Neutron localization induced by the pairing field in an inhomogeneous neutron matter

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It is shown that in an inhomogeneous neutron matter the pairing field bounds neutrons around the Fermi level leading to the formation of localized Andreev states. In the case of the inner crust of neutron stars the localization length has been determined as a function of the nuclear density.

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The inhomogeneous nuclear matter forms the inner crust of neutron stars at densities ranging from $10^{-4}$ fm$^{-3}$ to $10^{-1}$ fm$^{-3}$. The nuclear matter is thus diluted and the nucleons are strongly paired through the attraction in the s-wave channel. Since the proton fraction does not exceed a few percent in this density region the main contribution to the pairing energy comes from neutrons. The dependence of the pairing field on the neutron density has been a subject of an intensive theoretical studies (see [1] and references therein). Approaches based on the BCS theory predict the maximum of the s-wave pairing gap $\Delta$ to be around $3$ MeV [2], whereas the inclusion of polarization corrections decreases this value to about $1$ MeV [3]. Recently, the non-perturbative calculations for fermions in the unitary limit predict the pairing gap at $T = 0$ of about $0.6\epsilon_F$, where $\epsilon_F$ is the Fermi energy [4]. Since the neutron matter at the densities between $10^{-3}$ fm$^{-3}$ and $10^{-2}$ fm$^{-3}$ is close to the unitary limit, it suggests that the pairing gap is a considerable fraction of the Fermi energy.

The proton admixture drives the nuclear matter towards an inhomogeneous state and results in the appearance of nuclear impurities immersed in the superfluid neutron liquid. Inside impurities the density reaches the nuclear saturation density. The lattice of nuclei is stabilized by the Coulomb interaction and becomes denser as the average density increases, and eventually melts due to the interplay between Coulomb, surface and shell energies [5, 6, 7]. Consequently, the inhomogeneous neutron density distribution induces the spatial variations of the pairing potential [8, 9]. Because the structure of the proton density distribution induces the spatial variations of the pairing potential [8, 9], the Fermi energy.

The spatial variations of the pairing field may be simulated in the first approximation by the simple model consisting of the spherical pairing well:

$$\Delta(r) = \begin{cases} \Delta_{in} & \text{for } r < R \\ \Delta_{out} & \text{otherwise} \end{cases}, \quad (1)$$

and $|\Delta_{out}| > |\Delta_{in}|$. In the absence of superflow the phase of the pairing field is constant and can be chosen to be zero. The BdG equation takes the form:

$$\begin{pmatrix} T - \mu & \Delta(r) \\ \Delta(r) & -(T - \mu) \end{pmatrix} \begin{pmatrix} u(r) \\ v(r) \end{pmatrix} = E \begin{pmatrix} u(r) \\ v(r) \end{pmatrix}, \quad (2)$$

The inhomogeneity of the pairing field induces an additional scattering of quasiparticles. Because the pairing field is periodic, an additional Bragg scattering gives rise to corrections to the neutron band structure. Therefore the natural question appears to be whether this effect is large, or can be neglected. This article provides an answer to this question and discuss the impact of the inhomogeneity of the pairing field on the quasiparticle states. It is organized as follows: first, I describe the mechanism of bound state formation in the schematic model. Second, it will be shown, that contrary to the normal three-dimensional potential, in the case of a pairing well, there is always a bound state, no matter how shallow the well is. Finally, I will present the variational calculations for realistic pairing potentials through the inner crust and estimate the density region where the localized states appear.

In order to investigate the problem one must realize that the scattering of quasiparticles on the pairing field is qualitatively different than that on the normal potential. Namely, the scattering matrix contains in this case the normal part which comprises the scattering amplitude of hole into hole and particle into particle, but in addition it contains the amplitude of scattering of particle into hole and vice versa. This, so called Andreev reflection has an unusual property that the particle/hole is reflected almost exactly in the same direction as the incoming hole/particle [10]. In fact, it can be shown that the Andreev reflection dominates around the Fermi level and gives rise to the formation of the so-called Andreev states [10] [11].

The spatial variations of the pairing field may be simulated in the first approximation by the simple model consisting of the spherical pairing well:
where $T$ is the kinetic energy operator (for simplicity we disregard the normal potential), $E$ is the quasiparticle energy, $u(\mathbf{r})$ and $v(\mathbf{r})$ are the particle and hole amplitudes of the quasiparticle wave function. In the simplest Andreev approximation, which will admit the analytic solution, the wave function is decomposed into the rapidly oscillating part, and the smooth part, related to the changes of the pairing field \[ 12 \]:

$$
\begin{pmatrix}
  u(\mathbf{r}) \\
v(\mathbf{r})
\end{pmatrix}
= \exp(i\mathbf{k}_F \cdot \mathbf{r})
\begin{pmatrix}
  \bar{u}(\mathbf{r}) \\
  \bar{v}(\mathbf{r})
\end{pmatrix},
$$

where $\bar{u}(\mathbf{r})$ and $\bar{v}(\mathbf{r})$ changes slowly within the scale determined by the coherence length. Neglecting terms of the order of $1/(k_F \xi)$ the original equations transform into:

$$
\begin{pmatrix}
-2i\mathbf{k}_F \cdot \nabla - \mu & \Delta(r) \\
2i\mathbf{k}_F \cdot \nabla + \mu & -\Delta(r)
\end{pmatrix}
\begin{pmatrix}
  \bar{u}(\mathbf{r}) \\
  \bar{v}(\mathbf{r})
\end{pmatrix}
= E
\begin{pmatrix}
  \bar{u}(\mathbf{r}) \\
  \bar{v}(\mathbf{r})
\end{pmatrix}.
$$

(4)

We have chosen units: $\frac{k_F^2}{2\mu} = 1$, i.e. $T = -\nabla^2$ and $k_F^2 = \mu$. Although the Andreev approximation must be used with care for a very dilute neutron matter, it is instructive to study this limit (i.e. $k_F \xi \to \infty$, $\Delta/\mu \to 0$), as the consequences of this oversimplified approach remain valid for a more realistic cases.

The astonishing implication of eq. (4) is that the three-dimensional problem is transformed into a one dimensional, and for each path specified by the vector $\mathbf{k}_F$ exists a solution. This result is a direct consequence of the focusing property of the Andreev reflection\[ 10 \, 11 \]. In order to investigate the $L = 0$ state, one must consider the quantization condition along the diameter:

$$
A(\phi_+, \phi_-) \exp(2i\eta R) = 1,
$$

(5)

where

$$
A(\phi_+ , \phi_-) = \frac{(e^{-\phi_-} - e^{i\phi_+}) (e^{i\phi_-} - e^{\phi_+})}{(e^{-\phi_-} - e^{\phi_+}) (e^{i\phi_-} - e^{\phi_+})},
$$

$$
2qR = k_F \frac{\Delta_{out}}{\mu} \sqrt{(\frac{E}{\Delta_{out}})^2 - (\frac{\Delta_{in}}{\Delta_{out}})^2},
$$

(6)

and

$$
\cos(\phi_+) = \frac{E}{\Delta_{out}},
$$

$$
\cosh(\phi_-) = \frac{E}{\Delta_{in}}.
$$

The closer inspection of the above result reveals that the equation (5) always admits a solution for $E$ in the interval $(\Delta_{in}, \Delta_{out})$, no matter how small is the difference $|\Delta_{in}| - |\Delta_{out}|$. Namely, in the limit $|\Delta_{in}| < |\Delta_{out}| \to 0$ the quantization condition becomes:

$$
\exp(2i(qR + \Phi)) = 1,
$$

(7)

where

$$
\cos \Phi = \frac{2y - x(1 + y)}{x(1 - y)},
$$

$$
x = \frac{E}{\Delta_{out}},
$$

$$
y = \frac{\Delta_{in}}{\Delta_{out}}.
$$

(8)

Since $qR \to 0$ and $y \leq x \leq 1$ it is easy to find that the solution reads:

$$
x \approx \frac{1}{1 + (qR)^2 \frac{1 - y}{y}} \leq 1.
$$

(9)

Another limit relevant for the neutron star crust is when $\Delta_{in} \to 0$. In that case one gets:

$$
\frac{1}{2} k_F \Delta_{out} = \arccos \frac{E}{\Delta_{out}} = \pi m, \quad m = 0, \pm 1, \pm 2, \ldots
$$

(10)

The examination of both limits reveals that once $k_F \Delta_{out}$ and $\Delta_{in}$ are fixed, then $\Delta_{out}$ is constant as well. This property is worth mentioning in the context of the neutron star crust, because there $R \propto \xi$, as the size of nuclear impurities is small with respect to the coherence length. In this case clearly $k_F \Delta_{out}$ is approximately constant and thus $\frac{E}{\Delta_{out}}$ stays fixed as well, through the whole inner crust. Consequently, one may expect that the size of the tail of the wave function bound by the pairing field, $\eta$, will be proportional to $\xi$ and $\eta/\xi$ will be approximately constant through the crust. In the following we will show that this is indeed the case.

Let us turn now to the three-dimensional description, in order to study whether the above results survive in a more realistic case. In the variational formulation of the problem \[ 2 \], one must consider the expectation value of the square of the energy:

$$
\langle E^2 \rangle = \int \! d^3r \chi^+(\mathbf{r}) \hat{\Omega} \chi(\mathbf{r}),
$$

(11)

where

$$
\hat{\Omega} = \left( \begin{array}{cc}
T - \mu & \Delta(r) \\
\Delta^*(r) & -(T - \mu)
\end{array} \right),
$$

(12)

and $T = -\nabla^2$ and $k_F^2 = \mu$. We use the ansatz so that the wave function $\chi(\mathbf{r})$ can be decomposed similarly to (13):

$$
\chi(\mathbf{r}) = \phi_{k_F}(\mathbf{r}) \psi_{k_z}(\mathbf{r}),
$$

(13)

where $\psi_{k_z}(\mathbf{r})$ is a spinor, and the following conditions hold:

$$
-\nabla^2 \psi_{k_z}(\mathbf{r}) = k_F^2 \psi_{k_z}(\mathbf{r}),
$$

$$
-\nabla^2 \psi_{k_z}(\mathbf{r}) \propto k_F^2 \psi_{k_z}(\mathbf{r}),
$$

(14)

$$
\frac{k_F}{k_F} \ll 1.
$$
Because the problem is time-reversal invariant then \( \chi(r) \) can be chosen to be real, and consequently the term \( \langle [\nabla^2, \Delta(r)] \rangle = 0 \). Hence (11) can be rewritten as:

\[
\langle E^2 \rangle = \langle (T - \mu)^2 \rangle + \langle V^2 \rangle,
\]

where the first term can be estimated as follows:

\[
\langle (T - \mu)^2 \rangle = \mu k_f^2 \left( \alpha + 2 \beta \alpha \frac{k_\eta}{k_f} + \beta^2 \left( \frac{k_\eta}{k_f} \right)^2 \right), \tag{16}
\]

with \( \alpha, \beta \) being numerical factors of the order of unity.

In order to estimate \( \langle V^2 \rangle \) let us assume now that \( \Delta(r) = \Delta_{\text{out}} + \delta\Delta(r) \) and \( \delta\Delta(r) \neq 0 \) for \( r < R \). If we treat \( \delta\Delta(r) \) as a small perturbation on top of the homogeneous pairing field \( \Delta_{\text{out}} \) then:

\[
\langle V^2 \rangle \approx \Delta_{\text{out}}^2 + 2\Delta_{\text{out}} \int_{r < R} d^3 r \chi^T(r) \delta\Delta(r) \chi(r) = \Delta_{\text{out}}^2 + 2\Delta_{\text{out}} \Delta \int_{r < R} d^3 r \chi^T(r) \chi(r), \tag{17}
\]

where \( \delta\Delta = \delta\Delta(r_0) \) and \( r_0 < R \). Clearly, the last integral contains a smooth and a rapidly oscillating part. The smooth part can be extracted using the mean value theorem:

\[
\int_{r < R} d^3 r \chi^T(r) \chi(r) = \psi_{k_\eta}^T(r_1) \psi_{k_\eta}(r_1) \int_{r < R} d^3 r \phi_{k_f}(r)^2, \tag{18}
\]

where \( r_1 < R \). If \( k_f R \gg 1 \) then the right hand side integral is proportional to \( R \) instead of \( R^3 \) as one would expect. This fact can be understood in the classical picture since the probability of finding a free particle inside some area is proportional to its linear size. On the other hand, since the above integral is dimensionless it should be a function of \( k_f R \). Hence one may write:

\[
\langle V^2 \rangle \approx \Delta_{\text{out}}^2 + 2\Delta_{\text{out}} \delta\Delta f(k_\eta R), \tag{19}
\]

where \( f(k_\eta R) \rightarrow \gamma k_\eta R, \) if \( k_\eta \rightarrow 0 \) and \( \gamma > 0 \) is a numerical constant. Using the above estimates one may express \( \langle E^2 \rangle \) as a function of \( k_\eta \). Because we would like to answer the question whether the minimum of energy corresponds to \( k_\eta > 0 \), it is sufficient if we consider the limit of \( k_\eta \) being very small. In this case:

\[
\langle E^2 \rangle \approx \mu k_\eta^2 \alpha^2 + \Delta_{\text{out}}^2 + 2\Delta_{\text{out}} \delta\Delta k_\eta R. \tag{20}
\]

Note the striking similarity with the estimation for the ground state energy of a normal potential in one dimension. The condition for the minimum of \( \langle E^2 \rangle \) gives the relation for \( k_\eta \):

\[
k_\eta \approx -\frac{\gamma}{\alpha^2} \frac{\delta\Delta_{\text{out}} R}{\mu}. \tag{21}
\]

It shows that \( \langle E^2 \rangle \) is minimized by \( k_\eta > 0 \), if \( \Delta_{\text{out}} \delta\Delta < 0 \). The last condition is obvious as it corresponds to the pairing potential well, whereas a positive value would be equivalent to a potential barrier.

Hence the scattering on the pairing potential has a more severe effect on the quasiparticle spectrum as the pairing field turns effectively the three-dimensional problem into a one-dimensional, and thus the scattering is more effective.

In order to quantitatively study this effect we minimize \( \langle E^2 \rangle \) in a class of wave functions of the form:

\[
\chi(r) = \frac{\sin(k_f r)}{k_f r} \exp(-k_\eta^2 r^2) \left( \begin{array}{c} u \\ v \end{array} \right), \tag{22}
\]

with \( \int d^3 r \chi^T(r) \chi(r) = 1 \). The pairing field was assumed to have the form:

\[
\Delta(r) = \Delta_{\text{out}} + \delta\Delta \exp(-r/\xi^2), \tag{23}
\]

where \( \delta\Delta = \Delta_{\text{in}} - \Delta_{\text{out}} \), and \( \xi = \frac{1}{k_f \Delta_{\text{out}}} \). The actual parameters \( \Delta_{\text{out}}, \Delta_{\text{in}} \) requires some knowledge about the

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**FIG. 1:** The results of minimization of \( \langle E^2 \rangle \) with respect to localization length \( \eta \) (see text) plotted as a function of the nuclear density in the inner crust of neutron star. The dashed line corresponds to the BCS pairing and the solid line denotes the pairing with inclusion of polarization corrections. The lower subfigure shows the corresponding pairing gaps, used in the calculations.
density dependence of the pairing field. The two sets of calculation has been performed. In the first one, the pairing field $\Delta_{\text{out}}$ is assumed to have the density dependence obtained from the BCS approach with Argonne v18 interaction [2], whereas $\Delta_{\text{in}}$ corresponds to the neutron pairing at the nuclear saturation density. The second set of calculation was performed for pairing extracted from the s-wave N-N scattering data $\Delta \approx \left(\frac{2}{e}\right)^{7/3} \frac{\hbar^2 k_F^2}{2m_0} \exp \left(-\frac{\pi}{2 \tan \delta(k_F)}\right)$, which is supposed to include the polarization corrections [13]. The results are presented in the Fig. 1. In the upper subfigure the $R_C/\eta$ is plotted, where $\eta = 1/k_\eta$, and $R_C$ is the Wigner-Seitz cell taken according to the calculations from refs. [14].

The results indicate a strong dependence of $\eta$ on the pairing field. In the case of stronger pairing the quasiparticle states become practically localized for lower densities, i.e. $R_C/\eta > 1$. For weaker pairing the impact of the inhomogeneous pairing field is less pronounced although the ratio $R_C/\eta$ reaches 0.6. It has to be noted that the ratio $\sqrt{(E^2)/\Delta_{\text{out}}} \approx 0.9$ throughout the crust. Consequently $\eta/\xi \approx 50$. It supports the conclusion based on the schematic model.

Note that in the calculations the mean nuclear potential was set to zero. Otherwise the results would have been obscured by the existence of an additional scattering which would alter the rapidly oscillating part of the wave function. The current approach corresponds to the Thomas-Fermi approximation for the mean-field potential. Thus the results represent the average behavior of the quasiparticle states around the Fermi level. The solution of the full BdG or HFB equations can change the above results in the following way: around the Fermi level will appear resonances induced by the mean-field. However their pairing properties will be practically unaltered, as compared to our model, unless their energy will deviate from the Fermi level more than $|\Delta_{\text{out}}|$. The consequences of the presented results are twofold. First, they suggest that the scattering of quasiparticles has to be taken into account if the neutron band structure is to be determined. Consequently, for this purpose the BCS approach is too poor, as there the quasiparticle scattering is neglected. On the other hand BCS is sufficient if one is interested in the bulk properties of the inner crust: energy density, density distribution, proton to neutron fraction, etc. It is interesting to note that the need for HFB (or BdG) approach has been noticed in the context of nuclei close to the neutron drip line, long time ago [12]. There the argument originated from the nuclear density localization requirement. Here, the main argument comes from the realization that the pairing field exhibits spatial variations in the inhomogeneous nuclear matter. When the matter becomes diluted the scattering on the pairing potential becomes more effective and contribute to the band structure around the Fermi level.

Second, the consequence of neutron localization by the pairing field may turn out to be important for transport properties. Namely, in this case the effective mass has to be determined and thus the band structure has to be calculated [10]. Within the approach we presented here, one may estimate the dispersion relation for the band corresponding to bound neutrons. Taking into account only the nearest neighbour interaction of the localized neutrons one obtains: $E(k) \propto \text{const.} + \exp (-\frac{1}{2}(R_C/\eta)^2)k^2$. Note that in the limit of $R_C/\eta > 1$ the coefficient in front is practically zero and the effective mass tends to infinity. This corresponds to complete localization of neutron states at the Fermi level around nuclear impurities. Clearly the detailed calculations need to be performed in order to investigate this problem.

Summarizing, it was shown that the inhomogeneity of the pairing field in the neutron matter leads to the formation of neutron bound states around the Fermi level. They appear at an arbitrarily weak deviation from the homogeneous pairing field and are the consequence of the focusing property of Andreev reflection.

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