Special ramification loci on the double product of a general curve *

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Abstract

Let \( C \) be a general connected, smooth, projective curve of positive genus \( g \). For each integer \( i \geq 0 \) we give formulas for the number of pairs \((P, Q) \in C \times C\) off the diagonal such that \((g + i - 1)Q - (i + 1)P\) is linearly equivalent to an effective divisor, and the number of pairs \((P, Q) \in C \times C\) off the diagonal such that \((g + i + 1)Q - (i + 1)P\) is linearly equivalent to a moving effective divisor.

1 Introduction

Let \( C \) be a general connected, smooth, projective curve of genus \( g > 0 \). Put \( C^2 := C \times C \), and let \( \Delta \subset C^2 \) be the diagonal. For each integer \( i \geq 0 \) consider

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the following loci on $C^2$:

$$D_i := \{(P, Q) \in C^2 - \Delta \mid h^0(O_C((g + i - 1)Q - (i + 1)P)) > 0\},$$

$$E_i := \{(P, Q) \in C^2 - \Delta \mid h^0(O_C((g + i + 1)Q - (i + 1)P)) > 1\}.$$

Our Proposition 5.4 claims that $D_i$ and $E_i$ are finite, and our main result, Theorem 5.6, gives formulas for the number of points in $D_i$ and $E_i$.

A formula for the number of points in $D_i$ appeared already as Lemma 6.3 on page 24 of the seminal work by Diaz [6], where the unnecessary extra hypotheses that $g \geq 2$ and $i \geq 2$ are made. Diaz used this formula to compute the class in the moduli space of genus-$g$ stable curves $\overline{M}_g$ of the closure $\overline{D}_g$ of the locus of smooth curves $C$ having a Weierstrass point $P$ of type $g - 1$, i.e. such that $h^0(O_C((g - 1)P)) \geq 2$.

Later on, Cukierman [4] gave a formula for the class in $\overline{M}_g$ of the closure $\overline{E}_g$ of the locus of smooth curves $C$ containing a Weierstrass point $P$ of type $g + 1$, i.e. such that $h^0(O_C((g + 1)P)) \geq 3$. He did not follow in Diaz’s footsteps for this formula, but rather observed that the union $\overline{D}_g \cup \overline{E}_g$ is the branch locus of the Weierstrass divisor on the “universal” curve over $\overline{M}_g$, and used a Hurwitz formula with singularities to compute the class of this branch divisor.

Had Cukierman followed in Diaz’s footsteps, he would probably have found he needed a formula for the number of points in $E_i$. We give this formula here.

In fact, in a sense to be explained below, it is slightly easier to obtain the number of points in $E_i$ than in $D_i$, though we obtain both in a quite integrated form here. To obtain these numbers, the natural procedure is to use Porteous formula to compute the virtual classes of certain natural ramification schemes $D_i^+$ and $E_i^+$ of maps of vector bundles on $C^2$; see Subsection 5.2.

Set-theoretically, $D_i^+$ and $E_i^+$ are given exactly as $D_i$ and $E_i$, but without the restriction that the pair $(P, Q)$ lies off $\Delta$.

The problem is that $D_i^+$ and $E_i^+$ are both larger than $D_i$ and $E_i$. Indeed, $E_i^+$ is the union of $E_i$ with the set of points $(P, P)$ such that $P$ is a Weierstrass point of $C$ and, worse, $D_i^+$ is the union of $D_i$ and the whole diagonal $\Delta$. Since $E_i^+$ is finite, Porteous formula does give an expression for the number of points in $E_i^+$, with weights, and thus at least an upper bound for the number of points in $E_i$. But it does not a priori give any information on $D_i$.

To compute the number of points in $D_i$ and $E_i$, we use the fact that, by the Riemann-Roch Theorem, the union of $D_i$ and $E_i$ is the locus $SW_i$ of pairs
\((P, Q) \in C^2 - \Delta\) such that \(Q\) is a special ramification point of the complete linear system \(H^0(\omega_C((i + 1)P))\), where \(\omega_C\) is the canonical bundle of \(C\).

We give \(SW_i\) a scheme structure as follows. First, we consider the ramification divisor \(Z_i \subset C^2\) of the family of linear systems \(H^0(\omega_C((i + 1)P))\) parameterized by \(P \in C\). Our Proposition 2.2 implies that \(Z_i\) contains \(\Delta\) with multiplicity exactly \(g\). Set \(W_i := Z_i - g\Delta\). Our Proposition 4.3 gives an expression for the cycle \([W_i]\), and our Proposition 4.4 claims that \(W_i\) is nonsingular. Furthermore, in Subsection 5.2 we observe that the branch divisor of \(W_i\) with respect to the projection \(p_1: C^2 \to C\) over the first factor has support \(SW_i\). We give \(SW_i\) the structure of this branch divisor.

The advantage of considering \(SW_i\) is that it is quite easy to compute its degree. Indeed, \([SW_i]\) is the second Chern class of the bundle of first-order relative jets of \(p_1\) with coefficients in \(\mathcal{O}_{C^2}(W_i)\). Having an expression for \([W_i]\) we derive very quickly an expression for \(\int_{C^2}[SW_i]\) in Proposition 5.5.

Now, giving \(D_i\) and \(E_i\) the subscheme structures induced from \(D_i^+\) and \(E_i^+\), our Proposition 5.4 shows that, as 0-cycles,

\[
[D_i] + [E_i] = [SW_i].
\]

Actually, \(D_i\) and \(E_i\) are reduced. Indeed, in the proof of Theorem 5.6 we show that the weight of \((P, Q)\) in \([SW_i]\) is at most 2, and the maximum weight is achieved if and only if \((P, Q) \in D_i \cap E_i\).

Now, as we already know \(\int_{C^2}[SW_i]\), it is enough to compute either \(\int_{C^2}[D_i]\) or \(\int_{C^2}[E_i]\). As mentioned above, we compute the latter. In fact, we can get \(\int_{C^2}[E_i^+]\) using Porteous formula, and a local analysis, done in Proposition 5.4, shows that the weight of \((P, P)\) in \([E_i^+]\) is equal to \(g + 1\) for each Weierstrass point of \(C\). Thus \(\int_{C^2}[E_i]\) follows.

In a second article [5], we show how the knowledge of the number of points in \(E_i\) can be used to compute the class of \(\overline{E}_g\) in \(\overline{M}_g\). This computation is not straightforward as, following in Diaz’s footsteps, we have to determine the limits of special Weierstrass points of type \(g + 1\) on stable curves with just one node. This is the main result of [5].

The limits of special Weierstrass points of type \(g - 1\) were computed by Diaz, using admissible covers. However, the same method does not apply to points of type \(g + 1\). For those we apply in [5] the theory of limit linear series in a rather new way, using 2-parameter families. Actually, as in the present article, we use an integrated approach in [5] that yields simultaneously the limits of special Weierstrass points of both types, and also formulas for the classes of both \(\overline{D}_g\) and \(\overline{E}_g\).
Here is a layout of the article. In Section 2 we review the theory of linear systems and ramification on a smooth curve $C$, introduce the linear systems we will consider in the remaining of the article, $H^0(\omega_C((i+1)P))$ for $P \in C$, and prove a preliminary result about them. In Section 3, assuming $C$ is general, we obtain through degeneration methods results that bound the order sequence of $H^0(\omega_C((i+1)P))$ at any point of $C$. In Section 4, we describe the structure of the ramification divisor $Z_i \subset C^2$ of the family of linear systems $H^0(\omega_C((i+1)P))$ parameterized by $P \in C$. Finally, in Section 5 we define the loci $D_i$ and $E_i$ and compute their number of points, through the study of the locus $SW_i$ of special ramification points of the family $H^0(\omega_C((i+1)P))$ for $P \in C$.

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2 Setup

2.1 (Linear systems and ramification) Let $C$ be a smooth curve, that is, a projective, connected, smooth scheme of dimension 1 over $\mathbb{C}$. Denote by $\omega_C$ its canonical sheaf. Let $g := h^0(C, \omega_C)$, the genus of $C$.

Let $V$ be a $\mathbb{C}$-vector space of sections of a line bundle $L$ on $C$. We call $V$ a linear system. The linear system is called complete if $V = H^0(C, L)$. Let $r := \dim V - 1$ and $d := \deg L$. We call $r$ the rank of $V$ and $d$ its degree. We say as well that $\dim V$ is the dimension of $V$.

For each point $P$ of $C$, and each integer $j \geq 0$, let $V(-jP)$ denote the vector subspace of $V$ of sections of $L$ that vanish with order at least $j$ at $P$. We say that $j$ is an order of $V$ at $P$ if $V(-jP) \neq V(-(j+1)P)$. There are $r+1$ orders, which, in an increasing sequence, will be denoted by $\epsilon_0(V,P), \epsilon_1(V,P), \ldots, \epsilon_r(V,P)$.

For each integer $\ell \geq 0$ and each line bundle $\mathcal{M}$ on $C$, let $\mathcal{J}^\ell_C(\mathcal{M})$ be the bundle of jets, or principal parts, of $\mathcal{M}$ truncated in order $\ell$. Consider the map of rank-$r$ bundles,

$$V \otimes \mathcal{O}_C \longrightarrow \mathcal{J}^r_C(L),$$

locally obtained by differentiating up to order $r$ the sections of $L$ in $V$. The wronskian $w_V$ of $V$ is the (nonzero) section of

$$L^\otimes r+1 \otimes \omega_C^\otimes r(r+1)/2$$
induced by taking determinants in the above map of bundles.

For each point \( P \) of \( C \), the \textit{weight} \( w_{VT}(P) \) of \( P \) in \( V \) is the order of vanishing of \( w_V \) at \( P \). We call \( P \) a \textit{ramification point} of \( V \) if \( w_{VT}(P) > 0 \); otherwise we call \( P \) \textit{ordinary}. A local analysis yields the formula

\[
    w_{VT}(P) = \sum_{j=0}^{r} (\epsilon_j(V, P) - j).
\]

We call \( P \) a \textit{simple} ramification point if \( w_{VT}(P) = 1 \); otherwise we call \( P \) \textit{special}. The point \( P \) is special if and only if the section \( Dw_{VT} \in H^0\left(C, \mathcal{J}_C^1(\mathcal{L}^{\otimes (r+1)} \otimes \omega_C^{\otimes (r+1)/2})\right) \), locally obtained from \( w_V \) by differentiating, vanishes at \( P \).

The \textit{total weight} of the ramification points of \( V \) is the (finite) sum

\[
    w_T := \sum_{P \in C} w_{VT}(P).
\]

It is equal to the degree of the line bundle of which \( w_V \) is a section, that is,

\[
    w_T = (r + 1)(d + (g - 1)r),
\]

a formula usually referred to as the \textit{Brill–Segre} or \textit{Plücker formula}.

The \textit{canonical system} is the complete linear system of sections of \( \omega_C \). Its rank is \( g - 1 \), and its degree is \( 2(g - 1) \). For each point \( P \) of \( C \), its \textit{Weierstrass weight} \( wt(P) \) is its weight in the canonical system, and the \textit{Weierstrass order sequence} at \( P \) is the increasing sequence of orders at \( P \) of the canonical system.

For each integer \( i \geq -1 \) and each \( P \in C \), let \( V_C(i, P) \) denote the complete linear system of sections of \( \omega_C((i + 1)P) \).

\textbf{2.2 Proposition.} Let \( C \) be a smooth curve of genus \( g \). For each integer \( i \geq 0 \) and each \( P \in C \), the following two statements hold for \( V := V_C(i, P) \):

1. The weight \( w_{VT}(P) \) of \( P \) as a ramification point of \( V \) satisfies

\[
    w_{VT}(P) = g + wt(P),
\]

where \( wt(P) \) is the Weierstrass weight of \( P \).
2. The total weight $w_t V$ of the ramification points of $V$ satisfies

$$w_t V = g(g + i)^2. \quad (2)$$

**Proof.** From the Riemann–Roch theorem, for each $j = 0, \ldots, i$,

$$\dim V(-jP) = g + i - j.$$ 

In particular, comparing dimensions, we get

$$V(-iP) = V(-(i + 1)P) = H^0(C, \omega_C).$$

Hence, the order sequence of $V$ at $P$ is

$$0, 1, \ldots, i - 1, (i + 1) + \epsilon_0, (i + 1) + \epsilon_1, \ldots, (i + 1) + \epsilon_{g-1},$$

where $\epsilon_0, \epsilon_1, \ldots, \epsilon_{g-1}$ is the Weierstrass order sequence at $P$. Thus

$$w_t V(P) = \sum_{k=0}^{g-1} (i + 1 + \epsilon_k - i - k) = g + w_t(P).$$

The second statement is a direct application of the Brill–Segre formula, using that the rank of $V$ is $g + i - 1$ and its degree is $2g - 2 + (i + 1)$. ■

3 The general curve

3.1 Proposition. Fix an integer $i_0 \geq 0$. Let $C$ be a general smooth curve of genus $g \geq 1$. Then the following two statements hold for each nonnegative integer $i \leq i_0$:

1. For a general point $P$ of $C$, the linear system $V_C(i, P)$ ramifies at $P$ with weight $g$, and has otherwise at most simple ramification points.

2. For any two points $P$ and $R$ of $C$,

$$h^0(C, \omega_C((i + 1)P - (g + i)R)) \leq 1 \quad (3)$$

$$h^0(C, \omega_C((i + 1)P - (g + i + 2)R) = 0. \quad (4)$$
Proof. Let us first observe that the property required of \( C \) is open. Indeed, let \( f: X \to S \) be any family of smooth curves, that is, a projective, smooth map with connected fibers of dimension 1. Consider the fibered product \( X^{(2)} := X \times_S X \) of two copies of \( f \), and denote by \( p_1 \) and \( p_2 \) the projection maps. Denote by \( \Delta \) the diagonal subscheme of \( X^{(2)} \). Let \( \omega_f \) denote the relative canonical bundle of \( f \). Then \( \omega := p_2^* \omega_f \) is the relative canonical bundle of \( p_1 \). Let

\[
V := p_1^* (\omega((i + 1)\Delta)).
\]

A fiberwise analysis shows that \( V \) is a bundle of rank \( g + i \) with formation commuting with base change. For each integer \( \ell \geq 0 \), denote by \( \mathcal{J}^\ell \) the bundle of rank \( \ell + 1 \) of \( p_1 \)-relative jets of \( \omega((i + 1)\Delta) \) truncated in order \( \ell \), and denote by \( \psi_\ell: p_1^* V \to \mathcal{J}^\ell \) the map locally obtained by differentiating the sections of \( \omega((i + 1)\Delta) \) up to order \( \ell \) along the fibers of \( p_1 \). Let \( W_{i,1} \) (resp. \( W_{i,2} \)) be the closed subset of \( X^{(2)} \) where \( \psi_{g+i-1} \) (resp. \( \psi_{g+i+1} \)) has rank at most \( g + i - 2 \) (resp. \( g + i - 1 \)). Also, let \( W_i \) be the closed subset of \( X^{(2)} \) where \( \psi_{g+i-1} \) has rank at most \( g + i - 1 \). By Proposition 2.2, \( W_i \) contains \( \Delta \) with multiplicity \( g \). Let \( W_i' := W_i - g\Delta \) and \( Z_i := \Delta \cap W_i' \). Let \( W_i'' \subset W_i' \) be the ramification scheme of the map \( p_1|_{W_i} \). Let \( U_i \subseteq S \) be the intersection of \( S - f(p_1(W_{i,1} \cup W_{i,2})) \) with \( f(X - p_1(W''_i \cup Z_i)) \). Since \( p_1 \) is proper, and \( f \) is both proper and open, \( U_i \) is an open subscheme of \( S \). Let \( U := U_0 \cap \cdots \cap U_{i_0} \). The formation of \( U \) commutes with base change. Thus a fiberwise analysis reveals that \( U \) consists of the set of points \( s \in S \) such that the proposition holds for \( C := X(s) \).

Now, keeping in mind the existence of a versal family of smooth curves, it is enough to exhibit a single curve \( C \) for which the statement holds. We will actually show a somewhat stronger existence result:

3.2 Lemma. Fix nonnegative integers \( i_0 \) and \( j_0 \). Let \( g \) be a positive integer. Then there is a smooth pointed curve \( (C, Q) \) of genus \( g \) for which the following three statements hold for each nonnegative integers \( i \leq i_0 \) and \( j \leq j_0 \):

1. The linear system \( V_C(j, Q) \) ramifies at \( Q \) with weight \( g \), and has otherwise at most simple ramification points.

2. For each \( P \in C \) distinct from \( Q \), either \( Q \) is an ordinary point or a simple ramification point of \( V_C(i, P) \).

3. For each \( P \in C \) distinct from \( Q \), the linear system \( V \) of sections of
\( \omega_C((i + 1)P + (j + 1)Q) \) given by

\[
V := H^0(\omega_C((i + 1)P)) + H^0(\omega_C((j + 1)Q))
\]

satisfies

\[
\dim V(-(g + i + j)R) \leq 1 \quad \text{and} \quad V(-(g + i + j + 2)R) = 0
\]

for each \( R \in C \) distinct from \( P \) and \( Q \).

We will first see how the lemma implies the proposition. Set \( j_0 = i_0 \), and consider the pointed curve \((C, Q)\) given by the lemma. Then the two statements of Proposition 3.1 hold for \( C \). Indeed, the first statement holds for \( P = Q \), whence for \( P \) in a neighborhood of \( Q \), that is, for a general \( P \).

As for the second statement, first notice that (3) and (4) hold for \( P = Q \) and every \( R \in C \), a consequence of the first statement of the lemma for \( j := i \). They hold as well for \( R = Q \) and any \( P \in C \) distinct from \( Q \), a consequence of the second statement of the lemma. Furthermore, they hold for \( R = P \) and any \( P \in C \). Indeed, the first statement of the lemma for \( j := 0 \) implies that the canonical linear system has at most simple ramification points. Thus \( h^0(\omega_C((1 - g)P)) \leq 1 \) and \( h^0(\omega_C(-(g + 1)P)) = 0 \).

Finally, fix a point \( P \in C \) distinct from \( Q \) and a point \( R \in C \) distinct from \( P \) and \( Q \). For \( j := 0 \), the linear system \( V \) defined in the lemma is the system of sections of \( \omega_C((i + 1)P + Q) \) that are zero on \( Q \). Since \( R \neq Q \), the third statement of the lemma yields (3) and (4).

It is thus enough to prove the lemma, what we do below.

**Proof.** (Lemma 3.2) We will do induction on \( g \). The initial step is taken care of below.

Let \( C \) be any elliptic curve and \( Q \in C \) any point. Then the ramification points of the complete linear system of sections of \( \omega_C((j + 1)Q) \) are simple. (These are the \((j + 1)^2 \) points \( R \) for which \( Q - R \) is \((j + 1)\)-torsion, what includes \( Q \).) In fact, it follows from the Riemann–Roch theorem that every complete linear system has only simple ramification points. Thus Statements 1 and 2 of the lemma hold. Now, given \( P \in C \) distinct from \( Q \), since the vector subspace \( V \) of \( H^0(\omega_C((i + 1)P + (j + 1)Q)) \) defined in Statement 3 has codimension 1, the order sequence of \( V \) at a point \( R \) is obtained either from

\[
0, 1, \ldots, g + i + j - 1, g + i + j \quad \text{or} \quad 0, 1, \ldots, g + i + j - 1, g + i + j + 1
\]
by removing an order. In any case, there is at most one order of \( V \) at \( R \) above \( g + i + j - 1 \), that is \( \dim V(-(g + i + j)R) \leq 1 \), and all orders are at most \( g + i + j + 1 \), that is \( V(-(g + i + j + 2)R) = 0 \).

Assume from now on that \( g > 1 \), and that the claim holds for smaller genera and any integers \( i_0 \) and \( j_0 \). We will employ a degeneration technique in order to apply the induction hypothesis.

Let \((Y, A)\) and \((Z, B)\) be nonrational smooth pointed curves of genera \( g_Y \) and \( g_Z \), with \( g_Y + g_Z = g \). From the induction hypothesis, we may assume that the statements of the lemma hold for \((C, Q)\) replaced by \((Y, A)\) and all nonnegative integers \( i \leq i_0 \) and \( j \leq g_Z + i_0 + j_0 + 1 \), and for \((C, Q)\) replaced by \((Z, B)\) and all nonnegative integers \( i \leq i_0 \) and \( j \leq g_Y + i_0 + j_0 + 1 \).

Let \( C_0 \) be the curve of compact type that is the union of \( Y \), of \( Z \), and of a chain of rational curves \( E_1, \ldots, E_{n-1} \) connecting \( A \) to \( B \), where \( n \geq 2 \). Our convention is that \( E_1 \) contains \( A \) and \( E_{n-1} \) contains \( B \). Let \( v \) be any integer such that \( 0 < v < n \), and let \( Q_0 \) be any point of \( E_v \) that is not a node of \( C_0 \).

Let \( S := \text{Spec}(\mathbb{C}[[t]]) \), and denote its special point by 0 and generic point by \( \eta \). Since there are no obstructions to deforming pointed nodal curves, there are a projective, flat map \( f: X \to S \) and a section \( \lambda: S \to X \) of \( f \) such that \((X(0), \lambda(0)) = (C_0, Q_0)\) and \((X(\eta), \lambda(\eta))\) is a smooth pointed curve over the field of formal Laurent series \( \mathbb{C}[[t]][1/t] \).

Let \( C \) be the base extension of \( X(\eta) \) to the algebraic closure of \( \mathbb{C}[[t]][1/t] \). Set \( Q := \lambda(\eta) \). It is enough to see that the statements of the lemma hold for \((C, Q)\). Indeed, the argument is quite standard, and is summarized below. Though the pointed curve \((C, Q)\) is not defined over \( \mathbb{C} \), it is defined over a finitely generated extension \( L \) of \( \mathbb{Q} \). If the statements of the lemma hold for \((C, Q)\), they also hold for the base extension of \((C, Q)\) over any algebraically closed field containing \( L \). But, since \( \mathbb{C} \) has many transcendentals over \( \mathbb{Q} \), there is an algebraically closed field containing \( L \) which is isomorphic to \( \mathbb{C} \). So, if the statements of the lemma hold for \((C, Q)\), they hold as well for some pointed curve over \( \mathbb{C} \).

Now, any finite set of points of \( C \) is defined over a finite field extension of \( \mathbb{C}[[t]][1/t] \). Replacing \( S \) by its normalization in this field extension, we may assume that these are rational points of \( X(\eta) \), and thus that there are sections of \( f \) intersecting \( X(\eta) \) at them. By making a further base extension, if necessary, and a sequence of blowups at the singular points of the special fiber, we may assume that the total space \( X \) is regular, and that these sections factor through the smooth locus of \( f \). The compensation for this is a change of the special fiber. However, the special fiber will have the same specification.
as the $C_0$ we described above. Thus, no confusion will ensue if we keep calling by $C_0$ this new fiber. Also, the section $\lambda$ can be extended to a section of this new family.

Now, let $P$ and $R$ be points of $C$ with $P$ distinct from $Q$ and $R$ distinct from $P$ and $Q$. As we mentioned above, we may assume there are sections $\gamma: S \to X$ and $\rho: S \to X$ through the smooth locus of $f$ such that $\gamma(\eta) = P$ and $\rho(\eta) = R$. Set $P_0 := \gamma(0)$ and $R_0 := \rho(0)$. Let $\Gamma$ and $\Lambda$ be the images of $\gamma$ and $\lambda$, respectively.

Fix nonnegative integers $i \leq i_0$ and $j \leq j_0$. Let $\omega$ be the relative dualizing bundle of $f: X \to S$. Let $V_\eta$ be the linear system of sections of the line bundle $\omega(\eta)((i + 1)P + (j + 1)Q)$ given by

$$V_\eta := H^0(\omega(\eta)((i + 1)P)) + H^0(\omega(\eta)((j + 1)Q)).$$

Assume that $R$ is a ramification point of $V_\eta$. To prove the statements of the lemma hold for $(C, Q)$, it is enough to prove the following three statements:

1. For $i = 0$, the system $V_\eta$ ramifies at $Q$ with weight $g$, and $R$ is a simple ramification point of $V_\eta$.

2. For $j = 0$, the point $Q$ is a ramification point of $V_\eta$ of weight $g + i$ or $g + i + 1$.

3. $\dim V_\eta(-(g + i + j)R) \leq 1$ and $V_\eta(-(g + i + j + 2)R) = 0$.

We will employ techniques of limit linear series, from [7], to show the above three statements. There are two cases to consider:

**Case 1:** Assume that $P_0 \in E_u$ for some $u$.

Since $C_0$ is of compact type, there is an effective divisor $D$ of $X$ supported on $C_0$ such that, letting

$$L := \omega((i + 1)\Gamma + (j + 1)\Lambda + D),$$

we have $L|_{E_m} \cong \mathcal{O}_{E_m}$ for each $m = 1, \ldots, n - 1$,

$$L|_Z \cong \omega_Z((gY + i + j + 3)B) \quad \text{and} \quad L|_Y \cong \omega_Y((1 - gY)A).$$

Since, from the induction hypothesis, $A$ is a ramification point of weight $gY$ of the complete linear system of sections of $\omega_Y(A)$, the point $A$ is not a Weierstrass point of $Y$. Then $V := H^0(X, L) \cap V_\eta$ restricts to a linear system
$V_Z$ of dimension $g + i + j$ of sections of $\omega_Z((g_Y + i + j + 3)B)$. Also from the induction hypothesis, $B$ is not a Weierstrass point of $Z$. So the order sequence of $B$ in the complete linear system of sections of $\omega_Z((g_Y + i + j + 3)B)$ is

$$0, 1, \ldots, g_Y + i + j + 1, g_Y + i + j + 3, \ldots, g + i + j + 2.$$  

As a consequence, the weight $w_B$ of $B$ as a ramification point of the linear system $V_Z$ satisfies

$$w_B \leq 2(g_Y + i + j) + 3g_Z, \quad (5)$$

with equality if and only if $V_Z = H^0(\omega_Z((g_Y + i + j + 1)B))$.

Analogously, choosing an appropriate $D$, we obtain a linear system $V_Y$ of dimension $g + i + j$ of sections of $\omega_Y((g_Z + i + j + 3)A)$, and the weight $w_A$ of $A$ as a ramification point of $V_Y$ satisfies

$$w_A \leq 2(g_Z + i + j) + 3g_Y, \quad (6)$$

with equality if and only if $V_Y = H^0(\omega_Y((g_Z + i + j + 1)A))$.

Let $r := g + i + j - 1$. Using the Plücker formula, the number $N$ of ramification points of $V_Y$ and $V_Z$ on $(Y - A) \cup (Z - B)$, counted with their respective weights, satisfies

$$N = (r + 1)\left((2g_Z + g_Y + i + j + 1) + r(g_Z - 1)\right) - w_B$$
$$+ (r + 1)\left((2g_Y + g_Z + i + j + 1) + r(g_Y - 1)\right) - w_A$$
$$= N' + 5g + 4(i + j) - w_A - w_B,$$

where

$$N' := (r + 1)\left((2g + i + j) + r(g - 1)\right) - 2g - i - j.$$

Now, from the theory of limit linear series, each one of the ramification points of $V_Y$ or $V_Z$ on $(Y - A) \cup (Z - B)$ is a limit of ramification points of $V_\eta$, and its weight as a ramification point is the sum of the weights of the ramification points of $V_\eta$ converging to it. Besides those, since $P$ and $Q$ are ramification points of $V_\eta$ with weights at least $g + j$ and $g + i$, respectively, the points $P_0$ and $Q_0$ appear as limits of ramification points of $V_\eta$ with weights summing up to at least $2g + i + j$. Thus, from the Plücker formula, at most $N'$ ramification points of $V_\eta$, counted with their weights, converge to $(Y - A) \cup (Z - B)$. So

$$5g + 4(i + j) - w_A - w_B \leq 0.$$
However, Inequalities (5) and (6) for \( w_B \) and \( w_A \) yield the opposite inequality:

\[
5g + 4(i + j) - w_A - w_B \geq 0.
\]

Thus, equalities hold, and hence

\[
V_Y = H^0(\omega_Y((g\ Z + i + j + 1)A)) \quad \text{and} \quad V_Z = H^0(\omega_Z((g\ Y + i + j + 1)B)).
\]

In addition, \( P \) and \( Q \) are ramification points of \( V_\eta \) of weights \( g + j \) and \( g + i \), respectively, and all the other ramification points of \( V_\eta \) converge to \((Y-A)\cup(Z-B)\). In particular, Statement 2 and the first part of Statement 1 are shown.

Now, since \( R \) is a ramification point of \( V_\eta \), and \( R \) is distinct from \( P \) and \( Q \), we have \( R_0 \in (Y-A)\cup(Z-B) \). So \( R_0 \) is a ramification point of either \( V_Y \) or \( V_Z \). From the induction hypothesis, the complete linear systems of sections of \( \omega_Y((g\ Z + i + j + 1)A) \) and \( \omega_Z((g\ Y + i + j + 1)B) \) have at most simple ramification points, other than \( A \) or \( B \). Thus \( R \) is the unique ramification point of \( V_\eta \) converging to \( R_0 \) and its weight is 1. So the remainder of Statement 1 is shown.

As for Statement 3, assume, without loss of generality, that \( R_0 \in Z \).

Set \( n := \dim V_\eta(-(g + i + j)R) \), and let \( \sigma_1, \ldots, \sigma_n \) form a \( \mathbb{C}[[t]] \)-basis of \( V\cap V_\eta(-(g + i + j)R) \). Their restrictions to \( Z \) are sections of \( V_Z \) vanishing with multiplicity at least \( g + i + j \) on \( R_0 \). Assume, by contradiction, that \( n \geq 2 \).

Since \( R_0 \) is a simple ramification point of \( V_Z \), the sections \( \sigma_1|_Z, \ldots, \sigma_n|_Z \) are linearly dependent. Thus, there is a nonzero \( n \)-tuple \((c_1, \ldots, c_n) \in \mathbb{C}^n \) such that \( c_1\sigma_1 + \cdots + c_n\sigma_n \) vanishes on \( Z \), and hence on the whole \( C_0 \). Thus

\[
c_1\sigma_1 + \cdots + c_n\sigma_n = t\sigma
\]

for some \( \sigma \in H^0(X, L) \). Also \( \sigma \in V_\eta(-(g + i + j)R) \), and hence \( \sigma \) is a \( \mathbb{C}[[t]] \)-linear combination of \( \sigma_1, \ldots, \sigma_n \). Plugging this linear combination in (7) we obtain a nontrivial \( \mathbb{C}[[t]] \)-linear relation among the sections \( \sigma_i \), a contradiction. Thus \( n \leq 1 \). A similar analysis, using that \( V_Z(-(g + i + j + 2)R_0) = 0 \), shows that \( V_\eta(-(g + i + 2)R) = 0 \), finishing the proof of Statement 3.

Case 2: Assume \( P_0 \) belongs to either \( Y \) or \( Z \).

Without loss of generality, we may assume that \( P_0 \in Z \). Again, since \( C_0 \) is of compact type, there is an effective divisor \( D \) of \( X \) supported on \( C_0 \) such that, letting

\[
L := \omega((i + 1)\Gamma + (j + 1)\Lambda + D),
\]
we have $\mathcal{L}|_{E_m} \cong \mathcal{O}_{E_m}$ for each $m = 1, \ldots, n - 1$,

$$
\mathcal{L}|_Z \cong \omega_Z((i + 1)P_0 + (g_Y + j + 2)B) \quad \text{and} \quad \mathcal{L}|_Y \cong \omega_Y((1 - g_Y)A).
$$

As before, $V := H^0(X, \mathcal{L}) \cap V_\eta$ restricts to a linear system $V_Z$ of dimension $g + i + j$ of sections of $\omega_Z((i + 1)P_0 + (g_Y + j + 2)B)$.

Now,

$$
V \supseteq H^0(X, \omega((i + 1)\Gamma)) + H^0(X, \omega((j + 1)\Lambda + D)).
$$

Reasoning as in Case 1, we can show that $H^0(X, \omega((j + 1)\Lambda + D))$ restricts to $H^0(\omega_Z((g_Y + j + 1)B))$. On the other hand, the exact sequence

$$
0 \rightarrow H^0(\omega_Z((i + 1)P_0)) \rightarrow H^0(\omega((i + 1)\Gamma)|_{C_0}) \rightarrow H^0(\omega_Y(\Lambda))
$$

shows that $h^0(\omega((i + 1)\Gamma)|_{C_0}) = g + i$, and hence that $H^0(X, \omega((i + 1)\Gamma))$ restricts to a vector subspace of $H^0(\omega_Z((i + 1)P_0 + B))$ containing the subspace $H^0(\omega_Z((i + 1)P_0))$. Thus

$$
V_Z \supseteq H^0(\omega_Z((i + 1)P_0)) + H^0(\omega_Z((g_Y + j + 1)B)),
$$

and a dimension count shows that equality holds.

The weight $w_B$ of $B$ as a ramification point of $V_Z$ depends on its weight as a ramification point of $V_Z(i, P_0)$. Now, from the induction hypothesis, $B$ is either an ordinary point or a simple ramification point of $V_Z(i, P_0)$. Hence, the order sequence at $B$ of the linear system $V_Z$ is either

$$
1, 2, \ldots, g_Y + j, g_Y + j + 2, g_Y + j + 3, \ldots, g + i + j, g + i + j + 1
$$
or

$$
1, 2, \ldots, g_Y + j, g_Y + j + 2, g_Y + j + 3, \ldots, g + i + j, g + i + j + 2.
$$

At any rate,

$$
w_B \leq g_Y + j + 2(g_Z + i) + 1. \quad (8)
$$

Notice that, if $i = 0$, then $B$ is an ordinary point of $V_Z(0, P_0)$, as it is an ordinary point of $Z$, and thus Inequality (8) is strict.

On the other hand, let $D'$ be an effective divisor of $X$ supported on $C_0$ such that, letting

$$
\mathcal{M} := \omega((i + 1)\Gamma + (j + 1)\Lambda + D'),
$$

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we have $M|E_m \cong O_{E_m}$ for each $m = 1, \ldots, n - 1$,

$$M|_Y \cong \omega_Y((g_Z + i + j + 3)A) \text{ and } M|_Z \cong \omega_Z((i + 1)P_0 - (g_Z + i)B).$$

Since, as mentioned above, $B$ is either an ordinary point or a simple ramifica-
tion point of $V_Z(i, P_0)$, we have that $H^0(X, M) \cap V_\eta$ restricts to a linear
system $V_\eta$ of dimension $g + i + j$ of sections of $\omega_Y((g_Z + i + j + 3)A)$.

Since $A$ is not a Weierstrass point of $Y$, the sequence of orders at $A$ of
the complete linear system of sections of $\omega_Y((g_Z + i + j + 3)A)$ is

$$0, 1, \ldots, g_Z + i + j + 1, g_Z + i + j + 3, g_Z + i + j + 4, \ldots, g + i + j + 2.$$ 

Since $V_\eta$ has codimension 2 in $H^0(\omega_Y((g_Z + i + j + 3)A))$, the weight $w_A$ of
$V_\eta$ at $A$ satisfies

$$w_A \leq 2(g_Z + i + j) + 3g_Y, \quad (9)$$

with equality if and only if $V_\eta = H^0(\omega_Y((g_Z + i + j + 1)A))$.

As in Case 1, using the Plücker formula, the number $N$ of ramification
points of $V_\eta$ and $V_\xi$ on $(Y - A) \cup (Z - B)$, counted with their respective
weights, satisfies

$$N = N' + 4g + 4i + 3j - w_A - w_B,$$

where

$$N' := (g + i + j)(2g + i + j) + (g + i + j)(g + i + j - 1)(g - 1) - g - i.$$

As in Case 1, since $Q$ is a ramification point of $V_\eta$ with weight at least $g + i$,
there are at most $N'$ ramification points of $V_\eta$, counted with their respective
weights, converging to $(Y - A) \cup (Z - B)$. So

$$4g + 4i + 3j - w_A - w_B \leq 0.$$

On the other hand, Inequalities (8) and (9) yield

$$w_A + w_B \leq 4g + 4i + 3j + 1.$$

In particular, $w_A \geq 2(g_Z + i + j) + 3g_Y - 1$, whence

$$V_\eta \subset H^0(\omega_Y((g_Z + i + j + 2)A)).$$
Also, $Q$ has weight $g + i$ or $g + i + 1$ in $V_\eta$. Thus Statement 2 is shown. Furthermore, if $i = 0$ we have $w_A + w_B = 4g + 4i + 3j$. In this case, $V_\gamma = H^0(\omega_\gamma((gZ + i + j + 1)A))$ and $Q$ has weight $g + i$ in $V_\eta$, showing the first part of Statement 1.

If $Q$ has weight $g + i + 1$ in $V_\eta$, all other ramification points converge to $(Y - A) \cup (Z - B)$. If $Q$ has weight $g + i$, there is at most one ramification point of $V_\eta$, other than $Q$, converging outside $(Y - A) \cup (Z - B)$, and that point is simple. If $R$ is that point, then $\dim V_\eta(-(g + i + j)R) \leq 1$ and $V_\eta(-(g + i + j + 2)R) = 0$ because of the simplicity of $R$.

Assume now that $R_0 \in (Y - A) \cup (Z - B)$. Let us first consider the case $R_0 \in Y - A$. In this case, since, from the induction hypothesis, the complete linear system of sections of $\omega_\gamma((gZ + i + j + 2)A)$ has at most simple ramification points, other than $A$, we have

$$h^0(\omega_\gamma((gZ + i + j + 2)A - (g + i + j)R_0) \leq 1,$$
$$h^0(\omega_\gamma((gZ + i + j + 2)A - (g + i + j + 2)R_0) = 0.$$  

Thus $\dim V_\gamma(-(g + i + j)R_0) \leq 1$ and $V_\gamma(-(g + i + j + 2)R_0) = 0$ as well. It follows, as in Case 1, that

$$\dim V_\eta(-(g + i + j)R) \leq 1 \quad \text{and} \quad V_\eta(-(g + i + j + 2)R) = 0.$$  

Furthermore, if $i = 0$, since in this case $V_\gamma = H^0(\omega_\gamma((gZ + i + j + 1)A))$, all the ramification points of $V_\gamma$ distinct from $A$ are simple. Thus $R_0$ is simple in $V_\gamma$, and hence $R$ is simple in $V_\eta$.

Assume now that $R_0 \in Z - B$. There are two cases to consider. First, assume $R_0 = P_0$. Since, by induction hypothesis, the complete linear system of sections of $\omega_Z((gY + j + 1)B)$ has at most simple ramification points other than $B$, the weight of $P_0$ as a ramification point of $V_Z$ is either $g + j$ or $g + j + 1$. Since $P$ has at least weight $g + j$ in $V_\eta$, and $R \neq P$, the latter must hold, and $R$ must be a simple ramification point of $V_\eta$. In particular, $\dim V_\eta(-(g + i + j)R) \leq 1$ and $V_\eta(-(g + i + j + 2)R) = 0$.

Finally, assume $R_0 \neq P_0$. Then

$$\dim V_Z(-(g + i + j)R_0) \leq 1 \quad \text{and} \quad V_Z(-(g + i + j + 2)R_0) = 0$$  

from the induction hypothesis, and hence $\dim V_\eta(-(g + i + j)R) \leq 1$ and $V_\eta(-(g + i + j + 2)R) = 0$. Thus Statement 3 is shown. Also, if $i = 0$, then $V_Z = H^0(\omega_Z((gY + j + 1)B))$, and, since $R_0 \neq P_0$, the weight of $R_0$
in $V_Z$ is equal to its weight in the complete linear system of sections of $\omega_Z((g_Y + j + 1)B)$. By induction hypothesis, this weight is one, and thus $R$ is a simple ramification point of $V_\eta$. So Statement 1 is shown.

3.3 Corollary. If $C$ is a general smooth curve of genus $g \geq 1$, then all its Weierstrass points are simple.

**Proof.** Apply Statement 1 of Proposition 3.1 for $i_0 := 0$ and $i := 0$.

3.4 Proposition. Fix an integer $i_0 \geq 0$. Let $C$ be a general smooth curve of genus $g \geq 1$. Then for any two distinct points $P$ and $R$ of $C$, and any nonnegative integer $i \leq i_0$,

$$h^0(C, \omega_C((i+1)P - (g+i-2)R)) = 2.$$  

**Proof.** A line bundle of degree 2 on an elliptic curve has (at most) 2 linearly independent sections. Thus we may assume $g \geq 2$. Also, for $i = 0$,

$$h^0(C, \omega_C((i+1)P - (g+i-2)R)) = h^0(C, \omega_C(-(g-2)R)) = 2,$$

since $R$ is at most a simple Weierstrass point of $C$, a consequence of Corollary 3.3. So we need only show the stated equality for integers $i > 0$.

For each integer $j \geq 2$ (resp. $j \geq 1$), let $M_j$ be the moduli space of smooth curves (resp. let $M_{j,1}$ be the moduli space of smooth pointed curves) of genus $j$. Let $\overline{M}_j$ and $\overline{M}_{j,1}$ denote their respective compactifications by stable (resp. stable, pointed) curves. For each positive integer $i \leq i_0$, let $D^{(i)} \subseteq M_{g+i}$ be the subset parameterizing curves admitting a covering of degree at most $g+i-2$ of the projective line totally ramified at a point. By [1], Thm. 3.11, p. 333, the subvariety $D^{(i)}$ is irreducible of codimension 2. Let $\overline{D}^{(i)} \subset \overline{M}_{g+i}$ be the closure of $D^{(i)}$.

Let $\mu_i: M_{g,1} \times M_{i,1} \to \overline{M}_{g+i}$ be the natural map, associating to a pair of smooth pointed curves the stable uninodal curve which is the union of these curves identified at the marked points. Let $E^{(i)} := \mu_i^{-1}(\overline{D}^{(i)})$. Let $\rho_i: E^{(i)} \to M_g$ be the natural map, forgetting the second pointed curve and the marked point on the first curve. Since $C$ is general, we may assume that, for each $i = 1, \ldots, i_0$, the curve $C$ is parameterized by a point of $M_g$ over which the fiber of $\rho_i$ has minimum dimension. We claim this dimension is at most $3i - 3$, whence less than $\dim M_{i,1}$. Indeed, if the dimension were larger, then $E^{(i)}$ would have codimension at most 1 in $M_{g,1} \times M_{i,1}$, and
hence would dominate $D^{(i)}$ under $\mu_i$. So $D^{(i)}$ would be contained in the boundary $M_{g+i} - M_{g+i}$, an absurd. From the claim, for each $i = 1, \ldots, i_0$, the general smooth pointed curve $(Y_i, B_i)$ of genus $i$ is such that, for any $P \in C$, the pair of pointed curves $((C, P), (Y_i, B_i))$ is not parameterized by $E^{(i)}$. Consequently, the stable unimodal curve $X_i$, union of $C$ and $Y_i$ with $P$ and $B_i$ identified, is parameterized by a point of $\overline{M}_g$ off $D^{(i)}$, for each $i = 1, \ldots, i_0$.

Suppose, by contradiction, that for certain distinct points $P$ and $Q$ of $C$, and a certain positive integer $i \leq i_0$, we have

$$h^0(C, \omega_C((i+1)P - (g+i-2)Q)) \geq 3.$$  

Put $g' := g + i$. Since, by Riemann–Roch, $h^0(C, \omega_C((i+1)P - iQ)) = g$, there is an integer $j$ with $2 \leq j < g$ such that

$$h^0(C, \omega_C((i+1)P - (g' - j)Q)) = h^0(C, \omega_C((i+1)P - (g' - j - 1)Q)) = j + 1.$$  

Again by Riemann–Roch,

$$h^0(C, \mathcal{O}_C((g' - j)Q - (i+1)P)) > h^0(C, \mathcal{O}_C((g' - j - 1)Q - (i+1)P)).$$  

Thus, there is a map $\phi: C \to \mathbb{P}^1$ of degree $g' - j$ such that $\phi^*(0) = (g' - j)Q$ and $\phi^*(\infty) \geq (i+1)P$. Let $i'$ be the integer such that $i' + 1$ is the multiplicity of $P$ in $\phi^*(\infty)$. Then $i' \geq i$.

Set $Y := Y_i$ and $B := B_i$. Since $B$ is general, $B$ is not a Weierstrass point of $Y$. Thus, since $i' \geq i$, we have $h^0(Y, \mathcal{O}_Y(i'B)) < h^0(Y, \mathcal{O}_Y((i' + 1)B))$. So, there is a map $\psi: Y \to \mathbb{P}^1$ of degree $i' + 1$ such that $\psi^*(\infty) = (i' + 1)B$.

Putting together the maps $\phi$ and $\psi$, we may construct the covering with source $X_i$ depicted in Figure 1 below, which can be represented by a point $[X_i]$ of the (compactification of the) Hurwitz scheme parameterizing (pseudo)admissible coverings of the projective line of degree $(g' - j)$ totally ramified at a point; see Remark 3.5. Since coverings of $\mathbb{P}^1$ form a dense open subscheme of this compactification, the curve $X_i$ is limit of smooth curves equipped with a degree-$(g' - j)$ map to the projective line totally ramified at a point. Since $j \geq 2$, it follows that $[X_i]$ lies on the boundary of $D^{(i)}$, a contradiction.

3.5 Remark. The Hurwitz scheme we used in the proof of Proposition 3.4 is mentioned in [6], Section 5. It can be constructed following the same
reasoning used in the construction of the Hurwitz scheme of (simple) admissible coverings, given in the proof of [10], Thm. 4, p. 58. Also, the local descriptions of both schemes are the same, given on [10], p. 62. From this description we see that the Hurwitz scheme is equidimensional. Now, there is a natural forgetful map from the Hurwitz scheme to a corresponding moduli space of pointed genus-0 curves, taking a covering to its target. This map is finite and surjective, also by [10], Thm. 4, p. 58. Since the moduli spaces of pointed genus-0 curves are irreducible (see [12] or [11]), it follows that each irreducible component of the Hurwitz scheme covers the target. So coverings of $\mathbb{P}^1$ form a dense open subscheme of the Hurwitz scheme, a fact used in the proof of Proposition 3.4.

3.6 Remark. We tried to prove Proposition 3.4 using the same induction argument used in the proof of Lemma 3.2. However, we could not prove the initial step, that is, the following statement: Let $C$ be a general elliptic curve, $Q \in C$ a general point, and $P \in C \setminus \{Q\}$ any point. Let $i$ and $j$ be nonnegative integers. Then the linear system $V$ of sections of the line bundle $\omega_C((i+1)P+(j+1)Q)$ generated by $H^0(\omega_C((i+1)P))$ and $H^0(\omega_C((j+1)Q))$ satisfies $\dim V(-(i+j-1)R) = 2$ for each $R \in C \setminus \{P,Q\}$.
4 Weierstrass divisors

4.1 (Wronski maps) Let $C$ be a smooth curve of genus $g$. For each integer $j \geq 0$, consider the family of linear systems $V_C(j, P)$ for $P$ varying on $C$. More precisely, let $p_1$ and $p_2$ denote the projections of $C \times C$ onto the first and second factors, and $\Delta \subset C \times C$ the diagonal. The relative canonical bundle of $p_1$ is simply the pullback $p_2^* \omega_C$ of the canonical bundle $\omega_C$ of $C$. For each integer $j \geq 0$, let

$$\mathcal{L}_j := p_2^* \omega_C((j + 1)\Delta), \quad \mathcal{E}_j := p_1^* \mathcal{L}_j.$$  

Notice that, for each point $P$ of $C$, identifying $\{P\} \times C$ with $C$ in the natural way, $\mathcal{L}_j|_{\{P\} \times C} = \omega_C((j + 1)P)$. Also, as $h^0(\omega_C((j + 1)P)) = g + j$ for every $P \in C$, the sheaf $\mathcal{E}_j$ is a bundle of rank $g + j$ and $\mathcal{E}_j|_P = H^0(\omega_C((j + 1)P))$.

For each integer $\ell \geq 0$ and each line bundle $M$ on $C \times C$, let $J^\ell_{p_1}(M)$ be the bundle of rank $\ell + 1$ of $p_1$-relative jets of $M$ truncated in order $\ell$. Let

$$\rho_{j,\ell}: p_1^* \mathcal{E}_j \to J^\ell_{p_1}(\mathcal{L}_j)$$

be the map of bundles locally obtained by differentiating up to order $\ell$ along the fibers of $p_1$ the sections of $\mathcal{L}_j$. We call $\rho_{j,\ell}$ a Wronski map.

The map $\rho_{j,g+j-1}$ is a map of bundles of the same rank. Taking determinants, we get a section $z_j$ of the line bundle

$$\bigwedge^{g+j} J^g_{p_1}(\mathcal{L}_j) \otimes \bigwedge^{g+j} p_1^* \mathcal{E}_j^\vee,$$

which is naturally isomorphic, using the truncation sequence of the bundles of jets, to

$$p_2^* \omega_C((j + 1)\Delta)^{\otimes g+j} \otimes p_2^* \omega_C^{\otimes (g+j)(g+j-1)/2} \otimes \bigwedge^{g+j} p_1^* \mathcal{E}_j^\vee,$$

or more simply to

$$p_2^* \omega_C^{\otimes (g+j)(g+j+1)/2}((g + j)(j + 1)\Delta) \otimes \bigwedge^{g+j} p_1^* \mathcal{E}_j^\vee.$$

4.2 (Weierstrass divisors.) Keep the notation used in Subsection 4.1. Let $Z_j \subseteq C \times C$ denote the zero scheme of $z_j$. The section $z_j$ is a relative wronskian. More precisely, for each $P \in C$, on $\{P\} \times C$, identified with $C$ in
the natural way, the section $z_j$ restricts to the wronskian of the linear system $V_C(j, P)$. Hence, $Z_j$ consists of the pairs $(P, Q) \in C \times C$ such that $V_C(j, P)$ ramifies at $Q$. Now, since $z_j$ is nonzero, being so on each fiber, $Z_j$ is a Cartier divisor. By Proposition 2.2 the divisor $Z_j$ intersects each fiber $\{P\} \times C$ at $(P, P)$ with multiplicity $g + \text{wt}(P)$, where $\text{wt}(P)$ is the weight of $P$ in the canonical system of $C$. Thus $Z_j$ contains $\Delta$ with multiplicity exactly $g$. Let

$$W_j := Z_j - g\Delta.$$  

Then $W_j$ is, set-theoretically, the locus of pairs $(P, Q) \in C \times C$ such that either $P = Q$ and $P$ is a Weierstrass point of $C$, or $P \neq Q$ and $Q$ is a ramification point of $V_C(j, P)$. We call $W_j$ the $j$-th Weierstrass divisor of $C$.

4.3 Proposition. Let $C$ be a smooth curve of genus $g \geq 1$ and $j$ a nonnegative integer. Let $\Delta$ be the diagonal of $C \times C$, and $p_1$ and $p_2$ the projections of $C \times C$ onto the indicated factors. Let $\omega_C$ be the canonical bundle of $C$, and set $K_\ell := c_1(p_\ell^* \omega_C)$ for $\ell = 1, 2$. Let $W_j \subseteq C \times C$ be the $j$-th Weierstrass divisor of $C$. Then its class $[W_j]$ in the Chow group of $C \times C$ satisfies

$$[W_j] = \frac{1}{2}(g + j)(g + j + 1)K_2 + j(g + j + 1)[\Delta] + \frac{1}{2}j(j + 1)K_1.$$  

(11)

Proof. Use the notation in Subsections 4.1 and 4.2. Since $W_j = Z_j - g\Delta$, and $Z_j$ is the zero scheme of a section of the line bundle

$$p_2^* \omega_C^{\otimes (g+j)(g+j+1)/2} \left((g + j)(j + 1)\Delta\right)^{\otimes \frac{g+j}{2}} \bigwedge \text{p}_1^* \mathcal{E}_j^{\vee},$$

we get

$$[W_j] = \frac{1}{2}(g + j)(g + j + 1)K_2 + j(g + j + 1)[\Delta] - p_1^* c_1(\mathcal{E}_j).$$  

(12)

To finish, we need only show that

$$c_1(\mathcal{E}_j) = -\frac{1}{2}j(j + 1)c_1(\omega_C).$$  

(13)

We show (13) by induction on $j$. First of all,

$$\mathcal{E}_0 = p_1^* p_2^* \omega_C = H^0(\omega_C) \otimes O_C.$$  

Since $\mathcal{E}_0$ is free, $c_1(\mathcal{E}_0) = 0$.  

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Assume now that \( j > 0 \) and \( c_1(E_{j-1}) = -(j(j-1)/2)c_1(\omega_C) \). Consider the natural short exact sequence

\[
0 \to p^*_2\omega_C(j\Delta) \to p^*_2\omega_C((j+1)\Delta) \to p^*_2\omega_C((j+1)\Delta)|_\Delta \to 0.
\]

Since \( H^1(\omega_C(jP)) = 0 \) for each \( P \in C \), applying \( p_1^* \) to the sequence above, we get the exact sequence

\[
0 \to E_{j-1} \to E_j \to p_1^*p^*_2\omega_C((j+1)\Delta)|_\Delta \to 0.
\]

Now, \( p_\ell|_\Delta \) is an isomorphism for \( \ell = 1, 2 \). So \( p_1^*p^*_2\omega_C|_\Delta = \omega_C \). In addition, \( p_1^*O_{\Delta} = \omega_C \). Thus

\[
c_1(E_j) = c_1(E_{j-1}) + c_1(p_1^*p^*_2\omega_C((j+1)\Delta)|_\Delta)
\]

\[
= -(j(j-1)/2)c_1(\omega_C) + (1-(j+1))c_1(\omega_C)
\]

\[
= -(j(j+1)/2)c_1(\omega_C),
\]

as claimed. \( \blacksquare \)

4.4 Proposition. Let \( C \) be a general smooth curve of genus \( g \geq 1 \) and \( j \) a nonnegative integer. Let \( W_j \subseteq C \times C \) be the \( j \)-th Weierstrass divisor of \( C \). Then \( W_j \) is nonsingular and intersects the diagonal \( \Delta \) transversally, at the pairs \((P,P)\) such that \( P \) is a Weierstrass point of \( C \).

Proof. Let us show first that \( W_j \) intersects \( \Delta \) transversally. As pointed out in Subsection 4.2, the intersection \( W_j \cap \Delta \) is, set-theoretically, the set of pairs \((P,P)\) such that \( P \) is a Weierstrass point of \( C \). As \( C \) is general, by Corollary 3.3 all its Weierstrass points are simple, and number \( g^3 - g \) by Plücker Formula. Now, since the intersection \( W_j \cap \Delta \) is finite, the number of points of intersection, weighted by their intersection multiplicities, is equal to the degree of the product \([W_j][\Delta]\). Using the notation and Formula (11) of Proposition 4.3, and using the Formulas

\[
\int_{C \times C} K_2[\Delta] = \int_{C \times C} K_1[\Delta] = - \int_{C \times C} [\Delta]^2 = 2g - 2 \tag{14}
\]

and

\[
\int_{C \times C} K_1K_2 = 4(g-1)^2, \tag{15}
\]

we get

\[
\int_{C \times C} [W_j][\Delta] = (g+j)(g+j+1)(g-1) - 2j(g+j+1)(g-1) + j(j+1)(g-1),
\]

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which is exactly $g^3 - g$. Thus the intersection multiplicities are all one.

As a corollary of the transversal intersection, $W_j$ is nonsingular at its points on $\Delta$. So, let now $(P, Q) \in W_j$ for $P$ and $Q$ distinct, and let us show that $W_j$ is nonsingular at $(P, Q)$ as well.

Let $J := \text{Pic}^{g-1}(C)$, the component of the Picard scheme of $C$ parameterizing line bundles of degree $g - 1$. Let $\Theta \subseteq J$ be the theta divisor, parameterizing line bundles with nontrivial global sections. Let

$$
\mu: C \times C \to J
$$

be the map taking a pair $(R, S)$ to the point of $J$ representing the bundle $\omega_C((j + 1)R - (g + j)S)$.

We claim that $\mu(W_j) \subseteq \Theta$. Indeed, let $C^{(3)} := C \times C \times C$, and denote by $p_{1,2}$ and $p_3$ the projection maps of $C^{(3)}$ onto the indicated factors. Let $\Delta_{1,3}$ and $\Delta_{2,3}$ be the indicated diagonals of $C^{(3)}$. Set

$$
F := p_3^*\omega_C((j + 1)\Delta_{1,3} - (g + j)\Delta_{2,3}).
$$

Recall the notation of Subsection 4.1. From the construction of $\Theta$, to show that $\mu(W_j) \subseteq \Theta$, it is enough to show that the Wronski map $\rho_{j,g+j-1}$ represents universally the cohomology of $F$ or, put more simply, that $\rho_{j,g+j-1}$ can be viewed as a presentation of the right derived image $R^1(p_{1,2})_*F$.

Let $G := p_3^*\omega_C((j + 1)\Delta_{1,3})$. Then $F \subseteq G$. From the definition of the Wronski map $\rho_{j,g+j-1}$, we get that $\rho_{j,g+j-1}$ is the image under $(p_{1,2})_*$ of the quotient map $G \to G \to F$. Thus, the map $\rho_{j,g+j-1}$ is the first map in the following piece of the long derived sequence of $0 \to F \to G \to G \to R^1(p_{1,2})_*F \to 0$

$$
(p_{1,2})_*G \to (p_{1,2})_*(G/F) \to R^1(p_{1,2})_*F \to R^1(p_{1,2})_*G.
$$

Now, a fiberwise analysis shows that $R^1(p_{1,2})_*G = 0$. Thus $\rho_{j,g+j-1}$ is a presentation for $R^1(p_{1,2})_*F$, finishing the proof that $\mu(W_j) \subseteq \Theta$.

Let $\mathcal{L} := \omega_C((j + 1)P - (g + j)Q)$, and denote by $[\mathcal{L}]$ the point of $J$ representing $\mathcal{L}$. Since $(P, Q) \in W_j$, we have $[\mathcal{L}] \in \Theta$. By Proposition 3.1, $h^0(C, \mathcal{L}) = 1$. Thus, it follows from [2], Prop. (4.2), p. 189, that $[\mathcal{L}]$ is a nonsingular point of $\Theta$. Furthermore, identifying the cotangent space of $J$ at $[\mathcal{L}]$ with $H^0(C, \omega_C)$, the cotangent space of $\Theta$ at $[\mathcal{L}]$ is the quotient by the subspace $H^0(C, \omega_C(-F))$, where $F$ is the unique effective divisor of $C$ such that $\mathcal{L} = \mathcal{O}_C(F)$.
Identifying the cotangent space of $C \times C$ at $(P, Q)$ with $\omega_C|_P \oplus \omega_C|_Q$, the induced map of cotangent spaces $d\mu^*: T^*_J \rightarrow T^*_{C \times C, (P, Q)}$ is equivalent to the evaluation map,

$$\epsilon: H^0(C, \omega_C) \rightarrow \omega_C|_P \oplus \omega_C|_Q.$$ 

We claim that $\epsilon(H^0(C, \omega_C(-F))) \neq 0$. Indeed, if that were not the case, we would have $H^0(C, \omega_C(-F)) = H^0(C, \omega_C(-F - P - Q))$, that is,

$$h^0(C, \mathcal{O}_C((g + j - 1)Q - (j + 2)P)) = h^0(C, \mathcal{O}_C((g + j)Q - (j + 1)P)).$$

By the Riemann–Roch theorem,

$$h^0(C, \mathcal{O}_C((g + j)Q - (j + 1)P)) = h^0(C, \mathcal{L}) = 1,$$

and thus, also by the Riemann–Roch theorem,

$$h^0(C, \omega_C((j + 2)P - (g + j - 1)Q)) = 3.$$ 

However, this contradicts Proposition 3.4.

Since $\mu(W_j) \subseteq \Theta$, the image of $\epsilon(H^0(C, \omega_C(-F)))$ in the cotangent space of $W_j$ at $(P, Q)$ is zero. Since $\epsilon(H^0(C, \omega_C(-F))) \neq 0$, that cotangent space is a proper quotient of the cotangent space of $C \times C$ at $(P, Q)$, and thus has dimension at most 1. Since $W_j$ is a divisor, it follows that $W_j$ is nonsingular at $(P, Q)$.

\section{5 Special ramification classes}

\subsection{5.1 (Special ramification loci.)} Let $C$ be a smooth curve of genus $g \geq 1$. For each nonnegative integer $i$, consider the following loci in $C \times C$:

1. The locus $D_i^+$ of pairs $(P, Q) \in C \times C$ such that

$$(g + i - 1)Q - (i + 1)P$$

is linearly equivalent to an effective divisor.

2. The locus $E_i^+$ of pairs $(P, Q) \in C \times C$ such that

$$(g + i + 1)Q - (i + 1)P$$

is linearly equivalent to a moving effective divisor.
3. The locus $SW_i^+$ of pairs $(P,Q) \in C \times C$ such that $Q$ is a special ramification point of $V_C(i,P)$.

We claim that, set-theoretically,

$$SW_i^+ = D_i^+ \cup E_i^+.$$  \hfill (16)

Indeed, by the Riemann–Roch Theorem, for a pair $(P,Q) \in C \times C$, the divisor $(g+i-1)Q-(i+1)P$ is linearly equivalent to an effective one if and only if

$$h^0(\omega_C((i+1)P-(g+i-1)Q)) \geq 2.$$  \hfill (17)

while $(g+i+1)Q-(i+1)P$ is linearly equivalent to a moving effective divisor if and only if

$$h^0(\omega_C((i+1)P-(g+i+1)Q)) \geq 1.$$  \hfill (18)

At any rate, if $(P,Q) \in D_i^+ \cup E_i^+$, then $Q$ is a special ramification point of $V_C(i,P)$, that is, $(P,Q) \in SW_i^+$.

On the other hand, let $(P,Q) \in C \times C - (D_i^+ \cup E_i^+)$. Then

$$h^0(\omega_C((i+1)P-(g+i-1)Q)) = 1,$$

$$h^0(\omega_C((i+1)P-(g+i+1)Q)) = 0.$$

So, either $Q$ is an ordinary or a simple ramification point of $V_C(i,P)$, that is, $(P,Q) \notin SW_i^+$.

Let $\Delta$ be the diagonal subscheme of $C \times C$. Notice that $E_i^+ \cap \Delta$ consists of the pairs $(P,P)$ such that $P$ is a Weierstrass point of $C$. However, if $g > 1$, both $D_i^+$ and $SW_i^+$ contain $\Delta$. (If $g = 1$, then $D_i^+ = E_i^+ = SW_i^+ = \emptyset$.)

Let $D_i$, $E_i$ and $SW_i$ be the loci of points in $D_i^+$, $E_i^+$ and $SW_i^+$ that lie off $\Delta$. Of course, Expression (16) implies $SW_i = D_i \cup E_i$. Our Proposition 5.4 claims that, if $C$ is general, then $SW_i = D_i \cup E_i$ holds in a more refined way, in the cycle group of $C \times C$. Before stating it, we need to endow $D_i$, $E_i$ and $SW_i$ with natural subscheme structures.

5.2 (Special ramification schemes) Keep the notation of Subsection 5.1, and recall that of Subsections 4.1 and 4.2. Notice that the subsets $D_i^+$ and $E_i^+$ are the supports of the degeneracy schemes of $\rho_{i,g+i-2}$ and $\rho_{i,g+i}$, respectively. So we may give $D_i^+$ and $E_i^+$ the corresponding scheme structures. Give $D_i$ and $E_i$ the corresponding open subscheme structures. We say that $D_i$ and $E_i$ are the $i$-th special ramification schemes of type Diaz and Cukierman,
respectively. Call $E_i^+$ the \textit{i-th expanded special ramification scheme of type Cukierman}.

In addition, differentiating along the fibers of $p_1$ a section of $\mathcal{O}_{C \times C}(Z_i)$ defining $Z_i$, we obtain a section of $\mathcal{J}_{p_1}^1(\mathcal{O}_{C \times C}(Z_i))$, well-defined modulo $\mathbb{C}^*$. By functoriality, its zero scheme contains a pair $(P, Q)$ if and only if $Q$ is a special Weierstrass point of $V_C(i, P)$. Thus the zero scheme gives a scheme structure for $SW_i^+$. Give $SW_i$ the induced open subscheme structure. We say that $SW_i$ is the \textit{i-th special ramification scheme of $C$}.

Now, $Z_i = W_i + g\Delta$. As done for $Z_i$, we can differentiate along the fibers of $p_1$ a section of $\mathcal{O}_{C \times C}(W_i)$ defining $W_i$ to obtain a section of $\mathcal{J}_{p_1}^1(\mathcal{O}_{C \times C}(W_i))$. Its zero scheme $S$ coincides with the scheme $SW_i$ off $\Delta$, because $Z_i$ coincides with $W_i$ there. Moreover, if $C$ is general, then $S$ does not intersect $\Delta$, and hence $S = SW_i$ scheme-theoretically. Indeed, let $P$ be a point of $C$. If $(P, P) \in W_i$, then $P$ is a Weierstrass point of $C$. Moreover, as $C$ is general, by Corollary 3.3 the point $P$ is a simple Weierstrass point. So, it follows from Proposition 2.2 that $W_i$ intersects the fiber $\{P\} \times C$ transversally at $(P, P)$. Thus the derivative along $\{P\} \times C$ of a section defining $W_i$ does not vanish at $(P, P)$. So $S \cap \Delta = \emptyset$.

5.3 \textbf{Lemma.} Let $\mathcal{O}$ be a local ring, and $r$ a nonnegative integer. Let $M$ be a matrix with $r + 2$ rows and $r + 1$ columns and entries in $\mathcal{O}$. Let $M_1$ and $M_2$ be the submatrices obtained from $M$ by removing the last row, and the last two rows, respectively. Assume that the matrix obtained from $M_1$ by taking residues has rank at least $r$. Let $z$ denote the determinant of $M_1$. Then there are $u, v \in \mathcal{O}$ such that

1. $(z, u)$ is the ideal of all maximal minors of $M_2$,
2. $(z, v)$ is the ideal of all maximal minors of $M$,
3. $(z, uv)$ is the ideal generated by the two maximal minors of $M$ obtained by removing each of the last two rows.

\textbf{Proof.} We may write $M$ in the form

$$
M = \begin{bmatrix}
A & a & b \\
c & f_1 & f_2 \\
d & g_1 & g_2 \\
e & h_1 & h_2
\end{bmatrix},
$$

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where $A$ is a square matrix of size $r - 1$, where $a$ and $b$ are column vectors of size $r - 1$, where $c$, $d$ and $e$ are row vectors of size $r - 1$, and where $f_1, f_2, g_1, g_2, h_1$ and $h_2$ are elements of $O$.

Let $I$ and $J$ be the ideals of $O$ generated, respectively, by all maximal minors of the submatrices

\[ M_2 = \begin{bmatrix} A & a & b \\ c & f_1 & f_2 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} A & a & b \\ c & f_1 & f_2 \\ d & g_1 & g_2 \\ e & h_1 & h_2 \end{bmatrix}. \]

Also, let $K \subseteq O$ be the ideal generated by the determinants of the square submatrices

\[ M_1 = \begin{bmatrix} A & a & b \\ c & f_1 & f_2 \\ d & g_1 & g_2 \end{bmatrix} \quad \text{and} \quad M_1' := \begin{bmatrix} A & a & b \\ c & f_1 & f_2 \\ e & h_1 & h_2 \end{bmatrix}. \]

Notice that the determinant of the first matrix is $z$.

From the hypothesis, the matrix obtained from $M_2$ by taking residues has rank at least $r - 1$. Thus, performing row and column operations on $M$, including column and row exchanges, we may, without changing the ideals $I$, $J$ and $K$, assume that $A$ is the identity matrix, $a = b = 0$ and $c = d = e = 0$. Then $z = f_1 g_2 - f_2 g_1$ and

\[
I = (f_1, f_2), \quad J = (f_1 g_2 - f_2 g_1, f_1 h_2 - f_2 h_1, g_1 h_2 - g_2 h_1), \quad K = (f_1 g_2 - f_2 g_1, f_1 h_2 - f_2 h_1).
\]

Now, since the matrix obtained from $M_1$ by taking residues has rank at least $r$, at least one among $f_1, f_2, g_1, g_2$ is invertible.

If $f_1$ is invertible, then

\[
g_1 h_2 - g_2 h_1 = (g_1 / f_1)(f_1 h_2 - f_2 h_1) - (h_1 / f_1)(f_1 g_2 - f_2 g_1).
\]

Thus, the lemma holds for $u = 1$ and $v = f_1 h_2 - f_2 h_1$. The case where $f_2$ is invertible is similar.

If $g_1$ is invertible, then

\[
(f_1 h_2 - f_2 h_1) = (f_1 / g_1)(g_1 h_2 - g_2 h_1) + (h_1 / g_1)(f_1 g_2 - f_2 g_1),
\]

\[
f_2 = (g_2 / g_1) f_1 - (1 / g_1)(f_1 g_2 - f_2 g_1).
\]
Thus the lemma holds for \( u = f_1 \) and \( v = g_1 h_2 - g_2 h_1 \). A similar analysis holds if \( g_2 \) is invertible.

5.4 Proposition. Let \( C \) be a general smooth curve of genus \( g \geq 1 \) and \( i \) a nonnegative integer. Let \( \Delta \) be the diagonal of \( C \times C \) and \( W_i \) the \( i \)-th Weierstrass divisor. Let \( SW_i \) be the \( i \)-th special ramification scheme, and \( D_i \) and \( E_i \) the \( i \)-th special ramification schemes of type Diaz and Cukierman, respectively. Let \( E_i^+ \) be the \( i \)-th expanded special ramification scheme of type Cukierman. Then these ramification schemes are finite and satisfy, in the cycle group of \( C \times C \):

\[
[SW_i] = [D_i] + [E_i] \quad \text{and} \quad [E_i^+] = [E_i] + (g + 1)[W_i \cap \Delta].
\]

Proof. Since \( C \) is general, by Statement 1 of Proposition 3.1, the set \( SW_i \) is finite for each \( i \geq 0 \). Thus, so are \( D_i \) and \( E_i \) by Expression (16). It follows that \( E_i^+ \) is finite, because \( E_i^+ \cap \Delta \) is the set of points \((P, P)\) such that \( P \) is Weierstrass, whence is finite.

Recall the notation of Subsections 4.1, 4.2, 5.1 and 5.2. Set \( r := g + i - 1 \). Both equalities can be proved locally. Thus, let \((P, Q) \in C \times C\) and \( \mathcal{O} \) be the local ring of \( C \times C \) at \((P, Q)\). As a map of \( \mathcal{O} \)-modules, \( \rho_{i,r+1} \) is given by a matrix \( M \) of the form described in the proof of Lemma 5.3. Let us use the notation described in the statement of that lemma.

Let \( K \subseteq \mathcal{O} \) define \( SW_i^+ \). Then \( K = (z, z') \), where \( z \) (resp. \( z' \)) is the maximal minor obtained from \( M \) by removing the last (resp. last but one) row. Notice that, from the nature of \( M \) as a “wronskian matrix”, \( z' \) is also the derivative of \( z \) along \( p_1 \). Let \( I \) and \( J \) be the ideals of \( \mathcal{O} \) defining \( D_i^+ \) and \( E_i^+ \), respectively. Then \( I \) and \( J \) are the ideals of all the maximal minors of \( M_2 \) and \( M \), respectively.

Now, since \( C \) is a general curve, by Statement 2 of Proposition 3.1

\[
h^0(\omega_C((i + 1)P - (g + i)Q)) \leq 1.
\]

This translates in the matrix obtained from \( M_1 \) by evaluating at \((P, Q)\) having rank at least \( r \). Applying Lemma 5.3, there are \( u, v \in \mathcal{O} \) such

\[
I = (z, u), \quad J = (z, v), \quad K = (z, uv).
\]

Now, since \( E_i^+ \) is finite-dimensional and \( C \times C \) is smooth, the sequence \( z, v \) is regular. The same holds for the sequence \( z, u \) if \( P \neq Q \). It follows that

\[
[SW_i] = [D_i] + [E_i].
\]
The second equality in the statement of the proposition is obvious off \( \Delta \). Thus, assume \( Q = P \). Since \( E_i^+ \cap \Delta = W_i \cap \Delta \), we may also assume that \( P \) is a Weierstrass point of \( C \).

Let \( t \) be a local parameter of \( C \) at \( P \), and \( t_1, t_2 \in \mathcal{O} \) be its pullbacks with respect to the projections \( p_1 \) and \( p_2 \). Then \( t := t_2 - t_1 \) is a local equation for \( \Delta \). As we saw in Subsection 5.2, we have \( z = t^g w \), where \( w \in \mathcal{O} \) defines \( W_i \), and is not divisible by \( t \). Letting \( \partial \) denote the derivative with respect to \( t \), we have

\[
z' = \partial z = \partial(t^g w) = gt^{g-1}w + t^g \partial w.
\]

Thus \( t^{g-1} \) divides \( z \) and \( z' \), and hence each element of \( K \), in particular \( uv \).

Since \( E_i^+ \) is finite, \( t \) does not divide \( v \), and hence \( t^{g-1} \mid u \). Let \( L := t^{1-g} K \). Then there are two expressions for \( L \):

\[
L = (tw, uv/t^{g-1}) \quad \text{and} \quad L = (tw, gw + t \partial w).
\]   

(19)

Since \( W_i \cap \Delta \) is finite, the sequences \( gw + t \partial w, t \) and \( w, t \) are regular. Thus, from the second expression for \( L \) above, we get

\[
\ell(\mathcal{O}/L) = 2\ell(\mathcal{O}/(t, w)) + \ell(\mathcal{O}/(w, \partial w)).
\]

Now, \( \ell(\mathcal{O}/(w, \partial w)) = 0 \) because \( w \) and \( \partial w \) cut out \( SW_i \), and \( SW_i \) does not meet \( \Delta \). Also, by Lemma 4.4, \( W_i \) intersects \( \Delta \) transversally. Thus \( \ell(\mathcal{O}/(t, w)) = 1 \), and hence \( \ell(\mathcal{O}/L) = 2 \).

Now, since the sequence \( z, v \) is regular, and \( z = t^g w \), also the sequence \( tw, v \) is regular. Thus, from the first expression for \( L \) in (19), we get

\[
\ell(\mathcal{O}/L) = \ell(\mathcal{O}/(tw, u/t^{g-1})) + \ell(\mathcal{O}/(tw, v)),
\]

and whence \( \ell(\mathcal{O}/(tw, v)) \leq 2 \). Since \( \mathcal{O} \) is regular, and the sequence \( tw, v \) is regular, so is the sequence \( v, w \). Thus

\[
\ell(\mathcal{O}/(tw, v)) = \ell(\mathcal{O}/(t, v)) + \ell(\mathcal{O}/(w, v)).
\]

Since \( E_i^+ \) contains \( (P, P) \), the function \( v \) is zero on \( (P, P) \). Thus, since also \( t \) and \( w \) vanish on \( (P, P) \), we get \( \ell(\mathcal{O}/(t, v)) = \ell(\mathcal{O}/(w, v)) = 1 \). So, the multiplicity of \( E_i^+ \) at \( (P, P) \) is

\[
\ell(\mathcal{O}/(z, v)) = g\ell(\mathcal{O}/(t, v)) + \ell(\mathcal{O}/(w, v)) = (g + 1).
\]

Since, by Lemma 4.4, the multiplicity of \( W_i \cap \Delta \) at \( (P, P) \) is 1, we are done. 

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5.5 Proposition. Let $C$ be a general smooth curve of genus $g \geq 1$ and $i$ a nonnegative integer. Let $SW_i$ be the $i$-th special ramification scheme of $C$. Then
\[
\int_{C \times C} [SW_i] = 2i g (g - 1) \left( (i + 2)(g + i)^2 + 2(g + i) + 2 \right). \quad (20)
\]

Proof. Recall the notation of Subsections 4.1, 4.2, 5.1 and 5.2. Since $C$ is general, $SW_i$ is finite. Also, $SW_i$ is the zero scheme of a section of the rank-2 bundle $\mathcal{J}^1_{p_1}(\mathcal{O}_{C \times C}(W_i))$. Thus its class in the Chow group of $C \times C$ satisfies
\[
[SW_i] = c_2(\mathcal{J}^1_{p_1}(\mathcal{O}_{C \times C}(W_i))).
\]
Using the truncation sequence for bundles of jets, we get
\[
[SW_i] = [W_i](c_1(p_2^*\omega_C) + [W_i]).
\]
Now, $c_1(p_2^*\omega_C) = K_2$. Using Expression (11) for $j = i$, and taking into account that $K_2^2 = 0$ for $\ell = 1, 2$, we get
\[
[SW_i] = i(g + i + 1)\left( (g + i)^2 + g + i + 1 \right) K_2[\Delta]
+ \frac{1}{2}i(i + 1)\left( (g + i)^2 + g + i + 1 \right) K_1 K_2
+ i^2(2g - 2) + i^2(g + i + 1)^2[\Delta] + i^2(2g - 2) + i^2(g + i + 1)(i + 1)K_1[\Delta].
\]
Using Formulas (14) and (15), we get
\[
\int_{C \times C} [SW_i] = i(g + i + 1)\left( (g + i)^2 + g + i + 1 \right)(2g - 2)
+ \frac{1}{2}i(i + 1)\left( (g + i)^2 + g + i + 1 \right)4(g - 1)^2
- i^2(2g - 2) + i^2(2g - 2) + i^2(g + i + 1)(i + 1)(2g - 2).
\]
Simplifying, we get the claimed formula. \hfill \blacksquare

5.6 Theorem. Let $C$ be a general smooth curve of genus $g \geq 1$, and $i$ a nonnegative integer. Let $D_i$ and $E_i$ be the $i$-th special ramification schemes of type Diaz and Cukierman, respectively. Then $D_i$ and $E_i$ are reduced, and
\[
\int_{C \times C} [D_i] = g(g - 1)\left( (g + i - 1)^2(i + 1)^2 - (g - 1)^2 \right) \quad (21)
\]
and
\[
\int_{C \times C} [E_i] = g(g - 1)\left( (g + i + 1)^2(i + 1)^2 - (g - 1)^2 \right). \quad (22)
\]
Proof. Recall the notation of Subsections 4.1, 4.2, 5.1 and 5.2. We will first compute the degrees of $D_i$ and $E_i$. First of all, since $E_i^+$ is finite, and is the degeneracy scheme of $\rho_{i,g+i}$, applying Porteous formula ([8], Thm. 14.4, p. 254), we get the following expression for the class $[E_i^+]$ in the Chow group of $C \times C$:

$$[E_i^+] = c_2(\mathcal{J}_{p_1}^{g+i}(\mathcal{L}_i) - p_1^*\mathcal{E}_i).$$

Now, $c_2(\mathcal{E}_i) = c_1(\mathcal{E}_i)^2 = 0$, since $C$ is one-dimensional. Thus

$$[E_i^+] = c_2(\mathcal{J}_{p_1}^{g+i}(\mathcal{L}_i)) - c_1(\mathcal{J}_{p_1}^{g+i}(\mathcal{L}_i))c_1(p_1^*\mathcal{E}_i).$$

Using the truncation sequence of the bundles of jets, we get

$$c_1(\mathcal{J}_{p_1}^{g+i}(\mathcal{L}_i)) = \sum_{\ell=1}^{g+i+1} (\ell K_2 + (i + 1)[\Delta]);$$

$$c_2(\mathcal{J}_{p_1}^{g+i}(\mathcal{L}_i)) = \sum_{m=2}^{g+i+1} \sum_{\ell=1}^{m-1} (\ell K_2 + (i + 1)[\Delta])(m K_2 + (i + 1)[\Delta]).$$

Expanding, and using that $K_2^2 = 0$, we get

$$c_1(\mathcal{J}_{p_1}^{g+i}(\mathcal{L}_i)) = \frac{1}{2}(g + i + 1)(g + i + 2)K_2 + (i + 1)(g + i + 1)[\Delta];$$

$$c_2(\mathcal{J}_{p_1}^{g+i}(\mathcal{L}_i)) = \frac{1}{2}(i + 1)(g + i)(g + i + 1)(g + i + 2)K_2[\Delta]$$

$$+ \frac{1}{2}(i + 1)^2(g + i)(g + i + 1)[\Delta]^2.$$

Finally, using Formula (13) for $j = i$, and Formulas (14) and (15), we get

$$\int_{C \times C} [E_i^+] = (i + 1)^2 g(g - 1)(g + i + 1)^2.$$

Now, it follows from Proposition 4.4 that $W_i$ meets $\Delta$ transversally at $g^3 - g$ points. Thus, using Proposition 5.4, we get

$$\int_{C \times C} [E_i] = \int_{C \times C} [E_i^+] - (g + 1)(g^3 - g)$$

$$= g(g - 1)((g + i + 1)^2(i + 1)^2 - (g + 1)^2),$$

the stated formula for the degree of $[E_i]$. 

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Now, the expression for the degree of \([D_i]\) follows now from the equality
\[\text{[SW}_i] = [D_i] + [E_i]\] proved in Proposition 5.4 and Formula (20) for the degree
of \([SW_i]\) proved in Proposition 5.5.

Let us now show that \(D_i\) and \(E_i\) are reduced. Let \((P,Q) \in \text{SW}_i\). Let \(\hat{O}\) be the completion of the local ring of \(C \times C\) at \((P,Q)\). Let \(t_1\) and \(t_2\) be local equations in \(\hat{O}\) for \(\{P\} \times C\) and \(C \times \{Q\}\), respectively. Then \(\hat{O} = \mathbb{C}[[t_1,t_2]]\). Let \(w \in \mathcal{O}\) be a local equation for \(W_i\). Since \((P,Q)\) \in \(\text{SW}_i\), and since \(W_i\) is nonsingular by Proposition 4.4, we may assume that \(w = t_1 + u\), where \(u \in \mathbb{C}[[t_2]]\). Now, let \(w'\) and \(u'\) be the derivatives of \(w\) and \(u\) with respect to \(t_2\). Then the ideal defining \(\text{SW}_i\) at \((P,Q)\) is \((w,w')\), and the multiplicity of the cycle \([\text{SW}_i]\) at \((P,Q)\) is \(\ell(\hat{O}/(w,w'))\). Notice that \(w' = u'\), and

\[
\frac{\hat{O}}{(w,w')} \cong \frac{\mathbb{C}[[t_2]]}{(u,u')} \cong \frac{\mathbb{C}[[t_1,t_2]]}{(t_1,w,w')}.
\]

Thus the multiplicity of the cycle \([\text{SW}_i]\) at \((P,Q)\) is the multiplicity \(m\) of \(\text{SW}_i \cap (\{P\} \times C)\) at \((P,Q)\).

Since the formation of \(\text{SW}_i\) commutes with base change, this multiplicity \(m\) satisfies

\[m = \text{wt}_V(Q) - 1,\]

where \(V\) is the complete linear system of sections of \(\omega_{C \times \{i+1\}P}\). Now, by Propositions 3.1 and 3.4, the order sequence of \(V\) at \(Q\) satisfies

\[
\epsilon_j(V,Q) = j \quad (j = 0, 1, \ldots, g + i - 3),
\]

\[
\epsilon_{g+i-2}(V,Q) \leq g + i - 1,
\]

\[
\epsilon_{g+i-1}(V,Q) \leq g + i + 1.
\]

Thus \(m \leq 2\), with equality if and only if

\[h^0(\omega_{C \times (i+1)P - (g+i-1)Q}) = 2 \quad \text{and} \quad h^0(\omega_{C \times (i+1)P - (g+i+1)Q}) = 1,\]

that is, if and only if \((P,Q) \in D_i \cap E_i\). Since \([\text{SW}_i] = [D_i] + [E_i]\) by Proposition 5.4, it follows that \(D_i\) and \(E_i\) are reduced at \((P,Q)\).

\[\square\]

References

[1] E. Arbarello, Weierstrass points and moduli of curves. Compo. Math. 29, no. 3, (1974), 325–342.
[2] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of algebraic curves*. Grundlehren der mathematischen Wissenschaften, vol. 267, Springer, New York, 1985.

[3] CoCoATeam, CoCoA: a system for doing Computations in Commutative Algebra. Available at [http://cocoa.dima.unige.it](http://cocoa.dima.unige.it).

[4] F. Cukierman, *Families of Weierstrass points*. Duke Math. J. **58**, no. 2, (1989), 317–346.

[5] C. Cumino, E. Esteves, and L. Gatto, *Limits of special Weierstrass points*. To appear.

[6] S. Diaz, *Exceptional Weierstrass points and the divisor on moduli space that they define*. Mem. Amer. Math. Soc., vol. 56, no. 327, Amer. Math. Soc., Providence, 1985.

[7] E. Esteves, *Linear systems and ramification points on reducible nodal curves*. In “Algebra Meeting” (A. Garcia, E. Esteves and A. Pacheco, Eds.), 21–35, Mat. Contemp., vol. 14, Soc. Bras. Mat., Rio de Janeiro, 1998.

[8] W. Fulton, *Intersection theory*. Grundlehren der mathematischen Wissenschaften, vol. 2, Springer, Berlin Heidelberg, 1984.

[9] L. Gatto and F. Ponza, *Derivatives of wronskians with applications to families of Weierstrass points*. Trans. Amer. Math. Soc. **351**, no. 6, (1999), 2233–2255.

[10] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*. Invent. Math. **67** (1982), 23–86.

[11] S. Keel, *Intersection theory of moduli space of stable n-pointed curves of genus zero*. Trans. Amer. Math. Soc. **330** (1992), 545–574.

[12] F. Knudsen, *The projectivity of the moduli space of stable curves, II*. Math. Scand. **52** (1983), 161–199.