Algebraic constructions on the Nearly Frobenius category

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Abstract

In this article we determine the Frobenius dimension of the product and tensor product of two nearly Frobenius algebras from the Frobenius dimension of each of them. Also we prove that one of the conditions required in the original definition of nearly Frobenius algebra, the coassociativity, is redundant. We applying these results to semisimple algebras.

Keywords: nearly Frobenius algebras.

MSC: 16W99

1 Introduction

The concept of nearly Frobenius algebra is motivated by the result proved in [3], which states that: the homology of the free loop space $H^\ast(LM)$ has the structure of a Frobenius algebra without counit. These objects were studied in [4] and their algebraic properties were developed in [2], in particular the possible nearly Frobenius structures in gentle algebras were described.

In the framework of differential graded algebras, Abbaspour considers in [1] nearly Frobenius algebras that he calls open Frobenius algebras. He proves that if $A$ is a symmetric open Frobenius algebra of degree $m$, then $HH_\ast(A,A)[m]$ is an open Frobenius algebra.

In this work we determine the Frobenius dimension of the product and tensor product of two nearly Frobenius algebras from the Frobenius dimension of each of them. Also we prove that one of the conditions required in the original definition of nearly Frobenius algebra, the coassociativity, is redundant. This applies to the definition of Frobenius algebras too.

2 Nearly Frobenius algebras

In one of the definitions of Frobenius algebras it is required that the algebra $A$ admits a coalgebra structure $(A,\Delta,\varepsilon)$ where the coproduct $\Delta$ is a morphism of $A$-bimodules. In the next result we prove that the coassociativity condition is redundant.

**Proposition 1.** Let $A$ be a Frobenius algebra, then the coassociativity condition is a consequence of the $A$-bimodule morphism condition of $\Delta$, and the unit axiom.
Proof. In the next diagram we illustrate this affirmation.

\[
\begin{array}{c}
\Delta \\
\downarrow \\
A \otimes A \\
\downarrow \\
A \otimes A
\end{array}
\]

All the internal diagrams commute as a consequence of the \(A\)-bimodule condition, the unit axiom and the natural decomposition of the morphism \(\Delta \otimes \Delta\); then the external diagram commutes too.

The previous result allows us to give the next alternative definition of nearly Frobenius algebras.

Definition 2. An algebra \(A\) is a nearly Frobenius algebra if it admits a linear map \(\Delta : A \to A \otimes A\) such that

\[
\begin{array}{c}
A \otimes A \\
\downarrow \\
A \\
\downarrow \\
A \otimes A
\end{array}
\]

commute.

Definition 3. The Frobenius space associated to an algebra \(A\) is the vector space of all the possible coproducts \(\Delta\) that make it into a nearly Frobenius algebra \((\mathcal{E})\), see \([2]\). Its dimension over \(k\) is called the Frobenius dimension of \(A\), that is,

\[
\text{Frobdim } A = \dim_k \mathcal{E}.
\]

Definition 4. Let \((A, \Delta_A)\) and \((B, \Delta_B)\) be two nearly Frobenius algebras. A homomorphism \(f : A \to B\) is a nearly Frobenius homomorphism if it is a morphism of algebras and the following diagram commutes.

\[
\begin{array}{c}
A \\
\Delta_A \\
A \otimes A \\
\downarrow f \otimes f \\
B \otimes B
\end{array}
\]

If \(f\) is bijective then \(f\) is said to be an isomorphism between \(A\) and \(B\).

Notation: \(\text{nFrob}\) is the category of nearly Frobenius algebras.

Theorem 5. Let \((A, \Delta_A)\) be a nearly Frobenius algebra, \(B\) an algebra and \(f : A \to B\) an isomorphism of algebras. Then \(B\) admits a nearly Frobenius structure defined as

\[
\Delta_B = (f \otimes f) \circ \Delta_A \circ f^{-1}.
\]

In particular \(\text{Frobdim } A = \text{Frobdim } B\).
Proof. We need to check that $\Delta_B$ is a $B$-bimodule morphism. That is,

\[
\begin{array}{ccc}
B \otimes B & \xrightarrow{m} & B \\
\downarrow{\Delta_B \otimes 1} & & \downarrow{\Delta_B} \\
B \otimes B \otimes B & \xrightarrow{1 \otimes m} & B \otimes B \\
\end{array}
\]

commute. To prove this we only need to see that the dotted face of the next cube commutes.

Since $f$ is an isomorphism of algebras and $\Delta_A$ is a nearly Frobenius coproduct in $A$ all the other faces commute and then the dotted face commutes. \qed

Remark 6. Assume that $A_1$ and $A_2$ are $k$-algebras. The product of the algebras $A_1$ and $A_2$ is the algebra $A = A_1 \times A_2$ with the addition and the multiplication given by the formulas $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$ and $(a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 b_2)$, where $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$. The identity of $A$ is the element $1 = (1_{A_1}, 1_{A_2}) = e_1 + e_2 \in A_1 \times A_2$, where $e_1 = (1_{A_1}, 0)$ and $e_2 = (0, 1_{A_2})$. If $(A_1, \Delta_1)$ and $(A_2, \Delta_2)$ are nearly Frobenius algebras then $A$ admits a natural structure of Nearly Frobenius algebra. In the next paragraph we describe this structure.

First, we define $\Delta(e_1) = \sum (a_1, 0) \otimes (a_2, 0)$, where $\Delta_1(1_{A_1}) = \sum a_1 \otimes a_2$ and $\Delta(e_2) = \sum (0, b_1) \otimes (0, b_2)$, where $\Delta_2(1_{A_2}) = \sum b_1 \otimes b_2$. Then

\[
\Delta(1) = \sum (a_1, 0) \otimes (a_2, 0) + \sum (0, b_1) \otimes (0, b_2) \in A \otimes A.
\]

To prove that this defines a bimodule morphism it is necessary to guarantee that $\Delta(1)$ satisfies that

\[ (c \otimes 1) \Delta(1) = \Delta(1)(1 \otimes c), \quad \forall \ c \in A. \]

Denote $c = (c_1, c_2) \in A$, then

\[
(c \otimes 1) \Delta(1) = (c_1, c_2) \otimes (1, 1) \left[ \sum (a_1, 0) \otimes (a_2, 0) + \sum (0, b_1) \otimes (0, b_2) \right]
= \sum ((c_1, c_2) \otimes (1, 1)) ((a_1, 0) \otimes (a_2, 0)) + \sum ((c_1, c_2) \otimes (1, 1)) ((0, b_1) \otimes (0, b_2))
= \sum (c_1 a_1, 0) \otimes (a_2, 0) + \sum (0, c_2 b_1) \otimes (0, b_2).
\]

On the other hand

\[
\Delta(1)(1 \otimes c) = \left[ \sum (a_1, 0) \otimes (a_2, 0) + \sum (0, b_1) \otimes (0, b_2) \right] ((1, 1) \otimes (c_1, c_2))
= \sum ((a_1, 0) \otimes (a_2, 0)) ((1, 1) \otimes (c_1, c_2)) + \sum ((0, b_1) \otimes (0, b_2)) ((1, 1) \otimes (c_1, c_2))
= \sum (a_1, 0) \otimes (a_2 c_1, 0) + \sum (0, b_1) \otimes (0, b_2 c_2).
\]
Remember that $\Delta_{A_1}$ and $\Delta_{A_2}$ are bimodule morphisms, then

$$(c_1 \otimes 1)\Delta_{A_1}(1_{A_1}) = \sum c_1a_1 \otimes a_2 = \sum a_1 \otimes a_2c_1 = \Delta_{A_1}(1_{A_2})(1 \otimes c_1)$$

and

$$(c_2 \otimes 1)\Delta_{A_2}(1_{A_2}) = \sum c_2b_1 \otimes b_2 = \sum b_1 \otimes b_2c_2 = \Delta_{A_2}(1_{A_1})(1 \otimes c_2)$$

This proves that $(c \otimes 1)\Delta(1) = \Delta(1)(1 \otimes c)$. Then $A$ is a nearly Frobenius algebra.

Remark 7. Similarly, we can consider the tensor product $A \otimes B$ of the $k$-algebras $A$ and $B$. As before, we can define a nearly Frobenius coproduct on $A \otimes B$. In this case we take the transposition map $\tau : (A \otimes A) \otimes (B \otimes B) \rightarrow (A \otimes B) \otimes (A \otimes B)$ and the coproduct on $A$ and $B$ to define the coproduct on $A \otimes B$ as follows

$$\Delta := \tau \circ \Delta_A \otimes \Delta_B : A \otimes B \rightarrow (A \otimes B) \otimes (A \otimes B).$$

Since all the maps are linear, the map $\Delta$ is linear too. We will test only one of the two necessary conditions to guarantee that it is bimodule morphism, the other one is analogous.

Proposition 8. Consider $A$ and $B$ two $k$-algebras, then the following isomorphisms of vector spaces hold:

1. $E_{A \times B} \cong E_A \times E_B$. In particular $\text{Frobdim}(A \times B) = \text{Frobdim}(A) + \text{Frobdim}(B)$.

2. $E_{A \otimes B} \cong E_A \otimes E_B$. Therefore $\text{Frobdim}(A \otimes B) = \text{Frobdim}(A) \cdot \text{Frobdim}(B)$.

Proof. In Remark 6 we saw that there exist natural inclusions of $E_A \times E_B$ in $E_{A \times B}$ and, in Remark 7 of $E_A \otimes E_B$ in $E_{A \otimes B}$.

To finish the proof it is necessary to check that the maps are surjective.

1. We note the unit of $A \times B$ as $1 = e_1 + e_2$, where $e_1 = (1_A, 0)$ and $e_2 = (0, 1_B)$.

Let’s take $\Delta \in E_{A \times B}$ and express $\Delta(1)$ as follows:

$$\Delta(1) = \sum_{i,j} (\eta_i, \xi_i) \otimes (\rho_j, \nu_j)$$
with \( \eta_i, \rho_j \in \mathcal{A} \) and \( \xi_i, \nu_j \in \mathcal{B} \) for all \( i, j \). Since \( \Delta \) is a bimodule morphism we can prove that 
\[ \Delta(e_1) = \sum_{i,j}(\eta_i, 0) \otimes (\rho_j, 0) \]
and, in a similar way, that 
\[ \Delta(e_2) = \sum_{i,j}(0, \xi_i) \otimes (0, \nu_j). \]
Then, we conclude that the coproduct has the expression 
\[ \Delta(1) = \Delta(e_1) + \Delta(e_2) = \sum_{i,j}(\eta_i, 0) \otimes (\rho_j, 0) + \sum_{i,j}(0, \xi_i) \otimes (0, \nu_j). \]
This allows us to define \( \Delta_A(1_A) = \sum_{i,j} a_{ij} x_i \otimes y_j \) and, in a similar way, that 
\[ \Delta_B(1_B) = \sum_{i,j} b_{ij} y_i \otimes x_j. \]
Using again that \( \Delta \) is a bimodule morphism, we deduce that \( \Delta_A \) and \( \Delta_B \) are also bimodule morphisms, then 
\( (\mathcal{A}, \Delta_A) \) and \( (\mathcal{B}, \Delta_B) \) are nearly Frobenius algebras. In particular \( \Delta = \iota \circ (\Delta_A + \Delta_B). \)

2. Consider \( \Delta \in \mathcal{E}_{A \otimes B} \) and \( \{x_i\}, \{y_j\} \) bases of \( \mathcal{A} \) and \( \mathcal{B} \) respectively, where \( x_1 = 1_A \) and \( y_1 = 1_B \). Then 
\[ \Delta(1_A \otimes 1_B) = \sum_{i,j,k,l} a_{ijkl} x_i y_j x_k y_l, \]
where \( a_{ijkl} \in \mathbb{k} \).
Using that \( \Delta \) is bimodule morphism we have that 
\[ \Delta(x \otimes 1_B) = \sum_{i,j,k,l} a_{ijkl} x_i y_j x_k y_l = \sum_{i,j,k,l} a_{ijkl} x_i y_j x_k y_l. \]
As \( y_j = y_l = 1_B \) when \( j = l = 1 \) we can define 
\[ \Delta_A(1_A) = \sum_{i,k} a_{ik} x_i \otimes x_k, \]
analogously we can define 
\[ \Delta_B(1_B) = \sum_{j,l} b_{jl} y_j \otimes y_l. \]
Note that with these definitions the coproduct \( \Delta \) is 
\[ \Delta = (1 \otimes \tau \otimes 1) \circ (\Delta_A \otimes \Delta_B). \]

The next corollary is a consequence of Theorem 5 and Proposition 8.

**Corollary 9.** If \( A \) is a semisimple algebra over an algebraically closed field \( k \), then it is possible to determine completely its Frobenius dimension.

**Proof.** By the Artin-Wedderburn Theorem we have that \( A \cong M_{n_1}(k) \times \cdots \times M_{n_r}(k) \).
Then 
\[ \text{Frobdim}(A) = \text{Frobdim}(M_{n_1}(k) \times \cdots \times M_{n_r}(k)) = \sum_{i=1}^{r} \text{Frobdim}(M_{n_i}(k)) = \sum_{i=1}^{r} n_i^2. \]
Finally 
\[ \text{Frobdim}(A) = \sum_{i=1}^{r} n_i^2. \]

\[ \square \]
Corollary 10. Let $G$ be a finite group. If $\text{char}(k)$ does not divide the order of $G$ and $k$ is an algebraically closed field, then it is possible to determine completely the Frobenius dimension of $kG$.

Proof. Applying Maschke’s theorem we have that $kG$ is a semisimple algebra then, by the previous corollary, it is possible to determine completely its Frobenius dimension. \hfill \Box

In the next results we are going to use an example presented in [2], which has a small error in its calculation. We shall now present the result quoted and its correction.

Let $G$ be a cyclic finite group of order $n$ and the group algebra $kG$, with the natural basis $\{g^i : i = 1, \ldots, n\}$. This algebra is a nearly Frobenius algebra. Moreover, we can determine all the nearly Frobenius structures that it admits.

Using the bimodule condition of the coproduct, we can prove that a basis of the Frobenius space is

$$B = \{ \Delta_k : kG \to kG \otimes kG : k \in \mathbb{1}, \ldots, n \},$$

where $\Delta_1(1) = \sum_{i=1}^{n} g^i \otimes g^{n+1-i}$ and $\Delta_k(1) = \sum_{i=1}^{k-1} g^i \otimes g^{k-i} + \sum_{i=k}^{n} g^i \otimes g^{n+k-i}$ for $k = 2, \ldots, n$.

In particular, we have that \( \text{Frobdim}(kG) = |G| \).

The general expression of any nearly Frobenius coproduct in the unit is

$$\Delta(1) = a_1 \sum_{i=1}^{n} g^i \otimes g^{n+1-i} + \sum_{k=2}^{p} a_k \left( \sum_{i=1}^{k-1} g^i \otimes g^{k-i} + \sum_{i=k}^{n} g^i \otimes g^{n+k-i} \right),$$

where $a_i \in k$ for $i = 1, \ldots, n$.

Corollary 11. If $G$ is a finite abelian group, then it is possible to determine $\text{Frobdim}(kG)$.

Proof. If $G$ is a finite abelian group, then $G = G_1 \oplus G_2 \oplus \cdots \oplus G_p$, where $G_i$ is a finite cyclic group for $i \in \{1, \ldots, p\}$. The group algebra $kG$ is isomorphic, as a $k$-algebra, to $kG_1 \otimes kG_2 \otimes \cdots \otimes kG_p$. Therefore, applying Theorem 10 and Proposition 8,

$$\text{Frobdim}(kG) = \prod_{i=1}^{p} \text{Frobdim}(kG_i) = \prod_{i=1}^{p} |G_i| \cdot$$

Finally,

$$\text{Frobdim}(kG) = \prod_{i=1}^{p} |G_i| \cdot \hfill \Box

Examples 2.1. We illustrate the results given in Proposition 8 with a couple of examples.

1. Let’s consider the cyclic groups $G$ and $H$ where $|G| = 2$ and $|H| = 3$ and their corresponding group algebras $A_1 = kG$, $A_2 = kH$. Then, by Proposition 8, $B = A_1 \times A_2$ is a nearly Frobenius algebra of Frobenius dimension 5.

$$E_{A_1} = \text{span}_k \{ \Delta_1^1, \Delta_2^1 \}.$$
Consider the linear quiver $Q$. It is known that $E : A \rightarrow A \otimes A$, defined as $\Delta(1) = \eta \otimes e_1 + e_2 \otimes \eta$. On the other hand, if we consider the next quiver $C$. This algebra admits only one coproduct, and it is isomorphic to $B$, where $A$ is a vector space of dimension 1, and a generator is the coproduct $\Delta(1) = \eta \otimes e_1 + e_2 \otimes \eta$. Then, the general expression of any nearly Frobenius coproduct in the unit is $\Delta = \eta \otimes e_1 + e_2 \otimes \eta$. The isomorphism $\phi : B \rightarrow C$, defined as $\phi(e_1 \otimes e_1) = e_1, \phi(e_1 \otimes e_2) = e_2, \phi(e_2 \otimes e_1) = e_3, \phi(e_2 \otimes e_2) = e_4, \phi(e_1 \otimes \eta) = \alpha, \phi(\eta \otimes e_2) = \beta, \phi(\eta \otimes e_1) = \gamma, \phi(e_2 \otimes \eta) = \delta, \phi(\eta \otimes \eta) = \alpha \beta$. It is clear that the isomorphism $\phi$ respects the algebra structures. Then, we can conclude that $E_C$ has dimension one and a generator is $\Delta(1) = \alpha \beta \otimes e_1 + \beta \otimes \alpha + \delta \otimes \gamma + e_4 \otimes \alpha \beta$. 

2. Consider the linear quiver $Q : \bullet \xrightarrow{\eta} \bullet$ and its associated path algebra: $A = kQ = \langle e_1, e_2, \eta \rangle$. It is known that $E_A$ is a vector space of dimension 1, and a generator is the coproduct $\Delta : A \rightarrow A \otimes A$ defined as $\Delta(1) = \eta \otimes e_1 + e_2 \otimes \eta$. Now we will construct the tensor product of two copies of $A$: $B = A \otimes A = \langle e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2, e_1 \otimes \eta, e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes \eta, e_2 \otimes \eta \rangle$. This algebra admits only one coproduct, and it is $\Delta = (1 \otimes \tau \otimes 1) \circ (\Delta \otimes \Delta)$. On the other hand, if we consider the next quiver $\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\gamma} \bullet \xrightarrow{\delta} \bullet$, and the algebra $C = \frac{kQ}{\langle \alpha \beta - \gamma \delta \rangle} = \langle e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \Delta, \alpha \beta \rangle$, we can prove that this algebra is isomorphic to $B$. The isomorphism given on the basis is as follows:

$$
\varphi : B \rightarrow C
$$

$$
\varphi(e_1 \otimes e_1) = e_1, \quad \varphi(e_1 \otimes e_2) = e_2, \quad \varphi(e_2 \otimes e_1) = e_3, \quad \varphi(e_2 \otimes e_2) = e_4
$$

$$
\varphi(e_1 \otimes \eta) = \alpha, \quad \varphi(\eta \otimes e_2) = \beta, \quad \varphi(\eta \otimes e_1) = \gamma, \quad \varphi(e_2 \otimes \eta) = \delta
$$

$$
\varphi(\eta \otimes \eta) = \alpha \beta.
$$

It is clear that the isomorphism $\varphi$ respects the algebra structures. Then, we can conclude that $E_C$ has dimension one and a generator is $\Delta(1) = \alpha \beta \otimes e_1 + \beta \otimes \alpha + \delta \otimes \gamma + e_4 \otimes \alpha \beta$. 

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Remark 12. In these lines we want to make notice that we cannot establish a nice property that relates the Frobenius dimension of a quotient algebra with the original algebra. First, in the Example 7 of [2] a nontrivial coproduct is constructed in the quotient algebra $A/J$ from a nontrivial structure on $A$, but $\text{Frobdim}(A) = 1$ and $\text{Frobdim}(A/J) = 3$.

In addition, we can not always recover a nontrivial structure on the quotient from one on the original algebra, for example if we consider $A = k\mathbb{A}^4 = \langle e_1, e_2, e_3, e_4, \alpha, \beta, \gamma \rangle$ with all the arrows having the same orientation and the radical square zero algebra $B = k\mathbb{A}^4/I$, we know that $A$ admits only one nontrivial nearly Frobenius coproduct, that is

$$
\Delta(e_1) = \alpha \beta \gamma \otimes e_1, \Delta(e_2) = \beta \gamma \otimes \alpha, \Delta(e_3) = \gamma \otimes \alpha \beta, \Delta(e_4) = e_4 \otimes \alpha \beta \gamma,
$$

and this structure is trivial in $B$. But we can prove that $B$ admits nontrivial nearly Frobenius coproducts, moreover $\text{Frobdim} B = 5$.

References

[1] H. Abbaspour. On the Hochschild homology of open Frobenius algebras. J. Noncommut. Geom., 10(2) pp. 709-743 (2016).

[2] D. Artenstein, A. González, and M. Lanzilotta, Constructing Nearly Frobenius Algebras, Algebras and Representation Theory (2015) Volume 18, 339-367.

[3] Ralph L. Cohen and Véronique Godin, A polarized view of string topology, Topology, geometry and quantum field theory, London Math. Soc. Lecture Note Ser., vol. 308, Cambridge Univ. Press, Cambridge, 2004, pp. 127-154.

[4] A. González, E. Lupercio, C. Segovia and B. Uribe Orbifold Topological Quantum Field Theories in Dimension 2. Preprint.