Embedding of hyperbolic Coxeter groups into products of binary trees and aperiodic tilings

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Abstract

We prove that a finitely generated, right-angled, hyperbolic Coxeter group $\Gamma$ can be quasiisometrically embedded into the product of $n$ binary trees, where $n$ is the chromatic number of $\Gamma$. As application we obtain certain strongly aperiodic tilings of the Davis complex of these groups.

1 Introduction

We consider a finitely generated, right-angled Coxeter group $\Gamma$, i.e. a group $\Gamma$ together with a finite set of generators $S$, such that every element of $S$ has order two and that all relations in $\Gamma$ are consequences of relations of the form $st = ts$, where $s, t \in S$.

We prove embedding results of the Cayley graph $C(\Gamma, S)$ of hyperbolic right angled Coxeter groups into products of the binary tree $T = T_{\{0,1\}}$. On graphs and trees we consider always the simplicial metric, hence every edge has length 1. On a product of trees we consider the $l_1$-product metric, i.e. the distance is equal to the sum of the distances in the factors.

Theorem 1.1. Let $\Gamma$ be a finitely generated, hyperbolic, right-angled Coxeter group with chromatic number $n$. Then there exists a quasiisometric embedding $\psi : C(\Gamma, S) \to T \times \cdots \times T$ ($n$-factors).

We can apply Theorem 1.1 for special Coxeter groups operating compactly on the hyperbolic plane $\mathbb{H}^2$, e.g. the group generated by the reflections at the edges of the regular right-angled geodesic hexagon. This group has chromatic number 2 and we obtain:

Corollary 1.2. There exists a quasiisometric embedding of the hyperbolic plane into the product of two binary trees.
Combining this with a result of Brady and Farb [BF] we get the following higher dimensional version:

**Corollary 1.3.** For every integer \( n \) there exists a quasiisometric embedding \( \psi : \mathbb{H}^n \to T \times \cdots \times T \) of the hyperbolic space \( \mathbb{H}^n \) into the \( 2(n-1) \)-fold product of binary trees.

It is an open problem, if for \( n \geq 3 \) there is a quasiisometric embedding of \( \mathbb{H}^n \) into the \( n \)-fold product of binary trees. There are partial results in this direction. In [BS2] it is shown that there exists a quasiisometric embedding of \( \mathbb{H}^n \) into an \( n \)-fold product of locally infinite trees. On the other hand a recent construction of Januszkiewicz and Swiatkowski [JS] shows for every \( n \) the existence of a right angled Gromov hyperbolic Coxeter group with virtual cohomological dimension and coloring number equal to \( n \). Combining Theorem 1.1 with that result we obtain:

**Corollary 1.4.** For every \( n \) there exists a Gromov hyperbolic group \( \Gamma_n \) with virtual cohomological dimension \( n \) which admits a quasiisometric embedding into the product of \( n \) binary trees.

Corollary 1.4 can be used to determine the hyperbolic rank (compare [BS1]) of a product of trees:

**Corollary 1.5.** The hyperbolic rank of the product of \( n \) metric trees is \( (n-1) \).

**Corollary 1.6.** The hyperbolic rank of the product of \( n \) free groups is equal to \( (n-1) \).

With the methods of the proof we also obtain strongly aperiodic tilings of the Davis complex of right-angled, hyperbolic Coxeter groups.

**Theorem 1.7.** For every finitely generated, right-angled and hyperbolic Coxeter group \( \Gamma \) the Davis complex \( X \) admits a strongly aperiodic tiling \( \Phi \) with finitely many tiles. In addition, every limit tiling of \( \Phi \) is also strongly aperiodic.

In dimensions 2,3 and 4 there are right-angled reflection groups operating with compact fundamental domain on the hyperbolic space. Two dimensional examples come from the regular right-angled \( p \)-gons, \( p \geq 5 \) in the hyperbolic plane. In dimension 3 there exists a right-angled regular hyperbolic dodecahedron (compare also [A]). In dimension 4 there exists e.g. a right-angled hyperbolic 120-cell ([PV],[D2],[C]). Thus we obtain

**Corollary 1.8.** The hyperbolic spaces \( \mathbb{H}^2, \mathbb{H}^3 \) and \( \mathbb{H}^4 \) admit strongly aperiodic tilings such that every limit tiling is also strongly aperiodic.
A strongly aperiodic tiling of $\mathbb{H}^2$ was recently constructed by Goodman-Strauss (this tiling has even a finite strongly aperiodic set of tiles). Existence of aperiodic tilings in $\mathbb{H}^3$ and $\mathbb{H}^4$ follows from the results of Block-Weinberger. The aperiodicity of their tilings follow from the fact that the tilings are unbalanced. We construct strictly balanced aperiodic tilings of the Davis complex. Indeed we develop a general method how to modify tilings of the Davis complex to obtain strictly balanced tilings. For a precise statement see Theorem 6.18.

The structure of the paper is as follows. After the preliminaries we prove the main result for the two-colored case in section 3. In section 4 we generalize the argument to the $n$-colored case. In section 5 we prove the corollaries of this result. In section 6 we discuss aperiodic tilings of the Davis complex.

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2 Preliminaries

2.1 Right-angled Coxeter Groups

In this section we review the necessary facts from the theory of right-angled Coxeter groups.

A Coxeter matrix $(m_{s,t})_{s,t \in S}$ is a symmetric $S \times S$ matrix with 1 on the diagonal and with all other entries nonnegative integers different from 1. A Coxeter matrix defines a Coxeter group $\Gamma$ generated by the index set $S$ with relations $(st)^{m_{s,t}} = 1$ for all $s, t \in S$. Here we use the convention that $\gamma^0 = 1$ for all elements, thus if $m_{s,t} = 0$ then there is no relation between $s$ and $t$. A Coxeter group $\Gamma$ is finitely generated, if $S$ is finite. The group $\Gamma$ is called right-angled, if all entries of the corresponding Coxeter matrix are $0, 1, 2$. The Coxeter matrix of a right angled Coxeter group is completely described by a graph with vertex set $S$ where we connect two vertices $s$ and $t$ iff $m(s, t) = 2$.

Let $\Gamma$ be a right-angled Coxeter group with generating set $S$. The nerve $N = N(\Gamma, S)$ is the simplicial complex defined in the following way: the vertices of $N$ are the elements of $S$. Two different vertices $s, t$ are joined by an edge, if and only if $m(s, t) = 2$. In general $(k+1)$ different vertices $s_1, \ldots, s_{k+1}$ span a $k$-simplex, if and only $m(s_i,s_k) = 2$ for all pairs of different $i, j \in \{1, \ldots, k + 1\}$.

There is a simple characterization of hyperbolic right-angled Coxeter groups. A square in $S$ is a collection of four elements $t_1, t_2, s_1, s_2 \in S$, such that $s_i$ commutes with $t_1$ and $t_2$ for $i = 1, 2$, but $s_1$ and $s_2$ as well as $t_1$ and $t_2$ do not commute. Thus these elements are the vertices of a square.
in the nerve. The following characterization is known as the Siebenmann no-square condition:

**Lemma 2.1.** A right-angled Coxeter group \( \Gamma \) with generating set \( S \) is hyperbolic if and only if \( S \) contains no square.

The chromatic number of a graph is the minimal number of colors needed to color the vertices in such a way that adjacent vertices have different colors. The chromatic number of a simplicial complex is the chromatic number of its 1-dimensional skeleton.

Assume that the chromatic number of the nerve \( N(\Gamma) \) of a right-angled Coxeter group \( \Gamma \) equals \( n \) and let \( c : N^{(0)} \rightarrow \{1, \ldots, n\} \) be a corresponding coloring map. Then we can write \( S \) as a disjoint union \( S = \bigsqcup c S_c \) where \( S_c \) are the elements of color \( c \). Elements of a given color do not commute. We prefer to write \( S = A \bigsqcup B \bigsqcup C \ldots \), where \( A \subset S \) are the elements with color \( a \).

Right-angled Coxeter groups have a very simple deletion law. By the following two operations every word \( W \) in the generators \( S \) can be transformed to a reduced word and two reduced words representing the same element can be transformed by means only the second operation \([Br]\):

1. (i) delete a subword of the form \( ss \), \( s \in S \)
2. (ii) replace a subword \( st \) by \( ts \) if \( m_{s,t} = 2 \).

This deletion rule has the following consequences:

**Lemma 2.2.** (a) If \( W \) and \( W' \) represent the same element then the lengths of \( W \) and \( W' \) are either both even or both odd.

(b) Let \( W \) and \( W' \) be reduced representations of the same element \( \gamma \in \Gamma \), then \( W \) and \( W' \) are formed from the same set of letters and they have the same length.

We now investigate some properties of the Cayley graph \( C(\Gamma, S) \) of \( \Gamma \) with respect to the generating set \( S \) for a right-angled Coxeter group. Let \( \gamma \in \Gamma \) and let \( W \) be a reduced representation of \( \gamma \). The length of \( W \) is denoted by \( \ell(\gamma) \) and called the norm of \( \gamma \). This is well defined by (b). If \( a \) is a color and \( \gamma \in \Gamma \), then we denote with \( \ell_a(\gamma) \) the number of letters with color \( a \) in a reduced representation of \( \gamma \). This is well defined and we clearly have \( \sum_c \ell_c(\gamma) = \ell(\gamma) \).

On \( \Gamma \) we consider the distance function \( d(\gamma, \beta) = \ell(\gamma^{-1}\beta) \). Let \( \gamma \) and \( \beta \) be elements of \( \Gamma \) which are neighbors in the Cayley graph and let \( W \) be a reduced word representing \( \gamma \). Then there exists a generator \( s \in S \) such that \( \beta \) has the representation \( Ws \). It follows from (b) that \( \ell(\gamma) \neq \ell(\beta) \). Thus an edge in the Cayley graph connects two elements with different
norm. As usual one can define geodesics in the Cayley graph. A geodesic between two points $\alpha, \beta \in \Gamma$ is given by a sequence $\alpha = \gamma_0, \ldots, \gamma_k = \beta$ with $d(\gamma_i, \gamma_j) = |i - j|$

Let $\alpha, \beta, \gamma$ be elements of $\Gamma$. We say that $\gamma$ lies between $\alpha$ and $\beta$ if $d(\alpha, \gamma) + d(\gamma, \beta) = d(\alpha, \beta)$. The Cayley graphs of right angled Coxeter groups have the following property, which says that any three points in $\Gamma$ span a tripod:

**Lemma 2.3.** Let $\alpha, \beta, \gamma \in \Gamma$, then there exists $\delta \in \Gamma$ such that $\delta$ lies between $\gamma_i$ and $\gamma_j$ for any choice of distinct elements $\gamma_i, \gamma_j \in \{\alpha, \beta, \gamma\}$.

**Proof.** By the $\Gamma$-invariance of the metric it suffices to show this result for the case that $\gamma = 1$. Let $\alpha, \beta \in \Gamma$ and consider a geodesic path $\alpha = \alpha_0, \ldots, \alpha_k = \beta$ from $\alpha$ to $\beta$ and consider the sequence of norms $n_0 = \ell(\alpha_0), \ldots, n_k = \ell(\alpha_k)$. Note that by the properties discussed above $|n_i - n_{i+1}| = 1$. Assume that there is a subsequence $\alpha_{i-1}, \alpha_i, \alpha_{i+1}$ with $n_{i-1} < n_i > n_{i+1}$. Then one can represent $\alpha_i$ in a reduced way as $W_1 t = W_2 s$ where $W_1$ is a reduced word representing $\alpha_{i-1}$ and $W_2$ a reduced word representing $\alpha_{i+1}$ and $s, t \in S$. Since by the deletion law one can transform $W_1 t$ into $W_2 s$ by means of operations of type (b), we see that $st = ts$ and that one can represent $\alpha_i$ by a reduced word of the form $W st = W ts$ where $W s$ represents $\alpha_{i-1}$ and $W t$ represents $\alpha_{i+1}$. Replace now $\alpha_i$ by the element $\alpha'_i$ represented by $W$ and we obtain a new geodesic sequence between $\alpha$ and $\beta$ such that for the corresponding sequence of norms we have $n_{i-1} > n'_i < n_{i+1}$. Applying this procedure several times we obtain a geodesic path from $\alpha$ to $\beta$, such that the sequence of the norms has no local maximum any more and hence only a global minimum $n_{i_0}$. The corresponding element $\delta = \alpha_{i_0}$ lies between $\alpha$ and $\beta$ but also between 1 and $\alpha$ resp. 1 and $\beta$.

\[\Box\]

**2.2 Rooted Trees**

Let $\Omega$ be a set. We associate to $\Omega$ a rooted simplicial tree $T_\Omega$ in the following way: the set of vertices is the set of finite sequences $(\omega_1, \ldots, \omega_k)$ with $\omega_i \in \Omega$. The empty sequence defines the root vertex and is denoted by $\emptyset$.

Two vertices are connected by an edge in $T_\Omega$ if their length (as sequences) differ by one and the shorter can be obtained by erasing the last term of the longer. The root vertex has $|\Omega|$ neighbors and every other vertex has $|\Omega| + 1$ neighbors, one ancestor and $|\Omega|$ descendents. Here $|\Omega|$ denotes the cardinality of $|\Omega|$. The set $\Omega$ is also called the *label set* of $T_\Omega$. On trees we consider always the simplicial metric, hence every edge has length 1. On a product of trees we consider the $l_1$-product metric, i.e. the distance is equal to the sum of the distances in the factors.

The tree $T_{\{0,1\}}$ is also called the *binary tree*. If $\Omega$ is finite and $|\Omega| \geq 2$, then $T_\Omega$ is quasiisometric to the binary tree. This is easily proved using the
fact that the elements of $\Omega$ can be represented by binary sequences of length $\leq \log_2(|\Omega|) + 1$.

3 Two-colored Coxeter Groups

Since the case of chromatic number 2 is technically much easier, we first give the proof of our main result in this case. Thus we assume in this section, that $\Gamma$ is generated by a finite set $S$ which can be decomposed as $S = A \sqcup B$ such that elements within $A$ do not commute with each other and the same holds for elements in $B$.

3.1 Canonical $a$-presentation

Fix a color, say $a$. A reduced left $a$-representation of $\gamma \in \Gamma$ is a reduced word $W = W_1a_1W_2a_2\ldots W_ra_rW_{r+1}$ representing $\gamma$ in which the $W_i = w^i_1 \ldots w^i_{r_i}$ are words with letters in $B$ and the last entry of $W_i$ does not commute with $a_i$, i.e. the letters with color $a$ are moved as left as possible. The reduced $a$-representation of $\gamma$ is unique and thus we call it also the canonical $a$-presentation of $\gamma$. The word $W_i$ in the canonical $a$-representation is called the coefficient of $a_i$ if $i \leq r$ and it is called the free coefficient if $i = r + 1$. Note that $W_i$ could be empty. Let $W$ be a reduced word, then the reduction to the canonical $a$-representation permutes the letters of $W$. We call this permutation the canonical reduction map of $W$.

We now study in general the situation that we have two canonical $a$-representations $U$ and $V$ such that the composition $UV$ is a reduced word. We choose the notation such that

$$U = U_1a_1 \cdots U_pa_pU_{p+1},$$
$$V = V_{p+1}a_{p+1} \cdots V_{p+m}a_{p+m}V_{p+m+1}.$$

Let then

$$W = W_1a_1W_2a_2\ldots W_{p+m}a_{p+m}W_{p+m+1}$$

be the canonical $a$-presentation of the word $UV$ and let $\varphi : UV \to W$ be the canonical reduction map for $UV$. With this notation $\varphi(a_i) = a_i$ for all $1 \leq i \leq p + m$. Let $U_{p+1} = u_q \ldots u_1$ be the free coefficient of $U$ and $V_{p+1} = v_k \ldots v_1$ be the coefficient of $a_{p+1}$ in $V$ and let $U_R = \{u \in U|a_{p+1} < \varphi(u)\}$ denote the set of all letters $u \in U$ that are moved by $\varphi$ right of $a_{p+1}$. Then

**Proposition 3.1.** (1) $\varphi(v_i) = w_i$ for all $i \leq k$, where $W_{p+1} = w_1 \ldots w_1$ is the coefficient of $a_{p+1}$ in $W$.

(2) If $k \geq 1$ then $U_R = \emptyset$ and $UV = W$ is the canonical $a$-representation.
(3) If \( U_R \neq \emptyset \), then \( U_R \) are the last \(|U_R|\) letters of \( u_q \ldots u_1 \).

(4) If \( \Gamma \) is hyperbolic, then there is at most one \( u \in U_R \) with \( \varphi(u) > a_{p+2} \).

**Proof.** The first point is obvious. If \( k \geq 1 \), then (since no element \( u_i \in U_{p+1} \) can pass \( v_1 \)) \( U_R = \emptyset \). This proves (2). If \( U_R \neq \emptyset \), then \( k = 0 \) and clearly \( U_R \) consists out of the last letters of \( U_{p+1} \), hence (3). To prove the last point assume that there are two elements in \( U_R \) which are mapped by \( \Phi \) to the right of \( a_{p+2} \). Then these two elements commute with \( a_{p+1} \) and \( a_{p+2} \) and thus this four elements together form a square. Hence \( \Gamma \) is not hyperbolic by Lemma 2.1.

Consider two elements \( \gamma, \overline{\gamma} \in \Gamma \). We investigate the situation, how the canonical a-representations of these elements differ. By Lemma 2.3, \( \gamma \) and \( \overline{\gamma} \) span a tripod. Thus there are words \( U,V,\overline{V} \) given in canonical a-presentations, such that \( UV \) is a reduced representation of \( \gamma \) and \( U\overline{V} \) is a reduced representation of \( \overline{\gamma} \). Furthermore \( V^{-1}\overline{V} \) is a reduced representation of \( \gamma^{-1}\overline{\gamma} \). Let \( m = \ell_a(V) \) and \( \overline{m} = \ell_a(\overline{V}) \).

We then write

\[
U = U_1a_1 \ldots U_pa_pU_{p+1}
\]

\[
V = V_{p+1}a_p \ldots V_{p+m}a_{p+m}V_{p+m+1}
\]

\[
\overline{V} = \overline{V}_{p+1}\overline{a}_p \ldots \overline{V}_{p+m}\overline{a}_{p+m}\overline{V}_{p+m+1}
\]

Let

\[
W = W_1a_1 \ldots W_{p+m}a_{p+m}W_{p+m+1}
\]

\[
\overline{W} = \overline{W}_1\overline{a}_1 \ldots \overline{W}_{p+m}\overline{a}_{p+m}\overline{W}_{p+m+1}
\]

be the canonical a-representations of \( \gamma, \overline{\gamma} \). Clearly \( W_i\overline{a}_i = \overline{W}_i\overline{a}_i = U_ia_i \) for \( 1 \leq i \leq p \).

**Lemma 3.2.** \( W_{p+1}a_{p+1} \neq \overline{U}_{p+1}\overline{a}_{p+1} \).

**Proof.** Since \( V^{-1}\overline{V} \) is reduced, the first letter from \( V \) is different from the first letter of \( \overline{V} \). We start by considering the case, that the first letters are both of color \( a \), i.e. \( V_{p+1} = \emptyset = \overline{V}_{p+1} \). In this case we have \( a_{p+1} \neq \overline{a}_{p+1} \) and hence the claim. If (say) \( V_{p+1} = \emptyset \) and \( \overline{V}_{p+1} \neq \emptyset \), then we get from Proposition 3.1 that \( \overline{V}_{p+1} \) is longer then \( W_{p+1} \) and hence the claim. If both \( V_{p+1} \neq \emptyset \) and \( \overline{V}_{p+1} \neq \emptyset \), then by Proposition 3.1 \( W_{p+1} = U_{p+1}V_{p+1} \) and \( \overline{W}_{p+1} = \overline{U}_{p+1}\overline{V}_{p+1} \) and hence the two word differ at the first entry of \( V_{p+1} \) resp. \( \overline{V}_{p+1} \). This proves the result.

\[ \square \]
3.2 Alice’s diary

Let $\kappa \in \mathbb{N}$. Define $E = (A \cup B \cup \{\emptyset\})$. Let $T_a^{\text{dia}}$ be the rooted tree with label set $E^\kappa$. We call $T_a^{\text{dia}}$ the diary tree. We define a map $\psi_a^{\text{dia}} : \Gamma \to T_a^{\text{dia}}$ as follows:

Let $\gamma \in \Gamma$ be given by the canonical $a$-presentation

$$W = (w_{k_1}^1 \ldots w_1^1)a_1 \ldots (w_{k_r}^r \ldots w_1^r)a_r(w_{k_{r+1}}^{r+1} \ldots w_{1}^{r+1}).$$

By $W|_i$ we denote the $i$-cut

$$W|_i = (w_{k_i}^1 \ldots w_1^1)a_1 \ldots a_{i-1}(w_i^i \ldots w_1^i).$$

Recall that the vertices of the tree $T_a^{\text{dia}}$ are finite sequences of elements of $E^\kappa$. We define $\psi_a^{\text{dia}}(W) = (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_r)$ as a sequence of length $r$ by induction on $i$.

Let $\alpha_1$ be the string of the last $\kappa$ symbols in the chain

$$\emptyset \ldots \emptyset w_{k_1}^1 \ldots w_1^1$$

considered as a word in the alphabet $E$. We assume here that we have enough (say $\kappa$) symbols $\emptyset$ in front. We define $\alpha_i$ as the string of the last $\kappa$ symbols (E-letters) of the word in the alphabet $E$ that is obtained from the word

$$\emptyset \ldots \emptyset W|_i$$

by removing the E-letters from $\cup_{j<i} \alpha_j$.

**Remark 3.3.** One can intuitively describe this "diary" in the following way: Consider the word $W$ as description of a long journey of (say) Alice who wants to write a diary about her trip. Every letter of $W$ represents a day. There are two types of days during this journey. The days of color $b$ where the weather is so fine that Alice has no time to write her diary. Then there are more cloudy days of color $a$. In the morning of every $a$-day Alice writes $\kappa$ pages in her diary: she starts with yesterday and then the day before yesterday etc. Of course she skips the days which were already described earlier in the diary. If there is no day left to describe, she marks on the corresponding pages of her diary the symbol $\emptyset$.

It is not difficult to show and left as an exercise to prove the following; if $\psi_a^{\text{dia}}(W) = (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_r)$ and $\alpha_i$ contains the symbol $\emptyset$, then one can reconstruct the whole $i$-cut $W|_i$ from $(\alpha_1, \ldots, \alpha_i)$.

**Remark 3.4.** It is a subtle point that Alice writes her diary in the morning and not in the evening; i.e. the entry $\alpha_i$ does not contain $a_i$. This choice of the diary makes it possible to reconstruct from the diary the $i$-cut $W|_i$.

More generally we have
Proposition 3.5. Let \( W = W_1a_1 \ldots W_{j+r}a_{j+r}W_{j+r+1} \) be a canonical \( a \)-representation and assume that
\[
k := \kappa(r + 1) - \ell(W_{j+1}a_{j+1} \ldots W_{j+r}) \geq 1.
\]

Then one of the following holds:

1. We can reconstruct the subword \( W_ja_j \) from the diary entries \((\alpha_j, \ldots, \alpha_{j+r})\).
2. The word \( W_ja_j \) has \( \geq k \) letters and we can reconstruct from \((\alpha_j, \ldots, \alpha_{j+r})\) the last \( k \) letters of \( W_ja_j \).

Proof. If some entry \( \alpha_q \) for some \( j \leq q \leq j+r \) contains \( \emptyset \), then we can reconstruct the complete \( q \)-cut \( W|_q \) from the diary entries \((\alpha_1, \ldots, \alpha_q)\). In particular we can reconstruct in this case \( W_ja_j \) from the diary entries \( \alpha_s, j \leq s \leq j+r \).

Thus we can assume that the entries \( \alpha_q, j \leq q \leq j+r \) do not contain \( \emptyset \) and hence each entry contains \( \kappa \) letters from the word \( W_1a_1 \ldots a_{j+r-1}W_{j+r} \).

If we cannot reconstruct \( W_ja_j \) from these entries, then all entries from \( \alpha_q, j \leq q \leq j+r \) are from the word \( W_ja_j \ldots a_{q-1}W_q \) (if some entry is from \( W_{j-1}a_{j-1} \), then the first occurrence of this entry is \( a_{j-1} \) and in this moment we have all information to reconstruct \( W_ja_j \)). Thus all \( \kappa(r+1) \) diary entries are from \( W_ja_j \ldots a_{j+r-1}W_{j+r} \), and hence at least \( k \) of them from \( W_ja_j \).

We collect the maps \( \psi^\text{dia}_a \) and the corresponding \( \psi^\text{dia}_b \) to a common map
\[
\psi^\text{dia} : \Gamma \to T^\text{dia}_a \times T^\text{dia}_b
\]

On the product of the trees we consider the \( l_1 \)-product metric.

Lemma 3.6. The map \( \psi^\text{dia} \) is 1-Lipschitz.

Proof. Let \( \gamma, \gamma' \in \Gamma \) elements with \( d(\gamma, \gamma') = 1 \). Then \( |\ell(\gamma') - \ell(\gamma)| = 1 \) and we can assume that \( \ell(\gamma') = \ell(\gamma) + 1 \). In this case we can represent \( \gamma' \) in a reduced way as \( Us \), where \( U \) is a reduced word representing \( \gamma \). Assume w.l.o.g. that \( s \in A \). In this case clearly \( \psi^\text{dia}_b(\gamma) = \psi^\text{dia}_b(\gamma') \). Note that \( \psi^\text{dia}_a(\gamma') \) has one more entry as \( \psi^\text{dia}_a(\gamma) \), and all other entries agree. Thus \( \psi^\text{dia}(\gamma) \) and \( \psi^\text{dia}(\gamma') \) have distance 1.

The next Lemma shows that the diary map is a ”radial isometry”.

Lemma 3.7. The map \( \psi^\text{dia} \) is a radial isometry, i.e. \( |\psi^\text{dia}(\gamma)| = \ell(\gamma) \), where \(|.| \) denotes the distance from \((\text{root, root}) \in T^\text{dia}_a \times T^\text{dia}_b \).
Proof. It follows from the definition of \( \psi_a^\text{dia} \), that the sequence \( \psi_a^\text{dia}(\gamma) \) has \( \ell_a(\gamma) \) terms. It follows that

\[
|\psi_a^\text{dia}(\gamma)| = \ell_a(\gamma) + \ell_b(\gamma) = \ell(\gamma).
\]

\[
\square
\]

A small disadvantage of the diary constructed above is the fact that \( \alpha_i \) does not contain the information about the last letter \( a_i \) (compare remark 3.4). To obtain this additional information we define an "augmented" diary tree \( T_a^\text{dia} \) by the label set \( E^\kappa \times A \) and define

\[
\psi_a'^\text{dia}(\gamma) = \psi_a'^\text{dia}(W) = ((\alpha_1, a_1), \ldots, (\alpha_r, a_r))
\]

where \( W \) is a canonical \( a \)-representation of \( \gamma \).

The augmented diary map \( \psi'^\text{dia} = \psi'^{\text{dia}}_a \times \psi^{\text{dia}}_b \) carries some additional information. Also this map is 1-Lipschitz and a radial isometry.

Remark 3.8. The maps \( \psi^\text{dia} \) and \( \psi'^\text{dia} \) are (in most cases) not quasiisometric. Consider reduced words \( \overline{A} \) formed from \( a \)-letters, and \( \overline{B} \) formed from \( b \)-letters, such that the first and the last entry of \( \overline{B} \) are different. Then also the words \( \overline{B}^k \) are reduced for \( k \geq 1 \). Assume \( \ell(\overline{B}) \ll \ell(\overline{A}) \). Let \( k \in \mathbb{N} \) such that \( \ell(\overline{A}) \ll k \) and let also \( \kappa \ll k \). Now consider the two reduced words \( \gamma = (\overline{B})^k \overline{A} \) and \( \overline{\gamma} = (\overline{B})^{k+1} \overline{A} \). The conditions imply that \( \psi_a^\text{dia}(\gamma) = \psi_a^\text{dia}(\overline{\gamma}) \) and

\[
|\psi_b^\text{dia}(\gamma) - \psi_b^\text{dia}(\overline{\gamma})| = \ell(\overline{B}) \ll 2\ell(\overline{A}) + \ell(\overline{B}) = d(\gamma, \overline{\gamma}).
\]

Thus \( \psi^\text{dia} \) cannot be quasiisometric. It turns out that the essential point of this example is the periodicity within the words \( \gamma \) and \( \overline{\gamma} \). To exclude this periodicity, we use the Morse Thue decoration.

### 3.3 The Morse Thue decoration

The following sequence was studied by Thue [T] and later independently by Morse [M].

**Definition 3.9. (Morse-Thue Sequence \( t(i) \)).** Consider the substitution rule

\[
0 \to 01 \quad \text{and} \quad 1 \to 10.
\]

Then start from 0 to perform this substitutions

\[
0 \to 01 \to 0110 \to 01101001 \to \ldots
\]

to obtain a nested family of sequences of length \( 2^k \) in the alphabet \( \{0,1\} \).

The resulting limit sequence is called the Morse-Thue sequence.

The Morse-Thue sequence has the following remarkable property (see e.g. [HE]),

**Theorem 3.10.** The Morse-Thue sequence contains no string of type \( WWW \) where \( W \) is any word in 0 and 1.
3.4 The quasiisometry

We will use the Morse-Thue sequence to "decorate" elements of the generating set. To every generator $s \in S$ can be given a decoration 0 or 1. Let $E_a = \{b^0, b^1 | b \in B\} \cup A \cup \{\emptyset\}$. Thus $E_a$ is the alphabet $E$ where we in addition have decorations of the letters of color $b$. Let $T_a$ be the rooted tree with label set $(E_a)^{\kappa} \times A$.

We define the final map $\psi_a$ from $\Gamma$ into the tree $T_a$ in the following way. As above take the canonical $a$-representation $W$ of an element $\gamma \in \Gamma$ and decorate every element with color $b$ in $W$ by the Morse Thue sequence; i.e. the $i$-th letter of color $b$ gets the decoration $t(i)$. Then we apply the map $\psi_a^{\text{dia}}(W)$ to the decorated word.

Analogously we define $E_b$, $T_b$ and a map $\psi_b : \Gamma \rightarrow T_b$ by interchanging the roles of the two colors.

Let

$$\psi = (\psi_a, \psi_b) : \Gamma \rightarrow T_a \times T_b$$

In the same way as above we see that $\psi$ is 1-Lipschitz and a radial isometry.

**Theorem 3.11.** If $\Gamma$ is 2-colored, right angled and hyperbolic Coxeter group and let $\kappa \geq 160$ then the map $\psi$ is a quasiisometry.

**Proof.** To keep the proof more general, define $n = 2$ to be the chromatic number.

We know already that $\psi$ is 1-Lipschitz. To prove the opposite inequality let $\gamma, \overline{\gamma} \in \Gamma$ and let $c := d(\gamma, \overline{\gamma})$.

Claim: If $c \geq 60n$ then $|\psi(\gamma)\psi(\overline{\gamma})| \geq c/6n$.

Clearly the claim implies the desired lower estimate on the distance of image points. To prove the claim, we use the notation as in Lemma 3.2. Hence we have canonical $a$-representations

$$U = U_1a_1 \ldots U_pa_pU_{p+1}$$

$$V = V_{p+1}a_p \ldots V_{p+m}a_{p+m}V_{p+m+1}$$

$$\overline{V} = \overline{V}_{p+1}\overline{a}_p \ldots \overline{V}_{p+m}\overline{a}_{p+m}\overline{V}_{p+m+1}$$

such that

$$W = W_1a_1 \ldots W_{p+m}a_{p+m}W_{p+m+1}$$

represents $\gamma$ and $\overline{\gamma}$ is represented by

$$\overline{W} = \overline{W}_1\overline{a}_1 \ldots \overline{W}_{p+m}\overline{a}_{p+m}\overline{W}_{p+m+1}$$

We can assume without loss of generality that in a reduced representation of $\gamma^{-1}\overline{\gamma}$ the number of letters with color $a$ is larger or equal than the number of letters with color $b$. 

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This means that \( m + \overline{m} \geq c/n \).

We first show that the claim is true in the case \( m < c/3n \) or \( \overline{m} < c/3n \): Assume that (say) \( m < c/3n \). Since \( m + \overline{m} \geq c/n \) we have \( \overline{m} - m \geq c/n \). Since \( \psi \) is a radial isometry, we have

\[
|\psi_a(\gamma)| - |\psi_a(\gamma)| = (p + \overline{m}) - (p + m) = \overline{m} - m \geq c/3n
\]

and hence \( |\psi_a(\gamma)\psi_a(\gamma)| \geq c/3n \). Thus we can assume for the rest of the proof that \( m \geq c/3n \) and \( \overline{m} \geq c/3n \).

We will now prove the claim by contradiction. Therefore assume that \( |\psi_a(\gamma)\psi_a(\gamma)| < c/6n \). This assumption together with \( m \geq c/3n \) and \( \overline{m} \geq c/3n \) implies

\[
\psi_a(W|_{p+r}a_{p+r}) = \psi_a(\overline{W}|_{p+r}\overline{a}_{p+r}) \quad \text{for some } r > c/6n \quad \text{(1)}
\]

In particular the diary entries \( \psi_a^{\text{dia}}(W|_{p+r}a_{p+r}) = (\alpha_1, \ldots, \alpha_{p+r}) \) coincide with the corresponding entries \( (\overline{\alpha}_1, \ldots, \overline{\alpha}_{p+r}) \). Since \( c \geq 60n \) by assumption we have \( r > 10 \) and one computes elementary that also \( (r-1) \geq c/10n \) and \( (r-2) \geq c/10n \).

Consider now the three cases:

Case 1: \( V_{p+1} = \emptyset = \overline{V}_{p+1} \). Then it follows from the fact that \( V^{-1}\overline{V} \) is reduced that \( a_{p+1} \neq \overline{\alpha}_{p+1} \) and hence \( \psi_a(W|_{p+1}a_{p+1}) \neq \psi_a(\overline{W}|_{p+1}\overline{\alpha}_{p+1}) \) (by definition of the augmented map \( \psi_a^{\text{dia}} \)). This is a contradiction to equation (1).

Case 2: \( V_{p+1} = \emptyset \) and \( \overline{V}_{p+1} \neq \emptyset \).

In this case \( \overline{W}_j = \overline{V}_j \) for all \( j \geq p + 2 \) by Proposition 3.1 (2). In particular

\[
\ell(\overline{W}_{p+3}\overline{a}_{p+3}\ldots\overline{W}_{p+r}) \leq \ell(\overline{V}) \leq c
\]

and hence (since \( \kappa \geq 20n \) and \( (r-2) \geq c/10n \))

\[
k = \kappa(r - 2) - \ell(\overline{W}_{p+3}\overline{a}_{p+3}\ldots\overline{W}_{p+r}) \geq c
\]

By Proposition 3.5 we can reconstruct from \( (\overline{\alpha}_{p+2}, \ldots, \overline{\alpha}_{p+r}) \) either the whole word \( \overline{W}_{p+2}\overline{a}_{p+2} \) or at least the last \( k \geq c \) letters of this word. Since \( \overline{W}_{p+2}\overline{a}_{p+2} \) is a subword of \( \overline{V} \) it has length \( \leq c \) and hence we can reconstruct the whole word. Then, since \( (\overline{\alpha}_{p+2}, \ldots, \overline{\alpha}_{p+r}) = (\alpha_{p+2}, \ldots, \alpha_{p+r}) \) we conclude \( \overline{W}_{p+2}\overline{a}_{p+2} = W_{p+2}a_{p+2} \).

We make a similar computation at the place \( p + 1 \). In the same way as above we compute

\[
\ell(\overline{W}_{p+2}\overline{a}_{p+2}\ldots\overline{W}_{p+r}) \leq \ell(\overline{V}) \leq c
\]

and hence (since \( \kappa \geq 80n \) and \( (r-1) \geq c/10n \))

\[
k = \kappa(r - 1) - \ell(\overline{W}_{p+2}\overline{a}_{p+2}\ldots\overline{W}_{p+r}) \geq 7c.
\]
By Proposition 3.5 we can reconstruct from \((\alpha_{p+1}, \ldots, \alpha_{p+r})\) either the whole word \(W_{p+1}\alpha_{p+1}\) or at least the last \(k\) letters of this word. Assume for a moment that we can reconstruct the whole word. Then, since \((\alpha_{p+1}, \ldots, \alpha_{p+r}) = (\alpha_{p+1}, \ldots, \alpha_{p+r})\) we have \(W_{p+1}\alpha_{p+1} = W_{p+1}\alpha_{p+1}\) in contradiction to Lemma 3.2.

It follows that \(W_{p+1}\alpha_{p+1}\) and \(W_{p+1}\alpha_{p+1}\) have at least \(k \geq 7c\) letters and the last \(k\) letters of the words coincide.

Note that (using the terminology of Proposition 3.1): \(W_{p+1} = W_{p+1}U_{R_{p+1}}V_{p+1}\). By the last point of Proposition 3.1 and the just proved fact that \(W_{p+2} = W_{p+2}\) we see

\[
|U_{R}| \leq |W_{p+2}| + 1 = |W_{p+2}| + 1 \leq \ell(V) \leq c.
\]

It follows that \(|U_{R}V_{p+1}| \leq 2c\). Since the last \(6c\) letters of \(W_{p+1}\) and \(W_{p+1}\) coincide, we can write \(W_{p+2}\) in the form \(MHH\) and \(W_{p+2}\) in the form \(MHHH\) where \(H = U_{R}V_{p+1}\).

Since \(\psi_{a}\) also contains the Morse-Thue decoration of the elements of \(B\) and the property of the Morse-Thue sequence, there can not be any subsequence of decorated letters of the form \(HHH\). Thus we obtain a contradiction.

Case 3. \(V_{p+1} \neq \emptyset\) and \(V_{p+1} \neq \emptyset\).

Then \(W_{p+1} = U_{p+1}V_{p+1}\) and \(W_{p+1} = U_{p+1}V_{p+1}\). Since \(V_{p+1} \neq V_{p+1}\) but the last \(7c\) letters of \(W_{p+1}\) and \(W_{p+1}\) coincide by the same arguments as in Case 2, we see \(\ell(V_{p+1}) \neq \ell(V_{p+1})\). Let w.l.o.g. \(\ell(V_{p+1}) > \ell(V_{p+1})\). It is now completely elementary to see that again \(W_{p+2}\) is of the form \(MHHH\) and \(W_{p+2}\) of the form \(MHHH\) where \(H\) are the first \(\ell(V_{p+1}) - \ell(V_{p+1})\) letters of \(V_{p+1}\). We obtain now a contradiction as in Case 2.

\[\square\]

4 The n-colored case

In this section we modify the arguments of section 3 to prove the general case. We assume that \(\Gamma\) is a finitely generated right-angled Coxeter group with chromatic number \(n\). Thus the set \(S\) of generators has a decomposition \(S = A \cup B \cup \ldots\) into \(n\) subsets in such a way that elements of the same color do not commute. Fix a color, say \(a\). A reduced left \(a\)-representation of \(\gamma \in \Gamma\) is a reduced word \(W\) representing \(\gamma\), which has the form

\[
W = W_{1}a_{1}W_{1}a_{2} \ldots W_{r}a_{r}W_{r+1}
\]

in which for every \(i\) no letter \(x\) in \(W_{i}\) commutes with \(a_{i}\) and with all letters of \(W_{i}\) to the right of \(x\).

As discussed in 2.1 every set of mutually commutative generators \(R \subset S\) defines a simplex in the nerve \(N(\Gamma, S)\). For every element \(\gamma \in \Gamma\) we define
a right simplex presentation

\[ U = \Delta_q \ldots \Delta_1 \]

where the letters of each word \( \Delta_i \) form a simplex. The simplices \( \Delta_i \) are defined by induction on \( i \). We consider \( U \) in reduced presentation and define \( \Delta_1 \) as the set of all letters from \( U \) which can be placed at the very end of the word. So \( \Delta_1 \) consists of all letters \( x \) that commute with all letters to the right. Then \( U \) can be represented as \( U^1 \Delta_1 \). Then we apply this procedure to \( U^1 \) to obtain \( \Delta_2 \) and so on. We note that the right simplex presentation is unique in the sense that the sequence of simplices is uniquely determined. Thus, it gives a unique presentation of a group element up to permutation of the letters inside simplices.

A right simplex presentation \( U = \Delta_q \ldots \Delta_1 \) is a word in the alphabet \( S \). On the other hand it is a word in the alphabet \( \Sigma \), the set of all simplices. We will use the same notation for both.

We now define a canonical a-representation of \( \gamma \in \Gamma \) to be a reduced left a-presentation

\[ W = W_1 a_1 W_2 a_2 \ldots W_r a_r W_{r+1} \]

where each \( W_i = \Delta_{i_1} \ldots \Delta_{i_{r_i}} \) is in the right simplex presentation.

The word \( W_i \) in the left a-presentation is called the coefficient of \( a_i \) if \( i \leq r \) and it is called the free coefficient, if \( i = r + 1 \).

Let \( \Sigma_a \) be the set of nontrivial simplices of \( N(\Gamma, S) \) which do not contain letters with color \( a \). We can view \( W \) as a word in the alphabet \( \mathcal{A} = A \cup \Sigma_a \). Considered as a word in this alphabet it is unique and therefore also called the canonical a-representation. Note that considered as a word in \( S \) a simplex from \( \Sigma_a \) has length \( \leq n - 1 \) since it cannot contain letters with the same color.

Let \( W \) be a reduced word, a reduction to a canonical a-representation permutes the letters of \( W \). We call this permutation the canonical reduction map of \( W \).

As above we study the situation that we have

\[ U = U_1 a_1 \ldots U_p a_p U_{p+1}, \]

\[ V = V_{p+1} a_{p+1} \ldots V_{p+m} a_{p+m} V_{p+m+1}. \]

and

\[ W = W_1 a_1 W_2 a_2 \ldots W_{p+m} a_{p+m} W_{p+m+1} \]

the canonical a-presentation of the word \( UV \) and let \( \varphi : UV \to W \) be the canonical reduction map for \( UV \). With this notation \( \varphi(a_i) = a_i \) for all \( 1 \leq i \leq p + m \). Let \( U_{p+1} = \Delta_{q_{p+1}} \ldots \Delta_{r_{p+1}} \) be the free coefficient of \( U \) and \( V_{p+1} = \Delta_{q_{p+1}} \ldots \Delta_{r_{p+1}} \) be the coefficient of \( a_{p+1} \) in \( V \) and let \( U_R = \{ u \in U | a_{p+1} < \varphi(u) \} \) denote the set of all letters \( u \in U \) that are moved by \( \varphi \) right of \( a_{p+1} \). Then

**Proposition 4.1.** (1) \( \varphi(\Delta_k) \subset \varphi(\Delta_k^w) \), where \( \Delta_k^w \ldots \Delta_1^w = W_{p+1} \) is the coefficient of \( a_{p+1} \) of \( W \).
(2) If $\Gamma$ is hyperbolic, then the following holds: If $u \in U_R$ and $\varphi(u) > a_{p+2}$, then $u \in \Delta^u_1$.

(3) If $\Gamma$ is hyperbolic, and $V_{p+1} \neq \emptyset$, then $U_R \subset \Delta^u_1$.

Proof. (1). Let $U_{p+1}$ be the free coefficient of $U$ and $V_{p+1}$ be the first coefficient of $V$. By the definition of the canonical $a$-reduction the permutation $\Phi$ is a composition $\Phi = \Phi_2 \circ \Phi_1$ of two maps, where $\Phi_1$ moves the $a$’s to the left as far as possible. In other words $\Phi_1$ moves $U_R$ to the right of $a_{p+1}$. Then $\Phi_2$ forms the simplices $\Delta^u_1, \ldots \Delta^u_k = W_{p+1}$ out of $U_{p+1}V_{p+1}$ where $U_{p+1} = U_{p+1} \setminus U_R$. One now checks easily $\Phi(\Delta^u_k) \subset \Phi(\Delta^u_{k+1})$.

(2). Assume that there exists $u \in U_{p+1} \setminus \Delta^u_1$ with $\varphi(u) > a_{p+2}$. Since $u$ cannot go right of all elements of $\Delta^u_1$, there exists $u' \in \Delta^u_1$ which does not commute with $u$. Then $\varphi(u') > \varphi(u) > a_{p+2}$. If $a_{p+1} \neq a_{p+2}$ then $u, u', a_{p+1}, a_{p+2}$ form a square in contradiction to the hyperbolicity assumption. If $a_{p+1} = a_{p+2}$ then there exists $u''$ between $a_{p+1}$ and $a_{p+2}$ which does not commute with them. Then $u, u', u'', a_{p+1}$ form a square.

(3). If we assume the contrary, then similar to (2) we obtain a square formed by two elements from $U_R$, one element from $V_{p+1}$ and $a_{p+1}$. □

In the same way as in section 8, represent two given elements $\gamma, \gamma' \in \Gamma$ by means of words

$$U = U_1a_1 \ldots U_pa_pU_{p+1}$$
$$V = V_{p+1}a_p \ldots V_{p+m}a_{p+m}V_{p+m+1}$$
$$\overline{V} = \overline{V}_{p+1}a_p \ldots \overline{V}_{p+m}a_{p+m}\overline{V}_{p+m+1}$$

such that $\gamma$ and $\gamma'$ are represented by

$$W = W_1a_1 \ldots W_{p+m}a_{p+m}W_{p+m+1}$$
$$\overline{W} = \overline{W}_1a_1 \ldots \overline{W}_{p+m}a_{p+m}\overline{W}_{p+m+1}$$

Clearly $W_ia_i = \overline{W}_i\overline{a}_i = U_i a_i$ for $1 \leq i \leq p$ and we have in generalization of Lemma 4.2.

Lemma 4.2. $W_{p+1}a_{p+1} \neq \overline{W}_{p+1}a_{p+1}$.

Proof. Since $V^{-1}\overline{V}$ is reduced, the first letter from $V$ is different from the first letter of $\overline{V}$.

Case 1: $V_{p+1} = \emptyset = \overline{V}_{p+1}$. In this case we have $a_{p+1} \neq \overline{a}_{p+1}$ and hence the claim.

Case 2: $V_{p+1} = \emptyset$ and $\overline{V}_{p+1} \neq \emptyset$.

We can assume in addition that $a_{p+1} = \overline{a}_{p+1}$ since otherwise the result is trivially true. Note that no element from $\overline{V}_{p+1}$ can be moved right to $\overline{a}_{p+1} = a_{p+1}$. Therefore all elements in $U_{p+1}\overline{V}_{p+1}$ which can be moved right
of $a_{p+1}$ are elements of $U_{p+1}$ and hence elements from $U_R$ in the notation of Proposition 4.1. If follows that $\ell(W_{p+1}) < \ell(W_{p+1})$.

Case 3: $V_{p+1} \neq \emptyset$ and $V_{p+1} \neq \emptyset$. As above we can assume in addition that $a_{p+1} = a_{p+1}$.

We first investigate the case $U_R \neq U_R$. Assume that there exists $u \in U_R$ such that $u$ is not in $U_R$. Then there exists a "blocking element" $b \in V_{p+1}$ which does not commute with $u$. The letter $b$ is not contained in $V_{p+1}$, since $u \in U_R$. Furthermore $b$ is not contained in $U_R$ since $U_R \cap V_{p+1} = \emptyset$. It follows that the letter $b$ occurs more often in $W_{p+1}$ than in $W_{p+1}$. But this implies the claim.

Thus we can assume that $U_R = U_R$. Now by Lemma 4.1(3) $U_R = U_R \subset \Delta^1$ which implies that we can write $U_{p+1}$ in a reduced representation as $U'V_{p+1}$ and $W_{p+1}$ the same element as $U'V_{p+1}$. Since $V_{p+1}$ represents a different group element as $V_{p+1}$ we obtain the result.

We can now define the diary map. Let $\kappa \in \mathbb{N}$. Define $E_a = (A \cup \{\emptyset\})$. Then $E_a^{\kappa}$ is the label set $T_a^{\text{dia}}$, the diary tree.

In the same way as in section 3 we define the diary map

$$\psi_a^{\text{dia}}(W) = (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_r)$$

as a sequence of length $\ell_a(W)$. As above we obtain

**Proposition 4.3.** Let $W = W_1a_1 \ldots W_{i+m}a_{i+m}W_{i+m+1}$ be a canonical $a$-representation and assume that

$$k := \kappa m - \ell_A(W_{i+1}a_{i+1} \ldots W_{i+m}) \geq 1,$$

where $\ell_A$ denotes the length of a word in the alphabet $A$. Then one of the following holds:

1. we can reconstruct $W_{i}a_{i}$ from the diary entries $(\alpha_{i+1}, \ldots, \alpha_{i+m})$

2. The word $W_{i}a_{i}$ has $\geq k$ $A$-letters and we can reconstruct from $(\alpha_{i+1}, \ldots, \alpha_{i+m})$ the last $k$ $A$-letters of $W_{i}a_{i}$.

We collect the maps $\psi_c^{\text{dia}}$ to a common map $\psi^{\text{dia}} : \Gamma \to \prod_c T_c^{\text{dia}}$. and we also define the augmented diary map. As in chapter 3 we see that the diary map is Lipschitz and a radial isometry. Finally we decorate the simplices from $\Sigma_a$ by the Morse Thue sequence in the following way. Given a word $W$ in the canonical $a$-representation, we decorate for every color $b \neq a$ every $S$-letter with color $b$ by the Morse-Thue sequence, i.e. the $i$-th letter of color $b$ gets the decoration $t(i)$. Since within a given color the order of the elements cannot change, this is well defined. Thus a simplex is now a simplex of
decorated \(S\)-letters. To this decorated sequence we apply the augmented diary map to obtain the map \(\psi\) and finally we collect the maps to get

\[
\psi = \prod_c \psi_c : \Gamma \rightarrow \prod_c T_c
\]

Now we can prove Theorem 1.1 in general. The proof is essentially as in section \(\S\). There are however some significant differences in the discussion of Case 2 and Case 3.

The idea is to reduce the argument to the case of two colors. Therefore we use the following reduction. Let \(W\) be a word given in the canonical \(a\)-decomposition and let \(b\) be a color different from \(a\). We define a word \(W^b\) in the alphabet \(\{b^a, b^1 | b \in B\} \cup A \cup \{\star\}\) (where \(\star\) is some additional symbol) in the following way: Replace every decorated simplex \(\Delta \in \Sigma_a\) from \(W\) by the decorated letter with color \(b\), if \(\Delta\) contains a letter with color \(b\). If \(\Delta\) does not contain a letter with color \(b\) then replace \(\Delta\) by \(\star\).

\textbf{Lemma 4.4.} Let \(G\) and \(H\) be words given in the canonical \(a\)-representation and assume that \(GH\) is reduced. Assume that the first \(A\)-letter of \(H\) contains a letter of color \(b\). Then \((GH)^b\) can be obtained from \(Gb^bH^b\) by removing certain numbers of the letter \(\star\) from \(Gb^b\).

\textbf{Proof.} Let \(H = \Delta_l \ldots \Delta_1\) be the canonical \(a\)-decomposition of \(H\). Let \(b\) be a color that occurs in \(\Delta_l\). Since \(GH\) is reduced, there is no letter of color \(b\) in \(G\) which is mapped by the canonical reduction map right to a letter of color \(b\) in \(\Delta_l\). The result follows now easily. \(\square\)

We now proceed with the proof of the main theorem in the \(n\)-colored case.

Case 2: \(V_{p+1} = \emptyset\) and \(\overline{\nabla}_{p+1} \neq \emptyset\).

Similar in chapter \(\S\) we conclude that \(\overline{W}_{p+1}a_{p+1}\) and \(W_{p+1}a_{p+1}\) have the last \(k \geq 7c\) letters in common. These letters are decorated simplices. We furthermore see that in this case we can write \(\overline{W}_{p+1} = W_{p+1}H\) where \(H = U'R\) and \(U' = U_R \setminus U_R\). Let \(H = \Delta_l \ldots \Delta_1\) be the canonical \(a\)-decomposition of \(H\). Let \(b\) be a color that occurs in \(\Delta_l\). This implies that there is no letter with color \(b\) in \(\overline{U}_R\). Let \(\Delta_l^w \ldots \Delta_1^w\) be the last simplices of \(W_{p+1}\). Since the last \(A\)-letters of \(W_{p+1}\) and \(\overline{W}_{p+1} = W_{p+1}H\) coincide we have

\[
\Delta_i \subset \Delta_i^w
\]  \hspace{1cm} (2)

by Proposition 4.4 (1).

We use now the reduction to the 2-colored case. By Lemma 4.4 we obtain \((W_{p+1}H)^b\) from \(W_{p+1}^bH^b\) by deleting some \(\star\)-letters from \(W_{p+1}^b\).

Because of relation (2) and since \(U_R\) contains no letter of color \(b\), we see that the last \(k \geq 7c\) letters of \(W_{p+1}^b\) coincide with \(H^b\). It follows from Lemma 4.3 that there are words \(Q, H_1, H_2 \in \mathcal{A}\), such that \(W_{p+1} = QH_1\),
$W_{p+1} = QH_1H_0$ such that $H_0^b = H^b$ and $H_1^b$ can be obtained from $H^b$ by deleting some $\star$-letters. In particular the first letter of $H_1^b$ has color $b$. Again, since the last letters of $W_{p+1}^b = (QH_1)^b$ coincide with the last letters from $W_{p+1}^b = (QH_1H_0)^b$ we see that $W_{p+1}^b = \sigma H_2^b H_1^b H^b$ where $H_2^b$ is obtained from $H_1^b$ by deleting some $\star$-letters. Thus the sequence of simplices with a vertex of color $b$ has a subsequence of period 3 in contradiction to the properties of the decoration.

Case 3: $V_{p+1} \neq \emptyset$ and $V_{p+1} \neq \emptyset$.

We first show, that the length of $V_{p+1}$ differs from the length of $V_{p+1}$ where the length is measured in the alphabet $A$.

Assume to the contrary that $$V_{p+1} = \Delta_1^y \ldots \Delta_i^y$$ and $$V_{p+1} = \Delta_1^\sigma \ldots \Delta_i^\sigma$$ are of the same length.

Since $V V^{-1}$ is reduced, $\Delta_i^\sigma \cap \Delta_i^y = \emptyset$. Let $x \in \Delta_i^\sigma$ and $y \in \Delta_i^y$ letters (of $S$).

Since the last 7c $A$-letters of $W_{p+1}$ and $W_{p+1}$ coincide, we see $\Delta_i^\sigma = \Delta_i^\sigma$ and by Proposition 4.1 (1) $\Delta_i^\sigma \subset \Delta_i^y$ and $\Delta_i^y \subset \Delta_i^\sigma$. Thus the letters $x,y$ must also occur in $U$ in a way, that the letter $x$ can be moved in $W$ to $\Delta_i^\sigma$ and $y$ in $W$ to $\Delta_i^\sigma$. Since $UV$ is reduced, there exists $z \in U$ right to $x$ which does not commute with $x$. Since this $x$ can be moved to $\Delta_i^\sigma$ we see that also $z$ can be moved right to $\Delta_i^\sigma$ which implies that $y$ commutes with $z$. In the same way we see that there exists $u \in U$ right of $y$ which does not commute with $y$ but commutes with $x$. The letters $x, y, z, u$ form a square in contradiction to the hyperbolicity condition.

We therefore can assume w.l.o.g. that $\bar{V}_{p+1}$ is longer than $V_{p+1}$. Let $b$ be a color which occurs in the first $A$-letter of $\bar{V}_{p+1}$. With arguments as in case 2 one shows now that $\bar{W}_{p+1}^b$ is obtained from a word of the form $M^b H_2^b H_1^b H_0^b P^b$ by removing some $\star$-letters. Here $H_2^b$ is obtained from $H^b$ by removing $\star$-letters and $H$ is formed by the first $\ell_A(V_{p+1}) - \ell_A(V_{p+1})$ $A$-letters of $\bar{V}_{p+1}$. We obtain a contradiction in a similar way.

5 Proof of the Corollaries

Proof. (of Corollary 1.2)

Consider the right angled Coxeter group $\Gamma$ given by the generator set $S = \{s_1, \ldots, s_6\}$ and relations $s_i s_{i+1} = s_{i+1} s_i$ (indices mod 6). This group acts discretely on the hyperbolic plane such that a Dirichlet fundamental domain is bounded by the regular right angled hexagon in $\mathbb{H}^2$. By Theorem 1.1 we can embed the Cayley graph of $\Gamma$ quasiisometrically into the product.
Proof. (of Corollary 1.3)

By a result of Brady and Farb [BF] there exists a quasiisometric embedding of the hyperbolic space $\mathbb{H}^n$ into the $(n-1)$-fold product of hyperbolic planes. Actually the proof in [BF] gives a bilipschitz embedding (compare [F, section 2]). Combining Corollary 1.2 with this result, we are done.

Proof. (of Corollary 1.4)

A recent result of Januszkiewicz and Swiatkowski [JS] shows the existence of a Gromov-hyperbolic right angled Coxeter group $\Gamma_n$ of arbitrary given $n$, such that the virtual cohomological dimension of $\Gamma_n$ is $n$. The construction of these groups imply that they have chromatic number $n$. By Theorem 1.1 $\Gamma_n$ can be quasiisometrically embedded into the product of $n$ binary trees.

Proof. (of Corollary 1.5)

We recall the definition of the hyperbolic rank of a metric space. Given a metric space $M$ consider all locally compact Gromov-hyperbolic subspaces $Y$ quasiisometrically embedded into $M$. Then $\text{rank}_h(M) = \sup_Y \dim \partial_\infty Y$ is called the hyperbolic rank. (Compare [BS1] for a discussion of this notion). Let $\Gamma_n$ as in the proof of Corollary 1.4 above. $\Gamma_n$ can be embedded in a bilipschitz way into the $n$-fold product $T^n$ of the binary tree $T$. Since the virtual cohomological dimension of $\Gamma_n$ is $n$, we have $\dim \partial_\infty \Gamma_n = (n-1)$ by [BM]. Thus $\text{rank}_h(T^n) \geq (n-1)$ by the definition of the hyperbolic rank. The opposite inequality $\text{rank}_h(T^n) \leq (n-1)$ follows from standard topological considerations.

6 Strongly aperiodic tilings of the Davis Complex

As an application of the methods developed above, we construct certain aperiodic tilings of the Davis complex of $X$.

We recall the definition of tiling of a metric space from [BW]. Let $X$ be a metric space. A set of tiles $(T,F)$ is a finite collection of compact $n$-dimensional complexes $t \in T$ and a collection of subcomplexes $f \in F$ of dimension $< n$, together with an opposition function $o : F \to F$, $o^2 = \text{id}$. A space $X$ is tiled by the set $(T,F)$ if

1. $X = \bigcup_{\lambda} t_\lambda$ where each $t_\lambda$ is isometric to one of the tiles in $T$;
2. $t_\lambda \setminus \bigcup_{f \in t_\lambda} = \text{Int}(t_\lambda)$ in $X$ for every $\lambda$. 

(3) If \( \text{Int}(t_\lambda \cup t_\chi) \neq \text{Int}(t_\lambda) \cup \text{Int}(t_\chi) \) then \( t_\lambda \) and \( t_\chi \) intersect along \( f \in t_\lambda \) and \( o(f) \in t_\chi \);

(4) There are no free faces of \( t_\lambda \).

Remark 6.1. Strictly speaking the tiling is given by a collection \( \{ \varphi_\lambda \} \) of specified isometries \( \varphi_\lambda : t_\lambda \to t \), where \( t \in T \), such that for an \( n-1 \)-dimensional face \( \sigma \subset t_\lambda \) we have \( \varphi_\lambda(\sigma) \in F \). Point (3) says the following: if \( \sigma = t_\lambda \cap t_\chi \) is the "common" face, then \( o(\varphi_\lambda(\sigma)) = \varphi_\chi(\sigma) \).

Let \( g \) be an isometry of \( X \) preserving the decomposition structure, i.e. every \( t_\lambda \) is mapped by \( g \) to some \( t_\chi \). Then \( g \) induces a new tiling \( \{ g^* \varphi_\lambda \} \) with the same tiling set \( (T, F) \) by \( g^* \varphi_\lambda = \varphi_\chi \circ g \). We say that \( g \) is an isometry of the tiling, if \( \{ g^* \varphi_\lambda \} = \{ \varphi_\lambda \} \). We call a tiling \( \{ \psi_\lambda \} \) a limit tiling of \( \{ \varphi_\lambda \} \), if there exists a sequence \( g_i \) of decomposition preserving isometries of \( X \), such that for every \( \lambda \) there exists \( i_0(\lambda) \in \mathbb{N} \), such that \( \psi_\lambda = g_i^* \varphi_\lambda \) for all \( i \geq i_0(\lambda) \).

Definition 6.2. (1) A tiling is called aperiodic, if the isometry group of the tiling does not act cocompactly on \( X \).

(2) A tiling is called strongly aperiodic if it has a trivial group of isometries.

Theorem 6.3. For every finitely generated, right-angled and hyperbolic Coxeter group \( \Gamma \) the Davis complex \( X = X(\Gamma, S) \) admits a strongly aperiodic tiling with finitely many tiles. In addition every limit tiling is also strongly aperiodic.

Since in dimensions 2, 3, 4 there are right-angled reflection groups with compact fundamental domain we obtain

Corollary 6.4. The hyperbolic spaces \( \mathbb{H}^2, \mathbb{H}^3, \mathbb{H}^4 \) admit strongly aperiodic tilings such that all limit tilings are also and strongly aperiodic.

Before we prove this result, we recall the basic facts concerning the Davis complex.

6.1 The Davis Complex

Let \( \Gamma \) be a right angled Coxeter group with generating set \( S \) and let \( N = N(\Gamma, S) \) be the nerve of \( (\Gamma, S) \). By \( N' \) we denote the barycentric subdivision of \( N \). The cone \( C = \text{Cone} \ N' \) over \( N' \) is called a chamber for \( \Gamma \). The Davis complex \( X = X(\Gamma, S) \) is the image of a simplicial map \( q : \Gamma \times C \to X \) defined by the following equivalence relation on the vertices: \( a \times v_\sigma \sim b \times v_\sigma \) provided \( a^{-1}b \in \Gamma_\sigma \). Here \( \sigma \) is a simplex in \( N \), \( \Gamma_\sigma \) is the subgroup of \( \Gamma \) generated by the vertices of \( \sigma \), \( v_\sigma \) is the barycenter of \( \sigma \). We identify \( C \) with the image \( q(1 \times C) \) as a subset of \( X \). The group \( \Gamma \) acts simplicially on \( X \) by
Hence \( g \sigma \). Since the group is right-angled, \( \Gamma \) is obtained by gluing the chambers \( \gamma C \), \( \gamma \in \Gamma \) along the boundaries. Note that \( X \) admits an equivariant cell structure with the vertices \( X^{(0)} \) equal the cone points of the chambers and with the 1-skeleton \( X^{(1)} \) isomorphic to the Cayley graph of \( \Gamma \).

The condition \( s \in \mathcal{M} \) in Lemma 6.5. For every generator \( s \) in a right-angled Coxeter group \( \Gamma \) we have \( \mathcal{M}_s = \{ q(w \times x) \mid w \in Z_s, x \in St(s, N') \} \), where \( Z_s \) is the centralizer of \( s \) in \( \Gamma \).

Proof. \( " \supset " \): Let \( w \in Z_s \) and \( x \in St(s, N') \), i.e. \( x \) is an affine combination \( x = \sum_{s \in \sigma} x_{\sigma}v_{\sigma} \) one easily computes

\[
sg(w \times x) = wq(s \times x) = wq(1 \times x) = q(w \times x).
\]

\( " \subset " \): Let \( z \in M_s \). Then \( z = q(g \times x) \) for some \( g \in \Gamma \) and \( x \in cone(N') \). The condition \( s(z) = z \) can be rewritten as \( q(sg \times x) = q(g \times x) \). Hence \( g^{-1}sg \in \Gamma_\sigma \) for some simplex \( \sigma \) of \( N \) and \( x \in \cap_{v\in\sigma} St(v, N') \). By the deletion law \( s \in \Gamma_\sigma \), since the number of \( s \) in \( g^{-1}sg \) is odd. Hence \( s \in \sigma \) and \( x \in St(s, N') \).

Let \( s_1 \ldots s_k \) be a reduced presentation of \( g^{-1}sg \). We note that all \( s_i \in \sigma \). Since the group is right-angled, \( \Gamma_\sigma \) is commutative and hence all \( s_i \) are different. Let \( u_1 \ldots u_l \) be a reduced presentation of \( g \). Note that \( s_j \neq u_i \) for every \( u_i \neq s \), since \( u_i \) appears even number times in the word \( u_1 \ldots u_1 su_1 \ldots u_l \). Hence \( g^{-1}sg = s \), i.e. \( g \in Z_s \).

Lemma 6.6. Different mirrors of the same S-color are disjoint.

Proof. Let \( a, a' \in A \) and let \( M = M_a \) and \( M' = M_{a'} \) be the corresponding mirrors. Assume that \( \gamma_1(M) \cap \gamma_2(M') \neq \emptyset \). Therefore \( gM \cap M' \neq \emptyset \) where \( g = \gamma_2^{-1}\gamma_1 \). Let \( x \in gM \cap M' \). By Lemma 6.5 we have \( x = q(w \times y) = q(gu \times z) \), where \( w \in Z_{a'}, y \in St(t, N') \) and \( u \in Z_a, z \in St(s, N') \). Hence \( y = z = \sum_{a, a' \in \sigma} y_{a'}v_{a'} \). In particular \( a, a' \) are in a common simplex and hence commute. Thus \( a = a' \) since different elements in \( A \) do not commute.
Since \( q(w \times y) = q(gu \times z) \), we have \( w^{-1}gu \in \Gamma_\sigma \) for some simplex \( \sigma \) with \( a \in \sigma \). Thus \( w^{-1}gu \in Z_a \) which implies \( g \in Z_a \) and \( gM_a = M_a \).

The following Lemma is useful

**Lemma 6.7.** If a group \( G \) acts on the Davis complex \( X \) of a Coxeter group \( \Gamma \) in such a way that \( \Gamma \) takes walls to walls and respects the indices of the walls. Then \( G \) is a subgroup of \( \Gamma \).

**Proof.** Let \( g \in G \) and apply \( g \) to the base chamber \( C \subset X \). Since \( g \) preserves the walls of \( X \), also \( gC \) is a chamber, i.e. \( gC = \gamma C \) as a set. Since \( g \) and \( \gamma \) are isometries and both respect the indices of the walls, we clearly have \( g = \gamma \in \Gamma \). \( \Box \)

### 6.2 A tiling by color of the Davis Complex

Let \( X \) be the Davis complex of a right-angled Coxeter group \( \Gamma \).

By a **coloring** of mirrors by finitely many colors we mean a function \( \Phi : M \to F \), where \( F \) is a finite set. The coloring induces a coloring of the faces of all chambers, and chambers are glued together respecting the coloring.

**Proposition 6.8.** Every coloring \( \Phi : M \to F \) with a finite \( F \) defines a tiling \( (T, F) \) of the Davis complex \( X \) with \( o(f) = f \) for all \( f \in F \).

**Proof.** The set of tiles \( T \) is the set of chambers with all possible colorings of their faces. The set of faces \( F \) is the set of possible colored faces of the chambers. Set \( o(f) = f \). Then all conditions hold. \( \Box \)

In this situation we can identify \( F \) with \( F \). We call such a tiling as a **tiling by color**.

We note that for any tiling by color \( \Phi : M \to F \) there are limit tilings. We can describe them as follows. The group \( \Gamma \) acts (from the right) on the compact space \( F^M \) of \( F \)-sequences indexed by \( M \). Every limit point \( \Phi' \) of the sequence \( \Gamma(\Phi) \subset F^M \) is a limit for some sequence \( g_i \in \Gamma \), \( g_i(\Phi) \to \Phi' \). Then \( \Phi' \) is the limit tiling for \( \{g_i\} \).

It is easy to construct a tiling by color of the Davis complex which is strongly aperiodic. For that it suffices to paint all walls of the base chamber by different colors and all remaining wall by another color. Clearly, in this example every limit tiling is \( \Gamma \)-periodic. In this section we give a strongly aperiodic tilings by color of the Davis complex such that all of its limit tilings are strongly aperiodic.

The proof of Theorem \( \text{[11]} \) gives a coloring of the walls of the Davis complex in the following way. Let \( M_r \) be a wall, where \( r = \gamma a \gamma^{-1} \) is a reflection for (say) an element \( a \in A \subset S \). We assume that \( \ell(\gamma a) = \ell(\gamma) + 1 \).
(If not we replace $\gamma$ by $\gamma a$). Now we define the color of $\Phi(M_r)$ as the last entry in the sequence $\psi_a(\gamma a)$. We have to show that this is well defined: if we can also write $r = \beta a \beta^{-1}$ with some $\beta$ and $\ell(\beta) a = \ell(\beta) + 1$, then $\beta^{-1} \gamma$ commutes with $a$ and hence does not contain (in a reduced representation) a letter of $A$. Note that $\gamma a = \beta a \beta^{-1} \gamma$. By the definition of $\psi_a$ we see that $\psi_a(\gamma a) = \psi_a(\beta a \beta^{-1} \gamma) = \psi_a(\beta a)$, where the last inequality holds since a reduced word representing $\beta^{-1} \gamma$ does not contain letters with color $a$.

We investigate what information this coloring gives. We call two faces of the tiling adjacent, if the two faces are different but are contained in a common simplex.

**Definition 6.9.** A radial gallery of faces is a sequence $F_0, \ldots, F_k$ of faces, such that there is a geodesic $\gamma_0, \ldots, \gamma_k$ in $\Gamma$ starting from $\gamma_0 = 1$, such that $F_i$ is a face of the chamber $\gamma_i C$.

**Lemma 6.10.** Let $\Phi$ be the above tiling, and let $F$ be a face. Then we can reconstruct from the properties of the coloring $\Phi$ a radial gallery $F_0, \ldots, F_k$ of faces with $F = F_k$.

**Proof.** Let $F$ be the face between the chambers $\gamma C$ and $\gamma a C$ for some generator $a \in A \subset S$, such that $\ell(\gamma a) = \ell(\gamma) + 1$. Now the coloring gives as information the last entry of $\psi_a(\gamma a)$. This last entry contains in particular the diary entry, which itself consists of the last $\kappa$ entries (in the alphabet $A$) of the word $W$ which is the canonical $a$-representation of $\gamma$. The first of this $\kappa$ entries is a (decorated) simplex $\Delta \subset N(\Gamma, S)$. Note that by definition $s \in \Delta$ if and only if $\ell(\gamma s) < \ell(\gamma)$. Choose some $s \in \Delta$ and let $F'$ be the face between $\gamma s$ and $\gamma$. Inductively we will construct a gallery $F, F', F'', \ldots$ which finally will stop at the base chamber. Note that the diary entries of all faces of the base chamber $C$ are $\kappa$ times the symbol $\emptyset$ and a diary entry is of the form $(\emptyset, \ldots, \emptyset)$ if and only if this entry comes from a face of the base chamber. Thus the color tells us, when to stop. In this way we obtain the desired gallery.

**Remark 6.11.** (1) One easily sees from the proof, that one can reconstruct from the coloring all possible geodesics from $\gamma C$ to $C$.

(2) Two different faces of the gallery belong to different walls.

Now it is almost immediate that the tiling $\Phi$ is strongly aperiodic. Note that as mentioned in the proof of Lemma 6.10 the faces of the base chamber are characterized by coloring $\Phi$. Hence every isometry $g$ of the tiling has to fix the base chamber $C$. Since $g \in \Gamma$ by Lemma 5.7 we see $g = 1$.

We now consider the case of limit tilings. Indeed we need some additional properties for the coloring. We have to make the coloring in a way that "nearby" walls of the same $S$-color have different colors in $\Phi$.

To make this precise we define
Definition 6.12. Let \( M_{r_1} \) and \( M_{r_2} \) be two different mirrors with the same \( S \)-color (say) \( a \). Then the distance is defined to be the number
\[
d(M_{r_1}, M_{r_2}) = \inf \{d(\gamma_1, \gamma_2) + 1 \mid \exists \gamma_1, \gamma_2 \in \Gamma, \ a_1, a_2 \in A \text{ with } r_i = \gamma_i a_i \gamma_i^{-1} \}\]

Remark 6.13. We can view the mirrors \( M_{r_i} \) as subsets of the Cayley graph such that \( M_{r_i} \) is the set of midpoints of the edges with endpoints \( \gamma_i \) and \( \gamma_i s_i \) where \( \gamma_i s_i \gamma_i^{-1} \) is a representation of \( r_i \). Since by Lemma 6.6 two different mirrors with the same color do not intersect, the distance defined above is exactly the distance of the mirrors considered as subsets of the Cayley graph.

The following result is essential for our construction

Proposition 6.14. For every \( D \geq 0 \) there exists a map \( f_D: M \to G \) where \( G \) is a finite set such that for two reflections \( r_j, j = 1, 2 \) which are conjugates of the same element \( s \in S \) the following holds: \( f_D(M_{r_1}) = f_D(M_{r_2}) \) implies \( M_{r_1} = M_{r_2} \) or \( d(M_{r_1}, M_{r_2}) \geq D \).

Proof. Let \( \alpha_1, \ldots, \alpha_k \in \Gamma \) be the set of nontrivial elements with \( \ell(\alpha_i) \leq 2D + 2 \). Since a Coxeter group \( \Gamma \) is residually finite, there exists a finite group \( G \) and a homomorphism \( \sigma: \Gamma \to G \) such that \( \sigma(\alpha_i) \neq 1 \) for all \( i = 1, \ldots, k \). For a reflection \( r \) we define \( f_D(M_r) = \sigma(r) \). Let \( r_1, r_2 \) be as in the assumption and let \( d(M_{r_1}, M_{r_2}) < D \). Then there exists \( s \in S \) and \( \gamma_j \in \Gamma \) such that \( r_j = \gamma_j s \gamma_j^{-1} \) and \( d(\gamma_1, \gamma_2) \leq D \). Let \( \tau = \gamma_1^{-1} \gamma_2 \), hence \( \ell(\tau) \leq D \). By assumption \( r_1^{-1} r_2 \in \ker(\sigma) \). Since \( r_1^{-1} r_2 = \gamma_1 s \tau s_\tau^{-1} \gamma_1^{-1} \) we have \( s_\tau s^{-1} \in \ker(\sigma) \). Since \( \ell(s \tau s^{-1}) \leq 2D + 2 \) we have by construction that \( s_\tau s^{-1} \) is trivial and hence \( \tau \) commutes with \( s \) which implies that \( r_1 = r_2 \).

Proposition 6.15. For every finitely generated, right-angled and hyperbolic Coxeter group \( \Gamma \) the Davis complex \( X \) admits a tiling \( \Psi \) by colors with finitely many tiles such that \( \Psi \) and every limit tiling of \( \Psi \) is also strongly aperiodic.

Proof. We have to construct a coloring of the walls \( M \) by finitely many colors, such that the corresponding tiling is strongly aperiodic and also all limit tiling are strongly periodic. First we associate to every wall \( M_r \) the "color" \( \Phi(M_r) \). This gives already information about

- the \( S \)-color of \( M_r \)
- the diary entries in order that Lemma 6.10 holds.
- the Morse Thue decoration.

We need in addition a second "decoration", which we will call the distance decoration, of the walls with the following property: If \( M \) and \( M' \) are walls with the same \( S \)-color and the same distance decoration, then
\[
M = M' \text{ or } d(M, M') > 2C_h + 3,
\]

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where $C_h$ is the hyperbolicity constant of $\Gamma$. It follows directly from Proposition 6.14 that we can find such a decoration with a finite decorating set.

If we combine the coloring $\Phi$ with this additional decoration, we obtain a coloring $\Psi : \mathcal{M} \to F$, where $F$ is some finite set. Thus the induced tiling has again only finitely many tiles. As already shown above the tiling is strongly aperiodic and we have to consider a limit tiling. In the case of a limit tiling Lemma 6.10 then implies that given a face $F_0$ of the tiling, we can construct a gallery $F_0, F_{-1}, F_{-2} \ldots$ starting from $F_0$. We can assume that this gallery does not end, since we are in a limit case. Note that this gallery can be considered as a geodesic $\nu : \{0, 1, 2, \ldots\} \to C(\Gamma, S)$, where every point $\nu(i)$ correspond to a face of the tiling or an edge in $C(\Gamma, S)$. Assume that there is an isometry $g$ of the tiling. Then $g\nu$ is also a geodesic in $C(\Gamma, S)$. Since $\nu$ and $g\nu$ are limits of isometric images of geodesics segments starting at the base chamber, the geodesics $\nu$ and $g\nu$ are asymptotic. Thus there exists a "shift constant" $k$ (depending on $g$) such that $d(\nu(i), g\nu(i + k)) \leq C_h$ for all $i$ large enough, where $C_h$ is the hyperbolicity constant of $\Gamma$. By shifting the initial point of the geodesic and maybe interchanging the role of $\nu$ and $g\nu$, we can assume w.l.o.g. that $k \geq 0$ and $d(\nu(i), g\nu(i + k)) \leq C_h$ for all $i \geq 0$.

There exists some $S$-color (say) $a$, such that this color occurs infinitely often in this sequence. Now fix in addition a possible decoration. Let $j_0, j_1, \ldots$ be the subsequence such that the $\nu(j_i)$ are the faces of $S$-color $a$ and this given decoration. Note that this sequence may be finite. By Remark 6.11 (2) the walls belonging to the faces $\nu(j_i)$ are different. By the properties of the distance decoration there is a sequence $k_i$, with $k_i \leq j_i \leq k_{i+1}$, such that the faces $\nu(k_i)$ and $\nu(k_{i+1})$ have distance $\geq C_h$ from the walls belonging to the face $\nu(j_i)$ and these points lie on different sides of this wall. By triangle inequality also the points $g\nu(k_i)$ and $g\nu(k_{i+1})$ are on different sides of this wall which implies that $g\nu$ has to cross the wall. Thus the geodesic $g\nu$ also crosses all the faces with $S$-color $a$ and the given decoration which occur on the geodesic $\nu$. This argument holds for all possible decorations. Hence all walls of $S$-color $a$, which are intersected by $\nu$ are also intersected by $g\nu$ and (since walls of the same $S$-color do not intersect) in the same order. Thus the sequence of these faces on $\nu$ is a shift these faces on $g\nu$. By the properties of the Morse Thue decoration, there are no proper shifts. This implies that $g$ leaves these walls invariant. This implies $k = 0$ and hence the displacement of $g$ is $\leq C_h$ on $\nu(i)$. Thus there is a chamber of $X$ which is moved by $g$ at most by distance $C_h + 1$. Since $g$ preserves also the distance decoration, all the walls of this chamber are invariant under $g$ which implies that $g$ fixes the chamber and hence (since the coloring also contains the information of the element $s \in S$ of the wall $M_r$, where $r$ is a conjugate of $s$), $g$ is the identity.

\[\square\]
6.3 Balanced Tilings

Let \((T, F)\) be a set of tiles. A function \(w : F \to \mathbb{Z}\) is called a **weight function** if \(w(o(f)) = -w(f)\) for every \(f \in F\). We recall a definition from [BW].

**Definition 6.16.** A finite set of tiles \((T, F)\) is **unbalanced** if there is a weight function \(w\) such that \(\sum_{f \in t} w(f) > 0\) for all \(t \in T\).

It is called **semibalanced** if \(\sum_{f \in t} w(f) \geq 0\) for all \(t \in T\).

We call a set of tiles **strictly balanced** if for every nontrivial weight function \(w\) there are tiles \(t^+\) and \(t^-\) such that \(\sum_{f \in t^+} w(f) > 0\) and \(\sum_{f \in t^-} w(f) < 0\). A tiling called **unbalanced** (**strictly balanced**) if the corresponding set of tiles is unbalanced (**strictly balanced**).

In [BW] aperiodic tilings of some nonamenable metric spaces (such as the Davis complexes of hyperbolic Coxeter groups) are constructed where the aperiodicity follows from the fact that they are unbalanced. Here we show, how one can modify a tiling \(\Phi\) by color of the Davis complex in a way, that the new tiling is strictly balanced and also all limit tilings are strictly balanced. We start from a tiling by color, i.e. a function \(\Phi : M \to F\) where \(F\) is a finite set.

We now associate in addition to every wall an orientation. A wall divides the Davis complex into two components. Roughly speaking the orientation says which of the components is left and which is right. The orientation of the walls defines a new tiling \((T', F')\) where the new faces have in addition a sign \(+\) or \(−\). Thus \(F' = F_+ \cup F_-\), where \(F_+\) and \(F_-\) are copies of \(F\). The face \(f \in t_\lambda\) has sign \(+\), if \(Int(t_\lambda)\) is left of the wall and sign \(−\), if \(Int(t_\lambda)\) is right of the wall. In this case the opposition function \(o : F' \to F'\) maps \(f_+\) to \(f_-\). Geometrically this means that we deform all faces of given color and given sign in the same direction by the same pattern. For the faces of the same color but of opposite sign we take the opposite deformation. We call such tiling a **geometric resolution** of a tiling by color. This new tiling is not any more a tiling by color.

Note that in the Davis complex a wall has a canonical orientation, by deciding that the base chamber is in the left component. Thus we can indicate the chosen orientation itself by a sign. A wall gets the sign \(+\), if the orientation of the wall is the canonical one and \(−\) otherwise.

The following is obvious.

**Lemma 6.17.** (1) Assume that a coloring \(\Phi : M \to F\) defines a strongly aperiodic tiling of \(X\). Then any geometric resolution is also strongly aperiodic.

(2) If every limit tiling of \(\Phi\) is strongly aperiodic, then every limit tiling of any geometric resolution is strongly aperiodic.

We formulate and prove the main result of this section.
Theorem 6.18. For every coloring $\Phi : \mathcal{M} \to F$ with the property that walls of the same color do not intersect, there is a strictly balanced geometric resolution of this tiling. In addition also every limit tiling of this resolution is strictly balanced.

Proof. In the first step of the proof we construct a strictly balanced geometric resolution of $\Phi$. Consider the set of mirrors $\mathcal{M}_f = \Phi^{-1}(f)$ of the same color $f$. Since walls of the same color do not intersect by assumption, they are ordered by level from the base chamber. (The level is defined by induction. If one removes the mirrors $\mathcal{M}_f$ from $X$, the space is divided into components. Mirrors that bound the component of the base chamber are of level one. Then drop mirrors of level one and repeat the procedure to get new mirrors of level one and call them of level two and so on). We give the mirrors signs in alternate fashion by the level: $+ - + - + - \ldots$. Thus, elements of $\mathcal{M}_f$ have in alternate ways the $+$-orientation and the $-$-orientation.

The chosen orientation induces a geometric resolution of the tiling. We show that this resolution is strictly balanced.

Let $w : F_+ \cup F_- \to \mathbb{Z}$ be a nontrivial weight function with $w(f_+) = -w(f_-)$. We show that there is a chamber $C_+$ such that

$$\sum_{M \in C_+} w(M) > 0$$

and there is a chamber $C_-$ such that

$$\sum_{M \in C_-} w(M) < 0.$$  

We show the first.

Since the function $w$ is nontrivial there exists a face $f^0$ which is the common face of two adjacent complexes $t_\lambda$ and $t_{\lambda'}$ such that $w(f^0) \neq 0$. (By a slight abuse of notation we here identify the "real face" $\sigma = t_\lambda \cap t_{\lambda'} \subset X$ with the color $\Phi(\varphi_\lambda(\sigma)) = \Phi(\varphi_{\lambda'}(\sigma)) \in \mathcal{F}$.)

Now there are four cases corresponding to the parity of the sign of $w(f^0)$ and the orientation of the mirror $M_{f^0}$ defined by the face $f^0$. All this cases are treated in a similar way. We discuss only one, and to be fair not the easiest of these cases: let $w(f^+_\lambda) > 0$ and the orientation of the wall $M_{f^0}$ be negative.

By the convention how we use the words "left" and "right" this means that the base chamber is right of the wall $M_{f^0}$. Now $f^0 = t_{\lambda'} \cap t_\lambda$ where we assume that $t_\lambda$ is right, and $t_{\lambda'}$ is left of the wall. Thus $f^0$ considered as a face of $t_\lambda$ has sign $-$ and considered as a face of $t_{\lambda'}$ has sign $+$. Now we choose some number $k \in \mathbb{N}$ such that every geodesic from $Int(t_\lambda)$ to $Int(C)$ intersects $\leq k$ mirrors. (i.e. $k$ is larger than the combinatorial distance of $t_{\lambda'}$ and the base chamber.)
Now consider any color $f$. Then there are three possibilities:

(a) $w(f_+) > 0$ and $w(f_-) < 0$; then we call $f$ an even color for $w$,
(b) $w(f_+) < 0$ and $w(f_-) > 0$; then we call $f$ an odd color for $w$.
(c) $w(f_+)=w(f_-)=0$; then we call $f$ a 0-color for $w$.

Let

\[
M_{2k}^\text{ev} = \{ M_f | f \neq f^0, \text{ $f$ even for $w$ and level of $f = 2k$} \}
\]

\[
M_{2k+1}^\text{odd} = \{ M_f | f \neq f^0, \text{ $f$ odd for $w$ and level of $f = 2k+1$} \}
\]

\[
M_{2k+1}^{\text{0}} = \{ M_f | f \text{ 0-color for $w$ and level of $f = k+1$} \}
\]

Claim 1: The set of mirrors $M_{2k}^\text{ev} \cup M_{2k+1}^\text{odd} \cup M_{2k+1}^{\text{0}} \cup \{ M_{f^0} \}$ bound a bounded set $D$ containing the chamber $t_{\lambda'}$.

Clearly this set of mirrors bounds a convex set in the Hadamard space $X$. If the component containing $t_{\lambda'}$ is unbounded, then there is a ray from $t_{\lambda'}$ to the visual boundary which does not intersect any of our mirrors. Since we have only finitely many colors, there is a color $f$ such that this ray intersects infinitely many mirrors of this color. By the choice of $k$ the first of these intersected mirrors has level $\leq k+1$ and mirrors of the same color and level are intersected at most twice by convexity. Thus one of these intersected mirrors is contained in our set of mirrors. Contradiction.

Claim 2: If $f$ occurs as face of a tile $t_\mu \subset D$ such that $f \subset \partial D$, then $w(f) \geq 0$.

To prove Claim 2 we consider the cases

(i) If $f = f^0$, then $f$ as a face of $t_\mu$ has the orientation + since $t_\mu$ lies on the same side of $M_{f^0}$ as $t_{\lambda'}$. Thus $w(f) = w(f^0) > 0$.
(ii) If $f$ is a 0-face , then anyway $w(f) = 0$.
(iii) Let $f$ be an even color for $w$. Then $f$ is contained in a wall of $M_{2k}^\text{ev}$. By the choice of $k$ the tile $t_\mu$ is on the same side of this wall as the base chamber. The wall has level $2k$ hence orientation +, thus the base chamber and hence $t_\mu$ is on the left side of this wall. This implies that $f$ considered as a face of $t_\mu$ gets the sign +. Hence $w(f) = w(f_+) > 0$ since $f$ is an even color for $w$.
(iv) A similar argument applies for $f$, if $f$ is an odd color for $w$. This proves Claim 2.

According to the Claim 1 we have $D = \cup_{i=1}^{k} C_i$ where $C_1, \ldots, C_k$ is a finite collection of chambers. Then

\[
\sum_{i=1}^{k} \sum_{f \in C_i} w(f') = \sum_{f' \in \partial D} w(f') \geq 0.
\]

by Claim 2. Since also $f^0_+$ is in the last set of faces we see that the expression is indeed $> 0$. Therefore, $\sum_{f' \in C_i} w(f') > 0$ for some $i$.

To obtain the chamber $C_{-}$, we make a similar construction.
This finishes the proof of the first step. Thus we have constructed a strictly balanced geometric resolution of $\Phi$.

Actually the proof of the first step shows more: If we choose for any given color $f$ an orientations of the walls $M_f$ in alternate way $++-\ldots$ or $-+-\ldots$ (and maybe for different colors in a different way), then the resulting geometric resolution is strictly balanced. (This more general result follows from some obvious modifications of the above proof). Let us call such a choice of orientations an allowed orientation of walls. The levels of walls depend on the base chamber. If we define levels with respect to a different chamber, they are changed. The change has the property, that either all parities of the levels are preserved or all are changed. Thus, whether an orientation of the walls is allowed (or not), does not depend on the choice of the base chamber. As a consequence we have the following: if the orientation of a tiling by color $\{\varphi_\lambda\}$ is allowed, then also the orientation of $\{g^*\varphi_\lambda\}$ is allowed. Thus also all limit tilings of the tiling constructed in step 1 are strictly balanced.

\[
\square
\]

In two-dimensional jigsaw tiling puzzles a geometric resolution is usually realized by adding rounded tabs out on the sides of the pieces with a corresponding blank cut into the intervening sides to receive the tabs of adjacent pieces. This procedure destroys the convexity of the pieces. We show that in the case of the hyperbolic plane $\mathbb{H}^2$ we can modify this construction to obtain aperiodic and strictly balanced tilings with convex tiles. Compare also the papers [MaMo], [Mo].

**Theorem 6.19.** (1) For every $n \geq 3$ there is a strictly balanced strongly aperiodic tiling of $\mathbb{H}^2$ by convex $2n$-gons with finitely many tiles such that every limit tiling is strongly aperiodic.

(2) For every $n \geq 3$ there is a finite set of tiles $(T,F)$ that consists of convex $2n$-gons with a strongly aperiodic tiling of $\mathbb{H}^2$ with strongly aperiodic limit tilings such that every $(T,F)$-tiling of $\mathbb{H}^2$ is aperiodic.

**Proof.** (1) Identify $\mathbb{H}^2$ with the Davis complex for the right-angled Coxeter group generated by the reflections at a regular right angled $2n$-gon. This is a 2-colored group. We call these $S$-colors $a$ and $b$. We fix a strongly aperiodic tiling $\Phi$ by color as constructed above and an orientation of the walls in order that the geometric resolution of the tiling is strictly balanced. We now define a modification of the tiling defined by $\Phi$. Consider a vertex of the Davis complex. This is a point where some $a$-wall intersects some $b$-wall. Thus due to the map $\Phi$ the point correspond to a unique pair of colors $(a_i,b_j) \in F \times F$, where $F$ is the finite image of $\Phi$ and $a_i$ (resp. $b_j$) indicates that the corresponding $S$-color is $a$ (resp. $b$). The additional information about the orientation of the walls give a well defined lokal coordinate system around the vertex (by deciding that the positive quadrant is the quadrant
which is right to both walls, and by deciding that the $a$-wall correspond to the first and the $b$-wall to the second coordinate.) We now move the vertex by a small amount using these local coordinates. This move should only depend on $(a_i, b_j)$ and not on the vertex. E.g. we can choose a small $\mu(a_i, b_j) > 0$ and move the point to a distance $\mu(a_i, b_j)$ into the direction of the diagonal of the positive quadrant. After this deformation we obtain a finite number of new convex tiles, which (for generic deformations) only allow tilings of $\mathbb{H}^2$ compatible with the matching rules defined by $\Phi$ and the orientation of the walls. Roughly speaking the matching rules are now encoded in the length of the sides and the angles of the tiles. Our original aperiodic tiling is deformed to an aperiodic tiling with the desired properties.

(2) We start with the tiling by color $\Phi : M \to F$ defined earlier and take an unbalanced geometric resolution. One of the way to do it is to assign $+$ for passing through a wall from the component that contains the base chamber. Then for every chamber $C$ the faces whose walls do not separate $C$ and the base chamber obtain the sign $+$, all other $-$. In view of hyperbolicity the number of $+$ is greater than $-$ for every $C$. We define a weight function by sending a positive face to $+1$ and a negative face to $-1$. This geometric realization has all required properties except the last one by (1). Since the set of tiles $(T, F)$ is unbalanced, in view of the following proposition any other tiling by $(T, F)$ is aperiodic.

**Proposition 6.20.** Let $(T, F)$ denote the set of tiles of a geometric realization of a tiling by color of the Davis complex $X$ of a Coxeter group $\Gamma$ supplied with a left $\Gamma$-invariant metric. Suppose that the set of tiles $(T, F)$ is unbalanced. Then any $(T, F)$-tiling of $X$ is aperiodic.

**Proof.** This proposition can be derived formally from Proposition 4.1 [BW]. Since the proof there has some omissions we present a proof below.

Let $G$ be a group of isometries of a $(T, F)$-tiling of $X$ which acts cocompactly. Since the group of isometries of $X$ is a matrix group, by Selberg Lemma $G$ contains a torsion free subgroup $G'$ of finite index. Then the orbit space $X/G'$ is compact and admits a $(T, F)$-tiling (Note that by taking $X/G$ as in [BW] we cannot always obtain a tiling because of free faces). Then we obtain a contradiction:

$$0 < \sum_{t \in G/G'} \sum_{f \in t} w(f) = \sum_{f \in X/G'} w(f) + w(o(f)) = 0.$$

**References**

[A] E. Andreev, *On the intersection of faces of acute-angled polyhedra*, Mat. Zametki 8 (1970), 521-527. English translation: Math. Notes 8 (1971), 761-764.
[BM] M. Bestvina, G. Mess, The boundary of negatively curved groups, Journal AMS. 4, no. 3 (1991) 469-481.

[BF] N. Brady, B. Farb, Filling-invariants at infinity for manifolds of non-positive curvature, Trans. Amer. Math. Soc, 350 (1998), 3393-3405.

[BH] M. Bridson, A. Haefliger, Metric spaces of nonpositive curvature, Springer-Verlag 1999

[Br] K. Brown, Buildings, Springer Verlag, New York, 1989

[BS1] S. Buyalo, V. Schroeder, Hyperbolic rank and subexponential corank of metric spaces, GAFA 12 (2002), 293-306

[BS2] S. Buyalo, V. Schroeder, Embedding of hyperbolic spaces in the product of trees, arXive:math.GT/0311524, 2003, to appear in GAFA

[BW] J. Block, S. Weinberger Aperiodic tilings, positive scalar curvature and amenability of spaces, J. Amer. Math. Soc. 5 (1992) no.4, 907-921

[C] H. Coxeter, Regular honeycombs in hyperbolic space Twelve Geom. Essays, Southern Univ. Press, Carbondale, II (1968), 200-214.

[D1] M. Davis Groups generated by reflections and aspherical manifolds not covered by Euclidean space, Annals of Math. (2) 117, no. 2, (1983), 293-324

[D2] M. Davis A hyperbolic 4-manifold, Proc. Amer. Math. Soc. 93 (1985), 325-328

[DJ] A. Dranishnikov, T. Januszkiewicz Every Coxeter group acts amenabley on a compact space Proceedings of the 1999 Topology and Dynamics Conference (Salt Lake City, UT), p. 135-141 ,1999

[DS] A. Dranishnikov, V. Schroeder, Embedding of Coxeter groups in a product of trees, arXive:math.GR/0402398

[F] T. Foertsch, Euclidean Rank and Hyperbolic Rank of Hadamard Manifolds, Thesis, Univ. of Zurich 2002.

[G] C. Goodman-Strauss: A strongly aperiodic set of tiles in the hyperbolic plane Invent. math. 159 (2005), 119-132

[H] J. Humphreys: Reflection Groups and Coxeter groups Cambridge University press, 1990

[He] G.A. Hedlund: Remarks on the work of Axel Thue on sequences, Nordisk Mat. Tidskrift 15 (1967), 148-150
[JS] T. Januszkiewicz, J. Swiatkowski Hyperbolic Coxeter groups, Commentarii Math. Helvetici, 78, no. 3 (2003), 555-583.

[MaMo] G.A. Margulis, S. Mozes Aperiodic tilings of the hyperbolic plane by convex polygons Israel J. Math. 107 (1998), 319–325.

[M] M. Morse, Recurrent geodesics on a surface of negative curvature, Trans. Amer. Math. Soc. 22 (1921), 84-100

[Mo] S. Mozes Aperiodic tilings, Invent. Math. 128 no. 3, (1997), 603–611.

[PV] L. Potyagailo, E. Vinberg, On right-angled reflection groups in hyperbolic spaces, Comm. Math. Helvetic to appear

[T] A. Thue, Über unendliche Zeichenreihen, Norske vid. Selsk. Skr. Mat. Nat. Kl 7 (1906), 1-22

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