Binary functions, degeneracy, and alternating
dimaps

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Abstract

This paper continues the study of combinatorial properties of binary functions — that is, functions \( f : 2^E \to \mathbb{C} \) such that \( f(\emptyset) = 1 \), where \( E \) is a finite set. Binary functions have previously been shown to admit families of transforms that generalise duality, including a trinity transform, and families of associated minor operations that generalise deletion and contraction, with both these families parameterised by the complex numbers. Binary function representations exist for graphs (via the indicator functions of their cutset spaces) and indeed arbitrary matroids (as shown by the author previously). In this paper, we characterise degenerate elements — analogues of loops and coloops — in binary functions, with respect to any pair of minor operations from our complex-parameterised family. We then apply this to study the relationship between binary functions and Tutte’s alternating dimaps, which also support a trinity transform and three associated minor operations. It is shown that only the simplest alternating dimaps have binary representations of the form we consider, which seems to be the most direct type of representation. The question of whether there exist other, more sophisticated types of binary function representations for alternating dimaps is left open.

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1 Introduction

Duality is a pervasive theme in discrete mathematics. It runs strongly through planar graph theory, matroid theory, mathematical programming, and (via the Hadamard transform) information and coding theory. In some of these contexts, there are also minor operations (local “reductions” based on a specific element) that are dual to each other — the main example being deletion and contraction for graphs and matroids. These operations are important in characterising structures with various properties and in the theory of enumeration.

In some combinatorial systems, transforms of higher order than duality — triality or trinity, which have order three rather than two — are important. These may also have minor operations of some type, now three in number. The main object types with a trinity transform and associated minors known to the author are alternating dimaps (which go back to Tutte [19], with minor operations in [13]), binary functions [11], multimatroids (including isotropic systems) [2, 3] and the related transition matroids [18]. This paper continues our ongoing study of combinatorial trinitie s and associated minor operations [11, 13], by exploring the relationship between alternating dimaps and binary functions.

An alternating dimap is an orientably embedded directed graph in which, around each vertex, the incident edges are directed alternately into, and out of, the vertex. Alternating dimaps were introduced by Tutte [19] as part of his work on dissecting equilateral triangles into equilateral triangles.

A binary function is a complex-valued function defined on all subsets of a set that takes value 1 on the empty set. The prototypical example is the indicator function of the cutset space of a graph, or of a binary linear space (i.e., a cocircuit space of a binary matroid). Other examples come from indicator functions of powerful sets [14] or indeed any collection of finite sets that includes the empty set. In these cases, the binary functions only need to be \{0, 1\}-valued. But larger ranges allow binary functions to generalise all matroids [7], to support minor operations more general than just deletion and contraction [8], and to support transforms other than duality, such as trinity and higher order transforms [11].

One reason for our interest in binary functions is that they have their own Tutte-Whitney functions (not always strictly polynomials, since the exponents of the variables may not necessarily be integers). These were introduced in [7], and shown there to contain functions of independent interest such as the weight enumerator of an arbitrary (not necessarily linear) code, Oxley and Welsh’s clutter reliability (or percolation probability) [17], and Kung’s generalised chromatic polynomial [16]. A natural generalisation of the Potts model partition function to binary functions was similarly treated in [8]. The Tutte-Whitney polynomials (or functions) of binary functions and their duals (i.e., Hadamard transforms) were found in [8] to be just two members of a whole family of generalised Tutte-Whitney functions, and these were found in [10] to contain the partition function of the symmetric Ashkin-Teller model, which cannot be found from the usual Tutte-Whitney polynomials.
To better understand the combinatorics of binary functions, we need to understand their degenerate elements. An element is degenerate if all possible minor operations on it give the same result. For example, the degenerate elements of a graph or a matroid are its loops and coloops. These play a fundamental role: they constitute the singleton components of the matroid; they may be regarded as base cases for the recursive definitions of Tutte-Whitney polynomials; and they must be treated as special cases in numerous proofs. Degenerate elements of alternating dimaps are triloops, in the terminology of [13]. We characterise degenerate elements for binary functions in §5. We then apply the characterisation to study the relationship between binary functions and alternating dimaps.

Some alternating dimaps certainly cannot be represented by binary functions, or indeed any of the other types of objects mentioned in the second paragraph, since alternating dimap minor operations may not commute, unlike those in the other settings. In §6 we determine those alternating dimaps that can be represented faithfully by binary functions.

## 2 Alternating dimaps

An alternating dimap is a directed graph without isolated vertices whose components are each 2-cell-embedded in a separate orientable 2-manifold, such that for each vertex $v$, the edges incident with $v$ are directed alternately into, and out of, $v$ (going around $v$ in the embedding). So each vertex in an alternating dimap has equal indegree and outdegree. Loops and/or multiple edges are allowed, but coloops are not possible. The empty alternating dimap has no vertices, edges or faces. Alternating dimaps were introduced by Tutte and his collaborators, who developed their theory in [19, 5, 20, 22]; see also [4, §10.3] and [21, Ch. 4]. Work on the topic by others includes [11, 6, 15], with much of this work focusing on Tutte’s Tree Trinity Theorem. For a brief summary of the main elements of the theory that concern us here, and for a review of the history and related work, see [13].

If $G$ is an alternating dimap then $kG$ is the disjoint union of $k$ copies of $G$. Each of these copies is regarded as being embedded in separate surfaces.

A face is clockwise or anticlockwise according to the direction of the edges around it. (All faces are of one of these two types). If two faces share a common edge, then one of the faces is clockwise and the other is anticlockwise. The left successor (respectively, right successor) of an edge $e$ is the next edge after $e$, going around its anticlockwise (resp., clockwise) face in the direction given by $e$ (i.e., anticlockwise or clockwise, respectively).

If $G$ is an alternating dimap, then its trial $G^\omega$, introduced by Tutte [19], may be defined as follows. The vertices of $G^\omega$ correspond to the clockwise faces of $G$; write $C_u$ for the clockwise face in $G$ represented by vertex $u$ in $G^\omega$. There is a directed edge $(u, v)$ from $u$ to $v$ in $G^\omega$ whenever there is a vertex $a$ in $G$ belonging to both $C_u$ and $C_v$ such that $C_u$ is the next clockwise face after $C_v$, going clockwise around $a$. This edge is defined to be the image $e^\omega$, under triality, of the edge $e$ of $C_v$ that goes into $a$
and whose left successor is an edge of $C_u$ going out of $a$. See [13] for a more detailed treatment.

Tutte showed that $(G^\omega)^\omega = G$. The symbol $\omega$ is taken to satisfy $\omega^3 = 1$, so we can write, for example, $((G^\omega)^\omega)^\omega = (G^{\omega^2})^{\omega} = G^{\omega^3} = G^1 = G$ At times it is natural to put $\omega = \exp(2\pi i/3)$.

In [13], the author introduced three reductions, or minor operations, that may be done to any edge $e$ in an alternating dimap, and which are the analogues in this context of deletion and contraction in graphs. More specifically, they are the analogues of the surface minor versions of deletion and contraction, which apply to embedded graphs rather than abstract graphs. The first of these three reductions is 1-reduction or contraction, which behaves exactly like the standard contraction operation on embedded graphs. In particular, if $e$ is a loop at a vertex $v$ such that $e$ does not constitute a face in its own right, then contraction of $e$ causes its incident vertex to split into two copies of itself, one for each of the two sides into which $e$ divides the neighbourhood of $v$ in the surface. This split will either increase the number of components or reduce the genus. The second reduction is $\omega$-reduction, in which the left successor of $e$ is changed so that it starts at the tail of $e$ (and its head is unchanged) and $e$ is deleted. The third reduction is $\omega^2$ reduction, which is defined as for $\omega$-reductions except using the right successor rather than the left successor. A minor of an alternating dimap is another alternating dimap obtained from the first by some sequence of reductions.

The relationship between triality and minors is described by the following result.

**Theorem 1** [13, Theorem 2.2] If $e \in E(G)$ and $\mu, \nu \in \{1, \omega, \omega^2\}$ then 

$$G^{\mu\nu}[e]e^{\mu} = (G[\mu\nu]e)^{\mu}.$$

This extends the classical relationship between duality, deletion and contraction, under which $G^* \setminus e = (G/e)^*$ and $G^*/e = (G \setminus e)^*$.

The relationship between triality and minor operations for alternating dimaps given by Theorem 1 is reminiscent of properties of binary functions found by the author in [11]. One outcome of the present paper is to establish how close this connection is. We will determine those alternating dimaps which can be represented, in a certain faithful manner, by binary functions. It is found that alternating dimaps and binary functions actually do not have much in common.

A special role in the theory of alternating dimaps is played by various types of special edges, which are analogous to loops and coloops in graphs but more diverse. An ultraloop is a loop forming a component in its own right. A 1-loop is an edge whose head has indegree = outdegree = 1 (and which need not necessarily be a loop in the conventional sense). An $\omega$-loop is a loop forming an anticlockwise face of size 1, while an $\omega^2$-loop is a loop forming a clockwise face of size 1. An ultraloop is therefore also a 1-loop, an $\omega$-loop, and an $\omega^2$-loop. A triloop is an edge that is a 1-loop, an $\omega$-loop or an $\omega^2$-loop, and it is proper if it is not also an ultraloop. The triloops are precisely those edges such that the three reductions on it all give identical minors.
For abstract graphs, loops and coloops are the only types of special edges. But embedded graphs have other types of special edges too, which are not necessarily degenerate: a semiloop (respectively, a semicoloop) is either a loop (resp., coloop) or an edge whose contraction (resp., deletion) either increases the number of components or decreases the genus. These too have analogues for alternating dimaps.

A 1-semiloop is just an ordinary loop. An \( \omega \)-semiloop (respectively, an \( \omega^2 \)-semiloop) is an \( \omega^2 \)-loop (resp., an \( \omega \)-loop) or an edge for which \( \omega^2 \)-reduction (resp., an \( \omega \)-reduction) either increases the number of components or reduces the genus. A semiloop is proper if it is not also a triloop.

3 Binary functions

In this section we briefly summarise some of the theory of binary functions developed by the author in [7, 8, 9, 10, 11]. We restrict attention to aspects that are relevant to this paper, and so focus on [11].

Let \( E \) be a finite set, with \( m = |E| \). A binary function with ground set \( E \) and dimension \( m \) is a function \( f : 2^E \to \mathbb{C} \) such that \( f(\emptyset) = 1 \). Equivalently, we regard it as a \( 2^m \)-element complex vector \( f \) whose elements are indexed by the subsets of \( E \) and whose first element (indexed by \( \emptyset \)) is 1. (The restriction \( f(\emptyset) = 1 \) was not imposed as part of the definition in earlier work [7, 8, 9, 10], but all scalar multiples of a binary function are equivalent for our purposes, and we have always been most interested in the cases where \( f(\emptyset) \neq 0 \).) If \( f \) is a binary function then \( E(f) \) denotes its ground set.

We often represent a subset \( X \subseteq E \) by its characteristic vector \( x \in \{0, 1\}^E \), with \( x_e = 1 \) if \( e \in X \) and \( x_e = 0 \) otherwise; here, the set \( E \) indexes the positions in the characteristic vector. Since \( x \) may be thought of as a binary string, it may also be taken to be the binary representation of a number \( x \) such that \( 0 \leq x \leq 2^m - 1 \), using \( E = \{0, \ldots, m - 1\} \). These numbers give the order in which the subsets of \( E \) are listed, as indices of the entries of the vector \( f \). With this notation, \( f(X) \) may also be written \( f_k \) or \( f_x \). In particular, \( f(\emptyset) = f_{(0,\ldots,0)} = f_0 = 1 \). We write \( 0_k \) for the sequence of \( k \) 0s, and sometimes drop the subscript \( k \) when it is clear from the context.

The definition was motivated by indicator functions of linear spaces over \( \mathbb{GF}(2) \), especially of cutset spaces of graphs: if \( N \) is a matrix over \( \mathbb{GF}(2) \) whose columns are indexed by \( E \) (such as the incidence matrix of a graph, or the matrix representation of a binary matroid), then the indicator function of the rowspace of \( N \) takes value 1 on a set \( X \subseteq E \) if the characteristic vector of \( X \) belongs to the rowspace of \( N \), and takes value 0 otherwise.

If \( f, g : 2^E \to \mathbb{C} \) and there exists a constant \( c \in \mathbb{C} \setminus \{0\} \) such that \( f(X) = cg(X) \) for all \( X \subseteq E \), then we write \( f \simeq g \).

Define

\[
M(\mu) := \frac{1}{2\sqrt{2}} \begin{pmatrix}
\sqrt{2} + 1 + (\sqrt{2} - 1)\mu & 1 - \mu \\
1 - \mu & \sqrt{2} - 1 + (\sqrt{2} + 1)\mu
\end{pmatrix}.
\]
The \(\mu\)-transform of \(f\), denoted by \(L[^\mu]f\), is given by

\[
L[^\mu]f := M(\mu)^\otimes m f,
\]

where the \(2^m \times 2^m\) matrix on the right is the \(m\)-th Kronecker power of \(M(\mu)\).

When \(\mu = 1\), we have the identity transform, while when \(\mu = -1\), we have a scalar multiple of the Hadamard transform. It is well known that the Hadamard transform takes the indicator function of a linear space to a scalar multiple of the indicator function of its dual, from which it follows that the indicator functions of the cutset and circuit spaces of a graph are related by the Hadamard transform in the same way. It was shown in [7] that general matroid duality is also described by the Hadamard transform.

It is easy to show that \(M(\mu_1\mu_2) = M(\mu_1)M(\mu_2)\) for all \(\mu_1, \mu_2\), see [11]. It follows (using the mixed-product property for the Kronecker product) that composition of the \(L[^\mu]\) transforms corresponds to multiplication of their \(\mu\) parameters: \(L[^{\mu_1}]L[^{\mu_2}] = L^{[^{\mu_1}\mu_2]}\), from [11, Theorem 2]. At this point, readers may ask: what happens when \(\mu = \omega\)? We look at this shortly.

Suppose \(E = \{e_0, \ldots, e_{m-1}\}\).

Let \(b \in \{0, 1\}\). We use \(f_b\), as shorthand for the vector of length \(2^{m-1}\), with elements indexed by subsets of \(E \setminus \{e_0\}\), whose \(X\)-element is \(f(X)\), if \(b = 0\), or \(f(X \cup \{e_0\})\), if \(b = 1\) (for \(X \subseteq E \setminus \{e_0\}\)). We define \(f_{eb}\) in the same way, except that we use \(e_{m-1}\) instead of \(e_0\) throughout. The vectors \(f_{b0}\) and \(f_{b1}\) give the top and bottom halves, respectively, of \(f\), while \(f_{00}\) and \(f_{01}\) give the elements in even and odd positions, respectively, of \(f\).

Let \(I_l\) denote the \(l \times l\) identity matrix. If \(e \in E\), then the \([\mu]\)-minor of \(f\) by \(e\) is the \(2^{m-1}\)-element vector \(f\|_{[\mu]} e\), with entries indexed by subsets of \(E \setminus \{e\}\), given by

\[
f\|_{[\mu]} e_i := c \cdot (I_2^{\otimes i} \otimes \left(1 + \frac{1 + \mu}{\sqrt{2} + 1 - (\sqrt{2} - 1)\mu}\right) \otimes I_2^{\otimes (m-i-1)}) f,
\]

where \(c\) is such that the \(0\)-element of \(f\|_{[\mu]} e_i\) is 1.

For any \(\mu\), define \(\lambda = \lambda(\mu)\) by

\[
\lambda(\mu) := \frac{1 + \mu}{\sqrt{2} + 1 - (\sqrt{2} - 1)\mu}.
\]

Then \(f\|_{[\mu]} e_i\) is a scalar multiple of \((I_2^{\otimes i} \otimes (1 \lambda) \otimes I_2^{\otimes (m-i-1)}) f\).

When \(f\) is the indicator function of the cutset space of a graph, the minor \(f\|_{[\mu]} e\) amounts to deletion when \(\mu = 1\) and contraction when \(\mu = -1\). See [11, §2, §6], and also [8] for the first definition of generalised minor operations interpolating between deletion and contraction (albeit with a different parameterisation to that used here and in [11]). This work has its roots in [7], where deletion and contraction are expressed in terms of indicator functions of cutset spaces, and these operations are extended to general binary functions.

The relationship between transforms and minors for binary functions is as follows.
Theorem 2 \textit{[11, Theorem 6.1]} If $e \in E(f)$ and $\mu, \nu \in \mathbb{C}$ then
\[(L^{[\mu]}f) \parallel_{[\nu]} e \simeq L^{[\mu]}(f \parallel_{[\mu\nu]} e).\]

This may be compared with Theorem 1. In particular, we have
\[(L^{[\omega]}f) \parallel_{[1]} e \simeq L^{[\omega]}(f \parallel_{[\omega]} e),\]
\[(L^{[\omega]}f) \parallel_{[\omega]} e \simeq L^{[\omega]}(f \parallel_{[\omega]} e),\]
\[(L^{[\omega]}f) \parallel_{[\omega^2]} e \simeq L^{[\omega]}(f \parallel_{[1]} e).\]

4 Alternating dimaps and binary functions

The relationship described above between the transform $L^{[\omega]}$ (called the \textit{trinity transform} or \textit{triality transform}) and the minor operations for binary functions follows the same pattern as the relationships between triality and minors for alternating dimaps, given in Theorem 1. It is natural to ask what connection there may be between the two.

For binary functions, the minor operations always commute \textit{[8, Lemma 4]}. In fact, that result implies that every binary function is \textit{totally reduction-commutative}, meaning that for any set of reductions $\bullet \parallel_{[\mu_i]} e_i$ on distinct edges $e_i$, any ordering of the reductions gives the same minor (borrowing some terminology from \textit{[13]}). But, as we saw in \textit{[13, §3]}, the minor operations for alternating dimaps do not always commute. It follows that alternating dimaps, along with triality and minor operations, cannot be represented faithfully by binary functions with their trinity transform and minor operations described above.

Nonetheless, we can ask if there is a subclass of alternating dimaps which can be represented faithfully by binary functions in this way. For this to occur, this subclass must consist only of alternating dimaps that are totally reduction-commutative. Such alternating dimaps were characterised in \textit{[13, Theorem 17]}; the subclass we seek must be a subset of those.

Later we will give a definition of faithful representation by binary functions, and determine when such a representation is possible. To do the latter, it will help to characterise those binary functions for which any $\mu$-reduction ($\mu \in \{1, \omega, \omega^2\}$), on any element of the ground set, gives the same result.

5 Degeneracy for binary functions

We now extend the term “degenerate” to any type of combinatorial structure on which some kind of minor operations are defined: an element is \textit{degenerate} if all reductions of it, using minor operations, give the same object. Any real understanding of a particular type of combinatorial structure with minors can be expected to depend, in part, on understanding the degenerate elements.
For alternating dimaps, the degenerate elements are the triloops (including ultrlaloops).

In this section, we consider degenerate elements for binary functions under the three minor operations. To do this, we need some more notation.

Throughout, we write

\[ i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad H = (h_0, \ldots, h_{k-1}) \in \{i, j\}^{0, \ldots, k-1}. \]

For each \( i \), \( H^i = (h_0, \ldots, h_{i-1}, h_{i+1}, \ldots, h_{k-1}) \) is the sequence obtained from \( H \) by omitting the term indexed by \( i \).

For each \( H \), define the sequence \( G = G(H) = (g_0, \ldots, g_{k-1}) \) by

\[ g_i = \begin{cases} 0, & \text{if } h_i = i; \\ 1, & \text{if } h_i = j. \end{cases} \]

The sequence obtained from this by omitting the term indexed by \( i \) is

\[ G^i = (g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_{k-1}). \]

The subsequence \((g_{i_1}, \ldots, g_{i_2})\) of \( G \) is denoted by \( G[i_1..i_2] \).

If \( b \in \{0, 1\} \), then \( G : i \leftarrow b \) denotes the sequence obtained by inserting \( b \) between the \( i \)-th and \((i + 1)\)-th elements of \( G \):

\[ G : i \leftarrow b = (g_0, \ldots, g_{i-1}, b, g_i, \ldots, g_{k-1}). \]

The two-element vector \( f_{G,i} \) is defined by

\[ f_{G,i} = \begin{pmatrix} f_{G;i\leftarrow 0} \\ f_{G;i\leftarrow 1} \end{pmatrix}. \]

Write \( u \) for a \( 2^k \)-element vector indexed by the numbers \( 0, \ldots, 2^k-1 \) — or, equivalently, by vectors of \( k \) bits, or by subsets of \( \{0, \ldots, k-1\} \).

For a given \( G \), we write \( u_G \) for the entry of \( u \) whose index has binary representation given by \( G \), i.e., whose index is \( \sum_{i=0}^{k-1} g_i 2^{k-1-i} \).

It is routine to show that, if \( m \geq 1 \) and \( u \) is a (vector representation of a) binary function with ground set of size \( m - 1 \), then

\[ u = \sum_H (h_0 \otimes \cdots \otimes h_{m-2}) u_{G(H)}. \quad (3) \]

If \( m = 1 \) then there is a single \( H \) to sum over, consisting of the empty sequence, and the empty product \( h_0 \otimes \cdots \otimes h_{m-2} \) is the trivial single-element vector \( (1) \). Also \( G = G(H) \) is the empty bit-sequence, representing the number 0, and \( u_G = u_0 = 1 \), so \( u = (1) \), as expected.
Theorem 3 Let $\mu_1, \mu_2 \in \mathbb{C} \setminus \{3 + 2\sqrt{2}\}$ be distinct. Suppose $f$ and $u$ are binary functions of dimension $m$ and $m - 1$ respectively. Then

$$f||_{\mu_1} e_i = f||_{\mu_2} e_i = u$$

if and only if for all $G \in \{0, 1\}^{\{0, \ldots, m-2\}}$ and all $b \in \{0, 1\}$,

$$f_{G;b} = f_{0;b} u_G.$$  \hspace{1cm} (4)

Proof. Let us write the hypothesis as a set of equations, using (1) and (2). The condition that $f||_{\mu_j} e_i = u$ for all $j \in \{1, 2\}$ is equivalent to the assertion that, for each such $j$, there exists $c_{ij}$ such that

$$(I_2^{\otimes i} \otimes (1 \otimes I_2^{\otimes (m-i-1)}) f = c_{ij} u,$$  \hspace{1cm} (5)

where $\lambda_j := \lambda(\mu_j)$. Note that $\lambda_1 \neq \lambda_2$.

Put

$$R = \begin{pmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{pmatrix} \quad \text{and} \quad c_i = \begin{pmatrix} c_{i1} \\ c_{i2} \end{pmatrix}.$$  

The equations (5) may be written (using (3)),

$$(I_2^{\otimes i} \otimes R \otimes I_2^{\otimes (m-i-1)}) f = \sum_H (h_0 \otimes \cdots \otimes h_{i-1} \otimes c_i \otimes h_i \otimes \cdots \otimes h_{m-2}) u_G.$$  \hspace{1cm} (6)

Here and below we write $G = G(H)$ for brevity. We may write

$$f = \sum_H (h_0 \otimes \cdots \otimes h_{i-1} \otimes f_{G;i} \otimes h_i \otimes \cdots \otimes h_{m-2}),$$

so the left-hand side of (6) is

$$(I_2^{\otimes i} \otimes R \otimes I_2^{\otimes (m-i-1)}) \sum_H (\bigotimes_{k=0}^{i-1} h_k \otimes f_{G;i} \otimes \bigotimes_{k=i}^{m-2} h_k)$$

$$= \sum_H (I_2^{\otimes i} \otimes R \otimes I_2^{\otimes (m-i-1)}) (\bigotimes_{k=0}^{i-1} h_k \otimes f_{G;i} \otimes \bigotimes_{k=i}^{m-2} h_k)$$

$$= \sum_H (\bigotimes_{k=0}^{i-1} I_2 h_k \otimes R f_{G;i} \otimes \bigotimes_{k=i}^{m-2} I_2 h_k)$$

$$= \sum_H (\bigotimes_{k=0}^{i-1} h_k \otimes R f_{G;i} \otimes \bigotimes_{k=i}^{m-2} h_k).$$

Setting this equal to the right-hand side of (6) and equating appropriate elements, we find that (6) is equivalent to

$$R f_{G;i} = c_i u_G, \quad \text{for all } G \in \{0, 1\}^{\{0, \ldots, m-2\}}.$$  \hspace{1cm} (7)
When $G = 0$, (7) and $u_0 = 1$ give

$$R f_{0;i} = c_i. \quad (8)$$

For all $G \in \{0, 1\}^{0, \ldots, m-2}$, we have

$$R f_{G;i} = c_i u_G \iff R f_{G;i} = R f_{0;i} u_G \quad \text{(using (8))}$$

$$\iff f_{G;i} = f_{0;i} u_G \quad \text{(since $R$ is invertible, because the $\lambda_j$ are distinct)}$$

$$\iff \forall b \in \{0, 1\} : f_{G;i\leftarrow b} = f_{0;i\leftarrow b} u_G.$$

Using this result, we can characterise degenerate elements in a binary function as those elements $e_i$ such that the binary function entries whose index vectors differ only in position $i$ have constant ratio or are both zero.

**Corollary 4** Let $\mu_1, \mu_2 \in \mathbb{C} \setminus \{3 + 2\sqrt{2}\}$ be distinct. Let $f$ be a binary function of dimension $m$. The following are equivalent.

(a) The element $e_i$ is degenerate in $f$ with respect to the minor operations $\bullet ||_{[\mu_1]}$ and $\bullet ||_{[\mu_2]}$.

(b) For all $G \in \{0, 1\}^{0, \ldots, m-2}$

$$f_{G;i\leftarrow 1} f_{0;i\leftarrow 0} = f_{G;i\leftarrow 0} f_{0;i\leftarrow 1}. \quad (9)$$

(c) For all $G \in \{0, 1\}^{0, \ldots, m-2}$ either

$$f_{G;i\leftarrow 1} f_{0;i\leftarrow 0} = f_{0;i\leftarrow 1} f_{0;i\leftarrow 0} \quad (10)$$

or

$$f_{G;i\leftarrow 0} = f_{G;i\leftarrow 1} = 0. \quad (11)$$

**Proof.** ((a) $\iff$ (b)): Degeneracy means that

$$f ||_{[\mu_1]} e_i = f ||_{[\mu_2]} e_i =: u,$$

where $u$ has dimension $m - 1$. By Theorem 3 degeneracy is equivalent to the assertion that, for all $G \in \{0, 1\}^{0, \ldots, m-2}$ and all $b \in \{0, 1\}$,

$$f_{G;i\leftarrow b} = f_{0;i\leftarrow b} u_G. \quad (12)$$

Putting $b = 0$ we have $u_G = f_{G;i\leftarrow 0} / f_{0;i\leftarrow 0}$; substituting this into (12) with $b = 1$ gives (9). Conversely, if (9) holds for all $G \in \{0, 1\}^{0, \ldots, m-2}$, we put $u_G := f_{G;i\leftarrow 0} / f_{0;i\leftarrow 0}$ for each $G$, which gives $f_{G;i\leftarrow 0} = f_{0;i\leftarrow 0} u_G$ immediately, and $f_{G;i\leftarrow 1} = f_{0;i\leftarrow 1} u_G$ after
substitution into (9). Since (12) then holds for all $G$ and $b$, we conclude that $e_i$ is degenerate, by Theorem 3.

(c) ⇒ (b) is immediate.

((b) ⇒ (c)): Suppose (9) holds for all $G$. Then $f_{G;i-1} = f_{G;i-0} = 0$ or $f_{G;i-1} = f_{0;i-1} = 0$ or both sides of (9) are nonzero, in which case we have

$$\frac{f_{G;i-1}}{f_{G;i-0}} = \frac{f_{0;i-1}}{f_{0;i-0}}.$$ 

But this last equation subsumes $f_{G;i-1} = f_{0;i-1} = 0$, so we have (c).

We see from these results that any two distinct minor operations give the same notion of degeneracy, and hence the same goes for any set of minor operations.

It is instructive to consider the case of binary matroids, with $f$ now the indicator function of the cocircuit space. A loop is an element $e_i$ that belongs to no member of the cocircuit space. So, $f(X) = 0$ if $e_i \in X$; expressed in the above manner, this is $f_{G;i-1} = 0$ for all $G$, including the case $G = 0$ where $f_{0;i-1} = 0$ (i.e., a loop itself is not a member of the cocircuit space). This ensures that (9) holds. A coloop is an element $e_i$ such that, for all $X \subseteq E \setminus \{e_i\}$, $X$ belongs to the cocircuit space if and only if $X \cup \{e_i\}$ does. This includes the case $X = \emptyset$, which always belongs to the cocircuit space, so $\{e_i\}$ does too. It follows that $f_{G;i-1} = f_{G;i-0}$, for all $G$, which includes $f_{0;i-1} = f_{0;i-0} = 1$. If $f_{G;i-1} = f_{G;i-0} = 0$, then (9) holds since both sides are 0. If $f_{G;i-1} = f_{G;i-0} = 1$ then (9) holds since every quantity is 1. Conversely, suppose (9) holds for all $G$. If $f_{0;i-1} = 0$, then (9) implies that for all $G$ we have $f_{G;i-1} = 0$, so $e_i$ is a loop. If $f_{0;i-1} = 1$, then (9) implies $f_{G;i-1} = f_{G;i-0}$ for all $G$, so $e_i$ is a coloop.

6 Strict binary representations

We now define our notion of faithful representation, and then determine when it is possible.

Definition

A strict binary representation of a minor-closed set $\mathcal{A}$ of alternating dimaps is a triple $(F, \varepsilon, \nu)$ such that

(a) $F : \mathcal{A} \to \{\text{binary functions}\}$

(b) $\varepsilon = (\varepsilon_G \mid G \in \mathcal{A})$ is a family of bijections $\varepsilon_G : E(G) \to E(F(G))$;

(c) $\nu \in \mathbb{C}$ with $|\nu| = 1$;

(d) $F(G^\omega) \simeq L[\omega] F(G)$ for all $G \in \mathcal{A}$;

(e) $F(G[\mu]e) \simeq F(G) \|_{[\nu \mu]} \varepsilon_G(e)$ for all $G \in \mathcal{A}$, $e \in E(G)$ and $\mu \in \{1, \omega, \omega^2\}$. 

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We may interpret this definition as follows. For each alternating dimap $G$, its corresponding binary function is $F(G)$, and the correspondence between edges of $G$ and the elements of the ground set of $F(G)$ is given by $\varepsilon_G$. We require that triality of alternating dimaps corresponds to the trinity transform for binary functions (condition (d)) and that minor operations correspond too (condition (e)). The role of $\nu$ (condition (c)) is simply to allow us to be a little relaxed about which binary function reduction is used to represent each alternating dimap reduction. There is not much room to move here; $\nu$ captures what little room to move there is.

Let $C_1$ denote the ultraloop. We write $U_k = \{iC_1 \mid i = 0, \ldots, k\}$ and $U_\infty = \{iC_1 \mid i \in \mathbb{N} \cup \{0\}\}$, where $0C_1$ is the empty alternating dimap.

**Theorem 5** If $\mathcal{A}$ is a minor-closed class of alternating dimaps which has a strict binary representation then $\mathcal{A} = \emptyset$, or $\mathcal{A} = U_k$ for some $k$, or $\mathcal{A} = U_\infty$.

**Proof.** Suppose $(F, \varepsilon, \nu)$ is a strict binary representation of $\mathcal{A}$.

The theorem is immediately true if $\mathcal{A} = \emptyset$. So suppose $\mathcal{A} \neq \emptyset$.

If $|\mathcal{A}| \geq 1$ then, since it is minor-closed, it must contain the empty alternating dimap $C_0$, and the image $F(C_0)$, representing $C_0$ as a binary function, must be the binary function $f : 2^\emptyset \to \mathbb{C}$ defined by $f(\emptyset) = 1$.

So, if $|\mathcal{A}| = 1$ then $\mathcal{A} = U_0$, and the previous paragraph gives a strict binary representation of $\mathcal{A}$.

Similarly, if $|\mathcal{A}| \geq 2$, then it must contain the ultraloop $C_1$, since that is the only alternating dimap on one edge.

Claim 1: The image $F(C_1)$ of the ultraloop $C_1$ is given by

$$F(C_1) = \left( \frac{1}{1 - \sqrt{2}} \right).$$

Proof:

$F(C_1)$ must be some binary function $f$ on a singleton ground set, $E = \{e\}$ say, with $f(\emptyset) = 1$ and $f(\{e\}) = u$ for some $u \in \mathbb{C}$. Since $C_1$ is self trial, so must $f$ be (by (d) above). This means that its vector form $\mathbf{f} = \left( \begin{array}{c} 1 \\ u \end{array} \right)$ must be an eigenvector for eigenvalue 1 of the matrix $M(\omega)$. Now this matrix has eigenvalues 1 and $\omega$, and the eigenvectors for the former are the scalar multiples of

$$\left( \begin{array}{c} 1 \\ \sqrt{2} - 1 \end{array} \right).$$

So this is $F(C_1)$, and $u = \sqrt{2} - 1$. So Claim 1 is proved.

If $|\mathcal{A}| = 2$ then $\mathcal{A}$ consists of just the empty alternating dimap and the ultraloop. The $F$ given by Claim 1, together with appropriate identity maps $\varepsilon$ (and, in fact, any
ν), gives a strict binary representation. So we are done in this case.

It remains to deal with |A| ≥ 3, when A contains at least one alternating dimap on two edges.

Claim 2: The only binary function f with the property that every reduction, on any of the elements of its ground set, gives \( u = F(C_1)^{\otimes k} \), is \( f = F(C_1)^{\otimes (k+1)} \).

Proof:
Observe that, by Claim 1, \( u = (u_G \mid G \in \{0, 1\}^E) \) where \( u_G = (\sqrt{2} - 1)^{|G|} \), where \(|E| = k\) and |G| is the number of 1s in G.

Applying Theorem 3 for all \( i \in \{0, \ldots, k\} \), to \( u = F(C_1)^{\otimes k} \) gives

\[
G: i \leftarrow b \gets f(0; i) \leftarrow b u_G.
\]

Hence, for each \( i \) and each \( G \),

\[
f_G: i \leftarrow 0 = f_{0_k} u_G = u_G = (\sqrt{2} - 1)^{|G|}.
\]

Now consider \( f_G: i \leftarrow 1 \). Put \( j := 0 \) if \( i \neq 0 \) and \( j := 1 \) otherwise (so \( j \neq i \)). Then

\[
f_G: i \leftarrow 1 = f_{0_k} u_G = f_{0_k; i} u_G = f_{0_k}; j \leftarrow 0 u_{0_k; i} \leftarrow 1 u_G = f_{0_k} u_{0_k; i} \leftarrow 1 u_G = 1 \cdot (\sqrt{2} - 1) \cdot (\sqrt{2} - 1)^{|G|} = (\sqrt{2} - 1)^{|G|+1}.
\]

It follows that, for all \( G' \in \{0, 1\}^{k+1} \),

\[
f_G' = (\sqrt{2} - 1)^{|G'|}.
\]

Therefore

\[
f = F(C_1)^{\otimes (k+1)},
\]

proving the Claim.

Claim 3: If \( k \geq 2 \) and every reduction of \( G \) is \( kC_1 \), then \( G = (k + 1)C_1 \).

Before proving the claim, consider the case \( k = 1 \), which it does not cover. Then every alternating dimap on two edges (of which there are four) has the claimed property. Of these, the only self-trial one is \( 2C_1 \).

Proof:
Suppose \( k = 2 \). If \( G \) is connected, then there must be some \( e \in E(G) \) and some \( \mu \in \{1, \omega, \omega^2\} \) such that \( G[\mu]e = 2C_1 \) and is therefore disconnected. The only way in which \( \mu \)-reducing a single edge can disconnect a connected alternating dimap is if the edge is a proper \( \mu^{-1} \)-semiloop. It is easily determined that the only alternating dimaps on three edges which have this property are those consisting of two triloops and a semiloop. These do not have three proper semiloops. So, although they have
the specified property for one of their edges, they do not have it for all of their edges. So $G$ must be disconnected. Since $G$ has only three edges, some component of $G$ must be an ultraloop. But this disappears when reduced, so the rest of $G$ must be $2C_1$, so $G = 3C_1$.

Now suppose $k \geq 3$. It is impossible for $G$ to be connected, because no reduction of any edge of any connected alternating dimap can possibly break it up into three or more components. So consider the components of $G$. If any of these is not an ultraloop, then it has at least two edges, and also is left unchanged by reduction of any edge in any other component (of which there must be at least one), so we would have a reduction of $G$ that does not give $kC_1$, which is a contradiction. So every component of $G$ must be an ultraloop. Each of these just disappears on reduction, giving $kC_1$, as desired.

So Claim 3 is proved.

Claim 4: For all $k \geq 0$, either $\mathcal{A}$ has no members with $k$ edges, or it has just one such member which is $kC_1$.

Proof:
We prove the claim by induction on $k$.
We have seen that this is true already for $k \leq 1$.

Suppose $k = 2$. Every alternating dimap $G_2$ on two edges has the property that every reduction of it gives the ultraloop. Therefore, if $G_2 \in \mathcal{A}$ then $F(G_2) = F(C_1)^{\otimes 2}$, using Claim 2. But $F(C_1)^{\otimes 2}$ is self-trial, since $F(C_1)$ is. Therefore $G_2$ must be self-trial too. But the only self-trial alternating dimap on two edges is $2C_1$. So the only member of $\mathcal{A}$ with two edges is $2C_1$.

Now suppose it is true regarding members of $\mathcal{A}$ with $k - 1$ edges, where $k \geq 3$. We show that it is true for $k$ edges.

If $\mathcal{A}$ has no members with $k - 1$ edges, then it can have no members with $k$ edges either, since it is minor-closed.

If $\mathcal{A}$ has at least one member with $k - 1$ edges, then by the inductive hypothesis it can have only one such member, and this must be $(k - 1)C_1$. We must show that, if $\mathcal{A}$ has at least one member with $k$ edges, then it can have only one, and it is $kC_1$.

Let $G$ be a member of $\mathcal{A}$ with $k$ edges. Since $\mathcal{A}$ is minor-closed and has $(k - 1)C_1$ as its only member with $k - 1$ edges, all reductions of $G$ must give $(k - 1)C_1$. So, by the requirements of a strict binary representation, all reductions of $F(G)$ must give $F(C_1)^{\otimes (k-1)}$. This implies that $F(G) = F(C_1)^{\otimes k}$, by Claim 3. This completes the proof of Claim 4.

It follows from Claim 4 that $\mathcal{A}$ can only be one of the classes given in the statement of the theorem. It remains to establish that a strict binary representation is possible for each of those classes. This is routine, using

$$F(kC_1) = \left( \frac{1}{\sqrt{2} - 1} \right)^{\otimes k}$$
for every $k$ for which $kC_1 \in \mathcal{A}$. Let $\varepsilon$ consist just of identity maps. To show that this does indeed enable a strict binary representation, use Claims 1–3. The details are a routine exercise.

### 7 Future work

It is possible to develop broader definitions of binary representations of classes of alternating dimaps. For example, we could allow the edges of $G$ to be represented by disjoint subsets of the ground set of $F(G)$ instead of just by distinct single elements. This suggests the problem of characterising those minor-closed classes of alternating dimaps that have binary function representations of a more general type, such as that suggested above.

There may be other ways of representing alternating dimap reductions in a linear way, using matrices that do not necessarily commute. We have not ruled out the possibility that alternating dimaps may indeed be representable as binary functions, but using other ways of representing the alternating dimap reductions $\bullet [\mu] e$ (rather than the binary function reductions $\bullet [\mu] e$ we have studied so far).

### References

[1] K. A. Berman, A proof of Tutte’s trinity theorem and a new determinant formula, *SIAM J. Alg. Disc. Meth.* 1 (1980) 64–69.

[2] A. Bouchet, Isotropic systems. *European J. Combin.* 8 (1987) 231–244.

[3] A. Bouchet, Multimatroids. II. Orthogonality, minors and connectivity, *Electron. J. Combin.* 5 (1998), Research Paper 8, 25pp.

[4] R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte, The dissection of rectangles into squares. *Duke Math. J.* 7 (1940) 312–340.

[5] R. L. Brooks, C. A. B. Smith, A. H. Stone and W. T. Tutte, Leaky electricity and triangulated triangles, *Philips Res. Repts.* 30 (1975) 205–219.

[6] R. Cori and J. G. Penaud, The complexity of a planar hypermap and that of its dual, *Ann. Discrete Math.* 9 (1980) 53–62.

[7] G. E. Farr, A generalization of the Whitney rank generating function, *Math. Proc. Camb. Phil. Soc.* 113 (1993) 267–280.

[8] G. E. Farr, Some results on generalised Whitney functions, *Adv. in Appl. Math.* 32 (2004) 239–262.

[9] G. E. Farr, Tutte-Whitney polynomials: some history and generalizations, in: G. R. Grimmett and C. J. H. McDiarmid (eds.), *Combinatorics, Complexity and Chance: A Tribute to Dominic Welsh*, Oxford University Press, 2007, pp. 28–52.
[10] G. E. Farr, On the Ashkin-Teller model and Tutte-Whitney functions, *Combin. Probab. Comput.* **16** (2007) 251–260.

[11] G. E. Farr, Transforms and minors for binary functions, *Ann. Combin.* **17** (2013) 477–493.

[12] G. E. Farr, Minors for alternating dimaps, preprint, 2013, http://arxiv.org/abs/1311.2783.

[13] G. E. Farr, Minors for alternating dimaps, *Q. J. Math.*, to appear.

[14] G. E. Farr and A. Y. Z. Wang, Powerful sets: a generalisation of binary matroids, preprint, 2017, https://arxiv.org/abs/1705.07437.

[15] T. Kálmán, A version of Tutte’s polynomial for hypergraphs *Adv. Math.* **244** (2013) 823–873.

[16] J. P. S. Kung, The Rédei function of a relation, *J. Combin. Theory (Ser. A)* **29** (1980) 287–296.

[17] J. G. Oxley and D. J. A. Welsh, The Tutte polynomial and percolation, in: J. A. Bondy and U. S. R. Murty (eds.), *Graph Theory and Related Topics*, Academic Press, New York, 1979, pp. 329–339.

[18] L. Traldi, The transition matroid of a 4-regular graph: an introduction, *European J. Combin.* **50** (2015), 180–207.

[19] W. T. Tutte, The dissection of equilateral triangles into equilateral triangles, *Proc. Cambridge Philos. Soc.* **44** (1948) 463–482.

[20] W. T. Tutte, Duality and trinity, in: *Infinite and Finite Sets (Colloq., Keszthely, 1973), Vol. III*, Colloq. Math. Soc. Janos Bolyai, Vol. 10, North-Holland, Amsterdam, 1975, pp. 1459–1472.

[21] W. T. Tutte, *Graph Theory as I Have Known It*, Oxford University Press, 1998.

[22] W. T. Tutte, Bicubic planar maps, *Symposium à la Mémoire de François Jaeger* (Grenoble, 1998), *Ann. Inst. Fourier (Grenoble)* **49** (1999) 1095–1102.