THE BRYLINSKI BETA FUNCTION OF A SURFACE

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Abstract. An analogue of Brylinski’s knot beta function is defined for a submanifold of \(d\)-dimensional Euclidean space. This is a meromorphic function on the complex plane. The first few residues are computed for a surface in three dimensional space.

1. Introduction

In [1], Brylinski introduced the beta function of a geometric knot in \(\mathbb{R}^3\). He was partly motivated by the desire to give a definition of Möbius energy (see [2]) independent of an arbitrary “renormalization”. However, he also gave some beautiful formulae for the first few residues of his beta function. They turn out to be integrals of polynomials in the curvature, torsion and their derivatives.

In this note, we consider arbitrary submanifolds of \(\mathbb{R}^d\). Essentially the same definition (as in [1]) works in this situation, to define the beta function on the right half-plane, and it can be analytically continued to be a meromorphic function on \(\mathbb{C}\), with only simple poles. The location of the poles is dependent on the dimension of the submanifold, and if \(M\) is a hypersurface, the residues are integrals of polynomials in complete contractions of the covariant derivatives of the second fundamental form.

We consider surfaces in \(\mathbb{R}^3\) in more detail, and compute some residues. In particular, we characterize the spheres by the vanishing of the residue at \(s = -4\).

For a knot, the beta function does not have a pole at the Möbius invariant parameter \(s = -2\), and its value there coincides with the Möbius energy. However, for a surface, the Möbius invariant parameter is \(s = -4\), and the beta function has a pole there, in general. The value obtained after subtracting the pole may be considered the natural renormalization of Möbius energy (see [6]).

2. The Brylinski beta function

Let \(M\) be a compact smooth \(n\)-dimensional submanifold of \(\mathbb{R}^d\). Let \(dA\) denote the \(n\)-dimensional area element of \(M\). Observe that if \(\text{Re } s > -n\), and \(F(u, v) = \|v - u\|^s\), then \(F \in L^1(M \times M)\).
Definition 2.1. The Brylinski beta function of $M$ is the function $B_{M}(s)$, defined for $\text{Re } s > -n$ by

$$B_{M}(s) = \int_{M \times M} \|v - u\|^s \, dA(v) \, dA(u).$$

We also need to consider the pointwise version of the beta function. For fixed $u \in M$, we define

$$B_{M}^{u}(s) = \int_{M} \|v - u\|^s \, dA(v), \quad \text{Re } s > -n.$$

Thus

$$B_{M}(s) = \int_{M} B_{M}^{u}(s) \, dA(u).$$

Note that $B_{M}^{u}$ and $B_{M}$ are analytic in the half-plane $\text{Re } s > -n$.

Example 2.2. Let $M = S^2(r) := \{u \in \mathbb{R}^3 \mid \|u\| = r\}$ be the sphere of radius $r$ in $\mathbb{R}^3$. Then $B_{M}^{u}(s) = \frac{2^{s+3} \pi r^{s+2}}{s+2}$ for all $u \in M$, and so

$$B_{S^2(r)}(s) = \frac{2^{s+5} \pi^2 r^{s+4}}{s+2}.$$

Proof. By rotational invariance, it is clear that $B_{M}^{u}(s)$ is independent of $u$, so we take $u = (0,0,r)$ for the computation. We use Cartesian coordinates $(x, y, z)$ and spherical coordinates $(r, \theta, \varphi)$, thus

$$x = r \sin \theta \cos \varphi$$

$$y = r \sin \theta \sin \varphi$$

$$z = r \cos \theta.$$

So

$$\|v - u\|^s = (x^2 + y^2 + (z - r)^2)^{s/2}$$

$$= \left[r^2 \sin^2 \theta \cos^2 \varphi + r^2 \sin^2 \theta \sin^2 \varphi + r^2 (\cos \theta - 1)^2\right]^{s/2}$$

$$= \left[r^2 \sin^2 \theta + r^2 (\cos^2 \theta - 2 \cos \theta + 1)\right]^{s/2}$$

$$= r^s (2 - 2 \cos \theta)^{s/2}$$

$$= 2^s r^s \sin^s(\theta/2).$$
We recall that the area element on $M$ is $dA = r^2 \sin \theta \, d\theta \, d\varphi$, so
\[
B_M^u(s) = \int_M \| v - u \|^s \, dA(v)
\]
\[
= 2^s r^s \int_0^{2\pi} \int_0^\pi \sin^s(\theta/2) r^2 \sin(\theta) \, d\theta \, d\varphi
\]
\[
= 2^{s+2} \pi r^{s+2} \int_0^\pi \sin^{s+1}(\theta/2) \cos(\theta/2) \, d\theta
\]
\[
= 2^{s+3} \pi r^{s+2} \int_0^1 t^{s+1} \, dt
\]
\[
= \frac{2^{s+3} \pi r^{s+2}}{s + 2}.
\]
\[\square\]

**Example 2.3.** More generally, if $M = S^n(r) := \{ u \in \mathbb{R}^{n+1} \mid \| u \| = r \}$ is the sphere of radius $r$ in $\mathbb{R}^{n+1}$, then
\[
B_M^u(s) = 2^{s+n} \omega_{n-1} r^{s+n} B\left(\frac{s + n}{2}, \frac{n}{2}\right)
\]
for all $u \in M$, and so
\[
B_{S^n(r)}(s) = 2^{s+n} \omega_{n-1} \omega_n r^{s+2n} B\left(\frac{s + n}{2}, \frac{n}{2}\right),
\]
where $\omega_n$ denotes the $n$-dimensional “area” of $S^n(1)$, and $B(s,t)$ is Euler’s beta function.

**Proof.**
\[
B_M^u(s) = \int_0^\pi \left[2r \sin(\theta/2)\right]^s \omega_{n-1}(r \sin \theta)^{n-1} r \, d\theta
\]
\[
= 2^s \omega_{n-1} r^{s+n} \int_0^\pi \sin^s(\theta/2) \sin^{n-1}(\theta) \, d\theta
\]
\[
= 2^{s+n-1} \omega_{n-1} r^{s+n} \int_0^\pi \sin^{s+n-1}(\theta/2) \cos^{n-1}(\theta/2) \, d\theta
\]
\[
= 2^{s+n} \omega_{n-1} r^{s+n} B\left(\frac{s + n}{2}, \frac{n}{2}\right).
\]
\[\square\]

3. THE ANALYTIC CONTINUATION

We begin with two analytic lemmas that are used in the main argument. We try to imitate the arguments in §3.2 and §3.9 of [3]. There are two obstacles to this. Firstly, in our setting, the test-function is also varying (holomorphically); this is easily overcome using Lemma 3.1. The second obstacle is somewhat more serious: the test function is actually not smooth at the origin. This problem is resolved by “blowing-up” the origin, and applying the Malgrange preparation theorem to show that the pulled back test function extends smoothly across the exceptional divisor (Lemma 3.2).
Lemma 3.1. Let $X$ be a compact Hausdorff space. Let $\mu$ be a finite Baire measure on $X$. Suppose $G : X \times \mathbb{C} \to \mathbb{C}$ is continuous, and for each $x \in X$, the function $G(x, \cdot)$ is entire. Then

$$g(s) = \int_X G(x, s) \mu(x)$$

is entire.

Proof. Fix $R \in (0, \infty)$ and put $C = \sup\{G(x, s) \mid x \in X, |s| = R\}$. By the Cauchy estimates,

$$\left| \frac{\partial^k G}{\partial s^k}(x, 0) \right| \leq \frac{k!C}{R^k},$$

so

$$\left| \frac{d^k g}{ds^k}(0) \right| = \left| \int_X \frac{\partial^k G}{\partial s^k}(x, 0) \mu(x) \right| \leq \frac{k!C\mu(X)}{R^k},$$

and so the radius of convergence of the Taylor series of $g$ is at least $R$. Since $R$ is arbitrary, the result follows. \qed

Suppose $f \in C^\infty(\mathbb{R}^n)$ and $f$ vanishes to second order at 0. If $n > 1$, the function $F(w) = f(w)/\|w\|^2$ does not necessarily extend smoothly to 0. However, the singularity is mild, and may be resolved by “blowing up”. Define $P : \mathbb{R} \times S^{n-1}(1) \to \mathbb{R}^n$ by $P(r, w) = rw$.

Lemma 3.2. The function $F \circ P$ extends to a smooth function.

Proof. Let $\Phi : \mathbb{R}^{n-1} \to S^{n-1}(1)$ be a chart and define $g : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$g(r, w') = (f \circ P)(r, \Phi(w')) = f(r\Phi(w')).$$

By the Malgrange preparation theorem ([5, Theorem 7.5.6]), we can write

$$g(r, w') = q(r, w')r^2 + r_1(w')r + r_0(w'), \quad q \in C^\infty(\mathbb{R}^n), r_1, r_0 \in C^\infty(\mathbb{R}^{n-1}).$$

However, since $g$ vanishes to second order on the hyperplane $r = 0$, the functions $r_1$ and $r_2$ are identically zero, and we have $g(r, w') = q(r, w')r^2$. But $(F \circ P)(r, \Phi(w'))$ agrees with $q(r, w')$ when $r \neq 0$, so we can use $q$ to locally extend $F \circ P$. \qed

Theorem 3.3. The function $B^u_M$ can be analytically continued to a meromorphic function on $\mathbb{C}$ with simple poles at $-n - j$, $j = 0, 2, 4, \ldots$. Moreover, if $M$ is a hypersurface, the residues are polynomials in complete contractions of the covariant derivatives of the second fundamental form.

Proof. If $\psi \in C^\infty_c(\mathbb{R}^d)$ is identically 1 in a neighborhood of $u$ then the localized beta function

$$B^\psi_M(s) = \int_M \|v - u\|^s \psi(v) dA(v)$$

has the same principal part as $B^u_M$ because their difference extends to a holomorphic function on $\mathbb{C}$. So it suffices to prove the result with $B^u_M$ replaced by $B^\psi_M$ for an appropriate $\psi$. By rotating and translating $M$, we may assume that $u = 0$ and the tangent space to $M$ is $\mathbb{R}^n \subseteq \mathbb{R}^d$ (clearly this process does not affect the beta functions). Then, in a neighborhood
of 0, \( M \) is the graph of a function \( f : \mathbb{R}^n \to \mathbb{R}^{d-n} \) which vanishes to second order at 0. By making the neighborhood smaller, we may assume \( \|f(w)\| < \|w\| \). Choose \( \psi \) to have support in this neighborhood. Then

\[
B_{M}^{\psi}(s) = \int_{\mathbb{R}^n} \left( \|w\|^2 + \|f(w)\|^2 \right)^{s/2} \varphi(w) \, dw
\]

where \( \varphi(w) = \psi(w, f(w))A(w) \) and \( A(w) \) is the area-density (it may be expressed in terms of the partial derivatives of \( f \)).

Now,

\[
B_{M}^{\psi}(s) = \int_{\mathbb{R}^n} \|w\|^s \left( 1 + \frac{\|f(w)\|^2}{\|w\|^2} \right)^{s/2} \varphi(w) \, dw
= \int_0^\infty r^{s+n-1} S(r, s) \, dr
\]

where

\[
S(r, s) = \int_{S^{n-1}(1)} \left( 1 + \frac{\|f(rw)\|^2}{\|rw\|^2} \right)^{s/2} \varphi(w) \, d\sigma(w)
\]

and \( \sigma \) is the surface measure on \( S^{n-1}(1) \). Note that \( S(r, s) \) is the mean value of the function \( \left( 1 + \frac{\|f(rw)\|^2}{\|rw\|^2} \right)^{s/2} \varphi(w) \) on the sphere of radius \( r > 0 \). We can extend the definition of \( S(r, s) \) to all real values of \( r \) by writing

\[
S(r, s) = \int_{S^{n-1}(1)} G(r, w, s) \, d\sigma(w)
\]

where

\[
G(r, w, s) = \left( 1 + \frac{\|f \circ P(r, w)\|^2}{r^2} \right)^{s/2} (\varphi \circ P)(r, w), \quad r \in \mathbb{R}, \ s \in \mathbb{C}.
\]

By Lemma 3.2, \( G \in C^\infty(\mathbb{R} \times S^{n-1}(0) \times \mathbb{C}) \). Moreover, for each \( r \in \mathbb{R} \) and \( w \in S^{n-1}(0) \), the function \( G(r, w, \cdot) \) is entire, so by Lemma 3.1, \( S \in C^\infty(\mathbb{R} \times \mathbb{C}) \) and \( S(r, \cdot) \) is entire for each \( r \in \mathbb{R} \). Note that by the equality of mixed-partials, the functions \( \frac{\partial^i S}{\partial r^i}(r, \cdot) \) are also entire. Since \( G(-r, -w, s) = G(r, w, s) \), it follows that \( S(\cdot, s) \) is even, and so \( \frac{\partial^i S}{\partial r^i}(0, \cdot) = 0 \) for odd \( i \).

Now fix a positive integer \( k \). For \( Re s > -n \), we have

\[
B_{M}^{\psi}(s) = \int_0^1 r^{s+n-1} \left[ S(r, s) - S(0, s) - r \frac{\partial S}{\partial r}(0, s) - \cdots - \frac{r^{k-1}}{(k-1)!} \frac{\partial^{k-1} S}{\partial r^{k-1}}(0, s) \right] \, dr
+ \int_1^\infty r^{s+n-1} S(r, s) \, dr + \sum_{j=0}^{k-1} \frac{1}{j!(s+n+j)} \frac{\partial^j S}{\partial r^j}(0, s).
\]

By Taylor’s theorem, the first integral on the right is defined and holomorphic as a function of \( s \) for \( Re s > -n - k \), so the right hand side is a meromorphic function with only simple poles at \( -n - j \) on the half-plane \( Re s > -n - k \). By our remark about the odd-order
partial derivatives of $S$, it follows that $B_M^\psi$ does not actually have a pole at $-n - j$ for odd $j$. Since $k$ is arbitrary, this provides the desired analytic continuation of $B_M^\psi$ to $\mathbb{C}$. Observe that

$$\text{Res}_{s=-n-j} B_M^u = \frac{1}{j!} \frac{\partial^j S}{\partial r^j}(0, -n - j).$$

Now suppose $M$ is a hypersurface, i.e. $d = n + 1$. Then the area density $A(w) = (1 + \|\nabla f(w)\|^2)^{1/2}$, so for small $r$, the spherical mean

$$S(r, s) = \int_{S^{n-1}(1)} \left(1 + \frac{f(rw)^2}{r^2}\right)^{s/2} (1 + \|\nabla f(w)\|^2)^{1/2} d\sigma(w),$$

and so $\frac{\partial^j S}{\partial r^j}(0, s)$ may be expressed as a polynomial in the Taylor coefficients of $f$ at $0$ and the moment integrals $\int_{S^{n-1}(1)} w^\alpha d\sigma(w)$.

Let $I$ and $II$ denote the first and second fundamental forms of $M$. Using the local parametrization $(w, f(w))$ of $M$, we find

$$I_{ij} = \delta_{ij} + \partial_i f \partial_j f, \quad \text{and} \quad II_{ij} = \frac{\partial_i \partial_j f}{(1 + \|\nabla f\|^2)^{1/2}}.$$

Now, if $\alpha$ is a multi-index, the covariant derivative of the second fundamental form

$$(II_{ij}; \alpha) = (\partial^\alpha \partial_i \partial_j f)(0) + Q$$

where $Q$ is a polynomial in the Taylor coefficients (at $0$) of $f$ of order less than or equal to $|\alpha| + 1$. Such a “triangular” relation may be inverted to express the Taylor coefficients of $f$ as polynomials in $(II_{ij}; \alpha)(0)$.

It follows that $\text{Res}_{s=-n-j} B_M^u$ may be expressed as a polynomial in $(II_{ij}; \alpha)(0)$. However, $B_M^u$ is independent of the choice of $f$, so this polynomial is $O(n - 1)$-invariant. By Weyl’s First Fundamental Theorem on Orthogonal Invariants, it follows that the polynomial may be re-expressed as a polynomial in the complete contractions of $II_{ij; \alpha}(0)$. (cf. [7]. Also, [4] §4.2 and §4.6] give a modern exposition of the application of Weyl’s theorem in Geometry)

\[ \square \]

4. SURFACES IN $\mathbb{R}^3$

In this section, assume $M \subseteq \mathbb{R}^3$ is a surface. We will compute $\text{Res}_{s=k} B_M(s)$ for $k = -2, -4$ and $-6$ in terms of $I$, $II$ and the first two covariant derivatives of $II$. 
Fix $u \in M$ and choose coordinates such that $I_{ij} = \delta_{ij}$ at $x$. Let $\Pi_{ij,k}$ and $\Pi_{ij;kl}$ denote the components of the first and second covariant derivatives of $\Pi$. Define, at $u$

\[H_0 = \Pi_{ii}\]
\[H_1 = \Pi_{ij} \Pi_{ij}\]
\[H_2 = \Pi_{ij,k} \Pi_{ij;k}\]
\[H_3 = \Pi_{ij;jj}\]
\[H_4 = \Pi_{ij} \Pi_{ij;kk}\]
\[H_5 = \Pi_{ij} \Pi_{kk;ij}.\]

Here we are using the extended Einstein summation convention, where we sum over repeated indices, even if they are both covariant. This is justified because we are working in a coordinate system which is orthogonal at $u$. One can get formulae for the $H_\alpha$ in a general coordinate system by first raising one of the repeated indices using $I$ before summing.

Being complete contractions of tensors, $H_\alpha$, $\alpha = 0, \ldots, 5$ are smooth functions on $M$, independent of any coordinate choices. Note that $H_0$ is just the mean-curvature.

**Theorem 4.1.**

\[
\text{Res}_{s=-2}B_M^u = 2\pi \\
\text{Res}_{s=-4}B_M^u = \frac{\pi}{2}(2H_1 - H_0^2) \\
\text{Res}_{s=-6}B_M^u = \frac{\pi}{32}\left( -\frac{3}{16}H_0^4 + \frac{3}{4}H_1^2 + \frac{4}{3}H_2 - \frac{1}{2}H_0H_3 + \frac{3}{2}H_4 + \frac{1}{2}H_5 \right)
\]

**Proof.** Assume, as in the proof of Theorem 3.3 that $u = 0$ and the $xy$-plane is tangent to $M$ at 0. So locally $M$ is the graph of a function $z = f(w)$, where $w = (x,y)$, which vanishes to second order at 0. By the proof of Theorem 3.3

\[
\text{Res}_{s=-2-j}B_M^u = \frac{1}{j!}\frac{\partial^jS}{\partial r^j}(0,-2-j).
\]

where

\[
S(r,s) = \int_{S^1(1)} \left( 1 + \frac{f(rw)^2}{r^2} \right)^{s/2} (1 + \|\nabla f(rw)\|^2)^{1/2} d\sigma(w),
\]

for small $r$.

If we write

\[
f(x,y) = b_1x^2 + b_2xy + b_3y^2 + c_1x^3 + c_2x^2y + c_3xy^2 + c_4y^3 + d_1x^4 + d_2x^3y + d_3x^2y^2 + d_4xy^3 + d_5y^4 + O(\|w\|^5),
\]
The beta function of a knot

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