LETTER TO THE EDITOR

N-Soliton Solutions to a New (2 + 1) Dimensional Integrable Equation

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Abstract. We give explicitly N-soliton solutions of a new (2 + 1) dimensional equation, \( \phi_{xt} + \phi_{xzz}/4 + \phi_{zxx} + \phi_{xzx}/2 + \partial_{z}^{-1}\phi_{zzz}/4 = 0 \). This equation is obtained by unifying two directional generalization of the KdV equation, composing the closed ring with the KP equation and Bogoyavlenskii-Schiff equation. We also find the Miura transformation which yields the same ring in the corresponding modified equations.

Short title: LETTER TO THE EDITOR

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The study of higher dimensional integrable system is one of the central themes in integrable systems. A typical example of higher dimensional integrable systems is to modify the Lax operators of a basic equation, in this letter the potential KdV (p-KdV) equation. The Lax pair of the p-KdV equation have the form

\[ L(x,t) = \partial_x^2 + \phi_x(x,t), \]  
\[ T(x,t) = \left(L(x,t)^3\right)_t + \partial_t = \partial_x L(x,t) + T'(x,t) + \partial_t. \]

The p-KdV equation

\[ \phi_{xt} + \frac{1}{4} \phi_{xxxx} + \frac{3}{2} \phi_x \phi_{xx} = 0, \]

is equivalent to the Lax equation

\[ [L, T] = 0. \]

B. G. Konopelchenko and V. G. Dubovsky modified \( L \) operator from \((1 + 1)\)-dimensions to \((2 + 1)\)-dimensions and gave the Lax pair of the KP equation \([1]\). On the other hand, O. I. Bogoyavlenskii modified \( T \) operator and gave the Lax pair of the Bogoyavlenskii-Schiff (BS) equation \([2]\). We modified \( L \) and \( T \) operators searching for \((3 + 1)\)-dimensional integrable equation. However, the Lax equation was eventually reduced to \((2 + 1)\)-dimensional equation \([3]\).

\[ \phi_{xt} + \frac{1}{4} \phi_{xxxx} + \phi_x \phi_{xx} + \frac{1}{2} \phi_{xx} \phi_z + \frac{1}{4} \phi^{-1}_{xx} \phi_{zzz} = 0. \]

In this letter we will give explicitly \( N \)-soliton solutions to this new \((2 + 1)\)-dimensional equation. Moreover, we will give the modified equation of \((5)\) from the Miura transformation and its Lax pair.

Equation \((5)\) admits the Lax representation \([3]\); \([3]\):

\[ L = \partial_x^2 + \phi_x + \partial_z, \]
\[ T = \partial_x^2 \partial_z + \frac{1}{2} \phi_x \partial_z + \phi_x \partial_z + \frac{3}{2} \phi_{xx} - \frac{1}{4} \phi^{-1}_{xx} + \partial_t. \]

Equation \((5)\) has also the Painlevé property \([3]\) in the sense of WTC method \([4]\).

By the dependent variable transformation \( \phi \equiv 2 \frac{\tau_z}{\tau} \),

\[ \phi_{xt} + \frac{1}{4} \phi_{xxxx} + \frac{3}{2} \phi_x \phi_{xx} + \frac{1}{4} \phi^{-1}_{xx} \phi_{zzz} = 0. \]

The operators \( T, T^* \) are defined by \([5, 6]\)

\[ T^n f(z) \cdot g(z) \cdot h(z) \equiv (\partial_{z_1} + \cdots + \partial_{z_n}) f(z_1) g(z_2) h(z_3) \big|_{z_1=z_2=z_3=z}. \]
where $j$ is the cubic root of unity, $j = \exp(2i\pi/3)$. $T^*_z$ is the complex conjugate operator of $T_z$ obtained by replacing $(\partial_{z_1} + j\partial_{z_2} + j^2\partial_{z_3})$ by $(\partial_{z_1} + j^2\partial_{z_2} + j\partial_{z_3})$. Equation (20) was obtained by J. Hietarinta, B. Grammaticos and A. Ramani from the singularity analysis of the trilinear equation (6). The 1-soliton solution of equation (15) takes the form

$$\tau_1 = 1 + \exp(P_1 x + Q_1 z + R_1 t + S_1),$$

and the dispersion relation is

$$P_1^3Q_1 + Q_1^3 + 4P_1^2R_1 = 0. \tag{12}$$

Here $S_1$ is a constant. Nextly the form of 2-soliton solution is written as

$$\begin{align*}
\tau_2 &= 1 + \exp(P_1 x + Q_1 z + R_1 t + S_1) + \exp(P_2 x + Q_2 z + R_2 t + S_2) \\
&\quad + A_{12} \exp((P_1 + P_2)x + (Q_1 + Q_2)z + (R_1 + R_2)t + S_1 + S_2), \tag{13}
\end{align*}$$

and $A_{12}$ is

$$A_{12} = \frac{P_1^2P_2^2(P_1 - P_2)^2 - (P_1Q_2 - P_2Q_1)^2}{P_1^2P_2^2(P_1 + P_2)^2 - (P_1Q_2 - P_2Q_1)^2}. \tag{14}$$

As a result, we have the conjecture that $N$-soliton solutions of equation (10) have the form

$$\tau_N = 1 + \sum_{n=1}^{N} \sum_{C_n} A_{i_1\cdots i_n} \exp(\eta_{i_1} + \cdots + \eta_{i_n}), \tag{15}$$

$$\eta_j = P_j x + Q_j z + R_j t + S_j, \quad P_j^4Q_j + Q_j^3 + 4P_j^2R_j = 0 \tag{16}$$

$$A_{jk} = \frac{P_j^2P_k^2(P_j - P_k)^2 - (P_jQ_k - P_kQ_j)^2}{P_j^2P_k^2(P_j + P_k)^2 - (P_jQ_k - P_kQ_j)^2}, \tag{17}$$

$$A_{i_1\cdots i_n} = A_{i_1, i_2} \cdots A_{i_1, i_n} \cdots A_{i_{n-1}, i_n}. \tag{18}$$

Here the summation $\sum_{C_n}$ indicates summation over all possible combinations of $n$ elements taken from $N$, and symbols $S_j$ always denote arbitrary constants. However, this equation also allows the following the Wronskian type solution, which is easier for the analytic proof of solution than the form (15). We introduce new parameters $p_j$ and $q_j$ as follows,

$$p_j - q_j = P_j, \quad p_j^2 - q_j^2 = \pm Q_j, \quad p_j^4 - q_j^4 = \mp 2R_j. \tag{19}$$

We rewrite $N$-soliton solution $\tau_N$ (15) in the form of $N \times N$ Wronskian

$$\tau_N = \det \begin{pmatrix} f_1 & \cdots & \partial_x^{N-1}f_1 \\ \vdots & \ddots & \vdots \\ f_N & \cdots & \partial_x^{N-1}f_N \end{pmatrix}, \tag{20}$$

where

$$f_j = \exp(p_j x + p_j^2z + \frac{1}{2}p_j^4t + c_j) + \exp(q_j x + q_j^2z + \frac{1}{2}q_j^4t + d_j). \tag{21}$$
with constants, $c_j$ and $d_j$. We prove analytically that Wronskian form (20) is indeed the solution to a new $(2 + 1)$-dimensional trilinear equation (9). For later use, we represent the Wronskian solution (20) as follows,

$$
\tau_N = \det \left( \begin{array}{ccc}
  f_1 & \cdots & \partial_x^{N-1} f_1 \\
  \vdots & \ddots & \vdots \\
  f_N & \cdots & \partial_x^{N-1} f_N \\
\end{array} \right) \equiv [0, \cdots, N - 1],
$$

(22)

where the symbol $j$ in $[\cdots]$ denote the $j$th derivative of the column vector $^t(f_1, \cdots f_N)$. Then the derivatives of $\tau_N$ are described as

$$
\tau_{N,x} = [0, \cdots, N - 2, N],
$$

(23)

$$
\tau_{N,z} = \mp[0, \cdots, N - 3, N - 1, N] \pm [0, \cdots, N - 2, N + 1],
$$

(24)

$$
\tau_{N,t} = \pm \frac{1}{2}[0, \cdots, N - 5, N - 3, N - 2, N - 1, N] + \frac{1}{2}[0, \cdots, N - 4, N - 2, N - 1, N + 1] + \frac{1}{2}[0, \cdots, N - 3, N - 1, N + 2] + \frac{1}{2}[0, \cdots, N - 2, N + 3],
$$

(25)

Hence equation (9) becomes

$$
\pm 8[0, \cdots, N - 1][0, \cdots, N - 3, N, N + 3] - [0, \cdots, N - 2, N][0, \cdots, N - 3, N - 1, N + 3] + [0, \cdots, N - 2, N + 3][0, \cdots, N - 3, N - 1, N]
$$

$$
\mp 8[0, \cdots, N - 1][0, \cdots, N - 5, N - 3, N - 2, N, N + 1] - [0, \cdots, N - 2, N][0, \cdots, N - 5, N - 3, N - 2, N - 1, N + 1] + [0, \cdots, N - 2, N + 1][0, \cdots, N - 5, N - 3, N - 2, N - 1, N]
$$

$$
\mp 8[0, \cdots, N - 2, N][0, \cdots, N - 3, N, N + 2] - [0, \cdots, N - 2, N][0, \cdots, N - 3, N - 1, N + 2] + [0, \cdots, N - 2, N + 2][0, \cdots, N - 3, N - 1, N]
$$

$$
\pm 8[0, \cdots, N - 2, N][0, \cdots, N - 4, N - 2, N, N + 1] - [0, \cdots, N - 2, N][0, \cdots, N - 4, N - 2, N - 1, N + 1]
$$
\[\pm 8\{[0, \cdots, N-3, N-1, N] - [0, \cdots, N-2, N+1]\}
\times \left([0, \cdots, N-1][0, \cdots, N-3, N, N+1] - [0, \cdots, N-2, N][0, \cdots, N-3, N-1, N+1]
+ [0, \cdots, N-2, N+1][0, \cdots, N-3, N-1, N]\right) = 0 \quad (26)\]

Since \((\ )\) parts of equation(26) are nothing but the Plücker relations, each term of the left hand side of equation(26) become zero. This completes the proof.

Thus equation(3) is proved to be an completely integrable system and must have an infinite series of invariants. In this connection we proceed to discuss the Miura transformation of equation (5). The Miura transformation in the dependent variable of equation(5) is

\[\phi_x = \psi_x^2 + \sigma \psi_{xx} + \sigma \psi_z \quad (27)\]

with \(\sigma = \pm i\). This transformation is the same as the potential KP equation(p-KP)'s. Then we obtain the modified equation

\[\psi_{xt} + \frac{1}{4} \psi_{xxxz} + \sigma \psi_{xz} \psi_z + \left(\frac{1}{2} \psi_x + \frac{1}{2} \psi_{xx} \partial_x^{-1} - \frac{1}{4} \psi_z \psi_{zz} + \sigma \psi_{zz} \right) \left((\psi_x^2)_z + \sigma \psi_{zz}\right) = 0. \quad (28)\]

We have the Lax pair of equation(28) as follows,

\[L = \partial_x^2 + 2\sigma \psi_x \partial_x + \partial_z, \quad (29)\]
\[T = \partial_z^2 \partial_x + \sigma \partial_z \partial_x^2 + 2\sigma \psi_x \partial_x \partial_z + \frac{1}{2} \partial_z^2 + \left(-2\psi_x \psi_z + \frac{3}{2} \sigma \psi_{xz} + \frac{1}{2} \psi_z \psi_x \partial_x^{-1} \psi_x^2_z \right)
- \frac{1}{2} \sigma \partial_x^{-1} \psi_{zz} \right) \partial_x + \partial_t, \quad (30)\]

where we have replaced \(\sigma \rightarrow -\sigma\) in equation(28).

Here concluding remarks are in order. We have obtained the exact \(N\)-soliton solution(15) and \(N \times N\) Wronskian solution(20) of equation(5) and have constructed the modified equation(28) using the Miura transformation(27). Moreover, we have constructed the Lax pair of equation(28).

In the previous paper\[3\] we discussed the construction method for the higher dimensional integrable equations. We obtained the higher dimensional equations(the p-KP equation, the BS equation and equation(5)) from the p-KdV equation by this method, which is schematically depicted in the top part of Figure.1. The Miura transformation of the p-KdV equation and the BS equation is

\[\phi_x = \psi_x^2 + \sigma \psi_{xx}. \quad (31)\]

The corresponding modified equations are called the potential modified KdV(p-mKdV) equation and the modified BS(mBS) equation, respectively. The latter is

\[\psi_{xt} + \frac{1}{4} \psi_{xxxz} + \psi_x^2 \psi_{xx} + \frac{1}{2} \partial_x^{-1} (\psi_x^2)_z = 0. \quad (32)\]
The Miura transformation of the p-KP equation is same as equation(3)’s, i.e. equation(27). In this letter, we have given the modified equation(28) of equation(5) from the Miura transformation(27) (see Figure.1). B. G. Konopelchenko and V. G. Dubrovsky gave the Lax pairs of the mKdV equation and the mKP equation[1]. However, the Lax pairs of the mBS equation(32) and of equation(28) were not discussed. Using the construction method for higher dimensional equation, We have obtained the Lax pair of the mBS equation(32).

\[ L = \partial^2_x + 2\sigma \psi_x \partial_x \]
\[ T = \partial^2_x \partial_z + \sigma \psi_z \partial^2_x + 2\sigma \psi_x \partial_x \partial_z + \left( -2\psi_x \psi_z + \frac{3}{2} \sigma \psi_{xz} + \frac{1}{2} \sigma^{-1} (\psi^2_x)_z \right) \partial_x + \partial_t \]

and the Lax pair(29), (30) of the modified equation(28) (see the bottom part of Figure.1). Thus we have succeeded to construct the Lax pairs of the higher dimensional modified equations (the m-BS equation and equation(28)), composing the analogous ring of modified systems.

![Figure 1. Scheme of extensions of the KdV equation and the mKdV equation. There are two directional routes of extensions (dashed arrows): One leads us to the p-KP equation and the p-mKP equation. Another does us to the BS equation and the mBS equation. Equations (5) and (28) are given by unifying two routes of extensions. Four equations in the bottom part of this figure are induced by the Miura transformations (solid arrows).](image)

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