Graphs with Non-unique Decomposition and Their Associated Surfaces

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Abstract

The ideal (tagged resp.) triangulation of bounded surface with marked points are associated with skew-symmetric (skew-symmetrizable) exchange matrices. An algorithm is established to decompose the graph associated to such matrix. There are finite many graph with non-unique decomposition. We find all such graphs and their decompositions. In addition, we also find the associated ideal (tagged) triangulations to different decompositions.

1 Introduction

Triangulation is a useful tool to study the topology of surfaces. Ideal triangulation of bordered surfaces with marked points is of particular interests in cluster algebra. For example, in [?], the authors construct cluster algebra associated to an ideal triangulation.

Definition 1. We associate to each ideal triangulation $T$ the (generalized) signed adjacency matrix $B = B(T)$ that reflects the combinatorics of $T$. The rows and columns of $B(T)$ are naturally labeled by the arcs in $T$. For notational convenience, we arbitrarily label these arcs by the numbers $1, \ldots, n$, so that the rows and columns of $B(T)$ are numbered from 1 to $n$ as customary, with the understanding that this numbering of rows and columns is temporary rather than intrinsic. For an arc (labeled) $i$, let $\pi_T(i)$ denote (the label of) the arc defined as follows: if there is a self-folded ideal triangle in $T$ folded along $i$, then $\pi_T(i)$ is its remaining side (the enclosing loop); if there is no such triangle, set $\pi_T(i) = i$. For each ideal triangle $\triangle$ in $T$ which is not self-folded, define the $n \times n$ integer matrix $B_\triangle = (b_{ij}^\triangle)$ by settings:

$$b_{ij}^\triangle = \begin{cases} 1 & \text{if } \triangle \text{ has sides labeled } \pi_T(i) \text{ and } \pi_T(j) \\
 & \text{with } \pi_T(j) \text{ following } \pi_T(i) \text{ in the clockwise order;} \\
-1 & \text{if the same holds, with the counter-clockwise order;} \\
0 & \text{otherwise.}
\end{cases}$$

The matrix $B = B(T) = (b_{ij})$ is then defined by

$$B = \sum_\triangle B_\triangle$$
The sum is taken over all ideal triangles $\triangle$ in $T$ which are not self-folded. The $n \times n$ matrix $B$ is skew-symmetric, and all its entries $b_{ij}$ are equal to 0, 1, $-1, 2,$ or $-2$.

A quiver is defined as a finite oriented multi-graph without loops and 2-cycles.

**Definition 2.** Let $G$ be a quiver, $B(G) = (b_{ij})$ is the skew-symmetric matrix whose rows and columns are labeled by the vertices of $G$, and whose entry $b_{ij}$ is equal to the number of edges going from $i$ to $j$ minus the number of edges going from $j$ to $i$.

**Definition 3.** Suppose $B$ is a signed adjacency matrix associated to an ideal triangulation of a bordered surface with marked points $(S, M)$, and $G$ is a quiver. If $B(G) = B$, we say $G$ is the oriented adjacency graph associated to $(S, M)$.

The notion of *Block decomposition* plays an important role in determining the mutation class of a quiver. It is proved in [?] that a quiver is block-decomposable if and only if it is the associated adjacency graph of an ideal triangulations of a bordered surface with marked points. A quiver is a finite oriented multi-graph without loops and 2-cycles. In [?], we provide an algorithm that determines if a given quiver is block decomposable. In addition, we find all connected decomposable graphs with non-unique block-decomposition.

In [?], the authors generalize the property to the graph associated to ideal (tagged) triangulation of bordered surfaces with marked points. A new decomposability called *s-decomposable* is studied. It is proved in the same article that there is a one-to-one correspondence between s-decomposable skew-symmetrizable graphs with fixed block decomposition and ideal tagged triangulations of marked bordered surfaces with fixed tuple of conjugate pairs of edges. In [?], we provide a generalized algorithm that determines if a given graph is s-decomposable. In addition, we find that only two connected s-decomposable graphs that are not block-decomposable have non-unique decomposition.

## 2 Decomposition Rules and Blocks

For convenience, we denote an edge that connects nodes $x, y$ by $\overrightarrow{xy}$ if the orientation of this edge is unknown or irrelevant, $\overleftarrow{xy}$ if the edge is directed from $x$ to $y$, and $\overleftrightarrow{xy}$ otherwise.

**Definition 4.** We recall that a diagram (or graph) is block-decomposable (or decomposable) if it is obtained by gluing elementary blocks of Table[I]b by the following gluing rules:

1. Two white nodes of two different blocks can be identified. As a result, the graph becomes a union of two parts; the common node is colored black. A white node can neither be identified to itself nor with another node of the same block.

2. A black node can not be identified with any other node.

3. If two white nodes $x, y$ of one block (endpoints of edge $\overleftrightarrow{xy}$) are identified with two white nodes $p, q$ of another block (endpoints of edge $\overleftrightarrow{pq}$), $x$ with $p$, $y$ with $q$ correspondingly, then a multi-edge of weight 2 is formed, and nodes $x = p, y = q$ are black.
4. If two white nodes $x$, $y$ of one block (endpoints of edge $\hat{xy}$) are identified with two white nodes $p$, $q$ of another block (endpoints of edge $\hat{pq}$), $x$ with $q$, $y$ with $p$ correspondingly, then both edges are removed after gluing, and nodes $x = q$, $y = p$ are black.

**Definition 5.** If a graph $G$ can be obtained by gluing both elementary blocks and new blocks in Table 2 by the gluing rules in Definition 4 and the following new rules, we say the graph is $s$-decomposable:

1. If the graph has multiple edges containing $n$ parallel edges, replace the multiple edge by an edge of weight $2n$. For example, if we glue two parallel spikes of the same direction, we get an edge of weight 4 (see Figure 1).

   ![Figure 1: Edge of Weight 4](image1)

2. All single edges have weight 1.

   Gluing two blocks corresponding to gluing two pieces of triangulations of surfaces: gluing two white nodes means gluing the corresponding sides of the triangulations, (see Figure 2).

   ![Figure 2: Triangulation Gluing](image2)

   If a decomposable graph has a white node, we will glue a particular piece surface to that node in the corresponding triangulation to form the boundary, see Figure 3.

   ![Figure 3: Boundary Gluing](image3)
It is shown in [?] that there is a one-to-one correspondence between a decomposition of a graph and an ideal triangulation of a bordered surfaces with marked points. We show in next section that most graphs with non-unique decomposition correspond to unique bordered surfaces.

3 Results

All graphs with non-unique decompositions (s-decompositions) are given in Figure. 78 in [?] and Figure. 4 in [?]. We list all their block decomposition (s-decomposition) and corresponding ideal (tagged) triangulation of surfaces.

**Theorem 1.** If \( G \) is a decomposable or s-decomposable graph, \( G \) is associated to a unique bordered surface unless \( G \) is graph 5.
Table 1: Elementary Blocks
| New Blocks | Unfolding | Triangulation |
|------------|-----------|---------------|
| Ia:        | ![Diagram](image1) | ![Diagram](image2) |
| Ib:        | ![Diagram](image3) | ![Diagram](image4) |
| II:        | ![Diagram](image5) | ![Diagram](image6) |
| IIIa:      | ![Diagram](image7) | ![Diagram](image8) |
| IIIb:      | ![Diagram](image9) | ![Diagram](image10) |
| IV:        | ![Diagram](image11) | ![Diagram](image12) |
| V:         | ![Diagram](image13) | ![Diagram](image14) |
Graph 1

Decomposition

Surfaces
| Graph 2 |
|-------------------|
| Decomposition |
| Surfaces |

| |  |  |  |
|---|---|---|---|
| 1 | 1 | 2 | 2 |
| 3 | 4 | 3 | 4 |

| |  |  |  |
|---|---|---|---|
| p | p | p | p |

| |  |  |  |
|---|---|---|---|
| p | p | p | p |

| |  |  |  |
|---|---|---|---|
| 1 | 2 | 2 | 1 |
| 4 | 3 | 4 | 3 |

| |  |  |  |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 3 | 4 | 3 | 4 |

| |  |  |  |
|---|---|---|---|
| p | p | p | p |

| |  |  |  |
|---|---|---|---|
| p | p | p | p |

| |  |  |  |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 3 | 4 | 3 | 4 |

| |  |  |  |
|---|---|---|---|
| p | p | p | p |

| |  |  |  |
|---|---|---|---|
| p | p | p | p |

| |  |  |  |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 3 | 4 | 3 | 4 |
| Surface | Graph 3 |
|---------|---------|
| **Decomposition** | ![Decomposition Diagram](image) | ![Decomposition Diagram](image) |
| **Surfaces** | ![Surfaces Diagram](image) | ![Surfaces Diagram](image) |
| Graph 4 | Decomposition | Surfaces |
|--------|----------------|----------|
| ![Graph 4](image1.png) | ![Decomposition](image2.png) | ![Surfaces](image3.png) |
| Graph 5          | Decomposition | Surfaces  |
|-----------------|---------------|-----------|
| ![Graph 5](image) | ![Decomposition](image) | ![Surfaces](image) |
| Graph 6  | Decomposition | Surfaces    |
|---------|---------------|------------|
| ![Graph 6](image1.png) | ![Decomposition](image2.png) | ![Surfaces](image3.png) |
| Graph 7       | Decomposition | Surfaces     |
|--------------|---------------|--------------|
| ![Graph](attachment:image.png) | ![Decomposition](attachment:image.png) | ![Surfaces](attachment:image.png) |
| Decomposition | Surfaces |
|---------------|---------|
| ![Decomposition Diagram](graph1.png) | ![Surfaces Diagram](graph2.png) |

**Graph 7**

| Decomposition | Surfaces |
|---------------|---------|
| ![Decomposition Diagram](graph3.png) | ![Surfaces Diagram](graph4.png) |
| Graph 8 |
|--------|
| ![Graph 8 Diagram] |

| Decomposition |
|---------------|
| ![Decomposition Diagram] |

| Surfaces |
|----------|
| ![Surfaces Diagram] |
|   | Graph 9 |   |
|---|---------|---|
| Decomposition |   |   |
| Surfaces |   |   |
| Graph 10 | Decomposition | Surfaces |
|----------|----------------|----------|
| ![Graph 10](image1.png) | ![Decomposition](image2.png) | ![Surfaces](image3.png) |
| Graph 11 |
|---|
| Decomposition |
| Surfaces |

| | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 2 |
| 3 |
| 1 |

| | 1 | 2 |
|---|---|---|
| 2 | 3 |
| 1 |

| | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 2 |
| 3 |

| | 1 | 2 |
|---|---|---|
| 2 | 3 |
| 1 |
| Graph 12   | Decomposition | Surfaces     |
|-----------|---------------|--------------|
| ![Graph](#) | ![Decomposition](#) | ![Surfaces](#) |
| Graph 13 | Decomposition | Surfaces |
|---------|---------------|----------|
| ![Graph 13 Diagram](image1.png) | ![Decomposition Diagram](image2.png) | ![Surfaces Diagram](image3.png) |
| Graph 14 |  
|----------|
| ![Graph 14](image1.png) |

| Decomposition |  
|----------------|
| ![Decomposition](image2.png) |

| Surfaces |  
|----------|
| ![Surfaces](image3.png) |
| Graph 15 | Decomposition | Surfaces |
|---------|---------------|---------|
| ![Graph 15](image1.png) | ![Decomposition](image2.png) | ![Surfaces](image3.png) |
| Graph 16 |  
| --- |  
| **Decomposition** |  
| Surfaces |  
| Graph 17 |  
| **Decomposition** |  
| Surfaces |  

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