Finite BRST transformation and constrained systems

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We establish the connection between the generating functional for the first-class theories and the generating functional for the second-class theories using the finite field dependent BRST (FFBRST) transformation. We show this connection with the help of explicit calculations in two different models. The generating functional of Proca model is obtained from the generating functional of Stueckelberg theory for massive spin 1 vector field using FFBRST transformation. In the other example we relate the generating functionals for gauge invariant and gauge variant theory for self-dual chiral boson.

I. INTRODUCTION

The Dirac’s method for constraints analysis has been used in great extent in the Hamiltonian formalism to quantize the system with second-class constraints [1]. However, the Dirac brackets which are the main ingredient in such formulation are generally field dependent and nonlocal. Moreover, these lead to a serious ordering problem between the field operators. These are unfavorable circumstances for finding canonically conjugate dynamical variables. On the other hand, the quantization of the system with first-class constraints [2] has been well appreciated in a gauge invariant manner preserving BRST symmetry [3–5]. The system with second-class constraints can be quantized by converting these to a first-class theory in an extended phase space [2–6]. This procedure has been considered extensively by Batalin, Fradkin and Tyutin [7, 8] and has been applied to various models [9–12].

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The Proca model in (1+3) dimensions (4D) for massive spin 1 vector field is a system with second-class constraint as the gauge symmetry is broken by the mass term of the theory\cite{13-17}. However, Stueckelberg converted this theory to a first-class theory by introducing a scalar field\cite{18-20}. Such a gauge invariant description for massive spin 1 field has many application in gauge field theories as well as in string theories\cite{21-23}.

The gauge variant model for single self-dual chiral boson in (1+1) dimensions (2D) is another well known example of second-class theory\cite{24-29}. This model can be made gauge invariant by adding Wess-Zumino (WZ) term and has been studied using Batalin, Fradkin and Vilkovisky (BFV) formulation\cite{30, 31}. Such a model is very useful in the study of certain string theoretic models\cite{32} and plays a crucial role in the study of quantum Hall effect\cite{33}.

In this work we show that the theories with first-class constraints can be related to the theories with second-class constraint through FFBRST transformation introduced by Joglekar and Mandal\cite{34}. FFBRST transformation is a generalized version of usual BRST transformation where the anticommuting parameter is finite and field dependent. Such generalized BRST transformation is nilpotent and also the symmetry of the effective action. However being finite in nature such a transformation does not leave the path integral measure in the expression of generating functional invariant. Usefulness of such transformation is established through its wide applications in different field theoretic models\cite{34-42}. The finite field dependent anti-BRST (FF-anti-BRST) transformation also plays the exactly similar role as FFBRST transformation\cite{37, 39}. Here we show that the generating functional of Stueckelberg theory for massive spin 1 vector field can be related to the generating functional of Proca model for the same theory through FFBRST and FF-anti-BRST transformations. Similar relationship is also established between the gauge invariant and gauge variant models for single self-dual chiral boson through FF-BRST and FF-anti-BRST formalism. The complicacy arises due to the nonlocal and field dependent Dirac brackets in the quantization of second-class theories can thus be avoided by using FFBRST/FF-anti-BRST transformations which relate the Green’s functions of second-class theories to the first-class theories.

Here is the plan of the paper. In Sec.II we outline the idea of the FFBRST transformation. We review briefly the theories endowed with second-class and first-class constraints
for massive spin 1 field and self-dual chiral boson in Sec. III. The connection between first-class and second-class theories through FFBRST and FF-anti-BRST transformations is established in Sec. IV and Sec. V respectively. Sec. VI is reserved for discussion and conclusions.

II. FFBRST TRANSFORMATION

In this section, we start with the properties of the usual BRST transformations which do not depend on whether the infinitesimal BRST parameter $\Lambda$ is (i) finite or infinitesimal, (ii) field dependent or not, as long as it is anticommuting and space-time independent. These observations give us freedom to generalize the BRST transformations by making the parameter $\Lambda$, finite and field dependent without affecting its properties. To generalize the BRST transformations we start with the making of infinitesimal BRST parameter field dependent. For the sake of convenience we further introduce a numerical parameter $\kappa$ ($0 \leq \kappa \leq 1$) and make all the fields, $\phi(x, \kappa)$, $\kappa$ dependent in such a manner that $\phi(x, \kappa = 0) = \phi(x)$ and $\phi(x, \kappa = 1) = \phi'(x)$, the transformed field.

The usual infinitesimal BRST transformations, thus can be written generically as

$$d\phi(x, \kappa) = \delta_b[\phi(x, \kappa)]\Theta'[\phi(x, \kappa)]d\kappa,$$

where $\Theta'[\phi(x, \kappa)]d\kappa$ is the infinitesimal but field dependent parameter. The finite field dependent parameter then can be constructed by integrating such infinitesimal transformations from $\kappa = 0$ to $\kappa = 1$, such that

$$\phi'(x) = \phi(x) + \delta_b[\phi(x)]\Theta[\phi(x)],$$

where

$$\Theta[\phi(x)] = \int_0^1 d\kappa'\Theta'[\phi(x, \kappa')].$$

Such FFBRST transformations are nilpotent and symmetry of the effective action. However, the path integral measure in the definition of generating functional is not invariant under such transformations as the BRST parameter is finite. The Jacobian of the path integral measure for such transformations can be evaluated for some particular choice of the finite field dependent parameter, $\Theta[\phi(x)]$, as

$$D\phi = J(\kappa)\ D\phi(\kappa) = J(\kappa + d\kappa)\ D\phi(\kappa + d\kappa).$$

(2.4)
Now the transformation from $\phi(\kappa)$ to $\phi(\kappa + d\kappa)$ is infinitesimal in nature. Thus

$$\frac{J(\kappa)}{J(\kappa + d\kappa)} = \Sigma\phi \pm \frac{\delta\phi(x, \kappa)}{\delta\phi(x, \kappa + d\kappa)},$$

(2.5)

where $\Sigma\phi$ sums over all fields involved in the measure and $\pm$ sign refers to whether $\phi$ is a bosonic or a fermionic field. Using the above expression we calculate the infinitesimal change in the $J(\kappa)$ as

$$\frac{1}{J} \frac{dJ}{d\kappa} = -\int d^4x \left[ \pm \delta_b\phi(x, \kappa) \frac{\partial\Theta[\phi(x, \kappa)]}{\partial\phi(x, \kappa)} \right],$$

(2.6)

The Jacobian, $J(\kappa)$ can be replaced (within the functional integral) as

$$J(\kappa) \to \exp[iS_1[\phi(x, \kappa)]],$$

(2.7)

iff the following condition is satisfied

$$\int D\phi(x) \left[ \frac{1}{J} \frac{dJ}{d\kappa} - i \frac{dS_1[\phi(x, \kappa)]}{d\kappa} \right] \exp[i(S_{eff} + S_1)] = 0,$$

(2.8)

where $S_1[\phi]$ is local functional of fields.

By choosing appropriate $\Theta$, we can make the transformed generating functional $Z' \equiv \int D\phi \ e^{iS_{eff}+S_1}$ under FFBRST transformation as another effective theory.

**III. THE THEORIES WITH CONSTRAINTS: EXAMPLES**

In this section, we briefly outline the essential features of second-class and first-class theories. In particular we discuss the Proca theory for massive spin 1 vector field theory and gauge variant theory for self-dual chiral boson, which are second-class theories. Corresponding first-class theories i.e. the Stueckelberg theory for massive spin 1 vector fields and gauge invariant theory for self-dual chiral boson are also outlined in this section.

**A. Theory for massive spin 1 vector field**

1. **Proca model**

We start with the action for a massive charge neutral spin 1 vector field $A_\mu$ in 4D

$$S_P = \int d^4x \ L_P,$$

(3.1)
where the Lagrangian density is given as
\[ \mathcal{L}_P = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{M^2}{2} A_{\mu} A^{\mu}. \] (3.2)

The field strength tensor is defined as \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \). We choose the convention \( g^{\mu\nu} = \text{diagonal (1, -1, -1, -1)} \) in this case, where \( \mu, \nu = 0, 1, 2, 3 \). The canonically conjugate momenta for \( A_{\mu} \) field is
\[ \Pi_{\mu} = \frac{\partial \mathcal{L}}{\partial \dot{A}_{\mu}} = F_{0\mu}. \] (3.3)

This implies that the primary constraint of the theory is
\[ \Omega_1 \equiv \Pi^0 = 0. \] (3.4)

The Hamiltonian density of the theory is given by
\[ \mathcal{H} = \Pi_{\mu} \dot{A}^{\mu} - \mathcal{L} = \Pi_i \partial^i A^0 - \frac{1}{2} \Pi_i^2 + \frac{1}{2} F_{ij} F^{ij} - \frac{1}{2} M^2 A_{\mu} A^{\mu}. \] (3.5)

The time evolution for the dynamical variable \( \Pi^0 \) can be written as
\[ \dot{\Pi}^0 = [\Pi^0, H], \] (3.6)
where the Hamiltonian \( H = \int d^3 x \mathcal{H} \). The constraints of the theory should be invariant under time evolution and using (3.6) we obtain the secondary constraint
\[ \Omega_2 \equiv \partial_i \Pi^i + M^2 A^0 = 0. \] (3.7)

Constraint \( \Omega_2 \) contains \( A^0 \) which implies that \( [\Omega_1, \Omega_2] \neq 0 \). Hence, the Proca theory for massive spin 1 vector field is endowed with second-class constraint.

The propagator for this theory can be written in a simple manner
\[ iG_{\mu\nu}(p) = -\frac{i}{p^2 - M^2} \left( \eta_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{M^2} \right). \] (3.8)

Note the propagator in this theory does not fall rapidly for large values of the momenta. This leads to difficulties in establishing renormalizability of the (interacting) Proca theory for massive photons. Hence the limit \( M \to 0 \) of the Proca theory is clearly difficult to perceive.

The generating functional for the Proca theory is defined as
\[ Z_P \equiv \int D A_{\mu} e^{iS_P}. \] (3.9)
2. Stueckelberg theory

To remove the difficulties in Proca model, Stueckelberg considered the following generalized Lagrangian density

$$S_{ST} = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{M^2}{2} \left( A_\mu - \frac{1}{M} \partial_\mu B \right)^2 \right],$$

(3.10)

by introducing a real scalar field $B$.

This action is invariant under the following gauge transformation

$$A_\mu(x) \rightarrow A_\mu'(x) = A_\mu(x) + \partial_\mu \lambda(x),$$

(3.11)

$$B(x) \rightarrow B'(x) = B(x) + M \lambda(x),$$

(3.12)

where $\lambda$ is gauge parameter. For the quantization of such theory one has to choose a gauge condition. By choosing the 't Hooft gauge condition, $L_{gf} = -\frac{1}{2\chi} (\partial^\mu A_\mu + \chi M B)^2$ where $\chi$ is any arbitrary gauge parameter, it is easy to see that the propagators are well behaved at high momentum. As a result, there is no difficulty in establishing renormalizability for such theory. Now we turn to the BRST symmetry for the Stueckelberg theory. Introducing a ghost ($\omega$) and antighost fields ($\omega^*$) the effective Stueckelberg action can be written as

$$S_{ST} = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{M^2}{2} \left( A_\mu - \frac{1}{M} \partial_\mu B \right)^2 \
- \frac{1}{2\chi} (\partial^\mu A_\mu + \chi M B)^2 - \omega^* (\partial^2 + \chi M^2) \omega \right].$$

(3.13)

This action is invariant under following on-shell BRST transformation

$$\delta_b A_\mu = \partial_\mu \omega \quad \Lambda, \quad \delta_b B = M \omega \quad \Lambda, \quad \delta_b \omega = 0, \quad \delta_b \omega^* = -\frac{1}{\chi} (\partial_\mu A_\mu + \chi M B) \quad \Lambda,$$

(3.14)

where $\Lambda$ is infinitesimal, anticommuting and global parameter. The generating functional for Stueckelberg theory is defined as

$$Z_{ST} \equiv \int D\phi \ e^{iS_{ST}[\phi]},$$

(3.15)

where $\phi$ is the generic notation for all fields involved in the theory. All the Green’s functions in this theory can be obtain from $Z_{ST}$. 
B. Theory for self-dual chiral boson

Self-dual chiral boson can be described by gauge variant as well as gauge invariant model. The purpose of this section is to introduce such models for self-dual chiral boson.

1. Gauge variant theory for self-dual chiral boson

We start with the gauge variant model \[25\] in 2D for single self-dual chiral boson. The effective action for such a theory is given as

\[
S_{CB} = \int d^2 x \ L_{CB} = \int d^2 x \left[ \frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} \varphi'^2 + \lambda (\dot{\varphi} - \varphi') \right], \tag{3.16}
\]

where over dot and prime denote time and space derivatives respectively and \(\lambda\) is Lagrange multiplier. The field \(\varphi\) satisfies the self-duality condition \(\dot{\varphi} = \varphi'\) in this case. We choose the Lorentz metric \(g^{\mu\nu} = (1, -1)\) with \(\mu, \nu = 0, 1\). The associated momenta for the field \(\varphi\) and Lagrange multiplier are calculated as

\[
\pi_\varphi = \frac{\partial L_{CB}}{\partial \dot{\varphi}} = \dot{\varphi} + \lambda, \quad \pi_\lambda = \frac{\partial L_{CB}}{\partial \dot{\lambda}} = 0, \tag{3.17}
\]

which show that the model has following primary constraint \(\Omega_1 \equiv \pi_\lambda \approx 0\). The Hamiltonian density corresponding to the above Lagrangian density \(L_{CB}\) in Eq. (3.16) is

\[
H_{CB} = \pi_\varphi \dot{\varphi} + \pi_\lambda \dot{\lambda} - L_{CB} = \frac{1}{2} (\pi_\varphi - \lambda)^2 + \frac{1}{2} \varphi'^2 + \lambda \varphi'. \tag{3.18}
\]

Further we can write the total Hamiltonian density corresponding to \(L_{CB}\) by introducing Lagrange multiplier field \(\eta\) for the primary constraint \(\Omega_1\) as

\[
H_{CB}^T = \frac{1}{2} (\pi_\varphi - \lambda)^2 + \frac{1}{2} \varphi'^2 + \lambda \varphi' + \eta \Omega_1,
\]

\[
= \frac{1}{2} (\pi_\varphi - \lambda)^2 + \frac{1}{2} \varphi'^2 + \lambda \varphi' + \eta \pi_\lambda. \tag{3.19}
\]

Following the Dirac’s prescription \[1\], we obtain the secondary constraint in this case as

\[
\Omega_2 \equiv \dot{\pi}_\lambda = [\pi_\lambda, H_{CB}] = \pi_\varphi - \lambda - \varphi' \approx 0. \tag{3.20}
\]

The constraints \(\Omega_1\) and \(\Omega_2\) are of second-class as \([\Omega_1, \Omega_2] \neq 0\). This is an essential feature of a gauge variant theory.
This model is quantized by establishing the following commutation relations

\[
[\varphi(x), \pi_{\varphi}(y)] = [\varphi(x), \lambda(y)] = +i\delta(x - y), \tag{3.21}
\]
\[
2[\lambda(x), \pi_{\varphi}(y)] = [\lambda(x), \lambda(y)] = -2i\delta'(x - y), \tag{3.22}
\]

where prime denotes the space derivative. The rest of the commutator vanishes.

The generating functional for gauge variant theory for self-dual chiral boson is defined as

\[
Z_{CB} = \int D\phi \ e^{iS_{CB}}, \tag{3.23}
\]

where \( D\phi \) is the path integral measure and \( S_{CB} \) is the effective action for self-dual chiral boson.

2. Gauge invariant theory for self-dual chiral boson

To construct a gauge invariant theory corresponding to the gauge non-invariant model for chiral bosons, one generally introduces the WZ term in the Lagrangian density \( L_{CB} \). For this purpose we need to enlarge the Hilbert space of the theory by introducing a new quantum field \( \vartheta \), called as WZ field, through the redefinition of fields \( \varphi \) and \( \lambda \) as follows

\[
\varphi \rightarrow \varphi - \vartheta, \quad \lambda \rightarrow \lambda + \dot{\vartheta}.
\]

With these redefinition of fields the modified Lagrangian density becomes

\[
L_{CB}^I = L_{CB} + L_{CB}^{WZ}, \tag{3.24}
\]

where the WZ term

\[
L_{CB}^{WZ} = -\frac{1}{2} \dot{\vartheta}^2 - \frac{1}{2} \vartheta'^2 + \varphi' \vartheta' + \dot{\vartheta}\vartheta' - \dot{\vartheta}'\varphi' - \lambda(\dot{\vartheta} - \vartheta'). \tag{3.25}
\]

The above Lagrangian density in Eq. (3.24) is invariant under time-dependent chiral gauge transformation:

\[
\delta \varphi = \mu(x, t), \quad \delta \vartheta = \mu(x, t), \quad \delta \lambda = -\dot{\mu}(x, t),
\]
\[
\delta \pi_{\varphi} = 0, \quad \delta \pi_{\vartheta} = 0, \quad \delta p_{\lambda} = 0, \tag{3.26}
\]

where \( \mu(x, t) \) is an arbitrary function of the space and time.
The BRST invariant effective theory for self-dual chiral boson \([31]\) can be written as
\[
S_{\text{CB}}^{\text{II}} = \int d^2 x \mathcal{L}_{\text{CB}}^{\text{II}},
\]
where \(\mathcal{L}_{\text{CB}}^{\text{II}} = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \varphi'^2 + \lambda(\dot{\varphi} - \varphi') - \frac{1}{2} \dot{\vartheta}^2 - \frac{1}{2} \vartheta'^2 + \varphi' \vartheta' + \dot{\vartheta} \vartheta' - \dot{\varphi} \varphi' + \lambda(\dot{\vartheta} - \vartheta') - \frac{1}{2}(\dot{\lambda} - \varphi - \vartheta)^2 + \dot{c} \bar{c} - 2\bar{c}c.
\]
\(c\) and \(\bar{c}\) are ghost and antighost fields respectively. Corresponding generating functional for gauge invariant theory for self-dual chiral boson is given as
\[
Z_{\text{CB}}^{\text{II}} = \int D\phi e^{iS_{\text{CB}}^{\text{II}}},
\]
where \(\phi\) is generic notation for all fields involved in the effective action. The effective action \(S_{\text{CB}}^{\text{II}}\) and the generating functional \(Z_{\text{CB}}^{\text{II}}\) are invariant under the following nilpotent BRST transformation
\[
\delta_b \phi = \bar{c} \Lambda, \quad \delta_b \lambda = -\dot{c} \Lambda, \quad \delta_b \vartheta = c \Lambda, \\
\delta_b \bar{c} = -(\dot{\lambda} - \varphi - \vartheta) \Lambda, \quad \delta_b c = 0,
\]
where \(\Lambda\) is infinitesimal and anticommuting BRST parameter.

\[\text{IV. RELATING THE FIRST-CLASS AND SECOND-CLASS THEORIES THROUGH FFBRST FORMULATION: EXAMPLES}\]

In this section, we consider two examples to show the connection between the generating functionals for theories with first-class and second-class constraints. Firstly we show the connection between Stueckelberg theory and Proca theory for massive vector fields. In the second example we link the gauge invariant and gauge variant theory for self-dual chiral boson.

\[\text{A. Relating Stueckelberg and Proca theories}\]

We start with the linearize form of the Stueckelberg effective action \([3.13]\) by introducing a Nakanishi-Lautrup type auxiliary field \(B\) as
\[
S_{\text{ST}} = \int d^4 x \left[ -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} M^2 \left( A_\mu - \frac{1}{M} \partial_\mu B \right)^2 \right]
\]
\[ \frac{\chi}{2} B^2 - B(\partial_\mu A^\mu + \chi MB) - \omega^*(\partial^2 + \chi M^2) \omega] \] 

which is invariant under the following off-shell nilpotent BRST transformation

\[ \delta_b A_\mu = \partial_\mu \omega \Lambda, \quad \delta_b B = M \omega \Lambda, \quad \delta_b \omega = 0, \quad \delta_b \omega^* = B \Lambda, \quad \delta_b B = 0. \] (4.2)

The FFBRST transformation corresponding to the above BRST transformation is constructed as,

\[ \delta_b A_\mu = \partial_\mu \omega \Theta_1[\phi], \quad \delta_b B = M \omega \Theta_1[\phi], \quad \delta_b \omega = 0, \quad \delta_b \omega^* = B \Theta_1[\phi], \quad \delta_b B = 0, \] (4.3)

where \( \Theta_1 \) is an arbitrary finite field dependent parameter but still anticommuting in nature. To establish the connection we choose a finite field dependent parameter \( \Theta_1 \) obtainable from

\[ \Theta_1' = i \gamma \int d^4x \left[ \omega^* \left( \chi MB - \frac{\chi}{2} B + \partial_\mu A^\mu \right) \right], \] (4.4)

via Eq. (2.3), where \( \gamma \) is an arbitrary parameter.

Using Eq. (2.6) the infinitesimal change in nontrivial Jacobian can be calculated for this finite field dependent parameter as

\[ \frac{1}{J} \frac{dJ}{d\kappa} = i \gamma \int d^4x \left[ B \left( \chi MB - \frac{\chi}{2} B + \partial_\mu A^\mu \right) \right], \] (4.5)

where the equation of motion for antighost field, \((\partial^2 + \chi M^2) \omega = 0\), has been used.

We now make the following ansatz for \( S_1 \) as

\[ S_1 = \int d^4x [\xi_1(\kappa) B^2 + \xi_2(\kappa) B \partial_\mu A^\mu + \xi_3(\kappa) \chi MB \bar{B}], \] (4.6)

where \( \xi_i, (i = 1, 2, 3) \) are arbitrary \( \kappa \) dependent parameter and satisfy following initial conditions \( \xi_i(\kappa = 0) = 0 \). Now, using the relation in Eq. (2.1) we calculate \( \frac{dS_1}{d\kappa} \) as

\[ \frac{dS_1}{d\kappa} = \int d^4x \left[ B^2 \xi_1' + B \partial_\mu A^\mu \xi_2' + \chi MB \bar{B} \xi_3' \right], \] (4.7)

where prime denote the differentiation with respect to \( \kappa \). The Jacobian contribution can be written as \( e^{S_1} \) if the essential condition in Eq. (2.8) is satisfied. This leads to

\[ \int d^4x e^{i(S_{ST} + S_1)} \left[ iB^2(\xi_1' + \gamma \frac{\chi}{2}) + iB \partial_\mu A^\mu (\xi_2' - \gamma) + i\chi MB \bar{B} (\xi_3' - \gamma) \right] = 0. \] (4.8)

Equating the coefficient of terms \( iB^2, iB \partial_\mu A^\mu, \) and \( i\chi MB \bar{B} \) from both sides of above condition, we get following differential equations

\[ \xi_1 + \gamma \frac{\chi}{2} = 0, \quad \xi_2' - \gamma = 0, \quad \xi_3' - \gamma = 0. \] (4.9)
To obtain the solution of the above equations we put $\gamma = 1$ without any loose of generality. The solutions satisfying initial conditions are given as

$$\xi_1 = -\frac{\chi}{2} \kappa, \quad \xi_2 = \kappa, \quad \xi_3 = \kappa.$$  \hspace{1cm} (4.10)

The transformed action can be obtained by adding $S_1(\kappa = 1)$ to $S_{ST}$ as

$$S_{ST} + S_1 = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 \left( A_\mu - \frac{1}{M} \partial_\mu B \right)^2 - \omega^* (\partial^2 + \chi M^2) \omega \right].$$  \hspace{1cm} (4.11)

Now the generating functional under FFBRST transforms as

$$Z' = \int DA_\mu DB D\omega D\omega^* e^{i(S_{ST} + S_1)}.$$  \hspace{1cm} (4.12)

To remove the divergence due to gauge volume in the above expression we integrate over the $B, \omega,$ and $\omega^*$ fields which reduces it to the generating functional for Proca model upto some normalization constant as follows

$$Z' = \int DA_\mu e^{i \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{M^2}{2} A_\mu A^\mu \right]} = Z_P.$$  \hspace{1cm} (4.13)

Here we would like to point out that this action is divergence free as mass term breaks the gauge symmetry. Therefore,

$$Z_{ST} \left( \int D\phi e^{iS_{ST}} \right) \xrightarrow{FFBRST} Z_P \left( \int DA_\mu e^{iS_P} \right).$$  \hspace{1cm} (4.14)

Thus by constructing appropriate finite field dependent parameter (given in Eq. 4.3) we have shown that the generating functional for Stueckelberg theory is connected to the generating functional for Proca theory through FFBRST transformation. This indicates that the Green’s functions in these two theories are related through FFBRST formulation.

**B. Relating the gauge invariant and variant theory for chiral boson**

To see the connection between the gauge invariant and variant theories for chiral boson, we start with the effective action for the gauge invariant self-dual chiral boson theory as

$$S_{CB}^{II} = \int d^2x \left[ \frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} \varphi'^2 + \lambda(\varphi - \varphi') - \frac{1}{2} \dot{\vartheta}^2 - \frac{1}{2} \vartheta'^2 + \varphi' \vartheta' \right. \right.$$

$$\left. + \dot{\vartheta} \vartheta' - \dot{\varphi} \varphi' - \lambda(\vartheta - \vartheta') + \frac{1}{2} B^2 + B(\dot{\lambda} - \varphi - \vartheta) + \dot{\bar{c}} \bar{c} - 2 \bar{c} \bar{c} \right].$$  \hspace{1cm} (4.15)
where we have linearized the gauge fixing part of the effective action by introducing the extra auxiliary field $B$. This effective action is invariant under following infinitesimal BRST transformation

$$
\delta_b \varphi = c \Lambda, \quad \delta_b \lambda = -\dot{c} \Lambda, \quad \delta_b \vartheta = c \Lambda, \\
\delta_b \bar{c} = B \Lambda, \quad \delta_b B = 0, \quad \delta_b c = 0.
$$

(4.16)

Corresponding FFBRST transformation can be written as

$$
\delta_b \varphi = c \Theta_1[\phi], \quad \delta_b \lambda = -\dot{c} \Theta_1[\phi], \quad \delta_b \vartheta = c \Theta_1[\phi], \\
\delta_b \bar{c} = B \Theta_1[\phi], \quad \delta_b B = 0, \quad \delta_b c = 0.
$$

(4.17)

where $\Theta_1[\phi]$ is arbitrary finite field dependent parameter, which we have to constructed. In this case we construct the finite field dependent BRST parameter $\Theta_1[\phi]$ obtainable from

$$
\Theta'_1 = i\gamma \int d^2 x \left[ \bar{c}(\dot{\lambda} - \varphi - \vartheta + \frac{1}{2}B) \right],
$$

(4.18)

using Eq. (2.3) and demand that the corresponding BRST transformation will lead to the gauge variant theory for self-dual chiral boson.

To justify our claim we calculate the change in Jacobian, using equation of motion for antighost field, as

$$
\frac{1}{J} \frac{dJ}{d\kappa} = i\gamma \int d^2 x \left[ B(\dot{\lambda} - \varphi - \vartheta + \frac{1}{2}B) \right].
$$

(4.19)

We make an ansatz for local functional $S_1$ as,

$$
S_1 = \int d^2 x \left[ \xi_1(\kappa) B^2 + \xi_2(\kappa) B(\dot{\lambda} - \varphi - \vartheta) \right].
$$

(4.20)

The change in $S_1$ with respect to $\kappa$ is calculated as

$$
\frac{dS_1}{d\kappa} = \int d^2 x \left[ \xi'_1 B^2 + \xi'_2 B(\dot{\lambda} - \varphi - \vartheta) \right].
$$

(4.21)

Now, the necessary condition in Eq. (2.8) leads to the following equation

$$
\int d^2 x e^{i(\mathcal{L}_B + S_1)} \left[ iB^2 (\xi'_1 - \frac{\gamma}{2}) + iB(\dot{\lambda} - \varphi - \vartheta)(\xi'_2 - \gamma) \right] = 0.
$$

(4.22)

Equating the coefficient of terms $iB^2$ and $iB(\dot{\lambda} - \varphi - \vartheta)$ from both sides of above condition, we get following differential equations:

$$
\xi'_1 - \frac{\gamma}{2} = 0, \quad \xi'_2 - \gamma = 0.
$$

(4.23)
The solutions of above equations are $\xi_1 = -\frac{1}{2}\kappa$, $\xi_2 = -\kappa$, where we have taken the parameter $\gamma = -1$. The transformed action is obtained by adding $S_1(\kappa = 1)$ to $S_{CB}^{II}$ as

$$
S_{CB}^{II} + S_1 = \int d^2x \left[ \frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} \varphi'^2 + \lambda(\dot{\varphi} - \varphi') - \frac{1}{2} \dot{\varphi}'^2 
- \frac{1}{2} \varphi'^2 + \varphi' \varphi'' + \phi \varphi' - \lambda(\phi - \varphi') + \dot{c}\dot{\bar{c}} - 2\bar{c}c \right].
$$

(4.24)

Now the transformed generating functional becomes

$$
Z' = \int D\varphi D\lambda DcD\bar{c} e^{i(S_{CB}^{II} + S_1)}.
$$

(4.25)

Performing integration over fields $\vartheta, c$ and $\bar{c}$, the above generating functional reduces to the generating functional for self dual chiral boson up to some constant as

$$
Z' = \int D\varphi D\lambda e^{iS_{CB}} = Z_{CB}.
$$

(4.26)

Therefore,

$$
Z_{CB}^{II} \left( \int D\phi e^{iS_{CB}^{II}} \right) \xrightarrow{\text{FFBRST}} Z_{CB} \left( \int D\varphi D\lambda e^{iS_{CB}} \right).
$$

(4.27)

Thus, the generating functionals corresponding to the gauge invariant and gauge non-invariant theory for self-dual chiral boson are connected through the FFBRST transformation given in Eq. (4.17).

We end up this section by making conclusion that using FFBRST formulation the generating functional for the theory with second-class constraint can be achieved by generating functional for theory with first-class constraint.

V. RELATING THE FIRST-CLASS AND SECOND-CLASS THEORIES THROUGH FF-ANTI-BRST FORMULATION: EXAMPLES

In this section, we consider FF-anti-BRST formulation to show the connection between the generating functionals for theories with first-class and second-class constraints with same examples. The FF-anti-BRST transformation is also developed in same fashion as FFBRST transformation, the only key difference is the role of ghost fields are interchanged with anti-ghost fields and vice-versa.
A. Relating Stueckelberg and Proca theories

We start with anti-BRST symmetry transformation for effective action given in Eq. (4.1), as

\[
\delta_{ab} A_\mu = \partial_\mu \omega^* \Lambda, \quad \delta_{ab} B = M \omega^* \Lambda, \quad \delta_{ab} \omega = -\mathcal{B} \Lambda,
\]

\[
\delta_{ab} \omega^* = 0, \quad \delta_{ab} \mathcal{B} = 0,
\]

(5.1)

where \( \Lambda \) is infinitesimal, anticommuting and global parameter. The FF-anti-BRST transformation corresponding to the above anti-BRST transformation is constructed as,

\[
\delta_{ab} A_\mu = \partial_\mu \omega^* \Theta_2, \quad \delta_{ab} B = M \omega^* \Theta_2, \quad \delta_{ab} \omega = -\mathcal{B} \Theta_2,
\]

\[
\delta_{ab} \omega^* = 0, \quad \delta_{ab} \mathcal{B} = 0,
\]

(5.2)

where \( \Theta_2 \) is an arbitrary finite field dependent parameter but still anticommuting in nature. To establish the connection we choose a finite field dependent parameter \( \Theta_2 \) obtainable from

\[
\Theta_2' = -i\gamma \int d^4x \left[ \omega \left( \chi M \mathcal{B} - \frac{\chi}{2\mathcal{B}} \mathcal{B} + \partial_\mu A^\mu \right) \right],
\]

(5.3)

where \( \gamma \) is an arbitrary parameter.

Using Eq. (2.6) the infinitesimal change in nontrivial Jacobian can be calculated for this finite field dependent parameter as

\[
\frac{1}{J} \frac{dJ}{d\kappa} = -i\gamma \int d^4x \left[ -\mathcal{B} \left( \chi M \mathcal{B} - \frac{\chi}{2\mathcal{B}} \mathcal{B} + \partial_\mu A^\mu \right) \right].
\]

(5.4)

To Jacobian contribution can be expressed as \( e^{iS_2} \). To calculate \( S_2 \) we make following ansatz

\[
S_2 = \int d^4x [\xi_5(\kappa) \mathcal{B}^2 + \xi_6(\kappa) \mathcal{B} \partial_\mu A^\mu + \xi_7(\kappa) \chi M \mathcal{B} \mathcal{B}],
\]

(5.5)

where \( \xi_i, \ (i = 5, .., 7) \) are arbitrary \( \kappa \) dependent parameter and satisfy following initial conditions \( \xi_i(\kappa = 0) = 0 \). Now, infinitesimal change in \( S_2 \) is calculated as

\[
\frac{dS_2}{d\kappa} = \int d^4x \left[ \mathcal{B}^2 \xi_5' + \mathcal{B} \partial_\mu A^\mu \xi_6' + \chi M \mathcal{B} \mathcal{B} \xi_7' \right],
\]

(5.6)

where prime denotes the differentiation with respect to \( \kappa \).
Putting the expressions (5.4) and (5.6) in the essential condition given in Eq. (2.8), we obtain

\[ \int d^4 x \ e^{i(S_{ST}+S_2)} \left[ B^2(\xi'_5 + \gamma \frac{X}{2}) + B \partial_\mu A^\mu(\xi'_6 - \gamma) + \chi MBB(\xi'_7 - \gamma) \right] = 0. \] (5.7)

Equating the coefficient of terms \( iB^2, iB \partial_\mu A^\mu, \) and \( i\chi MBB \) from both sides of above condition, we get following differential equations

\[ \xi'_5 + \gamma \frac{X}{2} = 0, \quad \xi'_6 - \gamma = 0, \quad \xi'_7 - \gamma = 0. \] (5.8)

The solutions of the above differential equation for \( \gamma = 1 \) are \( \xi_5 = -\frac{1}{2} \kappa, \ \xi_6 = \kappa, \ \xi_7 = \kappa. \)

The transformed action can be obtained by adding \( S_2(\kappa = 1) \) to \( S_{ST} \) as

\[ S_{ST} + S_2 = \int d^4 x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 \left( A_\mu - \frac{1}{M} \partial_\mu B \right)^2 - \omega^*(\partial^2 + \chi M^2) \omega \right]. \] (5.9)

We perform integration over \( B, \omega \) and \( \omega^* \) fields to remove the divergence of transformed generating functional \( Z' = \int DA_\mu DBD\omega D\omega^* e^{i(S_{ST}+S_2)} \) and hence we get the generating functional for Proca model as

\[ Z' = \int DA_\mu e^{i \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 A_\mu A^\mu \right]} = Z_P, \] (5.10)

which is a divergence free theory. Therefore, \( Z_{ST}^{FF-anti-BRST} \rightarrow Z_P. \) Thus by constructing appropriate finite field dependent parameter (given in Eq. (5.2)) we have shown that the generating functional for Stueckelberg theory is related to the generating functional for Proca theory through FF-anti-BRST transformation also. This indicates that the Green’s functions in these two theories are related through FFBRST and FF-anti-BRST transformation.

B. Relating the gauge invariant and variant theory for chiral boson

To connect the gauge invariant and variant theories for chiral boson through FF-anti-BRST transformation, first of all we write the anti-BRST transformation for effective action (4.15) as

\[ \delta_{ab} \varphi = \bar{c} \Lambda, \quad \delta_{ab} \lambda = -\bar{c} \Lambda, \quad \delta_{ab} \vartheta = \bar{c} \Lambda, \]

\[ \delta_{ab} c = -B \Lambda, \quad \delta_{ab} B = 0, \quad \delta_{ab} \bar{c} = 0. \] (5.11)
Corresponding FF-anti-BRST transformation can be written as

\[
\begin{align*}
\delta_{ab}\varphi &= \bar{c} \Theta_2[\phi], \\
\delta_{ab}\lambda &= -\dot{c} \Theta_2[\phi], \\
\delta_{ab}\vartheta &= c \Theta_2[\phi], \\
\delta_{ab}c &= -B \Theta_2[\phi], \\
\delta_{ab}B &= 0, \\
\delta_{ab}\bar{c} &= 0,
\end{align*}
\]

(5.12)

where \(\Theta_2[\phi]\) is arbitrary finite field dependent parameter, which we have to construct. In this case we construct the finite field dependent anti-BRST parameter \(\Theta_2[\phi]\) obtainable from

\[
\Theta_2' = -i\gamma \int d^2x \left[ c(\lambda - \varphi - \vartheta + \frac{1}{2}B) \right],
\]

(5.13)

using Eq. (2.3) and demand that the corresponding anti-BRST transformation will lead to the gauge variant theory for self-dual chiral boson.

We make an ansatz for local functional \(S_2\) in this case as,

\[
S_2 = \int d^2x \left[ \xi_5(\kappa) B^2 + \xi_6(\kappa) B(\lambda - \varphi - \vartheta) \right].
\]

(5.14)

Now, the necessary condition in Eq. (2.8) leads to the following equation

\[
\int d^2x e^{i(S_{CB}^{\prime} + S_2)} \left[ iB^2(\xi_5' - \frac{\gamma}{2}) + iB(\lambda - \varphi - \vartheta)(\xi_6' - \gamma) \right] = 0.
\]

(5.15)

Equating the coefficient of different terms in the both sides of above equation, we get following differential equations:

\[
\xi_5' - \frac{\gamma}{2} = 0, \quad \xi_6' - \gamma = 0.
\]

(5.16)

The solutions of above equations are \(\xi_5 = -\frac{1}{2}\kappa, \xi_6 = -\kappa\), where the parameter \(\gamma = -1\).

The transformed action can be obtained by adding \(S_2(\kappa = 1)\) to \(S_{CB}^{\prime}\) as

\[
S_{CB}^{\prime} + S_2 = \int d^2x \left[ \frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} \varphi'^2 + \lambda(\dot{\varphi} - \varphi') - \frac{1}{2} \dot{\vartheta}^2 + \frac{1}{2} \vartheta'^2 + \varphi' \vartheta' - \dot{\varphi} \vartheta' - \lambda(\dot{\vartheta} - \vartheta') + \dot{c} \bar{c} - 2\bar{c}c \right].
\]

(5.17)

After functional integration over fields \(\vartheta, c\) and \(\bar{c}\) in the expression of transformed generating functional, we get the generating functional as

\[
Z' = \int D\varphi D\lambda e^{iS_{CB}} = Z_{ CB}.
\]

(5.18)

Therefore, \(Z_{CB}^{FF-anti-BRST} \rightarrow Z_{CB}\). Thus, the generating functionals corresponding to the gauge invariant and gauge non-invariant theory for self-dual chiral boson are also connected through the FF-anti-BRST transformation given in Eq. (5.12).
We end up this section by making comment that the generating functional for the theory with second-class constraint can be obtained from generating functional for theory with first-class constraint using both FFBRST and FF-anti-BRST transformations.

VI. CONCLUDING REMARKS

The Stueckelberg theory for the massive spin 1 field and gauge invariant theory for self-dual chiral boson are the first-class theories. On the other hand, the Proca theory for massive spin 1 field and gauge variant theory for self-dual chiral boson are theories with second-class constraint. We have shown that the generalized BRST transformation, where the BRST parameter is finite and field dependent, relates the generating functionals of second-class theory and first-class theory. The path integral measure in the definition of generating functionals are not invariant under such FFBRST transformation and are responsible for such connections. The Jacobian for path integral measure under such a transformation with appropriate finite parameter cancels the extra parts of the first-class theory. We have avoided the complicated calculations of nonlocal and field dependent Dirac brackets for second-class theory in the cost of calculating the Jacobian of nontrivial FFBRST transformation. Our result is supported by two explicit examples. In the first case we have related the generating functional of Stueckelberg theory to the generating functional of Proca model and in the second case the generating functionals corresponding to the gauge invariant theory and gauge variant theory for self-dual chiral boson have linked through FFBRST transformation with appropriate choices of finite field dependent parameter. Same goal has been achieved by using FF-anti-BRST transformation. These formulations can be applied to connect the generating functionals for any first-class (e.g. non-Abelian gauge theories) and second-class theories provided appropriate finite parameters are constructed.

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SR/S2/HEP-29/2007.

[1] P. A. M. Dirac, Lectures on Quantum Mechanics, (Yeshiva Univ. Press, New York, 1964).
[2] E. S. Fradkin and G. A. Vilkovisky, Phys. Lett. B 55, (1975) 224.
[3] C. Becchi, A. Rouet and R. Stora, Annals Phys. 98, (1976) 287.
[4] I. V. Tyutin, LEBEDEV 39 (1975).
[5] T. Kugo and I. Ojima, Prog. Theor. Phys. Suppl. 66, (1979) 1.
[6] A. Restuccia and J. Stephany, Phys. Lett. B 305, (1993) 348.
[7] I. A. Batalin and E. S. Fradkin, Nucl. Phys. B 279, (1987) 514; Phys. Lett. B 180, (1986) 157.
[8] I. A. Batalin and I. V. Tyutin, Int. J. Mod. Phys. A 6, (1991) 3255.
[9] T. Fujiwara, Y. Igarashi and J. Kubo, Nucl. Phys. B 341, (1990) 695.
[10] Y. W. Kim, S. K. Kim, W. T. Kim, Y. J. Park, K. Y. Kim and Y. Kim Phys. Rev. D 46, (1992) 4574.
[11] L. D. Faddeev and S. L. Shatashivili, Phys. Lett. B 167, (1986) 225.
[12] J. Wess and B. Zumino, Phys. Lett. B 37, (1971) 95.
[13] A. Proca, J. de Phys. et le Radium 7, (1936) 347.
[14] R. J. Glauber, Prog. Theor. Phys. 9, (1953) 295.
[15] A. Lahiri, ArXiv [hep-th] 9301060 (1993).
[16] A. Lahiri, Phys. Rev. D 55, (1997) 5045.
[17] A. Lahiri, Phys. Rev. D 63, (2001) 105002.
[18] E. C. G. Stueckelberg, Helv. Phys. Acta 11, (1938) 225.
[19] E. C. G. Stueckelberg, Helv. Phys. Acta 11, (1938) 299.
[20] E. C. G. Stueckelberg, Helv. Phys. Acta 11, (1938) 312.
[21] C. Marshall and P. Ramond, Nucl. Phys. B 85, (1975) 375.
[22] P. Ramond, Prog. Theor. Phys. Suppl. 86, (1986) 126.
[23] G. Aldazabal, L. E. Ibanez and F. Quevedo, JHEP 02, (2000) 015.
[24] D. S. Kulshreshtha and Muller-Kirsten, Phys. Rev. D 45 (1992) R 393.
[25] P. P. Srivastva, Phys. Rev. Lett. 63, (1989) 2791.
[26] R. Floreanini and R. Jackiw, Phys. Rev. Lett. 49, (1987) 1873.

[27] M. E. V. Costa and H. O. Girotti, Phys. Rev. Lett. 17, (1988) 1771.

[28] H. O. Girotti, M. Gomes, V. Kurak, V. O. Rivelles and A. J. da Silva, Phys. Rev. Lett. 60, (1988) 1913.

[29] H. O. Girotti, M. Gomes and V. O. Rivelles, Phys. Rev. D 45, (1992) R 3329.

[30] S. Ghosh, Phys. Rev. D 49, (1994) 2990.

[31] S. Upadhyay and B. P. Mandal, Eur. Phys. J. C 71, (2011) 1759.

[32] N. Marcus and J. Schwarz, Phys. Lett. B 115, (1982) 111.

[33] X. G. Wen, Phys. Rev. Lett. 64, (1990) 2206.

[34] S. D. Joglekar and B. P. Mandal, Phys. Rev. D 51, (1995) 1919.

[35] R. Banerjee and B. P. Mandal, Phys. Lett. B 27, (2000) 488.

[36] S. D. Joglekar and B. P. Mandal, Int. J. Mod. Phys. A 17, (2002) 1279.

[37] S. Upadhyay, S. K. Rai and B. P. Mandal, J. Math. Phys. 52, (2011) 022301.

[38] S. Upadhyay and B. P. Mandal, Eur. Phys. Lett. 93, (2011) 31001.

[39] S. Upadhyay and B. P. Mandal, Mod. Phys. Lett. A 40, (2010) 3347.

[40] B. P. Mandal, S. K. Rai and S. Upadhyay, Eur. Phys. Lett. 92, (2010) 21001.

[41] S. Upadhyay and B. P. Mandal, Eur. Phys. J. C 72, (2012) 2065.

[42] S. Upadhyay and B. P. Mandal, AIP Conf. Proc. 1444, 213 (2012).