TEAM SEMANTICS AND INDEPENDENCE NOTIONS IN QUANTUM PHYSICS

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Abstract

We study dependence and independence concepts found in quantum physics, especially those related to hidden variables and non-locality, through the lens of team semantics and probabilistic team semantics, adapting a relational framework introduced in [1]. This leads to new developments also in independence logic and probabilistic independence logic.

1 Introduction

The semantics of first order logic is based on the inductively defined concept of an assignment satisfying a given formula in a given model. In a more general approach, called team semantics, the basic concept is that of a set of assignments satisfying a given formula in a given model. This allows consideration of new atomic formulas such as “$x$ is totally determined by $y_1,\ldots,y_n$” and “$x_1,\ldots,x_n$ are independent of $y_1,\ldots,y_m$”. Such constraints on variables appear throughout sciences but in experimental sciences in particular. In this paper we apply team semantics to investigate determinism and independence concepts in quantum physics, following very closely [1]. In an independent development, R. Albert and E. Grädel have in their paper [3] come to many of the same conclusions.

The indeterministic nature of quantum mechanics, since its conception, has challenged the deterministic view of the world. To retain a more classical looking picture, several hidden-variable models for quantum mechanics—that would explain the probabilistic behaviour in terms of a deterministic theory—have been proposed since the 1920s. These models try to explain the predictions of quantum mechanics by adding unobservable hidden variables that play a role in determining

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the state of a quantum system. And indeed, if no constraints are posed on how the hidden variables can act—for instance, if the hidden variables are allowed to influence which measurements we make—then we can certainly come up with a hidden-variable explanation of anything. However, in order to form a reasonable, satisfactory theory, one needs to require that the hidden variable models satisfy some combination of natural properties such as Bell locality. A critical challenge for the hidden-variable program then emerged in the form of the famous no-go theorems by Bell and others [5, 11, 14, 20]: they showed that models satisfying what are generally regarded as reasonable assumptions could provably never account to the predictions of quantum mechanics.

The first author introduces in [1] a relational framework for developing the key notions and results on hidden variables and non-locality, which can be seen as a relational variant of the probabilistic setting of Brandenburger and Yanofsky [6]. He introduces what he calls “relational empirical models” and uses them to show that the basic results of the foundations of quantum mechanics, usually formulated in terms of probabilistic models, can be seen already on the level of mere (two-valued) relations. Our basic observation is that we can think of the relational empirical models of [1] as teams in the sense of team semantics. The basic quantum theoretical properties of relational empirical models can then be defined in terms of the independence atoms of independence logic [13]. We observe that the relationships between quantum theoretic properties of relational models become instances of logical consequence of independence logic in its team semantics. In fact, the existential-positive-conjunctive fragment suffices. The No-Go theorems become instances of failure of logical consequence between specific formulas of independence logic. This extends also to probabilistic models with independence logic replaced by the probabilistic independence logic of [10], capturing the probabilistic notions of [6].

Logical consequence in independence logic is in general non-axiomatizable. Even on the level of atoms no finite axiomatization exists [22]. This shows that the concept of logical consequence is here highly non-trivial and potentially quite complex. It should be emphasised that the logical consequences arising from the quantum theoretical examples are purely logical i.e. have a priori nothing to do with quantum mechanics, hence they apply to any other field where independence plays a role, e.g. the theory of social choice or biology. On the other hand, the first author introduces in [1] a concept which in team semantics characterizes teams which arise (potentially) from a quantum mechanical experiment. Presumably the most subtle relationships between quantum mechanical concepts are particular to such quantum theoretic teams. We introduce, following the example of [1], the concept “quantum mechanical” to probabilistic independence logic and propose questions it gives rise to.
We think translating [1] to the language and terminology of team semantics is interesting in itself from the point of view of team semantics. However, our paper goes beyond this. We use the language of independence logic to express hidden variable properties of empirical models. Respectively, we use probabilistic independence logic to express properties of probabilistic hidden variable models. This calls for some new developments in independence logic itself. For example, we use the existential quantifier of independence logic to guess values of hidden variables, but since the values may be outside the current domain (model) we introduce to independence logic the existential quantifier of sort logic, which allows the extension of the domain by new sorts. Relations between hidden variable properties can be seen as logical consequences in independence logic. In some cases these logical consequences are provable from the axioms. We use probabilistic independence logic to express probabilistic hidden variable properties and their mutual relationships. We prove the probabilistic validity of axioms and rules of independence logic, so the relationships of probabilistic hidden variable models that follow from the axioms of independence logic hold also probabilistically. We introduce an operator $\text{PR} \varphi$ which holds in a team if and only if the team is the possibilistic collapse of a probabilistic team satisfying $\varphi$. Adopting the concept of a quantum realizable team from [1] we introduce the operator $\text{QR} \varphi$ which holds in a team if and only if the team is the possibilistic collapse of a quantum realizable probabilistic team satisfying $\varphi$. We take the first step towards developing independence logic with the operators $\text{PR}$ and $\text{QR}$.

This paper is part of a program to find general principles that govern the uses of dependence and independence concepts in science and humanities.

We are grateful to Philip Dawid for suggesting to use the separoid axioms and their probabilistic validity to prove the probabilistic validity of our independence axioms.

1.1 Dependence Logic

The basic concept of the semantics of first order logic is the concept of an assignment. The concept of a team was introduced in [25] to make sense of the dependence atom $= (\vec{x}, \vec{y})$, “$\vec{x}$ totally determines $\vec{y}$”. One of the intuitions behind the concept of a team is a set of observations, such as readings of physical measurements. Let us consider a system consisting of experiments

$q_0, \ldots, q_{n-1}$.

Each experiment $q_i$ has an input $x_i$ and an output $y_i$. The input could be the exact time of the experiment, the exact location of a measurement, turning a nob to a certain position, pouring an amount of a liquid to a test tube, choosing the angle of a magnetic field, etc. The output of the experiment can be “true” or “false”, a
reading of a gauge, number of clicks in a detector, etc. After \( m \) rounds of making the experiments \( q_0, \ldots, q_{n-1} \) we have the data

\[
X = \begin{bmatrix}
  x_0 & y_0 & \ldots & x_{n-1} & y_{n-1} \\
  a_0^0 & b_0^0 & \ldots & a_0^{n-1} & b_0^{n-1} \\
  a_1^0 & b_1^0 & \ldots & a_1^{n-1} & b_1^{n-1} \\
  \vdots & \vdots & & \vdots & \vdots \\
  a_{m-1}^0 & b_{m-1}^0 & \ldots & a_{m-1}^{n-1} & b_{m-1}^{n-1}
\end{bmatrix}
\]

We can think of \( X \) as a team (in the sense of team semantics) consisting of assignments of values to the variables \( x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1} \). Even though the data has a clear structure dividing the elements of the table into “inputs” and “outputs”, we can also look at the table as a mere database of data irrespective of how it was created. We can ask what kind of dependences this matrix of data—team—manifests.

The meaning of the dependence atom \( = (\vec{x}, \vec{y}) \) of [25] on a team \( X \) is

\[
\forall s, s' \in X (s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})).
\]

Thus we can say that the team of data \( X \) supports strong determinism if it satisfies

\[
= (x_i, y_i)
\]

for all \( i < n \). Intuitively, in each experiment the input completely determines the output, that is, the output does not, in the light of \( X \), depend on anything else than the input. This limits the applicability of this concept. Probably the experiments \( q_0, \ldots, q_{n-1} \) each have to have quite substantial inputs. In systems arising from scientific experiments it is most common that the result of an experiment depends on a number of things. For example, if one takes measurements related to the weather, each component of the system (air pressure, wind direction, rainfall, cloudiness, thunder, location, time of the day, day of the month, month of the year) is likely to somehow be dependent on each other, perhaps bound by one set of complicated partial differential equations.

Respectively, we say that the team \( X \) supports weak determinism if it satisfies

\[
= (x_0, \ldots, x_{n-1}, y_i)
\]

for all \( i < n \). Intuitively, the inputs of all the experiments of the system collectively completely determine the output, that is, the output does not, in the light of \( X \), depend on anything else than the inputs of the system. In systems arising from scientific experiments this means that the system has enough “variables” to determine its outcome. The more experiments the more likely it is that the data
supports weak determinism. On the other hand, the bigger the data the more likely it is to reveal that not enough input variables have been designed into the system.

1.2 **Independence Logic**

There are important aspects of experimental data that cannot be expressed in terms of the dependence atom only. We therefore move on to a stronger concept, one that supersedes dependence and allows to express also independence.

In independence logic [13], we add a new atomic formula

$$\vec{y} \perp_{\vec{x}} \vec{z}$$

to first order logic. Intuitively this formula says that keeping $\vec{x}$ fixed, $\vec{y}$ and $\vec{z}$ are independent of each other. A team $X$ is defined to satisfy $\vec{y} \perp_{\vec{x}} \vec{z}$ if

$$\forall s, s' \in X [(s(\vec{x}) = s'(\vec{x})) \implies \exists s'' \in X (s''(\vec{x}) = s(\vec{x}) \wedge s''(\vec{y}) = s(\vec{y}) \wedge s''(\vec{z}) = s'(\vec{z})].$$

We may observe that, unlike $=\vec{(x_0, \ldots, x_{n-1}, y_i)}$, this is not closed downward, but it is closed under unions of increasing chains. Note that this atomic formula is first-order, in particular $\Sigma_1^1$ (and hence NP), in the same sense as formulas of dependence logic. Here is an example of a team satisfying $y_0 \perp_{x_0, x_1} y_1$:

| $x_0$ | $y_0$ | $x_1$ | $y_1$ |
|-------|-------|-------|-------|
| 0     | 0     | 0     | 0     |
| 0     | 1     | 1     | 2     |
| 0     | 1     | 1     | 7     |
| 0     | 5     | 1     | 2     |
| 0     | 5     | 1     | 7     |
| 1     | 5     | 1     | 1     |

For fixed $x_0$ and $x_1$, e.g. $x_0 = 0, x_1 = 1$, the values of $y_0$ and $y_1$ are independent of each other in the strong sense that if a value of $y_0$ occurs in combination with any value of $y_1$, e.g. 2, it occurs also with any other value of $y_1$, e.g. 7. Intuitively this says that in these experiments the individual experiments do not interfere with each other.

Note that the dependence atom can be defined in terms of the independence atom:

$$=\vec{(x, y)} \equiv \vec{y} \perp_{\vec{x}} \vec{y}.$$ 

We will thus use $=\vec{(x, y)}$ as a shorthand for $\vec{y} \perp_{\vec{x}} \vec{y}$ when dealing with independence logic.
To rigorously define the semantics of independence logic—an extension of first-order logic by the independence atom—we need to be more precise about our definitions. For the sake of some technical details later on, we consider team semantics in the context of many-sorted structures.

**Definition 1.1.** A (many-sorted) *vocabulary* $\tau$ is a tuple $(\mathrm{sor}_\tau, \mathrm{rel}_\tau, \mathbf{a}_\tau, \mathbf{s}_\tau)$ such that

(i) $\mathrm{rel}_\tau$ is a set of relation symbols\(^1\) and $\mathrm{sor}_\tau \subseteq \mathbb{N}$,

(ii) $\mathbf{a}_\tau : \mathrm{rel}_\tau \to \mathbb{N}$ and $\mathbf{s}_\tau : \mathrm{rel}_\tau \to \mathbb{N}^{<\omega}$ are functions with $\mathbf{s}_\tau(R) \in \mathbb{N}^{a_\tau(R)}$ for $R \in \mathrm{rel}_\tau$, and

(iii) if $n_i \in \mathbb{N}$, $i < k$, are such that $\mathbf{s}_\tau(R) = (n_0, \ldots, n_{k-1})$ for some $R \in \mathrm{rel}_\tau$, then $n_0, \ldots, n_{k-1} \in \mathrm{sor}_\tau$.

We call $\mathbf{a}_\tau(R)$ the arity of $R$ and $\mathbf{s}_\tau(R)$ the sort of $R$. For $n \notin \mathrm{sor}_\tau$, we say that a vocabulary $\tau'$ is the expansion of $\tau$ by the sort $n$ if

$$
\tau' = (\mathrm{sor}_\tau \cup \{n\}, \mathrm{rel}_\tau, \mathbf{s}_\tau, \mathbf{a}_\tau).
$$

A (many-sorted) $\tau$-*structure* is a function $\mathfrak{A}$ defined on the set $\mathrm{rel}_\tau \cup \mathrm{sor}_\tau$ such that

(i) $\mathfrak{A}(n)$ is a set $A_n$ for $n \in \mathrm{sor}_\tau$ and called the sort $n$ domain of $\mathfrak{A}$, and

(ii) $\mathfrak{A}(R) \subseteq A_{n_0} \times \cdots \times A_{n_{k-1}}$ for $R \in \mathrm{rel}_\tau$, where $\mathbf{s}_\tau(R) = (n_0, \ldots, n_{k-1})$.

If $\tau'$ is an expansion of $\tau$ by sort $n$, we call a $\tau'$-structure $\mathfrak{B}$ an expansion of $\mathfrak{A}$ by the sort $n$ if $\mathfrak{B} \upharpoonright (\mathrm{rel}_\tau \cup \mathrm{sor}_\tau) = \mathfrak{A}$.

We usually denote $\mathfrak{A}(R)$ simply by $R^{\mathfrak{A}}$ and $\mathfrak{A}(n)$ by $A_n$. If $\mathfrak{A}$ only has one sort, then we denote the domain of that sort by $A$ and call it the domain of $\mathfrak{A}$. When there is no risk for confusion, we write $\mathfrak{a}$ and $\mathfrak{s}$ for $\mathbf{a}_\tau$ and $\mathbf{s}_\tau$.

For each sort $n \in \mathbb{N}$, we designate a set $\{v_i^n \mid i \in \mathbb{N}\}$ of variables of sort $n$, although for simplicity of notation we usually use symbols like $x$, $y$ and $z$ for variables and indicate the sort by writing $\mathfrak{s}(x)$ for the sort of $x$.

**Definition 1.2** (Syntax of Independence Logic). The set of of $\tau$-formulas of independence logic is defined as follows.

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\(^1\)We do not consider function symbols, as we do not need them later. They can be handled similarly to relation symbols.
(i) First-order atomic and negated atomic formulas \( u = v, \neg u = v, R(\bar{x}) \) and \( \neg R(\bar{x}) \) are \( \tau \)-formulas, where \( R \in \text{rel}_\tau \), \( \bar{x} = (x_0, \ldots, x_{|\tau|-1}) \) and \( v, u \) and \( x_i \) are variables such that \( s(u), s(v), s(x_i) \in \text{sor}_\tau \), \( s(u) = s(v) \) and \( s(R) = (s(x_0), \ldots, s(x_{|\tau|-1})) \).

(ii) Independence atoms \( \bar{y} \perp_{\bar{x}} \bar{z} \) are \( \tau \)-formulas, where \( \bar{x} = (x_0, \ldots, x_{n-1}), \bar{y} = (y_0, \ldots, y_{m-1}) \) and \( \bar{z} = (z_0, \ldots, z_{l-1}) \), and \( x_i, y_j \) and \( z_k \) are variables such that \( s(x_i), s(y_j), s(z_k) \in \text{sor}_\tau \).

(iii) If \( \varphi \) and \( \psi \) are \( \tau \)-formulas, then so are \( \varphi \land \psi \) and \( \varphi \lor \psi \).

(iv) If \( \varphi \) is a \( \tau \)-formula and \( v \) is a variable with \( s(v) \in \text{sor}_\tau \), then also \( \forall v \varphi \) and \( \exists v \varphi \) are \( \tau \)-formulas.

(v) If \( v \) is a variable such that \( s(v) \notin \text{sor}_\tau \), \( \tau' \) is the expansion of \( \tau \) by the sort \( s(v) \) and \( \varphi \) is a \( \tau' \)-formula, then \( \forall v \varphi \) and \( \exists v \varphi \) are \( \tau \)-formulas.

We call dependence logic the fragment of independence logic where only independence atoms of the form \( \bar{y} \perp_{\bar{x}} \bar{z} \) are allowed.

Here we have introduced new quantifiers \( \forall \) and \( \exists \) which we will interpret as new sort quantifiers. Similar quantifiers, although second order, were defined by the third author in [26].

**Definition 1.3.** Let \( \mathfrak{A} \) be a \( \tau \)-structure and \( D \) a set of variables. An assignment \( s \) of \( \mathfrak{A} \) with domain \( D \) is a function \( D \to \bigcup_{n \in \text{sor}_\tau} A_n \) such that \( s(v) \in A_{s(v)} \) for all \( v \in D \). If \( s \) is an assignment of \( \mathfrak{A} \) with domain \( D \), we write \( s: D \to \mathfrak{A} \). A team \( X \) of \( \mathfrak{A} \) with domain \( D \) is a set of assignments of \( \mathfrak{A} \) with domain \( D \). We denote by \( \text{dom}(X) \) the set \( D \) and by \( \text{rng}(X) \) the set \( \{s(v) \mid v \in D, s \in X\} \). If \( X \) contains every assignment of \( \mathfrak{A} \), we call \( X \) the full team of \( \mathfrak{A} \).

For an assignment \( s: D \to \mathfrak{A} \), a variable \( v \) (not necessarily in \( D \)) and \( a \in A_{s(v)} \), we denote by \( s(a/v) \) the assignment \( D \cup \{v\} \to \mathfrak{A} \) that maps \( v \) to \( a \) and \( w \) to \( s(w) \) for \( w \in D \setminus \{v\} \). If \( \bar{x} = (x_0, \ldots, x_{n-1}) \) is a tuple of variables, we denote by \( s(\bar{x}) \) the tuple \( (s(x_0), \ldots, s(x_{n-1})) \).

Given a team \( X \) of \( \mathfrak{A} \), a variable \( v \) and a function \( F: X \to \mathcal{P}(A_{s(v)}) \setminus \{\emptyset\} \), we denote by \( X[F/v] \) the (supplemented) team \( \{s(a/v) \mid s \in X, a \in F(s)\} \) and by \( X[A_{s(v)}/v] \) the (duplicated) team \( \{s(a/v) \mid s \in X, a \in A_{s(v)}\} \).

**Definition 1.4** (Semantics of Independence Logic). Let \( \mathfrak{A} \) be a (possibly many-sorted) structure and \( X \) a team of \( \mathfrak{A} \). We then define the satisfaction of a formula \( \varphi \) in the structure \( \mathfrak{A} \) with the team \( X \), in symbols \( \mathfrak{A} \models_X \varphi \), as follows.\(^2\)

\(^2\)Strictly speaking, this definition is given in set theory for each quantifier rank separately. A uniform definition is possible if we put an upper bound on the cardinality of models considered.
(i) If \( \varphi \) is first-order atomic or negated atomic formula, then \( \mathfrak{A} \models_X \varphi \) if \( \mathfrak{A} \models_s \varphi \) for all \( s \in X \).

(ii) If \( \varphi = \vec{y} \perp_{\vec{x}} \vec{z} \), then \( \mathfrak{A} \models_X \varphi \) if for any \( s, s' \in X \) with \( s(\vec{x}) = s'(\vec{x}) \) there exists \( s'' \in X \) with \( s''(\vec{x}y) = s(\vec{x}y) \) and \( s''(\vec{z}) = s'(\vec{z}) \).

(iii) If \( \varphi = \psi \land \theta \), then \( \mathfrak{A} \models_X \varphi \) if \( \mathfrak{A} \models_Y \psi \) and \( \mathfrak{A} \models_Z \theta \) for some teams \( Y \) and \( Z \) such that \( Y \cup Z = X \).

(iv) If \( \varphi = \psi \lor \theta \), then \( \mathfrak{A} \models_X \varphi \) if \( \mathfrak{A} \models_Y \psi \) and \( \mathfrak{A} \models_Z \theta \) for some teams \( Y \) and \( Z \) such that \( Y \cup Z = X \).

(v) If \( \varphi = \forall v \psi \), then \( \mathfrak{A} \models_X \varphi \) if \( \mathfrak{A} \models_{X[A_{X(v)}]} \psi \).

(vi) If \( \varphi = \exists v \psi \), then \( \mathfrak{A} \models_X \varphi \) if \( \mathfrak{A} \models_{X[F/v]} \psi \) for some function \( F : X \to \mathcal{P}(A_{X(v)}) \setminus \{\emptyset\} \).

(vii) If \( \varphi = \tilde{\exists} v \psi \), then \( \mathfrak{A} \models_X \varphi \) if \( \mathfrak{B} \models_X \forall v \psi \) for all expansions \( \mathfrak{B} \) of \( \mathfrak{A} \) by the sort \( s(v) \).

(viii) If \( \varphi = \tilde{\exists} v \psi \), then \( \mathfrak{A} \models_X \varphi \) if \( \mathfrak{B} \models_X \exists v \psi \) for some expansion \( \mathfrak{B} \) of \( \mathfrak{A} \) by the sort \( s(v) \).

If we restrict our attention to vocabularies and structures with just one sort, we get exactly the ordinary team semantics of independence logic.

When the underlying structure \( \mathfrak{A} \) is clear from the context or is irrelevant to the discussion (e.g. when the formula \( \varphi \) does not contain any non-logical symbols or variables of multiple sorts), we simply write \( X \models \varphi \) instead of \( \mathfrak{A} \models_X \varphi \).

Although logical consequence in team semantics cannot be completely axiomatized, it makes sense to isolate axioms that suffice for proving as many of the interesting logical consequences as possible.

**Definition 1.5** (Axioms of Independence Atom, [13, 12]). The **axioms** of the independence atom are:

(i) \( \vec{y} \perp_{\vec{x}} \vec{y} \) entails \( \vec{y} \perp_{\vec{x}} \vec{z} \). (Constancy Rule)

(ii) \( \vec{x} \perp_{\vec{x}} \vec{y} \). (Reflexivity Rule)

(iii) \( \vec{z} \perp_{\vec{x}} \vec{y} \) entails \( \vec{y} \perp_{\vec{x}} \vec{z} \). (Symmetry Rule)

(iv) \( \vec{y}y' \perp_{\vec{x}} \vec{z}z' \) entails \( \vec{y} \perp_{\vec{x}} \vec{z} \). (Weakening Rule)

(v) If \( \vec{z} \) is a permutation of \( \vec{z}, \vec{x} \) is a permutation of \( \vec{x}, \vec{y} \) is a permutation of \( \vec{y} \), then \( \vec{y} \perp_{\vec{x}} \vec{z} \) entails \( \vec{y}' \perp_{\vec{x}} \vec{z}' \). (Permutation Rule)

(vi) \( \vec{z} \perp_{\vec{x}} \vec{y} \) entails \( \vec{y}x \perp_{\vec{x}} \vec{z}x \). (Fixed Parameter Rule)
(vii) $\vec{x} \perp_{\vec{z}} \vec{y} \land \vec{u} \perp_{\vec{z}\vec{x}} \vec{y}$ entails $\vec{u} \perp_{\vec{z}} \vec{y}$. (First Transitivity Rule)

(viii) $\vec{y} \perp_{\vec{z}} \vec{y} \land \vec{z}\vec{x} \perp_{\vec{z}} \vec{u}$ entails $\vec{x} \perp_{\vec{z}} \vec{u}$. (Second Transitivity Rule)

(ix) $\vec{x} \perp_{\vec{z}} \vec{y} \land \vec{x}\vec{y} \perp_{\vec{z}} \vec{u}$ entails $\vec{x} \perp_{\vec{z}} \vec{y}\vec{u}$. (Exchange Rule)

The so-called Armstrong’s Axioms for the dependence atom [4] follow from the above axioms.

**Definition 1.6.** The following is the elimination rule for existential quantifier:

If $\Sigma$ is a set of formulas, $\Sigma \cup \{\phi\}$ entails $\psi$ and $x$ does not occur free in $\psi$ or in any $\theta \in \Sigma$, then $\Sigma'$ entails $\psi$ for any $\Sigma' \supseteq \Sigma \cup \{\exists x\phi\}$.

The following is the introduction rule for existential quantifier:

If $y$ does not occur in the range of $Qx$ in $\phi$ for any $Q \in \{\exists, \forall, \tilde{\exists}, \tilde{\forall}\}$, then $\phi(y/x)$ entails $\exists x\phi$.

The following is the elimination rule for conjunction:

$\phi \land \psi$ entails both $\phi$ and $\psi$.

The following is the introduction rule for conjunction:

$\{\phi, \psi\}$ entails $\phi \land \psi$.

**Definition 1.7 (Dependence Introduction, [21]).** The following is the rule for dependence introduction:

$\exists x\phi$ entails $\exists x(=(\vec{z}, x) \land \phi)$ whenever $\phi$ is a formula of dependence logic, where $\vec{z}$ lists the free variables of $\exists x\phi$.

**Proposition 1.8** (Soundness Theorem). If $\phi$ entails $\psi$ by repeated applications of the rules of Definition 1.5, Definition 1.6 and Definition 1.7, then $\phi \models \psi$ in team semantics.

If $\phi$ entails $\psi$ by repeated applications of the above rules, we write $\phi \vdash \psi$.

## 2 Empirical and Hidden-Variable Teams

Quantum physics provides a rich source of highly non-trivial dependence and independence concepts. Some of the most fundamental questions of quantum physics are about independence of outcomes of experiments. The first author presented
in [1] a relational (possibilistic) approach to model these dependence and independence phenomena. His framework very naturally transforms into a team-semantic adaptation which we will perform here.

We consider teams with designated variables for measurements and separate variables for outcomes. An important role in models of quantum physics is played by the so-called hidden variables, variables that have an unobservable outcome and no input. The following terminology and notation is helpful in dealing with teams arising, intuitively, from quantum physical experiments.

We use a division of variables into three sorts, defined below. A priori there is no difference between the variables. The words “measurement”, “outcome”, “hidden variable” and “empirical” are just words that fit well with the intuition about quantum physical experiments, and could as well be “red”, “blue”, “black” and “colourful”. Also the division into three sorts is not relevant but fits our intended model well. Our purely abstract results about teams based on these variables help us organize quantum theoretical concepts.

**Definition 2.1.** Fix

- a set $V_m = \{x_0, \ldots, x_{n-1}\}$ of measurement variables,
- a corresponding set $V_o = \{y_0, \ldots, y_{n-1}\}$ of outcome variables, and
- a set $V_h = \{z_0, \ldots, z_{l-1}\}$ of hidden variables.

We say that a team $X$ is an empirical team if $\text{dom}(X) = V_m \cup V_o$. We say that a team $X$ is a hidden-variable team if $\text{dom}(X) = V_m \cup V_o \cup V_h$.

Throughout the paper, we will denote by $n$ the number of measurement and outcome variables and by $l$ the number of hidden variables.

Divisions of variables such as measurements vs. outcomes in the above definition are common in most application areas. In medical data we can have distinguished variables $x_i$ which describe various patient health parameters and distinguished variables $y_i$ describing symptom parameters. In biological data we can have distinguished variables $x_i$ describing genotype and distinguished variables $y_i$ describing phenotype. In social choice we can have distinguished variables $x_i$ describing voter preferences and a distinguished variable $y$ describing the social choice made on the basis of the vote data. In each of these cases there are certain expected dependences and certain expected independences. For example symptoms are expected to be more or less determined by the health parameters, the phenotype is expected to be more or less determined by the genotype, and the social welfare choice is supposed to be determined by the voter preferences. On the other hand, in an ideal health database patient health parameters are independent from each other (in order for scientific conclusions to be possible), in any
large genetic database there is certain (if not total) independence between the alleles manifested in the genotypes, and (ideally) the voters choose their preferences independently from each other. In each of these cases there is a lot of inaccuracy and uncertainty.

There does not seem to be any commonly used analog of the hidden variables outside quantum mechanics. The hidden variable question makes quantum physics in a sense special. Another sense in which quantum physics is special and ideal, with all its complications, is that there is a mathematical model provided by Hilbert spaces which is considered a perfect simulation of experiments.

We will pay special attention to definability of properties of teams. In other words, if \( P \) is a property of teams, especially of empirical or hidden-variable teams, we ask whether there is a formula \( \phi \) of independence logic with the free variables \( V_m \cup V_o \) (or \( V_m \cup V_o \cup V_h \)) which is satisfied in the sense of team semantics exactly by those teams that have the property \( P \).

A hidden-variable team is a team of the form

\[
Y = \begin{pmatrix}
  x_0 & y_0 & \ldots & x_n & y_{n-1} & z_0 & \ldots & z_{l-1} \\
  a_0^0 & b_0^0 & \ldots & a_0^0 & b_0^0 & \gamma_0^0 & \ldots & \gamma_0^{l-1} \\
  a_0^1 & b_0^1 & \ldots & a_0^1 & b_0^1 & \gamma_0^1 & \ldots & \gamma_0^{l-1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_0^{m-1} & b_0^{m-1} & \ldots & a_0^{m-1} & b_0^{m-1} & \gamma_0^{m-1} & \ldots & \gamma_0^{m-1}
\end{pmatrix}
\]

where \( \gamma_j^i \) indicate values which we cannot observe. Therefore we speak of “hidden” variables. Intuitively, a hidden variable is a variable which exists but has eluded us and may elude us forever and therefore we cannot measure it. A typical hidden variable is some kind of “state” of the system. From the point of view of the experiments \( q_0, \ldots, q_{n-1} \), the hidden variables are otherwise like the \( x_i \) and the \( y_i \), except that they do not arise from performing any experiments or measurements.

Every team has a background model from where the values of assignments come from. In a many-sorted context the background model has one universe for each sort. The universes may intersect. We assume a universe also for the hidden variable sort.

**Definition 2.2** ([1]). A hidden-variable team \( Y \) realizes an empirical team \( X \) if

\[
s \in X \iff \exists s' \in Y \bigwedge_{i<n} (s'(x_i) = s(x_i) \land s'(y_i) = s(y_i)).
\]

Two hidden-variable teams are said to be (empirically) equivalent if they realize the same empirical team.

The property of being the realization of a hidden-variable team is definable in independence logic. It can be defined simply by the existential quantifier: If
\[ \varphi(\vec{x}, \vec{y}, \vec{z}) \] is a formula of independence logic, and thereby defines a property of teams, then \( \exists z_0 \exists z_1 \ldots \exists z_{l-1} \varphi \) defines the class of empirical teams that are realized by some hidden-variable teams satisfying \( \varphi \). The “hidden” character of the hidden variables is built into the semantics to many-sorted quantification.

Realization of an empirical team by a hidden-variable team is a kind of projection where one projects away the hidden variables. Hidden-variable teams are divided into equivalence classes according to whether they project into the same empirical team or not. This phenomenon can of course be thought of more generally: for any set \( V \) of variables and \( V' \subseteq V \), one can define a projection mapping \( \Pr_{V'} \) such that if \( X \) is a team with domain \( V \), then \( \Pr_{V'}(X) = \{ s \upharpoonright V' \mid s \in X \} \).

### 3 Logical Properties of Teams

In this section, we use the resources of independence logic with its team semantics to express properties of empirical and hidden-variable teams. The possible benefits of expressing such properties in the formal language of independence logic are two-fold. First, the quantum theoretical concepts may suggest new interesting facts about independence logic in general, applicable perhaps also in other fields. Second, concepts, proofs and constructions of independence logic may bring new light to connection of concepts in quantum physics and may focus attention to what is particular to quantum physics and what is merely general facts about independence concepts.

#### 3.1 Properties of Empirical Teams

We observe that the definitions of the simpler properties of empirical teams treated by the first author in [1] can be expressed by a formula of independence logic, in fact a conjunction of independence atoms.

As discussed earlier, a team is said to support weak determinism if each outcome is determined by the combination of all the measurement variables.

**Definition 3.1** (Weak Determinism). An empirical team \( X \) supports weak determinism if it satisfies the formula

\[
\bigwedge_{i<n} = (\vec{x}, y_i). \tag{WD}
\]

Thus weak determinism is expressed simply with a dependence atom. In fact, the meaning of the dependence atom \( = (x, y) \) is that \( x \) completely determines \( y \). Therefore saying that teams supporting (WD) support weak determinism is appropriate. The only difference to the ordinary dependence atom is that in (WD) we separate the variables into the measurement \( x_i \) and the outcomes \( y_i \).
A team is said to support *strong determinism* if the outcome variable $y_i$ of any measurement is completely determined by the measurement variable $x_i$.

**Definition 3.2** (Strong Determinism). An empirical team $X$ supports strong determinism if it satisfies the formula

$$ \bigwedge_{i<n}=(x_i, y_i). \quad \text{(SD)}$$

A team $X$ is said to support *no-signalling* if the following holds: Suppose the team $X$ has two measurement-outcome combinations $s$ and $s'$ with input $x_i$ the same. So now $s(y_i)$ is a possible outcome of experiment $q_i$ in view of $X$. We demand that $s(y_i)$ is also a possible outcome if the inputs $s(x_j), j \neq i$, are changed to $s'(x_j)$.

**Definition 3.3** (No-Signalling). An empirical team $X$ supports no-signalling if it satisfies the formula

$$ \bigwedge_{i<n}\{x_j \mid j \neq i\} \perp_{x_i} y_i. \quad \text{(NS)}$$

In principle, supporting no-signalling means just satisfying an independence atom. But the atom is of a particular form because of our division of variables into difference sorts. The atom $\{x_j \mid j \neq i\} \perp_{x_i} y_i$ says that the outcome $y_i$ is meant to be related to the measurement $x_i$ and be totally independent of the measurements $x_j, j \neq i$.

### 3.2 Properties of Hidden-Variable Teams

For hidden-variable teams, the hidden variables are added in the definition of determinism as extra variables that determine the systems outcomes.

**Definition 3.4** (Weak Determinism). A hidden-variable team $X$ supports weak determinism if it satisfies the formula

$$ \bigwedge_{i<n}=(\vec{x}, \vec{z}, y_i). \quad \text{(WD)}$$

**Definition 3.5** (Strong Determinism). A hidden-variable team $X$ supports strong determinism if it satisfies the formula

$$ \bigwedge_{i<n}=(x_i \vec{z}, y_i). \quad \text{(SD)}$$

A team $X$ is said to support *single-valuedness* if each hidden variable $z_k$ can only take one value.
Definition 3.6 (Single-Valuedness). A hidden-variable team $X$ supports single-valuedness if it satisfies the formula

$$(\vec{z}).$$

A team $X$ is said to support $z$-independence if the following holds: Suppose the team $X$ has two measurement-outcome combinations $s$ and $s'$. Now the hidden variables $\vec{z}$ have some value $s(\vec{z})$ in the combination $s$. We demand that $s(\vec{z})$ should occur as the value of the hidden variable also if the inputs $s(\vec{x})$ are changed to $s'(\vec{x})$.

Definition 3.7 ($z$-Independence). A hidden-variable team $X$ supports $z$-independence if it satisfies the formula

$$\vec{z} \perp \vec{x}.$$  

A team $X$ is said to support parameter-independence if the following holds: Suppose the team $X$ has two measurement-outcome combinations $s$ and $s'$ with the same input data about $x_i$ and the same hidden variables $\vec{z}$, i.e. $s(x_i) = s'(x_i)$ and $s(\vec{z}) = s'(\vec{z})$. We demand that output $s(y_i)$ should occur as a possible output also if the inputs $s(\{x_j \mid j \neq i\})$ are changed to $s'(\{x_j \mid j \neq i\})$.

Definition 3.8 (Parameter Independence). A hidden-variable team $X$ supports parameter independence if it satisfies the formula

$$\bigwedge_{i<n} \{x_j \mid j \neq i\} \perp_{x_i,y_i}$$

A team $X$ is said to support outcome-independence if the following holds: Suppose the team $X$ has two measurement-outcome combinations $s$ and $s'$ with the same total input data $\vec{x}$ and the same hidden variables $\vec{z}$, i.e. $s(\vec{x}) = s'(\vec{x})$ and $s(\vec{z}) = s'(\vec{z})$. We demand that output $s(y_i)$ should occur as an output also if the outputs $s(\{y_j \mid j \neq i\})$ are changed to $s'(\{y_j \mid j \neq i\})$. In other words, the variables $y_i$, $i < n$, are mutually independent whenever $\vec{x}\vec{z}$ is fixed.

Definition 3.9 (Outcome Independence). A hidden-variable team $X$ supports outcome independence if it satisfies the formula

$$\bigwedge_{i<n} y_i \perp_{x_i} \{y_j \mid j \neq i\}.$$  

All the previous examples were, from the point of view of independence logic, atoms or conjunctions of atoms of the same kind with a certain organization of the variables. We now introduce a property which is slightly more complicated.

A team $X$ is said to support locality if the following holds: For any fixed value of the hidden variables $\vec{z}$ and any values $a_i,b_i$ of $x_i,y_i$ in $X$, if the tuple $\vec{a}$ occurs as a value of $\vec{x}$, then also the tuple $\vec{a}\vec{b}$ occurs as a value of $\vec{xy}$. 

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Definition 3.10 (Locality). A hidden-variable team $X$ supports locality if

$$\forall s_0, \ldots, s_{n-1} \in X \left[ \exists s \in X \bigwedge_{i<n} s(x_i\vec{z}) = s_i(x_i\vec{z}) \right] \Rightarrow \exists' s' \in X \bigwedge_{i<n} s'(x_i y_i \vec{z}) = s_i(x_i y_i \vec{z})$$

The definition of locality is not per se an expression of independence logic. However, in Lemma 3.14 below we prove that locality can be defined, after all, by a conjunction of independence atoms.

3.3 Relationships between the Properties

We present several logical consequences of independence logic and demonstrate how they can be interpreted in the context of empirical and hidden variables teams. In many cases we can derive the logical consequence relation from the axioms of Definition 1.5. Semantic proofs we adapt from [1].

It should be noted that logical consequence in independence logic is in principle a highly complex concept. For example, it cannot be axiomatized because the set of Gödel numbers of valid sentences (of even dependence logic\(^3\)) is non-arithmetical. Even the implication problem for the independence atoms is undecidable [17], while for dependence atoms it is decidable [4]. Part of the problem is that there is an infinite team satisfying a formula (e.g. $x \perp y \land y \perp z$) such that no finite subteam satisfies the particular formula. Logical implication between finite conjunctions of independence atoms is, however, recursively axiomatizable, as it can be reduced to logical consequence in first order logic by introducing a new predicate symbol.

Because of the complexity of logical consequence, it is important to accumulate good examples. We claim that the below examples arising from quantum mechanics are illustrative examples and guide us in finding a more systematic approach.

Lemma 3.11. $=(\vec{x}y, \vec{z}) \vdash \bigwedge_{i<n} y_i \perp_{\vec{z}} \{ y_j \mid j \neq i \}$.

In words, if a hidden-variable team supports weak determinism, then it supports outcome independence.

Proof. $=(\vec{x}y, \vec{z})$ means $\vec{y} \perp_{\vec{z}} \vec{y}$. Given any $i < n$, one obtains $y_i \perp_{\vec{z}} \{ y_j \mid j \neq i \}$ from $\vec{y} \perp_{\vec{z}} \vec{y}$ by a single application of the Weakening Rule of independence atoms.

Lemma 3.12. For all $i < n$, $=(x_i \vec{z}, y_i) \vdash \{ x_j \mid j \neq i \} \perp_{x_i\vec{z}} y_i$.

\(^3\)This observation is essentially due to A. Ehrenfeucht, see [16].
In words, if a hidden-variable team supports strong determinism, then it supports parameter independence.

**Proof.** Fix $i < n$. $=(x_i\vec{z}, y_i)$ means $y_i \perp_{x_i\vec{z}} \vec{z}$. Now the Constancy Rule of independence atoms gives $y_i \perp_{x_i\vec{z}} \vec{w}$ for any variable tuple $\vec{w}$, in particular when $\vec{w} = \{x_j \mid j \neq i\}$. Finally, we obtain $\{x_j \mid j \neq i\} \perp_{x_i\vec{z}} y_i$ by using the Symmetry Rule. \hfill $\square$

**Lemma 3.13.** $(\bigwedge_{i<n} \{x_j \mid j \neq i\} \perp_{x_i\vec{z}} y_i) \land (\vec{x}\vec{z}, \vec{y}) \models \bigwedge_{i<n} = (x_i\vec{z}, y_i)$.

In words, if a hidden-variable team supports parameter independence and weak determinism, then it supports strong determinism.

**Proof.** Fix $i < n$. $=(\vec{x}\vec{z}, \vec{y})$ means $\vec{y} \perp_{\vec{x}\vec{z}} \vec{z}$, from which we get $y_i \perp_{\vec{x}\vec{z}} \vec{z}$ using the Weakening Rule. Then we have

$$\{x_j \mid j \neq i\} \perp_{x_i\vec{z}} y_i \land y_i \perp_{\vec{x}\vec{z}} y_i.$$  

Finally, the First Transitivity Rule yields $y_i \perp_{x_i\vec{z}} y_i$, which means $=(x_i\vec{z}, y_i)$. \hfill $\square$

**Lemma 3.14.** Locality is equivalent to the formula

$$\bigwedge_{i<n} ((\{x_j \mid j \neq i\} \perp_{x_i\vec{z}} y_i) \land (y_i \perp_{\vec{x}\vec{z}} \{y_j \mid j \neq i\})).$$

In words, a hidden-variable team X supports locality if and only if it supports both parameter independence and outcome independence.

**Proof.** It is straightforward to show that locality implies both outcome and parameter independence.

We show that the conjunction of parameter and outcome independence implies locality. So suppose that X supports both parameter and outcome independence. For locality, let $s_0, \ldots, s_{n-1} \in X$ be such that there is $s \in X$ with $s(x_i\vec{z}) = s_i(x_i\vec{z})$ for $i < n$. Denote $a_i = s_i(x_i)$, $b_i = s_i(y_i)$ and $\vec{c} = s(\vec{z})$. Note that now $s(\vec{x}\vec{z}) = \vec{a}\vec{c}$. Our goal is to find $s' \in X$ with $s'(\vec{x}\vec{y}\vec{z}) = \vec{a}\vec{b}\vec{c}$.

Let $s_0' = s$. By parameter independence,

$$X \models \{x_j \mid j \neq 0\} \perp_{x_0\vec{z}} y_0,$$

so as both $s_0'$ and $s_0(x_0\vec{z})$ fix the value of $x_0\vec{z}$ to $a_0\vec{c}$, there is $s_1' \in X$, that also fixes the value of $x_0\vec{z}$ to $a_0\vec{c}$, such that $s_1'(x_j) = s_0'(x_j) = a_j$ for all $j \neq 0$ and $s_1'(y_0) = s_0(y_0) = b_0$. So now $s_1'(\vec{x}\vec{y}\vec{z}) = \vec{a}\vec{b}\vec{c}$. By parameter independence, again,

$$X \models \{x_j \mid j \neq 1\} \perp_{x_1\vec{z}} y_1.$$
so, as in turn, \( s_1^* \) and \( s_1 \) both fix \( x_1 \bar{z} \) to \( a_1 \bar{\gamma} \), we obtain \( s_2^* \in X \) that also fixes them so, with \( s_2^*(x_j) = s_1^*(x_j) = a_j \) for \( j \neq 1 \) and \( s_2^*(y_1) = s_1(y_1) = b_1 \). So now \( s_2^*(\bar{x}y_1 \bar{z}) = \bar{a}b_1 \bar{\gamma} \). Continuing like this we obtain assignments \( s_1^*, \ldots, s_n^* \) such that

\[
\phi(z) = \exists x \forall y \bar{z} = \bar{a}b_1 \bar{\gamma} \quad \text{for all} \quad i < n.
\]

Note that by outcome independence, when the value of \( \bar{x} \bar{z} \) is fixed, variables \( y_i \), \( i < n \), are mutually independent. Thus, as each \( s_{i+1}^* \) fixes \( \bar{x} \bar{z} \) to \( \bar{a} \bar{\gamma} \), we can find \( s' \in X \), that also fixes \( \bar{x} \bar{z} \) to \( \bar{a} \bar{\gamma} \), such that for all \( i < n \), \( s'(y_i) = s_{i+1}^*(y_i) = b_i \). But then \( s'((\bar{x} \bar{y} \bar{z})) = \bar{a} \bar{b} \bar{\gamma} \), so \( s' \) is as desired.

Next we indicate connections between empirical and hidden-variable teams with some properties, again following [1].

**Proposition 3.15.** The sentence \( \exists z_0 \ldots \exists z_{l-1} = (\bar{z}) \) is valid. More generally, if the variables \( \bar{z} \) are not free in \( \varphi \), then \( \varphi \vdash \exists z_0 \ldots \exists z_{l-1} = (\bar{z}) \land \varphi \).

Note that it then follows that \( \varphi \models \exists z_0 \exists z_1 \ldots \exists z_{l-1} = (\bar{z}) \land \varphi \). In words, every empirical team is realized by a hidden-variable team supporting single-valuedness.

**Proof.** By the Constancy Rule, we have \( \bar{z} \perp \bar{x} \bar{z} \). Using introduction of existential quantifier \( l \) times, we get \( \exists z_0 \ldots \exists z_{l-1} \forall y \bar{z} \perp \bar{x} \bar{z} \). Using elimination of existential quantifier and introduction of dependence \( l \) times we get \( \exists z_0 \ldots \exists z_{l-1} (\bigwedge_{k<l} = (z_k) \land \bar{z} \perp \bar{x} \bar{z}) \). As, from Armstrong’s axioms, one can infer \( \bigwedge_{k<l} = (z_k) \vdash = (\bar{z}) \), we then easily obtain \( \exists \bar{z} = (\bar{z}) \). Then by using elimination and introduction of existential quantifier \( l \) times, we get \( \exists \bar{z} = (\bar{z}) \land \varphi \). \( \square \)

**Proposition 3.16.** The following formulas are equivalent.

(i) \( \bigwedge_{i<n} \{ x_j \mid j \neq i \} \perp_{x_i} y_i \),

(ii) \( \exists z_0 \exists z_1 \ldots \exists z_{l-1} (\bar{z} \perp \bar{z} \land \bigwedge_{i<n} \{ x_j \mid j \neq i \} \perp_{x_i \bar{z}} y_i) \).

In words, an empirical team supports no-signalling if and only if it can be realized by a hidden-variable team supporting \( \bar{z} \)-independence and parameter independence.

**Proof.** Let \( X \) be an empirical team.

First suppose that

\( X \models \{ x_j \mid j \neq i \} \perp_{x_i} y_i \).

Let \( Y \) be any single-valuedness-supporting extension of \( X \) into a hidden-variable team. Then also

\( Y \models \{ x_j \mid j \neq i \} \perp_{x_i \bar{z}} y_i \),

and so, as \( \bar{z} \) is constant, we have

\( Y \models \{ x_j \mid j \neq i \} \perp_{x_i \bar{z}} y_i \)
for all $i < n$. Thus $Y$ supports parameter independence. As $\vec{z}$ is constant, it is trivially independent of $\vec{x}$ and hence $Y$ supports $z$-independence.

Then suppose that $X$ is realized by a hidden-variable team $Y$ supporting $z$-independence and parameter independence. Fix $i < n$. To show that

$$X \models \{x_j \mid j \neq i\} \perp_{x_i} y_i,$$

let $s, s' \in X$ be such that $s(x_i) = s'(x_i)$. Let $\tilde{s}, \tilde{s}' \in Y$ be any extensions of $s$ and $s'$, respectively. As by $z$-independence $Y \models \vec{z} \perp \vec{x}$, there is $\tilde{s}^* \in Y$ with $\tilde{s}^*(\vec{x}) = \tilde{s}(\vec{x})$ and $\tilde{s}^*(\vec{z}) = \tilde{s}'(\vec{z})$. As by parameter independence

$$Y \models \{x_j \mid j \neq i\} \perp_{x_i} \tilde{y}_i,$$

and as $\tilde{s}^*(x_i \vec{z}) = \tilde{s}(x_i)\tilde{s}'(\vec{z}) = \tilde{s}'(x_i \vec{z})$, there is $\tilde{s}'' \in Y$ with $\tilde{s}''(\vec{x}\vec{z}) = \tilde{s}^*(\vec{x}\vec{z})$ and $\tilde{s}''(y_i) = \tilde{s}'(y_i)$. Let $s'' = \tilde{s}'' | \text{dom}(X)$. Then $s'' \in X$, as $Y$ realizes $X$. Now

$$s''(\vec{x}) = s''(\vec{z}) = \tilde{s}^*(\vec{x}) = s(\vec{x})$$

and

$$s''(y_i) = \tilde{s}''(y_i) = \tilde{s}'(y_i) = s'(y_i).$$

Thus we have shown that $X$ supports no-signalling. \hfill \Box

**Proposition 3.17.** The formula $\exists z_0 \exists z_1 \ldots \exists z_{i-1} \land_{i<n} = (x_i \vec{z}, y_i)$ is valid. More generally, $\varphi \models \exists z_0 \exists z_1 \ldots \exists z_{i-1} (\land_{i<n} = (x_i \vec{z}, y_i) \land \varphi)$, when $\vec{z}$ does not occur free in $\varphi$.

In words, every empirical team is realized by a hidden-variable team supporting strong determinism.

*Proof.* Let $X$ be an empirical team such that $X \models \varphi$. We wish to find a hidden-variable team $Y$ that realizes $X$ and satisfies $\varphi$ and $= (x_i \vec{z}, y_i)$ for all $i < n$. We construct this $Y$ in pieces.

For each $i < n$ we do the following. For a given value $a$ of $x_i$, let $b_0^a, \ldots, b_{m_a-1}^a$ enumerate the set $\{s(y_i) \mid s(x_i) = a\}$ without repetition. Fix $m_a$ many fresh and distinct values $\vec{\gamma}_j^a$, $j < m_a$, for the hidden variables. Then let

$$Y_i = \bigcup_{a \in \text{rng}(X)} \bigcup_{j < m_a} \{s(\vec{\gamma}_j^a / \vec{z}) \mid s \in X, s(x_i) = a \text{ and } s(y_i) = b_j^a\}.$$ 

We then let $Y$ be the union of $Y_i$, $i < n$. Now clearly $Y$ realizes $X$. To show that $Y \models = (x_i \vec{z}, y_i)$, let $s, s' \in Y$ be such that $s(x_i \vec{z}) = s'(x_i \vec{z})$. Let $a$ be the value of $x_i$ under these assignments. Then by construction, as $s(\vec{z}) = s'(\vec{z})$, also $s(y_i) = s'(y_i)$. Again by locality of independence logic, $Y \models \varphi$. \hfill \Box
Proposition 3.18. The formula $\exists z_0 \exists z_1 \ldots \exists z_{l-1} (=(\vec{x}, \vec{y}) \land \vec{z} \perp \vec{x})$ is valid. More generally, $\varphi \models \exists z_0 \exists z_1 \ldots \exists z_{l-1} (=(\vec{x}, \vec{y}) \land \vec{z} \perp \vec{x} \land \varphi)$, when $\vec{z}$ does not occur free in $\varphi$.

In words, every empirical team is realized by a hidden-variable team supporting weak determinism and $z$-independence.

Proof. Let $X$ be an empirical team. We wish to find a hidden-variable team $Y$ that realizes $X$ and satisfies $=(\vec{x}, \vec{y})$, $\vec{z} \perp \vec{x}$ and $\varphi$. The satisfaction of $\varphi$ will again be taken care of by locality of independence logic, so we concentrate on the other two formulas. Without loss of generality we assume that $l = 1$ and $\vec{z} = z$.

Denote $M = \{s(\vec{x}) \mid s \in X\}$ and $O = \{s(\vec{y}) \mid s \in X\}$. Let $\Gamma$ be the set of all functions $\gamma: M \to O$ such that for any $\vec{a} \in M$ there is some $s \in X$ with $s(\vec{x}) = \vec{a}$ and $s(\vec{y}) = \gamma(\vec{a})$. We then let $Y$ be the set of all such $s$ that $s(\vec{x}) \in M$, $s(\vec{y}) \in O$, $s(z) \in \Gamma$ and $s(z)(s(\vec{x})) = s(\vec{y})$.

We now show that $Y$ is as desired. To show that $Y$ realizes $X$, let $s \in X$. Then let $\gamma \in \Gamma$ be a function mapping $s(\vec{x})$ to $s(\vec{y})$. Then $s(\gamma/z) \in Y$. For the converse, let $s \in Y$ and denote $s' = s \cup \text{dom}(X)$, $\gamma = s(z)$ and $\vec{a} = s(\vec{x})$. We wish to show that $s' \in X$. Now, by the definition of $Y$, $\gamma \in \Gamma$. Thus, as $\vec{a} \in M$, there is some $s'' \in X$ with $s''(\vec{x}) = \vec{a}$ and $s''(\vec{y}) = \gamma(\vec{a})$. But also by the definition of $Y$, we have $s'(\vec{y}) = s(\vec{y}) = \gamma(s(\vec{x})) = \gamma(\vec{a})$, and remembering that $s'(\vec{x}) = \vec{a}$, we get that $s'' = s'$. This implies that $s' \in X$, as desired.

To show that $Y \models = (\vec{x}, \vec{y})$, let $s, s' \in Y$ be such that $s(\vec{x}) = s'(\vec{x})$. Then by the definition of $Y$ and the facts that $s(z) = s'(z)$ and $s(\vec{x}) = s'(\vec{x})$,

$$s(\vec{y}) = s(z)(s(\vec{x})) = s'(z)(s(\vec{x})) = s'(z)(s'(\vec{x})) = s'(\vec{y}),$$

as desired.

To show that $Y \models z \perp \vec{x}$, let $s, s' \in Y$ be arbitrary. We wish to find $s'' \in Y$ with $s''(z) = s(z)$ and $s''(\vec{x}) = s'(\vec{x})$. Let $\gamma = s(z)$ and $\vec{a} = s(\vec{x})$. Now $\gamma \in \Gamma$ and $\vec{a} \in M$, so by the definition of $\Gamma$, there is some $s^* \in X$ with $s^*(\vec{x}) = \vec{a}$ and $s^*(\vec{y}) = \gamma(\vec{a})$. Let $s'' = s^*(\gamma/z)$. Then

$$s''(z)(s''(\vec{x})) = \gamma(s''(\vec{x})) = \gamma(s^*(\vec{x})) = \gamma(\vec{a}) = s^*(\vec{y}) = s''(\vec{y}).$$

Thus $s'' \in Y$. Also, $s''(z) = \gamma = s(z)$ and $s''(\vec{x}) = s^*(\vec{x}) = \vec{a} = s'(\vec{x})$, so $s''$ is as desired.

\[ \square \]

Proposition 3.19.

$$\exists z_0 \exists z_1 \ldots \exists z_{l-1} \left( \vec{z} \perp \vec{x} \land \bigwedge_{i<n} (\{x_j \mid j \neq i\} \perp_{\vec{x}, \vec{z}} y_i) \land (y_i \perp_{\vec{x}, \vec{z}} \{y_j \mid j \neq i\}) \right)$$

$$\models \exists z_0 \exists z_1 \ldots \exists z_{l-1} \left( \vec{z} \perp \vec{x} \land \bigwedge_{i<n} (= (\vec{x}, \vec{z}, y_i)) \right).$$
In words, any hidden-variable team supporting $z$-independence and locality is equivalent to a hidden-variable team supporting $z$-independence and strong determinism.

**Proof.** Let $X$ be a hidden-variable team supporting $z$-independence and locality. Denote $M = \{s(\vec{x}) \mid s \in X\}$ and $\Gamma = \{s(\vec{z}) \mid s \in X\}$, and, for $i < n$, denote $M_i = \{s(x_i) \mid s \in X\}$ and $O_i = \{s(y_i) \mid s \in X\}$. Furthermore, for $i < n$, $\vec{\gamma} \in \Gamma$ and $a \in M_i$, denote $O_{\vec{z}, \vec{\gamma}} = \{s(y_i) \mid s \in X$ and $s(x_i, \vec{z}) = a\vec{\gamma}\}$.

Let $\vec{\gamma}^* \in \text{rng}(X)$ be a constant. We now let $X'$ be the set of all $s$ such that

1. $s(\vec{x}) \in M$,
2. $s(z_1) = \cdots = (z_{i-1}) = \vec{\gamma}^*$,
3. there are $\vec{\gamma} \in \Gamma$ and functions $f_i : M_i \to O_i$ such that $s(z_0) = (\vec{\gamma}, f)$, where $f = (f_0, \ldots, f_{n-1})$,
4. for each $i < n$ and $a \in M_i$, $f_i(a) \in O_{\vec{z}, \vec{\gamma}}$,
5. for each $i < n$, $f_i(s(x_i)) = s(y_i)$.

The idea is that the value of the hidden variable functionally determines the outcome of the measurements, and does so in a way that depends on the value for the hidden variable given in the team $X$. Now we show that $X'$ and $X$ are equivalent and that $X'$ supports $z$-independence and strong determinism.

First, we show that $X'$ supports strong determinism, i.e. that, for all $i < n$, $X' \models = (x_i, z_i, y_i)$ (even $X' \models = (x_i z_0, y_i)$). Let $s, s' \in X'$ be such that $s(x_i) = s'(x_i)$ and $s(z) = s'(z)$. Denote $(\vec{\gamma}, f) = s(z_0)$. Now $s(y_i) = f_i(s(x_i)) = f_i(s'(x_i)) = s'(y_i)$, as desired.

Second, we show that $X'$ supports $z$-independence, i.e. that $X' \models = \vec{z} \perp \vec{x}$. Let $s, s' \in X'$. We wish to find $s'' \in X'$ with $s''(\vec{z}) = s(\vec{z})$ and $s''(\vec{x}) = s'(\vec{x})$. Denote $(\vec{\gamma}, f) = s(z_0)$ and $\vec{a} = s'(\vec{x})$. Now, simply define $s''$ by letting $s''(\vec{x}) = \vec{a}$, $s''(\vec{z}) = ((\vec{\gamma}, f), \vec{\gamma}^*, \ldots, \vec{\gamma}^*)$ and $s''(y_i) = f_i(a_i)$ for $i < n$. Now if $s'' \in X'$, we are done. So we need to show that $s''$ satisfies (i)–(v) above. First, $s''(\vec{z}) = \vec{a} \in M$, as $s'$ satisfies (i). Second, $s''(z_i) = \vec{\gamma}^*$ for $0 < i < l$ by definition. Third, $s''(z_0) = (\vec{\gamma}, f)$, where $\vec{\gamma} \in \Gamma$ and $f_i : M_i \to O_i$, as $s$ satisfies (iii). Fourth, for all $i < n$ and $a \in M_i$, $f_i(a) \in O_{\vec{z}, \vec{\gamma}}$, as $s$ satisfies (iv). Fifth, $f_i(s''(x_i)) = f_i(a_i) = s''(y_i)$ by the definition of $s''$. Thus $s'' \in X'$.

Finally, we show that $X$ and $X'$ are equivalent. First, let $s \in X$. We wish to find $s' \in X'$ with $s'(\vec{x}^\prime \vec{y}) = s(\vec{x}^\prime \vec{y})$. Denote $\vec{a} = s(\vec{x})$ and $\vec{b} = s(\vec{b})$ and $\vec{\gamma} = s(\vec{z})$. Now obviously $\vec{a} \in M$, $\vec{\gamma} \in \Gamma$ and $\vec{b} \in O_{\vec{z}, \vec{\gamma}}$. For any $i < n$, as $O_{\vec{z}, \vec{\gamma}} \neq \emptyset$ for all $a \in \Gamma_i$ (by $z$-independence of $X$), there is a function $f_i : M_i \to O_i$ with $f_i(a) \in O_{\vec{z}, \vec{\gamma}}$ such that $f_i(a_i) = b_i$ (then in particular $f_i(a_i) \in O_{\vec{z}, \vec{\gamma}}$). Let $f = (f_0, \ldots, f_{n-1})$. Define
Theorem 4.1. The EPR construction is an example of an interesting existential formula not being provable from our axioms. This is a subject of further study.

4 SOME COUNTEREXAMPLES

In previous sections we showed logical consequences and equivalences between team properties (or formulas). Here we construct, again on the footsteps of [1], teams that demonstrate failures of logical consequences. In the quantum mechanical interpretation they correspond to the classic no-go theorem constructions of Einstein–Podolsky–Rosen (EPR), Greenberger–Horne–Zeilinger (GHZ), Hardy and Kochen–Specker (KS).

In Tarski semantics of first-order logic, some existential formulas, such as $\exists z(x = z \land \neg y = z)$, are not valid while others, such as $\exists z(x = z \lor x = y)$, are. To decide which are valid and which are not is particularly simple, especially in the empty vocabulary because first order logic has in that case elimination of quantifiers. In team semantics where such quantifier elimination is not possible, non-valid existential-conjunctive formulas can be quite complicated, as the examples below show. As we shall see, the no-go results of quantum mechanics give rise to very interesting teams.

4.1 EPR TEAMS

The EPR construction is an example of an interesting existential formula not being valid in team semantics.

Theorem 4.1. $\exists z_0 \exists z_1 \ldots \exists z_{l-1} \{= z \land \bigwedge_{i < n} y_i \perp_{x\neq z} \{y_j \mid j \neq i\}\}$ is not valid.
In words, there is an empirical team which cannot be realized by any hiddenvariable team supporting single-valuedness and outcome independence. We will call the team constructed in the proof the EPR team.

**Proof.** We consider a case where $n = 2$. Let $X$ be the team presented in the following table.

|   | $x_0$ | $x_1$ | $y_0$ | $y_1$ |
|---|------|------|------|------|
| $s$ | 0    | 1    | 0    | 1    |
| $s'$| 0    | 1    | 1    | 0    |

Suppose that $Y$ is a hidden-variable team realizing $X$ and supporting single-valuedness. Thus $Y \models = (\vec{z})$. In consequence, there is some value $\vec{\gamma}$ such that $Y = \{ s(\vec{\gamma}/\vec{z}), s'(\vec{\gamma}/\vec{z}) \}$. But now $Y$ does not support outcome independence, as $Y \not\models y_0 \perp_{\vec{x}\vec{z}} y_1$, since $s(\vec{\gamma}/\vec{z})(\vec{x}\vec{z}) = s'(\vec{\gamma}/\vec{z})(\vec{x}\vec{z})$ and there is no $s'' \in Y$ with $s''(y_0) = s(y_0) = 0$ and $s''(y_1) = s'(y_1) = 0$.

4.2 GHZ teams

We define the concept of a GHZ team and then use GHZ teams to prove a nonlogical consequence result.

**Definition 4.2.** Assume that $n = 3$. Let $X$ be an empirical team with $\text{rng}(X) = \{0, 1\}$. Denote

- $P = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$
- $Q = \{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ and
- $R = \{(0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1)\}$.

We say that $X$ is a GHZ team if it satisfies the following conditions.

(i) $Q = \{ s(\vec{y}) \mid s \in X, s(\vec{x}) \in P \}$ and $P \subseteq \{ s(\vec{x}) \mid s \in X, s(\vec{y}) \in Q \}$.

(ii) $R = \{ s(\vec{y}) \mid s \in X, s(\vec{x}) = (0, 0, 0) \}$.

An example of a GHZ team would be the following minimal example:

|   | $x_0$ | $x_1$ | $x_2$ | $y_0$ | $y_1$ | $y_2$ |
|---|------|------|------|------|------|------|
| $s_0$| 0    | 0    | 0    | 0    | 0    | 1    |
| $s_1$| 0    | 0    | 0    | 0    | 1    | 0    |
| $s_2$| 0    | 0    | 0    | 1    | 0    | 0    |
| $s_3$| 0    | 0    | 0    | 1    | 1    | 1    |
| $s_4$| 0    | 1    | 1    | 0    | 0    | 0    |
| $s_5$| 0    | 1    | 1    | 0    | 1    | 1    |
| $s_6$| 1    | 0    | 1    | 1    | 0    | 1    |
| $s_7$| 1    | 1    | 0    | 1    | 1    | 0    |
Theorem 4.3. The formula
\[ \exists z_0 \exists z_1 \ldots \exists z_{l-1} \left( \bar{z} \perp \bar{x} \land \bigwedge_{i<n} ((\{x_j \mid j \neq i\} \perp_{x,i} y_i) \land (y_i \perp_{\bar{x},\bar{z}} \{y_j \mid j \neq i\})) \right) \]
is not valid, as demonstrated by any GHZ team.

In words, no GHZ team can be realized by a hidden-variable team supporting \( z \)-independence and locality.

Proof. Let \( X \) be a GHZ team and suppose for a contradiction that it is realized by a hidden-variable team supporting \( z \)-independence and locality. By Proposition 3.19, there is a hidden-variable team \( Y \) supporting \( z \)-independence and strong determinism that realizes \( X \).

As \( Y \) realizes \( X \) and \( R \neq \emptyset \), there is some \( s \in Y \) with \( s(\vec{x}) = (0,0,0) \). Let \( \vec{\gamma} \) be a value of \( \vec{z} \) occurring on a row where the value of \( \vec{x} \) is \( (0,0,0) \). Now for each \( \vec{a} \in P \) there is some \( s \in Y \) with \( s(\vec{x}) = \vec{a} \). Thus by \( z \)-independence of \( Y \), there is for any \( \vec{a} \in P \) some \( s \in Y \) with \( s(\vec{x}) = \vec{a} \vec{\gamma} \).

Now look at the sets
\[ S = \{s(\vec{y}) \mid s \in Y, s(\vec{x}) = (0,1,1,\vec{\gamma})\} \]
\[ T = \{s(\vec{y}) \mid s \in Y, s(\vec{x}) = (1,1,0,\vec{\gamma})\}. \]

As \( Y \) realizes \( X \) and \( (0,1,1) \in P \), at least one \( s \in Y \) such that \( s(\vec{y}) \in S \) must exist, so we can deduce that \( S \) is a non-empty subset of \( Q \). By strong determinism, as every element of \( S \) is a value of \( \vec{y} \) for some \( s \in Y \) such that \( s(\vec{x}) = (0,1,1,\vec{\gamma}) \), \( S \) must be a singleton. Similarly we can argue that \( T \) is a singleton subset of \( Q \).

Let \( \vec{b}_S := (b_0^S, b_1^S, b_2^S) \in S \) and \( \vec{b}_T := (b_0^T, b_1^T, b_2^T) \in T \).

Let \( s, s' \in Y \) be such that \( s(\vec{x}) = (0,1,1,\vec{\gamma}) \) and \( s'(\vec{x}) = (1,1,0,\vec{\gamma}) \). Now, as \( s(x_1\vec{z}) = (1,\vec{\gamma}) = s'(x_1\vec{z}) \), by strong determinism we can conclude that \( b_1^S = s(y_1) = s'(y_1) = b_1^T \). Thus, if \( S = \{(0,0,0)\} \) or \( S = \{(1,0,1)\} \), then \( T = \{(0,0,0)\} \) or \( T = \{(1,0,1)\} \); and if \( S = \{(0,1,1)\} \) or \( S = \{(1,1,0)\} \), then \( T = \{(0,1,1)\} \) or \( T = \{(1,1,0)\} \). Thus there are 8 options for assignments pairs \( s, s' \in Y \) with \( s(\vec{z}) = s'(\vec{z}) = \vec{\gamma} \) and \( s(\vec{x}) = (0,1,1) \) and \( s'(\vec{x}) = (1,1,0) \):

| \( s \)  | \( x_0 \) | \( x_1 \) | \( x_2 \) | \( y_0 \) | \( y_1 \) | \( y_2 \) | \( \vec{z} \) | \( s' \)  | \( x_0 \) | \( x_1 \) | \( x_2 \) | \( y_0 \) | \( y_1 \) | \( y_2 \) | \( \vec{z} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( s_0 \) | 0 | 1 | 1 | 0 | 0 | 0 | \( \vec{\gamma} \) | \( s_4 \) | 0 | 1 | 1 | 0 | 1 | 1 | \( \vec{\gamma} \) |
| \( s'_0 \) | 1 | 1 | 0 | 0 | 0 | \( \vec{\gamma} \) | \( s'_4 \) | 1 | 1 | 0 | 0 | 1 | 1 | \( \vec{\gamma} \) |
| \( s_1 \) | 0 | 1 | 1 | 0 | 0 | 0 | \( \vec{\gamma} \) | \( s_5 \) | 0 | 1 | 1 | 0 | 1 | 1 | \( \vec{\gamma} \) |
| \( s'_1 \) | 1 | 1 | 0 | 0 | 0 | \( \vec{\gamma} \) | \( s'_5 \) | 1 | 1 | 0 | 1 | 1 | 0 | \( \vec{\gamma} \) |
| \( s_2 \) | 0 | 1 | 1 | 1 | 0 | 1 | \( \vec{\gamma} \) | \( s_6 \) | 0 | 1 | 1 | 1 | 1 | 0 | \( \vec{\gamma} \) |
| \( s'_2 \) | 1 | 1 | 0 | 0 | 0 | \( \vec{\gamma} \) | \( s'_6 \) | 1 | 1 | 0 | 0 | 1 | 1 | \( \vec{\gamma} \) |
| \( s_3 \) | 0 | 1 | 1 | 1 | 0 | 1 | \( \vec{\gamma} \) | \( s_7 \) | 0 | 1 | 1 | 1 | 1 | 0 | \( \vec{\gamma} \) |
| \( s'_3 \) | 1 | 1 | 0 | 1 | 0 | 1 | \( \vec{\gamma} \) | \( s'_7 \) | 1 | 1 | 0 | 1 | 1 | 0 | \( \vec{\gamma} \) |
Let \( s'' \in Y \) be such that \( s''(\vec{x}\vec{z}) = (1, 0, 1, \vec{\gamma}) \) (such \( s'' \) exists as \( (1, 0, 1) \in P \)). By going through each case \( s = s_i \) and \( s' = s'_i, i < 8 \), of options we can show that \( s'' \) is unique.

For instance, if \( s = s_5 \) and \( s' = s'_5 \), i.e. \( \vec{y} \mapsto (0, 1, 1) \) and \( \vec{y} \mapsto (1, 1, 0) \), then, by strong determinism, as \( s'(x_0\vec{z}) = (1, \vec{\gamma}) = s''(x_0\vec{z}) \), we get \( s''(y_0) = s'(y_0) = 1 \), and as \( s(x_2\vec{z}) = (1, \vec{\gamma}) = s''(x_2\vec{z}) \), we get \( s''(y_2) = s(y_2) = 1 \). Thus \( s''(\vec{y}) = (1, b, 1) \) for some \( b \). As \( s''(\vec{x}) = (1, 0, 1) \in P \), we have \( s(\vec{y}) \in Q \), and the only element of \( Q \) of the form \( (1, b, 1) \) is \( (1, 0, 1) \). Thus \( s''(\vec{y}) = (1, 0, 1) \).

We can represent the values of \( y_i \) as a function of values of \( x_i \) as follows: we write a so called Mermin instruction

\[
\begin{array}{cccc}
b_0^0 & b_0^1 & b_0^2 \\
b_1^0 & b_1^1 & b_1^2 \\
\end{array}
\]

where, given an assignment and \( i < 3 \), if the value of \( x_i \) under the assignment is \( j < 2 \), then the value of \( y_i \) will be \( b_i^j \). In the example above (case 5), we have the instruction

\[
\begin{array}{c}
000 \\
111.
\end{array}
\]

Overall, we get as the complete set of Mermin instructions the following table. (On the first row are the instructions for the cases 0, 1, 2 and 3, and on the second row for the cases 4, 5, 6 and 7.)

\[
\begin{array}{cccc}
000 & 011 & 110 & 101 \\
000 & 100 & 001 & 101 \\
011 & 000 & 101 & 110 \\
011 & 111 & 010 & 110 \\
\end{array}
\]

Let \( \vec{b} \) be the top row of the instruction that corresponds to the case that we are in. Let \( \vec{a} = (0, 0, 0) \). Now, whichever the case was, we have for any \( i < 3 \) some \( s_i \in Y \) with \( s_i(x_iy_i\vec{z}) = (a_i, b_i, \vec{\gamma}) \). As we chose \( \vec{\gamma} \) so that it occurs with \( \vec{a} \), we can, by locality, find an assignment \( s \in Y \) with \( s(\vec{x}\vec{y}\vec{z}) = \vec{a}\vec{b}\vec{\gamma} \). This implies that \( \vec{b} \in R \). Now notice that every element of \( R \) contains an even number of 0s, but in every case the top row of the instruction contains an odd number of 0s. This is a contradiction, and the proof is finished.

### 4.3 Hardy teams

We define the concept of a Hardy team and then use Hardy teams to prove a non-logical consequence result.
Definition 4.4. Assume that \( n = 2 \). Let \( X \) be an empirical team with \( \text{rng}(X) = \{0, 1\} \). Let \( s_0, \ldots, s_3 \) be as in the following table.

|     | \( x_0 \) | \( x_1 \) | \( y_0 \) | \( y_1 \) |
|-----|---------|---------|---------|---------|
| \( s_0 \) | 0       | 0       | 0       | 0       |
| \( s_1 \) | 0       | 1       | 0       | 0       |
| \( s_2 \) | 1       | 0       | 0       | 0       |
| \( s_3 \) | 1       | 1       | 1       | 1       |

We say that \( X \) is a Hardy team if the following hold:

(i) \( s_0 \in X \) but \( s_1, s_2, s_3 \notin X \), and

(ii) for every pair \( \vec{a} \in \{0, 1\}^2 \) there is some \( s \in X \) with \( s(\vec{x}) = \vec{a} \).

A minimal example of a Hardy team would be the following:

|     | \( x_0 \) | \( x_1 \) | \( y_0 \) | \( y_1 \) |
|-----|---------|---------|---------|---------|
| \( s_0 \) | 0       | 0       | 0       | 0       |
| \( s_1' \) | 0       | 1       | 1       | 1       |
| \( s_2' \) | 1       | 0       | 1       | 1       |
| \( s_3' \) | 1       | 1       | 0       | 0       |

Theorem 4.5. No Hardy team can be realized by a hidden-variable team supporting \( z \)-independence and locality.

Proof of Theorem 4.5. Let \( X \) be a Hardy team and suppose for a contradiction that it is realized by some hidden-variable team that supports \( z \)-independence and locality. Again by Proposition 3.19 there is a hidden-variable team \( Y \) that supports \( z \)-independence and strong determinism and realizes \( X \). As strong determinism implies locality and locality implies parameter independence, \( Y \) supports parameter independence.

As \( Y \) realizes \( X \) and \( s_0 \in X \), there is some \( s \in Y \) with \( s(\vec{x}y) = (0, 0, 0, 0) \). Denote \( \vec{\gamma} = s(\vec{z}) \). By (i) in the definition of a Hardy team, there is \( s' \in Y \) with \( s'(\vec{x}) = (0, 1) \), and by \( z \)-independence we may assume that \( s'(\vec{z}) = \vec{\gamma} \). As \( s'(x_0\vec{z}) = (0, \vec{\gamma}) = s(x_0\vec{z}) \), we may apply parameter independence to \( s' \) and \( s \) to obtain some \( \hat{s} \in Y \) with \( \hat{s}(\vec{x}y_0\vec{z}) = (s'(x_0), s'(x_1), s(y_0), s'(\vec{z})) = (0, 1, 0, \vec{\gamma}) \). As \( s_1 \notin X \), we know that \( \hat{s}(y_1) \neq 0 \). Thus \( \hat{s}(y_1) = 1 \). This confirms the identity of \( \hat{s} \) to be the mapping \( \vec{x}y\vec{z} \mapsto (0, 1, 0, 1, \vec{\gamma}) \). Similarly, using the absence of \( s_2 \)

\[4\text{I.e. the team satisfies the so called universality atom } \forall (\vec{x}).\]
and \( s_3 \) in \( X \), we obtain assignments \( \hat{s}', \hat{s}'' \in Y \) with \( \hat{s}'(\vec{x}\vec{y}\vec{z}) = (1, 1, 0, 1, \vec{\gamma}) \) and \( \hat{s}''(\vec{x}\vec{y}\vec{z}) = (1, 0, 0, 1, \vec{\gamma}) \).

Now, by strong determinism \( Y \models = (x_1 \vec{z}, y_1) \), so as \( s(x_1 \vec{z}) = (0, \vec{\gamma}) = \hat{s}''(x_1 \vec{z}) \), we have also \( 0 = s(y_1) = \hat{s}''(y_1) = 1 \), which is a contradiction. \( \square \)

### 4.4 KS teams

We define the concept of a KS team and then use KS teams to prove a non-logical consequence result.

**Definition 4.6.** We define an action of the symmetry group \( S_n \) on a set \( A^n \) by setting
\[
\pi \cdot \vec{a} = (a_{\pi^{-1}(0)}, \ldots, a_{\pi^{-1}(n-1)})
\]
for all \( \vec{a} \in A^n \) and \( \pi \in S_n \). We say that an empirical team \( X \) is **equivariant** if, for all \( \vec{a} \) and \( \vec{b} \), and for all \( \pi \in S_n \),
\[
\exists s \in X (s(\vec{x}) = \vec{a} \land s(\vec{y}) = \vec{b}) \iff \exists s \in X (s(\vec{x}) = \pi \cdot \vec{a} \land s(\vec{y}) = \pi \cdot \vec{b}).
\]

Suppose that \( n = 4 \). Let \( P \) be the set of quarduples whose elements are
\[
(0, 1, 2, 3), \quad (0, 4, 5, 6), \quad (7, 8, 2, 9),
(7, 10, 6, 11), \quad (1, 4, 12, 13), \quad (8, 10, 13, 14),
(15, 16, 3, 9), \quad (15, 17, 5, 11), \quad (16, 17, 12, 14)
\]
and \( Q \) the set whose elements are
\[
(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1).
\]

We call a team \( X \) with \( \{s(x_i) \mid s \in X, i < 4\} \subseteq \{0, \ldots, 17\} \) and \( \{s(y_i) \mid s \in X, i < 4\} \subseteq \{0, 1\} \) a KS team if

1. there is a function \( f: P \to Q \) such that for all \( \vec{a} \in P \) and \( \vec{b} \in \{0, 1\}^4 \),
\[
\exists s \in X (s(\vec{x}) = \vec{a} \land s(\vec{y}) = \vec{b}) \iff f(\vec{a}) = \vec{b},
\]
and
2. \( X \) is equivariant.

**Theorem 4.7.** The formula
\[
\exists z_0 \exists z_1 \ldots \exists z_{l-1} \left( z \perp \vec{x} \land \bigwedge_{i<n} \{x_j \mid j \neq i\} \perp_{x_i \vec{z}} y_i \right)
\]
is not valid, as demonstrated by any KS team.
In words, there is an empirical team which cannot be realized by any hidden-variable team supporting $z$-independence and parameter independence.

The theorem gives an even stronger result than Theorems 4.3 and 4.5: locality is the conjunction of parameter and outcome independence, but even if we leave out outcome independence, we can construct a counter-example.

*Proof of Theorem 4.7.* Let $X$ be an KS team. Suppose for a contradiction that $X$ is realized by some hidden-variable team $Y$ supporting $z$-independence and parameter independence. Then by Proposition 3.16, $X$ supports no-signalling.

Fix the function $f$ from the definition of $X$, and let $c = (0, 1, 2, 3)$ for the unique $i < 4$ such that $(f(0, 1, 2, 3))_i = 1$. Then let $s \in X$ be such that $s(\vec{x}) = (0, 1, 2, 3)$ and $s(\vec{y}) = f(0, 1, 2, 3)$. Notice that every element of $\text{rng}(X)$ occurs in exactly two tuples in $P$. Let $\vec{a}$ be the other element of $P$ where $c$ occurs. Then by equivariance, there is $s' \in X$ with $s'(\vec{x}) = \pi.\vec{a}$ and $s'(\vec{y}) = \pi.f(\vec{a})$, where $\pi$ is such a permutation that $c = (\pi.\vec{a})_i$. By no-signalling, $X \models y_i \perp x_i \{x_j \mid j \neq i\}$. Thus, as $s(x_i) = c = s'(x_i)$, there exists $s'' \in X$ with $s''(\vec{x}) = \pi.\vec{a}$ and $s''(\vec{y}) = f(\vec{a})$. This means that $s''(x_i) = c$ and $s''(y_i) = (f(0, 1, 2, 3))_i = 1$. But by the definition of $X$, this means that $f$ assigns the value 1 to $c$ in both tuples that $c$ occurs in.

The same argument can be applied to every element of $P$, whence we get that $f$ assigns the same value to the same element in every tuple ($f$ is non-contextual). However, this is impossible: as every element occurs in exactly two tuples of $P$, the number $|\{\vec{a} \in P \mid f(\vec{a}) \text{ contains a 1}\}|$ is even, but clearly the number should be odd, as every element of $P$ has a unique element whose corresponding value under $f$ is 1 and $P$ has an odd number of elements. This is the desired contradiction.

5 Probabilistic Teams

So far we have only been looking at possibilistic (i.e. two-valued relational) versions of the independence notions of quantum physics while these notions are usually taken to be probabilistic. To be able to discuss the probabilistic notions in the point of view of team semantics, we need a suitable framework. For this, we consider probabilistic team semantics.

The study of a probabilistic variant of independence logic was first done in a multiteam setting in [9]. Prior to that, multiteams were studied in [18], [19] and [27]. *Probabilistic teams* were then introduced by Durand et al. in [10] as a way to generalize multiteams, and further investigated in [15]. They can be thought of as a special case of *measure teams*, another approach to probabilities in team semantics given in [19].

It should be observed that we are not introducing probabilistic logic in the sense that formulas have probabilities. In our approach, only the teams are probabilistic and everything else is two-valued.
Definition 5.1. Let \( A \) be a finite set and \( V \) a finite set of variables. A probabilistic team, with variable domain \( V \) and value domain \( A \), is a probability distribution \( X: A^V \rightarrow [0,1] \).

Let \( \mathfrak{A} \) be a (possibly many-sorted) finite structure, and let \( X \) be the full team of \( \mathfrak{A} \) with domain \( V \). Then a probabilistic team of \( \mathfrak{A} \) with variable domain \( V \) is a distribution \( X: X \rightarrow [0,1] \).

Ordinary teams of size \( k \) can be seen as probabilistic teams by giving each assignment in the team probability \( 1/k \) and assignments not in the team probability zero. This idea of treating ordinary teams as uniformly distributed probabilistic teams generalizes to multiteams i.e. teams in which assignments can have several occurrences. Then an assignment which occurs \( m \) times is given the probability \( m/k \). In fact, it is not difficult to see that any probabilistic team with rational probabilities corresponds to a multiteam.

We will call a probabilistic team with variable domain \( V_m \cup V_o \) a probabilistic empirical team and a probabilistic team with variable domain \( V_m \cup V_o \cup V_h \) a probabilistic hidden-variable team.

Definition 5.2. We say that a team \( X \) is the possibilistic collapse of a probabilistic team \( X \) if for any assignment \( s, s \in X \) if and only if \( X(s) > 0 \).

Note that if \( X \) is a probabilistic team of \( \mathfrak{A} \), then the possibilistic collapse \( X \) is a team of \( \mathfrak{A} \).

We may consider a probabilistic team a “possibilistic realization” of its collapse. Of course, an ordinary team has a multitude such probabilistic realizations.

The possibilistic collapse of a probabilistic team \( X \) is also called the support of \( X \) and denoted by \( \text{supp} X \).

We denote by \( |X_{\bar{u}=\bar{a}}| \) the number

\[
\sum_{s(\bar{u})=\bar{a}} X(s),
\]

i.e. the marginal probability of the variable tuple \( \bar{u} \) having the value \( \bar{a} \) in \( X \).

Next we define the probabilistic analogue for an empirical team being realized by a hidden-variable team.

Definition 5.3. A probabilistic hidden-variable team \( Y \) realizes a probabilistic empirical team \( X \) if for all \( \bar{a} \) and \( \bar{b} \) we have

(i) \( |X_{\bar{y}=\bar{a}}| = 0 \) if and only if \( |Y_{\bar{y}=\bar{a}}| = 0 \), and

(ii) \( |X_{\bar{y}=\bar{a}} \cdot |Y_{\bar{y}=\bar{a}}| = |Y_{\bar{y}=\bar{a}} \cdot |X_{\bar{y}=\bar{a}}| \).
Y uniformly realizes X if in addition $|X_{\vec{x} = \vec{a}}| = |Y_{\vec{x} = \vec{a}}|$ for all $\vec{a}$.

The intuition behind the definition is the following: Y realizes X if the probability of the event “$s(\vec{x}) = \vec{a}$” is non-zero in both teams exactly the same time, and in case the probability indeed is non-zero, the probability of the event “$s(\vec{y}) = \vec{b}$”, conditional to “$s(\vec{x}) = \vec{a}$”, is the same in both teams. Uniform realizability appears to be a stronger concept than realizability, but we do not have a proof for that yet.

**Proposition 5.4.** If Y realizes X, then the possibilistic collapse of Y realizes the possibilistic collapse of X.

Proposition 5.4 says that one obtains the same team by first projecting away hidden variables and then taking the possibilistic collapse as one gets by first taking the possibilistic collapse and then projecting away the hidden variables, i.e. the diagram in Figure 1 commutes.

![Diagram](image)

**Figure 1:** Probabilistic realization implies possibilistic realization.

**Proof of Proposition 5.4.** Suppose that Y realizes X, and denote by Y and X the respective possibilistic collapse. In order to prove that Y realizes X, we need to show that for all assignments $s$,

$$s \in X \iff \exists s' \in Y \left( \bigwedge_{i<n} (s'(x_i) = s(x_i) \land s'(y_i) = s(y_i)) \right).$$

We show only one direction, the other one is similar.

Suppose that $s \in X$. Denote $\vec{a} = s(\vec{x})$ and $\vec{b} = s(\vec{y})$. The aim is to show that there is some $s' \in Y$ with $s'(\vec{x}) = \vec{a}$ and $s'(\vec{y}) = \vec{b}$. Since X is the possibilistic
collapse of $X$ and $s \in X$, we have $X(s) > 0$, and as in addition $s(x) = \vec{a}$, we get $|X_{\vec{x}=\vec{a}}| > 0$. Thus, as $\mathcal{Y}$ realizes $X$, $|Y_{\vec{x}=\vec{a}}| > 0$. Similarly, $|X_{\vec{x}\vec{y}=\vec{a}\vec{b}}| > 0$, and as in addition $s(\vec{y}) = \vec{a}$, we get $|X_{\vec{x}\vec{y}=\vec{a}\vec{b}}| > 0$. Thus, as $\mathcal{Y}$ realizes $X$, $|Y_{\vec{x}\vec{y}=\vec{a}\vec{b}}| > 0$. Similarly, $|Y_{\vec{x}\vec{y}=\vec{a}\vec{b}}| > 0$. This means that there is some $s'$ with $Y(s') > 0$ and $s'(\vec{x}\vec{y}) = \vec{a}\vec{b}$. Since $Y$ is the possibilistic collapse of $\mathcal{Y}$, this means that $s' \in Y$, as desired.

5.1 Probabilistic independence logic

We now present the semantics of the probabilistic (conditional) independence atom $\vec{u} \perp \perp \vec{v} \vec{w}$, as defined in [10].

**Definition 5.5.** Let $\mathfrak{A}$ be a structure and $X$ a probabilistic team of $\mathfrak{A}$, and let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be tuples of variables. Then $\mathfrak{A}$ and $X$ satisfy the formula $\vec{u} \perp \perp \vec{v} \vec{w}$, in symbols $\mathfrak{A} \models X \vec{u} \perp \perp \vec{v} \vec{w}$, if for all $\vec{a}$, $\vec{b}$ and $\vec{c}$,

$$|X_{\vec{u}\vec{v}=\vec{a}\vec{b}}| \cdot |X_{\vec{v}\vec{u}=\vec{b}\vec{c}}| = |X_{\vec{u}\vec{v}\vec{u}=\vec{a}\vec{b}\vec{c}}| \cdot |X_{\vec{v}=\vec{b}}| .$$

The intention behind the atom is to capture the notion of conditional independence in probability theory: denoting a probability measure by $p$, two events $A$ and $B$ are conditionally independent over an event $C$ (assuming $p(C) > 0$) if

$$p(A \mid C)p(B \mid C) = p(A \cap B \mid C).$$

Recalling that $p(D \mid C) = p(D \cap C)/p(C)$, we can multiply both sides of the equation by $p(C)^2$ and obtain

$$p(A \cap C)p(B \cap C) = p(A \cap B \cap C)p(C),$$

which is exactly what the probabilistic dependence atom expresses. In the case when $p(C) = 0$, both sides of the new equation are 0, so in that case the independence atom is vacuously true.

Exactly the same way as in ordinary independence logic, we can define the probabilistic dependence atom $=(\vec{v},\vec{w})$ via the independence atom:

**Definition 5.6.** Let $\vec{v}$ and $\vec{w}$ tuples of variables. Then by $=(\vec{v},\vec{w})$ we mean the formula $\vec{w} \perp \perp \vec{v} \vec{w}$. By $=(\vec{w})$ we mean the formula $\vec{w} \perp \perp \vec{w}$ and call it the probabilistic constancy atom.
The syntax of probabilistic independence logic is the same as the syntax of ordinary independence logic, except that we use the symbol ⊥ instead of ⊥. Next we present the semantics of more complicated formulas of probabilistic independence logic, as defined in [10]. First we define the r-scaled union \( \mathcal{X} \sqcup_r \mathcal{Y} \) of two probabilistic teams \( \mathcal{X} \) and \( \mathcal{Y} \) with the same variable and value domain, for \( r \in [0,1] \), by setting
\[
(\mathcal{X} \sqcup_r \mathcal{Y})(s) := r\mathcal{X}(s) + (1-r)\mathcal{Y}(s).
\]
We define the (duplicated) team \( \mathcal{X}[A(s)/v] \) by setting
\[
\mathcal{X}[A(s)/v](s(a/v)) := \sum_{t \in \text{supp} \mathcal{X} \atop t(a/v) = s(a/v)} \frac{\mathcal{X}(t)}{|A(s)|}
\]
for all \( a \in A(s) \). If \( v \) is a fresh variable, then \( \mathcal{X}[A/v](s(a/v)) = \mathcal{X}(s)/|A(s)| \).
Finally, given a function \( F \) from the set \( \text{supp} \mathcal{X} \) to the set of all probability distributions on \( A(s) \), we define the (supplemented) team \( \mathcal{X}[F/v] \) by setting
\[
\mathcal{X}[F/v](s(a/v)) := \sum_{t \in \text{supp} \mathcal{X} \atop t(a/v) = s(a/v)} \mathcal{X}(t)F(t)(a)
\]
for all \( a \in A(s) \). Again, if \( v \) is fresh, then \( \mathcal{X}[F/v](s(a/v)) = \mathcal{X}(s)F(s)(a) \).

**Definition 5.7.** Let \( \mathfrak{A} \) be a structure and \( \mathcal{X} \) a probabilistic team of \( \mathfrak{A} \). Then

(i) \( \mathfrak{A} \models_{\mathcal{X}} \alpha \) for a first-order atomic or negated atomic formula \( \alpha \) if \( \mathfrak{A} \models_{X} \alpha \), where \( X \) is the possibilistic collapse of \( \mathcal{X} \).

(ii) \( \mathfrak{A} \models_{\mathcal{X}} \varphi \land \psi \) if \( \mathfrak{A} \models_{\mathcal{X}} \varphi \) and \( \mathfrak{A} \models_{\mathcal{X}} \psi \).

(iii) \( \mathfrak{A} \models_{\mathcal{X}} \varphi \lor \psi \) if \( \mathfrak{A} \models_{\mathcal{Y}} \varphi \) and \( \mathfrak{A} \models_{\mathcal{Z}} \psi \) for some probabilistic teams \( \mathcal{Y} \) and \( \mathcal{Z} \), and \( r \in [0,1] \) such that \( \mathcal{X} = \mathcal{Y} \sqcup_r \mathcal{Z} \).

(iv) \( \mathfrak{A} \models_{\mathcal{X}} \forall v \varphi \) if \( \mathfrak{A} \models_{\mathcal{X}[A(s)/v]} \varphi \).

(v) \( \mathfrak{A} \models_{\mathcal{X}} \exists v \varphi \) if \( \mathfrak{A} \models_{\mathcal{X}[F/v]} \varphi \) for some function \( F : \text{supp} \mathcal{X} \to \{p \in [0,1]^{A(s)} \mid p \text{ is a probability distribution}\} \).

(vi) \( \mathfrak{A} \models_{\mathcal{X}} \exists^* v \varphi \) if \( \mathfrak{B} \models_{\mathcal{X}} \forall v \varphi \) for all expansions \( \mathfrak{B} \) of \( \mathfrak{A} \) by the sort \( s(v) \).

(vii) \( \mathfrak{A} \models_{\mathcal{X}} \exists^* v \varphi \) if \( \mathfrak{B} \models_{\mathcal{X}} \exists v \varphi \) for some expansion \( \mathfrak{B} \) of \( \mathfrak{A} \) by the sort \( s(v) \).

Again, when it is clear what is meant, we write \( \mathcal{X} \models \varphi \) instead of \( \mathfrak{A} \models_{\mathcal{X}} \varphi \).

We next show that probabilistic independence atom satisfies the axioms of Definition 1.5. In order to do this, it is convenient to introduce the concept of a separoid.
Definition 5.8 (Separoid, [8]). Let $A$ be a set, $\leq$ a binary relation on $A$ and $\perp$ a ternary relation on $A$. The structure $(A, \leq, \perp)$ is a separoid if

(i) $(A, \leq)$ is a quasorder\footnote{That is, $\leq$ is reflexive and transitive but not necessarily anti-symmetric.} such that for any two elements $a, b \in A$, the set $\{a, b\}$ has a least upper bound $a \lor b \in A$ (the join of $a$ and $b$), and

(ii) the following axioms hold for all $a, b, c, d \in A$:

(P1) $a \perp c b$ implies $b \perp c a$.
(P2) $a \perp a b$.
(P3) $a \perp c b$ and $d \leq b$ imply $a \perp c d$.
(P4) $a \perp c b$ and $d \leq b$ imply $a \perp c d b$.
(P5) $a \perp c b$ and $a \perp b \lor c d$ imply $a \perp c (b \lor d)$.

Theorem 5.9 ([8]). Let $(A, \leq, \perp)$ be a separoid. Define a binary relation $\succeq$ on $A$ by

$$a \succeq b \iff a \perp b a.$$

Then also $(A, \succeq, \perp)$ is a separoid.

Lemma 5.10 ([8]). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, \mathcal{E})$ a measurable space, and denote by $\mathbf{F}$ the set of random variables $\Omega \to E$. Then the structure $(\mathbf{F}, \leq, \perp)$ is a separoid, where $X \leq Y$ if for all $a, b \in \Omega$,

$$Y(a) = Y(b) \implies X(a) = X(b),$$

and $X \perp Y$ if $X$ and $Y$ are stochastically conditionally independent, given $Z$.

Lemma 5.11. Let $V$ be a finite set of variables and $\mathcal{X}$ a probabilistic team with variable domain $V$. Then the structure $(V^{<\omega}, \succeq, \perp)$ is a separoid, where

$$\vec{x} \succeq \vec{y} \iff \mathcal{X} \models = (\vec{y}, \vec{x})$$

and

$$\vec{x} \perp \vec{y} \iff \mathcal{X} \models \vec{x} \perp \vec{y}.$$

Proof. Denote by $A$ the value domain of $\mathcal{X}$. We can consider a variable tuple $\vec{x}$ as a random variable $X_{\vec{x}}$ on the probability space $(A^{V}, \mathcal{X})$ with values in $A$ such that $X_{\vec{x}}(s) = s(\vec{x})$. Then $\mathcal{X}(X_{\vec{x}} = \vec{a}) = \mathcal{X}_{\vec{x}=\vec{a}}$. Thus, from Lemma 5.10, it follows that $(V^{<\omega}, \succeq, \perp)$ is a separoid. Then the rest follows from Theorem 5.9.

Proposition 5.12. Probabilistic independence atom satisfies the axioms of Definition 1.5.
Proof. Let $X$ be a probabilistic team. We show that $X$ satisfies each axiom. Denote by $S$ the separoid of Lemma 5.11.

(i) **Constancy Rule:** Suppose $X \models \vec{y} \perp \perp \vec{x}$. Then, in $S$, we have $\vec{y} \preceq \vec{x}$. By the separoid axiom (P2), we have $\vec{x} \perp \perp \vec{z}$ and using symmetry (P1) we get $\vec{z} \perp \perp \vec{x}$. From the assumptions

\[
\vec{x} \perp \perp \vec{z} \quad \text{and} \quad \vec{y} \preceq \vec{x}
\]

we can, using (P3), infer $\vec{z} \perp \perp \vec{y}$. Then the desired result, $\vec{y} \perp \perp \vec{z}$, is obtained by symmetry. Thus $X \models \vec{y} \perp \perp \vec{z}$.

(ii) **Reflexivity Rule:** This is the separoid axiom (P2).

(iii) **Symmetry Rule:** This is the separoid axiom (P1).

(iv) **Weakening Rule:** Suppose that $X \models \vec{y}' \perp \perp \vec{z}'$. Now clearly $X \models = (\vec{y}', \vec{y}) \wedge = (\vec{z}', \vec{z})$, so in $S$, we have $\vec{y} \preceq \vec{y}'$ and $\vec{z} \preceq \vec{z}'$. From the assumptions

\[
\vec{y}' \perp \perp \vec{z}' \quad \text{and} \quad \vec{x} \perceq \vec{z}
\]

we can, using the separoid axiom (P3), infer $\vec{y}' \perp \perp \vec{z}$. Using symmetry and applying (P3) to

\[
\vec{x} \perp \perp \vec{y}' \quad \text{and} \quad \vec{z} \preceq \vec{y}',
\]

we finally obtain $\vec{z} \perp \perp \vec{y}$, whence $X \models \vec{y} \perp \perp \vec{z}$.

(v) **Permutation Rule:** A similar argument to the proof of the Weakening rule (use the facts that if $\vec{x}'$ is a permutation of $\vec{x}$, then $\vec{x}' \preceq \vec{x}$ in $S$).

(vi) **Fixed Parameter Rule:** Suppose that $X \models \vec{z} \perp \perp \vec{y}$. Then by the separoid axiom (P2), $\vec{x} \perp \perp \vec{z}$ holds, and by symmetry we get $\vec{z} \perp \perp \vec{x}$. Notice that in $S$, we have $\vec{x} \vee \vec{x} = \vec{x}$. Thus $\vec{z} \perp \perp \vec{x}$ holds. Then from the assumptions

\[
\vec{z} \perp \perp \vec{x} \quad \text{and} \quad \vec{y} \perp \perp \vec{x} \vee \vec{y}
\]

we can, using the separoid axiom (P5), infer $\vec{z} \perp \perp (\vec{x} \vee \vec{y})$. Notice that $\vec{x} \vee \vec{y} = \vec{y} \vec{x}$, whence $\vec{z} \perp \perp \vec{y} \vec{x}$ holds. A symmetric argument now shows that $\vec{y} \vec{x} \perp \perp \vec{z} \vec{x}$, whence $X \models \vec{y} \vec{x} \perp \perp \vec{z} \vec{x}$.

(vii) **First Transitivity Rule:** Suppose that $X \models \vec{x} \perp \perp \vec{z} \wedge \vec{y} \perp \perp \vec{z} \vec{y}$. Using symmetry and noting that $\vec{z} \vec{x} = \vec{x} \vee \vec{z}$, we get that $\vec{y} \perp \perp \vec{x}$ and $\vec{y} \perp \perp \vec{z} \vec{y}$ hold in $S$. Then, applying the separoid axiom (P5) to the assumptions

\[
\vec{y} \perp \perp \vec{x} \quad \text{and} \quad \vec{y} \perp \perp \vec{z} \vec{y}
\]

we get $\vec{y} \perp \perp (\vec{x} \vee \vec{y})$. As $\vec{y} \preceq \vec{x} \vee \vec{y}$, by (P3) we get $\vec{y} \perp \perp \vec{u}$, whence $X \models \vec{u} \perp \perp \vec{y}$. 33
(viii) **Second Transitivity Rule:** Suppose that $X \models \vec{y} \perp \vec{x} \wedge \vec{z} \perp \vec{u}$. Now, obviously $X \models = (\vec{z} \vec{x}, \vec{z})$, so $\vec{z} \leq \vec{z} \vec{x}$ in $S$. Then from the assumptions

$$u \perp \vec{y} \vec{x} \quad \text{and} \quad \vec{z} \leq \vec{z} \vec{x}$$

we can, by the separoid axiom (P4), infer $\vec{u} \perp \vec{y} \vec{x}$. Then from the assumptions $\vec{u} \perp \vec{y} \vec{x}$ and $\vec{z} \leq \vec{z} \vec{x}$ we can, by the separoid axiom (P4), infer $\vec{u} \perp \vec{y} \vec{x}$. Then, as $X \models = (\vec{z}, \vec{y})$, we have $\vec{z} \leq \vec{y}$ in $S$, and thus $\vec{z} = \vec{y} \vec{x}$. But then $\vec{u} \perp \vec{z} \vec{x}$ holds in $S$, whence $X \models = (\vec{y}, \vec{u})$.

(ix) **Exchange Rule:** Suppose that $X \models = (\vec{x} \vec{y}, \vec{x})$. Then, by (P5), we finally obtain $X \models = (\vec{z} \vec{y}, \vec{u})$.

---

**Proposition 5.13** (Probabilistic Soundness Theorem). If $\varphi$ entails $\psi$ by repeated applications of the rules of Definition 1.5, Definition 1.6 and Definition 1.7, then $\varphi \models \psi$ in probabilistic team semantics.

**Proof.** Soundness of the axioms of the independence atom is shown in Proposition 5.12.

**Dependence introduction:** Follows from Corollary 5.20.

**Elimination of existential quantifier:** Follows from Lemma 5.21 and the observation that if $x \notin V$, then $X[F/x] \models V = X \models V$ for any function $F$.

**Introduction of existential quantifier:** Suppose that $y$ does not occur in the range of $\exists x$ or $\forall x$ in $\varphi$. Suppose that $X \models = \varphi(y/x)$. Then define a function $F$ by setting

$$F(s)(a) = \begin{cases} 1 & \text{if } s(y) = a, \\ 0 & \text{otherwise}. \end{cases}$$

Then clearly $X[F/x]$ is the same distribution on assignments $s(s(y)/x)$ as $X$ is on assignments $s$. Thus $X[F/x] \models = \varphi$ and hence $X \models = \exists x \varphi$.

By definition, first-order atomic formulas are satisfied by a probabilistic team if and only if the underlying possibilistic collapse satisfies them. This property is in the multiteam setting of [9] called weak flatness.

**Definition 5.14.** We say that a formula $\varphi$ of a probabilistic independence logic is **weakly flat** if for all probabilistic teams $X$, denoting by $X$ the possibilistic collapse, we have

$$X \models = \varphi \iff X \models = \varphi^*,$$
where $\varphi^*$ is the formula of independence logic obtained from $\varphi$ by replacing each occurrence of the symbol $\bot$ by the symbol $\bot$. A sublogic of probabilistic independence logic is weakly flat if every formula of the logic is.

Later on, we simply write $\varphi$ instead of $\varphi^*$, as it is obvious what is meant.

**Lemma 5.15.** The probabilistic dependence atom is weakly flat.

*Proof.* The proof given in the multiteam setting in [9] works also in the probabilistic team setting. 

**Lemma 5.16.** Logical operations of Definition 1.2 preserve weak flatness.

*Proof.* Suppose that $\varphi$ and $\psi$ are weakly flat. We then show that the formulas $\varphi \land \psi$, $\varphi \lor \psi$, $\exists v \varphi$, $\forall v \varphi$, $\exists v \varphi$ and $\forall v \varphi$ are weakly flat. Let $\mathfrak{A}$ be a structure and $X$ a probabilistic team of $\mathfrak{A}$, and let $X$ be the possibilistic collapse of $X$. Note that to show that a formula $\theta$ is weakly flat, we only need to show that $\mathfrak{A} \models X \theta = X \theta$, as the other direction is true for all formulas of independence logic (see Proposition 5.24 below).

The case for conjunction is trivial.

Suppose that $X \models \varphi \lor \psi$. Then there are $X_0$ and $X_1$ such that $X_0 \models \varphi$ and $X_1 \models \psi$ and $X = X_0 \cup X_1$. Let $X_i$ be a probabilistic team with collapse $X_i$ such that

$$X_i(s) = \begin{cases} X(s)/(2p_i + q) & \text{if } s \in X_0 \cap X_1, \\ X(s)/\left(p_i + q/2\right) & \text{otherwise}, \end{cases}$$

where $p_i = \sum_{s \in X_i \setminus X_{i-1}} X(s)$ and $q = \sum_{s \in X_0 \cap X_1} X(s)$. As $\varphi$ and $\psi$ are weakly flat, $X_0 \models \varphi$ and $X_1 \models \psi$. Now, if $s \in X_0 \setminus X_1$, then

$$X(s) = \frac{(p_0 + q/2)X_0(s)}{p_0 + q/2} = (p_0 + q/2)X_0(s) = (X_0 \cup X_1)(s),$$

and if $s \in X_1 \setminus X_0$, then

$$X(s) = \frac{(p_1 + q/2)X_1(s)}{p_1 + q/2} = (p_1 + q/2)X_1(s) = (1 - (p_0 + q/2))X_1(s) = (X_0 \cup X_1)(s),$$
and if \( s \in X_0 \cap X_1 \), then
\[
X(s) = \frac{X(s)}{2^2} + \frac{X(s)}{2^2} = \frac{(p_0 + \frac{q}{2})X(s) + (p_1 + \frac{q}{2})X(s)}{2(p_0 + \frac{q}{2})} + \frac{(p_1 + \frac{q}{2})X(s)}{2(p_1 + \frac{q}{2})}
\]
\[
= \frac{(p_0 + \frac{q}{2})X_0(s) + (p_1 + \frac{q}{2})X_1(s)}{2p_0 + q} = (p_0 + \frac{q}{2})X_0(s) + (p_1 + \frac{q}{2})X_1(s)
\]
\[
= (p_0 + \frac{q}{2})X_0(s) + (1 - (p_0 + \frac{q}{2}))X_1(s) = (X_{0} \uplus_{p_0 + \frac{q}{2}} X_{1})(s).
\]
Hence \( X = X_0 \uplus_{p_0 + \frac{q}{2}} X_1 \) and thus \( X \models \varphi \lor \psi \).

Suppose that \( X \models \exists v \varphi \). Then there is a function \( F : X \to A_{s(v)} \) such that \( X[F/v] \models \varphi \). Define a function \( G : X \to \{ p \in [0,1]^{A_{s(v)}} \mid p \text{ is a distribution} \} \) by setting
\[
G(s)(a) = \begin{cases} 
1/|F(s)| & \text{if } a \in F(s), \\
0 & \text{otherwise.}
\end{cases}
\]
Then \( X[F/v] \) is the possibilistic collapse of \( X[G/v] \), as
\[
X[G/v](s(a/v)) > 0 \iff \sum_{t \in X} X(t)G(t)(a) > 0
\]
\[
\iff \exists t \in X (G(t)(a) > 0 \text{ and } t(a/v) = s(a/v))
\]
\[
\iff \exists t \in X (a \in F(t) \text{ and } t(a/v) = s(a/v))
\]
\[
\iff s(a/v) \in X[F/v].
\]
Then as \( \varphi \) is weakly flat, \( X[G/v] \models \varphi \), so \( X \models \varphi \).

The universal quantifier case is similar.

Suppose that \( A \models_X \exists v \varphi \). Then there is an expansion \( B \) of \( A \) of the new sort \( s(v) \) such that \( B \models_X \exists v \varphi \). We already showed that \( \exists v \varphi \) is weakly flat, so thus \( B \models_X Qv \varphi \). Thus \( A \models_X Qv \varphi \).

The universal sort quantifier case is similar.

In ordinary team semantics, the dependence atom is **downwards closed**, meaning that if \( X \models = (\bar{v}, \bar{w}) \), then for any \( Y \subseteq X \) also \( Y \models = (\bar{v}, \bar{w}) \). We define an analogous concept of downwards closedness and show that dependence logic is downwards closed also in probabilistic team semantics.

**Definition 5.17.** We say that a probabilistic team \( Y \) is a **weak subteam** of a probabilistic team \( X \) if they have the same variable and value domain and, denoting by \( Y \) and \( X \) the respective possibilistic collapses, \( Y \subseteq X \). We say that \( Y \) is a **subteam** of \( X \) if it is a weak subteam of \( X \) and there is \( r \in (0,1] \) such that \( X(s) = rY(s) \) for all \( s \in Y \).
The concept of a weak subteam is the weakest notion of subteam that one would think of. Still, due to weak flatness, it seems to be enough.

Definition 5.18. We say that a formula \( \varphi \) of a probabilistic independence logic is \textit{downwards closed} if for all probabilistic teams \( X \) that satisfy \( \varphi \), every subteam of \( X \) also satisfies \( \varphi \). We say that a formula \( \varphi \) is \textit{strongly downwards closed} if for all probabilistic teams \( X \) that satisfy \( \varphi \), every weak subteam of \( X \) also satisfies \( \varphi \). A sublogic of probabilistic independence logic is (strongly) downwards closed if every formula of the logic is.

Lemma 5.19. Every weak flat formula that is downwards closed in ordinary team semantics is strongly downwards closed in probabilistic team semantics.

Proof. Let \( \varphi \) be a weak flat formula that is downwards closed in ordinary team semantics. Let \( X \) a probabilistic team, \( Y \) a weak subteam of \( X \) and \( X \) and \( Y \) the respective possibilistic collapses. Suppose that \( X \models \varphi \). By weak flatness, \( X \models \varphi \). By downwards closedness of \( \varphi \) in ordinary team semantics, \( Y \models \varphi \). Then by weak flatness again, \( Y \models \varphi \).

Notice that all atomic formulas of probabilistic dependence logic are weakly flat, as proved in [9]. Hence we get the corollary:

Corollary 5.20. Probabilistic dependence logic is weakly flat and thus also strongly downwards closed.

Next we show that logical operations preserve downwards closedness. To make it easier, we first show that one may change the name of a bound variable without affecting the truth of the formula.

Lemma 5.21 (Locality, [10]). Let \( \varphi \) be a formula of probabilistic independence logic, with its free variables among \( v_0, \ldots, v_{m-1} \). Then for all probabilistic teams \( X \) whose variable domain \( D \) includes the variables \( v_i \) and any set \( V \) such that \( \{v_0, \ldots, v_{m-1}\} \subseteq V \subseteq D \), we have

\[
X \models \varphi \iff X \restriction V \models \varphi,
\]

where \( X \restriction V \) is the probabilistic team \( Y \) with variable domain \( V \) defined by \( Y(s) = \sum_{t|V=s} X(t) \).

Lemma 5.22. Let \( \varphi \) be a formula of probabilistic independence logic, and let \( Q \in \{\forall, \exists\} \). Then the formulas \( Qv\varphi \) and \( Qw\varphi(w/v) \) are equivalent, where \( w \) is a variable that does not occur in \( \varphi \) and \( \varphi(w/v) \) denotes the formula one obtains by replacing every free occurrence of variable \( v \) in \( \varphi \) by variable \( w \).
Proof. Let $V$ be the set of free variables of $Qv\varphi$. Then in particular, $v, w \notin V$. By Lemma 5.21, for any probabilistic team $X$ whose variable domain contains $V$, we have

$$X \models Qx\varphi \iff X \upharpoonright V = Qx\varphi$$

and

$$X \models Qy\varphi(y/x) \iff X \upharpoonright V = Qy\varphi(y/x).$$

Thus it is enough to show the claim for teams $X$ whose variable domain is exactly $V$. Furthermore, it is enough to show that for any $X$ with variable domain $V$ and for any function $F$,

$$X[F/v] \models \varphi \iff X[F/w] \models \varphi(w/v),$$

as both the universal and existential quantifier cases follow from this.

However, we proceed by proving a more general claim, namely the following: for any formula $\varphi$ such that $w$ does not occur in $\varphi$, and for any $X$ with variable domain $V \cup \{v\}$ and $Y$ with variable domain $V \cup \{w\}$ such that $X(s(a/v)) = Y(s(a/w))$ for all $s$ and $a$, we have

$$X \models \varphi \iff Y \models \varphi(w/v),$$

where $V$ is the set of free variables of $\exists v\varphi$. This is sufficient, as $X[F/v]$ and $X[F/w]$ are such teams. We prove this by induction on $\varphi$.

First, notice that from $X(s(a/v)) = Y(s(a/w))$ it follows that

$$|X_{\bar{u} = \bar{b}}| = |Y_{\bar{u}(w/v) = \bar{b}}|$$

for any variable tuple $\bar{u} \in (V \cup \{v\})^n$. Thus, if $\varphi$ is an independence atom, we have $X \models \varphi$ if and only if $Y \models \varphi(w/v)$.

By flatness, the case for $\varphi$ being a first-order atom is clear. The case for $\varphi = \psi \land \theta$ follows directly from the induction hypothesis.

Then suppose that $\varphi = \psi \lor \theta$. Suppose that $X \models \varphi$. Then there are $X_0$, $X_1$ and $r$ such that $X_0 \models \psi$, $X_1 \models \theta$ and $X = X_0 \cup_r X_1$. Define $Y_i$, $i < 2$, by setting

$$Y_i(s(a/w)) = X_i(s(a/v))$$

for any $s$ and $a$. Now $Y = Y_0 \cup_r Y_1$, as

$$Y(s(a/w)) = X(s(a/v)) = rX_0(s(a/v)) + (1 - r)X_1(s(a/v))$$

$$= rY_0(s(a/w)) + (1 - r)Y_1(s(a/w)).$$
By the induction hypothesis, $\mathcal{Y}_0 \models \psi$ and $\mathcal{Y}_1 \models \theta$. Thus $\mathcal{Y} \models \varphi$. The other direction is similar.

Finally suppose that $\varphi = \exists u \psi$ (the cases for $\forall u \psi$, $\exists u \psi$ and $\forall u \psi$ are similar). Suppose that $\mathcal{X} \models \varphi$. Then there is a function $F$ such that $\mathcal{X}[F/u] \models \psi$. Define a function $G$ by setting
\[
G(s(a/w)) = F(s(a/v))
\]
for all $s$ and $a$. Now
\[
\mathcal{X}[F/u](s(a/v)(b/u)) = \mathcal{X}(s(a/v))F(s(a/v))(b) = \mathcal{Y}(s(a/w))G(s(a/w))(b) = \mathcal{Y}[G/u](s(a/w)(b/u))
\]
Thus by the induction hypothesis $\mathcal{Y}[G/u] \models \psi$ and thus $\mathcal{Y} \models \varphi$. The other direction is similar. \qed

**Proposition 5.23.** If all atomic formulas of a sublogic of probabilistic independence logic are downwards closed, then the whole sublogic is.

**Proof.** We show by induction that every formula is downwards closed. All atomic formulas are downwards closed by assumption. The case of conjunction follows immediately from the induction hypothesis.

Suppose that $\mathcal{Y}$ is a subteam of $\mathcal{X}$ and $\mathcal{X} \models \varphi \vee \psi$. Let $p \in (0, 1]$ be such that $p\mathcal{Y}(s) = \mathcal{X}(s)$ for $s \in \text{supp} \mathcal{Y}$. Note that then $p = \sum_{s \in \text{supp} \mathcal{Y}} \mathcal{X}(s)$. Now there are $\mathcal{X}_0$ and $\mathcal{X}_1$ and $q \in [0, 1]$ such that $\mathcal{X}_0 \models \varphi$ and $\mathcal{X}_1 \models \psi$ and $\mathcal{X} = \mathcal{X}_0 \cup_q \mathcal{X}_1$. Then let $\mathcal{Y}_i$ be such that
\[
\text{supp} \mathcal{Y}_0 = \text{supp} \mathcal{X}_0 \cap \text{supp} \mathcal{Y} \quad \text{and} \quad \mathcal{Y}_0(s) = \mathcal{X}_0(s)/p_0
\]
for $s \in \text{supp} \mathcal{Y}_i$, where $p_0 = \sum_{s \in \text{supp} \mathcal{Y}_0} \mathcal{X}_0(s)$, and
\[
\text{supp} \mathcal{Y}_1 = (\text{supp} \mathcal{X}_1 \setminus \text{supp} \mathcal{X}_0) \cap \text{supp} \mathcal{Y} \quad \text{and} \quad \mathcal{Y}_1(s) = \mathcal{X}_1(s)/p_1
\]
for $s \in \text{supp} \mathcal{Y}_i$, where $p_1 = \sum_{s \in \text{supp} \mathcal{Y}_1} \mathcal{X}_1(s)$. Now

(i) $\mathcal{Y}_i$ is well defined distribution for $i < 2$, as
\[
\sum_{s \in \text{supp} \mathcal{Y}_i} \mathcal{Y}_i(s) = \sum_{s \in \text{supp} \mathcal{Y}_i} \mathcal{X}_i(s)/p_i = \frac{\sum_{s \in \text{supp} \mathcal{Y}_i} \mathcal{X}_i(s)}{\sum_{s \in \text{supp} \mathcal{Y}_i} \mathcal{X}_i(s)} = 1.
\]

(ii) $\mathcal{Y}_i$ is a subteam of $\mathcal{X}_i$ for $i < 2$, as by definition $\text{supp} \mathcal{Y}_i \subseteq \text{supp} \mathcal{X}_i$ and $\mathcal{X}_i(s) = p_i \mathcal{Y}_i(s)$ for $s \in \text{supp} \mathcal{Y}_i$, where $p_i \in (0, 1]$. 39
(iii) $Y = Y_0 \sqcup Y_1$, where

$$r = \frac{qp_0}{qp_0 + (1-q)p_1},$$

as can be verified by a straightforward calculation.

Then by the induction hypothesis, $Y_0 \models \varphi$ and $Y_1 \models \psi$, so $Y \models \varphi \land \psi$.

Suppose that $Y$ is a subteam of $X$ and $X \models \exists v \varphi$. Let $w$ be a fresh variable outside of the variable domain of $X$. By Lemma 5.22, $X \models \exists w \varphi(w/v)$. Let $p \in (0,1]$ be such that $pY(s) = X(s)$ for $s \in \text{supp } Y$. Now $X[F/w] \models \varphi(w/v)$ for some $F$. Then $Y[F/w]$ is a subteam of $X[F/w]$, as

$$X[F/w](s(a/w)) = X(s)F(s)(a)$$
$$= pY(s)F(s)(a)$$
$$= pY[F/w](s(a/w))$$

for all $s \in \text{supp } Y$. But then by the induction hypothesis, $Y[F/w] \models \varphi(w/v)$, so $Y \models \exists w \varphi(w/v)$. Then by Lemma 5.22, $Y \models \varphi$.

The other quantifier cases are similar.

One might wonder whether ordinary independence logic is a result of “collapsing” probabilistic independence logic in the sense that a team $X$ satisfies a formula $\varphi$ if and only if it is the collapse of some probabilistic team $\tilde{X}$ such that $\tilde{X}$ satisfies the probabilistic version of $\varphi$. It turns out, in Proposition 5.24, that, indeed, given a probabilistic team that satisfies a formula, also the collapse will satisfy the (possibilistic version of the) formula. But given an ordinary team that satisfies a formula, there may not be any probabilistic realization of that team that would satisfy the (probabilistic version of the) formula. We will see in Proposition 5.31 that such a formula and a team can be quite simple.

As it is not the case for every formula of independence logic that a team satisfies it if and only if it has a probabilistic realization that satisfies the probabilistic version of the formula, we can add a new operation $\text{PR}$ to ordinary independence logic, defined by

$$X \models \text{PR } \varphi \text{ if } X \text{ has a probabilistic realization } \tilde{X} \text{ such that } \tilde{X} \models \varphi.$$ 

One can then ask whether this operation is downwards closed, closed under unions, $\Sigma_1^1$-definable etc.

By weak flatness, for a formula $\varphi$ of dependence logic, we have $\varphi \equiv \text{PR } \varphi$.

A closely related discussion is that about logical consequence. Logical consequence in ordinary team semantics is different from logical consequence of probabilistic team semantics: as was shown by Studený [24] and also pointed out in
team semantics context by Albert & Grädel [3], the following is an example of a rule that is valid in team semantics but not in probabilistic team semantics:

\[
\{ \vec{x} \perp \vec{y}, \vec{x} \perp \vec{u}, \vec{z} \perp \vec{x}, \vec{u}, \vec{y}, \vec{z} \perp \vec{x} \\vec{u} \vec{y} \} \text{ entails } \vec{x} \perp \vec{x} \vec{u} \vec{y},
\]

while the following is an example of a rule that is valid in probabilistic team semantics but not in ordinary team semantics:

\[
\{ \vec{x} \perp \vec{y}, \vec{z} \perp \vec{x}, \vec{z} \perp \vec{x}, \vec{u}, \vec{x} \perp \vec{y} \} \text{ entails } \vec{z} \perp \vec{u}.
\]

Certainly this means that such rules cannot be derived from the axioms of Definition 1.5 (or even the separoid axioms), as the axioms are satisfied by both ordinary and probabilistic independence logic.

**Proposition 5.24.** Let \( X \) be a probabilistic team and \( \varphi \) a formula of independence logic. Denote by \( X' \) the possibilistic collapse of \( X \). Then

\[ X \models \varphi \implies X' \models \varphi. \]

**Proof.** We prove the claim by induction on \( \varphi \). The first-order atomic and negated atomic cases are clear.

Suppose that \( X \models \vec{u} \perp \vec{v} \vec{w} \). To show that \( X \) satisfies \( \vec{u} \perp \vec{v} \vec{w} \), let \( s, s' \in X \) be such that \( s'(\vec{v}) = s'(\vec{v}) \). Denote \( \vec{a} = s'(\vec{u}), \vec{b} = s(\vec{v}) \) and \( \vec{c} = s'(\vec{w}) \). Then we wish to find \( s'' \in X \) with \( s'(\vec{u} \vec{v} \vec{w}) = \vec{a} \vec{b} \vec{c} \). Looking at the assignment \( s \), we see that

\[
|X_{\vec{u} \vec{v} \vec{w} = \vec{a} \vec{b} \vec{c}}| = \sum_{s''(\vec{u} \vec{v}) = \vec{a} \vec{b}} X(s'') \geq X(s) > 0
\]

and similarly looking at \( s' \) we get that \( |X_{\vec{u} \vec{v} \vec{w} = \vec{a} \vec{b} \vec{c}}| > 0 \). As \( X \) satisfies \( \vec{u} \perp \vec{v} \vec{w} \), we have

\[
|X_{\vec{u} \vec{v} \vec{w} = \vec{a} \vec{b} \vec{c}}| \cdot |X_{\vec{u} \vec{v} \vec{w} = \vec{a} \vec{b} \vec{c}}| = |X_{\vec{u} \vec{v} \vec{w} = \vec{a} \vec{b} \vec{c}}| \cdot |X_{\vec{v} = \vec{c}}|,
\]

which implies that also \( |X_{\vec{u} \vec{v} \vec{w} = \vec{a} \vec{b} \vec{c}}| > 0 \). But then, as

\[
|X_{\vec{u} \vec{v} \vec{w} = \vec{a} \vec{b} \vec{c}}| = \sum_{s''(\vec{u} \vec{v} \vec{w}) = \vec{a} \vec{b} \vec{c}} X(s''),
\]

there is at least one assignment \( s'' \) such that \( X(s'') > 0 \) and \( s''(\vec{u} \vec{v} \vec{w}) = \vec{a} \vec{b} \vec{c} \). But as \( X \) is the possibilistic collapse of \( X \), this means that \( s'' \in X \), as desired.

The claim follows for conjunction directly from the induction hypothesis and the truth definition.

Suppose that \( X \models \varphi \lor \psi \) and let \( X = Y \sqcup Z \) so that \( Y \models \varphi \) and \( Z \models \psi \). Denote by \( Y \) and \( Z \) the collapses of \( Y \) and \( Z \), respectively. By the induction hypothesis,
\( Y \models \varphi \) and \( Z \models \psi \), and thus \( Y \cup Z \models \varphi \lor \psi \). But now for any assignment \( s \), we have

\[
  s \in X \iff X(s) > 0 \\
  \iff rY(s) + (1 - r)Z(s) > 0 \\
  \iff Y(s) > 0 \lor Z(s) > 0 \\
  \iff s \in Y \lor s \in Z \\
  \iff s \in Y \cup Z.
\]

Thus \( X = Y \cup Z \), so \( X \models \varphi \lor \psi \).

Then suppose that \( X \models \exists v \varphi \). Then there is some function \( F : \text{supp } X \to \{ p \in [0, 1]^A \mid p \text{ is a probability distribution} \} \) such that \( X[F/v] \models \varphi \). Denote by \( X' \) the collapse of \( X[F/v] \). By the induction hypothesis \( X' \models \varphi \). Let \( s \) be an assignment and \( a \in A \). Let \( G \) be the function \( X \to \mathcal{P}(A) \setminus \{ \emptyset \} \) with \( G(s) = \{ a \in A \mid F(s)(a) > 0 \} \). Then

\[
  s(a/v) \in X' \iff X[F/v](s(a/v)) > 0 \\
  \iff \sum_{t \in \text{supp } X, \ t(a/v)=s(a/v)} X(t)F(t)(a) \\
  \iff X(t) > 0 \text{ and } F(t)(a) > 0 \text{ for some } t \\
  \text{ such that } t(a/v) = s(a/v) \\
  \iff t \in X \text{ and } a \in G(t) \text{ for some } t \\
  \text{ such that } t(a/v) = s(a/v) \\
  \iff s(a/v) \in X[G/v].
\]

Thus \( X' = X[G/v] \), so \( X[G/v] \models \varphi \), whence \( X \models \exists v \varphi \).

The universal quantifier case is similar. \( \square \)

Earlier we noted that in ordinary team semantics, being realization of a hidden-variable team is expressible via existential quantifiers. We show that this is also the case in probabilistic team semantics.

**Lemma 5.25.** \( \mathfrak{A} \) be a structure and \( \mathfrak{B} \) an expansion of \( \mathfrak{A} \) by the hidden-variable sort. Let \( X \) a probabilistic empirical team of \( \mathfrak{A} \) and \( Y \) a probabilistic hidden-variable team of \( \mathfrak{B} \). Then \( X \) is uniformly realized by \( Y \) if and only if

\[
  Y = X[F_0/z_0] \ldots [F_0/z_{l-1}]
\]

for some functions \( F_i \).
Proof. For simplicity we assume that \( l = 1 \) and \( \vec{z} = z \).

Suppose that \( Y \) uniformly realizes \( X \). Then for all \( \vec{a} \) and \( \vec{b} \),

(i) \[ |X_{\vec{x}=\vec{a}}| = |Y_{\vec{x}=\vec{a}}|, \]

(ii) \[ |X_{\vec{y}=\vec{a}}| |X_{\vec{x}=\vec{a}}| = |Y_{\vec{y}=\vec{a}}| |X_{\vec{x}=\vec{a}}|. \]

Define a function \( F \) by setting

\[ F(s)(\gamma) = \frac{\mathbb{Y}(s(\gamma/z))}{\mathbb{Y}_{\vec{y}=s(\vec{y})}} \]

for all \( s \in \text{supp} \ X \) and \( \gamma \). Now \( F(s) \) is a distribution, as

\[ \sum_{\gamma} F(s)(\gamma) = \sum_{\gamma} \frac{\mathbb{Y}(s(\gamma/z))}{\mathbb{Y}_{\vec{y}=s(\vec{y})}} = \frac{\mathbb{Y}_{\vec{y}=s(\vec{y})}}{\mathbb{Y}_{\vec{y}=s(\vec{y})}} = 1. \]

Then

\[ X[F/z](\vec{x}\vec{y}z \mapsto \vec{x}\vec{y} \mapsto \vec{a}\vec{b}) = \mathbb{X}(\vec{x}\vec{y} \mapsto \vec{a}\vec{b})F(\vec{x}\vec{y} \mapsto \vec{a}\vec{b})(\gamma) \]

\[ = \left| X_{\vec{x}=\vec{a}} \right| \mathbb{Y}(\vec{x}\vec{y}z \mapsto \vec{a}\vec{b}) \]

\[ = \left| Y_{\vec{x}=\vec{a}} \right| \mathbb{Y}(\vec{x}\vec{y} \mapsto \vec{a}\vec{b}) \]

\[ = \mathbb{Y}(\vec{x}\vec{y}z \mapsto \vec{a}\vec{b}), \]

whence \( Y = X[F/z] \). Thus \( X[F/z] = \varphi \), so \( X = \exists z \varphi \).

Conversely, suppose that \( Y = X[F/z] \) for some \( F \). Then it is clear that \( |X_{\vec{x}=\vec{a}}| = 0 \) if and only if \( |Y_{\vec{x}=\vec{a}}| = 0 \) for any \( \vec{a} \). Now

\[ \left| Y_{\vec{y}=\vec{a}} \right| \left| X_{\vec{x}=\vec{a}} \right| = \left| X_{\vec{x}=\vec{a}} \right| \sum_{\gamma} \mathbb{Y}(\vec{x}\vec{y}z \mapsto \vec{a}\vec{b}) \gamma \]

\[ = \left| X_{\vec{x}=\vec{a}} \right| \sum_{\gamma} \mathbb{X}(\vec{x}\vec{y} \mapsto \vec{a}\vec{b})F(\vec{x}\vec{y} \mapsto \vec{a}\vec{b})(\gamma) \]

\[ = \left| X_{\vec{x}=\vec{a}} \right| \mathbb{X}(\vec{x}\vec{y} \mapsto \vec{a}\vec{b}) \]

\[ = \left| X_{\vec{y}=\vec{a}} \right| \left| X_{\vec{x}=\vec{a}} \right| \left| Y_{\vec{x}=\vec{a}} \right|. \]

Thus \( Y \) uniformly realizes \( X \). \( \square \)
5.2 Properties of Probabilistic Teams

By simply replacing the symbol $\perp$ by the symbol $\perp \perp$, we get the probabilistic versions of the previously introduced possibilistic team properties of empirical and hidden-variable teams.

**Definition 5.26 (Probabilistic Team Properties).**

(i) A probabilistic empirical team $\mathbb{X}$ supports **probabilistic no-signalling** if it satisfies the formula

$$\bigwedge_{i<n}\{x_j \mid j \neq i\} \perp_{x_i} y_i.$$  \hspace{1cm} \text{(PNS)}

(ii) A probabilistic hidden-variable team $\mathbb{X}$ supports **probabilistic weak determinism** if it satisfies the formula

$$\bigwedge_{i<n}(\vec{x}, \vec{z}, y_i).$$  \hspace{1cm} \text{(PWD)}

(iii) A probabilistic hidden-variable team $\mathbb{X}$ supports **probabilistic strong determinism** if it satisfies the formula

$$\bigwedge_{i<n}(x_i \vec{z}, y_i).$$  \hspace{1cm} \text{(PSD)}

(iv) A probabilistic hidden-variable team $\mathbb{X}$ supports **probabilistic single-valuedness** if it satisfies the formula

$$=(\vec{z}).$$  \hspace{1cm} \text{(PSV)}

(v) A probabilistic hidden-variable team $\mathbb{X}$ supports **probabilistic $z$-independence** if it satisfies the formula

$$\vec{z} \perp \vec{x}.$$  \hspace{1cm} \text{(PzI)}

(vi) A probabilistic hidden-variable team $\mathbb{X}$ supports **probabilistic parameter independence** if it satisfies the formula

$$\bigwedge_{i<n}\{x_j \mid j \neq i\} \perp_{x_i \vec{z}} y_i.$$  \hspace{1cm} \text{(PPI)}

(vii) A probabilistic hidden-variable team $\mathbb{X}$ supports **probabilistic outcome independence** if it satisfies the formula

$$\bigwedge_{i<n} y_i \perp_{x_i \vec{z}} \{y_j \mid j \neq i\}.$$  \hspace{1cm} \text{(POI)}
As we did not have a syntactic formula for locality, we need to give an explicit semantic definition for probabilistic locality as well.

(viii) A probabilistic hidden-variable team $\mathcal{X}$ supports probabilistic locality if for all $\vec{a}$, $\vec{b}$ and $\vec{\gamma}$ we have

$$|X_{\vec{x}\vec{y}\vec{z}=\vec{a}\vec{b}\vec{\gamma}}| \prod_{i<n} |X_{x_i\vec{z}=a_i\vec{\gamma}}| = |X_{\vec{x}\vec{z}=\vec{a}\vec{\gamma}}| \prod_{i<n} |X_{x_iy_i\vec{z}=a_ib_i\vec{\gamma}}|.$$  

Lemma 3.14, stating that locality is equivalent to the conjunction of parameter and outcome independence, remains true in the probabilistic world. A similar proof is presented in [6], outside of team semantics context. For convenience, we first present a characterization of probabilistic mutual independence of more than two tuples of variables.

**Lemma 5.27.** A probabilistic team $\mathcal{X}$ satisfies the formula

$$\bigwedge_{i<n} \forall_{\vec{a}} \{v_j \mid j \neq i\}$$

if and only if for all $\vec{a}$ and $\vec{b}$,

$$|X_{\vec{v}\vec{u}=\vec{ab}}| \cdot |X_{\vec{u}=\vec{b}}|^{n-1} = \prod_{i<n} |X_{v_i\vec{a}=a_i\vec{b}}|.$$  

Each $v_i$ can also be replaced by a tuple of variables.

**Proof.** By induction on $n$.  

**Lemma 5.28.** Probabilistic locality is equivalent to the conjunction of probabilistic parameter independence and probabilistic outcome independence.

**Proof.** Denote by $\vec{x^-}$ the tuple $(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1})$, and similarly for other tuples ($i$ should be clear from context).

First suppose that $\mathcal{X}$ supports locality. We first show that $\mathcal{X}$ supports parameter independence, i.e.

$$\mathcal{X} \models \{x_j \mid j \neq i\} \perp_{-x_i} y_i$$

for all $i < n$. Let $\vec{a}$, $\vec{b}$ and $\vec{\gamma}$ be arbitrary. We now need to show that

$$|X_{\vec{x}\vec{z}=\vec{a}\vec{\gamma}}| \cdot |X_{x_iy_i\vec{z}=a_ib_i\vec{\gamma}}| = |X_{\vec{x}\vec{y}\vec{z}=\vec{a}\vec{b}\vec{\gamma}}|.$$  

If $|X_{x_j\vec{z}=a_j\vec{\gamma}}| = 0$ for any $j < n$, then both sides of the above equation are automatically 0 and thus equal, so we may assume that $|X_{x_j\vec{z}=a_j\vec{\gamma}}| > 0$ for all $j < n$. Then also their product is non-zero. The same holds for $|X_{\vec{x}\vec{z}=\vec{a}\vec{\gamma}}|$, so we may also
assume that it is non-zero. Therefore our equation can be transformed into the form
\[
\frac{|X_{xy, z=\bar{a} \gamma}|}{|X_{xz, z=\bar{a} \gamma}|} = \frac{|X_{x,y, z=\bar{a} \gamma}|}{|X_{x,z, z=\bar{a} \gamma}|}.
\]
By locality, for any \( \bar{c} \), we have
\[
|X_{x,y, z=\bar{a} \gamma} \prod_{j<n}|X_{x_j, z=a_j \gamma}| = |X_{x,y, z=\bar{a} \gamma}| \prod_{j<n}|X_{x_j, y_j, z=a_j \gamma}|,
\]
whence
\[
\frac{|X_{x,y, z=\bar{a} \gamma} \prod_{j<n}|X_{x_j, z=a_j \gamma}|}{|X_{xz, z=\bar{a} \gamma}|} = \prod_{j<n}\frac{|X_{x_j, y_j, z=a_j \gamma}|}{|X_{x,j, z=a_j \gamma}|}.
\]
Then
\[
\frac{|X_{x,y, z=\bar{a} \gamma} \prod_{j<n}|X_{x_j, z=a_j \gamma}|}{|X_{xz, z=\bar{a} \gamma}|} = \sum_{\bar{c} \neq b} \frac{|X_{x,y, z=\bar{a} \gamma} \prod_{j<n}|X_{x_j, z=a_j \gamma}|}{|X_{x,z, z=\bar{a} \gamma}|} = \sum_{\bar{c} \neq b} \prod_{j<n}\frac{|X_{x_j, y_j, z=a_j \gamma}|}{|X_{x,j, z=a_j \gamma}|} = \prod_{j<n}\frac{|X_{x,y, z=\bar{a} \gamma} \prod_{j\neq i}\frac{|X_{x_j, y_j, z=a_j \gamma}|}{|X_{x,j, z=a_j \gamma}|}}{|X_{x,z, z=\bar{a} \gamma}|} = \prod_{j<n}\frac{|X_{x,y, z=\bar{a} \gamma} \prod_{j\neq i}\frac{|X_{x_j, z=a_j \gamma}|}{|X_{x,j, z=a_j \gamma}|}}{|X_{x,z, z=\bar{a} \gamma}|},
\]
as desired.

Next we show that \( X \) supports outcome independence, i.e.
\[
X = y_i \perp_{\mathcal{E}} \{y_j \mid j \neq i\}
\]
for all \( i < n \), by invoking Lemma 5.27. Let \( \bar{a}, \bar{b} \) and \( \bar{\gamma} \) be arbitrary. We now need to show that
\[
|X_{x,y, z=\bar{a} \gamma} |X_{x,z, z=\bar{a} \gamma}|^{n-1} = \prod_{i<n}|X_{x,y, z=\bar{a} \gamma}|.
\]
Once more we may assume that \( |X_{x,z, z=\bar{a} \gamma}| \) and \( |X_{x,z, z=\bar{a} \gamma}|, j < n \), are non-zero. As we have already shown that \( X \) supports parameter independence, we know that the equation
\[
\frac{|X_{x,y, z=\bar{a} \gamma}|}{|X_{xz, z=\bar{a} \gamma}|} = \frac{|X_{x,y, z=\bar{a} \gamma}|}{|X_{x,z, z=\bar{a} \gamma}|}
\]
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holds for all $i < n$. By substituting each of these into the equation given by locality, we obtain

\[
\frac{X_{\vec{x}\vec{y}\vec{z}=\vec{a}\vec{b}\vec{\gamma}}}{|X_{\vec{x}\vec{z}=\vec{a}\vec{\gamma}}|} = \prod_{j<n} \frac{|X_{x_j y_j z=a_j b_j \gamma}|}{|X_{x_j \vec{z}=a_j \vec{\gamma}}|} = \prod_{j<n} \frac{|X_{x_j y_j z=a_j b_j \gamma}|}{|X_{x_j \vec{z}=a_j \vec{\gamma}}|},
\]

whence

\[
\left|\frac{X_{\vec{x}\vec{y}\vec{z}=\vec{a}\vec{b}\vec{\gamma}}}{|X_{\vec{x}\vec{z}=\vec{a}\vec{\gamma}}|}\right|^n = \prod_{j<n} |X_{x_j y_j z=a_j b_j \gamma}|,
\]

as desired.

Finally, we assume that $X$ supports parameter and outcome independence and proceed to show that then $X$ supports locality. We may assume, again, that $|X_{\vec{x}\vec{z}=\vec{a}\vec{\gamma}}|$ and $|X_{x_j \vec{z}=a_j \vec{\gamma}}|$, $j < n$, are non-zero. By outcome independence (and using Lemma 5.27), we have

\[
\left|\frac{X_{\vec{x}\vec{y}\vec{z}=\vec{a}\vec{b}\vec{\gamma}}}{|X_{\vec{x}\vec{z}=\vec{a}\vec{\gamma}}|}\right|^n = \prod_{i<n} |X_{x_i y_i z=a_i b_i \gamma}|,
\]

i.e.

\[
\frac{X_{\vec{x}\vec{y}\vec{z}=\vec{a}\vec{b}\vec{\gamma}}}{|X_{\vec{x}\vec{z}=\vec{a}\vec{\gamma}}|} = \prod_{j<n} \frac{|X_{x_j y_j z=a_j b_j \gamma}|}{|X_{x_j \vec{z}=a_j \vec{\gamma}}|}.
\]

By parameter independence, we have, for all $i < n$,

\[
\frac{|X_{x_i y_i z=a_i b_i \gamma}|}{|X_{\vec{x} \vec{z}=\vec{a}\vec{\gamma}}|} = \frac{|X_{x_i y_i z=a_i b_i \gamma}|}{|X_{x_i \vec{z}=a_i \vec{\gamma}}|}.
\]

By substituting each of these into the equation obtained from outcome independence, we get

\[
\frac{X_{\vec{x}\vec{y}\vec{z}=\vec{a}\vec{b}\vec{\gamma}}}{|X_{\vec{x}\vec{z}=\vec{a}\vec{\gamma}}|} = \prod_{j<n} \frac{|X_{x_j y_j z=a_j b_j \gamma}|}{|X_{x_j \vec{z}=a_j \vec{\gamma}}|} = \prod_{i<n} \frac{|X_{x_i y_i z=a_i b_i \gamma}|}{|X_{x_i \vec{z}=a_i \vec{\gamma}}|},
\]

whence

\[
\left|\frac{X_{\vec{x}\vec{y}\vec{z}=\vec{a}\vec{b}\vec{\gamma}}}{|X_{\vec{x}\vec{z}=\vec{a}\vec{\gamma}}|}\right| \prod_{i<n} |X_{x_i \vec{z}=a_i \vec{\gamma}}| = \left|\frac{X_{\vec{x}\vec{z}=\vec{a}\vec{\gamma}}}{|X_{\vec{x}\vec{z}=\vec{a}\vec{\gamma}}|}\right| \prod_{i<n} |X_{x_i y_i z=a_i b_i \gamma}|,
\]

as desired. \qed

**Corollary 5.29.** For any of the properties is Definition 5.26, if a probabilistic team supports it, then the possibilistic collapse supports the corresponding possibilistic property.
Proof. An immediate consequence of Proposition 5.24.

As by Proposition 5.13 the axioms presented in Section 1.2 are valid also in
the probabilistic setting, all the results from Section 3.3 that were proved from the
axioms are true also for probabilistic teams.

**Corollary 5.30.** The following hold for probabilistic teams:

(i) \((\vec{x}z, \vec{y}) \models \bigwedge_{i<n} y_i \perp \perp \{y_j \mid j \neq i\}\),

(ii) \((x_i z, y_i) \models \{x_j \mid j \neq i\} \perp \perp x_i y_i\),

(iii) \(\bigwedge_{i<n} \{x_j \mid j \neq i\} \perp \perp x_i y_i \land \neg(\vec{x}z, \vec{y}) \models \bigwedge \neg(x_i z, y_i)\),

(iv) \(\varphi \models \exists z_0 \ldots \exists z_{l-1} = \overline{(z)} \land \varphi\).

5.3 Building Probabilistic Teams

As we saw in section 5.2, properties of probabilistic teams are inherited by their
possibilistic collapses. Here we prove results concerning the question of when the
converse holds: when can one construct a probabilistic team out of a possibilistic
one, with the same properties?

Following [1], we proceed to show that some no-signalling te-
a ms have no prob-
abilistic realization that also supports probabilistic no-signalling.

This also shows that \(\phi \models \PR \phi\) is not true for all formulas \(\varphi\) of independence
logic.

**Proposition 5.31.** There is an empirical team \(X\) supporting no-signalling such
that there is no probabilistic team \(X\) that supports probabilistic no-signalling and
whose possibilistic collapse is \(X\), i.e.

\[
\bigwedge_{i<n} \{x_j \mid j \neq i\} \perp \perp x_i y_i \models \PR \bigwedge_{i<n} \{x_j \mid j \neq i\} \perp \perp x_i y_i.
\]

**Proof.** Suppose that \(n = 2\). We let \(X\) be the team represented in the following
table.

| x₀ | x₁ | y₀ | y₁ |
|----|----|----|----|
| s₀ | 0  | 0  | 0  |
| s₁ | 0  | 0  | 1  |
| s₂ | 0  | 1  | 1  |
| s₃ | 1  | 0  | 0  |
| s₄ | 0  | 1  | 0  |
| s₅ | 0  | 1  | 1  |

| x₀ | x₁ | y₀ | y₁ |
|----|----|----|----|
| s₆ | 1  | 0  | 0  |
| s₇ | 1  | 0  | 1  |
| s₈ | 1  | 1  | 1  |
| s₉ | 1  | 0  | 0  |
| s₁₀| 1  | 1  | 0  |
| s₁₁| 1  | 1  | 1  |

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It is straightforward to check that \( X \) supports no-signalling. Suppose for a contradiction that \( X \) is a probabilistic team that supports probabilistic no-signalling and whose possibilistic collapse is \( X \). Then there are positive numbers \( p_0, \ldots, p_{11} \) with \( \sum_{i < 12} p_i = 1 \) such that \( X(s_i) = p_i \) for all \( i < 12 \). By probabilistic no-signalling,

\[
X \models x_{1-i} \perp \perp x_i y_i,
\]

for all \( i \in \{0, 1\} \), so we have, for all \( a, b, c, i \in \{0, 1\} \),

\[
|X_{x_{1-i}, x_i = ab}| \cdot |X_{x_i, y_i = bc}| = |X_{x_{1-i}, x_i = abc}| \cdot |X_{x_i} = b|.
\]

Calculating the marginal probabilities and applying the above condition, we get the following four equations:

(i) \( p_2 p_3 = (p_0 + p_1)(p_4 + p_5) \),

(ii) \( p_0 p_8 = (p_1 + p_2)(p_6 + p_7) \),

(iii) \( p_6 p_{11} = (p_7 + p_8)(p_9 + p_{10}) \), and

(iv) \( p_5 p_9 = (p_3 + p_4)(p_{10} + p_{11}) \).

From this, using the third and the fourth equations, we get

\[
p_6 p_{11} p_5 p_9 = (p_7 + p_8)(p_9 + p_{10})(p_3 + p_4)(p_{10} + p_{11}) > p_8 p_9 p_3 p_{11},
\]

whence \( p_5 p_6 > p_3 p_8 \). Then by multiplying by \( p_2 \) and using the first equation, we get

\[
p_2 p_5 p_6 > p_2 p_3 p_8 = (p_0 + p_1)(p_4 + p_5)p_8 > p_0 p_5 p_8,
\]

whence \( p_2 p_6 > p_0 p_8 \). Then finally, using the second equation, we get

\[
p_2 p_6 > p_0 p_8 = (p_1 + p_2)(p_6 + p_7) > p_2 p_6,
\]

which is clearly a contradiction. \( \square \)

A minimal example of a possibilistic no-signalling team which is not a collapse of any probabilistic no-signalling team can be obtained by translating an example of [2]—which occurs as a part of discussion about the question whether there exists an intrinsic characterization of the class of no-signalling teams that are collapses of probabilistic no-signalling teams—into our team-semantic framework.

The next property of empirical and hidden-variable teams, measurement locality, was introduced by the first author in [1]. Measurement locality states that the measurement variables are mutually independent of each other.
Definition 5.32 (Measurement Locality). An empirical team \( X \) supports \textit{measurement locality} if it satisfies the formula

\[
\bigwedge_{i<n} x_i \perp \{ x_j \mid j \neq i \}. \tag{ML}
\]

A hidden-variable team \( X \) supports measurement locality if it satisfies the formula

\[
\bigwedge_{i<n} x_i \perp \vec{z} \{ x_j \mid j \neq i \}. \tag{ML}
\]

Definition 5.33 (Probabilistic Measurement Locality). A probabilistic empirical team \( X \) supports \textit{probabilistic measurement locality} if it satisfies the formula

\[
\bigwedge_{i<n} x_i \perp \perp \{ x_j \mid j \neq i \}. \tag{PML}
\]

A probabilistic hidden-variable team \( X \) supports probabilistic measurement locality if it satisfies the formula

\[
\bigwedge_{i<n} x_i \perp \perp \vec{z} \{ x_j \mid j \neq i \}. \tag{PML}
\]

Corollary 5.34. Whenever a probabilistic team supports probabilistic measurement locality, then the possibilistic collapse supports measurement locality.

\textit{Proof.} An immediate consequence of Proposition 5.24.

Definition 5.35. Given a sets \( A = \prod_{i<n} A_i \) and \( B = \bigcup_{i<n} B_i \) and a probability distribution \( p_{\vec{a}} \) on \( B \) for each \( \vec{a} \in A \), we say that a probabilistic empirical team \( \bar{X} \) is a \textit{uniform joint distribution} of the outcome distribution family \( \{ p_{\vec{a}} \mid \vec{a} \in A \} \) if the value domain of \( \bar{X} \) is \( \bigcup_{i<n} (A_i \cup B_i) \) and \( \bar{X}(s) = \frac{p_{s(\vec{x})}}{|A|} \) whenever \( s(x_i) \in A_i \) and \( s(y_i) \in B_i \) for all \( i < n \), and \( \bar{X}(s) = 0 \) otherwise.

Similarly, given outcome distributions \( p_{\vec{a}\vec{\gamma}} \) on \( B \) for \( \vec{a} \in A \) and \( \vec{\gamma} \in \Gamma \), we say that a probabilistic hidden-variable team \( \bar{X} \) is a uniform joint distribution of the outcome distribution family if \( \bar{X}(s) = \frac{p_{s(\vec{x}\vec{y})}}{|A \times \Gamma|} \).

Example. Quantum-mechanical teams, introduced in Section 6, are by definition uniform joint distributions of an outcome distribution family.

Lemma 5.36. A uniform joint distribution of an outcome distribution family automatically supports probabilistic measurement locality.
Proof. First observe that
\[
|X_{x_i=z_i=a_i|\gamma}| = \sum_{\vec{c} \in A} \sum_{\vec{b} \in B} \frac{p_{\vec{c}}(\vec{b})}{|A \times \Gamma|} = \frac{1}{|A| |\Gamma|} \sum_{\vec{c} \in A} \left| \left\{ \vec{c} \in A \mid c_i = a_i \right\} \right|
\]
\[
= \frac{1}{|\Gamma| \prod_{j<n} |A_j|} \prod_{j<n} |A_j|
\]
\[
= \frac{1}{|\Gamma| |A_i|}.
\]

Then we have
\[
|X_{x_i=x_i,a_i|\gamma} \cdot |X_{x_i=x_i|\gamma}|^{n-1} = \left( \sum_{\vec{b} \in B} \frac{p_{\vec{b}}(\vec{b})}{|A \times \Gamma|} \right)^{n-1} \left( \sum_{\vec{c} \in A} \sum_{\vec{b} \in B} \frac{p_{\vec{c}}(\vec{b})}{|A \times \Gamma|} \right)^{n-1}
\]
\[
= \frac{1}{|A| |\Gamma|^{n-1}} \prod_{i<n} \frac{1}{|A_i|} = \prod_{i<n} \frac{1}{|A_i| |\Gamma|^{n-1}}
\]
\[
= \prod_{i<n} |X_{x_i=x_i,a_i}|,
\]
which by Lemma 5.27 shows that \(X \models x_i \perp \perp \{x_j \mid j \neq i\}\). \(\square\)

Next we show that there is a canonical way of constructing a probabilistic hidden-variable team out of a possibilistic hidden-variable team that supports \(z\)-independence, and that such a probabilistic team will support locality, measurement locality and \(z\)-independence if its possibilistic collapse does.

**Definition 5.37.** Given a hidden-variable team \(X\) that supports \(z\)-independence, we define the probabilistic hidden-variable team \(\text{Prob}(X)\) as follows. Denote
\[
\Gamma = \{s(\vec{z}) \mid s \in X\},
\]
\[
M = \{s(\vec{x}) \mid s \in X\},
\]
\[
O_{\vec{a},\vec{\gamma}} = \{s(\vec{y}) \mid s \in X, s(\vec{x}\vec{z}) = \vec{a}\vec{\gamma}\},
\]
and \(m_h = |\Gamma|, m_m = |M|\) and \(m_o(\vec{a}, \vec{\gamma}) = |O_{\vec{a},\vec{\gamma}}|\). We then define \(\text{Prob}(X)\) by setting
\[
\text{Prob}(X)(s) = \begin{cases} 
1/(m_h \cdot m_m \cdot m_o(s(\vec{x}), s(\vec{z}))) & \text{if } s(\vec{x}) \in M \text{ and } s(\vec{z}) \in \Gamma, \\
0 & \text{otherwise}.
\end{cases}
\]
If $X$ is a well-defined probabilistic team, the possibilistic collapse of $X$ is clearly $X$. So we show that $X$ is well-defined.

**Lemma 5.38.** $\text{Prob}(X)$ is well-defined.

**Proof.** First, as $X$ supports $z$-independence, for every $s, s' \in X$ we can find $s'' \in X$ with $s''(\vec{x}) = s(\vec{x})$ and $s''(\vec{z}) = s'(\vec{z})$, and thus, given an assignment $s$, the condition $s(\vec{x}) \in M$ and $s(\vec{z}) \in \Gamma$ implies that there is some $s' \in X$ with $s'(\vec{x}) = s(\vec{x})$ and thus the number $m_o(s(\vec{x}), s(\vec{z}))$ is non-zero. Hence $\text{Prob}(X)$ is well-defined as a function. Left is to show that $\text{Prob}(X)$ is a probability distribution. Notice that for each $\vec{a} \vec{b} \vec{\gamma}$, the probability of the assignment $\vec{x} \vec{y} \vec{z} \mapsto \vec{a} \vec{b} \vec{\gamma}$ does not depend on $\vec{b}$, so each assignment $s$ with $s(\vec{x}) = \vec{a} \vec{\gamma}$ has an equal probability, which is $1/(m_h m_m m_o(\vec{a}, \vec{\gamma}))$, and thus the joint probability of such assignments is

$$\left| \text{Prob}(X)_{\vec{x}=\vec{a} \vec{\gamma}} \right| = \sum_{\vec{b}} \text{Prob}(X) (\vec{x} \vec{y} \vec{z} \mapsto \vec{a} \vec{b} \vec{\gamma}) = m_o(\vec{a}, \vec{\gamma}) \cdot \frac{1}{m_h m_m m_o(\vec{a}, \vec{\gamma})} = \frac{1}{m_h m_m}.$$ 

This, in turn, does not depend on $\vec{a}$ or $\vec{\gamma}$. Also, by $z$-independence we have $|\{s \mid s(\vec{x}) = \vec{a} \vec{\gamma}\}| = m_m \cdot m_h$. Thus

$$\sum_{s \in X} \text{Prob}(X) (s) = \sum_{\vec{a} \vec{\gamma}} \text{Prob}(X) (\vec{x} \vec{y} \vec{z} \mapsto \vec{a} \vec{b} \vec{\gamma}) = \sum_{\vec{a} \vec{\gamma}} \frac{1}{m_h m_m} = m_m m_h \cdot \frac{1}{m_h m_m} = 1.$$ 

Thus $\text{Prob}(X)$ is a well-defined distribution. $\square$

**Proposition 5.39.** Let $X$ be a hidden-variable team supporting measurement locality, $z$-independence and locality. Then there is a probabilistic hidden-variable team $\mathbb{X}$ supporting probabilistic measurement locality, probabilistic $z$-independence and probabilistic locality whose possibilistic collapse is $X$. In other words, the formula

$$\varphi := \vec{z} \perp \vec{x} \land \bigwedge_{i<n} x_i y_i \perp \perp_{\vec{z}} \{x_j y_j \mid j \neq i\}$$

satisfies $\varphi \models \text{PR} \varphi$. 

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Proof. As $X$ supports $z$-independence, we may let $\mathbb{X} = \text{Prob}(X)$. Now note the following. Denote
\[
m_{i,m} = |\{s(x_i) \mid s \in X\}|.
\]
and
\[
m_{i,o}(a, \vec{\gamma}) = |\{s(y_i) \mid s(x_i, z) = (a, \vec{\gamma}), s \in X\}|.
\]
As $X$ supports measurement locality, we have $X \models x_i \perp x_j \{x_j \mid j \neq i\}$ for all $i < n$. As $X$ supports also $z$-independence in addition to measurement locality, we have $X \models \vec{z} \perp \vec{x}$, whence $X \models x_i \perp \{x_j \mid j \neq i\}$. This implies that
\[
m_m = \prod_{i<n} m_{m,i}.
\tag{1}
\]
Similarly, measurement locality and locality together imply that $X \models x_i y_i \perp \vec{z} \{x_j y_j \mid j \neq i\}$, whence
\[
m_o(a, \vec{\gamma}) = \prod_{i<n} m_{o,i}(a_i, \vec{\gamma}).
\tag{2}
\]
Thus we have shown that the numbers $m_m$ and $m_o(a, \vec{\gamma})$ can, in some sense, be calculated locally.

Next we show that $\mathbb{X}$ has the desired properties, starting from probabilistic measurement locality. Since the variables $x_0, \ldots, x_{n-1}, z$ are all mutually independent in $X$, we know that the set $\{s(\vec{x}z) \mid s \in X\}$ is a Cartesian product $(\prod_{i<n} A_i) \times \Gamma$. As, in addition, $\mathbb{X}(s) = p_h(s(\vec{y}))/|\prod_{i<n} A_i| \times \Gamma|$, where $p_{a,\vec{\gamma}}(\vec{b}) = 1/m_o(a, \vec{\gamma})$, by Lemma 5.36, $\mathbb{X}$ supports probabilistic measurement locality.

Then consider probabilistic $z$-independence. We need to show that $X \models z \perp x$, i.e. $|\mathbb{X}_{\vec{x}=\vec{a}}| |\mathbb{X}_{\vec{x}=\vec{a}}| = |\mathbb{X}_{\vec{x}=\vec{a}}| |\mathbb{X}_{\vec{x}=\vec{a}}|$ for all $\vec{a}$ and $\vec{\gamma}$. In the proof of Lemma 5.38, we showed that $|\mathbb{X}_{\vec{x}=\vec{a}}| = 1/(m_h m_m)$. Notice that
\[
|\mathbb{X}_{\vec{x}=\vec{a}}| = \sum_{\vec{a}} \sum_{\vec{b}} \mathbb{X}(\vec{x} y \vec{z} \mapsto \vec{a} \vec{b} \vec{\gamma}) = \sum_{\vec{a}} 1/(m_h m_m) = 1/m_h
\]
and
\[
|\mathbb{X}_{\vec{x}=\vec{a}}| = \sum_{\vec{\gamma}} \sum_{\vec{b}} \mathbb{X}(\vec{x} y \vec{z} \mapsto \vec{a} \vec{b} \vec{\gamma}) = \sum_{\vec{\gamma}} 1/(m_h m_m) = 1/m_m.
\]
Thus $|\mathbb{X}_{\vec{x}=\vec{a}}| |\mathbb{X}_{\vec{x}=\vec{a}}| = |\mathbb{X}_{\vec{x}=\vec{a}}| |\mathbb{X}_{\vec{x}=\vec{a}}|$. Finally, consider probabilistic locality. We have to show that for all $\vec{a}$, $\vec{b}$ and $\vec{\gamma}$,
\[
|\mathbb{X}_{\vec{x}=\vec{a}}| \prod_{i<n} |\mathbb{X}_{\vec{x}_i=\vec{a}_i, \vec{\gamma}_i}| = |\mathbb{X}_{\vec{x}=\vec{a}}| \prod_{i<n} |\mathbb{X}_{\vec{x}_i=\vec{a}_i, \vec{b}_i, \vec{\gamma}_i}|.
\]
We note that (2) ensures that both sides of the equation are 0 exactly the same time. Now, using (1), we get

\[
|X_{x, \bar{z}, a, \gamma}| = \sum_{\mathcal{C} \subseteq \mathcal{M}} |X_{\bar{x}, \bar{z}, c, \gamma}| = \sum_{\mathcal{C} \subseteq \mathcal{M}} \frac{1}{m_h m_m} = \frac{1}{m_h m_m} \prod m^j_m
\]

\[
= \frac{1}{m_h m_m} \cdot \frac{m_m}{m^i_m} = \frac{1}{m_h m_m^i}.
\]

Then

\[
\prod_{i<n} |X_{x_i, \bar{z}, a_i, \gamma}| = \prod_{i<n} \frac{1}{m_h m_m^i} = \frac{1}{m_h m_m^i},
\]

so, as \(X_{\bar{x}, \bar{z}, \bar{a}, \gamma}\) is just \(X(\bar{x}, \bar{z}, \bar{a}, \gamma) \mapsto (\bar{a}, \gamma)\), i.e. \(1/(m_h \cdot m_m \cdot m_o(\bar{a}, \gamma))\), we have

\[
|X_{x, \bar{z}, a, \gamma}| \prod_{i<n} |X_{x_i, \bar{z}, a_i, \gamma}| = \frac{1}{m_h m_m m_o(\bar{a}, \gamma)} \cdot \frac{1}{m_h m_m} = \frac{1}{m_h m_m m_o(\bar{a}, \gamma)}.
\]

On the other hand, using (2), we get

\[
|X_{x, y, \bar{z}, a, b, \gamma}| = \sum_{\mathcal{C} \subseteq \mathcal{M}} \sum_{\mathcal{D} \subseteq \mathcal{O} \cap \mathcal{I}} X(\bar{x}, \bar{y}, \bar{z}, \bar{a}, \gamma) = \sum_{\mathcal{C} \subseteq \mathcal{M}} \sum_{\mathcal{D} \subseteq \mathcal{O} \cap \mathcal{I}} \frac{1}{m_h m_m m_o(\bar{c}, \gamma)}
\]

\[
= \sum_{\mathcal{C} \subseteq \mathcal{M}} \frac{1}{m_h m_m m_o(\bar{c}, \gamma)} \prod m^j_o(c_i, \gamma)
\]

\[
= \sum_{\mathcal{C} \subseteq \mathcal{M}} \frac{1}{m_h m_m m_o(\bar{c}, \gamma)} \cdot m^i_o(c_i, \gamma) = \sum_{\mathcal{C} \subseteq \mathcal{M}} \frac{1}{m_h m_m m_o(\bar{c}, \gamma)}
\]

\[
= \frac{1}{m_h m_m m^i_o(a_i, \gamma)} \prod_{j \neq i} m^j_m = \frac{1}{m_h m_m m^i_o(a_i, \gamma)} \cdot \frac{m^i_m}{m^i_m}
\]

Then

\[
\prod_{i<n} |X_{x, y_i, \bar{z}, a_i, b_i, \gamma}| = \prod_{i<n} \frac{1}{m_h m^i_m m^i_o(a_i, \gamma)} = \frac{1}{m_h m_m m_o(o_a, \gamma)}.
\]

so

\[
|X_{\bar{x}, \bar{z}, \bar{a}, \gamma}| \prod_{i<n} |X_{x_i, y_i, \bar{z}, a_i, b_i, \gamma}| = \frac{1}{m_h m_m} \cdot \frac{1}{m_h m_m m_o(o_a, \gamma)} = \frac{1}{m_h m_m^i m^i_m m_o(o_a, \gamma)}.
\]
Thus we can conclude that, indeed,

\[
|X_{\vec{x}z = \vec{a}\vec{b}\vec{c}}| \prod_{i < n} |X_{x_i z = a_i \vec{c}}| = |X_{\vec{x}z = \vec{a}\vec{c}}| \prod_{i < n} |X_{x_i y_i z = a_i \vec{b}_i \vec{c}}| .
\]

This finishes the proof. \(\square\)

5.4 Maximum Entropy Characterization

In this section we show that \(\text{Prob}(X)\) is a maximum entropy construction, again adapting [1]. Out of all probabilistic teams \(X\) such that \(X\) is the possibilistic collapse of \(\mathcal{X}\), \(\text{Prob}(X)\) is the one with maximal entropy.

Below is the standard definition of entropy of a probability distribution.

**Definition 5.40.** The entropy \(H(p)\) of a probability distribution \(p\) on a finite set \(A\) is the number

\[
H(p) = - \sum_{a \in \text{supp} \ p} p(a) \log p(a),
\]

where \(\text{supp} \ p = \{a \in A \mid p(a) > 0\}\).

Next we define two distributions related to a probabilistic team, one a sort of an initial probability of measurement combinations and the other a marginalization of probability conditional to a measurement combination. These will be used to characterize the team \(\text{Prob}(X)\) as a maximum entropy construction.

**Definition 5.41.** Let \(\mathcal{X}\) be a probabilistic hidden-variable team. Denote by \(A\) the set \(\{s(\vec{x}\vec{z}) \mid \mathcal{X}(s) > 0\}\), and by \(B\) the set \(\{s(\vec{y}) \mid \mathcal{X}(s) > 0\}\). Then the *measurement prior* of \(\mathcal{X}\), denoted by \(\theta_\mathcal{X}\), is the probabilistic team with variable domain \(V_m \cup V_h\) and value domain \(A\), defined by

\[
\theta_\mathcal{X}(\vec{x}\vec{z} \mapsto \vec{a}\vec{c}) = |X_{\vec{x}z = \vec{a}\vec{c}}| .
\]

For all \(\vec{a}\vec{c} \in A\) such that \(\theta_\mathcal{X}(\vec{x}\vec{z} \mapsto \vec{a}\vec{c}) > 0\), we define a probabilistic team \(\eta_\mathcal{X}^{\vec{a}\vec{c}}\) with variable domain \(V_o\) and value domain \(B\) by setting

\[
\eta_\mathcal{X}^{\vec{a}\vec{c}}(\vec{y} \mapsto \vec{b}) = \mathcal{X}(\vec{x}\vec{y}\vec{z} \mapsto \vec{a}\vec{b}\vec{c}) / \theta_\mathcal{X}(\vec{x}\vec{z} \mapsto \vec{a}\vec{c}) .
\]

Note that we can obtain the original team \(\mathcal{X}\) from \(\theta_\mathcal{X}\) and \(\eta_\mathcal{X}^{\vec{a}\vec{c}}\), \(\vec{a}\vec{c} \in A\), as

\[
\mathcal{X}(\vec{x}\vec{y}\vec{z} \mapsto \vec{a}\vec{b}\vec{c}) = \begin{cases} 
\theta_\mathcal{X}(\vec{x}\vec{z} \mapsto \vec{a}\vec{c}) \cdot \eta_\mathcal{X}^{\vec{a}\vec{c}}(\vec{y} \mapsto \vec{b}) & \text{if } \theta_\mathcal{X}(\vec{x}\vec{z} \mapsto \vec{a}\vec{c}) > 0, \\
0 & \text{otherwise}.
\end{cases}
\]
Lemma 5.42. Let $X$ be a probabilistic hidden-variable team. Then
\[ H(X) = H(\theta_X) + \sum_{\vec{a}, \vec{\gamma} \in A \mid X_{\vec{x}, \vec{z} = \vec{a} \vec{\gamma}} > 0} \theta_X(\vec{x} \vec{z} \mapsto \vec{a} \vec{\gamma}) H(\eta_{X_{\vec{x}, \vec{z} = \vec{a} \vec{\gamma}}}). \]

Proof. A straightforward calculation. \hfill \Box

Proposition 5.43 (Maximum Entropy Characterization). Let $X$ be a hidden-variable team supporting $z$-independence, and let $C$ be the class of probabilistic hidden-variable teams whose possibilistic collapse is $X$. Then for all $Y \in C$ and for all $s \in X$,
\[ H(\theta_{\text{Prob}(X)}) \geq H(\theta_Y) \quad \text{and} \quad H(\eta_{s(\vec{x} \vec{z})} \text{Prob}(X)) \geq H(\eta_{s(\vec{x} \vec{z})} Y). \]

Proof. We will use the known fact that a uniform distribution has maximal entropy. Denote $X = \text{Prob}(X)$ and let $Y \in C$. Note that as $X$ is the possibilistic collapse of both $X$ and $Y$, we have supp $\theta_X = \text{supp} \theta_Y$. As was seen in the proof of Lemma 5.38, $\theta_X(s) = |X_{\vec{x}, \vec{z} = s(\vec{x} \vec{z})}| = 1/(m_h m_m)$ for any assignment $s \in \text{supp} \theta_X$. Therefore $\theta_X$ is a uniform distribution and hence its entropy is maximal, so $H(\theta_X) \geq H(\theta_Y)$.

Fix $\vec{a}$ and $\vec{\gamma}$ such that $\theta_X(\vec{x} \vec{z} \mapsto \vec{a} \vec{\gamma}) > 0$. Again, as $X$ is the possibilistic collapse of both $X$ and $Y$, we have supp $\eta_{X_{\vec{x}, \vec{z} = \vec{a} \vec{\gamma}}} = \text{supp} \eta_{Y_{\vec{x}, \vec{z} = \vec{a} \vec{\gamma}}}$. Now for any $\vec{b}$ with $\eta_{X_{\vec{x}, \vec{z} = \vec{a} \vec{\gamma}}} (\vec{y} \mapsto \vec{b}) > 0$,
\[ \eta_{X_{\vec{x}, \vec{z} = \vec{a} \vec{\gamma}}} (\vec{y} \mapsto \vec{b}) = \frac{X(\vec{x} \vec{y} \vec{z} \mapsto \vec{a} \vec{b} \vec{\gamma})}{\theta_X(\vec{x} \vec{z} \mapsto \vec{a} \vec{\gamma})} = \frac{1}{(m_h m_m m_o(\vec{a}, \vec{\gamma}))} = \frac{1}{m_o(\vec{a}, \vec{\gamma})}. \]
Hence $\eta_{X_{\vec{x}, \vec{z} = \vec{a} \vec{\gamma}}}$ is a uniform distribution on its support, whence its entropy is maximal. Thus $H(\eta_{X_{\vec{x}, \vec{z} = \vec{a} \vec{\gamma}}}) \geq H(\eta_{Y_{\vec{x}, \vec{z} = \vec{a} \vec{\gamma}}})$. \hfill \Box

6 Empirical Teams Arising from Quantum Mechanics

A team, even what we call an empirical team, is in itself just an abstract set of assignments. It does not need to have any “provenance”, although in practical applications teams arise from concrete data. In our current context of quantum mechanics, we use the totally abstract concept of a team for implications which indeed are totally general and abstract. However, when it comes to counter-examples demonstrating that some implications are not valid, the question arises whether our example teams are “merely” abstract or whether they can actually arise in experiments. One of the beauties of quantum physics is that we have a perfect mathematical model of what the possible outcomes of experiments could be, namely the theory of operators of complex Hilbert spaces.

We say that an empirical team is quantum-mechanical if it arises from a quantum-mechanical experiment. This means being expressible via positive operators in a complex Hilbert space.
Definition 6.1. Let $M$ and $O$ be sets of $n$-tuples (the “set of measurements” and the “set of outcomes”), and, for $i < n$, denote $M_i = \{a_i \mid \vec{a} \in M\}$ and $O_i = \{b_i \mid \vec{b} \in O\}$. A quantum system of type $(M, O)$ is a tuple

$$(\mathcal{H}, (A_{a,b}^i)_{a \in M_i, b \in O_i, i<n}, \rho),$$

where

- $\mathcal{H}$ is the tensor product $\bigotimes_{i<n} \mathcal{H}_i$ of finite-dimensional Hilbert spaces $\mathcal{H}_i$, $i < n$,
- for all $i < n$ and $a \in M_i$, $\{A_{a,b}^i \mid b \in O_i\}$ is a positive operator-valued measure on $\mathcal{H}_i$, and
- $\rho$ is a density operator on $\mathcal{H}$, i.e.

$$\rho = \sum_{j<k} p_j |\psi_j\rangle \langle \psi_j|,$$

where $|\psi_j\rangle \in \mathcal{H}$ and $p_j \in [0, 1]$ for all $j < k$ and $\sum_{j<k} p_j = 1$.

For each measurement $\vec{a} \in M$, we define the probability distribution $p_{\vec{a}}$ of outcomes by setting $p_{\vec{a}}(\vec{b}) := \text{Tr}(A_{\vec{a},\vec{b}}^i \rho)$, where $A_{\vec{a},\vec{b}}^i$ denotes the operator $\bigotimes_{i<n} A_{a_i,b_i}^i$ and $\text{Tr}(L)$ denotes the trace of the matrix $L$.

Definition 6.2. Let $X$ be a probabilistic team with variable domain $V_m \cup V_o$ and denote $M = \{s(\vec{x}) \mid s \in \text{supp} X\}$ and $O = \{s(\vec{y}) \mid s \in \text{supp} X\}$. We say that $X$ is quantum-mechanical if there exists a quantum system

$$(\mathcal{H}, (A_{a,b}^i)_{a \in M_i, b \in O_i, i<n}, \rho)$$

of type $(M, O)$ such that for all assignments $s$, we have $X(s) = p_{s(\vec{x})}(s(\vec{y}))/|M|$.

We call a quantum-mechanical team $X$ a quantum realization of an empirical team $X$ if $X$ is the possibilistic collapse of $X$.

We can define a new atomic formula $QR$ such that $X \models QR$ if $X$ has a quantum realization. Then one can ask what kind of properties this atom has. More generally, we can define an operation $QR$ by

$$X \models QR \varphi \text{ if } X \text{ has a quantum realization } \exists X \text{ such that } \exists X \models \varphi.$$
One can also ask what kind of property of probabilistic teams being quantum-mechanical is. In [10], Durand et al. showed that probabilistic independence logic (with rational probabilities) is equivalent to a probabilistic variant of existential second order logic ESOf\(_Q\). Is being quantum-mechanical expressible in ESOf\(_R\) or do we need more expressibility?

We now observe that the set \(\{X \mid X \models QR\}\) is undecidable but recursively enumerable. For this purpose, we briefly introduce non-local games.

**Definition 6.3.** (i) Let \(I_A, I_B, O_A\) and \(O_B\) be finite sets and let \(V: O_A \times O_B \times I_A \times I_B \rightarrow \{0, 1\}\) be a function\(^7\). A (two-player one-round) non-local game \(G\) with question sets \(I_A\) and \(I_B\), answer sets \(O_A\) and \(O_B\) and decision predicate \(V\) is defined as follows: the first player (Alice) receives an element \(c \in I_A\) and the second player (Bob) receives an element \(d \in I_B\). Alice returns an element \(a \in O_A\) and Bob returns an element \(b \in O_B\). The players are not allowed to communicate the received inputs or their chosen outputs to each other. The players win if \(V(a, b \mid c, d) = 1\) and lose otherwise.

(ii) Let \(G\) be a non-local game. A strategy for \(G\) is a function \(p: O_A \times O_B \times I_A \times I_B \rightarrow [0, 1]\) such that for each pair \((c, d) \in I_A \times I_B\) the function \((a, b) \mapsto p(a, b \mid c, d)\) is a probability distribution. A strategy \(p\) is perfect if \(V(a, b \mid c, d) = 0\) implies \(p(a, b \mid c, d) = 0\).

(iii) Let \(G\) be a non-local game and \(p\) a strategy for \(G\). We say that \(p\) is a quantum strategy if there are finite-dimensional Hilbert spaces \(H_A\) and \(H_B\), a quantum state \(\rho\) of \(H_A \otimes H_B\), a POVM \((M^c_a)_{a \in O_A}\) on \(H_A\) for each \(c \in I_A\) and a POVM \((N^d_b)_{b \in O_B}\) on \(H_B\) for each \(d \in I_B\) such that

\[
p(a, b \mid c, d) = \text{Tr}(M^c_a \otimes N^d_b \rho)
\]

for all \((a, b, c, d) \in O_A \times O_B \times I_A \times I_B\).

The following was proved by Slofstra in [23].

**Theorem 6.4.** It is undecidable to determine whether a linear non-local game has a perfect quantum strategy.

**Proposition 6.5.** There is a many-one reduction from non-local games that have a perfect quantum strategy to teams that have a quantum realization.

**Proof.** Let \(G\) be a game with question sets \(I_A\) and \(I_B\) and answer sets \(O_A\) and \(O_B\) and decision predicate \(V\). We may assume that for each \(c \in I_A\) and \(d \in I_B\) there are some \(a \in O_A\) and \(b \in O_B\) such that \(V(a, b \mid c, d) = 1\), otherwise we may

\(^7\)We write \(V(a, b \mid c, d)\) for the function value.
just map $G$ into the empty team. We let $X_G$ be the set of all assignments $s$ with domain $\{x_0, x_1, y_0, y_1\}$ such that $s(x_0) \in I_A$, $s(x_1) \in I_B$, $s(y_0) \in O_A$, $s(y_1) \in O_B$ and $V(s(y_0), s(y_1) | s(x_0), s(x_1)) = 1$. Let $M = I_A \times I_B$ and

$$O = \{(a, b) \in O_A \times O_B \mid V(a, b | c, d) = 1 \text{ for some } c \in I_A \text{ and } d \in I_B\},$$

and denote by $M_0$, $M_1$, $O_0$ and $O_1$ the appropriate projections of $M$ and $O$. Then

$$M = \{s(x_0, x_1) \mid s \in X_G\} \text{ and } O = \{s(y_0, y_1) \mid s \in X_G\}.$$

We show that $G$ has a perfect quantum strategy if and only if $X_G$ is quantum-realizable. We only show one direction, the other is similar. Suppose that $p$ is a perfect quantum strategy for $G$. Then there are finite-dimensional Hilbert spaces $H_A$ and $H_B$, a quantum state $\rho$ of $H_A \otimes H_B$, a POVM $\{M_c^a \mid a \in O_A\}$ on $H_A$ for each $c \in I_A$ and a POVM $\{N_b^d \mid b \in O_B\}$ on $H_B$ for each $d \in I_B$ such that

$$p(a, b | c, d) = \text{Tr}(M_c^a \otimes N_b^d \rho)$$

for all $(a, b, c, d) \in O_A \times O_B \times I_A \times I_B$. We now define a quantum system

$$S = (\mathcal{H}, (A_i^{c,a})_{c \in M, a \in O_i, i < 2}, \rho)$$

of type $(M, O)$ by setting

- $\mathcal{H} = H_A \otimes H_B$, and
- $A_0^{c,a} = M_c^a$ and $A_1^{d,b} = N_b^d$ for all $a \in O_0$, $b \in O_1$, $c \in M_0$ and $d \in M_1$.

Now clearly

$$p_s^{(c,d)}(a, b) = \text{Tr}(A_i^{(c,d), (a,b)} \rho) = \text{Tr}(M_c^a \otimes N_b^d \rho) = p(a, b | c, d).$$

As $p$ is a perfect strategy, we have $p(a, b | c, d) = 0$ for any $a, b, c$ and $d$ such that $V(a, b | c, d) = 0$. Thus the probabilistic team $X$ arising from the quantum system $S$ is such that $X(s) > 0$ if and only if $V(s(y_0), s(y_1) | s(x_0), s(x_1)) = 1$. Hence the possibilistic collapse of $X$ is $X_G$, and thus $X$ is a quantum realization of $X_G$. \hfill \Box

**Corollary 6.6.** The set $\{X \mid X \models QR\}$ is undecidable but recursively enumerable.

**Proof.** Undecidability follows from Theorem 6.4 and Proposition 6.5.

It is not difficult to show that the problem of determining whether a team has a probabilistic realization which corresponds to a quantum system of dimension $d$ is reducible to the existential theory of the reals, which is known to be in PSPACE [7]. Hence one can check for each dimension $d$ whether a team has a quantum realization of dimension $d$, and thus we obtain an r.e. algorithm. \hfill \Box

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For the rest of this section, we will concentrate on cases where $\rho$ is a so called pure state, i.e. of the form $|\psi\rangle \langle \psi|$ for some vector $|\psi\rangle$, and the POVM elements $A_{i}^{a,b}$ are projectors $|\psi_{i}^{a,b}\rangle \langle \psi_{i}^{a,b}|$, where $|\psi_{i}^{a,b}\rangle$ is the basic vector corresponding to outcome $b$ of measurement $a$ in the space $\mathcal{H}_{i}$. Given $\vec{a} \in M$ and $\vec{b} \in O$, we write $|\psi_{\vec{a},\vec{b}}\rangle$ for $\bigotimes_{i<n} |\psi_{i}^{a_{i},b_{i}}\rangle$. In this case the expression for the probability distribution simplifies to $p_{a}(\vec{b}) = |\langle \psi_{\vec{a},\vec{b}} |\psi\rangle|^{2}$.

We now demonstrate that the teams we used above to prove the no-go theorems of quantum mechanics are all quantum realizable. We leave out in this presentation the KS team, as it is quantum realizable only in a more complex sense.

6.1 EPR teams

We show that EPR teams are quantum realizable.

**Proposition 6.7.** There is a quantum-mechanical team that is the quantum realization of the EPR team.

**Proof.** Look at the system type $(M,O)$, where $M = \{0\} \times \{1\}$ and $O = \{0,1\}^{2}$. Consider the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ in a 2-qubit quantum system (i.e. $\mathcal{H} = \mathbb{C}^{2} \otimes \mathbb{C}^{2}$), and 1-qubit measurements in the computational basis in both qubits: $|\psi_{0}^{0,0}\rangle = |\psi_{1}^{0,0}\rangle = |0\rangle$ and $|\psi_{0}^{0,1}\rangle = |\psi_{1}^{1,1}\rangle = |1\rangle$. Then

$|\psi_{0}^{0,0}\rangle = |00\rangle$,  
$|\psi_{0}^{0,1}\rangle = |01\rangle$,  
$|\psi_{0}^{1,0}\rangle = |10\rangle$ and  
$|\psi_{0}^{1,1}\rangle = |11\rangle$.

Then the probability distribution will be

$$p_{a}(\vec{b}) = \frac{1}{2} |\langle 01 | + \langle 10 | |\psi_{\vec{a},\vec{b}}\rangle|^{2}$$

$$= \frac{1}{2} |\langle 01 | + \langle 10 | |\psi_{\vec{a},\vec{b}}\rangle|^{2}$$

$$= \begin{cases} 
1/2 & \text{if } \vec{b} \in \{(0, 1), (1, 0)\}, \\
0 & \text{otherwise}.
\end{cases}$$

The probabilistic team defined by this quantum system is presented by the following table.
| $x_0$ | $x_1$ | $y_1$ | $y_1$ | probability |
|-------|-------|-------|-------|-------------|
| 0     | 1     | 0     | 0     | 0           |
| 0     | 1     | 0     | 1     | 1/2         |
| 0     | 1     | 1     | 0     | 1/2         |
| 0     | 1     | 1     | 1     | 0           |

Clearly the possibilistic collapse of this is the EPR team. \hfill \square

**Corollary 6.8.** There is a quantum-mechanical team which is not realized by any probabilistic hidden-variable team supporting probabilistic single-valuedness and probabilistic outcome independence, hence

$$QR \not\models \exists \vec{z} \left( \sum_{i<n} y_i \perp \{ y_j \mid j \neq i \} \right).$$

**Proof.** Let $X$ be a quantum team whose possibilistic collapse is the EPR team, and suppose for a contradiction that a probabilistic hidden-variable team $Y$ realizes it. Let $X$ and $Y$ be the respective possibilistic collapses. Then, by Proposition 5.4, $Y$ realizes $X$. By Corollary 5.29, as $Y$ supports probabilistic single-valuedness and probabilistic outcome independence, $Y$ supports single-valuedness and outcome independence. But then $X$ is a GHZ team realized by a hidden-variable team $Y$ that supports single-valuedness and outcome independence, contradicting Theorem 4.1. \hfill \square

### 6.2 GHZ Teams

We show that GHZ teams are quantum realizable.

**Proposition 6.9.** There is a quantum-mechanical team that is the quantum realization of a GHZ team.

**Proof.** We consider the state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)$$

in a 3-qubit quantum system (i.e. $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$), and two 1-qubit measurements in each qubit, with measurement eigen vectors

$$|\psi_k^{0,0}\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \quad \text{and} \quad |\psi_k^{0,1}\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

for one measurement (labeled by “0”) and

$$|\psi_k^{1,0}\rangle = \frac{1}{\sqrt{2}} (|0\rangle - i |1\rangle) \quad \text{and} \quad |\psi_k^{1,1}\rangle = \frac{1}{\sqrt{2}} (|0\rangle + i |1\rangle)$$

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for the other measurement (labeled by “1”) on each qubit \( k = 0, 1, 2 \), where \( i \) denotes the imaginary unit. Conveniently, we can write \( |\psi_{k}^{i,j}\rangle \) as \((|0\rangle + (-1)^{1-j}i|1\rangle)/\sqrt{2})

Then the tensor product of these for a measurement \( \vec{a} \) and outcome \( \vec{b} \) will be

\[
|\psi^{\vec{a},\vec{b}}\rangle = \bigotimes_{i<3} |\psi_{i}^{a_{i},b_{i}}\rangle = \frac{1}{\sqrt{8}} \bigotimes_{i<3} (|0\rangle + (-1)^{1-b_{i}}i^{a_{i}}|1\rangle) = \frac{1}{\sqrt{8}} (|000\rangle + (-1)^{3-(b_{0}+b_{1}+b_{2})}i^{a_{0}+a_{1}+a_{2}}|111\rangle + |\phi\rangle),
\]

where \( |\phi\rangle \in \text{span}(|001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle) \). Then

\[
p_{\vec{a}}(\vec{b}) = \left| \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) |\psi^{\vec{a},\vec{b}}\rangle \right|^{2} = \frac{1}{2} \left| (|000\rangle + |111\rangle) \frac{1}{\sqrt{8}} (|000\rangle + (-1)^{3-(b_{0}+b_{1}+b_{2})}i^{a_{0}+a_{1}+a_{2}}|111\rangle + |\phi\rangle) \right|^{2} = \frac{1}{16} \left| (|000\rangle + (-1)^{3-(b_{0}+b_{1}+b_{2})}i^{a_{0}+a_{1}+a_{2}}|111\rangle + |\phi\rangle) \right|^{2} = \frac{1}{16} \left| 1 + 0 + 0 + 0 + (-1)^{3-(b_{0}+b_{1}+b_{2})}i^{a_{0}+a_{1}+a_{2}} + 0 \right|^{2} = \frac{1}{16} \left| 1 + (-1)^{3-(b_{0}+b_{1}+b_{2})}i^{a_{0}+a_{1}+a_{2}} \right|^{2}.
\]

Thus, by calculating each case, we can conclude that \( p_{\vec{a}}(\vec{b}) \) equals

- 1/4 if \( a_{0} + a_{1} + a_{2} = 0 \) and \( b_{0} + b_{1} + b_{2} \) is odd, or if \( a_{0} + a_{1} + a_{2} = 2 \) and \( b_{0} + b_{1} + b_{2} \) is even,
- 0 if \( a_{0} + a_{1} + a_{2} = 0 \) and \( b_{0} + b_{1} + b_{2} \) is even, or if \( a_{0} + a_{1} + a_{2} = 2 \) and \( b_{0} + b_{1} + b_{2} \) is odd, and
- 1/8 otherwise.

Denote the sets \( P, Q \) and \( R \) as in the definition of a GHZ team, i.e.

\[
P = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\},
\]

\[
Q = \{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\} \text{ and }
\]

\[
R = \{(0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1)\}.
\]

Notice that \( P \) is the set of tuples \( \vec{a} \) such that \( a_{0} + a_{1} + a_{2} = 2 \), \( Q \) the set of tuples \( \vec{b} \) such that \( b_{0} + b_{1} + b_{2} \) is even and \( R \) the set of tuples \( \vec{b} \) such that \( b_{0} + b_{1} + b_{2} \) is odd. Now, the probabilistic team \( \mathcal{X} \) defined by the above quantum system is as follows.
• \(X(s) = 1/32\) when \(s(\vec{x}) = (0, 0, 0)\) and \(s(\vec{y}) \in R\), or when \(s(\vec{x}) \in P\) and \(s(\vec{y}) \in Q\),

• \(X(s) = 0\) when \(s(\vec{x}) = (0, 0, 0)\) and \(s(\vec{y}) \notin R\), or when \(s(\vec{x}) \in P\) and \(s(\vec{y}) \notin Q\), and

• \(X(s) = 1/64\) otherwise.

Let \(X\) be the possibilistic collapse of \(X\). Now \(X\) contains every assignment apart from the following:

\[
\begin{array}{cccccccc}
 x_0 & x_1 & x_2 & y_0 & y_1 & y_2 \\
 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 \\
 0 & 0 & 0 & 1 & 0 & 1 \\
 0 & 0 & 0 & 1 & 1 & 0 \\
 0 & 1 & 1 & 0 & 0 & 1 \\
 0 & 1 & 1 & 0 & 1 & 0 \\
 0 & 1 & 1 & 1 & 0 & 0 \\
 0 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
 x_0 & x_1 & x_2 & y_0 & y_1 & y_2 \\
 1 & 0 & 1 & 0 & 0 & 1 \\
 1 & 0 & 1 & 0 & 1 & 0 \\
 1 & 0 & 1 & 1 & 0 & 0 \\
 1 & 0 & 1 & 1 & 1 & 1 \\
 1 & 1 & 0 & 0 & 0 & 1 \\
 1 & 1 & 0 & 0 & 1 & 0 \\
 1 & 1 & 0 & 1 & 0 & 0 \\
 1 & 1 & 0 & 1 & 1 & 1 \\
\end{array}
\]

Then using the above observations, we see that

\[
R = \{s(\vec{y}) \mid X(s) > 0, s(\vec{x}) = (0, 0, 0)\} \\
= \{s(\vec{y}) \mid s \in X, s(\vec{x}) = (0, 0, 0)\},
\]

\[
Q = \{s(\vec{y}) \mid X(s) > 0, s(\vec{x}) \in P\} \\
= \{s(\vec{y}) \mid s \in X, s(\vec{x}) \in P\}
\]

and

\[
P \subseteq Q = \{s(\vec{y}) \mid s \in X, s(\vec{x}) \in P\} \\
\subseteq \{s(\vec{y}) \mid s \in X, s(\vec{x}) \in Q\}.
\]

Thus \(X\) is a GHZ team.

\[ \square \]

**Corollary 6.10.** There is a quantum-mechanical team which is not realized by any probabilistic hidden-variable team supporting probabilistic \(z\)-independence and probabilistic locality, hence

\[
QR \not\models \exists \vec{z} \left( \vec{z} \perp \vec{x} \land \bigwedge_{i<n} \left( \perp_{x_i \vec{z}} y_i \right) \land \left( y_i \perp \vec{z} \{y_j \mid j \neq i\} \right) \right).
\]

**Proof.** Similar to the proof of Corollary 6.8, but using Thereom 4.3. \[ \square \]
6.3 HARDY TEAMS

We show that Hardy teams are quantum realizable.

Proposition 6.11. There is a quantum-mechanical team that is the quantum realization of a Hardy team.

Proof. We consider the state

\[ |\psi\rangle = -\frac{1}{2}|00\rangle + \frac{\sqrt{3}}{8}|01\rangle + \frac{\sqrt{3}}{8}|10\rangle \]

in a 2-qubit quantum system, and two 1-qubit measurements in each qubit, with measurement eigen vectors

\[ |\psi_0^{0,0}\rangle = \sqrt{\frac{3}{5}}|0\rangle + \sqrt{\frac{2}{5}}|1\rangle \]

and

\[ |\psi_1^{0,1}\rangle = -\sqrt{\frac{2}{5}}|0\rangle + \sqrt{\frac{3}{5}}|1\rangle \]

and

\[ |\psi_0^{1,0}\rangle = |0\rangle \]

and

\[ |\psi_1^{1,1}\rangle = |1\rangle \]

for each qubit \( k = 0, 1 \). Then

\[ |\psi^{(0,0),(0,0)}\rangle = \frac{3}{5}|00\rangle + \frac{\sqrt{6}}{5}|01\rangle + \frac{\sqrt{6}}{5}|10\rangle + \frac{2}{5}|11\rangle , \]

\[ |\psi^{(0,0),(0,1)}\rangle = -\frac{\sqrt{6}}{5}|00\rangle + \frac{3}{5}|01\rangle - \frac{2}{5}|10\rangle + \frac{\sqrt{6}}{5}|11\rangle , \]

\[ |\psi^{(0,0),(1,0)}\rangle = -\frac{\sqrt{6}}{5}|00\rangle - \frac{2}{5}|01\rangle + \frac{3}{5}|10\rangle + \frac{\sqrt{6}}{5}|11\rangle , \]

\[ |\psi^{(0,0),(1,1)}\rangle = \frac{2}{5}|00\rangle - \frac{\sqrt{6}}{5}|01\rangle - \frac{\sqrt{6}}{5}|10\rangle + \frac{3}{5}|11\rangle , \]

\[ |\psi^{(1,0),(0,0)}\rangle = \sqrt{\frac{3}{5}}|i0\rangle + \sqrt{\frac{2}{5}}|i1\rangle , \]

\[ |\psi^{(1,0),(0,1)}\rangle = -\sqrt{\frac{2}{5}}|i0\rangle + \sqrt{\frac{3}{5}}|i1\rangle , \]

\[ |\psi^{(1,0),(1,0)}\rangle = \sqrt{\frac{3}{5}}|i0\rangle + \sqrt{\frac{2}{5}}|i1\rangle , \]

\[ |\psi^{(1,0),(1,1)}\rangle = -\sqrt{\frac{2}{5}}|i0\rangle + \sqrt{\frac{3}{5}}|i1\rangle , \] and

\[ |\psi^{(1,1),(i,j)}\rangle = |ij\rangle . \]
Then we can calculate the probabilities:

\[
p_{(0,0)}(0,0) = |\langle \psi^{(0,0),(0,0)} | \psi \rangle|^2 = \left| \frac{3}{5} \cdot \left( -\frac{1}{2} \right) + \frac{\sqrt{6}}{5} \cdot \frac{\sqrt{3}}{\sqrt{8}} + \frac{\sqrt{6}}{5} \cdot \frac{\sqrt{3}}{\sqrt{8}} \right|^2 \\
= \frac{9}{100},
\]

and others similarly. The probabilities are listed in the below table (measurements on rows, outcomes on columns).

|       | (0, 0) | (0, 1) | (1, 0) | (1, 1) |
|-------|--------|--------|--------|--------|
| (0, 0) | \frac{9}{100} | \frac{27}{200} | \frac{27}{200} | \frac{16}{25} |
| (0, 1) | 0      | \frac{9}{40}  | \frac{5}{8}  | \frac{3}{20}  |
| (1, 0) | 0      | \frac{5}{8}  | \frac{9}{40} | \frac{3}{20}  |
| (1, 1) | \frac{1}{4} | \frac{3}{8}  | \frac{3}{8}  | 0             |

Notable here is that the assignment \((x_0, x_1, y_0, y_1) \mapsto (0, 0, 0, 0)\) will get a positive probability and thus be in the possibilistic collapse while the assignments \((x_0, x_1, y_0, y_1) \mapsto (0, 1, 0, 0), (x_0, x_1, y_0, y_1) \mapsto (1, 0, 0, 0)\) and \((x_0, x_1, y_0, y_1) \mapsto (1, 1, 1, 1)\) do not, as per the definition of a Hardy team. The second requirement in the definition of a Hardy team is also clearly satisfied by the possibilistic collapse.

For concreteness, the below table presents the possibilistic collapse.

| \(x_0\) | \(x_1\) | \(y_0\) | \(y_1\) | \(x_0\) | \(x_1\) | \(y_0\) | \(y_1\) |
|--------|--------|--------|--------|--------|--------|--------|--------|
| 0      | 0      | 0      | 0      | 1      | 0      | 0      | 1      |
| 0      | 0      | 0      | 1      | 1      | 0      | 1      | 0      |
| 0      | 0      | 1      | 0      | 1      | 0      | 1      | 1      |
| 0      | 0      | 1      | 1      | 1      | 0      | 0      | 0      |
| 0      | 1      | 0      | 1      | 1      | 1      | 0      | 1      |
| 0      | 1      | 1      | 0      | 1      | 1      | 1      | 0      |
| 0      | 1      | 1      | 1      | 0      | 1      | 1      | 0      |

7 Open Questions

Questions left open include the following.

- Do the concepts of downwards closedness and strong downwards closedness in probabilistic team semantics coincide?
- What properties commonly found in team-based logics, such as downwards closedness, do the operations \(PR\) and \(QR\) have?
• Can we think of $\mathbf{PR}$ as a “modal” operator? If yes, what axioms does it satisfy? How about $\mathbf{QR}$?

• Is the property of a probabilistic team being quantum-mechanical definable in $\text{ESOf}_R$ or some similar logic?

• Is there a more general theorem behind Proposition 5.39? Why does $\varphi \models \mathbf{PR} \varphi$ hold there while in Proposition 5.31 it fails?

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