Wasserstein continuity of entropy and outer bounds for interference channels

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Abstract

It is shown that under suitable regularity conditions, differential entropy is a Lipschitz functional on the space of distributions on $\mathbb{R}^n$ with respect to the quadratic Wasserstein distance. Under similar conditions, (discrete) Shannon entropy is shown to be Lipschitz continuous in distributions over the product space with respect to Ornstein’s $\bar{d}$-distance (Wasserstein distance corresponding to the Hamming distance). These results together with Talagrand’s and Marton’s transportation-information inequalities allow one to replace the unknown multi-user interference with its i.i.d. approximations. As an application, a new outer bound for the two-user Gaussian interference channel is proved, which, in particular, settles the “missing corner point” problem of Costa (1985).

1 Introduction

Let $X$ and $\tilde{X}$ be random vectors in $\mathbb{R}^n$. We ask the following question: If the distributions of $X$ and $\tilde{X}$ are close in certain sense, can we guarantee that their differential entropies are close as well? For example, one can ask whether

$$D(P_X \| P_{\tilde{X}}) = o(n) \Rightarrow |h(X) - h(\tilde{X})| = o(n). \quad (1)$$

One motivation comes from multi-user information theory, where frequently one user causes interference to the other and in proving the converse one wants to replace the complicated non-i.i.d. interference by a simpler i.i.d. approximation. For example, the following question turns out to be central in establishing a missing corner point in the capacity region of the two-user Gaussian interference channels (GIC) [Cos85a] (see [RC15,Sas15] for a recent account of this problem): Given independent $n$-dimensional random vectors $X_1, X_2, G_2, Z$ with the latter two being Gaussian, is it true that

$$D(P_{X_2+Z} \| P_{G_2+Z}) = o(n) \Rightarrow |h(X_1 + X_2 + Z) - h(X_1 + G_2 + Z)| = o(n). \quad (2)$$

To illustrate the nature of the problem, we first note that the answer to (1) is in fact negative as the counterexample of $X \sim \mathcal{N}(0, 2I_n)$ and $\tilde{X} \sim \frac{1}{2}\mathcal{N}(0, I_n) + \frac{1}{2}\mathcal{N}(0, 2I_n)$ demonstrates, in which case the divergence is $D(P_X \| P_{\tilde{X}}) \leq \log 2$ but the differential entropies differ by $\Theta(n)$. Therefore

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even for very smooth densities the difference in entropies is not controlled by the divergence. The situation for discrete alphabets is very similar, in the sense that the gap of Shannon entropies cannot be bounded by divergence in general (with essentially the same counterexample as that in the continuous case: \( X \) and \( \tilde{X} \) being uniform on one and two Hamming spheres respectively).

The rationale of the above discussion is two-fold: a) Certain regularity conditions of the distributions must be imposed; b) Distances other than KL divergence might be more suited for bounding the entropy difference. Correspondingly, the main contribution of this paper is the following: Under suitable regularity conditions, the difference in entropy (in both continuous and discrete cases) can in fact be bounded by the Wasserstein distance, a notion originating from optimal transportation theory which turns out to be the main tool of this paper.

We start with the definition of the Wasserstein distance on the Euclidean space. Given probability measures \( \mu, \nu \) on \( \mathbb{R}^n \), define their \( p \)-Wasserstein distance (\( p \geq 1 \)) as

\[
W_p(\mu, \nu) \triangleq \inf (\mathbb{E}[\|X - Y\|^p])^{1/p},
\]

where \( \| \cdot \| \) denotes the Euclidean distance and the infimum is taken over all couplings of \( \mu \) and \( \nu \), i.e., joint distributions \( P_{XY} \) whose marginals satisfy \( P_X = \mu \) and \( P_Y = \nu \). The following dual representation of the \( W_1 \) distance is useful:

\[
W_1(\mu, \nu) = \sup_{\text{Lip}(f) \leq 1} \int f\mu - \int f\nu.
\]

Similar to (1), it is easy to see that in order to control \( |h(X) - h(\tilde{X})| \) by means of \( W_2(P_X, P_{\tilde{X}}) \), one necessarily needs to assume some regularity properties of \( P_X \) and \( P_{\tilde{X}} \); otherwise, choosing one to be a fine quantization of the other creates infinite gap between differential entropies, while keeping the \( W_2 \) distance arbitrarily small. Our main result in Section 2 shows that under moment constraints and certain conditions on the densities (which are in particular satisfied by convolutions with Gaussians), various information measures such as differential entropy and mutual information on \( \mathbb{R}^n \) are in fact \( \sqrt{n} \)-Lipschitz continuous with respect to the \( W_2 \)-distance. These results have natural counterparts in the discrete case where the Euclidean distance is replaced by Hamming distance (Section 4).

Furthermore, transportation-information inequalities, such as those due to Marton [Mar86] and Talagrand [Tal96], allow us to bound the Wasserstein distance by the KL divergence (see, e.g., [RS13] for a review). For example, Talagrand’s inequality states that if \( Q = \mathcal{N}(0, \Sigma) \), then

\[
W_2^2(P, Q) \leq \frac{2\sigma_{\text{max}}(\Sigma)}{\log e} D(P\|Q),
\]

where \( \sigma_{\text{max}}(\Sigma) \) denotes the maximal singular value of \( \Sigma \). Invoking this result in conjunction with the Wasserstein continuity of the differential entropy, we prove a new outer bound for the capacity region of the two-user GIC, which finally settles the missing corner point in [Cos85a]. See Section 3 for details.

One interesting by-product is an estimate that goes in the reverse direction of (5). Namely, under regularity conditions on \( P \) and \( Q \) we have

\[
D(P\|Q) \lesssim \sqrt{\int_{\mathbb{R}^n} \|x\|^2 (dP + dQ) \cdot W_2(P, Q)}.
\]

See Proposition 1 and Corollary 4 in the next section. We want to emphasize that there are a number of estimates of the form \( D(P_{X+Z}\|P_{\tilde{X}+Z}) \lesssim W_2^2(P_X, P_{\tilde{X}}) \) where \( X \perp Z \sim \mathcal{N}(0, I_n) \), cf. [Vil03, Chapter 9]. The key difference of (6) is that the \( W_2 \) distance is measured after convolving with \( P_Z \).
2 Wasserstein-continuity of information quantities

We say that a probability density function \( p \) on \( \mathbb{R}^n \) is \( (c_1, c_2) \)-regular if \( c_1 > 0, c_2 \geq 0 \) and
\[
\| \nabla \log p(x) \| \leq c_1 \| x \| + c_2, \quad \forall x \in \mathbb{R}^n.
\]
Notice that in particular, regular density is never zero and furthermore
\[
| \log p(x) - \log p(0) | \leq \frac{c_1}{2} \| x \|^2 + c_2 \| x \|
\]
Therefore, if \( X \) has a regular density and finite second moment then
\[
|h(X)| \leq | \log P_X(0) | + c_2 \mathbb{E}[\| X \|] + \frac{c_1}{2} \mathbb{E}[\| X \|^2] < \infty.
\]

**Proposition 1.** Let \( U \) and \( V \) be random vectors with finite second moments. If \( V \) has a \( (c_1, c_2) \)-regular density \( p_V \), then there exists a coupling \( P_{UV} \), such that
\[
\mathbb{E} \left[ \left| \log \frac{p_V(V)}{p_U(U)} \right| \right] \leq \Delta,
\]
where
\[
\Delta = \left( \frac{c_1}{2} \sqrt{\mathbb{E}[\| U \|^2]} + \frac{c_1}{2} \sqrt{\mathbb{E}[\| V \|^2]} + c_2 \right) W_2(P_U, P_V).
\]
Consequently,
\[
h(U) - h(V) \leq \Delta.
\]
If both \( U \) and \( V \) are \( (c_1, c_2) \)-regular, then
\[
|h(U) - h(V)| \leq \Delta
\]
\[
D(P_U \| P_V) + D(P_V \| P_U) \leq 2\Delta
\]

**Proof.** First notice:
\[
| \log p_V(v) - \log p_V(u) | = \left| \int_0^1 dt (\nabla \log p_V(tv + (1 - t)u), u - v) \right|
\]
\[
\leq \int_0^1 (c_2 + c_1 t \| v \| + c_1 (1 - t) \| u \|) \| u - v \|
\]
\[
= (c_2 + c_1 \| v \|/2 + c_1 \| u \|/2) \| u - v \|. \tag{11}
\]
Taking expectation of (13) with respect to \((u, v)\) distributed according to the optimal \( W_2 \)-coupling of \( P_U \) and \( P_V \) and then applying Cauchy-Schwartz and triangle inequality, we obtain (7).

To show (8) notice that by finiteness of second moment \( h(U) < \infty \). If \( h(U) = -\infty \) then there is nothing to prove. So assume otherwise, then in identity
\[
h(U) - h(V) + D(P_U \| P_V) = \mathbb{E} \left[ \log \frac{p_V(V)}{p_U(U)} \right]
\]
all terms are finite and hence (8) follows. Clearly, (8) implies (9) (when applied with \( U \) and \( V \) interchanged).

Finally, for (10) just add the identity (14) to itself with \( U \) and \( V \) interchanged to obtain
\[
D(P_U \| P_V) + D(P_V \| P_U) = \mathbb{E} \left[ \log \frac{p_V(V)}{p_U(U)} \right] + \mathbb{E} \left[ \log \frac{p_U(U)}{p_V(V)} \right]
\]
and estimate both terms via (7).  \( \square \)

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\(^1\)In this paper \( \log \) is to arbitrary base, which also specifies the units of differential entropy \( h(\cdot) \), entropy \( H(\cdot) \), mutual information \( I(\cdot; \cdot) \) and divergence \( D(\cdot; \cdot) \). The base-e logarithm is denoted by \( \ln \).
The key question now is what densities are regular. It turns out that convolution with sufficiently smooth density, such as Gaussians, produces a regular density.

**Proposition 2.** Let \( V = B + Z \) where \( B \perp Z \sim N(0, \sigma^2 I_n) \). Then the density of \( V \) is \((c_1, c_2)\)-regular with \( c_1 = \frac{3 \log e}{\sigma^2} \) and \( c_2 = \frac{4 \log e}{\sigma^2} \mathbb{E}[\|B\|] \).

**Proof.** First notice that whenever density \( p_Z \) of \( Z \) is differentiable and non-vanishing, we have:

\[
\nabla \log p_V(v) = \mathbb{E}[\nabla \log p_Z(v - B) | V = v],
\]

where \( p_V \) is the density of \( V \). For Gaussian noise we have

\[
\nabla \log p_Z(v - B) = \log e (B - v).
\]

So the proof is completed by showing

\[
\mathbb{E}[\|B - v\| | V = v] \leq 3\|v\| + 4\mathbb{E}[\|B\|].
\]

For this, we mirror the proof in [WV12, Lemma 4]. Indeed, we have

\[
\mathbb{E}[\|B - v\| | V = v] = \mathbb{E}[\|B - v\| \frac{p_Z(B - v)}{p_V(v)}] \leq 2\mathbb{E}[\|B - v\| 1\{a(B, v) \leq 2\}] + \mathbb{E}[\|B - v\| a(B, v) 1\{a(B, v) > 2\}],
\]

where we denoted

\[
a(B, v) \equiv \frac{p_Z(B - v)}{p_V(v)}.
\]

Next, notice that

\[
\{a(B, v) > 2\} = \{\|B - v\|^2 \leq -2\sigma^2 \ln((2\pi\sigma^2)^{n/2} 2p_V(v))\}.
\]

Thus since \( \mathbb{E}[p_Z(B - v)] = p_V(v) \) we have an upper bound for the second term in (18) as follows

\[
\mathbb{E}[\|B - v\| a(B, v) 1\{a(B, v) > 2\}] \leq \sqrt{2\sigma} \sqrt{\frac{\ln^{+} \frac{1}{(2\pi\sigma^2)^{n/2} 2p_V(v)}.}{}}
\]

From Markov inequality we have \( \mathbb{P}[\|B\| \leq 2\mathbb{E}[\|B\|]] \geq 1/2 \) and therefore

\[
p_V(v) \geq \frac{1}{2(2\pi\sigma^2)^{n/2}} e^{-\frac{\left(\|v\|^2 + 2\mathbb{E}[\|B\|]\right)^2}{2\sigma^2}}.
\]

Using this estimate in (19) we get

\[
\mathbb{E}[\|B - v\| a(B, v) 1\{a(B, v) > 2\}] \leq \|v\| + 2\mathbb{E}[\|B\|].
\]

Upper-bounding the first term in (18) by \( 2\mathbb{E}[\|B\|] + 2\|v\| \) we finish the proof of (16).

Another useful criterion for regularity is the following:

**Proposition 3.** If \( W \) has \((c_1, c_2)\)-regular density and \( B \perp W \) satisfies

\[
\|B\| \leq \sqrt{n\mathbb{P}} \quad \text{a.s.}
\]

then \( V = B + W \) has \((c_1, c_2 + c_1\sqrt{n\mathbb{P}})\)-regular density.
Proof. Apply (15) and the estimate:
\[ \mathbb{E}[\|\nabla \log p_W(v-B)\| | V=v] \leq c_1(\|v\| + \sqrt{nP}) + c_2. \]

As a consequence, we show that when smoothed by Gaussian noise, mutual information, differential entropy and divergence are Lipschitz with respect to the $W_2$-distance under average power constraints:

**Corollary 4.** Let $\mathbb{E}[\|X\|^2], \mathbb{E}[\|\tilde{X}\|^2] \leq nP$ and $Z \sim \mathcal{N}(0,I_n)$. Then for some universal constant $\kappa$,
\[
|I(X;X+Z) - I(\tilde{X};\tilde{X}+Z)| = |h(X+Z) - h(\tilde{X}+Z)| \leq \kappa \sqrt{nPW_2(P_{X+Z},P_{\tilde{X}+Z})},
\]
which implies
\[
D(P_{\tilde{X}+Z}\|P_{X+Z}) + D(P_{\tilde{X}+Z}\|P_{\tilde{X}+Z}) \leq 2\kappa \sqrt{nPW_2(P_{X+Z},P_{\tilde{X}+Z})}.
\]

Proof. By Proposition 2, the densities of $X+Z$ and $\tilde{X}+Z$ are $(3\log e, 4\sqrt{nP}\log e)$-regular. The desired statement then follows from (9)-(10).

**Remark 1.** The Lipschitz constant $\sqrt{n}$ is order-optimal as the example of Gaussian $X$ and $\tilde{X}$ with different variances (one of them could be zero) demonstrates. The linear dependence of $W_2$ with different variances (one of them could be zero) demonstrates. The linear dependence of $W_2$ is also optimal. To see this, consider $X \sim \mathcal{N}(0,1)$ and $\tilde{X} \sim \mathcal{N}(0,1+t)$ in one dimension. Then
\[
I(X;X+Z) - I(\tilde{X};\tilde{X}+Z) = 1/2 \log(1+t/2) = \Theta(t) \text{ and } W_2^2(X+Z,\tilde{X}+Z) = (\sqrt{2+t} - \sqrt{2})^2 = \Theta(t^2),
\]
as $t \to 0$.

In fact, to get the best constants for applications to interference channels it is best to circumvent the notion of regular density and deal directly with (15). Indeed, when the inputs has almost sure bounded norms, the next result gives a sharpened version of what can be obtained by combining Proposition 1 with 2.

**Proposition 5.** Let $B$ satisfying (21) and $G \sim \mathcal{N}(0,\sigma_G^2 I_n)$ be independent. Let $V = B + G$. Then for any $U$,\[
h(U) - h(V) \leq \mathbb{E}
\left[
\log \frac{p_V(V)}{p_U(U)}
\right]
\leq \frac{\log e}{2\sigma_G^2} \left(\mathbb{E}[\|U\|^2] - \mathbb{E}[\|V\|^2]\right) + \frac{\sqrt{nP}}{\sigma_G^2} W_1(P_U, P_V).
\]

Proof. Plugging Gaussian density into (15) we get
\[
\nabla \log p_V(v) = \frac{\log e}{\sigma_G^2} (\hat{B}(v) - v),
\]
where $\hat{B}(v) = \mathbb{E}[B|V=v] = \frac{\mathbb{E}[B|\varphi(v-B)]}{\mathbb{E}[\varphi(v-B)]}$ satisfies
\[
\|\hat{B}(v)\| \leq \sqrt{nP},
\]
since $\|B\| \leq \sqrt{nP}$ almost surely. Next we use
\[
\log \frac{p_V(v)}{p_U(u)} = \int_0^1 dt \left(\nabla \log p_V(tv + (1-t)u), v - u\right)
\leq \frac{\log e}{\sigma_G^2} \int_0^1 dt \left(\hat{B}(tv + (1-t)u), v - u\right) - \frac{\log e}{2\sigma_G^2} (\|v\|^2 - \|u\|^2)
\leq \frac{\sqrt{nP} \log e}{\sigma_G^2} \|v - u\| - \frac{\log e}{2\sigma_G^2} (\|v\|^2 - \|u\|^2).
\]
Taking expectation of the last equation under the $W_1$-optimal coupling yields (25).
To get slightly better constants in one-sided version of the above we can apply Proposition 5:

**Corollary 6.** Let $A, B, G, Z$ be independent, with $G \sim \mathcal{N}(0, \sigma_G^2 I_n)$, $Z \sim \mathcal{N}(0, \sigma_Z^2 I_n)$ and $B$ satisfying (21). Then for every $c \in [0, 1]$ we have:

$$h(B + A + Z) - h(B + G + Z) \leq \frac{\log e}{2(\sigma_G^2 + \sigma_Z^2)} (\mathbb{E}[\|A\|^2] - \mathbb{E}[\|G\|^2]) + \frac{2nP(\sigma_G^2 + c^2 \sigma_Z^2) \log e}{\sigma_G^2 + \sigma_Z^2} \sqrt{D(P_{A+cZ}\|P_{G+cZ})}$$

(30)

**Proof.** First, notice that by definition Wasserstein distance is non-decreasing under convolutions, i.e., $W_2(P_1 * Q, P_2 * Q) \leq W_2(P_1, P_2)$. Since $c \leq 1$, we have

$$W_2(P_{B+A+Z}, P_{B+G+Z}) \leq W_2(P_{A+Z}, P_{G+Z}) \leq W_2(P_{A+cZ}, P_{G+cZ}),$$

which, in turn, can be bounded via Talagrand’s inequality (5) by

$$W_2(P_{A+cZ}, P_{G+cZ}) \leq \sqrt{\frac{2(\sigma_G^2 + c^2 \sigma_Z^2)}{\log e} D(P_{A+cZ}\|G + cZ).}$$

From here we apply Proposition 5 with $G$ replaced by $G + Z$ (and $\sigma_Z^2$ by $\sigma_Z^2 + \sigma_G^2$). $\square$

### 3 Applications to Gaussian interference channels

#### 3.1 New outer bound

Consider the two-user Gaussian interference channel (GIC):

$$\begin{align*}
Y_1 &= X_1 + bX_2 + Z_1 \\
Y_2 &= aX_1 + X_2 + Z_2,
\end{align*}$$

(31)

with $a, b \geq 0$, $Z_i \sim \mathcal{N}(0, I_n)$ and a power constraint on the $n$-letter codebooks:

$$\|X_1\| \leq \sqrt{nP_1}, \quad \|X_2\| \leq \sqrt{nP_2} \quad \text{a.s.}$$

(32)

Denote by $\mathcal{R}(a, b)$ the capacity region of the GIC (31). As an application of the results developed in Section 2, we prove an outer bound for the capacity region.

**Theorem 7.** Let $0 < a \leq 1$. Then for any $b \geq 0$ and $\tilde{C}_2 \leq R_2 \leq C_2$, any rate pair $(R_1, R_2) \in \mathcal{R}(a, b)$ satisfies

$$R_1 \leq \frac{1}{2} \log \min \left\{ A - \frac{1}{a^2} + 1, A \frac{(1 + P_2)(1 - (1 - a^2) \exp(-2\delta)) - a^2}{P_2} \right\}$$

(33)

where $C_2 = \frac{1}{2} \log(1 + P_2)$, $\tilde{C}_2 = \frac{1}{2} \log(1 + \frac{P_2}{1 + a^2 P_1})$ and

$$A = (P_1 + a^{-2}(1 + P_2)) \exp(-2R_2),$$

$$\delta = C_2 - R_2 + a \sqrt{\frac{2P_1(C_2 - R_2) \log e}{1 + P_2}}.$$  

(34)

(35)

Consequently, $R_2 \geq C_2 - \epsilon$ implies that $R_1 \leq \frac{1}{2} \log(1 + \frac{a^2 P_2}{1 + P_1}) - \epsilon'$ where $\epsilon' \to 0$ as $\epsilon \to 0$. The latter statement continues to hold even if the almost sure power constraint (32) is relaxed to

$$\mathbb{E}[\|X_1\|^2] \leq nP_1, \quad \mathbb{E}[\|X_2\|^2] \leq nP_2.$$  

(36)
Proof. First of all, setting $b = 0$ (which is equivalent to granting the first user access to $X_2$) will not shrink the capacity region of the interference channel (31). Therefore to prove the desired outer bound it suffices to focus on the following Z-interference channel henceforth:

$$
Y_1 = X_1 + Z_1 \\
Y_2 = aX_1 + X_2 + Z_2.
$$

(37)

Let $(X_1, X_2)$ be $n$-dimensional random variables corresponding to the encoder output of the first and second user, which are uniformly distributed on the respective codebook. For $i = 1, 2$ define

$$
R_i \triangleq \frac{1}{n} I(X_i; Y_i).
$$

By Fano’s inequality there is no difference asymptotically between this definition of rate and the operational one. Define the entropy-power function of the $X_1$-codebook:

$$
N_1(t) \triangleq \exp\left\{ \frac{2}{n} h(X_1 + \sqrt{t}Z) \right\}, \quad Z \sim \mathcal{N}(0, I_n).
$$

We know the following general properties of $N_1(t)$:

- $N_1$ is monotonically increasing.
- $N_1(0) = 0$ (since $X_1$ is uniform over the codebook).
- $N_1'(t) \geq 2\pi e$ (entropy-power inequality).
- $N_1(t)$ is concave (Costa’s EPI [Cos85a]).
- $N_1(t) \leq 2\pi e(P_1 + t)$ (Gaussian maximizes differential entropy).

We can then express $R_1$ in terms of the entropy power function as

$$
R_1 = \frac{1}{2} \log \frac{N_1(1)}{2\pi e}.
$$

(38)

It remains to upper bound $N_1(1)$. Note that

$$
nR_2 = I(X_2; Y_2) = h(X_2 + aX_1 + Z) - h(aX_1 + Z) \leq \frac{1}{2} \log 2\pi e(1 + P_2 + a^2P_1) - h(aX_1 + Z),
$$

and therefore

$$
N_1\left(\frac{1}{a^2}\right) \leq 2\pi eA,
$$

(39)

where $A$ is defined in (34). This in conjunction with the slope property $N_1'(t) \geq 2\pi e$ yields

$$
N_1(1) \leq N_1\left(\frac{1}{a^2}\right) - 2\pi e(a^{-2} - 1) \leq 2\pi e(A - a^{-2} + 1),
$$

(40)

which, in view of (38), yields the first part of the bound (33).

To obtain the second bound, let $G_2 \sim \mathcal{N}(0, P_2I_n)$. Using $\mathbb{E}[\|X_2\|^2] \leq nP_2$ and $X_1 \perp \!\!\!\perp X_2$, we obtain

$$
nR_2 = I(X_2; Y_2) \leq I(X_2; Y_2|X_1) = I(X_2; X_2 + Z_2)
$$

$$
= nC_2 - h(G_2 + Z_2) + h(X_2 + Z_2) \leq nC_2 - D(P_{X_2+Z_2}\|P_{G_2+Z_2}),
$$

where $C_2$ is the capacity of the Gaussian channel with input covariance $P_{X_2+Z_2}$ and output covariance $P_{G_2+Z_2}$.
that is,
\[ D(P_{X_2+Z_2} || P_{G_2+Z_2}) \leq h(G_2 + Z_2) - h(X_2 + Z_2) \leq n(C_2 - R_2). \]  
(41)

Furthermore,
\[ nR_2 = I(X_2; Y_2) = h(aX_1 + X_2 + Z_2) - h(aX_1 + G_2 + Z_2) + h(aX_1 + G_2 + Z_2) - h(aX_1 + Z_2). \]  
(42)

(43)

Note that the second term (43) is precisely \( \frac{a}{2} \log \frac{N_1(\frac{a}{2})}{N_1(\frac{a}{2})}. \) The first term (42) can be bounded by applying Corollary 6 and (41) with \( B = aX_1, A = X_2, G = G_2 \) and \( c = 1: \)
\[ h(aX_1 + X_2 + Z_2) - h(aX_1 + G_2 + Z_2) \leq n\sqrt{\frac{2a^2 P_1(C_2 - R_2) \log e}{1 + P_2}}. \]  
(44)

Combining (42) – (44) yields
\[ N_1 \left( \frac{1}{a^2} \right) \leq \frac{\exp(2\delta)}{1 + P_2} N_1 \left( \frac{1 + P_2}{a^2} \right). \]  
(45)

where \( \delta \) is defined in (35). From the concavity of \( N_1(t) \) and (45)
\[ N_1(1) \leq \gamma N_1 \left( \frac{1}{a^2} \right) - (\gamma - 1)N_1 \left( \frac{1 + P_2}{a^2} \right) \]
\[ \leq N_1 \left( \frac{1}{a^2} \right) \left( \gamma - (\gamma - 1) \frac{1 + P_2}{\exp(2\delta)} \right) \],  
(46)
(47)

where \( \gamma = 1 + \frac{1-a^2}{P_2} > 1. \) In view of (38), upper bounding \( N_1(1/a^2) \) in (47) via (39) we get after some simplifications the second part of (33).

Finally, when \( R_2 \to C_2 \), we have \( \delta \to 0, A = \frac{1}{a^2} + \frac{P_2}{1 + P_1} \) and hence from (33) \( R_1 \leq C'_1. \) To show that this statement holds under the average power constraint (36), we only need to replace the use of Corollary 6 in the estimate (44) by Corollary 4 and Talagrand’s inequality
\[ W_2(P_{X_2+Z_2}, P_{G_2+Z_2}) \leq \sqrt{\frac{2(1 + P_2)}{\log e} D(P_{X_2+Z_2} || P_{G_2+Z_2})}. \]

\[ \square \]

**Remark 2.** Curiously, the first part of the bound (33) coincides with the bound of Kramer [Kra04, Theorem 2], which was obtained by reducing the Z-interference channel to the degraded broadcast channel. Note that our estimates on \( N_1(1) \) in the proof of Theorem 7 are tight in the sense that there exists a concave function \( N_1(t) \) satisfying the listed general properties, estimates (45) and (39) as well as attaining the minimum of (40) and (47) at \( N_1(1) \). Hence, tightening the bound via this method would require inferring more information about \( N_1(t) \).

**Remark 3.** The outer bound (33) relies on Costa’s EPI. To establish the second statement about corner point, it is sufficient to invoke the concavity of \( \gamma \mapsto I(X_2; \sqrt{\gamma}X_2 + Z_2) \), which is strictly weaker than Costa’s EPI.

The outer bound (33) is evaluated on Fig. 1 for the case of \( b = 0 \) (Z-interference), where we also plot (just for reference) the simple Han-Kobayashi inner bound for the Z-GIC (37) attained by choosing \( X_1 = U + V \) with \( U \perp V \) jointly Gaussian. This achieves rates:
\[
\begin{cases}
R_1 = \frac{1}{2} \log (1 + P_1 - s) + \frac{1}{2} \log \left( 1 + \frac{a^2 s}{1 + a^2 (P_1 - s) + P_2} \right), & 0 \leq s \leq P_1, \\
R_2 = \frac{1}{2} \log \left( 1 + \frac{P_2}{1 + a^2 (P_1 - s)} \right), & 0 \leq P_2.
\end{cases}
\]
(48)

For more sophisticated Han-Kobayashi bounds see [Sas04, Cos11].
3.2 Corner points of the capacity region

The two corner points of the capacity region are defined as follows:

\[ C'_1(a, b) \triangleq \max \{ R_1 : (R_1, C_2) \in \mathcal{R}(a, b) \} , \]
\[ C'_2(a, b) \triangleq \max \{ R_2 : (C_1, R_2) \in \mathcal{R}(a, b) \} , \]

where \( C_i = \frac{1}{2} \log(1 + P_i) \). As a corollary, Theorem 7 completes the picture of the corner points for the capacity region of GIC for all values of \( a, b \in \mathbb{R}_+ \) under the average power constraint (36). We note that the new result here is the proof of \( C'_1(a, b) = \frac{1}{2} \log \left( 1 + \frac{a^2 P_1}{1+P_2} \right) \) for \( 0 < a \leq 1 \) and \( b \geq 0 \).

The interpretation is that if one user desires to achieve its own interference-free capacity, then the other user must guarantee that its message is decodable at both receivers. The achievability of this corner point was previously known, while the converse was previously considered by Costa [Cos85b] but with a flawed proof, as pointed out in [Sas04]. The high-level difference between our proof and that of [Cos85b] is the replacement of Pinsker’s inequality by Talagrand’s and the use of a coupling argument.\(^2\)

Below we present a brief account of the corner points in various cases; for an extensive discussion see [Sas15]. We start with a few simple observations about the capacity region \( \mathcal{R}(a, b) \):

\(^2\)After posting the initial draft, we were informed by the authors of [BPS14] that their updated manuscript also verifies Costa’s conjecture.
• Any rate pair satisfying the following belongs to $\mathcal{R}(a, b)$:

$$R_1 \leq \frac{1}{2} \log(1 + P_1 \min(1, a^2))$$

$$R_2 \leq \frac{1}{2} \log(1 + P_2 \min(1, b^2))$$

$$R_1 + R_2 \leq \frac{1}{2} \log(1 + \min(P_1 + b^2 P_2, P_2 + a^2 P_1)),$$

which corresponds to the intersection of two Gaussian multiple-access (MAC) capacity regions, namely, $(X_1, X_2) \to Y_1$ and $(X_1, X_2) \to Y_2$. These rate pairs correspond to the case when each receiver decodes both messages.

• For $a > 1, b > 1$ the capacity region is known to coincide with (51). So, without loss of generality we assume $a \leq 1$ henceforth.

• Replacing either $a$ or $b$ with zero can only enlarge the region (genie argument).

• If $b \geq 1$ then for any $(R_1, R_2) \in \mathcal{R}(a, b)$ we have

$$R_1 + R_2 \leq \frac{1}{2} \log \left(1 + b^2 P_2 + P_1\right).$$

This follows from the observation that in this case $I(X_1, X_2; Y_1) = H(X_1, X_2) - o(n)$, since conditioned on $X_1, Y_2$ is a noisier observation of $X_2$ than $Y_1$.

• Similarly, if $a \geq 1$ then for any $(R_1, R_2) \in \mathcal{R}(a, b)$ we have

$$R_1 + R_2 \leq \frac{1}{2} \log \left(1 + a^2 P_1 + P_2\right).$$

For the top corner, we have the following:

$$C'_1(a, b) = \begin{cases} \frac{1}{2} \log \left(1 + \frac{a^2 P_1}{1 + P_2}\right), & 0 < a \leq 1, b \geq 0 \\ C_1, & a = 0, b = 0 \text{ or } b \geq \sqrt{1 + P_1} \\ \frac{1}{2} \log \left(1 + \frac{P_1 + (b^2 - 1)P_2}{1 + P_2}\right), & a = 0, 1 < b < \sqrt{1 + P_1} \\ \frac{1}{2} \log \left(1 + \frac{P_1}{1 + b^2 P_2}\right), & a = 0, 0 < b \leq 1. \end{cases}$$

(54)

Note that for any $b \geq 0$, $a \mapsto C'_1(a, b)$ is discontinuous as $a \downarrow 0$. To verify (54) we consider each case separately:

1. For $a > 0$ the converse bound follows from Theorem 7. For achievability, we consider two cases. When $b \leq 1$, we have $\frac{a^2 P_1}{1 + P_2} \leq \frac{P_1}{1 + b^2 P_2}$ and therefore treating interference $X_2$ as noise at the first receiver and using a Gaussian MAC-code for $(X_1, X_2) \to Y_2$ works. For $b > 1$, the achievability follows from the MAC inner bound (51). Note that since $\frac{1}{2} \log \left(1 + P_1 + b^2 P_2\right) \geq \frac{1}{2} \log \left(1 + P_2 + a^2 P_1\right)$, a Gaussian MAC-code that works for $(X_1, X_2) \to Y_2$ will also work for $(X_1, X_2) \to Y_1$. Alternatively, the achievability also follows from Han-Kobayashi inner bound (see, e.g., [EGK11, Theorem 6.4] with $(U_1, U_2) = (X_1, X_2)$ for $b \geq 1$ and $(U_1, U_2) = (X_1, 0)$ for $b \leq 1$).

2. For $a = 0$ and $b \geq \sqrt{1 + P_1}$ the converse is obvious, while for achievability we have that $\frac{b^2 P_2}{1 + P_1} \leq P_2$ and therefore $X_2$ is decodable at $Y_1$. 

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3. For \(a = 0\) and \(1 < b < \sqrt{1 + P_1}\) the converse is (52) and the achievability is just the MAC code \((X_1, X_2) \to Y_1\) with rate \(R_2 = C_2\).

4. For \(a = 0\) and \(0 < b \leq 1\) the result follows from the treatment of \(C'_2(a, b)\) below by interchanging \(a \leftrightarrow b\) and \(P_1 \leftrightarrow P_2\).

The bottom corner point is given by the following:

\[
C'_2(a, b) = \begin{cases} \frac{1}{2} \log \left( 1 + \frac{P_2}{1 + a^2 P_1} \right), & 0 \leq a \leq 1, b = 0 \text{ or } b \geq \sqrt{\frac{1 + P_1}{1 + a^2 P_1}} \\ \frac{1}{2} \log \left( 1 + \frac{b^2 P_2}{1 + P_1} \right), & 0 \leq a \leq 1, 1 < b < \sqrt{\frac{1 + P_1}{1 + a^2 P_1}} \\ \frac{1}{2} \log \left( 1 + \frac{b^2 P_2}{1 + P_1} \right), & 0 \leq a \leq 1, 0 < b \leq 1 \end{cases}
\]

which is discontinuous as \(b \downarrow 0\) for any fixed \(a \in [0, 1]\). We treat each case separately:

1. The case of \(C'_2(a, 0)\) is due to Sato [Sat78] (see also [Kra04, Theorem 2]). The converse part also follows from Theorem 7 (for \(a = 0\) there is nothing to prove). For the achievability, we notice that under \(b \geq \sqrt{\frac{1 + P_1}{1 + a^2 P_1}}\) we have \(\frac{b^2 P_2}{1 + P_1} > \frac{P_2}{1 + a^2 P_1}\) and thus \(X_2\) at rate \(C'_2(a, 0)\) can be decoded and canceled from \(Y_1\) by simply treating \(X_1\) as Gaussian noise (as usual, we assume Gaussian random codebooks). Thus the problem reduces to that of \(b = 0\). For \(b = 0\), the Gaussian random coding achieves the claimed result if the second receiver treats \(X_1\) as Gaussian noise.

2. The converse follows from (52) and for the achievability we use the Gaussian MAC-code \((X_1, X_2) \to Y_1\) and treat \(X_1\) as Gaussian interference at \(Y_2\).

3. If \(b \in (0, 1]\), we apply results on \(C'_1(a, b)\) in (54) by interchanging \(a \leftrightarrow b\) and \(P_1 \leftrightarrow P_2\).

4 Discrete version

Fix a finite alphabet \(\mathcal{X}\) and an integer \(n\). On the product space \(\mathcal{X}^n\) we define the Hamming distance

\[
d_H(x, y) = \sum_{j=1}^{n} 1_{\{x_j \neq y_j\}},
\]

and consider the Wasserstein distance \(W_1(\cdot, \cdot)\) with respect to this distance. In fact, \(\frac{1}{n} W_1(P, Q)\) is known as Ornstein’s \(\bar{d}\)-distance [GNS75,Mar86], namely,

\[
\bar{d}(P, Q) = \frac{1}{n} \inf \mathbb{E}[d_H(X, Y)],
\]

where the infimum is taken over all couplings \(P_{XY}\) of \(P\) and \(Q\).

For a pair of distributions \(P, Q\) on \(\mathcal{X}^n\) we may ask the following questions:

1. Does \(D(P\|Q)\) control the entropy difference \(H(P) - H(Q)\)?

2. Does \(\bar{d}(P, Q)\) control the entropy difference \(H(P) - H(Q)\)?

Recall that in the Euclidean space the answer to both questions was negative unless the distributions satisfy certain regularity conditions. For discrete alphabets the answer to the first question is still negative in general (see Section 1 for a counterexample); nevertheless, the answer to the second one turns out to be positive:
Proposition 8. Let $P, Q$ be distributions on $\mathcal{X}^n$ and let

$$F_X(x) \triangleq x \log(|\mathcal{X}| - 1) + x \log \frac{1}{x} + (1 - x) \log \frac{1}{1 - x}.$$ 

Then

$$|H(P) - H(Q)| \leq nF_X(\bar{d}(P, Q)).$$

(57)

Proof. In fact, the statement holds for any translation-invariant distance $d(\cdot, \cdot)$ on $\mathcal{X}^n$. Indeed, define

$$f_n(s) \triangleq \max \left\{ \frac{1}{n} H(X) : \mathbb{E}[d(X, x_0)] \leq ns \right\},$$

where $x_0 \in \mathcal{X}^n$ is an arbitrary fixed string. It is easy to see that $s \mapsto f_n(s)$ is concave and, furthermore,

$$f_n(s) = f_1(s).$$

Thus, letting $X, Y$ be distributed according to the $W_1$-optimal coupling of $P$ and $Q$, we get

$$H(X) - H(Y) \leq H(X, Y) - H(Y) = H(X|Y)$$

(58)

$$\leq n\mathbb{E} [f_n(\mathbb{E}[d(X, Y)|Y])]$$

(59)

$$\leq nf_n(\bar{d}(P, Q)),$$

(60)

where (59) is by definition of $f_n(\cdot)$ and (60) is by Jensen’s inequality. Finally, for the Hamming distance we have $f_1(\cdot) = F_X(\cdot)$ by Fano’s inequality. \qed

Remark 4. In the special case when $Q$ is a product distribution, we can further bound the $W_1$ distance by the KL divergence via Marton’s transportation inequality [Mar86, Lemma 1]:

$$\bar{d}(P, Q) \leq \sqrt{\frac{D(P\|Q)}{2n \log e}}.$$  

(61)

Assuming that $D(P\|Q) = \epsilon n$ for some $\epsilon \ll 1$, combining (57) and (61) gives

$$|H(P) - H(Q)| \leq nF_X \left( \sqrt{\frac{D(P\|Q)}{2n \log e}} \right),$$

which the upper bound behaves like $n\sqrt{\epsilon} \log \frac{1}{\epsilon}$ when $\epsilon \to 0$. This estimate has a one-sided improvement:

$$H(P) - H(Q) \leq \sqrt{\frac{2nD(P\|Q)}{\log e}} \log |\mathcal{X}|.$$  

(62)

The inequality (62) is proved by [CS07] for $n = 1$; for the general $n$ see [WV10, Appendix H].

Notice that the right-hand side of (57) behaves like $n\bar{d} \log \frac{1}{\epsilon}$ when $\bar{d}(P, Q)$ is small. This super-linear dependence is in fact sharp.\footnote{To see this, consider $Q = \text{Bern}(p)^\otimes n$ and choose $P$ to be the output distribution of the optimal lossy compressor for $Q$ at average distortion $\delta n$. By definition, $\bar{d}(P, Q) \leq \delta$. On the other hand, $H(P) = n(h(p) - h(\delta) + o(1))$ as $n \to \infty$ and hence $|H(P) - H(Q)| = n(h(\delta) + o(1))$, which asymptotically meets the upper bound (57) with equality.} Nevertheless, if certain regularity of distributions is assumed, the estimate (57) can be improved to be linear in $\bar{d}(P, Q)$. We formulate the result in the form suitable for applications in multi-user information theory. It is probably easiest to consider the special case of $X$ being deterministic at a first reading.
Theorem 9. Let $P_{Y|X,A}$ be a two-input blocklength-$n$ memoryless channel, namely $P_{Y|X,A}(y|x,a) = \prod_{j=1}^n P_{Y_1|X_1,A_1}(y_j|x_j,a_j)$. Let $X, A, \tilde{A}$ be independent $n$-dimensional discrete random vectors. Let $Y, \tilde{Y}$ be the outputs generated by $(X,A)$ and $(X,\tilde{A})$, respectively. Then

$$|H(Y) - H(\tilde{Y})| \leq cn\bar{d}(P_Y, P_{\tilde{Y}})$$

(63)

$$D(P_Y \| P_{\tilde{Y}}) + D(P_{\tilde{Y}} \| P_Y) \leq 2cn\bar{d}(P_Y, P_{\tilde{Y}})$$

(64)

$$|I(X;Y) - I(X;\tilde{Y})| \leq 2cn\bar{d}(P_Y, P_{\tilde{Y}})$$

(65)

where

$$c \triangleq \max_{x,a,y,y'} \log \frac{P_{Y_1|X_1,A_1}(y|x,a)}{P_{Y_1|X_1,A_1}(y'|x,a)}$$

(66)

Proof. The function $y \mapsto \log P_Y(y)$ is $c$-Lipschitz with respect to the Hamming distance, cf. [PV14, Eqn. (58)]. From Lipschitz continuity we conclude the existence of a coupling $P_{Y,\tilde{Y}}$, such that

$$\mathbb{E} \left[ \log \frac{P_{Y}(Y)}{P_{\tilde{Y}}(Y)} \right] \leq cn\bar{d}(P_Y, P_{\tilde{Y}}).$$

The rest is the same as in the proof of (8)–(10).

To get the inequality for mutual informations, just apply theorem for $X$ being constant thus estimating $|H(Y|X) - H(\tilde{Y}|X)|$. \qed

Remark 5. When $\tilde{A}$ has i.i.d. components $P_{\tilde{A}} = P_{\tilde{A}^n}$, it is possible to further convert the result of Theorem 9 to estimates in terms of divergence, similar to (62). Then we have

$$\bar{d}(P_Y, P_{\tilde{Y}}) \leq \eta_{TV}\bar{d}(P_A, P_{\tilde{A}}) \leq \eta_{TV}\sqrt{\frac{D(P_A \| P_{\tilde{A}})}{2n\log e}},$$

(67)

where the first inequality is by Dobrushin contraction [Dob70] (see [PW14, Proposition 18]) and we defined

$$\eta_{TV} \triangleq \max_{x,a,a'} \text{TV}(P_{Y_1|X_1=x,A_1=a}, P_{Y_1|X_1=x,A_1=a'}),$$

(68)

and the second inequality is via (61). An alternative to the estimate (67) is the following:

$$\bar{d}(P_Y, P_{\tilde{Y}}) \leq \sum_{x \in \mathcal{X}^n} P_X(x)\bar{d}(P_{Y|x=x}, P_{Y|x=x})$$

(69)

$$\leq \sum_{x \in \mathcal{X}^n} P_X(x)\sqrt{\frac{1}{2n\log e} D(P_{Y|x=x} \| P_{\tilde{Y}|x=x})}$$

(70)

$$\leq \sqrt{\frac{1}{2n\log e} D(P_Y \| P_{\tilde{Y}} \| P_X)}$$

(71)

$$\leq \sqrt{\frac{1}{2n\log e} \eta_{KL} D(P_A \| P_{\tilde{A}})}$$

(72)

where (69) is by the convexity of $\bar{d}$ distance as a Wasserstein distance, (70) is by (61), (71) is by Jensen’s inequality, and (72) is by the tensorization property of the strong data-processing constant [AG76]:

$$\eta_{KL} \triangleq \max_{x,P_A} \frac{D(P_{Y_1|x=x} \| P_{Y_1|x=x})}{D(P_A \| P_{\tilde{A}})}.$$  

\footnote{That is, $\eta_{TV}$ is the maximal Dobrushin contraction coefficients among all channels $P_{Y_1|A_1,X_1=x}$ indexed by $x \in \mathcal{X}$.}
In order to apply Theorem 9 to proving corner points of discrete memoryless interference channels (DMIC) we will need an auxiliary tensorization result, which is of independent interest.

**Proposition 10.** Given channels \( P_{A_1|X_1} \) and \( P_{B_1|A_1} \) on finite alphabets, define

\[
F_c(t) \triangleq \max \{ H(X_1|A_1,U_1) : H(X_1|B_1,U_1) \leq t, U_1 \to X_1 \to A_1 \to B_1 \}.
\]

(73)

Then the following hold:

1. (Property of \( F_c \)) The function \( F_c : \mathbb{R}_+ \to \mathbb{R}_+ \) is concave, non-decreasing and \( F_c(t) \leq t \). Furthermore, \( F_c(t) < t \) for all \( t > 0 \), provided that \( P_{B_1|A_1} \) and \( P_{A_1|X_1} \) satisfy

\[
P_{B_1|A_1 = a} \not\perp P_{B_1|A_1 = a'}, \quad \forall a \neq a' \tag{74}
\]

and

\[
P_{A_1|X_1 = x} \neq P_{A_1|X_1 = x'}, \quad \forall x \neq x' \tag{75}
\]

respectively.

2. (Tensorization) For any blocklength-\( n \) Markov chain \( X \to A \to B \), where \( P_{A_1|X} = P^{\otimes n}_{A_1|X_1} \) and \( P_{B_1|A} = P^{\otimes n}_{B_1|A_1} \) are \( n \)-letter memoryless channels, we have

\[
H(X|A) \leq n F_c \left( \frac{1}{n} H(X|B) \right). \tag{76}
\]

**Proof.** Basic properties of \( F_c \) follow from standard arguments. To show the strict inequality \( F_c(t) < t \) under the conditions \( (74) \) and \( (75) \), we first notice that \( F_c \) is simply the concave envelope of the set of achievable pairs \( (H(X_1|A_1), H(X_1|B_1)) \) obtained by iterating over all \( P_A \). By Caratheodory’s theorem, it is sufficient to consider a ternary-valued \( U_1 \) in the optimization defining \( F_c(t) \). Then the set of achievable pairs \( (H(X_1|A_1,U_1), H(X_1|B_1,U_1)) \) is convex and compact (as the continuous image of the compact set of distributions \( P_{U_1,X_1} \)). Consequently, to have \( F_c(t) = t \) there must exist a distribution \( P_{U_1,X_1} \), such that

\[
H(X_1|A_1,U_1) = H(X_1|B_1,U_1) = t. \tag{77}
\]

We next show that under the extra conditions on \( P_{B_1|A_1} \) and \( P_{A_1|X_1} \) we must have \( t = 0 \). Indeed, \( (74) \) guarantees the channel \( P_{B_1|A_1} \) satisfies the strong data processing inequality (see, e.g., [CK11, Exercise 15.12 (b)] and [PW14, Section 1.2] for a survey) that there exists \( \eta > 1 \) such that

\[
I(X_1;B_1|U_1) \leq \eta I(X_1;A_1|U_1). \tag{78}
\]

From \( (77) \) and \( (78) \) we infer that \( I(X_1;A_1|U_1) = 0 \), or equivalently

\[
D(P_{A_1|X_1}||P_{A_1|U_1}|P_{U_1,X_1}) = 0.
\]

On the other hand, the condition \( (75) \) ensures that then we must have \( H(X_1|U_1) = 0 \). Clearly, this implies \( t = 0 \) in \( (77) \).

To show the single-letterization statement \( (76) \), we only consider the case of \( n = 2 \) since the generalization is straightforward by induction. Let \( X \to A \to B \) be a Markov chain with blocklength-2 memoryless channel in between. We have

\[
H(X|B) = H(X_1|B_1B_2) + H(X_2|B_1B_2X_1) \tag{79}
\]

\[
= H(X_1|B_1B_2) + H(X_2|B_2X_1) \tag{80}
\]

\[
\geq H(X_1|B_1A_2) + H(X_2|B_2X_1) \tag{81}
\]
where (80) is because $B_2 \to X_2 \to X_1 \to B_1$ and hence $I(X_2; B_1 | X_2 B_2) = 0$, and (81) is because $B_1 \to X_1 \to A_2 \to B_2$. Next consider the chain

$$H(X|A_1 A_2) = H(X_1|A_1 A_2) + H(X_2|A_1 A_2 X_1)$$

$$= H(X_1|A_1 A_2) + H(X_2|A_2 X_1)$$

$$\leq F_c(H(X_1|B_1 A_2)) + F_c(H(X_2|B_2 X_1))$$

$$\leq 2F_c \left( \frac{1}{2} H(X_1|B_1 A_2) + \frac{1}{2} H(X_2|B_2 X_1) \right)$$

$$\leq 2F_c \left( \frac{1}{2} H(X|B) \right)$$

where (83) is by $A_2 \to X_2 \to X_1 \to A_1$ and hence $I(X_2; A_1 | X_1 A_2) = 0$, (84) is by the definition of $F_c$ and since we have both $A_2 \to X_1 \to A_1 \to B_1$ and $X_1 \to X_2 \to A_2 \to B_2$, (85) is by the concavity of $F_c$, and finally (86) is by the monotonicity of $F_c$ and (81).

The important consequence of Proposition 10 is the following implication:

**Corollary 11.** Let $X \to A \to B$, where the memoryless channels $P_{A|X}$ and $P_{B|A}$ of blocklength $n$ satisfy the conditions (74) and (75). Then there exists a continuous function $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $g(0) = 0$, such that

$$I(X; A) \leq I(X; B) + \epsilon n \implies H(X) \leq I(X; B) + g(\epsilon) n, \quad (87)$$

**Proof.** By Proposition 10, we have $F_c(t) < t$ for all $t > 0$. This together with the concavity of $F_c$ implies that $t \mapsto t - F_c(t)$ is convex, strictly increasing and strictly non-zero on $(0, \infty)$. Define $g$ as the inverse of $t \mapsto t - F_c(t)$, which is increasing and concave and satisfies $g(0) = 0$. Since $I(X; A) \leq I(X; B) + \epsilon n$, the tensorization result (73) yields

$$H(X|B) \leq H(X|A) + \epsilon n \leq nF_c \left( \frac{1}{n} H(X|B) \right) + \epsilon n,$$

i.e., $t \leq F_c(t) + \epsilon$, where $t \triangleq \frac{1}{n} H(X|B)$. Then $t \leq g(\epsilon)$ by definition, completing the proof.

We are now ready to state a non-trivial example of corner points for the capacity region of DMIC. The proof strategy mirrors that of Theorem 7, with Corollary 6 and Costa’s EPI replaced by Theorem 9 and (87), respectively.

**Theorem 12.** Consider the two-user DMIC:

$$Y_1 = X_1, \quad (88)$$

$$Y_2 = X_2 + X_1 + Z_2 \mod 3, \quad (89)$$

---

5This is the analog of the following property of Gaussian channels, exploited in Theorem 7 in the form of Costa’s EPI: For iid Gaussian $Z$ and $t_1 < t_2 < t_3$ we have

$$I(X; X + t_2 Z) = I(X; X + t_3 Z) + o(n) \implies I(X; X + t_1 Z) = I(X; X + t_3 Z) + o(n).$$

This also follows from the concavity of $\gamma \mapsto I(X; \sqrt{\gamma} X + Z)$. 
where $X_1 \in \{0,1,2\}^n$, $X_2 \in \{0,1\}^n$, $Z_2 \in \{0,1,2\}^n$ are independent and $Z_2 \sim P_2^{\otimes n}$ is iid for some non-uniform $P_2$ containing no zeros. The maximal rate achievable by user 2 is

\[ C_2 = \max_{\text{supp}(Q) \subset \{0,1\}} H(Q * P_2) - H(P_2). \] (90)

At this rate the maximal rate of user 1 is

\[ C_1' = \log 3 - \max_{\text{supp}(Q) \subset \{0,1\}} H(Q * P_2). \] (91)

**Remark 6.** As an example, consider $P_2 = [1 - \delta, \delta, \frac{\delta}{2}]$ where $\delta \neq 0, 1, \frac{1}{3}$. Then the maximum in (90) is achieved by $Q = [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$. Therefore $C_2 = H(P_3) - H(P_2)$ and $C_1'' = \log 3 - H(P_3)$, where $P_3 = [\frac{2 - \delta}{4}, \frac{2 - \delta}{4}, \frac{\delta}{2}]$. Note that in the case of $\delta = \frac{1}{3}$, where Theorem 12 is not applicable, we simply have $C_2 = 0$ and $C_1'' = \log 2$ since $X_2 \perp Y_2$. Therefore the corner point is discontinuous in $\delta$.

**Remark 7.** Theorem 12 continues to hold even if cost constraints are imposed. Indeed, if $X_2 \in \{0,1,2\}^n$ is required to satisfy

\[ \sum_{i=1}^{n} b(X_{2,i}) \leq nB \]

for some cost function $b : \{0,1,2\} \rightarrow \mathbb{R}$. Then the maximum in (90) and (91) is taken over all $Q$ such that $\mathbb{E}_Q[|b(U)|] \leq B$. Note that taking $B = \infty$ is equivalent to lifting the constraint $X_2 \in \{0,1\}^n$ in (90). In this case, $C_1'' = 0$ which can be shown by a similar argument not involving Theorem 9.

**Proof.** We start with the converse. Given a code with rate pairs $(R_1, R_2)$, where $R_2 = C_2 - \epsilon$, we show that $R_1 \leq C_1'' - \epsilon'$, where $\epsilon' \rightarrow 0$ as $\epsilon \rightarrow 0$. Let $Q_2$ be the maximizer of (90), i.e., the capacity-achieving distribution of $X_2 \rightarrow X_2 + Z_2$. Let $\tilde{X}_2 \in \{0,1\}^n$ be distributed according to $Q_2^\otimes n$. Then $\tilde{X}_2 + Z_2 \sim P_2^{\otimes n}$, where $P_3 = Q_2 * P_2$. By Fano’s inequality,

\[ n(C_2 - \epsilon + o(1)) = n(R_2 + o(1)) = I(X_2; Y_2) \]

\[ \leq I(X_2; Y_2 | X_1) = I(X_2; Y_2 | X_2 + Z_2) \]

\[ = nC_2 - D(P_{X_2+Z_2} \| P_{\tilde{X}_2+Z_2}), \] (92)

that is,

\[ D(P_{X_2+Z_2} \| P_{\tilde{X}_2+Z_2}) \leq n\epsilon + o(n). \] (93)

Then

\[ d(P_{X_1+X_2+Z_2}, P_{\tilde{X}_1+\tilde{X}_2+Z_2}) \leq d(P_{X_2+Z_2}, P_{\tilde{X}_2+Z_2}) \leq \sqrt{\frac{\epsilon}{2\log \epsilon}} + o(1), \]

where the first inequality follows from the fact that convolution reduces Wasserstein distance, and the second inequality follows from Marton’s inequality (61), since $P_{\tilde{X}_2+Z_2} = P_3^{\otimes n}$ is a product distribution. Applying (65) in Theorem 9, we obtain

\[ |I(X_1; Y_2) - I(X_1; X_1 + \tilde{X}_2 + Z_2)| = |I(X_1; X_1 + X_2 + Z_2) - I(X_1; X_1 + \tilde{X}_2 + Z_2)| \leq (\alpha \sqrt{\epsilon} + o(1))n, \]

where $\alpha = \frac{1}{2 \log \epsilon} \max_{z, z' \in \{0,1,2\}} \log \frac{P_2(z)}{P_2(z')}$.\] is finite since $P_2$ contains no zeros by assumption. On the other hand,

\[ I(X_1; X_1 + Z_2) = I(X_1; Y_2 | X_2) = I(X_1; Y_2) + I(X_1; X_2 | Y_2) = I(X_1; Y_2) + o(n), \]
where \( I(X_1; X_2|Y_2) \leq H(X_2|Y_2) = o(n) \) by Fano’s inequality. Combining the last two displays, we have
\[
I(X_1; X_1 + \tilde{X}_2 + Z_2) \leq I(X_1; X_1 + Z_2) + (\alpha \sqrt{\epsilon} + o(1))n.
\]
Next we apply Corollary 11, with \( X = X_1 \rightarrow A = X_1 + Z_2 \rightarrow B = A + \tilde{X}_2 \). To verify the conditions, note that the channel \( P_{A|X} \) is memoryless and additive with non-uniform noise distribution \( P_2 \), which satisfies the condition (75). Similar, the channel \( P_{B|A} \) is memoryless and additive with noise distribution \( Q_2 \), which is the maximizer of (90). Since \( P_2 \) is not uniform, \( Q_2 \) is not a point mass. Therefore \( P_{B|A} \) satisfies (74). Then Corollary 11 yields
\[
nR_1 = H(X_1) \leq I(X_1; X_1 + \tilde{X}_2 + Z_2) + g(\alpha \sqrt{\epsilon})n \leq nC'_1 + o(n),
\]
where the last inequality follows from the fact that \( \max_{X_1} I(X_1; X_1 + \tilde{X}_2 + Z_2) = nC'_1 \) attained by \( X_1 \) uniform on \( \{0, 1, 2\}^n \).

Finally, note that the rate pair \((C'_1, C_2)\) is achievable by a random MAC-code for \((X_1, X_2) \rightarrow Y_2\), with \( X_1 \) uniform on \( \{0, 1, 2\}^n \) and \( X_2 \sim Q_2^\otimes n \). \( \square \)

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