COMPLETE STABLE MINIMAL HYPERSURFACES IN POSITIVELY CURVED 4-MANIFOLDS

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Abstract. We show that the combination of non-negative sectional curvature (or 2-intermediate Ricci curvature) and strict positivity of scalar curvature forces rigidity of complete (non-compact) two-sided stable minimal hypersurfaces in a 4-manifold with bounded curvature.

Our work leads to new comparison results. We also construct various examples showing rigidity of stable minimal hypersurfaces can fail under other curvature conditions.

1. Introduction

Recall that a two-sided stable minimal hypersurface $M^{n-1} \to (X^n, g)$ is an immersed hypersurface with vanishing mean curvature and trivial normal bundle satisfying the inequality

$$\int_M (|A_M|^2 + \text{Ric}_g(\nu, \nu)) \varphi^2 \leq \int_M |\nabla \varphi|^2$$

for any $\varphi \in C^\infty_c(M \setminus \partial M)$. Here, $A_M$ is the second fundamental form of the immersion, $\text{Ric}_g$ is the ambient Ricci curvature, and $\nu$ is (any) choice of unit normal. Stable minimal hypersurfaces can be used in a similar manner to stable geodesics to probe the ambient geometry of $(X^n, g)$. The basic results along these lines are as follows. When $M^{n-1} \to (X^n, g)$ is a closed (compact without boundary) two-sided stable minimal hypersurface, it holds that

1. If $\text{Ric}_g \geq 0$ then $M$ is totally geodesic and $\text{Ric}_g(\nu, \nu) \equiv 0$ along $M$ \cite{Sim68} (cf. \cite{BT73}). In particular, when $\text{Ric} > 0$, there are no closed two-sided stable minimal hypersurfaces.
2. If $R_g \geq 1$ (scalar curvature) then $M$ also admits a metric of positive scalar curvature \cite{SY79a, SY79b}. In particular, when $n = 3$ and $X$ is oriented, each component of $M$ must be a 2-sphere.

The second result is a fundamental tool in the study of manifolds of positive scalar curvature.

When $M$ is now assumed to be complete with respect to the induced metric (in particular, $M$ has no boundary) and non-compact, the theory becomes considerably more complicated. As the following results show, it has been well-developed in 3-dimensions. Consider $M^2 \to (X^3, g)$ complete two-sided stable minimal immersion:

1. When $R_g \geq 0$, the induced metric on $M$ is conformal to either the plane or the cylinder \cite{FCSS0}. In the latter case, $M$ is totally geodesic, intrinsically flat, and $R_g \equiv 0, \text{Ric}_g(\nu, \nu) \equiv 0$ along $M$ (cf. \cite{CCE16}, Proposition C.1).
(2) When $\text{Ric}_g \geq 0$, $M$ is totally geodesic, intrinsically flat, and $\text{Ric}_g(\nu,\nu) \equiv 0$ along $M$ [SY82] (cf. [FCS80, dCP79, Pog81]).

(3) When $R_g \geq 1$, $M$ must be compact [SY83] (cf. [GL83]).

These results have had important applications to comparison geometry. In particular, we refer to the following (incomplete) list of results that rely specifically on (non-compact) stable minimal surfaces in 3-manifolds: [SY82, GL83, AR89a, Liu13, Wan19]. For most of these applications, it is essential that no assumption is made related to properness or volume growth of $M$.

The first- and second- named authors have recently resolved [CL21] the conjecture of Schoen that a complete two-sided stable minimal immersion $M^3 \rightarrow \mathbb{R}^4$ is flat (without any additional assumptions on $M$); see also the subsequent proofs in [CL23, CMR22]. It is natural to ask what happens in a curved ambient background, rather than flat Euclidean space. For example, a basic question is to determine which ambient curvature conditions suffice to show that a complete stable minimal hypersurface must be compact, or cannot exist. The methods used in [CL21] do not seem to extend to the curved setting.

1 (See [SSY75, SY96, CSZ97, LW04] for previous results in this direction.)

In fact, (as was discussed above) in a 3-manifold, $R_g \geq 1$ implies compactness and $\text{Ric}_g > 0$ implies non-existence; on the other hand, in 4-dimensions, neither of these results can hold:

**Example 1.1** (Non-compact stable minimal hypersurface in 4-manifold with $R_g \geq 1$).

Fix an oriented closed 2-dimensional Riemannian manifold $(X^2, g)$ admitting

$$\sigma : \mathbb{R} \rightarrow (X^2, g)$$

unit speed stable geodesic. Then, for $\varepsilon > 0$ sufficiently small $(X^2, g) \times S^2(\varepsilon)$ has $R_g \geq 1$, while $\sigma \times S^2(\varepsilon)$ is an unbounded two-sided stable minimal immersion.

**Example 1.2** (Stable minimal hypersurface in a 4-manifold with positive sectional curvature).

Consider a rotationally symmetric metric $g$ on $\mathbb{R}^4$ with strictly positive sectional curvatures. There are totally geodesic copies of $\mathbb{R}^3$ in such a metric, and as long as the Ricci curvature of $g$ decays to zero sufficiently fast, these will be (complete) two-sided stable minimal embeddings. Essentially, the point is that on flat Euclidean 3-space $(\mathbb{R}^3, \bar{g})$, any Schrödinger operator with non-negative, rapidly decaying potential will be stable, thanks to a classical Hardy inequality

$$\int_{\mathbb{R}^3} 4|\nabla \varphi|^2 d\bar{\mu} \geq \int_{\mathbb{R}^3} r^{-2} \varphi^2 d\bar{\mu}.$$ 

We explicitly construct such an example and check its stability in Section B.1.

1.1. **Main results.** It turns out that if one combines positivity of curvature with strict positivity of scalar curvature, such examples can be avoided. This is the main result of this article.

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1 In particular, the use of the $L^3$ Schoen–Simon–Yau inequality [SSY75] seems troublesome in most curved ambient manifolds.

2 Note that a Schrödinger operator with non-negative (but not identically zero) potential cannot be stable on $\mathbb{R}^2$, thanks to the log-cutoff trick.
Theorem 1.3. If \((X^4, g)\) is a complete 4-manifold with weakly bounded geometry, non-negative sectional curvature, and strictly positive scalar curvature \(R_g \geq R_0 > 0\), then any complete two-sided stable minimal immersion \(M^3 \to (X^4, g)\) is totally geodesic and has \(\text{Ric}_g(\nu, \nu) \equiv 0\) along \(M\).

Here, weakly bounded geometry means that around any point, there is a local diffeomorphism from a ball in \(\mathbb{R}^4\) of definite size so that the pullback metric is \(C^{1,\alpha}\)-close to Euclidean (see Definition 2.3). (Note that this allows for collapsing behavior at infinity.)

Remark 1.4. All results in this paper assume that \((X, g)\) at least has \(\text{Ric}_g \geq 0\), in which case “weakly bounded geometry” could be replaced by the assumption “\(\text{sec}_g \leq K\)” anywhere it occurs here. It would be interesting to understand if this condition could be removed.

Remark 1.5. Shen–Ye [SY96] proved that for \(n \leq 4\), if \((X^{n+1}, g)\) satisfies

\[
\text{biRic} \geq k > 0,
\]

then any complete two-sided stable minimal immersion \(M^n \to (X^{n+1}, g)\) is compact. Here biRic is called the bi-Ricci curvature, defined in [SY96] (biRic \(\geq k\) implies that \(R_g \geq c_n k\) for some dimensional constant \(c_n\)). (See Appendix A for precise definitions of these curvature conditions as well as Section 1.2 below.)

In particular, it follows that such a minimal immersion does not exist if \((X^{n+1}, g)\) is assumed to be closed with positive sectional curvature. After our paper appeared, the Shen–Ye result was rediscovered and extended to \(n = 5\) by Catino–Mastrolia–Roncoroni [CMR22], under the additional assumption of nonnegative sectional curvature (or even just nonnegative 4-intermediate Ricci curvature). The approach in [SY96, CMR22] is very different from that of this paper and is based on a conformal deformation technique first introduced by Fischer-Colbrie [FC85]. An interesting feature of our paper (see Theorem 1.10) is that we are able to replace (when \(n = 3\)) the condition biRic \(\geq k > 0\) by the weaker assumption on scalar curvature \(R_g \geq R_0 > 0\) in (1.1). We note that the condition considered in this paper allows for the existence of complete, non-compact stable minimal immersions (we prove they must be totally geodesic and have vanishing normal Ricci curvature). For example, one may consider \(S^2 \times \mathbb{R} \to S^2 \times \mathbb{R} \times S^1\).

Of course, any closed manifold \((X, g)\) has weakly bounded geometry, so we have the following result:

Corollary 1.6. If \((X^4, g)\) is a closed 4-manifold with non-negative sectional curvature and positive scalar curvature, then a complete two-sided stable minimal immersion \(M^3 \to (X^4, g)\) is totally geodesic and \(\text{Ric}_g(\nu, \nu) \equiv 0\) along \(M\).

This applies (for example) to the standard product metrics on \(S^4, S^1 \times S^3, S^2 \times S^2, \) and \(T^2 \times S^2\). We emphasize that the latter example does not have positive bi-Ricci curvature, and is thus not covered by the results in [SY96, CMR22].

Remark 1.7. The study of complete (non-compact) stable minimal immersions in a compact 3-manifold has played an important role in the study of embedded minimal surfaces with bounded Morse index [CKM17, Car17, LZ16]. One may hope that the Corollary 1.6 could lead to similar results in closed 4-manifolds.
Remark 1.8. Theorem 1.3 fails in 6-dimensional manifolds (and higher). See Appendix B.3 for a counterexample. Gromov’s conjecture that a 4-manifold with uniformly positive scalar curvature has macroscopic dimension 2 suggests that the 5-dimensional version of Theorem 1.3 would be the “critical” dimension (this is related to the log-cutoff trick for minimal surfaces in 3-manifolds).

1.2. Concerning the assumption of non-negative sectional curvature. It is natural to ask whether or not there exist complete two-sided stable minimal immersions $M^3 \rightarrow (X^4, g)$ where $(X^4, g)$ has weakly bounded geometry, $\text{Ric}_g \geq 0$ and $R_g \geq 1$. (Note that in Theorem 1.3 $\text{Ric}_g \geq 0$ is replaced by non-negative sectional curvature.)

To this end, we have the following example indicating that this is unlikely to be true.

Example 1.9 (Stable minimal hypersurface in a 4-manifold with strictly positive Ricci curvature). There exists a closed 4-manifold $(X^4, g)$ (so it automatically has bounded geometry) and a complete two-sided stable minimal immersion $M^3 \rightarrow (X^4, g)$ so that $\text{Ric}_g \geq 1$ (so, in particular $R_g \geq 1$) in some $\delta$-neighborhood of the image of $M$.

This example is constructed in detail in Section B.2 but we briefly describe it here. The minimal immersion $M^3 \rightarrow (X^4, g)$ is constructed by taking the universal cover of some embedded compact minimal submanifold $M_0 \subset (X^4, g)$. We choose $M_0$ so that it is diffeomorphic to $(S^3 \times S^2)\#(S^1 \times S^2)$, so $\pi_1(M_0) = F_2$ is a “non-amenable” group. This implies that the universal cover has positive first eigenvalue. As such, as long as $|A_M|^2 + \text{Ric}_g(\nu, \nu)$ is sufficiently small, the universal cover will be stable.

We note that it is unclear if one can construct such an example where $(X^4, g)$ is complete and satisfies $\text{Ric}_g > 0$ and $R_g \geq 1$ at all points (not just near the image of $M$). (We discuss the possibility of doing so further in Section B.2.) However, we emphasize that this example precludes any argument concerning stable minimal immersions in such $(X^4, g)$ that is based purely on the second variation of area.

On the other hand, it is possible to weaken non-negativity of sectional curvature to an intermediate condition. Namely, we say that $(X^4, g)$ has non-negative 2-intermediate Ricci curvature if the sectional curvatures satisfy $\sec_g(\Pi_1) + \sec_g(\Pi_2) \geq 0$ for any two 2-planes $\Pi_1, \Pi_2 \subset T_pX$ intersecting in a line at a right angle (cf. Definition 2.1). We will write $\text{Ric}_g^2 \geq 0$ for this condition. (Note that non-negative sectional curvature implies $\text{Ric}_g^2 \geq 0$ which implies $\text{Ric}_g \geq 0$.) The results discussed above all hold with non-negative sectional curvature replaced by $\text{Ric}_g^2 \geq 0$. More precisely, we have

**Theorem 1.10.** If $(X^4, g)$ is a complete 4-manifold with weakly bounded geometry, $\text{Ric}_g^2 \geq 0$, and $R_g \geq R_0 > 0$, then any complete two-sided stable minimal immersion $M^3 \rightarrow (X^4, g)$ is totally geodesic and has $\text{Ric}_g(\nu, \nu) \equiv 0$ along $M$.

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3Note that $M_0$ is necessarily unstable, since $\text{Ric}_g(\nu, \nu) > 0$; the instability occurs due to the “largeness” of the universal cover.

4Note that Li–Wang have used the Busemann function in their analysis of proper ends of stable minimal hypersurfaces in non-negatively curved manifolds [LW01]. As such, this argument relies on the ambient geometry outside of just a tubular neighborhood of the immersion. However, it is unclear how to extend such a argument to non-negative Ricci curvature (the Busemann function is subharmonic rather than convex so it is difficult to control the restriction to a minimal hypersurface). Along these lines, we note that the stable minimal immersion $M^3 \rightarrow (X^4, g)$ discussed in Example 1.3 has infinitely many ends.
We discuss the topological implications of this result in Section 6.

We remark that $\text{Ric}_g \geq 0$ (instead of $\text{Ric}_g^2 \geq 0$) suffices for most (but not all) steps of the proof of Theorem 1.10, so if one places additional topological assumptions on the minimal hypersurface, then the proof of Theorem 1.10 can be adapted in a straightforward way to show the following result.

**Theorem 1.11.** Let $(X^4, g)$ be a complete 4-manifold with weakly bounded geometry, $\text{Ric}_g^2 \geq -K$, $\text{Ric}_g \geq 0$, and $R_g \geq R_0 > 0$. Let $M^3 \to (X^4, g)$ be a complete two-sided stable minimal immersion with finitely many ends and $\text{b}_1(M) < \infty$. Then $M$ is totally geodesic and has $\text{Ric}_g(\nu, \nu) \equiv 0$ along $M$.

### 1.3. Topological applications.

Recall that Schoen–Yau used minimal surfaces to prove that a complete non-compact $(X^3, g)$ with $\text{Ric}_g > 0$ is diffeomorphic to $\mathbb{R}^3$ [SY82] (cf. [AR89b, Liu13]). They did this by showing that $\pi_2(X) = 0$ and that $X$ is simply connected at infinity. In this section, we observe that given Theorem 1.10, similar techniques can be used to prove new topological restrictions on $(X^4, g)$ with $\text{Ric}_g^2 > 0$, $R_g \geq R_0 > 0$, and weakly bounded geometry. (See [Mon] for a collection of results concerning the $\text{Ric}_g^2$ curvature condition.)

All homology groups and cohomology groups here will be taken with $\mathbb{Z}$-coefficients.

**Definition 1.12.** Let $X$ be a non-compact manifold, $k \in \mathbb{N}$. Define the $k$-th homology group of $X$ at infinity as the inverse limit

$$H_k^\infty(X) = \lim_{A \to X} H_k(X - A).$$

We say $X$ is $H_k$-trivial at infinity if the natural homeomorphism $H_k^\infty(X) \to H_k(X)$ is injective.

In section 6 we prove the following result.

**Theorem 1.13.** Suppose $(X^4, g)$ is a complete, non-compact, oriented 4-manifold with weakly bounded geometry, $\text{Ric}_g^2 > 0$, and strictly positive scalar curvature $R_g \geq R_0 > 0$. Then we have:

1. $H_1(X)$ only has torsion elements;
2. $X$ is $H_2$-trivial at infinity.

It would be interesting to see if the techniques from [AR89b, Liu13] (cf. [CCE16, CEM19]) could be adapted to study the $\text{Ric}_g^2 \geq 0$ rigidity version of this result.

**Remark 1.14.** If $(X^n, g)$ has $\text{Ric}_g > 0$, then $H_{n-1} = 0$ by [Yau76, SS01].

In light of Examples 1.1 and 1.2, it would be interesting to know if Theorem 1.13 held under weaker curvature conditions (e.g., $\text{Ric}_g^2 > 0$ but without $R_g \geq R_0 > 0$).

### 1.4. Idea of the proof and related results.

An interesting feature of Theorem 1.13 is that it combines different curvature conditions. Indeed, the different conditions come in at rather different places in the proof. As a consequence, the strategy of the proof is quite different from most previous results concerning stability of complete (non-compact) minimal surfaces (including the previous work of the first- and second-named authors on stable minimal immersions $M^3 \to \mathbb{R}^4$ [CL21]). A key feature of this paper...
is the use of $\mu$-bubbles, a technique introduced by Gromov [Gro96], allowing one to gain distance control (in certain settings) from positivity of scalar curvature.

The overall strategy is as follows, indicating which curvature conditions enter at which step. Consider $M^3 \to (X^4, g)$ a complete two-sided stable minimal hypersurface.

1. To show that $M$ is totally geodesic and has vanishing normal Ricci curvature, we would like to find a sequence of compactly supported functions $\varphi_i$ with $\varphi_i \to 1$ and $\int_M |\nabla \varphi_i|^2 = o(1)$. Taking such functions in the stability inequality then gives the desired conclusion (this just uses $\text{Ric}_g \geq 0$). Note that for minimal surfaces in a 3-manifold, this condition is usually guaranteed by proving that the surface is conformal to the plane and then using the log-cutoff trick. However, in the higher dimensions, this is not usually not possible (e.g., no such functions exist on flat $\mathbb{R}^3$).

2. The non-negativity of sectional curvature (or $\text{Ric}_g^g \geq 0$) ensures that $M$ has at most one nonparabolic end (see Definition 3.3). Nonparabolic ends are more difficult to handle, since parabolic ends automatically admit test good functions as described in (1).

3. The remaining issue is to construct a good test function along the nonparabolic end. In general, this is not possible with only a non-negative sectional curvature assumption (cf. Example 1.2). This is where the positivity of scalar curvature comes in. One knows (going back to the work of Schoen–Yau [SY79b]) that a stable minimal hypersurface in a positive scalar curvature manifold “acts” as if it itself has positive scalar curvature.

In particular, a 3-dimensional manifold with positive scalar curvature tends to be macroscopically 1-dimensional in various senses (cf. [GL83, MN12, Son18, LM20]). As such, one might imagine that a single end of $M$ has linear volume growth (this is where the parabolic/nonparabolic distinction from (2) is important; a metric tree is 1-dimensional but has exponential length growth, compare with the universal cover of $(S^1 \times S^1) \# (S^1 \times S^2)$). Note that if this end had linear volume growth, then the standard cutoff function that is 1 in $B_R$ and 0 in $B_{2R}$ would have vanishing Dirichlet energy as $R \to \infty$, so this would allow us to construct the desired cutoff function.

It seems difficult (or potentially impossible) to actually prove that the nonparabolic end $E$ has linear volume growth. Instead we construct an exhaustion of $E$ by bounded sets $\Omega_1 \subset \Omega_2 \subset \ldots$ so that $\partial \Omega_i \cap E$ has controlled diameter. To do so, we use the theory of $\mu$-bubbles (surfaces of carefully prescribed mean curvature) first introduced by Gromov [Gro96] (see also [Gro18]), allowing one to study metric quantities related to scalar curvature. (This application is similar in spirit to the use of $\mu$-bubbles in [CL20, Gro20, CLL21].) This is the step that uses the assumption of strictly positive scalar curvature.

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5The reality is somewhat more complicated, since the nonparabolic component could have infinitely many parabolic ends attached, but this captures the main idea.

6One can imagine that $E$ is a cylinder $S^2 \times (0, \infty)$ connected sum at integer points with spheres $S^3$ with volume tending to $\infty$ and scalar curvature $\geq 1$. It is unclear if this could be ruled out by stability.
Finally, to construct a good test function, it remains to show that the $L$-neighborhood of $\partial \Omega_i \cap E$ has bounded volume (for some constant $L$). To upgrade diameter control (obtained in (3)) to volume control, we need to control (from below) the intrinsic Ricci curvature of the minimal immersion $M$. This is achieved (thanks to the Gauss equations) by combining a lower bound on the sectional (or $\text{Ric}_g^2$) curvatures with an absolute bound on the second fundamental form of $M$ as follows from [CL21]. This is where the weakly bounded geometry assumption enters.

We remark that steps (3) and (4) are somewhat reminiscent of a recent result (but not the proof) of Munteanu–Wang proving that a complete 3-manifold with $\text{R} \geq 1$ and $\text{Ric}_g \geq 0$ has linear volume growth [MW22] (see also [Xu20, MW21, Zhu22]). See our recent work [CLS22] for an approach to this result following the methods of this paper.

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1.6. Organization of the paper. In Section 2 we discuss the $\text{Ric}_2 \geq 0$ and bounded geometry conditions used here. Section 3 contains some preliminary results concerning the ends of non-compact manifolds. We then study the nonparabolic ends in further detail in Section 4. We discuss the $\mu$-bubble exhaustion result and corresponding volume growth estimates and then prove the main results in Section 5. We discuss some topological applications of the main results in Section 6. Appendix A contains various curvature conditions referred to in the paper. Appendix B contains some examples relevant to the main results. Finally, Appendix C describes how to pullback immersions along local diffeomorphisms. Finally, Appendix D relates sectional curvature bounds to the weakly bounded geometry condition.

2. Curvature Preliminaries

If $(X, g)$ is a Riemannian manifold, we denote the Riemann curvature tensor by $R_g(\cdot, \cdot, \cdot, \cdot)$ with 4 arguments, the sectional curvature by $\text{sec}_g$, the Ricci curvature by $\text{Ric}_g$, and the scalar curvature by $R_g$ with no argument. If $M \to (X, g)$ is an immersion, we write $\text{Ric}_M$ and $R_M$ to denote respectively the Ricci curvature and scalar curvature of the pullback metric on $M$. We follow the curvature conventions used in [Pet16, Chapter 3]; in particular, if $\{u, v\}$ is an orthonormal basis for a 2-plane $\Pi \subset T_p X$, then $\text{sec}_g(\Pi) = R_g(v, u, u, v)$.

2.1. The $\text{Ric}_2 \geq 0$ Condition. Let $(X^4, g)$ be a complete Riemannian 4-manifold. The following curvature condition is tailored to exploit the Gauss equation for hypersurfaces in 4-manifolds, and sits between non-negative sectional curvature and non-negative Ricci curvature.

**Definition 2.1.** $(X^4, g)$ satisfies $\text{Ric}_2^g \geq 0$ if $R_g(v, u, u, v) + R_g(w, u, u, w) \geq 0$
for any orthonormal vectors \{u, v, w\} in \(TX\).

This condition has been called non-negativity of 2-intermediate Ricci curvature in the literature (cf. Appendix A). In particular, we note that there is a metric on \(S^2 \times S^2\) with strictly positive 2-intermediate Ricci curvature \[\text{Example 2.3}\) (see \[Mon21\]).

**Lemma 2.2.** Let \((X^4, g)\) be a 4-manifold with \(\text{Ric}^2 \geq 0\). If \(M^3 \to (X^4, g)\) is a minimal immersion, then

\[
\text{Ric}_M \geq -|A_M|^2.
\]

**Proof.** Pick an orthonormal basis \(\{e_i\}\) at a point \(p \in M\) with \(e_4\) normal to \(M\). The traced Gauss equation gives

\[
R_g(e_2, e_1, e_1, e_2) + R_g(e_3, e_1, e_1, e_3) - \text{Ric}_M(e_1, e_1) = \sum_{j=1}^3 A_M(e_1, e_j)^2.
\]

Rearranging and using \(\text{Ric}^2 \geq 0\) we have

\[
\text{Ric}_M(e_1, e_1) \geq R_g(e_2, e_1, e_1, e_2) + R_g(e_3, e_1, e_1, e_3) - |A_M|^2 \geq -|A_M|^2.
\]

Since \(e_1\) was an arbitrary unit vector in \(T_pM\), the conclusion follows. \(\square\)

### 2.2. Weakly Bounded Geometry.

**Definition 2.3.** In this paper, we say that a complete Riemannian manifold \((X^n, g)\) has \(Q\)-weakly bounded geometry if for any \(p \in X\), there is a \(C^{2,\alpha}\) local diffeomorphism \(\Phi : (B(0, Q^{-1}), 0) \subset \mathbb{R}^n \to (U, p) \subset X\) so that

1. we have \(e^{-2Q\delta} \leq \Phi^* g \leq e^{2Q\delta}\) in the sense of bilinear forms, and
2. it holds that \(\|\partial_k \Phi^* g_{ij}\|_{C^{\alpha}} \leq Q\).

We will often say that \((X, g)\) has “weakly bounded geometry” if the above definition holds for some \(Q\).

Note that this condition allows for some collapsing behavior of \((X, g)\) at infinity\(^7\). It is well-known that \(|\text{sec}_g| \leq K < \infty\) implies \(Q\)-weakly bounded geometry for \(Q = Q(K)\) (cf. \[RST10\]). We outline the proof of this fact in Proposition D.1. Note also that \(\text{Ric}_g \geq 0\) and \(\text{sec}_g \leq K\) imply that \(|\text{sec}_g| \leq (n - 2)K\), and in this paper the ambient space \((X^4, g)\) is always assumed to have \(\text{Ric}_g \geq 0\) (often a stronger condition is assumed).

Alternatively, we recall that \(|\text{Ric}_g| \leq K\) and \(\text{inj}(X, g) \geq i_0 > 0\) also implies this condition (and actually one can replace “local diffeomorphism” by “diffeomorphism” in this case); this follows from \[And90\] (cf. \[Pet16\] Theorem 11.4.3)).

The first- and second-named authors have recently proven that a two-sided complete stable minimal hypersurface \(M^3 \to \mathbb{R}^4\) is flat \[CL21\]. Using a standard blow-up argument, this yields curvature estimates for stable minimal hypersurfaces in (compact) 4-manifolds. Here, we observe that the same thing holds for non-compact 4-manifolds, assuming the weakly bounded geometry condition. The proof below is similar to the

\(^7\)For example, a hyperbolic cusp has weakly bounded geometry but the injectivity radius tends to zero at infinity.
argument used in [RST10] (see also [Coo10]) to prove curvature estimates for stable minimal (or more generally CMC) surfaces in 3-manifolds (cf. [Sch83]).

Lemma 2.4. Let \((X^4, g)\) be a complete 4-manifold with \(Q\)-weakly bounded geometry. Then there is a constant \(C = C(Q) < \infty\) such that every compact two-sided stable minimal immersion \(M^3 \to (X^4, g)\) satisfies
\[
\sup_{q \in M} |A_M(q)| \min\{1, d_M(q, \partial M)\} \leq C.
\]

Proof. This follows by combining a standard point picking argument (cf. [CL21, Theorem 3]) with the lifting argument described in Appendix C. For the sake of completeness we describe the full argument below.

For contradiction, we assume that the assertion fails. Then there is a sequence of compact two-sided stable minimal immersions \(M^3_i \to (X^4_i, g_i)\) such that
\[
\sup_{q \in M_i} |A_{M_i}(q)| \min\{1, d_{M_i}(q, \partial M_i)\} \to \infty
\]
as \(i \to \infty\). Since \(M_i\) is compact and the argument of the supremum is continuous and vanishes on \(\partial M_i\), there is \(p_i \in M_i \setminus \partial M_i\) with
\[
|A_{M_i}(p_i)| \min\{1, d_{M_i}(p_i, \partial M_i)\} = \sup_{q \in M_i} |A_{M_i}(q)| \min\{1, d_{M_i}(q, \partial M_i)\} \to \infty.
\]
Define \(r_i = |A_{M_i}(p_i)|^{-1} \to 0\) and \(x_i\) the image of \(p_i\) in \(X_i\). By the weakly bounded geometry assumption, there are local diffeomorphisms
\[
\Phi_i : (B(0, Q^{-1}), 0) \subset \mathbb{R}^4 \to (X_i, x_i)
\]
with \(e^{-2Q\delta} \leq \Phi_i^* g_i \leq e^{2Q\delta}\) and \(\|\partial_a \Phi_i^*(g_i)_{bc}\|_{C^\alpha} \leq Q\).

By the pullback operation described in Appendix C we can find a sequence of pointed 3-manifolds \((S_i, s_i)\), immersions \(F_i : (S_i, s_i) \to (B(0, Q^{-1}), 0)\), and local diffeomorphisms \(\Psi_i : (S_i, s_i) \to (M_i, p_i)\) so that the following diagram commutes (writing \(B = B(0, Q^{-1}) \subset \mathbb{R}^4\))

\[
\begin{array}{ccc}
S_i & \xrightarrow{F_i} & B \\
\Psi_i \downarrow & & \Phi_i \\
M_i & \xrightarrow{} & X_i
\end{array}
\]

and so that \(F_i : S_i \to (B, \Phi_i^* g_i)\) is a two-sided stable minimal immersion. (Two-sided and minimality follow immediately as they are local properties, while stability follows by lifting a positive first eigenfunction for the stability operator on \(M_i\) to \(S_i\).)

Define maps \(D_i : B(0, r_i^{-1} Q^{-1}) \to B(0, Q^{-1}), x \mapsto r_i x\) and then consider the metrics \(\tilde{g}_i := r_i^{-2} D_i^* \Phi_i^* g_i\) on \(B(0, r_i^{-1} Q^{-1})\). The bounded geometry condition ensures that \(\tilde{g}_i\) converges to the flat metric \(\delta\) on \(\mathbb{R}^4\) in \(C^{1,\alpha}_{\text{loc}}\) (in sense of \(C^{1,\alpha}_{\text{loc}}\) convergence of the metric coefficients in Euclidean coordinates). In particular, the Christoffel symbols of \(\tilde{g}_i\) with respect to the Euclidean coordinates on \(B(0, r_i^{-1} Q^{-1})\) converge to 0 in \(C^{0,\alpha}_{\text{loc}}\).

We now consider the rescaled immersions \(\tilde{F}_i := D_i^{-1} \circ F_i : S_i \to B(0, r_i^{-1} Q^{-1})\). Observe that \(\tilde{F}_i\) is a minimal immersion with respect to \(\tilde{g}_i\). Furthermore, by the point picking argument, if \(q \in S_i\) has \(d(s_i, q) \leq \rho\) (with respect to \(\tilde{F}_i^* g_i\)) then the second fundamental form of \(\tilde{F}_i\) with respect to \(\tilde{g}_i\) satisfies \(|A_{\tilde{F}_i}(q)| \leq C = C(\rho)\).
The bounded curvature condition ensures there is \( \mu > 0 \) (a numerical constant) so that for any \( q \in S_i \), for \( i \) sufficiently large, we can write \( B^i_{\mu} \) as a normal graph (in the Euclidean coordinates) over a subset of \( T_q S_i \) of a function \( f_i \) with (Euclidean) \( C^2 \)-norm \( \leq \mu \) (cf. [CM11] Lemma 2.4), except one should use that the Christoffel symbols are uniformly controlled. Geometric considerations show that the area of such a graph depends on the metric coefficients \( \tilde{g}_i \) (but not derivatives). Thus, \( f_i \) satisfies a non-divergence form elliptic PDE, whose coefficients depend on the coefficients of \( \tilde{g}_i \) and \( \tilde{\partial} \tilde{g}_i \) (in Euclidean coordinates). These coefficients are uniformly controlled in \( C^\alpha \), and thus Schauder estimates yield interior \( C^{2,\alpha} \)-estimates for \( f_i \). In particular, the injectivity radius of the pull back metric on \( h_i := \tilde{F}_i^* \tilde{g}_i \) is locally uniformly bounded. The Gauss equations imply that the sectional curvature of \( \tilde{h}_i \) is locally uniformly bounded.

In particular, the injectivity radius and curvature bounds imply that we can take a subsequential \( C^{1,\alpha} \) limit of \((S_i, \tilde{h}_i, s_i)\) in the pointed Cheeger–Gromov sense (see [Pet16] Theorem 11.4.7) to a limit \((S, h, s)\). Recall that this means that for any \( s \in \Omega \subset S \) there is \( i_0(\Omega) \) large so that for \( i \geq i_0(\Omega) \) there are embeddings

\[
G_i : (\Omega, s) \to (S, s_i)
\]

so that \( G_i^* \tilde{h}_i \to h \) in the \( C^{1,\alpha} \)-topology.

We now verify that we can also pass the immersions \( \tilde{F}_i \) to a subsequential \( C^{2,\beta}_\text{loc} \) limit for any \( \beta < \alpha \). To be precise, we claim that up to passing to a subsequence, for \( \Omega \subset S \) the maps \( \tilde{F}_i \circ G_i : (\Omega, s) \to B(0, r_i^{-1}Q^{-1}) \subset R^4 \) converge in the \( C^{2,\beta}_\text{loc} \)-topology. It is convenient prove this with respect to the flat metric on \( R^4 \); in particular, this is just a statement about \( R^4 \)-valued functions on \((\Omega, s)\) as opposed to a statement about maps between Riemannian manifolds.

To prove this claim, it suffices to obtain \( C^{2,\alpha} \) estimates of the functions \( \tilde{x}_k := x_k \circ \tilde{F}_i \circ G_i \) for \( k = 1, 2, 3, 4 \) (with respect to the metrics \( G_i^* \tilde{h}_i \)). Indeed, this follows from the fact that \( D_{G_i^* \tilde{F}_i}^2 \tilde{x}_k = (\tilde{\partial}_k \cdot \nu_{\tilde{g}})A_{G_i^* \tilde{F}_i} \) and that the (Euclidean) second fundamental form \( A_{G_i^* \tilde{F}_i} \) is locally uniformly in \( C^{\alpha} \) (this follows in an straightforward way from the fact that the local graphical functions \( f_i \) are locally uniformly bounded in \( C^{2,\alpha} \)).

Thus, up to passing to a subsequence, the immersions \( \tilde{F}_i \) limit to \( F : (S, s) \to (R^4, 0) \) in the \( C^{2,\beta}_\text{loc} \)-sense as described above. This (and the \( C^{1,\alpha}_\text{loc} \) convergence of \( \tilde{g}_i \) to \( \delta \)) implies that \( F \) is a complete stable minimal immersion with \( |A_F| = 1 \). By [CL21] Theorem 1], such an immersion must be flat. This contradiction completes the proof.

Proposition [D.1] shows that \( |\sec| \leq K \) implies \( Q \)-weakly bounded geometry for \( Q = Q(K) \), so we have also proven the following:

**Corollary 2.5.** Let \((X^4, g)\) be a complete 4-manifold with \( |\sec| \leq K \). Then there is a constant \( C(K) < \infty \) such that every compact two-sided stable minimal immersion \( M^3 \to (X^4, g) \) satisfies

\[
\sup_{q \in M} |A_M(q)| \min\{1, d_M(q, \partial M)\} \leq C(K).
\]
This corollary will not be used in this paper, but we have included it since it resolves the conjecture of Schoen stated in [CM11, Conjecture 2.13] (the analogous 3-dimensional result was proven by Schoen in [Sch83], see also [RST10]). As mentioned above, the result in [CL21, Theorem 3] established such an estimate only for closed $(X^4, g)$ where $C = C(X, g)$ (see also [Wei16, §3]).

3. Preliminary Results on Ends

We establish notation and collect some relevant facts about ends.

3.1. Ends adapted to geodesic balls. Let $M$ be a complete Riemannian manifold. Fix a point $x \in M$, and take a length scale $L > 0$.

Definition 3.1. A collection of open sets $\{E_k\}_{k \in \mathbb{N}}$ is an end of $M$ adapted to $x$ with length scale $L$ if each set $E_k$ is an unbounded connected component of $M \setminus B_{kL}(x)$ and satisfies $E_{k+1} \subset E_k$.

Proposition 3.2. If $M$ is simply connected and $\{E_k\}$ is an end adapted to $x$ with length scale $L$, then both $E_k \setminus E_{k+1}$ and $\partial E_k$ are connected for all $k$.

Proof. Suppose $\partial E_k$ has at least two components. Since $B_{kL}(x)$ and $E_k$ are connected, we can construct a loop in $M$ having nontrivial intersection number with two of the components of $\partial E_k$, which contradicts that $M$ is simply connected. The connectedness of $E_k \setminus E_{k+1}$ follows from the connectedness of $\partial E_k$ by taking segments of radial geodesics from $x$, which must intersect $\partial E_k$.

3.2. Parabolicity and nonparabolicity. We recall the notion of parabolicity for subsets.

Definition 3.3. Let $M$ be a complete Riemannian manifold. Let $K \subset M$ be a compact subset of $M$. Let $E \subset M$ be an unbounded component of $M \setminus K$ with smooth boundary. We say that $E$ is parabolic if it does not admit a positive harmonic function $f$ satisfying

$$f|_{\partial E} \equiv 1 \quad \text{and} \quad f|_{E} < 1.$$ 

Otherwise $E$ is nonparabolic.

We first state a well-known result about parabolic sets.

Proposition 3.4. Let $M$ be a complete Riemannian manifold. Let $K \subset M$ be a compact subset of $M$. Let $E \subset M$ be an unbounded component of $M \setminus K$ with smooth boundary. Suppose $E$ is parabolic. Let $f_i$ be harmonic functions on $E \cap B_{R_i}$ satisfying

$$f_i|_{\partial E} \equiv 1 \quad \text{and} \quad f_i|_{\partial B_{R_i}(x)} \equiv 0,$$

where $R_i \to \infty$ and $K \subset B_{R_i}(x)$ for all $i$. Then $f_i$ converges uniformly to 1 on compact subsets, and

$$\lim_{i \to \infty} \int_E |\nabla f_i|^2 = 0.$$
Proof. We follow [Li04, Theorem 10.1]. The maximum principle guarantees that $0 \leq f_i \leq 1$. Thus, up to a subsequence, $f_i$ limits to $f$ locally smoothly, with $\Delta f = 0$, $0 \leq f \leq 1$, and $f|_{\partial E} = 1$. By parabolicity, $f(x) = 1$ for some $x \in E$, so $f \equiv 1$ by the maximum principle. Thus, we see that $f_i$ limits locally smoothly to 1 on $E$. Finally, integrating by parts, we see that

$$\int_E |\nabla f_i|^2 = -\int_{\partial E} f_i \nabla \nu f_i \to 0.$$

This completes the proof. \qed

We now discuss when nonparabolicity is inherited by subsets.

**Proposition 3.5.** Let $M$ be a complete Riemannian manifold. Let $K \subset \tilde{K} \subset M$ be compact subsets of $M$ with smooth boundary. Let $E \subset M$ be an unbounded component of $M \setminus K$. If $E$ is nonparabolic, then there is a nonparabolic unbounded component $\tilde{E}$ of $E \setminus \tilde{K}$.

Proof. Since $E$ is nonparabolic, there is a positive harmonic function $f$ on $E$ such that $f|_{\partial E} \equiv 1$ and $f|_{\tilde{E}} < 1$.

By addition and scaling, we can assume without loss of generality that

$$\lim_{r \to \infty} \inf_{\partial B_r(x) \cap E} f = 0.$$

Since $\tilde{K}$ is compact with smooth boundary, $\partial \tilde{K}$ is a disjoint union of finitely many smooth closed surfaces. Since $E$ is connected, the boundary of each component of $E \setminus \tilde{K}$ is a union of at least one of the components of $\partial \tilde{K}$. Since distinct components of $E \setminus \tilde{K}$ have disjoint boundaries, the number of such components is finite. Hence, there is an unbounded component $\hat{E}$ of $E \setminus \tilde{K}$ such that

$$\lim_{r \to \infty} \inf_{\partial B_r(x) \cap E} f = 0.$$

We show that $\hat{E}$ is nonparabolic.

We take a sequence of harmonic functions $\hat{f}_i$ on $\hat{E} \cap B_{R_i}(x)$ satisfying

$$\hat{f}_i|_{\partial \hat{E}} \equiv 1 \quad \text{and} \quad \hat{f}_i|_{\partial B_{R_i}(x) \cap \hat{E}} \equiv 0,$$

where $R_i \to \infty$ such that $\tilde{K} \subset B_{R_i}(x)$ for all $i$. Then $\hat{f}_i$ converges locally uniformly to a positive harmonic function $\hat{f}$ on $\hat{E}$ with

$$\hat{f}|_{\partial \hat{E}} \equiv 1 \quad \text{and} \quad \hat{f}|_{\hat{E}} \leq 1.$$

Let $c = \inf_{\partial \hat{E}} f > 0$. Then $\hat{f}_i \leq c^{-1} f$ on $\hat{E} \cap B_{R_i}(x)$ for all $i$ by the maximum principle, which implies $\hat{f} \leq c^{-1} f$. In particular,

$$\lim_{r \to \infty} \inf_{\partial B_r(x) \cap \hat{E}} \hat{f} = 0,$$

so $\hat{f} \not\equiv 1$. Hence, $\hat{f}|_{\hat{E}} < 1$ by the maximum principle, so $\hat{E}$ is nonparabolic. \qed

By Proposition 3.5 it makes sense to define nonparabolicity for an end.
Definition 3.6. An end \( \{E_k\} \) adapted to \( x \) with length scale \( L \) is nonparabolic if there exist connected open sets \( F_k \subset E_k \) with smooth boundary such that

\[
E_k \setminus \overline{B}_{(k+1)L}(x) \subset F_k \subset E_k
\]

and \( F_k \) is nonparabolic for all \( k \) sufficiently large.

4. Nonparabolic Ends of Stable Hypersurfaces in 4-Manifolds with \( \text{Ric}_2 \geq 0 \)

In this section, we observe\footnote{One may observe that the same result holds (with the same proof) for \( M^{n-1} \to (X^n, g) \) when \( (X^n, g) \) has \( \text{Ric}_{n-2}^\rho \geq 0 \) (meaning that if \( \Pi_1, \ldots, \Pi_{n-2} \) is a set of planes in \( T_pX \) meeting pairwise orthogonally along a fixed line, then \( \sum_{i=1}^{n-2} \sec_{\rho}(\Pi_i) \geq 0 \)).} that a minor modification of the arguments used in [LW04, Li04, §11] restricts the number of nonparabolic ends of a stable hypersurface in a 4-manifold with \( \text{Ric}_2 \geq 0 \).

Theorem 4.1. Let \( (X^4, g) \) be a complete 4-manifold with \( \text{Ric}_2^g \geq 0 \). Let \( M^3 \to (X^4, g) \) be a complete two-sided stable minimal immersion. Let \( K \subset M \) be a compact subset of \( M \) with smooth boundary. Then \( M \setminus K \) admits at most one nonparabolic unbounded component. In particular, \( M \) has at most one nonparabolic end.

We give the proof in Section 4.3 after establishing some preliminary results.

4.1. Schoen-Yau inequality. We use the following inequality of Schoen and Yau, which is known in non-negative sectional curvature. We show that the same proof works under the weaker assumption \( \text{Ric}_2^g \geq 0 \). We include the proof for completeness.

Lemma 4.2 ([SY76]). Let \( (X^4, g) \) be a complete 4-manifold satisfying \( \text{Ric}_2^g \geq 0 \). Let \( M^3 \to (X^4, g) \) be a complete two-sided stable minimal immersion. Let \( u \) be a harmonic function on \( M \). Then

\[
\frac{1}{3} \int_M \phi^2 |A_M|^2 |\nabla u|^2 + \frac{1}{2} \int_M \phi^2 |\nabla |\nabla u||^2 \leq \int_M |\nabla \phi|^2 |\nabla u|^2
\]

for any compactly supported, nonnegative \( \phi \in W^{1,2}(M) \).

Proof. We choose a favorable test function in the stability inequality, combined with a rearrangement of the Bochner formula. Below, we assume that \( u \) is not a constant function (otherwise the desired inequality is trivial).

The favorable test function is \( \psi := \phi |\nabla u| \), where \( \phi \) is a compactly supported, non-negative function in \( W^{1,2}(M) \). Since \( \text{Ric}_2^g \geq 0 \) implies non-negative Ricci curvature, the stability inequality gives

\[
\int_M \phi^2 |A_M|^2 |\nabla u|^2 \leq \int_M |\nabla \phi|^2 |\nabla u|^2 + 2 \int_M \phi |\nabla u| (\nabla \phi, \nabla |\nabla u|) + \int_M \phi^2 |\nabla |\nabla u||^2
\]

(4.1)

\[
= \int_M |\nabla \phi|^2 |\nabla u|^2 + \frac{1}{2} \int_M (|\nabla \phi|^2)^2 + \int_M \phi^2 |\nabla |\nabla u||^2
\]

(4.2)

\[
= \int_M |\nabla \phi|^2 |\nabla u|^2 - \int_M \phi^2 \left( \frac{1}{2} |\Delta |\nabla u|^2 - |\nabla |\nabla u||^2 \right),
\]

where the divergence theorem is used in the third equality.
We now rearrange the Bochner formula. Recall that the usual Bochner formula gives
\begin{equation}
\frac{1}{2} \Delta |\nabla u|^2 = \text{Ric}_M(\nabla u, \nabla u) + |\text{Hess}_M u|^2. \tag{4.3}
\end{equation}

For now, consider a point \( p \) where \( \nabla u \neq 0 \). Choose a local orthonormal frame around \( p \) so that \( e_1 = \frac{\nabla u}{|\nabla u|} \). The improved Kato inequality (cf. \cite[Lemma 7.2]{Li04}) yields
\begin{equation}
|\text{Hess}_M u|^2 \geq \frac{3}{2} |\nabla |\nabla u||^2. \tag{4.4}
\end{equation}

Now we rearrange the Ricci curvature term. The Gauss equation and \( \text{Ric}_2 \geq 0 \) give
\begin{equation}
\text{Ric}_M(e_1, e_1) = \sum_{j=2}^{3} R(e_j, e_1, e_1, e_j) - \sum_{j=1}^{3} A(e_1, e_j)^2 \geq -\sum_{j=1}^{3} A(e_1, e_j)^2. \tag{4.5}
\end{equation}

On the other hand, Cauchy-Schwarz and minimality give
\begin{equation}
|A_M|^2 \geq A_M(e_1, e_1)^2 + 2 \sum_{j=2}^{3} A_M(e_1, e_j)^2 + \sum_{j=2}^{3} A_M(e_j, e_j)^2 \tag{4.6}
\end{equation}
\begin{equation*}
\geq A_M(e_1, e_1)^2 + 2 \sum_{j=2}^{3} A_M(e_1, e_j)^2 + \frac{1}{2} \left( \sum_{j=2}^{3} A_M(e_j, e_j) \right)^2
\end{equation*}
\begin{equation*}
= A_M(e_1, e_1)^2 + 2 \sum_{j=2}^{3} A_M(e_1, e_j)^2 + \frac{1}{2} A_M(e_1, e_1)^2
\end{equation*}
\begin{equation*}
\geq \frac{3}{2} \sum_{j=1}^{3} A_M(e_1, e_j)^2.
\end{equation*}

Together, (4.5) and (4.6) give
\begin{equation}
\text{Ric}_M(\nabla u, \nabla u) \geq -\frac{2}{3} |A_M|^2 |\nabla u|^2. \tag{4.7}
\end{equation}

Hence, (4.3), (4.4), (4.7) give the rearrangement
\begin{equation}
\frac{1}{2} \Delta |\nabla u|^2 \geq -\frac{2}{3} |A_M|^2 |\nabla u|^2 + \frac{3}{2} |\nabla |\nabla u||^2, \tag{4.8}
\end{equation}
so
\begin{equation}
\frac{1}{2} \Delta |\nabla u|^2 - |\nabla |\nabla u||^2 \geq -\frac{2}{3} |A_M|^2 |\nabla u|^2 + \frac{1}{2} |\nabla |\nabla u||^2. \tag{4.9}
\end{equation}

We derived (4.9) under the assumption \( |\nabla u| \neq 0 \) at the given point.

Now suppose that \( |\nabla u| \) vanishes at \( p \) but \( x \mapsto |\nabla u|(x) \) is differentiable at \( p \). In this case, the right hand side of (4.9) is \( 0 \) (since \( p \) is a local minimum of \( |\nabla u| \)). On the other hand, the left-hand side is \( \geq 0 \) since \( p \) is a local minimum of the smooth function \( |\nabla u|^2 \). In sum, we find that that (4.9) holds anytime that \( |\nabla u| \) is differentiable, which is almost everywhere (by Rademacher’s theorem). Thus, we can use (4.9) in (4.1) to complete the proof. \( \Box \)
4.2. Multiple nonparabolic ends. Under the assumption of multiple nonparabolic ends with respect to a fixed compact set, we produce a non-constant harmonic function.

Lemma 4.3 (cf. [Li04 Theorem 4.3]). Let $M$ be a complete Riemannian manifold. Let $K \subset M$ be a compact subset of $M$ with smooth boundary. Suppose $M \setminus K$ has at least two nonparabolic unbounded components. Then there exists a non-constant harmonic function with finite Dirichlet energy on $M$.

Proof. Let $E, F$ denote nonparabolic components of $M \setminus K$. There exists a harmonic function $0 \leq h \leq 1$ on $M \setminus K$ so that $h \equiv 0$ on $M \setminus (K \cup E \cup F)$, $h|_{\partial E} = 1$, $h|_{\partial F} = 0$, $\liminf_{x \in E, x \to \infty} h = 0$, and $\limsup_{x \in F, x \to \infty} h = 1$. By the argument in [Li04] Lemma 3.6, we can assume that $h$ has finite Dirichlet energy on $M \setminus K$.

For $R_i \to \infty$, define $f_i$ to be the harmonic function on $B_{R_i}(x)$ with $f_i|_{\partial B_{R_i}(x)} = h|_{\partial B_{R_i}(x)}$. Note that $0 \leq f_i \leq 1$. In particular, $f_i \leq h$ on $E \cap B_{R_i}(x)$, and $f_i \geq h$ on $F \cap B_{R_i}(x)$. Passing to a subsequence, the $f_i$ limit locally smoothly to a harmonic function $f$ on $M$ with $f \leq h$ on $E$ and $f \geq h$ on $F$. In particular, $f$ is non-constant.

Finally, observe that

$$\int_{B_{R_{i+1}}(x)} |\nabla f_{i+1}|^2 \leq \int_{B_{R_i}(x)} |\nabla f_i|^2 + \int_{B_{R_{i+1}}(x) \setminus B_{R_i}(x)} |\nabla h|^2.$$ 

This implies that $f$ has finite Dirichlet energy, completing the proof. \hfill \Box

4.3. Proof of Theorem 4.1

Proof. Suppose for contradiction that $\{E_k\}$ and $\{\tilde{E}_k\}$ are distinct nonparabolic ends of $M$ adapted to $x$ with length scale $L$. Let $F_k$ and $\tilde{F}_k$ be open subsets with smooth boundary given by Definition 3.6. Since the ends are distinct, there is a $k_0 \in \mathbb{N}$ such that $E_{k_0} \cap \tilde{E}_{k_0} = \emptyset$. Taking $k_0$ larger if necessary, $F_{k_0}$ and $\tilde{F}_{k_0}$ are disjoint and nonparabolic. Hence, it suffices to rule out two nonparabolic sets.

Suppose $E$ and $F$ are nonparabolic unbounded components of $M \setminus K$. By Lemma 4.3 there is a nonconstant harmonic function $u$ with finite Dirichlet energy on $M$.

We choose a nice cutoff function to use in Lemma 4.2. Let $\rho_i$ be a smoothing of $d_M(x, \cdot)$ with $\rho_i|_{\partial B((k_0+i)L)^i} = (k_0+i)L$, $\rho_i|_{\partial B((k_0+2i)L)^i} = (k_0+2i)L$, and $|\nabla \rho_i| \leq 2$. Then we define

$$\phi_i(y) := \begin{cases} 
1 & \quad y \in \overline{B}_{(k_0+i)L}(x) \\
\frac{(k_0+i)L - \rho_i(y)}{L} & \quad y \in \overline{B}_{(k_0+2i)L}(x) \setminus B_{(k_0+i)L}(x) \\
0 & \quad y \in M \setminus \overline{B}_{(k_0+2i)L}(x).
\end{cases}$$

Lemma 4.2 and the fact that $u$ has finite Dirichlet energy implies

$$\int_{B_{(k_0+i)L}(x)} \frac{1}{3} |A_M|^2 |\nabla u|^2 + \frac{1}{2} |\nabla |\nabla u||^2 \leq \int_{M} \frac{1}{3} \phi_i^2 |A_M|^2 |\nabla u|^2 + \frac{1}{2} \phi_i^2 |\nabla |\nabla u||^2$$

$$\leq \int_{M} |\nabla \phi_i|^2 |\nabla u|^2$$

$$\leq C \frac{1}{i^2}.$$
Taking $i \to \infty$, we conclude that
$$|A_M|^2|\nabla u|^2 \equiv 0 \quad \text{and} \quad |\nabla|\nabla u||^2 \equiv 0.$$ 
Since $|\nabla u|$ is constant and $u$ is non-constant, we have $|\nabla u| \neq 0$ everywhere. Then $|A_M| \equiv 0$, which implies $\text{Ric}_M \geq 0$ by Lemma 2.2. The assumption of at least two ends implies (by Cheeger-Gromoll splitting) that $M$ is a product $\mathbb{R} \times P$. Therefore, $M$ has infinite volume. However, $|\nabla u|^2$ is a nonzero constant and $\int_M |\nabla u|^2 < \infty$ by Lemma 4.3, which implies that $M$ has finite volume. Hence, we reach a contradiction. □

5. Stable Hypersurfaces in PSC 4-Manifolds with $\text{Ric}_2 \geq 0$

Our aim in this section is to prove Theorem 1.10 (which immediately implies Theorem 1.3 and Corollary 1.6).

Remark 5.1. If $M$ is parabolic, then the conclusion of Theorem 1.10 follows immediately, as parabolicity allows us to “plug 1 into the stability inequality.” The hard case is therefore the nonparabolic case. In this case, Theorem 4.1 implies that $M$ has precisely one nonparabolic end. Hence, the main difficulty of Theorem 1.10 is this troublesome end.

Remark 5.2. We can assume without loss of generality that $M$ is simply connected. In the general case, we pass to the universal cover, which inherits minimality, stability, two-sidedness, and completeness from the original immersion. The conclusion of Theorem 1.10 for the universal cover then descends to the original immersion.

5.1. $\mu$-bubbles. We first recall a diameter bound that uses the theory of warped $\mu$-bubbles for stable minimal hypersurfaces in PSC 4-manifolds.

Lemma 5.3 (Warped $\mu$-bubble diameter bound). Let $(X^4, g)$ be a complete 4-manifold with $R_g \geq 1$. Let $N^3 \to (X^4, g)$ be a two-sided stable minimal immersion with compact boundary. There are universal constants $L > 0$ and $c > 0$ such that if there is a $p \in N$ with $d_N(p, \partial N) > L/2$, then in $N$ there is an open set $\Omega \subset B_{L/2}(\partial N)$ and a smooth surface $\Sigma^2$ such that $\partial \Omega = \Sigma \sqcup \partial N$ and each component of $\Sigma$ has diameter at most $c$.

Proof. This follows from Gromov’s band-width estimate technique [Gro18]. For completeness, we sketch the proof here, with references to the relevant statements from [CL20]. Choose $L = 20\pi$. Since $N$ is two-sided and stable, we have
$$\int_N |\nabla \psi|^2 - \frac{1}{2}(R_g - R_N + |A_N|^2)\psi^2 \geq 0, \quad \forall \psi \in C^1_c(N).$$
Since $R_g \geq 1$, there exists $u \in C^\infty(N)$, $u > 0$ in $\bar{N}$ such that
$$(5.1) \quad \Delta_N u \leq -\frac{1}{2}(1 - R_N)u.$$ 
For instance, see [FCS80] Theorem 1].

Take $\rho_0 \in C^\infty(M)$ to be a smoothing of $d_N(\cdot, \partial N)$, such that $|\text{Lip}(\rho_0)| \leq 2$, and $\rho_0 = 0$ on $\partial N$. Choose $\varepsilon \in (0, \frac{1}{2})$ such that $\varepsilon$, $4\pi + \frac{3}{2}\varepsilon$, $8\pi + 2\varepsilon$ are regular values of $\rho_0$. Define
$$\rho = \frac{\rho_0 - \varepsilon}{8 + \frac{3}{2}\varepsilon} - \frac{\pi}{2},$$
\( \Omega_1 = \{ x \in N : -\frac{\pi}{2} < \rho < \frac{\pi}{2} \} \), and \( \Omega_0 = \{ x \in N : -\frac{\pi}{2} < \rho \leq 0 \} \). Clearly \( \text{Lip}(\rho) \) is simply connected and complete two-sided stable immersion. Let \( \Omega = \{ x \in \mathbb{N} : 0 \leq \rho_0(x) \leq \varepsilon \} \). We claim that \( \Omega \) satisfies the conclusion with \( c = \frac{4\pi}{\sqrt{3}} \).

\[ \mathcal{A}(\Omega) = \int_{\partial \Omega} u \, d\mathcal{H}^2 - \int_{\Omega} (\chi_\Omega - \chi_{\Omega_0}) hu \, d\mathcal{H}^3. \]

By [CL20, Proposition 12], there exists a minimizer \( \hat{\Omega} \) for \( \mathcal{A} \) such that \( \hat{\Omega} \Delta \Omega_0 \) is compactly contained in \( \Omega_1 \). Let \( \delta = \{ x \in \mathbb{N} : 0 \leq \rho_0(x) \leq \varepsilon \} \cup \hat{\Omega} \). We claim that \( \Omega \) satisfies the conclusion with \( c = \frac{4\pi}{\sqrt{3}} \).

\[ \text{Almost linear volume growth.} \]

We show that every end of a stable hypersurface in a \( \text{Ric} \geq 0 \) PSC 4-manifold has a core tube with linear volume growth.

**Lemma 5.4** (Almost linear volume growth of an end). Let \( (X^4, g) \) be a complete 4-manifold with weakly bounded geometry, \( \text{Ric}^g \geq 0 \), and \( \text{R}_g \geq 1 \). Let \( M^3 \to (X^4, g) \) be a simply connected complete two-sided stable minimal immersion. Let \( \{ E_k \}_{k \in \mathbb{N}} \) be an end of \( M \) adapted to \( x \in M \) with length scale \( L \), where \( L \) is the constant from Lemma 5.3. Let \( M_k := E_k \cap B_{(k+1)L}(x) \). Then there is a constant \( C > 0 \) such that

\[ \text{Vol}_M(M_k) \leq C \]

for all \( k \).

**Proof.** We apply Lemma 5.3 to \( E_k \), which supplies an open set \( \Omega_k \) in \( B_{L/2}(\partial E_k) \cap E_k \) with \( \partial E_k \subset \partial \Omega_k \). Note that \( \Omega_k \subset M_k \) and \( \partial E_k \cap \Omega_k = \emptyset \) by construction. Let \( \Sigma^{(i)}_k \) denote the components of \( \partial \Omega_k \setminus \partial E_k \).

First, we claim that \( M_k \) is connected. Indeed, this claim follows from the connectedness of \( \partial E_k \) by taking segments of radial geodesics from \( x \), just as in Proposition 3.2.

Second, we claim that some \( \Sigma^{(i)}_k \) separates \( \partial E_k \) from \( \partial E_{k+1} \). Let \( \gamma \) be any curve from \( \partial E_k \) to \( \partial E_{k+1} \) in \( M_k \), which exists by the connectedness of \( M_k \). Perturbing \( \gamma \) if necessary, we can guarantee that any intersection with \( \bigcup_i \Sigma^{(i)}_k \) is transverse. Since \( \partial E_k \subset \Omega_k \) and \( \partial E_{k+1} \cap \Omega_k = \emptyset \), \( \gamma \) intersects some \( \Sigma^{(i)}_k \) an odd number of times. Suppose for contradiction that \( \Sigma^{(i)}_k \) does not separate \( \partial E_k \) from \( \partial E_{k+1} \). Then there is another
curve $\gamma'$ from $\partial E_k$ to $\partial E_{k+1}$ in $\overline{M}_k \setminus \Sigma^{(i)}_k$. Since $\partial E_k$ and $\partial E_{k+1}$ are connected by Proposition 3.2, we have constructed a loop with nontrivial intersection number with $\Sigma^{(i)}_k$, contradicting the simple connectedness of $M$.

Let $\Sigma_k$ denote the component of $\partial \Omega_k \setminus \partial E_k$ provided by the above claim. By Lemma 5.3, we have $\text{diam}(\Sigma_k) \leq c$.

We claim that there is a constant $D > 0$ (independent of $k$) so that $\text{diam}(M_k) \leq D$. Let $y$ and $z$ be any points in $M_k$. Take the radial geodesic from $x$ to $y$. As in the proof of the connectedness of $M_k$, this curve intersects $\partial E_k$ and $\partial E_{k-1}$. By the choice of the components $\{\Sigma_k\}$, an arc of this curve of length at most $2L$ joins $y$ to $\Sigma_{k-1}$. The same argument applies to $z$. Hence, the diameter bound for $\Sigma_{k-1}$ implies
\[ d_M(y,z) \leq 4L + c =: D. \]

By Lemma 2.4, $M$ has bounded second fundamental form (e.g., use the compact immersion of $B^M_2(q)$ for any $q \in M$). By Lemma 2.2, $\text{Ric}_M$ is bounded from below. Hence, by Bishop–Gromov volume comparison, there is a constant $C > 0$ so that
\[ \text{Vol}(B^M_D(p)) \leq C \]
for all $p \in M$. Since $M_k \subset B^M_D(p)$ for any $p \in M_k$, the lemma follows. \qed

5.3. Decomposition of $M$. To construct a nice sequence of test functions for the stability inequality, we need to decompose $M$ into convenient building-blocks.

Let $M^3$ be a complete, simply connected, Riemannian 3-manifold such that the conclusions of Theorem 1.1 hold. Let $\{E_k\}_{k \in \mathbb{N}}$ be the nonparabolic end of $M$ adapted to $x \in M$ with length scale $L$ (where $L$ is the universal constant from Lemma 5.3).

We now decompose $M$ to handle the nonparabolic end. Let $k_0 \geq 1$ and $i \geq 1$.

- $M_k := E_k \cap B_{(k+1)L}(x)$ for all $k$.
- $\{P^{(\alpha)}_{k_0}\}_{\alpha=1}^{n_{k_0}}$ are the components of $M \setminus \overline{B}_{k_0 L}(x)$ besides $E_{k_0}$.
- $\{P^{(\alpha)}_k\}_{\alpha=1}^{n_k}$ are the components of $E_{k-1} \setminus \overline{B}_{kL}(x)$ besides $E_k$ for $k > k_0$.

Note that we have the decomposition
\[ M = \overline{B}_{(k_0+i)L}(x) \cup \bigcup_{k=k_0+i}^{k_0+2i-1} \left( \overline{M}_k \cup \bigcup_{\alpha=1}^{n_k} P^{(\alpha)}_k \right) \cup \left( \overline{E}_{k_0+2i-1} \setminus B_{(k_0+2i)L}(x) \right). \]

By Theorem 1.1, $P^{(\alpha)}_k$ is either bounded or $P^{(\alpha)}_k \setminus K$ is parabolic for all compact $K \subset M$ such that $P^{(\alpha)}_k \setminus K$ has smooth boundary.

5.4. Proof of Theorem 1.10.

Proof. Supposing it exists, let $\{E_k\}$ be the nonparabolic end of $M$. We use the decomposition of $M$ from 5.3.

The Core Tube. For each $k$, let $\rho_k$ be a smooth function on $\overline{M}_k$ such that $|\nabla \rho_k| \leq 2$, $\rho_k|_{\partial E_k} \equiv kL$, and $\rho_k|_{\partial \overline{M}_k \setminus \partial E_k} \equiv (k+1)L$.

For instance, we can take $\rho_k$ to be a smoothing of the distance function from $x$.

Let $\phi : [(k_0+i)L, (k_0+2i)L] \to [0,1]$ be the linear function satisfying $\phi((k_0+i)L) = 1$ and $\phi((k_0+2i)L) = 0$. 
The Extraneous Pieces. By Proposition 3.4 there is a compactly supported Lipschitz function \( u_{k,\alpha,i} \) on \( P_k^{(\alpha)} \) such that
\[
u u_{k,\alpha,i}|_{\partial P_k^{(\alpha)}} \equiv 1 \quad \text{and} \quad \int_{P_k^{(\alpha)}} |\nabla u_{k,\alpha,i}|^2 < \frac{1}{i^2 n_k}.
\]
Note that if \( \partial P_k^{(\alpha)} \) is not smooth, we apply Proposition 3.4 to \( P_k^{(\alpha)} \setminus K \) (where \( K \) is compact and \( P_k^{(\alpha)} \setminus K \) has smooth boundary) and extend the function by 1 on \( P_k^{(\alpha)} \cap K \).

Moreover, if \( P_k^{(\alpha)} \) is bounded, we can just take \( u_{k,\alpha,i} \equiv 1 \).

The Test Function. We now construct a test function \( f_i \) for the stability inequality as follows:
\[
f_i(y) := \begin{cases} 
1 & y \in B_{(k_0+i)L}(x) \\
0 & y \in E_{k_0+2i-1} \setminus B_{(k_0+2i)L}(x) \\
\phi(\rho_k(y)) & y \in M_k, \ k_0 + i \leq k < k_0 + 2i \\
\phi(kL)u_{k,\alpha,i} & y \in \overline{P_k^{(\alpha)}}, \ k_0 + i \leq k < k_0 + 2i.
\end{cases}
\]

By construction, \( f_i \) is compactly supported and Lipschitz. Therefore, \( f_i \) is an eligible test function for the stability inequality. Hence, the stability inequality and Lemma 5.4 give
\[
\int_M (\text{Ric}_g(\nu, \nu) + |A_M|^2) f_i^2 \leq \int_M |\nabla f_i|^2
\]
\[
= \sum_{k=k_0+i}^{k_0+2i-1} \int_{M_k} \phi(\rho_k)^2 |\nabla \rho_k|^2 + \sum_{k=k_0+i}^{k_0+2i-1} \sum_{\alpha=1}^{n_k} \phi(kL)^2 \int_{P_k^{(\alpha)}} |\nabla u_{k,\alpha,i}|^2
\]
\[
\leq \frac{4C}{L^2} + \frac{1}{i} = \frac{C'}{i}.
\]

Since \( f_i \) converges uniformly to the constant 1 function on compact subsets, the \( i \to \infty \) limit yields the desired conclusion. \( \square \)

6. Topology of PSC 4-Manifolds with \( \text{Ric}_2 > 0 \)

In this section, we prove Theorem 1.13. We begin with the following lemma.

Lemma 6.1. Suppose \( X^n \) is a non-compact oriented manifold, \( \sigma \subset X \) is a closed curve and \([\sigma] \in H_1(X)\) is a nontrivial, non-torsion element. For any pre-compact open set \( D \subset X \) containing a neighborhood of \( \sigma \), there exists a smooth compact submanifold \( M_0 \) of dimension \((n - 1)\), such that \( \partial M_0 \subset X \setminus D \), and the algebraic intersection number of \( M_0 \) and \( \sigma \) is 1.

Proof. By Poincaré duality for non-compact manifolds, \( H_1(X) \) is isomorphic to \( H^{n-1}_c(X) \). Thus, there exists a connected pre-compact open set \( A \) with \( D \subset A \) as well as \( \alpha \in H^{n-1}_c(X, X \setminus A) \) with \( \alpha \sim \mu_A = [\sigma] \). By assumption, \( \alpha \) is not a torsion element of \( H^{n-1}_c(X, X \setminus A) \). Thus, by the universal coefficient theorem, we can find \( \beta \in H_{n-1}(X, X \setminus A) \) with \( \alpha \sim \beta = 1 \in H_0(X, X \setminus A) \). By the excision theorem and Lefschetz duality, we have
\[
H_{n-1}(X, X \setminus A) \simeq H^1(A) \simeq \langle A, S^1 \rangle,
\]
where $\langle A, S^1 \rangle$ is the basepoint-preserving homotopy classes of maps to $S^1$. Take a smooth map $f$ in $\langle A, S^1 \rangle$ corresponding to $\beta$, and let $p \in S^1$ be a regular value of $f$. Then the smooth hypersurface $f^{-1}(p) \subset A$ represents $\beta$ and hence has algebraic intersection $1$ with $\sigma$. □

**Proof of Theorem 1.13.** We first prove that $H_1(X)$ consists only of torsion elements. If not, there exists a closed curve $\sigma$ with $[\sigma] \in H_1(X)$ non-torsion. Take $p \in X$, $R_0 > 0$ with $\sigma \subset B_{R_0}(p)$. By Lemma 6.1, for any integer $k > R_0$, there exists a smooth compact stable two-sided minimal immersion $M_k$ such that $\partial M_k \subset X \setminus B_k(p)$, and the algebraic intersection number of $M_k$ with $\sigma$ is $1$. Choose $\Omega_k \subset X$ a pre-compact region with smooth boundary, with $M_k \subset \Omega_k$. Form a metric $g_k$ so that $\Omega_k$ has mean-convex boundary and $g_k$ agrees with $g$ away from $\partial \Omega_k$.

Consider the area-minimizing problem

$$\inf \{ \mathcal{H}^3_{g_k}(M) : \partial M = \partial \tilde{M}_k, \text{ } M \text{ is homologous to } \tilde{M}_k \}.$$ 

It is standard to show that a compact smooth two-sided embedded minimizer $M_k$ exists. Moreover, since $M_k$ is homologous to $\tilde{M}_k$, the algebraic intersection number of $M_k$ with $\sigma$ is $1$. In particular, $M_k \cap \sigma \neq \emptyset$.

By Lemma 2.4, we have curvature estimates for $\{M_k\}$ on any compact subset of $X$. Thus, by passing to a subsequence (not relabeled), $\{M_k\}$ converges to a complete two-sided stable minimal immersion $M^3 \to (X, g)$. Note that $M$ is not empty, thanks to the condition that $M_k \cap \sigma \neq \emptyset$. This contradicts Theorem 1.10.

Now we prove that $X$ is $H_2$-trivial at infinity. Suppose the contrary, that there exists $\alpha \in H_2^\infty(X)$ whose image in $H_2(X)$ is zero. By definition, there exist nested bounded open sets $\{A_j\}_{j=1}^\infty$ such that $A_i \subset A_j$ when $i \leq j$ and $\cup A_j = X$, and non-trivial elements $\alpha_j \in H_2(X \setminus A_j)$, such that:

1. When $i \leq j$, $\iota_\ast(\alpha_j) = \alpha_i \neq 0$, here $\iota : X \setminus A_j \hookrightarrow X \setminus A_i$ is the inclusion.
2. For each $j$, $\iota_\ast(\alpha_j) = 0$, here $\iota : X \setminus A_j \hookrightarrow X$ is the inclusion.

Thus, for each $j$, there exists a $2$-cycle $\Sigma_j \subset X \setminus A_j$ such that $[\Sigma_j]$ is null homologous in $X$ and if $\tilde{M}$ is a $2$-chain with $\partial \tilde{M} = \Sigma_j$, then $M \cap A_1 \neq \emptyset$ (otherwise $[\Sigma_j]$ would be null-homologous in $X \setminus A_1$, contradiction). Consider the area-minimizing problem (for metrics $g_j$ as before)

$$\inf \{ \mathcal{H}^3_{g_j}(\tilde{M}) : \partial \tilde{M} = \Sigma_j \}.$$ 

A smooth compact two-sided minimizer $M_j$ exists. By construction, $M_j \cap A_1 \neq \emptyset$. Thus, by Lemma 2.4, a subsequence converges smoothly to a complete two-sided stable minimal immersion $M^3 \to (X^3, g)$, contradicting Theorem 1.10. This completes the proof. □

**Appendix A. Curvature conditions**

Here we review the various curvature conditions referred to in this paper. Fix $(X^{n+1}, g)$ a Riemannian manifold. We recall the convention used here: $\Pi \subset T_pX$ a $2$-plane with orthonormal basis $\{u, v\}$ then $\sec_g(\Pi) = R_g(v, u, u, v)$. 
For $k \in \{1, \ldots, n\}$ and $v_0, \ldots, v_k$ set of $k + 1$ orthonormal vectors, we define the $k$-th intermediate Ricci curvature by

$$\text{Ric}_k(v_0, \ldots, v_k) := \sum_{i=1}^{k} R(v_0, v_k, v_k, v_0).$$

Note that $\text{Ric}_n$ is the usual Ricci curvature while $\text{Ric}_1$ is the sectional curvature.

Following [SY95] we define the bi-Ricci curvature of an orthonormal set of vectors $\{u, v\}$ by

$$\text{biRic}(u, v) = \text{Ric}(u, u) + \text{Ric}(v, v) - R(u, v, v, u).$$

Note that when $n + 1 = 3$ then $\text{biRic}(u, v) = R_g/2$, but in higher dimensions $\text{biRic} \geq 0$ is a stronger curvature condition than $R_g \geq 0$.

**Appendix B. Examples**

**B.1. Positive sectional curvature.** Here we give the details of Example [1.2] concerning the existence of a complete stable minimal hypersurface in ambient positive sectional curvature.

For $\alpha \in (0, 1)$ to be chosen close to 1 below, define

$$\rho(r) = \alpha r + (1 - \alpha) \int_0^r e^{-s^2} ds.$$

Consider a metric $g$ on $\mathbb{R}^4$ given by

$$g = dr^2 + \rho(r)^2 g_{S^3}.$$

It is standard to compute (cf. [Pet16, §4.2.3]) that the sectional curvatures of $g_\alpha$ lie between

$$-\frac{\rho''(r)}{\rho(r)} = \frac{2(1 - \alpha)re^{-r^2}}{\alpha r + (1 - \alpha) \int_0^r e^{-s^2} ds}$$

and

$$1 - \frac{\rho'(r)^2}{\rho(r)^2} = \frac{1 - (\alpha + (1 - \alpha)e^{-r^2})^2}{(\alpha r + (1 - \alpha) \int_0^r e^{-s^2} ds)^2}.$$  

so $(\mathbb{R}^4, g)$ has positive sectional curvature. On the other hand, we can fix a totally geodesic $\mathbb{R}^3 \to (\mathbb{R}^4, g)$ (corresponding to $[0, \infty) \times S^2 \subset [0, \infty) \times S^3$ in the radial coordinates used above). We claim that this is a stable immersion (embedding), at least for $\alpha$ sufficiently close to 1. Indeed, for $\varphi \in C_c^\infty(\mathbb{R}^3)$, we compute (un-barred quantities denote the induced metric $dr^2 + \rho(r)^2 g_{S^2}$ and barred quantities denote the Euclidean metric $dr^2 + r^2 g_{S^2}$)

$$\int_{\mathbb{R}^3} |\nabla \varphi|^2 d\mu \geq \int_{\mathbb{R}^3} \alpha^2 |\nabla \varphi|^2 d\bar{\mu} \geq \int_{\mathbb{R}^3} \frac{\alpha^2}{4r^2} \varphi^2 d\bar{\mu} \geq \int_{\mathbb{R}^3} \frac{\alpha^2}{4r^2} \varphi^2 d\mu.$$
In the first and third lines we used \( \alpha r \leq \rho(r) \leq r \), in the second we used a well-known Hardy inequality. Finally, we note that
\[
\mathrm{Ric}_g(\nu,\nu) \leq 2 \left( \frac{1 - \alpha^2}{\alpha^2 r^2} + \frac{1 - \alpha}{\alpha} e^{-r^2} \right) \leq \frac{\alpha^2}{4r^2},
\]
where the second inequality holds for \( \alpha \) sufficiently close to 1. Hence, for this choice of \( \alpha \), we find that \( R^3 \to (R^4, g) \) is stable.

**B.2. Positive Ricci curvature and strictly positive scalar curvature.** The following example should be compared to the example of Schoen indicated in [MR06, Appendix]. We construct a complete two-sided stable minimal hypersurface \( M^3 \to (X^4, g) \) so that along \( M \), \( R_g \geq 1 \), \( \mathrm{Ric}_g \geq 1 \), and the sectional curvature is uniformly bounded below.

**Remark B.1.** We emphasize that \((X^4, g)\) will not be complete, but \( M^3 \) will have a tubular neighborhood of uniform diameter. We discuss this point further below.

**Remark B.2.** The example below can be directly generalized to produce \( M^{n-1} \to (X^n, g) \) as above, for any dimension \( n \geq 4 \).

Fix \( M_0 = (S^1 \times S^2) \# (S^1 \times S^2) \). Because \( \dim M_0 = 3 > 2 \), the Schoen–Yau/Gromov–Lawson surgery construction [SY79b, GL80] and the Kazdan–Warner trichotomy theorem [KW75a, KW75b, KW75c] (see e.g., [KW75a, Theorem 1.3]) imply that there is a scalar flat metric \( h_0 \) on \( M_0 \).

Write \((M, h)\) for the universal cover of \((M_0, h_0)\). Defining
\[
\lambda_1(M, h) = \inf_{u \in C_c^\infty(M) \setminus \{0\}} \frac{\int_M |\nabla u|^2}{\int_M u^2},
\]
we claim that \( \lambda_1(M, h) > 0 \). This is a consequence of a result of Brooks [Bro81] (see also [DK18, §16]) which says that \( \lambda_1(M, h) = 0 \) if and only if \( \pi_1(M_0) \) is amenable. The definition of an amenable group can be found in [Bro81, §1] (among other places), but all that matters here is that \( \pi_1(M_0) = F_2 \), the free group on two generators, which is not amenable (see e.g., [Tao09, Proposition 4]).

We now fix \( \epsilon = \frac{\lambda_1(M, h)}{6} \).

For \( \delta > 0 \) to be fixed below (depending only on \( h \)), define a family of metrics \( h_t \) on \( M_0 \) by:
\[
h_t = h_0 + t^2 (\mathrm{Ric}_{h_0} - 2\epsilon h_0), \quad t \in (-\delta, \delta).
\]
For \( \delta \) sufficiently small, \( h_t \) is a Riemannian metric. Finally, we define a metric \( g \) on \( X = M_0 \times (-\delta, \delta) \) by
\[
g = h_t + dt^2.
\]

\[^9\text{For any } 1 < p < n \text{ and } \varphi \in C_c^\infty(R^n) \text{ it holds that } \int_{R^n} \frac{\varphi^p}{\varphi} d\mu \leq \left( \frac{p}{n-p} \right)^p \int_{R^n} |\nabla \varphi|^p d\mu. \text{ This can be deduced from the usual one-variable Hardy inequality [HLP88, Theorem 327] via symmetrization (cf. [CF08, Section 2]). Alternatively, the } p = 2 \text{ case (all we need here) can be deduced via a simple integration by parts argument (cf. [Eva10, Section 5.8.4]).} \]
We make a few observations. First, since \( \partial_t h_1|_{t=0} = 0 \), the embedding \( M_0 \times \{0\} \subset (X, g) \) is totally geodesic. Moreover, by the Riccati equation (cf. [Gra04 Corollary 3.3]), we have that

\[
2\varepsilon h_0 - \text{Ric}_{h_0} = -\frac{1}{2} \partial_t^2 h_t|_{t=0} = \text{Ric}_g(\cdot, \partial_t, \partial_t, \cdot).
\]

Let \( \{e_i\}_{i=1}^4 \) be a local orthonormal frame on \( M \), where \( e_4 = \partial_t \). By the Gauss equations for \( M_0 \times \{0\} \) and (B.1), we have for \( i \in \{1, 2, 3\} \)

\[
\text{Ric}_g(e_i, e_i) = \text{Ric}_{h_0}(e_i, e_i) + \text{Ric}_g(e_i, e_i) = 2\varepsilon.
\]

Furthermore, by (B.1), we have

\[
\text{Ric}_g(e_4, e_4) = \sum_{i=1}^3 \text{Ric}_g(e_i, e_4, e_4, e_i) = 6\varepsilon - \text{Ric}_{h_0} = 6\varepsilon.
\]

In particular, \( \text{Ric}_g \geq 2\varepsilon \) along \( M_0 \times \{0\} \). If we take \( \delta > 0 \) even smaller, we find that \( \text{Ric}_g \geq \varepsilon \) on \( X \).

Now we verify that the immersion \( M^3 \to (X^4, g) \) from the universal cover is stable. Note that the potential term in the stability operator for \( M \) satisfies \( |A_M|^2 + \text{Ric}_g(\nu, \nu) = 6\varepsilon = \lambda_1(M) \). Thus, for any \( u \in C^\infty_c(M) \), we have

\[
\int_M |\nabla u|^2 \geq \lambda_1(M, h) \int_M u^2 = \int_M (|A_M|^2 + \text{Ric}_g(\nu, \nu)) u^2.
\]

This completes the proof (after scaling so that \( \text{Ric}_g \geq 1 \)). Note that because the image of \( M \) is an embedded submanifold, the sectional curvature is uniformly bounded below along a tubular neighborhood of \( M \).

Remark B.3. It is an interesting question as to whether or not one can find \( M^3 \to (X^4, g) \) stable minimal where \( (X^4, g) \) is a closed (or complete) manifold with \( \text{Ric}_g \geq 1 \) at all points. It is tempting to try to study the metric of positive Ricci curvature on \((S^2 \times S^2)\#(S^2 \times S^2)\) constructed in [SY91]. If the construction is carried out in the most symmetric way possible, there will be a totally geodesic submanifold diffeomorphic to \( M_0 := (S^1 \times S^2)\#(S^1 \times S^2) \). If one could modify the construction so as to keep positivity of the curvature, but to ensure that \( 0 \leq \text{Ric}_g(\nu, \nu) \to 0 \) along \( M_0 \), then the proof used above would show that the universal cover of \( M_0 \) was eventually stable.

B.3. Non-negative sectional curvature and strictly positive scalar curvature in six dimensions. Choose \((\mathbb{R}^3, g)\) as in Appendix B.1 and consider \((X^6, g_X) := (\mathbb{R}^3, g) \times (S^2, g_{S^2})\). Then, \((X^6, g_X)\) will have non-negative sectional curvature, scalar curvature \( R_{g_X} \geq 2 \), and strictly positive Ricci curvature. On the other hand, if we cross the totally geodesic \( \mathbb{R}^3 \to (\mathbb{R}^4, g) \) by \( S^2 \), we find \( M^5 := \mathbb{R}^3 \times S^2 \to (X^6, g_X) \) two-sided minimal immersion. The immersion \( M \to (X, g_X) \) will be stable. To see this, consider \( B_\rho(0) \times S^2 \subset M \) for \( \rho > 0 \). The first eigenfunction of the stability operator on this set will be \( S^2 \)-invariant, so the argument used in Appendix B.1 (the Hardy inequality) implies that the first eigenvalue of the stability operator on this set is positive. Letting \( \rho \to \infty \), we see that \( M \) is stable. On the other hand \( \text{Ric}_{g_X}(\nu, \nu) > 0 \) along \( M \).
Appendix C. Pulling back immersions along a local diffeomorphism

Suppose that $X, Y, M$ are smooth manifolds, $\Psi : Y \to X$ is a local diffeomorphism and $F : M \to X$ is an immersion. We describe here how to "pull back" $F$ along $\Psi$.

This construction is presumably well-known. Below, we will use the standard notation $f \pitchfork Z$ to mean that the map $f$ is transverse to the submanifold $Z$.

Consider the map $F \times \Psi : M \times Y \to X \times X$.

Write $\Delta = \{(x, x) : x \in X\}$ for the diagonal in $X \times X$.

**Lemma C.1.** $F \times \Psi \pitchfork \Delta$.

**Proof.** We note that $(F \times \Psi)^{-1}(\Delta) = \{(m, y) \in M \times Y : F(m) = \Psi(y)\}$.

As such, for $(m, y) \in (F \times \Psi)^{-1}(\Delta)$, we find
\[
\text{image } d(F \times \Psi)_{(m,y)} = \text{image } dF_m \times T_{\Psi(y)}X.
\]

Hence, for $(v_1, v_2) \in T_{F(m)}X \times T_{\Psi(y)}X$, we can write
\[
(v_1, v_2) = (v_1, v_1) + (0, v_2 - v_1) \in T_{(F(m),\Psi(y))}\Delta + \text{image } d(F \times \Psi)_{(m,y)}.
\]

This completes the proof. $\square$

Thus $S := (F \times \Psi)^{-1}(\Delta)$ is a submanifold of $M \times Y$. Recall that
\[
T_s S = \left( d(F \times \Psi)_s \right)^{-1}(T_{(F(s),\Psi(s))}\Delta)
\]

and
\[
\dim M + \dim Y - \dim S = \text{codim}(S \subset Y \times M) = \text{codim}(\Delta \subset X \times X) = \dim X,
\]

so (because $\dim Y = \dim X$) we have
\[
\dim S = \dim M.
\]

Write $F_S : S \to Y$ for the restriction of the projection map $M \times Y \to Y$ and similarly for $\Psi_S : S \to M$. In particular, the following diagram commutes
\[
\begin{array}{ccc}
S & \xrightarrow{F_S} & Y \\
\Psi_S \downarrow & & \downarrow \Psi \\
M & \xrightarrow{F} & X
\end{array}
\]

We now check that the maps $\Psi_S, F_S$ have the desired properties.

**Lemma C.2.** The map $F_S : S \to Y$ is an immersion and the map $\Psi_S : S \to M$ is a local diffeomorphism.

---

10This is a pullback in the category theoretic sense (in the category of smooth maps/manifolds). Note that the pullback of two maps need not always exist in this category, but it does when the maps are transversal.
Proof. We start with $F_S$. For $s \in S$, write $s = (m, y) \in Y \times M$. We have that $(dF_S)_s$ is the restriction of the projection onto the second factor map $\pi_2 : T_{m}M \times T_{y}Y \rightarrow T_yY$. As such,

$$\ker(dF_S)_s = (T_{m}M \times 0) \cap T_sS.$$ 

Hence, if $(t, 0) \in \ker(dF_S)_s$ then

$$(t, 0) \in (d(F \times \Psi)_s)^{-1}(T_{(F(s), \Psi(s))}T_S),$$

i.e., $dF_m(t) = 0$. Because $F$ is an immersion, we have $t = 0$. This proves that $F_S$ is an immersion. For $\Psi_S$, note that $\dim S = \dim M$, so it suffices to prove that $\Psi_S$ is an immersion. The proof is then identical to the one we just gave for $F_S$. \qed

Appendix D. Existence of local covering maps with good regularity

Recall the definition of $Q$-weakly bounded geometry from Definition [2.3]. The following result is well-known (cf. [RST10, NT18, Lemma 2.1] and [Pet16, Exercise 11.6.15]). We sketch the proof below, referring to [RST10, Pet16] for several crucial details.

**Proposition D.1.** If $(X^n, g)$ is a complete manifold with $|\sec| \leq K < \infty$, then there is $Q = Q(K)$ so that $(X^n, g)$ has $Q$-weakly bounded geometry.

**Proof.** Fix $x \in X$ and choose an orthonormal basis for $T_xX$. We will identify $T_xX$ with $\mathbb{R}^n$ (so $g_x$ agrees with the standard inner product on $\mathbb{R}^n$). Using Jacobi field estimates (and $|\sec| \leq K < \infty$), there is $r_0 = r_0(K, n) > 0$ so that

$$\exp_x : B(0, 4r_0) \subset \mathbb{R}^n \rightarrow X$$

is a local diffeomorphism, and $\tilde{g} := \exp_x^* g$ satisfies $\frac{1}{2} \delta \leq \tilde{g} \leq 2\delta$ on $B(0, 4r_0) \subset \mathbb{R}^n$ as quadratic forms. By [RST10, Lemma 2.2] it holds that $\operatorname{inj}_{\tilde{g}}(v) \geq i_0 = i_0(K, n) > 0$ for $v \in B(0, r_0)$. The assertion then follows by constructing harmonic coordinates for $\tilde{g}$ in a uniformly big neighborhood of 0 as in [Pet16, Theorem 11.4.3] (see also [And90] and [RST10, Theorem 2.1]) \qed

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