An Analytical Approach to Eddy Current in Electromagnetic Damping

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Abstract

This article explains and illustrates an analytical method of calculating eddy current in a metallic spinning gyroscope in external magnetic field. All the assumptions introduced to simplify the calculation are examined to be rational and the result is verified by comparing the theoretical deceleration process of the gyroscope due to the magnetic force on the rotator with the experimental one. The error between theory and experiment is only 8.61% and the possible cause of it is discussed towards the end of the article.
I. INTRODUCTION

The International Young Physicist Tournament (2019) contains a problem that a spinning gyroscope made from a conducting, yet non-ferromagnetic material slows down when placed in a magnetic field due to the interaction between induced eddy current and the field\textsuperscript{1}. There is a video that demonstrate it\textsuperscript{2}. Such a phenomenon, which is usually called ‘electromagnetic breaking’ or ‘electromagnetic damping’, is widely discussed qualitatively in electromagnetism courses in colleges and high schools, as a direct result of Faraday’s law of electromagnetic induction\textsuperscript{3}. However, the quantitative treatment of the deceleration process has not been seen. The calculations of eddy currents are difficult in two aspects. On one hand, the current is distributed in a block of conductor rather than a wire; On the other hand, the electromotive force, whose role is to drive the current, is also in the conductor, hence the system cannot be treated as a conventional electric circuit that consists of localized batteries, wires and resistors.

According to our research, in order to deal with such a problem properly, the best way is probably going back to the fundamental laws of electromagnetism. We illustrate a general method of calculating eddy current based on solving the Maxwell Equations with some reasonable assumptions, which can be further simplified to a boundary value problem of Poisson’s equation if the problem is time-independent. In the general case, the process of solving such equation(s) is often too complicated to be done by hand, hence numerical simulation is widely applied. However, in order to have a better understanding of the underlying physical process, deriving an analytical formula is of great significance, if it is achievable. Fortunately, in our gyroscope problem, we find a set up of the field and the gyroscope that the distribution of induced eddy current can be obtained analytically. Meanwhile, all the idealized conditions can be realized with simple apparatus in experiments. As it is hard to measure eddy currents in conductors directly, which is the rotator of our gyroscope, it is straightforward to compare the deceleration processes deduced in theory and observed in experiment to verify our analytical result of the eddy current.

To show the whole picture of our work, the basic logic of our theory to calculate eddy currents in a block of conductor is stated first. Then, we apply the theory to the gyroscope problem to get a theoretical result of the deceleration process. The comparison of the theoretical and experimental results shows that they fit each other extremely well. In the
last part of this article, we discuss the rationality of all the assumptions of our model and give a qualitative explanation of the remaining 8.61% error.

II. BASIC LOGIC OF THE THEORY AND ITS APPLICATION

The discussion in this section is about how to apply electromagnetism laws to calculate eddy currents, which starts with Maxwell equations:  

\[
\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}, \\
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \\
\nabla \cdot \vec{B} = 0, \\
c^2 \nabla \times \vec{B} = \vec{j} + \frac{\partial \vec{E}}{\partial t},
\]

where \(\vec{E}, \vec{B}\) are electric and magnetic field, while \(\rho, \vec{j}\) are the density of charge and current, respectively. (We may call them as 'charge' and 'current' in the rest of the article to be concise.)

Since we are dealing with current in a piece of conductor, Ohm’s law must be taken into account describe the nature of the conductor:  

\[
\vec{j} = \sigma (\vec{E} + \vec{f}),
\]

where \(\vec{f}\) is the so-called 'non-electrostatic force', which is usually inside the batteries. Its loop integral along the circuit is 'electromotive force', usually noted as 'emf'.

In the problem of the gyroscope, the magnetic field is produced by permanent magnetic blocks and hence independent of time. (Of course, the magnetic field produced by eddy current is omitted as usual.) As a result, the electric field in the conductor (gyroscope) obeys the reduced form of Equation (1) & (2):  

\[
\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}, \\
\nabla \times \vec{E} = 0.
\]

From the equations above, it is convenient to set up a scalar field: \(\varphi(\vec{r})\), which is called scalar potential in electromagnetism, whose minus gradient is the electric field \((-\nabla \varphi = \vec{E}\), and it satisfies (7) automatically). Plug this into Equation (6), we have:  

\[
-\nabla^2 \varphi = \frac{\rho}{\varepsilon_0}.
\]

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Now, it seems that Equation (8), together with proper boundary conditions, is sufficient to determine scalar potential $\varphi$ (and then electric field). However, a meticulous reader may point out that one cannot calculate scalar potential $\varphi$ from Equation (8), since we do not have a priori knowledge of the charge distribution. Therefore, a different perspective must be brought in to construct another relation between $\varphi$ and $\rho$, in order to get a set of solvable equations of scalar potential. It is worth noting that $\vec{f}$ together with $\vec{E}$ in Equation (5) drives the motion of charge, hence, the conservation law of charge:

$$\vec{\nabla} \cdot \vec{j} = \frac{\partial \rho}{\partial t},$$

(9)

can be used to establish the relation between current and charge. By plugging Equation (5) into (9), and then replacing $\vec{E}$ with $-\vec{\nabla} \varphi$, another equation linking electric field and charge density is derived:

$$\sigma(-\nabla^2 \varphi + \vec{\nabla} \cdot \vec{f}) = \frac{\partial \rho}{\partial t}.$$ 

(10)

Now, by solving Equations (8) & (10), a result of $\varphi$ and then $\vec{E}$ and finally current $\vec{j}$ can be obtained. Note that, in order to solve these equations, proper boundary and initial conditions are needed. Since initial conditions are unique and straightforward in different problems, we only list the boundary condition below, which is derived from the fact that current cannot go through the surface of the conductor:

$$\vec{j} \cdot \hat{n} \bigg|_{\text{boundary}} = j_n \bigg|_{\text{boundary}} = 0,$$

(11)

where $\hat{n}$ is the unit normal vector of the boundary.

In summary, we construct a general method to calculate current in a piece of conductor. For static magnetic field, the electric field has to be 'irrotational' ($\vec{\nabla} \times \vec{E} = -\partial \vec{B} / \partial t = 0$). Other parameters, such as the shape of the conductor, the conductivity or the distribution of non-electrostatic force can be arbitrary. Although difficulties may be encountered when solving the second order time-dependent partial differential equations:

$$\begin{cases}
-\nabla^2 \varphi = \frac{\rho}{\varepsilon_0}, \\
\sigma(-\nabla^2 \varphi + \vec{\nabla} \cdot \vec{f}) = \frac{\partial \rho}{\partial t}, \\
j_n \bigg|_{\text{boundary}} = 0.
\end{cases}$$

(12)

The problem we are going to investigate, which is about the spinning gyroscope in a magnetic field, is much simpler. We are going to introduce several assumptions here to simplify the calculation (the rationality of them will be discussed in the last section of this article):
The current varies so slow as time goes on that it can be treated as nearly time-independent, which means $\partial \rho / \partial t = 0$.

2. The non-electrostatic force here in the problem is due to the rotation of the gyroscope, which can be expressed as:

$$\vec{f} = \vec{v}_{rot} \times \vec{B} = (\vec{\omega} \times \vec{r}) \times \vec{B},$$

(13)

where $\vec{v}_{rot}$ represents the velocity of a particular point on the rotator, $\vec{\omega}$ is the angular velocity of the rotator and $\vec{r}$ is the position vector from the center of the gyroscope.

3. The magnetic field will give a force to the current, and this force acts on the gyroscope directly to make it slow down, but does not affect current distribution.

In conclusion, we can use a much more concise Poisson’s Equation to calculate the current in the gyroscope:

$$\nabla^2 \phi = \nabla \cdot [(\vec{\omega} \times \vec{r}) \times \vec{B}],$$

(14)

and then calculate the deceleration process by evaluating the magnetic torque on it.

It will be shown in the next section that in our problem, the current has rotational form and can be hence called ‘eddy current’. Additionally, it is worth pointing out that such ‘eddy current’ is a common phenomenon when non-electrostatic force is originally produced by magnetic field (both magnetic force and the force given by induced electric field). Such ‘eddy current’ is harmful in many cases since it cause extra loss of energy.

III. SET UP OF THE MODEL

We are now finding a simplest design of the model in order to solve the problem analytically. Here we only take the component of the field that is vertical to the plane of the rotator in to consideration, since the in-plane magnetic field, whose role will be discussed in the last part of the article, has a much smaller contribution to eddy current due to the fact that the radius of the rotator is much larger than its thickness.

Qualitative analysis shows that there will be no stable current in the gyroscope placed in an even magnetic field, because the non-electrostatic force distribution has a rotational symmetry (the free electrons will immediately form an electric field to cancel non-electrostatic
force precisely). Therefore, a magnetic field with anti-symmetrical distribution is the simplest form for us to both set up in experiments and analyze in theory.

(a) The schematic diagram of our model

(b) Vertical component of the field

FIG. 1. Picture (a) gives a schematic view of our model. It shows the relative position between the magnets (red-blue blocks) and the rotator of our gyroscope (yellow cylinder). Red and blue parts represent the north and south poles of the magnets respectively. Picture (b) is a plot of the vertical component (y component) of the field at the height of the rotator. We find that the magnetic field is antisymmetric about the x-y plane. (We will discuss the horizontal component in the last section, since it contributes to our rest 8.61% error.)

To produce such an anti-symmetrical field, we take two identical cuboid magnets and put them next to each other in opposite directions, and they attract each other and stick together. Such a configuration forms an anti-symmetrical magnetic field above it, in the vertical direction. (The picture of our model and the vertical component of field distribution is shown in FIG.1). The gyroscope is placed on the magnets with the brass rotator parallel to the surface of them and its center on the middle of the line between the two. The apparatus and experiment process are shown in FIG.2 and its illustration.

IV. DISTRIBUTION OF MAGNETIC FIELD

Before we deal with the problem of the gyroscope, it is beneficial to focus on the magnetic field in advance. In our model, the magnetic field is merely an external parameter that can be measured point by point via experiment. However, the work of measurement can be rather tough. The difficulty we faced here is that magnetic field is a vector field, hence it has to be ensured that the surface of the Hall sensor on the teslameter is precisely parallel to the surface of the magnets, in order to get the correct result of the vertical component.
FIG. 2. Here shows how the experiments were done. Above two strongly stuck magnets with a shape of square, a gyroscope made of brass is laid horizontally, and is fixed by a small pick of clay. The gyroscope is placed right on the magnets with the brass rotator parallel to the surface of them. The blue rectangular device was used for starting the gyroscope, and four basically identical pieces of adhesive tape are stuck as a crisscross on the top of the gyroscope. Then, the angular velocity of the rotator in the process of deceleration is able to be measured with a 20.00 Hz strobe light. If a static figure of crisscross is seen (one can see the crisscross rotates in one direction, stops for a instance, and then rotates backwards), the frequency of the rotator accords with the strobe light at the moment, that is, the revolution of the rotator is an integral multiple of 5.000 r/sec (a quarter of 20.00 Hz).

Notice: 20.00 Hz was chosen as an appropriate frequency because higher frequency causes difficulties in video recording, while a lower one leads to missing the exact turning point of the rotation of the figure.

Detailed information of our apparatus: 1. Magnets: 10.00cm * 10.00cm * 2.00cm; 2. Rotator: made of brass (type: H59), 5.30cm in diameter; 3. Distance between top surface of the magnets and the rotator: $h = 3.10\, cm$.

Unfortunately, due to the small size of the Hall sensor chip, it is nearly impossible to ensure that it is horizontal within the accuracy we need

To avoid such a problem, we decide to use a theoretical formula of the magnetic field
produced by magnet blocks, which can provide us a credible result of the distribution of magnetic field with only a few external parameters that are not hard to obtain. Schlueter and Marks have derived the distribution of magnetic field of a rectangular permanent magnet block using the uniform magnetic charge sheet model. For the specific problem we have in mind, we only consider the case that magnification is along the y axis. As a result, the charge sheets exist only on the surfaces that are parallel to the x-z plane, i.e., the normal of the charge sheets are of the same amplitude but opposite signs. For a single charge sheet, whose corners are \((x_1, y_1, z_1), (x_1, y_1, z_2), (x_2, y_1, z_1)\) and \((x_2, y_1, z_2)\), the components of magnetic field are:

\[
B_{1x} = \frac{B_r}{4\pi} \ln \left[\frac{z_2 - z + r(x_2, y_1, z_2)}{z_1 - z + r(x_1, y_1, z_1)}\right], \quad (15)
\]

\[
B_{1y} = \frac{B_r}{4\pi} \left\{ \tan^{-1}\left[\frac{(z_2 - z)(x_1 - x)}{(y_1 - y)r(x_1, y_1, z_2)}\right] + \tan^{-1}\left[\frac{(z_1 - z)(x_2 - x)}{(y_1 - y)r(x_2, y_1, z_1)}\right] - \tan^{-1}\left[\frac{(z_2 - z)(x_1 - x)}{(y_1 - y)r(x_1, y_1, z_2)}\right] - \tan^{-1}\left[\frac{(z_1 - z)(x_2 - x)}{(y_1 - y)r(x_2, y_1, z_1)}\right]\right\}, \quad (16)
\]

\[
B_{1z} = \frac{B_r}{4\pi} \ln \left[\frac{x_2 - x + r(x_2, y_1, z_2)}{x_1 - x + r(x_1, y_1, z_1)}\right], \quad (17)
\]

where,

\[
r(x_i, y_j, z_k) = \sqrt{(x_i - x)^2 + (y_j - y)^2 + (z_k - z)^2}, \quad (18)
\]

and quantity \(B_r\) is the remanence of the magnet, which can be treated as a fitting parameter in practical works. For our magnets, it is about 600 mT, which leads a field whose vertical component about 110 mT right on the top surface. (We use the word 'about' here because this parameter is checked before every time before an experiment is taken.)

For the other single charge sheet of the block, whose corners are \((x_1, y_2, z_1), (x_1, y_2, z_2), (x_2, y_2, z_1)\) and \((x_2, y_2, z_2)\), the field components are similar to the previous ones:

\[
B_{2x} = -\frac{B_r}{4\pi} \ln \left[\frac{z_2 - z + r(x_2, y_2, z_2)}{z_1 - z + r(x_1, y_2, z_1)}\right], \quad (19)
\]

\[
B_{2y} = -\frac{B_r}{4\pi} \left\{ \tan^{-1}\left[\frac{(z_2 - z)(x_1 - x)}{(y_2 - y)r(x_1, y_2, z_2)}\right] + \tan^{-1}\left[\frac{(z_1 - z)(x_2 - x)}{(y_2 - y)r(x_2, y_2, z_1)}\right] - \tan^{-1}\left[\frac{(z_2 - z)(x_1 - x)}{(y_2 - y)r(x_1, y_2, z_2)}\right] - \tan^{-1}\left[\frac{(z_1 - z)(x_2 - x)}{(y_2 - y)r(x_2, y_2, z_1)}\right]\right\}, \quad (20)
\]

\[
B_{2z} = -\frac{B_r}{4\pi} \ln \left[\frac{x_2 - x + r(x_2, y_2, z_2)}{x_1 - x + r(x_1, y_2, z_1)}\right], \quad (21)
\]

The total field is:

\[
\vec{B} = \vec{B}_1 + \vec{B}_2. \quad (22)
\]
The above result of magnetic field is obtained assuming $\mu_r = 1$, where $\mu_r$ is the relative permeability. In reality, $\mu_r\parallel = 1.04 - 1.08$ and $\mu_r\perp = 1.02 - 1.08$. This is the main source of error of this model. Yet our method of calibrating $B_r$ with measurement removes the error to a large extent, replacing it with measurement error. Of course the anisotropy of the real magnet is still unaccounted for, but it is in general smaller.

Now we receive an analytical formula of the magnetic field distribution, but it is still hard to apply to further calculation of eddy currents, since such a complicated formula will appear as a nonhomogeneous term in Equation (14). Consequently, it will be beneficial to derive a simpler formula of the magnetic field in the gyroscope region in place of the complicated one. By enter the external parameters of our magnets into the precise formula of magnetic field, we realize that in the region of gyroscope, the vertical component of the field is solely dependent on coordinate $z$, as shown in FIG.3.

![FIG. 3. The vertical component of the field is nearly solely dependent on coordinate z in the region of gyroscope.](image)

Therefore, it is reasonable to describe the magnetic field (vertical component) by a one-parameter function. A conventional way to find a simpler local formula of a function is the method of approximation, which is really much similar to polynomial curve fitting in experiments. To be specific, we use the following function as an approximation of the field
distribution:
\[ B_y(x, y = h, z) = B_z = k_1(z - z_0) + k_3(z - z_0)^3. \] (23)

where \( k_1 \) and \( k_3 \) are undetermined coefficients, and \( z_0 \), which is zero in our model, is the position of the boundary of the two magnets.

Then, with the help of Wolfram Mathematica, the values of \( k_1 \) and \( k_3 \) are obtained. The result shown in FIG.4 indicates that in the region of the rotator, the analytical and approximate formulas correspond to each other very well.

![FIG. 4. The blue line is the result of the original formula, while the orange one is that of Equation(23) with the correct values of \( k_1 = -45.67 mT \cdot cm^{-1} \) & \( k_3 = 2.785 mT \cdot cm^{-3} \). It can be seen that they fit each other very well.](image)

V. CALCULATION OF THE CURRENT

Based on the preparatory work, we can transform our ‘gyroscope teslameter’ problem into a boundary value problem of Poisson equation:

\[
\begin{align*}
\nabla^2 \varphi &= \nabla \cdot [(\vec{\omega} \times \vec{r}) \times \vec{B}], \\
\left. j_n \right|_{\text{boundary}} &= \sigma (\vec{E} + \vec{f}) \cdot \hat{n} \mid_{\text{boundary}} = 0.
\end{align*}
\] (24)

Since the rotator of our gyroscope is not simply a cylinder, but a combination of two rings and a circular plate (see FIG.5), we decide to solve the Poisson equation in the rings
and the plate separately, and then combine them together. As a result, the problem in each
region reduces into 2-dimension (since they are translationally-invariant along the y axis)
and is much easier to deal with. Therefore, the equation we are about to solve is:

\[
\nabla^2 \varphi = \tilde{\nabla} \cdot [\omega r (k_1 r \sin \theta + k_3 r^3 \sin^3 \theta) \hat{r}]
\]
\[
= \omega [3k_1 r \sin \theta + \frac{5}{4}k_3 r^3 (3 \sin \theta - \sin 3 \theta)],
\]
\[
\frac{\partial \varphi}{\partial r} |_{r=a} - \omega a (k_1 a \sin \theta + k_3 a^3 \sin^3 \theta) = 0,
\]
\[
\frac{\partial \varphi}{\partial r} |_{r=b} - \omega b (k_1 b \sin \theta + k_3 b^3 \sin^3 \theta) = 0,
\]

where \(a\) & \(b\) are inside and outside diameter of the ring. For the plane region, we just take
\(a = 0\). Here we make use of cylindrical coordinates (or can be called polar coordinates since
the problem is 2D) them. The relations between them and the original cartesian coordinates
are:

\[
\begin{align*}
x &= r \cos \theta, \\
y &= y, \\
z &= r \sin \theta.
\end{align*}
\]

FIG. 5. This is a close-up view of the gyroscope. It can be seen that the rotator is a combination
of two rings and a plate (there is only one ring can be seen on the picture, and the other one is on
the bottom side).

To solve the Equation (25), we use the superposition principle to split this Poisson equation
with the sophisticated boundary conditions into two equations. One of these equations
is the same Poisson equation with all boundary conditions degenerate to zero:

$$\begin{align*}
\nabla^2 \varphi_1 &= \omega[3k_1 r \sin \theta + \frac{5}{3} k_3 r^3(3 \sin \theta - \sin 3\theta)], \\
\frac{\partial \varphi_1}{\partial r} \bigg|_{r=a} &= 0, \\
\frac{\partial \varphi_1}{\partial r} \bigg|_{r=b} &= 0.
\end{align*}$$

(27)

Another equation then become a Laplace equation with the same boundary conditions of the initial equation (25):

$$\begin{align*}
\nabla^2 \varphi_2 &= 0, \\
\frac{\partial \varphi_2}{\partial r} \bigg|_{r=a} &= \omega a(k_1 a \sin \theta + k_3 a^3 \sin^3 \theta), \\
\frac{\partial \varphi_2}{\partial r} \bigg|_{r=b} &= \omega b(k_1 b \sin \theta + k_3 b^3 \sin^3 \theta).
\end{align*}$$

(28)

The solution to these two equations \( \varphi_1 \) and \( \varphi_2 \) can be added to get the solution to the initial equation \( \varphi \):

$$\varphi = \varphi_1 + \varphi_2$$

(29)

Finally, we obtain an analytical result of the scalar potential distribution. Furthermore, we can get the electric field by evaluating \( \vec{E} = -\nabla \varphi \), and plug \( \vec{E} \) into Equation (5) to get current distribution. Since the final formula is rather long, we list the current induced by the first and second term of the magnetic field (see Equation 23) separately, naming them \( \vec{j}_1 \& \vec{j}_3 \) respectively:

\[
\vec{j}_1(r, \theta) = \sigma \omega k_1 \left[ r^2 \sin \theta + \frac{1}{8} (a^2 + b^2 - \frac{a^2 b^2}{r^2} - 9r^2) \sin \theta \right] \hat{r} \\
+ \sigma \omega k_1 \frac{1}{8} (a^2 + b^2 - \frac{a^2 b^2}{r^2} - 3r^2) \cos \theta \hat{\theta},
\]

(30)

\[
\vec{j}_3(r, \theta) = \sigma \omega k_3 \frac{(a^2 - r^2)(r^2 - b^2)}{64(a^4 + a^2 b^2 + b^4)r^4} \left[ 2(a^4 + a^2 b^2 + b^4)(a^2 + b^2 + r^2)r^2 \sin \theta \\
- 9(r^4(a^4 + a^2 b^2 + b^4) + r^2 a^2 b^2(a^2 + b^2) + a^4 b^4) \sin 3\theta \right] \hat{r} \\
+ \frac{\sigma \omega k_3}{64r^4} \left[ 2r^2(a^2 b^2(a^2 + b^2) + r^2(a^4 + a^2 b^2 + b^4) - 5r^6) \cos \theta \\
- 9r^6(a^6 + a^4 b^2 + a^2 b^4 + b^6) - 15r^8(a^4 + a^2 b^2 + b^4) + 9a^6 b^6 \right] \cos 3\theta \hat{\theta}.
\]

(31)

The above two formulas are the analytical representation of current distributions in the rotator. To give a more intuitive picture, we use a built-in function of Wolfram Mathematica to draw the following stream plots of the currents (see FIG.6, the formulas of boundary...
conditions are also listed there). Here we should note that these stream plots are slightly different from the conventional ones that illustrate electric and magnetic fields, as we usually see in fundamental electromagnetism textbooks. The density of lines here does not represent the intensity of current, while only the directions of arrows have practical meanings (the direction of current at each point).

![Stream plots](image)

**FIG. 6.** These are stream plots of the currents given by Wolfram Mathematica. Picture (a) represents eddy current in a circle plate (or a cylinder) induced by a 'linear magnet field', that is, the first term of Equation (23) (Equation(29)). Picture (b) represents eddy current in a ring induced by the same field. And Pictures (c) and (d) show those currents induced by a 'cubic magnetic field', that is, the second term of Equation (23) (Equation(30)). One interesting feature of these plots is the degree of symmetry. There are left-right symmetry and top-down anti-symmetry, which reflects the symmetry of the magnetic field.
VI. COMPARISON WITH EXPERIMENT

With the formulas of current distribution, Equation (29) & (30), we can easily calculate the magnetic torque $\vec{M}$ on the rotator of gyroscope via the following integral:

$$\vec{M} = \int_{V} \vec{j} \times \vec{B} \, dv = -\lambda \vec{\omega}, \quad (32)$$

where coefficient $\lambda$ is independent of angular velocity. That the torque is proportional to angular velocity is obvious since the current in the integral is proportional to it as well.

From the following law of mechanics:

$$I \frac{d\omega}{dt} = M = -\lambda \omega, \quad (33)$$

whose solution is

$$\omega(t) = \omega_0 e^{-t/\tau}, \quad (34)$$

where $\tau = I/\lambda$ is usually named 'characteristic decaying time'.

Entering the dimensions and the conductivity of our gyroscope, we get the result that $\tau = 7.839 \, s$.

On the other hand, the measurement of deceleration of the gyroscope above magnets is done with the help of a strobe light (see FIG.2 for details). However, we realize that the experimental result is not an exponential decay, but with a minus constant added, as shown below:

$$\omega(t) = \omega_0 e^{-t/\tau_{\text{exp}}} - \omega_{\text{rest}}, \quad (35)$$

where the subscript 'exp' (refers to 'experiment') is to distinguish from the theoretical one.

Such a result is reasonable since an ideal exponential decay indicates that the gyroscope will not stop spinning until the time goes to infinity, which is unrealistic. After examining Equation (32) carefully, we found that if a constant frictional force is added to the right hand side, a decelerating curve of the same form with the experimental one is obtained.

Consequently, we turn to investigate the frictional force here. Through experiments, we measure the deceleration curve of the gyroscope without the magnets, which means the only torque on the rotator is caused by frictional force. Such a deceleration curve gives directly the form of the torque generated by the frictional force, which is a constant term with another term proportional to the angular velocity:

$$M_f = -M_0 - \alpha \omega. \quad (36)$$
However, this is not the end of the story. An intrinsic shortcoming of our gyroscope is that its shaft is made of steel, which is magnetic. The attraction between the shaft and the magnets exerts additional pressure between the rotator and the frame which enlarge frictional force. Therefore, the frictional torque we measured away from the magnets is not identical to that in the presence of the magnets. Fortunately, the empirical formula of frictional force: $F_f = \mu F_N$ indicates that it is proportional to pressure, hence the torque we are looking for is likely to be a multiple of the one we measured in absence of the magnets. In our model, we treat this multiplier as a fitting parameter. By choosing it properly, we are able to make the constant term in the theoretical deceleration curve the same as that of the experimental curve. In the end, we achieve good agreement between the theory and the experiment (see FIG.7), the error in the characteristic decaying time is about 8.61%.

FIG. 7. A comparison between the theory and the experiment. In our experiments, the initial angular velocity is $\omega_0 = 377.0 rad/s$, which is 60.00 r/s, the value of decaying time is $\tau_{\text{exp}} = 5.420s$, and that of the constant term is $\omega_{\text{rest}} = 32.57 rad/s$. Our calculation of the interaction between magnetic field and eddy current, together with the calibration of the frictional torque, gives that with the same initial conditions, the decaying time is $\tau = 5.887s$. The error in the characteristic decaying time is about 8.61%.

Notice: we do not draw error bars on the points since the error caused by measurement is too small to be shown on the plot.
VII. REASONABILITY OF THE ASSUMPTIONS

In this section, we give a brief review of the assumptions we introduced in Section II and discuss a possible cause of the sources of error.

The first assumption we introduced is that we treat the current (as well as electric field) as nearly time-independent. As all the fields, charge and current distribution varying slowly as time goes on, quantity $\partial \rho / \partial t$ will be small enough to be omitted, hence the divergence of the current is zero. To be more specific, the condition of nearly time-independent that can be expressed as the partial derivative of electric field (or current) is much less than that of electric field itself (of course with some constants to scale them to the same dimension):

$$\varepsilon \frac{\partial \vec{E}}{\partial t} \ll \sigma \vec{E}.$$  \hfill (37)

As for our brass rotator, $\frac{\varepsilon}{\sigma} = 1.613 \times 10^{18} \text{s}^{-1}$, which has a huge dimension so that the electric field and the current can be treated as time-independent.

Our second assumption is straight forward, which is what people usually take when calculating dynamic electromotive force.

Regarding the third assumption, which is that the magnetic field gives a force to the current, and this force acts on the gyroscope directly to make it slow down without affect current distribution. It is justified by the fact that under normal circumstance, where electrons move at unrelativistic velocity electric force on a moving charge will be much more significant than magnetic force, hence the distribution of the current will be determined primarily by the electric field. Being a higher order pertubation, magnetic force will drag the moving electrons aside slightly, but such a transversal movement will be prevented by rapid collision between electrons and lattice. As a result, the magnetic force is transfer to lattice of the conductor.

Finally, we would like to discuss a possible source of the 8.61 % error in $\tau$. Besides all the approximations we made in process of calculation, we think the most significant origin of the error is that the magnetic field is not actually vertical, that is, the horizontal component of the field has some influences on the rotation as well. In fact, there is a strong horizontal component of the same dimension with the vertical one, which may also induce a vertical component of current and then another torque. Due to the fact that the thickness of the gyroscope is small relative to its diameter, the energy loss is small compared to the main
effect. However, such a current is not easy to calculate, so only a limit of error is here given by coarse estimation, which is about 10% in torque and also in characteristic decaying time.

VIII. CONCLUSION

The piece of work introduced here provides a method which enables people to obtain an accurate solution of eddy current in a block of conductor and is verified to be consistent with measurement through comparing the deceleration process of the gyroscope. Moreover, it can be concluded that, such a method of calculating eddy current can be applied in more complex situations, thus providing a tool to solve a wider class of problems about electromagnetic damping. Although, for many cases, it may not be easy to find out the analytical expression as in the problem of a gyroscope, a numerical solution can be obtained.

The authors hope the work can assist students around the world to learn the eddy currents and give them a deeper understanding of Maxwell equations and how to apply them in real cases. We intentionally omit many details of solving the Poisson’s equation since we do not want the main logic of physics to be obstructed by mathematical details.

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