A general representation of \(\delta\)-normal sets to sublevels of convex functions

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Dedicated to Prof. Michel Théra on his 70th birthday

Abstract

The \((\delta\cdot)\)-normal cone to an arbitrary intersection of sublevel sets of proper, lower semicontinuous, and convex functions is characterized, using either \(\varepsilon\)-subdifferentials at the nominal point or exact subdifferentials at nearby points. Our tools include \((\varepsilon\cdot)\) calculus rules for sup/max functions. The framework of this work is that of a locally convex space, however, formulas using exact subdifferentials require some restriction either on the space (e.g. Banach), or on the function (e.g. epi-pointed).

Key words. \(\delta\)-normal set, normal cone, sublevel set, convex function, sup function, epi-pointed function, subdifferential.

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1 Introduction

In [6] the authors prove that for a convex, proper and lower semicontinuous (lsc) function \(\Phi : X \to \mathbb{R} \cup \{+\infty\}\), defined on a (reflexive) Banach space \(X\), the normal cone to the sublevel set \([\Phi \leq \Phi(\bar{x})] := \{x \in X \mid \Phi(x) \leq \Phi(\bar{x})\}\) at a point \(\bar{x}\) in the effective domain of \(\Phi\), is characterized as the (norm) upper-limit of directions in the Fenchel subdifferential of \(\Phi\) at sufficiently close points to \(\bar{x}\); that is,

\[N_{[\Phi \leq \Phi(\bar{x})]}(\bar{x}) = \limsup_{x \to \bar{x}} \mathbb{R}_+ \partial \Phi(x).\]  \(1\)

This relation is also valid in general Banach spaces but up to some specifications of both the \(\limsup\) and the convergence of \(x\) to \(\bar{x}\). The proof of (1) and its generalization to Banach spaces given in [6], is based on sequential calculus rules for the subdifferential...
of composite functions developed in [25]. In this work, assuming that \( X \) is a general locally convex (lc) space with a topological dual \( Y \), we propose another route to approach this problem and provide different characterizations of the \( \delta \)-normal set to an arbitrary intersection of sublevel sets

\[
N_{\cap_{t \in T} \Phi_t \leq \Phi_t(x)}(x^*) := \{ x^* \in Y \mid \langle x^*, x - \bar{x} \rangle \leq \delta \text{ for all } x \in \cap_{t \in T} \Phi_t \leq \Phi_t(\bar{x}) \}, \quad \delta \geq 0,
\]

for convex functions \( \Phi_t \) which are indexed in an arbitrary set \( T \). Compared to formula (1), the present characterization only involves the reference point \( \bar{x} \) rather than nearby points, and uses \( \varepsilon \)-subdifferentials instead of exact ones.

Owing to Brøndsted-Rockafellar’s theorem ([2]), these epsilon-like formulas for \( \delta \)-normal sets easily lead to characterizations in the line of formula (1). This passage from \( \varepsilon \)-subdifferentials to exact ones will require some natural conditions either on the underlying space or on the associated function; typically, that either \( X \) is Banach or that \( \Phi \) satisfies some continuity/coercivity conditions.

More generally, we shall prove that if \( \Phi = \sup_{t \in T} \Phi_t \), with \( \Phi_t : X \to \mathbb{R} \cup \{+\infty\} \), \( t \in T \), being convex, proper and lsc, then for every \( \delta \geq 0 \), \( \bar{x} \in \text{dom } \Phi \) and \( \lambda \in (-\infty, +\infty] \) we have that

\[
N_{\{\Phi \leq \lambda\} \cup \{\bar{x} + [\Phi \leq \Phi(\bar{x})]_\infty\}}(x^*) = \limsup_{\sum_{i \geq 1, k} \mu_i \partial_{x^*} \Phi_{t_i}(\bar{x}) \to \delta, \nu \uparrow \lambda} \sum_{i \in T, k} \mu_i \partial_{x^*} \Phi_{t_i}(\bar{x}), \tag{2}
\]

where the subscript \( \infty \) refers to the recession cone, and \( \text{dom } \Phi \) denotes the effective domain of \( \Phi \). The sets \( \text{dom } \Phi \) and \( \bar{x} + [\Phi \leq \Phi(\bar{x})]_\infty \) are superfluous when \( \Phi(\bar{x}) \leq \lambda < \infty \), so that the left-hand side in (2) reduces to the usual \( \delta \)-normal set of \( [\Phi \leq \lambda] \). However, these two extra sets are necessary to make formula (2) meaningful when \( \bar{x} \) may lie outside the set \( [\Phi \leq \lambda] \); for instance, when \( [\Phi \leq \lambda] = \emptyset \). The case \( \lambda = +\infty \) is also meaningful since it leads to a characterization of the normal cone to the domain of \( \Phi \). It is worth recalling that when \( \Phi_t \equiv \Phi \) and \( \Phi(\bar{x}) \leq \lambda \), formula (2) yields a well-known result, which was first established in [15] (see, also, [13]). Formula (2) also naturally simplifies under Slater’s type conditions, giving rise to familiar results (see [24]).

The passage from single to arbitrary intersections of sublevel sets will be made possible through the investigation in this work of new and general (\( \varepsilon \)-)subdifferential calculus rules for pointwise suprema, extending some previous results on this theme ([3, 7, 11, 12, 10, 18, 19]). These rules are extensively used at different stages of the proof of (2).

The going back and forth between (1) and (2) will be made clear through the use of Brøndsted-Rockafellar’s like-theorem, which allows rewriting (2) by means only of exact subdifferentials. However, this approach requires some extra conditions, in light of the example of proper, convex and lsc functions, defined in non-complete normed spaces, which have an empty subdifferential at every point ([4]). Despite this limitation, we
shall see that the following relation

\[
N_{[\Phi \leq \Phi(\bar{x})]}(\bar{x}) = \lim_{x \to \text{dom } \Phi^*} \mu \partial \Phi(x),
\]

still hold true for a wide class of functions, which includes for example functions defined on locally convex spaces which either they or their conjugates are finite and continuous at some point. This class, referred to as the class of epi-pointed functions (e.g., [9]), also contains convex functions defined in Banach spaces up to some appropriate localization.

The need for explicit characterizations of the normal cone to sublevel sets is fundamental in optimization theory, namely in the derivation of optimality conditions for convex programming problems (e.g., [23, 24]). It is also relevant in the investigation of stationarity and stability properties of different dynamic systems ([1, 5, 6]).

The previous formulas are applied at the end of this work to spectral functions ([16, 17]). As it is expected, the associated normal cone will only depend on the values of the function on the range of eigenvalues vectors.

The paper is organized as follows: In section 2 we give the necessary definitions and basic notations. In section 3 we develop some approximate subdifferential calculus rules for the max function, which are needed for our analysis. In section 4 we give in Theorem 5 the main characterization of the normal cone to sublevel sets, which uses the approximate subdifferential. This characterization will be rewritten in section 5, Theorem 15, by means only of exact subdifferentials. Finally, in section 6, we apply the previous result to investigate the case of spectral functions.

2 Notations and Definitions

We recall in this section some definitions and notations that will be used in the sequel. We consider a dual pair \((X, Y)\) of real locally convex (lc, for short) spaces \(X\) and \(Y\), defined via a bilinear form \(\langle x^*, x \rangle := \langle x^*, x \rangle := x^*(x), x^* \in Y, x \in X\). By \(N_X(x)\) we refer to the family of absolutely convex neighborhoods of \(x\). The origin vectors are denoted by \(\theta\).

Given a non-empty set \(S \subset X\), by \(\overline{S}\) (or \(\text{cl } S\)), and \(\text{aff } S\) we denote the closure, and the affine hull of \(S\), respectively. The relative interior of \(S\) is the interior of \(S\) relative to \(\text{aff } S\) when this set is closed, and the emptyset otherwise. The polar set, the dual cone, and the orthogonal space of \(S\) are the subsets of \(Y\) given by \(S^0 := \{x^* \in Y \mid \langle x^*, x \rangle \leq 1 \text{ for all } x \in S\}\), \(S^- := \{x^* \in Y \mid \langle x^*, x \rangle \leq 0 \text{ for all } x \in S\}\), and \(S^+ := \{x^* \in Y \mid \langle x^*, x \rangle = 0 \text{ for all } x \in S\}\), respectively. Given \(\delta \geq 0\), the \(\delta\)-normal set to \(S\) at \(\bar{x}\) is the set

\[
N^\delta_S(\bar{x}) := \{x^* \in Y \mid \langle x^*, x - \bar{x} \rangle \leq \delta, \forall x \in S\};
\]
we call \(N_S(\bar{x}) := N^0_S(\bar{x})\) the normal cone to \(S\) at \(\bar{x}\).

We fix a function \(\Phi : X \to \mathbb{R} \cup \{+\infty\}\). We say that \(\Phi\) is proper if its (effective) domain, \(\text{dom } \Phi := \{x \in X \mid \Phi(x) < \infty\}\), is nonempty; convex (lower semi-continuous
\( \text{epi } \Phi := \{ (x, \lambda) \in X \times \mathbb{R} \mid \Phi(x) \leq \lambda \}, \)

is convex (closed, resp.). If \( \Phi \) is proper, convex and lsc we write \( \Phi \in \Gamma_0(X) \). The restriction function of \( \Phi \) to \( S \) is denoted by \( \Phi |_S \). The closed convex hull of \( \Phi \) is defined as

\[
\sigma^\circ \Phi(x) = \liminf_{u \to x} \inf \left\{ \sum_{i \in 1,k} \lambda_i \Phi(x_i) \mid u = \sum_{i \in 1,k} \lambda_i x_i, (\lambda_i) \in \Delta_k, k \in \mathbb{N} \right\}, \ x \in X,
\]

where \( \Delta_k := \{ (\lambda_1, \ldots, \lambda_k) \mid \lambda_i > 0, \lambda_1 + \cdots + \lambda_k = 1 \} \). The sublevel set of \( \Phi \) at \( \lambda \in ]-\infty, +\infty[ \) is the set

\[
[\Phi \leq \lambda] := \{ x \in X \mid \Phi(x) \leq \lambda \}.
\]

Assume that \( \Phi \in \Gamma_0(X) \). For \( \varepsilon \geq 0 \) the \( \varepsilon \)-subdifferential of \( \Phi \) at \( x \in \text{dom } \Phi \) is

\[
\partial_\varepsilon \Phi(x) := \{ x^* \in Y \mid \langle x^*, u - x \rangle \leq \Phi(u) - \Phi(x) + \varepsilon, \forall u \in X \};
\]

we write \( \partial \Phi(x) := \emptyset \) if \( \varepsilon < 0 \) or if \( x \notin \text{dom } \Phi \). The subdifferential of \( \Phi \) at \( x \) is the set \( \partial \Phi(x) := \partial_0 \Phi(x) \). The directional \( \varepsilon \)-derivative of \( \Phi \) at \( x \) in a direction \( v \in X \) is defined as

\[
\Phi'_\varepsilon(x,v) := \inf_{t > 0} \frac{\Phi(x + tv) - \Phi(x) + \varepsilon}{t};
\]

again, if \( \varepsilon = 0 \), we just call it directional derivative and write \( \Phi'(x,v) \). Equivalently, for \( \varepsilon > 0 \) we have [26, Theorem 2.4.11]

\[
\Phi'_\varepsilon(x,v) = \sup_{x^* \in \partial_\varepsilon \Phi(x)} \langle x^*, v \rangle.
\]

The asymptotic function of \( \Phi \), \( \Phi^\infty : X \to \mathbb{R} \cup \{ +\infty \} \), is defined as

\[
\Phi^\infty(v) := \sup_{t > 0} t^{-1}(\Phi(x + tv) - \Phi(x)).
\]

The conjugate of \( \Phi \) is the proper, convex and lsc function defined on \( Y \) as

\[
\Phi^*(x^*) := \sup\{ \langle x^*, x \rangle - \Phi(x), \ x \in X \}.
\]

The indicator function of \( S \), \( I_S : X \to \mathbb{R} \cup \{ +\infty \} \), is defined by

\[
I_S(x) = \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{if } x \notin S, \end{cases}
\]
while the support function, $\sigma_S : X \to \mathbb{R} \cup \{+\infty\}$ (when $S \subset Y$), is defined as the conjugate of $I_S$. We shall frequently use the following relation, which holds for every function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ having a proper conjugate,

$$\varphi^{**} := (\varphi^*)^* = \overline{\partial \varphi}.$$  \hfill (3)

We know that

$$\Phi^\infty = \sigma_{\text{dom} \Phi^*},$$  \hfill (4)

$$[\Phi \leq \Phi(x)]_\infty = [\Phi^\infty \leq 0] = [\sigma_{\text{dom} \Phi^*} \leq 0] = (\text{dom} \Phi^*)^-;$$  \hfill (5)

where $S^\infty$ denotes the asymptotic cone of $S$ defined via the relation $I_{S^\infty} = (I_S)^\infty$.  

Recall that for any set $T$ and $\Phi_t \in I_0(X)$, $t \in T$, we have that (see [20])

$$\left(\sup_{t \in T} \Phi_t\right)^* = \overline{\partial \Phi}^* \left(\inf_{t \in T} \Phi_t^*\right), \quad \left(\inf_{t \in T} \Phi_t\right)^* = \sup_{t \in T} \Phi_t^*.$$  \hfill (6)

Finally, given a multifunction $M : U \rightrightarrows V$, defined between two topological spaces $U$ and $(V, \tau)$, the Painleve-Kuratowski upper limit of $M$ at $\bar{u} \in U$ is defined as

$$\tau\text{-} \limsup_{u \to \bar{u}} M(u) := \{v \in V \mid \forall W \in \mathcal{N}_V(v), \forall Z \in \mathcal{N}_V(\bar{u}), \exists u \in Z, M(u) \cap W \neq \emptyset\}.$$  

Equivalently, $v \in \tau\text{-} \limsup_{u \to \bar{u}} M(u)$ iff $v$ is the $\tau$-limit of a net $(v_\alpha)$ such that $v_\alpha \in M(u_\alpha)$ for some $(v_\alpha) \subset U$ converging to $\bar{u}$. If the sets $U$ and $V$ are first countable, then we take sequences instead of nets. We will often omit the reference to $\tau$ and just write $\limsup_{u \to \bar{u}} M(u)$ when the topology $\tau$ is understood.

### 3 $\varepsilon$-subdifferential calculus for pointwise suprema

In this section, we develop different rules for the $\varepsilon$-subdifferential mapping of pointwise suprema. The setting here is that of a dual pair $(X, Y)$ of (real) vector spaces with an associated separating bilinear form denoted by $\langle \cdot, \cdot \rangle$, so that $X$ and $Y$ are endowed with compatible topologies.

In the following we characterize the $\varepsilon$-subdifferential mapping of the conjugate function.

**Lemma 1** Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $f^*$ is proper, we have for all $\varepsilon > 0$ and $x^* \in Y$

$$\partial f^*(x^*) = \bigcap_{\delta > 0} \text{cl} \left( \sum_{i \in 1, k} \lambda_i (\partial \varepsilon_i f)^{-1}(x^*) \mid (\lambda_i) \in \Delta_k, \sum_{i \in 1, k} \lambda_i \varepsilon_i \leq \varepsilon + \delta, \varepsilon_i \geq 0, k \geq 1 \right).$$  \hfill (7)

**Proof.** To verify the inclusion "$\supset$", we fix $\delta > 0$ and take $x = \sum_{i=1}^k \lambda_i x_i$ with $x_i \in \text{dom} \partial \varepsilon_i f$. Then $x \in$
\((\partial_{\varepsilon_i} f)^{-1}(x^*), (\lambda_i) \in \Delta_k\) and \(\sum \lambda_i \varepsilon_i \leq \varepsilon + \delta \) \((\varepsilon_i \geq 0, k \in \mathbb{N})\). Then \(x^* \in \partial_{\varepsilon_i} f(x_i)\) and, so,
\[
\langle x^*, u - x_i \rangle \leq f(u) - f(x_i) + \varepsilon_i, \quad \forall u \in X, \forall i = 1, \ldots, k.
\]
Multiplying this inequality by \(\lambda_i\) and summing up over \(i\), we obtain (recall that \(f^{**} = (f^*)^* = \overline{\text{co}} f\), by (3))
\[
\langle x^*, u - x \rangle \leq f(u) + \sum_{i \in \mathbb{N}} \lambda_i (-f(x_i) + \varepsilon_i) \leq f(u) - f^{**}(x) + \varepsilon + \delta. \tag{8}
\]
Since \(x^* \in \partial_{\varepsilon_i} f(x_i)\) and \(f^*\) is proper, we have \(-\infty < f^{**}(x) = f^{**}(\sum_{i=1}^{k} \lambda_i x_i) \leq \sum_{i \in \mathbb{N}} \lambda_i f^{**}(x_i) \leq \sum_{i \in \mathbb{N}} \lambda_i f(x_i) + \infty\). Then, by taking the supremum over \(u\) in (8),
\[
f^*(x^*) + (f^*)^*(x) = f^*(x^*) + f^{**}(x) \leq \langle x^*, x \rangle + \varepsilon + \delta;
\]
that is, \(x \in \partial_{\varepsilon + \delta} f^*(x^*)\). Because \(\delta\) was arbitrarily chosen, it follows that \(x \in \partial_{\varepsilon} f^*(x^*)\).
Consequently, the inclusion \(\subset\) follows due to the closedness and the convexity of the set \(\partial_{\varepsilon} f^*(x^*)\).
To establish the inclusion \(\supset\) we take \(x \in \partial_{\varepsilon} f^*(x^*)\) so that, by (3), \(\overline{\text{co}} f(x) = f^{**}(x) \in \mathbb{R}\) and
\[
f^*(x^*) + \overline{\text{co}} f(x) \leq \langle x^*, x \rangle + \varepsilon. \tag{9}
\]
Given a \(\delta > 0\) we choose a \(V \in \mathcal{N}_X(\theta)\) such that \(\sigma_V(x^*) \leq \delta\). From the definition of \(\overline{\text{co}} f\) there are elements \(x_1, \ldots, x_k \in \text{dom} f\), \((\lambda_i) \in \Delta_k\), and \(k \in \mathbb{N}\), such that \(x - \sum_{i=1}^{k} \lambda_i x_i \in V\) and
\[
\overline{\text{co}} f(x) - \delta \leq \sum_{i=1}^{k} \lambda_i f(x_i) \leq \overline{\text{co}} f(x) + \delta \leq -f^*(x^*) + \langle x^*, x \rangle + \varepsilon + \delta. \tag{10}
\]
We put \(\varepsilon_i := f(x_i) - \overline{\text{co}} f(x) + \langle x^*, x - x_i \rangle + \varepsilon \in \mathbb{R}\). Observe that from the definition of \(f^*\) and the relation (9) we have
\[
\varepsilon_i = -((\langle x^*, x_i \rangle - f(x_i)) - \overline{\text{co}} f(x) + \langle x^*, x \rangle + \varepsilon \geq -f^*(x^*) - \overline{\text{co}} f(x) + \langle x^*, x \rangle + \varepsilon \geq 0.
\]
Also, using (10) and the choice of \(V\) we obtain (recall (3))
\[
\sum_{i=1}^{k} \lambda_i \varepsilon_i = \sum_{i=1}^{k} \lambda_i f(x_i) - \overline{\text{co}} f(x) + \left(\langle x^*, x - \sum_{i=1}^{k} \lambda_i x_i \rangle + \varepsilon \right) \leq \delta + \sigma_V(x^*) + \varepsilon \leq 2\delta + \varepsilon.
\]
Thus, since (recall (9))
\[
\varepsilon_i + \langle x^*, x_i \rangle = f(x_i) + (\langle x^*, x \rangle + \varepsilon - \overline{\text{co}} f(x)) \geq f(x_i) + f^*(x^*),
\]
it follows that \(x_i \in (\partial_{\varepsilon_i} f)^{-1}(x^*)\), and so \(x \in \sum_{i=1}^{k} \lambda_i (\partial_{\varepsilon_i} f)^{-1}(x^*) + V\). Finally, the arbitrariness of \(V\) and \(\delta > 0\) leads us to the desired inclusion \(\supset\). \(\blacksquare\)
We give now a formula for the $\varepsilon$-subdifferential of the supremum function, which extends and improves [19, Theorem 1].

**Theorem 2** Given set $T$ and functions $\Phi_t \in \Gamma_0(X)$, $t \in T$, we put $\Phi := \sup_{t \in T} \Phi_t$. Then for every $x \in X$ and $\varepsilon \geq 0$ we have

$$\partial_{\varepsilon} \Phi(x) = \bigcap_{\delta > \varepsilon} \text{cl} \left( \bigcup_{i \in T, k} \sum_{i \in T} \lambda_i \partial_{\delta, \varepsilon_i} \Phi_{t_i}(x) \right).$$

**Proof.** We fix $x \in X$ and $\delta > \varepsilon \geq 0$. If $g := \inf_{t \in T} \Phi_t^*$, then writing (by (3))

$$g^* = \sup_{t \in T} (\Phi_t^*)^* = \sup_{t \in T} \partial_{\varepsilon} \Phi_t = \sup_{t \in T} \Phi_t = \Phi,$$

according to Lemma 1 it follows that

$$\partial_{\varepsilon} \Phi(x) = \partial_{\varepsilon} g^*(x)$$

$$= \bigcap_{\delta > \varepsilon} \text{cl} \left( \bigcup_{i \in T, k} \sum_{i \in T} \lambda_i (\partial_{\varepsilon_i} g)^{-1}(x) \big| (\lambda_i) \in \Delta_k, \sum_{i \in T} \lambda_i \varepsilon_i \leq \delta, \varepsilon_i \geq 0, k \geq 1 \right). \quad (12)$$

To establish the inclusion “$\subset$” of the current theorem we pick $x_t^* \in (\partial_{\varepsilon_i} g)^{-1}(x)$, $i \in T, k$, where $\varepsilon_i \geq 0$ and $k \in \mathbb{N}$ are such that $\sum_{i \in T} \lambda_i \varepsilon_i \leq \delta$ for some $(\lambda_i) \in \Delta_k$. Then $x \in \partial_{\varepsilon_i} g(x_t^*)$ and

$$\inf_{t \in T} \Phi_t^*(x_t^*) + \Phi(x) = g(x_t^*) + g^*(x) \leq \langle x_t^*, x \rangle + \varepsilon_i.$$

By choosing $t_i \in T$ such that $\inf_{t \in T} \Phi_t^*(x_t^*) \geq \Phi_t^*(x_t^*) - \delta + \varepsilon$ we obtain

$$\Phi_{t_i}(x_t^*) + \Phi(x) \leq \inf_{t \in T} \Phi_t^*(x_t^*) + \Phi(x) + \delta - \varepsilon \leq \langle x_t^*, x \rangle + \varepsilon_i + \delta - \varepsilon,$$

and so $\Phi(x) \leq \langle x_t^*, x \rangle - \Phi_{t_i}(x_t^*) + \varepsilon_i + \delta - \varepsilon \leq (\Phi_{t_i}^*)^*(\Phi(x)) + \varepsilon_i + \delta - \varepsilon = \Phi_{t_i}(x) + \varepsilon_i + \delta - \varepsilon$. Also, since $\Phi_{t_i}^*(x_t^*) + \Phi(x) - \langle x_t^*, x \rangle \leq \varepsilon_i + \delta - \varepsilon$, we get

$$0 \leq \Phi_{t_i}^*(x_t^*) + \Phi_{t_i}(x) - \langle x_t^*, x \rangle \leq \varepsilon_i + \delta - \varepsilon + \Phi_{t_i}(x) - \Phi(x) =: \hat{\beta}_i;$$

that is, $x_t^* \in \partial_{\hat{\beta}_i} \Phi_{t_i}(x)$ and

$$\sum_{i \in T} \lambda_i (\hat{\beta}_i - \Phi_{t_i}(x)) = \sum_{i \in T} \lambda_i (\varepsilon_i + \delta - \varepsilon - \Phi(x)) \leq 2\delta - \varepsilon - \Phi(x).$$

Let $\gamma \geq 0$ such that $\sum_{i \in T} \lambda_i (\hat{\beta}_i - \Phi_{t_i}(x)) + \gamma = 2\delta - \varepsilon - \Phi(x)$ and denote $\beta_i := \hat{\beta}_i + \gamma$. Then $x_t^* \in \partial_{\beta_i} \Phi_{t_i}(x) \subset \partial_{\beta_i} \Phi_{t_i}(x)$, $\sum_{i \in T} \lambda_i (\beta_i - \Phi_{t_i}(x)) = 2\delta - \varepsilon - \Phi(x)$, and, consequently, the inclusion “$\subset$” follows thanks to (12).
To prove the inclusion “⊇” we take \(x^* = \sum_{i \in \overline{1,k}} \lambda_i x^*_i\), where \(x^*_i \in \partial_{\beta_i} \Phi_n(x)\) (i \(\in \overline{1,k}\)), \(\delta > 0\), \(k \in \mathbb{N}\), and \((\lambda_i) \in \Delta_k\), with \(\sum_{i \in \overline{1,k}} \lambda_i(\beta_i - \Phi_n(x)) + \Phi(x) = \varepsilon + \delta\). Then

\[
\Phi_n(x) + \Phi^*_n(x) \leq \langle x, x^* \rangle + \beta_i,
\]

and, by multiplying by \(\lambda_i\) and summing over \(i \in \overline{1,k}\),

\[
\sum_{i \in \overline{1,k}} \lambda_i \Phi^*_n(x^*_i) \leq \left( x, \sum_{i \in \overline{1,k}} \lambda_i x^*_i \right) + \sum_{i \in \overline{1,k}} \lambda_i(\beta_i - \Phi_n(x)) = \langle x, x^* \rangle + \varepsilon + \delta - \Phi(x).
\]

But \(\Phi^* = (g^*)^* = \overline{\text{co}}(\inf_{i \in T} \Phi_i^*)\) (as \(g^* = \Phi\) is proper, recall (11)), and so the last inequality yields \(\Phi^*(x^*) \leq \sum_{i \in \overline{1,k}} \lambda_i \overline{\text{co}}(\inf_{i \in T} \Phi_i^*)(x^*_i) \leq \langle x, x^* \rangle + \varepsilon + \delta - \Phi(x)\); that is, \(x^* \in \partial_{\varepsilon + \delta} \Phi(x)\). Thus, the desired inclusion “⊇” follows along the arbitrariness of \(\delta > 0\).

Theorem 2 becomes more explicit when the index set \(T\) is countable and the sequence \((\Phi_n)_n\) is non-decreasing.

**Corollary 3** With the assumptions of Theorem 2 we take \(T = \mathbb{N}\). If the sequence \((\Phi_n)_n\) is non-decreasing, then for every \(x \in X\) and \(\varepsilon \geq 0\) we have

\[
\partial_{\varepsilon} \Phi(x) = \bigcap_{\delta > \varepsilon} \limsup_{n \to +\infty} \partial_{\delta} \Phi_n(x) = \limsup_{n \to +\infty} \partial_{\delta} \Phi_n(x).
\]

**Proof.** Take \(\xi \in \partial_{\varepsilon} \Phi(x)\) and fix \(\delta > 0\), \(V \in \mathcal{N}_Y(\theta)\). According to Theorem 2, we have that \(\xi \in \sum_{i \in \overline{1,k}} \lambda_i \xi_i + V\) for some \(\xi_i \in \partial_{\beta_i} \Phi_n_i(x)\) and \((\lambda_i) \in \Delta_k\) \((k \in \mathbb{N})\), where \(\beta_i \geq 0\) and \(n_i \in \mathbb{N}\) \((i \in \overline{1,k})\) are such that \(\sum_{i \in \overline{1,k}} \lambda_i(\beta_i - \Phi_n_i(x) + \Phi(x)) = \varepsilon + \frac{\delta}{2}\).

Set \(m_0 := \max_{i \in \overline{1,k}} n_i \geq 1\). Then, from one hand, by the current assumption on the sequence \((\Phi_n)_n\), we obtain

\[
\Phi_{m_0}(x) \geq \sum_{i \in \overline{1,k}} \lambda_i \Phi_{n_i}(x) \geq \Phi(x) + \sum_{i \in \overline{1,k}} \lambda_i \beta_i - \varepsilon - \frac{\delta}{2} \geq \Phi(x) - \varepsilon - \delta.
\]

On the other hand, by writing the relation \(\xi_i \in \partial_{\beta_i} \Phi_{n_i}(x)\) into an inequality form and,
next, summing up over \( i \in \{1, \ldots, k\} \), we obtain, for all \( u \in x + \frac{\delta}{2}V^\circ \) (hence, \( \sigma_V(u - x) \leq \frac{\delta}{2} \)),

\[
\langle \xi, u - x \rangle \leq \sum_{i=1}^{k} \lambda_i \langle \xi_i, u - x \rangle + \frac{\delta}{2}
\]

\[
\leq \sum_{i=1}^{k} \lambda_i (\Phi_n(u) - \Phi_n(x) + \beta_i) + \frac{\delta}{2}
\]

\[
\leq \sum_{i=1}^{k} \lambda_i (\Phi_n(u) - \Phi(x)) + \varepsilon + \delta
\]

\[
\leq \Phi_{m_0}(u) - \Phi(x) + \varepsilon + \delta.
\]

Thus,

\[
\langle \xi, u - x \rangle \leq \Phi_n(y) - \Phi_n(x) + \varepsilon + \delta \quad \text{for all} \quad n \geq m_0,
\]

and, so, taking into account (14), by the sum rule of \( \varepsilon \)-subdifferentials (e.g., [14]) we get

\[
\xi \in \bigcap_{n \geq m_0} \partial_{\varepsilon + \delta} (\Phi_n + 1_{x + \frac{\delta}{2}V^\circ})(x) \subset \bigcap_{n \geq m_0, \eta > 0} \partial_{\varepsilon + \delta + \eta} \Phi_n(x) + \frac{3(\varepsilon + \delta + \eta)}{\delta}V.
\]

Hence, as \( V \) and \( \delta \) were arbitrarily chosen, we deduce that

\[
\xi \in \limsup_{n \to +\infty, \frac{\delta}{\varepsilon}} \partial_{\varepsilon} \Phi_n(x) = \bigcap_{n \to +\infty} \limsup_{\delta > 0} \partial_{\varepsilon + \delta} \Phi_n(x).
\]

This finishes the proof since the opposite inclusion \( \supset \) holds straightforwardly. □

We recover the subdifferential rule for the case of finitely many convex functions; see, e.g., [26, Corollary 2.8.11].

**Corollary 4** With the assumptions of Theorem 2 we take \( T = \{1, \ldots, n\} \). Then for every \( x \in X \)

\[
\partial_{\varepsilon} \Phi(x) = \bigcup \{ \partial_{\eta} \left( \sum_{i=1}^{n} \lambda_i \Phi_i(x) \right) \mid (\lambda_i) \in \Delta_n, \quad \eta \in [0, \varepsilon], \quad \sum_{i=1}^{n} \lambda_i \Phi_i(x) \geq \Phi(x) + \eta - \varepsilon \}.
\]

**Proof.** We may assume that \( x \in \text{dom} \Phi = \bigcap_{i=1}^{n} \text{dom} \Phi_i \) and \( \partial_{\varepsilon} \Phi(x) \neq \emptyset \). The inclusion
“⊃” is straightforward. For the other inclusion we have, by Theorem 2,
\[
\partial_\varepsilon \Phi(x) = \bigcap_{\delta > \varepsilon} \text{cl}\{\sum_{i=1}^{n} \lambda_i \partial_\beta_i \Phi_i(x) \mid (\lambda_i, \beta_i) \in \Delta_n, \beta_i \geq 0, \sum_{i=1}^{n} \lambda_i (\beta_i - \Phi_i(x)) + \Phi(x) = \delta\}
\]
\[
= \bigcap_{\delta > \varepsilon} \text{cl}\{\sum_{i=1}^{n} \lambda_i \partial_\beta_i \Phi_i(x) \mid (\lambda_i, \beta_i) \in \Delta_n, \lambda_i, \beta_i > 0, \sum_{i=1}^{n} \lambda_i (\beta_i - \Phi_i(x)) + \Phi(x) = \delta\},
\]
and so, setting \(\eta = \sum_{i=1}^{n} \lambda_i \beta_i\),
\[
\partial_\varepsilon \Phi(x) \subset \bigcap_{\delta > \varepsilon} \text{cl}\{\partial_\eta \left(\sum_{i=1}^{n} \lambda_i \Phi_i\right)(x) \mid (\lambda_i) \in \Delta_n, \eta \in [0, \varepsilon], \sum_{i=1}^{n} \lambda_i \Phi_i(x) \geq \Phi(x) + \eta - \delta\}. \quad (15)
\]
Then the desired inclusion follows due to the compactness of the set \(\Delta_n\).

4 \(\varepsilon\)-subdifferential approach

In this section, we give the desired characterization of \(\delta\)-normal sets to arbitrary intersections of sublevel sets. As in the previous section, the framework here is that of a dual pair \((X, Y)\) of (real) lc spaces \(X\) and \(Y\), endowed with compatible topologies with respect to a given dual pairing \(\langle \cdot, \cdot \rangle\). We consider a family of proper, lsc, and convex functions \(\Phi_t : X \to \mathbb{R} \cup \{+\infty\}, t \in T\), where \(T\) is an arbitrary index set, together with the associated supremum function
\[\Phi := \sup_{t \in T} \Phi_t.\]

The following theorem provides the main result of this section. Its proof is based on a series of lemmas that we postpone to the Appendix.

**Theorem 5** For every \(\bar{x} \in \text{dom} \Phi\), \(\delta \geq 0\) and \(\lambda \in (-\infty, +\infty]\) we have
\[
N^\delta_{\{(\Phi \leq \lambda) \cap \text{dom} \Phi\} \cup \{(\Phi \leq \Phi(\bar{x}))\}}(\bar{x}) = \limsup_{\sum_{i \in T} \mu_i (\nu - \Phi_{t_i}(\bar{x})) + \varepsilon_i \to \delta, \nu \uparrow \lambda} \sum_{i \in T} \mu_i \partial_{\varepsilon_i} \Phi_{t_i}(\bar{x}). \quad (16)
\]

**Proof.** Let us start with the proof of the inclusion “⊃”. We take \(x^* = \lim_j \sum_{i=1}^{k_j} \mu_{i,j} x_{i,j}^*\)
for \( k_j \in \mathbb{N}, \mu_{i,j} > 0, \varepsilon_{i,j} \geq 0, t_{i,j} \in T \) and \( x_{i,j}^* \in \partial_{\varepsilon_{i,j}} \Phi_{t_{i,j}}(\bar{x}) \) (\( i = 1, k_j \)) such that

\[
\sum_{i=0}^{k_j} \mu_{i,j}(\nu_j - \Phi_{t_{i,j}}(\bar{x}) + \varepsilon_{i,j}) \to \delta,
\]

where \( \nu_j \uparrow \lambda \) (observe that one can take \( \nu_j = \lambda \) when \( \lambda \) is finite).

We fix an element \( u \in [\Phi \leq \lambda] \cap \text{dom} \Phi \); hence, we may suppose that \( \Phi(u) \leq \nu_j \) for all \( j \). Then from the definition of \( x_{i,j}^* \) we get

\[
\langle x_{i,j}^*, u - \bar{x} \rangle \leq \Phi_{t_{i,j}}(u) - \Phi_{t_{i,j}}(\bar{x}) + \varepsilon_{i,j} \leq \nu_j - \Phi_{t_{i,j}}(\bar{x}) + \varepsilon_{i,j}.
\]

Multiplying this last inequality by \( \mu_{i,j} \) and summing up over \( i = 1, k_j \) gives us

\[
\left\langle \sum_{i=1}^{k_j} \mu_{i,j} x_{i,j}^*, u - \bar{x} \right\rangle \leq \sum_{i=0}^{k_j} \mu_{i,j}(\nu_j - \Phi_{t_{i,j}}(\bar{x}) + \varepsilon_{i,j}),
\]

which at the limit yields

\[
\left\langle x^*, u - \bar{x} \right\rangle \leq \delta.
\]

Now we take \( u \in [\Phi \leq \Phi(\bar{x})]_{[\infty}. \) Then for all \( \beta > 0 \) we have \( \bar{x} + \beta u \in \bar{x} + [\Phi \leq \Phi(\bar{x})]_{[\infty} \) and, so, by (17),

\[
\left\langle \sum_{i=1}^{k_j} \mu_{i,j} x_{i,j}^*, \beta u \right\rangle = \left\langle \sum_{i=1}^{k_j} \mu_{i,j} x_{i,j}^*, \bar{x} + \beta u - \bar{x} \right\rangle \leq \sum_{i=0}^{k_j} \mu_{i,j}(\Phi(\bar{x}) - \Phi_{t_{i,j}}(\bar{x}) + \varepsilon_{i,j}),
\]

implying that \( \left\langle \sum_{i=1}^{k_j} \mu_{i,j} x_{i,j}^*, u \right\rangle \leq 0 \). Hence, after taking the the limit on \( j \) we get \( \langle x^*, u \rangle \leq 0 \leq \delta. \) Combining this with (18) yields \( x^* \in N_{[\Phi \leq \lambda]}(\text{dom} \Phi \cup (\bar{x} + [\Phi \leq \Phi(\bar{x})]_{[\infty}) (\bar{x}). \) This proves the inclusion “\( \subset \)”.

To prove the inclusion “\( \supset \)” we pick an element \( \xi \in N_{[\Phi \leq \lambda]}(\text{dom} \Phi \cup (\bar{x} + [\Phi \leq \Phi(\bar{x})]_{[\infty} (\bar{x}). \) We proceed by investigating all the possible values of \( \lambda \) and \( \Phi(\bar{x}) \):

1. \( \Phi(\bar{x}) \leq \lambda < +\infty. \) In this case, the right-hand side in (16) reduces to \( N_{\Phi \leq \lambda}(\bar{x}) \), and it follows from Lemma 23 that \( \xi \in \limsup_{\mu(\lambda - \Phi(\bar{x}) + \varepsilon) \to \delta, \mu > 0} \mu \partial_{\varepsilon} \Phi(\bar{x}). \) Then, for every given \( \eta > 0 \) (small enough) and \( \theta \)-neighborhood \( V \subset Y (V \in \mathcal{N}_\theta(\theta)) \), there exist \( \mu > 0, \varepsilon \geq 0 \) and \( x^* \in \partial_{\varepsilon} \Phi(\bar{x}) \) such that

\[
\xi \in \mu x^* + \frac{1}{2} V, \ |\mu(\lambda - \Phi(\bar{x}) + \varepsilon) - \delta| < \frac{\eta}{2}.
\]

Thus, by Theorem 2 there are \( k \in \mathbb{N}, t_i \in T, \varepsilon_i \geq 0 \) and \( x_i^* \in \partial_{\varepsilon_i} \Phi_{t_i}(\bar{x}) \) (\( i \in \bar{1}, k \)), together with \( \langle \alpha_i \rangle \in \Delta_k \) such that

\[
\varepsilon - \frac{\eta}{2\mu} \leq \sum_{i=1}^{k} \alpha_i(\varepsilon_i - \Phi_{t_i}(\bar{x})) + \Phi(\bar{x}) \leq \varepsilon + \frac{\eta}{2\mu}, \ x^* \in \sum_{i=1}^{k} \alpha_i x_i^* + \frac{1}{2\mu} V.
\]
Therefore, for \( \mu_i := \mu \alpha_i (> 0) \) we obtain that

\[
\sum_{i=1}^{k} \mu_i (\lambda - \Phi_{t_i}(\bar{x}) + \varepsilon_i) = \mu \sum_{i=1}^{k} \alpha_i (\varepsilon_i - \Phi_{t_i}(\bar{x}) + \lambda) \leq \mu (\lambda - \Phi(\bar{x}) + \varepsilon) + \frac{\eta}{2} < \delta + \eta,
\]

\[
\sum_{i=1}^{k} \mu_i (\lambda - \Phi_{t_i}(\bar{x}) + \varepsilon_i) = \mu \sum_{i=1}^{k} \alpha_i (\varepsilon_i - \Phi_{t_i}(\bar{x}) + \lambda) \geq \mu (\lambda - \Phi(\bar{x}) + \varepsilon) - \frac{\eta}{2} > \delta - \eta,
\]

together with \( \xi \in \sum_{i=1}^{k} \mu_i x_i^* + V \). This yields the inclusion “\( \subset \)” in (16).

(2) \( \lambda = +\infty \). In this case, the right-hand side in (16) reduces to \( N_{\text{dom} \Phi}(\bar{x}) \) and so, from Lemma 21, for every given \( \eta > 0 \) (small enough) and \( \theta \)-neighborhood \( V \subset Y \) there exist \( \mu \in (0, \frac{\eta}{2}) \) and \( \varepsilon \geq 0 \) together with \( x^* \in \partial_x \Phi(\bar{x}) \) such that \( \mu \varepsilon \in (\delta - \frac{\eta}{2}, \delta + \frac{\eta}{2}) \) and

\[
\xi \in \mu x^* + V.
\]

We take \( \nu = \eta^{-1} + \Phi(\bar{x}) (> 0) \), so that \( \mu \nu = \frac{\eta}{2} \) and \( |\mu (\nu - \Phi(\bar{x}) + \varepsilon) - \delta| = |\mu \varepsilon - \delta| = \eta \). In other words, we also have (19) in the current case, and the proof follows by arguing as in point (1) above.

(3) \( \Phi(\bar{x}) > \lambda \) : In this case we appeal to Lemma 27, which ensures that \( \xi \in \bigcup_{\mu > 0} \partial_{\mu (\Phi(\bar{x}) - \lambda)} (\mu \Phi)(\bar{x}) = \bigcup_{\mu > 0} \partial_{\frac{\mu}{\mu + \Phi(\bar{x}) - \lambda}} \Phi(\bar{x}) \). Hence, for every \( \theta \)-neighborhood \( V \subset Y \), there exist \( \mu > 0 \) and \( x^* \in \partial_{\frac{\mu}{\mu + \Phi(\bar{x}) - \lambda}} \Phi(\bar{x}) \) such that \( \xi \in \mu x^* + \frac{1}{2} V \). Hence, (19) follows by taking \( \varepsilon = \frac{\delta}{\mu} + \Phi(\bar{x}) - \lambda \), and we proceed as in point (1) above.

The proof of the theorem is complete. \( \blacksquare \)

Let us say some words to explain the elements involved in formula (16); namely, the appealing to the set \( [\Phi \leq \Phi(\bar{x})]_\infty \), and the consideration of the value \( \lambda = +\infty \).

**Remark 1**

(i) It is clear that when \( \Phi(\bar{x}) \leq \lambda < +\infty \), the set \( [\Phi \leq \lambda] \cup (\bar{x} + [\Phi \leq \Phi(\bar{x})]_\infty) \) reduces to the sublevel set \( [\Phi \leq \lambda] \), and so (16) gives the required explicit characterization for the normal cone to \( [\Phi \leq \lambda] \). The resulting formula in this case can be compared, when all the \( \Phi_i \)‘s are all equal, to the one given in [13] (see, also, [15]). Original characterizations for the normal cone to sublevel sets remounts to [24].

(ii) Even with the lack of the nonemptiness of the set \( [\Phi \leq \lambda] \), formula (16) is still meaningful, since the vector \( \bar{x} \) always belongs to \( \bar{x} + [\Phi \leq \Phi(\bar{x})]_\infty \). The other interesting situation covered by (16) is that when \( [\Phi \leq \lambda] \) is non-empty, but \( \bar{x} \notin [\Phi \leq \lambda] \). In this case, the presence of the term \( \bar{x} + [\Phi \leq \Phi(\bar{x})]_\infty \) becomes necessary, since, for otherwise, the normal cone to \( [\Phi \leq \lambda] \) at \( \bar{x} \) can not be defined appropriately.

(iii) If \( \lambda = +\infty \), then \( [\Phi \leq \lambda] \cap \text{dom} \Phi = \text{dom} \Phi \), and the left-hand side in (16) reduces to the normal cone to the domain of \( \Phi \). In this case, the relation \( \lambda_i \uparrow \lambda \) is used to force the term \( \sum_{i \in \mathbb{T}} \mu_i \) to go to +\( \infty \).

(iv) Due to the relation

\[
[\Phi \leq \lambda] \cup (\bar{x} + [\Phi \leq \Phi(\bar{x})]_\infty) = \bigcap_{i \in \mathbb{T}} ([\Phi_t \leq \lambda] \cup (\bar{x} + [\Phi_t \leq \Phi_t(\bar{x})]_\infty)),
\]
formula (16) is indeed a characterization of the normal cone to the arbitrary intersection
\( \cap_{t \in T} [\Phi_t \leq \lambda] \cup (\bar{x} + [\Phi_t \leq \Phi_t(\bar{x})]_{\infty}) \).

We are going now to specify Theorem 5 to certain special cases, which lead to simpler
characterizations of the \( \delta \)-normal set.

Firstly, write formula (16) in its most frequent form, corresponding to \( \lambda = \Phi(\bar{x}) \).

**Corollary 6** For every \( \bar{x} \in \text{dom} \Phi \) and \( \delta \geq 0 \) we have that

\[
N^\delta_{[\Phi \leq \Phi(\bar{x})]}(\bar{x}) = \limsup_{\sum_{t \in T} \mu_t (\Phi(\bar{x}) - \Phi_t(\bar{x})) + \epsilon_i \rightarrow \delta} \sum_{t \in T, k} \mu_t \partial_{\epsilon_i} \Phi_t(\bar{x}),
\]

and, particularly, when \( \Phi \equiv \Phi_t \) for all \( t \in T \),

\[
N^\delta_{[\Phi \leq \Phi(\bar{x})]}(\bar{x}) = \limsup_{\mu \rightarrow \delta, \mu \geq 0} \mu \partial_{\epsilon_i} \Phi(\bar{x}).
\]

**Proof.** This follows easily from Theorem 5 due to the relation \([\Phi \leq \Phi(\bar{x})] = [\Phi \leq \Phi(\bar{x})] \cup (\bar{x} + [\Phi \leq \Phi(\bar{x})]_{\infty})\). ■

Formula (16) takes an algebraic form in the following corollary, giving rise to a known
formula ([15]).

**Corollary 7** For every \( \bar{x} \in \text{dom} \Phi \) and \( \delta \geq 0 \) we have that

\[
N^\delta_{[\Phi \leq \Phi(\bar{x})]}(\bar{x}) = \text{cl} \left( \bigcup_{\mu > 0} \partial_\delta (\mu \Phi)(\bar{x}) \right) \text{ for all } \delta > 0,
\]

and, consequently,

\[
N_{[\Phi \leq \Phi(\bar{x})]}(\bar{x}) = \bigcap_{\delta > 0} \text{cl} \left( \bigcup_{\mu > 0} \partial_\delta (\mu \Phi)(\bar{x}) \right).
\]

**Proof.** Since \([\Phi \leq \Phi(\bar{x})] = [\Phi \leq \lambda] \cup (\bar{x} + [\Phi \leq \Phi(\bar{x})]_{\infty})\), by Theorem 5 we obtain for
all $\delta > 0$

\[
N_\delta^{[\Phi \leq \lambda]}(\bar{x}) = \limsup_{\mu \to \delta, \mu > 0} \mu \partial_\varepsilon \Phi(\bar{x}) \\
= \limsup_{\mu \to \delta, \mu > 0} \partial_\varepsilon (\mu \Phi)(\bar{x}) \\
\subset \limsup_{\nu \to \delta} \bigcup_{\mu > 0} \partial_\varepsilon \left( \frac{\nu}{\delta} \mu \Phi \right)(\bar{x}) \quad \text{(taking $\nu = \mu \varepsilon$)}
\]

\[
= \limsup_{\nu \to \delta} \frac{\nu}{\delta} \left( \bigcup_{\mu > 0} \partial_\varepsilon (\mu \Phi)(\bar{x}) \right)
\]

\[
= \operatorname{cl} \left( \bigcup_{\mu > 0} \partial_\varepsilon (\mu \Phi)(\bar{x}) \right)
\]

\[
\subset \limsup_{\mu \to \delta, \mu > 0} \partial_\varepsilon (\mu \Phi)(\bar{x}) \quad \text{(taking $\varepsilon = \frac{\delta}{\mu}$)}.
\]

When $[\Phi \leq \lambda] \neq \emptyset$ but $\Phi(\bar{x}) > \lambda$ Theorem 5 simplifies to:

**Corollary 8** Given $\bar{x} \in \operatorname{dom} \Phi$ and $\lambda \in \mathbb{R}$ we assume that $[\Phi \leq \lambda] \neq \emptyset$ and $\Phi(\bar{x}) > \lambda$. Then we have

\[
N_{[\Phi \leq \lambda] \cup \{\bar{x} + [\Phi \leq \Phi(\bar{x})]\}}(\bar{x}) = \mathbb{R}_+ \partial_\varepsilon \Phi(\bar{x}).
\]

**Proof.** The inclusion “$\subset$” is immediate from Lemma 20 (see the Appendix), while the converse inclusion follows from Theorem 5. ■

In the following corollary we consider the case in which the sublevel set $[\Phi \leq \lambda]$ is empty.

**Corollary 9** If $\lambda \in \mathbb{R}$ is such that $[\Phi \leq \lambda] = \emptyset$, then for every $\bar{x} \in \operatorname{dom} \Phi$ we have

\[
N_{\bar{x} + [\Phi \leq \Phi(\bar{x})]}(\bar{x}) = \limsup_{\mu \to \lambda - \Phi(\bar{x}) + \varepsilon, \mu > 0} \mu \partial_\varepsilon \Phi(\bar{x})
\]

\[
= \mathbb{R}_+ \partial_\varepsilon \Phi(\bar{x}) \quad \text{(for every $\varepsilon \geq \Phi(\bar{x}) - \lambda$)}.
\]

**Proof.** The first equality is a direct consequence Theorem 5, while the inclusion “$\supset$” in the second equality holds easily. To show the converse inclusion, we fix $\varepsilon \geq \Phi(\bar{x}) - \lambda$ and take $v \in (\partial_\varepsilon \Phi(\bar{x}))^-$. Since $[\Phi \leq \Phi(\bar{x}) - \varepsilon] \subset [\Phi \leq \lambda] = \emptyset$ we deduce that

\[
0 \leq \inf_{t > 0} \Phi(\bar{x} + tv) - \Phi(\bar{x}) + \varepsilon = \Phi'_\varepsilon(\bar{x}; v) = \sup_{x^* \in \partial_\varepsilon \Phi(\bar{x})} \langle v, x^* \rangle \leq 0. \tag{20}
\]
Let \( t_n \) be a minimizing sequence for this last infimum. If \( \bar{t} \in \mathbb{R} \) is an accumulation point of \( t_n \), then the last relation above gives

\[
\Phi(\bar{x} + \bar{t}v) - \Phi(\bar{x}) + \varepsilon \leq \lim \inf_n \Phi(\bar{x} + t_nv) - \Phi(\bar{x}) + \varepsilon \leq 0,
\]

so that \( \bar{x} + \bar{t}v \in [\Phi \leq \lambda] = \emptyset \), a contradiction. Thus we may assume that \( t_n \to +\infty \), so that, by the convexity of \( \Phi \),

\[
\sup_{t>0} \frac{\Phi(\bar{x} + tv) - \Phi(\bar{x})}{t} = \lim \frac{\Phi(\bar{x} + t_nv) - \Phi(\bar{x}) + \varepsilon}{t_n} = \inf_{t>0} \frac{\Phi(\bar{x} + tv) - \Phi(\bar{x}) + \varepsilon}{t} = 0.
\]

This shows that \( \bar{x} + tv \in [\Phi \leq \Phi(\bar{x})] \) for all \( t > 0 \), and then \( v \in (N_{\bar{x} + [\Phi \leq \Phi(\bar{x})]}(\bar{x}))^- \). In other words, \( (\partial \varepsilon \Phi(\bar{x}))^- \subset (N_{\bar{x} + [\Phi \leq \Phi(\bar{x})]}(\bar{x}))^- \) and, so, using the bipolar Theorem, \( N_{\bar{x} + [\Phi \leq \Phi(\bar{x})]}(\bar{x}) \subset [(\partial \varepsilon \Phi(\bar{x}))^-]^- \subset \mathbb{R}_+ \partial \varepsilon \Phi(\bar{x}) \).

The following result puts in clear the different sets composing the right-hand side of (16).

**Corollary 10**

For every \( \bar{x} \in \text{dom} \Phi \) and \( \delta \geq 0 \) we have

\[
N^\delta_{[\Phi \leq \Phi(\bar{x})]}(\bar{x}) = \bigcup_{\mu \geq 0} \partial \varepsilon(\mu \Phi)(\bar{x}) \cup \limsup_{\mu \to 0, \mu \to +\infty} \mu \partial \varepsilon \Phi(\bar{x}),
\]

and, in particular,

\[
N_{[\Phi \leq \Phi(\bar{x})]}(\bar{x}) = \mathbb{R}_+ \partial \varepsilon \Phi(\bar{x}) \cup N_{\text{dom} \Phi}(\bar{x}) \cup \limsup_{\mu \to 0, \mu \to +\infty} \mu \partial \varepsilon \Phi(\bar{x}).
\]

Moreover, if Slater’s condition holds at \( \Phi(\bar{x}) \), then

\[
N^\delta_{[\Phi \leq \Phi(\bar{x})]}(\bar{x}) = \bigcup_{\mu \geq 0} \partial \varepsilon(\mu \Phi)(\bar{x}).
\]

**Proof.** To prove the first statement of the corollary we take \( x^* \in N^\delta_{[\Phi \leq \Phi(\bar{x})]}(\bar{x}) \) and, according to Theorem 5, we let \( \mu_i, \varepsilon_i \geq 0 \) and \( x^*_i \in \partial \varepsilon_i \Phi(\bar{x}) \) such that \( \mu_i \varepsilon_i \to \delta \) and \( \mu_i x^*_i \to x^* \). We may suppose without loss of generality that \( \mu_i \) converges in \( \mathbb{R}_+ \cup \{ \infty \} \).

If \( \mu_i \to \mu \in \mathbb{R} \), by taking the limit on \( i \) in the inequality

\[
\langle \mu_i x^*_i, y - \bar{x} \rangle \leq (\mu_i \Phi)(y) - (\mu_i \Phi)(\bar{x}) + \mu_i \varepsilon_i \quad \text{for all} \quad y \in X,
\]

we get \( \langle x^*, y - \bar{x} \rangle \leq (\mu \Phi)(y) - (\mu \Phi)(\bar{x}) + \delta \). So, \( x^* \in \partial \varepsilon(\mu \Phi)(\bar{x}) \) and the inclusion “\( \subset \)” in the first equality holds. The converse inclusion follows by Theorem 5. For the second
formula it suffices to observe that, when $\delta = 0$,

$$\bigcup_{\mu \geq 0} \partial(\mu \Phi)(\bar{x}) = \mathbb{R}_+ \partial \Phi(\bar{x}) \cup N_{\text{dom}} \Phi(\bar{x}).$$

To prove the last statement of the corollary we only need to verify that the set
$$\limsup_{\mu \to +\infty} \mu \partial \Phi(\bar{x})$$

is empty whenever Slater’s condition holds at $\Phi(\bar{x})$. Otherwise, if this were not the case, then there would exist $\mu_i, \varepsilon_i \geq 0$ such that $\mu_i \varepsilon_i \to \delta$, $\mu_i \to +\infty$, and $\mu_i x_i^* \to x^*$. Then $\varepsilon_i \to 0$ and $x_i^* \to 0$, so that $\theta \in \limsup_{x \to 0} \partial \Phi(\bar{x}) \cap \partial \Phi(\bar{x})$, which contradicts Slater’s assumption. \qed

Before we close this section we consider now the setting of Banach spaces and we ask whether formula (16) is still valid when the lim sup is taken with respect to the norm topology. As expected, the answer is affirmative in reflexive Banach spaces as the following corollary shows:

**Corollary 11** With the notation of Theorem 5 we assume that $(X, \| \|)$ is a reflexive Banach space. Then we have

$$N^\delta_{\{\Phi \leq \Phi(\bar{x})\}}(\bar{x}) = \| \|_\ast \cdot \limsup_{\sum_{t \in T} \mu_t (\Phi(\bar{x}) - \Phi_t(\bar{x}) + \varepsilon_t) \to \delta} \sum_{\mu \geq 1, \lambda \in \Lambda, t \in T, k \in \mathbb{N}} \lambda_k \partial \Phi(\bar{x}) \cdot (\Phi(\bar{x}) - \Phi_t(\bar{x}) + \varepsilon_t),$$

(21)

where $\| \|_\ast$ denotes the dual norm in $X^\ast$.

**Proof.** We consider the pair $(X, X^\ast)$ with $X^\ast$ being the topological dual space of $X$ associated with the norm topology. Assume, for simplicity, that $\Phi \equiv \Phi_t$ for all $t \in T$. So, according to Corollary 6, we only need to check that

$$\limsup_{\mu \to +\infty} \mu \partial \Phi(\bar{x}) \subset \| \|_\ast \cdot \limsup_{\mu \to +\infty} \mu \partial \Phi(\bar{x}).$$

(22)

Also, by Lemma 20 given in the Appendix, it suffices to suppose that $\bar{x} \in \text{argmin} \Phi$. Take $x^\ast$ in the left hand-side and let nets $(\varepsilon_\alpha)_\alpha$, $(\mu_\alpha)_\alpha$ and $(x_\alpha)_\alpha$ be such that $\varepsilon_\alpha \to 0$, $x_\alpha^\ast \in \partial \Phi(\bar{x})$, $\mu_\alpha \varepsilon_\alpha \to \delta$, and $\mu_\alpha x_\alpha^\ast \to x^\ast$. Fix $\eta > 0$ and let $A$ be the set of elements $\alpha$ such that $\mu_\alpha \varepsilon_\alpha \leq \delta + \eta$. Then $x_\alpha^\ast \in \overline{\text{co}} \{\mu_\alpha x_\alpha^\ast\}_{\alpha \in A} = \overline{\text{co}} \{\mu_\alpha x_\alpha^\ast\}_{\alpha \in A}$, due to the reflexivity of $X$, and there exists $x^\ast_\eta := \sum_{\alpha \in A} \lambda_\alpha \mu_\alpha x^\ast_\alpha$ for some $\lambda_\alpha \geq 1$ and $\lambda_\alpha \in \Delta_{\alpha \eta}$, such that $\|x^\ast - x^\ast_\eta\| \leq \eta$. Since $\bar{x} \in \text{argmin} \Phi$, for each $i \in \mathbb{N}$, we have, setting $\mu_\eta := \max_{j \in 1, n_\eta} \lambda_\alpha j$, $\varepsilon_\eta := \frac{\max_{k \in 1, \eta} \lambda_\alpha k \mu_\alpha}{\max_{j \in 1, \eta} \mu_\alpha j}$,

$$\left(\frac{\mu_\alpha}{\mu_\eta}\right) x^\ast_\alpha \in \partial \Phi(\bar{x}) \subset \partial \Phi(\bar{x}) \subset \partial \Phi(\bar{x}) \subset \partial \Phi(\bar{x}) \subset \partial \Phi(\bar{x}) \subset \partial \Phi(\bar{x}) \subset \partial \Phi(\bar{x}),$$

which shows that $x^\ast_\eta \in \mu_\eta \partial \Phi(\bar{x})$. But $\mu_\eta \varepsilon_\eta = \max_{k \in 1, \eta} \mu_\alpha \varepsilon_\alpha \leq \delta + \eta$ and, so, by
choosing a convergent subnet of \((\mu'_n, \epsilon'_n)_n\) we arrive at

\[ x^* \in \| \| \ast - \limsup_{\mu \to \delta, \mu \geq 0} \mu \partial_{\epsilon} \Phi(x). \]

5 Subdifferential approach

As in the previous sections, here we also work with a dual pair \((X, Y)\) of lcs \(X\) and \(Y\), which are endowed with compatible topologies. Our aim is to characterize the normal cone to sublevel sets by using exclusively the exact subdifferential of the nominal function.

We consider in this section some restrictions either on the underlying space, or on the nominal function. We proceed in this way because of the existence in every non-complete normed space of convex proper lsc functions (e.g., [4]), which have empty subdifferential mapping everywhere. For simplicity of the presentation, we only study the normal cone to the sublevel set \([\Phi \leq \Phi(\bar{x})]\).

In what follows, \(\Phi : X \to \mathbb{R} \cup \{+\infty\}\) is a proper lsc convex function defined on the lcs \(X\). First, we recall the definition of epi-pointed functions.

**Definition 1** Function \(\Phi\) is said to be epi-pointed if its conjugate \(\Phi^*\) is finite and Mackey-continuous at least at some point.

The following lemma gathers some useful properties of epi-pointed functions, which can be found in [9, Lemma 2.1.6 in and Theorem 4.2].

**Lemma 12** Assume that function \(\Phi\) is epi-pointed. Then the following assertions hold:

(i) For every \(\varepsilon > 0\) we have

\[ \partial_{\varepsilon} \Phi(x) \cap \text{int}(\text{dom } \Phi^*) = \partial_{\varepsilon} \Phi(x). \]

(ii) Assume that \(x^*_0 \in \partial_{\varepsilon} \Phi(x_0) \cap \text{int}(\text{dom } \Phi^*)\), for some \(\varepsilon \geq 0\) and \(x_0 \in X\). Then for every \(\beta \geq 0\), every continuous seminorm \(p\) in \(X\), and every \(\lambda > 0\), there are \(x_\varepsilon \in X\), \(y^*_\varepsilon \in [p \leq 1]^\circ\) and \(\lambda_\varepsilon \in [-1, 1]\) such that:

\[ p(x_0 - x_\varepsilon) + \beta |\langle x^*_0, x_0 - x_\varepsilon \rangle| \leq \lambda, \quad |\langle x^*_\varepsilon, x_0 - x_\varepsilon \rangle| \leq \varepsilon + \frac{\lambda}{\beta}, \quad |\Phi(x_0) - \Phi(x_\varepsilon)| \leq \varepsilon + \frac{\lambda}{\beta}, \quad \text{and} \]

\[ x^*_\varepsilon := x^*_0 + \frac{\varepsilon}{\lambda} (y^*_\varepsilon + \beta \lambda x^*_0) \in \partial \Phi(x_\varepsilon) \cap \partial_{2\varepsilon} \Phi(x_0) \]

(with the convention that \(\frac{1}{\infty} = 0\)).
Lemma 13 Assume that function $\Phi$ is epi-pointed. Then for every $\bar{x} \in \text{dom } \Phi$

$$N_{[\Phi \leq \Phi(\bar{x})]}(\bar{x}) = \limsup_{x \to \bar{x}, \mu \geq 0} \mu \partial \Phi(x);$$

that is, each $x^* \in N_{[\Phi \leq \Phi(\bar{x})]}(\bar{x})$ is the (weak*-limit) of a net $(\mu_i x^*_i)$, $\mu_i \geq 0$ and $x^*_i \in \partial \Phi(x_i)$, with $x_i \to \bar{x}$, $\mu_i(\Phi(x_i) - \Phi(\bar{x})) \to 0$, and $\mu \langle x^*_i, x - \bar{x} \rangle \to 0$.

Proof. The inclusion “$\supset$” being direct, we are going to prove the inclusion “$\subset$”. Given $\xi \in N_{[\Phi \leq \Phi(\bar{x})]}(\bar{x})$, we fix two open neighborhoods $U \in N_X(\theta)$ and $V \in N^*_X(\theta)$, and choose $W \in N_X(\theta)$ such that $W + W \subset V$. According to Theorem 5, for each $\varepsilon > 0$ there exist $\mu > 0$ and $x^*_0 \in \partial_\ell(\mu \Phi)(\bar{x})$ such that $x^*_0 \in \xi + W$. Let $p_V : X \to \mathbb{R}_+$ denote the Minkowski functional defined as

$$p_V(x) := \inf \left\{ t > 0 \mid x \in tU \right\},$$

which is a continuous seminorm on $X$. Similarly, we define the weak*-continuous seminorm $p_V : X^* \to \mathbb{R}_+$. By Lemma 12, we can assume that $x^*_0 \in \partial_\ell(\mu \Phi)(\bar{x}) \cap \text{int}(\text{dom}(\mu \Phi)^*)$, so that from Lemma 12, applied with $\lambda = \sqrt{\varepsilon}$ and $\beta = 1$, there exist $x_\varepsilon \in X$ and $x^*_\varepsilon \in \partial \Phi(x_\varepsilon)$ such that

$$p_V(\bar{x} - x_\varepsilon) \leq \sqrt{\varepsilon}, \quad \mu x^*_\varepsilon - (1 + \sqrt{\varepsilon})x^*_0 \leq \sqrt{\varepsilon} \sup_{y^* \in U^*} p_V(y^*), \quad (23)$$

$$|\mu \langle x^*_\varepsilon, x_\varepsilon - \bar{x} \rangle | \leq \varepsilon + \sqrt{\varepsilon}, \quad (24)$$

$$|\mu \Phi(x_\varepsilon) - \mu \Phi(\bar{x}) | \leq \varepsilon + \sqrt{\varepsilon}. \quad (25)$$

If $\varepsilon > 0$ is small enough such that $\sqrt{\varepsilon} < 1$, then condition (23) ensures that $x_\varepsilon \in \bar{x} + U$. Moreover, since $x^*_0 - \xi \in W$ we infer that $2p_V(x^*_0 - \xi) \leq p_V(x^*_0 - \xi) \leq 1$. But $U^*$ is weak*-compact, by the Banach-Alaoglu-Bourbaki theorem, and $p_V$ is weak*-continuous, and so the supremum in (24) is finite and we may assume that $p_V(x^*_0 - \mu x^*_\varepsilon) \leq \sqrt{\varepsilon}p_V(x^*_0) + \sqrt{\varepsilon} < \frac{1}{4}$. Consequently,

$$p_V(\mu x^*_\varepsilon - \xi) \leq p_V(x^*_0 - \mu x^*_\varepsilon) + p_V(x^*_0 - \xi) < 1,$$

and we get $\mu x^*_\varepsilon - \xi \in V$. Finally, with the use of conditions (25) and (26) we can easily see that $\xi$ belongs to the right-hand side of the desired formula. 

Corollary 14 With the notation of Lemma 13 we have

$$N_{[\Phi \leq \Phi(\bar{x})]}(\bar{x}) = \limsup_{\mu(\Phi(x) - \Phi(\bar{x})) \to 0, \mu \geq 0} \mu \partial \Phi(x).$$

Proof. Take $x^*$ in the right-hand side, so that $x^* = \lim_i \mu_i x^*_i$ for some $\mu_i \geq 0$ and $(\mu_i, x^*_i) \in \partial \Phi$ such that $\mu_i(\Phi(x_i) - \Phi(\bar{x})) \to 0$ and $\mu_i \langle \cdot, x_i - \bar{x} \rangle \to 0$. Then, given
This entails that $x^* \in N_{[\Phi \leq \Phi(\bar{x})]}(\bar{x})$, whereas Lemma 13 gives us

$$N_{[\Phi \leq \Phi(\bar{x})]}(\bar{x}) = \limsup_{x \to \bar{x}, \mu \geq 0} \mu \partial \Phi(x) \subset \limsup_{\mu(\Phi(x)-\Phi(\bar{x})) \to 0} \mu \partial \Phi(x) \subset N_{[\Phi \leq \Phi(\bar{x})]}(\bar{x}).$$

\[ \Box \]

Up to a little modification in the way that one perturbs the nominal point $\bar{x}$, Lemma 13 above is still valid if the epi-pointedness condition is weakened to the continuity of the conjugate relative to its domain. We obtain then the following result, which gives both 13 above is still valid if the epi-pointedness condition is weakened to the continuity of the conjugate relative to its domain. We obtain then the following result, which gives both primal and dual symmetric condition ensuring the same characterization of the normal cone. One can understand that these conditions preclude the subdifferential mapping from being empty everywhere.

The notation $x \to_A \bar{x}, A \subset X^*$, used below, refers to the convergence of the corresponding equivalence classes in the quotient space $X/A^\perp$.

**Theorem 15** Assume that either $\Phi_{\text{aff}(\text{dom } \Phi)}$ is finite and continuous in $\text{ri}(\text{dom } \Phi) (\neq \emptyset)$ or $\Phi^*_{\text{aff}(\text{dom } \Phi^*)}$ is finite and (Mackey-)continuous in $\text{ri}(\text{dom } \Phi^*) (\neq \emptyset)$. Then for every $\bar{x} \in \text{dom } \Phi$

$$N_{[\Phi \leq \Phi(\bar{x})]}(\bar{x}) = \limsup_{x \to \text{dom } \Phi \ast \bar{x}, \mu \geq 0} \mu \partial \Phi(x) = \limsup_{\mu(\Phi(x)-\Phi(\bar{x})) \to 0} \mu \partial \Phi(x).$$

**Proof.** We assume first that $\Phi^*_{\text{aff}(\text{dom } \Phi^*)}$ is finite and (Mackey-)continuous in the set $\text{ri}(\text{dom } \Phi^*) (\neq \emptyset)$. In view of Lemma 20 we may assume that $\theta \in \partial \Phi(\bar{x})$; otherwise, we are obviously done. Hence, $\theta \in \text{dom } \Phi^* \subset \text{aff}(\text{dom } \Phi^*)$ so that $Y^* := \text{aff}(\text{dom } \Phi^*)$ is a closed subspace of $X^*$. Denote $Y := X/\text{aff}(\text{dom } \Phi^*)^\perp$ (the quotient space) so that the pair $(Y, Y^*)$ becomes a dual pair when endowed with the quotient and trace topologies of $X$ and $X^*$, respectively. Consider the function $\Phi$ defined on $Y$ as $\Phi(\bar{x}) = \Phi(x)$ where $x$ is in the equivalent class of $\bar{x} \in Y$. It can be easily checked that $\Phi \in \Gamma_0(Y)$ and that $(\Phi)^*(y^*) = \Phi^*(y^*)$ for all $y^* \in \text{dom } \Phi^*(= \text{dom}(\Phi^*))$; hence, $\Phi$ is epi-pointed too. Take $x^* \in N_{[\Phi \leq \Phi(\bar{x})]}(\bar{x})$ so that, by Lemma 13, $x^* \in \text{cl}(\text{ri}(\text{dom } \Phi)) \subset \text{cl}(\text{ri}(\text{dom } \Phi^*)) \subset Y^*$ and, hence, $x^* \in N_{[\Phi \leq \Phi(\bar{x})]}(\bar{x})$. By applying again Lemma 13, and also Corollary 14, we get

$$x^* \in \limsup_{\bar{x} \to x^*, \mu \geq 0} \mu \partial \Phi(\bar{x}) = \limsup_{x \to \text{dom } \Phi \ast \bar{x}, \mu \geq 0} \mu \partial \Phi(x) \subset \limsup_{\mu(\Phi(x)-\Phi(\bar{x})) \to 0} \mu \partial \Phi(x).$$

\[ 19 \]
and we conclude.

Assume now that \( \Phi_{|\aff(\dom \Phi)} \) is finite and continuous in \( \ri(\dom \Phi) \). Let \( L \) be a finite-dimensional subspace of \( X \) which contains \( \bar{x} \) and denote by \( \Phi_L \) the restriction of \( \Phi \) to \( L \); hence, \( \Phi^*_L \in \Gamma_0(L^*) \) (\( L^* \) being the dual space of \( L \)) and \( \ri(\dom \Phi^*_L) \neq \emptyset \). From the first part of the proof applied in the dual pair \((L, L^*)\) we get

\[
\eta \leq \limsup_{x \to \dom \Phi^*_L, x \geq \bar{x}} \mu \partial \Phi_L(x).
\]

We pick an \( x^* \in N_{[\Phi \leq \Phi(x)]}(\bar{x}) \); hence, \( x^*_i \in N_{[\Phi \leq \Phi(x)]}(\bar{x}) \). From the last relation above we find \( x_i \in L \ (\subset X) \), \( \mu_i \geq 0 \) and \( \hat{x}_i^* \in \partial \Phi_L(x_i) \) such that \( x^*_i \in \partial \Phi_L(x_i) = w^*\text{-}\lim_i \mu_i \hat{x}_i^*, x_i \to_{\dom \Phi^*_L} \bar{x} \), and

\[
\mu_i (\hat{x}_i^*, x_i - \bar{x}) \to 0, \quad \mu_i (\Phi(x_i) - \Phi(\bar{x})) = \mu_i (\Phi_L(x_i) - \Phi(\bar{x})) \to 0.
\]

By Hahn-Banach we extend \( \hat{x}_i^* \) to \( x_i^* \in X^* \) such that \( x_i^* \mid_L \equiv \hat{x}_i^* \) and the net \( (\mu_i x_i^*)_i \) remains bounded, so, weak* convergent (w.lo.g.). It follows that \( x^*_i \in \partial (\Phi + I_L)(x_i) = \partial \Phi(x_i) + L^\perp \) (e.g., [8]), \( \mu_i (x_i^*, x_i - \bar{x}) \to 0 \) and

\[
x^* \in (w^*\text{-}\lim_i \mu_i x_i^*) + L^\perp.
\]

Finally, since \( (\dom \Phi^*)_L := \{ u^*_L \mid u^* \in \dom \Phi^* \} \subset \dom \Phi^*_L \), the finite-dimensionality of \( L \) implies that \( x_i \to_{\dom \Phi^*} \bar{x} \). In other words, due to the arbitrariness of \( L \), we conclude that \( x^* \in \limsup_{x \to \dom \Phi^*, x \geq \bar{x}} \mu \partial \Phi(x) \), which yields the left inclusion \( \subset \) of the required statement. The other inclusions are easy and have been proved in previous opportunities (see Corollary 14).

It is worth observing that in the Banach setting any convex function can be made epi-pointed via a penalization with the indicator function of a bounded set. Then we obtain the following result, given originally in [6].

**Corollary 16** Assume that \( X \) is Banach.Then for every \( \bar{x} \in \dom \Phi \) we have that

\[
\eta \leq \limsup_{x \to \bar{x}, x \geq \bar{x}} \mu \partial \Phi(x),
\]

where the limit is taken with respect to the weak*-topology.

**Proof.** We consider the dual pair \((X, X^*)\), where \( X^* \) is the topological dual of \( X^* \) endowed with the weak*-topology. Fix a non-zero element \( \xi \in N_{[\Phi \leq \Phi(x)]}(\bar{x}) \) and pick a \( \theta \)-neighborhood \( V \) (with respect to the Mackey-topology in \( X^* \)) together with an \( \eta > 0 \). Next, we choose a weakly compact (and symmetric) convex set \( K \subset X \) such that \( 8K^\circ \subset V \). We set \( \tilde{\Phi} := \Phi + I_{\bar{x} + K} \in \Gamma_0(X) \). It is easy to verify that the conjugate
function $\tilde{\Phi}^*$ is finite and bounded from above on $x_0^* + K^\circ$ for some $x_0^* \in \text{dom} \Phi^*$, and, so, is (Mackey-)continuous on $x_0^* + K$. It follows that $\tilde{\Phi}$ is epi-pointed, by definition, so that Lemma 13 applies and yields

$$\xi \in N_{[\Phi \leq \Phi(x)]}(\bar{x}) \subset N_{[\Phi \leq \Phi(x)] \cap (\bar{x} + K)}(\bar{x}) = N_{[-\Phi \leq \Phi(x)]}(\bar{x}) = \limsup_{x \to \bar{x}, \mu \geq 0} \partial(\mu \tilde{\Phi})(x).$$

Thus, for any $\theta$-neighborhood $U \in \mathcal{N}_X(\theta)$, there exist $\mu_i > 0$, $\hat{x}_i \in \bar{x} + U$, and $\hat{x}_i^* \in \partial(\mu_i \Phi)(\hat{x}_i)$ such that

$$\xi \in \hat{x}_i^* + \eta K^\circ, \ |\mu_i(\Phi(\hat{x}_i) - \Phi(\bar{x}))| \leq \eta,$$

$$|\langle \hat{x}_i^*, \hat{x}_i - \bar{x} \rangle| \leq \eta;$$

in particular, we have that

$$\hat{x}_i \in \bar{x} + U \cap K.$$  \hfill (27)

On another hand, by the sum rule in [25], for each $i$ we find

$$x_i \in \hat{x}_i + U, \ x_i^* \in \partial(\mu_i \Phi)(x_i),$$

$$y_i^* \in \partial_\eta(\mu_i 1_{x_i + K})(\hat{x}_i) = \mu_i N^{\eta \mu_i^{-1}}_{x_i + K}(\hat{x}_i)$$

such that

$$\hat{x}_i^* \in x_i^* + y_i^* + \eta K^\circ,$$

$$|\langle x_i^*, x_i - \hat{x}_i \rangle| \leq \eta,$$

$$\max\{|\mu_i(\Phi(\hat{x}_i) - \Phi(x_i))|, |\langle x_i^*, \hat{x}_i - x_i \rangle|\} \leq \eta;$$

hence, in particular,

$$\{ |\mu_i(\Phi(x_i) - \Phi(\bar{x}))|, |\langle x_i^*, x_i - \bar{x} \rangle| \} \leq 2\eta.$$  \hfill (34)

Now, for every $i$ and $z \in K$ it holds

\[
\begin{aligned}
\langle y_i^*, z \rangle & \leq \eta + \langle y_i^*, \hat{x}_i - \bar{x} \rangle \\
& = \eta + \langle x_i^* + y_i^* - x_i^*, \hat{x}_i - \bar{x} \rangle + \langle \hat{x}_i^*, \hat{x}_i - \bar{x} \rangle - \langle x_i^*, \hat{x}_i - \bar{x} \rangle \\
& \leq \eta + \eta \gamma K^\circ(\hat{x}_i - \bar{x}) + \langle \hat{x}_i^*, \hat{x}_i - \bar{x} \rangle - \langle x_i^*, \hat{x}_i - \bar{x} \rangle \\
& \leq \eta + \eta + \eta - \langle x_i^*, \hat{x}_i - \bar{x} \rangle \\
& \leq 3\eta - \langle x_i^*, \hat{x}_i - \bar{x} \rangle \quad \text{by (31)}
\end{aligned}
\]

\[
\begin{aligned}
& \leq 3\eta - \langle x_i^*, \hat{x}_i - \bar{x} \rangle + \langle x_i^*, \bar{x} - x_i \rangle \\
& \leq 3\eta + \eta + \langle x_i^*, \bar{x} - x_i \rangle \quad \text{by (33)}
\end{aligned}
\]

\[
\begin{aligned}
& \leq 4\eta + (\mu_i \Phi)(\bar{x}) - (\mu_i \Phi)(x_i) \\
& \leq 4\eta + 2\eta = 6\eta \quad \text{by (30)}
\end{aligned}
\]
so that

\[ y_i^* \in 6\eta K^\circ; \quad (35) \]

hence, it also follows from the seventh inequality in the table above that

\[ \langle x_i^*, x_i - \bar{x} \rangle \leq 4\eta + \sigma_K(y_i^*) \leq 10\eta. \quad (36) \]

Consequently, using successively (27), (32), together with the choice of \( V \) in the beginning of the proof, (35) gives us

\[ \xi \in \hat{x}_i^* + \eta K^\circ \subset x_i^* + y_i^* + 2\eta K^\circ \subset x_i^* + 8\eta K^\circ \subset x_i^* + V. \quad (37) \]

Finally, since \( x_i^* \in \partial(\mu_i \Phi)(x_i) \) (recall (30)) and we have that

\[ x_i \in \hat{x}_i + U \subset \bar{x} + U + U, \] by (30) and (29),

\[ |\mu(\Phi(x_i) - \Phi(\bar{x}))| \leq 2\eta, \] by (34),

\[ -2\eta \leq \mu(\Phi(x_i) - \Phi(\bar{x})) \leq \langle x_i^*, x_i - \bar{x} \rangle \leq 10\eta, \] by (36), (30), and (34),

together with (34), relation (37) leads us to \( \xi \in \limsup_{x \to \bar{x}, \mu > 0} \mu \partial \Phi(x) + V. \) The desired inclusion follows then by the arbitrariness of \( V. \)

**Remark 2** It is possible to obtain Corollary 16 directly from Lemma 13 by applying Borwein’s version of Brøndsted-Rockafellar’s theorem. Nevertheless, our approach permits to highlight the generality of Lemma 13 (and Theorem 15) in the sense that the epi-pointedness condition is not so restrictive as it may appear from a first glance. Let us, for completeness, give the direct proof:

Take \( x^* \) in \( N_{[\Phi \leq \Phi(\bar{x})]}(\bar{x}) \). By Corollary 7 for each \( \delta > 0 \) we have

\[ x^* \in N_{[\Phi \leq \Phi(\bar{x})]}(\bar{x}) = \bigcup_{\mu > 0} \partial_\delta(\mu \Phi)(\bar{x})^{\mu^*}. \quad (38) \]

Thus, for every weak* neighborhood \( W \) of \( x^* \), there are some \( \mu > 0 \) and \( x_0^* \in \partial_\delta(\mu \Phi)(\bar{x}) \cap W \). By Brøndsted-Rockafellar’s-like Theorem ([21]) applied to the function \( f := \mu \Phi \), we find \( x_\delta \in X \) and \( y_\delta^* = \mu x_\delta^* \in \mu \partial \Phi(x_\delta) = \partial(\mu \Phi)(x_\delta) \) such that

\[ \|x_\delta - \bar{x}\| \leq \sqrt{\delta}, \]

\[ \|x_\delta^* - x_0^*\| \leq \sqrt{\delta(1 + \sqrt{\delta})}, \]

\[ |\mu \langle x_\delta^*, x_\delta - \bar{x} \rangle| \leq \delta + \sqrt{\delta}, \]

\[ |\mu \Phi(x_\delta) - \mu \Phi(\bar{x})| \leq \delta + \sqrt{\delta}. \]

Since \( W \) is also open in the norm topology \( \tau_{\| \cdot \|_*} \), by taking \( \delta \) small enough if necessary,
the second inequality above guarantees that \( x^*_\delta \in W \). So, we get

\[
x^* \in w^* - \limsup_{\mu(\Phi(x) - \Phi(\bar{x}) \to 0} \frac{\mu \partial \Phi(x)}{\mu \geq 0}.
\]

6 Spectral functions

In this last section, we apply Theorems 5 and 15 to derive characterizations of the normal cone to sublevel sets of a (proper, convex and lsc) spectral function, by means of its restriction to the range of the eigenvalues vectors.

Here, we identify \( X \) to the Euclidean space of \( n \times n \) symmetric matrices with coefficients in \( \mathbb{R} \), \( S^n(\mathbb{R}) \), which is endowed with the trace inner product \( \langle X, Y \rangle := \text{tr}(XY) \). We denote by \( S^n_-(\mathbb{R}) \) the cone of semi-definite negative matrices.

An extended real-valued matrix function \( F: S^n(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\} \) is said to be spectral if it depends only on the eigenvalues of the corresponding matrix; that is,

\[
F(A) = F(U^T A U),
\]

for any \( A \in S^n(\mathbb{R}) \) and any orthogonal matrix \( U \) (i.e., \( U \in O^n(\mathbb{R}) \)); the superscript \( T \) denotes the transpose matrix. We shall denote by \( \mathcal{D} \) the set of (continuous) linear transformations \( A_U: \mathbb{R}^n \to S^n(\mathbb{R}), U \in O^n(\mathbb{R}) \), given by \( A_U x := U^T \text{diag}(x) U \), where \( \text{diag}(x) \) is the diagonal matrix whose main diagonal is formed by the elements of vector \( x \).

If \( F \in \Gamma_0(S^n(\mathbb{R})) \), then (see [16]) there is a symmetric function \( f \in \Gamma_0(\mathbb{R}^n) \) such that \( F = f \circ \lambda \), where \( \lambda(A) := (\lambda_1(A), \cdots, \lambda_n(A)) \) is the spectra of \( A \) arranged in a non-increasing order. We know that

\[
(f \circ \lambda)^* = f^* \circ \lambda, \tag{39}
\]

and, consequently, for every \( X \in S^n(\mathbb{R}) \) and \( \varepsilon \geq 0 \) it holds that

\[
\partial_{\varepsilon}(f \circ \lambda)(X) = A_U \partial_{\varepsilon} f(\lambda(X)),
\]

where \( U \in O^n(\mathbb{R}) \) satisfies \( X = A_U \lambda(X) \). Relation (39) also yields the following comparison between the recession functions of \( f \) and \( f \circ \lambda \).

**Lemma 17** Given a symmetric function \( f \in \Gamma_0(\mathbb{R}^n) \), we have \( (f \circ \lambda)^\infty = f^\infty \circ \lambda \).

**Proof.** Since both functions \( f \) and \( f \circ \lambda \) are proper, convex and lsc, (39) gives us
Proof. By applying Theorem 5 twice we obtain
\[(f \circ \lambda)\infty = \sigma_{\text{dom}(f \circ \lambda)}^* = \sigma_{\text{dom}(f^* \circ \lambda)}^*\] and \(f^\infty \circ \lambda = \sigma_{\text{dom} f^* \circ \lambda}\). But
\[
\text{dom}(f^* \circ \lambda) = \{X \in S^n(\mathbb{R}) \mid \lambda(X) \in \text{dom } f^*\} = \{X \in A_U \text{dom } f^*, U \in O^n(\mathbb{R}) \text{ st. } A_U \lambda(X) = X\} = \{A_U \text{dom } f^* \mid U \in D\},
\]
and, so, using Von Neumann’s trace inequality (see, e.g., [16, Theorem 2.1]), for every \(X \in S^n(\mathbb{R})\)
\[
(f \circ \lambda)^\infty(X) = \sigma_{A_U \text{dom } f^*, U \in D}(X) = \sup_{y \in \text{dom } f^*} \sup_{U \in O^n(\mathbb{R})} \langle A_U y, X \rangle = \sup_{y \in \text{dom } f^*} (y, \lambda(X)) = (f^\infty \circ \lambda)(X).
\]

We have the following result:

**Corollary 18** Given a symmetric function \(f \in \Gamma_0(\mathbb{R}^n)\) and \(\bar{X} \in \text{dom}(f \circ \lambda)\), we choose a matrix \(U \in O^n(\mathbb{R})\) such that \(\bar{X} = A_U \lambda(\bar{X})\). Then for every \(\delta \geq 0\) and \(\alpha \in ]-\infty, +\infty]\) we have
\[
N^\delta_{\{f \circ \lambda \leq \alpha\} \cap \text{dom}(f \circ \lambda)}(\bar{X}) = A_U N^\delta_{\{f \leq \alpha\} \cap \text{dom } f \cup (f^\infty \circ \lambda \leq 0)}(\lambda(\bar{X})) = A_U \limsup_{\mu(\beta - f(\lambda(\bar{X})) + \epsilon) \to \delta} \mu \partial_\epsilon f(\lambda(\bar{X})).
\]

In addition, if \(\delta = 0\) and \(f(\lambda(\bar{X})) = \alpha\), then we also have
\[
N_{\{f \circ \lambda \leq f(\lambda(\bar{X}))\}}(\bar{X}) = A_U \limsup_{x \to \lambda(\bar{X})} \mu \partial_\epsilon f(x).
\]

**Proof.** By applying Theorem 5 twice we obtain
\[
N^\delta_{\{f \circ \lambda \leq \alpha\} \cap \text{dom}(f \circ \lambda)}(\bar{X}) = \limsup_{\mu(\beta - f(\lambda(\bar{X})) + \epsilon) \to \delta} \mu \partial_\epsilon (f \circ \lambda)(\bar{X}) = \limsup_{\mu(\beta - f(\lambda(\bar{X})) + \epsilon) \to \delta} \mu A_U \partial_\epsilon f(\lambda(\bar{X})) = A_U \limsup_{\mu(\beta - f(\lambda(\bar{X})) + \epsilon) \to \delta} \mu \partial_\epsilon f(\lambda(\bar{X})) = A_U N^\delta_{\{f \leq \alpha\} \cap \text{dom } f \cup (f^\infty \circ \lambda \leq 0)}(\lambda(\bar{X})).
\]
which yields the first statement of the corollary. The last statement follows from Corollary 16 in a similar way. ■

In particular, when \( \alpha = f(\lambda(\bar{X})) \in \mathbb{R} \) and \( \delta = 0 \), the previous characterization gives us

\[
N_{[f \circ \lambda \leq f(\lambda(\bar{X}))]}(\bar{X}) = A_U N_{[f \leq f(\lambda(\bar{X}))]}(\lambda(\bar{X})) = A_U \limsup_{\mu \varepsilon \rightarrow 0, \mu \geq 0} \mu \partial_{\varepsilon} f(\lambda(\bar{X})),
\]

where \( U \in \mathcal{O}^n(\mathbb{R}) \) is such that \( \bar{X} = A_U \lambda(\bar{X}) \). The first equality can be easily obtained from a more general result given in [17, Corollary 5.3], by observing that the set \( C := [f \circ \lambda \leq f(\lambda(\bar{X}))] \) is an invariant set (see [17, Corollary 5.3]). We can proceed similarly when \( [f \circ \lambda \leq \alpha] = \emptyset \), since that \( N_{([f \circ \lambda \leq \alpha] \cap \text{dom}(f \circ \lambda)) \cup (\bar{X} + [f \circ \lambda \leq 0])}(\bar{X}) = N_{[f \circ \lambda \leq 0]}(0) \) and the set \( [f \circ \lambda \leq 0] \) is invariant too. However, this argument cannot be used, at least in an immediate way, to get the first equality in Corollary 18 when the set \( [f \circ \lambda \leq \alpha] \) is not empty and \( f(\lambda(\bar{X})) > \alpha \), in which case the set \( ([f \circ \lambda \leq \alpha] \cap \text{dom}(f \circ \lambda)) \cup (\bar{X} + [f \circ \lambda \leq 0]) \) is not necessarily invariant.

We consider in the following corollary the special and typical example of the largest eigenvalue function, \( \lambda_{\text{max}}(X) := \max_{i \in \Gamma_n} \lambda_i(X) \).

**Corollary 19** Given an \( \bar{X} \in S^m(\mathbb{R}) \) we choose a \( U \in \mathcal{O}^n(\mathbb{R}) \) such that \( \bar{X} = A_U \lambda(\bar{X}) \). Then for every \( \delta \geq 0 \) and \( \alpha \in \mathbb{R} \)

\[
N_{\lambda_{\text{max}} \leq \alpha}^\delta(\bar{X}) = \frac{A_U}{\sum_{i=1}^n \mu_i(\alpha - \lambda_{\text{max}}(\bar{X}) + \varepsilon) \rightarrow \delta \mu_i \geq 0} \left( \mu_1 \lambda_1(\bar{X}), \cdots, \mu_n \lambda_n(\bar{X}) \right)^T.
\]

**Proof.** Since \( \lambda_{\text{max}} = f \circ \lambda \), with \( f(x_1, \cdots, x_n) := \max\{x_1, \cdots, x_n\} \ (\in \Gamma_0(\mathbb{R}^n)) \), Lemma 17 ensures that \( \lambda_{\text{max}}(X) = f^\infty \circ \lambda = f \circ \lambda = \lambda_{\text{max}}(X) \). Then \( [f^\infty \circ \lambda \leq 0] = S_n(\mathbb{R}) \) and so, according to Corollary 18,

\[
N_{\lambda_{\text{max}} \leq \alpha}^\delta(\bar{X}) = N_{\lambda_{\text{max}} \leq \alpha}^\delta(\bar{X}) = A_U \limsup_{\mu(\alpha - f(\lambda(\bar{X})) + \varepsilon) \rightarrow \delta \mu \geq 0} \mu \partial_{\varepsilon} f(\lambda(\bar{X})).
\]

Thus, Corollary 4 leads us to

\[
N_{\lambda_{\text{max}} \leq \alpha}^\delta(\bar{X}) = A_U \limsup_{\mu(\alpha - \lambda_{\text{max}}(\bar{X}) + \varepsilon) \rightarrow \delta \mu_i \geq 0} \left( \mu_1 \gamma_1(\bar{X}), \cdots, \mu_n \gamma_n(\bar{X}) \right)^T
\]

\[
= A_U \limsup_{\sum_{i=1}^n \mu_i(\alpha - \lambda_{\text{max}}(\bar{X}) + \varepsilon) \rightarrow \delta \mu_i \geq 0} \left( \mu_1 \lambda_1(\bar{X}), \cdots, \mu_n \lambda_n(\bar{X}) \right)^T
\]

\[
\sum_{i=1}^n \mu_i \lambda_i(\bar{X}) - \lambda_{\text{max}}(\bar{X}) \varepsilon \geq 0
\]

■
7 A Technical Appendix: Auxiliary Results and Proofs

In this appendix we report the auxiliary results which were needed for the proof of Theorem 5. Recall that \((X, Y)\) denotes a dual pair of (real) vector spaces with an associated separating bilinear form (dual pairing) denoted by \(\langle \cdot, \cdot \rangle\), so that \(X\) and \(Y\) are endowed with compatible topologies.

In what follows we fix a lsc convex proper function \(\Phi : X \to \mathbb{R} \cup \{+\infty\}\), together with elements \(\bar{x} \in \text{dom } \Phi\), \(\delta \geq 0\), and \(\lambda \in [\infty, +\infty]\).

The following lemmas give estimations for \(N_\delta[\Phi \leq \lambda](\bar{x})\) under different conditions. The first one uses Slater’s condition at \(\lambda\), which means that for some \(x_0 \in X\) we have \(\Phi(x_0) < \lambda\).

Lemma 20 If \(\Phi(\bar{x}) \leq \lambda < +\infty\) and Slater’s condition holds at \(\lambda\), then
\[N_\delta[\Phi \leq \lambda](\bar{x}) \subset \bigcup_{\mu \geq 0} \partial_{\delta+\mu(\Phi(\bar{x})-\lambda)}(\mu \Phi)(\bar{x}).\]

**Proof.** Given \(\xi \in N_\delta[\Phi \leq \lambda](\bar{x})\), we define the proper lsc convex function \(\varphi : X \to \mathbb{R} \cup \{+\infty\}\) as
\[\varphi(x) := \max\{\Phi(x) - \lambda, \delta - \langle \xi, x - \bar{x} \rangle\}.
\]
From the definition of the \(\delta\)-normal set we have that \(\varphi(x) \geq 0\) for all \(x \in X\). Thus, since \(\varphi(\bar{x}) = \delta\), it follows that \(\bar{x}\) is a \(\delta\)-minimum of \(\varphi\) and we get, according to Corollary 4,
\[
\theta \in \partial_\delta \varphi(\bar{x}) = \bigcup_{\eta \leq \alpha \delta + (1-\alpha)(\Phi(\bar{x})-\lambda)} \partial_\eta((1-\alpha)(\Phi - \lambda) + \alpha(\delta - \xi + \langle \xi, \bar{x} \rangle))(\bar{x}).
\]
In other words, there exist \(\alpha \in [0, 1]\) and \(\eta \in [0, \alpha \delta + (1-\alpha)(\Phi(\bar{x})-\lambda)]\) such that \(\alpha \xi \in \partial_\eta(1-\alpha)(\Phi(\bar{x})-\lambda)\). If \(\alpha = 0\), then \(\eta = 0\) (as \(\Phi(\bar{x}) \leq \lambda\)) and we get \(\theta \in \partial \Phi(\bar{x})\), which contradicts Slater’s condition. So, \(\alpha > 0\) and the number \(\mu := \frac{1-\alpha}{\alpha}\) (\(\geq 0\)) is well-defined and satisfies \(\frac{\mu}{\alpha} \leq \delta + \mu(\Phi(\bar{x}) - \lambda)\), together with
\[
\xi \in \frac{1}{\alpha} \partial_\eta(1-\alpha)(\Phi(\bar{x}) \subset \partial_{\frac{\mu}{\alpha}}((1-\alpha)\alpha^{-1})(\Phi(\bar{x}) \subset \partial_{\delta+\mu(\Phi(\bar{x})-\lambda)}(\mu \Phi)(\bar{x}).
\]

The following lemma deals with the case \(\lambda = +\infty\).
Lemma 21 We have that

\[ N_{\text{dom } \Phi \cup (\bar{x} + [\Phi \leq \Phi(\bar{x})]_{\infty})}^\delta(\bar{x}) = N_{\text{dom } \Phi}^\delta(\bar{x}) \subset \limsup_{\mu \to \delta, \varepsilon \geq 0, \mu \downarrow 0} \mu \partial \varepsilon \Phi(\bar{x}). \]

**Proof.** The first equality is clear because \( \text{dom } \Phi \cup (\bar{x} + [\Phi \leq \Phi(\bar{x})]_{\infty}) = \text{dom } \Phi \). To verify the inclusion we take \( \xi \in N_{\text{dom } \Phi}^\delta(\bar{x}) \). We choose \( \theta \)-neighborhood \( V \subset Y \) and \( n_0 \geq 1 \) such that \( \bar{x}, 2\bar{x} \in V^\circ \) and

\[ \Phi(y) - \Phi(\bar{x}) \geq -n \quad \text{for all } y \in V^\circ \text{ and all } n \geq n_0; \]

the existence of such a \( V \) is due to the continuity of the dual pairing, while \( n_0 \) comes from the weak lower semicontinuity of \( \Phi \) and the weak compactness of \( V^\circ \). Fix \( n \geq n_0 \).

Hence, for all \( y \in V^\circ \cap \text{dom } \Phi \) and \( \mu \in (0, \frac{\delta}{n}) \) we get

\[ \langle \xi, y - \bar{x} \rangle \leq \delta \leq \mu(\Phi(y) - \Phi(\bar{x}) + n) + \delta. \]

The last inequality being also true when \( y \in V^\circ \setminus \text{dom } \Phi \), by using the sum rule for the approximate subdifferential (see, e.g., [14]) we obtain that

\[ \xi \in \partial_{\delta + \mu n} (\mu \Phi + I_{V^\circ})(\bar{x}) \subset \partial_{\delta + 2\mu n}(\mu \Phi)(\bar{x}) + \partial_{\delta + 2\mu n} I_{V^\circ}(\bar{x}) + V. \]

Since \( \bar{x}, 2\bar{x} \in V^\circ \) we easily verify that \( \partial_{\delta + 2\mu n} I_{V^\circ}(\bar{x}) \subset 2(\delta + 2\mu n)V \subset 2\delta(1 + \frac{2}{n})V \), and we get

\[ \xi \in \partial_{\delta(1 + \frac{2}{n})}(\mu \Phi)(\bar{x}) + 2\delta(1 + \frac{2}{n})V + V = \mu \partial_{\delta + 1}(\phi(\bar{x}) + 2\delta(1 + \frac{2}{n})V + V, \]

which proves that \( \xi \in \limsup_{\mu \to \delta, \varepsilon \geq 0, \mu \downarrow 0} \mu \partial \varepsilon \Phi(\bar{x}). \)

Lemma 22 Assume that \( \Phi(\bar{x}) \leq \lambda < +\infty \). Then

\[ N_{\Phi(\leq \lambda)}^\delta(\bar{x}) = \bigcap_{\alpha > \delta} \cl \left( \bigcup_{\alpha > \gamma > 0} N_{\Phi(\leq \lambda + \gamma)}(\bar{x}) \right). \]

**Proof.** We are going to apply Corollary 3 to the (proper, lsc, and convex) functions \( \varphi_n := I_{[\Phi(\leq \lambda + \frac{1}{n})]} \); \( n \geq 1 \). It is clear that \( (\varphi_n) \) non-decreases as \( n \) goes to \( +\infty \) to the function \( \varphi := I_{[\Phi(\leq \lambda)]} \). It is also clear, since \( \varphi_n(\bar{x}) = \varphi(\bar{x}) = 0 \), that for each \( \alpha > 0 \) the sequence of sets \( \partial_{\alpha} \varphi_n(\bar{x}) \) is non-decreasing. Hence, Corollary 3 applies and yields

\[ \partial_{\delta} \varphi(\bar{x}) = \bigcap_{\alpha > \delta} \limsup_{n \to +\infty} \partial_{\alpha} \varphi_n(\bar{x}) = \bigcap_{\alpha > \delta} \cl \left( \bigcup_{n \geq 1} \partial_{\alpha} \varphi_n(\bar{x}) \right). \]
Consequently, we write

\[ N^\delta_{[\Phi \leq \lambda]}(\bar{x}) = \partial \varphi(\bar{x}) = \bigcap_{\alpha > \delta} \text{cl} \left( \bigcup_{n \geq 1} \partial \varphi_n(\bar{x}) \right) = \bigcap_{\alpha > \delta} \text{cl} \left( \bigcup_{n \geq 1} N^\alpha_{[\Phi \leq \lambda + \frac{1}{n}]}(\bar{x}) \right) = \bigcap_{\alpha > \delta} \text{cl} \left( \bigcup_{\gamma > 0} N^\alpha_{[\Phi \leq \lambda + \gamma]}(\bar{x}) \right) , \]

as we wanted to prove. ■

**Lemma 23** If \( \Phi(\bar{x}) \leq \lambda < +\infty \), then

\[ N^\delta_{[\Phi \leq \lambda]}(\bar{x}) \subset \limsup_{\mu(\lambda - \Phi(\bar{x}) + \varepsilon) \to \delta, \mu > 0, \varepsilon \geq 0} \mu \partial \varepsilon \Phi(\bar{x}). \]

**Proof.** Take \( \xi \in N^\delta_{[\Phi \leq \lambda]}(\bar{x}) \). Suppose first that Slater’s condition holds at \( \lambda \). Then, by Lemma 20, there exists \( \mu \geq 0 \) such that

\[ \xi \in \partial \delta + \mu(\Phi(\bar{x}) - \lambda) (\mu \Phi)(\bar{x}). \]

Hence, we are done whenever \( \mu > 0 \). Otherwise, if \( \mu = 0 \), then the last relation reads

\[ \xi \in \partial \delta (0 \Phi)(\bar{x}) = N^\delta_{\text{dom} \Phi}(\bar{x}), \]

and Lemma 21 gives us

\[ \xi \in \limsup_{\mu \varepsilon \to 0, \mu \varepsilon \geq 0} \mu \partial \varepsilon \Phi(\bar{x}). \]

Suppose now that we don’t have Slater’s condition at \( \lambda \), so that \( \Phi(\bar{x}) = \lambda \leq \Phi(x) \) for all \( x \in X \), and \( \bar{x} \) is a minimum of \( \Phi \). Let us pick a \( \theta \)-neighborhood \( V \subset Y \). Since \( I_{[\Phi \leq \Phi(\bar{x})]} = \sup_{n \geq 1} I_{[\Phi \leq \Phi(\bar{x}) + \frac{1}{n}]} \) and the sequence \( (I_{[\Phi \leq \Phi(\bar{x}) + \frac{1}{n}]} \) is non-decreasing, by Lemma 22 we get

\[ N^\delta_{[\Phi \leq \Phi(\bar{x})]}(\bar{x}) = \bigcap_{\varepsilon > 0} \text{cl} \left( \bigcup_{n \in \mathbb{N}} N^\delta_{[\Phi \leq \Phi(\bar{x}) + \frac{1}{n}]}(\bar{x}) \right) . \]

Hence, for every \( \varepsilon > 0 \) small enough there exists \( n_0 \in \mathbb{N} \) such that \( \xi \in N^\delta_{[\Phi \leq \Phi(\bar{x}) + \frac{1}{n}]}(\bar{x}) + V \) for all \( n \geq 1 \). As Slater’s condition obviously holds at \( \Phi(\bar{x}) + \frac{1}{n} \), by Lemma 20 we obtain that

\[ \xi \in \bigcup_{\mu \geq 0} \partial_{\delta + \varepsilon - \frac{\varepsilon}{\mu}} (\mu \Phi)(\bar{x}) + V \subset \bigcup_{\mu \geq 0} \partial_{\delta + \varepsilon} (\mu \Phi)(\bar{x}) + V. \]

But \( \theta \in \partial \Phi(\bar{x}) \), since Slater’s condition does not hold at \( \lambda \), and so

\[ \xi \in \bigcup_{\mu \geq 0} \partial_{\delta + \varepsilon} (\mu \Phi)(\bar{x}) + V = \bigcup_{\mu \geq 0} \mu \partial_{\delta + \varepsilon} \Phi(\bar{x}) + V \subset \bigcup_{\mu \geq 0} \mu \partial_{\delta + \varepsilon} \Phi(\bar{x}) + V. \]

The conclusion follows then from the arbitrariness of \( \varepsilon > 0 \) and \( V \). ■
In Lemma 27 below we analyze the case where $\bar{x}$ does not belong to the set $[\Phi \leq \lambda]$. We shall need the following three lemmas, which may have their own interest, with the objective to prove that the operator $S$ given in Lemma 25 below is maximal monotone.

**Lemma 24** Fix $v \in X$. The function $\varepsilon \to R(\varepsilon) := \Phi_1'(\bar{x};v)$ is non-decreasing and continuous on $\mathbb{R}_+$. 

**Proof.** Take $0 \leq \varepsilon_0 < \varepsilon_1$. If $f(t) := \Phi(\bar{x} + tv) - \Phi(\bar{x})$, $t > 0$, then $t^{-1}(f(t) + \varepsilon_0) < t^{-1}(f(t) + \varepsilon_1)$ and, so, $R(\varepsilon_0) \leq R(\varepsilon_1)$; that is, $R$ is non-decreasing, and

$$
\lim_{\varepsilon \downarrow 0} R(\varepsilon) = \inf_{\varepsilon > 0} R(\varepsilon) = \inf_{t > 0} \inf_{\varepsilon > 0} \frac{f(t) + \varepsilon}{t} = \inf_{t > 0} \frac{f(t) + \varepsilon_0}{t} = R(\varepsilon_0),
$$

showing that $R$ is right-continuous at $\varepsilon_0$. If $\varepsilon_0 > 0$, given $n \geq 1$ and $\varepsilon < \varepsilon_0$ we choose $t_\varepsilon > 0$ such that $R(\varepsilon) \geq \frac{f(t_\varepsilon) + \varepsilon}{t_\varepsilon} - \frac{1}{n}$. Then there is some $\gamma > 0$ such that $t_\varepsilon \geq \gamma$ for all $\varepsilon \in (0, \varepsilon_0)$ close enough to $\varepsilon_0$; such a $\gamma$ exists because for otherwise, the relation $\lim_{\varepsilon \uparrow \varepsilon_0} t_\varepsilon = 0$ would yield the following contradiction (as $f$ is lsc at 0 and $f(0) = 0$)

$$
+\infty > R(\varepsilon_0) \geq \lim_{\varepsilon \uparrow \varepsilon_0} \inf R(\varepsilon) \geq \liminf_{\varepsilon \uparrow \varepsilon_0} \frac{f(t_\varepsilon) + \varepsilon}{t_\varepsilon} - \frac{1}{n} = +\infty.
$$

Now, writing

$$
R(\varepsilon) \geq \frac{f(t_\varepsilon) + \varepsilon}{t_\varepsilon} - \frac{1}{n} = \frac{f(t_\varepsilon) + \varepsilon_0}{t_\varepsilon} + \frac{\varepsilon - \varepsilon_0}{t_\varepsilon} - \frac{1}{n} \geq R(\varepsilon_0) + \frac{\varepsilon - \varepsilon_0}{\gamma} - \frac{1}{n},
$$

we infer that $R(\varepsilon_0) \geq \lim_{\varepsilon \uparrow \varepsilon_0} R(\varepsilon) \geq R(\varepsilon_0) - \frac{1}{n}$. Thus, the continuity of $R$ at $\varepsilon_0$ follows as $n$ goes to $+\infty$. ■

**Lemma 25** Given $v \in X$, we define the set-valued mapping $S : \mathbb{R} \to \mathbb{R}$ as $S(\varepsilon) := \emptyset$ for $\varepsilon < 0$, and

$$
S(\varepsilon) := \{- \lim_{n \to +\infty} t_n^{-1} \mid \lim_{n \to +\infty} t_n^{-1} (\Phi(\bar{x} + t_nv) - \Phi(\bar{x}) + \varepsilon) = \Phi_1'(\bar{x};v)\}, \text{ for } \varepsilon \geq 0.
$$

Then the following assertions hold:

(i) $S(\varepsilon) \neq \emptyset$ for all $\varepsilon > 0$;

(ii) The set $S(0)$ is a possibly empty closed interval of $\mathbb{R}_-$;

(iii) When $S(0) = \emptyset$ there exists $s_\varepsilon \in S(\varepsilon)$ such that $s_\varepsilon \to -\infty$ as $\varepsilon \downarrow 0$;

(iv) For every $\varepsilon \geq 0$ the set $S(\varepsilon)$ is convex and closed.

**Proof.** (i) If $\varepsilon > 0$, then each sequence $(t_n)$ realizing the infimum in $R(\varepsilon)$ (recall Lemma 24) must converge (up to a subsequence) to some $t > 0$ (including $t = +\infty$), so that $-\frac{1}{t} \in S(\varepsilon)$ (with the convention that $\frac{1}{+\infty} = 0$).

(ii) If $S(0)$ is a non-empty subset of $\mathbb{R}_-$, then it is closed, by the continuity of function $f = \Phi(\bar{x} + v) - \Phi(\bar{x})$. If $s_0 \in S(0) \subset \mathbb{R}_-$ is the maximum element in $S(0)$,
then there is a sequence \((t_n)\) of positive numbers such that \(s_0 = -\lim_{n \to +\infty} t_n^{-1}\) and \(\lim_{n \to +\infty} t_n^{-1}(\Phi(\bar{x} + t_n v) - \Phi(\bar{x})) = \Phi'(\bar{x}; v)\). If \(s_0 < 0\), then for \(t_0 := \frac{1}{s_0}\) we get
\[
\Phi'(\bar{x}; v) \leq t^{-1}(\Phi(\bar{x} + tv) - \Phi(\bar{x})) \leq t_0^{-1}(\Phi(\bar{x} + t_0 v) - \Phi(\bar{x})) = \Phi'(\bar{x}; v) \quad \text{for all } t \in [0, t_0];
\]

hence, \([-\infty, s_0] \subset S(0)\). If \(s_0 = 0\), then \(t_n \to \infty\) and we obtain
\[
\Phi'(\bar{x}; v) \leq \inf_{t>0} t^{-1}(\Phi(\bar{x} + tv) - \Phi(\bar{x})) \leq \sup_{t>0} t^{-1}(\Phi(\bar{x} + tv) - \Phi(\bar{x})) = \lim_{n \to +\infty} t_n^{-1}(\Phi(\bar{x} + t_n v) - \Phi(\bar{x})) = \Phi'(\bar{x}; v);
\]
that is, \([-\infty, 0] \subset S(0)\). Thus, in both cases we have \([-\infty, s_0] \subset S(0)\).

(iii) Assume that \(S(0)\) is empty. Then, as \(\varepsilon \downarrow 0\), there always exist positive numbers \(s_\varepsilon \in S(\varepsilon)\) such that \(s_\varepsilon \to -\infty\). In fact, given an \(\varepsilon > 0\), we choose \(t_\varepsilon > 0\) such that
\[
R(\varepsilon) + \varepsilon = \frac{f(t_\varepsilon) + \varepsilon}{t_\varepsilon} \geq \frac{f(t_\varepsilon)}{t_\varepsilon} > R(0);
\]
such an \(t_\varepsilon\) always exists because, for otherwise, there would exist \(\varepsilon_i \downarrow 0\) and \(t^*_i \to +\infty\) such that \(\lim_{n \to +}\infty (t^*_i)^{-1}(\Phi(\bar{x} + t^*_i v) - \Phi(\bar{x}) + \varepsilon_i) = \Phi'_{\varepsilon_i}((\bar{x}; v)\) for all \(i\), entailing that
\[
\sup_{t>0} t^{-1}(\Phi(\bar{x} + tv) - \Phi(\bar{x})) \leq \lim_{n \to +\infty} (t^*_i)^{-1}(\Phi(\bar{x} + t^*_i v) - \Phi(\bar{x}) + \varepsilon_i) = \Phi'_{\varepsilon_i}((\bar{x}; v)\) for all \(i\). In other words, we get \(\sup_{t>0} t^{-1}(\Phi(\bar{x} + tv) - \Phi(\bar{x})) \leq \Phi'(\bar{x}; v)\), which gives rise to \(S(0) = [-\infty, 0]\), a contradiction. Consequently, (40) makes sense, so that the vacuity of \(S(0)\) together with the continuity of \(f\) lead us to \(t_\varepsilon \to 0^+\) (recall Lemma 24). In other words, \(s_\varepsilon = -t_\varepsilon^{-1}\) goes to \(-\infty\) as \(\varepsilon\) goes to \(0\).

(iv) Since the function \(t \to t^{-1}(f(t) + \varepsilon)\) (for \(\varepsilon > 0\)) is quasi-convex (has convex sublevel sets) and continuous, the set \(A \subset [0, +\infty)\) defined as
\[
A := \{t \geq 0 \mid \exists t_n \to t \text{ s.t. } \lim_{n \to +\infty} t_n^{-1}(\Phi(\bar{x} + t_n v) - \Phi(\bar{x}) + \varepsilon) = R(\varepsilon)\}
\]
is convex and closed. Moreover, \(0 \notin A\) and the image of \(A\) by the function \(\rho(t) := \frac{1}{t}\) (\(t > 0\)) coincides with \(S(\varepsilon)\). Hence, since function \(\rho\) is monotone and continuous we conclude that \(S(\varepsilon)\) is convex and closed. \(\blacksquare\)

**Lemma 26** With the notation of Lemmas 24 and 25, \(S\) is a maximal monotone operator.

**Proof.** To show that \(S\) is monotone, we pick \((\varepsilon_i, s_i) \in S\) (the graph of \(S\), \(i = 0, 1\), with \(0 < \varepsilon_0 < \varepsilon_1\). Then for each \(i = 0, 1\) there is a sequence \((t^*_i)^{-1} \to -s_i\) such that
\[
\lim_{n \to +\infty} (t^n_\varepsilon)^{-1}(\Phi(\bar{x} + t^n_\varepsilon v) - \Phi(\bar{x}) + \varepsilon_i) = R(\varepsilon_1) \text{ (recall Lemma 24)}; \text{ hence } t^n_\varepsilon > 0. \text{ Writing }
\]
\[
R(\varepsilon_1) = \lim_{n \to \infty} (t^n_\varepsilon)^{-1}(f(t^n_\varepsilon) + \varepsilon_1)
= \lim_{n \to \infty} ((t^n_\varepsilon)^{-1}(f(t^n_\varepsilon) + \varepsilon_0) + (t^n_\varepsilon)^{-1}(\varepsilon_1 - \varepsilon_0))
\geq R(\varepsilon_0) + \liminf_{n \to \infty} (t^n_\varepsilon)^{-1}(\varepsilon_1 - \varepsilon_0)
= \lim_{n \to \infty} ((t^n_\varepsilon)^{-1}(f(t^n_\varepsilon) + \varepsilon_0) + \liminf_{n \to \infty} (t^n_\varepsilon)^{-1}(\varepsilon_1 - \varepsilon_0)
\geq \lim_{n \to \infty} ((t^n_\varepsilon)^{-1}(f(t^n_\varepsilon) + \varepsilon_1) + \liminf_{n \to \infty} (t^n_\varepsilon)^{-1}(\varepsilon_0 - \varepsilon_1) + \liminf_{n \to \infty} (t^n_\varepsilon)^{-1}(\varepsilon_1 - \varepsilon_0)
\geq R(\varepsilon_1) + \liminf_{n \to \infty} ((t^n_\varepsilon)^{-1} - (t^n_\varepsilon_1)^{-1})(\varepsilon_0 - \varepsilon_1),
\]
we deduce that \((\varepsilon_0 - \varepsilon_1)(s_0 - s_1) \geq 0\), and the monotonicity of \(S\) follows. To check the maximality of \(S\), we observe that the function \( \psi : \mathbb{R} \to \mathbb{R} \), defined as \( \psi(\varepsilon) := \inf \{ s \mid s \in S(\varepsilon) \} \) for \( \varepsilon \geq 0 \) and \( -\infty \) otherwise, is non-decreasing (and satisfies \( \lim_{\varepsilon \downarrow 0} \psi(\varepsilon) = -\infty \)); so, it possesses left and right-limits \( \psi_- \) and \( \psi_+ \) everywhere in \( \mathbb{R}_+ \). Then, given an \( \varepsilon_0 > 0 \), by using [26, Theorem 2.1.7] the function \( g \) defined on \( \mathbb{R} \) as \( g(t) := \int_0^t \psi(s)ds \) is a proper lsc convex function with \( \mathbb{R}_+ \subset \text{dom} \ g \subset \mathbb{R} \), and \( \partial g(t) = [\psi_-(t), \psi_+(t)] \) for every \( t > 0 \), while \( \partial g(0) = [-\infty, \psi_+(0)] \), and \( \partial g(t) = \emptyset \) for all \( t < 0 \). Since \( S(\varepsilon) \) is convex and closed for every \( \varepsilon \geq 0 \), by Lemma 25, we infer that \( \partial g \subset S \) and so, by Rockafellar’s theorem [24] we infer that \( S = \partial g \) and, in particular, \( S \) is maximal monotone.}

Now we are ready to study the set \( N^{\delta}_{[\Phi \leq \lambda] \cup \{ [x + [\Phi(\bar{x})]_{\varepsilon}] \}}(\bar{x}) \) when \( \Phi(\bar{x}) > \lambda \).

**Lemma 27** If \( \Phi(\bar{x}) > \lambda \), then
\[
N^{\delta}_{[\Phi \leq \lambda] \cup \{ [x + [\Phi(\bar{x})]_{\varepsilon}] \}}(\bar{x}) \subset \bigcup_{\mu > 0} \partial_{\delta + \mu(\Phi(\bar{x}) - \lambda)}(\mu\Phi)(\bar{x}).
\]

**Proof.** We pick an element \( \xi \in N^{\delta}_{[\Phi \leq \lambda] \cup \{ [x + [\Phi(\bar{x})]_{\varepsilon}] \}}(\bar{x}) \) such that
\[
\xi \notin \bigcup_{\mu > 0} \partial_{\delta + \mu(\Phi(\bar{x}) - \lambda)}(\mu\Phi)(\bar{x}). \tag{41}
\]

Since this last set is convex, nonempty and closed, by Hahn-Banach’s Theorem there exist \( v \in X \) and \( \alpha \in \mathbb{R} \) such that
\[
\langle \xi, v \rangle > \alpha \geq \langle x^*, v \rangle, \text{ for all } x^* \in \bigcup_{\mu > 0} \partial_{\delta + \mu(\Phi(\bar{x}) - \lambda)}(\mu\Phi)(\bar{x});
\]
moreover, because \( \partial_{\delta + \mu(\Phi(\bar{x}) - \lambda)}(\mu\Phi)(\bar{x}) \supset \partial_{\mu(\Phi(\bar{x}) - \lambda)}(\mu\Phi)(\bar{x}) = \mu\partial_{\Phi(\bar{x}) - \lambda}(\Phi(\bar{x}) \neq \emptyset, \text{ we have that } \alpha \geq 0 \). So, one may suppose that \( \alpha = \delta \), so that the inequalities above read, for all \( \mu > 0 \),
\[
\langle \xi, v \rangle > \delta \geq \langle x^*, v \rangle, \text{ for all } x^* \in \partial_{\delta + \mu(\Phi(\bar{x}) - \lambda)}(\mu\Phi)(\bar{x}) = \mu\partial_{\delta + \Phi(\bar{x}) - \lambda}(\Phi(\bar{x}); \tag{42}
\]
31
that is $\mu \Phi'_{\delta/\mu+\Phi(\bar{x})-\lambda}(\bar{x},v) \leq \delta$, or, equivalently, for all $\varepsilon \geq 0$ (setting $\varepsilon = \delta/\mu$)

$$\inf_{t>0} \frac{\Phi(\bar{x}+tv) - \lambda + \varepsilon}{t} \leq \varepsilon.$$  \hspace{1cm} (43)

Let the multifunction $S$ be defined as in Lemma 26. First, we assume that $S(\Phi(\bar{x}) - \lambda) \cap [-1,0] \neq \emptyset$. If $S(\Phi(\bar{x}) - \lambda)$ (ne $\neq \emptyset$ by Lemma 26) contains a point $s_0 \in [-1,0]$, then $t_0 := \frac{1}{s_0} \geq 1$ satisfies $\Phi(\bar{x} + t_0v) - \lambda \leq 0$, by (43), and this leads us to the following contradiction (recall (42)),

$$\langle \xi, v \rangle = t_0^{-1} \langle \xi, \bar{x} + t_0v - \bar{x} \rangle \leq t_0^{-1} \delta \leq \delta < \langle \xi, v \rangle.$$  \hspace{1cm} (44)

If $S(\Phi(\bar{x}) - \lambda)$ contains 0, there would exist $t_n \to +\infty$ such that $\lim_{n \to +\infty} t_n^{-1}(\Phi(\bar{x} + t_nv) - \lambda) = R(\Phi(\bar{x}) - \lambda) \leq 0$, which shows that

$$\sup_{t>0} t^{-1}(\Phi(\bar{x}+tv) - \Phi(\bar{x})) = \lim_{n \to +\infty} t_n^{-1}(\Phi(\bar{x}+t_nv) - \Phi(\bar{x})) = \lim_{n \to +\infty} t_n^{-1}(\Phi(\bar{x}+t_nv) - \lambda) \leq 0;$$

hence, $v \in [\Phi \leq \Phi(\bar{x})]_\infty$ and we get a contradiction along of (44).

Now, we suppose that $S(\Phi(\bar{x}) - \lambda) \cap [-1,0] = \emptyset$; that is, $s < -1$ for all $s \in S(\Phi(\bar{x}) - \lambda)$. Then two cases may occur:

(a) For every $\varepsilon > \Phi(\bar{x}) - \lambda$ and $s \in S(\varepsilon)$ we have $s < -1$. In this case we pick an $s_\varepsilon \in S(\varepsilon)$ and put $t_\varepsilon := \frac{1}{s_\varepsilon}$; hence, $t_\varepsilon < 1$, so that

$$t_\varepsilon^{-1}(\Phi(\bar{x} + t_\varepsilon v) - \lambda) + 1 \leq 1.$$  

Since $\Phi(\bar{x} + v)$ is bounded from below in $[0,1]$, this last inequality implies that $t_\varepsilon \to 1$ as $\varepsilon \to +\infty$, as well as $\varepsilon^{-1}(\Phi(\bar{x} + t_\varepsilon v) - \lambda) \leq 0$ for $\varepsilon$ large enough (because $t_\varepsilon < 1$). Then $\Phi(\bar{x} + v) = \lim_{\varepsilon \to +\infty} \Phi(\bar{x} + t_\varepsilon v) \leq \lambda$ and we get a contradiction as in (44).

(b) There exist some $\varepsilon_0 > \Phi(\bar{x}) - \lambda$ and $s_0 \in S(\varepsilon_0)$ such that $s_0 \geq -1$. Since $S$ is a maximal monotone operator (Lemma 26), it has a convex range and, so, because $s < -1$ for all $s \in S(\Phi(\bar{x}) - \lambda)$ while $s_0 \geq -1$, there must exist some $\varepsilon_1 > 0$ such that $-1 \in S(\varepsilon_1)$; that is,

$$\Phi(\bar{x} + v) - \lambda + \varepsilon_1 \leq \varepsilon_1,$$

and we get $\bar{x} + v \in [\Phi \leq \lambda]$, which leads us to a contradiction similar to the one in (44). Consequently, (41) is not true and we must have that $\xi \in \bigcup_{\mu \geq 0} \partial_{\delta+\mu(\Phi(\bar{x})-\lambda)}(\mu \Phi)(\bar{x})$. \hfill \blacksquare

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