Non-Markovian reservoir, Entanglement sudden death (ESD), Entanglement sudden birth (ESB)
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1. Introduction

The entanglement dynamics of open quantum systems has attracted considerable interests over recent years [1]. By investigating a system of two qubits interacting independently with their corresponding vacuum reservoirs, Ref. [2] has pointed out the sudden termination of the entanglement initially owned by the two qubits at a finite interval, which is called entanglement sudden death (ESD). This intriguing phenomenon has recently been demonstrated experimentally by linear optics systems [3, 4] and by atomic ensembles [5]. A great deal of theoretical investigations of ESD have so far been reported [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. For example, based on the same model as in Ref. [2], it was found that ESD regarding the qubits would always occur under the thermal and squeezing reservoirs [8, 9]. Meanwhile, studies of ESD have been extended to multi-particle systems [10], which indicated the delay of ESD time with more qubits involved. In addition, some studies have also showed that entanglement of qubits will revive in the case of a commonly shared reservoir [11, 12] or of independent non-Markovian reservoirs [13, 14, 15].

A question naturally arises: where has the lost entanglement gone when ESD occurs? An answer to the question has been recently given by [7]. Considering the systems and the reservoirs as a whole, Ref. [7] presented the lost entanglement to be transferred to reservoir degrees of freedom, which is called "entanglement sudden birth" (ESB) of the reservoirs. Intriguingly, ESD of the systems and ESB of the reservoirs are of some symmetry and ESB might be occurring before, simultaneously with or even after ESD, depending on different initial states. However, the discussions in [7] are restricted to the weak coupling regime under the Markovian approximation, which only works when the reservoir correlation time is small compared to the relaxation time of the system [20]. If the qubit is strongly coupled to its environment or the characteristic time of the system is shorter than the environmental correlation time [20, 21], such as in recent experiments with cavity QED system [22] or solid-state systems [23, 24], the situation would be much complicated and we have to give up the Markovian approximation in treatment.

The present paper is focused on the study of entanglement dynamics without the Markovian approximation for two qubits interacting, respectively, with two independent reservoirs. For the initial condition with the two qubits entangled, we will study ESD and ESB, and show their different variation from the Markovian treatment.

2. Physical model

We consider a two-qubit system coupling to two uncorrelated vacuum reservoirs at zero temperature. Since there is no interaction between the two pairs of qubit-reservoir, the dynamics of the whole system can be obtained simply from the evolution of the individual pairs [7, 13]. The Hamiltonian of the individual qubit-reservoir pair under the rotating wave approximation is given by ($\hbar = 1$),

$$H = \omega_0 \sigma_+ \sigma_- + \sum_{k=1}^{N} \omega_k a_k^\dagger a_k + \sum_{k=1}^{N} g_k (\sigma_- a_k^\dagger + \sigma_+ a_k),$$

where $\omega_0$ is the resonant transition frequency of the qubit between levels $|0\rangle$ and $|1\rangle$, with $\sigma_+ = |1\rangle \langle 0|$ and $\sigma_- = |0\rangle \langle 1|$. 

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1We are aware of two very recent papers: Phys. Rev. A 79 (2009) 012301 and Phys. Rev. A 79 (2009) 042302 which addressed similar topics in entanglement revival of two qubits shared with a common reservoir.
\( \omega_k, a_k^\dagger (a_k), g_k \) are the frequency, creation (annihilation) operator and the coupling constant for the \( k \)th mode of the reservoir. For simplicity in our treatment, only one excitation of the reservoir will be considered. Consequently, the initial state \( |\Psi(0)\rangle = C_0(0)|1\rangle_0 |\tilde{0}\rangle_r + \sum_{k=1}^{N} C_k(0)|0\rangle_q |1\rangle_k \) will evolve to

\[ |\Psi(t)\rangle = C_0(t)|1\rangle_0 |\tilde{0}\rangle_r + \sum_{k=1}^{N} C_k(t)|0\rangle_q |1\rangle_k, \]

where \( |0\rangle_r = \bigotimes_{k=1}^{N} |0\rangle_{k,r}, \) and \( |1\rangle_r \), an abbreviation of \( \bigotimes_{k=1}^{N} |1\rangle_{k,r} \), is the reservoir state with one excitation in the \( k \)th mode and other states in vacuum. Assuming the initial conditions \( C_0(0) = 1 \) and \( C_k(0) = 0 \), we could obtain the following equation by solving the Schrödinger equation of Eq. (2),

\[ C_0(t) = -\int_0^t d\tau \Gamma(t - \tau) C_0(\tau), \]

where the correlation function \( \Gamma(t - \tau) \) in the limit of \( N \rightarrow \infty \) is of the form \( \Gamma(t - \tau) = \int d\omega J(\omega) e^{i(\omega - \omega_0)(t - \tau)} \) with \( J(\omega) \) the spectral density of the reservoir. In what follows, we will consider in our calculation the Lorentzian spectral distribution \( J(\omega) = \frac{\gamma^2}{2 \pi} \frac{1}{(\omega - \omega_0)^2 + \gamma^2} \), in which \( \gamma_0 = \tau_0^{-1} \) and \( \gamma = \tau^{-1} \), with \( \tau_0 \) and \( \tau \) the qubit relaxation time and the reservoir correlation time, respectively [24]. In a non-Markovian regime, \( \gamma_0 > \gamma/2 \), i.e., the reservoir correlation time is longer than the relaxation time of the qubit, and \( \gamma_0 < \gamma/2 \) means a Markovian regime [20]. As a result, in the non-Markovian regime, the probability amplitude can be easily solved as

\[ C_0(t) = e^{-\frac{\gamma}{2} \left[ \cos \left( \frac{\Gamma t}{2} \right) + \frac{\gamma}{\Gamma} \sin \left( \frac{\Gamma t}{2} \right) \right]}. \]

with \( \Gamma = \sqrt{\frac{\gamma_0}{2} \left( 1 - \gamma/2 \right)} \). If we set \( \tilde{\Gamma}(t) = \sqrt{1 - C_0(t)} \), Eq. (2) can be rewritten as

\[ |\Psi(t)\rangle = C_0(t)|1\rangle_0 |\tilde{0}\rangle_r + \tilde{\Gamma}(t)|0\rangle_q |\tilde{1}\rangle_r, \]

with \( |\tilde{1}\rangle_r = \sum_{k=1}^{N} C_k(t)|1\rangle_k \), \( \tilde{\Gamma}(t) \) (\( N \rightarrow \infty \)).

3. Entanglement dynamics

We will employ concurrence [25] to assess entanglement. In the case of the density matrix with the "X" type

\[ \rho = \begin{pmatrix} a & 0 & 0 & w \\ 0 & b & 0 & z \\ 0 & z^* & c & 0 \\ w^* & 0 & 0 & d \end{pmatrix}, \]

the concurrence could be easily calculated by [26]

\[ C(\rho) = 2 \max[0, |z| - \sqrt{ad}, |w| - \sqrt{bc}]. \]

We consider that the two qubits are initially entangled and the two reservoirs are in zero-temperature vacuum states, which is described as [2],

\[ \rho(0) = \frac{1}{3} \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 - \alpha \end{pmatrix} \otimes |\tilde{0}\rangle_{q_1} |\tilde{0}\rangle_{q_2}. \]

Using Eq. (5), we could straightforwardly reach the reduced density matrices for the two qubits,

\[ \rho_{q_1 \rightarrow q_2} = \begin{pmatrix} c_{11} & 0 & 0 & 0 \\ 0 & c_{22} & 0 & 0 \\ 0 & 0 & c_{33} & 0 \\ 0 & 0 & 0 & 1 - c_{44} \end{pmatrix}, \]

for the two reservoirs,

\[ \rho_{r_1 \rightarrow r_2} = \begin{pmatrix} c_{11} & 0 & 0 & 0 \\ 0 & c_{22} & 0 & 0 \\ 0 & 0 & c_{33} & 0 \\ 0 & 0 & 0 & c_{44} \end{pmatrix}, \]

for the qubit-1 and reservoir-1,

\[ \rho_{q_1 \rightarrow r_1} = \begin{pmatrix} c_{11} & 0 & 0 & 0 \\ 0 & c_{22} & 0 & 0 \\ 0 & 0 & c_{33} & 0 \\ 0 & 0 & 0 & c_{44} \end{pmatrix}, \]

and for the qubit-1 and reservoir-2,

\[ \rho_{q_1 \rightarrow r_2} = \begin{pmatrix} c_{11} & 0 & 0 & 0 \\ 0 & c_{22} & 0 & 0 \\ 0 & 0 & c_{33} & 0 \\ 0 & 0 & 0 & c_{44} \end{pmatrix}. \]

As Eqs. (9)-(12) are all of the "X" types, the concurrence of the partitions \( q_1 - q_2, r_1 - r_2, q_1 - r_1 \) and \( q_1 - r_2 \) can be calculated directly using Eq. (7). In addition, Eq. (7) also implies that the ESD time of the qubits or the ESB time of the reservoirs can be obtained from \(|z| - \sqrt{ad} = 0 \) and \(|w| - \sqrt{bc} = 0 \). However, as we could not solve them in an explicitly analytical way, numerical calculations are given below instead.

4. Numerical results and discussions

Fig. 1 shows the entanglement dynamics of \( q_1 - q_2, r_1 - r_2, q_1 - r_1, \) and \( q_1 - r_2 \) in the non-Markovian regime with \( \gamma = 0.1 \gamma_0 \). In Fig. 1(a), there are three regimes for entanglement variance regarding the two qubits: for \( \alpha \lesssim 0.3 \), no ESD will occur; for \( 0.3 \lesssim \alpha \lesssim 0.4 \), there will be entanglement revival after ESD takes place; as for \( \alpha \gtrsim 0.4 \) entanglement will vanish forever at a finite interval [13]. In contrast to the behavior regarding the two qubits, there are only two \( \alpha \)-dependent regimes for entanglement variance regarding the two reservoirs: ESB or no ESB, as shown in Fig. 1 (b). But no matter in which regime, the two reservoirs will finally be entangled in a time-dependently oscillating manner. The entanglement transfer is done step by step from the qubits to the reservoirs, which is reflected in Figs. 1(c) and 1(d) with the entanglement variance regarding the partitions \( q_1 - r_1, q_1 - r_2 \). The partitions \( q_1 - r_1, q_1 - r_2 \) behave as something like relays, which get and release entanglement in an oscillating way. Meanwhile the \( r_1 - r_2 \) entanglement is enhanced also in the oscillating way.

\[ \text{To avoid misunderstanding, we would like to mention that the term "} \]

entanglement transfer\[ \text{" used in the present paper means the appearance of a lost}

entanglement in some other degrees of freedom, e.g., qubit-reservoir, reservoir-reservoir etc. due to complicated couplings between the qubits and the environment. This is quite different from in Refs. [27-28, 29-30, 31-32, etc., where "entanglement transfer" is referred to the movement of entanglement from the flying qubits to static qubits by some operations.} \]

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Figure 1: (Color online) Concurrence as a function of $\alpha$ and $\gamma_0 t$ for Eqs. (9)-(12) in the non-Markovian regime with $\gamma = 0.1\gamma_0$.

Figure 2: (Color online) Concurrence as a function of $\gamma_0 t$ with $\alpha = 0.35$ for Eqs. (9)-(12) in different coupling intensities where the green (grey) solid lines mean the Markovian case with $\gamma = 5\gamma_0$; the blue curves correspond to the non-Markovian cases with $\gamma = 0.1\gamma_0$ (solid lines) and $\gamma = 0.05\gamma_0$ (dashed lines) respectively.
To see more clearly and also to make comparison with the Markovian treatment, we have plotted Fig. 2 for the entanglement dynamics with $\alpha = 0.35 \ (t_{\text{ESD}} > t_{\text{ESR}})$. In the non-Markovian regime, e.g., $\gamma = 0.05\gamma_0$, when ESD occurs at $\gamma t\approx 8$, the entanglement of the qubits has been transferred to other partitions, e.g., $r_1 - r_2$, $q_1 - r_1$, $q_1 - r_2$ etc. In more details, we may first visualize that within an interval $\gamma t \in [0, 5.5]$, in which the entanglement transfer from qubit-qubit to qubit-reservoir is greater than that from qubit-reservoir to reservoir-reservoir degrees of freedom, and the entanglement of qubit-reservoir and reservoir-reservoir are both increasing. Then from the critical point $\gamma t \approx 5.5$, the qubit-reservoir entanglement begins to be decreasing until $\gamma t \approx 11$, at which the entanglement of qubit-reservoir has been totally transferred to the reservoir degrees of freedom. However, in contrast to the case of Markovian reservoirs with the entanglement fully transferred to the reservoirs at one time, due to the memory effect of the non-Markovian reservoirs, entanglement will partially be recoiled to the qubit-reservoir degrees of freedom, or even back to the qubit-qubit reservoirs, entanglement will partially be recoiled to the qubit-qubit reservoirs, the more evident the non-Markovian effect.

Moreover, Fig. 2 also presents that, the stronger the qubits couple to their reservoirs, the reservoirs are entangled. Finally, the reservoirs are entangled. Moreover, Fig. 2 also presents that, the stronger the qubits coupling to their reservoirs, the more evident the non-Markovian memory effects.

5. Conclusions

In summary, we have investigated two-qubit entanglement affected by the decoherence from non-Markovian reservoirs. In contrast to the smooth decay of the qubit entanglement in the case of Markovian reservoirs, the entanglement of the qubits will revive after ESD occurs in the non-Markovian case. Meanwhile, by means of the qubit-reservoir partitions as the relays, the reservoirs will be asymptotically entangled in an oscillating manner, which is a good interpretation of entanglement revival of qubits in the non-Markovian reservoirs. We argue that our present study would be useful for quantum information science. For example, the solid-state qubits in systems with strong correlation always experience strong decoherence \[21, 23, 24\]. So quantum information processing on any qubit in such systems should be affected by detrimental influence from the non-Markovian reservoir. Moreover, fast logic gating is necessary in quantum information processing. As the characteristic time of the system during the fast gating is very short, we have to consider the non-Markovian effect from environment in assessing the operational fidelity. We have noticed recent studies on non-Markovian reservoir from the microscopic viewpoint, however, could explain some experimental observations of entanglement loss regarding qubits more straightforwardly. On the other hand, entanglement is the unique resource in quantum information science. As no system could be completely isolated from external noise, it is of interest to understand what happens in the entanglement transfer from the system to the environment, although whether the reservoirs are entangled or not is actually of no practical application with current technology.

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