Graphical Mahonian Statistics on Words

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Abstract

Foata and Zeilberger defined the graphical major index, \( \text{maj}'_U \), and the graphical inversion index, \( \text{inv}'_U \), for words. These statistics are a generalization of the classical permutation statistics \( \text{maj} \) and \( \text{inv} \) indexed by directed graphs \( U \). They showed that \( \text{maj}'_U \) and \( \text{inv}'_U \) are equidistributed over all rearrangement classes if and only if \( U \) is bipartitional. In this paper we strengthen their result by showing that if \( \text{maj}'_U \) and \( \text{inv}'_U \) are equidistributed on a single rearrangement class then \( U \) is essentially bipartitional. Moreover, we define a graphical sorting index, \( \text{sor}'_U \), which generalizes the sorting index of a permutation. We then characterize the graphs \( U \) for which \( \text{sor}'_U \) is equidistributed with \( \text{inv}'_U \) and \( \text{maj}'_U \) on a single rearrangement class.

1 Introduction

Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) be a sequence of nonnegative integers. We will denote by \( R(\alpha) \) the set of permutations of the multiset \( \{1^{\alpha_1}, 2^{\alpha_2}, \ldots, n^{\alpha_n}\} \), i.e., \( R(\alpha) \) is the set of all words containing \( \alpha_i \) occurrences of the letter \( i \) for all \( i = 1, 2, \ldots, n \). For \( w = x_1x_2 \ldots x_m \in R(\alpha) \), the inversion number is defined as

\[
\text{inv } w = \sum_{1 \leq i<j \leq m} \mathcal{X}(x_i > x_j),
\]

and the major index is defined as

\[
\text{maj } w = \sum_{i=1}^{m-1} i \mathcal{X}(x_i > x_{i+1}).
\]

The set of all positions \( i \) such that \( x_i > x_{i+1} \) is known as the descent set of \( w \), \( \text{Des } w \), and its cardinality is denoted by \( \text{des } w \). So, \( \text{maj } w = \sum_{i \in \text{Des } w} i \).

The generating function for permutations by number of inversions goes back to Rodriguez \[19\] and the generalization to multisets is due to MacMahon \[14\]. MacMahon also showed \[13, 15\] that \( \text{maj} \) and \( \text{inv} \) are equidistributed on \( R(\alpha) \). Namely,

\[
\sum_{w \in R(\alpha)} q^{\text{inv } w} = \sum_{w \in R(\alpha)} q^{\text{maj } w} = \left[ \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{\alpha_1, \alpha_2, \ldots, \alpha_n} \right].
\]

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where

\[
\begin{bmatrix}
\alpha_1 + \alpha_2 + \ldots + \alpha_k \\
\alpha_1, \alpha_2, \ldots, \alpha_k
\end{bmatrix}
= \frac{[\alpha_1 + \alpha_2 + \cdots + \alpha_k]!}{[\alpha_1]![\alpha_2]![\cdots][\alpha_k]!}
\]

is the \(q\)-analog of the multinomial coefficient and \([n]! = (1+q)(1+q+q^2) \cdots (1+q+q^2+\cdots+q^{n-1})\) is the \(q\)-factorial.

In honor of MacMahon, all permutation statistics that share the same distribution are called Mahonian. These two classical Mahonian statistics have been generalized in various ways. Some examples are Kadell’s weighted inversion number [11], the \(r\)-major index introduced by Rawlings [18], the statistics introduced by Clarke [3], and the maj-inv statistics of Kasraoui [12]. They defined graphical statistics (graphical inversions and graphical major index) parameterized by a general directed graph \(U\) and they described the graphs \(U\) for which these statistics are equidistributed on all rearrangement classes.

**Theorem 1.1** ([7]). The statistics \(\text{inv}'_U\) and \(\text{maj}'_U\) are equidistributed on each rearrangement class \(\mathcal{R}(\alpha)\) if and only if the relation \(U\) is bipartitional.

A similar result was proved in [6], where the definition of graphical inversions and major index is modified to allow different behavior of the letters at the end of the word.

Here we do two different things. First, we strengthen Foata and Zeilberger’s result by showing that the equidistribution of \(\text{inv}'_U\) and \(\text{maj}'_U\) on a single rearrangement class \(\mathcal{R}(\alpha)\) implies that \(U\) is essentially bipartitional (Theorem 2.1). Second, we define a graphical sorting index on words, a statistics which generalizes the sorting index for permutations [16]. We then describe the directed graphs \(U\) for which \(\text{sort}'_U\) is equidistributed with \(\text{inv}'_U\) and \(\text{maj}'_U\) on a fixed class \(\mathcal{R}(\alpha)\) (Theorem 2.2).

In the next section we define the terminology we need and state the main results. Then we prove Theorem 2.1 and Theorem 2.2 in Section 3 and Section 4, respectively.

## 2 Preliminaries and Main Results

A **directed graph** on \(X = \{1, 2, \ldots, n\}\) is any subset \(U\) of the Cartesian product \(X \times X\). For each such directed graph \(U\), we have the following statistics defined on each word \(w = x_1x_2\ldots x_m\) with letters from \(X\):

\[
\begin{align*}
\text{inv}'_U w &= \sum_{1 \leq i < j \leq m} X((x_i, x_j) \in U), \\
\text{Des}'_U w &= \{i : 1 \leq i \leq m, (x_i, x_{i+1}) \in U\}, \\
\text{des}'_U w &= |\text{Des}'_U w|, \\
\text{maj}'_U w &= \sum_{i \in \text{Des}'_U w} i.
\end{align*}
\]

Since \(U\) is also a relation on \(X\), for convenience, in some places we will use the notation \(x >_U y\) to represent the edge \((x, y) \in U\). We will say \(x\) is related to \(y\) if \((x, y) \in U\) or \((y, x) \in U\).

An **ordered bipartition** of \(X\) is a sequence \((B_1, B_2, \ldots, B_k)\) of nonempty disjoint subsets of \(X\) such that \(B_1 \cup B_2 \cup \cdots \cup B_k = X\), together with a sequence \((\beta_1, \beta_2, \ldots, \beta_k)\) of elements equal to
0 or 1. If \( \beta_i = 0 \) we say the subset \( B_i \) is non-underlined, and if \( \beta_i = 1 \) we say the subset \( B_i \) is underlined.

A relation \( U \) on \( X \times X \) is said to be bipartitional, if there exists an ordered bipartition \( ((B_1, B_2, \ldots, B_k), (\beta_1, \beta_2, \ldots, \beta_k)) \) such that \((x, y) \in U\) if and only if either \( x \in B_i \), \( y \in B_j \) and \( i < j \), or \( x \) and \( y \) belong to the same underlined block \( B_i \). Bipartitional relations were introduced in [7] as an answer to the question “When are \( \text{inv}_U^I \) and \( \text{maj}_U^I \) equidistributed over all rearrangement classes?”. In particular, there the authors showed that if \( U \) is bipartitional with blocks \( ((B_1, \ldots, B_k), (\beta_1, \ldots, \beta_k)) \) then

\[
\sum_{w \in R(\alpha)} q^{\text{inv}_U^I} w = \sum_{w \in R(\alpha)} q^{\text{maj}_U^I} w = \left[ \frac{|\alpha|}{m_1, \ldots, m_k} \right] \prod_{j=1}^{\infty} \left( \frac{m_j}{\alpha(B_i)} \right) q^{\beta_j(m_j)},
\]

(2.1)

Here and later we use the notation

\[
|\alpha| = \alpha_1 + \cdots + \alpha_n,
\]

\[
m_i = |B_i|,
\]

\[
\alpha(B_i) = (\alpha_{i_1}, \ldots, \alpha_{i_l}) \text{ if } B_i = \{i_1 < \cdots < i_l\}.
\]

Han [9] showed that bipartitional relations \( U \) can also be characterized as relations \( U \) for which both \( U \) and its complement are transitive. Hetyei and Krattenthaler [10] showed that the poset of bipartitional relations ordered by inclusions has nice combinatorial properties.

In this paper we will be considering the distribution of \( \text{inv}_U^I \) and \( \text{maj}_U^I \) over a fixed rearrangement class \( R(\alpha) \). Notice that if the multiplicity \( \alpha_x \) of \( x \in X \) is 1, then the pair \((x, x)\) cannot contribute to neither \( \text{inv}_U^I \) nor \( \text{maj}_U^I \). Therefore, omitting or adding such pairs to \( U \) doesn’t change these two statistics over \( R(\alpha) \). For that purpose, we define \( U \) to be essentially bipartitional relative to \( \alpha \) if there are disjoint sets \( I \subseteq X \) and \( J \subseteq X \) such that

1. \( \alpha_x = 1 \) for all \( x \in I \cup J \) and
2. \( (U \setminus \{(x, x) : x \in I\}) \cup \{(x, x) : x \in J\} \) is bipartitional.

**Theorem 2.1.** The statistics \( \text{inv}_U^I \) and \( \text{maj}_U^I \) are equidistributed over \( R(\alpha) \) if and only if the relation \( U \) is essentially bipartitional relative to \( \alpha \).

In view of the comment preceding the theorem, the “if” part of Theorem 2.1 follows from Theorem 1.1. We prove the “only if” in Section 3.

The third Mahonian statistic we will consider is the sorting index introduced by Peterson [16] and also studied independently by Wilson [20]. Every permutation \( \sigma \in S_n \) can uniquely be decomposed as a product of transpositions, \( \sigma = (i_1, j_1)(i_2, j_2) \cdots (i_k, j_k) \), such that \( j_1 < j_2 < \cdots < j_k \) and \( i_1 < j_1, i_2 < j_2, \ldots, i_k < j_k \). The sorting index is defined by

\[
\text{sort} \sigma = \sum_{r=1}^{k} (j_r - i_r).
\]

The desired transposition decomposition can be found using the Straight Selection Sort algorithm. The algorithm first places \( n \) in the \( n \)-th position by applying a transposition, then places \( n - 1 \) in the \((n - 1)\)-st position by applying a transposition, etc. For example, for \( \sigma = 2413576 \), we have

\[
2413576 \rightarrow 2413567 \rightarrow 2314567 \rightarrow 2134567 \rightarrow 1234567
\]
and, therefore, \( \text{sor} \sigma = (2 - 1) + (3 - 2) + (4 - 2) + (7 - 6) = 5 \).

The sorting index has been extended to labeled forests by the authors [8]. It can also be naturally extended to words \( w \in R(\alpha) \) by a generalization of Straight Selection Sort which reorders the letters into a weakly increasing sequence. At each step transpositions are applied to place all the \( n \)'s at the end, then all the \( n - 1 \)'s to the left of them, etc, so that for each \( x \in X \), the \( \alpha_x \) copies of \( x \) stay in the same relative order they were right before they were “processed”. Then we define \( \text{sor} w \) to be the sum of the number of positions each element moved during the sorting. For example, applying this sorting algorithm to \( w = 143123123 \) yields

\[
143123123 \rightarrow 133123124 \rightarrow 123123134 \rightarrow 123123134 \rightarrow 121123334 \rightarrow 111223334
\]

and thus \( \text{sor} w = 7 + 2 + 3 + 4 + 2 = 18 \).

We define a graphical sorting index that depends on \( U \) using the same sorting algorithm but at each step, when sorting \( x \), we only count how many elements \( y \) such that \((x, y) \in U\) it “jumps over”. More formally, to compute \( \text{sor}'_U w \) for \( w = x_1 x_2 \ldots x_m \):

- Begin with \( i = m \), and \( \text{sor}'_U w = 0 \).
- Consider the largest element in \( w \) with respect to integer order. If there is a tie, pick the element with the largest subscript, and call this element \( x_j \).
- Interchange \( x_j \) with \( x_i \).
- For each \( h = j + 1, j + 2, \ldots, i \), if \((x_j, x_h) \in U\) increase \( \text{sor}'_U w \) by 1.
- Repeat this process for \( i = m - 1, \ldots, 1 \).

For example, consider the same word \( w = 143123123 \) with \( U = \{(4, 3), (3, 3), (3, 1), (2, 3), (1, 1)\} \). The sorting steps are given in (2.2) and thus \( \text{sor}'_U w = 3 + 1 + 1 + 2 + 0 = 7 \). In particular, if \( U \) is the natural integer order \( > \) then \( \text{sor} w = \text{sor}'_U w \). Our second main result is the following.

**Theorem 2.2.** The statistics \( \text{sor}'_U, \text{inv}'_U \) and \( \text{maj}'_U \) are equidistributed on a fixed rearrangement class \( R(\alpha) \) if and only if the relation \( U \) has the following properties.

1. \( U \) is bipartitional with no underlined blocks,
2. If \((x, y) \in U\) then \( x > y \),
3. All but the last block of \( U \) are of size at most 2,
4. If \( U \) has \( k \) blocks \( B_1, \ldots, B_k \) and \(|B_i| = 2\) for some \( 1 \leq i \leq k - 1 \) then \( \alpha_{\max B_i} = 1 \).

We give the proof of Theorem 2.2 in Section 4. We mention also that recently the question “When are \( \text{sor} \) and \( \text{inv} \) equidistributed on order ideals of the Bruhat order?” was recently addressed in [4].
3 The Proof of Theorem 2.1

We begin the proof with a simple observation.

Lemma 3.1. The statistics $\text{maj}_U'$ and $\text{inv}_U'$ are equidistributed on $\mathcal{R}(\alpha)$ if and only if $\text{maj}_{Uc}$ and $\text{inv}_{Uc}$ are equidistributed on $\mathcal{R}(\alpha)$.

Proof. This follows from the fact that for every $w \in \mathcal{R}(\alpha)$,

$$\text{maj}_U' w + \text{maj}_{Uc} w = \binom{|\alpha|}{2} = \text{inv}_U' w + \text{inv}_{Uc} w.$$ 

\[ \square \]

Lemma 3.2. For any $\alpha = (\alpha_1, \ldots, \alpha_n)$ and any relation $U$ on $X = \{1, 2, \ldots, n\}$,

$$\max_{w \in \mathcal{R}(\alpha)} \text{maj}_U' w \geq \max_{w \in \mathcal{R}(\alpha)} \text{inv}_U' w.$$ 

Proof. We will use induction on $|\alpha|$. It’s clear that the statement holds when $|\alpha| = 1$. Assume that it holds for all $\alpha$ with $|\alpha| \leq m$.

Consider a rearrangement class $\mathcal{R}(\alpha)$ such that $|\alpha| = m + 1$ and a relation $U$ on $[n]$. Let $(\alpha, U)$ be a directed graph with vertex set $\{1^{\alpha_1}, \ldots, n^{\alpha_n}\}$ and a directed edge $x \to y$ whenever $(x, y) \in U$. Let $x_1 \to x_2 \to \cdots \to x_n$ be a directed path in $(\alpha, U)$ of maximal possible length. This means we have a descending chain $x_1 >_U x_2 >_U \cdots >_U x_l$ of maximal possible length. Set $\alpha' = (\alpha_1', \ldots, \alpha_n')$ where

$$\alpha_i' = \alpha_i - \sum_{j=1}^{l} \chi(x_j = i).$$

Let $u'$ be a word that maximizes $\text{maj}_U'$ on the rearrangement class $\mathcal{R}(\alpha')$. One can easily verify that for the word $u = u'x_1x_2\cdots x_l$ in $\mathcal{R}(\alpha)$ we have

$$\text{maj}_U' u = \text{maj}_U' u' + \frac{(l - 1)(2m + 2 - l)}{2}. \quad (3.1)$$

To bound $\max_{w \in \mathcal{R}(\alpha)} \text{inv}_U' w$, first suppose there is an element $y \in (\alpha', U)$ such that for all $i = 1, 2, \ldots, l$ we have $(y, x_i) \in U$ or $(x_i, y) \in U$. If $(y, x_1) \in U$ then $y >_U x_1 >_U x_2 >_U \cdots >_U x_l$ is a longer chain in $(\alpha, U)$, therefore $(y, x_1) \notin U$ and $(x_1, y) \notin U$. Similarly, if $(x_l, y) \in U$ we can form the longer chain $x_1 >_U x_2 >_U \cdots >_U x_l >_U y$ in $(\alpha, U)$; thus we must have $(x_i, y) \notin U$ and $(y, x_l) \in U$. However, this implies that there are elements $x_i$ and $x_{i+1}$ such that $(x_i, y), (y, x_{i+1}) \in U$, which yields a longer chain $x_1 >_U x_2 >_U \cdots >_U x_l >_U y >_U x_{i+1} >_U \cdots >_U x_l$. Therefore, every $y \in (\alpha', U)$ is related to at most $l - 1$ elements in the chain $x_1 >_U \cdots >_U x_l$.

Now consider a word $v \in \mathcal{R}(\alpha)$ and the corresponding word $v' \in \mathcal{R}(\alpha')$ obtained by deleting $x_1, \ldots, x_l$. By the argument in the previous paragraph, the $m + 1 - l$ letters in $v'$ create at most $(m + 1 - l)(l - 1)$ graphical inversions with $x_1, \ldots, x_l$. Therefore, by (3.1) and the induction hypothesis,
Since the equality in (3.2) holds, the elements \( w \) be a maximal chain formed from the maximal chains \( w \) the chain \( w \) word which implies that \( w \) any \( w \) maximal chain word in \( \mathcal{R}(\alpha) \), \( U \), \( \alpha \). The following lemma shows that if \( \text{maj}_U^I \) is the property holds: if \( x \) in the maximal chains can be reordered, if necessary, so that within each of them the following equation holds: if \( maj^I_U(w) \geq \max_{v \in \mathcal{R}(\alpha)} \text{inv}^I_U(v) \) can be constructed by “peeling off” descending chains of maximal length from \( (\alpha, U) \) and ordering them from right to left, forming the subwords \( w_1, w_2, \ldots, w_k \) in that order. These kind of words will be used in the proof and for a fixed relation \( U \), we will call such words maximal chain words in \( \mathcal{R}(\alpha) \).

**Lemma 3.3.** Suppose \( \text{maj}_U^I \) and \( \text{inv}_U^I \) are equidistributed on \( \mathcal{R}(\alpha) \). Let \( w = w_k w_{k-1} \cdots w_1 \in \mathcal{R}(\alpha) \) be a maximal chain formed from the maximal chains \( w_1, w_2, \ldots, w_k \). Then

\[
\begin{align*}
(1) & \text{ For each of the maximal descending chains } w_j = x_{i_{j-1}+1} x_{i_{j-1}+2} \cdots x_{i_j} \\
& (x_r, x_s) \in U \text{ or } (x_s, x_r) \in U \text{ for all } i_{j-1}+1 \leq r < s \leq i_j. \\
(2) & \text{ Each letter } y \text{ in a maximal descending chain } w_i, i > j, \text{ is in relation with exactly } i_j - 1 \\
& \text{ elements from } w_j, \text{ i.e., there is a unique } r \in \{i_{j-1}+1, \ldots, i_j\} \text{ such that } (y, x_r) \notin U \text{ and } \\
& (x_r, y) \notin U. \text{ Moreover, } (x_s, y) \in U \text{ for } i_{j-1}+1 \leq s < r \text{ and } (y, x_s) \in U \text{ for } r < s \leq i_j.
\end{align*}
\]

**Proof.** Condition (i) is necessary for equality to hold in (3.2). The property (ii) also follows from the fact that equality holds in (3.2) and the definition of a maximal chain word which implies that the chain \( w_j \) is be the longest one that can be formed among the letters in \( w_k w_{k-1} \cdots w_j \). □

The following lemma shows that if \( \text{maj}_U^I \) and \( \text{inv}_U^I \) are equidistributed on \( \mathcal{R}(\alpha) \) the elements in the maximal chains can be reordered, if necessary, so that within each of them the following property holds: if \( x \) precedes \( y \) in the same chain of a maximal chain word then \( (x, y) \in U \).

**Lemma 3.4.** If \( \text{maj}_U^I \) and \( \text{inv}_U^I \) are equidistributed on \( \mathcal{R}(\alpha) \), then there exists a maximal chain word \( w = w_k w_{k-1} \cdots w_1 \in \mathcal{R}(\alpha) \) with subwords \( w_i \) formed from descending chains such that for any \( w_j = x_{i_{j-1}+1} x_{i_{j-1}+2} \cdots x_{i_j} \) we have

\[
(x_r, x_s) \in U \text{ for all } i_{j-1}+1 \leq r < s \leq i_j.
\]

**Proof.** Since the equality in (3.2) holds, the elements \( x_1, x_2, \ldots, x_l \) in the maximal chain can be arranged so that they form \( \binom{l}{2} \) graphical inversions, which implies the statement in the lemma. □

**Lemma 3.5.** Suppose \( \text{maj}_U^I \) and \( \text{inv}_U^I \) are equidistributed on \( \mathcal{R}(\alpha) \). Let \( w = w_k w_{k-1} \cdots w_1 \) be a maximal chain word in \( \mathcal{R}(\alpha) \) for \( U \) with maximal chains \( w_1, \ldots, w_k \). If \( (x, y) \in U \text{ and } (y, x) \in U \) for some \( x \neq y \), then the \( x \)’s and \( y \)’s are all in the same chain \( w_i \).
Lemma 3.3 implies that \((x, x)\) chain of a maximal chain word. In particular, since two elements of \(X\) elements of \(\cdot \cdot \cdot x\) implies \(\boxed{\text{Suppose}}\).

\[
\text{Suppose there exists an element } x \in X \text{ with } \alpha_x \geq 1 \text{ such that } U \subset (X \setminus \{x\}) \times (X \setminus \{x\}) \text{ and } U \neq \emptyset. \text{ Then } \text{inv}_U' \text{ and } \text{maj}_U' \text{ are not equidistributed over } \mathcal{R}(\alpha).
\]

\[
\text{Proof.} \text{ Set } \alpha'_U = \begin{cases} 
\alpha_i & \text{if } i \neq x, \\
0 & \text{if } i = x.
\end{cases}
\]

Let \(w' \in \mathcal{R}(\alpha')\) be the word such that \(\text{maj}_{U'} w' = \max_{w \in \mathcal{R}(\alpha')} \text{maj}_{U} u.\) Consider the word \(w = \underline{x_1 x_2 \cdots x_n} w'\) in \(\mathcal{R}(\alpha)\). Since \(x\) does not create any graphical inversions, by Lemma 3.2, we have

\[
\max_{u \in \mathcal{R}(\alpha)} \text{maj}_U u \geq \text{maj}_{U'} w = \text{maj}_{U} w' + \alpha_x \text{des}_{U} w'
\]

\[
= \max_{u \in \mathcal{R}(\alpha')} \text{maj}_{U} u + \alpha_x \text{des}_{U} w'
\]

\[
\geq \max_{u \in \mathcal{R}(\alpha')} \text{inv}_{U'} u + \alpha_x \text{des}_{U} w'
\]

\[
= \max_{u \in \mathcal{R}(\alpha')} \text{inv}_{U} u + \alpha_x \text{des}_{U} w'
\]

Consequently, if \(\text{inv}_{U'} \) and \(\text{maj}_{U'} \) are equidistributed on \(\mathcal{R}(\alpha)\), then \(\text{des}_{U} w' = 0\) and thus

\[
\max_{u \in \mathcal{R}(\alpha')} \text{maj} u = 0.
\]

This contradicts the fact that \(U \neq \emptyset\).

\[
\text{Lemma 3.7. Suppose } \text{maj}_{U'} \text{ and } \text{inv}_{U'} \text{ are equidistributed on } \mathcal{R}(\alpha). \text{ If } (x, y), (y, x) \in U \text{ and } \alpha_x > 1 \text{ then } (x, x) \in U.
\]

\[
\text{Proof. Since } (x, y), (y, x) \in U, \text{ by Lemma 3.5 all the } x's \text{ and } y's \text{ must be in the same maximal chain of a maximal chain word. In particular, since two } x's \text{ are in the same chain, part } (i) \text{ of Lemma 3.3 implies that } (x, x) \in U.
\]

\[
\text{Lemma 3.8. Suppose } \text{maj}_{U'} \text{ and } \text{inv}_{U'} \text{ are equidistributed over } \mathcal{R}(\alpha)\text{ and let } x \text{ and } y \text{ be two distinct elements of } X \text{ such that } (x, y), (y, x) \in U. \text{ For every } z \in \{1^\alpha_1, \ldots, n^\alpha_n\} \setminus \{x, y\}, \text{ we have }
\]

\[
(z, x) \in U \text{ if and only if } (z, y) \in U
\]

\[
(x, z) \in U \text{ if and only if } (y, z) \in U.
\]
Proof. If $z = x$ then $\alpha_x > 1$ and the claim follows from Lemma 3.7. The same is true if $z = y$. So, suppose $z \neq x, z \neq y$. Because of symmetry, it suffices to prove
\[
(z, x) \in U \implies (z, y) \in U \quad (3.8)
\]
\[
(x, z) \in U \implies (y, z) \in U \quad (3.9)
\]
To see (3.8), suppose that $(z, x) \in U, (z, y) \notin U$. We consider two cases.

Case 1: $(y, z) \notin U$. Let $w = w_1w_2\cdots w_1 \in R(\alpha)$ be a maximal chain word that satisfies (3.7). By Lemma 3.3, $x$ and $y$ are in the same chain $w_i$ of $w$. By Lemma 3.3, $z$ is in a different chain $w_j$ and by Lemma 3.5, $x, z \notin U$. If $j > i$, notice that, by Lemma 3.3, $x$ cannot precede $y$ in $w_i$, so $w_i$ must be of the form $w_i = b_1\cdots b_kb_{k+1}\cdots b_l x b_{l+1} \cdots b_m$. Then $b_1\cdots b_kb_{k+1} \cdots b_l x b_{l+1} \cdots b_m$ is a descending chain longer than $w_i$. If $j < i$, then $w_j = b_1\cdots b_kb_{k+1} \cdots b_l$. By part (ii) of Lemma 3.3, $(b_k, x), (y, b_{k+1}) \in U$, which implies that $b_1\cdots b_kb_{k+1} \cdots b_l$ is a descending chain longer than $w_j$.

Case 2: $(y, z) \in U$. By Lemma 3.1, $\text{maj}'_{U^c}$ and $\text{inv}'_{U^c}$ are equidistributed on $R(\alpha)$. Let $w = w_1w_2\cdots w_1 \in R(\alpha)$ be a maximal chain word for $U^c$ that satisfies (3.7). Suppose $x, y, z$ are in the chains $w_i, w_j, w_k$, respectively. By Lemma 3.3, $i \neq j, i \neq k$. If $i < j, k$ and $w_i = b_1\cdots b_kb_{k+1} \cdots b_m$ then a different maximal chain word $w'$ could be constructed by taking the same chains $w_1, \ldots, w_{i-1}$ as in $w$ and replacing $w_i$ by $b_1\cdots b_kb_{k+1} \cdots b_m$. Since $(z, y) \in U^c$, it follows from Lemma 3.3 that $z$ is not in relation $U^c$ with some $b_r, r \leq l$ and therefore $(z, x) \in U^c$, which contradicts $(z, x) \in U$. The similar argument holds if $j < i, k$. If $k < i, j$ and $w_k = b_1\cdots b_kb_{k+1} \cdots b_m$ then $y$ is not in relation $U^c$ with some $b_r, r > l$, and a different maximal chain word for $U^c$ could be formed by replacing $w_k$ with $b_1\cdots b_kb_{k+1} \cdots b_{l-1} y b_{l+1} \cdots b_m$. Part (ii) of Lemma 3.3 now implies that $(z, x) \in U^c$, which contradicts $(z, x) \in U$. Finally, if $j = k < i$, then since $(z, x) \notin U^c$ and $(x, y), (y, x) \notin U^c$, Lemma 3.3 implies that $(x, z) \in U^c$ and $y$ precedes $z$ in $w_j$. Therefore, $(y, z) \in U^c$, which contradicts $(y, z) \notin U$.

The implication (3.9) can be proved by considering completely analogous cases, so we omit it here.

For a relation $U$ on $X$, call $S(U) = \{(x, y) \in X \times X : (x, y), (y, x) \in U\}$, the symmetric part of $U$, and call $A(U) = U \setminus S(U)$ the asymmetric part of $U$. Let $X_U = \{x \in X : (x, y) \in S(U) \text{ for some } y \in X\}$.

Lemma 3.9. If $\text{maj}'_{U^c}$ and $\text{inv}'_{U^c}$ are equidistributed over a rearrangement class $R(\alpha)$, then $S(U) \cup \{(x, x) : x \in X_U, \alpha_x = 1\}$ is an equivalence relation on $X_U \times X_U$.

Proof. Let $x \in X_U$ and $y \in X$ such that $(x, y) \in U$. If $y = x$ then we have $(x, x) \in U$. If $y \neq x$ then we have $(x, y), (y, x) \in U$ and thus, if $\alpha_x > 1$ by Lemma 3.7 we have $(x, x) \in U$. $S(U)$ is symmetric by definition because $(x, y) \in S(U)$ implies $(y, x) \in S(U)$. Now consider $x, y, z \in X_U$ and assume $(x, y), (y, z) \in S(U)$. Then by definition of $S(U)$, $(y, x), (z, y) \in S(U)$ and Lemma 3.3 implies $(x, z) \in S(U)$.

Consequently, $X_U$ can be partitioned into blocks $B_1, \ldots, B_l$ such that
\[
S(U) \cup \{(x, x) : x \in X_U, \alpha_x = 1\} = (B_1 \times B_1) \cup (B_2 \times B_2) \cup \cdots \cup (B_l \times B_l).
\]

Lemma 3.10. Suppose that $\text{maj}'_{U^c}$ and $\text{inv}'_{U^c}$ are equidistributed over a rearrangement class $R(\alpha)$. Then either there is a block $B$ of $X_U$ such that
\[
\text{for all } x \in B \text{ and all } y \in X \setminus B \text{ we have } (x, y) \notin U,
\]
or there is an element 

\[ x \in X \setminus X_U \text{ such that for all } y \in X \setminus \{x\} \text{ we have } (x,y) \notin U. \]

Proof. Suppose that the lemma does not hold. In other words, assume that \( \text{maj}_U \) and \( \text{inv}_U \) are equidistributed over a rearrangement class \( \mathcal{R}(\alpha) \), but for all blocks \( B_i \) of \( X_U \) there exists a \( x \in B_i \) and \( y \in X \setminus B_i \) such that \( (x,y) \in U \), and for all \( x \in X \setminus X_U \) there exists a \( y \in X \setminus \{x\} \) such that \( (x,y) \in U \).

Consider \( x_0 \in X \). If \( x_0 \in B_{i_0} \) for some \( B_{i_0} \subset X_U \) there exists a \( x_1 \in B_{i_0} \) and \( x_2 \in X \setminus B_{i_0} \) such that \( (x_0,x_1),(x_1,x_0) \in U \), and \( (x_1,x_2) \in U \). Lemma 3.8 implies \( (x_0,x_2) \in U \), and \( (x_2,x_0) \notin U \) because if so \( x_2 \in B_{i_0} \). Note that if we began with \( x_0 \notin X_U \) our assumptions would still give an element \( x_2 \in X \setminus \{x\} \) such that \( (x_0,x_2) \in U \), and \( (x_2,x_0) \notin U \) because \( x_0 \notin X_U \). Now there are two cases to consider.

Case 1: \( x_2 \in B_{i_1} \) for some \( B_{i_1} \subset X_U \) and \( B_{i_1} \neq B_{i_0} \). Then there exists a \( x_3 \in B_{i_1} \) and \( x_4 \notin B_{i_1} \) such that \( (x_2,x_4) \in U \) and, by Lemma 3.8, \( (x_2,x_4) \in U \) and \( (x_4,x_2) \notin U \).

Case 2: \( x_2 \notin X_U \), and then there exists a \( x_4 \in X \setminus \{x\} \) such that \( (x_2,x_4) \in U \) and \( (x_4,x_2) \notin U \).

Continuing this process we can build a sequence \( x_0,x_2,x_4,x_6,\ldots \) with the properties \( x_0 >_U x_2 >_U x_4 >_U x_6 >_U \cdots \) and \( x_0 \neq U x_2 \neq U x_4 \neq U x_6 \neq U \cdots \) with \( x_{2i} \neq x_{2i+2} \) for all \( i \).

The set \( X \) is finite and thus this sequence can not be infinite with distinct terms. Therefore, with relabeling there is a finite sequence \( y_1, y_2, y_3, \ldots, y_{l+1} \) such that

- \( l \geq 2 \)
- all \( y_i \)'s are distinct for \( i = 1,2,\ldots,l \)
- \( y_1 >_U y_2 >_U (y_3 >_U y_4 >_U \cdots >_U y_l >_U y_{l+1} = y_1 \)
- \( y_1 \neq U y_2 \neq U y_3 \neq U y_4 \neq U \cdots \neq U y_l \neq U y_{l+1} = y_1 \)

If \( l = 2 \) then we have \( y_1 <_U y_2 <_U y_1 \) and \( y_1 \neq U y_2 \neq U y_1 \) which is a contradiction and hence
\( l \geq 3 \).

Let \( w \in \mathcal{R}(\alpha) \) be a maximal chain word for \( U \). If all \( y_1,\ldots,y_l \) appear in the same chain \( w_i \), then by Lemma 3.3 they can be relabeled to give a sequence \( z_1,\ldots,z_l \) such that \( (z_r,z_s) \in U \) for all \( 1 \leq r < s \leq l \). If \( z_l = y_i \), then this means that \( y_{i+1} >_U y_i \), which is a contradiction. If not all \( y_1,\ldots,y_l \) appear in the same chain let \( y_j \) be the one that appears in the rightmost chain of \( w \). Then either \( y_{j+1} \) is already in the same chain or its not related to an element to the right of \( y_j \). In the latter case, another maximal chain word can be constructed in which \( y_j \) and \( y_{j+1} \) are in the same chain, while the other \( y_i \)'s are either in the same chain or in chains to the left. Continuing this argument, we see that we can construct a maximal chain word in which all \( y_1,\ldots,y_l \) are in the same chain, which as we saw before is impossible.

Let \( C = \{ x \in X \setminus X_U : \alpha_x > 1, (x,y) \notin U, \forall y \in X \text{ or } \alpha_x = 1, (x,y) \notin U, \forall y \in X \setminus \{x\} \} \).

Lemma 3.11. Suppose that \( \text{maj}_U \) and \( \text{inv}_U \) are equidistributed over a rearrangement class \( \mathcal{R}(\alpha) \).
If \( C \) is nonempty then

\[ \text{for all } y \in X \setminus C \text{ and for all } x \in C \text{ we have } (y,x) \in U. \]
If \( C \) is empty and \( B \) is the block defined in Lemma 3.10 then

\[
\text{for all } y \in X \setminus B \text{ and for all } x \in B \text{ we have } (y, x) \in U.
\]

**Proof.** Suppose \( C \) is nonempty, and that the claim does not hold. In other words assume that there exists a \( y \in X \setminus C \) and \( x \in C \) such that \( (y, x) \notin U \). Now \( y \notin C \) so there exists a \( z \in X \) such that \( (y, z) \in U \). Notice that we may have \( y = z \) if \( y \in X_U \), but then \( \alpha_y > 1 \), and \( z \neq x \) by assumption. Now we have \( (y, z) \in U \), \((y, x) \notin U \), and \((x, y), (x, z) \notin U \) since \( x \in C \). Since \((x, y), (y, x) \notin U \), and \((x, z) \notin U \), Lemma 3.8 applied to \( U^c \) yields \((y, z) \notin U \), which is a contradiction.

Now suppose \( C \) is empty, \( B \) is the block defined in Lemma 3.10 and the claim does not hold. In other words, there exists a \( y \in X \setminus B \) and \( x \in B \) such that \((y, x) \notin U \). Since \( y \notin B \), and \( C \) is nonempty there must be an element \( z \) such that \((y, z) \in U \). Now we have \((y, z) \in U \), \((y, x) \notin U \), and \((x, y), (x, z) \notin U \) since \( x \in B \). Therefore, the same argument as above gives a contradiction.

**Proof of Theorem 2.7.** Theorem 2.1 can be proved using induction on the size of the set \( X \). Suppose first that \( C \neq \emptyset \). Then, by Lemma 3.1 \( C \times X = \emptyset \) and \((X \setminus C) \times C \subset U \). Consider \( \inv_U' \) and \( \maj_U' \) over the rearrangement class \( R(\alpha') \) of the permutations of the multiset \( \{x^{\alpha_x} : x \in X \setminus C \} \). Inserting the elements from \( C \) in all possible ways among the letters of a word \( w' \in R(\alpha') \) results in a set of words \( S(w') \subset R(\alpha') \). It is not hard to see that as \( w \) ranges over \( S(w') \), the difference \( \inv_U' w - \inv_U' w' \) ranges over the multiset \( \{i_1 + i_2 + \cdots + i_r : 0 \leq i_1 \leq i_2 \leq \cdots \leq i_s \leq s\} \) where \( r = |\alpha(C)| \) and \( s = |\alpha(X \setminus C)| \). The same is true for the difference \( \maj_U' w - \maj_U' w' \). This is less obvious but follows from a similar property of the classical major index for words (see e.g., [2, Lemma 4.6] for a proof). Let \( U_1 = U \cap ((X \setminus C) \times (X \setminus C)) \) be the restriction of \( U \) on \( X \setminus C \). For \( w \in R(\alpha') \), \( \inv_U' w = \inv_{U_1} w \) and \( \maj_U' w = \maj_{U_1} w \). So,

\[
\sum_{w \in R(\alpha)} q^{\inv_U' w} = \frac{|\alpha|}{|C|} \left( \frac{|C|}{\alpha(C)} \right) \sum_{w \in R(\alpha')} q^{\inv_{U_1} w},
\]

\[
\sum_{w \in R(\alpha)} q^{\maj_U' w} = \frac{|\alpha|}{|C|} \left( \frac{|C|}{\alpha(C)} \right) \sum_{w \in R(\alpha')} q^{\maj_{U_1} w}.
\]

Thus if \( \inv_U' \) and \( \sort_U' \) are equidistributed on \( R(\alpha) \) then \( \inv_{U_1} \) and \( \maj_{U_1} \) are equidistributed on \( R(\alpha') \). By the induction hypothesis, \( U_1 \) is essentially bipartitional relative to \( \alpha' \) and thus \( U \) is essentially bipartitional relative to \( \alpha \) with one more non-underlined block \( C \).

In the case when \( C = \emptyset \) there is a block \( B \) such that \( B \times (X \setminus B) \) is empty and \( X \times B \subset U \). Then we consider the relation \( U_1 = U \cap ((X \setminus B) \times (X \setminus B)) \) on \( X \setminus B \). Similar reasoning as above yields

\[
\sum_{w \in R(\alpha)} q^{\inv_U' w} = \frac{|\alpha|}{|B|} \left( \frac{|B|}{\alpha(B)} \right) \sum_{w \in R(\alpha')} q^{\inv_{U_1} w},
\]

\[
\sum_{w \in R(\alpha)} q^{\maj_U' w} = \frac{|\alpha|}{|B|} \left( \frac{|B|}{\alpha(B)} \right) \sum_{w \in R(\alpha')} q^{\maj_{U_1} w}.
\]

So, \( U_1 \) is essentially bipartitional relative to \( \alpha' \) and thus \( U \) is essentially bipartitional relative to \( \alpha \) with one more underlined block \( B \).
4 Graphical Sorting Index

In this section we will prove Theorem 2.2. The “if” part follows from the following proposition and (2.1), while the “only if” part follows from Lemma 4.3 and Lemma 4.5.

**Proposition 4.1.** If the relation $U$ satisfies the properties of Theorem 2.2 and has blocks $B_1, \ldots, B_k$ then

$$
\sum_{w \in R(\alpha)} q^{\text{sort}_w} = \left[ \left| \alpha \right| m_1, \ldots, m_k \right] \prod_{j=1}^{k} \left( m_j \alpha(B_j) \right).
$$

**Proof.** We will prove the statement using a B-code for the words in $R(\alpha)$ that we define. Let $w = x_1x_2\ldots x_l \in R(\alpha)$. B-code $w$ is a pair of two sequences: a sequence of partitions and a sequence of nonnegative integers. Precisely, we define B-code $w$ to be

$$
((b_{1,1} \geq b_{1,2} \geq \ldots \geq b_{1,m_1}; b_{2,1} \geq b_{2,2} \geq \ldots \geq b_{2,m_2}; \ldots; b_{k,1} \geq b_{k,2} \geq \ldots \geq b_{k,m_k}), (p_1, p_2, \ldots, p_k))
$$

where

1° for $i < k$ each partition $b_{i,1} \geq \ldots \geq b_{i,m_i} \geq 0$ each part has size $b_{i,j} \leq m_{i+1} + m_{i+2} + \cdots + m_k,$

1 \leq j \leq m_i, while $b_{k,j} = 0$ for $1 \leq j \leq m_k$,

2° $p_i = 0$ if $|B_i| = 1$ and $1 \leq p_i \leq m_i$ if $|B_i| = 2$.

B-code $w$ is computed as follows.

1. Set $j = 1$.

2. If $B_j = \{y_1, y_2\}$ has two integers $y_2 > y_1$ then let $p_j = i$ be the position of $y_2$ in the subword of $w$ formed by the elements of $B_j$. Otherwise set $p_j = 0$.

3. Sort the elements of the block $B_j$ and form the partition $b_{j,1} \geq \ldots \geq b_{j,m_j} \geq 0$ from the contributions to $\text{sort}_w$ (listed in nonincreasing order) by the elements of $B_j$. Keep calling the partially sorted word $w$.

4. If $j < k$ increase $j$ by 1 and go to step (2). Otherwise stop.

Consider, for example, the relation $U = \{(5, 3), (5, 2), (5, 1), (4, 3), (4, 2), (4, 1), (3, 2), (3, 1)\}$ which is bipartitional with blocks $B_1 = \{5, 4\}, B_2 = \{3\}, B_3 = \{2, 1\}$ and $\beta_1 = \beta_2 = \beta_3 = 0$. Let $w = 42345411 \in R(2, 1, 1, 3, 1)$. Since the subword formed by the 4’s and the 5 is 4454, we have $p_1 = 3$. The steps for sorting the 4’s and the 5 are

$$
4234541 \longrightarrow 42344145 \longrightarrow 42341145 \longrightarrow 42311445 \longrightarrow 12314445
$$

and, therefore, the first partition in B-code $w$ is $4 \geq 2 \geq 1 \geq 1$. Then $p_2 = 0$ and sorting the 3 yields 12134445, therefore the second partition is 1. Finally, $p_3 = 2$ and

$$
\text{B-code } w = ((4 \geq 2 \geq 1 \geq 1; 1; 0 \geq 0 \geq 0), (3, 0, 2)).
$$
Since the parts of the partitions in the B-code represent contributions to the sorting index, the bound for their size \(b_{i,j} \leq m_{i+1} + m_{i+2} + \cdots + m_k\) easily follows. Therefore, the B-code is clearly a map from \(R(\alpha)\) to the set of pairs of sequences of partitions and integers which satisfy (1\(^°\)) and (2\(^°\)), which we claim is a bijection. For describing the inverse, the crucial observation is that for blocks of size 2, \(B_j = \{y_1 < y_2\}\), the contribution to the sorting index is given by \(b_{j,p_j}\). Then given

\[
((b_{1,1} \geq b_{1,2} \geq \cdots \geq b_{1,m_1}; b_{2,1} \geq b_{2,2} \geq \cdots \geq b_{2,m_2}; \ldots; b_{k,1} \geq b_{k,2} \geq \cdots \geq b_{k,m_k}),(p_1,p_2,\ldots,p_k))
\]

which satisfies (1\(^°\)) and (2\(^°\)), the corresponding word \(w \in R(\alpha)\) is constructed as follows.

1. Let \(j = k\) and \(w\) be the empty word.
2. Add to the end of \(w\) the elements of \(B_j\) with their multiplicities, listed in nondecreasing order \(x_j,1x_j,2 \cdots x_j,m_j\).
3. If \(|B_j| = 1\), then for \(i = 1, \ldots, m_j\), swap \(x_{j,i}\) with the element of \(w\) which is \(b_{j,i}\) places to the left of \(x_{j,i}\).
4. If \(B_j = \{y_1 < y_2\}\), then let \(b_{j,1}' \geq \cdots \geq b_{j,m_j-1}'\) be the partition obtained from \(b_{j,1} \geq \cdots \geq b_{j,m_j}\) by deleting the part \(b_{j,p_j}\). Then for \(i = 1, \ldots, m_j - 1\), swap \(x_{j,i}\) with the element of \(w\) which is \(b_{j,i}'\) places to the left of \(x_{j,i}\). Finally, swap \(x_{j,m_j} = y_2\) with the element in \(w\) which is \(b_{j,p_j} + m_j - p_j\) positions to its left. (After this step there are \(b_{j,p_j}\) elements from \(B_{j+1}, \ldots, B_k\) and \(m_j - p_j\) elements from \(B_j\) to the right of \(y_2\).)
5. If \(j > 1\) decrease \(j\) by 1 and go to step (2). Otherwise stop.

The B-code is designed so that \(\operatorname{sort}^L_w = \sum_{i=1}^k \sum_{j=1}^{m_i} b_{i,j}\). The bijection described above then yields the generating function for \(\operatorname{sort}^L_w\). Let \(p(j,k,n)\) denote the number of partitions of \(n\) into at most \(k\) parts, with largest part at most \(j\). It is known that \(\sum_{n \geq 0} p(j,k,n)q^n = \binom{j+k}{j}\). The block \(B_j\) contributes

\[
\binom{m_j}{\alpha(B_j)} \sum_{n \geq 0} p(m_{j+1} + m_{j+2} + \cdots + m_n, m_j, n)q^n = \binom{m_j}{\alpha(B_j)} \left[ m_j + m_{j+1} \cdots + m_n \right] \left[ m_j \right]^{-1}
\]

to \(\sum_{w \in R(\alpha)} q^{\operatorname{sort}^L_w}\), where the leading binomial coefficient counts the number of possible values of \(p_j\). Thus we have

\[
\sum_{w \in R(\alpha)} q^{\operatorname{sort}^L_w} = \prod_{j=1}^k \binom{m_j}{\alpha(B_j)} \left[ m_j + m_{j+1} \cdots + m_n \right] \left[ m_j \right]^{-1} \prod_{j=1}^k \binom{m_j}{\alpha(B_j)} = \binom{|\alpha|}{m_1, \ldots, m_k}.
\]

In particular, we get the generating function for the standard sorting index for words.

**Corollary 4.2.**

\[
\sum_{w \in R(\alpha)} q^{\operatorname{sort}_w} = \binom{|\alpha|}{m_1, \ldots, m_k}.
\]
Finally, we prove the “only if” part of Theorem 2.2 via the following few lemmas.

**Lemma 4.3.** If \( \text{sor}_U', \text{maj}_U', \) and \( \text{inv}_U' \) are equidistributed over a fixed rearrangement class \( \mathcal{R}(\alpha) \) then the relation \( U \) must be a subset of the integer order modulo relations \( (x, x) \).

**Proof.** Suppose \( \text{sor}_U', \text{maj}_U', \) and \( \text{inv}_U' \) are equidistributed on \( \mathcal{R}(\alpha) \). By Theorem 2.1 \( U \) must be essentially bipartitional relative to \( \alpha \). That means that there are subsets \( I, J \subset \{x : \alpha_x = 1\} \) such that \( U' = (U \setminus \{(x, x) : x \in I\}) \cup \{(x, x) : x \in J\} \) is bipartitional. Without loss of generality we may assume that \( I, J \) are chosen so that \( U' \) does not have underlined blocks \( \{x\} \) of size 1 such that \( \alpha_x = 1 \). We claim that \( U' \) is a subset of the natural order.

First we will show that there are no underlined blocks in \( U' \). Suppose the contrary. Then there exist elements \( x \) and \( y \) such that \( (x, y), (y, x) \in U' \) \( (x \neq y \text{ or } y \text{ is a second copy of the same element with } \alpha_x > 1) \). Because we have both \( (x, y) \) and \( (y, x) \) in \( U' \) every word \( w \in \mathcal{R}(\alpha) \) has at least one \( U' \)-inversion. Therefore the minimum \( \text{inv}_U' \) over the rearrangement class \( \mathcal{R}(\alpha) \) is 1. On the other hand, \( \text{sor}_U'1 \cdot 122 \cdots 2 \cdot nn \cdots n = 0 \). This is a contradiction, and thus there are no underlined blocks in \( U' \).

Now assume that \( U' \) is not a subset of the natural integer order. Then there exist at least two elements such that \( (x, y) \in U' \), but \( y > x \) with respect to the natural order. Let \( B_1, B_2, \ldots, B_k \) be the blocks of \( U' \). Now consider the words created by placing the elements of \( B_1 \) in some order followed by the elements of \( B_2 \) placed to the right of \( B_1 \) and continue the process until the elements of \( B_k \) in some order are the last elements of the word. The words of this type will have \( \text{inv}_U' \) equal to the number of edges in the graph \( (\alpha, U') \) as defined in the proof of Lemma 3.2. Therefore, the maximum \( \text{inv}_U' \) is bounded below by the number of edges in \( (\alpha, U') \) (it is in fact equal to the number of edges in \( (\alpha, U') \)). In the sorting algorithm, however, elements are only sorted over elements that are smaller than them with respect to the natural order. Therefore \( x \) will never jump over \( y \), and thus the relation \( (x, y) \) will never contribute to the sorting index. Since each edge of the graph \( (\alpha, U') \) contributes at most 1 to \( \text{sor}_U' \), we conclude that the maximum \( \text{sor}_U' \) on \( \mathcal{R}(\alpha) \) is less than the maximum \( \text{inv}_U' \). This is a contradiction, and \( U' \) must be a subset of the natural order. \( \Box \)

The next inequality will be used to prove the remaining part of Theorem 2.2.

**Lemma 4.4.** For \( a, b \in \mathbb{Z}_{\geq 1} \)

\[
\sum_{i=0}^{\min\{a, b\}} \binom{a}{i} \leq \binom{a+b}{b}
\]

and equality holds if and only if \( b = 1 \).

**Proof.** If \( a \leq b \) then using the Vandermonde’s Identity we have

\[
\sum_{i=0}^{\min\{a, b\}} \binom{a}{i} = \sum_{i=0}^{a} \binom{a}{i} \leq \sum_{i=0}^{a} \binom{a}{i} \binom{b}{b-i} = \binom{a+b}{b}
\]

and equality holds if and only if \( a = b = 1 \). Similarly, if \( a > b \) then

\[
\sum_{i=0}^{\min\{a, b\}} \binom{a}{i} = \sum_{i=0}^{b} \binom{a}{i} \leq \sum_{i=0}^{b} \binom{a}{i} \binom{b}{b-i} = \binom{a+b}{b}
\]

\( \Box \)
Lemma 4.5. Suppose $U$ is a bipartitional relation with blocks $B_1, \ldots, B_k$, none of which are underlined, such that $\text{sort}'_U, \text{maj}'_U$, and $\text{inv}'_U$ are equidistributed over $\mathcal{R}(\alpha)$. Then for every $1 \leq i < k$, $|B_i| \leq 2$ and if the equality $|B_1| = 2$ holds then $\alpha_{\text{max}}B_i = 1$.

Proof. By Lemma 4.3, the blocks $B_1, \ldots, B_k$ are consecutive intervals with $n \in B_1$ and $1 \in B_k$. If $k = 1$ there is nothing to prove, so suppose $k > 1$.

Let $i(B_1, \ldots, B_k)$ and $s(B_1, \ldots, B_k)$ denote the number of words in $\mathcal{R}(\alpha)$ that maximize $\text{inv}'_U$ and $\text{sort}'_U$, respectively. Let $B_1 = \{s, s + 1, \ldots, n\}, s \leq n - 1$. The words in $\mathcal{R}(\alpha)$ that maximize $\text{inv}'_U$ are exactly those formed by a permutation of the elements of $B_1$ (with their multiplicities) followed by a permutation of the elements from $B_2$, etc. So, $i(B_1, \ldots, B_k) = \prod_{i=1}^{k} (m_{i})_{\alpha(B_i)}$.

On the other hand, if $w \in \mathcal{R}(\alpha)$ maximizes $\text{sort}'_U$, then after sorting the $n$’s, one obtains a word $w' \in \mathcal{R}(\alpha')$ that maximizes $\text{sort}'_U$ for $\alpha' = (\alpha_1, \ldots, \alpha_{n-1})$. The map $w \to w'$ is not one-to-one. One can write $w' = uw$ where $u$ is the longest prefix of $w'$ formed by elements of $B_1$. Then the number of words $w$ that yield $w'$ is at most $\sum_{i=0}^{\min\{|u|, \alpha_n\}} (|u|_i)$. Namely, such a $w$ can be obtained by appending the $\alpha_n$ copies of $n$ to $w'$ and then swapping the leftmost $i$ copies of $n$ with $i$ letters from $u$ and the remaining $\alpha_n - i$ copies of $n$ with the first $\alpha_n - i$ letters of $v$.

Since, by Lemma 4.3

$$\sum_{i=0}^{\min\{|u|, \alpha_n\}} (|u|_i) \leq \binom{|u| + \alpha_n}{\alpha_n} \leq \binom{\alpha_n + \alpha_{n-1} + \cdots + \alpha_s}{\alpha_n}$$

with equality when $\alpha_n = 1$, we have

$$s(B_1, \ldots, B_k) \leq \binom{\alpha_n + \alpha_{n-1} + \cdots + \alpha_s}{\alpha_n} s(B_1 \setminus \{n\}, \ldots, B_k),$$

where $s(B_1 \setminus \{n\}, \ldots, B_k)$ is the number of words in $\mathcal{R}(\alpha')$ that maximize $\text{sort}'_U$. So, inductively, we get

$$s(B_1, \ldots, B_k) \leq \binom{\alpha_n + \alpha_{n-1} + \cdots + \alpha_s}{\alpha_s, \ldots, \alpha_{n-1}, \alpha_n} s(B_2, \ldots, B_k) \leq \prod_{i=1}^{k} (m_{i})_{\alpha(B_i)} = i(B_1, \ldots, B_k).$$

Since we have equalities everywhere, $\alpha_n = 1$. We also get that $s(B_1 \setminus \{n\}, \ldots, B_k) = i(B_1 \setminus \{n\}, \ldots, B_k)$ and by the same argument, $\alpha_n = \alpha_{n-1} = \cdots = \alpha_{s+1} = 1$.

Now consider a permutation $p$ of the multiset $\{1^{\alpha_1}, 2^{\alpha_2}, \ldots, s-1^{\alpha_{s-1}}\}$ which maximizes $\text{sort}'_U$. By appending $\alpha_s$ copies of $s$ to $p$ and then swapping them with the first $\alpha_s$ letters of $p$ we get the word

$$\underbrace{ss \cdots s}_{\alpha_s} p'.$$

One can readily see that the word

$$w' = (n-1) \underbrace{ss \cdots s}_{\alpha_s} p' s(s+1)(s+2) \cdots (n-2) \in \mathcal{R}(\alpha')$$

14
maximizes $\text{sor}'_U$ over $R(\alpha')$. Also, there are exactly $\alpha_s + 1$ words $w$ in $R(\alpha)$ that maximize $\text{sor}'_U$ which can be obtained from $w'$, namely,

$$\begin{align*}
&\frac{n}{\alpha_{s-1}} ss \cdots sp's(s+1)(s+2) \cdots (n-2)(n-1), \\
&(n-1)\frac{n}{\alpha_{s-2}} ss \cdots sp's(s+1)(s+2) \cdots (n-2)s, \\
&(n-1)\frac{n}{\alpha_{s-3}} ss \cdots sp's(s+1)(s+2) \cdots (n-2)s, \\
&\cdots \\
&(n-1)\frac{n}{\alpha_{s-2}} ss \cdots sp'p's(s+1)(s+2) \cdots (n-2)s, \\
&(n-1)\frac{n}{\alpha_{s-1}} ss \cdots sp'p'p's(s+1)(s+2) \cdots (n-2)a,
\end{align*}$$

where $a$ is the first letter of $p'$. However, as we saw above, if $\text{sor}'_U$ and $\text{inv}'_U$ are equidistributed on $R(\alpha)$, each word $w'$ corresponds to exactly $\left(\frac{\alpha_n+\alpha_{n-1}+\cdots+\alpha_s}{\alpha_n}\right)$ words $w$. So,

$$\left(\frac{\alpha_n+\alpha_{n-1}+\cdots+\alpha_s}{\alpha_n}\right) = \alpha_s + 1$$

and therefore $s = n - 1$.

This proves that either $B_1 = \{n-1, n\}$ with $\alpha_n = 1$ or $B_1 = \{n\}$. Since the block is of this form, reasoning as in the proof of Proposition 4.1 one can see that

$$\sum_{w \in R(\alpha)} q^{\text{sor}'_U} w = \left(\frac{m_1}{\alpha(B_1)}\right) \left[\frac{m_1 + m_2 \cdots + m_n}{m_j}\right] \sum_{w \in R(\alpha'')} q^{\text{sor}'_U} w,$$

where $R(\alpha'')$ is the set of all permutations of the elements of $B_2, \ldots, B_k$ with the multiplicities given by $\alpha$. Since

$$\sum_{w \in R(\alpha)} q^{\text{inv}'_U} w = \left(\frac{m_1}{\alpha(B_1)}\right) \left[\frac{m_1 + m_2 \cdots + m_n}{m_j}\right] \sum_{w \in R(\alpha'')} q^{\text{inv}'_U} w,$$

we conclude that $\text{sor}'_U$ and $\text{inv}'_U$ are equidistributed on $R(\alpha'')$ and inductively, we get that each of the remaining blocks $B_2, \ldots, B_{k-1}$ has either size 1 or size 2 with the multiplicity of the largest element being 1.

This completes the proof of Theorem 2.2.

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