NONLINEAR DECOMPOSITION PRINCIPLE
AND FUNDAMENTAL MATRIX SOLUTIONS
FOR DYNAMIC COMPARTMENTAL SYSTEMS

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Abstract. A decomposition principle for nonlinear dynamic compartmental systems is introduced in the present paper. This theory is based on the novel mutually exclusive and exhaustive system and subsystem decomposition methodologies. A deterministic mathematical method is developed for dynamic analysis of nonlinear compartmental systems based on the proposed theory. This dynamic method enables tracking the evolution of all initial stocks, external inputs, and arbitrary intercompartmental flows as well as the associated storages derived from these inputs, flows and stocks individually and separately within the system. Various system flows and associated storages transmitted from one compartment directly or indirectly to any other or along a given flow path are then analytically characterized, systematically classified, and mathematically formulated. Thus, the dynamic influence of one compartment, in terms of flow and storage transfer, directly or indirectly on any other is ascertained. Consequently, new mathematical system analysis tools are formulated as quantitative system indicators. The proposed mathematical method is then applied to various models from literature to demonstrate its efficiency and wide applicability.

Key words. nonlinear decomposition principle, fundamental matrix solutions, dynamic system and subsystem decomposition, nonlinear dynamic compartmental systems, diact flows and storages

AMS subject classifications. 34A34, 70G60, 37N25, 92C42, 92D40, 92D30, 91B74

1. Introduction. Compartmental systems are mathematical abstractions of networks composed of discrete, homogeneous, interconnected components that approximate the behavior of continuous physical systems. The system compartments are interrelated through the flow of a conserved quantity, such as energy, matter, or currency between them and their environment based on conservation principles. Therefore, for an accurate quantification of the compartmental system functions, analytic and explicit formulation of system flows and the associated storages generated by these flows are of paramount importance.

Today’s major natural problems involve change, and this makes the need for dynamic and analytical methods of nonlinear system analysis not only appropriate, but also urgent. Dynamic methods for nonlinear compartmental system analysis have remained a long-standing, open problem. Sound rationales are offered in literature for compartmental system analysis, but they are for special cases, such as linear models and steady state conditions [12, 13, 8, 14]. Various mathematical aspects of compartmental systems are studied in literature [10, 1].

This is the first manuscript in literature that potentially addresses the mismatch between the needs for dynamic nonlinear compartmental system analysis and current static and computational simulation methods. The manuscript is structured in three levels: theory, methods, and applications. The underlying novel mathematical theory will be called the nonlinear decomposition principle. The theory is based on the dynamic system and subsystem decomposition methodologies. A deterministic mathematical method is then developed for the dynamic analysis of nonlinear compartmental systems.

The system decomposition methodology explicitly generates mutually exclusive and exhaustive subsystems, each driven by a single external input and initial condition, that are running within the original system and have the same structures and

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dynamics as the system itself. Therefore, the system decomposition methodology yields the subthroughflow and substorage matrix functions, which respectively represent the throughflows and storages generated by external inputs. Equipped with these matrix measures, the system partitioning ascertains the dynamic distribution of external inputs and initial stocks as well as the organization of the associated storages generated by these inputs and stocks individually and separately within the system. Consequently, the composite compartmental throughflows and storages are dynamically decomposed into the subcompartmental subthroughflow and substorage segments based on the constituent external sources. In other words, the system decomposition enables dynamically tracking the evolution of the initial compartmental stocks (initial conditions) and external inputs as well as the associated storages (state variables) individually and separately within the system.

The subsystems are then further decomposed along a set of mutually exclusive and exhaustive directed subflow paths. The subsystem decomposition methodology yields the transient and the dynamic direct, indirect, acyclic, cycling, and transfer (diact) flows and the associated storages generated by these flows. The transient subflows and associated substorages determine the dynamic distribution of arbitrary intercompartmental flows and the organization of the associated storages generated by these flows along given subflow paths within the subsystems. Consequently, arbitrary composite intercompartmental flows and associated storages are dynamically decomposed into the constituent transient subflow and substorage segments along the given subflow paths. In other words, the subsystem decomposition enables dynamically tracking the fate of arbitrary intercompartmental flows and associated storages within the subsystems. Moreover, a history of compartments visited by arbitrary system flows and storages can also be compiled. Based on the concept of transient flow and storage, the dynamic diact flows and storages transmitted from one compartment, directly or indirectly, to any other in the system are analytically characterized, systematically classified, and mathematically formulated for the quantification of intercompartmental flow and storage dynamics. In summary, the proposed mathematical method as a whole decomposes the system flows and storages to the utmost level.

The direct influence of one compartment on another, in terms of flow and storage transfer, can be determined through the state of the art techniques. The proposed methodology, however, makes the dynamic analysis of both direct or indirect influence of one compartment on any other possible in a complex nonlinear system. The methodology, therefore, constructs a base for the development of dynamic system analysis tools as quantitative system indicators. Multiple such dynamic system measures are formulated and used in the analysis of illustrative models in Section 3 and Appendix E to demonstrate the wide applicability and efficiency of the proposed methodology. The results indicate that the proposed methodology provides significant advancements in the theory, methodology, and practicality of nonlinear dynamic compartmental system analysis.

The applicability of the proposed method extends to various realms regardless of their naturogenic and anthropogenic nature, such as ecology, economics, pharmacokinetics, chemical reaction kinetics, epidemiology, biomedical systems, neural networks, and information science. Essentially, the methodology is applicable to real world phenomena where compartmental models of conserved quantities can be constructed. Considering a hypothetical complex network with several interacting compartments, for example, the compartments can be species in an ecosystem, financial institutions in a financial system, organs in an organism, molecules in a chemical reaction, or neurons in a neural network. The conserved quantity that needs to be investigated
within this system, then, would be a nutrient, money, a certain drug, a specific type of atom, or particular ions, respectively.

An illustrative SIRS model from epidemiology is analyzed in detail in Section 3, and more case studies are presented from ecosystem ecology in the Appendices. The SIRS model consists of three compartments that represent the populations of three groups: the susceptible or uninfected, $S$, infectious, $I$, and recovered or immune, $R$. The model determines the number of individuals infected with a contagious illness over time [11]. It is shown that, the proposed dynamic system decomposition methodology enables tracking the evolution of the health states of the newborn or initial $SIR$ populations individually and separately within the total population. The proposed dynamic subsystem decomposition methodology then enables tracking the evolution of the health states of an arbitrary population in any of the $SIR$ groups along a given infection path. Therefore, the effect of an arbitrary population on any other group in terms of the spread of the disease, through not only direct but also indirect interactions, can be ascertained. Consequently, the spread of the disease from an arbitrary population to the entire system can be determined and monitored. It is worth noting for a comparison that the solution to the SIRS model through the state of the art techniques can only provide the composite $SIR$ populations without distinguishing their previous health states.

The paper is organized as follows: the mathematical method is introduced in Sections 2.1, 2.2, and 2.5, the nonlinear fundamental matrix solutions and decomposition principle are introduced in Sections 2.3 and 2.4, system analysis is discussed in Section 2.6, and results, examples, discussions, and conclusions follow at the end of the manuscript.

2. Method. The nonlinear decomposition principle for dynamic compartmental systems is introduced in this section based on the novel dynamic system and subsystem decomposition methodologies. A deterministic mathematical method is then developed for the dynamic analysis of nonlinear compartmental systems. The proposed theory and method construct a base for the formulation of new system analysis tools, such as the fundamental matrices and the diact flows and storages, as system indicators. These new concepts and quantities are developed and formulated in this section.

2.1. Compartmental Systems. We assume that components of a physical system are modeled as compartments that are interconnected through flow of energy, matter, or currency. In such a compartmental model, the state variable $x_i(t)$ represents the amount of storage in compartment $i$, and $f_{ij}(t, x)$ represents the non-negative flow rate from compartment $j$ to $i$ at time $t$.

Let the governing equations formulated based on conservation principles for this system of nonlinear dynamic compartmental network be given as

$$(2.1) \dot{x}(t) = \tau(t, x)$$

with the initial conditions $x(t_0) = x_0$. The state vector $x(t) = [x_1(t), \ldots, x_n(t)]^T$ is a differentiable function for time, $t$. The function $\tau(t, x) = [\tau_1(t, x), \ldots, \tau_n(t, x)]^T$ will be called the net throughflow rate vector and expressed as

$$(2.2) \tau(t, x) = \tau^{in}(t, x) - \tau^{out}(t, x)$$

where the respective inward and outward throughflow rate vector functions are

$\tau^{in}(t, x) = [\tau^{in}_1(t, x), \ldots, \tau^{in}_n(t, x)]^T$ and $\tau^{out}(t, x) = [\tau^{out}_1(t, x), \ldots, \tau^{out}_n(t, x)]^T$. 
The components of the throughput vectors can further be expanded as

\[ \tau_i^{in}(t, x) := \sum_{j=0}^{n} f_{ij}(t, x) \quad \text{and} \quad \tau_i^{out}(t, x) := \sum_{j=0}^{n} f_{ji}(t, x) \]

for \( i = 1, \ldots, n \). Index \( j = 0 \) represents the system exterior (see Fig. 1).

Let \( \Omega \subset \mathbb{R}^n \) be a domain (connected, open set) and \( \mathcal{I} \subset \mathbb{R} \) be some open interval. We assume that \( f_{ij}(t, x) \) is a continuous and continuously differentiable function for \( x \) on \( \mathcal{I} \times \Omega \). Because of being linear combinations of \( f_{ij}(t, x) \), \( \tau_i^{in}(t, x) \), \( \tau_i^{out}(t, x) \), and \( \tau_i(t, x) \) have also the same properties. These conditions imply the existence and uniqueness of the solutions to the governing system, Eq 2.2.

We assume the following conditions on the flow rate functions:

\[ f_{ij}(t, x) := q_{ij}(t, x) x_j(t), \quad f_{ij}(t, x) \geq 0, \quad \text{and} \quad f_{ii}(t, x) = 0, \quad \forall i, j \]

where \( q_{ij}(t, x) \) has the same properties as \( f_{ij}(t, x) \). The first condition guarantees non-negativity of the state variables, that is, \( x_j(t) \geq 0 \) for all \( j \). The external input and output flow rates, \( z_i(t, x) \) and \( y_i(t, x) \), into and from compartment \( i \) are denoted by

\[ z_i(t, x) := f_{i0}(t, x) \quad \text{and} \quad y_i(t, x) := f_{0i}(t, x). \]

The system can then be rewritten componentwise as

\[ \dot{x}_i(t) = \tau_i^{in}(t, x) - \tau_i^{out}(t, x) \]

for \( i = 1, \ldots, n \). When the external input and output are separated, the system Eq 2.5 takes the following standard form:

\[ \dot{x}_i(t) = \left( z_i(t, x) + \sum_{j=1}^{n} f_{ij}(t, x) \right) - \left( y_i(t, x) + \sum_{j=1}^{n} f_{ji}(t, x) \right) \]

with the initial conditions \( x_i(t_0) = x_{i,0}, \) for \( i = 1, \ldots, n \). If \( z_i(t, x) > 0 \) or \( x_{i,0} > 0 \) for all \( i \), then these positive inputs or initial conditions ensure that the state variables are always strictly positive, \( x_i(t) > 0 \), for all \( i \).

The proposed methodology is designed for conservative compartmental systems, as defined below.

**Definition 2.1.** A dynamical system will be called compartmental if it can be expressed in the form of Eq. 2.6 with the conditions given in Eq. 2.4. The compartmental system will be called conservative if all internal flow rates add up to zero when the system is closed, that is, when there is neither external input nor output. Formally,

\[ \sum_{i=1}^{n} \dot{x}_i(t) = 0 \quad \text{when} \quad z(t, x) = y(t, x) = 0 \quad \text{on} \ \mathcal{I} \]

where \( 0 \) is the zero vector of size \( n \).

We define the state, input, and output matrix functions as

\[ X(t) := \text{diag} (x(t)), \quad Z(t, x) := \text{diag} (z(t, x)), \quad \text{and} \quad Y(t, x) := \text{diag} (y(t, x)), \]

respectively. The notation \( \text{diag} (x(t)) \) represents the diagonal matrix whose diagonal elements are the elements of vector \( x(t) \), and \( \text{diag} (X(t)) \) represents the diagonal
matrix whose diagonal elements are the same as the diagonal elements of matrix $X(t)$. The *external input* and *output vectors* are

$$z(t, x) = [z_1(t, x), \ldots, z_n(t, x)]^T \text{ and } y(t, x) = [y_1(t, x), \ldots, y_n(t, x)]^T,$$

respectively. Clearly,

$$x(t) = \mathcal{X}(t) \mathbf{1}, \quad z(t, x) = \mathcal{Z}(t, x) \mathbf{1}, \quad \text{and} \quad y(t, x) = \mathcal{Y}(t, x) \mathbf{1}$$

where $\mathbf{1}$ is the vector of size $n$ whose entries are all equal to 1. Excluding the external input and output, we define the *flow rate matrix* as the matrix of intercompartmental direct flows:

(2.8)  \[ F(t, x) := (f_{ij}(t, x)). \]

Using these notations, $\tau^{in}(t, x)$ and $\tau^{out}(t, x)$, defined in Eq. 2.3, can be expressed in compact form as

\[
\begin{align*}
\tau^{in}(t, x) &= \mathcal{Z}(t, x) \mathbf{1} + F(t, x) \mathbf{1} = z(t, x) + F(t, x) \mathbf{1}, \\
\tau^{out}(t, x) &= \mathcal{Y}(t, x) \mathbf{1} + F^T(t, x) \mathbf{1} = y(t, x) + F^T(t, x) \mathbf{1}.
\end{align*}
\]

The governing equation, Eq. 2.6, then becomes

(2.10)  \[ \dot{x}(t) = (z(t, x) + F(t, x) \mathbf{1}) - (y(t, x) + F^T(t, x) \mathbf{1}) \]

with the initial conditions $x(t_0) = x_0$. Separating external inputs from the intercompartmental flows and outputs, the governing equation, Eq. 2.10, takes the following form:

(2.11)  \[ \dot{x}(t) = z(t, x) + F(t, x) \mathbf{1} \]

where

(2.12)  \[ F(t, x) := F(t, x) - \mathcal{Y}(t, x) - \text{diag} (F^T(t, x) \mathbf{1}) = F(t, x) - \mathcal{T}(t, x), \]

and $\mathcal{T}(t, x) := \text{diag} (\tau^{out}(t, x)) = \mathcal{Y}(t, x) + \text{diag} (F^T(t, x) \mathbf{1})$.

**2.2. Dynamic System Decomposition.** We introduce the *dynamic system decomposition* methodology in this section for partitioning the system into mutually exclusive and exhaustive subsystems. This decomposition enables the determination of the distribution of external inputs and initial stocks as well as the organization of the associated storages generated by the inputs and derived from the stocks individually and separately within the system. Therefore, the system decomposition methodology dynamically decomposes composite compartmental throughflows and storages into subcompartmental segments based on their constituent external inputs and initial stocks.

The initial subsystem is driven by the initial stocks, and each of all the other subsystems is driven by an external input. Therefore, since we assumed that all external inputs and initial stocks are positive, there are $n$ subsystems, one for each input, and 1 initial subsystem for the initial stocks. These $n + 1$ subsystems are indexed by $k = 0, \ldots, n$, where $k = 0$ represents the initial subsystem. The dynamic system decomposition methodology has two components: subcompartmentalization or state decomposition and flow rate decomposition, as introduced in this section.

The initial subsystem will be further decomposed into initial subsystems for a similar analysis of the distribution and organization of the initial stocks. This dynamic initial system decomposition methodology is introduced in Appendix A (see Fig. 1 and 2).
2.2.1. State Decomposition. In this section, we will introduce the subcompartmentalization or state decomposition methodology.

![Fig. 1. Schematic representation of input-oriented dynamic subcompartmentalization in a three-state model system. Each subsystem is colored differently; the second subsystem (k=2) is blue, for example. Only the subcompartments in the same subsystem (x₁(t), x₂(t), and x₃(t) in the second subsystem, for example) interact with each other. Subsystem k receives external input only at subcompartment k. The initial subsystem (gray) has no external input. Compare this figure with Fig. 2, in which the subcompartmentalization and the corresponding flow rate decomposition are illustrated for x₁(t) only.]

We use the notation of \( x_{i_k}(t) \) to represent the \( k^{th} \) substate of the \( i^{th} \) state variable. Each substate \( x_{i_k}(t) \) represents the storage in subcompartment \( i_k \) at time \( t \), which identifies the portion of the storage in compartment \( i \), \( x_i(t) \), that is derived from external input into compartment \( k \neq 0 \), \( z_k(t, x) \), during \([t_0, t]\) (see Fig. 1). Therefore, \( x_{i_k}(t) \) will also be called substorage function. The 0\(^{th}\) substate of the \( i^{th} \) state, \( x_{i_0}(t) \), will be called the initial substate (or substorage) of \( x_i(t) \), and initially it is equal to the initial condition \( x_i,0 \). The initial substates then represent the evolution of the initial stocks for \( t > t_0 \). Therefore, all substate variables are assumed to be zero initially, except the initial substate. Consequently, due to the mutually exclusiveness and exhaustiveness of the system decomposition, we have

\[
(2.13) \quad x_i(t) = \sum_{k=0}^{n} x_{i_k}(t)
\]

for \( i = 1, \ldots, n \), and the initial conditions are

\[
(2.14) \quad x_{i_k}(t_0) = \begin{cases} 
  x_i(t_0) = x_{i,0}, & k = 0 \\
  0, & k \neq 0
\end{cases}
\]

Similar to the original system, the initial subcompartments will further be decomposed into \( n \) subcompartments, as explained in Appendix A.1 (see Fig. 2). We will use the notation \( x_{i_k,0}(t) \) for the \( k^{th} \) substate of the \( i^{th} \) initial substate function or \( \dot{x}_{i_k}(t) := x_{i_k,0}(t) \) for notational convenience. Based on this further decomposition
of the initial substates, we have
\begin{equation}
\label{2.15}
x_{i_0}(t) = \tilde{x}_i(t) = \sum_{k=1}^{n} \bar{x}_{ik}(t),
\end{equation}
for \( i = 1, \ldots, n \), and the corresponding initial conditions become
\begin{equation}
\label{2.16}
\bar{x}_{ik}(t_0) = \delta_{ik} x_{i_0}(t_0) = \begin{cases} x_{i_0}(t_0) = x_{i_0}, & i = k \\ 0, & i \neq k \end{cases}
\end{equation}

Let the \( k \)-th substate and initial substate vector functions be defined as
\begin{equation}
\label{2.17}
x_k(t) := [x_{1k}(t), \ldots, x_{nk}(t)]^T, \quad k = 0, \ldots, n \quad \text{and} \quad \bar{x}_k(t) := [\bar{x}_{1k}(t), \ldots, \bar{x}_{nk}(t)]^T, \quad k = 1, \ldots, n.
\end{equation}
The vector function \( \bar{x}(t) \) of all initial substate and substate variables for the decomposed system will be denoted by
\begin{equation}
\label{2.18}
\bar{x}(t) := \begin{bmatrix} x_{11}(t), \ldots, x_{1n}(t), x_{21}(t), \ldots, x_{2n}(t), \ldots, x_{n1}(t), \ldots, x_{nn}(t) \end{bmatrix}^T 
\end{equation}
\begin{equation}
\label{2.19}
= [\bar{x}_1(t), \ldots, \bar{x}_n(t)], \quad \bar{x}(t_0) = \bar{x}_k(t_0), \quad k = 1, \ldots, n.
\end{equation}

The state decompositions formulated in Eqs. 2.13 and 2.15 and corresponding intial values in Eqs. 2.14 and 2.16 can then be expressed in vector form as
\begin{equation}
\label{2.20}
\begin{aligned}
\bar{x}(t) &= \dot{\bar{x}}(t) + \ddot{\bar{x}}(t), \quad \bar{x}(t_0) = \bar{x}_0 \\
\dot{\bar{x}}(t) &= x_k(t) = \bar{x}_1(t) + \ldots + \bar{x}_n(t), \quad \bar{x}_0(t_0) = \bar{x}_k(t_0) = \bar{x}_0, \quad \text{and} \quad \ddot{\bar{x}}(t) := \begin{bmatrix} x_1(t) & \ldots & x_n(t) \end{bmatrix} \\
for \quad k = 1, \ldots, n, \text{where } \bar{e}_k \text{ is the standard elementary unit vector of size } n. \text{ The vector functions } \bar{x}(t) \text{ and } \ddot{\bar{x}}(t) \text{ are the partitions of the state variable } \bar{x}(t), \text{ which represent the storages derived from the initial stocks and generated by external inputs within the system, respectively. Equations 2.13 and 2.15 also imply that}
\end{aligned}
\end{equation}
\begin{equation}
\label{2.21}
\begin{aligned}
x_i(t) &= \sum_{k=1}^{n} \bar{x}_{ik}(t) + x_{i_0}(t) \quad \Rightarrow \quad \dot{x}_i(t) = \sum_{k=1}^{n} \dot{\bar{x}}_{ik}(t) + \dot{x}_{i_0}(t).
\end{aligned}
\end{equation}
In vector notation, that is,
\begin{equation}
\label{2.22}
\dot{x}(t) = \ddot{x}(t) + \dot{x}(t) = (\dot{\bar{x}}_1(t) + \ldots + \dot{\bar{x}}_n(t)) + (x_1(t) + \ldots + x_n(t)).
\end{equation}

We define the substate and \( k \)-th substate matrix functions, \( X(t) \) and \( X_k(t) \), as
\begin{equation}
\label{2.23}
X(t) := (x_{i_0}(t)) = [x_1(t) \ldots x_n(t)] \quad \text{and} \quad X_k(t) := \text{diag}(x_k(t))
\end{equation}
for \( k = 0, \ldots, n \), together with the initial conditions given in Eq. 2.14,
\begin{equation}
\label{2.24}
X(t_0) = \mathbf{0}, \quad X_k(t_0) = \mathbf{0} \quad \text{for } k \neq 0, \quad \text{and} \quad X_0(t_0) = \text{diag}(x_0).
\end{equation}
These matrices will, alternatively, be called the substorage and \( k \)-th substorage matrix functions, respectively. Note that we use the notation \( \mathbf{0} \) for both the \( n \times 1 \) zero vector and \( n \times n \) zero matrix, which should be distinguished from the context. We then have
\begin{equation}
\label{2.25}
\begin{aligned}
\mathbf{x}(t) &= \bar{x}(t) + \ddot{x}(t) = \mathbf{x}_0(t) + X(t) \mathbf{1} \quad \text{and} \quad x_k(t) = X_k(t) \mathbf{1}.
\end{aligned}
\end{equation}
The dynamic state decomposition methodology can be schematized as follows:
\[ x(t) = x_0(t) + X(t) 1 \]

\[ x_1(t) = \sum_{k=0}^{n} x_{ik}(t) \]

\[ x_i(t) = \begin{bmatrix} x_{10} \\ x_{20} \\ \vdots \\ x_{n0} \end{bmatrix} + \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ \vdots \end{bmatrix} \]

2.2.2. Flow Rate Decomposition. For a coherent system decomposition, flow rates are also decomposed into flow segments that will be called the subflow rate functions. These subflow rates represent the rate of flow segments between the subcompartments (see Fig. 2).

![Fig. 2. Schematic representation of a dynamic flow rate decomposition in a three-compartment model system. The figure illustrates the subcompartamentalization of compartment 1 and the corresponding flow rate decomposition from compartment 1 to others, \( j \), \( f_{j1}(t, x) \). The figure also illustrates further decomposition of the initial subcompartment and the corresponding initial subflow rate function, \( f_{j010}(t, x) \) (both dark gray).](image)

We assume that external input \( z_i(t, x) \) enters into the system at subcompartment \( i_1 \) (see Fig. 1). Therefore, the input decomposition can be expressed as

\[ z_{ik}(t, x) := \delta_{ik} z_i(t, x) = \begin{cases} z_i(t, x) = z_i(t, x), & i = k \\ 0, & i \neq k \end{cases} \]

\( \quad (2.24) \)

The flow rates, \( f_{ij}(t, x) \), and output functions, \( y_j(t, x) \), will also be decomposed into the subflow rate functions. First, we define the flow intensity directed from compartment \( j \) to \( i \) at time \( t \) as

\[ q_{ij}(t, x) = \frac{f_{ij}(t, x)}{x_j(t)} \]

\( \quad (2.25) \)

for \( i = 0, \ldots, n, \ j = 1, \ldots, n \), as formulated in Eq. 2.4. Note that \( q_{ij}(t, x) \) are sometimes called transfer coefficients, technical coefficients in economics, or stoichiometric coefficients in chemistry. The subflow rates are then defined to be the flow segments proportional to the flow intensities with the proportionality factors of \( x_{jk}(t) \). That is,

\[ f_{ik,jk}(t, x) := x_{jk}(t) \frac{f_{ij}(t, x)}{x_j(t)} = x_{jk}(t) \frac{f_{ij}(t, x)}{x_i(t)} \]

\( \quad (2.26) \)
for $i, k = 0, \ldots, n$ and $j = 1, \ldots, n$. The index $0_k$ is equivalent to the index 0, and both represent the system exterior. We will use index 0 in both cases for notational convenience. Similar to Eq. 2.4, the functions $f_{i_k, j_k}(t, x) \geq 0$ represent nonnegative subflow rates from subcompartment $j_k$ to $i_k$ and $f_{i_k, i_k}(t, x) = 0$. Due to the mutually exclusiveness and exhaustiveness of the system decomposition and Eq. 2.13, we have

$$
\tag{2.27} \quad f_{ij}(t, x) = \sum_{k=0}^{n} f_{i_k, j_k}(t, x)
$$

for $i, j = 1, \ldots, n$. It can be seen from Eq. 2.26 that flow and subflow intensities between the same compartments in the same flow direction are the same, that is,

$$
\tag{2.28} \quad \frac{f_{i_k, j_k}(t, x)}{x_{j_k}(t)} = \frac{f_{i_j}(t, x)}{x_j(t)}
$$

for $i, k = 0, \ldots, n$ and $j = 1, \ldots, n$ (see Fig. 2).

In Eq. 2.26 above,

$$
\tag{2.29} \quad d_{jk}(x) := \frac{x_{jk}(t)}{x_j(t)}
$$

will be called the decomposition factors. It is worth emphasizing that, due to the state decomposition, Eq. 2.13, the decomposition factors form a continuous partition of unity:

$$
\tag{2.30} \quad 0 \leq d_{jk}(x) \leq 1 \quad \text{and} \quad \sum_{k=0}^{n} d_{jk}(x) = 1.
$$

The decomposition and $k^{th}$ decomposition matrices, $D(x) := (d_{jk}(x))$ and $D_k(x) = \text{diag} ([d_{1k}(x), \ldots, d_{nk}(x)])$, can be formulated, accordingly, as

$$
\tag{2.31} \quad D(x) = \mathcal{X}^{-1}(t) X(t) \quad \text{and} \quad D_k(x) := \mathcal{X}^{-1}(t) \mathcal{X}_k(t)
$$

for $k = 0, \ldots, n$. Equations 2.23, 2.29 and 2.30 imply that

$$
\tag{2.32} \quad 1 = \mathcal{X}^{-1}(t) x(t) = \mathcal{X}^{-1}(t) X_0(t) + \mathcal{X}^{-1}(t) X(t) 1 = D_0(x) 1 + D(x) 1.
$$

We define the $k^{th}$ subflow rate matrix function as

$$
\tag{2.33} \quad F_k(t, x) := (f_{i_k j_k}(t, x))
$$

for $k = 0, \ldots, n$. Using Eq. 2.26, $F_k(t, x)$ can be expressed in matrix form as

$$
\tag{2.34} \quad F_k(t, x) = F(t, x) D_k(x) = F(t, x) \mathcal{X}^{-1}(t) \mathcal{X}_k(t).
$$

That is, the $k^{th}$ decomposition matrix, $D_k(x)$, decomposes the compartmental direct flow matrix, $F(t, x)$, into the subcompartmental subflow matrices, $F_k(t, x)$. Similarly, the $k^{th}$ output function

$$
\tag{2.35} \quad \mathcal{Y}_k(t, x) = \text{diag} ([f_{01_k}(t, x), \ldots, f_{0n_k}(t, x)])
$$

can be expressed in matrix form as

$$
\tag{2.36} \quad \mathcal{Y}_k(t, x) = \mathcal{Y}(t, x) D_k(x) = \mathcal{Y}(t, x) \mathcal{X}^{-1}(t) \mathcal{X}_k(t),
$$
and the \( k^{th} \) input matrix function can be written as

\[
Z_k(t, \mathbf{x}) := \text{diag} (z_k(t, \mathbf{x}) e_k)
\]

for \( k = 0, \ldots, n \). We set \( e_0 = \mathbf{0} \). The respective \( k^{th} \) output and input vector functions, \( y_k(t, \mathbf{x}) \) and \( z_k(t, \mathbf{x}) \), can be defined as

\[
y_k(t, \mathbf{x}) := Y_k(t, \mathbf{x}) \mathbf{1} \quad \text{and} \quad z_k(t, \mathbf{x}) := Z_k(t, \mathbf{x}) \mathbf{1}.
\]

Using these notations, the flow rate decompositions given in Eq. 2.27 and input decomposition formulated in Eq. 2.24 can be written in matrix form as follows:

\[
F(t, \mathbf{x}) = \sum_{k=0}^{n} F_k(t, \mathbf{x}), \quad Y(t, \mathbf{x}) = \sum_{k=0}^{n} Y_k(t, \mathbf{x}), \quad Z(t, \mathbf{x}) = \sum_{k=0}^{n} Z_k(t, \mathbf{x}).
\]

The equivalence of the flow and subflow rate intensities given in Eq. 2.28 can also be expressed in matrix form as

\[
F(t, \mathbf{x}) \mathcal{X}^{-1}(t) = F_k(t, \mathbf{x}) \mathcal{X}^{-1}_k(t)
\]

for \( k = 0, \ldots, n \).

The flow rate decomposition methodology given in Eq. 2.26 can then be schematized as follows:

\[
\begin{bmatrix}
    f_{11} & \cdots & f_{1n} \\
    f_{21} & \cdots & f_{2n} \\
    \vdots & \ddots & \vdots \\
    f_{n1} & \cdots & f_{nn}
\end{bmatrix}
\]

flow rate decomposition

\[
\begin{bmatrix}
    f_{1k,1k} & \cdots & f_{1k,n_k} \\
    f_{2k,1k} & \cdots & f_{2k,n_k} \\
    \vdots & \ddots & \vdots \\
    f_{nk,1k} & \cdots & f_{nk,n_k}
\end{bmatrix}
\]

2.2.3. Domain Decomposition. The dynamic system decomposition methodology uniquely yields new substate variables in a higher dimensional domain from the original ones. We will show in this section that the properties guarantee the existence and uniqueness of the solutions to the decomposed system on the new domain are inherited from those of the original system on the original domain.

There exists a unique decomposition \( \mathbf{x} \) for each \( \mathbf{x} \in \Omega \) with the relationship

\[
x_i(t) = \sum_{k=1}^{n} \bar{x}_{ik}(t) + x_{ik}(t), \quad \text{as given in Eq. 2.19}.
\]

The new domain that includes these substate variables is denoted by \( \bar{\mathcal{O}} \subset \mathbb{R}^{2n^2} \) and will be called the decomposed domain. This process can be represented as

\[
\mathbb{R}^n \supset \Omega \ni \mathbf{x} \xrightarrow{\text{domain decomposition}} \mathbf{x} \in \bar{\mathcal{O}} \subset \mathbb{R}^{2n^2}.
\]

There is a one-to-one correspondence between the original and decomposed domains. This correspondence is due to the existence and uniqueness of the governing systems in both the original and decomposed forms as shown further below in this section.

**Proposition 2.2.** The subflow rate functions \( f_{ik,jk}(t, \mathbf{x}) \) and \( \bar{f}_{ik,jk}(t, \mathbf{x}) \) are continuous and continuously differentiable in \( \mathbf{x} \) on domain \( \mathcal{I} \times \bar{\mathcal{O}} \subset \mathbb{R} \times \mathbb{R}^{2n^2} \).
Proof. We know that \( f_{ij}(t, x) \) is continuous and continuously differentiable in \( x \) on \( I \times \Omega \subset \mathbb{R} \times \mathbb{R}^n \). Note that, due to Eq. 2.13, \( x_j(t) \leq x_j(t) \). The decomposition factors \( d_j(x) = x_j(t)/x_j(t) \) are, therefore, well-defined even if \( x_j(t) \to 0 \). Note also that the decomposition factors are continuous and continuously differentiable with respect to \( x_j \) on \( I \times \overline{\Omega} \). Therefore,

\[
(2.42) \quad f_{ik,jk}(t, x) = \frac{x_{jk}(t)}{x_j(t)} f_{ij}(t, x)
\]

is also continuous and continuously differentiable in \( x_{jk} \) on \( I \times \overline{\Omega} \).

By construction, the subthroughflow functions, \( \tau_{ik}^{in}(t, x) \) and \( \tau_{ik}^{out}(t, x) \), as well as the net subthroughflow function, \( \tau_{ik}(t, x) \), are linear combination of \( f_{ik,jk}(t, x) \), as formulated in Eq. 2.44. Therefore, they have the same properties as \( f_{ik,jk}(t, x) \). That is, they are continuous and continuously differentiable in \( x \) on domain \( I \times \overline{\Omega} \subset \mathbb{R} \times \mathbb{R}^{2n^2} \). The same arguments with the same conclusions are also valid for \( \tilde{f}_{ik,jk}(t, x) \), \( \tilde{\tau}_{ik}^{in}(t, x) \), \( \tilde{\tau}_{ik}^{out}(t, x) \), and \( \tilde{\tau}_{ik}(t, x) \).

2.2.4. Subsystems. Using the dynamic system decomposition methodology composed of the analytic state and flow rate decomposition components, the system can explicitly be decomposed into mutually exclusive and exhaustive subsystems, each of which is driven by a single external input, except the initial subsystem (see Fig. 1 and 2).

The \( k \)th subcompartments of each compartment together with the corresponding \( k \)th subsystems, subflow rates, inputs, and outputs constitute the \( k \)th subsystem. Therefore, the system decomposition methodology generates mutually exclusive and exhaustive subsystems that are running within the original system and have the same structure and dynamics as the system itself, except for their external inputs and initial conditions. By mutual exclusiveness we mean that transactions are possible only within corresponding subcompartments of the system. By exhaustiveness we mean that all the generated subsystems sum to the entire system so partitioned. These otherwise-decoupled subsystems are coupled through the decomposition factors. Therefore, assuming positive external input and presence of initial stocks for each compartment in a system with \( n \) compartments, each compartment has \( n \) subcompartments and 1 initial subcompartment. The substorage of each of these subcompartments is derived from a single external input, and that of the initial subcompartment represents the evolution of the initial stocks. Therefore, the system has \( n + 1 \) subsystems indexed by \( k = 0, \ldots, n \). The initial subsystem \((k = 0)\) has no external input and has the same initial conditions as the original system. The initial subsystem is further decomposed into \( n \) subsystems, as formulated in Appendix A.3.

In Section 2.1, the governing equations are formulated for the original system, Eq. 2.10. In what follows, we will similarly introduce the governing equations for each subsystem. The governing equations for the \( k \)th subsystem can be written in vector form as

\[
(2.43) \quad \dot{x}_k(t) = \tau_{ik}^{in}(t, x) - \tau_{ik}^{out}(t, x)
= (z_k(t, x) + F_k(t, x) 1) - (y_k(t, x) + F_k^T(t, x) 1)
\]

for \( k = 0, \ldots, n \). The initial conditions are \( x_0(t_0) = x_0 \) and \( x_k(t_0) = 0 \) for \( k \neq 0 \).
the \( k^{th} \) subsystem given in Eq. 2.43 can then be expressed as

\[
\tau_{k}^{in}(t, x) := z_{k}(t, x) + F_{k}(t, x) 1
\]
\[
= Z_{k}(t, x) 1 + F(t, x) \mathcal{X}^{-1}(t) x_{k}(t),
\]
(2.44)

\[
\tau_{k}^{out}(t, x) := y_{k}(t, x) + F_{k}^{T}(t, x) 1
\]
\[
= \mathcal{Y}(t, x) \mathcal{X}^{-1}(t) x_{k}(t) + x_{k}(t) \mathcal{X}^{-1}(t) F^{T}(t, x) 1
\]
\[
= \left( \mathcal{Y}(t, x) + \text{diag} \left( F^{T}(t, x) 1 \right) \right) \mathcal{X}^{-1}(t) x_{k}(t)
\]
\[
= T(t, x) \mathcal{X}^{-1}(t) x_{k}(t).
\]

The \( k^{th} \) net subthroughflow rate vector, \( \tau_{k}(t, x) = [\tau_{1k}(t, x), \ldots, \tau_{nk}(t, x)]^{T} \), then becomes

\[
\tau_{k}(t, x) := \tau_{k}^{in}(t, x) - \tau_{k}^{out}(t, x) = z_{k}(t, x) + A(t, x) x_{k}(t)
\]
where, using the definition of \( F(t, x) \) given in Eq. 2.12,

\[
A(t, x) := F(t, x) \mathcal{X}^{-1}(t) = (F(t, x) - T(t, x)) \mathcal{X}^{-1}(t)
\]
\[
= Q^{x}(t, x) - \mathcal{R}^{-1}(t, x),
\]
(2.45)

\( Q^{x}(t, x) := F(t, x) \mathcal{X}^{-1}(t) \), and \( \mathcal{R}^{-1}(t, x) := T(t, x) \mathcal{X}^{-1}(t) \). Note that, \( A(t, x) \) is the difference of two matrices \( Q^{x}(t, x) \) and \( \mathcal{R}^{-1}(t, x) \) whose entries are the intercompartmental flow intensities and outward throughflow intensities, respectively. We will, therefore, call \( A(t, x) \) the flow intensity matrix. It is sometimes called the compartmental matrix. As indicated earlier in Eq. 2.4, \( Q^{x}(t, x) \) is called the coefficient matrix in general, but it will be called the storage distribution matrix in the context of the proposed methodology. The new matrix measure introduced in this work, \( R(t, x) \), will be called the residence time matrix [5].

The \( k^{th} \) inward and outward subthroughflow matrices, \( T_{k}^{in}(t, x) := \text{diag} \left( \tau_{k}^{in}(t, x) \right) \) and \( T_{k}^{out}(t, x) := \text{diag} \left( \tau_{k}^{out}(t, x) \right) \), can then be expressed as

\[
T_{k}^{in}(t, x) := Z_{k}(t, x) + \text{diag} \left( F(t, x) \mathcal{X}^{-1}(t) x_{k}(t) 1 \right)
\]
\[
T_{k}^{out}(t, x) := T(t, x) \mathcal{X}^{-1}(t) x_{k}(t).
\]
(2.47)

Note that, using Eq. 2.47, \( F_{k}(t, x) \) formulated in Eq. 2.34 can, alternatively, be written in terms of the system flows only:

\[
F_{k}(t, x) = F(t, x) \mathcal{R}^{-1}(t, x) T_{k}^{out}(t, x).
\]
(2.48)

We will call \( Q^{x}(t, x) := F(t, x) \mathcal{R}^{-1}(t, x) \) the flow distribution matrix [5].

We define the inward and outward subthroughflow matrices, \( T^{in}(t, x) \) and \( T^{out}(t, x) \), as the matrices whose \( k^{th} \) columns are the \( k^{th} \) inward and outward subthroughflow vectors, \( \tau_{k}^{in}(t, x) \) and \( \tau_{k}^{out}(t, x) \), \( k = 1, \ldots, n \), respectively:

\[
T^{in}(t, x) := \begin{bmatrix} \tau_{1}^{in}(t, x) & \cdots & \tau_{n}^{in}(t, x) \end{bmatrix},
\]
\[
T^{out}(t, x) := \begin{bmatrix} \tau_{1}^{out}(t, x) & \cdots & \tau_{n}^{out}(t, x) \end{bmatrix}.
\]
(2.49)

Using the relationships in Eq. 2.44, these subthroughflow matrices can be expressed in matrix form as

\[
T^{in}(t, x) = Z(t, x) + F(t, x) \mathcal{X}^{-1}(t) X(t),
\]
\[
T^{out}(t, x) = T(t, x) \mathcal{X}^{-1}(t) X(t).
\]
(2.50)
We then define the net subthroughflow matrix, \( T(t, x) \), as
\[
T(t, x) := T^{in}(t, x) - T^{in}(t, x) = \mathcal{Z}(t, x) + A(t, x) X(t).
\]

Due to Eq. 2.50, the decomposition matrix \( D(x) \) can be expressed in terms of the subthroughflow functions, instead of the substate functions, as
\[
D(x) = \mathcal{X}^{-1}(t) X(t) = T(t, x)^{-1} T^{out}(t, x).
\]

Note that, the subthroughflow matrices can be written in various forms as follows:
\[
\begin{align*}
T^{in}(t, x) - \mathcal{Z}(t, x) &= F(t, x) D(x) = Q^{r}(t, x) X(t) = Q^{r}(t, x) T^{out}(t, x) \\
T^{out}(t, x) &= R^{-1}(t, x) X(t).
\end{align*}
\]

These different forms prove useful particularly in the analyses of static systems as introduced by [5, 6]. At steady state, the flow and storage distribution matrices, \( Q^{r} \) and \( Q^{s} \), can be considered as linear transformations acting by left multiplication on subthroughflow and substorage matrices, \( T^{out} \) and \( X \), respectively, and map these matrices to the internal subthroughflow matrix, \( T^{in} - \mathcal{Z} \). The diagonal residence time matrix, \( R \), also acts on the subthroughflow matrix, \( T^{out} \), by left multiplication and maps this matrix to the substorage matrix, \( X \) [5].

For each fixed \( j \), Eqs. 2.28 or 2.50 imply that
\[
\tau^{out}_{jk}(t, x) = \sum_{i=0}^{n} f_{ij}(t, x) \frac{x_{j}(t)}{x_{j}(t)} = \sum_{i=0}^{n} f_{ijk}(t, x) \frac{\tau^{out}_{jk}(t, x)}{x_{jk}(t)}
\]
for \( k = 0, \ldots, n \). This equivalence between the outward throughflow and subthroughflow intensities given in the first and last equalities of Eq. 2.54 can be expressed in matrix form as
\[
\mathcal{R}^{-1}(t, x) = T(t, x) \mathcal{X}^{-1}(t) = T^{out}(t, x) X^{-1}(t)
\]
where the last equality is derived from Eq. 2.53. This relationship agrees with Eq. 2.47. This proportionality is used in the derivation of the static direct flows and storages in matrix form by [5] as listed in Table 1. The residence time matrix, \( R(t, x) \), has a central role in the integration of various system components, and, thus, the holistic analysis of the static systems as introduced by [5].

Equations 2.28 and 2.54 also imply that
\[
\frac{\tau^{out}_{jk}(t, x)}{\tau^{out}_{jk}(t, x)} = \frac{x_{jk}(t)}{x_{jk}(t)} = \frac{\mathcal{X}^{out}_{jk}(t, x)}{\mathcal{X}^{out}_{jk}(t, x)}
\]
for \( k, \ell = 0, \ldots, n \). This relationship indicates the proportionality of the parallel subflows and corresponding subthroughflows and substorages. By parallel subflows, we mean the intercompartmental flows that transit through different subcompartments of the same compartment along the same flow path at the same time. The flow path terminology is developed in Appendix C.1.

Using Eq. 2.55, the \( k^{th} \) decomposition matrix, \( D_{k}(x) \), can be expressed as
\[
D_{k}(x) = \mathcal{X}^{-1}(t) X_{k}(t) = T(t, x)^{-1} T^{out}_{k}(t, x),
\]
similar to the decomposition matrix formulated in Eq. 2.52. It is worth noting that, the decomposition and \( k^{th} \) decomposition matrices, \( D(x) \) and \( D_{k}(x) \), decompose the
HUSEYIN COSKUN

compartmental throughflow matrix, \( T(t, x) \), into the outward subthroughflow and \( k^{th} \) subthroughflow matrices as indicated in Eqs. 2.53 and 2.57, similar to the decomposition of \( F(x) \) as formulated in Eq. 2.34. That is,

\[
(2.58) \quad T^{out}(t, x) = T(t, x) D(x) \quad \text{and} \quad T^{out}_k(t, x) = T(t, x) D_k(x).
\]

At steady state, based on the relationships given in Eqs. 2.48, 2.53, and 2.58, \( D \) and \( D_k \) can be considered as linear transformations that map the system flows and throughflows, acting on them by right multiplication, to the subflows and subthroughflows.

It is worth noting also the relationships given below between the flow and subthroughflow matrices:

\[
\begin{align*}
    z(t, x) + F(t, x) 1 &= \sum_{k=0}^{n} z_k(t, x) + F_k(t, x) 1 = \sum_{k=0}^{n} \tau^{in}_k(t, x) = \tau^{in}(t, x) \\
    &= \tau^{in}_0(t, x) + T^{in}(t, x) 1, \\
    y(t, x) + F^T(t, x) 1 &= \sum_{k=0}^{n} y_k(t, x) + F^T_k(t, x) 1 = \sum_{k=0}^{n} \tau^{out}_k(t, x) = \tau^{out}(t, x) \\
    &= \tau^{out}_0(t, x) + T^{out}(t, x) 1.
\end{align*}
\]

The governing equations for the subsystems of the decomposed system can then be written in vector form as

\[
(2.59) \quad \dot{x}_k(t) = z_k(t, x) + A(t, x) x_k(t)
\]

with the initial conditions \( x_0(t_0) = x_0 \) and \( x_k(t_0) = 0 \) for \( k = 1, \ldots, n \). The governing equations for the decomposed system, Eq. 2.62, can similarly be expressed in matrix form using the matrix functions introduced above as follows:

\[
(2.60) \quad \dot{X}(t) = T(t, x) = T^{in}(t, x) - T^{out}(t, x), \quad X(t_0) = 0,
\]

\[
\dot{x}_0(t) = \tau_0(t, x) = \tau^{in}_0(t, x) - \tau^{out}_0(t, x), \quad x_0(t_0) = x_0.
\]

This system can also be expressed in terms of the flow intensity matrix, \( A(t, x) \):

\[
(2.61) \quad \dot{X}(t) = Z(t, x) + A(t, x) X(t), \quad X(t_0) = 0,
\]

\[
\dot{x}_0(t) = A(t, x) x_0(t), \quad x_0(t_0) = x_0.
\]

The system decomposition mechanism yielding governing equations from the original system for each subsystem in vector form, or for the entire system in matrix form, can be schematized as follows:
\[ \dot{x}(t) = \tau(t, x) \quad \dot{x}_k(t) = \tau_k(t, x), \quad k = 0, \ldots, n \]

The governing equations for the decomposed system, Eqs. 2.62 and 2.63, are already expressed in vector forms in Eqs. 2.43 and A.41 as follows:

\[ \dot{x}_k(t) = z_k(t, x) + A(t, x) \dot{x}_k(t), \quad x_k(t_0) = 0 \]

\[ \dot{x}_k(t) = A(t, x) \ddot{x}_k(t), \quad x_k(t_0) = x_{k,0} e_k \]

for \( k = 1, \ldots, n \). Summing up the governing equations over \( k \) separately for both subsystems and initial subsystems formulated in Eq. 2.64 yields the system

\[ \dot{x}(t) = z(t, x) + A(t, x) \dot{x}(t), \quad x(t_0) = 0, \]

This system enables the analysis of the evolution of external inputs and initial conditions within the system separately. Adding these two equations side by side gives back the original system, Eq. 2.11, in the following form:

\[ \dot{x}(t) = z(t, x) + A(t, x) x(t), \quad x(t_0) = x_0 \]
as \( F(t, x) \mathbf{1} = A(t, x) x(t) \). The decomposition formulated in Eq. 2.65 could directly be obtained from the original system by defining a decomposition with two subsystems—one for external inputs and the other for the initial conditions.

The governing equations, Eqs. 2.62 and 2.63, for the decomposed system are already expressed in matrix form in Eqs. 2.61 and A.43 as follows:

\[
\begin{align*}
\dot{X}(t) &= Z(t, x) + A(t, x) X(t), \quad X(t_0) = 0, \\
\dot{\bar{X}}(t) &= A(t, x) \bar{X}(t), \quad \bar{X}(t_0) = \bar{X}_0.
\end{align*}
\]  

Let a new matrix \( X(t) \) be defined component-wise as \( X_{ik}(t) := \bar{x}_{ik}(t) + x_{ik}(t) \). The governing equation for \( X(t) = \dot{X}(t) + \bar{X}(t) \) then becomes

\[
\dot{X}(t) = Z(t, x) + A(t, x) X(t), \quad X(t_0) = X_0.
\]  

Note that Eq. 2.67 is the decomposed form of Eq. 2.65, and Eq. 2.68 is the decomposed form of Eq. 2.66.

2.3. Nonlinear Fundamental Matrix Solutions. In this section, the nonlinear fundamental matrix solutions will be introduced for dynamic compartmental models. They will be called the fundamental matrix solutions due to the common properties outlined in Theorem 2.4 with fundamental matrix solutions to systems of linear ordinary differential equations. We will first show the existence and uniqueness of the decomposed system in vector form.

**Theorem 2.3.** Let \((t_0, x_0) \in \mathcal{I} \times \Omega\). There exists a positive integer \( r > 0 \) and an interval \( \mathcal{I}' = (t_0 - r, t_0 + r) \subset \mathcal{I} \) such that the governing equations Eq. 2.64 for the decomposed system has a unique solution passing through \((t_0, x_0)\) on \( \mathcal{I}' \).

**Proof.** The net subthroughflow functions \( \tau_{ik}(t, x) \) and \( \bar{\tau}_{ik}(t, x) \) on the right hand side of the decomposed system Eq. 2.64 are continuous and continuously differentiable for \( x \) on \( \mathcal{I} \times \Omega \), as shown in Proposition 2.2. The existence and uniqueness of solutions to Eq. 2.64 on \( \mathcal{I}' \) is an immediate consequence of Picard’s local existence and uniqueness theorem.

The definitions of nonlinear fundamental matrix solutions and their main properties are outlined in the following theorem.

**Theorem 2.4.** Let \( X(t) \) and \( \bar{X}(t) \) be the matrix functions defined in Eqs. 2.21 and A.3.

1. \( X(t) \) and \( \bar{X}(t) \) are the unique matrix solutions to the decomposed system, Eq. 2.67. They will be called the nonlinear fundamental matrix solutions of Eq. 2.67.
2. For any given \((t_0, x_0) \in \mathcal{I} \times \Omega\), the unique solution to the original system, Eq. 2.11, is given by

\[
x(t) = \bar{X}(t) \mathbf{1} + X(t) \mathbf{1}.
\]

That is, \( x(t) \) is the linear combination of the columns of \( \bar{X}(t) \) and \( X(t) \), where all the combination coefficients are 1.

3. Let \( x_{i,0} > 0 \) and \( z_i(t, x) > 0 \), \( t \in \mathcal{I} \), for all \( i \). The column vectors of \( \bar{X}(t) \) and \( X(t) \) are linearly independent vectors in \( \mathbb{R}^n \). Therefore, both \( \bar{X}(t) \) and \( X(t) \) are invertible matrices at any time \( t \in \mathcal{I} \) under given conditions.

**Proof.**

1. The existence and uniqueness of the solution to the decomposed system in vector form, Eq. 2.64, is shown in Thm. 2.3. The existence and
uniqueness of the system in matrix form, Eq. 2.67, follows those of the system Eq. 2.64, by a column-wise comparison of both sides of the matrix equation, Eq. 2.67.

2. By the principle of nonlinear decomposition stated in Thm. 2.5,

\[ \mathbf{x}(t) = \mathbf{X}(t) \mathbf{1} = \bar{\mathbf{X}}(t) \mathbf{1} + \mathbf{X}(t) \mathbf{1} = \sum_{k=1}^{n} \bar{x}_k(t) + \sum_{k=1}^{n} x_k(t) \]

is a solution of the original system Eq. 2.11. The uniqueness of this solution follows the uniqueness of the decomposed system Eq. 2.64 as shown in part (1) of this theorem.

3. Let \( \bar{x}_i(t) \) and \( x_i(t) \) be solution of the decomposed system Eq. 2.64. We would like to show that, for each fixed \( t \), the set of vectors \( \{\bar{x}_1(t), \ldots, \bar{x}_n(t)\} \) and \( \{x_1(t), \ldots, x_n(t)\} \) in \( \mathbb{R}^n \) are linearly independent.

We will first show that \( \{\bar{x}_1(t), \ldots, \bar{x}_n(t)\} \) is linearly independent set in \( \mathbb{R}^n \) for each fixed \( t \in \mathcal{I} \). Suppose that, there exists a \( t_1 \in \mathcal{I} \) such that the column vectors in \( \{x_1(t_1), \ldots, x_n(t_1)\} \) are linearly dependent. There exists then a combination constants \( c_1, \ldots, c_n \) not all zero, such that

\[ 0 = c_1 x_1(t_1) + \ldots + c_n x_n(t_1) = X(t_1) c \]

where \( c = (c_1, \ldots, c_n)^T \in \mathbb{R}^n \). Let

\[ \alpha(t) := c_1 x_1(t) + \ldots + c_n x_n(t) = X(t) c, \quad t \in \mathcal{I}. \]

Therefore, \( \dot{\alpha}(t) = \dot{X}(t) c \). From the governing matrix equation for \( X(t) \) in Eq. 2.67 and also Eq. 2.70, we have

\[ \dot{\alpha}(t) = \mathcal{Z}(t, x) c + A(t, x) \alpha(t), \quad \alpha(t_0) = 0. \]

Equation 2.69 implies that \( \alpha(t_1) = 0 \). Without loss of generality, assume that \( c_i > 0 \) for some \( i \). We then have

\[ \alpha_i(t_0) = 0, \quad \dot{\alpha}_i(t_0) = c_i z_i(t_0, x) > 0, \quad \text{and} \]

\[ \alpha_i(t_1) = 0, \quad \dot{\alpha}_i(t_1) = c_i z_i(t_1, x) > 0, \quad \forall i, \]

as \( z_i(t, x) > 0 \). Since \( \alpha(t) \) is a differentiable and, therefore, is a continuous function, Eq. 2.72 implies that there exists at least one \( t^* \in (t_0, t_1) \) such that \( \alpha_i(t^*) = 0 \) and \( \dot{\alpha}_i(t^*) < 0 \). Due to Eq. 2.71, this result implies that \( z_i(t^*, x) < 0 \). This contradiction completes the first part of the proof.

Now, we would like to show that \( \{\bar{x}_1(t), \ldots, \bar{x}_n(t)\} \) is a linearly independent set in \( \mathbb{R}^n \) for each fixed \( t \in \mathcal{I} \). The condition for this case is that \( x_{i,0} > 0, \forall i \). Suppose now that at the same \( t_1 \in \mathcal{I} \), the column vectors in \( \{\bar{x}_1(t_1), \ldots, \bar{x}_n(t_1)\} \) are linearly dependent in \( \Omega \). There then exists constants \( \bar{c}_1, \ldots, \bar{c}_n \) not all zero, such that

\[ 0 = \bar{c}_1 \bar{x}_1(t_1) + \ldots + \bar{c}_n \bar{x}_n(t_1) = \bar{X}(t_1) \bar{c} \]

where \( \bar{c} = (\bar{c}_1, \ldots, \bar{c}_n)^T \in \mathbb{R}^n \). Let

\[ \bar{\alpha}(t) := \bar{c}_1 x_1(t) + \ldots + \bar{c}_n x_n(t) = \bar{X}(t) \bar{c}, \quad t \in \mathcal{I}. \]
This implies that \( \dot{\alpha}(t) = X(t) \overline{c} \). From the governing matrix equation, Eq. 2.67, for \( \overline{X} \), and Eq. 2.73, we have

\[
\dot{\alpha}(t) = A(t, x) \alpha(t), \quad \alpha(t_1) = 0.
\]

Due to the uniqueness theorem, Thm. 2.3, \( \dot{\alpha}(t) = 0 \), \( t \in I \). In particular, \( \alpha(t_0) = \overline{X}(t_0) \overline{c} = X_0 \overline{c} = 0 \Rightarrow \overline{c} = 0 \)

as \( x_{t_0} > 0 \), \( \forall i \), which is a contradiction.

These two contradictions for each part of the system Eq. 2.67 indicate that neither the set of the column vectors of \( \overline{X}(t) \) nor that of \( X(t) \) can be linearly dependent at any \( t_1 \in I \). Therefore, the column vectors of \( \overline{X}(t) \) and \( X(t) \) form linearly independent sets, and consequently, the matrices are invertible for all \( t \in I \).

**2.4. Nonlinear Decomposition Principle.** We state the decomposition principle for nonlinear dynamic compartmental systems in the following theorem. It essentially asserts that the solution for each subsystem also solves the original system, so is any arbitrary combination of these solutions as specified in the theorem.

**Theorem 2.5.** Let \( d_i(x) \) and \( \overline{d}_i(x) \) be the decomposition factors of subsystems and initial subsystems, respectively, based on which the original system, Eq. 2.10, is decomposed into Eq. 2.64. Let also \( x_k(t) \) and \( \overline{x}_k(t) \) be the respective solutions on \( \Omega \) to the \( k \)th subsystem and initial subsystem of the decomposed system with the corresponding external inputs and initial conditions given in Eq. 2.64. The following combination of the vector functions

\[
x(t) = \sum_{k=1}^{n} \alpha_k x_k(t) + \beta_k \overline{x}_k(t), \quad \alpha_k, \beta_k \in \{0, 1\},
\]

then is a solution to the original system with the following external inputs and initial conditions on \( \Omega \):

\[
z(t, x) = \sum_{k=1}^{n} \alpha_k z_k(t, x) = \sum_{k=1}^{n} \alpha_k z_k(t, x) e_k \quad \text{and} \quad x_0 = x(t_0) = \sum_{k=1}^{n} \beta_k \overline{x}_k(t_0) = \sum_{k=1}^{n} \beta_k x_{k,0} e_k.
\]

**Proof.** Note that, if \( \alpha_k = 0 \) or \( \beta_k = 0 \) for some \( k \), the corresponding solutions to Eq. 2.64 become \( x_k(t) = 0 \) or \( \overline{x}_k(t) = 0 \), respectively, with the conditions given in Eq. 2.75. This is because of the fact that the subsystems are driven either by external inputs or initial conditions and, therefore, if there is no external input or the initial condition is zero for a subsystem or initial subsystem, respectively, the corresponding subsystem becomes null—the substate variables (and the subflows rates) for that subsystem or initial subsystem become zero.

Due to the construction of the system decomposition, we have

\[
x(t) = X(t) 1 = \overline{X}(t) 1 + X(t) 1 = \sum_{k=1}^{n} x_k(t) + x_k(t).
\]
Therefore, multiplying both sides of the governing equation, Eq. 2.68, by $1$ yields the original system in the form of Eq. 2.66 because of the fact that $F(t, x) = A(t, x) \mathbf{x}(t)$.

Consequently, if $\mathbf{x}_k$ and $\bar{\mathbf{x}}_k$ are the respective solutions to the $k^{th}$ subsystem and initial subsystem of the decomposed system, Eq. 2.64, on $\bar{\Omega}$, $\mathbf{x}$ is the solution to the original system, Eq. 2.10, on $\Omega$.

This nonlinear decomposition principle corresponds to the superposition principle for the linear ordinary differential equations. It is in the sense that, the solution to a nonlinear system can be decomposed into subsolutions, each of which, as well as any arbitrary combination of them as specified in Thm. 2.5, solves the original system.

2.5. Dynamic Subsystem Decomposition. We introduce the dynamic subsystem decomposition methodology in this section for further partitioning or segmentation of subsystems along a given set of mutually exclusive and exhaustive subflow paths. This decomposition enables the determination of the distribution of arbitrary intercompartental flows and the organization of the associated storages generated by these flows within the subsystems. Therefore, the subsystem decomposition methodology dynamically decomposes arbitrary composite intercompartental flows and the associated storages generated by these flows into the constituent transient subflow and substorage segments along given subflow paths.

The dynamic subsystem methodology will be formulated below using the directed subflow paths terminology introduced in Appendix C. The subsystems can further be decomposed into subflows and associated substorages along a set of mutually exclusive and exhaustive directed subflow paths. By mutually exclusive subflow paths we mean that no given subflow path in a subsystem is a subpath, that is, completely inside of another path in the same subsystem. The exhaustiveness, in this context, means that such mutually exclusive subflow paths all together sum to the entire subsystem subflows and associated substorages. We will use the notation $P_{i_k,j_k}$ for a set of mutually exclusive and exhaustive subflow paths from subcompartment $j_k$ to $i_k$ in subsystem $k$ and the number of subflow paths in $P_{i_k,j_k}$ will be denoted by $w_k$. The natural subsystem decomposition defined in Appendix C yields a mutually exclusive and exhaustive decomposition of the entire system.

We will first introduce the transient flows and associated storages below. The transient flows and storages will then be used for the formulation of the dialect flows and storages in the next section.

2.5.1. Transient Flows and Storages. The transient and cumulative transient subflows along a subflow path between two subcompartments will be defined in what follows. Along a given subflow path $p_{n_k,i_k} = i_k \rightarrow j_k \rightarrow \ell_k \rightarrow \cdots \rightarrow n_k$, the transient inflow at subcompartment $\ell_k$, $f_{i_k,j_k}^w(t)$, is the subflow segment transmitted to $\ell_k$ at time $t$, which is generated by the local input from subcompartment $i_k$ (local source) into the first subcompartment of the path, $j_k$, (connection) during $[t_1, t]$, $t_1 \geq t_0$.

Similarly, the transient outflow at subcompartment $\ell_k$, $f_{n_k,\ell_k}^w(t)$, is the subflow segment transmitted from $\ell_k$ to the next subcompartment, $n_k$, along the path at time $t$, which is generated by the transient inflow into $\ell_k$ during $[t_1, t]$. The associated transient substorage, $x_{n_k,\ell_k,j_k}^w(t)$, is then the substorage segment in subcompartment $\ell_k$ at time $t$, which is derived from the transient inflow and governed by the transient inflow and outflow balance during $[t_1, t]$.

The transient outflow at subcompartment $\ell_k$ at time $t$, from $j_k$ to $n_k$ along subflow
path \( p_{nk}^{w} \), can be formulated as

\[
(2.76) \quad f_{nk,\ell,jk}^{w}(t) = \frac{f_{nk,\ell}(t, x)}{x_{\ell}(t)} x_{nk,\ell,jk}^{w}(t),
\]

similar to Eq. 2.26, where the transient substorage \( x_{nk,\ell,jk}^{w}(t) \) is determined by the governing equation

\[
(2.77) \quad \dot{x}_{nk,\ell,jk}^{w}(t) = f_{nk,\ell,jk}^{w}(t) - \frac{\tau_{\ell,k}^{\text{out}}(t, x)}{x_{\ell}(t)} x_{nk,\ell,jk}^{w}(t), \quad x_{nk,\ell,jk}^{w}(t) = 0.
\]

The equivalence of the throughflow and subthroughflow intensities as well as the flow and subflow intensities in the same direction, that is

\[
g_{\ell}^{\text{out}}(t, x) = \frac{f_{nk,\ell}(t, x)}{x_{\ell}(t)} = \frac{f_{nk,\ell}(t, x)}{x_{\ell}(t)} \quad \text{and} \quad r_{\ell,k}^{-1}(t, x) = \frac{\tau_{\ell,k}^{\text{out}}(t, x)}{x_{\ell}(t)} = \frac{\tau_{\ell,k}^{\text{out}}(t, x)}{x_{\ell}(t)}
\]

are given by Eqs. 2.40 and 2.55, for \( \ell, n = 1, \ldots, n \), and \( k = 0, 1, \ldots, n \). Therefore, since the rational expressions in Eqs. 2.76 and 2.77 can be expressed at both compartmental and subcompartmental levels, the subsystem decomposition is actually independent from the state decomposition. Note that the initial condition given in Eq. 2.77 for the initial subsystem \((k = 0)\) is \( x_{nk,\ell,jk}^{w}(t_1) = x_{\ell,0} \), and this initial value of \( x_{\ell,0} \) is not considered as a transient substorage. The governing equations, Eqs. 2.76 and 2.77, establishes the foundation of the dynamic subsystem decomposition (see Fig. 3). These equations for each subcompartment along a given flow path of interest will then be coupled with the decomposed system, Eqs. 2.62 and 2.63, or the original system, Eq. 2.6, and be solved simultaneously.

If the original system, Eq. 2.6, is linear and the decomposed system, Eqs. 2.62 and 2.63, is also linear and can be solved analytically as formulated in Appendix B, Eq. 2.77 can be solved explicitly for \( x_{nk,\ell,jk}^{w}(t) \) as well. The solution becomes

\[
(2.78) \quad x_{nk,\ell,jk}^{w}(t) = \int_{t_1}^{t} e^{-\int_{t_1}^{s} r_{\ell,k}^{-1}(s', x) \, ds'} f_{nk,\ell,jk}^{w}(s) \, ds
\]

where the outward throughflow intensity \( r_{\ell,k}^{-1}(t, x) = \frac{\tau_{\ell,k}^{\text{out}}(t, x)}{x_{\ell}(t)} \) is defined in Eq. 2.46. Equation 2.78 formulates the transient storage \( x_{nk,\ell,jk}^{w}(t) \) generated by the transient inflow, \( f_{nk,\ell,jk}^{w}(t) \), at subcompartment \( \ell_k \) dynamically (see Fig. 3).

The sum of the transient inflows from subcompartment \( j_k \) to \( \ell_k \) and the outflows from \( \ell_k \) to \( n_k \) at subcompartment \( \ell_k \) along a given self-intersecting path \( p_{nk}^{w} \) will.
be called the cumulative transient inflow, \( f^w_{i_k,j_k}(t) \), and outflow, \( f^w_{n_k \ell_k}(t) \), respectively, and the associated total substorage will be called the cumulative transient substorage, \( x^w_{\ell_k}(t) \). They can be formulated as

\[
\begin{align*}
\tau_{i_k,j_k}^w(t) := & \sum_{m=1}^{m_w} f^w_{i_k,j_k i_k}(t), \quad \tau_{n_k \ell_k}^w(t) := \sum_{m=1}^{m_w} f^w_{n_k \ell_k j_k}(t), \quad \text{and} \\
x^w_{\ell_k}(t) := & \sum_{m=1}^{m_w} x^w_{n_k \ell_k j_k}(t)
\end{align*}
\]

(2.79)

where the superscript \( m \) represents the cycle number, and \( m_w \) is the number of cycles, that is, the number of times the path \( p^w_{n_k i_k} \) intersects itself.

The transient subflows and substorages along a given subflow path within the initial subsystems can be defined similarly.

### 2.5.2. The diact Flows and Storages

Five important transaction types are introduced in this section based on the subsystem decomposition methodology: the diact flows and associated storages. The transfer flows (denoted by \( t \)) and storages will be formulated in detail below, and parallel derivation for direct (\( d \)), indirect (\( i \)), cycling (\( c \)), and acyclic (\( a \)) flows and storages can be found in Appendix D.

The transfer flow will be defined as the total intercompartamental transient flow from one compartment, directly or indirectly through other compartments, to another. The direct and indirect flow will be defined as the transfer flows from one compartment to another directly and indirectly through other compartments, respectively. The cycling flow will be defined as the transfer flow from a compartment, indirectly through other compartments, back into itself. Lastly, the acyclic flow at a compartment will be defined as the non-cyclic segment of the compartmental throughput flow at that compartment. The diact storage is then defined as the storage generated by the corresponding diact flow. The diact flows and storages at both subcompartamental and compartmental levels are formulated below and in Appendix D (see Fig. 4).

The transfer subflow will be defined as the total intercompartamental transient flow from one subcompartment, directly or indirectly through other subcompartments, to another in the same subsystem. Let \( P^w_{i_k,j_k} \) be the set of mutually exclusive and exhaustive subflow paths \( p^w_{i_k,j_k} \) from subcompartment \( j_k \) directly or indirectly to \( i_k \) in subsystem \( k \). The transfer inflow from subcompartment \( j_k \) to \( i_k \), \( \tau_{i_k,j_k}^w(t) \), is defined as the sum of the cumulative transient inflows generated by local inputs initiated at \( j_k \) during \([t_0, t]\) and transmitted to \( i_k \) at time \( t \) along all subflow paths in \( P^w_{i_k,j_k} \). The associated transfer substorage, \( x^w_{i_k,j_k}(t) \), at subcompartment \( i_k \) at time \( t \) is the sum of the cumulative transient substorages derived from transfer inflow \( \tau_{i_k,j_k}^w(t) \) during \([t_0, t]\). The transfer inflow and substorage can then be formulated as

\[
\tau_{i_k,j_k}^w(t) = \sum_{w=1}^{w_k} \sum_{t=1}^{n} f^w_{i_k,j_k}(t) \quad \text{and} \quad x^w_{i_k,j_k}(t) = \sum_{w=1}^{w_k} x^w_{i_k}(t)
\]

(2.80)
For notational convenience, we define $n \times n$ matrix functions $T_k^x(t)$ and $X_k^x(t)$ whose $(i,j)$-elements are $\tau_{ij,k}^x(t)$ and $x_{ij,k}^x(t)$, respectively. That is,

\begin{equation}
T_k^x(t) = (\tau_{ij,k}^x(t)) \quad \text{and} \quad X_k^x(t) = (x_{ij,k}^x(t)).
\end{equation}

These matrix measures $T_k^x(t)$ and $X_k^x(t)$ are called the $k^{th}$ transfer subflow and associated substorage matrix functions. The corresponding transfer flow and associated storage matrix measures are $T^x(t) = (\tau_{ij}^x(t))$ and $X^x(t) = (x_{ij}^x(t))$, respectively.

Let $P_k^{d,ij,k}$ and $P_k^{l,ij,k}$ be defined as the sets of subflow paths $p_{ij,k}^d$ from subcompartment $j_k$, directly and indirectly, to $i_k$, respectively; $P_k^{s}$ be the set of subflow paths $p_{ik}^w$ from subcompartment $i_k$ indirectly back to itself; and $P_k^{x}$ be the set of linear subflow paths $p_{ik}^w$ from subcompartment $k_k$, directly or indirectly, to $i_k$ in subsystem $k$. All these diact subflow sets are assumed to be mutually exclusive and exhaustive. The transfer flows, associated storages, and corresponding matrix functions are formulated in Eqs. 2.80, 2.81, and 2.82 using the subflow set $P_k^{a,ij,k}$. The other diact flows, associated storages, and matrix functions can then be formulated similarly by substituting the corresponding diact flows and storages for their transfer counterparts in these equations and by using the corresponding diact subflow sets instead. Figure 9 depicts the complementary nature of the direct, indirect and cycling flows.

The direct and indirect subflow will be defined as the transfer subflows from one subcompartment, directly and indirectly through other subcompartments, to another in the same subsystem, respectively. The indirect subflow, $\tau_{ik,jk}^i(t)$, from subcompartment $j_k$ to $i_k$ can be considered as the transfer subflow diminished by the direct subflow from $j_k$ to $i_k$ at time $t$ (see Fig. 4). Therefore, it can also be formulated as

\begin{equation}
\tau_{ik,jk}^i(t) = \tau_{ik,jk}^x(t) - f_{ik,jk}(t,x).
\end{equation}

Consequently, we have

\begin{equation}
T^x(t) = T^d(t) + T^i(t) \quad \text{and} \quad X^x(t) = X^d(t) + X^i(t).
\end{equation}

There is a functional similarity between $T^i(t)$ and $T^d(t) = F(t,x)$; the $(i,k)$-element of $T^i(t)$, $\tau_{ik}^i(t)$, is the indirect flow, while that of $F(t,x)$, $\tau_{ik}^d(t) = f_{ik}(t,x)$, is the direct flow from compartment $k$ to $i$ at time $t$ (see Fig. 4).

The cycling subflow will be defined as the transfer subflow from a subcompartment, indirectly through other subcompartments in the same subsystem, back into itself. The cycling subflow and associated substorage matrices, $T^c(t)$ and $X^c(t)$, are
formulated in Appx. D. Due to the construction of cycling flow as reflexive transfer or indirect flow, we have

\begin{align}
T^c(t) &= \text{diag} (T^c(t)) = \text{diag} (T^i(t)), \\
X^c(t) &= \text{diag} (X^c(t)) = \text{diag} (X^i(t)).
\end{align}

Note that, the cycling flow and subflow as well as the cycling storage and substorage matrices are related as

\( T^c(t) = \text{diag} (T^c(t) 1) \) and \( X^c(t) = \text{diag} (X^c(t) 1). \)

Lastly, the acyclic subflow at a subcompartment will be defined as the non-cyclic segment of the subthroughflow at that subcompartment. In other words, the acyclic flows and associated storages are generated by external inputs directly or indirectly through linear subflow paths. In that sense, they quantify through (non-cyclic) influence of environment on system compartments. The acyclic subflow and associated substorage matrices can be formulated as

\begin{align}
T^a(t) &= \tilde{T}(t, x) - T^c(t) \quad \text{and} \quad X^a(t) = X(t) - X^c(t).
\end{align}

Note that, the \((i, k)\) element of \( T^a(t) \) and \( T^c(t) \), \( \tau^a_{ik}(t) \) and \( \tau^c_{ik}(t) \), represent the acyclic flow through linear subflow paths from compartment \( k \) to \( i \) and cycling flow at subcompartment \( i_k \) at time \( t \), respectively, generated by external input \( z_k(t) \) during \([t_0, t]\).

It is worth mentioning that the indirect subflow from an input-receiving subcompartment \( k_k \) to \( i_k \) can, alternatively, be formulated as

\begin{align}
\tau^i_{ik,k_k}(t) &= \sum_{j=1, j \neq k}^n f_{i_k j_k} (t, x) = \tau^i_{ik}(t, x) - f_{i_k k_k} (t, x) - z_{ik}(t, x)
\end{align}

for \( i, k = 1, \ldots, n \). Similarly, the transfer subflow from \( k_k \) to \( i_k \) becomes

\begin{align}
\tau^t_{ik,k_k}(t) &= \sum_{j=1}^n f_{i_k j_k} (t, x) = \tau^t_{ik}(t, x) - z_{ik}(t, x).
\end{align}

The cycling subflow from an input-receiving subcompartment \( i_i \) to itself, \( \tau^c_{ii}(t) \), can then be formulated in terms of the indirect or transfer subflows as

\begin{align}
\tau^c_{ii}(t) &= \tau^i_{ii}(t) = \tau^t_{ii}(t) = \sum_{j=1}^n f_{i j} (t, x) = \tau^i_{ii}(t, x) - z_i(t, x).
\end{align}

Consequently,

\begin{align}
\tau^a_{ii}(t) = \tau^i_{ii}(t, x) - \tau^c_{ii}(t) = z_i(t, x).
\end{align}

The diact flows and storages can similarly be defined for initial subsystems.

2.6. System Analysis and Measures. The dynamic system decomposition methodology yields the subthroughflow and substorage matrices that measure the external influence on system compartments in terms of the flow and storage generation. For the quantification of intercompartmental flow and storage dynamics, the dynamic
The transient and dynamic diact flows and associated storages. These mathematical system analysis tools and their interpretation as quantitative system indicators will be discussed in this section.

The elements of the fundamental matrix solutions, that is, those of the substate and initial substate matrices, \( \bar{X}(t) \) and \( X(t) \), represent the organization of storages within the system generated by the initial stocks and external inputs, respectively. More specifically, \( \bar{x}_{ik}(t) \) represents the storage value in compartment \( i \) at time \( t \), derived from the initial stock in compartment \( k \) during time interval \([t_0, t] \). Similarly, \( x_{ik}(t) \) represents the storage in compartment \( i \) at time \( t \) generated by the external input into compartment \( k \), \( z_k(t) \), during \([t_0, t] \) (see Fig. 1). In other words, the proposed methodology can dynamically partition composite compartmental storages into subcompartmental segments based on their constituent external sources. This decomposition enables tracking the evolution of the storages generated by the external inputs and initial stocks individually and separately within the system. The state variable, \( x_i(t) \), which represents the composite compartmental storage, cannot be used to distinguish the portions of this storage derived from different individual external inputs or initial stocks. Therefore, the solution to the decomposed system brings out inferences that cannot be obtained through the analysis of the original system by the state of the art techniques.

The elements of net subthroughflow and initial subthroughflow rate matrices, \( \bar{T}(t, x) \) and \( T(t, x) \), represent the distribution of the subthroughflows within the system generated by the initial storages and external inputs, respectively. More specifically, \( \bar{\tau}_{ik}(t, x) \) represents the net subthroughflow rate at compartment \( i \) at time \( t \) generated by the initial stock in compartment \( k \) during \([t_0, t] \). Similarly, \( \tau_{ik}(t, x) \) represents the net subthroughflow rate at compartment \( i \) at time \( t \), generated by the external input into compartment \( k \) during \([t_0, t] \) (see Fig. 2). In other words, the proposed methodology can dynamically partition composite compartmental throughflows into subcompartmental segments based on their constituent external sources. This decomposition enables tracking the evolution of the initial stocks and external inputs within the system individually and separately. Thus, the subthroughflow and initial subthroughflow functions of the decomposed system, \( \tau_{ik}(t, x) \) and \( \bar{\tau}_{ik}(t, x) \), provide more detailed information than the composite throughflow function of the original system, \( \tau_i(t, x) \), similar to the state and substate variables, as explained above.

The transient flows and associated storages transmitted along given flow paths are also formulated systematically, through subsystem decomposition methodology. Therefore, the dynamic subsystem decomposition determines the distribution of arbitrary intercompartmental flows and the organization of the associated storages generated by these flows along given subflow paths within the subsystems. Consequently, arbitrary composite intercompartmental flows and storages are dynamically decomposed into the constituent transient subflow segments along a given set of subflow paths and the transient substorage segments generated by the transient flows in each compartment along these paths. In other words, the subsystem decomposition enables dynamically tracking the fate of arbitrary intercompartmental flows and associated storages within the subsystems. The proposed methodology allows for the determination of the dynamic influence of one compartment, through direct or indirect interactions, on any other in a complex network. Moreover, a history of compartments visited by arbitrary system flows and storages can also be compiled. The dynamic direct, indirect, acyclic, cycling, transfer (diact) flows and storages transmitted from one compartment, directly or indirectly, to any other—including itself—within the system are also formulated for the quantification of intercompartmental flow and storage...
The proposed methodology constructs a base for the formulation of many other dynamic and static system analysis tools of matrix, vector, and scalar types as quantitative system indicators. The system measures and indices for the diact effect, utility, exposure, residence time, as well as the corresponding system efficiency, stress, and resilience have recently been developed by [3, 4] in the context of ecosystem ecology. The static versions of these system analysis tools have also been introduced in separate works [5, 6].

3. Results. The proposed methodology is applied to various compartmental models from literature in this section and in Appendix E. The results and their interpretations are presented.

3.1. Case Study. The SIR model is one of the simplest compartmental models in epidemiology which consists of three compartments that represent the populations of three groups: the susceptible or uninfected, \( x_1 = S \), infectious, \( x_2 = I \), and recovered or immune, \( x_3 = R \). The model determines the number of individuals infected with a contagious illness over time. It is reasonably predictive for infectious diseases transmitted from individual to individual. The first SIR model was proposed in its simplest form by [11].

In this section, we will analyze a modified version of SIR model. More specifically, it is called the SIRS model for waning immunity with demographics. The model parameters are adopted from [2] and the modeling assumptions can be deduced from the model formulation below or can be readily found in the literature [2, 7].

The proposed methodology is applied to the following dynamic compartmental system, governing a laboratory population of mice infected with microbes:

\[
\begin{align*}
\frac{dx_1}{dt} &= \alpha + \nu x_3 - \beta x_1 x_2 - \mu x_1 \\
\frac{dx_2}{dt} &= \beta x_1 x_2 - (\gamma + \sigma + \mu) x_2 \\
\frac{dx_3}{dt} &= \gamma x_2 - (\nu + \mu) x_3
\end{align*}
\]

(3.1)

with the initial conditions \( x(t_0) = [10, 10, 0]^T \). The total initial population is given to be 20 by [2], but the initial population for each group is not specified individually. They are, therefore, arbitrarily chosen in this work. The model parameters are the birth rate (or daily rate of input of susceptible mice) \( \alpha = 0.33 \), the natural mortality rate \( \mu = 0.006 \), the mortality rate caused by the disease \( \sigma = 0.06 \), the infection rate \( \beta = 0.0056 \), the recovery rate \( \gamma = 0.04 \), and the immunity loss rate \( \nu = 0.021 \). All parameters are in units of [day\(^{-1}\)] (see Fig. 5).

The system flow regime can be expressed in matrix form as

\[
F(t, x) = \begin{bmatrix} 0 & 0 & \nu x_3 \\ \beta x_1 x_2 & 0 & 0 \\ 0 & \gamma x_2 & 0 \end{bmatrix}, \quad z(t, x) = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}, \quad y(t, x) = \begin{bmatrix} \mu x_1 \\ (\mu + \sigma) x_2 \\ \mu x_3 \end{bmatrix}.
\]

The decomposed system can then be expressed in the following matrix form:

\[
\begin{align*}
\dot{X}(t) &= Z(t, x) + A(t, x) X(t), \quad X(t_0) = 0, \\
\dot{X}(t) &= A(t, x) \dot{X}(t), \quad \dot{X}(t_0) = X_0.
\end{align*}
\]

(3.2)
The fundamental substate matrices, $X(t) = (x_{ik}(t))$ and $\bar{X}(t) = (\bar{x}_{ik}(t))$, the state, external output, and input matrices, $\mathcal{X}(t) = \text{diag} \left( x(t) \right)$, $\mathcal{Y}(t, x) = \text{diag} \left( y(t, x) \right)$, and $\mathcal{Z}(t, x) = \text{diag} \left( z(t, x) \right)$, as well as the flow intensity matrix,

$$A(t, x) = (F(t, x) - T(t, x)) \mathcal{X}^{-1}(t)$$

where $T(t, x) = \mathcal{Y}(t, x) + \text{diag} \left( F^T(t, x) \mathbf{1} \right)$ are defined in the Methods Section.

The numerical results for compartmental state variables $x(t)$, $\bar{x}(t)$, and $x^z(t)$ are presented in Fig. 6. It can be observed that the oscillatory behavior of the real data is better approximated by the total population function, $x_1(t) + x_2(t) + x_3(t)$, presented in Fig. 6, than the corresponding graph presented by [2] (cf. solid dots in Fig. 1(d) in [2]). Moreover, unlike the state of the art techniques, the proposed methodology enables tracking the evolution of the initial populations and the populations generated by the external input individually and separately within the system, as presented in Fig. 6.

The fundamental matrices, that is, the substate and initial substate matrix functions, $X(t)$ and $\bar{X}(t)$, are also presented in Fig. 7. Note that the substate functions for the 3rd initial subsystem and 2nd and 3rd subsystems are identically zero because of the zero initial condition, $x_{30} = 0$, and external inputs, $z_2(t) = z_3(t) = 0$. That is,

$$\bar{x}_{i3}(t) = 0 \quad \text{and} \quad x_{ik}(t) = 0 \quad \text{for} \quad k = 2, 3 \quad \text{and} \quad i = 1, 2, 3.$$  

Among the elements of these matrices, the initial substate and substate variables $\bar{x}_{21}(t)$ and $x_{21}(t)$, for example, represent the population in compartment 2 at time...
which is derived from the initial population in compartment 1, \(x_{1,0}\), and external input into compartment 1, \(z_1(t)\), during \([t_0, t]\), respectively. Biologically, \(\bar{x}_{2,1}(t)\) can be interpreted as the population of the infected mice at time \(t\), that had been initially susceptible and then infected sometime during \([t_0, t]\). Similarly, \(x_{2,1}(t)\) represents the population of the infected mice at time \(t\), which were born (or introduced) susceptible and then infected during \([t_0, t]\).

\[
\begin{align*}
\text{FIG. 7.} & \quad \text{Graphical representations of the initial substate and substate functions} \ \bar{x}_{i_k}(t) \text{ and } x_{i_k}(t) \\
& \quad \text{for all } i, k. \quad \text{The substates that are equal to zero are not labeled. (Example 3.1).}
\end{align*}
\]

In general terms, the state variable of the original system, \(x_i(t)\), for SIR group dynamics, Eq. 3.1, represents the population in group \(i\) at time \(t\) with the initial population, \(x_i(t_0)\). It cannot be used to distinguish the subpopulations generated by either newborn (the only external input in this model, \(z_1(t)\)) or derived from the initial SIR populations (initial conditions, \(x_{i_0}\)). On the other hand, the state variable of the decomposed system, \(x_{i_1}(t)\), for SIR subgroup dynamics, Eq. 3.2, represents the subpopulation in group \(i\) at time \(t\), which is transferred from the newborn population in group \(S\) during \([t_0, t]\). Similarly, the state variable of the decomposed system, \(\bar{x}_{i_k}(t)\), represents the subpopulation in group \(i\) at time \(t\), which is transferred from the initial population in group \(k\), \(x_k(t_0)\), during \([t_0, t]\). Parallel interpretations are true for the throughflow function of the original system, \(\tau(t, x)\), and the subthroughflow functions of the decomposed system, \(\tau_{i_k}^{in}(t, x)\) and \(\tau_{i_k}^{out}(t, x)\), as well.

The proposed dynamic system decomposition methodology, consequently, enables compiling a health history of the newborn or initial SIR populations by tracking the evolution of their health states individually and separately. Note that, the solution to the original system through the state of the art techniques can only provide the composite SIR populations without distinguishing their constituent sources, that is, their previous health states.

The transient substorage functions at compartment 2 along the closed subflow path \(p^1_{2,2_1} = 0_1 \leftrightarrow 1_1 \leftrightarrow 2_1 \leftrightarrow 3_1 \leftrightarrow 1_1 \leftrightarrow 2_1\) are computed as an application of the proposed dynamic subsystem decomposition methodology. The links on this path that directly contribute to the cumulative transient storage \(\bar{x}^1_{2_1}(t)\) are numbered with the red cycle numbers, \(m\), in the extended subflow path diagram below:

\[
p^1_{2,2_1} = 0_1 \rightarrow 1_1 \rightarrow 2_1 \rightarrow 3_1 \rightarrow 1_1 \rightarrow 2_1 \rightarrow 3_1 \rightarrow 1_1 \rightarrow 2_1 \rightarrow 3_1 \rightarrow \cdots
\]

The cumulative transient substorage function \(\bar{x}^1_{2_1}(t)\) at subcompartment \(2_1\) along \(p^1_{2,2_1}\)
will be approximated by three terms \((m_1 = 3)\) using Eq. 2.79:

\[
x_{2i}^1(t) \approx \sum_{m=1}^{3} x_{3i,2i,1}^{1,m}(t) = x_{3i,2i,1}^{1,1}(t) + x_{3i,2i,1}^{1,2}(t) + x_{3i,2i,1}^{1,3}(t).
\]

The governing equations Eqs. 2.76 and 2.77 for the transient substorage functions, \(x_{2i,1}^{1,m}(t)\), are solved simultaneously with the decomposed system Eq. 2.67. Numerical results are presented in Fig. 8. Since the subflow path \(p_{2i,1}^1\) covers the entire flow regime in subsystem 1, \(x_{2i}(t)\) and \(x_{2i}^1(t)\) must be the same. They, however, are approximately equal as presented in Fig. 8, that is, \(x_{2i}^1(t) \approx x_{2i}(t)\). The difference is caused by the truncation errors in the computation of cumulative transient subflows, and larger \(m_1\) values improve the approximation. Biologically, these transient substorage functions, \(x_{3i,2i,1}^{1,m}(t)\), represent the population of the mice at time \(t\) that are infected \(m\) times after being recovered during \([t_0, t]\). These populations are decreasing with increasing \(m\) values, as expected. This classification and characterization of the subpopulations in each SIR group as presented above are not available through the application of the state of the art techniques.

Along the following subflow paths of finite length,

\[
\begin{align*}
p_{01,01}^2 &:= 0_1 \rightarrow 1_1 \rightarrow 2_1 \rightarrow 3_1 \rightarrow 1_1 \rightarrow 2_1 \rightarrow 3_1 \rightarrow 1_1 \rightarrow 2_1 \rightarrow 0_1, \\
p_{01,01}^3 &:= 0_1 \rightarrow 1_1 \rightarrow 2_1 \rightarrow 3_1 \rightarrow 1_1 \rightarrow 2_1 \rightarrow 3_1 \rightarrow 1_1 \rightarrow 2_1 \rightarrow \mu \rightarrow 0_1,
\end{align*}
\]

the transient subflow rate functions can be computed using Eq. 2.79, similar to the transient storages as discussed above (see Fig. 5). Note that the paths represent death (external output) of the newborn mice (external input) following two complete infection cycles. The only difference between them are the last links which represent the death due to the disease (\(\sigma\)) and the natural death (\(\mu\)). The numerical results for these transient external output rates at compartment 2, \(f_{0,2i,1}^2(t)\) and \(f_{0,2i,1}^3(t)\), which correspond to these last two links, are depicted in Fig. 8. Because of the corresponding parameter values, \(\sigma\) and \(\mu\), the death rate due to the disease is 10 times higher than that due to the natural death. These rate functions can biologically be interpreted as the number of mice that were born during \([t_0, t]\), which die per day at time \(t\) after being recovered and getting infected again for the third time. More
specifically, one of these transient output rates at \( t = 500 \) days, \( f_{2,1}^0(500) = 0.027 \), for example, indicates that 2.7 out of 100 mice that were born during \([0,500]\) die per day due to the disease, after getting infected three times.

The proposed dynamic subsystem decomposition methodology, consequently, enables compiling a history of the health states of an arbitrary population in any of the \( SIR \) groups along a given infection path. Therefore, the effect of the arbitrary population on any other group in terms of the spread of the disease, through not only direct but also indirect interactions, can be determined.

The system approaches an epidemic equilibrium as presented in the graphs of Fig. 7 after about 450 days. At this steady state, the system information becomes

\[
F = \begin{bmatrix}
0 & 0 & 0.20 \\
0.53 & 0 & 0 \\
0 & 0.20 & 0
\end{bmatrix},
\quad x = \begin{bmatrix}
18.93 \\
5.00 \\
9.52
\end{bmatrix},
\quad y = \begin{bmatrix}
0 \\
0.33 \\
0
\end{bmatrix},
\quad z = \begin{bmatrix}
0.33 \\
0 \\
0
\end{bmatrix}.
\]

The components of the output vector implies that, at steady state, the only external output occurs from compartment 2; \( y_2 = 0.33 \). This biologically means that, at steady state, the model accounts for only death caused by the disease.

The static \( \text{diact} \) flows and storages in matrix form are listed in Table 1 as formulated by [5]. For this SIRS model, the indirect flow rate and storage matrices become

\[
T^i = \begin{bmatrix}
0.20 & 0.20 & 0 \\
0 & 0.20 & 0.20 \\
0.20 & 0 & 0.08
\end{bmatrix}
\quad \text{and} \quad X^i = \begin{bmatrix}
7.14 & 7.14 & 0 \\
0 & 1.89 & 1.89 \\
9.52 & 0 & 3.59
\end{bmatrix}.
\]

The zero entries of the matrices indicate that there is no indirect flow, and therefore, associated storage generation in the corresponding flow direction. For example, the \((1,3)\)–entry of \( T^i \) is \( \tau^i_{13} = 0 \), which indicates that although there is a direct flow from the recovered population to the susceptible, \( f_{13} = 0.20 \), there is no indirect flow in the same direction through the infectious population at the epidemic equilibrium. Moreover, the diagonal elements of these matrices represent the cycling flows and storages. The cycling subflow rate \( \tau^i_{33} = 0.08 \) indicates that 0.08 recovered mice lose immunity and become susceptible, then get infected and recovered again per day. In other words, about 8 mice complete such a recovery cycle within a 100 day period. This cycling flow constantly maintains the cycling storage of \( x^i_{33} = 3.59 \) mice in the recovered mice population. The other static \( \text{diact} \) flows and associated storages can be computed and interpreted similarly.

Evidently, the detailed information and inferences enabled by the proposed methodology cannot be obtained through analysis of the original system utilizing the state of the art techniques.
Realistically, nature is always on the move and its systems are constantly changing to meet ever-renewing circumstances. Therefore, the need for dynamic analysis of nonlinear compartmental systems has always been present.

This is the first manuscript in literature that proposes a theory and develops a comprehensive method for holistic analysis of nonlinear dynamic compartmental systems. The proposed theory is based on the dynamic system and subsystem decomposition methodologies. The original nonlinear compartmental system is decomposed into mutually exclusive and exhaustive subsystems deterministically through the governing equations for each subsystem by the system decomposition methodology. The subsystems are then further decomposed along a given set of mutually exclusive and exhaustive subflow paths through an additional set of coupled governing equations by the subsystem decomposition methodology. While the dynamic system decomposition formulates the distribution of external inputs and initial stocks as well as the organization of the associated storages generated by the inputs and stocks individually and separately within the system, the dynamic subsystem decomposition formulates the distribution of intercompartmental flows and the organization of associated storages within the subsystems. The proposed mathematical method, therefore, as a whole, yields the dynamic decomposition of system flows and storages to the utmost level.

The system decomposition methodology yields the subthroughflow and substorage matrix functions that respectively represent the flows and storages generated by individual external inputs at each compartment separately. More specifically, the composite compartmental storage and throughflow, \( x_i(t) \) and \( \tau_i(t) \), are dynamically partitioned into the subcompartmental substorage and subthroughflow segments, \( x_{ik}(t) \) and \( \tau_{ik}(t) \) \((x_{i0}(t) \text{ and } \tau_{i0}(t))\), respectively, based on their constituent external (initial) sources, \( z_k(t) \) \((x_{i0})\). In other words, this methodology enables tracking the evolution of external inputs and initial stocks individually and separately within the system. The subsystem decomposition methodology then yields the transient and the dynamic direct, indirect, acyclic, cycling, and transfer (diact) flows and associated storages transmitted along a given flow path or from one compartment, directly or indirectly, to any other. The subsystem partitioning, therefore, enables tracking the fate of arbitrary intercompartmental flows and the associated storages generated by these flows at each compartment along a given flow path within the subsystems. Moreover, a history of compartments visited by arbitrary system flows and storages can also be compiled.

The proposed dynamic methodology also constructs a base for the development of new dynamic system analysis tools, as it determines the influence of one compartment, in terms of flow and storage transfer, on any other in the system. Multiple measures, such as the substorage, subthroughflow, diact flow and storage matrices, are formulated in the present manuscript. The illustrative case studies in Section 3 and Appendix E provide analytic solutions for some specific models, such as linear and static systems, and demonstrate that the proposed measures are rigorous and efficient mathematical system analysis tools that can be used as quantitative system indicators. More measures and indices have recently been introduced in the context of ecosystem ecology by [4, 6].

In summary, we consider that this new mathematical theory and methodology brings a novel, formal, deterministic, complex system theory to the service of urgent natural problems of the day.

**Appendices.** The dynamic system and subsystem decomposition methodologies for the initial subsystem in parallel to the decomposition of the original system, the
Appendix A. Initial System Decomposition.

The dynamic system decomposition methodology is presented in Section 2.2. In order to analyze the distribution of the initial stocks and the organization of the associated storages derived from these stocks individually and separately within the system, the initial subsystem will further be decomposed into initial subsystems in parallel to the system decomposition. With some abuse of terminology, we will call the subsystems of the initial subsystem the initial subsystems, instead of the initial sub-sub-systems (see Fig. 1 and 2). The initial system decomposition methodology dynamically decomposes composite initial subthroughflows and storages into segments based on their constituent initial stocks. In other words, the initial system decomposition enables dynamically tracking the evolution of the initial stocks as well as the associated storages derived form these stocks individually and separately within the system.

Each initial subsystem is driven by an initial stock. The number of initial subsystems, therefore, is equal to the number of positive initial conditions. Since we assumed that all initial conditions are positive, there are \( n \) initial subsystems, one for each initial condition. These \( n \) initial subsystems are indexed by \( k = 1, \ldots, n \). The dynamic initial subsystem decomposition methodology is introduced in this section.

A.1. Initial State Decomposition. Similar to the original system, the initial subcompartments can further be decomposed into \( n \) subcompartments (see Fig. 2). We will use the notation \( x_{i_k,0}(t) \) for \( k \)th substate of \( i \)th initial substate or \( \tilde{x}_i(t) := \sum_{k=1}^{n} \tilde{x}_{ik}(t) \) for notational convenience. Based on this further decomposition of the initial substates, we have

\[
(A.1) \quad x_{i0}(t) = \tilde{x}_i(t) = \sum_{k=1}^{n} \tilde{x}_{ik}(t)
\]

for \( i = 1, \ldots, n \), and the corresponding initial conditions are

\[
(A.2) \quad \tilde{x}_{ik}(t_0) = \delta_{ik} x_{i,0}(t_0) = \begin{cases} 
  x_{i0}(t_0) = x_i, & i = k \\
  0, & i \neq k 
\end{cases}
\]

The initial substate matrix function is then defined as

\[
(A.3) \quad \tilde{X}(t) := (\tilde{x}_{ik}(t)) = [\tilde{x}_1(t) \ldots \tilde{x}_n(t)].
\]

We also define the initial state, \( \tilde{X}(t) \), and the \( k \)th initial substate matrix functions, \( \tilde{X}_k(t) \), as

\[
(A.4) \quad \tilde{X}(t) := \tilde{X}_0(t) = \text{diag} (\tilde{x}(t)) \quad \text{and} \quad \tilde{X}_k(t) := \text{diag} (\tilde{x}_k(t))
\]

for \( k = 1, \ldots, n \). These matrices will, alternatively, be called the initial substorage and \( k \)th initial substorage matrix functions, respectively. The initial conditions of these matrices given in Eq. A.2 can be expressed in matrix form as

\[
(A.5) \quad \tilde{X}(t_0) = \tilde{X}_0(t_0) = \text{diag}(x_0) \quad \text{and} \quad \tilde{X}_k(t_0) := \text{diag}(x_{k,0}e_k).
\]

Note that

\[
(A.6) \quad \tilde{x}(t) = \tilde{X}(t) \mathbf{1} \quad \text{and} \quad \tilde{x}_k(t) = \tilde{X}_k(t) \mathbf{1}.
\]
The state decomposition methodology for the initial subsystem can be summarized as follows:

\[
\mathbf{x}_0 = \tilde{\mathbf{x}}(t) = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix} \xrightarrow{\text{state decomposition}} \tilde{X}(t) = \begin{bmatrix} \bar{x}_{11} & \bar{x}_{12} & \cdots & \bar{x}_{1n} \\ \bar{x}_{21} & \bar{x}_{22} & \cdots & \bar{x}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{x}_{n1} & \bar{x}_{n2} & \cdots & \bar{x}_{nn} \end{bmatrix}
\]

A.2. Initial Flow Rate Decomposition. Similar to the flow rate decomposition of the original system in Section 2.2.2, the initial flow rates are also decomposed into initial subflow rates that represent the rate of subflow segments between the initial subcompartments (see Fig. 2).

For notational convenience, we set

\[
(A.7) \quad \bar{f}_{ij}(t, \mathbf{x}) := f_{i0j0}(t, \mathbf{x}), \quad \bar{z}_i(t, \mathbf{x}) := \bar{f}_{i0}(t, \mathbf{x}), \quad \text{and} \quad \bar{y}_j(t, \mathbf{x}) := \bar{f}_{0j}(t, \mathbf{x})
\]

for \(i, j = 1, \ldots, n\). We define the initial subflow rate, input, and output matrix functions as

\[
(A.8) \quad \bar{F}(t, \mathbf{x}) := F_0(t, \mathbf{x}), \quad \bar{Z}(t, \mathbf{x}) := Z_0(t, \mathbf{x}) = \mathbf{0}, \quad \text{and} \quad \bar{Y}(t, \mathbf{x}) := Y_0(t, \mathbf{x}).
\]

We will use the notation \(f_{i0j0,k0}(t, \mathbf{x})\) for flow rate from initial substate \(\bar{x}_{jk}(t) = x_{jk,0}(t)\) to \(\bar{x}_{ik}(t) = x_{ik,0}(t)\) or \(\bar{f}_{i0j0,k0}(t, \mathbf{x})\) for notational convenience, for \(i, j, k = 1, \ldots, n\). In particular, \(\bar{z}_{ik}(t, \mathbf{x}) := f_{i0k0}(t, \mathbf{x}) = 0\) and \(\bar{y}_{jk}(t, \mathbf{x}) := f_{j0k0}(t, \mathbf{x})\).

The initial subflow rate decomposition can then be formulated as

\[
(A.9) \quad \bar{f}_{ij}(t, \mathbf{x}) := \frac{\bar{x}_{jk}(t)}{\bar{x}_{jk}(t)} \tilde{f}_{ij}(t, \mathbf{x}) = \frac{\bar{x}_{jk}(t)}{\bar{x}_{jk}(t)} f_{ij}(t, \mathbf{x})
\]

for \(i, j, k = 1, \ldots, n\), because of Eq. 2.28. Due to the state decomposition, Eq. A.1, we also have

\[
(A.10) \quad \bar{f}_{ij}(t, \mathbf{x}) = \sum_{k=1}^{n} \bar{f}_{ij}(t, \mathbf{x})
\]

for \(i, j = 1, \ldots, n\). It can be seen from Eq. A.9 that the initial flow and subflow rate intensities between the same compartments in the same flow direction are the same, that is

\[
(A.11) \quad \frac{\bar{f}_{i0j0}(t, \mathbf{x})}{\bar{x}_{jk}(t)} = \frac{\bar{f}_{i0}(t, \mathbf{x})}{\bar{x}_{j0}(t)} = \frac{\bar{f}_{0j}(t, \mathbf{x})}{\bar{x}_{0j}(t)}
\]

for \(i, j, k = 1, \ldots, n\) (see Fig. 2). The last two equalities are due to Eq. 2.28.

The decomposition factors, \(\bar{d}_{ik}(\mathbf{x})\), for the initial subflow rates with the following definition and properties

\[
(A.12) \quad \bar{d}_{ik}(\mathbf{x}) := \frac{\bar{x}_{jk}(t)}{\bar{x}_{jk}(t)}, \quad 0 \leq \bar{d}_{ik}(\mathbf{x}) \leq 1, \quad \text{and} \quad \sum_{k=0}^{n} \bar{d}_{jk}(\mathbf{x}) = 1,
\]

form another continuous partition of unity. Note also that, due to Eq. A.1, \(\bar{x}_{jk}(t) \leq \bar{x}_{jk}(t)\). The decomposition factors are, therefore, well-defined even if \(\bar{x}_{jk}(t) \to 0\). The respective initial decomposition and \(k^{th}\) initial decomposition matrices, \(\bar{D}(\mathbf{x}) := (\bar{d}_{jk}(\mathbf{x}))\)
and \( \mathcal{D}_k(x) = \text{diag} (\hat{d}_{k1}(x), \ldots, \hat{d}_{kn}(x)) \), for the initial subsystems can then be formulated, accordingly, as
\[
A.13 \quad \hat{D}(x) = \mathcal{X}^{-1}(t) \tilde{X}(t) \quad \text{and} \quad \mathcal{D}_k(x) := \mathcal{X}^{-1}(t) \mathcal{X}_k(t)
\]
for \( k = 1, \ldots, n \). Equations A.6 and A.12 imply that
\[
A.14 \quad 1 = \tilde{X}(t) x_0(t) = \mathcal{X}^{-1}(t) \tilde{X}(t) 1 = \hat{D}(x) 1.
\]
From Eqs. 2.32 and A.14, we also have
\[
A.15 \quad x(t) = \tilde{x}(t) + x^z(t) = \tilde{X}(t) 1 + X(t) 1,
\]
similar to Eq. 2.23.

We define the \( k \)th \textit{initial subflow rate matrix function} as
\[
A.16 \quad \bar{F}_k(t, x) := \left( \bar{f}_{ik,jk}(t, x) \right)
\]
for \( k = 1, \ldots, n \). Using Eq. A.9, \( \bar{F}_k(t, x) \) can be expressed in matrix form as
\[
A.17 \quad \bar{F}_k(t, x) = \bar{F}(t, x) \bar{D}_k(x) = \bar{F}(t, x) \mathcal{X}^{-1}(t) \mathcal{X}_k(t) = F(t, x) \mathcal{X}^{-1}(t) \mathcal{X}_k(t).
\]
That is, the \( k \)th initial decomposition matrix, \( \bar{D}_k(x) \), decomposes the direct initial flow matrix, \( \bar{F}(t, x) \), into the initial subflow matrices, \( \bar{F}_k(t, x) \). The last equality is derived from Eqs. A.4 and 2.34, as these equations imply that
\[
A.18 \quad \bar{F}(t, x) = F_0(t, x) = F(t, x) \mathcal{X}^{-1}(t) x_0(t) = F(t, x) \mathcal{X}^{-1}(t) \mathcal{X}_k(t).
\]
Similarly, the \( k \)th \textit{initial output matrix function},
\[
A.19 \quad \bar{Y}_k(t, x) := \text{diag} \left( \bar{f}_{01k}(t, x), \ldots, \bar{f}_{0nk}(t, x) \right)
\]
can be expressed in matrix form as
\[
A.20 \quad \bar{Y}_k(t, x) = \bar{Y}(t, x) \bar{D}_k(x) = \bar{Y}(t, x) \mathcal{X}^{-1}(t) \mathcal{X}_k(t) = \bar{Y}(t, x) \mathcal{X}^{-1}(t) \mathcal{X}_k(t)
\]
and the \( k \)th \textit{initial input matrix function} as
\[
A.21 \quad \bar{Z}_k(t, x) := 0
\]
for \( k = 1, \ldots, n \). The \( k \)th initial output and input vectors, \( \bar{y}_k(t, x) \) and \( \bar{z}_k(t, x) \), for the \( k \)th initial subsystem can be defined as
\[
A.22 \quad \bar{y}_k(t, x) := \bar{Y}_k(t, x) 1 \quad \text{and} \quad \bar{z}_k(t, x) := \bar{Z}_k(t, x) 1 = 0.
\]

Using these notations, the flow rate decompositions given in Eq. A.10 can be expressed in matrix form as
\[
A.23 \quad \bar{F}(t, x) = \sum_{k=1}^{n} \bar{F}_k(t, x), \quad \bar{Y}(t, x) = \sum_{k=1}^{n} \bar{Y}_k(t, x), \quad \bar{Z}(t, x) = \sum_{k=1}^{n} \bar{Z}_k(t, x) = 0.
\]
The equivalence of the flow, initial flow, and initial subflow rate intensities given in Eq. A.11 can also be expressed in matrix form as
\[
A.24 \quad \bar{F}_k(t, x) \mathcal{X}_k^{-1}(t) = \bar{F}(t, x) \mathcal{X}^{-1}(t) = F_0(t, x) \mathcal{X}_0^{-1}(t) = F(t, x) \mathcal{X}^{-1}(t)
\]
for \( k = 1, \ldots, n \).

The initial flow rate decomposition methodology for the initial subsystem given in Eq. A.9 can be schematized as follows:
The governing equations for the initial subsystem can be written in the following vector form

\[ \dot{\bar{\mathbf{x}}}(t) = \bar{\mathbf{\tau}}^{\text{in}}(t, \mathbf{x}) - \bar{\mathbf{\tau}}^{\text{out}}(t, \mathbf{x}), \quad \bar{\mathbf{x}}(t_0) = \bar{\mathbf{x}}_0 \]

as given in Eq. 2.43 where \( \bar{\mathbf{x}}(t) = \bar{\mathbf{x}}_0, \bar{\mathbf{\tau}}^{\text{in}}(t, \mathbf{x}) = \bar{\mathbf{\tau}}^{\text{in}}_0(t, \mathbf{x}), \) and \( \bar{\mathbf{\tau}}^{\text{out}}(t, \mathbf{x}) = \bar{\mathbf{\tau}}^{\text{out}}_0(t, \mathbf{x}). \) The governing equations for the \( k^{\text{th}} \) initial subsystem can then be written in vector form as

\[ \dot{\mathbf{\bar{x}}}_k(t) = \bar{\mathbf{\tau}}^{\text{in}}_k(t, \mathbf{x}) - \bar{\mathbf{\tau}}^{\text{out}}_k(t, \mathbf{x}) = \bar{F}_k(t, \mathbf{x}) \mathbf{1} - (\bar{\mathbf{y}}_k(t, \mathbf{x}) + \bar{F}_k^T(t, \mathbf{x}) \mathbf{1}) \]

for \( k = 1, \ldots, n. \) The corresponding initial conditions become \( \bar{\mathbf{x}}_k(t_0) = x_{k,0} \mathbf{e}_k. \)

The \( k^{\text{th}} \) inward and outward subthroughflow vectors, \( \bar{\mathbf{\tau}}^{\text{in}}_k(t, \mathbf{x}) \) and \( \bar{\mathbf{\tau}}^{\text{out}}_k(t, \mathbf{x}), \) for the \( k^{\text{th}} \) initial subsystem given in Eq. A.26 can then be expressed in the following forms:

\[ \bar{\mathbf{\tau}}^{\text{in}}_k(t, \mathbf{x}) := \bar{F}_k(t, \mathbf{x}) \mathbf{1} = F(t, \mathbf{x}) \bar{\mathbf{\Lambda}}^{-1}(t) \bar{\mathbf{\Lambda}}_k(t) = F(t, \mathbf{x}) \bar{\mathbf{\Lambda}}^{-1}(t) \bar{\mathbf{\Lambda}}_k(t), \]

\[ \bar{\mathbf{\tau}}^{\text{out}}_k(t, \mathbf{x}) := \bar{\mathbf{y}}_k(t, \mathbf{x}) + \bar{F}_k^T(t, \mathbf{x}) \mathbf{1} = \tilde{\bar{\mathbf{y}}}(t, \mathbf{x}) \bar{\mathbf{\Lambda}}^{-1}(t) \bar{\mathbf{\Lambda}}_k(t) \mathbf{1} + \bar{\mathbf{\Lambda}}_k(t) \bar{\mathbf{\Lambda}}^{-1}(t) \bar{F}_k^T(t, \mathbf{x}) \mathbf{1} = (\mathcal{Y}(t, \mathbf{x}) + \text{diag}(F^T(t, \mathbf{x}) \mathbf{1}) \bar{\mathbf{\Lambda}}^{-1}(t) \bar{\mathbf{\Lambda}}_k(t) = \mathcal{T}(t, \mathbf{x}) \bar{\mathbf{\Lambda}}^{-1}(t) \bar{\mathbf{\Lambda}}_k(t). \]
The $k^{th}$ net subthroughflow rate vector, $\bar{\tau}_k(t, x) = [\bar{\tau}_{1k}(t, x), \ldots, \bar{\tau}_{nk}(t, x)]^T$, for the initial subsystem then becomes

\begin{equation}
\bar{\tau}_k(t, x) := \bar{\tau}_{in}^k(t, x) - \bar{\tau}_{out}^k(t, x) = A(t, x) \bar{x}_k(t).
\end{equation}

The $k^{th}$ inward and outward subthroughflow matrices, $\bar{T}_{in}^k(t, x) := \text{diag}(\bar{\tau}_{in}^k(t, x))$ and $\bar{T}_{out}^k(t, x) := \text{diag}(\bar{\tau}_{out}^k(t, x))$, for the $k^{th}$ initial subsystem can be expressed as

\begin{align}
\bar{T}_{in}^k(t, x) &= \text{diag}(F(t, x) X^{-1}(t) \bar{X}_k(t) 1), \\
\bar{T}_{out}^k(t, x) &= T(t, x) X^{-1}(t) \bar{X}_k(t).
\end{align}

Note that, using Eq. A.29, $\tilde{F}_k(t, x)$ given in Eq. A.17 can, alternatively, be written in terms of the system flows only:

\begin{equation}
\tilde{F}_k(t, x) = F(t, x) T^{-1}(t, x) \bar{T}_{out}^k(t, x).
\end{equation}

We define the inward and outward subthroughflow matrices, $\tilde{T}_{in}^k(t, x)$ and $\tilde{T}_{out}^k(t, x)$, for the initial subsystems as the matrices whose $k^{th}$ columns are the inward and outward initial subthroughflow vectors, $\bar{\tau}_{in}^k(t, x)$ and $\bar{\tau}_{out}^k(t, x)$, $k = 1, \ldots, n$, respectively:

\begin{align}
\tilde{T}_{in}^k(t, x) &= (\bar{\tau}_{in}^k(t, x)) = [\bar{\tau}_{1in}^k(t, x) \ldots \bar{\tau}_{nin}^k(t, x)], \\
\tilde{T}_{out}^k(t, x) &= (\bar{\tau}_{out}^k(t, x)) = [\bar{\tau}_{1out}^k(t, x) \ldots \bar{\tau}_{nout}^k(t, x)].
\end{align}

Using the relationships Eq. A.27, these subthroughflow matrices for the initial subsystems can be expressed in matrix form as

\begin{align}
\tilde{T}_{in}^k(t, x) &= F(t, x) X^{-1}(t) \bar{X}(t), \\
\tilde{T}_{out}^k(t, x) &= T(t, x) X^{-1}(t) \bar{X}(t).
\end{align}

We then define the net subthroughflow matrix, $\tilde{T}(t, x)$, for the initial subsystems as

\begin{equation}
\tilde{T}(t, x) := \tilde{T}_{in}^k(t, x) - \tilde{T}_{in}^k(t, x) = A(t, x) \bar{X}(t).
\end{equation}

Due to Eq. A.32, the decomposition matrix $\tilde{D}(x)$ can be expressed in terms of the subthroughflow functions instead of the substate functions as

\begin{equation}
\tilde{D}(x) = X^{-1}(t) \bar{X}(t) = T(t, x)^{-1} \tilde{T}_{out}^k(t, x).
\end{equation}

Note that, the subthroughflow matrices can be written in various forms as follows:

\begin{align}
\tilde{T}_{in}^k(t, x) &= F(t, x) \tilde{D}(x) = Q^T(t, x) \bar{X}(t) = Q^T(t, x) \tilde{T}_{out}^k(t, x), \\
\tilde{T}_{out}^k(t, x) &= R^{-1}(t, x) \bar{X}(t).
\end{align}

For each fixed $j$, Eqs. A.9 or A.32 implies that

\begin{equation}
\frac{\tau_{out}^j(t, x)}{x_{j0}(t)} = \frac{\tilde{\tau}_{out}^j(t, x)}{\bar{x}_j(t)} = \sum_{i=0}^{n} \tilde{f}_{ij}(t, x) \bar{x}_j(t) = \sum_{i=0}^{n} \tilde{f}_{ikj}(t, x) \bar{x}_j(t) = \frac{\tilde{\tau}_{out}^j(t, x)}{\bar{x}_{jk}(t)}
\end{equation}

for $k = 1, \ldots, n$. This equivalence of outward throughflow and subthroughflow intensities for the initial subsystems given in the second and last equalities of Eq. A.36 can be expressed in matrix form as

\begin{align}
R^{-1}(t, x) &= T(t, x) X^{-1}(t) = T_0^{-out}(t, x) X_0^{-1}(t) \\
&= \tilde{T}_{out}^k(t, x) X^{-1}(t) = \tilde{T}_{out}^k(t, x) X^{-1}(t) = \tilde{T}_{out}^k(t, x) X^{-1}(t).
\end{align}
where the last equality is derived from Eq. A.35. This relationship agrees with Eq. A.29. Equations A.9 and A.36 also imply that
\[
(A.38) \quad \bar{\tau}_{jk}(t, x) = \frac{\bar{x}_{jk}(t)}{f_{jk}(t, x)} = \bar{f}_{ik} \bar{f}_{jk}(t, x)
\]
for \(k, \ell = 1, \ldots, n\). This relationship indicates the proportionality of the parallel initial subflows and corresponding subthroughflows and substorages.

Using Eq. A.37, the \(k^{th}\) initial decomposition matrix, \(\bar{D}_k(x)\), can be written as
\[
(A.39) \quad \bar{D}_k(x) = \mathcal{X}^{-1}(t) \bar{X}_k(t) = \mathcal{T}(t, x)^{-1} \bar{T}_k^{\text{out}}(t, x),
\]
similar to the decomposition matrix formulated in Eq. A.34. It is worth noting that the initial decomposition and \(k^{th}\) initial decomposition matrices, \(\bar{D}(x)\) and \(\bar{D}_k(x)\), decompose the compartmental throughflow matrix, \(\mathcal{T}(t, x)\), into the outward initial subthroughflow and \(k^{th}\) initial subthroughflow matrices as indicated in Eqs. A.35 and A.39, similar to the decomposition of \(\bar{F}(t, x)\) as formulated in Eq. A.17. That is,
\[
(A.40) \quad \bar{T}^{\text{out}}(t, x) = \mathcal{T}(t, x) \bar{D}(x) \quad \text{and} \quad \bar{T}_k^{\text{out}}(t, x) = \mathcal{T}(t, x) \bar{D}_k(x).
\]

It is worth noting also the relationships given below between the flow and subthroughflow matrices for the initial subsystems:
\[
\bar{F}(t, x) = \sum_{k=1}^n \bar{F}_k(t, x) \mathbf{1} = \sum_{k=1}^n \bar{\tau}_{ik}^{\text{in}}(t, x) = \bar{\tau}^{\text{in}}(t, x) = \bar{T}^{\text{in}}(t, x) \mathbf{1},
\]
\[
\bar{y}(t, x) + \bar{F}^T(t, x) \mathbf{1} = \sum_{k=1}^n \bar{y}_k(t, x) + \bar{F}_k^T(t, x) \mathbf{1} = \sum_{k=1}^n \bar{\tau}_{k}^{\text{out}}(t, x) = \bar{\tau}^{\text{out}}(t, x)
\]
\[
= \bar{T}^{\text{out}}(t, x) \mathbf{1}.
\]

The governing equations for the initial subsystems of the decomposed system can then be written in vector form as
\[
(A.41) \quad \dot{\bar{x}}_k(t) = A(t, x) \bar{x}_k(t), \quad \bar{x}_k(t_0) = x_{k,0} e_k
\]
for \(k = 1, \ldots, n\). The governing equations for the decomposed system, Eq. 2.63, can similarly be expressed in matrix form using the matrix functions introduced above as follows:
\[
(A.42) \quad \dot{\bar{X}}(t) = \bar{T}(t, x) = \bar{T}^{\text{in}}(t, x) - \bar{T}^{\text{out}}(t, x), \quad \bar{X}(t_0) = \mathcal{X}_0.
\]

This system can also be expressed in terms of the flow intensity matrix, \(A(t, x)\):
\[
(A.43) \quad \dot{\bar{X}}(t) = A(t, x) \bar{X}(t), \quad \bar{X}(t_0) = \mathcal{X}_0.
\]

The system decomposition mechanism yielding governing equations for each initial subsystem in vector form, or for the entire initial system in matrix form, can be schematized as follows:
\[ \dot{x}(t) = \bar{r}(t, x) \]
\[ \dot{x}_k(t) = \bar{r}_k(t, x), \; k = 1, \ldots, n \]

Nonlinear Decomposition Principle

Appendix B. Analytic Solution to Linear Compartmental Systems.

In this section, we analyze linear dynamic compartmental systems. For linear systems, the original system of governing equations, Eq. 2.66, takes the following compact form:

(B.1) \[ \dot{x}(t) = \mathbf{z}(t) + A(t) \mathbf{x}(t), \; \mathbf{x}(t_0) = \mathbf{x}_0. \]

The decomposed system, Eq. 2.67, is also linear, and, in matrix form, it becomes

(B.2) \[ \dot{X}(t) = \mathbf{Z}(t) + A(t) \mathbf{X}(t), \; \mathbf{X}(t_0) = \mathbf{0}, \]
\[ \dot{\mathbf{X}}(t) = A(t) \mathbf{X}(t), \quad \mathbf{X}(t_0) = \mathbf{X}_0. \]

Let \( U(t) \) be the fundamental matrix solution of the system Eq. B.1, that is the unique solution of the system

(B.3) \[ \dot{U}(t) = A(t) U(t), \quad U(t_0) = I. \]

The solutions for \( X(t) \) and \( \mathbf{X}(t) \) in terms of \( U(t) \) become

(B.4) \[ X(t) = \int_{t_0}^{t} U(t) U^{-1}(\tau) \mathbf{Z}(\tau) d\tau \quad \text{and} \quad \dot{\mathbf{X}}(t) = U(t) \mathbf{X}_0. \]

Therefore, we have the following observation.

Remark B.1. The initial substate matrix of the decomposed system, \( \mathbf{X}(t) \), scaled by nonzero initial conditions, \( \mathbf{X}_0 > 0 \), is the fundamental matrix solution to the original linear system, Eq. B.1. That is, \( U(t) = \dot{\mathbf{X}}(t) \mathbf{X}_0^{-1}. \)

The solution for \( X(t) \) can then be expressed in terms of \( \mathbf{X}(t) \) as

(B.5) \[ X(t) = \int_{t_0}^{t} \dot{\mathbf{X}}(t) \mathbf{X}^{-1}(\tau) \mathbf{Z}(\tau) d\tau. \]

For linear systems, Equation 2.68 becomes

(B.6) \[ \dot{X}(t) = \mathbf{Z}(t) + A(t) \mathbf{X}(t), \quad \mathbf{X}(t_0) = \mathbf{X}_0. \]
The solution for \( X(t) \) can be written as

\[
X(t) = \bar{X}(t) + X(t) + \int_{t_0}^{t} \bar{X}(t) \bar{X}^{-1}(\tau) Z(\tau) d\tau.
\]

Multiplying both sides by 1, we get

\[
x(t) = x_0(t) + \int_{t_0}^{t} \bar{X}(t) \bar{X}^{-1}(\tau) z(\tau) d\tau,
\]

which is the general solution to the original system, Eq. B.1.

For the particular case of constant diagonalizable flow intensity matrix \( A(t) = A \), the fundamental matrix solution can be written as

\[
U(t) = \exp \left( \int_{t_0}^{t} A ds \right) = \Omega e^{(t-t_0)A} \Omega^{-1}
\]

where \( \Omega \) is the matrix whose columns are eigenvectors of \( A \), and \( \Lambda \) is the diagonal matrix whose diagonal elements are the corresponding eigenvalues of \( A \). The solution for the matrix equation given in Eq. B.7 then becomes

\[
X(t) = \bar{X}(t) + X(t) = e^{(t-t_0)A}X_0 + \int_{t_0}^{t} e^{(t-\tau)A} Z(\tau) d\tau.
\]

Consequently, Eq. B.8 takes the following form:

\[
x(t) = e^{(t-t_0)A} x_0 + \int_{t_0}^{t} e^{(t-\tau)A} z(\tau) d\tau.
\]

**B.1. Static Compartmental System Analysis.** At steady state, the time derivatives of the state variables are zero. That is,

\[
\dot{X}(t) = \dot{\bar{X}}(t) = 0.
\]

Clearly, if the decomposed system, Eq. 2.67, is at steady state, the original system, Eq. 2.11, is also at steady state, due to Eq. 2.19. Since \( A \) is a strictly diagonally dominant constant matrix, it is invertible, and

\[
T = \bar{T} = 0 \Rightarrow X = -A^{-1}Z \quad \text{and} \quad \bar{X} = 0
\]

because of the relationships given in Eq. 2.67. The static systems can be decomposed and analyzed based on both external inputs and outputs. The proposed methodology for static systems in both input- and output-orientations have recently been formulated and their duality is demonstrated by [5, 6].

**Appendix C. Dynamic Subsystem Decomposition.**

The dynamic subsystem decomposition methodology is introduced in Section 2.5. The method formulation is based on the concept of directed subflow paths, which will be detailed below.

**C.1. Subflow Paths.** A link will be defined as the connection between two system compartments, which represents direct transactions between them. A link constitutes a step along a given flow path from one subcompartment to another. A directed subflow path in a subsystem will then be defined as a chain of connected links.
initiated at one subcompartment and ending in another of the same subsystem. The environment can be taken as the initial or terminal subcompartment. The connection of a subflow path to the ambient subsystem is defined as the initial and only subcompartment on the path that receives inflow. The transient subflow and substorage computations along a subflow path should start at its connection. The first link of the path will be called the local input, which represents the only source of inflow into the path at its connection. The outflow from the terminal subcompartment of the path will be called the local output.

A subflow path $j_k \rightarrow \ell_k \leadsto n_k \leadsto \ldots$ in subsystem $k$ with connection $j_k$ and the local source $i_k$ will be represented by $i_k \rightarrow j_k^* \rightarrow \ell_k \leadsto n_k \leadsto \ldots$. The connection is marked with red superscript ($*$), and the local input with red arrow. A subflow link that does not directly contribute to the particular subflow or substorage in question will be represented by $(\leadsto)$ symbol. If the local input or output are external input or output, the corresponding subcompartments will be denoted by $0_k$.

The subsystems can be decomposed into subflows and associated substorages generated by these subflows along a set of mutually exclusive and exhaustive directed subflow paths. By mutually exclusive subflow paths, we mean that no given subflow path in a subsystem is a subpath, that is, completely inside of another path in the same subsystem. The exhaustiveness, in this context, means that such mutually exclusive subflow paths all together sum to the entire subsystem subflows and associated substorages. A subflow path that does not self intersect will be called the linear path, and the one with the same initial and terminal subcompartment will be called the closed path. A subflow path composed of linear and closed subpaths will be called the mixed type subflow path. We will use the notation $P_{i_k}^{i_k}$ for a set of mutually exclusive and exhaustive subflow paths from subcompartment $j_k$ to $i_k$ in subsystem $k$ and $P_{i_k}$ for closed paths from $i_k$ to itself. The number of subflow paths in $P_{i_k}$ will be denoted by $w_k$.

Each subsystem can be partitioned along a set of mutually exclusive and exhaustive subflow paths of linear, closed, and mixed type as follows. For subsystem $k$, the connections of all subflow paths can be taken to be subcompartment $k$, with the local input being external input $z_k(t)$. The terminal subcompartments of the linear paths are the ones with external output, and those of the closed and mixed type paths can be taken as the first (and the last) subcompartments of the closed subpaths. Consequently, the number of subflow paths in a subsystem obtained by this partitioning is the number of local outputs or terminal subcompartments. This subsystem partitioning will be called the natural subsystem partitioning. The natural subsystem partitioning of all subsystems yields a mutually exclusive and exhaustive partitioning of the entire system.

C.2. Transient Flows and Storages. The transient subflows and substorages are defined in Section 2.5. Additional relationships will be formulated in this section.

The transient subflows and substorages are defined for linear subflow paths above. Self-intersecting directed flow paths are also common in compartmental systems. The
transient inflow, $f_{\ell_k i_k k}^{w,m}(t)$, and outflow, $f_{n_k \ell_k j_k}(t)$, at subcompartment $\ell_k$ and the associated transient substorage, $x_{n_k \ell_k j_k}(t)$, change with each cycle along the path, where the superscript $m$ represents the cycle number. The cumulative values are obtained by summing up all transient subflows and storages at the corresponding subcompartments with each cycle. The number of terms in these summations, $m_w$, depends on the number of times the directed path intersects itself. In particular, transient subflows cycle along directed closed paths repeatedly and indefinitely. Therefore, in this case, the summations yield infinite series of functions that are convergent pointwise at any time $t$ due to their construction.

The sum of the transient inflows from subcompartment $j_k$ to $\ell_k$ and the outflows from $\ell_k$ to $n_k$ at subcompartment $\ell_k$ along a given self-intersecting path $p_{n_k i_k}^w$ will be called the cumulative transient inflow, $f_{i_k j_k}^w(t)$, and outflow, $f_{n_k \ell_k}^w(t)$, respectively, and associated total substorage will be called the cumulative transient substorage, $x_{n_k \ell_k j_k}^w(t)$. They can be formulated as

\[(C.1)\]

\[
x_{n_k \ell_k j_k}^w(t) = \sum_{m=1}^{m_w} x_{n_k \ell_k j_k}^{w,m}(t), \quad f_{i_k j_k}^w(t) = \sum_{m=1}^{m_w} f_{i_k j_k}^{w,m}(t), \quad f_{n_k \ell_k}^w(t) = \sum_{m=1}^{m_w} f_{n_k \ell_k}^{w,m}(t)
\]

where the superscript $m$ represents the cycle number, and $m_w$ is the number of cycles, that is, the number of times the path $p_{n_k i_k}^w$ intersects itself. Large number of terms, $m_w$, in computation of these summations reduce truncation errors and, thus, improve the approximations. We will not use superscript $m$ for linear subflow paths ($m = 1$).

Therefore, if a given path is not self-intersecting at $x_{\ell_k}$, then

\[
x_{n_k \ell_k j_k}^w(t) = x_{n_k \ell_k j_k}^w(t), \quad f_{i_k j_k}^w(t) = f_{i_k j_k}^w(t), \quad f_{n_k \ell_k}^w(t) = f_{n_k \ell_k}^w(t).
\]

The sum of the cumulative transient inflows from and outflows to all subcompartments generated at subcompartment $\ell_k$ at time $t$ by local input into the connection of a given subflow path $p_{n_k i_k}^w$ during $[t_1, t]$, $t_1 \geq t_0$, are then defined as

\[(C.2)\]

\[
\tau_{\ell_k}^{in,w}(t) = \sum_{j=1}^{n} f_{i_k j_k}^w(t) \quad \text{and} \quad \tau_{\ell_k}^{out,w}(t) = \sum_{j=1}^{n} f_{n_k \ell_k}^w(t)
\]

for $k = 0, 1, \ldots, n$. Therefore, $\tau_{\ell_k}^{in,w}(t)$ and $\tau_{\ell_k}^{out,w}(t)$ will be called the respective inward and outward transient subthroughflow at subcompartment $\ell_k$ at time $t$ along path $p_{n_k i_k}^w$ in subsystem $k$. Similar to Eq. 2.21 and 2.49, the inward and outward transient subthroughflow, $T_k^{in,w}(t)$ and $T_k^{out,w}(t)$, as well as the transient substorage, $X_k^w(t)$, matrix functions, whose entries are the transient subthroughflows and associated substorages along a given subflow path $p_{n_k i_k}^w$, can be formulated.

Note that we have the relationships

\[
x_{n_k \ell_k j_k}^w(t) \leq x_{\ell_k}(t), \quad f_{i_k j_k}^w(t) \leq f_{j_k}(t, x), \quad \text{and} \quad f_{n_k \ell_k}^w(t) \leq f_{n_k \ell_k}(t, x)
\]

as the transient subflows and substorages are the segments of the corresponding subflows and substorages (see Fig. 3). We also have

\[
f_{i_k j_k}^w(t, x) = \sum_{u=1}^{u_k} f_{i_k j_k}^w(t) \quad \text{and} \quad f_{n_k \ell_k}^w(t, x) = \sum_{u=1}^{u_k} f_{n_k \ell_k}^w(t)
\]

where $P_k$ is a set of mutually exclusive and exhaustive subflow paths in subsystem $k$. 

and \( w_k \) is the number of paths in this set. For the same set, we also have

\[
x_{\ell_k}(t) = \sum_{w=1}^{w_k} x_{\ell_k}^w(t), \quad \tau_{\ell_k}^{in}(t, x) = \sum_{w=1}^{w_k} \tau_{\ell_k}^{in,w}(t), \quad \text{and} \quad \tau_{\ell_k}^{out}(t, x) = \sum_{w=1}^{w_k} \tau_{\ell_k}^{out,w}(t).
\]

That is, due to the mutually exclusiveness and exhaustiveness of the subsystem decomposition, the sum of the cumulative transient subflows along all subflow paths in subsystem \( k \) are equal to the subflows, and that of the transient subthroughflows and substorages are equal to the subthroughflows and substorages, respectively.

**C.3. Static Subsystem Decomposition.** The static versions of the dynamic subsystem decomposition introduced in Eqs. 2.76 and 2.77 are formulated by [5]. Since the time derivatives of the state variables are zero at steady state, we set \( \dot{x}_{\tau_k}^w(t) = 0 \) in Eq. 2.77. Then, the static transient outflow \( f_{n_k,\ell_k,j_k}^w \) at subcompartment \( \ell_k \), from \( j_k \) to \( n_k \) along subflow path \( p_{n_k,i_k}^w \), and the transient substorage \( x_{n_k,\ell_k,j_k}^w \) generated at \( \ell_k \) by the transient inflow \( f_{n_k,\ell_k,i_k}^w \) are formulated as

\[
(C.3) \quad x_{n_k,\ell_k,j_k}^w = \frac{x_{\ell_k}}{\tau_{\ell_k}} f_{n_k,\ell_k,i_k}^w \quad \text{and} \quad f_{n_k,\ell_k,j_k}^w = \frac{\tau_{\ell_k}}{x_{\ell_k}} x_{n_k,\ell_k,j_k}^w = \frac{\tau_{\ell_k}}{x_{\ell_k}} f_{n_k,\ell_k,i_k}^w.
\]

**Appendix D. The diact Flows and Storages.**

The diact transaction types are introduced in Sec. 2.5.2 based on the proposed dynamic subsystem decomposition methodology. The detailed formulation of diact subflows and the associated substorages generated by these flows are presented in this section.

**D.1. Transfer Flows and Storages.** The transfer subflow, \( \tau_{i_k,j_k}^{t}(t) \), is defined in Sec. 2.5.2. From subcompartment \( j_k \) to \( i_k \), it can, alternatively, be defined as the sum of the transient subthroughflows initiated at subcompartment \( j_k \) during \([t_0, t]\) and transmitted into \( i_k \) at time \( t \) along all subflow paths in \( P_{i_k,j_k}^{t} \). The associated transfer substorage, \( x_{i_k,j_k}^{t}(t) \), at subcompartment \( i_k \) at time \( t \) is then the sum of the cumulative transient substorages derived from transfer inflow, \( \tau_{i_k,j_k}^{t}(t) \), during \([t_0, t]\). This alternative formulation can be expressed as

\[
\tau_{i_k,j_k}^{t}(t) = \sum_{w=1}^{w_k} \tau_{i_k}^w(t) \quad \text{and} \quad x_{i_k,j_k}^{t}(t) = \sum_{w=1}^{w_k} x_{i_k}^w(t)
\]

where \( w_k \) is the number of subflow paths \( p_{i_k,j_k}^w \in P_{i_k,j_k}^{t} \). The transfer flow and storage and \( k^{th} \) transfer subflow and substorage matrices are defined in Sec. 2.5.2. The transfer throughflow and compartmental storage matrices can also be formulated as

\[
T^{t}(t) = \text{diag}(T^{t}(t) \mathbf{1}) \quad \text{and} \quad X^{t}(t) = \text{diag}(X^{t}(t) \mathbf{1}).
\]

**D.2. Direct Flows and Storages.** We define the direct subflow as the transfer subflow from one subcompartment directly to another in the same subsystem. Let \( P_{i_k,j_k}^{d} \) be the set of mutually exclusive and exhaustive subflow paths \( p_{i_k,j_k}^w \) from subcompartment \( j_k \) directly to \( i_k \) in subsystem \( k \). The direct inflow from subcompartment \( j_k \) to \( i_k \), \( \tau_{i_k,j_k}^{d}(t) \), is the sum of the cumulative transient inflows generated by local inputs, initiated at \( j_k \) during \([t_0, t]\) and transmitted to \( i_k \) at time \( t \) along all subflow paths in \( P_{i_k,j_k}^{d} \). Therefore, the direct inflow from subcompartment \( j_k \) to \( i_k \) at time \( t \), \( \tau_{i_k,j_k}^{d}(t) \), is equal to \( f_{i_k,j_k}(t, x) \), that is \( \tau_{i_k,j_k}^{d}(t) = f_{i_k,j_k}(t, x) \). The associated direct
substorage, $x_{ik}^d(t)$, at subcompartment $i_k$ at time $t$ is the sum of the cumulative transient substorages derived from direct inflow, $\tau_{ik}^d(t)$, during $[t_0, t]$.

The direct subflows and substorages, $\tau_{ik}^d(t)$ and $x_{ik}^d(t)$, and corresponding direct flows and storages, $\tau_{ij}^d(t)$ and $x_{ij}^d(t)$, can be formulated similar to their transfer counterparts as given in Eqs. 2.80 and 2.81, using subflow set $P_{ik}$ instead. The $k^{th}$ direct subflow and associated substorage matrix functions, $T_k^d(t) = F_k(t, x)$ and $X_k^d(t)$, and the corresponding direct flow and associated storage matrices, $T^d(t) = F(t, x)$ and $X^d(t)$, can also be formulated as their transfer counterparts given in Eq. 2.82, using the corresponding direct flows and storages. The direct throughflow and compartmental storage matrices can then be expressed as

$$T^d(t) = \text{diag}(T^d(t)1) \quad \text{and} \quad X^d(t) = \text{diag}(X^d(t)1).$$

**D.3. Indirect Flows and Storages.** We define the indirect subflow as the transfer subflow from one subcompartment, indirectly through other subcompartments to another in the same subsystem. Let $P_{ik}$ be the set of mutually exclusive and exhaustive subflow paths $p_{ik}$ from subcompartment $j_k$ indirectly to $i_k$ in subsystem $k$. The indirect inflow from subcompartment $j_k$ to $i_k$, $\tau_{ik}^c(t)$, is defined as the sum of the cumulative transient inflows generated by local inputs, initiated at $j_k$ during $[t_0, t]$ and transmitted to $i_k$ at time $t$ along all subflow paths in $P_{ik}$. The associated indirect substorage, $x_{ik}^c(t)$, at subcompartment $i_k$ at time $t$ is the sum of the cumulative transient substorages derived from indirect inflow, $\tau_{ik}^c(t)$, during $[t_0, t]$.

The indirect subflows and substorages, $\tau_{ik}^c(t)$ and $x_{ik}^c(t)$, and corresponding indirect flows and storages, $\tau_{ij}^c(t)$ and $x_{ij}^c(t)$, can be formulated similar to their transfer counterparts as given in Eqs. 2.80 and 2.81, using subflow set $P_{ik}$ instead. The $k^{th}$ indirect subflow and associated substorage matrix functions, $T_k^c(t) = F_k(t, x)$ and $X_k^c(t)$, and the corresponding indirect flow and associated storage matrices, $T^c(t)$ and $X^c(t)$, can also be formulated as their transfer counterparts given in Eq. 2.82, using the corresponding indirect flows and storages. The indirect throughflow and compartmental storage matrices can then be expressed as

$$T^c(t) = \text{diag}(T^c(t)1) \quad \text{and} \quad X^c(t) = \text{diag}(X^c(t)1).$$

**D.4. Cycling Flows and Storages.** We define the cycling subflow as the transfer subflow from one subcompartment, indirectly through other subcompartments in the same subsystem, back into itself. Let $P_{ik}$ be the set of mutually exclusive and exhaustive subflow paths $p_{ik}$ from subcompartment $i_k$ indirectly back to itself in subsystem $k$. The cycling inflow from subcompartment $i_k$ to itself, $\tau_{ik}^c(t)$, will be defined as the sum of the cumulative transient inflows generated by local inputs, initiated at $i_k$ during $[t_0, t]$ and transmitted indirectly back into itself at time $t$ along all subflow paths in $P_{ik}$. The associated cycling substorage, $x_{ik}^c(t)$, at subcompartment $i_k$ at time $t$ is then the sum of the cumulative transient substorages derived from the cycling inflow, $\tau_{ik}^c(t)$, during $[t_0, t]$. Figure 9 depicts the complementary relationships among direct, indirect and cycling flows.

The cycling subflows and substorages, $\tau_{ik}^c(t)$ and $x_{ik}^c(t)$, and corresponding cycling flows and storages, $\tau_{ij}^c(t)$ and $x_{ij}^c(t)$, can be formulated similar to their transfer counterparts as given in Eqs. 2.80 and 2.81, using subflow set $P_{ik}$ instead. The diagonal $k^{th}$ cycling subflow and associated substorage matrix functions, $T_k^c(t) = F_k(t, x)$ and $X_k^c(t)$, and the corresponding cycling flow and associated storage matrices, $T^c(t)$ and $X^c(t)$,
can also be formulated as their transfer counterparts given in Eq. 2.82, using the corresponding cycling flows and storages.

We also construct the **cycling subflow** and associated **substorage matrices**, \( T^c(t) \) and \( X^c(t) \), so that the \( k \)th column vector of these matrices are the \( k \)th cycling subflow and associated substorage vectors, respectively. That is,

\[
(D.1) \quad T^c(t) = [T_1^c(t) \ 1 \ \cdots \ T_n^c(t) \ 1] \quad \text{and} \quad X^c(t) = [X_1^c(t) \ 1 \ \cdots \ X_n^c(t) \ 1].
\]

Note that, the **cycling flow** (or throughflow) and subflow, and (compartmental) storage and substorage matrices can then be expressed as

\[
T^c(t) = \text{diag} (T^c(t) \ 1) \quad \text{and} \quad X^c(t) = \text{diag} (X^c(t) \ 1).
\]

**D.5. Acyclic Flows and Storages.** Lastly, we define the **acyclic subflow** at a subcompartment as the non-cyclic segment of the subthroughflow at that subcompartment. Let \( P^a_{ik} \) be the set of mutually exclusive and exhaustive linear subflow paths \( p^a_{ik} \) from subcompartment \( k \) directly or indirectly to \( i_k \) in subsystem \( k \). The **acyclic inflow** from subcompartment \( k \) to \( i_k \), \( \tau^a_{ik}(t) \), will be defined as the sum of the cumulative transient inflows generated by external input \( z_k(t) \) during \([t_0, t]\) and transmitted to \( i_k \) at time \( t \) along all subflow paths in \( P^a_{ik} \). The associated **acyclic substorage**, \( x^a_{ik}(t) \), at subcompartment \( i_k \) at time \( t \) is then the sum of the cumulative transient storages derived from the acyclic inflow, \( \tau^a_{ik}(t) \), during \([t_0, t]\).

The acyclic subflows and storages, \( \tau^a_{ik}(t) \) and \( x^a_{ik}(t) \), and corresponding **acyclic flows and storages**, \( \tau^a(t) \) and \( x^a(t) \), can be formulated similar to their transfer counterparts as given in Eqs. 2.80 and 2.81, using subflow set \( P^a_{ik} \) instead. The diagonal \( k \)th **acyclic subflow** and associated **substorage** matrix functions, \( T^a_k(t) \) and \( X^a_k(t) \), and the corresponding **acyclic flow** and associated **storage** matrices, \( T^a(t) \) and \( X^a(t) \), can also be formulated as their transfer counterparts given in Eq. 2.82, using the corresponding acyclic flows and storages.

We also construct the **acyclic subflow** and associated **substorage matrices**, \( T^a(t) \) and \( X^a(t) \), so that, the \( k \)th column vector of these matrices are the \( k \)th acyclic subflow and associated substorage vectors, respectively. That is,

\[
(D.2) \quad T^a(t) = [T_1^a(t) \ 1 \ \cdots \ T_n^a(t) \ 1] \quad \text{and} \quad X^a(t) = [X_1^a(t) \ 1 \ \cdots \ X_n^a(t) \ 1].
\]

Therefore, the **acyclic subflow** and associated **substorage matrices** can be formulated as

\[
(D.3) \quad T^a(t) = \hat{T}(t, x) - T^c(t) \quad \text{and} \quad X^a(t) = X(t) - X^c(t).
\]
Note that, the acyclic flow (or throughflow) and (compartmental) storage matrices can then be expressed as

\[ T^a(t) = \text{diag}(T^a(t) \mathbf{1}) \quad \text{and} \quad X^a(t) = \text{diag}(X^a(t) \mathbf{1}). \]

Moreover, due to its construction, the diagonal entries of \( T^a(t) \) are external inputs. That is, \( \text{diag}(T^a(t)) = \mathbf{Z}(t, \mathbf{x}). \)

**D.6. Static diact Flows and Storages.** Based on the static subsystem decomposition presented above in Section C.3, the static diact flows and storages are also introduced by [5] using Eq. 2.56, as presented in matrix form in Table 1. All quantities in the table are the static counterparts of their dynamic versions introduced in the present paper and \( N = \text{diag}(N) \). See Example E.2 for application of these formulations to a static ecological model.

**Appendix E. Case Studies.**

Two additional case studies are presented in this section to demonstrate various other aspects of the proposed methodology. Both models are from ecosystem ecology. The first case study is a linear dynamic model, which is solved analytically through the proposed methodology. The second model is a static model, which is used for an application of the static version of the proposed dynamic methodology.

**E.1. Case Study.** In this example, a linear model introduced by [9] is solved analytically. This linear dynamic model has two state variables, \( x_1(t) \) and \( x_2(t) \) (see Fig. 10). The external input, \( z(t) = [z_1(t, \mathbf{x}), z_2(t, \mathbf{x})]^T \), output, \( y(t, \mathbf{x}) = [y_1(t, \mathbf{x}), y_2(t, \mathbf{x})]^T \), and the rate functions, \( F(t, \mathbf{x}) \), are given as

\[
E.1 \quad F(t, \mathbf{x}) = \begin{bmatrix}
0 & 2/3 x_2 \\
4/3 x_1 & 0
\end{bmatrix}, \quad y(t, \mathbf{x}) = \begin{bmatrix} 1/3 x_1 \\ 5/3 x_2 \end{bmatrix}, \quad \text{and} \quad z(t, \mathbf{x}) = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.
\]

From this system information, we define input, output, and the state matrices as

\[
E.2 \quad \mathbf{Z}(t, \mathbf{x}) = \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}, \quad \mathbf{Y}(t, \mathbf{x}) = \begin{bmatrix} 1/3 x_1 & 0 \\ 0 & 5/3 x_2 \end{bmatrix}, \quad \text{and} \quad \mathbf{X}(t, \mathbf{x}) = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}.
\]

Componentwise, the governing equations, Eq. 2.6, take the following form:

\[
E.3 \quad \dot{x}_1(t) = z_1(t) + \frac{2}{3} x_2(t) - \left( \frac{4}{3} + \frac{1}{3} \right) x_1(t)
\]

\[
\dot{x}_2(t) = z_2(t) + \frac{4}{3} x_1(t) - \left( \frac{2}{3} + \frac{5}{3} \right) x_2(t)
\]
with the initial conditions \( x_0 = [x_{1.0}, x_{2.0}]^T = [3, 3]^T \). In vector form, using the notation of Eq. 2.10, the governing equations can be expressed as

\[
\begin{aligned}
\dot{x}(t) &= (z(t, x) + F(t, x)) 1 - (y(t, x) + F^T(t, x)) 1 \\
x(t_0) &= x_0
\end{aligned}
\]  

(E.4)

In matrix form, as given in Eq. 2.66, the system of governing equations becomes

\[
\begin{aligned}
\dot{x}(t) &= z(t) + A x(t), \quad x(t_0) = x_0
\end{aligned}
\]  

(E.5)

where \( A \) is the constant flow intensity matrix given in Eq. 2.46:

\[
A = (F - \text{diag} \left( \tau^{out} \right) ) X^{-1}.
\]

(E.6)

It can be written explicitly as

\[
A = \begin{bmatrix}
-\left(\frac{4}{3} + \frac{1}{3}\right) & \frac{2}{3} \\
\frac{4}{3} & -\left(\frac{2}{3} + \frac{5}{3}\right)
\end{bmatrix} = \begin{bmatrix}
-\frac{5}{3} & \frac{2}{3} \\
\frac{4}{3} & -\frac{7}{3}
\end{bmatrix}.
\]

(E.7)

For the following state decomposition,

\[
x_i(t) = \sum_{k=0}^{2} x_{i_k}(t),
\]

(E.8)

the substate and subflow rate functions become

\[
\begin{aligned}
X(t) &= \begin{bmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22}
\end{bmatrix}, \\
F_k(t, x) &= \begin{bmatrix}
F_{1k} \\
F_{2k}
\end{bmatrix} = \begin{bmatrix}
0 & \frac{2}{3} d_{2k} & x_2 \\
\frac{4}{3} d_{1k} x_1 & 0
\end{bmatrix} = \begin{bmatrix}
0 & \frac{2}{3} x_{2k} \\
\frac{4}{3} x_{1k} & 0
\end{bmatrix}, \\
z_k(t, x) &= \begin{bmatrix}
z_{1k} \\
z_{2k}
\end{bmatrix} = \begin{bmatrix}
\delta_{1k} x_1 \\
\delta_{2k} x_2
\end{bmatrix}, \\
y_k(t, x) &= \begin{bmatrix}
y_{1k} \\
y_{2k}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{3} d_{1k} x_1 \\
\frac{5}{3} d_{2k} x_2
\end{bmatrix} = \begin{bmatrix}
\frac{1}{3} x_{1k} \\
\frac{5}{3} x_{2k}
\end{bmatrix}.
\end{aligned}
\]

(E.9)

The decomposed system, Eq. 2.62, can componentwise be expressed as

\[
\begin{aligned}
\dot{x}_{1k}(t) &= z_{1k}(t) + \frac{2}{3} x_{2k}(t) - \left(\frac{4}{3} + \frac{1}{3}\right) x_{1k}(t) \\
\dot{x}_{2k}(t) &= z_{2k}(t) + \frac{4}{3} x_{1k}(t) - \left(\frac{2}{3} + \frac{5}{3}\right) x_{2k}(t)
\end{aligned}
\]  

(E.10)
with the initial conditions \( x_{i_k}(t_0) = 0 \) for \( i, k = 1, 2 \).

Similarly, for the following decomposition of the initial subsystem \((k = 0)\),

\[
\bar{x}_i(t) = \sum_{k=1}^{2} \bar{x}_{i_k}(t),
\]

the initial subflow rate functions become

\[
\begin{align*}
\bar{X}(t) &= \begin{bmatrix} \bar{x}_{11} & \bar{x}_{12} \\ \bar{x}_{21} & \bar{x}_{22} \end{bmatrix}, \\
\bar{F}_k(t, \bar{x}) &= \begin{bmatrix} \bar{f}_{1k,1} & \bar{f}_{1k,2} \\ \bar{f}_{2k,1} & \bar{f}_{2k,2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} \bar{d}_{2k} \bar{x}_2 \\ \frac{4}{3} \bar{d}_{1k} \bar{x}_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} \bar{x}_{2k} \\ \frac{4}{3} \bar{x}_{1k} & 0 \end{bmatrix}, \\
\bar{z}_k(t, \bar{x}) &= \begin{bmatrix} \bar{z}_{1k} \\ \bar{z}_{2k} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
\bar{y}_k(t, \bar{x}) &= \begin{bmatrix} \bar{y}_{1k} \\ \bar{y}_{2k} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \bar{d}_{1k} \bar{x}_1 \\ \frac{5}{3} \bar{d}_{2k} \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \bar{x}_{1k} \\ \frac{5}{3} \bar{x}_{2k} \end{bmatrix}.
\end{align*}
\]

The initial subsystem decomposition yields the following governing equations, as formulated in Eq. 2.63:

\[
\begin{align*}
\dot{\bar{x}}_{1k}(t) &= \frac{2}{3} \bar{x}_{2k}(t) - \left( \frac{4}{3} + \frac{1}{3} \right) \bar{x}_{1k}(t) \\
\dot{\bar{x}}_{2k}(t) &= \frac{4}{3} \bar{x}_{1k}(t) - \left( \frac{2}{3} + \frac{5}{3} \right) \bar{x}_{2k}(t)
\end{align*}
\]

with the initial conditions \( \bar{x}_{i_k}(t_0) = x_{i_k,0}(t_0) = 3 \delta_{ik} \) for \( i, k = 1, 2 \). Thus, there are \( 2n^2 = 8 \) equations in the decomposed system; \( n^2 = 4 \) of them are for the substates and the other \( n^2 \) equations are for the initial substates.

The governing equations for the decomposed system can be written in vector form, as given in Eq. 2.64:

\[
\begin{align*}
\dot{x}_k(t) &= z_k + A x_k(t), \quad x_k(t_0) = 0, \\
\dot{x}_k(t) &= A x_k(t), \quad x_k(t_0) = x_{k,0} e_k,
\end{align*}
\]

for \( k = 1, 2 \) or in matrix form, as given in Eq. 2.67:

\[
\begin{align*}
\dot{X}(t) &= Z + AX(t), \quad X(t_0) = 0, \\
\dot{X}(t) &= AX(t), \quad X(t_0) = X_0.
\end{align*}
\]

The governing system, Eq. E.15, is linear. It can, therefore, be solved analytically as formulated in Section B. Since \( A \) is a constant flow intensity matrix, we have the following fundamental solution as given in Eq. B.9

\[
U(t) = \begin{bmatrix} \frac{2e^{-t}}{3} + \frac{e^{-3t}}{3} & \frac{2e^{-t}}{3} - \frac{2e^{-3t}}{3} \\ \frac{e^{-t}}{3} - \frac{e^{-3t}}{3} & \frac{2e^{-t}}{3} + \frac{e^{-3t}}{3} \end{bmatrix}.
\]

The solution for the matrix equation, Eq. E.15, as formulated in Eq. B.4 then becomes

\[
\begin{align*}
\bar{X}(t) &= \begin{bmatrix} 2e^{-t} + e^{-3t} \\ 2e^{-t} - 2e^{-3t} \end{bmatrix}, \\
X(t) &= \begin{bmatrix} \frac{7}{9} + \frac{e^{-3t}}{9} - 2e^{-t} \\ \frac{2}{3} + \frac{2e^{-3t}}{3} - 2e^{-t} \end{bmatrix}.
\end{align*}
\]
Therefore, the solutions to the original system is

(E.18) \[ x(t) = X(t) \mathbf{1} = \begin{bmatrix} 2e^{-t} + 1 \\ 2e^{-t} + 1 \end{bmatrix} \]

where

(E.19) \[ X(t) = \tilde{X}(t) + X(t) = \begin{bmatrix} \frac{7}{9} + \frac{4e^{-t}}{3} + \frac{8e^{-3t}}{9} \\ \frac{2}{9} + \frac{2e^{-t}}{3} - \frac{8e^{-3t}}{9} \end{bmatrix} \]

as given in Eqs. B.10 and B.11.

The steady state solutions as formulated in Eq. B.13 are

(E.20) \[ X = -A^{-1} = X \left( \text{diag}(\tau^{in}) - F \right)^{-1} = \begin{bmatrix} \frac{7}{9} \\ \frac{2}{9} \end{bmatrix} \quad \text{and} \quad \tilde{X} = \mathbf{0}. \]

It can be seen easily that the dynamic solution, Eq. E.17, converges to this steady state solution as \( t \to \infty \).

We also analyze the system with a time dependent input \( z(t) = [3 + \sin(t), 3 + \sin(2t)]^T \). The initial substate matrix, \( \tilde{X}(t) \), is the same as the one given in Eq. E.17 for the constant input case above. Similar computations leads us to the following initial substate vector and substate matrix components:

(E.21) \[
\begin{align*}
    x_{1a}(t) &= x_{2a}(t) = 3e^{-t}, \\
    x_{11}(t) &= \frac{7}{3} - \frac{11 \cos(t)}{30} + \frac{13 \sin(t)}{30} - \frac{5e^{-t}}{3} - \frac{3e^{-3t}}{10}, \\
    x_{12}(t) &= \frac{2}{3} - \frac{16 \cos(2t)}{195} - \frac{2 \sin(2t)}{195} - \frac{13e^{-t}}{15} + \frac{11e^{-3t}}{39}, \\
    x_{21}(t) &= \frac{4}{3} - \frac{4 \cos(t)}{15} + \frac{2 \sin(t)}{15} - \frac{5e^{-t}}{3} + \frac{3e^{-3t}}{5}, \\
    x_{22}(t) &= \frac{5}{3} - \frac{46 \cos(2t)}{195} - \frac{43 \sin(2t)}{195} - \frac{13e^{-t}}{15} - \frac{22e^{-3t}}{39}.
\end{align*}
\]

The solutions to the system given in Eq. E.17 for the constant input are the same as the ones given by [9]. The authors, however, did not provide an explicit solution for the time-dependent input case for a comparison. At steady state, the solution for the substate matrix becomes

(E.22) \[ \lim_{t \to \infty} X(t) = \begin{bmatrix} \frac{7}{3} - \frac{11 \cos(t)}{30} + \frac{13 \sin(t)}{30} & \frac{2}{3} - \frac{16 \cos(2t)}{195} + \frac{2 \sin(2t)}{195} \\ \frac{4}{3} - \frac{4 \cos(t)}{15} + \frac{2 \sin(t)}{15} & \frac{5}{3} - \frac{46 \cos(2t)}{195} + \frac{43 \sin(2t)}{195} \end{bmatrix}. \]

Similarly, the elements of the initial subthroughflow vector and the subthroughflow matrix are

(E.23) \[
\begin{align*}
    \tau^{in}_{1a}(t) &= 2e^{-t}, \quad \tau^{in}_{2a}(t) = 4e^{-t}, \\
    \tau^{in}_{11}(t) &= \frac{35}{9} - \frac{8 \cos(t)}{45} + \frac{49 \sin(t)}{45} - \frac{10e^{-t}}{9} + \frac{2e^{-3t}}{5}, \\
    \tau^{in}_{12}(t) &= \frac{742}{585} - \frac{184 \cos^2(t)}{585} + \frac{86 \sin(2t)}{585} - \frac{26e^{-t}}{45} - \frac{44e^{-3t}}{117}, \\
    \tau^{in}_{21}(t) &= \frac{28}{9} - \frac{22 \cos(t)}{45} + \frac{26 \sin(t)}{45} - \frac{20e^{-t}}{9} - \frac{2e^{-3t}}{5}, \\
    \tau^{in}_{22}(t) &= \frac{2339}{585} - \frac{128 \cos^2(t)}{585} + \frac{577 \sin(2t)}{585} - \frac{52e^{-t}}{45} + \frac{44e^{-3t}}{117}.
\end{align*}
\]
Graphical representations of the substate and the inward subthroughflow matrices, $X(t)$ and $T^{in}(t)$, given in Eqs. E.21 and E.23, are depicted in Fig. 11. Using these matrices, the dynamic distribution of environmental inputs as inward throughflows and the organization of the associated storages generated by the inputs within the system can be analyzed individually and separately.

The transfer flow and associated storage function from compartment 2 to 1, $\tau_{12}^t(t)$ and $x_{12}^t(t)$, are computed below as an application of the proposed subsystem decomposition methodology. They can be expressed as

$$(E.24) \quad \tau_{12}^t(t) = \sum_{k=0}^{2} \tau_{1k2k}^t(t) \quad \text{and} \quad x_{12}^t(t) = \sum_{k=0}^{2} x_{1k2k}^t(t),$$

as formulated in Eqs. 2.81. The sets of mutually exclusive and exhaustive subflow paths from $2_k$ to $1_k$, $P_{1k2k}$, for $k = 0, 1, 2$, can be formulated as follows: $P_{1020} = \{p_{1020}^1, p_{1020}^2\}$, $P_{1121} = \{p_{1121}^1\}$, $P_{1222} = \{p_{1222}^1\}$ where $p_{1020}^1 = 10 \rightarrow 10 \rightarrow 20 \rightarrow 10$, $p_{1020}^2 = 20 \rightarrow 20 \rightarrow 20 \rightarrow 20$, $p_{1121}^1 = 11 \rightarrow 11 \rightarrow 21 \rightarrow 11$, and $p_{1222}^1 = 22 \rightarrow 22 \rightarrow 22$.

There are two subflow paths in the initial subsystem, $p_{1020}^1$ and $p_{1020}^2$, and, therefore, $w_0 = 2$. Since $f_{ii}(t) = 0$, the corresponding transfer subflow and associated substorage functions as formulated in Eq. 2.80 become

$$(E.25) \quad \tau_{1020}^t(t) = \sum_{w=1}^{2} \sum_{\ell=1}^{2} f_{10\ell w}^t(t) = f_{1020}^1(t) + f_{1020}^2(t) \quad \text{and} \quad x_{1020}^t(t) = \sum_{w=1}^{2} x_{10\ell w}^t(t) = x_{10}^1(t) + x_{10}^2(t).$$

Similarly, we have

$$(E.26) \quad \tau_{1k2k}^t(t) = \sum_{w=1}^{1} \sum_{\ell=1}^{1} f_{1k\ell w}^t(t) = f_{1k2k}^1(t) \quad \text{and} \quad x_{1k2k}^t(t) = \sum_{w=1}^{1} x_{1k\ell w}^t(t) = x_{1k}^1(t),$$
since there is only one subflow path in these subsystems ($w_k = 1$ for $k = 1, 2$). The links on this path $p_{1,12}^1$ that directly contribute to the inflow are numbered with the red cycle numbers, $m$, in the extended subflow path diagram below:

$$p_{1,12}^1 = 0_1 \to 1_1 \to 2_1 \to 1_1 \to 2_1 \to 1_1 \to 2_1 \to 1_1 \to \cdots$$

The cumulative transient inflow $f_{1,12}^1(t)$ and substorage $x_{1,1}^1(t)$ at subcompartment 11 along $p_{1,12}^1$ will be approximated by two terms ($m = 2$) using Eq. 2.79:

$$x_{1,1}^1(t) = \sum_{m=1}^{2} x_{2,1;1,2}^{1,m}(t) \approx x_{2,1;1,2}^{1,1}(t) + x_{2,1;1,2}^{1,2}(t),$$

(E.27)

$$f_{1,12}^1(t) = \sum_{m=1}^{2} f_{1,2;1,1}^{1,m}(t) \approx f_{1,2;1,1}^{1,1}(t) + f_{1,2;1,1}^{1,2}(t).$$

The governing equations, Eqs. 2.76 and 2.77, for the transient subflow and associated substorage functions, $f_{1,12;1,1}^{1,m}(t)$ and $x_{2,1;1,2}^{1,m}(t)$, and other transient subflows and substages involved in Eqs. E.25 and E.26, are solved simultaneously, together with the decomposed system, Eq. 2.67. Numerical results for the transfer subflow and associated substorage functions are presented in Fig. 12.

The subflow paths in $P_{1,2k}$ for each subsystem $k$ are mutually exclusive and exhaustive. Therefore, $x_1(t)$ and $x_{12}(t) + x_{21;1,0}(t)$ must be the same, as well as $f_{12}(t)$ and $\tau_{12}^x(t)$. The term added to $x_{12}^1(t)$ for the comparison, $x_{2,1;1,0}(t)$, is the storage generated by environmental input $z_1(t)$ in 11 $(0_1 \to 1_1)$ and, therefore, is not part of transfer storage $x_{12}^1(t)$. They, however, are approximately equal, as presented in Fig. 12, that is, $x_{12}^1(t) + x_{2,1;1,0}(t) \approx x_1(t)$ and $\tau_{12}^x(t) \approx f_{12}(t)$. The difference is caused by the truncation errors in the computation of cumulative transient subflows as indicated in Eq. E.27, and larger $m_w$ values improve the approximations. Closed paths are approximated by an infinite series of functions due to the method construction. These close approximations by just two terms of the series justify the validity and indicate the accuracy of the proposed subsystem decomposition methodology.

The cycling flows and the associated storages generated by these flows are also calculated below along the following subflow paths. The sets of mutually exclusive and exhaustive subflow paths from subcompartment 11 to itself, $P_{1,1}^x$, are given as $P_{1,1}^x = \{p_{1,1}^1, p_{1,1}^2\}$, $P_{1,1}^2 = \{p_{1,1}^1, p_{1,1}^2\}$, where $p_{1,1}^1 = 0_1 \to 1_0 \to 2_0 \to 1_0$, $p_{1,1}^2 = 2_0 \to 2_0 \to 1_0 \to 2_0 \to 1_0$, $p_{1,1}^1 = 0_1 \to 1_1 \to 2_1 \to 1_1$, and $p_{1,1}^2 = 2_2 \to 1_2 \to 2_2 \to 1_2$ (see Fig. 10). The sets of subflow paths for $P_{2,k}^x$, $k = 0, 1, 2$, can similarly be defined.
The cycling subflow at subcompartment \( 1_2 \) along the only subflow path \((w_2 = 1)\) \( p_{12}^1 \in P_{12}^1 \) and associated substorages are

\[
\tau_{12}^c(t) = \sum_{w=1}^{1} \sum_{\ell=1}^{2} f_{12\ell}^w(t) = f_{1222}^1(t) \quad \text{and} \quad x_{12}^c(t) = \sum_{w=1}^{1} x_{12}^w(t) = x_{12}^1(t),
\]

as formulated in Eq. 2.80. The links contributing to the cycling flow along the path are marked with red cycle numbers in the extended subflow diagram below:

\[
p_{12}^1 = 0 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow \cdot \cdot \cdot
\]

The cumulative transient inflow \( f_{1222}^1(t) \) and substorage \( x_{12}^1(t) \) can be approximated by two terms \((m_1 = 2)\) as formulated in Eq. 2.79:

\[
x_{12}^1(t) = \sum_{m=1}^{2} x_{21222}^{1,m}(t) \approx x_{21222}^{1,1}(t) + x_{21222}^{1,2}(t),
\]

\[
f_{1222}^1(t) = \sum_{m=1}^{2} f_{12222}^{1,m}(t) \approx f_{12222}^{1,1}(t) + f_{12222}^{1,2}(t).
\]

The transient subflows and associated substorage functions \( f_{12222}^{w,m}(t) \) and \( x_{21222}^{w,m}(t) \) and the other transient subflows and substorages involved in Eq. 2.28, as formulated in Eqs. 2.76 and 2.77, are solved simultaneously together with the decomposed system, Eqs. 2.62 and 2.63. The numerical results for the cycling flow and associated storage functions

\[
\tau_{12}^c(t) = \sum_{k=0}^{2} \tau_{12}^c(t) \quad \text{and} \quad x_{12}^c(t) = \sum_{k=0}^{2} x_{12}^c(t)
\]

for \( i = 1, 2 \), are presented in Fig. 12.

**E.2. Case Study.** To demonstrate an application of the static version of the proposed methodology, a commonly studied ecosystem network, first proposed by [16], is used as an example in this section. This ecosystem was already analyzed by [5, 6] in detail and ecological interpretations of the results are also presented in that paper.

Cone Spring is a small, shallow spring-brook located in Louisa County, Iowa. The study area consists of 116 \( m^2 \). The network has 5 compartments representing 1—Plants, 2—Detritus, 3—Bacteria, 4—Detritus Feeders, and 5—Carnivores. These compartments are connected by the transaction of energy between them. The conserved quantity needs to be investigated within the system is energy. The system flow information is given as follows:

\[
F = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
8881 & 0 & 1600 & 200 & 167 \\
0 & 5205 & 0 & 0 & 0 \\
0 & 2309 & 75 & 0 & 0 \\
0 & 0 & 0 & 370 & 0 \\
\end{bmatrix}, \quad y = \begin{bmatrix}
2303 \\
3969 \\
3530 \\
1814 \\
203 \\
\end{bmatrix}, \quad z = \begin{bmatrix}
11184 \\
635 \\
0 \\
0 \\
0 \\
\end{bmatrix}.
\]

The unit for energy flows and storages are \( \text{kkal} \times m^{-2} \times y^{-1} \) and \( \text{kkal} \times m^{-2} \), respectively.
The system decomposition methodology yields the subthroughflow matrix as

\[
T = \begin{bmatrix}
11184 & 0 & 0 & 0 & 0 \\
10717 & 766 & 0 & 0 & 0 \\
4858 & 347 & 0 & 0 & 0 \\
2225 & 159 & 0 & 0 & 0 \\
345 & 25 & 0 & 0 & 0
\end{bmatrix}.
\]

The subsystem decomposition methodology yields the direct flows and storages as given in Table 1. The direct flow matrix is \( T^d = F \). The indirect flow matrix becomes

\[
T^i = \begin{bmatrix}
1835.74 & 1967.00 & 63.02 & 226.41 & 31.04 \\
4857.67 & 0 & 753.81 & 193.28 & 89.76 \\
2224.91 & 75.00 & 334.40 & 88.52 & 41.11 \\
345.31 & 370.00 & 63.53 & 0 & 6.38
\end{bmatrix}.
\]

Note the \( \tau_{12}^1 = \tau_{54}^1 = 0 \). This is because of the fact that there is no indirect flow from compartment 2 to 3 or from 4 to 5 (see Fig. 13). There is no indirect flow to compartment 1 from any other compartments either, so the first row vector of \( T^i \) is zero.

The transfer flow matrix as formulated in Table 1 becomes

\[
(T^e)^{E.29} = \begin{bmatrix}
10716.74 & 1967.00 & 1663.02 & 426.42 & 198.04 \\
4857.67 & 5205.00 & 753.81 & 193.28 & 89.76 \\
2224.91 & 2384.00 & 409.40 & 88.53 & 41.12 \\
345.31 & 370.00 & 63.54 & 370.00 & 6.38
\end{bmatrix}.
\]

The Cone Spring model is also studied by [15] for similar purposes. The authors defined a matrix called the total flow matrix. Although the derivation rationales are different, the transfer flow matrix given in Eq. \( E.29 \) is equivalent to this total flow matrix. As listed in Table 1, the cycling and acyclic subflow matrices can also be
expressed as follows:

\[
T^c = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
1835.74 & 131.26 & 0 & 0 & 0 \\
703.51 & 50.30 & 0 & 0 & 0 \\
82.62 & 5.91 & 0 & 0 & 0 \\
5.96 & 0.43 & 0 & 0 & 0 \\
\end{bmatrix}
\quad \text{and} \quad
T^a = \begin{bmatrix}
11184 & 0 & 0 & 0 & 0 \\
8881 & 635 & 0 & 0 & 0 \\
4154 & 635 & 0 & 0 & 0 \\
2142 & 313 & 0 & 0 & 0 \\
339 & 35 & 24 & 0.91 & 0 & 0 \\
\end{bmatrix}
\]

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REFERENCES

[1] D. Anderson, *Compartmental Modeling and Tracer Kinetics*, Springer Berlin Heidelberg, 2013.
[2] R. M. Anderson and R. M. May, *Population biology of infectious diseases: Part i*, Nature, 280 (1979), pp. 361 EP –, [http://dx.doi.org/10.1038/280361a0](http://dx.doi.org/10.1038/280361a0).
[3] H. Coskun, *Dynamic ecological system analysis*. Preprint, 2018, [https://doi.org/10.31219/osf.io/35xkb](https://doi.org/10.31219/osf.io/35xkb).
[4] H. Coskun, *Dynamic ecological system measures*. Preprint, 2018, [https://doi.org/10.31219/osf.io/j2pd3](https://doi.org/10.31219/osf.io/j2pd3).
[5] H. Coskun, *Static ecological system analysis*. Preprint, 2018, [https://doi.org/10.31219/osf.io/zqxc5](https://doi.org/10.31219/osf.io/zqxc5).
[6] H. Coskun, *Static ecological system measures*. Preprint, 2018, [https://doi.org/10.31219/osf.io/g4xzt](https://doi.org/10.31219/osf.io/g4xzt).
[7] L. Edelstein-Keshet, *Mathematical Models in Biology*, SIAM, 2004.
[8] B. Hannon, *The structure of ecosystems*, Journal of theoretical biology, 41 (1973), pp. 535–546.
[9] P. W. Hipp, *Environ analysis of linear compartmental systems: The dynamic, time-invariant case*, Ecological Modelling, 19 (1983), pp. 1–26, [https://doi.org/10.1016/0304-3800(83)90067-4](https://doi.org/10.1016/0304-3800(83)90067-4).
[10] J. Jacquez and C. Simon, *Qualitative theory of compartmental systems*, SIAM Review, 35 (1993), pp. 43–79, [https://doi.org/10.1137/1035003](https://doi.org/10.1137/1035003).
[11] W. O. Kermack and A. G. McKendrick, *A contribution to the mathematical theory of epidemics*, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 115 (1927), pp. 700–721, [https://doi.org/10.1098/rspa.1927.0118](https://doi.org/10.1098/rspa.1927.0118), [http://rspa.royalsocietypublishing.org/content/115/772/700](http://rspa.royalsocietypublishing.org/content/115/772/700), [https://arxiv.org/abs/http://rspa.royalsocietypublishing.org/content/115/772/700.full.pdf](https://arxiv.org/abs/http://rspa.royalsocietypublishing.org/content/115/772/700.full.pdf).
[12] W. W. Leontief, *Quantitative input and output relations in the economic systems of the united states*, The review of economic statistics, 18 (1936), pp. 105–125.
[13] W. W. Leontief, *Input-output economics*, Oxford University Press on Demand, New York, 1986.
[14] B. C. Patten, *Systems approach to the concept of environment*, Ohio Journal of Science, 78 (1978), pp. 206–222.
[15] J. Szyrmer and R. E. Ulanowicz, *Total flows in ecosystems*, Ecological Modelling, 35 (1987), pp. 123–136.
[16] L. J. Tilly, *The structure and dynamics of Cone Spring*, Ecological Monographs, 38 (1968), pp. 169–197.