The Principle of Least Action as Interpreted by Nature and by the Observer

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Abstract

In this paper, we show that the difficulties of interpretation of the principle of least action concerning "final causes" or "efficient causes" are due to the existence of two different actions, the "Euler-Lagrange action" (or classical action) and the "Hamilton-Jacobi action". These two actions are solutions to the same Hamilton-Jacobi equation, but with very different initial conditions: smooth conditions for the Hamilton-Jacobi action, singular conditions for the Euler-Lagrange action. We propose a clear-cut interpretation of the principle of least action: the Hamilton-Jacobi action does not use the "final causes" and seems to be the action used by Nature: it is the ontological action; the Euler-Lagrange action uses the "final causes" and is the action used by an observer to retrospectively determine the trajectory of the particle: it is the epistemological action.

1 Introduction

In 1744, Pierre-Louis Moreau de Maupertuis (1698-1759) introduced the action and the principle of least action into classical mechanics [1]: "Nature, in the production of its effects, does so always by the simplest means [...] the path it takes is the one by which the quantity of action is the least," and in 1746, he states [2]: "This is the principle of least action, a principle so wise and so worthy of the supreme Being, and intrinsic to all natural phenomena [...] When a change occurs in Nature, the quantity of action necessary for change is the smallest possible. The quantity of action is the product of the mass of the body times its velocity and the distance it moves." Maupertuis understood that, under certain conditions, Newton’s equations imply to the minimization of a certain quantity. He dubbed this quantity as the action.
Euler [3], Lagrange [4], Hamilton [5, 7], Jacobi [8] and others, turned this principle of least action into one of the most powerful tools for discovering the laws of nature [9][10]. This principle serves to determine the equations of motion of a particle (if we minimize the trajectories) and the laws of nature (if we minimize the parameters defining the fields).

However, when applied to the study of particle trajectories, this principle has often been viewed as puzzling by many scholars, including Henri Poincaré, who was nonetheless one of its most intensive users [11]: "The very enunciation of the principle of least action is objectionable. To move from one point to another, a material molecule, acted on by no force, but compelled to move on a surface, will take as its path the geodesic line, i.e., the shortest path. This molecule seems to know the point to which we want to take it, to foresee the time it will take to reach it by such a path, and then to know how to choose the most convenient path. The enunciation of the principle presents it to us, so to speak, as a living and free entity. It is clear that it would be better to replace it by a less objectionable enunciation, one in which, as philosophers would say, final effects do not seem to be substituted for acting causes."

We will show that the difficulties of interpretation of the principle of least action concerning the "final causes" or the "efficient causes" come from the existence of two different actions: the Euler-Lagrange action (or classical action) \( S_{cl}(x, t; x_0) \), which links the initial position \( x_0 \) and its position \( x \) at time \( t \), and the Hamilton-Jacobi action \( S(x, t) \), which depends on an initial action \( S_0(x) \).

These two actions are solutions to the same Hamilton-Jacobi equation, but with very different initial conditions: smooth conditions for the Hamilton-Jacobi action, singular conditions for the Euler-Lagrange action. These initial conditions are not taken into account in the classical mechanics textbooks [10][12]. We show that they are the key to understanding the principle of least action.

In section 2, we recall the main properties of the Euler-Lagrange action and we propose a new interpretation. In section 3, we propose a novel way to look at the Hamilton-Jacobi action, in explaining the principle of least action that it satisfies. In conclusion, we respond to Poincaré by providing a clear-cut interpretation of this principle.
2 The Euler-Lagrange equation and the Euler-lagrange action

Let us consider a system evolving from the position $x_0$ at initial time to the position $x$ at time $t$ where the variable of control $u(s)$ is the velocity:

$$\frac{dx(s)}{ds} = u(s), \quad \forall s \in [0, t]$$  \hspace{1cm} (1)

$$x(0) = x_0, \quad x(t) = x.$$  \hspace{1cm} (2)

If $L(x, \dot{x}, t)$ is the Lagrangian of the system, when the two positions $x_0$ and $x$ are given, the Euler-Lagrange action $S_{cl}(x, t; x_0)$ is the function defined by:

$$S_{cl}(x, t; x_0) = \min_{u(s), 0 \leq s \leq t} \int_0^t L(x(s), u(s), s) ds,$$  \hspace{1cm} (3)

where the minimum (or more generally an extremum) is taken on the controls $u(s), s \in [0, t]$, with the state $x(s)$ given by equations (1) and (2). This is the principle of least action defined by Euler \[3\] in 1744 and Lagrange \[4\] in 1755.

The solution $(\tilde{x}(s), \tilde{u}(s))$ of (3), if the Lagrangian $L(x, \dot{x}, t)$ is twice differentiable, satisfies the Euler-Lagrange equations on the interval $[0, t]$:

$$\frac{d}{ds} \frac{\partial L}{\partial \dot{x}}(x(s), \dot{x}(s), s) - \frac{\partial L}{\partial x}(x(s), \dot{x}(s), s) = 0, \quad \forall s \in [0, t]$$  \hspace{1cm} (4)

$$x(0) = x_0, \quad x(t) = x.$$  \hspace{1cm} (5)

For a non-relativistic particle in a linear potential field $V(x) = -K \cdot x$ with the Lagrangian $L(x, \dot{x}, t) = \frac{1}{2} m \ddot{x}^2 - V(x) = \frac{1}{2} m \ddot{x}^2 + K \cdot x$, equation (4) yields $\frac{d}{ds}(m\ddot{x}(s)) - K = 0$. We successively obtain $\ddot{x}(s) = v_0 + \frac{K}{m} s$ and $x(s) = x_0 + v_0 s + \frac{K}{2m} s^2$. The initial velocity $v_0$ is defined using $x(t) = x$ (equation (5)): $v_0 = \frac{x - x_0}{t} - \frac{K}{2mt}$. Finally, the trajectory minimizing the action is $\tilde{x}(s) = x_0 + \frac{x - x_0}{t} s - \frac{K}{2mt} s^2 + \frac{K}{2m} s^2$ and the Euler-Lagrange action is equal to

$$S_{cl}(x, t; x_0) = m \left( \frac{(x - x_0)^2}{2t} + \frac{K \cdot (x + x_0)}{2} - \frac{K^2}{24tm^3} \right).$$  \hspace{1cm} (6)

Figure 1 shows different trajectories going from $x_0$ at time $s = 0$ to $x$ at final time $s = t$. The parabolic trajectory $\tilde{x}(s)$ corresponds to this which realizes the minimum in the equation (3).

Equation (3) seems to show that, among the trajectories which can reach $(x, t)$ from the initial position $x_0$, the principle of least action allows to choose the velocity at each time. In reality, the principle of least action used in equation (3) does not choose the velocity at each time $s$ between
Figure 1: Different trajectories \( x(s) \) (0 ≤ s ≤ t) between \((x_0, 0)\) and \((x, t)\) and the optimal trajectory \( \tilde{x}(s) \) with \( \tilde{v}_0 = \frac{x - x_0}{t} - \frac{Kt}{2m} \).

0 and \( t \), but only when the particle arrives at \( x \) at time \( t \). The knowledge of the velocity at each time \( s \) (0 ≤ s ≤ t) requires the resolution of the Euler-Lagrange equations (4) and (5) on the whole trajectory. In the case of a non-relativistic particle in a linear potential field, the velocity at time \( s \) (0 ≤ s ≤ t) is \( \tilde{v}(s) = \frac{x - x_0}{t} - \frac{Kt}{2m} + \frac{Ks}{m} \) with the initial velocity

\[
\tilde{v}_0 = \frac{x - x_0}{t} - \frac{Kt}{2m}.
\] (7)

Then, \( \tilde{v}_0 \) depends on the position \( x \) of the particle at the final time \( t \). This dependence of the "final causes" is general. This is the Poincaré’s main criticism of the principle of least action: "This molecule seems to know the point to which we want to take it, to foresee the time it will take to reach it by such a path, and then to know how to choose the most convenient path."

One must conclude that, without knowing the initial velocity, the Euler-Lagrange action answers a problem posed by an observer, and not by Nature: "What would be the velocity of the particle at the initial time to attained \( x \) at time \( t \)?" The resolution of this problem implies that the observer solves the Euler-Lagrange equations (4) and (5) after the observation of \( x \) at time \( t \). This is an \textit{a posteriori} point of view.
3 The Hamilton-Jacobi action

Let us now consider that an initial action $S_0(x)$ is given, then the Hamilton-Jacobi action $S(x, t)$ is the function defined by:

$$S(x, t) = \min_{x_0; u(s), 0 \leq s \leq t} \left\{ S_0(x_0) + \int_{0}^{t} L(x(s), u(s), s) \, ds \right\}$$

(8)

where the minimum is taken on all initial positions $x_0$ and on the controls $u(s), s \in [0, t]$, with the state $x(s)$ given by the equations (1) and (2).

The introduction of the Hamilton-Jacobi action highlights the importance of the initial action $S_0(x)$, while textbooks do not well differentiate these two actions.

Noting that $S_0(x_0)$ does not play a role in (8) for the minimization on $u(s)$, we obtain a new relation between the Hamilton-Jacobi action and Euler-Lagrange action:

$$S(x, t) = \min_{x_0} (S_0(x_0) + S_{cl}(x, t; x_0)).$$

(9)

It is an equation which generalizes the Hopf-Lax and Lax-Oleinik formula [13, 14].

For a particle in a linear potential $V(x) = -K \cdot x$ with the initial action $S_0(x) = mv_0 \cdot x$, we deduce from the equation (9) that the Hamilton-Jacobi action is equal to $S(x, t) = mv_0 \cdot x - \frac{1}{2}mv_0^2 t + K \cdot xt - K^2 \frac{t^3}{6m^3}$.

Figure 2 shows the classical trajectories (parabols) going from different starting points $x_0$ at time $t = 0$ to the point $x$ at final time $t$. The Hamilton-Jacobi action is compute with these trajectories in the equation (9).

For the Lagrangian $L(x, \dot{x}, t) = \frac{1}{2}m\dot{x}^2 - V(x, t)$, we deduce the well-known result [14]:

The velocity of a non-relativistic classical particle is given for each point $(x, t)$ by:

$$v(x, t) = \frac{\nabla S(x, t)}{m}$$

(10)

where $S(x, t)$ is the Hamilton-Jacobi action, solution of the Hamilton-Jacobi equations:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V(x, t) = 0$$

(11)

$$S(x, 0) = S_0(x).$$

(12)

Equation (10) shows that the solution $S(x, t)$ of the Hamilton-Jacobi equations yields the velocity field for each point $(x, t)$ from the velocity field $\nabla S_0(x)$ at initial time. In particular, if at initial time, we know the initial position $x_0$ of a particle, its velocity at this time is equal to $\frac{\nabla S_0(x_0)}{m}$. From the solution $S(x, t)$ of the Hamilton-Jacobi equations, we deduce with (10)
Figure 2: Classical trajectories $\tilde{x}(s) \ (0 \leq s \leq t)$ between the different initial positions $x_0$ and the position $x$ at time $t$. We obtain $\tilde{v}_0 = \frac{x-x_0}{t} - \frac{Kt}{2m}$. 

The trajectories of the particle. The Hamilton-Jacobi action $S(x, t)$ is then a field which "pilots" the particle. 

For a particle in a linear potential $V(x) = -K \cdot x$ with the initial action $S_0(x) = m v_0 \cdot x$, the initial velocity field is constant, $v(x, 0) = \nabla S_0(x) = v_0$, and the velocity field at time $t$ is also constant, $v(x, t) = \frac{\nabla S(x, t)}{m} = v_0 + \frac{Kt}{m}$. Figure 3 shows these velocity fields.

Equation (10) seems to show that, among the trajectories which can reach $(x, t)$ from an unknown initial position and a known initial velocity field, Nature chooses the initial position and at each time the velocity that yields the minimum (or the extremum) of the Hamilton-Jacobi action. 

Equations (10), (11) and (12) confirm this interpretation. They show that the Hamilton-Jacobi action $S(x, t)$ does not solve only a given problem with a single initial condition $(x_0, \frac{\nabla S_0(x_0)}{m})$, but a set of problems with an infinity of initial conditions, all the pairs $(y, \frac{\nabla S_0(y)}{m})$. It answers the following question: "If we know the action (or the velocity field) at the initial time, can we determine the action (or the velocity field) at each later time?" This problem is solved sequentially by the (local) evolution equation (11). This is an a priori point of view. It is the problem solved by Nature with the principle of least action.

Let us note that the Euler-lagrange action $S_{cl}(x; t; x_0)$ also satisfies the Hamilton-Jacobi equation (11), but with an initial condition

$$S_0(x) = \begin{cases} 0 & \text{if} \quad x = x_0, \\ +\infty & \text{otherwise} \end{cases} \quad (13)$$
Figure 3: Velocity field that corresponds to the Hamilton-Jacobi action: 

\[ S(x, t) = m v_0 \cdot x - \frac{1}{2} m v_0^2 t + K \cdot x t - \frac{1}{2} K \cdot v_0 t^2 - \frac{K^2 t^3}{6m} \]

then \( v(x, t) = \nabla S(x, t) \), \( m \) = \( v_0 + K t \) and three trajectories of particles piloted by this field.

which is a very singular function. It is the mathematical difference with the Hamilton-Jacobi action.

4 Conclusion

The introduction of the Hamilton-Jacobi action highlights the importance of the initial action \( S_0(x) \), while textbooks do not clearly differentiate these two actions.

We are now in a position to solve Poincaré’s puzzle: the Hamilton-Jacobi action does not use the "final causes" and seems to be the action used by Nature; the Euler-Lagrange action uses the "final causes" and is the action used by an observer to retrospectively determine the trajectory of the particle and its initial velocity. It is as if Nature solved each time the Hamilton-Jacobi equation. The Hamilton-Jacobi action is ontological while the Euler-Lagrange action is epistemological.

The principle of least action used by the Nature is represented by equation (8), the principle of least action used by the observer is represented by equation (3). (9) is the equation between these two actions.
References

[1] de Maupertuis, P.L.: Accord de différentes lois de la nature qui avaient jusqu’ici paru incompatibles. Mémoires de l’Académie Royale des Sciences, p.417-426 (Paris,1744); reprint in: Oeuvres, 4, 1-23 Reprografischer Nachdruck der Ausg. Lyon (1768).

[2] de Maupertuis, P.L.: Les Loix du mouvement et du repos déduites d’un principe métaphysique. Histoire de l’Académie Royale des Sciences et des Belles Lettres [de Berlin], p.267-294, 1746.

[3] Euler, L.: Methodus Inveniendi Lineas Curvas Maximi Minive Proprietate Gaudentes. Bousquet, Lausanne et Geneva (1744). Reprint in: Leonhardi Euleri Opera Omnia: Series I vol 24 C. Cartheodory (ed.) Orell Fuessli, Zurich (1952).

[4] Lagrange, J.L.: Mécanique Analytique. Gauthier-Villars, 2nd ed., Paris (1888); translated in: Analytic Mechanics, Klumer Academic, Dordrecht (2001).

[5] Hamilton, W.R.: On a general method in dynamics, by which the study of the motions of all free systems of attracting or repelling points is reduced to the search and differentiation of one central Relation or characteristic Function. Philosophical Transactions of the Royal Society of London, p.247-308, vol.II, 1834.

[6] Hamilton, W. R.: On a general method in dynamics, by which the study of the motions of all free systems of attracting or repelling points is reduced to the search and differentiation of one central Relation or characteristic Function. Philos. Trans; R. Soci. PartII, 247-308 (1834).

[7] Hamilton, W.R.: Second essay on a general method in dynamics. Philosophical Transactions of the Royal Society of London, p.95-144, vol.I, 1835.

[8] Jacobi, C.G.J.: Vorlesungen über Dynamik, Reimer, Berlin, 1866.

[9] Feynman R.P., Leighton R.B., and Sands, Matthew": The Feynman Lectures on Physics, chap.19, p.8, vol.II, Addison-Wesley, Reading, MA, 1964, 1964

[10] Landau, L.D., Lifshitz, E.M.: Mechanics, Course of Theoretical Physics. chap.1, Butterworth-Heinemann, London (1976).

[11] Poincaré, H.: La Science et l’Hypothèse. Flammarion, (1902); Translated in: The Foundations of Sciences: Science and Hypothesis, The value of Science, Science and Method. New York: Science Press (1913).

[12] Goldstein, H.: Classical mechanics. Addison-Wesley (1966).

[13] Lions, P. L.: Generalized Solutions of Hamilton-Jacobi Equations. Pitman (1982).

[14] Evans, L.C.: Partial Differential Equations, p123-124, Graduate Studies in Mathematics 19, American Mathematical Society, 1998