HYPERKÄHLER VARIETIES AS BRILL-NOETHER LOCI ON CURVES

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Abstract. Consider the moduli space $M_C(r; K_C)$ of stable rank $r$ vector bundles on a curve $C$ with canonical determinant, and let $h$ be the maximum number of linearly independent global sections of these bundles. If $C$ embeds in a K3 surface $X$ as a generator of Pic$(X)$ and the genus of $C$ is sufficiently high, we show the Brill-Noether locus $BN_C \subset M_C(r; K_C)$ of bundles with $h$ global sections is a smooth projective Hyperkähler manifold of dimension $2g - 2r[2]$, isomorphic to a moduli space of stable vector bundles on $X$.

1. Introduction

In this paper, we show that specific Brill-Noether loci of rank $r$-stable vector bundles on K3 curves are isomorphic to moduli spaces of stable vector bundles on K3 surfaces. This generalises Mukai’s program [Muk01, ABS14, Fey20a, Fey20b] to Brill-Noether loci on K3 curves of dimension higher than 2.

Fix $(r, k) \in \mathbb{Z}_{>0} \oplus \mathbb{Z}_{>0}$ such that $\gcd(r, k) = 1$ and $k < r$. Then take $g \gg 0$. Let $(X, H)$ be a polarised K3 surface with Pic$(X) = \mathbb{Z} \cdot H$ and let $C$ be any curve in the linear system $|H|$ of genus $g$. There is a unique $s \in \mathbb{Z}$ such that

$$-2 \leq k^2H^2 - 2rs < -2 + 2r.$$  

We further assume $\gcd(s, k) = 1$. Consider the Brill-Noether locus $BN_C(r, k(2g - 2), r + s)$ of semistable\(^1\) rank $r$-vector bundles on $C$ having degree $kH^2 = k(2g - 2)$ and possessing at least $r + s$ linearly independent global sections. Also $M_X(v)$ denotes the moduli space of $H$-Gieseker semistable sheaves on $X$ with Mukai vector $v = (r, kH, s)$. It is a (non-empty) smooth quasi-projective variety\(^2\) of dimension $v^2 + 2 = k^2H^2 - 2rs + 2$.

Theorem 1.1. There is an isomorphism

$$\Psi: M_X(v) \to BN_C(r, k(2g - 2), r + s)$$

which sends a vector bundle\(^3\) $E$ on $X$ to its restriction $E|_C$.

The above Theorem says that for any vector bundle $F$ on the curve $C$ in the Brill-Noether locus $BN_C(r, k(2g - 2), r + s)$, there exists a unique vector bundle on the surface $X$ whose restriction to $C$ is isomorphic to $F$. In particular, we obtain the following:

Corollary 1.2. Any vector bundle $F \in BN_C(r, k(2g - 2), r + s)$ is stable (cannot be strictly semistable) and $\wedge^r F = \omega_C^\otimes k$ with $h^0(C, F) = r + s$.

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\(^1\)See Definition 2.1.

\(^2\)See for instance [Huy16, Chapter 10, Corollary 2.1 & Theorem 2.7].

\(^3\)Any $H$-Gieseker semistable sheaf of Mukai vector $v$ is locally free, see Lemma 3.5.
Proof ideas. To prove Theorem 1.1, we go through the following steps:

(a) Consider the embedding $\iota: C \hookrightarrow X$. The push-forward $\iota_* F$ for any vector bundle $F \in \text{BN}_C(r, kH^2, r+ s)$ is semistable in the large volume limit of the space of Bridgeland stability conditions on $X$. We study walls for objects of class $\text{ch}(\iota_* F)$ between the large volume limit and the Brill-Noether wall that the structure sheaf $\mathcal{O}_X$ is making. The first wall is made by stable sheaves with Mukai vector $v = (r, kH, s)$. We show $\iota_* F$ gets destabilised along this wall if and only if $F = E|_C$ for a sheaf $E \in M_X(v)$.

(b) The assumption (1) implies that any stable coherent sheaf of Mukai vector $v$ is locally-free. We show there is no wall for objects with Mukai vector $v$ between the large volume limit and the Brill-Noether wall. That’s why we get $h^0(X, E) = r + s$ for any $E \in M_X(v)$. Hence, there is a well-defined map

$$\Psi: M_X(v) \rightarrow \text{BN}_C(r, kH^2, r + s)$$

$$E \mapsto E|_C.$$ 

Then a usual wall-crossing argument gives injectivity of $\Psi$ due to the uniqueness of Harder-Narasimhan filtration.

(c) To prove $\Psi$ is surjective we apply the technique developed in [Fey20a] which bounds the number of global sections of sheaves on $X$ in terms of the length of a convex polygon (which is the Harder-Narasimhan polygon in the Brill-Noether region). This implies that if $h^0(X, \iota_* F) \geq r + s$, then $F$ is the restriction of a vector bundle $E \in M_X(v)$ to the curve $C$, so in particular, $\Psi$ is surjective.

(d) Finally, any vector bundle $E \in M_X(v)$ is generated by global sections and the kernel

$$K_E := \ker(\mathcal{O}_X^{\oplus r+s} \xrightarrow{ev} E).$$

is a slope-stable vector bundle. By applying a wall-crossing argument, we show that $\text{Hom}(K_E, E(-H)[1]) = 0$. Then a usual deformation theory argument implies that the derivative $d\Psi$ is surjective, and so $\Psi$ is an isomorphism.

Note that in Theorem 1.1, the assumption $g \gg 0$ is necessary for all the above steps. Remark 4.3 gives a list of inequalities that $g$ must satisfy to get Theorem 1.1.

Outlook. In this paper, we only work on curves on K3 surfaces of Picard rank one. More generally one can consider a polarised K3 surfaces $(X, H)$ of arbitrary Picard rank and pick a curve $C \in |H|$ of genus $g(C) \gg 0$. For any $r, k \in \mathbb{Z}^+$, we define

$$h_{r,k} := \max \left\{ h^0(C, F) : \text{rank } r\text{-stable vector bundles } F \text{ on } C \text{ with } \Lambda^r F = \omega_C^{\otimes k} \right\}.$$

Consider the Brill-Noether locus $\text{BN}_{\text{st}}(r, \omega_C^{\otimes k}, h_{r,k})$ of stable rank $r$-vector bundles $F$ on $C$ of fixed determinant $\omega_C^{\otimes k}$ and having $h_{r,k}$ linearly independent global sections. Then the natural question is whether this Brill-Noether locus is a Hyperkähler variety when $g(C)$ is

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One can replace the structure sheaf $\mathcal{O}_C$ with any other vector bundle on $C$ and consider the corresponding Brill-Noether locus.
high enough. I expect to answer this question in the future by applying a more general wall-crossing argument.

Note that the assumption $g(C) \gg 0$ is necessary as there are examples of the above Brill-Noether loci on K3 curves of genus 7 [Muk01] and genus 12 [Fey22] which are smooth Fano varieties. A similar technique as in this paper can be applied to generalise Theorem 1.1 to

(a) polarised K3 surfaces $(X, H)$ such that $H^2$ divides $H.D$ for all curve classes $D$ on $X$, and

(b) curves $C \hookrightarrow X$ which are not necessarily in the linear system $|H|$.

Plan. In Section 2, we review Bridgeland stability conditions on the bounded derived category of coherent sheaves on a K3 surface $X$. Section 3 analyses walls for the push-forward of vector bundles in our Brill-Noether locus to the K3 surface $X$ and as a result, proves Theorem 1.1. The computations for the location of walls are all postponed to Section 4.

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2. Review: Bridgeland stability conditions on K3 surfaces

In this section, we review the description of a slice of the space of stability conditions on the bounded derived category of coherent sheaves on a K3 surface given in [Bri08, Section 1.7].

Let $X$ be a projective scheme over $\mathbb{C}$ of dimension $\dim(X) \geq 1$, and let $H$ be an ample line bundle on $X$. The Hilbert polynomial $P(E, m)$ is given by $m \mapsto \chi(O_X, E \otimes O_X(mH))$. It can be uniquely written in the form

$$P(E, m) = \sum_{i=0}^{\dim(E)} \alpha_i(E) m^i/d!$$

with integral coefficients $\alpha_i(E)$. The reduced Hilbert polynomial $p(E, m)$ of a coherent sheaf $E$ of dimension $d$ is defined by

$$p(E, m) = \frac{P(E, m)}{\alpha_d(E)}.$$

Definition 2.1. [HL10, Definition 1.2.4] A coherent sheaf $E$ of dimension $d$ is (semi)stable if $E$ is pure and for any proper subsheaf $F \subset E$ one has $p(F, m) (\leq) p(E, m)$ for $m \gg 0$. Here ($\leq$) denotes $<$ for stability and $\leq$ for semistability.
From now on, we always assume \((X, H)\) is a smooth polarized K3 surface over \(\mathbb{C}\) with \(\text{Pic}(X) = \mathbb{Z} \cdot H\). Denote the bounded derived category of coherent sheaves on \(X\) by \(\mathcal{D}(X)\) and its Grothendieck group by \(K(X) := K(\mathcal{D}(X))\). Given an object \(E \in \mathcal{D}(X)\), we write \(\text{ch}(E) = (\text{ch}_0(E), \text{ch}_1(E), \text{ch}_2(E)) \in H^\ast(X, \mathbb{Z})\) for its Chern characters. The Mukai vector of an object \(E \in \mathcal{D}(X)\) is an element of the numerical Grothendieck group \(\mathcal{N}(X) = \mathbb{Z} \oplus \text{NS}(X) \oplus \mathbb{Z} \cong \mathbb{Z}^3\) defined via

\[
v(E) := (\text{ch}_0(E), \text{ch}_1(E), \text{ch}_0(E) + \text{ch}_2(E)) = \text{ch}(E) \sqrt{\text{td}(X)}.
\]

The Mukai bilinear form

\[
\langle v(E), v(E') \rangle = \text{ch}_1(E) \text{ch}_1(E') + \text{ch}_0(E) (\text{ch}_2(E') + \text{ch}_2(E')) + \text{ch}_0(E') (\text{ch}_0(E) + \text{ch}_2(E))
\]

makes \(\mathcal{N}(X)\) into a lattice of signature \((2, 1)\). The Riemann-Roch theorem implies that this form is the negative of the Euler form, defined as

\[
\chi(E, E') = \sum_i (-1)^i \dim_{\mathbb{C}} \text{Hom}_X^i(E, E') = - \langle v(E), v(E') \rangle.
\]

The slope of a coherent sheaf \(E \in \text{Coh}(X)\) is defined by

\[
\mu_H(E) := \begin{cases} \frac{H \cdot \text{ch}_1(E)}{H^2 \cdot \text{ch}_0(E)} & \text{if } \text{ch}_0(E) > 0 \\ +\infty & \text{if } \text{ch}_0(E) = 0. \end{cases}
\]

This leads to the usual notion of \(\mu_H\)-stability. Associated to it every sheaf \(E\) has a Harder-Narasimhan filtration. Its graded pieces have slopes whose maximum we denote by \(\mu_\mu^+(E)\) and minimum by \(\mu_\mu^-(E)\).

For any \(b \in \mathbb{R}\), let \(\mathcal{A}(b) \subset \mathcal{D}(X)\) denote the abelian category of complexes

\[
\mathcal{A}(b) = \{ E^{-1} \xrightarrow{d} E^0 : \mu_\mu^+(\ker d) \leq b, \ \mu_\mu^-(\text{coker } d) > b \}.
\]

Then \(\mathcal{A}(b)\) is the heart of a bounded t-structure on \(\mathcal{D}(X)\) by [Bri08, Lemma 6.1]. For any pair \((b, w) \in \mathbb{R}^2\), we define the group homomorphism \(Z_{b, w} : K(X) \to \mathbb{C}\) by

\[
Z_{b, w}(E) := - \text{ch}_2(E) + w \text{ch}_0(E) + i \left( \frac{H \cdot \text{ch}_1(E)}{H^2} - b \text{ch}_0(E) \right).
\]

Define the function \(\Gamma : \mathbb{R} \to \mathbb{R}\) as

\[
\Gamma(b) := \begin{cases} b^2 \left( \frac{H^2}{2} + 1 \right) - 1 & \text{if } b \neq 0 \\ 0 & \text{if } b = 0. \end{cases}
\]

By abuse of notations, we also denote the graph of \(\Gamma\) by curve \(\Gamma\) (see Figure 1). We define

\[
U := \{(b, w) \in \mathbb{R}^2 : w > \Gamma(b)\}.
\]

In figures, we will plot the \((b, w)\)-plane simultaneously with the image of the projection map

\[
\Pi : K(X) \setminus \{ E : \text{ch}_0(E) = 0 \} \to \mathbb{R}^2, \quad E \mapsto \left( \frac{\text{ch}_1(E) \cdot H}{\text{ch}_0(E) \cdot H^2}, \frac{\text{ch}_2(E)}{\text{ch}_0(E)} \right).
\]
Consider the slope function $\nu_{b,w}: \mathcal{A}(b) \to \mathbb{R} \cup \{+\infty\}$, $\nu_{b,w}(E) := \begin{cases} \frac{\text{Re}[Z_{b,w}(E)]}{\text{Im}[Z_{b,w}(E)]} & \text{if } \text{Im}[Z_{b,w}(E)] > 0 \\ +\infty & \text{if } \text{Im}[Z_{b,w}(E)] = 0. \end{cases}$

This defines our notion of stability in $\mathcal{A}(b)$:

**Definition 2.2.** Fix $w > \Gamma(\mathcal{A}(b))$. We say $E \in \mathcal{D}(\mathcal{X})$ is (semi)stable with respect to the pair $\sigma_{b,w} = (\mathcal{A}(b), Z_{b,w})$ if and only if

- $E[k] \in \mathcal{A}(b)$ for some $k \in \mathbb{Z}$, and
- $\nu_{b,w}(F) (\leq) \nu_{b,w}(E[k]/F)$ for all non-trivial subobjects $F \hookrightarrow E[k]$ in $\mathcal{A}(b)$.

Let $E$ be a semistable coherent sheaf, or more generally a $\sigma_{b,w}$-semistable object for some $(b, w) \in U$. By [FL21, Remark 2.3(a)]\(^5\), the projection $\Pi(E)$ does not lie in $U$. Thus the same argument as in [Bri08, Lemma 6.2] shows that the pair $\sigma_{b,w} = (\mathcal{A}(b), Z_{b,w})$ is a Bridgeland stability condition on $\mathcal{D}(\mathcal{X})$, see [Bri07, Definition 1.1 & Proposition 5.3]\(^6\). This in particular implies that every object $E \in \mathcal{A}(b)$ admits a Harder–Narasimhan filtration which is a finite sequence of objects in $\mathcal{A}(b)$:

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_k = E$$

whose factors $E_i/E_{i-1}$ are $\sigma_{b,w}$-semistable and $\nu_{b,w}(E_1) > \nu_{b,w}(E_2) > \cdots > \nu_{b,w}(E/E_{k-1})$. We denote $\nu_{b,w}^+(E) := \nu_{b,w}(E_1)$ and $\nu_{b,w}^-(E) := \nu_{b,w}(E_k)$. Summarizing, we have the following:

**Theorem 2.3** ([Bri08, Section 1]). For any pair $(b, w) \in U$, the pair $\sigma_{b,w} = (\mathcal{A}(b), Z_{b,w})$ defines a stability condition on $\mathcal{D}(\mathcal{X})$. Moreover, the map from $U$ to the space of stability conditions $\text{Stab}(\mathcal{X})$ on $\mathcal{D}(\mathcal{X})$ given by $(b, w) \mapsto \sigma_{b,w}$ is continuous.

\(^5\)Note that the curve $\Gamma$ defined in this paper lies above the curve $\Gamma$ considered in [FL21].

\(^6\)Up to the action of $\widetilde{\text{GL}}^+(2; \mathbb{R})$, the stability conditions $\sigma_{b,w}$ are the same as the stability conditions defined in [Bri08, section 6].
The second part of Theorem 2.3 implies that the two-dimensional family of stability conditions $\sigma_{b,w}$ satisfies wall-crossing as $b$ and $w$ vary, see for instance [FT21, Proposition 4.1] or [FL21, Proposition 2.4].

**Proposition 2.4 (Wall and chamber structure).** Fix a non-zero object $E \in \mathcal{D}(X)$. There exists a collection of line segments (walls) $\mathcal{W}_E^i$ in $U$ (called “walls”) which are locally finite and satisfy

(a) The extension of each line segment $\mathcal{W}_E^i$ passes through the point $\Pi(E)$ if $c_0(E) \neq 0$, or has fixed slope $\frac{c_2(E)H^2}{c_1(E)H}$ if $c_0(E) = 0$.

(b) An endpoint of the segment $\mathcal{W}_E^i$ is either on the curve $\Gamma$ or on the line segment through $(0,0)$ to $(0,-1)$.

(c) The $\sigma_{b,w}$-(semi)stability of $E$ is unchanged as $(b,w)$ varies within any connected component (called a “chamber”) of $U \setminus \bigcup_{i \in I} \mathcal{W}_E^i$.

(d) For any wall $\mathcal{W}_E^i$ there is $k_i \in \mathbb{Z}$ and a map $f : F \to E[k_i]$ in $\mathcal{D}(X)$ such that

- for any $(b,w) \in \mathcal{W}_E^i$, the objects $E[k_i]$, $F$ lies in the heart $A(b)$,

- $E[k_i]$ is $\nu_{b,w}$-semistable with $\nu_{b,w}(E) = \nu_{b,w}(F) = \text{slope}(\mathcal{W}_E^i)$ constant on $\mathcal{W}_E^i$, and

- $f$ is an injection $F \subset E[k_i]$ in $A(b)$ which strictly destabilises $E[k_i]$ for $(b,w)$ in one of the two chambers adjacent to the wall $\mathcal{W}_E^i$.

![Figure 2. Walls $\mathcal{W}_E^i$ for $E$.](image)

### 3. Wall-crossing

Let $(X,H)$ be a smooth polarized K3 surface over $\mathbb{C}$ with $\text{Pic}(X) = \mathbb{Z} \cdot H$, and let $C \in |H|$ be any curve of genus $g = \frac{1}{2}H^2 + 1$. The Brill-Noether loci of vector bundles on $C$ when genus $g$ is low have been highly studied, see e.g. [Muk96, Muk88, Muk02, Muk01, Muk95]. But in this section, we analyse these loci when genus $g$ is high.
As in the Introduction, fix \((r, k) \in \mathbb{Z}_{>0} \oplus \mathbb{Z}_{>0}\) such that \(\gcd(r, k) = 1\) and \(k < r\). Then we take \(g \gg 0\) and consider polarised K3 surfaces \((X, H)\) of genus \(g = \frac{1}{2}H^2 + 1\). Consider the Mukai vector \(v := (r, kH, s)\) such that \(\gcd(s, k) = 1\) and the assumption (1) holds:

\[-2 \leq v^2 = k^2H^2 - 2rs < -2 + 2r.\]

This implies

\[-1 + \frac{1}{r} + \frac{k^2}{2r}H^2 < s \leq \frac{1}{r} + \frac{k^2}{2r}H^2.\]

We set Mukai vector \(\alpha := (s, -kH, r)\), so

\[
\Pi(v) = \left(\frac{k}{r}, \frac{s-r}{r}\right), \quad \Pi(\alpha) = \left(\frac{k}{s}, \frac{r-s}{s}\right),
\]

and

\[
\Pi(v(-H)) = \Pi \left( r, (k-r)H, s - kH^2 + \frac{r}{2}H^2 \right) = \left( \frac{k-r}{r}, \frac{s-r}{r} - \frac{k}{r}H^2 + \frac{1}{2}H^2 \right).
\]

**Definition 3.1.** Let \(\ell^*\) be the lowest line of slope \(H^2\left(\frac{k}{r} - \frac{1}{2}\right)\) which intersects the curve \(\Gamma\) at two points with \(b\)-values \(b_1^* < b_2^*\) satisfying

\[
\max \left\{ b_1^* - \frac{k-r}{r}, \frac{k-r}{r} - b_2^* \right\} \leq \frac{1}{r^2(r+1)}.
\]

**Proposition 3.2.** Suppose \(H^2 = 2g - 2 \gg 0\).

(a) The origin point \((0, 0)\) lies below \(\ell^*\) and

\[
\frac{k-r}{r} < b_1^* < b_2^* < \frac{k}{r}.
\]

(b) The line \(\tilde{\ell}\) passing through \(\Pi(v)\) and \(\Pi(v(-H))\) is parallel to \(\ell^*\) and lies above it. It intersects the curve \(\Gamma\) at two points with \(b\)-values \(\tilde{b}_1 < \tilde{b}_2\) such that

\[
\frac{k-r}{r} \leq \tilde{b}_1 < b_1^* \quad \text{and} \quad b_2^* < \tilde{b}_2 \leq \frac{k}{r}.
\]

(c) The line \(\ell_v\) passing through \(\Pi(v)\) and the origin \((0, 0)\) intersects \(\Gamma\) at two points (except the origin) with \(b\)-values \(b_1^* < b_2^*\) satisfying

\[
b_1^* < \frac{k}{s} + \frac{1}{s(s-1)} \quad \text{and} \quad b_2^* < b_2^* \leq \frac{k}{r}.
\]

(d) The line \(\ell_\alpha\) passing through \(\Pi(v(-H))\) and \(\Pi(\alpha)\) intersects \(\Gamma\) at two points with \(b\)-values \(b_1^* < b_2^*\) so that

\[
\frac{k-r}{r} \leq b_1^* < b_1^* \quad \text{and} \quad \frac{k}{s} - \frac{1}{s(s-1)} < b_2^*.
\]
(e) The line $\ell_v(-H)$ passing through $\Pi(v(-H))$ and the origin intersects the curve $\Gamma$ at two points (except the origin) with $b$-values $b_1^{v(-H)} < 0 < b_2^{v(-H)}$ such that

$$\frac{k-r}{r} \leq b_1^{v(-H)} < b_2^{v(-H)}.$$

**Lemma 3.3.** Let $\ell^*$ intersects the vertical lines $b = \frac{k}{r}$ and $b = \frac{k-r}{r}$ at $p_1 = \left(\frac{k}{r}, w_1\right)$ and $p_2 = \left(\frac{k-r}{r}, w_2\right)$, respectively. The line $\ell_1$ that passes through $p_1$ and the origin intersects the vertical line $b = \frac{k}{r} - \frac{1}{r(r-1)}$ at a point inside $U$. Similarly, the line $\ell_2$ through $p_2$ and the origin intersects $b = \frac{k-r}{r} + \frac{1}{r(r-1)}$ at a point inside $U$.

We postpone the proof of Proposition 3.2 and Lemma 3.3 to Section 4. The next Lemma gives vertical lines on which we can rule out walls of instability for objects of certain classes.
Lemma 3.4. Take an object $E \in \mathcal{D}(X)$ with $\chi_{\leq 1}(E) = (r', k'H)$ such that $\gcd(r', k') = 1$. There are unique $m^\pm, n^\pm \in \mathbb{Z}$ such that $|n^\pm| < r'$, $n^+ r' > 0$ and $m^r - n^k' = -1$ and $m^r + n^r' = 1$.

We set $b^\pm_E := m^\pm n^\pm = k'^{r'} + 1$, then there is no wall for $E$ crossing the vertical lines $b^\pm = b^\pm_E$, i.e., if $E$ is $\sigma_{b^\pm_E, w}$-semistable for some $w > \Gamma(b^\pm_E)$, then it is $\sigma_{b^\pm_E, w}$, stable for any $w > \Gamma(b^\pm_E)$.

Proof. The first part is trivial and the second part follows by the fact that the imaginary part $\left| \Im[Z_{b^\pm_E, w}(E)] \right| = k'^{r'} - \frac{m}{n_i} r'$ is minimal, see [Fey21, Lemma 3.5] for more details. \(\square\)

Let $M_X(v)$ be the moduli space of $H$-Gieseker semistable sheaves on the surface $X$ with Mukai vector $v = (r, kH, s)$.

Lemma 3.5. Any coherent sheaf $E \in M_X(v)$ is a $\mu_H$-stable locally free sheaf.

Proof. Since $\gcd(r, k) = 1$, any $H$-Gieseker-semistable sheaf of class $v$ is $\mu_H$-stable. Assume $E$ is not locally-free, then taking its reflexive hull gives the exact sequence

$$E \hookrightarrow E^{\vee \vee} \twoheadrightarrow Q$$

where $Q$ is a torsion sheaf supported in dimension zero. We know the reflexive sheaf $E^{\vee \vee}$ is also slope-stable, so

$$-2 \leq v(E^{\vee \vee})^2 = (r, kH, s + \text{ch}_3(Q))^2 = k^2H^2 - 2r(s + \text{ch}_3(Q)) \leq k^2H^2 - 2rs - 2r$$

which is not possible by our assumption (1). \(\square\)

Define the object $K_E[1] \in \mathcal{D}(X)$ as the cone of the evaluation map:

$$\mathcal{O}^h_{X} \xrightarrow{\text{ev}} E \rightarrow K_E[1].$$

(6)

Proposition 3.6. Let $E \in M_X(v)$ be a $\mu_H$-stable vector bundle on the surface $X$. Then

(a) $\Hom(E, E(-H)[1]) = 0$.

(b) For any curve $C \in |H|$, the restriction $E|_C$ is a stable\(^7\) vector bundle on $C$ and $h^0(C, E|_C) = r + s$.

(c) The object $K_E$ is a $\mu_H$-stable locally free sheaf on $X$ of Mukai vector $v(K_E) = \alpha = (s, -kH, r)$ and $\Hom(K_E, E(-H)[1]) = 0$.

\(^7\)Stability on the curve $C$ is defined as in Definition 2.1
Proof. Step 1. We first do wall-crossing for the bundle $E$. By [Bri08, Proposition 14.2], $E$ is $\sigma_{b,w}$-stable for $b < \mu_H(E)$ and $w > 0$. As in Lemma 3.4 consider the vertical line $b = b_E$. We know
\[
\frac{k}{r} - \frac{1}{r} \leq b_E^+ \leq \frac{k}{r} - \frac{1}{r(r - 1)}.
\]
By Proposition 3.2 (c), the line $\ell_v$ intersects $\Gamma$ at a point with positive $b$-value $b^+_v$ satisfying
\[
\frac{k}{r} - \frac{1}{r^2(r + 1)} \leq b^+_v < b^+_2.
\]
Thus the line segment $\ell_v \cap U$ intersects the vertical line $b = b_E^+$ at a point inside $U$. Hence Lemma 3.4 implies that there is no wall for $E$ above $\ell_v$.

We have $\text{Hom}(\mathcal{O}_X, E[2]) = \text{Hom}(E, \mathcal{O}_X)^* = 0$, so
\[
\chi(\mathcal{O}_X, E) = r + s = h^0(E) - h^1(E).
\]
Thus $h^0(E) \neq 0$ and $\mathcal{O}_X$ makes a wall for $E$. This wall is indeed $\ell_v \cap U$ for $b \in (\ell^+_v, 0)$. Take a point $(b, w)$ on this wall and consider the evaluation map
\[
\mathcal{O}^0_{X, E} \overset{ev}{\rightarrow} E \rightarrow K_E[1]
\]
We know $\mathcal{O}_X$ and $E$ are $\sigma_{b,w}$-semistable of the same $\nu_{b,w}$-slope and $\mathcal{O}_X$ is strictly $\sigma_{b,w}$-stable. Thus the map $ev$ is injective in the abelian subcategory of $\sigma_{b,w}$-semistable objects in $\mathcal{A}(b)$ of $\nu_{b,w}$-slope equal to $\nu_{b,w}(\mathcal{O}_X)$. Therefore the co-kernel $K_E[1]$ is also $\sigma_{b,w}$-semistable.

By [Fey20a, Lemma 3.2], we know
\[
h^0(X, E) \leq \frac{r + s}{2} + \frac{\sqrt{(r - s)^2 + k^2(2H^2 + 4)}}{2} = h
\]
We claim $h < r + s + 1$, i.e.
\[
(r - s)^2 + k^2(2H^2 + 4) < (r + s + 2)^2.
\]
This is equivalent to $k^2H^2 - 2rs < 2 + (r + s) - 2k^2$. Since $\nu^2 = k^2H^2 - 2rs < 2r - 2$ by (1), we only need to show
\[
2r - 2 \leq 2 + (r + s) - 2k^2
\]
i.e. $-2 + k^2 \leq s$ which holds by (4) if $H^2 > 2r$. Therefore $h^0(E) = r + s, h^1(E) = 0$ and $v(K_E[1]) = -\alpha = (-s, KH, -r)$.

Step 2. The next step is to examine walls for $K_E[1]$. By Proposition 3.2 (c), our Brill-Noether wall $\ell_v \cap U$ intersects $\Gamma$ at a point with $b$-value $b^+_v < -\frac{k}{s} + \frac{1}{s(s - 1)}$. We know $\gcd(s, k) = 1$, so if $k > 1$, the value $b^+_{K_E}$ in Lemma 3.4 satisfies
\[
-\frac{k}{s} + \frac{1}{s(s - 1)} \leq b^+_{K_E} \leq -\frac{k}{s} + \frac{1}{s} < 0.
\]
Thus the vertical line $b = b^+_{K_E}$ intersects $\ell_v \cap U$ at a point inside $U$. Hence $\sigma_{b,w}$-semistability of $K_E[1]$ for $(b, w) \in \ell_v \cap U$ implies that it is $\sigma_{b=b^+_{K_E},w}$-stable for any $w > \Gamma(b^+_{K_E})$ by Lemma 3.4.
If \( k = 1 \), then we show directly that \( K_E[1] \) is \( \sigma_{b,w} \)-stable for \((b, w) \in \ell_v \cap U \) when \( b < 0 \). Assume otherwise, and let \( K_1 \) be a \( \sigma_{b,w} \)-stable factor of \( K_E \). There are \( t_1, s_1 \in \mathbb{Q} \) such that 
\[
\text{ch}(K_1) = s_1 \text{ch}(\mathcal{O}_X) + t_1 \text{ch}(E),
\]
see for instance [Fey20a, Remark 2.5]. Taking \( \text{ch}_1 \) implies \( t_1 \in \mathbb{Z} \). Moving along the wall \( \ell_v \) towards the origin implies that 
\[
0 \leq \lim_{b \to 0} \text{Im}[Z_{(b,w)}(K_1)] = t_1 \leq \lim_{b \to 0} \text{Im}[Z_{(b,w)}(K_E[1])] = 1.
\]
Thus \( t_1 = 0 \) or \( 1 \), so the structure sheaf \( \mathcal{O}_X \) is a subobject or a quotient of \( K_E[1] \). By definition \( \text{Hom}(\mathcal{O}_X, K_E[1]) = 0 \) and since \( \text{Hom}(E, \mathcal{O}_X) = 0 \), we have \( \text{Hom}(K_E[1], \mathcal{O}_X) = 0 \), a contradiction. Therefore \( K_E[1] \) is \( \sigma_{b,w} \)-stable along the wall \( \ell_v \cap U \) for \( b < 0 \). Openness of stability implies that it is \( \sigma_{b=0,w} \)-stable for \( 0 < w \ll 1 \). Then Lemma 3.4 implies that \( K_E[1] \) is \( \sigma_{b=0,w} \)-stable for any \( w > 0 \).

Hence in both cases, \( K_E[1] \) is \( \sigma_{b,w} \)-stable for \( b > \mu(K_E) \) and \( w \gg 0 \). [MS17, Lemma 6.18] implies that \( \mathcal{H}^{-1}(K_E[1]) \) is a \( \mu_H \)-semistable torsion-free sheaf and \( \mathcal{H}^{0}(K_E[1]) \) is supported in dimension zero. Since \( \gcd(s, k) = 1 \), \( \mathcal{H}^{-1}(K_E[1]) \) is \( \mu_H \)-stable. If \( \mathcal{H}^{0}(K_E[1]) \) is non-zero, then
\[
-2 \leq v(\mathcal{H}^{-1}(K_E[1]))^2 = (s - kH, r + \text{ch}_3(\mathcal{H}^{0}(K_E[1])))^2
\]
\[
= v(E)^2 - 2s \text{ch}_3(\mathcal{H}^{0}(K_E[1]))
\]
\[
\leq 2r - 2s .
\]

This implies \( s < r \), thus (4) gives
\[
-1 + \frac{1}{r} + \frac{k^2}{2r}H^2 < r
\]
which is not possible as \( H^2 > 2r(r + 1) \). Hence \( K_E \) is a \( \mu_H \)-stable sheaf, and so is \( \sigma_{b,w} \)-stable for \( b > -\frac{k}{s} \) and \( w \gg 0 \) by [Bri08, Proposition 14.2]. By taking the reflexive hull and doing the same computations as in (7), one can easily check that \( K_E \) is indeed locally-free.

**Step 3.** The final step is to analyse walls for \( K_E \) and \( E(-H)[1] \) when \( b < \mu(K_E) \). Consider the vertical line \( b = b_{K_E}^\alpha \) as in Lemma 3.4. By Proposition 3.2 (d), the line \( \ell_{\alpha} \) intersects \( \Gamma \) at two points with \( b \)-values \( b_1^\alpha < b_2^\alpha \) such that
\[
b_1^\alpha < -\frac{k}{s} - \frac{1}{s} \leq b_{K_E}^\alpha \leq -\frac{k}{s} - \frac{1}{s(s - 1)} < b_2^\alpha.
\]
The first inequality follows from
\[
b_1^\alpha < b_1^\alpha \leq \frac{k - r}{r} + \frac{1}{r^2(r + 1)}
\]
and the point that for \( g \gg 0 \), we have
\[
k - 1 + \frac{1}{r^2(r + 1)} < -\frac{(k + 1)}{\frac{k^2}{2r}H^2 - 1}
\]
\[
< -\frac{k}{s} - \frac{1}{s}. 
\]
Thus Lemma 3.4 shows that by moving down along the vertical line $b = b_K^-$ we get $K_E$ is $\sigma_{b,w}$-stable for any $(b, w) \in \ell_\alpha \cap U$.

On the other hand, $E$ is a $\mu_H$-stable vector bundle, thus [Fey21, Lemma 3.3] implies that $E(-H)[1]$ is $\sigma_{b,w}$-stable for $b > \frac{k-r}{r}$ and $w \gg 0$. If $k \neq r-1$, consider the vertical line $b = b_E^{+(-H)}$; then by Proposition 3.2 (d),

$$b_0^b < b_1^b \leq \frac{k-r}{r} + \frac{1}{r(r-1)} \leq b^{+(-H)}_E \leq \frac{k-r}{r} + \frac{1}{r} \leq -\frac{1}{r}.$$ 

If $k = r-1$, then we consider the vertical line $b = -\frac{1}{r+1}$ and the same argument as in Lemma 3.4 shows that there is no wall for $E(-H)[1]$ crossing this vertical line. For $g \gg 0$, we have

$$\frac{1}{r+1} < \frac{-k+1}{s} < \frac{1}{s} < b_2^0,$$

Hence Proposition 3.2 implies the lines $\tilde{\ell}, \ell_{\nu(-H)}$ and $\ell_\alpha$ intersect the vertical line $b = b_{E(-H)}^+$ if $k < r-1$ or $b = -\frac{1}{r+1}$ in case $k = r-1$ at points inside $U$. Hence by Lemma 3.4, $E(-H)[1]$ is $\sigma_{b,w}$-stable along these three lines and so we get the following:

(a) $E$ and $E(-H)[1]$ are stable of the same $\nu_{b,w}$-slope for $(b, w) \in \ell \cap U$, therefore $\text{Hom}(E, E(-H)[1]) = 0$ as claimed in part (a).

(b) We know $\iota_* E|_C$ lies in the exact sequence

$$E \hookrightarrow \iota_* E|_C \twoheadrightarrow E(-H)[1]$$

in $\mathcal{A}(b = 0)$. Applying the same argument as in [Fey21, Corollary 4.3] implies that $E|_C$ is stable. Since $E(-H)[1]$ and $\mathcal{O}_X$ are $\sigma_{b,w}$-stable of the same $\nu_{b,w}$-slope for $(b, w) \in \ell_{\nu(-H)} \cap U$, we get $\text{Hom}(\mathcal{O}_X, E(-H)[1]) = 0$. Thus $h^0(C, E|_C) = h^0(X, E) = r + s$. This completes the proof of (b).

(c) We have shown that $K_E$ is a $\mu_H$-stable locally free sheaf. Moreover $E(-H)[1]$ and $K_E$ are $\sigma_{b,w}$-stable of the same slope with respect to $(b, w) \in \ell_\alpha \cap U$, so $\text{Hom}(K_E, E(-H)[1]) = 0$ as claimed in part (c).

Wall-crossing for the push-forward of vector bundles on curves. As before $C \hookrightarrow X$ is a curve in the linear system $|H|$. Let $F$ be a semistable vector bundle on $C$ of rank $r$ and degree $kH^2$. Then $\iota_* F$ is of Mukai vector

$$v(\iota_* F) = \left(0, rH, kH^2 - \frac{r}{2}H^2\right).$$

Semistability of $F$ implies that $\iota_* F$ is $H$-Gieseker-semistable, so it is $\sigma_{b,w}$-semistable for $w \gg 0$ [Bri08, Proposition 14.2]. The walls of instability for $\iota_* F$ are parallel lines of slope $H^2 \left(\frac{r}{k} - \frac{1}{2}\right)$. 
Proposition 3.7. Let \( \ell \) be a wall for \( \iota_* F \) which lies above or on the line \( \ell^* \) (see Definition 3.1). Let \( F_1 \rightarrow \iota_* F \rightarrow F_2 \) be a destabilising sequence along \( \ell \) with \( \nu_{b,w}(F_1) = \nu_{b,w}(\iota_* F) \) for \( (b,w) \in \ell \cap U \) and \( \nu_{b,w}^{-}(F_1) > \nu_{b,w}^{-}(\iota_* F) \) for \( (b,w^-) \in U \) below \( \ell \cap U \). Then

\begin{itemize}
\item[(a)] \( F_1 \) is a \( \mu_H \)-stable sheaf with \( \text{ch}_{\leq 1}(F_1) = (r,kH) \).
\item[(b)] \( \mathcal{H}^{-1}(F_2) \) is a \( \mu_H \)-stable sheaf with \( \text{ch}_{\leq 1}(\mathcal{H}^{-1}(F_2)) = (r, (k-r)H) \) and \( \mathcal{H}^0(F_2) \) is either zero or a torsion sheaf supported in dimension zero.
\item[(c)] The sequence \( F_1 \rightarrow \iota_* F \rightarrow F_2 \) is the HN filtration of \( \iota_* F \) with respect to \( \sigma_{b=0,w} \) for \( 0 < w \ll 1 \).
\item[(d)] The wall \( \ell \) lies below or on \( \tilde{\ell} \). If it coincides with \( \tilde{\ell} \), then \( \text{ch}_2(F_1) \) is maximum (equal to \( s-r \)) and \( F = F_1|_C \).
\end{itemize}

Proof. The first part of the argument is similar to [Fey20a, Proposition 4.2]; we add it for completeness. Taking cohomology of the destabilising sequence gives a long exact sequence of coherent sheaves

\[
0 \rightarrow \mathcal{H}^{-1}(F_1) \rightarrow 0 \rightarrow \mathcal{H}^{-1}(F_2) \rightarrow \mathcal{H}^0(F_1) \xrightarrow{d_0} \iota_* F \xrightarrow{d_1} \mathcal{H}^0(F_2) \rightarrow 0.
\]

Thus \( \mathcal{H}^{-1}(F_1) = 0 \) and \( \mathcal{H}^0(F_1) \cong F_1 \). Let \( \nu(F_1) = (r',k'H,s') \). If \( r' = 0 \), then \( F_1 \) and \( \iota_* F \) have the same \( \nu_{b,w} \)-slope with respect to any \( (b,w) \in U \), so \( F_1 \) will not destabilise \( \iota_* F \) below the wall. Hence \( r' > 0 \). The first step is to show that \( r' = r \).

Let \( T(F_1) \) be the maximal torsion subsheaf of \( F_1 \) and \( F_1/T(F_1) \) be its torsion-free part. Let \( \nu(T(F_1)) = (0, \tilde{r}H, \tilde{s}) \). Right-exactness of the underived pull-back \( \iota^* \) applied to the short exact sequence

\[
T(F_1) \rightarrow F_1 \rightarrow F_1/T(F_1)
\]

implies that

\[
\text{rank}(\iota^* F_1) \leq \text{rank}(\iota^* T(F_1)) + \text{rank}(\iota^* (F_1/T(F_1))).
\]

Take a point \( (b,w) \in \ell \cap U \). By definition of \( A(b) \), the sequence (12) is an exact triangle in \( A(b) \). Consider the composition of injections \( s : T(F_1) \rightarrow F_1 \rightarrow \iota_* F \) in \( A(b) \). Then the cokernel \( c(s) \) in \( A(b) \) is also a rank zero sheaf because if \( \mathcal{H}^{-1}(c(s)) \neq 0 \), the it must be a torsion-free sheaf which is not possible. Therefore, \( T(F_1) \) is a subsheaf of \( \iota_* F \). Since \( F \) is a vector bundle on an irreducible curve \( C \), we get \( \text{rank}(\iota^* T(F_1)) = \tilde{r} \). Thus (13) gives

\[
\text{rank}(\iota^* F_1) \leq \tilde{r} + r'.
\]

Let \( \nu(\mathcal{H}^0(F_2)) = (0,k''H,s'') \). The right-exactness of \( \iota^* \) implies

\[
\text{rank}(\iota^* F_1) \leq \text{rank}(\iota^* k d_1) = \text{rank}(\iota^* \text{im} d_0) \leq \text{rank}(\iota^* F_1) \leq \tilde{r} + r' \quad (13)
\]

Since \( \ell \) lies above or on \( \ell^* \), it intersects \( \Gamma \) at two points with \( b \)-values \( b_1 < b_2 \) such that \( b_1 \leq b_1^* \) and \( b_2^* \leq b_2 \). Note that the point \((0,0)\) lies below \( \ell \cap U \) by Proposition 3.2 (a). Therefore

\[
\mu^+(\mathcal{H}^{-1}(F_2)) \leq b_1^* \quad \text{and} \quad b_2^* \leq \mu^-(F_1). \quad (15)
\]

This implies

\[
\frac{r - k'' - \tilde{r}}{r'} = \mu_H(F_1/T(F_1)) - \mu_H(\mathcal{H}^{-1}(F_2)) \geq b_2^* - b_1^*
\]
and so

\[ r' \leq \frac{r - k'' - \tilde{r}'}{b_2' - b_1'} < (r - k'' - \tilde{r}) + 1. \tag{16} \]

Here the right hand side inequality comes from

\[ \frac{r - k'' - \tilde{r}}{r - k'' - \tilde{r} + 1} \leq \frac{r}{r + 1} < 1 - \frac{1}{r^2(r + 1)} \leq b_2' - b_1'. \tag{5} \]

Comparing (14) with (16) implies \( r' = r - k'' - \tilde{r} \). Then (15) implies

\[ b_2' \leq \mu_H(F_1/T(F_1)) = \frac{k' - \tilde{r}}{r - k'' - \tilde{r}} = \frac{\chi_1(H^{-1}(F_2))}{H^2} + \frac{r - k'' - \tilde{r}}{r - k'' - \tilde{r}} \leq b_1' + 1. \tag{17} \]

Moreover, Proposition 3.2 (a) and (5) give

\[ b_2' < \frac{k}{r} < b_1' + 1 \quad \text{and} \quad b_1' + 1 - b_2' < \frac{2}{r^2(r + 1)}. \tag{18} \]

Thus comparing (17) and (18) shows

\[ \left| \frac{k' - \tilde{r}}{r - k'' - \tilde{r}} - \frac{k}{r} \right| < \frac{1}{r(r + 1)} \]

which is possible only if we have equality of two fractions \( \frac{k' - \tilde{r}}{r - k'' - \tilde{r}} = \frac{k}{r} \). Since \( \gcd(r, k) = 1 \), we get \( k'' = \tilde{r} = 0 \), \( (r', k') = (r, k) \) and \( v(T(F_1)) = (0, 0, \tilde{s}) \). However \( T(F_1) \) cannot be a skyscraper sheaf because \( T(F_1) \) is a subsheaf of \( \iota_* F \) as explained above. Thus \( T(F_1) = 0 \) and \( F_1 \) is torsion-free. Then (15) and (5) give

\[ \frac{k}{r} - \frac{1}{r^2(r + 1)} \leq b_2' \leq \mu_H(F_1) \leq \mu_H(F_1) = \frac{k}{r} \]

which implies \( F_1 \) is \( \mu_H \)-stable. This completes the proof of (a).

Since \( k'' = 0 \), we have \( \chi_{\leq 1}(H^{-1}(F_2)) = (r, (k - r)H) \), so the first inequality in (15) implies

\[ \frac{k - r}{r} \leq \mu_H(H^{-1}(F_2)) \leq b_1' \leq \frac{k - r}{r} + \frac{1}{r^2(r + 1)}. \]

Therefore \( H^{-1}(F_2) \) is a \( \mu_H \)-stable sheaf as \( \gcd(r, k - r) = 1 \). This completes the proof of part (b).

Since \( F_1 \) is \( \mu_H \)-stable, we get

\[ v(F_1)^2 = k^2 H^2 - 2r(v + \chi_2(F_1)) \geq -2. \]

Thus \( \chi_2(F_1) \leq s - r \). We know \( \ell \) passes through \( \Pi(F_1) \), so if \( \chi_2(F_1) = s - r \), then \( \ell \) coincides with \( \ell \) and if \( \chi_2(F_1) < s - r \), it lies below \( \ell \).

We claim \( F_1 \) is \( \sigma_{b=0,w} \)-stable for any \( w > 0 \). Consider the vertical line \( b = b_{F_1} \) as in Lemma 3.4; it satisfies

\[ 0 \leq \frac{k}{r} - \frac{1}{r} < b_{F_1} \leq \frac{k}{r} - \frac{1}{r(r - 1)}. \tag{19} \]
Let \( \ell_{F_1} \) be the line passing through \( \Pi(F_1) \) and the origin. The line segment \( \ell_{F_1} \cap U \) lies above \( \ell_1 \cap U \) considered in Lemma 3.3, see Figure 4. Thus combining (19) with Lemma 3.3 implies that \( \ell_{F_1} \cap U \) intersects the vertical line \( b = b_{F_1}^- \) at a point in the closure \( \overline{U} \). We know the wall \( \ell \) lies above \( \ell_{F_1} \) as the origin point \((0,0)\) lies below \( \ell^* \) and so \( \ell \) by Proposition 3.2(a). Thus \( \ell \) intersects the vertical line \( b = b_{F_1}^- \) at a point \((b_{F_1}^-, w)\) inside \( U \), so semistability of \( F_1 \) along the wall and Lemma 3.4 imply that \( F_1 \) is \( \sigma_{b_{F_1}^-} \)-stable for any \( w > \Gamma(b_{F_1}^-) \). Hence the structure of walls (which all pass through \( \Pi(F_1) \)) implies that \( F_1 \) is \( \sigma_{b,w} \)-stable for any \((b, w)\) above \( \ell_{F_1} \) when \( b < \frac{k}{r} \), so in particular, it is \( \sigma_{b=0,w} \)-stable for any \( w > 0 \).

Similarly, we know

\[
\frac{k - r}{r} + \frac{1}{r(r-1)} \leq b_{F_2}^+ \leq \frac{k - r}{r} + \frac{1}{r} \leq 0.
\]

Thus by the same argument as for \( F_1 \), Lemma 3.3 implies that the line \( \ell_{F_2} \) passing through \( \Pi(F_2) \) and the origin intersects the vertical line \( b = b_{F_2}^+ \) at a point inside \( \overline{U} \), see Figure 4. Thus by Lemma 3.4, \( F_2 \) is also \( \sigma_{b=0,w} \)-stable for any \( w > 0 \). We know \( \nu_{b,w} \)-slope of \( F_1 \) is bigger than \( F_2 \) for \((b, w)\) below the wall \( \ell \), thus the sequence \( F_1 \to i_*F \to F_2 \) is the HN filtration of \( i_*F \) with respect to \( \sigma_{0,w} \) for \( 0 < w \ll 1 \) as claimed in part (c).

In case of equality \( \text{ch}_3(F_1) = s - r \), Proposition 3.6 implies that \( i_*F_1|_C \) is a stable sheaf. We know the non-zero morphism \( d_0 \) in the long exact sequence (11) factors via the morphism \( d_0' : i_*F_1|_C \to i_*F \). The sheaves \( i_*F_1|_C \) and \( i_*F \) are both stable and have the same Mukai vector, hence \( d_0' \) is an isomorphism. This completes the proof of part (d). \( \square \)

Proposition 3.7 only describes walls for class \( v(i_*F) \) which are above \( \ell^* \). Instead of classifying walls below \( \ell^* \), we jump over the Brill-Noether region (neighbourhood of \( \Pi(\mathcal{O}_X) \)) and find an upper bound for the number of global sections of stable vector bundles on the
curve $C$ of rank $r$ and degree $kH^2$. We first recall definition of the Harder-Narasimhan polygon.

**Definition 3.8.** Given a stability condition $\sigma_{(b,w)}$ and an object $E \in \mathcal{A}(b)$, the Harder-Narasimhan polygon of $E$ with respect to $(b,w)$ is the convex hull of the points $Z_{b,w}(E')$ for all subobjects $E' \subset E$ of $E$.

If the Harder-Narasimhan filtration of $E$ is the sequence

$$0 = E_0 \subset E_1 \subset ... \subset E_{n-1} \subset E_n = E,$$

then the points $\{p_i := Z_{b,w}(E_i)\}_{i=0}^n$ are the extremal points of the Harder-Narasimhan polygon of $E$ on the left side of the line segment $oZ_{b,w}(E)$, see Figure 5.

![Figure 5. HN polygon](image)

We want to consider the limit of HN polygon when $(b,w)$ is close to the origin. Define the function $\overline{Z}: K(X) \to \mathbb{C}$ as

$$\overline{Z}(E) := \lim_{w \to 0} Z_{b=0,w}(E) = -\text{ch}_2(E) + i \frac{\text{ch}_1(E) \cdot H}{H^2}.$$  

Take an object $E \in \mathcal{A}(b=0)$ which has no subobject $E' \subset E$ in $\mathcal{A}(0)$ with $\text{ch}_1(E') = 0$, i.e. $\nu_{b,w}^+(E) < +\infty$. [Fey20a, Proposition 3.4] implies that there exists $w^* > 0$ such that the Harder-Narasimhan filtration of $E$ is a fixed sequence

(20)

$$0 = E_0 \subset E_1 \subset ... \subset E_{n-1} \subset E_n = E,$$

with respect to all stability conditions $\sigma_{0,w}$ where $0 < w < w^*$. We define $P_E$ to be the polygon with extremal points $p_i = \overline{Z}(E_i)$ and sides $\overline{p_0p_n}$ and $\overline{p_ip_{i+1}}$ for $i \in [0, n-1]$. Since $P_E$ is the limit of HN polygon when $w \to 0$, it is also a convex polygon.

**Lemma 3.9.** Let $F$ be a rank $r$-semistable vector bundle on $C$ with degree $kH^2$ as before. The polygon $P_{\iota_* F}$ is contained in the triangle $\triangle oz_1z_2$ where

$$z_1 := \overline{Z}(v) = -s + r + ik \quad \text{and} \quad z_2 := \overline{Z}(\iota_* F) = H^2 \left(\frac{r}{2} - k\right) + ir.$$

If $P_{\iota_* F}$ coincides with $\triangle oz_1z_2$, then $F = E|_C$ for a vector bundle $E \in \mathcal{M}_X(v)$. 

Proof. Consider the HN filtration (20) for $E = \iota_* F$ with respect to $\sigma_{0,w}$ where $0 < w \ll 1$. Since $P_{\iota_* F}$ is a convex polygon, to prove the first claim we only need to show
\[
-\frac{\text{Re}[Z(E_1)]}{\text{Im}[Z(E_1)]} \leq -\frac{\text{Re}[z_1]}{\text{Im}[z_1]} \quad \text{and} \quad -\frac{\text{Re}[z_2 - z_1]}{\text{Im}[z_2 - z_1]} \leq -\frac{\text{Re}[Z(\iota_* F/E_{n-1})]}{\text{Im}[Z(\iota_* F/E_{n-1})]},
\]
i.e.
\[
\frac{\text{ch}_2(E_1)H^2}{\text{ch}_1(E_1)H} \leq \frac{\text{ch}_2(\psi(-H))H^2}{\text{ch}_1(\psi(-H))H} \quad \text{and} \quad \frac{\text{ch}_2(\psi(-H))H^2}{\text{ch}_1(\psi(-H))H} \leq \frac{\text{ch}_2(\iota_* F/E_{n-1})H^2}{\text{ch}_1(\iota_* F/E_{n-1})H}.
\]

We first show that $\text{ch}_0(E_1) > 0$ and $\text{ch}_0(\iota_* F/E_{n-1}) < 0$. Taking cohomology from the exact sequence $E_1 \hookrightarrow \iota_* F \twoheadrightarrow \iota_* F/E_1$ in $A(b = 0)$ implies that $E_1$ is a sheaf. If $E_1$ is of rank zero, then $\mathcal{H}^{-1}(\iota_* F/E_1)$ is zero as it is must be a torsion-free sheaf, so $F_1$ is a subsheaf of $\iota_* F$ of bigger $\nu_{0,w}$-slope which is not possible as $F$ is semistable\footnote{Recall that $\nu_{b,w}$-slope of any rank zero sheaf $E$ is equal to $\frac{\text{ch}_2(E)H^2}{\text{ch}_1(E)H^2}$ which is independent of $(b,w)$.}, thus $\text{ch}_0(E_1) > 0$. Similarly, the exact triangle $E_{n-1} \hookrightarrow \iota_* F \twoheadrightarrow \iota_* F/E_{n-1}$ implies $\text{ch}_0(E_{n-1}) > 0$ and so $\iota_* F/E_{n-1}$ is of negative rank. Since $E_1, \iota_* F/E_{n-1} \in A(b = 0)$ we get
\[
0 \leq \frac{1}{\text{ch}_0(E_1)} \text{Im}[Z_{0,w}(E_1)] = \mu(E_1) \quad \text{and} \quad 0 \leq \frac{1}{\text{ch}_0(\iota_* F/E_{n-1})} \text{Im}[Z_{0,w}(\iota_* F/E_{n-1})] \leq 0.
\]
Therefore (21) is equivalent to the claim that $\Pi(E_1)$ lies below or on the line $\ell_\psi$ and $\Pi(\iota_* F/E_{n-1})$ lies below or on $\ell_{\psi}(-H)$, see Figure 7.

If there exists a wall for $\iota_* F$ above or on $\ell^*$, Proposition 3.7 gives a complete description of the HN factors of $\iota_* F$ with respect to $\sigma_{0,w}$ for $0 < w \ll 1$ and so the claim follows. Thus we may assume there is no wall above or on $\ell^*$ for $\iota_* F$ and so $\iota_* F$ is $\sigma_{b,w}$-semistable for any $(b, w) \in U$ above $\ell^*$.

Consider the line $\ell$ parallel to $\ell^*$ passing through $\Pi(E_1)$. We know $\iota_* F$ and $E_1$ have the same $\nu_{b,w}$-slope for $(b, w) \in \ell \cap U$ and $E_1$ has bigger slope then $\iota_* F$ below the line as the slope function $\nu_{b,w}(E_1)$ is a decreasing function with respect to $w$. Thus $\iota_* F$ is $\sigma_{b,w}$-unstable below $\ell$, hence $\ell$ and so $\Pi(E_1)$ lie below $\ell^*$. On the other hand, $\sigma_{0,w}$-semistability

\[
\begin{align*}
\text{Im}[Z] &= \frac{H \text{ch}_1}{H^2} \\
\text{Re}[Z] &= -\text{ch}_2
\end{align*}
\]
of \(E_1\) implies that \(\Pi(E_1)\) does not lie above \(\Gamma\), so \(\Pi(E_1)\) lies below the line passing through the origin and \((b^*_2, \Gamma(b^*_2))\). Then Proposition 3.2(c) implies \(\Pi(E_1)\) lies below \(\ell_v\) as claimed. A similar argument shows that \(\Pi(\iota_*F/E_{n-1})\) lies below the line passing through the origin and \((b^*_1, \Gamma(b^*_1))\) and so it lies below \(\ell_v(-H)\) by Proposition 3.2(c). This completes the proof of (21) and so \(P_{\iota_*F}\) is contained in the triangle \(\triangle oz_1z_2\).

If HN polygon coincides with the triangle \(\triangle oz_1z_2\), then we have equality in both inequalities of (21), i.e. \(\Pi(E_1)\) lies on \(\ell_v\) and \(\Pi(\iota_*F/E_{n-1})\) lies on \(\ell_v(-H)\). Since \(\Pi(E_1)\) cannot lie inside \(U\), it lies above or on \(\ell^*\). Thus there exits a wall for \(\iota_*F\) above or on \(\ell^*\) and so Proposition 3.7 implies \(v(F_1) = v\) and \(F = E_1|_C\) as claimed. \(\square\)

Let \(p_i = \mathcal{Z}(E_i)\) be the extremal points of \(P_{\iota_*F}\) where \(E_i\)'s are the factor in the HN filtration of \(E = \iota_*F\) as in (20). Then [Fey20a, Proposition 3.4] implies that

\[
(22) \quad h^0(X, \iota_*F) \leq \frac{\chi(X, \iota_*F)}{2} + \frac{1}{2} \sum_{i=1}^{n} \|p_ip_{i-1}\|
\]

where \(\|\cdot\|\) is the non-standard norm defined in [Fey20a, Equation (3.2)], i.e. \(\|x + iy\| = \sqrt{x^2 + (2H^2 + 4)y^2}\).

**Proposition 3.10.** Let \(F\) be a rank \(r\)-semistable vector bundle on \(C\) with degree \(kH^2\) as before. If the polygon \(P_{\iota_*F}\) lies strictly inside \(\triangle oz_1z_2\), then \(h^0(C, F) < r + s\).

**Proof.** Since the extremal points of \(P_{\iota_*F}\) have integral coordinates, \(P_{\iota_*F}\) lies inside \(oz_0'z_1'z_2'z_2\) where \(z_0' = (-s + r)\frac{k-1}{k} + i(k-1)\), \(z_1' = -s + r + 1 + i\) \(k\) and

\[
z_2' = (-kH^2 + \frac{r}{2}H^2 + s - r)\frac{1}{r - k} - s + r + i(k + 1),
\]

see Figure 8.
Thus (23) gives
\[ 2 \varepsilon_{\text{out}} := k H^2 - \frac{r}{2} H^2 + Q_{\text{out}} - 2(r + s) \]
\[ = \sqrt{(s - r)^2 + (2H^2 + 4)k^2 + \left(-k H^2 + \frac{r}{2} H^2 + s - r\right)^2 + (2H^2 + 4)(r - k)^2} \]
\[ - (s + r) - (-k H^2 + \frac{r}{2} H^2 + s + r) \]
\[ = \frac{-4rs + (2H^2 + 4)k^2}{\sqrt{(s - r)^2 + (2H^2 + 4)k^2 + (s + r)}} + \frac{-4r(-k H^2 + \frac{r}{2} H^2 + s) + (2H^2 + 4)(r - k)^2}{\sqrt{(-k H^2 + \frac{r}{2} H^2 + s - r)^2 + (2H^2 + 4)(r - k)^2 + (-k H^2 + \frac{r}{2} H^2 + s + r)}}. \]

By our assumption (1), \(2k^2H^2 - 4rs + 4k^2 \geq -4 + 4k^2 \geq 0\) so
\[ (s + r)^2 \leq (s + r)^2 - 4rs + k^2(2H^2 + 4) = (s - r)^2 + k^2(2H^2 + 4). \]

Thus (23) \(\leq \frac{-4rs + (2H^2 + 4)k^2}{2(r + s)}.\)
The numerator of (24) is \(2k^2H^2 - 4rs + 4(r - k)^2 \geq -4 + 4(r - k)^2 \geq 0\) by (1), so
\[
(-kH^2 + \frac{r}{2}H^2 + s + r) \leq \left(-kH^2 + \frac{r}{2}H^2 + s + r\right)^2 + 2k^2H^2 - 4rs + 4(r - k)^2 = \left(-kH^2 + \frac{r}{2}H^2 + s - r\right)^2 + (2H^2 + 4)(r - k)^2.
\]
This implies (24) \(\leq \frac{2k^2H^2 - 4rs + 4(r - k)^2}{2(-kH^2 + \frac{r}{2}H^2 + s + r)}\), thus
\[
2\epsilon_{\text{out}} < \frac{k^2(H^2 + 2) - 2rs}{r + s} + \frac{k^2H^2 - 2rs + 2(r - k)^2}{-kH^2 + \frac{r}{2}H^2 + s + r} =: M_1
\]
When \(H^2 = 2g - 2 \to +\infty\), we know \(s \to \frac{k^2}{2r}H^2 \to +\infty\) by (4), so one gets \(M_1 \to 0\). On the other hand,
\[
(25) \quad Q_{\text{out}} - Q_{\text{in}} = \sqrt{\left(\frac{1}{k}(s - r)\right)^2 + (2H^2 + 4)} - \sqrt{\left(\frac{1}{k}(s - r) - 1\right)^2 + (2H^2 + 4)} + \sqrt{\left(\frac{1}{r - k}(-kH^2 + \frac{r}{2}H^2 + s - r)\right)^2 + (2H^2 + 4)} - \sqrt{\left(\frac{1}{r - k}(-kH^2 + \frac{r}{2}H^2 + s - r) - 1\right)^2 + (2H^2 + 4)}
\]
By (4), when \(H^2 \to +\infty\), we have \(\frac{1}{k}(s - r) > 1\) and
\[
\frac{1}{r - k}(-kH^2 + \frac{r}{2}H^2 + s - r) > \frac{1}{r - k}\left(-kH^2 + \frac{r}{2}H^2 + \frac{k^2}{2r}H^2 - r - 1 + \frac{1}{r}\right) = \frac{r - k}{2r}H^2 + \frac{1}{r - k}\left(-r - 1 + \frac{1}{r}\right)
\]
> 1.
Thus (25) gives
\[
Q_{\text{out}} - Q_{\text{in}} > \frac{s - r - \frac{1}{k}}{\sqrt{\frac{1}{k}(s - r)^2 + (2H^2 + 4)}} + \frac{-kH^2 + \frac{r}{2}H^2 + s - r - \frac{1}{2}}{\sqrt{\frac{1}{(r - k)^2}(-kH^2 + \frac{r}{2}H^2 + s - r)^2 + (2H^2 + 4)}}
\]
When \(s \to \frac{k^2}{2r}H^2 \to +\infty\), the right hand side goes to 2, thus for \(g \gg 0\), we obtain
\[
(26) \quad 2\epsilon_{\text{out}} < Q_{\text{out}} - Q_{\text{in}}
\]
as claimed. \(\square\)

**Proof of Theorem 1.1.** By Proposition 3.6(b), there is a well-defined map
\[
\Psi: M_X(v) \to BN_C(r, kH^2, r + s)
\]
\(E \mapsto E|_C\).
We know the exact sequence $E \to i_*E|_C \to E(-H)[1]$ in $\mathcal{A}(b = 0)$ is the HN filtration of $i_*E|_C$ below the wall $\tilde{\ell}$, so the uniqueness of HN factors implies $\Psi$ is injective. Combining Lemma 3.9 with Proposition 3.10 shows $\Psi$ is surjective. Thus Proposition 3.6(b) implies that any vector bundle $F$ in the Brill-Noether locus $\text{BN}_C(r, k(g - 2), r + s)$ is stable and $h^0(F) = r + s$. The Zariski tangent space to the Brill-Noether locus at the point $[F]$ is the kernel of the map

$$k_1: \text{Ext}^1(F, F) \to \text{Hom}(H^0(C, F), H^1(C, F)),$$

where any $f: F \to F[1] \in \text{Ext}^1(F, F) = \text{Hom}_C(F, F[1])$ goes to

$$k_1(f) = H^0(f): \text{Hom}_C(\mathcal{O}_C, F) \to \text{Hom}_C(\mathcal{O}_C, F[1]),$$

see [BS13, Proposition 4.3] for more details. Note that the proof in [BS13] is valid for any family of simple sheaves on a variety.

The moduli space $M_X(v)$ is a (non-empty) smooth quasi-projective variety, see for instance [Huy16, Chapter 10, Corollary 2.1 & Theorem 2.7]. Hence to prove Theorem 1.1, we only need to show the derivative of the restriction map

$$d\Psi: T[E]M_X(v) \to T[E|_C]\text{BN},$$

is surjective. It sends any $f: E \to E[1] \in \text{Hom}_X(E, E[1])$ to its restriction $i_*f: i_*E \to i_*E[1] \in \ker(k_1)$. By Proposition 3.6(c), we can apply the same argument as in the proof of [Fey20a, Theorem 1.2] to show surjectivity of $d\Psi$, hence $\Psi$ is an isomorphism. $\square$

4. LOCATION OF THE FIRST WALL

In this section, we prove Proposition 3.2 and Lemma 3.3 for $g \gg 0$. The first step is to control the position of the lines $\ell_v$ and $\ell_\alpha$.

**Proposition 4.1.** Fix $0 < \epsilon, \epsilon' < \frac{1}{2r}$. If $g \gg 0$, then

(a) The line $\ell_v$ passing through $\Pi(v)$ and the origin intersects $\Gamma$ at two points (except the origin) with $b$-values $b_v^1 < b_v^2$ such that

$$b_v^1 < -\frac{k}{s} + \frac{1}{s(s - 1)} \quad \text{and} \quad \frac{k}{r} - \epsilon < b_v^2.$$

(b) The line $\ell_\alpha$ passing through $\Pi(v(-H))$ and $\Pi(\alpha)$ intersects $\Gamma$ at two points with $b$-values $b_\alpha^1 < b_\alpha^2$ such that

$$b_\alpha^1 < \frac{k - r}{r} + \epsilon' \quad \text{and} \quad -\frac{k}{s} - \frac{1}{s(s - 1)} < b_\alpha^2.$$

**Proof.** The line $\ell_v$ which is of equation $w = \frac{s - r}{k}b$ connects the point $\Pi(v)$ which is outside or on $\Gamma$ to the point $(0, 0)$, so it intersects $\Gamma$ at two other points except the origin with $b$-values $b_1^v < b_2^v$. The equation of $\Gamma$ for $b \neq 0$ is $\Gamma(b) = (1 + \frac{H^2}{2})b^2 - 1$, thus

$$b_1^v, b_2^v = \frac{\frac{s - r}{k} \pm \sqrt{(\frac{s - r}{k})^2 + 2H^2 + 4}}{H^2 + 2}.$$
We claim if \( g \gg 0 \), then

\[
b_1^v < -\frac{k}{s} + \frac{1}{s(s-1)}
\]

i.e.

\[
\frac{s-r}{k} + \frac{(k(s-1)-1)(H^2+2)}{s(s-1)} < \sqrt{\left(\frac{s-r}{k}\right)^2 + 2H^2 + 4}.
\]

Taking the power 2 of both sides shows that we only need to prove

\[
(k(s-1)-1)\left(2\frac{s-r}{k}s(s-1) + (k(s-1)-1)(H^2+2)\right) < 2s^2(s-1)^2
\]

which is equivalent to

\[
(27) \quad \frac{1}{2}\left(k - \frac{r}{k}\right)^2 s - r + (H^2 + 2)\left(k^2(s-1) + \frac{1}{s-1} - 2k\right) < 2rs(s-1).
\]

When \( H^2 \to +\infty \), then \( s \to \frac{k^2}{2r}H^2 \to +\infty \) by (4), so the limit of the left hand side is \((H^2)^2\frac{k^3}{2r}(k - \frac{r}{k})\) but the limit of the right hand side is \((H^2)^2\frac{k^4}{2r}\), thus the claim follows.

The next step is to show for \( g \gg 0 \), we have

\[
\frac{s-r}{k} + \sqrt{(\frac{s-r}{k})^2 + 2H^2 + 4} > \frac{k}{r} - \epsilon.
\]

This holds if

\[
\frac{1}{2}\left(k - \frac{r}{k}\right)^2 s - r + (H^2 + 2)\left(k^2(s-1) + \frac{1}{s-1} - 2k\right) < 2rs(s-1).
\]

The limit of the right hand side when \( s \to \frac{k^2}{2r}H^2 \to +\infty \) is \( k - \frac{r}{k}(k - \frac{r}{k})H^2 \), so the above holds when \( g \gg 0 \). This completes the proof of part (a).

The line \( \ell_\alpha \) in part (b) is of equation \( w = \theta b + \beta - 1 \) for

\[
\theta := \frac{s^2 - r^2 - skH^2 + \frac{rs}{2}H^2}{s(k-r) + kr}, \quad \beta := \frac{r(k-r) + sk - k^2H^2 + \frac{k^2}{2}rH^2}{s(k-r) + kr}.
\]

When \( s \to \frac{k^2}{2r}H^2 \to +\infty \), we know

\[
(30) \quad \theta \to \frac{k-r}{2r}H^2 \quad \text{and} \quad \beta \to \frac{k-r}{k}.
\]

Hence if \( g \gg 0 \), we have

\[
(31) \quad \theta^2 + 2\beta(H^2 + 2) \geq 0.
\]

Therefore \( \ell_\alpha \) intersects the parabola \( w = b^2(H^2 + 1) - 1 \) at two points with \( b \)-values

\[
b_1^a, b_2^a = \frac{\theta \pm \sqrt{\theta^2 + 2\beta(H^2 + 2)}}{H^2 + 2}.
\]
We claim if \( g \gg 0 \), then
\[
-k - \frac{1}{s(s-1)} < b_2^g = \frac{\theta + \sqrt{\theta^2 + 2\beta(H^2 + 2)}}{H^2 + 2}
\]
which holds if
\[
\frac{H^2 + 2}{2} \left( \frac{k}{s} + \frac{1}{s(s-1)} \right)^2 < \beta - \theta \left( \frac{k}{s} + \frac{1}{s(s-1)} \right).
\]
The right hand side is equal to
\[
\beta - \theta \left( \frac{k}{s} + \frac{1}{s(s-1)} \right) = \frac{H^2}{s-1} \left( k - \frac{r}{2} \right) + r(k - r) - \frac{s}{s-1} + \frac{kr^2}{s} + \frac{r^2}{s(s-1)} \frac{s(k - r)}{kr}
\]
thus its limit when \( s \to \frac{\delta^2}{2r}H^2 \to +\infty \) is equal to \( \frac{\frac{2r^2}{kr} + \frac{2r(r-k)}{kr^2}}{H^2r^2} \). Since the limit of the left hand side of (33) is equal to \( \frac{2r^2}{H^2r^2} \), the claim (33) follows.

The other intersection point of \( \ell_\alpha \) with \( \Gamma \) satisfies
\[
b_1^g = \frac{\theta - \sqrt{\theta^2 + 2\beta(H^2 + 2)}}{H^2 + 2} < \frac{k - r}{r} + \epsilon'
\]
if and only if
\[
\theta - (H^2 + 2) \left( \frac{k - r}{r} + \epsilon' \right) < \sqrt{\theta^2 + 2\beta(H^2 + 2)}.
\]
Taking power 2 from both sides shows that (34) holds if
\[
\frac{H^2 + 2}{2} \left( \frac{r - k}{r} - \epsilon' \right)^2 < \beta - \theta \left( \frac{r - k}{r} - \epsilon' \right)
\]
which is satisfied by (30) for \( g \gg 0 \) and \( \epsilon' < \frac{1}{2\epsilon} \).

If \( g \gg 0 \), (30) implies
\[
\beta - 1 < 0
\]
where \( \beta \) is defined as in (29). Thus the origin point \((0,0)\) and so the line segment connecting \( \Pi(\nu(-H)) \) to the origin lies above \( \ell_\alpha \). Thus it intersects the curve \( \Gamma \) at a point with \( b \)-value \( b_1^\nu(-H) \) satisfying
\[
\frac{k - r}{r} \leq b_1^\nu(-H) \leq b_1^\alpha.
\]
Consider the line \( \tilde{\ell} \) passing through \( \Pi(\nu) \) and \( \Pi(\nu(-H)) \). It is of equation
\[
w = H^2 \left( \frac{k}{r} - \frac{1}{2} \right) b + \frac{s}{r} - H^2 \frac{k}{r} \left( \frac{k}{r} - \frac{1}{2} \right) - 1.
\]
When \( s \to \frac{k^2}{2r} H^2 \to +\infty \), we get

\[
(37) \quad \frac{s}{r} - H^2 \frac{k}{r} \left( \frac{k}{r} - \frac{1}{2} \right) - 1 > 0.
\]

This implies the origin \((0, 0)\) lies below \( \bar{\ell} \). Therefore the line segment connecting \( \Pi(v) \) to \( \Pi(v(-H)) \) (which is part of \( \bar{\ell} \)) lies above \( \ell_v \) and \( \ell_v(-H) \), so \( \bar{\ell} \) intersects \( \Gamma \) at two points with \( b \)-values \( \bar{b}_1 < \bar{b}_2 \) satisfying

\[
\frac{k-r}{r} \leq \bar{b}_1 \leq b_1^{\nu(-H)} \quad \text{and} \quad \bar{b}_2 \leq b_2 \leq \frac{k}{r}.
\]

Thus \((36)\) together with Proposition 4.1 implies

\[
(38) \quad \frac{k-r}{r} \leq \bar{b}_1 < \frac{k-r}{r} + \epsilon' \quad \text{and} \quad \frac{k-r}{r} - \epsilon < \bar{b}_2 \leq \frac{k}{r}.
\]

Thus if we assume \( H^2 \) is large enough, the line \( \bar{\ell} \) intersects \( \Gamma \) at two points which are arbitrary close to \( \Pi(v) \) and \( \Pi(v(-H)) \). Consider the line \( \ell^* \) as in Definition 3.1. We set

\[
(39) \quad \epsilon' = b_1^* - \frac{k-r}{r} \quad \text{and} \quad \epsilon = \frac{k}{r} - b_2^*.
\]

Then the line \( \ell^* \) lies below the parallel line \( \bar{\ell} \).

**Lemma 4.2.** The origin point \((0, 0)\) lies below \( \ell^* \) and

\[
(40) \quad \frac{k-r}{r} < b_1^* < 0 < b_2^* < \frac{k}{r}.
\]

**Proof.** The equation of \( \ell^* \) is given by \( w = H^2 \left( \frac{k}{r} - \frac{1}{2} \right) b + \alpha - 1 \) for some \( \alpha \in \mathbb{R} \). To prove the first claim we need to show \( \alpha > 1 \). We know the intersection points of \( \ell^* \) with \( \Gamma \) satisfies

\[
b_2^* - b_1^* = \frac{2\sqrt{(H^2)^2 \left( \frac{k}{r} - \frac{1}{2} \right)^2 + 2\alpha(H^2 + 2)}}{H^2 + 2} \geq \frac{k-r}{r} - \frac{1}{r^2(r+1)} - \frac{k-r}{r} - \frac{1}{r^2(r+1)}
\]

This implies for \( g \gg 0 \), we have

\[
(41) \quad 2\alpha \geq (H^2 + 2) \left( \frac{1}{2} - \frac{1}{r^2(r+1)} \right)^2 - \frac{(H^2)^2}{H^2 + 2} \left( \frac{k}{r} - \frac{1}{2} \right)^2 > 2.
\]

The second claim \((40)\) follows from \((38)\) and the point that by the choice of \( \epsilon \) and \( \epsilon' \) as in \((39)\), we know \( \bar{\ell} \) lies above \( \ell^* \). \( \square \)

The above arguments completes the proof of Proposition 3.2. The final step is to prove Lemma 3.3.

**Proof of Lemma 3.3.** The equation of the line \( \ell^* \) is given by \( w = H^2 \left( \frac{k}{r} - \frac{1}{2} \right) b + \alpha - 1 \) where \( \alpha = (b_2^*)^2 \left( \frac{H^2}{2} + 1 \right) - H^2 \left( \frac{k}{r} - \frac{1}{2} \right) b_2^* \) and equation of the line \( \ell_1 \) is given by \( w = \frac{k}{r} w_1 b \). We
know $b_2^* = \frac{k}{r} - \epsilon$, so
\[
\alpha = \left( \frac{H^2}{2} + 1 \right) \left( \frac{k}{r} - \epsilon \right)^2 - H^2 \left( \frac{k}{r} - \frac{1}{2} \right) \left( \frac{k}{r} - \epsilon \right).
\]
Thus when $H^2 \to +\infty$, we know
\[
\alpha \to \frac{H^2}{2} \left( \frac{k}{r} - \epsilon \right) \left( -\frac{k}{r} - \epsilon + 1 \right)
\]
and so
\[
(42) \quad \frac{r}{k} w_1 \to H^2 \left( \frac{k}{r} - \frac{1}{2} \right) + \frac{H^2 r}{2k} \left( \frac{k}{r} - \epsilon \right) \left( -\frac{k}{r} - \epsilon + 1 \right).
\]
The line $\ell_1$ intersects the vertical line $b = \frac{k}{r} - \frac{1}{r(r-1)}$ at a point with $w = \frac{r}{k}w_1(\frac{k}{r} - \frac{1}{r(r-1)})$. We claim if $g \gg 0$, this point lies in $U$, i.e.
\[
(43) \quad \frac{r}{k} w_1 > \Gamma \left( \frac{k}{r} - \frac{1}{r(r-1)} \right) \frac{1}{\frac{k}{r} - \frac{1}{r(r-1)}}.
\]
The limit of the right hand side when $H^2 \to +\infty$ is $\frac{H^2}{2} (\frac{k}{r} - \frac{1}{r(r-1)})$, so (42) shows that (43) holds if
\[
\left( \frac{k}{r} - \epsilon \right) \left( -\frac{k}{r} - \epsilon + 1 \right) > \frac{k}{r} \left( 1 - \frac{k}{r} - \frac{1}{r(r-1)} \right).
\]
This is equivalent to
\[
\epsilon^2 - \epsilon^2 < \frac{k}{r^2(r-1)}
\]
which is satisfied by our assumption on $\epsilon = \frac{k}{r} - b_2^*$ in Definition 3.1.

Similarly, we have
\[
w_2 = H^2 \frac{k-r}{r} \left( \frac{k}{r} - \frac{1}{2} \right) + \alpha - 1
\]
We require
\[
(44) \quad \frac{r}{k-r} w_2 < \Gamma \left( \frac{k-r}{r} + \frac{1}{r(r-1)} \right) \frac{\frac{k-r}{r} + \frac{1}{r(r-1)}}{\frac{k-r}{r} + \frac{1}{r(r-1)}}.
\]
The limit of the left hand side is
\[
\lim_{H^2 \to +\infty} \frac{r}{k-r} w_2 = H^2 \left( \frac{k}{r} - \frac{1}{2} \right) + \frac{H^2 r}{2(k-r)} \left( \frac{k}{r} - \epsilon \right) \left( -\frac{k}{r} - \epsilon + 1 \right)
\]
and the limit of the right hand side is $\frac{H^2}{2} \left( \frac{k-r}{r} + \frac{1}{r(r-1)} \right)$. Thus (44) holds for $g \gg 0$ if
\[
\frac{r}{k-r} \left( \frac{k}{r} - \epsilon \right) \left( -\frac{k}{r} - \epsilon + 1 \right) < \frac{1}{r(r-1)} - \frac{k}{r}.
\]
This is equivalent to
\[ \epsilon - \epsilon^2 < \frac{r - k}{r^2(r - 1)} \]
which holds again by our assumptions on \( \epsilon = \frac{k}{r} - b^*_2 \).

\[
\square
\]

**Remark 4.3.** The genus \( g = \frac{H^2}{2} + 1 \) is large enough for Theorem 1.1 if \( H^2 > 2r(r+1) \) and inequalities (9), (10), (26), (27), (31), (32), (35), (37), (41), (43), (44), and for the chosen values of \( \epsilon, \epsilon' \) in (39), inequalities (28) and (34) are satisfied.

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