Implicit One-Step Legendre Polynomial Hybrid Block Method for the Solution of First-Order Stiff Differential Equations

J. Sunday¹*, Y. Skwame¹ and I. U. Huoma²

¹Department of Mathematics, Adamawa State University, Mubi, Nigeria.
²Department of Mathematics, University of Nigeria, Nsukka, Nigeria.

Abstract

We formulate an implicit hybrid block method for the numerical solution of stiff first-order Ordinary Differential Equations (ODEs) using the Legendre polynomial as our basis function via interpolation and collocation techniques. The paper further investigates the basic properties of the implicit hybrid block method and found it to be zero-stable, consistent and convergent. The method was also tested on some sampled stiff problems and found to perform better than some existing ones with which we compared our results.

Keywords: Hybrid; implicit; Legendre Polynomial; one-step; Stiff.

AMS Subject Classification (2010): 65L05, 65L06, 65D30.

1 Introduction

In this paper, we are concerned with the solution of stiff first-order differential equations of the form,

\[ y' = f(x, y), \quad y(a) = \eta, \quad x \in [a, b] \] (1)

*Corresponding author: joshuasunday2000@yahoo.com;
where $f$ is assumed to be Lipchitz continuous in $y$ and $\eta$ is a given initial value.

Different methods have been proposed for the solution of (1) ranging from predictor-corrector methods to hybrid methods. Despite the success recorded by the predictor-corrector methods, its major setback is that the predictors are in reducing order of accuracy especially when the value of the step-length is high and moreover the results are at overlapping interval [2]. Hybrid methods have advantage of incorporating function evaluation at off-step points which affords the opportunity of circumventing the ‘the Dahlquist zero stability barrier’ and it is actually possible to obtain convergent k-step methods with order $2k + 1$ up to $k = 7$, [3]. The method is also useful in reducing the step number of a method and still remains zero-stable, [1].

Stiff differential equations were first encountered in the study of the motion of springs varying stiffness, from which the problem derives its name. Stiffness occurs when some components of the solution decay much more rapidly than others. These problems are of frequent occurrence in the mathematical formulation of physical situations in control theory and mass action kinetics, where processes with widely varying time constants are usually encountered. Historically, two chemical engineers, Curtis and Hirschfelder in 1952 proposed the first set of numerical integration formulas (both implicit backward differentiation formulas) that are well suited for stiff differential equations. Stiff equations pose stability problems for most numerical integrators, [4].

**Definition 1** [5]: Legendre polynomials $y^m_n(x)$ defined on $[-1,1]$ are given by,

$$y^m_n(x) = \sum_{m=0}^{n} (-1)^{m} \frac{(2n-2m)!x^{2-2m}}{2^m m!(n-m)!(n-2m)!}$$

where $M = n/2$ or $(n-1)/2$ whichever is an integer.

In particular, $y^0_0(x) = 1$, $y^1_1(x) = x$, $y^2_2(x) = (3x^2 - 1)/2$, ... The polynomials satisfy the following properties;

- $y^m_n(x)$ is an even polynomial if $n$ is even and an odd polynomial if $n$ is odd,
- $y^m_n(x)$ are orthogonal polynomials and satisfy

$$\int_{-1}^{1} y^m_n(x)y^n_m(x)dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$$

- $y^m_n(-x) = (-1)^n y^n_m(x)$

**Definition 2** [6]: A differential equation is said to be stiff if $Re(\lambda_i) < 0, i = 1(1)m$, where $\lambda$ is the eigen value of the differential equation.

Many scholars have proposed various forms of methods for the solution of (1) by adopting power series, Chebyshev and Lagrange polynomials as basis functions. In this paper, we develop an implicit hybrid block method that gives better stability condition by using Legendre polynomial as our basis function.
2 Formulation of the Implicit One-Step Hybrid Block Method

We consider the first six terms of the Legendre polynomial as our basis function. This is given by,

\[(n+1)y_{n+1}(x) = y_0(x) + y_1(x) + \sum_{\eta=1}^{4} \left[ (2n+1)xy_\eta(x) - ny_{\eta-1}(x) \right] \]  \hspace{1cm} (3)

Interpolating (3) at point \(x_{n+s}, s = 0\) and collocating its first derivative at points \(x_{n+r}, r = 0 \left(\frac{1}{4}\right) 1\), where \(s\) and \(r\) are the numbers of interpolation and collocation points respectively, leads to the following system of equations,

\[XA = U\]  \hspace{1cm} (4)

where

\[A = [a_0, a_1, a_2, a_3, a_4, a_5]^T\]

\[U = [y_n, f_n, f_{n+1}, f_{n+\frac{1}{2}}, f_{n+\frac{3}{4}}, f_{n+1}]^T\]

and

\[
X = \begin{bmatrix}
7 & 11x_n & -18x_n^2 & -50x_n^3 & 35x_n^4 & 63x_n^5 \\
0 & 11 & -36x_n & -150x_n^2 & 140x_n^3 & 315x_n^4 \\
0 & 11 & -36x_n & -150x_n^2 & 140x_n^3 & 315x_n^4 \\
0 & 11 & -36x_n & -150x_n^2 & 140x_n^3 & 315x_n^4 \\
0 & 11 & -36x_n & -150x_n^2 & 140x_n^3 & 315x_n^4 \\
0 & 11 & -36x_n & -150x_n^2 & 140x_n^3 & 315x_n^4 \\
\end{bmatrix}
\]

Solving (4), for \(a_j, j = 0(1)5\) and substituting back into (3) gives a continuous linear multistep method of the form,

\[y(x) = a_0(x)y_n + h \sum_{j=0}^{s} \beta_j(x)f_{n+j}, \quad j = 0 \left(\frac{1}{4}\right) 1\]  \hspace{1cm} (5)

where the coefficients of \(y_n\) and \(f_{n+j}\) are given by,
\[
\begin{align*}
\alpha_0(t) &= 1 \\
\beta_0(t) &= -\frac{1}{90}(192t^4 - 600t^3 + 700t^2 - 375t^2 + 90t) \\
\beta_\frac{\pi}{4}(t) &= \frac{1}{45}(-384t^4 + 1080t^3 - 1040t^2 + 360t^2) \\
\beta_\frac{\pi}{2}(t) &= -\frac{1}{15}(192t^4 - 480t^3 + 380t^2 - 90t^2) \\
\beta_\frac{3\pi}{4}(t) &= \frac{1}{45}(-384t^4 + 840t^3 - 560t^2 + 120t^2) \\
\beta_\pi(t) &= -\frac{1}{90}(192t^4 - 360t^3 + 220t^2 - 45t^2)
\end{align*}
\]  

and \( t = (x - x_n)/h \). Evaluating (5) at \( t = \left(\frac{1}{4}, \frac{1}{4}\right) \) gives a discrete hybrid block method of the form,

\[
A^{(0)}Y_m = EY_n + hdf(y_n) + hbF(Y_m)
\]  

Where

\[
Y_m = \begin{bmatrix} y_{n+\frac{1}{4}}, y_{n+\frac{1}{2}}, y_{n+\frac{3}{4}}, y_{n+1} \end{bmatrix}^T, \quad Y_n = \begin{bmatrix} y_{n-\frac{1}{4}}, y_{n-\frac{1}{2}}, y_{n-\frac{3}{4}}, y_n \end{bmatrix}^T,
\]

\[
F(Y_m) = \begin{bmatrix} f_{n+\frac{1}{4}}, f_{n+\frac{1}{2}}, f_{n+\frac{3}{4}}, f_{n+1} \end{bmatrix}^T, \quad f(Y_n) = \begin{bmatrix} f_{n-\frac{1}{4}}, f_{n-\frac{1}{2}}, f_{n-\frac{3}{4}}, f_n \end{bmatrix}^T
\]

\[
A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \end{bmatrix}, \quad d = \begin{bmatrix} 251 \\
2880 \\
29 \\
360 \end{bmatrix}, \quad b = \begin{bmatrix} 323 & -11 & 53 & -19 \\
1440 & 120 & 1440 & 2880 \\
31 & 1 & 1 & -1 \\
90 & 15 & 90 & 360 \\
51 & 90 & 21 & -3 \\
160 & 40 & 160 & 320 \\
16 & 2 & 16 & 7 \\
45 & 15 & 45 & 90 \end{bmatrix}
\]
3 Analysis of Basic Properties of the Implicit One-Step Hybrid Block Method

3.1 Order of the Implicit One-step Hybrid Block Method

Let the linear operator \( L \{ y(x); h \} \) associated with the block (7) be defined as,

\[
L \{ y(x); h \} = A^{(0)}Y_m - Ey_n - hdf(Y_n) - hhF(Y_n)
\]  \( (8) \)

Expanding (8) using Taylor series and comparing the coefficients of \( h \) gives,

\[
L \{ y(x); h \} = c_0y(x) + c_1hy'(x) + c_2h^2y''(x) + \ldots + c_p h^p y^p(x) + c_{p+1} h^{p+1} y^{p+1}(x) + \ldots
\]  \( (9) \)

**Definition 3** [4]: The linear operator \( L \) and the associated continuous linear multistep method (5) are said to be of order \( p \) if \( c_0 = c_1 = c_2 = \ldots = c_p = 0 \) and \( c_{p+1} \neq 0 \).

\( c_{p+1} \) is called the error constant and the local truncation error is given by,

\[
et_{n+k} = c_{p+1} h^{(p+1)} y^{(p+1)}(x_n) + o(h^{p+2})
\]  \( (10) \)

For implicit one-step hybrid block method,

\[
L \{ y(x); h \} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
y_{n+\frac{1}{4}} \\
y_{n+\frac{1}{2}} \\
y_{n-\frac{1}{2}} \\
y_{n} \\
\end{bmatrix}
- \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
y_{n+\frac{1}{4}} \\
y_{n+\frac{1}{2}} \\
y_{n-\frac{1}{2}} \\
y_{n} \\
\end{bmatrix}
\begin{bmatrix}
f_n \\
f_{n+\frac{1}{4}} \\
f_{n+\frac{1}{2}} \\
f_{n+\frac{1}{2}} \\
\end{bmatrix}
\]  \( (11) \)

\[
- \begin{bmatrix}
251 & 323 & -11 & 53 & -19 \\
2800 & 1440 & 120 & 1440 & 2880 \\
29 & 31 & 1 & 1 & -1 \\
360 & 90 & 15 & 90 & 360 \\
-2997 & 51 & 90 & 21 & -3 \\
80 & 160 & 40 & 160 & 320 \\
7 & 16 & 2 & 16 & 7 \\
90 & 45 & 15 & 45 & 90 \\
\end{bmatrix}
\begin{bmatrix}
f_n \\
f_{n+\frac{1}{4}} \\
f_{n+\frac{1}{2}} \\
f_{n+\frac{1}{2}} \\
f_{n+\frac{1}{2}} \\
\end{bmatrix}
\]
Expanding (11) in Taylor series gives,

\[
\sum_{j=0}^{\infty} \frac{1}{j!} y^{(j)} - y^{0} = \frac{251 h}{2880} y^{0} - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y^{(j+1)} \left\{ \frac{323}{1440} \left( \frac{1}{4} \right)^{\prime} - \frac{11}{120} \left( \frac{1}{2} \right)^{\prime} + \frac{53}{1440} \left( \frac{3}{4} \right)^{\prime} - \frac{19}{2880} \right\}
\]

\[
\sum_{j=0}^{\infty} \frac{\left( \frac{1}{2} h \right)^{j}}{j!} y^{(j)} - y^{0} = \frac{29 h}{360} y^{0} - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y^{(j+1)} \left\{ \frac{31}{90} \left( \frac{1}{4} \right)^{\prime} - \frac{1}{15} \left( \frac{1}{2} \right)^{\prime} + \frac{7}{45} \left( \frac{3}{4} \right)^{\prime} - \frac{1}{360} \right\}
\]

\[
\sum_{j=0}^{\infty} \frac{\left( \frac{3}{4} h \right)^{j}}{j!} y^{(j)} - y^{0} = -\frac{2997 h}{80} y^{0} - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y^{(j+1)} \left\{ \frac{51}{160} \left( \frac{1}{4} \right)^{\prime} + \frac{9}{40} \left( \frac{1}{2} \right)^{\prime} + \frac{21}{160} \left( \frac{3}{4} \right)^{\prime} - \frac{3}{320} \right\}
\]

\[
\sum_{j=0}^{\infty} \left( \frac{h}{j!} \right)^{j} y^{(j)} - y^{0} = \frac{7 h}{90} y^{0} - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y^{(j+1)} \left\{ \frac{16}{45} \left( \frac{1}{4} \right)^{\prime} + \frac{2}{15} \left( \frac{1}{2} \right)^{\prime} + \frac{16}{45} \left( \frac{3}{4} \right)^{\prime} + \frac{7}{90} \right\}
\]

(12)

Equating the coefficients of the Taylor series expansion to zero yields,

\[
\tilde{c}_{0} = \tilde{c}_{1} = \tilde{c}_{2} = \tilde{c}_{3} = \tilde{c}_{4} = \tilde{c}_{5} = 0, \quad \tilde{c}_{6} = [4.5776(-06) \ 1.2418(-11) \ 4.5776(-06) \ -5.1670(-07)]^{T}
\]

Therefore, the implicit one-step hybrid block method is of uniform order five.

### 3.2 Zero Stability of the Implicit One-Step Hybrid Block Method

**Definition 4** [4]: The implicit one-step hybrid block method (7) is said to be zero-stable, if the roots \( z_{s}, s = 1, 2, ..., k \) of the first characteristic polynomial \( \rho(z) \) defined by \( \rho(z) = \text{det}(zA^{(0)} - E) \) satisfies \( |z_{s}| \leq 1 \) and every root satisfying \( |z_{s}| = 1 \) have multiplicity not exceeding the order of the differential equation. Moreover, as \( h \to 0 \), \( \rho(z) = z^{-r}(z-1)^{\mu} \) where \( \mu \) is the order of the differential equation, \( r \) is the order of the matrices \( A^{(0)} \) and \( E \), see [7] for details.

For our method,

\[
\rho(z) = z^{4} - z^{3} = z^{2}(z-1) = 0 \quad \Rightarrow \quad z_{1} = z_{2} = z_{3} = 0, \quad z_{4} = 1.
\]

Hence, the implicit one-step hybrid block method is zero-stable.

### 3.3 Consistency and Convergence of the Implicit One-Step Hybrid Block Method

The implicit one-step hybrid block method (7) is consistent since it has order \( p = 5 \geq 1 \) and it is also convergent by consequence of Dahlquist theorem stated below.
Theorem 2 [8]: The necessary and sufficient conditions that a continuous LMM be convergent are that it be consistent and zero-stable.

3.4 Convergence and Region of Absolute Stability of the Implicit One-Step Hybrid Block Method

Definition 5 [9]: Region of absolute stability is a region in the complex $z$ plane, where $z = \lambda h$. It is defined as those values of $z$ such that the numerical solutions of $y' = -\lambda y$ satisfy $y_j \to 0$ as $j \to \infty$ for any initial condition.

We shall adopt the boundary locus method to determine the region of absolute stability of the implicit one-step hybrid block method. This gives the stability polynomial below,

$$\bar{h}(w) = h^4\left(\frac{1}{1280}w^4 + \frac{5}{8}w^3\right) - h^3\left(\frac{5}{384}w^4 + \frac{6433}{1920}w^3\right) + h^2\left(\frac{7}{64}w^4 + \frac{4237}{320}w^3\right) - h\left(\frac{1}{2}w^4 + \frac{1}{2}w^3\right) + w^4 - w^3$$

The stability region is shown in the figure below.

![Stability Region](image)

**Fig. 1. Stability Region of the Implicit One-Step Hybrid Block Method**

The stability region obtained in Fig. 1 above is A-Stable, since it contains the whole of the left-half complex plane, [6].
4 Numerical Experiments

The implicit one-step hybrid block method formulated shall be tested on some set of stiff differential equations and compare the results with solutions from some methods of similar derivation. The numerical results were obtained using MATLAB. The following notations shall be used in the tables below;

- **ERR**: |Exact Solution-Computed Solution|
- **EJA**: Error in [10]
- **EOK**: Error in [11]

**Problem 1**

A certain radioactive substance is known to decay at the rate proportional to the amount present. A block of this substance having a mass of 100g originally is observed. After 40 hours, its mass reduced to 90g. Test for the consistency of the implicit one-step hybrid block method on this problem for \( t \in [0,1] \).

This stiff problem is modeled by the differential equation,

\[
\frac{dN}{dt} = \alpha N, \quad \alpha = -0.0026, \quad N(0) = 100, \quad t \in [0,1]
\]  

(15)

where \( N \) represents the mass of the substance at any time \( t \) and \( \alpha \) is a constant which specifies the rate at which this particular substance decays. The theoretical solution to (15) is given by;

\[
N(t) = 100e^{-0.0026t}
\]

(16)

Source: [12]

This problem was solved using Half-step hybrid method of order six. We present the result for problem 1 using the newly derived method in Table 1 below.

**Problem 2**

Consider the highly stiff ODE

\[
y' = -10(y-1)^2, \quad y(0) = 2
\]

(17)

with the exact solution,

\[
y(t) = 1 + \frac{1}{1+10t}
\]

(18)

Source: [11].

This problem was earlier discussed by [9], he showed that many predictor-corrector and block methods become unstable with this problem, including the hybrid methods. However, the newly
The derived block method is used for the integration of this problem within the interval $0 \leq t \leq 0.1$. The authors in [10] solved this stiff problem by adopting a new 3-point block method with step size ratio at $r = 1$ and of order five. We present the result for problem 2 using the newly derived method in Table 2 below.

| $t$   | Exact solution | Computed solution | ERR        | EJA          | $t / \text{sec}$ |
|-------|----------------|-------------------|------------|--------------|------------------|
| 0.0000| 1.9999993954763210 | 1.9999995349009182 | 1.597500e-009 | 1.739147e-009 | 0.0637           |
| 0.0200| 1.8333333333333333 | 1.8333333333333333 | 3.33333e-008  | 2.38462e-008  | 0.0644           |
| 0.0300| 1.7692307692307692 | 1.7692307692307692 | 2.30769e-008  | 4.51693e-008  | 0.0649           |
| 0.0400| 1.7142857142857143 | 1.7142857142857143 | 3.14286e-008  | 6.00000e-008  | 0.0653           |
| 0.0500| 1.6666666666666667 | 1.6666666666666667 | 2.85714e-008  | 8.46154e-008  | 0.0657           |
| 0.0600| 1.6250000000000000 | 1.6250000000000000 | 2.63636e-008  | 1.03636e-008  | 0.0661           |
| 0.0700| 1.5862559411764706 | 1.5862559411764706 | 2.39506e-008  | 1.27950e-008  | 0.0665           |
| 0.0800| 1.5555555555555555 | 1.5555555555555555 | 2.13750e-008  | 1.53750e-008  | 0.0670           |
| 0.0900| 1.5238095238095238 | 1.5238095238095238 | 1.97404e-008  | 1.75404e-008  | 0.0674           |
| 0.1000| 1.5000000000000000 | 1.5000000000000000 | 1.79686e-008  | 1.81686e-008  | 0.0678           |

### 5 Conclusion

We formulated an A-stable implicit one-step hybrid block method for the solution of stiff first-order ordinary differential equations where we adopted Legendre polynomial as our basis function. The method developed was also found to be zero-stable, consistent and convergent. The hybrid block method was also found to perform better than some existing methods in view of the numerical results obtained.

### Competing Interests

Authors have declared that no competing interests exist.

### References

[1] Adesanya AO, Udoh MO, Ajileye AM. A new hybrid method for the solution of general third-order initial value problems of ordinary differential equations. International J. of Pure and Applied Mathematics. 2013;86(2):37-48.
Adesanya AO, Odekuine MR, Udoh MO. Four-steps continuous method for the solution of $y''' = f(x, y, y', y'')$. American J. of Computational Mathematics. 2013;3:169-174.

Awoyemi DO, Ademiluyi RA, Amuseghan W. Off-grids exploitation in the development of more accurate method for the solution of ODEs. Journal of Mathematical Physics. 2007;12:379-386.

Fatunla SO. Numerical methods for initial value problems in ordinary differential equations. Academic Press Inc, New York; 1988.

Jain MK, Iyengar SRK, Jain RK. Numerical methods (for scientific and engineering computation). New age international (p) limited publishers, New Delhi (5th edition); 2007.

Lambert JD. Computational methods in ordinary differential equations. John Willey and Sons, New York; 1973.

Butcher JC. Numerical methods for ordinary differential equations in the 20th century. Journal of Computational and Applied Mathematics. 2000;125:1-29.

Dahlquist GG. Convergence and Stability in the Numerical Integration of Ordinary Differential Equations, Math. Scand. 1956;4:33-50.

Yan YL. Numerical methods for differential equations. City University of Hong Kong, Kowloon; 2011.

James AA, Adesanya AO, Sunday J, Yakubu DG. Half-step continuous block method for the solution of modeled problems of ordinary differential equations. American Journal of Computational Mathematics. 2013;3:261-269.

Okunuga SA, Sofoluwe AB, Ehigie JO. Some block numerical schemes for solving initial value problems in ODEs. Journal of Mathematical Sciences. 2013;2(1):387-402.

Sunday J. On Adomian decomposition method for numerical solution of ODEs arising from the natural laws of growth and decay. The Pacific Journal of Science and Technology. 2011;12(1):237-243.

© 2015 Sunday et al.; This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
www.sciencedomain.org/review-history.php?id=1035&id=6&aid=9104