Given a Wilson action invariant under global chiral transformations, we can construct current composite operators in terms of the Wilson action. The short distance singularities in the multiple products of the current operators are taken care of by the exact renormalization group. The Ward-Takahashi identity is compatible with the finite momentum cutoff of the Wilson action. The exact renormalization group and the Ward-Takahashi identity together determine the products. As a concrete example, we study the Gaussian fixed-point Wilson action of the chiral fermions to construct the products of current operators.
1 Introduction

It is a principle of quantum field theory that the invariance of a theory under a continuous transformation implies the conservation of a current. When a theory is expressed by a Wilson action with a finite momentum cutoff, the principle holds for the Wilson action. In [1] an energy-momentum tensor was constructed from the invariance of the Wilson action under translations and rotations. In this paper we would like to consider the Wilson action of chiral fermions with global flavor symmetry to construct multiple products of the conserved current operator.

To build the Wilson action, we use the exact renormalization group (ERG) formalism. (See for example [2–5] and references therein.) The Wilson action satisfies a well defined differential equation under the continuous change of scale. We adopt a convention that each time we integrate more of the high-momentum fluctuations, we introduce a change of scale to restore the same cutoff function. The continuum limit corresponds to a trajectory parametrized by a logarithmic scale parameter $t$ so that a fixed point is reached in the limit $t \to -\infty$.

The Wilson action of a theory in the continuum limit has all the short distance physics incorporated into the vertices of the action. The full theory is obtained by further integration of the fields with momenta below the cutoff. The Wilson action is determined by the ERG differential equation whose solution is parametrized by the relevant variables of the theory.

Composite operators can be considered as infinitesimal changes of the Wilson action, and they also obey well defined differential equations under the change of logarithmic scale. The general properties of products of composite operators have been discussed in [6]. We follow and extend the discussions there by considering the multiple products of current operators.

ERG is good at handling the short distance singularities via ERG differential equations. Well defined ERG differential equations admit only the solutions consistent with locality, i.e., the vertices of the action and composite operators must be analytic at zero momenta. This is the guiding principle we follow throughout the paper.

Though we consider only chiral fermion fields as dynamical fields, our discussion of current operators is easy to modify in the presence of other dynamical fields, for example in the case of QCD with massless quarks.

Our subject obviously overlaps with the construction of chiral gauge theories using the ERG formalism. (See, for example, [7] and references therein.) For example, the derivation of chiral anomaly using the ERG formalism was done in the context of gauge theory. (See, for example, Sec. 9 of [4] and [8].) The multiple products of current operators require a much lighter formalism.
The paper is organized as follows. In Sec. II, we introduce a current operator for a generic Wilson action of chiral fermions under the assumption of global continuous symmetry. In Sec. III, we introduce multiple products of current operators and derive the ERG equations satisfied by them. By coupling the current with an external gauge field, we construct a composite operator in terms of which we can consider all the products of current operators at once. In Sec. IV, we introduce the Ward-Takahashi (WT) identity for the multiple products of currents. This amounts to the commutation relation of currents. The single current operator, introduced in Sec. II, satisfies the WT identity by construction. The corresponding identity for the products is very plausible, but we are unable to derive it solely from the assumption of global continuous symmetry. We here introduce it as a working hypothesis. The ERG differential equation and the WT identity, thus introduced, are mutually consistent, and they together characterize the products of current operators. In Sec. V we discuss the changes to the ERG equation and the WT identity caused by the short-distance singularities of the operator products. In Sec. VI we consider the products of current operators for the free theory. Though this section is all about 1-loop diagrams, the example elucidates the general formalism given in the preceding sections.

Please note that we use the following condensed notation for momentum integrals:

\[ \int_p \equiv \int \frac{d^D p}{(2\pi)^D}, \quad \delta(p) \equiv (2\pi)^D \delta^D(p). \quad (1) \]

2 Current composite operators

We consider a theory of chiral fermion fields \( \psi, \bar{\psi} \) satisfying

\[ a_R \psi(p) = \psi(p), \quad \bar{\psi}(-p) a_L = \bar{\psi}(-p), \quad (2) \]

where

\[ a_R \equiv \frac{1 + \gamma_5}{2}, \quad a_L \equiv \frac{1 - \gamma_5}{2}. \quad (3) \]

The theory is determined by its Wilson action with a fixed UV cutoff. The cutoff is given in terms of a smooth momentum cutoff function \( K(p) \), such as \( e^{-p^2} \), that is 1 at \( p = 0 \) and vanishes as \( p \to \infty \). We parametrize the Wilson action by a logarithmic scale parameter \( t \).
and demand that it obey the ERG differential equation

\[
\partial_t e^{S_t[\psi, \bar{\psi}]} = \int p \left[ \left( \frac{\Delta(p)}{K(p)} + \frac{D + 1}{2} + p \cdot \partial_p \right) \bar{\psi}(-p) \frac{\delta}{\delta \psi(-p)} e^{S_t} \right. \\
+ e^{S_t} \frac{\delta}{\delta \bar{\psi}(p)} \left( \frac{\Delta(p)}{K(p)} + \frac{D + 1}{2} + p \cdot \partial_p \right) \psi(p) \\
- \Delta \left( \frac{\Delta(p)}{K(p)} - 2\gamma_t K(p) (1 - K(p)) \right) \right],
\]

(4)

where

\[
\Delta(p) \equiv -p \cdot \partial_p K(p).
\]

(5)

\(\gamma_t\) is an anomalous dimension of the chiral fermion field, the trace is for both spinor and flavor indices, and the minus in front of the trace is due to the Fermi statistics.

We assume that the Wilson action \(S_t\) describes a continuum limit; as we take \(t \to -\infty\), we obtain a UV fixed-point:

\[
\lim_{t \to -\infty} S_t = S^*.
\]

(6)

All the physics beyond the fixed cutoff scale of 1 has been incorporated into the action. By integrating the fluctuations of momenta less than 1, we get full correlation functions of the fields.

We define the correlation functions by

\[
\langle \psi(p_1) \cdots \psi(p_n) \bar{\psi}(-q_1) \cdots \bar{\psi}(-q_n) \rangle_t \equiv \prod_{i=1}^n \frac{1}{K(p_i)K(q_i)} \\
\times \left\{ \psi(p_1) \cdots \psi(p_n) \exp \left( -\int_0^t \frac{\delta}{\delta \psi(p)} K(p) h_F(p) \frac{\delta}{\delta \bar{\psi}(-p)} \right) \bar{\psi}(-q_1) \cdots \bar{\psi}(-q_n) \right\}_{S_t},
\]

(7)

where

\[
h_F(p) \equiv a_R \frac{1 - K(p)}{\bar{p}}
\]

(8)

is the high-momentum propagator. The correction involving the cutoff function is a technicality typical in the ERG formalism. Thanks to the correction, though, the correlation functions satisfy the simple scaling relation

\[
\langle \psi(p_1 e^{t-t'}) \cdots \bar{\psi}(-q_n e^{t-t'}) \rangle_t \\
= \exp \left( -n(D - 1)(t - t') + 2n \int_{t'}^t d\tau \gamma_{\tau} \right) \langle \psi(p_1) \cdots \bar{\psi}(-q_n) \rangle_{t'}.
\]

(9)
Another technicality is necessary before we move on to discuss symmetry. A composite operator $O_t(p)$ is a functional whose correlation functions are defined by

$$
\left\langle \left\langle O_t(p) \psi(p_1) \cdots \psi(-q_n) \right\rangle \right\rangle_t = \prod_{i=1}^{n} \frac{1}{K(p_i)K(q_i)}
$$

$$
\equiv \left\langle O_t(p) \left( \psi(p_1) \cdots \exp \left( - \int \frac{\delta}{\delta \psi(q)} K(q) h_F(q) \frac{\delta}{\delta \psi(-q)} \right) \right) \cdots \psi(-q_n) \right\rangle_{S_t}
$$

(10)

where the exponentiated differential operator does not act on $O_t(p)$. We define $O_t(p)$ so that its correlation functions satisfy the scaling relation

$$
\left\langle \left\langle O_t(p e^{t-t'}) \psi(p_1 e^{t-t'}) \cdots \psi(-q_n e^{t-t'}) \right\rangle \right\rangle_t = e^{-y(t-t')} \exp \left( -n(D-1)(t-t') + n \int_{t'}^t d\tau \gamma_\tau \right) \left\langle \left\langle O_t(p) \psi(p_1) \cdots \psi(-q_n) \right\rangle \right\rangle_{t'}
$$

(11)

For simplicity we have taken $-y$, the scale dimension of $O_t(p)$, independent of $t$. For (11) to be valid, $O_t(p)$ must satisfy the ERG differential equation

$$
(\partial_t + y + p \cdot \partial_p - D_t) O_t(p) = 0,
$$

(12)

where $D_t$, acting on functionals, is defined by

$$
D_t \mathcal{O} \equiv \int q \left[ \left( \frac{\Delta(q)}{K(q)} + \frac{D+1}{2} - \gamma_t + q \cdot \partial_q \right) \psi(-q) \cdot \frac{\delta}{\delta \psi(-q)} \mathcal{O}
\right.
\left. + \mathcal{O} \frac{\delta}{\delta \psi(q)} \left( \frac{\Delta(q)}{K(q)} + \frac{D+1}{2} - \gamma_t + q \cdot \partial_q \right) \psi(q)
\right.
\left. + S_t \frac{\delta}{\delta \psi(q)} \left( a_R \frac{\Delta(q)}{q} - 2 \gamma_t K(q) h_F(q) \right) \frac{\delta}{\delta \psi(-q)} \mathcal{O}
\right.
\left. + \mathcal{O} \frac{\delta}{\delta \psi(q)} \left( a_R \frac{\Delta(q)}{q} - 2 \gamma_t K(q) h_F(q) \right) \frac{\delta}{\delta \psi(-q)} S_t
\right.
\left. - \text{Tr} \left( a_R \frac{\Delta(q)}{q} - 2 \gamma_t K(q) h_F(q) \right) \frac{\delta}{\delta \psi(-q)} \mathcal{O} \frac{\delta}{\delta \psi(q)} \right].
$$

(13)

The simplest example of a composite operator is

$$
\Psi(p) \equiv \frac{1}{K(p)} \left[ \psi(p) + h_F(p) \frac{\delta}{\delta \psi(-p)} S_t \right],
$$

(14a)

$$
\bar{\Psi}(-p) \equiv \frac{1}{K(p)} \left[ \bar{\psi}(-p) + S_t \frac{\delta}{\delta \bar{\psi}(p)} h_F(p) \right].
$$

(14b)
Though they are composite operators, they have the same correlation functions as the elementary fields $\psi(p), \bar{\psi}(-p)$:

$$\left\langle \Psi(p_1)\Psi(p_2)\cdots\Psi(-q_n) \right\rangle_t = \left\langle \psi(p_1)\cdots\bar{\psi}(-q_n) \right\rangle_t, \quad (15)$$

$$\left\langle \psi(p_1)\cdots\bar{\psi}(-q_{n-1})\bar{\Psi}(-q_n) \right\rangle_t = \left\langle \psi(p_1)\cdots\bar{\psi}(-q_n) \right\rangle_t. \quad (16)$$

We are now ready to discuss symmetry. We assume that the correlation functions have global symmetry:

$$\left\langle \left( U\psi(p_1)\cdots U\psi(p_n) \right) \bar{\psi}(-q_1)U^\dagger \cdots \bar{\psi}(-q_n)U^\dagger \right\rangle_t = \left\langle \psi(p_1)\cdots\bar{\psi}(-q_1)\cdots\bar{\psi}(-q_n) \right\rangle_t, \quad (17)$$

where $U$ is an arbitrary unitary matrix that acts on the flavor indices of $\psi$ and $\bar{\psi}$. ($U$ may be a $U(N)$ matrix if we have $N$ flavors.) For infinitesimal transformations we obtain

$$\sum_{i=1}^n \left( -\left\langle \psi(p_1)\cdots T^a\psi(p_i)\cdots\bar{\psi}(-q_1)\cdots\bar{\psi}(-q_n) \right\rangle_t \right)$$

$$+ \left\langle \psi(p_1)\cdots\bar{\psi}(p_n)\bar{\psi}(-q_1)\cdots\bar{\psi}(-q_i)T^a\cdots\bar{\psi}(-q_n) \right\rangle_t = 0, \quad (18)$$

where $T^a$ are hermitian matrices normalized by

$$\text{Tr} T^a T^b = \delta^{ab} \quad (19)$$

and satisfying the commutation relation

$$\left[ T^a, T^b \right] = i \sum_c f^{abc} T^c. \quad (20)$$

(We will omit the summation symbol for the repeated indices $c$ from now on.)

To express (17) as an operator equation, we introduce an equation-of-motion composite operator by

$$\mathcal{E}^a(p) \equiv e^{-S_t} \int_q K(q) \text{Tr} \left[ \frac{\hat{\delta}}{\delta \bar{\psi}(-q)} \left( \bar{\Psi}(-q+p)T^a e^{S_t} \right) - \left( e^{S_t T^a \Psi(q+p)} \right) \frac{\delta}{\delta \bar{\psi}(q)} \right], \quad (21)$$

where $\Psi, \bar{\Psi}$ are defined by (14). $\mathcal{E}^a$ is a total derivative of the exponentiated Wilson action, and it has correlation functions

$$\left\langle \left\langle \mathcal{E}^a(p_1)\cdots\psi(p_n)\bar{\psi}(-q_1)\cdots\bar{\psi}(-q_n) \right\rangle_t \right\rangle_t$$

$$= \sum_{i=1}^n \left[ -\left\langle \psi(p_1)\cdots T^a\psi(p_i+p)\cdots\bar{\psi}(p+n)\bar{\psi}(-q_1)\cdots\bar{\psi}(-q_n) \right\rangle_t \right]$$

$$+ \left\langle \psi(p_1)\cdots\bar{\psi}(p_n)\bar{\psi}(-q_1)\cdots\bar{\psi}(p-q_i)T^a\cdots\bar{\psi}(-q_n) \right\rangle_t. \quad (22)$$
The symmetry (17) is equivalent to
\[ \mathcal{E}^a(p = 0) = 0 . \] (23)

In fact this is equivalent to what we usually consider as the invariance of the action
\[ \int_p \left( \bar{\psi}(-p) T^a \frac{\delta}{\delta \psi(-p)} S_t - S_t \frac{\delta}{\delta \psi(p)} T^a \bar{\psi}(p) \right) = 0 . \] (24)

In Appendix A we show that this is equivalent to (23).

Since \( \mathcal{E}^a(p) \) is a local operator, it must be proportional to the momentum:
\[ \mathcal{E}^a(p) = p_\mu J^a_\mu(p) , \] (25)

where the current \( J^a_\mu(p) \) must be a local composite operator. Unless there is a local operator \( j^a_\mu(p) \) orthogonal to \( p_\mu \)
\[ p_\mu j^a_\mu(p) = 0 , \] (26)

(25) defines the current \( J^a_\mu(p) \) unambiguously. Since \( \mathcal{E}^a(p) \) has scale dimension 0, \( J^a_\mu(p) \) must have scale dimension \(-1\). In coordinate space \( J^a_\mu(x) = \int_p e^{ipx} J^a_\mu(p) \) has scale dimension \( D - 1 \).

As an example, let us consider the Gaussian fixed-point theory
\[ S_G = -\int_p \frac{1}{K(p)} \bar{\psi}(-p) \phi a_R \psi(p) , \] (27)

for which
\[ \Psi(p) = \psi(p) , \quad \bar{\Psi}(-p) = \bar{\psi}(-p) . \] (28)

We find
\[ \mathcal{E}^a(p) = \int_q K(q) \left[ -\bar{\psi}(-q + p) T^a \frac{\delta}{\delta \psi(-q)} S + S \frac{\delta}{\delta \psi(q)} T^a \psi(q + p) \right] \]
\[ = \int_q \left( \bar{\psi}(-q + p) T^a a_R \phi \psi(q) - \bar{\psi}(-q) a_R \phi T^a \psi(q + p) \right) \]
\[ = \int_q \bar{\psi}(-q) T^a a_R \phi \psi(q + p) . \] (29)

This implies
\[ J^a_\mu(p) = \int_q \bar{\psi}(-q) T^a \gamma_\mu a_R \psi(q + p) . \] (30)
3 Products of current operators

We wish to define multiple products of currents. The product of two currents is defined as

\[
J^a_\mu(p)J^b_\nu(q) \equiv J^a_\mu(p)J^b_\nu(q) + P_{\mu\nu}(p, q),
\]  

(31)

where \(P\) is a local counterterm necessary to make the product a composite operator; the bare product \(J^a_\mu(p)J^b_\nu(q)\) is not a composite operator in the sense introduced in the previous section. \(P\) also takes care of the short-distance singularity occurring when the two currents come close together. For the product to be a composite operator of scale dimension \(-2\), it must satisfy

\[
(\partial_t + p \cdot \partial_p + q \cdot \partial_q + 2 - D_t) \left[ J^a_\mu(p)J^b_\nu(q) \right] = 0,
\]  

(32)

where \(D_t\) is given by (13). This implies

\[
(\partial_t + p \cdot \partial_p + q \cdot \partial_q + 2 - D_t) P_{\mu\nu}(p, q)
= \int_r \left[ J^a_\mu(p) \frac{\Delta(r)}{\delta \bar{\psi}(r)} \left( a_R \frac{\Delta(r)}{4} - 2\gamma_1 K(r) h_F(r) \right) \frac{\delta}{\delta \bar{\psi}(r)} J^b_\nu(q) + \left( J^a_\mu(p) \leftrightarrow J^b_\nu(q) \right) \right].
\]  

(33)

Similarly, we define the product of three currents as

\[
\left[ J^a_\mu_1(p_1)J^a_\mu_2(p_2)J^a_\mu_3(p_3) \right] \equiv J^a_\mu_1(p_1)J^a_\mu_2(p_2)J^a_\mu_3(p_3)
+ P_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3)J^a_\mu_3(p_3),
\]  

(34)

and so on for the higher order products. We note that \(P_{\mu_1\cdots\mu_n}(p_1, \cdots, p_n)\) gives the short-distance singularity due to all the \(n\) currents coming together simultaneously, and it is proportional to the delta function in momentum space

\[
P_{\mu_1\cdots\mu_n}(p_1, \cdots, p_n) \propto \delta \left( \sum_{i=1}^n p_i / 2 \right),
\]  

(35)

unless there is a composite operator of scale dimension \(-n\) or less available. (That means scale dimension \(D - n\) or less in coordinate space.) The ERG equation for \(P_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3)\)
is given by
\[
\left( \partial_t + \sum_{i=1}^{3} p_i \cdot \partial_{p_i} + 3 - D_t \right) \mathcal{P}^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3)
\]
\[
= \int_q \left[ \mathcal{P}^{a_1a_2}_{\mu_1\mu_2}(p_1, p_2) \left( a_R \frac{\Delta(q)}{q} - 2 \gamma_t K(q) h_F(q) \right) \mathcal{P}^{a_3}_{\mu_3}(p_3) \right.
\]
\[
+ J^{a_3}_{\mu_3}(p_3) \left( a_R \frac{\Delta(q)}{q} - 2 \gamma_t K(q) h_F(q) \right) \mathcal{P}^{a_1a_2}_{\mu_1\mu_2}(p_1, p_2)
\]
\[
+ (4 \text{ other terms}) \right].
\] (36)

The ERG equations for the higher order counterterms are given similarly.

To consider all the local products of current operators simultaneously, we introduce a classical gauge field coupled to the current:
\[
W_t[A] \equiv \int_p A_{\mu}^a(-p) J_{\mu}^a(p) + \frac{1}{n!} \int_{p_1, \ldots, p_n} A_{\mu_1}^{a_1}(-p_1) \cdots A_{\mu_n}^{a_n}(-p_n) \mathcal{P}_{\mu_1, \ldots, \mu_n}^{a_1, \ldots, a_n}(p_1, \ldots, p_n)
\] (37)

so that its exponential
\[
e^{W_t[A]} \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \int_{p_1, \ldots, p_n} A_{\mu_1}^{a_1}(-p_1) \cdots A_{\mu_n}^{a_n}(-p_n) \left[ J_{\mu_1}^{a_1}(p_1) \cdots J_{\mu_n}^{a_n}(p_n) \right]
\] (38)
is a composite operator. We assign the scale dimension \(-D + 1\) to the source field \(A_{\mu}^a\) so that \(e^{W_t[A]}\) becomes a composite operator of scale dimension 0, satisfying the ERG equation
\[
\left( \partial_t + \int_p (-p \cdot \partial_{p} - D + 1) A_{\mu}^a(p) \cdot \frac{\delta}{\delta A_{\mu}^a(p)} - D_t \right) e^{W_t[A]} = 0,
\] (39)
where \(D_t\) is defined by [13].

4 Commutation relation — Ward-Takahashi identity

We now wish to consider the “commutation relation” of two currents. The quotation mark is put because it needs to be explained. Our commutation relation is an operator equation
\[
p_{\mu} \left[ J_{\mu}^a(p), J_{\nu}^b(q) \right] = i f^{abc} J_{\nu}^c(p + q) + \mathcal{E}^a(p) \ast J_{\nu}^b(q)
\] (40)
which amounts to the Ward-Takahashi (WT) identity
\[ p_\mu \left\langle \left[ J^a_\mu (p) J^b_\nu (q) \right] \psi (p_1) \cdots \tilde{\psi} (-q_n) \right\rangle_t = i f^{abc} \left\langle J^c_\nu (p + q) \psi (p_1) \cdots \tilde{\psi} (-q_n) \right\rangle_t \]
\[ + \sum_{i=1}^n \left( - \left\langle J^b_\nu (q) \psi (p_1) \cdots T^a \psi (p + p_i) \cdots \tilde{\psi} (-q_n) \right\rangle_t \right. \]
\[ \left. + \left\langle J^b_\nu (q) \psi (p_1) \cdots \tilde{\psi} (p - q_i) T^a \cdots \tilde{\psi} (-q_n) \right\rangle_t \right) . \] (41)

We wish to explain the above and its generalization to higher order products in this section.

We define an equation-of-motion composite operator by
\[ \mathcal{E}^a (p) \ast J^b_\alpha (q) \equiv e^{-S} \int_r K (r) \text{Tr} \left[ \frac{\delta}{\delta \bar{\psi} (-r)} \left( \left[ \bar{\Psi} (-r + p) T^a J^b_\alpha (q) \right] e^{S_t} \right) \right. \]
\[ \left. - \left( e^{S_t} \left[ T^a \Psi (q + p) J^b_\alpha (q) \right] \right) \right] , \] (42)

where
\[ \left[ \bar{\Psi} (-r) J^b_\alpha (q) \right] \equiv \bar{\Psi} (-r) J^b_\alpha (q) \]
\[ - \frac{\delta}{\delta \bar{\psi} (r)} h_F (r) , \] (43)
\[ \left[ \Psi (r) J^b_\alpha (q) \right] \equiv \Psi (r) J^b_\alpha (q) + h_F (r) \frac{\delta}{\delta \bar{\psi} (r)} J^b_\alpha (q) \] (44)

are the composite operators satisfying
\[ \left\langle \psi (p_1) \cdots \tilde{\psi} (-q_{n-1}) \left[ \bar{\Psi} (-r) J^b_\alpha (q) \right] \right\rangle_t = \left\langle J^b_\alpha (q) \psi (p_1) \cdots \tilde{\psi} (-q_n) \tilde{\psi} (-r) \right\rangle_t , \] (45)
\[ \left\langle \left[ \Psi (r) J^b_\alpha (q) \right] \psi (p_2) \cdots \tilde{\psi} (-q_n) \right\rangle_t = \left\langle J^b_\alpha (q) \psi (r) \psi (p_2) \cdots \tilde{\psi} (-q_n) \right\rangle_t . \] (46)

Hence, we obtain
\[ \left\langle \mathcal{E}^a (p) \ast J^b_\alpha (q) \psi (p_1) \cdots \tilde{\psi} (-q_n) \right\rangle_t \]
\[ = \sum_{i=1}^n \left( - \left\langle J^b_\nu (q) \psi (p_1) \cdots T^a \psi (p + p_i) \cdots \tilde{\psi} (-q_n) \right\rangle_t \right. \]
\[ \left. + \left\langle J^b_\nu (q) \psi (p_1) \cdots \tilde{\psi} (p - q_i) T^a \cdots \tilde{\psi} (-q_n) \right\rangle_t \right) . \] (47)

This gives the second term of the right-hand side of (41).

Let
\[ \mathcal{O}^{ab}_\nu (p + q) \equiv p_\mu \left[ J^a_\mu (p) J^b_\nu (q) \right] - \mathcal{E}^a (p) \ast J^b_\nu (q) . \] (48)

[41] then amounts to
\[ \mathcal{O}^{ab}_\nu (p + q) = i f^{abc} J^c_\nu (p + q) . \] (49)

This equality is plausible but not obvious, and it needs an explanation. We will check this later explicitly for the Gaussian theory, but we have not been able to derive it on the basis
of the global symmetry \([17]\). Here we satisfy ourselves by checking the consistency of \((49)\) with Bose symmetry of the current operator, which requires the product

\[
p_{\mu} q_{\nu} \left[ J^{\mu}_{\nu}(p) J^{\mu}_{\nu}(q) \right]
\]

be symmetric under the interchange. The product may depend on which divergence we calculate first. Calculating \(p_{\mu} J^{\mu}_{\nu}(p)\) first, \((48)\) gives

\[
p_{\mu} q_{\nu} \left[ J^{\mu}_{\nu}(p) J^{\mu}_{\nu}(q) \right] = q_{\nu} \mathcal{O}_{\nu}^{ab}(p + q) + \mathcal{E}^{a}(p) * q_{\nu} J^{\mu}_{\nu}(q)
\]

\[
= q_{\nu} \mathcal{O}_{\nu}^{ab}(p + q) + \mathcal{E}^{a}(p) * \mathcal{E}^{b}(q) .
\]

Calculating \(q_{\nu} J^{\mu}_{\nu}(q)\) first, we obtain

\[
p_{\mu} q_{\nu} \left[ J^{\mu}_{\nu}(p) J^{\mu}_{\nu}(q) \right] = p_{\mu} \mathcal{O}_{\mu}^{ba}(p + q) + \mathcal{E}^{b}(q) * p_{\mu} J^{\mu}_{\nu}(p)
\]

\[
= p_{\mu} \mathcal{O}_{\mu}^{ba}(p + q) + \mathcal{E}^{b}(q) * \mathcal{E}^{a}(p) .
\]

Hence, for consistency, we must find

\[
p_{\mu} \mathcal{O}_{\mu}^{ba}(p + q) - q_{\mu} \mathcal{O}_{\mu}^{ab}(p + q) = \mathcal{E}^{a}(p) * \mathcal{E}^{b}(q) - \mathcal{E}^{b}(q) * \mathcal{E}^{a}(p) .
\]

To compute the right-hand side, we consider correlation functions:

\[
\left\langle \mathcal{E}^{a}(p) * \mathcal{E}^{b}(q) \psi(p_{1}) \cdots \psi(p_{n}) \bar{\psi}(-q_{1}) \cdots \bar{\psi}(-q_{n}) \right\rangle_{t}
\]

\[
= \sum_{i=1}^{n} \left[ - \left\langle \mathcal{E}^{b}(q) \cdots T^{a}(p + q) \cdots \bar{\psi}(-q_{1}) \cdots \bar{\psi}(-q_{n}) \bar{\psi}(p_{1}) \cdots \right\rangle_{t} + \left\langle \mathcal{E}^{b}(q) \cdots \bar{\psi}(p + q) \bar{\psi}(q_{1}) \cdots T^{a} \cdots \right\rangle_{t} \right]
\]

\[
= \sum_{i=1}^{n} \left[ \left\langle \cdots T^{a}(p + q + p_{i}) \bar{\psi}(p + q + p_{i}) \cdots \right\rangle_{t} + \left\langle \cdots \bar{\psi}(p + q - q_{i}) T^{a} \cdots \right\rangle_{t} \right]
\]

\[
- \sum_{j=1}^{n} \left( \left\langle \cdots T^{a}(p + p_{i}) \bar{\psi}(q - q_{i}) T^{b} \cdots \right\rangle_{t} + \left\langle \cdots T^{b}(q + p_{i}) \bar{\psi}(p - q_{i}) \cdots \right\rangle_{t} \right)
\]

\[
+ \sum_{j \neq i} \left( \left\langle \cdots T^{a}(p + p_{i}) T^{b}(q + p_{j}) \bar{\psi}(q - q_{i}) \cdots \right\rangle_{t} + \left\langle \cdots \bar{\psi}(p - q_{i}) T^{a} \cdots \bar{\psi}(q - q_{i}) T^{b} \cdots \right\rangle_{t} \right)
\]

(53)

Hence, we obtain

\[
\mathcal{E}^{a}(p) * \mathcal{E}^{b}(q) - \mathcal{E}^{b}(q) * \mathcal{E}^{a}(p) = -i f^{abc} \mathcal{E}^{c}(p + q) .
\]

(54)

Then, the consistency condition \((52)\) gives

\[
p_{\mu} \mathcal{O}_{\mu}^{ba}(p + q) - q_{\mu} \mathcal{O}_{\mu}^{ab}(p + q) = -i f^{abc} \mathcal{E}^{c}(p + q) = (-i f^{abc})(p + q)_{\mu} J^{\mu}_{\nu}(p + q) ,
\]

which is indeed satisfied by \((49)\).
We have thus checked at least that (40) is consistent with the Bose symmetry of the current. We adopt (40) and its generalization to higher orders as our working hypothesis:

\[ p_\mu \left[ J^{a_1}_\mu(p)J^{a_2}_{\mu_1}(p_1) \cdots J^{a_k}_{\mu_k}(p_k) \right] = \sum_{i=1}^{k} i f^{a_i a_i} \left[ J^{a_1}_{\mu_1}(p_1) \cdots J^{a_i}_{\mu_i}(p + p_i) \cdots J^{a_k}_{\mu_k}(p_k) \right] + \mathcal{E}^a(p) \star \left[ J^{a_1}_{\mu_1}(p_1) \cdots J^{a_k}_{\mu_k}(p_k) \right]. \]  

(56)

For the correlation functions, this gives

\[
p_\mu \left\langle \left[ J^{a_1}_\mu(p)J^{a_2}_{\mu_1}(p_1) \cdots J^{a_k}_{\mu_k}(p_k) \right] \psi(q_1) \cdots \psi(q_n) \bar{\psi}(-r_1) \cdots \bar{\psi}(-r_n) \right\rangle_t
= \sum_{i=1}^{k} i f^{a_i a_i} \left\langle \left[ J^{a_1}_{\mu_1}(p_1) \cdots J^{a_i}_{\mu_i}(p + p_i) \cdots J^{a_k}_{\mu_k}(p_k) \right] \psi(q_1) \cdots \psi(q_n) \bar{\psi}(-r_1) \cdots \bar{\psi}(-r_n) \right\rangle_t
+ \sum_{j=1}^{n} \left\{ - \left\langle \left[ J^{a_1}_{\mu_1}(p_1) \cdots J^{a_k}_{\mu_k}(p_k) \right] \cdots T^a \psi(q_j + p) \cdots \right\rangle_t + \left\langle \left[ J^{a_1}_{\mu_1}(p_1) \cdots J^{a_k}_{\mu_k}(p_k) \right] \cdots \bar{\psi}(-r_j + p) T^a \cdots \right\rangle_t \right\}. \]

(57)

The WT identity (56) we just introduced is compactly expressed in terms of the composite operator \( e^{W_i[A]} \) as

\[
\int_q \left( p_\mu \delta_{ab} \delta(p - q) + i f^{abc} A^c_\mu(-q + p) \right) \frac{\delta}{\delta A^b_\mu(-q)} e^{W_i[A]} = \mathcal{E}^a(p) \star e^{W_i[A]}.
\]

(58)

Expanding this in powers of the external source \( A \), we can easily check the equivalence to (56). Multiplying an infinitesimal \( e^a(-p) \) and integrating over \( p \), we can rewrite this as

\[
\delta_\epsilon e^{W_i[A]} \equiv e^{W_i[A^\epsilon]} - e^{W_i[A]} = \int_p e^a(-p)\mathcal{E}^a(p) \star e^{W_i[A]},
\]

(59)

where

\[
(A^\epsilon)^a_\mu(-p) \equiv A^a_\mu(-p) + p_\mu e^a(-p) + i f^{abc} \int_q A^b_\mu(q - p) e^c(-q)
\]

(60)

is an infinitesimal gauge transformation.

5 Corrections to the ERG equation and the WT identity

We have identified two important properties of \( e^{W_i[A]} \). One is the ERG differential equation (39), and the other is the gauge invariance (59). Both may receive corrections due to short distance singularities. Since the nature of singularities depends on the space dimension \( D \), we specify \( D = 4 \) in the following discussion.
We first consider possible corrections to the ERG equation. The product of $n$ current operators has scale dimension $-n$, and it can mix with operators of the same scale dimension. As for the mixing with the delta function $\delta(\sum_i p_i)$, we only need to consider
\[
\left[ J^a_\alpha(p_1) J^b_\beta(p_2) \right], \quad \left[ J^a_\alpha(p_1) J^b_\beta(p_2) J^c_\gamma(p_3) \right], \quad \left[ J^a_\alpha(p_1) J^b_\beta(p_2) J^c_\gamma(p_3) J^d_\delta(p_4) \right]
\]
which mix with the delta function $\delta(\sum_i p_i)$ with appropriate powers (quadratic, linear, none) of momenta. This gives a new ERG differential equation:
\[
\left( \partial_t + \int_p (-p \cdot \partial_p - D + 1) A^a_\mu(p) \cdot \frac{\delta}{\delta A^a_\mu(p)} - D_t \right) e^{W_t[A]} = \int d^D x f(t; A(x)) e^{W_t[A]},
\]
where $f$ is a linear combination of the products of two $A$'s with two derivatives, three $A$'s with one derivative, and four $A$'s with no derivative. Consistency with (59) gives the gauge invariance of $f$. Hence, we obtain
\[
f(t; A) = b(t) \frac{1}{4} \text{Tr} \left( \partial_\alpha A_\beta - \partial_\beta A_\alpha - i[A_\alpha, A_\beta] \right)^2,
\]
where
\[
A_\mu \equiv T^a A^a_\mu.
\]
In fact the gauge invariance (59) itself may also get corrected as
\[
\delta e^{W_t[A]} \equiv e^{W_t[A'] - W_t[A]} = \int_p e^a(-p) (\mathcal{E}^a(p) \star + F^a(p; A)) e^{W_t[A]},
\]
where $F^a(p; A)$ is a polynomial of $A$ with scale dimension $-4$. This is the familiar chiral anomaly. [9] [10] Please note that we have used gauge invariance to derive (61) and (62). For (64) to be consistent with (61), $F^a(p; A)$ must be independent of $t$, i.e., the anomaly must be scale independent. In other words the $t$-dependence of $W_t[A]$ is still gauge invariant, as is given by (61).

The algebraic structure of the anomaly is well known. [11] For completeness, let us derive it using the ERG formalism. By definition of $\delta_\epsilon$, we must obtain
\[
(\delta_\eta \delta_\epsilon - \delta_\epsilon \delta_\eta) e^{W_t[A]} = \delta_{[\eta, \epsilon]} e^{W_t[A]},
\]
where
\[
[\eta, \epsilon] = \eta^a \epsilon^b \left[ T^a, T^b \right] = i f^{abc} \eta^a \epsilon^b T^c.
\]
Using (64) twice, we obtain
\[
(\delta_\epsilon \delta_\eta - \delta_\eta \delta_\epsilon) e^{W_t[A]} = \int_p e^a(-p) \int_q \eta^b(-q)(-i) f^{abc} \mathcal{E}^c(p + q) \star e^{W_t[A]}
\]
\[
+ \int_p (e^a(-p) \delta_\eta F^a(p; A) - \eta^a(-p) \delta_\epsilon F^a(p; A)) e^{W_t[A]},
\]
where we have used (54). Hence, (65) gives the desired algebraic constraint
\[
\int_p (e^a(-p)\delta_\eta F^\mu(p;A) - \eta^a(-p)\delta_e F^\alpha(p;A)) = -i f^{abc} \int_p e^a(-p)\eta^b(-q)F^c(p + q; A) .
\] (68)
A well-known nontrivial solution to this is given by [12]
\[
\int_p e^a(-p)F^\alpha(p) = \text{const} \times \epsilon_{\alpha \beta \gamma \delta} \int d^4x \text{Tr} \partial_\alpha \epsilon \cdot (A_\beta \partial_\gamma A_\delta + \frac{1}{2i} A_\beta A_\gamma A_\delta) .
\] (69)
(A trivial solution is \(\delta_\epsilon\) of a polynomial of \(A\).)

Concluding this section, we have explained that the ERG equation for \(e^{W_A}[A]\) can be modified to (61) and (62), and that the WT identity can get an anomaly (64) where \(F^\alpha\) is given by (69). Differentiating these with respect to the source \(A\), we can get the ERG equation and WT identity satisfied by the products of the current operators. Since their expressions are lengthy, we give them in Appendix C.

6 Free theory in \(D = 4\)

As a concrete example, we construct \(W[A]\) for the Gaussian fixed-point theory in \(D = 4\):
\[
W[A] = \int_p A^\alpha_\mu(-p)J^\mu_\alpha(p) + \sum_{n=2}^{\infty} \frac{1}{n!} \int_{p_1, \cdots, p_n} A^{a_1}_{\mu_1}(-p_1) \cdots A^{a_n}_{\mu_n}(-p_n) \mathcal{P}^{a_1 \cdots a_n}_{\mu_1 \cdots \mu_n}(p_1, \cdots, p_n) .
\] (70)
The construction of \(e^{W[A]}\) is guided by two equations. One is the ERG differential equation
\[
\left( \int_p (-p \cdot \partial_p - D + 1) A^\alpha_\mu(p) \cdot \frac{\delta}{\delta A^\alpha_\mu(p)} \right) W[A] = \frac{b}{4} \int d^4x \text{Tr} \left( \partial_\alpha A_\beta - \partial_\beta A_\alpha - i [A_\alpha, A_\beta] \right)^2 ,
\] (71)
where \(b\) is a constant, and \(\mathcal{D}\) is defined by
\[
\mathcal{D} \equiv \int_q \left[ \left( \frac{D + 1}{2} + q \cdot \partial_q \right) \bar{\psi}(-q) \cdot \frac{\delta}{\delta \bar{\psi}(-q)} \mathcal{O} + \mathcal{O} \left( \frac{\delta}{\delta \bar{\psi}(-q)} \right) \left( \frac{D + 1}{2} + q \cdot \partial_q \right) \psi(q) \right.
\]
\[
\left. \left. - \text{Tr} a_R \frac{\Delta(q)}{q} \frac{\delta}{\delta \bar{\psi}(-q)} \mathcal{O} \frac{\delta}{\delta \bar{\psi}(q)} \right] .
\] (72)
The other is the WT identity with anomaly
\[
\delta_\epsilon e^{W[A]} = \left[ \int_p e^a(-p)\mathcal{E}^\alpha(p) \ast A \epsilon_{\alpha \beta \gamma \delta} \int d^4x \text{Tr} \partial_\alpha \epsilon \cdot (A_\beta \partial_\gamma A_\delta + \frac{1}{2i} A_\beta A_\gamma A_\delta) \right] e^{W[A]} ,
\] (73)
where \(A\) is a constant. Both \(b\) and \(A\) are determined as we construct \(\mathcal{P}^{a_1 \cdots a_n}_{\mu_1 \cdots \mu_n}(p_1, \cdots, p_n)\) from \(n = 2\) to higher \(n\). \(b\) is determined by locality. Locality implies the analyticity of \(\mathcal{P}\)’s
at zero momenta. We must choose \( b \) appropriately to guarantee that \( \tau \) admits a solution satisfying locality. Similarly, the coefficient \( \mathcal{A} \) of the chiral anomaly is determined by locality. The solution to \( \tau \) admits a couple of free parameters consistent with locality. We tune them to satisfy \( \mathcal{I} \) as much as possible. What is left is the anomaly.

At the end of Sec. II, the current was derived as

\[
J^a_\mu(p) = \int_q \bar{\psi}(-q) T^a \gamma_\mu a_R \psi(q + p) .
\] (74)

The counterterms \( \mathcal{P} \) are quadratic in fields, and we can write them in the form

\[
\mathcal{P}^{a_1 \cdots a_n}_{\mu_1 \cdots \mu_n}(p_1, \cdots, p_n) = \int_q \bar{\psi}(-q) c^{a_1 \cdots a_n}_{\mu_1 \cdots \mu_n}(p_1, \cdots, p_n; -q, q + p_1 + \cdots + p_n) \psi(q + p_1 + \cdots + p_n)
\]

\[
+ d^{a_1 \cdots a_n}_{\mu_1 \cdots \mu_n}(p_1, \cdots, p_n) \delta \left( \sum_{i=1}^n p_i \right),
\] (75)

where

\[
c^{a_1 \cdots a_n}_{\mu_1 \cdots \mu_n}(p_1, \cdots, p_n; -q, q + p_1 + \cdots + p_n)
= \sum_{\sigma \in S_n} T^{a_1} T^{a_2} \cdots T^{a_n} \gamma_{\mu_1(1)} h_F(q + p_\sigma(1)) \gamma_{\mu_2(2)} h_F(q + p_\sigma(1) + p_\sigma(2))
\]

\[
\cdots \gamma_{\mu_{\sigma(n-1)}} h_F(q + p_\sigma(1) + \cdots + p_\sigma(n-1)) \gamma_{\mu_{\sigma(n)}}
\] (76)

The sum is taken over all the permutations of \( 1, \cdots, n \).

Similarly, we can write

\[
d^{a_1 \cdots a_n}_{\mu_1 \cdots \mu_n}(p_1, \cdots, p_n)
= \sum_{\sigma \in S_{n-1}} \text{Tr} \left( T^{a_1} T^{a_2} \cdots T^{a_{n-1}} T^{a_n} \right) . d^{a_1 \cdots a_n}_{\mu_1 \mu_2 \cdots \mu_{n-1}}(p_1, p_\sigma(2), \cdots, p_\sigma(n)) ,
\] (77)

where the sum is taken over all the permutations of \( 2, \cdots, n \). \( d \)'s satisfy the ERG equations

\[
\left( \sum_{i=1}^n p_i \cdot \partial_{p_i} + n - 4 \right) d_{a_1 \cdots a_n}(p_1, \cdots, p_n)
= \left( - \right) \int_q \text{Tr} f_F(q) \left[ \gamma_{a_1} h_F(q + p_1) \gamma_{a_2} \cdots \gamma_{a_{n-1}} h_F(q + p_1 + \cdots + p_{n-1}) \gamma_{a_n}
\right.
\]

\[
+ \gamma_{a_2} h_F(q + p_2) \gamma_{a_3} \cdots \gamma_{a_n} h_F(q + p_2 + \cdots + p_n) \gamma_{a_1} + \cdots \right] ,
\] (78)
where

\[ h(p) \equiv \frac{1 - K(p)}{p^2}, \quad (79a) \]

\[ f(p) \equiv (p \cdot \partial_p + 2)h(p) = \frac{\Delta(p)}{p^2}, \quad (79b) \]

\[ f_F(p) \equiv f(p)a_R\phi = a_R\frac{\Delta(p)}{p}. \quad (79c) \]

For \( n \geq 5 \), the solutions are given by the finite loop integrals:

\[
d_{\mu_1\cdots\mu_n}(p_1,\cdots,p_n) \]

\[ = (-) \int_q \text{Tr} \left[ \gamma_{\mu_1} h_F(q + p_1) \gamma_{\mu_2} h_F(q + p_1 + p_2) \cdots h_F(q - p_n) \gamma_{\mu_n} h_F(q) \right] \quad (80) \]

For \( n = 2, 3, 4 \), however, the above loop integrals are UV divergent, and we must define \( d \)'s as solutions of (78). We emphasize that there is no need to introduce an additional UV cutoff to regularize the loop integrals. In fact we need to modify (78) first by adding local terms proportional to the coefficient \( b \) so that the solutions become analytic at zero momenta. ERG then determines \( d_2 \) up to \( t \)-independent terms quadratic in momenta, \( d_3 \) up to terms linear in momenta, and \( d_4 \) up to a constant. To remove the ambiguities we can resort to the WT identity, which would be given by

\[
p_{1\alpha} d_{\alpha_2\cdots\alpha_n}(p_1,\cdots,p_n) = d_{\alpha_2\cdots\alpha_n}(p_1 + p_2, p_3, \cdots, p_n) - d_{\alpha_2\cdots\alpha_n}(p_2, \cdots, p_{n-1}, p_n + p_1)
+ \int_q K(q) \text{Tr} \left[ h_F(q - p_1) \gamma_{\alpha_2} h_F(q + p_2) \cdots h_F(q + p_2 + \cdots + p_{n-1}) \gamma_{\alpha_n}
-h_F(q + p_1) \gamma_{\alpha_2} h_F(q + p_1 + p_2) \cdots h_F(q + p_1 + \cdots + p_{n-1}) \gamma_{\alpha_n} \right], \quad (81) \]

if there were no anomaly. This is satisfied by (80) for \( n \geq 5 \). But this is corrected for \( n = 3, 4 \) by the anomaly, proportional to \( \mathcal{A} \). We can obtain \( \mathcal{A} \) by expanding the WT identity in powers of small momenta. This is a straightforward calculation.

In the following we sketch the calculation of \( d_{\alpha_1\cdots\alpha_n} \) for \( n = 2, 3, 4 \). The case \( n = 2 \) is sufficient to determine the coefficient \( b \), but we need the case \( n = 3 \) to determine \( \mathcal{A} \). We calculate the case \( n = 4 \) for completeness and check of our formalism. The essential steps are
expansions of $d$'s in small momenta. The calculations are all straightforward, and thanks to the presence of a finite cutoff there is no hidden subtlety. Perhaps we could have condensed this section into a smaller number of pages, but we have decided to give all the details for the reader unfamiliar with calculations with cutoff functions. The more experienced reader may skip what seems trivial or redundant.

6.1 Product of Two $n = 2$

$d_{\alpha\beta}^{ab}(p_1, p_2) = \delta_{ab}d_{\alpha\beta}(p, -p)$ satisfies the ERG equation

$$(p \cdot \partial_p - 2) d_{\alpha\beta}(p, -p) = (-) \int_q \text{Tr} \ f_F(q) \left( \gamma_\alpha h_F(q + p) \gamma_\beta + \gamma_\beta h_F(q - p) \gamma_\alpha \right) + b(p^2 \delta_{\alpha\beta} - p_\alpha p_\beta),$$

and the Ward identity

$$p_\alpha d_{\alpha\beta}(p, -p) = \int_q K(q) \text{Tr} \left[ h_F(q - p) \gamma_\beta - h_F(q + p) \gamma_\beta \right].$$

The analyticity of $d_{\alpha\beta}(p, -p)$ at $p = 0$ demands that the rhs (82) be free of quadratic terms in $p$. (If there were, we would obtain a nonlocal $p^2 \ln p$ dependence.) To expand the integral on the rhs of (82) in powers of $p$, we use

$$\text{Tr} a_R d b d e = 2 \left[ (ab)(cd) + (ad)(bc) - (ac)(bd) + \epsilon_{\alpha\beta\gamma\delta} a_\alpha b_\beta c_\gamma d_\delta \right],$$

where $\epsilon_{1234} = 1$, and

$$h(q + p) = h(q) + (2qp + p^2)h'(q) + \frac{1}{2}(2qp)^2h''(q) + O(p^3),$$

where $h'(q) \equiv \frac{d}{dq}h(q)$, etc. We obtain

$$(-) \int_q \text{Tr} f_F(q) \left( \gamma_\alpha h_F(q + p) \gamma_\beta + \gamma_\beta h_F(q - p) \gamma_\alpha \right) \xrightarrow{p \to 0} 2\delta_{\alpha\beta} \int_q f(q)h(q)q^2 - b_2(p^2 \delta_{\alpha\beta} - p_\alpha p_\beta),$$

where

$$b_2 \equiv -4 \int_q f(q) \left( q^2h'(q) + \frac{1}{3} q^4h''(q) \right) = \frac{1}{(4\pi)^2} \frac{4}{3}. $$

The integrand is a total derivative, and the value of the integral is independent of the choice of the cutoff function $K(p)$. (See SubSec. 3 of Appendix B for the calculation.)
Hence, with the choice
\[ b = b_2 = \frac{1}{(4\pi)^2} \frac{4}{3}, \]  
the general solution of (82) is given by
\[
d_{\alpha\beta}(p, -p) = -\delta_{\alpha\beta} \int_q f(q) h(q) q^2
\]
\[ + \int_0^1 dt e^{-2t} \left[ (-) \int_q \text{Tr} f_F(q) \left( \gamma_\alpha h_F(q + pe^t) \gamma_\beta + \gamma_\beta h_F(q - pe^t) \gamma_\alpha \right) \right. \]
\[ + \int_q \text{Tr} f_F(q) \left( \gamma_\alpha h_F(q) \gamma_\beta + (\alpha \leftrightarrow \beta) \right) + 4b \left( p^2 \delta_{\alpha\beta} - p_\alpha p_\beta \right) e^{2t} \left. \right] \]
\[ + Ap^2 \delta_{\alpha\beta} + B \left( p_\alpha p_\beta - p^2 \delta_{\alpha\beta} \right), \]  
(89)
where \( A, B \) are free parameters. The subtractions make the integrand of order \( e^{2t} \) as \( t \to -\infty \), and the integral is convergent.

We can fix \( A \) using the WT identity (83). First note
\[
p_\alpha d_{\alpha\beta}(p, -p) \xrightarrow{p \to 0} -p_\alpha d_{\alpha\beta}(0, 0) + A p^2 p_\beta. \]  
(90)
To determine \( A \) we compute the rhs of (83):
\[
\int_q K(q) \text{Tr} \left( h_F(q - p) \gamma_\beta - h_F(q + p) \gamma_\beta \right) = -2 \int_q K(q) h(q + p) \text{Tr} \left( q + p \right) \gamma_\beta a_R \xrightarrow{p \to 0} p_\beta (-4) \int_q K(q) \left( h(q) + \frac{1}{2} q^2 h'(q) \right) \]
\[ + p^2 p_\beta (-4) \int_q K(q) \left\{ h'(q) + q^2 h''(q) + \frac{1}{6} q^4 h'''(q) \right\}. \]  
(91)
Consistency with (90) demands
\[
\int_q f(q) h(q) q^2 = \int_q K(q) \left( 4h(q) + 2q^2 h'(q) \right), \]  
(92)
and
\[ A = -4 \int_q K(q) \left\{ h'(q) + q^2 h''(q) + \frac{1}{6} q^4 h'''(q) \right\}. \]  
(93)
The first equation must hold since the WT identity is an operator equation consistent with ERG. We verify it explicitly in SubSec. 5 of Appendix B. There, in SubSec. 4, we also compute
\[ A = \frac{1}{(4\pi)^2} \frac{1}{3}. \]  
(94)
\( B \) is left arbitrary.
Let us stop here to examine the asymptotic behavior of \( d_{\alpha\beta}(p,-p) \) for large \( p \). In principle we could obtain the asymptotic behavior using the solution (89). Instead, it is easier to go back to (82) and (83), which give

\[
(p \cdot \partial_p - 2) d_{\alpha\beta}(p,-p) \xrightarrow{p \to \infty} b \left( p^2 \delta_{\alpha\beta} - p_\alpha p_\beta \right) ,
\]

\[
p_\alpha d_{\alpha\beta}(p,-p) \xrightarrow{p \to \infty} 0 .
\]

Hence, we obtain the asymptotic behavior

\[
d_{\alpha\beta}(pe^t,-pe^t) \xrightarrow{t \to +\infty} b t e^{2t} \left( p^2 \delta_{\alpha\beta} - p_\alpha p_\beta \right) ,
\]

determined by the constant \( b \). Using this we can construct the continuum limit as

\[
D_{\alpha\beta}(p,-p) \equiv \lim_{t \to \infty} e^{-2t} \left( d_{\alpha\beta}(pe^t,-pe^t) - b t e^{2t} \left( p^2 \delta_{\alpha\beta} - p_\alpha p_\beta \right) \right) .
\]

This satisfies

\[
(p \cdot \partial_p - 2) D_{\alpha\beta}(p,-p) = b \left( p^2 \delta_{\alpha\beta} - p_\alpha p_\beta \right) .
\]

Since \( D_{\alpha\beta} \) depends on the constant \( B \), we can rewrite this as

\[
(p \cdot \partial_p - 2 + b \partial_B) D_{\alpha\beta}(p,-p) = 0 .
\]

\( D_{\alpha\beta}(p,-p) \) is also transverse:

\[
p_\alpha D_{\alpha\beta}(p,-p) = 0 .
\]

The two-point function of the current is now obtained as

\[
\left\langle \left\langle J^a_\alpha(p) J^b_\beta(q) \right\rangle \right\rangle_B = \delta^{ab} \delta(p + q) D_{\alpha\beta}(p,-p) ,
\]

which is transverse, and satisfies the scaling relation

\[
(p \cdot \partial_p + q \cdot \partial_q + 2 + b \partial_B) \left\langle \left\langle J^a_\alpha(p) J^b_\beta(q) \right\rangle \right\rangle_B = 0 .
\]

6.2 Product of Three \( n = 3 \)

\( d_{\alpha\beta\gamma}(p_1,p_2,p_3) \) satisfies the ERG equation

\[
\left( \sum_{i=1}^3 p_i \cdot \partial_{p_i} - 1 \right) d_{\alpha\beta\gamma}(p_1,p_2,p_3)
= (-) \int_q \text{Tr} f_F(q) \left[ \gamma_\alpha h_F(q + p_1) \gamma_\beta h_F(q + p_1 + p_2) \gamma_\gamma \\
+ \gamma_\beta h_F(q + p_2) \gamma_\gamma h_F(q + p_2 + p_3) \gamma_\alpha + \gamma_\gamma h_F(q + p_3) \gamma_\alpha h_F(q + p_3 + p_1) \gamma_\beta \\
+ b \left[ \delta_{\alpha\beta}(p_1 - p_2) \gamma + \delta_{\beta\gamma}(p_2 - p_3) \alpha + \delta_{\gamma\alpha}(p_3 - p_1) \beta \right] \right] ,
\]
where $b$ is given by (88), and the WT identity
\[
p_{1\alpha}d_{\alpha\beta\gamma}(p_1, p_2, p_3) = d_{\beta\gamma}(p_1 + p_2, p_3) - d_{\beta\gamma}(p_2, p_3 + p_4) + \int_q K(q) \text{Tr} \left[ h_F(q - p_1) \gamma_\beta h_F(q + p_2) \gamma_\gamma - h_F(q + p_1) \gamma_\beta h_F(q + p_2) \gamma_\gamma \right] - \frac{1}{2}A\epsilon_{\alpha\beta\gamma\delta}p_{1\alpha} (p_2 - p_3)_\delta,
\]
where $A$ is to be determined.

Analyticity of $d_{\alpha\beta\gamma}$ at $p_i = 0$ requires the absence of terms linear in $p_i$ from the rhs of (104). ($p_i$ would imply nonlocal $p_i \ln p_i$.) Let us check it. Expanding the integral in momenta, we obtain the linear terms as
\[
(-)2 \int_q f(q) h(q) \gamma_\beta \gamma_\gamma - \frac{1}{2} \frac{1}{(4\pi)^2} \frac{4}{3} = -b.
\]
(See SubSec. 1 of Appendix [1]). Hence, (104) is consistent with locality.

The general solution is given by
\[
d_{\alpha\beta\gamma}(p_1, p_2, p_3)
= \int_{-\infty}^{0} dt e^{-t} \left[ (-) \int_q \text{Tr} f_F(q) \left\{ \gamma_\alpha h_F(q + p_1 e^t) \gamma_\beta h_F(q + (p_1 + p_2) e^t) \gamma_\gamma 
+ \gamma_\beta h_F(q + p_2 e^t) \gamma_\gamma h_F(q + (p_2 + p_3) e^t) \gamma_\alpha + \gamma_\gamma h_F(q + p_3 e^t) \gamma_\alpha h_F(q + (p_3 + p_1) e^t) \gamma_\beta 
- b \left\{ \delta_{\alpha\beta}(p_1 - p_2) \gamma + \delta_{\beta\gamma}(p_2 - p_3) \gamma_\alpha + \delta_{\gamma\alpha}(p_3 - p_1) \gamma_\beta \right\} e^t \right\}
+ c_{\alpha\beta\gamma\delta}p_{1\delta} + c_{\beta\gamma\alpha\delta}p_{2\delta} + c_{\gamma\alpha\beta\delta}p_{3\delta},
\]
where $c_{\alpha\beta\gamma\delta}$ are arbitrary constants, not determined by (104). We note that the integrand behaves as $e^t$ as $t \to -\infty$, and the integral is convergent. The particular form of the linear terms is required by the cyclic symmetry:
\[
d_{\alpha\beta\gamma}(p_1, p_2, p_3) = d_{\beta\gamma\alpha}(p_2, p_3, p_1) = d_{\gamma\alpha\beta}(p_3, p_1, p_2).
\]
The most general form of $c_{\alpha\beta\gamma\delta}$ is given by
\[
c_{\alpha\beta\gamma\delta} = s \delta_{\alpha\beta} \delta_{\gamma\delta} + t \delta_{\alpha\gamma} \delta_{\beta\delta} + u \delta_{\alpha\delta} \delta_{\beta\gamma},
\]
where $s, t, u$ are constants.\footnote{Cyclic symmetry allows a term proportional to $\epsilon_{\alpha\beta\gamma\delta}$, but it does not contribute to $d_{\alpha\beta\gamma}$. Similarly, $c_{\alpha\beta\gamma\delta}$ does not change if we change $s, t, u$ by the same amount. So, we could set $u$ to zero.
We now wish to show that we can choose $s, t, u$, and $A$ so that (105) is valid. Since (105) is consistent with (104), we only need to check the terms quadratic in momenta. Using
\[ d_{\alpha\beta}(p, -p) \overset{p \rightarrow 0}{\longrightarrow} -\delta_{\alpha\beta} \int q f(q)h(q)q^2 + Ap^2\delta_{\alpha\beta} + B (p_\alpha p_\beta - p^2\delta_{\alpha\beta}) , \] (111)
we obtain
\[ d_{\beta\gamma}(p_1 + p_2, p_3) - d_{\beta\gamma}(p_2, p_3 + p_1) \overset{p \rightarrow 0}{\longrightarrow} A(p_3^2 - p_2^2)\delta_{\beta\gamma} + B \{ p_{3\beta}p_{3\gamma} - p_3^2\delta_{\beta\gamma} - p_{2\beta}p_{2\gamma} + p_2^2\delta_{\beta\gamma} \} , \] (112)
where $A$ is given by (94).

We next consider the small momentum behavior of the integral on the rhs of (105):
\[
\int_q K(q) \text{Tr} \left[ h_F(q - p_1)\gamma_\beta h_F(q + p_2) - h_F(q + p_1)\gamma_\beta h_F(q - p_3) \right] \gamma_\gamma \\
= \int_q K(q) \text{Tr} \left[ h_F(q + p_1)\gamma_\beta (h_F(q - p_2) - h_F(q - p_3)) \gamma_\gamma \right] \\
= \int_q K(q) \left[ h(q + p_1)h(q - p_2)\text{Tr} a_R(q + p_1)\gamma_\beta (q - p_2)\gamma_\gamma \\
- h(q + p_1)h(q - p_3)\text{Tr} a_R(q + p_1)\gamma_\beta (q - p_3)\gamma_\gamma \right] \\
\overset{p \rightarrow 0}{\longrightarrow} \left( -2\epsilon_{\alpha\beta\delta}p_{1\alpha}(p_2 - p_3)\delta - 2(p_3^2 - p_2^2)\delta_{\beta\gamma} \right) \int_q K(q) \left( h(q)q^2 + h(q)q^2h'(q) \right) \\
+ (p_{3\beta}p_{3\gamma} - p_3^2\delta_{\beta\gamma} - p_{2\beta}p_{2\gamma} + p_2^2\delta_{\beta\gamma}) \\
\times \int_q K(q) \left( -4h(q)^2 - 6h(q)q^2h'(q) - \frac{4}{3}(q^2h'(q))^2 - \frac{2}{3}h(q)q^4h''(q) \right) , \] (113)
where the first integral, whose integrand is a total derivative, can be calculated as
\[
\int_q K(q) \left( h(q)^2 + h(q)q^2h'(q) \right) = \frac{1}{(4\pi)^2} \frac{1}{6} . \] (114)
(See SubSec. 2 of Appendix D) Hence, we obtain the rhs of (105) as
\[
\text{rhs} \overset{p \rightarrow 0}{\longrightarrow} \left( p_{3\beta}p_{3\gamma} - p_3^2\delta_{\beta\gamma} - p_{2\beta}p_{2\gamma} + p_2^2\delta_{\beta\gamma} \right) \\
\times \left[ B - \int_q K \left( 4h^2 + 6hq^2h' + \frac{4}{3}(q^2h')^2 + \frac{2}{3}hq^4h'' \right) \right] \\
+ \left( \frac{1}{(4\pi)^2} \frac{1}{6} - \frac{1}{2} A \right) \epsilon_{\alpha\beta\gamma\delta}p_{1\alpha}(p_2 - p_3)\delta . \] (115)

We next compute the small momentum behavior of the lhs of (105). From
\[ d_{\alpha\gamma}(p_1, p_2, p_3) \overset{p \rightarrow 0}{\longrightarrow} \delta_{\alpha\beta}(s p_1 + t p_2 + up_3)\gamma + \delta_{\beta\gamma}(u p_1 + s p_2 + t p_3)\alpha + \delta_{\gamma\alpha}(t p_1 + u p_2 + s p_3)\beta , \] (116)
we obtain
\[
p_{1\alpha}d_{\alpha\beta\gamma}(p_1, p_2, p_3) \xrightarrow{p_i \to 0} p_{1\alpha} \left( c_{\alpha\beta\gamma\delta} p_{1\delta} + c_{\beta\gamma\alpha\delta} p_{2\delta} + c_{\gamma\alpha\beta\delta} p_{3\delta} \right)
\]
\[
= \left( p_{2\beta} p_{2\gamma} - \delta_{\beta\gamma} p_2^2 \right) \left( s - u \right) + \left( p_{3\beta} p_{3\gamma} - p_3^2 \delta_{\beta\gamma} \right) \left( t - u \right)
\]
\[
+ \left( s + t - 2u \right) \left( p_{2\beta} p_{3\gamma} - \delta_{\beta\gamma} (p_2 p_3) \right) .
\tag{117}
\]
Matching this with (115), we obtain
\[
u = \frac{1}{2} (s + t) ,
\tag{118a}
\]
\[
\frac{1}{2} (s - t) = -B + \int_q K(q) \left( 4q^2 + 6hq^2 h' + \frac{4}{3} (q^2 h')^2 + \frac{2}{3} h q h'' \right) ,
\tag{118b}
\]
which determine the low momentum behavior
\[
d_{\alpha\beta\gamma}(p_1, p_2, p_3) \xrightarrow{p_i \to 0} \frac{1}{2} (s - t) \left( \delta_{\alpha\beta} (p_1 - p_2)_\gamma + \delta_{\beta\gamma} (p_2 - p_3)_\alpha + \delta_{\gamma\alpha} (p_3 - p_1)_\beta \right) .
\tag{119}
\]
We also obtain the coefficient of the anomaly as\footnote{It was first pointed out in [14] that the chiral anomaly comes from the short-distance singularity of three currents. The calculation following this suggestion was completed in coordinate space in [15].}
\[
\mathcal{A} = \frac{1}{(4\pi)^2} \frac{2}{3}.
\tag{120}
\]
Let us stop here to examine the asymptotic behavior of $d_{\alpha\beta\gamma}(p_1, p_2, p_3)$ for large momenta.

Instead of taking the asymptotic limit of (108), we go back to (104) and (105). For large momenta, (104) gives
\[
\left( \sum_{i=1}^3 p_i \cdot \partial_{p_i} - 1 \right) d_{\alpha\beta\gamma}(p_1, p_2, p_3) \xrightarrow{p_i \to \infty} b \left[ \delta_{\alpha\beta} (p_1 - p_2)_\gamma + \delta_{\beta\gamma} (p_2 - p_3)_\alpha + \delta_{\gamma\alpha} (p_3 - p_1)_\beta \right] ,
\tag{121}
\]
and (105) gives
\[
p_{1\alpha}d_{\alpha\beta\gamma}(p_1, p_2, p_3) \xrightarrow{p_i \to \infty} d_{\beta\gamma}(-p_3, p_3) - d_{\beta\gamma}(p_2, -p_2) - \frac{1}{2} \mathcal{A} \epsilon_{\alpha\beta\gamma\delta} p_{1\alpha} (p_2 - p_3)_\delta .
\tag{122}
\]
(121) gives the dominant asymptotic behavior
\[
d_{\alpha\beta\gamma}(p_1 e^t, p_2 e^t, p_3 e^t) \xrightarrow{t \to \infty} b t e^t \left[ \delta_{\alpha\beta} (p_1 - p_2)_\gamma + \delta_{\beta\gamma} (p_2 - p_3)_\alpha + \delta_{\gamma\alpha} (p_3 - p_1)_\beta \right] ,
\tag{123}
\]
which is proportional to the coefficient $b$. Hence, we can construct the continuum limit as
\[
D_{\alpha\beta\gamma}(p_1, p_2, p_3) \equiv \lim_{t \to +\infty} e^{-t} \left[ d_{\alpha\beta\gamma}(p_1 e^t, p_2 e^t, p_3 e^t) \right.
\]
\[
- b t e^t \left\{ \delta_{\alpha\beta} (p_1 - p_2)_\gamma + \delta_{\beta\gamma} (p_2 - p_3)_\alpha + \delta_{\gamma\alpha} (p_3 - p_1)_\beta \right\} .
\tag{124}
\]
This satisfies the scaling relation

\[
\left( \sum_{i=1}^{3} p_i \cdot \partial_{p_i} - 1 \right) D_{\alpha\beta\gamma}(p_1, p_2, p_3) = b \left\{ \delta_{\alpha\beta}(p_1 - p_2)\gamma + \delta_{\beta\gamma}(p_2 - p_3)\alpha + \delta_{\gamma\alpha}(p_3 - p_1)\beta \right\} \\
= -b \partial_B D_{\alpha\beta\gamma}(p_1, p_2, p_3),
\]

(125)

and the WT identity:

\[
p_{1\alpha} D_{\alpha\beta\gamma}(p_1, p_2, p_3) = D_{\beta\gamma}(-p_3, p_3) - D_{\beta\gamma}(p_2, -p_2) - \frac{1}{2} A \epsilon_{\alpha\beta\gamma\delta} p_{1\alpha} (p_2 - p_3)\delta.
\]

(126)

The continuum limit of the connected three-point function defined by

\[
\left\langle \left\langle J_{\alpha}^a(p_1) J_{\beta}^b(p_2) J_{\gamma}^c(p_3) \right\rangle \right\rangle_B^{\text{conn}}
\]

\[
\equiv \delta(p_1 + p_2 + p_3) \left[ \text{Tr} \, T^a T^b T^c D_{\alpha\beta\gamma}(p_1, p_2, p_3) + \text{Tr} \, T^a T^b D_{\alpha\beta\gamma}(p_1, p_2, p_3) \right]
\]

(127)

satisfies the scaling relation

\[
\left( \sum_{i=1}^{3} p_i \cdot \partial_{p_i} + 3 + b \partial_B \right) \left\langle \left\langle J_{\alpha}^a(p_1) J_{\beta}^b(p_2) J_{\gamma}^c(p_3) \right\rangle \right\rangle_B^{\text{conn}} = 0,
\]

(128)

and the WT identity

\[
p_{1\alpha} \left\langle \left\langle J_{\alpha}^a(p_1) J_{\beta}^b(p_2) J_{\gamma}^c(p_3) \right\rangle \right\rangle_B^{\text{conn}}
\]

\[
= if^{abcd} \left\langle \left\langle J_{\beta}^d(p_1 + p_2) J_{\gamma}^c(p_3) \right\rangle \right\rangle_B^{\text{conn}} + if^{acdb} \left\langle \left\langle J_{\beta}^d(p_2) J_{\gamma}^c(p_1 + p_3) \right\rangle \right\rangle_B^{\text{conn}}
\]

\[
- \frac{1}{2} A \text{Tr} \left\{ T^b, T^c \right\} \epsilon_{\alpha\beta\gamma\delta} p_{1\alpha} (p_2 - p_3)\delta (p_1 + p_2 + p_3).
\]

(129)

6.3 Product of Four \(n = 4\)

\(d_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)\) must satisfy the ERG equation

\[
\sum_{i=1}^{4} p_i \cdot \partial_{p_i} d_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)
\]

\[
= (-) \int \text{Tr} f_F(q) \left[ \gamma_\alpha h_F(q + p_1) \gamma_\beta h_F(q + p_1 + p_2) \gamma_\gamma h_F(q + p_1 + p_2 + p_3) \gamma_\delta \\
+ \gamma_\beta h_F(q + p_2) \gamma_\gamma h_F(q + p_2 + p_3) \gamma_\delta h_F(q + p_2 + p_3 + p_4) \gamma_\alpha \\
+ \gamma_\gamma h_F(q + p_3) \gamma_\delta h_F(q + p_3 + p_4) \gamma_\alpha h_F(q + p_3 + p_4 + p_1) \gamma_\beta \\
+ \gamma_\delta h_F(q + p_4) \gamma_\alpha h_F(q + p_4 + p_1) \gamma_\beta h_F(q + p_4 + p_1 + p_2) \gamma_\gamma \right] \\
+ b \left( \delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\beta\gamma}\delta_{\alpha\delta} - 2 \delta_{\alpha\gamma}\delta_{\beta\delta} \right),
\]

(130)
where \( b \) is given by (88), and the WT identity

\[
p_{1\alpha} d_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) = d_{\beta\gamma\delta}(p_1 + p_2, p_3, p_4) - d_{\beta\gamma\delta}(p_2, p_3, p_4 + p_1) \\
+ \int_q K(q) \text{Tr} \left[ h_F(q - p_1) \gamma_\beta h_F(q + p_2) \gamma_\gamma h_F(q + p_2 + p_3) \right. \\
- h_F(q + p_1) \gamma_\beta h_F(q + p_1 + p_2) \gamma_\gamma h_F(q + p_1 + p_2 + p_3) \left. \right] \gamma_\delta \\
- \frac{1}{2} A_{p_1\alpha} \epsilon_{\alpha\beta\gamma\delta}, \tag{131}
\]

where \( A \) is given by (120).

We would like to check two things. As for (130), we would like to check the vanishing of the rhs at zero momenta. (A constant would imply nonlocal ln \( p \).) As for (131), we would like to check its validity at the first order in momenta.

The rhs of (130) gives

\[
\text{(rhs)} \overset{p_i \to 0}{\longrightarrow} \left[ (-) \frac{1}{6} \int_q f(q) h(q)^3 q^4 \times 16 + b \right] \left( \delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\beta\gamma} \delta_{\alpha\delta} - 2 \delta_{\alpha\gamma} \delta_{\beta\delta} \right). \tag{132}
\]

The integrand is a total derivative, and we obtain

\[
\frac{8}{3} \int_q f(q) h(q)^3 q^4 = \frac{1}{(4\pi)^2} \frac{4}{3} = b. \tag{133}
\]

(See SubSec. 1 of Appendix B) Hence, the rhs vanishes at zero momenta as desired.

We now wish to check (131) to first order in momenta. (130) determines only the momentum dependence of \( d_{\alpha\beta\gamma\delta} \), but its value at \( p_i = 0 \) is left undetermined. The most general form, consistent with cyclic symmetry, is

\[
d_{\alpha\beta\gamma\delta}(0, 0, 0, 0) = s_4 \left( \delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\beta\gamma} \delta_{\alpha\delta} \right) + t_4 \delta_{\alpha\gamma} \delta_{\beta\delta}, \tag{134}
\]

where \( s_4, t_4 \) are constants so that

\[
p_{1\alpha} d_{\alpha\beta\gamma\delta}(0, 0, 0, 0) = s_4 \left( p_1 \delta_{\gamma\delta} + p_1 \delta_{\beta\gamma} \right) + t_4 p_1 \delta_{\gamma\delta}. \tag{135}
\]

To compare this with the rhs, we first compute

\[
d_{\beta\gamma\delta}(p_1 + p_2, p_3, p_4) - d_{\beta\gamma\delta}(p_2, p_3, p_4 + p_1) \overset{p_i \to 0}{\longrightarrow} \frac{1}{2} \left( s - t \right) \left( p_1 \delta_{\beta\gamma} + p_1 \delta_{\gamma\delta} - 2p_1 \delta_{\beta\delta} \right). \tag{136}
\]
Hence, the rhs of (131) is
\[
\int_q K(q) \text{Tr} \left[ h_F(q - p_1) \gamma_\beta h_F(q + p_2) \gamma_\gamma h_F(q + p_2 + p_3) \right] \gamma_\delta 
\]
\[
= \int_q K(q) \left[ h(q - p_1) h(q + p_2) h(q + p_2 + p_3) \text{Tr} (\mathbf{q} - \mathbf{p}_1) \gamma_\beta (\mathbf{q} + \mathbf{p}_2) \gamma_\gamma (\mathbf{q} + \mathbf{q} + \mathbf{p}_3) \right. 
\]
\[
- h(q + p_1) h(q + p_1 + p_2) h(q + p_1 + p_2 + p_3) 
\]
\[
\times \text{Tr} (\mathbf{q} + \mathbf{p}_1) \gamma_\beta (\mathbf{q} + \mathbf{p}_1 + \mathbf{p}_2) \gamma_\gamma (\mathbf{q} + \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \right] \gamma_\delta a_R 
\]
\[
p_{t\rightarrow 0} \int_q K(q) h(q)^2 q^2 \left( h(q) + q^2 h'(q) \right) 
\]
\[
+ (p_{1\beta} \delta_\gamma \delta + p_{1\delta} \delta_\beta \gamma) \int_q K(q) h(q)^2 q^2 \left( 2h(q) + \frac{4}{3} q^2 h'(q) \right) 
\]
\[
+ p_{1\gamma} \delta_\beta \delta \frac{4}{3} \int_q K(q) h(q)^2 q^4 h'(q) 
\]
\[
+ p_{1\alpha} \delta_\alpha_\beta \delta a_R \left( 4 \int_q K(q) h(q)^2 q^2 \left( h(q) + q^2 h'(q) \right) - \frac{1}{2} A \right) . 
\]

Hence, the rhs of (131) is
\[
\text{rhs} \xrightarrow{p_{t\rightarrow 0}} \frac{1}{2} (s - t) \left( p_{1\beta} \delta_\gamma \delta + p_{1\delta} \delta_\beta \gamma - 2 p_{1\gamma} \delta_\beta \delta \right) 
\]
\[
+ (p_{1\beta} \delta_\gamma \delta + p_{1\delta} \delta_\beta \gamma) \int_q K(q) h(q)^2 q^2 \left( 2h(q) + \frac{4}{3} q^2 h'(q) \right) 
\]
\[
+ p_{1\gamma} \delta_\beta \delta \frac{4}{3} \int_q K(q) h(q)^2 q^4 h'(q) 
\]
\[
+ p_{1\alpha} \delta_\alpha_\beta \delta a_R \left( 4 \int_q K(q) h(q)^2 q^2 \left( h(q) + q^2 h'(q) \right) - \frac{1}{2} A \right) . 
\]

The last term vanishes because
\[
\int_q K(q) h(q)^2 q^2 \left( h(q) + q^2 h'(q) \right) = \frac{1}{(4\pi)^2} \frac{1}{12} . 
\]

(See SubSec. 2 of Appendix [B]) We can make (138) match with (135) by choosing
\[
s_4 = \frac{1}{2} (s - t) + \int_q K(q) h(q)^2 q^2 \left( 2h(q) + \frac{4}{3} q^2 h'(q) \right) 
\]
\[
= -B + \int_q K \left( 4h^2 + 6h^2 q^2 h' + \frac{4}{3} (q^2 h')^2 + \frac{2}{3} h(q)^4 h'' + 2h^3 q^2 + \frac{4}{3} h^2 q^4 h' \right) , 
\]
\[
t_4 = -(s - t) + \frac{4}{3} \int_q K(q) h(q)^2 q^4 h'(q) 
\]
\[
= -2s_4 + 4 \int_q K(q) h(q)^2 q^2 \left( h(q) + q^2 h'(q) \right) = -2s_4 + \frac{1}{(4\pi)^2} \frac{1}{3} . 
\]

We have thus checked the validity of (131).
Hence, the connected four-point function defined by (130) and (131) give
\[
\sum_{i=1}^{4} p_i \cdot \partial p_i d_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) \xrightarrow{p_i \to \infty} b \left( \delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\beta\gamma} \delta_{\alpha\delta} - 2\delta_{\alpha\gamma} \delta_{\beta\delta} \right), \tag{141}
\]
and the WT identity
\[
p_{1\alpha} d_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) \xrightarrow{p_i \to \infty} d_{\beta\gamma\delta}(-p_3 - p_4, p_3, p_4) - d_{\beta\gamma\delta}(p_2, p_3, -p_2 - p_3) - \frac{1}{2} A p_{1\alpha} \epsilon_{\alpha\beta\gamma\delta}. \tag{142}
\]
The first equation gives the asymptotic behavior
\[
d_{\alpha\beta\gamma\delta}(p_1 e^t, p_2 e^t, p_3 e^t, p_4 e^t) \xrightarrow{t \to \infty} b t \left( \delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\beta\gamma} \delta_{\alpha\delta} - 2\delta_{\alpha\gamma} \delta_{\beta\delta} \right). \tag{143}
\]
Hence, a continuum limit is obtained as
\[
D_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)
\equiv \lim_{t \to +\infty} \left[ d_{\alpha\beta\gamma\delta}(p_1 e^t, p_2 e^t, p_3 e^t, p_4 e^t) - b t \left( \delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\beta\gamma} \delta_{\alpha\delta} - 2\delta_{\alpha\gamma} \delta_{\beta\delta} \right) \right], \tag{144}
\]
which satisfies the scaling relation
\[
\sum_{i=1}^{4} p_i \cdot \partial p_i D_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) = b \left( \delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\beta\gamma} \delta_{\alpha\delta} - 2\delta_{\alpha\gamma} \delta_{\beta\delta} \right)
= -b \partial_B D_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4), \tag{145}
\]
and the WT identity
\[
p_{1\alpha} D_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) = D_{\beta\gamma\delta}(-p_3 - p_4, p_3, p_4) - D_{\beta\gamma\delta}(p_2, p_3, -p_2 - p_3) - \frac{1}{2} A p_{1\alpha} \epsilon_{\alpha\beta\gamma\delta}. \tag{146}
\]
Hence, the connected four-point function defined by
\[
\left\langle J_{\alpha}^a(p_1) J_{\beta}^b(p_2) J_{\gamma}^c(p_3) J_{\delta}^d(p_4) \right\rangle_{B}^{\text{conn}} \equiv \delta(p_1 + p_2 + p_3 + p_4) \left[ \text{Tr} T^a T^b T^c T^d D_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) + \cdots \right] \tag{147}
\]
satisfies the scaling relation
\[
\left( \sum_{i=1}^{4} p_i \cdot \partial p_i + 4 + b \partial_B \right) \left\langle J_{\alpha}^a(p_1) J_{\beta}^b(p_2) J_{\gamma}^c(p_3) J_{\delta}^d(p_4) \right\rangle_{B}^{\text{conn}} = 0 \tag{148}
\]
and the WT identity
\[
p_{1\alpha} \left\langle J_{\alpha}^a(p_1) J_{\beta}^b(p_2) J_{\gamma}^c(p_3) J_{\delta}^d(p_4) \right\rangle_{B}^{\text{conn}} = i f^{abc} \left\langle J_{\beta}^c(p_1 + p_2) J_{\gamma}^c(p_3) J_{\delta}^d(p_4) \right\rangle_{B}^{\text{conn}} + \cdots
\quad - \frac{1}{2} A p_{1\alpha} \epsilon_{\alpha\beta\gamma\delta} \delta(p_1 + p_2 + p_3 + p_4) \times \text{Tr} T^a \left( T^b[T^c, T^d] + T^c[T^d, T^b] + T^d[T^b, T^c] \right). \tag{149}
\]
6.4 Recapitulation

Let us recapitulate the results of this section by writing down equations for $e^{W[A]}$, a composite operator of scale dimension 0. The ERG differential equation is given by

$$\left(\int_p (-p \cdot \partial_p - D + 1) A^a_\mu(p) \cdot \frac{\delta}{\delta A^a_\mu(p)} - D\right) e^{W[A]} = \frac{1}{(4\pi)^2} \frac{4}{3} \int d^4x \, \text{Tr} \left( \partial_\alpha A_\beta - \partial_\beta A_\alpha - i[A_\alpha, A_\beta]\right) \left( \partial_\alpha A_\beta - \partial_\beta A_\alpha - i[A_\alpha, A_\beta]\right) e^{W[A]} .$$

(150)

The WT identity is given by

$$\delta e^{W[A]} \equiv \int_p \left(-p_\mu e^a(p) + i f^{abc} \int_q A^b_\mu(p + q) e^c(-q)\right) \frac{\delta}{\delta A^a_\mu(p)} e^{W[A]}$$

$$= \left[\int_p e^a(-p) \mathcal{E}^a(p) \ast + \frac{1}{(4\pi)^2} \frac{2}{3} \int d^4x \, \epsilon_{\alpha\beta\gamma\delta} \text{Tr} \partial_\alpha \epsilon \left( A_\beta \partial_\gamma A_\delta - \frac{i}{2} A_\beta A_\gamma A_\delta\right)\right] e^{W[A]} .$$

(151)

$W[A]$ is determined uniquely by the above two equations up to a constant multiple of the gauge invariant

$$\frac{1}{4} \int d^4x \, \text{Tr} \left( \partial_\alpha A_\beta - \partial_\beta A_\alpha - i[A_\alpha, A_\beta]\right) \left( \partial_\alpha A_\beta - \partial_\beta A_\alpha - i[A_\alpha, A_\beta]\right) .$$

If we define

$$W_g[A] = -\frac{1}{4g} \int d^4x \, \text{Tr} \left( \partial_\alpha A_\beta - \partial_\beta A_\alpha - i[A_\alpha, A_\beta]\right) \left( \partial_\alpha A_\beta - \partial_\beta A_\alpha - i[A_\alpha, A_\beta]\right) + W[A] ,$$

(152)

we can rewrite the ERG equation as

$$\left(\beta(g) \partial_g + \int_p (-p \cdot \partial_p - D + 1) A^a_\mu(p) \cdot \frac{\delta}{\delta A^a_\mu(p)} - D\right) e^{W_g[A]} = 0$$

(153)

where

$$\beta(g) = -\frac{1}{(4\pi)^2} \frac{4}{3} g^2$$

(154)

is the 1-loop beta function.

7 Conclusions

In this paper we have discussed the multiple products of current operators using the exact renormalization group (ERG) formalism. The multiple products are characterized by
two mutually consistent equations: one is the ERG differential equation and the other is the Ward-Takahashi (WT) identity. We have argued that these two equations suffer changes due to the short-distance singularities of the products, and the revised equations are given by (61) for ERG and (64) for the WT identity. In Sec. VI we have calculated the multiple products explicitly by solving these equations for the Gaussian fixed-point. The guiding principle in these calculations is the locality of the operators. Since the momenta below the cutoff have not been integrated, the coefficient functions for the products of the current are analytic at zero momenta.

There are some future directions we can consider. We may consider a theory such as QCD with fields other than the chiral fermions. Or we may consider a more nontrivial fixed-point Wilson action. We also think it interesting to study the multiple products of other composite operators such as the energy-momentum tensor.

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A  Invariance of the Wilson action

Given a Wilson action $S[\psi, \bar{\psi}]$, its invariance under global flavor transformations is most straightforwardly given by

$$\int_p \left[ \bar{\psi}(-p) T^a \frac{\delta}{\delta \psi(-p)} S_t - S_t \frac{\delta}{\delta \psi(p)} T^a \psi(p) \right] = 0. \quad (A1)$$

We wish to show that this is equal to (23) which is

$$\mathcal{E}^a(0) \equiv e^{-S_t} \int_p K(p) \text{Tr} \left[ \frac{\delta}{\delta \psi(-p)} \left( \bar{\Psi}(-p) e^{S_t} - (e^{S_t} T^a \Psi(p)) \frac{\delta}{\delta \psi(p)} \right) \right] = 0, \quad (A2)$$

where

$$\Psi(p) = \frac{1}{K(p)} \left( \psi(p) + h_F(p) \frac{\delta}{\delta \psi(-p)} S_t \right), \quad (A3a)$$

$$\bar{\Psi}(-p) = \frac{1}{K(p)} \left( \bar{\psi}(-p) + S_t \frac{\delta}{\delta \psi(p)} h_F(p) \right). \quad (A3b)$$

Substituting (A3) into (A2), we obtain

$$\mathcal{E}^a(0) = \int_p \left[ - \left( \bar{\psi}(-p) + S_t \frac{\delta}{\delta \psi(p)} h_F(p) \right) T^a \frac{\delta}{\delta \psi(-p)} S_t \right.$$

$$+ S_t \frac{\delta}{\delta \psi(p)} \left( \psi(p) + h_F(p) \frac{\delta}{\delta \psi(-p)} S_t \right) \left. \right]$$

$$+ \int_p \text{Tr} \left[ \delta(0) T^a - \delta(0) T^a \right]$$

$$+ \int_p \text{Tr} \left[ \frac{\delta}{\delta \psi(-p)} \frac{\delta}{\delta \psi(p)} S_t \right]$$

$$= \int_p \left[ - \bar{\psi}(-p) T^a \frac{\delta}{\delta \psi(-p)} S_t + S_t \frac{\delta}{\delta \psi(p)} T^a \frac{\delta}{\delta \psi(-p)} \right] = 0, \quad (A4)$$

which is (A1).

B  Universal Cutoff Integrals

We give four integrals involving a cutoff function $K(p)$. The values of these integrals are universal in the sense that they do not depend on the choice of $K(p)$ as long as $K(0) = 1$
and $K(p)$ vanishes asymptotically as $p^2 \to \infty$. The functions $h$ and $f$ are defined by

$$h(p) \equiv \frac{1 - K(p)}{p^2},$$  \hspace{1cm} (B1a)

$$f(p) \equiv (p \cdot \partial_p + 2) h(p) = \frac{\Delta(p)}{p^2},$$  \hspace{1cm} (B1b)

$$\Delta(p) \equiv -p \cdot \partial_p K(p).$$  \hspace{1cm} (B1c)

**B.1**

For $n = 0, 1, 2, \cdots$, we obtain

$$\int_q f(q) h(q) (q^2 h(q))^n = \int_q (q \cdot \partial_q + 2) h(q) \cdot \frac{1}{q^2} (q^2 h(q))^{n+1}$$

$$= \int_q \frac{1}{q^4} q \cdot \partial_q \left\{ \frac{(q^2 h(q))^{n+2}}{n+2} \right\}$$

$$= \frac{2\pi^2}{(2\pi)^4} \int_0^\infty dq^2 \frac{d}{dq^2} \left\{ \frac{(q^2 h(q))^{n+2}}{n+2} \right\}$$

$$= \frac{1}{(4\pi)^2} \frac{2}{n+2}.  \hspace{1cm} (B2)$$

**B.2**

For $n = 0, 1, 2, \cdots$, we obtain

$$\int_q K(q) h(q) (q^2 h(q))^n (h(q) + q^2 h'(q)) = \int_q (1 - q^2 h(q)) h(q) (q^2 h(q))^n \frac{d}{dq^2} (q^2 h(q))$$

$$= \int_q \frac{1}{q^2} (1 - q^2 h(q)) (q^2 h(q))^{n+1} \frac{d}{dq^2} (q^2 h(q))$$

$$= \frac{1}{(4\pi)^2} \int_0^\infty dq^2 \frac{d}{dq^2} \left( \frac{(q^2 h(q))^{n+2}}{n+2} - \frac{(q^2 h(q))^{n+3}}{n+3} \right)$$

$$= \frac{1}{(4\pi)^2} \frac{1}{(n+2)(n+3)}.  \hspace{1cm} (B3)$$
\[ B.3 \]
\[
\int_q f(q) \left( q^2 h'(q) + \frac{1}{3} q^4 h''(q) \right) = \frac{1}{(4\pi)^2} \int_0^\infty x dx \left( x \frac{d}{dx} + \frac{1}{3} x^2 \frac{d^2}{dx^2} \right) h(x) = 2(x + 1)h(x)
\]
\[
= \frac{2}{(4\pi)^2} \int_0^\infty dx x \left( 1 + x \frac{d}{dx} \right) h(x) \cdot x \left( x \frac{d}{dx} + \frac{1}{3} x^2 \frac{d^2}{dx^2} \right) h(x)
\]
\[
= \frac{2}{(4\pi)^2} \int_0^\infty dx \frac{d}{dx} \left( \frac{1}{3} x^3 h'(x) + \frac{1}{6} x^4 h'(x)^2 \right)
\]
\[
= \frac{2}{(4\pi)^2} \left( -\frac{1}{3} + \frac{1}{6} \right) = -\frac{1}{(4\pi)^2} \frac{1}{3}.
\] (B4)

\[ B.4 \]
\[
\int_q K(q) \left( h'(q) + q^2 h''(q) + \frac{1}{6} q^4 h'''(q) \right)
\]
\[
= \frac{1}{(4\pi)^2} \int_0^\infty dq^2 q^2 K(q) \left( h'(q) + q^2 h''(q) + \frac{1}{6} h'''(q) \right)
\]
\[
= \frac{1}{(4\pi)^2} \int_0^\infty dq^2 \frac{d}{dq^2} \left[ -\frac{1}{6} q^4 K(q) K''(q) + \frac{1}{6} \left( q^4 \frac{1}{2} K'(q)^2 - q^2 K(q) K'(q) \right) + \frac{1}{12} K(q)^2 \right]
\]
\[
= -\frac{1}{(4\pi)^2} \frac{1}{12}.
\] (B5)

\[ B.5 \] Check of (92)

We wish to check (92) in sec. 6, which can be written as
\[
\int_q (f(q) h(q) q^2 - K(q)(2h(q) + f(q))) = 0.
\] (B6)

The integrand is a total derivative:
\[
(q \cdot \partial_q + 4) (K(q) h(q)) = -\Delta(q) h(q) + K(q) f(q) + 2K(q) h(q)
\]
\[
= -q^2 f(q) h(q) + K(q) (2h(q) + f(q)).
\] (B7)

Since \( K(q) h(q) \) vanishes at \( q^2 = 0, \infty \), the integral vanishes.

C Corrections to the ERG equation and the WT identity

Differentiating (61) and (64) with respect to the source \( A \), we obtain the ERG equation and the WT identity for the products of current operators.
C.1 Product of Two

The ERG differential equation is

\[
(\partial_t + p_1 \cdot \partial_{p_1} + p_2 \cdot \partial_{p_2} + 2 - D_t) \left[ J^a_\alpha(p_1) J^b_\beta(p_2) \right] = b(t) \delta(p_1 + p_2) \delta^{ab} (p_1 \delta_{\alpha \beta} - p_2 \delta_{\alpha \beta}).
\]

(C1)

The WT identity has no anomaly.

\[
p_\alpha \left[ J^a_\alpha(p_1) J^b_\beta(p_1) \right] = if^{abc} J^c_\gamma(p_1) + \mathcal{E}^a(p_1) \star J^b_\beta(p_1).
\]

(C2)

C.2 Product of Three

ERG differential equation

\[
\left( \sum_{i=1}^3 p_i \cdot \partial_{p_i} + 3 - D_t \right) \left[ J^a_\alpha(p_1) J^b_\beta(p_2) J^c_\gamma(p_3) \right] = b(t) \left[ \delta \left( \sum_{i=1}^3 p_i \right) \text{Tr} T^a \left[ T^b, T^c \right] \left\{ \delta_{\alpha \beta} (p_1 - p_2)_\gamma + \delta_{\beta \gamma} (p_2 - p_3)_\alpha + \delta_{\gamma \alpha} (p_3 - p_1)_\beta \right\} 
- \left\{ \delta(p_1 + p_2) (p_{1 \alpha} p_{1 \beta} - p_{1}^2 \delta_{\alpha \beta}) \delta^{ab} J^c_\gamma(p_3) + \delta(p_2 + p_3) (p_{2 \beta} p_{2 \gamma} - p_{2}^2 \delta_{\beta \gamma}) \delta^{bc} J^a_\alpha(p_1) 
+ \delta(p_3 + p_1) (p_{3 \gamma} p_{3 \alpha} - p_{3}^2 \delta_{\gamma \alpha}) \delta^{ca} J^b_\beta(p_2) \right\} \right].
\]

(C3)

The WT identity can be anomalous:

\[
p_\alpha \left[ J^a_\alpha(p_1) J^b_\beta(q) J^c_\gamma(r) \right] = if^{abcd} J^d_\beta(p + q) J^c_\gamma(r) + if^{acd} J^b_\beta(q) J^d_\gamma(p + r) + \mathcal{E}^a(p_1) \star J^b_\beta(q) J^c_\gamma(r) 
- \frac{A}{2} \delta(p + q + r) \text{Tr} T^a \left[ T^b, T^c \right] \epsilon^{\alpha \beta \gamma \delta} p_\alpha (q - r) \delta.
\]

(C4)
C.3 Product of Four

ERG differential equation

\[
\left( \sum_{i=1}^{4} p_i \cdot \partial_{p_i} + 4 - D_t \right) \left[ J^a_\alpha(p_1)J^b_\beta(p_2)J^c_\gamma(p_3)J^d_\delta(p_4) \right]
\]

\[= b(t) \left[ \delta(p_1 + p_2 + p_3 + p_4) \left( \text{Tr} T^{aTbTcTd} \left( \delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\beta\gamma}\delta_{\delta\alpha} - 2\delta_{\alpha\gamma}\delta_{\beta\delta} \right) + \text{Tr} T^{aTbTcTd} \left( \delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\beta\gamma}\delta_{\delta\alpha} - 2\delta_{\alpha\gamma}\delta_{\beta\delta} \right) + \text{Tr} T^{aTcTbd} \left( \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\beta\delta}\delta_{\alpha\gamma} - 2\delta_{\alpha\beta}\delta_{\gamma\delta} \right) + \text{Tr} T^{aTcTbd} \left( \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\beta\delta}\delta_{\alpha\gamma} - 2\delta_{\alpha\beta}\delta_{\gamma\delta} \right) \right) \]

\[+ \delta(p_1 + p_2 + p_3) \text{Tr} T^a \left( T^bT^c \right) \left( \delta_{\alpha\beta}(p_1 - p_2)\gamma + \delta_{\beta\gamma}(p_2 - p_3)\alpha + \delta_{\gamma\alpha}(p_3 - p_1)\beta \right) J^d_\delta(p_4) \]

\[+ \delta(p_1 + p_2 + p_4) \text{Tr} T^a \left( T^bT^d \right) \left( \delta_{\alpha\beta}(p_1 - p_2)\delta + \delta_{\beta\delta}(p_2 - p_4)\alpha + \delta_{\delta\alpha}(p_4 - p_1)\beta \right) J^c_\gamma(p_3) \]

\[+ \delta(p_1 + p_3 + p_4) \text{Tr} T^a \left( T^cT^d \right) \left( \delta_{\alpha\gamma}(p_1 - p_3)\delta + \delta_{\gamma\delta}(p_3 - p_4)\alpha + \delta_{\delta\alpha}(p_4 - p_1)\gamma \right) J^b_\beta(p_2) \]

\[+ \delta(p_2 + p_3 + p_4) \text{Tr} T^b \left( T^cT^d \right) \left( \delta_{\beta\gamma}(p_2 - p_3)\delta + \delta_{\gamma\delta}(p_4 - p_4)\beta + \delta_{\delta\beta}(p_4 - p_2)\gamma \right) J^a_\alpha(p_1) \]

\[+ \delta(p_1 + p_2)(p_2^2\delta_{\alpha\beta} - p_1\alpha_1p_1\beta) \delta^{ab} \left[ J^c_\gamma(p_3)J^d_\delta(p_4) \right] + \delta(p_1 + p_3)(p_2^2\delta_{\alpha\gamma} - p_1\alpha_1p_1\gamma) \delta^{ac} \left[ J^b_\beta(p_2)J^d_\gamma(p_4) \right] \]

\[+ \delta(p_1 + p_4)(p_2^2\delta_{\alpha\delta} - p_1\alpha_1p_1\delta) \delta^{ad} \left[ J^b_\beta(p_2)J^c_\gamma(p_3) \right] + \delta(p_2 + p_3)(p_2^2\delta_{\beta\delta} - p_2\beta_2p_2\delta \delta^{bc} \left[ J^a_\alpha(p_1)J^d_\delta(p_4) \right] \]

\[+ \delta(p_2 + p_4)(p_2^2\delta_{\beta\delta} - p_2p_2\delta\delta^{bd} \left[ J^a_\alpha(p_1)J^c_\gamma(p_4) \right] + \delta(p_3 + p_4)(p_2^2\delta_{\gamma\delta} - p_3\gamma_3p_3\delta) \delta^{cd} \left[ J^a_\alpha(p_1)J^b_\beta(p_2) \right] \]

(C5)

The WT identity can be anomalous:

\[p_\alpha \left[ J^a_\alpha(p)J^b_\beta(q)J^c_\gamma(r)J^d_\delta(s) \right] \]

\[= \epsilon^{abc} \left[ J^c_\beta(q + p)J^c_\gamma(r)J^d_\delta(s) \right] + \epsilon^{ace} \left[ J^b_\beta(q)J^c_\gamma(r + p)J^d_\delta(s) \right] + \epsilon^{ade} \left[ J^b_\beta(q)J^c_\gamma(r)J^d_\delta(s + p) \right] \]

\[+ \epsilon^{abc} \left[ J^c_\beta(q)J^c_\gamma(r)J^d_\delta(s) \right] \]

\[- \frac{A}{2} \left[ \delta(p + q + r + s) p_1\epsilon_{\alpha\beta\gamma\delta}\text{Tr} T^a \left( T^bT^cT^dT^e \right) \right] \]

\[+ \delta(p + q + r) \text{Tr} T^a \left( T^bT^c \right) \epsilon_{\alpha\beta\gamma\delta} p_\alpha(q - r) \epsilon^d_\delta(s) \]

\[+ \delta(p + q + s) \text{Tr} T^a \left( T^bT^d \right) \epsilon_{\alpha\beta\delta\epsilon} p_\alpha(q - s) \epsilon^c_\epsilon(r) \]

\[+ \delta(p + r + s) \text{Tr} T^a \left( T^cT^d \right) \epsilon_{\alpha\gamma\delta\epsilon} p_\alpha(r - s) \epsilon^b_\epsilon(q) . \]