ON IWASAWA THEORY OVER FUNCTION FIELDS

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Abstract. Let \( k_\infty \) be a \( \mathbb{Z}_p \)-extension of a global function field \( k \) of characteristic \( p \). Let \( \text{Cl}_{k_\infty,p} \) be the \( p \) completion of the class group of \( k_\infty \). We prove that the characteristic ideal of the Galois module \( \text{Cl}_{k_\infty,p} \) is generated by the Stickelberger element of Gross which calculates the special values of \( L \) functions.

1. Introduction

We prove the \( p \) part of Iwasawa’s main conjecture over global function fields. We show that the characteristic ideal of the Iwasawa module obtained from the class groups is generated by the Stickelberger element defined by Gross. Our proof begins with a theorem of Tate and the rest of the proof is based on some results on the leading term of the Stickelberger element first conjectured by Gross ([Gro88]) and on the properties of pseudo-null Iwasawa modules studied by Greenberg ([Grn78]).

Let \( k \) be a global function field of characteristic \( p \). Let \( k_\infty/k \) be a \( \mathbb{Z}_p \)-extension unramified outside a non-empty finite set \( S \) of places of \( k \). For each non-negative integer \( n \) let \( k_n \) be the fixed field of \( \Gamma \). Denote the \( p \)-Sylow subgroup of the class group \( \text{Cl}_{k_n} \) of divisors of degree zero of \( k_n \) by \( \text{Cl}_{k_n,p} \). Let \( \text{Cl}_{k_\infty,p} \) be the projective limit \( \lim_{\leftarrow} \text{Cl}_{k_n,p} \) taken over the norm maps \( \text{Cl}_{k_m,p} \rightarrow \text{Cl}_{k_n,p} \) with \( m \geq n \).

Write \( \Gamma \) for the Galois group \( \text{Gal}(k_\infty/k) \). Let \( \Lambda_\Gamma \) denote the complete group ring \( \mathbb{Z}_p[[\Gamma]] \). Let \( \sigma_1, \ldots, \sigma_d \) be a \( \mathbb{Z}_p \)-basis of \( \Gamma \) viewed as a \( \mathbb{Z}_p \)-module. Put \( t_j = \sigma_j - 1 \). Then \( \Lambda_\Gamma = \mathbb{Z}_p[[t_1, \ldots, t_d]] \). The ring \( \Lambda_\Gamma \) is a unique factorization domain ([Bou72], Chap. 7, Sec. 3.9, Prop. 8) and the class group \( \text{Cl}_{k_\infty,p} \) is a finitely generated torsion \( \Lambda_\Gamma \) module ([Iwa59, Ser58]). Moreover \( \text{Cl}_{k_\infty,p} \) is pseudo-isomorphic to \( \bigoplus_{j=1}^d \Lambda_\Gamma / \wp_j^{n_j} \) where the non-negative integers \( n_j \) and the height 1 prime ideals \( \wp_j \) of \( \Lambda_\Gamma \) are uniquely determined ([Bou72], Chap. VII, Sec. 4.4, Definition 3 and Theorem 5). We define the characteristic ideal \( \chi_\Gamma(\text{Cl}_{k_\infty,p}) \) of the \( \Lambda_\Gamma \)-module by

\[
\chi_\Gamma(\text{Cl}_{k_\infty,p}) := \prod_{j=1}^d \wp_j^{n_j}.
\]

Fix another finite set \( T \) of places of \( k \) so that \( S \cap T = \emptyset \). Given a finite abelian extension \( K/k \) unramified outside \( S \) with Galois group \( G \), let \( [\nu] \in G \) be the Frobenius

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at $v$ and $q$ be the order of the constant field of $k$. Put
\begin{equation}
\Theta_{K/k,S,T}(u) = \prod_{v \in S} \left( 1 - [v]u^{\deg v} \right)^{-1} \cdot \prod_{v \in T} \left( 1 - [v](qu)^{\deg v} \right).
\end{equation}
Let $\theta_{K/k,S,T} = \Theta_{K/k,S,T}(1)$. The result of Weil (see [Tat84] Chapter V) shows that $\Theta_{K/k,S,T}(u)$ is a polynomial in $u$ with coefficients in $\mathbb{Z}[G]$, in particular $\theta_{K/k,S,T} \in \mathbb{Z}[G]$ and is the same as the Stickelberger element $\theta_G$ defined in [Gro88].

If $K'$ is an intermediate field of $K/k$ with $\text{Gal}(K/K') = H$, then the Stickelberger elements enjoy the functorial property that under the ring homomorphism $pr : \mathbb{Z}[G] \to \mathbb{Z}[G/H]$ induced from the natural projection from $G$ onto $G/H$ we have
\begin{equation}
pr(\theta_{K/k,S,T}) = \theta_{K'/k,S,T}.
\end{equation}
We use these maps to define $\theta_{k_{\infty}/k,S,T} \in \Lambda_{\Gamma}$ as the projective limit over $n$ of $\theta_{k_n/k,S,T} \in \mathbb{Z}_p[\Gamma_n]$.

We can now state our main result.

**Main Theorem.** Let $k$ be a global function field of characteristic $p$. Let $k_{\infty}/k$ be a $\mathbb{Z}_p^d$-extension with Galois group $\Gamma = \text{Gal}(k_{\infty}/k)$. We assume that $k_{\infty}/k$ is unramified outside a non-empty finite set $S$ of places $k$ and that no place in $S$ splits completely in $k_{\infty}/k$. Let $S_0 \subset S$ denote the subset consisting of unramified places in $k_{\infty}/k$. If $S_0 \neq S$, put
\[
\delta = \prod_{v \in S_0} (1 - [v]),
\]
where $[v] \in \Gamma$ is the Frobenius at $v$; if $S_0 = S$, then $k_{\infty}/k$ is the constant $\mathbb{Z}_p$-extension, and we put
\[
\delta = (1 - Fr)^{-1} \prod_{v \in S_0} (1 - [v]),
\]
where $Fr \in \Gamma$ is the Frobenius element. Fix a non-empty finite set $T$ of places outside $S$. Then
\[
(\theta_{k_{\infty}/k,S,T}) = (\delta) \cdot \chi_{\Gamma}(\text{Cl}_{k_{\infty},p}).
\]

**Remarks:** Over a global function field of characteristic different from $p$, the constant field extension is the only $\mathbb{Z}_p^d$-extension. In this case $d = 1$ and Iwasawa’s main conjecture is known to be a consequence of Weil’s theory. Therefore, we only deal with those $\mathbb{Z}_p^d$-extensions with $p$ equals the characteristic of the ground field. Also, we shall see in Section 2.5 that, unlike the counterpart in the cases where $k$ is a number field, for a fixed $S$ the rank $d$ of the Galois group $\Gamma$ can be arbitrary large. The next point is that the assumption in the theorem is necessary, since we know that if some $v \in S$ splits completely then $\theta_{k_{\infty}/k,S,T} = 0$ ([Gro88], [Tan95]). The appearance of the extra factor $\delta$ is consistent with the fact that if $v \in S_0$ and $S' = S - \{v\}$, then
\[
\theta_{k_{\infty}/k,S,T} = (1 - [v])\theta_{k_{\infty}/k,S',T}.
\]
We note that if $S_0 = S$, then
\[
[v] = Fr^{\deg_k(v)}
\]
and hence $(1 - Fr)^{-1}(1 - [v]) \in \Lambda_{\Gamma}$. Finally, if a different $T'$ is chosen, then the new Stickelberger element $\theta_{k_{\infty}/k,S,T'}$ will be a multiple of $\theta_{k_{\infty}/k,S,T}$ by a unit in $\Lambda_{\Gamma}$, and the
two Stickelberger elements generate the same ideal. This explains the dependence on $T$ in the last equation in the main Theorem.

Here is an outline of the paper. The proof of the Main theorem will be completed in section 4. In section 3 we set up the induction machine which allows us to bootstrap one $\mathbb{Z}_p$ extension down each step. This is summarized in Lemma 3.13. In section 2 we prove the results which are needed to begin the induction, namely we use Tate’s theorem to prove the Main Theorem in the case of a $\mathbb{Z}_p$ extension whose the Stickelberger element is monomial (in the sense of Definition 2.1) and we construct an independent extensions (see Definition 2.2) $L_\infty$ over $k_\infty$ which contains an intermediate $\mathbb{Z}_p$ extension whose Stickelberger element is monomial. We shall keep the notations of this section in the rest of the paper.

2. The class groups and the regulators

In this section we first draw some consequences from Tate’s theorem. Next we recall Weil’s theory on the zeta-function associated to a global function field, and then we apply it together with Tan’s theorem [Tan95] on the exact form of the leading term of the Stickelberger element $\theta_{k_\infty/k,S,T}$ as conjectured by Gross ([Gro88]) and the computation of $p$-adic values of special values of $L$ function to prove the Main Theorem in the case when the Stickelberger element is monomial in the sense of Definition 2.1.

We shall see that by enlarging the set $S$ of places of $k$ we can make the order of the class group $\text{Cl}_{k,S,T}$ relatively prime to $p$. It seems that we cannot simplify the refined regulator by just enlarging the set $S$. But we can do so by extending the field $k_\infty$. The other main result of this section is the existence of an independent extension of the given pair $(k, S)$ (see Definition 2.2 and Lemma 2.9). Then it follows immediately from Tate’s theorem that if $(L_\infty, \tilde{S})$ is an independent extension of $(k_\infty, S)$ then

$$
\chi_{\text{Gal}(L_\infty/k)}(\text{Cl}_{L_\infty,p}) = (\theta_{L_\infty/k,S,T}^m),
$$

for some positive integer $m$.

The group of divisors of $k$, denoted by $\text{Div}_k$, is the free $\mathbb{Z}$-module generated by all the places of $k$ (see [Tat84] Chap. V). There is a group homomorphism $k^\times \rightarrow \text{Div}_k$ such that an element $\alpha \in k^\times$ is sent to the divisor $\sum \text{ord}_v(\alpha) \cdot v$, and the image of this homomorphism is denoted as $P_k$. Let $\text{Div}_k^0$ denote the subgroup of $\text{Div}_k$ consisting of zero-degree divisors. The class group of $k$ is $\text{Cl}_k := \text{Div}_k^0 / P_k$.

2.1. Some consequences of Tate’s theorem. Our proof of the Main Theorem begins with a theorem of Tate ([Tat84]), which is in fact the function-field version of the Brumer-Stark conjecture. We shall only quote the main part of this splendid theorem. Let $S$ and $T$ be as before. For a finite abelian extension $K/k$ unramified outside $S$ with Galois group $G = \text{Gal}(K/k)$, put

$$
\Theta_{K/k,S}(u) = \prod_{v \not\in S} (1 - [v]u^{\deg v})^{-1} \in 1 + u\mathbb{Q}[G][[u]],
$$

where $[v] \in G$ denotes the Frobenius at $v$ and $q_k$ denotes the order of the constant field $\mathbb{F}_k$ of $K$. 
Theorem 2.1. (Tate [Tat84]) We have \((q_K - 1) \cdot \Theta_{K/k,S}(1) \in \mathbb{Z}[G]\) and 
\((q_K - 1) \cdot \Theta_{K/k,S}(1) \cdot \text{Cl}_K = 0.\)

Obviously, we have 
\[\Theta_{K/k,S,T}(u) = \Theta_{K/k,S}(u) \cdot \prod_{v \in T}(1 - [v](qu)^{\text{deg}_v}). \quad (2.3)\]

Since the number \(q_K - 1\) is relatively prime to \(p\), it follows from Tate’s theorem and 
the equation (2.3) that 
\[\theta_{k/\infty,k,S,T} \cdot \text{Cl}_{k,p} = 0 \quad (2.4)\]
and so 
\[\theta_{k/\infty,k,S,T} \cdot \text{Cl}_{k/\infty,p} = 0. \quad (2.5)\]

Since \(\Lambda_\Gamma\) is a unique factorization domain, every prime ideal \(\wp_i\) appearing in the char-
acteristic ideal \(\chi_\Gamma(\text{Cl}_{k/\infty,p})\) (1.1) is generated by a prime element \(\pi_i \in \Lambda_\Gamma\). Therefore,
the characteristic ideal \(\chi_\Gamma(\text{Cl}_{k/\infty,p})\) is generated by the product 
\[\pi_i^{n_i} = \prod_{i \in J} \pi_i^{n_i}. \quad (2.11)\]

Now the equation (2.5) is actually equivalent to saying that \(\theta_{k/\infty,k,S,T}\) is divisible by the 
least common multiple of all the \(\pi_i^{n_i}\).

Corollary 2.1. If \(\theta_{k/\infty,k,S,T}\) is irreducible in \(\Lambda_\Gamma\), then 
\[\chi_\Gamma(\text{Cl}_{k/\infty,p}) = (\theta_{k/\infty,k,S,T}^m), \quad (2.6)\]
for some non-negative integer \(m\).

We will prove that for suitable choice of \(S\), the Stickelberger element \(\theta_{k/\infty,k,S,T}\) 
atually gives rise to a generator of the characteristic ideal of \(\text{Cl}_{k/\infty,p}\).

2.2. The class number formula of Gross. Consider a finite extension \(K/k\) un-
ramified outside \(S\). For each place \(w\) of \(K\) let \(F_w\) denote the residue field of \(w\). Put 
\(q_w = \left|F_w\right|\). The zeta-function \(\zeta_K(s) = \prod_w (1 - q_w^{-s})^{-1}\) can be written as
\[\zeta_K(s) = \frac{P_1(q_K^{-s})}{(1 - q_K^{-s})(1 - q_K^{1-s})} \quad (2.7)\]
with \(P_1(u) \in \mathbb{Z}[u]\) and \(P_1(1) = |\text{Cl}_K|\). \quad (2.8)

Use \(S(K)\) (resp. \(T(K)\)) to denote the set consisting of places of \(K\) sitting over \(S\) 
(resp. \(T\)), and define
\[\zeta_{K,S,T}(s) = \prod_{w \in S(K)}(1 - q_w^{-s}) \cdot \prod_{w \in T(K)}(1 - q_w^{1-s}) \cdot \zeta_K(s). \quad (2.9)\]

Recall that the degree \(\text{deg}_K(w)\) of \(w\) equals the degree \(|F_w : F_K|\) of the field extension 
\(F_w/F_K\). It is easy to see from equalities (2.7) and (2.8) that if \(r_K = |S(K)| - 1,\) 
then the Taylor expansion begins as 
\[\zeta_{K,S,T}(s) \equiv B_{K,S,T} \cdot s^{r_K} \pmod{s^{r_K+1}}, \quad (2.10)\]
where
\[B_{K,S,T} = |\text{Cl}_K| \cdot (\log q_K)^{r_K} \cdot \left(\prod_{w \in S(K)} \text{deg}_K(w) \right) \prod_{w \in T(K)} (1 - |F_w|) \quad (2.11)\]
For an intermediate field $K'$ of $K/k$ we define the ratio $\rho_{K/K'}(s)$ of zeta functions as

$$\rho_{K/K'}(s) := \zeta_{K,S,T}(s)/\zeta_{K',S,T}(s).$$

The next lemma is an immediate consequence of (2.10) and (2.11).

**Lemma 2.1.** If $K'$ is an intermediate field of $K/k$ with $r_{K'} = r_K$, then the function $\rho_{K/K'}(s)$ is regular at $s = 0$ with

$$\rho_{K/K'}(0) = \epsilon_0 \cdot \frac{|\text{Cl}_K|}{|\text{Cl}_{K'}|} \cdot \left(\frac{\log q_K}{\log q_{K'}}\right)^{r_K} \cdot \prod_{w \in S(K)} \deg_{K}(w) \cdot \prod_{w' \in S(K')} \deg_{K'}(w'),$$

for an $\epsilon_0 \in \mathbb{Z}_p^\times$.

We would like to point out that (2.10) is a class number formula since the coefficient $B_{K,S,T}$ can be expressed as the product of some class number and regulator.

To see this, we consider $O_{K,S}$, the ring of $S$-integers of $K$. Let $\text{Cl}_{K,S}$ denote the ideal class group of the ring $O_{K,S}$ and let $U_{K,S,T}$ denote the subgroup of $O_{K,S}$ consisting of elements which are congruent to 1 modulo $T(K)$. Define $\text{Div}_{K,S,T}$ as the group whose elements are of the form $(\mathcal{I}, (a_w)_{w \in T(K)})$ where $\mathcal{I}$ is an ideal of $O_{K,S}$ and $a_w$ is a local generator of $\mathcal{I}$ at $w$. And define $\text{Cl}_{K,S,T}$ as the quotient of $\text{Div}_{K,S,T}$ by the subgroup generated by principle elements. Then we have the exact sequence

$$1 \to U_{K,S,T} \to O_{K,S}^\times \to \prod_{w \in T(K)} \mathbb{F}_p^\times \to \text{Cl}_{K,S,T} \to \text{Cl}_{K,S} \to 1. \quad (2.12)$$

Let $R_{K,S,T}$ denote the classical regulator formed by $U_{K,S,T}$. Namely, if $w_1, \ldots, w_{r_K}$ are distinct places in $S(K)$ and $u_1, \ldots, u_{r_K}$ is a $\mathbb{Z}$-basis of $U_{K,S,T}$, which is in fact free over $\mathbb{Z}$ (see [Gro88]), then $R_{K,S,T} = |\det_{1 \leq i, j \leq r_K} (\text{ord}_{w_i}(u_j) \cdot \deg_K(w_i))|$. This implies (see [Gro88])

$$B_{K,S,T} = (-1)^{|T(K)|-1} \cdot |\text{Cl}_{K,S,T}| \cdot R_{K,S,T}. \quad (2.13)$$

We conclude this section by recalling another type of class number formula proposed by Gross. It involves the Stickelberger element and a refined regulator ([Gro88]). This formula will be useful for us.

Assume that $K/k$ is a pro-$p$ abelian extension with $G = \text{Gal}(K/k)$. If $G$ is finite, $I_G$ (resp. $I_{G,p}$), the augmentation ideal of $\mathbb{Z}[G]$ (resp. $\mathbb{Z}_p[G]$), is defined as the kernel of the ring homomorphism onto $\mathbb{Z}$ (resp. $\mathbb{Z}_p$) sending $\sum_{g \in G} a_g \cdot g$ to $\sum_{g \in G} a_g$. If $G$ is pro-finite and $n \in \mathbb{Z}_+$, the $n$’th power augmentation ideal $I^n_G$ (resp. $I^n_{G,p}$) is defined as the projective limit of the corresponding $n$’th powers $I^n_G$ (resp. $I^n_{G,p}$) where $G$ runs through all finite quotients of $G$.

For a place $v \in S$ let

$$\psi_{v,G} : k_v^\times \to G_v \subset G \quad (2.14)$$

be the local norm residue map. Write $S = \{v_0, v_1, \ldots, v_r\}$ and $r = r_K = |S| - 1$. We choose a $\mathbb{Z}$-basis $u_1, \ldots, u_r$ of $U_{k,S,T}$. Assume that the ordering of this basis is chosen such that the number

$$\det_{1 \leq i, j \leq r} (\text{ord}_{v_i}(u_j) \cdot \deg_K(v_i))$$
is positive. Then the refined regulator of Gross is defined as the residue class modulo $I_G^{r+1}$ of the element

$$\det_{K/k,S,T} = \det_{1 \leq i, j \leq r} (\psi_{v_i, G}(u_j)) \in I_G^r \subset \mathbb{Z}[[G]]. \quad (2.15)$$

The following pro-$p$ version of a conjecture of Gross ([Gro88]) was proved in [Tan95] (the case where $r = 1$ was first proved in [Hay88]).

**Theorem 2.2.** If $K/k$ is a pro-$p$ abelian extension unramified outside $S$, then $\theta_{K/k,S,T} \in I_{G,p}$ and $\theta_{K/k,S,T} \equiv |\Cl_{K,S,T}| \cdot \det_{K/k,S,T} \pmod{I_{G,p}^{r+1}}$.

2.3. **Special values of $L$ function.** Let $S$ and $T$ be as before. Write $q_v = q^{\deg v}$ for the order of the residue field at a place $v$ of $k$. Let $K/k$ be a finite abelian extension unramified outside $S$ with Galois group $G$. Write $[v] \in G$ for the Frobenius at $v$. For each character $\varphi$ of $G$ define

$$L_{\varphi,S,T}(s) = \prod_{v \in S} \frac{1}{1 - \varphi([v])q_v^{-s}} \cdot \prod_{v \in T} (1 - \varphi([v])q_v^{1-s}).$$

We extend $\varphi$ linearly to a homomorphism $\varphi : \mathbb{Z}[G][[q^{-s}]] \rightarrow \overline{\mathbb{Q}}[[q^{-s}]]$. It follows from (2.3) that

$$\varphi(\Theta_{K/k,S,T}(q^{-s})) = L_{\varphi,S,T}(s).$$

If we view the dual group $G/H$ as a subgroup of $\hat{G}$ then the ratio $\rho_{K/K'}(s)$ of zeta functions can be written as a product of $L$ functions, namely,

$$\rho_{K/K'}(s) = \prod_{\varphi \in G/H} L_{\varphi,S,T}(s),$$

and so

$$\prod_{\varphi \in G/H} \varphi(\Theta_{K/k,S,T}(q^{-s})) = \rho_{K/K'}(s). \quad (2.16)$$

Now Lemma 2.1 implies the following.

**Lemma 2.2.** If $K'$ is an intermediate field of $K/k$ with $r_{K'} = r_K$, then

$$\prod_{\varphi \in G/H} \varphi(\theta_{K/k,S,T}) = \epsilon_0 \cdot \frac{|\Cl_K|}{|\Cl_{K'}|} \cdot \left(\frac{\log q_K}{\log q_{K'}}\right)^{r_K} \cdot \prod_{w \in S(K)} \frac{\deg_K(w)}{\prod_{w \in S(K')} \deg_{K'}(w)}, \quad \epsilon_0 \in \mathbb{Z}_p^\times.$$

A continuous character $\varphi \in \hat{G}$ can also be extended linearly to a ring homomorphism $\varphi : \Lambda_1 \rightarrow \overline{\mathbb{Q}}_p$. If $\varphi \in \hat{G}_p \subset \hat{G}_\infty$ then it has order dividing $p^n$, and vice versa. From the functorial property (1.3), we have $\varphi(\theta_{k_\infty/k,S,T}) = \varphi(\theta_{k_\infty/k,S,T})$.

Now we assume that $k_\infty/k$ is a $\mathbb{Z}_p$-extension and let $\text{ord}_p$ be the unique valuation on $\mathbb{Q}_p$ with $\text{ord}_p(p) = 1$. If $\varphi, \varphi' \in \hat{G}_\infty$ are of the same order, then they are conjugate under the action of $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ and hence

$$\text{ord}_p(\varphi(\theta_{k_\infty/k,S,T})) = \text{ord}_p(\varphi(\theta_{k_\infty/k,S,T})). \quad (2.17)$$

Let $\sigma$ be a $\mathbb{Z}_p$-basis of the Galois group $\Gamma$ of $k_\infty/k$ viewed as a $\mathbb{Z}_p\sigma$-module. Put $t = \sigma - 1$. Then $\Lambda_1 = \mathbb{Z}_p[[t]]$. The prime element $\pi_i$ generating the prime ideal $\varphi_i$ appearing in the characteristic ideal $\chi_\Gamma(\Cl_{k_\infty})$ can be chosen to be either the
prime number $p$ or an irreducible distinguished polynomial $p_i(t) \in \Lambda_\Gamma$. In this case we have the following pseudo-isomorphism
\[
\text{Cl}_{k_\infty,p} \rightarrow \bigoplus_{i=1}^{M} \Lambda_\Gamma/p^{m_i}\Lambda_\Gamma \bigoplus \bigoplus_{j=1}^{L} \Lambda_\Gamma/p_j(t)^{l_j}\Lambda_\Gamma.
\]
(2.18)

If we put
\[
\mu = \sum_{i=1}^{M} m_i, \quad f = \prod_{j=1}^{L} p_j(t)^{l_j} \quad \text{and} \quad \lambda = \sum_{j=1}^{L} l_j \cdot \deg(p_j(t)),
\]
then ([Ser58, Iwa59]) there is a constant $\nu$ such that $|\text{Cl}_{k_n,p}| = p^n$ with
\[
e_n = \mu p^n + \lambda n + \nu.
\]
(2.20)

**Lemma 2.3.** Assume that $k_\infty/k$ is a $\mathbb{Z}_p$-extension and no place in $S$ splits completely under $k_\infty/k$. Let $\mu$, $\lambda$, $f$ be as in (2.18) and (2.19) and let $\delta$ be as in the Main Theorem. Then $\theta_{k_\infty/k,S,T} = \epsilon_1 \cdot p^e \cdot g$, where $\epsilon_1 \in \Lambda_\Gamma$ and $g$ is a distinguished polynomial of degree $\lambda + \deg(\delta)$ which is equal to $\deg(f \cdot \delta)$.

**Proof.** If an $\eta \in \Lambda_\Gamma$ is expressed as
\[
\eta = \epsilon_2 \cdot p^m \cdot h,
\]
where $\epsilon_2 \in \Lambda_\Gamma$ and $h = t^l + b_{l-1}t^{l-1} + \cdots + b_0$ is the associated distinguished polynomial with $b_0, \ldots, b_{l-1} \in \mathfrak{p} \cdot \Lambda_\Gamma$, then for a $\varphi \in \hat{\Gamma}_\infty$ of order $p^n$ with $p^n \geq 1$, we have
\[
\text{ord}_p(\varphi(\eta)) = m + \frac{l}{p^n - 1}.
\]
In view of this, we only need to show that for $n$ large
\[
\text{ord}_p(\varphi(\theta_{k_\infty/k,S,T})) = \mu + \frac{\lambda + \deg(\delta)}{p^n - 1}.
\]
From the equation (2.20), Lemma 2.2 and the equation (2.17), we see that it is enough to verify that for large $n$
\[
p^{\deg(\delta)} = \left(\frac{q_{k_n}}{q_{k_{n-1}}}\right)^{r_{k_n}} \prod_{w \in S(k_n)} \frac{\deg_{k_n}(w)}{\prod_{w' \in S(k_{n-1})} \deg_{k_{n-1}}(w')}.
\]
Since no place in $S$ splits completely, the cardinality of the set $S(k_n)$ will eventually be stable, and every $w$ in $S(k_{n-1})$ will be either ramified or inert under $k_n/k_{n-1}$. In the case where $k_\infty/k$ is the constant $\mathbb{Z}_p$-extension and $n$ is large enough, we have $q_{k_n} = q_{k_{n-1}}^{r_{k_n}}$, $r_{k_n} = \deg(\delta)$ and $\deg_{k_n}(w) = \deg_{k_{n-1}}(w')$ if $w'$ sits below $w$. Therefore the lemma holds. In other cases, when $n$ is large enough $q_{k_n} = q_{k_{n-1}}$, for $w'$ sitting below $w$ the ratio $\deg_{k_n}(w)/\deg_{k_{n-1}}(w')$ equals 1 (resp. $p$) if $k_n/k_{n-1}$ ramified (resp. inert) at $w'$, and the cardinality of inert places in $S(k_{n-1})$ equals $\deg(\delta)$. The lemma also holds in these situations. \qed

**Definition 2.1.** Assume that $k_\infty/k$ is a $\mathbb{Z}_p$-extension and no place in $S$ splits completely under $k_\infty/k$. The Stickelberger element $\theta_{k_\infty,S,T}$ is called monomial if for some $\epsilon_2 \in \mathbb{Z}_p$
\[
\theta_{k_\infty,S,T} \equiv \epsilon_2 t^r \pmod{(t^{r+1})}.
\]
Lemma 2.4. Assume that \( k_{\infty}/k \) is a \( \mathbb{Z}_p \)-extension and no place in \( S \) splits completely under \( k_{\infty}/k \).

1. If the order of \( \text{Cl}_{k,S,T} \) is prime to \( p \) and there exists a unit \( \epsilon_2 \in \mathbb{Z}_p^\times \) such that
   \[
   \det_{k_{\infty}/k,S,T} \equiv \epsilon_2 \cdot t^r \pmod{(t^{r+1})},
   \]
   then \( \theta_{k_{\infty},S,T} \) is monomial.

2. If \( \theta_{k_{\infty},S,T} \) is monomial, then the Main Theorem holds.

Proof. The augmentation ideal \( I_{\Gamma,p} \) is just the ideal generated by \( t \). The statement (1) is a consequence of Theorem 2.2. If the Stickelberger element is monomial, then \( \theta_{k_{\infty}/k,S,T} = t^r \cdot \xi \), where the formal series \( \xi \) begins with the constant term which is contained in \( \mathbb{Z}_p^\times \). This means that \( \xi \) is itself a unit in \( \Lambda_{\Gamma} \). Lemma 2.3 says that \( \mu = 0 \) and \( \deg(f) + \deg(\delta) = r \). From equations (2.5), (1.4) and (1.5), we see that every prime factor of \( \mathfrak{w}\delta \) divides \( \theta_{k_{\infty}/k,S,T} \). Thus \( t \) is the only prime factor of both side. □

2.4. Order of the group \( \text{Cl}_{k,S,T} \).

Lemma 2.5. Let \( \tilde{S} \) be a finite set of places of \( k \) satisfying the following conditions:

1. The subgroup of \( \text{Cl}_k \) generated by the set of all the zero-degree divisors which are supported on \( \tilde{S} \) contains the \( p \)-Sylow subgroup \( \text{Cl}_{k,p} \) of the class group.

2. The greatest common divisor of the degrees of the places in \( \tilde{S} \) is one.

Then the order of the group \( \text{Cl}_{k,\tilde{S},T} \) is prime to \( p \).

Proof. Let \( X_{k,\tilde{S}} \subset \text{Div}_k^0 \) be the subgroup formed by divisors supported on \( \tilde{S} \). Use \( \text{Div}_{k,\tilde{S}} \) to denote the group of ideals of the ring \( \mathcal{O}_{\tilde{S}} \). Then we have the exact sequence

\[
0 \rightarrow X_{k,\tilde{S}} \xrightarrow{i} \text{Div}_k^0 \xrightarrow{\pi} \text{Div}_{k,\tilde{S}} \rightarrow 0
\]

(2.21)

with \( \pi \) taking \( \sum_v a_v \cdot v \) to \( \sum_{v \in \tilde{S}} a_v \cdot v \). The surjectivity of \( \pi \) is from the condition (2). Consequently, we have the induced exact sequence

\[
0 \rightarrow \overline{X}_{k,\tilde{S}} \xrightarrow{i} \text{Cl}_k \xrightarrow{\pi} \text{Cl}_{k,\tilde{S}} \rightarrow 0,
\]

(2.22)

where \( \overline{X}_{k,\tilde{S}} \) is the subgroup of \( \text{Cl}_k \) formed by divisor classes obtained from \( X_{k,\tilde{S}} \).

The condition (1) says that \( \tilde{i} \) is actually surjective on the \( p \)-part. Therefore \( \text{Cl}_{k,\tilde{S}} \) has order prime to \( p \) and hence so is \( \text{Cl}_{k,\tilde{S},T} \) (see (2.12) ). □

Corollary 2.2. There are infinitely many finite sets \( \tilde{S} \) of places of \( k \) with the following properties:

1. the order of the group \( \text{Cl}_{k,\tilde{S},T} \) is prime to \( p \),
2. the intersection \( \tilde{S} \cap T = \emptyset \),
3. \( \tilde{S} \nsubseteq S \) and no place in \( \tilde{S} \) splits completely over \( k_{\infty}/k \).

Proof. Suppose the greatest common divisor of the degrees of the places in \( S \) is \( N \). Let \( L/k \) be the constant field extension of degree \( N \). Choose a generator \( \tau \in \text{Gal}(L/k) \). Tchebotarev’s density theorem says that there is a place \( v \) of \( k \) outside \( S \cup T \) such that the element \( \tau \) equals the Frobenius \([v] \in \text{Gal}(L/k) \). We
know from the class field theory that if $Fr_q : x \mapsto x^q$ is the Frobenius substitution on the constant fields, then $Fr_q$ generate $Gal(L/k)$ and $[v] = Fr_{q^\deg(v)}$. Since $Fr_{q^\deg(v)}$ is also a generator of $Gal(L/k)$, the degree of $v$ must be relatively prime to $N$. Therefore the greatest common divisor of the degrees of places in $S' := S \cup \{v\}$ equals 1.

Consider the finite extension $L'/k$ which is the composite of all the everywhere unramified cyclic extensions of order $p$ and choose a set of generators $\tau'_1, ..., \tau'_s \in Gal(L'/k)$. Again, Tchebotarev’s density theorem says that there are places $v'_1, ..., v'_s$ outside $S' \cup T$ such that each $\tau'_i$ equals the Frobenius $[v'_i] \in Gal(L'/k)$. And we know from the class field theory that the Galois group $Gal(L'/k)$ is identified with $(\text{Div}_k / P_k) \otimes \mathbb{Z}/p\mathbb{Z}$ and the condition on $\tau'_1, ..., \tau'_s$ implies that the divisor classes of $v'_1, ..., v'_s$ generate $\text{Div}_k / P_k \otimes \mathbb{Z}/p\mathbb{Z}$. Therefore, the divisor classes of $v'_1, ..., v'_s$ generate the $\mathbb{Z}_p$-module $(\text{Div}_k / P_k) \otimes \mathbb{Z}_p$. Take $\tilde{S} = S' \cup \{v'_1, ..., v'_s\}$. Then the classes of all zero-degree divisors supported on $\tilde{S}$ generate $Cl_k \otimes \mathbb{Z}_p$. Therefore the conditions (1) and (2) of Lemma 2.5 are satisfied. Tchebotarev’s theorem also ensures us that the places $v, v'_1, ..., v'_s$ can be chosen so that none of them splits completely over $k_{\infty}/k$. □

2.5. Local norm residue maps. Let $\tilde{S} \supseteq S$ be a finite set of places and let $\mathbb{H}$ be the Galois group of the maximal pro-$p$ abelian extension over $k$ unramified outside $\tilde{S}$. It is known that ([Kis93, Tan95]) $\mathbb{H}$ is isomorphic, as a topological group, to a countable infinite product of $\mathbb{Z}_p$. This is actually due to the following simple Lemma which can be viewed as the function-field version of the local Leopoldt conjecture.

Lemma 2.6. If at some place $v$ a global element $a \in k^\times$ equals $b^p$ for some $b \in k_v^\times$, then $b \in k_\infty^\times$.

Proof. Since the field extension $k(a^{\frac{1}{p}})/k = k(a^{\frac{1}{p}}) \cap k_v/k$ is both purely inseparable and separable. □

Suppose that $K/k$ is a $\mathbb{Z}_p$-extension unramified outside $\tilde{S}$. Then choosing a topological generator of $Gal(K/k)$ is the same as choosing an isomorphism $Gal(K/k) \cong \mathbb{Z}_p$. Thus the extension $K/k$ together with a topological generator of $Gal(K/k)$ gives rise to a continuous homomorphism $\varphi : \mathbb{H} \rightarrow \mathbb{Z}_p$, and vice versa. In particular, taking $\tilde{S} = \{v\}$, we see that there exists a $\mathbb{Z}_p$-extension, associated to a $\varphi(v) : \mathbb{H} \rightarrow \mathbb{Z}_p$, which is ramified at $v$ and unramified at other places.

Lemma 2.7. There exists a $\mathbb{Z}_p$-extension over $k$, which is unramified outside $\tilde{S}$ but ramified at every place in $\tilde{S}$.

Proof. The $\mathbb{Z}_p$-extension associated to $\sum_{v \in \tilde{S}} \varphi(v)$ satisfies the required condition. □

Let $\tilde{r} + 1$ be the cardinality of $\tilde{S}$. Choose distinct places $v_1, ..., v_{\tilde{r}} \in \tilde{S}$. As in (2.14), for each $i$ let

$$\psi_{v_i, \mathbb{H}} : k_{v_i}^\times \rightarrow \mathbb{H}_{v_i} \subset \mathbb{H}$$
be the local norm residue map. Let \( \varphi_i \) be the composition of the natural embedding \( U_{k,S,T} \rightarrow k_v^\times \) with \( \psi_{v_i,H} \). Since \( H \) is a \( \mathbb{Z}_p \)-module and \( \psi_{v_i} \) is continuous, we can extend linearly \( \varphi_i \) to a homomorphism \( \varphi_i : \mathcal{U} \rightarrow H \) where \( \mathcal{U} := U_{k,S,T} \otimes \mathbb{Z}_p \).

**Lemma 2.8.** Let \( \mathcal{W} := \mathcal{U} \times \mathcal{U} \times \cdots \times \mathcal{U} \) be the direct sum of \( \tilde{r} \) copies of \( \mathcal{U} \). Then the homomorphism \( \Psi : \mathcal{W} \rightarrow H \) sending \((w_1, ..., w_{\tilde{r}}) \in \mathcal{W}\) to \( \sum_{i=1}^{\tilde{r}} \varphi_i(w_i) \in H \) is injective and the quotient group \( H/\Psi(\mathcal{W}) \) is torsion free.

**Proof.** We need to show that if \( w_1, ..., w_{\tilde{r}} \) are elements in \( U_{k,\tilde{S},T} \) with \( \sum_{i=1}^{\tilde{r}} \varphi_i(w_i) \) divisible by \( p \) in \( H \), then every \( w_i \) is divisible by \( p \) in \( U_{k,\tilde{S},T} \). Let \( x = (x_v)_v \in \tilde{A}_k^\times \) be the image such that \( x_v = w_i \) if \( v = v_i \); \( x_v = 1 \) otherwise. If \( \sum_{i=1}^{\tilde{r}} \varphi_i(w_i) \) is divisible by \( p \), then there are \( y \in \tilde{A}_k^\times \), \( \alpha \in k^\times \) and \( z \in \prod_{v \in \tilde{S}} \mathcal{O}_v^\times \) such that

\[
x = \alpha \cdot y^p \cdot z.
\]

As \( \tilde{S} \) contains \( \tilde{r} + 1 \) elements, there is a place \( v_0 \in \tilde{S} \setminus \{ v_1, ..., v_{\tilde{r}} \} \). Then we see from the equality (2.23) that \( \alpha \) is divisible by \( p \) in \( k_v^\times \), and hence by Lemma 2.6 there is an element \( \beta \in k^\times \) such that \( \alpha = \beta^p \). The equality (2.23) implies that each \( w_i \) is divisible by \( p \) in \( k_v^\times \) and hence also divisible by \( p \) in \( U_{k,\tilde{S},T} \). \( \square \)

The following is a consequence of Lemma 2.8 and the fact that \( H \) is the direct product of countable infinite many copies of \( \mathbb{Z}_p \).

**Corollary 2.3.** Let \( \tilde{S} \) be a set of places of \( k \) with \( \tilde{r} + 1 \) elements. If \( K/k \) is any given abelian extension unramified outside \( \tilde{S} \) with Galois group isomorphic to \( \mathbb{Z}_p^{d_0} \) for some non-negative integer \( d_0 \), then there exists a field extension \( L/K \) with the following properties:

1. \( L/k \) is also an abelian extension unramified outside \( \tilde{S} \) with Galois group isomorphic to \( \mathbb{Z}_p^c \) for some non-negative integer \( c \).

2. If \( u_1, ..., u_{\tilde{r}} \) is a \( \mathbb{Z} \)-basis of \( U_{k,\tilde{S},T} \) and \( \psi_{v_i,\text{Gal}(L/k)} \) is the local norm residue map at \( v_i \), then the subset \( \{ \psi_{v_i,\text{Gal}(L/k)}(u_i) \mid 1 \leq i, j \leq \tilde{r} \} \) of \( \text{Gal}(L/k) \) is linearly independent over \( \mathbb{Z}_p \) and it generates a direct summand of \( \text{Gal}(L/k) \).

2.6. **Independent extensions.** In this section, we show that if the set \( S \) and the field extension \( k_\infty \) are enlarged in a suitable way then the Stickelberger element will become irreducible.

**Definition 2.2.** Let \((K,S)\) be a pair where \( K/k \) is a \( \mathbb{Z}_p^{d_0} \)-extension unramified outside \( S \). A pair \((L,\tilde{S})\) is said to be an independent extension of \((K,S)\) if the following conditions hold:

1. \( \tilde{S} \) is a finite set of places of \( k \), satisfying Corollary 2.2.

2. The field extension \( L/k \) is a \( \mathbb{Z}_p^c \)-extension which ramifies at every place in \( \tilde{S} \) and satisfies Corollary 2.3.

**Lemma 2.9.**

1. There exist independent extensions \((L_\infty,\tilde{S})\) of \((k_\infty,S)\) with \( \tilde{S} \) arbitrarily large.

2. If \((L_\infty,\tilde{S})\) is an independent extension of \((k_\infty,S)\), then \( \theta_{L_\infty/k,\tilde{S},T} \) is irreducible in \( \Lambda_{\text{Gal}(L_\infty/k)} \).
(3) If \((L_{\infty}, \tilde{S})\) is an independent extension of \((k_{\infty}, S)\) then there is an intermediate \(\mathbb{Z}_p\)-extension \(k'_{\infty}/k\) of \(L_{\infty}/k\) such that \(k'_{\infty}/k\) ramifies at every place in \(\tilde{S}\) and the Stickelberger element \(\theta_{k'_{\infty}/k, \tilde{S}, T}\) is monomial.

**Proof.** The existence of arbitrarily large \(\tilde{S}\) follows from Corollary 2.2. Let \(K'/k\) be a \(\mathbb{Z}_p\)-extension unramified outside \(\tilde{S}\) but ramified at every place of \(\tilde{S}\) (see Lemma 2.7). Replace \(K\) by the composite \(K'K\) if necessary, and we can assume that \(K/k\) is ramified at every place of \(\tilde{S}\). Then the existence of \(L_{\infty}\) is from Corollary 2.3. This proves (1).

To prove (2) we first recall the notations in Corollary 2.3. For \(1 \leq i, j \leq \tilde{r}\), we set \(\sigma_{ij} = \psi_{\sigma_{ij}, \text{Gal}(L_{\infty}/k)}(u_i)\) and \(t_{ij} = \sigma_{ij} - 1\). Corollary 2.3 (2) says the set \(\{\sigma_{ij}\}_{1 \leq i, j \leq \tilde{r}}\) can be extended to a basis \(\{\sigma_1, ..., \sigma_r\}\) of \(\text{Gal}(L_{\infty}/k)\) over \(\mathbb{Z}_p\). If \(t_i = \sigma_i - 1\), then the augmentation quotient \(I_{\tilde{r}}^{\tilde{S}, T}/I_{\tilde{r}}^{1, T}\) can be identified with the \(\mathbb{Z}_p\)-module of \(\tilde{r}\)-degree homogeneous polynomials in \(t_1, ..., t_c\). The refined regulator \(\text{det}_{L_{\infty}/k, \tilde{S}, T}\) determines a residue class in the above augmentation quotient, and from (2.15) we see that this residue class is identified as the polynomial \(\text{det}(t_{ij})_{1 \leq i, j \leq \tilde{r}}\). It is well-known (see [Vas70]) that this polynomial is irreducible. Corollary 2.2 says that the order of \(\text{Cl}_{k, \tilde{S}, T}\) is prime to \(p\), and then Theorem 2.2 says that the Taylor expansion of \(\theta_{L_{\infty}, \tilde{S}, T} \in \mathbb{Z}_p[[t_1, ..., t_c]]\) begins with the irreducible polynomial \(|\text{Cl}_{k, \tilde{S}, T} \cdot \text{det}(t_{ij})|\) in \(\mathbb{Z}_p[t_1, ..., t_c]\). Suppose \(\theta_{L_{\infty}, \tilde{S}, T} = \theta_1 \theta_2\) in \(\mathbb{Z}_p[[t_1, ..., t_c]]\) and the Taylor expansions of \(\theta_1\) and \(\theta_2\) begin with the leading homogeneous polynomials \(\vartheta_1, \vartheta_2 \in \mathbb{Z}_p[t_1, ..., t_c]\). Then the product \(\vartheta_1 \vartheta_2\) is irreducible in \(\mathbb{Z}_p[t_1, ..., t_c]\). Therefore one of them, say \(\vartheta_1\) must be in the units group \(\mathbb{Z}_p^\times\) of the polynomial ring \(\mathbb{Z}_p[t_1, ..., t_c]\). This implies that \(\vartheta_1\), beginning with a unit in its Taylor expansion, must be a unit in \(\mathbb{Z}_p[t_1, ..., t_c]\). This argument shows that \(\theta_{L_{\infty}, \tilde{S}, T}\) is an irreducible element in \(\Lambda_{\text{Gal}(L_{\infty}/k)}\).

The \(\mathbb{Z}_p\)-module \(\text{Hom}_{\mathbb{Z}_p}(\text{Gal}(L_{\infty}/k), \mathbb{Z}_p)\) is isomorphic to \(\mathbb{Z}_p^c\). Every element in it is a continuous map with respect to the pro-finite topologies on \(\text{Gal}(L_{\infty}/k)\) and \(\mathbb{Z}_p\). Also, the pro-finite topology on \(\mathbb{Z}_p\) coincides with the compact-open topology on \(\text{Hom}_{\mathbb{Z}_p}(\text{Gal}(L_{\infty}/k), \mathbb{Z}_p)\). Since the subset \(\mathbb{Z}_p^\times\) is open in \(\mathbb{Z}_p\), the subset \(O\) of the group \(\text{Hom}_{\mathbb{Z}_p}(\text{Gal}(L_{\infty}/k), \mathbb{Z}_p)\) consisting of those \(\psi\) such that

\[
\text{det}_{\psi} := \text{det}(\psi(\sigma_{ij}))_{1 \leq i,j \leq \tilde{r}} \in \mathbb{Z}_p^\times
\]

is open in \(\text{Hom}_{\mathbb{Z}_p}(\text{Gal}(L_{\infty}/k), \mathbb{Z}_p)\). Since \(\{\sigma_{ij}\}_{1 \leq i,j \leq \tilde{r}}\) is a subset of the basis \(\{\sigma_1, ..., \sigma_r\}\), there is at least one \(\psi\) such that \(\psi(\sigma_i) = 1\) for \(i = 1, ..., \tilde{r}\), and \(\psi(\sigma_{ij}) = 0\) for \(i \neq j\). And for this \(\psi\) the determinant \(\text{det}_{\psi} = 1\). Therefore, the open set \(O \neq \emptyset\). At each place \(v \in \tilde{S}\), let \(O(v) \subset \text{Hom}_{\mathbb{Z}_p}(\text{Gal}(L_{\infty}/k), \mathbb{Z}_p)\) be the subset consisting of those \(\psi\) whose restriction to the inertia subgroup at \(v\) is non-zero. Since this inertia subgroup is non-trivial, the set \(O(v)\) is a nonempty open subset of \(\text{Hom}_{\mathbb{Z}_p}(\text{Gal}(L_{\infty}/k), \mathbb{Z}_p)\).

Let \(\psi \in \bigcap_{v \in \tilde{S}} O(v) \cap O\) and let \(k'_{\infty}\) be the fixed field of the kernel of \(\psi\). Then \(k'_{\infty}/k\) is ramified at each place in \(\tilde{S}\) and \(\theta_{k'_{\infty}/k, \tilde{S}, T}\) is monomial (Lemma 2.4). \(\square\)

3. The maximal pseudo-null sub-module

Suppose that \(\mathcal{R}_{\infty}/k\) is a \(\mathbb{Z}_p\)-extension with \(\text{Gal}(\mathcal{R}_{\infty}/k) = Y\) and \(\Xi\) is a rank one \(\mathbb{Z}_p\)-submodule of \(Y\) with \(Y/\Xi \simeq \mathbb{Z}_p^{-1}\). Let \(\mathcal{R}_{\Xi}\) be the fixed field of \(\Xi\) and let
pr : \Lambda_T \longrightarrow \Lambda_T/\mathbb{Z} be as usual the natural projection. We shall find a relation between \( pr(\chi_T(Cl_{\kappa_{\infty,p}})) \) and \( \chi_T/\mathbb{Z}(Cl_{\kappa_{\infty,p}}) \) by using the fact that the characteristic ideals are multiplicative (see [Bou72], Chap. 7, Sec. 4.5).

Fix an element \( s \in \Lambda_\Xi \) such that \( \xi := s + 1 \) is a topological generator of \( \Xi \). For an abelian Galois group \( G \), we use \( G_v, G^0_v \) to denote the decomposition subgroup and the inertia subgroup at a place \( v \). We choose a set \( \mathcal{S} \) of places of \( k \) so that \( \kappa_{\infty/k} \) is unramified outside of \( \mathcal{S} \) and no place in \( \mathcal{S} \) splits completely in \( \kappa_{\infty/k} \). Let \( \mathcal{S}_1 = \{ v \in \mathcal{S} \mid \mathcal{Y}_v \cap \Xi \neq \{0\} \} \). Then \( \kappa_{\infty/\Xi} \) is unramified outside \( \mathcal{S}_1 \). We set \( \mathcal{S}_2 = \{ v \in \mathcal{S}_1 \mid |\mathcal{Y}_v/\mathcal{Y}_v^0| < \infty \} \). Thus, a place \( v \in \mathcal{S} \) is in \( \mathcal{S}_2 \) if and only if the extension \( \kappa_{\infty/\Xi} \) is ramified at every place sitting over \( v \) and the corresponding residue field extension for \( \kappa_{\infty/k} \) is of finite degree.

3.1. The group \( \mathcal{M}_{\kappa_{\infty,p}} \). For a global function field \( K \) let \( A_K^\times \) be the idele group of \( K \). Write \( \mathcal{M}_K \) for the group \( \mathbb{K}^\times/\prod_v \mathcal{O}_{K,v}^\times \). Let \( \mathcal{M}_{K,p} \) be the \( p \)-completion of \( \mathcal{M}_K \). We define \( \mathcal{M}_{\kappa_{\infty,p}} \) as the projective limit

\[
\mathcal{M}_{\kappa_{\infty,p}} = \lim_{\longrightarrow} \mathcal{M}_{K,p},
\]

where \( K \) runs through all finite intermediate extensions of \( \kappa_{\infty/k} \).

The group \( \mathcal{M}_K \) is the group of divisor classes \( \text{Div}_K/\text{P}_K \). We have the exact sequence \( 0 \rightarrow \text{Cl}_K \longrightarrow \mathcal{M}_K \xrightarrow{\text{deg}_K} \mathbb{Z} \longrightarrow 0 \), where \( \text{deg}_K \) is the degree map. From the class field theory we see that \( \mathcal{M}_K \) is a dense subgroup of the Galois group of the maximal unramified abelian extension of \( K \) and the fixed field of \( \text{Cl}_K \) is the maximal constant field extension of \( K \). Therefore the \( p \)-completion \( \mathcal{M}_{K,p} \) of \( \mathcal{M}_K \) is the Galois group of the maximal unramified pro-\( p \) abelian extension of \( K \) and the fixed field of \( \text{Cl}_{K,p} \) is the constant \( \mathbb{Z}_p \)-extension of \( K \). Since we also have \( \mathcal{M}_{K,p} = \mathcal{M}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p \), an element in \( \mathcal{M}_{K,p} \) can be viewed as an equivalent class of \( \mathbb{Z}_p \)-divisors, which are \( \mathbb{Z}_p \) linear combinations of places in \( K \), where two such divisors are equivalent if and only if they are \( \mathbb{Z}_p \)-linear equivalent in the sense that they differ by a divisor of some element in the \( p \)-completion \( \mathbb{K}^\times \) of the multiplicative group \( K^\times \). We should note that if \( \text{P}_K \) is the \( p \)-completion of the group \( \text{P}_K \), then from the exact sequence

\[
0 \longrightarrow \mathbb{F}_K^\times \longrightarrow \mathbb{K}^\times \longrightarrow \text{P}_K \longrightarrow 0
\]

and the fact that \( |\mathbb{F}_K^\times| \) is prime to \( p \), we get an isomorphism \( \mathbb{K}^\times \simeq \text{P}_K \).

If \( K' \) is another finite intermediate field containing \( K \), then we have the commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Cl}_{K',p} & \longrightarrow & \mathcal{M}_{K',p} & \xrightarrow{\text{deg}_{K'}} \mathbb{Z}_p & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Cl}_{K,p} & \longrightarrow & \mathcal{M}_{K,p} & \xrightarrow{\text{deg}_K} \mathbb{Z}_p & \longrightarrow & 0
\end{array}
\]

where the two left down arrows are norm maps and the right down arrow is the multiplication by \( [F_{K'} : F_K] \). The diagram is commutative because for a divisor \( D \) of \( K' \) we have \( \text{deg}_K(\mathcal{N}_{K'/K}(D)) = [F_{K'} : F_K] \cdot \text{deg}_{K'}(D) \). Taking projective limit, we get the exact sequence

\[
0 \longrightarrow \text{Cl}_{\kappa_{\infty,p}} \longrightarrow \mathcal{M}_{\kappa_{\infty,p}} \xrightarrow{\iota} \mathbb{Z}_p
\]
where \( \dagger \) is induced from the degree maps.

**Lemma 3.1.** The map \( \dagger \) is the zero map if the constant \( \mathbb{Z}_p \)-extension \( \mathbb{F}_{p^\infty} \) is contained in \( \mathfrak{R}_\infty \); otherwise the map is surjective.

**Proof.** Suppose \( K' \) and \( K \) are the \( m \)'th and the \( n \)'th layers of \( \mathfrak{R}_\infty \), with \( m > n \gg 0 \). If the constant \( \mathbb{Z}_p \)-extension is contained in \( \mathfrak{R}_\infty \) then we have \( [\mathbb{F}_{K'} : \mathbb{F}_K] = p^{m-n} \). Since \( m \) can be arbitrary large, we must have \( \dagger = 0 \). On the other hand, if the constant \( \mathbb{Z}_p \)-extension is not contained in \( \mathfrak{R}_\infty \), then \( [\mathbb{F}_{K'} : \mathbb{F}_K] = 1 \) and hence \( \dagger \) is surjective. \( \square \)

**Corollary 3.1.** The group \( \mathfrak{M}_{\mathfrak{R}_\infty,p} \) is a torsion finitely generated module over \( \Lambda_\Gamma \). If \( e = 1 \) and \( \mathfrak{R}_\infty \) is not a constant field extension, then

\[
\chi_\Gamma(\mathfrak{M}_{\mathfrak{R}_\infty,p}) = \chi_\Gamma(\text{Cl}_{\mathfrak{R}_\infty,p}) \cdot (s);
\]

otherwise, we have

\[
\chi_\Gamma(\mathfrak{M}_{\mathfrak{R}_\infty,p}) = \chi_\Gamma(\text{Cl}_{\mathfrak{R}_\infty,p}).
\]

**Proof.** Suppose we have chosen a basis of \( \Upsilon \) and use it to identify \( \Lambda_\Gamma \) with the formal power series ring \( \mathbb{Z}_p[[t_1, \ldots, t_e]] \). Then the \( \Lambda_\Gamma \)-module \( \mathbb{Z}_p \) is isomorphic to \( \Lambda_\Gamma/(t_1, \ldots, t_e) \), which is pseudo-null unless \( e = 1 \). If \( e = 1 \), then \( \Upsilon = \Xi \) and \( \chi_\Gamma(\mathbb{Z}_p) = (s) \). \( \square \)

To find the characteristic ideal of \( \mathfrak{M}_{\mathfrak{R}_\infty,p} \) we consider a pseudo-isomorphism

\[
\phi : \mathfrak{M}_{\mathfrak{R}_\infty,p} \rightarrow \bigoplus_{j=1}^{\alpha} \Lambda_\Gamma/(\zeta_j^{m_j})
\]

where each \( \zeta_j \) is a prime element in \( \Lambda_\Gamma \). Since any annihilator of a given element in \( \Lambda_\Gamma/(\zeta_j^{m_j}) \) must be inside the prime ideal \( (\zeta_j) \) which is of height one, the module \( \Lambda_\Gamma/(\zeta_j^{m_j}) \) contains no non-trivial pseudo-null sub-module. Thus the kernel of \( \phi \) is the maximal pseudo-null submodule of \( \mathfrak{M}_{\mathfrak{R}_\infty,p} \). We use \( \mathcal{N}_{\mathfrak{R}_\infty} \) to denote it.

We have chosen \( s \in \Lambda_\Xi \) such that \( \xi := s + 1 \) is a topological generator of \( \Xi \). Since \( \mathcal{N}_{\mathfrak{R}_\infty} \) is the maximal pseudo-null submodule, we must have \( \mathcal{N}_{\mathfrak{R}_\infty} \cap s\mathfrak{M}_{\mathfrak{R}_\infty,p} = s\mathcal{N}_{\mathfrak{R}_\infty} \), and hence

\[
\mathcal{N}_{\mathfrak{R}_\infty} / s\mathcal{N}_{\mathfrak{R}_\infty} \hookrightarrow \mathfrak{M}_{\mathfrak{R}_\infty,p} / s\mathfrak{M}_{\mathfrak{R}_\infty,p}.
\]

**3.2.** The \( \Xi \)-invariant part of \( \mathfrak{M}_{\mathfrak{R}_\infty,p} \). For each open subgroup \( \alpha \subset \Upsilon \) let \( \mathfrak{R}_\alpha \) be the fixed field of \( \alpha \). Then \( \mathfrak{R}_\infty \) is nothing but the \( n \)'th layer \( \mathfrak{R}_n \). And the sub-system \( \{ \mathfrak{R}_n \}_{n} \) is cofinal in the system \( \{ \mathfrak{R}_\alpha \}_{\alpha} \). For an element \( x \in \mathfrak{M}_{\mathfrak{R}_\infty,p} \), we use \( x_\alpha \in \mathfrak{M}_{\mathfrak{R}_\alpha,p} \) and \( x_n \in \mathfrak{M}_{\mathfrak{R}_n,p} \) to denote its images under the corresponding natural maps.

**Lemma 3.2.** If \( x \in \mathfrak{M}_{\mathfrak{R}_\infty,p} \), then for each open subgroup \( \alpha \) of \( \Upsilon \) the divisor class \( x_\alpha \) is represented by a \( \Xi \)-invariant divisor of \( \mathfrak{R}_\alpha \).

**Proof.** For an open subgroup \( \alpha \), let \( D_\alpha \) be a \( \mathbb{Z}_p \)-divisor of \( \mathfrak{R}_\alpha \) representing the class \( x_\alpha \). Then there is an \( a_\alpha \) in the \( p \)-completion \( \tilde{\mathfrak{R}}_\alpha^\times \) of \( \mathfrak{R}_\alpha^\times \) such that \( (a_\alpha) = \tilde{\xi} D_\alpha - D_\alpha \). (\( \xi := s + 1 \) is a topological generator \( \Xi \).) This means that the image of \( a_\alpha \) under the norm map from \( \mathfrak{R}_\alpha^\times \) to \( \mathfrak{R}_{\alpha+\Xi}^\times \) gives rise to a trivial element of \( \mathfrak{P}_{\mathfrak{R}_\alpha+\Xi} \). As we have...
observed that $\tilde{\mathcal{R}}_{n+z}$ and $\mathcal{P}_{\tilde{\mathcal{R}}_{n+z}}$ are isomorphic this norm is the trivial element in $\tilde{\mathcal{R}}_{n+z}$. By Hilbert’s Theorem 90 there is a element $b_\alpha$ in the $p$-completion $\tilde{\mathcal{R}}_{\alpha}$ such that $(a_\alpha) = (b_\alpha)$. This shows that $x_\alpha$ is represented by the divisor $D_\alpha - (b_\alpha)$ which is invariant by the action of $\Xi$.

\textbf{Lemma 3.3.} If $x \in \mathbb{W}_{\tilde{\mathcal{R}}_{n,p}}$, then for each open subgroup $\alpha$ of $\Upsilon$ the divisor class $x_\alpha$ is represented by a $\Xi$-invariant divisor which is supported on the set $\mathcal{S}_1(\tilde{\mathcal{R}}_\alpha)$ consisting of places of $\tilde{\mathcal{R}}_\alpha$ sitting over $\mathcal{S}_1$.

\textit{Proof.} Choose $v_0 \in \mathcal{S}$ such that $v_0 \in \mathcal{S}_1$ if $\mathcal{S}_1 \neq \emptyset$. Suppose that $v \in \mathcal{S}$ is outside $\mathcal{S}_1$ and $\mathcal{S}' := \mathcal{S} \setminus \{v\}$. Then we set a basis $\sigma_1 = \xi, \sigma_2, ..., \sigma_e$ of $\Upsilon$ over $\mathbb{Z}_p$ such that $\mathcal{Y}_v^0 \subset \mathbb{Z}_p\sigma_2 + \cdots + \mathbb{Z}_p\sigma_e$. For simplicity, denote

$$(m, n) = p^m\mathbb{Z}_p\sigma_1 + p^n(\mathbb{Z}_p\sigma_2 + \cdots + \mathbb{Z}_p\sigma_e).$$

Then we have $\tilde{\mathcal{R}}_n = \tilde{\mathcal{R}}_{(m,n)}$. If $m \geq n$, then

$$\text{Gal}(\tilde{\mathcal{R}}_{(m,n)}/\tilde{\mathcal{R}}_n) \simeq \frac{(n, n)}{(m, n)} \simeq \frac{p^n\mathbb{Z}_p}{p^m\mathbb{Z}_p} \simeq \mathbb{Z}/\mathbb{Z}. $$

is a cyclic group generated by the restriction of $\xi$ on $\tilde{\mathcal{R}}_{(m,n)}$. Since $\mathcal{Y}_v^0 \cap (n, n) = \mathcal{Y}_v \cap (m, n)$, the extension $\tilde{\mathcal{R}}_{(m,n)}/\tilde{\mathcal{R}}_n$ is unramified at every place sitting over $v$ and hence unramified outside $\mathcal{S}'(\tilde{\mathcal{R}}_n)$.

There is a natural embedding $i : \text{Div}(\tilde{\mathcal{R}}_n) \otimes \mathbb{Z}\mathbb{P} \hookrightarrow \text{Div}(\tilde{\mathcal{R}}_{(m,n)}) \otimes \mathbb{Z}\mathbb{P}$, sending a place (a prime divisor) $w_0$ of $\tilde{\mathcal{R}}_n$ to the divisor $\sum_{w|w_0} e(w)w$ of $\tilde{\mathcal{R}}_{(m,n)}$, where $w$ runs through places of $\tilde{\mathcal{R}}_{(m,n)}$ sitting over $w_0$ and $e(w)$ is the ramification index of $w$ over $w_0$. In particular, we have $e(w) = 1$ if $w \notin \mathcal{S}_1(\tilde{\mathcal{R}}_{(m,n)})$. Let $E = \sum_{w \notin \mathcal{S}_1(\tilde{\mathcal{R}}_{(m,n)})} a(w)w$ be a $\Xi$-invariant $\mathbb{Z}_p$-divisor of $\tilde{\mathcal{R}}_{(m,n)}$ supported outside $\mathcal{S}_1(\tilde{\mathcal{R}}_{(m,n)})$. Then the action of the Galois group $\text{Gal}(\tilde{\mathcal{R}}_{(m,n)}/\tilde{\mathcal{R}}_n)$, which is the restriction of the action of $\Xi$, fixes $E$ and we have $a(w) = a(w')$ for $w, w'$ sitting over the same place of $\tilde{\mathcal{R}}_n$. Therefore, $E$ is in the image of the natural embedding $i$. Also, since $E$ is fixed by the action of the Galois group $\text{Gal}(\tilde{\mathcal{R}}_{(m,n)}/\tilde{\mathcal{R}}_n)$ the norm $N_{\tilde{\mathcal{R}}_{(m,n)}/\tilde{\mathcal{R}}_n}(E)$ just equals $p^{m-n}E$. Let $D_{(m,n)}$ be a $\Xi$-invariant $\mathbb{Z}_p$-divisor of $\tilde{\mathcal{R}}_{(m,n)}$ representing $x_{(m,n)}$, and put $D_{(m,n)} = D_{(m,n)}^{(1)} + D_{(m,n)}^{(2)}$ where $D_{(m,n)}^{(1)}$ is supported on $\mathcal{S}'(\tilde{\mathcal{R}}_{(m,n)})$ and $D_{(m,n)}^{(2)}$ is supported outside $\mathcal{S}'(\tilde{\mathcal{R}}_{(m,n)})$. Both $D_{(m,n)}^{(1)}$ and $D_{(m,n)}^{(2)}$ are $\Xi$-invariant. Put $E = D_{(m,n)}^{(2)}$, and from the above discussion we see that if $D_n = N_{\tilde{\mathcal{R}}_{(m,n)}/\tilde{\mathcal{R}}_n}(D_{(m,n)})$ then

$$D_n = D_{(1)}^{(1)} + p^{m-n}E,$$

where $D_{(1)}^{(1)}$ is a $\Xi$-invariant divisor of $\tilde{\mathcal{R}}_n$ supported on $\mathcal{S}'(\tilde{\mathcal{R}}_n)$.

Let $\iota, \zeta \in \mathbb{Z}$ and $\kappa \in \mathbb{Z}'$ be such that $\deg(v_0) = \kappa \cdot p' \cdot \deg_{\mathcal{R}_{0,n,p}}(E)$. Since these numbers are independent of the choice of $m$. The divisor $\kappa p' E - \deg_{\tilde{\mathcal{R}}_{0,n,p}}(E) \cdot v_0$ is of degree zero, and hence its multiple by $p'$ is in the trivial divisor class. Thus, there is an element $a$ in the $p$-completion $\tilde{\mathcal{R}}_{\alpha}$ such that

$$a = p^\zeta(\kappa p'E - \deg_{\tilde{\mathcal{R}}_{0,n,p}}(E) \cdot v_0).$$

(3.7)
If $m$ is chosen to be greater than the integer $n + \iota + \zeta$, then from (3.6) and (3.7) we find that the divisor $D_n$ is $\mathbb{Z}_p$-linearly equivalent to a $\mathcal{G}'(\mathcal{R}_n)$-supported divisor which is also invariant under the action of $\Xi$.

We can replace $\mathcal{G}$ by $\mathcal{G}'$ and repeat the above argument, if necessary. In the case where $\mathcal{G}_1$ is non-empty, this will lead to the conclusion that $x_\iota$ is represented by a $\Xi$-invariant divisor supported on $\mathcal{G}_1(\mathcal{R}_n)$ and hence the proof is completed. If $\mathcal{G}_1 = \emptyset$, then the above method shows that $x_\iota$ is represented by a $\Xi$-invariant divisor supported on places sitting over $v_0$. We then apply the above argument again by taking $\mathcal{G} = \{v_0\}$ and $v = v_0$. This time in the equation (3.6) the divisor $D^{(1)}$ is trivial and $D_n$ is divisible by $p^{m-n}$. Since $m$ can be chosen arbitrary large, the class $x_\iota$ is $p$-divisible in $\mathcal{M}_\mathcal{R}_n$, which is a finite $\mathbb{Z}_p$-module. Therefore, we must have $x_\iota = 0$. The proof is completed.

Using a similar method, we can make some further reduction.

**Lemma 3.4.** If $x \in \mathcal{M}_{\mathcal{R}_n,p}$, then for each open subgroup $\alpha \subset \Upsilon$ the divisor class $x_\alpha$ is represented by a $\Xi$-invariant divisor which is supported on $\mathcal{G}_2(\mathcal{R}_n)$.

**Proof.** Choose $v_0$ in $\mathcal{G}_1$ such that $v_0 \in \mathcal{G}_2$ if $\mathcal{G}_2 \neq \emptyset$. For $v \in \mathcal{G}_1 \setminus \mathcal{G}_2$, we choose a basis $\sigma_1, \ldots, \sigma_e$ of $\Upsilon$ such that $\Upsilon_0 \subset \mathbb{Z}_p \sigma_2 + \cdots + \mathbb{Z}_p \sigma_e$ and the index

$$[\Upsilon_0 \cap (\mathbb{Z}_p \sigma_2 + \cdots + \mathbb{Z}_p \sigma_e) : \Upsilon_0] < \infty.$$ 

Then under the natural projection

$$\Upsilon \twoheadrightarrow \Upsilon/\mathbb{Z}_p \sigma_2 + \cdots + \mathbb{Z}_p \sigma_e$$

the image of $\Upsilon_0$ must be nontrivial. In particular, it is generated by $p^{n'} \bar{\sigma}_1$ for some $n'$. Define the subgroup $(m, n)$ of $\Upsilon$ as in the proof of the previous lemma. Since we have $\Upsilon_0 \cap (m, n) = \Upsilon_0 \cap (0, n')$ the extension $\mathcal{R}_n(m, n)/\mathcal{R}_n(0, n')$ is unramified at $v$. Its Galois group is generated by the restriction of $\bar{\sigma}_1$ to $\mathcal{R}_n(m, n)$ and the decomposition subgroup is generated by the restriction of $p^{n'} \bar{\sigma}_1$. We also consider the field extension $\mathcal{R}_n(m, n)/\mathcal{R}_n$. This extension is unramified at $v$, since $\mathcal{R}_n(0, n')$ is a sub-field of $\mathcal{R}_n$. If $m > n > n'$, then for every place of $\mathcal{R}_n$ sitting over $v$ there is only one place of $\mathcal{R}_n(m, n)$ sitting over it. From this we make the key observation that if $E$ is a divisor of $\mathcal{R}_n(m, n)$ supported on places sitting over $v$ then $E$ is in fact a divisor of $\mathcal{R}_n$. In this case, we have $N_{\mathcal{R}_n(m, n)/\mathcal{R}_n}(E) = p^{m-n}E$.

Now $x_{(m, n)}$ is represented by a $\Xi$-invariant divisor $D_{(m, n)}$ in $\mathcal{R}_n(m, n)$, which is supported on $\mathcal{G}_1(\mathcal{R}_n(m, n))$. Let $E$ be the part of $D_{(m, n)}$ supported on places sitting over $v$ and let $D_{(m, n)}^{(1)} = D_{(m, n)} - E$ which is supported on $\mathcal{G}_1'(\mathcal{R}_n(m, n))$ where $\mathcal{G}_1' := \mathcal{G}_1 \setminus \{v\}$. Then both $E$ and $D_{(m, n)}^{(1)}$ are $\Xi$-invariant. Let $D_{(m, n)}^{(1)}$ be the image of $D_{(m, n)}^{(1)}$ under the norm map from $\mathcal{R}_n(m, n)$ to $\mathcal{R}_n$. Then $D_{(m, n)}^{(1)}$ is a divisor of $\mathcal{R}_n$ supported on $\mathcal{G}_1'(\mathcal{R}_n)$ and

$$D_n := N_{\mathcal{R}_n(m, n)/\mathcal{R}_n}(D_{(m, n)}) = D_{(m, n)}^{(1)} + p^{m-n}E. \quad (3.8)$$

Then we finish the proof in the same way as the last part of the proof of Lemma 3.3.
3.3. Some special modules. We will express the module \( \mathcal{M}^\mathbb{Z}_{\mathcal{R}_\infty} \) in terms of some special modules. Suppose that \( v \in \mathcal{S}_1 \). For each open subgroup \( \alpha \subset \Upsilon \), let \( (\Upsilon/\alpha)_v \) be the decomposition sub-group of \( \Upsilon/\alpha = \text{Gal}(\mathcal{R}_\alpha/k) \). Put

\[
\mathcal{G}_{\Upsilon/\alpha,v} = \Lambda_{\Upsilon/\alpha}/\mathfrak{S}_{\Upsilon/\alpha,v},
\]

where \( \Lambda_{\Upsilon/\alpha} = \mathbb{Z}_p[\Upsilon/\alpha] \) and \( \mathfrak{S}_{\Upsilon/\alpha,v} \) is the ideal generated by the set of all \( \sigma - 1 \) with \( \sigma \in (\Upsilon/\alpha)_v + (\Xi + \alpha)/\alpha \).

For \( \alpha \subset \beta \) we have the obvious homomorphism \( \rho_{\alpha,\beta,v} : \mathcal{G}_{\Upsilon/\alpha,v} \to \mathcal{G}_{\Upsilon/\beta,v} \) which we use to form the projective limit

\[
\mathcal{G}_v = \lim_{\alpha \in \mathcal{S}_{\Upsilon/\alpha,v}} \mathcal{G}_{\Upsilon/\alpha,v}.
\]

We have \( \mathcal{G}_v = \Lambda_{\Upsilon}/\mathfrak{S}_v \), where the ideal \( \mathfrak{S}_v \) is generated by the set \( \{ \sigma - 1 \mid \sigma \in \Upsilon_v + \Xi \} \).

The group of \( \Xi \)-invariant divisors which are supported on \( \mathcal{G}_v(\mathcal{R}_\alpha) \) can be easily determined. For each \( v \in \mathcal{S}_2 \), we choose a place \( v_{(\infty)} \) of \( \mathcal{R}_\infty \) sitting over \( v \), and for each \( \alpha \) let \( v_\alpha \) be the place of \( \mathcal{R}_\alpha \) sitting below \( v_{(\infty)} \). The orbit \( \mathcal{T}_{\alpha,v} \) of \( v_\alpha \) under the action of \( \Xi \) is finite. Define the divisor \( D_{\alpha,v} = \sum_{w \in \mathcal{T}_{\alpha,v}} w \). Under the action of \( \Upsilon/\alpha \) the stabilizer of \( D_{\alpha,v} \) is the subgroup \( (\Upsilon/\alpha)_{v_\alpha} + (\Xi + \alpha)/\alpha \). Then every \( \Xi \)-invariant divisor of \( \mathcal{R}_\alpha \) supported on places sitting over \( v \) can be express as \( \bar{y}_\alpha D_{\alpha,v} \) for some \( \bar{y}_\alpha \) in the ring \( \mathcal{G}_{\Upsilon/\alpha} \). The assignment sending \( 1 \) to \( [D_{\alpha,v}] \), the divisor class of \( D_{\alpha,v} \), induces a \( \Lambda_{\Upsilon/\alpha} \)-homomorphism \( \varphi_{\alpha,v} : \mathcal{G}_{\Upsilon/\alpha,v} \to \mathcal{M}_p^{\mathbb{Z}}_{\mathcal{R}_\alpha,p} \). Taking projective limit, we get a \( \Lambda_{\Upsilon} \)-homomorphism \( \varphi_v : \mathcal{G}_v \to \mathcal{M}_p^{\mathbb{Z}}_{\mathcal{R}_\alpha,p} \).

Lemma 3.5. Then map

\[
\varphi = \sum_{v \in \mathcal{S}_2} \varphi_v : \bigoplus_{v \in \mathcal{S}_2} \mathcal{G}_v \to \mathcal{M}_p^{\mathbb{Z}}_{\mathcal{R}_\alpha,p}
\]

is an isomorphism.

Proof. Lemma 3.4 implies that \( \varphi \) is surjective. Suppose that \( \bar{y} = \sum_{v \in \mathcal{S}_2} \bar{y}_v \) is in the kernel of \( \varphi \). Let \( \rho_{\alpha,v} : \mathcal{G}_v \to \mathcal{G}_{\Upsilon/\alpha,v} \) be the natural map, and denote the image \( \rho_{\alpha,v}(\bar{y}_v) \) as \( \bar{y}_{\alpha,v} \). We lift it through (3.9) to an element \( y_{\alpha,v} \) of the group ring \( \Lambda_{\Upsilon/\alpha} \). Then the divisor \( E_{\alpha,v} = \sum_{v \in \mathcal{S}_2} y_{\alpha,v} D_{\alpha,v} \) is \( \mathbb{Z}_p \)-linear equivalent to zero. We shall note that \( E_{\alpha} \) is independent of the choice of the above lifting. To prove the injectivity of \( \varphi \), we only need to show that every \( E_{\alpha} \) is the trivial divisor, since this will imply that each \( y_{\alpha,v} D_{\alpha,v} \) is trivial and hence \( y_{\alpha,v} \) is in the ideal \( \mathfrak{S}_{\Upsilon/\alpha,v} \).

For \( \alpha = p^n \Upsilon \) with \( n \) large enough, each of the intersections \( \Upsilon_v \cap \alpha, v \in \mathcal{S}_2 \), contains \( \Xi \cap \alpha \) which is a direct summand of \( \alpha \). For a large \( m \) let \( \beta \subset \alpha \) be such that the natural map \( \Xi \cap \alpha \to \alpha/\beta \) is surjective with kernel equal \( p^m(\Xi \cap \alpha) \). Then \( \mathcal{R}_\beta/\mathcal{R}_\alpha \) is a cyclic extension of degree \( p^m \) and the decomposition subgroup at each \( v_\alpha \), \( v \in \mathcal{S}_2 \), is the whole Galois group \( \text{Gal}(\mathcal{R}_\beta/\mathcal{R}_\alpha) \). There are \( a_\alpha \in U_\alpha := \mathcal{O}^\times_{\mathcal{R}_\alpha,\mathcal{S}_2} \otimes \mathbb{Z}_p \), and \( a_\beta \in U_\beta := \mathcal{O}^\times_{\mathcal{R}_\beta,\mathcal{S}_2} \otimes \mathbb{Z}_p \) (the units groups \( \mathcal{O}^\times_{\mathcal{R}_\alpha,\mathcal{S}_2} \) and \( \mathcal{O}^\times_{\mathcal{R}_\beta,\mathcal{S}_2} \) are as those defined in Section 2.2) such that \( E_\alpha = (a_\alpha) \) and \( E_\beta = (a_\beta) \). Now \( U_\alpha \) and \( U_\beta \) are of the same rank over \( \mathbb{Z}_p \). If \( U_\alpha \subsetneq U_\beta \), then there would be an \( u \in \mathcal{O}^\times_{\mathcal{R}_\alpha,\mathcal{S}_2} \setminus \mathcal{O}^\times_{\mathcal{R}_\beta,\mathcal{S}_2} \) such that \( u_\alpha := u^\alpha \in \mathcal{O}^\times_{\mathcal{R}_\alpha,\mathcal{S}_2} \). But this means that \( u \in \mathcal{R}_\alpha(u_1^\frac{1}{p}) \cap \mathcal{R}_\beta = \mathcal{R}_\alpha \), since \( \mathcal{R}_\alpha(u_1^\frac{1}{p})/\mathcal{R}_\alpha \) is purely inseparable while \( \mathcal{R}_\beta/\mathcal{R}_\alpha \) is separable. This would lead to the
contradiction that $u \in \mathcal{O}_{\mathcal{R}_\infty}^\times$. Therefore $U_\alpha = U_\beta$ and this implies that $E_\beta$ is a divisor of $\mathcal{R}_\alpha$. Since $E_\alpha$ is the norm of the divisor $E_\beta$ and hence equals $p^mE_\beta$, we conclude that $E_\alpha$ and $a_\alpha$ are divisible by $p^m$. As $m$ can be arbitrary large, in the finite $\mathbb{Z}_p$-module $U_\alpha$ the element $a_\alpha$ must be trivial. Therefore we have $E_\alpha = 0$. □

**Corollary 3.2.** If no place in $\mathcal{S}$ splits completely in $\mathcal{R}_\infty$, then $\mathcal{W}_{\mathcal{R}_\infty}^\xi$ equals the kernel $\mathcal{N}_{\mathcal{R}_\infty}^\xi$ of the pseudo-isomorphism $\phi$ (3.3). And we have an isomorphism $\mathcal{N}_{\mathcal{R}_\infty}^\xi \simeq \bigoplus_{v \in \mathcal{E}_2} \mathcal{H}_v$.

**Proof.** Since for $v \in \mathcal{S}$ the decomposition group $\Gamma_v$ is not contained in $\mathcal{E}$, a prime ideal containing $\mathcal{I}_v$ for some $v \in \mathcal{E}_2$ must be of height greater than one. □

3.4. **The $\Xi$-co-invariant part of $\mathcal{W}_{\mathcal{R}_\infty}^\xi$.** Let $N_{\mathcal{R}_\infty/\mathcal{E}} : \mathcal{W}_{\mathcal{R}_\infty}^\xi \rightarrow \mathcal{W}_{\mathcal{R}_\infty}^\xi$ be the norm map. Let $\mathcal{M}_\infty$ be the submodule $\ker(N_{\mathcal{R}_\infty/\mathcal{E}}/\mathcal{W}_{\mathcal{R}_\infty}^\xi)$ of the quotient $\mathcal{W}_{\mathcal{R}_\infty}^\xi/\mathcal{M}_{\mathcal{R}_\infty}^\xi$. Define $\mathcal{E}_v$ as in (3.10). Our goal is to establish the exact sequence of Lemma 3.3

$$0 \rightarrow \mathcal{M}_\infty \rightarrow \bigoplus_{v \in \mathcal{E}_2} \mathcal{H}_v \rightarrow \mathbb{Z}_p \rightarrow 0.$$ 

For a topological group $G$ let $G^\wedge$ denote the Pontryagin dual group $\text{Hom}_{\text{cont}}(G, \mathbb{Q}_p/\mathbb{Z}_p)$ consisting of continuous homomorphisms. As the duality in the case of the dual $\mathcal{W}_{\mathcal{R}_\infty}^\xi$ of $\mathcal{W}_{\mathcal{R}_\infty}^\xi$ respects the actions of $\Gamma$, it is a duality between $\Lambda_{\Gamma}$-modules. In particular, an element $\phi \in \mathcal{W}_{\mathcal{R}_\infty}^\xi$ annihilates $s\mathcal{W}_{\mathcal{R}_\infty}^\xi$ if and only if for all $x \in \mathcal{W}_{\mathcal{R}_\infty}^\xi$ we have

$$0 = \phi((\xi - 1)x) = (\xi^{-1}\phi)(x) - \phi(x) = (\xi^{-1}\phi - \phi)(x).$$

($\xi = s + 1$ is the chosen topological generator of $\Xi$). Therefore the annihilators of $s\mathcal{W}_{\mathcal{R}_\infty}^\xi$ are the elements of $(\mathcal{W}_{\mathcal{R}_\infty}^\xi)^\wedge$. And the Pontryagin dual of the quotient $\mathcal{W}_{\mathcal{R}_\infty}^\xi/\mathcal{M}_{\mathcal{R}_\infty}^\xi$ is the $\Xi$-invariant sub-module $(\mathcal{W}_{\mathcal{R}_\infty}^\xi)^\wedge$. Using this, we deduce in a similar way that

$$\mathcal{M}_\infty^\wedge = (\mathcal{W}_{\mathcal{R}_\infty}^\xi)^\wedge / i^*(\mathcal{W}_{\mathcal{R}_\infty}^\xi)^\wedge.$$ 

(3.11)

Since $\mathcal{W}_{\mathcal{R}_\infty}^\xi$ is identified with the Galois group of the maximal everywhere unramified pro-$p$ abelian extension over $\mathcal{R}_\infty$, the module $\mathcal{W}_{\mathcal{R}_\infty}^\xi$ is identified as a sub-module of $\text{Gal}(\bar{k}/\mathcal{R}_\infty)^\wedge$ where $\bar{k}$ is a fixed separable closure of $k$.

Since $\mathcal{W}_{\mathcal{R}_\infty}^\xi$ is compact, the image of an $\omega \in \mathcal{W}_{\mathcal{R}_\infty}^\xi$ is a finite, and hence cyclic, subgroup of $\mathbb{Q}_p/\mathbb{Z}_p$. Therefore, the fixed field of $\ker(\omega)$ is a finite cyclic extension $\mathcal{R}_{\infty}^{(\omega)}$ over $\mathcal{R}_\infty$. If $\omega$ in $(\mathcal{W}_{\mathcal{R}_\infty}^\xi)^\wedge$, then the extension $\mathcal{R}_{\infty}^{(\omega)}/\mathcal{R}_\infty$ is invariant under the action of the $\xi$. This implies that the extension $\mathcal{R}_{\infty}^{(\omega)}/\mathcal{R}_\mathcal{E}$ is also abelian and the Galois group $G := \text{Gal}(\mathcal{R}_{\infty}^{(\omega)}/\mathcal{R}_\mathcal{E})$ is an extension of the Galois group $\text{Gal}(\mathcal{R}_\infty/\mathcal{R}_\mathcal{E}) = \Xi \simeq \mathbb{Z}_p$ by the finite cyclic group $\text{Gal}(\mathcal{R}_{\infty}^{(\omega)}/\mathcal{R}_\infty)$. Therefore, $G$ is the direct product of a subgroup $H \simeq \mathbb{Z}_p$ with the finite $p$-torsion subgroup $\text{Gal}(\mathcal{R}_{\infty}^{(\omega)}/\mathcal{R}_\infty)$. Let $\mathcal{R}'$ be the fixed field of $H$. Then $\mathcal{R}_{\infty}^{(\omega)} = \mathcal{R}_\infty \mathcal{R}'$ and $\text{Gal}(\mathcal{R}_{\infty}^{(\omega)}/\mathcal{R}_\infty) \simeq \text{Gal}(\mathcal{R}'/\mathcal{R}_\mathcal{E})$. Thus, if we identify these two Galois groups, then $\mathcal{R}$ can be obtained form a character of $\text{Gal}(\mathcal{R}'/\mathcal{R}_\mathcal{E})$.

Let $i^* : \text{Gal}(\bar{k}/\mathcal{R}_\mathcal{E})^\wedge \rightarrow \text{Gal}(\bar{k}/\mathcal{R}_\infty)^\wedge$ be the homomorphism dual to the restriction map of the Galois groups. We have just shown that $(\mathcal{W}_{\mathcal{R}_\infty}^\xi)^\wedge$ is contained in the image of $i^*$. Let $F_\infty := i^*^{-1}((\mathcal{W}_{\mathcal{R}_\infty}^\xi)^\wedge)$. Let $\mathcal{L}_\mathcal{E}/\mathcal{R}_\mathcal{E}$ be the maximal everywhere
unramified pro-$p$ abelian extension and let $\mathcal{G}_{\xi,p} = \text{Gal}(\mathfrak{K}_\infty \mathcal{L}_\xi / \mathfrak{K}_\xi)$. Put $H_\infty := (\mathcal{G}_{\xi,p})^\wedge$. Then $F_\infty$ contains $H_\infty$ and the map $i^*$ induces an isomorphism $F_\infty / H_\infty \simeq (\mathfrak{M}_{\xi,p})^\wedge / i^*(\mathfrak{M}_{\xi,p})$. This and (3.11) imply the following lemma.

**Lemma 3.6.** The $\Lambda_T$-module $\mathcal{M}_\infty$ is dual to the quotient $F_\infty / H_\infty$.

Let $\alpha$ be an open subgroup of $\alpha$ containing $\Xi$. Write $\mathfrak{K}_\alpha$ for the fixed field of $\alpha$. Let $\mathfrak{A}_{\mathfrak{A}}^\times$ be the idele group of $\mathfrak{K}_\alpha$ and let

$$\psi_\alpha : \mathfrak{A}_{\mathfrak{A}}^\times \longrightarrow \text{Gal}(\mathfrak{K}_\infty / \mathfrak{K}_\alpha) = \alpha$$

be the global norm residue map. Then for each place $w$ of $\mathfrak{K}_\alpha$ the local norm residue map can be viewed as the composition

$$\psi_{w,\alpha} : \mathfrak{C}_w^\times \longrightarrow \mathfrak{A}_{\mathfrak{A}}^\times \longrightarrow \mathfrak{K}_\alpha \longrightarrow \alpha.$$Let $\mathcal{C}_w \subset \mathcal{O}_w^\times$ be the intersection $\ker(\psi_{w,\alpha}) \cap \mathcal{O}_w^\times$. Then through the local norm residue map $\mathcal{O}_w^\times / \mathcal{C}_w$ is identified with the inertia subgroup $\alpha_0^\wedge$ of $\alpha$. Denote $\gamma_w = \Xi \cap \alpha_0^\wedge$ and put $\mathcal{B}_w = \psi_{w,\alpha}^{-1}(\gamma_w)$. Then we have the exact sequence

$$0 \longrightarrow \mathcal{B}_w / \mathcal{C}_w \longrightarrow \gamma_w \subset \Xi. \quad (3.12)$$

Let $\mathcal{G}_\alpha$ be the $p$-completion of the idele class group

$$\mathfrak{K}_\alpha^\times \backslash \mathfrak{A}_{\mathfrak{A}}^\times / \prod_{w \in \mathcal{S}_1(\mathfrak{K}_\alpha)} \mathcal{C}_w \cdot \prod_{w \not\in \mathcal{S}_1(\mathfrak{K}_\alpha)} \mathcal{O}_w^\times$$

and let $\mathcal{G}_\alpha$ be that of the idele class group

$$\mathfrak{K}_\alpha^\times \backslash \mathfrak{A}_{\mathfrak{A}}^\times / \prod_{w \in \mathcal{S}_1(\mathfrak{K}_\alpha)} \mathcal{B}_w \cdot \prod_{w \not\in \mathcal{S}_1(\mathfrak{K}_\alpha)} \mathcal{O}_w^\times.$$It is not difficult to see that the kernel of the natural map $Q : \mathcal{G}_\alpha \longrightarrow \mathcal{G}_\alpha$ is exactly the image of the natural embedding $\prod_{w \in \mathcal{S}_1(\mathfrak{K}_\alpha)} \mathcal{B}_w / \mathcal{C}_w \longrightarrow \mathcal{G}_\alpha$. And in view of (3.12) it is obvious that $\psi_\alpha(\ker(Q)) \subset \Xi$. In other words, we can define a map $\mathcal{J}_\alpha : \prod_{w \in \mathcal{S}_1(\mathfrak{K}_\alpha)} \mathcal{B}_w / \mathcal{C}_w \longrightarrow \Xi$ and incorporate these in the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{G}_\alpha & \xrightarrow{\psi_\alpha} & \alpha \\
\uparrow & \circ & \cup \\
\prod_{w \in \mathcal{S}_1(\mathfrak{K}_\alpha)} \mathcal{B}_w / \mathcal{C}_w & \xrightarrow{\mathcal{J}_\alpha} & \Xi
\end{array}$$

The next lemma involves the projective limit of modules of the form $\mathcal{A}_\alpha := \ker(\mathcal{J}_\alpha)$. If $\Xi \subset \beta \subset \alpha$ then the norm map on ideles induces a surjective homomorphism $\mathcal{A}_\beta \rightarrow \mathcal{A}_\alpha$ and we denote the projective limit by: $\mathcal{A}_\infty = \lim_{\alpha} \mathcal{A}_\alpha$. It is easy to see that $\mathcal{A}_\alpha = \ker(Q) \cap \ker(\psi_\alpha)$.

**Lemma 3.7.** The $\Lambda_T$-module $F_\infty / H_\infty$ is dual to $\mathcal{A}_\infty$.

**Proof.** Firstly we express $F_\infty$ as a direct limit. For each open sub-group $\alpha$ of $\alpha$ containing $\Xi$, we define $F_\alpha$ as the subgroup of the group $\text{Gal}(k / \mathfrak{K}_\alpha)^\wedge$ consisting of elements $\omega$ such that $j_\alpha^*(\omega) \in F_\infty$, where $j_\alpha^* : \text{Gal}(k / \mathfrak{K}_\alpha)^\wedge \rightarrow \text{Gal}(k / \mathfrak{K}_\Xi)^\wedge$ is the dual of the restriction of Galois group. Since every cyclic extension over $\mathfrak{K}_\Xi$ is obtained
from some cyclic extension over $\mathfrak{K}_\alpha$ for some $\alpha$, the modules $F_\infty$ is a direct limit: $F_\infty = \lim_{\alpha} F_\alpha$ where $\alpha$ runs through all the open subgroups of $\mathfrak{T}$ containing $\Xi$. Let $\mathfrak{R}'/\mathfrak{K}_\alpha$ be an abelian extension with $\text{Gal}(\mathfrak{R}'/\mathfrak{K}_\alpha) = \mathfrak{T}$, and let $\psi : \mathfrak{K}_\alpha^x \rightarrow \mathfrak{T}$ and $\psi_w : \mathfrak{K}_\alpha^x \rightarrow \mathfrak{T}_w$ be the corresponding global and local norm residue maps. Under this setting, the condition $\mathfrak{R}_\infty \subset \mathfrak{R}'$ is equivalent to $\ker(\psi) \subset \ker(\psi_\alpha)$. If this holds, then the condition that $\mathfrak{R}'/\mathfrak{K}_\infty$ is unramified at places sitting over $w$ is equivalent to the condition that $\mathfrak{C}_w \subset \ker(\psi_w)$. Therefore $G_\alpha$ is the Galois group over $\mathfrak{K}_\alpha$ of the maximal pro-$p$ abelian extension containing $\mathfrak{R}_\infty$ such that it is everywhere unramified over $\mathfrak{K}_\infty$. It is obvious that $G_\alpha^\wedge \subset F_\alpha$. If $\omega \in F_\alpha$ and $\mathfrak{K}_\alpha(\omega)$ is the fixed field of its kernel, then the extension $\mathfrak{R}_\infty^\omega/\mathfrak{K}_\alpha$ is abelian and the extension $\mathfrak{R}_\infty^\omega/\mathfrak{K}_\infty$ is everywhere unramified. This means that $\omega \in G_\alpha^\wedge$. Hence we see that $F_\alpha = G_\alpha^\wedge$.

Secondly we consider $H_\alpha$. As before for an open subgroup $\alpha$ of $\mathfrak{T}$ containing $\Xi$, we put $H_\alpha$ as the subgroup of the group $\text{Gal}(\mathfrak{K}/\mathfrak{K}_\alpha)^\wedge$ consisting of elements $\omega$ such that $j^\alpha_\omega(\omega) \in H_\alpha$. Then we have $H_\alpha = \lim_{\alpha} H_\alpha$. Again, let $\mathfrak{R}'/\mathfrak{K}_\alpha$ be an abelian extension with $\text{Gal}(\mathfrak{R}'/\mathfrak{K}_\alpha) = \mathfrak{T}$ and let $\psi : \mathfrak{K}_\alpha^x \rightarrow \mathfrak{T}$ and $\psi_w : \mathfrak{K}_\alpha^x \rightarrow \mathfrak{T}_w$ be the corresponding global and local norm residue maps. The condition that $\mathfrak{R}'$ contains the field $\mathfrak{R}_\Xi$ is equivalent to the condition that $\ker(\psi) \subset \psi^{-1}(\Xi)$. If this holds, then the condition that $\mathfrak{R}'/\mathfrak{R}_\Xi$ is unramified at places sitting over $w$ is equivalent to the condition that $\mathfrak{B}_w \subset \ker(\psi_w)$. This implies that $\mathfrak{G}_\alpha$ is the Galois group over $\mathfrak{K}_\alpha$ of the maximal pro-$p$ abelian extension containing $\mathfrak{R}_\Xi$ such that it is everywhere unramified over $\mathfrak{R}_\Xi$. Denote this field extension as $\mathfrak{L}'/\mathfrak{K}_\alpha$. Then $H_\alpha$ is dual to the Galois group $\text{Gal}(\mathfrak{R}_\Xi^\wedge/\mathfrak{R}_\alpha)$. But since $k_\infty$ is the fixed field of $\ker(\psi_\alpha) \subset G_\alpha$ and $\mathfrak{L}'$ is that of $\ker(Q) \subset G_\alpha$ we have the natural isomorphism $\text{Gal}(\mathfrak{R}_\Xi^\wedge/\mathfrak{R}_\alpha) = G_\alpha/\ker(Q) \cap \ker(\psi_\alpha)$. This shows that $H_\alpha = (G_\alpha/\ker(Q) \cap \ker(\psi_\alpha))^\wedge$. In particular, we see that inside $F_\alpha = G_\alpha^\wedge$ the subgroup $H_\alpha$ is the annihilator of the subgroup $\ker(Q) \cap \ker(\psi_\alpha) \subset G_\alpha$, and hence $F_\alpha/H_\alpha = (\ker(Q) \cap \ker(\psi_\alpha))^\wedge$. But we have seen that $A_\alpha$ equals to the intersection $\ker(Q) \cap \ker(\psi_\alpha)$. The proof of the lemma is completed. □

Finally we relate $A_\infty$ to the special modules $\mathfrak{S}_v$ of section 3.3.

**Lemma 3.8.** We have an exact sequence of $\Lambda_\mathfrak{T}$-modules

$$0 \longrightarrow A_\infty \longrightarrow \bigoplus_{v \in \mathfrak{S}_1} \mathfrak{S}_v \longrightarrow \mathbb{Z}_p \longrightarrow 0,$$

where $\mathbb{Z}_p$ is endowed with the trivial action of $\mathfrak{T}$.

**Proof.** We first note that if $w \notin \mathfrak{S}_1(\mathfrak{R}_\alpha)$ then the group $\mathfrak{T}_w$ is trivial and $\mathfrak{B}_w = \mathfrak{C}_w$; otherwise $\mathfrak{T}_w$ is non-trivial and hence isomorphic to $\mathbb{Z}_p$. And we also observe that the group $\mathfrak{T}_w$ only depends on the place $v \in \mathfrak{S}_1$ sitting below $w$. In fact, if $\Xi \subset \beta \subset \alpha$ and $w'$ is a place of $\mathfrak{R}_\beta$ sitting over $v$, from the definition, we see that $\mathfrak{T}_{w'} = \mathfrak{T}_w$. (3.13)

For simplicity, we denote $\mathfrak{T}_w = \mathfrak{T}_w$. To treat the groups $\mathfrak{T}_w$, $v \in \mathfrak{S}_1$ in a consistent way, we let $d_v$ denote the integer such that $p^{d_v} \xi$ is a generator of $\mathfrak{T}_w$. As before, for each place $v \in \mathfrak{S}_1$, choose a place $v_{(\infty)}$ of $\mathfrak{R}_\infty$ sitting over $v$ and for each $\alpha$ denote by $v_\alpha$ the place of $\mathfrak{R}_\alpha$ sitting below $v_{(\infty)}$. Every place $w \in \mathfrak{S}_1(\mathfrak{R}_\alpha)$ sitting over $v$ is in
the orbit of $v_\alpha$ under the action of $\Upsilon$. Thus, there is a $\sigma \in \Upsilon$ such that $w = \sigma(v_\alpha)$. In this case, we have the commutative diagram:

$$
\begin{array}{ccc}
B_{v_\alpha}/C_{v_\alpha} & \overset{\overline{\psi}_{v_\alpha}}{\longrightarrow} & \Upsilon_v \\
\downarrow \sigma & \bigcirc & \| \\
B_w/C_w & \overset{\overline{\psi}_{w_\alpha}}{\longrightarrow} & \Upsilon_w.
\end{array}
$$

(3.14)

Recall that the homomorphism $\overline{\psi}_{v_\alpha,\alpha}$ is the one in (3.12). Put $b_{v_\alpha} = \overline{\psi}_{v_\alpha,\alpha}^{-1}(\rho^{d_v})$. Since under the action of $\Upsilon$, the stabilizer of $v_\alpha$ is $\Upsilon_{v_\alpha} + \alpha$, where it is assumed that $\Xi \subset \alpha$, we have an isomorphism of $\Upsilon$-modules:

$$
\mathcal{B}_{v_\alpha} : \mathcal{G}_{\Upsilon/v_\alpha} \cong \prod_{w|w_\alpha} B_w/C_w
$$

(3.15)

where the ring $\mathcal{G}_{\Upsilon/v_\alpha}$ is defined in Section 3.3, the left down-arrow is the map $F_{v_\alpha} : \mathcal{G}_{\Upsilon/v_\alpha} \cong \prod_{w|w_\alpha} B_w/C_w$ given by $z_v \mapsto z_v b_{v_\alpha}$, and a commutative diagram

$$
\begin{array}{ccc}
\bigoplus_{v \in \mathcal{O}_1} \mathcal{G}_{\Upsilon/v_\alpha} & \overset{\Sigma_\alpha}{\longrightarrow} & \mathbb{Z}_p \\
\downarrow & \bigcirc & \downarrow \\
\prod_{w \in \mathcal{O}_1(k_\alpha)} B_w/C_w & \overset{\lambda_w}{\longrightarrow} & \Xi
\end{array}
$$

Now we consider the projective limits of the above objects. First it is easy to see that

$$
\bigoplus_{v \in \mathcal{O}_1} \mathcal{G}_{\Upsilon/v_\alpha} = \lim_{\alpha} \bigoplus_{v \in \mathcal{O}_1} \mathcal{G}_{\Upsilon/v_\alpha}
$$

where $\alpha$ runs through all open sub-group of $\Upsilon$ containing $\Xi$. Since $\bigoplus_{v \in \mathcal{O}_1} \mathcal{G}_v$ is compact and the image of $\Sigma_\alpha$ is an open sub-group of $\mathbb{Z}_p$ independent of $\alpha$, we have an exact sequence $0 \rightarrow \lim_{\alpha} \ker(\Sigma_\alpha) \rightarrow \bigoplus_{v \in \mathcal{O}_1} \mathcal{G}_v \rightarrow \mathbb{Z}_p \rightarrow 0$. If $\Xi \subset \beta \subset \alpha$ and $w'$ is a place of $\mathcal{K}_\beta$ sitting over $w$, then we have the commutative diagram

$$
\begin{array}{ccc}
B_{w'}/C_{w'} & \overset{\overline{\psi}_{w',\beta}}{\longrightarrow} & \Upsilon_{w'} \\
\downarrow \sigma & \bigcirc & \| \\
B_w/C_w & \overset{\overline{\psi}_{w,\alpha}}{\longrightarrow} & \Upsilon_w.
\end{array}
$$

(3.17)

where the left down-arrow is from the local norm map. This together with the isomorphism (3.16) and the diagram (3.15) implies $\mathcal{A}_\infty \simeq \lim_{\alpha} \ker(\Sigma_\alpha)$. □

We summarize the above discussions in the following lemma.

**Lemma 3.9.** Let $N : \mathcal{W}_{\mathcal{K}_\infty}/\mathcal{W}_{\mathcal{K}_\Xi} \longrightarrow \mathcal{W}_{\mathcal{K}_\Xi}$ be the natural map induced from the norm $N_{\mathcal{K}_\infty}/\mathcal{K}_\Xi$. If $\mathcal{K}_\infty/\mathcal{K}_\Xi$ is everywhere unramified, then we have the exact sequence of $\Lambda_{\Upsilon}$-modules

$$
0 \rightarrow \mathcal{W}_{\mathcal{K}_\infty}/\mathcal{W}_{\mathcal{K}_\Xi} N \rightarrow \mathcal{W}_{\mathcal{K}_\Xi} \rightarrow \mathbb{Z}_p \rightarrow 0;
$$

otherwise, we have exact sequences of $\Lambda_{\Upsilon}$-modules

$$
0 \rightarrow \mathcal{M}_\infty \rightarrow \mathcal{W}_{\mathcal{K}_\infty}/\mathcal{W}_{\mathcal{K}_\Xi} N \rightarrow \mathcal{W}_{\mathcal{K}_\Xi} \rightarrow \mathcal{Z} \rightarrow 0
$$
and

\[ 0 \rightarrow \mathcal{M}_\infty \rightarrow \bigoplus_{v \in \mathcal{S}_1} \mathcal{G}_v \rightarrow \mathbb{Z}_p \rightarrow 0, \]

where \( \mathbb{Z} \) is of finite cardinality and \( \mathbb{Z}_p \) is endowed with the trivial action of \( \Upsilon \).

**Proof.** If \( \mathfrak{A}_\infty / \mathfrak{A}_\Xi \) is everywhere unramified, then \( \mathcal{S}_1 \) is empty and \( \mathcal{M}_\infty \) is trivial. We need to determine the cokernel of \( N \). From the duality, we see that it is dual to the kernel of \( i_* |_{\mathcal{M}^{\wedge}_{\mathfrak{A}_\Xi,p}}: \mathcal{M}^{\wedge}_{\mathfrak{A}_\Xi,p} \rightarrow \text{Gal}(\bar{k}/\mathcal{K}_\Xi)^\wedge \). It is easy to see that this kernel is just \( \mathfrak{A}_\Xi \cap \mathcal{M}^{\wedge}_{\mathfrak{A}_\Xi,p} \). Here we consider both \( \mathfrak{A}_\Xi \) and \( \mathcal{M}^{\wedge}_{\mathfrak{A}_\Xi,p} \) as subgroups of \( \text{Gal}(\bar{k}/\mathcal{K}_\Xi)^\wedge \). Since \( \mathcal{M}^{\wedge}_{\mathfrak{A}_\Xi,p} \) is the Galois group of the maximal everywhere unramified pro-\( p \) abelian extension of \( \mathcal{K}_\Xi \), the intersection is the Pontryagin dual of the quotient \( \mathfrak{A}_\Xi / \mathfrak{A}_\Xi' \) where \( \mathfrak{A}_\Xi' \subset \mathfrak{A}_\Xi \) is the subgroup generated by all the inertia groups at all the places. Consequently, if \( \mathfrak{A}_\infty / \mathfrak{A}_\alpha \) is everywhere unramified then \( \text{coker}(N) \simeq \mathbb{Z}_p \); otherwise, it is a finite set. □

**Corollary 3.3.** If \( e > 1 \) and no place in \( \mathcal{S}_1 \) splits completely under \( \mathcal{K}_\Xi / k \), then the \( \Lambda_\Upsilon \)-modules \( \mathcal{M}_\infty \) and \( \mathcal{M}^{\wedge}_{\mathfrak{A}_\Xi,p} / s\mathcal{M}^{\wedge}_{\mathfrak{A}_\Xi,p} \) are pseudo-null \( \Lambda_\Upsilon \)-modules.

**Proof.** Since \( \mathfrak{A}_\Xi \subseteq \Upsilon \), a finitely generated torsion module over \( \Lambda_\Upsilon / \mathfrak{A}_\Xi \) is pseudo-null if it is considered as a \( \Lambda_\Upsilon \)-module. Since \( \mathbb{Z}_p \) and \( \mathcal{M}^{\wedge}_{\mathfrak{A}_\Xi,p} \) are finitely generated torsion over \( \Lambda_\Upsilon / \mathfrak{A}_\Xi \), in view of Lemma 3.9, we only need to show that for each \( v \in \mathcal{S}_1 \), the module \( \mathcal{G}_v \) is also finitely generated torsion over \( \Lambda_\Upsilon / \mathfrak{A}_\Xi \). But we have

\[ \mathcal{G}_v = \Lambda_\Upsilon / \mathfrak{G}_v \simeq \Lambda_\Upsilon / \bar{\mathfrak{G}}_v \]

where \( \bar{\mathfrak{G}}_v \) is the ideal of \( \Lambda_\Upsilon / \mathfrak{A}_\Xi \) generated by the set \( \{ \sigma - 1 \mid \sigma \in \mathfrak{G}_v + \mathfrak{A}_\Xi / \Xi \} \). Since \( v \) does not completely split over \( \mathfrak{A}_\Xi / k \), the quotient group \( \mathfrak{G}_v + \Xi / \Xi \) is non-trivial and hence \( \mathcal{G}_v \) is torsion over \( \Lambda_\Upsilon / \mathfrak{A}_\Xi \). □

### 3.5. Greenberg’s lemma.

**Lemma 3.10.** (Greenberg) Let \( \Upsilon \simeq \mathbb{Z}_p^e \) for some \( e \) and let \( \mathcal{Y} \) be a finitely generated torsion \( \Lambda_\Upsilon \)-module. Then the following are true.

1. Assume that \( \mathcal{Y} \) has an annihilator \( \Phi \in \Lambda_\Upsilon \) such that \( p \nmid \Phi \). Then \( \Upsilon \) contains at least one sub-group \( \Upsilon' \) such that \( \Upsilon / \Upsilon' \simeq \mathbb{Z}_p \) with the property that \( \mathcal{Y} \) is finitely generated over \( \Lambda_\Upsilon / \Upsilon' \). Furthermore, for every \( e - 1 \)-dimensional \( \mathbb{F}_p \)-subspace \( \bar{\Upsilon}' \) of \( \bar{\Upsilon} := \Upsilon \otimes \mathbb{Z}_p / p\mathbb{Z}_p \), the sub-group \( \Upsilon' \) can be chosen such that its image under \( \Upsilon \rightarrow \bar{\Upsilon} \) equals \( \bar{\Upsilon}' \).

2. If \( \mathcal{Y} \) is pseudo-null, then (1) holds. In this case \( \mathcal{Y} \) is a torsion module over \( \Lambda_\Upsilon / \Upsilon' \).

**Proof.** The first part of statement (1) is actually Lemma 2 in [Grn/8], and its proof actually proves the last part of (1). Statement (2) is proved in the discussion after the proof of Lemma 2. □

### 3.6. The characteristic ideals.

In this subsection we shall compare the characteristic ideals \( \chi_\Upsilon(\text{Cl}_{\mathfrak{A}_\Xi,p}) \) and \( \chi_{\Upsilon/\Xi}(\text{Cl}_{\mathfrak{A}_\Xi,p}) \). We assume that no place in \( \mathfrak{G} \) splits
completely under $R_E/k$. As before we let $\phi : \mathcal{M}_{R_{\infty},p} \rightarrow \bigoplus_{j=1}^{J} \Lambda_T/(\zeta_j^{m_j})$ be a pseudo-isomorphism with $\ker(\phi) = N_{R_{\infty}}$. We also denote $T_{R_{\infty}} = \coker(\phi)$. Then we have exact sequences

$$0 \rightarrow N_{R_{\infty}} \rightarrow \mathcal{M}_{R_{\infty},p} \rightarrow \text{Im}(\phi) \rightarrow 0,$$

and

$$0 \rightarrow \text{Im}(\phi) \rightarrow \bigoplus_{j=1}^{J} \Lambda_T/(\zeta_j^{m_j}) \rightarrow T_{R_{\infty}} \rightarrow 0.$$  

(3.18)

By Corollary 3.2 and Corollary 3.3 the $\Lambda$-module $\mathcal{Y} = N_{R_{\infty}} \oplus T_{R_{\infty}} \oplus \mathcal{M}_{R_{\infty},p}/s \mathcal{M}_{R_{\infty},p} \oplus \mathcal{M}_{R_{\infty},p}^{\bar{Y}}$ is pseudo-null. Applying Lemma 3.10 we can find a subgroup $\mathcal{Y}'$ such that $\mathcal{Y}$ is a finitely generated torsion $\Lambda_T$-module. We can choose $\mathcal{Y}'$ such that the subspaces $\bar{Y}'$ and $\bar{Z} := Z \otimes Z_p/pZ_p$ span the space $\bar{Y}$. This means that $\bar{Y}$ is the direct sum of $\bar{Z}$ and $\bar{Y}'$.

The action of $s = \xi - 1$ commutes with (3.18) and (3.19) and from the snake lemma, we have exact sequences

$$0 \rightarrow N_{R_{\infty}}^{\bar{Y}} \rightarrow \mathcal{M}_{R_{\infty},p}^{\bar{Y}} \rightarrow \text{Im}(\phi)^{\bar{Y}} \rightarrow N_{R_{\infty}}/sN_{R_{\infty}}$$

$$0 \leftarrow \text{Im}(\phi)/s \text{Im}(\phi) \leftarrow \mathcal{M}_{R_{\infty},p}/s \mathcal{M}_{R_{\infty},p}$$

(3.20)

and

$$0 \rightarrow \text{Im}(\phi)^{\bar{Y}} \rightarrow \bigoplus_{j=1}^{J} \Lambda_T/(\zeta_j^{m_j})^{\bar{Y}} \rightarrow T_{R_{\infty}}^{\bar{Y}}$$

$$0 \leftarrow T_{R_{\infty}}/sT_{R_{\infty}} \leftarrow \bigoplus_{j=1}^{J} \Lambda_T/\bar{Z}^{\bar{Y}} \leftarrow \text{Im}(\phi)/s \text{Im}(\phi),$$

(3.21)

where $\bar{\zeta}$ is the image of $\zeta$ under the projection $\Lambda_T \rightarrow \Lambda_T/\bar{Z}$.

**Lemma 3.11.** Under the condition that no place in $\mathcal{S}$ splits completely over $R_E/k$, we have $\bar{\zeta} \neq 0$, for each $j \in J$.

**Proof.** If $\bar{\zeta}_i = 0$ for some $i$, then $\Lambda_T/\bar{Z}^{\bar{Y}}/(\zeta_i^{m_i})$ is a free $\Lambda_T/\bar{Z}$-module. Since $T_{R_{\infty}}/sT_{R_{\infty}}$, $T_{R_{\infty}}^{\bar{Y}}$ and $N_{R_{\infty}}/sN_{R_{\infty}}$ are all torsion $\Lambda_T/\bar{Z}$-modules, the above exact sequences say that neither $\text{Im}(\phi)/s \text{Im}(\phi)$ nor $\mathcal{M}_{R_{\infty},p}/s \mathcal{M}_{R_{\infty},p}$ is a torsion $\Lambda_T/\bar{Z}$-module. The homomorphism $Y' \rightarrow Y \rightarrow Y/\bar{Z}$ induces an identification of $\Lambda_T$ with $\Lambda_T/\bar{Z}$. This implies that $\mathcal{M}_{R_{\infty},p}/s \mathcal{M}_{R_{\infty},p}$ is not a torsion $\Lambda_T$-module. On the other hand, we know that $Y$ is a torsion $\Lambda_T$-module and hence so is $\mathcal{M}_{R_{\infty},p}/s \mathcal{M}_{R_{\infty},p}$. This is a contradiction. 

This leads to the following obvious corollary.

**Corollary 3.4.** Under the condition that no place in $\mathcal{S}$ splits completely over $R_E/k$, every object involved in the exact sequences (3.20) and (3.21) is a finitely generated torsion $\Lambda_T/\bar{Z}$-module.

**Corollary 3.5.** Under the condition that no place in $\mathcal{S}$ splits completely over $R_E/k$, the map $T_{R_{\infty}}^{\bar{Y}} \rightarrow \text{Im}(\phi)/s \text{Im}(\phi)$ in the exact sequence (3.21) is injective.

**Proof.** Since $s$ is relatively prime to every $\zeta_i$, the group $(\bigoplus_{j=1}^{J} \Lambda_T/(\zeta_j^{m_j}))^{\bar{Z}}$ is in fact trivial.
From this corollary and the exact sequence (3.21), we get
\[
\chi_{\mathfrak{T}/\Xi}(\text{Im}(\phi)/s \text{Im}(\phi)) \cdot \chi_{\mathfrak{T}/\Xi}(T_{\mathfrak{C}_\infty}/sT_{\mathfrak{C}_\infty}) = \chi_{\mathfrak{T}/\Xi}(T_{\mathfrak{C}_\infty}/sT_{\mathfrak{C}_\infty}) \cdot \prod_{j=1}^{J}(\tilde{\zeta}_j^{m_j}).
\] (3.22)

Also, from the injection (3.4) and the exact sequence (3.20), we get
\[
\chi_{\mathfrak{T}/\Xi}(\mathfrak{M}_{\mathfrak{C}_\infty,p}/s\mathfrak{M}_{\mathfrak{C}_\infty,p}) = \chi_{\mathfrak{T}/\Xi}(\text{Im}(\phi)/s \text{Im}(\phi)) \cdot \chi_{\mathfrak{T}/\Xi}(\mathfrak{N}_{\mathfrak{C}_\infty}/s\mathfrak{N}_{\mathfrak{C}_\infty}).
\] (3.23)

We use the following lemma to make further simplification of the above relations.

**Lemma 3.12.** We have
\[
\chi_{\mathfrak{T}/\Xi}(T_{\mathfrak{C}_\infty}/sT_{\mathfrak{C}_\infty}) = \chi_{\mathfrak{T}/\Xi}(T_{\mathfrak{C}_\infty}/sT_{\mathfrak{C}_\infty}),
\]
and
\[
\chi_{\mathfrak{T}/\Xi}(\mathfrak{N}_{\mathfrak{C}_\infty}/s\mathfrak{N}_{\mathfrak{C}_\infty}) = \chi_{\mathfrak{T}/\Xi}(\mathfrak{N}_{\mathfrak{C}_\infty}/s\mathfrak{N}_{\mathfrak{C}_\infty}).
\]

**Proof.** We have the obvious exact sequence
\[
0 \rightarrow T_{\mathfrak{C}_\infty}/sT_{\mathfrak{C}_\infty} \rightarrow T_{\mathfrak{C}_\infty}/sT_{\mathfrak{C}_\infty} \rightarrow T_{\mathfrak{C}_\infty}/sT_{\mathfrak{C}_\infty} \rightarrow 0,
\]
where each term is actually a finitely generated torsion \( \Lambda_{\mathfrak{T}/\Xi} \)-module. The first equality is proved by using the fact that the characteristic ideals are multiplicative. The second equality is proved by a similar argument. \(\square\)

Finally we prove the following key lemma.

**Lemma 3.13.** Suppose that \( \mathfrak{K}_\infty/k \) is a \( \mathbb{Z}_p \)-extension with \( \text{Gal}(\mathfrak{K}_\infty/k) = \mathfrak{T} \) and \( \Xi \) is a rank one \( \mathbb{Z}_p \)-submodule of \( \mathfrak{T} \) with \( \mathfrak{T}/\Xi \cong \mathbb{Z}_p^{-1} \). Let \( \mathfrak{K}_\Xi \) be the fixed field of \( \Xi \). We define \( \mathfrak{M}_{\mathfrak{K}_\infty,p} \) and \( \mathfrak{M}_{\mathfrak{K}_\Xi,p} \) as in (3.1) and \( \mathfrak{G}_\pi \) as in (3.10). Let \( \mathfrak{S} \) be a set of places of \( k \) so that \( \mathfrak{K}_\infty/k \) is unramified outside of \( \mathfrak{S} \) and no place in \( \mathfrak{S} \) splits completely in \( \mathfrak{K}_{\Xi}/k \). Let \( \mathfrak{S}_1 = \{ v \in \mathfrak{S} \mid \mathfrak{T}_v^0 \cap \Xi \neq \{0\} \} \) and \( \mathfrak{S}_2 = \{ v \in \mathfrak{S}_1 \mid |\mathfrak{T}_v^0/\mathfrak{T}_v^0| < \infty \} \). Let \( \tilde{\zeta} \) denotes the image of \( \zeta \) under the projection \( \Lambda_{\mathfrak{T}/\Xi} \rightarrow \Lambda_{\mathfrak{T}/\Xi} \). Then
\[
\chi_{\mathfrak{T}/\Xi}(\mathfrak{M}_{\mathfrak{K}_\infty,p}) \cdot \prod_{v \in \mathfrak{S}_1 \setminus \mathfrak{S}_2} \chi_{\mathfrak{T}/\Xi}(\mathfrak{G}_v) = \chi_{\mathfrak{T}/\Xi}(\mathfrak{M}_{\mathfrak{K}_\Xi,p}) \cdot \chi_{\mathfrak{T}/\Xi}(\mathbb{Z}_p).
\]

Here the module \( \mathbb{Z}_p \) is endowed with the trivial action of \( \mathfrak{T} \).

**Proof.** Lemma 3.12 Corollary 3.2 equations (3.22) and (3.23) imply that
\[
\chi_{\mathfrak{T}/\Xi}(\mathfrak{M}_{\mathfrak{K}_\infty,p}/s\mathfrak{M}_{\mathfrak{K}_\infty,p}) = \prod_{v \in \mathfrak{S}_2} \chi_{\mathfrak{T}/\Xi}(\mathfrak{G}_v) \cdot \prod_{j=1}^{J}(\tilde{\zeta}_j^{m_j}).
\]

From Lemma 3.9 we obtain
\[
\chi_{\mathfrak{T}/\Xi}(\mathfrak{M}_{\mathfrak{K}_\Xi,p}/s\mathfrak{M}_{\mathfrak{K}_\Xi,p}) \cdot \chi_{\mathfrak{T}/\Xi}(\mathbb{Z}_p) = \chi_{\mathfrak{T}/\Xi}(\mathfrak{M}_{\mathfrak{K}_\Xi,p}) \cdot \prod_{v \in \mathfrak{S}_1} \chi_{\mathfrak{T}/\Xi}(\mathfrak{G}_v),
\]
and this gives the lemma. \(\square\)
4. Proof of the Main Theorem

The first step of the proof is to apply Lemma 2.9. It allows us to find an independent extension (\(L_\infty/k, S\)) of the given pair (\(k_\infty/k, S\)) with \(|S|\) much larger than \(|S|\). Let \(k'_\infty/k\) be a \(\mathbb{Z}_p\)-extension satisfying Lemma 2.9 (3). Since \(k'_\infty/k\) is ramified at every place in \(S\) while the extension \(k_\infty/k\) is unramified at each \(v \in \tilde{S} \setminus S\), we must have \(k'_\infty \nsubseteq k_\infty\). Therefore, the Galois group Gal(\(k_\infty/k'_\infty/k\)) is isomorphic to \(\mathbb{Z}_p^{d+1}\).

Denote Gal(\(L_\infty/k\)) = \(\Delta \cong \mathbb{Z}_p^c\). By Lemma 2.9, the Stickelberger element \(\theta_{L_\infty/k, \tilde{S}, T}\) is an irreducible element in \(\Lambda_\Delta\) and hence by Corollary 2.1 and Corollary 3.1, we have

\[
\chi_\Delta(M_{c, p}) = (\theta^m_{L_\infty/k, \tilde{S}, T}), \text{ for some } m \in \mathbb{Z}_+.
\]  

To apply the results obtained in the previous sections, we choose a chain of ascending Galois groups

\[
\Xi_0 \subset \Xi_1 \subset \cdots \subset \Xi_{c-1} = \text{Gal}(L_\infty/k'_\infty)
\]  

such that \(\Xi_0 = 0\), \(\Xi_i/\Xi_{i-1} \cong \mathbb{Z}_p\) for \(i > 0\).

Let \(L_{\Xi_i}, i = 0, \ldots, c - 1\), be the fixed fields of \(\Xi_i\) acting on \(L_\infty\). Then we have

\[
k \subset k'_\infty = L_{\Xi_{c-1}} \subset \cdots \subset L_{\Xi_1} \subset L_{\Xi_0} = L_\infty.
\]  

We shall also make the choice so that

\[
L_{\Xi_{c-d-1}} = k_\infty k'_\infty.
\]

Now consider the sequence (4.2) and for \(i = 0, \ldots, c - 2\) set \(\mathcal{S} = \tilde{S}, \mathcal{T} = \Delta/\Xi_i, \Xi = \Xi_{i+1}/\Xi_i, \mathcal{R}_\Xi = L_{\Xi_i}\) and \(\mathcal{R}_\Xi = L_{\Xi_{i+1}}\). We first note that the field \(\mathcal{R}_\Xi\) always contains the field \(k'_\infty\), and since the extension \(k'_\infty/k\) is monomial it is ramified at every place in \(\mathcal{S}\). In particular, no place in \(\mathcal{S}\) splits completely in \(\mathcal{R}_\Xi/k\) and we can apply the key Lemma 3.13 to these cases. Also, if a place \(v \in \mathcal{S}_{1} \setminus \mathcal{S}_{2}\), then the corresponding residue extension for \(\mathcal{R}_\Xi/k\) is of infinite degree, hence the \(\mathbb{Z}_p\)-rank of decomposition subgroup \((\mathcal{T}/\Xi)_{v}\) of \(\mathcal{T}/\Xi\) is at least 2. But for \(v \in \mathcal{S}_{1}\) we have \(\mathcal{S}_{v} = L_{\Delta/\Xi}/\mathcal{S}_{v}\) where \(\mathcal{S}_{v}\) is the ideal of \(\Lambda_{\Delta/\Xi}\) generated by the set \(\{\sigma - 1 | \sigma \in (\mathcal{T}/\Xi)_{v}\}\). Therefore, for \(v \in \mathcal{S}_{1} \setminus \mathcal{S}_{2}\), the module \(\mathcal{S}_{v}\) is in fact pseudo-null over \(\Lambda_{\Delta/\Xi}\). We also note that in the case where \(i \leq c - 3\), the \(\mathbb{Z}_p\)-rank of \(\mathcal{T}/\Xi\) is at least 2. Consequently, the module \(\mathbb{Z}_p\) is pseudo-null over \(\Lambda_{\Delta/\Xi}\) which is the ring of formal power series in at least two variables. Therefore its characteristic ideal is \(\Lambda_{\Delta/\Xi}\).

Lemma 3.13 implies that if we choose for each \(i\) a generator \(\zeta_i\) for the characteristic ideal \(\chi_{\Delta/\Xi}(M_{\Xi_i, p})\), then for \(i = 0, \ldots, c - 3\), the generator \(\zeta_{i+1}\) can be chosen as \(\zeta_j\) which is the image of \(\zeta_i\) under the projection \(\Lambda_{\Delta/\Xi} \rightarrow \Lambda_{\Delta/\Xi_{i+1}}\). Here we should remind the readers that according to the functoriality (1.3), this ring homomorphism actually sends the Stickelberger element \(\theta_{\Xi_i/k, \tilde{S}, T}\) to the corresponding \(\theta_{\Xi_{i+1}/k, \tilde{S}, T}\). This simple fact turns out very useful, since the equation (4.1) says that \(\zeta_0\) can be chosen as \(\theta^m_{\Xi_0/k, \tilde{S}, T}\). And from this we can deduce step by step that for \(j = 1, \ldots, c - 2\),

\[
\chi_{\Delta/\Xi}(M_{\Xi_j, p}) = (\zeta_j) = (\theta^m_{\Xi_j/k, \tilde{S}, T}).
\]  

(4.4)
In the case where \( i = c - 2 \) the situation is a little different from the above. This time the Galois group \( \Upsilon / \Xi = \Delta / \Xi_{c-1} = \text{Gal}(k_\infty'/k) \) is of rank one over \( \mathbb{Z}_p \). Then Lemma \( \text{[1.13]} \) says that

\[
\chi_{\text{Gal}(k_\infty'/k)}(\mathcal{M}_{k_\infty,p}) = (\zeta_{c-1}) = (\zeta_{c-2}) \cdot \chi_{\text{Gal}(k_\infty'/k)}(\mathbb{Z}_p).
\]

Since \( \text{Gal}(k_\infty'/k) \) acts trivially on \( \mathbb{Z}_p \), if \( \sigma' \) is a topological generator of this Galois group then the ideal \( \chi_{\text{Gal}(k_\infty'/k)}(\mathbb{Z}_p) = (\sigma' - 1) \). From this and Corollary \( \text{[3.1]} \) we get

\[
\chi_{\text{Gal}(k_\infty'/k)}(\text{Cl}_{k_\infty,p}) = (\zeta_{c-2}).
\]

Then equation \( \text{(4.4)} \), for \( j = c - 2 \), and the functoriality \( \text{[1.3]} \) allow us to conclude that

\[
\chi_{\text{Gal}(k_\infty'/k)}(\text{Cl}_{k_\infty,p}) = (\theta_{k_\infty,k,S,T}^m).
\] (4.5)

On the other hand, the condition we set at the beginning that \( k_\infty'/k \) is monomial implies that

\[
\chi_{\text{Gal}(k_\infty'/k)}(\text{Cl}_{k_\infty,p}) = (\theta_{k_\infty,k,S,T}) = (\sigma' - 1)^{\tilde{r}}
\]

where \( \tilde{r} \) is \( |\tilde{S}| - 1 \). Comparing this with the equation \( \text{(4.5)} \) and taking into account the fact that the element \( \sigma' - 1 \) is irreducible in \( \Lambda_{\text{Gal}(k_\infty'/k)} \), we deduce that \( m = 1 \).

To complete the proof, we apply a similar argument by setting \( \mathcal{S} = \tilde{S}, \ U = \text{Gal}(k_\infty,k'/k), \ V = \text{Gal}(k_\infty,k'/k_\infty), \ R_\infty = k_\infty k_\infty' \) and \( R = k_\infty \). From the equation \( \text{(1.3)} \) and \( \text{(4.4)} \), with \( j = c - d - 1 \), we already have

\[
\chi_{\text{Gal}(k_\infty,k_\infty/k)}(\mathcal{M}_{k_\infty,k_\infty',p}) = (\theta_{k_\infty,k_\infty,k,S,T}).
\] (4.6)

It is known (Corollary \( \text{[2.2]} \)) that no place in \( \mathcal{S} \) splits completely in \( R/k = k_\infty/k \). Let \( S_0 \subset S \) be the subset consisting of unramified places in \( k_\infty/k \) and let \( S_1 = (S \cap \mathcal{S}_1) \setminus (S_0 \cup \mathcal{S}_2) \). It is easy to see that \( \mathcal{S}_1 \cap \mathcal{S}_2 = S_0 \cup S_1 \cup (\tilde{S} \setminus S) \) and if \( v \in S_1 \) then \( \mathcal{S}_v \) is pseudo-null over \( \Lambda_{U/V} \) (the decomposition group \( (U/V)_v \) is at least of rank two over \( \mathbb{Z}_p \)). In this situation, equations \( \text{(1.3)}, \text{(4.6)} \) and Lemma \( \text{[1.13]} \) imply that

\[
\chi_{\Gamma}(\mathcal{M}_{k_\infty,p}) \cdot \prod_{v \in S_0 \cup S \setminus (S_0 \cup \mathcal{S}_S)} (1 - [v]) = (\theta_{k_\infty,k,S,T}) \cdot \chi_{\Gamma}(\mathbb{Z}_p),
\]

where \([v] \in \Gamma\) is the Frobenius at \( v \). By \( \text{(1.4)} \), this reduces to

\[
\chi_{\Gamma}(\mathcal{M}_{k_\infty,p}) \cdot \prod_{v \in S_0} (1 - [v]) = (\theta_{k_\infty,k,S,T}) \cdot \chi_{\Gamma}(\mathbb{Z}_p).
\]

If \( d \geq 2 \), then \( \chi_{\Gamma}(\mathbb{Z}_p) = \Lambda_{\Gamma} \) and the proof is completed. If \( d = 1 \), then \( \chi_{\Gamma}(\mathbb{Z}_p) = (\sigma - 1) \), where \( \sigma \) is a generator of \( \Gamma \), and we can apply Corollary \( \text{[3.1]} \). This completes the proof of the Main Theorem.

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