Hierarchies of quantum explicitly solvable and integrable models

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Abstract. Realizing bosonic field \( \psi(x) \) as current of massless (chiral) fermions we derive hierarchy of quantum polynomial interactions of the field \( \psi(x) \) that are completely integrable and lead to linear evolutions for the fermionic field. It is proved that in the classical limit this hierarchy reduces to the dispersionless KdV hierarchy. Application of our construction to quantization of generic completely integrable interaction is demonstrated by example of the mKdV equation.

1. Introduction

Special quantum fields that first appeared in the literature (see, e.g. [1]) under the name “massless two-dimensional fermionic fields,” are known for decades to be useful tool of investigation of completely integrable models in quantum (fermionization procedure [2]–[4]) and in classical (symmetry approach to KP hierarchy [5]) cases. Already in [2] it was shown that when bosonic field of the quantum version of some integrable model is considered as a composition of fermions, the most nonlinear parts of the quantum bosonic Hamiltonian becomes bilinear in terms of these Fermi fields. In [6]–[8] the same property was proved for the Nonlinear Schrödinger equation and some integrable models of statistical physics, where fermionic fields naturally appeared in the so called limit of the infinite interaction, i.e., again as describing the most nonlinear part of the Hamiltonian. Quantization of the KdV equation is based on analogy of the Gardner–Zakharov–Faddeev (GZF) [9] and Magri [10] Poisson brackets with the current and Virasoro algebras [4], [11], [12]. In [4] we proved that quantization of any of these brackets for the KdV equation by means of fermionization procedure can be performed on the entire \( x \)-axis and the Hamiltonian is given as sum of two terms, bilinear with respect to either fermionic or current operators. We also proved that the quantum dispersionless KdV equation generates linear evolution equation for the Fermi field. Thus this equation is explicitly and uniquely solvable for any instant of time (in contrast to the classical case).

In this article we construct hierarchy of nonlinear interactions for the bosonic quantum field \( \psi(x) \) that obey the following properties:

- All equation of this hierarchy are completely integrable in the sense that they have infinite set of local, polynomial (with respect to \( \psi \) and its derivatives) commuting integrals of motion.
- All equations of this hierarchy are explicitly solvable in the following sense. Let \( \psi \) be realized as current of fermionic field \( \tilde{\psi} \). Then all these nonlinear equations for \( \psi \) lead to linear evolution equations for \( \tilde{\psi} \).
- In the limit \( \hbar \to 0 \) this hierarchy reduces to the dispersionless KdV hierarchy.

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The paper is as follows. In Sec. 2 we present some well-known results on the “two-dimensional massless” fermions. In the Sec. 3 the hierarchy is derived and its properties are studied. In Sec. 4 we demonstrate by means of the modified KdV equation that results of our construction can be applied to quantization of the generic integrable models. Discussion of the classical limit of the hierarchy and some concluding remarks are given in the Sec. 5.

2. Massless two-dimensional fermions

Here we introduce notations and list some standard properties of the massless Fermi fields (see, e.g.,[1]). Let $H$ denote the fermionic Fock space generated by operators $\psi(k)$ and $\psi^*(k)$, where $\ast$ means Hermitian conjugation, and that obey canonical anticommutation relations,

$$\{\psi^*(k), \psi(p)\}_+ = \delta(k - p), \quad \{\psi(k), \psi(p)\}_+ = 0.$$ (2.1)

Let $\Omega \in H$ denote vacuum vector and $\psi(k < 0)$ and $\psi^*(k > 0)$ be annihilation operators,

$$\psi(k)\Omega \bigg|_{k < 0} = 0, \quad \psi^*(k)\Omega \bigg|_{k > 0} = 0,$$ (2.2)

whereas $\psi(k > 0)$ and $\psi^*(k < 0)$ are creation operators. Fermionic field is the Fourier transform,

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int dk e^{ikx} \psi(k),$$ (2.3)

and obeys relations

$$\{\psi^*(x), \psi(y)\}_+ = \delta(x - y), \quad \{\psi(x), \psi(y)\}_+ = 0,$$ (2.4)

where we denoted $\varepsilon = (\sqrt{2\pi})^{-1}$. This notation is convenient as in order to restore the Plank constant $\hbar$ we need not only to substitute all commutators and anticommutators $[\cdot, \cdot] \rightarrow [\cdot, \cdot]\hbar^{-1}$, but also put

$$\varepsilon = \sqrt{\frac{\hbar}{2\pi}}.$$ (2.6)

The current of the massless two-dimensional fermionic field is given by the bilinear combination

$$v(x) = \varepsilon^{-1} : \psi^* \psi : (x),$$ (2.7)

where the sign $:\ldots:$ denotes the Wick ordering with respect to the fermionic creation–annihilation operators, for example, $:\psi^*(x)\psi(y): = \psi^*(x)\psi(y) - (\Omega, \psi^*(x)\psi(x)\Omega)$ and $:\psi^*\psi: (x) = \lim_{y \rightarrow x} : \psi^*(x)\psi(y) :$, etc. Current is a self-adjoint operator-valued distribution in the space $H$ obeying the following commutation relations:

$$[v(x), v(y)] = \varepsilon^{-1} \delta(x - y)\psi(x),$$ (2.8)

$$[\psi(x), v(y)] = i\delta'(x - y).$$ (2.9)

The charge of the fermionic field, $\Lambda = \int dx v(x)$, is self-adjoint operator with spectrum $\varepsilon^{-1}\mathbb{Z}$.

Commutation relation (2.9) suggests interpretation of $v(x)$ as bosonic field that obeys quantized version of the GZF bracket ([3], see also (4.2) below). In what follows, we use the decomposition

$$v(x) = v^+(x) + v^-(x)$$ (2.10)
of this field, where positive and negative parts equal

\[(2.11)\quad v^\pm(x) = \pm \frac{1}{2\pi i} \int \frac{dy \, v(y)}{y - x + i\theta}\]

and admit analytic continuation in the upper and bottom half-planes of variable \(x\), correspondingly. They are mutually conjugate and

\[(2.12)\quad v^-(x)\Omega = 0.\]

Let

\[(2.13)\quad v(k) = \int dx \, e^{-ikx} v(x),\]

so that \(v^\pm(x) = (2\pi)^{-1} \int dk \, e^{ikx}\theta(\pm k)v(k)\), where \(\theta(k)\) is step function. Then

\[(2.14)\quad v^*(k) = v(-k), \quad v(k)\Omega|_{k<0} = 0.\]

Thus \(v(k > 0)\) and \(v(k < 0)\) are bosonic creation and annihilation operators, correspondingly, that are bilinear with respect to fermionic ones. One can introduce the bosonic Wick ordering for the products of currents, which we denote by the symbol \(\ldots:\); that means that all positive components of the currents are placed to the left from the negative components, for instance,

\[(2.15)\quad \ldots v(x)v(y)\ldots = v^+(x)v^+(y) + v^+(x)v^-(y) + v^+(y)v^-(x) + v^-(x)v^-(y)\]

and again \(\ldots v^2(x)\ldots = \lim_{y\to x}\ldots v(x)v(y)\ldots\). We can also use equality

\[(2.16)\quad \ldots v(x)v(y)\ldots = v(x)v(y) - (\Omega, v(x)v(y)\Omega),\]

where

\[(2.17)\quad (\Omega, v(x)v(y)\Omega) = \left( i\varepsilon \frac{x - y}{x - y - i\theta} \right)^2.\]

Fermionization procedure is essentially based on the relation between these two normal orderings. The bosonic ordering \(\ldots:\) can be extended for expressions that include fermionic field:

\[(2.18)\quad \ldots v(x)\psi(y)\ldots = v^+(x)\psi(y) + \psi(y)v^-(x).\]

Then by (2.8) and (2.11),

\[(2.19)\quad \ldots v(x)\psi(y)\ldots = \ldots v(x)\psi(y)\ldots + i\varepsilon \frac{\psi(x) - \psi(y)}{x - y},\]

so that this expression as well as its derivatives w.r.t. \(x\) and \(y\) are well-defined in the limit \(y \to x\). In this limit one uses the obvious fact that under the sign of the fermionic normal product any expression of the kind \(\ldots \psi(x)\ldots \psi(x)\ldots \) equals to zero. In particular, we get relation

\[(2.20)\quad \ldots v\psi\ldots (x) = i\varepsilon \psi_\pm(x),\]

that results in the bosonization of fermions [2–3]. More exactly, one can integrate this equality and write (at least formally) that

\[(2.21)\quad \psi(x) = \ldots e^{-i\varepsilon^{-1} \int^x v(x)dx}\ldots = e^{-i\varepsilon^{-1} \int^x v^+(x)dx} e^{-i\varepsilon^{-1} \int^x v^-(x)dx},\]
where in the second equality definition of the bosonic normal product was used for the exponent. Relation (2.21) needs special infrared regularization of the primitive of the current, ∫x v(x)dx, and its positive and negative components. This procedure can be performed, say, like in [3], and it leads to a special constant operator conjugated to the charge Λ, that must be included in the r.h.s. of (2.21).

3. Hierarchy of explicitly solvable models

Problems of interpretation of Eq. (2.21) do not appear if we deal with bilinear combinations of fermionic fields of the type (2.7). In this case neither infrared regularization, nor the above mentioned auxiliary operator are needed and product of Fermi fields is given directly in terms of the current. An analog of such relation is known in the literature on the symmetries of the KP and KdV hierarchies (see [5]) in the sense of formal series.

Theorem. For any real x and y the identity

\[ (2.3) \quad \varepsilon^{-1}: \psi^*(x + y)\psi(x - y) = \exp \left( i\varepsilon^{-1} \int_{x+y}^{-} dx' v(x') \right) \frac{1}{2i(y-x+1)} \]

holds in the sense of operator-valued distributions of spatial variable x that smoothly depend on parameter y.

Proof. Thanks to (2.18) and (2.20) we have that

\[ -i \frac{\partial}{\partial y} \psi^*(x + y)\psi(x - y) = \]

\[ = v^+(x + y)\psi^*(x + y)\psi(x - y) + \psi^*(x + y)\psi(x - y)v^-(x - y) + \psi^*(x + y)v^-(x - y)\psi(x - y) + \psi^*(x + y)v^+(x + y)v^-(x - y) + \psi^*(x + y)v^+(x + y)\psi(x - y) + \psi^*(x + y)v^-(x + y)(v^-(x + y) + v^-(x - y)) + \]

Taking (2.8) and (2.11) into account we get

\[ (3.2) \quad -i\varepsilon \frac{\partial}{\partial y} \psi^*(x + y)\psi(x - y) = \]

\[ = \left( v^+(x + y) + v^+(x - y) + \frac{i\varepsilon}{2(y - i0)} \right) \psi^*(x + y)\psi(x - y) + \psi^*(x + y)\psi(x - y) \left( v^-(x + y) + v^-(x - y) + \frac{i\varepsilon}{2(y - i0)} \right) \]

Let

\[ (3.3) \quad F(x, y) = \varepsilon^{-1}: \psi^*(x + y)\psi(x - y) : , \]

i.e., F denotes the l.h.s. of (3.1). Then by definition of the fermionic Wick ordering we get

\[ \psi^*(x + y)\psi(x - y) = \varepsilon F(x, y) - i\varepsilon^2(2y - i0)^{-1} . \]

We substitute ψ*ψ in (3.2) by means of this equality and take into account that powers of distributions (y - i0)^{-1} are well defined. Then we get the following differential equation for F(x, y):

\[ \frac{\partial F(x, y)}{\partial y} = \frac{i}{\varepsilon} : (v(x + y) + v(x - y)) F(x, y) : + \frac{v(x + y) + v(x - y) - 2F(x, y)}{2(y - i0)} \]
At the same time by (2.7) and (3.3) at \( y = 0 \)
(3.4) \[ F(x, 0) = v(x), \]
so the \( i\theta \) term in the denominator can be omitted and we get
(3.5) \[ \frac{\partial F(x, y)}{\partial y} = \frac{i}{\varepsilon} \left( v(x + y) + v(x - y) \right) F(x, y) + \frac{v(x + y) + v(x - y) - 2F(x, y)}{2y}. \]

It is easy to check directly that the r.h.s. of (3.1) obeys the same differential equation with respect to \( y \) and the same boundary condition (3.4), that proves (3.1).

Thanks to (3.3) it is obvious that \( F(x, y) \), indeed, is operator-valued distribution with respect to \( x \) that is smooth, infinitely differentiable function of \( y \). Then the same are properties of this function when it is given by the r.h.s. of (3.1) as
(3.6) \[ F(x, y) = \varepsilon \exp \left( i\varepsilon^{-1} \int_{x-y}^{x+y} dx' v(x') \right) - 1, \]
that completes proof of the theorem.

Let us introduce
(3.7) \[ F_n(x) \equiv \left( \frac{\varepsilon \partial_y}{2i} \right)^n F(x, y) \bigg|_{y=0} = \varepsilon^{n-1} (2i)^n D^n(\psi^* \cdot \psi)(x), \]
where in the second equality we used notation for the Hirota derivative \([3]\), that in the generic case of two functions \( f(x) \) and \( g(x) \) reads as
(3.8) \[ D^n(f \cdot g)(x) = \lim_{y \to 0} \frac{\partial^n}{\partial y^n} f(x + y)g(x - y), \quad n = 1, 2, \ldots. \]

In particular, by (3.4) we get that
(3.9) \[ F_0(x) = v(x), \]
(3.10) \[ F_1(x) = \frac{1}{2i} D(\psi^* \cdot \psi)(x), \]
that are current and energy–momentum density of the massless fermi-field, correspondingly. Thus Eq. (3.1) gives relation of the Hirota derivatives of the fermionic fields with polynomials of the current and its derivatives. All \( F_n(x) \) by (3.7) are self-adjoint operator-valued distributions on the Fock space \( \mathcal{H} \) and by (3.6) we get recursion relations

(3.11) \[ F_{2n+1}(x) = \frac{1}{2n+2} \sum_{m=0}^{n} \frac{(-i\varepsilon/2)^{2m}(2n+1)!}{(2m)!(2(n-m))!} :v^{(2m)}(x)F_{2(n-m)}(x):, \]

for \( n = 0, 1, 2, \ldots, \)

and

(3.12) \[ F_{2n}(x) = \frac{1}{2n+1} \sum_{m=0}^{n-1} \frac{(-i\varepsilon/2)^{2m}(2n)!(2(n-m) - 1)!}{(2m)!(2(n-m))!} :v^{(2m)}(x)F_{2(n-m)-1}(x): + \]

\[ + \left( \frac{\varepsilon}{2i} \right)^n \frac{v^{(2n)}(x)}{2n+1}, \quad n = 1, 2, 3, \ldots, \]
where \( F_0 \) is given in (3.3). The lowest simplest examples are as follows:

\[
\begin{align*}
F_1(x) &= \frac{1}{2}v_{xx}(x), \\
F_2(x) &= \frac{1}{3}v_{xx}(x) - \frac{\varepsilon^2 u_x(x)}{12}, \\
F_3(x) &= \frac{1}{4}v_{xx}(x) - \frac{\varepsilon^2 v(x)v_{xx}(x)}{4}, \\
F_4(x) &= \frac{1}{5}v_{xx}(x) - \frac{\varepsilon^2 u^2(x)v_{xx}(x)}{2} + \frac{\varepsilon^4 u_{xxx}(x) v_{xx}(x)}{80}.
\end{align*}
\]

By definition (3.3) operator \( F(x, y) \) obeys commutation relation

\[
[F(x, y), F(x', y')] = -\varepsilon^{-1}\delta(x - x' + y + y')F(x + y', y + y') + \varepsilon^{-1}\delta(x - x' - y - y')F(x', y + y') + \delta(x - x' + y + y') - \delta(x - x' - y - y'),
\]

that generates corresponding commutation relations for \( F_m \) (closely related with a representation of the \( \mathfrak{gl}_\infty \)-algebra). Only the lowest terms, \( F_0 \) and \( F_1 \), form closed subalgebras:

\[
\begin{align*}
[F_0(x), F_0(x')] &= i\delta'(x - x'), \\
[F_0(x), F_1(x')] &= i\delta'(x - x')F_0(x'), \\
[F_1(x), F_1(x')] &= i\{F_1(x) + F_1(x')\}\delta'(x - x') - \frac{i\varepsilon^2}{12}\delta''(x - x'),
\end{align*}
\]

while commutators of the type \([F_m, F_n]\) include \( F_j \)'s till \( F_{m+n-1} \).

Operator \( F(x, y) \) admits integration with respect to \( x \) along the entire axis and result of integration is well defined operator in the fermionic Fock space \( \mathcal{H} \). Indeed, by (2.3) and (3.3)

\[
\int dx \ F(x, y) = \frac{1}{\varepsilon} \int_0^\infty dk \ \left( e^{2iky} \psi^*(-k)\psi(-k) - e^{-2iky} \psi(k)\psi^*(k) \right),
\]

where expression in the r.h.s. is normally ordered and has creation×annihilation form, so that thanks to (2.2)

\[
\int dx \ F(x, y) \ \Omega = 0
\]

for any \( y \). From here we derive that all operators

\[
H_n = \int dx \ F_n(x) = \frac{1}{\varepsilon} \int_0^\infty dk \ (\varepsilon k)^n (\psi^*(-k)\psi(-k) - (-1)^n \psi(k)\psi^*(k))
\]

are well defined and self-adjoint. For odd \( n \) they are positively defined. At the same time by (3.17) we get

\[
\left[ \int dx \ F(x, y), \int dx' \ F(x', y') \right] = 0
\]

for any \( y \) and \( y' \). This means in particular that all

\[
[H_m, H_n] = 0, \quad m, n = 0, 1, \ldots
\]
In other words, these operators define commuting flows on the space \( \mathcal{H} \) and we can introduce hierarchy of integrable time evolutions by means of commutation relation

\[
(3.26) \quad v_{tm}(x) = i[H_m, v(x)], \quad m = 0, 1, \ldots,
\]

so that by \( (3.23) \): \( (\partial_{t_m} \partial_{t_n} - \partial_{t_n} \partial_{t_m}) v(x) = 0 \) for any \( m \) and \( n \) (we do not indicate the time dependence in all cases where it is not necessary). On the other side, by \( (3.17) \)

\[
(3.27) \quad \left[ \int dx \, F(x, y), v(x') \right] = \varepsilon^{-1} \left[ F(x' + y, y) - F(x' - y, y) \right] = \frac{1}{2iy} \left\{ : \exp \left( \int_{x'}^{x' + 2y} dx' \varepsilon v(\xi) \right) : - : \exp \left( \int_{x'}^{x' - 2y} dx' \varepsilon v(\xi) \right) : \right\},
\]

that leads to highly nonlinear (polynomial) dynamic equations for \( v(x) \) in all cases with exception to \( t_0 \) and \( t_1 \). Thanks to \( (3.7) \), \( (3.23) \), and \( (3.26) \) we have:

\[
(3.28) \quad v_{t_0}(x) = 0,
\]

\[
(3.29) \quad v_{t_1}(x) = v_x(x),
\]

and in the generic situation

\[
(3.30) \quad v_{t_n}(x) = \frac{\partial}{\partial x} \sum_{m=0}^{\frac{n+1}{2}} \frac{(i\varepsilon/2)^{2m}!}{(n - 2m - 1)!(2m + 1)!} \varepsilon^{2m} F_{n-2m-1}(x), \quad n = 1, 2, \ldots.
\]

The simplest examples are as follows:

\[
(3.31) \quad v_{t_2}(x) = \partial_x \psi_x^2(x),
\]

\[
(3.32) \quad v_{t_3}(x) = \partial_x \left( \psi_x^3(x) - \frac{\varepsilon^2}{2} v_{xx}(x) \right),
\]

\[
(3.33) \quad v_{t_4}(x) = \partial_x \left( \psi_x^4(x) - 2\varepsilon^2 v_{xx} \psi_x^2(x) - \varepsilon^2 v_x^2 \psi_x(x) \right).
\]

These polynomial interactions are closely related to the KdV hierarchy: the second evolution is just dispersionless quantum KdV (cf. \[\text{(3.4)}\]), the third evolution coincide with the modified KdV equation for some specific value of the interaction constant, and so on. In the next section we discuss the case of mKdV equation in more detail. Here we emphasize that in spite of the highly nonlinear form of all these equations in terms of the field \( v \), all of them give linear evolutions for fermions. Indeed, introducing the time dependence of \( \psi(x) \) in analogy with \( (3.26) \) as \( \psi_{tm} = i[H_m, \psi] \), we get by \( (3.23) \)

\[
(3.34) \quad \psi_{tm}(x) = \frac{1}{i\varepsilon} (i\varepsilon \partial_x)^m \psi(x),
\]

or by \( (3.3) \) \( \psi_{tm}(k) = (i\varepsilon)^{-1} (-\varepsilon k)^m \psi(k) \). Let now \( \psi(t_m, x), v(t_m, x) \), and \( F(t_m, x, y) \) be operators with time evolution given by some \( H_m \) and determined by the condition that at \( t_m = 0 \) they equal to \( \psi(x) \), \( v(x) \), and \( F(x, y) \), correspondingly. Thanks to \( (3.22) \) the definitions of the both normal products do not depend on time. This means that these operators are related at arbitrary value of \( t_m \) by means of the same Eqs. \( (3.7) \), \( (3.3) \), \( (3.4) \), and \( (3.9) \) as at \( t_m = 0 \). In particular, by \( (3.3) \)

\[
(3.35) \quad F(t_m, x, y) = \frac{1}{\varepsilon} : \psi^*(t_m, x + y) \psi(t_m, x - y) :.
\]
Then, thanks to (2.3), (3.3), and (3.34) we get explicit expression for \( F(t_m, x, y) \) in terms of its initial value \( F(x, y) \):

\[
F(t_m, x, y) = \frac{2}{(2\pi)^2} \int dx' \int dy' \int dk \int dp F(x - x', y - y') \times \\
\times \exp\left(i(k - p)x' + i(k + p)y' + i\varepsilon^{m-1}(k^m - p^m)t_m\right).
\]

Thanks to (3.7) and (3.9) we obtain for \( y = 0 \):

\[
v(t_m, x) = \frac{2}{(2\pi)^2} \int dx' \int dy' \int dk \int dp F(x - x', y') \times \\
\times \exp\left(i(k - p)x' - i(k + p)y' + i\varepsilon^{m-1}(k^m - p^m)t_m\right).
\]

Substituting here \( F(x, y) \) by means of (3.6) we get solution of the \( m \)'s equation of the hierarchy (3.26) in terms of the initial data \( v(x) \):

\[
v(t_m, x) = \frac{1}{(2\pi)^2} \int dx' \int dy' \int dk \int dp F(x - x', y') \times \\
\times \exp\left(i(k - p)x' - i(k + p)y' + i\varepsilon^{m-1}(k^m - p^m)t_m\right).
\]

Generalization to the case where time evolution is determined by a linear combination of Hamiltonians \( \mathcal{H}_m \) is straightforward.

Thus we see, that all these models are not only completely integrable, but also explicitly solvable in the fermionic Fock space \( \mathcal{H} \). On the other side, taking into account that thanks to (3.25) and (3.28) the charge operator \( \Lambda = H_0/\sqrt{2\pi} \) commutes with all Hamiltonians and \( v(x) \), one can reduce bosonic equations to the zero (or any other, fixed) charge sector of \( \mathcal{H} \), that is exactly the standard bosonic Fock space. In that case all relations of the type (3.6) and (3.37) remain valid and give explicit solution of the hierarchy (3.26) in the bosonic Fock space.

4. THE MODIFIED KdV EQUATION

The modified Korteweg–de Vries (mKdV) equation

\[
v_t = \partial_x \left( gv^3 - \frac{v_{xx}}{2} \right)
\]

for the real function \( v(t, x) \) is well known example of the completely integrable differential equation. If \( v(x) \) is a smooth real function that decays rapidly enough when \( |x| \to \infty \), the Inverse Spectral Transform (IST) method (see [13, 14] and references therein) is applicable to Eq. (4.1). Constant \( g \) in this equation is an arbitrary real parameter and properties of solutions essentially depend on its sign. In particular, the soliton solutions exist only if \( g < 0 \).

The mKdV equation is Hamiltonian system with respect to the GZF bracket [1]

\[
\{ v(x), v(y) \} = \delta'(x - y),
\]

so that Eq. (4.1) can be written in the form \( v_t = -\{ H, v \} \), where Hamiltonian

\[
H = \frac{1}{4} \int dx \left( gv^4(x) + v_x^2(x) \right)
\]

The direct quantization of the mKdV equation on the whole axis requires some regularization (e.g., space cut-off) of the Hamiltonian in order to supply it with operator meaning.
Any such regularization is incompatible with the IST already in the classical case: the continuous and discrete spectra of corresponding linear (Zakharov–Shabat) problem become mixed and the most interesting, soliton solutions cease to exist.

Here we show that realizing $v(x)$ as in (2.7), i.e., as a composition of fermionic fields we can avoid any cut-off procedure in (4.3), because the Hamiltonian becomes well-defined in the fermionic Fock space $\mathcal{H}$.

We choose the quantum Hamiltonian to be bosonically ordered expression (4.3),

$$H = \frac{1}{4} \int dx \dot{g}v^4(x) + v^2_2(x)\dot{v}.$$

Then, thanks to (3.15) we get

$$H = gH_3 + \frac{1 - g\varepsilon^2}{4} \int dx \dot{v}^2_2(x),$$

where (3.23) for $n = 3$ was used. Thus, in analogy with the KdV case (see [4]), the most singular part of the Hamiltonian (4.4) that was of the fourth order with respect to bosonic operators is only of the second order with respect to fermions. Taking into account that by (2.13)

$$\int dx \dot{v}^2_2(x) = 2 \int_0^\infty dk k^2 v(k) v(-k)$$

we get that both terms in (4.5) are bilinear in either fermionic, or bosonic creation–annihilation operators, they are normally ordered and have a diagonal form, i.e., they include “creation×annihilation” terms only. Correspondingly, both these terms are well defined self-adjoint operators in $\mathcal{H}$ and under our quantization procedure no any regularization of the Hamiltonian is needed. In particular, by (2.14) and (3.22)

$$H\Omega = 0$$

and by (3.23) and (4.6) the Hamiltonian (4.5) is positively defined when $\varepsilon^{-2} \geq g \geq 0$.

It is clear that time evolution given by the Hamiltonian (4.4),

$$v_t = i[H, v] \equiv \partial_x \left( g\dot{v}^3 - \frac{v_{xx}}{2} \right),$$

is exactly the quantum version of the Eq. (4.1) normally ordered with respect to the bosonic operators. Thanks to (2.14) we can exclude the $v^3$-term and get the quantum bilinear form of the mKdV equation in terms of the fermionic fields:

$$v_t(x) = \partial_x \left( 3gF_2(x) + \frac{g\varepsilon^2 - 2}{4} v_{xx}(x) \right),$$

that can be considered as a quantum Hirota form of the mKdV equation.

In order to derive time evolution of the fermionic field $\psi$ it is reasonable to rewrite the second term of (4.3) by means of the fermionic normal ordering. Omitting details we get by definitions of the both normal orderings and Eqs. (2.7) and (3.3) the equality

$$\dot{v}(x)v(y) = v(x)v(y) + \varepsilon \frac{F(x + y/2, y - y/2) - F(x + y/2, y + x/2)}{i(x - y)},$$

that after differentiation gives in the limit $y \to x$

$$\dot{v}(x)v(y) = v^2_2(x) + \frac{1}{2} \partial^2_x F_1(x) + \frac{2}{3\varepsilon^2} F_3(x),$$
where (3.7) was used and where by (2.7) : \( v_z^2 : (x) = 2\varepsilon^{-2} : \psi_x^* \psi_x : \). Thus we can write (4.3) as

\[
H = \frac{5g + \varepsilon^{-2}}{6} H_3 + \frac{\varepsilon^{-2} - g}{2} \int dx : \psi_x^* \psi_x : (x),
\]

and thus time evolution of the fermionic field, \( \psi_t = i[H, \psi] \) is given by equation

\[
\psi_t(x) = -\frac{5g\varepsilon^2 + 1}{6} \psi_{xxx}(x) + \frac{g\varepsilon^2 - 1}{2i\varepsilon} \psi_{xx} : (x),
\]

that is, of course, nonlinear when \( g \neq \varepsilon^{-2} \).

Investigation of the spectrum of the quantum Hamiltonian deserves the separate studying. But like in (4) it can be shown that in the fermionic Fock space \( \mathcal{H} \) for \( g < 0 \) there exists one-soliton state, i.e., such state that the average of the field \( v \) with respect to it equals to the classical one-soliton solution at least at zero (or any fixed) instant of time. This state does not belong to the zero charge sector of \( \mathcal{H} \), so it cannot exist in the standard (bosonic) quantization of the mKdV equation. Again, like in (4) it can be shown that existence of this state implies quantization of the soliton action variable.

5. Conclusion

We derived hierarchy of nonlinear integrable and at the same time solvable evolutions of the bosonic field \( v(x) \) realized as composition of the fermionic fields—current. By (3.3) this means that \( F_0(x) \) was chosen to be a dynamical variable. But the closed subalgebra of commutation relations (3.13)–(3.20) is given also by \( F_0(x) \) and \( F_1(x) \). Moreover, the linear combination

\[
\bar{F}(x) = F_1(x) + a \partial_x F_0(x)
\]

with real constant coefficient \( a \) also obeys closed commutation relation,

\[
[\bar{F}(x), \bar{F}(x')] = i\{\bar{F}(x) + \bar{F}(x')\} \delta'(x - x') - i \left( a^2 + \frac{\varepsilon^2}{12} \right) \delta'''(x - x'),
\]

as follows from (3.18)–(3.20). This means that \( \bar{F}(x) \) gives another possible choice of a dynamical variable. In (4) we proved that the dispersionless KdV in this case is also solvable, while—in contrast to the above—it was \( v(x) \) that evolved linearly. It is natural to expect that the same property is valid for the entire hierarchy (3.30) generated by the quantum version (5.2) of the Magri bracket.

Coefficients of the r.h.s. of the bosonic equations of motion (3.29)–(3.33) are uniquely (up to a common factor) fixed by recursion relations (3.11)–(3.12). Indeed, transformation

\[
v(x) \to av(ax),
\]

is the only canonical scaling transformation that is unitary implemented in \( \mathcal{H} \). Here constant \( a > 0 \) in order to preserve definition (2.11) of positive and negative parts of \( v \). This transformation generates:

\[
\psi(x) \to \sqrt{a} \psi(ax), \quad F(x, y) \to aF(ax, ay), \quad F_n(x) \to a^n F_n(ax),
\]

that is compatible with (3.9)–(3.12). Thus by (3.23) \( H_n \to a^{n-1} H_n \), and thanks to (3.30) transformation (5.3) can be compensated by rescaling of times: \( t_n \to a^{1-n} t_n \).

Flows given in (3.3)–(3.16) are close to the flows of the KdV hierarchy [13]; they are polynomial with respect to \( v(x) \) and its derivatives and have the same leading terms. On the other side, the lowest nontrivial example (3.14) shows that some essential terms that are involved in the KdV case are absent in (3.30). In fact, as it was natural to expect by (3.3) Eq. (3.14) is the dispersionless KdV equation: the term \( v_{xxx}(x) \) is absent. The
higher equations, like (3.13), (3.16), and so on already include terms with derivatives, so these equations are not the dispersionless ones. On the other side, coefficients of all such terms of all commuting flows introduced in Sec. 3 are proportional to powers of $\varepsilon^2$, i.e., of $\hbar$ by (2.6). Thanks to (3.9) and (3.11), (3.12) it is easy to see that in the limit $\hbar \rightarrow 0$

\[ F_m(x) \rightarrow v^{m+1}(x) / m + 1, \]

so that by (3.30) we get in the classical limit equations

\[ \partial_{t_m} v(t_m, x) = m v^{m-1}(t_m, x) v_x(t_m, x), \]

i.e., the dispersionless KdV hierarchy. Solution of the initial problem for the $m$th equation can be written in the parametric form as

\[ x = s - m t_m v^{m-1}_0(s), \quad v(t_m, x) = v(s), \]

where $v(x)$ is initial data. This solution is known to describe overturn of the front, so the initial problem for the Eqs. (5.6) has no global solution. On the other side, Eq. (3.38) gives global solution of the quantum hierarchy (3.30). It is easy to see that before the overturn of the front we get from (3.38) in the limit $\varepsilon \rightarrow 0$ (i.e., $\hbar \rightarrow 0$) that

\[ v(t_m, x) = \frac{1}{(2\pi)^2} \int dx' \int dy' \int dk \int dp \frac{e^{i y' v(x-x')}}{y'} - 1 + e^{ikx'-ipy'+imt_m kp^{m-1}}, \]

so that for the classical solution of (3.7) we get representation

\[ v(t_m, x) = \int dp \left[ \theta(v(x + m t_m p^{m-1}) - p) - \theta(-p) \right], \]

that in this region coincides with (5.7) (here $\theta(p)$ denotes the step function). Summarizing, it is natural to call the hierarchy introduced in the Sec. 3 the quantum dispersionless KdV hierarchy. Dispersionless limits of integrable hierarchies attract now essential attention in the literature, see [15, 16].

Our construction here is essentially based on the representation 3.1 valid for the standard massless fermionic fields. Thanks to this relation we got description of the quantum dispersionless KdV hierarchy. It is natural to hypothesize that anyonic generalization [17] of the fermions leads to more generic integrable bosonic systems.

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