The Gołąb-Schinzel and Goldie functional equations in Banach algebras
by
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Abstract. We are concerned below with the characterization in a unital commutative real Banach algebra $A$ of continuous solutions of the Gołąb-Schinzel functional equation (below), the general Popa groups they generate and the associated Goldie functional equation. This yields general structure theorems involving both linear and exponential homogeneity in $A$ for both these functional equations and also explicit forms, in terms of the recently developed theory of multi-Popa groups [BinO3,4], both for the ring $C[0,1]$ and for the case of $\mathbb{R}^d$ with componentwise product, clarifying the context of recent developments in [RooSW]. The case $A = \mathbb{C}$ provides a new viewpoint on continuous complex-valued solutions of the primary equation by distinguishing analytic from real-analytic ones.

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1 Equations and groups

1.1 Functional equations linked to that of Cauchy

The Cauchy, Gołąb-Schinzel, Goldie and Levi-Civita functional equations. General regular variation [BinO1] has recently emerged as embracing three kinds of univariate regular variation (RV) due in turn to: Karamata (classical), Bojanic-Karamata-de Haan, and Beurling (for the Beurling Tauberian Theorem), for which see [BinGT]. Underlying this unification is the Gołąb-Schinzel functional equation and an associated group structure (below). The equation reads

$$S(x + S(x)y) = S(x)S(y) \quad (GS)$$

with $x, y$ ranging over a half-line in $\mathbb{R}$ (‘$S$ for survival probability’) and has positive (continuous) solutions which necessarily take the ‘canonical’ form

$$S(x) = S_\rho(x) := 1 + \rho x \quad (\rho \geq 0), \quad (Can)$$

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so with a maximal connected domain $G_\rho := \{ x : 1 + \rho x > 0 \} = (-\rho^{-1}, \infty)$: cf. Cor. 5.1. (For $\rho = 0$, interpret $-\rho^{-1}$ as $-\infty$.) It is significant that the associated group operation $\circ$ on $G_\rho$ (the Popa group, below) allows a re-statement of $(GS)$ as a homomorphism equation

$$S(x \circ y) = S(x)S(y),$$

so is fundamentally the multiplicative variant of the Cauchy functional equation. We study only continuous solutions, leaving aside the issues of automatic continuity of homomorphisms, for which see e.g. [Ros], [Dal], cf. [Ost1, §1].

A further key tool in generalized regular variation is the Goldie functional equation:

$$K(x + y) = K(x) + g(x)K(y)$$

(GFE)

in the pair $(K, g)$ of real-valued functions; this is very closely related to $(GS)$ (see [BinO1]). (Equation (GFE) is a special case of a Levi-Civita equation [Lev] – see [Stet, Ch. 5]; cf. [AczD, Ch.14].) Here $g$ is necessarily multiplicative:

$$g(x + y) = g(x)g(y),$$

and so is termed the multiplicative auxiliary. It is either trivial ($g \equiv 1$) or exponential on $\mathbb{R}$. Correspondingly, $K$ is additive and so linear, or in the non-trivial case (characterized by $g = 1$ only for $x = 0$), it is monotone (cf. [BinO4, Lemma 1]). Then, as $+$ is commutative and $K, g$ are real-valued,

$$K(y) + g(y)K(x) = K(x) + g(x)K(y) : [g(y) - 1]K(x) = [g(x) - 1]K(y) :$$

$$K(x) = \text{const} \times (1 - g(x)) = c\frac{1 - e^{-\gamma x}}{1 - e^{-\gamma}}, \quad (Exp)$$

with $c$ a constant. It is thematic here and below that $(Exp)$ reduces to $K(x) = cx$ for $\gamma = 0$ under the L’Hospital convention: we shall see curvilinear variants of this linear-versus-exponential dichotomy in a more general setting.

We note that any decreasing solution $g$ contributes a notable solution to (GFE), namely

$$K(x) = 1 - e^{-\gamma x} \quad \text{with } \gamma > 0,$$

one that is an exponential probability distribution on $[0, \infty)$; for the background here, see [BinGT, Ch. 3] and [BinO1].

As pointed out in [Ost5] (cf. [Ost3]), assuming the solution $S$ of $(GS)$ to be injective, as will be the case for $S = S_\rho$ with $\rho > 0$, replacement of $S(x)$
by $e^x$ and of $S^{-1}(e^x)$ by $K(x)$ yields $(GFE)$ with $g(x) = e^x$. That is, $(GFE)$ and $(GS)$ are then formally equivalent.

**Infinite-dimensional settings.** Both $(GS)$ and $(GFE)$ are capable of a natural interpretation when $x, y$ range over a topological vector space $X$ and the functions $S, K, g$ are real-valued. Here, by the Brillouët-Dhombres-Brzdęk theorem [BriD, Prop. 3], [Brz1, Th. 4], the continuous solutions of $(GS)$ take the form, for $x \in X$

$$S(x) = S_\rho(x) := 1 + \rho(x) \quad (\rho \in X^*), \quad \text{(Can}_X)$$

with $X^*$ the dual space of continuous linear functionals on $X$. Here too there are: an analogous maximal connected open domain

$$\mathbb{G}_\rho(X) := \{x \in X : 1 + \rho(x) > 0\} \quad (\mathbb{G}_\rho(X))$$

and an associated abelian group structure $\circ_\rho$ on this domain. In this context, Goldie’s equation generalizes to

$$K(x \circ_\rho y) = K(x) + g(x)K(y), \quad \text{(GGE}_\rho g)$$

$g$ is again necessarily multiplicative on $\mathbb{G}_\rho$:

$$g(x \circ_\rho y) = g(x)g(y), \quad \text{(G)}$$

and, again by the commutativity of $\circ$, for some constant $c$

$$K(x) = c(1 - g(x)).$$

It further emerges [BinO3,4] that with $K$ injective, as here, $(GFE)$ may be equivalently rewritten as

$$K(x + y) = K(x) \circ_\sigma K(y) \quad (x, y \in X) \quad \text{for } \sigma(z) := g(K^{-1}(z)),$n

with $\sigma = \sigma_g \in X^*$. So again a Cauchy equation. Its (continuous) solutions are explicitly characterized in [BinO3,4] in the more general context

$$K(x \circ_\rho y) = K(x) \circ_\sigma K(y) \quad (x, y \in \mathbb{G}_\rho(X))$$

by reference to the fundamental homomorphisms: linear maps, exponentials, and the two $GS$-type functions: $(1 + \rho(x))$ and $(1 + \sigma(x))$, these being the basic building blocks.
Results. Below we pursue both equations in the more general setting of finite- and infinite-dimensional Banach algebras. There are six main results in this general setting: the Decomposition Theorem, Th. 2.1 (into linear and non-linear parts); Phantom Linear Characterization, Th. 2.2; Pencil Theorem, Th. 4.1, showing that neighbourhoods of the origin are spanned by a pencil through 0 of otherwise disjoint subgroups isomorphic to canonical (abelian) Popa groups; a First Characterization Theorem, Th. 5.1, from which it emerges that \((GS)\) and \((GGE_{SS})\), another generalized form of \((GFE)\), are as inseparable as husband and wife; a Second Characterization Theorem, Th. 5.2, showing that the solution of \((GFE)\) in a Banach algebra exhibits the exponential homogeneity familiar in the real line setting \((Exp)\), albeit in curvilinear form; finally, a Third Characterization Theorem, Th. 5.3, giving under technical assumptions a differential characterization to a region where \(S\) takes the linear-plus-one form \(\left(1_A + \text{Linear}\right)\).

As a preliminary, the analysis will begin with the finite-dimensional Banach algebras provided by Euclidean space. Here the main result is the Structure Theorem, Th. 3.3, where the general \(d\)-dimensional Popa group is broken down into irreducible building blocks (as with the decomposition of finite groups into finite simple groups). There is a consequent analogue for the Banach algebra \(C[0,1]\), based on the Stone-Weierstrass Theorem. The signature of the Euclidean decomposition is the partition \(P\) of \(\mathbb{N}_d = \{1, \ldots, d\}\). This breaks down the \(d\) coordinates of the group elements into parts, whose coordinates are interchangeable with each other but not with those from other parts. This ‘signature decomposition’ reveals structure at two levels for each factor, \(i \in I\) and \(I \in P\), and so three levels altogether (cf., say, ‘continents, countries and counties’). Such structure may not be previously visible.

Connections with statistics. The emergence of such ‘unsuspected structure’ can be very important, even in two dimensions. To give a classic instance: in Silverman’s book on density estimation [Sil, §4.2.3, Fig. 4.7] he gives an account of a study of a certain disease. A two-dimensional contour plot of an estimated density revealed (in the manner of an Ordnance Survey map) two ‘peaks’. Medical investigation showed that the disease under study occurred in two forms, corresponding to these peaks. With this difference identified, it emerged that the two forms were best treated in different ways.

Connections with probability. In probability theory, the theory of independent sums is of central importance (infinitely-divisible laws, Lévy processes,
Lévy-Itô decomposition, Lévy-Khintchine formula, etc.) A central role here is played by the stable laws – those obtainable from a single sequence of independent copies of a random variable, rather than a doubly-indexed array of distributions (infinitely divisible laws) or a singly-indexed one (self-decomposability). The role played by functional equations in the study of stability directly (rather than by specialisation from infinite-divisibility) has been considered by Pitman and Pitman [PitP] and the second author [Ost4]. The functional equations relevant here are those of Goldie, Cauchy and Levi-Civita.

In extreme-value theory (EVT), the role of sums above is played instead by maxima (cf. [BinGT, §8.15]). A survey of the regular-variation aspects of EVT was recently given by the authors ([BinO5]; cf. [BinO4]). It emerges that the key functional equation there is the Goldie equation. It is striking that the Goldie equation plays a central role in the theory of both sums and maxima, two important areas whose similarities are striking but whose differences are even more so.

This is in one dimension; in multidimensional situations in probability and statistics, it is interesting to see the effect of dimensionality on how the relevant limits are parametrised. In EVT in one dimension, the limits are parametric (Fisher-Tippett theorem: three families classically, one if one uses generalised extreme-value laws, GEV). But as soon as the dimension $d$ is at least two, limits become non-parametric (more precisely, semi-parametric: one scalar radial parameter, one spectral measure on the unit sphere). If one specialises to vines (cf. [BinO5, §2 Dependence structure]), matters decompose into bivariate copulas (non-parametric), linked by nested trees ($O(d)$ parameters). In Popa groups, there is no such abrupt discontinuity as the dimension increases through $d = 1, 1 < d < \infty$ and $d = \infty$, and it is striking that the link with Popa groups is lost as soon as $d > 1$. Here the multidimensional feature permits alternative interpretations of $(GS)$ according to the side-conditions (e.g. collinearity or other co-dependencies) imposed on its two free variables: see the comments preceding Prop. 1.1 below.

### 1.2 Associated Groups and Banach algebras

**Popa groups.** Following Popa’s analysis [Pop] of (Lebesgue) measurable solutions of $(GS)$, we equip $G_\rho \subseteq \mathbb{R}$ above with the operation

$$x \circ y = x \circ_\rho y := x + (1 + \rho x)y,$$
turning $G_\rho$ into a group, which under $S$ is isomorphic to $(\mathbb{R}_+, \times)$. One may also follow Javor [Jav] by applying $\circ \rho$ to $G_\rho^* := \{ x : 1 + \rho x \neq 0 \} = \mathbb{R} \setminus \{-\rho^{-1}\}$. We term $-\rho^{-1}$ the *Popa centre*.

This *univariate* Popa group-structure provides the group theory, previously lacking, with which to export transparently Karamata theory (whose underlying group structure was explicitly recognized by Bajšanski-Karamata [BajK] and Balkema [Bal, Ch. 9]) to the other RV theories.

The equation $(GS)$, and likewise the corresponding group structure, refers to the *ring* structure of $\mathbb{R}$ and so extends to the context of a unital commutative *real* Banach algebra $A$. Three examples here are: the complex numbers $\mathbb{C}$, the setting for the study of complex regularly varying functions, for which see [BinGT, A1.2] (in Corollary 5.6 below we characterize the continuous solutions of $(GS_\mathbb{C})$); the Euclidean algebra $\mathbb{R}^d$ equipped with componentwise (Hadamard) product, corresponding to the statistics of sea-level measurement at $d$ locations – see e.g. [RooSW], [KirRSW]; the ring of continuous functions $C[0, 1]$, corresponding to measurements of sea-levels along a coastline parametrized by $[0, 1]$. The latter two cases, characterized in §3, provide a setting for the *location and scale* standardization of their statistics: the vector-space structure allows for the translation (re-location) of each component random variable according to its mean, together with *uniform* scaling (i.e. a scale common to all components); furthermore, the componentwise product structure permits *individual* scaling of each component of a random variable or stochastic process by its variance (equivalently its precision).

For a unital Banach algebra $A$, we denote by $A^{-1}$ the open subset of invertible elements of $A$, viewed as a multiplicative group, and by $A_1$ the connected component of the identity (the *principal component* [Ric, Def. 1.4.9]), a multiplicative subgroup of $A^{-1}$, coinciding with $\exp(A)$, the exponential elements [Rud, 10.34]. For a solution $S : A \rightarrow A$ of $(GS)$, we equip the sets

$$G_{S}^*(A) := \{ x \in A : S(x) \in A^{-1} \}, \text{ and } G_S(A) := \{ x \in A : S(x) \in A_1 \}$$

with the operation $\circ_S$,

$$x \circ_S y := x + S(x)y,$$

($\circ_S$)

generating the *Popa group* corresponding to $S$. The case $S(x) := 1 - x$ yields the *circle operation* of the well established group of ‘quasi-regular’ elements of $A$ [Ric, Ch. 1 §4]; cf. [Ost2]. Since $\mathbb{R}_1 = (0, \infty)$, $G_\rho^*(\mathbb{R}) = G_{S}^*(\mathbb{R})$ and
\( \mathbb{G}_\rho(\mathbb{R}) = \mathbb{G}_S(\mathbb{R}) \) correspond to \( S = S_\rho \) as in \((Can)\). For further examples see §7 (Appendix).

We show in §4 that \( \mathbb{G}_S(\mathbb{A}) \) usually contains, as abelian subgroups, copies of \( \mathbb{G}_\rho(\mathbb{A}) \) for \( \rho \in \mathbb{A}^{-1} \); we also derive in Theorem 5.1 a characterization for \( S \) by reference to an auxiliary function that solves an equation of Goldie type, \((GGE_{SS}),\) generalizing \((GFE)\), by assuming the differentiability of its multiplicative auxiliary \( S \). This complements the Wołdzko approach [Wol] as encapsulated in a theorem in [Jav] and cited later in [BriD]; that earlier approach readily extends to the present context provided the invertible elements of the commutative field there are interpreted as referring to \( \mathbb{A}^{-1} \); see §6.1 with further details in §7 (Appendix).

Other re-interpretations of \((GS)\) are possible. Of particular interest below are the multivariate Popa groups of [BinO3,4] over a topological vector space \( X \) (note the larger category involved here), briefly the \textit{multi-Popa groups} (to maintain a clear distinction), with an operation defined via a continuous linear functional \( \rho \in X^* \), by

\[
x \circ y := x + (1 + \rho(x))y
\]

for \( x, y \) ranging over the half-space \( \mathbb{G}_\rho(X) := \{ x \in X : 1 + \rho(x) > 0 \} \). So here \( S(x) = 1 + \rho(x) \) generalizes the canonical form \((Can)\) but with \( S : X \to \mathbb{R} \), i.e. mapping to \( \mathbb{R} \) rather than back to its domain; nevertheless, here \( x \) acts affinely on \( y \), thus allowing location and uniform scaling. The groups \( \mathbb{G}_\rho(X) \) helpfully contribute a structure theorem describing the more general Popa groups \( \mathbb{G}_S(\mathbb{R}^d) \) and \( \mathbb{G}_S(C[0,1]) \).

We do not pursue yet another interpretation, studied in [BriD], in which \( S : X \to GL(X) \), for \( X \) a Banach space, so that \( x \) acts on \( y \) via \( S(x)y \), but we do note below occasional similarities. Such similarities are inevitable since \( S(a) \in GL(\mathbb{A}) \) for \( a \in \mathbb{G}_S(\mathbb{A}) \) (the map \( x \mapsto S(a)x \) being a continuous automorphism of \( \mathbb{A} \) since \( ||S(a)x|| \leq ||S(a)|| ||x|| \)). It is all the more unsurprising given that the componentwise product \( x \cdot y \) of two vectors in \( \mathbb{R}^d \) may be presented as a matrix product \( P(x)y \) in which \( P(x) := \text{diag}(x) \) is the diagonal matrix generated by \( x \) (mapping the product to composition: \( P(x \cdot y) = P(x)P(y) \)). Corresponding results from [BriD, Ths 7, 8] will be seen from the present context as ‘degenerate’ variants of our results (while sometimes our results are special cases of theirs): see the Remark after Th.4.2.

\textit{Banach algebras.} We are concerned below with the characterization, in a
unital commutative real Banach algebra $A$, of those solutions $S : A \to A$ of the Gołąb-Schinzel equation

$$S(x + yS(x)) = S(x)S(y) \quad (x, y \in A) \quad (GS_A)$$

that, when restricted to $G_S^* = G_S^*(A) := \{x : S(x) \in A^{-1}\}$, are Fréchet differentiable at the points of $G_S(A)$ relative to its range (in contrast to $A$-differentiability: see §5). In this case, with $1_A$ denoting the identity element of $A$ (under multiplication), a significant role is played by the adjustor, defined in Th. 5.1 as the map

$$N(x) := S(x) - 1_A - (S(1_A) - 1_A)x \quad (x \in G_S^*).$$

We assume here and below that

$$1_A \in G_S^*,$$

i.e. that $S(1_A) \in A^{-1}$. Thus $N$ measures divergence from the canonical affine form. This is the central theme of §5.

Here and below $A$ is always a unital commutative real Banach algebra (and so below ‘linear’ means ‘$\mathbb{R}$-linear’, unless otherwise indicated), and the quantifier over $x, y$ in $(GS_A)$ is restricted to $G_S^*(A)$. It is noteworthy that, unlike in the case of $A = \mathbb{R}$ (cf. [BinO1]), in a multi-dimensional context quantifier weakening (which we do not pursue here) will broaden the nature of a solution function $S$, as was pointed out by Marshall and Olkin in [MarO1] for the similar context of the multivariate Cauchy functional equation; see also [MarO2, MarO3] and §6.2.

Henceforth we view $S$ as a homomorphism. Our starting point is to establish the group structure it generates on its domain, the significance of its kernel $N$ (‘$N$ for null’) for its image, and ‘invariance of openness’ under $\circ_S$-shifts. Later in §5 we will be concerned with the adjustor function $N$ above, which takes values in $N$.

**Proposition 1.1.** Suppose $S : A \to A$ satisfies $(GS_A)$.

(i) $(G_S^*, \circ_S)$ is a group and $G_S$ a subgroup of $G_S^*$.

(ii) Furthermore,

$$N = N_S := \{a \in A : S(a) = 1_A\} \subseteq G_S^*$$

is a subgroup of $G_S$ on which $+$ and $\circ_S$ agree, so an additive subspace of $A$. 

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(iii) If 0 is an interior point of $\mathbb{G}_S(\mathbb{A})$, then so is $w$ for any $w \in \mathbb{G}_S(\mathbb{A})$, and, conversely, if $w$ is an interior point of $\mathbb{G}_S(\mathbb{A})$, then so also is 0. The same holds relativized to a line $\langle u \rangle$.

(iv) For $S$ continuous, both $(\mathbb{G}_S, \circ_S)$ and $(\mathbb{G}_S^*, \circ_S)$ are topological groups (in the subspace topology induced by $\mathbb{A}$) and $\mathcal{N}$ is closed.

**Proof.** For parts (i) and (ii), which are routine, see §7 (Appendix).

(iii) We know from (i) that for $w \in \mathbb{G}_S(\mathbb{A})$ the map $x \mapsto w \circ_S x$ takes $\mathbb{G}_S(\mathbb{A})$ into $\mathbb{G}_S(\mathbb{A})$. So if $B_\delta(0) \subseteq \mathbb{G}_S(\mathbb{A})$, then

$$w + S(w)B_\delta(0) = w \circ_S B_\delta(0) \subseteq \mathbb{G}_S(\mathbb{A}).$$

Here $S(w)B_\delta(0)$ is an open neighbourhood of 0, since multiplication by an invertible element of $\mathbb{A}$ is a homeomorphism of $\mathbb{A}$. So $w$ is interior to $\mathbb{G}_S(\mathbb{A})$.

It now also follows that, for $w \in \langle u \rangle$, if $x \in \langle u \rangle \cap B_\delta(0) \subseteq \langle u \rangle \cap \mathbb{G}_S(\mathbb{A})$, then $w + (S(w)B_\delta(0) \cap \langle u \rangle) \subseteq \mathbb{G}_S(\mathbb{A}) \cap \langle u \rangle$ and, as $S(w)B_\delta(0)$ is open, its intersection with $\langle u \rangle$ is relatively open.

Conversely, for $w \in \mathbb{G}_S(\mathbb{A})$, by (i) the inverse $w_S^{-1} \in \mathbb{G}_S(\mathbb{A})$, so if $(w + B_\delta(0)) \subseteq \mathbb{G}_S(\mathbb{A})$, then again by (i),

$$w_S^{-1} + S(w_S^{-1})(w + B_\delta(0)) = w_S^{-1} \circ (w + B_\delta(0)) \subseteq \mathbb{G}_S(\mathbb{A});$$

$$S(w)^{-1}B_\delta(0) = w_S^{-1} + S(w)^{-1}(w + B_\delta(0)) \subseteq \mathbb{G}_S(\mathbb{A}),$$

as $S(w)^{-1} = S(w_S^{-1})$, $S$ being a homomorphism. Here $S(w)^{-1}B_\delta(0)$ is an open neighbourhood of 0, again as multiplication by an invertible element of $\mathbb{A}$ is a homeomorphism of $\mathbb{A}$.

A similar argument also holds under relativization to $\langle u \rangle$.

(iv) Now take $S$ continuous. Here $a \mapsto -aS(a)^{-1}$ is continuous, since inversion is continuous on $\mathbb{A}^{-1}$ [Rud, p. 268, Th. 10.34], [Con, VII Th.2.2]; clearly $(a, b) \mapsto a + S(a)b$ is continuous: $\mathbb{G}_S$ is a topological group. □

**Corollary 1.1.** The multiplicative group $(S(\mathbb{G}_S(\mathbb{A})), \cdot)$ is isomorphic to $\mathbb{G}_S^* / \mathcal{N}$.

**Proof.** This is immediate, since $S(a \circ_S b) = S(a)S(b)$; so by Prop. 1.1 $S$ is a homomorphism from $\mathbb{G}_S^*$ to $(S(\mathbb{G}_S(\mathbb{A})), \cdot)$ with kernel $\mathcal{N}$. □

**Remarks.** 1. As $\mathbb{A}^{-1}$ and $\mathbb{A}_1$ are open [Con, Ch. 7 Th. 2.2], so too is $\mathbb{G}_S^*(\mathbb{A}) = S^{-1}(\mathbb{A}^{-1})$ and $\mathbb{G}_S(\mathbb{A}) = S^{-1}(\mathbb{A}_1)$, for $S$ continuous. Following [DalF] say that $\mathbb{A}$ has *dense invertibles* if $\mathbb{A}^{-1}$ is dense in $\mathbb{A}$, a convenient property whenever
invertibility is needed. In the present circumstances this condition holds iff the topological stable rank of $A$ is 1; this in turn is equivalent to the existence of a dense set of points whose spectra have empty interior: see [CorS, Cor. 1.10]. Spectra emerge in §5.

2. $S[G_S^*(A)]$ is an (abelian) multiplicative subgroup of $A^{-1}$, as $S$ is a homomorphism (by $(GS)$).

3. $A_\lambda$, the connected component of the identity, coincides with the subgroup generated by the set of elements which have a logarithm [Ric, Th.1.4.10], [Rud, Th. 10.34(c)], with connection from $1_\lambda$ to $g = e^h$ provided by $e^{th}$.

4. If the operation $\circ_S$ is commutative, then for $a, b \in A^{-1} \cap G^*_S(A)$

$$a+S(a)b = b+S(b)a = (1_\lambda - S(b))b^{-1} = (1_\lambda - S(a))a^{-1} = \text{constant} = -\rho,$$

and so

$$S(a) = 1_\lambda + \rho a \quad (a \in A^{-1} \cap G_S(A)),$$

hence for all $a \in G_S$ if $A$ has dense invertibles. For $\mathbb{R}$, this is implicit in [GolS, Lemma 5] and explicit for Popa [Pop, Prop. 3], where it is key. If $1_\lambda \in G_S^*(A)$, this Remark is non-vacuous. Evidently, the operation $x \circ_\rho y$ applied to $x, y$ in any commutative ring is commutative.

The Popa groups $G_\rho(A)$ below emerge in Prop 4.4 as subgroups of $G_S(A)$.

**Proposition 1.2.** The multiplicative group $(S(G_S(A)), \cdot)$ is isomorphic to the Popa group $G_\rho(A) := (\rho^{-1}(\text{ran}S - 1_\lambda), \circ_\rho)$, for each $\rho \in A^{-1}$. The latter contains $\{\rho^{-1}(e^{t\rho} - 1_\lambda) : t \in \mathbb{R}\}$ as a one-parameter connected subgroup of $G_\rho(A)$.

**Proof.** Put $c = \rho^{-1}$; then $y = \eta_\rho(x) := 1_\lambda + \rho x$ iff $\eta_\rho^{-1}(y) = c(y - 1_\lambda)$. So

$$\eta_\rho^{-1}(\text{ran}S) = c(\text{ran}S - 1_\lambda); \quad \eta_\rho(g)\eta_\rho(h) = \eta_\rho(g \circ_\rho h); \quad g \circ_\rho h = \eta_\rho^{-1}(\eta_\rho(g)\eta_\rho(h)).$$

The final assertion is clear, since each $e^{t\rho} \in A^{-1}$. $\square$

## 2 Beyond Brillouët-Dhombres-Brzdek and linear-plus-1

The theorems of this section are motivated by the observation that $(GS)$ may be solved with $S$ a Fréchet differentiable function in the form

$$S(x) = 1_\lambda + \gamma_S(x),$$
for $\gamma = \gamma_S$ linear and continuous, provided $\gamma$ has the following property which we may term $A$-homogeneity over $G_S$:

$$\gamma(u\gamma(v)) = \gamma(u)\gamma(v) \quad (u, v \in G_S).$$

A weakened version of the property, considered at the end of the section, is also relevant in describing solutions to the tilting equation of Section 5. To illustrate the property consider the following examples:

(i) $\gamma(x) = \gamma \cdot x$ for some $\gamma \in A$,

(ii) $\gamma(x) = \gamma(x) \cdot 1_A$ for some continuous linear $\gamma : A \to \mathbb{R}$: which includes the case:

(iii) $\gamma(x) = x(\theta)$ for $x \in A = C[0, 1]$ and some fixed $\theta$ with $0 \leq \theta \leq 1$.

We will see later in Cor. 5.4 that in $\mathbb{R}^d$ the property (ii) may hold 'partwise'. (That is, there is a (fixed) partition of $\{1, ..., d\}$ such that (ii) holds for pairs of vectors restricted to the subspace generated by the natural base vectors corresponding to any one part.) In such cases $\gamma$ is $\mathbb{R}^d$-homogeneous.

Our main result in this section gives two decompositions of a Fréchet differentiable solution of $(GS)$ into a linear part and a part that is orthogonal relative to both of the symmetric bilinear forms generated by $\gamma$:

$$\langle a, b \rangle = \gamma(ab) \text{ and } \langle a, b \rangle_\gamma = \gamma(a\gamma(b)),$$

though we have yet to prove (see below) the symmetry of the latter form. When working relative to the latter, we speak of $\gamma$-orthogonality. One decomposition yields a linear and a non-linear part, the other an $A$-differentiable part (in the sense of §5) – ultimate source here of the distinction between analytic and real-analytic solutions. We need a preliminary calculation.

**Proposition 2.1.** For $S$ Fréchet differentiable, satisfying $(GS)$

$$S'(c) = S(c)S'(0)S(c)^{-1} \quad (c \in G_S^*) .$$

**Proof.** For fixed $a \in G_S^*$, the ‘affine’ map $b \mapsto a \circ_S b = a + S(a)b$ is Fréchet differentiable on $A$ and is onto (as $S(a)$ is invertible and multiplication is continuous) with derivative $S(a)$. For $b \in G_S^*$, as $a + S(a)b = a \circ_S b \in G_S^*$ and $G_S^*$ is open, with $S$ Fréchet differentiable, the Chain Rule applies [Ber, Th. 2.1.15]: differentiating $(GS)$ with respect to $b$ and setting $a = b_S^1$ yields

$$S'(a + bS(a))S(a) = S(a)S'(b) :$$

$$S'(b) = S(b)S'(0)S(b)^{-1} ,$$

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since \( S(a)^{-1} = S(a_S^{-1}) = S(b) \).

The next result for \( S \) real-valued and \( \gamma \) injective can apply to \( \mathbb{R}^d \) only for \( d = 1 \), as then \( \gamma \) has rank 1 and nullity 0. This points up the essential difference between the linear form in Theorem 2.1 and that of the Brillouët-Dhombres-Brzdek characterization appropriate to Euclidean spaces (of \( 1 + \rho(x) \) with \( \rho \) in (the dual of) \( \mathbb{R}^d \).

**Theorem 2.1 (Decomposition Theorem).** For \( S \) Fréchet differentiable satisfying \((GS)\) with \( \gamma = S'(0) \), there exist two functions \( m, n \) both with range \( N_\gamma = \{ x : \gamma(x) = 0 \} \) with

- \( (i) \) \( n(x) \) orthogonal in either of the above senses to \( \gamma(x) \) and with \( n(x) = o(x) \) as \( x \to 0 \),
- \( (ii) \) \( m(x) \) orthogonal to \( \gamma(x) \) and \( \gamma \)-orthogonal to both \( \gamma(x) \) and \( \gamma(1_A)x \), and

\[
S(x) = 1_A + \gamma(x) + n(x) = 1_A + \gamma(1_A)x + m(x).
\]

In particular,

\[
S(N) \subseteq \{ y : \gamma(y - 1_A) = 0 \} = 1_A + N_\gamma,
\]

and if \( \gamma \) is injective

\[
S(x) = 1_A + \gamma(1_A)x.
\]

This will be a corollary of the following result in which we prove an \( A \)-homogeneity property weakened by placing an additional \( \gamma \), ‘like a mask’, over the desired relation, as in (pH) below.

**Remark.** For the special case of \( S \) real-valued on \( A = \mathbb{C} \), the map \( \gamma \) is homogeneous, being real-valued, so \( \gamma(1+\zeta) \) is complex, implying compensation by a necessarily complex component \( m(\zeta) \). So for this case, the decomposition is uninformative. See Cor. 5.5.

**Theorem 2.2 (Phantom characterization of linearity-plus-1).** For \( S \) Fréchet differentiable with \( \gamma = S'(0) : S \) satisfies

\[
S(a + bS(a))S(a) = S(a)S(b) \quad (a, b \in \mathbb{G}_S) \quad (GS)
\]

iff both

- \( (i) \)
  \[
  \gamma(S(c)h) = \gamma((1_A + \gamma(c))h) \quad (h \in A, c \in \mathbb{G}_S),
  \]
- \( (ii) \)
  \[
  \gamma(\gamma(k)h) = \gamma(k\gamma(h)) \quad (k, h \in A),
  \]

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subject to the similarity relations $S'(c) = S(c)\gamma S(c)^{-1}$, in which case for all $a \in \text{ran}(S)$,
\[
\gamma((a^{-1}\gamma a)(k)h) = \gamma(k\gamma(h)) \quad (k, h \in A).
\]  

(***)

In particular, (*) implies the phantom (or masked) linearity-plus-1
\[
\gamma(S(c)) = \gamma(1_A + \gamma(c)) \quad (c \in G_S)
\]

(pL)

(‘p for phantom, L for linear’), while (**) implies symmetry
\[
\langle a, b \rangle \gamma = \langle b, a \rangle \gamma,
\]

and (***) implies
\[
\gamma(\gamma(h) - \gamma(1_A)h) = 0 \quad (h \in A).
\]

This last implies the following phantom homogeneity:
\[
\gamma(\gamma(a\gamma(b))) = \gamma(\gamma(a)\gamma(b)) = \gamma(\gamma(b\gamma(a))).
\]

(pH)

**Proof.** Relegating the routine details to §7 (Appendix), increment both $a$ and $b$ in $(GS)$ by $h$. Expansion of $S(a + h)$ and $S(b + h)$ to order $o(h)$ and use of Prop 2.1 give
\[
\gamma(S(c)h) = \gamma(h + \gamma(c\gamma h)),
\]

(*)

with $c$ for the inverse of $b$ under $\circ$, all steps being reversible. Differentiating with respect to $c$ in direction $k$ leads to
\[
\gamma(k\gamma h) = \gamma(S'(c)(k)h) = \gamma(S(c)\gamma S(c)^{-1}(k)h),
\]

which for $a = S(c)$ yields the claim (***) and for $c = 0$ the claim (**). Furthermore, writing $S(c)k$ for $k$ gives via (*) and (**) that
\[
\gamma(S(c)h) = \gamma(h + \gamma(c\gamma h) = \gamma(h + \gamma(h\gamma(c)) = \gamma(h(1 + \gamma(c))).
\]

That is, (*) holds. Evidently (*) and (**) yield (*'), and so the conjunction of (*) and (**) yields $(GS)$. The remaining conclusions are routine. $\Box$

**Proof of Theorem 2.1.** Since $\gamma(\gamma(1_A)x) = \gamma(1_A\gamma(x))$,
\[
\gamma(S(x) - 1_A - \gamma(1_A)x) = \gamma(S(x) - 1_A - \gamma(x)) = 0.
\]
Now take \( m(x) := S(x) - 1_A - \gamma(1_A)x \in \mathcal{N}_\gamma = \{ x : \gamma(x) = 0 \} \). By (**) 
\[
\langle \gamma(x), m(x) \rangle = \gamma(\gamma(x)m(x)) = \gamma(x\gamma(m(x))) = \gamma(0) = 0,
\]
so \( m(x) \) is orthogonal to \( \gamma(x) \). Again by (**) 
\[
\langle \gamma(1_A)x, m(x) \rangle = \gamma(\gamma(1_A)x\gamma(m(x))) = 0 = \gamma(\gamma(1_A)x)m(x)
\]
\[
= \gamma(\gamma(\gamma(x))m(x)) = \gamma(\gamma(x)\gamma(m(x))) = \langle \gamma(x), m(x) \rangle
\]
so \( m(x) \) is \( \gamma \)-orthogonal to both \( \gamma(1_A)x \) and to \( \gamma(x) \).
Put \( n(x) = S(x) - 1_A - \gamma(x) \); then \( n(x) = o(x) \), and by (*) 
\[
\langle \gamma(x), n(x) \rangle = \gamma(\gamma(x)(S(x) - 1_A - \gamma(x))) = 0,
\]
\[
\langle \gamma(x), n(x) \rangle = \gamma(\gamma(x))(S(x) - 1_A - \gamma(x))) = 0.
\]
If \( S(x) = 1_A \), then clearly \( \gamma(S(x) - 1_A) = 0 \), proving the final claim. \( \square \)

**Corollary 2.1.** With the assumptions of Theorem 2.1, for \( c \in \mathcal{N} \), both 
\( \gamma(c) \in \mathcal{N}_\gamma \) and \( \gamma(1_A)c \in \mathcal{N}_\gamma \). If \( \gamma \) is injective, then 
\[
S(c) = 1_A + \gamma(c) = 1_A + \gamma(1_A)c \quad (c \in \mathcal{G}_S),
\]
so \( \gamma \) is \( A \)-homogeneous and \( \mathcal{N} = \mathcal{N}_\gamma = \{ 0 \} \).

**Proof.** Since \( S(c) = 1_A \) for \( c \in \mathcal{N} \), by Theorem 2.1 
\[
\gamma(1_A) = \gamma(S(c)) = \gamma(1_A) + \gamma(\gamma(c)),
\]
so \( 0 = \gamma(\gamma(c)) = \gamma(\gamma(1_A)c) \), yielding both \( \gamma(c) \in \mathcal{N}_\gamma \) and \( \gamma(1_A)c \in \mathcal{N}_\gamma \).

For \( \gamma \) injective, \( \mathcal{N}_\gamma = \{ 0 \} \) and \( S(c) = 1_A + \gamma(c) \), for \( c \in \mathcal{G}_S \). Likewise \( \gamma(\gamma(c)) = \gamma(\gamma(1_A)c) \) implies \( \gamma(c) = \gamma(1_A)c \), so \( \gamma \) is \( A \)-homogeneous on \( \mathcal{G}_S \).
Now if \( c \in \mathcal{N} \), then \( \gamma(c) = 0 \), as \( S(c) = 1_A \), so \( \{ 0 \} \subseteq \mathcal{N} \subseteq \mathcal{N}_\gamma = \{ 0 \} \), giving \( \mathcal{N} = \mathcal{N}_\gamma \). \( \square \)

**Remarks.** 1. When \( S \) is real-valued and \( \gamma \neq 0 \), the Cor. 2.1 captures the traditional and well established fact that \( S(x) = 1 + \gamma(x) \) with \( \gamma \) linear. The relevant statistical literature includes Oakes and Dasu [OakD].
2. Later Theorem 5.3 identifies \( \mathcal{N}_\gamma \) as the maximal vector subspace \( \mathcal{H} \) of \( \mathcal{N} \), whence above \( \gamma(c) \in \mathcal{H} \) and \( \gamma(1_A)c \in \mathcal{H} \) for \( c \in \mathcal{N} \).
Definition. Recalling, from e.g. [Kec], the set-theoretic notation \( \omega = \{0, 1, 2, \ldots\} \), say that \( \gamma \) is \( \omega \)-homogeneous if the homogeneity property \( \gamma(v\gamma(u)) = \gamma(v)\gamma(u) \) holds for any \( u \) and \( v \in \{u\gamma(u)^k : k \in \omega\} \). This is equivalent to a power-raising (-shifting) multiplicative effect of \( u \) under \( \gamma \):

Proposition 2.2. \( \gamma \) is \( \omega \)-homogeneous iff
\[
\gamma(u\gamma(u)^k) = \gamma(u)^{k+1} \text{ for all } u \text{ and all } k = 0, 1, \ldots
\]

Proof. A routine induction establishes this. See §7 (Appendix).

Below \( f \) denotes both a real-analytic function and its natural extension to \( A \). See §5 for applications (Prop. 5.1) and an extension of the domain of validity (Theorem S).

Corollary 2.2. For \( f \) with all Taylor coefficients non-zero and radius \( R > 0 \):
the linear continuous \( \gamma \) satisfies \((\times)\) for \( \gamma(u) \in A^{-1} \) iff
\[
f(t\gamma(u)) = \gamma(f(t\gamma(u)) \cdot u/\gamma(u)) \quad (0 \leq t||\gamma(u)|| < R).
\]
Thus this equivalence holds both for \( f(x) := e^x - 1_A \) and its inverse:
\[
e^{t\gamma(u)} - 1_A = \gamma((e^{t\gamma(u)} - 1_A) \cdot u/\gamma(u)) \quad (t \geq 0),
\]
\[
\log(1_A + t\gamma(v)) = \gamma(\log(1_A + t\gamma(v)) \cdot v/\gamma(v)) \quad (0 \leq t||\gamma(v)|| < 1).
\]

Proof. Assuming \((\times)\), apply \( \gamma \) term by term to the series expansion of \( f \):
\[
\gamma uf(t\gamma(u))/\gamma(u) = \sum_{n=0}^{\infty} a_n t^n \gamma(u^\gamma(u)^n)/\gamma(u) = f(t\gamma(u)) \quad (0 \leq t||\gamma(v)|| < R).
\]
Conversely, compare coefficients at \( t^k \) to obtain \((\times)\).

Proposition 2.3. Suppose that \( S \) continuous satisfies \((GS)\) and, for some continuous linear \( \gamma \), takes the form
\[
S(x) = 1_A + \gamma(x) + e(x).
\]
Then
\[
\lim_{x \to 0} e(x)/||x||^2 = 0 \text{ implies the power-raising property } (\times) \text{ above.}
\]
The converse holds for \( e(x \circ x)/2e(x) \) bounded away from \( 1_A \) (as \( x \to 0 \)).

Proof. For the sake of continuity, we defer this to §7 (Appendix).

Remark. The proof is somewhat reminiscent of the Hyers-Ulam stability theorem with its near additivity: see e.g. [CabC]. We conjecture that non-additivity of \( e(.) \) implies boundedness away from unity. Example 7.3 in the Appendix is illuminating here.
3 Finite dimensions and $C[0, 1]$

Here we denote by $\cdot$ (rather than by $\odot$) the componentwise Hadamard-Schur product applied to vectors in $\mathbb{R}^d$, which turns the $d$-vectors into a Banach algebra under the Euclidean norm. For $S$ a continuous solution of $(GS)$ on this Banach space, $\mathbb{G}_S^*(\mathbb{R}^d)$ is a topological group under the Euclidean norm topology by Prop. 1.1, hence by the Montgomery-Zippin theorem this is a Lie group [MonZ] or [Tao, Th. 1.1.13]. Then $S$ is $C^\infty$ in the real-variable sense, cf. [BriD]. We use this fact in Corollary 5.6 below to give a new treatment of $(GS_C)$ based on the differentiability of the adjustor $N$ of §1, viewing $C$ as a two-dimensional real Banach algebra. Our first result below characterizes which ‘linear’ functions solve $(GS)$ in $\mathbb{G}_S^*(\mathbb{R}^d)$. That these are indeed the only non-degenerate continuous solutions (i.e. truly $d$-variate, below) is asserted thereafter in Th. 3.2. It is convenient here to use the language of partitions $P$ of $\{1, ..., d\}$ into (disjoint) subsets, termed parts $I$. When needed, $|I|$ denotes the cardinality of the part $I$. Later, in the context of $C[0, 1]$, we use partitions of $[0, 1]$ into compact parts $K$.

**Theorem 3.1 (Euclidean Characterization Theorem).**

(i) The continuous solutions $S : \mathbb{G}_S^*(\mathbb{R}^d) \to \mathbb{R}^d$ of

$$S(x + S(x) \cdot y) = S(x) \cdot S(y)$$

for $S(x) = (..., 1 + \sigma_i(x), ...) \text{ taking the form}$

$$S(x) := 1 + \Sigma x \text{ with } \Sigma = (\sigma_{ij}),$$

where $1 := (1, 1, ..., 1)'$, have matrices $\Sigma = (\sigma_{ij})$ satisfying, for

$$\sigma_i(x) := \Sigma_j \sigma_{ij} x_j;$$

$$\sigma_{ij} = 0 \text{ or } \sigma_i(x) \equiv \sigma_j(x) \quad (1 \leq i, j \leq d).$$

(ii) Hence there are: a linear map $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ with

$$S(x) := 1 + \sigma(x),$$

a ‘generator’ functional $\rho : \mathbb{R}^d \to \mathbb{R}$

$$\rho(x) := \Sigma_i \rho_i x_i,$$
and a partition $\mathcal{P}$ of $\{1, \ldots, d\}$ with parts $I$ so that, with $e_1$ the projection onto the span $\langle\{e_i : i \in I\}\rangle$ of the corresponding natural base vectors $e_i = (\delta_{ij})$,

$$\sigma(x) := \sum_{I \in \mathcal{P}} \sum_{i \in I} \rho(e_i x) e_i.$$ (†)

(iii) For instance, partitioning of $\{1, \ldots, d\}$ into two parts $I, J$ generates the solutions

$$\sigma_i(x) = \sigma_I(x) = \sum_{k \in I} \rho_k x_k \quad (i \in I), \quad \sigma_j(x) = \sigma_J(x) := \sum_{k \in J} \rho_k x_k \quad (j \in J).$$

(iv) In particular, in $\mathbb{R}^3$ the solutions $S = (S_1, S_2, S_3)^T$ take one of the following three forms:

$$S_i(x) = 1 + \rho_i x_i \quad (i = 1, 2, 3);$$

or with $(i, j, k)$ a permutation of $(1, 2, 3)$:

$$S_i(x) = S_j(x) = 1 + \rho_i x_i + \rho_j x_j \quad \text{and} \quad S_k(x) = 1 + \rho_k x_k;$$

or

$$S_1 = S_2 = S_3 = 1 + \rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3.$$

Thus the set $N_S = \{x \in \mathbb{R}^3 : S(x) = 1\}$ is a vector subspace of corresponding dimension $0, 1, 2$.

**Proof.** (i) We compute the two sides of $(GS)$:

$$S(x) \cdot S(y) = (1 + \Sigma x) \cdot (1 + \Sigma y) = 1 + \Sigma x + \Sigma y + \Sigma x \cdot \Sigma y,$$

$$S(x + S(x)y) = 1 + \Sigma(x + y + (\Sigma x) \cdot y) = 1 + \Sigma x + \Sigma y + \Sigma((\Sigma x) \cdot y).$$

On comparing, $(GS)$ reduces to

$$\Sigma((\Sigma x) \cdot y) = \Sigma x \cdot \Sigma y.$$

We compute the $i$-th component on each side:

$$RHS_i = \Sigma_j \sigma_{ij} x_j (\Sigma_k \sigma_{ik} y_k) = \Sigma_{jk} \sigma_{ij} \sigma_{ik} x_j y_k,$$

$$LHS_i = \Sigma_k \sigma_{ik} \Sigma_j \sigma_{kj} x_j y_k = \Sigma_{jk} \sigma_{ik} \sigma_{kj} x_j y_k.$$ 

Comparison of the coefficient of $x_j y_k$ on each side yields that, for all $ijk$,

$$\sigma_{ij} \sigma_{ik} = \sigma_{ik} \sigma_{kj} : \quad \sigma_{ik} = 0 \text{ or } \sigma_{ij} = \sigma_{kj},$$

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as asserted.

(ii) The partition statements are corollaries of (i) as follows. For \( \rho \in \mathbb{R}^d \) and \( K \subseteq \{1, \ldots, d\} \) put

\[
\sigma_K(x) := \sum_{k \in K} \rho_k x_k.
\]

Thus \( (\sigma_K)_k = 0 \) for \( k \notin K \). Partition \( \{1, \ldots, d\} \) into \( I, J \) and take \( \sigma_i = \sigma_I \) for \( i \in I \), \( \sigma_j = \sigma_J \) for \( j \in J \).

Then \( \sigma_{ij} = 0 \) for \( i \in I, j \in J \); indeed, \( \sigma_{ij} = 0 \) for \( i \in I \), as \( j \notin I \). Similarly for \( i \in J \) and \( j \in I \). So

\[
\begin{align*}
\sigma_i &= \sigma_k \text{ for } i, k \in I \text{ and } \sigma_j = \sigma_k \text{ for } j, k \in J, \\
\sigma_{ij} &= 0 \text{ for } (i, j) \in (I \times J) \cup (J \times I).
\end{align*}
\]

Thus \( \sigma_{ik} = 0 \) or \( \sigma_i = \sigma_k \).

(iii) and (iv) are now immediate corollaries of (ii). \( \square \)

**Remark.** For \( A = \mathbb{R}^2 \), \( S \) either takes the *independent form*:

\[
S(x) = S(x_1, x_2) = 1 + \rho x = (1 + \rho_1 x_1, 1 + \rho_2 x_2), \quad \text{with } \rho_1 \rho_2 \neq 0,
\]

and then \( \mathcal{N}(\rho) := \{(x_1, x_2) : (\rho_1 x_1, \rho_2 x_2) = (0, 0)\} = \{(0, 0)\} \) has co-dimension 2; or it takes the *co-dependent form* with its two components given by \( S_1(x) = S_2(x) = 1 + \rho_1 x_1 + \rho_2 x_2 \), where the corresponding \( \mathcal{N}(\rho) \) has co-dimension 1.

In \( \mathbb{R}^3 \) the co-dimensions can be 3, 2, or 1:

\[
\mathcal{N}(1, 1, 1) = \{0, 0, 0\}, \mathcal{N}(0, 1, 1) = \mathbb{R} \times \{(0, 0)\}, \mathcal{N}(0, 0, 1) = \mathbb{R}^2 \times \{0\}.
\]

Here the corresponding ranges \( \mathcal{N}(\rho) \) are of dimensions 0, 1, 2.

We complement Th. 3.1 by proving that it is *exhaustive*, i.e. includes all the continuous solutions, as qualified below. The proof is in principle straightforward: it reduces the analysis of \((GS)\) to a series of interconnected scalar equations. This is unavoidably lengthy and messy. For the sake of simplicity we confine ourselves to the case \( d = 2 \). For the sake of continuity, we defer the proof to §7 (Appendix).
Theorem 3.2 (Exhaustivity). The non-trivial continuous solutions for \( s : \mathbb{R}^2 \to \mathbb{R}^2 \) to the Gołąb-Schinzel equation below in the algebra of \( \mathbb{R}^2 \)
\[
s(a + s(a)b) = s(a)s(b)
\]
(GS)
take for some \( \rho \in \mathbb{R}^2 \) either the ‘co-dependent form’:
\[
s(x_1, x_2) = (1 + \rho_1 x_1 + \rho_2 x_2, 1 + \rho_1 x_1 + \rho_2 x_2) = (\eta_\rho(x), \eta_\rho(x)) = 1 + x(\rho^T, \rho^T) \quad \text{(matrix form)},
\]
or the ‘independent form’:
\[
s(x_1, x_2) = (1 + \rho_1 x_1, 1 + \rho_2 x_2),
\]
or the ‘degenerate’ (univariate) form:
\[
s(x_1, x_2) = (\eta_\rho(x_1), \tau(x_1)) \text{ or } (\tau(x_2), \eta_\rho(x_2)),
\]
with \( \rho \in \mathbb{R} \), where \( \tau \) solves the homomorphism equation
\[
\tau(x_1 \circ_{\rho} y_1) = \tau(x_1)\tau(y_1).
\]
(\text{Hom})

For instance, corresponding to \( \rho = 0, \rho \in (0, \infty) \) and \( \rho = \infty \), for some \( \gamma \in \mathbb{R} \)
\[
s(x_1, x_2) = (1, e^{\gamma x_1}) \text{ or } (1 + \rho x_1, (1 + \rho x_1)^\gamma) \text{ or } (x_1, x_1^\gamma), \text{ respectively.}
\]

Remark. Degenerate solutions of (GS) in the context \( S : \mathbb{R}^d \to \mathbb{R}^d \) occur when the \( d \) components of \( x \) may be partitioned, as \( x = (u, v) \) say, and \( S(x) = S(u, 0) \) so that \( S \) is not truly \( d \)-variate; then for the corresponding partition \( S(x) = (s(u), t(u)) \), the (GS) equation reduces to the two equations
\[
s(a + bs(a)) = s(a)s(b), \quad t(a + bs(a)) = t(a)t(b).
\]
A degenerate solution thus departs from the ‘1-plus-linear’ form and couples a lower-dimensional solution \( s \) of (GS) (of 1-plus-linear form’) with a sequence of components \( s(u)^{\gamma_i} \) (”\( s \)-homomorphism”), or in the maximally degenerate case when \( s \equiv 1 \) with a sequence of exponential components \( e^{\langle \gamma_i, u \rangle} \) as in [BriD, Th. 8].
We turn now to the Structure Theorem, Th. 3.3 below, and its proof. We use ‘tilde on tilde’ \( \approx \) for ‘is isomorphic to’ and \( d_I := |I| \) (the cardinality of a part \( I \) of the partition \( \mathcal{P} \) of \( \{1, \ldots, d\} \)).

**Theorem 3.3 (Structure Theorem).** (i) The Banach-algebra Popa group \( \mathbb{G}_S^*(\mathbb{R}^d) \) generated by a continuous solution \( S \) (with invertible values) of the Gołąb-Schinzel equation (GS) on \( \mathbb{R}^d \) equipped with Hadamard product is isomorphic to a direct product of multi-Popa groups:

\[
\mathbb{G}_S^*(\mathbb{R}^d) \approx \otimes_{I \in \mathcal{P}} \mathbb{G}_{\sigma_I}(\mathbb{R}^{d_I})
\]

for a partition \( \mathcal{P} \) of \( \{1, \ldots, d\} \) and some linear maps \( \sigma_I : \mathbb{R}^{d_I} \rightarrow \mathbb{R} \) with \( I \in \mathcal{P} \).

(ii) Likewise, the Popa group \( \mathbb{G}_S^*([0, 1]) \) is isomorphic to a direct product of multi-Popa groups:

\[
\mathbb{G}_S^*([0, 1]) \approx \otimes_{K \in \mathcal{P}} \mathbb{G}_{S_K}([K])
\]

for some partition \( \mathcal{P} \) of \([0, 1]\) into compact subsets and some 1-plus-linear maps \( S_K : [K] \rightarrow \mathbb{R} \) for \( K \in \mathcal{P} \).

**Proof.** (i) For \( \mathbb{G}_S^*(\mathbb{R}^d) \): by Theorem 3.1 for each \( I \) the relevant affine map is

\[
S_I := S|_{\mathbb{R}^I} : \mathbb{R}^{d_I} \rightarrow \mathbb{R}.
\]

Furthermore, projecting \( x + y \cdot S_I(y) \) to \( \mathbb{R}^{d_I} \) yields for \( x_I := \langle x_i : i \in I \rangle \) and \( y_I := \langle y_i : i \in I \rangle \)

\[
\langle x_i : i \in I \rangle + \langle y_i S_I(x_i) : i \in I \rangle = \langle x_i : i \in I \rangle + S_I(x_I)\langle y_i : i \in I \rangle = x_I + y_I S_I(x_I),
\]

which is the same binary operation as that of the multivariate Popa group \( \mathbb{G}_{S_I}(\mathbb{R}^{d_I}) \). The functions \( \{S_I : I \in \mathcal{P}\} \) may then be ‘merged’, as in Th. 3.1, to yield a single linear generator function \( \rho : \mathbb{R}^d \rightarrow \mathbb{R} \) satisfying (†), say with

\[
\rho(x) := \Sigma_{i=1}^d \sigma_i x_i.
\]

(ii) This follows from (i) by Stone-Weierstrass approximation; see §7 (Appendix). \( \square \)
Remark. In Th. 3.3 for $x \in \mathbb{G}^*_S(C[0,1])$ domain restriction $x \mapsto x_K$ represents the isomorphism map via

$$x \mapsto (x_K)_{K \in \mathcal{P}}, \quad S(x) \mapsto (S(x_K))_{K \in \mathcal{P}}.$$

Thus, if $\{k\} = K \in \mathcal{P}$ with $k \in [0,1]$, then $x_K = x(k)$, giving $C(K) = \mathbb{R}$ and $\mathbb{G}_{\sigma_K}(C[K]) = \mathbb{G}_{\sigma_k}(\mathbb{R})$, where $\sigma_K(t) = 1 + \sigma_k t$, say; so

$$\sigma_K(x) = 1 + \sigma_k x(k),$$

$$S(x)(k) \mapsto S(x)_K = 1 + \sigma_k x(k).$$

In particular, for $\mathcal{P} = \{\{k\} : k \in [0,1]\}$, take $\rho(k) := \sigma_k$ for $k \in [0,1]$; then

$$S(x) = 1 + \rho x$$

with $\rho \in C[0,1]$ (since $\rho = S(1) - 1 \in C[0,1]$). Here $\rho \neq 0$: for any $s \neq t$, since $K_t = \{t\}$, there is $x_s$ with $\rho(t)x_s(t) \neq \rho(s)x_s(s)$.

Remark. Our results subsume and extend the similar results in [BriD], concerned with $S(x)$ in diagonal form, specifically their Th. 7(i) for $\mathbb{R}^2$; likewise their Th. 8 refers to those functions $S(x_1,\ldots,x_d)$ in diagonal form which, like the univariate types above, are from our perspective degenerate solutions to $(GS)$ through not being properly $d$-variate.

4 Spanning Pencil Theorem

Our main theorem here, Th. 4.1 below, shows that neighbourhoods of the origin are spanned by a pencil of disjoint (modulo 0) subgroups isomorphic to canonical Popa groups. As a preliminary, we study general properties of the set $\mathcal{N} = \mathcal{N}_S = S^{-1}(1_A)$ for $S$ a continuous solution of $(GS_a)$ in the context of a unital commutative real Banach algebra $A$. We have seen in Prop 1.1 that $\mathcal{N}$ is additive. A significant issue, addressed both in this and in the next section when discussing $(GFE)$ in the Banach algebra setting, is whether $\mathcal{N}$ is a vector subspace; see e.g. Lemma 4.2 and Lemma 5.4. This is not an easy matter to verify for $A$ of dimension higher than 2 and seems to require additional hypotheses either on the range of $S$ or on appropriate solubility of the tilting equation $(T)$ in §5 (see Prop. 5.3). Even for an image isomorphic to a (commutative) field such as $\mathbb{R}$ or $\mathbb{C}$, quite some effort may be required: see the verification in [BriD, Prop. 3] and within the proof of [Brz, Th. 3].
(There the function is not assumed continuous: instead of being open, the associated Popa group $G^*_S(X)$ is assumed to have the algebraic interior point property [Lyu, §2.2].) That said, note that passing to $\mathbb{C}$-valued functions on $\mathbb{C}$, $(GFE)$ is satisfied by $(K,g)$ with multiplicative auxiliary $g(\zeta) = e^{\zeta}$ whose level set $g = 1$ is discrete. (Taking $K(\zeta) := (1 - e^{\zeta})/2$, the level set $K = 1$ is also discrete, being the solution set of $e^{\zeta} = -1$.)

When $S$ is Fréchet differentiable at $0$ and $N$ is a vector subspace, Theorem 5.3 gives a differential characterization of $N$ as

$$N = \{ u : DS(0)u = 0 \},$$

with the adjustor $N$ (of §1) linear on $N$. As a result this allows decomposition of any continuous solution of $(GEE)$ into the sum of a linear function and one that is exponential in a curvilinear sense (as a corollary of Th. 5.2).

Below, for $\Sigma \subseteq A$, $\langle \Sigma \rangle$ denotes the vector subspace of $A$ generated by $\Sigma$. Some of the initial results here, like Lemma 4.1, are known for $\mathbb{R}^+$: see e.g. [Mur, Lemma 1] (the proof needs only invertibility) which generalizes to the invertible elements $A^{-1}$.

**Lemma 4.1** ([GolS, Lemma 1], [Mur, Lemma 1]). For $a, b \in G^*_S$, if $S(a) = S(b)$, then:

(i) $S(c) = S(c + a - b)$ for any $c \in G^*_S$, so in particular, $S(a - b) = S(0) = 1_A$;

(ii) $S(c) = S(c + S(z)(a - b))$ for any $z \in G^*_S$, so in particular $S(z)a \in G^*_S$, so $S(\xi)N = N$.

**Proof:** See §7 (Appendix). \Box

**Theorem B** ([Bou, VII §2, Prop. 3]). Every non-discrete closed additive subgroup of $\mathbb{R}^n$ contains a one-dimensional vector subspace.

The proof rests on local compactness. Notice the inherent limitation: the subgroup $\mathbb{R} \times \mathbb{Z}$ is closed and non-discrete and contains a vector subspace but is not itself a vector space; we shall exclude this for $N$ below in the case $A = \mathbb{R}^2$.

**Proposition 4.1.** For $S$ a continuous solution of $(GS)$, $N$ is closed and either the singleton $\{0\}$ or dense-in-itself. For

$$N_0 := N \cap G_S,$$
\[ N_0 \text{ and so } N \text{ is closed and either the singleton } \{0\} \text{ or dense-in-itself.} \]

**Proof.** For the details see §7 (Appendix). In brief: for \( b \) close enough to 0 and any \( 0 \neq a \in N \), \( S(b)a \in N \) by Lemma 4.1(ii). Then \( ||a - aS(b)|| \leq ||a|| - S(b)|| \), so \( a \) is an accumulation point of \( N \), unless \( S = 1_A \) near 0. But then \( B_\varepsilon(0) \subseteq N \) for some \( \varepsilon > 0 \) and so \( N \), being an additive subgroup, contains \( \bigcup_{n \in \mathbb{N}} B_{n\varepsilon}(0) = A \), and so is dense-in-itself (and also \( S \equiv 1 \)). \( \square \)

**Corollary 4.1.** For \( 0 \neq a \in N \), either \( \langle a \rangle \subseteq N \) or there is \( b \in G_S^* \), with \( S(b) \notin \langle 1_A \rangle \).

**Proof.** W.l.o.g. \( N \neq \{0\} \). Following [Dal], put \( B_\varepsilon^*(0) := B_\varepsilon(0) \setminus \{0\} \). Note first that \( (\langle a \rangle \setminus \{a\}) \cap S(B_\varepsilon^*(0))a \neq \emptyset \), provided \( S(b)a = ta \) or \( S(b) = t1_A \), for some \( t \neq 1 \).

By Prop. 4.1, for each \( a \in N \), one of two cases may arise:
(i) \( (\langle a \rangle \setminus \{a\}) \cap S(B_\varepsilon(0))a \neq \emptyset \) for each \( \varepsilon > 0 \), i.e. \( S(B_\varepsilon(0)) \cap \langle 1_A \rangle \) contains points other than \( 1_A \);
(ii) for some \( \varepsilon > 0 \), \( S(B_\varepsilon^*(0))a \cap \langle a \rangle = \emptyset \).

If case (ii) never arises, then \( \langle a \rangle \cap N \) is closed and dense and so \( \langle a \rangle \subseteq N \) for each \( a \in N \) (and then \( N \) is a vector subspace, being an additive subgroup).

Otherwise, there is some \( \varepsilon > 0 \) and \( a_0 \in N \) such that \( S(b)a_0 \notin \langle a_0 \rangle \) for all \( b \in B_\varepsilon^*(0) \). So for each \( t \in \mathbb{R} \), \( S(b)a_0 \neq ta_0 \), implying \( S(b) \neq t1_A \) (otherwise \( S(b) = t1_A \) implies \( S(b)a = ta \)), and then \( S(b) \notin \langle 1_A \rangle \). \( \square \)

**Lemma 4.2.** If \( S(G_S^*(A)) \supseteq \mathbb{R} + 1_A \), then \( N \) is a vector space.

**Proof.** For \( 0 \neq a \in N \) and each \( t > 0 \) there is \( b_t \) with \( S(b_t) = t \), so \( ta = t1_Aa = S(b_t)a \in S(G_S^*(A))N \subseteq N \). By Theorem B \( \langle a \rangle \cap N \), being up to the isomorphism \( t \mapsto ta \) a dense subgroup of \( \mathbb{R} \), is all of \( \mathbb{R} \), so \( \langle a \rangle \subseteq S(G_S^*(A))N \subseteq N \) and so \( N \) is a vector subspace. \( \square \)

The condition in Lemma 4.2 is not fulfilled in Example 7.3 in the Appendix. We return to this matter below.

A simple corollary is that for a two-dimensional \( A \) the subgroup \( N \) is a vector subspace. For this we need the following

**Lemma G** (cf. [Geb]). For \( a \in A^{-1} \) with \( \langle a \rangle \subseteq N \) and \( b \in A \), if \( S(b) \notin \langle 1_A \rangle \), then \( \langle a, S(b)a \rangle \) is a two-dimensional vector subspace of \( N \). Likewise, for
if \( a \in \mathbb{A} \) with \( \langle a \rangle \subseteq \mathcal{N} \) and \( b \in \mathbb{A} \), if \( S(b)a \notin \langle a \rangle \), then \( \langle a, S(b)a \rangle \) is a two-dimensional vector subspace of \( \mathcal{N} \).

**Proof.** Two-dimensionality of \( \langle a, S(b)a \rangle \) is clear, as otherwise \( S(b)a = ta \) for some \( t \in \mathbb{R} \), and then \( S(b) = t1_\mathbb{A} \), a contradiction. Evidently, \( tS(b)a = S(b)ta \in \mathcal{N} \) as by Lemma 4.1(ii) \( S(b)\langle a \rangle \subseteq \mathcal{N} \), i.e. \( S(b)\langle a \rangle \subseteq \mathcal{N} \), and so \( \langle a, S(b)a \rangle \subseteq \mathcal{N} \).

**Corollary 4.2.** If \( \mathbb{A} \) has dimension \( \leq 2 \), then \( \mathcal{N} \) is a vector subspace.

**Proof.** W.l.o.g. \( \mathcal{N} \neq \{0\} \). By Prop. 4.1 and Theorem B there is \( a \in \mathcal{N} \) with \( \langle a \rangle \subseteq \mathcal{N} \). So either \( \mathcal{N} = \langle a \rangle \), a vector subspace, or otherwise by Lemma G \( \mathcal{N} \) contains a two-dimensional subspace, and so \( \mathcal{N} = \mathbb{A} \), again a vector subspace. \( \square \)

Lemma G has a natural extension.

**Lemma 4.3.** If \( \mathcal{N} \) contains an \( m \)-dimensional subspace generated by \( \{S(b_1)a, \ldots, S(b_m)a\} \) and \( \text{ran} S(B_\varepsilon(0)) \) is topologically at least \( (m + 1) \)-dimensional, then \( \mathcal{N} \) contains an \( (m + 1) \)-dimensional subspace \( \langle a, S(b_1)a, \ldots, S(b_{m+1})a \rangle \).

**Proof.** W.l.o.g. \( \langle 1_\mathbb{A} \rangle \cap \text{ran} S(B_\varepsilon(0)) = \emptyset \). Take \( \Sigma = \{S(b_1), \ldots, S(b_m)\} \); then \( \langle \Sigma \rangle \) is \( m \)-dimensional, so there is \( b \) with

\[
S(b)a \notin \langle \Sigma a \rangle,
\]

and so \( \langle S(b)a, \Sigma \rangle \) is \( (m + 1) \)-dimensional. Then as \( S(G)\mathbb{R}a \subseteq S(G)\mathcal{N} \subseteq \mathcal{N} \), for each such \( b \), \( \langle S(b)a, \Sigma \rangle \subseteq \mathcal{N} \), i.e. \( \mathcal{N} \) contains an \( (m + 1) \)-dimensional subspace. \( \square \)

**Corollary 4.3.** If \( \text{ran} S \) and \( \langle \mathcal{N} \rangle \) are both \( n \)-dimensional, then \( \mathcal{N} \) is a vector subspace.

**Proof.** As before by Prop. 4.1 and Theorem B, \( \mathcal{N} \) contains a 1-dimensional subspace. Now apply the preceding Lemma \( n - 1 \) times. \( \square \)

We now study the condition \( \text{ran} S \supseteq \mathbb{R}_+1_\mathbb{A} \). Our main tool is the functional equation for \( g : \mathbb{R}_+ \to \mathbb{A} \)

\[
g(s) + sg(t) = g(st) \quad (s, t \in \mathbb{R}_+),
\]

\( (g_\mathbb{A}) \)
which, as we shall see below, has solution

\[ g(t) = c_g(t - 1) \quad (t \in \mathbb{R}^+) \]

for some \( c_g \in \mathbb{A} \). We return to a more general variant of \((g_R)\) in Prop. 4.4 and again in §5 below where we see the more general appearance of the Popa group \( G_\rho(\mathbb{A}) \), here in the form \( \rho^{-1}(\mathbb{R}^+ - 1_A) = (-1, \infty) \rho^{-1} \), isomorphic to \( G_1(\mathbb{R}) \) and so to \((\mathbb{R}^+, \times)\).

**Proposition 4.2.** For \( S \) a solution of \((GS)\), if \( \text{ran} S \supseteq \mathbb{R}^+ 1_A \), then for some \( c \in \mathbb{A} \)

\[ S(c(t-1)) = t \quad (t \in \mathbb{R}^+) \]

So if \( c \in A^{-1} \), then for \( \rho = c^{-1} \)

\[ S(w) = 1_A + \rho w \quad (w \in \rho^{-1}(\mathbb{R}^+ - 1_A) = (-1, \infty) \rho^{-1}) \]

in particular \( G_\rho(\mathbb{A}) = \rho^{-1}(\mathbb{R}^+ - 1_A) \) is a subgroup of \( G_S(\mathbb{A}) \).

Conversely, if \( S(w) = 1_A + \rho w \) for some \( \rho \in A^{-1} \) and all \( w \in (-1, \infty) \rho^{-1} \), then

\[ \text{ran} S \supseteq \mathbb{R}^+ 1_A. \]

**Proof.** Suppose that \( \text{ran} S \supseteq \mathbb{R}^+ 1_A \). For \( t > 0 \), select \( w(t) \) with \( S(w(t)) = t 1_A \). Then

\[ S(w(s) + w(t)) = S(w(s) + w(t)) = S(w(s))S(w(t)) = st 1_A = S(w(st)), \]

\[ w(s) + sw(t) = w(st) \mod \mathcal{N}, \]

by Lemma. 4.1(i). Thus \( w \) satisfies \((g_R) \mod \mathcal{N} \). Put \( c := w(2) \). Then for \( t > 0 \), with \( s = 2 \),

\[ w(2) + 2w(t) = w(2t) = w(t) + tw(2) \mod \mathcal{N}, \]

\[ w(t) = w(2)(t - 1) \mod \mathcal{N} = c(t - 1) \mod \mathcal{N}, \]

\[ w(t) = c(t - 1) + n(t), \text{ say, with } n(t) \in \mathcal{N}. \]

So, as \( S(n(t)) = 1_A \),

\[ S(w(t)) = S(n(t) + c(t - 1)) = S(n(t) + c(t - 1)S(n(t))) = S(n(t))S(c(t - 1)) = S(c(t - 1)) = t 1_A. \]
For $c \in A^{-1}$, as $w = c(t - 1)$ iff $1_A + c^{-1}w = t1_A$, by Prop. 1.2 the remaining assertions are clear. □

**Proposition 4.3** (cf. [BriD, Prop. 3]). For $S$ a solution of $(GS)$, if $1_A \in G_S$, and $S(G_S)$ contains an interval on $\langle 1_A \rangle$ contiguous with $1_A$, then $\mathcal{N}$ is a vector subspace; this is so when $S(G_S) \cap \langle 1_A \rangle$ is non-meagre on $\langle 1_A \rangle$. If $S(G_S) \supseteq A_1$, it is also an ideal; this is so when $S(G_S)$ is non-meagre.

**Proof.** The two assertions follow from $S(G_S)\mathcal{N} = \mathcal{N}$. For the case of $\langle 1_A \rangle$, as $\mathcal{N}$ is additive and $\mathbb{R}$ is the union of all the iterated vector sums of any non-empty open interval, it follows that $a\mathbb{R}1_A \subseteq \mathcal{N}$, for each $a \in \mathcal{N}$. As $\mathcal{N}$ is additive, this in turn implies that $\mathcal{N}$ is a vector subspace. The second assertion is similarly proved, since for any $a \in \mathcal{N}$ the linear span of $B_{\delta}(1_A)$ is $A$, and so $A\mathcal{N} \subseteq \mathcal{N}$, i.e. $\mathcal{N}$ is a closed ideal.

The two particular cases asserted follow from the Interior-point Theorem for category (Steinhaus-Piccard-Pettis theorem [Oxt, Th. 4.8] or [BinO2]): indeed, $G_S$, as an open subset of a Banach space, is analytic, so $S(G_S)$, being the continuous image of an analytic set, is analytic, so has the Baire property, by Nikodym’s theorem [Rog]. So $1_A \in \text{int}(S(G_S)S(G_S)^{-1}) \subseteq S(G_S)$, the latter inclusion holding because $S(G_S)$ is a group. □

We recall that $G_S := S^{-1}(A_1)$ with $A_1$ the connected component of the identity $1_A$. Below our assumptions imply that $1_A$ is not isolated in ran$S = S(G_S(A))$. The latter is a natural property, as otherwise $S^{-1}(\{1_A\})$ is an open neighbourhood of 0 on which $S \equiv 1_A$. The analysis below is modelled after that of Prop. 4.2, but with some significant differences which uncover a spanning pencil of isomorphic abelian Popa subgroups $G_{\rho}(A)$ continuously covering an open set contiguous to 0. An illustrative example follows.

**Theorem 4.1 (Spanning Pencil Theorem).** For $S$ a solution of $(GS)$, if $1_A$ is an accumulation point of ran$S \cap (1_A - \text{ran}S)$, then for some $c \in A$

$$S(c(g - 1_A)) = g \quad (g \in S(G_S(A))). \quad \text{(Scale(c))}$$

So if $c \in A^{-1}$, then for $\rho := c^{-1}$

$$S(w) = 1_A + cw \quad (w \in G_{\rho}(A)) = \rho^{-1}(\text{ran}S - 1_A)).$$

In particular, $G_{\rho}(A)$ under $\circ_{\rho}$ is an abelian subgroup of $G_S(A)$. 26
Furthermore, if also Scale(d) (that is, Scale(c)) with c replaced by d) holds for some d ∈ ℂ, then c − d ∈ ℂ. So, the sets \{G_ρ(ℂ) \cap ℂ^{-1} : ρ ∈ ℂ^{-1}\} are mutually disjoint and induce a continuous partition of \( G_S(ℂ) \cap ℂ^{-1} \) in some neighbourhood of 0.

Conversely, if \( S(w) = 1_ℂ + ρw \) for some \( ρ ∈ ℂ^{-1} \) and all \( w ∈ G_ρ(ℂ) \), then \( 1_ℂ \) is an accumulation point of ran\( S \cap (1_ℂ − \text{ran}S) \).

**Proof.** By assumption we may choose \( k ∈ S(ℂ) \) with both \(||k|| < 1\) and \( 1_ℂ − k ∈ S(ℂ) \); then \( 1_ℂ − k \) is an accumulation point of ran\( S \cap (1_ℂ − \text{ran}S) \).

For \( g ∈ S(ℂ) ⊆ ℂ^{-1} \), select \( W(g) ∈ ℂ \) with \( S(W(g)) = g \) and put \( c := W(k) (1_ℂ − k)^{-1} \). For \( g, h ∈ \text{ran}S \)
\[
S(W(g) + W(h) S(W(g)) = S(W(g) S(W(h))
= gh = S(W(gh)) \mod ℂ,
W(g) + g W(h) = W(gh) \mod ℂ,
\]
by Lemma 4.1(i). Thus \( W \) satisfies \((g_ℂ) \mod ℂ\). By commutativity of \( ℂ \)
\[
S(S(W(g) + W(h) S(W(g))) = gh = h g = S(W(h) + W(g) S(W(h)).
\]
So again by Lemma 4.1(i), writing \( =_\mathcal{N} \) for equality mod \( \mathcal{N} \), with \( k \) for \( h \)
\[
W(h) + k W(g) = \mathcal{N} W(g) + g W(k),
W(g)[k − 1_ℂ] = W(k)[g − 1_ℂ] + n(g), \text{ with } n(g) ∈ \mathcal{N} \text{ say:}
W(g) = [k − 1_ℂ]^{-1} W(k)[g − 1_ℂ] + [k − 1_ℂ]^{-1} n(g) =_\mathcal{N} c(g − 1_ℂ).
\]
Write
\[
W(g) = c(g − 1_ℂ) + n_k(g), \text{ with } n_k(g) ∈ \mathcal{N};
\]
then, as \( S(n_k(g)) = 1_ℂ \),
\[
g = S(W(g)) = S(n_k(g) + c(g − 1_ℂ)) = S(n_k(g) + S(n_k(g)) c(g − 1_ℂ))
= S(n_k(g)) S(c(g − 1_ℂ)) = S(c(g − 1_ℂ)).
\]
So (Scale(c)) holds.
For \( c \in \mathbb{A}^{-1} \), as \( w = c(g - 1_\mathbb{A}) \) iff \( 1_\mathbb{A} + c^{-1}w = g \),
\[
S(w) = 1_\mathbb{A} + c^{-1}w.
\]
Hence with \( \rho = c^{-1} \), \( \mathbb{G}_\rho(\mathbb{A}) \) is an abelian subgroup of \( \mathbb{G}_S(\mathbb{A}) \).

Furthermore, if also \( (\text{Scale}(d)) \) holds for some \( d \), then
\[
S(c(g - 1_\mathbb{A})) = g = S(d(g - 1_\mathbb{A})) \quad (g \in S(\mathbb{G}_S(\mathbb{A}))).
\]
As \( 1_\mathbb{A} \) is not isolated in \( \text{ran} S \cap (1_\mathbb{A} - \text{ran} S) \), we may take \( g \) with \( ||g|| < 1 \) and \( 1_\mathbb{A} - g \in S(\mathbb{G}_S(\mathbb{A})) \). As \( g - 1_\mathbb{A} \in \mathbb{A}^{-1} \), by Lemma 4.1(i) there is \( n \in \mathbb{N} \) with
\[
c(g - 1_\mathbb{A}) = d(g - 1_\mathbb{A}) + n : \quad c = d + n(g - 1_\mathbb{A})^{-1} = d + n_g,
\]
with \( n_g \in \mathbb{N} \) say, as \( S(\mathbb{G}_S(\mathbb{A})), \mathbb{N} = \mathbb{N} \) by Lemma 4.1(ii) (and also as \( (g - 1_\mathbb{A})^{-1} \in S(\mathbb{G}_S(\mathbb{A})) \), which is a multiplicative group). So \( c - d \in \mathbb{N} \).

If \( w \in c(\text{ran} S - 1_\mathbb{A}) \cap d(\text{ran} S - 1_\mathbb{A}) \cap \mathbb{A}^{-1} \) for \( c, d \in \mathbb{A}^{-1} \), then
\[
1_\mathbb{A} + c^{-1}w = S(w) = 1_\mathbb{A} + d^{-1}w : \quad c = d.
\]
First take \( V \) an open neighbourhood of 0 with \( V \subseteq S^{-1}(B_1(1_\mathbb{A})) \). Now take \( w \in \mathbb{G}_S \cap \mathbb{A}^{-1} \cap V \); then \( y = S(w) \in B_1(1_\mathbb{A}) \). Put \( c := w(y - 1_\mathbb{A})^{-1} \in \mathbb{A}^{-1} \).

Then \( y - 1_\mathbb{A} = c^{-1}w = \rho w \), say. Then \( w = c(y - 1_\mathbb{A}) \in c(\text{ran} S - 1_\mathbb{A}) = \mathbb{G}_\rho(\mathbb{A}) \),
and so \( w \in \mathbb{G}_\rho(\mathbb{A}) \cap \mathbb{A}^{-1} \). The map \( w \mapsto c = c(w) := w(S(w) - 1_\mathbb{A})^{-1} \) is continuous on \( \mathbb{G}_S \cap \mathbb{A}^{-1} \cap V \).

For the converse with \( c = \rho^{-1} \in \mathbb{A}^{-1} \), note that \( \mathbb{G}_w := S^{-1}(\mathbb{A}_1) \) is a non-empty open neighbourhood of 0. Choose in \( \mathbb{G}_S \) non-zero \( h_n \to 0 \) with \( h_n \in c(\text{ran} S - 1_\mathbb{A}) \); this is possible since \( 1_\mathbb{A} \) is not isolated and for some \( \delta > 0 \), \( S(B_\delta(0)) \) is connected and contains \( 1_\mathbb{A} \) (because \( B_\delta(0) \subseteq S^{-1}(\mathbb{A}_1) \) for some \( \delta > 0 \)). Put
\[
g_n := S(h_n) = 1_\mathbb{A} + \rho h_n \neq 1_\mathbb{A}; \quad k_n := -\rho h_n \to 0.
\]
Then, for \( w_n := -\rho^{-1} - h_n = -\rho^{-1} + \rho^{-1}k_n = c(k_n - 1) \in c(\text{ran} S - 1_\mathbb{A}) \),
\[
S(w_n) = 1 + \rho(-\rho^{-1} - h_n) = k_n = 1 - g_n : \quad g_n = 1 - k_n.
\]
So \( g_n, k_n \in \text{ran} S \) and \( g_n \to 1_\mathbb{A} \). So \( 1_\mathbb{A} \) is a limit point of \( \text{ran} S \cap (1_\mathbb{A} - \text{ran} S) \).

**Corollary 4.4 (Illustrative example).** Up to isomorphism, \( \mathbb{G}_S(\mathbb{C}) \) is either \( \mathbb{G}_\rho(\mathbb{C}) \) or a Popa product \( \mathbb{G}_\alpha^*(\mathbb{R}) \times \sigma \mathbb{G}_\beta^*(\mathbb{R}) \).
Proof. See §7 (Appendix).

In $\mathbb{G}_\rho(\mathbb{R})$, recall that $\mathcal{N} = \{0\}$, so that $S(a) = 1$ for $a \in \mathcal{N}$. Our final result generalizes this to the Banach-algebra setting. A similar observation in a different context, and with an altogether different proof, is made in [BriD]. In [BriD, Lemma 6], the lower bound $||S(a)|| \geq ||1_A||$ holds for all $a$, because the context of $GL(\mathbb{A})$ forces $S(a)$ to be an automorphism (invertible).

**Proposition 4.4 (A dichotomy)** (cf. [BriD, Lemma 6]). If $S$ is defined on $\mathbb{A}$ and satisfies $(GS)$ on $\mathbb{A}$, then for any $a \in \mathbb{A}$, if $1 - S(a)$ is invertible, then $S(a(1_A - S(a))^{-1}) = 0$. In particular, either $||S(a)|| \geq ||1_A||$ or $S(a(1_A - S(a))^{-1}) = 0$.

**Proof.** Assume w.l.o.g. $||1_A|| = 1$, and take $a \in \mathbb{A}$. Suppose $1_A - S(a)$ is invertible in $\mathbb{A}$, so that $S(a) \neq 1_A$. Take $b := a(1_A - S(a))^{-1}$, then

$$b = a + bS(a) = a \circ_S b : \quad S(b) = S(a)S(b) :$$

$$0 = S(b)(1_A - S(a)) : \quad S(b) = 0.$$

So $b \notin \mathbb{G}_S(\mathbb{A})$, i.e. $S(b)$ is not invertible, as claimed.

Now suppose $||S(a)|| < 1$. Then [Rud, 10.7] $1_A - S(a)$ is invertible in $\mathbb{A}$, so $S(b) = 0$. $\square$

**Remarks.**

1. The final argument above fails if $(GS)$ holds only on $\mathbb{G}_S(\mathbb{A})$, as $b \notin \mathbb{G}_S(\mathbb{A})$.

2. It is instructive to consider the case of $\mathbb{G}_\rho(\mathbb{R})$ with $\rho > 0$. If $|S(a)| = |1 + \rho a| < 1$, then automatically $a \neq 1_{\rho} = 0$, and $-2\rho^{-1} < a < 0$. Here $1 - S(a) = 1 - (1 + \rho a) = -\rho a$ and so $b := a(-\rho a)^{-1} = -\rho^{-1}$, the Popa centre, the only real that is not a member of $\mathbb{G}_\rho^+(\mathbb{R})$. So $S(b) = 0$, as $0$ is the only non-invertible here. So also $S$ satisfies $(GS)$ on all of $\mathbb{R}$. More generally, prompted by the case of $\mathbb{R}$- and $\mathbb{C}$-valued functions considered in [Brz1]:

**Corollary 4.5 ([Brz1, Cor. 2]).** If $S$ is as in Prop. 4.4 with $S$ taking values only in $\mathbb{A}^{-1}$, then $1_A - S$ is never invertible. In particular, if $S : \langle u \rangle \to \mathbb{R}1_A$ with $S$ taking only non-zero values, then $S|\langle u \rangle \equiv 1_A$.

**Proof.** If $1_A - S(a)$ were invertible for some $a$, then, for $b := a(1 - S(a))^{-1}$, $S(b) \notin \mathbb{A}^{-1}$, a contradiction. So $1_A - S$ is never invertible. In particular, for $S : \langle u \rangle \to \mathbb{R}1_A$, since $uS(\langle u \rangle) \subseteq \langle u \rangle$, take $A$ to be $\langle u \rangle$ and $S$ to be $uS|\langle u \rangle$ to conclude that as $\text{ran}(S) \subseteq 1_A\mathbb{R}$, $1_A - S \equiv 0$. $\square$
The next result is distilled from [Brz1, Th. 3] and included here, as it pursues the linearity theme of $\mathcal{N}$. Key here is a density argument. Exceptionally, we do not assume that $S$ is continuous; instead, following [Brz1], we assume that, for each $w$, if $\langle w \rangle \cap G_S(\mathbb{A})$ is non-empty, then it has an interior point – the algebraic interior point property [Lyu, §2.2], weaker than $0$ being in the interior of $G_S(\mathbb{A})$. By Prop. 1.1 (ii) w.l.o.g. we may assume below that $0$ is the relevant algebraic interior point.

**Corollary 4.6.** Suppose $S : \mathbb{A} \to \mathbb{R}_1^\mathbb{A}$ satisfies (GS) and that, for each $w$, a non-empty intersection $\langle w \rangle \cap G_S(\mathbb{A})$ has an interior point. Then $\mathcal{N}$ is a vector space.

**Proof.** Suppose otherwise, then $S$ is not identically $1_\mathbb{A}$ on some line $\langle u \rangle$ with $u \in \mathcal{N}$ (otherwise for $u, v \in \mathcal{N}$, $\langle u \rangle, \langle v \rangle \subseteq \mathcal{N}$, and so $su + tv \in \mathcal{N}$ for $s, t \in \mathbb{R}$, by additivity of $\mathcal{N}$). So by Cor. 4.5 $S(tu)$ vanishes for some $t \in \mathbb{R}$.

By the algebraic interior point property, $\langle u \rangle \cap B_\delta \subseteq G_S(\mathbb{A})$ for some $\delta > 0$. By Lemma 4.1(ii), $\mathbb{R}\mathcal{N} = \mathcal{N}$ and so, as $\mathcal{N} \neq \{0\}$, $\mathcal{N}$ is dense-in-itself (as in Prop. 4.1). So there is $u \in \mathcal{N} \cap \langle u \rangle$ with $|(-tu) - su| < \delta$ and so $(tu + su) \in G_S(\mathbb{A})$. But $S(su + tu) = S(su + S(su)tu) = S(su)S(tu) = 0$, contradicting that $S(su + tu)$ is invertible. So $\mathcal{N}$ is a vector space. \qed

## 5 Banach algebra characterisations

Our first main result, Theorem 5.1, is an analogue in the Banach-algebra context of Th. 3.1, which characterises in the Euclidean context the continuous solutions $S$ of (GS). Here this characterizes continuous solutions of (GS) over the Popa group $\mathbb{G}_S^\times$ of a Banach algebra as $1 + \rho x + N(x)$, where the adjustor $N(x)$ satisfies a Goldie equation: this incorporation of the Goldie equation bestows on the results of Section 2 a more satisfactory presentation of the non-linear contribution $n(x)$ of $S(x)$ as $N(x)$. Our second main result, Theorem 5.2, identifies a dichotomy in the behaviour of $N(x)$: it is either linear or exhibits a curvilinear exponential homogeneity. We term the latter exponential tilting. (It is further studied in Proposition 5.1.) Finally, in Theorem 5.3 with the hypothesis that $\mathcal{N}$ is a vector space and $N$ is Fréchet differentiable, we give a differential characterization of $\mathcal{N}$ and show that $N$ is linear on $\mathcal{N}$ but not beyond. However, it is more appealing not to assume $\mathcal{N}$ is a vector space: the focus then shifts to the maximal vector subspace of $\mathcal{N}$, denoted $\mathcal{H}$ below on account of its defining homogeneity property, on which
is linear. In any case, Prop. 5.3 below connects circumstances of solubility of the tilting equation \((T)\) (below Th. 5.3) to whether \(N\) is a vector subspace.

We will need Lemma 4.1 and the following two lemmas.

**Lemma 5.1.** For normed vector spaces \(X,Y\), if \(F : X \to Y\) satisfies:

(i) \(F\) is Fréchet differentiable at every \(x \in X\) with derivative \(F'(x)\),

(ii) \(F(0) = 0\), and

(iii) for some continuous linear \(L : X \to Y\) and

\[
F'(x)(h) = L(h) \quad (x, h \in X)
\]

then \(F\) is linear and \(F = L\).

**Proof.** Fix \(u \in X\). For \(t \in \mathbb{R}\) take \(f(t) := F(tu)\). Then, with \(D_u\) the directional derivative,

\[
f'(t) = \lim_{s \to 0} (F((t + s)u) - F(tu))/s = D_uF(tu) = L(u).
\]

Integrating from 0 to \(t\),

\[
f(t) - f(0) = L(u)t : \quad F(tu) = L(u)t.
\]

Now take \(t = 1\).

The result below shows that in general the solution of \((GS)\) involves not only the ‘canonical example’ of a ‘GS function’ from the context of \(\mathbb{R}\), namely \(1 + \rho x\), but also an adjustor \(N(x)\) whose characterising equation is a particular case of the generalized Goldie equation \((GGE_{SS})\) below (with the subscript indicating that the auxiliaries on the inside and outside of \(N\) are \(S\), cf. [Jab1,2]). The adjustor is at best Fréchet differentiable and need not be linear – see Example 7.3 in the Appendix. (Given associativity of the circle operation, \((GGE_{SS})\) implies for non-trivial \(N\) that \(S\), like \(g\) in §1, is multiplicative: it satisfies \((GS)\).)

**Definition.** Following the notion of \(\mathbb{C}\)-differentiability, say that \(f : \mathbb{A} \to \mathbb{A}\) is \(\mathbb{A}\)-differentiable at \(a \in \mathbb{A}\) if for some \(m \in \mathbb{A}\)

\[
\lim_{h \to 0} h^{-1}[f(a + h) - f(a)] = m, \quad (h \in \mathbb{A}^{-1})
\]

in which case we will write \(m = f'(a)\).
This is a far stronger property than Fréchet differentiability: by Th. 5.1 below \( G^*_S(A) \) has to be abelian if \( S \) is \( A \)-differentiable (cf. Example 7.1-3 in the Appendix); the preceding definition involves not just the vector but also the ring structure, as Lemma 5.2 clarifies.

**Lemma 5.2.** For \( A \) with dense invertibles, if \( f : A \to A \) is continuous near \( a \) and \( A \)-differentiable at \( a \) with \( f'(a) = m \), then \( f \) is Fréchet differentiable at \( a \) with derivative \( f'(a)h = mh \).

**Proof.** The map \( L : h \mapsto mh \) is linear and bounded (as \( ||mh|| \leq ||m||.||h|| \)). For \( \varepsilon > 0 \) and \( ||h|| \) small enough (and in \( A^{-1} \))

\[
||h^{-1}[f(a + h) - f(a) - mh]|| \leq \varepsilon
\]

holds, which implies for \( h \in A^{-1} \) that

\[
||hh^{-1}[f(a + h) - f(a) - mh]|| \leq ||h||.||h^{-1}[f(a + h) - f(a) - mh]|| : ||f(a + h) - f(a) - mh|| \leq \varepsilon||h||,
\]

the latter extending to all small enough \( h \), by density of \( A^{-1} \) and continuity of \( f \) near \( a \). That is, \( f \) is Fréchet differentiable at \( a \) with derivative \( L \). \( \square \)

In the following theorem the adjustor \( N \) is typically linear on \( N \); see Th. 5.3. Its derivative behaves as does that of \( S \) in Prop. 2.1.

**Theorem 5.1 (First Banach Algebra Characterization Theorem).** If \( S : G^*_S \to A \) satisfies \( (GS)_A \), and \( 1_A \in G^*_S(A) \), then with \( \rho := S(1_A) - 1_A \) there is \( N : G^*_S \to N \) such that

\[
S(x) = 1_A + \rho x + N(x), \quad (\dagger)
\]

where \( N \) satisfies the adjustor equation

\[
N(x + S(x)y) = N(x) + S(x)N(y) \quad (x, y \in G^*_S). \quad (GGE_{SS})
\]

In particular,

\[
N(0) = N(1_A) = 0.
\]

Moreover, if \( N \) is Fréchet differentiable as a map into \( N \), then its derivative satisfies the similarity relation

\[
N'(x) = S(x)N'(0)S(x)^{-1} = S(x)N'(0)S(x_S^{-1}) \quad (x \in G^*_S).
\]
If $N$ is linear over $\mathbb{G}_S^*$, then for some projection $\pi : \mathbb{G}_S^* \to N$ and some linear map $L$, with $L\pi$ linear and injective (into $N$),

$$S(x) = 1_A + \rho x + L(\pi(x)).$$

Furthermore, for $A$ with dense invertibles, if $S$ is $A$-differentiable at 0 (equivalently everywhere), then $N(x) \equiv 0$ and $\mathbb{G}_S^*(A)$ is abelian, since

$$S(x) = 1_A + \rho x.$$

**Proof.** The argument here is an extension of that in Prop. 4.2 and 4.4. As $S(x \circ 1_A) = S(1_A)S(x) = S(1_A \circ x)$, by Lemma 4.1(i), for each $x \in \mathbb{G}_S^*$ there is $N(x) \in N$ with

$$x + S(x) = 1_A + xS(1_A) + N(x),$$

$$S(x) = 1_A + x[S(1_A) - 1_A] + N(x) = 1_A + \rho x + N(x).$$

It now follows that $N(0) = N(1_A) = 0$. Substituting for $S$ into $(GS):$

$$1_A + \rho(x \circ_y y + N(x \circ_S y) = S(x \circ_S y) = [1_A + \rho x + N(x)][1_A + \rho y + N(y)].$$

Now $\rho(x \circ_S y) = \rho(x + yS(x)) = \rho(x + y + \rho xy + N(x)y)$, so multiplying out

$$1_A + \rho x + \rho y + \rho^2 xy + N(x)\rho y + N(x \circ_S y) = [1_A + \rho x + N(x)][1_A + \rho y + N(y)].$$

Re-arrangement gives the asserted equation.

Now proceed, as in Prop. 2.1, differentiating $(GGE_{SS})$ w.r.t. $y$, then setting $y = 0$. Recalling that $S$ takes invertible values, this gives for $x, y, h \in \mathbb{G}_S^*$:

$$N'(x + S(x)y)(S(x)h) = S(x)N'(y)h; \quad S(x)^{-1}N'(x + S(x)y)(S(x)h) = N'(y)h,$$

$$S(x)^{-1}N'(x)(S(x)h) = N'(0)h.$$

Equivalently,

$$N'(x) = S(x)N'(0)S(x)^{-1} = S(x)N'(0)S(x_S^{-1}) \quad (x \in \mathbb{G}_S^*),$$

the latter since $1_A = S(x \circ_S x_S^{-1}) = S(x)S(x_S^{-1})$.

If $N$ is linear, then ran $N$ is a vector subspace of $N$. Take $V_1 := \ker(N)$ and $V_0$ a complementary subspace to $V_1$. Take $\pi$ to be projection onto $V_0$ parallel to $V_1$ and define $L : V_0 \to V_0$ to be $N|_{V_0}$. For $x \in A,$

$$N(x) = N(\pi x + (x - \pi x)) = N(\pi x) = L(\pi(x)),$$

$$S(x) = 1_A + \rho x + N(x) = 1_A + \rho x + L(\pi(x)).$$
with \( L\pi : V_0 \rightarrow \mathcal{N} \) linear and injective.

Finally, suppose that \( \mathbb{A}^{-1} \) is dense in \( \mathbb{A} \) and \( S \) is \( \mathbb{A} \)-differentiable at 0. For \( x \in \mathcal{G}_S^0(\mathbb{A}) \) and \( h \in \mathbb{A}^{-1} \) take \( k = hS(x)^{-1} \in \mathbb{A}^{-1} \); take \( h \to 0 \), then as \( ||k|| \leq ||h|| \cdot ||S(x)^{-1}|| \), \( k \to 0 \) and so \((GS)\) gives

\[
h^{-1}[S(x+h)-S(x)] = h^{-1}S(x)[S(hS(x)^{-1}) - 1_{\mathbb{A}}] = k^{-1}[S(k) - S(0)] \to S'(0).
\]

Thus \( S'(x) = S'(0) \). So Lemma 5.1 applies at each \( x \in \mathcal{G}_S^0(\mathbb{A}) \) to \( F(x) := S(x) - S(0) \), since \( S \) is Fréchet differentiable (by continuity of \( S \) and Lemma 5.2) with \( S'(x)h = S'(0)h \), giving

\[
S(x) = S(0) + S'(0)x.
\]

Taking \( x = 1_{\mathbb{A}} \) gives \( S'(0) = \rho \) and so \( S(x) = 1 + \rho x \), which is equivalent to commutativity of \( \circ S \), as noted in the Remark 4 after Cor. 1.1. \( \square \)

**Corollary 5.2.** (i) If \( \mathcal{N} = \{0\} \), then \( N(x) \equiv 0 \) and so

\[
S(x) = 1_{\mathbb{A}} + \rho x \quad \text{for} \quad \rho := S(1_{\mathbb{A}}) - 1_{\mathbb{A}}
\]

Thus, for \( \mathbb{A} = \mathbb{R} \) positive solutions of \((GS_\mathbb{R})\) take this form with \( \rho \geq 0 \).

(ii) The adjustor \( N \) takes the form \( kx \) with \( k \in \mathbb{A} \) iff \( \circ S \) is commutative.

**Proof.** (i) The first assertion is clear. For \( \mathbb{A} = \mathbb{R} \), since \( S \) takes values in \( \mathbb{R}^{-1} = \mathbb{R} \setminus \{0\} \), by Cor 4.2 either \( \mathcal{N} = \mathbb{R} \) and then \( S(x) \equiv 1_{\mathbb{A}} \) or \( \mathcal{N} = \{0\} \).

(ii) If \( \circ S \) is commutative, then by Remark 4 after Cor. 1.1. above, \( S(x) = 1_{\mathbb{A}} + \sigma x \) for some \( \sigma \); then \( 1_{\mathbb{A}} + \rho x + N(x) = 1_{\mathbb{A}} + \sigma x \), and so \( N(x) = (\sigma - \rho)x \) is linear. (This also follows directly from \((GGE_\mathbb{SS})\).) Conversely, if \( N(x) = kx \) for some \( k \), then \( S(x) = 1_{\mathbb{A}} + \sigma x \) for \( \sigma := (\rho + k) \), so \( \circ S \) is commutative. \( \square \)

Our next result, Th. 5.2, draws on a result in [BinO4] where the context refers to real-valued functions on a topological vector space \( X \); however, the calculations there may be reinterpreted upon replacing \( X \) there by \( \mathbb{A} \) here, thereby introducing also \( \mathbb{A} \)-valued functions and using symbolic calculus (Riesz-Dunford functional calculus). This requires relevant elements of \( \mathbb{A} \), such as \( e^\gamma(u) - 1_{\mathbb{A}} \) below for \( \gamma(u) := S'_u(0) \), the directional derivative, to be invertible; see e.g. [Rud, Ch. 10]. Below \( e^\gamma(u) - 1_{\mathbb{A}} \) will be invertible iff \( e^\lambda \neq 1 \) for all \( \lambda \) in the spectrum of \( \gamma(u) \) [Rud, Th. 10.28]. See Example 7.3 in the
Appendix. However, the analysis here departs from [BinO4] in requiring the stronger assumption of Fréchet differentiability. To use symbolic calculus in Theorem 5.2 below, we need the following convergence result.

**Lemma 5.3.** (i) For $C^\dagger$ an unbounded, connected open subset of the complex plane containing the origin and for a sequence of positive reals $t(n) \to t > 0$:

$$\frac{(1 + z/n)^{nt(n)} - 1}{(1 + z/n)^n - 1} \to \frac{e^{tz} - 1}{e^z - 1}, \text{ or } t \text{ if } e^z = 1,$$

convergence being uniform on compact subsets of $C^\dagger$.

(ii) The map $z \mapsto \mu(z) := (e^z - 1)/z$, or 1 if $z = 0$, is holomorphic and invertible near $(w, z) = (1, 0)$.

**Proof.** (i) The assumptions allow the use of a logarithm on $C^\dagger$. So w.l.o.g assume $C^\dagger := C\langle -\infty, 0 \rangle$. For $t > 0$, the function $g_t$, defined for $\zeta \in C^\dagger$ by

$$g_t(\zeta) = \frac{e^{t\log \zeta} - 1}{\zeta - 1} \text{ with } g_t(1) = t,$$

is differentiable, because it is differentiable at $\zeta = 1$ : using L’Hospital twice,

$$\lim_{w \to 0} \frac{g_t(1 + w) - t}{w} = \frac{t(t - 1)}{2}.$$

Hence

$$f(\zeta) := g_t(e^\zeta) = \frac{e^{t\zeta} - 1}{e^z - 1}, \text{ or } t \text{ if } e^\zeta = 1,$$

is holomorphic, as is for each $n = 1, 2, ...$

$$f_n(\zeta) := g_{t(n)}((1 + \zeta/n)^n) = \frac{(1 + \zeta/n)^{nt(n)} - 1}{(1 + \zeta/n)^n - 1}, \text{ or } t(n) \text{ if } (1 + \zeta/n)^n = 1.$$

Furthermore, since $(1 + \zeta/n)^n \to e^\zeta$ and $t(n) \to t > 0$, then for each $\zeta \in C^\dagger$

$$f_n(\zeta) \to f(\zeta) = \frac{e^{t\zeta} - 1}{e^z - 1}, \text{ or } t \text{ if } e^\zeta = 1,$$

convergence being uniform on compact subsets of $C^\dagger$.
(ii) The first part is clear from the preceding paragraph. As for invertibility of \( w = w(\zeta) \) near \( w = 1 \), take \( F(\omega, \zeta) := \omega - \mu(\zeta) \), which is analytic (in the two complex variables sense) with \( F(1, 0) = 0 \), and note that \( F_\zeta = -(1 + e^{\zeta}(\zeta - 1))/\zeta^2 \) or \(-1/2\) if \( \zeta = 0 \). By the Implicit Function Theorem [Gam1, Ch. 3] there is a solution \( z = z(\omega) \) near \( \omega = 1 \) with \( z(1) = 0 \).  

Corollary 5.3. With \( \mathbb{C}^\dagger \) as in Lemma 5.3, in any Banach algebra with dense invertibles, for \( a \) an element with spectrum satisfying \( \text{spec}(a) \subseteq \mathbb{C}^\dagger \cup \{0\} : \)

\[
\frac{(1 + a/n)^{nt(n)} - 1}{(1 + a/n)^n - 1} \rightarrow \frac{e^{ta} - 1}{e^a - 1}, \text{ or } t \text{ if } 1 \in \exp(\text{spec}(a)).
\]

Proof. If \( 0 \notin \text{spec}(a) \), the result follows from Lemma 5.3 by [Rud, Th. 10.27]. If \( 0 \in \text{spec}(a) \), then \( a \) is not invertible. Assuming dense invertibles, choose invertible elements with \( a_k \rightarrow a \). Again by Lemma 5.3 and [Rud, Th. 10.27] for each \( k \)

\[
\frac{(1 + a_k/n)^{nt(n)} - 1}{(1 + a_k/n)^n - 1} \rightarrow \frac{e^{ta_k} - 1}{e^{a_k} - 1}, \text{ or } t \text{ if } 1 \in \exp(\text{spec}(a_k)).
\]

Now choose \( k(n) \) so that \( (1 + a_k(n)/n)^n \rightarrow a \). So if \( \text{spec}(a) \subseteq \mathbb{C}^\dagger \), then

\[
\frac{(1 + a_k(n)/n)^{nt(n)} - 1}{(1 + a_k(n)/n)^n - 1} \rightarrow \frac{e^{ta} - 1}{e^a - 1}, \text{ or } t \text{ if } 1 \in \exp(\text{spec}(a)).
\]

and so

\[
\frac{(1 + a/n)^{nt(n)} - 1}{(1 + a/n)^n - 1} \rightarrow \frac{e^{ta} - 1}{e^a - 1}, \text{ or } t \text{ if } 1 \in \exp(\text{spec}(a)).
\]

Indeed, if \( 1 \notin \exp(\text{spec}(a)) \), then \( e^a - 1 \) is invertible and so also for large \( n \) is \( (1 + a/n)^n - 1 \), and so also is its approximand \( (1 + a_k(n)/n)^n - 1 \).  

Recall that the spectrum of an element is compact. So its complement is open: in order to have a logarithm available for Lemma 5.3 to hold, the connected component of \( 0 \) must be unbounded; one may term this a no encirclement condition.

Theorem 5.2 (Second Characterization Theorem: Curvilinear exponential homogeneity). Suppose that the solution \( S \) to \((GS)\) is Fréchet differentiable and that \( N \) solves the Goldie equation in \( \mathbb{A} : \)

\[
N(x + S(x)y) = N(x) + S(x)N(y).
\]

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For any $u$ with $\gamma(u) := S_u'(0)$ having a spectrum not separating 0 from $\infty$, the radiality formulae below hold for $t \geq 0$:

$$N(u(e^{t\gamma(u)} - 1)/\gamma(u)) = \lambda_u(t)N(u(e^{\gamma(u)} - 1)/\gamma(u)), \quad \text{(Rad)}$$

for

$$\lambda_u(t) := (e^{t\gamma(u)} - 1)/(e^\gamma(u) - 1),$$

with the L’Hospital convention that, when $1 \in \exp(\text{spec}(\gamma(u)))$,

$$N(tu) = t1_\mathbb{A}N(u) = tN(u) \quad (t \geq 0).$$

Furthermore, the exponential tilting map (with $\mu$ as above)

$$T(u) := u\mu(\gamma(u)) = u(e^{\gamma(u)} - 1)/\gamma(u) : T(tu) = \lambda_u(t)T(u) \quad (t \geq 0)$$

has invertible multiplier $\mu(\gamma(u))$ for $u \in \mathbb{G}_S(\mathbb{A})$, and exhibits the ‘curvilinear exponential homogeneity’ under $N$:

$$N(T(tu)) = N(\lambda_u(t)T(u)) = \lambda_u(t)N(T(u)) \quad (t \geq 0).$$

**Proof.** Referring to the polynomials $\varphi_n(x) = 1 + x + \ldots + x^{n-1}$ and rational polynomials $[\varphi_m/\varphi_n](x)$, [BinO4] Lemma 5 gives, taking $g = h = S$, that

$$N(\varphi_m(S(u/n))u/n) = [\varphi_m/\varphi_n](S(u/n))N(\varphi_n(S(u/n))u/n), \quad \text{(PreRad)}$$

for any $u$. By Fréchet differentiability, write

$$S(u/n) - S(0) = S'(0)(u/n) + \varepsilon_n(u),$$

with $n\varepsilon_n(u) \to 0$. Put $\xi_n = S(u/n)$ and $\gamma(u) := S'(0)u$. Note that $\gamma(u)/n \to 0$, as $||\gamma(u)|| \leq ||S'(0)||.||u||$. For all $n$ large enough $1_A + \gamma(u)/n + \varepsilon_n(u) \in A_1$, the connected component of unity; then $\eta_n(u) := n\log[1_A + \gamma(u)/n + \varepsilon_n(u)]$ is well defined ([Rud,10.43c], [Ric, 1.4.12]) and $\eta_n(u) \to 0$. Fix $t > 0$ and choose $m = m(n)$ so that $m/n = m(n)/n = t(n) \to t$. By Lemma 5.3,

$$[\varphi_m/\varphi_n](\xi_n) = \frac{\exp\{t(n)n\log[1_A + S_u'(0)(1/n) + \varepsilon_n(u)]\} - 1}{\exp\{n\log[1_A + S_u'(0)(1/n) + \varepsilon_n(u)]\} - 1} = \frac{\exp(t(n)\eta_n(u)) - 1}{\exp(\eta(n)) - 1} \to \frac{e^{\gamma(u)t} - 1}{e^{\gamma(u)} - 1}, \text{ or } t \in \exp(\text{spec}(\gamma(u))).$$
Likewise, provided $\xi_n \neq 1$

$$\varphi_m(\xi_n)/n = \frac{\exp\{t(n)[n\log[1 + S'_u(0)(1/n) + \varepsilon_n(u)]] - 1}{n(\xi_n - 1)}$$

$$= \frac{\exp(t(n)\eta_n(u)) - 1}{S'(0)u + n\varepsilon_n(u)} \rightarrow \mu(\gamma(u)) = \frac{e^{\gamma(u)t} - 1}{\gamma(u)}, \text{ or } t \text{ if } 1 \in \exp(\text{spec}(\gamma(u))).$$

The equations now follow from (PreRad) and Corollary 1, the case $t = 0$ being trivial.

As $\gamma(u)$ is (linear and so) homogeneous,

$$T(tu) = u(e^{t\gamma(u)} - 1)/\gamma(u) = u\mu(\gamma(u))\lambda_u(t) = \lambda_u(t)T(u),$$

$$N(u(e^{t\gamma(u)} - 1)/\gamma(u)) = N(u\lambda_u(t)(e^{\gamma(u)} - 1)/\gamma(u)) = \lambda_u(t)N(u(e^{\gamma(u)} - 1)/\gamma(u)).$$

By Lemma 5.4(ii) $\mu(\gamma(u))$ is invertible in $A$. □

Remarks: 1. Exponential tilting. One may interpret the adjustor $N$ as comprising a linear action on $N$ and, on complementary directions, a homogeneous action after the tilting $T$ with (exponential) scaling $\lambda_u$ – so a kind of shearing. (The term exponential tilting here is borrowed from probability theory, where it is used as a synonym for the Esscher transform of collective risk theory [GerS].)

2. Interestingly, $\lambda_u$ satisfies a pair of Goldie equations with parameter $\gamma(u)$:

$$\lambda_u(s + t) = \lambda_u(s) + e^{\gamma(u)s}\lambda_u(t) \quad (s, t \in \mathbb{R});$$

$$\lambda_u(s + t) = \lambda_u(t) + e^{\gamma(u)t}\lambda_u(t) \quad (t \in \mathbb{R}).$$

We now study the interplay between $N$ and the set of $N$-invariant directions, which we denote by $\mathcal{H}$ for ‘homogeneous’:

$$\mathcal{H} := \{x : (\forall t \in \mathbb{R})N(tx) = tN(x)\}.$$ 

This is closed by continuity of $N$. If $x \in \mathcal{H}$, then, for $s, t \in \mathbb{R}$, $N(t(sx)) = tsN(x) = tN(sx)$, so $sx \in \mathcal{H}$, i.e. $\langle x \rangle \subseteq \mathcal{H}$: so $\mathcal{H}$ is homogeneous.

Lemma 5.4. (i) For $S$ the solution to $(GS)$ and $N$ its adjustor, by $(\dagger)$

$$\mathcal{N} = \{x : N(x) = -\rho x\},$$

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so $N$ is additive on $N$, and so when $1 \in \exp(\text{spec}(\gamma(u)))$, then

$$N(tu) = tN(u) \quad (u \in N, \ t \in \mathbb{R}).$$

(ii) Furthermore,

$$N \subseteq H \iff N \text{ is a vector subspace.}$$

That is, $N$ is a vector subspace iff $N$ acts homogeneously on $N$ iff $N$ is linear on $N$.

(iii) If $\langle u \rangle \subseteq N$, then $\langle u \rangle \subseteq H$.

(iv) $H \cap N$ is a vector subspace (so if $u \in H \cap N$, then $\langle u \rangle \subseteq N$).

**Proof.** (i) Indeed $S(x) = 1_A + \rho x + N(x)$ implies the equivalence of $S(x) = 1$ and $N(x) = -\rho x$. For $u, v \in N$, as $u + v \in N$,

$$N(u + v) = \rho(u + v) = \rho u + \rho v = N(u) + N(v).$$

In particular, for $u \in N$, $0 = N(u - u) = N(u) + N(-u)$, i.e. $N(-u) = -N(u)$, and so since $1 \in \exp(\text{spec}(\gamma(u)))$ implies $1 \in \exp(\text{spec}(\gamma(-u)))$, the claim of linearity follows from Th. 5.2.

(ii) Suppose that $N \subseteq H$. For $u, v \in N$ and positive integers $p, q, r$, additivity gives $pu + qv \in N$, from which it follows that $(pu + qv)/r \in H$, by homogeneity of $H$, i.e. that $su + tv \in H$ for $s, t$ in $\mathbb{Q}_+$, and so, as $H$ is closed, that $su + tv \in H$ for $s, t$ in $\mathbb{R}_+$. Further, with $u, v, p, q, r$ as above, as $pu + qv \in H$, by additivity of $N$ on $N$,

$$N((pu + qv)/r) = N(pu + qv)/r = (pN(u) + qN(v))/r :$$

$$N((pu + qv)/r) = (p/r)N(u) + (q/r)N(v) :$$

$$N(su + tv) = sN(u) + tN(v), \text{ for } s, t \in \mathbb{R}_+,$$

by continuity of $N$. So, as $N(u) = -\rho u$ and $N(v) = -\rho v$, for $s, t \in \mathbb{R}_+$,

$$S(su + tv) = 1_A + \rho (su + tv) + N(su + tv)$$

$$= 1_A + s(\rho u + N(u)) + t(\rho v + N(v)) = 1_A.$$

That is, $su + tv \in N$. But $-N = N$, so $N$ is a vector space.

Conversely, suppose $N$ is a vector space. For $u \in N$, and $t \in \mathbb{R}$, as $tu \in N$, $N(tu) = -\rho tu = t(-\rho u) = tN(u)$ and so $u \in H$. That is $N \subseteq H$. 39
If $N$ acts homogeneously on $x \in \mathcal{N}$, i.e. $N \subseteq \mathcal{H}$, then for $x \in \mathcal{N}$ and $t \in \mathbb{R}$, $N(tx) = tN(x) = -t\rho x$ and so $tx \in \mathcal{N}$; so $N$, being an additive subgroup, is a vector subspace. Conversely, if $\mathcal{N}$ is a vector subspace, then, for $t \in \mathbb{R}$ and $x \in \mathcal{N}$, $tN(x) = -t\rho x = N(tx)$, as $tx \in \mathcal{N}$; so $N$ acts homogeneously on $\mathcal{N}$.

As for the final condition, $N$, being additive on $\mathcal{N}$ by (GGE)SS, is linear on $\mathcal{N}$ iff it acts homogeneously on $\mathcal{N}$.

(iii) If there is $u$ with $\langle u \rangle \subseteq \mathcal{N}$ (e.g. in a finite-dimensional setting where this holds by Theorem B of §4), then, for $t \in \mathbb{R}$, $N(tu) = -t\rho u = tN(u)$, since $u$ and $tu \in \mathcal{N}$. That is, $\langle u \rangle \subseteq \mathcal{H}$ and so $\mathcal{H}$ is non-empty.

(iv) Take $u, v \in \mathcal{H} \cap \mathcal{N}$ and $t \in \mathbb{R}$; then, as $N(tu) = tN(u)$ and by (i),

\[
S(tu) = 1_\mathcal{H} + \rho tu + N(tu) = 1_\mathcal{H} + t(\rho u + N(u)) = 1_\mathcal{H},
\]

\[
N(t(u + v)) = t\rho (u + v) = t(N(u) + N(v)) = t(N(u + v)),
\]

so $\langle u \rangle \subseteq \mathcal{H} \cap \mathcal{N}$ and $u + v \in \mathcal{H} \cap \mathcal{N}$, which is thus a vector subspace. □

It is routine to show, by reference to additivity and closure of $\mathcal{N}$, using rational scalars as above, that $\langle \mathcal{N} \rangle = \{tx : x \in \mathcal{N}, t \in \mathbb{R}\}$ (cf. [Brz1, Proof of Th. 3]), hence the interplay of $\mathcal{H}$ and $\mathcal{N}$. The next two results again require the hypothesis of Th. 5.2 on non-encirclement of the origin by the spectrum. Here we are about to see that $\mathcal{H} \subseteq \mathcal{N}$, i.e. the reverse inclusion to that of Lemma 5.4. It emerges below that $\mathcal{H} = \mathcal{N}$ if $\mathcal{N}$ is a vector subspace upon which $N$ is linear. Since $N(\mathcal{N}) \subseteq \mathcal{N}$ (Th. 5.1), this is in line with a similar result in the First Popa Homomorphism Theorem, Th. 4A of [BinO3], where the homomorphism is linear on $\mathcal{N} = \mathcal{N}(\rho)$. The result below should be compared to the identity $\mathcal{N} = \{x : N(x) = -\rho x\}$ of Lemma 5.4. That $\mathcal{H} = \mathcal{N}$ when $A = \mathbb{R}^d$ will eventually follow. For $\mathcal{N}_\gamma$ below, see Cor 2.1.

**Theorem 5.3 (Third – differential – characterization of $\mathcal{N}$).** For $S$ Fréchet differentiable at 0 and $\gamma = S'(0)$, if each element $\gamma u$ has spectrum not separating 0 from $\infty$, then

\[
\mathcal{H} = \mathcal{N}_\gamma = \{u : S'(0)u = 0\} = \{u : N'(0)u = -\rho u\} \subseteq \mathcal{N},
\]

so that $\mathcal{H}$ is a vector subspace and $N$ is linear on $\mathcal{H}$; furthermore, $\mathcal{H}$ is the maximal vector subspace of $\mathcal{N}$ such that if $\mathcal{N}$ is a vector subspace, then $\mathcal{N} = \mathcal{H}$ and $N$ is linear on $\mathcal{N}$.
For \( M \) any subspace complementary to \( N \):

\[
N(x) = N(\pi_N(x)) + N(\pi_M(x)),
\]

where \( \pi \) refers to the corresponding projections.

**Proof.** By Lemma 5.4 and Theorem 5.2 and since \( \gamma(u) = S'(0)u \) (Fréchet differentiability at 0):

\[
\mathcal{H} = \{ u : N(tu) = tN(u) (\forall t) \} = \{ u : \lambda_u(t) = t1 (\forall t) \} = \{ u : \gamma(u) = 0 \}
\]

So for \( u \in \mathcal{H} \), as \( DS(0)u = 0 \),

\[
0 = \lim_{t \to 0} S(tu) - 1_A \quad \text{and} \quad \lim_{t \to 0} \frac{ptu + N(tu)}{t} = pu + N(u),
\]

and so \( u \in N \) by Lemma 5.4. Thus \( \mathcal{H} \subseteq N \). Since \( \gamma \) is linear, \( \mathcal{H} \) as its kernel is a vector subspace. (Also by Lemma 5.4(iv), as \( \mathcal{H} = \mathcal{H} \cap N \).)

We may now deduce from its additivity on \( N \) that \( N \) is linear on \( \mathcal{H} \): for \( s, t \in \mathbb{R} \) and \( u, v \in \mathcal{H} \), as \( su, tv \in \mathcal{H} \subseteq N \) by additivity on \( N \)

\[
N(su + tv) = N(su) + N(tv) = sN(u) + tN(v),
\]

the last since \( \mathcal{H} \) is homogeneous for \( N \).

For \( N \) a vector subspace, \( \mathcal{N} \subseteq \mathcal{H} \) (Lemma 5.4); but \( \mathcal{H} \subseteq N \), so \( \mathcal{N} = \mathcal{H} \).

Given \( M \) complementary to \( N \) in \( A \), for \( x \in \mathcal{N} \) and \( y \in M \), (+) follows from \((GGE)_{SS}\), as \( S(x) = 1 \) and \( x + y = x \circ_S y \).

In the Euclidean case, the spectrum of \( \gamma(u) \) is finite, hence does not separate 0 from infinity; however, the proof that \( \mathcal{H} = N \) when \( A = \mathbb{R}^d \), must wait until we have studied the tilting map of Th. 5.3 and the equation

\[
v = T(u) := u(e^{\gamma(u)} - 1_A)/\gamma(u).
\]

If \( \gamma \) is \( A \)-homogeneous (for which see Section 2 above), the map \( T \) is invertible on a part of its range and with the explicit inverse of Prop. 5.1.

As for the conclusion below about the range of \( T \) covering a nhd of the origin, the simplest case of \( A = \mathbb{R} \) and \( \gamma(u) = \gamma u \), with \( \gamma > 0 \) say, is illuminating. For given \( v \), \( v = T(u) \) is soluble only for \(-1 < \gamma v \), as

\[
(e^{\gamma v} - 1)/\gamma = v : \quad e^{\gamma v} = 1 + \gamma v.
\]
So the range of $T$ is bounded in one direction.

This uni-directional scenario above emerges in a more general setting, enabled by a uniqueness/identity result, kindly established for us by Amol Sasane. For the proof see §7 (Appendix). We apply it to $f(\zeta) = \log(1 + \zeta)$ with domain $\mathbb{C}\setminus(-\infty,0]$ to validate a condition placed on $f(\gamma(u))$. Recall that $\mathbb{A}$ is termed *semisimple* if its Gelfand transform has trivial kernel, i.e. is injective [Rud, 11.9], [Con, Ch. VII §8], an instance being $C[0,1]$ (loc. cit.).

**Theorem S (Uniqueness/Identity).** For $\mathbb{A}$ semisimple, $f(\zeta)$ holomorphic on its domain, $u \in \mathbb{A}$, and

$$D_u := \{ \zeta \in \mathbb{C} : \text{spec}(\zeta \gamma(u)) \subseteq \text{dom } f \}$$

non-empty, open and connected: if, for some set $\Sigma \subseteq D_u$ with a limit point in $D_u$, the identity

$$f(\zeta \gamma(u)) = \gamma(f(\zeta \gamma(u) \cdot u/\gamma(u))$$

holds for all $\zeta \in \Sigma$, then it holds also for all $\zeta \in D_u$.

**Proposition 5.1.** For a derivative $\gamma = S'(0)$ that is $\omega$-homogeneous: if $v = T(u)$ for $u$ and $v$ with $\gamma(u), \gamma(v) \in \mathbb{A}^{-1}$, and $t > 0$, then: (i)

$$u = u(v) := v \log(1 + \gamma(v))/\gamma(v),$$

so that $u$ is uniquely determined, and further: (ii) $1 + \gamma(v) \in \exp(\mathbb{A}) = \mathbb{A}_1$ and

$$\gamma(u) = \log(1 + \gamma(v)) \in \text{Ran}(\gamma).$$

Conversely, for a general derivative $\gamma$, if (ii) holds for some $v$ with $u = u(v)$ from (i), then $v = T(v(u))$. This condition holds with $\mathbb{A}$ semi-simple for an $\omega$-homogeneous $\gamma$, and, furthermore, the range of $T$ contains the neighbourhood \{v : ||\gamma(v)|| < 1\} of 0.

**Proof.** By Corollary 2.2, if $v = T(u)$, then

$$\gamma(v) = \gamma(u(e^{\gamma(u)} - 1_A)/\gamma(u)) = e^{\gamma(u)} - 1_A :$$

$$\gamma(u) = \log(1_A + \gamma(v)) :$$

$$u = v \gamma(u)/(e^{\gamma(u)} - 1_A) : u = v \log(1_A + \gamma(v))/\gamma(v),$$

giving (i); the first line above giving (ii).
Conversely, if \( \gamma(u(v)) = \log(1 + \gamma(v)) \), then \( e^{\gamma(u(v))} - 1 \) and so

\[
T(u(v)) = u(v) \frac{e^{\gamma(u(v))} - 1}{\gamma(u(v))} = v, \\
\frac{\log(1 + \gamma(v))}{\gamma(v)} \cdot \frac{(e^{\gamma(u(v))} - 1)}{\gamma(u(v))} = v.
\]

In particular this holds for \( 1 + \gamma(v) \in A^{-1} \) in the \( \omega \)-homogeneous case, as Cor. 2.2 gives \( \gamma(u(tv)) = \log(1 + t\gamma(v)) \) for \( 0 \leq t||\gamma(v)|| < 1 \) and Theorem S extends the domain of validity. The final claim follows, since \( \gamma \) is continuous and the set in question is an open neighbourhood of 0.

\( \square \)

**Remark.** In the particular case of \( \gamma \) real, i.e. with values in \((1_A, 1_A)\), if \( 1_A + \gamma(v) > 0 \), then \( \log(1_A + \gamma(v)) \) is real, as may be verified from either the familiar series or that for \( \log(1_A - \gamma(v)/(1_A + \gamma(v))) \), when \( \gamma(v) \geq 1 \); then

\[
\gamma(u(v)) = \gamma(\log(1_A + \gamma(v)) \cdot v/\gamma(v)) = \log(1_A + \gamma(v)) \cdot \gamma(v)/\gamma(v).
\]

**Remark (Standardized tilting).** Guided by the alignment of \( u \) and \( v \) in the formula \((T\text{-inv})\) above, for \( A \) with dense invertibles \((\S 1.2)\) it is natural to measure the ‘tilt’ of \( v = T(u) \) relative to \( u \) (scaling included) when \( u, v \in A^{-1} \) by \( \theta := v^{-1}u \in A^{-1} \). Then \( u = \theta v \) solves \((T)\) for a given \( v \), provided \( w = w(\theta) := \gamma(\theta v)\theta^{-1} \) satisfies the apparently simpler equation:

\[
e^{\theta w} = 1 + w. \quad (ST_A)
\]

(ST for ‘Standardized Tilting’.) Sufficiency of this condition (when holding for some \( \theta \)) is proved exactly as in Prop. 5.1 (Converse part).

It is thematic that \((ST_A)\) equates a canonical (affine) Popa function with a degenerate (exponential) one, and its solubility may perhaps depend on the geometry of \( A \).

For \( \gamma \) real-valued and \( \theta \in \mathbb{R} \), both sides of \((ST_A)\) are real; the resulting formula for \( u \) coincides with \((T\text{-inv})\), since \( w = \gamma(v) \) here, so that \( \theta = \log(1 + \gamma(v))\gamma(v) \), as before (assuming \( 1 + \gamma(v) \) has a logarithm). In general, one wants \( \theta \) to induce a tilt aligning \( \log(1 + w(\theta)) \) with \( \gamma(\theta v) \), with \( \theta \) pointing in the direction of norm-increase of the exponential function (equivalently of \( \gamma(\theta v) \) – see Lemma 5.5 below), to yield a solution for \((ST_A)\), giving

\[
u = \theta v \log(1 + \gamma(\theta v)\theta^{-1})/\gamma(\theta v).
\]

**Example 5.1.** For \( A = \mathbb{C} \), it emerges that simple collinearity can be effected. With \( \theta = 1 \) \((ST_A)\) takes on its simplest form:

\[
e^{\omega} = 1 + \omega. \quad (ST_C)
\]
The example is instructive: while this has no solutions in \( \text{Re}(\omega) < 0 \), there exists a sequence (necessarily unbounded, on account of the Identity Theorem \([\text{Gam2, V.7}], [\text{Rem, Ch. 8}], \) as in the Great Picard Theorem \([\text{Gam2, XII.2}]\) of solutions in \( \text{Re}(\omega) > 0 \). (In the half-space of unbounded growth of \( e^{\text{Re}(\omega)} \), the factor \( e^{\text{Im}(\omega)} \) can dampen exponential growth to linear in \( \omega \) on the sequence, by taking \( \text{Im}(\omega) \) arbitrarily close to \( \pi/2 \mod 2\pi \mathbb{Z} \), by taking \( \text{Re}(\omega) \) suitably large: for the details see §7 (Appendix).) This resonates with a similar behaviour in a complex Banach algebra when it contains elements \( h \) with \( ||e^{t h}|| = 1 \) for all \( t \in \mathbb{R} \), the ‘hermitian’ elements – see \([\text{Pal}]\) for their particular relevance to \( C^* \)-algebras.

The next result, on uni-directional unboundedness, has two exceptions, just as in \( \mathbb{C} \): the unit sphere and the vector subspace \( \{a \in A : \lim_{t \to \infty} e^{\pm a t} = 0\} \). See \([\text{BohK}]\), where the bound \( ||e^{ta}|| \leq e^{t\varphi(a)} \) for \( t > 0 \), arises from the function:

\[
\varphi(a) := \sup_{t > 0} (t^{-1} \log ||e^{ta}||).
\]

**Lemma 5.5 (One-sided unboundedness).** Assume \( ||1_A|| = 1 \). Unless \( ||e^a|| = 1 \), then as \( s \to \infty \) only one of the two sets \( E_\pm := \{e^{\pm s a} : s > 0\} \) is unbounded, the other being bounded by 0. In particular, unless \( ||e^{\gamma(a)}|| = 1 \), the range of the map

\[
s \mapsto T(su) = u(e^{s\gamma(u)} - 1)/\gamma(u),
\]

for \( u \) fixed, is unbounded as \( s \to \infty \) in only one of the directions \( \pm u \), with limit \( -u/\gamma(u) \) in the other.

**Proof.** This comes from a routine induction, based on noting that if \( ||e^{-a}|| < 1 \), then \( ||e^a|| > 1 \): see §7 (Appendix).

**Corollary 5.4.** For \( A = \mathbb{R}^d \): (i) \( \mathcal{H} = \mathcal{N} \);
(ii) in the context of Theorem 3.1, \( \gamma \) is \( A \)-homogeneouse and equation \((T)\) is soluble for \( v + 1_A \in \exp(A) \)
(iii) Likewise, with \( A = \mathbb{C} \), equation \((T)\) is soluble for \( v + 1 \in \exp(\mathbb{C}) \).

**Proof.** First we prove (ii) (whence (i) will follow). For \( I \subseteq \{1, \ldots, d\} \) we write \( v|_I \) for the projection \( e_I(v) \) of \( v \) on the span \( \langle \{e_i : i \in I\} \rangle \) of the corresponding base vectors, as in Theorem 3.1. As there, we partition \( d \) into parts \( I_i \) of cardinality \( d_i \) (with \( d_i \) summing to \( d \)) for \( i = 1, \ldots k \), obtaining, as equivalent
to \((T)\) on \(\mathbb{R}^d\), the de-coupled simultaneous system below of \(k\) equations \((T)\) on \(\mathbb{R}^d_i\) in terms of \(\gamma_i(u) = \gamma(u)1_{I_i}\) with corresponding solutions.

\[
\begin{align*}
 v|_{I_i} &= u|_{I_i}(e^{\gamma_i(u|_{I_i})} - 1_k)/\gamma_i(u|_{I_i})1_{I_i} \\
 u|_{I_i} &= (v|_{I_i}/\gamma_i(v))\log(1_{A} + v|_{I_i})
\end{align*}
\]

\((i = 1, \ldots, k)\).

Re-combining yields a solution in \(\mathbb{R}^d\) (with \(\gamma\) here \(A\)-homogeneous).

(i) This now follows by partitioning \(\{1, \ldots, d\}\) into the part where the ‘degenerate’ components of \(S\) take value 1 (Th. 3.2) and applying (ii) to the complementary part. Compare the final assertion of Th. 3.1.

(iii) This follows from the solubility of \((ST_{\gamma})\) above. \(\Box\)

Even if \(\gamma\) is not \(A\)-homogeneous, the iteration

\[u_{n+1} = u_n + v - T(u_n)\]

establishes more generally the final conclusion of Prop. 5.1:

**Proposition 5.2.** The range of the tilting map \(T\) contains a nhd of 0.

We begin with a technical

**Lemma 5.6.** For \(S\) Fréchet differentiable, take

\[H(u) := (e^{\gamma(u)} - 1 - \gamma(u))/\gamma(u) = \gamma(u)(1 + 1/2\gamma(u) + \ldots).\]

Then, for some \(\delta > 0\),

\[||H(u)|| \leq ||\gamma||.||u||e^{||\gamma||.||u||},\]

\[||aH(a) - bH(b)|| \leq 1/2||a - b|| \quad (a, b \in B_\delta(0)).\]

**Proof.** Below we write 1 for \(1_A\). By hypothesis, \(S\) is Fréchet differentiable, so \(||\gamma(u)|| \leq ||\gamma||.||u||\). By the triangle inequality applied termwise to the series defining \(H\),

\[||H(u)|| \leq ||\gamma||.||u||e^{||\gamma||.||u||}.
\]

A similar approach gives for any \(u, w\)

\[||wH'(u)|| \leq ||w||.||\gamma||e^{||\gamma||.||u||}.
\]
Indeed
\[ H'(u)h = \left( \frac{1}{2} + \frac{2}{3!}\gamma(u) + \ldots \right)\gamma(h). \]

Now
\[ aH(a) - bH(b) = a(H(a) - H(b)) + (a - b)H(b) = aH'(a)(a - b) + o(a - b) + (a - b)H(b). \]

Thus for \( a, b \) small enough, say for \( a, b \in B_{\delta}(0) \),
\[ ||aH(a) - bH(b)|| \leq \frac{1}{2}||a - b||. \]

Indeed, \( ||H(x)|| \leq ||x||.||\gamma||.e^{||\gamma||.||x||} < 1/3 \) and \( ||xH'(x)|| \leq ||x||.||\gamma||.e^{||\gamma||} < \frac{1}{3} \)
provided \( ||x|| < \min\{1, 1/(3||\gamma||.e^{||\gamma||})\} \).

\textbf{Proof of Proposition 5.2.} We assume \( ||\gamma|| > 0 \), otherwise \( T(u) = u \) and
the result is immediate. With \( H \) as in Lemma 5.6
\[ T(u) - u = u(e^{\gamma(u)} - 1 - \gamma(u))/\gamma(u) = uH(u). \]

With \( \delta \) as in Lemma 5.6, put
\[ \eta = \min\{1, \delta/2, \delta/(2||\gamma||.e^{||\gamma||})\} < \delta/2. \]

Take \( v \in B_{\eta}(0) \) and \( u_1 = v \), so \( ||u_1|| < \delta/2 \). Define a recurrence by
\[ u_{n+1} = v - u_nH(u_n) = v - \left( \frac{1}{2}u_n\gamma(u_n) + \ldots \right). \]

Then \( ||u_2|| < \delta \), since by Lemma 5.6,
\[ ||u_2 - u_1|| = ||vH(v)|| \leq ||\gamma||.||v||.e^{||\gamma||} \leq ||v||.||\gamma||.e^{||\gamma||} < \delta/4. \]

Apply Lemma 5.6 inductively, with the inductive hypothesis
\[ u_n, u_{n-1} \in B_{\delta}(0) \quad \text{and} \quad ||u_n - u_{n-1}|| < \delta/2^n, \]
which holds for \( n = 2 \). Since \( u_n, u_{n-1} \in B_{\delta}(0) \), by Lemma 5.6
\[ ||u_{n+1} - u_n|| = ||u_{n-1}H(u_{n-1}) - u_nH(u_n)|| \leq \frac{1}{2}||u_n - u_{n-1}|| < \delta/2^n: \]
\[ ||u_{n+1} - u_1|| < (\delta/4) + (\delta/8) + \ldots < \delta/2; \]
so \( u_{n+1}, u_n \in B_\delta(0) \), as \( ||u_1|| < \delta/2 \), completing the induction.

So the sequence \( \{u_n\} \) is Cauchy. Say \( u = \lim u_n \); then

\[
u = v - uH(u), \text{ i.e. } v = T(u).
\]

As \( v \) was arbitrary, \( B_\eta(0) \) is in the range of \( T \).

The next result connects the solubility of \((T)\) to whether \(N\) is a vector subspace and is motivated by Example 5.1.

**Proposition 5.3.** With the assumptions and notation of Theorem 4.2:

(i) the following hold for \( t \geq 0 \):

\[
N'(0)u = (\gamma(u)/(e^{\gamma(u)} - 1))N(u(e^{\gamma(u)} - 1)/\gamma(u)) \quad \text{for any } u, \tag{1}
\]

\[
N(T(tu)) = \lambda_u(t)N'(0)T(u) \quad \text{for } u \in N : \tag{2}
\]

\[
N(tu) = tN'(0)u \quad \text{for } u \in H. \tag{3}
\]

So \( N \) is linear on \( H \) and \( N'(0)u = -\rho u \) for \( u \in H \).

(ii) Furthermore, for \( u \in N \),

\[
\{\lambda_u(t)N'(0)T(u) : t \geq 0\} \subseteq N.
\]

In particular, for \( u \in N \) and any \( t \geq 0 \), provided \( \lambda_u(t) \) is invertible, \( T(tu) \in N \) iff \( T(u) \in H \). Furthermore, \( T(u) \in N \) iff \( u \in H \). So for \( u \in N \):

\[
T(u) \in H \text{ iff } T(u) \in N \text{ iff } u \in H.
\]

(iii) \( N \) is a vector subspace if, for all large \( v \), one of \( \pm v = T(u) \) is soluble.

**Proof.** (i)(1) For any \( u \) and subject to the L’Hospital convention, since \( N \) is Fréchet differentiable, recalling \((Rad)\) of Theorem 5.2:

\[
N(u(e^{\gamma(u)} - 1)/\gamma(u)) = \lambda_u(t)N(u(e^{\gamma(u)} - 1)/\gamma(u)) \quad \text{(i.e. \((Rad)\))},
\]

\[
N(tu) = t\gamma(u)/[e^{\gamma(u)} - 1]N(u(e^{\gamma(u)} - 1)/\gamma(u)) + o(t),
\]

\[
N(tu)/t = \gamma(u)/[e^{\gamma(u)} - 1]N(u(e^{\gamma(u)} - 1)/\gamma(u)) + o(t)/t.
\]

Now passage to the limit \( t \to 0 \) yields the claimed formula (as \( N(0) = 0 \)).

(2) Differentiating the radiality formula (left to right) with respect to \( t \) and using commutativity and post-multiplication yields

\[
N(u(e^{\gamma(u)} - 1)/\gamma(u))\gamma(u)e^{t\gamma(u)}/(e^{\gamma(u)} - 1) = N'(u(e^{t\gamma(u)} - 1)/\gamma(u))ue^{t\gamma(u)},
\]

47
\[ N(u(e^{\gamma(u)} - 1)/\gamma(u)) = N'(u(e^{\gamma(u)} - 1)/\gamma(u))u(e^{\gamma(u)} - 1)/\gamma(u). \]

Setting \( t = 0 \) gives

\[ N(T(u)) = N'(0)T(u) \quad (= N'(u)T(u) \text{ for } u \in \mathcal{N}), \]

the last since by (\( \dagger \)), \( N'(u) = N'(0) \) for \( u \in \mathcal{N} \). Formula (2) now follows from \( N(T(tu)) = \lambda_u(t)N(T(u)) \).

(3) Since \( N(0) = 0 \), for \( u \in \mathcal{H} \),

\[ N'(0)u = \lim(N(tu) - N(0))/t = N(u). \]

(ii) Since \( N \) maps into \( \mathcal{N} \), the first assertion follows from formula (2), since \( \lambda_u(t)N'(0)T(u) = N(T(tu)) \in \mathcal{N} \), for \( u \in \mathcal{N} \). Here \( \lambda_u(t) \sim t\gamma(u)/[e^{\gamma(u)} - 1] \), so \( \mathcal{N} \) is tangentially dense along \( \gamma(u)/[e^{\gamma(u)} - 1]N'(0)T(u) \).

Suppose \( \lambda_u(t) \) is invertible and \( u \in \mathcal{N} \), then \( T(tu) \in \mathcal{N} \) iff

\[
\begin{align*}
-\rho T(tu) &= N(T(tu)) : \text{(Lemma 4.4; use Th. 4.2)} \\
-\rho \lambda_u(t)T(u) &= \lambda_u(t)N'(0)T(u) : \text{(using (2) for } u \in \mathcal{N}) \\
-\rho T(u) &= N'(0)T(u) \quad \text{(cancelling) } \text{ iff } T(u) \in \mathcal{H}.
\end{align*}
\]

As \( \mu(\gamma(u)) \) is invertible, \( T(u) = u(e^{\gamma(u)} - 1)/\gamma(u) \in \mathcal{N} \) iff

\[
\begin{align*}
N(u(e^{\gamma(u)} - 1)/\gamma(u)) &= -\rho u(e^{\gamma(u)} - 1)/\gamma(u) \quad \text{(Lemma 4.4)} \\
(\gamma(u)/(e^{\gamma(u)} - 1))N(u(e^{\gamma(u)} - 1)/\gamma(u)) &= -\rho u \quad \text{(cross multiply)} \\
N'(0)u &= -\rho u \quad \text{ (using (1)) } \text{ iff } u \in \mathcal{H}.
\end{align*}
\]

The final claim comes, since \( \lambda_u(1) = 1_\mathcal{A} \), and so we may combine \( u \in \mathcal{H} \) iff \( T(u) \in \mathcal{N} \) with \( T(u) \in \mathcal{N} \) iff \( T(u) \in \mathcal{H} \).

(iii) As to the last claim, take \( v \in \mathcal{N} \). Hence \( \pm nv \in \mathcal{N} \) for \( n \in \mathbb{N} \), as \( \mathcal{N} \) is an additive subgroup. Say \( +nv = T(u) \), for some \( n \in \mathbb{N} \) and \( u \). Then \( nv = T(u) \in \mathcal{H} \) since \( nv = T(u) \in \mathcal{N} \). Then \( v \in \mathcal{H} \), as \( \mathcal{H} \) is a vector subspace. Thus \( \mathcal{N} \subseteq \mathcal{H} \) and so \( \mathcal{N} = \mathcal{H} \); that is, \( \mathcal{N} \) is a vector subspace.

To clarify a first application of Th. 5.1 below (to the case of \( \mathbb{C} \)) we offer

**Example 5.2.** If \( S(\zeta) = 1 + a \Re(\zeta) + b \Im(\zeta) \) with \( a, b \) real and \( b \neq 0 \), then \( \mathcal{N} := \{ \zeta \in \mathbb{C} : S(\zeta) = 1 \} = \langle b - ai \rangle \). For, writing \( \zeta = x + iy \),

\[
a x + by = 0 \text{ iff } z = x - i(ax)/b = (x/b)[b - ai].
\]

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Here $S(1) = 1 + a$, so $\rho = a$ and the adjustor for $1 + \rho \zeta$ is

$$N(\zeta) = -a\zeta + a \Re(\zeta) + b \Im(\zeta) = (b - ai) \Im(\zeta),$$

which is $\mathbb{R}$-linear on $\mathbb{C}$ but not $\mathbb{C}$-differentiable.

We now use the adjustor of Theorem 5.1 to characterize the continuous solutions of $(GS_C)$. This analysis shows that the two kinds of solution correspond to the distinction between analyticity and real analyticity, for which see [KraP]. Like Baron [Bar], we rely on Theorem B of §4 above (albeit through a corollary), which is key also to the Gebert proof [Geb]. (The latter source also uses the Wołodźko-Javor characterization, for which see §6.1, to infer very elegantly from Th. B that, unless $S$ is real-valued, $\mathcal{N}$ is discrete, and so $\mathcal{N}$ is the trivial vector subspace $\{0\}$, since $S(\mathbb{C})$ is connected and $S(\mathbb{C}) \cap \mathcal{N} = \mathcal{N}$; so Corollary 5.2 applies here.)

**Corollary 5.6** (cf. [Bar]). For $A = \mathbb{C}$, if $S$ solves $(GS_C)$ and is $\mathbb{C}$-differentiable, then for some $\rho \in \mathbb{C}$

$$S(z) = 1 + \rho z. \quad \text{(Can}_C)$$

If $S$ is only continuous, so that $N : \mathbb{C} \to \mathcal{N}$ is Fréchet differentiable, then for some $a, b \in \mathbb{R}$

$$S(z) = 1 + a \Re(z) + b \Im(z). \quad \text{(Re-Im)}$$

**Proof.** The special case of a $\mathbb{C}$-differentiable $S$ is covered by Th. 5.1.

We now assume only that $S$ is continuous. As noted in §3, the Popa group $G_z^\times(\mathbb{R}^2)$ is a Lie group, so interpreting $z = x + iy$ as $(x, y)$ and $S(z)$ as $S(x, y)$, this, and so also its adjustor $N$, is differentiable in the usual Euclidean sense. By Prop. 4.2 the closed subgroup $\mathcal{N} := \{z : S(z) = 1\}$ is a vector subspace of $\mathbb{R}^2$. If the subspace $\mathcal{N}$ is two-dimensional, then $\mathcal{N} = \mathbb{C}$ and $S \equiv 1$. Otherwise $\mathcal{N} = \alpha \mathbb{R}$ for some $\alpha \in \mathbb{C}$. W.l.o.g. we take $\alpha = 1$ (otherwise apply to $\mathbb{C}$ the transformation $z \mapsto \alpha^{-1}z$).

We now use the fact that $N : \mathbb{C} \to \mathbb{R}$ is Fréchet (= ordinarily) differentiable. Note that if $L : \mathbb{C} \to \mathbb{R}$ is linear, then for some $a_L, b_L \in \mathbb{R}$

$$L(z) = a_L \Re(z) + b_L \Im(z)$$

(since $L(x + iy) = L(x) + L(iy)$). Fix $z$ and put $L := N'(z)$ and $L_0 := N'(0)$. Then, with $a := a_{L_0}$ and $b := b_{L_0}$, by Theorem 5.1,

$$S(z)^{-1}[a_L \Re(S(z)h) + b_L \Im(S(z)h)] = a \Re(h) + b \Im(h) \quad (h \in \mathbb{C}).$$
First suppose that \(ab \neq 0\). Then \(S(z)^{-1}\) is real (take e.g. \(h = 1\)). We may cancel through the real and imaginary parts, giving

\[
[a_L \text{Re}(h) + b_L \text{Im}(h)] = a \text{Re}(h) + b \text{Im}(h) \quad (h \in \mathbb{C}).
\]

So \(a_L = a\) and \(b_L = b\). Now suppose \(a = b = 0\); then

\[
[a_L \text{Re}(S(z)h) + b_L \text{Im}(S(z)h)] = 0 \quad (h \in \mathbb{C}).
\]

Take \(h = \overline{S(z)}\) so that \(S(z)h > 0\) (as \(S\) takes invertible values), and similarly take \(h = i \overline{S(z)}\), to deduce that \(a_L = b_L = 0\).

So, in either case \(a_L = a\) and \(b_L = b\), and so \(N'(z) = N'(0)\) for all \(z \in \mathbb{C}\).

Then, since \(N(0) = 0\), again by Lemma 5.1,

\[
N(z) = L_0(z) = a \text{Re}(z) + b \text{Im}(z) : \quad S(z) = 1 + \rho z + a \text{Re}(z) + b \text{Im}(z).
\]

But \(\rho := S(1) - 1 = \rho + a\); so \(a = 0\), and so

\[
S(z) = 1 + \rho z + b \text{Im}(z).
\]

For \(b = 0\) we again obtain \((\text{Can}_C)\). Substitution into \((G\!S)\) shows that \(\rho = 0\), if \(b \neq 0\). Now the transformation \(z \mapsto (u + iv)z\) of \(\mathbb{C}\) with \(u, v\) real yields

\[
\text{Im}((u + iv)z) = v \text{Re}(z) + u \text{Im}(z), \quad (\text{Im})
\]

since with \(x, y\) real

\[
\text{Im}((u + iv)(x + iy)) = \text{Im}((ux - vy) + i(vx + uy)).
\]

As \(b\) above is real, this yields \((\text{Re-Im})\) with \(bv\) and \(ub\) for \(a\) and \(b\). Thus, we have obtained both \((\text{Can}_C)\) and \((\text{Re-Im})\) assuming continuity of \(S\).  

The next result corresponds to \(S(1_A) = 1_A\) with \(N\) arbitrary except for the condition \(N(1_A) = 0\). Take a linear \(\sigma : A \to \mathbb{R}\) with \(\sigma(1_A) = 0\). Then for \(\nu : A \to \mathcal{N} := \langle e_i : i \in I \rangle\) as below, \(\nu(1_A) = 0\); indeed, adjoin \(e_0 = 1_A\), as a further orthogonal idempotent, and then \(\sigma(e_i 1_A) = 0\) all \(i\).

Mutatis mutandis, the idempotents \(e_i\) below may also be interpreted as orthogonal projections onto one-dimensional vector subspaces: compare \(\text{Re}\) and \(\text{Im}\) in Example 5.2.
**Proposition 5.4.** For \( e_i \in A \) mutually orthogonal idempotents (i.e. with \( e_i e_j = \delta_{ij} e_i \)) and \( \sigma : A \to \mathbb{R} \) linear, take

\[
\nu(x) := \sum_i \sigma(e_i x) e_i,
\]

assumed convergent, then \( \nu : A \to (e_i : i \in I) \) is linear and \((GS)\) has solution

\[
S(x) := 1_A + \nu(x).
\]

**Proof.** See §7 (Appendix). \( \square \)

**Remark.** Assume that the \( e_i \) are projections with one-dimensional ranges, spanning \( A \), with \( 1_A = \sum_i e_i \). Then the Proposition above also includes the case \( S(x) = 1 + \rho x \) for some \( \rho \in A \). For, take \( \rho_i := e_i \rho, \ x_i = e_i x, \) and, interpreting these as scalar multipliers (as \( e_i x = \xi_i e_i \) for some scalar \( \xi_i \)), put

\[
f_\rho(x) := \sum_i \rho_i x_i,
\]

so that \( f_\rho : A \to \mathbb{R} \) is linear. Then

\[
\nu(x) := \sum_i f_\rho(e_i x) e_i = \sum_i \rho_i x_i e_i.
\]

But \( \rho = \rho 1_A = \sum_i \rho e_i e_i = \sum_i \rho_i e_i \) and so

\[
\rho x = \sum_i \rho_i e_i \sum_j x_j e_j = \sum_{i,j} \rho_i e_i x_i e_j = \sum_i \rho_i x_i e_i = \nu(x).
\]

This may be viewed as the totally independent case in that

\[
\nu_i(x) = \nu(x) e_i = \rho_i x_i : \quad S(x)e_i = 1 + \rho_i x_i.
\]

### 6 Complements

#### 6.1 Wołodźko-Javor theory

The following result of Wołodźko [Wol] was presented in 1968 as a ‘construction’ yielding, for \( F \) a commutative field, all \( F \)-valued solutions \( S \) of \((GS)\) over a vector space \( X \) and was cited as a theorem first in [Jav], also in 1968, and again later in the textbook [AczD, Ch. 19 Th. 5] (attributed there to [Jav], but see the comment in [BriD, Prop. 4]). The idea, however, may be traced back to [GolS, Th. 4]. We check that this characterization of solutions
of $S$ continues to hold also over $A$ by interpreting the invertible elements of $F$ there by $A^{-1}$ here. This yields, mutatis mutandis, a characterization of the restriction $S|G_{S}(A) : A \rightarrow A^{-1}$ and a connection with the Goldie equation, noted in the Remark below.

**Theorem 6.1 (Wołodźko-Javor Theorem).** $S : A \rightarrow A$ solves $(GS)$ iff there exist: an additive subgroup $N$ of $A$, a multiplicative subgroup $\Lambda$ of $A^{-1}$, and a function $W : \Lambda \rightarrow A$ such that:

i) $\Lambda N = N$,

ii) $W(\lambda) \in N$ iff $\lambda = 1$,

iii) for all $\lambda_1, \lambda_2 \in \Lambda$, the Wołodźko equation holds:

\[
W(\lambda_1 \lambda_2) = W(\lambda_1) + \lambda_1 W(\lambda_2) \mod N, \quad (W)
\]

iv) $S(x) = \begin{cases} 
\lambda, & \text{if } x = W(\lambda) \mod N, \text{ for some } \lambda \in \Lambda, \\
0, & \text{otherwise.}
\end{cases}$

**Proof.** See §7 (Appendix). $\square$

**Remark.** $K(u) := W(\exp(u)) (u \in A^{-1})$ converts $(W)$ to a $(GFE)$ variant:

\[
K(u_1 + u_2) = K(u_1) + e^{u_1} K(u_2) \mod N.
\]

### 6.2 Functional equations and probability theory

We mention briefly some of the probability background to some of the functional equations we use here. For a monograph treatment of the intersection of these two areas, see [BalL] and [KagLR].

1. **The Cauchy functional equation, (CFE).** The Cauchy functional equation of §1.1 is ubiquitous, for example in regular variation [BinGT, §1.1], and in the lack-of-memory property of the exponential law, and so in the Markov property in continuous time (see e.g. [GriS, §6.9]). In the setting of e.g. renewal theory, survival times $X$ are in $\mathbb{R}_+ := [0, \infty)$, survival probabilities $S(x) := P(X > x)(x \geq 0)$ are monotone, so solutions $S$ to the $(CFE)$ here are exponential (see e.g. [GriS, 4.14.5]), $e^{-\lambda x} (\lambda \geq 0, \text{ and } \lambda > 0 \text{ in the non-trivial case})$.

2. **The Golab-Schinzel equation (GS).** Replacing $(\mathbb{R}_+, +)$ here by $(G_\rho(\mathbb{R}), \circ_\rho)$ takes $(CFE)$ to $(GS)$, with solutions the generalised Pareto laws of EVT.
([BinO5], and for higher dimensions, [RooT]). A relative of (GS),

\[ S(x + \theta(x)y) = S(x)S(y) \quad (\theta(x) = 1 + cx), \]

appears in the probability literature in [OakD] (cf. [AsaRS, Th. 3.4]) in connection with characterisation of the Hall-Wellner laws [HalW, Prop. 6].

3. **(CFE) in higher dimensions.** This is sensitive to quantifier weakening. See Marshall and Olkin [MarO1]; for further developments, see [MarO2, §4], [MarO3, Ex. 5.1].

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7 Appendix (arXiv only):

To lighten the burden of the main text above, the four sections here serve as a repository: first for some further examples, second for an independent derivation of the radiality formula of Theorem 5.2 in the Euclidean case that illuminates the role of spectral conditions, third for the fulfilment of earlier promised referrals to some lengthier, straightforward, albeit necessary, arguments, and lastly to reproduce Sasane’s proof of his Theorem S.

7.1 Further Examples

Example 7.1. Take \( A = \mathbb{R}^2 \); then the \( A \)-differentiable solutions are of the form

\[
S(x) = 1 + \rho x = (1 + \rho_1 x_1, 1 + \rho_2 x_2).
\]

Here provided \( \rho \) is invertible, \( t1_A = S((t - 1)/\rho_1, (t - 1)/\rho_2) \).

We consider the alternative Fréchet differentiable (but not \( A \)-differentiable) solution given by Th.3.1(ii) (with \( d = 2 \) and trivial partition)

\[
S(x) : = (1 + \sigma_1 x_1 + \sigma_2 x_2, 1 + \sigma_1 x_1 + \sigma_2 x_2) \in \langle (1, 1) \rangle.
\]

\[
\gamma(u) = DS(0)u = \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_1 & \sigma_2 \end{bmatrix} u = \begin{bmatrix} \tau(u) \\ \tau(u) \end{bmatrix} \quad \text{for } \tau(u) := \sigma_1 u_1 + \sigma_2 u_2,
\]

\[
N : = \langle (\sigma_2, -\sigma_1) \rangle = \{ u : \tau(u) = 0 \}.
\]

So \( e^{\gamma(u)} - 1 = (e^{\tau(u)} - 1, e^{\tau(u)} - 1) \) is non-invertible for \( \tau(u) = 0 \), implying \( N \) is homogeneous on \( N \) and so linear, as we shall see.

\[
\rho := S(1) - 1 = (\sigma_1 + \sigma_2, \sigma_1 + \sigma_2) = (\sigma_1 + \sigma_2)(1, 1).
\]

Adjusting \( 1 + \rho x \) into agreement with \( S(x) \) gives

\[
N(x) := (\sigma_2(x_2 - x_1), \sigma_1(x_1 - x_2)) = (x_2 - x_1)(\sigma_2, -\sigma_1) \in N,
\]

which is indeed linear. Then \( \text{ran}N = \langle (\sigma_2, -\sigma_1) \rangle = N, \text{ker}N = \langle (1, 1) \rangle \) and

\[
N = \text{ker}(S'(0)) = \{ (x_1, x_2) : \sigma_1 x_1 + \sigma_2 x_2 = 0 \} = \langle (\sigma_2, -\sigma_1) \rangle.
\]

Tilting here is :

\[
T(u) = u(e^{\gamma(u)} - 1)/\gamma(u) = u \left( \frac{e^{(\sigma_1 u_1 + \sigma_2 u_2)} - 1}{\sigma_1 u_1 + \sigma_2 u_2}, \frac{e^{(\sigma_1 u_1 + \sigma_2 u_2)} - 1}{\sigma_1 u_1 + \sigma_2 u_2} \right) \quad \text{or } u \text{ if } \tau(u) = 0.
\]
Example 7.2. As in Example 7.1, take $A = \mathbb{R}^3$; then the $A$-differentiable solutions are of the form

$$S(x) = 1 + \rho x = (1 + \rho_1 x_1, 1 + \rho_2 x_2, 1 + \rho_3 x_3).$$

We consider the alternative (Fréchet differentiable) solution given by Th. 3.3(iv):

Here, provided $(\sigma_1, \sigma_2) \neq 0$ and $\sigma_3 \neq 0$, if $(x_1, x_2)$ solves $\sigma_1 x_1 + \sigma_2 x_2 = (t - 1)$, then $t 1_A = S(x_1, x_2, (t - 1)/\sigma_3)$, in which case

$$(\text{ran}S)u \supseteq \langle 1_A \rangle u = \langle u \rangle.$$

$$S(x) : = (1 + \sigma_1 x_1 + \sigma_2 x_2, 1 + \sigma_1 x_1 + \sigma_2 x_2, 1 + \sigma_3 x_3) :$$

$$\gamma(u) = DS(0)u = \begin{bmatrix} \sigma_1 & \sigma_2 & 0 \\ \sigma_1 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} u = \begin{bmatrix} \sigma_1 u_1 + \sigma_2 u_2 \\ \sigma_1 u_1 + \sigma_2 u_2 \\ \sigma_3 u_3 \end{bmatrix},$$

$$(e^{\gamma(u)} - 1)/\gamma(u) = \left( (e^{\tau(u)} - 1)/\tau(u), (e^{\tau(u)} - 1)/\tau(u), \frac{e^{\sigma_3 u_3} - 1}{\sigma_3 u_3} \right),$$

$$\mathcal{N} : = \{(x_1, x_2, 0) : \sigma_1 x_1 + \sigma_2 x_2 = 0 \} = \{(\sigma_2, -\sigma_1, 0)\},$$

$$\rho : = S(1) - 1 = (\sigma_1 + \sigma_2, \sigma_1 + \sigma_2, \sigma_3).$$

Adjusting $1 + \rho x$ into agreement with $S(x)$ gives

$$N(x) := (\sigma_2 (x_2 - x_1), \sigma_1 (x_1 - x_2), 0) = (x_2 - x_1)(\sigma_2, -\sigma_1, 0) \in \mathcal{N},$$

which is linear; then $\text{ran} N = \mathcal{N}$, $\text{ker} N = \langle (1, 1, 0), (0, 0, 1) \rangle$.

Example 7.3. (Non-linear adjustor for a degenerate $S$). In the previous examples the solution function $S$ had an underlying linear form. This is not so for $A = \mathbb{R}^2 = G_S^*(\mathbb{R}^2)$, when $S(x_1, x_2) := (1, e^{x_1})$. Note that $e(x) = (0, e^{x_1} - 1 - x_1) = (0, \frac{1}{2} x_1^2 + ...) \text{ and } e(x \circ x) = (0, 2 x_1^2 + ...)$. Here

$$e(x)/||x||^2 = \frac{1}{2} \frac{x_1^2}{x_1^2 + x_2^2} \to \text{ depends on } x_2/x_1$$

with variable limit as $x \to 0$.

$$S(x)(0, u) = (0, e^{x_1} u) : \quad (\text{ran} S)(0, u) = \langle (0, u) \rangle.$$
\( S(x_1, x_2) = (1, e^{x_1}) = 1 + (0, e^{x_1} - 1) : \)
\[ \mathcal{N} = \{ x : S(x) = 1 \} = \{ x : x_1 = 0 \} = \langle e_2 \rangle, \]
\[ \gamma(u) = DS(0)u = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} u = \begin{bmatrix} 0 \\ u_1 \end{bmatrix} = 0 \text{ iff } u_1 = 0. \]
\[ \rho = S(1) - 1 = (0, e - 1) : \rho x = (0, (e - 1)x_2) : \]
\[ N(x_1, x_2) = S(x_1, x_2) - 1 - \rho x = (0, (e^{x_1} - 1) - (e - 1)x_2) \in \mathcal{N}. \]

Applying the L'Hospital convention:
\[ \text{spec}(\gamma(u)) = \{ 0, u_1 \} : \lambda_u(t) = (t, \frac{e^{tu_1} - 1}{e^{u_1} - 1}), \text{ or } (t, t) \text{ if } u_1 = 0 : \]
\[ N(te_2) = t\mathcal{N}(e_2) = t\mathcal{N}(e_2) \quad (\text{for } u_1 = 0); \]

but for \( u_1 \neq 0 : \)
\[ N((t, e^t - 1)e_1) = (t, \frac{e^t - 1}{e - 1})\mathcal{N}((1, e^t - 1)e_1) : \]
\[ N(te_1) = (t, \frac{e^t - 1}{e - 1})\mathcal{N}(e_1) : \]
\[ N(t(1,0)) = (0, e^t - 1) = \frac{e^t - 1}{e - 1}(0, e - 1) = \lambda(t)N(1,0). \]

Here
\[ S(x)(0, u) = (0, e^{x_1}u) : \quad (\text{ran}S)(0, u) = \langle (0, u) \rangle. \]

and tilting is given by
\[ T(u) := u(e^{\gamma(u)} - 1)/\gamma(u) = u(1, \frac{e^{u_1} - 1}{u_1}) \text{ or } u \text{ if } u_1 = 0. \]

**Example 7.4.** \( S(x) = (1, 1, e^{x_1 + x_2}). \) Here \( \mathcal{N} = \{ x : x_1 + x_2 = 0 \}, \) which is 2-dimensional. As in Cor. 4.4:

\[ S(x)(u_1, u_2, u_3) = (u_1, u_2, e^{x_1 + x_2}u_3) : \ u \in (\text{ran}S)u : \]
\[ (-u) + (\text{ran}S)u \supseteq \{ (0, 0, (e^t - 1)u_3) : t \in \mathbb{R} \}; \]

\[ (\text{ran}S)\mathcal{N} = \mathcal{N} : \quad \mathcal{N} \supseteq \{ (0, 0, \lambda u_3) : \lambda > -1 \} \text{ for } u \in \mathcal{N} \text{ and so } \mathcal{H} \neq \emptyset. \]

\[ \{ u : (\text{ran}S)u = \{ u \} \} = \{ u : u_3 = 0 \} \text{ orthogonal to } (0, 0, 1). \]
$DS(0)u = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} u = \begin{bmatrix} 0 \\ 0 \\ u_1 + u_2 \end{bmatrix}$ = 0 iff $u_1 + u_2 = 0,$

$\lambda_u(t) = \left(t, t, \frac{e^{t(u_1+u_2)} - 1}{e^{u_1+u_2} - 1} \right)$ or $(t, t, t)$ if $u_1 + u_2 = 0.$

So $\gamma(e_1 - e_2) = \gamma(e_1) - \gamma(e_2) = 0$ and $\gamma(e_3) = 0,$ confirming that $N = \langle e_1 - e_2, e_3 \rangle.$

So tilting is given by

$T(u) = u(e^{\gamma(u)} - 1)/\gamma(u) = u(1, 1, \frac{e^{(u_1+u_2)} - 1}{u_1 + u_2})$ or $u$ if $u_1 + u_2 = 0.$

We may check this directly:

$\rho = S(1) - 1 = (0, 0, e^2 - 1): \quad \rho x = (e^2 - 1)x_3 :$

$N(x) = (0, 0, (e^{x_1+x_2} - 1) = (e^2 - 1)x_3).$

Here

$N(tx) = (0, 0, (e^{tx_1+tx_2} - 1) - (e^2 - 1)tx_3),$

$N(tx_1, -tx_1, tx_3) = (0, 0, -(e^2 - 1)tx_3) = tN(x_1, -x_1, x_3):$

$N(tx_1, -tx_1, 0) = tN(x_1, -x_1, 0)$ and $N(te_3) = tN(e_3),$

yielding the two directions along which $N$ is homogenous:

$N(t(1, -1, 0) = tN(1, -1, 0),$ and $N(te_3) = tN(e_3).$

**Remark.** Along the natural base directions (with $\lambda$ subscript $i$ for $e_i$):

$\lambda_1(t) = \lambda_2(t) = (t, t, (e^t - 1)/(e - 1)) \quad \lambda_3(t) = (t, t, t).$

So

$N(u(e^{\gamma(u)} - 1)/\gamma(u)) = \lambda_u(t)N(u(e^{\gamma(u)} - 1)/\gamma(u)) :$

$N(e_i(t, t, (e^t - 1)/(e - 1))) = (t, t, (e^t - 1)/(e - 1))N(e_i(1, 1, (e^t - 1)): \quad (i = 1, 2)$

$N(te_i) = (t, t, (e^t - 1)/(e - 1))N(e_i) \quad (i = 1, 2),$

$N(te_3) = t(1, 1, 1)N(e_3) = tN(e_3).$
7.2 Radiality formula in \( \mathbb{R}^d \)

We give a direct proof of Theorem 4.2 when \( A = \mathbb{R}^d \). We begin with two Lemmas. In \( \mathbb{R}^d \) we will write \( x_i \) or \((x)_i\) for the \( i \)-th component of \( x \); of course \( x_i = xe_i \) where the \( e_i \) are the natural base vectors. We write \( \gamma(u) = S'_u(0) \).

Recall that the spectrum of \( x \) [Rud, 10.10] is defined by

\[
\text{spec}(x) := \{ \lambda : (\lambda 1_A - x) \notin A^{-1} \}.
\]

So in \( A = \mathbb{R}^d \), \( \text{spec}(\gamma(u)) \) comprises the components \( \gamma(u)_i \).

**Lemma 7.1.** In \( \mathbb{R}^d \), \( e^{\gamma(u)} - 1_A \) is invertible iff \( 0 \notin \text{spec}(\gamma(u)) \).

**Proof.** \( \gamma(u) \) has real components, so \( \text{spec}(\gamma(u)) \subseteq \mathbb{R} \). So by [Rud, 10.28], \( e^{\gamma(u)} - 1_A \) is invertible iff \( e^{\lambda} - 1 \neq 0 \) for all \( \lambda \in \text{spec}(\gamma(u)) \subseteq \mathbb{R} \) iff \( 0 \notin \text{spec}(\gamma(u)) \).

**Lemma 7.2.** In \( \mathbb{R}^d \), if \( S'_u(0)_i = 0 \) for some \( i \) and \( u = e_j \), then \( S(te_j)_i = 1 \) for all \( t \in \mathbb{R} \), i.e. \( \langle e_j \rangle \subseteq \{ x : S(x)_i = 1 \} \). In particular, if \( S'_u(0)_i = 0 \), for \( u = e_j \) and for all \( i \), then \( S(te_j) = 1 \) for all \( t \in \mathbb{R} \), i.e. \( \langle e_j \rangle \subseteq N = \{ x : S(x) = 1 \} \).

**Proof.** Fix \( i \). Working in direction \( e_j \), \( (GS) \) implies that for \( s, t \in \mathbb{R} \)

\[
S(se_j + tS(se_j)e_j)_i = (S(se_j))_i(S(te_j))_i.
\]

This is a pexiderized form of the real-valued univariate version of \( (GS) \) (see e.g. [Jab]). Since \( t \mapsto S(te_j)_j \) is continuous for all \( j \), it follows that, for some \( \sigma_{ij} \in \mathbb{R} \),

\[
S(te_j)_j = S(te_j)_i = 1 + t\sigma_{ij} \quad (t \in \mathbb{R}).
\]

Then for \( u = e_j \)

\[
S(tu)_i - S(0)_i = t\sigma_{ij} : \quad \sigma_{ij} = S'_u(0)_i = 0.
\]

So \( S(te_j)_j = S(te_j)_i = 1 + t\sigma_{ij} = 1 \).

**Theorem 7.1** Euclidean. For \( S \) Fréchet differentiable, the solution \( (N,S) \) to the Goldie equation in \( \mathbb{R}^d \),

\[
N(x + S(x)y) = N(x) + S(x)N(y),
\]

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satisfies, for \( u \) any natural base vector; the radiality formulae below (with the L’Hospital convention); that is, with \( \gamma(u) = S'_u(0) \) and

\[
\lambda_u(t) := [e^{t\gamma(u)} - 1]/[e^{\gamma(u)} - 1]: \quad \lambda_u(t)_i = [e^{t\gamma(u)_i} - 1]/[e^{\gamma(u)_i} - 1],
\]

\[
N(u(e^{\gamma(u)} - 1)/\gamma(u)) = \lambda_u(t)N(u(e^{\gamma(u)} - 1)/\gamma(u)).
\]

In particular, if \( \gamma(u)_i = 0 \) for all \( i \), then

\[
N(tu) = t1N(u) = tN(u).
\]

**Proof.** Referring again to the polynomials \( \varphi_n(x) = 1 + x + ... + x^{n-1} \) and rational polynomials \( [\varphi_m/\varphi_n](x) \), [BinO4] Theorem 2 gives with \( N \) here for \( K \) there that for any \( u \)

\[
N(\varphi_m(S(u/n))u/n) = [\varphi_m/\varphi_n](S(u/n))N(\varphi_n(S(u/n))u/n).
\]

By Gateaux differentiability, write

\[
S(u/n) - S(0) = S'_u(0)(1/n) + \varepsilon_n(u),
\]

with \( n\varepsilon_n(u) \rightarrow 0 \). Fix \( u = e_j \). We work with odd integers \( n \) below and take \( \gamma(u) := S'_u(0) \) as above and \( \xi_n := S(u/n) \).

Case 1. \( \gamma(u)_i \neq 0 \) for all \( i \). Provided \( (\xi_n)_i \neq 1 \) for infinitely many (odd) \( n \):

\[
[\varphi_m/\varphi_n](\xi_n)_i = [(S(u/n)_i)^m - 1]/[(S(u/n)_i)^n - 1]
\]

\[
= [(1 + S'_u(0)_i/n + \varepsilon_n(u))^{mt(n)} - 1]/[(1 + S'_u(0)_i/n + \varepsilon_n(u))^n - 1]
\]

\[
\rightarrow [e^{\gamma(u)_i} - 1]/[e^{\gamma(u)_i} - 1],
\]

as \( n \rightarrow \infty \) with \( m(n)/n \rightarrow t \in \mathbb{R} \). If for some \( i, (\xi_n)_i = 1 \) for all large (odd) \( n \), then

\[
0 = S(u/n)_i - S(0)_i = S'_u(0)_i(1/n) + \varepsilon_n(u)_i:
\]

\[
S'_u(0)_i = -n\varepsilon_n(u) \rightarrow 0 : \quad \gamma(u)_i = 0,
\]

a contradiction to this case.

Case 2. For some \( i, S'_u(0)_i = 0 \). Here \( u = e_j \), so by Lemma 2 \( (\xi_n)_i = \varphi_n(S(0)/n)_i = 1 \) for all \( n \); then, again as \( m/n \rightarrow t \) and as \( \varphi_n(1) = n \),

\[
[\varphi_m/\varphi_n](\xi_n)_i = m(n)/n \rightarrow t.
\]

So we may interpret the earlier displayed formula using the L’Hospital convention. \( \square \)
7.3 Expanded arguments

Below we make good on the promise to set out certain routine arguments.

7.3.1 Proof of Prop 1.1

(i) $G^*_S(\mathbb{A})$ is closed under the operation $\circ_S$ since

$$S(a \circ_S b) = S(a)S(b),$$

so that $S(a)S(b)$ is invertible when $S(a)$ and $S(b)$ are invertible. Note that for $a \in G_S(\mathbb{A})$

$$a = a \circ_S b = a + S(a)b \text{ iff } b = 0.$$  

Also for $a \in G_S$, as $S(a)$ is invertible,

$$S(a) = S(a \circ_S 0) = S(a)S(0) : S(0) = 1_{\mathbb{A}}.$$  

Thus the neutral element for $\circ_S$, i.e. $0$ is in $G^*_S$. The $\circ_S$-inverse of $a$ is $b = -aS(a)^{-1}$. As in [Jav], the operation is associative:

$$(a \circ_S b) \circ_S c = (a + S(a)b + S(a)b)c = a + S(a)b + S(a)S(b)c,$$

$$a \circ_S (b \circ_S c) = a + S(a)[b + S(b)c] = a + S(a)b + S(a)S(b)c.$$  

As the elements of $\mathbb{A}_1$ are invertible, $G_S := S^{-1}(\mathbb{A}_1) \subseteq G^*_S$. As $S(0) = 1_{\mathbb{A}}$, $0 \in G_S$, and furthermore for $a, b \in G_S$ : as $S(a), S(b) \in \mathbb{A}_1$,

$$S(a \circ_S b) = S(a)S(b) \in \mathbb{A}_1$$

(as $\mathbb{A}_1$ is a multiplicative group), so $a \circ_S b \in G_S$, and also $a^{-1}_S \in G_S$ because

$$S(a^{-1}_S) = S(a)^{-1} \in \mathbb{A}_1,$$

since

$$1_{\mathbb{A}} = S(0) = S(a \circ_S a^{-1}_S) = S(a)S(a^{-1}_S).$$  

So $G_S$ is a subgroup of $G^*_S$.

(ii) Evidently, for $a, b \in \mathbb{N}$

$$a + b = a + S(a)b = a \circ_S b.$$  

So if $a, b \in \mathbb{N}$, then $a + b \in \mathbb{N}$ and $-a \in \mathbb{N}$, since

$$S(a + b) = S(a + S(a)b) = S(a)S(b) = 1_{\mathbb{A}},$$

$$1_{\mathbb{A}} = S(0) = S(a - a) = S(a - S(a)a) = S(a)S(-a) = S(-a).$$  

Also, for $a \in \mathbb{N}$, $-a = -aS(-a) = -aS(a)^{-1}$ is the $\circ_S$-inverse of $a$, so $\mathbb{N}$ is both an additive subgroup of $\mathbb{A}$ and a subgroup of $G^*_S$.  

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7.3.2 Proof of Th. 2.2

Below we use the notation \( a \circ b := a + bS(a) \) on \( \mathbb{G}_S := \{ a \in \mathbb{A} : S(a) \in \mathbb{A}^{-1} \} \).

Differentiation of \((GS)\) w.r.t. to \( b \) gives

\[
S'(a + bS(a))S(a) = S(a)S'(b); 
\]

now take \( a = b^{-1}_S \) so \( S(a)^{-1} = S(b) \) to obtain the similarity relations:

\[
S(a)^{-1}S'(0)S(a) = S'(b); \quad S(b)\gamma S(b)^{-1} = S'(b). 
\]

Below we repeatedly write

\[
S(a + h) = S(a) + S'(a)h + o(h). 
\]

We consider \((GS)\) after \( a \) and \( b \) are equally incremented by \( h \):

\[
S((a + h) + (b + h)(S(a) + S'(a)h + o(h)) \\
= S((a + bS(a)) + h + bS'(a)h + hS(a) + o(h)) \\
= S(a \circ b) + S'(a \circ b)[h + bS'(a)h + S(a)h + o(h)] + o(h) \\
= (S(a) + S'(a)h + o(h))(S(b) + S'(b)h + o(h)) \\
= S(a)S(b) + S(a)S'(b)h + S(b)S'(a)h + o(h). 
\]

Comparison of the two sided gives to within \( o(h) \)

\[
S'(a \circ b)[1 + bS'(a) + S(a)] = S(a)S'(b) + S(b)S'(a). 
\]

Applying the similarity relations gives

\[
S(a \circ b)\gamma S(a \circ b)^{-1}[1 + bS'(a) + S(a)] \\
= S(a)S(b)\gamma S(b)^{-1} + S(b)S(a)\gamma S(a)^{-1}. 
\]

Cancelling \( S(a)S(b) \) on the left-hand side:

\[
\gamma S(a)^{-1}S(b)^{-1}[1 + bS(a)\gamma S(a)^{-1} + S(a)] = \gamma S(b)^{-1} + \gamma S(a)^{-1}. 
\]

Absorbing \( S(a)^{-1} \) on the left-hand side:

\[
\gamma S(b)^{-1}[S(a)^{-1} + b\gamma S(a)^{-1} + 1\gamma] = \gamma S(b)^{-1} + \gamma S(a)^{-1}. 
\]

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Cancelling the term $\gamma S(b)\gamma$ appearing on each side:

$$\gamma[S(b)^{-1}S(a)^{-1} + S(b)^{-1}b\gamma S(a)^{-1}] = \gamma S(a)^{-1}.$$ 

Cancelling $S(a)^{-1}$ on the right on both sides gives:

$$\gamma[S(b)^{-1} - b\gamma] = \gamma,$$

with $b^{-1}$ the $\circ_S$-inverse of $b$. Put $c = b^{-1}$; then, on rearranging,

$$\gamma(S(c)h) = \gamma(h) + \gamma(c\gamma h). \quad (\ast')$$

All the steps above are reversible. We now improve on the last equation.

Differentiating $(\ast')$ with respect to $c$ in direction $k$:

$$\gamma(S'(c)(k)h) = \gamma(k\gamma h).$$

Using the similarity relations gives

$$\gamma(S(c)\gamma S(c)^{-1}(k)h) = \gamma(k\gamma h),$$

which for $a = S(c)$ yields the claim $(\ast \ast \ast)$ and for $c = 0$ yields the claim $(\ast \ast)$.

Writing $S(c)k$ for $k$ in the last equation gives

$$\gamma(S(c)\gamma(k)h) = \gamma(S(c)k\gamma h). \quad (\tilde{\ast})$$

It now follows from $(\ast')$ and $(\ast \ast)$ that

$$\gamma(S(c)h) = \gamma(h) + \gamma(c\gamma h) = \gamma(h) + \gamma(h\gamma(c)) = \gamma(h(1 + \gamma(c))).$$

That is, $(\ast)$ holds. Evidently $(\ast)$ and $(\ast \ast)$ yield $(\ast')$, and so the conjunction of $(\ast)$ and $(\ast \ast)$ yields $(GS)$.

It is immediate from $(\ast)$ that

$$\gamma(S(c)) = \gamma(1_A + \gamma(c)).$$

Furthermore, $(\ast \ast)$ with $k = 1_A$ yields

$$\gamma(\gamma(h)) = \gamma(1_A\gamma(h)).$$

Differentiating $(\tilde{\ast})$ with respect to $c$ in direction $u$ and setting $c = 0$ gives

$$\gamma(\gamma(u)\gamma(k)h) = \gamma(\gamma(u)k\gamma(h)).$$

Taking $k = 1_A$ yields

$$\gamma(\gamma(1_A)\gamma(u)h) = \gamma(\gamma(u)\gamma(h)) : \quad \gamma(\gamma(\gamma(u)h) = \gamma(\gamma(u)\gamma(h)). \quad \square$$
7.3.3 Proof of Prop. 2.2.

Assume the weaker homogeneity property and suppose for some $n$ that

$$ \gamma(x\gamma(x)^n) = \gamma(x)^{n+1}, $$

which is valid for $n = 0$; the inductive step is provided by

$$ \gamma(x\gamma(x)^{n+1}) = \gamma((x\gamma(x)^n).\gamma(x)) = \gamma(x\gamma(x)^n).\gamma(x) = \gamma(x)^{n+1}\gamma(x). $$

Conversely, the case of $(\times)$ for $k$ implies for $y = x\gamma(x)^k$ that

$$ \gamma(y\gamma(x)) = \gamma(x\gamma(x)^k\gamma(x)) = \gamma(x\gamma(x)^{k+1}) = \gamma(x)^{k+1}\gamma(x) $$

$$ = \gamma(x\gamma(x)^k)\gamma(x) = \gamma(y)\gamma(x). \quad \Box $$

7.3.4 Proof of Proposition 2.3.

We first prove the case $k = 1$ of $(\times)$, which case also establishes the converse. By linearity of $\gamma$, it is enough to prove this identity for any $u$ of norm 1. Fix $u$ of norm 1, and take $x = tu$ with $t = ||x|| \to 0$. Substitution into $GS$ leads to

$$ \gamma(x\gamma(x)) - \gamma(x)^2 + \gamma(xe(x)) - 2e(x)\gamma(x) = 2e(x) - e(x \circ x). \quad (\dagger) $$

Since $e(x) = o(x)$,

$$ ||\gamma(xe(x))||/||x||^2 = ||\gamma(u \cdot e(x)/||x||)|| \leq ||\gamma|| ||u|| ||e(x)/||x|||| \to 0. $$

Furthermore,

$$ ||x \circ x||/||x|| = ||2u + tu\gamma(u) + ue(tu)|| \to 2, $$

and so RHS of $(\dagger)$ gives

$$ [2e(x) - e(x \circ x)]/||x||^2 = 2e(x)/||x||^2 - \frac{||x \circ x||^2}{||x||^2} \cdot e(x \circ x)/||x \circ x||^2 \to 0. $$

Hence

$$ \gamma(u\gamma(u)) = \gamma(u)^2, $$

and holds for all $u$. This last identity implies the converse: from $(\dagger)$,

$$ \gamma(u\gamma(u)) - \gamma(u)^2 + \gamma(ue(tu)/t) - 2\gamma(u)e(tu)/t = \frac{2e(x)}{t^2}(1_A - \frac{e(x \circ x)}{2e(x)}). $$

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Passage to the limit yields \( e(x)/||x||^2 \to 0 \).

To deduce the cases \( k \geq 2 \) of \((\ast)\), it again suffices to establish them for \( u \) of norm 1. Comparing the two sides of \( S(x \circ y) = S(x)S(y) \) yields:

\[
\gamma(y\gamma(x)) - \gamma(x)\gamma(y) + \gamma(ye(x)) - e(x)\gamma(y) - e(y)\gamma(x)
= \quad e(x) + e(y) - e(x \circ y).
\]

With \( x = tu \) and \( y = x\gamma(x)^k = t^{k+1}u\gamma(u)^k \), the analogous estimates of ratios against \( t^2 \) are similar and easier because the power of \( t \) in \( y \) (strictly) exceeds 2 and \( e(x) = o(x) \). Note that \( ||x \circ y||/t \to 1 \).

\[ \square \]

### 7.3.5 Proof of Th. 3.2 (Exhaustivity)

Put \( s(x, y) = (s_1(x, y), s_2(x, y)) = (\sigma(x, y), \tau(x, y)) \); evaluating components on right- and left-hand sides, the two of the right are

\[
RHS_1 = \sigma(a_1, a_2)\sigma(b_1, b_2) : \quad RHS_2 = \tau(a_1, a_2)\tau(b_1, b_2).
\]

Likewise the first component on the left is

\[
LHS_1 = \sigma(a + s(a))b = \sigma(a_1 + \sigma(a)b_1, a_2 + \tau(a)b_2).
\]

So

\[
\sigma(a_1 + \sigma(a)b_1, a_2 + \tau(a)b_2) = \sigma(a_1, a_2)\sigma(b_1, b_2). \tag{S1}
\]

Similarly,

\[
\tau(a_1 + \sigma(a)b_1, a_2 + \tau(a)b_2) = \tau(a_1, a_2)\tau(b_1, b_2). \tag{S2}
\]

In (S1), taking \( a_2 = b_2 = 0 \) gives

\[
\sigma(a_1 + \sigma(a), 0)b_1, 0) = \sigma(a_1, 0)\sigma(b_1, 0).
\]

So \( \varphi(x) := \sigma(x, 0) \) solves the standard \((GS)\) equation

\[
\varphi(x + y\varphi(x)) = \varphi(x)\varphi(y),
\]

and so for some \( \sigma_1 \)

\[
\sigma(x, 0) = 1 + \sigma_1 x.
\]

Likewise, working with (S2) with \( a_1 = b_1 = 0 \) gives

\[
\tau(0, a_2 + \tau(0, a_2)b_2) = \tau(0, a_2)\tau(0, b_2) : \quad \tau(0, y) = 1 + \tau_2 y.
\]

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Now taking $a_2 = b_2 = 0$ in $(S2)$ gives

$$\tau(a_1 + \sigma(a_1, 0)b_1, 0) = \tau(a_1, 0)\tau(b_1, 0).$$

So $\varphi(x) := \tau(x, 0)$ and $h(x) := \sigma(x, 0)$ solve the pexiderized version of the (scalar) $(GS)$:

$$\varphi(x + h(x)y) = \varphi(x)\varphi(y),$$

for which see [Jab, Cor. 1(iv)]. Here there are two possibilities

$$h = 1 \text{ and } \varphi = e^{\gamma x},$$
$$h = 1 + cx \text{ and } \varphi = (1 + cx)^{\gamma}. $$

That is,

$$\sigma(x, 0) = 1 + \sigma_1 x = 1 \text{ and } \tau(x, 0) = e^{\gamma x}, \text{ i.e. } \sigma_1 = 0.$$
$$\sigma(x, 0) = 1 + \sigma_1 x = 1 + cx \text{ and } \tau(x, 0) = (1 + cx)^{\gamma}. $$

The latter case should be interpreted as either $\gamma \neq 0$ and $\tau_1 = c = \sigma_1$, or $\gamma = 0$ and so $\tau_1 = 0$.

Also taking $a_1 = b_1 = 0$ in $(S1)$ gives

$$\sigma(0, a_2 + \tau(0, a_2)b_2) = \sigma(0, a_2)\sigma(0, b_2).$$

So $\varphi(x) := \sigma(0, y)$ and $h(x) := \tau(0, y)$ solve the pexiderized version

$$\varphi(x + h(x)y) = \varphi(x)\varphi(y).$$

Here again there are two possibilities:

$$\tau(0, y) = 1 + \tau_2 y = 1 \text{ and } \sigma(0, y) = e^{\gamma y}, \text{ i.e. } \tau_2 = 0;$$
$$\tau(0, y) = 1 + \tau_2 y = 1 + cy \text{ and } \sigma(0, y) = (1 + cy)^{\gamma}, \text{ i.e. } \sigma(0, y) = (1 + \tau_2 y)^{\gamma}. $$

In summary,

$$\sigma(x, 0) = 1 + \sigma_1 x, \quad \tau(0, y) = 1 + \tau_2 y,$$

and the following ‘side’ conditions hold.

If $\tau_2 \neq 0$, then $\sigma_2 := \tau_2$ and $\sigma(0, y) = (1 + \sigma_2 y)^{\gamma}$; otherwise $\sigma(0, y) = e^{\gamma y}$.

If $\sigma_1 \neq 0$, then $\tau_1 := \sigma_1$ and $\tau(x, 0) = (1 + \tau_1 x)^{\delta}$; otherwise $\tau(x, 0) = e^{\delta x}$.
Finally in (S1) and in (S2) take \( a_2 = b_1 = 0 \). Then
\[
\sigma(a_1, \tau(a_1, 0)b_2) = \sigma(a_1, 0)\sigma(0, b_2)
\]
\[
\tau(a_1, \tau(a_1, 0)b_2) = \tau(a_1, 0)\tau(0, b_2).
\]
Thus
\[
\tau(x, y) = \tau(x, 0)(1 + \tau_2y/\tau(x, 0)) \quad \text{(taking } y = \tau(x, 0)b_2) \]
\[
= \tau(x, 0) + \tau_2y = \begin{cases} 
(1 + \tau_1x)^\delta + \tau_2y & \text{if } \tau_1 = \sigma_1 \neq 0 \\
\exp^\delta x + \tau_2y & \text{if } \sigma_1 = 0.
\end{cases}
\]
Similarly
\[
\sigma(a_1, \tau(a_1, 0)b_2) = \sigma(a_1, 0)\sigma(0, b_2)
\]
\[
\sigma(x, \tau(x, 0)b_2) = (1 + \sigma_1x)\sigma(0, y/\tau(x, 0)) \quad \text{(take } y = \tau(x, 0)b_2) \]
\[
= \begin{cases} 
(1 + \sigma_1x)(1 + \sigma_2y/\tau(x, 0))^\gamma & \text{if } \sigma_2 = \tau_2 \neq 0 \\
(1 + \sigma_1x)e^{\gamma y/\tau(x, 0)} & \text{if } \sigma_2 = 0.
\end{cases}
\]
We examine all the possible cases. The first and second special cases below emerge as the only viable ones, i.e. leading to a solution. The remaining general cases turn out to be impossible on asymptotic growth grounds.

\textbf{First Special Case:} Take \( \sigma_1 \neq 0 \neq \sigma_2 \) and \( \gamma = \delta = 1 \); then \( \sigma(x, 0) = \tau(x, 0) = 1 + \sigma_1x \) and \( \sigma(0, y) = \tau(0, y) = 1 + \sigma_2y \) with:
\[
\sigma(a_1, (1 + \sigma_1a_1)b_2) = (1 + \sigma_1a_1)(1 + \sigma_2b_2) \quad \text{(} z - b_2 := \sigma_1a_1b_2),
\]
\[
\sigma(a_1, z) = 1 + \sigma_1a_1 + \sigma_2b_2 + \sigma_1\sigma_2a_1b_2
\]
\[
= 1 + \sigma_1a_1 + \sigma_2b_2 + \sigma_2[z - b_2]
\]
\[
= 1 + \sigma_1a_1 + \sigma_2z.
\]

We verify that this formula with \( \sigma = \tau \) satisfies (GS). It will suffice to check the first component:
\[
LHS_1 = \sigma(a_1 + \sigma(a)b_1, a_2 + \tau(a)b_2)
\]
\[
= 1 + \rho_1(a_1 + (1 + \rho_1a_1 + \rho_2a_2)b_1) + \rho_2(a_2 + (1 + \rho_1a_1 + \rho_2a_2)b_2)
\]
\[
= 1 + \rho_1a_1 + \rho_1b_1 + \rho_1^2a_1b_1 + \rho_1\rho_2a_2b_1 + \rho_2a_2 + \rho_2b_2 + \rho_2\rho_1a_1b_2 + \rho_2^2a_2b_2
\]
\[
= 1 + \rho_1a_1 + \rho_2a_2 + \rho_1b_1 + \rho_2b_2 + (\rho_1a_1 + \rho_2a_2)(\rho_1b_1 + \rho_2b_2).
\]
This agrees with

\[ \text{RHS}_1 = (1 + \rho_1 a_1 + \rho_2 a_2)(1 + \rho_1 b_1 + \rho_2 b_2) \]
\[ = 1 + \rho_1 a_1 + \rho_2 a_2 + \rho_1 b_1 + \rho_2 b_2 \]
\[ + (\rho_1 a_1 + \rho_2 a_2)(\rho_1 b_1 + \rho_2 b_2). \]

The calculation of the second component is exactly the same in these circumstances.

Second Special Case. Take \( \tau_2 \neq 0 \), and \( \gamma = 0 \), equivalently \( \sigma_2 = 0 \). Then \( \sigma(x, y) = 1 + \sigma_1 x \). Take \( \delta = 0 \), then \( \tau(x, 0) = 1 \), equivalently \( \tau_1 = 0 \). Then \( \tau(x, y) = 1 + \tau_2 y \). We check the proposed solution:

\[ s(x_1, x_2) = (1 + \sigma_1 x_1, 1 + \tau_2 x_2). \]

The first component on the left is

\[ \text{LHS}_1 \]
\[ = \sigma(a_1 + \sigma(a)b_1, a_2 + \tau(a)b_2) \]
\[ = 1 + \sigma_1 (a_1 + b_1 + \sigma_1 a_1 b_1) = 1 + \sigma_1 a_1 + \sigma_1 b_1 + \sigma_1^2 a_1 b_1, \]

and this matches

\[ \text{RHS}_1 \]
\[ = \sigma(a_1, a_2)\sigma(b_1, b_2) \]
\[ = (1 + \sigma_1 a_1)(1 + \sigma_1 b_1) \]
\[ = 1 + \sigma_1 a_1 + \sigma_1 b_1 + \sigma_1^2 a_1 b_1. \]

The calculation of the second component (with \( \tau \)) is similar.

It turns out that these special cases are the only possible ones. So it now remains to eliminate the remaining (ostensibly ‘general’) cases.

General Cases (non-viable).

We first compute \( \sigma \), according to the side condition below (summary) above, treating it ‘disjunctively’ as Cases A and B.

Case A (If \( \tau_2 \neq 0 \) then \( \sigma_2 = \tau_2 \) and \( \sigma(0, y) = (1 + \sigma_2 y)^\gamma \)). Here

\[ \sigma(\sigma(0, a_2)b_1, a_2) = \sigma(0, a_2)\sigma(b_1, 0). \]

Substitution under this case yields

\[ \sigma((1 + \sigma_2 y)^\gamma b_1, y) = (1 + \sigma_2 y)^\gamma (1 + \sigma_1 b_1) \]
\[ \sigma(x, y) = \sigma_1 x + (1 + \sigma_2 y)^\gamma \text{ with } \sigma_2 = \tau_2 \neq 0. \]
Case B ($\sigma(0, y) = e^{\gamma y}$). Substituting as in Case A

$$\sigma(e^{\gamma y}b_1, y) = e^{\gamma y}(1 + \sigma_1 b_1) = e^{\gamma y} + \sigma_1 e^{\gamma y}b_1$$
$$\sigma(x, y) = e^{\gamma y}(1 + \sigma_1 b_1) = \sigma_1 x + e^{\gamma y} \text{ with } \tau_2 = 0.$$

Next we compute $\tau$ analogously to $\sigma$, again ‘disjunctively’.

Case A (If $\sigma_1 \neq 0$ then $\sigma_1 = \tau_1$ and $\tau(x, 0) = (1 + \sigma_1 x)^\delta$). Substituting under this case into

$$\tau(a_1, \tau(a_1, 0)b_2) = \tau(a_1, 0)\tau(0, b_2)$$

yields

$$\tau(x, (1 + \sigma_1 x)^\delta b_2) = (1 + \sigma_1 x)^\delta (1 + \tau_2 b_2)$$
$$\tau(x, y) = (1 + \sigma_1 x)^\delta + \tau_2 b_2(1 + \sigma_1 x)^\delta$$
$$= (1 + \tau_1 x)^\delta + \tau_2 y \text{ with } \tau_1 := \sigma_1 \neq 0.$$

Case B ($\tau(x, 0) = e^{\delta x}$). Substituting as in the preceding Case A yields:

$$\tau(x, e^{\delta x}b_2) = \tau(x, 0)\tau(0, b_2) = e^{\delta x}(1 + \tau_2 b_2)$$
$$\tau(x, y) = e^{\delta x} + \tau_2 b_2 e^{\delta x}$$
$$= e^{\delta x} + \tau_2 y \text{ with } \tau_1 = \sigma_1 = 0.$$

In summary, we have the following possibilities for $\sigma$ and $\tau$.

$$\sigma(x_1, x_2) = \begin{cases} 
\sigma_1 x_1 + (1 + \sigma_2 x_2)^\gamma & \text{with } \sigma_2 = \tau_2 \neq 0, \\
\sigma_1 x_1 + e^{\gamma x_2} & \text{with } \sigma_2 = \tau_2 = 0,
\end{cases}$$

$$\tau(x_1, x_2) = \begin{cases} 
(1 + \tau_1 x_1)^\delta + \tau_2 x_2 & \text{with } \tau_1 := \sigma_1 \neq 0, \\
e^{\delta x_1} + \tau_2 x_2 & \text{with } \tau_1 = \sigma_1 = 0.
\end{cases}$$

We now rule out all four possible pairings of $\sigma$ and $\tau$ by their asymptotic behaviour for large values of the arguments on both sides of the first component equation ($S1$). This last asserts that

$$\sigma(a_1 + \sigma(a)b_1, a_2 + \tau(a)b_2) = \sigma(a_1, a_2)\sigma(b_1, b_2).$$

Case 1. Consider pairing the first choices available to $\sigma$ and $\tau$.

$$\sigma(x) = \sigma_1 x_1 + (1 + \sigma_2 x_2)^\gamma, \tau(x) = (1 + \tau_1 x_1)^\delta + \tau_2 x_2.$$
On the LHS of \((S1)\) the two arguments for \(\sigma\) are

\[
x_1 := a_1 + b_1[\sigma_1 a_1 + (1 + \sigma_2 a_2)^\gamma], \quad x_2 := a_2 + b_2(1 + \tau_1 a_1)^\delta + \tau_2 a_2 b_2.
\]

So

\[
LHS = \sigma_1 x_1 + (1 + \sigma_2 x_2)^\gamma = \sigma_1 a_1 + \sigma_1 \sigma_1 a_1 b_1 + \sigma_1 b_1(1 + \sigma_2 a_2)^\gamma + (1 + \sigma_2 a_2 + \sigma_2 \tau_2 a_2 b_2 + \sigma_2 b_2 (1 + \tau_1 a_1)^\delta)^\gamma.
\]

But

\[
RHS = [\sigma_1 a_1 + (1 + \sigma_2 a_2)^\gamma][\sigma_1 b_1 + (1 + \sigma_2 b_2)^\gamma] = \sigma_1 a_1 \sigma_1 b_1 + \sigma_1 a_1 (1 + \sigma_2 b_2)^\gamma + \sigma_1 b_1 (1 + \sigma_2 a_2)^\gamma + (1 + \sigma_2 a_2)^\gamma (1 + \sigma_2 b_2)^\gamma.
\]

Equating sides and setting \(b_1 = a_2 = 0\) gives

\[
\sigma_1 a_1 + (1 + \sigma_2 b_2(1 + \tau_1 a_1)^\delta)^\gamma = \sigma_1 a_1(1 + \sigma_2 b_2)^\gamma + (1 + \sigma_2 b_2)^\gamma.
\]

Letting \(a_1, b_2 \to \infty\) yields

\[
\sigma_1 a_1 + (\sigma_2 b_2)^\gamma (1 + \tau_1 a_1)^\delta + \gamma \sim \sigma_1 a_1(\sigma_2 b_2)^\gamma + (\sigma_2 b_2)^\gamma,
\]

a contradiction unless \(\sigma_1 = \tau_1 = 0\). This yields \(\sigma = (1 + \sigma_2 x_2)^\gamma\) and \(\tau = 1 + \tau_2 x_2\) in the univariate format. Clearly \((S2)\) is satisfied, whereas \((S1)\) requires that

\[
[1 + \sigma_2(x_2 + y_2(1 + \tau_2 x_2))]^\gamma = [(1 + \sigma_2 x_2)(1 + \sigma_2 y_2)]^\gamma.
\]

The case \(\gamma = 0\) gives the ‘independent’ format with \(\sigma = 1\), as does \(\sigma_2 = 0\). Otherwise one has \(\sigma_2 = \tau_2\), a univariate type.

Case 2. Consider now pairing second choices, so that

\[
\sigma = \sigma_1 x_1 + e^{\gamma x_2}, \quad \tau = e^{\delta x_1} + \tau_2 x_2.
\]

Here the arguments on the left of \((S1)\) are

\[
x_1 = a_1 + b_1[\sigma_1 a_1 + e^{\gamma a_2}], \quad x_2 = a_2 + b_2[e^{\delta a_1} + \tau_2 a_2].
\]
So

\[ LHS = \sigma_1 x_1 + e^{\gamma x_2} = \sigma_1 a_1 + \sigma_1 b_1 \sigma_1 a_1 + \sigma_1 b_1 e^{\gamma a_2} + \exp[\gamma a_2 + \gamma b_2 e^{\delta a_1 + \tau a_2}]. \]

But

\[ RHS = [\sigma_1 a_1 + e^{\gamma a_2}] [\sigma_1 b_1 + e^{\gamma b_2}] = \sigma_1 a_1 \sigma_1 b_1 + \sigma_1 b_1 e^{\gamma a_2} + \sigma_1 a_1 e^{\gamma b_2} + e^{\gamma (a_2 + b_2)}. \]

Again the asymptotic behaviours of both sides do not match unless \( \sigma_1 = \gamma = 0 \). when both sides reduce to 1. So here \( \sigma = 1 \) and so by \((S2)\)

\[ e^{\delta(x_1 + y_1)} + \tau_2 (x_2 + y_2 (e^{\delta x_1} + \tau_2 x_2)) = (e^{\delta x_1} + \tau_2 x_2) (e^{\delta y_1} + \tau_2 y_2) \]

leading to the solutions \( \sigma = 1, \tau = 1 + \tau_2 x_2 \) covered by the ‘independent’ format and finally the ‘univariate’ type

\[ \sigma = 1, \tau = e^{\delta x_1}. \]

The remaining Cases 3 and 4 (cross-choices) are similar. \( \square \)

### 7.3.6 Th. 3.3 – Stone-Weierstrass argument

Using the Structure Theorem of the Euclidean case of \( G_\infty^*(\mathbb{R}^d) \) established in §3, we may now describe \( S \) by reference to \( C(T) \) as follows.

Denote by \( \delta \) and \( \delta_T \) the Dirac mass function respectively for \([0,1]^2\) and for \( T^2 \) so that \( \delta(t)(s) = 1 \) iff \( t = s \); then by Th. 3.1, there are numbers \( \sigma_{ij} \) with

\[ \sigma_T(x_T)(t_i) = \Sigma_j \sigma_{ij} x_T(t_j) \text{ so that } \sigma_{ij} = \sigma_T(t_i)_j = S(\delta_T(t_j))(t_i) - 1. \]

The corresponding ‘generator’ will be represented here by \( \rho_T(x_T) \in \mathbb{R}^{n+1} = C(T) \) with

\[ \rho_T(x_T)(t_i) := \Sigma_j \sigma_{ij} x_T(t_j) = \Sigma_{i=0}^n \sigma_T(x_T)(t_i) \to S(x^T)(t_i), \]

with the limit here again under refinement of subdivisions.

Define the partition \( \mathcal{P}_T \) of \( T \) to comprise all the distinct sets \( I_T(t) \) for \( t \in T \) with

\[ I_T(t) := \{ t_i \in T : (\forall x)S(x)(t_i) = S(x)(t) \}. \]
Then for $s, t \in I_T(t)$
\[ S(\delta(t_j))(s) = S(\delta(t_j))(t) \quad (j = 0, 1, ..., n). \]

Here taking limits under refinement of subdivisions $T$ yields
\[ I_T(t) \to K_t. \]

For $s, t \in I \in P_T$, as $S(\delta(t_j))(t) = 0$ iff $S(\delta(t_j))(s) = 0$, we may define
\[ J_T(I) := \{ t_j \in T : S(\delta(t_j))(t) \neq 0, t \in I \}. \]

Then for $x = x_T$
\[ \sigma_I(x) := \rho_T(e_I \cdot x) e_I \text{ for } e_I := \Sigma_{t \in J(I)} \delta_T(t) \text{ and } I \in P_T. \]

Here $x \mapsto e_I \cdot x$ is the projection from $\mathbb{R}^{n+1}$ onto the span of $\{ \delta_T(t) : t \in J(I) \}$.

Taking limits over subdivisions $T$ under refinement with inclusion of $T$ yields
\[ \rho_T(x_T)(t) \to \rho(x)(t); \]
this limit generator gives a continuous linear map $\rho : C[0, 1] \to C[0, 1]$. For $s \in [0, 1]$, put $\rho_s(x) := \rho(x)(s)$, a continuous linear functional. Then
\[ \{ t : \rho_s = \rho_t \} = K_s. \]

Let $e_K$ denote the map $x \mapsto e_K \cdot x$ projecting from $C[0, 1]$ onto $C(K)$. Now define $\sigma_K : C(K) \to \mathbb{R}$ for $K_t = K \in P$ by
\[ \sigma_K(x) := \rho_t(e_K \cdot x), \]
thereby completing the analysis of the action of $S$. \hfill \square

7.3.7 Proof of Lemma 4.1

With restrictions on $a, b, c, z$ as above, using $(GS)$ and invertibility of $S$,
\[ S(c + (a - b)) = S(a + (c - b)S(a)S(b)^{-1}) = S(a)S((c - b)S(b)^{-1}) \]
\[ = S(b)S((c - b)S(b)^{-1}) = S(b + (c - b)S(b)^{-1}) = S(c). \]

Since
\[ S(z + S(z)a) = S(z)S(a) = S(z)S(b) = S(z + S(z)b) \in \mathbb{A}^{-1}, \]
replacing in (i) $a$ by $z + S(z)a$ and $b$ by $z + S(z)b$ yields (ii). In particular, for any $z \in \mathbb{G}_S^*$
\[ 1_{\mathbb{A}} = S(0) = S(S(z)a) \]
(take $b = c = 0$), giving $S(\mathbb{G}_S^*)\mathcal{N} \subseteq \mathcal{N}$ and so also the last assertion, as $S(0)\mathcal{N} = \mathcal{N}$. \hfill \square

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7.3.8 Proof of Corollary 4.4

Note first that since $C_1 = \mathbb{C} \setminus \{0\} = \mathbb{C}^{-1}$, $G_\rho^*(\mathbb{C}) = G_S(\mathbb{C})$. It is known [Bar] and reproved in Prop. 4.2 that in the case $A = \mathbb{C}$ any continuous solution $S$ of $(GS)$ is either canonical, i.e. of the form $S(z) = 1 + \rho z$ so that $G_S(\mathbb{C}) = G_\rho(\mathbb{C})$, or takes the ‘non-canonical’ form $S(z) := 1 + a \Re(z) + b \Im(z)$ with $a, b$ real constants. Here if $a = b = 0$, this leads to $\text{ran} S = \{1\}$; then 1 is an isolated point, and the preceding result cannot be applied, although its conclusion still holds with $c = 0$. The alternative is that $\text{ran} S = \langle 1 \rangle$.

This presents two possibilities:

(i) $S(z) = 1 + az$ for $z \in \langle 1 \rangle$ with $a \neq 0$, yielding a Popa subgroup $G_\alpha^*(\langle 1 \rangle) = G^*_\alpha(\mathbb{R})$ with $\alpha = 1/\beta$;

(ii) $S(z) = 1 - ibz$ for $z \in \langle i \rangle$ with $b \neq 0$, yielding a Popa subgroup $G_\beta^*(\langle i \rangle) \approx G^*_\beta(\mathbb{R})$ with $\beta = 1/(ib)$.

These two separate restrictions of $S$, both in ‘canonical’ form, correspond to $S((z - 1)/a) = z$ for $z \in \langle 1 \rangle$ (with $c = 1/\alpha$), and $S((iz - 1)/(ib)) = iz$ for $z \in \langle i \rangle$ (with $c = i/b$).

Write $z_1 = u + iv$ and $z_2 = x + iy$; then, since $S(z_1) = 1 + au + bv$,

$$z_1 \circ_S z_2 = (u + iv) + (x + iy) + (1 + au + bv)(x + iy).$$

This corresponds to a Popa operation $o_\sigma$ on the set $G_\alpha^*(\mathbb{R}) \times G_\beta^*(\mathbb{R})$ with $\sigma(u, v) := 1 + au + bv$ and

$$(u, v) \circ_\sigma (x, y) = (u + x + \sigma(u, v)x, v + y + \sigma(u, v)y).$$

We shall identified this in §3 as $G^*_\sigma(\mathbb{R}^2)$. \hfill \Box

7.3.9 Example 5.1 (Standardized Tilting in $\mathbb{C}$)

Writing $\omega = x + iy$, the real and imaginary parts give:

$$e^x \cos y = 1 + x, \quad e^x \sin y = y;$$

$$y^2 = y^2(x) := e^{2x} - (1 + x)^2 > 0, \text{ for } x > 0.$$

For $x > 0$, $y(x)$ is monotonic and unbounded, whereas

$$x \mapsto e^{-x} y(x) = \sqrt{1 - (e^{-x}(1 + x))^2}$$

increases on $[0, \infty)$ strictly from 0 to 1, yielding solutions in $x$ to the equation

$$\sin y(x) = e^{-x} y(x),$$

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one within each consecutive interval in which \( \sin y(x) \) traces the interval \([-1, 1]\) with \( x \) near the solution point of \( y(x) = \pi/2 \mod 2\pi\mathbb{Z} \). It follows that, in alternate intervals where \( \cos y(x) > 0 \),

\[
e^x \cos y(x) = \sqrt{e^{2x} - y(x)^2} = 1 + x,
\]

thus satisfying \((ST)\).

**Remark.** The function \( y(x) \) is also defined in a maximal interval \([-\xi, 0]\), with \( \xi \) satisfying \( e^{-2\xi} = (1 - \xi)^2 \), equivalently \( e^{-\xi} = \xi - 1 \). From here \( \xi > 1 \), and in fact

\[
\xi = 1.27846...;
\]

it follows that \( y(-\xi) = 0 \). (Evidently \( y(-1) = 1 \).) So \( \sin y(-\xi) = e^{\xi}y(-\xi) = 0 \); however,

\[
e^{-\xi} \cos y(-\xi) = e^{-\xi} = \xi - 1 \neq 1 - \xi,
\]

since \( \xi \neq 1 \).

### 7.3.10 Proof of Prop. 5.4

Using orthogonality,

\[
RHS_{GS} = (1_A + \sum_i \sigma(e_i x) e_i)(1_A + \sum_j \sigma(e_i y) e_j)
\]

\[
= 1_A + \sum_i \sigma(e_i x) e_i + \sum_j \sigma(e_i y) e_j
\]

\[
+ \sum_i \sigma(e_i x) \sigma(e_i y) e_i.
\]

Noting that

\[
x + [1_A + \sum_i \sigma(e_i x) e_i]y = x + y + \sum_i \sigma(e_i x) e_i y,
\]

we compute, using orthogonality, that

\[
\sigma(e_j[x + y + \sum_i \sigma(e_i x) e_i y]) = \sigma(e_j x + e_j y + \sum_i \sigma(e_i x) e_j e_i y)
\]

\[
= \sigma(e_j x) + \sigma(e_j y) + \sigma[\sigma(e_j x) e_j y]
\]

\[
= \sigma(e_j x) + \sigma(e_j y) + \sigma(e_j x) \sigma[e_j y].
\]

So

\[
LHS_{GS} = 1_A + \sum_j \sigma(e_j[x + y + \sum_i \sigma(e_i x) e_i y]) e_j
\]

\[
= 1_A + \sum_j \sigma(e_j x) e_j + \sum_i \sigma(e_j y) e_j + \sigma(e_j x) \sigma(e_j y) e_j,
\]

and the two sides match. □
7.3.11 Proof of Lemma 5.5

W.l.o.g., \(|e^{-a}| < 1\). Then \(|e^{a}| > 1\), for otherwise \(|e^{a}| \leq 1\), leading to

\[ 1 = |e^{-a}e^{a}| \leq |e^{-a}| \cdot |e^{a}| < 1, \]

a contradiction. Furthermore, for \(n \in \mathbb{Z}\),

\[ ||e^{na}|| = ||e^{(n+1)a}e^{-a}|| \leq |e^{-a}| \cdot ||e^{(n+1)a}|| < ||e^{(n+1)a}||. \]

The sequence \(\{||e^{na}||\}_{n \in \mathbb{N}}\) is thus monotonically increasing. If \(|e^{na}| \to c\) for some finite \(c > 0\), then, by the preceding inequality

\[ c \leq ||e^{-a}||c < c, \]

also a contradiction. So \(|e^{na}|\) is unbounded. Similarly, \(\{||e^{-na}||\}_{n \in \mathbb{N}}\) is monotonically decreasing, this time with limit 0.

As for the final statement, provided \(|e^{\gamma(a)}|| \neq 1\),

\[ T(\pm us) = u(e^{\pm s\gamma(u)} - 1)/\gamma(u) \]

is unbounded as \(s \to \infty\) in one of the directions \(\pm u\). \(\square\)

7.3.12 Proof of Th. 6.1 (Wołodźko-Javor Theorem)

Given \(S\), take \(\mathcal{N} := \{u \in G_{S}^{*} : S(u) = 1_A\}\), \(\Lambda := S(G_{S}^{*})\), and choose any right inverse \(W\) with \(S(W(a)) \equiv a\). Then use Lemma 2.1(i) and (ii).

For the reverse direction, as \(\mathcal{N}\) is a subgroup, we may work mod \(\mathcal{N}\), indicating this now with \(\equiv_{\mathcal{N}}\). First note that \(S\) is well-defined. For if \(W(\lambda_{1}) = W(\lambda)\), then, taking \(\lambda_{2} = \lambda_{1}\lambda_{1}^{-1}\), so that \(\lambda = \lambda_{1}\lambda_{2}\) by (iii),

\[ \lambda_{1}W(\lambda_{2}) = \lambda_{1}W(\lambda_{2}) + [W(\lambda_{1}) - W(\lambda_{1}\lambda_{2})] \equiv_{\mathcal{N}} 0. \]

So \(\lambda_{1}W(\lambda_{2}) \in \mathcal{N}\), or \(W(\lambda_{2}) \in \lambda_{1}^{-1}\mathcal{N} \subseteq \Lambda\mathcal{N} = \mathcal{N}\); so by (ii), \(\lambda_{2} = 1\), i.e. \(\lambda = \lambda_{1}\), as required.

We check that (iv) satisfies \((GS)\) with \(x = x_{1}\) and \(y = x_{2}\).

If \(S(x_{1}) = 0\), then \((GS)\) holds trivially.

If \(S(x_{1}) \neq 0\) and \(S(x_{1} + S(x_{1})x_{2}) \neq 0\), then pick \(\lambda_{1}\) and \(\lambda\) with \(W(\lambda_{1}) = x_{1}\) and \(W(\lambda) \equiv x_{1} + S(x_{1})x_{2}\). Take \(\lambda_{2} = \lambda \lambda_{1}^{-1}\); then

\[ W(\lambda_{1}\lambda_{2}) = W(\lambda) \equiv_{\mathcal{N}} x_{1} + S(x_{1})x_{2} = W(\lambda_{1}) + \lambda_{1}x_{2} : \]

\[ \lambda_{1}x_{2} \equiv_{\mathcal{N}} W(\lambda_{1}\lambda_{2}) - W(\lambda_{1}) \equiv_{\mathcal{N}} \lambda_{1}W(\lambda_{2}) : \]

\[ x_{2} \equiv_{\mathcal{N}} W(\lambda_{2}). \]
So here $S(x_2) = \lambda_2 \neq 0$, i.e. passing to the contrapositive: if $S(x_2) = 0$, then $S(x_1 + S(x_1)x_2) = 0$ and $(GS)$ holds.

Now consider $x_i$ with both $S(x_i) \neq 0$. Write $x_i \equiv_N W(\lambda_i)$. Then

$$x_1 + S(x_1)x_2 \equiv A W(\lambda_1) + \lambda_1 W(\lambda_2) \equiv_N W(\lambda_1 \lambda_2) :$$

$$S(x_1 + S(x_1)x_2) = \lambda_1 \lambda_2 = S(x_1)S(x_2).$$

This completes the check that $(GS)$ holds. \qed

7.4 Proof of Theorem S

We are grateful to Amol Sasane for the following proof.

For fixed $\theta \in M_h$, the maximal ideal space of $A$ (viewed as comprising characters in $L(A, \mathbb{C})$), define for $\zeta \in D_u$

$$\varphi_{\theta u}(\zeta) : = \theta(f(\zeta\gamma(u))),$$

$$\psi_{\theta u}(\zeta) : = \theta(\gamma(f(\zeta\gamma(u)) \cdot u/\gamma(u)));$$

both are holomorphic on $D_u$, since $\theta$ is linear. Furthermore, by the assumed identity,

$$\varphi_{\theta u}(\zeta) = \psi_{\theta u}(\zeta) \quad (\zeta \in \Sigma).$$

So, by Riemann’s Uniqueness (Identity) theorem [Gam2, V.7], [Rem, Ch. 8], also $\varphi_{\theta u} = \psi_{\theta u}$ on $D_u$. As $\theta$ was arbitrary, this may be restated using Gelfand transforms as

$$f(\zeta\gamma(u)^\sim) = \gamma(f(\zeta\gamma(u) \cdot u/\gamma(u))^\sim).$$

For $A$ semisimple, the Gelfand transform is injective and so

$$f(\zeta\gamma(u)) = \gamma(f(\zeta\gamma(u) \cdot u/\gamma(u))) \quad (\zeta \in D_u),$$

thus extending an identity from $\Sigma$ to $D_u$. \qed