Equivariant autoequivalences for finite group actions

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Abstract The familiar Fourier-Mukai technique can be extended to an equivariant setting where a finite group \( G \) acts on a smooth projective variety \( X \). In this paper we compare the group of invariant autoequivalences \( \text{Aut}(D^b(X))^G \) with the group of autoequivalences of \( D^G(X) \). We apply this method in three cases: Hilbert schemes on K3 surfaces, Kummer surfaces and canonical quotients.

1 Introduction and Setup

It often proves useful to consider analogues of classical settings, adding the presence of a group action. Instances of this in algebraic geometry are e.g. equivariant intersection theory or the McKay type theorems. There already is a theory for derived categories of varieties with actions by algebraic groups [2]. In this article, we study the behaviour of automorphism groups of such derived categories in the case when the group is finite. This paper grew out of my Ph.D. thesis [16]. I would like to thank Georg Hein, Daniel Huybrechts, Manfred Lehn, and Richard Thomas for their help and valuable suggestions.

We always work with varieties over \( \mathbb{C} \). A kernel \( P \in D^b(X \times Y) \) gives rise to a Fourier-Mukai transform which we denote by \( \text{FM}_P : D^b(X) \to D^b(Y) \). We also introduce special notation for composition of such transforms: Let us write \( \text{FM}_Q \circ \text{FM}_P = \text{FM}_{Q \star P} : D^b(X) \to D^b(Z) \) for \( P \in D^b(X \times Y) \) and \( Q \in D^b(Y \times Z) \), i.e. \( Q \star P = \mathbb{P}P_{XZ} \cdot \mathbb{P}P_{XY} \). This works for morphisms as well: \( f : P \to P' \) and \( g : Q \to Q' \) give rise to \( g \circ f : P \to Q \). Put \( D^G(X) := D^b(\text{Coh}^G(X)) \) for its derived category.

1.1 Linearisations and \( D^G(X) \)

Let \( X \) be a smooth projective variety on which a finite group \( G \) acts. A \( G \)-linearisation of a sheaf \( E \) on \( X \) is given by isomorphisms \( \lambda_g : E \to g^* E \) for all \( g \in G \) satisfying \( \lambda_1 = \text{id}_E \) and \( \lambda_{gh} = h^* \lambda_g \circ \lambda_h \). A morphism \( f : (E_1, \lambda_1) \to (E_2, \lambda_2) \) is \( G \)-invariant, if \( f = g : f := \lambda_2^{-1} \circ g^* f \circ \lambda_1 \) for all \( g \in G \). The category of \( G \)-linearised coherent sheaves on \( X \) with \( G \)-invariant morphisms is denoted by \( \text{Coh}^G(X) \); note that it is abelian and contains enough injectives, see [3]. Put \( D^G(X) := D^b(\text{Coh}^G(X)) \) for its derived category.

There is an equivalent point of view on \( D^G(X) \): let \( \mathcal{T} \) be the category consisting of \( G \)-linearised objects of \( D^b(X) \), i.e. complexes \( E^* \in D^b(X) \) together with isomorphisms \( \lambda_g : E^* \to g^* E^* \) in \( D^b(X) \) satisfying the same cocycle condition as above. The canonical functor \( D^G(X) \to \mathcal{T} \) is fully faithful in view of \( \text{Hom}_{D^G(X)}(E_1, \lambda_1^*), (E_2, \lambda_2^*) = \text{Hom}_{D^b(X)}(E_1^*, E_2^*)^G \) for objects \( (E_1, \lambda_1^*), (E_2, \lambda_2^*) \in D^G(X) \). To show that the functor is essentially surjective, take \( (E^*, \lambda) \in \mathcal{T} \). Choosing an injective bounded resolution \( E^* \to I^* \), we can assume that \( \lambda \) corresponds to genuine complex maps \( \lambda_g : I^* \to g^* I^* \). Hence \( (I, \lambda)^* \) is a complex with a linearisation of each sheaf, and using \( D^b(X) \cong D^b_{\text{Coh}(X)}(\text{Qcoh}(X)) \cong (I, \lambda)^* \), we get \( \mathcal{T} \cong D^b(X) \).
1.2 Equivariant push-forwards and Fourier-Mukai transforms

Now consider two such varieties with finite group actions, \((X, G)\) and \((X', G')\). A map between them is given by a pair of morphisms \( \Phi: X \to X' \) and \( \varphi: G \to G' \) such that \( \Phi \circ g = \varphi(g) \circ \Phi \) for all \( g \in G \). Then, we have the pull-back \( \Phi^*: \text{Coh}^{G'}(X') \to \text{Coh}^{G}(X) \) (and its derived functor \( R\Phi^*: \text{D}^{G'}(X') \to \text{D}^{G}(X) \)) which just means equipping the usual pull-back \( \Phi^*E' \) with the \( G_1\)-linearisation \( \Phi^*\lambda \varphi(g)^*E' = g^*\Phi^*E' \).

Suppose that \( \varphi \) is surjective and put \( K := \ker(\varphi) \). Then, there is also an equivariant push-forward defined for \((E, \lambda) \in \text{Coh}^{G}(X)\) in the following way: the usual push-forward \( \Phi_*E \) is canonically \( G \)-linearised since \( \varphi \) is surjective. Now \( K \) acts trivially on \( X' \), thus is it possible to take \( K \)-invariants. Then the subsheaf \( \Phi^K(E, \lambda) := [\Phi_*E]^K \subset \Phi_*E \) is still \( G' \)-linearised and \( R\Phi^K_*: \text{D}^{G}(X) \to \text{D}^{G'}(X') \) is the correct push-forward.

For objects \((E, \lambda) \in \text{Coh}^{G}(X)\) and \((E', \lambda') \in \text{Coh}^{G'}(X')\) and a \( G \)-invariant morphism \( \Phi^*E' \to E \), the adjoint morphism \( E' \to \Phi_*E \) has image in \( \Phi^K_*E \). Hence, the functors \( \Phi^* : \text{Coh}^{G'}(X') \to \text{Coh}^{G}(X) \) and \( \Phi^K_* : \text{Coh}^{G}(X) \to \text{Coh}^{G'}(X') \) are adjoint; analogously for \( \Phi_* \) and \( R\Phi^K_* \).

As a consequence, the usual Fourier-Mukai calculus extends to the equivariant setting if we use these functors (the tensor product of two linearised objects is obviously again linearised). Explicitly, for an object \((P, \rho) \in \text{D}^{G\times G'}(X \times X')\) we get a functor

\[
\text{FM}_{(P, \rho)}: \text{D}^{G}(X) \to \text{D}^{G'}(X'), \quad (E, \lambda) \mapsto R\rho_X^* (P \otimes^L p_X^*(E))
\]

whit the projections \( p_{X'} : X \times X' \to X' \) and \( p_X : X \times X' \to X \) (and similar projections on the group level).

1.3 Inflation and restriction

There is an obvious forgetful functor \( \text{for}: \text{D}^{G}(X) \to \text{D}^{b}(X) \). In the other direction, we have the inflation functor \( \text{inf}: \text{D}^{b}(X) \to \text{D}^{G}(X) \) with \( \text{inf}(E) := \bigoplus_{g \in G} g^* E \) and the \( G \)-linearisation comes from permuting the summands\(^1\). A generalisation of \( \text{inf} \) to the case of a subgroup \( H \subset G \) is given by

\[
\text{inf}^H_{G}: \text{D}^H(X) \to \text{D}^{G}(X), \quad (E, \lambda) \mapsto \bigoplus_{[g] \in H \setminus G} g^* E
\]

and the \( G \)-linearisation of the sum is a natural combination of \( \lambda \) and permutations.

See Bernstein/Lunts [2] for generalisations of \( \text{D}^{G}(X) \) and \( \text{inf}^G_{H} \) to the case of algebraic groups (neither of which is straightforward).

\(^1\) In a similar vein, every symmetric polynomial \( \sigma \in \mathbb{Z}[x_1, \ldots, x_n]^S_n \) (where \( n := \#G \)) gives rise to a functor \( \text{inf}^\sigma : \text{D}^b(X) \to \text{D}^{G}(X) \). For example, \( \text{inf}_{x_1 + \cdots + x_n}^1 : \text{D}^b(X) \to \text{D}^{G}(X) \). As an application, for an ample line bundle \( L \) on \( X \) we get \( \text{inf}_{x_1 \cdots x_n}^1 (L) \), an ample \( G \)-linearised line bundle.
1.4 Invariant vs linearised objects

Obviously, a $G$-linearised object has to be $G$-invariant, i.e. fixed by all pullbacks $g^*$. It is a difficult question under which conditions the other direction is true. For us the following fact ([16, Lemma 3.4]) will suffice.

**Lemma 1.** Let $E \in D^b(X)$ be simple and $G$-invariant. Then there is a group cohomology class $[E] \in H^2(G, \mathbb{C}^*)$ such that $E$ is $G$-linearisable if and only if $[E] = 0$. Furthermore, if $[E] = 0$, then the set of $G$-linearisations of $E$ is canonically a $\hat{G}$-torsor.

**Proof.** Note that the $G$-action on Aut$(E) = \mathbb{C}^*$ is trivial. There are isomorphisms $\mu_g : E \rightarrow g^*E$ for all $g \in G$. As $E$ is simple, we can define units $c_{g,h} \in \mathbb{C}^*$ by $\mu_{gh} = h^* \mu_g \circ \mu_h \cdot c_{g,h}$.

It is a straightforward check that the map $c : G^2 \rightarrow \mathbb{C}^*$ is a 2-cocycle of $G$ with values in $\mathbb{C}^*$, i.e. $c \in Z^2(G, \mathbb{C}^*)$. Replacing the isomorphisms $\mu_g$ with some other $\mu'_g$ yields the map $e : G \rightarrow \mathbb{C}^*$ such that $\mu'_g = c_g \cdot e_g$. The two cocycles $c, c' : G^2 \rightarrow \mathbb{C}^*$ derived from $\mu$ and $\mu'$ differ by the boundary coming from $e$ by another easy computation. Hence, $c' / c = d(e)$ and thus $c = c' \in H^2(G, \mathbb{C}^*)$. Thus the $G$-invariant object $E$ gives rise to a unique class $[E] := c \in H^2(G, \mathbb{C}^*)$. In these terms, $E$ is $G$-linearisable if and only if $c \equiv 1$, i.e. $[E]$ vanishes.

For the second statement, we write $\hat{G} := \text{Hom}(G, \mathbb{C}^*)$ for the group of characters and $\text{Lin}_G(E)$ for the set of non-isomorphic $G$-linearisations of $E$. Consider the $G$-action $\hat{G} \times \text{Lin}_G(E) \rightarrow \text{Lin}_G(E)$, $(\chi, \lambda) \mapsto \chi \cdot \lambda$ on $\text{Lin}_G(E)$. First take $\chi \in \hat{G}$ and $\lambda \in \text{Lin}_G(E)$ such that $\chi \cdot \lambda = \lambda$. Then, there is an isomorphism $f : (E, \lambda) \Rightarrow (E, \chi \cdot \lambda)$ which in turn immediately implies $\chi = 1$ using $f \in \text{Aut}(E) = \mathbb{C}^*$. Thus, the action is effective. Now take two elements $\lambda, \lambda' \in \text{Lin}_G(E)$ and consider $\lambda^{-1} \circ \lambda' : E \rightarrow g^*E \Rightarrow E$. As $E$ is simple, we have $\lambda^{-1} \circ \lambda' = \chi(g) \cdot \text{id}_E$. It follows from the cocycle condition for linearisations that $\chi$ is multiplicative, i.e. $\chi \in \hat{G}$. In other words, $\lambda = \chi \cdot \lambda$ and the action is also transitive. Altogether $\hat{G}$ acts simply transitive on $\text{Lin}_G(E)$. \qed

For our use of group cohomology, refer for example [17]. The second cohomology $H^2(G, \mathbb{C}^*)$ of the finite group $G$ acting trivially (on an algebraically closed field of characteristic 0) is also known as the Schur multiplier of $G$ (refer [8, §25]).

Two relevant facts about it are: $H^2(G, \mathbb{C}^*)$ is a finite abelian group; its exponent is a divisor of $\#G$. Examples are given by $H^2((\mathbb{Z}/n\mathbb{Z})^k, \mathbb{C}^*) = (\mathbb{Z}/n\mathbb{Z})^{k(k-1)/2}$ for copies of a cyclic group: $H^2(D_{2n}, \mathbb{C}^*) = \mathbb{Z}/2\mathbb{Z}$ and $H^2(D_{2n+1}, \mathbb{C}^*) = 0$ for the dihedral groups with $n > 1$; and $H^2(S_n, \mathbb{C}^*) = \mathbb{Z}/2\mathbb{Z}$ for the symmetric groups with $n > 3$.

Note that a group with vanishing Schur multiplier has the following property: every simple $G$-invariant object of $D^b(X)$ is $G$-linearisable, no matter how $G$ acts on $X$.

**Remark 2.** The condition that $E$ be simple in the Lemma is important. Consider an abelian surface $A$ with the action of $G = \mathbb{Z}/2\mathbb{Z} = \{ \pm \text{id}_A \}$. Then the sheaf $E := k(a) \oplus k(-a) = \inf(k(a))$ is $G$-invariant but not simple. Yet it is uniquely $\mathbb{Z}/2\mathbb{Z}$-linearisable as an easy computation shows [16, Example 3.9] (in contrast to $G$-invariant simple sheaves, which have precisely two non-isomorphic $G$-linearisations according to $H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*) = 0$). This behaviour is expected from geometry: by derived McKay correspondence ([3]) one has
\(D^G(A) \cong D^b(X)\), where \(X\) is the Kummer surface of \(A\), a crepant resolution \(\psi : X \to A/G\). Under this equivalence, skyscraper sheaves of points \(x \in X\) outside of exceptional fibres of \(\psi\) are mapped to \(k(\psi(x)) \oplus k(-\psi(x))\).

**Example 3.** If \(G\) acts on \(X\), then the canonical sheaf \(\omega_X\) is simple (as it is a line bundle) and \(G\)-invariant (because it is functorial). Due to this functoriality, it is actually \(G\)-linearisable: the morphism \(g : X \to X\) induces a morphism of cotangent bundles \(b_* : g^*\Omega_X \to \Omega_X\). Going to determinants and using adjunction yields the desired isomorphisms \(\lambda_g := \det(g_*^{-1}) : \omega_X \to g^*\omega_X\).

### 2 Groups of autoequivalences

We are interested in comparing the automorphism group \(\text{Aut}(D^G(X))\) with the group \(\text{Aut}(D^b(X))^G := \{F \in \text{Aut}(D^b(X)) : g^* \circ F = F \circ g^* \forall g \in G\}\). It turns out that a useful intermediate step is to look at Fourier-Mukai equivalences on \(D^b(X)\) which are diagonally \(G\)-linearised.

To make this precise, consider a Fourier-Mukai transform \(\text{FM}_P : D^b(X) \to D^b(X')\). Suppose that \(G\) acts on both \(X\) and \(X'\). Then we have the diagonal action \(G \times X \times X' \to X \times X'\), \(g \cdot (x, x') := (gx, gx')\) which we sometimes (especially in the case \(X = X'\)) for emphasis call the \(G_\Delta\)-action of \(G\) on \(X \times X'\).

Now we are in a position to study objects \((P, \rho) \in D^{G_\Delta}(X \times X')\) which give Fourier-Mukai equivalences \(\text{FM}_P : D^b(X) \to D^b(X')\). In other words, these are ordinary kernels for equivalences \(D^b(X) \Rightarrow D^b(X')\) which additionally have been equipped with a \(G_\Delta\)-linearisation.

Not every kernel in \(P \in D^b(X \times X')\) has the latter property. A necessary condition is that \(P\) must be \(G_\Delta\)-invariant, i.e. \((g, g)^* P \cong P\) for all \(g \in G\), or, equivalently, \(g^* \circ \text{FM}_P = \text{FM}_P \circ g^*\). Now we apply the following general fact:

**Lemma 4.** If \(P \in D^b(X \times Y)\) is the Fourier-Mukai kernel of an equivalence \(\text{FM}_P : D^b(X) \Rightarrow D^b(Y)\) then \(P\) is simple, i.e. \(\text{Hom}_{D^b(X \times Y)}(P, P) = \mathbb{C}\).

**Proof.** Fix \(f \in \text{Hom}_{D^b(X \times Y)}(P, P)\) and let \(Q\) be a quasi-inverse kernel for \(\text{FM}_P\), i.e. \(P \ast Q \cong \mathcal{O}_{X'}\). By \(\text{Hom}_{D^b(X \times Y)}(\mathcal{O}_{X'}, \mathcal{O}_{X'}) = \text{Hom}_{X \times Y}(\mathcal{O}_{X'}, \mathcal{O}_{X'}) = \mathbb{C}\) we have \(f \ast \text{id}_{\mathcal{O}_{X'}} = c \ast \text{id}_{\mathcal{O}_{X'}}\) for a \(c \in \mathbb{C}\). Composing again with \(\text{id}_P : P \to P\) gives \(f = f \ast \text{id}_{\mathcal{O}_{X'}} = f \ast (Q \ast \text{id}_P) = (f \ast Q) \ast \text{id}_P = (c \ast \text{id}_{\mathcal{O}_{X'}}) \ast \text{id}_P = c \ast \text{id}_P\). \(\square\)

Combining Lemmas 1 and 4, we see that \(G_\Delta\)-invariant kernels for equivalences are \(G_\Delta\)-linearisable, provided that \(H^2(G, \mathbb{C}^*) = 0\) (or more generally, if the obstruction class in \(H^2(G, \mathcal{C}^*)\) vanishes).

Now suppose we have an arbitrary object \((P, \rho) \in D^{G_\Delta}(X \times X')\) and the accompanying functor \(\text{FM}_P : D^b(X) \to D^b(X')\). The general device of inflation allows us to produce the following equivariant Fourier-Mukai transform from \((P, \rho)\):

\[
\text{FM}^G_{(P, \rho)} := \text{FM}_{\text{inf}^G_{\Delta}(P, \rho)} : D^G(X) \to D^G(X').
\]

For brevity, we set \(G \cdot P := \text{inf}^G_{G_\Delta}(P, \rho) \in D^G(X \times X')\) and call it the inflation of \((P, \rho)\). The following lemma states the main properties of this assignment.

**Lemma 5.** Let \(X, X', X''\) be smooth projective varieties with \(G\)-actions and let \((P, \rho) \in D^{G_\Delta}(X \times X')\) and \((P', \rho') \in D^{G_\Delta}(X' \times X'')\).

1. \(\text{FM}_{\mathcal{O}_{X \times X} \text{-can}} \cong \text{id} : D^G(X) \to D^G(X)\).
(2) For \((P, \rho) \in D^G(X \times X')\) there is a commutative diagram

\[
\begin{array}{ccc}
D^G(X) & \xrightarrow{\text{FM}^G_{(P, \rho)}} & D^G(X') \\
\downarrow\text{for} & & \downarrow\text{for} \\
D^B(X) & \xrightarrow{\text{FM}_P} & D^B(X')
\end{array}
\]

(3) \(\text{FM}^G_{(P', \rho')} \circ \text{FM}^G_{(P, \rho)} \cong \text{FM}^G_{(P' \star P, \rho' \star \rho)}\) where \((\rho' \star \rho)_g := \rho'_g \ast \rho_g\).

(4) \(\text{FM}_P\) fully faithful \(\implies\) \(\text{FM}^G_{(P, \rho)}\) fully faithful.

(5) \(\text{FM}_P\) equivalence \(\implies\) \(\text{FM}^G_{(P, \rho)}\) equivalence.

Proof. (1) The structure sheaf \(\mathcal{O}_\Delta\) of the diagonal \(\Delta \subset X \times X\) has a canonical \(G_\Delta\)-linearisation, as \((g, g)^*\mathcal{O}_\Delta = \mathcal{O}_\Delta\). The inflation of \(\mathcal{O}_\Delta\) is \(G : \mathcal{O}_\Delta = \bigoplus_{g \in G} \mathcal{O}_{(g, 1)\Delta}\), and its \(G^2\)-linearisation is given by the permutation of summands via \(G \rightarrow G, g \mapsto kg^{-1}\).

Using this one can check by hand that \(\text{FM}^G_{(\mathcal{O}_{\Delta, \text{can}})}\) maps any \((E, \lambda) \in D^G(X)\) to itself (see [16, Example 3.14]).

(2) Take any object \((E, \lambda) \in D^G(Y)\). Then, we have by definition of equivariant Fourier-Mukai transforms \(\text{FM}_P^G : D^G(Y) \rightarrow D^G(Y), (E, \lambda) \mapsto [\mathbb{R}_{P_{23}}(G \cdot P \otimes L p_1^* E)]^{G \times 1}\). The \(G \times 1\)-linearisation of \(G \cdot P\) is given by permutations (the \(G_\Delta\)-linearisation of \(P\) does not enter). Since \(\mathbb{R}_{P_{23}}((g, 1)^* P \otimes L p_1^* E) = \mathbb{R}_{P_{23}}(g, 1)^* (P \otimes L p_1^* g^{-1} E) = \mathbb{R}_{P_{23}}(P \otimes L p_1^* E),\) we see that \(\mathbb{R}_{P_{23}}(G \cdot P \otimes L p_1^* E) \cong \bigoplus_{G, h \in G^1} \mathbb{R}_{P_{23}}(P \otimes L p_1^* E)\) and \(G \times 1\) acts with permutation matrices where the \(1\)'s are replaced by \(p_1^* \lambda_g\)'s. Taking \(G \times 1\)-invariants singles out a subobject of this sum isomorphic to one summand.

A morphism \(f : E_1 \rightarrow E_2\) in \(D^G(X)\) is likewise first taken to a \(G\)-fold direct sum. The final taking of \(G \times 1\)-invariants then leaves one copy of \(\text{FM}_P(f)\).

(3) The composite \(\text{FM}^G_{G, \rho} \circ \text{FM}_G\) has the kernel

\[
(G \cdot P') \star (G \cdot P) = \left[\mathbb{R}_{P_{13+}}(p_{12}^* (G \cdot P) \otimes L p_{23}^* (G \cdot P'))\right]^{1 \times G \times 1}
\]

\[
= \left[\mathbb{R}_{P_{13+}}(p_{12}^* \bigoplus_{h \in G} (g, 1)^* P \otimes L p_{23}^* (1, h^{-1})^* P')\right]^{1 \times G \times 1}
\]

\[
\cong \rho' \left[\mathbb{R}_{P_{13+}}(p_{12}^* \bigoplus_{h \in G} (g, 1)^* P \otimes L p_{23}^* (1, h^{-1})^* P')\right]^{1 \times G \times 1}
\]

\[
= \left[\bigoplus_{g, h \in G} \mathbb{R}_{P_{13+}}(g, 1, h^{-1})^* (p_{12}^* P \otimes L p_{23}^* P')\right]^{1 \times G \times 1}
\]

Now note that \((1, c, 1) \in 1 \times G \times 1\) acts on \((G \cdot P') \star (G \cdot P)\) via permutations (inverse multiplications from left) and \(\rho\) on \(P\), and \((1, c, 1)\) acts purely by permutations (which are multiplications from right) on \(P'\). Plugging this into the last equation, we find that taking invariants we end up with \(\bigoplus_{d \in G} (d^{-1}, d)^* \mathbb{R}_{P_{13+}}(p_{12}^* P \otimes L p_{23}^* P')\). Since the \((d^{-1}, d)\)'s give all classes in \(G_\Delta \setminus G^2\), we find that \((G \cdot P') \star (G \cdot P) \cong G \cdot (P' \star P)\).

(4) Fix two objects \((E_1, \lambda_1), (E_2, \lambda_2) \in D^G(X)\). The injectivity of the natural map \(\text{Hom}_{D^G(X)}((E_1, \lambda), (E_2, \lambda_2)) \rightarrow \text{Hom}_{D^G(X)}(\text{FM}^G_P(E_1, \lambda_1), \text{FM}^G_P(E_2, \lambda_2))\) is
Let the finite group $G$ act on a smooth projective variety $X$.

(1) The construction of inflation gives a group homomorphism $\inf$ which fits in the following exact sequence, where $Z(G) \subset G$ is the centre of $G$:

$$
0 \to Z(G) \to \Aut^{G\Delta}(D^b(X)) \xrightarrow{\inf} \Aut(D^b(X))
$$

$$(P, \rho) \mapsto \FM_{(P, \rho)}^G
$$

(2) Forgetting the $G\Delta$-linearisation gives a group homomorphism for which sits in the following exact sequence; here $G_{ab} := G/[G, G] \cong \Hom(G, \mathbb{C})^* = H^1(G, \mathbb{C}^*)$ is the abelianisation:

$$
0 \to G_{ab} \to \Aut^{G\Delta}(D^b(X)) \xrightarrow{\for} \Aut(D^b(X))^G \to H^2(G, \mathbb{C}^*) \to [P]
$$

The first identity uses the action $G \times \Aut(D^b(X)) \to \Aut(D^b(X))$ given by $g \cdot F := (g^{-1})^* \circ F \circ g^*$ and the formula $\FM_{(g, g)\ast} \circ P = g^* \circ \FM_P \circ (g^{-1})^*$, and finally Orlov’s result on the existence of Fourier-Mukai kernels [14, Theorem 2.2]. This has been extended by Kawamata to smooth stacks associated to normal projective varieties with quotient singularities [11]. In view of $\text{Coh}([X/G]) \cong \text{Coh}^G(X)$, this implies the second relation. Finally we have to turn $\Aut^{G\Delta}(D^b(X))$ into a group. This is done by Lemma 5: (3) settles the composition, (1) the neutral element and (5) the inverses.

The following theorem [16, Proposition 3.17] is an attempt to compare these groups.
At first consider two smooth projective varieties $X$ and $Y$ with their natural $S_n$-actions. Let $P \in D^b(X \times Y)$ be the kernel
of a Fourier-Mukai transform $\text{FM}_\rho : D^b(X) \rightarrow D^b(Y)$. Then, the exterior tensor product $p^n_{\mathbb{G}m} = p^n_1 P \otimes \cdots \otimes p^n_n P \in D^b(X^n \times Y^n)$ yields the functor $F^n = \text{FM}_{p^n_{\mathbb{G}m}} : D^b(X^n) \rightarrow D^b(Y^n)$. Furthermore, $P^n_{\mathbb{G}m}$ has an obvious $(S_n)_{\Delta}$-linearisation via permutation of tensor factors. Hence, using inflation, we get the new functor $F^{[n]} := \text{FM}_{(p^n_{\mathbb{G}m})^\sim} : D^{S_n}(X^n) \rightarrow D^{S_n}(Y^n)$. If we restrict to the case $X = Y$ and autoequivalences $F : D^b(X) \Rightarrow D^b(X)$, we get a group homomorphism by Theorem 6

$$\text{Aut}(D^b(X)) \rightarrow \text{Aut}^{(S_n)_{\Delta}}(D^b(X^n)) \rightarrow \text{Aut}(D^{S_n}(X)), \quad \text{FM}_\rho \mapsto \text{FM}^{S_n}_{p^n_{\mathbb{G}m}, \text{perm}}.$$}

From now on we need the provision $\dim(X) = \dim(Y) = 2$. It is well-known that for surfaces $\text{Hilb}_n(X)$ is a crepant resolution of $X^n/S_n$. Furthermore, a theorem of Haiman states $\text{Hilb}_n(X) \cong S_n\text{-Hilb}(X^n)$, see [7]. If we additionally assume $\omega_X \cong \mathcal{O}_X$ and $\omega_Y \cong \mathcal{O}_Y$, then we can invoke Theorem 7 in order to obtain $D^n(\text{Hilb}_n(X)) \cong D^{S_n}(X^n)$ because in this case $X^n$ and $Y^n$ are symplectic manifolds and the condition $\dim(\text{Hilb}_n(X) \times X^n/S_n, \text{Hilb}_n(X)) < 1 + n \dim(X)$ of Theorem 7 is automatically fulfilled; see [3, Corollary 1.3]. However, a posteriori this inequality is true for general surfaces as the dimension of the fibre product is a local quantity which may be computed with any (e.g. affine or symplectic) model. The above homomorphism of groups of autoequivalences is now

$$\text{Aut}(D^b(X)) \rightarrow \text{Aut}(D^b(\text{Hilb}_n(X))).$$

It is always injective: this is clear for $n > 2$ since the centre of $S_n$ is trivial in this case. For $n = 2$ one can check that the sheaf $\mathcal{O}_{X \times X}$ with the non-canonical $(S_2)_{\Delta}$-linearisation is not in the image of $\text{Aut}(D^b(X)) \rightarrow \text{Aut}^{(S_2)_{\Delta}}(D^b(X^2))$.

Let us introduce the shorthand $X^{[n]} := \text{Hilb}_n(X)$, so that $D^b(X^{[n]}) \cong D^{S_n}(X^n)$ by the above and $\text{Aut}(D^b(X)) \hookrightarrow \text{Aut}(D^b(X^{[n]}))$. The above technique shows

**Proposition 8.** If $X$ and $Y$ are two smooth projective surfaces with $D^b(X) \cong D^b(Y)$, then $D^b(X^{[n]}) \cong D^b(Y^{[n]}).$

**Remark 9.** A birational isomorphism $f : X \dashrightarrow Y$ of smooth projective surfaces induces a birational map $f^{[n]} : X^{[n]} \dashrightarrow Y^{[n]}$ between their Hilbert schemes. There is a derived analogue: a Fourier-Mukai transform (resp. equivalence) $F = \text{FM}_\rho : D^b(X) \rightarrow D^b(Y)$ induces a functor (resp. equivalence) $F^{[n]} = \text{FM}_{\rho^{S_n}} : D^b(X^{[n]}) \rightarrow D^b(Y^{[n]})$. Since for the time being it is unknown whether every functor $F : D^b(X) \rightarrow D^b(Y)$ is of Fourier-Mukai type, we have to restrict to those (which include equivalences by [14, Theorem 2.2]).

Finally, we specialise to K3 surfaces.

**Proposition 10.** Let $X_1$ and $X_2$ be two projective K3 surfaces. If there is a birational isomorphism $X_1^{[n]} \dashrightarrow X_2^{[n]}$ of their Hilbert schemes, then the derived categories are equivalent: $D^b(X_1^{[n]}) \cong D^b(X_2^{[n]})$.

**Proof.** A birational isomorphism $f : X_1^{[n]} \dashrightarrow X_2^{[n]}$ induces an isomorphism on second cohomology, $f^* : H^2(X_1^{[n]}) \Rightarrow H^2(X_2^{[n]})$, respecting the Hodge structures, because Hilbert schemes of symplectic surfaces are symplectic manifolds. From the crepant resolution $X_1^{[n]} \rightarrow X_1^n/S_n$, we find $H^2(X_1) \subset H^2(X_1^{[n]})$, and only the
exceptional divisor $E_1 \subset X_1^{[n]}$ is missing: $H^2(X_1^{[n]}) = H^2(X_1) \oplus \mathbb{Z} \cdot \delta_1$ with $2\delta_1 = E_1$. In particular, as $[E_1]$ is obviously an algebraic class, the transcendental sublattices coincide: $T(X_1) = T(X_1^{[n]})$. Hence, the birational isomorphism furnishes an isometry $T(X_1) \cong T(X_2)$. Orlov’s derived Torelli theorem for K3 surfaces [14] then implies $D^b(X_1) \cong D^b(X_2)$. But now we apply the above result on lifting equivalences from K3 surfaces to Hilbert schemes and deduce $D^b(X_1^{[n]}) \cong D^b(X_2^{[n]})$, as claimed. 

**Remarks 11.** (1) follows from Remark 9 and (2) from Proposition 10.

(1) D-equivalent abelian or K3 surfaces $X_1$ and $X_2$ have D-equivalent Hilbert schemes $X_1^{[n]}$ and $X_2^{[n]}$.

(2) Considering only birational equivalence classes of Hilbert schemes on K3 surfaces, we find that each such class is finite, since K3 surfaces have only finitely many Fourier-Mukai partners [4].

(3) Markman [12] gives an example of non-birational Hilbert schemes $X_1^{[n]}$ and $X_2^{[n]}$ with $H^2(X_1^{[n]}) \cong H^2(X_2^{[n]})$. The above arguments still yield $D^b(X_1) \cong D^b(X_2)$ and $D^b(X_1^{[n]}) \cong D^b(X_2^{[n]})$, i.e. $X_1^{[n]}$ is D-equivalent to $X_2^{[n]}$.

### 3.2 Kummer surfaces

Let $A$ be an abelian variety and consider the action of $G := \{ \pm \text{id}_A \} \cong \mathbb{Z}/2\mathbb{Z}$. In order to investigate $\text{Aut}(D^b(A))$, we start with the exact sequence (see Orlov’s article [15]):

$$
0 \longrightarrow \mathbb{Z} \times A \times \hat{A} \overset{\eta}{\longrightarrow} \text{Aut}(D^b(A)) \overset{\gamma}{\longrightarrow} \text{Sp}(A \times \hat{A}) \longrightarrow 0.
$$

the first morphism $\eta$ maps a triple $(n, a, \xi)$ to the autoequivalence $t_a^* \circ M_\xi[n]$, where $t_a : A \rightarrow A$ denotes the translation by $a$ and $M_\xi : D^b(A) \rightarrow D^b(A)$ the line bundle twist with $\xi$. Note that shifts, translations and twists by degree 0 line bundles commute. Before turning to the second morphism $\gamma$, we set

$$
\text{Sp}(A \times \hat{A}) := \left\{ \left( \frac{f_1}{f_2}, \frac{f_3}{f_4} \right) \in \text{Aut}(A \times \hat{A}) : \left( \frac{f_1}{f_2}, \frac{f_3}{f_4} \right) \left( \begin{array}{cc} -f_2 & f_1 \\ f_4 & f_3 \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \right\}.
$$

Now given $F \in \text{Aut}(D^b(X))$, there is a functorial way to attach an equivalence $\Phi_F : D^b(A \times \hat{A}) \rightarrow D^b(A \times \hat{A})$ which sends skyscraper sheaves to skyscraper sheaves. Hence $\Phi_F$ yields an automorphism $\gamma(F) : A \times \hat{A} \rightarrow A \times \hat{A}$ which turns out to be in $\text{Sp}(A \times \hat{A})$ (see the original [15, §2] by Orlov or [16, §4] for a slightly different presentation).

Note that $G$ acts on $\text{Aut}(D^b(A))$ via conjugation, i.e. $(-1) \cdot F := (-1)^* \circ F \circ (-1)^*$. This induces an action on $\text{Sp}(A \times \hat{A})$, which is trivial since $\gamma((-1)^*) = -\text{id}_{A \times \hat{A}}$ is central. Taking $G$-invariants of Orlov’s exact sequence, we get

$$
0 \longrightarrow \mathbb{Z} \times A[2] \times \hat{A}[2] \overset{\eta^G}{\longrightarrow} \text{Aut}(D^b(A))^G \overset{\gamma^G}{\longrightarrow} \text{Sp}(A \times \hat{A}) \longrightarrow 0.
$$

Here, $A[2] \subset A$ and $\hat{A}[2] \subset \hat{A}$ denote the 2-torsion subgroups; the surjectivity of $\gamma^G$ uses $H^1(G, \mathbb{Z} \times A \times \hat{A}) = 0$; see [16, Proposition 4.8]. Hence, any autoequivalence $F \in \text{Aut}(D^b(A))$ differs from a $G$-invariant one just by translations and degree 0 line bundle twists.
Assume from now on that $A$ is an abelian surface. Let $X$ be the corresponding Kummer surface, which is a crepant resolution of $A/G$. A realisation is $X = G$-Hilb($A$) and we use derived McKay correspondence (Theorem 7) to infer $\text{Db}(X) \cong \text{D}^G(A)$. From Theorem 6 we get group homomorphisms

\[ \text{Aut}(\text{Db}(A))^G \xrightarrow{\text{for}} \text{Aut}^G(\text{Db}(A)) \xrightarrow{\text{inf}} \text{Aut}(\text{D}^G(A)) \]

In our situation both for and inf are 2:1, and for is surjective as $H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*) = 0$. There does not seem to be a homomorphism between the lower groups making the triangle commutative.

However, when going to cohomology (here with $\mathbb{Q}$ coefficients throughout) the roof can be completed to a diagram (refer [16, Proposition 4.14]):

\[ \begin{array}{ccc}
\text{Aut}(\text{Db}(A)) & \xrightarrow{(\cdot)^H} & \text{Aut}(\text{H}^2(A)) \\
\xrightarrow{\text{inf}} & & \xrightarrow{\text{res}} \\
\text{Aut}^G(\text{Db}(A)) & & \\
\xrightarrow{\text{for}} & & \\
\text{Aut}(\text{Db}(X)) & \xrightarrow{(\cdot)^H} & \text{Aut}(\text{H}^*(X)) \\
\end{array} \]

Here, $(\cdot)^H : \text{Aut}(\text{Db}(A)) \to \text{Aut}(\text{H}^2(A))$ sends a Fourier-Mukai equivalence to the corresponding isomorphism on cohomology. Further we use that the image of $(\cdot)^H \circ \text{inf}$ lies inside the subgroup of isometries preserving the exceptional classes,

\[ \text{Aut}(\text{H}^*_\text{ex}(X)) := \{ \varphi \in \text{Aut}(\text{H}^*(X)) : \varphi(\Lambda) = \Lambda \} \]

where $\Lambda \cong Q_{16}$ is the lattice spanned by the $(-2)$-classes arising from the Kummer construction; the morphism $\text{res} : \varphi \mapsto \varphi|_\Lambda$ is then the obvious restriction.

### 3.3 Canonical quotients

Let $X$ be a smooth projective variety whose canonical bundle is of finite order. Suppose that $n > 0$ is minimal with $\omega_X^n \cong \mathcal{O}_X$. Then there is an étale covering $\tilde{X} \to X$ of degree $n$ with $\omega_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}$. One concrete definition is $\tilde{X} = \text{Spec}(\mathcal{O}_X \oplus \omega_X \cdots \oplus \omega_X^{n-1})$, see [1, §1.17]. The group $G := \mathbb{Z}/n\mathbb{Z}$ then acts freely on $\tilde{X}$ with $\tilde{X}/G = X$. Fix a generator $g \in G$ and note that a $G$-linearisation for some $E \in \text{D}^b(\tilde{X})$ is completely determined by an isomorphism $\lambda : E \to g^*E$ subject to $(g^{n-1})^*\lambda \circ \cdots \circ g^*\lambda \circ \lambda = \text{id}_E$ as $G$ is cyclic. Here we have an equivalence $\text{Coh}(X) \cong \text{Coh}^G(\tilde{X})$ already on the level of abelian categories (see [13, §7]). Hence, $\text{D}^b(X) \cong \text{D}^G(\tilde{X})$ as well, a fact which also follows from the derived McKay correspondence using the trivially crepant resolution $X \xrightarrow{\text{id}_X} X$. Then,

\[ \text{Aut}(\text{D}^b(\tilde{X}))^G \xrightarrow{\text{for}} \text{Aut}^G(\text{D}^b(\tilde{X})) \xrightarrow{\text{inf}} \text{Aut}(\text{D}^G(\tilde{X})) = \text{Aut}(\text{D}^G(\tilde{X})). \]

Bridgeland and Maciocia consider in [5] canonical quotients from the point of view of derived categories. They introduce the set of all equivariant equivalences
by
\[ \text{Aut}_{\text{eq}}(D^b(\bar{X})) := \{(F, \mu) \in \text{Aut}(D^b(\bar{X})) \times \text{Aut}(G) : g_\ast \circ \bar{F} \cong \bar{F} \circ \mu(g), \forall g \in G\}; \]
is actually a group. There is an exact sequence
\[ 0 \rightarrow \text{Aut}(D^b(\bar{X}))^G \rightarrow \text{Aut}_{\text{eq}}(D^b(\bar{X})) \rightarrow \text{Aut}(G) \rightarrow 0 \]
where the latter morphism maps \((\bar{F}, \mu) \mapsto \mu \) and \(\text{Aut}(D^b(\bar{X}))^G\) is by definition the group of all equivariant equivalences with \(\mu = \text{id}_G\). Note that \(G \cong \mathbb{Z}/n\mathbb{Z}\) implies \(\text{Aut}(G) \cong \mathbb{Z}/\varphi(n)\mathbb{Z}\). Furthermore, we have a subgroup \(G \hookrightarrow \text{Aut}(D^b(\bar{X}))^G, \ g \mapsto g_\ast = (g^{-1})^*, \) or also \(G \hookrightarrow \text{Aut}_{\text{eq}}(D^b(\bar{X})), \ g \mapsto (g_\ast, \text{id}_G). \) The latter is a normal divisor in view of \((\bar{F}, \mu)^{-1} \circ (g_\ast, \text{id}_G) \circ (\bar{F}, \mu) = (\bar{F}^{-1}, \mu^{-1}) \circ (g_\ast \circ \bar{F}, \mu) = (\mu(g)_\ast, \text{id}_G). \) By [5, Theorem 4.5] every equivalence \(F \in \text{Aut}(D^b(\bar{X}))\) has an equivariant lift \(\bar{F} \in \text{Aut}(D^b(\bar{X})), \ i.e. \ \pi_\ast \circ \bar{F} \cong F \circ \pi_\ast \) and \(\pi^\ast \circ F \cong F \circ \pi^\ast. \) If \(\bar{F}_1\) and \(\bar{F}_2\) are two lifts of \(F\), then \(\bar{F}_2^{-1} \circ \bar{F}_1 \cong g_\ast\) for a \(g \in G\) ([5, Lemma 4.3(a)]). Thus the equivariant lift \(\bar{F} \in \text{Aut}_{\text{eq}}(D^b(\bar{X}))\) is unique up to the action of \(G\) and we get a group homomorphism \(\text{lift} : \text{Aut}(D^b(\bar{X})) \rightarrow \text{Aut}_{\text{eq}}(D^b(\bar{X}))/G. \) [5, Lemma 4.3(b)] states that if \(F, F' \in \text{Aut}(D^b(\bar{X}))\) both lift to \(\bar{F} \in \text{Aut}_{\text{eq}}(D^b(\bar{X})), \) then they differ by a line bundle twist: \(F \cong F' \circ \omega_{\text{def}}\) with \(0 \leq i < n. \) Thus \(\text{lift}\) is \(n : 1, \) and we propose the following commutative pentagram

\[
\begin{array}{ccc}
\text{Aut}^{G \Delta}(D^b(\bar{X})) & \xrightarrow{\text{inf}} & \text{Aut}_{\text{eq}}(D^b(\bar{X}))/G \\
\xrightarrow{\text{lift}} \downarrow & & \downarrow \xrightarrow{\text{lift}} \\
\text{Aut}(D^b(\bar{X}))/G & \rightarrow & \text{Aut}(D^b(\bar{X})) \\
\xrightarrow{1: \varphi(n)} & & \xrightarrow{n:1} \\
\text{Aut}(D^b(X)) & \xrightarrow{\text{inf}} & \text{Aut}_{\text{eq}}(D^b(\bar{X}))/G \\
\end{array}
\]

The commutativity of this diagram boils down to the following question: given a kernel \((P, \rho) \in \text{Aut}^{G \Delta}(D^b(\bar{X})), \) is \(\text{FM}_{\rho} : D^b(\bar{X}) \Rightarrow D^b(\bar{X}) \) a lift of \(\text{FM}_{(P, \rho)}^G : D^G(\bar{X}) \Rightarrow D^G(\bar{X})\) (where we identify \(D^G(\bar{X}) \cong D^b(\bar{X})\))? However, this is clear from \(\pi_\ast : D^b(\bar{X}) \rightarrow D^G(\bar{X}), E \mapsto \text{inf}(E)\) and \(\pi^\ast : D^G(\bar{X}) \rightarrow D^b(\bar{X}), (F, \lambda) \mapsto [F, \lambda]^G. \)

**Example 12.** In the case of a double covering (e.g. \(X\) an Enriques surface), we have \(n = 2\) and hence \(\text{Aut}(D^b(\bar{X}))^G = \text{Aut}_{\text{eq}}(D^b(\bar{X})).\) Then the diagram looks like

\[
\begin{array}{ccc}
\text{Aut}^{G \Delta}(D^b(\bar{X})) & \xrightarrow{\text{inf}} & \text{Aut}_{\text{eq}}(D^b(\bar{X}))/G \\
\xrightarrow{\text{lift}} \downarrow & & \downarrow \xrightarrow{\text{lift}} \\
\text{Aut}(D^b(\bar{X}))/G & \rightarrow & \text{Aut}(D^b(\bar{X})) \\
\end{array}
\]
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