The André-Oort conjecture for products of Drinfeld modular curves

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Abstract. Let $Z = X_1 \times \cdots \times X_n$ be a product of Drinfeld modular curves. We characterize those algebraic subvarieties $X \subset Z$ containing a Zariski-dense set of CM points, i.e. points corresponding to $n$-tuples of Drinfeld modules with complex multiplication (and suitable level structure). This is a characteristic $p$ analogue of a special case of the André-Oort conjecture.

1 Introduction

The aim of this paper is to prove an analogue of the André-Oort conjecture for the case of subvarieties of a product of Drinfeld modular curves.

This conjecture states, in general,

Conjecture 1 (André-Oort) Let $S$ be a Shimura variety, and $X \subset S$ an irreducible algebraic subvariety. Then $X$ contains a Zariski-dense set of special points if and only if $X$ is a subvariety of Hodge type.

For the relevant definitions, and known results, we refer the reader to [1, 4, 8, 9, 11, 22, 23, 24, 25, 29, 30].

The special case of Conjecture 1 of interest to us is the following. Consider $\mathbb{A}^n_C$ as the moduli space of $n$-tuples of elliptic curves, by associating the tuple $(E_1, \ldots, E_n)$ to the point $(j(E_1), \ldots, j(E_n))$. Then $\mathbb{A}^n_C$ is a Shimura variety, and the special points correspond to tuples for which each $E_i$ has complex multiplication, hence they are also called CM points. The subvarieties of Hodge type of $\mathbb{A}^n_C$, which we call modular subvarieties, are defined by imposing isogeny conditions between some coordinates and setting other coordinates equal to CM $j$-invariants. We make this definition precise below.

Denote by $\mathcal{H}$ the Poincaré upper half-plane, on which the group $\text{GL}_2^+(\mathbb{R})$ acts. Then a point $\tau \in \mathcal{H}$ corresponds to the complex elliptic curve $E_\tau = \mathbb{C}/\mathbb{Z}\tau \oplus \mathbb{Z}$. Two points $\tau_1, \tau_2 \in \mathcal{H}$ correspond to isogenous elliptic curves if and only if $\tau_1 = \sigma(\tau_2)$ for some $\sigma \in \text{GL}_2^+(\mathbb{Q})$. We now let $(\sigma_1, \ldots, \sigma_n) \in \text{GL}_2^+(\mathbb{Q})^n$ and consider the map

$$
\begin{align*}
\mathcal{H} & \longrightarrow \mathbb{A}^n(\mathbb{C}) \\
\tau & \mapsto (j(\sigma_1(\tau)), \ldots, j(\sigma_n(\tau))).
\end{align*}
$$

The image lies on an irreducible algebraic curve $Y \subset \mathbb{A}^n$, which we call a modular curve in $\mathbb{A}^n$. Now we can define the modular varieties in $\mathbb{A}^n$ as products (up to permutation of coordinates) of copies of $\mathbb{A}^1$, CM points in $\mathbb{A}^1$ and modular curves in $\mathbb{A}^m$ for $m \leq n$. 

It is clear that a modular curve contains a dense (in the complex topology) set of CM points (as all the coordinates are isogenous), and hence so does a modular variety. Conjecture 1 claims the converse, more precisely

Conjecture 2 (André-Oort for $\mathbb{A}^n$) Let $X \subset \mathbb{A}^n$ be an irreducible algebraic variety. Then $X$ contains a Zariski-dense set of CM points if and only if $X$ is a modular variety in the above sense.

Yves André [2] has proved Conjecture 2 for $n = 2$, and Bas Edixhoven [8, 10] has shown that Conjecture 2 holds for all $n$ if one assumes the Generalized Riemann Hypothesis (GRH) for quadratic imaginary fields.

The aim of this paper is to adapt Edixhoven’s techniques to function fields, and thereby prove Conjecture 2 with elliptic curves replaced by rank 2 Drinfeld modules.

More precisely, let $q$ be a power of the odd prime $p$ and set $A = \mathbb{F}_q[T]$ and $k = \mathbb{F}_q(T)$. Denote by $\infty$ the place of $k$ with uniformizer $1/T$, and let $\mathbb{C} = \mathbb{F}_\infty$ denote the completion of the algebraic closure of $k_\infty$. (Here $A$, $k$, $k_\infty$ and $\mathbb{C}$ play the roles of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively). Then we may view $\mathbb{A}^n_C$ as the moduli space of $n$-tuples of rank 2 Drinfeld $A$-modules, via the $j$-invariant, and a point $(x_1, \ldots, x_n) \in A^n(C)$ is called a CM point if every $x_i$ is the $j$-invariant of a Drinfeld module with complex multiplication. We prove the following results.

**Theorem 1** Assume that $q$ is odd. Let $X \subset \mathbb{A}^n_C$ be an irreducible variety. Then $X(C)$ contains a Zariski-dense subset $S$ of CM points if and only if $X$ is a modular variety.

When $X$ is a curve, we have an effective result.

**Theorem 2** Assume that $q$ is odd. Let $d, m$ and $n$ be given positive integers, and $g$ a given non-negative integer. Then there exists an effectively computable constant $B = B(n, m, d, g)$ such that the following holds. Let $X$ be an irreducible algebraic curve in $\mathbb{A}^n_C$ of degree $d$, defined over a finite extension $F$ of $k$ of degree $[F : k] = m$ and genus $g(F) = g$. Then $X$ is a modular curve if and only if $X(C)$ contains a CM point of height at least $B$.

As level structures play no role here, one may replace $\mathbb{A}^n = (\mathbb{A}^1)^n$ by a product $X_1 \times \cdots \times X_n$ of Drinfeld modular curves, and obtain a similar result (see Corollaries 3.12 and 3.17 for the exact statements). The definition of modular curves and modular varieties, in $\mathbb{A}^n$ or in $X_1 \times \cdots \times X_n$, will be given in §1.4.

The proofs of Theorems 1 and 2 may be divided into two parts. Firstly, in the topological part (§2) one shows that a variety which is stabilized by a certain Hecke operator must be modular. In this part we follow an approach similar to Edixhoven’s, but the translation from number fields to function fields is not automatic, as problems specific to finite characteristic arise (e.g. one can no longer use arguments from Lie theory, $k_\infty$ has many non-trivial automorphisms, and $\mathbb{C}$ has infinite dimension over $k_\infty$).

In the second, arithmetic part (§3), one shows that varieties containing suitable CM points are stabilized by certain Hecke operators, and the translation is easier. Here one uses GRH twice, once for a strong version of the Čebotarev Theorem, and once to obtain effective bounds on class numbers of quadratic imaginary fields (one only needs GRH for such bounds if one wants an effective result, such as Theorem 2, the classical analogue of which appears in [5]).
But as we’re working over function fields, GRH is known (Hasse-Weil), so our results are unconditional.

We have assumed that $q$ is odd for technical reasons (notably concerning the arithmetic of quadratic extensions of $k$), but we expect a similar result to hold in characteristic 2.

In the rest of this introduction we will briefly recall some facts about Drinfeld modules and Drinfeld modular curves, to fix notation, and we will define the notion of modular varieties in §1.4.

Acknowledgments. The results presented here made up my Ph.D thesis at l’Université Denis Diderot (Paris 7), and I am deeply indebted to my supervisor, Marc Hindry, for his cheerful advice and guidance. I would also like to thank Bas Edixhoven for his many patient explanations, and for making a preliminary version of [10] available to me. The idea of replacing elliptic curves by Drinfeld modules in the André-Oort conjecture was first suggested to me by Hans-Georg Rück, and I am also grateful to Henning Stichtenoth for providing me with Proposition 3.1 which allowed me to remove the condition $q \geq 5$. Lastly, I wish to thank the National Center for Theoretical Sciences in Hsinchun, Taiwan, and the Max-Planck-Institut für Mathematik in Bonn, Germany, for their hospitality.

1.1 Drinfeld modules

We give here a very brief introduction to Drinfeld modules, the aim being rather to fix our notation than to initiate the reader in this fascinating topic. For details, we refer the reader to [16] and [17].

Let $\tau$ denote the $q$-th power Frobenius acting on the additive group $\mathbb{G}_a, \mathbb{C}$. Then the $\mathbb{F}_q$-linear endomorphisms of $\mathbb{G}_a, \mathbb{C}$ are given by $\text{End}_{\mathbb{F}_q}(\mathbb{G}_a, \mathbb{C}) = \mathbb{C}\{\tau\}$, the ring of twisted polynomials, i.e. non-commutative polynomials in $\tau$ subject to the relations $\tau x = x^{q} \tau$ for all $x \in \mathbb{C}$. Then a Drinfeld $A$-module of rank $r$ (and defined over $\mathbb{C}$) is an injective ring homomorphism

$$\phi : A \rightarrow \mathbb{C}\{\tau\}$$

$$a \mapsto \phi_a = a \tau^0 + a_1 \tau + \ldots + a_n \tau^n, \ a_n \neq 0, \ n = r \deg(a).$$

A morphism of Drinfeld modules, written $f : \phi \rightarrow \phi'$, is an element $f \in \mathbb{C}\{\tau\}$ such that $f \phi_a = \phi'_a f$ for all $a \in A$. If $f$ is non-zero then $\phi$ and $\phi'$ have the same rank, and we call $f$ an isogeny. $f$ is an isomorphism if and only if $f \in \mathbb{C}^*$. The isogeny $f$ is called cyclic of degree $N \in A$ if $\ker(f) \cong A/NA$ as $A$-modules.

There is also an analytic construction of Drinfeld modules, similar to the construction of complex elliptic curves as quotients of $\mathbb{C}$ by a lattice. A lattice of rank $r$ in $\mathbb{C}$ is a discrete $A$-submodule $\Lambda$ of $\mathbb{C}$ such that $k_\infty \Lambda$ has dimension $r$ over $k_\infty$. As $\mathbb{C}$ has infinite dimension over $k_\infty$, it contains lattices of any rank (unlike $\mathbb{C}$). Then there is an equivalence between the categories (Drinfeld modules of rank $r$ over $\mathbb{C}$, morphisms) and (Lattices of rank $r$ in $\mathbb{C}$, homotheties).

From now on, by a Drinfeld module $\phi$ we will always mean a Drinfeld $A$-module of rank $r = 2$.

Clearly $\text{End}(\phi) = \{ f \in \mathbb{C}\{\tau\} \mid f \phi_a = \phi_a f, \forall a \in A \}$ is the centralizer of $\phi(A)$ in $\mathbb{C}\{\tau\}$. Generically, $\text{End}(\phi) \cong A$, but sometimes $\text{End}(\phi)$ is strictly larger than $A$, and we say that

1Beware, this is not a function between sets!
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p has complex multiplication (CM). In this case, the endomorphism ring is of the form \( \mathcal{O} = A[\sqrt{\Delta}] \), for some non-square \( d \in A \). Write \( d = f^2 D \), with square-free \( D \in A \), then \( \mathcal{O} \) is an order of conductor \( f \) in the CM field \( K = k(\sqrt{\Delta}) \). \( K/k \) is an imaginary quadratic extension, which means that the place \( \infty \) does not split in \( K/k \). Equivalently, \( K \) has no embedding into \( k_\infty \), hence the terminology. We distinguish two cases. Either \( \infty \) ramifies in \( K/k \), in which case \( \deg(D) \) is odd, or \( \infty \) is inert in \( K/k \), in which case \( \deg D \) is even and the leading coefficient of \( D \) is not a square in \( F_q \).

A Drinfeld module \( \phi \) is uniquely determined by \( \phi_T = T^{r_0} + g_T + \Delta T^2 \), where \( g, \Delta \in C \) and \( \Delta \neq 0 \). We define the \( j \)-invariant of \( \phi \) by \( j = j(\phi) = g^{q+1}/\Delta \), and one verifies easily that two Drinfeld modules \( \phi \) and \( \phi' \) are isomorphic if and only if \( j(\phi) = j(\phi') \). Moreover, \( \phi \) is isomorphic to a Drinfeld module defined over \( k(j) \).

The \( j \)-invariant induces a bijection between the set of isomorphism classes of Drinfeld modules and \( A^1(C) \), hence we view \( A^n_C \) as the moduli space of \( n \)-tuples of Drinfeld modules. A point \( x = (x_1, \ldots, x_n) \in A^n_C \) is called a CM point if each \( x_i \) is the \( j \)-invariant of a CM Drinfeld module. As in the classical case, CM points have remarkable arithmetical properties (see \([13]\)):

**Theorem 3 (CM Theory)** Let \( \phi \) be a CM Drinfeld module, with \( \mathcal{O} = \text{End}(\phi) \) an order of conductor \( f \) in the CM field \( K = k(\sqrt{\Delta}) \). Then \( j = j(\phi) \in k^{sep} \) is integral over \( A \), and \( K(j) \) is the ring class field of \( \mathcal{O} \). This means that \( K(j)/K \) is unramified outside \( f \), split completely at the unique place \( \infty \) of \( K \), and \( \text{Gal}(K(j)/K) \cong \text{Pic}(\mathcal{O}) \) via class-field theory.

In particular, suppose that \( p \in A \) is a prime which splits in \( K/k \) and does not divide \( f \), we say that \( p \) splits in \( \mathcal{O} \). Let \( \sigma_p = (p, K(j)/K) \) be the associated Frobenius element. Then there is a cyclic isogeny of degree \( p \) from \( \phi \) to \( \phi^{\sigma_p} \).

### 1.2 Drinfeld modular curves

Let \( \Omega = C \setminus k_\infty \) denote Drinfeld’s upper half-plane, which plays the role of \( \mathbb{H}^+ \) in the classical case. The group \( \text{PGL}_2(k_\infty) \) acts on \( \Omega \) by fractional linear transformations, but unlike the classical case, this action is not transitive, as \( C \) has infinite dimension over \( k_\infty \). A point \( z \in \Omega \) is called quadratic if \( [k_\infty(z) : k_\infty] = 2 \), in which case the stabilizer of \( z \) in \( \text{PGL}_2(k_\infty) \) is a one dimensional Lie group over \( k_\infty \). Otherwise we call \( z \) non-quadratic, and its stabilizer is trivial.

Similarly to the classical case, the quotients of \( \Omega \) by congruence subgroups of \( \text{PGL}_2(A) \) give rise to affine Drinfeld modular curves, which may be compactified by adding finitely many cusps. See \([15]\) for details.

A point \( z \in \Omega \) gives rise to a Drinfeld module \( \phi^z \) associated to the lattice \( \langle 1, z \rangle \). Notice that \( \phi^z \) has CM if and only if \( [k(z) : k] = 2 \). The \( j \)-invariant induces a rigid analytic isomorphism \( j : \text{PGL}_2(A) \setminus \Omega \to A^1(C) \).

We will here define three special Drinfeld modular curves. Let \( N \in A \) and set

\[
\Gamma(N) = \{ \gamma \in \text{GL}_2(A) \mid \gamma \equiv 1 \text{ mod } N \}/Z(F_q^*) \subset \text{PGL}_2(A),
\]

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(A) \mid c \equiv 0 \text{ mod } N \right\}/Z(F_q^*) \subset \text{PGL}_2(A),
\]

\[
\Gamma_2(N) = \{ \gamma \in \text{GL}_2(A) \mid (\gamma \text{ mod } N) \in Z((A/NA)^*) \}/Z(F_q^*) \subset \text{PGL}_2(A).
\]
Here $Z(R^*) \cong R^*$ denotes the subgroup of scalar matrices in $GL_2(R)$, for a ring $R$. Then we define the curves\footnote{There also exist Drinfeld modular curves denoted by $Y_1(N)$, which is why we introduce the notation $Y_2(N)$.}

$$Y(N) = \Gamma(N) \setminus \Omega, \quad Y_0(N) = \Gamma_0(N) \setminus \Omega, \quad Y_2(N) = \Gamma_2(N) \setminus \Omega.$$ 

The curve $Y_0(N)$ is the coarse moduli space parameterizing isomorphism classes of triples $(\phi, \phi', f)$, where $\phi$ and $\phi'$ are Drinfeld modules and $f : \phi \to \phi'$ is a cyclic isogeny of degree $N$. We have $Y(1) = Y_0(1) = Y_2(1) = A^1$.

**Proposition 1.1** The curve $Y_2(N)$ covers $Y_0(N)$ and is Galois over $Y(1)$. Suppose that $N$ is square-free and that every prime factor of $N$ has even degree. Then $\text{Gal}(Y_2(N)/Y(1)) \cong \text{PSL}_2(A/NA)$.

Notice that we use the definition

$$\text{PSL}_2(R) := SL_2(R)/(Z(R^* \cap SL_2(R)) \cong SL_2(R)/\{x \in R^* \mid x^2 = 1\}$$

for a ring $R$.

**Proof.** Firstly, it is clear that $Y_2(N)$ covers $Y_0(N)$, as $\Gamma_2(N) \subset \Gamma_0(N)$.

We note that $Y(N)/Y(1)$ is Galois \cite{14} with Galois group

$$\text{Gal}(Y(N)/Y(1)) \cong G(N)/Z(F_q^*),$$

where we have set

$$G(N) = \{\alpha \in GL_2(A/NA) \mid \det(\alpha) \in F_q^*\}.$$

Next, we consider the coverings $Y(N) \to Y_2(N) \to Y(1)$. Here $Y_2(N)$ corresponds to the normal subgroup $H(N)/Z(F_q^*)$ of $\text{Gal}(Y(N)/Y(1))$, where

$$H(N) = Z((A/NA)^*) \cap G(N).$$

Hence $Y_2(N)$ is Galois over $Y(1)$, with Galois group

$$\text{Gal}(Y_2(N)/Y(1)) \cong G(N)/H(N) \subset \text{PGL}_2(A/NA).$$

This is the subgroup of $\text{PGL}_2(A/NA)$ of those elements with determinant in $F_q^*$. As $N$ is square-free, composed of prime factors of even degree, it follows that every $\alpha \in F_q^*$ is a square in $A/NA$. Hence $G(N)/H(N) = \text{PSL}_2(A/NA)$. \hfill $\square$

### 1.3 Pure modular curves in $A^n$

For a subset $I \subset \{1, \ldots, n\}$ we denote by $p_I : \mathbb{A}^n \to \mathbb{A}^I$ the projection onto the coordinates listed in $I$. We also write $p_{i,j} = p_{(i,j)}$.

We map $Y_0(N)$ into $\mathbb{A}^2$ by sending the pair $(\phi, \phi')$ of isogenous Drinfeld modules to the point $(j(\phi), j(\phi'))$. The image, which we denote by $Y'_0(N)$, is the locus of an irreducible polynomial $\Phi_N(t_1, t_2) \in A[t_1, t_2]$ (see \cite{J}). This polynomial is symmetrical and of degree $\psi(N) = |N| \prod_{p|N} (1 + |p|^{-1})$ in $t_1$ and $t_2$. \hfill $\square$
Two points \( z_1, z_2 \in \Omega \) correspond to isogenous Drinfeld modules \( \phi^{z_1} \) and \( \phi^{z_2} \) if and only if \( z_1 = \sigma(z_2) \) for some \( \sigma \in \text{PGL}_2(k) \). In this case, \( \phi^{z_1} \) and \( \phi^{z_2} \) are linked by a cyclic isogeny of degree \( N = \det(m\sigma) \), where \( m \in A \) is chosen such that the entries of \( m\sigma \) are in \( A \) and have no factor in common. We call this \( N \) the degree of \( \sigma \) (it is unique up to the square of an element of \( \mathbb{P}^* \)).

Let \( (\sigma_1, \ldots, \sigma_n) \in \text{PGL}_2(k)^n \) and consider the map

\[
\rho : \Omega \rightarrow \mathbb{A}^n(C) \quad z \mapsto (j(\sigma_1(z)), \ldots, j(\sigma_n(z))).
\]

The image lies on an irreducible algebraic curve \( Y \subset \mathbb{A}^n \), which we call a pure modular curve. When \( n = 2 \), we obtain again the curve \( Y'_0(N) \), where \( N \) is the degree of \( \sigma_2\sigma_1^{-1} \). We note that a curve \( Y \subset \mathbb{A}^n \) is a pure modular curve if and only if \( p_{i,j}(Y) = Y'_0(N_{i,j}) \) for some \( N_{i,j} \in A \) and every pair of coordinates \( i \neq j \).

Clearly \( (\sigma_1, \ldots, \sigma_n) \) and \( (\sigma'_1, \ldots, \sigma'_n) \) in \( \text{PGL}_2(k)^n \) define the same modular curve via the map \( \rho \) if and only if there exist \( \sigma \in \text{PGL}_2(k) \) and \((\gamma_1, \ldots, \gamma_n) \in \text{PGL}_2(A)^n \) such that \( \gamma_i\sigma\gamma = \sigma'_i \) for \( i = 1, \ldots, n \). So the set of pure modular curves in \( \mathbb{A}^n \) is in bijection with the double cosets

\[
(\text{PGL}_2(A) \setminus \text{PGL}_2(k))^n / \text{PGL}_2(k),
\]

where the \( \text{PGL}_2(k) \) acts diagonally on \( \text{PGL}_2(k)^n \). We may transpose the actions, and write \( \text{PGL}_2(k)^n \) as the restricted product, over all primes \( p \in A \), of

\[
\text{PGL}_2(k_p) / (\text{PGL}_2(k_p) / \text{PGL}_2(A_p))^n,
\]

where \( k_p \) and \( A_p \) denote the completions at \( p \) of \( k \) and \( A \), respectively. Now each \( \text{PGL}_2(k_p) / \text{PGL}_2(A_p) \) may be identified with the Bruhat-Tits tree \( \mathcal{T}_p \) of \( \text{PGL}_2(k_p) \) (see \[24\]).

We recall that the vertices of \( \mathcal{T}_p \) correspond to homothety classes of \( A_p \)-lattices in the vector space \( k_p^2 \), and two vertices \( v_1 \) and \( v_2 \) are adjacent if we may find representative lattices \( L_i \in v_i \) such that \( L_1/L_2 \cong A_p/pA_p \). So a pure modular curve \( Y \) corresponds to an \( n \)-tuple \( (v_{p,1}, \ldots, v_{p,n}) \) of vertices of \( \mathcal{T}_p \), up to \( \text{PGL}_2(k_p) \)-action, for every prime \( p \in A \), with the condition that the vertices \( v_{p,1}, \ldots, v_{p,n} \) coincide for almost all \( p \).

We now give a special description of the pure modular curves in \( \mathbb{A}^3 \). An end of \( \mathcal{T}_p \) is an equivalence class of infinite paths \( v_1 \to v_2 \cdots \) of distinct vertices in \( \mathcal{T}_p \), two such paths being equivalent if they differ by a finite subgraph. The set of ends of \( \mathcal{T}_p \) is in a natural bijection with \( \mathbb{P}^1(k_p) \).

Given a triple of distinct ends \( (E_1, E_2, E_3) \) we may define a unique vertex \( v_c \) of \( \mathcal{T}_p \), called the center of \( (E_1, E_2, E_3) \), such that each \( E_i \) is represented by a path starting at \( v_c \). Similarly, for a triple of (not necessarily distinct) vertices \( (v_1, v_2, v_3) \), we may define the center \( v_c \) as the unique vertex such that there exist disjoint paths from \( v_c \) to each \( v_i \). As the action of \( \text{PGL}_2(k_p) \) on \( \mathbb{P}^1(k_p) \) is 3-transitive, we may map any given triple \( (E_1, E_2, E_3) \) of distinct ends with center \( v_c \) to any other given triple \( (E'_1, E'_2, E'_3) \) with center \( v'_c \) via an element of \( \text{PGL}_2(k_p) \). It follows that the \( \text{PGL}_2(k_p) \)-class of triples of vertices \( (v_1, v_2, v_3) \) of \( \mathcal{T}_p \) is uniquely determined by the triple \( (n_1, n_2, n_3) \) of distances from the vertices \( v_i \) to the center of \( (v_1, v_2, v_3) \).

Let \( Y \subset \mathbb{A}^3 \) be a pure modular curve, corresponding to triples of vertices \( (v_{p,1}, v_{p,2}, v_{p,3}) \) in \( \mathcal{T}_p \) for each prime \( p \in A \). To each triple we associate the triple of non-negative integers \( (n_{p,1}, n_{p,2}, n_{p,3}) \) of distances to the center, as above. We set \( N_i = \prod_{p \in A} p^{n_{p,i}} \in A \) for \( i = 1, 2, 3 \). Then we have shown that the modular curve \( Y \) is uniquely determined by the triple \( (N_1, N_2, N_3) \). We also see that \( p_{i,j}(Y) = Y'_0(N_iN_j) \) for every pair of coordinates \( 1 \leq i < j \leq 3 \).
For $n \geq 4$ such a combinatorial description of pure modular curves becomes more complicated, as the $\text{PGL}_2(k_p)$-action is not $n$-transitive.

1.4 Modular varieties in $\mathbb{A}^n$

Let $\pi \in S_n$ be a permutation on $n$ letters, then $\pi$ acts as a permutation of coordinates on $\mathbb{A}^n$.

**Definition 1.4** An irreducible algebraic variety $X$ in $\mathbb{A}^n$ is said to be a modular variety if it is isomorphic, via some permutation of coordinates $\pi \in S_n$, to a variety of the form

$$\mathbb{A}^{n_0} \times \prod_{i=1}^{g} Y_i \times \{x\}$$

where each $Y_i$ is a pure modular curve in $\mathbb{A}^{n_i}$ and $x$ is a CM point in $\mathbb{A}^{n_{g+1}}$, and $n = n_0 + \cdots + n_{g+1}$. The data

$$(\pi, n_0, Y_1, \ldots, Y_g)$$

is called the type of $X$.

A reducible variety is modular if all its irreducible components are modular, and its type is the set of types of the irreducible components.

A modular variety is pure if all of the projections $p_i : X \to \mathbb{A}^1$ are dominant on every irreducible component of $X$.

Thus a modular curve $X$ in $\mathbb{A}^n$ is a modular variety of dimension one, i.e. either a pure modular curve, or the product of a pure modular curve in $\mathbb{A}^m$ and a CM point in $\mathbb{A}^{n-m}$.

In the more general case, let $Z = \prod_{i=1}^{n} X_i$ be a product of Drinfeld modular curves $X_i (= \text{compactification of } \Gamma_i \less \Omega)$, where the $\Gamma_i$’s are congruence subgroups of $\text{PGL}_2(A)$). A point $x = (x_1, \ldots, x_n)$ in $Z$ is a CM point if each $x_i$ corresponds to a CM Drinfeld module (with $\Gamma_i$-level structure, of course).

A special curve in $Z$ is the set of points represented by $(\sigma_1(z), \ldots, \sigma_n(z)) \in \Omega^n$ for some $(\sigma_1, \ldots, \sigma_n) \in \text{PGL}_2(k)^n$ and all $z \in \Omega$.

**Definition 1.6** An irreducible subvariety $X$ of $Z$ is modular if there is a partition $\{1, \ldots, n\} = \coprod_{i=0}^{g+1} S_i$, and $X$ is given by

$$X = \prod_{i \in S_0} X_i \times \prod_{j=1}^{g} Y_j \times \{x\}$$

where each $Y_j$ is a special curve in $\prod_{i \in S_j} X_i$ and $x$ is a CM point in $\prod_{i \in S_{g+1}} X_i$. As before, a reducible subvariety is modular if all its irreducible components are modular, and is pure if the projections $p_i : X \to X_i$ are dominant for each $i$ from every irreducible component of $X$.

But as level structures play no role in these phenomena, any result concerning $\mathbb{A}^n$ automatically implies the corresponding result for $Z = X_1 \times \cdots \times X_n$.

2 Hecke operators

Throughout this section, $m$ denotes a monic square-free element of $A$. 

7
2.1 Hecke operators and Hecke orbits

Definition 2.1 The Hecke operator $T_{\mathbb{A}^n,m}$ on $\mathbb{A}^n$ is the correspondence given by the image of

$$Y_0^m(m)^n \rightarrow \mathbb{A}^n \times \mathbb{A}^n$$

$$((x_1, y_1), \ldots, (x_n, y_n)) \mapsto ((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)).$$

We may also view $T_{\mathbb{A}^n,m}$ as a map from subsets of $\mathbb{A}^n$ to subsets of $\mathbb{A}^n$, generated by its action on single points:

$$T_{\mathbb{A}^n,m} : (x_1, \ldots, x_n) \mapsto \{(y_1, \ldots, y_n) \mid \text{There exist cyclic isogenies } x_i \rightarrow y_i \text{ of degree } m \text{ for all } i = 1, \ldots, n \}.$$

We also use the notation $T_m$ when the $\mathbb{A}^n$ is clear. We notice that the operator $T_m$ is symmetric, in the sense that $x \in T_m(y) \iff y \in T_m(x)$, and that $T_m$ is defined over $k$.

Let $X = \bigcup_{i=1}^r X_i$ be a variety in $\mathbb{A}^n$, with irreducible components $X_1, \ldots, X_r$. Then $T_m(X)$ is also a variety in $\mathbb{A}^n$, and $T_m(X) = \bigcup_{i=1}^r T_m(X_i)$.

Definition 2.2 Let $X$ be a variety in $\mathbb{A}^n$, and suppose all of its irreducible components have the same dimension. If $X \subset T_{\mathbb{A}^n,m}(X)$, then we say that $X$ is stabilized by $T_m$, and we define the Hecke operator restricted to $X$ by

$$T_{X,m} := \text{union of components of } T_{\mathbb{A}^n,m} \cap (X \times X) \text{ of maximal dimension}.$$ 

Whenever we use the notation $T_{X,m}$, then it is implicit that $X$ is stabilized by $T_{\mathbb{A}^n,m}$. The correspondence $T_{X,m}$ is still surjective in the sense that the two projections $p_X : T_{X,m} \rightarrow X$ are surjective. We may compose Hecke correspondences, and we have the standard property $T_{m_1} \circ T_{m_2} = T_{m_1,m_2}$ for $m_1, m_2 \in A$ relatively prime.

Definition 2.3 Let $X \subset \mathbb{A}^n$ be a variety (possibly $X = \mathbb{A}^n$), and $S \subset X$ a subset. Then the Hecke orbit of $S$ under $T_{X,m}$ is given by

$$T_{X,m}^\infty(S) = \bigcup_{d=1}^\infty T_{X,m}^d(S)$$

(Here $T_{X,m}^d$ means $T_{X,m}$ iterated $d$ times.)

As there are only finitely many correspondences on the finite set of irreducible components of $X$, we may decompose $X$ into a finite disjoint union of Hecke orbits, each orbit being generated by each of its irreducible components. If $S \subset X_i$ is Zariski-dense, then $T_{X,m}^\infty(S)$ is Zariski-dense in all of $T_{X,m}^\infty(X_i)$.

Let $x \in \mathbb{A}^1(C)$, and suppose that $x \in T_{\mathbb{A}^1,m}(x)$. Then $x$ has a cyclic endomorphism, hence is a CM point. For fixed $m \in A$ there are only finitely many such stable points for $T_{\mathbb{A}^1,m}$, namely the roots of the polynomial $\Phi_m(t, t)$. On the other hand, let $x \in \mathbb{A}^1(C)$ be a given CM point with $\mathcal{O} = \text{End}(x)$ an order of conductor $f$ in the CM field $K$. Then for every prime $p \in A$, which does not divide $f$ and decomposes into two principal primes of $K$, we have $x \in T_{\mathbb{A}^1,p}(x)$. These are precisely the primes which split completely in the ring class field of $\mathcal{O}$, hence, by Čebotarev, they have density at least $1/2 \# \text{Pic}(\mathcal{O})$. 
2.2 Some intersection theory

We define the degree of an irreducible variety $X \subset A^d$ of dimension $d$ as the number of points in the intersection of $X$ with a generic linear subspace of codimension $d$ in $A^n$. If $X$ is not irreducible, then we define its degree to be the sum of the degrees of its irreducible components of maximal dimension.

We have the following properties, which are easily verified.

**Proposition 2.4** Let $X \subset A^n$ be a variety of dimension $d$.

1. $X$ has at most $\deg(X)$ irreducible components of maximal dimension.
2. (Bézout) If $Y \subset A^n$ is another variety, then $\deg(X \cap Y) \leq \deg(X) \deg(Y)$.
3. $\psi(m) \leq \deg(Y_i^d(m)) \leq 2\psi(m)$.
4. $\deg(T_{k^n,m}(X)) \leq 2^n \psi(m)^n \deg(X)$.

**Proposition 2.5** Let $B > 0$ and $n \in \mathbb{N}$ be given. Then there are only finitely many different types (recall Definition 1.4) of modular varieties $X \subset A^n$ with $\deg(X) \leq B$.

**Proof.** It suffices to show that there are only finitely many pure modular curves $Y \subset A^n$ with degree less than a given bound. Let $p_{\{i,i+1\}}(Y) = Y_i^d(N_i)$ for $i = 1, \ldots, n - 1$. Now $\deg(Y) \geq \deg(p_{\{i,i+1\}}(Y)) \geq \psi(N_i)$ for all $i$. But $\psi(N_i) \to \infty$ as $N_i$ varies, and the result follows. \qed

2.3 Preimages in $\Omega^n$

We have a rigid analytic map $\pi = (j \times \cdots \times j) : \Omega^n \to A^n(C)$. For each irreducible component $X_i$ of $X$ we choose an irreducible component $Z_i$ of the rigid analytic variety $\pi^{-1}(X_i) \subset \Omega^n$. We set $Z = \bigcup_i Z_i$. The group $\text{PGL}_2(k^\infty)^n$ acts on $\Omega^n$ and the $\text{PGL}_2(A)^n$-orbit of $Z_i$ is all of $\pi^{-1}(X_i)$. We want to describe the Hecke operators acting in the space $\Omega^n$.

For the following discussion of matrices, see [3]. We let $\Delta_m^*$ denote the set of $2 \times 2$ matrices over $A$ with determinant in $F_m^* \mathfrak{m}$ and whose entries have no factor in common. Then $\text{GL}_2(A)$ acts from the right on $\Delta_m^*$ and the representatives of $\Delta_m^*/\text{GL}_2(A)$ may be chosen of the form

$$t_i = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix}, \quad i \in \mathcal{I} := \{1, \ldots, \psi(m)\}$$

where $a_i, d_i$ are monic, $a_i d_i = m$ and $|b_i| < |a_i|$. For $i = (i_1, \ldots, i_n)$ ranging through $\mathcal{T}^n$ we denote by $t_i = (t_{i_1}, \ldots, t_{i_n})$ the resulting representatives of $(\Delta_m^*)/\text{GL}_2(A)^n$.

For each $Z_i$ we define $\mathcal{J}_{Z_i} \subset \mathcal{T}^n$ as the set of those indices $j$ for which $t_j(Z_i) \subset \pi^{-1}(X)$. Let $x \in X_i(C)$, and choose some $z \in Z_i$ with $\pi(z) = x$. Then the action of $T_{k^n,m}$ and $T_{X,m}$ on $x$ are given by $T_{k^n,m}(x) = \{\pi(t_j(z)) \mid j \in \mathcal{T}^n\}$ and $T_{X,m}(x) = \{\pi(t_j(z)) \mid j \in \mathcal{J}_{Z_i}\}$. In particular, $\mathcal{J}_{Z_i}$ is non-empty, as $T_{X,m}$ is a surjective correspondence.
2.4 Surjectivity of projections

Theorem 4 Let \( X \subseteq \mathbb{A}^n \) be a variety all of whose irreducible components have the same dimension, and suppose that \( X \subseteq T_m(X) \) for some square-free \( m \in A \) which is a product of distinct primes \( p \in A \) of even degree satisfying \( |p| \geq \max(13, \deg X) \). Let \( X_i \) be an irreducible component of \( X \) for which the projection \( p_I : X_i \to \mathbb{A}^1 \) is dominant. Then the projection

\[
(2.6) \quad p_I : J_{Z_i} \to T^I
\]

is surjective. In particular, let \( x \in X_i(C) \). Then the projection of finite sets

\[
(2.7) \quad p_I : T_{X,m}(x) \to T_{A^I,m}(p_I(x))
\]

is surjective.

Proof. Clearly, the surjectivity of (2.7) follows from the surjectivity of (2.6), which in turn follows from the surjectivity of (2.7) for a generic point \( x \in X_i \).

So we suppose \( x \in X_i \) is generic. Denote by \( T_{X,m,i} = T_{X,m} \cap (X_i \times X) \) the restriction of the source of the Hecke correspondence \( T_{X,m} \) to the component \( X_i \). Consider the following diagram

\[
\begin{array}{c}
\begin{array}{ccc}
T_{X,m,i} & \xrightarrow{p_I \times p_I} & T_{A^I,m} \\
\downarrow{p_{X,i}} & & \downarrow{p_{A^I}} \\
X_i & \xrightarrow{f_i} & T_{A^I,m} \\
\downarrow{p_I} & & \downarrow{p_I} \\
\mathbb{A}^I,
\end{array}
\end{array}
\]

where the vertical arrows are projections onto the sources of the respective correspondences, and the horizontal arrows are projections onto the coordinates in \( I \). There exists a canonical map \( f_i \) from \( T_{X,m,i} \) to the fibered product \( X_i \times_{A^I} T_{A^I,m} \), which is generically finite, as \( p_{X,i} : T_{X,m,i} \to X_i \) is generically finite. Clearly \( X_i \times_{A^I} T_{A^I,m} \) has the same dimension as \( T_{X,m} \) and \( X \), and it is irreducible, as we will show below. It follows that \( f_i \) is dominant.

Now let \( x_I = p_I(x) \), and let \( y_I \in T_{A^I,m}(x_I) \). Then \( (x, (x_I, y_I)) \) is a generic point on the fibered product, hence has a preimage \((x, y)\) under \( f_i \). We see that \( y \in T_{X,m,i}(x) = T_{X,m}(x) \) and \( p_I(y) = y_I \), so we have shown that (2.7) is surjective.

It remains to show that \( X_i \times_{A^I} T_{A^I,m} \) is irreducible. This will follow if the function fields of \( X_i \) and \( T_{A^I,m} \cong (Y_0(m))^I \) over \( C \) are linearly disjoint over the function field of \( A^I \) over \( C \). Recall from Proposition 14 that the modular curve \( Y_2(m) \) covers \( Y_0(m) \) and is Galois over \( Y(1) = A^1 \) with Galois group

\[
\text{Gal}(Y_2(m)/Y(1)) \cong \text{PSL}_2(A/mA) \cong \prod_{p \mid m} \text{PSL}_2(A/pA).
\]

On the other hand, let \( L \) be an intermediate field \( C(A^I) \subset L \subset C(X_i) \) which is purely transcendental over \( C(A^I) \) and for which \( [C(X_i) : L] \leq \text{deg}(X_i) \leq \text{deg}(X) \) is finite. Then \( L \cap C(Y_2(m)^I) = C(A^I) \), and \( C(Y_2(m)^I) \) is Galois over \( C(A^I) \), so it follows that \( L \) and
\[ C(Y_2(m)^I) \] are linearly disjoint over \( C(\mathbb{A}^I) \). Denote by \( L_m \) the field \( L \otimes_{C(\mathbb{A}^I)} C(Y_2(m)^I) \). Then we have
\[
C(X_i) \otimes_{C(\mathbb{A}^I)} C(T_{\mathbb{A}^I,m}) \subset C(X_i) \otimes_{C(\mathbb{A}^I)} C(Y_2(m)^I) = C(X_i) \otimes_L L_m.
\]

But now \( L_m \) is Galois over \( L \), with group \( \text{Gal}(L_m/L) \cong \text{PSL}_2(A/mA)^I \). Moreover, as \( |p| \geq 13 \) for all \( p|m \), it follows that this group has no proper subgroup of index less than \( |p| + 1 \) (see for example [20], Hauptsatz 8.27 and Satz 8.28).

On the other hand, \( |C(X_i) : L| \leq \deg(X) < |p| + 1 \), so \( C(X_i) \cap L_m = C(\mathbb{A}^I) \). It follows that \( C(X_i) \) and \( L_m \) are linearly disjoint, hence \( X_i \times_{\mathbb{A}^I} T_{\mathbb{A}^I,m} \) is irreducible, as required. \( \square \)

For the next two corollaries, we assume \( X \subset \mathbb{A}^n \) is a variety, with irreducible components \( X_i, \ i = 1, \ldots, r \), which are all of the same dimension. We assume further that \( X \subset T_{\mathbb{A}^n,m}(X) \) for some square-free \( m \in A \), composed of distinct primes \( p \in A \), each of even degree and satisfying \( |p| \geq \max(13, \deg X) \).

**Corollary 2.8** Suppose that the projection \( p_1 : X_i \to \mathbb{A}^1 \) onto the first coordinate is dominant for all \( i = 1, \ldots, r \). Let \( x_1 \in \mathbb{A}^1 \) such that \( x_1 \in T_{\mathbb{A}^1,m}(x_1) \). Let \( X_{x_1} = X \cap \{x_1\} \times \mathbb{A}^{n-1} \).

Then \( X_{x_1} \subset T_{X,m}(X_{x_1}) \).

**Proof.** Let \( x \in X_{x_1} \). Then setting \( I = \{1\} \) in Theorem 4 we see that
\[
p_1 : T_{X,m}(x) \to T_{\mathbb{A}^1,m}(x_1)
\]
is surjective. Let \( y \in T_{X,m}(x) \) be a preimage of \( x_1 \in T_{\mathbb{A}^1,m}(x_1) \). Then \( y \in X_{x_1} \) and \( x \in T_{X,m}(y) \), hence \( x \in T_{X,m}(X_{x_1}) \), as required. \( \square \)

**Corollary 2.9** Let \( x \in X_i \). Then the Hecke orbit \( T_{X,m}^\infty(x) \) is Zariski-dense in the Hecke orbit \( T_{X,m}^\infty(X_i) \).

**Proof.** Clearly, we may suppose that \( \dim(X) \geq 1 \). Let \( I \subset \{1, \ldots, n\} \) be such that \( \#I = \dim(X) \) and the projection \( p_I : X_i \to \mathbb{A}^I \) is dominant.

We claim that \( p_I : X_j \to \mathbb{A}^I \) is also dominant for every irreducible component \( X_j \) of \( T_{X,m}^\infty(X_i) \). By induction, it suffices to prove the claim for \( X_j \subset T_{X,m}(X_i) \).

Let \( x_I \in \mathbb{A}^I \) be a generic point. Then there is some \( x \in X \) with \( p_I(x) = x_I \). At least one point \( y \in T_{X,m}(x) \) lies on \( X_j \), and \( p_I(y) = y_I \in T_{\mathbb{A}^I,m}(x_I) \). So it follows that every generic \( x_I \in \mathbb{A}^I \) is m-isogenous to some \( y_I \) coming from \( X_j \), in other words, \( T_{\mathbb{A}^I,m}(p_I(X_j)) \) is Zariski-dense in \( \mathbb{A}^I \). It follows that \( \dim(p_I(X_j)) = \dim(T_{\mathbb{A}^I,m}(p_I(X_j))) = \dim(\mathbb{A}^I) \), and so \( p_I(X_j) \) is Zariski-dense in \( \mathbb{A}^I \), which proves the claim.

Now we apply Theorem 4 to obtain a surjection
\[
p_I : T_{X,m}^\infty(x) \to T_{\mathbb{A}^I,m}^\infty(p_I(x)).
\]
This last set is Zariski-dense in \( \mathbb{A}^I \), as \( T_{\mathbb{A}^I,m}^\infty(p_I(x)) = \prod_{j \in I} T_{\mathbb{A}^I,m}^\infty(x_j) \) is a product of infinite subsets of \( \mathbb{A}^I \). As the projection \( p_I : X \to \mathbb{A}^I \) is generically finite, it follows that \( T_{X,m}^\infty(x) \) must be Zariski-dense on at least one component \( X_j \) of \( T_{X,m}^\infty(X_i) \), hence on all of \( T_{X,m}^\infty(X_i) \). \( \square \)
Remark 2.10 As $T_m$ is defined over $k$, we may replace the word “irreducible” by “$F$-irreducible” everywhere in the preceding sections, for any field $F \supset k$ over which the relevant varieties are defined. In particular, it follows from Corollary 2.9 above, that if $X$ is a variety defined over $F$, $X_i$ is an $F$-irreducible component of $X$, and $x \in X_i$, then the Hecke orbit $T_{X,m}(x)$ is Zariski-dense on $X_i$.

2.5 Curves stabilized by Hecke operators

We are now ready to prove a fundamental result: a characterization of the modular curves $Y'_0(N)$ in terms of Hecke operators.

Theorem 5 Let $X \subset \mathbb{A}^2$ be an irreducible algebraic curve, and suppose $X \subset T_{k^2,m}(X)$ for some square-free $m \in A$, $|m| > 1$, composed of primes $p \in A$ of even degree satisfying $|p| \geq \max(13, \deg X)$. Then $X = Y'_0(N)$ for some $N \in A$.

The proof will occupy the next three sections.

If $X = \{x\} \times \mathbb{A}^1$ or $X = \mathbb{A}^1 \times \{x\}$, then $x$ is a CM point (as it is stabilized by $T_{k^2,m}$), and so $X$ is modular. So we may assume that the projections $p_i : X \to \mathbb{A}^1$ are dominant, and have degree $1 \leq d_i \leq \deg(X)$, for $i = 1, 2$.

The group $G := \text{PGL}_2(k_\infty)^2$ acts on $\Omega^2$, and we also define the following groups: $S := \text{PSL}_2(k_\infty)^2$, $\Gamma := \text{PGL}_2(A)^2$, and $\Sigma := \text{PSL}_2(A)^2$. As before, we choose an irreducible component $Z$ of the rigid analytic variety $\pi^{-1}(X)$. Let $G_Z$ be the stabilizer of $Z$ under the action of $G$, it is a closed analytic subgroup of $G$. We also define $S_Z := G_Z \cap S$, $\Gamma_Z := G_Z \cap \Gamma$, and $\Sigma_Z := G_Z \cap \Sigma$. Our aim is to investigate the structure of $S_Z$, under the hypothesis that $X \subset T_m(X)$, and hence conclude that $X$ must be a modular curve.

So our whole approach is similar to that of Edixhoven [S], but with slightly different details, for example the action of $G$ on $\Omega^2$ is not transitive, the topology is ultrametric, and Lie theory works a bit differently in characteristic $p$, so we replace it by explicit calculations.

We denote by $p_{ri} : G \to \text{PGL}_2(k_\infty)$ the two projections, $i = 1, 2$. The following lemma holds for an arbitrary curve $X$ (with non-constant projections).

Lemma 2.11

1. The two projections $p_{ri} : G_Z \to \text{PGL}_2(k_\infty)$ are injective.

2. $p_{ri}(\Gamma_Z)$ has index at most $d_i$ in $\text{PGL}_2(A)$.

Proof. (1) Let $K = \ker(p_{r2} : G_Z \to \text{PGL}_2(k_\infty))$. Then $K$ is the stabilizer of $Z$ in $\text{PGL}_2(k_\infty)^2 \times \{1\}$, and stabilizes $Z_z = Z \cap (\Omega \times \{z\})$, for any $z \in \Omega$. But $Z_z$ is discrete, and we may choose $z$ in such a way that $Z_z$ contains a non-quadratic element, whose stabilizer is trivial, so it follows that $K$ is discrete. Now $K \triangleleft \text{PGL}_2(k_\infty)^2 \times \{1\}$, which has no non-trivial discrete normal subgroups, thus $K = \{1\}$. The same holds for the other projection.

(2) We factor the map $\pi$ as follows:

$$
\begin{array}{ccc}
\Omega \times \Omega & \xrightarrow{\pi_1} & \mathbb{A}^1 \times \Omega \\
\downarrow & & \downarrow \\
Z & \xrightarrow{p_{r2}} & W \\
& \downarrow & \downarrow \\
& \mathbb{A}^1 \times \mathbb{A}^1 & \xrightarrow{\pi_2} & X
\end{array}
$$
Here $W = \pi_1(Z)$ is an irreducible component of $Y = \pi_2^{-1}(X) = \text{PGL}_2(A) \cdot W$. Let $S$ be the set of all $x$’s in $X$ for which every (equivalently at least one) point of $\pi^{-1}(x)$ lies in more than one component of $\pi^{-1}(X)$. Then $S$ lies in the finite set consisting of the singular points of $X$ as well as those with at least one coordinate equal to 0.

Let $X' = X - S$, and let $Z'$ and $W'$ be the corresponding preimages, then the map $\pi : Z' \to X'$ is a quotient for the action of $\Gamma_Z$, and $\pi_2 : W' \to X'$ is a quotient for the action of $\text{pr}_2(\Gamma_Z)$. It follows that $\text{pr}_2(\Gamma_Z)$ is the stabilizer of $W$ for the action of $\text{PGL}_2(A)$, hence the irreducible components of $Y$ correspond to the cosets $\text{PGL}_2(A)/\text{pr}_2(\Gamma_Z)$, so the index is the number of these components.

On the other hand, $Y = \pi_2^{-1}(X)$ is the fibered product of the maps $p_2 : X \to \mathbb{A}^1$ and $j : \Omega \to \mathbb{A}^1$, hence it has at most $d_2$ irreducible components. Again, the same holds for the other projection. \hfill \square

### 2.6 The structure of $S_Z$

Now we make use of the fact that $X \subset T_m(X)$. Let $i \in J_Z$, then $t_i(Z) \subset \pi^{-1}(X)$ (by definition; remember that $X$ is irreducible), and thus there is some $\gamma_i \in \Gamma$ such that $g_i := \gamma_i t_i \in G_Z$. Moreover, applying Theorem 4 to the projection $p_i : X \to \mathbb{A}^1$ we see that the projection $p_i : J_Z \to \mathcal{I} = \{1, \ldots, \psi(m)\}$ is surjective. This gives us many non-trivial elements in $G_Z$. More precisely, we will study the projections $H_i = \text{pr}_i(G_Z)$, and show that they each contain $\text{PSL}_2(k_\infty)$.

From Lemma 2.11 follows that $\text{PGL}_2(A) \cap H_1$ has finite index in $\text{PGL}_2(A)$. Let $R$ be a finite set of representatives of $\text{PGL}_2(A)/(\text{PGL}_2(A) \cap H_1)$. The group $\text{GL}_2(A)$ acts from the right on the set of left cosets $\text{GL}_2(A) \setminus \Delta^*_m$. We claim that for any string $i_1 \ldots i_n$ of elements in $\mathcal{I}$, and any $a \in \text{GL}_2(A)$, we can construct an element of the form $\gamma_{i_n} \cdots \gamma_{i_1} a$ in $H_1$, for some $\gamma \in R$ depending on the string and on $a$. Indeed, by induction we need only show that, given $a_1 \in \text{GL}_2(A)$ and $i_1 \in \mathcal{I}$, we can construct an element of the form $\gamma_{i_1} a_1$ in $H_1$. This element is constructed as follows. Let $a_1$ act from the right on the coset $\text{GL}_2(A) \cdot t_{i_1}$, to obtain another coset $\text{GL}_2(A) \cdot t_{i_1} a_1 = \text{GL}_2(A) \cdot t_j$. Then $t_{i_1} a_1 = \gamma_j t_j$, and multiplying on the left with a suitable element $\gamma_1$ of $R$ gives $\gamma_1 t_{i_1} a_1 = \gamma_1 t_{i_1} = \gamma_1 g_j \in H_1$, with $\gamma_1 \in H_1 \cap \text{PGL}_2(A)$. This proves the claim.

Multiplying by a suitable power of the scalar $m$, we see that for any $x \in A[1/m]$ and any $a \in \text{GL}_2(A)$, there exists $\gamma_{x,a} \in R$ such that $\gamma_{x,a} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} a \in H_1$.

The group $\text{PSL}_2(A[1/m])$ is generated by $\text{PSL}_2(A)$ and elements of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, hence for any $g \in \text{PSL}_2(A[1/m])$, we can construct an element $\gamma g \in H_1$, for some $\gamma \in R$, obtained by multiplying together suitable elements of the form $\gamma a \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} a \in H_1$. It follows that $H_1 \cap \text{PSL}_2(A[1/m])$ has finite index in $\text{PSL}_2(A[1/m])$, which is dense in $\text{PSL}_2(k_\infty)$.

**Lemma 2.12** $G_Z$ is not discrete

**Proof.** Assume that $G_Z$ is discrete. Choose a non-quadratic point $z = (z_1, z_2)$ in $Z$. Then its orbit $G_Z \cdot z$ is discrete in $Z$, so $\pi(G_Z \cdot z)$ is discrete in $X$, as $\Gamma_Z \subset G_Z$. Next, $p_1(\pi(G_Z \cdot z))$ is discrete in $\mathbb{A}^1$ (as $p_1 : X \to \mathbb{A}^1$ is finite), and thus $j^{-1}(p_1(\pi(G_Z \cdot z)))$ is discrete in $\Omega$. But
from above we see that this set contains the orbit \((H_1 \cap PSL_2(A[1/m])) \cdot z_1\), which is not discrete. This is a contradiction.

So we see that \(G_Z\) is a closed analytic subgroup of \(G\) which is not discrete, hence of dimension at least one. The projection \(pr_1 : S_Z \rightarrow PGL_2(k_{\infty})\) is injective, so we see that \(H_1 = A_1 \setminus B_1\), where \(A_1, B_1\) are analytic sets and \(\dim(A_1) = \dim(G_Z) > \dim(B_1)\). Thus, there exists a point \(x \in H_1\) and a closed (in the non-archimedean topology) neighborhood \(A_x\) of \(x\) in \(PGL_2(k_{\infty})\) such that \(A_x \cap H_1\) is closed in \(PGL_2(k_{\infty})\), i.e. \(H_1\) is locally closed at \(x\). Since \(H_1\) is a topological group, it is closed in \(PGL_2(k_{\infty})\).

Now, \(PSL_2(A[1/m])\) is dense in \(PSL_2(k_{\infty})\), so \(H_1 \cap PSL_2(k_{\infty})\) has finite index in \(PSL_2(k_{\infty})\), which is simple, hence \(PSL_2(k_{\infty}) \subset H_1\). In particular, \(H_1\) has finite index in \(PGL_2(k_{\infty})\). Of course, the same holds for \(H_2 = pr_2(G_Z)\).

Goursat’s lemma says that \(G_Z\) is of the form

\[
G_Z = \{(g, \rho(g)) \mid g \in H_1\}
\]

for some isomorphism \(\rho : H_1 \rightarrow H_2 \subset PGL_2(k_{\infty})\). Now, as \(PSL_2(k_{\infty})\) is simple and the image of \(\rho\) has finite index in \(PGL_2(k_{\infty})\), it follows that \(\rho\) restricts to an automorphism on \(PSL_2(k_{\infty})\). Thus we have shown

\[
G_Z \cap (PSL_2(k_{\infty}))^2 = S_Z = \{(g, \rho(g)) \mid g \in PSL_2(k_{\infty})\}
\]

for some \(\rho \in \text{Aut}(PSL_2(k_{\infty}))\).

Every automorphism of \(PSL_2(k_{\infty})\) is of the form \(g \mapsto hg^n h^{-1}\) for some \(h \in PGL_2(k_{\infty})\) and \(\sigma \in \text{Aut}(k_{\infty})\), see [19].

By the definition of \(\Sigma_Z\) and the structure of \(S_Z\), we see that \(h \cdot pr_1(\Sigma_Z)^\sigma \cdot h^{-1} \subset PSL_2(A)\). On the other hand, Lemma [2, 11] tells us that \(pr_1(\Sigma_Z)\) has finite index in \(PSL_2(A)\). This in turn severely restricts \(h\) and \(\sigma\):

**Proposition 2.13** Let \(G\) be a subgroup of finite index in \(PSL_2(A)\), and suppose that \(hG^n h^{-1} \subset PGL_2(k)\), for some \(h \in PGL_2(k_{\infty})\) and \(\sigma \in \text{Aut}(k_{\infty})\). Then \(h \in PGL_2(k)\) and \(\sigma(T) = uT + v\) for some \(u \in \mathbb{F}_q\), \(v \in \mathbb{F}_q\), and \(\sigma(\mathbb{F}_q) = \mathbb{F}_q\).

**Proof.** Firstly, let \(h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) and let \(r = \det(h)\). As \(k_{\infty}^n/k_{\infty}^2\) may be represented by \(\{1, \alpha, T, \alpha T\}\), for some non-square \(\alpha \in \mathbb{F}_q\), we may assume that \(r \in k\).

Denote by \(B_1 = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\) and \(B_2 = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\) the two Borel subgroups of \(PSL_2(A)\). The group \(G\) has finite index in \(PSL_2(A)\), so it follows that \(G \cap B_1\) and \(G \cap B_2\) are of finite index in \(B_1\) and \(B_2\), respectively. Hence

\[
A_0^+ := \{x \in A \mid \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in G\}
\]

has finite index in the additive group \(A^+\) of \(A\). Now for every \(x \in A_0^+\) we have

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 - \frac{ac}{r} \sigma(x) & \frac{bc}{r} \sigma(x) \\ -\frac{cd}{r} \sigma(x) & 1 + \frac{ac}{r} \sigma(x) \end{pmatrix} \in PGL_2(k),
\]
and it follows that

\[ a c \sigma(x), \ a^2 \sigma(x), \ c^2 \sigma(x) \in k. \]

Likewise, from \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \in \text{PGL}_2(k) \) follows

\[ b d \sigma(x), \ b^2 \sigma(x), \ d^2 \sigma(x) \in k. \]

From (2.14), (2.15) and \( ad - bc \in k \), we may deduce, in that order,

\[ \frac{a}{c}, \frac{b}{d}, \frac{c d}{b}, \frac{a b}{c d} \sigma(x), \frac{c^2}{d^2} \in k \]

from which follows that

\[ \sigma(x)^2 \in k \quad \forall x \in A_0^+. \]

Now as \( A_0^+ \) has finite index in \( A^+ \) it follows that there exists a pair \( x_1 \neq x_2 \in A_0^+ \) such that

\[ y = x_1^2 - x_2^2 \in A_0^+. \]

We have \( \sigma(y) = \sigma(x_1)^2 - \sigma(x_2)^2 \in k \). Substituting this \( y \) for \( x \) in (2.16) shows that in fact

\[ \frac{a}{b} \in k, \ \frac{c}{d} \in k \quad \text{and} \quad \frac{a}{b} = \frac{a}{d} \in k. \]

It follows firstly that \( h \in \text{PGL}_2(k) \), and secondly that \( \sigma(x) \in k \) for all \( x \in A_0^+ \). As the elements of \( A_0^+ \) generate \( k \) as a ring, \( \sigma(x) \in k \) for all \( x \in k \). It remains to characterize those automorphisms \( \sigma \) for which \( \sigma(k) \subset k \).

Let \( R = \mathbb{F}_q[[1/T]] = \{ x \in k_\infty \mid |x| \leq 1 \} \). Then \( R \) is the unique valuation ring of \( k_\infty \).

It is characterized by the property: \( x \in R \) or \( x^{-1} \in R \) for all \( x \in k_\infty \) and \( R \neq k_\infty \). This property must be preserved by \( \sigma \), so \( \sigma(R) \subset R \). So \( \sigma \) also preserves \( k \cap R = A \), and the only automorphisms that send polynomials to polynomials are of the form \( \sigma(T) = uT + v \), for some \( u \in \mathbb{F}_q^*, \ v \in \mathbb{F}_q \), and \( \sigma(\mathbb{F}_q) = \mathbb{F}_q \). \( \square \)

### 2.7 Completing the proof of Theorem 5

**Proof of Theorem 5.** From Proposition 2.18 follows that

\[ S_Z = \{(g, h g^\sigma h^{-1}) \mid g \in \text{PSL}_2(k_\infty)\}, \]

where \( h \in \text{PGL}_2(k) \), \( \sigma(T) = uT + v \) and \( \sigma(\mathbb{F}_q) = \mathbb{F}_q \). There is some \( t \in \mathbb{N} \) such that \( \sigma(\alpha) = \alpha^{p^t} \) for all \( \alpha \in \mathbb{F}_q \), as \( \sigma(\mathbb{F}_q) \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_p) \).

We let \( f = (T^q - T)^{p^t-1} \), then \( \sigma(f) = f \). Let \( F = \mathbb{F}_p((1/f)) \). This is a complete subfield of \( k_\infty \) and \( \sigma \) acts trivially on \( F \).

Now fix some non-square \( \alpha \in \mathbb{F}_q \), and define the set

\[ P = \{ z \in \Omega \mid z^2 = \alpha e, \ e \in F \}. \]

This is an uncountable subset of \( \Omega = \mathbb{C} \setminus k_\infty \), as \( \sqrt{\alpha} \notin k_\infty \).

Next, we notice that \( \sigma(\alpha e) = \alpha^{p^t} e = \beta^2 \alpha e \), where we set \( \beta = \alpha^{(p^t-1)/2} \in \mathbb{F}_q^* \) (remember that \( p \) is odd).

Let \( z_1 = \sqrt{\alpha e} \in P \) and

\[ S_1 = \text{Stab}_{\text{PSL}_2(F)}(z_1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a = d, \ b = c e, \ ad - bc = 1 \right\} / \{ \pm 1 \}, \]

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which is a one-dimensional Lie-group over \( F \).

Now let \( z_2 \in \Omega \) such that \( (z_1, z_2) \in Z \), and consider the “\( S_1 \)-orbit” of \( (z_1, z_2) \):

\[
\{(g(z_1), h g^a h^{-1}(z_2)) \mid g \in S_1 \} \subseteq Z \cap (\{z_1\} \times \Omega).
\]

This set is discrete, but the group \( S_1 \) is not, hence there exists some non-trivial \( g \in S_1 \) such that \( g \) fixes \( z_1 \) (by definition of \( S_1 \)) and \( h g^a h^{-1} \) fixes \( z_2 \). But \( g^a \) fixes the point \( z_1'' \) such that \( \sqrt{\sigma(ae)} = \beta z_1 \), so we see that \( h g^a h^{-1} \) fixes both \( z_2 \) and \( h(\beta z_1) = h'(z_1) \), where we have written \( h' = h \circ \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \in \text{PGL}_2(k) \). However, any non-trivial element of \( \text{PGL}_2(k) \) fixes at most two points of \( \Omega \), namely a conjugate pair of quadratic points. So \( z_2 \) and \( h'(z_1) \) are conjugate. So we get either \( (z_1, z_2) \) or \( (z_2, h'(z_1)) \) or \( (j(z_1), j(h'(z_1))) \) or \( (j(-z_1), j(h'(-z_1))) \) on the curve \( X \) in \( \mathbb{A}^2(\mathbb{C}) \).

Let \( N \) be the degree of \( h' \). Then we see that the points \( (j(z_1), j(h'(z_1))) \) and \( (j(-z_1), j(h'(-z_1))) \) lie on \( Y_0'(N) \) (which is independent of \( z_1 \)). We get such a point for each \( z_1 \in P \), and \( P \) is uncountable whereas the fibers of \( j \) are countable, so it follows that \( X(\mathbb{C}) \cap Y_0'(N)(\mathbb{C}) \) is infinite, hence \( X = Y_0'(N) \). This completes the proof of Theorem \( 5 \). \( \square \)

By considering various projections onto pairs of coordinates, we immediately get

**Corollary 2.19** Let \( X \subset \mathbb{A}^n \) be an irreducible algebraic curve, and suppose that \( X \subset \text{ker}(\text{m})_\infty(X) \) for some square-free \( m \in A \), \( |m| > 1 \), and composed of primes \( p \) of even degree and satisfying \( |p| \geq \max(13, \deg X) \). Then \( X \) is a modular curve.

### 2.8 Varieties stabilized by Hecke operators

In this section we generalize Theorem \( 5 \) to subvarieties of higher dimensions.

**Theorem 6** Let \( F \) be a field lying between \( k \) and \( \mathbb{C} \). Let \( X \subset \mathbb{A}^n \) be an \( F \)-irreducible variety, containing a CM point \( x \in X(\mathbb{C}) \). Suppose that \( X \subset \text{ker}(\text{m})_\infty(X) \) where \( m \in A \), \( |m| > 1 \) is square-free, composed of primes \( p \) of even degree and satisfying \( |p| \geq \max(13, \deg X) \). Then \( X \) is a modular variety.

**Proof.** We know from Corollary \( 2.19 \) and the subsequent Remark, that the Hecke orbit \( S = \text{ker}(\text{m})_\infty(x) \) is Zariski-dense in \( X \). In particular, it is Zariski-dense on every (geometrically) irreducible component. So now we assume that \( X \) is geometrically irreducible (but not necessarily stabilized by \( \text{ker}(\text{m})_\infty \) - indeed, all we need is a dense Hecke orbit of CM points). All the points in \( S \) are CM points, isogenous coordinate-wise to \( x \). As CM points are defined over \( k^{\text{sep}} \), so is \( X \). So we may also assume that \( X \) is defined over a finite Galois extension (again denoted \( F \)) of \( k \).

**Step 1.** Write \( x = (x_1, \ldots, x_n) \), and let \( O_i = \text{End}(x_i) \) be an order of conductor \( f_i \) in the imaginary quadratic field \( K_i \), for each \( i = 1, \ldots, n \). Set \( K = K_1 \cdots K_n \) and \( f = f_1 \cdots f_n \), and define

\[
P = \{ l \in A \mid \text{monic prime, of even degree, split completely in } FK \text{ and } l \nmid fm \}.
\]

This set is infinite (Čebotarev).
Let \( x' = (x'_1, \ldots, x'_n) \in S \), then each \( \mathcal{O}'_i = \text{End}(x'_i) \) is an order of conductor \( f'_i \) in \( K_i \) (the CM fields are the same, as \( x_1 \) and \( x'_1 \) are isogenous). Furthermore, all the prime factors of \( f'_i \) are factors of \( f_i \) and of \( \mathfrak{m} \). It follows that every \( l \in \mathcal{P} \) splits also in \( \mathcal{O}'_i \). Set \( M = K(x'_1, \ldots, x'_n) \) and let \( \mathfrak{L} \) be a prime of \( FM \) lying over \( l \). Denote by \( \mathfrak{L}_i \) the restriction of \( \mathfrak{L} \) to the field \( K_i(x'_i) \).

From CM Theory (Theorem 3) follows that \( l \) is unramified in \( M \), hence also in \( FM \). Let \( \sigma = (\mathfrak{L}, FM/k) \) be the Frobenius element. Set \( \sigma_i = \sigma|_{K_i(x'_i)} = (\mathfrak{L}_i, K_i(x'_i)/k) \). As \( l \) splits in \( K_i \), we have in fact \( \sigma_i = (\mathfrak{L}_i, K_i(x'_i)/K_i) \). CM theory then tells us that there is a cyclic isogeny \( x'_i \to \sigma_i(x'_i) \) of degree \( l \). Now \( \sigma \) fixes \( F \), and we have

\[ x' \in X \cap T_{h^n,l}(X^\sigma) = X \cap T_{h^n,l}(X). \]

This holds for every \( x' \) in the Zariski-dense set \( S \), so it follows that

\[ (2.20) \quad X \subset T_{h^n,l}(X). \]

Moreover, \( (2.20) \) holds for every \( l \in \mathcal{P} \).

**Step 2.** Now we use induction on \( d = \dim(X) \), and suppose \( d \geq 2 \).

We may assume without loss of generality that the projection \( p_1 : X \to \mathbb{A}^1 \) is dominant. Now we may choose an infinite subset \( \{x^1, x^2, \ldots \} \subset S \) of points, written \( x^j = (x^j_1, \ldots, x^j_n) \), such that the first coordinates \( x^j_1 \) are distinct, for \( j \in \mathbb{N} \). For each \( j \) we may find \( l_j \in \mathcal{P} \) such that \( x^j_1 \in T_{h^n,l_j}(x^j_1) \) and \( |l_j| \geq \max(13, \deg X) \). In fact, \( \mathcal{P} \) contains infinitely many such primes, namely those which split completely in the ring class field of \( \text{End}(x^j_1) \).

For each \( j \in \mathbb{N} \) we consider the “slice”

\[ X_j = X \cap (\{x^j_1\} \times \mathbb{A}^{n-1}), \]

which satisfies \( X_j \subset T_{X,l_j}(X_j) \) (Corollary 2.8), \( \dim(X_j) = d - 1 \) and \( x^j \in X_j \). Let \( X'_j \) be an irreducible component of \( X_j \) containing \( x^j \). Then the Hecke orbit \( T_{X,l_j}(x^j) \) is Zariski-dense in \( X'_j \). As in Step 1 above, we can find infinitely many primes \( \mathfrak{p} \) such that \( T_{h^n,\mathfrak{p}} \) stabilizes \( X'_j \), so from the induction hypothesis follows that \( X'_j \) is modular.

Now \( \deg(X'_j) \leq \deg(X) \), and there are only finitely many types of modular varieties of bounded degree (Proposition 2.5), so it follows that we have an infinite subset \( I \subset \mathbb{N} \) and some \( \pi \in S_n \) such that, after permutation of coordinates by \( \pi \),

\[ X'_j = Y \times \{y_j\} \quad \forall j \in I, \]

where \( Y \subset \mathbb{A}^{n-m} \) is a fixed modular variety, and \( y_j \in \mathbb{A}^m \) is a CM point, for some \( m \geq 1 \). Let \( Y' \subset \mathbb{A}^m \) be the Zariski-closure of \( \{y_j \mid j \in I\} \), then \( \dim(Y') \geq 1 \). Now the Zariski-closure of \( \{X'_j \mid j \in I\} \) is equal to \( Y \times Y' \), is contained in \( X \) and has dimension at least \( \dim(Y) + 1 = \dim(X) \). It follows that \( X = Y \times Y' \), with \( Y' \) an irreducible curve.

Moreover, \( Y' \) is stabilized by the Hecke operators \( T_{h^n,l} \) for all \( l \in \mathcal{P} \), hence is itself modular. It follows that \( X \) is modular, which is what we set out to prove. \( \square \)

3 Heights of CM points

3.1 Estimating class numbers

We now want to derive a lower bound for the class number of an order in an imaginary quadratic function field. Our standard reference to facts about function fields is [28].
Let $F$ be a global function field of genus $g$ and exact field of constants $\mathbb{F}_q$, and denote by $h = h(F) = \#\text{Pic}^0(F)$ its class number. We want upper and lower bounds for $h(F)$. Using the Hasse-Weil theorem, one easily obtains $|\sqrt{q} - 1|^{2g} \leq h \leq |\sqrt{q} + 1|^{2g}$. Unfortunately, the lower bound is only useful when $q \geq 5$, and so for general $q$ we have the following bound, which was shown to me by Henning Stichtenoth.

**Proposition 3.1** We have

$$h(F) \geq \frac{(q - 1)(q^{2g} - 2qg^g + 1)}{2g(q^{g+1} - 1)}.$$

**Proof.** We consider the constant field extension $F' = \mathbb{F}_{q^{2g}}$, of $F$ of degree $2g$. The exact field of constants of $F'$ is $\mathbb{F}_q$. Let $N'$ denote the number of rational (that is, $\mathbb{F}_q^{2g}$-rational) places of $F'$. The Hasse-Weil bound gives us $N' \geq q^{2g} - 2qg^g + 1$. Let $Q$ be one such rational place of $F'$, lying over the place $P$ of $F$. As $Q$ has degree one, we get $2g = f(Q|P) \cdot \text{deg}(P)$, and so $\text{deg}(P)$ divides $2g$. It follows that $(2g/\text{deg}(P)) \cdot P$ is an effective divisor of degree $2g$ of $F$. As there are at most $2g$ places $Q$ above $P$, we see that in this way we have constructed at least $N'/2g$ effective divisors of degree $2g$ of $F$. On the other hand, there are exactly $h(q^{g+1} - 1)/(q - 1)$ such places, so we get

$$\frac{h}{q - 1}(q^{g+1} - 1) \geq \frac{N'}{2g} \geq \frac{q^{2g} - 2qg^g + 1}{2g},$$

from which the result follows. \qed

We now let $F = K = k(\sqrt{D})$ be an imaginary quadratic extension of $k = \mathbb{F}_q(T)$, where $D \in A$ is square-free. Then the genus of $K$ is given by

$$g = \begin{cases} (\text{deg}(D) - 1)/2 & \text{if } \text{deg}(D) \text{ is odd} \\ (\text{deg}(D) - 2)/2 & \text{if } \text{deg}(D) \text{ is even}. \end{cases}$$

Let $\mathcal{O}$ be an order of conductor $f$ in $K$. Then, as in the classical case, one may express $\#\text{Pic}(\mathcal{O})$ in terms of $h(K)$ and $f$ (e.g. [26], Proposition 17.9), which, combined with our bounds on $h(K)$, gives us

$$B_\varepsilon|Df^2|^\frac{g}{1+\varepsilon} \leq \#\text{Pic}(\mathcal{O}) \leq C_\varepsilon|Df^2|^\frac{g}{1-\varepsilon}$$

for every $\varepsilon > 0$ and effectively computable positive constants $B_\varepsilon$ and $C_\varepsilon$.

### 3.2 Estimating the $j$-invariant

In this section we estimate the $j$-invariant using analytic methods, following the first part of [7]. We point out that later parts of that paper (the part concerning supersingular reduction) have been shown to contain errors, but we will only use results from the first (and supposedly correct) part.

**Definition 3.3** Let $z \in \Omega$. Then we define

$$|z|_A = \inf_{a \in A} |z - a|, \quad \text{and} \quad |z|_i = \inf_{x \in k} |z - x|.$$

The *imaginary modulus* $|z|_i$ plays the role of $|\Im(z)|$ in the classical case.
Let \( \phi \) be a CM Drinfeld module. Then \( \text{End}(\phi) = \mathcal{O} = A[\sqrt{d}] = A[f \sqrt{D}] \) is an order of conductor \( f \) in \( K = k(\sqrt{D}) \), where \( D \) is the square-free part of \( d = Df^2 \).

A non-zero ideal \( \mathfrak{a} \) in \( \mathcal{O} \) is a rank 2 lattice in \( \mathcal{O} \). It follows that \( \mathfrak{a} \) is homothetic to the lattice \( \Lambda_z = \langle z, 1 \rangle \), for some \( z \in \Omega \). This \( z \) is determined up to \( \text{PGL}_2(A) \)-action, so we would like to have a fundamental domain for this action. Unfortunately, a perfect analogue of the classical fundamental domain for the \( \text{SL}_2(\mathbb{Z}) \)-action on \( \mathcal{H} \) does not seem to exist, but if we’re only interested in \( z \in \Omega \) quadratic over \( k \), then we do have the next best thing.

**Definition 3.4** The quadratic fundamental domain is

\[
\mathcal{D} = \{ z \in \Omega \mid z \text{ satisfies an equation of the form } az^2 + bz + c = 0, \text{ where } a, b, c \in A, a \text{ is monic, } |b| < |a| \leq |c|, \text{ and } \gcd(a, b, c) = 1 \}.
\]

In general we’re only interested in \( \mathcal{D} \cap K \), which we denote \( \mathcal{D}_K \). Then, as in the classical case, one may show that any rank 2 lattice in \( K \) is homothetic to \( \Lambda_z \) for some \( z \in \mathcal{D}_K \). Moreover, we have

**Proposition 3.5** If \( z \in \mathcal{D}_K \), then \(|z|_i = |z|_A = |z| \geq 1\).

**Proof.** It suffices to show that \(|z|_i = |z| \geq 1\). Write \( z = (-b + \sqrt{d})/2a \), where \( d = b^2 - 4ac \). Then \(|d| = |ac| \geq |a^2| \) and \(|d| \leq |c^2|\). Hence \(|z| = |\sqrt{d}/2a| \geq 1\).

We distinguish two cases.

(a) If \( \infty \) is ramified in \( K/k \), then \( \deg(d) \) is odd and \( v_\infty(\sqrt{d}/2a) \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z} \) is half integral, so \(|x| \neq |z| \) and \(|z - x| \geq |z| \quad \forall x \in k_\infty\).

(b) If \( \infty \) is inert, then \( \deg(d) \) is even, but its leading coefficient is not a square in \( \mathbb{F}_q \). So the leading coefficient of \( \sqrt{d}/2a \) as a Laurent series in \( 1/T \) is not in \( \mathbb{F}_q \), and the leading terms of \( x \in k_\infty \) and \( \sqrt{d}/2a \) cannot cancel. Hence \(|z - x| \geq |z| \) in this case, too. \( \square \)

Now we may estimate \(|j(z)| = |j(\phi^z)|\).

**Theorem 7** Suppose \( q \) is odd. Let \( z = (-b + \sqrt{d})/2a \in \mathcal{D}_K \). Then

1. If \(|z| = 1\) then \(|j(z)| \leq 1/q\).
2. If \(|z| > 1\) then \(|j(z)| = B_q^{|z|} \), where

\[
B_q = \begin{cases} 
q^{|z|} & \text{if } \deg(d) \text{ is even} \\
q^{\sqrt{q(q+1)/2}} & \text{if } \deg(d) \text{ is odd.}
\end{cases}
\]

**Proof.** One just follows the proof of [7], Theorem 2.8.2, using the fact that \(|z|_A = |z|_i = |z| \geq 1\) when \( z \in \mathcal{D}_K \). Then all the calculations of [7] work and we do not need to assume that \( d \) be square-free (i.e. that \( z \) correspond to a Drinfeld module with complex multiplication by the full ring of integers \( \mathcal{O}_K \) of \( K \)). \( \square \)

**Corollary 3.6** Let \( \mathcal{O} \) be an order in \( K \), and let \( \mathfrak{a} \subset \mathcal{O} \) be an invertible ideal. Then \(|j(\mathfrak{a})| \leq |j(\mathcal{O})|\), with equality if and only if \( \mathfrak{a} \) is principal.
Definition 3.7 Let \( \phi \) be a CM Drinfeld module, with \( \text{End}(\phi) = A[\sqrt{d}] = A[f\sqrt{D}] \) and \( j \)-invariant \( j = j(\phi) \). Then we define the CM height of \( \phi \) and of \( j \) to be

\[
H_{CM}(j) = H_{CM}(\phi) = |d| = |f^2 D|.
\]

If \( x \in \mathbb{C} \) is not a CM point, then we set \( H_{CM}(x) = 1 \). If \( x = (x_1, \ldots, x_n) \in \mathbb{A}^n(\mathbb{C}) \) then we define

\[
H_{CM}(x) = \max\{H_{CM}(x_1), \ldots, H_{CM}(x_n)\}.
\]

This height is not to be confused with the usual notion of the height of a Drinfeld module of finite characteristic. In fact, all the Drinfeld modules here have generic characteristic. This definition is analogous to the definition given in [5] for elliptic curves. The CM height is so-named because it forms a true counting function on the CM points of \( \mathbb{A}^1(\mathbb{C}) \) (and thus also of \( \mathbb{A}^n(\mathbb{C}) \)).

Proposition 3.8 For every \( \varepsilon > 0 \) we have

\[
\#\{j \in \mathbb{C} \mid j \text{ is CM and } H_{CM}(j) \leq t\} = O(t^{3/2+\varepsilon}).
\]

Proof. For every order \( \mathcal{O}_d = A[\sqrt{d}] \), there are exactly \( \#\text{Pic}(\mathcal{O}_d) \) isomorphism classes of Drinfeld modules \( \phi \) with \( \text{End}(\phi) = \mathcal{O}_d \), namely those corresponding to the ideal classes \([a] \in \text{Pic}(\mathcal{O}_d)\). So we have

\[
\#\{j \in \mathbb{C} \mid j \text{ is CM and } H_{CM}(j) \leq t\} = \sum_{|d| \leq t} \#\text{Pic}(\mathcal{O}_d)
\leq \sum_{|d| \leq t} C_{\varepsilon}|d|^{1/2+\varepsilon}
= O(t^{3/2+\varepsilon}).
\]

Now that we may view the CM height as a height function, one may ask how this compares to the usual (i.e. arithmetic) height in \( \mathbb{P}^1 \) (see [18], Part B, or [21], Chapter 3, for definitions). We have

Proposition 3.9 Let \( j \in k^{sep} \) be a CM point, with \( \text{End}(j) = \mathcal{O} = A[\sqrt{d}] \) and \( H_{CM}(j) = |d| \). Then the (logarithmic) height of \( j \in \mathbb{P}^1(k^{sep}) \) is bounded by \( h(j) \leq H_{CM}(j)^{1/2} + C_q \), where

\[
C_q = \begin{cases} 
q & \text{if } \deg(d) \text{ is even} \\
\sqrt{q(q+1)/2} & \text{if } \deg(d) \text{ is odd}.
\end{cases}
\]
Proof. Let $K = \mathcal{O} \otimes_A k$ denote the CM field and set $F = K(j)$. We recall that $j$ is integral over $A$, so that $|j|_v \leq 1$ for any place $v$ of $F$ that does not lie over the (unique) place $\infty$ of $K$. On the other hand, the place $\infty$ splits completely in $F/K$, so for any place $v|\infty$ of $F$ we have $|j|_v = |\sigma_v(j)|^2$, where $\sigma_v : F \hookrightarrow \mathbb{C}$ is the embedding of $F$ into $\mathbb{C}$ corresponding to the place $v$, and $|\cdot|$ denotes the unique absolute value of $\mathbb{C}$. This gives us

$$h(j) = \frac{1}{2[F : K]} \sum_{v|\infty} \log_q (\max \{|j|_v, 1\}) \quad (as \ j \ is \ integral)$$

$$= \frac{1}{2[F : K]} \sum_{\sigma \in \text{Gal}(F/K)} \log_q (\max\{|\sigma(j)|^2, 1\})$$

$$\leq \log_q |j(O)| \quad (from \ Corollary \ 3.6)$$

$$\leq |z| + \log_q(B_q) \quad (from \ Theorem \ 7)$$

$$= |d|^{1/2} + C_q.$$

The result follows. \hfill \Box

3.4 CM points on curves

We are now ready to prove our first main result: the effective André-Oort conjecture for the product of two Drinfeld modular curves.

Proof of Theorem 2. We will prove Theorem 2 for $n = 2$, the extension to general $n$ then follows by considering projections to pairs of coordinates.

Let $X \subset \mathbb{A}^2$ be a curve of degree $d$, as in Theorem 2. Firstly, it is clear that the modular curves $Y_0^\prime(N)$ contain CM points of arbitrary height. We want to prove the converse. Let $x = (x_1, x_2) \in X(\mathbb{C})$ be a CM point. From Proposition 3.9 it follows that it suffices to show that $X$ is modular if $x$ has a large CM height. We may assume that both projections $p_i : X \to \mathbb{A}^1$ are dominant (otherwise the result is trivial). We want to use Theorem 5 so we must show that $X$ is stabilized by a suitable Hecke operator.

Let $O_i = \text{End}(x_i) = A[f_i, \sqrt{D_i}]$ be orders of conductors $f_i$ in the imaginary quadratic fields $K_i$, for $i = 1, 2$, and let $K = K_1K_2$. Denote by $g_i$ the genus of $K_i$. Denote by $F_i$ the separable closure of $K$ in $F$, and let $L$ be the Galois closure of $F_iK(x_1, x_2)$ over $k$.

Let $p$ be a prime of even degree in $k$ which splits completely in $F_iK$ and does not divide $f_1f_2$. Let $\mathfrak{P}$ be a prime of $L$ lying over $p$, and denote by $\mathfrak{P}_i$ its restriction to the field $K_i(x_i)$.

From CM theory (Theorem 3) it follows that $\text{Gal}(K_i(x_i)/K_i) \cong \text{Pic}(O_i)$ and $p$ is unramified in $L/k$. Denote by $\sigma \in \text{Aut}(FL/FK)$ an extension of the Frobenius element $(\mathfrak{P}_i, L/k) \in \text{Gal}(L/k)$, and let $\sigma_i = \sigma|_{K_i(x_i)} = (\mathfrak{P}_i, K_i(x_i)/k) = (\mathfrak{P}_i, K_i(x_1)/K_i)$, as $p$ splits in $K_i$. Moreover, we have cyclic isogenies $x_i \to x_i^{\sigma_i}$ of degree $p$, so $(x_1, x_2) \in X \cap T_{x_1, x_2}(X) = X \cap T_{x_1, x_2}(X)$, as $\sigma$ acts trivially on $F$.

On the one hand, from Proposition 2.3 it follows that $\deg(X \cap T_{x_1, x_2}(X)) \leq 4d^2(|p| + 1)^2$. On the other hand, the whole $\text{Gal}(FK(x_1, x_2)/F)$-orbit of the point $(x_1, x_2)$ lies in this intersection, and there are at least $\#\text{Pic}(O_i)/m$ points in this orbit (for $i = 1$ and $i = 2$). We must show that $\#\text{Pic}(O_i) > 4md^2(|p| + 1)^2$, as then the intersection will be improper, giving $X \subset T_{x_1, x_2}(X)$, as $X$ is irreducible. Then the result will follow from Theorem 5 if $|p| \geq \max(13, d)$. 

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It remains to show that there exists a prime $p$ which has the desired properties. For this we use the Čebotarev Theorem (see [12], Proposition 5.16). Let $M$ be the Galois closure of $F,K$ over $k$, and set
\[ \pi_M(t) = \# \{ \mathfrak{p} \in A \mid \text{prime split in } M, \text{ and } |\mathfrak{p}| = q^t \}. \]

Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_q$ in $M$, let $n_c = [\mathbb{F} : \mathbb{F}_q]$ be the constant extension degree and $n_g = [M : \mathbb{F}k]$ be the geometric extension degree. If $n_c \nmid t$ then $\pi_M(t) = 0$. If $n_c | t$ then
\[ |\pi_M(t) - \frac{1}{n_g} q^t/t| < 4(g(M) + 2)q^{t/2}. \]

Here $g(M)$ is the genus of $M$, which can be bounded with the Castelnuovo inequality ([28], Theorem III.10.3) to give $g(M) \leq C_1(g_1 + g_2) + C_2g + C_3$, and we also have $n_g n_c \leq C_4$. Here the $C_i$’s are computable constants depending only on $m$.

Now we want both $\pi_M(t) > \deg(f_1 f_2) = \log_q |f_1 f_2|$ (so that we have a split prime $p$ not dividing $f_1 f_2$) and $\# \text{Pic}(O_i) > 4md^2(q^t + 1)^2$ (so that $X \subset T_{k^2,p}(X)$).

Summarizing, we want a simultaneous solution $t \in 2n_c \mathbb{N}$ to the inequalities
\begin{align}
\frac{1}{C_4} q^t/t - 4(C_1(g_1 + g_2) + C_2g + C_3 + 2)q^{t/2} &> \log_q |f_1 f_2| \quad \text{(3.10)} \\
B_\varepsilon(q^{q_i}|f_i|)^{1-\varepsilon} &> 4md^2(q^t + 1)^2 \quad \text{from (3.2)} \quad \text{(3.11)}
\end{align}
for some $\varepsilon > 0$ and at least one of $i = 1$ or $i = 2$.

These inequalities hold with $q^t \geq \max(13,d)$ if $H_{CM}(x_1, x_2) = \max(|D_1 f_1^2|, |D_2 f_2^2|)$ is larger than some computable constant $B$, which depends on $d, m$ and $g$. \hfill $\Box$

As we may equip our Drinfeld modules with arbitrary level structures - which play no role - we may replace each copy of $\mathbb{A}^1$ by a Drinfeld modular curve to obtain

**Corollary 3.12** Let $X_1, \ldots, X_n$ be Drinfeld modular curves. Let $Z = X_1 \times \cdots \times X_n$, and let $X \subset Z$ be an irreducible algebraic curve. Then the following are equivalent:

1. $X$ contains infinitely many CM points
2. $X$ contains at least one CM point of height larger than some effectively computable constant which depends only on $Z$, $\deg(X)$ and the field of definition of $X$.
3. There exists a non-empty subset $S \subset \{1, \ldots, n\}$ for which we may write
   \[ Z \cong Z_S \times Z'_S = \left( \prod_{i \in S} X_i \right) \times \left( \prod_{i \notin S} X_i \right), \]
   \[ X = X' \times \left( \prod_{i \notin S} \{x_i\} \right), \]
   where the $x_i \in X_i$ are CM points and $X'$ is a special curve in $Z_S$.

### 3.5 CM points on varieties

In this last section we prove our other main result: the André-Oort conjecture for subvarieties of the product of $n$ Drinfeld modular curves.
Proof of Theorem 1. As CM points are defined over $k^{sep}$, so is $X$. Hence there exists a finite Galois extension $F$ of $k$ such that $X$ is defined over $F$.

Set $d = \dim(X)$. We will use induction on $d$. From Theorem 2 and Corollary 3.2 we know that the result already holds for $d = 1$. We now suppose $d \geq 2$, $n \geq 3$ and that the result is already known for dimensions less than $d$. We claim that we may assume that $X \subset \mathbb{A}^n$ is a hypersurface.

Indeed, $X$ is an irreducible component of

$$\bigcap_{I \subset \{1, \ldots, n\}} \bigcap_{\#I=d+1} p_I^{-1}(p_I(X)).$$

The CM points are Zariski-dense in the hypersurfaces $p_I(X) \subset \mathbb{A}^j$, and if these are modular, then so are the $p_I^{-1}p_I(X)$, and thus also $X$. This proves our claim. Furthermore, we may assume that all the projections $p_i : X \to \mathbb{A}^1$ are dominant.

For a given constant $B > 0$ we may assume that every point $x = (x_1, \ldots, x_n) \in S$ satisfies $H_{CM}(x_i) > B$ for all $i = 1, \ldots, n$, as the set

$$\{x \in X(C) \mid H_{CM}(x_i) \leq B \text{ for some } i = 1, \ldots, n\}$$

is contained in a proper closed subvariety of $X$.

Step 1. Choose a point $x = (x_1, \ldots, x_n) \in S$. Suppose that we have primes $p_1, \ldots, p_{d-1}$ of $k$ of even degree, satisfying the following conditions:

1. Each $p_j$ splits completely in every $O_i = \text{End}(x_i)$ for $i = 1, \ldots, n$ and in $F$.
2. $|p_1| \geq \max\{13, \deg X\}$
3. $|p_{j+1}| \geq (\deg X)^{2j} \prod_{m=1}^{j} (2|p_m| + 2)^{n2j-m}$ for $j = 1, \ldots, d-2$
4. We have $\#\text{Pic}(O_i) > [F : k](\deg X)^{d-1} \prod_{m=1}^{d-1} (2|p_m| + 2)^{n2j-m-1}$ for each $i = 1, \ldots, n$, for which it suffices to assume

$$\#\text{Pic}(O_i) > [F : k]|p_{d-1}|^2 (2|p_{d-1}| + 2)^n. \quad (3.13)$$

Then, as in the proof of Theorem 2 it follows that

$$\text{Gal}(F^{sep}/F) \cdot x \subset X \cap T_{\mathbb{A}^n, p_j}(X), \quad j = 1, \ldots, d-1.$$

Let $X_1$ be an $F$-irreducible component of $X \cap T_{p_1}(X)$ containing $x$. Now either $X_1 = X$, in which case $X_1 \subset T_{p_1}(X_1)$ and $X_1$ is modular (Theorem 3), or $\dim(X_1) < \dim(X)$. In the latter case we repeat the procedure: We let $X_2$ be an $F$-irreducible component of $X_1 \cap T_{p_1}(X_1)$ containing $x$, and so on. We thus produce a sequence $X_1, X_2, \ldots$ of $F$-irreducible subvarieties of $X$ of strictly decreasing dimension. But, as the $X_j$ are defined over $F$, the full $\text{Gal}(F^{sep}/F)$-orbit of $x$ is contained in each $X_j$. Moreover, after at most $d-1$ steps we arrive at $\dim(X_j) \leq 1$, and

$$\deg X_j \leq (\deg X)^{2j} \prod_{m=1}^{j} (2|p_m| + 2)^{n2j-m} \quad \text{(using Proposition 2.4)}$$

$$< \#\text{Pic}(O_i)/[F : k] \leq \#\text{Gal}(F^{sep}/F) \cdot x \quad \text{(as } j \leq d-1).$$
Hence $X_j$ must have dimension at least 1. In summary, this process must terminate, after at most $d - 1$ steps, with some $X_j$ of dimension at least 1, satisfying $X_j \subset T_{p_{j+1}}(X_j)$. Hence $X_j$ is modular.

By varying $x \in S$, we see that we have covered $X$ by a Zariski-dense family of modular subvarieties $X_x$ for $x \in S$. We now show that the $X_x$’s are in fact pure modular. Suppose not. Recall that each $X_x$ is $F$-irreducible, so if it’s not pure then it is the $\text{Gal}(\bar{F}/F)$-orbit of a modular variety of the form $Y_x \times \{y_x\}$, where $y_x$ is a CM point (in fact a projection of $x$) and $Y_x$ a pure modular variety. But as the $\text{Gal}(\bar{F}/F)$-orbit of the point $y_x$ is larger than the degree of $X_x$, by construction, this would mean that the number of (geometrically) irreducible components of $X_x$ of maximal dimension is larger than $\text{deg}(X_x)$, which is impossible. So each $X_x$ is in fact pure modular. Now each $X_x$ contains a Zariski-dense family of pure modular curves, hence so does $X$.

**Step 2.** We want to show that $X$ is modular, using the fact that $X$ contains a Zariski-dense family of pure modular curves. For ease of notation we will denote this family by $S$ and the pure modular curves by $s \in S$.

Choose a CM point $x_1 \in \mathbb{A}^1(\mathbb{C})$ and consider the intersection

$$X_1 = X \cap (\{x_1\} \times \mathbb{A}^{n-1}).$$

As each curve $s \in S$ is pure modular, it intersects $X_1$ in at least one CM point. We denote by $X'$ the Zariski closure of these points:

$$X' = \bigcup_{s \in S} (s \cap X_1)^{\text{Zar}}.$$

Now if $\dim(X') = \dim(X)$, then $X \subset \{x_1\} \times \mathbb{A}^{n-1}$, which is impossible: we had assumed in the beginning that all projections $p_1 : X \to \mathbb{A}^1$ are dominant. Hence $\dim(X') < \dim(X)$. Then it follows from the induction hypothesis that all the (geometrically) irreducible components of $X'$ are modular. Write $X' = X'_1 \cup \cdots \cup X'_{r}$ as the union of $r$ irreducible components. Then the points of $s \cap X_1$ distribute amongst these components. By restricting $S$ to a Zariski-dense subfamily, and renumbering the components of $X'$, we may assume that $X'_1$ contains at least 1/r of the points of $s \cap X_1$ for every $s \in S$.

If, up to permutation of coordinates, $X'_1$ is of the form $\{y\} \times \mathbb{A}^m$ for some $m < n - 1$ and $y$ a CM point in $\mathbb{A}^{n-m}$, then it follows that $X$ is of the form (again up to permutation of coordinates) $Y \times \mathbb{A}^{n-2}$, where $Y$ is an irreducible curve in $\mathbb{A}^2$. But then $Y$ contains infinitely many CM points, hence is modular. In this case we see that $X$ is modular.

So we may now assume that at least one modular curve appears as a factor of $X'_1$. Then there exists some pair of coordinates $1 < i < j$ such that

$$p_{i,j}(X'_1) = Y'_0(m),$$

for some fixed $m \in A$.

Let $s' = p_{i,i,j}(s) \subset \mathbb{A}^3$ be characterized by the triplet $(N_{s,1}, N_{s,i}, N_{s,j}) \in A^3$ as in \[10.3\] and assume, by restricting $S$ to a Zariski-dense subfamily and permuting coordinates, that we always have $|N_{s,i}| \leq |N_{s,j}|$. Fix $s \in S$ and fix also $x_i$ such that we have a point $(x_1, x_i, x_j) \in s'$. We want to find many points $x_j$ with this property.

For each prime $p \in A$, consider the tree $T_p$. Then a generic point $(x_1, x_i, x_j)$ of $s'$ corresponds to a triple of vertices $(v_{p1}, v_{pi}, v_{pj})$, at distances $(n_{p1}, n_{pi}, n_{pj})$ from the center $v_{pc}$.
The family of vertices \((v_{p,c})_{p \in A}\) corresponds to a point \(x_c \in C\) which we call the *center* of \((x_1, x_i, x_j)\). The possible choices of \(v_{p,j}\) correspond to the length \(n_{p,j}\) paths leading out from \(v_{p,c}\) and disjoint from the two paths leading to \(v_{p,1}\) and \(v_{p,i}\). This gives \(| |p| - 1|(|p| + 1)^{n_{p,j} - 1}\) possibilities if \(n_{p,j} \geq 1\) (and just one if \(n_{p,j} = 0\)). Multiplying over all primes \(p \in A\) then shows that there are \(\prod_{p \nmid N_{s,j}} (|p| - 1)(|p| + 1)^{n_{p,j} - 1}\) possible choices for \(x_j\). So we have counted the number of suitable cyclic degree \(N_{s,j}\) isogenies from \(x_c\) to \(x_j\). But \((x_1, x_i, x_j)\) is a CM point, not a generic point, so some of these isogenies will produce the same point \(x_j\). But \((x_1, x_i, x_j)\) is a CM point, not a generic point, so some of these isogenies will produce the same point \(x_j\). So we have counted the number of distinct values of \(x_j\) satisfying \((x_1, x_i, x_j) \in S'\) tends to infinity as \(N_{s,j}\) increases.

But \(1/r\) of these points also satisfy \((x_i, x_j) \in X_0'(m)\), of which there can be at most \(\psi(m)\), for fixed \(x_i\). So we have shown that \(N_{s,j}\), and thus also \(N_{s,i}\), is bounded as \(s\) ranges through \(S\).

It follows that there are only finitely many possibilities for \(p_{i,j}(s) = Y'_0(N_{s,j}N_{s,i})\). By replacing \(S\) with a Zariski-dense subfamily, we may assume there is only one: \(p_{i,j}(s) = Y'_0(N_0)\) for all \(s \in S\). Now, after a permutation \((i, j) \mapsto (n - 1, n)\) of coordinates, we see that

\[
S \subset A^{n-2} \times Y'_0(N_0), \quad \text{and so}
\]

\[
X = S^{\mathbb{Z}^{ar}} \subset A^{n-2} \times Y'_0(N_0).
\]

But \(X\) is a hypersurface, so we have in fact \(X = A^{n-2} \times Y'_0(N_0)\), which is modular. This is what we set out to prove.

**Step 3.** It remains to show that we can find primes \(p_j\) with the desired properties. Recall that \(x = (x_1, \ldots, x_n)\) and each \(\mathcal{O}_i = \text{End}(x_i)\) is an order of conductor \(f_i\) in the imaginary quadratic field \(K_i\) of genus \(g_i\).

Set \(|p_j| = q_j^j\) for \(j = 1, \ldots, d - 1\). Firstly, we need \((3.13)\), which, combined with the lower bound for the class number \(3.2\), gives

\[
B_x(q^{q_0} |f_i|)^{1-\varepsilon} > [F : k] q^{2d-1}(2q^{d-1} + 2)^n.
\]

Secondly, the \(p_j\)'s must be well spaced out, i.e. we need

\[
q^{j+1} \geq (\deg X)^{2j} \prod_{m=1}^{j} (2q^{tm} + 2)^{n2j-m}.
\]

Thirdly, each \(p_j\) must split completely in \(FK\), where \(K = K_1 \cdots K_n\), and not divide \(f_1 \cdots f_n\). Here the Čebotarev Theorem says

\[
|\pi_{FK}(t_j) - \frac{1}{n_g} q^{t_j} / t_j| < 4(g(FK) + 2)q^{t_j/2} \quad \text{and} \quad n_c|t_j,
\]

where \(n_g\) denotes the geometric extension degree of \(FK/k\), \(n_c\) denotes the constant extension degree, and \(g(FK)\) is the genus of \(FK\). We have \(n_gn_c = |FK : k| \leq 2^n |F : k|\) and we may bound \(g(FK)\) from above via the Castelnuovo Inequality to obtain \(g(FK) \leq C_1(g_1 + \cdots + g_n) + C_2\) for some computable constants \(C_1\) and \(C_2\) depending on the field \(F\). We need \(\pi_{FK}(t_j) > \log_q |f_1 \cdots f_n|\), to obtain a split prime \(p_j\) that does not divide any of the conductors.
In summary, we want \( d - 1 \) simultaneous solutions \( t_1, \ldots, t_{d-1} \in 2n_c\mathbb{N} \) to the inequalities
\[ q^{t_1} \geq \max(13, \deg X), \quad (3.14), \quad (3.15) \]
and
\[ \frac{1}{2^n[F:k]} q^{t_j}/t_j - 4(C_1(g_1 + \cdots + g_n) + C_2 + 2)q^{t_j/2} > \log_2 |f_1 \cdots f_n|. \quad (3.16) \]

If we choose the constant \( B \) sufficiently large then, as \( B < H_{CM}(x_i) \leq q^{2g+1}|f_i|^2 \) for all \( i = 1, \ldots, n \), such a set of solutions \( (t_1, \ldots, t_{d-1}) \) exists. \( \square \)

**Corollary 3.17** Let \( X_1, \ldots, X_n \) be Drinfeld modular curves. Let \( Z = X_1 \times \cdots \times X_n \), and let \( X \subset Z \) be an irreducible algebraic subvariety. Then the following are equivalent:

1. \( X \) contains a Zariski-dense set of CM points
2. There exists a partition \( \{1, \ldots, n\} = \bigsqcup_{i=0}^{g+1} S_i \) for which we may write

\[ Z \cong \prod_{i=0}^{g+1} Z_i = \prod_{i=0}^{g+1} \left( \prod_{j \in S_i} X_j \right), \]

\[ X = Z_0 \times \prod_{i=1}^{g} Y_i \times \{x\}, \]

where each \( Y_i \) is a special curve in \( Z_i \) (for \( i = 1, \ldots, g \)) and \( x \) is a CM point in \( Z_{g+1} \).

We remark that Corollary 3.17 with \( X_1 = \cdots = X_n = X_0(MN) \) has an application to Heegner points on elliptic curves over \( k \) with conductor \( N \cdot \infty \), see [6].

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