t-Dual Baer Modules and t-Lifting Modules

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Abstract

We introduce the notions of t-lifting modules and t-dual Baer modules, which are generalizations of lifting modules. It is shown that an amply supplemented module \( M \) is t-lifting if and only if \( M \) is t-dual Baer and a t-\( K \)-module. We also prove that, over a right perfect ring \( R \), every noncosingular \( R \)-module is injective if and only if every \( R \)-module is t-dual Baer if and only if every \( R \)-module is t-lifting if and only if every injective \( R \)-module is t-lifting.

Keywords: t-dual Baer module; t-lifting module; noncosingular module.

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1 Introduction

Throughout this paper, \( R \) will denote an arbitrary associative ring with identity, \( M \) a unitary right \( R \)-module and \( S = \text{End}(M) \) the ring of all \( R \)-endomorphisms of \( M \). We will use the notation \( N \leq_e M \) to indicate that \( N \) is essential in \( M \) (i.e., \( N \cap L \neq 0 \ \forall 0 \neq L \leq M \)); \( N \ll M \) means that \( N \) is small in \( M \) (i.e. \( \forall L \leq M, L + N \neq M \)). The notation \( N \leq^\oplus M \) denotes that \( N \) is a direct summand of \( M \). We also denote \( D_S(N) = \{ \phi \in S | \text{Im} \phi \subseteq N \} \), for \( N \subseteq M \).

Recall that an \( R \)-module \( M \) is an extending module if for every submodule \( A \) of \( M \) there exists a direct summand \( B \) of \( M \) such that \( A \leq_e B \). Dually,
a module $M$ is called a lifting module if, for every submodule $A$ of $M$ there exists a direct summand $N$ of $M$ with $N \subseteq A$ and $A/N \ll M/N$. $M$ is lifting if and only if $M$ is amply supplemented and every coclosed submodule of $M$ is a direct summand (see [2, 22.3]).

In [7], Talebi and Vanaja defined $Z(M)$ as follows:

$$Z(M) = \text{Re}(M, S) = \bigcap \{ \text{Ker}(g) \mid g \in \text{Hom}(M, L), L \in S \},$$

where $S$ denotes the class of all small modules. Note that any module is called small if it is small in its injective hull.

They called $M$ a cosingular (noncosingular) module if $Z(M) = 0$ ($Z(M) = M$). Note that $Z^2(M)$ is defined as $Z(Z(M))$.

In [3], Kaplansky introduced the concept of a Baer ring. A ring $R$ is called right Baer (resp. left Baer) if the right (resp. left) annihilator of any nonempty subset of $R$ is generated by an idempotent. Rizvi and Roman introduced the concept of Baer modules in [6]. According to [6], $M$ is called a Baer module if the right annihilator in $M$ of any left ideal of $S$ is a direct summand of $M$. In [1], Keskin-Tütüncü and Tribak introduced the concept of dual Baer modules. A module $M$ is called a dual Baer module if for every right ideal $I$ of $S$, $\sum_{\phi \in I} \text{Im} \phi$ is a direct summand of $M$, equivalently, $D_S(N)$ is a direct summand of $M$ for every submodule $N$ of $M$. Asgari and Haghany introduced t-extending and t-Baer modules in [1] as two generalizations of extending modules. In this paper, motivated by this nice work, we introduce t-lifting modules and t-dual Baer modules to generalize lifting modules and obtain several dual results.

Let $M$ be a module and $A \leq M$. We say that $A$ is t-small (written $A \ll_t M$) if for every submodule $B$ of $M$, $Z^2(M) \leq A + B$ implies that $Z^2(M) \leq B$. Some equivalent conditions for a t-small submodule are given in Proposition 2.2. A submodule $C$ of a module $M$ is called t-coclosed if $C/C' \ll_t M/C'$ implies that $C = C'$. We say that a module $M$ is t-lifting if for every submodule $A$ of $M$ there exists a direct summand $N$ of $M$ with $N \leq A$ and $A/N \ll_t M/N$. In section 2, after giving some properties of t-coclosed submodules, we get some equivalent statements for a t-lifting module. We show that an amply supplemented module is t-lifting if and only if every t-coclosed submodule is a direct summand of $M$ and $Z^2(M)$ is lifting (Theorem 2.2). Let $M$ be a module. We say that $M$ is a t-dual Baer module if $tZ^2(M)$ is a direct summand of $M$, for every right ideal
I of End(M). We study t-dual Baer modules and prove in section 3 that a module M is t-dual Baer if and only if $A\mathcal{Z}^2(M)$ is a direct summand of M for every subset A of End(M) if and only if $\mathcal{Z}^2(M)$ is a dual Baer direct summand of M (Theorem 3.2). In addition, a closed connection exists between t-lifting modules and t-dual Baer modules; in fact, an amply supplemented module is t-lifting if and only if it is t-dual Baer and a t-$\mathcal{K}$-module (Theorem 3.9).

Finally, we prove the following:

Let R be a right perfect ring. Then the following statements are equivalent:

1. Every noncosingular R-module is injective;
2. For every R-module M, $\mathcal{Z}^2(M)$ is a direct summand of M and $\mathcal{Z}^2(M)$ is injective;
3. Every R-module is t-dual Baer;
4. Every R-module is t-lifting;
5. Every injective R-module is t-lifting;
6. Every noncosingular R-module is dual Baer and $\mathcal{Z}^2(M)$ is a direct summand of M for every R-module M;
7. Every noncosingular R-module is lifting and $\mathcal{Z}^2(M)$ is a direct summand of M for every R-module M (Theorem 3.12).

For the undefined notions in this paper we refer to [2].

2 t-coclosed submodules and t-lifting modules

Definition 2.1 A submodule A of M is called t-small in M, denoted by $A \ll_t M$, if for every submodule B of M, $\mathcal{Z}^2(M) \leq A + B$ implies that $\mathcal{Z}^2(M) \leq B$.

It is clear that if A is a submodule of a noncosingular module M, then A is t-small in M if and only if A is small in M.

The concept of amply supplemented modules will be used significantly in the paper. So we prefer to give its definition. Any module M is called amply supplemented if for any two submodules A and B with $M = A + B$, A contains a supplement of B. Note that a submodule X of any module M is called a supplement of any submodule Y in M if $M = X + Y$ and $X \cap Y$ is small in X.

Proposition 2.2 Let M be an amply supplemented module and A a submodule of M. Then the following statements are equivalent:
Proof. (1) We have $B \subseteq Z^2(M)$ and $(A \cap Z^2(M)) + B = Z^2(M)$. Then $Z^2(M) \subseteq A + B$. Since $A \ll_t M$, $Z^2(M) \subseteq B$. Therefore $B = Z^2(M)$ and so $A \cap Z^2(M) \ll Z^2(M)$.

(2) It is clear.

(3) It is obvious that every t-coclosed submodule is coclosed in amply supplemented modules and if $C$ is t-coclosed in $M$ of $Z$, then $C \subseteq A \cap Z^2(M) \ll M$. Hence $Z^2(A)$ is non-cosingular. On the other hand, by [7, Theorem 3.5], $Z^2(A)$ is non-cosingular. Hence $Z^2(A) = 0$.

(4) Let $Z^2(A) = 0$ and $Z^2(M) \subseteq A + B$ for some submodule $B$ of $M$. By [7, Theorem 3.5], $Z^2(M) = \bar{Z}^2(A + B)$ and $Z^2(A/(A \cap B)) = (\bar{Z}^2(A) + (A \cap B))/A \cap B$. Since $Z^2(A) = 0$, $Z^2(A/(A \cap B)) = 0$. Then $Z^2((A + B)/B) = 0$. Again by [7, Theorem 3.5], $\bar{0} = \bar{Z}^2((A + B)/B) = (\bar{Z}^2(A + B) + B)/B = (Z^2(M) + B)/B$ and so $Z^2(M) \subseteq B$. \hfill \Box

By Proposition 2.2, every small submodule of an amply supplemented module $M$ and every supplement to $Z^2(M)$ is t-small.

**Definition 2.3** A submodule $C$ of $M$ is called t-coclosed in $M$ and denoted by $C \leq_{tcc} M$ if $C/C' \ll_t M/C'$ implies that $C = C'$.

It is obvious that every t-coclosed submodule is coclosed in amply supplemented modules and if $C$ is a submodule of a noncosingular module $M$, then $C$ is t-coclosed in $M$ if and only if $C$ is coclosed in $M$.

**Lemma 2.4** Let $M$ be an amply supplemented module. Then:

(1) If $C \leq_{tcc} M$, then $C \leq Z^2(M)$.

(2) $M \leq_{tcc} M$ if and only if $M$ is noncosingular.

(3) If $A \subseteq C$ and $C \leq_{tcc} M$, then $C/A \leq_{tcc} M/A$.

(4) If $A \subseteq C$, $C/A \leq_{tcc} M/A$ and $A \leq_{tcc} M$, then $C \leq_{tcc} M$.

(5) If $A \subseteq C$ and $C$ is amply supplemented, then $A \leq_{tcc} M \iff A \leq_{tcc} C$.

Proof. (1) We have $C/(C \cap Z^2(M)) \cap Z^2(M/(C \cap Z^2(M))) = C/(C \cap Z^2(M)) \cap Z^2(M)/(C \cap Z^2(M)) = 0 \ll Z^2(M/(C \cap Z^2(M)))$. By Proposition 2.2, $C/(C \cap$
\( \mathcal{Z}^2(M) \) \( \triangleleft_t M/(C \cap \mathcal{Z}^2(M)) \). But \( C \leq_{tcc} M \), thus \( C = C \cap \mathcal{Z}^2(M) \). Hence \( C \leq \mathcal{Z}^2(M) \).

(2) Let \( M \leq_{tcc} M \). By (1), \( M \subseteq \mathcal{Z}^2(M) \). Then \( M = \mathcal{Z}^2(M) \). The converse is clear.

(3) Let \( C \leq_{tcc} M \). Let \( \frac{C/A}{T/A} \triangleleft_t \frac{M/A}{T/A} \) for some submodule \( T/A \) of \( M/A \) with \( T/A \leq C/A \). Then \( \mathcal{Z}^2(C/T) = 0 \) by Proposition 2.2 and hence \( C/T \triangleleft_t M/T \) by Proposition 2.2 again. Thus \( T = C \) since \( C \leq_{tcc} M \).

(4) Let \( C/T \triangleleft_t M/T \) for some submodule \( T \) of \( M \) with \( T \leq C \). By Proposition 2.2 \( \mathcal{Z}^2(C/T) = 0 \). Hence \( \mathcal{Z}^2(C) \leq T \) by [7, Theorem 3.5]. Now, \( \mathcal{Z}^2\left(\frac{C}{(C \cap (A+T))}\right) = \mathcal{Z}^2\left(\frac{C/(C+T)}{C/(A+T)}\right) = 0 \). Hence \( \mathcal{Z}^2\left(\frac{C/A}{(C/(A+T))/A}\right) = 0 \). By Proposition 2.2 \( \frac{C/A}{(C/(A+T))/A} \triangleleft_t \frac{M/A}{(C/(A+T))/A} \). Then \( C = C \cap (A+T) \) and so \( C = A + T \). Since \( \mathcal{Z}^2(C/T) = 0 \), then \( \mathcal{Z}^2(A/(A \cap T)) = 0 \). By Proposition 2.2 \( A/(A \cap T) \triangleleft_t M/(A \cap T) \). So, \( A = A \cap T \) and hence \( A \subseteq T \). Thus \( C = T \).

(5) By Proposition 2.2 \( \square \)

**Proposition 2.5** Let \( C \) be a submodule of an amply supplemented module \( M \). Then the following are equivalent:

1. There exists a submodule \( S \) such that \( C \) is minimal with respect to the property that \( \mathcal{Z}^2(M) \subseteq C + S \).
2. \( C \) is \( t \)-coclosed in \( M \).
3. \( C \) is contained in \( \mathcal{Z}^2(M) \) and \( C \) is a coclosed submodule of \( \mathcal{Z}^2(M) \).
4. \( C \) is contained in \( \mathcal{Z}^2(M) \) and \( C \) is a coclosed submodule of \( M \).
5. \( C \) is noncosingular.

Proof. (1) \( \Rightarrow \) (2) Let (1) hold and \( C/C'' \triangleleft_t M/C'' \). Then \( \mathcal{Z}^2(M) \subseteq C + C'' + S \). Then \( \mathcal{Z}^2(M/C'') = \mathcal{Z}^2(M) + C''/C'' \subseteq C/C'' + (C'' + S)/C'' \). Since \( C/C'' \triangleleft_t M/C'' \), \( \mathcal{Z}^2(M) + C''/C'' \subseteq (C'' + S)/C'' \) and so \( \mathcal{Z}^2(M) \subseteq C'' + S \). Hence \( C = C'' \).

(2) \( \Rightarrow \) (3) By Lemma 2.4 \( C \) is contained in \( \mathcal{Z}^2(M) \). Let \( C/C'' \triangleleft \mathcal{Z}^2(M)/C'' \). Then \( C/C'' \cap \mathcal{Z}^2(M)/C'' = C/C'' \triangleleft \mathcal{Z}^2(M)/C = \mathcal{Z}^2(M)/C \). By Proposition 2.2 \( C/C'' \triangleleft_t M/C'' \). By hypothesis, \( C = C'' \).

(3) \( \Rightarrow \) (4) By [7, Corollary 3.4], \( \mathcal{Z}^2(M) \) is coclosed in \( M \). By [2, 3.7(6)], \( C \) is coclosed in \( M \).

(4) \( \Rightarrow \) (3) By [2, 3.7(6)].

(3) \( \Leftrightarrow \) (5) By [7, Lemma 2.3(3) and Corollary 3.4].

(3) \( \Rightarrow \) (1) Let \( C \) be a coclosed submodule of \( \mathcal{Z}^2(M) \). Then \( C \) is supplement in \( \mathcal{Z}^2(M) \). Now, there exists a submodule \( S \) of \( M \) such that \( \mathcal{Z}^2(M) = C + S \).
and $C$ is minimal with $Z^2(M) = C + S$. For any submodule $X$ of $M$ with $X \subseteq C$, let $Z^2(M) \subseteq X + S$. Then by [7, Theorem 3.5], $Z^2(M) = Z^2(X + S)$. Hence $C + S = Z^2(M) = X + S$. By minimality of $C$ in $Z^2(M)$, $X = C$. □

Note that the conditions (3) – (5) of Lemma 2.4 are satisfied from Proposition 2.5 as well.

**Corollary 2.6** Let $M$ be an amply supplemented module. Then:

1. $Z^2(M)$ is $t$-coclosed in $M$.
2. If $\phi$ is an endomorphism of $M$ and $C$ is a $t$-coclosed submodule of $M$, then $\phi(C)$ is $t$-coclosed in $M$.

**Proof.** (1) Since $Z^2(M)$ is noncosingular, $Z^2(M)$ is $t$-coclosed in $M$ by Proposition 2.5.

(2) Since $C$ is noncosingular, $\phi(C)$ is noncosingular. Thus $\phi(C)$ is $t$-coclosed. □

The sum of two coclosed submodules need not be coclosed (see [2, 21.5]), but this term is always true if we replace coclosed with $t$-coclosed, as the following proposition shows.

**Corollary 2.7** Let $M$ be an amply supplemented module. Then an arbitrary sum of $t$-coclosed submodules of $M$ is $t$-coclosed.

**Proof.** Since arbitrary sum of noncosingular submodules is noncosingular, it is clear. □

**Definition 2.8** A module $M$ is called $t$-lifting if every submodule $A$ of $M$ contains a direct summand $B$ of $M$ such that $A/B \preccurlyeq_t M/B$.

The next result gives us several equivalent conditions for a $t$-lifting amply supplemented module.

**Theorem 2.9** Let $M$ be an amply supplemented module. Then the following are equivalent:

1. $M$ is $t$-lifting.
2. For every submodule $A$ of $M$, there exists a decomposition $A = N \oplus N'$ such that $N$ is a direct summand of $M$ and $N' \preccurlyeq_t M$.
3. Every $t$-coclosed submodule of $M$ is a direct summand.
(4) For every submodule $A$ of $M$, $\mathbb{Z}^2(A)$ is a direct summand of $M$.

(5) For every coclosed submodule $A$ of $M$, $\mathbb{Z}^2(A)$ is a direct summand of $M$.

(6) $\mathbb{Z}^2(M)$ is a direct summand of $M$ and $\mathbb{Z}^2(M)$ is lifting.

(7) Every submodule $A$ of $M$ which is contained in $\mathbb{Z}^2(M)$, contains a direct summand $N$ of $M$ such that $A/N \ll M/N$.

Proof. (1) $\Rightarrow$ (2) Let $A \leq M$. Then there exists a decomposition $M = N \oplus L$ such that $A/N \ll_t M/N$. Then $A = N \oplus (L \cap A)$. By Proposition 2.2, $\mathbb{Z}^2(A/N) = 0$ and so $\mathbb{Z}^2(L \cap A) = 0$. Again by Proposition 2.2 $L \cap A \ll_t M$.

(2) $\Rightarrow$ (3) Let $C$ be a t-coclosed submodule of $M$. By assumption, $C = N \oplus N'$ such that $N \leq \oplus M$ and $N' \ll_t M$. By Proposition 2.2, $\mathbb{Z}^2(N') = 0$, thus $\mathbb{Z}^2(C/N) = 0$. Again by Proposition 2.2, $C/N \ll_t M/N$. Since $C$ is t-coclosed, $C = N$ is a direct summand of $M$.

(3) $\Rightarrow$ (4) Since $\mathbb{Z}^2(A)$ is noncosingular, by Proposition 2.5, $\mathbb{Z}^2(A)$ is t-coclosed in $M$ and so $\mathbb{Z}^2(A)$ is a direct summand of $M$.

(4) $\Rightarrow$ (5) It is clear.

(5) $\Rightarrow$ (6) Since $\mathbb{Z}^2(M)$ is coclosed in $M$, $\mathbb{Z}^2(\mathbb{Z}^2(M)) = \mathbb{Z}^2(M)$ is a direct summand of $M$. Now, let $C$ be a coclosed submodule of $\mathbb{Z}^2(M)$. Thus, by [7, Lemma 2.3], $C$ is noncosingular. Hence $\mathbb{Z}^2(C) = C$ and so $C$ is a direct summand of $M$. Therefore $C$ is a direct summand of $\mathbb{Z}^2(M)$.

(6) $\Rightarrow$ (7) Let $A \leq \mathbb{Z}^2(M)$. Then there exists a direct summand $N$ of $\mathbb{Z}^2(M)$ such that $A/N \ll \mathbb{Z}^2(M)/N$. Thus $A/N \ll M/N$. It is clear that $N \leq \oplus M$.

(7) $\Rightarrow$ (1) Let $A \leq M$. By hypothesis, there exists a direct summand $N$ of $M$ such that $(A \cap \mathbb{Z}^2(M))/N \ll M/N$. By Proposition 2.2, $A/N \ll_t M/N$. Therefore $M$ is t-lifting. □

It is clear that if $\mathbb{Z}^2(M) = 0$, then $M$ is t-lifting, where $M$ is amply supplemented. Every lifting module is t-lifting since every t-coclosed submodule is coclosed in any amply supplemented module.

Example 2.10 (1) It is well known that the $\mathbb{Z}$-module $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$ is lifting, where $p$ is any prime. So $M$ is t-lifting.

(2) It is well known that the $\mathbb{Z}$-module $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ is not lifting, but it is amply supplemented. Let $A$ be a t-coclosed submodule of $M$. By Proposition 2.4, $A$ is noncosingular. On the other hand, $A$ is cosingular since
$M$ is cosingular. Thus $A = 0$, and hence it is a direct summand of $M$. Thus $M$ is t-lifting by Theorem 2.9.

**Proposition 2.11** Let $M$ be a t-lifting amply supplemented module. Then:

1. Every amply supplemented submodule of $M$ is t-lifting.
2. For every fully invariant submodule $L$ of $M$, $M/L$ is t-lifting.

**Proof.** (1) Let $A \leq M$ and $A$ be amply supplemented. Let $L \leq A$. Since $M$ is t-lifting, there exists a direct summand $N$ of $M$ such that $N \subseteq L$ and $L/N \ll_t M/N$. Then $N$ is a direct summand of $A$ and by Proposition 2.2, $L/N \ll_t A/N$. Hence $A$ is t-lifting.

(2) Let $L$ be a fully invariant submodule of $M$. Let $K/L \leq M/L$. Since $M$ is t-lifting, $M = N \oplus N'$, $N \subseteq K$ and $K/N \ll_t M/N$ for some submodule $N'$ of $M$. Note that $L = (N \cap L) \oplus (N' \cap L) = (N + L) \cap (N' + L)$ since $L$ is fully invariant in $M$. Hence $M/L = ((N + L)/L) \oplus ((N' + L)/L)$. By Proposition 2.2, $\mathbb{Z}^2(K) \leq N$. Then $\mathbb{Z}^2(K/(N+L)) = 0$. Again by Proposition 2.2, $K/(N+L) \ll_t M/(N+L)$. Hence $M/L$ is t-lifting. \qed

## 3 t-Dual Baer Modules

**Definition 3.1** A module $M$ is said to be t-dual Baer if $I(\mathbb{Z}^2(M))$ is a direct summand of $M$ for every right ideal $I$ of $S$, where $S = \text{End}(M)$.

It is clear that for a noncosingular module $M$, we have $M$ is dual Baer if and only if it is t-dual Baer.

Recall that a module $M$ is said to have strongly summand sum property if the sum of every number of direct summand of $M$ is a direct summand of $M$.

**Theorem 3.2** Let $M$ be a module with $S = \text{End}(M)$. Then the following are equivalent:

1. $M$ is t-dual Baer.
2. $\mathbb{Z}^2(M)$ is a direct summand of $M$ and $\mathbb{Z}^2(M)$ is a dual Baer module.
3. $M$ has the strongly summand sum property for direct summands which are contained in $\mathbb{Z}^2(M)$ and $\phi(\mathbb{Z}^2(M))$ is a direct summand of $M$ for every $\phi \in S$.
4. $\sum_{\phi \in A} \phi(\mathbb{Z}^2(M))$ is a direct summand of $M$ for every subset $A$ of $S$. 


Proof. (1) ⇒ (2) Since $M$ is t-dual Baer, $\overline{Z}^2(M) = S(\overline{Z}^2(M))$ is a direct summand of $M$. Let $I$ be a right ideal of $\overline{S} = \text{End}(\overline{Z}^2(M))$, $A = \{i\phi\pi \mid \phi \in I\}$ where $\pi$ is the canonical projection onto $\overline{Z}^2(M)$, $i$ is the inclusion map from $\overline{Z}^2(M)$ to $M$ and $I' = AS$. It is clear that $I(\overline{Z}^2(M)) = I'(\overline{Z}^2(M))$. Since $M$ is t-dual Baer, $I'\overline{Z}^2(M)$ is a direct summand of $M$. Thus $I'\overline{Z}^2(M)$ is a direct summand of $\overline{Z}^2(M)$. Therefore $\overline{Z}^2(M)$ is dual Baer.

(2) ⇒ (1) Let $I$ be a right ideal of $S$, $A' = \{\pi'\phi|\overline{Z}^2(M) : \phi \in I\}$ where $\pi'$ is the canonical projection onto $\overline{Z}^2(M)$, $\overline{S} = \text{End}(\overline{Z}^2(M))$ and $I' = A'S$. Since $\overline{Z}^2(M)$ is dual Baer, $I'\overline{Z}^2(M) \leq \oplus \overline{Z}^2(M)$. It is clear that $I\overline{Z}^2(M) = I'\overline{Z}^2(M)$. Since $\overline{Z}^2(M) \leq \oplus M$, $I\overline{Z}^2(M) \leq \oplus M$.

(1) ⇒ (3) Let $\phi \in S$. Since $\phi(\overline{Z}^2(M)) = \phi S(\overline{Z}^2(M))$ and $M$ is t-dual Baer, $\phi(\overline{Z}^2(M))$ is a direct summand of $M$. Take $e_i^2 = e_i \in S$, $i \in \Lambda$ and $e_i(M) \subseteq \overline{Z}^2(M)$. Let $I = \sum_{i \in \Lambda} e_i S$. Then $I(\overline{Z}^2(M)) = \sum_{\phi \in I} \phi(\overline{Z}^2(M)) \leq \sum_{i \in \Lambda} e_i M$. It is clear that $e_i(M) \subseteq \sum_{\phi \in I} \phi(\overline{Z}^2(M))$. Thus $\sum_{i \in \Lambda} e_i M = \sum_{\phi \in I} \phi(\overline{Z}^2(M)) = I(\overline{Z}^2(M)) \leq \oplus M$ because $M$ is t-dual Baer.

(3) ⇒ (4) It is obvious, since $\phi(\overline{Z}^2(M)) \subseteq \overline{Z}^2(M)$ for every $\phi \in S$.

(4) ⇒ (1) It is clear. □

Recall that a module $M$ is called a regular module if every cyclic submodule of $M$ is a direct summand of $M$.

**Corollary 3.3** If $M$ has the strongly summand sum property for direct summands which are contained in $\overline{Z}^2(M)$ and $M$ is regular, then $M$ is t-dual Baer.

Proof. By Theorem 3.2, it suffices to show that $\phi(\overline{Z}^2(M))$ is a direct summand of $M$ for every $\phi \in S$. Let $\phi \in S$ and $N = \phi(\overline{Z}^2(M))$. Suppose that $N = \sum_{x \in N} xR$. By hypothesis, $N$ is a direct summand of $M$. □

**Corollary 3.4** If $M$ is regular t-dual Baer, then $\overline{Z}^2(M)$ is semisimple.

Proof. Let $N \leq \overline{Z}^2(M)$. Suppose that $N = \sum_{x \in N} xR$. By Theorem 3.2, $N$ is a direct summand of $M$ and so it is a direct summand of $\overline{Z}^2(M)$. □

Now we give a relation between the properties of dual Baer and t-dual Baer modules.

**Proposition 3.5** A module $M$ is dual Baer and $\overline{Z}^2(M)$ is a direct summand of $M$ if and only if $M$ is t-dual Baer and $\sum_{\phi \in A} \phi(M) / \sum_{\phi \in A} \phi(\overline{Z}^2(M))$ is a direct summand of $M / \sum_{\phi \in A} \phi(\overline{Z}^2(M))$ for every subset $A$ of $S$.  

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Proof. By Theorem 3.2 and [11 Corollary 2.5 and Theorem 2.1]. □

**Theorem 3.6** Every direct summand of a t-dual Baer module is t-dual Baer.

Proof. Let $M = N \oplus N'$ and for every $i \in \Lambda$, $K_i$ be a direct summand of $N$ such that $K_i \subseteq Z^2(N)$. Then $K_i \subseteq Z^2(M)$ and since $M$ is t-dual Baer, we have $\sum_{i \in \Lambda} K_i \leq^\oplus M$. Thus $\sum_{i \in \Lambda} K_i \leq^\oplus N$. Let $f : N \to N$ be a homomorphism. Consider the homomorphism $f + 0_{N'} : N \oplus N' \to N \oplus N'$ defined by $(f + 0_{N'})(n + n') = f(n)$. Then $(f + 0_{N'})(Z^2(M)) = (f + 0_{N'})(Z^2(N) \oplus Z^2(N')) = f(Z^2(N))$. As $M$ is t-dual Baer, $f(Z^2(N)) \leq^\oplus M$ and hence it is a direct summand of $N$. Therefore $N$ is t-dual Baer. □

Recall that a module $M$ is a $K$-module if for every submodule $N$ of $M$, $D_S(N) = 0$ implies that $N$ is small in $M$.

Let $M$ be an $R$-module and $S = \text{End}(M)$. For a submodule $N$ of $M$ we denote $T_S(N) = \{ \phi \in S : \phi(Z^2(M)) \subseteq N \}$.

**Definition 3.7** A module $M$ is called a $t$-$K$-module if for every submodule $N$ of $M$, $T_S(N) = T_S(0)$ implies that $N$ is $t$-small in $M$. Moreover, a module $M$ is called a strongly $t$-$K$-module if for every submodule $N$ of $M$, $T_S(N) = T_S(0)$ implies that $N$ is small in $M$.

It is clear that every strongly $t$-$K$-module is a $t$-$K$-module. Obviously, for noncosingular modules the notions of $K$-modules and $t$-$K$-modules and strongly $t$-$K$-modules are equivalent.

**Proposition 3.8** Let $M$ be an amply supplemented module. Then:

1. $M$ is a $t$-$K$-module if and only if for every submodule $N$ of $M$ which is contained in $Z^2(M)$, $T_S(N) = T_S(0)$ implies that $N$ is small in $M$.

2. If $M$ is a $t$-$K$-module, then $Z^2(M)$ is a $K$-module.

Proof. (1) The implication ($\Rightarrow$) follows by Proposition 2.2(3). For ($\Leftarrow$), let $N$ be a submodule of $M$ and $T_S(N) = T_S(0)$. Since $T_S(N \cap Z^2(M)) = T_S(N) = T_S(0)$, by hypothesis, $N \cap Z^2(M)$ is small in $M$. Hence $N \ll_t M$.

(2) Let $S = \text{End}(Z^2(M))$ and $N$ be a submodule of $Z^2(M)$ such that $D_S(N) = 0$. Then $T_S(N) = T_S(0)$. For, let $\phi \in T_S(N)$, then $\overline{\phi} = \phi|_{Z^2(M)} : Z^2(M) \to Z^2(M)$ is a homomorphism such that $\overline{\phi}(Z^2(M)) \subseteq N$, thus $\overline{\phi} \in D_S(N) = 0$ and so $\phi \in T_S(0)$; hence, $T_S(N) = T_S(0)$. By hypothesis, $N$ is $t$-small in $M$. Therefore $N \ll Z^2(M)$ by Proposition 2.2. □
Theorem 3.9 Let $M$ be an amply supplemented module. Then the following are equivalent:

1. $M$ is $t$-lifting.
2. $M$ is $t$-dual Baer and $t$-$K$-module.
3. $M$ is $t$-dual Baer and $C = T_S(C)(\overline{Z}^2(M))$ for every $t$-coclosed submodule $C$ of $M$.
4. $M$ is $t$-dual Baer and for every $t$-coclosed submodule $C$ of $M$ if $T_S(C) = T_S(0)$, then $C = 0$.

Proof. (1) $\Rightarrow$ (2) By Theorem 2.9, $\overline{Z}^2(M)$ is a direct summand of $M$ and $\overline{Z}^2(M)$ is lifting. By [4, Theorem 2.14], every noncosingular lifting module is dual Baer and so $\overline{Z}^2(M)$ is dual Baer. By Theorem 3.2, $M$ is $t$-dual Baer. Now, by proposition 3.8, it suffices to show that if $N$ is a submodule of $M$ which is contained in $\overline{Z}^2(M)$, then $T_S(N) = T_S(0)$ implies that $N \ll M$. As $M$ is $t$-lifting, there exists a direct summand $K$ of $M$ such that $N/K \ll M/K$. By Proposition 2.2, $N/K \cap \overline{Z}^2(M) \ll M/K$. But $N/K \subseteq (\overline{Z}^2(M) + K)/K = \overline{Z}^2(M/K)$, thus $N/K \ll M/K$. Let $M = K \oplus K'$ and $K \neq 0$. Then $\overline{Z}^2(K) \neq 0$ since if $\overline{Z}^2(K) = 0$, then $0 \neq K \subseteq N \subseteq \overline{Z}^2(M) = \overline{Z}^2(K') \subseteq K'$. But $K \cap K' = 0$, contradiction. Now consider the canonical projection $\pi_K : M \to K$. Then $\pi_K \in T_S(N)$ and $\pi_K \notin T_S(0)$, which is a contradiction. Therefore $K = 0$ and so $N \ll M$.

(1) $\Rightarrow$ (3) By the proof of (1) $\Rightarrow$ (2), $M$ is $t$-dual Baer. Let $C$ be a $t$-coclosed submodule of $M$. Obviously, $T_S(C)(\overline{Z}^2(M)) \subseteq C$. By hypothesis, $C$ is a direct summand of $M$, say $M = C \oplus C'$. Consider the canonical projection $\pi$ onto $C$. It is clear that $\pi \in T_S(C)$. By Proposition 2.5, $C \subseteq \overline{Z}^2(M)$, thus $C = \pi(C) \subseteq \pi(\overline{Z}^2(M)) \subseteq T_S(C)(\overline{Z}^2(M))$. Hence $C = T_S(C)(\overline{Z}^2(M))$.

(2) $\Rightarrow$ (4) Let $C$ be a $t$-coclosed submodule of $M$ such that $T_S(C) = T_S(0)$. By assumption, $C = T_S(C)(\overline{Z}^2(M)) = T_S(0)(\overline{Z}^2(M)) = 0$.

(4) $\Rightarrow$ (1) By Theorem 2.9, it suffices to show that for any submodule $N$ of $M$ which is contained in $\overline{Z}^2(M)$, there exists a direct summand $A$ of $M$ such that $N/A \ll M/A$. Let $N$ be such a submodule of $M$. Since $M$ is $t$-dual Baer, $eM = \sum_{\phi \in T_S(N)} \phi(\overline{Z}^2(M)) = T_S(N)(\overline{Z}^2(M)) \subseteq N$ for some idempotent $e \in S$. If $N/eM$ is not small in $M/eM$, then there exists a proper submodule $K/eM$ of $M/eM$ with $eM \subseteq K$ such that $M/eM = K/eM + N/eM$. Restrict $N$ to a supplement $C$ of $K$ in $M$. $C$ is a coclosed submodule of $M$ and
\( C \subseteq \mathbb{Z}^2(M) \), and so by Proposition 2.5, \( C \) is t-coclosed. Now we show that \( T_S(C) = T_S(0) \). Let \( \phi \in T_S(C) \). Then \( \phi(\mathbb{Z}^2(M)) \subseteq C \), and so \( \phi(\mathbb{Z}^2(M)) \subseteq N \), hence \( \phi \in T_S(N) \). As \( eM = \sum_{\phi \in T_S(N)} \phi(\mathbb{Z}^2(M)) \), we have \( \phi(\mathbb{Z}^2(M)) \subseteq eM \). Thus \( \phi(\mathbb{Z}^2(M)) \subseteq K \). Consequently, \( \phi(\mathbb{Z}^2(M)) \subseteq K \cap C \). But \( K \cap C \ll M \) implies that \( \phi(\mathbb{Z}^2(M)) \ll M \). Hence \( \phi(\mathbb{Z}^2(M)) = 0 \). Hence \( \phi \in T_S(0) \). Thus \( T_S(C) = T_S(0) \). By hypothesis \( C = 0 \), and so \( M = K \), which is a contradiction. Therefore \( N/eM \ll M/eM \). \( \square \)

**Corollary 3.10** The following are equivalent for an amply supplemented module \( M \):

1. \( M \) is noncosingular lifting.
2. \( M \) is t-dual Baer and strongly t-\( K \)-module.
3. \( M \) is t-dual Baer and \( C = T_S(C)(\mathbb{Z}^2(M)) \) for every coclosed submodule \( C \).
4. \( M \) is t-dual Baer and for any coclosed submodule \( C \) of \( M \), if \( T_S(C) = T_S(0) \), then \( C = 0 \).

Proof. \((1) \Rightarrow (2) \) and \((1) \Rightarrow (3) \) By Theorem 3.9

\((2) \Rightarrow (4) \) This is clear.

\((3) \Rightarrow (4) \) Similar to the proof of Theorem 3.9 \((3) \Rightarrow (4) \).

\((4) \Rightarrow (1) \) By Theorem 3.2, \( M = \mathbb{Z}^2(M) \oplus K \) for some submodule \( K \) of \( M \) and \( \mathbb{Z}^2(M) \) is dual Baer. Clearly \( K \) is closed submodule and \( T_S(K) = T_S(0) \). By \( (4) \), \( K = 0 \) and so \( M = \mathbb{Z}^2(M) \). Hence \( M \) is noncosingular. By Theorem 3.9 \( M \) is lifting. \( \square \)

**Example 3.11** (1) By Theorem 3.9, every lifting module is t-dual Baer. But there exists t-dual Baer modules which are not lifting. Consider the \( \mathbb{Z} \)-module \( M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z} \) in Example 2.10(2). It is amply supplemented and t-lifting. By Theorem 3.9, it is t-dual Baer. But it is not lifting.

(2) Let \( R \) be a semiperfect ring which is not semisimple. Then the right \( R \)-module \( R_R \) is lifting by [5, Corollary 4.42]. Hence it is t-lifting. Then by Theorem 3.9, \( R_R \) is t-dual Baer. On the other hand, \( R_R \) is not dual Baer by [4, Corollary 2.9].

(3) If \( R \) is a right \( H \)-ring, then every injective \( R \)-module is t-lifting and t-dual Baer.

**Theorem 3.12** Let \( R \) be a right perfect ring. Then the following statements are equivalent:
(1) Every noncosingular R-module is injective.
(2) For every R-module M, \( \overline{Z}^2(M) \) is a direct summand of M and \( \overline{Z}^2(M) \) is injective.
(3) Every R-module is t-dual Baer.
(4) Every R-module is t-lifting.
(5) Every injective R-module is t-lifting.
(6) Every noncosingular R-module is dual Baer and \( \overline{Z}^2(M) \) is a direct summand of M for every R-module M.
(7) Every noncosingular R-module is lifting and \( \overline{Z}^2(M) \) is a direct summand of M for every R-module M.

Proof. (1) ⇒ (2) Since \( \overline{Z}^2(M) \) is noncosingular, by (1), \( \overline{Z}^2(M) \) is injective. Thus \( \overline{Z}^2(M) \) is a direct summand of M.

(2) ⇒ (1) Clear.

(2) ⇒ (3) Let M be any R-module. By (2), \( \overline{Z}^2(M) \) is a direct summand of M. Let C be a coclosed submodule of \( \overline{Z}^2(M) \). By [7, Lemma 2.3(2)], C is noncosingular. By (2), C is injective, and so it is a direct summand of \( \overline{Z}^2(M) \). Consequently, \( \overline{Z}^2(M) \) is lifting. By [4, Theorem 2.14], \( \overline{Z}^2(M) \) is dual Baer. Therefore M is t-dual Baer by Theorem 3.2.

(4) ⇒ (5) Clear.

(5) ⇒ (1) Let M be a noncosingular module and E(M) be the injective hull of M. Since M is noncosingular, \( \overline{Z}^2(M) = M \). By (5), E(M) is t-lifting. Then by Theorem 2.9(2), \( \overline{Z}^2(M) \leq E(M) \). Thus M is injective.

(7) ⇒ (4) Let M be any R-module. By (7), \( \overline{Z}^2(M) \) is lifting and \( \overline{Z}^2(M) \leq M \). Thus M is t-lifting by Theorem 2.9.

(3) ⇒ (6) Let X be a noncosingular module. By (3), X is t-dual Baer and hence it is dual Baer. Let M be any R-module. By (3) and Theorem 3.2, \( \overline{Z}^2(M) \leq M \).

(3) ⇒ (4) Let M be any R-module. Let \( K \leq M \) and define \( \phi : M \oplus K \to M \oplus K \) by \( \phi(m, k) = (k, 0) \). Note that \( M \oplus K \) is t-dual Baer by (3). Then by Theorem 3.2, \( \phi(\overline{Z}^2(M \oplus K)) = \phi(\overline{Z}^2(M) \oplus \overline{Z}^2(K)) = \overline{Z}^2(K) \oplus 0 \leq M \oplus K \). Thus \( \overline{Z}^2(K) \leq M \). By Theorem 2.9, M is t-lifting.

(6) ⇒ (7) Let M be a noncosingular R-module. Let K be a coclosed submodule of M. By [7, Lemma 2.3(3)], K is noncosingular, and so \( M \oplus K \) is noncosingular. Then by (6), \( M \oplus K \) is dual Baer. Define \( \phi : M \oplus K \to M \oplus K \) by \( \phi(m, k) = (k, 0) \). By [4, Theorem 2.1], \( \phi(M \oplus K) = K \oplus 0 \leq M \oplus K \).
Then $K \leq^\oplus M$. So $M$ is lifting. □

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