CONCERNING THE SHAPE OF A GEOMETRIC LATTICE

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Abstract. A well-known conjecture states that the Whitney numbers of the second kind of a geometric lattice (simple matroid) are logarithmically concave. We show this conjecture to be equivalent to proving an upper bound on the number of new copoints in the free erection of the associated simple matroid $M$. A bound on the number of these new copoints is given in terms of the copoints and colines of $M$. Also, the points-lines-planes conjecture is shown to be equivalent to a problem concerning the number subgraphs of a certain bipartite graph whose vertices are the points and lines of a geometric lattice.

1. Introduction

In this section we introduce some standard terminology concerning matroids and their erections, state conjectures concerning the shape of a geometric lattice and highlight results to date. In Section 2 we define the free erection of a matroid via constructs from Knuth's [9] random matroids algorithm, and prove in Theorem 2.1 that the free erection of a simple matroid is the erection containing maximum number of new copoints. The relevance of the free erection to the log-concavity conjecture is made clear in Theorem 2.3 while in Theorem 2.6 we give an upper bound on the number of copoints in the free erection of geometric lattice. In Section 3, using a recent result from Dukes [7], we give an equivalent formulation of the points-lines-planes conjecture in terms of the number of subgraphs of a bipartite graph.

Let $M$ be a simple matroid on the $n$-element set $S_n$ with rank function $\rho$. The rank-$i$ flats are denoted by $\mathcal{F}_i(M)$ for all $0 \leq i \leq \rho(M)$. The collections of copoints, colines and coplanes of $M$ are denoted $\mathcal{F}_{\rho(M)-1}(M)$, $\mathcal{F}_{\rho(M)-2}(M)$ and $\mathcal{F}_{\rho(M)-3}(M)$, respectively. The flats of a simple matroid, ordered by inclusion, form a geometric lattice. Where there is no confusion, $\mathcal{F}_i$ will be used instead of $\mathcal{F}_i(M)$. The numbers $W_i(M) := |\mathcal{F}_i(M)|$ are called the Whitney numbers (of the 2nd kind) of the geometric lattice $M$. The $k$-truncation of $M$ is the rank-$(k + 1)$ matroid $U_k(M)$ with flats $\mathcal{F}(U_k(M)) := \{S_n\} \cup \{F \in \mathcal{F}_i(M) : 0 \leq i \leq k\}$. The closure of a set $A \subset S_n$ is the intersection of all flats in $\mathcal{F}$ containing $A$. A set $A$ is $k$-closed if it contains the closures of all its $j$-element subsets, for all $j \leq k$.

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For matroids $M$ and $N$ on $S_n$, call $N$ an \emph{erection} of $M$ if the flats of $M$ are precisely those flats of $N$ not of rank $\rho(M)$. Allow $M$ to be an erection of itself, called the \emph{trivial erection}. Notice that a matroid $N$ is a non-trivial erection of $M$ if and only if $M$ is the $\rho(M) − 1$ truncation of $N$.

\textbf{Conjecture 1.1} (Mason [12]). \textit{Let $M$ be a simple matroid with $\rho(M) > 2$, then for all $0 < k < \rho(M)$,}

\begin{itemize}
\item[(i)] $W_k(M)^2 \geq W_{k-1}(M)W_{k+1}(M)$,
\item[(ii)] $W_k(M)^2 \geq \frac{k+1}{k}W_{k-1}(M)W_{k+1}(M)$,
\item[(iii)] $W_k(M)^2 \geq \frac{(k+1)(n-k+1)}{k(n-k)}W_{k-1}(M)W_{k+1}(M)$.
\end{itemize}

Of course (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). The idea behind the coefficient in (iii) is that the maximum of the ratio $W_k(M)^2/W_{k-1}(M)W_{k+1}(M)$ is thought to be attained when $W_i(M) = \binom{n}{i}$. The particular case of (iii) with $k = 2$ is known as the ‘points-lines-planes conjecture’. (The term ‘points-lines-planes’ conjecture was originally coined in Welsh [19, p.289] for (ii) with $k = 2$, however Seymour’s more recent paper deals with the more general inequality (iii) with $k = 2$.) The above conjectures are strengthenings of Rota’s [16] conjecture that the Whitney numbers of a matroid are unimodal.

It is well known that $W_1(M) \leq W_k(M)$ for all $k > 1$, but recently Kung [10] has shown that $W_2(M) \leq W_3(M)$ for simple matroids $M$ of rank greater than 5 and in which all lines contain the same number of points. Stone-sifer [18] showed the points-lines-planes conjecture to be true for all graphic matroids. Seymour [17] generalized the result and proved the conjecture to be true for all simple matroids $M$ such that $|F| \leq 4$, for all $F \in F_2(M)$ (i.e. no lines contain more than four points). His ‘localized’ proof relies on exhibiting a function $k(x, y)$ which satisfies a collection of inequalities relating the number of lines to the number of planes.

In what follows we deal with the log-concavity conjecture (i). An almost trivial fact which ties up both ends of (1.1) (i) is

\textbf{Lemma 1.2.} \textit{Let $M$ be a simple matroid on $S_n$ of rank $r \geq 2$. Then $W_1(M)^2 \geq W_0(M)W_2(M)$ and $W_{r-1}(M)^2 \geq W_{r-2}(M)W_r(M)$.}

\textit{Proof.} The first inequality follows since $W_0(M) = 1$, $W_1(M) = n$ and $W_2(M) \leq \binom{n}{2}$. The second inequality holds because $W_r(M) = 1$ and the collection of intersections of all copoints in $F_{r-1}$ generates all flats of lower rank, which includes the collection $F_{r-2}$. \hfill \Box

Erections of matroids (geometries) were introduced by Crapo [3,11], where the following fundamental result was first proved.

\textbf{Theorem 1.3} (Crupo[3]). \textit{A set $\mathcal{H}$ of subsets of $S_n$ is the set of copoints of an erection of $M$ if and only if}

\begin{itemize}
\item[(i)] each $H \in \mathcal{H}$ spans $M$
\item[(ii)] each $H \in \mathcal{H}$ is $(\rho(M) − 1)$-closed
\item[(iii)] each basis for $M$ is contained in a unique $H \in \mathcal{H}$.
\end{itemize}
Nguyen [13] resolved the problem of characterising when a matroid has a non-trivial erection, and gave an explicit construction of the free erection (defined in Theorem 1.4). Roberts [15] exhibited a different procedure to construct all erections and from which the automorphism groups were characterised.

Independently, Knuth [9] gave an algorithm for constructing all matroids on a finite set. Within the algorithm, a certain procedure takes the copoints of a matroid $M$, a collection of random sets and produces an erection of $M$ relative to these random sets. If no random sets are added then the resulting erection is the free erection.

An attractive aspect of Knuth’s construction is that the free erection is easily accessible (one simply adds no random sets). The relation of the free erection to the conjectures above will be made clear in Theorem 2.3.

We paraphrase Knuth’s results in the next theorem. For a collection $T$ of subsets of $S_n$, let $\text{Expand}(T) := \{A \cup \{a\} : A \in T \text{ and } a \in S_n - A\}$ and $\text{Filter}^*(T) := \{A \subseteq S_n : A \supset T \text{ for all } T \in T\}$, essentially the strict filter generated by $T$.

Let $\text{Refine}(T, T')$, where $T'$ is an arbitrary collection of sets, be the collection resulting from: Repeat until $A \cap B \subseteq C$ for some $C \in T'$ for all $A, B \in T$: if $A, B \in T$, $A \neq B$ and $A \cap B \not\subseteq C$ for any $C \in T'$ then replace $A$ and $B$ in $T$ by the set $A \cup B$.

Theorem 1.4 (Knuth[9]). Let $M$ be a simple matroid on $S_n$ with rank function $\rho$ and collection of copoints $\mathcal{H}_M$.

(i) If $N$ is an erection of $M$ then there exists a clutter $A \subseteq \text{Filter}^*(\mathcal{H}_M)$, not necessarily unique and possibly empty, such that

\[ \mathcal{F}_{\rho(M)}(N) = \text{Refine}(\text{Expand}(\mathcal{H}_M) \cup A, \mathcal{H}_M). \]

(ii) Let $\mathcal{F}_{\rho(M)}(\text{Free}(M)) = \text{Refine}(\text{Expand}(\mathcal{H}_M), \mathcal{H}_M)$. If

\[ \mathcal{F}_{\rho(M)}(\text{Free}(M)) = \{S_n\} \]

then $\text{Free}(M) = M$, otherwise the copoints of the free erection are $\mathcal{F}_{\rho(M)}(\text{Free}(M))$ and $\mathcal{F}_{\rho(M)+1}(\text{Free}(M)) := \{S_n\}$.

It was also proven in [9] that the order of the replacements within the Refine function make no difference. Formally this may be stated as:

\[ \text{Refine}(A, T) = \text{Refine}(A - \{A, B\} \cup \text{Refine}(\{A, B\}, T), T) \]

and will be used implicitly in the next section.

Notice that if the free erection of a matroid $M$ is the trivial erection, then $M$ has no other erections and the ranks of $\text{Free}(M)$ and $M$ are the same. Otherwise, if the free erection of $M$ is not the trivial erection, then the free erection has rank $\rho(M) + 1$, as do all other erections of $M$ except the trivial erection.

2. Inequalities concerning the free erection

The following theorem shows that the free erection of a matroid $M$ is the matroid which contains the largest number of copoints amongst all erections of $M$. 
Theorem 2.1. Let $M$ be a simple rank-$r$ matroid and $N$ an erection of $M$, then $W_r(N) \leq W_r(\text{Free}(M))$.

Proof. Let $F_{r-1} = F_{r-1}(M)$ be the copoints of $M$ and let $R_{r-1} \subseteq \text{Filter}^*(F_{r-1})$ be the collection of subsets of $S_n$ such that $F_r(N) = \text{Refine}(\text{Expand}(F_{r-1}) \cup R_{r-1})$. Since $N$ is an erection of $M$, the existence of such a collection $R_{r-1}$ is guaranteed due to Theorem 1.4.

Let $\text{Expand}(F_{r-1}) = \{E_1, \ldots, E_m\}$ and define $E'_i := E_i$ for all $1 \leq i \leq m$. Notice that if $R_1, R_2 \in R_{r-1}$ and $R_1, R_2 \supset E_i$, then

$$\text{Refine}(\{E_i, R_1, R_2\}, F_{r-1}) = \{R_1 \cup R_2\}.$$ 

If there are more than two such $R$’s, then the Refine operation results in the single-element set containing their union. This permits us to do the following: for $i = 1$ to $m$, if $X \in R_{r-1}$ and $X \supset E_i$, then let $E'_i := E'_i \cup X$ and remove $X$ from $R_{r-1}$. Repeat the previous operation until $R_{r-1}$ is empty. Consequently, $E_i \subseteq E'_i$ for all $1 \leq i \leq m$. For distinct $E_i, E_j \in \text{Expand}(F_{r-1})$, if $\text{Refine}(\{E_i, E_j\}, F_{r-1}) = \{E_i \cup E_j\}$ then $\text{Refine}(\{E'_i, E'_j\}, F_{r-1}) = \{E'_i \cup E'_j\}$ because $E_i \cap E_j \subseteq E'_i \cap E'_j$.

Hence $\text{Refine}(\{E_1, \ldots, E_m\}, F_{r-1}) = \text{Refine}(\text{Expand}(F_{r-1}), F_{r-1})$ has at least as many sets as

$$\text{Refine}(\{E'_1, \ldots, E'_m\}, F_{r-1}) = \text{Refine}(\text{Expand}(F_{r-1}) \cup R_{r-1}, F_{r-1}) = F_r(N).$$

An alternative proof may be given using the results and terminology of Crapo [3], in which it was shown that the lattice of all erections of a matroid $M$ contains a least element Free$(M)$, and from which all other erections of $M$ may be obtained by partitioning (an anti-chain with respect to another anti-chain according to certain covering relations).

It would be misleading to think of the number of copoints in the free erection as possessing a monotone property. The next example shows how slightly changing the copoints results in a completely different free erection. Intuition would suggest the opposite to happen, however this is not the case.

Example 2.2. Brylawski [2, p.171] gave the following example in which, by removing four copoints from a matroid and replacing them by their union, the number of copoints in the free erection actually increases. Let $M_1$ and $M_2$ be the matroids on $S_8$ with

$$F_3(M_1) = \{123, 124, 1256, 127, 128, 134, 1357, 136, 138, 1458, 146, 147, 167, 168, 178, 234, 235, 2367, 238, 245, 246, 247, 248, 257, 258, 268, 278, 345, 346, 3478, 356, 358, 368, 456, 457, 467, 468, 467, 468, 567, 568, 578, 678\},$$

$$F_3(M_2) = F_3(M_1) - \{246, 248, 268, 468\} \cup \{4268\}.$$ 

One finds that $F_4(\text{Free}(M_1)) = \{S_8\}$ whereas

$$F_4(\text{Free}(M_2)) = \{123567, 1234, 1238, 1278, 124568, 1247, 134578, 1346, 1368, 1467, 1678, 2345, 234678, 2358, 2578, 2578, 3456, 3568, 4567, 4567, 5678\}.$$
In Dukes [7] it was shown that the points-lines-planes conjecture is equivalent to bounding the number of copoints in the free erection of a rank-3 matroid. We now generalize this to show the relation of free erection to the log-concavity conjecture for the Whitney numbers of a matroid. In this sense, the log-concavity conjecture may be considered a copoints-colines-coplanes conjecture.

**Theorem 2.3.** The log-concavity conjecture (1.1)(i) is true if and only if for all simple matroids \( M \),

\[
W_{\rho(M)-1}(M)^2 \geq W_{\rho(M)-2}(M)W_{\rho(M)}(\text{Free}(M)).
\]  

(2.1)

**Proof.** First assume the conjecture to be true and let \( N = \text{Free}(M) \) for some simple matroid \( M \). Since \( N \neq M \) (for otherwise it is trivial),

\[
W_{\rho(N)-2}(N)^2 \geq W_{\rho(N)-3}(N)W_{\rho(N)-1}(N)
\]

and \( F_{\rho(N)-3}(N) = F_{\rho(M)-2}(M), F_{\rho(N)-2}(N) = F_{\rho(M)-1}(M), F_{\rho(N)-1}(N) = F_{\rho(M)}(\text{Free}(M)) \), the inequality (2.1) holds.

Conversely, assume that for all simple matroids the inequality (2.1) is true. Suppose there exists a matroid \( N \) such that \( W_i(N)^2 < W_{i-1}(N)W_{i+1}(N) \) for some \( 2 < i < \rho(N) - 1 \). Let \( M = U_i(N) \) and from Theorem 2.1 since \( F_{i+1}(N) \) are the copoints of an erection of \( M \),

\[
W_{\rho(M)-1}(M)^2 = W_i(N)^2,
\]

\[
< W_{i-1}(N)W_{i+1}(N),
\]

\[
= W_{\rho(M)-2}(M)W_{\rho(M)}(\text{Free}(M)),
\]

contradicting Equation (2.1). \( \square \)

One might be tempted to insert the clause ‘such that \( \text{Free}(M) \neq M \)’ in the statement of the previous theorem, however this is not necessary due to Lemma 1.2.

Nguyen’s [13] construction of the free erection can be viewed as wrapping the collection \( \text{Expand}(F_{r-1}) \) into a more compact collection \( \text{Pair}(F_{r-1}, F_{r-2}) \) to the effect that

\[
\text{Refine}(\text{Expand}(F_{r-1}), F_{r-1}) = \text{Refine}(\text{Pair}(F_{r-1}, F_{r-2}), F_{r-1})
\]

whereby \( \text{Pair}(X, Y) := \{ X_1 \cup X_2 : X_1, X_2 \in X \text{ and } X_1, X_2 \supset Y \in Y \} \). So \( \text{Pair}(F_{r-1}, F_{r-2}) \) is the set containing the unions of pairs of copoints which have a common coline. We give a short proof of the equivalence of the two collections under the \text{Refine} operation.

**Proposition 2.4.** Let \( M \) be a simple rank-\( r \) matroid on \( S_n \) with copoints \( F_{r-1} \) and colines \( F_{r-2} \). Then

\[
\text{Refine}(\text{Expand}(F_{r-1}), F_{r-1}) = \text{Refine}(\text{Pair}(F_{r-1}, F_{r-2}), F_{r-1}).
\]

**Proof.** The first point to note is that in the definition of the \text{Refine} function, the requirement that two sets be removed from \( \text{Expand}(F_{r-1}) \), if their union is to be included, is unnecessary. Should the two sets remain contained, then the \text{Refine} function will absorb both into their union as a final step.

Suppose \( A, B \in F_{r-1} \) and \( A \cap B \in F_{r-2} \). The existence of these pairs is guaranteed since for all \( C \in F_{r-2} \), the sets \( \{ X - C : X \in F_{r-1} \text{ and } X \supset C \} \) partition the set \( S_n - C \).
Choose \( a \in A \setminus B \) and \( b \in B \setminus A \). Then \( \{a\} \cup B, \{b\} \cup A \in \text{EXPAND}(\mathcal{F}_{r-1}) \) and \((\{a\} \cup B) \cap (\{b\} \cup A) = \{a, b\} \cup (A \cap B)\), but the set \( \{a, b\} \cup (A \cap B) \) is not contained in any \( C \in \mathcal{F}_{r-2} \) because \( A \cap B \in \mathcal{F}_{r-2} \). So the \text{REFINE} function will replace the sets \( \{a\} \cup B, \{b\} \cup A \) by the set \( A \cup B \).

Let us suppose this removal does not occur until all sets \( A, B \in \mathcal{F}_{r-1} \) such that \( A \cap B \in \mathcal{F}_{r-2} \) have been identified and their union inserted. Thus \( \text{EXPAND}(\mathcal{F}_{r-1}) \) contains all sets \( A \cup B \) such that \( A \cap B \in \mathcal{F}_{r-2} \) and \( A \cap B \in \mathcal{F}_{r-2} \), which is precisely the collection \( \text{PAIR}(\mathcal{F}_{r-1}, \mathcal{F}_{r-2}) \). Now as the final operation, absorb all sets of the type \( \{a\} \cup B \in \text{EXPAND}(\mathcal{F}_{r-1}) \) into the unions of \( \text{PAIR}(\mathcal{F}_{r-1}, \mathcal{F}_{r-2}) \).

\[ \square \]

**Example 2.5.** The Whitney numbers of a paving matroid are log-concave. A rank-(\( r + 1 \)) paving matroid on \( S_n \) is the free erection of \( N_r \), the truncation of the boolean algebra on \( S_n \) (i.e. \( N_r \) is the matroid with copoints \( (S_n, r) \)). Thus

\[
W_i(M) = \begin{cases} 
W_i(N) = \binom{n}{i}, & \text{if } i \leq r - 1, \\
\leq W_{i-1}(\text{FREE}(N_r)) = \binom{n}{i}, & \text{if } i = r, \\
1, & \text{if } i = r + 1.
\end{cases}
\]

The outstanding case is \( W_r(M)^2 \geq W_{r-1}(M)W_{r+1}(M) \), but this follows immediately from Lemma 1.2. Hence the sequence of Whitney numbers is log-concave.

For a simple matroid \( M \) on \( S_n \) and \( \pi \) a permutation of \( S_n \), let \( M(\pi) \) be the matroid obtained from \( M \) by permuting all elements of \( M \) by \( \pi \) in the natural way. We present the following bound on the number of copoints in the free erection. The bound is very case specific, as will be seen in the examples mentioned after the proof. For matroids whose free erection is the trivial erection the bound becomes an equality, unsurprisingly, if there exist two copoints containing a common coline, whose union is the ground set of the matroid.

**Theorem 2.6.** Let \( M \) be a simple rank-\( r \) matroid on \( S_n \). For each coline \( F \), let \( a_F(M) := |\{X \in \mathcal{F}_{r-1}(M) : \min_M(X) = F\}| \) where \( \min_M(X) \) denotes the lexicographically smallest coline contained in the copoint \( X \). Then

\[
W_r(\text{FREE}(M)) \leq \min_F \sum_{\pi \in \mathcal{F}_{r-2}(M(\pi))} \left( \frac{a_F(M(\pi))}{2} \right).
\]  

(2.2)

**Proof.** From Proposition 2.3

\[
\mathcal{F}_r(\text{FREE}(M)) = \text{REFINE}(\text{PAIR}(\mathcal{F}_{r-1}, \mathcal{F}_{r-2}), \mathcal{F}_{r-1}).
\]

Let \( F(X_1, X_2) \) be the unique copoint containing the two distinct colines \( X_1, X_2 \) of \( M \) and let \( \min_M(X) \) denote the lexicographically smallest coline containing the copoint \( X \). Define

\[
\text{PAIR}^*(M) := \{\text{distinct copoints } (X, Y) \text{ of } M : \min_M(X) = \min_M(Y) = F, \text{ for all colines } F \text{ of } M\}.
\]

Given \( H \in \mathcal{F}_r(\text{FREE}(M)) \), let \( \alpha_1(H) < \alpha_2(H) < \alpha_3(H) \) be the three lexicographically smallest colines of \( M \) contained in \( H \). For every such \( H \in \mathcal{F}_r(\text{FREE}(M)) \), the pair of copoints \( (F(\alpha_1(H), \alpha_2(H)), F(\alpha_1(H), \alpha_3(H))) \) (of
$M$) uniquely determine $H$. Thus $|\mathcal{F}_r(\text{Free}(M))| \leq |\text{PAIR}^*(\mathcal{F}_{r-1}, \mathcal{F}_{r-2})|$. To evaluate $\text{PAIR}^*$ notice that if

$$a_F(M) = |\{\text{copoints } (X,Y) \text{ of } M : \min_M(X) = \min_M(Y) = F\}|,$$

then

$$|\text{PAIR}^*(\mathcal{F}_{r-1}, \mathcal{F}_{r-2})| = \sum_{F \in \mathcal{F}_{r-2}(M)} \binom{a_F(M)}{2}.$$  

(Of course $\sum_{F \in \mathcal{F}_{r-2}} a_F(M) = |\mathcal{F}_{r-1}(M)|$.) Since the number of copoints in the free erection is invariant under permutations of the ground set, we can take the minimum over all such permutations. \hfill \square

**Example 2.7.** The varying performance of the bound may be seen through the following examples:

(i) Let $M_1$ and $M_2$ be the matroids in Example 2.2. We have that $W_4(\text{Free}(M_1)) = 1$ and $W_4(\text{Free}(M_2)) = 20$. The permutation $\pi = 12456837$ (not unique) minimizes both sums in (2.2), which are 32 and 30 for $M_1$ and $M_2$, respectively.

(ii) Let $M_3$ be the rank-4 matroid on $S_8$ with $\mathcal{F}_3(M_3) = \binom{S}{4}$. Clearly $M(\pi) = M$ for all permutations $\pi$ so the minimum in (2.2) is $\sum_{i=1}^{5} i \binom{7}{2-i} = 70$. The new copoints are $\mathcal{F}_4(\text{Free}(M_3)) = \binom{S_8}{4}$, giving $W_4(\text{Free}(M_3)) = 70$, and there is equality in (2.2).

(iii) Let $M_4$ be the rank-3 matroid on $S_8$ with

$$\mathcal{F}_2(M_4) = \{12, 13, 14, 15, 16, 16, 17, 18, 2345678\}.$$ 

The minimum is attained at any permutation $\pi$ with $\pi_1 = 8$, so that $\mathcal{F}_2(M_4(\pi)) = \{1234567, 18, 28, 38, 48, 58, 68, 78\}$ and $a_{\{1\}}(M_4(\pi)) = 2$, $a_{\{2\}}(M_4(\pi)) = \ldots = a_{\{8\}}(M_4(\pi)) = 1$ and so the sum in (2.2) is 1. Notice $\text{Free}(M_4) = M_4$ and again there is equality in (2.2).

(iv) Let $M_5$ be the rank-3 matroid on $S_8$ with copoints (from Nguyen 13 Example 1)

$$\mathcal{F}_2(M_5) = \{123, 14, 15, 16, 17, 18, 24, 258, 26, 27, 34, 35, 368, 37, 45, 46, 478, 56, 57, 67\}.$$ 

The copoints of the free erection are

$$\mathcal{F}_4(\text{Free}(M_5)) = \{1234, 123568, 1237, 145, 146, 1478, 157, 167, 24578, 246, 267, 345, 34678, 357, 456, 567\}.$$ 

Thus $W_4(\text{Free}(M_5)) = 16$ and the minimum of the sum is 26, attained at $\pi = 18234567$.

(v) Let $M_6$ be the rank-3 matroid on $S_9$ with copoints (from Nguyen 13 Example 2)

$$\mathcal{F}_2(M_6) = \{123, 14, 156, 17, 18, 19, 24, 25, 26, 279, 28, 34, 35, 36, 37, 38, 39, 45, 46, 47, 48, 49, 57, 58, 59, 678, 69, 89\}.$$ 

The free erection of $M_6$ is the trivial erection so $W_4(\text{Free}(M_6)) = 1$ and the minimum of the sum is 44, attained at $\pi = 12678459$. 




3. The free erection of a rank-3 geometry

The definition of the free erection, both in the present paper and previous papers addressing the subject [13, 14, 15, 3], is the outcome of a stringent sequence of joining operations on the copoints. Attempts to describe the free erection any further lead to either a reformulation of the \( \text{Refine} \) function, or a new collection \( A \) formed from copoints \( F_r \) such that the free erection is \( \text{Refine}(A, F_r) \). Indeed, in what is described below we do not achieve any further insights into the free erection. However, the formulation below has the benefit that, in the rank-3 case, one can ‘visualize’ the free erection, thereby making it accessible to graph theorists.

The first class of matroids with non-trivial erections are rank-3 matroids. The relationship of free erections of rank-3 matroids to the points-lines-planes conjecture was shown in [7] and we restate it here. The points-lines-planes conjecture states that for all simple rank-4 matroids \( M \) on \( S^n \),

\[
W_2(M)^2 \geq \frac{3(n-1)}{2(n-2)} W_1(M) W_3(M).
\]

Recall that the points-lines-planes conjecture is a statement about matroids of rank (at least) 4, whereas the statement of the next theorem concerns matroids of rank-3.

**Theorem 3.1** (Dukes [7, Theorem 4.4]). The points-lines-planes conjecture is true if and only if it is true for the class of rank-4 matroids that are the free erection of some rank-3 matroid.

The copoints of a simple rank-3 matroid are a 2-partition of the set \( S_n \) (see for example Aigner [1, p.258]). We use the notation of Diestel [5]. Let us say a bipartite graph \( G(V, E) \) with bipartition \( \{V_1, V_2\} \) has property \( \beta \) if

- \( V_1 \) and \( V_2 \) are non-empty,
- for all \( v_1, v'_1 \in V_1 \) there exists a unique \( v_2 \in V_2 \) such that \( (v_1, v_2), (v'_1, v_2) \in E(G) \),
- \( \delta(G) \geq 2 \).

Bipartite graphs with property \( \beta \) are in one-to-one correspondence with simple rank-3 matroids (up to isomorphism) since they represent the covering relations of the colines to copoints. In the rank 3 case the colines are the points and the copoints are the lines. For \( v \in V_2 \), let \( N(v) = \{v_1 \in V_1 : (v_1, v) \in E(G)\} \) and similarly for \( A \subseteq V_2 \) let \( N(A) \) be the union of the sets \( N(v) \), for all \( v \in A \). Denote by \( G[A \cup N(A)] \) the subgraph of \( G \) induced by restriction to the vertex set \( A \cup N(A) \).

Define \( \beta(G) := |\{A \subseteq V_2 : G[A \cup N(A)] has property \beta\}| \). Since the number of copoints in the free erection (for matroids which have a non-trivial erection) is precisely the number \( \beta(G) \), we have

**Proposition 3.2.** The points-lines-planes conjecture is true if and only if for all bipartite graphs \( G(V, E) \) with property \( \beta \)

\[
\beta(G) \leq \frac{2m^2(n-2)}{3n(n-1)},
\]

where \( n = |V_1|, m = |V_2| \) and \( \{V_1, V_2\} \) is the bipartition of \( G \).
Example 3.3. Consider the rank-3 matroid $M_5$ as mentioned in Example 2.7. Figure 1 shows the covering relations between the copoints and colines of $M_5$, thereby defining the associated bipartite graph $G_M$. The vertices of $G_M$ are $V_1 = \{1, 2, \ldots, 8\}$ and $V_2 = \{123, 14, \ldots, 67\}$.

If we choose $A = \{24, 258, 27, 45, 478, 57\}$ then we find that the subgraph $G[A \cup N(A)]$ induced by the vertices $\{24, 258, 27, 45, 478, 57, 2, 4, 5, 7, 8\}$ has property $\beta$, hence $N(A) = \{24578\} \in \mathcal{F}_4(M_5)$ (see Figure 2).

However, if $A = \{258, 26, 368, 56\}$ then we find that the subgraph $G[A \cup N(A)]$ induced by the vertices $\{258, 26, 368, 56, 2, 3, 5, 6, 8\}$ is such that there does not exist $v$ with both $(2, v), (3, v) \in E(G[A \cup N(A)])$, and similarly for the pair $\{3, 5\} \in V_1$. So $G[A \cup N(A)]$ does not have property $\beta$ and hence $N(A) = \{2568\} \notin \mathcal{F}_4(M_5)$ (see Figure 3).
A tempting case to first resolve is the situation of Kung [10], in which every line contains an equal number of points (say). In the setting of Proposition 3.2 we find that $d(v) = (n - 1)/(t - 1)$ for $v \in V_1$ and $d(v) = t$ for $v \in V_2$. This is the Steiner system $S(2, t, n)$ with $m = n(n - 1)/t(t - 1)$ lines (blocks). As mentioned in the introduction, the cases for $t = 3, 4$ have been resolved. It is possible that the theory of Steiner system can be applied to this case in Proposition 3.2 to resolve the conjecture for $t \geq 5$.

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