Generalized thermostatistics based on multifractal phase space

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Abstract

We consider the self-similar phase space with reduced fractal dimension $d$ being distributed within domain $0 < d < 1$ with spectrum $f(d)$. Related thermostatistics is shown to be governed by the Tsallis’ formalism of the non-extensive statistics, where role of the non-additivity parameter plays inverted value $\hat{\tau}(q) \equiv 1/\tau(q) > 1$ of the multifractal function $\tau(q) = q d(q) - f(d(q))$, being the specific heat, $q \in (1, \infty)$ is multifractal parameter. In this way, the equipartition law is shown to take place. Optimization of the multifractal spectrum $f(d)$ derives the relation between the statistical weight and the system complexity.

Key words: Phase space; fractal dimension spectrum; deformed exponential

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1 Introduction

Generalization of the thermostatistics is known to be based on the deformation procedure of both logarithm and exponential [1,2,3]. The simplest way to introduce these functions into the thermostatistics scheme is to consider the equation of motion for dimensionless volume $\gamma = \Gamma/(2\pi\hbar)^6N$ of the supported phase space ($\hbar, N$ being Dirac-Planck constant and particle number). In the course of evolution of the ensemble with statistical weight $w = w(\gamma)$ and entropy $S$, the variation rate of the phase space volume is governed by the following equation [3]

$$\frac{d\gamma}{dt} = w(\gamma) \frac{dS}{dt}, \quad (1)$$

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From this, one follows the relation \( dS = d\gamma/w(\gamma) \) whose integration gives the entropy related to the whole statistical weight \( W \):

\[
S(W) = \int_{\gamma(1)}^{\gamma(W)} \frac{d\gamma}{w(\gamma)}.
\]  

(2)

Here, we take into account that entropy of a single state \( S(w = 1) \) vanishes.

In the case of the smooth phase space, one has trivial relation \( w(\gamma) = \gamma \) whose insertion into the relation (2) derives to the Boltzmann entropy \( S = \ln W \). However, complex systems has fractal phase space with the dimension \( D < 6N \), so that relation between the statistical weight and the corresponding volume should be replaced by the power-law dependence

\[
w(\gamma) = \gamma^{d/d}
\]

(3)

where reduced fractal dimension \( d \equiv D/6N \leq 1 \) is introduced. Insertion of Eq. (3) into integral (2) gives the expression

\[
S(W) = \ln_{\bar{d}}(dW), \quad \ln_{\bar{d}}(x) \equiv \frac{x^{d-1} - 1}{d - 1}
\]

(4)

which is reduced to the Tsallis’ logarithm \( \ln_q(x) \) where non-additivity parameter \( q \) is replaced by inverted value \( \bar{d} \equiv 1/d \geq 1 \) of the reduced fractal dimension \( d \) of the phase space. Naturally, this expression is reduced to the Boltzmann entropy in the limit \( d \to 1 \).

Adduced formalism is based on the proposition that the phase space is a monofractal determined by single dimension \( d \). The aim of this letter is to generalize Tsallis’ thermostatistics onto the multifractal phase space with a spectrum \( f(d) \). Such a generalization for arbitrary distribution \( f(d) \) is carried out in Section 2. Discussion in Section 3 shows that overall representation of the thermostatistics based on the multifractal phase space demands of the consistent consideration of input, escort and ‘physical’ distributions. An optimization of the spectrum \( f(d) \) is considered in Section 4. Section 5 concludes our treatment.

2 Multifractal phase space

According to the usual recipe [4], the passage to multifractal phase space is provided with replacement of the power function (3) by the expression \( \gamma^{qd-f(d)} \) where \( q \) is multifractal index, multiplier \( \gamma^{-f(d)} \) takes into account specific number of monofractals with dimension \( d \) within the multifractal. As
a result, the statistical weight, playing the role of the multifractal measure, takes the form

\[ w_q(\gamma) = \int_0^1 \gamma^{qd-f(d)} \rho(d) \, dd \]  

(5)

where \( \rho(d) \) is the density distribution over dimensions \( d \). Using the method of steepest descent, we arrive at the modified power law

\[ w_q(\gamma) \simeq \gamma^{\tau(q)} \]  

(6)

which generalizes the simplest relation (3) due to replacement of the bare fractal dimension \( d \) by the multifractal function \( \tau(q) = qd(q) - f(d(q)) \). The conditions of application of the steepest descent method

\[
\left. \frac{df}{dd} \right|_{d=d(q)} = q, \quad \left. \frac{d^2f}{dd^2} \right|_{d=d(q)} < 0
\]  

(7)

allow us to find special fractal dimension \( d(q) \) related to given parameter \( q \).

Above consideration shows that the passage from monofractal phase space to multifractal one is obtained by replacement of the single dimension \( d \) by the function \( \tau(q) \) that monotonically increases, taking value \( \tau = -1 \) at \( q = 0 \) and \( \tau = 0 \) at \( q = 1 \). Limit behaviour of the function \( \tau(q) \) is characterized by the asymptotics

\[ \tau \propto (q-1) \quad \text{at} \quad 0 < q - 1 \ll 1, \quad \tau \simeq 1 \quad \text{at} \quad q \to \infty. \]  

(8)

Physical domain of the \( q \) parameter variation is bounded by condition \( q > 1 \) which ensures positive values of the function \( \tau(q) < 1 \).

As a result, we can use well-known Tsallis’ formalism of the non-extensive statistics where role of the non-additivity parameter plays inverted value \( \bar{\tau}(q) \equiv 1/\tau(q) > 1 \) of the multifractal function \( \tau(q) = qd(q) - f(d(q)) \). Thus, the entropy in dependence of the probability distribution \( P_i \) has the form [1]

\[ S_q = -\sum_{i=1}^{W_q} P_i \ln_{\bar{\tau}(q)}(P_i), \quad \ln_{\bar{\tau}}(x) \equiv \frac{x^{\bar{\tau}-1} - 1}{\bar{\tau}-1}. \]  

(9)

With accounting the conditions

\[
\sum_{i=1}^{W_q} P_i = 1, \quad E_q = \sum_{i=1}^{W_q} P_i \bar{\tau}(q),
\]  

(10)

this expression arrives at the generalized distribution over energy levels \( \varepsilon_i \):

\[ P_i = Z_q^{-1} \exp_{\bar{\tau}(q)}(-\beta \varepsilon_i) \]  

(11)
where $Z_q$ is the partition function and the deformed exponential is determined in the usual manner:

$$
\exp_q(x) \equiv \begin{cases} 
[1 + (\bar{\tau} - 1)x]^{\frac{1}{\bar{\tau} - 1}} & \text{at } 1 + (\bar{\tau} - 1)x > 0, \\
0 & \text{otherwise.}
\end{cases}
$$

(12)

The normalization condition (10) gives the relation

$$
\sum_{i=1}^{W_q} P_i^{\tau(q)} = Z_q^{-[\bar{\tau}(q)-1]} - [\bar{\tau}(q) - 1] \beta E_q.
$$

(13)

In spite of the formal similarity of above expressions with the same of the Tsallis’ statistics, a principle difference takes place: in the last case, the limit of extensive systems is reached, when non-additivity parameter takes value $q = 1$, whereas the multifractal phase space is reduced to monofractal one with dimension $\bar{\tau}(q) \to 1$ in the opposite limit $q \to \infty$. In this way, the step-function spectrum $\tau(q)$, being $\tau = 1$ at $q > 1$ and $\tau = 0$ otherwise, relates to the smooth phase space.

Thermodynamic functions of the model under consideration can be found analogously to the Tsallis’ non-extensive scheme [1]. However, related expressions are very cumbersome even in the simplest case of the ideal gas [5,6,7] and amount to the usual form only within the slightly non-extensive limit [8]. At the same time, developed scheme allows one to use thermodynamic formalism of multifractal objects [9]. Within the latter, the role of a state parameter plays the multifractal index $q$, whose variation may arrive at phase transitions if the dependence $\tau(q)$ has some singularities. It is worthwhile to stress that developed scheme arrives directly at related singularities of thermodynamic functions type of the internal energy (27) (see below), the entropy (cf. Eq.(4))

$$
S_q = \bar{\tau}(q) \ln_{\bar{\tau}(q)} (W_q), \quad \bar{\tau}(q) \equiv 1/\tau(q)
$$

(14)

and the free energy

$$
F_q = E_q - T S_q.
$$

(15)

3 Discussion

In accordance with the non-extensive thermostatistics [1,5] the distribution (11) plays the role of the escort probability related to the input one

$$
p_i \equiv P_i^{\tau(q)} = Z_q^{-\bar{\tau}(q)} \left[ \exp_{\tau(q)} (-\beta \varepsilon_i) \right]^{\bar{\tau}(q)},
$$

(16)
being normalized by the condition
\[ \sum_{i=1}^{W_q} p_i^{\tau(q)} = 1. \]

(17)

Easily to convince, the expression (16) can be reduced to pseudo-Gibbs form
\[ p_i = \exp_\tau [\beta_q (F_q - \varepsilon_i)] \]

(18)

if one introduces effective value of the inverted temperature
\[ \beta_q \equiv \bar{\tau}(q)Z_q^{[\bar{\tau}(q)-1]} \]

(19)

and determines the free energy \( F_q \) in the usual manner:
\[ F_q \equiv -T \ln_{\bar{\tau}(q)}(Z_q). \]

(20)

It is characteristically, the escort distribution (11), determined with the physical temperature \( T \equiv \beta^{-1} \), is normalized by the standard condition (10), whereas the input probability (16), being deformed exponential (18), is determined with non-physical temperature (19) to derive to the deformed free energy (20). Thus, both input and escort distributions appear as complementary ones. In this way, the escort distribution (11) is deformed with the inverted index \( \bar{\tau}(q) \), whereas the input probability (18) – with the index \( \tau(q) \) itself. This exhibits the known \( \tau(q) \leftrightarrow \bar{\tau}(q) \)-duality of Tsallis’ thermostatistics [2].

According to [5] the physical distribution is neither escort nor input probabilities, but the following one
\[ P(\varepsilon_i) \equiv \frac{P^{\bar{\tau}(q)}(\varepsilon_i)}{\sum_{i=1}^{W_q} P^{\bar{\tau}(q)}(\varepsilon_i)}. \]

(21)

It corresponds to the condition
\[ \sum_{i=1}^{W_q} (\varepsilon_i - E_q)P^{\bar{\tau}(q)}(\varepsilon_i) = 0 \]

(22)

instead of the second equation (10). Easily to show, the probabilities (16), (21) are connected by the relation
\[ P(\varepsilon_i) = Z_q^{[\bar{\tau}(q)-1]}p(\varepsilon_i - E_q) \]

(23)

to derive the distribution
\[ P(\varepsilon_i) = Z_q^{-1} \exp_{\bar{\tau}(q)} \left[-\bar{\tau}(q)\beta (\varepsilon_i - E_q)\right]. \]

(24)

In the case of continuous energy spectrum characterized with the density distribution \( \rho(\varepsilon) \), the internal energy related to the condition (22) takes the form
\[ E_q = \int_{-\infty}^{\infty} \varepsilon \mathcal{P}(\varepsilon) \rho(\varepsilon) d\varepsilon. \]  

(25)

Extreme value of \( E_q \) is reached at the condition

\[ \frac{\rho'(\varepsilon)}{\rho(\varepsilon)} \simeq -\frac{\mathcal{P}'(\varepsilon)}{\mathcal{P}(\varepsilon)} \]  

(26)

where prime denotes differentiation over \( \varepsilon \). Usually, the density function is reduced to the power law \( \rho(\varepsilon) \sim \varepsilon^cN \), \( c \sim 1 \), so that \( \rho'(\varepsilon)/\rho(\varepsilon) \simeq cN/\varepsilon \). Then, with using the distribution (24), the condition (26) taken at \( \varepsilon = E_q \) arrives at the equipartition law

\[ E_q = c\tau(q)NT \]  

(27)

according to which the value \( c\tau(q) \) is the specific heat.

Let us stress that above scheme is related to generalized definition of the deformed logarithm [2]

\[ \ln_\phi(x) \equiv \int_1^x \frac{dy}{\phi(y)} \]  

(28)

where the function \( \phi(y) \) is reduced to the statistical weight distribution \( w_q = p_i^{\tau(q)} \) type of Eq.(6). However, this function is obeyed the condition \( p_i' = -\beta_q w_q(p_i) \) which includes deformed temperature \( \beta_q^{-1} \), being inconvenient at physical considerations.

4 Optimization of multifractal spectrum

Up to now, we suppose that the multifractal spectrum \( f(d) \) is arbitrary. If it is optimized at normalization condition

\[ \int_0^1 f(d) dd = 1, \]  

(29)

one has to minimize the expression

\[ \tilde{S}_q\{f(d)\} = \int_{\gamma(1)}^{\gamma(W_q)} \left[ \int_0^1 \gamma^{\eta d - f(d)} \rho(d) dd \right]^{-1} d\gamma - \frac{\Sigma^2}{2} \left[ \int_0^1 f(d) dd - 1 \right], \]  

(30)
written with accounting Eqs.(2), (5), where $\Sigma$ is Lagrange multiplier. As a result, we arrive at the equality

$$ \gamma^{(W_q)} \int_\gamma^{(1)} \gamma^{([\nu_d-f(d)]-2\tau(q))} \ln \gamma \, d\gamma = \frac{\Sigma^2}{2} $$

(31)

whose integration gives, with accounting Eq.(6), the transcendental equation

$$ \frac{1}{2} [\Sigma \tau(q) T_q(d)]^2 - W_q^{T_q(d)} [T_q(d) \ln(W_q) - 1] - 1 = 0, $$

(32)

$$ T_q(d) \equiv \tilde{\tau}(q) \{1 - 2\tau(q) + [qd - f(d)]\}. $$

(33)

This equation is written in the form, when can be used either given spectrum function $f(d)$ or the index dependence $\tau(q)$. In the latter case, we find initially the dependence $q(d)$ from the equation

$$ \frac{d\tau}{dq} \bigg|_{q=q(d)} = d, $$

(34)

being conjugated to Eq.(7). Then, substituting this dependence into Eq.(33), we arrive at the trivial expression

$$ \mathcal{T}(q) \equiv T_q(d(q)) = \tilde{\tau}(q) - 1 $$

(35)

whose using in Eq.(32) allows us to determine the dependence of the statistical weight $W_q$ on the complexity $\Sigma$ at given function $\tau(q)$.

With passage to the smooth phase space, when $q \to \infty$, $d \to 1$, $\mathcal{T}(q) \to 0$, one obtains the statistical weight

$$ W_\infty = e^{\Sigma_\infty}, \quad \Sigma_\infty \equiv \sigma_\infty N $$

(36)

which is determined by the specific complexity $\sigma_\infty$ per one particle. At small deviation off the minimum complexity $(\Sigma - \Sigma_\infty \ll \Sigma_\infty)$ and light multifractality $(1 - \tau(q) \ll \Sigma_\infty^{-1})$, linearized equation (32) gives

$$ W_q \simeq W_{\tau(q)} \left\{1 + \tau(q) \left[1 - \frac{2}{3} (1 - \tau(q)) \Sigma_\infty \right] (\Sigma - \Sigma_\infty) \right\}, $$

$$ W_{\tau(q)} \equiv \exp \left\{\tau(q) \Sigma_\infty \left[1 - \frac{1}{3} (1 - \tau(q)) \Sigma_\infty \right] \right\}, \quad 1 \ll q < \infty. $$

(37)

In the opposite case $\tau(q) \ll 1$, one has with logarithmic accuracy

$$ W_q \simeq \left[\alpha \tau(q) \Sigma^2\right]^{\tau(q)}, \quad \alpha \sim 1, \quad q - 1 \ll 1. $$

(38)
5 Conclusion

As shows above consideration, the thermostatistics of complex systems with phase space, whose reduced fractal dimension $d$ is distributed with spectrum $f(d)$, is governed by the Tsallis’ formalism of the non-extensive statistics. In this way, the role of non-additivity parameter plays inverted value of the multifractal function $\tau(q) = q d(q) - f(d(q))$ which monotonically increases, taking value $\tau = 0$ at $q = 1$ and $\tau \simeq 1$ at $q \to \infty$ (the latter limit is related to the smooth phase space). The multifractal function $\tau(q)$ is reduced to the specific heat to determine, together with the inverted value $\bar{\tau}(q) \equiv 1/\tau(q) > 1$, both statistical distributions and thermodynamic functions of the system under consideration. At given function $\tau(q)$, optimization of the normalized multifractal spectrum $f(d)$ arrives at the dependence of the statistical weight on the system complexity.

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