The Modular Group, Operator Ordering, and Time in (2+1)-dimensional Gravity

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Abstract

A choice of time-slicing in classical general relativity permits the construction of time-dependent wave functions in the “frozen time” Chern-Simons formulation of (2 + 1)-dimensional quantum gravity. Because of operator ordering ambiguities, however, these wave functions are not unique. It is shown that when space has the topology of a torus, suitable operator orderings give rise to wave functions that transform under the modular group as automorphic functions of arbitrary weights, with dynamics determined by the corresponding Maass Laplacians on moduli space.

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Despite decades of research, physicists have not yet managed to construct a workable quantum theory of gravity. The considerable effort that has gone into this venture has not been wasted, however: we have gained a much better insight into the questions that must be addressed, and we now know many of the ingredients that are likely to be important in the final theory. The purpose of this Letter is to bring together three such fragments — the “problem of time,” operator orderings, and the mapping class group — in the simplified context of (2 + 1)-dimensional gravity.

The “problem of time” in quantum gravity appears in many guises, but it takes its sharpest form in “frozen time” formulations such as Chern-Simons quantization in 2 + 1 dimensions. Translations in (coordinate) time are diffeomorphisms, which are exact symmetries of the action of general relativity. Operators that commute with the constraints — for instance, the holonomies of the Chern-Simons formulation — are consequently time-independent. In this context, the basic problem can be posed quite simply: how does one describe dynamics when all observables are constants of motion?

Attempts to address this problem, even in very simple models, have been plagued by ambiguities in operator ordering [1, 2]. As Kuchař has stressed [3], such ambiguities can hide a multitude of sins, and no theory should be considered complete unless it offers a clear ordering prescription. So far, the only hint of such a prescription has come from the behavior of the mapping class group, the group of “large” diffeomorphisms. In (2 + 1)-dimensional gravity with a sufficiently simple topology, it is known that the requirement of good behavior under the action of this group places strong restrictions on possible operator orderings and quantizations [4]. In this aspect, (2 + 1)-dimensional quantum gravity resembles two-dimensional rational conformal field theory, where the representation theory of the mapping class group plays an important role in limiting the range of possible models [5].

In this Letter, we shall explore these restrictions in more detail. As we shall see, even the simplest nontrivial topology, [0, 1] × T², is rich enough to illustrate both the importance of the mapping class group in determining operator orderings and its limitations.

1. Chern-Simons and ADM Quantization

A systematic exploration of the relationship between Chern-Simons and Arnowitt-Deser-Misner (ADM) quantization of (2+1)-dimensional gravity was begun in references [4] and [6]. In this section we briefly summarize the results; for more details, the reader is referred to the original papers [7, 8].

In the ADM formulation of canonical (2 + 1)-dimensional gravity [7, 8], one begins by specifying a time-slicing. A convenient choice is York’s “extrinsic time,” in which spacetime is foliated by surfaces of constant mean extrinsic curvature \( \text{Tr} K = T \). For a spacetime with the topology \([0, 1] \times T^2\), a slice of constant \( T \) is a torus with an intrinsic geometry that can be characterized by a complex modulus \( \tau = \tau_1 + i \tau_2 \) and a conformal factor. Moncrief has shown that the conformal factor is uniquely determined by the constraints [7], so the

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*The notation in this Letter has been changed slightly to conform to the mathematical literature. The modulus of a torus \( T^2 \), previously denoted by \( m \), is now \( \tau \), while the trace of the extrinsic curvature, previously \( \tau \), is now \( T \).*
physical phase space is parameterized by $\tau(T)$, its conjugate momentum $\bar{p}(T)$, and their complex conjugates. This behavior is characteristic of (2 + 1)-dimensional gravity — all but a finite number of degrees of freedom are fixed by constraints, and the dynamics takes place on a finite-dimensional reduced phase space of global geometric variables. For the torus, in particular, it is not hard to show that the classical dynamics is determined by the Hamiltonian

$$H = T^{-1} \left( \tau^2 \bar{p} \right)^{1/2}. \quad (1.1)$$

In the Chern-Simons formulation [9], in contrast, a classical solution of the Einstein field equations is characterized by a flat ISO(2, 1) connection

$$A_\mu = e^a_\mu P_a + \omega^a_\mu J_a, \quad (1.2)$$

where $e^a_\mu$ is the triad, $\omega^a_\mu$ is the spin connection, and $\{P_a, J_a\}$ generate the (2 + 1)-dimensional Poincaré group. Up to gauge transformations, such a flat connection is completely determined by its holonomy group $\Gamma \subset$ ISO(2, 1). $\Gamma$ is a group of isometries of the Minkowski metric, and has a simple geometric interpretation: any classical (2 + 1)-dimensional spacetime is $M$ flat, and if the topology and causal structure are sufficiently simple, $M$ may be “uniformized” by $\Gamma$ [10] — that is, $M = F/\Gamma$, where $F$ is some region of Minkowski space on which $\Gamma$ acts properly discontinuously as a group of isometries.

In particular, if $M$ has the topology $[0, 1] \times T^2$, its fundamental group is $\pi_1(M) = \mathbb{Z} \oplus \mathbb{Z}$, so $\Gamma$ is generated by two commuting Poincaré transformations. In the relevant topological component of the space of flat connections [10], these can be chosen to be of the form

$$\Lambda_1 : (t, x, y) \rightarrow (t \cosh \lambda + x \sinh \lambda, x \cosh \lambda + t \sinh \lambda, y + a)$$

$$\Lambda_2 : (t, x, y) \rightarrow (t \cosh \mu + x \sinh \mu, x \cosh \mu + t \sinh \mu, y + b). \quad (1.3)$$

A spacetime is thus characterized by four time-independent parameters $a, b, \lambda$, and $\mu$, which in a sense already provide a “frozen time” picture at the classical level. Of course, there is nothing paradoxical here: as in ordinary Hamilton-Jacobi theory, we have simply described the classical solutions of the field equations in terms of a set of constants of motion.

For the torus topology, the quotient space $F/\langle \Lambda_1, \Lambda_2 \rangle$ can be worked out explicitly and compared to the ADM metric [4]. One finds that the ADM moduli at York time $T$ are

$$\tau = \left( a + i\frac{\lambda}{T} \right)^{-1} \left( b + i\frac{\mu}{T} \right), \quad (1.4)$$

and

$$p = -iT \left( a - i\frac{\lambda}{T} \right)^2. \quad (1.5)$$

Equations (1.4–1.5) can be viewed as time-dependent canonical transformations; the Poisson brackets

$$\{a, \mu\} = -\{b, \lambda\} = \frac{1}{2} \quad (1.6)$$
induce corresponding brackets
\[ \{ \tau, \bar{p} \} = \{ \bar{\tau}, p \} = 2 . \quad (1.7) \]

In terms of the holonomy variables, the Hamiltonian (1.1) is
\[ H = \frac{a\mu - \lambda b}{T} , \quad (1.8) \]
and it is easy to check that the moduli and momenta (1.4–1.5) obey Hamilton’s equations of motion,
\[ \frac{d\tau}{dT} = -\{ H, \tau \} , \quad \frac{dp}{dT} = -\{ H, p \} . \quad (1.9) \]

Note that the relation (1.4) of moduli and holonomies respects the action of the mapping class group of the torus (also known as the modular group). At the level of holonomies, this group is generated by the two transformations
\[ S : (a, \lambda) \to (b, \mu), \quad (b, \mu) \to (-a, -\lambda) \]
\[ T : (a, \lambda) \to (a, \lambda), \quad (b, \mu) \to (b + a, \mu + \lambda) , \quad (1.10) \]
whose form follows in a straightforward manner from the action of the mapping class group on \( \pi_1(T^2) \). If \( \tau \) is defined as in equation (1.4), the induced transformations are then
\[ S : \tau \to -\frac{1}{\tau} \]
\[ T : \tau \to \tau + 1 , \quad (1.11) \]
which may be recognized as the standard generators of the modular group.

It is natural to try to extend these classical relationships to the quantum theory. In Chern-Simons quantization, the wave function \( \psi(\lambda, \mu) \) is time-independent, and the absence of dynamics provides a clear illustration of the “problem of time.” One solution is to interpret the Chern-Simons quantum theory as a Heisenberg picture, in which wave functions should be time-independent, and to look for appropriate time-dependent operators. In particular, we might expect the operators representing the ADM moduli \( \tau \) and \( \bar{\tau} \) to depend explicitly on a “time” parameter \( T \), with dynamics described by the appropriate Heisenberg equations of motion corresponding to (1.9). If we pass to a Schrödinger picture by simultaneously diagonalizing \( \hat{\tau} \) and \( \hat{\tau}^\dagger \), we will obtain wave functions that depend on \( T \) as well. Different choices of classical time-slicing will lead to different wave functions, of course, but this may simply reflect the fact that they describe different aspects of the physics. This approach to quantization has been advocated by Rovelli [4], who argues that it provides a natural solution to the problem of time in quantum gravity.

2. Operator Ordering and the Modular Group

As might be anticipated, the basic difficulty with such a program is one of operator ordering. A classical variable like \( \tau \) does not uniquely determine an operator in the quantum
theory: given a candidate operator $\hat{\tau}$, any other operator of the form $\hat{\tau}' = \hat{V}^{-1}\hat{\tau}\hat{V}$ will have the same classical limit. Of course, if $\psi$ is an eigenfunction of $\hat{\tau}$, $\psi' = \hat{V}^{-1}\psi$ will be an eigenfunction of $\hat{\tau}'$; but unless $\hat{V}$ happens to be unitary, $\psi'$ will no longer be a simultaneous eigenfunction of the adjoint $(\hat{\tau}')^\dagger$. We thus run the risk of obtaining not a single Schrödinger picture, but many.

This ambiguity can alternatively be expressed — at least in this simple context — as an ambiguity in the definition of the inner product. Given an inner product $\langle \psi | \chi \rangle$, we can define a new $\langle \psi | \chi \rangle' = \langle \hat{V}^{-1}\psi | \hat{V}^{-1}\chi \rangle$. The adjoint of $\hat{\tau}$ then becomes $\hat{\tau}^\dagger = (\hat{V}\hat{\tau}\hat{V}^\dagger)^\dagger(\hat{V}\hat{V}^\dagger)^{-1}$. (2.1)

This observation connects our approach to quantization to the $C^*$-algebra methods developed in quantum field theory [11]; in particular, the relationship between ADM and Chern-Simons operator algebras found in [4] is not complete until the choice of adjoint is specified.

For our simple $[0,1] \times T^2$ topology, a natural choice of ordering is that of equations (1.4–1.5),

$$\hat{\tau} = \left( \hat{a} + \frac{i\lambda}{T} \right)^{-1} \left( \hat{b} + \frac{i\mu}{T} \right), \quad \hat{p} = -iT \left( \hat{a} - \frac{i\lambda}{T} \right)^2.$$ (2.2)

It is not hard to check that this ordering preserves the classical correspondence between modular transformations in the Chern-Simons and ADM pictures; among simple (rational) orderings, it seems to be essentially unique in this respect. Once these definitions have been chosen, the Hamiltonian is fixed up to an additive constant by the requirement that the Heisenberg equations of motion hold; it is

$$\hat{H} = \frac{\hat{a}\hat{\mu} - \hat{\lambda}\hat{b}}{T}.$$ (2.3)

In reference [4], it was shown that these choices lead to simultaneous eigenfunctions of $\hat{\tau}$ and $\hat{\tau}^\dagger$ that are automorphic forms of weight $1/2$, that is, spinors on moduli space.

To generalize this result, let us try to diagonalize $\hat{V}^{-1}\hat{\tau}\hat{V}$ and its adjoint for some arbitrary operator $\hat{V}$. Equivalently, we shall look for the simultaneous eigenfunctions of $\hat{\tau}$ and the adjoint $\hat{\tau}''$ defined in (2.1). As we shall see, this task is manageable because of the simple form of the eigenfunctions of $\hat{\tau}$.

From the Poisson brackets (1.6), we can represent $\hat{a}$ and $\hat{b}$ as

$$\hat{a} = \frac{i}{2} \frac{\partial}{\partial \mu}, \quad \hat{b} = -\frac{i}{2} \frac{\partial}{\partial \lambda},$$ (2.4)

so

$$\hat{\tau} = -\left( \frac{1}{2} \frac{\partial}{\partial \mu} + \frac{\lambda}{T} \right)^{-1} \left( \frac{1}{2} \frac{\partial}{\partial \lambda} - \frac{\mu}{T} \right).$$ (2.5)

The general eigenfunction of $\hat{\tau}$ with eigenvalue $\tau = \tau_1 + i\tau_2$ is then of the form

$$K(\tau, \bar{\tau}; \lambda, \mu, T) = f \left[ \frac{\mu - \tau \lambda}{\tau^2 T} \right] \exp \left\{ -\frac{i|\mu - \tau \lambda|^2}{\tau^2 T} \right\},$$ (2.6)
where \( f \) is an arbitrary analytic function and the dependence of the prefactor on \( T \) has been chosen so that
\[
- \frac{i}{2} \frac{\partial}{\partial T} K(\tau, \bar{\tau}; \lambda, \mu, T) = \hat{H} K(\tau, \bar{\tau}; \lambda, \mu, T). \tag{2.7}
\]
Taylor expanding \( f \), we are thus led to consider functions of the form
\[
K^{(n)}(\tau, \bar{\tau}; \lambda, \mu, T) = \left( \frac{\mu - \tau \lambda}{\tau_2^{1/2} T} \right)^n \exp \left\{ - \frac{i |\mu - \tau \lambda| \tau_2^{1/2}}{\tau T} \right\}. \tag{2.8}
\]
These functions have a number of useful properties. First, under the simultaneous modular transformations (1.10) of \( \mu \) and \( \lambda \) and (1.11) of \( \tau \) and \( \bar{\tau} \), we have
\[
S : K^{(n)} \rightarrow \left( \frac{\bar{\tau}}{\tau} \right)^{n/2} K^{(n)} \tag{2.9}
\]
\[
T : K^{(n)} \rightarrow K^{(n)},
\]
which means that \( K^{(n)} \) is an automorphic form of weight \(-n/2\) [12, 13], essentially a tensor on moduli space. Second, it is not hard to show that
\[
\hat{H}^2 K^{(n)} = T^{-2} \left( \Delta_{-n/2} - \frac{(n-1)^2}{4} \right) K^{(n)}, \tag{2.10}
\]
where \( \hat{H} \) is the Hamiltonian (2.3) and
\[
\Delta_k = -\tau_2^2 \left( \frac{\partial^2}{\partial r_1^2} + \frac{\partial^2}{\partial r_2^2} \right) + 2ik\tau_2 \frac{\partial}{\partial r_1} + k(k+1) \tag{2.11}
\]
is the Maass Laplacian acting on automorphic forms of weight \( k \). Together, these characteristics allow us to employ a large body of mathematical work on representations of the modular group; for instance, the Maass Laplacian is simply an \( \text{SL}(2, \mathbb{R}) \) Casimir operator [12].

Let us now expand a Chern-Simons wave function \( \psi(\mu, \lambda) \) in terms of the eigenfunctions \( K^{(n)}(\tau, \bar{\tau}; \lambda, \mu, T) \):
\[
\psi(\mu, \lambda) = \int \frac{d^2 \tau}{\tau^2} K^{(n)}(\tau, \bar{\tau}; \lambda, \mu, T) \tilde{\psi}^{(n)}(\tau, \bar{\tau}, T). \tag{2.12}
\]
For \( \psi(\mu, \lambda) \) to be invariant under modular transformations, \( \tilde{\psi}^{(n)}(\tau, \bar{\tau}, T) \) must transform with a phase that cancels that of \( K^{(n)}(\tau, \bar{\tau}; \lambda, \mu, T) \), i.e., it must be an automorphic form of weight \( n/2 \). Moreover, since \( \psi(\mu, \lambda) \) is independent of \( T \), equation (2.7) gives
\[
0 = \left( T \frac{\partial}{\partial T} \right)^2 \psi(\mu, \lambda) = \int \frac{d^2 \tau}{\tau^2} \left[ -T^2 \left( \hat{H}^2 K^{(n)}(\tau, \bar{\tau}; \lambda, \mu, T) \right) \tilde{\psi}^{(n)}(\tau, \bar{\tau}, T) \right.
\]
\[
+ K^{(n)}(\tau, \bar{\tau}; \lambda, \mu, T) \left( T \frac{\partial}{\partial T} \right)^2 \tilde{\psi}^{(n)}(\tau, \bar{\tau}, T) \right]. \tag{2.13}
\]
Using (2.10) and integrating by parts, we find that

$$\left(T \frac{\partial}{\partial T}\right)^2 \tilde{\psi}^{(n)}(\tau, \bar{\tau}, T) = \left[\Delta_{n/2} - \frac{(n+1)^2}{4}\right] \tilde{\psi}^{(n)}(\tau, \bar{\tau}, T) ,$$

which is the Klein-Gordon equation for an automorphic form of weight $n/2$. Our diagonalization procedure thus produces wave functions that obey a simple wave equation of the type one might expect from direct ADM quantization.

For $n = 1$, this analysis reduces to that of [6]. In particular, it is not hard to check that

$$\hat{\tau}^{\dagger} K^{(1)}(\tau, \bar{\tau}; \lambda, \mu, T) = \bar{\tau}K^{(1)}(\tau, \bar{\tau}; \lambda, \mu, T) ,$$

so the diagonalization of $\hat{\tau}$ and $\hat{\tau}^{\dagger}$ with the ordering (2.2) leads to ADM-type wave functions that transform as forms of weight 1/2. Naive ADM quantization, on the other hand, gives modular invariant wave functions, i.e., forms of weight 0 [8]. To understand such functions, observe that

$$\hat{\tau}^{\dagger} K^{(0)}(\tau, \bar{\tau}; \lambda, \mu, T) = \left(\hat{a} - i\hat{\lambda} \frac{T}{\hat{b}}\right)^{-1} \left(\hat{b} - i\hat{\mu} \frac{T}{\hat{b}}\right) K^{(0)}(\tau, \bar{\tau}; \lambda, \mu, T) = \bar{\tau}K^{(0)}(\tau, \bar{\tau}; \lambda, \mu, T) ,$$

so $n = 0$ corresponds to the simultaneous diagonalization of $\hat{\tau}$ and $\hat{\tau}^{\ast}$. Moreover, it is easy to check that

$$\hat{\tau}^{\ast} = (\hat{p}^{\dagger})^{-1/2} \hat{\tau}^{\dagger} (\hat{p}^{\dagger})^{1/2} .$$

Standard ADM quantum theory thus corresponds to the operator ordering (2.1) with $\hat{V} = \hat{p}^{-1/2}$. Other orderings will give rise to automorphic forms with other weights, but at least for orderings of the form $\hat{\tau}' = \hat{V}^{-1} \hat{\tau} \hat{V}$, we have now seen that this is the only ambiguity.

3. Implications

Kuchař has described the approach to quantization taken here as one of “evolving constants of motion” [3]. He points out a number of potential problems, including operator ordering ambiguities and possible dependence on the choice of time-slicing.

The York time-slicing is the only choice that has been studied in any detail in (2 + 1)-dimensional gravity, so we cannot yet address the latter issue. As for the former, we have seen that operator ordering ambiguities do indeed make the quantum theory nonunique. But the range of possible theories is surprisingly limited, and is essentially determined by the representation theory of the modular group. The modular group thus plays a role roughly analogous to that of the Poincaré group in ordinary free quantum field theory, determining a small family of admissible models.

Of course, the topology $[0, 1] \times T^2$ is exceptionally simple even for 2 + 1 dimensions, and one must worry about whether these results can be extended to less trivial spacetimes. In general, the Hamiltonian (1.1) will become much more complicated — it will typically be

1More precisely, he suggests that the dynamical operators corresponding to different choices of time-slicing may not all be self-adjoint under any choice of inner product.
nonpolynomial — and it will be difficult to express the ADM moduli explicitly in terms of holonomies. It is possible that useful results can be obtained for spaces of genus two, for which the hyperelliptic representation provides a major simplification. In any case, we can at least be confident that the representation theory of the mapping class group will continue to play a key role.

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