Deformation spaces of one-dimensional formal modules and their cohomology

Matthias Strauch

Department of Pure Mathematics and Mathematical Statistics
Centre for Mathematical Sciences, University of Cambridge
Wilberforce Road, Cambridge, CB3 0WB, United Kingdom
e-mail: M.Strauch@dpmms.cam.ac.uk

Abstract. Let $M_m$ be the formal scheme which represents the functor of deformations of a one-dimensional formal module over $\bar{\mathbb{F}}_p$ equipped with a level-$m$-structure. By work of Boyer (in mixed characteristic) and Harris and Taylor, the $\ell$-adic étale cohomology of the generic fibre $M_m$ of $M_m$ realizes simultaneously the local Langlands and Jacquet-Langlands correspondences. The proofs given so far use Drinfeld modular varieties or Shimura varieties to derive this local result. In this paper we show without the use of global moduli spaces that the Jacquet-Langlands correspondence is realized by the Euler-Poincaré characteristic of the cohomology. Under a certain finiteness assumption on the cohomology groups, it is shown that the correspondence is realized in only one degree. One main ingredient of the proof consists in analyzing the boundary of the deformation spaces and in studying larger spaces which can be considered as compactifications of the spaces $M_m$.

CONTENTS

1. Introduction 2
2. Deformation spaces with level structures 7
2.1. Deformation functors 7
2.2. Group actions 12
2.3. Algebraization of the formal schemes and the group action 16
2.4. Associated analytic spaces 19
2.5. $\ell$-adic cohomology and statement of the main result 20
2.6. The period morphism and fixed points 23
3. The characteristic-$\varpi$-boundary 27
3.1. Quasi-Compactifications 27
3.2. Consequences for formal models 31
3.3. The trace of regular elliptic elements 37
4. The Jacquet-Langlands correspondence realized on the cohomology 42
4.1. The trace on the Euler-Poincaré characteristic 42
4.2. The $\varpi$-adic boundary 46
4.3. Non-cuspidalness outside the middle degree 54
5. Appendix 61
1. Introduction

Let $F$ be a local non-Archimedean field and denote by $\mathfrak{o}$ its ring of integers. In [Dr1] V. G. Drinfeld introduced the notion of a formal $\mathfrak{o}$-module which generalizes the concept of a formal group. A formal $\mathfrak{o}$-module is a smooth formal group over an $\mathfrak{o}$-algebra $R$ which is equipped with an action of $\mathfrak{o}$ such that the induced $\mathfrak{o}$-module structure on the tangent space comes from the morphism $\mathfrak{o} \to R$. In this paper we consider the deformations of a fixed formal $\mathfrak{o}$-module $X$ over the algebraic closure $\overline{F}$ of the residue field of $\mathfrak{o}$, which has the property that multiplication by a uniformizer $\varpi$ of $\mathfrak{o}$ on $X$ is an isogeny of degree $q^n$ where $q$ is the number of elements of the residue field of $\mathfrak{o}$. Extending work of Lubin and Tate who considered the case $\mathfrak{o} = \mathbb{Z}_p$, Drinfeld had shown that the functor of deformations is representable by an affine formal scheme. Moreover, he introduced the important concept of a level structure for finite height one-dimensional formal modules and showed that the functor which associates to some $\hat{\mathfrak{o}}^{nr}$-algebra $R$ the set of pairs $(X, \phi)$ of deformations $X$ of $\overline{X}$ to $R$, together with a level-$m$-structure $\phi$, is representable by an affine formal scheme $\mathcal{M}_m$. To such a formal scheme one can attach its generic fibre $\mathcal{M}_m$, which is an analytic space over $\hat{F}^{nr}$ (the completion of the maximal unramified extension of $F$). These spaces can be considered as rigid analytic spaces, or as non-Archimedean analytic spaces in the sense of Berkovich, or as adic spaces in the sense of Huber. We are interested in the $\ell$-adic étale cohomology groups of the spaces $\mathcal{M}_m$, where $\ell$ is a prime number different from the residue characteristic of $F$.

Historical overview. We will give a brief historical overview of the development of the study of these spaces and their cohomology. Their significance, in the context of Shimura curves, was first pointed out by H. Carayol, who studied the bad reduction of these curves in [Ca1]. In [Ca2] he constructed a representation of $GL_2(F) \times B^\times \times W_F$ in terms of vanishing cycles at singular points in the special fibre at primes of bad reduction of these curves. Here $B$ is the quaternion division algebra over $F$, and $W_F$ is the Weil group. This representation, which he calls $représentation locale fondamentale$, is constructed purely locally and its existence rests solely on the representability of the deformation spaces with level structures, and can hence be carried out in the equal characteristic case too. The importance of this representation is that it realizes simultaneously the Jacquet-Langlands and the Langlands correspondence. It is, very roughly speaking, of the form

$$\bigoplus \pi \otimes \rho \otimes \sigma$$
where \( \pi \) is a representation of \( GL_2(F) \), \( \rho \) a representation of \( B^\times \) and \( \sigma \) a representation of \( W_F \), and a tensor product \( \pi \otimes \rho \otimes \sigma \) occurs if and only if \( \pi \) and \( \rho \) correspond under the Jacquet-Langlands correspondence and \( \pi \) and \( \sigma \) are related by the Langlands correspondence (cf. [Ca2] for a precise statement). Later, in the seminal article Non-abelian Lubin-Tate theory, cf. [Ca3], Carayol generalized the construction of this representation to \( GL_n \) and explained how to find this representation in the cohomology of Shimura varieties. He exhibited in fact another local representation whose construction rests on the existence of certain non-trivial coverings of \( p \)-adic symmetric spaces, which is another discovery of Drinfeld, [Dr2]. Following Carayol we call this latter setting the rigid setting, and the former, relying on deformation spaces of formal modules, the vanishing cycle setting. In each case he conjectured that the construction realizes simultaneously the Jacquet-Langlands and Langlands correspondence\(^1\), and he outlined a program how to prove these conjectures using the cohomology of Shimura varieties. The first to take up these conjectures was G. Faltings, who proved in [F1], in the context of the rigid setting, that the Euler-Poincaré characteristic of the cohomology of the coverings realizes the Jacquet-Langlands correspondence. His work is important because it does not use global arguments. On the other hand his method does not prove much about the representation of the Weil group (except, e.g., its dimension and compatibility with twisting). But it explains nicely the occurrence of the Jacquet-Langlands correspondence as being the consequence of a Lefschetz trace formula for rigid analytic spaces. In [H1] M. Harris considered the rigid setting too, but used the theory of \( p \)-adic uniformization of Shimura varieties which was shortly before generalized by Rapoport and Zink, cf. [RZ]. In this context and as consequences of that paper he was able to prove the existence of the local Langlands conjecture for \( n < p \), cf. [H2]. Around that time the equal-characteristic case began to be studied and it was already mentioned by Carayol that the modular varieties of Drinfeld and Stuhler furnish the global context in which to place the local construction. Then P. Boyer proved Carayol’s conjecture in the vanishing cycle setting for local fields of equal characteristic, cf. [Bo], thereby reproving the local Langlands correspondence which had been shown earlier, without Carayol’s construction but with the use of the Drinfeld-Stuhler varieties, by Laumon, Rapoport and Stuhler. Somewhat later Th. Hausberger showed that such modular varieties possess at certain places rigid-analytic uniformization, and could subsequently prove Carayol’s conjecture in the rigid setting in positive characteristic ([Ha]). Finally, M. Harris and R. Taylor accomplished Carayol’s program in the vanishing cycle setting in characteristic zero and proved for the first time the local Langlands correspondence for a local field of characteristic zero, cf. [HT]. Among the more recent papers pursuing investigations in this field we would like to mention T. Yoshida’s local study of the cohomology of the tame-level space \( M_1 \), cf. [Yo], and Laurent Fargue’s thesis [Fa2], where he considers the cohomology of other Rapoport-Zink spaces.

\(^1\)Carayol attributes these conjectures partly to P. Deligne and V. G. Drinfeld. Furthermore, I was told by Y. Varshavsky that Drinfeld gave the construction of the representation and stated the conjectures in an unpublished manuscript. For the rigid setting Drinfeld made already a less precise conjecture in [Dr2].
Deformation spaces of one-dimensional formal modules and their cohomology

The spaces $M_m$ as well as Drinfeld’s coverings of $p$-adic symmetric domains are examples of such spaces, cf. [RZ]). Most recently S. Wewers analyzed the ‘stable reduction’ of the deformation spaces $M_m$ in the height-2-case, gives a precise description of the action of the Weil group and proves Carayol’s conjecture in this case (cf. [W1], [W2]).

The present approach. In this paper we will study the spaces $M_m$ and their cohomology groups by local means. We prove first, without the use of global modular varieties, that, for supercuspidal representations, the Jacquet-Langlands correspondence is realized by the Euler-Poincaré characteristic of vanishing cycles in all degrees, for $F$ of arbitrary characteristic. Then we prove that there cannot be any supercuspidal representations except in one degree. At this step we have to assume that the cohomology groups of certain boundary strata are finite-dimensional. This should certainly be the case but has not been proved yet. So we will work under the assumption of the finiteness of these cohomology groups.

Following Faltings’ approach, we already considered in [St] the Euler-Poincaré characteristic and showed that a suitable Lefschetz trace formula for the group action of elements $(g, b) \in GL_n(F) \times B^\times$ on the $\ell$-adic cohomology of the spaces $M_m$ would show that the Jacquet-Langlands correspondence is realized by that representation. However, at that time we could only prove that such a formula holds in the case $n = 2$, using a trace formula proved by R. Huber, cf. [Hu5] (who’s work also rests on Faltings’ paper [F1]). Whereas it is clear that it suffices to consider pairs $(g, b)$ with $b$ being regular elliptic, our work was for a long time blocked by the fact that, even for regular elliptic $b$, the pair $(g, b)$ may have fixed points ‘at the boundary’ of the spaces $M_m$. To make this precise we found out that one can in fact speak of the boundary of these spaces, and it is a consequence of the ‘quasi-compactifications’ we construct, that for pairs $(g, b)$ with both elements being regular elliptic, a trace formula with the desired properties is indeed provable. Due to a remark Laurent Fargues made to me, it is actually possible to show that one can restrict oneself to pairs $(g, b)$ with $g$ and $b$ being regular elliptic. Fargues studied the deformation spaces too, cf. [Fa1], where he introduced generalized canonical subgroups for $\varphi$-modules of any height. Thereby he is able to define a stratification in a neighborhood of what we call the ‘boundary’, and he found out independently that regular elliptic $g$ act without fixed points in a neighborhood of the boundary.

Summary of content. We are going to give a brief description of the content of the paper. In the second section we recall the definitions of the deformation functors, following [Dr1], and define the group actions. We show that the formal schemes $\mathcal{M}_m$, which represent the deformation functors with level structures, are algebraizable. This uses only Artin’s
theorem on the algebraization of formal moduli and no Shimura or Drinfeld-Stuhler varieties. Then we introduce the generic fibres of the formal schemes $\mathcal{M}_m$, namely the spaces $M_m$. Because of later considerations, when we study the boundary of these spaces, we consider $M_m$ as an adic space. The first essential ingredient of our proof is the fixed point formula which gives a group-theoretical expression for the number of fixed points on a space $M_m$. This result is taken from [St] and uses in a crucial way the period map of Hopkins and Gross, which was generalized by Rapoport and Zink for other moduli spaces of $p$-divisible groups. For the sake of completeness we include it here again. In section three we introduce the main tool of the whole paper, namely the spaces $\overline{M}_m$ which we call 'quasi-compactifications', for lack of a better term. We refrain from calling them compactifications, because they are no longer spaces over the non-archimedean field $\hat{F}$. A compactification in the usual sense should be a space which is proper over $\hat{F}$ and contains $M_m$ as a dense open subspace. However, the boundary $\partial M_m = \overline{M}_m - M_m$ of the spaces $M_m$ is no longer defined over $\hat{F}$, it is a space where $\varpi = 0$ on the structure sheaf. We therefore call it the characteristic-$\varpi$-boundary. But $\overline{M}_m$ as well as $\partial M_m$ are quasi-compact analytic adic spaces, which have a natural interpretation in terms of formal models. Namely, $\overline{M}_m$ can be identified with Fujiwara’s Zariski-Riemann space associated to $M_m$. This is the projective limit of all admissible blow-ups of $M_m$. The special fibre of each such blow-up is a projective scheme over $\mathbb{F}$, and $\overline{M}_m$ is therefore, as a topological space, a projective limit of schemes which are proper over $\mathbb{F}$. On $\overline{M}_m$ there is still the universal level structure $\phi : \varpi^{-m}o^n/o^n \to X^{univ}[\varpi^m]$ ($X^{univ}$ is the universal deformation of $X$), and we can consider the subsets $\partial_A M_m$ of $\overline{M}_m$ where the kernel of this level structure is equal to some direct summand $A \subset \varpi^{-m}o^n/o^n$ which is free over $o'/\varpi^m$. These sets define a nice stratification of $\overline{M}_m$ by adic subspaces, and this stratification is respected by the group action. Then it is easy to see that regular elliptic $g$ will permute the boundary strata, mapping none of them to itself, for $m$ sufficiently large (provided $g$ maps $M_m$ to itself, which we assume here, for simplicity). This in turn has the consequence that there are suitable formal models such that the fixed point locus of $g$ on the special fibre is contained in the 'interior', i.e. in the complement of the image of the boundary under the specialization map. Then we can use Fujiwara’s techniques and his result on the specialization of local terms in the trace formula, cf. [Fu2], to conclude that the trace of $(g, b)$ on the cohomology of $M_m$\footnote{in this introduction, when we speak of the cohomology of $M_m$ we rather mean the Euler-Poincaré characteristic of the cohomology of $M_m \times \hat{p}_{nr} \hat{F}$} is equal to the number of rigid-analytic fixed points. In the forth section we use this result, together with a dévissage to the regular elliptic locus, to prove that the Jacquet-Langlands correspondence is realized on the Euler-Poincaré characteristic of the cohomology, as far as supercuspidal representations are concerned. The last step consists then in proving that there are no supercuspidal representations except in the middle degree. To this end we consider another kind of compactification, which is this time a space which lives over $\hat{F}$. One can describe it as follows. Let $\mathcal{M}_m$ be an algebraization of $\mathcal{M}_m$, i.e. a scheme of finite type over $\hat{o}_{nr}$ such that its completion at a
closed point \( \mathfrak{r}_m \) is isomorphic to \( \mathcal{M}_m \). Then one associates an analytic adic space \( \mathcal{M}^{ad}_m \) over \( \bar{F}^\wedge \) to it and considers the preimage \( \mathcal{M}^{\infty}_m \subset \mathcal{M}^{ad}_m \) of \( \mathfrak{r}_m \) under the specialization map. This is a pseudo-adic space which is proper over \( \bar{F}^\wedge \), and it contains \( M_m \times \hat{F}^{nr} \bar{F}^\wedge \) as an open subset (which can be shown to be dense in it). We call

\[
\partial^{\infty} M_m = \mathcal{M}^{\infty}_m - M_m \times \hat{F}^{nr} \bar{F}^\wedge
\]

the \( \varpi \)-adic boundary. There is a canonical map from \( \mathcal{M}^{\infty}_m \) to \( \mathcal{M}_m \) and we can pull back the previously defined strata. Combining theorems of Berkovich and Huber, it follows that the cohomology of \( \mathcal{M}^{\infty}_m \) is canonically isomorphic to the cohomology of \( M_m \). Because the cohomology of \( M_m \) vanishes in degree \( \geq n \), in the long exact sequence relating the cohomology with compact support of \( M_m \) to the cohomology of \( \mathcal{M}^{\infty}_m \) and the cohomology of the \( \varpi \)-adic boundary \( \partial^{\infty} M_m \), we find surjections

\[
H^i((\partial^{\infty} M_m) \times \hat{F}^{nr} \bar{F}^\wedge, \mathbb{Q}_\ell) \twoheadrightarrow H^i_c(M_m \times \hat{F}^{nr} \bar{F}^\wedge, \mathbb{Q}_\ell),
\]

if \( i \geq n \). From the stratification of \( \partial^{\infty} M_m \) we deduce that its cohomology is a successive extension of parabolically induced representations, provided that the cohomology groups of the strata of \( \partial^{\infty} M_m \) are finite-dimensional. Although this should definitely be the case, we can unfortunately not prove it yet. Hence we will work under the assumption of finiteness of the cohomology. At the end of section 4.2 we will make some remarks that should make it plausible why this assumption should be true. Then one can conclude that there are no supercuspidal representations in the cohomology except in degree \( n - 1 \). In the appendix we prove in a rather elementary way that the deformation rings are algebraizable in the equal-characteristic case. It is based on the fact that in this case the multiplication of \( \varpi \) on the universal deformation can be described by a polynomial. Lastly, we show that two endomorphisms of an affinoid rigid space have the same number of fixed points, if one of them has only finitely many simple fixed points and if the endomorphisms are sufficiently close to another.

At some instances it would have been possible to shorten the exposition a little. This would have been the effect of a systematic use of correspondences throughout the paper. As general group elements \( g \) will not map a space \( M_m \) to itself, we were led to consider ‘intermediate’ spaces \( M_K \), for compact subgroups \( K \subset \text{GL}_n(F) \). If one works with correspondences, that would have been superfluous. Another point to improve on would be the use of proper algebraizations of \( \mathcal{M}_m \), i.e. schemes \( \mathfrak{M}_m \) which are proper over \( \hat{O}^{nr} \) and such that \( \mathcal{M}_m \) is the completion of \( \mathfrak{M}_m \) at a closed point. Then we could have applied immediately Fujiwara’s results on the topological trace formula, instead of proving the existence of suitable formal models explicitly. However, where it seemed to possible for us to do so, we preferred a more direct method.
Acknowledgements. From what has been said above it is plain how much this paper rests on the work of Berkovich, Drinfeld, Carayol, Faltings, Fujiwara and Huber, to name just a few. I am very thankful to Laurent Fargues for many discussions about the topic of this paper. He made the crucial remark that it suffices to consider only regular elliptic group elements, and he explained to me how to algebraize the formal schemes. I am very grateful to Roland Huber for equally many discussions about adic spaces, which were always very helpful. V.G. Berkovich helped me to find a proof of some technical result (cf. 5.2.3), and I am very thankful to him for his quick response to my question. Part of this paper was written during a stay at IHÉS, Bures-sur-Yvette, and I would like to thank heartily C. Breuil and L. Lafforgue for their invitation, and IHÉS for its hospitality and support. I studied this problem and learned about the material connected with this paper while being at the department of Mathematics of the University of Münster, and I am thankful to S. Bosch and P. Schneider and their study groups for the framework they provided. The SFB 478 ”Geometrische Strukturen in der Mathematik” at the University of Münster financially supported travels to various conferences and seminars where I had the opportunity to discuss this and related topics or give talks about it.

Notation. In this paper, \( F \) will be a non-Archimedean local field, with ring of integers \( \mathfrak{o} \), and \( \varpi \) will be a uniformizer of \( F \). The number of elements of the residue field will be denoted by \( q \), and the residue field itself by \( \mathbb{F}_q \). The valuation \( v \) on \( F \) will be normalized by \( v(\varpi) = 1 \). This is the only valuation which we write additively; all valuations which appear in the context of adic spaces will be written multiplicatively. We denote by \( \mathbb{F} \) an algebraic closure of \( \mathbb{F}_q \). \( \hat{\mathbb{F}}^{nr} \) is the completion of the maximal unramified extension of \( F \), \( \hat{\mathfrak{o}}^{nr} \) its ring of integers, and \( \hat{\mathbb{F}}^\wedge \supset \hat{\mathbb{F}}^{nr} \) is the completion of an algebraic closure of \( F \) whose ring of integers we denote by \( \mathfrak{o}_{\hat{\mathbb{F}}^\wedge} \). If \( X \) is a scheme over \( \hat{\mathfrak{o}}^{nr} \) we often write \( X \times \hat{\mathfrak{o}}^{nr} \rightarrow \hat{\mathfrak{o}}^{nr} \) instead of \( X \times \text{Spec}(\hat{\mathfrak{o}}^{nr}) \text{Spec}(\hat{\mathfrak{o}}^{nr}) \), and we use similar abuse of notation in other instances of base change. If \( R \) is a local ring we denote by \( m_R \) its maximal ideal. \( G \) always stands for the group \( GL_n(F) \) (the number \( n \) will be fixed throughout the paper), \( K_0 \) is its maximal compact subgroup \( GL_n(\mathfrak{o}) \), and \( K_m = 1 + \varpi^m M_n(\mathfrak{o}) \) is the \( m \)th principal congruence subgroup. \( B \) will denote a central division algebra over \( F \) with invariant \( \frac{1}{n} \), and \( N : B \rightarrow F \) the reduced norm. \( \ell \) denotes a prime number not dividing \( q \).

2. Deformation spaces with level structures

2.1. Deformation functors.

2.1.1. Let \( X \) be a one-dimensional formal group over \( \mathbb{F} \) that is equipped with an action of \( \mathfrak{o} \), i.e. we assume given a homomorphism \( \mathfrak{o} \rightarrow \text{End}_F(X) \) such that the action of \( \mathfrak{o} \) on the tangent space is given by the reduction map \( \mathfrak{o} \rightarrow \mathbb{F}_q \subset \mathbb{F} \). Such an object is called a formal \( \mathfrak{o} \)-module over \( \mathbb{F} \). Moreover, we assume that \( X \) is of \( F \)-height \( n \), which means that the kernel of multiplication by \( \varpi \) is a finite group scheme of rank \( q^n \) over \( \mathbb{F} \).
It is known that for each $n \in \mathbb{Z}_{>0}$ there exists a formal $\mathfrak{o}$-module of $F$-height $n$ over $F$, and that it is unique up to isomorphism [Dr1], Prop. 1.6, 1.7.

Let $\mathcal{C}$ be the category of complete local noetherian $\hat{\mathfrak{o}}^{nr}$-algebras with residue field $F$. A deformation of $X$ over an object $R$ of $\mathcal{C}$ is a pair $(X, \iota)$, consisting of a formal $\mathfrak{o}$-module $X$ over $R$ which is equipped with an isomorphism $\iota : X \to X_F$ of formal $\mathfrak{o}$-modules over $F$, where $X_F$ denotes the reduction of $X$ modulo the maximal ideal $\mathfrak{m}_R$ of $R$. Sometimes we will omit $\iota$ from the notation.

Following Drinfeld [Dr1], sec. 4B, we define a structure of level $m$ or level-m-structure on a deformation $X$ over $R \in \mathcal{C}$ ($m \geq 0$) as an $\mathfrak{o}$-module homomorphism

$$\phi : (\varpi^{-m}\mathfrak{o}/\mathfrak{o})^n \to \mathfrak{m}_R,$$

such that, after having fixed a coordinate $T$ on the formal group $X$, the power series $[\varpi]_X(T) \in R[[T]]$, which describes the multiplication by $\varpi$ on $X$, is divisible by

$$\prod_{a \in (\varpi^{-1}\mathfrak{o}/\mathfrak{o})^n} (T - \phi(a)).$$

Here, $\mathfrak{m}_R$ is given the structure of an $\mathfrak{o}$-module via $X$.

For each $m \geq 1$ let $K_m = 1 + \varpi^m M_n(\mathfrak{o})$ be the $m$th principal congruence subgroup inside $K_0 = GL_n(\mathfrak{o})$. Define the following set-valued functor $\mathcal{M}^{(0)}_{K_m}$ on the category $\mathcal{C}$. For an object $R$ of $\mathcal{C}$ put

$$\mathcal{M}^{(0)}_{K_m}(R) = \{(X, \iota, \phi) \mid (X, \iota) \text{ is a def. over } R, \phi \text{ is a level-m-structure on } X\}/ \simeq,$$

where $(X, \iota, \phi) \simeq (X', \iota', \phi')$ if and only if there is an isomorphism $(X, \iota) \to (X', \iota')$ of formal $\mathfrak{o}$-modules over $R$, which is compatible with the level structures. For $0 \leq m' \leq m$ one gets by restricting a level-m-structure

$$\phi : (\varpi^{-m}\mathfrak{o}/\mathfrak{o})^n \to \mathfrak{m}_R$$

to $(\varpi^{-m'}\mathfrak{o}/\mathfrak{o})^n \subset (\varpi^{-m}\mathfrak{o}/\mathfrak{o})^n$ a level-$m'$-structure

$$\phi' = \phi \mid_{(\varpi^{-m'}\mathfrak{o}/\mathfrak{o})^n} : (\varpi^{-m'}\mathfrak{o}/\mathfrak{o})^n \to \mathfrak{m}_R,$$

and hence a natural transformation

$$\mathcal{M}^{(0)}_{K_m} \to \mathcal{M}^{(0)}_{K_{m'}}.$$
Theorem 2.1.2. (i) The functor \( \mathcal{M}_{K_m}^{(0)} \) is representable by a regular local ring \( R_m \) of dimension \( n \). Hence there is a universal formal \( \mathfrak{o} \)-module \( X^{univ} \) over \( R_0 \) which defines on the maximal ideal \( \mathfrak{m}_{R_0} \) of \( R_m \) the structure of an \( \mathfrak{o} \)-module. There is a universal level-\( m \)-structure

\[
\phi_m^{univ} : (\varpi^{-m}\mathfrak{o}/\mathfrak{o})^n \longrightarrow \mathfrak{m}_{R_m}
\]

such that, if \( a_1, \ldots, a_n \) is a basis of \( (\varpi^{-m}\mathfrak{o}/\mathfrak{o})^n \) over \( \mathfrak{o}/(\varpi^m) \), then

\[
\phi_m^{univ}(a_1), \ldots, \phi_m^{univ}(a_n)
\]

is a regular system of parameters for \( R_m \).

(ii) Let \( 0 \leq m' \leq m \). The ring homomorphism \( R_{m'} \rightarrow R_m \) which corresponds to the natural transformation \( \mathcal{M}_{K_m}^{(0)} \rightarrow \mathcal{M}_{K_{m'}}^{(0)} \), described above makes \( R_m \) a finite and flat \( R_{m'} \)-algebra. Moreover, \( R_m[\frac{1}{\varpi}] \) is étale and galois over \( R_{m'}[\frac{1}{\varpi}] \) with Galois group isomorphic to \( K_{m'}/K_m \).

(iii) \( R_0 \) is (non-canonically) isomorphic to \( \hat{o}[u_1, \ldots, u_{n-1}] \).

Proof. (i) This result is [Dr1], Prop. 4.3.

(ii) That \( R_m \) is finite and flat over \( R_{m'} \) is again [Dr1], Prop. 4.3. For the second assertion it suffices to treat the case \( m' = 0 \) (by [FK], Ch. 1, §1, Prop. 1.7. (3)).

We show first that the universal level-\( m \)-structure is injective. Suppose \( \phi_m^{univ}(a) = 0 \) for some non-zero \( a \in (\varpi^{-m}\mathfrak{o}/\mathfrak{o})^n \). Then there is also a non-zero \( a' \) in the subgroup

\[
(\varpi^{-1}\mathfrak{o}/\mathfrak{o})^n \subset (\varpi^{-m}\mathfrak{o}/\mathfrak{o})^n
\]

which is mapped to zero by \( \phi_m^{univ} \). But the restriction of \( \phi_m^{univ} \) to the subgroup \( (\varpi^{-1}\mathfrak{o}/\mathfrak{o})^n \) is equal to the composition of \( \phi_m^{univ} \) with the map \( R_1 \rightarrow R_m \). If \( a' \in (\varpi^{-1}\mathfrak{o}/\mathfrak{o})^n \) is non-zero, \( a' \) is part of a basis, and hence \( \phi_m^{univ}(a') \) is part of a regular system of parameters, in particular \( \phi_m^{univ}(a') \neq 0 \).

Denote by \( X^{univ} \) the universal deformation of \( X \) over \( R_0 \). We fix a coordinate \( T \) and denote for any \( \alpha \in \mathfrak{o} \) by \( [\alpha]_{X^{univ}}(T) \) the power series with coefficients in \( R_0 \) which describes the multiplication of \( \alpha \) on \( X^{univ} \). For a given \( m \geq 0 \) write

\[
[\varpi^m]_{X^{univ}}(T) = P_m(T)e_m(T),
\]

with a polynomial \( P_m(T) \in R_0[T] \) and a unit \( e_m(T) \in R_0[[T]] \), by the Weierstrass Preparation Theorem. Because \( X \) was assumed to be of \( F \)-height \( n \), \( P_m(T) \) is a polynomial of degree \( q^m \) whose zeros in \( R_m \) are exactly the elements \( \phi_m^{univ}(a) \), \( a \in (\varpi^{-m}\mathfrak{o}/\mathfrak{o})^n \). Therefore, as the universal level-\( m \)-structure is injective, \( P_m(T) \) is separable over \( \text{Frac}(R_0) \).
Moreover, for any prime ideal $q \in R_0$ the zeros of the image of $P_m(T)$ in $\text{Frac}(R_0/q)[T]$ are the images of the elements $\phi_{m}^{\text{univ}}(a)$ in $\text{Frac}(R_0/q)$. We will show that the image of $P_m(T)$ in $\text{Frac}(R_0/q)[T]$ is separable if $q$ does not contain $\varpi$. This means that for any such prime ideal $q$ and any zero $\xi = \phi_{m}^{\text{univ}}(a)$ one has $P_m'(/xi) \notin q$, which is equivalent to $[\varpi^m]_{X^{\text{univ}}}(/xi) \notin q$. Let us calculate this derivative. For a given $r \in \mathbb{Z}_{>0}$ we can write

$$X^{\text{univ}}(\xi, \varpi^r) = \xi + \varpi^r \tau(\xi, r)$$

with $\tau(\xi, r) \in 1 + m_{R_m}$, and the limit $\tau(\xi) = \lim_{r \to \infty} \tau(\xi, r)$ exists in $R_m$ and is again an element of $1 + m_{R_m}$. Then we compute:

$$[\varpi^m]_{X^{\text{univ}}}(\xi) = \lim_{r \to \infty} \frac{1}{\varpi^r}[\varpi^m]X^{\text{univ}}(\xi + \varpi^r)$$

$$= \lim_{r \to \infty} \frac{1}{\varpi^r \tau(\xi, r)}[\varpi^m]X^{\text{univ}}(\xi + \varpi^r \tau(\xi, r))$$

$$= \lim_{r \to \infty} \frac{1}{\varpi^r \tau(\xi, r)}X^{\text{univ}}([\varpi^m](\xi), [\varpi^m](\varpi^r))$$

$$= \lim_{r \to \infty} \frac{1}{\varpi^r \tau(\xi, r)}[\varpi^m](\varpi^r) = \lim_{r \to \infty} \frac{1}{\tau(\xi) \varpi^m} = \frac{1}{\tau(\xi)} \varpi^m.$$

It follows that the image of $P_m(T)$ in $\text{Frac}(R_0/q)[T]$ is separable if $q$ does not contain $\varpi$. We are going to use this to show that $R_m[1/\varpi]$ is unramified over $R_0[1/\varpi]$.

Let us recall from the proof of [Dr1], Prop. 4.3, how the ring $R_m$ can be build up explicitly. Suppose first that $m = 1$, and put $L_0 = R_0$. Then, for $0 \leq i < n$, there is a map

$$\varphi_i : (\varpi^{-1} \mathfrak{o} / \mathfrak{o})^i \to m_{L_i}$$

such that, in $L_i[[T]]$, $[\varpi]_{X^{\text{univ}}}(T)$ is divisible by

$$\prod_{a \in (\varpi^{-1} \mathfrak{o} / \mathfrak{o})^i} (T - \varphi_i(a)).$$

Put

$$f_i(T) = \frac{[\varpi]_{X^{\text{univ}}}(T)}{\prod_{a \in (\varpi^{-1} \mathfrak{o} / \mathfrak{o})^i} (T - \varphi_i(a))}, \text{ and } L_{i+1} = L_i[[\theta]]/(f_i(\theta)).$$

Then we have $L_n = R_1$. The power series $f_i$ all divide the power series $[\varpi]_{X^{\text{univ}}}(T)$, and because the polynomial $P_1(T)$ from above is separable over $R_0[1/\varpi]$ the successive extensions $L_i|L_{i-1}$ are étale after inverting $\varpi$ (cf. [Mi], Ch. I, Ex. 3.4). Hence $R_1[1/\varpi]$ is étale.
over $R_0[\frac{1}{\bar{w}}]$.

For $m > 1$ Drinfeld describes $R_m$ as follows: let $\phi_{m-1}^{univ}$ be the universal level-$(m-1)$-structure, and let $a_1, \ldots, a_n$ a basis of $(\bar{w}^{-(m-1)}\mathcal{o}/\mathcal{o})^n$ over $\mathcal{o}/(\bar{w}^{m-1})$. Then

$$R_m = R_{m-1}[[y_1, \ldots, y_n]]/[[\bar{w}]_{X_{univ}}(y_1) - \phi_{m-1}^{univ}(a_1), \ldots, [\bar{w}]_{X_{univ}}(y_n) - \phi_{m-1}^{univ}(a_n)].$$

Write for $j = 1, \ldots, n$

$$[\bar{w}]_{X_{univ}}(T) - \phi_{m-1}^{univ}(a_j) = P_{m,j}(T)e_{m,j}(T)$$

with a polynomial $P_{m,j}(T) \in R_{m-1}[T]$ and a unit $e_{m,j}(T) \in R_{m-1}[T]$. We will show that $P_{m,j}$ has only simple zeros outside the vanishing locus of $\bar{w}$. If $P_{m,j}(\xi) = 0$ then $[\bar{w}]_{X_{univ}}'(\xi) = 0$, and if furthermore $P_{m,j}'(\xi) = 0$ then $[\bar{w}]_{X_{univ}}'(\xi) = 0$. As

$$[\bar{w}]_{X_{univ}}'(T) = ([\bar{w}]_{X_{univ}}^{-1}([\bar{w}]_{X_{univ}}(T)))' = [\bar{w}]_{X_{univ}}^{-1}([\bar{w}]_{X_{univ}}(T)) : [\bar{w}]_{X_{univ}}'(T),$$

$[\bar{w}]_{X_{univ}}'(\xi) = 0$ would imply $[\bar{w}]_{X_{univ}}'(\xi) = 0$, and so any multiple zero of $P_{m,j}(T)$ would be a multiple zero of $[\bar{w}]_{X_{univ}}(T)$, which do not exist outside the vanishing locus of $\bar{w}$. This means that $R_m[\frac{1}{\bar{w}}]$ is étale over $R_0[\frac{1}{\bar{w}}]$ for any $m$.

From the description of $R_m$ just recalled it follows that the degree of $R_m$ over $R_0$ is equal to the cardinality of $GL_n(\mathcal{o}/(\bar{w}^m))$. On the covering $R_m[\frac{1}{\bar{w}}]$ over $R_0[\frac{1}{\bar{w}}]$ there is an obvious action of $GL_n(\mathcal{o}/(\bar{w}^m))$. To prove that the covering is galois with this group we only have to show that no non-trivial elements acts trivially. Suppose $g \in GL_n(\mathcal{o}/(\bar{w}^m))$ would act trivially on $R_m[\frac{1}{\bar{w}}]$. Then it would follow that $\hat{\phi}_m^{univ}(g(a_i)) = \hat{\phi}_m^{univ}(a_i)$ for $i = 1, \ldots, n$. But the universal level-$m$-structure is injective, hence $g = 1$.

(iii) This is [Dr1], Prop. 4.2. \hfill \Box

Remark. One can also show the statement about étaleness by showing that the finite flat group scheme of $\bar{w}^m$-torsion points is unramified over the base outside the vanishing locus of $\bar{w}$. This is the case because the $\mathcal{O}$-action on the module of relative differentials $\Omega$, which is induced by the structure of an formal $\mathcal{O}$-module, coincides with the usual action via the inclusion $\mathcal{O} \hookrightarrow R_0$. As, on the one hand, $\bar{w}^m$ acts trivial on $X_{univ}[\bar{w}^m]$, it acts as 0 on $\Omega$, but, on the other hand, $\bar{w}$ is invertible outside the vanishing locus of $\bar{w}$, hence the module $\Omega$ has to be zero.

The fact that $\hat{\phi}^m[[u_1, \ldots, u_{m-1}]]$ represents $\mathcal{M}_{K_0}^{(0)}$ is due to Lubin and Tate (for $F = \mathbb{Q}_p$), cf. [LT]. For this reason $\mathcal{M}_{K_0}^{(0)}$, the deformation space without level structures, is sometimes called the Lubin-Tate moduli space, cf. [HG], [Ch].
2.2. Group actions.

2.2.1. Let $X$ be a formal $\mathfrak{o}$-module over $R \in \mathcal{C}$ such that $X_{\mathbb{F}}$ has $F$-height $n$, in which case we say that the formal $\mathfrak{o}$-module $X$ has height $n$. As pointed out above, $X_{\mathbb{F}}$ is then isomorphic to $X$. Denote $\text{End}_{\mathfrak{o}}(X)$ by $\mathfrak{b}_B$; this $\mathfrak{o}$-algebra is the maximal compact subring of $B := \mathfrak{b}_B \otimes_{\mathfrak{o}} F$, which is a central division algebra over $F$ with invariant $\frac{1}{n}$, cf. [Dr1], Prop. 1.7.

Any non-zero element of $\text{Hom}_{\mathfrak{o}}(X, X_{\mathbb{F}}) \otimes_{\mathfrak{o}} F$ is called an $\mathfrak{o}$-quasi-isogeny from $X$ to $X_{\mathbb{F}}$. For such an element $\iota$ we define its $F$-height by

$$F\text{-height}(\iota) = F\text{-height}(\varpi^r \iota) - nr,$$

where we choose some $r \in \mathbb{Z}$ such that $\varpi^r \iota$ lies in $\text{Hom}_{\mathfrak{o}}(X, X_{\mathbb{F}})$, and for an element $\iota'$ of this latter set, its $F$-height is $h$ if $\ker(\iota')$ is a group scheme of rank $q^h$ over $F$.

Define for $j \in \mathbb{Z}$ a set-valued functor $\mathcal{M}^{(j)}_{K_m}$ on $\mathcal{C}$ as follows: for $R \in \mathcal{C}$ the set $\mathcal{M}^{(j)}_{K_m}(R)$ consists of equivalence classes of triples $(X, \iota, \phi)$, where $X$ is a formal $\mathfrak{o}$-module of height $n$ over $R$, $\iota$ is an $\mathfrak{o}$-quasi-isogeny from $X$ to $X_{\mathbb{F}}$ of $F$-height $j$, and $\phi$ is a level-$m$-structure on $X$. Now put

$$\mathcal{M}_{K_m} = \coprod_{j \in \mathbb{Z}} \mathcal{M}^{(j)}_{K_m}.$$

By the uniqueness of $X$ (up to isomorphism), we have $\mathcal{M}^{(j)}_{K_m} \simeq \mathcal{M}^{(0)}_{K_m}$, but there is no distinguished isomorphism.

2.2.2. There is an action of $B^\times$ from the right on the functors $\mathcal{M}_{K_m}$ given by

$$[X, \iota, \phi].b = [X, \iota \circ b, \phi],$$

where we denote by $[X, \iota, \phi]$ the equivalence class of $(X, \iota, \phi)$, and where $b \in B^\times$. If $[X, \iota, \phi]$ belongs to $\mathcal{M}^{(j)}_{K_m}(R)$, then $[X, \iota, \phi].b$ is an element of $\mathcal{M}^{(j+\nu(N(b)))}_{K_m}(R)$, where $N : B \to F$ denotes the reduced norm.

Next we will describe the ‘action’ of the group $G = GL_n(F)$ on the tower $(\mathcal{M}_{K_m})_m$. Let $g \in G$ and suppose first that $g^{-1} \in M_n(\mathfrak{o})$. For integers $m \geq m' \geq 0$ such that

$$g \mathfrak{o}^n \subset \varpi^{-(m-m')} \mathfrak{o}^n$$

(this inclusion is meant to be inside $F^n$) we will define a natural transformation

$$g_{m, m'} : \mathcal{M}_{K_m} \to \mathcal{M}_{K_{m'}}.$$
Let \([X, \iota, \phi] \in \mathcal{M}_{K_m}(R), R \in \mathcal{C}\). The following construction gives an element \([X', \iota', \phi']\) of \(\mathcal{M}_{K_{m'}}(R)\) that is the image under the corresponding map
\[
(g_{m,m'})_R : \mathcal{M}_{K_m}(R) \to \mathcal{M}_{K_{m'}}(R)
\]
on \(R\)-valued points and it will be denoted by \([X, \iota, \phi].g\).

The conditions imposed on \(g\) show that \(g\sigma^n\) contains \(\mathfrak{o}^n\) and that \(g\sigma^n/\mathfrak{o}^n\) can naturally be regarded as a subgroup of \(\varpi^{-m'}\mathfrak{o}^n/\mathfrak{o}^n\), so we may define a formal \(\mathfrak{o}\)-module \(X'\) over \(R\) by taking the quotient of \(X\) by the finite subgroup \(\phi(g\sigma^n/\mathfrak{o}^n)\) (cf. [Dr1], Prop. 4.4):
\[
X' = X/\phi(g\sigma^n/\mathfrak{o}^n).
\]
Moreover, left multiplication with \(g\) induces an injective homomorphism
\[
\varpi^{-m'}\mathfrak{o}^n/\mathfrak{o}^n \rightarrow \varpi^{-m}\mathfrak{o}^n/\mathfrak{o}^n = (\varpi^{-m\mathfrak{o}^n}/\mathfrak{o}^n)/(g\sigma^n/\mathfrak{o}^n)
\]
and the composition with the morphism induced by \(\phi\),
\[
(\varpi^{-m\mathfrak{o}^n}/\mathfrak{o}^n)/(g\sigma^n/\mathfrak{o}^n) \rightarrow X/\phi(g\sigma^n/\mathfrak{o}^n) = X',
\]
gives by [Dr1], Prop. 4.4, a level-\(m'\)-structure
\[
\phi' : \varpi^{-m'}\mathfrak{o}^n/\mathfrak{o}^n \rightarrow X' / [\varpi^{m'}].
\]
Finally define \(\iota'\) to be the composition of \(\iota\) with the projection
\[
X_{\varpi} \rightarrow (X')_{\varpi}.
\]
This construction is independent of the representative \((X, \iota, \phi)\) and gives a morphism of functors. If \([X, \iota, \phi]\) lies in \(\mathcal{M}_{K_m}(R)\) then \([X, \iota, \phi].g\) is an element of \(\mathcal{M}_{K_{m'}}(R)\).

For an arbitrary element \(g \in G\), choose \(r \in \mathbb{Z}\) such that \((\varpi^{-r})^{-1} \in M_n(\mathfrak{o})\). Then, for \(m \geq m' \geq 0\) with
\[
\varpi^{-r} g \sigma^n \subset \varpi^{-(m-m')} \mathfrak{o}^n
\]
and \([X, \iota, \phi] \in \mathcal{M}_{K_m}(R)\), define \([X', \iota', \phi'] = [X, \iota, \phi].(\varpi^{-r}g)\) as above and put
\[
[X, \iota, \phi].g = [X', \iota' \circ \varpi^{-r}, \phi'] .
\]
This construction gives a natural transformation
\[
g_{m,m'} : \mathcal{M}_{K_m} \rightarrow \mathcal{M}_{K_{m'}} ,
\]
which depends neither on $\varpi$ nor on the integer $r$ (among all $r$’s with $o^n \subset \varpi^{-r} g o^n \subset \varpi^{-(m'-m')} o^n$). In particular, one gets for each $m$ an action from the right of $K_0 = GL_n(o)$ on $\mathcal{M}_K$ which commutes with the action of $B^\times$. If $g_i \in G, i = 1, 2$, satisfy

$$o^n \subset \varpi^{-r_1} g_1 o^n \subset \varpi^{-(m_0-m_1)} o^n, \quad o^n \subset \varpi^{-r_2} g_2 o^n \subset \varpi^{-(m_1-m_2)} o^n$$

for integers $r_1, r_2$ and $0 \leq m_2 \leq m_1 \leq m_0$ then one has clearly

$$o^n \subset \varpi^{-(r_1+r_2)} g_1 g_2 o^n \subset \varpi^{-(m_0-m_2)} o^n$$.

Therefore, we have morphisms

$$(g_1)_{m_0, m_1} : \mathcal{M}_{K_{m_0}} \to \mathcal{M}_{K_{m_1}}$$

$$(g_2)_{m_1, m_2} : \mathcal{M}_{K_{m_1}} \to \mathcal{M}_{K_{m_2}}$$

$$(g_1 g_2)_{m_0, m_2} : \mathcal{M}_{K_{m_0}} \to \mathcal{M}_{K_{m_2}},$$

and it is easy to check that the composition of the two former morphisms is identical to the latter one:

$$(2.2.3) \quad (g_2)_{m_1, m_2} \circ (g_1)_{m_0, m_1} = (g_1 g_2)_{m_0, m_2}.$$

Note that we defined the morphisms $g_{m_i, m_j}$ so as to obtain a right action.

**2.2.4.** We need to consider also quotients of the formal schemes $\mathcal{M}_{K_m}$. For an open subgroup $K \subset K_0$ we choose $m \geq 0$ so that $K_m \subset K$. Then let $\mathcal{M}_{K_m}^{(j)} = \text{Spf}(R_{m}^{(j)})$ and put

$$R_{K}^{(j)} = (R_{m}^{(j)})^{K}, \quad \mathcal{M}_{K}^{(j)} = \text{Spf}(R_{K}^{(j)}), \quad \mathcal{M}_{K} = \coprod_{j \in \mathbb{Z}} \mathcal{M}_{K}^{(j)},$$

where $(R_{m}^{(j)})^{K}$ denotes the subring of $K$-invariant elements in $R_{m}^{(j)}$.

**Proposition 2.2.5.** Fix $j \in \mathbb{Z}$ and $0 \leq m' \leq m$ such that $K_m \subset K \subset K_{m'}$. Whenever $0 \leq m_1 \leq m_2$ we identify $R_{m_1}^{(j)}$ with a subring of $R_{m_2}^{(j)}$ (cf. Theorem 2.1.2).

(i) $(R_{m'}^{(j)})^{K} = (R_{m}^{(j)})^{K}$ for all $m''$ with $K_{m''} \subset K$, and hence $R_{K}^{(j)}$ is well-defined. In particular, if $K = K_{m'}$ then $R_{K}^{(j)} = R_{m}^{(j)}$.

(ii) $R_{K}^{(j)}$ is a local noetherian ring, which is finite over $R_{m}^{(j)}$. It is complete with respect to the topology defined by the maximal ideal.
(iii) $R^{(j)}_K$ is the integral closure of $R^{(j)}_m$ in the field $\text{Frac}(R^{(j)}_K)$. It is thus integrally closed. The set of prime ideals of $R^{(j)}_K$ is in bijection with the orbits of $K/K_m$ on the set of prime ideals of $R^{(j)}_m$: 

$$\text{Spec}(R^{(j)}_m)/K \cong \text{Spec}(R^{(j)}_K).$$

(iv) $R^{(j)}_K[\frac{1}{m}]$ is étale over $R^{(j)}_m[\frac{1}{m}]$, and it is galois with Galois group $K_m'/K$ if $K$ is normal in $K_m'$. $R^{(j)}_m[\frac{1}{m}]$ is étale and galois over $R^{(j)}_K[\frac{1}{m}]$ with Galois group $K/K_m$.

Proof. (i) Suppose $m \geq m''$. Then $K_m \subset K_{m''} \subset K$ and

$$(R^{(j)}_m)^K = ((R^{(j)}_m)^{K_{m''}})^{K/K_{m''}} = (R^{(j)}_m)^K.$$  

(ii) Denote by $\mathfrak{m}$ the maximal ideal of $R^{(j)}_m$. The set of $K$-invariant elements $\mathfrak{m}^K$ in $\mathfrak{m}$ is a proper ideal of $R^{(j)}_K$. If now $f \in R^{(j)}_K$ is not in $\mathfrak{m}^K$, it is invertible in $R^{(j)}_m$, $gf = 1$, say, with $g \in R^{(j)}_K$. Then, for any $k \in K$ we have $g - k(g) = gf(g - k(g)) = g(fg - k(fg)) = 0$, hence $g \in R^{(j)}_K$, and it follows that $R^{(j)}_K$ is local. $R^{(j)}_m$ is submodule of the finite $R^{(j)}_{m''}$-module $R^{(j)}_m$, and as $R^{(j)}_{m''}$ is noetherian, $R^{(j)}_K$ is finite over $R^{(j)}_m$ too. This implies that $R^{(j)}_K$ is a noetherian ring. The completeness of $R^{(j)}_K$ with respect to the topology defined by the maximal ideal follows from the fact that the automorphisms $k \in K$ act continuously on $R^{(j)}_m$.

(iii) An element of $\text{Frac}(R^{(j)}_K) = \text{Frac}(R^{(j)}_m)^K$ which is integral over $R^{(j)}_m$ belongs to $R^{(j)}_m$ because the latter ring is integrally closed (as it is regular, hence factorial). So it is an element of $R^{(j)}_m \cap \text{Frac}(R^{(j)}_m)^K = R^{(j)}_K$. The assertion about the orbits of $K$ on $\text{Spec}(R^{(j)}_m)$ is a general fact, cf. [Bou], Ch. V, §2.1, Thm. 1, §2.2, Thm. 2.

(iv) $R^{(j)}_m[\frac{1}{m}]$ is the integral closure of $R^{(j)}_m[\frac{1}{m}]$ in $\text{Frac}(R^{(j)}_m)$, and by Thm. 2.1.2 it is flat and unramified over $R^{(j)}_m[\frac{1}{m}]$. Further, $R^{(j)}_K[\frac{1}{m}]$ is the integral closure of $R^{(j)}_m[\frac{1}{m}]$ in $\text{Frac}(R^{(j)}_m)$. By [Bou], Ch. V, Ex. 19 (c) to §2, $R^{(j)}_K[\frac{1}{m}]$ is unramified over $R^{(j)}_m[\frac{1}{m}]$ and $R^{(j)}_m[\frac{1}{m}]$ is unramified over $R^{(j)}_K[\frac{1}{m}]$. By [FK], Ch. 1, §1, Lemma 1.5., $R^{(j)}_K[\frac{1}{m}]$ is étale over $R^{(j)}_m[\frac{1}{m}]$, and by [FK], Ch. 1, §1, Prop. 1.7., $R^{(j)}_m[\frac{1}{m}]$ is étale over $R^{(j)}_K[\frac{1}{m}]$.  

We equip $R^{(j)}_K$ with the adic topology defined by the maximal ideal.

2.2.6. Consider $g \in G$ and suppose that $K' = g^{-1}Kg$ is contained in $K_0$. Choose $0 \leq m' \leq m$ and $r \in \mathbb{Z}$ such that

- $a^n \subset \mathfrak{m}^{-r}g^na \subset \mathfrak{m}^{-(m-m')}a^n$,
- $K_m \subset K, K_{m'} \subset g^{-1}Kg$. 
Put $d = v(\det(g))$. Then we have for any $k \in K$ and $j \in \mathbb{Z}$ a commutative diagram (by 2.2.3)

\[
\begin{array}{ccc}
\mathcal{M}^{(j)}_{K_m} & \xrightarrow{g_{m,m'}} & \mathcal{M}^{(j-d)}_{K_{m'}} \\
\downarrow{k} & & \downarrow{g^{-1}kg} \\
\mathcal{M}^{(j)}_{K_m} & \xrightarrow{g_{m,m'}} & \mathcal{M}^{(j-d)}_{K_{m'}}
\end{array}
\]

This shows that the ring homomorphism

\[ g_{m,m'}^* : R^{(j-d)}_m \longrightarrow R^{(j)}_m \]

maps the $g^{-1}Kg$-invariants in $R^{(j-d)}_m$ to the $K$-invariants in $R^{(j)}_m$, and defines hence a morphism

\[ g^*_K : R^{(j-d)}_{g^{-1}Kg} \longrightarrow R^{(j)}_K, \]

which in turn induces a morphism of formal schemes

\[ g_K : \mathcal{M}_K \longrightarrow \mathcal{M}_{g^{-1}Kg}. \]

### 2.3. Algebraization of the formal schemes and the group action.

We will show that the formal schemes $\mathcal{M}^{(j)}_K$ are completions of schemes of finite type over $\hat{\mathcal{O}}^{nr}$ at a closed point. This implies that the cohomology groups of the rigid-analytic spaces attached to $\mathcal{M}^{(j)}_K$ are finite-dimensional. Moreover, we will need to know that the action of endomorphisms of these formal schemes can be approximated. Later we will need such an algebraicity statement when we are going to apply a result of K. Fujiwara on the specialization of local terms.

**Theorem 2.3.1.** Fix an open subgroup $K \subset K_0$ and an integer $j \in \mathbb{Z}$.

There is an affine scheme of finite type $\mathfrak{M}^{(j)}_K$ over $\hat{\mathfrak{O}}^{nr}$ and a closed point $\mathfrak{r}_K$ on $\mathfrak{M}^{(j)}_K$ such that the following assertions hold:

(i) there is an isomorphism of formal schemes over $\hat{\mathfrak{O}}^{nr}$

\[ \mathfrak{M}^{(j)}_{K,s_K} \cong \mathcal{M}^{(j)}_K \]

between the completion $\mathfrak{M}^{(j)}_{K,s_K}$ of $\mathfrak{M}^{(j)}_K$ at $\mathfrak{r}_K$ and $\mathcal{M}^{(j)}_K$;

(ii) the scheme $\mathfrak{M}^{(j)}_K$ carries an action of $K_0$, and the subgroup $K$ acts trivially on $\mathfrak{M}^{(j)}_K$.
(iii) the point \( \mathfrak{r}_K \) is a fixed point of the action of \( K_0 \), and the isomorphism between \( \mathcal{M}^{(j)}_{K, \mathfrak{r}} \) and \( \mathcal{M}^{(j)}_K \) is \( K_0 \)-equivariant.

Proof. We may assume without loss of generality that \( j = 0 \), and we identify \( R_0^{(0)} \) with a power series ring in the variables \( u_1, \ldots, u_{n-1} \) over \( \hat{\sigma}^{nr} \). For the proof we drop the superscripts.

(a) We show first that the group scheme \( X^{univ}[\varpi^m] \) is defined over some subring \( \mathcal{R} \subset R_0 \) which is of finite type over \( \hat{\sigma}^{nr} \), and whose completion at a maximal ideal is isomorphic to \( R_0 \). To do this we will reason along the same lines as explained after Thm. 1.7. of [A2]. In [F3] Faltings introduces what one may call ‘truncated Barsotti-Tate \( \sigma \)-modules’ (of level \( m \)). This is the analogue of the notion of a truncated Barsotti-Tate group (in the sense of [Me], and [Il]) in the context of group schemes with strict \( \sigma \)-action. Such an object is in particular of finite presentation over the base scheme. Put \( A = \hat{\sigma}^{nr}[u_1, \ldots, u_{n-1}] \), \( S = \text{Spec}(A) \), and consider the functor \( BT_m^\sigma/S \) on the category of \( S \)-schemes which associates to \( T/S \) the set of isomorphism classes of truncated Barsotti-Tate \( \sigma \)-modules of level \( m \) over \( T \). It is a functor which is locally of finite presentation over \( S \). We let \( \hat{A} \) denote the completion of \( A \) with respect to the ideal \( \mathfrak{m} = (\varpi, u_1, \ldots, u_{n-1}) \), i.e. \( \hat{A} = R_0 \), and put \( \hat{S} = \text{Spec}(\hat{A}) \). We have \( X^{univ}[\varpi^m] \in BT_m^\sigma(\hat{S}) \). By Artin’s approximation theorem, [A1], Cor. 2.2, there is an \( \acute{e} \text{tale} \) neighborhood (which we may assume to be affine)

\[
S' = \text{Spec}(A') \quad \longrightarrow \quad S = \text{Spec}(A) \quad \longrightarrow \quad \text{Spec}(A/\mathfrak{m})
\]

of the maximal ideal \( \mathfrak{m} \) and an element \( \mathfrak{x}_m \in BT_m^\sigma(S') \) such that \( \mathfrak{x}_m \) and \( X^{univ}[\varpi^m] \) have the same images in \( \hat{A}/\mathfrak{m}^2 \hat{A} = A'/\mathfrak{m}^2 A' \). Now we use the fact that the pair \( (\hat{A}, X^{univ}[\varpi^m]) \) is an effective formal versal deformation of \( X[\varpi^m] \) over \( \mathbb{F} \) in the sense of [A1], sec. 1. In the case \( \sigma = \mathbb{Z}_p \) this statement is Cor. 4.8 (ii) in [Il]. For arbitrary \( \sigma \) the versality of \( X[\varpi^m] \) follows from the results of Faltings, cf. [F3], p. 276, with the same arguments as in [Il]. We conclude as explained after Thm. 1.7. of [A2]: there is an \( \hat{\sigma}^{nr} \)-linear automorphism of \( \hat{A} \) such that \( \mathfrak{x}_m \times_{\hat{S}} \hat{S} \) is isomorphic to the pull back of \( X^{univ}[\varpi^m] \) via this isomorphism. We put \( \mathcal{R} = A' \) and \( \mathcal{M} = S' \). This is the algebraization we are looking for. We denote the point corresponding to the maximal ideal \( \mathfrak{m} \subset \mathcal{R} \) by \( \mathfrak{r} \).

(b) We define the notion of a Drinfeld level structure for truncated Barsotti-Tate \( \sigma \)-modules as in [HT], sec. II.2. By [HT], Lemma II.2.1 (6.), there exists a scheme \( \mathcal{M}_m \) which classifies level-\( m \)-structures on \( \mathfrak{x}_m \times_{\mathcal{M}} T \) over schemes \( T \) over \( \mathcal{M} \). As \( \mathcal{M}_m \) is finite over \( \mathcal{M} \), it is affine, and we put \( \mathcal{M}_m = \mathcal{O}_{\mathcal{M}_m}(\mathcal{M}_m) \). \( \mathcal{M}_m \) carries a natural action of \( GL_n(\sigma/\varpi^m) \) (action on the level structure). As there is only one level-\( m \)-structure on \( X[\varpi^m] \), there is only one point \( \mathfrak{r}_m \) of \( \mathcal{M}_m \) over \( \mathfrak{r} \). Passing to the completion at \( \mathfrak{r}_m \), we
get that $\widehat{\mathcal{M}}_{m,\ell_m}$ classifies level-$m$-structures on $X^{univ}[\varpi^m]$, hence is isomorphic to $\mathcal{M}_m$. Finally, if $K \subset K_0$ is an open subgroup containing $K_m$, we put $\mathfrak{N}_K = (\mathfrak{N}_m)_K$ and $\mathfrak{M}_K = \text{Spec}(\mathfrak{N}_K)$. The assertions (i)-(iii) are now clear.

Remark. This proof is due to L. Fargues, cf. [Fa1], sec. 9.2.1, Prop. 5, who proved an even stronger statement which gives a better control on the action of $GL_n(\mathfrak{o}/\varpi^m)$ on the 'boundary' (in the sense of loc. cit., Prop. 5).

Whereas the action of $GL_n(\mathfrak{o}/\varpi^m)$ on $\mathcal{M}^{(j)}_K$ extends, by our construction, to the algebraizations $\mathfrak{M}^{(j)}_K$, this is not the case for the action of elements in $\mathfrak{o}_B^\times$. For our purposes, it is enough to know that any given endomorphism $\gamma$ of one of our formal schemes $\mathfrak{M}^{(j)}_K$ can be approximated by a suitable correspondence. More precisely, we can find an étale neighborhood $\widetilde{\mathfrak{M}}^{(j)}_K$ of the point $\mathfrak{x}_K$ (cf. Prop. 2.3.1) in $\mathfrak{M}^{(j)}_K$ and a morphism from $\widetilde{\mathfrak{M}}^{(j)}_K$ to $\mathfrak{M}^{(j)}_K$ such that such that the induced endomorphism on the completions, both being isomorphic to $\mathfrak{M}^{(j)}_K$, approximate the given endomorphism $\gamma$ up to prescribed order.

**Theorem 2.3.2.** Suppose $\gamma : \mathfrak{M}^{(j)}_K \to \mathfrak{M}^{(j)}_K$ is a morphism of formal schemes over $\mathfrak{o}^{nr}$. Then, for any $c \in \mathbb{Z}_{\geq 0}$ there is an étale neighborhood $\widetilde{\mathfrak{M}}^{(j)}_K$ of the point $\mathfrak{x}_K \in \mathfrak{M}^{(j)}_K$

\[
\begin{array}{ccc}
\text{Spec}(\kappa(\mathfrak{x}_K)) & \longrightarrow & \widetilde{\mathfrak{M}}^{(j)}_K \\
\downarrow \text{id} & & \downarrow \\
\text{Spec}(\kappa(\mathfrak{x}_K)) & \longrightarrow & \mathfrak{M}^{(j)}_K
\end{array}
\]

and a morphism of schemes over $\mathfrak{o}^{nr}$

$\gamma_c : \widetilde{\mathfrak{M}}^{(j)}_K \longrightarrow \mathfrak{M}^{(j)}_K$

such that $\gamma_c(\mathfrak{x}_K) = \mathfrak{x}_K$, and $\gamma_c$ induces a morphism

$\hat{\gamma}_c : \mathcal{M}^{(j)}_K \longrightarrow \mathcal{M}^{(j)}_K$

such that $\hat{\gamma}_c \equiv \gamma (m_K^c)$ where $m_K$ is the maximal ideal of $R^{(j)}_K$, i.e. if

\[
\gamma^{\sharp}, \hat{\gamma}_c^{\sharp} : R^{(j)}_K \longrightarrow R^{(j)}_K
\]

are the corresponding ring homomorphisms, then, for all $f \in R^{(j)}_K$

\[
\gamma^{\sharp}(f) \equiv \hat{\gamma}_c^{\sharp}(f) (m_K^c).
\]

**Proof.** This is [A1], Cor. 2.5. \qed
2.4. Associated analytic spaces.

The next step is to introduce the analytic spaces whose $\ell$-adic étale cohomology groups we are going to study. There are different possible methods how to construct such spaces: as rigid-analytic spaces, as non-Archimedean analytic spaces as defined and studied by V. G. Berkovich [Be1], as Zariski-Riemann spaces ([Fu1], [Bom]) or finally as adic spaces in the sense of R. Huber [Hu1]. For each of these kinds of spaces there has been defined an étale cohomology theory ([dJvdP], [Be2], [Fu2], [Hu3]) and there are comparison theorems assuring that the resulting cohomology groups for the spaces considered by us are the same ([Hu3], sec. 8.3). In section 3 we make use of the theory of adic or Zariski-Riemann spaces. Therefore, we define the spaces we are going to work with directly as adic spaces. Nevertheless, we will now give brief references concerning the other constructions.

There is a construction of P. Berthelot, generalizing Raynaud’s construction for $\varpi$-adic formal schemes, which associates to any formal scheme which is locally formally of finite type over a discrete valuation ring $V$ a rigid analytic space ([Ber], 0.2.6, [RZ], sec. 5.5) over the field of fractions. Secondly, Berkovich has defined for so-called special formal schemes over $V$ (these possess affine coverings of the form Spf($R$) where $R$ is a quotient of some algebra $V(x_1,...,x_m)[[y_1,...,y_n]]$) an associated non-Archimedean analytic space [Be3], sec.1. Finally, R. Huber associates to a locally noetherian formal scheme $X$ an adic space $t(X)$ (cf. [Hu2], sec. 4, and, more generally, [Hu3], sec. 1.9). If $X = \text{Spf}(R)$ is affine, the set of points of the underlying topological space of $t(X) = \text{Spa}(R,R)$ consists of all equivalence classes of continuous valuations $|\cdot|_v$ on $R$ such that $|f|_v \leq 1$ for all $f \in R$. (We recall that all valuations occurring in the context of adic spaces will be written multiplicatively.) If $R = R_K^{(j)}$, the set of valuations $|\cdot|_v$ with $|\varpi|_v = 0$ is a closed subset which we denote by $V(\varpi)$. The open complement inherits the structure of an adic space and we put

$$M_K^{(j)} = t(M_K^{(j)}) - V(\varpi), \quad M_K = \prod_{j \in \mathbb{Z}} M_K^{(j)}.$$

If $K \subset K'$ are open subgroups of $K_0$ there is a morphism of adic spaces

$$M_K \longrightarrow M_{K'},$$

induced from the corresponding morphism of formal schemes $M_K \rightarrow M_{K'}$ (cf. 2.2.5). This morphism is always étale, and it is galois with Galois group $K'/K$ if $K$ is normal in $K'$, cf. 2.2.5 (iv). In particular the Galois group of $M_{K_m}$ over $M_{K_0}$ is $GL_n(\mathcal{o}/(\varpi^m))$. By 2.1.2 each space $M_{K_0}^{(j)}$ is isomorphic to an open polydisc of dimension $n-1$; in particular:

$$M_{K_0}^{(j)}(\bar{F}^\wedge) \simeq \{(z_1,\ldots,z_{n-1}) \in (\bar{F}^\wedge)^{n-1} | \text{for all } i : |z_i| < 1 \}.$$
Moreover, we get induced group actions on the analytic spaces. For any $g \in G$ and an open subgroup $K \subset K_0$ such that $g^{-1}Kg \subset K_0$ there is a morphism of analytic spaces

$$g : M_K \to M_{g^{-1}Kg},$$

and these morphisms, for varying $g$ and $K$, are compatible with each other whenever composition is defined. All the spaces $M_K$ also come with an induced action of $B^\times$ which commutes with the morphisms induced by elements of $G$.

### 2.5. $\ell$-adic cohomology and statement of the main result.

We are going to introduce the cohomology groups. We use the étale cohomology theory as developed by Huber ([Hu3]), respectively Berkovich ([Be2]). Because of the comparison theorems in [Hu3], sec. 8.3, we can and will use results of Berkovich for the étale cohomology of non-Archimedean analytic spaces. From now on, we fix a prime number $\ell$ which is different from the residue characteristic of $F$. So far, the cohomology theories and the results concern mostly the cohomology of torsion sheaves, and a general theory of $\ell$-adic cohomology has not been developed yet. But for the spaces considered by us we can show the following

**Lemma 2.5.1.** For any open subgroup $K \subset K_0$ and any $j \in \mathbb{Z}$, the $\mathbb{Q}_\ell$-vector spaces

$$H^i(M^{(j)}_K \times_{\hat{F}^\text{nr}} \hat{F}^\wedge, \mathbb{Q}_\ell) := \left( \lim_{\leftarrow r} H^i(M^{(j)}_K \times_{\hat{F}^\text{nr}} \hat{F}^\wedge, \mathbb{Z}/\ell^r\mathbb{Z}) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

and

$$H^i_c(M^{(j)}_K \times_{\hat{F}^\text{nr}} \hat{F}^\wedge, \mathbb{Q}_\ell) := \left( \lim_{\leftarrow r} H^i_c(M^{(j)}_K \times_{\hat{F}^\text{nr}} \hat{F}^\wedge, \mathbb{Z}/\ell^r\mathbb{Z}) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

are finite-dimensional and the induced action of $\Theta_B^\times$ on these spaces is smooth. The cohomology groups $H^i(M^{(j)}_K \otimes_{\hat{F}^\text{nr}} \hat{F}^\wedge, \mathbb{Q}_\ell)$ vanish for $i > n - 1$, and the cohomology groups $H^i_c(M^{(j)}_K \otimes_{\hat{F}^\text{nr}} F^\wedge, \mathbb{Q}_\ell)$ vanish for $i < n - 1$ and $i > 2(n - 1)$.

**Proof.** (Cf. [HT], Lemma I.5.1) By [Be3], Prop. 2.4, the cohomology group $H^i(M^{(j)}_K \otimes_{\hat{F}^\text{nr}} \hat{F}^\wedge, \mathbb{Z}/\ell^r\mathbb{Z})$ is isomorphic to the stalk of $R^i\Theta_F^*(\mathbb{Z}/\ell^r\mathbb{Z})$ at the single point in the special fibre of $M^{(j)}_K$, where $F^s$ denotes the separable closure of $\hat{F}^\text{nr}$ and $\Theta_F^*$ is Berkovich’s vanishing cycle functor, cf. [Be3], sec. 2. By 2.3.1, the formal scheme $M^{(j)}_K$ is algebraizable. By Berkovich’s comparison theorem, [Be3], Thm. 3.1, the stalk of $R^i\Theta_F^*(\mathbb{Z}/\ell^r\mathbb{Z})$ at the closed point is canonically isomorphic to the stalk of the corresponding vanishing cycle sheaf for the scheme $\mathfrak{M}^{(j)}_K$ at the closed point $x_K$, cf. 2.3.1. By Nagata’s theorem on compactifications, together with Thm. 3.2 in [De] and Prop. 5.3.1 in Exp. V in [SGA5], the
sheaf of vanishing cycles \( R^i \Phi(Q_\ell) \) on the special fibre of \( \mathcal{M}_K^{(j)} \) is constructible, and hence its stalk at \( r_K \) is finite-dimensional. Similarly, for the cohomology groups with compact support, we use [Be3], Cor. 2.5 and Thm. 3.1, and the finiteness theorem Thm. 3.2 of [De].

The spaces \( \mathcal{M}_K^{(j)} \) are quasi-affine in the sense of [Be3], Def. 5.5., and hence the statement about the vanishing of cohomology in degree \( > n - 1 \) (resp. in degree \( < n - 1 \)) follows from [Be3], Thm. 6.1, Cor. 6.2. The smoothness of the action of \( o_B^+ \) follows from [Be3], Cor. 4.5. \( \square \)

Next we put

\[
H^i_c(M_K) = H^i_c(M_K \times \hat{F}_{\text{nr}}, \hat{\mathcal{F}}^\wedge, Q_\ell) = \bigoplus_{j \in \mathbb{Z}} H^i_c(M_K^{(j)} \times \hat{F}_{\text{nr}}, \hat{\mathcal{F}}^\wedge, Q_\ell) \otimes_{Q_\ell} \overline{Q_\ell}.
\]

On each \( \overline{Q_\ell} \)-vector space \( H^i_c(M_K) \) there is an induced action of \( K_0 \times B^\times \) and for each \( g \in G \) there is an isomorphism

\[
H^i_c(M_{g^{-1}Kg}) \to H^i_c(M_K),
\]

as soon as \( g^{-1}Kg \subset K_0 \). These give rise to a representation of \( G \times B^\times \) on

\[
H^i_c := \lim_{\rightarrow} H^i_c(M_K),
\]

where the limit is taken over all compact-open subgroups \( K \) of \( G \) contained in \( K_0 \).

Remark. The inertia group \( \text{Gal}(\overline{F^{\text{sep}}}/F^{\text{nr}}) \) acts also on \( H^i_c(M_K) \), and this action can be extended to an action of the Weil group \( W_F \), cf. [Bo], Prop. 2.3.2, [RZ], sec. 3.48. Then one gets a smooth/continuous action of \( G \times B^\times \times W_F \) on \( H^i_c \). In this paper however we pay only attention to the representations of \( G \) and \( B^\times \).

The main result of this paper is the following theorem. Its content is implied by Boyer’s Theorem, [Bo], Thm. 3.2.4, in the equal characteristic case, and it follows from the work of Harris and Taylor [HT] in the mixed characteristic case. Whereas in these works Shimura or Drinfeld-Stuhler varieties play a decisive role, the proof given here does not use modular varieties. The second assertion of the following theorem is Thm. 4.1.3. For the last assertion, which follows immediately from Thm. 4.3.2 (iv), we have to assume that certain cohomology groups are finite-dimensional, cf. hypothesis (H) in 4.2.7.

**Theorem 2.5.2.** Let \( \pi \) be an irreducible supercuspidal representation of \( G \), supposed to be realized over \( \overline{Q_\ell} \).

(i) For each \( i \) the representation \( \text{Hom}_G(H^i_c, \pi) \) of \( B^\times \) is finite-dimensional and smooth.

(ii) In the Grothendieck group of admissible representations of \( B^\times \) one has
Deformation spaces of one-dimensional formal modules and their cohomology

\[ \sum_i (-1)^{i+n-1} \text{Hom}_G(H^i_c, \pi) = n \cdot J\mathcal{L}(\pi), \]

where \( J\mathcal{L}(\pi) \) is the representation of \( B^\times \) associated to \( \pi \) by the Jacquet-Langlands correspondence.

(iii) Suppose hypothesis (H) in 4.2.7 does hold. Then, if \( i \neq n-1 \) one has \( \text{Hom}_G(H^i_c, \pi) = 0 \), and

\[ \text{Hom}_G(H^{n-1}_c, \pi) \simeq J\mathcal{L}(\pi)^{\oplus n}. \]

\( (J\mathcal{L}(\pi)^{\oplus n} \) is the direct sum of \( n \) copies of \( J\mathcal{L}(\pi). \)

Proof of (i). The element \( \varpi \) of the center of \( G \) acts as a scalar on \( \pi \), and this scalar we can write as \( c^n \) for some \( c \in \mathbb{Q}_l \). Put \( \zeta(g) = c^{-v(\text{det}(g))}. \) Then:

\[ \text{Hom}_G(H^i_c, \pi) = \text{Hom}_G(H^i_c \otimes \zeta, \pi \otimes \zeta) \]

\[ = \text{Hom}_G((H^i_c \otimes \zeta)/\langle v - c^{-n} \cdot \varpi \cdot v | v \in H^i_c \rangle, \pi \otimes \zeta), \]

where \( \varpi \cdot v \) denotes the action of \( \varpi \), considered as an element of \( G \), on \( v \), considered as an element of \( H^i_c \).

Next, \( (H^i_c \otimes \zeta)/\langle v - c^{-n} \cdot \varpi \cdot v | v \in H^i_c \rangle \) is isomorphic, as a representation of \( G \times B^\times \), to the natural representation of \( G \times B^\times \) on

\[ \left( \lim_{\mathcal{K}} H^i_c(M_K/\varpi^Z) \right) \otimes \xi, \]

where

\[ H^i_c(M_K/\varpi^Z) = H^i_c((M_K/\varpi^Z) \times_{\text{f.w.r.}} \overline{F}^\times, \overline{\mathbb{Q}}_l) \]

and the limit is taken over all compact-open subgroups \( K \subset K_0 \) of \( G \), and \( \xi \) is the character of \( B^\times \) given by \( \xi(b) = c^{-v(N(b))}. \) The map is defined as follows: an element \( v \in H^i_c(M_K^{(j)}, \overline{\mathbb{Q}}_l) \) is mapped to \( c^{nk} \varpi^{-k} \cdot v \in H^i_c(M_K^{(j_0)}, \overline{\mathbb{Q}}_l) \), where \( j = j_0 + nk \) with \( 0 \leq j_0 < n \). It is not difficult to check that this map gives a \( G \times B^\times \)-equivariant isomorphism

\[ (H^i_c \otimes \zeta)/\langle v - c^{-n} \cdot \varpi \cdot v | v \in H^i_c \rangle \xrightarrow{\sim} \left( \lim_{\mathcal{K}} H^i_c(M_K/\varpi^Z) \right) \otimes \xi. \]

Hence we get the following identity of representations of \( B^\times \):
\[ \text{Hom}_G(H^i_c, \pi) = \text{Hom}_G \left( H^i_c(M\infty/\mathcal{O}, \pi \otimes \zeta) \otimes \xi^{-1} \right), \]

where

\[ H^i_c(M\infty/\mathcal{O}) := \lim_{\longrightarrow} K \text{Hom}^i_c((M_K/\mathcal{O}, \times F\text{or } \hat{F}^\wedge, \mathbb{Q}_\ell). \]

The representation \( H^i_c(M\infty/\mathcal{O}) \) is admissible if we regard it as a representation of \( G \), because if \( K \subset K_0 \) is an open subgroup, its subspace of \( K \)-invariant vectors is just \( H^i_c(M_K/\mathcal{O}) \), which is finite-dimensional, by 2.5.1. Therefore, if we denote by \( H^i_c(M\infty/\mathcal{O})_{\text{cusp}} \) its cuspidal part, we have:

\[ H^i_c(M\infty/\mathcal{O})_{\text{cusp}} = \bigoplus_{i \in I} \pi_i \otimes \rho_i, \]

where \( I \) is some countable set (because \( G/\mathcal{O} \) has compact center), the representations \( \pi_i \) are supercuspidal and pairwise non-isomorphic, and \( \rho_i \) is a finite-dimensional smooth representation of \( B^\times \), because \( \pi_i \) has finite multiplicity in \( H^i_c(M\infty/\mathcal{O}) \). This proves the first assertion. \( \square \)

2.6. The period morphism and fixed points.

2.6.1. To count fixed points we will use the period map from the moduli spaces \( M_K \) to a projective space of dimension \( n - 1 \). This map was first studied by M. Hopkins and B. Gross [HG]. Later on, M. Rapoport and Th. Zink introduced these morphisms for moduli spaces for \( p \)-divisible groups [RZ], thereby giving a unified account of \( p \)-adic period maps that have been studied before. The set-up of Gross and Hopkins is insofar closer to our situation as they work with formal \( o \)-modules (hence treat the mixed and equal characteristic case simultaneously), herein following Drinfeld. On the other hand, Gross and Hopkins only work with one component of the moduli space \( \mathcal{M}_{K_0} \), namely the component \( \mathcal{M}^{(0)}_{K_0} = \text{Spf}(R^{(0)}_0) \) where the quasi-isogeny on the special fibre has height zero. After recalling the main results of [HG] in the next section, we will explain how to define the period map on the whole space \( M_{K_0} \).

2.6.2. Let \( X^{\text{univ}} \) be the universal formal \( o \)-module over the formal scheme \( \mathcal{M}^{(0)}_{K_0} \), and denote by \( \mathcal{E} \) the universal extension of \( X^{\text{univ}} \) with additive kernel. This is a formal \( o \)-module of dimension \( n \) which sits in an exact sequence

\[ 0 \to \mathcal{V} \to \mathcal{E} \to X^{\text{univ}} \to 0, \]

where \( \mathcal{V} = \mathbb{G}_a \otimes \text{Hom}_{K_0}(\text{Ext}(\mathcal{A}, \mathbb{G}_a), R^{(0)}_0) \). This exact sequence furnishes an exact sequence
Deformation spaces of one-dimensional formal modules and their cohomology

\[ 0 \to \text{Lie}(\mathcal{V}) \to \text{Lie}(\mathcal{E}) \to \text{Lie}(X^{\text{univ}}) \to 0, \]

of vector bundles on the formal scheme \( \mathcal{M}^{(0)}_{K_0} \), and an analogous sequence

\[ 0 \to \text{Lie}(\mathcal{V})^{\text{ad}} \to \text{Lie}(\mathcal{E})^{\text{ad}} \to \text{Lie}(X^{\text{univ}})^{\text{ad}} \to 0, \]

on the generic fibre of this formal scheme, i.e. on the space \( M^{(0)}_{K_0} \).

**Proposition 2.6.3.** ([HG], Prop. 22.4, 23.2, 23.4) (i) There is a basis \( c_0, \ldots, c_{n-1} \) of \( \text{Lie}(\mathcal{E})^{\text{ad}} \) such that the \( \hat{F}^{\text{nr}} \)-subspace generated by these global sections is stable by the action of \( \mathfrak{o}_B^\times \). More precisely, the canonical map of vector bundles on \( M^{(0)}_{K_0} \)

\[ \langle c_0, \ldots, c_{n-1} \rangle \hat{F}^{\text{nr}} \otimes \mathcal{O}_{M^{(0)}_{K_0}} \to \text{Lie}(\mathcal{E})^{\text{ad}} \]

is an \( \mathfrak{o}_B^\times \)-equivariant isomorphism, where \( \mathfrak{o}_B^\times \) acts diagonally on the left hand side. The representation of \( \mathfrak{o}_B^\times \) on \( \langle c_0, \ldots, c_{n-1} \rangle \hat{F}^{\text{nr}} \) is equivalent to the representation of \( \mathfrak{o}_B^\times \) on \( B \otimes_{F_B} \hat{F}^{\text{nr}} \) given by left multiplication (where \( F_n/F \) is the unramified extension of degree \( n \) in \( \hat{F}^{\text{nr}} \)).

(ii) Let \( w_i \) be the image of \( c_i \) in \( \text{Lie}(X^{\text{univ}})^{\text{ad}}, i = 0, \ldots, n-1 \), and denote by \( W \) the space generated by these global sections over \( \hat{F}^{\text{nr}} \). Then the sections \( w_i \) have no common zeroes, and they are linearly independent over \( \hat{F}^{\text{nr}} \).

(iii) Denote by \( \mathbb{P}(W) \) the projective space of hyperplanes in \( W \), and by \( \mathbb{P}(W)^{\text{ad}} \) the associated analytic adic space. Define

\[ \pi^{(0)}_{K_0} : M^{(0)}_{K_0} \to \mathbb{P}(W)^{\text{ad}} \]

by sending \( x \in M^{(0)}_{K_0} \) to the hyperplane

\[ \{ w = \alpha_0 w_0 + \ldots + \alpha_{n-1} w_{n-1} \in W \otimes \hat{F}^{\text{nr}}(x) \mid \alpha_0 w_0(x) + \ldots + \alpha_{n-1} w_{n-1}(x) = 0 \} \]

This map is an étale morphism. It is \( \mathfrak{o}_B^\times \)-equivariant and surjective on \( \hat{F}^{\text{nr}} \)-valued points.

**2.6.4.** Choose an element \( \varpi_B \in \mathfrak{o}_B \) whose reduced norm is a uniformizer of \( F \). The action of \( B^\times \) on \( \mathcal{M}_{K_0} \) furnishes for each \( j \in \mathbb{Z} \) an isomorphism

\[ \varpi_B^j : \mathcal{M}^{(0)}_{K_0} \to \mathcal{M}^{(j)}_{K_0}. \]
Define $\pi_{K_0}^{(j)} : M_{K_0}^{(j)} \to \mathbb{P}(W)$ by $\pi_{K_0}^{(j)} = \varpi_B^{j} \circ \pi_{K_0}^{(0)} \circ \varpi_B^{-j}$. Because of the $\mathcal{O}_B^\times$-equivariance of $\pi_{K_0}^{(0)}$, this map does not depend on the choice of $\varpi_B$. Finally we get the period map on the whole space $M_{K_0}$ by putting

$$\pi_{K_0} = \prod_{j \in \mathbb{Z}} \pi_{K_0}^{(j)} : M_{K_0} = \prod_{j \in \mathbb{Z}} M_{K_0}^{(j)} \to \mathbb{P}(W)^{ad}.$$ 

More generally, for any open subgroup $K \subset K_0$ we let $\pi_K$ be the composition of the projection $M_K \to M_{K_0}$ with $\pi_{K_0}$, and refer to $\pi_K$ as a period morphism. The proposition above gives immediately the following assertion about the morphisms $\pi_K$.

**Proposition 2.6.5.** For any open subgroup $K \subset K_0$

$$\pi_K : M_K \to \mathbb{P}(W)^{ad}$$

is an étale morphism of analytic adic spaces over $\mathcal{F}^{nr}$. Moreover, $\pi_K$ is equivariant with respect to the action of $N_G(K) \times B^\times$, where the normalizer $N_G(K)$ of $K$ in $G$ acts trivially on $\mathbb{P}(W)^{ad}$ and the action of $B^\times$ on $\mathbb{P}(W)^{ad}$ is the one that is induced by the action of $B^\times$ on $W$.

2.6.6. Now we are in a position to count fixed points. Let $b \in B^\times$ be an element which is regular elliptic. Hence $b$ has $n$ distinct simple fixed points on $\mathbb{P}(W)^{ad} \times \mathcal{F}^\wedge$. Let $K$ be a compact-open subgroup of $G$ that is contained in $K_0$, and let $g$ be an element of the normalizer of $K$ in $G$. By Proposition 2.6.5, the action of the pair $(g, b^{-1})$ on $M_K \otimes \mathcal{F}^\wedge$ stabilizes the fibre of $\pi_K$ over a fixed point of $b^{-1}$ on $\mathbb{P}(W)^{ad} \times \mathcal{F}^\wedge$. Hence we need a description of the fibres of $\pi_K$ together with the action of $(g, b^{-1})$. The next proposition gives such a description.

**Proposition 2.6.7.** (i) Let $x \in M_{K_0}(\mathcal{F}^\wedge)$, and let $[X, \iota]$ be the deformation of $\mathbb{X}$ corresponding to $x$. Then, the fibre of $\pi_{K_0}$ through $x$ consists of all deformations which are quasi-isogenous to $X$. More precisely, it consists of those pairs $[X', \iota']$ such that there exists a quasi-isogeny $f : X' \to X$ with the property that $f_\mathbb{F} \circ \iota' = \iota$, where $f_\mathbb{F}$ is the the reduction of $f$.

(ii) The fibre of $\pi_{K_0}$ through $x$ can be identified with the set of lattices in the rational Tate module $V(X) = T(X) \otimes_{\mathcal{O}} F$, where

$$T(X) = \lim_{\leftarrow} X[\varpi^m](\mathcal{F}^\wedge).$$

By fixing an isomorphism $\phi : F^n \to V(X)$, this set gets identified with $G/K_0$. More generally, let $K \subset K_0$ be an open subgroup, and let $[X, \iota, \phi]$ be a point of $M_K(\mathcal{F}^\wedge)$. Then, the fibre of $\pi_K$ through this point can be identified with the coset $G/K$. 
(iii) Consider an $\bar{F}^\wedge$-valued fixed point of $b$ on $\mathbb{P}(W)^{ad}$, and choose a base point of the set of $\bar{F}^\wedge$-valued points of the fibre of $\pi_K$ over this point. Using this fixed point, identify this set with $G/K$, as in (ii). Then there exists $g_b \in G$ with the same characteristic polynomial as $b$ such that the action of $(g, b^{-1})$ on the (set of $\bar{F}^\wedge$-valued points of the) fibre is given, in terms of this identification, by

$$hK \mapsto g_b hgK.$$ 

**Proof.** The first assertion follows from Prop. 23.28 of [HG]. The relationship between lattices in the rational Tate module and quasi-isogenies in the mixed characteristic case can be found in Lubin’s paper [Lu], Theorem 2.2. The same holds true also in the equal characteristic case, cf. [Yu], sec. 3. The second assertion of (ii) follows immediately.

Now we are going to prove part (iii). Fix an $\bar{F}^\wedge$-valued point of $M_K$, given by a triple $[X, \iota, \phi]$. We can consider $\phi$ as an isomorphism $\sigma^n \to T(X)$ which is determined up to multiplication (from the right) by elements from $K$. Suppose this point is mapped by $\pi_K$ onto a fixed point of $b$. Then it follows from [HG], Prop. 23.28, that $\tilde{b}$ lifts to an endomorphism $\tilde{b} : X \to X$ of the formal $\sigma$-module $X$ such that $\tilde{b}_\sigma \circ \iota = \iota \circ b$, where $\tilde{b}_\sigma$ is the quasi-isogeny induced on the special fibre. $\tilde{b}$ is mapped to $b$ under the canonical map $\text{End}_\sigma(X) \otimes F \to \text{End}_\sigma(X) \otimes F$. Therefore the characteristic polynomial of $\tilde{b}$ is the same as that of $b$. Let $g_b \in G$ be such that the following diagram is commutative:

$$\begin{array}{ccc}
F^n & \xrightarrow{\phi} & V(X) \\
g_b \downarrow & & \downarrow V(\tilde{b}) \\
F^n & \xrightarrow{\phi} & V(X)
\end{array}$$

Let $[X', \iota', \phi']$ be an element in the fibre of $\pi_K$. Hence there is a quasi-isogeny $f : X' \to X$ and an element $h \in G$ such that the following diagram commutes:

$$\begin{array}{ccc}
F^n & \xrightarrow{\phi'} & V(X') \\
h \downarrow & & \downarrow V(f) \\
F^n & \xrightarrow{\phi} & V(X)
\end{array}$$

The class $hK \in G/K$ corresponds to the point $[X', \iota', \phi']$. This point is mapped by $b^{-1}$ to $[X', \iota' \circ b^{-1}, \phi']$. The map $\tilde{b} \circ f : X' \to X$ is then a quasi-isogeny, if we equip $X'$ with the map $\iota' \circ b^{-1} : X \to (X')_{\sigma}$. Moreover, we see that the following diagram commutes:

$$\begin{array}{ccc}
F^n & \xrightarrow{\phi'} & V(X') \\
g_bh \downarrow & & \downarrow V(\tilde{b})V(f) \\
F^n & \xrightarrow{\phi} & V(X)
\end{array}$$
The action of \( b^{-1} \) on the fibre of \( \pi_K \) is thus given by sending \( hK \) to \( g_bhK \). It is straightforward to check that the action of some \( g \in N_G(K) \) on this fibre is given by sending \( hK \) to \( hgK \). This proves the third assertion. \( \square \)

**Theorem 2.6.8.** Let \( g_b \) be in the conjugacy class corresponding to \( b \). Then the number of (intersection theoretic) fixed points of the pair \((g,b^{-1})\) on the space \((M_K/\varpi^Z) \times \bar{F}^\wedge \) is equal to

\[
n \cdot \# \{ h \in G/\varpi^ZK \mid h^{-1}g_bh = g^{-1} \},
\]

and each fixed point is simple. The identity \( h^{-1} = g_bhg^{-1} \) means that for some (and hence any) representative \( \dot{h} \) of \( h \in G/\varpi^ZK \) the cosets \( \dot{h}^{-1}g_b\dot{h}\varpi^ZK \) and \( g^{-1}\varpi^ZK \) are equal. By the fact pointed out in 4.1.1, the number of such \( h \in G/\varpi^ZK \) is always finite.

**Proof.** Let \( b \in B^\times \) be regular elliptic and consider the fibre of the induced map

\[
(M_K/\varpi^Z)(\bar{F}^\wedge) \to \mathbb{P}(W)(\bar{F}^\wedge)
\]

over a fixed point of \( b^{-1} \). By the preceding proposition, we may identify this set with \( G/\varpi^ZK \) and the action of \((g,b^{-1})\), \( g \) in the normalizer of \( K \) in \( G \), is given by

\[
h\varpi^ZK \mapsto g_bhg\varpi^ZK,
\]

where \( g_b \in G \) has the same characteristic polynomial as \( b \). Hence the number of fixed points on such a fibre is

\[
\# \{ h \in G/\varpi^ZK \mid h^{-1}g_bh = g^{-1} \}.
\]

Because there are \( n \) simple fixed points of \( b \) on \( \mathbb{P}(W) \) and the morphism \( \pi_K \) is étale, all fixed points are simple and the total number of fixed points is

\[
n \cdot \# \{ h \in G/\varpi^ZK \mid h^{-1}g_bh = g^{-1} \}.
\]

\( \square \)

3. **The characteristic-\( \varpi \)-boundary**

3.1. **Quasi-Compactifications.**

3.1.1. For any open subgroup \( K \subset K_0 \) and integer \( j \in \mathbb{Z} \) we consider the adic spaces

\[
\mathcal{M}^{(j)}_K = t(M^{(j)}_K)_a = \text{Spa}(R^{(j)}_K; R^{(j)}_K) - V(m^{(j)}_K),
\]
where $V(\mathfrak{m}_{R^{(j)}})$ is the one-point-set consisting of the single valuation of $R^{(j)}_K$ which factorizes over the residue field. This valuation is also the single non-analytic point of $\text{Spa}(R^{(j)}_K, R^{(j)}_K)$. We put

$$\overline{M}_K = \coprod_{j \in \mathbb{Z}} M^{(j)}_K.$$ 

The group action extends to these spaces. Namely, if $g \in G$ is such that $g^{-1}Kg \subset K_0$, then the induced map $M_K \to M_{g^{-1}Kg}$ induces a morphism of adic spaces

$$\overline{M}_K \to \overline{M}_{g^{-1}Kg}.$$ 

We will show below that these spaces contain the previously defined spaces $M^{(j)}_K$ and $M_K$ as open subspaces, and we denote by

$$\partial M^{(j)}_K = \overline{M}^{(j)}_K - M^{(j)}_K, \quad \partial M_K = \overline{M}_K - M_K$$

their complements. These complements have natural stratifications. In order to introduce them, we denote for a given integer $h$, $0 \leq h \leq n-1$, and $m > 0$, by $S_{m,h}$ the set of direct summands $A \subset (\varpi^{-m}\mathfrak{o}/\mathfrak{o})^n$ which are free over $\mathfrak{o}/(\varpi^m)$ of rank $h$. If $K \subset K_0$ is an open subgroup, fix $m \in \mathbb{Z}_{>0}$ such that $K$ contains $K_m$. Then $K$ acts (from the left) on $S_{m,h}$, and we put

$$S_{K,h} = K \backslash S_{m,h}, \quad S_K = \coprod_{0 \leq h \leq n-1} S_{K,h}.$$ 

The so defined sets do not depend on $m$. Note that even for $K = K_0$ we have chosen $m > 0$ so that $S_{K_0}$ is the disjoint union of $n$ one-element-sets. Consider the universal level-$m$-structure

$$\phi^\text{univ}_m : (\varpi^{-m}\mathfrak{o}/\mathfrak{o})^n \to X^\text{univ}[[\varpi^m]],$$

where we consider the universal deformation $X^\text{univ}$ as a formal $\mathfrak{o}$-module over $\overline{M}_{K_0}$, which is possible, because the universal deformation is already an object over the schemes $\text{Spec}(R^{(j)}_K)$ (cf. Prop. 2.1.2). By base change we consider it as an object over any of the spaces $\overline{M}_K, K \supset K_m$. For any point $x \in \overline{M}_{K_m}$, we can consider the formal $\mathfrak{o}$-module

$$X^\text{univ} \otimes \kappa(x)$$

over the residue field $\kappa(x)$ at that point. We can also consider the associated $\varpi$-divisible module, i.e. the ind-group scheme

$$(X^\text{univ} \otimes \kappa(x))[\varpi^\infty] = \lim_{\overset{\longrightarrow}{r}} (X^\text{univ} \otimes \kappa(x))[\varpi^r].$$
The étale part of this ind-group scheme has $F$-rank $n-h$ for some $h$ between 0 and $n-1$. (The point which corresponds to the maximal ideal is not in our spaces, and this is the only point where the universal deformation is connected.) It follows that the kernel of the induced level-$m$-structure

$$\phi_m^{univ} : (\varpi^{-m} o / o)^n \to (X^{univ} \otimes \kappa(x))[\varpi^m]$$

is a free submodule $A_x \subset (\varpi^{-m} o / o)^n$ of rank $h$. More generally, if $x \in \overline{M}_K$, there is always a point $x' \in \overline{M}_{K,m}$ lying over $x$ (cf. [En], 13.2), and if $x', x''$ both lie over $x$, then there is a $k \in K$ such that $x'k = x''$ (cf. [En], 14.1). As the group action is via the level structure, it is easily seen that for the kernels of the induced level structures at $x'$ and $x''$ one has:

$$A_{x''} = k^{-1} A_{x'}.$$

Hence we define the kernel of the universal level structure at $x$ to be the class of $A_{x'}$ in $S_{K,h}$ for any $x' \in \overline{M}_{K,m}$ lying over $x$. Then, for a given element $A \in S_{K,h}$ we denote by

$$\partial_A M_K \subset \overline{M}_K, \quad \partial_A M^{(j)}_K \subset \overline{M}^{(j)}_K$$

the set of points $x$ of $\overline{M}_K$ (resp. $\overline{M}^{(j)}_K$) such that the kernel of the universal level-$m$-structure at $x$ is equal to $A$. (Again we point out that $m$ was chosen to be positive.) Then we have a (set-theoretical) decomposition of $\overline{M}_K$:

$$\overline{M}_K = \bigsqcup_{A \in S_K} \partial_A M_K.$$

The group action respects this decomposition. For $g \in G$ choose $r \in \mathbb{Z}$ and $m \geq m' \in \mathbb{Z}_{\geq 0}$ such that

$$o^n \subset \varpi^{-r} g o^n \subset \varpi^{-(m-m')} o^n,$$

and $K_{m'} \subset g^{-1} K g \subset K_0$. Put $g_1 = \varpi^{-r} g$. Then we have the following commutative diagram:

$$\begin{array}{cccc}
\varpi^{-m} o^n / o^n & \xrightarrow{g_1} & \varpi^{-m} o^n / g_1 o^n & \xrightarrow{\varpi^{(m-m')}} \\
\varpi^{-m'} o^n / o^n & & \varpi^{-m'} o^n / o^n &
\end{array}$$

The composition of the arrows in the bottom line is an isomorphism which we denote by $g'$. Now consider an element $A \in S_{K,h}$, and choose a representative $A_1 \subset \varpi^{-m} o^n / o^n$. Then $\varpi^{(m-m')} A_1 \subset \varpi^{-m'} o^n / o^n$ is a direct summand, and $(g')^{-1} \varpi^{(m-m')} A_1$ is an element
Deformation spaces of one-dimensional formal modules and their cohomology

\[ \mathcal{S}_{K,h} \to \mathcal{S}_{g^{-1}K,g,h}, \quad A \mapsto g^{-1}A. \]

It is easy to check that the morphism \( g : \overline{M}_K \to \overline{M}_{g^{-1}K} \) induces maps

\[ g : \partial A M_K^{(j)} \to \partial g^{-1}A M_{g^{-1}K}^{(j-v(\det(g)))}. \]

If \( K = K_m \), we identify elements of \( \mathcal{S}_K \) with submodules of \((\omega^{-m} o / o)^n\), and hence there is a partial ordering on this set defined by \( A_1 \preceq A_2 \) if and only if \( A_1 \supset A_2 \). For \( K \subset K_m \) as above we define analogously \( A_1 \preceq A_2 \) for classes \( A_1, A_2 \in \mathcal{S}_K \) if and only if there are representatives \( A_1', A_2' \in \mathcal{S}_{K_m} \) such that \( A_1' \preceq A_2' \). This defines a partial ordering on \( \mathcal{S}_K \). Note that for this ordering the class of the zero module is the maximal element.

**Proposition 3.1.2.** Let \( K \subset K_0 \) be an open subgroup.

(i) For any \( j \in \mathbb{Z} \) and any \( A \in \mathcal{S}_K \) there is a prime ideal \( p_A \subset R_K^{(j)} \) with the following property: \( \partial A M_K^{(j)} \) consists of exactly those valuations \( v \in \overline{M}_K^{(j)} \) such that the support of \( v \) contains \( p_A \) but does not contain \( p_{A'} \) for any \( A' \preceq A, A' \neq A \). With the notation of adic spaces we have:

\[ \partial A M_K = \text{Spa}(R_K^{(j)}/p_A, R_K^{(j)}/p_A)_a - \bigcup_{A' \preceq A, A' \neq A} V(p_{A'}). \]

If \( A \neq A' \) then \( \partial A M_K \cap \partial A' M_K = \emptyset \).

(ii) For any element \( A \in \mathcal{S}_K \) the subspace \( \partial A M_K \subset \overline{M}_K \) is open in its closure, and this closure is equal to the union of all \( \partial A M_K \) with \( A' \preceq A \).

**Proof.** It suffices to prove this for the case of a principal congruence subgroup \( K = K_m \), \( m > 0 \). The general case follows by passage to the quotient.

(i) Consider the universal level-\( m \)-structure

\[ \phi_m^{\text{univ}} : (\omega^{-m} o / o)^n \to m_{R_K^{(j)}}. \]

Let

\[ p_A = (\phi_m^{\text{univ}}(a))_{a \in A} \]

be the ideal generated by the images of the elements of \( A \) under \( \phi_m^{\text{univ}} \). The points in \( \partial A M_K \) are clearly those whose support contains \( p_A \), but does contain \( p_{A'} \) for any \( A' \preceq A, A' \neq A \). To show that \( p_A \) is a prime ideal we may assume (as the situation is homogeneous under the group action) that \( A \) is generated by the first \( h \) standard basis vectors of
(\varpi^{-m} \mathfrak{o}/\mathfrak{o})^n. Then \( p_A \) is generated by the images of the first \( h \) standard basis vectors under the universal level-\( m \)-structure. By 2.1.2, these form part of a system of parameters, and hence generate a prime ideal.

(ii) We show more generally that if \( R \) is a noetherian local complete domain, and if \( \{0\} \subseteq \mathfrak{a} \) is any non-zero ideal of \( R \), then \( \text{Spa}(R,R)_a - V(\mathfrak{a}) \) is an open dense subset of \( \text{Spa}(R,R)_a \).

Firstly, \( V(\mathfrak{a}) \) is the intersection of the complements of the sets \( \{ v \mid |f|_v \leq |f|_v \neq 0 \} \) which are open in \( \text{Spa}(R,R)_a \). Hence \( V(\mathfrak{a}) \) is a closed subset. Suppose that \( U \) is an open subset of \( \text{Spa}(R,R)_a \) which is contained in \( V(\mathfrak{a}) \). We may assume that \( U \) is a rational subset of the form

\[ \{ v \in \text{Spa}(R,R)_a \mid |f_i|_v \leq |s|_v \neq 0 \text{ for } i = 1, \ldots, n \} \]

with elements \( f_1, \ldots, f_n \in R \) generating an open ideal. We consider the corresponding ring of functions on \( U \) which is \( R[\frac{f_1}{s}, \ldots, \frac{f_n}{s}] \). As \( U \) was supposed to lie in \( V(\mathfrak{a}) \), the images of the elements of \( \mathfrak{a} \) in \( R[\frac{f_1}{s}, \ldots, \frac{f_n}{s}] \) vanish. By its very definition, \( R[\frac{f_1}{s}, \ldots, \frac{f_n}{s}] \) is the completion of \( R[\frac{a}{s}] \) with respect to the topology having the sets \( \mathfrak{m}_R R[\frac{f_1}{s}, \ldots, \frac{f_n}{s}], r \geq 0 \), as a fundamental system of neighborhoods of zero. Suppose \( U \) is non-empty, and let \( | \cdot | \) be a continuous valuation of \( R \) in \( U \). Then \( s \neq 0 \) and we have \( |f| < 1 \) for all elements \( f \in \mathfrak{n} = \mathfrak{m}_R R[\frac{f_1}{s}, \ldots, \frac{f_n}{s}] \). Hence \( \mathfrak{n} \) is a proper ideal in \( R[\frac{f_1}{s}, \ldots, \frac{f_n}{s}] \). As this ring is a noetherian domain, the intersection \( \bigcap \mathfrak{n} \) is the zero ideal. Therefore, the elements in \( \mathfrak{a} \) already map to zero in \( R[\frac{1}{s}] \), which is impossible if \( R \) is a domain and \( \mathfrak{a} \) is non-zero.

3.2. Consequences for formal models.

3.2.1. In this subsection we fix a regular elliptic element \( g \in G \) and an element \( b \in B^\times \).

We assume \( v(det(g)) + v(N(b)) = 0 \), so that the action of the pair \((g, b^{-1})\) does not change the height of the quasi-isogeny. This implies that \((g, b^{-1})\) acts on the space \( M^{(j)}_K \) as soon as \( g \) normalizes \( K \). We are going to study the fixed point locus of this pair, and begin with the following

**Lemma 3.2.2.** (i) There is a fundamental system of neighborhoods of \( 1 \) in \( G \) consisting of compact open subgroups \( K \subset K_0 \) which are normalized by \( g \).

(ii) There is an open subgroup \( K \subset K_0 \), depending only on \( g \), with the property that if \( K \) is an open subgroup of \( K' \) and normalized by \( g \), then \((g, b^{-1}) \) maps for any \( A \in S_{K,K'} \) the stratum \( \partial_AM^{(j)}_K \) to \( \partial_g b^{-1}A_M^{(j)} \) and the intersection of these sets is empty if \( h > 0 \). In particular, \((g, b^{-1}) \) does not have any set-theoretical fixed points on \( \partial_M^K \).

**Proof.** (i) (Cf. [He2], sec. 6.1) The ring \( E = F[g] \subset M_n(F) \) is a separable field extension of \( F \) of degree \( n \). Denote by \( \mathfrak{o}_E \) its ring of integers, and by \( p_E \subset \mathfrak{o}_E \) its maximal ideal. We identify \( G \) with \( \text{End}_F(E)^\times \). For \( r \in \mathbb{Z}_{>0} \) put
\[ K(r) = 1 + \bigcap_{r' \geq 0} \text{Hom}_o(p_{E'}^r, p_{E}^{r+r'}). \]

These subgroups form a fundamental system of neighborhoods of 1 in \( G \), and a simple calculation shows that \( g \) normalizes \( K(r) \).

(ii) Fix \( h \) with \( 1 \leq h \leq n - 1 \). We can identify the set \( S_{K,h} \) of labels of boundary components of \( \overline{M_K^{(j)}} \) with \( K\backslash K_0/P(o) \), where \( P(o) \) is the stabilizer of \( o^h \subset o^n \). The quotient \( K_0/P(o) \) is the same as \( G/P \), where \( P \) is the stabilizer of \( F^h \subset F^n \), and the action by \( g \) on \( S_{K,h} \), cf. diagram 3.1.1, is given on \( K\backslash G/P \) by multiplication by \( g^{-1} \) from the left:

\[ KxP \mapsto (g^{-1}Kg)xP = Kg^{-1}xP. \]

An element of \( G \) is regular elliptic if and only if its characteristic polynomial is separable and irreducible over \( F \). It follows that the set of regular elliptic elements is open in \( G \). Therefore, if \( K \) is sufficiently small and normalized by \( g \), cf. 3.2.2, the set \( Kg^{-1} \) will have empty intersection with any conjugate of \( P \). Now, if \( KxP \) would be a fixed point of the action of \( g \), we would have \( kg^{-1} \in xPx^{-1} \) for some \( k \in K \), contradicting what we just stated. Hence there are no fixed points of \( g \) on \( S_{K,h} \). As \( (g, b^{-1}) \) maps \( \partial AM_K^{(j)} \) to \( \partial g^{-1}AM_K^{(j)} \) (the action of \( b \) leaves the boundary components unchanged, because \( b \) does not act on the level structure), and as different boundary components have empty intersection, we have proved the second assertion.

Let \( K \subset K_0 \) be an open subgroup normalized by \( g \) which has the property that \( \gamma = (g, b^{-1}) \) has no fixed points on the boundary of \( M_K^{(j)} \). We want to show that there is a formal model for the analytic space \( \overline{M_K^{(j)}} \), i.e. an admissible blow-up (in the sense of [Fu1], Def. 4.1.1) of \( M_K^{(j)} = \text{Spf}(R_K^{(j)}) \), to which the action of \( \gamma \) extends and such that the fixed point locus on the special fibre does not meet the image of the boundary under the specialization map. We recall the definition of the specialization map in the affine case (cf. [Hu2], Prop. 4.1). Let \( \mathcal{X} = \text{Spf}(A) \) be an affine noetherian formal scheme and \( v \in t(\mathcal{X}) = \text{Spa}(A,A) \) a continuous valuation. Then the set

\[ sp_{\mathcal{X}}(v) = \{ f \in A \mid ||f||_v < 1 \} \]

is an open prime ideal of \( A \), hence a point in \( \mathcal{X} = \text{Spa}(A) \). The so defined map

\[ sp_{\mathcal{X}} : t(\mathcal{X}) = \text{Spa}(A,A) \to \mathcal{X} = \text{Spf}(A) \]

is continuous, and even a morphism of topologically ringed spaces ([Hu2], Prop. 4.1). We denote its restriction to the subspace \( t(\mathcal{X})_a \subset t(\mathcal{X}) \) of analytic points by \( sp_{\mathcal{X}} \) as well. In order to deduce the existence of formal models from topological properties of the adic
space, we are going to use the theory of Fujiwara’s Zariski-Riemann spaces. The sheaf $\mathcal{O}_{t(\mathcal{X})_a}^+$ which appears below has been defined in [Hu2], sec. 1.

**Proposition 3.2.3.** If $\mathcal{X}$ is a noetherian formal scheme, then the locally ringed space

$$(t(\mathcal{X})_a, \mathcal{O}_{t(\mathcal{X})_a}^+)$$

having the open subspace $t(\mathcal{X})_a \subset t(\mathcal{X})$ as underlying topological space, which is equipped with the sheaf $\mathcal{O}_{t(\mathcal{X})_a}^+$ is canonically isomorphic, via the specialization maps, to the projective limit of all admissible blow-ups of $\mathcal{X}$:

$$\lim_{\leftarrow} \text{sp}_{\mathcal{X}'} : (t(\mathcal{X})_a, \mathcal{O}_{t(\mathcal{X})_a}^+) \sim \lim_{\leftarrow} \mathcal{X}' ,$$

where $\mathcal{X}'$ runs through the set of admissible blow-ups of $\mathcal{X}$ and the projective limit on the right carries the projective limit topology and structure of a locally ringed space (cf. [Fu1], 4.1.3).

**Proof.** If $\mathcal{X}$ is of topologically finite type over a discrete valuation ring, then the assertion is known by [vdPS]. For the general case we refer to [Hu0], 3.9.25. $\square$

We come back to the assertion about formal models that we want to prove.

**Proposition 3.2.4.** Let $\gamma = (g, b^{-1}) \in G \times B^\times$ be as in 3.2.1. For any open subgroup $K'$ of $K_0$ there exists an open subgroup $K \subset K'$ which is normalized by $g$, and such the following assertions do hold:

(i) There exists an admissible blow-up $(\mathcal{M}_K^{(j)})'$ of $\mathcal{M}_K^{(j)}$ to which the action of $\gamma$ extends, and such that the fixed point locus of the extended morphism

$$\gamma' : (\mathcal{M}_K^{(j)})' \to (\mathcal{M}_K^{(j)})'$$

has no set-theoretical fixed points on the image of $\partial\mathcal{M}_K^{(j)}$ under the map $\text{sp}_{(\mathcal{M}_K^{(j)})'}$.

(ii) Denote by $\tilde{\mathcal{M}}_K^{(j)}$ the scheme of finite type over $\mathcal{O}^{\text{nr}}$ from Thm. 2.3.1. For a given integer $c \in \mathbb{Z}_{\geq 0}$ denote by $\tilde{\mathcal{M}}_K^{(j)}$ the étale neighborhood of the point $x_K$ and by $\gamma_c$ the morphism

$$\gamma_c : \tilde{\mathcal{M}}_K^{(j)} \to \mathcal{M}_K^{(j)}$$

from Thm. 2.3.2. Then the blow-up $(\mathcal{M}_K^{(j)})'$ of $\mathcal{M}_K^{(j)}$ in (i) induces a blow-up $(\tilde{\mathcal{M}}_K^{(j)})'$ of $\tilde{\mathcal{M}}_K^{(j)}$, and $\gamma_c$ lifts to a blow-up $(\tilde{\mathcal{M}}_K^{(j)})'$ of $\tilde{\mathcal{M}}_K^{(j)}$: 
Now, if $J$ is any ideal of definition of $(\mathcal{M}_K^{(j)})'$, we can choose $c$ sufficiently large, so that the completion of $(\mathcal{M}_K^{(j)})'$ along the closed subscheme $\tilde{\pi}^{-1}(\tilde{x})$ will be equal to $(\mathcal{M}_K^{(j)})'$ and the induced morphism

$$\gamma_c' : (\mathcal{M}_K^{(j)})' \rightarrow (\mathcal{M}_K^{(j)})'$$

is congruent to $\gamma'$ modulo $J$. Moreover, if $c$ is large enough, the correspondence $\gamma_c'$ will also have no set-theoretical fixed points on $\text{sp}_{(\mathcal{M}_K^{(j)})'}(\partial\mathcal{M}_K^{(j)})$.

Proof. (i) We note first that the set of admissible blow-ups to which the action of $\gamma$ lifts is cofinal in the set of all admissible blow-ups. Namely, the maximal ideal $m_R$ of $R_K^{(j)}$ is mapped by $\gamma^\sharp : R_K^{(j)} \rightarrow R_K^{(j)}$ to itself, as $\gamma^\sharp$ is continuous. Furthermore, the element $\gamma$ generates topologically a profinite subgroup of $G \times B^\infty/(\varpi, \varpi)^\mathbb{Z}$. To see this, we consider the fields $E_g = F[g] \subset M_n(F)$ and $E_b = F[b] \subset B$. The element $\gamma$ lies in the subgroup $H \subset E_g^\times \times E_b^\times$ which consists of all elements $(\alpha, \beta)$ with $v(\alpha) - v(\beta) = 0$. The group $H/(\varpi, \varpi)^\mathbb{Z}$ is compact and hence profinite. Therefore, there is for any $c > 0$ some $N \in \mathbb{Z}_{>0}$ such that $\gamma^N$ is congruent to the identity modulo $m^c_R$. Hence, if the admissible ideal $\mathcal{I}$ of $R_K^{(j)}$ contains $m^c_R$, then the ideal

$$\mathcal{I}' = \mathcal{I} \cdot \gamma^\sharp(\mathcal{I}) \cdot \ldots \cdot (\gamma^\sharp)^N(\mathcal{I})$$

is admissible again and invariant under $\gamma^\sharp$. The action of $\gamma$ lifts to the blow-up of $\mathcal{I}'$, and this blow-up dominates the blow-up of $\mathcal{I}$.

Write $\partial_A$ instead of $\partial_A M_K^{(j)}$, and let $A_K$ be the class of $\{0\}$ in $S_{K,0}$ (which is the unique maximal element of $S_K$). We show that there is a covering $(U_A)_{A \in S_K}$ of $\overline{M}_K^{(j)}$ by constructible subsets $U_A$ with the property that for all $A \in S_K$:

$$\partial_A \subset \bigcup_{A' \prec A} U_A$$
and $\gamma(U_A) \cap U_A = \emptyset$ for any $A \neq A_K$. Note first that the complement of each of the closed subsets

$$\overline{\partial_A} = \bigcup_{A' \prec A} \partial_A$$

is closed under specialization, because the support of a point in an analytic adic space does not change under specialization. This implies that for each of these subsets there is a descending family

$$V_A^{(1)} \supset V_A^{(2)} \supset V_A^{(3)} \supset \ldots$$

of open quasi-compact neighborhoods of $\overline{\partial_A}$ whose intersection is $\overline{\partial_A}$. Now let $A \neq A_K$ be a minimal element in $S_K$. Then $\partial_A$ is just a closed point. We have

$$\bigcap_i \left( \gamma(V_A^{(i)}) \cap V_A^{(i)} \right) = \emptyset,$$

because $\gamma$ is bijective and $\gamma(\partial_A) \cap \partial_A = \emptyset$, and if the intersection of constructible sets in a spectral space is empty, then the intersection of already finitely many of them is empty. Namely, if we equip the spectral space with its constructible topology, then it becomes compact and the constructible subsets become closed. Hence there is some $i$ such that $\gamma(V_A^{(i)}) \cap V_A^{(i)} = \emptyset$ and we put $U_A = V_A^{(i)}$. We proceed by induction with respect to the partial ordering on $S_K$ and assume that for a given $A \neq A_K$ we have already constructible subsets $U_B$ for all $B \prec A, B \neq A$, such that

$$\partial_B \subset \bigcup_{B' \prec B} U_B$$

and $\gamma(\partial_B) \cap \partial_B = \emptyset$. Next we note that

$$\bigcap_i \left( V_A^{(i)} - \bigcup_{B \prec A, B \neq A} U_B \right)$$

is contained in $\overline{\partial_A} - \bigcup_{B \prec A} \partial_B = \partial_A$ and hence

$$\bigcap_i \left( \gamma(V_A^{(i)}) - \bigcup_{B \prec A, B \neq A} U_B \right) \cap \left( V_A^{(i)} - \bigcup_{B \prec A, B \neq A} U_B \right) = \emptyset.$$

By the same reasoning as above, there is some $i$ such that

$$\gamma(V_A^{(i)} - \bigcup_{B \prec A, B \neq A} U_B) \cap (V_A^{(i)} - \bigcup_{B \prec A, B \neq A} U_B) = \emptyset,$$
and we put $U_A = V^{(i)}_A - \bigcup_{B < A, B \neq A} U_B$. Thus we have proved the assertion about the covering $(U_A)_A$.

We show next that there is a formal model $(\mathcal{M}_K^{(j)})'$ such that the images of $\gamma(U_A)$ and $U_A$ under the specialization map have empty intersection for $A \neq A_K$. This is a very general fact. Put $\mathcal{X} = \mathcal{M}_K^{(j)}$ and $X = \overline{M}_K^{(j)}$. Namely, if $Z \subset X$ is a constructible subset one sees easily that

$$
\bigcap_{\mathcal{X}'} sp^{-1}_{\mathcal{X}'}(sp_{\mathcal{X}'}(Z)) = Z,
$$

where the intersection is over all models $\mathcal{X}'$. If now $Z, Z' \subset X$ are two constructible subsets having empty intersection, one has

$$
\bigcap_{\mathcal{X}'} sp^{-1}_{\mathcal{X}'}(sp_{\mathcal{X}'}(Z) \cap sp_{\mathcal{X}'}(Z')) = \emptyset,
$$

and by the same reasoning as above, there is a model $\mathcal{X}'$ such that $sp_{\mathcal{X}'}(Z) \cap sp_{\mathcal{X}'}(Z')$ is empty. If now $(\mathcal{M}_K^{(j)})'$ is a model such that the images of $\gamma(U_A)$ and $U_A$ under the specialization map have empty intersection (for $A \neq A_K$), then this is the case for any model dominating this one. Hence there is a model with this property for all $A \neq A_K$, and we can even find a model $(\mathcal{M}_K^{(j)})'$ having this property and such that the action of $\gamma$ lifts to

$$
\gamma' : (\mathcal{M}_K^{(j)})' \longrightarrow (\mathcal{M}_K^{(j)})'.
$$

If now $x \in (\mathcal{M}_K^{(j)})'$ lies in the image of $\partial M_K^{(j)}$, it is equal to $sp_{\mathcal{M}_K^{(j)}}(z)$ for some $z$ in some $U_A$ with $A \neq A_K$. Then

$$
\gamma'(x) = sp_{\mathcal{M}_K^{(j)}}(\gamma(z)) \in sp_{\mathcal{M}_K^{(j)}}(\gamma(U_A)),
$$

and hence $x \neq \gamma'(x)$.

(ii) Let $\mathcal{I} \subset R^{(j)}_K$ be the ideal that is blown up to give the formal scheme $(\mathcal{M}_K^{(j)})'$. Let $\mathfrak{M}_K^{(j)} = \text{Spec}(\mathfrak{R}_K^{(j)})$, and denote by $m_{\mathfrak{R}}$ the maximal ideal corresponding to $\mathfrak{r}_K$. Then $R^{(j)}_K$ is the completion of $\mathfrak{R}_K^{(j)}$ at $m_{\mathfrak{R}}$. Put

$$
\mathcal{I}_{\mathfrak{R}} = \mathcal{I} \cap \mathfrak{R}_K^{(j)};
$$

and let

$$
pr : (\mathfrak{M}_K^{(j)})' \longrightarrow \mathfrak{M}_K^{(j)}
$$
be the blow-up of \( \mathcal{I}_R \) on \( M_{K}^{(j)} \). As \( \mathcal{I} \) is admissible, it contains a power of the maximal ideal \( m_R \) of \( R_K^{(j)} \), and so \( \mathcal{I}_R \) contains a power of \( m_R \). It is hence invertible outside \( \mathfrak{r}_K \), and can actually be generated by finitely many elements of \( R_K^{(j)} \) which generate \( \mathcal{I} \) over \( R_K^{(j)} \). It follows that the completion of \( (M_{K}^{(j)})' \) along the preimage \( pr^{-1}(\mathfrak{r}_K) \) is isomorphic to \( (M_{K}^{(j)})' \). Let \( \widehat{M}_{K}^{(j)} = \text{Spec}(\widehat{R}_{K}^{(j)}) \). Put

\[
\mathcal{I}_{\widehat{R}} = \gamma_{e}^{*}(\mathcal{I}_R) \cdot \widehat{R}_{K}^{(j)}.
\]

Recall that \( R_K^{(j)} \) is also the completion of \( R_{K}^{(j)} \) at the unique closed point \( \bar{r}_K \) lying over \( \mathfrak{r}_K \). If we now choose \( c \) sufficiently large so that \( \mathcal{I}_{\widehat{R}} \) generates \( \mathcal{I} \) over \( R_K^{(j)} \), the blow-up

\[
p_{\mathcal{I}} : (\widehat{M}_{K}^{(j)})' \longrightarrow M_{K}^{(j)}
\]

of \( \mathcal{I}_{\widehat{R}} \) on \( \widehat{M}_{K}^{(j)} \) will be an étale neighborhood of \( pr^{-1}(\mathfrak{r}_K) \). Hence the completion of \( (\widehat{M}_{K}^{(j)})' \) along \( p_{\mathcal{I}}^{-1}(\bar{r}_K) \) will be isomorphic to \( (M_{K}^{(j)})' \). The next assertion finally follows from a very general fact. Namely, let \( \varphi_1, \varphi_2 \) be an endomorphism of a noetherian formal scheme \( X', \) let \( X' \rightarrow X \) be an admissible blow-up of \( X \), such that \( \varphi_i \) lifts to an endomorphism \( \varphi'_i \) of \( X' \), \( i = 1, 2 \). Then, for any ideal \( J \) of definition of \( X' \), there is an ideal of definition \( I \) of \( X \), such that, if \( \varphi_1 \equiv \varphi_2 \) modulo \( I \) then \( \varphi'_1 \equiv \varphi'_2 \) modulo \( J \).

Finally, if \( \gamma'_e \) approximates \( \gamma' \) sufficiently well, then the induced morphisms on the special fibre will be the same, hence \( \gamma'_e \) has no fixed points on \( sp_{(M_{K}^{(j)})'}(\partial M_{K}^{(j)}) \). \( \Box \)

Remark. We think that it would also be possible to prove the statements about the appropriate formal models by applying [Fu2], Prop. 2.2.5. Fujiwara however is working in an algebraic (not only formal) setting, and he supposes that the ambient algebraic space is proper. Although our formal schemes are also algebraizable, we preferred to prove the assertions directly.

3.3. The trace of regular elliptic elements.

The following theorem gives an expression of the trace of pairs \( (g, b^{-1}) \in G \times B^\infty \), both regular elliptic, on the cohomology in terms of the number of fixed points.

**Theorem 3.3.1.** Let \( g \in G, b \in B^\infty \) be both regular elliptic elements such that \( v(\det(g)) + v(N(b)) = 0 \). Then there is an open subgroup \( K' \subset K_0 \) such that the following holds: if \( K \subset K' \) is an open subgroup normalized by \( g \), then the alternating sum of traces of the endomorphism induced by \( (g, b^{-1}) \) on the cohomology

\[
tr((g, b^{-1})|H^j_c(M_{K}^{(j)})) = \sum_{i} (-1)^i tr((g, b^{-1})|H^i_c(M_{K}^{(j)} \times_{\bar{F}^nr} \bar{F}^\infty, \mathbb{Q}_\ell))
\]
is equal to the number of (intersection theoretic) fixed points of \((g, b^{-1})\) on \(M^{(j)}_K \times F_{nr} \hat{F}^\wedge\), which is finite.

**Proof.** Put \(\gamma = (g, b^{-1})\). We choose the following objects such that the assertions of Prop. 3.2.4 are fulfilled: subgroups \(K \subset K' \subset K_0\), the blow-ups

\[
(M^{(j)}_K)' \xrightarrow{pr} M^{(j)}_K, \quad (M^{(j)}_K)' \xrightarrow{pr} \tilde{M}^{(j)}_K, \quad (\tilde{M}^{(j)}_K)' \xrightarrow{pr} \tilde{M}^{(j)}_K,
\]

the endomorphism

\[
\gamma' : (M^{(j)}_K)' \to (M^{(j)}_K)',
\]

and the morphism of schemes

\[
\gamma_c' : (\tilde{M}^{(j)}_K)' \to (M^{(j)}_K)',
\]

which approximates \(\gamma'\). Put

\[
M' := (M^{(j)}_K)' \times_{\hat{F}_{nr}} F_{nr} \hat{F}^\wedge,
\]

and

\[
M'_\eta := (M^{(j)}_K)' \times_{\hat{F}_{nr}} \hat{F}^\wedge.
\]

Let \((M')^{an}\) be the non-Archimedean analytic space associated by Berkovich to the formal scheme \(\tilde{M}'\), \(\tilde{M}'\) being the completion of \(M'\) along its special fibre ([Be3], sec. 1). Let \((M')^{ad}\) be the analytic adic space associated by Huber to the formal scheme \(\hat{M}'\) ([Hu3], sec. 1.9). It is known that \((M')^{an}\) is the maximal Hausdorff quotient of \((M')^{ad}\). Put

\[
M = M^{(j)}_K, \quad \partial M = \partial M^{(j)}_K.
\]

Let

\[
sp^{ad} : (M')^{ad} \to \widetilde{M}'_s, \quad sp^{an} : (M')^{an} \to \widetilde{M}'_s, \quad sp : M \to (M^{(j)}_K)'\]

be the specialization maps, and

\[
j : M'_\eta \to M', \quad i : M'_s \to M'
\]

the open respectively closed embeddings. Put

\[
M = M^{(j)}_K \times F_{nr} \hat{F}^\wedge \subset (M')^{ad}\]

and let \(M^{an}\) be its image in \((M')^{an}\). Let \(Ypr^{-1}(x_K) \to \tilde{M}'_s\) be the inclusion. Then we have the following commutative diagram:
Denote by $\mathcal{F} = \mathbb{Z}/\ell'^*\mathbb{Z}$ the constant torsion sheaf on the scheme $(\mathbb{M}'_η)′$, on the analytic spaces $(\mathbb{M}'_s)′$, and on the closed subschemes defined to be the preimage of $(\mathbb{M}'_η)′$ under the map $\mathcal{G} → (\mathbb{M}'_η)′$. We show below in Lemma 3.3.3 that $\mathcal{G}′$ is open in $\mathcal{G}$.

It follows that there is a proper subscheme $\mathcal{D}⊂\mathcal{M}'$ which is contained and open in the fixed point locus of the correspondence $\gamma'_c$ and whose special fibre is equal to the fixed point locus of $\gamma'_c$ on $\mathcal{M}'_s$. Fujiwara’s theorem on the specialization of local terms, [Fu2], Prop. 1.7.1, then tells us that

$$\text{loc}_{\mathcal{D}_s}((\gamma'_c)_s, i^*_Y i^* R^+_j_*.\mathcal{F}) = \sum_{D'⊂\pi_0(D_\eta)} \text{loc}_{D'}((\gamma'_c)_\eta, \mathcal{F}).$$

We will show that the connected components $D'$ of $D_\eta$ are just points with multiplicity one, and the number of these points is equal to the number of fixed points of $\gamma$ on $\mathcal{M}$, provided $c$ is large enough. As we are now working only on $\mathcal{M}$, we can work with $\gamma_c$.
instead of $\gamma'_c$ (cf. Prop. 3.2.4), because the correspondences defined by $\gamma_c$ and $\gamma'_c$ on the analytic spaces are the same. $\gamma_c$ induces a morphism of formal schemes

$$\hat{\gamma}_c : M^{(j)}_K \rightarrow M^{(j)}_K$$

and the restriction of the correspondence $\gamma_c$ to $M$ is just an endomorphism, which we denote by $\gamma_c$ as well. Let

$$\hat{\gamma}_c^\sharp : R^{(j)}_K \rightarrow R^{(j)}_K$$

be the induced morphism on complete local rings, let $\mathfrak{m}$ be the maximal ideal of $R^{(j)}_K$, and let $f_1, \ldots, f_r$ be a system of generators of $\mathfrak{m}$. Because $\hat{\gamma}_c^\sharp$ maps $\mathfrak{m}$ to itself, the subsets

$$U_\nu = \{ v \in M \mid |f_i|_v \leq |\varpi|_v, i = 1, \ldots, r \}$$

of $M$ are stable under $\gamma_c$ and $\gamma$ as soon as $\nu$ is sufficiently large. $sp^{-1}(Y - \partial Y)$ is a quasi-compact open subset of $M$, and is hence contained in $U_\nu$, for $\nu \gg 0$. $\gamma_c$ and $\gamma$ do not have fixed points on $sp^{-1}(\partial Y) \cap M$. By Prop. 5.2.3, if $c$ is large enough, $\gamma_c$ will have only finitely many fixed points on $sp^{-1}(Y - \partial Y)$, these fixed points are all of multiplicity one, and their number is equal to the number of fixed points of $\gamma$. The connected components $D'$ of $D_\eta$ are hence just points of multiplicity one. By [SGA5], Exp. III, 4.12, each local term corresponding to a fixed point is then equal to 1 (as an element of $\mathbb{Z}/\ell^r \mathbb{Z}$). Now we pass to the limit $r \to \infty$. Hence we conclude that the trace of $\gamma$, which is equal to the trace of $\gamma'_c$, is given by the number of fixed points of $\gamma$ on $M$. $\square$

**Corollary 3.3.2.** Let $g \in G, b \in B^\times$ be both regular elliptic elements. Then there is an open subgroup $K' \subset K_0$ such that the following holds: if $K \subset K'$ is an open subgroup normalized by $g$, then the alternating sum of traces of the endomorphism induced by $(g, b^{-1})$ on the cohomology

$$tr((g, b^{-1})|H^i_c(M_K/\varpi^Z) = \sum_i (-1)^i tr((g, b^{-1})|H^i_c((M_K/\varpi^Z) \times \hat{F}^\wedge, \mathbb{Q}_\ell))$$

is equal to

$$n \cdot \# \{ h \in G/\varpi^Z K \mid h^{-1}gh = g^{-1} \}.$$

**Proof.** We identify the cohomology of $(M_K/\varpi^Z) \times \hat{F}^\wedge$ with the direct sum of the cohomology of the $M^{(j)}_K \times \hat{F}^\wedge$ for $j = 0, \ldots, n - 1$. If $\nu(det(g)) + \nu(N(b))$ is not a multiple of $n$ then the cohomology groups

$$H^i_c(M^{(j)}_K \times \hat{F}^\wedge, \mathbb{Q}_\ell)$$

are permuted, none of them is mapped to itself, and hence the trace on the cohomology is zero. Similarly, for any $h$ the cosets
Lemma 3.3.3. Let $\mathfrak{X}$ be a scheme of finite type over $\hat{\mathfrak{o}}^m$, let $Y \subset \mathfrak{X}$ be a closed subscheme and $\mathcal{Y}$ the completion of $\mathfrak{X}$ along $Y$. Let $t(\mathcal{Y})_a$ be the analytic adic space associated to $\mathcal{Y}$, and $V(\varpi)_{\mathcal{Y}} \subset t(\mathcal{Y})_a$ the subspace of points whose support contains $\varpi$. Put $Y' = sp_Y(V(\varpi)_{\mathcal{Y}}) \subset Y$, where $sp_Y : \mathcal{Y} \to Y$ is the specialization map. Then $Y - Y'$ is open in $\mathfrak{X}_s$. 

Proof. Denote by $\mathcal{X} = \hat{\mathfrak{X}}$ the $\varpi$-adic completion of $\mathfrak{X}$. There is a canonical continuous map $\varphi : t(\mathcal{Y}) \to t(\mathcal{X})$. To describe it, assume $\mathfrak{X}$ is affine, $\mathfrak{X} = \text{Spec}(\mathfrak{R})$ say. Denote by $\hat{\mathfrak{R}}$ the $\varpi$-adic completion of $\mathfrak{R}$, by $\mathfrak{a}$ the ideal corresponding to $Y$, and by $\mathfrak{R}$ the $\mathfrak{a}$-adic completion of $\mathfrak{R}$. Then $t(\mathcal{X}) = \text{Spa}(\hat{\mathfrak{R}}, \mathfrak{R})$ and $t(\mathcal{Y}) = \text{Spa}(\mathfrak{R}, \mathfrak{R})$. The canonical continuous morphism $\hat{\mathfrak{R}} \to \mathfrak{R}$ induces the map $\varphi$, which is injective. Without loss of generality we continue to assume that $\mathfrak{X}$ is affine. $\varphi$ maps $V(\varpi)_{\mathcal{Y}} \subset t(\mathcal{Y})$ to the corresponding subset $V(\varpi)_X \subset t(\mathcal{X})$, hence induces an injection

$$\varphi^{an} : t(\mathcal{Y}) - V(\varpi)_{\mathcal{Y}} \longrightarrow t(\mathcal{X})_a = t(\mathcal{X}) - V(\varpi)_X,$$

which commutes with the specialization maps $sp_Y$ and $sp_X$ to give a commutative diagram

$$
\begin{array}{ccc}
t(\mathcal{Y}) - V(\varpi)_{\mathcal{Y}} & \xrightarrow{\varphi^{an}} & t(\mathcal{X})_a \\
sp_Y & & sp_X \\
Y' & \xrightarrow{\varphi^{an}} & \hat{\mathfrak{X}}_s
\end{array}
$$

Let $f_1, \ldots, f_r$ be generators of $\mathfrak{a}$. Then $t(\mathcal{Y}) - V(\varpi)_{\mathcal{Y}}$ is the union of the rational subsets

$$U_n := \{v \in t(\mathcal{Y}) \mid \text{ for all } i : |f_i|^n_v \leq |\varpi|_v \neq 0\},$$

which are open rational subsets in $t(\mathcal{X})_a$ as well. We see in particular, that $\varphi^{an}$ is an open embedding. As $t(\mathcal{X})_a$ is homeomorphic to the projective limit over all admissible blow-ups, equipped with its projective limit topology, it is enough to show that the subset $sp_X^{-1}(Y - Y')$ is open in $t(\mathcal{X})_a$. Now $sp_Y^{-1}(Y - Y')$ is open in $t(\mathcal{Y}) - V(\varpi)_{\mathcal{Y}}$, so we have to show that $sp_X^{-1}(Y - Y')$ is equal to $\varphi^{an}(sp_Y^{-1}(Y - Y'))$. Let $v \in t(\mathcal{X})_a$ be a point such that $sp_X(v) \in Y$. Suppose $v$ does not lie in $\varphi^{an}(t(\mathcal{Y}) - V(\varpi)_{\mathcal{Y}})$. Then $v$ is a continuous valuation of $\hat{\mathfrak{R}}$ such that $|f|_v < 1$ for all $f \in \mathfrak{a}$ but $v$ is not $\mathfrak{a}$-adically continuous. Let $\Gamma_v$ be the value group of $v$ (supposed to be generated by the non-zero images of $v$), and let $\Gamma'_v \subset \Gamma_v$ be the largest convex subgroup such that for all $f \in \mathfrak{a}$ the value $|f|_v$ is

$$h^{-1}g(h)w^{-1}K \quad \text{and} \quad g^{-1}w^{-1}K$$

will be distinct. So suppose that $v(det(g)) + v(N(b))$ is $kn$ for some integer $k$. Then we can change $b$ to $b w^{-k}$ and assume that $v(det(g)) + v(N(b))$ is zero. Then the claimed identity follows immediately from Thm. 2.6.8 and Thm. 3.3.1. $\blacksquare$
cofinal for $\Gamma'_v$ (which means that for any $\delta \in \Gamma'_v$ there is an $t > 0$ such that $|f|_v^t < \delta$), cf. [Hu1], Lemma 2.4. By this Lemma, there is some $f \in \mathfrak{a}$ such that $|f|_v \in \Gamma'_v$. The valuation $w = v|\Gamma'_v$ (cf. [Hu1], sec. 2) is then $\mathfrak{a}$-adically continuous and analytic for the $\mathfrak{a}$-adic topology (i.e., its support does not contain $a$). But we have $|\varpi|_w = 0$, as otherwise $|\varpi|_v$ would be contained in $\Gamma'_v$, contradicting our assumption that $v$ is not $\mathfrak{a}$-adically continuous. Because $sp_X(v) = sp_Y(w) \in sp_Y(V(\varpi)_Y) = Y''$, we find that all elements in $sp_X^{-1}(Y - Y'')$ lie in the image of $\varphi^{an}$, and hence $sp_X^{-1}(Y - Y'') = \varphi^{an}(sp_Y^{-1}(Y - Y''))$ which is open in $t(X)_a$.

4. The Jacquet-Langlands correspondence realized on the cohomology

4.1. The trace on the Euler-Poincaré characteristic.

4.1.1. Let $\pi$ be an irreducible supercuspidal representation of $G$. By the fundamental result of Bushnell-Kutzko [BK] and Corwin [Co], we know that $\pi$ is induced from a finite-dimensional smooth irreducible representation $\lambda$ of some open subgroup $K_\pi \subset G$ that contains and is compact modulo the centre of $G$, cf. [BK], Thm. 8.4.1, for a more precise statement. Hence we may write

$$\pi = c \cdot \text{Ind}_{K_\pi}^G(\lambda) = \text{Ind}_{K_\pi}^G(\lambda),$$

where the second equality holds by [Bu], Thm.1. Here, $\text{Ind}_{K_\pi}^G(\lambda)$ is by definition the subspace of smooth vectors in the space

$$\{ f : G \to \lambda \mid \text{for all } k \in K_\pi, g \in G : f(kg) = \lambda(k)f(g) \}.$$

Moreover, the character of $\pi$ is a locally constant function on the set of elliptic regular elements in $G$ (i.e. those whose characteristic polynomial is separable and irreducible), and for such an element $g \in G$ we have

$$\chi_\pi(g) = \sum_{h \in G/K_\pi} \chi_\lambda(h^{-1}gh).$$

For regular elliptic $g$ the number of elements $h \in G/K_\pi$ such that $h^{-1}gh \in K_\pi$ is finite. This formula is due to Harish-Chandra, proofs can be found in [He1] and [Sa]. For the rest of this section we fix $\pi, K_\pi,$ and $\lambda$ with this property.

4.1.2. We recall the Jacquet-Langlands correspondence. For $\pi$ as above, the representation $\rho = JL(\pi)$ of $B^\times$ that corresponds to $\pi$ via the Jacquet-Langlands correspondence is characterized by the following identity. Let $g \in G$ and $b \in B^\times$ be regular elliptic elements with the same characteristic polynomial. Then the following character relation holds
\[ \chi_\rho(b) = (-1)^{n-1} \cdot \chi_\pi(g), \]

cf. [DKV], introduction, [Ro]. Thm. 5.8., [Ba].

For an irreducible supercuspidal representation \( \pi \) we know by 2.5.2, part (i), that the representation \( \text{Hom}_G(H^i_c, \pi) \) is a finite-dimensional smooth representation of \( B^\times \). We consider

\[
\text{Hom}_G(H^*_c, \pi) = \sum_i (-1)^i \text{Hom}_G(H^i_c, \pi).
\]
as an element of the Grothendieck group of admissible representations of \( B^\times \).

**Theorem 4.1.3.** Let \( \pi \) be an irreducible supercuspidal representation of \( G \). Then, in the Grothendieck group of admissible representations of \( B^\times \) we have:

\[
\text{Hom}_G(H^*_c, \pi) = n \cdot (-1)^{n-1} \mathcal{J}\mathcal{L}(\pi),
\]

**Proof.** In the proof of 2.5.2, part (i), we have seen that as a representation of \( B^\times \)

\[
\text{Hom}_G(H^i_c, \pi) = \text{Hom}_G(H^i_c(M_\infty/\varpi^Z), \pi \otimes \zeta) \otimes \xi^{-1},
\]

where the character \( \zeta \) of \( G \) is such that \( \varpi \in G \) acts as the identity on \( \pi \otimes \zeta \) (notations as introduced in 2.5.2). As the Jacquet-Langlands correspondence is compatible with twisting by characters, we can assume from now on that \( \varpi \) acts as the identity on \( \pi \). We consider for any \( i \geq 0 \) the admissible representation \( V^i = H^i_c(M_\infty/\varpi^Z) \) of \( G/\varpi^Z \times B^\times/\varpi^Z \).

We put \( V^i(\pi) = \text{Hom}_G(V^i, \pi) \).

We want to compute the traces of Hecke operators on \( B^\times \) on the \( V^i(\pi) \). With the notations as in 4.1.1 we put

\[
f_\pi = \frac{1}{\text{vol}(K_\pi/\varpi^Z)} \chi_\lambda \cdot 1_{K_\pi/\varpi^Z},
\]

where \( 1_{K_\pi/\varpi^Z} \) is the characteristic function of \( K_\pi/\varpi^Z \). Define for any function \( f \) on \( G \) or \( B^\times \) the function \( f^* \) by \( f^*(x) = f(x^{-1}) \). Then the multiplicity of \( \pi \) in \( V^i \) is \( \text{tr}(f^*_\pi|V^i) \), because

\[
\text{tr}(f^*_\pi|\pi) = 1,
\]

and \( \text{tr}(f^*_\pi|V) = 0 \) for any admissible representation \( V \) not containing \( \pi \). It follows that for any compactly supported function \( f \) on \( B^\times/\varpi^Z \) one has
Let $K \subset K_0$ be an open subgroup normalized by $K_\pi$ and such that $K$ lies in the kernel of $\lambda$. Then the trace of $f_\pi$ on $V^i$ is the same as the trace of $f_\pi$ on $H^i_c(M_K/\bar{\omega}^Z)$. Let $W^i$ be the $(G/\bar{\omega}^Z \times B^\times/\bar{\omega}^Z)$-subrepresentation of $V^i$ generated by $H^i_c(M_K/\bar{\omega}^Z)$. By [Cas], Thm. 6.3.10, the representation $W^i$ is of finite length, and the trace of $f_\pi^* \circ f^*$ on $W^i$ is the same as the trace of that function on $V^i$. For the following arguments we fix some isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, and consider all occurring representations by base-change as representations over $\mathbb{C}$. As $W^i$ is of finite length, its character $\chi_{W^i}$ is a locally integrable function on $G/\bar{\omega}^Z \times B^\times/\bar{\omega}^Z$ which is locally constant on the subset of regular elements (cf. [Le], Thm. 5.2.4). Further, by [Ka], Thm. A, the orbital integrals of $f_\pi$ over regular non-elliptic conjugacy classes all vanish (by [Le], Kazhdan’s results hold also in the equal-characteristic case). We get therefore:

$$tr(f_\pi^* \cdot f^*|V^i) = tr(f|V^i(\pi)).$$

where $G^e$ denotes the open set of regular elliptic elements in $G$. Note that $\operatorname{supp}(f_\pi) \cap G^e/\bar{\omega}^Z$ is an open, but in general not a compact subset of $G/\bar{\omega}^Z$. Let

$$C_1 \subset C_2 \subset \ldots \subset G^e/\bar{\omega}^Z$$

be an ascending sequence of compact-open subsets whose union is $\operatorname{supp}(f_\pi) \cap G^e/\bar{\omega}^Z$. Let $(f_j)$ be a sequence of locally constant functions on $G$, such that $\operatorname{supp}(f_j) \subset C_j$ and $f_j|c_j = f_\pi|c_j$. Then we have

$$tr(f_\pi^* \cdot f^*|W^i) = \lim_{j \to \infty} \int_{(G/\bar{\omega}^Z) \times (B^\times/\bar{\omega}^Z)} \chi_{W^i}(g,b) f_j^*(g) f^*(b) dgdb.$$

(The only reason we switched to $\mathbb{C}$ is to make sense of this limit.) The next step is to compute the integrals appearing in the formula above. Fix $j$. For a given element $g \in C_j$ there is a compact open subgroup $K_g$ of $G/\bar{\omega}^Z$ such that:

- $g^{-1}K_g g = K_g$
- $K_g g \subset C_j$, and
- $f_j^*$ is constant on the coset $K_g g$.

$K_g$ can be taken to be arbitrarily small (i.e. contained in any given open subgroup). As $C_j$ is compact, there are finitely many $g_1, g_2, \ldots, g_r \in C_j$ and corresponding compact-open subgroups $K_{g_r}$ such that $C_j$ is covered by the cosets $K_{g_r}$. We may assume that none of these cosets is contained in another coset. Then we can choose a normal compact-open subgroup $K_{g_1}'$ of $K_{g_1}$ which is contained in all the other subgroups $K_{g_r}$ and for which $g_1^{-1}K_{g_1}' g_1 = K_{g_1}'$. Then there are finitely many cosets $K_{g_1} g_{1,\alpha} \subset K_{g_1} g_1$ whose intersection
with any of the cosets $K_{g\nu}, \nu > 1$, is empty, and whose union is the complement of the union of the $K_{g\nu}, \nu > 1$, in $C_j$. Proceeding inductively we then can actually assume that the cosets $K_{g\nu}$ are mutually disjoint. Remember that the subrepresentation $W^i$ depends on some compact open subgroup $K$. Chose $K$ to be contained in all $K_{g\nu}$. Then we have

$$\sum_i (-1)^i \int_{(G/\mathbb{Z})\times (B^\times/\mathbb{Z})} \chi_{W^i}(g, b) f_\nu^* g f_\nu^* b db$$

$$= \sum_i (-1)^i \sum_{1 \leq \nu \leq r} \int_{(K_{g\nu}g\nu)\times (B^\times/\mathbb{Z})} \chi_{W^i}(g, b^{-1}) f_\nu^* g f_\nu^* b db$$

$$= \sum_{1 \leq \nu \leq r} \int_{(K_{g\nu}g\nu)\times (B^\times/\mathbb{Z})} f_\nu^* g tr((g, b^{-1})|H_\nu^* (M_{g\nu}/\mathbb{Z})) f_\nu^* b db$$

Now suppose that $\text{supp}(f)$ is contained in the set of regular elliptic elements of $B^\times$. Then we can use 3.3.2 which states that

$$tr((g, b^{-1})|H_\nu^* (M_{g\nu}/\mathbb{Z})) = n \cdot \# \{ h \in G/\mathbb{Z}K_{g\nu} | h^{-1} g h = g^{-1} \},$$

where $g_\nu$ is any element in $G$ having the same characteristic polynomial as $b$. Plugging this trace formula into the previously derived expression gives:

$$\sum_{1 \leq \nu \leq r} \int_{(K_{g\nu}g_\nu)\times B^\times} f_\nu^* g tr((g, b^{-1})|H_\nu^* (M_{g\nu}/\mathbb{Z})) f_\nu^* b db$$

$$= \sum_{1 \leq \nu \leq r} n \cdot \text{vol}(K_{g\nu}) \int_B \times f_\nu(g_\nu^{-1}) \# \{ h \in G/\mathbb{Z}K_{g\nu} | h^{-1} g h = g^{-1} \} f(b) db$$

$$= \sum_{1 \leq \nu \leq r} n \cdot \int_B \times \int_{G/\mathbb{Z}} f_\nu(h^{-1} g h) f(b) dh db$$

$$= \int_B \times \int_{G/\mathbb{Z}} f_\nu(h^{-1} g h) f(b) dh db,$$

where $f_{j, \nu}!$ denotes the extension by zero of the function $f_j|_{g_\nu^{-1} K_{g\nu}}$. As $f$ has compact support, we can pass to the limit as $j$ tends to infinity, and take the alternating sum over all $i$ to get:
Deformation spaces of one-dimensional formal modules and their cohomology

\[ tr(f | \text{Hom}_G(H^*_e, \pi)) = \sum_i (-1)^i tr(f_i^* \cdot f^* | V^i) \]

\[ = \lim_{j \to \infty} n \cdot \int_B^\times \left( \int_{G/\varpi^j} f_j(h^{-1}g_{bh})dg \right) f(b)db \]

\[ = n \cdot \int_B^\times \left( \int_{G/\varpi^j} f_{\pi}(h^{-1}g_{bh})dg \right) f(b)db \]

\[ = n \cdot \int_B^\times \left( \text{vol}(K_{\pi}/\varpi) \int_{G/K_{\pi}} f_{\pi}(h^{-1}g_{bh})dg \right) f(b)db \]

\[ = n \cdot \int_B^\times \left( \sum_{h \in G/K_{\pi}, h^{-1}g_{bh} \in K_{\pi}} \chi_{\lambda}(h^{-1}g_{bh}) \right) f(b)db \]

\[ = n \cdot \int_B^\times \chi_{\pi}(g_{bh})f(b)db. \]

If we now fix a regular elliptic element \( b \in B^\times \), and if we replace \( f \) by a sequence of compactly supported functions on \( B^\times \) whose support converges to \( \{ b \} \) and whose integral is 1, we get

\[ tr(b | \text{Hom}_G(H^*_e, \pi)) = n \cdot \chi_{\pi}(g_{bh}) = n \cdot (-1)^{n-1} \chi_{JL}(\pi)(b), \]

where the last equality is the character identity of the Jacquet-Langlands correspondence. Because a virtual representation of \( B^\times \) is already determined by the restriction of its character to the dense subset of regular elliptic elements, the theorem is proved. \( \square \)

4.2. The \( \varpi \)-adic boundary.

4.2.1. Let \( \pi \) be an irreducible supercuspidal representation of \( G \). The aim of the next section is to show that \( \text{Hom}_G(H^i_e, \pi) = 0 \) for \( i \neq n-1 \). As we have shown in the proof of 2.5.2, part (i), we can assume that \( \varpi \) acts on \( \pi \) as the identity. Therefore, we consider the admissible representation

\[ H^i_e(M_{\infty}/\varpi) = \lim_{K} H^i_e(M_K/\varpi), \]

with the notation as in the proof of 2.5.2, and show that no subquotient of this representation is supercuspidal. To this end we will introduce in this section yet another kind of compactifications of the spaces \( M^{(j)}_K \), to be denoted by \( M^{(\varpi, j)}_K \) and the boundary

\[ \partial^{\varpi} M^{(j)}_K = M^{(\varpi, j)}_K \times_{\hat{F}_{nr}} \hat{F}^\times \]
are analytic pseudo-adic spaces, with \(\varpi\) being invertible on the structure sheaf. We call \(\partial^\varpi M_K\) the \(\varpi\)-adic boundary.

4.2.2. Let \(M_K(j) = \text{Spec}(\mathcal{O}_K^{(j)})\) be one of the affine schemes of finite type over \(\hat{o}^{\text{nr}}\) from Thm. 2.3.1. There is a closed point \(x_K\) in the special fibre of \(M_K^{(j)}\) such that the completion of \(M_K\) at \(x_K\) is isomorphic to \(M_K^{(j)}\). Let \(\widehat{M}_K^{(j)}\) be the completion of

\[
\mathcal{O}_K^{(j)} \times_{\text{Spec}(\hat{o}^{\text{nr}})} \text{Spec}(\mathcal{O}_{F^\wedge})
\]

along the closed subscheme where \(\varpi\) is zero. Denote by \((\widehat{M}_K^{(j)})^{\text{ad}}\) the analytic adic space associated to \(\widehat{M}_K^{(j)}\), cf. [Hu3], Prop. 1.9.1, and by

\[
s_p\mathcal{O}_K^{(j)} : (\mathcal{O}_K^{(j)})^{\text{ad}} \longrightarrow \widehat{M}_K^{(j)}
\]

the specialization map. Put

\[
\overline{M}_K^{(\varpi, j)} = s_p^{-1}(x_K).
\]

This is a pseudo-adic subspace of \((\mathcal{O}_K^{(j)})^{\text{ad}}\), cf. [Hu3], sec. 1.10, and Prop. 4.2.5 below. \(\overline{M}_K^{(\varpi, j)}\) consists of all \(\varpi\)-adically continuous valuations \(v\) of the ring \(\mathcal{O}_K^{(j)} \otimes_{\hat{o}^{\text{nr}}} \mathcal{O}_{F^\wedge}\) such that

\[
s_p\mathcal{O}_K^{(j)}(v) = \{ f \in \mathcal{O}_K^{(j)} \otimes_{\hat{o}^{\text{nr}}} \mathcal{O}_{F^\wedge} \mid |f|_v < 1 \}
\]

is equal to the maximal ideal of \(\mathcal{O}_K^{(j)} \otimes_{\hat{o}^{\text{nr}}} \mathcal{O}_{F^\wedge}\) corresponding to the closed point \(x_K\). The canonical map

\[
\mathcal{O}_K^{(j)} \otimes_{\hat{o}^{\text{nr}}} \mathcal{O}_{F^\wedge} \longrightarrow R_K^{(j)} \otimes_{\hat{o}^{\text{nr}}} \mathcal{O}_{F^\wedge}
\]

induces an open embedding

\[
M_K^{(j)} \times_{\hat{F}^\wedge} \hat{F}^\wedge \hookrightarrow \overline{M}_K^{(\varpi, j)}
\]

(cf. the proof of Lemma 3.3.3). Most important for our following reasoning is that the cohomology of these two spaces coincides:

**Proposition 4.2.3.** For any \(i \geq 0\) and \(r \geq 0\) the canonical map

\[
H^i(\overline{M}_K^{(\varpi, j)}, \mathbb{Z}/\ell^r \mathbb{Z}) \longrightarrow H^i(M_K^{(j)} \times_{\hat{F}^\wedge} \hat{F}^\wedge, \mathbb{Z}/\ell^r \mathbb{Z})
\]

is an isomorphism.
Proof. Let
\[(\mathfrak{M}^{(j)}_K \times_{\mathfrak{F}^\wedge} \mathfrak{F}^\wedge)_s \rightarrow \mathfrak{M}^{(j)}_K \times_{\mathfrak{F}^\wedge} \mathfrak{F}^\wedge \leftarrow (\mathfrak{M}^{(j)}_K \times_{\mathfrak{F}^\wedge} \mathfrak{F}^\wedge)_g\]
be the closed (resp. open) embedding of the closed (resp. generic) fibre.

By [Hu3], Thm. 3.5.15, one has a canonical isomorphism
\[H^i(\overline{\mathcal{M}}^{(\omega,j)}_K, \mathbb{Z}/\ell^n \mathbb{Z}) \simeq (R^i j_*(\mathbb{Z}/\ell^n \mathbb{Z}))_{\overline{\mathcal{Y}}_K},\]

The maximal Hausdorff quotient \((\mathfrak{M}^{(j)}_K \times_{\mathfrak{F}^\wedge} \mathfrak{F}^\wedge)_{\text{an}}\) of the adic space \(\mathfrak{M}^{(j)}_K \times_{\mathfrak{F}^\wedge} \mathfrak{F}^\wedge\) is the non-Archimedean analytic space that Berkovich associates to the completion of \(\mathfrak{M}^{(j)}_K \times_{\mathfrak{F}^\wedge} \mathfrak{F}^\wedge\) at \(x\), cf. [Be3], sec. 1. By [Be3], Cor. 3.5, one has a canonical isomorphism
\[H^i((\mathfrak{M}^{(j)}_K \times_{\mathfrak{F}^\wedge} \mathfrak{F}^\wedge)_{\text{an}}, \mathbb{Z}/\ell^n \mathbb{Z}) \simeq (R^i j_*(\mathbb{Z}/\ell^n \mathbb{Z}))_{\overline{\mathcal{Y}}_K}.\]

Hence the cohomology of \(\mathfrak{M}^{(\omega,j)}_K\) is canonically isomorphic to the cohomology of \((\mathfrak{M}^{(j)}_K \times_{\mathfrak{F}^\wedge} \mathfrak{F}^\wedge)_{\text{an}}\). By [Hu3], Thm. 8.3.5, the cohomology of \((\mathfrak{M}^{(j)}_K \times_{\mathfrak{F}^\wedge} \mathfrak{F}^\wedge)_{\text{an}}\) is canonically isomorphic to the cohomology of \(\mathfrak{M}^{(j)}_K \times_{\mathfrak{F}^\wedge} \mathfrak{F}^\wedge\).

Remark. One can show that \(\mathfrak{M}^{(j)}_K \times_{\mathfrak{F}^\wedge} \mathfrak{F}^\wedge\) is the interior of \(\mathfrak{M}^{(\omega,j)}_K\) in \((\mathfrak{M}^{(j)}_K)_{\text{ad}}\). If \(F\) has characteristic zero then the assertion of the preceding proposition is [Hu4], Cor. 3.9.

4.2.4. Recall from 3.1.1 the adic space
\[\overline{\mathcal{M}}^{(j)}_K = t(R^{(j)}_{\text{K}}(\mathbb{Z}/\ell^n \mathbb{Z})).\]

There is a canonical continuous map of topological spaces
\[sp : \overline{\mathcal{M}}^{(\omega,j)}_K \rightarrow \overline{\mathcal{M}}^{(j)}_K\]
which is defined as follows, cf. [Hu1], Prop. 2.6. For a point \(v \in \overline{\mathcal{M}}^{(\omega,j)}_K\) with value group \(\Gamma_v\) let \(c\Gamma_v(m)\) be the largest convex subgroup of \(\Gamma_v\) such that \(|f|_v\) is cofinal for all \(f\) in the maximal ideal of \(\mathfrak{M}^{(j)}_K \otimes_{\mathfrak{F}^\wedge} \mathfrak{F}^\wedge\) corresponding to \(\mathfrak{Y}_K\). Then \(v\mid c\Gamma_v(m)\) (cf. [Hu1], sec. 2) is a valuation of \(\mathfrak{M}^{(j)}_K \otimes_{\mathfrak{F}^\wedge} \mathfrak{F}^\wedge\) which is continuous for the topology defined by the maximal ideal of \(\mathfrak{M}^{(j)}_K \otimes_{\mathfrak{F}^\wedge} \mathfrak{F}^\wedge\) corresponding to \(\mathfrak{Y}_K\). It extends therefore to a valuation of \(R^{(j)}_{\text{K}}(\mathbb{Z}/\ell^n \mathbb{Z})\) which is continuous for the topology defined by the maximal ideal of \(R^{(j)}_{\text{K}}\), and we put
\[sp(v) = v\mid c\Gamma_v(m) \in \overline{\mathcal{M}}^{(j)}_K.\]
If the valuation \( v \) is in \( M_K^{(j)} \times \mathcal{F}^\wedge \), then \( v \) is continuous with respect to the topology defined by the maximal ideal of \( \mathfrak{m}_K^{(j)} \otimes \mathcal{O}_{\mathcal{F}^\wedge} \) corresponding to \( \mathfrak{r}_K \), and it extends then to \( R_K^{(j)} \). By [Hu1], Prop. 2.6, the map \( sp \) is continuous. (\( sp \) is the restriction of the map \( r \) in [Hu3], Prop. 2.6.) We now consider the preimages of the strata defined in sec. 3.1.1. For an element \( A \in \mathcal{S}_K \) put

\[
\partial_A^{\sigma} M_K^{(j)} = sp^{-1}(\partial_A M_K^{(j)}),
\]

where \( \partial_A^{\sigma} M_K^{(j)} \) is defined as in 3.1.1. Put for \( h \in \{0, \ldots, n-1\} \)

\[
\partial_h^{\sigma} M_K^{(j)} = \bigcup_{A \in \mathcal{S}_K,h} \partial_A^{\sigma} M_K^{(j)}, \quad \partial_{\geq h}^{\sigma} M_K^{(j)} = \bigcup_{h' \geq h, A \in \mathcal{S}_K,h'} \partial_A^{\sigma} M_K^{(j)}.
\]

\( \partial_{\geq h}^{\sigma} M_K^{(j)} \) is a closed subspace of \( \overline{M}_K^{(\sigma, j)} \) because it is the preimage of all strata \( \partial_A M_K^{(j)} \) with \( A \in \mathcal{S}_K,h' \) and \( h' \geq h \), which is a closed subset by 3.1.2. Hence we have a descending sequence of closed subspaces

\[
\overline{M}_K^{(\sigma, j)} = \partial_{\geq 0}^{\sigma} M_K^{(j)} \supset \partial_{\geq 1}^{\sigma} M_K^{(j)} \supset \cdots \supset \partial_{\geq n-1}^{\sigma} M_K^{(j)}
\]

with

\[
\partial_{\geq h} M_K^{(j)} - \partial_{\geq h+1} M_K^{(j)} = \partial_h^{\sigma} M_K^{(j)}.
\]

Denote by

\[
\partial_A^{\sigma} M_K^{(j)} \rightarrow \partial_{\geq h}^{\sigma} M_K^{(j)}, \quad j_A : \partial_A^{\sigma} M_K^{(j)} \rightarrow \partial_{\geq h}^{\sigma} M_K^{(j)}
\]

the inclusions, where \( A \in \mathcal{S}_K,h \).

**Proposition 4.2.5.** (i) The subsets \( \partial_A^{\sigma} M_K^{(j)} \), \( \partial_h^{\sigma} M_K^{(j)} \), and \( \partial_{\geq h}^{\sigma} M_K^{(j)} \) are pseudo-adic subspaces of \( (\mathcal{M}_K^{(j)})^{ad} \).

(ii) \( \partial_{\geq h} M_K^{(j)} \) is proper over \( \text{Spa}(\mathcal{F}^\wedge, \mathcal{O}_{\mathcal{F}^\wedge}) \). \( \partial_A^{\sigma} M_K^{(j)} \) and \( \partial_h^{\sigma} M_K^{(j)} \) are partially proper over \( \text{Spa}(\mathcal{F}^\wedge, \mathcal{O}_{\mathcal{F}^\wedge}) \). If \( \mathcal{F} \) is a sheaf of abelian groups on \( \partial_A^{\sigma} M_K^{(j)} \), \( A \in \mathcal{S}_K,h \), then

\[
H^i_c(\partial_A^{\sigma} M_K^{(j)}, \mathcal{F}) = H^i(\partial_{\geq h} M_K^{(j)}, (j_A)_! \mathcal{F}),
\]

and

\[
H^i_c(\partial_h^{\sigma} M_K^{(j)}, \mathcal{F}) = H^i(\partial_{\geq h} M_K^{(j)}, (j_h)_! \mathcal{F}).
\]
Proof. (i) \( \overline{M}_K^{(\omega,j)} \) is a closed subspace of \((\mathfrak{M}_K^{(j)})^{ad} \) and hence convex and pro-constructible in \((\mathfrak{M}_K^{(j)})^{ad} \). So \((\mathfrak{M}_K^{(j)})^{ad}, \overline{M}_K^{(\omega,j)} \) is a pseudo-adic space in the sense of [Hu3], Def. 1.10.3. From now on we will always take \((\mathfrak{M}_K^{(j)})^{ad} \) as the ambient adic space and call a subset \( Z \subset (\mathfrak{M}_K^{(j)})^{ad} \) a pseudo-adic space if it is convex and locally pro-constructible in \((\mathfrak{M}_K^{(j)})^{ad} \).

The map \( sp : \overline{M}_K^{(\omega,j)} \rightarrow \overline{M}_K^{(j)} \) is not only continuous but even a spectral map, cf. [Hu1], Prop. 2.6. The preimage of a locally pro-constructible set under a spectral map is again locally pro-constructible, and the subsets \( \partial_A M_K^{(j)} \) of \( \overline{M}_K^{(j)} \) are locally closed, hence locally pro-constructible. \( \partial_{\omega h} M_K^{(j)} \) is closed and hence pro-constructible and convex. So it is a pseudo-adic space. Because in an analytic adic space the generalizations of a point form a chain, cf. [Hu2], Lemma 1.1.10 (i), a subset \( Z \subset (\mathfrak{M}_K^{(j)})^{ad} \) which contains all specializations of \( z \) in \((\mathfrak{M}_K^{(j)})^{ad} \) for any point \( z \in Z \) will be convex. We call such a set closed under specialization. If \( z \) lies in \( \partial_A M_K^{(j)} \), and if \( z' \) is a specialization of \( z \), then \( sp(z') \) is a specialization of \( sp(z) \). The adic space \( \overline{M}_K^{(j)} \) is analytic, and hence all specializations are secondary specializations, cf. [Hu3], sec. 1.1.9. The support of a secondary specialization of a point does not change, and so \( sp(z') \) lies in the same stratum as \( sp(z) \). This means in particular that the set \( \partial_A M_K^{(j)} \) is convex and hence a pseudo-adic subspace of \((\mathfrak{M}_K^{(j)})^{ad} \). The same is then the case for the union \( \partial_{\omega h} M_K^{(j)} \).

(ii) We show first that any valuation ring \((v, V)\) of \((\mathfrak{M}_K^{(j)})^{ad} \), in the sense of [Hu3], Def. 1.3.5, with support \( v \in \overline{M}_K^{(\omega,j)} \) has a unique center on \((\mathfrak{M}_K^{(j)})^{ad} \). By [Hu3], Lemma 1.3.6, there is at most one center on \((\mathfrak{M}_K^{(j)})^{ad} \), and \((v, V)\) has a center if and only if the image of \( R = \mathfrak{M}_K^{(j)} \otimes_{\mathcal{O}_{F^h}} \mathcal{O}_{F^h}^{x} \) in the residue field \( k(v) \) is contained in \( V \). \( V \) is by definition contained in \( k(v)^+ \). Let \( m \) be the maximal ideal of \( R \) corresponding to \( F_k \). If \( v \) is an element of \( \overline{M}_K^{(\omega,j)} \) then \( |f|_v < 1 \) for all \( f \in m \). Hence \( f(v), \) the image of \( f \) in \( k(v) \), is contained in the maximal ideal of \( k(v)^+ \), and therefore in the maximal ideal of \( V \). If \( f \) is any element of \( R - m \), there is a unit \( \alpha \in (\mathcal{O}_{F^h})^x \) and an element \( f_1 \in m \) such that \( f = \alpha + f_1 \). Then \( f(v) = \alpha + f_1(v) \in V \). Hence, if \( v \) is in \( \overline{M}_K^{(\omega,j)} \), and if \((v, V)\) is a valuation ring of \((\mathfrak{M}_K^{(j)})^{ad} \), it has a center on \((\mathfrak{M}_K^{(j)})^{ad} \). This center is a specialization of \( v \) and has to lie in \( \overline{M}_K^{(\omega,j)} \). \((\mathfrak{M}_K^{(j)})^{ad} \) is an affinoid adic space and in particular quasi-compact. \( \overline{M}_K^{(\omega,j)} \) is closed in \((\mathfrak{M}_K^{(j)})^{ad} \), and hence quasi-compact too. By the valuative criterion for properness [Hu3], Cor. 1.10.21, the space \( \overline{M}_K^{(\omega,j)} \) is proper over \( \text{Spa}(F^h, \mathcal{O}_{F^h}) \). Any pseudo-adic subspace of \( \overline{M}_K^{(\omega,j)} \) which is closed under specialization will then also verify the valuative criterion for partially properness, and if such a subspace is quasi-compact it will be proper over \( \text{Spa}(F^h, \mathcal{O}_{F^h}) \). For the partially proper spaces the cohomology with compact is defined as the derived functor of the functor ‘sections with compact support’. The last assertion of (ii) follows immediately from [Hu3], Lemma 5.4.2. \( \square \)
4.2.6. Next we want to show that the action of $G = GL_n(F)$ on the spaces $M^{(j)}_K$ extends to the spaces $\overline{M}^{(\varpi,j)}_K$, respects the strata and induces an action on the cohomology of the strata. This is not automatic because we defined $\overline{M}^{(\varpi,j)}_K$ as a subspace of $(M^{(j)}_K)^{ad}$ which depends on the algebraic scheme $M^{(j)}_K$ that we have chosen as in Thm. 2.3.1. The choice of such an algebraization is not unique, and so, in fact, the pseudo-adic spaces $\overline{M}^{(\varpi,j)}_K$ will not be isomorphic for different choices of algebraizations. However, by Prop. 4.2.3, the cohomology of the $\overline{M}^{(\varpi,j)}_K$, for different algebraizations, is isomorphic, and in the proof of Thm. 4.3.2 we will show that the cohomology of the strata is also isomorphic, for different algebraizations. But we will need to know that they are canonically isomorphic. To this end we will use a compatible system of algebraizations.

Let $G^{(0)} \subset G$ be the group of elements for which the valuation of the determinant $v(det(g))$ is a multiple of $n$. It is easily seen that with the notation of 4.2.1

$$H^i(M_\infty/\varpi^\infty) = Ind^{G}_{G^{(0)}} H^i_c(M^{(0)}_\infty)$$

where

$$H^i_c(M^{(0)}_\infty) = \lim_{\longrightarrow} K H^i_c(M^{(0)}_K \times _{F_{nr}} \tilde{F}^{\wedge}, \mathbb{Q}_\ell).$$

Hence it suffices to extend the action of elements of $G^{(0)}$ to the $\varpi$-adic boundary and its strata, and from now on we will therefore consider only the 'height-0-components' of the various spaces we defined. Moreover, we will consider only principal congruence subgroups $K_m$, and instead of the subscript $K_m$ we will write $m$. Let $\mathcal{S}^{(0)}_m$ be a scheme of finite type over $\hat{o}^{nr}$ over which there is given a truncated Barsotti-Tate $\mathfrak{o}$-module $\mathfrak{X}_m$ of level $m$, and a closed point $\eta_m$ such that the completion of $\mathcal{S}^{(0)}_m$ at $\eta_m$ is isomorphic to $\mathcal{M}^{(0)}_{K_0}$ (the universal deformation space of $\mathfrak{X}$), and such that $\mathfrak{X}_m$ corresponds to $X_{univ}[\varpi^m]$ under this isomorphism. Then, for $m' \leq m$, $\mathfrak{X}_{m'}[\varpi^{m'}]$ will be isomorphic to $X_{univ}[\varpi^{m'}]$. By the uniqueness theorem of algebraizations, cf. [A2], Thm. 1.7, there is a scheme $\mathcal{S}'$, a closed point $\eta'$, and a diagram of étale morphisms

$$\mathcal{S}' \ar{r}{\psi'} \ar{r}{\psi} \ar{r}{\psi} & \mathcal{S}^{(0)}_{m'} \ar{r}{\psi} \ar{r}{\psi} & \mathcal{S}^{(0)}_m$$

such that $\psi'(\eta') = \eta_{m'}$ and $\psi(\eta') = \eta_m$, and $(\psi')^* \mathfrak{X}_{m'} \simeq (\psi)^* \mathfrak{X}_m[\varpi^{m'}]$. Then, when we take the pair $(\mathcal{S}', (\psi)^* \mathfrak{X}_m)$ instead of $\mathcal{S}^{(0)}_m$, we find inductively a tower of schemes of finite type over $\hat{o}^{nr}$.
with closed points \( y_m \in S_m^{(0)} \) in the special fibre. The morphisms in this sequence are all \( \acute{e}tale \) and map \( y_m \) to \( y_m - 1 \), and on \( S_m^{(0)} \) there is a truncated Barsotti-Tate \( \mathfrak{a} \)-module \( X_m \) of level \( m \) which has the property that the pull-back of \( X_{m'} \) for \( m' \leq m \) is \( X_{m}[\varpi^{m'}] \).

We may clearly assume the schemes \( S_m^{(0)} \) to be affine. Then we define, as in the proof of Thm. 2.3.1, the scheme \( M_m^{(0)} \) as the scheme over \( S_m^{(0)} \) which parameterizes Drinfeld level-\( m \)-structures on \( X_m \). This gives a tower of affine schemes of finite type over \( \hat{o}^{nr} \):

\[
\begin{align*}
S_0^{(0)} & \leftarrow S_1^{(0)} \leftarrow S_2^{(0)} \leftarrow \ldots \\
M_0^{(0)} & \leftarrow M_1^{(0)} \leftarrow M_2^{(0)} \leftarrow \ldots 
\end{align*}
\]

with closed points \( x_m \in M_m^{(0)} \) such that the completion of \( M_m^{(0)} \) at \( x_m \) is isomorphic to \( M_m^{(0)} \).

Now we can explain how the action of the group \( G^{(0)} \) extends to the spaces \( \overline{M}_K^{(\varpi,0)} \) and that it respects the subsets \( \partial \overline{M}_K^{(0)} \). Given an element \( g \in G^{(0)} \), we can assume that the valuation of its determinant is zero, because \( \varpi \) acts anyway trivially on \( H^i_c(M_\infty/\varpi) \).

Then we define, as in sec. 2.2.2, for any sufficiently large \( m \) a morphism

\[
g : M_m^{(0)} \rightarrow M_{m'}^{(0)},
\]

where \( m' \) depends on \( m \) and \( g \). Let \( 0 \leq m' \leq m \) and \( r \in \mathbb{Z} \) be integers such that

\[
\mathfrak{o}^n \subset \varpi^{-r} g \mathfrak{o}^n \subset \varpi^{-(m-m')} \mathfrak{o}^n.
\]

Then we have an isomorphism

\[
g' : \varpi^{-m'} \mathfrak{o}^n / \mathfrak{o}^n \xrightarrow{\varpi^{-r} g} \varpi^{-m} \mathfrak{o}^n / \mathfrak{o}^n \xrightarrow{\varpi^{-m-m'}} \varpi^{-m'} \mathfrak{o}^n / \mathfrak{o}^n
\]

as in sec. 3.1.1. Let \( f : \mathcal{T} \rightarrow \mathfrak{G}_m^{(0)} \) be a morphism of schemes over \( \hat{o}^{nr} \), and let \( \phi : \varpi^{-m} \mathfrak{o}^n / \mathfrak{o}^n \rightarrow f^* X_m \) be a level-\( m \)-structure. Let

\[
\phi' = \phi|_{\varpi^{-m'} \mathfrak{o}^n / \mathfrak{o}^n} \circ g' : \varpi^{-m'} \mathfrak{o}^n / \mathfrak{o}^n \rightarrow f^* X_m.
\]

\( \phi' \) is then a level-\( m' \)-structure on \( (f^* X_m)[\varpi^{m'}] \). If we let \( f' : \mathcal{T} \rightarrow \mathfrak{G}_m^{(0)} \) be the composition of \( f \) with \( \mathfrak{G}_m^{(0)} \rightarrow \mathfrak{G}_m^{(0)} \), then \( (f^* X_m)[\varpi^{m'}] = (f')^* X_{m'} \), and \( \phi' \) is a level-\( m' \)-structure on \( (f')^* X_{m'} \). This defines the morphism

\[
g : M_m^{(0)} \rightarrow M_{m'}^{(0)},
\]

which maps the closed point \( x_m \) to \( x_{m'} \) and is, by its very construction, compatible with the action on the completions. Then there is a commutative diagram
It follows that $\overline{M}_m^{(\omega,0)}$ is mapped by $g$ to $\overline{M}_{m'}^{(\omega,0)}$, and this map is compatible with specialization maps to the spaces $\overline{M}_m^{(0)}$ and $\overline{M}_{m'}^{(0)}$, because these were constructed from the formal schemes $\mathcal{M}_m^{(0)}$ and $\mathcal{M}_{m'}^{(0)}$. Hence there is a commutative diagram

$$
\begin{array}{cccc}
\mathcal{M}_m^{(0)} & \longrightarrow & \mathcal{M}_m^{(0)} & \longrightarrow \\
\downarrow g & & \downarrow g & \longrightarrow \\
\mathcal{M}_{m'}^{(0)} & \longrightarrow & \mathcal{M}_{m'}^{(0)} & \longrightarrow \\
\end{array}
$$

and $g$ maps therefore the subspace $\partial_1^\omega M_m^{(0)}$ to $\partial_1^\omega A M_m^{(0)}$, for any $A \in S_m := S_{K,m}$. $\partial_1^\omega M_m^{(0)}$ is hence mapped to $\partial_1^\omega M_{m'}^{(0)}$, and we get induced maps on the cohomology (or cohomology with compact support) of these spaces.

4.2.7. We define the $\ell$-adic cohomology groups (with compact support) for the spaces $\partial_1^\omega M_m^{(0)}$ and $\partial_1^\omega M_{m'}^{(0)}$ by firstly taking cohomology with coefficients $\mathbb{Z}/\ell^r\mathbb{Z}$, then passing to the limit for $r \to \infty$ and finally tensoring with $\overline{\mathbb{Q}_\ell}$. Unfortunately, we are lacking here some basic finiteness results for the cohomology of such spaces. We believe that these cohomology groups are finite-dimensional, and the discussion in the next section will show that otherwise there would occur some very strange phenomena. Namely, it will be shown that the cohomology of the $\omega$-adic boundary $\partial_1^\omega M_m^{(0)} = \partial_1^\omega M_{m'}^{(0)}$ is indeed finite-dimensional, and the limit for $m \to \infty$ gives an admissible representation which sits in an exact sequence

$$
H^i_c(\partial_1^\omega M_m^{(0)}, \overline{\mathbb{Q}_\ell}) \longrightarrow H^i(\partial_1^\omega M_m^{(0)}, \overline{\mathbb{Q}_\ell}) \longrightarrow H^i(\partial_1^\omega M_{m'}^{(0)}, \overline{\mathbb{Q}_\ell}),
$$

where the subscript $\infty$ indicates that the limit is taken for $m \to \infty$. The representation on the left can be shown to be induced from a representation $\rho$ of a parabolic subgroup $P$. However, if the cohomology of the strata is not finite-dimensional, the representation $\rho$ would not be admissible, and the unipotent radical of $P$ may not act trivially. Similarly, the cohomology group $H^i(\partial_1^\omega M_{m'}^{(0)}, \overline{\mathbb{Q}_\ell})$ is an extension of subquotients of parabolically induced representations, which would not be admissible if the cohomology groups of the strata are not finite-dimensional. It would follow, that the representation in the middle
of the above exact sequence, which is admissible, is an extension of subquotients of rep-
resentations which are themselves not admissible. Theoretically this is of course possible,
but in our situation it seems to us very unlikely that something like that may happen. So
we will be working from now on under the hypothesis

(H) The cohomology groups \( H^i_c(\partial_0^\infty M_m^{(0)}, \overline{\mathbb{Q}_\ell}), \ H^1(\partial_0^\infty M_m^{(0)}, \overline{\mathbb{Q}_\ell}) \) and \( H^i(\partial_{\geq h}^\infty M_m^{(0)}, \overline{\mathbb{Q}_\ell}) \) are finite-dimensional \( \overline{\mathbb{Q}_\ell} \)-vector spaces, for any \( i, m, A \) and \( h \).

We conclude with two remarks on how to prove this hypothesis, which we conjecture to
be true. Firstly, the subspaces \( \partial_0^\infty M_m^{(0)} \) are of relatively simple type, and we expect that
a good theory of sub-constructible subsets and their étale cohomology should imply the
finiteness of the cohomology. This is in fact a conjecture of R. Huber, cf. the introduction
of [Hu4]. We understood that Huber is presently working on such a theory, and our
hypothesis may then be true by the results of this theory. Secondly, one can also compute
the cohomology groups of the \( \partial_A^\infty M_m^{(0)} \) by means of the morphism

\[ sp : \partial_A^\infty M_m^{(0)} \rightarrow \partial_A M_m^{(0)}. \]

The essential point to show would be that the sheaves \( R^i sp_* (\mathbb{Z}/\ell^r \mathbb{Z}) \) are constructible on
\( \partial_A M_m^{(0)} \). The stalk of this sheaf at a geometric point \( x \in \partial_A M_m^{(0)} \) can be identified with the
cohomology of the preimage \( sp^{-1}(x) \). But this set has a kind of modular interpretation.
Namely, if the rank of \( A \) is \( h \), then the connected part

\[ (X^{univ}[\varpi^\infty] \otimes k(x))^o \subset X^{univ}[\varpi^\infty] \otimes k(x) \]

is of rank \( h \). Suppose now that \( h > 0 \) (otherwise we are in the interior). The set
\( sp^{-1}(x) \) then looks like the generic fibre of the deformation space of \( (X^{univ}[\varpi^\infty] \otimes k(x))^o \)
together with an level-\( m \)-structure \( \phi : A \rightarrow (X^{univ}[\varpi^\infty] \otimes k(x))^o \). \( sp^{-1}(x) \) is a space over
\( Frac(W_0(k(x))) \), but the field \( k(x) \) carries a non-trivial valuation, and so \( Frac(W_0(k(x))) \)
is a higher-dimensional local field, equipped with a valuation of rank at least two. A
further analysis of \( sp^{-1}(x) \) reveals that it (or rather its interior) can be thought of as
a rigid analytic space over this higher-dimensional local field. But, as far as I know, a
theory of analytic spaces over higher-dimensional local fields has not yet been developed.

4.3. Non-cuspidalness outside the middle degree.

4.3.1. For \( h \in \{1, \ldots, n-1\} \) let \( F^h \subset F^n \) be the subspace which is generated by the
first \( h \) standard basis vectors. For \( m > 0 \) put

\[ A_{m,h} = (F^h/\mathfrak{o}^n) \cap (\varpi^{-m}\mathfrak{o}^n/\mathfrak{o}^n) \subset \varpi^{-m}\mathfrak{o}^n/\mathfrak{o}^n, \]
i.e. \( A_{m,h} \) is the submodule of \( \varpi^{-m}\mathfrak{o}^n/\mathfrak{o}^n \) generated by the first \( h \) standard generators. Let
\( P_h \subset G \) be the stabilizer of \( F^h \) and \( P_h^{(0)} = P_h \cap G^{(0)} \), cf. sec. 4.2.6. Recall the action of \( G \)
on the sets $S_{m,h} = S_{K_{m,h}}$ of labels of boundary components. Firstly, $S_{m,h}$ can naturally be identified with $K_{m} \backslash G / P_{h}$, and if $g \in G$ is such that $g^{-1}K_{m}g \subset K_{m'}$ then the map from $S_{m,h}$ to $S'_{m',h}$ can be identified with the map

$$K_{m} \backslash G / P_{h} \rightarrow K_{m'} \backslash G / P_{h}, \quad K_{m} x P_{h} \mapsto K_{m'} g^{-1} x P_{h}.$$ 

The element $A_{m,h} \in S_{m,h}$ corresponds to the double coset $K_{m} \cdot 1 \cdot P_{h}$, and is therefore mapped to $K_{m'} \cdot 1 \cdot P_{h}$, if $g$ is in $P_{h}$. If $g$ is an element of $P^{(0)}_{h}$, and if $g^{-1}K_{m}g \subset K_{m'}$ then $g$ maps $\partial_{A_{m,h},m}^{\varphi} M_{m}^{(0)}$ to $\partial_{A_{m',h},m}^{\varphi} M_{m}^{(0)}$, cf. 4.2.6. We have therefore an action of $P^{(0)}_{h}$ on

$$W^{(h,i,0)}_{m} = \lim_{\rightarrow} H^{i}_{c}(\partial_{A_{m,h},m}^{\varphi} M_{m}^{(0)}, \overline{\mathbb{Q}_{\ell}}),$$

and we put

$$W^{(h,i)}_{m} = Ind_{P^{(0)}_{h}}^{P_{h}} W^{(h,i,0)}_{m}.$$ 

Similarly, for any $h \in \{1, \ldots, n-1\}$ there is an action of $G^{(0)}$ on

$$V^{(h,i,0)}_{m} = \lim_{\rightarrow} H^{i}(\partial_{h}^{\varphi} M_{m}^{(0)}, \overline{\mathbb{Q}_{\ell}}) \quad \text{and} \quad V^{(\geq h,i,0)}_{m} = \lim_{\rightarrow} H^{i}(\partial_{\geq h}^{\varphi} M_{m}^{(0)}, \overline{\mathbb{Q}_{\ell}})$$

and we put

$$V^{(h,i)} = Ind_{G^{(0)}}^{G} V^{(h,i,0)} \quad \text{and} \quad V^{(\geq h,i)} = Ind_{G^{(0)}}^{G} V^{(\geq h,i,0)}.$$ 

**Theorem 4.3.2.** We assume that the hypothesis (H) in 4.2.7 is fulfilled. Then the following assertions do hold.

(i) For $h \in \{1, \ldots, n-1\}$ and any $i$ the representation $W^{(h,i)}$ of $P_{h}$ is admissible. The action of the unipotent radical of $P_{h}$ on $W^{(h,i)}$ is trivial.

(ii) For $h \in \{1, \ldots, n-1\}$ and any $i$ the representation of $G$ on $V^{(h,i)}$ is canonically isomorphic to the representation $Ind_{P_{h}}^{G} W^{(h,i)}$.

(iii) For $h \in \{1, \ldots, n-1\}$ and any $i$ the representation of $G$ on $V^{(\geq h,i)}$ does not have any supercuspidal representation as a subquotient.

(iv) For any $i \neq n-1$ the representation of $G$ on

$$H^{i}_{c}(M_{\infty} / \varpi^{\infty}) := \lim_{\rightarrow} H^{i}_{c}((M_{K} / \varpi^{K}) \times_{F_{w}} F^{\wedge}, \overline{\mathbb{Q}_{\ell}}),$$

does not have any supercuspidal representation as a subquotient.
Proof. (i) Fix $0 < m' \leq m$. We want to compute the invariants of $K_{m'} \cap P_h$ on $H^i_c(\partial_{A_{m',h}} M_{m'}^{(0)} \otimes \overline{\mathbb{Q}_\ell})$, and we want to show that it is equal to the image of the map

$$H^i_c(\partial_{A_{m',h}} M_{m'}^{(0)} \otimes \overline{\mathbb{Q}_\ell}) \to H^i_c(\partial_{A_{m,h}} M_{m}^{(0)} \otimes \overline{\mathbb{Q}_\ell}).$$

From now on we will drop the superscript '(0)' everywhere in this part of the proof. So we write $\mathcal{M}_m$ instead of $\mathcal{M}_m^{(0)}$. Let $\mathcal{M}_m = \text{Spec}(\mathfrak{M}_m)$. Recall that there is a action of $K_0$ on this scheme, via the level structure, which is trivial on $K_m$. Put $\mathfrak{M}_m' = (\mathfrak{M}_m)^{K_m'}$ and $\mathcal{M}_m' = \text{Spec}(\mathfrak{M}_m')$. This scheme classifies Drinfeld level-$m'$-structures on $\mathfrak{F}_m[\varpi^{m'}]$ over the scheme $\mathfrak{S}_m$, cf. 4.2.6. Let $\mathfrak{r}_m'$ be the closed point in $\mathfrak{M}_m'$, which is the image of $\mathfrak{r}_m$ under $\mathcal{M}_m \to \mathcal{M}_m'$. The morphism $pr : \mathcal{M}_m \to \mathcal{M}_m'$ induces a morphism

$$pr' : \mathfrak{M}_m' \to \mathfrak{M}_m'$$

because the action of $K_{m'}$ on $\mathfrak{M}_m'$ is trivial, and this morphism induces an isomorphism on the completions, even on the henselizations at the closed points $\mathfrak{r}_m'$ resp. $\mathfrak{r}_m'$. Let $\overline{\mathfrak{M}}_m'$ be the completion of $\mathfrak{M}_m' \times_{B_{m'}} \mathfrak{F}_m$ along the subscheme where $\varpi$ is zero, let $(\mathfrak{M}_m')^{\text{ad}}$ be the analytic adic space associated to $\overline{\mathfrak{M}}_m'$, and denote by

$$sp_{\overline{\mathfrak{M}}_m'} : (\mathfrak{M}_m')^{\text{ad}} \to \overline{\mathfrak{M}}_m'$$

the specialization map. Let $\overline{\mathfrak{M}}_m^{(\varpi)}$ the preimage of $\mathfrak{r}_m'$ under the specialization map. Then we have again another specialization map

$$sp' : \overline{\mathfrak{M}}_m^{(\varpi)} \to \overline{\mathfrak{M}}_m'$$

where $\overline{\mathfrak{M}}_m' = \overline{\mathfrak{M}}_m^{(0)}_{K_m'}$, in the notation of 3.1.1. For $B \in \mathfrak{S}_m'$ define $\partial_B \mathfrak{M}_m' \subset (\mathfrak{M}_m')^{\text{ad}}$ as $\left(sp'\right)^{-1}(\partial_B \mathfrak{M}_m')$. The morphism $pr' : \mathfrak{M}_m' \to \mathfrak{M}_m'$ induces a morphism

$$pr' : \overline{\mathfrak{M}}_m^{(\varpi)} \to \overline{\mathfrak{M}}_m'$$

which maps $\partial_B \mathfrak{M}_m'$ to $\partial_B \mathfrak{M}_m'$. Our aim is to show that

$$(\star) \quad pr' \text{ induces an isomorphism } H^i_c(\partial_B \mathfrak{M}_m', \mathbb{Z}/\ell^r \mathbb{Z}) \to H^i_c(\partial_B \mathfrak{M}_m', \mathbb{Z}/\ell^r \mathbb{Z}).$$

Define $\partial_h \mathfrak{M}_m'$ and $\partial_{\leq h} \mathfrak{M}_m'$ as in sec. 4.2.4. Because the cohomology with compact support of $\partial_h \mathfrak{M}_m'$ is the direct sum of the cohomology groups with compact support of the spaces $\partial_B \mathfrak{M}_m'$ for $B \in \mathfrak{S}_m', h$ it suffices to show that the cohomologies of the $\partial_h \mathfrak{M}_m'$ and $\partial_{\leq h} \mathfrak{M}_m'$ are isomorphic. Let $\mathcal{F}$ be the constant sheaf associated to $\mathbb{Z}/\ell^r \mathbb{Z}$ on any of our spaces. Consider the diagram with exact rows
Then we have an ascending sequence of ideals in $R$.

Put $m$ where the maximal ideal $P$ corresponds to the ring homomorphism, where $Q$ is pro-special in the sense of [Hu3], Def. 3.1.6. Then, by the remark before Lemma 3.1.7 in [Hu3], the preimage $Z_{h',\nu}$ of $Z_{h',\nu}$ under the map

$$pr' : \mathcal{O}(\overline{m'} \cap \overline{m}) \to \overline{m'}$$

is pro-special too, and the union of all $Z_{h',\nu}$ covers $\mathcal{O}(\overline{m'} \cap \overline{m})$. If $pr'$ induces an isomorphism between the cohomology groups of the pseudo-adic spaces $Z_{h',\nu}$ and $Z_{h',\nu}'$, then it maps the cohomology of $\mathcal{O}(\overline{m'} \cap \overline{m})$ isomorphically to the cohomology of $\mathcal{O}(\overline{m'} \cap \overline{m})$, because the cohomology of these spaces can be computed by the spectral sequence associated to the coverings $(Z_{h',\nu})_{h',\nu}$ and $(Z_{h',\nu})_{h',\nu}'$, cf. [Hu3], Cor. 2.6.10.

Now we will define the subsets $Z_{h',\nu}$. To this end, consider the universal level-$m'$-structure

$$\phi_{m'} : \overline{m'} \cap \overline{m} \to \mathfrak{X}_{m'},$$

and let $\mathfrak{X} \subset \mathcal{O}(\mathfrak{X}_{m'})$ be the ideal defined by the zero section. For $a \in \overline{m'} \cap \overline{m}$ let $\phi_{m'}(a) : \mathfrak{X}_{m'} \to \mathfrak{X}_{m'}$ be the corresponding section and $\phi_{m'}(a)^\sharp : \mathcal{O}(\mathfrak{X}_{m'}) \to \mathfrak{X}_{m'}$ the corresponding ring homomorphism, where $\mathfrak{M}_{m'}$ is $\text{Spec}(\mathfrak{X}_{m'})$. For $B \in \mathcal{S}_{m'}$ let $\mathfrak{P}_B$ be the ideal of $\mathfrak{X}_{m'}$ generated by $\phi_{m'}(a)^\sharp(\mathfrak{X})$ for all $a \in B$. Over the completion $R_{m'}$ of $\mathfrak{X}_{m'}$ at $\mathfrak{X}_{m'}$ the level structure $\phi_{m'}$ gives the universal level structure on $X_{univ}[\overline{m'}]$. Therefore, $\mathfrak{P}_B$ generates the ideal $\mathfrak{P} \subset R_{m'}$, cf. Prop. 3.1.2, and it follows that $\mathfrak{P} \cap \mathfrak{X}_{m'} = \mathfrak{P}_B$. Put

$$\mathfrak{Q}_h = \prod_{B \in \mathcal{S}_{m',h}} \mathfrak{P}_B, \quad \mathfrak{q}_h = \prod_{B \in \mathcal{S}_{m',h}} \mathfrak{P}.$$

Then we have an ascending sequence of ideals in $\mathfrak{X}_{m'}$:

$$\mathfrak{Q}_1 \subset \mathfrak{Q}_2 \subset \ldots \subset \mathfrak{Q}_{n-1} \subset m_{\mathfrak{X}_{m'}},$$

where the maximal ideal $m_{\mathfrak{X}_{m'}}$ corresponds to the closed point $\mathfrak{X}_{m'}$, and the ideal generated by $\mathfrak{Q}_h$ in $R_{m'}$ is equal to $\mathfrak{q}_h$. Choose inductively generators $f_{1,1}, \ldots, f_{1,t_1}, \ldots, f_{h,1}, \ldots, f_{h,t_h}$.
of $\Omega_h$ such that $f_{1,1}, \ldots, f_{1,t_1}, \ldots, f_{h-1,1}, \ldots, f_{h-1,t_{h-1}}$ generate $\Omega_{h-1}$. Define for $\nu = 1, \ldots, t_{h+1}$:

$$Z_{h,\nu} = \overline{M}_{m'} \cap \bigcap_{r > 0, 1 \leq h' \leq h, 1 \leq \mu \leq t_{h'}} \{ v \in (M_{m'})^{ad} \mid |\varpi|_v < |f_{h+1,\nu}|_v^r, |f_{h',\mu}|_v < |f_{h+1,\nu}|_v^r \}.$$ 

The subset $\overline{M}_{m'}$ is pro-special, because it is equal to

$$\{ v \in (M_{m'})^{ad} \mid \text{for all } f \in m_{\overline{R}_{m'}} : |f|_v < 1 \},$$

and $m_{\overline{R}_{m'}}$ contains $\varpi$, hence is open. ($\overline{M}_{m'}$ is even special because $m_{\overline{R}_{m'}}$ is finitely generated.) The set

$$\{ v \in (M_{m'})^{ad} \mid |\varpi|_v < |f_{h,\nu}|_v^r, |f_{h',\mu}|_v < |f_{h,\nu}|_v^r \}$$

is also special (because of the condition $|\varpi|_v < |f_{h,\nu}|_v^r$), and it is closed because its complement is

$$\{ v \in (M_{m'})^{ad} \mid |f_{h,\nu}|_v \leq |\varpi|_v \neq 0 \} \cup \{ v \in (M_{m'})^{ad} \mid |f_{h,\nu}|_v^r \leq |f_{h',\mu}|_v \neq 0 \}$$

which is an open subset by the very definition of the topology, cf. [Hu1], sec. 2. We claim that

$$\partial_{h'} M_{m'} \subset \bigcup_{1 \leq \nu \leq t_{h+1}} Z_{h,\nu} \subset \partial_{\geq h} M_{m'}.$$ 

If $v$ is an element of $Z_{h,\nu}$ then, for any $f \in \Omega_h$, $|f|_v$ is not in $c\Gamma_v(m)$ ($m := m_{\overline{R}_{m'}}$), cf. [Hu1], sec. 2, hence $\Omega_h \subset \text{supp}(sp(v))$, and so $q_h \subset \text{supp}(sp(v))$, which in turn implies that $p_B \subset \text{supp}(sp(v))$ for some $B \in S_{m',h}$. Consequently, $v$ lies in $\partial_{\geq h} M_{m'}$. If $v$ is an element of $\partial_{h'} M_{m'}$, then $\text{supp}(sp(v))$ contains $q_h$ but does not contain $p_B$ for any $B \in S_{m',h+1}$, so $\text{supp}(sp(v))$ does not contain $q_{h+1}$, and there is hence some $\nu$ with $|f_{h+1,\nu}|_v \neq 0$. Then $|f_{h+1,\nu}|_v$ is in $c\Gamma_v(m)$, hence $|f_{h+1,\nu}|_v^r > |f|_v$ for all $f \in \Omega_h$. So $v$ is in $Z_{h,\nu}$. Therefore

$$\bigcup_{h' \geq h, 1 \leq \nu \leq t_{h'+1}} Z_{h',\nu} = \partial_{\geq h} M_{m'}.$$ 

The morphism of schemes $\mathcal{M}_{m'}' \to \mathcal{M}_{m'}$ corresponds to a ring homomorphism $\mathcal{R}_{m'}' \to \mathcal{R}_{m'}$, and the preimage $Z_{h,\nu}$ of $Z_{h,\nu}$ is defined by the images of the functions $f_{h,\nu}$ in $\mathcal{R}_{m'}'$. By Thm. 3.2.1 in [Hu3], the cohomology of $Z_{h,\nu}$ is the same as the cohomology of the scheme Spec$(A(Z_{h,\nu}))$, where $A(Z_{h,\nu})$ is the henselization of the affinoid ring

$$(\mathcal{R}_{m'}' \otimes_{\mathcal{R}_{m'}} \overline{F}^\wedge, \mathcal{R}_{m'}' \otimes_{\mathcal{R}_{m'}} \overline{\mathcal{O}_F})$$.
along the pro-special subset \(Z_{h,\nu}\), cf. [Hu3], 3.1.12. From the very definition of the ring \(A(Z_{h,\nu})\) it is easily seen that it depends only on the henselization of \(\mathfrak{R}_{m'}\) at the maximal ideal \(m_{\mathfrak{R}_{m'}}\). The ring homomorphism \(\mathfrak{R}_{m'} \to \mathfrak{R}_{m'}'\) induces an isomorphism on the henselizations, and therefore it induces an isomorphism \(A(Z_{h,\nu}) \to A(Z'_{h,\nu})\). So Thm. 3.2.1 in [Hu3] shows that the map between the cohomology groups of \(Z'_{h,\nu}\) and \(Z_{h,\nu}\) is an isomorphism. This completes the proof (\(*\)).

Let \(P_{m,h}\) be the stabilizer of \(A_{m,h}\) in \(K_0/K_m\). Because the set \(S_{K_{m,h}}\) can be identified with \(K_0/P_{m,h}\), the group \(P_{m,h}\) is the subgroup of \(K_0/K_m\) which stabilizes \(\partial_{A_{m,h}} M_m\). And because \(K_m'/K_m\) acts transitively on the geometric fibres of the canonical map

\[
\mathcal{M}_m^{ad} \to (\mathcal{M}_{m'})^{ad}
\]

the subgroup \((K_m'/K_m) \cap P_{m,h}\) acts transitively on the geometric fibres of

\[
pr_{m,m'} : \partial_{A_{m,h}} M_m \to \partial_{A_{m',h}}' M'_m,
\]

This is a quasi-finite morphism, and the functor \((pr_{m,m'})^*\) is therefore exact ([Hu3], Prop. 2.6.4). \(K_m'/K_m\) is a finite \(p\)-group and has therefore no cohomology on \(\ell\)-torsion groups. Hence we can conclude

\[
H_c^i(\partial_{A_{m,h}} M_m, \mathcal{F})(K_{m'/K_m}) \cap P_{m,h} = H_c^i\left(\partial_{A_{m',h}}' M'_m, \left((pr_{m,m'})^*(\mathcal{F})\right)(K_{m'/K_m}) \cap P_{m,h}\right)
= H_c^i(\partial_{A_{m',h}}' M'_m, \mathcal{F}) = H_c^i(\partial_{A_{m',h}}' M'_m, \mathcal{F})
\]

Where the last equality holds by (\(*\)). It follows from this that

\[
H_c^i(\partial_{A_{m,h}} M_m, \mathbb{Q}_\ell)(K_{m'/K_m}) \cap P_{m,h} = H_c^i(\partial_{A_{m',h}}' M'_m, \mathbb{Q}_\ell),
\]

and when we assume (H), then these spaces are finite-dimensional. This shows that \(W_{(h,i)}^\vee\) is an admissible representation of \(P_h\). That the unipotent radical acts trivially is a general fact that was proved and used in this context by P. Boyer, [Bo], Lemme 13.2.3.

(ii) \(\partial_h M_m(0)\) is the disjoint union of the spaces \(\partial_{A_m} M_{m}(0)\), for \(A \in S_{m,h}\). The action of \(K_0/K_m\) on the set of labels \(S_{m,h}\) is transitive and the stabilizer of \(A_{m,h}\) is by definition \(P_{m,h}\). Hence we see that

\[
H_c^i(\partial_{A_{m,h}} M_m(0), \mathbb{Q}_\ell) Ind_{P_{m,h}}^{K_0/K_m} = H_c^i(\partial_{A_{m',h}}' M_m(0), \mathbb{Q}_\ell).
\]

When passing to the limit for \(m \to \infty\) we get

\[
V^{(h,i,0)} = Ind_{P(0)}^{G(0)} W^{(h,i,0)}
\]
and then
\[ V^{(h,i)} = \text{Ind}_G^{G(0)} \text{Ind}_P^{G(0)} W^{(h,i,0)} = \text{Ind}_P^{G(0)} W^{(h,i,0)} = \text{Ind}_P^{G(0)} W^{(h,i)}. \]

(iii) From the long exact sequences
\[ \ldots \to H^i_c(\partial_{\geq h} M^{(0)}, \overline{\mathbb{Q}_\ell}) \to H^i(\partial_{\geq h} M^{(0)}, \overline{\mathbb{Q}_\ell}) \to H^i(\partial_{\geq h+1} M^{(0)}, \overline{\mathbb{Q}_\ell}) \to \ldots \]
we deduce an exact sequence
\[ \ldots \to V^{(h,i)} \to V^{(\geq h,i)} \to V^{(\geq h+1,i)} \to \ldots , \]
where we have put
\[ V^{(\geq h,i)} = \text{Ind}_G^{G(0)} \left( \lim_{m} H^i(\partial_{\geq h} M^{(0)}, \overline{\mathbb{Q}_\ell}) \right) . \]
By definition we have
\[ \partial_{\geq n-1} M^{(0)} = \partial_{n-1} M^{(0)} \]
and hence \( V^{(\geq n-1,i)} = V^{(n-1,i)} \). By descending induction on \( h \), starting with \( h = n - 1 \) and using (i) and (ii), we conclude that \( V^{(\geq h,i)} \) is a successive extension of parabolically induced representations (with the unipotent radical acting trivially), hence does not have a supercuspidal representation as a subquotient.

(iv) By Prop. 4.2.3 and Prop. 4.2.5 (ii), there is a long exact sequence
\[ \ldots \to H^i_c(M^{(0)}_m \times \hat{\mathbb{F}}_n, \overline{\mathbb{F}}^\wedge, \overline{\mathbb{Q}_\ell}) \to H^i((M^{(0)}_m \times \hat{\mathbb{F}}_n, \overline{\mathbb{F}}^\wedge, \overline{\mathbb{Q}_\ell}) \to H^i(\partial_{\geq 1} M^{(0)}_m, \overline{\mathbb{Q}_\ell}) \to \ldots . \]
Put
\[ H^i(M_{\infty}/\mathcal{W}^Z) = \lim_{m} H^i((M_{m}/\mathcal{W}^Z) \times \hat{\mathbb{F}}_n, \overline{\mathbb{F}}^\wedge, \overline{\mathbb{Q}_\ell}). \]
Then we have an exact sequence
\[ \ldots \to H^i_c(M_{\infty}/\mathcal{W}^Z) \to H^i(M_{\infty}/\mathcal{W}^Z) \to V^{(\geq 1,i)} \to \ldots . \]
If \( i < n - 1 \) the cohomology with compact support \( H^i_c(M_{\infty}/\mathcal{W}^Z) \) vanishes, and if \( i > n - 1 \) the cohomology \( H^i_c(M_{\infty}/\mathcal{W}^Z) \) vanishes, cf. Lemma 2.5.1. Hence, if \( i > n - 1 \), there is in the preceding sequence a surjection
\[ V^{(\geq 1,i-1)} \to H^i_c(M_{\infty}/\mathcal{W}^Z) . \]
By part (iii) we can conclude that $H^i_t(M_\infty/\omega^2)$ does not have a supercuspidal representation as a subquotient if $i \neq n - 1$.

**4.3.3.** We want to conclude with some remarks concerning the spaces $\overline{M}_{K}^{(\omega,j)}$. In the definition of the space $\overline{M}_{K}^{(\omega,j)}$ we made use of the scheme of finite type $\mathfrak{M}_{K}^{(j)}$. However it is possible to consider also closely related spaces which are defined only in terms of the formal schemes $\mathcal{M}_{K}^{(j)} = \text{Spf}(R_{K}^{(j)})$. For the rest of this paragraph we will suppress the superscript '(j)'. We equip $R_{K}$ the $\omega$-adic topology, and denote this topological ring by $R_{K}^{\omega}$. To the formal scheme $\text{Spf}(R_{K}^{\omega})$ there is an associated analytic adic space $\overline{M}_{K}^{\omega} = t(\text{Spf}(R_{K}^{\omega}))$ and a specialization map

$$sp : M_{K}^{\omega} \longrightarrow \text{Spf}(R_{K}^{\omega}).$$

Let $\overline{M}_{K}^{\omega}$ be the preimage of the closed point of $\text{Spf}(R_{K}^{\omega})$ under the specialization map. It is in this way that the author had considered the $\omega$-adic 'compactifications' for the first time. By taking preimages one gets subspaces $\partial_{\omega}^A M_{K} \subset \overline{M}_{K}$, and the action of $GL_n(F)$ naturally extends to the spaces $\overline{M}_{K}^{\omega}$ and respects the subspaces $\partial_{\omega}^A M_{K} \subset \overline{M}_{K}$. Note that we did not use algebraizations to define these spaces, and we think that it would be desirable to work only with these spaces, and do without algebraizations. However, there are some foundational problems one is facing when working with them. First, one has to show that the fibre product $\overline{M}_{K}^{\omega} \times_{F_{nr}} \overline{F}^{\wedge}$ does exist as a pseudo-adic space. (This is a case that has not been dealt with so far, and is not covered by [Hu2] or [Hu3].) Let us suppose that it is well defined. Then one has to show that Prop. 4.2.3 is true for $\overline{M}_{K}^{\omega} \times_{F_{nr}} \overline{F}^{\wedge}$. We think that this is true but could not prove it yet. If one can show that Prop. 4.2.3 continues to hold for $\overline{M}_{K}^{\omega} \times_{F_{nr}} \overline{F}^{\wedge}$, then one could work entirely with the complete local rings $R_{K}$ and it would not be necessary to work with algebraizations in section 4.2. In the same vein, we think that it would be nice and also possible to prove a Lefschetz type trace formula for the space $\overline{M}_{K}$ studied in sec. 3, under the assumption that there are no fixed points on the boundary. Then one could dispense with algebraizations completely.

**5. Appendix**

**5.1. Algebraicity of the deformation rings in equal characteristic.**

If $F$ has positive characteristic, the fact that the deformation rings $R_m$ are completions of finitely generated $\hat{o}^{nr}$-algebras follows from the following

**Proposition 5.1.1.** If $\text{char}(F) > 0$ there is a regular system $u_0 = \omega, u_1, ..., u_{n-1}$ of parameters of $R_0$, a formal $\hat{o}$-module $\hat{X}$ over $\mathfrak{R}_0 = \hat{o}^{nr}[u_1, ..., u_{n-1}] \subset R_0$ and an isomorphism of formal $\hat{o}$-modules over $\mathbb{F}$
Deformation spaces of one-dimensional formal modules and their cohomology

\[ \tilde{i} : X \to \tilde{X} \times_{R_0} F \]

such that

(i) \((\tilde{X} \times_{R_0} R_0, \tilde{i})\) is a universal deformation of \(F\) over \(R_0\),
(ii) there is a coordinate \(T\) on \(\tilde{X}\) such that the multiplication by \(\varpi\) on \(\tilde{X}\) is given by the polynomial

\[ \varpi T + u_1 T^q + \ldots + u_{n-1} T^{q^{n-1}} + T^{q^n}. \]

Proof. Let \(F(u) = F(u_1, \ldots, u_{n-1})\) be the so-called universal \(\mathfrak{o}\)-typical formal \(\mathfrak{o}\)-module described in [HG], sec. 12. It is defined over \(\mathfrak{o}[u_1, \ldots, u_{n-1}]\), but we consider it over \(\hat{\mathfrak{o}}[u_1, \ldots, u_{n-1}]\), and we identify the universal deformation ring \(R_0\) with \(\hat{\mathfrak{o}}[u_1, \ldots, u_{n-1}]\).

We take the universal deformation \(X_{\text{univ}}\) to be \(F(u)\). \(F(u)\) has the property that

\[ F(u)(T_1, T_2) = T_1 + T_2 \]

and for \(j = 1, \ldots, n:\)

\[ [\varpi]_{F(u)}(T) \equiv u_j T^{q^j} \mod (\varpi, u_1, \ldots, u_{j-1}), \deg(q^j + 1), \]

where \(u_n = 1\). The reduction of \(F(u)\) modulo the maximal ideal of \(R_0\) is the formal \(\mathfrak{o}\)-module \(X\) with:

\[ X(T_1, T_2) = T_1 + T_2, \ [\varpi]_X(T) = T^{q^n}, \]

cf. [HG], (12.5). However, it is not the case that \( [\varpi]_{F(u)}(T) \) is a polynomial in \(T\). We define a deformation \(\tilde{X}\) of \(X\) over \(R_0 = \hat{\mathfrak{o}}[u_1, \ldots, u_{n-1}]\) by

\[ \tilde{X}(T_1, T_2) = T_1 + T_2, \ [\varpi]_{\tilde{X}}(T) = \varpi T + u_1 T^q + \ldots + u_{n-1} T^{q^{n-1}} + T^{q^n}. \]

The reduction of \(\tilde{X}\) modulo the ideal \((\varpi, u_1, \ldots, u_{n-1})\) is \(X\), and we take \(\tilde{i}\) to be the identity map. By the universal property of \(F(u)\), there are elements \(v_1, \ldots, v_{n-1}\) in the maximal ideal \(m_{R_0} = (\varpi, u_1, \ldots, u_{n-1})\) of \(R_0 = \hat{\mathfrak{o}}[u_1, \ldots, u_{n-1}]\), such that there exists an isomorphism

\[ \psi : F(v_1, \ldots, v_{n-1}) \xrightarrow{\sim} \tilde{X} \times_{R_0} R_0 \]

of formal \(\mathfrak{o}\)-modules over \(R_0\), and the reduction of \(\psi\) modulo \((\varpi, u_1, \ldots, u_{n-1})\) is the identity. We will show that the elements \(\varpi, v_1, \ldots, v_{n-1}\) form a regular system of parameters. This in turn implies that \(\tilde{X} \times_{R_0} R_0\) is a universal deformation too.

Write
\[
\psi(T) = a_1 T + a_2 T^2 + a_3 T^3 + \ldots
\]
with \(a_1 \equiv 1 \mod m_{R_0}\) and \(a_i \equiv 0 \mod m_{R_0}\) if \(i > 1\). Consider the identity of power series in \(T\):

\[
\psi([\varpi]_F(T)) = [\varpi]_X(\psi(T)).
\]

(5.1.2)

Computing modulo \(\varpi\) and degree \(q + 1\) we find that

\[
a_1 v_1 \equiv u_1 a_1^q \mod (\varpi),
\]

hence \(v_1 = a_1^{q-1} u_1 + \varpi \xi_{1,0}\) with \(\xi_{1,0} \in R_0\). Fix \(1 < i < n\) and assume we had already proven that for \(j < i\) there exist \(\xi_{j,0}, \ldots, \xi_{j,j-1} \in R_0\) such that

\[
v_j = a_1^{q-1} u_j + \varpi \xi_{j,0} + \ldots + u_{j-1} \xi_{j,j-1}.
\]

Then we consider equation 5.1.2 modulo \((\varpi, u_1, \ldots, u_{i-1})\) and degree \(q^j + 1\) and find:

\[
a_1 v_j \equiv u_j a_1^q \mod (\varpi, u_1, \ldots, u_{i-1}).
\]

This shows that the map defined by \(u_i \mapsto v_i\) induces on the tangent space \(m_{R_0}/(m_{R_0})^2\) a map which is given by an upper triangular matrix with respect to the basis \((\varpi, u_1, \ldots, u_{n-1})\) with units on the diagonal. Hence \((\varpi, v_1, \ldots, v_{n-1})\) is a regular system of parameters too. □

From the description of the rings \(R_m\) as given in the proof of [Dr1], Prop. 4.3, which we recalled in the proof of 2.1.2, and the fact that \(X^{univ}\) may be chosen to be defined over \(\hat{o}^{nr}[u_1, \ldots, u_{n-1}]\) and such that \([\varpi]_X^{univ}(T)\) is a polynomial it follows immediately that for any there is a regular system of parameters

\[
(\varpi, u_1, \ldots, u_{n-1})
\]

of \(R_0\) with the following properties:

(i) If we put \(\mathcal{R}_0 = \hat{o}^{nr}[u_1, \ldots, u_{n-1}]\), and consider it as a subring of \(R_0\), the image of universal level-\(m\)-structure

\[
\hat{\phi}_m^{univ} : (\varpi^{-m} o/\varpi o)^n \longrightarrow m_{R_0}
\]

consists of elements which are integral over \(\mathcal{R}_0\). For \(m \geq 0\) we let \(\mathcal{R}_m \subset R_m\) be the subring \(\mathcal{R}_m \subset R_m\) which is generated by the image of \(\hat{\phi}_m^{univ}\) over \(\mathcal{R}_0\). It is a free \(\mathcal{R}_0\)-module of rank equal to \(#GL_n(o/\varpi m)\). The ideal generated by the image of \(\hat{\phi}_m^{univ}\) is the unique maximal ideal \(m_m \subset \mathcal{R}_m\) over \((\varpi, u_1, \ldots, u_{n-1})\). The \(m_m\)-adic completion of \(\mathcal{R}_m\) is isomorphic to \(R_m\).
(ii) The maximal ideal $\mathfrak{m}_m$ is stable under the action of $K_0$, and the isomorphism

$$\mathcal{R}_m \xrightarrow{\sim} R_m$$

is $K_0$-equivariant. The completion on the left means $\mathfrak{m}_m$-adic completion.

The reason why there is only one maximal ideal of $\mathcal{R}_m$ over $(\varpi, u_1, \ldots, u_{n-1})$ is the following: any $\varpi^m$-torsion point $\xi$ of $X^{univ}$ satisfies an integral equation of the form

$$\xi^r + a_1\xi^{r-1} + \ldots + a_r = 0$$

with all coefficients $a_1, \ldots, a_r$ in $(\varpi, u_1, \ldots, u_{n-1})$. Hence, any prime ideal over $(\varpi, u_1, \ldots, u_{n-1})$ contains the ideal generated by all $\varpi^m$-torsion points, which is clearly a maximal ideal.

5.2. **A continuity property of isolated fixed points.**

5.2.1. In this section $E$ denotes an algebraically closed non-Archimedean field with a non-trivial valuation, and $\varpi$ is a non-zero element of the maximal ideal of $\mathfrak{o}_E$. Let $A$ be an affinoid $E$-algebra, and put $X = \text{Spa}(A, A^\circ)$. Suppose $X$ is smooth over $\text{Spa}(E, \mathfrak{o}_E)$. Let $\varphi : X \to X$ be an endomorphism of $X$ over $E$. We will assume that $\varphi$ has only a finite number of fixed points and that these are all of multiplicity one. Our aim in this section is to show that any endomorphism $\psi$ of $X$ which is sufficiently close to $\varphi$, in the sense of [Be4], sec. 6, has also only finitely many fixed points, that these are all of multiplicity one and their total number is equal to the number of fixed points of $\varphi$.

To make the relation of being close concrete, we choose an affinoid generating system $f_1, \ldots, f_r$ of $A$ over $E$. Then, by Cor. 6.3 of [Be4], for any $\varepsilon \in \mathfrak{C}(X)$ (with the notation of [Be4], sec. 6), there is a $t \in \mathbb{Z}_{>0}$ such that, if

$$d(\varphi, \psi) < \varepsilon \quad \text{where} \quad \varphi, \psi : A \to A$$

then $d(\varphi, \psi) < \varepsilon$. Here $\varphi, \psi : A \to A$ denote the corresponding ring homomorphisms, and $A^\circ$ is the subring of power bounded elements. By definition, we write $d(\varphi, \psi) < \varepsilon_t$ if the above relation holds for the fixed set of affinoid generators and the element $\varpi$.

**Lemma 5.2.2.** Let $x \in X$ be a fixed point of $\varphi$. Then there are open affinoid neighborhoods $U' = \text{Spa}(B', (B')^\circ) \subset U = \text{Spa}(B, B^\circ)$ of $x$, with the following properties:

(i) $\varphi(U') \subset U$, and $x$ is the only fixed point of $\varphi$ in $U'$.

(ii) $U$ and $U'$ are isomorphic to polydiscs:
\[ B' \simeq E\langle T_1, \ldots, T_n \rangle, \]

and \( B \) corresponds under the ring homomorphism \( B \to B' \) to

\[ \{ f = \sum \alpha_\nu T^\nu \in E\langle T_1, \ldots, T_n \rangle \mid \lim_{|\nu| \to \infty} |\alpha_\nu| |\nu|^{-\mu} = 0 \} \]

for some \( \mu \in \mathbb{Z}_{>0} \).

(iii) There is a \( t > 0 \) such that any morphism \( \psi : X \to X \) with \( d(\varphi, \psi) < \varepsilon_t \) maps \( U' \) to \( U \) and has a single fixed point of multiplicity one on \( U' \).

**Proof.** Because \( X \) is smooth, there is for any point \( x \in X \) an étale map from an open neighborhood of \( x \) to an affine space, [Hu3], Cor. 1.6.10. By [Be2], Thm. 3.4.1, if \( x \) is \( E \)-rational, there is even an isomorphism of a neighborhood of \( x \) with a neighborhood of \( 0 \) in an affine space. So we may assume \( U \) is isomorphic to a polydisc. Then there is another polydisc \( U' \subset (U \cap \varphi^{-1}(U)) \). This shows (i) and (ii).

Put \( a_i = \varphi^*(T_i) - T_i \in B' \), \( i = 1, \ldots, n \). Let \( I_\varphi \subset B' \) be the ideal generated by \( a_1, \ldots, a_n \). Let \( \text{diag} : U \to U \times_E U \) be the diagonal morphism and \( (1, \varphi) : U' \to U \times_E U \) be the graph of \( \varphi \) on \( U' \). Then

\[ U \times_{\text{diag}, U \times E U, (1, \varphi)} U' \simeq \text{Spa}(B'/I_\varphi, (B'/I_\varphi)^\circ) \]

is the fixed point locus of \( \varphi \) on \( U' \), hence equal to the single point \( x \), which is of multiplicity one, and so we have \( B'/I_\varphi \simeq E \). Denote by

\[ \delta_\varphi = \det \left( \frac{\partial a_i}{\partial T_j} \right)_{1 \leq i, j \leq n} \in B' \]

the Jacobi determinant. The image of \( \delta_\psi \) in \( B'/I_\varphi \) is invertible, because of our assumption that the fixed point \( x \) be of multiplicity one. By [Be4], Cor. 6.3, there is a \( t \) such that \( \psi^{-1}(U) = \varphi^{-1}(U) \) if \( d(\varphi, \psi) < \varepsilon_t \). Hence \( \psi^t \) maps \( B \) to \( B' \), and we can consider the elements \( b_i = \psi^t(T_i) - T_i \in B' \). Define \( I_\psi \) to be the ideal generated by \( b_1, \ldots, b_n \). By the generalized Krasner lemma, cf. Cor. 1.7.2 in [Hu3] or [Be3], Thm. 5.1, it is known that if the element \( b_i \) is sufficiently close to the element \( a_i \) for all \( i = 1, \ldots, n \), then the image of \( \delta_\psi \) (defined as above but with \( a_i \) being replaced by \( b_i \)) in \( B'/I_\psi \) is invertible and there is a (continuous) isomorphism of \( E \)-algebras \( B'/I_\psi \simeq B'/I_\varphi \). By increasing \( t \) we may assume that this is the case. Then \( \psi \) too has exactly one fixed point on \( U' \), which is of multiplicity one. \( \square \)
Proposition 5.2.3. Let $X$ be a smooth affinoid space, and $\varphi : X \to X$ be as above, i.e. with finitely many simple fixed points. Then there is a $t > 0$ such that any morphism $\psi : X \to X$ with $d(\varphi, \psi) < \varepsilon_t$ has finitely many fixed points on $X$, each fixed point is of multiplicity one, and their number is equal to the number of fixed points of $\varphi$.

Proof. Let $f_1, \ldots, f_r$ be the affinoid generating system of $A$ over $E$ from 5.2.1. Then the fixed points of $\varphi$ are exactly those $x \in X$ with $|\varphi^\sharp(f_i) - f_i|_x = 0$, $i = 1, \ldots, r$. Let $x_1, \ldots, x_s$ be the fixed points of $\varphi$. Choose for any $j$ an open neighborhood $U'_j$ of $x_j$ with the properties of the lemma above, and such that $U'_j \cap U'_{j'} = \emptyset$ for $j \neq j'$. Then there is a $k \in \mathbb{Z}_{>0}$ such that

$$\max_{1 \leq i \leq r} \{ |\varphi^\sharp(f_i) - f_i|_x \} > |\omega|_x^k$$

for all $x \not\in \bigcup_{1 \leq j \leq s} U'_j$. Suppose $d(\varphi, \psi) < \varepsilon_{2k}$. Then it follows that for all $x \not\in \bigcup_{1 \leq j \leq s} U'_j$ one has also

$$\max_{1 \leq i \leq r} \{ |\psi^\sharp(f_i) - f_i|_x \} > |\omega|_x^k.$$

Hence $\psi$ does not have any fixed point on the complement of the $U'_j$. If now $t \geq 2k$ is sufficiently large, such that part (iii) of the lemma holds for all fixed points of $\varphi$, then $\psi$ has on each $U'_j$ a single fixed point of multiplicity one. This proves the assertion. \qed

5.3. A picture of the two boundaries.

The following picture is about the boundaries of the space $M^{(0)}_{K_1}$ of deformations of a formal $\mathfrak{a}$-module of height $n = 3$ with level-1-structure, and the residue field of $\mathfrak{a}$ is $\mathbb{F}_2$. We denote this space here simply by $M_1$. The characteristic-$\omega$-boundary $\partial M_1$ has fourteen strata; seven one-dimensional strata and seven one-point-strata. $\partial M_1$ is drawn in the center of the picture. Let $\varphi_i = \varphi^{\text{univ}}(e_i)$, $i = 1, 2, 3$, be the images of the standard basis vectors of $\omega^{-1}\mathfrak{a}^{3}/\mathfrak{a}^{3}$ under the universal level structure. The seven one-dimensional strata are

$$\{ |\varphi_i|_v = 0 \}, i = 1, 2, 3;$$

$$\{ |\varphi_i + \varphi_j|_v = 0 \}, 1 \leq i < j \leq 3;$$

$$\{ |\varphi_1 + \varphi_2 + \varphi_3|_v = 0 \}.$$

The last stratum is drawn with a triangle shape. The strata of the $\omega$-adic boundary $\partial^\omega M_1$ are by definition the preimages under the specialization map

$$sp : \partial^\omega M_1 \to \partial M_1.$$. 
If $A \subset \varpi^{-1}\mathfrak{o}^3/\mathfrak{o}^3$ is of rank two over $\mathfrak{o}/(\varpi)$, then $\partial A M_1$ is just a point and its preimage $\partial^{\varpi}_A M_1$ possesses itself a stratification. Namely, the closure of each stratum $\partial^{\varpi}_B M_1$, with $B \subset A$ a rank-1-submodule, intersects $\partial^{\varpi}_A M_1$ in exactly one point, which we have drawn as a bold black dot. So, in fact, $\partial^{\varpi}_B M_1$ has a natural stratification indexed by the set of all flags in $\varpi^{-1}\mathfrak{o}^3/\mathfrak{o}^3$. This is of course not something that is special to $M_1$. For any $n$ and any $m \geq 1$ the space $\partial^{\varpi}_n M_m$ has a natural stratification whose strata are indexed by flags

$$A : \varpi^{-m}\mathfrak{o}^n/\mathfrak{o}^n = A_0 \supseteq A_1 \supseteq A_2 \ldots \supseteq A_r = \{0\},$$

where each $A_i$ is free over $\mathfrak{o}/(\varpi^m)$ and a direct summand of $A_{i-1}$. In order to define these refined strata intrinsically, let

$$\phi^{\text{univ}} : \varpi^{-m}\mathfrak{o}^n/\mathfrak{o}^n \rightarrow \mathfrak{m}_{R_m}$$

be the universal level-$m$-structure. For a point $v \in \partial^{\varpi}_n M_m$, we consider the set of $\varpi^m$-torsion points

$$\{\phi^{\text{univ}}(a)_v \mid a \in \varpi^{-m}\mathfrak{o}^n/\mathfrak{o}^n\} = X^{\text{univ}}[\varpi^m](k(v)),$$

where $\phi^{\text{univ}}(a)_v$ denotes the image of $\phi^{\text{univ}}(a)$ in the residue field $k(v)$ at $v$. We introduce the following notation:

$$|\phi^{\text{univ}}(a)|_v \ll |\phi^{\text{univ}}(a')|_v$$

if and only if for any $r \in \mathbb{Z}_{>0}$ one has $|\phi^{\text{univ}}(a)|_v < |\phi^{\text{univ}}(a')|_v$, and

$$|\phi^{\text{univ}}(a)|_v \sim |\phi^{\text{univ}}(a')|_v$$

if and only if there are $r, r' \in \mathbb{Z}_{>0}$ such that

$$|\phi^{\text{univ}}(a)|_v^{r'} < |\phi^{\text{univ}}(a')|_v$$

and

$$|\phi^{\text{univ}}(a')|_v^{r'} < |\phi^{\text{univ}}(a)|_v.$$

Then, for a given $v \in \partial^{\varpi}_n M_m$, put $A_0(v) = \varpi^{-m}\mathfrak{o}^n/\mathfrak{o}^n$ and define inductively, for $i > 0$, the subgroup $A_i(v) \subset A_{i-1}(v)$ to be the largest subgroup satisfying

$$\text{for all } a \in A_i(v) \text{ and for all } a' \in A_{i-1}(v) - A_i(v) : |\phi^{\text{univ}}(a)|_v \ll |\phi^{\text{univ}}(a')|_v.$$

Then it is not difficult to see that $A_i(v)$ is free over $\mathfrak{o}/(\varpi^m)$ and a direct summand of $A_{i-1}(v)$. (One reduces modulo the ideal generated by $\phi^{\text{univ}}(a)_v$ for $a \in A_i(v)$.) This defines a flag $A(v)$ in $\varpi^{-m}\mathfrak{o}^n/\mathfrak{o}^n$, and $\partial^{\varpi}_n M_m$ consists of all $v$ with $A(v) = A$. Moreover, the flag $A(v)$ defines a sequence of canonical subgroups
Deformation spaces of one-dimensional formal modules and their cohomology

\[ X^{univ}[\omega^m](k(v)) \supseteq X^{univ}[\omega^m]_{k(v),1} \supseteq \ldots \supseteq X^{univ}[\omega^m]_{k(v),r} \]

where

\[ X^{univ}[\omega^m]_{k(v),i} = \{ \phi^{univ}(a)_v \mid a \in A_i(v) \} \subset X^{univ}[\omega^m](k(v)) . \]
Deformation spaces of one-dimensional formal modules and their cohomology

REFERENCES

[A1] M. Artin, Algebraic approximation of structures over complete local rings. Inst. Hautes Études Sci. Publ. Math. No. 36, 1969, 23–58.

[A2] M. Artin, Algebraization of formal moduli: I. Global Analysis (Papers in Honor of K. Kodaira) pp. 21–71, Univ. Tokyo Press, Tokyo, 1969.

[Ba] A. I. Badulescu, Orthogonalité des caractères pour $\text{GL}_n$ sur un corps local de caractéristique non nulle. Manuscripta Math. 101 (2000), no. 1, 49–70.

[Be1] V. G. Berkovich, Spectral theory and analytic geometry over non-Archimedean fields. Mathematical Surveys and Monographs, 33. American Mathematical Society, Providence, RI, 1990.

[Be2] V. G. Berkovich, Étale cohomology for non-Archimedean analytic spaces. Inst. Hautes Études Sci. Publ. Math. No. 78(1993), 5–161 (1994).

[Be3] V. G. Berkovich, Vanishing cycles for formal schemes. II. Invent. Math. 125 (1996), no. 2, 367–390.

[Be4] V. G. Berkovich, Vanishing cycles for formal schemes. Invent. Math. 115 (1994), no. 3, 539–571.

[Ber] P. Berthelot, Cohomologie rigide et cohomologie rigide à support propre. Première partie. Prépublication IRMAR 96-03, 1996.

[BK] C. J. Bushnell, P. C. Kutzko, The admissible dual of $\text{GL}(N)$ via compact open subgroups. Annals of Mathematics Studies, 129. Princeton University Press, Princeton, NJ, 1993.

[Bo] P. Boyer, Mauvaise réduction des variétés de Drinfeld et correspondance de Langlands locale. Invent. Math. 138 (1999), no. 3, 573–629.

[Bom] U. Bombosch, Der Zariski-Riemann-Raum und die étale Kohomologie rigider Räume. Schriftenreihe Math. Inst. Univ. Münster 3. Ser., 20, Univ. Münster, Münster, 1997.

[Bou] N. Bourbaki, Commutative algebra. Chapters 1-7. Elements of Mathematics. Springer-Verlag, Berlin, 1989.

[Bu] C. J. Bushnell, Induced representations of locally profinite groups. J. Algebra 134 (1990), no. 1, 104–114.

[Ca1] H. Carayol, Sur la mauvaise réduction des courbes de Shimura. Compositio Math. 59 (1986), no. 2, 151–230.

[Ca2] H. Carayol, Sur les représentations $\ell$-adiques associées aux formes modulaires de Hilbert. Ann. Sci. École Norm. Sup. (4) 19 (1986), no. 3, 409–468.

[Ca3] H. Carayol, Nonabelian Lubin-Tate theory. Automorphic forms, Shimura varieties, and $L$-functions, Vol. II (Ann Arbor, MI, 1988), 15–39, Perspect. Math., 11, Academic Press, Boston, MA, 1990.

[Cas] W. Casselman, Introduction of the theory of admissible representations of $p$-adic reductive groups, unpublished manuscript, draft May 1, 1995, available at http://www.math.ubc.ca/people/faculty/cass/research.html.

[Ch] C.-L. Chai, The group action on the closed fiber of the Lubin-Tate moduli space. Duke Math. J. 82 (1996), no. 3, 725–754.

[Co] L. Corwin, A construction of the supercuspidal representations of $\text{GL}_n(F)$, $F$ $p$-adic. Trans. Amer. Math. Soc. 337 (1993), no. 1, 1–58.

[De] P. Deligne, Théorèmes de finitude en cohomologie $\ell$-adique. In: Cohomologie étale. Séménaire de Géométrie Algébrique du Bois-Marie SGA 4 1/2. Lecture Notes in Mathematics, Vol. 569. Springer-Verlag, Berlin-New York, 1977.
Matthias Strauch

[DKV] P. Deligne, D. Kazhdan, M.-F. Vignéras, Représentations des algèbres centrales simples $p$-adiques. *Representations of reductive groups over a local field*, 33–117, Travaux en Cours, Hermann, Paris, 1984.

[Dr1] V. G. Drinfeld, Elliptic modules. English translation: Math. USSR-Sb. 23 (1974), no. 4, 561–592.

[Dr2] V. G. Drinfeld, Coverings of $p$-adic symmetric domains, Funct. Anal. Appl., 10 (1976), 107–115.

[En] O. Endler, *Valuation theory*. Universitext. Springer-Verlag, New York-Heidelberg, 1972.

[F1] G. Faltings, The trace formula and Drinfeld’s upper halfplane. Duke Math. J. 76 (1994), no. 2, 467–481.

[F2] G. Faltings, A relation between two moduli spaces studied by V. G. Drinfeld. Algebraic number theory and algebraic geometry, 115–129, Contemp. Math., 300, Amer. Math. Soc., Providence, RI, 2002.

[F3] G. Faltings, Group schemes with strict $O$-action. Mosc. Math. J. 2 (2002), no. 2, 249–279.

[Fa1] L. Fargues, Manuscript on generalized canonical subgroups, 2004.

[Fa2] L. Fargues, Cohomologie des espaces de modules de groupes $p$-divisibles et correspondances de Langlands locales. In: Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales. Astérisque No. 291 (2004), 1–199.

[Fu1] K. Fujiwara, Theory of tubular neighborhood in étale topology. Duke Math. J. 80 (1995), no. 1, 15–57.

[Fu2] K. Fujiwara, Rigid geometry, Lefschetz-Verdier trace formula and Deligne’s conjecture. Invent. Math. 127 (1997), no. 3, 489–533.

[FK] E. Freitag, R. Kiehl, *Étale cohomology and the Weil conjecture*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 13. Springer-Verlag, Berlin, 1988.

[H1] M. Harris, Supercuspidal representations in the cohomology of Drinfeld upper half spaces; elaboration of Carayol’s program. Invent. Math. 129 (1997), no. 1, 75–119.

[H2] M. Harris, The local Langlands conjecture for GL$(n)$ over a $p$-adic field, $n < p$. Invent. Math. 134 (1998), no. 1, 177–210.

[Ha] Th. Hausberger, Uniformisation des variétés de Laumon-Rapoport-Stuhler et conjecture de Drinfeld-Carayol. Ann. Inst. Fourier (Grenoble) 55 (2005), no. 4, 1285–1371.

[He1] G. Henniart, Correspondance de Langlands-Kazhdan explicite dans le cas non ramifié. Math. Nachr. 158 (1992), 7–26.

[He2] G. Henniart, Correspondance de Jacquet-Langlands explicite. I. Le cas modéré de degré premier. Séminaire de Théorie des Nombres, Paris, 1990–91, 85–114, Progr. Math., 108, Birkhäuser Boston, Boston, MA, 1993.

[HG] M. J. Hopkins, B. H. Gross, Equivariant vector bundles on the Lubin-Tate moduli space. Topology and representation theory (Evanston, IL, 1992), 23–88, Contemp. Math., 158, Amer. Math. Soc., Providence, RI, 1994.

[HT] M. Harris, R. Taylor, *The geometry and cohomology of some simple Shimura varieties*. Annals of Mathematics Studies, 151. Princeton University Press, Princeton, NJ, 2001.

[Hu0] R. Huber, *Bewertungspektren und rigide Geometrie*. Regensburger Mathematische Schriften, 23. Universität Regensburg, Fachbereich Mathematik, Regensburg, 1993.

[Hu1] R. Huber, Continuous valuations. Math. Z. 212 (1993), no. 3, 455–477.

[Hu2] R. Huber, A generalization of formal schemes and rigid analytic varieties. Math. Z. 217 (1994), no. 4, 513–551.
Deformation spaces of one-dimensional formal modules and their cohomology

[Hu3] R. Huber, *Étale cohomology of rigid analytic varieties and adic spaces*. Aspects of Mathematics, E30. Friedr. Vieweg & Sohn, Braunschweig, 1996.

[Hu4] R. Huber, A finiteness result for direct image sheaves on the étale site of rigid analytic varieties. *J. Algebraic Geom.* 7 (1998), no. 2, 359–403.

[Hu5] R. Huber, Swan representations associated with rigid analytic curves. *J. Reine Angew. Math.* 537 (2001), 165–234.

[Il] L. Illusie, Déformations de groupes de Barsotti-Tate (d’après A. Grothendieck). In: Séminaire sur les pinceaux arithmétiques: la conjecture de Mordell. Edited by Lucien Szpiro. Astérisque No. 127 (1985), 151–198.

[dJvdP] A. J. de Jong, M. v. d. Put, Étale cohomology of rigid analytic spaces. *Doc. Math.* 1 (1996), No. 01, 1–56.

[Ka] D. Kazhdan, Cuspidal geometry of $p$-adic groups. *J. Analyse Math.* 47 (1986), 1–36.

[Le] B. Lemaire, Intégrabilité locale des caractères-distributions de $GL_N(F)$ où $F$ est un corps local non-archimédien de caractéristique quelconque. *Compositio Math.* 100 (1996), no. 1, 41–75.

[LT] J. Lubin, J. T. Tate, Formal moduli for one-parameter formal Lie groups. *Bull. Soc. Math. France* 94 (1966), 49–59.

[Lu] J. Lubin, Finite subgroups and isogenies of one-parameter formal Lie groups. *Ann. of Math.* (2) 85 1967 296–302.

[Ma] H. Matsumura, *Commutative Algebra*. Second edition. Mathematics Lecture Note Series, 56. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980.

[Me] W. Messing, *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*. Lecture Notes in Mathematics, Vol. 264. Springer-Verlag, Berlin-NewYork, 1972.

[Mi] J. S. Milne, *Étale cohomology*. Princeton Mathematical Series, 33. Princeton University Press, Princeton, N.J., 1980.

[vdPS] M. v. d. Put, P. Schneider, Points and topologies in rigid geometry. *Math. Ann.* 302 (1995), no. 1, 81–103.

[Ro] J. D. Rogawski, Representations of $GL(n)$ and division algebras over a $p$-adic field. *Duke Math. J.* 50 (1983), no. 1, 161–196.

[RZ] M. Rapoport, Th. Zink, Period spaces for $p$-divisible groups. *Annals of Mathematics Studies*, 141. Princeton University Press, Princeton, NJ, 1996.

[Sa] P. J. Sally, Some remarks on discrete series characters for reductive $p$-adic groups. *Representations of Lie groups, Kyoto, Hiroshima, 1986*, 337–348, Adv. Stud. Pure Math., 14, Academic Press, Boston, MA, 1988.

[SGA5] *Cohomologie ét-adique et fonctions L*. Séminaire de Géométrie Algébrique du Bois-Marie 1965–1966 (SGA 5). Édité par Luc Illusie. Lecture Notes in Mathematics, Vol. 589. Springer-Verlag, Berlin-NewYork, 1977.

[St] M. Strauch, On the Jacquet-Langlands correspondence in the cohomology of the Lubin-Tate deformation tower. *Automorphic forms. I.* Astérisque No. 298, (2005), 391–410.

[W1] S. Wewers, Swan conductors on the boundary of Lubin-Tate spaces, arXiv: math.NT/0511434.

[W2] S. Wewers, Non-abelian Lubin-Tate theory via stable reduction. Preprint, January 2006.

[Yo] T. Yoshida, On non-abelian Lubin-Tate theory via vanishing cycles. 2004, to appear in Annales de l’Institut Fourier.

[Yu] J.-K. Yu, On the moduli of quasi-canonical liftings. *Compositio Math.* 96 (1995), no. 3, 293–321.