Nonexistence, existence and symmetry of normalized ground states to Choquard equations with a local perturbation

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ABSTRACT
We study the Choquard equation with a local perturbation

$$-\Delta u = \lambda u + (I_\alpha * |u|^p)|u|^{p-2}u + \mu |u|^{q-2}u, \quad x \in \mathbb{R}^N$$

having prescribed mass

$$\int_{\mathbb{R}^N} |u|^2 dx = a^2.$$ 

For a $L^2$-critical or $L^2$-supercritical perturbation $\mu |u|^{q-2}u$, we prove nonexistence, existence and symmetry of normalized ground states, by using the mountain pass lemma, the Pohožaev constraint method, the Schwartz symmetrization rearrangements and some theories of polarizations. In particular, our results cover the Hardy-Littlewood-Sobolev upper critical exponent case $p = (N + \alpha)/(N - 2)$ for $N \geq 3$. Our results are a nonlocal counterpart of the results in Li [Studies of normalized solutions to Schrödinger equations with Sobolev critical exponent and combined nonlinearities. 2021 Apr 28, arXiv:2104.12997v2]; Soave [Normalized ground states for the NLS equation with combined nonlinearities: the Sobolev critical case. J Funct Anal. 2020;279(6):Article 108610]; Wei and Wu [Normalized solutions for Schrödinger equations with critical Sobolev exponent and mixed nonlinearities. 2021. arXiv:2102.04030v1].

1. Introduction and main results
We consider the equation

$$-\Delta u = \lambda u + (I_\alpha * |u|^p)|u|^{p-2}u + \mu |u|^{q-2}u, \quad x \in \mathbb{R}^N,$$

where $N \geq 1$, $\alpha \in (0, N)$, $I_\alpha$ is the Riesz potential defined for every $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = \frac{A_\alpha(N)}{|x|^{N-\alpha}}, \quad A_\alpha(N) = \frac{\Gamma \left( \frac{N-\alpha}{2} \right)}{\Gamma \left( \frac{q}{2} \right) \pi^{N/2} 2^\alpha}$$

with $\Gamma$ denoting the Gamma function (see [1, p. 19]), $\lambda$ and $\mu$ are real numbers, $p$ and $q$ will be defined later. The Equation (1) is usually called the nonlinear Choquard equation.
For the physical case $N = 3$, $p = 2$, $\alpha = 2$ and $\mu = 0$, (1) was investigated by Pekar in [2] to study the quantum theory of a polaron at rest. In [3], Choquard applied it as an approximation to Hartree-Fock theory of one component plasma. It also arises in multiple particles systems [4] and quantum mechanics [5].

When looking for solutions to (1), a possible choice is to fix $\lambda < 0$ and to search for solutions to (1) as critical points of the action functional

$$ J(u) := \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla u|^2 - \frac{\lambda}{2} |u|^2 - \frac{1}{2p} (I_\alpha * |u|^p) |u|^p - \frac{\mu}{q} |u|^q \right), $$

see for example [6–9] and the references therein.

Alternatively, from a physical point of view, it is interesting to find solutions of (1) having prescribed mass

$$ \int_{\mathbb{R}^N} |u|^2 = a^2. \quad (3) $$

In this direction, define on $H^1(\mathbb{R}^N, \mathbb{C})$ the energy functional

$$ E(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p - \frac{\mu}{q} \int_{\mathbb{R}^N} |u|^q. $$

It is standard to check that $E \in C^1$ under some assumptions on $p$ and $q$ and a critical point of $E$ constrained to

$$ S(a) := \left\{ u \in H^1(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} |u|^2 = a^2 \right\} $$
gives rise to a solution to (1), satisfying (3). Such solution is usually called a normalized solution of (1). In this method, the parameter $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier, which depends on the solution and is not a priori given. In this paper, we will focus on the normalized ground state of (1), defined as follows:

**Definition 1.1:** We say that $u$ is a normalized ground state to (1) on $S(a)$ if

$$ E(u) = \varepsilon^a := \inf \left\{ E(v) : v \in S(a), (E|_{S(a)})'(v) = 0 \right\}. $$

The set of the normalized ground states will be denoted by $\mathcal{G}$.

When studying normalized solutions of the Choquard equation, three exponents play an important role: the Hardy-Littlewood-Sobolev upper critical exponent $\tilde{p}$, the Hardy-Littlewood-Sobolev lower critical exponent $p$ and the $L^2$-critical exponent $p^*$ defined by

$$ \tilde{p} := \begin{cases} \infty, & N = 1, 2, \\ N + \alpha, & N \geq 3, \end{cases} \quad p := \frac{N + \alpha}{N}, \quad p^* := 1 + \frac{2 + \alpha}{N}. $$

Recently, researchers pay much attention to the normalized solutions to the Choquard equation (1) for the case $\mu = 0$, i.e.

$$ -\Delta u = \lambda u + (I_\alpha * |u|^p) |u|^{p-2} u, \quad x \in \mathbb{R}^N. \quad (4) $$

For $\tilde{p} < p < p^*$, Ye [10] obtained a normalized ground state to (4) by considering the minimizer of $E$ constrained on $S(a)$. For $p^* < p < \tilde{p}$, the functional $E$ is no longer bounded.
from below on $S(a)$, Luo [11] obtained a normalized ground state to (4) by considering the minimizer of $E$ constrained on $\mathcal{P}$ defined as in (9). For $p = p^*$, by scaling invariance, the result is delicate, see [10,12]. In the case $N \geq 3$, Li and Ye in [13] considered the general equation

$$- \Delta u = \lambda u + (I_\alpha * F(u))f(u), \quad x \in \mathbb{R}^N$$  \hspace{1cm} (5)

under a set of assumptions on $f$, which when $f$ takes the special form $f(s) = C_1 |s|^{r-2}s + C_2 |s|^{p-2}s$ requires that $p^* < r \leq p < \tilde{p}$. Bartsch et al. [14] further considered the existence of a normalized ground state and the existence of infinitely many normalized solutions to (5) in all dimensions $N \geq 1$. Recently, Yuan et al. [15] reconsidered (5) with more general $f \in C(\mathbb{R}, \mathbb{R})$. When $p = p^*$ and $2 < q < q^* := 2 + 4/N$, [16] considered the existence and orbital stability of the normalized ground state to (1). Most existing results considered similar equations to (1) with one positive nonlinearity and one negative nonlinearity, see [17–20] for the study of the Schrödinger-Poisson system.

When $N \geq 3$, to our knowledge, there are no papers considering the normalized solutions to the Choquard equation with the Hardy-Littlewood-Sobolev upper critical exponent $p = \tilde{p}$. By Moroz and Van Schaftingen [21], for fixed $\lambda < 0$, (4) has no solutions in $H^1(\mathbb{R}^N)$ under the range $p \geq \tilde{p}$. However, the equation

$$- \Delta u = (I_\alpha * |u|^{\tilde{p}})|u|^{\tilde{p}-2}u, \quad x \in \mathbb{R}^N$$  \hspace{1cm} (6)

has solutions in $D^{1,2}(\mathbb{R}^N)$, see [22]. So it is interesting to study the existence of normalized solutions to (4) with $p = \tilde{p}$ under a local perturbation $\mu |u|^q - 2u$, namely equation (1). In this paper, we will give an affirmative answer to the problem. We should point out that most of the citations concern the real case, while in this paper, we consider (1) in the complex value setting ($u : \mathbb{R}^N \to \mathbb{C}$).

Now, we present our first main result.

**Theorem 1.2:** Assume $N \geq 1$, $\alpha \in (0, N)$, $a > 0$, $\mu > 0$, $q^* = 2 + 4/N$,

$$q^* \leq q < 2^* := \begin{cases} \infty, & N = 1, 2, \\ \frac{2N}{N-2}, & N \geq 3, \end{cases} \quad p^* \leq p \begin{cases} < \tilde{p}, & N = 1, 2, \\ \leq \tilde{p}, & N \geq 3. \end{cases}$$

If $q = q^*$, we further assume that $\mu a^{4/N} < (a_N^*)^{4/N}$, where $a_N^*$ is defined in (10). Then the Equation (1) has a mountain pass type normalized ground state, $c^g > 0$ if $p < \tilde{p}$, and

$$0 < c^g < \frac{2 + \alpha}{2(N + \alpha)} \frac{N + \alpha}{S_{\alpha}^{N+\alpha}}$$

if $p = \tilde{p}$, where $S_\alpha$ is defined in (12). Moreover, every $u \in G$ solves (1) with some $\lambda = \lambda(u) < 0$. 
Remark 1.3: Recently, Yang [23] considered the fractional equation
\[ (-\Delta)^\sigma u = \lambda u + |u|^{q-2}u + \mu (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N \]  \tag{7}
with \( N \geq 2, \sigma \in (0, 1), \mu > 0 \) and \( \alpha \in (N-2\sigma, N) \). Under the assumptions
\[ 2 + \frac{4\sigma}{N} < q < \frac{2N}{N-2\sigma} \quad \text{and} \quad 1 + \frac{2\sigma + \alpha}{N} \leq p \leq \frac{q}{2} + \frac{\alpha}{N}, \]
they obtained a mountain pass type positive radial normalized ground state to (7). Note that the case \( p > \frac{q}{2} + \frac{\alpha}{N} \) was left in [23]. In this paper, by using the Schwartz symmetrization rearrangements, we can complement the interval for \( \sigma = 1 \). Note that our methods are also applied to Equation (7) with \( \sigma \in (0, 1) \).

Our second main result is about the positivity and radial symmetry of the normalized ground states.

**Theorem 1.4:** Assume the conditions in Theorem 1.2 hold. Let \( u \) be a normalized ground state to (1) on \( S(a) \), then
\begin{enumerate}
  \item \( |u| > 0 \) is a normalized ground state to (1);
  \item there exist \( x_0 \in \mathbb{R}^N \) and a non-increasing positive function \( v : (0, \infty) \to \mathbb{R} \) such that \( |u| = v(|x - x_0|) \) for almost every \( x \in \mathbb{R}^N \);
  \item \( u = e^{i\theta}|u| \) for some \( \theta \in \mathbb{R} \).
\end{enumerate}

Now we outline the methods used in this paper to prove Theorems 1.2 and 1.4. For the interaction of \((I_\alpha * |u|^p)|u|^{p-2}u \) and \(|u|^{q-2}u \), and the inequality for the Schwartz symmetrization rearrangements [24, Theorem 3.7]
\[ \int_{\mathbb{R}^N} (I_\alpha * (|u|^*)^p)(|u|^*)^p \geq \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p, \]
where \( |u|^* \) is the Schwartz symmetrization rearrangement of \(|u| \), the methods used in [11,17] or [25] can not solve our problems for the optimal range of parameters. In this paper, we combine the methods used in [26,27] (see also [7,28]) and [25](see also [47]) to prove Theorems 1.2 and 1.4. Using this method, we can treat the existence and symmetry of the normalized ground states to (1) simultaneously. Precisely, we first use the mountain pass lemma to obtain a Palais-Smale sequence \( \{u_n\} \) of \( E \) on \( S(a) \cap H^1_r(\mathbb{R}^N) \) with \( P(u_n) \to 0 \) and \( E(u_n) \to c^{mp}_r \) as \( n \to \infty \), where
\[ P(u) := \int_{\mathbb{R}^N} |\nabla u|^2 - \eta_p \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p - \mu_\gamma q \int_{\mathbb{R}^N} |u|^q, \]
\[ \eta_p := \frac{N}{2} - \frac{N + \alpha}{2p}, \quad \gamma_q := \frac{N}{2} - \frac{N}{q} \]
and \( c^{mp}_r \) is defined in (30). Secondly, by using the Pohožaev constraint method and the Schwartz symmetrization rearrangements, we can show that \( c^{mp}_r = c^{mp} = c^{po} \), where
\[ c^{po} := \inf_{u \in \mathcal{P}} E(u), \quad \mathcal{P} := \{ u \in S(a) : P(u) = 0 \} \]
and \( c^{mp} \) is defined in (33). Thirdly, by using the radial symmetry of \( \{u_n\} \), the bounds of \( c^{po} \), and the relationship of \( c^{po} \) and \( c^8 \), we can show that \( \{u_n\} \) converges to a normalized
ground state of (1). Lastly, by associating any normalized ground state \( u \) to a special path in \( \Gamma \) defined in (34), and using \( c^{mp} = c^{po} \) and the theories of polarizations, we can obtain the radial symmetry of \(|u|\).

By using the methods used in [29], we can obtain the following nonexistence result, see [30] for a different proof.

**Theorem 1.5:** Let \( N \geq 1 \), \( \alpha \in (0, N) \), \( a > 0 \), \( \mu > 0 \), \( q = q^* = 2 + 4/N \),

\[
p^* < p \left\{ \begin{array}{ll}
< \tilde{p}, & N = 1, 2, \\
\leq \tilde{p}, & N \geq 3,
\end{array} \right.
\]

and \( \mu a^{4/N} \geq (a_N^*)^{4/N} \) with \( a_N^* \) defined in (10). Then \( c^{po} = 0 \) and thus \( c^{po} \) can not be attained and (1) has no normalized ground states, where \( c^{po} \) is defined in (9).

**Remark 1.6:** The proof of Theorem 1.5 can be done by modifying the proof of Theorem 1.5 in [29] done to the Schrödinger equation, so we omit the proof here. The most difference is in Case 2 of Lemma 3.2 in [29] choosing \( \tilde{u} \) such that

\[
\| \tilde{u} \|_2^2 = a^2 \quad \text{and} \quad \| \tilde{u} \|_{q^*} = \left( \int_{\mathbb{R}^N} (I_\alpha * |\tilde{u}|^p) |\tilde{u}|^{p^*_q} \right)^{1/p^*_q},
\]

where \( \eta_p \) is defined in (8).

This paper is organized as follows. In Section 2, we cite some preliminaries. Sections 3 and 4 are devoted to the proof of Theorems 1.2 and 1.4, respectively.

**Notation:** In this paper, it is understood that all functions, unless otherwise stated, are complex valued, but for simplicity we write \( L^t(\mathbb{R}^N), H^1(\mathbb{R}^N), D^{1,2}(\mathbb{R}^N), \ldots \). For \( 1 \leq t < \infty \), \( L^t(\mathbb{R}^N) \) is the usual Lebesgue space endowed with the norm \( \| u \|_t := \int_{\mathbb{R}^N} |u|^t \), \( H^1(\mathbb{R}^N) \) is the usual Sobolev space endowed with the norm

\[
\| u \|_2^2 := \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2),
\]

and \( D^{1,2}(\mathbb{R}^N) := \{ u \in L^{2^*}(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N) \} \). \( H^1_1(\mathbb{R}^N) \) denotes the subspace of functions in \( H^1(\mathbb{R}^N) \) which are radially symmetric with respect to zero. \( S_{a,r} := S(a) \cap H^1_1(\mathbb{R}^N) \), \( C, C_1, C_2, \ldots \) denote positive constants, whose values can change from line to line.

**2. Preliminaries**

The following Gagliardo-Nirenberg inequality can be found in [31].

**Lemma 2.1:** Let \( N \geq 1 \) and \( 2 < q < 2^* \), then the following sharp Gagliardo-Nirenberg inequality

\[
\| u \|_q \leq C_{N,q}\| u \|_2^{1-\gamma_q}\| \nabla u \|_2^{\gamma_q}
\]

holds for any \( u \in H^1(\mathbb{R}^N) \), where \( \gamma_q \) is defined in (8), the sharp constant \( C_{N,q} \) is

\[
C_{N,q}^q = \frac{2q}{2N + (2-N)q} \left( \frac{2N + (2-N)q}{N(q-2)} \right)^{\frac{N(q-2)}{4}} \frac{1}{\| Q_q \|_2^{q-2}}
\]
and $Q_q$ is the unique positive radial solution of equation

$$-\Delta Q + Q = |Q|^{q-2}Q.$$  

In the special case $q = 2 + 4/N$, $C_{N,q}^q = q/(2\|Q_q\|_2^{4/N})$, or equivalently,

$$\|Q_q\|_2 = \left(\frac{q}{2C_{N,q}^q}\right)^{N/4} =: a_N^q.$$  

(10)

The following well-known Hardy-Littlewood-Sobolev inequality can be found in [24].

**Lemma 2.2:** Let $N \geq 1$, $p, r > 1$ and $0 < \alpha < N$ with $1/p + (N - \alpha)/N + 1/r = 2$. Let $u \in L^p(\mathbb{R}^N)$ and $v \in L^r(\mathbb{R}^N)$. Then there exists a sharp constant $C(N, \alpha, p)$, independent of $u$ and $v$, such that

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u(x)v(y)}{|x - y|^{N-\alpha}} \, dx \, dy \right| \leq C(N, \alpha, p) \|u\|_p \|v\|_r.$$  

If $p = r = \frac{2N}{N+\alpha}$, then

$$C(N, \alpha, p) = C_{\alpha}(N) = \pi^{\frac{N-\alpha}{2}} \frac{\Gamma \left(\frac{\alpha}{2}\right)}{\Gamma \left(\frac{N+\alpha}{2}\right)} \left\{ \frac{\Gamma \left(\frac{N}{2}\right)}{\Gamma(N)} \right\}^{-\frac{\alpha}{N}}.$$  

**Remark 2.3:**

1. By the Hardy-Littlewood-Sobolev inequality above, for any $v \in L^s(\mathbb{R}^N)$ with $s \in (1, N/\alpha)$, $I_\alpha \ast v \in L^{\frac{N\alpha}{N-\alpha}}(\mathbb{R}^N)$ and

$$\|I_\alpha \ast v\|_{L^{\frac{N\alpha}{N-\alpha}}} \leq C\|v\|_{L^s},$$

where $C > 0$ is a constant depending only on $N$, $\alpha$ and $s$.

2. By the Hardy-Littlewood-Sobolev inequality above and the Sobolev embedding theorem, we obtain

$$\int_{\mathbb{R}^N} (I_\alpha \ast |u|^p)|u|^p \leq C \left(\int_{\mathbb{R}^N} |u|^{2Np/(N+\alpha)}\right)^{1+\alpha/N} \leq C\|u\|_{H^1(\mathbb{R}^N)}^{2p}$$

(11)

for any $p \in [1 + \alpha/N, (N + \alpha)/(N - 2)]$ if $N \geq 3$ and $p \in [1 + \alpha/N, +\infty)$ if $N = 1, 2$, where $C > 0$ is a constant depending only on $N, \alpha$ and $p$.

The following Gagliardo-Nirenberg inequality for the convolution problem can be found in [21,32].
Lemma 2.4: Let $N \geq 1$, $0 < \alpha < N$, $1 + \alpha/N < p < \infty$ if $N = 1, 2$, and $1 + \alpha/N < p < (N + \alpha)/(N - 2)$ if $N \geq 3$, then

$$\left( \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p \right)^{\frac{1}{p}} \leq C_{\alpha, p} \|\nabla u\|_2^{\eta_p} \|u\|_2^{1-\eta_p},$$

where $\eta_p$ is defined in (8) and the best constant $C_{\alpha, p}$ is defined by

$$(C_{\alpha, p})^{2p} = \frac{2p}{2p - Np + N + \alpha} \left( \frac{2p - Np + N + \alpha}{Np - N - \alpha} \right)^{(Np - N - \alpha)/2} \|W_p\|_2^{2-2p}$$

with $W_p$ being a radially ground state solution of the elliptic equation

$$-\Delta W + W = (I_\alpha * |W|^p)|W|^{p-2}W.$$

In particular, in the $L^2$-critical case, i.e. $p = 1 + (2 + \alpha)/N$, $(C_{\alpha, p})^{2p} = p\|W_p\|_2^{2-2p}$.

Remark 2.5: Note that $W_p$ may be not unique, but it has the same $L^2$-norm, see [32]. Hence, if we define $R_p = \|W_p\|_2$, then $R_p$ is a constant.

The following result is cited from [22].

Lemma 2.6: Let $N \geq 3$, $\alpha \in (0, N)$ and $\tilde{p} = \frac{N+\alpha}{N-2}$. Then $S_\alpha$ defined by

$$S_\alpha := \inf_{u \in D^{1,2} (\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left( \int_{\mathbb{R}^N} (I_\alpha * |u|^{\tilde{p}}) |u|^{\tilde{p}} \right)^{1/\tilde{p}}}$$

is attained by

$$U_1(x) := \frac{(N(N-2))^{\frac{N-2}{4}}}{(1 + |x|^2)^{\frac{N-2}{2}}}.$$

Moreover, $S_\alpha = \frac{S}{(A_{\alpha}(N)C_{\alpha}(N))^{1/\tilde{p}}}$, where $A_{\alpha}(N)$ is defined in (2), $C_{\alpha}(N)$ is in Lemma 2.2 and

$$S := \inf_{u \in D^{1,2} (\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left( \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}}}.$$

The following fact is used in this paper, which can be proved by modifying the methods used in [7,27]. For the readers’ convenience, we give the proof here.

Lemma 2.7: Assume that $N \geq 1$, $\alpha \in (0, N)$, $1 + \alpha/N \leq p < \infty$ if $N = 1, 2$, and $1 + \alpha/N \leq p \leq (N + \alpha)/(N - 2)$ if $N \geq 3$. Let $\{u_k\} \subset H^1(\mathbb{R}^N)$ be a sequence satisfying that $u_k \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$. Then, for any $\varphi \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (I_\alpha * |u_k|^p) |u_k|^{p-2} u_k \varphi \to \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^{p-2} u \varphi$$

as $k \to \infty$.  

Proof: We split the proof into three cases.

Case 1 ($N \geq 3$ and $p = (N + \alpha)/(N - 2)$). Up to a subsequence, $\{u_k\}$ is bounded in $H^1(\mathbb{R}^N)$, $u_k \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ and $u_k \to u$ a.e. in $\mathbb{R}^N$. By the Sobolev embedding theorem, $\{u_k\}$ is bounded in $L^2(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N)$. Therefore, the sequence $\{|u_k|^p\}$ is bounded in $L^{2N/\alpha}(\mathbb{R}^N)$, and then

$$|u_k|^p \rightharpoonup |u|^p \quad \text{weakly in } L^{2N/\alpha}(\mathbb{R}^N).$$

By the Rellich theorem, $u_k \to u$ strongly in $L^r_{\text{loc}}(\mathbb{R}^N)$ for $r \in [1, 2^*)$ and then $|u_k|^{p-2}u_k \to |u|^{p-2}u$ strongly in $L^{2N/\alpha\delta}_{\text{loc}}(\mathbb{R}^N)$ with $\delta \in ((N + \alpha)/(2N), 1)$ (see Theorem A.2 in [33]). Hence, $|u_k|^{p-2}u_k \varphi \to |u|^{p-2}u\varphi$ strongly in $L^{2N/\alpha}(\mathbb{R}^N)$ for any $\varphi \in C_c^\infty(\mathbb{R}^N)$. By Remark 2.3, we have

$$I_{\alpha} \ast (|u_k|^{p-2}u_k \varphi) \to I_{\alpha} \ast (|u|^{p-2}u\varphi)$$

strongly in $L^{2N/\alpha}(\mathbb{R}^N)$. Thus,

$$\int_{\mathbb{R}^N} (I_{\alpha} \ast |u_k|^p)|u_k|^{p-2}u_k \varphi - \int_{\mathbb{R}^N} (I_{\alpha} \ast |u|^p)|u|^{p-2}u\varphi$$

$$= \int_{\mathbb{R}^N} |u_k|^p \left( I_{\alpha} \ast (|u_k|^{p-2}u_k \varphi) \right) - \int_{\mathbb{R}^N} |u|^p \left( I_{\alpha} \ast (|u|^{p-2}u\varphi) \right)$$

$$= \int_{\mathbb{R}^N} |u_k|^p \left( I_{\alpha} \ast (|u_k|^{p-2}u_k \varphi) - I_{\alpha} \ast (|u|^{p-2}u\varphi) \right)$$

$$+ \int_{\mathbb{R}^N} (|u_k|^p - |u|^p) \left( I_{\alpha} \ast (|u|^{p-2}u\varphi) \right)$$

$$\to 0$$

as $k \to \infty$. Since $C_c^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, the proof for this case is complete.

Case 2 ($N \geq 3$ and $1 + \alpha/N \leq p < (N + \alpha)/(N - 2)$). This case was considered in Lemma 2.4 in [7], whose proof can be done just by choosing $\delta = 1$ in Case 1.

Case 3 ($N = 1, 2$). Similarly to Case 2, noting that $2^* = \infty$, we have $\{u_k\}$ is bounded in $L^t(\mathbb{R}^N)$ for $t \in [2, \infty)$ and $u_k \rightharpoonup u$ strongly in $L^t_{\text{loc}}(\mathbb{R}^N)$ for $r \in [1, \infty)$. So the rest of the proof of this case can be done by repeating word by word the proof of Case 2.

The following Pohožaev identity is cited from [6], where the proof is given for $N \geq 3$ and $\lambda > 0$ but it clearly extends to $N = 1, 2$ and $\lambda \in \mathbb{R}$.

**Lemma 2.8:** Let $N \geq 1, \alpha \in (0, N), \lambda \in \mathbb{R}, \mu \in \mathbb{R}, p \in [1 + \alpha/N, +\infty)$ and $q \in [2, +\infty)$ for $N = 1, 2$, $p \in [1 + \alpha/N, (N + \alpha)/(N - 2)]$ and $q \in [2, 2N/(N - 2)]$ for $N \geq 3$. If $u \in H^1(\mathbb{R}^N)$ is a solution to (1), then $u$ satisfies the Pohožaev identity

$$\frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 = \frac{N\lambda}{2} \int_{\mathbb{R}^N} |u|^2 + \frac{N + \alpha}{2p} \int_{\mathbb{R}^N} (I_{\alpha} \ast |u|^p)|u|^p + \frac{\mu N}{q} \int_{\mathbb{R}^N} |u|^q.$$

**Lemma 2.9:** Assume the conditions in Lemma 2.8 hold. If $u \in H^1(\mathbb{R}^N)$ is a solution to (1), then $P(u) = 0$. 
Lemma 3.1: Assume the conditions in Theorem 1.2. We have the following lemma.

Proof: Multiplying (1) by $\bar{u}$ (the complex conjugate of $u$) and integrating over $\mathbb{R}^N$, we derive

$$\int_{\mathbb{R}^N} |\nabla u|^2 = \lambda \int_{\mathbb{R}^N} |u|^2 + \int_{\mathbb{R}^N} (I_{\alpha} * |u|^p)|u|^p + \mu \int_{\mathbb{R}^N} |u|^q,$$

which combined with the Pohožaev identity from Lemma 2.8 gives that $P(u) = 0$. 

3. Proof of Theorem thm1.2

In this section, we first study the properties of $c^{p_0}$ defined in (9), and then give the proof of Theorem 1.2.

For $u \in S(a)$ and $s \in \mathbb{R}$, define

$$(s \ast u)(x) := e^{N/N s} u(e^x), x \in \mathbb{R}^N. \quad (13)$$

Then $s \ast u \in S(a)$. Consider the fiber maps

$$\Psi_u(s) := E(s \ast u) = \frac{1}{2} e^{2s} \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2p} e^{(Np-N-\alpha)s} \int_{\mathbb{R}^N} (I_{\alpha} * |u|^p)|u|^p$$

$$- \frac{\mu}{q} e^{(Np-Nq-N)s} \int_{\mathbb{R}^N} |u|^q \quad (14)$$

and

$$P(s \ast u) = e^{2s} \int_{\mathbb{R}^N} |\nabla u|^2 - \eta_p e^{(Np-N-\alpha)s} \int_{\mathbb{R}^N} (I_{\alpha} * |u|^p)|u|^p$$

$$- \mu \gamma_q e^{(Np-Nq-N)s} \int_{\mathbb{R}^N} |u|^q.$$

We have the following lemma.

Lemma 3.1: Assume the conditions in Theorem 1.2 hold. Then for every $u \in S(a)$, there exists a unique $s_u \in \mathbb{R}$ such that $P(s_u \ast u) = 0$. $s_u$ is the unique critical point of the function $\Psi_u$, and is a strict maximum point at positive level. Moreover, $P(u) \leq 0$ is equivalent to $s_u \leq 0$.

Proof: Set $P(s \ast u) = e^{2s}g_u(s)$, where

$$g_u(s) = \int_{\mathbb{R}^N} |\nabla u|^2 - \eta_p e^{(Np-N-\alpha-2)s} \int_{\mathbb{R}^N} (I_{\alpha} * |u|^p)|u|^p$$

$$- \mu \gamma_q e^{(Np-Nq-N-2)s} \int_{\mathbb{R}^N} |u|^q.$$

If $q > q^*$, we have $\frac{N}{2}q - N - 2 > 0$. If $q = q^*$ and $\mu a^{4/N} < (a_N^*)^4/N$, we have $\frac{N}{2}q - N - 2 = 0$ and by the Gagliardo-Nirenberg inequality (Lemma 2.1),

$$\mu \gamma_q \int_{\mathbb{R}^N} |u|^q \, dx \leq \mu \gamma_q C_{N,q} q^{1-q} \|\nabla u\|_2^2 < \|\nabla u\|_2^2.$$

Since $Np - N - \alpha - 2 > 0$, so in both cases, $g_u(s) > 0$ for $s \ll 0$, $g_u(s) < 0$ for $s \gg 0$, and $g_u'(s) < 0$ for $s \in \mathbb{R}$. Thus, $g_u(s)$ has a unique zero $s_u$ as well as $P(s \ast u)$. It is obvious that $P(u) \leq 0 \implies s_u \leq 0$. 

By direct calculations, we have $\Psi'(s) = P(s \ast u)$, $\lim_{s \to -\infty} \Psi_u(s) = 0$, $\Psi_u(s) > 0$ for $s \ll 0$ and $\lim_{s \to +\infty} \Psi_u(s) = -\infty$. Thus, $s_u$ is the unique critical point of $\Psi_u(s)$ and $\Psi_u(s_u) = \max_{s \in \mathbb{R}} \Psi_u(s) > 0$.

**Lemma 3.2:** Assume the conditions in Theorem 1.2 hold. Then $c^{po} > 0$.

**Proof:** By Lemma 3.1, $P \neq \emptyset$.

**Case 1** ($p \neq \tilde{p}$). For any $u \in P$, by the Gagliardo-Nirenberg inequality (Lemmas 2.1 and 2.4), we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 = \eta_p \int_{\mathbb{R}^N} (I_\alpha |u|^p) |u|^p + \mu \gamma_q \int_{\mathbb{R}^N} |u|^q$$

$$\leq \eta_p C_{\alpha,p}^2 \|u\|_2^{2p(1-\eta_p)} \|\nabla u\|_2^{2p\eta_p} + \mu \gamma_q C_{N,q}^q \|u\|_2^{q(1-\gamma_q)} \|\nabla u\|_2^{q\gamma_q}$$

$$= \mu \gamma_q C_{N,q}^q \|u\|_2^{q\gamma_q} + \eta_p C_{\alpha,q}^{2p(1-\eta_p)} \|\nabla u\|_2^{2p\eta_p}. \quad (15)$$

If $q > q^*$, then $q\gamma_q > 2$. Since $2\eta_p > 2$, (15) implies that there exists a constant $C > 0$ such that $\|\nabla u\|_2^2 \geq C$. Consequently,

$$\eta_p \int_{\mathbb{R}^N} (I_\alpha |u|^p) |u|^p + \mu \gamma_q \int_{\mathbb{R}^N} |u|^q \geq C.$$  

If $q = q^*$ and $\mu a^{4/N} < (a_N^{4/N})^q$, then $q\gamma_q = 2$ and $\mu \gamma_q C_{N,q}^q a^{q(1-\gamma_q)} < 1$. Since $2\eta_p > 2$, (15) implies that there exists a constant $C > 0$ such that $\|\nabla u\|_2^2 \geq C$. Thus, it follows from (15) that

$$\eta_p \int_{\mathbb{R}^N} (I_\alpha |u|^p) |u|^p \geq \left(1 - \mu \gamma_q C_{N,q}^q a^{q(1-\gamma_q)}\right) \|\nabla u\|_2^2$$

$$\geq C \left(1 - \mu \gamma_q C_{N,q}^q a^{q(1-\gamma_q)}\right).$$

Any way, there always exists $C_1 > 0$ such that for any $u \in P$,

$$E(u) = \left(\frac{\eta_p}{2} - \frac{1}{2\tilde{p}}\right) \int_{\mathbb{R}^N} (I_\alpha |u|^\tilde{p}) |u|^\tilde{p} + \left(\frac{\gamma_q}{2} - \frac{1}{q}\right) \mu \int_{\mathbb{R}^N} |u|^q \geq C_1, \quad (16)$$

which implies $c^{po} > 0$.

**Case 2** ($p = \tilde{p}$). Similarly to Case 1, just in (15), we estimate the term $\int_{\mathbb{R}^N} (I_\alpha |u|^\tilde{p}) |u|^\tilde{p}$ by using (12), i.e.

$$\int_{\mathbb{R}^N} (I_\alpha |u|^\tilde{p}) |u|^\tilde{p} \leq \left(\frac{\int_{\mathbb{R}^N} |\nabla u|^2}{S_\alpha}\right)^{\tilde{p}}. \quad (17)$$

**Lemma 3.3:** Assume the conditions in Theorem 1.2 hold. Then there exists $k > 0$ sufficiently small such that

$$0 < \sup_{\bar{A}_k} E < c^{po} \quad \text{and} \quad u \in \bar{A}_k \Rightarrow E(u), \ P(u) > 0,$$

where $A_k = \{u \in S(a) : \|\nabla u\|_2^2 < k\}$. 

Proof: Case 1 ($p \neq \bar{p}$). By the Gagliardo-Nirenberg inequality, we have

\[ E(u) \geq \frac{1}{2} \|\nabla u\|_2^2 - \frac{\mu}{q} C_{N,q} a^q(1-\gamma_q) \|\nabla u\|_{2}^q - \frac{1}{2p} C_{\alpha,p} a^{2p(1-\eta_p)} \|\nabla u\|_{2}^{2p\eta_p} > 0, \]

\[ P(u) \geq \|\nabla u\|_2^2 - \mu \gamma_q C_{N,q} a^q(1-\gamma_q) \|\nabla u\|_{2}^q - \eta_p C_{\alpha,p} a^{2p(1-\eta_p)} \|\nabla u\|_{2}^{2p\eta_p} > 0, \]

if $u \in \mathcal{A}_k$ with $k$ small enough, see the proof of Lemma 3.2 for more details. If necessary replacing $k$ with a smaller quantity, recalling that $c^{p_0} > 0$ by Lemma 3.2, we also have

\[ E(u) \leq \frac{1}{2} \|\nabla u\|_2^2 < c^{p_0}. \]

Case 2 ($p = \bar{p}$). Similarly to Case 1, just in (18), we estimate the term $\int_{\mathbb{R}^N} (I_* |u|^{|\bar{p}}|u|^{\bar{p}}$ by using (17).

Lemma 3.4: Let $N \geq 3$, $\alpha \in (0, N)$, $a > 0$, $\mu > 0$, $p = \bar{p}$ and $q^* \leq q < 2^*$. If $q = q^*$, we further assume that $\mu a^{4/N} < (a^*_N)^{4/N}$. Then

\[ c^{p_0} < \frac{2 + \alpha}{2(N + \alpha)} S_\alpha^N. \]

Proof: Let $U_1(x)$ be the extremal function of $S_\alpha$ defined in Lemma 2.6, and $\varphi(x) \in C^\infty_0(\mathbb{R}^N)$ be a cut off function satisfying: (a) $0 \leq \varphi(x) \leq 1$ for any $x \in \mathbb{R}^N$; (b) $\varphi(x) \equiv 1$ in $B_1$; (c) $\varphi(x) \equiv 0$ in $\mathbb{R}^N \setminus B_2$. Here, $B_s$ denotes the ball in $\mathbb{R}^N$ of centre at origin and radius $s$. For any $\epsilon > 0$, we define

\[ U_\epsilon(x) := \epsilon^{-\frac{N-2}{2}} U_1(\epsilon^{-1} x) = \frac{(N(N-2)\epsilon^2)^{\frac{N-2}{4}}}{(\epsilon^2 + |x|^2)^{\frac{N-2}{2}}} \]

and

\[ u_\epsilon(x) = \varphi(x) U_\epsilon(x). \]

By Brezis and Nirenberg [34] (see also [33]), we have the following estimates:

\[ \int_{\mathbb{R}^N} |\nabla u_\epsilon|^2 = S_\alpha^N + O(\epsilon^{N-2}), \quad N \geq 3, \]

and

\[ \int_{\mathbb{R}^N} |u_\epsilon|^2 = \begin{cases} K_2 \epsilon^2 + O(\epsilon^{N-2}), & N \geq 5, \\ K_2 \epsilon^2 |\ln \epsilon| + O(\epsilon^2), & N = 4, \\ K_2 \epsilon + O(\epsilon^2), & N = 3, \end{cases} \]

where $K_2 > 0$. By direct calculations, for $t \in (2, 2^*)$, there exists $K_1 > 0$ such that

\[ \int_{\mathbb{R}^N} |u_\epsilon|^t \geq (N(N-2))^{\frac{N-2}{t}} \epsilon^{N - \frac{N-2}{t}} \int_{B_1(0)} \frac{1}{(1 + |x|^2)^{\frac{N-2}{2} t}} \, dx \]
\[
K_1 e^{N - \frac{N-2}{2} t}, \quad (N - 2) t > N,
\]
\[
K_1 e^{N - \frac{N-2}{2} | \ln \epsilon |}, \quad (N - 2) t = N,
\]
\[
K_1 e^{N-2 - \frac{1}{2} t}, \quad (N - 2) t < N.
\]

Moreover, similarly as in [22,35], by direct computations, we have
\[
\int_{\mathbb{R}^N} \left( I_{\alpha} * |u_\epsilon|^\tilde{p} \right) |u_\epsilon|^\tilde{p} \geq (A_\alpha(N)C_\alpha(N))^\frac{N}{2} S_\alpha^{\frac{N+\alpha}{2}} + O(e^{-\frac{N+\alpha}{2}}).
\]

Define \( \nu_\epsilon(x) = (a^{-1}||u_\epsilon||_2)^{\frac{N-2}{2}} u_\epsilon (a^{-1}||u_\epsilon||_2 x) \). Then
\[
\int_{\mathbb{R}^N} |\nu_\epsilon|^2 = a^2, \quad \int_{\mathbb{R}^N} |\nabla \nu_\epsilon|^2 = \int_{\mathbb{R}^N} |\nabla u_\epsilon|^2,
\]
\[
\int_{\mathbb{R}^N} (I_{\alpha} * |\nu_\epsilon|^\tilde{p}) |\nu_\epsilon|^\tilde{p} = \int_{\mathbb{R}^N} (I_{\alpha} * |u_\epsilon|^\tilde{p}) |u_\epsilon|^\tilde{p},
\]
and for \( q \in [q^*, 2^*) \),
\[
\int_{\mathbb{R}^N} |\nu_\epsilon|^q = (a^{-1}||u_\epsilon||_2)^{\frac{N-2}{2} q-N} \int_{\mathbb{R}^N} |u_\epsilon|^q
\]
\[
\geq a^{N-\frac{N-2}{2} q} ||u_\epsilon||_2^{\frac{N-2}{2} q-N} K_1 \epsilon^{N-\frac{N-2}{2} q}
\]
\[
\geq \frac{1}{2} a^{N-\frac{N-2}{2} q} K_1 K_2 \epsilon^{N-\frac{N-2}{2} q-N} \times \begin{cases} 1, & N \geq 5, \\ |\ln \epsilon|^{\frac{N-2}{4} q-N}, & N = 4, \\ \epsilon^{-\frac{N-2}{4} q}, & N = 3. \end{cases}
\]

Next we use \( \nu_\epsilon \) to estimate \( c^{p_0} \). By Lemma 3.1, there exists a unique \( s_\epsilon \) such that \( P(s_\epsilon * \nu_\epsilon) = 0 \) and \( E(s_\epsilon * \nu_\epsilon) = \max_{s \in \mathbb{R}} E(s * \nu_\epsilon) \). Thus, \( c^{p_0} \leq \max_{s \in \mathbb{R}} E(s * \nu_\epsilon) \). By direct calculations, one has
\[
E(s * \nu_\epsilon) = \frac{1}{2} e^{2s} \int_{\mathbb{R}^N} |\nabla \nu_\epsilon|^2 - \frac{1}{2p} e^{(Np-N-\alpha)\epsilon} \int_{\mathbb{R}^N} (I_{\alpha} * |\nu_\epsilon|^\tilde{p}) |\nu_\epsilon|^\tilde{p} - \frac{\mu}{q} e^{(\frac{N}{2} q-N)\epsilon} \int_{\mathbb{R}^N} |\nu_\epsilon|^q
\]
\[
\leq \frac{1}{2} e^{2s} \left( S_\frac{N}{2} + O(e^{-N-2}) \right) - \frac{1}{2p} e^{(Np-N-\alpha)\epsilon} \left( (A_\alpha(N)C_\alpha(N))^{\frac{N}{2}} S_\alpha^{\frac{N+\alpha}{2}} + O(e^{-\frac{N+\alpha}{2}}) \right)
\]
\[
- \frac{\mu}{q} e^{(\frac{N}{2} q-N)\epsilon} \frac{1}{2} a^{N-\frac{N-2}{2} q} K_1 K_2 \epsilon^{N-\frac{N-2}{2} q-N} \times \begin{cases} 1, & N \geq 5, \\ |\ln \epsilon|^{\frac{N-2}{4} q-N}, & N = 4, \\ \epsilon^{-\frac{N-2}{4} q}, & N = 3. \end{cases} \quad (19)
\]

We claim that there exist \( s_0, s_1 > 0 \) independent of \( \epsilon \) such that \( s_\epsilon \in [-s_0, s_1] \) for \( \epsilon > 0 \) small. Suppose by contradiction that \( s_\epsilon \to -\infty \) or \( s_\epsilon \to +\infty \) as \( \epsilon \to 0 \). (19) implies that \( \max_{s \in \mathbb{R}} E(s * \nu_\epsilon) \leq 0 \) as \( \epsilon \to 0 \) and then \( c^{p_0} \leq 0 \), which contradicts Lemma 3.2. Thus, the claim holds.

In (19), \( O(e^{-N-2}) \) and \( O(e^{-\frac{N+\alpha}{2}}) \) can be controlled by the last term for \( \epsilon > 0 \) small enough. Hence,
\[
\max_{s \in \mathbb{R}} E(s * \nu_\epsilon) < \sup_{s \in \mathbb{R}} \left( \frac{1}{2} e^{2s} S_\frac{N}{2} - \frac{1}{2p} e^{(Np-N-\alpha)\epsilon} (A_\alpha(N)C_\alpha(N))^{\frac{N}{2}} S_\alpha^{\frac{N+\alpha}{2}} \right)
\]
By Lemma 3.5, we have an alternative: either (i)

\begin{align*}
\text{Proof:}

\text{Lemma 3.6: Assume the conditions in Theorem 1.2 hold. If } u \in P \text{ such that } E(u) = c^0, \text{ then } u \text{ satisfies the Equation (1) with some } \lambda < 0.

\text{Proof:} \text{ By Lemma 3.5, we have an alternative: either (i) } P'(u) \text{ and } u \text{ are linearly dependent, or (ii) there exist } \lambda \text{ and } \eta \text{ such that } u \text{ satisfies}

\begin{align*}
-\Delta u - (I_{\alpha} \ast |u|^p)|u|^{p-2}u \leq -\mu |u|^{q-2}u u
\end{align*}

\begin{align*}
= \lambda u + \eta [-2\Delta u - 2\eta p (I_{\alpha} \ast |u|^p)|u|^{p-2}u - \mu q \gamma_q |u|^{q-2}u].
\end{align*}

(20)

If (i) holds, then \( u \) satisfies

\begin{align*}
-2\Delta u - 2\eta p (I_{\alpha} \ast |u|^p)|u|^{p-2}u - \mu q \gamma_q |u|^{q-2}u = \xi u
\end{align*}

for some \( \xi \). Similarly to the definition of \( P(u) \) in Lemma 2.9, we have

\begin{align*}
2 \int_{\mathbb{R}^N} |\nabla u|^2 - 2\eta p^2 \int_{\mathbb{R}^N} (I_{\alpha} \ast |u|^p)|u|^p - \mu q \gamma_q^2 \int_{\mathbb{R}^N} |u|^q = 0,
\end{align*}

which combined with \( P(u) = 0 \) gives that

\begin{align*}
2\eta p (p\eta p - 1) \int_{\mathbb{R}^N} (I_{\alpha} \ast |u|^p)|u|^p = \mu \gamma_q (2 - q \gamma_q) \int_{\mathbb{R}^N} |u|^q.
\end{align*}

Since \( \eta_p > 0, p\eta_p - 1 > 0, \mu > 0, \gamma_q > 0 \) and \( (2 - q \gamma_q) \leq 0 \), we obtain that

\begin{align*}
\int_{\mathbb{R}^N} (I_{\alpha} \ast |u|^p)|u|^p \leq 0.
\end{align*}

That is an contradiction. This implies that (i) does not occur and (ii) is true. Rewrite (20) in the following form

\begin{align*}
-(1 - 2\eta)\Delta u = \lambda u + (1 - \eta 2\eta p) (I_{\alpha} \ast |u|^p)|u|^{p-2}u + \mu (1 - \eta q \gamma_q) |u|^{q-2}u.
\end{align*}

Next we show \( \eta = 0 \). Similarly to the definition of \( P(u) \) (see Lemma 2.9), we have

\begin{align*}
(1 - 2\eta) \int_{\mathbb{R}^N} |\nabla u|^2 - (1 - \eta 2\eta p) \eta p \int_{\mathbb{R}^N} (I_{\alpha} \ast |u|^p)|u|^p - \mu (1 - \eta q \gamma_q) \gamma q \int_{\mathbb{R}^N} |u|^q = 0,
\end{align*}

\begin{align*}
\leq \frac{2 + \alpha}{2(N + \alpha)} S_{\alpha}^{N+\alpha}.
\end{align*}

The proof is complete.  \( \blacksquare \)
which combined with $P(u) = 0$ gives that
\[ \eta \left( 2 \int_{\mathbb{R}^N} |\nabla u|^2 - 2p\eta_p^2 \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p - \mu q\gamma_q^2 \int_{\mathbb{R}^N} |u|^q \right) = 0. \]
If $\eta \neq 0$, then
\[ 2 \int_{\mathbb{R}^N} |\nabla u|^2 - 2p\eta_p^2 \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p - \mu q\gamma_q^2 \int_{\mathbb{R}^N} |u|^q = 0, \]
which combined with $P(u) = 0$ gives that
\[
\begin{cases}
\mu \gamma_q (2p\eta_p - q\gamma_q) \int_{\mathbb{R}^N} |u|^q = (2p\eta_p - 2) \int_{\mathbb{R}^N} |\nabla u|^2, \\
\eta_p (q\gamma_q - 2p\eta_p) \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p = (q\gamma_q - 2) \int_{\mathbb{R}^N} |\nabla u|^2.
\end{cases}
\]
Since $2p\eta_p - 2 > 0$, $q\gamma_q - 2 \geq 0$, $\gamma_q > 0$, $\eta_p > 0$ and $\mu > 0$, then
\[ \int_{\mathbb{R}^N} |u|^q \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \leq 0 \]
for the case $2p\eta_p \neq q\gamma_q$ and $\int_{\mathbb{R}^N} |\nabla u|^2 = 0$ for the case $2p\eta_p = q\gamma_q$. These contradict the fact that $u \in S(a)$ with $a > 0$. So $\eta = 0$.

From (20) with $\eta = 0$, $P(u) = 0$, $0 < \gamma_q < 1$, $0 < \eta_p \leq 1$ and $\mu > 0$, we obtain
\[
\lambda a^2 = \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p - \mu \int_{\mathbb{R}^N} |u|^q
\]
\[ = (\eta_p - 1) \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p + \mu (\gamma_q - 1) \int_{\mathbb{R}^N} |u|^q < 0, \]
which implies $\lambda < 0$. The proof is complete. 

**Lemma 3.7:** Assume the conditions in Theorem 1.2 hold. If $\mathcal{C} \neq \emptyset$, then $c^{\mathcal{C}}_\mathcal{P} = c^\mathcal{G}$ and $\mathcal{C} = \mathcal{G}$, where $\mathcal{C} := \{ u \in \mathcal{P} : E(u) = c^{\mathcal{C}}_\mathcal{P} \}$.

**Proof:** For any $u \in \mathcal{C}$, by Lemma 3.6, $u$ is a solution to (1) and thus $E(u) \geq c^\mathcal{G}$. So $c^{\mathcal{C}}_\mathcal{P} \geq c^\mathcal{G}$ holds. On the other hand, for any normalized solution $v$ of (1) on $S(a)$, by Lemma 2.9, $P(v) = 0$ and thus $E(v) \geq c^{\mathcal{C}}_\mathcal{P}$, which implies that the reverse inequality $c^\mathcal{G} \geq c^{\mathcal{C}}_\mathcal{P}$ holds. Hence $c^\mathcal{G} = c^{\mathcal{C}}_\mathcal{P}$ and $\mathcal{C} = \mathcal{G}$. 

**Proposition 3.8:** Assume the conditions in Theorem 1.2 hold. Let $\{u_n\} \subset S_{a,r}$ be a Palais-Smale sequence for $E|_{S(a)}$ at level $c^{\mathcal{C}}_\mathcal{P}$ with $P(u_n) \to 0$ as $n \to \infty$. If $N = 1$, we further assume that there exists $\{v_n\} \subset S_{a,r}$, with $v_n$ radially non-increasing, such that $\|v_n - u_n\| \to 0$ as $n \to \infty$. Then up to a subsequence, $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$, and $u \in S(a)$ is a radial solution to (1) with some $\lambda < 0$.

**Proof:** The proof is divided into four steps.

**Step 1.** We show $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Since $\{u_n\} \subset S(a)$, it is enough to show that $\left(\|\nabla u_n\|_2^2\right)$ is bounded.
Case $q > q^*$, it follows from $P(u_n) = o_n(1)$ and $E(u_n) = c^{po} + o_n(1)$ that

$$E(u_n) = \left( \frac{\eta_p}{2} - \frac{1}{2p} \right) \int_{\mathbb{R}^N} (I_\alpha |u_n|^p) |u_n|^p + \left( \frac{\gamma q}{2} - \frac{1}{q} \right) \mu \int_{\mathbb{R}^N} |u_n|^q + o_n(1),$$

and then

$$\left( \frac{\eta_p}{2} - \frac{1}{2p} \right) \int_{\mathbb{R}^N} (I_\alpha |u_n|^p) |u_n|^p + \left( \frac{\gamma q}{2} - \frac{1}{q} \right) \mu \int_{\mathbb{R}^N} |u_n|^q \leq C.$$ 

Using $P(u_n) = o_n(1)$ again yields that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 = \eta_p \int_{\mathbb{R}^N} (I_\alpha |u_n|^p) |u_n|^p + \mu \gamma_q \int_{\mathbb{R}^N} |u_n|^q + o_n(1) \leq C.$$

Case $q = q^*$, from the equality

$$E(u_n) = \left( \frac{\eta_p}{2} - \frac{1}{2p} \right) \int_{\mathbb{R}^N} (I_\alpha |u_n|^p) |u_n|^p + o_n(1),$$

we know that $\{\int_{\mathbb{R}^N} (I_\alpha |u_n|^p) |u_n|^p\}$ is bounded. Suppose by contradiction that $\{\|\nabla u_n\|^2_2\}$ is not bounded, then by $P(u_n) = o_n(1)$, we obtain that

$$\lim_{n \to \infty} \frac{\mu \gamma_q \int_{\mathbb{R}^N} |u_n|^q}{\int_{\mathbb{R}^N} |\nabla u_n|^2} = 1,$$

which contradicts the fact

$$\mu \gamma_q \int_{\mathbb{R}^N} |u_n|^q \leq \mu \gamma_q C_{N,q} q^q (1 - \gamma_q) \|\nabla u_n\|^2_2 < \|\nabla u_n\|^2_2.$$

Hence, $\{\|\nabla u_n\|^2_2\}$ is bounded.

If $N \geq 2$, then there exists $u \in H^1_0(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, $u_n \to u$ strongly in $L^t(\mathbb{R}^N)$ with $t \in (2, 2^*)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$.

If $N = 1$, then $\{v_n\}$ is bounded in $H^1_0(\mathbb{R}^N)$ as well and there exists $u \in H^1_0(\mathbb{R}^N)$ such that $v_n \rightharpoonup u$ strongly in $L^t(\mathbb{R}^N)$ with $t \in (2, 2^*)$ by Proposition 1.7.1 in [37]. So up to a subsequence, $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, $u_n \to u$ strongly in $L^t(\mathbb{R}^N)$ with $t \in (2, 2^*)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$.

**Step 2.** We claim that $u \not= 0$. Suppose by contradiction that $u \equiv 0$.

Case $p < \tilde{p}$. By the assumptions on $p$ and $q$, we have

$$E(u_n) = \left( \frac{\eta_p}{2} - \frac{1}{2p} \right) \int_{\mathbb{R}^N} (I_\alpha |u_n|^p) |u_n|^p + \left( \frac{\gamma q}{2} - \frac{1}{q} \right) \mu \int_{\mathbb{R}^N} |u_n|^q + o_n(1)$$

$$= o_n(1),$$

which contradicts $E(u_n) \to c^{po} > 0$.

Case $p = \tilde{p}$. By using $E(u_n) = c^{po} + o_n(1)$, $P(u_n) = o_n(1)$, $\int_{\mathbb{R}^N} |u_n|^q = o_n(1)$ and (17), we get that

$$E(u_n) = \left( \frac{1}{2} - \frac{1}{2p\eta_p} \right) \int_{\mathbb{R}^N} |\nabla u_n|^2 + o_n(1)$$

and

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 = \eta_p \int_{\mathbb{R}^N} (I_\alpha |u_n|^p) |u_n|^p + o_n(1)$$

respectively.
and the boundedness of $c_p^0$ for any $\eta$.

Consequently,

$$c_p^0 = \lim_{n \to \infty} \left\{ \left( \frac{1}{2} - \frac{1}{2p\eta_p} \right) \int_{\mathbb{R}^N} |\nabla u_n|^2 + o_n(1) \right\} \geq \frac{2 + \alpha}{2(N + \alpha) S_{\alpha}^{N+\alpha}},$$

which contradicts Lemma 3.4. So $u \neq 0$.

**Step 3. We show $u$ is a solution to (1) with some $\lambda < 0$.** Since $\{u_n\}$ is a Palais-Smale sequence of $E|_{S(\alpha)}$, by the Lagrange multipliers rule, there exists $\lambda_n \in \mathbb{R}$ such that

$$\text{Re} \int_{\mathbb{R}^N} \left( \nabla u_n \cdot \nabla \bar{\varphi} - \lambda_n u_n \bar{\varphi} - (I_\alpha * |u_n|^p)|u_n|^{p-2}u_n \bar{\varphi} - \mu |u_n|^{q-2}u_n \bar{\varphi} \right) = o_n(1) \|\varphi\|$$

for every $\varphi \in H^1(\mathbb{R}^N)$, where Re stands for the real part. The choice $\varphi = u_n$ provides

$$\lambda_n a^2 = \int_{\mathbb{R}^N} |\nabla u_n|^2 - \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p - \mu \int_{\mathbb{R}^N} |u_n|^q + o_n(1)$$

(24)

and the boundedness of $\{u_n\}$ in $H^1(\mathbb{R}^N)$ implies that $\lambda_n$ is bounded as well; thus, up to a subsequence $\lambda_n \to \lambda \in \mathbb{R}$. Furthermore, by using $P(u_n) = o_n(1)$, (24), $\mu > 0$, $\eta_p \in (0, 1]$, $\gamma_q \in (0, 1)$ and $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, we have

$$-\lambda_n a^2 = (1 - \eta_p) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p + \mu (1 - \gamma_q) \int_{\mathbb{R}^N} |u_n|^q + o_n(1)$$

and then

$$-\lambda a^2 \geq (1 - \eta_p) \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p + \mu (1 - \gamma_q) \int_{\mathbb{R}^N} |u|^q > 0,$$

which implies that $\lambda < 0$. By using (23) and Lemma 2.7, we obtain that

$$\text{Re} \int_{\mathbb{R}^N} \left( \nabla u \cdot \nabla \bar{\varphi} - \lambda u \bar{\varphi} - (I_\alpha * |u|^p)|u|^{p-2}u \bar{\varphi} - \mu |u|^{q-2}u \bar{\varphi} \right) = \lim_{n \to \infty} \text{Re} \int_{\mathbb{R}^N} \left( \nabla u_n \cdot \nabla \bar{\varphi} - \lambda_n u_n \bar{\varphi} - (I_\alpha * |u_n|^p)|u_n|^{p-2}u_n \bar{\varphi} - \mu |u_n|^{q-2}u_n \bar{\varphi} \right) = \lim_{n \to \infty} o_n(1) \|\varphi\| = 0$$

(25)

for any $\varphi \in H^1(\mathbb{R}^N)$, which implies that $u$ satisfies the equation

$$-\Delta u = \lambda u + (I_\alpha * |u|^p)|u|^{p-2}u + \mu |u|^{q-2}u.$$  

(26)

Thus, $P(u) = 0$ by Lemma 2.9.
Step 4. We show $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$.

Case $p < \bar{p}$. Choosing $\varphi = u_n - u$ in (23) and (25), and subtracting, we obtain that

$$\int_{\mathbb{R}^N} (|\nabla (u_n - u)|^2 - \lambda|u_n - u|^2) \to 0.$$ 

Since $\lambda < 0$, we have $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$.

Case $p = \bar{p}$. Set $v_n := u_n - u$. Then we have

$$\|u_n\|_2^2 = \|u\|_2^2 + \|v_n\|_2^2 + o_n(1), \quad \|\nabla u_n\|_2^2 = \|\nabla u\|_2^2 + \|\nabla v_n\|_2^2 + o_n(1),$$

$$\|u_n\|_q^q = \|u\|_q^q + \|v_n\|_q^q + o_n(1) = \|u\|_q^q + o_n(1) \quad \text{(see}[38])$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p + \int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|v_n|^p + o_n(1) \quad \text{(see}[39]),$$

which combined with $P(u_n) = o_n(1)$ and $P(u) = 0$ gives that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 = \eta_p \int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|v_n|^p + o_n(1). \quad \text{(27)}$$

Similarly to (22), we infer that

$$\liminf_{n \to \infty} \|\nabla v_n\|_2^2 \geq \frac{N+\alpha}{2}\eta_p S_\alpha \quad \text{or} \quad \liminf_{n \to \infty} \|\nabla v_n\|_2^2 = 0.$$ 

If $\liminf_{n \to \infty} \|\nabla v_n\|_2^2 \geq \frac{N+\alpha}{2}\eta_p S_\alpha$, then by $2\eta_p \geq 2, q\gamma_q \geq 2, \mu > 0$ and (27),

$$E(u_n) = \left(\frac{\eta_p}{2} - \frac{1}{2p}\right) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p + \left(\frac{\gamma_q}{2} - \frac{1}{q}\right) \mu \int_{\mathbb{R}^N} |u_n|^q + o_n(1)$$

$$\geq \left(\frac{\eta_p}{2} - \frac{1}{2p}\right) \int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|v_n|^p + o_n(1)$$

$$\geq \frac{2 + \alpha}{2(N + \alpha)} \frac{N+\alpha}{2}\eta_p S_\alpha + o_n(1),$$

which contradicts $E(u_n) = c^{p\alpha} + o_n(1)$ and Lemma 3.4. Thus $\liminf_{n \to \infty} \|\nabla v_n\|_2^2 = 0$ holds. So up to a subsequence, $\nabla u_n \to \nabla u$ in $L^2(\mathbb{R}^N)$. Hence, similarly to the case $p < \bar{p}$, we infer that $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$. The proof is complete.$\blacksquare$

Now we are ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2: Let $k > 0$ be defined by Lemma 3.3. Following the strategy in [25,26] (see also [40]), we consider the augmented functional $\tilde{E} : \mathbb{R} \times H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\tilde{E}(s,u) := E(s \star u) = \frac{1}{2} e^{2s} \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2p} e^{(Np-N-\alpha)s} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p$$
and consider the restriction $\tilde{E}|_{\mathbb{R} \times S_{a,r}}$. Note that $\tilde{E} \in C^{1}$. Denoting by $E^{c}$ the closed sub-level set $\{u \in S(a) : E(u) \leq c\}$, we introduce the mini-max class

$$\Gamma_r := \{ \gamma = (\kappa, \beta) \in C([0,1], \mathbb{R} \times S_{a,r}) : \gamma(0) \in (0, \overline{A_k}), \gamma(1) \in (0, E^0) \}$$

(29)

with associated mini-max level

$$c^{mp}_{r} := \inf_{\gamma \in \Gamma_r} \max_{(s,u) \in \gamma([0,1])} \tilde{E}(s,u).$$

(30)

Let $u \in S_{a,r}$. Since $\int_{\mathbb{R}^N} |\nabla (s \star u)|^2 \rightarrow 0^+$ as $s \rightarrow -\infty$ and $E(s \star u) \rightarrow -\infty$ as $s \rightarrow +\infty$, there exist $s_0 \ll 1$ and $s_1 \gg 1$ such that

$$\gamma_{u} : \tau \in [0,1] \mapsto (0, ((1-\tau)s_0 + \tau s_1) \star u) \in \mathbb{R} \times S_{a,r}$$

(31)

is a path in $\Gamma_r$. The continuity of $\gamma_u$ follows from the fact that

$$(s,u) \in \mathbb{R} \times H^1(\mathbb{R}^N) \mapsto (s \star u) \in H^1(\mathbb{R}^N)$$

is continuous,

see Lemma 3.5 in [41]. Hence $c^{mp}_{r}$ is well defined.

To study the value of $c^{mp}_{r}$, we also consider the mini-max level

$$c^{mp} := \inf_{\gamma \in \Gamma} \max_{(s,u) \in \gamma([0,1])} \tilde{E}(s,u)$$

(33)

with

$$\Gamma := \{ \gamma = (\kappa, \beta) \in C([0,1], \mathbb{R} \times S(a)) : \gamma(0) \in (0, \overline{A_k}), \gamma(1) \in (0, E^0) \}.$$  

(34)

Obviously, $c^{mp}_{r} \geq c^{mp}$.

For any $\gamma = (\kappa, \beta) \in \Gamma$, consider the function

$$P_{\gamma} : \tau \in [0,1] \mapsto P(\kappa(\tau) \star \beta(\tau)) \in \mathbb{R}.$$  

We have $P_{\gamma}(0) = P(\beta(0)) > 0$ by Lemma 3.3, and by Lemma 3.1, $P_{\gamma}(1) = P(\beta(1)) < 0$ since $E(\beta(1)) \leq 0$. Moreover, $P_{\gamma}$ is continuous by (32), and hence there exists $\tau_{\gamma} \in (0,1)$ such that $P_{\gamma}(\tau_{\gamma}) = 0$, namely $\kappa(\tau_{\gamma}) \star \beta(\tau_{\gamma}) \in \mathcal{P}$; this implies that

$$\max_{\gamma([0,1])} \tilde{E} \geq \tilde{E}(\gamma(\tau_{\gamma})) = E(\kappa(\tau_{\gamma}) \star \beta(\tau_{\gamma})) \geq \inf_{\mathcal{P}} E = c^{po},$$

(35)

and consequently $c^{mp} \geq c^{po}$. 
For any $u \in \mathcal{P}$, let $|u|^*$ be the Schwartz symmetrization rearrangement of $|u|$. Since $\|u^*\|_t = \|u\|_t$ with $t \in [1, \infty)$, $\|\nabla(|u|^*)\|_2 \leq \|\nabla u\|_2$ and

$$\int_{\mathbb{R}^N} (I_{\alpha} * (|u|^*)^p)(|u|^*)^p \geq \int_{\mathbb{R}^N} (I_{\alpha} * |u|^p)|u|^p,$$

we obtain that $\Psi_{|u|^*}(s) \leq \Psi_u(s)$ for any $s \in \mathbb{R}$, where $\Psi_u(s)$ is defined in (14). Let $s_n$ be defined by Lemma 3.1 be such that $P(s_n^* u) = 0$. Then

$$E(u) = \Psi_u(0) = \Psi_u(s_n) \geq \Psi_u(s_{|u|^*}) \geq \Psi_{|u|^*}(s_{|u|^*}).$$

Since $s_{|u|^*} * |u|^* \in \mathcal{P} \cap H^1_r(\mathbb{R}^N)$, we have that

$$E(u) \geq \inf_{\mathcal{P} \cap H^1_r(\mathbb{R}^N)} E(u),$$

which implies that

$$c^{po} = \inf_{\mathcal{P}} E(u) \geq \inf_{\mathcal{P} \cap H^1_r(\mathbb{R}^N)} E(u). \quad (36)$$

For any $u \in \mathcal{P} \cap H^1_r(\mathbb{R}^N)$, $\gamma_u$ defined in (31) is a path in $\Gamma_r$ with

$$E(u) = \max_{\gamma_u([0,1])} \tilde{E} \geq c_r^{mp}, \quad (37)$$

which implies

$$\inf_{\mathcal{P} \cap H^1_r(\mathbb{R}^N)} E(u) \geq c_r^{mp}.$$ 

Now, we have proved that

$$c^{po} = c^{mp} = c_r^{mp} > \sup_{(\overline{A_k \cup E^0}) \cap S_{a,r}} E = \sup_{((0, \overline{A_k \cup (0,E^0)}) \cap (\mathbb{R} \times S_{a,r}))} \tilde{E}.$$

Using the terminology of Section 5 in [42], this means that $\{\gamma([0,1]) : \gamma \in \Gamma_r\}$ is a homotopy stable family of compact subsets of $\mathbb{R} \times S_{a,r}$ with extended closed boundary $(0, \overline{A_k}) \cup (0, E^0)$, and that the super-level set $\{\tilde{E} \geq c^{po}\}$ is a dual set for $\Gamma_r$, in the sense that assumptions $(P'1)$ and $(P'2)$ in Theorem 5.2 of [42] are satisfied. Therefore, taking any minimizing sequence $\{\gamma_n = (\kappa_n, \beta_n)\} \subset \Gamma_r$ for $c^{po}$ with the property that $\kappa_n \equiv 0$ and $\beta_n(\tau) \geq 0$ a.e. in $\mathbb{R}^N$ for every $\tau \in [0,1]$ (If $N = 1$, we further replace $\gamma_n$ by $(0, (\beta_n)^*)$, see Remark 5.2 in [25]), there exists a Palais-Smale sequence $\{(s_n, w_n) \subset \mathbb{R} \times S_{a,r}\}$ for $\tilde{E}|_{\mathbb{R} \times S_{a,r}}$, at level $c^{po}$, that is, $\tilde{E}(s_n, w_n) \to c^{po}$,

$$\partial_u \tilde{E}(s_n, w_n) \to 0 \quad \text{and} \quad \|\partial_u \tilde{E}(s_n, w_n)\|_{(T_{w_n} S_{a,r})^*} \to 0 \quad \text{as} \quad n \to \infty, \quad (38)$$

with the additional property that

$$|s_n| + \text{dist}_{H^1_r(\mathbb{R}^N)}(w_n, \beta_n([0,1])) \to 0 \quad \text{as} \quad n \to \infty. \quad (39)$$

By (38), the first condition in (38) reads $P(s_n^* w_n) \to 0$, while the second condition gives that

$$e^{2s_n} \int_{\mathbb{R}^N} \nabla w_n \cdot \nabla \phi - e^{(Np - N - \alpha)s_n} \int_{\mathbb{R}^N} (I_{\alpha} * |w_n|^p)|w_n|^p - 2w_n \phi$$
for every $\phi \in T \cap S_{a,r}$. Since $\{s_n\}$ is bounded due to (39), this is equivalent to
\[
d E(s_n \ast w_n)\|s_n \ast \phi\| = o_n(1)\|\phi\| \quad \text{as } n \to \infty.
\] (40)

Let $u_n := s_n \ast w_n$. By Lemma 5.8 in [25], Equation (40) establishes that $\{u_n\} \subset S_{a,r}$ is a Palais-Smale sequence for $E|_{S_{a,r}}$ (thus a Palais-Smale sequence for $E|_{S(a)}$, since the problem is invariant under rotations) at level $c_{\text{cpo}}$, with $P(u_n) \to 0$ as $n \to \infty$. Moreover, if $N = 1$, $\{u_n\}$ also satisfies $\|u_n - v_n\| \to 0$ as $n \to \infty$ for some radially non-increasing $\{v_n\} \subset S_{a,r}$.

By Proposition 3.8, up to a subsequence, $u_n \rightharpoonup u$ strongly in $H^1(\mathbb{R}^N)$. Thus, $u \in S(a)$ is a mountain pass type normalized solution to (1) with $\lambda < 0$ and $E(u) = c_{\text{cpo}}$. By Lemma 3.7, $u$ is a normalized ground state. The proof is complete. $\blacksquare$

4. Proof of Theorem 1.4

Firstly, we study the positivity of the normalized ground states to (1). By Lemma 3.7, it is enough to prove the following fact.

**Proposition 4.1:** Assume the conditions in Theorem 1.2 hold. If $u \in \mathcal{P}$ such that $E(u) = c_{\text{cpo}}$, then $|u| \in \mathcal{P}$, $E(|u|) = c_{\text{cpo}}$ and $\|\nabla |u|\|_2 = \|\nabla u\|_2$. Moreover, $|u| > 0$ in $\mathbb{R}^N$.

**Proof:** It follows from
\[
\int_{\mathbb{R}^N} |\nabla |u||^2 \leq \int_{\mathbb{R}^N} |\nabla u|^2
\]
that $P(|u|) \leq 0$. By Lemma 3.1, there exists $s_{|u|} \leq 0$ such that $s_{|u|} \ast |u| \in \mathcal{P}$. Thus,

- \[
E(s_{|u|} \ast |u|) = \left(\frac{\eta_p}{2} - \frac{1}{2p}\right) \int_{\mathbb{R}^N} (I_\alpha \ast |s_{|u|} \ast |u|^p|) |s_{|u|} \ast |u||^p + \left(\frac{\gamma_q}{2} - \frac{1}{q}\right) \mu \int_{\mathbb{R}^N} |s_{|u|} \ast |u||^q
\]
- \[
= \left(\frac{\eta_p}{2} - \frac{1}{2p}\right) \int_{\mathbb{R}^N} (I_\alpha \ast |u|^p) |u|^p + \left(\frac{\gamma_q}{2} - \frac{1}{q}\right) \mu \int_{\mathbb{R}^N} |u|^q
\]
- \[
\leq \left(\frac{\eta_p}{2} - \frac{1}{2p}\right) \int_{\mathbb{R}^N} (I_\alpha \ast |u|^p) |u|^p + \left(\frac{\gamma_q}{2} - \frac{1}{q}\right) \mu \int_{\mathbb{R}^N} |u|^q
\]
- \[
= E(u) = c_{\text{cpo}}.
\]

By the definition of $c_{\text{cpo}}$, we have $s_{|u|} = 0$, $|u| \in \mathcal{P}$, $E(|u|) = c_{\text{cpo}}$ and $\|\nabla |u||_2 = \|\nabla u\|_2$.

By Lemma 3.6, there exists $\lambda < 0$ such that $|u|$ satisfies the equation
\[
- \Delta u = \lambda u + (I_\alpha \ast |u|^p)|u|^{p-2}u + \mu |u|^{q-2}u.
\]
Since $|u|$ is continuous by Theorem 2.1 in [6], the strong maximum principle implies that $|u| > 0$ in $\mathbb{R}^N$. $\blacksquare$

Nextly, we study the radial symmetry of the normalized ground states to (1). We follow the strategy of [27]. The argument relies on polarizations. So we first recall some theories of polarizations [21,43,44].
Assume that \( H \subset \mathbb{R}^N \) is a closed half-space and that \( \sigma_H \) is the reflection with respect to \( \partial H \). The polarization \( u^H : \mathbb{R}^N \to \mathbb{R} \) of \( u : \mathbb{R}^N \to \mathbb{R} \) is defined for \( x \in \mathbb{R}^N \) by

\[
u^H(x) = \begin{cases} \max\{u(x), u(\sigma_H(x))\}, & \text{if } x \in H, \\ \min\{u(x), u(\sigma_H(x))\}, & \text{if } x \notin H. \end{cases}
\]

We will use the following standard property of polarizations (Lemma 5.3 in [43]).

**Lemma 4.2 (Polarization and Dirichlet integrals):** If \( u \in H^1(\mathbb{R}^N) \), then \( u^H \in H^1(\mathbb{R}^N) \) and

\[
\int_{\mathbb{R}^N} |\nabla u^H|^2 = \int_{\mathbb{R}^N} |\nabla u|^2.
\]

We shall also use a polarization inequality with equality cases (Lemma 5.3 in [21]).

**Lemma 4.3 (Polarization and nonlocal integrals):** Let \( \alpha \in (0, N) \), \( u \in L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N) \) and \( H \subset \mathbb{R}^N \) be a closed half-space. If \( u \geq 0 \), then

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u(x)u(y)}{|x-y|^{N-\alpha}} \, dx \, dy \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^H(x)u^H(y)}{|x-y|^{N-\alpha}} \, dx \, dy,
\]

with equality if and only if either \( u^H = u \) or \( u^H = u \circ \sigma_H \).

The last tool that we need is a characterization of symmetric functions by polarizations (Proposition 3.15 in [44], Lemma 5.4 in [21]).

**Lemma 4.4 (Symmetry and polarization):** Assume that \( u \in L^2(\mathbb{R}^N) \) is nonnegative. There exist \( x_0 \in \mathbb{R}^N \) and a non-increasing function \( v : (0, \infty) \to \mathbb{R} \) such that for almost every \( x \in \mathbb{R}^N \), \( u(x) = v(|x - x_0|) \) if and only if for every closed half-space \( H \subset \mathbb{R}^N \), \( u^H = u \) or \( u^H = u \circ \sigma_H \).

Now we are ready to prove the radial symmetry of the positive normalized ground states to (1).

**Proposition 4.5:** Assume that the conditions in Theorem 1.2 hold. Let \( u \) be a positive normalized ground state to (1), then there exist \( x_0 \in \mathbb{R}^N \) and a non-increasing positive function \( v : (0, \infty) \to \mathbb{R} \) such that \( u(x) = v(|x - x_0|) \) for almost every \( x \in \mathbb{R}^N \).

**Proof:** By Lemma 3.7, \( E(u) = c^{po} \) and \( P(u) = 0 \). Let \( \bar{E} \) and \( \Gamma \) be defined in (28) and (34), respectively, and let \( \gamma_u(\tau) = (0, ((1 - \tau)s_0 + \tau s_1) \ast u) \in \Gamma \) be a path defined in (31). Denote \( \beta_u(\tau) = ((1 - \tau)s_0 + \tau s_1) \ast u \). Then, \( \beta_u(-s_0/(s_1 - s_0)) = u, \beta_u(\tau) \geq 0 \) for every \( \tau \in [0, 1], \bar{E}(\gamma_u(\tau)) = E(\beta_u(\tau)) < E(u) = c^{po} \) for any \( \tau \in ([0, 1] \setminus \{-s_0/(s_1 - s_0)\}) \).

For every closed half-space \( H \) define the path \( \gamma_u^H : [0, 1] \to \mathbb{R} \times S(a) \) by \( \gamma_u^H(\tau) = (0, (\beta_u(\tau))^H) \). By Lemma 4.2 and \( \|u^H\|_r = \|u\|_r \) with \( r \in [1, \infty) \), we have \( \gamma_u^H \in \)
C([0, 1], \mathbb{R} \times S(a)). By Lemmas 4.2 and 4.3, we obtain that \( \tilde{E}(\gamma_u^H(\tau)) \leq \tilde{E}(\gamma_u(\tau)) \) for every \( \tau \in [0, 1] \) and then \( \gamma_u^H(\tau) \in \Gamma \). Hence,

\[
\max_{\tau \in [0, 1]} \tilde{E}(\gamma_u^H(\tau)) \geq c^{po}.
\]

Since for every \( \tau \in ([0, 1] \setminus \{-s_0/(s_1 - s_0)\}) \),

\[
\tilde{E}(\gamma_u^H(\tau)) \leq \tilde{E}(\gamma_u(\tau)) < E(u) = c^{po},
\]

we deduce that

\[
\tilde{E}\left(\gamma_u^H\left(\frac{-s_0}{s_1 - s_0}\right)\right) = E(u^H) = c^{po}.
\]

Hence \( E(u^H) = E(u) \), which implies that

\[
\int_{\mathbb{R}^N} (I_\alpha * |u^H|^p) |u^H|^p = \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p.
\]

By Lemma 4.3, we have \( u^H = u \) or \( u^H = u \circ \sigma_H \). By Lemma 4.4, we complete the proof. \( \square \)

**Proof of Theorem 1.4:** (1) and (2) are the direct results of Lemma 3.7, Propositions 4.1 and 4.5.

Next we prove (3). We follow the arguments of Theorem 4.1 in [45]. Let \( u \in \mathcal{G} \). First note that \( e^{\theta|u|} \in \mathcal{G} \) for any \( \theta \in \mathbb{R} \). Next we show \( u = e^{\theta|u|} \) for some \( \theta \in \mathbb{R} \). Denote \( u(x) = u_1(x) + iu_2(x) \). By Lemma 3.7 and Proposition 4.1, we know \( |u| = \sqrt{u_1^2 + u_2^2} > 0 \) and \( \|\nabla |u|\| = \|\nabla u\| \). Hence,

\[
\int_{\mathbb{R}^N} \sum_{j=1}^N \frac{(u_1 \partial_j u_2 - u_2 \partial_j u_1)^2}{u_1^2 + u_2^2} = 0. \quad (41)
\]

By Lemmas 3.6 and 3.7, \( u \) satisfies (1) with some \( \lambda < 0 \). Hence, \( u_1 \) and \( u_2 \) satisfy

\[
\begin{cases}
-\Delta u_1 = \lambda u_1 + (I_\alpha * |u|^p)|u|^{p-2}u_1 + \mu |u|^{q-2}u_1, \quad x \in \mathbb{R}^N, \\
-\Delta u_2 = \lambda u_2 + (I_\alpha * |u|^p)|u|^{p-2}u_2 + \mu |u|^{q-2}u_2, \quad x \in \mathbb{R}^N.
\end{cases} \quad (42)
\]

By the regularity theory, \( u_1, u_2 \in C(\mathbb{R}^N) \). Indeed, since \( |u| \) is a solution to (1) by Lemmas 3.6, 3.7 and Proposition 4.1, it follows from the proof of Theorem 2.1 in [6] that \( I_\alpha * |u|^p \in L^\infty(\mathbb{R}^N) \) and \( |u| \in W^{2,t}_0(\mathbb{R}^N) \) for any \( t > 1 \). Hence, \( |u| \in C(\mathbb{R}^N) \) and then \( |u| \in L^\infty_{loc}(\mathbb{R}^N) \). By (42), \( u_1, u_2 \in H^1(\mathbb{R}^N) \) are solutions to the equation

\[
-\Delta w - \lambda w = f
\]

with

\[
f := (I_\alpha * |u|^p)|u|^{p-2}w + \mu |u|^{q-2}w
\]

\[
= (I_\alpha * |u|^p)|u|^{p-1} \frac{w}{|u|} + \mu |u|^{q-1} \frac{w}{|u|} \in L^\infty_{loc}(\mathbb{R}^N).
\]
So \( u_1, u_2 \in C(\mathbb{R}^N) \) by Theorem 4.13 in \([46]\).

Let \( \Omega := \{ x \in \mathbb{R}^N : u_2(x) = 0 \} \). The continuity of \( u_2 \) implies that \( \Omega \) is closed. Suppose now that \( x_0 \in \Omega \). Since \( |u(x_0)| > 0 \), there exists an open ball \( B \) with centre \( x_0 \) such that \( u_1(x) \neq 0 \) for all \( x \in B \). Hence, for \( x \in B \),

\[
\frac{(u_1 \partial_j u_2 - u_2 \partial_j u_1)^2}{u_1^2 + u_2^2} = \left[ \frac{\partial_j \left( \frac{u_2}{u_1} \right) - \frac{u_2 u_1}{u_1^2}}{u_1^2 + u_2^2} \right] \to 1, \quad j = 1, 2, \ldots, N,
\]

which combined with (41) gives that

\[
\int_B \left[ \nabla \left( \frac{u_2}{u_1} \right) \right]^2 \frac{u_1^4}{u_1^2 + u_2^2} = 0.
\]

Thus \( \nabla \left( \frac{u_2}{u_1} \right) \equiv 0 \) on \( B \) and so there exists a constant \( C \) such that \( \frac{u_2}{u_1} \equiv C \) on \( B \). Since \( x_0 \in B \) and \( u_2(x_0) = 0 \), we have that \( C = 0 \), showing that \( \Omega \) is also an open subset of \( \mathbb{R}^N \). Hence either \( u_2 \equiv 0 \) or \( u_2(x) \neq 0 \) for all \( x \in \mathbb{R}^N \).

If \( u_2 \equiv 0 \) on \( \mathbb{R}^N \), then \( |u| = |u_1| > 0 \) on \( \mathbb{R}^N \) and so \( u(x) = u_1(x) = e^{i\theta} |u| \) where \( \theta = 0 \) if \( u_1 > 0 \) and \( \theta = \pi \) if \( u_1 < 0 \) on \( \mathbb{R}^N \). Otherwise, \( u_2(x) \neq 0 \) for all \( x \in \mathbb{R}^N \). Thus,

\[
\frac{(u_1 \partial_j u_2 - u_2 \partial_j u_1)^2}{u_1^2 + u_2^2} = \left[ \frac{\partial_j \left( \frac{u_1}{u_2} \right) - \frac{u_1 u_2}{u_1^2}}{u_1^2 + u_2^2} \right] \to 0, \quad j = 1, 2, \ldots, N,
\]

for all \( x \in \mathbb{R}^N \), which combined with (41) gives that \( \nabla \left( \frac{u_1}{u_2} \right) \equiv 0 \) on \( \mathbb{R}^N \). Hence there exists a constant \( C \in \mathbb{R} \) such that \( u_1 \equiv Cu_2 \) on \( \mathbb{R}^N \). Thus, in complex notation, \( u = u_1 + i u_2 = (C + i)u_2 \) and \( |u| = |C + i| |u_2| \). Let \( \eta \in \mathbb{R} \) be such that \( C + i = |C + i| e^{i\eta} \) and let \( \psi = 0 \) if \( u_2 > 0 \) and \( \psi = \pi \) if \( u_2 < 0 \) on \( \mathbb{R}^N \). Setting \( \theta = \eta + \psi \), we have that

\[
u = (C + i)u_2 = |C + i| e^{i\eta} e^{i\psi} |u_2| = e^{i\theta} |u|.
\]

This proves (3). The proof is complete. \( \blacksquare \)

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