Good formal structure for meromorphic flat connections on smooth projective surfaces

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Abstract

We prove the algebraic version of a conjecture of C. Sabbah on the existence of the good formal structure for meromorphic flat connections on surfaces after some blow up.

Keywords: meromorphic flat connection, irregular singularity, p-curvature, resolution of turning points

MSC: 14F10, 32C38

1 Introduction

1.1 Main result

Let $X$ be a smooth complex projective surface, and let $D$ be a normal crossing divisor of $X$. Let $(\mathcal{E}, \nabla)$ be a flat meromorphic connection on $(X, D)$, i.e., $\mathcal{E}$ denotes a locally free $\mathcal{O}_X(D)$-module, and $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ denotes a flat connection. We discuss a conjecture of C. Sabbah under the algebraicity assumption.

Theorem 1.1 There exists a regular birational morphism $\pi : \tilde{X} \to X$ such that $\pi^{-1}(\mathcal{E}, \nabla)$ has the good formal structure.

See Subsection 2.4 for good formal structure. For explanation of the meaning of the theorem, let us recall the very classical result in the curve case. (See the introduction of [14] for more detail, for example.) Let $C$ be a smooth projective curve, and let $Z \subset C$ be a finite subset. Let $(\mathcal{E}, \nabla)$ be a meromorphic connection on $(C, Z)$, i.e., $\mathcal{E}$ is a locally free $\mathcal{O}(Z)$-module with a connection $\nabla$. Let $P$ be any point of $Z$, and let $(U, t)$ be a holomorphic coordinate neighbourhood around $P$ such that $t(P) = 0$. The local structure of $\mathcal{E}$ around $P$ can be understood by the formal structure at $P$ and the Stokes structure around $P$. Namely, take a ramified covering $\varphi_P : (\tilde{U}, t_d) \to (U, t)$ given by $t = t_d^d$, where $d$ is a large integer divided by $(\text{rank}(\mathcal{E})!)^3$, for example. Let $\tilde{P} \in \tilde{U}$ be the inverse image of $P$. The formal completion of $\varphi_P^*(\mathcal{E}, \nabla)$ at $\tilde{P}$ is decomposed into the direct sum $\bigoplus_{a \in C(t_d)/C(t_d)} (\mathcal{E}_a, \nabla_a)$, where $\nabla_a - da$ are regular. (In the curve case, we do not have to assume that the base space is algebraic. In fact, the decomposition can be obtained for any connections on formal curves.) Then, the formal decomposition can be lifted to the decomposition on any small sectors by the asymptotic analysis, which leads us the Stokes structure.

It is a challenging and foundational problem to obtain the generalization in the higher dimensional case. The systematic study was initiated by H. Majima, who developed the asymptotic analysis in the higher dimensional case. (See [14], for example.) Briefly speaking, his result gives the lifting of a formal decomposition to the decomposition on small sectors. Inspired by Majima’s work, Sabbah ([21]) developed the asymptotic analysis in the other framework. He observed the significance of the understanding on the formal structure of the irregular connection. He proposed the conjecture which says that Theorem 1.1 may hold without the algebraicity assumption, and he established it in the case $\text{rank}(\mathcal{E}) \leq 5$. He also reduced the problem to the study of the turning points contained in the smooth part of the divisor $D$, without any assumption on the rank.

Sabbah gave some interesting applications of the conjecture, one of which is a conjecture of B. Malgrange on the absence of the confluence phenomena for flat meromorphic connections. Recently, Y. André ([1]) proved Malgrange’s conjecture motivated by Sabbah’s conjecture.

In this paper, we will give a proof of the algebraic version of Sabbah’s conjecture. In [17], the author intends to establish the correspondence of semisimple algebraic holonomic $\mathcal{D}$-modules and polarizable wild pure twistor
$D$-modules through wild harmonic bundles, on smooth projective varieties and the higher dimensional surfaces, which is related with a conjecture of M. Kashiwara [S]. Theor em 1.1 has the foundational importance for the study.

1.2 Main ideas

Let $k$ be an algebraically closed field, and let $(\mathcal{E}, \nabla)$ be a meromorphic flat connection on $k[[s]][(t)]$. If the characteristic number $p$ of $k$ is positive, we always assume that $p$ is much larger than rank $\mathcal{E}$ and the Poincaré rank of $\mathcal{E}$ with respect to $t$. We introduce the notion of $\mathcal{E}$ as the induced relative connection $\nabla = \nabla = \mathcal{E} \otimes \Omega^1_{k[[s]][(t)]/k[[s]]}$. The induced connection $(\mathcal{E}, \nabla_i) \otimes k((s))((t))$ is denoted by $(\mathcal{E}_i, \nabla_i)$. The specialization of $(\mathcal{E}, \nabla_i)$ at $s = 0$ is denoted by $(\mathcal{E}_0, \nabla_0)$. We have the set of the irregular values $\text{Irr}(\mathcal{E}_0, \nabla_0) \subset k((t)) \otimes k[[t]]$ and $\text{Irr}(\mathcal{E}_1, \nabla_1) \subset k((s))((t))/k((s))\otimes k[[t]]$, where $t_d$ and $s_d$ denote $d$-th roots of $t$ and $s$, respectively. Briefly and imprecisely speaking, one of the main issues is how to compare $\text{Irr}(\mathcal{E}_i, \nabla_i)$ for any $i = 0, 1$. Ideally, we hope that $\text{Irr}(\mathcal{E}_i, \nabla_i)$ is contained in $k[[s]]((t))/k[[s, t]]$, and that the specialization at $s = 0$ gives $\text{Irr}(\mathcal{E}_0, \nabla_0)$. However, they are not true, in general.

In the case $p > 0$, we have the $p$-curvature $\psi$ (resp. $\psi_i$) of the connection $\nabla$ (resp. $\nabla_i$).

$$\psi \in \text{End}(\mathcal{E}) \otimes F^*\Omega^1_{k[[s]][(t)]/k}, \quad \psi_i \in \text{End}(\mathcal{E}_i) \otimes F^*\Omega^1_{k((s))((t))/k[[s]]}, \quad \psi_0 \in \text{End}(\mathcal{E}_0) \otimes F^*\Omega^1_{k((t))/k}$$

Here, $F$ denotes the absolute Frobenius map. In the following, we use the notation $\psi(t\partial_t)$ to denote $\psi(F^*t\partial_t)$, for simplicity. Let $Sp(\psi(t\partial_t))$ denote the set of the eigenvalues of $\psi(t\partial_t)$, which is contained in $A_1$, where $A$ denotes a finite extension of $k[s, t]$ and $A_1$ denotes a localization of $A$ with respect to $t$. Similarly, let $Sp(\psi_i(t\partial_t))$ denote the set of the eigenvalues of $\psi_i(t\partial_t)$ for $i = 0, 1$, and then $Sp(\psi_0(t\partial_t)) \subset k((t))$ and $Sp(\psi(t\partial_t)) \subset k((s))((t))/k((s))\otimes k[[t]]$ for some appropriate $d \in \mathbb{Z}_{>0}$. We may have the natural inclusion $\kappa : A_1 \to k((s))((t))/k((s))\otimes k[[t]]$ and the specialization $\kappa_0 : A_1 \to k((t))$ at $s = 0$. Clearly, $\psi_i(t\partial_t)$ ($i = 0, 1$) are naturally obtained from $\psi(t\partial_t)$ by $\kappa_i$, and hence $Sp(\psi_i(t\partial_t))$ are obtained from $Sp(\psi(t\partial_t))$ by $\kappa_i$. Recall that the irregular value of $\nabla_i$ can be related with the negative part of the eigenvalues of $\psi_i(t\partial_t)$ (Lemma 2.3), where the negative part of $f = \sum f_j \cdot t_1^d \in R((t))$ is defined to be $f_- = \sum_{j < 0} f_j \cdot t_1^d$. Hence, we have the following diagram:

$$\begin{array}{ccc}
Sp(\psi_0(t\partial_t)) & \xrightarrow{\kappa_0} & Sp(\psi(t\partial_t)) \\
\downarrow & & \downarrow \\
\text{Irr}(\mathcal{E}_1, \nabla_1) & \xrightarrow{\kappa_i} & \text{Irr}(\mathcal{E}_0, \nabla_0)
\end{array}$$

But, we should remark that $\kappa_0(\alpha)_-$ and $\kappa_1(\alpha)_-$ cannot be directly related, in general.

Let us consider the simplest case where the ramification of $A$ over $k[s, t]$ may occur only at the divisor $(t = 0)$. Then, $Sp(\psi(t\partial_t))$ is contained in $k[[s]]((t))$, and $\kappa_0(\alpha)_-$ is the specialization of $\kappa_1(\alpha)_-$ at $s = 0$ for any $\alpha \in Sp(\psi(t\partial_t))$. Thus, we can compare the irregular values of $\nabla_i$ ($i = 0, 1$) in this simplest case.

Then, we have to consider what happens if the ramification of $A$ may be non-trivial. As the second simplest case, we assume that the ramification may occur only at the normal crossing divisor $(t) \cup (s')$ of $\text{Spec}^f k[s, t]$, where $s' = s + t \cdot h(t)$. Then, $Sp(\psi(t\partial_t))$ are contained in $k[[s]][(t)]$, and $s_d$ denotes a $d$-th root of $s'$. We assume, moreover, that $Sp(\psi(t\partial_t))$ are contained in $k[s][((t)) + k[s_d, t_d]]$. Then, the negative part of the eigenvalues behave well with respect to the specialization, i.e., $\kappa_1(\alpha)_- = \kappa_0(\alpha)_-$ for any $\alpha \in Sp(\psi(t\partial_t))$. Hence, we can compare the irregular values of $\nabla_i$ ($i = 0, 1$) in this mildly ramified case (Lemma 3.2).

We would like to apply such consideration to our problem. Essentially, the problem is the following, although we will discuss it in a different way. Let $(\mathcal{E}, \nabla)$ be a meromorphic flat connection with a lattice $E$ on $(X, D)$. For simplicity, we assume everything is defined over $\mathbb{Z}$. Then, we have the mod $p$-reductions $(\mathcal{E}_p, \nabla_p) := (\mathcal{E}, \nabla) \otimes \overline{\mathbb{F}}_p$ over $(X_p, D_p) := (X, D) \otimes \overline{\mathbb{F}}_p$ with the lattice $E_p = E \otimes \overline{\mathbb{F}}_p$. Let $\psi_p \in \text{End}(E_p) \otimes F^*\Omega^1_{X_p}(ND_p)$ denote the $p$-curvature. Then, we have the spectral manifold $\Sigma_p(\psi_p) := \{ (x, \omega) \mid \omega \text{ eigenvalues of } \psi_{p|_x} \} \subset F^*\Omega^1_{X_p} \otimes \mathcal{O}(ND_p)$.

For simplicity, we assume that $\psi_p$ has the distinct eigenvalues at the generic point. Then, we hope that the ramification of the projection $\pi_p$ of $\Sigma_p(\psi_p)$ to $X_p$ may happen at normal crossing divisor, after some blow ups, i.e., $R(\pi_p) := \{ x \in X_p \mid \pi_p \text{ is not etale at } x \}$ is normal crossing. If we fix $p$, it is easy to obtain such birational map because we are considering the surface case. But, for our problem, we would like to control the ramification for almost all $p$ at once. So we need something more.
Here, we recall the important observation of J. Bost, Y. Laszlo and C. Pauly [12] which says that we have \( \Sigma_p \) contained in \( \Omega_{X_p} \otimes O(ND) \), such that \( \Sigma_p(\psi_p) \) is the pull back of \( \Sigma_p \). So, we have only to control the ramification curves \( R(\pi'_p) \) of the projection \( \pi'_p \) of \( \Sigma_p \) to \( X_p \). Then, it is not difficult to see that the arithmetic genus of \( R(\pi'_p) \) are dominated, independently of \( p \). So, the complexity of the singularities of these ramification curves are bounded, and thus we can control them uniformly. (See Section 4)

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It is an extremely great pleasure for the author to dedicate this paper to Masaki Kashiwara with admiration for his great works and his leading role in the development of current mathematics.

2 Preliminary

2.1 Notation

Let \( R \) be a ring, and let \( t \) be a formal variable. We use the notation \( R[[t]] \) (resp. \( R((t)) \)) to denote the ring of formal power series (resp. the ring of formal Laurent power series) over \( R \). Let \( R((t))_{<0} \) denote the subset \( \{ \sum_{j<0} a_j \cdot t^j \in R((t)) \} \). For any \( f = \sum \sum_{j<0} a_j \cdot t^j \in R((t)) \), we put \( \text{ord}_d(f) := \min \{ j \mid a_j \neq 0 \} \). If we are given two variables \( s \) and \( t \), we use the notation \( R[s)((t)) \) to denote the ring of formal Laurent power series over \( R[s] \). The notation \( R((t))[s] \) is used to denote the ring of formal power series over \( R((t)) \). We have \( R[s][((t))] \subseteq R((t))[s] \).

For a given integer \( d > 0 \) and a formal variable \( t \), we use the notation \( t_d \) as a \( d \)-th root of \( t \), i.e., \( t_d^d = t \).

For any \( f = \sum f_j \cdot t_d^j \in R((t_d)) \), we put \( f_- := \sum_{j<0} f_j \cdot t_d^j \), which is called the negative part of \( f \). If \( d' \) is a factor of \( d \), we regard \( R((t_{d'})) \) as the subring of \( R((t_d)) \). For any \( f \in R((t_d)) \), we put \( \text{ord}_d(f) := d^{-1} \cdot \text{ord}_{d'}(f) \).

The definition is consistent for the inclusions \( R((t)) \subset R((t_{d'})) \subset R((t_d)) \). Let us consider the case where \( R \) is a ring over \( \mathbb{Z}/p\mathbb{Z} \) for some prime \( p \). If \( d \) is prime to \( p \), the derivation \( t \partial_t \) of \( R((t)) \) has the natural lift to \( R((t_d)) \), which is same as \( d^{-1} \cdot t_d \partial_{t_d} \). We put \( I_t(g) := \sum (d/j) \cdot g_j \cdot t_d^j \) for any \( g = \sum_{j \not\equiv 0 \mod p} g_j \cdot t_d^j \in R((t)) \). We have \( t \partial_t I_t(g) = g \) and \( I_t(t \partial_t g) = g \).

When \( R \) is a subring of \( C \) finitely generated over \( \mathbb{Z} \), let \( S(R, p) \) denote the set of the generic points of the irreducible components of \( \text{Spec}(R \otimes \mathbb{Z}/p\mathbb{Z}) \) for each prime number \( p \), and we put \( S(R) := \bigcup_p S(R, p) \). For each \( \eta \in S(R) \), let \( k(\eta) \) denote the corresponding field, and let \( \eta \rightarrow \eta \) denote a morphism such that \( k(\eta) \) is an algebraic closure of \( k(\eta) \).

We use the notation \( M_r(R) \) to denote the set of the \( r \)-th square matrices over \( R \), in general.

2.2 Irregular value

2.2.1 Definition

Let \( k \) be a field, whose characteristic number is denoted by \( p \). Let \( E \) be a locally free \( k[[t]] \)-module of rank \( r \). We use the notation \( E((t)) \) to denote \( E \otimes k((t)) \). Let \( E \) be a meromorphic connection of \( E((t)) \) such that \( \nabla(\partial_t)E \subset E \cdot t^{-\mu} \) for some non-negative integer \( \mu \).

Assumption 2.1 If \( p > 0 \), we assume that \( r \) and \( \mu \) are sufficiently smaller than \( p \), say \( 10 \cdot r! \cdot \mu < p \).
Let \( \overline{k} \) denote an algebraic closure of \( k \). Then, it is known (see [1], for example) that we have the unique subset \( \text{Irr}(E((t)), \nabla) \subset k((t_d))/k[t_d] \) and the unique decomposition
\[
(E((t)), \nabla) \otimes \overline{k}((t_d)) \cong \bigoplus_{a \in \text{Irr}(E((t)), \nabla)} (E_a((t_d)), \nabla_a)
\]
for some appropriate factor \( d \) of \( r! \), such that the following holds:

- For any element \( a \in \text{Irr}(E((t)), \nabla) \), take a lift \( \tilde{a} \in k((t_d)) \), and then \( \nabla_a - d\tilde{a} \cdot \text{id}_{E_a} \) is a logarithmic connection of \( E_a \). The elements of \( \text{Irr}(E((t)), \nabla) \) or their lifts are called the irregular values of \((E, \nabla)\).

The decomposition is called the irregular decomposition in this paper. We usually use the natural lifts of \( a \) in \( k((t_d))_{<0} \), and denote them by the same letter \( a \). We have \( \text{ord}_a(a) \geq -\mu + 1 \), and \( d \) is a factor of \( r! \), and hence \( \text{ord}_a(a) > -p \) under the assumption 2.1.

If the irregular decomposition exists on \( k((t)) \), then we say that \((E, \nabla)\) is unramified. The following lemma easily follows from the uniqueness of the irregular decomposition.

**Lemma 2.2** Let \( k' \) be an algebraic extension of \( k \), and let \( d' \) be a divisor of \( d \). If all the irregular values are contained in \( k'((t_{d'})) \), then \((E, \nabla) \otimes k'((t_{d'})) \) is unramified.

### 2.2.2 Connection form of Deligne-Malgrange lattice

We have another characterization of the irregular values. For simplicity, we assume that \((E((t)), \nabla)\) is unramified and that \( k \) is algebraically closed.

**Definition 2.3** We say that \( E \) is a Deligne-Malgrange lattice of \( E((t)) \), if the irregular decomposition [1] is given on \( k[[t]] \) not only on \( k((t)) \), i.e., \( E = \bigoplus E_a \).

If \( E \) is Deligne-Malgrange, we have the logarithmic connection \( \nabla^{\text{reg}} = \bigoplus \nabla_a^{\text{reg}} \), where \( \nabla_a^{\text{reg}} = \nabla_a - d\tilde{a} \cdot \text{id}_{E_a} \). We say \( E \) is a strict Deligne-Malgrange lattice, if \( \alpha - \beta \) are not integers for any two distinct eigenvalues \( \alpha, \beta \) of \( \text{Res}(\nabla^{\text{reg}}) \).

Let \( \nu \) be any frame of \( E \). Let \( A \in M_r(k((t))) \) be determined by \( \nabla(t\partial_t)v = \nu \cdot A \). Let \( \mathcal{S}p(A) \in k((t_d)) \) denote the set of the eigenvalues of \( A \) for some appropriate \( d \). For any \( \alpha \in \mathcal{S}p(A) \), we have the negative part \( \alpha_- \in k((t_d))_{<0} \) and \( I_\alpha(\alpha_-) \in k((t_d))_{<0} \) as explained in Subsection 2.1.

**Lemma 2.4** If \( E \) is Deligne-Malgrange, we have \( \text{Irr}(E((t)), \nabla) = \{ I_\alpha(\alpha_-) \mid \alpha \in \mathcal{S}p(A) \} \).

**Proof** We take a frame \( \nu_1 \) of \( E \) compatible with the irregular decomposition, and \( A_1 \) is determined as above. Then, we have \( A_1 \) has the decomposition corresponding to the irregular decomposition, \( A_1 = \bigoplus (t\partial_t a + R_a) \), where \( R_a \in M_r(k[[t]]) \). Hence the claim of the lemma clearly holds for the frame \( \nu_1 \).

For any frame \( \nu \) of \( E \), we have \( G \in \text{GL}(k[[t]]) \) such that \( \nu = \nu_1 \cdot G \). We have the relation \( A = G^{-1} \cdot A_1 \cdot G + G^{-1} \cdot t\partial_t G, \) i.e., \( G \cdot A \cdot G^{-1} = A_1 + (t\partial_t G) \cdot G^{-1} \), where \( t\partial_t G \cdot G^{-1} \in M_r(k[[t]]) \).

Hence, the claim is reduced to the following general lemma.

**Lemma 2.5** Let \( \Gamma \in M_r(k[[t]]) \) be a diagonal matrix whose \((i, i)\)-entry is given by \( \alpha_i \). Let \( B \) be any element of \( t^m \cdot M_r(k[[t]]) \) for a positive integer \( m > 0 \). Then, any eigenvalue \( \beta \in k[[t_d]] \) of \( \Gamma + B \) satisfies \( \text{ord}_i(\beta - \alpha_i) \geq m \) for some \( \alpha_i \).

**Proof** Let \( e_1, \ldots, e_r \) denote the canonical base of \( k((t))^r \). Let \( v = \sum f_i \cdot e_i \) be an eigenvector of \( \Gamma + B \) corresponding to the eigenvalue \( \beta \). We may assume \( \text{ord}_i(f_{i_0}) = 0 \) for some \( i_0 \). We obtain \( \text{ord}_i((\alpha_i - \beta) \cdot f_i) \geq m \) for any \( i \), and hence \( \text{ord}_i(\alpha_{i_0} - \beta) \geq m \). Thus, we obtain Lemma 2.5 and Lemma 2.4.
2.2.3 $p$-curvature

In the case $p > 0$, we have the other characteristic of the irregular values. For simplicity, we assume $k = \mathbb{T}$. Let $\text{Fr}: k((t)) \rightarrow k((t))$ be the absolute Frobenius morphism, i.e., $\text{Fr}(f) = f^p$. Applying $\text{Fr}$ to the coefficients, we obtain the homomorphism $k((t))[T] \rightarrow k((t))[T]$, which is also denoted by $\text{Fr}$. Let $\psi$ be the $p$-curvature of $\nabla$. (See [9] and [10] for example). Due to the observation of Bost-Laszlo-Panly ([12]), there exists a polynomial $P(\nabla)(T) \in k((t))[T]$ of degree $r$, such that $\det(T - \psi(t\partial_t)) = \text{Fr}(P(\nabla))(T)$. Let $\text{Sol}(P(\nabla))$ denote the set of the solutions of $P(\nabla)(T) = 0$. Then $\text{Sol}(P(\nabla)) \subset k((t)[u])$ for some appropriate factor $d$ of $r!$. Because of $\nabla(t\partial_t)(E) \subset E \cdot t^{-\mu}$, we have $\psi(t\partial_t)(E) \subset E \cdot t^{-\mu}$. Hence we have $\text{ord}_\alpha(\alpha) \geq -\mu + 1$ for any solution $\alpha \in \text{Sol}(P(\nabla))$. Under the assumption 2.1, we obtained $\text{ord}_\alpha(\alpha) > -p$ for any $\alpha \in \text{Sol}(P(\nabla))$.

Lemma 2.6 Under the assumption 2.1 we have $\text{Irr}(E((t)), \nabla) = \{ I_\alpha(\alpha_-) \mid \alpha \in \text{Sol}(P(\nabla)) \}$.

Proof We may assume that $(E, \nabla)$ is unramified and Deligne-Malgrange. Hence, we have only to consider the case where $(E, \nabla)$ has the unique irregular value, i.e., $\nabla = d\cdot \text{id}_E + \nabla_{\text{reg}}$, where $d \in k((t)), \text{ord}_t(d) > -p$, and $\nabla_{\text{reg}}$ is logarithmic. Let $\psi_{\text{reg}}$ denote the $p$-curvature of $\nabla_{\text{reg}}$. By a general formula ([20]. See also Lemma 3.4 of [22]), we have $\psi(t\partial_t) = \psi_{\text{reg}}(t\partial_t) + (t\partial_t)^p$, where $\psi_{\text{reg}}(t\partial_t) \in M_r(k[[t]])$. Then the claim of the lemma follows from Lemma 2.5.

2.3 Preliminary from elementary algebra

The following arguments are standard and well known. We would like to be careful about some finiteness, and we give just an outline. Let $R$ be a regular ring. Let $P_i(T) \in R[[t]][T]$ be a monic polynomial: $P_i(T) = T^r + \sum_{j=0}^{r-1} a_j(t) \cdot T^j$. The specialization at $t = 0$ is denoted by $P_0(T)$.

Lemma 2.7 Assume that $P_0(T) = \overline{h}_1(T) \cdot \overline{h}_2(T)$ in $R[T]$ such that $\overline{h}_1$ and $\overline{h}_2$ are monic polynomials and coprime in $K[T]$. Then, we have the decomposition $P(T) = h_1(T) \cdot h_2(T)$ in $R'[T][T]$, where $R'$ is the localization of $R$ with respect to some $f_1, \ldots, f_m$ in $R$ depending on $P_0$, and $h_i(T)$ are monics such that $h_i(T)|_{t=0} = \overline{h}_i(T)$.

Proof There exist $F_i \in K[T]$ $(i = 1, 2)$ such that $1 = \overline{h}_1(T) \cdot F_1(T) + \overline{h}_2(T) \cdot F_2(T)$. We may take a finite localization $R'$ of $R$ so that $F_i \in R'[T]$. For any $Q(T) \in R'[T]$ such that $\text{deg}_T Q < r$, we have $\overline{h}_1 \cdot (F_i Q) + \overline{h}_2 \cdot (F_2 Q) = Q$. Take $H, G \in R'[T]$ such that $\text{deg}_T H < \text{deg}_T (\overline{h}_1)$ and $F_2 Q = \overline{h}_1 \cdot G + H$. We put $\alpha = F_1 Q + \nabla \overline{h}_2$, and then we have $\overline{h}_1 \cdot \alpha + \overline{h}_2 \cdot H = Q$. Note $\text{deg}_T (\overline{h}_1) + \text{deg}_T (\overline{h}_2) = \text{deg}_T P_0 = r$. Then, we have $\text{deg}_T (\alpha) + \text{deg}_T (\overline{h}_1) \leq \max(\text{deg}_T Q, \text{deg}_T (\overline{h}_2) + \text{deg}_T H) < r$. Hence, $\text{deg}_T (\alpha) < r - \text{deg}_T (\overline{h}_1) = \text{deg}_T (\overline{h}_2)$.

Assume we are given $a_{i,j}(T)$ $(a = 1, 2; j = 1, \ldots, L)$ such that $\text{deg}_T h_{1,j} < \text{deg}_T (\overline{h}_1)$, $\text{deg}_T h_{2,j} < \text{deg}_T (\overline{h}_2)$ and

$$\left( \overline{h}_1(T) + \sum_{j=1}^{L} h_{1,j}(T)t^j \right) \cdot \left( \overline{h}_2(T) + \sum_{j=1}^{L} h_{2,j}(T)t^j \right) - \sum_{j=0}^{L} P_j(T)t^j \equiv 0 \mod t^{L+1}$$

By using the above remark, it is easy to show that we can take $h_{a,L+1}(a = 1, 2)$ such that $\text{deg}_T h_{1,L+1} < \text{deg}_T (\overline{h}_1)$, $\text{deg}_T h_{2,L+1} < \text{deg}_T (\overline{h}_2)$ and

$$\left( \overline{h}_1(T) + \sum_{j=1}^{L+1} h_{1,j}(T)t^j \right) \cdot \left( \overline{h}_2(T) + \sum_{j=1}^{L+1} h_{2,j}(T)t^j \right) - \sum_{j=0}^{L+1} P_j(T)t^j \equiv 0 \mod t^{L+2}$$

Thus, by an inductive argument, we can construct the desired $h_1$ and $h_2$.

Lemma 2.8 Let $P_i(T) \in R[[t]][T]$ (resp. $R((t))[T]$) be a monic polynomial. There exists an appropriate number $e$, such that it has a roots in $R'[t_e]$ (resp. $R'(t_e)$) where $R'$ is obtained from $R$ by finite algebraic extensions and localizations.

Proof Let $P_i(T) = \sum_{j=0}^{n} a_j(t) \cdot T^j$. We may assume that $n!$ is invertible in $R$. Let $\nu(P_i)$ denote the number $\min_j \{ \text{ord}_t(a_j)/(n-j) \}$. We use the induction on the numbers $\text{deg}_T P_i$ and $\nu(P_i)$. For simplicity, we use $\nu$
instead of $\nu(P_i)$, and let $d$ be the minimal positive integer such that $\nu \in d^{-1} \cdot \mathbb{Z}$. We formally use the notation $t^\nu$ to denote $t^d\nu$. We have the following monic polynomial:

$$Q_i(T') := t^{-\nu}P_i(t^\nu T') = \sum_{j=0}^n a_j(t) t^{-(n-j)\nu} T'^j = \sum_{j=0}^n b_j(t) T'^j \in R[[t]][T']$$

We have $d^{-1} \cdot \ord_d(b_j) = \ord_d(a_j) - (n - j) \cdot \nu \geq 0$, and we have $\ord_d(b_{j_0}) = 0$ for some $j_0$. We put $Q_0(T') = \sum_{j=0}^n b_j(0) T'^j \in R[T']$.

**Case 1** Assume $Q_0(T')$ has at least two different roots. Then, there exists a finite algebraic extension $R_1$ of $R$ such that we have the decomposition $Q_0(T') = R_1(T') R_2(T')$ in $R[T']$, and $R_1$ and $R_2$ are coprime in $K[T']$, where $K$ denotes the quotient field of $R_1$. Because of Lemma 2.7, we have $Q_i(T') = h_1(T') \cdot h_2(T')$ in $R'[t][T']$, where $R'_1$ is a localization of $R_1$ with respect to some finite elements. By the hypothesis of the induction on the degree with respect to $T'$, $h_i(T')$ ($i = 1, 2$) have the roots in $R_2[[t]]$, where $R_2$ is obtained from $R_1'$ by finite algebraic extensions and localizations. And $t^\nu \alpha$ gives the roots of $R_i(T)$.

**Case 2** In the case $Q_0(T') = (T' - \alpha)^n$, we have $n\alpha \in R$, and hence $\alpha \in R$. We have $\ord_d(a_{n-1})/n = \ord_d(a_{n-1})/(n - 1) = \nu$, and hence $\nu \in \mathbb{Z}$ and $d = 1$. We put $H_i(T) := P_i(T + t^\nu \alpha) = \sum_{j=0}^n c_j(t) T^j$. We have $\text{min}(\ord(c_j)(n - j)^{-1}) > \nu$.

We continue the process. If we reach the case 1, we can reduce the degree with respect to $T$. If we do not reach the case 1, it is shown that $P_i(T) = (T - \alpha)^n$ for some $\alpha \in R[t]$ (resp. $\alpha \in R[[t]]$). Thus we are done.

**Corollary 2.9** Any $P(s,t)(T) \in R[s,t][T]$ has the roots in $R_p((s_d))[[t]]$, and any $P(s,t)(T) \in R[s][[t]][T]$ has the roots in $R_p((s_d))(t_d)$. Here $R_p$ is obtained from $R$, depending on $P$, by finite algebraic extensions and localizations, and $d$ denotes an appropriate positive integer.

Let $R$ be an integral domain such that $\mathbb{Z} \subset R$. Let $K$ denote the quotient field of $R$. Let $(\mathcal{E}, \nabla)$ be a meromorphic connection on $R[[t]]$.

**Lemma 2.10** There exists an extension $R'$ obtained from $R$ by finite algebraic extensions and localizations, with the following property:

- The irregular values of $(\mathcal{E}, \nabla) \otimes K[[t]]$ are contained in $R'(t_d)$.
- The irregular decomposition and a Deligne-Malgrange lattice are defined on $R'(t_d)$.

**Proof** We need only a minor modification for the argument given in [13], and hence we give just an outline. We put $D = \nabla(t \partial_t)$. Let $K$ be the quotient field of $R(t)$. By applying the argument of Deligne [11] to $E \otimes K$ with the derivation $D$, we can take $e \in E$ such that $e, D(e), \ldots, D^{r-1}(e)$ give a base of the $K$-vector space $E \otimes K$. We have the relation $D^r e + \sum_{j=0}^{r-1} a_j \cdot D^j e = 0$ where $a_j \in K$. There exists a finite localization $R_1$ of $R$ such that $a_j \in R_1[[t]]$.

We put $\nu := \text{min}\{\ord(a_j)/r - j\}$. Note that $\nu \geq 0$ implies the regularity of the connection. Let $d$ denote the minimal positive integer such that $d \cdot \nu \in \mathbb{Z}$. We put $f_{i+1} := t^{-(i+1)} D^i e$ ($i = 0, \ldots, r - 1$), and $f = (f_i | i = 1, \ldots, r)$. Let $A \in M_r(R_{1}(t_d))$ be determined by $DF = fA$. Then, $A$ is of the form $t^\nu (A_0 + t_d \cdot A_1(t_d))$ such that (i) $A_1 \in M_r(R_{1}[t_d])$, (ii) $A_0 \in M_r(R_1)$ whose $(i, j)$-entries are as follows:

$$(A_0)_{i,j} = \begin{cases} 1 & (i = j + 1) \\ -(t^{(-r+i-1)\nu} a_{i-1})_{i,j} = 0 & (j = r) \\ 0 & \text{otherwise} \end{cases}$$

By the choice, one of $(A_0)_{i,r}$ not 0.

**Case 1** Let us consider the case where $A_0$ has at least two distinct eigenvalues. There exists a finite extension $R_2$ such that (i) we have $G \in \text{GL}_r(R_2)$ for which $G^{-1} A_0 G$ is Jordan, (ii) the difference of any two distinct eigenvalues of $A_0$ are invertible in $R_2$. By a standard argument (See [13], or the proof of Lemma 2.10 below), we can show that there exists $G_1 \in \text{GL}_r(R_2[t_d])$ such that (i) $G_1|_{t_d=0} = G$, (ii) let $g = f \cdot G_1$ and $Dg = g \cdot B$,
then \( B \) is decomposed into a direct sum of matrices with smaller sizes. Hence, we obtain a decomposition into connections with lower ranks. Thus, we can reduce the problem to the lower rank case.

**Case 2** If \( A_0 \) has the unique eigenvalues \( \alpha \in R_1 \), it can be shown that \( d = 1 \) and \( \nu \in \mathbb{Z}_{<0} \), as in the proof of Lemma 2.8 We put \( \nabla' = \nabla - t^\nu \alpha \cdot dt/t \) and \( \nabla' = \nabla'(t \partial_t) \). Let \( K_1 \) be the quotient field of \( R_1((t)) \). It can be shown that \( e, D'e, \ldots, (D')^{r-1}e \) give a base of \( \nabla \otimes K_1 \). Let \( a'_j \) be determined by \( D'^r e + \sum a'_j \cdot D'^j e = 0 \). Then, we have \( a'_j \in R_1((t)) \) and \( \nu(\nabla') = \min \{ \text{ord}_d(a'_j)/(r - j) \} \geq \nu + |\nu|/r \). We continue the process. After the finite steps, we will arrive at the case 1 or the case \( \nu(\nabla') \geq 0 \). □

**Corollary 2.11** Let \( (E, \nabla) \) be a meromorphic connection on \( R[[s]]((t)) \). Then, there exists an extension \( R' \), which is obtained from \( R \) by finite algebraic extensions and localizations, with the following property:

- The irregular values of \( (E, \nabla) \otimes K((s))((t)) \) are contained in \( R'(\langle s \rangle)((t_d)) \).
- The irregular decomposition and a Deligne-Malgrange lattice are defined on \( R'(\langle s \rangle)((t_d)) \). □

### 2.4 Good formal structure

Let \( X \) be a complex algebraic surface, with a simple normal crossing divisor \( D \). Let \( (E, \nabla) \) be a meromorphic flat connection on \( (X, D) \). We recall the notion of good formal structure, by following [21].

If \( P \) is a smooth point of \( D \), we take a holomorphic coordinate \( (U, t, s) \) around \( P \) such that \( t^{-1}(0) = U \cap D \). For a positive integer \( d \), we take a ramified covering \( \varphi_d : U_d \rightarrow U \) given by \( (t_d, s) \mapsto (t_d^r, s) \). We put \( D_d := \{ t_d = 0 \} \subset U_d \). Let \( M(U_d, D_d) \) (resp. \( H(U_d) \)) denote the space of meromorphic (resp. holomorphic) functions on \( U_d \) whose poles are contained in \( D_d \). For any element \( a \) of \( M(U_d, D_d)/H(U_d) \), we have the natural lift to \( M(U_d, D_d) \) which is also denoted by \( a \). Let \( \tilde{D}_d \) denote the formal space obtained as the completion of \( U_d \) along \( D_d \). (See [3], for example.)

**Definition 2.12** We say that \( (E, \nabla) \) has the good formal structure at \( P \), if the following holds for some \( (U, t, s) \) and some \( d \in \mathbb{Z}_{>0} \):

- We have the finite subset \( \text{Irr}(E, \nabla) \subset M(U_d)/H(U_d) \) and the decomposition:

\[
\varphi_d^*(E, \nabla)|_{\tilde{D}_d} = \bigoplus_{a \in \text{Irr}(E, \nabla)} (E_a, \nabla_a)
\]

Here \( \nabla_a^{\text{reg}} := \nabla_a - da \cdot \text{id}_{E_a} \) are regular.

- For any non-zero \( a \in \text{Irr}(E, \nabla) \), the 0-divisor of \( a \) has no intersection with \( D_d \).

- For any two distinct \( a, b \in \text{Irr}(E, \nabla) \), the 0-divisor of \( a - b \) has no intersection with \( D_d \). □

If \( P \) is a cross point of \( D \), we take a holomorphic coordinate \( (U, t, s) \) such that \( D \cap U = \{ t \cdot s = 0 \} \). For each \( d \in \mathbb{Z}_{>0} \), we take a ramified covering \( \varphi_d : U_d \rightarrow U \) given by \( (t_d, s_d) \mapsto (t_d^r, s_d) \). We put \( D_d := \{ t_d = s_d = 0 \} \) and \( P_d := (0, 0) \). Let \( \tilde{D}_d \) denote the formal space obtained as the completion of \( U_d \) along \( P_d \).

Let \( M(U_d, D_d) \) (resp. \( H(U_d) \)) denote the space of the meromorphic (holomorphic) functions on \( U_d \) whose poles are contained in \( D_d \). For any element \( a \) of \( M(U_d, D_d)/H(U_d) \), we have the natural lift to \( M(U_d, D_d) \), which is also denoted by \( a \).

We use the partial order \( \leq_{\mathbb{Z}^2} \) on \( \mathbb{Z}^2 \) given by \( (a_1, a_2) \leq_{\mathbb{Z}^2} (a'_1, a'_2) \iff a_i \leq a'_i \ (i = 1, 2) \). For any element \( f = \sum f_{i,j} \cdot s^i \cdot t^j \in M(U_d, D_d) \), let \( \text{ord}(f) \) denote the minimum of the set \( \min \{ \min \{ (i, j) \mid f_{i,j} \neq 0 \} \} \), if it exists.

**Definition 2.13** We say that \( (E, \nabla) \) has the good formal structure if the following holds:

- We have the finite subset \( \text{Irr}(E, \nabla) \subset M(U_d)/H(U_d) \) and the decomposition for some \( d \in \mathbb{Z}_{>0} \):

\[
\varphi_d^*(E, \nabla)|_{\tilde{D}_d} = \bigoplus_{a \in \text{Irr}(E, \nabla)} (E_a, \nabla_a)
\]

Here \( \nabla_a^{\text{reg}} := \nabla_a - da \cdot \text{id}_{E_a} \) are regular.
• \( \text{ord}(a) \) exists in \( \mathbb{Z}_{\leq 0} - \{(0, 0)\} \) for each non-zero \( a \in \text{Irr}(E, \nabla) \).

• \( \text{ord}(a - b) \) exists in \( \mathbb{Z}_{\leq 0} - \{(0, 0)\} \) for any two distinct \( a, b \in \text{Irr}(E, \nabla) \). And the set \( \{\text{ord}(a - b) \mid a, b \in \text{Irr}(E, \nabla)\} \) is totally ordered with respect to the above order \( \leq_{\mathbb{Z}} \).

**Definition 2.14** A point \( P \) is called turning with respect to \( (E, \nabla) \), if \( (E, \nabla) \) does not have a good formal structure at \( P \).

### 2.5 A sufficient condition for the existence of the good formal structure

#### 2.5.1 Preliminary

Let \( E \) be a free \( C[[s, t]] \)-module. Let \( \nabla_t : E \rightarrow E \otimes \Omega^1_{C[[s, t]]/C[[s]]}(st) \) be a connection such that the following holds for some \( k \geq 1 \) and \( p \geq 0 \):

\[
\nabla_t(t^{k+1} s^p \partial_t)(E) \subset E
\]

In that case, \( \nabla_t(t^{k+1} s^p \partial_t) \) induces the endomorphism of \( E_0 := E_{[t=0]} \), which is denoted by \( F_0 \).

**Lemma 2.15** If \( F_0 \) is invertible, any meromorphic flat section \( f = \sum_{j \geq -N} f_j \cdot t^j \) of \( E \) is 0.

**Proof** Let \( f \) be a meromorphic flat section of \( E \). Assume \( f \neq 0 \). We may assume that \( -N = \min\{j \mid f_j \neq 0\} \). From \( \nabla_t(t^{k+1} s^p \partial_t)f = 0 \), we have \( F_0(f_{-N}) = 0 \). Because \( F_0 \) is invertible, we obtain \( f_{-N} = 0 \), which contradicts with the choice of \( N \).

**Lemma 2.16** Assume the following:

- We have the decomposition \( (E_0, F_0) = (E_0^{(1)}, F_0^{(1)}) \oplus (E_0^{(2)}, F_0^{(2)}) \).
- The eigenvalues of \( F_0^{(i)} \) are contained in \( C[[s]] \). If \( b_i \) (\( i = 1, 2 \)) are the eigenvalues of \( F_0^{(i)} \), we have \( (b_1 - b_2)|_{s = 0} \neq 0 \).

Then, we have the unique \( \nabla_t \)-flat decomposition \( E = E^{(1)} \oplus E^{(2)} \) such that the restriction to \( t = 0 \) is the same as \( E_0 = F_0^{(1)} \oplus F_0^{(2)} \).

**Proof** We closely follow the argument in \[13\]. Let \( \nu \) be a frame of \( E \) such that \( \nu_{[t=0]} \) is compatible with the decomposition \( E_0 = E_0^{(1)} \oplus E_0^{(2)} \). Then, \( \nu \) is divided as \( (\nu^{(1)}, \nu^{(2)}) \), where \( \nu^{(i)} \) are the frames of \( E_0^{(i)} \). Let \( A = \sum_{j=0}^\infty A_j(s) \cdot t^j \) be determined by the following:

\[
\nabla(t^{k+1} s^p \partial_t)\nu = \nu \cdot A.
\]

We have the following decomposition corresponding to the decomposition of the frame \( \nu = (\nu^{(1)}, \nu^{(2)}) \):

\[
A_j = \begin{pmatrix}
A_j^{(11)} & A_j^{(12)} \\
A_j^{(21)} & A_j^{(22)}
\end{pmatrix}
\]

By the assumption, we have \( A_0^{(1)} = 0 \) and \( A_0^{(2)} = 0 \). For the change of the frame from \( \nu \) to \( \nu \cdot G \), we have the following:

\[
\nabla(t^{k+1} s^p \partial_t)(\nu \cdot G) = (\nu \cdot G) \cdot \tilde{A}(G), \quad \tilde{A}(G) := G^{-1}AG + t^{k+1} s^p G^{-1} \partial_t G
\]

We consider the formal transform \( G \) of the following form:

\[
G = I + \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}, \quad X = \sum_{j=1}^\infty X_j(s) \cdot t^j, \quad Y = \sum_{j=1}^\infty Y_j(s) \cdot t^j
\]

Here the entries of \( X_j(s) \) and \( Y_j(s) \) are contained in \( C[[s]] \). We want to determine \( X_j \) and \( Y_j \) by the following condition:
• The $(1, 2)$-component and the $(2, 1)$-component of $\tilde{A}(G)$ are $0$.

• The $(1, 1)$-component of $\tilde{A}(G)$ is of the form $A^{(11)}_0 + B^{(11)}$, where the entries of $B^{(11)}$ are contained in $t \cdot C[s, t]$. Similarly, the $(2, 2)$-component is of the form $A^{(22)}_0 + B^{(22)}$, where the entries of $B^{(22)}$ are contained in $t \cdot C[s, t]$.

We obtain the following equations for $Y$ and $B^{(11)}$:

$$A^{(11)} + A^{(12)}Y - A^{(11)}_0 - B^{(11)} = 0, \quad A^{(21)} + A^{(22)}Y + t^{k+1}s\partial_t Y - Y(A^{(11)}_0 + B^{(11)}) = 0$$

Then, we obtain the following equation for $Y$:

$$A^{(22)}_0Y - YA^{(11)}_0 - Y(A^{(11)} - A^{(11)}_0) + (A^{(22)} - A^{(22)}_0)Y - YA^{(12)}Y + t^{k+1}s\partial_t Y + A^{(21)} = 0$$

For the expansion $Y = \sum_{j=1}^\infty Y_j(s) \cdot t^j$, we obtain the following equations:

$$A^{(22)}_0 Y_j - Y_j A^{(11)}_0 + \sum_{l+m=j} Y_l A^{(11)}_m + \sum_{l+m=j} A^{(22)}_l Y_m - \sum_{l+m+n=j} Y_l A^{(12)}_m Y_n + (j-k)s^pY_{j-k} \cdot \chi_{j \geq k} + A^{(21)} = 0 \quad (3)$$

Here $\chi_{j \geq k} = 0$ if $j < k$ and $\chi_{j \geq k} = 1$ if $j \geq k$. When we are given $Y_m$ ($1 \leq m \leq j - 1$) whose entries are contained in $C[s]$, we have the unique solution $Y_j$ of (3), whose entries are contained in $C[s]$. Hence, we have appropriate $Y$ and $B^{(11)}$. Similarly, we have appropriate $X$ and $B^{(22)}$. Thus, we can conclude the existence of the desired decomposition $E = E^{(1)} \oplus E^{(2)}$. The uniqueness follows from Lemma 2.15.

Let us consider the case where $\nabla_t$ comes from a flat meromorphic connection $\nabla : E \to E \otimes \Omega^1_{C[s][((t))]/C((st))}$.

**Lemma 2.17** Assume the hypothesis in Lemma 2.16. The decomposition $E = E^{(1)} \oplus E^{(2)}$ is $\nabla$-flat.

**Proof** We may assume $v = (v_1, v_2)$ is compatible with the decomposition $E = E^{(1)} \oplus E^{(2)}$. Let $A$ and $B$ be determined by the following:

$$\nabla(t^{k+1}\partial_t) v = v \cdot A, \quad A = \begin{pmatrix} A^{(11)} & 0 \\ 0 & A^{(22)} \end{pmatrix}$$

$$\nabla(\partial_s) v = v \cdot B, \quad B = \begin{pmatrix} B^{(11)} & B^{(12)} \\ B^{(21)} & B^{(22)} \end{pmatrix}$$

From the relation $[\nabla(\partial_s), \nabla(t^{k+1}\partial_t)]$, we have the following equation for $B^{(12)}$:

$$A^{(11)}B^{(12)} - B^{(12)}A^{(22)} + t^{k+1}\partial_t B^{(12)} = 0$$

Assume $B^{(12)} \neq 0$. We have the expression $B^{(12)} = \sum_{j \geq -N} B^{(12)}_j \cdot t^j$, and we may assume $B^{(12)}_N \neq 0$. But, we have the relation $B^{(12)}_N A^{(11)}_0 - A^{(22)}_0 B^{(12)}_N = 0$, and hence $B^{(12)}_N = 0$. Thus, we arrive at the contradiction, and we can conclude $B^{(12)} = 0$. Similarly, we obtain $B^{(21)} = 0$. 

**2.5.2 A condition**

Let $E$ be a free $C[s, t]$-module with a flat meromorphic connection $\nabla : E \to E \otimes \Omega^1_{C[s][((t))]/C((st))}$. We have the induced relative connection $\nabla_t : E \to E \otimes \Omega^1_{C[s][((t))]/C[s, t]}$. We put $\mathfrak{K} := C(\langle s \rangle)\langle \langle t \rangle \rangle$. We put $(E_{\mathfrak{K}}, \nabla_{\mathfrak{K}}) := (E, \nabla_t) \otimes \mathfrak{K}$ and $E_R := E \otimes C(\langle s \rangle)[[t]]$. We assume that $E_R$ is a strict Deligne-Malgrange lattice. The intersection of $E_R$ and $E((st))$ in $E_R$ is the same as $E$, which gives a characterization of $E$.

**Proposition 2.18** Assume the following:

- $\text{Irr}(E_{\mathfrak{K}}, \nabla_{\mathfrak{K}})$ is contained in $C[s][\langle \langle t \rangle \rangle]/C[s, t]$. 

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• $\text{Irr}(E, \nabla) \subset \mathbb{C}$ is good, in the following sense:
  
  - Let $a \in \text{Irr}(E, \nabla)$. For the expression $a = \sum_{j} \text{ord}_t(a_j) t^j$, we have $a_{\text{ord}_t(a)}(0) \neq 0$.
  
  - Similarly, we have $(a - b)_{\text{ord}_t(a-b)}(0) \neq 0$ for any two distinct $a, b \in \text{Irr}(E, \nabla)$.

Then, $(E(t), \nabla)$ has the good formal structure.

**Proof** We put $k(E) := -\min\{\text{ord}_t(a) | a \in \text{Irr}(\nabla)\}$. Assume $k(E) \geq 1$. Because $E_R$ is a Deligne-Malgrange lattice of $E$, we have $\nabla((E^{(k(E)+1)} + t\partial_t)E) \subset E$. Let $F_0$ denote the endomorphism of $E_0$ induced by $\nabla((E^{(k(E)+1)} + t\partial_t)E)$. The eigenvalues of $F_0$ are given by $(t^{(k(E)+1)} + t\partial_t)a)_{t=0} (a \in \text{Irr}(\nabla))$. By using Lemma 2.10 and Lemma 2.17, we obtain the decomposition:

\[
(E, \nabla) = \bigoplus_{b \in S} (E_b, \nabla_b), \quad S := \left\{ b = (t^{k(E)}a)_{t=0} \in \mathbb{C} \mid a \in \text{Irr}(\nabla) \right\}
\]

We put $\nabla'_b := \nabla_b - d(t^{-k(E)}b)$. Then, $(E_b, \nabla'_b)$ also satisfy the assumption of this lemma, and we have $k(E_b) \leq k(E) - 1$. If $k(E_b) \geq 1$, we may apply the above argument to $(E_b, \nabla'_b)$ by the inductive argument, we obtain the flat decomposition $(E, \nabla) = \bigoplus_{a \in \text{Irr}(\nabla)} (E_a, \nabla_a)$. Assume $N > 0$. We recall the following standard lemma.

**Lemma 2.19** We can take a lattice $E'$ of $E \otimes k[[t]]$ such that (i) $\nabla$ is logarithmic with respect to $E'$, (ii) $\alpha - \beta \notin \mathbb{Z}$ for any two distinct $\alpha, \beta \in \text{Irr}(\nabla)$.

**Proof** We give only an outline. Let $S$ denote the set of the eigenvalues of $\text{Res}_E(\nabla)$. We say $\alpha < \beta$ for $\alpha, \beta \in S$ if $\beta - \alpha \in \mathbb{Z}_{>0}$. We say $\alpha \leq \beta$ if $\alpha = \beta$ or $\alpha < \beta$. It determines the partial order on $S$. We use $\rho(E) := \max\{\beta - \alpha \mid \beta \leq \alpha, \alpha, \beta \in S\}$. If $\rho(E) = 0$, we have nothing to do. We will reduce the number $\rho(E)$ by replacing $E$.

Let $S_0$ denote the maximal elements $\beta$ of $S$ such that there exists $\alpha \in S$ with $\alpha < \beta$. Let $\overline{k}$ denote the algebraically closure of $k$. We have the generalized eigen decomposition $E_0 \otimes \overline{k} = \bigoplus_{\alpha \in S} E_0$. Note that $S_0$ is preserved by the action of the Galois group of $\overline{k}$ over $k$. It is easy to see that $\bigoplus_{\alpha \in S_0} E_\alpha$ comes from the subspace $V$ of $E_0$. Let $E^{(1)} := t^{-1} E$. The specialization $E_0^{(1)}$ of $E^{(1)}$ at $t = 0$ is naturally isomorphic to $E_0$ up to constant multiplication. Hence, $V$ determines the subspace $V^{(1)} \subset E_0^{(1)}$. Let $E^{(2)}$ denote the kernel of the naturally defined morphism $E^{(1)} \to E_0^{(1)}/V^{(1)}$. Then, it can be checked $\rho(E^{(2)}) \leq \rho(E) - 1$. 

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2.6 Adjustment of the residue of a logarithmic connection

Let $k$ be a field whose characteristic number is 0. Let $E$ be a free module over $k[[t]]$ with a meromorphic connection $\nabla$ such that $t\nabla(\partial_t)(E) \subset E$. Let $E_0$ denote the specialization of $E$ at $t = 0$. We have the well defined endomorphism $\text{Res}_E(\nabla)$ of $E_0$. To distinguish the dependence on $E$, we denote it by $\text{Res}_E(\nabla)$. We recall the following standard lemma.

**Lemma 2.19** We can take a lattice $E'$ of $E \otimes k[[t]]$ such that (i) $\nabla$ is logarithmic with respect to $E'$, (ii) $\alpha - \beta \notin \mathbb{Z}$ for any two distinct $\alpha, \beta \in \text{Irr}(\nabla)$.

**Proof** We give only an outline. Let $S$ denote the set of the eigenvalues of $\text{Res}_E(\nabla)$. We say $\alpha < \beta$ for $\alpha, \beta \in S$ if $\beta - \alpha \in \mathbb{Z}_{>0}$. We say $\alpha \leq \beta$ if $\alpha = \beta$ or $\alpha < \beta$. It determines the partial order on $S$. We use $\rho(E) := \max\{\beta - \alpha \mid \beta \leq \alpha, \alpha, \beta \in S\}$. If $\rho(E) = 0$, we have nothing to do. We will reduce the number $\rho(E)$ by replacing $E$.

Let $S_0$ denote the maximal elements $\beta$ of $S$ such that there exists $\alpha \in S$ with $\alpha < \beta$. Let $\overline{k}$ denote the algebraically closure of $k$. We have the generalized eigen decomposition $E_0 \otimes \overline{k} = \bigoplus_{\alpha \in S} E_\alpha$. Note that $S_0$ is preserved by the action of the Galois group of $\overline{k}$ over $k$. It is easy to see that $\bigoplus_{\alpha \in S_0} E_\alpha$ comes from the subspace $V$ of $E_0$. Let $E^{(1)} := t^{-1} E$. The specialization $E_0^{(1)}$ of $E^{(1)}$ at $t = 0$ is naturally isomorphic to $E_0$ up to constant multiplication. Hence, $V$ determines the subspace $V^{(1)} \subset E_0^{(1)}$. Let $E^{(2)}$ denote the kernel of the naturally defined morphism $E^{(1)} \to E_0^{(1)}/V^{(1)}$. Then, it can be checked $\rho(E^{(2)}) \leq \rho(E) - 1$. 

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3 Mildly ramified connection

3.1 Positive characteristic case

Let $k$ be an algebraically closed field whose characteristic number $p$ is positive. Let $C$ be a smooth divisor of Spec $k[s,t]$, which intersects with the divisor $\{ t = 0 \}$ transversally. We can take a morphism Spec $k[u] \cong C \subset$ Spec $k[s,t]$ given by $(s(u), t(u))$. We may assume $t(u) = u$ and $s(u) = u \cdot h(u)$. We put $s' = s - h(t) \cdot t$. Then, $C$ is given by the ideal generated by $s'$. We also have $k[s', t] \cong k[s, t]$. For any positive integer $d$, we use the notation $s'_d$ to denote a $d$-th root of $s'$.

Let $E$ be a free $k[s](t)$-module. Let $\nabla : E \rightarrow E \otimes \Omega^1_{k[s](t)/k}$ be a flat meromorphic connection. Let $\psi$ denote the $p$-curvature of $\nabla$. Let $F_r$ denote the absolute Frobenius map $k[s](t) \rightarrow k[s](t)$. It induces the ring homomorphism $F_r : k[s](t)(T) \rightarrow k[s](t)(T)$ by $F_r(\sum a_j \cdot T^j) = \sum \text{Fr}(a_j) \cdot T^j$. Due to the observation of Bost-Laszlo-Pauly [12], we have $P_r(T), P_t(T) \in k[s](t)(T)$ such that $\det(T - \psi(T)) = F_r(P_r(T))$ and $\det(T - \psi(T)) = F_r(P_t(T))$. In general, the polynomials $P_r(T)$ and $P_t(T)$ have the roots in $k((s_d(t_d))]$ for some appropriate integer $d$.

**Definition 3.1** We say that $(E, \nabla)$ is mildly ramified at $\{ t = 0 \} \cup C$, if the following conditions are satisfied:

1. The roots of the polynomials $P_r(T) = 0$ and $P_t(T) = 0$ are contained in $k[s_d', (t_d)]$ for some $d \in \mathbb{Z}_{>0}$, where $s_d'$ is taken for $C$ as above.

2. The roots are of the form $\alpha + \beta$, where $\alpha \in k[s']((t_d))$ and $\beta \in k[s_d', t_d]$.

We say that $(E, \nabla)$ is mildly ramified, if it is mildly ramified at $\{ t = 0 \} \cup C$ for some $C$.

The connection $\nabla$ induces the relative connection $\nabla_t : E \rightarrow E \otimes \Omega^1_{\mathbb{A}^1/k}[t]/k[s]$. We put $\mathcal{R} := k(s)((t))$ and $\mathfrak{t} := k(t)$. Both of them are equipped with the differential $\partial_t$. We have the natural inclusion $k[s](t) \subset \mathcal{R}$, and the specialization $k[s](t) \rightarrow \mathfrak{t}$ at $s = 0$. The morphisms are equivariant with respect to $\partial_t$. Therefore, we have the induced connections of $E \otimes \mathcal{R}$ and $E \otimes \mathfrak{t}$, which are also denoted by $\nabla_t$.

**Lemma 3.2** Assume that $(E, \nabla)$ is mildly ramified at $\{ t = 0 \} \cup C$. Then, the irregular values of $(E \otimes \mathcal{R}, \nabla_t)$ are contained in $k[s](t_d)$, and their specialization at $s = 0$ give the irregular values for $(E \otimes \mathfrak{t}, \nabla_t)$. The induced map $\text{Irr}(E \otimes \mathcal{R}, \nabla_t) \rightarrow \text{Irr}(E \otimes \mathfrak{t}, \nabla_t)$ is surjective.

**Proof** Let $\text{Sol}(P_t)$ denote the set of the solutions of $P_t(T) = 0$. By assumption, any element of $\text{Sol}(P_t)$ is of the form $\alpha + \beta$ as above. We have the natural map $k[s_d](t_d) \rightarrow k((s_d'))((t_d)) \cong k((s_d)(t_d))$. The image of $\text{Sol}(P_t)$ via $k_1$ gives the set of the solutions of $P_t(T) = 0$ in $k((s_d)(t_d))$. We remark that the image of $k[s_d', t_d]$ via $k_1$ is contained in $k((s_d)(t_d))$. Hence, we have $k_1(\alpha + \beta -) = k_1(\alpha -) \in k[s](t_d)$ for any $\alpha + \beta \in \text{Sol}(P_t)$.

The first claim follows from the characterization of the irregular value given in Lemma 2.16.

On the other hand, let us take the specialization of $P_t(T)$ to $s = 0$, which are denoted by $P_{t,0}(T) \in \mathfrak{t}[T]$. Let $\text{Sol}(P_{t,0})$ denote the solution of the equation $P_{t,0}(T) = 0$, which is contained in $k((t_d))$ for some appropriate $d$. Then, $\text{Sol}(P_{t,0})$ is the image of $\text{Sol}(P_t)$ by the composite $k_2$ of the following morphisms:

$$k[s_d']((t_d)) \simeq k[s]((t_d)) / (U - s'(s, t)) \rightarrow k((s_d)(t_d)) / (U - s'(0, t)) \rightarrow k((t_d)).$$

The last map is given by the substitution $U = s'(0, t)^{1/d} \in k((t_d))$ for some choice of $s'(0, t)^{1/d}$. Any element of $k[s_d', t_d]$ is mapped into $k[t_d]$ via $k_2$, and the image of any element of $k[s]((t_d)) = k[s']((t_d))$ via $k_2$ is given by the natural specialization at $s = 0$. Hence, for any $k_2(\alpha + \beta -) \in \text{Sol}(P_{t,0}(T))$, we have $k_2(\alpha + \beta -) = k_2(\alpha -)$. Then, the second and third claims follow from the characterization of the irregular values in Lemma 2.16.

Let $\varphi : \text{Spec}^f k[v] \rightarrow \text{Spec}^f k[s,t]$ be a morphism given by $\varphi^*(v) = v \cdot \overline{h}_0(v)$ and $\varphi^*(t) = v^a$ for some $a > 0$. We assume $a$ is sufficiently smaller than $p$. We consider the morphism $\Phi : \text{Spec}^f k[u, v] \rightarrow \text{Spec}^f k[s, t]$ given by $\Phi^* s = u + v \overline{h}_0(v)$ and $\Phi^* t = v^a$. Then, we have $\Phi^* s' = \Phi^*(s - h(t) \cdot t) = u + v \overline{h}_0(v) - h(t^a) \cdot v^a = u + v \cdot h_1(v)$. In particular, the divisor $\Phi^*(s') = 0$ is smooth and transversal to the divisor $\{ v = 0 \}$.

**Lemma 3.3** $\Phi^*(E, \nabla)$ is mildly ramified at $\{ v = 0 \} \cup \{ \Phi^*(s') = 0 \}$.

[11]
Proof Let $\Phi^*(ds) = a_{1,1} \cdot du + a_{1,2} \cdot dv/v$ and $\Phi^*(dt/t) = a_{2,1} \cdot du + a_{2,2} \cdot dv/v$, where $a_{i,j}$ are contained in $k[u,v]$. Then, due to a formula of O. Gabber (Appendix of [11]) we have the following:

$$
\Phi^*(\psi)(\partial_u) = a_{1,1}^p \cdot \Phi^*(\psi(\partial_u)) + a_{2,1}^p \cdot \Phi^*(\psi(t\partial_t)), \quad \Phi^*(\psi(\partial_u)) = a_{1,1}^p \cdot \Phi^*(\psi(\partial_u)) + a_{1,2}^p \cdot \Phi^*(\psi(t\partial_t))
$$

Then, it is easy to check the claim of the lemma because of the commutativity of $\psi(\partial_u)$ and $\psi(t\partial_t)$.

3.2 Mixed characteristic case

Let $R$ be a subring of $C$ finitely generated over $\mathbb{Z}$. Let $\mathcal{E}_R$ be a free $R[s]\langle(t)\rangle$-module, and let $\nabla : \mathcal{E}_R \to \mathcal{E}_R \otimes \Omega^1_{\mathcal{E}_R(s)\langle(t)\rangle/R}$ be a meromorphic flat connection. For each $\eta \in S(R)$, we put $\mathcal{E}_{\eta} := \mathcal{E}_R \otimes_R k(\eta)$, and we have the induced meromorphic flat connection $\nabla$ of $\mathcal{E}_{\eta}$.

Definition 3.4 We say that $(\mathcal{E}_R, \nabla)$ is mildly ramified, if $(\mathcal{E}_{\eta}, \nabla)$ is mildly ramified for any $\eta \in S(R)$. Note that the ramification curves may depend on $\eta$.

If $(\mathcal{E}_R, \nabla)$ is mildly ramified, it is easy to show that $(\mathcal{E}_R, \nabla) \otimes R'$ is also mildly ramified for any $R' \subset C$ finitely generated over $R$.

3.3 Complex number field case

Let $\mathcal{E}_C$ be a free $C[s]\langle(t)\rangle$-module with a meromorphic connection $\nabla : \mathcal{E}_C \to \mathcal{E}_C \otimes \Omega^1_{\mathcal{E}_C(s)\langle(t)\rangle/C}$.

Definition 3.5 We say that $(\mathcal{E}_C, \nabla)$ is algebraic, if there exists a subring $R \subset C$ finitely generated over $\mathbb{Z}$, a free $R[s]\langle(t)\rangle$-module $\mathcal{E}_R$ with a meromorphic connection $\nabla : \mathcal{E}_R \to \mathcal{E}_R \otimes \Omega^1_{\mathcal{E}_R(s)\langle(t)\rangle/R}$ such that $(\mathcal{E}_R, \nabla) \otimes_R C \simeq (\mathcal{E}_C, \nabla)$. Such $(\mathcal{E}_R, \nabla)$ is called an $R$-model of $(\mathcal{E}_C, \nabla)$.

Definition 3.6 Let $(\mathcal{E}_C, \nabla)$ be algebraic. We say $(\mathcal{E}_C, \nabla)$ is mildly ramified, if an $R$-model of $(\mathcal{E}_C, \nabla)$ is mildly ramified for some $R$.

We put $\mathfrak{R}_C := C(s)\langle(t)\rangle$ and $\mathfrak{t}_C := C\langle(t)\rangle$. We have the induced relative connection $\nabla_t : \mathcal{E}_C \to \mathcal{E}_C \otimes \Omega^1_{\mathcal{E}_C(s)\langle(t)\rangle/C}$. We put $(\mathcal{E}_{\mathfrak{R}_C}, \nabla_t) := (\mathcal{E}_C, \nabla_t) \otimes_{\mathcal{E}_C} \mathfrak{R}_C$ and $(\mathcal{E}_{\mathfrak{t}_C}, \nabla_t) := (\mathcal{E}_C, \nabla_t) \otimes_{\mathcal{E}_C} \mathfrak{t}_C$.

Proposition 3.7 Assume that $(\mathcal{E}_C, \nabla)$ is algebraic and mildly ramified. Then the irregular values of $(\mathcal{E}_{\mathfrak{R}_C}, \nabla_t)$ are contained in $C[s]\langle(t_d)\rangle_{<0}$ for some $d \in \mathbb{Z}_{>0}$, and their specializations at $s = 0$ give the irregular values of $(\mathcal{E}_{\mathfrak{t}_C}, \nabla_t)$. The induced map $\text{Irr}(\mathcal{E}_{\mathfrak{R}_C}, \nabla_t) \to \text{Irr}(\mathcal{E}_{\mathfrak{t}_C}, \nabla_t)$ is surjective.

Proof We take a subring $R \subset C$ finitely generated over $\mathbb{Z}$, and an $R$-model $(\mathcal{E}_R, \nabla)$ such that $(\mathcal{E}_R, \nabla) \otimes_R C \simeq (\mathcal{E}_C, \nabla)$. We may assume that the irregular decomposition of $(\mathcal{E}_{\mathfrak{R}_C}, \nabla_t)$ is defined on $R((s_d))(t_d)$ (Corollary 2.11):

$$
(\mathcal{E}_{\mathfrak{R}_C}, \nabla_t) \otimes_R R((s_d))(t_d) = \bigoplus_{a \in \text{Irr}(\mathcal{E}_{\mathfrak{R}_C}, \nabla_t)} (\mathcal{E}_a, \nabla_{a,t})
$$

(4)

We may also have a Deligne-Malgrange lattice $\bigoplus E_a \subset \bigoplus \mathcal{E}_a$.

Let $p$ be a sufficiently large prime, and let $\eta$ be any point of $S(R, p)$. We put $\mathfrak{R}_{\eta} := k(\eta)((s_d))(t_d)$ and $\mathfrak{t}_{\eta} := k(\eta)((t_d))$. We have the decomposition of $(\mathcal{E}_{\mathfrak{R}_{\eta}}, \nabla_t) := (\mathcal{E}_{\mathfrak{R}_C}, \nabla_t) \otimes_{\mathcal{E}_{\mathfrak{R}_C}} \mathfrak{R}_{\eta}$ induced by $[11]$:

$$
(\mathcal{E}_{\mathfrak{R}_{\eta}}, \nabla_t) = \bigoplus_{a \in \text{Irr}(\mathcal{E}_{\mathfrak{R}_C}, \nabla_t)} (\mathcal{E}_a, \nabla_{a,t})
$$

We use the notation $\mathcal{F}_{\mathfrak{R}}$ to denote the naturally induced morphism $R((s_d))(t_d) \to \mathcal{E}_{\mathfrak{R}_C}$ and $R((t_d)) \to \mathfrak{t}_{\eta}$. Since $\nabla_t \otimes \text{id}_{\mathcal{E}_{\mathfrak{R}_C}}$ are logarithmic with respect to the lattice $E_a \mathfrak{R}_{\eta}$, we can conclude that $\text{Irr}(\mathcal{E}_{\mathfrak{R}_{\eta}}, \nabla_t)$ is the image of $\text{Irr}(\mathcal{E}_{\mathfrak{R}_C}, \nabla_t)$ via the map $\mathcal{F}_{\mathfrak{R}}$. Due to Lemma 3.2 $\text{F}_{\mathfrak{R}}(a)$ are contained in $k(\eta)[s](t_d)_{<0}$ for any $a \in \text{Irr}(\mathcal{E}_{\mathfrak{R}_C}, \nabla_t)$, then it follows that $a$ are contained in $R[s](t_d)_{<0}$.

Moreover, $\text{F}_{\mathfrak{R}}(a|_{s=0}) = \text{F}_{\mathfrak{R}}(a)_{|s=0}$ give the irregular values of $(\mathcal{E}_{\mathfrak{R}_{\eta}}, \nabla_t)$ for any $a \in \text{Irr}(\mathcal{E}_{\mathfrak{R}_C}, \nabla_t)$, due to Lemma 3.2. To conclude that $a|_{s=0}$ gives the irregular values of $(\mathcal{E}_{\mathfrak{R}_{\eta}}, \nabla_t)$, we use the following lemma.
Lemma 3.8 Let $(\mathcal{E}, \nabla)$ be a meromorphic connection on $R((t))$. Let $a \in R((t_d))_{< 0}$. If $\mathcal{F}_{\mathcal{E}}(a)$ are the irregular values for $(\mathcal{E}, \nabla)(\mathcal{E}, \nabla)$ on $k(\eta)((t))$ for any $\eta$, then $a$ is an irregular value for $(\mathcal{E}, \nabla) \otimes C((t))$.

Proof We may assume to have the irregular decomposition $(\mathcal{E}, \nabla) = \bigoplus_{i} (\mathcal{E}_i, da_i + \nabla_{i}^{\text{reg}})$ on $R((t_d))$, due to Corollary 2.11. Then, for some $i$, there are infinitely many $\eta \in S(R)$ such that $\mathcal{F}_{\mathcal{E}}(a_i) = \mathcal{F}_{\mathcal{E}}(b_i)$ in $k(\eta)((t_d))_{< 0}$. It implies $a_i = a_0$. Thus, we obtain Lemma 3.8.

Let us return to the proof of Proposition 3.7. Let $a \in \text{Irr}(\mathcal{E}_{AC}, \nabla_i)$. Because of the surjectivity in Lemma 3.2, there exists $a \in \text{Irr}(\mathcal{E}_{AC}, \nabla_i)$ such that $\mathcal{F}_{\mathcal{E}}(a_{i,0}) = \mathcal{F}_{\mathcal{E}}(b)$ in $k(\eta)((t_d))_{< 0}$ for infinitely many $\eta \in S(R)$. It implies $a_{i,0} = b$. Hence, we obtain the surjectivity of the induced map $\text{Irr}(\mathcal{E}_{AC}, \nabla_i) \hookrightarrow \text{Irr}(\mathcal{E}_{AC}, \nabla_i)$. Thus the proof of Proposition 3.7 is finished.

Let $\varphi_C : \text{Spec}^f C[v] \rightarrow \text{Spec}^f C[s, t]$ be an algebraic morphism, i.e., there exist a morphism $\text{Spec} A_1 \rightarrow \text{Spec} A_2$ for some regular rings $A_i$ ($i = 1, 2$) finitely generated over $C$, such that the completion at some closed points is isomorphic to $\varphi_C$. We assume $\varphi_C(t) \neq 0$. We have the induced map $\varphi_{<0}^* : \mathcal{E}[s, t]/\mathcal{E}[s, t] \rightarrow \mathcal{C}[u]/\mathcal{C}[u]$ for any $d$.

Proposition 3.9 If $(\mathcal{E}, \nabla)$ is algebraic and mildly ramified, the set of the irregular values of $\varphi_{<0}^*(\mathcal{E}, \nabla)$ is given by the image of $\text{Irr}(\mathcal{E}, \nabla_i)$ via $\varphi_{<0}^*$.

Proof By extending $R$, we may assume that $\varphi$ is induced from $\varphi_R : \text{Spec}^f R[v] \rightarrow \text{Spec}^f R[s, t]$ given by $\varphi_R^*(t) = v^a$ and $\varphi_R^*(s) = v \cdot h(v)$. We have the induced map $\varphi_{<0}^* : R((s_d))(t_d) \rightarrow R((t_d))$. Let $\Phi : \text{Spec}^f R[u, v] \rightarrow \text{Spec}^f R[u, v]$ be given by $t = v^a$ and $s = u + v \cdot h(v)$. Then, $\Phi^*(\mathcal{E}, \nabla)$ is mildly ramified because of Lemma 3.3.

We put $\mathcal{E}_{\mathcal{E}_{AC} \otimes C[s]}(t_d, \nabla_i)$ and $\mathcal{E}_{\mathcal{E}_{AC} \otimes R(u, v)}$ on which the relative connection $\nabla_v$ is induced. We have the induced map $R[s]/(t_d)/R[s, t_d] \rightarrow R[u]/(v_d)/R[u, v_d]$, which is denoted by $\Phi_{<0}^*$. Then, we have only to show that $\text{Irr}(\mathcal{E}_{\mathcal{E}_{AC} \otimes R(u, v)}, \nabla_v)$ is the same as the image of $\text{Irr}(\mathcal{E}_{\mathcal{E}_{AC} \otimes \nabla_i})$ via the map $\Phi_{<0}^*$ due to Proposition 3.7. Since both of them are contained in $C[u]/(v_d)/C[u, v_d]$, we have only to compare them in $C[u]/(v_d)/C[u, v_d]$.

The meromorphic connection $(\mathcal{E}_{\mathcal{E}_{AC} \otimes C[s]}(t_d, \nabla_i))$ is unramified, because the irregular values are contained in $C[s]/(t_d)/C[s, t_d]$. By Lemma 2.11, we have a strict Deligne-Malgrange lattice $E_{\mathcal{E}_{AC}}$ which is free $C[s]/(t_d)$-module, and the irregular decomposition with respect to the relative connection $\nabla_v$:

$$(E_{\mathcal{E}, \nabla_i} = \bigoplus_{a \in \text{Irr}(\mathcal{E}_{\mathcal{E}_{AC} \otimes \nabla_i})} (E_a, \nabla_{a, t})$$

Due to the uniqueness of the irregular decomposition and the commutativity of $\nabla(\partial_s)$ and $\nabla(t \partial_t)$, it is standard to show that $\nabla(\partial_s)(E_a(t_d)) \subset E_a(t_d)$. (See the proof of Lemma 2.11 for example.) Hence, it is the decomposition of the meromorphic flat connection:

$$(E_{\mathcal{E}, \nabla}) = \bigoplus (E_a, \nabla_a).$$

We put $\nabla_a := \nabla_a - da$. By construction, we have $\nabla_a(t_d \partial_t)(E_a) \subset E_a$. Since $E_{\mathcal{E}_{AC}}$ is assumed to be strict Deligne-Malgrange, it can be shown that $\nabla_a(\partial_s)(E_a) \subset E_a$ by a standard argument. (See the last part of the proof of Proposition 2.14 for example.) We put $\nabla_a = \bigoplus \nabla_a$. Let $v$ be a frame of $E_{\mathcal{E}_{AC}}$ compatible with the irregular decomposition. Let $A$ and $B$ be determined by $\nabla^a - v \cdot (A \cdot dt_d/t_d + B \cdot ds)$. Then, $A, B \in M_v(C((s))\{t_d\})$. We remark $\Phi^*(s)^{-k} \in C((u))/v$ for any integer $k$. Then, it is easy to see that $E_{\mathcal{E}_{AC} \otimes C((u))}(v_d)$ gives a Deligne-Malgrange lattice of $\mathcal{E} \otimes C((u))(v_d)$ with respect to $\nabla(v_d \partial_d)$, and the irregular decomposition of $\Phi^*(\mathcal{E}, \nabla)$ is given as follows:

$$(\mathcal{E} \otimes C((u))(v_d) = \bigoplus_{v \in C((u))(v_d)/C((u))(v_d)} \left( \bigoplus_{\Phi_{<0}^*(a) = b} E_a \otimes C((u))(v_d) \right).$$

Thus, we are done.


4 Resolution of turning points

4.1 Resolution of the discriminants of polynomials

Let $R$ be a regular subring of $\mathbf{C}$ which is finitely generated over $\mathbf{Z}$. Let $X_R$ be a smooth projective surface over $R$. Let $D_R$ be a simply effective normal crossing divisor of $X_R$. We assume that $X_R \otimes_{\mathbf{Z}} \mathbf{Z}/p\mathbf{Z}$ is smooth or empty for each $p$. Let $N$ be a positive integer.

Take $q \in S(R, p)$. We put $X_\eta := X_R \otimes_{\mathbf{Z}} \mathbf{Z}/p\mathbf{Z}$. We denote the function field of $X_\eta$ by $K(X_\eta)$. Let $\mathcal{P}(a)(T) \in \bigoplus_{j=0}^r H^0(X_\eta, \mathcal{O}_X(jND_\eta)) \cdot T^{r-j}$ be monic polynomials $(a = 1, \ldots, L)$. The tuple $(\mathcal{P}(a)|_{a = 1, \ldots, L})$ is denoted by $\mathcal{P}$. We regard them as elements of $K(X_\eta)[T]$. Let $\mathcal{P}(a) = \prod_{i=1}^{a(m)} (P_i^{(a)})^{(a)}$ be the irreducible decomposition. The monic polynomials $P_i^{(a)}$ are contained in $\bigoplus_{j=0}^r H^0(X_\eta, \mathcal{O}_X(jND_\eta)) \cdot T^{r(a)-j}$, where $n_i(a) := \deg_T P_i^{(a)}$. We regard the discriminants $\text{disc}(P_i^{(a)})$ as the elements of the function field $K(X_\eta)$. There exists a constant $M_1 > 0$, which is independent of the choice of $\eta$ and $p$, such that $\text{disc}(P_i^{(a)})$ are contained in $H^0(X_\eta, \mathcal{O}_X(M_1 : D_\eta))$. We put as follows:

$$\text{disc}(\mathcal{P}) := \prod_{a=1}^L \prod_{j=1}^{m(a)} \text{disc}(P_j^{(a)}) \in K(X_\eta)$$

There exists a constant $M_2 > 0$, which is independent of the choice of $\eta$ and $p$, such that $\text{disc}(\mathcal{P})$ is contained in $H^0(X_\eta, \mathcal{O}_X(M_2 : D_\eta))$. Let $Z(\mathcal{P})$ denote the 0-set of $\text{disc}(\mathcal{P})$, when we regard $\text{disc}(\mathcal{P})$ as a section of the line bundle $\mathcal{O}_X(M_2 : D_\eta)$. We may assume $D_\eta \subset Z(\mathcal{P})$, by making $M_2$ larger. Since $Z(\mathcal{P})$ is a member of some bounded family, the following lemma immediately follows from the flattening lemma (see [18]) and the semi-continuity theorem (see [19]) for the flat family.

**Lemma 4.1** There exists a constant $M_3$, which is independent of $\eta$ and $p$, such that the arithmetic genus of $Z(\mathcal{P})$ is smaller than $M_3$.

Let $P$ be any closed point of $X_\eta$. We put $(X_\eta^0, P^0) := (X_\eta, P)$. Inductively, let $\pi(i) : X_\eta^i \longrightarrow X_\eta^{i-1}$ be the blow up at $P^{(i-1)}$, and let us take a point $P^{(i)} \in \pi(i)(P^{(i-1)})$. Let $\pi_i$ denote the naturally induced map $X^{(i)} \longrightarrow X$. By the classical arguments (see Section V.3 in [20], for example), we can show the following lemma.

**Lemma 4.2** There exists some $i_0$, independent of the choice of $p$, $\eta$ and the points $P^{(i)}$, such that the divisor $(\pi(i))^{-1}Z(\mathcal{P})$ is normal crossing around the exceptional divisor $(\pi(i))^{-1}(P^{(i-1)})$ for any $i \geq i_0$.

**Proof** We give only an outline. We use the notation $p_a$ to denote the arithmetic genus. Let $Y$ denote the reduced scheme associated to $Z(\mathcal{P})$. Let $Y = \bigcup Y_j$ denote the irreducible decomposition. We have $p_a(Y) \leq M_3$ and $p_a(Y_j) \leq M_3$. Let $\tilde{Y}_i$ denote the inverse image of $Y$ via $\pi_i$ with the reduced structure. Let $\tilde{Y}_i$ denote the strict transform of $Y_j$ via $\pi_i$. Let $C_i$ denote the strict transform of $(\pi^{(a)})^{-1}(P^{(a-1)})$ via the natural map $X^{(i)} \longrightarrow X^{(q)}$. We have $\tilde{Y}_i = \bigcup \tilde{Y}_{i,j} \cup \bigcup C_i$. Let $r_{P^{(q)}}(\tilde{Y}_q)$ denote the multiplicity of $P^{(q)}$ in $\tilde{Y}_q$. We use the notation $r_{P^{(q)}}(\tilde{Y}_{i,j})$ in a similar meaning. We have the equality (Section V.3 of [20]):

$$p_a(\tilde{Y}_{i,j}) = p_a(Y_j) - \sum_{q \leq i-1} \frac{1}{2} r_{P^{(q)}}(\tilde{Y}_{q,j}) \cdot (r_{P^{(q)}}(\tilde{Y}_{q,j}) - 1)$$

By our choice, $P^{(i)}$ is a smooth point of $\tilde{Y}_{i,j}$ for any $i \geq i(1)$ if $P^{(i-1)}$ is a smooth point of $\tilde{Y}_{i-1,j}$. Hence, we obtain $r_{P^{(i)}}(\tilde{Y}_{i,j}) \leq 1$ if $i$ is sufficiently large. We also have the following equality (Section V.3 of [20]):

$$p_a(\tilde{Y}_i) = p_a(Y) - \sum_{q \leq i-1} \frac{1}{2} (r_{P^{(q)}}(\tilde{Y}_q) - 1) \cdot (r_{P^{(q)}}(\tilde{Y}_q) - 2)$$

Assume $r_{P^{(q)}}(\tilde{Y}_q) = 2$. Then, as explained in the proof of Theorem 3.9 in Section V of [20], there are three possibility:
• \( \tilde{Y}_q \) is normal crossing around \( P(q) \).

• Let \( \tilde{Y}_{q+1} \) denote the strict transform of \( \tilde{Y}_q \) via \( \pi^{(q+1)} \). Then, it is nonsingular in a neighbourhood of \( (\pi^{(q+1)})^{-1}(P(q)) \), and \( \tilde{Y}_{q+1} \) and \( (\pi^{(q+1)})^{-1}(P(q)) \) intersect at one point with multiplicity 2. If \( P(q+1) \) and \( P(q+2) \) are also singular points of \( \tilde{Y}_{q+1} \) and \( \tilde{Y}_{q+2} \) respectively, we have \( r_{P(q+2)}(\tilde{Y}_{q+2}) = 3 \).

• \( \tilde{Y}_{q+1} \) and \( (\pi^{(q+1)})^{-1}(P(q)) \) intersects at one point, whose multiplicity in \( \tilde{Y}_{q+1} \) is 2. If \( P(q+1) \) is singular point of a \( \tilde{Y}_{q+1} \), we have \( r_{P(q+1)}(\tilde{Y}_{q+1}) = 3 \).

Hence, we obtain that \( \tilde{Y}_i \) are normal crossing around \( P(i) \) for sufficiently large \( i \).

Let \( i \geq i_0 \). Let \( C_i(\mathcal{P}) \) denote the closure of \( Z(\mathcal{P}) \cap (X_{\eta} - D_{\eta}) \) in \( X^{(i)}_{\eta} \). We take a local coordinate neighbourhood \( (U^{(i)}, s^{(i)}, t^{(i)}) \) around \( P(i) \) such that (i) \( (t^{(i)})^{-1}(0) = U^{(i)} \cap (\pi^{(i)})^{-1}(P(i-1)) \), (ii) if \( P(i) \) is contained in \( C_i(\mathcal{P}) \), then \( (s^{(i)})^{-1}(0) = U^{(i)} \cap C_i(\mathcal{P}) \), (ii)' if \( P(i) \) is not contained in \( C_i(\mathcal{P}) \), then \( s^{(i)} \) may be anything. Because of generalized Abhyankar’s lemma (see Expose XIII Section 5 of [23]), any solutions of the equations \( \pi^{(a)}_s \mathcal{P} \mathcal{J}^{(a)}(T) = 0 \) \( (a = 1, \ldots, L, j = 1, \ldots, m(a)) \) are contained in \( k(\eta)[s^{(i)}_d][t^{(i)}_d] \) for some appropriate \( d \), which is a factor of \( r! \).

Lemma 4.3 There exists an \( i_1 \), which is independent of the choice of \( \eta, p \) and the points \( P(i) \), such that the following holds for any \( i \geq i_1 \):

• Any solutions of the equations \( \pi^{(a)}_s \mathcal{P} \mathcal{J}^{(a)}(T) = 0 \) \( (a = 1, \ldots, L, j = 1, \ldots, m(a)) \) are contained in the following:

\[
k(\eta)[s^{(i)}_d][t^{(i)}_d] + k(\eta)[s^{(i)}][t^{(i)}]
\]

Proof If \( P(i_0) \) is not contained in \( C_{i_0}(\mathcal{P}) \), then \( P(i) \notin C_i(\mathcal{P}) \) for any \( i \geq i_0 \), and the claim is obvious in this case. Assume \( P(i_0) \) is contained in \( C_{i_0}(\mathcal{P}) \). Let \( \alpha^{(i_0)}_l \in k(\eta)[s^{(i_0)}_d][t^{(i_0)}_d] \) be any solution of \( \pi^{(a)}_s \mathcal{P} \mathcal{J}^{(a)}(T) = 0 \) for some \( (a, j) \). Note that there exists a constant \( M_4 \), which is independent of the choice of \( \eta, p \), the sequence of the points \( P(i) \), with the following property:

• The orders of the poles of the coefficients of \( \pi^{(a)}_s \mathcal{P} \mathcal{J}^{(a)}(T) \) with respect to \( t^{(i)}_d \) are dominated by \( M_4 \).

Hence, there exists a constant \( M_5 \), which is independent of the choice of \( \eta, p \), the sequence of the points \( P(i) \), \( (a, j) \) and \( \alpha^{(i_0)}_l \), with the following property:

• The order of the pole of \( \alpha^{(i_0)}_l \) with respect to \( t^{(i_0)}_d \) are dominated by \( M_5 \).

If \( P(i) \) are contained in \( C_i(\mathcal{P}) \) for \( i \geq i_0 \), we may assume \( \pi^{(a)}(s^{(i-1)}) = s^{(i)} \cdot t^{(i)} \) and \( \pi^{(a)}(t^{(i-1)}) = t^{(i)} \). Hence, the pull back of \( \alpha^{(i_0)}_l \) via \( X^{(i)}_{\eta} \longrightarrow X^{(i_0)}_{\eta} \) are contained in \( k[s^{(i)}_d][t^{(i)}_d] \) for sufficiently large \( i \).

4.2 Proof of Theorem 1.1

If we take a sufficiently large \( R \), then \( E, \nabla, X \) and \( D \) come from \( E_R, \nabla_R, X_R \) and \( D_R \) which are defined over \( R \). We may also assume that we have the canonical lattice \( E_R \subset E_R \) defined over \( R \). (See [15]) By applying a theorem of Sabbah ([21]), we may assume that any cross points of \( D \) are not turning. Let \( P \) be a turning point contained in a smooth part of \( D \). Let \( U \) be a neighbourhood of \( P \) with an étale morphism \( (x, y) : U \longrightarrow A^2 \) such that \( x^{-1}(0) = D \cap U \). For simplicity, \( U \) does not contain any other turning points than \( P \). We may assume \( P \) and \( (U, x, y) \) are also defined over \( R \). We have only to take a proper birational map \( \pi : U' \longrightarrow U \) such that \( \pi^{-1}(E, \nabla) \) has no turning points. On \( U \), we have the vector field \( t d x \) and \( s d y \). By taking blow up of \( X \) outside of \( U \), and by extending \( D \), we may assume that \( x \partial x \) and \( \partial y \) are sections of \( \Theta_X(M_0 D) \). We have a positive number \( M'_0 \) such that \( \nabla(E_R) \subset E_R(M'_0 D_R) \). Hence, we have the constant \( M_1 \) such that \( \nabla(x \partial x)(E_R) \) and \( \nabla(\partial y)(E_R) \) are contained in \( E_R(M_1 D_R) \).

Let \( p \) be a large prime. For \( \eta \in S(R, p) \), let \( E_{\eta}, \nabla_{\eta}, X_{\eta}, D_{\eta}, P_{\eta} \) and \( U_{\eta} \) denote the induced objects over \( k(\eta) \). Let \( \psi \) be the \( p \)-curvature of \( (E_{\eta}, \nabla_{\eta}) \). We put \( \psi_x := \psi(x \partial x) \) and \( \psi_y := \psi(\partial y) \). Because of \( \nabla(\partial x)(E_{\eta}) \subset k(\eta)[s^{(i)}_d][t^{(i)}_d] \) for sufficiently large \( i \).
Because of the Cartier descent, we have only to show finitely generated over $\text{Spec } F$. Thus $T\Gamma(\Psi^{\kappa,j})$ are defined over $\mathbf{Q}$.) Let $\eta \in S(R,p)$. We take geometric points $\overline{\eta}(i)$ of $S(R^{(i)}, p)$ for any $i$ with the morphisms $\overline{\eta}(i) \to \overline{\eta}(i - 1) \to \overline{\eta}$ compatible with $\text{Spec } R^{(i)} \to \text{Spec } R^{(i - 1)} \to \text{Spec } R$. For $j \leq i$, $X^{(j)}$ are defined over $R^{(i)}$, and we have $X^{(j)}_{R^{(i)}} \subseteq R^{(i)} k(\overline{\eta}(i)) \simeq X^{(j)}_{\overline{\eta}(i)} \otimes_{\overline{\eta}(i)} k(\overline{\eta}(i))$. And the objects over them are naturally related by the pull backs.

Let $P_{x}(T) = \prod P_{x,j}(T)^{\kappa,j}$ denote the irreducible decomposition of the polynomials $P_{x}(T)$ above $(\kappa = x, y)$. Applying Lemma 4.4 and Lemma 4.3, we can show that there exist $i_{1}$ and $p_{1}$ such that the following claims hold for any $i \geq i_{1}, p \geq p_{1}$ and any $\overline{\eta}(i)$:

- Let $C$ be any exceptional divisor with respect to $\pi^{(i)}_{\overline{\eta}(i)}$. Then, $\bigcup_{j,\kappa} T_{\overline{\eta}(i)}(\text{disc}(P_{x,j}) \cup D)$ are normal crossing around $C$.
- Let $Z^{(i)}_{\kappa,j}$ denote the closure of $\text{disc}(P_{x,j}) \cap (X^{(i)}_{\overline{\eta}(i)} - D_{\overline{\eta}(i)})$ in $X^{(i)}_{\overline{\eta}(i)}$. If $C$ intersects at $Q$ with $Z^{(i)}_{\kappa,j}$ for some $(\kappa, j)$, we take a coordinate neighbourhood $(U_{Q}, z, w)$ such that $w^{-1}(0) = C \cap U_{Q}$ and $z^{-1}(0)$ is $Z^{(i)}_{\kappa,j} \cap U_{Q}$. Then, any solutions of $P_{x,j}(T) = 0$ are contained in $k(\overline{\eta}(i))[z, w] + k(\overline{\eta}(i))[z](w)$. We remark that the completion of $\pi^{*}_{\overline{\eta}(i)}(E, \nabla)$ at such $Q$ is mildly ramified, which can be shown by the same argument as the proof of Lemma 4.3.

Due to a theorem of Sabbah in [21], we can take a regular birational map $F : \overline{X} \to X^{(i)}$ as follows:

- $F$ is the blow up along the ideal supported at the cross points of the divisor $\pi_{i_{1}^{*}}(P)$.
- Any cross points of the divisor $G^{-1}(P)$ are not turning points for $G^{*}(E, \nabla)$, where $G := \pi_{i_{1}} \circ F$. Let $Q$ be a point of the smooth part of $G^{-1}(P) \subseteq \overline{X}$ which is a turning point for $\overline{E, \nabla}$. We remark that $F(Q) \in X^{(i)}$ is contained in some exceptional divisor with respect to $\pi^{(i)}$. We take a subring $R_{0} \subseteq C$ finitely generated over $R^{(i)}$, on which $Q$ is defined. We may also have a neighbourhood $U_{Q}$ with an étale map $(u, v) : U_{Q} \to A^{2}$ around $Q$ such that $u^{-1}(0) = G^{-1}(P) \cap U_{Q}$. By considering the completion at $Q$, we obtain the free $R_{0}[u][v]$-module $\hat{E}_{R_{0}}$ with a meromorphic connection $\nabla_{R_{0}}$. 

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Lemma 4.5 \((\mathcal{E}_{R_0}, \nabla_{R_0})\) is mildly ramified.

Proof Let \(\eta_0\) be a geometric point of \(\text{Spec } R_0\) over some \(\pi(i_1) \in S(\mathcal{E}(i_1))\). We have only to show that \((\mathcal{E}_{\eta_0}, \nabla_{\eta_0})\) is mildly ramified. Assume \(F(Q)\) is a cross point of the divisor \(\pi_{i_1}^{-1}(P)\). Then, \(F_{\eta_0}(\mathcal{E}_{\eta_0})\) is not contained in any \(Z_{i,j}\), and hence the ramification around \(Q_{\eta_0}\) may occur only at \(G_{\eta_0}^{-1}(P_{\eta_0})\). In the case where \(F(Q)\) is contained in the smooth part of \(\pi_{i_1}^{-1}(P)\), the claim follows from our choice of \(i_1\).

Then, we can control the irregular values for \((\mathcal{E}, \nabla)\).

Lemma 4.6 Let \(S\) denote the set of the irregular values of \((\mathcal{E}, \nabla) \otimes C((u)\!(v))\).

- \(S\) is contained in \(C[u]/(v_d)/C[u,v_d]\) for some appropriate \(d\).
- For any curve \(\varphi : C \to \hat{X}\) such that \(\varphi(C) \cap D_Q = \{Q\}\), where \(D_Q\) denotes the exceptional divisor containing \(Q\), the irregular values of \(\varphi^*(\mathcal{E}, \nabla)\) are given by the negative parts of \(\varphi^*a\) (\(a \in S\)).

Proof It follows from Proposition 3.7, Proposition 3.9 and Lemma 4.5.

Now, we use the classical topology. Let \(U\) be a neighbourhood of \(Q\) in \(\hat{X}\). We will shrink \(U\) without mention in the following argument, if it is necessary. Let \(\varphi : U \to \hat{U}\) be the ramified covering given by \((u, v_d) \mapsto (u, v_d^2)\) for some appropriate \(d\). We put \(G := Z/dZ\) which naturally acts \(\hat{U}\). We put \(D_d := \{v_d = 0\}\). Let \(M(U)\) (resp. \(H(U)\)) denote the space of meromorphic (resp. holomorphic) functions whose poles are contained in \(D_d\). For each \(a \in M(U)/H(U)\), we use the same notation to denote the natural lift to \(M(U)_c\). Because of Lemma 4.6, there exists the finite subset \(S \subset M(U)/H(U)\) which gives the irregular values of \((\mathcal{E}, \nabla) \otimes C((u)\!(v))\). (Meromorphic property of the irregular values is shown in Theorem 2.3.1 of [21], for example.) Let \(S_1\) denote the set of pairs \((a, b) \in S^2\) such that \(a \neq b\).

We put \(\varphi(a) := \prod_{\sigma \in G^*} \sigma^*a\) for any \(a \in S\) which give the meromorphic functions \(\varphi(a)\) on \(U\). For any \((a, b) \in S_1\), we have the meromorphic functions \(\varphi(a - b)\) on \(U\), similarly. The union of the zero and the pole of \(\varphi(a)\) is denoted by \(|\varphi(a)|\). We use the notation \(|\varphi(a - b)|\) in a similar meaning.

We can take the resolution \(\kappa : U_1 \to U\) such that the following holds:

- \(\kappa^{-1}(|\varphi(a)| \cup D_Q)\) and \(\kappa^{-1}(|\varphi(a - b)| \cup D_Q)\) are normal crossing for any \(a \in S\) and \((a, b) \in S_1\). Here \(D_Q\) denotes the component of \(G^{-1}(P)\) such that \(Q \in D_Q\).

- The zero and the pole of \(\kappa^{-1}(\varphi(a))\) have no intersections for any \(a \in S\). The zero and the pole of \(\kappa^{-1}(\varphi(a - b))\) have no intersections for any \((a, b) \in S_1\).

- For any \((a, b), (a', b') \in S_1\), the ideals generated by \(\kappa^{-1}(\varphi(a - b))\) and \(\kappa^{-1}(\varphi(a' - b'))\) are principal.

Applying Sabbah’s theorem, we can take \(\nu : U'' \to U\) such that any cross points of the divisor \((\kappa \circ \nu)^{-1}(Q)\) are not turning. We put \(\hat{\kappa} := \kappa \circ \nu\), for which the above three conditions are satisfied. For any point \(Q'\) of the smooth part of \(\hat{\kappa}^{-1}(Q)\), the irregular values of \(\hat{\kappa}^{-1}((\mathcal{E}, \nabla))\) around \(Q'\) are given by the negative parts of \(\hat{\kappa}^{-1}(a)\) due to Lemma 4.6. By using Proposition 2.18, we can conclude that \(Q'\) is not a turning point. Therefore, we have no turning points in \(\hat{\kappa}^{-1}(Q)\). Applying the procedure to any turning points for \((\mathcal{E}, \nabla)\) contained in \(G^{-1}(U)\), we can resolve them. Thus the proof of Theorem 1.1 is finished.

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