Links and Quantum Entanglement

Allan I. Solomon
Department of Physics and Astronomy, Open University, Milton Keynes, MK7 6AA, U.K.
E-mail: a.i.solomon@open.ac.uk
and
LPTMC, University of Paris VI, 75252, Paris, France

Choon-Lin Ho
Department of Physics, Tamkang University, Tamsui 251, Taiwan, R.O.C.
and
Department of Physics, and Center for Quantum Technologies, National University of Singapore, 117543 Singapore
E-mail: hcl@mail.tku.edu.tw

We discuss the analogy between topological entanglement and quantum entanglement, particularly for tripartite quantum systems. We illustrate our approach by first discussing two clearly (topologically) inequivalent systems of three-ring links: The Borromean rings, in which the removal of any one link leaves the remaining two non-linked (or, by analogy, non-entangled); and an inequivalent system (which we call the NUS link) for which the removal of any one link leaves the remaining two linked (or, entangled in our analogy). We introduce unitary representations for the appropriate Braid Group ($B_3$) which produce the related quantum entangled systems. We finally remark that these two quantum systems, which clearly possess inequivalent entanglement properties, are locally unitarily equivalent.

Keywords: quantum entanglement, braid groups, topological links

1. Introduction: The Borromean Rings and the NUS Link

In this note we shall explore the analogy between topological links and the quantum entanglement of tripartite systems. In the figures Fig. 1(a) and Fig. 1(b), we give examples of two different three-ring links.

The first, Fig. 1(a), represents the celebrated Borromean rings. This link has the property that removing any ring leaves the remaining two rings unlinked (non-entangled). The second, Fig. 1(b), which we call for brevity the NUS link as it is part of the logo of the National University of Singapore, has the converse property; removing any ring still leaves the two remaining linked (entangled).

These two links recall the following tripartite quantum states: The Greenberger-
Horne-Zeilinger (GHZ) state, which is simply a tripartite extension of the bipartite Bell state \((1/\sqrt{2})(|0, 0, \rangle + |1, 1, \rangle)\).

\[ |\text{GHZ} \rangle = (1/\sqrt{2})(|0, 0, 0, \rangle + |1, 1, 1, \rangle), \quad (1) \]

and

\[ |\phi \rangle = (1/2)(|0, 0, 0, \rangle + |0, 1, 1, \rangle + |1, 0, 1, \rangle + |1, 1, 0, \rangle). \quad (2) \]

In the first case, measuring any subspace state as \(|0\rangle\) (resp. \(|1\rangle\)) leads to the non-entangled state \(|0, 0, 0, \rangle\) (resp. \(|1, 1, 1, \rangle\)); while in the second case a similar determination always leads to a (maximally) entangled bipartite state (Bell state).

The mathematical representation of links is made via Braid Groups, introduced by Artin. To pursue the quantum entanglement analogy further, we first discuss braid groups, with an introductory reminder of a presentation of the closely-related symmetric group. Then, in order to apply these ideas in quantum theory, we discuss their unitary representations, which we take to act on the qubit spaces.

2. Braid Groups and Links

2.1. Symmetric Group

The symmetric group \(S_n\) (sometimes called the permutation group) is defined as the set of \(n!\) permutations on \(n\) distinct objects, combining according to the rule illustrated by

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
3 & 1 & 2 & 4 \\
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 3 & 2 & 4 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 1 & 3 & 4 \\
\end{pmatrix}
\]  

for the case of \(S_4\). A diagrammatic representation of the resultant permutation is found in Figure 2(a). The symmetric group \(S_n\) has a presentation in terms of \(n-1\) adjacent transpositions \(\{s_i \mid i = 1 \ldots n-1\}\) where \(s_i\) sends the \(i\) to \(i+1\) and \(i+1\) to \(i\).

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*The right-hand side of Eq. (3) is an adjacent transposition.*

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(a) (b)

Fig. 1. Two three-ring links. (a) Borromean Rings (b) NUS Link
This rather mysterious presentation is:

\[ s_i s_j = s_j s_i \quad |i - j| > 1 \quad (4) \]

\[ s_i s_i = I \quad (5) \]

\[ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (6) \]

where Eq.(6) plays an important role in the generalization to the Braid group, in which context it is known as the *braiding relation* or the *Yang-Baxter condition*.

### 2.2. Braid group

The braid group is like the symmetric group, but in three dimensions, so one must imagine the arrows joining the elements of a permuted set of points to go “over” or “under” each other. Intuitively, each element of the braid group \( B_n \) is one way of joining \( n \) points to another \( n \) points by strings. (For an expanded version of this intuitive definition see Reference\[1\].)

The braid group \( B_n \) has a presentation in terms of \( n - 1 \) generators \( \sigma_i \). This (defining) presentation is:

\[ \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i - j| > 1 \quad (7) \]

\[ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad (8) \]

Note that the constraint Eq.(5) is absent; this absence leads to all the Braid groups being of infinite order. Eq.(8) is known as the *braiding relation* or the *Yang-Baxter condition*. A diagrammatic representation of the elements \( \sigma_1 \) and \( \sigma_1^{-1} \), as well as the second generator \( \sigma_2 \), of \( B_3 \) is given in Figure 2(b). This group is the main example that we discuss in this note, although for simplicity and illustration we start by discussing the group \( B_2 \), which has only one generator, and no braiding condition to satisfy; it is isomorphic to the infinite cyclic group, equivalently \( \mathbb{Z} \), the set of integers under addition.

### 2.3. Knots and Links

Of particular interest to us is the fact that, as shown by Alexander, all knots and links may be obtained from elements of a braid group by the simple expedient of joining the the “dots”; that is, join 1 to 1, 2 to 2, and so on.

![Fig. 2. Elements of \( S_4 \) and \( B_3 \). (a) A transposition in \( S_4 \). (b) Elements \( \sigma_1, \sigma_1^{-1}, \sigma_2 \) of \( B_3 \).](image-url)
For the braid group $B_2$ with one generator $\sigma_1$, we can see that performing the action using the element $\sigma_1^2$ gives the Hopf Link, as in Figure 3.

For the braid group $B_3$ with two generators $\sigma_1$ and $\sigma_2$, we can see that performing this action with the braid element

$$\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$$

produces the Borromean rings, as in Figure 4(a).

On the other hand, the braid element

$$\sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2$$

(10)

corresponds to the NUS link, as in Figure 4(b).
3. Unitary Representations of braid groups and entanglement

In order to relate the action of the braid group to unitary transformations on quantum systems, we adopt the following procedure:

(i) we associate each initial point of the braid group with a qubit (e.g. for $B_3$ there are 3 initial points and therefore we may represent unitary action on a three-qubit system);
(ii) for a braid word of the form $g^n$ we shall assume that the quantum entanglement is generated by the unitary representative $\hat{g}$;
(iii) to simulate the closure of the action of a braid word, say $g^n$, to form a link, the unitary matrix $\hat{g}^n$ must equal $I$ (up to a phase factor).

A generic unitary representation of the braid group which satisfies the relation Eq.(7) can in principle be obtained from the following:

$$\hat{\sigma}_i = I \times \cdots \times U \times I \cdots \times I$$ (11)

where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $U$ is a $4 \times 4$ unitary matrix occupying the $(i, i+1)$ position in the product. Of course it is more difficult to satisfy Eq.(8), the braiding, or Yang-Baxter, relation. We describe representations for $B_2$ and $B_3$ in the following.

3.1. The Hopf link and entanglement

In a sense finding a unitary representation for $B_2$ is a trivial exercise, as in this case there are effectively no relations on the single generator $\sigma_1$. Thus any unitary matrix will do. For our purpose we require a $4 \times 4$ unitary matrix - since it is acting on the two-qubit space. We define a unitary transformation matrix as follows:

$$\hat{\sigma}_1 \equiv \frac{e^{i\theta}}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \quad (\theta/\pi \text{ irrational}).$$ (12)

The braid word word corresponding to the Hopf link is $\sigma_1^2$ so following the procedure as in (3(ii) outlined above, our choice of unitary representative $\hat{\sigma}_1$ is the generator of entanglement, and produces a maximally entangled (Bell) state from a (generic) non-entangled state,

$$\hat{\sigma}_1 |0,0\rangle = \frac{\exp(i\theta)}{\sqrt{2}} (|0,0\rangle + |1,1\rangle).$$ (13)

Note that $\hat{\sigma}_1^2 = e^{2i\theta} I$, satisfying condition $\exists$iii).

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$^b$The multiplicative phase factor is necessary to ensure a genuine representation of $B_2$, as in its absence the representation would be non-faithful, and finite dimensional ($\mathbb{Z}_2$).
3.2. Unitary representations for $B_3$

3.2.1. The NUS link and entanglement

Using the matrix $U$ of Reference 5 (where it is defined however without the phase factor) we define

$$U = \frac{e^{i\theta}}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(14)

where $\theta/\pi$ is irrational but otherwise arbitrary, as above. The representation for $B_3$ is

$$\hat{\sigma}_1 = U \times I, \quad \hat{\sigma}_2 = I \times U.$$  

(15)

One may verify that the braiding relation Eq.(8) is satisfied. As in Eq.(10), the braid word $(\hat{\sigma}_1 \hat{\sigma}_2)^3$ produces the NUS link. Following the recipe above, we note that $(\hat{\sigma}_1 \hat{\sigma}_2)^3$ is indeed the $8 \times 8$ unit matrix (up to a non-vanishing phase factor); and the generator of entanglement for this link $\hat{\sigma}_1 \hat{\sigma}_2$ produces the state $|\phi\rangle$ of Eq.(2) (up to the phase factor $e^{2i\theta}$)

$$\hat{\sigma}_1 \hat{\sigma}_2 |0, 0, 0, 0\rangle = \exp(2i\theta)|\phi\rangle.$$  

(16)

3.2.2. Entanglement and the Borromean rings

We use a different representation for the Borromean rings in order to obtain the GHZ state directly. Following the procedure detailed in Reference 6 we use the Jones representation:

$$\hat{\sigma}_i = Ah_i + A^{-1}I,$$

$$\hat{\sigma}_i^{-1} = A^{-1}h_i + AI.$$  

(17)

We choose $A = \exp(3\pi i/8)$, and the matrices $h_1$ and $h_2$ as follows:

$$h_1 = \sqrt{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$  

(18)

\footnote{In what follows we omit the explicit irrational phase factor needed to ensure the faithfulness of the representation.}
and

\[
    h_2 = \frac{1}{\sqrt{2}} \begin{pmatrix}
        1 & 0 & 0 & 0 & 0 & 0 & -1 \\
        0 & 1 & 0 & 0 & 0 & -1 & 0 \\
        0 & 0 & 1 & 0 & 0 & 1 & 0 \\
        0 & 0 & 0 & 1 & -1 & 0 & 0 \\
        0 & 0 & 0 & -1 & 1 & 0 & 0 \\
        0 & 0 & -1 & 0 & 0 & 1 & 0 \\
        0 & -1 & 0 & 0 & 0 & 0 & 1 \\
        -1 & 0 & 0 & 0 & 0 & 0 & 1
    \end{pmatrix}.
\]  

Then it may be verified that \(\hat{\sigma}_1\) and \(\hat{\sigma}_2\) satisfy Eq.(8). The Borromean link is defined by the braid word given in Eq.(9), and additionally the criterion of 3(iii) is satisfied, since \((\hat{\sigma}_1\hat{\sigma}_2^{-1})^3\) equals the identity up to a phase factor.

Applying the braid word entanglement generator, in this case \(\hat{\sigma}_1\hat{\sigma}_2^{-1}\), to the fiducial ground state \(|0, 0, 0\rangle\), we obtain the GHZ state

\[
    \hat{\sigma}_1\hat{\sigma}_2^{-1}|0, 0, 0\rangle = \frac{1 + i}{2}(|0, 0, 0\rangle + |1, 1, 1\rangle).
\]  

4. Conclusions: Local Unitary Equivalence

This note has emphasized the analogy between topological entanglement in the form of links, and quantum entanglement.

We introduced a recipe whereby we could relate a topological link to an appropriate entangled quantum state, via a unitary representation of the braid word producing the link. For the two cases of links produced by \(B_3\), the Borromean rings link and the one we dubbed the NUS link, we used two different unitary representations of \(B_3\). It should come as no surprise that different unitary representations produce different pictures of entanglement, as quantum entanglement is not invariant under unitary transformations. And indeed, from our description of the Borromean rings link and the NUS link in the Introduction, we can see that the topological entanglement properties of these two links are quite different. Similarly, from the discussion following Eqs.(1) and (2) we also see that the quantum entanglement properties of the states \(|GHZ\rangle\) and \(|\phi\rangle\) are similarly distinct.

Further, it would appear that the entanglement properties in the 3-qubit case are not invariant under local unitary transformations either. It has been pointed out\(^{[4]}\) that in fact the two states \(|GHZ\rangle\) and \(|\phi\rangle\) are locally unitarily equivalent, since by use of the local transformation \(V = v \otimes v \otimes v\) where \(v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\),

\[
    V \frac{1}{\sqrt{2}}(|0, 0, 0\rangle + |1, 1, 1\rangle) = -\frac{1}{2}(|0, 0, 0\rangle + |0, 1, 1\rangle + |1, 0, 1\rangle + |1, 1, 0\rangle).
\]

\(^{[4]}\)This analogy has also been remarked upon by, among others, Kauffman and Lomonaco\(^{[5]}\) and one of the authors.\(^{[6]}\)
Thus, in the case of *tripartite* states, at least, local unitary equivalence does not preserve the entanglement properties.

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   (This note presents in preliminary form some of the ideas of this paper. However, due to an algebraic error the generator of entanglement for the Borromean Rings system was taken to be \((\hat{\sigma}_1\hat{\sigma}_2^{-1})^3\) instead of \(\hat{\sigma}_1\hat{\sigma}_2^{-1}\).)