REVERSED DICKSON POLYNOMIALS OF THE FOURTH KIND
OVER FINITE FIELDS

KAIMIN CHENG\textsuperscript{A, B}, SHAOFANG HONG\textsuperscript{A, *} AND XIAOER QIN\textsuperscript{C}
\textsuperscript{A}MATHEMATICAL COLLEGE, SICHUAN UNIVERSITY, CHENGDU 610064, P.R. CHINA
\textsuperscript{B}DEPARTMENT OF MATHEMATICS, SICHUAN UNIVERSITY JINJIANG COLLEGE,
PENGSHAN 620860, P.R. CHINA
\textsuperscript{C}SCHOOL OF MATHEMATICS AND STATISTICS, YANGTZE NORMAL UNIVERSITY,
CHONGQING 408100, P.R. CHINA

Abstract. In this paper, we obtain several results on the permutational behavior of
the reversed Dickson polynomial $D_{n,3}(1, x)$ of the fourth kind over the finite field $\mathbb{F}_q$.
Particularly, we present the explicit evaluation of the first moment $\sum_{a \in \mathbb{F}_q} D_{n,3}(1, a)$.

1. Introduction

Let $\mathbb{F}_q$ be the finite field of characteristic $p$ with $q$ elements. Associated to any integer
$n \geq 0$ and a parameter $a \in \mathbb{F}_q$, the $n$-th Dickson polynomials of the first kind and of the
second kind, denoted by $D_n(x, a)$ and $E_n(x, a)$, are defined for $n \geq 1$ by

$D_n(x, a) := \sum_{i=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-i} \binom{n-i}{i} (-a)^i x^{n-2i}$

and

$E_n(x, a) := \sum_{i=0}^{\left[\frac{n}{2}\right]} \binom{n-i}{i} (-a)^i x^{n-2i}$,

respectively, and $D_0(x, a) := 2, E_0(x, a) := 1$, where $\left[\frac{n}{2}\right]$ means the largest integer no
more than $\frac{n}{2}$. In 2012, Wang and Yucas \cite{WangYucas} further defined the $n$-th Dickson polynomial
of the $(k+1)$-th kind $D_{n,k}(x, a) \in \mathbb{F}_q[x]$ for $n \geq 1$ by

$D_{n,k}(x, a) := \sum_{i=0}^{\left[\frac{n}{2}\right]} \frac{n-k}{n-i} \binom{n-i}{i} (-a)^i x^{n-2i}$

and $D_{0,k}(x, a) := 2 - k$.

Hou, Mullen, Sellers and Yucas \cite{HouMullenSellersYucas} introduced the definition of the reversed Dickson
polynomial of the first kind, denoted by $D_n(a, x)$, as follows

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Emails: ckm20@126.com, cheng.km@stu.scu.edu.cn (K. Cheng); sfhong@scu.edu.cn, s-f.hong@tom.com,
hongsf02@yahoo.com (S. Hong); qincn328@sina.com (X. Qin).
of the fourth kind which is defined by Dickson polynomial

\[ D_n(a, x) := \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{n-i} \binom{n-i}{i} (-x)^i a^{n-2i} \]

if \( n \geq 1 \) and \( D_0(a, x) = 2 \). To extend the definition of reversed Dickson polynomials, Wang and Yucas [6] defined the \( n \)-th reversed Dickson polynomial of \((k + 1)\)-th kind \( D_{n,k}(a, x) \) ∈ \( \mathbb{F}_q[x] \), which is defined for \( n \geq 1 \) by

\[ D_{n,k}(a, x) := \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n - ki}{n-i} \binom{n-i}{i} (-x)^i a^{n-2i} \]

and \( D_{0,k}(a, x) = 2 - k \).

It is well known that \( D_n(x, 0) \) is a permutation polynomial of \( \mathbb{F}_q \) if and only if \( \gcd(n, q-1) = 1 \), and if \( a \neq 0 \), then \( D_n(x, a) \) induces a permutation of \( \mathbb{F}_q \) if and only if \( \gcd(n, q^2 - 1) = 1 \). Besides, there are lots of published results on permutational properties of Dickson polynomial \( E_n(x, a) \) of the second kind (see, for example, [1]). In [6], Wang and Yucas investigated the permutational properties of Dickson polynomial \( D_{n,2}(x, 1) \) of the third kind. They got some necessary conditions for \( D_{n,2}(x, 1) \) to be a permutation polynomial of \( \mathbb{F}_q \).

Hou, Mullen, Sellers and Yucas [4] considered the permutational behavior of reversed Dickson polynomial \( D_n(a, x) \) of the first kind. Actually, they showed that \( D_n(a, x) \) is closely related to almost perfect nonlinear functions, and obtained some families of permutation polynomials from the revered Dickson polynomials of the first kind. In [3], Hou and Ly found several necessary conditions for the revered Dickson Polynomials \( D_n(1, x) \) of the first kind to be a permutation polynomial. Recently, Hong, Qin and Zhao [2] studied the revered Dickson polynomial \( E_n(a, x) \) of the second kind that is defined for \( n \geq 1 \) by

\[ E_n(a, x) := \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} (-x)^i a^{n-2i} \]

and \( E_0(a, x) = 1 \). In fact, they gave some necessary conditions for the revered Dickson polynomial \( E_n(1, x) \) of the second kind to be a permutation polynomial of \( \mathbb{F}_q \). Regarding the revered Dickson polynomial \( D_{n,2}(a, x) \) of the third kind, from its definition one can derive that

\[ D_{n,2}(a, x) = E_{n-1}(a, x) \quad (1.1) \]

for each \( x \in \mathbb{F}_q \). Using [13], we can deduce immediately from [2] the similar results on the permutational behavior of the reversed Dickson polynomial \( D_{n,2}(a, x) \) of the third kind. Actually, for the results in [2], we need just to replace \( E_n(1, x) \) by \( D_{n,2}(1, x) \) and replace all other \( n \) by \( n-1 \), then we can obtain the corresponding results on the reversed Dickson polynomial \( D_{n,2}(a, x) \) ∈ \( \mathbb{F}_q[x] \) of the third kind. We here do not list these results.

In this paper, our main goal is to investigate the revered Dickson polynomial \( D_{n,3}(a, x) \) of the fourth kind which is defined by

\[ D_{n,3}(a, x) := \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n-3i}{n-i} \binom{n-i}{i} (-x)^i a^{n-2i} \quad (1.2) \]

if \( n \geq 1 \) and \( D_{0,3}(a, x) := -1 \). For \( a \neq 0 \), we write \( x = y(a - y) \) with an indeterminate \( y \neq \frac{a}{2} \). Then \( D_{n,3}(a, x) \) can be rewritten as
\[ D_{n,3}(a, x) = \frac{(2a - y)y^n - (y + a)(a - y)^n}{2y - a}. \]  

(1.3)

We have

\[ D_{n,3}(a, x^2) = \frac{(3n - 1)a^n}{2^n}. \]  

(1.4)

In fact, (1.3) and (1.4) follows from Theorem 2.2 (i) and Theorem 2.4 (i) below. It is easy to see that \( D_{n,3}(a, x^2) = E_n(a, x) \) if \( \text{char}(\mathbb{F}_q) = 2 \), and \( D_{n,3}(a, x) = D_n(a, x) \) if \( \text{char}(\mathbb{F}_q) = 3 \). Thus we always assume \( p = \text{char}(\mathbb{F}_q) > 3 \) in what follows.

The paper is organized as follows. First in section 2, we study the properties of the reversed Dickson polynomial \( D_{n,3}(a, x) \) of the fourth kind. Subsequently, in Section 3, we prove a necessary condition for the reversed Dickson polynomial \( D_{n,3}(1, x) \) of the fourth kind to be a permutation polynomial of \( \mathbb{F}_q \) and then introduce an auxiliary polynomial to present a characterization for \( D_{n,3}(1, x) \) to be a permutation of \( \mathbb{F}_q \).

From the Hermite criterion \( [5] \) one knows that a function \( f : \mathbb{F}_q \rightarrow \mathbb{F}_q \) is a permutation polynomial of \( \mathbb{F}_q \) if and only if the \( i \)-th moment

\[ \sum_{a \in \mathbb{F}_q} f(a)^i = \begin{cases} 0, & \text{if } 0 \leq i \leq q - 2, \\ -1, & \text{if } i = q - 1. \end{cases} \]

Thus to understand well the permutational behavior of the reversed Dickson polynomial \( D_{n,3}(1, x) \) of the fourth kind, we would like to know if the \( i \)-th moment \( \sum_{a \in \mathbb{F}_q} D_{n,3}(1, a)^i \) is computable. We are able to treat with this sum when \( i = 1 \). The final section is devoted to the computation of the first moment \( \sum_{a \in \mathbb{F}_q} D_{n,3}(1, a) \).

2. Revered Dickson polynomials of the fourth kind

In this section, we study the properties of the reversed Dickson polynomials \( D_{n,3}(a, x) \) of the fourth kind. Clearly, if \( a = 0 \), then

\[ D_{n,3}(0, x) = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ (-1)^{\frac{n}{2} + 1}x^{\frac{n}{2}}, & \text{if } n \text{ is even}. \end{cases} \]

Therefore, \( D_{n,3}(0, x) \) is a PP (permutation polynomial) of \( \mathbb{F}_q \) if and only if \( n \) is an even integer with \( \gcd\left(\frac{n}{2}, q - 1\right) = 1 \). In what follows, we always let \( a \in \mathbb{F}_q^* \). First, we give a basic fact as follows.

**Lemma 2.1.** \[ \text{Let } f(x) \in \mathbb{F}_q[x]. \text{ Then } f(x) \text{ is a PP of } \mathbb{F}_q \text{ if and only if } cf(dx) \text{ is a PP of } \mathbb{F}_q \text{ for any given } c, d \in \mathbb{F}_q^*. \]

Then we can deduce the following result.

**Theorem 2.2.** Let \( a, b \in \mathbb{F}_q^* \). Then the following are true.

(i). One has \( D_{n,3}(a, x) = \frac{a^n}{b^n} D_{n,3}(b, \frac{x}{a^2}). \)

(ii). We have that \( D_{n,3}(a, x) \) is a PP of \( \mathbb{F}_q \) if and only if \( D_{n,3}(1, x) \) is a PP of \( \mathbb{F}_q \).

**Proof.** (i). By the definition of \( D_{n,3}(a, x) \), we have
\[
\frac{a^n}{b^n} D_{n,3}(b, \frac{b^2}{a^2} x)
\]
\[
= \frac{a^n}{b^n} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n-3i}{n-i} \binom{n-i}{i} (-1)^i b^{n-2i} \frac{b^{2i}}{a^{2i}} x^i
\]
\[
= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n-3i}{n-i} \binom{n-i}{i} (-1)^i a^{n-2i} x^i
\]
\[
= D_{n,3}(a, x)
\]
as required. Part (i) is proved.

(ii). Taking \( b = 1 \) in part (i), we have
\[
D_{n,3}(a, x) = a^n D_{n,3}(1, x).
\]

It then follows from Lemma 2.1 that \( D_{n,3}(a, x) \) is a PP of \( \mathbb{F}_q \) if and only if \( D_{n,3}(1, x) \) is a PP of \( \mathbb{F}_q \). This completes the proof of part (ii). So Theorem 2.2 is proved.

Theorem 2.2 tells us that to study the permutational behavior of \( D_{n,3}(a, x) \) over \( \mathbb{F}_q \), one only needs to consider that of \( D_{n,3}(1, x) \). In the following, we supply several basic properties on the revered Dickson polynomial \( D_{n,3}(1, x) \) of the fourth kind. The following result is given in [2] and [4] without proof. For the completeness, we here present a proof.

Lemma 2.3. [2] [4] Let \( n \geq 0 \) be an integer. Then we have \( D_n(1, x(1-x)) = x^n + (1-x)^n \) and \( E_n(1, x(1-x)) = \frac{x^{n+1} - (1-x)^{n+1}}{2x-1} \).

Proof. Since \( D_0(1, x(1-x)) = 2 \), the first formula is true for the case \( n = 0 \). Let now \( n \geq 1 \) be an integer. Then
\[
D_n(1, x(1-x)) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i} \binom{n-i}{i} (x+1-x)^{n-2i} (-x(1-x))^i.
\]
It then follows from Waring’s formula (see, for instance, Theorem 1.76 of [5]) that for any integer \( n \geq 1 \), we have
\[
D_n(1, x(1-x)) = x^n + (1-x)^n.
\]
as desired. The first formula is proved.

Since \( E_0(1, x(1-x)) = E_1(1, x(1-x)) = 1 \), the second formula holds when \( n = 0 \) and 1. Now let \( n \geq 2 \) be an integer. Then we have
$E_n(1, x(1-x))$

$= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n-i}{n-i} \binom{n-i}{i} (-x(1-x))^i$

$= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i} \binom{n-i}{i} (-x(1-x))^i + x(1-x) \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{i}{n-i} \binom{n-i}{i} (-x(1-x))^{i-1}$

$= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i} \binom{n-i}{i} (-x(1-x))^i + x \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1-i}{i-1} (-x(1-x))^{i-1}$

$= D_n(1, x(1-x)) + x E_{n-2}(1, x(1-x)).$

It follows that

$E_n(1, x(1-x))$

$= \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} x^i (1-x)^i D_{n-2i}(1, x(1-x)) + x^{\lfloor \frac{n}{2} \rfloor} (1-x)^{\lfloor \frac{n}{2} \rfloor} E_{n-2}^{\lfloor \frac{n}{2} \rfloor}(1, x(1-x)).$  (2.2)

From (2.1) and (2.2) one can deduce that if $n \geq 1$ is odd, then we have

$E_n(1, x(1-x))$

$= \sum_{i=0}^{\lfloor \frac{n-3}{2} \rfloor} x^i (1-x)^i D_{n-2i}(1, x(1-x)) + y^{\lfloor \frac{n-1}{2} \rfloor} (1-x)^{\lfloor \frac{n-1}{2} \rfloor} E_1(1, x(1-x))$

$= \sum_{i=0}^{\lfloor \frac{n-3}{2} \rfloor} x^i (1-x)^i (x^{n-2i} + (1-x)^{n-2i}) + x^{\lfloor \frac{n-1}{2} \rfloor} (1-x)^{\lfloor \frac{n-1}{2} \rfloor} (x + 1 - x)$

$= \sum_{i=0}^{\lfloor n-1 \rfloor} (x^{n-i}(1-x)^i + x^i(1-x)^{n-i})$

$= \sum_{i=0}^{n} x^{n-i}(1-x)^i$

$= \frac{x^{n+1} - (1-x)^{n+1}}{2x - 1},$

and if $n \geq 0$ is even, then one has
as expected. So the second formula is proved.

This concludes the proof of Lemma 2.3. \qed

\textbf{Theorem 2.4.} Each of the following is true.

(i) For any integer \( n \geq 0 \), we have \( D_{n,3}(1, \frac{1}{4}) = \frac{3n-1}{2^n} \) and \( D_{n,3}(1, x(1-x)) = (2-x)x^{n-2}(x+1)(1-x)^n \) if \( x \neq \frac{1}{2} \).

(ii) If \( n_1 \) and \( n_2 \) are positive integers such that \( n_1 \equiv n_2 \pmod{q^2-1} \), then one has \( D_{n_1,3}(1, x_0) = D_{n_2,3}(1, x_0) \) for any \( x_0 \in \mathbb{F}_q \setminus \{ \frac{1}{4} \} \).

\textit{Proof.} (i). First of all, it is easy to see that \( D_{0,3}(1, \frac{1}{4}) = -1 = \frac{3\cdot0-1}{2^0} \) and \( D_{1,3}(1, \frac{1}{4}) = 1 = \frac{3\cdot1-1}{2^1} \). the first identity is true for the cases that \( n = 0 \) and \( 1 \). Now let \( n \geq 2 \). Then one has

\[
D_{n,3}(1, \frac{1}{4}) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n-3i}{n-i} \binom{n-i}{i} \left( -\frac{1}{4} \right)^i \\
= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n-2i}{n-i} \binom{n-i}{i} \left( -\frac{1}{4} \right)^i + \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{-i}{n-i} \binom{n-i}{i} \left( -\frac{1}{4} \right)^i \\
= D_{n,2}(1, \frac{1}{4}) + \frac{1}{4} \sum_{i=0}^{n-2} \binom{n-2-i}{i} \left( -\frac{1}{4} \right)^i \\
= D_{n,2}(1, \frac{1}{4}) + \frac{1}{4} E_{n-2}(1, \frac{1}{4}).
\]

But (1.1) gives us that \( D_{n,2}(1, \frac{1}{4}) = E_{n-1}(1, \frac{1}{4}) \). Hence Theorem 2.2 of [2] implies that

\[
D_{n,3}(1, \frac{1}{4}) = E_{n-1}(1, \frac{1}{4}) + \frac{1}{4} E_{n-2}(1, \frac{1}{4}) \\
= \frac{n}{2^{n-1}} + \frac{1}{4} \cdot \frac{n-1}{2^{n-2}} \\
= \frac{3n-1}{2^n}
\]

as desired. So the first identity is proved.
Now we turn our attention to the second identity. Let \( x \neq \frac{1}{2} \). Then by the definition of the \( n \)-th reversed Dickson polynomial of the fourth kind, one has

\[
D_{n,3}(1, x(1 - x)) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n - 3i}{n - i} \binom{n - i}{i} (-x(1 - x))^i
\]

\[
= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{3(n - i) - 2n}{n - i} \binom{n - i}{i} (-x(1 - x))^i
\]

\[
= 3 \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n - i}{i} (-x(1 - x))^i - 2 \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n - i}{n - i} \binom{n - i}{i} (-x(1 - x))^i
\]

\[
= 3E_n(1, x(1 - x)) - 2D_n(1, x(1 - x)). \tag{2.3}
\]

But Lemma 2.3 gives us that

\[
D_n(1, x(1 - x)) = x^n + (1 - x)^n
\]

and

\[
E_n(1, x(1 - x)) = \sum_{i=0}^{n} x^{n-i}(1 - x)^i = \frac{x^{n+1} - (1 - x)^{n+1}}{2x - 1}. \tag{2.4}
\]

Thus it follows from (2.3) to (2.5) that

\[
D_{n,3}(1, x) = D_{n,3}(1, y(1 - y))
\]

\[
= 3E_n(1, y(1 - y)) - 2D_n(1, y(1 - y))
\]

\[
= \frac{3y^{n+1} - 3(1 - y)^{n+1}}{2y - 1} - 2(y^n + (1 - y)^n)
\]

\[
= \frac{(2 - y)2y^n - (y + 1)(1 - y)^n}{2y - 1}
\]

as required. So the second identity holds. Part (i) is proved.

(ii). For each \( x_0 \in F_q \setminus \{ \frac{1}{2} \} \), one can choose an element \( y_0 \in F_{q^2} \setminus \{ \frac{1}{2} \} \) such that \( x_0 = y_0(1 - y_0) \). Since \( n_1 \equiv n_2 \pmod{q^2 - 1} \), one has \( y_0^{n_1} = y_0^{n_2} \) and \( (1 - y_0)^{n_1} = (1 - y_0)^{n_2} \). It then follows from part (i) that

\[
D_{n_1,3}(1, x_0) = D_{n_1,3}(1, y_0(1 - y_0))
\]

\[
= \frac{(2 - y_0)y_0^{n_1} - (y_0 + 1)(1 - y_0)^{n_2}}{2y_0 - 1}
\]

\[
= \frac{(2 - y_0)y_0^{n_2} - (y_0 + 1)(1 - y_0)^{n_2}}{2y_0 - 1}
\]

\[
= D_{n_2,3}(1, x_0)
\]

as desired. This ends the proof of Theorem 2.4. \( \Box \)

Evidently, by Theorem 2.2 (i) and Theorem 2.4 (i) one can derive that (1.3) and (1.4) are true.

**Proposition 2.5.** Let \( n \geq 2 \) be an integer. Then the recursion

\[
D_{n,3}(1, x) = D_{n-1,3}(1, x) - xD_{n-2,3}(1, x)
\]

holds for any \( x \in F_q \).
Proof. We consider the following two cases.

Case 1. $x \neq \frac{1}{4}$. For this case, one may let $x = y(1 - y)$ with $y \in \mathbb{F}_{q^2} \setminus \{\frac{1}{2}\}$. Then by Theorem 2.4 (i), we have

\[
D_{n-1,3}(1, x) - xD_{n-2,3}(1, x) = D_{n-1,3}(1, y(1 - y)) - y(1 - y)D_{n-2,3}(1, y(1 - y)) = (2 - y)y^{n-1} - (y + 1)(1 - y)^{n-1} - y(1 - y) \frac{(2 - y)y^{n-2} - (y + 1)(1 - y)^{n-2}}{2y - 1} = \frac{(2 - y)y^n - (y + 1)(1 - y)^n}{2y - 1} = D_{n,3}(1, x)
\]

as required.

Case 2. $x = \frac{1}{4}$. Then by Theorem 2.4 (i), we have

\[
D_{n-1,3}\left(1, \frac{1}{4}\right) - \frac{1}{4}D_{n-2,3}\left(1, \frac{1}{4}\right) = \frac{3n - 4}{2^{n-1}} - \frac{13n - 7}{4 \cdot 2^{n-2}} = \frac{3n - 1}{2^n} = D_{n,3}\left(1, \frac{1}{4}\right).
\]

This concludes the proof of Proposition 2.5.

By Proposition 2.5, we can obtain the generating function of the reverend Dickson polynomial $D_{n,3}(1, x)$ of the fourth kind as follows.

Proposition 2.6. The generating function of $D_{n,3}(1, x)$ is given by

\[
\sum_{n=0}^{\infty} D_{n,3}(1, x)t^n = \frac{2t - 1}{1 - t + xt^2}.
\]

Proof. By the recursion presented in Proposition 2.5, we have

\[
(1 - t + xt^2) \sum_{n=0}^{\infty} D_{n,3}(1, x)t^n = \sum_{n=0}^{\infty} D_{n,3}(1, x)t^n - \sum_{n=0}^{\infty} D_{n,3}(1, x)t^{n+1} + x\sum_{n=0}^{\infty} D_{n,3}(1, x)t^{n+2} = 2t - 1 + \sum_{n=0}^{\infty} (D_{n+2,3}(1, x) - D_{n+1,3}(1, x) + xD_{n,3}(1, x))t^{n+2} = 2t - 1.
\]

Thus the desired result follows immediately.

Now we can use Theorem 2.4 to present an explicit formula for $D_{n,3}(1, x)$ when $n$ is a power of the characteristic $p$. Then we show that $D_{n,3}(1, x)$ is not a PP of $\mathbb{F}_q$ in this case.
Proposition 2.7. Let $p = \text{char}(\mathbb{F}_q) > 3$ and $k$ be a positive integer. Then
\[
2^p D_{p^k,3}(1, x) + 1 = 3(1 - 4x)^{k - \frac{1}{2}}.
\]

**Proof.** Putting $x = y(1 - y)$ in Theorem 2.4 (i) gives us that
\[
D_{p^k,3}(1, x) = D_{p^k,3}(1, y(1 - y))
\]
\[
= \frac{(2 - y)y^{p^k} - (y + 1)(1 - y)^{p^k}}{2y - 1}
\]
\[
= \frac{3 - u(u + 1)^{p^k} - 3 + u(1 - u)^{p^k}}{u}
\]
\[
= \frac{1}{2^{p^k + 1}u} \left( (3 - u)(u + 1)^{p^k} - (u + 3)(1 - u)^{p^k} \right)
\]
\[
= \frac{1}{2^{p^k}}(3u^{p^k - 1} - 1),
\]
where $u = 2y - 1$. So we obtain that
\[
2^p D_{p^k,3}(1, x)
\]
\[
= 3(u^2)^{k - \frac{1}{2}} - 1
\]
\[
= 3((2y - 1)^2)^{k - \frac{1}{2}} - 1,
\]
which infers that
\[
2^p D_{p^k,3}(1, x) + 1 = 3(1 - 4x)^{k - \frac{1}{2}}
\]
as desired. So Proposition 2.7 is proved. \hfill \Box

It is well known that every linear polynomial over $\mathbb{F}_q$ is a PP of $\mathbb{F}_q$ and that the monomial $x^a$ is a PP of $\mathbb{F}_q$ if and only if $\gcd(a, q - 1) = 1$. Then by Proposition 2.7, we have the following result.

**Corollary 2.8.** Let $p > 3$ be a prime and $q = p^e$. Let $e$ and $k$ be positive integers with $k \leq e$. Then $D_{p^k,3}(1, x)$ is not a PP of $\mathbb{F}_q$.

**Proof.** By Proposition 2.7, we know that $D_{p^k,3}(1, x)$ is a PP of $\mathbb{F}_q$ if and only if
\[
(1 - 4x)^{k - \frac{1}{2}}
\]
is a PP of $\mathbb{F}_q$ which is equivalent to
\[
\gcd \left( \frac{p^k - 1}{2}, q - 1 \right) = 1.
\]
The latter one is impossible since $\frac{p^k - 1}{2} | \gcd \left( \frac{p^k - 1}{2}, q - 1 \right)$ implies that
\[
\gcd \left( \frac{p^k - 1}{2}, q - 1 \right) \geq \frac{p^k - 1}{2} > 1.
\]
Thus $D_{p^k,3}(1, x)$ is not a PP of $\mathbb{F}_q$. \hfill \Box

**Lemma 2.9.** \[4\] Let $x \in \mathbb{F}_{q^2}$. Then $x(1 - x) \in \mathbb{F}_q$ if and only if $x^q = x$ or $x^q = 1 - x$.

Let $V$ be defined by
\[
V := \{ x \in \mathbb{F}_{q^2} : x^q = 1 - x \}.
\]
Clearly, $\mathbb{F}_q \cap V = \{ \frac{1}{2} \}$. Then we obtain a characterization for $D_{n,3}(1, x)$ to be a PP of $\mathbb{F}_q$ as follows.
Theorem 2.10. Let $q = p^e$ with $p > 3$ being a prime and $e$ being a positive integer. Let 
$$f : y \mapsto \frac{(2 - y)y^n - (y + 1)(1 - y)^n}{2y - 1}$$
be a mapping on $(\mathbb{F}_q \cup V) \setminus \{\frac{1}{2}\}$. Then $D_{n,3}(1, x)$ is a PP of $\mathbb{F}_q$ if and only if $f$ is 2-to-1 and $f(y) \neq \frac{3n - 1}{2n}$ for any $y \in (\mathbb{F}_q \cup V) \setminus \{\frac{1}{2}\}$.

**Proof.** First, we show the sufficiency part. Let $f$ be 2-to-1 and $f(y) \neq \frac{3n - 1}{2n}$ for any $y \in (\mathbb{F}_q \cup V) \setminus \{\frac{1}{2}\}$. Let $D_{n,3}(1, x_1) = D_{n,3}(1, x_2)$ for $x_1, x_2 \in \mathbb{F}_q$. To show that $D_{n,3}(1, x)$ is a PP of $\mathbb{F}_q$, it suffices to show that $x_1 = x_2$ that will be done in what follows.

First of all, one can find $y_1, y_2 \in \mathbb{F}_q^2$ satisfying $x_1 = y_1(1 - y_1)$ and $x_2 = y_2(1 - y_2)$. By Lemma 2.9, we know that $y_1, y_2 \in \mathbb{F}_q \cup V$. We divide the proof into the following two cases.

**Case 1.** At least one of $x_1$ and $x_2$ is equal to $\frac{1}{2}$. Without loss of any generality, we may let $x_1 = \frac{1}{2}$. So by Theorem 2.4 (i), one derives that 
$$D_{n,3}(1, x_2) = D_{n,3}(1, x_1) = D_{n,3}\left(1, \frac{1}{4}\right) = \frac{3n - 1}{2n}, \quad (2.6)$$

We claim that $x_2 = \frac{1}{4}$. Assume that $x_2 \neq \frac{1}{4}$. Then $y_2 \neq \frac{1}{2}$. Since $f(y) \neq \frac{3n - 1}{2n}$ for any $y \in (\mathbb{F}_q \cup V) \setminus \{\frac{1}{2}\}$, by Theorem 2.4 (i), we get that 
$$D_{n,3}(1, x_2) = \frac{(2 - y_2)y_2^n - (y_2 + 1)(1 - y_2)^n}{2y_2 - 1} = f(y_2) \neq \frac{3n - 1}{2n},$$
which contradicts to (2.6). Hence the claim is true, and so we have $x_1 = x_2$ as required.

**Case 2.** Both of $x_1$ and $x_2$ are not equal to $\frac{1}{2}$. Then $y_1 \neq \frac{1}{2}$ and $y_2 \neq \frac{1}{2}$. Since $D_{n,3}(1, x_1) = D_{n,3}(1, x_2)$, by Theorem 2.4 (i), one has 
$$\frac{(2 - y_2)y_1^n - (y_1 + 1)(1 - y_1)^n}{2y_1 - 1} = \frac{(2 - y_2)y_2^n - (y_2 + 1)(1 - y_2)^n}{2y_2 - 1},$$
which is equivalent to $f(y_1) = f(y_2)$. However, $f$ is a 2-to-1 mapping on $(\mathbb{F}_q \cup V) \setminus \{\frac{1}{2}\}$, and $f(y_2) = f(1 - y_2)$ by the definition of $f$. It then follows that $y_1 = y_2$ or $y_1 = 1 - y_2$. Thus $x_1 = x_2$ as desired. Hence the sufficiency part is proved.

Now we prove the necessity part. Let $D_{n,3}(1, x)$ be a PP of $\mathbb{F}_q$. Choose two elements $y_1, y_2 \in (\mathbb{F}_q \cup V) \setminus \{\frac{1}{2}\}$ such that $f(y_1) = f(y_2)$, that is, 
$$\frac{(2 - y_1)y_1^n - (y_1 + 1)(1 - y_1)^n}{2y_1 - 1} = \frac{(2 - y_2)y_2^n - (y_2 + 1)(1 - y_2)^n}{2y_2 - 1}, \quad (2.7)$$
Since $y_1, y_2 \in (\mathbb{F}_q \cup V) \setminus \{\frac{1}{2}\}$, it follows from Lemma 2.9 that $y_1(1 - y_1) \in \mathbb{F}_q$ and $y_2(1 - y_2) \in \mathbb{F}_q$. So by Theorem 2.4 (i), (2.7) implies that 
$$D_{n,3}(1, y_0(1 - y_0)) = D_{n,3}(1, y(1 - y)).$$
Thus $y_1(1 - y_1) = y_2(1 - y_2)$ since $D_{n,3}(1, x)$ is a PP of $\mathbb{F}_q$, which infers that $y_1 = y_2$ or $y_1 = 1 - y_2$. Since $y_2 \neq \frac{1}{2}$, one has $y_2 \neq 1 - y_2$. Therefore $f$ is a 2-to-1 mapping on $(\mathbb{F}_q \cup V) \setminus \{\frac{1}{2}\}$.

Now take $y' \in (\mathbb{F}_q \cup V) \setminus \{\frac{1}{2}\}$. Then from Lemma 2.9 it follows that $y'(1 - y') \in \mathbb{F}_q$ and 
$$y'(1 - y') \neq \frac{1}{2}\left(1 - \frac{1}{2}\right).$$
Notice that $D_{n,3}(1, x)$ is a PP of $\mathbb{F}_q$. Hence one has 
$$D_{n,3}(1, y'(1 - y')) \neq D_{n,3}\left(1, \frac{1}{2}\left(1 - \frac{1}{2}\right)\right).$$
But Theorem 2.4 (i) tells us that

\[ D_{n,3}\left(1, \frac{1}{2}\left(1 - \frac{1}{2}\right)\right) = \frac{3n - 1}{2^n}. \]

Then by Theorem 2.4 (i) and noting that \( y' \neq \frac{1}{2} \), we have

\[ \frac{(2 - y')y^n - (y' + 1)(1 - y')^n}{2y' - 1} \neq \frac{3n - 1}{2^n}, \]

which infers that \( f(y') \neq \frac{3n-1}{2^n} \) for any \( y' \in (\mathbb{F}_q \cup V) \setminus \{\frac{1}{2}\} \). So the necessity part is proved.

The proof of Theorem 2.10 is complete. \( \square \)

3. A NECESSARY CONDITION FOR \( D_{n,3}(1, x) \) TO BE PERMUTATIONAL AND AN AUXILIARY POLYNOMIAL

In this section, we study some necessary conditions on \( n \) for \( D_{n,3}(1, x) \) to be a PP of \( \mathbb{F}_q \). It is easy to check that

\[ D_{n,3}(1, 0) = 1, D_{0,3}(1, 1) = -1, D_{1,3}(1, 1) = 1, D_{0,3}(1, -2) = -1, D_{1,3}(1, -2) = 1. \]

Then by Proposition 2.5, we have the following recursion relations

\[
\begin{cases}
D_{0,3}(1, 1) = -1, \\
D_{1,3}(1, 1) = 1, \\
D_{n+2,3}(1, 1) = D_{n+1,3}(1, 1) - D_{n,3}(1, 1),
\end{cases}
\]

and

\[
\begin{cases}
D_{0,3}(1, -2) = -1, \\
D_{1,3}(1, -2) = 1, \\
D_{n+2,3}(1, -2) = D_{n+1,3}(1, -2) + 2D_{n,3}(1, -2).
\end{cases}
\]

From these recursive formulas, one can easily show that the sequences

\[ \{D_{n,3}(1, 1)|n \in \mathbb{N}\} \text{ and } \{D_{n,3}(1, -2)|n \in \mathbb{N}\} \]

are periodic with the smallest periods 6 and 2, respectively. In fact, one has

\[ D_{n,3}(1, 1) = \begin{cases}
1, & \text{if } n \equiv 1, 3 \pmod{6}, \\
-1, & \text{if } n \equiv 0, 4 \pmod{6}, \\
2, & \text{if } n \equiv 2 \pmod{6}, \\
-2, & \text{if } n \equiv 5 \pmod{6}
\end{cases} \]

and

\[ D_{n,3}(1, -2) = \begin{cases}
1, & \text{if } n \equiv 1 \pmod{2}, \\
-1, & \text{if } n \equiv 0 \pmod{2},
\end{cases} \]

**Theorem 3.1.** Assume that \( D_{n,3}(1, x) \) is a PP of \( \mathbb{F}_q \) with \( q = p^e \) and \( p > 3 \). Then \( n \equiv 2 \pmod{6} \).

**Proof.** Let \( D_{n,3}(1, x) \) be a PP of \( \mathbb{F}_q \). Then \( D_{n,3}(1, 0), D_{n,3}(1, 1) \) and \( D_{n,3}(1, -2) \) are distinct. Since \( D_{n,3}(1, 0) = 1 \), one has \( D_{n,3}(1, 1) \neq 1 \) and \( D_{n,3}(1, -2) \neq 1 \). Then the above results tells us that \( n \not\equiv 1, 3, 5 \pmod{6} \). Further, we have \( D_{n,3}(1, -2) = -1 \) which means that \( n \) must be even, and so \( D_{n,3}(1, 1) \neq -2 \). But \( D_{n,3}(1, 1) \neq D_{n,3}(1, -2) \). So \( D_{n,3}(1, 1) \neq -1 \). Hence \( D_{n,3}(1, 1) = 2 \). Finally, the desired result \( n \equiv 2 \pmod{6} \) follows immediately. \( \square \)
Evidently, Corollary 2.8 can be easily deduced from Theorem 3.1. Furthermore, by Theorem 3.1, we know that $D_{n,3}(1, x)$ is not a PP of $\mathbb{F}_q$ if $n$ is odd.

In what follows, we investigate $D_{n,3}(1, x)$ with $n$ being an even number. We define the following auxiliary polynomial $f_n(x) \in \mathbb{Z}[x]$ by

$$f_n(x) := -x^n + \sum_{j=0}^{\frac{n}{2}-1} \frac{3n-8j-1}{n+1} (n+1) x^j.$$ 

Then we have the following relation between $D_{n,3}(1, x)$ and $f_n(x)$.

**Theorem 3.2.** Let $p > 3$ be a prime and $n \geq 0$ be an even integer. Then

(i). One has

$$D_{n,3}(1, x) = \frac{1}{2^n} f_n(1 - 4x).$$

(ii). We have that $D_{n,3}(1, x)$ is a PP of $\mathbb{F}_q$ if and only if $f_n(x)$ is a PP of $\mathbb{F}_q$.

**Proof.** (i). First, let $x \in \mathbb{F}_q \setminus \{\frac{1}{4}\}$. Then there exists $y \in \mathbb{F}_q \setminus \{\frac{1}{2}\}$ such that $x = y(1 - y)$. Let $u = 2y - 1$. Since for any integer $j$ with $0 \leq j \leq \frac{n}{2} - 1$, one has

$$3\left(\binom{n}{2j+1} - \binom{n}{2j}\right) = \frac{3n-8j-1}{n+1} (n+1) \left(\binom{n}{2j+1} - \binom{n}{2j}\right),$$

it then follows from Theorem 2.4 (i) that

$$D_{n,3}(1, x) = D_{n,3}(1, y(1 - y))$$

$$= \frac{(2-y)y^n -(y+1)(1-y)^n}{2y-1}$$

$$= \frac{3-u\left(\frac{u+1}{2}\right)^n - 3+u\left(\frac{1-u}{2}\right)^n}{u}$$

$$= \frac{1}{2^{n+1}u} ((3-u)(u+1)^n -(u+3)(1-u)^n)$$

$$= \frac{1}{2^n} (-u^n + \sum_{j=0}^{\frac{n}{2}-1} (3\left(\binom{n}{2j+1} - \binom{n}{2j}\right) u^{2j})$$

$$= \frac{1}{2^n} f_n(u^2)$$

$$= \frac{1}{2^n} f_n(1 - 4y(1 - y))$$

$$= \frac{1}{2^n} f_n(1 - 4x)$$

as desired. So (3.1) holds in this case.

Consequently, we let $x = \frac{1}{4}$. Then by Theorem 2.4 (i), we have

$$D_{n,3}\left(1, \frac{1}{4}\right) = \frac{3n-1}{2^n}.$$ 

On the other hand, we can easily check that $f_n(0) = 3n - 1$. Therefore

$$D_{n,3}\left(1, \frac{1}{4}\right) = \frac{1}{2^n} f_n(0) = \frac{1}{2^n} f_n\left(1 - 4 \times \frac{1}{4}\right)$$

as one desires. So (3.1) is proved.

(ii). Notice that $\frac{1}{2^n} \in \mathbb{F}_q$ and $1 - 4x$ is linear. So $D_{n,3}(1, x)$ is a PP of $\mathbb{F}_q$ if and only if $f_n(x)$ is a PP of $\mathbb{F}_q$. This ends the proof of Theorem 3.2. □
Moreover, by Theorem 2.4 (ii), it follows that for any $x$ with $\equiv (\frac{2}{t})$, one has

$$D_{n,3}(1, x^q) = D_{n,3}(1, x)$$

when $n_1 \equiv n_2 \pmod{q^2 - 1}$. Thus if $x \neq \frac{2}{t}$, one has

$$\sum_{n=0}^{\infty} D_{n,3}(1, x)t^n = 1 + \sum_{n=1}^{q^2-1} \sum_{\ell=0}^{\infty} D_{n+\ell(q^2-1),3}(1, x)t^{n+\ell(q^2-1)}$$

$$= 1 + \sum_{n=1}^{\infty} D_{n,3}(1, x) \sum_{\ell=0}^{q^2-1} t^{n+\ell(q^2-1)}$$

$$= 1 + \frac{1}{1 - tq^{q-1}} \sum_{n=1}^{q^2-1} D_{n,3}(1, x)t^n. \tag{4.2}$$

Then (4.1) together with (4.2) gives that for any $x \neq \frac{2}{t}$, we have

$$\sum_{n=1}^{q^2-1} D_{n,3}(1, x)t^n \tag{4.3}$$

$$= \left( \sum_{n=0}^{\infty} D_{n,3}(1, x)t^n - 1 \right)(1 - tq^{q-1})$$

$$= \left( \frac{2t - 1}{1 - t} - 1 \right)(1 - tq^{q-1}) + \frac{(1 - tq^{q-1})(2t - 1)}{1 - t} \sum_{k=1}^{q-1} \frac{t^{q-1-k} t^{2k}}{(t - 1)q^{q-1} - t^{2q-1}x^k} (\mod x^q - x)$$

$$= \frac{(3t - 2)(1 - tq^{q-1})}{1 - t} + h(t) \sum_{k=1}^{q-1} (t - 1)^{q-1-k} t^{2k} x^k, \tag{4.4}$$

where

$$h(t) := \frac{(tq^{q-1} - 1)(2t - 1)}{(t - 1)q - (t - 1)t^{2q-1}}.$$
Lemma 4.1. Let \( u_0, u_1, \ldots, u_{q-1} \) be the list of all elements of \( \mathbb{F}_q \). Then

\[
\sum_{i=0}^{q-1} u_i = \begin{cases} 
0, & \text{if } 0 \leq k \leq q-2, \\
-1, & \text{if } k = q-1.
\end{cases}
\]

Now by Theorem 2.4 (i), Lemma 4.1 and (4.4), we derive that

\[
\sum_{n=1}^{q^2-1} \sum_{a \in \mathbb{F}_q} D_{n,3}(1, a) t^n
= \sum_{n=1}^{q^2-1} D_{n,3}(1, a) t^n + \sum_{n=1}^{q^2-1} \sum_{a \in \mathbb{F}_q \setminus \left\{ \frac{1}{4} \right\}} D_{n,3}(1, a) t^n
= \sum_{n=1}^{q^2-1} \frac{3n-1}{2n} t^n + \sum_{a \in \mathbb{F}_q \setminus \left\{ \frac{1}{4} \right\}} \frac{(3t-2)(1-t^{q^2-1})}{1-t} + h(t) \sum_{k=1}^{q-1} (t-1)^{q-1-k} t^{2k} \sum_{a \in \mathbb{F}_q \setminus \left\{ \frac{1}{4} \right\}} a^k
= \sum_{n=1}^{q^2-1} \frac{3n-1}{2n} t^n + (q-1) \frac{(3t-2)(1-t^{q^2-1})}{1-t} + h(t) \sum_{k=1}^{q-1} (t-1)^{q-1-k} t^{2k} \sum_{a \in \mathbb{F}_q} a^k
= \sum_{n=1}^{q^2-1} \frac{3n-1}{2n} t^n - \frac{(3t-2)(1-t^{q^2-1})}{1-t} - h(t) t^{2(q-1)} - h(t) \sum_{k=1}^{q-1} (t-1)^{q-1-k} t^{2k} \left( \frac{1}{4} \right)^k.
\]

(4.5)

Since \( (t-1)^q = t^q - 1 \) and \( q \) is odd, one has

\[
h(t) = \frac{(t^q - 1)(2t-1)}{(t-1)^q - (t-1)t^{2(q-1)}}
= \frac{(t^q - 1)(2t-1)}{(1-t^q)(t^q - t^{q-1} - 1)}
= \frac{(t^q - t)(2t-1)}{(t-t^q)(t^q - t^{q-1} - 1)}
= \frac{(t^q - t)^2 + t^q - t - 2t - 1}{t^q - t^{q-1} - 1}
= \frac{(-1 - (t-t^q)^{q-1})(2t-1)}{t^q - t^{q-1} - 1}
= \frac{(2t-1) \sum_{i=0}^{q^2-q} b_i t^i}{t^q - t^{q-1} - 1}.
\]

(4.6)

where

\[
\sum_{i=0}^{q^2-q} b_i t^i := -1 - (t-t^q)^{q-1}.
\]

Then by the binomial theorem applied to \( (t-t^q)^{q-1} \), we can derive the following expression for the coefficient \( b_i \).
Proposition 4.2. For each integer $i$ with $0 \leq i \leq q^2 - q$, write $i = \alpha + \beta q$ with $\alpha$ and $\beta$ being integers such that $0 \leq \alpha, \beta \leq q - 1$. Then

$$b_i = \begin{cases} (-1)^{\beta+1} \left(\frac{q-1}{q}\right), & \text{if } \alpha + \beta = q - 1, \\ -1, & \text{if } \alpha = \beta = 0, \\ 0, & \text{otherwise}. \end{cases}$$

For convenience, let

$$a_n := \sum_{a \in \mathbb{F}_q} D_{n,3}(1,a).$$

Then by (4.5) and (4.6), we arrive at

$$\sum_{n=1}^{q^2-1} \left( a_n - \frac{3n-1}{2^n} \right) t^n = - \frac{(3t-2)(1-t^{q^2-1})}{1-t} - \frac{(2t-1) \sum_{i=1}^{q^2-q} b_i t^i}{t^q - t^{q-1} - 1} \left( t^{2(q-1)} + \sum_{k=1}^{q-1} (t-1)^{q-1-k} t^{2k} \left( \frac{1}{4} \right)^k \right) t^i,$$

which implies that

$$(t^q - t^{q-1} - 1) \sum_{n=1}^{q^2-1} \left( a_n - \frac{3n-1}{2^n} \right) t^n = - (t^q - t^{q-1} - 1)(3t-2) \sum_{i=0}^{q^2-2} t^i - (2t-1) \left( t^{2(q-1)} + \sum_{k=1}^{q-1} (t-1)^{q-1-k} t^{2k} \left( \frac{1}{4} \right)^k \right) \sum_{i=0}^{q^2-q} b_i t^i.$$ (4.8)

Let

$$\sum_{i=1}^{q^2+q-1} c_i t^i$$

denote the right-hand side of (4.9) and let

$$d_n := a_n - \frac{3n-1}{2^n}$$

for each integer $n$ with $1 \leq n \leq q^2 - 1$. Then (4.9) can be reduced to

$$(t^q - t^{q-1} - 1) \sum_{n=1}^{q^2-1} d_n t^n = \sum_{i=1}^{q^2+q-1} c_i t^i.$$ (4.10)

Then by comparing the coefficient of $t^i$ with $1 \leq i \leq q^2+q-1$ of the both sides in (4.10), we derive the following relations:

$$\begin{cases} c_j = -d_j, & \text{if } 1 \leq j \leq q - 1, \\ c_q = -d_1 - d_q, \\ c_{q+j} = d_j - d_{j+1} - d_{q+j}, & \text{if } 1 \leq j \leq q^2 - q - 1, \\ c_{q^2+j} = d_{q^2-j} - d_{q^2+q+j+1}, & \text{if } 0 \leq j \leq q - 2, \\ c_{q^2+q-1} = d_{q^2-1}. \end{cases}$$
from which we can deduce that
\[
\begin{aligned}
    d_j &= -c_j, & \text{if } 1 \leq j \leq q - 1, \\
    d_q &= c_1 - c_q, \\
    d_{(\ell-1)q+j} &= d_{(\ell-1)q+j-1} - c_{\ell q + j}, & \text{if } 1 \leq \ell \leq q - 2 \text{ and } 1 \leq j \leq q - 1, \\
    d_{\ell q} &= d_{(\ell-1)q} - d_{(\ell-1)q+1} - c_{\ell q}, & \text{if } 2 \leq \ell \leq q - 2, \\
    d_{q^2 - q + j} &= \sum_{i=j}^{q-1} c_{q^2 + i}, & \text{if } 0 \leq j \leq q - 1.
\end{aligned}
\] (4.11)

Finally, (4.11) together with the following identity
\[
\sum_{a \in F_q} D_{n,3}(1, a) = d_n + \frac{3n - 1}{2n}
\]
shows that the last main result of this paper is true:

**Theorem 4.3.** Let $c_i$ be the coefficient of $t^i$ in the right-hand side of (4.9) with $i$ being an integer such that $1 \leq i \leq q^2 + q - 1$. Then we have
\[
\sum_{a \in F_q} D_{j,3}(1, a) = -c_j + \frac{3j - 1}{2^j} \quad \text{if } 1 \leq j \leq q - 1,
\]
\[
\sum_{a \in F_q} D_{q,3}(1, a) = c_1 - c_q - \frac{1}{2},
\]
\[
\sum_{a \in F_q} D_{\ell q,3}(1, a) = \sum_{a \in F_q} D_{(\ell-1)q,3}(1, a) - \sum_{a \in F_q} D_{(\ell-1)q+1,3}(1, a) - c_{\ell q + j} + \frac{3}{2^{\ell+j}} \quad \text{if } 1 \leq \ell \leq q - 2 \text{ and } 1 \leq j \leq q - 1,
\]
\[
\sum_{a \in F_q} D_{q,3}(1, a) = \sum_{a \in F_q} D_{(\ell-1)q,3}(1, a) - \sum_{a \in F_q} D_{(\ell-1)q+1,3}(1, a) - c_{\ell q} + \frac{3}{2^\ell} \quad \text{if } 2 \leq \ell \leq q - 2
\]
and
\[
\sum_{a \in F_q} D_{q^2 - q + j,3}(1, a) = \sum_{i=j}^{q-1} c_{q^2 + i} + \frac{3j - 1}{2^j} \quad \text{if } 0 \leq j \leq q - 1.
\]

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