Variational cohomology and topological solitons in Yang–Mills–Chern–Simons theories

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Abstract
In cohomological formulations of the calculus of variations obstructions to the existence of (global) solutions of the Euler–Lagrange equations can arise in principle. It seems, however, quite common to assume that such obstructions always vanish, at least in the cases of interest in theoretical physics. This is not so: for Yang–Mills–Chern–Simons theories in odd dimensions > 5 we find a non trivial obstruction which leads to a quite strong non existence theorem for topological solitons/instantons. Applied to holographic QCD this reveals then a possible mathematical inconsistency. For solitons in the important Sakai–Sugimoto model this inconsistency takes the form that their $u_1$-component cannot decay sufficiently fast to “extend to infinity” like the $su_n$-component.

Key words: cohomological formulation of the calculus of variations, Chern–Weil theory, Yang–Mills–Chern–Simons theories, topological solitons
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1 Introduction
By “variational cohomology” we refer, somewhat loosely, to any formulation of the calculus of variations in terms of a (co-)chain complex or double complex derived from the de Rham complex of a fibre bundle $Y \xrightarrow{\pi} X$ such that
the objects and operations of the calculus of variations (Lagrangians, equations, conservation laws and conserved quantities, taking the variation, etc.) are expressed in terms of the elements and the (co-)boundary operator and the complex still calculates de Rham cohomology. Variational cohomology is part of the general approach to mathematical/theoretical physics for which the expression “Cohomological Physics” has been coined (see [58]).

Since de Rham cohomology detects obstructions to glue local solutions of differential equations together to global solutions, it is natural to ask if in the cohomological formulations of the calculus of variations de Rham cohomology classes appear as cohomological obstructions to the existence of solutions of Euler–Langrange equations. Regarding this, the common conviction seems to be that for variational problems relevant in theoretical physics such obstructions vanish always. This is not the case. In [50] (see also section 2.4 below) we introduced a family of cohomological obstructions $[\Xi, \eta_\lambda]$ and showed that the well known fact that 3–dimensional Chern–Simons theories admit solutions only on flat bundles can be expressed by means of these.

Chern–Simons theories are Lagrangian gauge field theories for principal connections derived from the Chern–Simons secondary characteristic class corresponding to some Chern–Weil characteristic class. They are local variational problems, i.e. they consist of a collection of local Lagrangians, e.g. one for each coordinate patch, which, nevertheless, have unique global Euler–Langrange equations; in other words, they are related like a closed form and a set of local potentials (see e.g. [23], see also sections 2.2 and 2.3.1 below).

Here, we continue the investigation of non trivial obstructions arising in variational cohomology for Chern–Simons theories in higher dimensions. More specifically, we analyze Lagrangian gauge field theories derived from the $p + 1$–component of the Chern character for principal connections on a $U(n)$–principal bundle over a $2p + 1$-dimensional manifold, $n \geq p$ (for the case of theories derived from the $p + 1$th Chern class see [51]). We find a non trivial obstruction which vanishes if and only if the $p$–component of the Chern character vanishes (as a cohomology class).

This obstruction has its natural field of application in Yang–Mill–Chern–Simons theories. YMCS theories are Lagrangian gauge field theories the Lagrangian of which is a sum of the YM– and CS–Lagrangians; this construction seems to be mainly motivated by the rôle of the Chern–Simons form in anomaly physics (see e.g. [1], [8] and [67]). We derive a a quite strong non existence theorem for solitons/instantons. This raises doubts about the mathematical consistency of their use in theories of nuclear matter in five di-
Mensional (holographic) QCD, (see e.g. [7], [9], [35], [36], [52] and [54]). More specifically, the obstruction cannot vanish for $SU(n)$–theories and in the case of $U(n)$–theories its vanishing imposes a qualitative difference between the asymptotic behaviour at infinity of the $SU(n)$–component of the fields on one hand and the $U(1)$–component or the metric on the other. In particular, in the case of the Sakai–Sugimoto model, widely considered the most important example of YMCS theory for holographic QCD, the $U(1)$–component cannot decay sufficiently fast to “extend to infinity” like the $SU(n)$–component. The case of a non extendable metric is more intricate, it might even be the key to construct a theory where all components of the fields extend to infinity.

The paper is organized as follows. In section 2 we set up the calculus of variations on fibre bundles, the variational sequence on finite order jet bundles, i.e. the particular cohomological formulation of the calculus of variations we will use, and introduce our obstructions; we also introduce the concept of local variational problem.

Since variational cohomology requires variational problems to be formulated in terms of sections of fibre bundles, in section 3 we indicate how principal connections can be viewed as sections of a bundle, the so called bundle of (principal) connections of a principal bundle, and discuss some of its main features.

In section 4 we set up the formulation of Chern–Simons theories on the bundle of connections via Chern–Weil theory and Chern–Simons secondary characteristic classes.

Section 5 deals with the $p + 1$–component of the Chern character for principal connections on a $U(n)$–principal bundle over a $2p + 1$-dimensional manifold, $n \geq p$. First, we identify a certain cohomological obstruction in Chern–Simons theories and then we analyze its effects in Yang–Mill–Chern–Simons theories. Here, the power of variational cohomology is on display: it extracts from the Yang–Mills Euler–Lagrange form a specific counterpart to the obstruction. For the existence of solutions it is then necessary that the counterpart and the corresponding obstruction annihilate each other. This leads to a rather strong non existence theorem for solitons/instantons in Yang–Mill–Chern–Simons theories. We conclude the section by analyzing the consequences of this non existence theorem for the Yang–Mill–Chern–Simons theories in five dimensional (holographic) QCD in general and the Sakai-Sugimoto model in particular.

This article is rather lengthy because we need to review a lot of material from mathematics and theoretical physics which, while hardly esoteric, is not
2 Variational Cohomology

We will now outline the construction and the characteristics of the variational sequence on finite order jet bundles, the particular variational cohomology theory which we will use.

The common starting point for the construction of variational cohomologies is the fact that the pullback of $T^*(J^kY)$ to $J^{k+1}Y$ has a canonical decomposition in horizontal and vertical subbundles ($J^kY$ is the $k$th-order jet bundle of a fibre bundle $Y \to X$, see below). This decomposition can be used to define the so-called contact forms, i.e., those forms for which the pullback with (the prolongation of) a section of $Y \to X$ vanishes. The core of any such construction consists then in eliminating contact forms in the right way; not all contact forms, since for $p > \dim X$, $\Omega^p(J^kY)$ consists, of course, entirely of contact forms for dimensional reasons.

The various approaches are divided in two groups according to whether they are constructed using finite order jet bundles or the infinite order jet bundle $J^\infty Y$, the projective limit of the $J^kY$. Using the infinite order jet bundle avoids flipping forth and back between jet bundles of different orders which characterizes the finite order approach, since $J^\infty Y$ is already equipped with a decomposition of the cotangent bundle in horizontal and vertical subbundles. Thus, the constructions based on this approach are rather more elegant. But this elegance comes at a price: one has to deal with the intricacies and ambiguities of smooth structures on infinite dimensional manifolds. Infinite order jet bundles can be equipped only with the structure of a Frechet manifold for which there is not even a unique concept of smooth function, see e.g. [57] chapter 7. The standard approach of working with finite order representatives for the equivalence class in the projective limit is therefore fraught with problems. It is not at all clear which manipulations carried out for these representatives will still be valid for the classes which they represent in the limit. Because of the above mentioned ambiguity of the definition of smooth function, already partition of unity arguments or arguments based on the continuity of smooth functions cannot by considered valid by default and have instead to be checked with utmost care (for more information about the infinite order approach see [2] and the literature therein).

For this reason we prefer the finite order approach. In particular, we will
use Krupka’s variational sequence on finite order jet bundles following [40] (see [39] for a slightly different approach). Here, the chain complex comes equipped by construction with an explicit representation as a complex of differential forms on some $J^kY$. It is, in particular, quite comfortable to work with when dealing with the first or second order field theories prevalently used in theoretical physics.

The origins of variational cohomology may be traced back at least to the work of Helmholtz on what is now called the Helmholtz condition. However, in view of our predilection for working in finite dimensions, we wish to acknowledge the fundamental importance of T. Lepage’s work (see [41]) with respect to the idea of formulating the calculus of variations in terms of actual differential forms and exterior differentials.

For those interested in the history and the ramifications of the cohomological formulations of the calculus of variations the best starting point is again [2] and the literature therein.

In this section we will proceed as follows: first we have a technical section about jet bundles and the particularities of their differential structure relevant to us; next we present the basics of the calculus of variations on fibre bundles; then we turn to the variational sequence proper. We will conclude with the cohomological obstructions introduced in [50].

2.1 Jet bundles

For a fibre bundle $Y \xrightarrow{\pi} X$ the $k$th order jet manifold is defined as the set $J^kY := \cup_{x \in X} J^k\sigma$. $J^k\sigma$ are the equivalence classes of germs of sections $\sigma : X \to Y$ of $\pi$ at $x \in X$, where two germs are equivalent if and only if their Taylor expansions at $x$ coincide up to the $k$-th order (note that $J^0Y = Y$).

$J^kY$ has a natural manifold structure (see [57]). The derivatives up to order $k$ of $\sigma$ at $x$ define coordinate charts:

$$(x^\mu, y^i, y^i_{\alpha}, \ldots, y^i_{\alpha_1 \cdots \alpha_k}) := \left(x^\mu, \sigma^i(x), \frac{\partial \sigma^i(x)}{\partial x_\alpha}, \ldots, \frac{\partial^k \sigma^i(x)}{\partial x_{\alpha_1 \cdots \alpha_k}}\right)$$

where $(x^\mu, y^i)$ are fibered coordinates on $Y$ and it is to be understood that the indices are ordered: $\alpha_1 \leq \cdots \leq \alpha_k$. While in a coordinate expression like $y^i_{\alpha_1 \cdots \alpha_k \mu}$ it is to be understood that the index $\mu$ is placed on its proper place in the ordering of the indices.

$J^kY$, the $k$th order jet bundle, is a fibre bundle over $Y$ and we denote its projection by $\pi^k : J^kY \to Y$. Furthermore, we have the following
topological (but not linear) isomorphism

\[ J^kY = \pi^* \left( \sum_{i=1}^{k} S^i (T^*X) \right) \otimes V(Y) \]

Here, \( S^i (T^*X) \) is the \( i \)th symmetric power of \( T^*X \) and \( V(Y) \) is the vertical bundle with respect to \( \pi \).

The natural projections \( J^kY \xrightarrow{\pi^k} J^{k-1}Y \) are affine bundles. For every section \( \sigma : X \to Y \), local or global, there is by construction a section \( j^k \sigma : X \to J^kY \), the \( k \)th order prolongation of \( \sigma \):

\[ j^k \sigma := \left( x^\mu, \sigma^i(x), \frac{\partial \sigma^i(x)}{\partial x_\alpha}, \ldots, \frac{\partial^k \sigma^i(x)}{\partial x_{\alpha_1} \ldots \partial x_{\alpha_k}} \right) \]

(2)

Note that \( Y \) is a retract of \( J^kY \) and, thus, their cohomologies are isomorphic; furthermore, the images in cohomology of \( \pi^k \) and any section of \( J^kY \to Y \) introduce a canonical isomorphism.

### 2.1.1 The contact structure

The differential at some point \( x \in X \) of the \( k \)th order prolongation of some section \( \sigma \)

\[ d(j^r \sigma(x)) : T_x X \to ((\pi^{r+1})^* T(J^r Y)_{j^r \sigma(x)}) \]

has the coordinate expression

\[ d(j^r \sigma(x))(\xi) = \sum_{\mu} \xi^\mu(x) \left( \frac{\partial}{\partial x^\mu} + \sum_{k=0}^{r} \sum_{\alpha_1 \leq \ldots \leq \alpha_k} \frac{\partial^{k+1} \sigma^i(x)}{\partial x_{\alpha_1} \ldots \partial x_{\alpha_k}} \frac{\partial}{\partial y_{\alpha_1 \ldots \alpha_k}} \right) \]

Comparing this to the expressions [1] and [2] above, we see that the image of \( T_x X \) can be naturally seen as a subspace of the pullback of the tangent bundle of \( J^r Y \) to \( J^{r+1} Y \) at \( j^{r+1} \sigma(x) \). Collecting all these subspaces into a subbundle we obtain the canonical horizontal complement to the pullback of the \( \pi^r \)-vertical subbundle

\[ (\pi_r^{r+1})^* T(J^r Y) = (\pi_r^{r+1})^* V(J^r Y) \oplus H(\pi_r^{r+1}) \]

(3)

We have the canonical projection \( h : T(J^r Y) \to H(\pi_r^{r+1}) \), the horizontalization. For a vector field on \( J^r Y \) with the local coordinate expression

\[ \xi = \sum_{\mu} \xi^\mu \cdot \frac{\partial}{\partial x^\mu} + \sum_{k=0}^{r} \sum_{\alpha_1 \leq \ldots \leq \alpha_k} \xi_{\alpha_1 \ldots \alpha_k} \cdot \frac{\partial}{\partial y_{\alpha_1 \ldots \alpha_k}} \]
where $\xi^\mu$ and $\Xi^{i\alpha_1...\alpha_k}$ are functions on the respective coordinate patch of $J^rY$, it has the local coordinate expression

$$h(\xi) = \sum_\mu \xi^\mu \cdot \left( \frac{\partial}{\partial x^\mu} + \sum_{k=0}^r \sum_{\alpha_1\leq...\leq\alpha_k} y_{\alpha_1...\alpha_k}^i \cdot \frac{\partial}{\partial y_{\alpha_1...\alpha_k}^i} \right)$$

Likewise we get the complementary canonical projection $p_1 : T(J^rY) \mapsto (\pi_{r+1}^r)^*V(J^rY)$ with the local coordinate expression

$$p_1(\xi) = \sum_{k=0}^r \sum_{\alpha_1\leq...\leq\alpha_k} (\Xi^{i\alpha_1...\alpha_k} - \sum_\mu y_{\alpha_1...\alpha_k}^i : \xi^\mu(x)) \cdot \frac{\partial}{\partial y_{\alpha_1...\alpha_k}^i}$$

For differential 1-forms $\gamma$ the horizontalization $h$ is defined by

$$h(\gamma)(\pi_{r+1}^r \circ \xi) = (\pi_{r+1}^r)^*(\gamma)(h(\xi))$$

We have

$$h(dx^\mu) = dx^\mu \quad h(dy_{\alpha_1...\alpha_k}^i) = \sum_\mu y_{\alpha_1...\alpha_k}^i dx^\mu$$

The $\theta$ for which $h(\theta) = 0$ are called contact 1-forms of order $r$. They are characterized by the fact that their pullback with the $r$th-order prolongation of a section $j^r\sigma : X \rightarrow J^rY$ vanishes, i.e. being contact does not depend on the choice of a coordinate chart. The complementary projection $p_1$ of the horizontalization is then defined by

$$p_1(\gamma)(\pi_{r+1}^r \circ \xi) = (\pi_{r+1}^r)^*(\gamma)(p_1(\xi))$$

Locally, we have

$$p_1(dx^\mu) = 0 \quad p_1(dy_{\alpha_1...\alpha_k}^i) = dy_{\alpha_1...\alpha_k}^i - \sum_\mu y_{\alpha_1...\alpha_k}^i : dx^\mu =: \theta_{\alpha_1...\alpha_k}^i$$

with $0 \leq k \leq r$. The case $k = 0$ is understood to be

$$p_1(dy^i) = dy^i - \sum_\mu y_{\alpha_1...\alpha_k}^i : dx^\mu =: \theta^i$$

Let $1 \leq \mu \leq \text{dim}X$, $0 \leq k \leq r - 1$ and $1 \leq i \leq \text{dim}Y - \text{dim}X$, the collection

$$dx^\mu, \theta_{\alpha_1...\alpha_k}^i, dy_{\alpha_1...\alpha_r}^i$$

is a local basis of $\Omega^1(J^rY)$. Only in $(\pi_{r+1}^r)^*\Omega^1(J^rY)$ we have

$$(\pi_{r+1}^r)^*dy_{\alpha_1...\alpha_r}^i = \sum_\mu y_{\alpha_1...\alpha_r}^i : dx^\mu + \theta_{\alpha_1...\alpha_r}^i$$
Therefore, \((\pi^{r+1}_r)^*\Omega^1(J^r Y)\) has a basis
\[ dx^\mu, \theta^i_{\alpha_1...\alpha_k} \]
where instead \(0 \leq k \leq r\). For differential \(q\)-forms \(\gamma\) in \(\Omega^k(J^r Y)\) we have then
\[ (\pi^{r+1}_r)^*(\gamma)(\pi^{r+1}_r \circ \xi_1, \ldots, \pi^{r+1}_r \circ \xi_q) = (\pi^{r+1}_r)^*(\gamma)(h(\xi_1) + p_1(\xi_1), \ldots, h(\xi_k) + p_1(\xi_q)) \]
Therefore, there is a canonical decomposition
\[ (\pi^{r+1}_r)^*(\gamma) = \gamma_0 + \cdots + \gamma_q \]
\(\gamma_0 = h(\gamma)\) and for \(1 \leq k \leq q\)
\[ \gamma_k = \sum_{i_1 \leq \cdots \leq i_k, i_{k+1} \leq \cdots \leq i_q} |I| \cdot (\pi^{r+1}_r)^*(\gamma)(p_1(\xi_{i_1}), \ldots, p_1(\xi_{i_k}), h(\xi_{k+1}), \ldots, h(\xi_q)) \]
here \(|I|\) is the sign of the permutation between \((1, \ldots, q)\) and \(i_1, \ldots, i_k, i_{k+1}, \ldots, i_q\).
\(\gamma_k\) is called the \(k\)-contact component. Locally it is a linear combination of forms of type
\[ \delta \land \theta^i_{\alpha_1...\alpha_j} \land \cdots \land d\theta^i_{\alpha_1...\alpha_{r-1}} \]
where \(\delta\) is a degree \(q - k\) horizontal form and the exterior product of the \(\theta^i_{\alpha_1...\alpha_k}\) and the \(d\theta^i_{\alpha_1...\alpha_{r-1}}\) is of degree \(k\)
\[ p_k(\gamma) := \gamma_k \]
is the projection of \(\gamma\) on its \(k\)-contact component. The fact that for \(q \geq n + 1\) all \(q\)-forms are contact is reflected and refined by the following property of the canonical decomposition: let \(q = n + k\) and \(k \geq 1\), we have then
\[ \gamma_0 = \cdots = \gamma_{k-1} = 0 \quad (\pi^{r+1}_r)^*(\gamma) = \gamma_k + \cdots + \gamma_q \]
A differential form is \(k\)-contact if and only if \(\gamma_0 = \cdots = \gamma_{k-1} = 0\) and \(\gamma_k \neq 0\). The horizontalization allows also to define the ”prolongation” of the partial derivatives of \(X\), the total or formal derivative \(d_\mu\) by
\[ h(df) = \sum_\mu d_\mu f \cdot dx^\mu \]
explicitly, we have
\[ d_\mu = h(\frac{\partial}{\partial x^\mu}) = \frac{\partial}{\partial x^\mu} + \sum_{k=0}^r \sum_{i, \alpha_1 \leq \cdots \leq \alpha_k} y^i_{\alpha_1...\alpha_k \mu} \frac{\partial}{\partial y^i_{\alpha_1...\alpha_k}} \]
2.2 Variational problems on fibre bundles

Before turning to the variational sequence we need to sketch out the basics of the variational calculus on fibre bundles. First of all, from now on we set \( \dim X = n \) and \( \dim Y = n + m \). A \( r \)-th order Lagrangian is a horizontal \( n \)-form \( \lambda \) on \( J^r Y \).

\[
\lambda = \mathcal{L}(x^\mu, y^i, y_{a_1}^i, \ldots, y_{a_r}^i) \cdot dx_1 \wedge \cdots \wedge dx_n
\]

in coordinates. The corresponding variational problem consists in finding the critical sections \( \sigma \) of \( Y \xrightarrow{\pi} X \), i.e. the sections for which the integral

\[
\int_X j^* \sigma^* \mathcal{L}
\]

over \( X \) itself or the integrals

\[
\int_B j^* \sigma^* \mathcal{L}
\]

over all \( n \)-dimensional compact submanifolds with boundary \( B \) of \( X \) take their extremal values, i.e. in the calculus of variations on fibre bundles the variation of the functional is induced by the variation of the section of the bundle. On \( J^{2r+1} Y \) we have then the \( n + 1 \)-form \( E_\lambda \), the Euler-Lagrange form, locally

\[
E_\lambda = \sum_i E(\mathcal{L})_i \theta^i \wedge dx_1 \wedge \cdots \wedge dx_n
\]

with

\[
E(\mathcal{L})_i = \frac{\partial \mathcal{L}}{\partial y^i} + \sum_{k=0}^{r+1} (-1)^k d_{a_1} \cdots d_{a_k} \frac{\partial \mathcal{L}}{\partial y_{a_1}^i \cdots a_k}
\]

(here \( \theta^i = dy^i - \sum_{\mu} y_{\mu}^i dx^\mu \), see above). The critical sections satisfy then the Euler-Lagrange equations:

\[
E_\lambda \circ j^{2r+1} \sigma = 0
\]

This is, of course, not the pullback \( j^{2r+1} \sigma^* (E_\lambda) \) of the differential \( n + 1 \)-form \( E_\lambda \), which would be zero for any section \( \sigma \) for dimensional reasons, but the restriction \( E_\lambda |_{j^{2r+1} \sigma} \). Analogously, one defines Helmholtz conditions, the first variation formula, etc. for the calculus of variations on fibre bundles. A local variational problem of order \( r \) on \( Y \) consists in an open covering \( \mathcal{U} \) of \( Y \) and a \( r \)-th-order Lagrangian \( \lambda_i \) on each \( U_i \in \mathcal{U} \) such that there is a globally well defined \( n + 1 \)-form \( \eta_\lambda \) on \( J^{2r+1} Y \) with

\[
\eta_\lambda |_{j^{2r+1} U_i} = E_\lambda_i
\]
2.3 The Variational Sequence

The observation that contact forms do not contribute to the action integral, see equations (5) and (6) above, is the starting point for the construction of any cohomological formulation of the calculus of variations. In the case of the variational sequence the construction consists in taking the quotient of the de Rham complex \((\Omega^*(J^{r-1}Y), d)\) of \(J^{r-1}Y\) by a certain cohomologically trivial subcomplex \((\Theta_{r-1}^*, d)\) of contact forms and finding a representation of the quotient \((\Omega^*(J^{r-1}Y)/\Theta_{r-1}^*, \widehat{d})\mapsto (\mathcal{V}_r^*, E_i)\) as a complex of differential forms \((\mathcal{V}_r^*, E_i)\) on higher order jet bundles of \(Y\). Both the quotient and its representation will still calculate the de Rham cohomology of \(J^{r-1}Y\) (canonically isomorphic to that of \(Y\)). Regarding the fundamental objects of the calculus of variations: the \(\lambda \in \mathcal{V}_r^n\) are the Lagrangians and \(E_n(\lambda) \in \mathcal{V}_r^{n+1} \subset \Omega^{n+1}(J^{2r-1}Y)\) are the Euler-Lagrange forms. We start from \(J^{r-1}Y\) instead of \(J^{r}Y\), because in this way the construction covers only the type of \(r\)th-order Lagrangians we are interested in (see the remarks at the end).

The construction is sheaf theoretic in nature (to prove the vanishing of the cohomology of \((\Theta_{r-1}^*, d)\) and to link coordinate expressions to global objects) and not very enlightening: many steps involve rather cumbersome calculations and are intelligible only with hindsight in as far as they make everything work.

We will only sketch the final result, i.e. the complex \((\mathcal{V}_r^*, E_i)\), as far as needed and present it in terms of the projections \(\Omega^k(J^{r-1}Y)\mapsto \mathcal{V}_r^*\), the interested reader will find the details of the construction in [40]. For \(0 \leq k \leq n\) we have

\[ \mathcal{V}_r^k := h(\Omega^k(J^{r-1}Y)) \]

where \(h\) is the horizontalization. The differentials for \(1 \leq k \leq n-1\) are then defined by

\[ E_k : \mathcal{V}_r^k \to \mathcal{V}_r^{k+1} \quad E_k(h(\gamma)) = h(d\gamma) \]

In addition we set

\[ \mathcal{V}_r^0 := C^\infty(J^{r-1}Y) \]

and

\[ E_0 : C^\infty(J^{r-1}Y) \to \mathcal{V}_r^1 \quad E_0(f) = h(df) \]
$E_k, \ 0 \leq k \leq n,$ is sometimes also denoted by $d_h$ to underscore that it is just the horizontalization of the exterior derivative. The $\lambda \in \mathcal{V}_r^n$ are the Lagrangians in the variational sequence.

For $k \geq n + 1$ all $k$-forms are contact. To construct $\mathcal{V}_r^k$ for $k \geq n + 1$ we need, thus, to distinguish the contact forms which are “essential” for the calculus of variations from those which are “inessential”.

This is done by means of a family of operators $I_k, k \geq 1$, called the $k$th internal Euler operator. $I_1(\gamma)$ is characterized by the fact that the terms of its local expressions contain only the contact factors $\theta^i$ of first jet order and if we have for a $n + 1$-form $\gamma$ locally the coordinate expression

$$\gamma = \sum_{i=1}^m \left( A^i \cdot \theta^i + \sum_{l=1}^r \Sigma_{\alpha_1, \ldots, \alpha_l} A^i_{\alpha_1, \ldots, \alpha_l} \cdot \theta^i_{\alpha_1, \ldots, \alpha_l} \wedge dx^1 \wedge \cdots \wedge dx^n \right)$$

for $\gamma$, then $I_1(\gamma)$ is defined by

$$I_1(\gamma) = \sum_{i=1}^m \left( A^i + \sum_{l=1}^r \Sigma_{\alpha_1, \ldots, \alpha_l} (-1)^l d_{\alpha_1} \ldots d_{\alpha_l} A^i_{\alpha_1, \ldots, \alpha_l} \right) \cdot \theta^i \wedge \omega^0 \quad (7)$$

here we have set $\omega^0 := dx^1 \wedge \cdots \wedge dx^n$ and $d_{\alpha_l}$ is the total derivative with respect to the coordinate $x^{\alpha_l}$, see equation (4) for its local expression.

$I_k(\gamma)$ is a $k$-contact $n + k$-form for which each term of any of its local expressions contains at least one contact factor $\theta^i$ of first jet order. All further information can be found in [40]. For our purposes only $I_1$ will be relevant. We set then for $k \geq 1$

$$\mathcal{V}_r^{n+k} := I_k \circ p_k \left( \Omega^k(J^{r-1}Y) \right) \quad \mathcal{V}_r^{n+1} \subset \Omega^{n+1}(J^{2r-1}Y)$$

Note that there is an integer $M := m \left( \frac{n+r-1}{n} \right) + 2n - 1$ such that for $n+k > M$ we have $\mathcal{V}_r^{M+l} = \Omega^{M+l}(J^{r-1}Y)$ for $l \geq 1$. For $\lambda \in \mathcal{V}_r^n$ and $\rho \in \Omega^n(J^{r-1}Y)$ with $\lambda = h(\rho)$ (note that no matter if we start with $\lambda$ or $\rho$ there is always a pedant to complete the pair), we set then

$$E_n(\lambda) := I_1(d\lambda) = I_1(dh(\rho)) = I_1 \circ p_1(d\rho) \quad E_n : \mathcal{V}_r^n \rightarrow \mathcal{V}_r^{n+1} \quad (8)$$

$E_n$ is called the Euler–Lagrange operator. Analogously, for $\gamma \in \Omega^{n+k}(J^{r-1}Y)$ with $1 \leq k \leq M - n - 1$, we set

$$E_{n+k}(I_k \circ p_k(\gamma)) := I_{n+k+1}(dp_k(\gamma)) \quad E_n : \mathcal{V}_r^{n+k} \rightarrow \mathcal{V}_r^{n+k+1}$$

For $\gamma \in \Omega^M(J^{2r-1}Y)$, we have $E_M(I_M \circ p_M(\gamma)) = d\gamma$ and, obviously, for $i > M$ we have simply $E_i = d$. In the variational sequence formulation of
the calculus of variations $E_n(\lambda)$ is the **Euler–Lagrange form** and $E_n(\lambda) \circ j^{2r-1}\sigma = 0$ are the **Euler–Lagrange equations**.

To see that this construction actually provides a cohomological formulation of the calculus of variations, one simply compares the relevant forms and expressions in $(\mathcal{V}^\ast_r, E_i)$ with the corresponding forms and expressions arising from variational problems on fibre bundles; in particular, on checks that $E_n(\lambda) = E_\lambda$.

Starting from $(\Omega^\ast(J^{r-1}Y), d)$ the $r$th-order Lagrangians $\lambda = h(\gamma) \in \mathcal{V}_r^n$ can be at most polynomial in the $y^i_{\alpha_1...\alpha_r}$. To include all $r$th-order Lagrangians one would have to start the construction from $(\Omega^\ast(J^rY), d)$. But we are exclusively interested in the Lagrangians of Chern–Simons and Yang–Mills theories which are polynomial in the $y^i_{\alpha_1...\alpha_r}$. So we will not go into this.

### 2.3.1 The inverse problem and local variational problems

One of the principal motivations for the development of variational cohomology was the so called inverse problem, i.e. to find criteria as to when a given set of equations are the Euler-Lagrange equations of a variational problem. Variational cohomology theories provide both a clear cut formulation and the solution of this problem (see, among others, [62], [3]). In terms of the variational sequence, for $\eta \in \mathcal{V}_{r+1}^{n+1}$, a **source or dynamical form**, the equations $\eta \circ j^{2r-1}\sigma = 0$ are **locally variational** if and only if $E_{n+1}(\eta) = 0$, this is the **Helmholtz condition in variational cohomology**, and they are **globally variational** if and only if the cohomology class defined by a locally variational $\eta$ vanishes, i.e. $\eta = E_n(\lambda)$ for some $\lambda \in \mathcal{V}_r^n$.

This clarifies also the concept of local variational problem (see the end of section 2.2 above): For any $\eta \in \mathcal{V}_r^{n+1}$ with $E_{n+1}(\eta) = 0$ and any open covering $(U_i)_{i \in I}$, there is a collection of local lagrangians $\lambda_i \in \mathcal{V}_r^n|_{U_i}$ such that $\eta|_{j^{2r-1}U_i} = E_n(\lambda_i)$. Every local variational arises in this way. Of course, by setting $\lambda_i := \lambda|_{U_i}$ again for some open cover $(U_i)_{i \in I}$ any variational problem is also canonically a local variational problem.

Local variational problems are, of course, very interesting in their own right whenever the cohomology class $[\eta]$ is non zero. But much of their importance arises in situations when a source form in principal admits a global Lagrangian: global Lagrangians may not be easy to find or have undesirable properties, systems of local Lagrangians may be anyhow better to work with or a specific system may be the standard choice for historical reason. Chern–Simons gauge theories are prime examples for all these cases.
2.4 The obstruction

One of the interesting features of variational cohomology is that it makes the importance of the real cohomology of $X$ and $Y$ in the calculus of variations explicit. Somewhat surprisingly, this aspect seems not to have been explored very much over the years. In part this may depend on the fact that it is quite hard to explicitly describe variational cohomology classes in terms of the de Rham cohomology of $X$ and $Y$. Also, the folk theorem, mentioned in the introduction, that for variational problems relevant in physics all the cohomology classes vanish may have had its impact. Anyhow, the following theorem apparently has been overlooked until recently ([50], see also [49]); we give here a (slightly different) proof to illustrate the workings of the variational sequence.

**Theorem 1** Let $\eta_\lambda$ be the dynamical form of a local variational problem on a fibre bundle $Y \hookrightarrow X$ with $\dim Y > \dim X = n$ and let $\pi^* : H^*_dR(X) \hookrightarrow H^*_dR(Y)$ be an isomorphism. Let $\Xi$ be a vertical vector field such that $E_n(\Xi|\eta_\lambda) = 0$. Then both $[\Xi|\eta_\lambda] \in H^*_dR(J^{2r-1}Y)$ and $[(j^{2r-1}\sigma)^*(\Xi|\eta_\lambda)] \in H^*_dR(X)$ for an arbitrary section $\sigma : X \hookrightarrow Y$ are obstructions to the existence of (global) solutions; $(j^{2r-1}\sigma)^*(\Xi|\eta_\lambda) \in H^*_dR(X)$ is independent of the section.

**Proof.** Since $\eta$ is a 1-contact form in $\Omega^{n+1}_{J^{2r-1}Y}$, $\Xi|\eta$ is horizontal in $\Omega^n_{J^{2r-1}Y}(Y)$. Thus, we find a

$$\beta \in h^{-1}(\Xi|\eta_\lambda) \subset \Omega^n_{J^{2r-2}Y}(Y)$$

with $d\beta = 0$ and which represents the same cohomology class as $\Xi|\eta_\lambda$. Since $(\pi^{2r-1})^*\beta = \Xi|\eta_\lambda + \theta$ ($\theta$ is a contact n-form), we have

$$j^{2r-2}\sigma^*\beta = j^{2r-1}\sigma^*(\Xi|\eta_\lambda)$$

If we denote by $[\gamma]$ the cohomology class corresponding to the closed differential form $\gamma$ we have

$$j^{2r-1}\sigma^*[\Xi|\eta_\lambda] = [j^{2r-1}\sigma^*(\Xi|\eta_\lambda)] = [j^{2r-2}\sigma^*\beta] = j^{2r-2}\sigma^*\beta$$

Since $H^*_dR(Y) \sim H^*_dR(J^{2r-1}Y)$ via the jet bundle projection, any jet prolongation of a section induces an inverse isomorphism to $(\pi^k)^*$:

$$(j^k)^* \circ (\pi^k)^* = 1_{H^*_dR(X)}$$
and
\[(\pi^k)^* \circ (j^k)^* = 1_{H^2_dR(J\cdot Y)}\]
Thus, \((j^{2r-1}\sigma)^*(\Xi|\eta_\lambda) \in H^n_dR(X)\) does not dependent on the section. Therefore, if \([\Xi|\eta_\lambda] \neq 0\) then also \([j^{2r-1}\sigma^*(\Xi|\eta_\lambda)] \neq 0\) for all sections. Hence, \(j^{2r-1}\sigma^*(\Xi|\eta_\lambda)\) does not vanish along any section.

From the Euler–Lagrange equations \(\eta_\lambda \circ j^{2r-1}\sigma = 0\) we see, however, that if \(\sigma\) is a solution also \(j^{2r-1}\sigma^*(\Xi|\eta_\lambda) = 0\). Therefore, if \([\Xi|\eta_\lambda] \neq 0\) there can be no solutions.

There is also a more general version dropping the requirement of \(\pi^* : H^n_dR(X) \mapsto H^n_dR(Y)\) to be an isomorphism, then \((j^{2r-1}\sigma)^*(\Xi|\eta_\lambda) \in H^n_dR(X)\) depends on the homotopy class of the section and is an obstruction only within this class ([50]). The cohomology class \([\Xi|\eta_\lambda]\) arises naturally in the context of the **Noether-Bessel-Hagen theorem**, a kind of combination of the two Noether theorems ([45]) in variational cohomology. It is there an obstruction to existence of global conserved quantities (see e.g. [23], [50], also [49]).

General results regarding these obstructions seem virtually impossible to come by: partially, because of the already mentioned difficulties of identifying variational cohomology classes in de Rham cohomology, but mainly because they are defined via contractions and contractions behave badly with cohomology. Regarding this, note that in the case of \(0 = [\eta_\lambda] \in H^{n+1}_{dR}(Y)\) we may still have \(0 \neq [\Xi|\eta_\lambda] \in H^n_{dR}(X)\); in particular, this is true for Chern–Simons theories.

Being in the top dimensional cohomology of \(X\), the obstruction \([j^{2r-1}\sigma^*(\Xi|\eta_\lambda)]\) can be non zero only if \(X\) is closed, i.e. compact without boundary. This condition is not “unphysical”, theoretical physicists work actually quite often with (open sets in) closed manifolds, but this fact may then appear in the disguise of boundary conditions, long time behaviour or behaviour at infinity.

On the other hand, in section 5.2.1 we will work with a specific (local) variational problem on a non compact manifold. This will illustrate how to pass from the compact to the non compact case when the compact case is well understood.

At this point, one word of warning is in order: while almost trivial, it is nevertheless easy to overlook that one cannot work exclusively with \((j^{2r-1}\sigma)^*(\Xi|\eta_\lambda)\). Even if it represents a non trivial cohomology class for
some $\sigma$, this class may not be an obstruction as long as $\Xi \eta_\lambda$ is not closed (in the variational sequence).

### 3 The bundle of connections

Yang–Mills and Chern–Simons gauge theories are a classical field theories for principal connections on a principal bundle $P \to X$ over some $n$–dimensional smooth manifold $X$. To study them by means of a variational cohomology theory like the variational sequence, it is necessary to describe principal connections as sections of a bundle. To do so we need to introduce the bundle of connections (see [13] and the references therein for a more thorough treatment). To make its construction transparent, we start recalling three ways of defining principal connections and the relations between them (see e.g. [37]).

A principal connection can be defined as decomposition of $TP$, the tangent bundle of $P$, into a direct sum of the vertical bundle $VP$ and a horizontal complement $\mathcal{H}$ which is invariant under the right action of $G$ on $P$, i.e.

$$TP = \mathcal{H} \oplus VP \quad \mathcal{H}_y = dR_g(\mathcal{H}_y)$$

with $y \in P, g \in G$ and $dR_g$ the differential of the right action $R_g$. A second possibility is to define a principal connection as a right invariant Lie algebra valued one form $\omega$ on $P$ which maps the fundamental vector fields on their corresponding Lie algebra elements, i.e.

$$\omega : TP \to \mathfrak{g} \quad \omega(\tilde{A}) = A \quad \omega(dR_g(X)) = ad(g^{-1})\omega(X)$$

with $\mathfrak{g}$ the Lie algebra of $G$, $\tilde{A} \in VP$, the fundamental vector field corresponding to $A \in \mathfrak{g}$, and $X \in TP$. Or one can define a principal connection as a collection of $\mathfrak{g}$-valued one forms $\omega_U$ relative to an open cover $\mathcal{U}$ of $X$ over which $P$ can be trivialized, i.e.

$$\omega_U : TU \to \mathfrak{g} \quad U \in \mathcal{U} \quad P|_U \sim U \times G$$

such that

$$(\omega_U(x))|_{U \cap V}(X) = (ad(h_{UV}(x)^{-1})(\omega_V(x)) + h_{UV}^{-1}(x)dh_{UV}(x))|_{U \cap V}(X) \quad (9)$$

with $U, V \in \mathcal{U}, x \in U \cap V$ and $h_{UV} : U \cap V \to G$ the transition function of the change of trivialisation from $P|_V$ to $P|_U$. 

These different definitions are related by $H = \ker \omega$ and $\omega_U = (\gamma|_U)^*\omega$ with $\gamma|_U$ a section of $P|_U$ over $U \in \mathcal{U}$. For the local coordinate expressions, note that each $\gamma|_U$ defines a local trivialization $U \times G \sim P|_U$ by $(x, g) \mapsto \gamma|_U(x) \cdot g$ and with respect to this trivialization we have $\gamma|_U(x) = (x, 1_G)$.

The bundle of connections of $P$ may now be defined by

$$\mathcal{C}_P := J^1P/G \hookrightarrow P/G \sim X$$

The sections $\sigma$ of this bundle are in one to one correspondence with the right invariant decompositions of $TP$ above by virtue of the canonical decomposition, see (3) at the beginning of section 2.1.1. $\pi: \mathcal{C}_P \hookrightarrow X$, is an affine bundle modelled on the vector bundle $T^*X \otimes VP/G \hookrightarrow X$, i.e. the bundle of $VP/G$ valued 1-forms on $X$. An affine bundle $\mathcal{V}_a$ is, roughly speaking, a bundle for which the difference of two sections is defined and is a section of a vector bundle $\mathcal{V}$, but which is distinguished from a vector bundle in that it has no zero section. One says then $\mathcal{V}_a$ is modelled on $\mathcal{V}$; $\mathcal{V}_a$ and $\mathcal{V}$ are isomorphic as topological bundles (see e.g. [57]).

$\mathcal{C}_P$ inherits its affine structure from $J^1P$. It reflects the fact that the difference of two connections is a $VP/G$ valued 1-forms on $X$, but the connections themselves are not. Note also that, since $\mathcal{C}_P$ is a (strong) deformation retract of $X$, we have $H^*_dR(\mathcal{C}_P) \sim (H^*_dR(X))$ via $\pi^*$ and $\sigma^*$ for an arbitrary section $\sigma$ of $\mathcal{C}_P$.

Locally $\mathcal{C}_P$ can be trivialized as $\mathcal{C}_P|_U \sim U \times \mathbb{R}^n \otimes \mathfrak{g}$ with $\mathfrak{g}$ the Lie algebra of $G$. If $(x^\mu)_{1 \leq i \leq n}$ are coordinates on $X$ and $e_i$ is a base of $\mathfrak{g}$, we have coordinates $(x^\mu, A^i)$ ($A^i$ is the coefficient of the component $dx^\mu \otimes e_i$) on $\mathcal{C}_P|_U$. The transition functions of $\mathcal{C}_P$ take the form

$$k_{UV}(x) = ad(h_{UV}(x)^{-1}) \otimes J^*_{UV} + h_{UV}^{-1}(x)dh_{UV}(x)$$

i.e. the transition functions derive from the change of trivialization formula of the local connection forms ([3]) and they are affine in view of the additive “displacement term” $h_{UV}^{-1}(x)dh_{UV}(x)$; here $J^*_{UV}: T^*V|_{U \cap V} \hookrightarrow T^*U|_{U \cap V}$ indicates the pullback with the change of coordinate Jacobian $J_{UV}: TU|_{U \cap V} \hookrightarrow TV|_{U \cap V}$.

The contact structure of $J^1P$ defines the canonical connection on the principal bundle $J^1P \hookrightarrow \mathcal{C}_P$ (see [13], note, however, that we use a different
sign convention); at the point \( q \in J^1P \) with coordinates \((x^\mu, A^i_\mu, g)\) with respect to a local trivialization \( \mathcal{C}_P|_U \times G \sim U \times \mathbb{R}^n \otimes \mathfrak{g} \times G \) the connection form \( \phi \) of this connection can be written as

\[
\phi_{(x^\mu, A^i_\mu, g)} = \text{ad}(g^{-1}) \left( \Sigma_i \mathcal{e}_i \otimes (dg_i - \Sigma_{\mu} A^i_\mu dx^\mu) \right)
\]

\( dg_i \) is defined at the point \( q \) by \((\mathcal{e}_i \otimes dg_i)_q(\tilde{\mathcal{e}}_i(q)) = \mathcal{e}_i \); here \( \tilde{\mathcal{e}}_i \) is the fundamental vector field corresponding to \( \mathcal{e}_i \); restricted to the fibre \( G \) of \( J^1P \mapsto \mathcal{C}_P \), \( \Sigma_i \mathcal{e}_i \otimes dg_i \) is, of course, the Maurer–Cartan form of \( G \).

Let \( \gamma|_U : \mathcal{C}_P|_U \mapsto J^1P|_U \) be the section defined by \( \gamma|_U(x, A^i_\mu) = (x, A^i_\mu, 1_G) \).

For the local connection form \( \phi_U \) we have then

\[
(\phi_U)_{(x^\mu, A^i_\mu)} = (\phi \circ \gamma|_U)_{(x^\mu, A^i_\mu)} = -\Sigma_i \mathcal{e}_i \otimes \Sigma_{\mu} A^i_\mu dx^\mu
\]

\( \omega_\sigma := \phi \circ \sigma_P \) is the connection one form corresponding to the principal connection defined by the section \( \sigma : X \mapsto \mathcal{C}_P \), where \( \sigma_P \) is the lift of \( \sigma \) to \( P \) defined by the following commutative diagram

\[
\begin{array}{ccc}
P & \xrightarrow{\sigma_P} & J^1P \\
\downarrow & & \downarrow \\
X & \xrightarrow{\sigma} & \mathcal{C}_P 
\end{array}
\]

The curvature of a connection \( \omega \) is \( \Omega = d\omega + \frac{1}{2} [\omega, \omega] \). Because of the right invariance of \( \omega \) its curvature can be interpreted as either a \( \mathfrak{g} \) valued two form on \( P \) or a \( VP/G \) valued two form on \( X \). For the curvature \( \mathcal{F} \) of \( \phi \) we have the following local expression

\[
\mathcal{F} = \Sigma_k \mathcal{e}_k \otimes \left( \Sigma_{\mu,\nu,\kappa} \left( dx^\mu \wedge dA^k_\nu + \Sigma_{i,j} \frac{1}{2} c^k_{ij} A^i_\nu A^j_\kappa dx^\mu \wedge dx^\kappa \right) \right)
\]

\( (c^k_{ij}) \) are the structural constants of \( \mathfrak{g} \).

For the horizontalization of \( \mathcal{F} \) on \( J^1\mathcal{C}_P \) we have then

\[
h(\mathcal{F}) = \Sigma_k \mathcal{e}_k \otimes \left( \Sigma_{\mu,\nu} \left( (A^k_\nu - A^k_\mu) dx^\mu \wedge dx^\nu + \Sigma_{i,j} \frac{1}{2} c^k_{ij} A^i_\nu A^j_\mu dx^\mu \wedge dx^\nu \right) \right)
\]
and we have the decomposition
\[(\pi_0^1)^* \mathcal{F} = \Theta_\phi + h(F)\]
where $\Theta_\phi$ is the contact component of $(\pi_0^1)^* \mathcal{F}$; locally
\[\Theta_\phi = \Sigma_k e_k \otimes \Sigma_{\mu} dx^\mu \wedge \theta^k\]  \hspace{1cm} (11)
($\theta^k = dA^k - \Sigma_{\nu} A^k_{\nu} dx^\nu$ are the local contact one forms on $J^1 \mathcal{C}_P$).

Depending on the interpretation of $\mathcal{F}$ we have either
\[\Theta_\phi, h(F) \in \Lambda^2(J^1 \mathcal{C}_P) \otimes V_P/G\]
or
\[\Theta_\phi, h(F) \in \Lambda^2((\pi_0^1)^* J^1 P) \otimes g\]
$(\pi_0^1)^* J^1 P$ is the total space of the pullback of the principal bundle $J^1 P \rightarrow \mathcal{C}_P$ to $J^1 \mathcal{C}_P$ via the jet bundle projection $\pi_0^1 : J^1 \mathcal{C}_P \rightarrow \mathcal{C}_P$.

Finally, for the curvature $\Omega_\sigma$ of the connection $\omega_\sigma$ with $\sigma(x) = (x, A^k_\mu(x))$
locally, we have
\[\Omega_\sigma = (j^1 \sigma)^* h(F) = \sigma^* \mathcal{F}\]  \hspace{1cm} (12)
and the coordinate expression
\[\Omega_\sigma = \Sigma_k e_k \otimes \left( \Sigma_{\mu,\nu} \left( \left( \frac{\partial}{\partial x^\nu} A^k_\mu(x) - \frac{\partial}{\partial x^\mu} A^k_\nu(x) \right) dx^\mu \wedge dx^\nu + \Sigma_{i,j} \frac{1}{2} c^k_{ij} A^i_\mu(x) A^j_\nu(x) dx^\mu \wedge dx^\nu \right) \right)\]

4 Chern–Simons theories on the bundle of connections

We will now outline the construction of the Chern–Simons lagrangians on the bundle of connections and the corresponding field theories. Chern–Simons theories are gauge theories for connections. Starting point is the work by Chern and Simons on secondary characteristic classes, [16] and [17].
Let $\mathcal{P}_k$ be an $\text{adG}$-invariant polynomial of degree $k$ on $\mathfrak{g}$ the Lie algebra of $G$, let $\omega$ be a connection on the principal bundle $P$ over $X$ and $\Omega$ its curvature form $\mathcal{P}_k(\Omega)$ is a closed form of degree $2k$ on $P$. Horizontal and invariant under the right action of $G$, it can also be considered a closed form on $X$. The cohomology classes thus defined are independent of the connection $\omega$ (this is the starting point of the Chern–Weil theory of characteristic classes).

Before continuing, we will introduce some operations on Lie algebra– or matrix–valued forms and the conventions with which we use them. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ or $\mathbb{C}$, $(e_i)_{i \in I}$ a basis of it and $\alpha$ and $\beta \ R$– or $\mathbb{C}$–valued differential forms. We set now

$$[e_i \otimes \alpha, e_j \otimes \beta] := [e_i, e_j] \otimes \alpha \wedge \beta$$

and

$$e_i \otimes \alpha \wedge e_j \otimes \beta := (e_i \otimes e_j) \otimes \alpha \wedge \beta$$

and extend the Lie bracket $[,]$ and the exterior product $\wedge$ by linearity to arbitrary $\mathfrak{g}$–valued differential forms. Note that the Lie bracket of $\mathfrak{g}$–valued forms is $\mathfrak{g}$–valued, while their exterior product takes values in $\mathfrak{g} \otimes \mathfrak{g}$.

The product of matrices $A = (\alpha_{ij})_{1 \leq i,j \leq n}$ and $B = (\beta_{ij})_{1 \leq i,j \leq n}$ of differential forms is the matrix of differential forms defined by

$$AB = C \quad C = (\gamma_{ij})_{1 \leq i,j \leq n}$$

with

$$\gamma_{ij} = \sum_{k=1}^{n} \alpha_{ik} \wedge \beta_{kj}$$

To not overload the notation even further we will allow the following ambiguity for the expression $\Upsilon^i$ for a $\mathfrak{g}$–valued form $\Upsilon$

- wherever $\mathfrak{g}$ appears as an abstract Lie algebra, e.g. when dealing with invariant polynomials on $\mathfrak{g}$ in general terms, we will set $\Upsilon^i = \Upsilon \wedge \ldots \wedge \Upsilon$ ($i$-fold wedge product)

- if instead $\Upsilon$ is given with respect to a specific representation of $\mathfrak{g}$, $\Upsilon^i$ will be the $i$-fold matrix product of $\Upsilon$ with itself.
Chern and Simons found in [17] a 2\(k - 1\)-form \(T\mathcal{P}_k(\omega)\) on \(P\) such that
\[
P_k(\Omega) = dT\mathcal{P}_k(\omega)
\]
on \(P\). One can write explicitly ([17])
\[
T\mathcal{P}_k(\omega) = \sum_{i=0}^{k-1} \kappa_i \cdot \mathcal{P}_k(\omega \wedge [\omega, \omega]^i \wedge \Omega^{k-i-1})
\]
where \(\kappa_i = \frac{(-1)^i i! (k-1)!}{2^i (k+i)! (k-1-i)!}\) is a rational numerical factor; if \(\mathcal{P}_k(\Omega) = 0\) we have \(0 = dT\mathcal{P}_k(\omega)\) and, thus, \(T\mathcal{P}_k(\omega)\) defines a cohomology class on \(P\). These cohomology classes are the **Chern–Simons secondary characteristic classes**. In contrast to Chern–Weil theory they may depend on the connection \(\omega\). The expository material about Chern–Weil theory and Chern–Simons secondary characteristic classes, their variants and ramifications is vast and of easy access, so we indicate only a few classics: [14], [15], [16], [17], [18] chapter 19 of [34], chapter XII and appendix 20 of Volume II of [37] and appendix C of [44]; to them we add also the recent [65].

If \(\dim X = 2p + 1\), \(T\mathcal{P}_{2p+2}(\omega)\) depends on the connection. The central idea behind Chern–Simons gauge theories is now to use this dependence to construct a local variational principle for connections. From now on \(P \mapsto X\) will be a \(G\)-principal bundle over the \(2p + 1\)-dimensional manifold \(X\). On \(J^1 P\) we have
\[
dT\mathcal{P}_{p+1}(\phi) = \mathcal{P}_{p+1}(\mathcal{F})
\]
for the canonical connection \(\phi\) on the principal bundle \(J^1 P \mapsto C_P\).

Let \(U\) be now an open cover of \(C_P\). Using a family of local section \(\alpha_U : U \mapsto J^1 P|_U \ U \in U\), we get a system of local connection forms \(\phi_U := \phi \circ \alpha_U\). Thus, on \(C_P\) we have locally
\[
dT\mathcal{P}_{p+1}(\phi_U) = \mathcal{P}_{p+1}(\mathcal{F})|_U.
\]
The horizontalization of these local potentials of \(\mathcal{P}_{p+1}(\mathcal{F})\) is now the system of local Lagrangians on \(J^1 C_P\) of the Chern–Simons theory
\[
\lambda_{CS}^U := h(T\mathcal{P}_{p+1}(\phi_U)) = \sum_{i=0}^{p} \kappa_i \cdot \mathcal{P}_{p+1} \left( \phi_U \wedge [\phi_U, \phi_U]^i \wedge (h(\mathcal{F})|_U)^{p-i} \right)
\]
The Euler-Lagrange form of this local variational problem is (see (8) above)
\[
\eta_{CS} = I_1 \circ p_1 (\mathcal{P}_{p+1}(\mathcal{F})) = I_1 (\mathcal{P}_{p+1}(\Theta_\phi \wedge h(\mathcal{F})^p)) = \mathcal{P}_{p+1}(\Theta_\phi \wedge h(\mathcal{F})^p)
\]
Note that the last equality holds because \(\mathcal{P}_{p+1}(\Theta_\phi \wedge h(\mathcal{F})^p)\) contains only first order contact terms. We summarize this:
Proposition 1 Let $P \mapsto X$ be a $G$ principal bundle over the $(2p + 1)$-dimensional manifold $X$; let $P_{p+1}$ be an $\text{ad}(G)$-invariant polynomial of degree $p+1$ on $g$, the Lie algebra of $G$. On an open cover $U$ of the respective bundle of connections $C_P \mapsto X$ the system of local Lagrangians

$$\lambda_{CS}^U := \sum_{i=0}^p \kappa_i P_{p+1} \left( \phi_U \wedge [\phi_U, \phi_U]^i \wedge (h(F))_{U}^{p-i} \right)$$

defines a first order variational problem, the Chern–Simons theory of $(P, P_{p+1})$. Its Euler-Lagrange form is

$$\eta_{CS} = P_{p+1}(\Theta_{\phi} \wedge h(F)^p)$$

Note that in this case not only the Langrangian, but also the Euler-Lagrange form is of first order, i.e. defined on $J^1 CP$.

Since $H^{n+1}(J^r CP) \sim H^{n+1}(CP) \sim H^{n+1}(X) = 0$, Chern–Simons theories are in principal globally variational, see 2.3.1. But a global Lagrangian seems to depend always on fixing a physical quantity, a background connection, a priori (see [12] for the 3-dimensional case and [25] for higher dimensions). Anyhow, the questions of what is the correct or best Chern–Simons Lagrangian may be important for physicists for “the way Feynman diagrams can be read off from a Lagrangian” ([24] introduction to chapter 9 and paragraphs 6.3–6.6), but it is completely irrelevant to us, since we will occupy ourselves exclusively with cohomological invariants extracted from the Euler–Lagrange form.

5 Unitary Chern–Simons theories and Yang–Mills–Chern–Simons theories

We will now turn to unitary Chern–Simons theories derived from the polynomial $ch_{p+1}$; the polynomial $ch_k$ corresponds via Chern–Weil theory to the $k$th component of the Chern character.

We will introduce and analyze a specific obstruction of type $\Xi|\eta_{CS}$ and apply it in the context of the theory of solitons (instantons) in Yang–Mills–Chern–Simons theories, i.e. Lagrangian gauge field theories obtained adding Yang–Mills and Chern–Simons Lagrangians. Here the variational cohomology comes into focus: for existence of solutions it is necessary that the obstructions is annihilated by a Yang–Mills counterterm and the cohomological mechanism selects the precise counterterm.
At present, Yang–Mill–Chern–Simons theories seem to be under consideration as physical theories exclusively in five dimensions, in holographic QCD. However, in view of the long standing and ongoing interest in higher dimensional gauge theories in theoretical physics and mathematics (see e.g. [21], [22], [27], [46], [47], [63], [64]), we will work in arbitrary dimensions until we deal explicitly with holographic QCD.

5.1 Chern–Simons theories for $U(n)$ or $SU(n+1)$, $(n \geq p)$, and the polynomial $ch_{p+1}$

$\mathcal{P} \mapsto \mathbf{X}$ will now be a $U(n)$ or $SU(n+1)$ principal bundle over the $(2p+1)$-dimensional manifold $\mathbf{X}$ with $(n \geq p)$. As before, all matrix expressions are relative to the standard representations of $U(n)$ and $SU(n+1)$ and their Lie algebras $\mathfrak{u}_n$ or $\mathfrak{su}_{n+1}$. For an arbitrary matrix $A$ denote by $trA$ its trace; we define then an invariant polynomial $ch_k$ on $(\mathfrak{s})_{\mathfrak{u}_{n(1)}}$

$$ch_k(X) = \left(\frac{i}{2\pi}\right)^k trX^k \quad X \in (\mathfrak{s})_{\mathfrak{u}_{p(1)}}$$

If $\Omega$ is the curvature 2-form of a connection $\omega$ on $\mathcal{P}$, we have by Chern–Weil theory

$$[ch_k(\Omega)] = ch_k(\mathcal{P}) \in H^{2k}(\mathbf{X}, \mathbb{R})$$

where $ch_k(\mathcal{P})$ is the $k$th component of the Chern character of $\mathcal{P}$. In algebraic topology, the Chern character is a formal power series in the Chern classes (see e.g. [34] chapter 20, paragraphs 4 and 11 and [34] paragraph 16 problem B; see also [34] Appendix C and [18] for the Chern character in Chern–Weil theory); $\left(\frac{i}{2\pi}\right)^k$ is again a normalization factor such that $[ch_k(\Omega)]$ lies in the image of $H^{2k}(\mathbf{X}, \mathbb{Z})$ in $H^{2k}(\mathbf{X}, \mathbb{R})$ induced by the inclusion $\mathbb{Z} \subset \mathbb{R}$. By proposition 1, we have then

$$ch_{p+1}(\Theta_\phi + h(\mathcal{F})) = \left(\frac{i}{2\pi}\right)^{p+1} tr(\Theta_\phi + h(\mathcal{F}))^{p+1}$$

and

$$\eta_{CS} = ch_{p+1}(\Theta_\phi \wedge h(\mathcal{F})^p) = \left(\frac{i}{2\pi}\right)^{p+1} tr(\Theta_\phi h(\mathcal{F})^p) =$$
for the Chern–Simons gauge field theory derived from $ch_{p+1}$; note that the latter two expressions are again globally well defined on $(\pi_1^0)^* J^1 P$.

**Proposition 2** Let $P \mapsto X$ be a $U(n)$ or $SU(n+1)$ principal bundle over the $(2p+1)$-dimensional manifold $X$, $(n \geq p)$; let $\omega_\sigma$ be the principal connection on $P$ corresponding to the section $\sigma$ of $\pi_{CP} : CP \mapsto X$ and let $\Omega_\sigma$ be its curvature. $\sigma$ is then a solution of the Chern–Simons gauge field theory derived from $ch_{p+1}$ admits solutions if and only if

$$(\Omega_\sigma)^p = 0$$

($p$-fold matrix product of $\Omega_\sigma$ with itself taken with respect to the standard representation of $(s)u_{n(+1)}$).

**Proof.** We have to find the conditions under which $\eta_{CS} \circ j^1 \sigma = 0$. The key observation is that $\Theta_\phi$ does not vanish along any section (see equation \[(11)\]) In particular, none of the $(\Theta_\phi)_{kj}$ vanish along sections. Thus, equation \[(15)\] implies that all the $(h(F)^{\phi})_{jk}|_\sigma = 0$. Since $\Omega_\sigma = (j^1 \sigma)^* h(F)$, see equation \[(12)\], this means that all the $(\Omega_\sigma)^p_{jk}$ vanish (everywhere locally), i.e. $(\Omega_\sigma)^p = 0$.

For a solution $\omega_\sigma$ this implies, of course, $ch_p(\Omega_\sigma) = tr(\Omega_\sigma)^p = 0$ and, thus, $0 = [ch_p(\Omega_\sigma)] = ch_p(P) \in H^{2k}(X, \mathbb{R})$. Hence, $ch_p(P)$ is an obstruction to the existence of solutions of the Euler–Lagrange equations.

For the applications to Yang–Mills–Chern–Simons theories in the next section, we will now construct an obstruction of type $\Xi|\eta_{CS}$ that vanishes if and only if $ch_p(P)$ vanishes.

For the rest of this section and for the non existence results of the next (see theorem \[2\]) will restrict ourselves to closed manifolds of dimension $2p+1$. This compactness hypothesis serves to streamline the following exposition somewhat. In section \[5.2.1\] we will show how it can be relaxed in a specific case.

We note that on a closed $(2p+1$–dimensional) manifold $X$ there exists, by Poincaré duality, a closed one form $\beta$ such that $0 \neq [\beta \wedge ch_p(\Omega_\sigma)] \in H^{2p+1}(X, \mathbb{R})$ whenever $0 \neq [ch_p(\Omega_\sigma)] \in H^{2p}(X, \mathbb{R})$. This means, we need to find a vertical vector field $\Xi$ on $CP$ such that

$$j^1 \sigma ^* (\Xi|\eta_{CS}) = \beta \wedge ch_p(\Omega_\sigma)$$
First we will deal with the case of an $U(n)$–principal bundle $\mathbf{P} \hookrightarrow X$, $n \geq p$. Since $\pi_{\mathcal{P}} : \mathcal{C}_{\mathbf{P}} \hookrightarrow X$, is an affine bundle modeled on the vector bundle

$$T^*X \otimes \mathcal{V}P / U(n) \hookrightarrow X$$

we have

$$V\mathcal{C}_{\mathbf{P}} \sim \mathcal{C}_{\mathbf{P}} \times_X T^*X \otimes \mathcal{V}P / U(n)$$

and, in particular, we can identify

$$\frac{\partial}{\partial A^\mu} = \epsilon_k \otimes dx^\mu$$

Since $i \cdot 1_n \in \mathfrak{u}(n)$ is $ad U(n)$–invariant, $-i \cdot 1_n \otimes \alpha$ is then identified with a global vertical vector field for any differential one form $\alpha$ on $X$ and we have by equation (15)

$$j^1\sigma^* \left( (\left. -i \cdot 1_n \otimes \alpha \right|_{\eta_{CS}}) \right) = \frac{-1}{2\pi} \cdot \left( \frac{i}{2\pi} \right)^p \cdot \alpha \wedge tr(\Omega^p_\sigma) = \frac{-1}{2\pi} \cdot \alpha \wedge ch_p(\Omega_\sigma)$$

In particular, $j^1\sigma^* \left( (\left. -i \cdot 1_n \otimes \alpha \right|_{\eta_{CS}}) \right)$ is closed if and only if $\alpha$ is and we can choose $\alpha$ to be the Poincaré dual $\beta$ above if $0 \neq \left[ ch_p(\Omega_\sigma) \right]$. It remains to be shown that $\left. ( -i \cdot 1_n \otimes \alpha \right|_{\eta_{CS}})$ is closed whenever $j^1\sigma^* \left( (\left. -i \cdot 1_n \otimes \alpha \right|_{\eta_{CS}}) \right)$ is. This follow immediately from the fact that

$$\frac{-1}{2\pi} \cdot \alpha \wedge ch_p(\Omega_\sigma) = \sigma^* \left( \frac{-1}{2\pi} \cdot \pi^*(\alpha) \wedge ch_p(\mathcal{F}) \right)$$

and

$$\left( -i \cdot 1_n \otimes \alpha \right|_{\eta_{CS}} = h \left( \frac{-1}{2\pi} \cdot \pi^*(\alpha) \wedge ch_p(\mathcal{F}) \right)$$

i.e. if $j^1\sigma^* \left( (\left. -i \cdot 1_n \otimes \alpha \right|_{\eta_{CS}}) \right)$ is closed, then $\left. ( -i \cdot 1_n \otimes \alpha \right|_{\eta_{CS}})$ is the horizontalization of a closed differential form on $\mathcal{C}_{\mathbf{P}}$ and, therefore, closed by construction of the variational sequence, see section[2].

For a $SU(n+1)$–principal bundle $\mathbf{P} \hookrightarrow X$ ($n \geq p$), we note that the inclusion $SU(n+1) \subset U(n+1)$ induces the inclusion $\mathbf{P} \subset \mathbf{P}_{U(n+1)}$, with $\mathbf{P}_{U(n+1)} \sim \mathbf{P} \times_{SU(n+1)} U(n+1)$; recall that $\mathbf{P}_{U(n+1)}$ is a $U(n+1)$–principal bundle.
Since \( P_{U(n+1)} \sim P \times_X U(1) \) where \( U(1) \) is the trivial \( U(1) \)-principal bundle over \( X \) (and \( \times_X \) denotes the fibre product over \( X \), we have \( C_{P_{U(n+1)}} \sim C_P \times_X C_{U(1)} \)

Hence,

- every \( SU(n+1) \)-principal connection on \( P \) extents canonically to a \( U(n+1) \)-principal connection on \( P_{U(n+1)} \) by adding the canonical flat connection on \( U(1) \)
- the Chern–Simons gauge field theory derived from \( ch_{p+1} \) on \( C_P \) extents obviously to \( C_{P_{U(n+1)}} \); however, only the \( U(n+1) \)-principal connections on \( P_{U(n+1)} \) with \( u(1) \)-component the canonical flat connections are the “fields” of the \( SU(n+1) \)-Chern–Simons theory
- a solution of the Euler–Lagrange equations on \( C_P \) is also a solution of the Euler–Lagrange equations on \( C_{P_{U(n+1)}} \)

Thus, \((-i \cdot 1_n \otimes \alpha) \eta_{CS}\), respectively \( j^1 \sigma^* ((-i \cdot 1_p \otimes \alpha) \eta_{CS})\), is also an obstruction in the \( SU(n+1) \) case. We summarize this:

**Proposition 3** Let \( P \rightarrow X \) be a \( U(n) \) or \( SU(n+1) \) principal bundle over the closed \((2p+1)\)-dimensional manifold \( X \), \((n \geq p)\) and let \( \eta_{CS} \) be the Euler-Lagrange form of the Chern–Simons gauge field theory derived from \( ch_{p+1} \). Let \( \alpha \) be a closed one form on \( X \) and let furthermore \( \Xi_\alpha \) be the vertical vector field on \( C_P \) corresponding to \(-i \cdot 1_n \otimes \alpha\).

The cohomology classes \([\Xi_\alpha, \eta_{CS}] \in H^{2p+1}(J^1C_P, IR)\) and \([j^1 \sigma^* ((\Xi_\alpha, \eta_{CS})]) \in H^{2p+1}(X, IR)\) are then obstructions to the existence of solutions of the Euler–Lagrange equations. These obstructions vanish for all \( \Xi_\alpha \) if and only if \( 0 = ch_p(P) \in H^{2p+1}(X, IR) \).

Regarding the \( SU(n+1) \) case some remarks are in order. That \( ch_p(P) \) is an obstruction, is already implicit in the Euler–Lagrange form, avoiding the above construction of an extension to the \( U(n+1) \) case does not change this. The point of this construction is that it expresses this fact in terms of the obstructions \([\Xi_\alpha, \eta_{CS}]\) and makes it, thus, explicit.

For "coupled" Lagrangians this allows also to identify the precise counterterms which would need to annihilate the obstructions, as we will see in the next section for Yang–Mills–Chern–Simons theories. For \( SU(n+1) \) theories
one will have to keep track of effect of the vanishing of the $u(1)$-component of the curvature on these counterterms.

More generally, the $SU(n+1)$ case suggests that the obstructions of type $[\Xi_\alpha | \eta]$ may not exhaust the cohomological obstruction that can be found analyzing the Euler–Lagrange forms.

5.2 On the non existence of solitons in Yang–Mills–Chern–Simons theories

We will now explicate the how the obstructions $\Xi_\alpha | \eta_{CS}$ of proposition 3 essentially impede the existence of solitons in Yang–Mills–Chern–Simons theories. Adopting fairly common terminology (see e.g. [42]), by “topological soliton” or simply “soliton” we will refer to any solution of the Euler–Lagrange equations of a (locally) Lagrangian quantum field theory. The solitons of Yang–Mills theory in four dimensions are, of course, usually called “instantons”.

The literature about Yang–Mills theory and instantons is extraordinary vast, so we will give only some pointers. As a general introductions to Yang–Mills theories may serve [28] and [43]. As introduction to instantons we refer to [4], [5] and [19]. A good survey of instantons in theoretical physics is [48]. And, finally, for the rôle of instantons in differential geometry, see [20] in four dimensions and [21], [22] and [64] for the possibility of extending these ideas to higher dimensions.

Yang–Mills–Chern–Simons theories are (locally) Lagrangian quantum field theories on odd dimensional manifolds, the Langrangian of which is a sum of the Yang–Mills and Chern–Simons Lagrangians:

$$\lambda_{YMCS} = \lambda_{YM} + \kappa \cdot \lambda_{CS}$$

($\kappa$ is a constant factor, the so called coupling constant). This construction seems to motivated by the rôle of the Chern–Simons form in anomaly physics (see e.g. [1], [8] and [67]); for holographic QCD this is made explicit e.g. in [31] and [32]. There the idea is roughly as follows. The Yang–Mills–Chern–Simons theory is defined on $\mathbb{R}^5$ (with Lorentzian metric). This $\mathbb{R}^5$ is the interior of a manifold with boundary $M$ with the boundary $\partial M$ considered to be at “infinity”. The point of Yang–Mills–Chern–Simons theories is then that the Chern–Simons term causes anomaly cancellation in the Yang–Mills theory on the boundary; an anomaly is a conserved quantity corresponding to
a symmetry of the classical theory which is not a symmetry of the quantum theory.

Such relation between Field theories on the interior (physicists seem to call this the “bulk”) and their restrictions to the boundary of a manifold with boundary is what physicists refer to when using the term “holography”; the boundary is also often referred to as “holographic”, apparently [66], section 2, was the first explicit description of such an interplay between field theories on the bulk and on the boundary. for mathematicians it is still an excellent starting point.

Of course, the Chern–Simons Lagrangians are, by construction, only locally defined and, thus, is \( \lambda_{YMC} \). But since Chern–Simons theories are local variational problems (see sections [2,3,1] and [8]), also Yang–Mills–Chern–Simons theories are. The corresponding Euler–Lagrange form (on \( J^2 \mathcal{C}_P \)) is

\[
\eta_{YMC} = E_n(\lambda_{YM}) + \kappa \cdot (\pi^2) \cdot \eta_{CS}
\]

Let \( X \) be an oriented Riemannian manifold or an oriented Pseudo-Riemannian manifold of Lorentzian signature and let \( P \) be a \( U(n) \)-principal bundle over \( X \). The unitary Yang–Mills theory on \( J^1 \mathcal{C}_P \) is the Lagrangian field theory with Lagrangian

\[
\lambda_{YM} = \langle h(F), h(F) \rangle > d_m X
\]

where \( d_m X \) is the metric volume form on \( X \) and \( <,> \) is induced in the usual way by the (Pseudo-)Riemannian metric on \( X \) and the \( adU(n) \)-invariant metric on \( u_n \) defined by \( < A, B > = -\text{tr}(AB) \), with \( A, B \in u_n \) in the standard representation. Of course, this is completely equivalent to the usual formulation of unitary Yang–Mills theories on the (affine) space of connections \( \mathcal{A} \).

\( U(n) \) is canonically isomorphic to \( U(1) \times SU(n) \) (and, of course, \( u_n \) to \( u_1 \oplus su_n \)). Thus, every principal connection \( \omega \) on \( P \) and its curvature \( \Omega \), as well as the canonical connection \( \phi \) on \( J^1 P \mapsto \mathcal{C}_P \) and its curvature \( \mathcal{F} \) decompose into a sum of an \( u_1 \) and a \( su_n \) component. For the Yang–Mills Lagrangian we have then

\[
\lambda_{YM} = \langle h(F_{u_1}), h(F_{u_1}) \rangle > d_m X + \langle h(F_{su_n}), h(F_{su_n}) \rangle > d_m X
\]

Hence, by linearity, also its Euler–Lagrange form decomposes into an \( u_1 \) and a \( su_n \) component

\[
E_n(\lambda_{YM}) = E_n(\lambda_{YM})_{u_1} + E_n(\lambda_{YM})_{su_n}
\]
Note that the existence of a Pseudo-Riemannian metric of Lorentzian

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For a hypothetical solution $\sigma$ of the Euler–Lagrange equations we would have
\[ j^2 \sigma^* (\Xi_\alpha \eta_{YMCs}) = 0 \]
and, thus,
\[ \alpha \wedge d^* (\Omega_{\sigma})_{u_1} = \kappa \cdot \left( \frac{i}{2\pi} \right)^p \cdot \alpha \wedge \text{ch}_p(\Omega_{\sigma}) \] (16)

For $SU(n+1)$-theories (a $SU(n+1)$-theory can always be viewed as an $U(n+1)$-theory, see section 5.1 above) this is equivalent to
\[ \frac{\kappa}{2\pi} \cdot \left( \frac{i}{2\pi} \right)^p \cdot \alpha \wedge \text{ch}_p(\Omega_{\sigma}) = 0 \]
and this is, of course, possible if and only if $0 = \text{ch}_p(\mathcal{P}) \in H^{2p}(\mathcal{X}, \mathbb{R})$. In the case of $U(n)$-theories, since $\alpha$ is closed, we have
\[ \alpha \wedge d^* (\Omega_{\sigma})_{u_1} = -d (\alpha \wedge (\Omega_{\sigma})_{u_1}) \]

Thus, the Yang–Mills component of $j^2 \sigma^* (\Xi_\alpha \eta_{YMCs})$ is always a coboundary and cohomologically trivial. For the Chern–Simons component, instead, exists always a vector field $\Xi_\alpha$ such that it is cohomologically non trivial as long as $0 \neq \text{ch}_p(\mathcal{P}) \in H^{2p}(\mathcal{X}, \mathbb{R})$ (see section 5.1 above).

This means that equation (16) cannot have solutions if $0 \neq \text{ch}_p(\mathcal{P}) \in H^{2p}(\mathcal{X}, \mathbb{R})$. As a consequence, the vanishing of $\text{ch}_p(\mathcal{P}) \in H^{2p}(\mathcal{X}, \mathbb{R})$ is a necessary condition for the Euler–Lagrange equations $j^2 \sigma \circ \eta_{YMCs} = 0$ to have solutions. Hence, we have the following non existence theorem.

**Theorem 2** Let $\mathcal{P}$ be a $U(n)$– or $SU(n+1)$–principal bundle over the closed, $2p + 1$-dimensional oriented Riemannian or oriented Pseudo-Riemannian manifold $\mathcal{X}$ of Lorentzian signature, $n \geq p$. Let also $0 \neq \text{ch}_p(\mathcal{P}) \in H^{2p}(\mathcal{X}, \mathbb{R})$.

The Euler–Lagrange equations $\eta_{YMCs} \circ j^2 \sigma = 0$ of the corresponding unitary Yang–Mills–Chern–Simons theory on $J^1\mathcal{C}_\mathcal{P}$ do not admit solutions.

For Yang–Mills theories on 4-dimensional Riemannian manifolds, $\int \text{ch}_2(\Omega)$ is up to a numerical factor the instanton number. Since this integral vanishes if and only if $\text{ch}_2(\mathcal{P})$ is cohomologically trivial, this indicates, by analogy, how restrictive this non existence theorem is. However, we will not define a “soliton number” for Yang–Mills–Chern–Simons theories at the present level of generality. But we will address the issue in the next section for the Yang–Mills–Chern–Simons theories of 5-dimensional QCD.
5.2.1 On solitons in the Yang–Mills–Chern–Simons theories of 5-dimensional (holographic) QCD

We will now apply the ideas of section 5.2 above to the Yang–Mills–Chern–Simons theories of 5-dimensional (holographic) QCD. Solitons are a central part in many of these theories. They represent a particular class of particles called baryons on which relies the treatment of nuclear matter (see e.g. [35], [36], [54] and [38] for a more recent example).

Probably the most important among the YMCS theories of holographic QCD is the Sakai–Sugimoto model (see [55] and [56], see also [60] for a concise and recent exposition). Its solitons have been extensively studied (see e.g. [7], [10], [29], [30], [33]). Note, however, that the study of the solitons in the the Sakai–Sugimoto model is based on approximations or numerical methods which require the implicit assumption that the solitons exist, while our observations regard the question if such solitons can exist in a mathematically consistent way.

Constructions of such Yang–Mills–Chern–Simons theories start usually with an $\mathbb{IR}^5$ with coordinates $x_0, x_1, x_2, x_3, z$, where the $x_i$ are the coordinates of Minkowski spacetime, $x_0$ being time, and $z$ is the so called holographic coordinate, and equipped with a not necessarily flat Pseudo-Riemannian metric $\mu$ of signature $(-, +, +, +, +)$ (or $(+, -,-, -,-)$), see e.g. [7] and [10].

This $\mathbb{IR}^5$ (or some subset of it, defined by so called infrared and ultraviolet cut offs somewhere on the holographic axis) is then the Pseudo-Riemannian base manifold $X$ of a principal bundle $P$ on which the Yang–Mills–Chern–Simons theories will be set up. One considers mainly theories with group $U(n)$, but also $SU(n+1)$ (see e.g. [31]), $n \geq 2$.

In analogy with the instantons of 4–dimensional Yang–Mills theory, the solitons of holographic QCD are required to satisfy the finiteness condition that the integral

$$\tau \cdot \int_{x_0=\text{const.}} ch_2(\Omega)$$ (17)

converges (see e.g. [10]); here, $\Omega$ is the curvature of an principal connection $\omega$ on the unitary principal bundle $P \mapsto \mathbb{IR}^5$ and $\tau$ is a numerical factor. For this $\Omega$ needs, of course, to go sufficiently fast to 0 for $r \mapsto \infty$ with $r = \sqrt{x_1^2 + x_2^2 + x_3^2 + z^2}$.

The most natural and mathematical consistent way to guarantee that the integral (17) is well defined, is to require that $P \mapsto \mathbb{IR}^5$ is the restriction of
a principal bundle $\hat{\mathcal{P}} \mapsto \mathbb{R} \times S^4$ and the connections $\omega$ are the restrictions of the principal connections $\hat{\omega}$ on $\hat{\mathcal{P}}$. The restriction is with respect to the embedding $s_+ : \mathbb{R}^5 \mapsto \mathbb{R} \times S^4$ which is defined by being the identity on the coordinate $x_0$ and the stereographic projection $\mathbb{R}^4 \mapsto S^4$ (with respect to $(0,0,0,0,1) \in \mathbb{R}^5$) on $(x_1,x_2,x_3,z)$.

At the beginning of section 5.2 we mentioned that one of the key features of YMCS theories of holographic QCD is the interplay between the field theory in the “bulk” and the field theory on the boundary at infinity (see [31] and [32]). That behaviour at infinity and the one we have just discussed have got nothing to do with each other, each results from a specific, independent theoretical requirement; we will return to this point.

It is known from Morse theory that every closed, smooth $n$–manifold has a CW-decomposition with a single $n$-cell. Thus, $S^4$ could be replaced by any closed, smooth 4–manifold mapping the $\mathbb{R}^4$ corresponding to $(x_1,x_2,x_3,z)$ diffeomorphically onto the single open $n$-cell. However, this would be equivalent to imposing additional conditions “at infinity”. In this sense, the above construction is “canonical”.

Our goal is now to analyze equation (16) in this context. To prepare this, we will start by reviewing the classification of $U(n)$– principal bundles over $\mathbb{R} \times S^4$ up to isomorphism and, in particular, the rôle played by $ch_2(\Omega)$ in this classification.

$\mathbb{R} \times S^4$ can be covered by $s_+$ (see above) and $s_-$; $s_-$ is defined like $s_+$ but using the opposite stereographic projection, i.e. the one with respect to $(0,0,0,0,-1) \in \mathbb{R}^5$, instead. Every smooth $U(n)$– principal bundle, $n \geq 2$, $\hat{\mathcal{P}}$ over $\mathbb{R} \times S^4$ is then isomorphic to

$$\left( \mathbb{R}_+^5 \times U(n) \cup \mathbb{R}_-^5 \times U(n) \right) / \sim \mapsto \left( \mathbb{R}_+^5 \cup \mathbb{R}_-^5 \right) / \sim \sim \mathbb{R} \times S^4 \quad (18)$$

for $x \in \mathbb{R}_+^5$ and $y \in \mathbb{R}_-^5$, the latter equivalence relation is simply that $x \sim y$ if and only if $x = s_+^{-1}(s_-(y))$. The former instead is that $(x,g) \sim (y,h)$ if and only if $x \sim y$ and $g = \nu_+(x)h$ with a smooth map $\nu_+ : \mathbb{R} \times (\mathbb{R}^4 - 0) \mapsto U(n)$.

The isomorphism classes are then in one to one correspondence with the homotopy classes of the maps $\nu_+$. Since $\mathbb{R} \times (\mathbb{R}^4 - 0)$ can be contracted onto $S^3$, the set of isomorphism classes can be identified with $\pi_3(U(n)) \sim \mathbb{Z}$. In particular, we can choose $\nu_+$ to be constant along the $x_0$-direction and along the radial direction in $\mathbb{R}^4 - 0$, i.e.

$$\nu_+ = \bar{\nu}_+ \circ \pi_{S^3}(\mathbb{R} \times (\mathbb{R}^4 - 0)) \quad (19)$$
with $\tilde{\nu}_{+-} : S^3 \mapsto S^3$ and $\pi_{S^3}^R \times (R^4 - 0) : R \times (R^4 - 0) \mapsto S^3$ is the projection along the radial and $x_0$-direction. From now on $\nu_{+-}$ will be of this type.

Pulling back the (positive) generator of $H^3(U(n), \mathbb{Z})$ with $\nu_{+-}$ one can identify the set of isomorphism classes also with $H^3(R \times (R^4 - 0), \mathbb{Z}) \sim H^3(S^3, \mathbb{Z}) \sim \mathbb{Z}$. Note that the inclusion $\mathbb{Z} \subset R$ induces an inclusion $H^3(R \times (R^4 - 0), \mathbb{Z}) \subset H^3(R \times (R^4 - 0), R)$, since $H^3(R \times (R^4 - 0), \mathbb{Z})$ is torsion free. The isomorphism classes can then be represented by the forms

$$\kappa \cdot \text{tr} \left( \phi \phi^\dagger \right)$$

(20)

here, $\phi = \nu_{+-}^{-1} d\nu_{+-}$ is the pullback of the Maurer–Cartan form of $U(n)$ by $\nu_{+-}$ and $\kappa$ is the usual normalization factor such that the cohomology class is in $H^3(R \times (R^4 - 0), \mathbb{Z}) \subset H^3(R \times (R^4 - 0), R)$.

Via the connecting homomorphism in the Mayer-Vietoris sequence for $R^5_+, R^5_-$ and $R \times S^4 \sim (R^5_+ \cup R^5_-) / \sim$ the set of isomorphism classes can also be identified with $H^4(R \times S^4, \mathbb{Z}) \sim H^4(S^4, \mathbb{Z}) \sim \mathbb{Z}$; $ch_2(\hat{P})$ can then be interpreted as the cohomology class which represents the isomorphism class of $\hat{P}$.

Again, the inclusion $\mathbb{Z} \subset R$ induces an inclusion $H^4(R \times S^4, \mathbb{Z}) \subset H^4(R \times S^4, R)$, since $H^4(R \times S^4, \mathbb{Z})$ is torsion free.

The numerical factor $\tau$ can be chosen such that the integral (17) takes integer values and since $[ch_2(\Omega)] = ch_2(\hat{P}) \in H^4(R \times S^4, R)$, the integral can then be viewed as the identification $H^4(R \times S^4, \mathbb{Z}) \sim \mathbb{Z}$. Thus, also the values of the integral (17) classify the $U(n)$– principal bundles over $R \times S^4$ up to isomorphism. This value is then identified as the baryon number and is the analog of the Instanton number of 4–dimensional Yang–Mills theory.

Regarding the physical meaning, for us the following fairly naive interpretation will suffice. Baryons are the particles of which matter is made. They are composed of three quarks “[...] and a “sea” of quark-antiquark pairs” ([53] p. 253). The respective antiparticles are antibaryons with three antiquarks instead of the three quarks. In physics, the baryon number is then defined as a third of the difference of the number of quarks and antiquarks. We won’t bother with specific particles and their names, but apparently it is a lot more complicated than just protons and neutrons ([53] chapter 16).

The solitons of Yang–Mills–Chern–Simons theories of 5-dimensional QCD are interpreted as (multi-) baryons or antibaryons and the value of the integral (17) indicates the number of baryons present if it is positive and the number of anti baryons if it is negative.
Technically, this is done by adapting the construction by Atiyah and Manton of deriving skyrmions in 3 dimensions from instantons in 4 dimensions by calculating the holonomy along lines parallel to the time axis. In this way the solitons in 5 dimensions produce “holographic” skyrmions, i.e. defined on the holographic boundary of the 5-dimensional theory mentioned at the beginning of section 5.2, and these holographic skyrmions are then the model for baryons in holographic QCD (see [61] for an extensive review of holographic skyrmions; see [6] and [42] for the original construction by Atiyah and Manton).

Our observations, however, do not touch upon this, but are only concerned with the existence of solutions for the 5-dimensional theory, i.e. we can and will continue to ignore the holographic boundary, albeit it is maybe the most important feature of holographic QCD.

That the baryon number be not only finite, but also conserved is a fundamental requisite of particle physics ([53] chapter 8.1). This is guaranteed by the fact that different baryon numbers belong to topologically non isomorphic principal bundles and there is no way to smoothly pass from one to the other. A way to meet these requirements without the asymptotic behaviour at infinity imposing this topologically non trivial situation is yet to be found.

Thus, mathematical consistency would require that only solitons on $\mathbb{R}^5$ which extend to solitons on $\mathbb{R} \times S^4$ are admissible for “producing” holographic skyrmions.

For the study of baryonic matter, of course, only the cases with non zero baryon number are relevant. Thus, $0 \neq ch_2(\hat{P}) \in H^4(\mathbb{R} \times S^4, \mathbb{R})$ and $\hat{P}$ is, therefore, non trivial.

As a special case of theorem 2 we know from equation (16) that solitons with non zero baryon number do not exist on a closed, oriented Pseudo-Riemannian 5-manifold of Lorentzian signature. We will now discuss what changes in the present setting where our YMCS-theory is defined on the non compact manifold $\mathbb{R}^5$ as a restriction of a YMCS-theory on $\mathbb{R} \times S^4$ via the embedding $s_+: \mathbb{R}^5 \hookrightarrow \mathbb{R} \times S^4$, see above.

Let then $\hat{\mathcal{P}}$ be a smooth $U(n)$– principal bundle, $n \geq 2$, over $\mathbb{R} \times S^4$ with $ch_2(\hat{\mathcal{P}})$ cohomologically non trivial and let $\Xi$ be the vertical vector field on $\mathcal{C}_{\mathcal{P}}$ corresponding to $dx_0 \otimes (-i \cdot 1_n)$. Equation (16) becomes then on the $\mathbb{R}^5 \subset \mathbb{R} \times S^4$
\[ dx_0 \wedge d \ast (\Omega_\sigma)_{u_1} = \frac{\kappa}{2\pi} \cdot \left( \frac{i}{2\pi} \right)^p \cdot dx_0 \wedge ch_2(\Omega_\sigma) \]  

(21)

For a \( k \)-form \( \alpha \) on \( \mathbb{R}^5 \) we define \( \alpha_x \) by \( \frac{\partial}{\partial x_0} \mid \alpha_x = 0 \) and \( dx_0 \wedge \alpha = dx_0 \wedge \alpha_x \).

We can then derive from equation (21)

\[ (d \ast (\Omega_0^0)_{u_1})_x = \frac{\kappa}{2\pi} \cdot \left( \frac{i}{2\pi} \right)^p \cdot ch_2(\Omega_\sigma)_x \]  

(22)

Let now \( \varepsilon_t : \mathbb{R}^4 \hookrightarrow \mathbb{R}^5 \) be the embedding which identifies \((x_1, x_2, x_3, z) \in \mathbb{R}^4 \) with \((t, x_1, x_2, x_3, z) \in \mathbb{R}^5 \). We have then

\[ \varepsilon^*_t (d \ast (\Omega_\sigma)_{u_1})_x = \varepsilon^*_t d \ast (\Omega_\sigma)_{u_1} = d (\varepsilon^*_t \ast (\Omega_\sigma)_{u_1}) = d (\varepsilon^*_t \ast (\Omega_\sigma)_{u_1})_x \]

and

\[ \varepsilon^*_t ch_2(\Omega_\sigma)_x = \varepsilon^*_t ch_2(\Omega_\sigma) \]

Thus on \( \mathbb{R}^4 \) equation (22) becomes

\[ d (\varepsilon^*_t \ast (\Omega_\sigma)_{u_1})_x = \frac{\kappa}{2\pi} \cdot \left( \frac{i}{2\pi} \right)^p \cdot \varepsilon^*_t ch_2(\Omega_\sigma) \]  

(23)

To have solutions of the Euler–Lagrange equations, it is necessary that equations (21), (22) and (23) are satisfied. If \( ch_2(\Omega_\sigma) \) is cohomologically non trivial, so is \( \varepsilon^*_t ch_2(\Omega_\sigma) \). By Equation (23) this means that \( d (\varepsilon^*_t \ast (\Omega_\sigma)_{u_1})_x \) has to extend to \( S^4 \), but \( \ast (\Omega_\sigma)_{u_1} \) must not, else \( d (\varepsilon^*_t \ast (\Omega_\sigma)_{u_1})_x \) would be cohomologically trivial on \( S^4 \).

By the definition of \( \varepsilon_t \) this means in turn that \( d \ast (\Omega_\sigma^0)_{u_1} \) needs to extend to \( \mathbb{R} \times S^4 \), but \( \ast (\Omega_\sigma)_{u_1} \) and, thus, \( \ast (\Omega_\sigma)_{u_1} \) must not.

We can draw now our first conclusions. \( SU(n) \)-theories do not admit solitons because the \( u_1 \)-component of the curvature of their fields vanishes identically (see section 5.1 above). For \( U(n) \)-theories instead, the existence of solitons would require that at least one between the metric \( \mu \) and \( (\Omega_\sigma)_{u_1} \) does not extend to \( \mathbb{R} \times S^4 \), because else \( \ast (\Omega_\sigma)_{u_1} \) would extend to \( \mathbb{R} \times S^4 \) as well.

In particular, the latter possibility would impose on the \( u_1 \)-component of the curvature an “asymptotic behaviour at infinity” qualitatively different
from the one of the $\mathfrak{su}_n$-component. Such a violation of the requirement that solitons extend to $\mathbb{R} \times S^4$ can hardly be acceptable.

To gain a more precise understanding of the situation, we start by analyzing the restriction $ch_2(\Omega)|_{\mathbb{R}^5}$. With our discussion of the classification of principal bundles over $\mathbb{R} \times S^4$ in mind we arrive then, e.g. from the construction of the connecting homomorphism in the Mayer–Vietoris sequence for de Rham cohomology (see e.g. \cite{11}), at the following relation between the forms (20) and the restriction $ch_2(\Omega)|_{\mathbb{R}^5}$:

\[
ch_2(\Omega)|_{\mathbb{R}^5} = \iota \cdot d \left( \beta + d\alpha + \frac{r^2}{r^2 + 1} \cdot \text{tr} \left( \phi[\phi, \phi] \right) \right) \tag{24}
\]

$\iota$ is some numerical factor and $r = \sqrt{x_1^2 + x_2^2 + x_3^2 + z^2}$; $\beta$ is a 3-form on $\mathbb{R}^5$ which decays faster than $\frac{1}{r}$ for $r \to \infty$ so that it extends to $\mathbb{R} \times S^4$; $\alpha$ is an arbitrary 2-form on $\mathbb{R}^5$.

By inserting equation (24) into equation (23) we can compare $(*(\Omega^0_\sigma)_{u_1})_x$ and the local potential of $ch_2(\Omega)|_{\mathbb{R}^5}$:

\[
d (\varepsilon^*_i (*(\Omega^0_\sigma)_{u_1})_x) = \frac{\kappa}{2\pi} \cdot \left( \frac{i}{2\pi} \right)^p \cdot \iota \cdot \varepsilon^*_i d \left( \beta + d\alpha + \frac{r^2}{r^2 + 1} \cdot \text{tr} \left( \phi[\phi, \phi] \right) \right) \]

\[
= \frac{\kappa}{2\pi} \cdot \left( \frac{i}{2\pi} \right)^p \cdot \iota \cdot d \left( \varepsilon^*_i \beta + \varepsilon^*_i d\alpha + \frac{r^2}{r^2 + 1} \cdot \text{tr} \left( \phi[\phi, \phi] \right) \right)_{x_0=1} \tag{25}
\]

From this we derive that for $r \to \infty$

\[
(*(\Omega^0_\sigma)_{u_1})_x \sim \delta \cdot tr \left( \tilde{\phi}[\tilde{\phi}, \tilde{\phi}] \right) + \lim_{r \to \infty} d\rho \tag{26}
\]

here $\tilde{\phi} = \tilde{\nu}^{-1}_+ \cdot d\nu$, see equation (19) above, $\delta$ is a numerical normalization factor and $\rho$ is some 2-form. Since we are only interested the effect of $[ch_2(\Omega)]$ being non zero, $d\rho$ is irrelevant; the relevant consequence of equation (26) is that some component of $(*(\Omega^0_\sigma)_{u_1})_x$ has to converge to $\delta \cdot tr \left( \tilde{\phi}[\tilde{\phi}, \tilde{\phi}] \right)$ for $r \to \infty$.

We will see now that in the case of the Sakai–Sugimoto model this means that $(\Omega^0_\sigma)_{u_1}$ cannot extend to $\mathbb{R} \times S^4$ and therefore is qualitatively different from the $\mathfrak{su}_n$-component.

The metric in the Sakai–Sugimoto model is

\[
\mu = \left( 1 + \frac{z^2}{L^2} \right)^\frac{4}{3} \cdot \left( -(dx_0)^2 + \sum_{i=1}^3 (dx_i)^2 \right) + \frac{1}{(1 + \frac{z^2}{L^2})^\frac{2}{3}} \cdot (dz)^2
\]
where $L$ is a positive constant which we normalize to 1. If we define the coefficients of $(\Omega_\sigma)_{u_1} - ((\Omega_\sigma)_{u_1})_x$ by

$$(\Omega_\sigma)_{u_1} - ((\Omega_\sigma)_{u_1})_x =$$

$$(\Omega_\sigma)_{u_1}^0 dx_0 \wedge dx_1 + (\Omega_\sigma)_{u_1}^2 dx_0 \wedge dx_2 + (\Omega_\sigma)_{u_1}^3 dx_0 \wedge dx_3 + (\Omega_\sigma)_{u_1}^\sigma dx_0 \wedge dz$$

(note that in the Ansatz of [10] we have $((\Omega_\sigma)_{u_1})_x = 0$), then we have

$$(\star(\Omega_\sigma)_{u_1})_x =$$

$$\frac{-1}{(1 + z^2)^{\frac{1}{2}}} ((\Omega_\sigma)_{u_1}^0 dx_2 \wedge dx_3 \wedge dz - (\Omega_\sigma)_{u_1}^2 dx_1 \wedge dx_3 \wedge dz + (\Omega_\sigma)_{u_1}^3 dx_1 \wedge dx_2 \wedge dz)$$

$$+ (1 + z^2)(\Omega_\sigma)_{u_1}^\sigma dx_1 \wedge dx_2 \wedge dx_3$$

Since $\nu_{+-}$ and $\tilde{\nu}_{+-}$ do not depend on $r$, see equation (19) above, $\tilde{\phi}$ and $\tilde{\phi} \cdot tr(\tilde{\phi}[\tilde{\phi}, \tilde{\phi}])$ do not depend on $r$.

Therefore, equations (25) and (26) imply that $(\Omega_\sigma)_{u_1}$ must contain components which do not depend on $r$. But, since $\frac{-1}{(1 + z^2)^{\frac{1}{2}}}$ depends on $r$ like $\frac{1}{r^\frac{3}{2}}$ for $r \to \infty$, this implies that the coefficients $(\Omega_\sigma)_{u_1}^i, i = 1, 2, 3$ need to have components which depend on $r$ like $r^\frac{3}{2}$ for $r \to \infty$.

But for $(\Omega_\sigma)_{u_1}$ to extend to $IR \times S^4$, it is instead necessary that the coefficients of $(\Omega_\sigma)_{u_1} - ((\Omega_\sigma)_{u_1})_x$ decrease faster than $\frac{1}{r}$ (note that $x_0$ does not depend on $r$). In view of this contradiction, we have

**Corollary 1** In the case of non zero Baryon number, the $u_1$-component of the curvature of a solution of the Euler–Lagrange equations of the the Sakai–Sugimoto model cannot extend to $IR \times S^4$.

This begs the question, if the set up of the Sakai–Sugimoto model is mathematically sufficiently consistent for a physical theory.

The consequences of a non extending metric, on the other hand, are rather more subtle: it might cause the breakdown of the Yang–Mills part of the theory, because its Langrangian and its Euler–Lagrange form might not extend as well. This is, however, by no means necessary: the metric of four dimensional Yang–Mills theory does not extend either, strictly speaking (it
is, of course, conformally related to a metric which extends and the Yang–Mills Lagrangian is in four dimensions invariant under conformal changes of the metric). Moreover, for holographic QCD the only relevant requirement is that the metric behaves well with the holographic boundary.

Thus, a non extending metric could be actually the means to construct a theory with solitons which show the same behaviour at infinity in all their components. As mentioned above, $S^4$ could be replaced by any closed, smooth 4–manifold $M$ and in the present situation this may very well be necessary. So we consider an embedding $\iota : \mathbb{R}^5 \hookrightarrow \mathbb{R} \times M$ mapping the $\mathbb{R}^4$ corresponding to $(x_1, x_2, x_3, z)$ diffeomorphically onto the single open $n$-cell of a suitable CW–decomposition of $M$.

We need then a YMCS theory on $\mathbb{R}^5$ with respect to a metric $\mu_{\mathbb{R}^5}$ which cannot extend to $\mathbb{R} \times M$ and a local variational problem $\lambda$ (see section 2.3.1) on $\mathbb{R} \times M$ such that for the Euler–Lagrange forms we have $\eta_{YMCS} = \eta_\lambda \circ \iota^*$. Any restriction to $\mathbb{R}^5$ of a soliton on $\mathbb{R} \times M$ will then be a soliton on $\mathbb{R}^5$ with consistent behaviour at infinity and being derived from a local variational problem should be enough to consider the restriction of this set up to $\mathbb{R}^5$ a bone fide Lagrangian field theory. Note that the Yang–Mills part of the Lagrangian YMCS theory on $\mathbb{R}^5$ cannot extend, because the metric $\mu_{\mathbb{R}^5}$ cannot extend.

This construction can, of course, be modified in various ways: for example, since the metric on $\mathbb{R}^5$ must not extend, one may replace $\mathbb{R} \times M$ with $s^1 \times M$ or even a more general compact 5-dimensional Manifold.

Note, however, that there is no guarantee that any of this can be actually carried out in practice.

Trying to modify the Sakai–Sugimoto model may not be the best starting point. It is derived from string theory by a rather involved procedure (see [55]). This appears to be its most important theoretical justification and, presumably, is the source of many desirable properties. Therefore, a serious reworking of the model would probably require to start again from string theory.

There are, however, other models for Yang–Mills–Chern–Simons theories of 5-dimensional QCD (see e.g. [9], [52]) which are quite similar, but the metrics of which might already suffice to guarantee the extendability of the $u_1$-component of the curvature. Unfortunately, the models of [9] and [52] are constructed on subsets of $\mathbb{R}^5$ defined by the “infrared” and “ultraviolet” cutoffs and the exposition leaves it somewhat obscure how they are supposed to be embedded into $\mathbb{R} \times S^4$ or some $\mathbb{R} \times M$; cf. the discussion of equation...
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