SURFACES CONTAINING A FAMILY OF PLANE CURVES NOT FORMING A FIBRATION

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Abstract. Extending some previous results of [20] and [4], we complete the classification of smooth surfaces swept out by a 1-dimensional family of plane curves that do not form a fibration. As a consequence, we characterize manifolds swept out by a 1-dimensional family of hypersurfaces that do not form a fibration.

Introduction

In a classical paper [18], Segre characterized the Veronese surface, the rational cubic scroll and the cones as the only surfaces in $\mathbb{P}^N$ containing a 2-dimensional family of plane curves when $N \geq 4$. In the late 80’s, the influential article [8] on surfaces in $\mathbb{P}^4$ led to the study of surfaces swept out by a 1-dimensional family of plane curves. Surfaces in $\mathbb{P}^4$ ruled in conics were classified in [7] (see also [5]) and [1], and surfaces in $\mathbb{P}^4$ fibred by plane curves of arbitrary degree were classified in [16]. In that paper it was suggested to study surfaces in $\mathbb{P}^4$ containing a family of plane curves not forming a fibration (see [16] Remark 0.3). The classification of such surfaces has recently been obtained in [20] and [4].

To complete the picture, in this note we focus on surfaces in $\mathbb{P}^N$ containing a 1-dimensional family of plane curves when $N \geq 5$. Fibrations can be obtained by taking a divisor on a scroll of planes over a curve, so these surfaces do not exhibit any special property. However, the situation is different when the surface is swept out by a family of plane curves not forming a fibration. This is due to the fact that families of planes in $\mathbb{P}^N$ such that any two of them intersect are special if $N \geq 5$, as Morin already noticed in [13] (cf. Theorem 3). Our results, combined with those of [20] and [4] for $N = 4$, can be summarized in the following way:

Theorem 1. Let $X \subset \mathbb{P}^N$ be a non-degenerate smooth linearly normal surface containing a 1-dimensional family of plane curves not forming a fibration. If $N \geq 4$ then one of the following holds:

(i) $X \subset \mathbb{P}^N$ is contained in a rational normal scroll $S := S_{0,a,b} \subset \mathbb{P}^{N=a+b+2}$, with $a \in \{0,1\}$, and it is linked to $N - 3$ rulings of $S$ by the complete intersection of $S$ and a hypersurface in $\mathbb{P}^N$;

(ii) $X \subset \mathbb{P}^5$ is the second symmetric product of a smooth plane curve, and the embedding is given by the embedding of the second symmetric product of the plane as the secant variety of the Veronese surface;

(iii) $X$ is a ruled surface with invariant $e = -1$ over an elliptic curve, and it is embedded either in $\mathbb{P}^4$ by $|C_0 + Lf|$, where $\deg(L) = 2$, or in $\mathbb{P}^5$ by $|2C_0 + Lf|$, where $\deg(L) = 1$.

As a consequence of Theorem 1 we obtain the following:

Corollary 1. Let $Y \subset \mathbb{P}^N$ be a non-degenerate manifold of dimension $n \geq 3$ containing a 1-dimensional family of hypersurfaces not forming a fibration. If $N$ –
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\[ n \geq 2 \text{ then } n = 3, N = 5 \text{ and } Y \subset \mathbb{P}^5 \text{ is linked to a linear } \mathbb{P}^3 \text{ by the complete intersection of a quadric cone of rank } 4 \text{ and a hypersurface in } \mathbb{P}^5. \]

Corollary I can be seen as the counterpart to the problem of classifying manifolds of dimension \( n \) in \( \mathbb{P}^N \) fibred by hypersurfaces (or even linear spaces) of dimension \( n - 1 \). According to the Barth-Larsen theorem, the Picard group of a manifold is generated by the hyperplane section if \( N \leq 2n - 2 \). On the other hand, hypersurface fibrations are not special when \( N \geq 2n + 1 \) so this problem makes sense only if \( 2n - 1 \leq N \leq 2n \). Manifolds in \( \mathbb{P}^{2n-1} \) fibred by linear spaces of dimension \( n - 1 \), usually called \textit{scrolls}, were classified in [12], and manifolds in \( \mathbb{P}^{2n-1} \) swept out by hypersurfaces (either forming a fibration or not) have recently been classified in [19]. However, the situation seems to be more complicated for \( N = 2n \), where the irregularity of the manifold needs not to be zero. In fact, the classification of scrolls in \( \mathbb{P}^{2n} \) has only been obtained for \( n \leq 4 \) (see [11]), and, to the best of the author’s knowledge, hypersurface fibrations in \( \mathbb{P}^{2n} \) remain unexplored for \( n \geq 3 \).

The paper is organized as follows. In the preliminary section we study surfaces embedded in singular rational normal scrolls of planes, and we also recall some facts on families of intersecting planes that will be crucial in the sequel. In Section 2 we prove Theorem I and Corollary I. Finally, in Section 3 we point out some other consequences of Morin’s work on families of intersecting linear spaces.

1. Preliminaries

1.1. Notation and conventions. We work over the field of complex numbers. We adopt the following notation and conventions:

\( \mathbb{P}^N \): projective space of dimension \( N \)
\( \mathbb{P}^N^* \): dual projective space
\( \mathcal{G}(k, N) \): Grassmann variety of \( k \)-planes in \( \mathbb{P}^N \)
\( a_0, \ldots, a_n \): sequence of non-negative integers satisfying \( 0 \leq a_0 \leq \cdots \leq a_n \)
\( S := S_{a_0, \ldots, a_n} \): rational normal scroll of \( n \)-planes in \( \mathbb{P}^{d_{a_0} + \cdots + n} \), where \( d_S := \sum_{i=0}^{n} a_i \)
\( \tilde{S} \): projectivization of the vector bundle \( E := \mathcal{O}_{\mathbb{P}^1}(a_0) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n) \) over \( \mathbb{P}^1 \)
\( X \): integral subvariety of \( \mathbb{P}^N \)
\( X \): strict transform in \( \tilde{S} \) of a variety \( X \subset S \)
\( T_x X \): embedded tangent space to \( X \subset \mathbb{P}^N \) at \( x \)
\( \sim \): rational equivalence of cycles
\( \{ C_b \}_{b \in B} \): 1-dimensional family of plane curves on \( X \) not forming a fibration
\( \Sigma \): 1-dimensional family of planes in \( \mathcal{G}(2, N) \) corresponding to \( \{ C_b \}_{b \in B} \)
\( V_\Sigma \): union of the planes of \( \Sigma \) in \( \mathbb{P}^N \)
\( S^2 Z \): second symmetric product of a variety \( Z \)
\( x + y \): point of \( S^2 Z \) corresponding to the unordered pair of points \( x, y \in Z \)
\( v_2(\mathbb{P}^k) \): 2-Veronese manifold in \( \mathbb{P}^{k(k+3)/2} \) given by \( |\mathcal{O}_{\mathbb{P}^2}(2)| \)
\( \phi_k \): map from \( S^2 \mathbb{P}^k \) to \( \mathbb{P}^{k(k+3)/2} \), whose image is the secant variety of \( v_2(\mathbb{P}^k) \)
\( v_k(\mathbb{P}^2) \): \( k \)-Veronese surface in \( \mathbb{P}^{k(k+3)/2} \) given by \( |\mathcal{O}_{\mathbb{P}^2}(k)| \)

1.2. Singular rational normal scrolls and Roth varieties. Let \( S := S_{a_0, \ldots, a_n} \subset \mathbb{P}^{N = d_{a_0} + \cdots + n} \) be a rational normal scroll of \( n \)-planes of degree \( d_S := \sum_{i=0}^{n} a_i = N - n \). In this paper we are interested in two particular cases, namely \( 0 = a_0 < a_1 \) and \( 0 = a_0 = a_1 < a_2 \). In both cases \( S \subset \mathbb{P}^N \) is an \((n + 1)\)-dimensional cone of vertex \( p \) (a point) and \( L \) (a line), respectively. We set some notation that will be used throughout the paper. Let \( \tilde{S} := \mathcal{P}(E) \) be the projectivization of the vector bundle \( E := \mathcal{O}_{\mathbb{P}^1}(a_0) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n) \) over \( \mathbb{P}^1 \). So \( \tilde{S} \) is a desingularization of \( S \) with natural
Let $F$ denote a fibre of $\pi_1$, and let $H$ denote the pullback by $\pi_2$ of a hyperplane section of $S \subset \mathbb{P}^N$. These two divisors generate the Chow ring $A(\tilde{S})$ of $\tilde{S}$, with relations $F^2 \sim 0$ and $H^{n+1} \sim d_3 H^n \cdot F$. For any effective Weil divisor $X$ on $S$, let $\tilde{X}$ denote the strict transform of $X$ on $\tilde{S}$. So $\tilde{X} \sim \alpha H + \beta F$ for some integers $\alpha, \beta$.

Manifolds $X$ of dimension $n \geq 2$ contained in $S_{0, a_2, \ldots, a_n}$ such that $L \subset X$ were widely studied in [10], extending to higher dimensions the study of surfaces contained in rational normal scrolls of planes initiated in Roth’s classical paper [17, §3.5], and they were called Roth varieties (see [10] Definition 3.1). Most of the results obtained in Ilic’s paper also hold for manifolds $X$ of dimension $n \geq 2$ contained in $S_{0, a_1, a_2, \ldots, a_n}$ such that $p \in X$ (see [10] Remark 5.7]). However, there is one case that does not seem to be considered there. More precisely, for the results obtained in Ilic’s paper also hold for manifolds $X$ of dimension $n \geq 2$ contained in rational normal scrolls of planes initiated in Roth’s classical paper [17, §3.5], and they were called Roth varieties (see [10] Definition 3.1). Most of the results obtained in Ilic’s paper also hold for manifolds $X$ of dimension $n \geq 2$ contained in $S_{0, a_1, a_2, \ldots, a_n}$ such that $p \in X$ (see [10] Remark 5.7]). However, there is one case that does not seem to be considered there. More precisely, for the results obtained in Ilic’s paper also hold for manifolds $X$ of dimension $n \geq 2$ contained in rational normal scrolls of planes initiated in Roth’s classical paper (as announced in [10] Remark 5.7] concerning Theorem 3.7). So we state [10] Remark 5.7] in the following way:

**Theorem 2.** Let $X \subset \mathbb{P}^N$ be a manifold of dimension $n$ contained in a rational normal scroll $S := S_{0, a_1, \ldots, a_n}$ with $a_i \geq 1$ and $\sum_{i=0}^n a_i = N - n$ such that $p \in X$.

Then either $\pi_2|\tilde{X} : \tilde{X} \rightarrow X$ is an isomorphism and $\tilde{X} \in [\alpha H + F]$, or $n = 2$, $\pi_2|\tilde{X} : \tilde{X} \rightarrow X$ is the blowing-up of $X$ at $p$, $S = S_{0, 1, b} \subset \mathbb{P}^{N-b+3}$, $\tilde{X} \in [\alpha H - b F]$ and $X \subset \mathbb{P}^N$ is linked to $N - 3$ rulings of $S$ by the complete intersection of $S$ and a hypersurface of degree $\alpha$ in $\mathbb{P}^N$. Furthermore, in both cases, such a manifold exists for every integer $\alpha \geq 1$.

*Proof.* Let $\xi := \pi_2^{-1}(p) \subset \tilde{S}$. If $\pi_2|\tilde{X} : \tilde{X} \rightarrow X$ is an isomorphism then $\tilde{X} : \xi = 1$, so we get $1 = \tilde{X} : \xi = (\alpha H + \beta F) : \xi = \beta$ and therefore $\tilde{X} \in [\alpha H + F]$, as claimed in [10] Remark 5.7. Otherwise, $\pi_2|\tilde{X} : \tilde{X} \rightarrow X$ contracts the curve $\xi$ by Zariski’s main theorem. Hence we deduce that $n = 2$ (if $n \geq 3$ then $\pi_2|\tilde{X}$ would be a small contraction and $X$ would be singular at $p$). This happens if and only if $\xi \equiv [\mathbb{P}^1]$ is a $(-1)$-curve on $\tilde{X}$, i.e. if and only if $\xi^2 = -1$. The exact sequence of normal bundles

$$0 \rightarrow N_\xi/\tilde{X} \rightarrow N_\xi/\tilde{S} \rightarrow (N_{\tilde{X}/\tilde{S}})_\xi \rightarrow 0$$

turns out to be

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b) \rightarrow \mathcal{O}_{\mathbb{P}^1}(\beta) \rightarrow 0,$$

as $\xi \sim H^2 - (a+b)H \cdot F$ and $(N_{\tilde{X}/\tilde{S}})_\xi = \tilde{X} : \xi = (\alpha R + \beta F) : (H^2 - (a+b)H \cdot F) = \beta$.

Therefore, $\beta = 1 - a - b$ and $\mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b) \in \text{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(\beta), \mathcal{O}_{\mathbb{P}^1}(-1)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1 - \beta)) = 0$, so $a = 1$ and $b = -\beta$, whence $\tilde{X} \in [\alpha H - b F]$ and $X$ is linked to $b = N - 3$ rulings of $S$ by the complete intersection of $S$ and a hypersurface of degree $\alpha$ in $\mathbb{P}^N$.

Let us show now the existence of $X$ for every integer $\alpha \geq 1$. Since $H + F$ is very ample on $\tilde{S}$ we deduce that $\alpha H + F$ is also very ample for any $\alpha \geq 1$, whence a general $\tilde{X} \in [\alpha H + F]$ is smooth by Bertini’s theorem (and hence irreducible). Assume now $n = 2$, $a = 1$ and $\tilde{X} \in [\alpha H - b F]$. Since $\xi \subset \tilde{X}$, we cannot directly deduce from Bertini’s theorem that a general $\tilde{X} \in [\alpha H - b F]$ is smooth. However, we can argue as in [8] pp. 60–61. Let $\mathbb{P}^2 \subset \mathbb{P}^N$ be the plane corresponding to the embedding $S_{0, 1} \subset S_{0, 1, b}$ (which is unique if $b > 1$). Hence $\mathbb{P}^2 \in [H - b F]$ and $\mathbb{P}^2 \equiv \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$, so in particular the statement for $\alpha = 1$ is proved. Let
Let \( \sigma : Bl_\xi(\tilde{S}) \to \tilde{S} \) be the blowing-up of \( \tilde{S} \) along \( \xi \), and let \( Bl(\tilde{X}) \) denote the strict transform of \( \tilde{X} \) on \( Bl_\xi(\tilde{S}) \). Note that \( \sigma_{|\sigma^{-1}(\xi)} : \sigma^{-1}(\xi) \to \xi \) is a \( \mathbb{P}^1 \)-bundle over \( \xi \), whose fibre is denoted by \( f \), and that \( Bl(\tilde{P}^2) \cdot f = 1 \). If \( \alpha \geq 2 \), we remark that \( \dim((\alpha H - bF)) \geq 1 \) and that \( Bl(X) \) cuts out the complete linear system \( \mathcal{O}_Y(1) \), and so it has no base points on \( \sigma^{-1}(\xi) \). It follows that \( |Bl(\tilde{X})| \) is a base-point-free linear system of positive dimension, whence its general element \( V \) is smooth by Bertini’s theorem. If \( V \) would be reducible, say \( V = V' + V'' \), then \( V' \cap V'' \neq \emptyset \) and \( V \) would be singular. Therefore \( V \) is irreducible. Furthermore,
\[
Bl(\tilde{X}) \cdot f = (Bl(\tilde{P}^2) + (\alpha - 1)Bl(H)) \cdot f = 1,
\]
as \( \tilde{X} \sim \tilde{\mathbb{P}}^2 + (\alpha - 1)H \) and \( H \cdot \xi = 0 \). So \( V \cdot f = 1 \) and therefore \( \sigma|_V : V \to \sigma|_V(V) \) is an isomorphism. In particular \( \sigma|_V(V) \in |\alpha H - bF| \), and hence \( X := \pi_2(\sigma|_V(V)) \) is also smooth and irreducible.

**Remark 1.** In Theorem 2 there is a more geometric way to see that \( a = 1 \) when \( \pi_2|_X : \tilde{X} \to X \) is the blowing-up of \( X \) at \( p \). Note that the embedded tangent lines at \( p \) to the plane curves on \( X \) sweep out the embedded tangent plane \( T_pX \), so we deduce that \( T_pX \subset S \) and that \( T_pX \) is not a ruling of \( S \). Therefore \( a = 1 \), since for \( a \geq 2 \) the only planes contained in \( S \) are the rulings.

**Remark 2.** Surfaces \( X \subset \mathbb{P}^N \) contained in singular rational normal scrolls of planes whose vertex is a point \( p \) were studied by Roth in \( \mathbb{P}[3.5] \). In \( \mathbb{P}[3.5] \) p. 156, it was stated that surfaces containing \( p \) are linked to \( N - 3 \) planes by the complete intersection of the scroll and a hypersurface in \( \mathbb{P}^N \). We would like to remark that this statement is correct when \( \pi_2|_X : \tilde{X} \to X \) is the blowing-up of \( X \) at \( p \), i.e. when \( \tilde{X} \in |\alpha H - bF| \), but it is no longer true when \( \pi_2|_X : \tilde{X} \to X \) is an isomorphism, i.e. when \( \tilde{X} \in |\alpha H + F| \).

**Remark 3.** According to [10, Definition 3.1], it is natural to extend the definition of Roth varieties to smooth \( n \)-dimensional subvarieties of \( S_{0,\ldots,0,a_{l+1},\ldots,a_n} \) containing the vertex of the scroll. For the sake of completeness, we point out that \( n \)-dimensional subvarieties \( X \) of \( S := S_{0,\ldots,0,a_{l+1},\ldots,a_n} \) are always singular as soon as \( l \geq 2 \). This follows from Zak’s theorem on tangencies [24, Ch. I, Corollary 1.8], arguing as in the proof of Corollary [11] if \( X \) is smooth then the tangent space \( T_pS \) to \( S \) at a general point \( s \in S \) is tangent to \( X \) along \( X_s := (\mathbb{P}^2, s) \cap X \), where \( \mathbb{P}^2 \) denotes the vertex of \( S \), and \( \dim(X_s) = l \). Therefore, \( l = \dim(X_s) \leq \dim(T_pS) - n = n + 1 - n = 1 \). So the definition of (smooth) Roth varieties only makes sense if \( l \in \{0,1\} \).

Let us look more closely at surfaces with a family of plane curves not forming a fibration contained in rational normal scrolls:

**Example 1.** Let \( S := S_{0,0,\alpha} \subset \mathbb{P}^{N = b + 2} \), and let \( L \) be its vertex. For every integer \( \alpha \geq 1 \), let \( \tilde{X} \in |\alpha H - F| \) be a general divisor on \( \tilde{S} \). Then \( \tilde{X} \) is smooth, \( \pi_2|_{\tilde{X}} : \tilde{X} \to X \) is an isomorphism and \( L \subset X \) (see [10, Proposition 3.4]). Furthermore, since \( \tilde{X} \in |\alpha H - F| \sim |(\alpha + 1)H - (b - 1)F| \) it follows that \( X \subset \mathbb{P}^N \) is linked to \( b - 1 = N - 3 \) rulings of \( S \) by the complete intersection of \( S \) and a hypersurface of degree \( \alpha + 1 \) in \( \mathbb{P}^N \). We point out that \( L \) is a base component of the pencil \( \{C_F\}_{F \in \mathbb{P}^2} := \{L + \pi_2|_{\tilde{X}}(F)|_{\tilde{X}}\}_{F \in \mathbb{P}^2} \) of plane curves on \( X \) of degree \( \alpha + 1 \). Moreover, we remark that \( X \) is also fibred by the pencil \( \{C_F - L\}_{F \in \mathbb{P}^2} \) of plane curves of degree \( \alpha \).

**Example 2.** Let \( S := S_{0,1,\alpha} \subset \mathbb{P}^{N = b + 3} \), and let \( p \) be its vertex. For every integer \( \alpha \geq 1 \), let \( \tilde{X} \in |\alpha H - bF| \) be a general divisor on \( \tilde{S} \). Then \( \tilde{X} \) is smooth, \( \pi_2|_{\tilde{X}} : \tilde{X} \to X \) is the blowing-up of \( X \) at \( p \), and \( X \subset S \) is a smooth surface linked to \( b = N - 3 \) rulings of \( S \) by the complete intersection of \( S \) and a hypersurface of
degree $\alpha$ in $\mathbb{P}^N$ by Theorem 2. In this case, $p$ is the base point of the pencil $\{C_F\}_{F \in \mathcal{P}} := \{\pi_2(\xi(F))\}_{F \in \mathcal{P}}$ of plane curves on $X$ of degree $\alpha$, and $(C_F)^2 = (\pi_2(\xi(F))^2 = (\xi + F|\xi|^2 = 1$.

1.3. Families of intersecting planes. The main ingredient of the proof of Theorem 1 is a classical result of Morin [13] on families of planes with the property that any two of them intersect. We include it here for the reader’s convenience, as it might be difficult to find it in the literature:

**Definition 1.** A subvariety $\Sigma \subset G(2, N)$ such that any two planes of $\Sigma$ intersect is said to be **elementary** (see [13] p. 908) if one of the following holds:

- $N \leq 4$;
- there exists a point $p \in \mathbb{P}^N$ such that $p \in \Pi$ for every $\Pi \in \Sigma$;
- there exists some plane $\Lambda \subset \mathbb{P}^N$ such that $\dim(\Lambda \cap \Pi) \geq 1$ for every $\Pi \in \Sigma$.

**Theorem 3** (Morin [13]). Let $\Sigma \subset G(2, N)$ be a non-degenerate (i.e. not contained in $\Sigma$) non-elementary integral subvariety of positive dimension such that any two planes of $\Sigma$ intersect. Then $N = 5$ and $\Sigma$ is contained in one of the following families:

(i) the $\mathcal{C}^2$ planes of the conics of the Veronese surface $\nu_2(\mathbb{P}^2)$ in $\mathbb{P}^5$;

(ii) the $\mathcal{C}^2$ tangent planes to the Veronese surface $\nu_2(\mathbb{P}^2)$ in $\mathbb{P}^5$;

(iii) one of the two $\mathcal{C}^3$ planes contained in a smooth quadric $Q$ in $\mathbb{P}^5$.

**Remark 4.** (a) The family of tangent planes to the Veronese surface $\nu_2(\mathbb{P}^2)$ in $\mathbb{P}^5$ is isomorphic to $\mathbb{P}^2$, and the embedding $\varphi_2 : \mathbb{P}^2 \hookrightarrow G(2, 5)$ is given by six (non-general) sections of $\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3}$. This follows from the definition of the bundle of principal parts and Euler’s exact sequence.

(b) The family of planes of the conics of the Veronese surface $\nu_2(\mathbb{P}^2)$ in $\mathbb{P}^5$ is isomorphic to $\mathbb{P}^2$, and the embedding in $G(2, 5)$ corresponds to a vector bundle given by a resolution $0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2} \to E \to 0$. In particular, the family (i) corresponds to the family (ii) by the identification $G(2, 5) \cong G(2, 5^*)$, and vice versa.

(c) The family of planes contained in a smooth quadric $Q$ of $\mathbb{P}^5$ is isomorphic to $\mathbb{P}^3$, and the embedding $\varphi_3 : \mathbb{P}^3 \hookrightarrow G(2, 5)$ is given by the vector bundle $\Omega_{\mathbb{P}^3}(2)$. In particular, the family (iii) is self-dual by the identification $G(2, 5) \cong G(2, 5^*)$.

Let us show some examples of surfaces swept out by plane curves not forming a fibration coming from the families in Theorem 3.

**Example 3.** Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree $d$. Then $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ is swept out by the family of conics $\{\nu_2(L)\}_{L \in C}$, and a general point of $\nu_2(\mathbb{P}^2)$ is contained in $d$ conics of the family (see [20] Example 2.1)). The same property holds for the rational normal scroll of $\mathbb{P}^d$ (see [20] Example 2.3)).

**Example 4.** Let $S^2\mathbb{P}^2$ denote the second symmetric product of $\mathbb{P}^2$, and let the map $\phi_2 : S^2\mathbb{P}^2 \to \text{Sec}(v_2(\mathbb{P}^2)) \subset \mathbb{P}^5$ denote the embedding defined by

$$(X_0 : X_1 : X_2) + (Y_0 : Y_1 : Y_2) \mapsto (Z_{00} : Z_{01} : Z_{02} : Z_{11} : Z_{12} : Z_{22}),$$

where $Z_{ij} := X_iY_j + X_jY_i$. Let us recall that the secant variety and the tangent variety of $\nu_2(\mathbb{P}^2)$ in $\mathbb{P}^5$ coincide, as $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ is 1-defective (see [21] Ch. I, Theorem 1.4)). Geometrically, $\varphi_2$ is given by $\varphi_2(x + y) = T_{\nu_2(x)}(\nu_2(y)) \cap T_{\nu_2(x)}(\nu_2(y))$ and $\varphi_2(x + y) = v_2(x) \in v_2(\mathbb{P}^2)$. In general, one can define in a similar way an isomorphism $\phi_k : S^2\mathbb{P}^k \to \text{Sec}(v_2(\mathbb{P}^k)) \subset \mathbb{P}^{k(k+3)/2}$ with the same geometric property (cf. Proposition 1). For every smooth curve $C \subset \mathbb{P}^2$ of degree $d$ we obtain a smooth surface $X := \phi_2(S^2C) \subset \mathbb{P}^5$ containing a family of plane curves $\{C_x\}_{x \in C}$ not forming a fibration, where $C_x := \phi_2(x + C) \cong C$ is a plane curve.
on $X$ of degree $d$ for every $x \in C$ and $\langle C_x \rangle = T_{v_2(x)}v_2(\mathbb{P}^2)$. Furthermore, we get $O_X(1) \sim C_{x_1} + \cdots + C_{x_d}$ for every set of collinear points $x_1, \ldots, x_d \in C$ in $\mathbb{P}^2$. In fact, if $L := \langle x_1, \ldots, x_d \rangle$ then the hyperplane $H$ in $\mathbb{P}^5$ corresponding to $2L \in H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(2))$ is tangent to $v_2(\mathbb{P}^2)$ along the conic $v_2(L)$, and $H \cap X = C_{x_1} + \cdots + C_{x_d}$ as $H \cap \text{Sec}(v_2(\mathbb{P}^2)) = \cup_{x \in L} T_{v_2(x)}v_2(\mathbb{P}^2)$.

**Example 5.** Identifying the planes (of a fixed family) contained in a smooth quadric $Q \cong G(1,3)$ in $\mathbb{P}^6$ with the set of lines passing through a point of $\mathbb{P}^3$, we obtain the following two examples (see [2] cases 4) and 15) in p. 44):

(i) Let $C \subset \mathbb{P}^3$ be a twisted cubic, and let $X \subset G(1,3)$ be the family of secant lines to $C$. Then $X$ is the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$.

(ii) Let $C \subset \mathbb{P}^3$ be an elliptic curve of degree 4, and let $X \subset G(1,3)$ be the family of secant lines to $C$. Then $X$ is a ruled surface with invariant $e = -1$ over $C$ and the embedding in $\mathbb{P}^5$ is given by $|2C_0 + L|$, where $\text{deg}(L) = 1$.

2. **Proof of the main results**

Let $X \subset \mathbb{P}^N$ be a smooth surface swept out by an algebraic family $\{C_b\}_{b \in B}$ of plane curves. Since plane curves are linearly normal, we can suppose without loss of generality that $X \subset \mathbb{P}^N$ is linearly normal and non-degenerate. According to [18], we also assume hereafter that $B$ is an integral curve. We say that $\{C_b\}_{b \in B}$ does not form a fibration on $X$ if there is no regular morphism $\pi : X \to B$ such that $\pi^{-1}(b) = C_b$ for every $b \in B$. As $X$ is smooth and $C_b, C_b'$ are algebraically equivalent for any two $b, b' \in B$, this happens if and only if $C_b \cap C_b' \neq \emptyset$ for any two $b, b' \in B$. Let $\varphi : B \to G(2, N)$ be the map which associates to each $b \in B$ the linear span $\langle C_b \rangle$ of $C_b$ in $\mathbb{P}^N$, and let $\Sigma := \varphi(B)$. The map $\varphi$ is constant only if $X = \mathbb{P}^2$, so $\Sigma \subset G(2, N)$ is an integral curve as soon as $N \geq 3$. Let $V_2 \subset \mathbb{P}^N$ denote the 3-dimensional subvariety swept out by the planes of $\Sigma$. We divide the proof of Theorem 1 into two cases, as in [20].

2.1. $V_2$ is a cone. We study separately two cases, according to the dimension of the vertex:

**Proposition 1.** Let $X \subset \mathbb{P}^N$ be a smooth surface containing a 1-dimensional family of plane curves not forming a fibration, $N \geq 4$. If $V_2 \subset \mathbb{P}^N$ is a cone of vertex a line $L$, then $V_2 = S_{0,0,N-2}$ and $X$ is linked to $N-3$ rulings of $V_2$ by the complete intersection of $V_2$ and a hypersurface in $\mathbb{P}^N$.

**Proof.** Replacing $G(2,4)$ by $G(2, N)$ in the proof of [20] Lemma 3.1] we get that $\Sigma \subset G(2, N)$ is a rational curve. Therefore $V_2 = S_{0,0,N-2}$, as $X \subset \mathbb{P}^N$ is non-degenerate and linearly normal by assumption. Suppose first $L \subset X \subset S_{0,0,N-2}$. It follows from [10] Theorem 3.7] that $\tilde{X} \subset (\alpha H + F)$ and that $\pi_2|\tilde{X} : \tilde{X} \to X$ is an isomorphism. Since $H \sim (N-2)F$ we deduce that $\tilde{X} \subset (\alpha + 1)H - (N-3)F$, i.e. $X$ is linked to $N-3$ rulings of $S_{0,0,N-2}$ by the complete intersection of $S_{0,0,N-2}$ and a hypersurface of degree $\alpha + 1$ in $\mathbb{P}^N$.

Assume now $L \not\subset S$, and let $\{q_1, \ldots, q_r\} := X \cap L$. Let us see that $X$ is singular at these points. Note that $\pi_2|\tilde{X} : \tilde{X} \to X$ contracts $\xi_i := \pi^{-1}(q_i)$, so we get $\sum_{i=1}^r \xi_i \sim r(H^2 - b(H \cdot F))$. Let $\mathbb{P}^3 \times L := \pi_2^{-1}(L) \subset S$, so $\mathbb{P}^3 \times L \sim H - bF$ (cf. [10] Lemma 3.2]). Moreover $(\mathbb{P}^3 \times L) \cdot \tilde{X} = \sum_{i=1}^r \xi_i$, and hence $(H - bF) \cdot (\alpha H + \beta F) = \sum_{i=1}^r (H - bF) = r(H^2 - b(H \cdot F))$. Therefore $\alpha H^2 - (\alpha H + \beta F) = rH^2 - rb(H \cdot F)$, whence $\alpha = r$ and $\beta = 0$. So $X$ is the complete intersection of $S_{0,0,N-2}$ and a hypersurface of degree $\alpha$ in $\mathbb{P}^N$, and hence $X$ is singular at $\{q_1, \ldots, q_r\}$. \hfill \Box

On the other hand, if $V_2 \subset \mathbb{P}^N$ is a cone of vertex a point we get the following:
Proposition 2. Let \( X \subset \mathbb{P}^N \) be a smooth surface containing a 1-dimensional family of plane curves not forming a fibration. If \( V_\Sigma \subset \mathbb{P}^N \) is a cone of vertex a point \( p \), then \( V_\Sigma = S_{0,1,N-3} \) and \( X \) is linked to \( N-3 \) rulings of \( V_\Sigma \) by the complete intersection of \( V_\Sigma \) and a hypersurface in \( \mathbb{P}^N \).

Proof. Arguing as in Proposition 1, we get that \( V_\Sigma = S_{0,a,b} \subset \mathbb{P}^{N=a+b+2} \). According to Theorem 2 there are two possibilities. On the one hand, if \( \pi_2|\tilde{X} : \tilde{X} \to X \) is an isomorphism then \( \tilde{X} \in |\alpha H + F| \). In this case, the pencil \( \{C_F\}_{F \in \mathbb{P}^1} : \{\pi_2|\tilde{X}(F|\tilde{X})\}_{F \in \mathbb{P}^1} \) of plane curves on \( X \) forms a fibration, since \((C_F)^2 = (\pi_2|\tilde{X}F))^2 = (F|\tilde{X})^2 = 0 \). On the other hand, if \( \pi_2|\tilde{X} : \tilde{X} \to X \) is the blowing-up of \( X \) at \( p \) then \( a = 1 \) and \( \tilde{X} \in |\alpha H - bF| \). Hence \( V_\Sigma = S_{0,1,b} \) and \( X \) is linked to \( b = N-3 \) rulings of \( V_\Sigma \) by the complete intersection of \( V_\Sigma \) and a hypersurface of degree \( \alpha \) in \( \mathbb{P}^N \). In this case, the pencil \( \{C_F\}_{F \in \mathbb{P}^1} : \{\pi_2|\tilde{X}F|\tilde{X}\}_{F \in \mathbb{P}^1} \) of plane curves on \( X \) does not form a fibration as we showed in Example 2.

Remark 5. Surfaces in Propositions 1 and 2 achieve the greatest possible geometric genus \( \rho \) (see [9, p. 65 i]), where \( \epsilon = 0 \) in both cases).

2.2. \( V_\Sigma \) is not a cone. In this case, we use Morin’s result [13] on families of intersecting planes quoted in Subsection 1.3.

Proposition 3. Let \( X \subset \mathbb{P}^N \) be a smooth surface containing a 1-dimensional family of plane curves not forming a fibration, \( N \geq 5 \). If \( V_\Sigma \) is not a cone then \( N = 5 \) and one of the following holds:

(i) \( X \subset \mathbb{P}^5 \) is the Veronese surface;

(ii) \( X \subset \mathbb{P}^5 \) is the second symmetric product of a smooth curve in \( \mathbb{P}^2 \), and the embedding is given by the embedding \( \phi_2 : S^2 \mathbb{P}^2 \hookrightarrow \mathbb{P}^5 \);

(iii) \( X \subset \mathbb{P}^5 \) is a ruled surface with invariant \( \epsilon = -1 \) over an elliptic curve, and the embedding is given by \( |2C_0 + \mathcal{L}f| \), where \( \deg(\mathcal{L}) = 1 \).

Proof. Since the family \( \{C_b\}_{b \in B} \) of plane curves does not form a fibration on \( X \), \( N \geq 5 \) and \( V_\Sigma \) is not a cone, we deduce that \( \Sigma \subset \mathbb{G}(2,N) \) is a non-elementary family of intersecting planes. Therefore, \( N = 5 \) and \( \Sigma \subset \mathbb{G}(2,5) \) is contained in one of the families of Theorem 3.

If \( \Sigma \subset \mathbb{G}(2,5) \) is as in Theorem 3(i), then \( X = v_3(\mathbb{P}^2) \). This is due to the fact that any two planes of \( \Sigma \) intersect in a point of \( v_3(\mathbb{P}^2) \).

Assume now that \( \Sigma \subset \mathbb{G}(2,5) \) is as in Theorem 3(ii). Let \( \phi_2 : \mathbb{P}^2 \hookrightarrow \mathbb{G}(2,5) \) denote the embedding defined in Remark 4 and let \( \phi_2^{-1}(\Sigma) \subset \mathbb{P}^2 \). Then \( X = \phi_2(\phi_2^{-1}(\Sigma)) \subset \mathbb{P}^5 \), where \( \phi_2 : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5 \) is the embedding given in Example 4.

In particular, \( \Sigma \) is necessarily smooth.

Finally, if \( \Sigma \subset \mathbb{G}(2,5) \) is as in Theorem 3(iii), we can suppose that every plane of \( \Sigma \) corresponds to the set of lines passing through a point of \( \mathbb{P}^3 \) via \( \varphi_3 : \mathbb{P}^3 \hookrightarrow \mathbb{G}(2,5) \) (cf. Remark 4). Therefore, \( X \subset \mathbb{G}(1,3) \) is a smooth congruence given by the family of secant lines to the (necessarily smooth) curve \( \varphi_3^{-1}(\Sigma) \subset \mathbb{P}^3 \). Moreover, \( \varphi_3^{-1}(\Sigma) \subset \mathbb{P}^3 \) cannot have trisecant lines as otherwise \( X \subset \mathbb{G}(1,3) \) would be singular. Then it is well known that either \( \varphi_3^{-1}(\Sigma) \subset \mathbb{P}^3 \) is the twisted cubic (and hence \( X \subset \mathbb{P}^5 \) is the Veronese surface), or else \( \varphi_3^{-1}(\Sigma) \subset \mathbb{P}^3 \) is an elliptic quartic. In that case, \( X \subset \mathbb{P}^5 \) is a ruled surface with invariant \( \epsilon = -1 \) over an elliptic curve and the embedding is given by \( |2C_0 + \mathcal{L}f| \), where \( \deg(\mathcal{L}) = 1 \), as we pointed out in Example 5.

Summing up:

Proof of Theorem 4. If \( V_\Sigma \subset \mathbb{P}^N \) is a cone then we get (i) by Propositions 1 and 2.

If \( V_\Sigma \subset \mathbb{P}^N \) is not a cone and \( N = 4 \), then \( X \) is either the rational normal scroll,
or a quintic elliptic scroll, or the projected Veronese surface by [4 Theorem 0.3] (cf. [20] Theorem 4.10 and Conjecture 4.13). The rational normal scroll and the Veronese surface actually contain a 2-dimensional family of plane curves and they can be described as in case (i). More precisely, $S_{1,2} \subseteq P^4$ is linked to a plane by the complete intersection of $S_{0,1,1}$ (resp. $S_{0,0,2}$) and a quadric of $P^4$, and $V_2(P^2) \subseteq P^5$ is linked to two planes by the complete of $S_{0,1,2}$ and a quadric of $P^5$. If $V_3 \subseteq P^N$ is not a cone and $N \geq 5$, then we get the remaining cases by Proposition 3.

Remark 6. Ruled surfaces with invariant $e = -1$ over elliptic curves also appear in Theorem 1(ii) when the plane curve is a smooth cubic, and the embedding in $P^5$ is given by $|3C_0|$ (cf. Example 3).

We now prove Corollary 1 by reducing the problem to the case of surfaces:

Proof of Corollary 1. Let $Y \subseteq P^N$ be a non-degenerate manifold of dimension $n$ swept out by a 1-dimensional family $\{D_b\}_{b \in B}$ of hypersurfaces, i.e. $\langle D_b \rangle = P^N$, not forming a fibration.

Let us show first that a surface as in cases (ii)-(iii) of Theorem 1 cannot be a linear section of $Y$. To this aim we can assume $n = 3$. Let $X := Y \cap H$ be a smooth non-degenerate surface obtained by intersecting $Y$ with a general hyperplane $H$ in $P^N$. Then $C_b := D_b \cap H$ is a plane curve for every $b \in B$, and the family $\{C_b\}_{b \in B}$ does not form a fibration on $X$. To get a contradiction, assume $X$ as in cases (ii)-(iii) of Theorem 1. Then $D_b \cap D_{b'} \cap H = C_b \cap C_{b'} = 1$ for any two $b, b' \in B$, whence $D_b \cap D_{b'}$ is a line in $P^N$ and $Y$ is swept out by a 2-dimensional family $F \subseteq G(1, N)$. Therefore $Y \subseteq P^N$ is a cone over $X$, as otherwise $H$ contains some line $L \in F$ (and hence $L \subseteq X$, which is not possible). Let us prove this statement. Consider the dual family $F^* \subseteq G(N - 2, N)$, and let $V_{F^*} \subseteq P^{N^*}$ denote the union of the 2-dimensional family of $P^{N-2}$'s. If $H$ does not contain any line $L \in F$ then $\dim V_{F^*} \leq N - 1$, and hence $V_{F^*} = P^{N-1}$. Consequently, there exists a point of $P^N$ contained in every line $L \subseteq F$ and $Y \subseteq P^N$ is a cone over $X$.

So we deduce by intersecting $Y \subseteq P^N$ with $n - 2$ general hyperplanes in $P^N$ that $X \subseteq P^{N-n+2}$ is contained in a 3-dimensional cone as in Theorem 1(i). Hence $Y \subseteq P^N$ is contained in an $(n+1)$-dimensional cone $V \subseteq P^N$ with vertex $L$, and $\dim(L) = l$, where $n - 2 \leq l \leq n - 1$. We claim that $n = 3$ and $l = 1$. We argue as in [19] Lemmas 3.4 and 3.5. The embedded tangent space $T_v V = P^{n+1} \subseteq P^N$ to $V$ at a general $v \in V$ is tangent to $V$ along $(L, v)/L$, whence $T_v V$ is tangent to $Y$ along $V_v = (L, v) \cap Y$. Since $\dim(Y_v) = t$, Zak’s theorem on tangencies [21] Ch. I, Corollary 1.8 yields $l \leq n + 1 - n = 1$. So $n = 3$ and $l = 1$. Therefore, the linearly normal embedding of $Y$ is contained in a rational normal scroll $S_{0,0,a,b} \subseteq P^{a+b+3}$ and $D_b \cap D_{b'} = L \subseteq Y$. Hence $\pi_{2|F} : \tilde{Y} \rightarrow Y$ cannot be an isomorphism, so [10] Claim 3.6 yields $a = b = 1$. In that case, $Y \subseteq P^5$ is linked to a $P^3$ by the complete intersection of the rank-$4$ quadric cone $S_{0,0,1,1}$ and a hypersurface in $P^5$.

In particular, Corollary 1 immediately yields the following characterization:

Corollary 2. The Segre embedding $P^1 \times P^2 \subseteq P^5$ is the only non-degenerate manifold $X \subseteq P^N$ of dimension $n \geq 3$ and codimension $N - n \geq 2$ swept out by a 1-dimensional family of hypersurfaces $\{D_b\}_{b \in B}$ such that $D_b \cap D_{b'}$ moves on $X$.

3. Final remarks: a tribute to Ugo Morin

The purpose of this section is to point out some consequences of Morin’s results on families of intersecting linear spaces. In [14], Morin extended his previous result on families of intersecting planes in the following way:
Theorem 4 (Morin [14]). Let $\Sigma \subset \mathbb{G}(k, N)$ be a non-degenerate (i.e. not contained in any $\mathbb{G}(k, N - 1)$) integral subvariety of positive dimension such that any two $k$-planes of $\Sigma$ intersect. Then (up to some elementary exceptions) $N \leq k(k+3)/2$, with equality if and only if one of the following holds:

(i) the $\infty^2$ $k$-planes containing the rational normal curves of the $k$-Veronese surface $v_k(\mathbb{P}^2)$ in $\mathbb{P}^{k(k+3)/2}$;

(ii) the $\infty^h$ tangent spaces to the Veronese manifold $v_2(\mathbb{P}^k)$ in $\mathbb{P}^{k(k+3)/2}$;

(iii) the $\infty^{k+1}$ $k$-planes contained in $\mathbb{G}(1, k+1) \subset \mathbb{P}^{k(k+3)/2}$.

In view of Theorem 4, we can definitely say that the following result was already known by Morin:

**Proposition 4.** Let $X \subset \mathbb{P}^N$ be a (maybe singular) non-degenerate surface swept out by a 1-dimensional family of curves, each of them spanning a $k$-plane, such that any two of them intersect. Then either $V_2 \subset \mathbb{P}^N$ is a cone, or $N \leq k(k+3)/2$, with equality if and only if one of the following holds:

(i) $X$ is the $k$-Veronese surface $v_k(\mathbb{P}^2) \subset \mathbb{P}^{k(k+3)/2}$;

(ii) $X$ is the second symmetric product of an irreducible curve in $\mathbb{P}^k$, and the embedding in $\mathbb{P}^{k(k+3)/2}$ is given by $\phi_k : S^2 \mathbb{P}^k \rightarrow \mathbb{P}^{k(k+3)/2}$ (cf. Example 2);

(iii) $X$ is given by the secant lines to an irreducible curve of $\mathbb{P}^{k+1}$, embedded by the Plücker embedding of $\mathbb{G}(1, k+1)$ in $\mathbb{P}^{k(k+3)/2}$.

**Proof.** If any two curves of the family intersect then we obtain a family of intersecting $k$-planes, and we can apply Theorem 4. The elementary cases in Theorem 4 are the following (see [14, p. 186]): either $k$-planes contained in a $\mathbb{P}^N$ with $N \leq 2k$, or $k$-planes containing, respectively, a $(k-m)$-plane with $1 \leq m \leq k$ such that any two $(k-m)$-planes intersect. In the second case, either $k = m$ and hence $V_2 \subset \mathbb{P}^N$ is a cone, or else $X \subset \mathbb{P}^N$ is degenerate. Otherwise, if the family is non-elementary, we get $N \leq k(k+3)/2$, with equality if and only if $\Sigma \subset \mathbb{G}(k, N)$ is as in cases (i)-(iii) of Theorem 4. Now we can argue as in Proposition 3. □

**Remark 7.** We actually obtain smooth surfaces in Proposition 4 if and only if either $C \subset \mathbb{P}^k$ is smooth in (ii), or $C \subset \mathbb{P}^{k+1}$ is smooth and has no trisecant lines in (iii).

**Remark 8.** Concerning Proposition 4, Morin actually stated a result in a more general setting at the end of [13] (cf. [8, Proposition 0.18]):

"In conclusion: An irreducible algebraic surface containing an $\infty^1$ algebraic system of (irreducible) algebraic curves intersecting pairwise in just one variable point:

a) such that for a general point of the surface there pass more than two of these curves, is a rational surface and the algebraic system of curves is contained in a homaloidal net;

b) such that for a general point of the surface there pass exactly two of these curves, is the surface obtained as the pairs of points of a general curve of the system."

**Remark 9.** A further application of Theorem 4 was given by Beauville in the study of rank-3 vector bundles and theta functions on curves of genus 3 (see [3]). A more recent application has been given by O’Grady in [15].

We would like to finish this section by showing the connection between families of intersecting linear spaces and the theory of secant defective varieties:

**Remark 10.** Theorem 4 appears to be related to the classification of Scorza varieties in the particular case of secant defect $\delta = 1$ (see [21, Ch.VI, §2]), where Severi’s
characterization of the Veronese surface was extended to higher dimensions. More precisely, Zak proved that if \( X \subset \mathbb{P}^N \) is a non-degenerate manifold of dimension \( n \) that can be isomorphically projected into \( \mathbb{P}^{2n} \) then \( N \leq n(n + 3)/2 \), and equality holds if and only if \( X = v_2(\mathbb{P}^n) \). According to Terracini's Lemma, if \( X \subset \mathbb{P}^N \) can be isomorphically projected into \( \mathbb{P}^{2n} \) then the family of embedded tangent spaces to \( X \subset \mathbb{P}^N \) intersect pairwise. So one could deduce Zak's result from Theorem 4 after excluding the elementary cases described in the proof of Proposition 4.

References

[1] H. Abo, W. Decker, and N. Sasakura, An elliptic conic bundle in \( \mathbb{P}^4 \) arising from a stable rank-3 vector bundle, Math. Z. 229 (1998), 725–741.

[2] E. Arrondo and I. Sols, On congruences of lines in the projective space, Mém. Soc. Math. France (N.S.) (1992), no. 50.

[3] A. Beauville, Vector bundles and theta functions on curves of genus 2 and 3, Amer. J. Math. 128 (2006), 607–618.

[4] V. Beorchia and G. Sacchiero, Surfaces in \( \mathbb{P}^4 \) with a family of plane curves, J. Pure Appl. Algebra 213 (2009), 1750–1755.

[5] R. Braun and K. Ranestad, Conic bundles in projective fourspace, Algebraic geometry (Catania, 1993/Barcelona, 1994), Lecture Notes in Pure and Appl. Math., vol. 200, Dekker, New York, 1998, pp. 331–339.

[6] F. Catanese, C. Ciliberto, and M. Mendes Lopes, On the classification of irregular surfaces of general type with nonbirational bicanonical map, Trans. Amer. Math. Soc. 350 (1998), 275–308.

[7] Ph. Ellia and G. Sacchiero, Smooth surfaces of \( \mathbb{P}^4 \) ruled in conics, Algebraic geometry (Catania, 1993/Barcelona, 1994), Lecture Notes in Pure and Appl. Math., vol. 200, Dekker, New York, 1998, pp. 49–62.

[8] G. Ellingsrud and Ch. Peskine, Sur les surfaces lisses de \( \mathbb{P}_4 \), Invent. Math. 95 (1989), 1–11.

[9] J. Harris, A bound on the geometric genus of projective varieties, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 8 (1981), 35–68.

[10] B. Ilie, Geometric properties of the double-point divisor, Trans. Amer. Math. Soc. 350 (1998), 1643–1661.

[11] P. Ionescu and M. Toma, On very ample vector bundles on curves, Internat. J. Math. 8 (1997), 633–643.

[12] S. L. Kleiman, Plane forms and multiple-point formulas, Algebraic threefolds (Varenna, 1981), Lecture Notes in Math., vol. 947, Springer, Berlin, 1982, pp. 287–310.

[13] U. Morin, Sui sistemi di piani a due a due incidenti, Atti del R. Istituto Veneto di Scienze, Lettere ed Arti 89 (1929-30), 907–926.

[14] Ph. O'Grady, Su sistemi di \( S_2 \) a due a due incidenti e sulla generazione proiettiva di alcune varietà algebriche, Atti del R. Istituto Veneto di Scienze, Lettere ed Arti 101 (1941-42), 183–196.

[15] K. G. O'Grady, EPW-sextics: taxonomy, arXiv:1007.3882v2 [math.AG] To appear in Manuscripta Math. DOI: 10.1007/s00229-011-0472-7.

[16] K. Ranestad, On smooth plane curve fibrations in \( \mathbb{P}^4 \), Geometry of complex projective varieties (Cetraro, 1990), Sem. Conf., vol. 9, Mediterranean, Rende, 1993, pp. 243–255.

[17] L. Roth, On the projective classification of surfaces, Proc. Lond. Math. Soc. (2) 42 (1937), 142–170.

[18] C. Segre, Le superficie degli iperspazi con una doppia infinità di curve piane o spaziali, Atti R. Accad. Sci. Torino 56 (1920-21), 75–89.

[19] J. C. Sierra, Smooth \( n \)-dimensional subvarieties of \( \mathbb{P}^{2n-1} \) containing a family of very degenerate divisors, Internat. J. Math. 20 (2009), 109–122.

[20] J. C. Sierra and A. L. Tironi, Some remarks on surfaces in \( \mathbb{P}^4 \) containing a family of plane curves, J. Pure Appl. Algebra 2009 (2007), 361–369.

[21] F. L. Zak, Tangents and secants of algebraic varieties, Translations of Mathematical Monographs, vol. 127, American Mathematical Society, Providence, RI, 1993.

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