Research Article

Coefficient Estimates for Certain Subclasses of Biunivalent Functions Defined by Convolution

R. Vijaya,1 T. V. Sudharsan,2 and S. Sivasubramanian3

1Department of Mathematics, SDNB Vaishnav College for Women, Chrompet, Chennai, Tamil Nadu 600044, India
2Department of Mathematics, SIVET College, Gowrivakkam, Chennai, Tamil Nadu 600073, India
3Department of Mathematics, University College of Engineering Tindivanam, Anna University, Tindivanam, Tamil Nadu 604001, India

Correspondence should be addressed to S. Sivasubramanian; sivasaisastha@rediffmail.com

Received 7 July 2016; Accepted 25 October 2016

We introduce two new subclasses of the function class Σ of biunivalent functions in the open disc defined by convolution. Estimates on the coefficients |a2| and |a3| for the two subclasses are obtained. Moreover, we verify Brannan and Clunie’s conjecture |a2| ≤ √2 for our subclasses.

1. Introduction

Let A denote the class of functions of the form

\[ f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad (a_j \geq 0) \]  \hspace{1cm} (1)

which are analytic in the open disc \( \Delta = \{ z : |z| < 1 \} \) and normalized by \( f(0) = 0 \), \( f'(0) = 1 \). Let \( \mathcal{S} \) be the subclass of \( A \) consisting of univalent functions \( f(z) \) of form (1).

For \( f(z) \) defined by (1) and \( h(z) \) defined by

\[ h(z) = z + \sum_{j=2}^{\infty} h_j z^j, \quad (h_j \geq 0), \] \hspace{1cm} (2)

the Hadamard product (or convolution) of \( f \) and \( h \) is defined by

\[ (f \ast h)(z) = z + \sum_{j=2}^{\infty} a_j h_j z^j = (h \ast f)(z). \] \hspace{1cm} (3)

It is well known that every function \( f \in S \) has an inverse \( f^{-1} \) defined by

\[ f^{-1}(f(z)) = z \quad (z \in \Delta), \]
\[ f(f^{-1}(w)) = w \quad (|w| < r_0(f) : r_0(f) \geq \frac{1}{4}). \] \hspace{1cm} (4)

Indeed, the inverse function may have an analytic continuation to \( \Delta \), with

\[ f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_3^2 - 5a_2a_3 + a_4\right) w^4 + \cdots. \] \hspace{1cm} (5)

A function \( f \in \mathcal{A} \) is said to be univalent in \( \Delta \) if \( f(z) \) and \( f^{-1}(z) \) are univalent in \( \Delta \). Let \( \Sigma \) denote the class of biunivalent functions in \( \Delta \) given by (1). In 1967, Lewin [1] investigated the biunivalent function class \( \Sigma \) and showed that \( |a_2| < 1.51 \). Brannan and Clunie [2] conjectured that \( |a_2| \leq \sqrt{2} \). Netanyahu [3] introduced certain subclasses of biunivalent function class \( \Sigma \) similar to the familiar subclasses \( \mathcal{S}^*(\alpha) \) and \( \mathcal{K}(\alpha) \) of starlike and convex functions of order \( \alpha \) \((0 < \alpha \leq 1)\). Brannan and Taha [4] defined \( f \in \mathcal{A} \) in the class \( \mathcal{S}_2^*(\alpha) \) of strongly bistarlike functions of order \( \alpha \) \((0 < \alpha \leq 1)\) if each of the following conditions is satisfied:

\[ f \in \Sigma, \]
\[ \left| \arg \left( \frac{zf^{-1}(z)}{f(z)} \right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, \ z \in \Delta), \]
\[ \left| \arg \left( \frac{w f^{-1}(w)}{f(w)} \right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, \ w \in \Delta), \] \hspace{1cm} (6)
where \( g \) is as defined by (5). They also introduced the class of all bistarlike functions of order \( \beta \) defined as a function \( f \in \mathcal{A} \), which is said to be in the class \( \Delta(\beta) \) if the following conditions are satisfied:

\[
f \in \Sigma, \quad \Re \left( \frac{zf'(z)}{f(z)} \right) > \beta, \quad (z \in \Delta; \ 0 \leq \beta < 1), \tag{7}
\]

\[
\Re \left( \frac{wg'(w)}{g(w)} \right) > \beta, \quad (w \in \Delta; \ 0 \leq \beta < 1),
\]

where the function \( g \) is as defined in (5). The classes \( \Delta(\alpha) \) and \( \mathcal{K}(\alpha) \) of bistarlike functions of order \( \alpha \) and biconvex functions of order \( \alpha \), corresponding to the function classes \( \Delta^*(\alpha) \) and \( \mathcal{K}^*(\alpha) \), were introduced analogously. For each of the function classes \( \Delta(\alpha) \) and \( \mathcal{K}(\alpha) \), they found nonsharp estimates on the first two Taylor-Maclaurin coefficients \([a_k] \) for each of the following theorems (see [2, 5]). Some examples of biunivalent functions are \( z/(1 - z) \), \( (1/2) \log(1 + z)/(1 - z) \), and \( -\log(1 - z) \) (see [6]). The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients \([a_k] \) (see [7–11]), is still open ([16]). Various subclasses of biunivalent function class \( \Sigma \) were introduced and nonsharp estimates on the first two coefficients \([a_k] \) in the Taylor-Maclaurin series (1) were found in several investigations (see [7–11]).

In this present investigation, motivated by the works of Brannan and Taha [2] and Srivastava et al. [6], we introduce two new subclasses of biunivalent functions involving convolution. The first two initial coefficients of each of these new subclasses are obtained. Further, we prove that Brannan and Clunie’s conjecture is true for our subclasses.

In order to derive our main results, we have to recall the following lemmas.

**Lemma 1** (see [12]). If \( p \in \mathcal{P} \), then \( |p_k| \leq 2 \) for each \( k \), where \( \mathcal{P} \) is the family of all functions \( p(z) \) analytic in \( \Delta \) for which \( \Re[p(z)] > 0 \):

\[
p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \quad \forall z \in \Delta. \tag{8}
\]

**2. Coefficient Bounds for the Classes \( S_\Sigma(h, \alpha, \lambda) \) and \( S_\Sigma^*(h, \beta, \lambda) \)**

**Definition 2.** A function \( f(z) \) given by (1) is said to be in the class \( S_\Sigma(h, \alpha, \lambda) \), if the following conditions are satisfied:

\[
f \in \Sigma, \quad \left| \arg \left( \frac{z(f + h)'(z) + \lambda z^2 (f + h)''(z)}{(f + h)(z)} \right) \right| < \frac{\alpha \pi}{2}, \quad 0 < \alpha \leq 1, \ \lambda \geq 0, \ z \in \Delta, \tag{9}
\]

\[
\left| \arg \left( \frac{w((f + h)^{-1})'(w) + \lambda w^2 ((f + h)^{-1})''(w)}{((f + h)^{-1})(w)} \right) \right| < \frac{\alpha \pi}{2}, \quad 0 < \alpha \leq 1, \ \lambda \geq 0, \ w \in \Delta,
\]

where the function \( h(z) \) is defined by (2) and \( (f \ast h)^{-1}(w) \) is defined by

\[
(f \ast h)^{-1}(w) = w - a_2 h w^2 + \left(2a_2 h^2 - a_3 h_3 \right) w^3
\]

\[
- \left(5a_2^3 h^3 - 5a_2 h a_3 h_3 + a_4 h_4 \right) w^4
\]

\[
+ \cdots
\]

\[
((f \ast h)^{-1}(w))' = 1 - 2a_2 h w + 3 \left(2a_2^2 h^2 - a_3 h_3 \right) w^2
\]

\[
- \cdots.
\]

Clearly, \( S_\Sigma(z/(1 - z), \alpha, 0) \equiv S_\Sigma(\alpha) \), the class of all strong bistarlike functions of order \( \alpha \) introduced by Brannan and Taha [2].

**Definition 3.** A function \( f(z) \) given by (1) is said to be in the class \( S_\Sigma^*(h, \beta, \lambda) \), if the following conditions are satisfied:

\[
f \in \Sigma, \quad \left| \arg \left( \frac{z(f + h)'(z) + \lambda z^2 (f + h)''(z)}{(f + h)(z)} \right) \right| > \beta,
\]

\[
0 \leq \beta < 1, \ \lambda \geq 0, \ z \in \Delta, \tag{12}
\]

\[
\left| \arg \left( \frac{w((f + h)^{-1})'(w) + \lambda w^2 ((f + h)^{-1})''(w)}{((f + h)^{-1})(w)} \right) \right| > \beta,
\]

\[
0 \leq \beta < 1, \ \lambda \geq 0, \ w \in \Delta,
\]

where \( h(z) \) and \( (f \ast h)^{-1}(w) \) are defined, respectively, as in (2) and (10).

Clearly, \( S_\Sigma(z/(1 - z), \beta, 0) \equiv S_\Sigma^*(\beta) \), the class of all strong bistarlike functions of order \( \beta \) introduced by Brannan and Taha [2].

**Theorem 4.** Let \( f(z) \) given by (1) be in the class \( S_\Sigma(h, \alpha, \lambda) \), \( 0 < \alpha \leq 1 \) and \( \lambda \geq 0 \). Then

\[
|a_2| \leq \frac{2\alpha}{h_2 \sqrt{(\alpha + 1)(4\lambda + 1) + 4\lambda^2(\alpha - 1)}}, \tag{13}
\]

\[
|a_3| \leq \frac{4\alpha^2}{h_3 (1 + 2\lambda)^2} + \frac{\alpha}{h_3 (1 + 3\lambda)}.
\]

Further, for the choice of \( h(z) = z/(1 - z)^2 = z + \sum_{n=2}^{\infty} n w^n \), one gets

\[
|a_2| \leq \frac{\alpha}{\sqrt{(\alpha + 1)(4\lambda + 1) + 4\lambda^2(\alpha - 1)}},
\]

\[
|a_3| \leq \frac{4\alpha^2}{3 (1 + 2\lambda)^2} + \frac{\alpha}{3 (1 + 3\lambda)}, \tag{14}
\]
Proof. It follows from (9) that
\[
\frac{z (f * h)'(z) + \lambda z^2 (f * h)''(z)}{(f * h)(z)} = [p(z)]^\alpha,
\]
\[
\frac{w((f * h)^{-1})(w) + \lambda w^2 ((f * h)^{-1})''(w)}{(w)} = [q(w)]^\alpha,
\]
where \(p(z)\) and \(q(w)\) satisfy the following inequalities:
\[
\text{Re} \{p(z)\} > 0 \quad (z \in \Delta),
\]
\[
\text{Re} \{q(w)\} > 0 \quad (w \in \Delta).
\]
Furthermore, the functions \(p(z)\) and \(q(w)\) have the forms
\[
p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots,
\]
\[
q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots.
\]
Now, equating the coefficients in (15), we get
\[
(1 + 2\lambda) a_2 h_2 = \alpha p_1
\]
\[
2(1 + 3\lambda) a_3 h_3 = p_2 \alpha + \frac{\alpha (\alpha - 1)}{2} p_1^2 + \frac{\alpha^2 p_1^3}{(1 + 2\lambda)}
\]
\[
- (1 + 2\lambda) a_1 h_2 = \alpha q_1,
\]
\[
2(1 + 3\lambda) \left(2a_2^2 h_2^2 - a_3 h_3\right)
\]
\[
= q_2 \alpha + \frac{\alpha (\alpha - 1)}{2} q_1^2 + \frac{\alpha^2 q_1^3}{(1 + 2\lambda)}.
\]
From (19) and (21), we get
\[
p_1 = -q_1,
\]
\[
2(1 + 2\lambda)^2 a_2^2 h_2^2 = \alpha^2 \left(p_1^2 + q_1^2\right).
\]
Now, from (20), (22), and (24), we obtain
\[
4(1 + 3\lambda) a_2^2 h_2^2 = \alpha (p_2 + q_2) + \frac{\alpha (\alpha - 1)}{2} (p_1^2 + q_1^2)
\]
\[
+ \frac{\alpha^2 (p_1^2 + q_1^2)}{(1 + 2\lambda)}.
\]
Applying Lemma 1 for the coefficients \(p_2\) and \(q_2\), we immediately have
\[
|a_2| \leq \frac{2\alpha}{h_2 \sqrt{(\alpha + 1)(4\lambda + 1) + 4\lambda^2(1 - \alpha)}}.
\]
This gives the bound on \(|a_2|\).

Next, in order to find the bound on \(|a_3|\), by subtracting (20) from (22), we get
\[
4(1 + 3\lambda) \left(a_3 h_3 - a_2^2 h_2^2\right)
\]
\[
= \alpha (p_2 - q_2) + \frac{\alpha (\alpha - 1)}{2} (p_1^2 - q_1^2)
\]
\[
+ \frac{\alpha^2 (p_1^2 - q_1^2)}{(1 + 2\lambda)}.
\]
Upon substituting the value of \(a_2^2\) from (24) and observing that \(p_1^2 = q_1^2\), it follows that
\[
a_3 = a_2^2 + \frac{\alpha (p_2 - q_2)}{4h_3 (1 + 3\lambda)}
\]
\[
= \frac{\alpha^2 (p_1^2 + q_1^2)}{2h_3 (1 + 2\lambda)^2} + \frac{\alpha (p_2 - q_2)}{4h_3 (1 + 3\lambda)}.
\]
Applying Lemma 1 once again for the coefficients \(p_1, p_2, q_1,\) and \(q_2\), we get
\[
|a_3| \leq \frac{4\alpha^2}{h_3 (1 + 2\lambda)^2} + \frac{\alpha}{h_3 (1 + 3\lambda)}.
\]
This completes the proof. \(\square\)

Remark 5. When \(h(z) = z/(1 - z)\) and \(\lambda = 0\), in (13), we get the results obtained due to [4].

Remark 6. When \(\lambda = 0, \alpha = 1,\) and \(h_2 = 1\), we obtain Brannan and Clunie’s [2] conjecture \(|a_1| \leq \sqrt{2}\).

Theorem 7. Let \(f(z)\) given by (1) be in the class \(S_\lambda^\alpha(h, \beta, \lambda),\) \(0 \leq \beta < 1\) and \(\lambda \geq 0.\) Then
\[
|a_2| \leq \frac{1}{h_2} \sqrt{\frac{2(1 - \beta)}{(1 + 4\lambda)}},
\]
\[
|a_3| \leq \frac{1}{h_3} \left[\frac{4(1 - \beta)^2}{(1 + 2\lambda)^2} + \frac{(1 - \beta)}{(1 + 3\lambda)}\right].
\]
Further, for the choice of \(h(z) = z/(1 - z)^2 = z + \sum_{n=2}^{\infty} n z^n\), we get
\[
|a_2| \leq \sqrt{\frac{(1 - \beta)}{(1 + 4\lambda)}},
\]
\[
|a_3| \leq \frac{4(1 - \beta)^2}{(1 + 2\lambda)^2} + \frac{(1 - \beta)}{(1 + 3\lambda)}.
\]
Proof. It follows from (12) that there exist \( p(z) \) and \( q(w) \), such that
\[
\frac{z (f * h)'(z) + \lambda z^2 (f * h)''(z)}{(f * h)(z)} = \beta + (1 - \beta) p(z),
\]
and
\[
\frac{w ((f * h)^{-1})'(w) + \lambda w^2 ((f * h)^{-1})''(w)}{((f * h)^{-1})(w)} = \beta + (1 - q) q(w)
\]
where \( p(z) \) and \( q(w) \) have forms (17) and (18), respectively. Equating coefficients in (33) we obtain
\[
(1 + 2\lambda) a_2 h_2 = p_1 (1 - \beta) + q_1 (1 - \beta),
\]
\[
2 (1 + 3\lambda) a_2 h_3 = p_2 (1 - \beta) + \frac{p_1^2 (1 - \beta)^2}{1 + 2\lambda},
\]
\[
- (1 + 2\lambda) a_2 h_2 = q_1 (1 - \beta),
\]
\[
2 (1 + 3\lambda) (2a_2^2 h_2^2 - a_3 h_3) = q_2 (1 - \beta) + \frac{q_1^2 (1 - \beta)^2}{1 + 2\lambda}.
\]
From (34) and (36), we get
\[
|p_1| = -q_1.
\]
\[
2 (1 + 2\lambda)^2 a_2^2 h_2^2 = (1 - \beta)^2 (p_1^2 + q_1^2).
\]
Now from (35), (37), and (39), we obtain
\[
4 (1 + 3\lambda) a_2^2 h_2^2 = (1 - \beta) (p_2 + q_2)
\]
\[
+ \frac{(1 - \beta)^2 (p_1^2 + q_1^2)}{1 + 2\lambda}.
\]
Therefore, we have
\[
a_2^2 = \frac{(1 - \beta) (p_2 + q_2)}{2h_2^2 (1 + 4\lambda)}.
\]
Applying Lemma 1 for the coefficients \( p_1, p_2, q_1, \) and \( q_2 \), we readily get
\[
|a_2| \leq \frac{1}{h_2} \left[ \frac{4 (1 - \beta)^2}{(1 + 2\lambda)^2 + (1 - \beta)} \right]. \tag{44}
\]

Remark 8. When \( h(z) = z/(1 - z) \) and \( \lambda = 0 \) in (30), we have the following result due to [4]. The bounds are
\[
|a_2| \leq \sqrt{2 (1 - \beta)},
\]
\[
|a_3| \leq (1 - \beta) + 4 (1 - \beta)^2.
\]

Remark 9. When \( h(z) = z/(1 - z) \) and \( \lambda = 0 \) in (30), we have the following result due to [4]. The bounds are
\[
|a_2| \leq \sqrt{2 (1 - \beta)},
\]
\[
|a_3| \leq (1 - \beta) + 4 (1 - \beta)^2.
\]

Remark 10. When \( \lambda = 0, \beta = 0, \) and \( h_2 = 1 \) we obtain Brannan and Clunie's [2] conjecture \( |a_2| \leq \sqrt{2} \).

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

The work of the third author is supported by a grant from Department of Science and Technology, Government of India; vide Ref SR/FTP/MS-022/2012 under fast track scheme.

References

[1] M. Lewin, “On a coefficient problem for bi-univalent functions,” Proceedings of the American Mathematical Society, vol. 18, pp. 63–68, 1967.

[2] “Aspects of contemporary complex analysis,” in Proceedings of the NATO Advanced Study Institute, D. A. Brannan and J. G. Clunie, Eds., University of Durham, July 1979.

[3] E. Netanyahu, “The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in \( |z| < 1 \),” Archive for Rational Mechanics and Analysis, vol. 32, pp. 100–112, 1969.

[4] D. A. Brannan and T. S. Taha, “On some classes of bi-univalent,” Studia Universitatis Babeş-Bolyai Mathematica, vol. 31, no. 2, pp. 70–77, 1986.

[5] T. S. Taha, Topics in univalent functions theory [Ph.D. thesis], University of London, 1981.

[6] H. M. Srivastava, A. K. Mishra, and P. Gochurchay, “Certain subclasses of analytic and bi-univalent functions,” Applied Mathematics Letters, vol. 23, no. 10, pp. 1188–1192, 2010.

[7] R. M. El-Ashwah, “Subclasses of bi-univalent functions defined by convolution,” Journal of the Egyptian Mathematical Society, vol. 22, no. 3, pp. 348–351, 2014.
[8] B. A. Frasin and M. K. Aouf, “New subclasses of bi-univalent functions,” *Applied Mathematics Letters*, vol. 24, no. 9, pp. 1569–1573, 2011.

[9] T. Hayami and S. Owa, “Coefficient bounds for bi-univalent functions,” *Panamerican Mathematical Journal*, vol. 22, no. 4, pp. 15–26, 2012.

[10] Q.-H. Xu, Y.-C. Gui, and H. M. Srivastava, “Coefficient estimates for a certain subclass of analytic and bi-univalent functions,” *Applied Mathematics Letters*, vol. 25, no. 6, pp. 990–994, 2012.

[11] Q.-H. Xu, H.-G. Xiao, and H. M. Srivastava, “A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems,” *Applied Mathematics and Computation*, vol. 218, no. 23, pp. 11461–11465, 2012.

[12] C. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, Germany, 1975.
Submit your manuscripts at http://www.hindawi.com