Safe Learning for Near Optimal Scheduling

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Abstract. In this paper, we investigate the combination of synthesis techniques and learning techniques to obtain safe and near optimal schedulers for a preemptible task scheduling problem. We study both model-based learning techniques with PAC guarantees and model-free learning techniques based on shielded deep Q-learning. The new learning algorithms have been implemented to conduct experimental evaluations.

1 Introduction

In this paper, we show how to combine verification and learning techniques (model-based and model-free) to solve a scheduling problem featuring both hard and soft constraints. We investigate solutions to this problem both from a theoretical and from a more pragmatic point of view. On the theoretical side, we show how safety guarantees (as understood in formal verification) can be combined with guarantees offered by the Probably Approximately Correct (PAC) learning framework [18]. On the pragmatic side, we show how safety guarantees obtained from automatic synthesis can be combined with techniques based on deep Q-learning [2] to offer a scalable and practical solution to solve the scheduling problem at hand.

The scheduling problem that we consider has been introduced in [10] and is defined as follows. A task system is composed of a set of n preemptible tasks \((\tau_i)_{i \in [n]}, F; H)\) partitioned into a set \(F\) of soft tasks and a set \(H\) of hard tasks. Time is assumed to be discrete and measured e.g. in CPU ticks. Each task \(\tau_i\) generates an infinite number of instances \(\tau_{i,j}\), called jobs, with \(j = 1, 2, \ldots\). Jobs generated by both hard and soft tasks are equipped with deadlines, which are relative to the respective arrival times of the jobs in the system. The computation time requirements of the jobs follow a discrete probability distribution, and are unknown to the scheduler but upper bounded by their relative deadline. Jobs generated by hard tasks must complete before their respective deadlines. While this is not mandatory for jobs generated by soft tasks, deadline misses result in a penalty/cost. The tasks are assumed to be independent and generated stochastically: the occurrence of a new job of one task does not depend on the occurrences of jobs of other tasks, and both the inter-arrival time and the computation time of jobs are independent random variables. The scheduling problem consists in finding a scheduler, i.e. a function that associates, to all CPU ticks, a task that must run at that moment; in order to: (i) avoid deadline misses by hard tasks; and (ii) minimise the mean cost of deadline misses by soft tasks.
The semantics of the task system has been modelled using an MDP in [10], and the task there is to compute an optimal and safe scheduler. However, it assumes that the distribution of all tasks is known a priori which may be unrealistic. In the current paper, we investigate learning techniques in order to build algorithms that are able to schedule safely and optimally a set of hard and soft tasks if only the deadlines and the domains of the distributions describing the tasks of the system are known a priori and not the exact distributions. We believe that this is a more realistic assumption. Our motivation was also to investigate the joint application of both synthesis techniques coming from the field of formal verification and learning techniques on an easily understandable, yet challenging, setting.

Contributions. First, we consider model-based learning. We show that the distributions underlying a task system with only soft tasks are efficiently PAC learnable: by executing the task system for a polynomial number of steps, enough samples can be collected to infer $\varepsilon$-accurate approximations of the distributions with high probability (Thm. 1).

Then, we consider the general case of systems with both hard and soft tasks. Here, safe PAC learning is not always possible, and we identify two algorithmically-checkable sufficient conditions for task systems to be safely learnable (Thms. 2 and 3). These crucially depend on the underlying MDP being a single maximal end-component, as is the case in our setting (Lemma 2). Subsequently, we use robustness results on MDPs to compute or learn near-optimal safe strategies from the learnt models (Thm. 4). For the learning part, we apply shielded $Q$-learning in the sense of [1].

Third, in order to evaluate the relevance of the different algorithms defined in the paper, we present experiments of a prototype implementation. These empirically validate the efficient PAC guarantees proved in the theory we have developed. Unfortunately, the models that are learnt by the learning algorithms are often too large for the probabilistic model-checking tools. In contrast, the shielded deep $Q$-learning algorithm scales to larger examples: e.g. we learn safe scheduling strategies for systems with more than $10^{13}$ states. The experiments also show that a strategy that is learnt by assigning high costs to missing deadlines of hard tasks does not respect safety even if the learning happens for a reasonably long period of time, and the costs assigned for missing the deadlines of hard tasks are very high (cf. [1]).

Related works The authors of [10] introduce the scheduling problem considered here but make the assumption that the underlying distributions of the tasks are known. We drop this assumption here and provide learning algorithms. In [1], the framework to combine safety via shielding and model-free reinforcement learning is introduced and applied to several examples using table-based $Q$-learning as well as deep RL. Our shielded version of deep $Q$-learning fits the framework of post-posed shielding of [1]. In [3], shield synthesis is studied for long-run objective guarantees instead of rather than safety requirements. Unlike our work, the transition probabilities on MDPs in both [1] and [3] are assumed to be known. We observe that [1] and [3] do not provide model-based learning and PAC guarantees. Moreover, neither considers scheduling problems.
A framework to mix reactive synthesis and model-based reinforcement learning for mean-payoff with PAC guarantees has been studied in [11]. There, the learning algorithm estimates the probabilities on the transitions of the MDP. In our approach, we do not estimate these probabilities directly from the MDP, but learn probabilities for the individual tasks in the task system. The efficient PAC guarantees that we have obtained for the model-based part cannot be obtained from their framework. In [5] the combination of shielding with model-predictive control using MCTS has been introduced. However, that paper does not consider learning.

2 Preliminaries

We denote by \( \mathbb{N} \) the set of natural numbers; by \( \mathbb{Q} \), the set of rational numbers; and by \( \mathbb{Q}_{\geq 0} \) the set \( \{ q \in \mathbb{Q} \mid q \geq 0 \} \) of all non-negative rational numbers. Given \( n \in \mathbb{N} \), we denote by \([n]\) the set \{1, \ldots, n\}. Given a finite set \( A \), a (rational) probability distribution over \( A \) is a function \( p: A \rightarrow [0, 1] \cap \mathbb{Q} \) such that \( \sum_{a \in A} p(a) = 1 \). We call \( A \) the domain of \( p \), and denote it by \( \text{Dom}(p) \). We denote the set of probability distributions on \( A \) by \( \mathcal{D}(A) \). The support of the probability distribution \( p \) on \( A \) is \( \text{Supp}(p) = \{ a \in A \mid p(a) > 0 \} \). A distribution is called Dirac if \( |\text{Supp}(p)| = 1 \).

For a probability distribution \( p \), the minimum probability assigned by \( p \) to the elements in \( \text{Supp}(p) \) is \( \pi_{\text{min}}^p = \min_{a \in \text{Supp}(p)} (p(a)) \). We say two distributions \( p \) and \( p' \) are structurally identical if \( \text{Supp}(p) = \text{Supp}(p') \). Given two structurally identical distributions \( p \) and \( p' \), for \( 0 < \varepsilon < 1 \), we say that \( p \) is \( \varepsilon \)-close to \( p' \), denoted \( p \sim_\varepsilon p' \), if \( \text{Supp}(p) = \text{Supp}(p') \), and for all \( a \in \text{Supp}(p) \), we have that \( |p(a) - p'(a)| \leq \varepsilon \).

Scheduling problem An instance of the scheduling problem studied in [10] consists of a task system \( \mathcal{T} = ((\tau_i)_{i \in [n]}, F, H) \), where \((\tau_i)_{i \in [n]}\) are \( n \) preemptible tasks partitioned into hard and soft tasks \( H \) and \( S \) respectively. The latter need to be scheduled on a single processor. Formally, the work of [10] relies on a probabilistic model for the computation times of the jobs and for the delay between the arrival of two successive jobs of the same task. For all \( i \in [n] \), task \( \tau_i \) is defined as a tuple \( (\mathcal{C}_i, D_i, \mathcal{A}_i) \), where: (i) \( \mathcal{C}_i \) is a discrete probability distribution on the (finitely many) possible computation times of the jobs generated by \( \tau_i \); (ii) \( D_i \in \mathbb{N} \) is the deadline of all jobs generated by \( \tau_i \) which is relative to their arrival time; and (iii) \( \mathcal{A}_i \) is a discrete probability distribution on the (finitely many) possible inter-arrival times of the jobs generated by \( \tau_i \). We denote by \( \pi_{\text{max}}^p \) the maximum probability appearing in the definition of \( \mathcal{T} \), that is, across all the distributions \( \mathcal{C}_i \) and \( \mathcal{A}_i \), for all \( i \in [n] \). It is assumed that \( \max(\text{Dom}(\mathcal{C}_i)) \leq D_i \leq \min(\text{Dom}(\mathcal{A}_i)) \) for all \( i \in [n] \); hence, at any point in time, there is at most one job per task in the system. Also note that when a new job of some task arrives at the system, the deadline for the previous job of this task is already over. The potential degradation in the quality when a soft task misses its deadline is modelled by a cost function \( \text{cost} : F \rightarrow \mathbb{Q}_{\geq 0} \) that associates to each soft task \( \tau_j \) a cost \( c(j) \) that is incurred every time a job of \( \tau_j \) misses its deadline.
Given a task system \( T = ((\text{struct}(v)), F, H) \) with \( n \) tasks, the structure of \( T \) is \( ((\text{struct}(v)), F, H) \) where \( \text{struct}(C,D,A) = (\langle \text{Dom}(C), D, \text{Dom}(A) \rangle) \). We denote by \( C_{\max} \) and \( A_{\max} \) respectively the maximum computation time, and the maximum inter-arrival time of a task in \( T \). Formally, \( C_{\max} = \max_i \text{Dom}(C_i) \), and \( A_{\max} = \max_i \text{max}(\text{Dom}(A_i)) \). Note that \( A_{\max} \geq C_{\max} \). We also let \( D = \max_i \text{max}(\text{Dom}(A_i)) \).

Consider two task systems \( T_1 = ((\tau_i^1), F, H) \), and \( T_2 = ((\tau_i^2), F, H) \), with \( |T_1| = |T_2| \), \( \tau_i^1 = (C_i^1, D_i^1, A_i^1) \) for all \( i \in [n] \) and \( j \in [2] \). The two task systems \( T_1 \) and \( T_2 \) are said to be \( \varepsilon \)-close, denoted \( T_1 \approx \varepsilon T_2 \), if (i) \( \text{struct}(T_1) = \text{struct}(T_2) \), (ii) for all \( i \in [n] \), we have \( A_i^1 \approx \varepsilon A_i^2 \), and (iii) for all \( i \in [n] \), we have \( C_i^1 \approx \varepsilon C_i^2 \).

Markov decision processes Let us now introduce Markov Decision Process (MDP) as they form the basis of the formal model of [10], which we recall later. A finite Markov decision process is a tuple \( \Gamma = (V,E,L,(V_\delta,V_\circ),A,\delta,\text{cost}) \), where: (i) \( A \) is a finite set of actions; (ii) \( (V,E) \) is a finite directed graph and \( L \) is an edge-labelling function (we denote by \( E(v) \) the set of outgoing edges from vertex \( v \)); (iii) the set of vertices \( V \) is partitioned into \( V_\delta \) and \( V_\circ \); (iv) the graph is bipartite i.e. \( E \subseteq (V_\delta \times V_\circ) \cup (V_\circ \times V_\delta) \), and the labelling function is s.t. \( L(v,v') \in A \) if \( v \in V_\delta \), and \( L(v,v') \in Q \) if \( v \in V_\circ \); and (v) \( \delta \) assigns to each vertex \( v \in V_\circ \) a rational probability distribution on \( E(v) \). For all edges \( e \), we let \( \text{cost}(e) = L(e) \) if \( L(e) \in Q \), and \( \text{cost}(e) = 0 \) otherwise. We further assume that, for all \( v \in V_\circ \), for all \( e, e' \in E(v) \): \( L(e) = L(e') \) implies \( e = e' \), i.e. an action identifies uniquely an outgoing edge. Given \( v \in V_\circ \), and \( a \in A \), we define \( \text{Post}(v,a) = \{ v' \in V_\circ \mid (v,v') \in E \text{ and } L(v,v') = a \} \cup \{ v'' \in V_\circ \mid \exists v' : (v,v') \in E \text{, } L(v,v') = a \text{ and } \delta(v',v'') > 0 \} \). For all vertex \( v \in V_\circ \), we denote by \( A(v) \), the set of actions \( \{ a \in A \mid \text{Post}(v,a) \cap V_\delta \neq \emptyset \} \). The size of an MDP \( \Gamma \), denoted \( |\Gamma| \), is the sum of the number of vertices and the number of edges, that is, \( |V| + |E| \). An MDP \( \Gamma' = (V,E,L,(V_\delta,V_\circ),A,\delta,\text{cost}) \) is said to be structurally identical to another MDP \( \Gamma'' = (V,E,L'',(V_\delta,V_\circ),A,\delta'',\text{cost}) \) if for all \( v \in V_\circ \), we have that \( \text{Supp}(\delta(v)) = \text{Supp}(\delta'(v)) \). For two structurally identical MDPs \( \Gamma' \) and \( \Gamma'' \) with distribution assignment functions \( \delta \) and \( \delta' \) respectively, we say that \( \Gamma' \) is \( \varepsilon \)-approximate to \( \Gamma'' \), denoted \( \Gamma' \approx \varepsilon \Gamma'' \), if for all \( v \in V_\circ \): \( \delta(v) \approx \varepsilon \delta'(v) \).

An MDP \( \Gamma \) can be interpreted as a game \( G_\Gamma \) between two players: \( \square \) and \( \circ \), who own the vertices in \( V_\delta \) and \( V_\circ \) respectively. A play in an MDP is a path in its underlying graph \( (V,E,A \cup Q) \). We say that a prefix \( \pi(n) \) of a play \( \pi \) belongs to player \( i \in \{ \square, \circ \} \), iff its last vertex \( \text{Last}(\pi(n)) \) is in \( V_i \). The set of prefixes that belong to player \( i \) is denoted by \( \text{Pref}_i(G_\Gamma) \). A play in the MDP is then obtained by the interaction of the two players as follows: if the current play prefix \( \pi(n) \) belongs to \( \square \), she plays by picking an edge \( e \in E(\text{Last}(\pi(n))) \) (or, equivalently, an action that labels a necessarily unique edge from \( \text{Last}(\pi(n)) \)). Otherwise, when \( \pi(n) \) belongs to \( \circ \), the next edge \( e \in E(\text{Last}(\pi(n))) \) is chosen randomly according to \( \delta(\text{Last}(\pi(n))) \). In both cases, the plays prefix is extended by \( e \) and the game goes ad infinitum.

A (deterministic) strategy of \( \square \) is a function \( \sigma_\square : \text{Pref}_\square(G) \rightarrow E \), such that \( \sigma_\square(\rho) \in E(\text{Last}(\rho)) \) for all prefixes. A strategy \( \sigma_\square \) is memoryless if for all finite
prefixes \( \rho_1 \) and \( \rho_2 \in \text{Prefs}(\mathcal{G}) \): \( \text{Last}(\rho_1) = \text{Last}(\rho_2) \) implies \( \sigma_\rho(\rho_1) = \sigma_\rho(\rho_2) \). For memoryless strategies, we will abuse notations and assume that such strategies \( \sigma \) are of the form \( \sigma : V_\square \to E \) (i.e., the strategy associates the edge to play to the current vertex and not to the full prefix played so far). From now on, we will consider memoryless deterministic strategies unless otherwise stated. Let \( \Gamma = \langle V, E, L,(V_G,\gamma),A,\delta,\text{cost} \rangle \) be an MDP, and let \( \sigma_\square \) be a memoryless strategy. Then, assuming that \( \square \) plays according to \( \sigma_\square \), we can express the behaviour of \( \Gamma \) as a Markov chain \( \Gamma[\sigma_\square] \), where the probability distributions reflect the stochastic choices of \( \square \) (see [10] for the details).

**End components** An end-component (EC, for short) \( M = (T,A') \) with \( T \subseteq V \), and \( A' : T \cap V_\square \to 2^A \) is a sub-MDP of \( \Gamma \) (for all \( v \in T \cap V_\square \), \( A'(v) \) is a subset of the actions available to \( \square \) from \( v \); and for all \( a \in A'(v) \) : \( \text{Post}(v,a) \subseteq T \) that is strongly connected. A maximal EC (MEC) is an EC that is not included in any other EC.

**MDP model for the scheduling problem** Given a system \( \Upsilon = \{ \tau_1, \tau_2, \ldots, \tau_n \} \) of tasks, we describe below the modelling of the scheduling problem by a finite MDP \( \Upsilon_\Gamma = \langle V, E, L, (V_G,\gamma), A, \delta, \text{cost} \rangle \) as it appears in [10]. The two players \( \square \) and \( \circ \) correspond respectively to the Scheduler and the task generator (TaskGen) respectively. The vertices of the MDP correspond to the system states. Since there is at most one job of each task that is active at all times, we maintain, in all vertices, the following information about each task \( \tau_i \): (i) a distribution \( c_i \) over the job’s possible remaining computation times (rc\( t \)); (ii) the time \( d_i \) up to its deadline; and (iii) a distribution \( a_i \) over the possible times up to the next arrival of a new job of \( \tau_i \). We also tag vertices with either \( \square \) or \( \circ \) to remember their respective owners and we have a special vertex \( \perp \) that will be reached when a hard task misses a deadline. For a vertex \( v = (c_1,d_1,a_1),\ldots,(c_n,d_n,a_n) \), for \( \Delta \in \{ \square, \circ \} \), let \( \text{active}(v) = \{ i \mid c_i(0) \neq 1 \text{ and } d_i > 0 \} \) be the tasks that have an active job in \( v \), and \( \text{dmiss}(v) = \{ i \mid c_i(0) = 0 \text{ and } d_i = 0 \} \) be the tasks that have missed a deadline for sure in \( v \).

**Possible moves of Scheduler** The possible actions of Scheduler are to schedule an active task or to idle the CPU. We model this by having, from all vertices \( v \in V_\square \) one transition labelled by some element from \( \text{active}(v) \), or by \( \varepsilon \) (no job gets scheduled). Such transitions model the elapsing of one clock tick.

**Possible moves of TaskGen** The moves of TaskGen consist in selecting, for each task one possible action out of four: either (i) nothing (\( \varepsilon \)); or (ii) to finish the current job without submitting a new one (\( \text{fin} \)); or (iii) to submit a new job while the previous one is already finished (\( \text{sub} \)); or (iv) to submit a new job and kill the previous one, in the case of a soft task (\( \text{killANDsub} \)), which will incur a cost.

**Expected mean-cost threshold synthesis** Let us first associate a value, called the mean-cost MC(\( \pi \)) to all plays \( \pi \) in an MDP \( \Gamma = \langle V, E, L,(V_G,\gamma),A,\delta,\text{cost} \rangle \). First, for a prefix \( \rho = e_0 e_1 \ldots e_{n-1} \), we define \( \text{MC}(\rho) = \frac{1}{n} \sum_{i=0}^{n-1} \text{cost}(e_i) \) (recall that \( \text{cost}(\varepsilon) = 0 \) when \( L(\varepsilon) \) is an action). Then, for a play \( \pi = e_0 e_1 \ldots \), we have
MC(\pi) = \limsup_{n \to \infty} MC(\pi(n)). Observe that MC is a measurable function. A strategy \sigma_0 is optimal for the mean-cost from some initial vertex \(v_{\text{init}} \in V\) if \(E^\Gamma_{v_{\text{init}}} (MC) = \inf_{\pi_0} E^\Gamma_{v_{\text{init}}} (MC)\). Such optimal strategy always exists, and it is well-known that there is always one which is memoryless. Moreover, this problem can be solved in polynomial time through linear programming [9] or in practice using value iteration (as implemented, for example, in the tool Storm [11]). We denote by \(E^\Gamma_{v_{\text{init}}} (MC)\) the optimal value \(\inf_{\pi_0} E^\Gamma_{v_{\text{init}}} (MC)\).

**Safety synthesis** Given an MDP \(\Gamma = \langle V, E, (V, V), A, \delta, \text{cost} \rangle\), an initial vertex \(v_{\text{init}} \in V\), and a strategy \(\sigma_0\), we define the set of possible outcomes in the Markov chain \(\Gamma[\sigma_0]\) as the set of paths \(v_{\text{init}} = v_0 v_1 v_2 \ldots \) in \(\Gamma[\sigma_0]\) s.t., for all \(i \geq 0\), there is non-null probability to go from \(v_i\) to \(v_{i+1}\) in \(\Gamma[\sigma_0]\). Let \(V_{\text{Outs}}[\Gamma[\sigma_0](v_{\text{init}})] \subseteq V\) denote the set of vertices visited in the set of possible outcomes \(\text{Outs}[\Gamma[\sigma_0](v_{\text{init}})]\).

Given \(\Gamma\) with a set \(V\) of vertices, and an initial vertex \(v_{\text{init}} \in V\), and a set \(V_{\text{bad}} \subseteq V\) of so-called bad vertices, the safety synthesis problem is to decide whether \(\square\) has a strategy \(\sigma_0\) ensuring to visit the safe vertices only, i.e.: \(V_{\text{Outs}}[\Gamma[\sigma_0](v_{\text{init}})] \cap V_{\text{bad}} = \emptyset\) (in our scheduling problem, such vertices will model the situations where a hard task has never missed a deadline). If this is the case, we call such a strategy safe. The safety synthesis problem is decidable in polynomial time for MDPs. Indeed, since probabilities do not matter for this problem, the MDP can be regarded as a plain two-player game played on graphs (like in [16]), and the classical attractor algorithm can be used. We briefly describe below the attractor algorithm for completeness.

Given \(v \in V\), let \(\text{Succ}(v) = \{v' \in V \mid \exists (v, v') \in E\}\) be its set of successors, and \(E(v) = \{e \in E \mid \text{src}(e) = v\}\) be its set of outgoing edges. We assume that for all \(v \in V\): \(\text{Succ}(v) \neq \emptyset\), i.e. there is no deadlock.

The algorithm consists of computing all the vertices from which \(\square\) cannot avoid reaching the unsafe vertices. To this end, the algorithm computes a sequence of sets of vertices \(\langle A_i \rangle_{i \geq 0}\) defined as follows: (i) \(A_0 = V \setminus V_{\text{safe}}\); and (ii) for all \(i \geq 0\): \(A_{i+1} = A_i \cup \{v \in V \mid \text{Succ}(v) \subseteq A_i\} \cup \{v \in V \mid \text{Succ}(v) \cap A_i \neq \emptyset\}\). That is, the sequence \(\langle A_i \rangle_{i \geq 0}\) is initialised to the set of unsafe vertices. Then, the algorithm grows this set of vertices by adding: (i) vertices belonging to \(\square\) whose set of successors has been entirely identified as unsafe in a previous step; and (ii) vertices belonging to \(\square\) having at least one unsafe successor.

It is easy to check that this sequence converges after at most \(|V|\) steps (the graph of the MDP being finite) and returns the set of vertices \(\text{Attr}(V \setminus V_{\text{safe}})\) from which \(\square\) has no strategy to stay within \(V_{\text{safe}}\). Hence, \(\square\) has a strategy \(\sigma_0\) to stay within \(V_{\text{safe}}\) from all vertices in which is s.t. \(\sigma_0(v) \notin \text{Attr}(V \setminus V_{\text{safe}})\) (any successor of \(v\) satisfying this criterion yields a safe strategy).

Moreover, it is well known that, if a safe strategy exists, then there is in particular a memoryless safe strategy: so, from now on, we will consider safe strategies that are memoryless only. We say that a vertex \(v\) is safe if \(\square\) has a safe strategy from \(v\), and that an edge \(e = (v, v') \in E \cap (V \times V)\) is safe if there is a safe strategy \(\sigma_0\) s.t. \(\sigma_0(v) = v'\). So, the safe edges from some node \(v\), denoted \(\text{safe}(v)\), correspond to the choices that \(\square\) can safely make from \(v\). The set of safe edges exactly correspond to the set of safe actions that \(\square\) can make from \(v\). Then, we let
the safe region of $\Gamma$ be the MDP $\Gamma^{s_afe}$ obtained from $\Gamma$ by applying the following transformations: (i) remove from $\Gamma$ all unsafe edges; (ii) remove from $\Gamma$ all vertices and edges that are not reachable from $v_{s_{\text{init}}}$. Note $\Gamma^{s_afe}$ is an MDP, since we have removed edges from Player one vertices only.

Most general safe scheduler A task system $\Upsilon$ is said to be schedulable for the hard tasks if Scheduler has a winning strategy to avoid $\perp$ in $\Gamma_{\Upsilon}$. This strategy corresponds to a scheduler that prevents hard tasks from ever missing a deadline. We say that a scheduler is the most general safe scheduler for the hard tasks if from all vertices $v$ of Scheduler, it allows all possible safe edges from $v$.

Example 1. Consider a system with one hard task $\tau_h = \langle C_h, 2, A_h \rangle$ s.t. $C_h(1) = 1$ and $A_h(3) = 1$; one soft task $\tau_s = \langle C_s, 2, A_s \rangle$ s.t. $C_s(1) = 0.4$, $C_s(2) = 0.6$, and $A_s(3) = 1$; and the cost function $c$ s.t. $c(\tau_h) = 10$. Fig. 1 presents an excerpt of the MDP $\Gamma_{\Upsilon}$ built from the set of tasks $\tau = \{\tau_h, \tau_s\}$ of Example 1. A distribution $p$ with support $\{x_1, x_2, \ldots, x_n\}$ is denoted by $[x_1 : p(x_1), x_2 : p(x_2), \ldots, x_n : p(x_n)]$. When $p$ is s.t. $p(x) = 1$ for some $x$, we simply denote $p$ by $x$. Vertices from $V_{\Upsilon}$ and $V_{\square}$ are depicted by rectangles and rounded rectangles respectively. Each vertex is labelled by $(c_h, d_h, a_h)$ on the top, and $(c_s, d_s, a_s)$ below.

A strategy to avoid missing a deadline of $\tau_h$ consists in first scheduling $\tau_s$, then $\tau_h$. One then reaches the left-hand part of the graph from which $\sqcap$ can avoid $\perp$ whatever $\Box$ does. Note that other safe strategies are possible: the first step of the algorithm in [10] is actually to compute all the safe nodes (i.e. those from which $\Box$ can ensure to avoid $\perp$), and then to look for an optimal one w.r.t to missed-deadline costs.

There are two optimal memoryless strategies, one in which Scheduler first chooses to execute $\tau_h$, then $\tau_s$; and another where $\tau_s$ is scheduled for 1 time unit.

3 The existence of the most general safe scheduler is a direct consequence of the fact that the most general strategy exists for safety objectives [13].
and then preempted to let \( \tau_h \) execute. Since the time difference between the arrival of two consecutive jobs of the soft task \( \tau_s \) is 3 and the cost of missing a deadline is 10, for both of these optimal strategies, the soft task's deadline is missed with probability 0.6 over this time duration of 3, and hence the mean-cost is 2. Observe that there is another safe schedule that is not optimal is one in which only \( \tau_h \) is granted CPU access, and \( \tau_s \) is never scheduled thus giving a mean-cost of \( \frac{10}{3} \).

3 Model-Based Learning

We now investigate the case of model-based learning of task systems. First, we consider the simpler case of task systems with only soft tasks. We show that those systems are always efficiently PAC learnable. Second, we consider learning task systems with both hard and soft tasks. In that case, we study two conditions for learnability. The first condition allows us to identify task systems that are safely PAC learnable, i.e. learnable while enforcing safety for the hard tasks. The second condition is stronger and allows us to identify task systems that are safely and efficiently PAC learnable. Our learning algorithms on (safely) sampling the distributions underlying the behaviour of tasks.

Learning setting We consider a setting in which we are given the structure of a task system \( T = ((\tau_i)_{i \in I}, F, H) \) to schedule. While the structure is known, the actual distributions that describe the behaviour of the tasks are unknown and need to be learnt to behave optimally or near optimally. The learning must be done only by observing the jobs that arrive along time. When the task system contains some hard tasks \( (H \neq \emptyset) \), all deadlines of such tasks must be enforced.

For learning the inter-arrival time distribution of a task, a sample corresponds to observing the time difference between the arrivals of two consecutive jobs of that task. For learning the computation time distribution, a sample corresponds to observing CPU time that a job of the task has been assigned up to completion. Thus if a job does not finish execution before its deadline, we do not obtain a valid sample for the computation time. Given a class of task systems, we say:

- the class is probably approximately correct (PAC) learnable if there is an algorithm \( L \) such that for all task systems \( T \) in this class, for all \( \varepsilon, \gamma \in (0, 1) \) given \( \text{struct}(T) \), the algorithm \( L \) can execute the task system \( T \), and can compute \( T^M \) such that \( T \approx T^M \), with probability at least \( 1 - \gamma \).

- the class is safely PAC learnable if it is PAC learnable, and \( L \) can ensure safety for the hard tasks while computing \( T^M \).

- the class is called (safely) efficiently PAC learnable if it is (safely) PAC learnable, and there exists a polynomial \( q \) in the size of the task system, in \( \frac{1}{\varepsilon} \), and in \( \frac{1}{\gamma} \), s.t. \( L \) can obtain enough samples and compute \( T^M \) in a time which is bounded by \( q \).

Note that our notion of efficient PAC learning is slightly stronger than the definition used in classical PAC learning terminology [17] since we take into account the time that is needed to get samples and not only the number of samples needed. We will see later in this section why this distinction is important.
Learning discrete finite distributions by sampling We analyse the number of samples needed to closely approximate a discrete distribution with high probability. Towards this, we first introduce Hoeffding’s inequality.

Hoeffding’s inequality Let $X_1, \ldots, X_\ell$ be independent random variables with domain bounded by the interval $[0, 1]$, and let $L \overset{\text{def}}{=} \frac{1}{\ell} \sum_{i=1}^{\ell} X_i$. For all $0 < \varepsilon < 1$ the following hold.

- $\mathbb{P}[E[L] - L \geq \varepsilon] \leq \exp(-2\ell \varepsilon^2)$
- $\mathbb{P}[L - E[L] \geq \varepsilon] \leq \exp(-2\ell \varepsilon^2)$
- $\mathbb{P}[|L - E[L]| \geq \varepsilon] \leq 2 \exp(-2\ell \varepsilon^2)$

To learn an unknown discrete distribution $p$ defined on a finite domain $\text{Dom}(p)$, we collect i.i.d. samples from that distribution and infer a model of the distribution from those samples. Formally, given a sequence $S = (s_j)_{j \in J}$ of samples drawn i.i.d. from the distribution $p$, we denote by $p(S) : \text{Dom}(p) \rightarrow [0, 1]$, the function that maps every element $a \in \text{Dom}(p)$ to its relative frequency in $S$, i.e. to $\frac{|\{j \in J : s_j = a\}|}{|J|}$.

The following lemma tells us that if the size of $S$ is large enough then the model $p(S)$ is close to the actual $p$ with high probability.

**Lemma 1.** For all finite discrete distributions $p$ with $|\text{Dom}(p)| = r$, for all $\varepsilon, \gamma \in (0, 1)$ such that $\pi_{\text{min}}^p > \varepsilon$, if $S$ is a sequence of at least $r \cdot \left[ \frac{\varepsilon}{2\gamma} (\ln 2r - \ln \gamma) \right]$ i.i.d. samples drawn from $p$, then $p \sim^\varepsilon p(S)$ with probability at least $1 - \gamma$.

**Proof.** For a distribution $p$, and an element $e$ in $\text{Dom}(p)$, let $X_1^p, \ldots, X_m^p$ be independent and identically distributed Bernoulli random variables with $\mathbb{E}[X_j^p] = \mu$ for $j \in [m]$. Recall that a Bernoulli random variable takes two values, 1 and 0. In our case, the value 1 denotes witnessing the element $e$ in the domain of the distribution $p$. Thus we have $p(e) = \mu$. Let $\overline{X}_m^p = \frac{1}{m} \sum_{j \in [m]} X_j^p$. Here $m$ is the number of samples required to learn the probability of occurrence of the element $e$ of the support of the distribution.

By Hoeffding’s two sided inequality, for the special case of Bernoulli random variables, we have,

$$\mathbb{P}(|\overline{X}_m^p - \mu| \geq \varepsilon) \leq 2 \exp(-2m \varepsilon^2).$$

Now we want that the probability of $|\overline{X}_m^p - \mu| \geq \varepsilon$ for all $e \in \text{Dom}(p)$ is at most $\frac{\gamma}{2}$, so that the probability of $|\overline{X}_m^p - \mu| \geq \varepsilon$ for some element $e$ in the domain of the distribution $p$ is at most $\gamma$.

Thus we have $2 \exp(-2m \varepsilon^2) \leq \frac{\gamma}{2}$ leading to $m \geq \left\lceil \frac{\ln 2r - \ln \gamma}{2\varepsilon^2} \right\rceil$. Since there are $r$ elements in the domain, we need a total of at least $f(r, \varepsilon, \gamma) = m \cdot r$ samples, and hence the result. \(\square\)

For all distributions $p$, we say that we ‘PAC learn’ the distribution $p$ if for all $\varepsilon, \gamma \in (0, 1)$ such that $\pi_{\text{min}}^p \geq \varepsilon$, by drawing a sequence $S$ of i.i.d. samples from $p$, we have $p \sim^\varepsilon p(S)$ with probability at least $1 - \gamma$. Informally, we also refer to this as learning a distribution with strong guarantees. We note that given a task system $\Upsilon$, if we can learn the distributions corresponding to all the tasks in $\Upsilon$, and
hence a model $\mathcal{Y}^M$ of the task system, such that each learnt distribution in $\mathcal{Y}^M$ is structurally identical to its corresponding distribution in $\mathcal{Y}$, the corresponding MDP are structurally identical.

**Efficient PAC learning with soft tasks only** Let $\mathcal{Y} = ((\tau_i)_{i \in I}, F, H)$ be a task system with soft tasks only, and let $\varepsilon, \gamma \in (0, 1)$. We assume that for all distributions $p$ occurring in the models of the tasks in $\mathcal{Y}$: $\pi^p_{\min} > \varepsilon$. To learn a model $\mathcal{Y}^M$ which is $\varepsilon$-close to $\mathcal{Y}$ with probability at least $1 - \gamma$, we apply Lemma 1 in the following algorithm:

1. for all tasks $i = 1, 2, \cdots \in F$, repeat the following learning phase:
   - Always schedule task $\tau_i$ when a job of this task is active. Collect the samples $S(A_i)$ of $A_i$, and $S(C_i)$ of $C_i$ as observed. Collect enough samples to apply Lemma 1 and obtain the desired accuracy as fixed by $\varepsilon$ and $\gamma$.
2. the models of inter-arrival time distribution and computation time distribution for task $\tau_i$ are $p(S(A_i))$ and $p(S(C_i))$ respectively.

It follows that task systems with only soft tasks are efficiently PAC learnable:

**Theorem 1.** There exists a learning algorithm such that for all task systems $\mathcal{Y} = ((\tau_i)_{i \in I}, F, H)$ with $H = \emptyset$, for all $\varepsilon, \gamma \in (0, 1)$, the algorithm learns a model $\mathcal{Y}^M$ such that $\mathcal{Y}^M \approx^\varepsilon \mathcal{Y}$ with probability at least $1 - \gamma$ after executing $\mathcal{Y}$ for $|F| \cdot A_{\max} \cdot D \cdot \lceil \frac{1}{2\varepsilon^2} (\ln 4D |F| - \ln \gamma) \rceil$ steps.

**Proof.** Using Lemma 1, given $\varepsilon, \gamma' \in (0, 1)$, for every distribution $p$ of the task system, a sequence $S$ of $D \cdot \lceil \frac{1}{2\varepsilon^2} (\ln 2D - \ln \gamma') \rceil$ i.i.d. samples suffices to have $p(S) \sim^\varepsilon p$ with probability at least $1 - \gamma'$. Since in the task system $\mathcal{Y}$, there are $2|F|$ distributions, with probability at least $1 - 2|F|\gamma'$, we have that the learnt model $\mathcal{Y}^M \approx^\varepsilon \mathcal{Y}$. Thus for $\gamma' = \frac{1}{2\varepsilon^2}$, and using $2 \exp(-2\varepsilon^2 m^2) \leq \frac{1}{2\varepsilon^2}$, we have that for each distribution, a sequence of $D \cdot \lceil \frac{1}{2\varepsilon^2} (\ln 4D |F| - \ln \gamma) \rceil$ samples suffices so that $\mathcal{Y}^M \approx^\varepsilon \mathcal{Y}$ with probability at least $1 - \gamma$.

Since samples for computation time distribution and inter-arrival time distribution for each soft task can be collected simultaneously, and observing each sample takes a maximum of $A_{\max}$ time steps, and we collect samples for each soft task by scheduling one soft task after another, the result follows. □

**Safe learning in the presence of hard tasks** We now turn to the more challenging case of safely learning a task systems $\mathcal{Y} = ((\tau_i)_{i \in I}, F, H)$ with both hard and soft tasks (i.e. $F \neq \emptyset$ and $H \neq \emptyset$). The learning algorithm must ensure that all the jobs of hard tasks meet their deadlines while learning the task distributions. This algorithm for systems with only soft tasks is clearly not valid for that more general case.

From now on, we assume that the task system $\mathcal{Y} = ((\tau_i)_{i \in I}, F, H)$ for which we want to safely learn a model $\mathcal{Y}$ is schedulable for the hard task $\emptyset$. This is a necessary condition for safe learning but it is not a sufficient condition. Indeed, to apply Lemma 1, we need enough samples for all tasks $i \in H \cup F$.

\footnote{Note that safety synthesis already identifies task systems that violate this condition.}
First, we note that when executing any safe schedule for the hard tasks, we will observe enough samples for the hard tasks. Indeed, under a safe schedule for the hard tasks, any job of a hard task that enters the system will be executed to completion before its deadline (as the scheduler is safe). Then clearly, we observe the value of the inter-arrival time as well as the value of the computation time requested by all the jobs of hard tasks that enter the system. Unfortunately, this is not necessarily the case for soft tasks when they execute in the presence of hard tasks. As an example, consider a task system such that a job of a soft task with a computation time of 2 and deadline 4 arrives at a time when there already exists an active job of a hard task with remaining 3 units of computation time and a remaining time of 4 before the deadline. The job of the hard task needs to complete execution before its deadline and thus the job of the soft task cannot execute to completion before its deadline. Thus, considering as samples only those jobs that have been fully scheduled to completion and ignoring those that do not, would lead to samples that are biased towards smaller computation times, and would not allow us to draw conclusions about the real computation time distribution.

We thus need stronger conditions than hard task schedulability to safely learn task systems, in order to be able to learn the distributions of the soft tasks while ensuring safety. We develop two such conditions in the rest of this section. The first condition ensures PAC learnability but does not provide polynomial time guarantee on the learning time, while the second condition is stricter but ensures efficient PAC learnability.

**PAC guarantees for safe learning** Our condition to ensure safe PAC learnability relies on properties of the safe region $\Gamma_\text{safe}^\mathcal{T}$ in the MDP $\Gamma_\mathcal{T}$ associated to the task system $\mathcal{T}$. First, note that $\Gamma_\text{safe}^\mathcal{T}$ is guaranteed to be non-empty as the task system $\mathcal{T}$ is guaranteed to be schedulable for its hard tasks by hypothesis. Our condition will exploit the following property of its structure:

**Lemma 2.** Let $\mathcal{T} = ((\tau_i)_{i \in I}, F, H)$ be a task system and let $\Gamma_\text{safe}^\mathcal{T}$ be the safe region of its MDP. Then $\Gamma_\text{safe}^\mathcal{T}$ is a single MEC.

**Proof.** We first assume that the task system $\mathcal{T} = ((\tau_i)_{i \in I}, F, H)$ is schedulable. Otherwise, $\Gamma_\text{safe}^\mathcal{T}$ is empty and the Lemma is trivially true. Let $V$ and $E$ be the set of vertices and the set of edges of $\Gamma_\text{safe}^\mathcal{T}$ respectively. First, observe that, since we want to prove that the whole MDP $\Gamma_\text{safe}^\mathcal{T}$ corresponds to an MEC, we only need to show that its underlying graph $(V, E)$ is strongly connected. Indeed, since $(V, E)$ contains all vertices and edges from $\Gamma_\text{safe}^\mathcal{T}$, it is necessarily maximal, and all choices of actions from any vertex will always lead to a vertex in $V$.

In order to show the strongly connected property, we fix a vertex $v \in V$, and show that there exists a path in $\Gamma_\text{safe}^\mathcal{T}$ from $v$ to $v_{\text{init}}$. Since all vertices in $V$ are, by construction of $\Gamma_\text{safe}^\mathcal{T}$, reachable from the initial vertex $v_{\text{init}}$, this entails that all vertices $v'$ are also reachable from $v$, hence, the graph is strongly connected.

Let us first assume that $v \in V_{\text{S}}$, i.e., $v$ is a vertex where Scheduler has to take a decision. Let $v_{\text{init}} = v_0, v'_0, v_1, v'_1, \ldots, v'_{n-1}, v_n = v$ be the path $\pi$ leading to $v$, where all vertices $v_j$ belong to Scheduler, and all $v'_j$ are are vertices that belong to TaskGen.
Then, from path $\pi$, we extract, for all tasks $\tau_i$ the sequence of actual inter-arrival times $\sigma_i = t^i(1), t^i(2), \ldots, t^i(k_i)$ defined as follows: for all $1 \leq j \leq k_i$, $t^i(j) \in \text{Supp}(\mathcal{A}_i)$ is the time elapsed (in CPU ticks) between the arrival of the $j$-th job of task $i$ along $\pi$ (assuming the initial release occurring in the initial state $v_{init}$ is the 0-th release). In other words, letting $T^i(j) = \sum_{k=1}^j t^i(k)$, the $j$-th job of $\tau_i$ is released along $\pi$ on the transition between $v_{T^i(j-1)}$ and $v_{T^i(j)}$.

Observe thus that all tasks $i \in [n]$ are in the same state in vertex $v_{init}$ and in vertex $v_{T^i(j)}$, i.e. the time to the deadline, and the probability distributions on the next arrival and computation times are the same in $v_{init}$ and $v_{T^i(j)}$. However, the vertices $v_{T^i(j)}$ can be different for all the different tasks, since they depend on the sequence of job releases of $\tau_i$ along $\pi$. Nevertheless, we claim that $\pi$ can be extended, by repeating the sequence of arrivals of all the tasks along $\pi$, in order to reach a vertex where all tasks have just submitted a job (i.e. $v_{init}$). To this aim, we first extend, for all tasks $i \in [n]$, $\sigma_i$ into $\sigma'_i = \sigma_i, t^i(k_i + 1)$, where $t^i(k_i + 1) \in \text{Supp}(\mathcal{A}_i)$ ensures that the $k_i + 1$ arrival of a $\tau_i$ occurs after $v$.

For all $i \in [n]$, let $\Delta_i$ denote $\sum_{j=1}^{k_i+1} t^i(j)$, i.e. $\Delta_i$ is the total number of CPU ticks needed to reach the first state after $v$ where task $i$ has just submitted a job (following the sequence of arrival $\sigma'_i$ defined above). Further, let $\Delta = \text{lcm}(\Delta_i)_{i \in [n]}$.

Now, let $\pi'$ be a path in $I_{\mathcal{Y}}^{\text{safe}}$ that respects the following properties:

1. $\pi$ is a prefix of $\pi'$;
2. $\pi'$ has a length of $\Delta$ CPU ticks;
3. $\pi'$ ends in a $\square$ vertex $v'$; and
4. for all tasks $i \in [n]$: $\tau_i$ submits a job at time $t$ along $\pi'$ iff it submits a job at time $t \mod \Delta_i$ along $\pi$.

Observe that, in the definition of $\pi'$, we do not constrain the decisions of Scheduler after the prefix $\pi$. First, let us explain why such a path exists. Observe that the sequence of task arrival times is legal, since it consists, for all tasks $i$, in repeating $\Delta/\Delta_i$ times the sequence $\sigma'_i$ of inter-arrival times which is legal since it is extracted from path $\pi$ (remember that nothing that Scheduler player does can restrict the times at which TaskGen introduces new jobs in the system). Then, since $\mathcal{Y}$ is schedulable, we have the guarantee that all $\square$ vertices in $I_{\mathcal{Y}}^{\text{safe}}$ have at least one outgoing edge. This is sufficient to ensure that $\pi'$ indeed exists. Finally, we observe $\pi'$ visits $v$ (since $\pi$ is a prefix of $\pi'$), and that the last vertex $v'$ of $\pi'$ is a $\square$ vertex obtained just after all tasks have submitted a job, by construction. Thus $v' = v_{init}$, and we conclude that, from all $v \in V_{\mathcal{Y}}$ which is reachable from $v_{init}$, one can find a path in $I_{\mathcal{Y}}^{\text{safe}}$ that leads back to $v_{init}$.

This reasoning can be extended to account for the nodes $v \in V_{\mathcal{Y}}$: one can simply select any successor $v' \in V_{\mathcal{Y}}$ of $v$, and apply the above reasoning from $v'$ to find a path going back to $v_{init}$.

**Good for sampling and learnability** The safe region $I_{\mathcal{Y}}^{\text{safe}}$ of the task system $\mathcal{T} = (\{\tau_i\}_{i \in I}, F, H)$ is *good for sampling* if for all soft tasks $i \in F$, there exists a vertex $v_i \in I_{\mathcal{Y}}^{\text{safe}}$ such that:

1. a new job of task $i$ enters the system in $v_i$; and
2. there exists a strategy $\sigma_i$ of Scheduler that is compatible with the set of safe schedules for the hard tasks so that from $v_i$, under schedule $\sigma_i$, the new job associated to task $\tau_i$ is guaranteed to reach completion before its deadline, no matter how TaskGen behaves.

There is an algorithm that executes in polynomial time in the size of $\Gamma_T^{safe}$ and which decides if $\Gamma_T^{safe}$ is good for sampling. Also, remember that only the knowledge of the structure of the task system is needed to compute $\Gamma_T^{safe}$.

Given a task system $\Gamma_T^{safe}$ that is good for sampling, given any $\varepsilon, \gamma \in (0, 1)$, we safely learn a model $\Gamma_M$ which is $\varepsilon$-close to $\Gamma$ with probability at least $1 - \gamma$ (PAC guarantees) by applying the following algorithm:

1. Choose any safe strategy $\sigma_H$ for the hard tasks, and apply this strategy until enough samples ($S(A_i), S(C_i)$) for each $i \in H$ have been collected according to Lemma 1. The models for tasks $i \in H$ are $p(S(A_i))$ and $p(S(C_i))$ respectively.

2. Then for each $i \in F$, apply the following phase:
   (a) from the current vertex $v$, play uniformly at random among the safe edges in $\Gamma$ up to reaching some $v_i$ (while playing uniformly at random, we reach some $v_i$ with probability 1 by the hypothesis that $\Gamma_T^{safe}$ is good for sampling).
   (b) from $v_i$, apply the schedule $\sigma_i$ as defined in the good for sampling condition. This way we are guaranteed to observe the computation time requested by the new job of task $i$ that entered the system in vertex $v_i$, no matter how TaskGen behaves. At the completion of this job of task $i$, we have collected a valid sample of task $i$.
   (c) go back to (a) until enough samples ($S(A_i), S(C_i)$) have been collected for task $i$ according to Lemma 1. The models for task $i$ is given by $p(S(A_i))$ and $p(S(C_i))$.

The properties of the learning algorithm above are used to prove that:

**Theorem 2.** There exists a learning algorithm such that for all task systems $\Gamma = (\tau_i)_{i \in I}, F, H)$ with a safe region $\Gamma_T^{safe}$ that is good for sampling, for all $\varepsilon, \gamma \in (0, 1)$, the algorithm learns a model $\Gamma_M$ such that $\Gamma_M \approx^\varepsilon \Gamma$ with probability at least $1 - \gamma$.

**Proof.** For the hard tasks, as mentioned above, we can learn the distributions by applying the safe strategy $\sigma_H$ to collect enough samples ($S(A_i), S(C_i)$) for each $i \in H$.

We assume an order on the set of soft tasks. First for all $\tau_i$ for $i \in F$, since $\Gamma_T^{safe}$ is good for sampling, we note that the set $V_i$ of vertices $v_i$ (as defined in the definition of good for sampling condition) is non-empty. Recall from Lemma 2 that $\Gamma_T^{safe}$ has a single MEC. Thus from every vertex of $\Gamma_T^{safe}$, Scheduler by playing uniformly at random reaches some $v_i \in V_i$ with probability 1, and hence can visit the vertices of $V_i$ infinitely often with probability 1. Now given $\varepsilon$ and $\gamma$, using Theorem 1, we can compute an $m$, the number of samples corresponding to each distribution required for safe PAC learning of the task system. Since by playing uniformly at random, Scheduler has a strategy to visit the vertices of $V_i$ infinitely

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5 This follows from the fact that since there is a single MEC in the MDP by Lemma 2.
often with probability 1, it is thus possible to visit these vertices at least \( m \) times with arbitrarily high probability.

Also after we safely PAC learn the distributions for task \( \tau_i \), since there is a single MEC in \( \Gamma_T^{\text{safe}} \), there exists a uniform memoryless strategy to visit a vertex \( v_{i+1} \) corresponding to task \( \tau_{i+1} \) with probability 1. Hence the result. \( \square \)

In the algorithm above, to obtain one sample of a soft task, we need to reach a particular vertex \( v_i \) from which we can safely schedule a new job for the task \( i \) up to completion. As the underlying MDP \( \Gamma_T^{\text{safe}} \) can be large (exponential in the description of the task system), we cannot bound by a polynomial the time needed to get the next sample in the learning algorithm. So, this algorithm does not guarantee efficient PAC learning. We develop in the next paragraph a stronger condition to guarantee the existence of an efficient PAC learning algorithm.

**Good for efficient sampling and efficient PAC learning** The safe region \( \Gamma_T^{\text{safe}} \) of the task system \( \Gamma = (\{\tau_i\}_{i \in I}, F, H) \) is *good for efficient sampling* if there exists \( K \in \mathbb{N} \) which is bounded polynomially in the size of \( \Gamma = (\{\tau_i\}_{i \in I}, F, H) \), and if, for all soft tasks \( i \in F \) the two following conditions hold:

1. let \( V^{\text{safe}}_i \) be the set of Scheduler vertices in \( \Gamma_T^{\text{safe}} \). There is a non-empty subset \( \text{Safe}_i \subseteq V^{\text{safe}}_i \) of vertices from which there is a strategy \( \sigma_i \) for Scheduler to always schedule safely the set of tasks \( H \cup \{i\} \) (i.e. all hard tasks and the task \( i \)); and

2. for all \( v \in V^{\text{safe}}_i, i \in F \), there is a uniform memoryless strategy \( \sigma_{i,\text{Safe}_i} \) such that:
   (a) \( \sigma_{i,\text{Safe}_i} \) is compatible with the safe strategies (for the hard tasks) of \( \Gamma_T^{\text{safe}} \); and
   (b) when \( \sigma_{i,\text{Safe}_i} \) is executed from any \( v \in V^{\text{safe}}_i \), then the set \( \text{Safe}_i \) is reached within \( K \) steps. By Lemma 2, since \( \Gamma_T^{\text{safe}} \) has a single MEC, we have that \( \text{Safe}_i \) is reachable from every \( v \in V^{\text{safe}}_i \).

Here again, the condition can be efficiently decided: there is a polynomial-time algorithm in the size of \( \Gamma_T^{\text{safe}} \) that decides if \( \Gamma_T^{\text{safe}} \) is good for efficient sampling.

Given a task system \( \Gamma_T^{\text{safe}} \) that is *good for efficient sampling*, given \( \varepsilon, \gamma \in (0, 1) \), we safely and efficiently learn a model \( \mathcal{T} \) which is \( \varepsilon \)-close of \( \Gamma \) with probability at least than \( 1 - \gamma \) (efficient PAC guarantees) by applying:

1. Choose any safe strategy \( \sigma_H \) for the hard tasks, and apply this strategy until enough samples \( (\mathcal{S}(A_i), \mathcal{S}(C_i)) \) for each \( i \in H \) have been collected according to Lemma 3. The models for tasks \( i \in H \) are \( p(\mathcal{S}(A_i)) \) and \( p(\mathcal{S}(C_i)) \) respectively.

2. Then for each \( i \in F \), apply the following phase:
   (a) from the current vertex \( v \), play \( \sigma_{i,\text{Safe}_i} \) to reach the set \( \text{Safe}_i \).
   (b) from the current vertex in \( \text{Safe}_i \), apply the schedule \( \sigma_i \) as defined in the ‘good for efficient sampling’ condition. This way we are guaranteed to observe the computation time requested by all the jobs of task \( i \) that enter the system, no matter how TaskGen behaves. Hence, at the completion of each new instance of a job of task \( i \), we have collected a valid sample of task \( i \).
(c) go back to (b) until enough samples \((S(A_i), S(C_i))\) have been collected for task \(i\) according to Lemma \[\text{Lemma 1}\]. The models for task \(i\) is given by \(p(S(A_i))\) and \(p(S(C_i))\).

For a task system \(\mathcal{Y}\), let \(T = A_{\text{max}} \cdot D \cdot \lceil \frac{1}{2\varepsilon} (\ln 4D|\mathcal{Y}| - \ln \gamma) \rceil\). The properties of the learning algorithm above are used to prove the following theorem:

**Theorem 3.** There exists a learning algorithm such that for all task systems \(\mathcal{Y} = ((\tau_i)_{i \in I}, F, H)\) with a safe region \(\Gamma_{\mathcal{Y}}^{\text{safe}}\) that is good for efficient sampling, for all \(\varepsilon, \gamma \in (0, 1)\), the algorithm learns a model \(\mathcal{Y}^M\) such that \(\mathcal{Y}^M \approx \varepsilon \mathcal{Y}\) with probability at least than \(1 - \gamma\) after scheduling \(\mathcal{Y}\) for \(T + |F| \cdot (T + K)\) steps.

**Proof.** Consider the algorithm described above. Since \(\sigma_H\) is a safe schedule for the hard tasks, we can observe the samples corresponding to the computation time distribution and the inter-arrival time distribution for all the hard tasks simultaneously while scheduling the system. Following the proof of Theorem \[\text{Theorem 1}\], the samples required to learn the distributions of the hard tasks can be observed in time \(T\).

Now consider an order on the set of tasks. Under the good for efficient sampling condition, again from the proof of Theorem \[\text{Theorem 1}\], we need to execute the system for \(|F|T\) time steps for collecting samples to PAC learn the computation time distributions and the inter-arrival time distributions for all soft tasks in \(F\). Further, for every soft task \(\tau_i\) with \(i \in F\), from a vertex in \(V_{\text{safe}}^2\), by using the strategy \(\sigma \circ \text{Safe}_i\), we reach \(\text{Safe}_i\) in at most \(K\) steps. Hence the result.

We now provide an example of a task system below that satisfies the good for sampling condition, but not the stronger good for efficient sampling condition.

**Example 2.** Consider the following task system with one hard and one soft task that we want to learn. More specifically, we want to learn the distributions associated to the tasks in the system. For the hard task, the computation time distribution is Dirac with support \([2]\), the relative deadline is 2, and the inter-arrival time distribution is also Dirac with support \([4]\). For the soft task, the computation time distribution has the support \([1, 2]\), the relative deadline is 2, and the inter-arrival time distribution is also Dirac and has the support \([3]\). We assume that the domain of each distribution is the same as its support.

We can see that during the execution of the task system, for every time \(t\), Scheduler does not have a safe schedule from \(t\) that also ensures that the soft task will never miss a deadline. This implies that considering the good for efficient sampling condition, we have \(\text{Safe}_i = \emptyset\) for \(i \in F\), and hence the good for efficient sampling condition is not satisfied by this task system. Thus we cannot ensure safe and efficient PAC learning for this task system.

On the other hand, there exists a schedule such that for all the jobs of the soft task that arrive at time \(\text{lcm}(4, 3) \cdot n + 6 = 12n + 6\) (assuming that the system starts executing at time 0) for \(n \geq 0\) can be scheduled to completion, and thus by Theorem \[\text{Theorem 2}\] there exists an algorithm to safely PAC learn the task system. \(\square\)
Using the learnt model to behave optimally

Given a system $\mathcal{Y}$ of tasks, and parameters $\varepsilon, \gamma \in (0, 1)$, once we have learnt a model $\mathcal{Y}^\ast$, we can compute an optimal scheduling strategy that minimise the expected mean-cost of missing deadlines of soft tasks. Such an algorithm is given in [10]. Then we execute the actual task system $\mathcal{Y}$ under schedule $\sigma$. However, since $\sigma$ has been computed using the model $\mathcal{Y}^\ast$, it might not be optimal in the original, unknown tasks system $\mathcal{Y}$. Nevertheless, we can bound the difference between the optimal value obtained from $I_{\mathcal{Y}^\ast}^{\text{safe}}$ and the optimal value for $I_{\mathcal{Y}}^{\text{safe}}$.

The following lemma relates the model that is learnt with the approximate distribution that we have in the MDP corresponding to the learnt model for the system of tasks. Given $\varepsilon \in (0, 1)$, let $s = \min\{1, s^T_{\max} + \varepsilon\}$ and $\eta = s^{2n} - (s - \varepsilon)^{2n}$, where $n = |\mathcal{Y}|$.

**Lemma 3.** Let $\mathcal{Y}$ be a task system, let $\varepsilon, \gamma \in (0, 1)$, let $\mathcal{Y}^\ast$ be the learnt model such that $\mathcal{Y}^\ast \approx \mathcal{Y}$ with probability at least $1 - \gamma$. Then we have that $\Gamma_{\mathcal{Y}^\ast} \approx_n \Gamma_{\mathcal{Y}}$ with probability at least $1 - \gamma$.

**Proof.** Since we have that $\mathcal{Y}^\ast \approx \mathcal{Y}$ with probability at least $1 - \gamma$, by definition, we have that the probability that all the distributions of $\mathcal{Y}^\ast$ are $\varepsilon$-close to their corresponding distributions in $\mathcal{Y}$ is at least $1 - \gamma$. Let $|\mathcal{Y}| = n$, and there are a total of $2n$ distributions. Let $\delta$ and $\delta^\ast$ be the distribution assignment functions of $\Gamma$ and $\Gamma^\ast$ respectively. Thus corresponding to $\delta$ in $\mathcal{M}$, if an edge has probability $p = p_1p_2 \cdots p_{2n}$, and for $\delta^\ast$ we have the corresponding probability as $p^\ast$, then $\sum_{v \in \mathcal{V}} |p^\ast(v) - p(v)| \leq \prod_{i=1}^{2n} |p_i^\ast - p_i|$, where $p_i^\ast$ is the estimation of $p_i$ in $\delta^\ast$, and is such that $p_i^\ast \leq \min\{1, p_i + \varepsilon\}$, since each estimated probability in the distribution $\delta^\ast$ is also bounded above by $1$. Now $p_i^\ast \leq s$ for all $i \in [2n]$, and we have that $\prod_{i=1}^{2n} |p_i^\ast - p_i| \leq s^{2n} - (s - \varepsilon)^{2n}$, and thus $\delta^\ast \approx_n \delta$ with probability at least $1 - \gamma$.

A strategy $\sigma$ is said to be (uniformly) expectation-optimal if for all $v \in \mathcal{V}$, we have $E^{[\sigma]}(\mathcal{MC}) = \inf_{\pi} E^{[\pi]}(\mathcal{MC})$. The following Lemma captures the idea that some expectation-optimal strategies for MDPs whose transition functions have the same support as that of $\Gamma$ are ‘robust’.

**Lemma 4 (From [14] Theorem 5).** Consider $\beta \in (0, 1)$, and MDPs $\Gamma$ and $\Gamma'$ such that $\Gamma \approx_{10\beta} \Gamma'$ with $\eta_\beta = 10\beta_{\text{min}}$, where $\pi_{\text{min}}$ is the minimum probability appearing in $\Gamma$. For all memoryless deterministic expectation-optimal strategies $\sigma$ in $\Gamma'$, for all $v \in \mathcal{V}$, it holds that $\left| E^{[\sigma]}(\mathcal{MC}) - \inf_{\pi} E^{[\pi]}(\mathcal{MC}) \right| \leq \beta$.

**Proof.** Recall that in our case, we have that the cost of missing the deadlines of the soft tasks are known, and thus we have the same cost function cost in both $\Gamma$ and $\Gamma'$. The bounds for $\eta_\beta$ is obtained directly from Solan’s inequality [14] Theorem 6] as adapted by Chatterjee [14] Proposition 1):

$$\left| \inf_{\tau_1} E^{[\tau_1]}(\mathcal{MC}) - \inf_{\tau_2} E^{[\tau_2]}(\mathcal{MC}) \right| \leq \frac{4|\mathcal{V}|(\eta_\beta/\pi_{\text{min}})}{1 - 2|\mathcal{V}|(\eta_\beta/\pi_{\text{min}})}$$

(1)
In the proof of [4, Theorem 5], it has been shown that if the optimal expected values of two structurally identical MDPs differ by at most $\lambda$, then a memoryless expectation-optimal strategy for one MDP is $2\lambda$-expectation-optimal for the other one.

Thus, $4V_2(\eta_{\beta}/\pi_{\min}) \leq \frac{\beta}{2}$ that gives us $\eta_{\beta} \leq \frac{\beta \pi_{\min}}{8V_2(1+2|V_0|)} \leq \frac{\beta \pi_{\min}}{8|V_0|}$. The result thus follows.

One of the results we cite, i.e. [4, Theorem 5], focuses on stochastic parity games with the same support, i.e., for structurally identical MDPs. There, they derive robustness bounds for MDPs with the discounted-sum function and use them to obtain robustness bounds for MDPs with a parity objective. We are, however, extending those results to MDPs with the mean-cost function (cf. [6]) making use of an observation by Solan [14]: robustness bounds for discounted-sum MDPs extend directly to mean-cost MDPs if they do not depend on the discount factor.

Finally, using both Lemma 3 and Lemma 4, we obtain the following guarantees on the quality of the scheduler that our model-based learning algorithm outputs:

**Theorem 4.** Suppose we are given a task system $\Upsilon$ (with minimal probability $\pi_{\min}$) and a robustness precision $\beta \in (0,1)$. Let $\gamma, \epsilon \in (0,1)$ be s.t. $\epsilon \leq \frac{\beta \pi_{\min}}{8V_2}$. Let $\Upsilon^M$ be the model that is learnt using the above algorithms such that $\Upsilon^M \approx^{\epsilon} \Upsilon$ with probability at least $1-\gamma$, and let $\sigma$ be a memoryless deterministic expectation-optimal strategy of $\Gamma_{\Upsilon^M}$. Then, with probability at least $1-\gamma$, the expected mean-cost of playing $\sigma$ in $\Gamma_{\Upsilon}$ (i.e. in the task system $\Upsilon$) can be bounded as follows: for all $v \in V_2$: $\left| E_{\Gamma_{\Upsilon}[\sigma]}(MC) - \inf_{\tau} E_{\Gamma_{\Upsilon}[\tau]}(MC) \right| \leq \beta$.

### 4 On shielded model-free learning

Instead of first learning a model of the environment (here the task system) and then computing an $\epsilon$-optimal strategy for this model, as in the model-based learning approach described in the previous section, model-free learning aims at directly learning a (near) optimal strategy while interacting with the environment. We recall sufficient information to present an application of [1] with model-free learning to our problem.

**Q-learning, discounted sum and mean-cost** One of the most-successful model-free learning algorithms is the Q-learning algorithm, due to Watkins and Dayan [19]. It aims at learning (near) optimal strategies in a (partially unknown) MDP for the discounted sum objective: Let $d \in \mathbb{R}$ be a discount factor such that $0 < d < 1$. For a sequence of edges $\rho = e_0 e_1 \ldots e_{n-1}$, we define $DS(\rho) = \sum_{i=0}^{n-1} \text{cost}(e_i) \cdot d^i$. Then, for a play $\pi = e_0 e_1 \ldots$, we set $DS(\pi) = \lim_{n \to \infty} DS(\pi(n))$.

In our scheduling problem, we search for (near) optimal strategies for the mean-cost and not for the discounted sum, as we want to minimise the limit average of the cost of missing deadlines of soft tasks. However the two measures are closely related [14][12] since, for all MDPs $\Gamma$ with initial state $\upsilon_{\text{init}}$, we have that
$E_{\text{train}}^f (\text{MC}) = \lim_{d \to 1} ((1 - d) \cdot E_{\text{train}}^f (\text{DS}))$. Thus, if $d$ is large enough, both values coincide.

A way to optimize Q-learning is to approximate the “Q-table” it computes (essentially a map from states to actions) with neural networks. Such a variant of the Q-learning algorithm is called deep Q-learning [2]. We refer the reader to [19,15,2] for the technical details of the algorithms and the guarantees that they provide. We use an implementation of deep Q-learning made available in the OpenAI repository [8].

**Shielded deep Q-learning for safety** We will make use of shielding [5,13]: during the online model-free learning, only actions that are safe for the hard tasks can be used. More precisely, our shielded deep Q-learning algorithm is as follows: (i) compute the most general safe scheduler for the hard tasks. (ii) apply deep Q-learning online while restricting actions to those allowed by the most general safe scheduler.

This shielded deep Q-learning algorithm is able to scale to problems that are clearly beyond the reach of safe model-based learning at the cost of losing the PAC guarantees.

## 5 Experimental Results

In this section, we report on experiments realised with (non-optimised) implementations of our learning algorithms. We use STORM [7] to compute optimal strategies from known models and we use the baseline Q-learning implementation from OpenAI. We report both on model-based learning and on model-free learning, and comment on how the observed results corroborate our theoretical developments.

### Learning models with only soft tasks
Recall that in model-based learning, given the structure of a task system $\mathcal{T}$, and $\varepsilon, \gamma \in (0, 1)$, we learn a model $\mathcal{T}^M$ such that $\mathcal{T}^M \approx^\varepsilon \mathcal{T}$ with probability at least $1 - \gamma$. 

![Fig. 2. $L_1$ (sum), $L_2$ (Euclidean), $L_\infty$ (max) norm distances between learnt and actual distributions for 4 soft tasks.](image)

![Fig. 3. A system with 6 soft tasks; “exe” and “arr” stand for computation time distribution and inter-arrival time distribution resp.](image)
We report the $L_1$, $L_2$, and $L_\infty$ norms between the distributions of the actual task system and the model that is learnt. The figures give the evolution of these distances against the number of training steps. We consider two different task systems, one with four soft tasks (Fig. 2), and the other with six soft tasks (Fig. 3). Our experimental results support the theory: with a reasonable number of samples, we are able to learn models of the distributions that are close to the actual distributions. Each point is the result of averaging over 50 simulations.

**Online safe model-based learning** Now we consider safe model-based learning of systems with both hard and soft tasks. In this benchmark, we consider a task system that has one hard and two soft tasks. The limited size of the task system allows us to compare the value of the strategy obtained on the learnt model and the one obtained on the actual model of the task system using STORM.

First, we compute the most general safe scheduler. We verify that the task system satisfies the good for efficient sampling condition, and hence admits safe efficient PAC learning. We report the value of the optimal expected mean-cost strategy as computed by STORM on the learnt model as a function of the number of steps for which the system is executed (training steps). This value converges towards the expected value of the optimal expected value of the actual task system, roughly equal to 0.06 (see Fig 4). We observe that even with few training steps, the model-based learning algorithm learns a strategy that gives a value that is close to the optimal one.

In this approach, the main bottleneck towards scalability is the extraction of an optimal strategy from the learnt model using probabilistic model-checkers like STORM since the underlying MDP grows exponentially with the number of tasks.

**Input/output of the neural network** In the model-free approach, we train a neural network ($NN$). At every step, the $NN$ receives inputs from the task system and suggests scheduling actions. The learning algorithm only observes the following events: (i) the arrival of a new job for a given task; (ii) the end of a job that has finished its computation before its deadline; and (iii) the violation of a deadline by a job, which triggers a cost. Additionally, the learning algorithm can provide the $NN$s with the following computed values, for each job:
Fig. 6. Comparison of shielded and offline model-free learning for 1 hard task and 2 soft tasks for input variants NN and NN-ED.

Fig. 7. Comparison of shielded and offline model-free learning for 2 hard tasks and 1 soft task for input variants NN and NN-ED.

1. (NN) the maximum among the remaining computation time, the time before deadline, and the minimum among the remaining time before arrival of the next job of the same task;
2. (NN-ED) the maximum among the remaining computation time and the time before deadline;
3. (NN-DL) remaining time before deadline; or
4. (NN-lax) difference between the remaining time before deadline and the maximum remaining computation time.

Additionally, after each decision, the NN gets a feedback in term of cost.

Comparison between the different sets of inputs to the NN To compare the ability of the deep Q-learning algorithm to learn with the different variants of inputs described above, we consider a task system with 4 soft tasks. Our experiments results in Fig. 5 show that the first two variants (NN and NN-ED) that give richer information to the NN lead to more efficient learning procedure. We see that the performances of the model-free learning for these two variants are similar to the performances of the model-based learning, but model-free learning needs more training steps.

Variants of shielded deep Q-learning Fig. 6 and 7 report on results obtained for two task systems and compare the performances obtained by two shielding strategies:

– (NN-shielded and NN-ED-shielded): the NN is trained online with the safety shield as proposed in this paper.
– (NN-offline and NN-ED-offline): the NN is trained offline without the safety shield. During the offline learning phase, missing a deadline for a hard task triggers a large cost (≈ 1,000 times that of the cost of missing a deadline of a soft task). Then, when the NN is executed online, if the action proposed by the schedule that is learnt is unsafe, the shield replaces it by a safe action taken uniformly at random among the set of safe actions (cf. 3).
In the first benchmark, the two approaches have comparable performances and converge to the optimal value (as computed with STORM). On the second benchmark, in which there are two hard tasks instead of one, the first approach works substantially better than the second one. Only the first proposed approach converges to the optimal value.

**Scalability of shielded deep Q-learning** We now consider benchmarks that show that the shielded deep Q-learning algorithm scales well. Fig. 8 reports on results obtained for a task system with two hard tasks and five soft tasks, and Fig. 9 reports on results for a task systems with three hard tasks and six soft tasks. These two task systems have state spaces of approximately $10^{10}$ and $10^{13}$ states respectively. In both the benchmarks we observe that the learnt safe scheduler performs substantially better than a uniform random safe scheduler. Further, the operation needed to update the NNs at each training step is fast enough and requires only a few milliseconds.

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