SOME QUANTITATIVE ESTIMATES FOR THE NONLINEAR FOURIER TRANSFORM

GEVORG MNATSAKANYAN

Abstract. We retract the main theorem from the last versions since we have found a mistake. This paper was trying to quantify the methods used by Alexei Poltoratski [Pol21] to prove pointwise convergence of the non-linear Fourier transform. However, [Pol21] also contains the same mistake. We leave the estimates that are still true in the form of Propositions.

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1. Introduction

One can write the exponential of the classical Fourier transform \( \hat{f} \) of an integrable function \( f \) in terms of a solution to a differential equation. Namely, given \( f \) consider the equation

\[
\partial_t G(t, x) = e^{-2i\omega t} f(t) G(t, x).
\]

Given initial datum at any point, a unique solution \( G \) exists. With the initial condition \( G(-\infty, x) = 1 \), we have

\[
\exp(\hat{f}(x)) = G(\infty, x).
\]

One can consider matrix-valued analogs (1.1) such as

\[
\partial_t G(t, x) = \begin{pmatrix} 0 & e^{-2i\omega t} f(t) \\ e^{2i\omega t} f(t) & 0 \end{pmatrix} G(t, x),
\]

where \( G \) is a 2 \times 2 matrix-valued function. It is not difficult to check that \( G(t, x) \) takes values in \( SU(1,1) \), that is

\[
G(t, x) = \begin{pmatrix} a(t, x) & b(t, x) \\ b(t, x) & a(t, x) \end{pmatrix},
\]

where

\[
|a(t, x)|^2 - |b(t, x)|^2 = 1.
\]

In analogy to the scalar case above, with the initial condition \( G(-\infty, x) \) equals the identity matrix, we call the matrix \( G(\infty, \cdot) \) the non-linear Fourier transform (NLFT) of \( f \). The NLFT and its kin have long been studied in analysis under various names such as orthogonal polynomials [Sim05], Krein systems [Den06], scattering transforms [BC84] and AKNS systems [AKNS74]. An \( SU(2) \) version of the above model in which the lower-left entry of the matrix in (1.2) gets an extra minus sign was studied in [Tsa05] and has recently been rediscovered [AMT23] in the context of quantum computing and is called quantum signal processing [LC17].

The NLFT shares many similarities with the classical Fourier transform. For example, it takes translations in \( f \) to certain modulations in \( b \). While the classical Fourier transform has independent symmetries under scaling of the argument and of the value of the function \( f \), the NLFT has only a one parameter scaling symmetry as follows. If

\[
\hat{f}(t) = \lambda f(\lambda t),
\]

for some \( \lambda > 0 \), then

\[
\hat{G}(t, z) = G(\lambda^{-1} t, \lambda^{-1} z).
\]

A more surprising analog is the following Plancherel identity. We have, for \( f \in L^2(\mathbb{R}) \),

\[
\|\sqrt{\log |a(\infty, \cdot)|}\|_{L^2(\mathbb{R})} = \sqrt{\frac{\pi}{2}} \|f\|_{L^2(\mathbb{R})}.
\]

For \( f \in L^p(\mathbb{R}) \), \( 1 \leq p < 2 \) we have the analog of the Hausdorff-Young inequality

\[
\|\sqrt{\log |a(\infty, \cdot)|}\|_{L^{p'}(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})},
\]

and the analog of the Menshov-Paley-Zygmund theorem

\[
\|\sup_t \sqrt{\log |a(t, \cdot)|}\|_{L^{p'}(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})}.
\]
Formula (1.5) is proven by a contour integral and in the discrete case goes back to Verblunsky in 1936 [Ver35]. Formulas (1.6) and (1.7) follow from the work of Christ and Kiselev [CK01a, CK01b, CK02] as summarized in [TT12, Lecture 1] or by an application of the variation norm Carleson theorem [OST+12, Appendix C]. Recent variation norm analogs and related results can be found in [Sil18, KOESR22]. All of the above asymptotically become the classical inequalities for small $f$, if one uses

$$a(x) := a(\infty, x) = 1 + O(\|f\|_1^2),$$

$$b(x) := b(\infty, x) = \hat{f}(x) + O(\|f\|_3^2).$$

An open problem [MTT01, MTT02] in nonlinear Fourier analysis is the natural analog of the weak-$L^2$ bound for Carleson’s operator, that is whether for $f \in L^2(\mathbb{R})$ the following inequality holds

$$\{|x \in \mathbb{R} : \sup_t \sqrt{\log |a(t, x)|} > \lambda\} \lesssim \frac{1}{\lambda^2} \|f\|_2^2. \tag{1.10}$$

Like in the linear setting, the weak bound (1.10) implies almost everywhere convergence of $|a|$. However, unlike in the linear setting where Stein’s principle [Ste61] applies, the reverse implication is not known to be true. Furthermore, the convergence of arg($a$) is not known to us even for the Cantor group model considered in [MTT02], for which (1.10) is proved.

The analysis in Poltoratski’s paper [Pol21], and in ours, relies on the study of the zeros of the de Branges function associated with the NLFT,

$$E(t, z) := e^{-itz} (a(t, z) + b(t, z)). \tag{1.11}$$

The function $E$ is a continuous analog of the orthogonal polynomials associated to the NLFT in the discrete setting [TT12, Lecture 5]. For fixed $t$, $E$ is an entire function of exponential type $t$ with zeros in the lower-half plane. The proofs in [Pol21] can be broken down into two parts. First, the function $|E|$, $|a|$ and $|b|$ converge at a point $x \in \mathbb{R}$ if $t$ times the distance to $x$ from the closest zero of $E(t, \cdot)$ diverges to infinity. In the context of the orthogonal polynomials this has been established by Bessonov and Denisov in [BD21, Theorem 3] using different methods. The second step is to show that a zero of $E(t, \cdot)$ for which the limit inferior of $t$ times the distance to the real line remains bounded, then it accumulates too much $L^2$ norm for the potential. In [Pol21], the mistake is in the the second part, in particular, in the proof of Lemma 11, on page 48 where it is claimed that $\eta(t_k, \cdot)$ has zero integral and arg($2\phi(t_k, u)$) has integral which is $o(1)$. The same mistake appeared in the proof of Proposition 7 of the previous version (version 3) of this paper as equations (7.45) and (7.48).

We make a quantitative analysis of the de Branges function $E$ and its zeros. We use the Hardy-Littlewood maximal function of the spectral measure to bound error terms for approximations of the function $E$. This is demonstrated in Proposition 2 below. In addition, we use Lemma 3 to compare large error terms to the $L^2$ norm of the potential.

We remark that the conjectured estimate (1.10), even with the restriction $\|f\|_1 \lesssim 1$, is strong enough to imply the linear Carleson theorem. We discuss this in the appendix as Lemma 12.

Notation. We write $A \lesssim B$ if $A \leq CB$ for some absolute constant $C$ and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. Throughout the paper $d > 0$ will denote an absolute constant.
that can be different at each occurrence. \( \psi \) will denote a complex number or a holomorphic function that is bounded by an absolute constant, i.e. \( |\psi| \leq d \), and can again be different at each occurrence. We will use subscripts and write \( d_0, d_1 \) to fix the value of the constant inside a section.

2. Overview of main results

We state the main propositions here, so that all the consecutive sections are independent of each other and only refer to the results and notation of this section. We fix a potential \( f \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+) \) with \( \|f\|_1 \leq \frac{1}{2} \). All our inequalities will be independent of \( f \). Furthermore, the constructions in the first subsection below are also true for general \( f \in L^2(\mathbb{R}_+) \).

2.1. The function \( E \). Recall, that we put \( E(t, z) := e^{-itz}(a(t, z) + b(t, z)) \).

There is an ODE for \( E \) that can be easily obtained from (1.2). We have

\[
\partial_t E(t, z) = -iezE(t, z) + f(t)E^\#(t, z).
\]

For the scattering function \( \tilde{E}(t, z) = e^{itz}E(t, z) \) we have

\[
\frac{\partial}{\partial t} \tilde{E}(t, z) = f(t)e^{2itz}E^\#(t, z).
\]

These differential equations lead to Gronwall’s inequalities.

Lemma 1.

\[
|E(t, z)| \leq e^{t|\Re z| + \int_0^t |f(\xi)| \, d\xi},
\]

and

\[
|E(t_1, z) - E(t_2, z)| \leq |E(t_1, \bar{z})|e^{(t_2 - t_1)((|z| - |\Re z|) + \int_{t_1}^{t_2} |f|)}\int_{t_1}^{t_2} |f|.
\]

The proof is given in the appendix.

Lemma 2. If \( \|f\|_1 \leq 1/2 \), then there are no zeros of \( E(t, \cdot) \) in the region \( \{z : |\Re z| \leq \frac{1}{2}\} \).

Proof. Let \( z_0 \) be a zero of \( E(t, \cdot) \) for some \( t \). Then,

\[
|E(0, z_0) - E(t, z_0)| = 1 \leq e^{2|z_0|+\|f\|_1\|f\|_1},
\]

So, if \( \Re z \geq -\frac{1}{4} \), then

\[
e^{\|f\|_1\|f\|_1} \geq e^{-2},
\]

and \( \|f\|_1 \geq 1/2 \). \( \square \)

Multiplying the equation (1.2) from left and right by the matrix \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) we see that, \( a \) is even and \( b \) is odd in \( f \). So we define \( \tilde{E} \) as the function \( E \) for the potential \( -f \), i.e.

\[
\tilde{E}(t, z) := -ie^{-itz}(a(t, z) - b(t, z)).
\]
We introduce the extra factor \(-i\) to be consistent with the notation of [Pol21]. Then, the transition from the pair \(a, b\) to the pair \(E, \tilde{E}\) is a sort of change of coordinates and one can recover \(a\) and \(b\) from \(E\) and \(\tilde{E}\). In particular,

\[
a(t, z) = \frac{e^{it\bar{z}}}{2} (E(t, z) + i\tilde{E}(t, z)).
\]

The identity (1.4) is equivalent to

\[
\det \left( \begin{array}{cc} E & \tilde{E} \\ E^\# & \tilde{E}^\# \end{array} \right) = 2i.
\]

The Krein-de Branges function \(E\) has rich structure and is well suited for analysis. In particular, as mentioned in the introduction, it is the continuous analog of the orthogonal polynomials on the unit circle. Alternative and more direct ways to define it are through the solutions of the Dirac system [Pol21, Sections 1, 5] or the Krein system [Den06, Sections 6, 13]. Let us sum up some of the basic properties of \(E\) that also demonstrate the similarity with the orthogonal polynomials. Most of the facts and constructions below can be found in [Rom14, Rem18].

\(E(t, \cdot)\) is an entire function of exponential type \(t\). \(t\) is the analog of the degree \(n\) of the orthogonal polynomial. \(E\) is a Hermite-Biehler function, that is for \(z \in \mathbb{C}^+ \quad |E(t, z)| > |E(t, \bar{z})|\).

In particular, \(E\) does not have zeros in the upper half-plane.

For an entire function \(g\), \(g^\#(z) = \bar{g}(\bar{z})\) denotes its holomorphic reflection with respect to the real axis. The function

\[
\theta(t, z) := \frac{E^\#(t, z)}{E(t, z)},
\]

is a meromorphic and inner, and \(E(t, \cdot)\) can be uniquely recovered from \(\theta(t, \cdot)\) up to a constant multiple. This is analogous to polynomials having product representation by their roots. Indeed a similar infinite representation is true for meromorphic inner functions.

Let us discuss "the orthogonality" of continuous analogs of orthogonal polynomials, namely, the construction of de Branges spaces. We consider the Weyl \(m\) function

\[
m(z) := \frac{1 - b(\infty, z)/a(\infty, z)}{1 + b(\infty, z)/a(\infty, z)}.
\]

It is a holomorphic function in the upper half-plane with positive real part. By Herglotz representation theorem there is a positive Poisson-finite measure \(d\mu\), the Poisson extension of which to the upper half-plane is \(\mathbb{H}m\). Let \(w\) be the absolutely continuous part of \(d\mu\). [Den06, Theorem 11.1] states that if \(f \in L^2(\mathbb{R}^+)\), then \(\log w\) is Poisson finite. This is the continuous analog of the theorem of Szegő. In addition if \(f \in L^1\), then by [Den06, Theorem 12.1], \(d\mu\) is absolutely continuous with respect to the Lebesgue measure.

Next, we define the following Hilbert spaces of holomorphic functions

\[
B(E(t, \cdot)) := \{ F \text{ entire} : F/E(t, \cdot) \in H^2(\mathbb{C}^+), \quad F^\# / E(t, \cdot) \in H^2(\mathbb{C}^+) \},
\]

with the scalar product defined by

\[
\langle F, G \rangle_{E(t, \cdot)} := \int_{\mathbb{R}} F(x) \overline{G(x)} \frac{dx}{|E(t, x)|^2}.
\]
From Grönwall’s inequality (2.4), one can see that the norm of \( B(E(t, \cdot)) \) is equivalent to the norm of the Paley-Wiener space

\[
PW_t := \{ g \in L^2(\mathbb{R}) : \exists h \in L^2(-t, t) \text{ with } f(x) = \int_{-t}^{t} h(\xi)e^{i\xi x}d\xi \}
\]

with constants depending on the \( L^1(0, t) \) norm of \( f \). Moreover, it is known that as a set \( B(E(t, \cdot)) \) is equal to the Paley-Wiener space \( PW_t \). The spaces \( B(E(t, \cdot)) \) are isometrically embedded in and grow to \( L^2(d\mu) \). In particular, for \( F \in PW_t \)

\[
\int_{\mathbb{R}} |F(x)|^2 \frac{dx}{|E(t,x)|^2} = \int_{\mathbb{R}} |F(x)|^2 d\mu(x).
\]

These spaces possess reproducing kernels

\[
K(t, \lambda, z) := \frac{1}{2\pi i} \frac{E(t, z)E^{\#}(t, \lambda) - E^{\#}(t, z)E(t, \lambda)}{\lambda - z}.
\]

That is, \( K(t, \lambda, \cdot) \in PW_t \), and for any \( F \in PW_t \) we have

\[
\langle F, K(t, \lambda, \cdot) \rangle_{E(t, \cdot)} = F(\lambda).
\]

When \( f \equiv 0 \), then \( E(t, z) = e^{-itz} \) and \( B(E(t, \cdot)) \) just coincides with the Paley-Wiener space \( PW_t \), for which the reproducing kernel is the sinc function

\[
sinc(t, \lambda, z) := \frac{1}{\pi} \frac{\sin t(z - \lambda)}{z - \lambda}.
\]

2.2. Approximations for \( E \). It turns out that \( K(t, s, s) \), \( s \in \mathbb{R} \), is a sort of Fejer mean for the NLFT. Indeed, in this setting Christoffel-Darboux formula [Den06, Lemma 3.6] takes the form

\[
K(t, \lambda, z) = 2e^{-it(z-\lambda)} \int_{0}^{2t} e^{i\xi(z-\lambda)} E(\xi, x)\overline{E(\xi, \lambda)} d\xi
\]

In linear approximation (1.8) and (1.9), we see that

\[
K(t, s, s) = 2t(1 + 2\frac{1}{t} \int_{0}^{t} \Re \hat{f}(\xi, s)d\xi + O(||f||^2)),
\]

where \( \hat{f}(\xi, s) = \int_{0}^{\xi} f(y)e^{-isy}dy \) is the partial Fourier integral. The main term \( \int_{0}^{t} \Re \hat{f}(\xi, s)d\xi \) is, indeed, a Fejer mean of Fourier transform of \( f \). So in retrospect it is somewhat understandable that proving convergence or bounds for this object should be easier.

Before we state our first proposition let us introduce some notation. We will denote by \( Mh \) the Hardy-Littlewood maximal function of a locally integrable function \( h \). The following quantities measure the location of a point \( \lambda \in \mathbb{C} \) with respect to a point \( s \) at a time \( t \). Let

\[
\mathcal{X}_{s, t}(\lambda) := \mathcal{X}(\lambda) := \mathcal{X} := \max \left( 1, \frac{t|R\lambda - s|}{t^2 |\lambda| + 1} \right),
\]

and

\[
\mathcal{V}_{t, s}(\lambda) := \mathcal{V}(\lambda) := \mathcal{X}_{t, s}(\lambda)sinc(t, \lambda, \lambda).
\]

We suppress the dependencies whenever they are clear from the context.
Proposition 1. There exist $D_3$, such that for any $\lambda, z \in \mathbb{C}$,

$$K(t, \lambda, z) - \frac{1}{w(s)} \text{sinc}(t, \lambda, z) \leq D_3 \sqrt{V(z) V(\lambda) M(w - 1)(s)}.$$  \hfill (2.21)

Let us introduce notation for rectangles, squares and balls.

$$R(s, a, b) := \{z \in \mathbb{C} : |\Re z - s| \leq a \text{ and } |\Im z| \leq b\},$$
$$Q(s, a) := R(s, a, a),$$
$$B(s, a) := \{z \in \mathbb{C} : |z - s| \leq a\}.$$  

The above proposition is a quantitative analog of [Pol21, Lemma 4] that proves uniform convergence in $z, \lambda \in Q(s, C/t)$ of the left-hand side of (2.21) to 0 for almost every $s$. Originally, the case $\lambda = z = s \in \mathbb{R}$ in the setting of orthogonal polynomials on the unit circle was proved by Máté, Nevai and Totik [MNT91]. In the setting of Krein systems their arguments was recently adapted by Gubkin [Gub20]. Poltoratski’s proof of [Pol21, Lemma 2] leading to [Pol21, Lemma 4] is based on compactness arguments and is qualitative in nature. In previous versions of this paper we based the proof of proposition 1 on Gubkin’s arguments. However, we provide a much simpler proof here that utilizes the fact that $\frac{1}{w}$ is Poisson integrable if $w \sim 1$. Sometimes similar qualitative statements are referred to as universality type results. We also refer to [Bes21] for related results.

Throughout the paper we will denote

$$\epsilon(s) := \epsilon := M(w - 1)(s).$$  \hfill (2.22)

To put the estimate (2.21) into perspective, let us mention that if $f \equiv 0$, then $w - 1 = 0$. Furthermore, for $\|f\|_1 \lesssim 1$ we have the estimate

$$\|w - 1\|_\infty \lesssim \|f\|_1.$$  

More precisely, for $\|f\|_1 \leq \frac{1}{6}$, Grönwall’s inequality from Lemma 1 implies

$$|1 - |E|| \leq e^{\frac{1}{6}} - \frac{1}{4} \leq \frac{1}{4}.$$  

So,

$$|1 - \frac{1}{|E|^2}| = \frac{|1 - |E||(1 + |E|)}{|E|^2} \leq \frac{3}{4}.$$  

Passing to a limit through a subsequence, we deduce

$$\|w - 1\|_\infty \leq \frac{3}{4}. \hfill (2.23)$$  

On the other end, there is the following substitute for the non-linear Plancherel identity (1.5).

Lemma 3. If $\|f\|_1 \leq \frac{1}{6}$, then

$$\|w - 1\|_2 \leq \sqrt{\frac{1}{\pi}} \|f\|_2.$$  \hfill (2.24)

Proof. By the convergence in measure of $\frac{1}{|E(t, \cdot)|^2}$ to $w$, $\frac{1}{|E(t, \cdot)|^2}$ to $\tilde{w}$ and the determinant identity (2.10), we easily see that

$$|a| = \frac{1}{2} \sqrt{\frac{1}{w} + \frac{1}{\tilde{w}} + 2}. \hfill (2.25)$$  

By (2.12) for $E$ and $\tilde{E}$ we see that

$$w(x) \tilde{w}(x) = \frac{(|a|^2 - |b|^2)^2}{|a^2 - b|^2} \leq 1. \hfill (2.26)$$
Then, we can write
\[ |a| = \frac{1}{2} \sqrt{\frac{1}{w} + \frac{1}{\overline{w}} + 2} \geq \frac{1}{2} \sqrt{\frac{1}{w} + w + 2} \geq 1 + \frac{1}{8(w - 1)^2}. \]
As \( \log(1 + x) \geq x/2 \) for \( 0 \leq x \leq \frac{1}{2} \), we have
\[ \log |a| \geq \frac{1}{16}(w - 1)^2. \]
By Plancherel’s identity (1.5) we deduce (2.24).

Similarly, for \( \tilde{E} \), we have \( \|\tilde{w} - 1\|_2 \leq \sqrt{8\pi}\|f\|_2 \). This lemma is important for controlling the large error terms.

Next, we can use the bounds from Proposition 1 and the reproducing kernel formula (2.15) to obtain an approximation for \( E(t, \cdot) \) through its zeros. Let us denote
\[ \gamma(p) := \frac{\sqrt{2}}{\sqrt{\sinh(2p)}} \]
and
\[ \alpha = \alpha(t, z) = -ie^{-i\arg E^\#(t, z)}. \]
If for \( (t, s) \) there exists points \( x \in \mathbb{R} \) with the property that \( E(t, x) \) is positive, then let \( x_0(t, s) \) be the closest with that property. We will see that those points always exist. Put
\[ \beta = \beta(t, s) := e^{itx_0(t, s)}. \]
Note that \( \alpha \) and \( \beta \) are unimodular.

**Proposition 2.** There exist \( D_4 > 0 \) such that the following holds. Let \( s \in \mathbb{R}, t > 0 \) and \( z_0 \) be a complex number with \( E(t, z_0) = 0 \). For all \( z \in \mathbb{C} \)
\begin{equation}
(2.27)
\left| E(t, z) - \frac{\alpha(t, z_0)\gamma(t3z_0)}{\sqrt{w(s)}} \sin[t(z - z_0)] \right| \leq D_4 \left| z - z_0 \right| \frac{\sinh(2t|z|)}{\sqrt{|3z_0| \cdot |3z|}} \epsilon(s) \frac{1}{\sqrt{|x(z)\overline{x}(z_0)|}}.
\end{equation}
If \( z_0 \) is also the closest zero of \( E(t, \cdot) \) to \( s \), then \( x_0(t, s) \) exists and
\begin{equation}
(2.28)
\sup_{z \in B(s, 1/t)} \left| E(t, z) - \frac{\beta(t, s)}{\sqrt{w(s)}} e^{-it\overline{z}} \right| \leq D_4 \left( \sqrt{\epsilon(s)} + \frac{1}{\sqrt{t|z_0 - s|}} \right).
\end{equation}

2.3. **Consequences of the approximation.** The first inequality of the above proposition reveals a lot of structure for the zeros of \( E \). For example, if the error on the right-hand side of (2.27) is small, then Rouché’s theorem implies the existence of zeros of \( E \) near the points \( z_0 \pm \pi k \). Also, writing the same approximation for \( \tilde{E} \), one can use the determinant identity (2.10), to align the zeros of \( E \) and \( \tilde{E} \). These observations are the content of the next proposition.

Let us define what we mean by small errors. For any \( s \in \mathbb{R} \) and \( t \in \mathbb{R}_+ \), we denote by \( z(t, s) \) and \( \tilde{z}(t, s) \) the closest zeros to \( s \) of \( E(t, \cdot) \) and \( \tilde{E}(t, \cdot) \), respectively. For \( D \geq 1 \), let
\[ \Omega^D := \{(t, s) : e^{2t\Im z(t, s)} \geq D(t|z(t, s) - s|)\overline{\epsilon}(s)\}, \]
and
\[ \tilde{\Omega}^D := \{(t, s) : e^{2t\Im \tilde{z}(t, s)} \geq D(t|\tilde{z}(t, s) - s|)\overline{\epsilon}(s)\}. \]
Note, that if \( d_1 \geq d_2 \), then \( \Omega^{d_1} \subset \Omega^{d_2} \) and \( \tilde{\Omega}^{d_1} \subset \tilde{\Omega}^{d_2} \).
Also denote,
\[ \mu = \mu(s) := \epsilon(s) + \bar{\epsilon}(s). \]

**Proposition 3.** There exists \( D_5 > 0 \) such that the following holds. Let \((t, s) \in \Omega^{D_5} \).

A. We have
\[
|t|\Re z(t, s) - s| \leq \frac{\pi}{2} \leq \frac{1}{100}.
\]
Furthermore, if \( z_j \) is the \( j + 1 \)th closest zero of \( E(t, \cdot) \) to \( s \) with \( \Re z_j \geq \Re z(t, s) \) for \( 1 \leq j \leq 10 \), then
\[
|t|z_j - z(t, s) - j\pi| \leq D_5 e^{2t|z(t, s)|} \epsilon(s).
\]
The same holds from the left of \( \Re z(t, s) \) with a plus sign. Also
\[
|\alpha(s, t, z_j) - \alpha(s, t, z(t, s))| \leq D_5 e^{2t|z(t, s)|} \epsilon(s).
\]

B. If
\[
e^{2t|z(t, s)|} \geq \min \left( \max(e^{2t|\bar{z}(t, s)|}, D_5(t|\bar{z}(t, s) - s|)\frac{2}{\pi}), \epsilon(t|\bar{z}(t, s) - s|) - \frac{2}{\pi} \right),
\]
then \((t, s) \in \Omega^{D_5/10} \).

C. If \((t, s) \in \Omega^{D_5} \cap \bar{\Omega}^{D_5} \), then we have
\[
|t\Im (z(t, s) - \bar{z}(t, s))| \leq D_5 e^{2t|z(t, s)|} \mu(s),
\]
\[
|\sin^2(t\Re (z(t, s) - \bar{z}(t, s))) - w(s)\bar{w}(s)| \leq D_5 e^{2t|z(t, s)|} \mu(s),
\]
and
\[
|\alpha + \text{sign} (\sin(t\Re (z(t, s) - \bar{z}(t, s)))| \bar{\alpha}| \leq D_5 e^{2t|z(t, s)|} \mu(s).
\]

To estimate \(|a(t, s)|\), we will separate two cases. If the errors on the right-hand side of (2.27) are large for \( E \) or \( \bar{E} \), then we will use the \( L^2 \) bound of the Hardy-Littlewood maximal function of \( w - 1 \) together with Lemma 3. Otherwise, if the errors are small for both \( E \) and \( \bar{E} \), i.e. \((t, s) \in \Omega^D \cap \bar{\Omega}^D \), then we will apply part C of Proposition 3 to reveal more structure for the approximations. In fact, we have the following estimate.

**Lemma 4.** There exists a \( D_6 > 0 \) such that for \((s, t) \in \Omega^{D_6} \cap \bar{\Omega}^{D_6} \), we have
\[
||a(t, s)| - |a(s)|| \leq D_6 e^{2t|z(t, s)|}\sqrt{t|\Im (z(t, s))|} \mu(s) + 36 e^{2t|z(t, s)|}.
\]

**Proof.** Let \( z_0 := z(t, s) \) and \( \bar{z}_0 := \bar{z}(t, s) \). Let us assume \( \sin(|\Re (z_0 - \bar{z}_0)|) > 0 \).

The other case is handled analogously. Let \( Y := t|\Im z_0|, X := t(R z_0 - s) \) and \( \varphi : = \arcsin \sqrt{w(s)\bar{w}(s)} \). We apply (2.27) and use part C of Proposition 3.

\[ E(t, s) = \frac{a\sqrt{2}}{\sqrt{w(s)\sinh 2Y}} \sin t(s - z_0) + \psi \sqrt{Y}, \]

and
\[ \bar{E}(t, s) = \frac{-a\sqrt{2}}{\sqrt{w(s)\sinh 2Y}} \sin \left[ t(s - z_0 - \varphi) \right] + \psi e^{2Y} \mu \sqrt{Y}. \]

We estimate
\[
||a(t, s)| - |a(s)|| = \frac{1}{2} \left| E(t, s) + \bar{E}(t, s) \right| - \sqrt{\frac{1}{w} + \frac{1}{\bar{w}} + 2}.
\]
\[ \leq D_6e^{2Y} \sqrt{Y} \mu + \frac{1}{2} \left| \frac{\sqrt{2}}{\sin \sqrt{2} \theta} \left| \sin t(s - z_0) \right| - \frac{1}{\sqrt{w(s)}} - \frac{i}{\sqrt{w(s)}} \right| \leq \frac{1}{w} + \frac{1}{w} + 2 \]

\[ \leq D_6e^{2Y} \sqrt{Y} \mu + 8e^{-2Y} + \left| \frac{1}{\sqrt{e^{2Y} - e^{-2Y}}} \left| e^{iX + Y} \right| - \frac{1}{\sqrt{w(s)}} - \frac{i}{\sqrt{w(s)}} \right| \leq \frac{1}{w} + \frac{1}{w} + 2 \]

\[ = D_6e^{2Y} \sqrt{Y} \mu + 8e^{-2Y} + \left| \frac{1}{w} + \frac{1}{w} + 2 \left| \frac{e^{Y}}{\sqrt{e^{2Y} - e^{-2Y}}} - 1 \right| \right| \leq D_6e^{2Y} \sqrt{Y} \mu + 9e^{-2Y}. \]

\[ \forall \theta, \zeta \in R, \frac{1}{\sin \theta} \leq 1 \]

Proposition 3 and Lemma 4 motivate the following definition of the set

\[ \Xi^D := \{(t, s) \in \Omega^D \cap \Omega^D : e^{4t3z(t, s)} \geq D(t|3z(t, s))|z(t)|^{\frac{1}{2}} \}. \]

A more subtle and technical argument connects the displacement of the real part of the zero and the change of the argument of \( a \). For this we need to study the movement of the zeros.

2.4. Zeros of \( E \). From the equation (2.2), we can deduce a differential equation for \( \theta \).

\[ \partial_t \theta = 2iz\theta + f - \overline{f} \theta^2. \]

Let \( z(t) \) be a continuous curve of zeros of \( E \), i.e. \( \theta(t, z(t)) = 0 \), then (2.37) implies

\[ z'(t) = -\frac{f(t)}{\theta_z(t, z(t))}, \]

and

\[ \frac{d}{dt} \theta_z(t, z(t)) = 2iz(t)\theta_z(t, z(t)) - f(t) \frac{\theta_z(t, z(t))}{\theta_z(t, z(t))}. \]

These equations will be used in the case when \( z(t) \) is close to the real line, so \( \theta_z(t, z(t)) \) is non-zero by (2.27).

Another corollary of Proposition 2 is an estimate for \( \theta \) and its derivatives.

Proposition 4. There exists a \( D_T > 0 \) such that the following holds.

Let \((t_0, s) \in \Omega^D \) and \( \xi_0 \) is at most 10th closest zero of \( E(t_0, \cdot) \) to \( s \). Then, for \( z \in R, 3/t_0, |3z(t, s)| + 2/t_0 \) we have

\[ \left| \theta(t_0, z) - \overline{\theta}(t_0, s, \xi_0) \right| \leq D_T \left\| e^{iX(3z(t, s))} \right\| \epsilon(s) \sqrt{|3z(t, s)|}. \]

Furthermore, if \( \xi_0 \) is a continuous curve of zeros of \( E(t, \cdot) \) for \( t \in [t_0 - \delta, t_0 + \delta] \) for some small \( \delta \) such that \( (t, s) \in \Omega^D \) and \( \xi_{t_0} = \xi_0 \), then we have

\[ \left| \left( \frac{\partial}{\partial t} \arg \theta_z(t, \xi) \right)(t_0) - s \right| \leq 3|f(t_0)| \cos(2t|3z(t, s)|) + \frac{20\pi}{t_0}. \]

While (2.41) controls the change of the argument of \( \theta(t, \xi) \), by differentiating (2.40) we can immediately obtain an estimate on its magnitude. In particular, under the assumptions of the above proposition, we get

\[ |\theta_z(t, \xi)| \sim e^{-2t|3z(t, s)|} \] and \[ |\theta_z(t, \xi)| \sim e^{-2t|3z(t, s)|}. \]
3. Proof of Proposition 1

3.1. The diagonal case: $z = \lambda$. Fix $\lambda$ and put

$$s_t(x) := \frac{|\text{sinc}(t, \lambda, x)|^2}{\|\text{sinc}(t, \lambda, \cdot)\|_2^2}.$$  

It is easy to see

$$s_t(x) \lesssim \frac{t}{t^2|x - \lambda|^2 + 1}$$

and

$$\int_{\mathbb{R}} s_t(x)^2 dx = 1,$$

hence

$$\int_{\mathbb{R}} s_t(x)(w - 1)(x) dx \leq X_{s,t}(\lambda)M(w - 1)(s).$$

We write two Cauchy-Schwarz inequalities, one for the lower bound and one for the upper bound.

For the lower bound, write

$$|\text{sinc}(t, \lambda, \lambda)|^2 = \int_{\mathbb{R}} |\text{sinc}(t, \lambda, x)K(t, \lambda, x)w(x) dx|^2$$

$$\leq \int_{\mathbb{R}} |\text{sinc}(t, \lambda, x)|^2 w(x) dx \int_{\mathbb{R}} |K(t, \lambda, x)|^2 w(x) dx$$

$$= K(t, \lambda, \lambda) \int_{\mathbb{R}} |\text{sinc}(t, \lambda, x)|^2 w(x) dx.$$  

$$\leq K(t, \lambda, \lambda)s(t, \lambda, \lambda) \int_{\mathbb{R}} s_t(x)w(x) dx$$

$$\leq K(t, \lambda, \lambda)s(t, \lambda, \lambda)(w(s) + XM\rho(s)).$$

Thus,

$$K(t, \lambda, \lambda) - \frac{1}{w(s)}|\text{sinc}(t, \lambda, \lambda)| \geq -dXM(w - 1)(s)|\text{sinc}(t, \lambda, \lambda).$$

For the upper bound,

$$K(t, \lambda, \lambda)^2 = \left| \int_{\mathbb{R}} K(t, \lambda, x)s(t, \lambda, x) dx \right|^2$$

$$\leq \int_{\mathbb{R}} |K(t, \lambda, x)|^2 w(x) dx \int_{\mathbb{R}} |\text{sinc}(t, \lambda, x)|^2 \frac{dx}{w(x)}$$

$$= K(t, \lambda, \lambda)s(t, \lambda, \lambda) \int_{\mathbb{R}} s_t(x) \frac{dx}{w(x)}$$

$$= K(t, \lambda, \lambda)s(t, \lambda, \lambda) \left( \int_{\mathbb{R}} s_t(x) \frac{dx}{w(x)} - \frac{1}{w(s)} + \frac{1}{w(s)} \right)$$

$$\leq K(t, \lambda, \lambda)s(t, \lambda, \lambda)(\frac{1}{w(s)} + XM(w - 1)(s)).$$
3.2. The off-diagonal case. First, note
\[
\langle K(t, \lambda, \cdot) - \frac{1}{w(s)} \operatorname{sinc}(t, \lambda, \cdot), K(t, \lambda, \cdot) - \frac{1}{w(s)} \operatorname{sinc}(t, \lambda, \cdot) \rangle_\mu = \\
= K(t, \lambda, \lambda) + \frac{\|\operatorname{sinc}(t, \lambda, \cdot)\|_2^2}{w(s)^2} \int s_1 d\mu - \frac{2}{w(s)} \operatorname{sinc}(t, \lambda, \lambda)
\]
\[
= K(t, \lambda, \lambda) - \frac{1}{w(s)} \operatorname{sinc}(t, \lambda, \lambda) + \frac{\operatorname{sinc}(t, \lambda, \lambda)}{w(s)^2} \left( \int s_1 d\mu - w(s) \right)
\]
\[
\lesssim X(\lambda) \operatorname{sinc}(t, \lambda, \lambda) M(w - 1)(s),
\]
Hence, we conclude
\[
(3.1) \quad \|K(t, \lambda, \cdot) - \frac{1}{w(s)} \operatorname{sinc}(t, \lambda, \cdot)\|_\mu^2 \lesssim V_t(s)(\lambda) M(w - 1)(s).
\]
Next,
\[
K(t, \lambda, z) - \frac{1}{w(s)} \operatorname{sinc}(t, \lambda, z) = \langle K(t, \lambda, \cdot), K(t, z, \cdot) \rangle_\mu - \frac{1}{w(s)} \operatorname{sinc}(t, \lambda, z)
\]
\[
= \langle K(t, \lambda, \cdot) - \frac{1}{w(s)} \operatorname{sinc}(t, \lambda, \cdot), K(t, \cdot, \cdot) - \frac{1}{w(s)} \operatorname{sinc}(t, \cdot, \cdot) \rangle_\mu + \frac{1}{w(s)} \operatorname{sinc}(t, \lambda, z)
\]
\[
+ \frac{1}{w(s)} \operatorname{sinc}(t, z, \lambda) - \frac{1}{w(s)^2} \int \operatorname{sinc}(t, \lambda, x) \operatorname{sinc}(t, z, x) d\mu(x) - \frac{1}{w(s)} \operatorname{sinc}(t, \lambda, z)
\]
\[
= \langle K(t, \lambda, \cdot) - \frac{1}{w(s)} \operatorname{sinc}(t, \lambda, \cdot), K(t, \cdot, \cdot) - \frac{1}{w(s)} \operatorname{sinc}(t, \cdot, \cdot) \rangle_\mu
\]
\[
+ \frac{1}{w(s)} \operatorname{sinc}(t, \lambda, z) - \frac{1}{w(s)^2} \int \operatorname{sinc}(t, \lambda, x) \operatorname{sinc}(t, z, x) d\mu(x).
\]
For the first term above, we use Cauchy-Schwarz and (3.1).
\[
\left| \langle K(t, \lambda, \cdot) - \frac{1}{w(s)} \operatorname{sinc}(t, \lambda, \cdot), K(t, z, \cdot) - \frac{1}{w(s)} \operatorname{sinc}(t, z, \cdot) \rangle_\mu \right| \leq \\
\|K(t, \lambda, \cdot) - \frac{1}{w(s)} \operatorname{sinc}(t, \lambda, \cdot)\|_\mu \|K(t, z, \cdot) - \frac{1}{w(s)} \operatorname{sinc}(t, z, \cdot)\|_\mu
\]
\[
\lesssim \sqrt{V(\lambda)V(z)} M(w - 1)(s).
\]
For the second term, by the reproducing kernel property (2.16) for \(\operatorname{sinc}\) and by Cauchy-Schwarz we write
\[
\left| \frac{1}{w(s)} \operatorname{sinc}(t, \lambda, z) - \frac{1}{w(s)^2} \int \operatorname{sinc}(t, \lambda, x) \operatorname{sinc}(t, z, x) d\mu(x) \right| = \\
\left| \frac{1}{w(s)^2} \int \operatorname{sinc}(t, \lambda, x) \operatorname{sinc}(t, z, x) (w(x) - w(s)) dx \right| \leq t \sqrt{V(\lambda)V(z)} \epsilon.
4. Proof of Proposition 2

Let
\[ A(t, z) := \frac{E^\#(t, z) + E(t, z)}{2} \quad \text{and} \quad C(t, z) := \frac{E^\#(t, z) - E(t, z)}{2i}, \]
so that
\[ E(t, z) = A(t, z) - iC(t, z), \]
and \( A \) and \( C \) are entire functions that are real on \( \mathbb{R} \). Let us denote the numerator of \( K(t, \lambda, z) \) by
\[ D(t, \lambda, z) := E(t, z)E^\#(t, \bar{\lambda}) - E^\#(t, z)E(t, \bar{\lambda}) = 2i(A(t, z)C(t, \bar{\lambda}) - A(t, \bar{\lambda})C(t, z)). \]

For the proof of (2.28) we will need the following simple lemma that is [Pol21, Lemma 1].

**Lemma 5.** Let \( F \) be a meromorphic inner function and let \( 1 > \varepsilon > 0 \). Let \( x, y \in \mathbb{R} \) be such that
\[ \frac{|F'(x)|}{|F'(y)|} > 1 + \varepsilon. \]
Then, the disk \( B(x, \frac{|x-y|}{\varepsilon}) \) contains at least one zero of \( F \).

4.1. **Proof of (2.27).** We have \( E(t, z_0) = 0 \). Let
\[ \kappa := E^\#(z_0). \]
Then,
\[ D(t, z_0, z) = \kappa E(t, z). \]
Hence, by Proposition 1
\[ (4.1) \quad \left| \frac{\kappa E(t, z) + \frac{1}{w(s)} \sin[t(z - z_0)]}{2} \right| \lesssim |z - z_0| |\sqrt{V(z)}V(z_0)|. \]

Our goal is to get a good estimate of \( \kappa \). Let \( Y := t|z_0| \) and plug in \( z = z_0 \) above
\[ D(t, z_0, z_0) = \kappa E(t, z_0) = |E(t, z_0)|^2 \]
\[ = \frac{2}{w(s)} \sinh 2Y + \frac{Y}{t} V(z_0) \varepsilon(s) = \frac{2}{w(s)} \sinh 2Y + \psi \sinh(2Y)X(z_0) \varepsilon(s), \]
where recall that \( |\psi| \lesssim 1 \). Hence,
\[ |\kappa|^2 = \frac{2}{w(s)} \sinh 2Y + \psi \sinh(2Y)X(z_0) \varepsilon. \]

If \( X(z_0) \varepsilon \gtrsim 1 \) then there is nothing to prove, otherwise
\[ |\kappa|^2 \gtrsim \sinh 2Y, \]
and
\[ \left| \frac{1}{|\kappa|} \sqrt{\frac{\sinh 2Y}{w(s)}} \right| = \left| \frac{1}{\sqrt{\frac{\sinh 2Y + \psi \sinh(2Y)X(z_0) \varepsilon}{w(s)}}} \right| - \frac{1}{\sqrt{\frac{\sinh 2Y}{w(s)}}} \]
\[ = \left| \frac{\sqrt{\frac{2}{w(s)} \sinh 2Y + \psi \sinh(2Y)X(z_0) \varepsilon} - \sqrt{\frac{2}{w(s)} \sinh 2Y}}{\sqrt{\frac{2}{w(s)} \sinh 2Y} \sqrt{\frac{2}{w(s)} \sinh 2Y + \psi \sinh(2Y)X(z_0) \varepsilon}} \right| \lesssim \frac{X(z_0) \varepsilon}{\sqrt{\sinh 2Y}}. \]
Combining these two inequalities for \( \kappa \) and recalling that \( \alpha = -ie^{-i\arg \kappa} \), we continue from (4.1)

\[
\left| E(t, z) - \frac{\alpha \gamma(Y)}{\sqrt{w(s)}} \sin[t(z - z_0)] \right| \leq \left| E(t, z) + \frac{2i}{\kappa w(s)} \sin[t(z - z_0)] \right|
\]

\[
+ \left| \frac{2i}{\kappa w(s)} \sin[t(z - z_0)] + \frac{\alpha \gamma(Y)}{\sqrt{w(s)}} \sin[t(z - z_0)] \right|
\]

\[
\lesssim \frac{|z - z_0| \epsilon \sqrt{V(z) V(z_0)}}{\sqrt{\sinh 2Y}} + |\sin t(z - z_0)| \left| \frac{i}{\kappa} + \frac{\alpha \sqrt{w(s)}}{\sqrt{2 \sinh 2Y}} \right|
\]

\[
\lesssim \frac{|z - z_0| \epsilon \sqrt{V(z) V(z_0)}}{\sqrt{3|z_0|}} + |\sin t(z - z_0)| |X(z_0)| \epsilon(s) / \sqrt{|z_0|}
\]

4.2. Proof of (2.28). Let \( I = (s - \frac{10\pi}{11}, s + \frac{10\pi}{11}) \). For \( x, y \in I \), recall that \((A(t, x), C(t, x))\) and \((C(t, y), -A(t, y))\) are real vectors, so we can write by Proposition 1

\[
\frac{D(t, x, y)}{2i} = (A(t, x), C(t, x))^T \cdot (C(t, y), -A(t, y))
\]

(4.2)

\[
= \frac{1}{w(s)} \sin t(y - x) + \epsilon \psi_0(x, y, t),
\]

with \( |\psi_0(x, y, t)| \leq dt|x - y| \). If \( \epsilon \) is larger than some absolute constant then there is nothing to prove. Hence, we can assume \( \epsilon \leq \frac{1}{\pi} \), to be specified later, so that for \( x, y \in 2I \)

\[
\epsilon |\psi_0(x, y, t)| \leq \frac{1}{100}.
\]

By Grönwall’s inequality (2.4), we have for all \( x \)

\[
|A(t, x), C(t, x))| \leq 2.
\]

Thus, for any \( y \in I \), taking \( x = y + \frac{\pi}{11} \) in (4.2), we get

\[
2 |(A(t, y), C(t, y))| \geq |(|A(t, x), C(t, x))| \cdot |(A(t, y), C(t, y))| \geq \frac{9}{11},
\]

so we get for any \( x \in I \)

(4.3)

\[
\frac{9}{11} \leq |(A(t, x), C(t, x))| \leq 2.
\]

From (4.2) and the above inequality, in particular, follows that the vector \((A(t, x), C(t, x))\) makes a full rotation on \((s - \frac{2\pi}{11}, s + \frac{2\pi}{11})\). Hence, there exist points \( x_0 \) and \( x_1 \) on \((s - \frac{2\pi}{11}, s + \frac{2\pi}{11})\) such that \( E(t, x_0) \) is positive and \( E(t, x_1) \) is negative imaginary. We put \((A(t, x_0), C(t, x_0)) =: (c_1, 0)\) and \((A(t, x_1), C(t, x_1)) =: (0, c_2)\) with \( c_{1,2} \) positive. Furthermore, from (4.3) it follows that \( x_0 \) and \( x_1 \) can be chosen so that

\[
\pi - \frac{1}{100} > t|x_0 - x_1| > \frac{1}{100}.
\]

Then, again from (4.2) we see that

(4.4)

\[
c_1c_2 = \frac{1}{w(s)} \sin t(x_0 - x_1) + \epsilon \psi.
\]
Now let us plug in \( \lambda = x_0 \) and \( \lambda = x_1 \) in (4.2) to get
\[
C(t, z) = \frac{1}{c_1 w(s)} \sin t(z - x_0) + \frac{\epsilon}{c_1} \psi_0(t, z, x_0),
\]
and
\[
A(t, z) = \frac{1}{c_2 w(s)} \cos t(z - x_2) + \frac{\epsilon}{c_2} \psi_0(t, z, x_2),
\]
for \( z \in B(s, 4\pi/t) \), where \( x_2 := x_1 + \frac{\pi}{t} \). Let \( \psi_1(z) = \psi_0(t, z, x_0) \) and \( \psi_2(z) = \psi_0(t, z, x_2) \). Then, \( \psi_1(x_0) = 0, \psi_2(x_2 - \frac{\pi}{t}) = 0 \) and \( \psi_1, \psi_2 \) are holomorphic. We write
\[
E(t, z) = \frac{1}{c_2 w(s)} \cos t(z - x_2) - i \frac{1}{c_1 w(s)} \sin t(z - x_0) + \left( \frac{\psi_2(x)}{c_2} - \frac{i \psi_1(x)}{c_1} \right) \epsilon.
\]

We want to prove that \( c_1 \) is equal to \( c_2 \) and \( x_0 \) is equal to \( x_2 \) up to \( \epsilon \) and \( \frac{1}{t} \) terms. We want to compute \( \frac{\theta'(x)}{\theta'(y)} \) to apply Lemma 5. Let us start with
\[
E'_x(t, x) = -\frac{t \sin t(x - x_2)}{c_2 w(s)} - \frac{it \cos t(x - x_0)}{c_1 w(s)} + \left( \frac{\psi_2(x)}{c_2} - \frac{i \psi_1(x)}{c_1} \right) \epsilon,
\]
and \( |\psi'_j(x)| \lesssim t \) by holomorphicity of \( \psi_j \)’s.

\[
|E(t, x)|^2 = \frac{\cos^2 t(x - x_2)}{c_2^2 w(s)^2} + \frac{\sin^2 t(x - x_0)}{c_1^2 w(s)^2} + 2\epsilon(s) \frac{\psi_2(x) \cos t(x - x_2)}{c_2^2 w(s)}
\]
\[
+ \epsilon(s)^2 \frac{\psi_2(x)^2}{c_2^2} + \epsilon(s)^2 \frac{\psi_1(x)^2}{c_1^2} + 2\epsilon(s) \psi_1(x) \sin t(x - x_0).
\]

Then, we calculate
\[
|E|^2 \frac{\theta'(x)}{\theta'(y)} = E'E - E'E
\]
\[
= \frac{2it}{c_1 c_2 w(s)^2} \left[ \sin t(x - x_2) \sin t(x - x_0) + \cos t(x - x_0) \cos t(x - x_2) + \psi(x) \epsilon \right]
\]
\[
= \frac{2it}{c_1 c_2} \left[ \cos t(x_0 - x_2) + \psi(x) \epsilon \right].
\]

As \( t|x_0 - x_2| < \pi/2 - \frac{1}{10t} \), \( \cos^2 t(x - x_0) \) dominates \( \psi(x) \epsilon \) if \( d_1 \) is small enough. We consider
\[
\frac{|\theta'(x_0)|}{|\theta'(x_2 - \frac{\pi}{2t})|} = \frac{|E(t, x_2 - \frac{\pi}{2t})|^2 |\cos t(x_0 - x_2) + \psi(x)|}{|E(t, x_0)|^2 |\cos t(x_0 - x_2) + \psi(x)|}
\]
\[
= \frac{c_2^2}{c_1^2} \frac{\cos^2 t(x_2 - x_0) + \psi(x)}{\cos^2 t(x_2 - x_0) + \psi(x)} \frac{|\psi(t_0 - x_2)| + \psi(x)}{|\cos t(x_0 - x_2) + \psi(x)|}.
\]

So we have
\[
\left| \frac{|\theta'(x_0)|}{|\theta'(x_2 - \frac{\pi}{2t})|} - \frac{c_2^2}{c_1^2} \right| \lesssim \frac{c_2^2}{c_1^2}.
\]

On the other hand, applying Lemma 5 with the assumption that there are no zeros of \( E \) at distance \( C/t \) from \( s \) we get
\[
\left| \frac{|\theta'(x_0)|}{|\theta'(x_2 - \frac{\pi}{2t})|} - 1 \right| \leq \frac{1}{C}.
\]

Thus, we deduce
\[
\frac{c_2^2}{c_1^2} - 1 \lesssim \frac{1}{C} + \epsilon.
\]
Thus, again by Rouché’s theorem we conclude (we again have

\[ \frac{|\theta'(x_0 + \frac{z}{Y})|}{|\theta'(x_0)|} = \frac{|E(x_0)|}{|E(x_0 + \frac{z}{Y})|} \leq \left| \frac{\cos t(x_0 - x_2) + \psi \epsilon}{\cos t(x_0 - x_2) + \psi \epsilon} \right| \]

\[ = \frac{\cos^2 t(x_2 - x_0) + \frac{d \epsilon}{e^2} \frac{c_2}{c_2}}{\frac{c_2}{c_2} w(s)^2 + \frac{1}{c_2} w(s)^2 + \frac{c_2}{c_2} + \frac{c_2}{c_2}} \frac{\cos t(x_0 - x_2) + \psi \epsilon}{\cos t(x_0 - x_2) + \psi \epsilon}. \]

Then, we can compute from

\[ \frac{1}{C} \geq \left| \frac{\theta'(x_0 + \frac{z}{Y})}{|\theta'(x_0)|} - 1 \right| \gtrsim (t|x_2 - x_0|)^2 - \frac{d}{C} - d \epsilon. \]

So for small enough \( \delta_5 \) and large enough \( C \), we conclude

\[ (t|x_2 - x_0|)^2 \lesssim \frac{1}{C} + \epsilon. \]

Combining the two estimates (4.9) and (4.8) with (4.4), we get

\[ \left| \frac{1}{c_j} - \sqrt{w(s)} \right| \leq \frac{d_1}{C} + d_2 \epsilon, \]

for \( j = 1, 2 \). And from the initial approximation formula (4.7), we conclude the proof.

5. PROOF OF PROPOSITION 3

Let us put \( z_0 := z(t, s) =: s + \frac{\bar{X}}{Y} - i \frac{\bar{Y}}{Y} \) and \( \tilde{z}_0 := \bar{z}(t, s) =: s + \frac{\bar{X}}{Y} + i \frac{\bar{Y}}{Y} \).

5.1. Proof of A. Let us apply Proposition 2 for \( z \) such that \( |z - (z_0 + \frac{\bar{X}}{Y})| \leq \frac{1}{200} \).

Recall by Lemma 2, \( t|\Im z_0| = Y \geq 1 \).

\[ |E(t, z) - \frac{\alpha \gamma(Y)}{\sqrt{w(s)}} \sin[t(z - z_0)]| \leq D_5 \frac{e^Y}{Y} \frac{\chi(z_0) \epsilon}{Y} \leq D_5 \frac{e^Y}{D_6 \epsilon} \]

\[ \leq \frac{1}{300} \frac{\alpha \gamma(Y)}{\sqrt{w(s)}} \sin[t(z - z_0)]|, \]

Here the last inequality holds as \( \frac{1}{4} \leq w \leq \frac{7}{4} \) and if \( D_6 \) is large enough. In the penultimate inequality we used \( (t, s) \in \Omega^{D_6} \). We apply Rouché’s theorem to conclude that \( E \) has a zero inside the circle \( \{ z : |z - (z_0 + \frac{\bar{X}}{Y})| \leq \frac{1}{200} \} \). So if \( z_0 \) was the closest zero to \( s \) then this circle should be farther from \( s \) than \( z_0 \). Similarly, the circle from the left of \( z_0 \), i.e. \( \{ z : |z - (z_0 - \frac{\bar{X}}{Y})| \leq \frac{1}{200} \} \), should be farther than \( z_0 \) and (2.29) follows.

We already have \( \chi(z_0) = 1 \). Considering the circle \( \{ z : |z - z_0 - j\pi| = D_6 \frac{e^{2Y}}{Y} \epsilon(s) \} \), we again have

\[ |E(t, z) - \frac{\alpha \gamma(Y)}{\sqrt{w(s)}} \sin[t(z - z_0)]| \leq D_5 \frac{e^Y}{Y} \frac{\epsilon \epsilon}{Y} \leq \frac{1}{300} \frac{\alpha \gamma(Y)}{\sqrt{w(s)}} \sin[t(z - z_0)]|. \]

Thus, again by Rouché’s theorem we conclude (2.30).

Now plug into (2.27), \( z = s \) with zeros \( z_0 \) and \( z_j \).

\[ |E(t, s) - \frac{\alpha \gamma(Y)}{\sqrt{w(s)}} \sin[t(s - z_0)]| \leq D_5 \epsilon \sqrt{Y}, \]
and
\[ |E(t, s) - \frac{\alpha_j \gamma(Y)}{\sqrt{w(s)}} \sin t(s - z_j)| \leq D_5 \epsilon \sqrt{Y}, \]
where \( \alpha_j = \alpha(s, t, z_j) \) and \( Y_j = -t \Im z_j \). By (2.30),
\[ |\gamma(Y) - \gamma(Y_j)| \lesssim e^{-2Y} |Y - Y_j| \lesssim e^{-2Y} \frac{e^{2Y}}{Y} \epsilon = \epsilon. \]
Thus, by two triangle inequalities
\[ |\alpha_0 - \alpha_j| \lesssim \epsilon \sqrt{Y} + \frac{\epsilon}{Y} \leq D_6 \frac{\epsilon}{Y}. \]

5.2. Proof of B. If \( e^{-2Y} \geq D_6(t|z_0 - s|)^{4/5} \epsilon \), then there is nothing to prove. So we assume to contrary. There are two cases. First, assume \( D_6(t|z_0 - s|)^{4/5} \epsilon > \epsilon^{1/2} + (t|z_0 - s|)^{-1/2} \) and \( e^{-2Y} \geq \epsilon^{1/2} + (t|z_0 - s|)^{-1/2} \). We want to arrive at a contradiction.

Let us use (2.27) for \( E(t, z) \) with the zero \( z_0 \) and (2.28) for \( \tilde{E} \). We have for \( z \in Q(s, 1/t) \)
\[ E(t, z) = \frac{\alpha \sqrt{2}}{\sqrt{w(s) \sinh(2Y)}} \sin t(z - z_0) + \psi \sqrt{Y}, \]
and
\[ \tilde{E}(t, z) = \frac{\beta}{\sqrt{w(s)}} e^{-itz} + \psi (\epsilon^{1/2} + (t|z_0 - s|)^{-1/2}), \]
for \( z \in Q(s, 1/t) \). Applying the determinant identity (2.10) for \( x \in (s - 1/t, s + 1/t) \) we write
\[ 2i = \frac{\sqrt{2}}{\sqrt{w(s) \sinh(2Y)}} \left( \alpha \beta \sin t(x - z_0) e^{itx} - \overline{\alpha \beta} \sin t(x - z_0) e^{-itx} \right) \]
\[ + \psi (\epsilon \sqrt{Y} + \epsilon^{1/2} + (t|z_0 - s|)^{-1/2}) \]
\[ = \frac{2i \sqrt{2}}{\sqrt{w(s) \sinh(2Y)}} \Im \left( \alpha \beta \sin t(x - z_0) e^{itx} \right) \]
\[ + \psi (\epsilon \sqrt{Y} + \epsilon^{1/2} + (t|z_0 - s|)^{-1/2}) \]
\[ = \frac{i \sqrt{2}}{\sqrt{w(s) \sinh(2Y)}} \Im \left( \alpha \beta (e^{2itx - itz_0} - e^{itz_0}) \right) \]
\[ + \psi (\epsilon \sqrt{Y} + \epsilon^{1/2} + (t|z_0 - s|)^{-1/2}). \]
Let us plug in \( x = x_1, x_2 \) to \( (s - 1/t, s + 1/t) \) and subtract the resulting two equations.
\[ \psi (\epsilon \sqrt{Y} + \epsilon^{1/2} + (t|z_0 - s|)^{-1/2}) = \frac{i \sqrt{2}}{\sqrt{w(s) \sinh(2Y)}} \Im \left( e^{-itz_0} \alpha \beta (e^{2itx_1} - e^{2itx_2}) \right). \]
By choosing appropriate \( x_1 \) and \( x_2 \), we can force
\[ |e^{2itx_1} - e^{2itx_2}| \geq \frac{1}{2} \]
and
\[ \left| \arg(e^{2itx_1} - e^{2itx_2}) + \arg(\alpha \beta e^{-itz_0}) - \frac{\pi}{2} \right| \leq \frac{\pi}{4}. \]
Then, we estimate
\[ d_0 \epsilon \sqrt{Y} + \epsilon^{1/2} + (t|z_0 - s|)^{-1/2} \geq \frac{1}{10} e^{-2Y}, \]
which is a contradiction to the assumption of the lemma if \( D_6 \) is large enough. 
Next, assume \( e^{-2Y} \geq \max(e^{-2Y}, D_6(t|z_0 - s|)\tilde{\xi}) \) and \( e^{-2Y} \geq e^{-2Y} \). In this case we apply (2.27) for \( \tilde{E} \) and the zero \( \tilde{z}_0 \). At \( z = \tilde{z}_0 \) and \( z = z_0 \), we write
\[
E(t, \tilde{z}_0) = -\frac{\alpha \sqrt{2}}{\sqrt{w(s) \sinh(2Y)}} \sin(2Y) + \psi e \sqrt{\sinh(2Y)},
\]
and
\[
\tilde{E}(t, z_0) = \frac{\tilde{\alpha} \sqrt{2}}{\sqrt{\tilde{w}(s) \sinh(2Y)}} \sin(tz_0 - \tilde{z}_0) + \psi e^Y \tilde{\xi} \sqrt{Y}.
\]
As \( 3z_0 \leq 3\tilde{z}_0 \), taking into account (2.29) of Proposition 3 we have \( |z_0 - \tilde{z}_0| \leq 2|\tilde{z}_0 - s| \). Hence,
\[
\tilde{E}(t, z_0) = \frac{\tilde{\alpha} \sqrt{2}}{\sqrt{\tilde{w}(s) \sinh(2Y)}} \sin(tz_0 - \tilde{z}_0) + \psi e^Y (t|z_0 - s|)\tilde{\xi} \tilde{\xi}.
\]
The determinant identity (2.10) at \( z = z_0 \) reads
\[
2i = \det \begin{pmatrix} E(t, z_0) & \tilde{E}(t, z_0) \\ E^\#(t, z_0) & \tilde{E}^\#(t, z_0) \end{pmatrix} = -\tilde{E}(t, z_0)E^\#(t, z_0).
\]
Plugging in the above estimates we get
\[
2 \equiv 2\tilde{\alpha} \tilde{\alpha} \sin(tz_0 - \tilde{z}_0) \sqrt{\sinh(2Y)} \sqrt{\sinh(2Y)} + \psi (\epsilon + |s - \tilde{z}_0|2\tilde{\xi}) e^{2Y}.
\]
As the second term can be smaller than \( \frac{1}{1000} \) if \( D_6 \) is large enough, we deduce
\[
\frac{|\sin(tz_0 - \tilde{z}_0)|\sqrt{\sinh(2Y)}}{\sqrt{\sinh(2Y)}} \leq 2.1,
\]
and
\[
0 \leq \tilde{Y} - Y \leq 1.
\]
So \( e^{-2Y} \geq e^2 e^{-2Y} \geq D_6(t|z_0 - s|)^2 \) and \( (t, s) \in \Omega_{D/e^2} \).

5.3. Proof of C. We know by (2.29) that \( X_{s, t}(z_0) = X_{s, t}(\tilde{z}_0) = 1 \).

First, we want to prove
(5.1) \( |Y - \tilde{Y}| \leq de^{2Y} \mu(s) \).

Let us assume w.l.o.g. \( Y \geq \tilde{Y} \) so by the proof of part B, \( |Y - \tilde{Y}| \leq 1 \). Apply Proposition 2 with \( z = z_0, \tilde{z}_0 \), we write
\[
E(t, \tilde{z}_0) = -i \frac{\alpha \sqrt{2}}{\sqrt{w(s) \sinh(2Y)}} \sinh(2Y) + \psi e \sqrt{\sinh(2Y)},
\]
and
\[
\tilde{E}(t, z_0) = \frac{\tilde{\alpha} \sqrt{2}}{\sqrt{\tilde{w}(s) \sinh(2Y)}} \sin(tz_0 - \tilde{z}_0) + \psi e^Y \tilde{\xi}.
\]
We have analogous estimates for $\tilde{z}_0$. Again we apply the determinant identity (2.10) at $z = z_0$ and $z = \tilde{z}_0$.

\[ \frac{\alpha \tilde{\alpha} \sin(t(\tilde{z}_0 - z_0)) \sqrt{\sinh 2Y}}{\sqrt{w(s)\bar{w}(s)}} \frac{\sqrt{\sinh 2Y}}{\sqrt{\sinh 2Y}} = 1 - \psi_1 \mu e^{2Y}, \]

\[ \frac{\alpha \tilde{\alpha} \sin(t(\tilde{z}_0 - z_0)) \sqrt{\sinh 2Y}}{\sqrt{w(s)\bar{w}(s)}} \frac{\sqrt{\sinh 2Y}}{\sqrt{\sinh 2Y}} = 1 - \psi_2 \mu e^{2Y}. \]

Divide the two equations and take absolute values.

\[ e^{Y - \tilde{Y}} \leq \frac{\sinh 2Y}{\sinh 2Y} \leq 1 + \psi \mu e^{2Y}. \]

This concludes the proof of (5.1). Plugging it back into the above two equations we get

\[ \frac{\alpha \tilde{\alpha} \sin(\tilde{X} - X)}{\sqrt{w(s)\bar{w}(s)}} = 1 + \psi_3 \mu e^{2Y}, \]

\[ \frac{\alpha \tilde{\alpha} \sin(\tilde{X} - X)}{\sqrt{w(s)\bar{w}(s)}} = 1 + \psi_4 \mu e^{2Y}. \]

Multiplying the two equations, we have

\[ \frac{\sin^2(\tilde{X} - X)}{w(s)\bar{w}(s)} = 1 + \psi e^{2Y} \mu, \]

so

\[ |\sin^2(\tilde{X} - X) - w(s)\bar{w}(s)| \leq e^{2Y} \mu, \]

and

\[ |\alpha + \text{sign}(\tilde{X} - X))\tilde{\alpha}| \leq e^{2Y} \mu. \]

6. Proof of Proposition 4

Proof of (2.40). Let $\xi_0 = s + \frac{\lambda_0}{t_0} - \frac{1}{t_0}$. By part A of Proposition 3, $X(\xi_0) = 1$. Let $z \in R(s, 3/t_0, (Y_0 + 4)/t)$, then we can estimate the error coming from (2.27) of Proposition 2

\[ |z - \xi_0| \frac{\sqrt{\sinh(2t|\xi_0|)}}{\sqrt{|\xi_0|}} \frac{\sqrt{\chi_{s,t}(z)\chi_{s,t}(\xi_0)}}{\sqrt{|\xi_0|}} \leq \sqrt{Y_0} e^{t|\xi_0|}. \]

We apply Proposition 2 to $E$. We write for $z$ as above and $\Im z \geq 0$.

\[ |\theta(t_0, z) - \tilde{\alpha}^2 \sin t_0(z - \xi_0)| = \left| \frac{E^\#(t_0, z)}{\sin t_0(z - \xi_0)} - E(t_0, z) \right| \]

\[ = \left| \frac{E^\#(t_0, z)\alpha \sin t_0(z - \xi_0) - \tilde{\alpha} \alpha E(t_0, z) \sin t_0(z - \xi_0)}{\alpha E(t_0, z) \sin t_0(z - \xi_0)} \right| \]

\[ \leq D_5 \sqrt{Y_0} e^{t|\xi_0|} \leq \frac{e^{t|\xi_0|} - Y_0 \sqrt{Y_0}}{e^{t|\xi_0|}}. \]

To see that this is exactly (2.40) one again needs to recall that part A of Proposition 3 aligns the heights of the first 10 zeros of $E$ closest to $s$. \qed
Proof of (2.41). Let $\xi_t := s + \frac{X_t}{t} - \frac{Y_t}{t}$, and $Y_{t_0} = Y_0$. For a small enough $\delta$, $\xi_t$ is still at most 11th closest zero of $E(t, \cdot)$ and the previous estimate applies on $\theta$. For $z \in R_+ (s, 1/t, (Y_t + 2)/t)$

$$|\theta(t, z) - \alpha^2 \frac{\sin t(z - \xi_t)}{\sin t(z - \xi_t)}| \leq D_8 e^{t|z - Y_t|} \sqrt{Y_t} \epsilon.$$  

By holomorphicity of the function on the left-hand side, we can differentiate at $z = \xi_t$ and obtain

$$\left|\frac{d}{dt} \theta(t, \xi_t) - \frac{t\alpha^2}{\sin t(\xi_t - \xi_t)}\right| \leq 2D_8 t \sqrt{Y_t} \epsilon. \tag{6.2}$$

Similarly, for the second derivative we compute

$$\left|\frac{d^2}{dt^2} \theta(t, \xi_t) + \frac{2t^2 \alpha^2 \cos(2Y_t)}{\sin^2(2Y_t)}\right| \leq 3D_8 t^2 \sqrt{Y_t} \epsilon. \tag{6.3}$$

For a function $g : (t_0 - \delta, t_0 + \delta) \to \mathbb{C}$ it is easy to compute

$$[\arg g(t)]' = \frac{\text{Proj}_{g(t)} g'(t)}{ig},$$

where $\text{Proj}_w$, for two complex numbers $z = x + iy$ and $w = u + iv$, stands for the complex number $p + iq$ such that the vector $(p, q)$ is the orthogonal projection of $(u, v)$ onto the direction of $(x, y)$. So

$$\frac{\partial}{\partial t} \arg \theta(t, \xi_t) = \frac{\text{Proj}_{i\theta(t, \xi_t), \xi_t} \frac{\partial}{\partial \xi} \theta(t, \xi_t)}{i\theta(t, \xi_t)}.$$  

Then, we apply the equation (2.39).

$$\frac{\text{Proj}_{i\theta(t, \xi_t)} \frac{\partial}{\partial \xi} \theta(t, \xi_t)}{i\theta(t, \xi_t)} = \Re \xi_t - \frac{\text{Proj}_{i\theta(t, \xi_t), \xi_t} f(t) \theta(t, \xi_t)}{i\theta(t, \xi_t)} = s + (\Re \xi_t - s) - \kappa(t)$$

for some real function $\kappa(t)$. From equations (6.2) and (6.3), we have

$$|\kappa(t)| \leq |f(t)| \left|\frac{\theta_z(t, \xi_t)}{\theta_z(t, \xi_t)}\right|^2 \leq 2|f(t)| \cosh(2Y_t)(1 + d e^{-2Y_t} \sqrt{Y_t} \epsilon) \leq 3|f(t)| \cosh(2Y_0)).$$

The statement follows since by part A of Proposition 3

$$|\Re \xi_t - s| \leq \frac{20\pi}{t}$$

7. Miscellaneous Lemmas

Let

$$\Phi_{s, t, z_0}(z) := \Phi_{z_0}(z) := \Phi(z) := \frac{|z - z_0| \sqrt{\sinh(2t |z|)}}{\sqrt{|z_0| \cdot |z|}} \sqrt{X_{s, t}(z)},$$

$$\Phi(z) := \frac{|z - z_0| \sqrt{\sinh(2t |z|)}}{\sqrt{|z_0| \cdot |z|}} \sqrt{X_{s, t}(z)},$$

$\square$
7.1. **Aligning the errors.** The following lemma makes a joint approximation for $E$ and $\tilde{E}$.

**Lemma 6.** There exists a $D_{12} > 0$ such that the following holds.

Let $(t, s) \in \Omega^{D_{12}} \cap \tilde{\Omega}^{D_{12}}$. Let $\xi_0$ be at most 10th closest zero of $E(t, \cdot)$ to $s$ and let $\tilde{\xi}_0 := \xi_0 \mp \arcsin \sqrt{\frac{w(s)\tilde{w}(s)}{t}}$, where the sign is chosen so that there is a zero of $\tilde{E}(t, \cdot)$ in the disk $\{ z : |z - \tilde{\xi}_0| \leq D_6 e^{2t|\Im\xi_0|} \sqrt{t|\Im\xi_0|} \mu(s) \}$.

Then, for each $u \in (s - \frac{\pi}{2}, s + \frac{\pi}{2})$ there exist a pair of complex numbers $z_0, \tilde{z}_0 \in \mathbb{C}^-$ and a unimodular number $\alpha_0$ such that

$$\begin{equation}
(7.1) \quad t|z_0 - \xi_0| + t|\tilde{z}_0 - \tilde{\xi}_0| + |\alpha_0 - \alpha(s, t, \xi_0)| < D_{12} e^{2t|\Im\xi_0|} \sqrt{t|\Im\xi_0|} \mu(s),
\end{equation}$$

and

$$\begin{equation}
(7.2) \quad E(t, u) = \frac{\alpha_0 \gamma(t \Im \xi_0)}{\sqrt{w(s)}} \sin t(u - z_0), \quad \tilde{E}(t, u) = \frac{\alpha_0 \gamma(t \Im \xi_0)}{\sqrt{w(s)}} \sin t(u - \tilde{z}_0).
\end{equation}$$

Furthermore, the approximation formulas continue to hold, i.e.

$$\begin{align}
|E(t, z) - \frac{\alpha_0 \gamma(t \Im \xi_0)}{\sqrt{w(s)}} \sin t(z - z_0)| &\leq D_{12} \Phi_{z_0}(z) e^{2t|\Im\xi_0|} \mu(s), \\
|\tilde{E}(t, z) - \frac{\alpha_0 \gamma(t \Im \xi_0)}{\sqrt{w(s)}} \sin t(z - \tilde{z}_0)| &\leq D_{12} \Phi_{z_0}(z) e^{2t|\Im\xi_0|} \mu(s).
\end{align}$$

**Proof.** We will assume that $\tilde{\xi}_0 = \xi_0 + \frac{\arcsin \sqrt{w(s)\tilde{w}(s)}}{t}$. The existence of the zero of $\tilde{E}$ in the appropriate disk is ensured by part C of Proposition 3. The other case is identical.

If (7.1) is true, then the last two inequalities follow from Propositions 3 and 2. Indeed, $\Phi_{z_0}(z) \sim \Phi_{\tilde{z}_0}(z)$ and we can write

$$|E(t, z) - \frac{\alpha_0 \gamma(t \Im \xi_0)}{\sqrt{w(s)}} \sin t(z - z_0)| \leq |E(t, z) - \frac{\alpha_0 \gamma(t \Im \xi_0)}{\sqrt{w(s)}} \sin t(z - \xi_0)|$$

$$+ \left| \frac{\alpha_0 \gamma(t \Im \xi_0)}{\sqrt{w(s)}} \sin t(z - \xi_0) - \frac{\alpha_0 \gamma(t \Im \xi_0)}{\sqrt{w(s)}} \sin t(z - z_0) \right|$$

$$\leq 2D_7 \Phi(\xi_0)(z) \epsilon + 100e^{t|\Im\xi_0|} e^{2t|\Im\xi_0|} \sqrt{t|\Im\xi_0|} \mu$$

$$\lesssim D_{12} \Phi(z) e^{2t|\Im\xi_0|} \sqrt{t|\Im\xi_0|} \mu(s).$$

The same calculation works for $\tilde{E}$. Put

$$g(z) = \frac{\sqrt{w(s)} \sin t(z - \xi_0)}{\sqrt{w(s)}} \sin t(z - \tilde{\xi}_0).$$

We know that $\sqrt{w(s)\tilde{w}(s)} \geq \frac{1}{4}$ so $\frac{1}{4} \leq t(\tilde{\xi}_0 - \xi_0) \leq \pi/2$ and $g$ is not constant. We have

$$\left| \frac{E(t, u)}{E(t, u)} - f(u) \right| \leq d_0 \sqrt{t|\Im\xi_0|} e^{2t|\Im\xi_0|} \mu.$$
holds for some \( w \in \{ |z - s| = \frac{d_1 \sqrt{t|3\xi_0|} e^{2it|3\xi_0|} \mu}{t} \} \). From approximations of \( E \) and \( \tilde{E} \),

\[
|g'(w)| \geq \frac{t}{d_2}.
\]

Then, we can estimate

\[
|g(z) - g(u)| \geq d_2^{-1} |z - u| \geq d_2^{-1} d_1 \sqrt{t|3\xi_0|} e^{2it|3\xi_0|} \mu > d_0 \frac{E(t, u)}{E(t, u)} - g(u)|,
\]

where the last inequality holds if \( d_1 \) is large enough. We can apply Rouche’s theorem on \( \{ |z - u| = \frac{d_1 \sqrt{t|3\xi_0|} e^{2it|3\xi_0|} \mu}{t} \} \) to functions \( g(z) - g(u) \) and \( g(u) - \frac{E(t, u)}{E(t, u)} \) to conclude that there is \( a \in \{ z : |z - u| \leq \frac{d_1 \sqrt{t|3\xi_0|} e^{2it|3\xi_0|} \mu}{t} \} \) such that

\[
g(a) = \frac{E(t, u)}{E(t, u)}.
\]

Putting \( z_0 := \xi_0 + u - a, \quad \tilde{z}_0 := \xi_0 + u - a \), (7.1) is immediately satisfied. Moreover, we have

\[
(7.4) \quad \frac{E(t, u)}{E(t, u)} = \frac{\sqrt{w(s)} \sin t(u - z_0)}{\sqrt{w(s)} \sin t(u - \tilde{z}_0)}.
\]

Plugging this into the determinant identity (2.10) and performing simple trigonometric manipulations, we see

\[
|E(t, u)| = \left| \frac{\gamma(t\xi_0)}{\sqrt{w(s)} \sin t(u - z_0)} \right|, \quad |\tilde{E}(t, u)| = \left| \frac{\gamma(t\xi_0)}{\sqrt{w(s)} \cos t(u - \tilde{z}_0)} \right|.
\]

The two identities with (7.4) imply

\[
E(t, u) = \frac{\alpha_0 \gamma(t\xi_0)}{\sqrt{w(s)} \sin t(u - z_0)}, \quad \tilde{E}(t, u) = \frac{\alpha_0 \gamma(t\xi_0)}{\sqrt{w(s)} \cos t(u - \tilde{z}_0)},
\]

for some unimodular \( \alpha_0 \). Comparing the above formulas with the approximation (2.27) of Proposition 2 for the zero \( \xi_0 \) and the first two estimates of (7.1) we conclude

\[
|\alpha(s, t, \xi_0) - \alpha_0| \leq D_{10} \sqrt{t|3\xi_0|} e^{2it|3\xi_0|} \mu(s).
\]

\[ \square \]

### 7.2. Short Movement of Zeros. Let \( 0 < \sigma < 1 \). We will say that an interval \( I \subset \mathbb{R}_+ \) is a \( \sigma \)-interval for \( f \) if

\[
\left| \int_I f \right| \geq (1 - \sigma) \int_I |f|.
\]

As \( f \in L^1 \), for almost every point \( s \in \mathbb{R} \) with \( f(s) \neq 0 \), there is a small neighborhood of \( s \) that is a \( \sigma \)-interval. If \( f \) is real valued, then \( f \) almost maintains the sign on a \( \sigma \)-interval. For complex valued \( f \) we should talk about the argument rather than the sign. The following simple geometric lemma gives the necessary control.

**Lemma 7.** Let \( I \) be a \( \sigma \)-interval for \( f \), then there exists \( \varphi \in [0, 2\pi] \) such that

\[
|\int_I f 1_{\varphi \leq \arg f \leq \varphi + \frac{\pi}{4}}| \geq (1 - 3 \cdot 10^4 \sigma) \int_I |f|.
\]
Proof. Let \( v_j := \int_I f^1_{j \leq \arg f \leq (j+1)} \) for \( 0 \leq j \leq 15 \). Then,

\[
|\sum_j v_j| > (1 - \sigma) \sum_j |v_j|.
\]

(7.5)

Without loss of generality let \( |v_0| \) be the largest. If for all \( 2 \leq k \leq 15 \),

\[
|v_k| \leq 2 \cdot 10^3 \sigma \sum_j |v_j|,
\]

then we are done as

\[
|\int_I f^1_{0 \leq \arg f \leq \frac{\pi}{3}}| > |\int_I f| - \sum_{2 \leq j \leq 15} |v_j|,
\]

\[
\geq (1 - \sigma) \int_I |f| - 28 \cdot 10^3 \sigma \int_I |f| = (1 - 29 \cdot 10^3 \sigma) \int_I |f|.
\]

Similarly, as we are on the circle, we are done if for all \( 1 \leq k \leq 14 \), \( |v_k| \leq 2 \cdot 10^4 \sigma \sum_j |v_j| \).

On the other hand, if for some \( k \),

\[
|v_k| > 2 \cdot 10^3 \sigma \sum_j |v_j| \text{ and } \arg(v_k \tilde{e}_1) > \frac{\pi}{16},
\]

then this contradicts to (7.5). Indeed, (7.5) is equivalent to

\[
(2\sigma - \sigma^2) \sum_j |v_j|^2 > \sum_{i \neq j} (2 - 2\sigma - 2 \cos(\arg v_i \tilde{e}_1)) |v_i||v_j|.
\]

\[
\geq \frac{1}{40} |v_1||v_k| - 2\sigma \sum_j |v_j|^2 > \frac{2 \cdot 10^3 \sigma}{320} (\sum_j |v_j|)^2 - 2\sigma \sum_j |v_j|^2.
\]

\[
\geq 2\sigma \sum_j |v_j|^2.
\]

So the only possible value of \( k \) for which in (7.6) does not hold is either 1 or 15 but not both, so we are done. \( \square \)

Henceforth we put \( \sigma = \frac{1}{10^7} \) to be an absolute constant. We will say that the collection of a point \( s \) a time interval \([t_1, t_2]\), continuous paths \( \xi, \tilde{\xi} : [t_1, t_2] \to \mathbb{C} \) are \((\delta, A)\)-admissible, if

(i) \( t_2 - t_1 \leq \min \left( \frac{1}{\|\tilde{e}_1\| + \delta}, e^{-3A}\delta t_1 \right) \),

(ii) \( [t_1, t_2] \) is a \( \sigma \)-interval,

(iii) \( \int_{t_1}^{t_2} |f| < \delta e^{-6A} \),

(iv) \( (t, s) \in \Xi^{1/\delta} \),

(v) \( \xi_t \) is a zero of \( E(t, \cdot) \) and \( \tilde{\xi}_t \) is a zero of \( \tilde{E}(t, \cdot) \) that are at most 5th closest to \( s \) and

\[
A - 2 \leq |t| |\Re \xi_t| \leq A + 2.
\]

In this case we denote \( \xi_1 := \xi_{t_1} \) and \( \xi_2 := \xi_{t_2} \). We always assume \( A \geq 1 \). If a collection is \((\delta, A)\)-admissible, then it is also \((\delta', A)\)-admissible for any \( \delta' > \delta \).

**Lemma 8.** There is a small \( \delta_{12} > 0 \) such that if we have a \((\delta_{12}, A)\)-admissible collection, then

\[
t_1|\xi_2 - \xi_1| \sim e^{2A} \int_{t_1}^{t_2} |f|.
\]

(7.8)
Proof. By the Riccati equation (2.38) and the estimate (2.42),

$$|\xi_2 - \xi_1| \leq \int_{t_1}^{t_2} \frac{|f|}{|\theta_z(t, \xi_1)|} \leq e^{2A} \int_{t_1}^{t_2} |f|/|t_1|.$$  

By Proposition 4,

$$(7.10) \quad |\arg \theta_z(t, \xi_2) - \arg \theta_z(t_1, \xi_1)| \leq (t_2 - t_1)s + \cosh 2A \int_{t_1}^{t_2} |f| + 10 \frac{t_2 - t_1}{t_1} \leq 0.1,$$

if $\delta_{12}$ is small enough. On the other hand, by Lemma 7 there is a $\varphi$ such that

$$|\int_{t_1}^{t_2} f \mathbf{1}_{\varphi \leq \arg f \leq \varphi + \frac{\pi}{4}}| > (1 - 3 \cdot 10^4 \sigma) \int_{t_1}^{t_2} |f|.$$

Let $f_1 := f \mathbf{1}_{\varphi \leq \arg f \leq \varphi + \frac{\pi}{4}}$ and $f_2 = f - f_1$. Then,

$$|\xi_2 - \xi_1| = \left| \int_{t_1}^{t_2} \frac{f}{\theta_z(t, \xi_1)} \right| \geq \left| \int_{t_1}^{t_2} \frac{f_1}{\theta_z(t, \xi_1)} \right| - \left| \int_{t_1}^{t_2} \frac{f_2}{\theta_z(t, \xi_1)} \right|$$

$$\geq \left| \int_{t_1}^{t_2} e^{i(\varphi + \arg \theta_z(t, \xi_1))} \frac{|f_1(t)|}{\theta_z(t, \xi_1)} e^{i\varphi(t)} dt \right| - d_0 \sigma e^{2A} \int_{t_1}^{t_2} |f|/|t_1|,$$

by above estimates $-\frac{\pi}{4} \leq \eta \leq \frac{\pi}{4}$, so we continue

$$\geq \cos \left( \frac{\pi}{4} \right) \int_{t_1}^{t_2} \frac{|f_1|}{|\theta_z(t, \xi_1)|} - d_0 \sigma e^{2A} \int_{t_1}^{t_2} |f|/|t_1| \geq e^{2A} \int_{t_1}^{t_2} |f|/|t_1|.$$

\qed

The next lemma aligns the displacement of the zeros of $E$ and $\tilde{E}$.

Lemma 9. For any $\varepsilon$ there exists a $\delta = \delta(\varepsilon)$ such that if we have a $(\delta, A)$-admissible collection, then

$$(7.10) \quad |(\xi_2 - \xi_1) - (\tilde{\xi}_2 - \tilde{\xi}_1)| \leq e^{-2A} \varepsilon |\xi_2 - \xi_1|.$$  

Proof. By Proposition 4, we have for $E$ and $\tilde{E}$ at $t$,

$$|\theta_z(t, \xi_1) - \frac{t \alpha^2}{\sin t(\xi_1 - \xi)}| \lesssim t \sqrt{A} \varepsilon$$

and

$$|\tilde{\theta}_z(t, \tilde{\xi}_1) - \frac{t \tilde{\alpha}^2}{\sin t(\tilde{\xi}_1 - \tilde{\xi})}| \lesssim t \sqrt{A} \tilde{\varepsilon}.$$  

Then, part C of Proposition 3 implies

$$|\theta_z(t, \xi_1) - \tilde{\theta}_z(t, \tilde{\xi}_1)| \lesssim t \sqrt{A} \mu.$$  

Furthermore, from the above approximations, we see

$$(7.11) \quad |\theta_z(t, \xi_1)| \sim e^{-2A} t$$

and $|\tilde{\theta}_z(t, \tilde{\xi}_1)| \sim e^{-2A} t$.

Since the velocities of zeros of $E$ and $\tilde{E}$ satisfy (2.38), we deduce applying Lemma 8,

$$|(\xi_2 - \xi_1) - (\tilde{\xi}_2 - \tilde{\xi}_1)| \lesssim e^{2A} \sqrt{A} \mu \xi_2 - \xi_1 \leq e^{-2A} \varepsilon |\xi_2 - \xi_1|,$$

if $\delta(\varepsilon)$ is small enough, where we used condition (iv) of admissibility.  \qed

The following lemma will be used in the proof of Lemma 11.
Lemma 10. For any \( \varepsilon \) there is a \( \delta = \delta(\varepsilon) \) such that the following holds. Let \( A \geq 1, \ s \in \mathbb{R} \) and \( [t_1, t_2] \subset \mathbb{R}_+ \) such that the conditions (i)-(iii) of admissibility hold. Let \( S(t, z) \), \( t \in [t_1, t_2] \) and \( z \in \mathbb{C} \), be the solution to the differential equation (2.2) with the initial condition
\[
S(t_1, z) := c_1 \gamma(t_1 \Re z_1) \sin t_1 (z - z_1),
\]
where \( \frac{1}{4} \leq |c_1| \leq 2 \) is a complex number and \( z_1 \in \mathbb{C}_- \) with \(-A - 1 \leq t \Im z_1 \leq -A\).
Let \( z_t \) be the continuous path of zeros of \( S(t, \cdot) \) such that \( z_{t_1} = z_1 \), and put \( z_2 := z_{t_2} \), then
\[
|S(t_2, z) - c_2 \sin t_1 (z - z_2)| \leq e^{-2A} \varepsilon t_1 |z_2 - z_1|,
\]
for \( z \in R(s, 1/t_2, A/t_2) \), where \( c_2 \) is some complex number such that \( |c_2| \sim e^{-A} \).

Proof. Let \( z^n_t := z_1 + \pi n / t_1 \) and \( z^n_2 \) be the zero of \( S(t, \cdot) \) evolved from \( z^n_1 \). Let \( S(t, z) = e^{it} S(t, z) \). Then, by inequality (2.5) of Lemma 1, we can estimate for \( z \in \mathbb{C}_- \)
\[
|S(t_1, z) - S(t, z)| < |S(t_1, z)| e^{2 \Im(z_1)} |z_2 - z_1| + e^{t_2} \int_{t_1}^{t_2} |f|
\]
We want to estimate \( |z^n_2 - z^n_1| \). First, let us see that \( |z^n_t - z^n_1| \leq \frac{2}{|t_1|} \) for all \( t_1 \leq t \leq t_2 \).
Indeed, assume on the contrary there is \( t \) such that \( |z^n_t - z^n_1| = \frac{1}{2} \). Then, plugging in \( z = z^n_2 \) in (7.13) we get
\[
e^A |S(t_1, z^n_2)| < 2e^{A} e^{A+1} \int_{t_1}^{t_2} |f|,
\]
given \( t_2 - t_1 \leq \frac{1}{|t_1|} \). This is a contradiction, as \( \int_{t_1}^{t_2} |f| \lesssim e^{-6A} \) and \( |S(t_1, z^n_2)| \sim 1 \) due to the assumed location of \( z^n_2 \). Let us again look at (7.14). We deduce
\[
|\sin t_1 (z^n_t - z^n_1)| \lesssim e^{2A} \int_{t_1}^{t_2} |f|,
\]
so that
\[
|z^n_2 - z^n_1| \lesssim e^{2A} \int_{t_1}^{t_2} |f|/t_2.
\]

Let \( I(t, z) = \frac{S(t, z)}{S(t_1, z)} \) be the inner function corresponding to \( S \) which solves the Riccati equation (2.37) with the initial condition
\[
I(t_1, z) = I_{S(t_1, z)}.
\]
We claim
\[
|I_z(t, z^n_t)| \sim e^{-2At} \text{ and } |I_{zz}(t, z^n_t)| \sim e^{-2A} t^2
\]
for \( t \in (t_1, t_2) \). This would follow from a direct computation for \( t = t_1 \) and the following estimate. Let \( v = \Re z^n_1 \), then
\[
|I(t, z) - \frac{I(t, v)}{I(t_1, v)} I(t_1, z)| \lesssim |I(t_1, z)| e^{2A} \int_{t_1}^{t_2} |f|,
\]
for all \( z \in R(v, 1/t_2, (A + 2)/t_2) \). (7.16) follows from above by differentiation near \( z^n_t \) taking into account that \( I(t, v)/I(t_1, v) \) is unimodular.
To prove (7.17), we should utilize the closeness of the zeros of \( I(t, z) \) and \( I(t_1, z) \), namely the inequality (7.15), and the product representation of meromorphic inner functions. Let us first put \( z_1^n = v + x_1 - iy_1 \) and \( z_2^n = v + x_2 - iy_2 \) and compute

\[
\frac{(z - z_1^n)(s - z_1^n)(z - z_1^n)(v - z_2^n)(v - z_1^n) - 1}{(z - z_2^n)(s - z_2^n)(z - z_2^n)(v - z_1^n)(v - z_2^n)} = \frac{2i(z - v)(v - z_1^n)(v - z_2^n)(v - z_1^n)(v - z_2^n)}{(v - z_1^n)(z - z_1^n)(v - z_2^n)(z - z_2^n)}
\]

\[
= \frac{1}{t_2} \left| (x_1 - x_2)y_2 + (x_2 - y_2) \right| + \left| (x_2 - z_2^n)y_1 + x_2^2|y_2 - y_1| + |y_1y_2(y_2 - y_1)|
\]

\[
\leq \frac{1}{(k - t_2|s - v|)^2} e^{2A} \int_{t_1}^{t_2} |f|.
\]

Then, we estimate

\[
\left| \frac{I(t, z)I(t_1, v)}{I(t_1, z)I(t, v)} - 1 \right| = \left| \prod_k \left( \frac{(z - z_1^n)(s - z_1^n)(z - z_1^n)(v - z_2^n)(v - z_1^n) - 1}{(z - z_2^n)(s - z_2^n)(z - z_2^n)(v - z_1^n)(v - z_2^n)} \right) \right|
\]

\[
\lesssim \sum_k \frac{1}{(k - t_2|s - v|)^2} e^{2A} \int_{t_1}^{t_2} |f| \leq e^{2A} \int_{t_1}^{t_2} |f|.
\]

We rewrite (2.39) for \( I \) as

\[
I_z(t, z^n) = I_z(t_1, z^n) e^{\int_{t_1}^{t_2} h_n(t) dt}, \quad h_n(t) = 2i z^n + f(t) \frac{I_{zz}(t, z^n)}{I_z(t, z^n)}.
\]

Utilizing (7.11), (7.16), (7.28) and (7.29), we obtain

\[
|I_z(t, z^n) - I_z(t_1, z^n)| \leq |I_z(t_1, z_1^n)| e^{\int_{t_1}^{t_2} h_n - e^{\int_{t_1}^{t_2} h_0}}
\]

\[
\lesssim t_1 e^{-2A} \int_{t_1}^{t_2} |h_n - h_0| \lesssim t_1 e^{-2A} \left( \int_{t_1}^{t_2} |z^n - z_1| + e^{2A} \int_{t_1}^{t_2} |f| \right)
\]

\[
\lesssim t_1 e^{-2A} \left( \frac{2\pi n(t_2 - t_1)}{t_1} + e^{2A} \int_{t_1}^{t_2} |f| \right),
\]

for all \( t \in (t_1, t_2) \). The last inequality, together with (7.11), (7.16), (7.8) and the Riccati equation (2.38) for \( z_1 \) and \( z_2^n \), gives

\[
(7.18) \quad \left| (z_2^n - z_1^n) - (z_2 - z_1) \right| \lesssim e^{2A} \left( e^{2A} \int_{t_1}^{t_2} |f| + \frac{n(t_2 - t_1)}{t_1} \right) |z_2 - z_1|.
\]

On the other hand, for large \( n \), we can write by (7.8) and (7.15),

\[
|z_2^n - z_1^n| \lesssim |z_2 - z_1|.
\]

By Grönwall’s inequality (2.4), \( |S(t_2, z)| \sim 1 \). So to prove (7.12) it suffices to show

\[
(7.20) \quad \left| \frac{S(t_2, z)}{S(t_2, s)} \sin[t_1 (s - z)] \right| \sim 1.
\]

We prove this analogously to (7.17). Let us write the product representations of \( S \) and the sin through their zeros. The fraction on the left-hand side above becomes

\[
\prod_{n=-\infty}^{\infty} \frac{(1 - \frac{z}{z_2^n})(1 - \frac{s - z}{z_2^n + z_2^n})}{(1 - \frac{z}{z_2^n})(1 - \frac{s - z}{z_2^n + z_1^n})} = \prod_{n=-\infty}^{\infty} \left( 1 + \frac{s - z}{z_2^n + \frac{z_2^n}{z_2^n + z_2^n} - z_2^n} \right)
\]
Let us estimate the $\Sigma$.

\[
= \prod_{n=-\infty}^{\infty} \left(1 + (s - z) \frac{(z_2 - z_1) - (z_2^n - z_1^n)}{(z_2^n - s)(z_2 + \frac{n\pi}{t_1} - z)} \right).
\]

LHS of (7.20) $\lesssim |s - z| \cdot \sum_{n=-\infty}^{\infty} \left| \frac{(z_2 - z_1) - (z_2^n - z_1^n)}{(z_2^n - s)(z_2 + \frac{n\pi}{t_1} - z)} \right| \leq \frac{2A}{t_2} \sum_{|n| \leq t_1/(t_2-t_1)} \left| \frac{(z_2 - z_1) - (z_2^n - z_1^n)}{(z_2^n - s)(z_2 + \frac{n\pi}{t_1} - z)} \right| + \frac{2A}{t_2} \sum_{|n| > t_1/(t_2-t_1)} \left| \frac{(z_2 - z_1) - (z_2^n - z_1^n)}{(z_2^n - s)(z_2 + \frac{n\pi}{t_1} - z)} \right| =: \Sigma_1 + \Sigma_2.

Let us estimate the $\Sigma_{1,2}$ separately using (7.19) and (7.18) correspondingly.

\[
\Sigma_1 \leq \frac{2A}{t_2} \sum_{|n| \leq t_1/(t_2-t_1)} e^{2A(t_2-t_1)} \left| f(t_2) + \frac{n(t_2-t_1)}{t_1} \right| \frac{|z_2 - z_1|}{n^2/t_1^2} \lesssim At_1 |z_2 - z_1| \sum_{|n| \leq t_1/(t_2-t_1)} \left( e^{2A(t_2-t_1)} \frac{|f(t_1)|}{n^2} + e^{2A(t_2-t_1)} \frac{t_2-t_1}{t_1 n} \right) \lesssim t_2 |z_2 - z_1| (e^{-2A}\delta + e^{2A} t_2^{-1} \log \frac{t_2-t_1}{t_1}) \leq \varepsilon e^{-2A} t_1 |z_2 - z_1|,
\]

if $\delta(\varepsilon)$ is small enough. As for $\Sigma_2$,

\[
\Sigma_2 \leq \frac{4A |z_2 - z_1|}{t_2} \sum_{n > t_1/(t_2-t_1)} \frac{1}{n^2/t_1^2} \lesssim A(t_2-t_1) |z_2 - z_1| \leq \varepsilon e^{-2A} t_1 |z_2 - z_1|.
\]

We will need differential equations for the absolute value and the argument of $E$ that are consequences of (2.2). We have

\[
\frac{\partial}{\partial t} |E(t, x)| = f(t) |E(t, x)| \cos [2 \arg E(t, x)],
\]

so that

\[
|E(t, x)| = |E(t_0, x)| \exp \left[ \int_{t_0}^{t} f(t) \cos [2 \arg E(t, x)] dt \right],
\]

and, for the continuous branch of $\arg E$,

\[
\frac{\partial}{\partial t} \arg E(t, x) = -x - f(t) \sin [2 \arg E(t, x)].
\]

**Lemma 11.** For any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that the following holds.

If we have a $(\delta, A)$-admissible collection, then for any $u \in (s - \frac{1}{t_2}, s + \frac{1}{t_2})$, there exist $\omega_1, \omega_2, \tilde{\omega}_1, \tilde{\omega}_2 \in \mathbb{C}^-$ and unimodular $\alpha_1, \alpha_2$ such that

1) $\Re(\omega_1 - \tilde{\omega}_1) = \Re(\omega_2 - \tilde{\omega}_2)$, $\Im \omega_j = \Im \tilde{\omega}_j$ for $j = 1, 2$,
2) $\cos(\Re(\omega_1 - \tilde{\omega}_1)) = \sqrt{\nu(s) \omega(s)}$,
3) $|\omega_1 + t_1 (\xi_1 - u)| \lesssim e^{2A} \sqrt{A \mu(s)}$,
4) For \( j = 1, 2 \)
\[
E(t, u) = \frac{\alpha_j \gamma(3\omega_j)}{\sqrt{w(s)}} \sin \omega_j \quad \text{and} \quad \bar{E}(t, u) = \frac{\alpha_j \gamma(3\omega_j)}{\sqrt{w(s)}} \sin \bar{\omega}_j,
\]
5)
\[
|\omega_2 - \omega_1| - t_1 (\xi_2 - \xi_1)| \leq e^{-2A\epsilon t_1}|\xi_2 - \xi_1|,
\]
and
\[
|\bar{\omega}_2 - \bar{\omega}_1| - t_1 (\xi_2 - \xi_1)| \leq e^{-2A\epsilon t_1}|\xi_2 - \xi_1|.
\]

Proof. By Lemma 6, there exist \( \omega_1, \bar{\omega}_1 \) and \( \alpha_1 \) such that for \( t = t_1 \) conclusions 1)-4) of the lemma hold for \( j = 1 \). Our goal is to find \( \omega_2, \bar{\omega}_2 \) and \( \alpha_2 \) to satisfy the rest of the conclusions.

We introduce the auxiliary function \( S(t, z) \) to be the solution of the equation (2.2) for \( t \in [t_1, t_2] \) with the initial condition
\[
S(t_1, z) = \frac{\alpha_1 \gamma(3\omega_1)}{\sqrt{w(s)}} \sin[t_1(z + \omega_1)].
\]
Let \( \zeta_e \) be the zero of \( S(t, z) \) evolved from \( \zeta_1 := \zeta_{t_1} := u - \frac{\omega_1}{t_1} \) and let \( \zeta_2 := \zeta_{t_2} \).

Conclusion 3) of the Lemma, that is true by Lemma 6, can be rewritten as
\[
|\xi_1 - \zeta_1| \lesssim \mu e^{2A} A^{-1/2} t_1.
\]
By Lemma 10, there is complex numbers \( C_1 \) so that
\[
|S(t_2, z) - C_1 \sin[t_1(z - \zeta_2)]| \leq e^{-2A\epsilon t_1}|\xi_2 - \zeta_1|.
\]
Lemma 10 also implies \( |C_1| \sim e^{-A} \).

Next, we show
\[
|\bar{\zeta}_2 - \zeta_1| - (\bar{\zeta}_2 - \xi_1)| \leq e^{-2A\epsilon \xi_2 - \xi_1}.
\]
Let \( I(t, z) \) be the inner function corresponding to \( S \) as in the proof of Lemma 10.

We have
\[
|\theta_z(t, \xi_1) - I(t_1, \xi_1)| < |\theta_z(t, \xi_1) - I_z(t_1, \xi_1)| + |I_z(t_1, \xi_1) - I(t_1, \xi_1)| \lesssim t_1 e^{2A}. \mu.
\]
Also by Lemma 8 and (7.15),
\[
|\xi_t - \zeta_1| < 2A \int_{t_1}^{t_2} |f|/t_2 + \mu e^{2A} A^{-1/2} t_2.
\]
We rewrite (2.39) for \( I \) and \( \theta \) as
\[
I_z(t, \zeta_1) = I_z(t_1, \xi_1)e^{\int_{t_1}^{t} h(t)dt}, \quad h(t) = 2\mu \xi_1 + f(t) \frac{\theta_z(t_1, \xi_1)}{I_z(t_1, \xi_1)},
\]
and
\[
\theta_z(t, \xi_1) = \theta_z(t_1, \xi_1)e^{\int_{t_1}^{t} g(t)dt}, \quad g(t) = 2\mu \xi_1 + f(t) \frac{\theta_z(t_1, \xi_1)}{\theta_z(t_1, \xi_1)}.
\]
We have
\[
\int_{t_1}^{t_2} (|g| + |h|) \lesssim \frac{t_2 - t_1}{t_1} + (t_2 - t_1)s + e^{2A} \int_{t_1}^{t_2} |f| \leq 2
\]
So utilizing (7.11), (7.16), (7.28) and (7.29), we obtain

\[ |I_z(t, \zeta_1) - \theta_z(t, \xi_1)| \leq |\theta_z(t_1, \xi_1)| \leq |I_z(t_1, \zeta_1)| e^{c_1 t_1 |g|} + |I_z(t_1, \zeta_1)| e^{c_1 t_1 |g - e^{c_1 h}|} \]

\[ \lesssim \sqrt{A} \mu t + t_1 e^{-2A} \int_{t_1}^{t} |g - h| \]

\[ \lesssim \sqrt{A} \mu t + t_1 e^{-2A} \left( \int_{t_1}^{t_2} |\xi_t - \zeta_t| + e^{2A} \int_{t_1}^{t_2} |f| \right) \]

\[ \lesssim \sqrt{A} \mu t + (t_2 - t_1) e^{-2A} \left( e^{2A} \int_{t_1}^{t_2} |f| + \mu \sqrt{A} \right) \]

\[ \lesssim t_1 \sqrt{A} \mu + (t_2 - t_1) \int_{t_1}^{t_2} |f|, \]

for all \( t \in (t_1, t_2) \). Applying the Riccati equation (2.37), the last inequality, the estimates on \( \theta_z \) and \( I_z \) and Lemma 8, we get

\[ |(\zeta_2 - \zeta_1) - (\xi_2 - \xi_1)| \lesssim e^{2A} \left( \frac{t_2 - t_1}{t_1} \int_{t_1}^{t_2} |f| + \sqrt{A} \mu \right) |\zeta_2 - \xi_2| \leq e^{-2A} |\zeta_2 - \xi_2|, \]

if \( \delta \) is small enough.

Using (7.27) and that \( |C_1| \sim e^{-A} \), we can rewrite (7.26) as

\[ |S(t_2, z) - C_1 \sin[t_1(z - (\zeta_1 + (\xi_2 - \xi_1)))| \leq e^{-2A} \epsilon t_1 |\zeta_2 - \xi_2|, \]

for \( z \in R(s, 3/t_2, (A + 3)/t_2) \).

We consider the solution \( S(t, z) \) of (2.2) with the initial condition

\[ S(t_1, z) = \frac{\gamma(3\omega_1)}{\sqrt{w(s)}} \sin[t_1(z - u) + \tilde{\omega}_1], \]

By the same considerations as above there is a complex number \( C_2 \) with \( |C_2| \sim e^{-A} \), so that

\[ |S(t_2, z) - C_2 \sin[t_1(z - (u + \tilde{\omega}_1 + \xi_2 - \xi_1))]| \leq e^{-2A} \epsilon t_1 |\zeta_2 - \xi_2|, \]

for \( z \in R(s, 1/t_2, A/t_2) \), where we have also used Lemma 9.

We want to get nice representations for \( C_1 \) and \( C_2 \). \( S(t_1, z) \) and \( \tilde{S}(t_1, z) \) satisfy the determinant identity

\[ \det \begin{pmatrix} S(t_1, z) & S^\#(t_1, z) \\ \tilde{S}(t_1, z) & \tilde{S}^\#(t_1, z) \end{pmatrix} \equiv 2i. \]

Furthermore, the differential equation (2.2) preserves this relation, so it also holds at \( t = t_2 \). Let us plug into the determinant the approximation formulas (7.30) and (7.31) and \( z = u - \tilde{\omega}_1 + (\xi_2 - \xi_1) \in R(s, 1/t_2, A/t_2) \). We compute

\[ 2i - \psi e^{-2A} \epsilon t_1 |\zeta_2 - \xi_1| = C_1 \tilde{C}_2 \sin[(t_1(\omega_1 - \tilde{\omega}_1)] [\sin[2i \bar{\mathfrak{m}} - \tilde{\omega}_1 + (\xi_2 - \xi_1)]], \]

so

\[ C_1 \tilde{C}_2 \frac{\cosh w(s)}{w(s)} \sinh 2\mathfrak{m}(\omega_1 - t_1(\xi_2 - \xi_1)) = 2 + \psi e^{-2A} \epsilon t_1 |\xi_2 - \xi_1|. \]

Hence, changing \( C_1 \) and \( C_2 \) by at most \( e^{-2A} |\xi_2 - \xi_1| \) we can guarantee that the approximations (7.30) and (7.31) still hold and

\[ (7.32) \quad C_1 \tilde{C}_2 > 0, \quad |C_1 C_2| = \frac{\gamma(3(\omega_1 - t_1(\xi_2 - \xi_1)))^2}{\sqrt{w(s)}w(s)}. \]
Let the point $s_1$ be such that $t_1 s_1 = t_1 (s + \Re(\omega_1 - \omega_1)) - \pi / 2$

$$\sqrt{w(s)} \tilde{S}(t_1, s_1) = \sqrt{w(s)} S(t_1, s).$$

Notice that since $|s - s_1| < 2\pi / t_2$, (7.22) implies

$$\phi(t) = \arg(\tilde{S}(t, s_1) / S(t, s)) < |s - s_1|(t_2 - t_1) + 2 \int_{t_1}^{t_2} |f| < 2\pi(t_2 - t_1) / t_2 + 2 \int_{t_1}^{t_2} |f|$$

for $t \in (t_1, t_2)$. For the absolute values, if we take into account the initial condition

$$\sqrt{w(s)} |S(t_1, s_1)| = \sqrt{w(s)} |S(t_1, s)| = C \leq 2,$$

(7.21) implies

$$\sqrt{w(s)} |\tilde{S}(t_1, s_1)| - \sqrt{w(s)} |S(t_1, s)| = C \left( e^{2 \frac{\int_{t_1}^{t_2} f(t) \cos [2 \arg \tilde{S}(t, s_1)] - \int_{t_1}^{t_2} f(t) \cos [2 \arg S(t, s)]}{t_2 - t_1} - 2 \int_{t_1}^{t_2} |f|} \right).$$

Therefore,

$$\sqrt{w(s)} |\tilde{S}(t_2, s_1)| - \sqrt{w(s)} |S(t_2, s)| \lesssim \left( \left( \int_{t_1}^{t_2} |f| \right)^2 + (t_2 - t_1) \int_{t_1}^{t_2} |f| / t_2 \right).$$

Combining the last relation with (7.32) we obtain that the constants $C_1, C_2$ can be chosen so that

$$C_1 = \alpha_2 \frac{\gamma(\Im(\omega_1 - t_1 (\xi_2 - \xi_1)))}{\sqrt{w(s)}}, \quad C_2 = \alpha_2 \frac{\gamma(\Im(\omega_1 - t_1 (\xi_2 - \xi_1)))}{\sqrt{w(s)}}$$

for some unimodular constant $\alpha_2$.

Finally, note, that by the choice of $\omega_1, \omega_1$, and $E(t_1, u) = S(t_1, u), \tilde{E}(t_1, u) = \tilde{S}(t_1, u)$. And by uniqueness of the solution of differential equation (2.2) at $t = t_2$ and $z = u$, we have

$$S(t_2, u) = E(t_2, u), \quad \tilde{S}(t_2, u) = \tilde{E}(t_2, u).$$

By the approximation formulas (7.31) and (7.30) with new $C_1$ and $C_2$, we have

$$\left| \frac{\sqrt{w(s)} \sin[t_1 (u - (u - \frac{\omega_1}{t_1} + \xi_2 - \xi_1))]}{\sqrt{w(s)} \sin[t_1 (u - (u - \frac{\omega_1}{t_1} + \xi_2 - \xi_1))]} - \frac{E(t_2, u)}{E(t_2, u)} \right| \lesssim e^{-2A} \varepsilon t_1 |\xi_2 - \xi_1|.$$

By an application of Roucé’s theorem with a lower bound on derivative of the fraction with the sines on the left, one can find a constant $\Delta$ with $|\Delta| \lesssim e^{-2A} \varepsilon |\xi_2 - \xi_1|$ such that

$$
\frac{\sqrt{w(s)} \sin[t_1 (u - (u - \frac{\omega_1}{t_1} + \xi_2 - \xi_1) + \Delta)]}{\sqrt{w(s)} \sin[t_1 (u - (u - \frac{\omega_1}{t_1} + \xi_2 - \xi_1) + \Delta)]} = \frac{E(t_2, u)}{E(t_2, u)}.
$$

We put

$$\omega_2 := \omega_1 - t_1 (\xi_2 - \xi_1) - t \Delta$$

and

$$\tilde{\omega}_2 := \tilde{\omega}_1 - t_1 (\xi_2 - \xi_1) - t \Delta.$$
Then, using trigonometric identities one can show that
\[
\begin{align*}
\left| \frac{\gamma(3\omega_2)}{\sqrt{w(s)}} \sin \omega_2 \right| & + \left| \frac{\gamma(3\omega_2)}{\sqrt{w(s)}} \sin \omega_2 \right| = 2i.
\end{align*}
\]

Since \( E, \tilde{E} \) must satisfy (2.10) and (7.34), it follows that
\[
\left| \frac{\gamma(3\omega_2)}{\sqrt{w(s)}} \sin \omega_2 \right| = |E(t_2, u)| \quad \text{and} \quad \left| \frac{\gamma(3\omega_2)}{\sqrt{w(s)}} \cos \omega_2 \right| = |\tilde{E}(t_2, u)|.
\]

The unimodular constant \( \alpha_2 \) is determined automatically. \( \square \)

APPENDIX A.

Lemma 12. Let \( \delta > 0 \). Assume the inequality (1.10) holds for all \( f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \) with \( \|f\|_1 \leq \delta \). Then, we have the weak-\( L^2 \) estimate of the classical Carleson operator, i.e.

(A.1) \[ |\{ x \in \mathbb{R} : \sup_t |\mathcal{F}(f1_{(-\infty, t)})| > \lambda \} | \lesssim \frac{1}{\lambda^2} \|f\|_2^2. \]

Proof. Fix \( 0 < \lambda < 1 \). The case of \( \lambda > 1 \) is trivial due to \( L^1 - L^\infty \) estimate (1.6).

By the asymptotic formula (1.9) and (1.4) we have
\[
2 \log |a(t, x)| = \log(1 + |b(t, x)|^2) \lesssim |\mathcal{F}(f1_{[0, t]})(x)|^2 + \|f\|_1^2.
\]

We take \( \epsilon > 0 \) and plug in \( \epsilon f \) and \( \epsilon \lambda \) into (1.10) and use the last estimate to get
\[
|\{ x : \sup_t |a_{\epsilon f}(t, x)| > \epsilon^2 \lambda^2 \}| \leq |\{ x : \epsilon^2 \sup_t |\mathcal{F}(f1_{[0, t]})(t, x)|^2 + \epsilon^3 > \epsilon^2 \lambda^2 \}|
\[ \lesssim \frac{1}{\epsilon^2 \lambda^2} \|\epsilon^2 f\|_2^2 = \frac{1}{\lambda^2} \|f\|_2^2. \]

Choosing \( \epsilon \lesssim \lambda^2 \), we deduce (A.1), for potentials with small \( L^1 \) norm. However, the scaling in the linear case allows us to drop that restriction. \( \square \)

Proof of Lemma 1. We start with (2.4). Let \( E(t, z) = g_1(t)e^{i\phi_1} \) and \( E^\#(t, z) = g_2e^{i\phi_2} \). Then, considering (2.2), we can write
\[
g_1' + ig_1\phi' = -izg_1 + f g_2e^{i(\phi_2 - \phi_1)}. \]

Taking the real part of the above equation, we get
\[
g_1' = yg_1 + g_2(\Re f) \cos(\phi_2 - \phi_1) - g_2(\Im f) \sin(\phi_2 - \phi_1).
\]

Since \( g_1 \) is away from 0, this is equivalent to
\[
(\log g_1)' = y + \frac{g_2}{g_1}(\Re f) \cos(\phi_2 - \phi_1) - \frac{g_2}{g_1}(\Im f) \sin(\phi_2 - \phi_1).
\]

As \( E \) is Hermite-Biehler, \( g_2 \leq g_1 \) and we get the desired estimate. To get (2.5), we integrate (2.3).
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Mathematical Institute, University of Bonn, Endenicher Allee 60, 53115 Bonn, Germany
Email address: gevorg@math.uni-bonn.de

Institute of Mathematics at the National Academy of Sciences of Armenia, Marshal Baghramyan Ave. 24/5, Yerevan 0019, Armenia
Email address: mnatsakanyan.g@yahoo.com