A one-variable bracket polynomial for some Turk’s head knots

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Abstract

We compute the Kauffman bracket polynomial of the three-lead Turk’s head, the chain sinnet and the figure-eight chain shadow diagrams. Each of these knots can in fact be constructed by repeatedly concatenating the same 3-tangle, respectively, then taking the closure. The bracket is then evaluated by expressing the state diagrams of the concerned 3-tangle by means of the Kauffman monoid diagram’s elements.

Keywords: bracket polynomial, tangle shadow, Kauffman state, flat sinnet.

1 Introduction

The present paper is a follow-up on our previous work which aims at collecting statistics on knot shadows [5]. We would like to establish the bracket polynomial for knot diagram generated by the 3-tangle shadows below:

\begin{equation}
\begin{array}{ccc}
T & C & E
\end{array}
\end{equation}

The knot diagrams under consideration are those obtained by repeatedly multiplying (or concatenating) the same 3-tangle, then taking the closure of the resulting 3-tangle (i.e., connecting the endpoints in a standard way, without introducing further crossings between the strands). Knots obtained from the 3-tangles pictured in (1) belong to the Ashley’s Turk’s head family [1, p. 226, Chap. 17]: the three-lead Turk’s head [1, #1305], the chain sinnet [1, #1374] and the figure-eight chain [1, #1376], respectively (e.g. see Figure 1).
Three-lead Turk’s head  
Chain sinnet  
Figure-eight chain

Figure 1: Some flat Turk’s-head knot diagrams.

The remainder of this paper is arranged as follows. In section 2, we establish the expression of the bracket polynomial for any 3-tangle shadow diagram. Then in section 3, we apply those results to the flat sinnet Turk’s heads mentioned earlier.

2 The Kauffman bracket of a 3-tangle shadow

In this paper, the Kauffman bracket maps a shadow diagram $D$ to $\langle D \rangle \in \mathbb{Z}[x]$ and is constructed from the following rules:

(K1): $\langle \bigcirc \rangle = x$;

(K2): $\langle \bigcirc \sqcup D \rangle = x \langle D \rangle$;

(K3): $\langle \bigtimes \rangle = \langle \bigcirc \rangle + \langle \bigcirc \sqcup \bigcirc \rangle$.

The diagram $\bigcirc$ in (K1) represents that of a single loop, and the symbol $\sqcup$ in (K2) denotes the disjoint union operation. Formula in (K3) expresses the splitting of a crossing. Recall that the choice of such splittings for any single crossing is referred to as the so-called Kauffman state. Rules (K1), (K2) and (K3) can be summarized by the summation which is taken over all the states for $D$, namely $\langle D \rangle = \sum_{S} x^{|S|}$, where $|S|$ gives the number of loops in the state $S$. Kauffman shows that the states elements of a 3-tangle diagram $B := \begin{array}{c}
\end{array}$ are generated by the product of a loop and the following 5 elements of the 3-strand diagram monoid $D_3$ [2, 8]:

In other words, given a state $S$, there exist a nonnegative integer $k$ and an element $U$ in $D_3$ such that one writes $S = \bigcirc^k \sqcup U$, where $\bigcirc^k = \bigcirc \sqcup \bigcirc \sqcup \cdots \sqcup \bigcirc$ denotes the disjoint
union of $k$ loops [3, p. 100]. The bracket of the 3-tangle $B$ becomes
\[
\langle B \rangle = \sum_s \langle S \rangle,
\]
where $\langle S \rangle = x^{[S]} \langle U \rangle$ for certain $U \in \mathcal{D}_3$.

Therefore $\langle B \rangle$ is a linear combination of the brackets $\langle 1_3 \rangle$, $\langle U_1 \rangle$, $\langle U_2 \rangle$, $\langle r \rangle$ and $\langle s \rangle$, i.e., there exist five polynomials $a, b, c, d, e$ in $\mathbb{Z}[x]$ such that
\[
\langle B \rangle = a \langle 1_3 \rangle + b \langle U_1 \rangle + c \langle U_2 \rangle + d \langle r \rangle + e \langle s \rangle.
\]

\textbf{Lemma 1.} Given two 3-tangles $B$ and $D$, we have
\[
\langle BD \rangle = a_B a_D \langle 1_3 \rangle + (b_B a_D + (a_B + b_B x + d_B) b_D + (d_B x + b_B) e_D) \langle U_1 \rangle
+ (c_B a_D + (a_B + c_B x + e_B) c_D + (c_B + e_B x) d_D) \langle U_2 \rangle
+ (d_B a_D + (d_B x + b_B) c_D + (a_B + b_B x + d_B) d_D) \langle r \rangle
+ (e_B a_D + (e_B + e_B x) b_D + (a_B + c_B x + e_B) e_D) \langle s \rangle.
\]

\textbf{Proof.} We first establish the states of $B$ leaving $D$ intact, and then in $D$:
\[
\langle BD \rangle = a_B a_D \langle 1_3 \rangle + a_B b_D \langle 1_3 U_1 \rangle + a_B c_D \langle 1_3 U_2 \rangle + a_B D \langle 1_3 r \rangle + a_B e_D \langle 1_3 s \rangle
+ b_B a_D \langle U_1 1_3 \rangle + b_B b_D \langle U_1^2 \rangle + b_B c_D \langle U_1 U_2 \rangle + b_B D \langle U_1 r \rangle + b_B e_D \langle U_1 s \rangle
+ c_B a_D \langle U_2 1_3 \rangle + c_B b_D \langle U_2 U_3 \rangle + c_B c_D \langle U_2^2 \rangle + c_B d_D \langle U_2 r \rangle + c_B e_D \langle U_2 s \rangle
+ d_B a_D \langle r 1_3 \rangle + d_B b_D \langle r U_1 \rangle + d_B c_D \langle r U_2 \rangle + d_B d_D \langle r^2 \rangle + d_B e_D \langle r s \rangle
+ e_B a_D \langle s 1_3 \rangle + e_B b_D \langle s U_1 \rangle + e_B c_D \langle s U_2 \rangle + e_B d_D \langle s r \rangle + e_B e_D \langle s^2 \rangle.
\]

The brackets for the pairs in the right-hand side can be evaluated by applying the following multiplication table.

|   | $1_3$ | $U_1$ | $U_2$ | $r$ | $s$ |
|---|------|------|------|----|----|
| $1_3$ | $1_3$ | $U_1$ | $U_2$ | $r$ | $s$ |
| $U_1$ | $U_1$ | $\Box \Box U_1$ | $s$ | $U_1$ | $\Box \Box s$ |
| $U_2$ | $U_2$ | $r$ | $\Box \Box U_2$ | $\Box \Box r$ | $U_2$ |
| $r$ | $r$ | $\Box \Box r$ | $U_2$ | $r$ | $\Box \Box U_2$ |
| $s$ | $s$ | $\Box \Box s$ | $\Box \Box U_1$ | $s$ | $s$ |

Table 1: Multiplication of elements in $\mathcal{D}_3$.

The proof is then completed by factoring with respect to the resulting brackets, eventually simplified according to (K2).

\textbf{Notation 2.} Let $B_n := BB \cdots B$ denote the 3-tangle obtained by multiplying the 3-tangle $B$ $n$ times, with $B_0 := 1_3$. For convenience, we shall identify the bracket formal expression in (2) by the 5-tuple $[a, b, c, d, e]^T$. Similarly, assume that $\langle B_n \rangle$ is identified by $[a_n, b_n, c_n, d_n, e_n]^T$. 

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Lemma 3. The bracket 5-tuple for $B_n$ is given by

\[
\begin{bmatrix}
a_n \\
b_n \\
c_n \\
d_n \\
e_n
\end{bmatrix} = \begin{bmatrix}
a & 0 & 0 & 0 & 0 \\
b & a + bx + d & 0 & 0 & dx + b \\
c & 0 & a + cx + e & c + ex & 0 \\
d & 0 & dx + b & a + bx + d & 0 \\
e & c + ex & 0 & 0 & a + cx + e
\end{bmatrix}^n \begin{bmatrix}
a_0 \\
b_0 \\
c_0 \\
d_0 \\
e_0
\end{bmatrix}.
\]

Proof. We write $B_{n+1} = BB_n$, then from Lemma 1 we have

\[
\begin{bmatrix}
a_{n+1} \\
b_{n+1} \\
c_{n+1} \\
d_{n+1} \\
e_{n+1}
\end{bmatrix} = \begin{bmatrix}
a & 0 & 0 & 0 & 0 \\
b & a + bx + d & 0 & 0 & dx + b \\
c & 0 & a + cx + e & c + ex & 0 \\
d & 0 & dx + b & a + bx + d & 0 \\
e & c + ex & 0 & 0 & a + cx + e
\end{bmatrix}\begin{bmatrix}
a_n \\
b_n \\
c_n \\
d_n \\
e_n
\end{bmatrix}.
\]

We conclude by unfolding the recurrence and taking into consideration the initial condition $[a_0, b_0, c_0, d_0, e_0]^T = [1, 0, 0, 0, 0]^T$. □

We let $M_B$ denote the $5 \times 5$ matrix in (3), and we will later refer to it as the states matrix for the 3-tangle $B$. Using the standard method for computing (3) we obtain the characteristic polynomial for $M_B$

\[
\chi(M_B, \lambda) = - (\lambda - a) \left( \lambda - \frac{1}{2} (p - q) \right)^2 \left( \lambda - \frac{1}{2} (p + q) \right)^2,
\]

then

\[
a_n = a^n, \\
b_n = - \frac{1}{2q(x^2 - 1)} \left( 2a^n q x + \left( \frac{p - q}{2} \right)^n \left( (b - c)x^2 + (-d - e - q)x - 2b \right) + \left( \frac{p + q}{2} \right)^n \left( -b + c)x^2 + (d + e - q)x + 2b \right) \right),
\]

\[
c_n = - \frac{1}{2q(x^2 - 1)} \left( 2a^n q x + \left( \frac{p + q}{2} \right)^n \left( (b - c)x^2 + (d + e - q)x + 2c \right) + \left( \frac{p - q}{2} \right)^n \left( -b + c)x^2 + (-d - e - q)x - 2c \right) \right),
\]

\[
d_n = \frac{1}{2q(x^2 - 1)} \left( 2a^n q x + \left( \frac{p - q}{2} \right)^n \left( -2d x^2 + (-b - c)x + d - e - q \right) + \left( \frac{p + q}{2} \right)^n \left( 2d x^2 + (b + c)x - d + e - q \right) \right),
\]

\[
e_n = \frac{1}{2q(x^2 - 1)} \left( 2a^n q x + \left( \frac{p - q}{2} \right)^n \left( -2e x^2 + (-b - c)x - d + e - q \right) + \left( \frac{p + q}{2} \right)^n \left( 2e x^2 + (b + c)x + d - e - q \right) \right),
\]
where

\[ p := (b + c)x + 2a + d + e, \]  
\[ q := \sqrt{(b^2 - 2bc + c^2 + 4de)x^2 + (2bd + 2cd + 2be + 2ce)x + 4bc + d^2 - 2de + e^2}. \]

Now let \( \overline{B_n} \) denote the tangle closure of \( B_n \). In order to evaluate \( \langle \overline{B_n} \rangle \) from formula (3) we need to apply the closure to the elements of \( \mathcal{D}_3 \).

**Lemma 4.** The expression of the bracket polynomial for the closure \( \overline{B_n} \) is given by

\[ \langle \overline{B_n} \rangle = x^3a_n + x^2(b_n + c_n) + x(d_n + e_n). \]  

The splitting at each crossing do not conflict with the closing process, hence the only point remaining concerns the evaluation of the brackets to the closure of the elements of \( \mathcal{D}_3 \), namely

\[ \langle \overline{T_3} \rangle = x^3, \quad \langle \overline{U_1} \rangle = x^2, \quad \langle \overline{U_2} \rangle = x^2, \quad \langle \overline{\tau} \rangle = x, \quad \langle \overline{\varsigma} \rangle = x. \]

Next, combining (3), (5)–(9) and (12), we obtain a better expression of the bracket:

**Lemma 5.** The bracket polynomial for the knot \( \overline{B_n} \) is given by

\[ \langle \overline{B_n} \rangle = x a^n (x^2 - 2) + x \left( \left( \frac{p - q}{2} \right)^n + \left( \frac{p + q}{2} \right)^n \right), \]  

where \( p \) and \( q \) are expressions defined in (10) and (11).

Finally, we let \( \overline{B}(x; y) := \sum_{n \geq 0} \langle \overline{B_n} \rangle y^n \) denote the generating function of \( (\langle \overline{B_n} \rangle)_n \). By (13) we deduce

\[ \overline{B}(x; y) = \frac{((b + c)x + 2a + d + e)y - 2}{((de - bc)x^2 + (-ac - ab)x + (-d - a)e - ad + bc - a^2)y^2 + ((b + c)x + 2a + d + e)y - 1 + \frac{x(x^2 - 2)}{1 - ay}}. \]

### 3 Application

Throughout this section, let us refer to the 3-tangles in (1) as *generators*. Recall that in the expression \( \langle \overline{B_n} \rangle = \sum_{k>0} s_B(n, k)x^k \) we have \( b_{n,k} = \# \{ S \mid S \text{ is a state of } B_n \text{ and } |S| = k \} \), with \( B \in \{ T, C, E \} \). For each flat sinnet Turk’s head below, we will give the corresponding distribution \( (s_B(n, k))_{n,k} \) for small values of \( n \) and \( k \).
1. Three-lead Turk’s head. Let $\sum_{k \geq 0} s_T(n, k)x^k := \langle T_n \rangle$.

- Bracket for the generator $T$:

\[
\langle T \rangle = \langle 1 \rangle + \langle U_1 \rangle + \langle U_2 \rangle + \langle s \rangle.
\]

- States matrix:

\[
M_T = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & x + 1 & 0 & 0 & 1 \\
1 & 0 & x + 2 & x + 1 & 0 \\
0 & 0 & 1 & x + 1 & 0 \\
1 & x + 1 & 0 & 0 & x + 2
\end{bmatrix}.
\]

- Bracket for $T_n$:

\[
\langle T_n \rangle = x \left( x^2 - 2 \right) + x \left( \left( \frac{2x + 3 - \sqrt{4x + 5}}{2} \right)^n + \left( \frac{2x + 3 + \sqrt{4x + 5}}{2} \right)^n \right).
\]

- Generating function:

\[
T(x; y) = \frac{x \left(\frac{-2x - 3}{2} y + 2\right)}{(x^2 + 2x + 1) y^2 + (-2x - 3) y + 1} + \frac{x \left( x^2 - 2 \right)}{1 - y}.
\]

- Distribution of $(s_T(n, k))_{n,k}$: [6, A316659]

| $n \setminus k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|-----------------|---|---|---|---|---|---|---|---|---|---|----|----|
| 0               | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |    |    |
| 1               | 0 | 1 | 2 | 1 |   |   |   |   |   |   |    |    |
| 2               | 0 | 5 | 8 | 3 |   |   |   |   |   |   |    |    |
| 3               | 0 | 16| 30| 16| 2 |   |   |   |   |   |    |    |
| 4               | 0 | 45| 104|81 |24 |2  |   |   |   |   |    |    |
| 5               | 0 | 121|340|356|170|35 |2  |   |   |   |    |    |
| 6               | 0 | 320|1068|1411|932|315|48 |2  |   |   |    |    |
| 7               | 0 | 841|3262|5209|4396|2079|532|63 |2  |   |    |    |
| 8               | 0 | 2205|9760|18281|18784|11440|4144|840|80 |2   |    |    |
| 9               | 0 | 5776|28746|61786|74838|55809|26226|7602|1260|99 |2   |    |
| 10              | 0 | 15125|83620|202841|282980|249815|144488|54690|13080|1815|120 |2   |

Table 2: Values of $s_T(n, k)$ for $0 \leq n \leq 10$ and $0 \leq k \leq 11.$
2. Chain sinnet. Let $\sum_{k \geq 0} s_C(n, k)x^k := \langle C_n \rangle$.

- Bracket for the generator $C$:
  \[
  \langle C \rangle = \langle 1_3 \rangle + \langle 2U_1 \rangle + \langle 2U_2 \rangle + \langle s \rangle.
  \]

- States matrix:
  \[
  M_C = \begin{bmatrix}
  x+2 & 0 & 0 & 0 & 0 \\
  x+2 & x^2+3x+2 & 0 & 0 & x+2 \\
  1 & 0 & 2x+3 & x+1 & 0 \\
  0 & 0 & x+2 & x^2+3x+2 & 0 \\
  1 & x+1 & 0 & 0 & 2x+3
  \end{bmatrix}.
  \]

- Bracket for $C_n$:
  \[
  \langle C_n \rangle = x \left( x^2 - 2 \right) \left( x+2 \right)^n + x \left( \frac{x^2 + 5x + 5 - \sqrt{x^4 + 2x^3 + 3x^2 + 10x + 9}}{2} \right)^n + x \left( \frac{x^2 + 5x + 5 + \sqrt{x^4 + 2x^3 + 3x^2 + 10x + 9}}{2} \right)^n.
  \]

- Generating function
  \[
  \overline{C}(x; y) = \frac{x \left( (-x^2 - 5x - 5) y + 2 \right)}{(2x^3 + 8x^2 + 10x + 4) y^2 + (-x^2 - 5x - 5) y + 1} + \frac{x \left( x^2 - 2 \right)}{1 - (x+2) y}.
  \]

- Distribution of $(s_C(n, k))_{n,k}$:

| $n \setminus k$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 |
|-----------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 0               | 0  | 0  | 0  | 1  |    |    |    |    |    |    |    |    |    |    |
| 1               | 0  | 1  | 3  | 3  | 1  |    |    |    |    |    |    |    |    |    |
| 2               | 0  | 9  | 22 | 21 | 10 | 2  |    |    |    |    |    |    |    |    |
| 3               | 0  | 49 | 141| 164| 105| 42 | 10 | 1  |    |    |    |    |    |    |
| 4               | 0  | 225| 796| 1186| 1008| 569| 232| 67 | 12 | 1  |    |    |    |    |
| 5               | 0  | 961| 4115| 7677| 8400| 6205| 3393| 1435| 461| 105| 15 | 1  |    |    |
| 6               | 0  | 3969| 20106| 45481| 61630| 57078| 39298| 21239| 9198| 3151| 822| 153| 18 | 1  |

Table 3: Values of $s_C(n, k)$ for $0 \leq n \leq 6$ and $0 \leq k \leq 13$. 

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3. **Figure-eight chain.** Let \( \sum_{k \geq 0} s_E(n, k) x^k := \langle E_n \rangle \).

- **Bracket for the generator \( E \):**
  \[
  \langle E \rangle = \langle \emptyset \rangle + \langle 8 \rangle = (x + 1) \langle \emptyset \rangle + \langle C \rangle
  = (x + 1) \left( \langle \emptyset \rangle + \langle \emptyset \rangle + \langle \emptyset \rangle + \langle \emptyset \rangle \right) + \langle C \rangle
  \]
  \[
  \langle E \rangle = (x^2 + 4x + 4) \langle 1 \rangle + (x + 2) \langle U \rangle + (x + 2) \langle U \rangle + \langle s \rangle.
  \]

- **States matrix:**
  \[
  M_E = \begin{bmatrix}
  x^2 + 4x + 4 & 0 & 0 & 0 & 0 \\
  x + 2 & x^2 + 6x + 4 & 0 & 0 & x + 2 \\
  x + 2 & 0 & x^2 + 6x + 5 & 2x + 2 & 0 \\
  0 & 0 & x + 2 & x^2 + 6x + 4 & 0 \\
  1 & 2x + 2 & 0 & 0 & 2x^2 + 6x + 5
  \end{bmatrix}.
  \]

- **Bracket for \( E_n \):**
  \[
  \langle E_n \rangle = x(x^2 - 2) (x^2 + 4x + 4)^n + x \left( \frac{4x^2 + 12x + 9 - \sqrt{8x^2 + 24x + 17}}{2} \right)^n
  + \left( \frac{4x^2 + 12x + 9 + \sqrt{8x^2 + 24x + 17}}{2} \right)^n.
  \]

- **Generating function**
  \[
  \overline{E}(x; y) = \frac{x ((-4x^2 - 12x + 9) y + 2)}{(4x^4 + 24x^3 + 52x^2 + 48x + 16) y^2 + (-4x^2 - 12x - 9) y + 1}
  + \frac{x (x^2 - 2)}{1 - (x^2 + 4x + 4) y}.
  \]

- **Distribution of \( s_E(n, k) \):**

| \( n \backslash k \) | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  |
|---------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0                   | 0   | 0   | 0   | 0   | 1   |     |     |     |     |     |     |     |     |     |
| 1                   | 0   | 1   | 4   | 6   | 4   | 1   |     |     |     |     |     |     |     |     |
| 2                   | 0   | 17  | 56  | 80  | 64  | 30  | 8   | 1   |     |     |     |     |     |     |
| 3                   | 0   | 169 | 660 | 1120| 1096| 684 | 280 | 74  | 12  | 1   |     |     |     |     |
| 4                   | 0   | 1377| 6640| 14112| 17504| 14128| 7808| 3008| 800 | 142 | 16  | 1   |     |     |
| 5                   | 0   | 10201| 59660| 156624| 244280| 252460| 182544| 94960| 35904| 9800| 1880| 242 | 20  | 1   |

Table 4: Values of \( s_E(n, k) \) for \( 0 \leq n \leq 5 \) and \( 0 \leq k \leq 13 \).
Remark 6. Column 1 in Table 2 is sequence A004146 in the OEIS [6], the sequence of alternate Lucas numbers minus 2, which is the determinant of the Turk’s Head Knots THK(3, n) [4]. Column 2 is the x-coefficients of a generalized Jaco-Lucas polynomials for even indices [7] (see column 1 in triangle A122076) and is also a subsequence of a Fibonacci-Lucas convolution A099920 for odd indices. Column 1 in Table 3 is A060867 with a leading 0.

Rows 1 in Table 2, Table 3, Table 4 match the coefficients of the bracket for the 2-twist loop (see row 1 in A300184, A300192 and row 0 in A300454), the 3-twist loop and the 4-twist loop modulo planar isotopy and move on the 2-sphere [5], respectively (see Figure 2 (a), (b) and (d)). Row 2 in Table 2 gives those of the figure-eight knot (see Figure 2 (b) and row 1 in A300454).

\[
\begin{array}{cccc}
(a) \ T_1 & \Rightarrow & \bullet \bullet \bullet \bullet \\
(b) \ T_2 & \Rightarrow & \circ \circ \circ \circ \circ \circ \\
(c) \ C_1 & \Rightarrow & \bullet \bullet \bullet \bullet \bullet \\
(d) \ E_1 & \Rightarrow & \circ \circ \circ \circ \circ \circ \circ \\
\end{array}
\]

Figure 2: Equivalent knot shadow diagrams.

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