SUBCANONICAL GRADED RINGS WHICH ARE NOT COHEN-MACAUŁAY

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This article is dedicated to Rob Lazarsfeld on the occasion of his 60-th birthday.

Abstract. We answer a question by Jonathan Wahl, giving examples of regular surfaces (so that the canonical ring is Gorenstein) with the following properties:
1) the canonical divisor $K_X \equiv rL$ is a positive multiple of an ample divisor $L$
2) the graded ring $R := \mathcal{R}(X, L)$ associated to $L$ is not Cohen-Macaulay.

In the appendix Wahl shows how these examples lead to the existence of Cohen-Macaulay singularities with $K_X \mathbb{Q}$-Cartier which are not $\mathbb{Q}$-Gorenstein, since their index one cover is not Cohen-Macaulay.

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The situation that we shall consider in this paper is the following: $L$ is an ample divisor on a complex projective manifold $X$ of complex dimension $n$, and we assume that $L$ is subcanonical, i.e., there exists an integer $h$ such that we have the linear equivalence $K_X \equiv hL$, where $h \neq 0$.

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There are then two cases: $h < 0$, and $X$ is a Fano manifold, or $h > 0$ and $X$ is a manifold with ample canonical divisor (in particular $X$ is of general type).

Assume that $X$ is a Fano manifold, and that $-K_X = rL$, with $r > 0$: then, by Kodaira vanishing

$$H^j(mL) := H^j(O_X(mL)) = 0 \forall m \in \mathbb{Z}, \forall 1 \leq j \leq n - 1.$$  

For $m < 0$ this follows from Kodaira vanishing (and holds for $j \geq 1$), while for $m \geq 0$ Serre duality gives $h^j(mL) = h^{n-j}(K - mL) = h^{n-j}((-r - m)L) = 0$.

At the other extreme, if $K_X$ is ample, and $K_X \equiv rL$, (thus $r > 0$) by the same argument we get vanishing outside of the interval

$$0 \leq m \leq r.$$  

To $L$ we associate as usual the finitely generated graded $\mathbb{C}$-algebra

$$R(X,L) := \bigoplus_{m \geq 0} H^0(X,O_X(mL)).$$  

Therefore in the Fano case, the divisor $L$ is arithmetically Cohen-Macaulay (see [Hart77]) and the above graded ring is a Gorenstein ring.

The question is whether also in the case where $K_X$ is ample one may hope for such a good property.

The above graded ring is integral over the canonical ring $A := R(X,K_X)$, which is a Gorenstein ring if and only if we have pluriregularity, i.e., vanishing

$$H^j(O_X) = 0 \forall 1 \leq j \leq n - 1.$$  

Jonathan Wahl asked the following question (which makes only sense for $n \geq 2$):

**Question 1. (J. Wahl)** Are there examples of subcanonical pluriregular varieties $X$ such that the graded ring $R(X,L)$ is not Cohen-Macaulay ?

We shall show that the answer is positive, also in the case of regular subcanonical surfaces with $K_X$ ample, where by the assumption we have the vanishing

$$H^1(mL) = 0 \forall m \leq 0, \text{ or } r \leq m$$  

and the question boils down to requiring the vanishing also for $1 \leq m \leq r - 1$.

The following theorem answers the question by J. Wahl:

**Theorem 2.** For each $r = n - 3$, where $n \geq 7$ is relatively prime to 30, and for each $m$, $1 \leq m \leq r - 1$ there are Beauville type surfaces $S$ with $q(S) = 0$ ($q(S) := \dim H^1(S,O_S)$) s.t. $K_S = rL$, and $H^1(mL) \neq 0$.
We get therefore examples of the following situation: \( \mathcal{A} := \mathcal{R}(S, K_S) \) is a Gorenstein graded ring, and a subring of the ring \( \mathcal{R} := \mathcal{R}(S, L) \), which is not arithmetically Cohen-Macaulay; hence we have constructed examples of non Cohen-Macaulay singularities (\( \text{Spec} (\mathcal{R}) \)) with \( K_Y \) Cartier which are cyclic quasi-étale covers of a Gorenstein singularity (\( \text{Spec} (\mathcal{A}) \)).

In the Appendix, J. Wahl uses these to construct Cohen-Macaulay singularities with \( K_X \) \( \mathbb{Q} \)-Cartier whose index one cover is not Cohen-Macaulay.

In fact, we can consider three graded rings, two of which are subrings of the third, and which are cones associated to line bundles on the surface \( S \):

- \( Y := \text{Spec} (\mathcal{R}) \), the cone associated to \( L \), which is not Cohen-Macaulay, while \( K_Y \) is Cartier;
- \( Z := \text{Spec} (\mathcal{A}) \), the cone associated to \( K_S \), which is Gorenstein;
- \( X := \text{Spec} (\mathcal{B}) \), the cone associated to \( K_S + L \) (for instance), which is Cohen-Macaulay with \( K_X \) \( \mathbb{Q} \)-Cartier, but whose index 1 (or canonical) cover \( Y = \text{Spec} (\mathcal{R}) \) is not Cohen-Macaulay.

2. The special case of even surfaces

Recall: a smooth projective surface \( S \) is said to be \textbf{even} if there is a divisor \( L \) such that \( K_S \equiv 2L \).

This is a topological condition, it means that the second Stiefel Whitney class \( w_2(S) = 0 \), or, equivalently, the intersection form
\[
H^2(S, \mathbb{Z}) \to \mathbb{Z}
\]
is even (takes only even values).

In particular, an even surface is a minimal surface.

In particular, if \( S \) is of general type and even, the self intersection
\[
K_S^2 = 4L^2 = 8k
\]
for some integer \( k \geq 1 \).

The first numerical case is therefore the case \( K_S^2 = 8 \).

**Proposition 3.** Assume that \( S \) is an even surface of general type with \( K_S^2 = 8 \) and \( p_g(S) = h^0(K_S) = 0 \). Then, if \( K_S \equiv 2L \), then \( H^1(L) = 0 \).

**Proof.** We assume that \( S \) is even, \( K \equiv 2L \), and \( p_g = 0 \).

Since the intersection form is even, and \( K^2 \leq 9 \) by the Bogomolov-Miyaoka-Yau inequality, we obtain that \( L^2 = 2 \).

The Riemann Roch theorem tells us: \( \chi(L) = 1 + \frac{1}{2}L(L - K) = 1 + \frac{1}{2}L(-L) = 0 \).

On the other hand, by Serre duality \( \chi(L) = 2h^0(L) - h^1(L) \), so if \( H^1(L) \) is different from zero, then \( H^0(L) \neq 0 \), contradicting \( p_g = 0 \). \( \square \)
Our construction for \( n = 5 \) shall show in particular that the ‘Beauville surface’, constructed by Beauville in [Bea78] is an even surface with \( K_S^2 = 8, q(S) = p_g(S) = 0 \), but with \( H^1(L) = 0 \).

3. Canonical linearization on Fermat curves

Fix a positive integer \( n \geq 5 \), and let \( C \) be the degree \( n \) Fermat curve

\[
C := \{(x, y, z) \in \mathbb{P}^2 \mid f(x, y, z) := x^n + y^n + z^n = 0\}.
\]

Let as usual \( \mu_n \) be the group of \( n \)-roots of unity. The group \( G := \mu_n^2 = \mu_n^3/\mu_n \) acts on \( C \), and we obtain a natural linearization of \( \mathcal{O}_C(1) \) by letting \((\zeta, \eta) \in \mu_n^2 \) act as follows:

\[
z \mapsto z, \quad x \mapsto \zeta x, \quad y \mapsto \eta y.
\]

In other words, \( H^0(\mathcal{O}_C(1)) \) splits as a direct sum of one dimensional eigenspaces (respectively generated by \( x, y, z \)) corresponding to the characters \( (1, 0), (0, 1), (0, 0) \in (\mathbb{Z}/n)^2 \cong \text{Hom}(G, \mathbb{C}^*) \).

Similarly, for \( m \leq n - 1 \), the monomial \( x^a y^b z^{m-a-b} \in H^0(\mathcal{O}_C(m)) \) generates the unique eigenspace for the character \((a, b)\) (we identify here \( \mathbb{Z}/n \cong \{0, 1, \ldots, n-1\} \) and we obviously require \( a + b \leq m \)).

However, any two linearizations differ (see [Mum70]) by a character of the group.

**Definition 4.** Assume that \( n \) is not divisible by 3.

We call the canonical linearization on \( H^0(\mathcal{O}_C(1)) \) the one obtained from the natural one by twisting with the character \((n-3)^{-1}(1,1)\).

Thus \( x \) corresponds to the character \( v_1 := (1,0) + (n-3)^{-1}(1,1) = (-3)^{-1}(-2,1) \), \( y \) corresponds to the character \( v_2 := (0,1) + (n-3)^{-1}(1,1) = (-3)^{-1}(1,-2) \), \( z \) corresponds to the character \( v_3 := (-3)^{-1}(1,1) \).

**Remark 5.** (I) Observe that \( v_1, v_2 \) are a basis of \((\mathbb{Z}/n)^2\) as soon as \( n \) is not divisible by 3.

Indeed, \( v_1 + v_2 = \frac{1}{3}(1,1) = 3^{-1}(1,1) \), hence

\[
(1,0) = v_1 + 3^{-1}(1,1) = 2v_1 + v_2, \quad (0,1) = 2v_2 + v_1.
\]

(II) Observe that the above linearization induces a linearization on all multiples of \( L \), and, in the case where \( m = (n - 3) \), we obtain the natural linearization on the canonical divisor of \( C \), \( \mathcal{O}_C(n-3) \cong \Omega^1_C \).

Since, if we take affine coordinates where \( z = 1 \), and we let \( f \) the equation of \( C \), we have

\[
H^0(\Omega^1_C) = \{P(x,y) \frac{dx}{fy} = -P(x,y) \frac{dy}{fx}\}
\]

and the monomial \( P = x^a y^b \) corresponds under this isomorphism to the character \((a+1, b+1)\).
(III) In particular, Serre duality

\[ H^0(\mathcal{O}_C(m)) \times H^1(\Omega^1_C(-m)) \rightarrow H^1(\Omega^1_C) \cong \mathbb{C}, \]

where \( \mathbb{C} \) is the trivial \( G \)-representation, is \( G \)-invariant.

From the previous discussion follows also

**Lemma 6.** The monomial \( x^ay^bz^c \in H^0(\mathcal{O}_C(m)) \) (here \( a, b, c \geq 0, a + b + c = m \)) corresponds to the character \( \chi \) equal to:

\[ (a, b) + (-3)^{-1}(m, m) = (a - c)v_1 + (b - c)v_2. \]

**Proof.** \( v_1 + v_2 = \frac{1}{3}(1, 1) \), hence \( (a, b) + (-3)^{-1}(m, m) = av_1 + bv_2 + (-m + a + b)(3)^{-1}(1, 1) = (a - c)v_1 + (b - c)v_2. \)

\[ \square \]

4. **Abelian Beauville Surfaces and their subcanonical divisors**

We recall now the construction (see also [Cat00], or [BCG05]) of a Beauville surface with Abelian group \( G \cong (\mathbb{Z}/n)^2 \), where \( n \) is not divisible by 2 and by 3.

**Definition 7.** (1) Let \( \Sigma \subset G \) be the union of the three respective subgroups generated by \((1, 0),(0, 1),(1, 1)\).

(2) Let \( \psi : G \rightarrow G \) a homomorphism such that, setting \( \Sigma^* := \Sigma \setminus \{(0, 0)\} \), \( \psi(\Sigma^*) \cap \Sigma^* = \emptyset \) (equivalently, \( \psi(\Sigma) \cap \Sigma = \{(0, 0)\} \)).

(3) Let \( C \) be the degree \( n \) Fermat curve and let \( S = (C \times C)/(\text{Id} \times \psi)(G), \)

i.e., the quotient of \( C \times C \) by the action of \( G \) such that \( g(P_1, P_2) = (g(P_1), \psi(g)(P_2)) \).

**Remark 8.** (i) By property (2) \( G \) acts freely and \( S \) is a projective smooth surface with ample canonical divisor.

(ii) The line bundle \( \mathcal{O}_{C \times C}(1, 1) \) is \( G \times G \) linearized, in particular it is \( G \cong (\text{Id} \times \psi)(G) \)-linearized, therefore it descends to \( S \), and we get a divisor \( L \) on \( S \) such that the pull back of \( \mathcal{O}_S(L) \) is the above \( G \)-linearized bundle.

(iii) By the previous remarks, we have a linear equivalence

\[ K_S \equiv (n - 3)L. \]

5. **Cohomology of multiples of the subcanonical divisor \( L \)**

We consider now an integer \( m \) with

\[ 1 \leq m \leq n - 4, \]

and we shall determine the space \( H^1(\mathcal{O}_S(mL)) \). Observe first of all that

\[ H^1(\mathcal{O}_S(mL)) \cong H^1(\mathcal{O}_{C \times C}(m, m))^G. \]
By the Künneth formula
\[ H^1(\mathcal{O}_{C \times C}(m,m)) \cong \]
\[ \cong [H^0(\mathcal{O}_C(m)) \otimes H^1(\mathcal{O}_C(m))] \oplus [H^1(\mathcal{O}_C(m)) \otimes H^0(\mathcal{O}_C(m))]. \]

We want to decompose the right hand side as a representation of \( G \cong (\text{Id} \times \psi)(G). \)

Explicitly, \( H^0(\mathcal{O}_C(m)) = \oplus \chi V_{\chi}, \) where if we write the character \( \chi = (a,b)+(-3)^{-1}(m,m) \) \( (\chi = (a-(m-a-b))v_1+(b-(m-a-b))v_2 \) as we saw) then \( V_{\chi} \) has dimension equal to one and corresponds to the monomial \( x^a y^b z^{m-a-b}, \) where \( a,b \geq 0, \ a+b \leq m. \)

By Serre duality, \( H^1(\mathcal{O}_C(m)) = \oplus \chi V_{-\chi'}, \) where if we write as above \( \chi' = (a',b')+(-3)^{-1}(m',m'), \) then \( V_{-\chi'} \) is the dual of \( V_{\chi'}, \) corresponding to the monomial \( x^{a'} y^{b'} z^{m'-(m'-b')}, \) where \( m' = n-3-m, \) so \( 1 \leq m' \leq n-4 \) also, and where \( a',b' \geq 0, \ a'+b' \leq m'. \)

Now, the homomorphism \( \psi: G \rightarrow G \) induces a dual homomorphism \( \phi := \psi^\vee: G^\vee \rightarrow G^\vee, \) therefore we can finally write \( H^1(\mathcal{O}_{C \times C}(m,m)) \) as a representation of \( G \cong (\text{Id} \times \psi)(G): \)
\[ H^1(\mathcal{O}_{C \times C}(m,m)) = \bigoplus_{\chi,\chi'} (V_{\chi} \otimes V_{-\phi(\chi')}) \oplus (V_{-\chi'} \otimes V_{\phi(\chi)}). \]

We have proven therefore the

**Lemma 9.** \( H^1(\mathcal{O}_S(mL)) \neq 0 \) if and only if there are characters \( \chi = (a-c)v_1+(b-c)v_2 \) and \( \chi' = (a'-c')v_1+(b'-c')v_2 \) with \( a,b \geq 0, \ a+b \leq m, \ a',b' \geq 0, \ a'+b' \leq m' = n-3-m \) such that
\[ \chi = \phi(\chi') \] or \( \chi' = \phi(\chi). \)

**Proof of theorem**

We take now \( \phi \) to be given by a diagonal matrix in the basis \( v_1,v_2, \) i.e., such that
\[ \phi(v_j) = \lambda_j v_j, \quad j = 1,2, \ \lambda_j \in (\mathbb{Z}/n)^*. \]

For further use we also set \( \lambda := \lambda_1, \mu := \lambda_2. \)

Given \( n \) relatively prime to \( 30 \) and \( 1 \leq m \leq n-4, \) we want to find \( \lambda_1 \) and \( \mu \) such that the equations
\[ (a-c) = \lambda (a'-c') \]
\[ (b-c) = \mu (b'-c') \]
have solutions with \( a,b,c \geq 0, \ a+b+c = m, \) and \( a',b',c' \geq 0, \ a'+b'+c' = m'. \)

The first idea is simply to take \( b = c \) and \( b' = c' \), so that \( \mu \) can be taken arbitrarily.
For the first equation some care is needed, since we want that \( \lambda \) be a unit: for this it suffices that \((a - c), (a' - c')\) are both units, for instance they could be chosen to be equal to one of the three numbers 1, 2, 3, according to the congruence class of \( m \), respectively \( m' \), modulo 3.

With this proviso we have to verify that we have a free action on the product.

**Lemma 10.** If \( n \geq 7 \), given \( \lambda \) a unit, there exists a unit \( \mu \) such that \( \psi = \phi^\nu \) satisfies the condition \( \psi(\Sigma) \cap \Sigma = \{(0,0)\} \).

*Proof.* Since \((1,0) = 2v_1 + v_2\) and \((0,1) = v_1 + 2v_2\), the matrix of \( \phi \) in the standard basis is the matrix
\[
\phi = \frac{1}{3} \begin{pmatrix} 4\lambda - \mu & 2(\lambda - \mu) \\ 2(\mu - \lambda) & 4\mu - \lambda \end{pmatrix}
\]
while the matrix of \( \psi \) is the matrix
\[
\psi = \frac{1}{3} \begin{pmatrix} A := 4\lambda - \mu & B := 2(\mu - \lambda) \\ C := 2(\lambda - \mu) & D := 4\mu - \lambda \end{pmatrix}
\]
The conditions for a free action boil down to:
\[
A, B, C, D, A + B, C + D
\]
are units in \( \mathbb{Z}/n \), and moreover \( A \neq B, C \neq D, A + B \neq C + D \).

These are in turn equivalent to the condition that
\[
\lambda, \mu, \lambda - 4\mu, \lambda - \mu, \mu - 4\lambda, \lambda + 2\mu, 2\lambda + \mu \in (\mathbb{Z}/n)^*.
\]

Given \( \lambda \in (\mathbb{Z}/n)^* \), consider its direct sum decomposition given by the Chinese remainder theorem and the primary factorization of \( n \). For each prime \( p \) dividing \( n \), the residue classes modulo \( p \) which are excluded by the above condition are at most five values inside \((\mathbb{Z}/p)^*\), hence we are done if \((\mathbb{Z}/p)^*\) has at least six elements.

Now, since \( n \) is relatively prime to 30, each prime number dividing it is greater or equal to \( p = 7 \). \( \square \)

**Proposition 11.** Consider the Beauville surface \( S \) constructed in [Bea78], corresponding to the case \( n = 5 \).

Then \( S \) is an even surface and \( K_S \equiv 2L \), where \( H^1(L) = 0 \).

*Proof.* We observe that \( L \) is unique, because the torsion group of \( S \) is of exponent 5 (see [BC04]).

The existence of \( L \) follows exactly as in the proof of the main theorem, where the condition \( n \geq 7 \) was not used. That \( H^1(L) = 0 \) follows directly from proposition 3. \( \square \)
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6. Appendix by Jonathan Wahl: A non-$\mathbb{Q}$-Gorenstein Cohen-Macaulay cone $X$ with $K_X \mathbb{Q}$-Cartier

A germ $(X, 0)$ of an isolated normal complex singularity of dimension $n \geq 2$ is called $\mathbb{Q}$-Gorenstein if

1. $(X, 0)$ is Cohen-Macaulay
2. The dualizing sheaf $K_X$ is $\mathbb{Q}$-Cartier (i.e., the invertible sheaf $\omega_{X-\{0\}}$ has finite order $r$)
3. The corresponding cyclic index one (or canonical) cover $(Y, 0) \rightarrow (X, 0)$ is Cohen-Macaulay, hence Gorenstein.

Alternatively, $(X, 0)$ is the quotient of a Gorenstein singularity by a cyclic group acting freely off the singular point. Some early definitions did not require the third condition, which is of course automatic for $n = 2$.

If $(X, 0)$ is $\mathbb{Q}$-Gorenstein, a one-parameter deformation $(\mathcal{X}, 0) \rightarrow (\mathbb{C}, 0)$ is called $\mathbb{Q}$-Gorenstein if it is the quotient of a deformation of the index one cover of $(X, 0)$; this is exactly the condition that $(\mathcal{X}, 0)$
is itself $\mathbb{Q}$-Gorenstein. These notions were introduced by Kollár and Shepherd-Barron [2], who made extensive use of the author’s explicit smoothings of certain cyclic quotient surface singularities in [3] (5.9); these deformations were patently $\mathbb{Q}$-Gorenstein, and it was important to name this property.

Recently, the author and others considered rational surface singularities admitting a rational homology disk smoothing (i.e., with Milnor number 0). The three-dimensional total space of the smoothing had a rational singularity with $K_{\mathbb{Q}}$-Cartier, but it was not initially clear whether the smoothings were $\mathbb{Q}$-Gorenstein. (This was later established [5] by proving the stronger result that the total spaces were log-terminal.) In fact, one needs to be careful because of the examples of A. Singh:

**Example.** [4]: There is a three-dimensional isolated rational (hence Cohen-Macaulay) complex singularity $(X, 0)$ with $K_X$ $\mathbb{Q}$-Cartier which however is not $\mathbb{Q}$-Gorenstein.

The purpose of this note is to use F. Catanese’s result to provide other examples; they are not rational, but are cones over a smooth projective variety, which could for instance be assumed to be projectively normal with ideal generated by quadrics.

**Proposition 12.** Let $S$ be a surface as in Theorem 2 of Catanese’s paper, with $h^1(S, \mathcal{O}_S) = 0$, $L$ ample, $K_S = rL$ (some $r > 1$), and $h^1(mL) \neq 0$ for some $m > 0$. Let $t$ be greater than $r$ and relatively prime to it. Then

1. The cone $R = \mathcal{R}(S, tL) := \oplus_{m \geq 0} H^0(S, \mathcal{O}_S(mtL))$ is Cohen-Macaulay.
2. The dualizing sheaf of $R$ is torsion, of order $t$.
3. The index one cover is $\mathcal{R}(S, L) := \oplus_{m \geq 0} H^0(S, \mathcal{O}_S(mL))$, and is not Cohen-Macaulay.

In particular, $R$ is not $\mathbb{Q}$-Gorenstein.

**Proof.** The Cohen-Macaulayness for $R$ follows because $h^1(itL) = 0$, all $i$, thanks to Kodaira Vanishing. Let $\pi : V \to S$ be the geometric line bundle corresponding to $-tL$; then $H^0(V, \mathcal{O}_V) \equiv R$. Since $K_V \equiv \pi^*(K_S + tL)$, one has that $JK_R \equiv \oplus_{n \in \mathbb{Z}} H^0(S, j(K_S + tL) + nL)$; since $tK_S = r(tL)$ with $r$ and $t$ relatively prime, $K_R$ has order $t$. Making a cyclic $t$-fold cover and normalizing gives that $\mathcal{R}(S, L)$ is the index one cover, which as Catanese has noted is not Cohen-Macaulay.

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