Long cycles in graphs through fragments

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Abstract

Four basic Dirac-type sufficient conditions for a graph $G$ to be hamiltonian are known involving order $n$, minimum degree $\delta$, connectivity $\kappa$ and independence number $\alpha$ of $G$: (1) $\delta \geq n/2$ (Dirac); (2) $\kappa \geq 2$ and $\delta \geq (n+\kappa)/3$ (by the author); (3) $\kappa \geq 2$ and $\delta \geq \max\{(n+2)/3, \alpha\}$ (Nash-Williams); (4) $\kappa \geq 3$ and $\delta \geq \max\{(n+3\kappa)/4, \alpha\}$ (by the author). In this paper we prove the reverse version of (4) concerning the circumference $c$ of $G$ and completing the list of reverse versions of (1)-(4): (R1) if $\kappa \geq 2$, then $c \geq \min\{n, 2\delta\}$ (Dirac); (R2) if $\kappa \geq 3$, then $c \geq \min\{n, 3\delta - \kappa\}$ (by the author); (R3) if $\kappa \geq 3$ and $\delta \geq \alpha$, then $c \geq \min\{n, 3\delta - 3\}$ (Voss and Zuluaga); (R4) if $\kappa \geq 4$ and $\delta \geq \alpha$, then $c \geq \min\{n, 4\delta - 2\kappa\}$. To prove (R4), we present four more general results centered around a lower bound $c \geq 4\delta - 2\kappa$ under four alternative conditions in terms of fragments. A subset $X$ of $V(G)$ is called a fragment of $G$ if $N(X)$ is a minimum cut-set and $V(G) - (X \cup N(X)) \neq \emptyset$.

Keywords: Hamilton cycle, circumference, Dirac-type result, connectivity, fragment.

1 Introduction

The classic hamiltonian problem asks to check whether a given graph has a spanning cycle. Such cycles are called Hamilton cycles in honor of Sir William Rowan Hamilton, who, in 1856, described an idea for a game. The hamiltonian problem is based entirely on two genuine concepts "graph" and "Hamilton cycle". Since this problem is NP-complete, generally it is senseless to expect nontrivial results in this area within these two initial concepts and it is natural to look for conditions for the existence of a Hamilton cycle either involving quite new concepts or transforming the initial ones.

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In 1952, Dirac [2] obtained the first sufficient condition for a graph to be hamiltonian based on "minimum degree $\delta$". Actually, this successful combination of three genuine concepts "graph", "Hamilton cycle" and "minimum degree" marked the beginning of a new period in hamiltonism generating a wide class of various problems and ideas for fruitful explorations. Further, these concepts continually were transformed in a way of various limitations, generalizations, extensions and manipulations based on:

(i) structural limitations on graphs: regular and bipartite graphs, graphs with forbidden subgraphs (for example, claw-free graphs and planar graphs) and so on,

(ii) quantitative limitations (relations) on graphs: 2-connected graphs, 1-tough graphs, graphs with $\delta \geq n/2$ and so on,

(iii) generalized Hamilton cycles: long cycles, Hamilton paths and their generalizations (for example, spanning trees with minimum number of leaves), 2-factors, large cycles (for example, dominating cycles and generalized dominating cycles with complements of certain structures) and so on,

(iv) generalized minimum degree notions: degree sequences, degree sums, neighborhood unions, generalized degrees and so on.

Due to transformations (i)-(iv), the frames of a concept "hamiltonian problem" were expanded rapidly involving various related concepts and occupying the major directions in so called "hamiltonian graph theory".

As for minimum degree (Dirac-type) approach, it has been inspired by a couple of well-known results (direct and reverse versions) due to Dirac [2], determining how small the minimum degree $\delta$ of a graph $G$ must be to guarantee the existence of a Hamilton cycle and how large is the circumference $c$ (the length of a longest cycle) depending on $\delta$. Although the corresponding starting bounds $n/2$ and $\min\{n, 2\delta\}$ in these theorems are best possible, since 1952 a number of other analogous best possible theorems appeared essentially lowering the bound $n/2$ and enlarging the bound $2\delta$ due to direct incorporation of some additional graph invariants into these bounds.

At present, four basic Dirac-type hamiltonian sufficient conditions are known directly involving order $n$, minimum degree $\delta$, connectivity $\kappa$ and independence number $\alpha$ with minimum additional limitations and transformations of the initial conceptions due to Dirac [2], the author [8],[9], Nash-Williams [7] and the author [10], respectively.

**Theorem A** [2]. Every graph with $\delta \geq \frac{1}{2}n$ is hamiltonian.

**Theorem B** [9]. Every 2-connected graph with $\delta \geq \frac{1}{3}(n + \kappa)$ is hamiltonian.

**Theorem C** [7]. Every 2-connected graph with $\delta \geq \max\{\frac{1}{3}(n + 2), \alpha\}$ is hamiltonian.

**Theorem D** [10]. Every 3-connected graph with $\delta \geq \max\{\frac{1}{4}(n + 2\kappa), \alpha\}$ is hamiltonian.
A short proof of Theorem B was given in [3] due to Häggkvist.

The reverse versions of Theorems A-C concerning long cycles in graphs, are due to Dirac [2], the author [8],[9] and Voss and Zuluaga [14], respectively. In this paper we present the detailed proof of the last reverse version corresponding to Theorem D (it was announced still in 1985 with a short outline of the proof [11]) completing the list of reverse versions of Theorems A-D.

**Theorem E** [2]. Every 2-connected graph has a cycle of length at least \( \min\{n, 2\delta\} \).

**Theorem F** [9]. Every 3-connected graph has a cycle of length at least \( \min\{n, 3\delta - \kappa\} \).

**Theorem G** [14]. Every 3-connected graph with \( \delta \geq \alpha \) has a cycle of length at least \( \min\{n, 3\delta - 3\} \).

**Theorem 1** [11]. Every 4-connected graph with \( \delta \geq \alpha \) has a cycle of length at least \( \min\{n, 4\delta - 2\kappa\} \).

To prove Theorem 1, we present four more general Dirac-type results centered around a lower bound \( c \geq 4\delta - 2\kappa \) under four alternative conditions in terms of fragments.

If \( X \subset V(G) \), then \( N(X) \) denotes the set of all vertices of \( G - X \) adjacent to vertices in \( X \). Furthermore, \( \hat{X} \) is defined as \( V(G) - (X \cup N(X)) \). Following Hamidoune [6], we define a subset \( X \) of \( V(G) \) to be a fragment of \( G \) if \( N(X) \) is a minimum cut-set and \( \hat{X} \neq \emptyset \). If \( X \) is a fragment then \( \hat{X} \) is a fragment too and \( \hat{\hat{X}} = X \). For convenience, we will use \( X^\uparrow \) and \( X^\downarrow \) to denote \( X \) and \( \hat{X} \), respectively. An endfragment is a fragment that contains no other fragments as a proper subset.

**Theorem 2.** Let \( G \) be a 3-connected graph with \( \delta \geq \alpha \). If \( |A^\uparrow| \leq 3\delta - \kappa - 4 \) and \( |A^\downarrow| \leq 3\delta - 3\kappa \) for an endfragment \( A^\downarrow \) of \( G \), then \( c \geq \min\{n, 4\delta - 2\kappa\} \).

**Theorem 3.** Let \( G \) be a 4-connected graph with \( \delta \geq \alpha \). If \( |A^\uparrow| \leq 3\delta - \kappa - 4 \), \( |A^\downarrow| \geq 3\delta - 3\kappa + 1 \) and \( |A^\downarrow| \geq |A^\downarrow| \) for an endfragment \( A^\downarrow \) of \( G \), then \( c \geq \min\{n, 4\delta - 2\kappa\} \).

**Theorem 4.** Let \( G \) be a 4-connected graph with \( \delta \geq \alpha \). If \( |A^\uparrow| \geq 3\delta - \kappa - 3 \) and \( |A^\downarrow| \leq 3\delta - 3\kappa \) for an endfragment \( A^\downarrow \) of \( G \), then \( c \geq \min\{n, 4\delta - 2\kappa\} \).

**Theorem 5.** Let \( G \) be a 4-connected graph with \( \delta \geq \alpha \). If \( |A^\uparrow| \geq 3\delta - \kappa - 3 \) and \( |A^\downarrow| \geq 3\delta - 3\kappa + 1 \) for an endfragment \( A^\downarrow \) of \( G \), then \( c \geq \min\{n, 4\delta - 2\kappa\} \).

Observe that the bounds \( n/2 \) and \( \min\{n, 2\delta\} \) in Theorems A and E were improved to \((n + \kappa)/3\) and \( \min\{n, 3\delta - \kappa\} \) (Theorems B and F), respectively, by direct incorporation of connectivity \( \kappa \) into these bounds. We conjecture that
the last two bounds are the best in a sense that they cannot be improved by an analogous way within graph invariants determinable in polynomial time.

**Conjecture 1.** The bounds $\frac{1}{3}(n + \kappa)$ and $3\delta - \kappa$ in Theorems B and F, respectively, can not be improved by direct incorporation of any graph invariants determinable in polynomial time.

## 2 Definitions and notations

By a graph we always mean a finite undirected graph $G$ without loops or multiple edges. A good reference for any undefined terms is [1]. For $H$ a subgraph of $G$ we will denote the vertices of $H$ by $V(H)$ and the edges of $H$ by $E(H)$. For every $S \subset V(G)$ we use $G - S$ short for $(V(G) - S)$, the subgraph of $G$ induced by $V(G) - S$. In addition, for a subgraph $H$ of $G$ we use $G - H$ short for $G - V(H)$. If $X \subseteq V(G)$, then $N(X)$ denotes the set of all vertices of $G - X$ adjacent to vertices in $X$.

Let $\delta$ denote the minimum degree of vertices of $G$. The connectivity $\kappa$ of $G$ is the minimum number of vertices whose removal from $G$ results in a disconnected or trivial graph. We say that $G$ is $s$-connected if $\kappa \geq s$. A set $S$ of vertices is independent if no two elements of $S$ are adjacent in $G$. The cardinality of maximum set of independent vertices is called the independence number and denoted by $\alpha$.

Paths and cycles in a graph $G$ are considered as subgraphs of $G$. If $Q$ is a path or a cycle of $G$, then the length of $Q$, denoted by $|Q|$, is $|E(Q)|$. Throughout the paper the vertices and edges of a graph can be interpreted as cycles of lengths 1 and 2, respectively. A graph $G$ is hamiltonian if it contains a Hamilton cycle (a cycle containing every vertex of $G$).

Let $C$ be a cycle of $G$ with a fixed cyclic direction. In that context, the $h$-th successor and the $h$-th predecessor of a vertex $u$ on $C$ are denoted by $u^{+h}$ and $u^{-h}$, respectively. If $h = 1$, we abbreviate $u^+$ and $u^-$ to $u^+$ and $u^-$, respectively. For a subset $S$ of $V(C)$, we define $S^+ = \{u^+ \mid u \in S\}$ and $S^- = \{u^- \mid u \in S\}$. For two vertices $u$ and $v$ of $C$, let $u \overrightarrow{C} v$ denote the segment of $C$ from $u$ to $v$ in the chosen direction on $C$ and $u \overleftarrow{C} v$ denote the segment in the reverse direction. We also use similar notation for a path $P$ of $G$. For $P$ a path of $G$, denote by $F(P)$ and $L(P)$ the first and the last vertices of $P$, respectively.

Let $Q$ be a cycle or a path of a graph $G$, $r \geq 2$ a positive integer and $Z_1, Z_2, \ldots, Z_p$ are subsets of $V(Q)$ with $p \geq 2$. A collection $(Z_1, \ldots, Z_p)$ is called a $(Q, r)$-scheme if $d_Q(x, y) \geq 2$ for each distinct $x, y \in Z_i$ (where $i \in \{1, \ldots, p\}$) and $d_Q(x, y) \geq r$ for each distinct $x \in Z_i$ and $y \in Z_j$ (where $i, j \in \{1, \ldots, p\}$ and $i \neq j$). A $(Q, r)$-scheme is nontrivial if $(Z_1, \ldots, Z_p)$ has a system of distinct representatives. The definition of $(Q, r)$-scheme was first introduced by Nash-Williams [7] for $p = 2$.

Given four integers $a, b, t, \kappa$ with $\kappa \leq t$, we will use $H(a, b, t, \kappa)$ as a limit.
example for Theorem 1 to denote the graph obtained from $tK_n + \overline{K}_t$ by taking any $\kappa$ vertices in subgraph $\overline{K}_t$ and joining each of them to all vertices of $K_t$.

**Definition A** $\{Q_1, \ldots, Q_m; Q'_1, \ldots, Q'_m; V_1, \ldots, V_m; V'\}$. Let $A^i$ be a fragment of $G$ with respect to a minimum cut-set $S$. Define $Q_1, \ldots, Q_m$ as a collection of vertex disjoint paths in $(A^i \cup S)$ with terminal vertices in $S$ such that $|V(Q_i)| \geq 2$ ($i = 1, \ldots, m$) and $\sum_{i=1}^m |V(Q_i)|$ is as great as possible. Abbreviate $V_i = V(Q_i)$ ($i = 1, \ldots, m$) and $V' = \bigcup_{i=1}^m V'_i$. Form a united path $Q'$ with vertex set $V'$ consisting of $Q_1, \ldots, Q_m$ and some appropriate extra-edges added in.

**Definition B** $\{Q_1, \ldots, Q_m; Q'_0; V_1, \ldots, V_m; V'\}$. Let $A^i$ be a fragment of $G$ with respect to a minimum cut-set $S$ and $Q_1, \ldots, Q_m$ are as defined in Definition A. Denote by $Q_1, \ldots, Q_n$, a collection of paths (if exist) in $(A^i \cup S)$ with $\sum_{i=1}^m |V(Q_i')|$ as great as possible such that combining $Q_1, \ldots, Q_m$ with $Q_1, \ldots, Q_n$, results in a simple cycle. Abbreviate $V_i = V(Q_i')$ ($i = 1, \ldots, m$) and $V' = \bigcup_{i=1}^m V'_i$. For the special case $|V| = 2$ and $G \not\in \emptyset$, say $z \in S - V'$, we will use $Q^i$ to denote a longest path in $(A^i \cup \{F(Q^i), L(Q^i), z\})$ connecting $F(Q^i)$ and $L(Q^i)$ and passing through $z$.

**Definition C** $\{C^*; C^{**}\}$. Denote by $C^*$ the cycle (if exist) consisting of $Q_1, \ldots, Q_m$ and $Q'_1, \ldots, Q'_m$. Assume w.l.o.g. that $Q_1, \ldots, Q_m$ is chosen such that $C^*$ has a maximal length. Denote by $C^{**}$ a longest cycle of $G$ with $V(C^*) \subseteq V(C^{**})$.

### 3 Preliminaries

In [7], Nash-Williams proved the following result concerning $(C, r)$-schemes for a cycle $C$ and a pair $(Z_1, Z_2)$ of subsets of $V(C)$.

**Lemma A** [7]. Let $C$ be a cycle and $(Z_1, Z_2)$ be a nontrivial $(C, r)$-scheme. Then

$$|V(C)| \geq \min \left\{ 2(|Z_1| + |Z_2|) + 2r - 6, \frac{1}{2}r(|Z_1| + |Z_2|) \right\}.$$ 

Basing on proof technique used in [7], we prove two analogous results for the families $(Z_1, Z_2, Z_3)$ and $(Z_1, Z_2, Z_3, Z_4)$ of subsets of $V(C)$ under additional limitations $|Z_1| = 1$ and $|Z_1| = |Z_2| = 1$, respectively.

**Lemma 1.** Let $C$ be a cycle and $(Z_1, Z_2, Z_3)$ be a nontrivial $(C, r)$-scheme with $|Z_1| = 1$. Then

$$|V(C)| \geq \min \left\{ 2 \sum_{i=1}^3 |Z_i| + 3r - 12, \frac{1}{2}r \left( \sum_{i=1}^3 |Z_i| - 1 \right) \right\}.$$
Lemma 2. Let $C$ be a cycle and $(Z_1, Z_2, Z_3, Z_4)$ be a nontrivial $(C, r)$-scheme with $|Z_1| = |Z_2| = 1$. Then

$$|V(C)| \geq \min \left\{ 2 \sum_{i=1}^{4} |Z_i| + 4r - 18, \frac{1}{2} r \left( \sum_{i=1}^{4} |Z_i| - 2 \right) \right\}.$$ 

In this paper a number of path-versions of Lemmas 1, 2 and 3 will be used for the path $Q$ and the families $(Z_1, Z_2)$, $(Z_1, Z_2, Z_3)$, $(Z_1, Z_2, Z_3, Z_4)$ of subsets of $V(Q)$ under additional limitations $|Z_1| = 1$, $|Z_1| = |Z_2| = 1$ and $|Z_1| = |Z_2| = |Z_3| = 1$ in some of them.

Lemma 3. Let $Q$ be a path and $(Z_1, Z_2)$ be a nontrivial $(Q, r)$-scheme. Then

$$|V(Q)| \geq \min \left\{ 2(|Z_1| + |Z_2|) + r - 5, \frac{1}{2} r (|Z_1| + |Z_2| - 2) + 1 \right\}.$$ 

Lemma 4. Let $Q$ be a path and $(Z_1, Z_2)$ be a nontrivial $(Q, r)$-scheme with $|Z_1| = 1$ and $|Z_2| \geq 2$. Then $|V(Q)| \geq 2|Z_2| + r - 3$.

Lemma 5. Let $Q$ be a path and $(Z_1, Z_2, Z_3)$ be a nontrivial $(Q, r)$-scheme with $|Z_1| = 1$. Then

$$|V(Q)| \geq \min \left\{ 2 \sum_{i=1}^{3} |Z_i| + 2r - 11, \frac{1}{2} r \left( \sum_{i=1}^{3} |Z_i| - 3 \right) + 1 \right\}.$$ 

Lemma 6. Let $Q$ be a path and $(Z_1, Z_2, Z_3)$ be a nontrivial $(Q, r)$-scheme with $|Z_1| = |Z_2| = 1$ and $|Z_3| \geq 3$. Then $|V(Q)| \geq 2|Z_3| + 2r - 5$.

Lemma 7. Let $Q$ be a path and $(Z_1, Z_2, Z_3, Z_4)$ be a nontrivial $(Q, r)$-scheme with $|Z_1| = |Z_2| = 1$. Then

$$|V(Q)| \geq \min \left\{ 2 \sum_{i=1}^{4} |Z_i| + 3r - 17, \frac{1}{2} r \left( \sum_{i=1}^{4} |Z_i| - 4 \right) + 1 \right\}.$$ 

Lemma 8. Let $Q$ be a path and $(Z_1, Z_2, Z_3, Z_4)$ be a nontrivial $(Q, r)$-scheme with $|Z_1| = |Z_2| = |Z_3| = 1$ and $|Z_4| \geq 4$. Then $|V(Q)| \geq 2|Z_4| + 3r - 7$.

Using Woodall’s proof technique [15] known as ”hopping”, we obtain the next result concerning cycles through specified edges.

Lemma 9. Let $G$ be a graph, $A^1$ be a fragment of $G$ with respect to a minimum cut-set $S$ and the connectivity $k$ is even. Let $L$ be a set of $k/2$ independent (vertex disjoint) edges in $(S)$ and let $v_1v_2v_3v_4$ be a path in $G$ with $v_1, v_4 \in A^1$ and $v_2, v_3 \in S$. If a subgraph $(S \cup A^1) - \{v_2, v_3, v_4\}$ contains a cycle $C$ that uses all the edges in $L - \{v_2v_3\}$, then $(S \cup A^1)$ contains a cycle that uses all the edges in $L$. 

Using Woodall’s proof technique [15] known as ”hopping”, we obtain the next result concerning cycles through specified edges.
In [12, Theorem 1], Veldman proved the following.

**Lemma B** [12]. If $G$ is a graph with $\delta > 3\kappa/2 - 1$, then no endfragment of $G$ contains a vertex $v$ with $\kappa(G - v) = \kappa - 1$.

We shall use Lemmas 9 and B to prove the following useful lemma.

**Lemma 10.** Let $G$ be a 2-connected graph, $A^\uparrow$ be an endfragment of $G$ with respect to a minimum cut-set $S$ and let $L$ be a set of independent edges in $\langle S \rangle$. If $\delta > 3\kappa/2 - 1$, then $\langle A^\uparrow \cup V(L) \rangle$ contains a cycle that uses all the edges in $L$.

For the special case $\alpha \leq \delta \leq 3\kappa/2 - 1$ the main lower bound $c \geq \min \{n, 4\delta - 2\kappa\}$ will be proved by an easy way.

**Lemma 11.** Every 3-connected graph with $\alpha \leq \delta \leq 3\kappa/2 - 1$ has a cycle of length at least $\min \{n, 4\delta - 2\kappa\}$.

We need also the following result from [13].

**Lemma C** [13]. Let $G$ be a hamiltonian graph with $\{v_1, ..., v_r\} \subseteq V(G)$ and $d(v_i) \geq r$ ($i = 1, ..., r$). Then any two vertices of $V(G)$ are connected by a path of length at least $r$.

Let $V^\uparrow$ and $V^\downarrow$ are as defined in Definitions A and B. Using above lemmas, we shall prove the following four basic lemmas that are crucial for the proofs of Theorems 2-5.

**Lemma 12.** Let $G$ be a 3-connected graph with $\delta \geq \alpha$. If $|A^\uparrow| \leq 3\delta - \kappa - 4$ for a fragment $A^\uparrow$ of $G$, then $A^\uparrow \subseteq V^\uparrow$.

**Lemma 13.** Let $G$ be a 4-connected graph with $\delta \geq \alpha$. If $|A^\uparrow| \geq 3\delta - \kappa - 3$ for a fragment $A^\uparrow$ of $G$, then either $A^\uparrow \subseteq V^\uparrow$ or $|V^\uparrow| \geq 3\delta - 5$.

**Lemma 14.** Let $G$ be a 3-connected graph with $\delta > 3\kappa/2 - 1$. If $|A^\downarrow| \leq 3\delta - 3\kappa$ for an endfragment $A^\downarrow$ of $G$, then $\langle A^\downarrow - V^\downarrow \rangle$ is edgeless.

**Lemma 15.** Let $G$ be a 3-connected graph with $\delta > 3\kappa/2 - 1$ and $|A^\downarrow| \geq 3\delta - 3\kappa + 1$ for an endfragment $A^\downarrow$ of $G$ with respect to a minimum cut-set $S$. If $f = 2$ and $S \subseteq V^\downarrow$, then either $\langle A^\downarrow - V^\downarrow \rangle$ is edgeless or $|V^\downarrow| \geq 2\delta - 2\kappa + 3$, where $f = |V^\downarrow \cap S|$. If $f = 2$ and $S \not\subseteq V^\downarrow$, then either $\langle A^\downarrow - V(Q^\downarrow_1) \rangle$ is edgeless or $|V^\downarrow| \geq 3\delta - 3\kappa + 1$. If $f \geq 3$, then either $\langle A^\downarrow - V^\downarrow \rangle$ is edgeless or $|V^\downarrow| \geq 3\delta - 3\kappa + f - 1$. 
4 Proofs of lemmas

Proof of Lemma 1. Put $Z = \bigcup_{i=1}^{3} Z_i$. For each $\xi \in Z$, let $f(\xi)$ be the smallest positive integer $h$ such that $\xi + h \in Z$ and let $g(\xi) = |\{i| \xi \in Z_i\}|$. Clearly

$$|C| = \sum_{\xi \in Z} f(\xi), \quad \sum_{i=1}^{3} |Z_i| = \sum_{\xi \in Z} g(\xi). \quad (1)$$

Since $(Z_1, Z_2, Z_3)$ is a nontrivial $(C, r)$-scheme, we have $f(\xi) \geq r$ for each $\xi \in Z$ when $g(\xi) \geq 2$ and $f(\xi) \geq 2$ when $g(\xi) = 1$. Let $(\xi_1, \xi_2, \xi_3)$ be a system of distinct representatives of $Z_1, Z_2, Z_3$. Since $|Z_1| = 1$, we have $g(\xi) \leq 2$ for each $\xi \in Z - \{\xi_1\}$. In particular, $g(\xi_i) \leq 2$ ($i = 2, 3$). Assume first that $r \leq 4$. Clearly $\xi_1 \notin Z_1$ and hence $f(\xi_1) \geq r \geq r(g(\xi_1) - 1)/2$. Further, for each $\xi \in Z - \{\xi_1\}$, either $g(\xi) = 2$ implying that $f(\xi) \geq r = rg(\xi))/2$ or $g(\xi) = 1$ implying that $f(\xi) \geq 2 \geq rg(\xi)/2$. By summing and using (1), we get

$$|C| \geq \sum_{\xi \in Z} f(\xi) \geq \left(\sum_{i=1}^{3} |Z_i|\right)r/2 - r/2$$

and the result follows. Now assume that $r \geq 5$.

Case 1. $f(\xi) \geq r$ for each $\xi \in Z$.

By (1), $|C| \geq r|Z| \geq r(|Z_1| + |Z_2| + |Z_3| - 1)/2$ and the result follows.

Case 2. $f(\xi) \leq r - 1$ for some $\xi \in Z$.

Since $g(\xi_2) \leq 2$ and $g(\xi_3) \leq 2$, we can distinguish three subcases.

Case 2.1. $g(\xi_2) = g(\xi_3) = 1$.

Let $\tau_i$ be the smallest positive integer such that $\xi_i^{+(r_i+1)} \in Z - Z_i$ and $g(\xi_i^{+(r_i)}) = 1$ ($i = 2, 3$). Then

$$f(\xi_i^{+(r_i+1)}) \geq r \geq 2g(\xi_i^{+(r_i)}) + r - 2 \quad (i = 2, 3),$$

$$f(\xi_1) \geq r \geq 2g(\xi_1) + r - 6.$$  

For each $\xi \in Z - \{\xi_2^{r_2}, \xi_3^{r_3}, \xi_1\}$, either $g(\xi) = 2$ implying $f(\xi) \geq r \geq 5 > 2g(\xi)$ or $g(\xi) = 1$ and again implying $f(\xi) \geq 2 = 2g(\xi)$. By (1), $|C| \geq 2 \sum_{i=1}^{3} |Z_i| + 3r - 10$ and the result follows.

Case 2.2. Either $g(\xi_2) = 1$, $g(\xi_3) = 2$ or $g(\xi_2) = 2$, $g(\xi_3) = 1$.

By symmetry, we can assume that $g(\xi_2) = 1$ and $g(\xi_3) = 2$. If $g(\xi) = 1$ for some $\xi \in Z_3$, then $(\xi_1, \xi_2, \xi)$ is a system of distinct representatives for $(Z_1, Z_2, Z_3)$ and we can argue as in Case 2.1. Let $g(\xi) \geq 2$ for all $\xi \in Z_3$ and let $\tau_2$ be the smallest positive integer such that $\xi_2^{+(\tau_2+1)} \in Z - Z_2$ and $g(\xi_2^{+(\tau_2)}) = 1$. Then
\[ f(\xi_1) \geq r \geq 2g(\xi_1) + r - 6, \]
\[ f(\xi_2^+)^2 \geq r \geq 2g(\xi_2^+) + r - 2, \quad f(\xi_3) \geq 2g(\xi_3) + r - 4. \]

For each \( \xi \in Z - \{\xi_1, \xi_2^+, \xi_3\} \), either \( g(\xi) = 2 \) which implies \( f(\xi) \geq r \geq 5 > 2g(\xi) \) or \( g(\xi) = 1 \) again implying \( f(\xi) \geq 2 = 2g(\xi) \). By (1), \(|C| \geq 2 \sum_{i=1}^{3} |Z_i| + 3r - 12 \) and the result follows.

**Case 2.3.** \( g(\xi_2) = g(\xi_3) = 2 \).

If \( g(\xi) = 1 \) for some \( \xi \in Z_2 \cup Z_3 \), say \( \xi \in Z_2 - Z_3 \), then \((\xi_1, \xi, \xi_3)\) is a system of distinct representatives and we can argue as in Case 2.2. Otherwise \( f(\xi) \geq r \) for all \( \xi \in Z \) and we can argue as in Case 1. \( \triangle \)

**Proof of Lemma 2.** Let \( Z = \bigcup_{i=1}^{4} Z_i \). For each \( \xi \in Z \), let \( f(\xi) \) be the smallest positive integer \( h \) such that \( \xi^+ h \in Z \) and let \( g(\xi) = |\{i|\xi \in Z_i\}|. \) Then

\[ |C| = \sum_{\xi \in Z} f(\xi), \quad \sum_{i=1}^{4} |Z_i| \sum_{\xi \in Z} g(\xi). \tag{2} \]

Let \((\xi_1, \xi_2, \xi_3, \xi_4)\) be a system of distinct representatives of \( Z_1, Z_2, Z_3, Z_4 \). Since \(|Z_1| = |Z_2| = 1\), we have \( g(\xi) \leq 3 \) for each \( \xi \in Z \). In particular, \( g(\xi_3) \leq 2 \) and \( g(\xi_4) \leq 2 \). Assume first that \( r \leq 4 \). Clearly \( \xi_1^+ \notin Z_i \) \((i = 1, 2)\) and hence,

\[ f(\xi_i) \geq r \geq r(g(\xi_i) - 1)/2 \quad (i = 1, 2). \]

For each \( \xi \in Z - \{\xi_1, \xi_2\} \), either \( g(\xi) = 2 \) implying \( f(\xi) \geq r = rg(\xi)/2 \) or \( g(\xi) = 1 \) again implying \( f(\xi) \geq 2 \geq rg(\xi)/2 \).

By (2), \(|C| \geq r(\sum_{i=1}^{4} |Z_i|)/2 - r \) and we are done. Now assume that \( r \geq 5 \).

**Case 1.** \( f(\xi) \geq r \) for each \( \xi \in Z \).

By (2), \(|C| \geq r|\xi| \geq r(\sum_{i=1}^{4} |Z_i| - 2)/2 \) and the result follows.

**Case 2.** \( f(\xi) \leq r - 1 \) for some \( \xi \in Z \).

**Case 2.1.** Either \( g(\xi_1) = 1 \) or \( g(\xi_2) = 1 \).

Assume w.l.o.g. that \( g(\xi_1) = 1 \). Let \( p \) be the smallest positive integer such that \( \xi_1^+ p \in Z \). Consider two new cycles \( C_1 \) and \( C_2 \), obtained from \( C \) by identifying \( \xi_1 \) and \( \xi_1^+ p \). Since \( f(\xi_1) \geq r \), we have \(|C_1| \geq r \) and \(|C_2| \leq |C| - r + 1 \). Clearly \((Z_2, Z_3, Z_4)\) is a nontrivial \((C_2, r)\)-scheme with \(|Z_2| = 1 \). Since \( \sum_{i=1}^{4} |Z_i| - 1 = \sum_{i=2}^{4} |Z_i| \) and \(|C| \geq |C_2| + r - 1 \), we can obtain the desired result by Lemma 1.

**Case 2.2.** \( g(\xi_1) \geq 2 \) and \( g(\xi_2) \geq 2 \).

Clearly \( f(\xi_i) \geq r \geq 2g(\xi_i) + r - 6 \quad (i = 1, 2) \). If \( g(\xi) \geq 2 \) for each \( \xi \in Z_3 \cup Z_4 \), then \( f(\xi) \geq r \) for each \( \xi \in Z \) and we can argue as in Case 1. Let \( g(\xi) = 1 \) for some \( \xi \in Z_3 \cup Z_4 \). Assume w.l.o.g. that \( g(\xi_3) = 1 \).

**Case 2.2.1.** \( g(\xi_4) = 1 \).
Let \( \tau_i \) be the smallest positive integer such that \( \xi_i^{+(\tau_i+1)} \in Z - Z_i \) and \( g(\xi_i^{+(\tau_i)}) = 1 \) \((i = 3, 4)\). Then

\[
f(\xi_i^{+(\tau_i)}) \geq r \geq 2g(\xi_i^{+(\tau_i)}) + r - 2 \quad (i = 3, 4).
\]

For each \( \xi \in Z - \{\xi_3^*, \xi_4^*, \xi_1, \xi_2\} \), either \( g(\xi) = 2 \) implying \( f(\xi) \geq r \geq 5 > 2g(\xi) \) or \( g(\xi) = 1 \) again implying \( f(\xi) \geq 2 = 2g(\xi) \). By (2), \( |C| \geq 2 \sum_{i=1}^4 |Z_i| + 4r - 16 \) and the result follows.

**Case 2.2.2.** \( g(\xi_i) = 2 \).

Let \( \tau_i \) be the smallest positive integer such that \( \xi_i^{+(\tau_i+1)} \in Z - Z_i \) and \( g(\xi_i^{+(\tau_i)}) = 1 \). Then \( f(\xi_i^{+(\tau_i)}) \geq r \geq 2g(\xi_i^{+(\tau_i)}) + r - 2 \). If \( g(\xi) = 1 \) for some \( \xi \in Z_4 \), then we can argue as in Case 2.2.1. Otherwise \( g(\xi) = 2 \) and \( f(\xi) \geq r \geq 2g(\xi) + r - 4 \) for each \( \xi \in Z_4 - \{\xi_1, \xi_2\} \). By (2), \( |C| \geq 2 \sum_{i=1}^4 |Z_i| + 3r - 18 \) and the result follows. \( \Delta \)

**Proofs of Lemmas 3-8.** To prove Lemma 3, form a cycle \( C \) consisting of \( Q \) and an arbitrary path of length \( r \) having only \( F(Q) \) and \( L(Q) \) in common with \( Q \). Since \((Z_1, Z_2)\) is a nontrivial \((C,r)\)-scheme, the desired result follows from Lemma A immediately. Lemmas 5 and 7 can be proved by a similar way using Lemmas 3 and 2, respectively. The proofs of Lemmas 4, 6 and 8 are straightforward. \( \Delta \)

**Proof of Lemma 9.** We use a variant of an important proof technique known as "hopping" [15]. For the case \( v_1 \notin V(C) \), we can argue exactly as in [15, proof of Theorem 2]. Let \( v_1 \in V(C) \). Put \( G^* = G - \{v_2, v_3\} \) and \( L' = L - \{v_2v_3\} \). If \( X \subseteq V(C) \), we consider all maximal segments of \( C - L' \) connecting two vertices of \( X \). Following [4], the union of the vertex sets of these segments is denoted \( Cl(X) \), the endvertices of the segments constitute \( Fr(X) \) and finally \( Int(X) = Cl(X) - Fr(X) \). The sequence \( A_{-1} \subseteq A_0 \subseteq A_1 \subseteq \ldots \) of subsets of \( V(C) \) is defined as follows: \( A_{-1} = \emptyset \) and \( A_0 \) is the set of vertices \( z \) of \( C \) such that \( G^* \) has a path from \( v_4 \) to \( z \) having only \( z \) in common with \( C \). For each \( p \geq 1 \), \( A_p \) is the union of \( A_{p-1} \) and the set of vertices \( z \) such that \( G^* \) contains a path \( P \) from \( Int(A_{p-1}) \) to \( z \) having only its ends in common with \( C \). Let \( A = \bigcup_{i=0}^\infty A_i \) and \( B = \{v_1\} \). Consider the following statement:

\( X(P) \): There exists a path \( R_p \) in \( G^* - \{v_4\} \) starting at \( v_p \) in \( A_p \) and terminating at \( v_1 \) such that conditions (a) – (c) below are satisfied.

(a) \( R_p \) contains all the edges of \( L' \) and all the vertices of \( Int(A_{p-1}) \).

(b) If \( Q \) is a segment of \( R_p \) from \( u \) to \( v \) say, having precisely \( u \) and \( v \) in common with \( C \), then one of \( u \) and \( v \) is outside \( A_p \) and the other is outside \( \{v_1\} \).

(c) If \( y \in Int(X) \cap R_p \), where \( X = A_{p'}, p' \leq p - 1 \) and \( M \) denotes the segment of \( C - L' \) which starts and terminates at \( Fr(X) \) and contains \( y \), then \( R_p \) contains \( M \).

Prove that \( X(P) \) holds for some \( p \). For suppose this is not the case. Then
Proof of Lemma 10. The proof is by induction on \( \kappa \). For \( \kappa = 2 \) the result follows easily. Let \( \kappa \geq 3 \). Suppose first that \( S - V(L) \neq \emptyset \) and choose any \( u \in S - V(L) \). Clearly \( A^1 \) is an endfragment for \( G - u \) too with respect to \( S - u \) and \( \delta(G - u) \geq \delta - 1 > 3\kappa(G - u)/2 - 1 \). By the induction hypothesis, \( G - u \) (as well as \( G \)) contains the desired cycle. Now let \( S - V(L) = \emptyset \), i.e. \( |L| = \kappa/2 \), and choose any \( vw \in L \). It follows from \( \delta > 3\kappa/2 - 1 \) that \( |A^1| \geq 2 \). Further, it is not hard to see that there exist two edges \( vv' \) and \( uw' \) such that \( v', w' \in A^1 \) and \( v' \neq w' \). Put \( G^* = G - \{v, w, w'\} \) and \( S' = S - \{v, w\} \). By Lemma B, \( \kappa(G - w') = \kappa \), i.e. \( \kappa(G^*) = \kappa - 2 \). Also, \( \delta(G^*) \geq \delta - 3 > 3\kappa(G^*)/2 - 1 \). If \( A^1 - \{w'\} \) is an endfragment of \( G^* \) (with respect to \( S' \)), then by the induction hypothesis, \( (S' \cup A^1) - \{w'\} \) contains a cycle that uses all the edges in \( L - \{vw\} \) and the result follows from Lemma 9 immediately. Otherwise choose an endfragment \( A_0^1 \subset A^1 - \{w'\} \) in \( G^* \) with respect to a minimum cut-set \( S'' \) of order \( \kappa - 2 \). Let \( P_1, \ldots, P_{\kappa - 2} \) be the vertex disjoint paths connecting \( S' \) and \( S'' \), where \( |V(P_i)| = 1 \) if and only if \( F(P_i) = L(P_i) \in S' \cap S'' \) \( (i = 1, \ldots, \kappa - 2) \). By the induction hypothesis, \( (A_0^1 \cup S'') \) contains a cycle that uses all the independent edges in \( (S'') \) chosen beforehand. Then using \( P_1, \ldots, P_{\kappa - 2} \), we can form a cycle in \( (S' \cup A^1 - \{w'\}) \) that uses all the edges in \( L - \{vw\} \) and the result follows from Lemma 9. \( \Delta \)

Proof of Lemma 11. By Theorem G, \( c \geq \min\{n, 3\delta - 3\} \). If \( c = n \), then we are done. So, assume that \( c \geq 3\delta - 3 \). If \( \kappa \geq 4 \), then \( c \geq 3\delta - 3 \geq 4\delta - 3\kappa/2 - 2 \geq 4\delta - 2\kappa \). Finally, if \( \kappa = 3 \), then from \( \delta \leq 3\kappa/2 - 1 \) we get \( \delta = 3 \) implying that \( c \geq 3\delta - 3 = 4\delta - 2\kappa \). \( \Delta \)

Proof of Lemma 12. Let \( S,Q_i^1,V_i^1 \) \( (i = 1, \ldots, m) \), \( Q^1 \) and \( V^1 \) are as defined in Definition A. Assume w.l.o.g. that

\[
F(Q_i^1) = u_i, \quad L(Q_i^1) = v_i \quad (i = 1, \ldots, m),
\]

\[
Q^1 = u_1Q_1^1v_1u_2Q_2^1v_2u_3\ldots v_{m-1}u_mQ_m^1v_m,
\]

where \( u_1u_2, v_2u_3, \ldots, v_{m-1}u_m \) are extra edges in \( G \). Assume the converse, that is, \( A^1 \not\subseteq V^1 \). Let \( P = y_1y_2\ldots y_p \) be a longest path in \( (A^1 - V^1) \). Set \( Z_1 = N(y_1) \cap V^1 \) and \( Z_2 = N(y_p) \cap V^1 \). Clearly \( p + |V^1| \leq |A^1| + |S| \leq 3\delta - 4 \) and hence

\[
|V^1| \leq 3\delta - p - 4. \tag{3}
\]
Case 1. Every path between $V(P)$ and $S - V^1$, intersects $V^1$.

Case 1.1. $p = 1$.

In this case, $N(y_1) \subseteq V^1$. Set $M = \{u_1, \ldots, u_m\} \cup \{v_1, \ldots, v_m\}$ and $M^* = M \cap N(y_1)$. Since $Q_1^1, \ldots, Q_m^1$ is extreme, $|M^*| \leq 2$. Moreover, $|M^*| = 2$ if and only if $M^* = \{u_i, v_i\}$ for some $i \in \{1, \ldots, m\}$. If $|M^*| \leq 1$, then by standard arguments either $N(y_1)^+ \cup \{y_1\}$ or $N(y_1)^- \cup \{y_1\}$ is an independent set of order at least $\delta + 1$, contradicting $\delta \geq \alpha$. So, $|M^*| = 2$. Assume w.l.o.g. that $M^* = \{u_1, v_1\}$, that is, $y_1$ is adjacent to both $u_1$ and $v_1$. Put $B = N(y_1) - v_1$. Since $y_1v_1 \notin E(G)$ ($i = 2, \ldots, m$), we have $|B^+| \geq \delta - 1$. If $B^+ \cap S = \emptyset$, then among standard arguments we can show that $B \cup \{y_1, w\}$ for each $w \in A^1$ is an independent set of order at least $\delta + 1$, contrary to $\delta \geq \alpha$. Hence, we can choose any $z \in B^+ \cap S$. If $z \in V_1^1$, then the collection of paths obtained from $Q_1^1, \ldots, Q_m^1$ by deleting $Q_1^1$ and adding $u_1 Q_1^1$ $z$ $y_1v_1 Q_1^1 z$, contradicts the definition of $Q_1^1, \ldots, Q_m^1$. Therefore, $z \notin V_1^1$. Assume w.l.o.g. that $z \in V_2^1$. If $z \neq v_2$, then we get a new collection of paths obtained from $Q_1^1, \ldots, Q_m^1$ by deleting $Q_1^1$ and $Q_2^1$, and adding $u_1 Q_1^1 v_1 y_1 z Q_2^1 v_2$ and $z Q_2^1 v_2$, contrary to $Q_1^1, \ldots, Q_m^1$. So, let $z = v_2$ implying that $v_2 \in N(y_1)$. Taking the reverse direction on $Q_2^1$, we can state in addition that $u_2^* \in N(y_1)$. By standard arguments, $B^+ \cup \{y_1\}$ is an independent set of vertices of order at least $\delta$. Now we claim that $u_2$ has no neighbors in $B^+ \cup \{y_1\}$. Assume, to the contrary, that is, $u_2w \in E(G)$ for some $w \in B^+ \cup \{y_1\}$. If $w = y_1$, then deleting $Q_1^1$ and $Q_2^1$ from $Q_1^1, \ldots, Q_m^1$ and adding $u_1 Q_1^1 v_1 y_1 u_2 Q_2^1 v_2$ we obtain a new collection of paths, contrary to $Q_1^1, \ldots, Q_m^1$. Next, if $w \in V_1^1$, then deleting $Q_1^1$ and $Q_2^1$ and adding $u_1 Q_1^1 w^- y_1 u_2 Q_2^1 v_2$ and $u_2 w Q_1^1 v_1$ we obtain a new collection, contrary to $Q_1^1, \ldots, Q_m^1$. Further, if $w \in V_2^1$, then deleting $Q_1^1$ and $Q_2^1$ and adding $u_1 Q_1^1 v_1 y_1 w^- Q_2^1 v_2 w Q_2^1 v_2$ we obtain a collection, contrary to $Q_1^1, \ldots, Q_m^1$. Finally, if $w \in V_1^1$ for some $i \geq 3$, say $i = 3$, then deleting $Q_3^1$ and $Q_3^1$ and adding $u_2w Q_1^1 v_3$ and $u_3 Q_3^1 w^- y_1 u_2 Q_2^1 v_2$, we obtain a collection, contrary to $Q_1^1, \ldots, Q_m^1$. So, $B^+ \cup \{y_1, u_2\}$ is an independent set of order at least $\delta + 1$, contrary to $\delta \geq \alpha$.

Case 1.2. $p \geq 2$.

If $p = 2$, then $|Z_1| \geq \delta - 1$, $|Z_2| \geq \delta - 1$ and $(Z_1, Z_2)$ is a nontrivial $(Q_1^1, 3)$-scheme. By Lemma 3, $|V^1| \geq \min\{4\delta - 6, 3\delta - 5\} = 3\delta - 5$, contradicting (3). So, we can assume that $p \geq 3$. Let $w_1, w_2, \ldots, w_s$ be the elements of $(N(y_p) \cap V(P))^+$. In a consecutive order, where $w_s = y_p$. Put $P_0 = w_1 P_1 w_2$, and $p_0 = |P_0|$. For each $w_i \in V(P)$ ($i \in \{1, \ldots, s\}$) there is a path $y_i P_0 w_1 P_1 w_2 \ldots P_s w_i$ in $(\langle V(P) \rangle$) of length $p$ connecting $y_1$ and $w_i$. Hence, we can assume w.l.o.g. that $P$ is chosen such that for each $i \in \{1, \ldots, s\}$,

$$|Z_1| \geq |N(w_i) \cap V^1|, \quad N(w_i) \cap V(P) \subseteq V(P_0).$$

In particular, $|Z_1| \geq |Z_2|$. Clearly $p_0 \geq 2$. If $p_0 = 2$, then $|Z_1| \geq |Z_2| \geq \delta - 1,$ and we can argue as in case $p = 2$. Let $p_0 \geq 3$. Since $G$ is 3-connected, there
are vertex disjoint paths $R_1, R_2, R_3$ connecting $P_0$ and $V^\uparrow$. Let $F(R_i) \in V(P_0)$ and $L(R_i) \in V^\uparrow$ ($i = 1, 2, 3$).

**Case 1.2.1.** $|Z_1| \leq 3$.

By (4), $|N(w_i) \cap V^\uparrow| \leq |Z_1| \leq 3$ and therefore, $|N(w_i) \cap V(P_0)| \geq \delta - 3$ ($i = 1, \ldots, s$). In particular, for $i = s$, we have $s = |N(y_p) \cap V(P_0)| \geq \delta - 3$, implying that $p \geq \delta - 2$. By (3), $|V^\uparrow| \leq 3\delta - p - 4 = 2\delta - 2$. Furthermore, by Lemma C, in $(V(P_0))$ any two vertices are joined by a path of length at least $\delta - 3$. Due to $R_1, R_2, R_3$, we have $|V^\uparrow| \geq 2\delta - 1$, a contradiction.

**Case 1.2.2.** $|Z_1| \geq 4$.

Choose $w \in \{w_1, \ldots, w_s\}$ as to maximize $|N(w_i) \cap V^\uparrow|$, $i = 1, \ldots, s$. Set $Z_3 = N(w) \cap V^\uparrow$. By (4), $|Z_1| \geq |Z_3| \geq |Z_2|$. If $|Z_3| \leq 3$, then we can argue as in Case 1.2.1. So, we can assume that $|Z_3| \geq 4$. Clearly $|N(w_i) \cap V(P_0)| \geq \delta - |Z_3|$ for each $i \in \{1, \ldots, s\}$. In particular, $s = |N(w_i) \cap V(P_0)| \geq \delta - |Z_3|$ implying that $p \geq \delta - |Z_3| + 1$. By Lemma C, in $(V(P_0))$ any two vertices are joined by a path of length at least $\delta - |Z_3|$.

**Case 1.2.1.** $\delta - |Z_3| \geq 1$.

Assume w.l.o.g. that $w \notin V(R_1 \cup R_2)$. If $R_1 \cup R_2$ does not intersect $y_1Pw_1^-$, then $(Z_1, \{L(R_1)\}, Z_3)$ is a nontrivial $(Q^\uparrow, \delta - |Z_3| + 2)$-scheme. Otherwise, let $t$ be the smallest integer such that $y_t \in V(R_1 \cup R_2)$. Assume w.l.o.g. that $y_t \in V(R_2)$. Then due to $y_tP \cap R_2F(R_2)$, we again can state that $(Z_1, \{L(R_1)\}, Z_3)$ is a nontrivial $(Q^\uparrow, \delta - |Z_3| + 2)$-scheme. By Lemma 3,

$$|V^\uparrow| \geq 2\delta + |Z_3| - 4 + \min\{|Z_3| - 1, (\delta - |Z_3| - 1)(|Z_3| - 3)\} \geq 2\delta + |Z_3| - 4.$$ 

On the other hand, using (3) and the fact that $p \geq \delta - |Z_3| + 1$, we get $|V^\uparrow| \geq 2\delta + |Z_3| - 5$, a contradiction.

**Case 1.2.2.** $\delta - |Z_3| \leq 0$.

In this case, $|Z_1| \geq |Z_3| \geq \delta$ and $(Z_1, Z_3)$ is a nontrivial $(Q^\uparrow, p + 1)$-scheme. By Lemma 3, $|V^\uparrow| \geq 3\delta - p - 3$, contradicting (3).

**Case 2.** There is a path between $V(P)$ and $S - V^\uparrow$ avoiding $V^\uparrow$.

Since $Q^\uparrow_1, \ldots, Q^\uparrow_m$ is extreme, all the paths connecting $V(P)$ and $S - V^\uparrow$ and not intersecting $V^\uparrow$, end in a unique vertex $z \in S - V^\uparrow$.

**Case 2.1.** $p = 1$.

In this case, $y_1z \in E(G)$ and $N(y_1) - z \subseteq V^\uparrow$. Put $B = (N(y_1) - z)^+ \cup \{y_1\}$. By standard arguments, $B$ is an independent set of order at least $\delta$. Now we claim that $u_1$ has no neighbors in $B$. Assume, to the contrary, that is, $u_1w \in E(G)$ for some $w \in B$. First, if $w = y_1$, then deleting $Q^\uparrow_1$ from $Q^\uparrow_1, \ldots, Q^\uparrow_m$ and adding $zy_1u_1Q^\uparrow_1v_1$ we obtain a new collection of paths, contrary to $Q^\uparrow_1, \ldots, Q^\uparrow_m$. Next, if $w \in V^\uparrow_1$, then deleting $Q^\uparrow_1$ and adding $v_1Q^\uparrow_1wu_1Q^\uparrow_1w^\uparrow y_1z$
we obtain another collection of paths, contrary to $Q_1^1, \ldots, Q_m^1$. Finally, if $w \in V_i^1$ for some $i \geq 2$, say $i = 2$, then deleting $Q_1^1$ and $Q_2^1$ and adding $u_2 \overrightarrow{Q_2} w u_1 \overrightarrow{Q_1} v_1$ and $u_2 \overrightarrow{Q_2} y_1 z \overrightarrow{Q_1} w$ we again obtain a collection, contrary to $Q_1^1, \ldots, Q_m^1$. So, $B \cup \{u_1\}$ is an independent set of order at least $\delta + 1$, contrary to $\delta \geq \alpha$.

**Case 2.2.** $p \geq 2$.

Divide $Q^1$ into three consecutive segments $I_1 = \xi_1 Q^1 \xi_2$, $I_2 = \xi_2 Q^1 \xi_3$ and $I_3 = \xi_3 Q^1 \xi_4$ such that $I_2$ contains $Z_1 \cup Z_2$ and is as small as possible. Denote by $R_1$ ($R_2$, respectively) a longest path joining $\xi_2$ ($\xi_3$, respectively) to $z$ and passing through $A^1 \setminus V^1$. Since $Q_1^1, \ldots, Q_m^1$ is extreme, $|I_1| \geq |R_1|$ and $|I_2| \geq |R_2|$. 

Further, we can first estimate $|I_2|$ by Lemma 3 as in Case 1.2, and observing that $|Q^1| = |I_1| + |I_2| + |I_3| \geq |I_2| + |R_1| + |R_2|$, we can argue exactly as in Case 1.2. Lemma 5 can be applied by a similar way with respect to $\cup_{i=1}^3 Z_i$.

**Proof of Lemma 13.** Let $S, Q^1, V^1 (i = 1, \ldots, m)$, $Q^1$ and $V^1$ are as defined in Definition A. In addition, let $P = y_1 y_2 \ldots y_p, p_0, Z_1, Z_2, Z_3$ are as defined in Lemma 12.

**Case 1.** Every path between $V(P)$ and $S - V^1$, intersects $V^1$.

If $p = 0$, then $A^1 \subseteq V^1$ and we are done. If $p = 1$, then $\alpha \geq \delta + 1$ (see the proof of Lemma 12, Case 1.1) contradicting the hypothesis. Further, if $p = 2$, then $(Z_1, Z_2)$ is a nontrivial $(Q^1, 3)$-scheme with $|Z_1| \geq \delta - 1$, $|Z_2| \geq \delta - 1$ and, by Lemma 3, $|V^1| \geq 3\delta - 5$. Now let $p = 3$. If $y_1 y_3 \notin E$ then $|Z_1| \geq \delta - 1$, $|Z_2| \geq \delta - 1$ and as above, $|V^1| \geq 3\delta - 5$. Let $y_1 y_3 \in E$. This means that $|Z_1| \geq \delta - 2$, $|Z_2| \geq \delta - 2$ and $(Z_1, Z_2)$ is a nontrivial $(Q^1, 4)$-scheme. By Lemma 3, $|V^1| \geq 4\delta - 11$. For $\delta \geq 6$ the inequality $|V^1| \geq 3\delta - 5$ holds immediately. Let $4 \leq \delta \leq 5$. Since $\kappa \geq 4$, there are four paths connecting $P$ and $Q^1$ and three of them are pairwise disjoint. Then it is easy to see that $|V^1| \geq 10 \geq 3\delta - 5$. So, we can assume that $p \geq 4$. Suppose first that $|V(P_0)| \leq 3$ whence $|Z_1| \geq |Z_2| \geq \delta - 2$. Clearly $(Z_1, Z_2)$ is a nontrivial $(Q^1, 5)$-scheme and by Lemma 3, $|V^1| \geq \min\{4\delta - 8, 5\delta - 14\}$. If $\delta \geq 5$ then $|V^1| \geq 3\delta - 5$ holds immediately. Otherwise, using 4-connectedness of $G$, it is easy to see that $|V^1| \geq 7 \geq 3\delta - 5$. So, assume that $|V(P_0)| \geq 4$. Then $P$ and $Q^1$ are connected by at least four pairwise disjoint paths $R_1, R_2, R_3, R_4$. If $|Z_1| \leq 3$, then as in Lemma 12 (Case 1.2.1), in $(V(P_0))$ each two vertices are connected by a path of length at least $\delta - 3$ and due to $R_1, R_2, R_3, R_4$, $|V^1| \geq 3(\delta - 1) + 1 > 3\delta - 5$. Let $|Z_1| \geq 4$. By similar arguments, $|Z_3| \geq 4$. Clearly $|N(w_i) \cap V(P_0)| \geq \delta - |Z_3|$ $(i = 1, \ldots, s)$ and by Lemma C, in $(V(P_0))$ any two vertices are joined by a path of length at least $\delta - |Z_3|$. If $\delta - |Z_3| \geq 1$, then we can assume w.l.o.g. (see the proof of Lemma 12, Case 1.2.2.1) that $(\{L(R_1)\}, \{L(R_2)\}, Z_1, Z_3)$ is a nontrivial $(Q^1, \delta - |Z_3| + 2)$-scheme and by Lemma 7,

$$|V^1| \geq 3\delta - 5 + \min\{|Z_3| - 4, (\delta - |Z_3| - 1)(|Z_3| - 4)\} \geq 3\delta - 5.$$ 

Otherwise $|Z_1| \geq |Z_3| \geq \delta$ and $(Z_1, Z_3)$ is a nontrivial $(Q^1, p + 1)$-scheme.
By Lemma 3,

\[ |V^1| \geq 3\delta - 5 + \min\{\delta + p + 1, (\delta - 1)(p - 2) + 3\} \geq 3\delta - 5. \]

**Case 2.** There is a path between \( V(P) \) and \( S - V^1 \) avoiding \( V^1 \).

We can argue exactly as in proof of Lemma 12 (Case 2).

**Proof of Lemma 14.** Let \( S, Q_1^1, ..., Q_m^1 \) and \( V_1^1, ..., V_m^1, V^1 \) are as defined in Definition B. The existence of \( Q_1^1, ..., Q_m^1 \) follows from Lemma 10. Put \( |V^1 \cap S| = f \). Clearly \( f \geq 2m \). We can assume that \( \delta - \kappa \geq 2 \) since otherwise \( |A^1| \leq 3 \) (by the hypothesis) and it is not hard to see that \( (A^1 - V^1) \) is edgeless. Let \( P = w_1w_2...w_p \) be a longest path in \( (A^1 - V^1) \). By the hypothesis, \( p + |V^1| - f \leq |A^1| \leq 3\delta - 3\kappa \) implying that

\[ |V^1| \leq 3\delta - 3\kappa - p + f. \]  \hspace{1cm} (5)

Put

\[ Z_1 = N(y_1) \cap V^1, \quad Z_2 = N(y_p) \cap V^1, \]
\[ Z_{1,i} = Z_1 \cap V_i^1, \quad Z_{2,i} = Z_2 \cap V_i^1 \quad (i = 1, ..., m). \]

Clearly \( Z_1 = \cup_{i=1}^m Z_{1,i} \) and \( Z_2 = \cup_{i=1}^m Z_{2,i} \). If \( p \leq 1 \), then \( (A^1 - V^1) \) is edgeless and we are done. Let \( p \geq 2 \).

**Case 1.** \( p = 2 \).

In this case, \( |Z_i| \geq \delta - \kappa + f - 1 \) \((i = 1, 2)\). We claim that

\[ |V_i^1| \geq \frac{3}{2}(|Z_{1,i}| + |Z_{2,i}|) - 2 \quad (i = 1, ..., m). \]  \hspace{1cm} (6)

Indeed, if \((Z_{1,i}, Z_{2,i})\) is a nontrivial \((Q_i^1, 3)\)-scheme, then (6) holds by Lemma 3, immediately. Otherwise it can be checked easily. By summing,

\[ |V^1| = \sum_{i=1}^m |V_i^1| \geq \frac{3}{2}(|Z_1| + |Z_2|) - 2m \geq 3\delta - 3\kappa - p + f + 1, \]
contradicting (5).

**Case 2.** \( p \geq 3 \).

Let \( w_1, w_2, ..., w_s \) be the elements of \((N(y_p) \cap V(P))^+\) occurring on \( \overrightarrow{P} \) in a consecutive order. Put \( P_0 = w_1 \overrightarrow{P} w_s \) and \( p_0 = |V(P_0)| \). As in proof of Lemma 12 (see (4)), we can assume w.l.o.g. that for each \( i \in \{1, ..., s\} \),

\[ |Z_i| \geq |N(w_i) \cap V^1|, \quad N(w_i) \cap V(P) \subseteq V(P_0). \]  \hspace{1cm} (7)

Choose \( w \in \{w_1, ..., w_s\} \) as to maximize \( |N(w_i) \cap V^1|, \ i = 1, ..., s \). Set

\[ Z_3 = N(w) \cap V^1, \quad Z_{3,i} = Z_3 \cap V_i^1 \quad (i = 1, ..., m). \]
Clearly \(|Z_1| \geq |Z_2| \geq \delta - \kappa + f - p_0 + 1\) and \(|Z_3| \geq |Z_2| \geq \delta - \kappa + f - p_0 + 1\).

We claim that

\[ |V_i^1| \geq 2(|Z_{i,1}| + |Z_{2,i}|) - 3 \quad (i = 1, \ldots, m). \tag{8} \]

Indeed, if \((Z_{1,i}, Z_{2,i})\) is a nontrivial \((Q_1^1, p + 1)\)-scheme, then (8) holds by Lemma 3 and the fact that \(p \geq 3\). Otherwise, it can be checked easily. Analogously,

\[ |V_i^1| \geq 2(|Z_{i,1}| + |Z_{3,i}|) - 3 \quad (i = 1, \ldots, m). \tag{9} \]

**Case 2.1.** \(p_0 \leq m + 1\).

Using (8) and summing, we get

\[ |V^1| = \sum_{i=1}^m |V_i^1| \geq 2(|Z_1| + |Z_2|) - 3m \geq 4(\delta - \kappa + f + p_0 + 1) - 3m \]

\[ = (3 \delta - 3 \kappa - p + f + 1) + (\delta - \kappa) + (p - p_0) + 3(m - p_0 + 1) + 3(f - 2m). \]

Recalling that \(\delta - \kappa \geq 2\), \(p \geq p_0\), \(m - p_0 + 1 \geq 0\) and \(f \geq 2m\), we get

\[ |V^1| \geq 3 \delta - 3 \kappa - p + f + 1, \]

contradicting (5).

**Case 2.2.** \(p_0 \geq m + 2\).

Assume first that \(\delta - \kappa + f - |Z_3| \leq 1\). Then \(|Z_1| \geq |Z_3| \geq \delta - \kappa + f - 1\).

Using (8) and summing, we get

\[ |V^1| = \sum_{i=1}^m |V_i^1| \geq 4(\delta - \kappa + f - 1) - 3m \]

\[ \geq (3 \delta - 3 \kappa - p + f + 1) - (\delta - \kappa - 2) + 3(f - m - 1) \geq 3 \delta - 3 \kappa - p + f + 1, \]

contradicting (5). Now assume that \(\delta - \kappa + f - |Z_3| \geq 2\). Clearly \(|N(w_i) \cap V(P_0)| \geq \delta - \kappa + f - |Z_3| (i = 1, \ldots, s)\). In particular, for \(i = s\), we have \(s \geq \delta - \kappa + f - |Z_3|\). By Lemma C, in \(\langle V(P_0) \rangle\) any two vertices are joined by a path of length at least \(\delta - \kappa + f - |Z_3|\). Observing that \(p \geq s + 1 \geq \delta - \kappa + f - |Z_3| + 1\) and combining it with (5), we get

\[ |V^1| \leq 2 \delta - 2 \kappa + |Z_3| - 1. \tag{10} \]

**Case 2.2.1.** \(p_0 \leq f\).

Since \(\kappa \geq f \geq p_0\), there are vertex disjoint paths \(R_1, \ldots, R_{p_0}\) connecting \(V(P_0)\) and \(V^1\). Let \(F(R_i) \in V(P_0)\) and \(L(R_i) \in V^1 (i = 1, \ldots, p_0)\). Since \(p_0 \geq m + 2\), we can assume w.l.o.g. that either \(L(R_i) \in V^1_1 (i = 1, 2)\) and \(L(R_i) \in V^1_2 (i = 3, 4)\) or \(L(R_i) \in V^1_1 (i = 1, 2, 3)\).

**Case 2.2.1.1.** \(L(R_i) \in V^1_1 (i = 1, 2)\) and \(L(R_i) \in V^1_2 (i = 3, 4)\).

Assume w.l.o.g. that \(R_1 \cup R_3\) does not intersect \(y_1 \overline{F} w_1^-\) (see the proof of Lemma 12, Case 1.2.2.1.1). Because of the symmetry, we can distinguish the
following six subcases.

**Case 2.2.1.1.** $|Z_{11}| \leq 2, |Z_{12}| \leq 2, |Z_{31}| \leq 2, |Z_{32}| \leq 2$.

Due to $R_1, R_2$ and $R_3, R_4$ we have $|V_i^1| \geq \delta - \kappa + f - |Z_3| + 3 \ (i = 1, 2)$. Using (9) for each $i \in \{3, ..., m\}$ and summing, we get

$$|V^1| = |V_1^1| + |V_2^1| + \sum_{i=3}^{m} |V_i^1| \geq 2(\delta - \kappa + f - |Z_3| + 3) + 2(|Z_1| - |Z_{11}| - |Z_{12}| + |Z_3| - |Z_{31}| - |Z_{32}|) - 3(m - 2) = (2\delta - 2\kappa + |Z_1|) + (|Z_1| - |Z_{11}| - |Z_{12}|) + (2f - 3m) + 12 - (|Z_{11}| + |Z_{12}| + 2|Z_{31}| + 2|Z_{32}|) > 2\delta - 2\kappa + |Z_3|,$$

contradicting (10).

**Case 2.2.1.1.2.** $|Z_{11}| \geq 3, |Z_{12}| \leq 2, |Z_{31}| \leq 2, |Z_{32}| \leq 2$.

Clearly either $\{L(R_1), Z_{11}\}$ or $\{L(R_2), Z_{11}\}$ is a nontrivial $(Q_1, \delta - \kappa + f - |Z_3| + 2)$-scheme. By Lemma 4, $|V_1^1| \geq (\delta - \kappa + f - |Z_3| + 2) + 2|Z_{11}| - 3$. Due to $R_1$ and $R_2$ we have $|V_2^1| \geq \delta - \kappa + f - |Z_3| + 3$. Using also (9) for each $i \in \{3, ..., m\}$ and summing, we get

$$|V^1| \geq 2(\delta - \kappa + f - |Z_3| + 2) + 2|Z_{11}| - 2 + \sum_{i=3}^{m} (2(|Z_{1i}| + |Z_{3i}|) - 3) = (2\delta - 2\kappa + |Z_1|) + (|Z_1| - |Z_{11}| - |Z_{12}|) + 2f - 3m + (|Z_{11}| + 8) - (|Z_{12}| + 2|Z_{31}| + 2|Z_{32}|) > 2\delta - 2\kappa + |Z_3|,$$

contradicting (10).

**Case 2.2.1.1.3.** $|Z_{11}| \geq 3, |Z_{12}| \geq 3, |Z_{31}| \leq 2, |Z_{32}| \leq 2$.

Clearly either $\{L(R_1), Z_{11}\}$ or $\{L(R_2), Z_{11}\}$ is a nontrivial $(Q_1, \delta - \kappa + f - |Z_3| + 2)$-scheme. By the same reason, either $\{L(R_3), Z_{12}\}$ or $\{L(R_4), Z_{12}\}$ is a nontrivial $(Q_2, \delta - \kappa + f - |Z_3| + 2)$-scheme too. By Lemma 4,

$$|V_i^1| \geq (\delta - \kappa + f - |Z_3| + 2) + 2|Z_{1i}| - 3 \quad (i = 1, 2).$$

Using (9) for each $i \in \{3, ..., m\}$ and summing,

$$|V^1| \geq |V_1^1| + |V_2^1| + \sum_{i=3}^{m} |V_i^1| \geq 2(\delta - \kappa + f - |Z_3| + 2) + 2(|Z_{11}| + |Z_{12}|) - 6 + 2(|Z_1| - |Z_{11}| - |Z_{12}| + |Z_3| - |Z_{31}| - |Z_{32}|) - 3(m - 2) = (2\delta - 2\kappa + |Z_1|) + 2f - 3m + (|Z_1| + 4) - 2(|Z_{31}| + |Z_{32}|) \geq 2\delta - 2\kappa + |Z_3|,$$
contradicting (10).

**Case 2.2.1.4.**  \(|Z_{11}| \geq 3, |Z_{12}| \leq 2, |Z_{31}| \geq 3, |Z_{32}| \leq 2\).

Since \((Z_{11}, Z_{31})\) is a nontrivial \((Q_1^*, \delta - \kappa + f - |Z_3| + 2)\)-scheme, we can apply Lemma 3,

\[
|V_1^i| \geq 2(|Z_{11}| + |Z_{31}|) + (\delta - \kappa + f - |Z_3|) - 5 + \\
\min\{4, \frac{1}{2}(\delta - \kappa + f - |Z_3| - 2)(|Z_{11}| + |Z_{31}| - 6)\}
\]

\[
\geq (\delta - \kappa + f - |Z_3|) + 2(|Z_{11}| + |Z_{31}|) - 5.
\]

Due to \(R_3\) and \(R_4\), we have \(|V_2^i| \geq \delta - \kappa + f - |Z_3| + 3\). Using (9) for each \(i \in \{3, ..., m\}\) and summing, we get

\[
|V^i| \geq 2(\delta - \kappa + f - |Z_3|) + 2(|Z_{11}| + |Z_{31}|) - 2 \\
+ 2(|Z_1| - |Z_{11}| - |Z_{12}| + |Z_3| - |Z_{31}| - |Z_{32}|) - 3(m - 2)
\]

\[
= (2\delta - 2\kappa + |Z_1|) + 2f - 3m + (|Z_1| + 4) - 2(|Z_{12}| + |Z_{32}|) \geq 2\delta - 2\kappa + |Z_3|,
\]

contradicting (10).

**Case 2.2.1.5.**  \(|Z_{11}| \geq 3, |Z_{12}| \geq 3, |Z_{31}| \geq 3, |Z_{32}| \leq 2\).

As in Case 2.2.1.4, \(|V_1^i| \geq (\delta - \kappa + f - |Z_3|) + 2(|Z_{11}| + |Z_{31}|) - 5\). By Lemma 4, \(|V_2^i| \geq (\delta - \kappa + f - |Z_3| + 2) + 2|Z_{12}| - 3\). Using (9) for each \(i \in \{3, ..., m\}\) and summing, we get

\[
|V^i| \geq 2(\delta - \kappa + f - |Z_3|) + 2(|Z_{11}| + |Z_{31}|) + 2|Z_{12}| - 6 \\
+ 2(|Z_1| - |Z_{11}| - |Z_{12}| + |Z_3| - |Z_{31}| - |Z_{32}|) - 3(m - 2)
\]

\[
\geq (2\delta - 2\kappa + |Z_1|) + 2f - 3m + |Z_1| - |Z_{32}| > 2\delta - 2\kappa + |Z_3|,
\]

contradicting (10).

**Case 2.2.1.6.**  \(|Z_{11}| \geq 3, |Z_{12}| \geq 3, |Z_{31}| \geq 3, |Z_{32}| \geq 3\).

As in Case 2.2.1.4,

\[
|V_1^i| + |V_2^i| \geq 2(\delta - \kappa + f - |Z_3|) + 2(|Z_{11}| + |Z_{31}|) + 2(|Z_{12}| + |Z_{32}|) - 10.
\]

Using (9) for each \(i \in \{3, ..., m\}\) and summing, we get

\[
|V^i| \geq 2(\delta - \kappa + f - |Z_3|) + 2(|Z_{11}| + |Z_{31}| + |Z_{12}| + |Z_{32}|) - 10
\]
+2(|Z_1| - |Z_{11}| - |Z_{12}| + |Z_3| - |Z_{31}| - |Z_{32}|) - 3(m - 2)

= (2\delta - 2\kappa + |Z_{11}|) + 2f - 3m + |Z_1| - 4 \geq 2\delta - 2\kappa + |Z_3|,

ccontrary to (10).

**Case 2.2.1.2.** $L(R_i) \in V_i^1$ ($i = 1, 2, 3$).

Assume w.l.o.g. that $R_1 \cup R_2$ does not intersect $y_1 \overrightarrow{P} w_1$ (see the proof of Lemma 12, Case 1.2.2.1).

**Case 2.2.1.2.1.** $|Z_{11}| \leq 3$, $|Z_{31}| \leq 3$.

Due to $R_1, R_2, R_3$ we have $|V_1^1| \geq 2(\delta - \kappa + f - |Z_3| + 2) + 1$. Using (9) for each $i \in \{2, \ldots, m\}$ and summing, we get

$$|V_1^1| \geq |V_1^1| + \sum_{i=2}^m |V_i^1|$$

$$
\geq 2(\delta - \kappa + f - |Z_3| + 2) + 1 + 2(|Z_1| - |Z_{11}| + |Z_3| - |Z_{31}|) - 3(m - 1)
$$

$$
= (2\delta - 2\kappa + |Z_{11}|) + (|Z_1| - |Z_{11}|) + (2f - 3m)
$$

$$
+ 8 - (|Z_{11}| + 2|Z_{31}|) \geq 2\delta - 2\kappa + |Z_3|,
$$

which contradicts (10).

**Case 2.2.1.2.2.** $|Z_{11}| \leq 3$, $|Z_{31}| \geq 4$.

Since $|Z_{31}| \geq 4$, we can suppose w.l.o.g. that $F(R_3) = w$. Then clearly $\{(L(R_1)), \{L(R_2)\}, Z_{31}\}$ is a nontrivial $(Q_{11}, \delta - \kappa + f - |Z_3| + 2)$-scheme and by Lemma 6, $|V_1^1| \geq 2(\delta - \kappa + f - |Z_3| + 2) + 2|Z_{31}| - 5$. Using (9) for each $i \in \{2, \ldots, m\}$ and summing, we get

$$|V_1^1| \geq 2(\delta - \kappa + f - |Z_3| + 2) + 2|Z_{31}| - 5 + 2(|Z_1| - |Z_{11}| + |Z_3| - |Z_{31}|) - 3(m - 1)
$$

$$
= (2\delta - 2\kappa + |Z_{11}|) + (|Z_1| - |Z_{11}|) + 2f - 3m + 2 - |Z_{11}| \geq 2\delta - 2\kappa + |Z_3|,
$$

contradicting (10).

**Case 2.2.1.2.3.** $|Z_{11}| \geq 4$, $|Z_{31}| \geq 4$.

For some $R_1, R_2, R_3$, say $R_1$, we have $F(R_1) \notin \{y_1, w\}$. Then clearly $\{(L(R_1)), Z_{11}, Z_{31}\}$ is a nontrivial $(L_{11}, \delta - \kappa + f - |Z_3| + 2)$-scheme. By Lemma 5,

$$|V_1^1| \geq \min\{2(\delta - \kappa + f - |Z_3| + 2) + 2(|Z_{11}| + |Z_{31}| + 1) - 11,
$$

$$
(\delta - \kappa + f - |Z_3| + 2)(|Z_{11}| + |Z_{31}| - 2)/2 + 1\}$$

19
\[ = 2(\delta - \kappa + f - |Z_3|) + 2(|Z_{11}| + |Z_{31}|) - 7 \]
\[ + \min\{2, (\delta - \kappa + f - |Z_3| - 2)(|Z_{11}| + |Z_{31}| - 6)/2\} \]
\[ \geq 2(\delta - \kappa + f - |Z_3|) + 2(|Z_{11}| + |Z_{31}|) - 7. \]

Further, applying (9) for each \( i \in \{2, \ldots, m\} \) and summing, we get
\[ |V^1| \geq 2(\delta - \kappa + f - |Z_3|) + 2(|Z_{11}| + |Z_{31}|) - 7 \]
\[ + 2(|Z_1| - |Z_{11}| + |Z_3| - |Z_{31}|) - 3(m-1) \]
\[ = (2\delta - 2\kappa + |Z_1|) + |Z_1| - 4 + 2f - 3m \geq 2\delta - 2\kappa + |Z_3|, \]
contradicting (10).

**Case 2.2.2.** \( p_0 \geq f + 1 \).

Let \( S = \{v_1, \ldots, v_\kappa\} \) and \( V^1 \cap S = \{v_1, \ldots, v_\kappa\} \). Consider a new graph \( G' = G - \{v_{f+1}, v_{f+2}, \ldots, v_\kappa\} \). Add new vertices \( a_1, a_2 \) in \( G' \) and join \( a_1 \) to all vertices of \( V^1 \), and join \( a_2 \) to all vertices of \( V(P_0) \). Set \( G'' = \langle V(G') \cup \{a_1, a_2\} \rangle \).

Clearly \( G'' \) is \( f \)-connected. Let \( V_0 \) be a minimum cut-set in \( G'' \) that separates \( a_1 \) and \( a_2 \). Since \( a_1 \) and \( a_2 \) are connected in \( G'' - \{v_2, \ldots, v_f\} \), we have \( V_0 \neq \{v_1, \ldots, v_f\} \). Observing also that \( A^1 \) is an endfragment for \( G' \), we can suppose that \( |V_0| \geq f + 1 \) and therefore there exist \( f + 1 \) internally disjoint paths in \( G'' \) joining \( a_1 \) and \( a_2 \). This means that in \( G' \) (as well as in \( G \)) there exist vertex disjoint paths \( R_1, R_2, \ldots, R_{f+1} \) connecting \( V^1 \) and \( V(P_0) \). Then using the fact that \( f + 1 \geq m + 2 \), we can argue exactly as in Case 2.2.1. \( \Delta \)

**Proof of Lemma 15.** Let \( Q^1_1, \ldots, Q^1_m \) and \( V^1_1, \ldots, V^1_m \). \( V^1 \) be as defined in Definition B. The existence of \( Q^1_1, \ldots, Q^1_m \) follows from Lemma 10. Further, let \( P = y_1 \ldots y_p, Z_1, Z_2 \) and \( Z_{1,i}, Z_{2,i} \ (i = 1, \ldots, m) \) be as defined in Lemma 14.

**Case 1.** \( f \geq 3 \).

Suppose first that \( \delta - \kappa \leq 1 \). Combining it with \( \delta > 3\kappa/2 - 1 \), we get \( \kappa = 3, \delta = 4, f = 3 \) and \( m = 1 \). Then it is easy to show that \( |V^1| \geq 5 = 3\delta - 3\kappa + f - 1 \) and we are done. Now let \( \delta - \kappa \geq 2 \). If \( p \leq 1 \), then clearly \( < A^1 - V^1 \) is edgeless and we are done. Further, if \( p = 2 \), then \( |V^1| \geq 3\delta - 3\kappa + f - 1 \) (see the proof of Lemma 14, Case 1). Let \( p \geq 3 \).

**Case 1.1.** \( p = 3 \).

In this case, \( P = y_1 y_2 y_3 \). If \( y_1 y_3 \notin E(G) \), then we can argue as in case \( p = 2 \). Let \( y_1 y_3 \in E(G) \) implying that \( |Z_i| \geq \delta - \kappa + f - 2 \ (i = 1, 2) \). Applying (8) (see the proof of Lemma 14) and summing, we get
\[ |V^1| \geq \sum_{i=1}^m |V^1_i| \geq 2(|Z_1| + |Z_2|) - 3m \geq 4(\delta - \kappa + f - 2) - 3m \]
= (3\delta - 3\kappa + f - 1) + (3f - 3m - 5) + \delta - \kappa - 2.

If \( f = 3 \), then \( m = 1 \) and \( 3f - 3m - 5 \geq 1 \). If \( f \geq 4 \), then \( 3f - 3m - 5 \geq f + m - 5 \geq 0 \). In both cases the desired result follows immediately.

**Case 1.2.** \( p \geq 4 \).

Let \( P_0, p_0, w, Z_3 \) and \( Z_{3,i} \) \((i = 1, \ldots, m)\) are as defined in Lemma 14 (Case 2). Clearly \(|Z_1| \geq |Z_2| \geq \delta - \kappa + f - p_0 + 1 \) and \(|Z_1| \geq |Z_3| \geq \delta - \kappa + f - p_0 + 1 \). By the definition of \( Q^1_1, \ldots, Q^1_m \), we have \(|V^1_i| \geq 3 \) \((i = 1, \ldots, m)\). We claim that

\[
|V^1_i| \geq 2(|Z_{1,i}| + |Z_{2,i}|) - 2 \quad (i = 1, \ldots, m). \tag{11}
\]

Indeed, if \((Z_{1,i}, Z_{3,i})\) is not a nontrivial \((Q^1_i, p + 1)\)-scheme, then (11) can be checked easily. Otherwise, by Lemma 3,

\[
|V^1_i| \geq \min\{2(|Z_{1,i}| + |Z_{2,i}|) + p - 4, \frac{1}{2}(p + 1)(|Z_{1,i}| + |Z_{2,i}| - 2) + 1\}
= 2(|Z_{1,i}| + |Z_{2,i}|) - 2 + \min\{p - 2, \frac{1}{2}(p - 3)(|Z_{1,i}| + |Z_{2,i}| - 4) + p - 4\}.
\]

If \(|Z_{1,i}| + |Z_{2,i}| \geq 4\), then (11) holds immediately. If \(|Z_{1,i}| + |Z_{2,i}| = 3\), then (11) follows from \(|V^1_i| \geq 2(|Z_{1,i}| + |Z_{2,i}|) - 5/2\). Finally, if \(|Z_{1,i}| + |Z_{2,i}| \leq 2\), then (11) follows from \(|V^1_i| \geq 3\). By a similar way,

\[
|V^1_i| \geq 2(|Z_{1,i}| + |Z_{3,i}|) - 2 \quad (i = 1, \ldots, m). \tag{12}
\]

If \( p_0 \leq m + 1 \), then using (11) and summing, we get

\[
|V^1| = \sum_{i=1}^{m} |V^1_i| \geq 2(|Z_1| + |Z_2|) - 2m \geq 4(\delta - \kappa + f - p_0 + 1) - 2m
= (3\delta - 3\kappa + f - 1) + \delta - \kappa + 3f - 4p_0 - 2m + 5
\geq (3\delta - 3\kappa + f - 1) + 4(m - p_0 + 1) + (\delta - \kappa + 1) > 3\delta - 3\kappa + f - 1.
\]

Now let \( p_0 \geq m + 2 \). If \( \delta - \kappa + f - |Z_3| \leq 1 \), then \(|Z_1| \geq |Z_3| \geq \delta - \kappa + f - 1\) and by (11),

\[
|V^1| = \sum_{i=1}^{m} |V^1_i| \geq 2(|Z_1| + |Z_2|) - 2m \geq 4(\delta - \kappa + f - 1) - 2m
= (3\delta - 3\kappa + f - 1) + (\delta - \kappa - 2) + 3f - 2m - 2 > 3\delta - 3\kappa + f - 1.
\]

Let \( \delta - \kappa + f - |Z_3| \geq 2 \). By the choice of \( w \),

\[
|N(w_i) \cap V(P_0)| \geq \delta - \kappa + f - |Z_3| \quad (i = 1, \ldots, s).
\]
In particular, for \( i = s \), we have \( s \geq \delta - \kappa + f - |Z_3| \). By Lemma C, in \( \langle V(P_0) \rangle \) any two vertices are joined by a path of length at least \( \delta - \kappa + f - |Z_3| \).

**Case 1.2.1.** \( p_0 \leq f \).

Let \( G' = G - (S - V^1) \). Since \( G' \) is \( p_0 \)-connected, there exist vertex disjoint paths \( R_1, R_2, \ldots, R_{p_0} \) connecting \( V(P_0) \) and \( V^1 \). Let \( F(R_i) \in V(P_0) \) and \( L(R_i) \in V^1 \) (\( i = 1, \ldots, p_0 \)). Since \( p_0 \geq m + 2 \), we can assume w.l.o.g. that either \( L(R_i) \in V^1_i \) (\( i = 1, 2 \)) and \( L(R_i) \in V^2_i \) (\( i = 3, 4 \)) or \( L(R_i) \in V^1_i \) (\( i = 1, 2, 3 \)).

**Case 1.2.1.1.** \( L(R_i) \in V^1_i \) (\( i = 1, 2 \)) and \( L(R_i) \in V^2_i \) (\( i = 3, 4 \)).

Assume w.l.o.g. that \( R_1 \cup R_2 \) does not intersect \( w_1 P_0 w_s \) (see the proof of Lemma 12, Case 1.2.2.1).

**Case 1.2.1.1.1.** \(|Z_{11}| \leq 2, |Z_{12}| \leq 2, |Z_{31}| \leq 2, |Z_{32}| \leq 2 \).

Due to \( R_1, R_2 \) and \( R_3, R_4 \), we have \(|V^1_i| \geq \delta - \kappa + f - |Z_3| + 3 \) (\( i = 1, 2 \)). Using (12) for each \( i \in \{3, \ldots, m\} \) and summing, we get

\[
|V^1| = |V^1_1| + |V^2_1| + \sum_{i=3}^{m} |V^1_i|
\geq 2(\delta - \kappa + f - |Z_3| + 3) + 2(|Z_1| - |Z_{11}| - |Z_{12}| + |Z_3| - |Z_{31}| - |Z_{32}|) - 2(m - 2)
\geq (3\delta - 3\kappa + f - 1) + (|Z_1| - |Z_{11}| - |Z_{12}|) + (|Z_3| - \delta + \kappa - f + p_0 - 1) + (|Z_1| - |Z_3|) + (2f - 2m - p_0) + 12 - |Z_{11}| - |Z_{12}| - 2|Z_{31}| - 2|Z_{32}|.
\]

Since \( |Z_3| \geq \delta - \kappa + f - p_0 + 1 \) and \( 2f - 2m - p_0 \geq f - p_0 \geq 0 \), we have \(|V^1| \geq 3\delta - 3\kappa + f - 1 \).

**Case 1.2.1.1.2.** Either \(|Z_{11}| \geq 3 \) or \(|Z_{12}| \geq 3 \) or \(|Z_{31}| \geq 3 \) or \(|Z_{32}| \geq 3 \).

In this case we can argue exactly as in proof of lemma 14 (Cases 2.2.1.1.2-2.2.1.1.6).

**Case 1.2.1.2.** \( L(R_i) \in V^1_i \) (\( i = 1, 2, 3 \)).

Assume w.l.o.g. that \( R_1 \cup R_2 \) does not intersect \( y_1 P w_1 \) (see the proof of Lemma 12, Case 1.2.2.1).

**Case 1.2.1.2.1.** \(|Z_{11}| \leq 3, |Z_{31}| \leq 3 \).

Due to \( R_1, R_2, R_3 \), we have \(|V^1_i| \geq 2(\delta - \kappa + f - |Z_3| + 2) + 1 \). Using (12) for each \( i \in \{2, \ldots, m\} \) and summing, we get

\[
|V^1| = |V^1_1| + \sum_{i=2}^{m} |V^1_i|
\geq 2(\delta - \kappa + f - |Z_3| + 2) + 1 + 2(|Z_1| - |Z_{11}| + |Z_3| - |Z_{31}|) - 2(m - 1)
\]

22
\[= (3\delta - 3\kappa + f - 1) + (|Z_3| - \delta + \kappa - f + p_0 - 1)\]
\[+ (|Z_1| - |Z_3|) + (|Z_1| - |Z_{11}|) + 9 - |Z_{11}| - 2|Z_{31}| + 2f - 2m - p_0\]
\[\geq (3\delta - 3\kappa + f - 1) + 2f - 2m - p_0.\]

Since \(2f - 2m - p_0 \geq f - p_0 \geq 0\), we have \(|V^i| \geq 3\delta - 3\kappa + f - 1.\)

**Case 1.2.1.2.2.** Either \(|Z_{11}| \geq 4\) or \(|Z_{31}| \geq 4.\)

In this case we can argue exactly as in proof of Lemma 14 (Cases 2.2.1.2.2-2.2.1.2.3).

**Case 1.2.2.** \(p_0 \geq f + 1.\)

As in proof of Lemma 14 (Case 2.2.2), there are \(f + 1\) vertex disjoint paths \(R_1, R_2, ..., R_{f+1}\) connecting \(V^i\) and \(V(P_0)\). Let \(F(R_i) \in V(P_0)\) and \(L(R_i) \in V^i (i = 1, ..., f + 1).\) Since \(f \geq 3\) and \(f + 1 \geq 2m + 1\), we can assume w.l.o.g. that either \(L(R_i) \in V^i_1 (i = 1, 2, 3, 4)\) or \(L(R_i) \in V^i_1 (i = 1, 2, 3)\) and \(L(R_i) \in V^i_2 (i = 4, 5).\)

**Case 1.2.2.1.** \(L(R_i) \in V^i_1 (i = 1, 2, 3, 4).\)

Assume w.l.o.g. that \(R_1 \cup R_2\) dose not intersect \(y_1 \overline{P} w_1^{-}\) (see the proof of Lemma 12, Case 1.2.2.1).

**Case 1.2.2.1.1.** \(\delta - \kappa + f - |Z_3| \leq 1.\)

Clearly \(|Z_1| \geq |Z_3| \geq \delta - \kappa + f - 1.\) Using (12) and summing,
\[|V^i| = \sum_{i=1}^m |V^i_1| \geq 2(|Z_1| + |Z_3|) - 2m \geq 4(\delta - \kappa + f - 1) - 2m\]
\[= (3\delta - 3\kappa + f - 1) + \delta - \kappa + 3f - 2m - 3 \geq 3\delta - 3\kappa + f - 1.\]

**Case 1.2.2.1.2.** \(\delta - \kappa + f - |Z_3| \geq 2.\)

By the choice of \(w,\) \(|\{w_i \cap V(P_0)\}| \geq \delta - \kappa + f - |Z_3| (i = 1, ..., s).\) In particular, when \(i = s,\) we have \(s \geq \delta - \kappa + f - |Z_3|).\) By Lemma C, in \(\langle V(P_0)\rangle\) any two vertices are joined by a path of length at least \(\delta - \kappa + f - |Z_3|).\) If \(|Z_3| \leq 3,\) then due to \(R_1, R_2, R_3, R_4,\)
\[|V^i| \geq |V^i_1| \geq 3(\delta - \kappa + f - |Z_3| + 2) + 1\]
\[= (3\delta - 3\kappa + f - 1) + 2f + 8 - 3|Z_3|.\]

If \(|Z_3| \leq 3,\) then clearly we are done. Otherwise, we have \(|Z_1| \geq |Z_3| \geq 4.\)

**Case 1.2.2.1.2.1.** \(|Z_{11}| \leq 3, \ |Z_{31}| \leq 3.\)

As in previous case, \(|V^i_1| \geq 3(\delta - \kappa + f - |Z_3| + 2) + 1.\) Using (12) for each \(i \in \{2, ..., m\},\) we get
|V^1| ≥ 3(δ - κ + f - |Z_3| + 2) + 1 + 2(|Z_1| - |Z_{11}| + |Z_3| - |Z_{31}|) - 2(m - 1)
= (3δ - 3κ + f - 1) + 2|Z_1| - |Z_3| + 2f - 2m + 10 - 2|Z_{11}| - 2|Z_{31}| > 3δ - 3κ + f - 1.

**Case 1.2.2.1.2.2.** |Z_{11}| ≥ 4, |Z_{31}| ≤ 3.
We can assume w.l.o.g. that (Q_1, δ - κ + f - |Z_3| + 2)-scheme. By Lemma 8,

|V^1_i| ≥ 3(δ - κ + f - |Z_3| + 2) + 2|Z_{11}| - 7.

Using (11) for each i ∈ {2, ..., m} and summing, we get

|V^1| ≥ 3(δ - κ + f - |Z_3| + 2) + 2|Z_{11}| - 7 + 2(|Z_1| - |Z_{11}| + |Z_3| - |Z_{31}|) - 2(m - 1)
= (3δ - 3κ + f - 1) + (|Z_1| - |Z_3|) + |Z_1| + 2f - 3m + 2 - 2|Z_{31}| > 3δ - 3κ + f - 1.

**Case 1.2.2.2.** |Z_{11}| ≥ 4, |Z_{31}| ≥ 4.
We can assume w.l.o.g. that (Q_1, δ - κ + f - |Z_3| + 2)-scheme. By Lemma 7,

|V^1_i| ≥ 3(δ - κ + f - |Z_3|) + 2(|Z_{11}| + |Z_{31}|) - 9
+ min\{2, \frac{1}{2}(δ - κ + f - |Z_3| - 2)(|Z_{11}| + |Z_{31}| - 8)\}
≥ 3(δ - κ + f - |Z_3|) + 2(|Z_{11}| + |Z_{31}|) - 9.

Using (12) for each i ∈ {2, ..., m} and summing, we get

|V^1| ≥ 3(δ - κ + f - |Z_3|) + 2(|Z_{11}| + |Z_{31}|) - 9
+ 2(|Z_1| - |Z_{11}|) + |Z_3| - |Z_{31}|) - 2(m - 1)
= (3δ - 3κ + f - 1) + 2|Z_1| - |Z_3| + 2f - 2m - 6 ≥ 3δ - 3κ + f - 1.

**Case 1.2.2.2.** L(R_i) ∈ V^1_i (i = 1, 2, 3) and L(R_i) ∈ V^1_i (i = 4, 5)
In this case we can argue as in Case 1.2.2.1.

**Case 2.** f = 2 and S ∉ V^1.
Suppose first that δ - κ ≤ 1. Since δ > 3κ/2 - 1, we have |A^1| ≥ δ - κ + 1 ≥ 2.
Then it is easy to show that |V^1| ≥ 4 ≥ 3δ - 3κ + 1 and we are done. So, we can assume that δ - κ ≥ 2. Let S = {v_1, ..., v_κ} and F(Q^1_i) = v_1, L(Q^1_i) = v_2. Assume w.l.o.g. that v_3 ∈ S - V^1. Clearly v_3 ∉ V^1. Consider the graph G' = G - {v_4, v_5, ..., v_κ}. If there are two paths in (A^1 ∪ {v_1, v_2, v_3}) joining v_3 to Q^1_i and having only v_3 in common, then the existence of Q^1_0 follows easily.
Otherwise there is a cut-set in \( A \cup \{v_1, v_2, v_3\} \) consisting of a single vertex \( z \) that separates \( v_3 \) and \( V(Q_1) - z \). Then \( \{v_1, v_2, z\} \) is an another cut-set of \( G' \), contradicting the definition of \( A \). So, the existence of \( Q_0 \) is proved. By the definition of \( Q_1 \), we have \(|Q_1| \leq |Q_1|\). The notation \( P = y_1...y_p \), \( Z_1 \), \( Z_2 \) defined for \( Q^i_1 \), we will use here for \( Q_0^i \). Let \( M_1 = v_1Q_{0}^{-1}v_3 \) and \( M_2 = v_3Q_{0}^{-1}v_2 \). In addition, put \( Z_{1,i} = Z_1 \cap V(M_i) \) and \( Z_{2,i} = Z_2 \cap V(M_i) \) \((i = 1, 2)\). If \( p \leq 1 \), then \( \langle A - V(Q_1) \rangle \) is edgeless. Let \( p \geq 2 \). If \( v_3 \in Z_1 \cup Z_2 \), then we can argue as in Case 1. Let \( v_3 \notin Z_1 \cup Z_2 \). Denote by \( M_1' \) and \( M_2' \) the minimal subsegments in \( M_1 \) and \( M_2 \), respectively, such that \( Z_{1,1} \cup Z_{2,1} \subseteq V(M_1') \) and \( Z_{1,2} \cup Z_{2,2} \subseteq V(M_2') \).

**Case 2.1.** \( p = 2 \).
Clearly \(|Z_i| \geq \delta - \kappa + 2 \) \((i = 1, 2)\). Applying (6) to \( M_1' \) and \( M_2' \), we get

\[
|V| = |V_1'| \geq |V(Q_1)| \geq |V(M_1)| + |V(M_2)| + 1
\]

\[
\geq 3(|Z_1| + |Z_2|)/2 - 3 \geq 3(\delta - \kappa + 2) - 3 > 3\delta - 3\kappa + 1.
\]

**Case 2.2.** \( p = 3 \).
Clearly \(|Z_i| \geq \delta - \kappa + 1 \) \((i = 1, 2)\). Applying (8) to \( M_1' \) and \( M_2' \), we get

\[
|V| \geq |V(Q_1)| \geq |V(M_1')| + |V(M_2')| + 1
\]

\[
\geq 2(|Z_1| + |Z_2|) - 5 \geq 4(\delta - \kappa + 1) - 5 \geq 3\delta - 3\kappa + 1.
\]

**Case 2.3.** \( p \geq 4 \).
Let \( P_0, P_1, w, Z_3 \) are as defined in Lemma 14 (Case 2). If \( p_0 \leq 3 \), then \(|Z_1| \geq |Z_2| \geq \delta - \kappa + 1 \) and we can argue as in Case 2.2. Let \( p_0 \geq 4 \). Further, if \( \delta - \kappa + 3 - |Z_3| \leq 1 \), then \(|Z_1| \geq |Z_3| \geq \delta - \kappa + 2 \) and we can argue as in Case 2.1. Let \( \delta - \kappa + 3 - |Z_3| \geq 2 \). Since \( p_0 \geq 4 \), there are vertex disjoint paths \( R_1, R_2, R_3, R_4 \) in \( G' \) connecting \( P_0 \) and \( Q_0^i \) \((i = 2)\), otherwise, there exist a cut-set of \( G' \) of order 3 contradicting the definition of \( A \). Let \( F(R_i) \in V(P) \) and \( L(R_i) \in V(Q_0) \) \((i = 1, 2, 3, 4)\). Clearly, \(|N(w_i) \cap V(P_0)| \geq \delta - \kappa + 3 - |Z_3| \) for each \( i \in \{1, ..., s\} \). In particular, for \( i = s \), we have \( s \geq \delta - \kappa + 3 - |Z_3| \). By Lemma C, in \((V(P_0))\) any two vertices are joined by a path of length at least \( \delta - \kappa + 3 - |Z_3| \). Assume w.l.o.g. that \( R_1 \cup R_2 \) does not intersect \( y_1Pw_1^{-1} \) \((see the proof of Lemma 12, Case 1.2.2.1)\) and does not contain \( w \). Let \( I_1, ..., I_t \) be the minimal segments of \( Q_0 \) connecting two vertices of \( Z_1 \cup Z_3 \cup Z_4 \cup Z_5 \), where \( Z_4 = \{L(R_1)\} \) and \( Z_5 = \{L(R_2)\} \). Assume w.l.o.g. that \( v_3 \) belongs to \( I_1 \). Put \( I_1 = v'\overline{Q_1}v'' \). If \( v_3 = v' \) or \( v_3 = v'' \), then we can argue as in Case 1. Let \( v_3 \neq v' \) and \( v_3 \neq v'' \). Choose a longest path \( Q_0 \) joining \( v' \) and \( v'' \) and passing through \( V(R_i) \cup \bigcup_{i=1}^{t} V(R_i) \). Clearly, \(|Q_0| \geq \delta - \kappa + 5 - |Z_3| \) if \( v', v'' \) belong to different \( Z_1, Z_3, Z_4, Z_5 \) and \(|Q_0| \geq 2 \), otherwise. Since \( Q_1 \) is extreme with ends \( v_1, v_2 \) and intermediate vertex \( v_3 \), we have \(|I_i| \geq \delta - \kappa + 5 - |Z_3| \) if the ends of \( I_i \) belong to different \( Z_1, Z_3, Z_4, Z_5 \) and \(|I_i| \geq 2 \) for each \( i \in \{2, ..., t\} \), otherwise.
Form a new path $Q_{10}$ from $Q_{1}^1$ by replacing $I_1$ with $Q_0$. Clearly $(Z_1, Z_3, Z_4, Z_5)$ is a nontrivial $(Q_{10}, \delta - \kappa + 5 - |Z_3|)$-scheme and we can argue as in Case 1.2.2.1.

**Case 3.** $f = 2$ and $S \subseteq V^\dagger$.

Let $P_0$, $p_0$, $w$ and $Z_3$ are as defined in Lemma 14 (Case 2). If $p_0 \leq 2$, then $|Z_1| \geq |Z_2| \geq \delta - \kappa + 1$ and we can argue as in proof of Lemma 14 (Case 1). Let $p_0 \geq 3$. Since $\kappa \geq 3$, there are vertex disjoint paths $R_1, R_2, R_3$ connecting $V^\dagger$ and $V(P_0)$ (see the proof of Lemma 14, Case 2.2.2). Let $F(R_i) \in V(P_0)$ and $L(R_i) \in V^\dagger$ $(i = 1, 2, 3)$. If $\delta - \kappa - |Z_3| + 2 \leq 1$, then $|Z_1| \geq |Z_3| \geq \delta - \kappa + 1$ and again we can argue as in the case $p_0 \leq 2$. Let $\delta - \kappa - |Z_3| + 2 \geq 2$. By the choice of $w$, $|N(w_i^-) \cap V(P_0)| \geq \delta - \kappa + f - |Z_3|$ $(i = 1, \ldots, s)$. In particular, for $i = s$, $s \geq \delta - \kappa - |Z_3| + 2$. By Lemma C, in $(V(P_0))$ any two vertices are joined by a path of length at least $\delta - \kappa - |Z_3| + 2$. If $|Z_1| \leq 2$, then $|Z_3| \leq 2$. Since $\{L(R_1)\}$, $\{L(R_2)\}$, $\{L(R_3)\}$ is a nontrivial $(Q^1_1, \delta - \kappa - |Z_3| + 4)$-scheme, we have $|V^\dagger| \geq 2(\delta - \kappa - |Z_3| + 4) + 1 \geq 2(\delta - \kappa + 2) + 1 \geq 2\delta - 2\kappa + f + 1$. Let $|Z_1| \geq 3$. Analogously, $|Z_2| \geq 3$. Assume w.l.o.g. that $F(R_1) = w$. In addition, we can assume w.l.o.g. that $R_2 \cup R_3$ does not intersect $y_1 \overline{Pw\overline{w}}$ (see the proof of Lemma 12, Case 1.2.2.1). So, $(Z_1, Z_3, \{L(R_1)\})$ is a nontrivial $(Q^1_1, \delta - \kappa - |Z_3| + 4)$-scheme and by Lemma 5,

\[
|V^\dagger| \geq \min\{2(|Z_1| + |Z_3| + 1) + 2(\delta - \kappa - |Z_3| + 4) - 11, \\
\frac{1}{2}(\delta - \kappa - |Z_3| + 4)(|Z_1| + |Z_3| - 2) + 1\}
\]

\[
\geq \min\{2(|Z_3| + 1) + 2(\delta - \kappa) - 2|Z_3| - 3, (\delta - \kappa - |Z_3| + 4)(|Z_3| - 1) + 1\}
\]

\[
\geq 2\delta - 2\kappa + 2|Z_3| - 3 + \min\{2, (\delta - \kappa - |Z_3|)(|Z_3| - 3)\} \geq 2\delta - 2\kappa + f + 1.
\]

Lemma 15 is proved.$\Delta$

5 Proofs of theorems

**Proof of Theorem 2.** If $\delta \leq 3\kappa/2 - 1$, then we are done by Lemma 11. Let $\delta > 3\kappa/2 - 1$. We will use the notation defined in Definitions A and B. In addition, let $A^\dagger$ is defined with respect to a minimum cut-set $S = \{v_1, \ldots, v_\kappa\}$. The existence of $Q^1_1, \ldots, Q^\dagger_\kappa$ and $C^*, C^{**}$ follows from Lemma 10. By Lemmas 12 and 14, $A^\dagger \subseteq V^\dagger$ and $(A^\dagger - V^\dagger)$ is edgeless. Since $Q^1_1, \ldots, Q^\dagger_\kappa$ is extreme, $\langle S - V(C^*) \rangle$ is edgeless too. Recalling the definition of $C^{**}$, we can state that $A^\dagger \subseteq V(C^{**})$ and in addition, $A^\dagger - V(C^{**})$ and $S - V(C^{**})$ both are edgeless.

**Case 1.** $A^\dagger - V(C^{**}) = \emptyset$.

If $S - V(C^{**}) = \emptyset$, then $C^{**}$ is a Hamilton Cycle. Let $S - V(C^{**}) \neq \emptyset$ and choose any $w \in S - V(C^{**})$. Clearly $N(w) \subseteq V(C^{**})$. If $N(w)^+ \cup \{w\}$ is independent, then $\alpha \geq \delta + 1$, contradicting the hypothesis. Otherwise we can form
(by standard arguments) a cycle with vertex set $V(C^*) \cup \{w\}$, contradicting the definition of $C^*$.

**Case 2.** $A^\uparrow - V(C^*) \neq \emptyset$.

Let $z \in A^\uparrow - V(C^*)$. If $N(z) \subseteq V(C^*)$, then we can argue as in Case 1. Otherwise $N(z) = D_1 \cup D_2$, where $D_1 \subseteq V(C^*)$ and $D_2 \subseteq S - V(C^*)$. If $|A^\uparrow| \leq \kappa$, then $n \leq |A^\uparrow| + |A^\uparrow| + \kappa \leq 3\delta - \kappa$ and by Theorem B, $G$ is hamiltonian.

Let $|A^\uparrow| \geq \kappa + 1$. If $N(v_i) \cap A^\uparrow = \emptyset$ for some $i \in \{1, \ldots, \kappa\}$, then $S - v_i$ is a cut-set of order $\kappa - 1$, a contradiction. Therefore,

$$N(v_i) \cap A^\uparrow \neq \emptyset \quad (i = 1, \ldots, \kappa).$$

Put $N_i = N(v_i) \cap A^\uparrow$ ($i = 1, \ldots, \kappa$). If $|\cup_{i \in J} N_i| < |J| \leq \kappa$ for a subset $J \subseteq \{1, \ldots, \kappa\}$, then $(\cup_{i \in J} N_i) \cup \{v \in S | i \notin J\}$ is a cut-set of $G$ with at most $\kappa - 1$ vertices, a contradiction. So, we can assume that $|\cup_{i \in J} N_i| \geq |J|$ for each $J \subseteq \{1, \ldots, \kappa\}$. By Hall’s [5] Theorem, the collection $N_1, \ldots, N_\kappa$ has a system of distinct representatives. Set $D_2 = \{v_{i_1}, v_{i_2}, \ldots, v_{i_\kappa}\}$ and let $w_{i_1}, \ldots, w_{i_\kappa}$ is a system of distinct representatives of $N_{i_1}, \ldots, N_{i_\kappa}$. Put $D_3 = \{w_{i_1}, \ldots, w_{i_\kappa}\}$. Since $A^\uparrow \subseteq V(C^*)$, we have $D_3 \subseteq V(C^*)$ and it is easy to see that $(D_2 \cup D_3)^+ \cup \{z\}$ is an independent set of order at least $\delta + 1$, a contradiction.

**Proof of Theorem 3.** If $\delta \leq 3\kappa/2 - 1$, then we are done by Lemma 11. Let $\delta > 3\kappa/2 - 1$. By Lemma 12, $A^\uparrow \subseteq V^\uparrow$. The existence of $Q_1^1, \ldots, Q_m^1$ and $C^*, C^*$ (see Definition B) follows from Lemma 10. Let $A^\uparrow$ is defined with respect to a minimum cut-set $S = \{v_1, \ldots, v_\kappa\}$. Put $f = |V^\uparrow \cap S|$. By Theorem G, $c \geq \min\{n, 3\delta - 3\}$. If $c \geq n$, then we are done. Let $c \geq 3\delta - 3$. Further, if $3\delta - 3 \geq 4\delta - 2\kappa$, i.e. $\delta \leq 2\kappa - 3$, then $c \geq 3\delta - 3 \geq 4\delta - 2\kappa$. Let $\delta \geq 2\kappa - 2$ which implies $|A^\uparrow| \geq 3\delta - 3\kappa + 1 \geq 2\delta - \kappa - 1$. Recalling also that $|A^\uparrow| \geq |A^\uparrow|$, we obtain

$$|A^\uparrow| \geq 3\delta - 3\kappa + 1 \geq 2\delta - \kappa - 1.$$  

If $A^\uparrow \subseteq V^\uparrow$, then by (14), $c \geq |A^\uparrow| + |A^\uparrow| + 2 \geq 4\delta - 2\kappa$. Let

$$A^\uparrow \not\subseteq V^\uparrow.$$  

**Case 1.** $f \geq 3$.

By Lemma 15, either $(A^\uparrow - V^\uparrow)$ is edgeless or $|V^\uparrow| \geq 3\delta - 3\kappa + f - 1$. If $(A^\uparrow - V^\uparrow)$ is edgeless, then we can argue as in proof of Theorem 2. Let $|V^\uparrow| \geq 3\delta - 3\kappa + f - 1$. By (14),

$$c \geq |A^\uparrow| + |V^\uparrow| \geq (2\delta - \kappa - 1) + (3\delta - 3\kappa + f - 1)$$

$$= (4\delta - 2\kappa) + (\delta - 2\kappa + 2) + f - 4 \geq 4\delta - 2\kappa + f - 4.$$

If $f \geq 4$, then we are done. Let $f = 3$. Then $c \geq |A^\uparrow| + |V^\uparrow \cap S| - 2 + |V^\uparrow|$ and the desired result can be obtained by a similar calculation as above, if either
$|A^1| \geq 2\delta - \kappa$ or $|V^1 \cap S| \geq 3$. So, we can assume that $|A^1| = |A^1| = 2\delta - \kappa - 1$ and $|V^1 \cap S| = 2$. Let $V^1 \cap S = \{v_1, v_2\}$ and $V^1 \cap S = \{v_1, v_2, v_3\}$. Consider the graph $G' = G - \{v_4, ..., v_\kappa\}$. By (15), there exists a connected component $H$ of $G' - Q_1^1$ intersecting $A^1$. Put $M = N(V(H))$ and let $x_1, ..., x_q$ be the elements of $M \cap V^1$ occurring on $\overline{Q}_1^1$ in a consecutive order. Since $G'$ is 3-connected, we have $q \geq 3$. Further, $|x_i \overline{Q}_1^1 x_{i+1}| \geq 2$ ($i = 1, ..., q - 1$), since $Q_1^1$ is extreme. Put

$$M^* = \{x \in M \cap V^1 | x^- \notin S \text{ or } x^+ \notin S\}.$$ 

If $M^* = \emptyset$, then it is easy to check that $|V^1 \cap S| \geq 4$, contradicting the fact that $f = 3$. Let $M^* \neq \emptyset$ and choose any $u \in M^*$. Assume w.l.o.g. that $u^+ \notin S$. Choose $w \in V(H)$ such that $uw \in E(G)$. Then by deleting $uw^+$ from $Q_1^1$ and adding $uw$, we obtain a pair of vertex disjoint paths $v_1 \overline{Q}_1^1 w$ and $v_2 \overline{Q}_1^1 u^+$ both having one end in $S$ and the other in $A^1$. These two paths can be extended from $w$ and $u^+$ (using only vertices of $A^1$) to a pair of maximal vertex disjoint paths $R_1 = v_1 R_1 w_1$ and $R_2 = v_2 R_2 w_2$. Add an extra edge $v_1 v_2$ to $G$ and consider the path $L = w_1 R_1 v_1 R_2 w_2$ in $G^*$. Label $L = \xi_1 \xi_2 ... \xi_h$ according to the direction on $L$ and put

$$d_1 = |N(\xi_1) \cap V(L)|, \quad d_2 = |N(\xi_2) \cap V(L)|.$$ 

If $\xi_1$ and $\xi_h$ have a common neighbor $v_i$ in $\{v_4, ..., v_\kappa\}$, then $v_1 R_1 \xi_1 v_i \xi_h R_2 v_2$ is a path contradicting the choice of $Q_1^1$. Otherwise, $d_1 + d_2 \geq 2\delta - \kappa + 3$. Since $|V(L)| \leq |A^1| + 3 = 2\delta - \kappa + 2$, i.e. $d_1 + d_2 \geq |V(L)| + 1$, it can be shown by standard arguments that $\xi_1 \xi_{i+1} \in E(G)$ and $\xi_h \xi_1 \in E(G)$ for some distinct $i = i_1, i_2$. Assume w.l.o.g. that $\xi_i, \xi_{i+1} \neq v_1 v_2$ and $\xi_i \in \xi_1 L v_1$. Then $v_1 L \xi_{i+1} \xi_1 L \xi_i \xi_h L v_2$ is a path longer than $Q_1^1$ contradicting the definition of $Q_1^1$.

Case 2. $f = 2$ and $S \subseteq V^1$.

By Lemma 15, either $\langle A^1 - V^1 \rangle$ is edgeless or $|V^1| \geq 2\delta - 2\kappa + 3$. In the first case we can argue as in proof of Theorem 2. In the second case,

$$c \geq |A^1| + |S| + |V^1| - f \geq (2\delta - \kappa - 1) + \kappa + (2\delta - 2\kappa + 3) - 2 = 4\delta - 2\kappa.$$ 

Case 3. $f = 2$ and $S \not\subseteq V^1$.

By Lemma 15, either $\langle A^1 - V(Q_0^1) \rangle$ is edgeless or $|V^1| \geq 3\delta - 3\kappa + 1$.

Case 3.1. $\langle A^1 - V(Q_0^1) \rangle$ is edgeless.

By the definition of $C_0^*$, we have $A^1 \subseteq V(C_0^*)$. Besides, $A^1 - V(C_0^*)$ and $S - V(C_0^*)$ both are independent in $G$. If $A^1 - V(C_0^*) \neq \emptyset$, then we can argue as in proof of Theorem 2 (Case 2). Otherwise

$$c \geq |A^1| + |A^1| + 3 \geq 2(2\delta - \kappa - 1) + 3 > 4\delta - 2\kappa.$$
Case 3.2. \(|V^1| \geq 3\delta - 3\kappa + 1\). 

Let \(R_1 = v_1 \overrightarrow{R_1} v_2, R_2 = v_2 \overrightarrow{R_2} v_3, L = \xi_1 \ldots \xi_4\) and \(d_1, d_2\) be as defined in Case 1 with respect to \(Q_1\). Put \(|V^1 \cap S| = f\). Using (14), we get

\[
c \geq |A^1| + |V^1| + f' - 2 \geq 2\delta - \kappa - 1 + 3\delta - 3\kappa + 1 + f' - 2
\]

\[
= (4\delta - 2\kappa) + \delta - 2\kappa + 2 + f' - 4 \geq 4\delta - 2\kappa + f' - 4.
\]  

(16)

If \(f' \geq 4\), then we are done. Let \(f' \leq 3\). Similar to (16), we can state that

\[
\text{if } |A^1| \geq 2\delta - \kappa, \text{ then } c \geq 4\delta - 2\kappa + f' - 3,
\]  

(17)

\[
\text{if } |A^1| \geq 2\delta - \kappa + 1, \text{ then } c \geq 4\delta - 2\kappa
\]  

(18)

Case 3.2.1. \(f' = 3\).

If \(|A^1| \geq 2\delta - \kappa\), then by (17) we are done. Let \(|A^1| = 2\delta - \kappa - 1\), implying also \(|A^1| = 2\delta - \kappa - 1\). If \(\xi_1\) and \(\xi_4\) have a common neighbor \(v_i\) in \(\{v_4, \ldots, v_\kappa\}\), then \(v_1 \overrightarrow{R_1} v_4 \overrightarrow{R_2} v_3\) is a path contradicting the choice of \(Q_1\). Otherwise we have \(d_1 + d_2 \geq 2\delta - \kappa + 1\). In addition, \(|V(L)| \leq |A^1| + 2 = 2\delta - \kappa + 1\). If \(|V(L)| < 2\delta - \kappa + 1\), then as in Case 1, \(d_1 + d_2 \geq |V(L)| + 1\) and we can form a path longer than \(Q_1\), connecting \(v_1, v_2\) and passing through \(A^1\), contrary to the definition of \(Q_1\). Hence, \(|V(L)| = 2\delta - \kappa + 1\). This means that \(v_3\) is adjacent to both \(\xi_1\) and \(\xi_4\). Besides, each \(v_i\) \((i \in \{4, \ldots, \kappa\})\) is adjacent either to \(\xi_1\) or \(\xi_4\). Assume w.l.o.g. that \(v_4 \overrightarrow{R_2} v_3\) and \(v_2 \overrightarrow{R_1} v_1\). Since \(|A^1| = |A^1|\), we can state that \(A^1\) and \(A^i\) are both endfragments. Then taking \(\{Y_1, Y_2\}\) instead of \(\{Q_1, \ldots, Q_m\}\) and \(A^1\) instead of \(A^1\), we can argue as in case \(f' \geq 4\).

Case 3.2.2. \(f' = 2\).

If \(|A^1| \geq 2\delta - \kappa + 1\), then we are done by (18). Let \(|A^1| \leq 2\delta - \kappa\) implying also \(|A^1| \leq 2\delta - \kappa + 2\). Further, we have \(d_1 + d_2 \geq 2\delta - \kappa + 2\).

Case 3.2.2.1. \(|A^1| = 2\delta - \kappa - 1\).

In this case, \(|A^1| = 2\delta - \kappa - 1\). Clearly \(|V(L)| \leq |A^1| + 2 = 2\delta - \kappa + 1\) implying that \(d_1 + d_2 \geq |V(L)| + 1\). Then we can form (as above) a path contradicting the definition of \(Q_1\).

Case 3.2.2.2. \(|A^1| = 2\delta - \kappa\).

In this case, \(2\delta - \kappa - 1 \leq |A^1| \leq 2\delta - \kappa\). If \(|A^1| = 2\delta - \kappa - 1\), then \(|V(L)| \leq |A^1| + 2 = 2\delta - \kappa + 1\) and hence \(d_1 + d_2 \geq |V(L)| + 1\). Then again we can form a path contradicting the choice of \(Q_1\). Now let \(|A^1| = 2\delta - \kappa\). It follows that \(A^1\) and \(A^i\) are both endfragments. On the other hand, \(|V(L)| \leq |A^1| + 2 = 2\delta - \kappa + 2\).
implying that \( d_1 + d_2 \geq |V(L)| \). Thus we can argue as in Case 3.2.1. \( \Delta \)

**Proof of Theorem 4.** If \( \delta \leq 3k/2 - 1 \), then we are done by Lemma 11. Let \( \delta > 3k/2 - 1 \). By Lemma 14, \( \langle A^1 - V^1 \rangle \) is edgeless.

**Case 1.** \( A^1 \subseteq V^1 \).
If \( A^1 \subseteq V^1 \), then

\[
c \geq |A^1| + |A^1| + 2 \geq (3\delta - \kappa - 3) + (\delta - \kappa + 1) + 2 = 4\delta - 2\kappa.
\]

Let \( A^1 \not\subseteq V^1 \). By Lemma 13, \( |V^1| \geq 3\delta - 5 \) and hence

\[
c \geq |V^1| + |A^1| \geq 3\delta - 5 + \delta - \kappa + 1 \geq 4\delta - 2\kappa.
\]

**Case 2.** \( A^1 \not\subseteq V^1 \).
By the definition of \( C^{**} \), \( \langle A^1 - V(C^{**}) \rangle \) is edgeless and hence \( N(z) \subseteq V(C^{**}) \subseteq S \) for each \( z \in A^1 - V(C^{**}) \). If \( N(z) \subseteq V(C^{**}) \), then by standard arguments, \( \alpha \geq \delta + 1 \), a contradiction. Let \( N(z) = D_1 \cup D_2 \), where \( D_1 \subseteq V(C^{**}) \) and \( D_2 \subseteq S - V(C^{**}) \). Set \( D_2 = \{ v_i_1, ..., v_i_t \} \) and \( N_i = N(v_i) \cap A^1 (i = i_1, ..., i_t) \). As in proof of Theorem 2, the collection \( N_i \) has a system of distinct representatives \( w_i_1, ..., w_i_t \). Put \( D_3 = \{ w_i_1, ..., w_i_t \} \). Let \( D_4 = D_1 \cup D_5 \), where \( D_4 \subseteq V(C^{**}) \) and \( D_3 = D_3 - D_4 \). By the definition of \( Q^1 \) and \( Q^1_m \), it is easy to see that \( (D_1 \cup D_4) \cup D_3 \cup \{ z \} \) is an independent set with at least \( \delta + 1 \) vertices, contradicting \( \delta \geq \alpha \). \( \Delta \)

**Proof of Theorem 5.** If \( \delta \leq 3k/2 - 1 \), then we are done by Lemma 11. Let \( \delta > 3k/2 - 1 \). The existence of \( Q^1 \) and \( C^*, C^{**} \) follows from Lemma 10. As in proof of theorem 3, \( \delta - 2\kappa + 2 \geq 0 \), implying in particular that \( \delta - \kappa \geq 2 \). By Lemma 13, either \( A^1 \subseteq V^1 \) or \( |V^1| \geq 3\delta - 5 \).

**Case 1.** \( A^1 \subseteq V^1 \).
If \( A^1 \subseteq V^1 \), then

\[
c \geq |A^1| + |V^1| + 2 \geq (3\delta - \kappa - 3) + (3\delta - 3\kappa + 1) + 2 = 4\delta - 2\kappa.
\]

If \( |V^1| \geq 3\delta - 5 \), then

\[
c \geq |V^1| + |A^1| \geq 3\delta - 5 + 3\delta - 3\kappa + 1 \geq 4\delta - 2\kappa.
\]

**Case 2.** \( A^1 \not\subseteq V^1 \).
If either \( \langle A^1 - V^1 \rangle \) or \( \langle A^1 - V(Q^1) \rangle \) is edgeless, then we can argue as in proof of Theorem 4. Otherwise, by Lemma 15, either \( f = 2 \) and \( |V^1| \geq 2\delta - 2\kappa + 3 \) or \( f \geq 3 \) and \( |V^1| \geq 3\delta - 3\kappa + f - 1 \geq 2\delta - 2\kappa + 3 \), where \( f = |V^1 \cap S| \). If \( A^1 \subseteq V^1 \), then
\[ c \geq |A^\uparrow| + |V^\downarrow| \geq 3\delta - \kappa - 3 + 2\delta - 2\kappa + 3 \geq 4\delta - 2\kappa. \]

Let \( A^\uparrow \not\subseteq V^\uparrow \). By Lemma 13, \(|V^\uparrow| \geq 3\delta - 5 \). If \( f = 2 \), then
\[ c \geq |V^\uparrow| + |V^\downarrow| - 2 \geq (3\delta - 5) + (2\delta - 2\kappa + 3) - 2 \]
\[ = (4\delta - 2\kappa) + \delta - 4 \geq 4\delta - 2\kappa. \]

If \( f \geq 3 \), then
\[ c \geq |V^\uparrow| + |V^\downarrow| - f \geq (3\delta - 5) + 3\delta - 3\kappa + f - 1 - f \]
\[ = (4\delta - 2\kappa) + (\delta - 2\kappa + 2) + (\delta + \kappa - 8) \geq 4\delta - 2\kappa. \]

\[ \Delta \]

**Proof of Theorem 1.** If \( \delta \leq 3\kappa/2 - 1 \), then we are done by Lemma 11. Let \( \delta > 3\kappa/2 - 1 \). Let \( A^\uparrow \) be an endfragment of \( G \) with \(|A^\uparrow| \geq |A^\downarrow| \). Then the desired result follows from Theorems 2-5. \( \Delta \)

**Remark.** The limit examples below show that Theorem 1 is best possible in all respects. The limit example 4\( K_2 + K_3 \) shows that 4-connectedness can not be replaced by 3-connectedness. Further, the limit example \( H(1, 1, 5, 4) \) shows that the condition \( \delta \geq \alpha \) cannot be replaced by \( \delta \geq \alpha - 1 \). Finally, the limit example 5\( K_2 + K_3 \) shows that the bound 4\( \delta - 2\kappa \) cannot be replaced by 4\( \delta - 2\kappa + 1 \).

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## References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications. Macmillan, London and Elsevier, New York (1976).

[2] G.A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. 2(1952) 69-81.

[3] R. Häggkvist, G.G. Nicoghossian (now - Zh.G. Nikoghosyan), A remark on hamiltonian cycles, J. Combin. Theory, Ser. B 30 (1981) 118-120.

[4] R. Häggkvist, C. Thomassen, Circuits through specified edges, Discrete Math. v.41 (1982) 29-34.

[5] Ph. Hall, On representatives of subsets, J. Lond. Math. Soc. 10 (1935) 26-30.

[6] Y.O. Hamidone, On critically h-connected graphs, Discrete Math. 32 (1980) 257-262.
[7] C.St.J.A. Nash-Williams, Edge-disjoint hamiltonian cycles in graphs with vertices of large valency, in: L. Mirsky (Ed), Studies in Pure Mathematics, Academic Press, San Diego, London (1971) 157-183.

[8] Zh.G. Nikoghosyan, On maximal cycle of a graph, DAN Arm.SSR v.LXXII 2 (1981) 82-87 (in Russian).

[9] Zh.G. Nikoghosyan, On maximal cycle of a graph, Studia Sci. Math. Hungar. 17 (1982) 251-282 (in Russian).

[10] Zh.G. Nikoghosyan, A sufficient condition for a graph to be hamiltonian, Matematicheskie voprosy kibernetiki I vichislitelnoy tekhniki v. XIV (1985) 34-54 (in Russian).

[11] Zh.G. Nikoghosyan, On maximal cycles in graphs, DAN Arm.SSR v.LXXXI 4 (1985) 166-170 (in Russian).

[12] H.J. Veldman, Non-κ-critical vertices in graphs, Discrete Math. v.44 1 (1983) 105-110.

[13] H.-J. Voss, Bridges of longest circuits and of longest paths in graphs, Beiträge zur graphen theorie and deren Anwendungen, Vorgetr. auf dem int. Kolloq., Oberhof DDR (1977) 275-286.

[14] H.-J. Voss, C. Zuluaga, Maximale gerade und ungerade Kreise in Graphen I, Wiss. Z. Tech. Hochschule Ilmenau, 23 (1977) 57-70.

[15] D.R. Woodall, Circuits containing specified edges, J. Combin. Theory, B22 3 (1977) 274-278.