Bottomonium spectrum in the relativistic flux tube model

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The bottomonium spectrum is far from being established. The structures of higher vector states, including the \(\Upsilon(10580), \Upsilon(10860)\), and \(\Upsilon(11020)\) states, are still in dispute. In addition, whether the \(\Upsilon(10750)\) signal which was recently observed by the Belle Collaboration is a normal \(b\bar{b}\) state or not should be examined. Faced with such situation, we carried out a systematic investigation of the bottomonium spectrum in the scheme of the relativistic flux tube (RFT) model. A Chew-Frautschi like formula was derived analytically for the spin average mass of bottomonium states. We further incorporated the spin-dependent interactions and obtained a complete bottomonium spectrum. We found that the most established bottomonium states can be explained in the RFT scheme. The \(\Upsilon(10750), \Upsilon(10860),\) and \(\Upsilon(11020)\) could be predominantly the \(3^3D_1, 5^1S_1,\) and \(4^4D_1\) states, respectively. Our predicted masses of \(1^1F\) and \(1^1G\) \(b\bar{b}\) states are in agreement with the results given by the method of lattice QCD, which can be tested by experiments in future.

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I. INTRODUCTION

The toponium system \((t\bar{t})\) can hardly exist in the nature due the very short lifetime of top quark \((\approx 0.5\times 10^{-24}\text{ s})\) [1]. Then the bottomonium is the heaviest meson system which have been researched by experiments for many years. This fact makes the bottomonium family occupy an important position in the hadron zoo and play a special role in the study of the strong interactions. A prominent feature of the bottomonium spectrum is that many excited states are below the threshold \(BB\), which provides a good platform to test the different kinds of effective theories and phenomenological models.

Comparing with the theoretical expectations, however, the complete bottomonium spectrum is far from being established. The first three bottomonium states, namely \(\Upsilon(1S), \Upsilon(2S),\) and \(\Upsilon(3S)\), were observed by the E288 Collaboration at Fermilab in 1977 [2, 3]. Since then nearly twenty bottomonium states have been established [4]. The experimental history of the \(b\bar{b}\) states has been reviewed in Ref. [5]. Here, we just briefly review some important measurements of bottomonium in the past fifteen years.

As shown in Fig. 1, after the discovery of \(\Upsilon(4S), \Upsilon(10860),\) and \(\Upsilon(11020)\) states [6, 7], no progress has been made in searching for the excited \(b\bar{b}\) states for a long time until the CLEO Collaboration observed a \(1^3D_2\) candidate in the cascade process, \(\Upsilon(3S) \rightarrow \gamma \chi_{b}(2P) \rightarrow \gamma \Upsilon(1^3D_2) \rightarrow \gamma\gamma\chi_{b}(1P) \rightarrow \gamma\gamma\Upsilon(1S)\), in 2004 [8]. This \(1D\) state was later confirmed by BABAR through the \(\Upsilon(1^3D_2) \rightarrow \pi^{+}\pi^{-}\Upsilon(1S)\) decay mode [9]. Furthermore, the BABAR sample may contain the \(\Upsilon(1^3D_2)\) and \(\Upsilon(1^3D_2)\) events though the significances of these two states were very low [9].

The spin singlet states of \(S\)- and \(P\)-wave \(b\bar{b}\) mesons, i.e., \(\eta_b(1S), \eta_b(2S), h_0(1P),\) and \(h_0(2P)\), have also been found by experiments in the recent years. As a long-sought state, the \(\eta_b(1S)\) state was first observed by BABAR in the decay channel \(\Upsilon(3S) \rightarrow \gamma \eta_b(1S)\) [10], and subsequently confirmed in the decay channel \(\Upsilon(2S) \rightarrow \gamma \eta_b(1S)\) [11]. The \(\eta_b(1S)\) has also been observed by the CLEO Collaboration in the channel \(\Upsilon(3S) \rightarrow \gamma \eta_b(1S)\) [12], and by the Belle Collaboration in the channels \(h_0(nP) \rightarrow \gamma \eta_b(1S)\) \((n=1,2)\) [13, 14].

The first probable signal of the \(\eta_b(2S)\) state was detected by the BABAR Collaboration [15] although their result was largely inconclusive. A clear evidence of \(\eta_b(2S)\) was achieved by the Belle Collaboration in the processes \(e^{+}e^{-} \rightarrow \Upsilon(5S) \rightarrow h_0(2P)\pi^{+}\pi^{-} \rightarrow \gamma \eta_b(2S)\pi^{+}\pi^{-}\) [13]. There the mass of \(\eta_b(2S)\) was measured by Belle as 9999.0 ± 3.5^{+2.8}_{-1.9} \text{ MeV}.\footnote{Dobbs et al. analysed \((9.32 \pm 0.19) \times 10^{6} \Upsilon(2S)\) recorded with the CLEO III detector and announced the observation of \(\eta_b(2S)\) in the reaction \(\Upsilon(2S) \rightarrow \gamma \eta_b(2S)\) [16]. However, their result was not confirmed by Belle with a larger sample of \(\Upsilon(2S)\) decays [17].}

The first evidence of spin-singlet state \(h_0(1P)\) was reported by BABAR in the sequential decays \(\Upsilon(3S) \rightarrow \pi^{+}\pi^{-}h_0(1P)\) →
There the mass value of $h_b(1P)$ was measured as $9902 \pm 4 \pm 1$ MeV though the effective signal significance was only $3.0 \sigma$. The significant signal of $h_b(1P)$ was achieved by Belle [13, 19] in the $π^+π^−$ missing spectrum of the reaction $e^+e^− \to Υ(5S) \to h_b(1P)π^+π^−$. Meanwhile, the radial excited $h_b(2P)$ was also observed in this measurements. The $h_b(1P)$ state was also found in the transition $Υ(4S) \to η h_b(1P)$ [14].

A $χ_b(3P)$ state was first discovered by the ATLAS Collaboration in the radiative decay modes of $χ_b(3P) \to Υ(1S,2S)γ$ [21], and subsequently confirmed by the D0 [22] and LHCb Collaborations [23, 24]. However, their measured masses were a little different from each other (see Table I).

Very recently, the Belle Collaboration discovered a new candidate of the upsilon resonance in the shape of cross sections of $e^+e^− \to Υ(nS)π^+π^− (n = 1, 2, 3)$ [25]. Belle denoted this state as the $Υ(10750)$ and determined the mass and width as

$$M = 10752.7 \pm 5.9^{+0.7}_{−1.1}$ MeV, \quad \Gamma = 35.5^{+17.6}_{−11.3}^{+3.9}_{−3.3}$ MeV, \quad (1)$$

respectively, by the Breit-Wigner parameterization. Surely, more experimental confirmations are required for the $Υ(10750)$ state.

Obviously, it is not an easy task to establish the bottomonium spectrum completely because even many $b\bar{b}$ states below the $B\bar{B}$ threshold have not been discovered. However, the situation may be changed especially because of the running of Belle II [26]. It is expected that more excited bottomonium states will be detected in the near future. So it is time to investigate the spectrum of $b\bar{b}$ by different approaches which incorporate the spirits of QCD.

So far, different types of quark potential model have been applied in studying the bottomonium spectrum, including the nonrelativistic [5, 27–30], the semirelativistic [31, 32], the relativized [33–35], and the relativistic [36, 37] versions. The bottomonium spectrum also has been studied by the Bethe-Salpeter equation [38], the coupled channel model [39–42], the QCD sum rule [43, 44], the Regge phenomenology [45–48], the lattice QCD [49–51], and other method [52].

In this work, we will explore bottomonium spectrum in the scheme of the RFT model which can be rigorously derived from the Wilson area law in QCD [53]. The investigation of $b\bar{b}$ spectrum here by the RFT model could be regarded as an extension of our previous work [54]. There we have shown that the RFT model can describe the masses of single heavy baryons well. Especially, the predicted mass of $1D\Lambda_b^+$ and $Λ_b^0$ states in Ref. [54] are in good agreement with the later measurements by the LHCb Collaboration [55, 56].

The manuscript is organized as follows. The RFT model is introduced in Sec. II where a spin average mass formula of the heavy quarkonia is derived. In Sec. III, we test the mass formula by the well measured $b\bar{b}$ states. In Sec. IV, the spin-dependent interactions are incorporated and the complete bottomonium spectrum is presented. Finally, the paper ends with the discussion and conclusion.

### II. Spin Average Mass Formula of the Heavy Quarkonia in the RFT Model

The idea of RFT model stemmed from the Nambu-Goto QCD string model [57–59]. Different aspects of the RFT model have been investigated by Olsson and the collaborators [60–65]. The deep relationship between the RFT model and QCD has been verified in Refs. [53, 66]. The basic assumption of the RFT model is that the gluon field connecting the largely separated quarks in the QCD dynamical ground state could be regarded as a rigid straight tube-like color flux configuration. Thus the angular momentum of gluon field is taken into account by the RFT model, which is qualitatively different from the usual quark potential models. The RFT model has been applied to study the masses of heavy-light mesons [67–69], charmonium states [70], single heavy baryons [54, 71], glueballs [72], and other exotic hadrons [73].

As shown in Fig. 2, the Lagrangian of a $q_1\bar{q}_2$ meson in the RFT model is written as [74]

$$\mathcal{L}(r, \hat{\theta}) = -\sum_{i=1}^{2} \left[ m_i \sqrt{1 - (r_i \hat{\theta})^2} + \int_{0}^{r_i} \tau \sqrt{1 - (\rho \hat{\theta})^2} d\rho \right], \quad (2)$$

where $m_i$ and $r_i$ denote the mass of $i$ ($i = 1, 2$) quark and its distance from the center of gravity (see Fig. 2). $\tau$ represents the string (flux tube) tension. Here, we only consider the transverse velocity of the quark and antiquark, i.e., $\hat{r}_i = 0$. from the Wilson area law in QCD [53]. The investigation of $b\bar{b}$ spectrum here by the RFT model could be regarded as an extension of our previous work [54]. There we have shown that the RFT model can describe the masses of single heavy baryons well. Especially, the predicted mass of $1D\Lambda_b^+$ and $Λ_b^0$ states in Ref. [54] are in good agreement with the later measurements by the LHCb Collaboration [55, 56].

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As shown in Fig. 2, the Lagrangian of a $q_1\bar{q}_2$ meson in the RFT model is written as [74]
Then the total orbital angular momentum \( L \) is defined by
\[
L = \frac{\partial L}{\partial \theta} = \sum_{i=1}^{2} \left[ \frac{m_i \rho_i^2 \theta}{\sqrt{1 - (r_i \theta)^2}} + \int_0^{\tau} \frac{\tau \rho_i^2 \theta}{\sqrt{1 - (\rho_i^2)^2}} d\rho_i \right].
\] (3)

The Hamiltonian of \( q_1 \bar{q}_2 \) meson is given by
\[
H = \hat{\theta}L - L = \sum_{i=1}^{2} \left[ \frac{m_i}{\sqrt{1 - u_i^2}} + \int_0^{\tau} \frac{\tau}{\sqrt{1 - (\rho_i^2)^2}} d\rho_i \right].
\] (4)

When we set \( r_i = r \theta \), the energy and orbital angular momentum can be written as
\[
\epsilon = \sum_{i=1}^{2} \left[ \frac{m_i}{\sqrt{1 - u_i^2}} + \frac{\tau}{\omega} \int_0^{\lambda} \frac{d\nu}{\sqrt{1 - v^2}} \right],
\] (5)
and
\[
L = \sum_{i=1}^{2} \left[ \frac{m_i \mu_i^2}{\omega} \frac{1}{\sqrt{1 - u_i^2}} + \frac{\tau}{\omega} \int_0^{\lambda} \frac{\nu^2 d\nu}{\sqrt{1 - v^2}} \right].
\] (6)

We have set \( \epsilon = 1 \) in natural unit for simplicity. Eqs. (5) and (6) have also been obtained by the Wilson area law [53]. With Eqs. (5) and (6), a mass formula for the heavy-light hadrons has been derived analytically in our previous work [54].

For the bottomonium system, the masses of \( b \) and \( \bar{b} \) quarks are denoted as \( m \). Then Eqs. (5) and (6) become as
\[
\epsilon = \frac{2m}{\sqrt{1 - u^2}} + \frac{2\tau}{\omega} \arcsin u,
\] (7)
and
\[
L = \frac{2m \mu^2}{\omega} \frac{1}{\sqrt{1 - u^2}} + \frac{\tau}{\omega^2} \left[ \arcsin u - u \sqrt{1 - u^2} \right].
\] (8)

Combing with the following relationship in the RFT model
\[
\frac{\tau}{\omega} = \frac{\mu}{1 - u^2},
\] (9)
we have
\[
\epsilon = \frac{2m}{\sqrt{1 - u^2}} + \frac{2m \mu}{1 - u^2} \arcsin u,
\] (10)
and
\[
\tau L = \frac{2m^2 \mu^2}{(1 - u^2)^{3/2}} + \frac{m^2 \mu^2}{(1 - u^2)^2} \left[ \arcsin u - u \sqrt{1 - u^2} \right].
\] (11)

Since the Eqs. (7) and (8) can be derived from the QCD [53], the \( m \) in the above equations could be regarded as the “current quark masses” of bottom quark. In practice, the constituent quark mass is more suitable for the phenomenological analysis. To this end, we assume
\[
m_b = \frac{m}{\sqrt{1 - u^2}}
\] (12)

From Eqs. (7)–(11), we have
\[
\epsilon = 2m_b (1 + f_1(u)); \quad \tau L = 2m_b^2 f_2(u).
\] (13)

In above equations, we set the following functions
\[
f_1(u) = \frac{u}{\sqrt{1 - u^2}} \arcsin u,
\] (14)
and
\[
f_2(u) = \frac{u^3}{\sqrt{1 - u^2}} + \frac{u^2}{2(1 - u^2)} \left[ \arcsin u - u \sqrt{1 - u^2} \right].
\] (15)

Since \( m_b \) has included the relativistic effect, we may treat it as the constituent quark mass of \( b \) quark. In this way, the value of \( m_b \) can be fixed by the experimental data, directly. The treatment of \( m_b \) which includes the relativistic effect is different from the work [70] where the RFT model has been applied to investigate the assignment of \( X(3872) \). As shown later, the velocity of bottom quark in the \( b \bar{b} \) meson is no more than 0.50. The Eqs. (13) can be expanded as
\[
\epsilon - \frac{2m_b}{2m_b} = f_1(u) = u^3 + O(u^4) + \cdots,
\] (16)
\[
\tau L = \frac{2m_b^2}{2m_b} = f_2(u) = u^3 + O(u^4) + \cdots.
\]

If we ignore the higher order of \( u \), the following relationship can be obtained
\[
\epsilon_L = 2m_b + \left( \frac{2}{m_b} \right)^{1/3} (\tau L)^{2/3}.
\] (17)

We replace the string tension \( \tau \) by the parameter \( \sigma \) with the relationship \( \sigma \equiv 2\pi \tau \). As done in Ref. [54], we further extend Eq. (17) to include the radial excited \( b \bar{b} \) states,
\[
\epsilon_{Lr} = 2m_b + \left( \frac{\sigma^2}{2\pi^2 m_b} \right)^{1/3} (\lambda n + L)^{2/3}.
\] (18)

The coefficient \( \lambda \) will also be determined by the experimental data. Eq. (18) is a Chew-Frautschi like formula of the mass of \( b \bar{b} \) states. When the distance between the \( b \) and \( \bar{b} \) quarks in a \( b \bar{b} \) meson is denoted as \( r \), we have the relationship: \( r = 2u/\omega \). Combing with Eq. (9), we get
\[
r = \frac{4\pi m_b}{\sigma} \frac{u^2}{\sqrt{1 - u^2}}.
\] (19)

In the region of \( u \in 0.3c \sim 0.6c, \) we find \( u^2/\sqrt{1 - u^2} \approx (0.95 \pm 0.02) \times f_1(u) \). With equations (13) and (18), we obtain the expression of \( r \) as
\[
r = \left( \frac{10.8}{\sigma m_b} \right)^{1/3} (\lambda n + L)^{2/3}.
\] (20)

In next Section, we shall test the Eq. (18) by the measured masses of \( b \bar{b} \) states. In Section IV, we will incorporate the spin-dependent interactions and present a complete bottomonium spectrum.

3 One can check that the error arising from the neglect of higher order is no more than 3\%.
III. TESTING EQ. (18) BY THE MEASURED MASSES OF BOTTOMONIUM STATES

Three parameters in Eq. (18), namely the mass of bottom quark $m_b$, the string tension $\sigma$, and the dimensionless $\lambda$, should be fixed by the experimental data. We will used the spin average masses of the $1S$, $2S$, and $1P$ $b\bar{b}$ states to fix the $m_b$, $\sigma$, and $\lambda$. The spin average mass of $1S$ $b\bar{b}$ is

$$\bar{M}(1S) = \frac{9398.7 + 9460.3 \times 3}{4} = 9444.9\,\text{MeV},$$

and the averaged mass of $2S$ $b\bar{b}$ is

$$\bar{M}(2S) = \frac{9999 + 10023.3 \times 3}{4} = 10017.2\,\text{MeV}. \quad (22)$$

Here, the masses of $1S$ and $2S$ $b\bar{b}$ states are taken from the latest “Review of Particle Physics” (RPP) [4] by the Particle Data Group (PDG). Since the average mass of $1^3P_0$, $1^3P_1$, and $1^3P_2$ $b\bar{b}$ states is quite close to the $1^1P_1$ state (see Ref. [75] for more discussions), we take the mass of $h_b(1P)$ as the average mass of $1P$ $b\bar{b}$ states. Specifically, the world average mass of $h_b(1P)$ state, i.e., 9899.3 MeV [4], is used to fix the parameters in Eq. (18). With the masses of $\bar{M}(1S)$, $\bar{M}(2S)$, and $h_b(1P)$, the parameters are fixed as

$$m_b = 4.7224\,\text{GeV}, \quad \sigma = 2.96\,\text{GeV}^2, \quad \lambda = 1.41. \quad (23)$$

Comparing the value of $m_b$ with the current mass of $b$ quark, i.e., $4.18_{-0.02}^{+0.03}$ GeV [4], the velocity of $b$ quark is estimated to be $0.46 \pm 0.01$. With the values of $m_b$, $\sigma$, and $\lambda$, the center of gravity of other $n^3S^1 \ell_J$ multiplet can be calculated directly. At present, the masses of $\Upsilon(3S)$, $h_\ell(2P)$ and $\Upsilon_2(1D)$ have been well measured by different experiments [4]. A comparison of the predictions by Eq. (18) with the measured results is given in the Table II.

| $n\ell$ | State   | Measured mass | Prediction |
|--------|---------|---------------|------------|
| $1D$  | $\Upsilon(1D)$ | $10163.7 \pm 1.4$ | $10166$  |
| $2P$  | $h\ell(2P)$  | $10259.8 \pm 1.2$ | $10262$  |
| $3S$  | $\Upsilon(3S)$ | $10355.2 \pm 0.5$ | $10352$  |

The mass of $h\ell(2P)$ is predicted to be $10262$ MeV which is consistent with the experimental result. The $\eta_b(3S)$ state has not been discovered by experiment. Nevertheless, the spin average mass of the $2S$ bottomonium states is about 6 MeV below the $\Upsilon(2S)$ state (see Eq. (22)). So one could reasonably expect the average mass of $3S$ states to be about $10350$ MeV which is also close to our prediction. As argued in Ref. [15], two $D$-wave $b\bar{b}$ states, namely the $\Upsilon(10152)$ and $\Upsilon_3(10173)$, may have been detected in the experimental data by the CLEO [8] and BABAR [9] Collaborations. Although the measured masses of these two states need more confirmations, the average mass of the $\Upsilon(10152)$, $\Upsilon_2(10164)$, and $\Upsilon_3(10173)$ states

$$\frac{10152 \times 3 + 10163.7 \times 5 + 10173 \times 7}{15} = 10165.7\,\text{MeV}, \quad (24)$$

is quite consistent with our result (see Table II).

As shown above, the predicted average masses of $\Upsilon(3S)$, $h\ell(2P)$ and $\Upsilon_2(1D)$ multiplets are well comparable with the experimental results. For completeness, we will incorporate the spin-dependent interactions and give a whole bottomonium spectrum in the next section.

IV. THE COMPLETE BOTTOMONIUM SPECTRUM BY INCORPORATING THE SPIN-DEPENDENT INTERACTIONS

For simplicity, we consider the color hyperfine interaction

$$H_{hyp} = \frac{4\alpha_s}{3m_b^2} \left( \frac{8\pi}{3} \delta^3(\mathbf{r})s_b \cdot s_b + \frac{1}{r} \hat{S}_{b\bar{b}} \right), \quad (25)$$

which arises from the one gluon exchange (OGE) forces, and the following spin-orbit term

$$H_{so} = \frac{1}{m_b} \left( \frac{2\alpha_s}{r} - \frac{b}{2r} \right) \mathbf{S} \cdot \mathbf{L}, \quad (26)$$

which includes the OGE spin-orbit and the longer-ranged spin-orbit terms. This type of spin-dependent interactions has been used to study the mass spectrum of $cc$ spectrum [76]. The $\hat{S}_{b\bar{b}}$ denotes the tensor operator. The “$\delta^3(\mathbf{r})$” function which comes from a contact hyperfine interaction can be simulated by the different smearing functions [33, 77]. In our calculations, we take the following smearing function

$$f(r) = \frac{4}{\pi^{1/2} r_0} e^{-\sqrt{27}\pi r_0}, \quad (27)$$

to reproduce the mass splitting of $nS$ ($n \geq 2$). Here, we take the $r_0$ as $0.94$ GeV$^{-1}$. Due to the heavy masses, the distance between $b$ and $\bar{b}$ quarks in the low-lying bottomonium states is much small. Therefore, one should treat the running coupling constant $\alpha_s$ in Eqs. (25) and (26) seriously. We use the following

$$\alpha_s(r) = \alpha_0 \text{Erf}\left( \left( \frac{m_b r}{0.72\pi^2} \right)^{1/2} \right), \quad (28)$$

to simulate the running coupling constant, where the Erf$[\cdot]$ refers to the error function. In our calculations, the running coupling constant is assumed to saturate at 0.68, i.e., $\alpha_0 = 0.68$. To reduce the free parameters, we take the value of $b$
TABLE III: The mass splitting of $n^2S_1$ and $n^1S_0$ states (in MeV).

| $\Delta M(nS)$ | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ | $n = 6$ |
|---------------|---------|---------|---------|---------|---------|
| Our           | 23.9    | 20.7    | 13.2    | 9.3     | 7.0     |
| Ref. [34]     | 27      | 18      | 12      | 9       | 5       |
| Ref. [27]     | 25      | 17      | 13      | 11      | 9       |

in Eq. (26) as the string tension $\tau$ in the RFT model, i.e., $b = \sigma r^2/2 \pi = 0.471$ GeV$^2$. With Eqs. (20), (25), (27), and (28), the splitting masses of $n^1S_1$ and $n^1S_0$ states ($n \geq 2$) are presented in Table III.

Obviously, our results are comparable with these from Refs. [27, 34]. As shown later, the masses of observed excited $b\bar{b}$ states will be well reproduced though our method is quite phenomenological.

A. $nS$ ($n \geq 2$) states

With the predicted splitting masses in Table III, the masses of $n^1S_0$ and $n^1S_1$ bottomonium states ($n \geq 2$) are predicted in Table IV where the experimental data [4] and the results from other works [29, 34, 41] are also listed for comparison.

TABLE IV: The masses of the $nS$ ($n \geq 2$) $b\bar{b}$ states (in MeV).

| States          | Expt. [4] | Our | Ref. [34] | Ref. [29] | Ref. [41] |
|-----------------|-----------|-----|-----------|-----------|-----------|
| $0^-(2S)$       | 9999±4    | 9999| 9976      | 9955      | 10005     |
| $1^-(2S)$       | 10023.3±0.3 | 10023| 10003     | 9979      | 10026     |
| $0^+(3S)$       | 10337     | 10336| 10338     | 10338     |           |
| $1^+(3S)$       | 10355.2±0.5 | 10357| 10354     | 10359     | 10352     |
| $0^-(4S)$       | 10627     | 10623| 10663     | 10593     |           |
| $1^-(4S)$       | 10579.4±1.2 | 10637| 10635     | 10683     | 10603     |
| $0^+(5S)$       | 10878     | 10869| 10956     | 10813     |           |
| $1^+(5S)$       | 10889.9±3.2 | 10887| 10878     | 10975     | 10820     |
| $0^+(6S)$       | 11111     | 11097| 11226     | 11008     |           |
| $1^-(6S)$       | 10992.9±10.0 | 11118| 11102     | 11243     | 11023     |

As shown in Table IV, the well measured masses of $\eta_b(2S)$, $\Upsilon(2S)$, and $\Upsilon(3S)$ are reproduced in our scheme. The predicted mass of unknown $\eta_b(3S)$ state is 10337 MeV which is comparable with these results from Refs. [29, 34, 41].

The masses of the $\Upsilon(4S)$, $\Upsilon(5S)$, and $\Upsilon(6S)$ obtained by the RFT model are quite close to the results given by the Godfrey-Isgur model [34]. Our results favor the $\Upsilon(10860)$ as a predominantly $5^3S_1$ state. Interestingly, a recent work based on the lattice QCD also suggested the $\Upsilon(10860)$ as a $5^3S_1$ state [78]. The mass of $\Upsilon(4S)$ predicted by the RFT model is about 60 MeV higher than the measured mass of $\Upsilon(10580)$ (see Table IV). The mass of $\Upsilon(4S)$ state predicted in Refs. [5, 27–29, 34, 41] was also larger than the $\Upsilon(10580)$ state. In the quark potential models, the mass gap between the $3^3S_1$ and $4^3S_1$ $b\bar{b}$ states is expected to be larger than the gap between the $4^3S_1$ and $5^3S_1$ states. However, the experimental measurement is contrary to the expectation, i.e.,

$$\Delta M(\Upsilon(10580) - \Upsilon(10355)) = 224.2 \text{ MeV},$$

which is smaller than

$$\Delta M(\Upsilon(10860) - \Upsilon(10580)) = 310.5 \text{ MeV}.$$ (30)

It indicates that the mass of $\Upsilon(4S)$ shifts down about 40–50 MeV due to a particular mechanism. This anomalously mass gaps of “$\Upsilon(4S) - \Upsilon(3S)$” and “$\Upsilon(5S) - \Upsilon(4S)$” cannot be simply solved by the naïve quark model. Törnqvist proposed a solution to this puzzle. Specifically, it may be disentangled by considering the coupled-channel effects [40]. More importantly, the masses of $\Upsilon(5S)$ and $\Upsilon(6S)$ were well predicted in the scheme of coupled-channel model [40] before the observations of candidates $\Upsilon(10860)$ and $\Upsilon(11020)$ [6, 7]. The scheme suggested by Törnqvist was supported by the recent work [42].

As a pure $6^3S_1$ $b\bar{b}$ state, the measured mass of $\Upsilon(11020)$ is about 100–200 MeV lower than the predictions by the RFT model and other methods [29, 34, 41, 79]. These results indicate that the $\Upsilon(11020)$ is not a pure $6S$' epsilon resonance. This conclusion is partially supported by the analysis of its dielectron widths [79] (see subsection IV C).

B. $nP$ states

The masses of $nP$ ($n = 1 \sim 5$) states which are predicted by the RFT model are listed in Table V with the experimental data [4] and other theoretical results from Refs. [29, 34, 36]. Up to now, the $1P$ and $2P$ bottomonium states are well established [4]. Obviously, the masses of these states are well reproduced by the RFT model.

The candidates of $3P$ bottomonium states have been detected by the ATLAS [21], D0 [22], and LHCb [23, 24] collaborations (see Table I). The masses of the $\chi_{b1}(3P)$ and $\chi_{b2}(3P)$ collected by the PDG are listed in Table V. The experimental results are about 20–40 MeV smaller than the theoretical results. One notices that the predicted masses $3P$ $b\bar{b}$ states are about 30–100 MeV above the thresholds of $BB$, $BB^* + B^*B$, and $B^*B$ decay channels. So the coupled-channel channel effect may affect the properties of $3P$ bottomonium states including their masses. More theoretical and experimental efforts are desirable for the $3P$ $b\bar{b}$ states in future.

The $4P$ and $5P$ bottomonium states are predicted around 10800 MeV and 11050 MeV, respectively, which means these states locate above the open-bottom thresholds. Then the Okubo-Zweig-Iizuka (OZI) allowed decays are probable for these states. In Ref. [34], the investigation of strong decays by the $3P_0$ model indicated that the $\chi_{b0}(4P)$ state mainly decays through the $BB$ and $B^*B$ channels while the $BB^* + B^*B$ is

$^5$ However, the practical calculations in Ref. [80] did not support this conjecture. There the $\chi_b(3P)$ state was suggested to be the (almost) pure bottomonium.
the largest decay channel for the $\chi_{bc}(4P), \chi_{bc}(2P)$, and $h_{bc}(4P)$ states. Different from the $4P$ bottomonium states, the largest decay channel of $5P$ states is the $B^+\bar{B}^−$. The total decay widths of $4P$ and $5P$ bottomonium states were predicted to be $30\sim70$ MeV. The decays predicted in Ref. [35] were slight different from these results in Ref. [34]. Of course, discovery of these high $P$-wave bottomonium states is a great challenge for present experiments.

### C. $nD$ states

So far only one $D$-wave $b\bar{b}$ state, namely $\Upsilon_2(1D)$, was listed in the summary table of PDG [4]. Its measured mass, i.e., $10163.7\pm1.7$ MeV, is quite in agreement with our prediction (see Table VI). The visible evidence of the $1^3D_1$ and $1^3D_3$ bottomonium states at 10152 MeV and 10173 MeV [8, 9], respectively, was pointed out in Ref. [15]. Our predictions in Table VI are comparable with these preliminary results. Our results are also consistent with the predicted masses of $1D b\bar{b}$ states by Lattice QCD [49].

None of the $2D b\bar{b}$ states have been announced by any experiments. Nevertheless, Beveren and Rupp found the $\Upsilon(2D)$ signal with 10.7 standard deviations [81] by reanalyzing the BABAR data [82]. There the mass of $\Upsilon(2D)$ was fitted to be $10495 \pm 5$ MeV, which is a bit larger than the predictions in Table VI.

As mentioned before, a $1^{−}\!−\!$ structure $\Upsilon(10750)$ which was discovered by the Belle collaboration [25] is still unclear. Since the $3^3D_1 b\bar{b}$ state is expected to has the masse around 10740 MeV, the $\Upsilon(10750)$ could be a good $3D$ candidate. Due to the significant mixing between the $(n + 1)^3S_1$ and $n^3D_1$ states ($n \geq 3$), the magnitude of dielectron widths of the mixed $\tilde{\Upsilon}(n^3D_1)$ resonances ($n = 3, 4, 5$) can increase by 2 orders [79]. For the $\tilde{\Upsilon}(3D)$ state, the dielectron width was obtained to be $0.095^{+0.028}_{−0.025}$ keV. The result shows that the predominantly $3^3D_1 b\bar{b}$ state can be produced in the $e^+e^−$ annihilation process with the high statistics data. Furthermore, the decay width of the $3^3D_1 b\bar{b}$ state was obtained as $54.1$ MeV [35] which is comparable with the measurement by the Belle Collaboration [25] (see Eq. (1)). So the $\Upsilon(10750)$ could be predominantly a $3^3D_1$ b\bar{b} state in our scheme. However, the other explanations suggested in Refs. [83, 84] are also possible for the $\Upsilon(10750)$ state. For revealing the inner structure of $\Upsilon(10750)$, more precise measurements including the dielectron width and the branching ratios of $\Gamma(B\bar{B}) : \Gamma(B\bar{B}^+ + B^−\bar{B}) : \Gamma(B^+\bar{B}^-)$ are needed in future.

### TABLE VI: The masses of the $nD b\bar{b}$ states (in MeV).

| States | Expt. [4] | Our Ref. [34] | Ref. [36] | Ref. [29] |
|--------|-----------|----------------|-----------|-----------|
| $1^{−}(1D)$ | 10136 | 10138 | 10154 | 10074 |
| $2^{−}(1D)$ | 10163.7±1.7 | 10164 | 10147 | 10161 | 10075 |
| $2^{−}(1D)$ | 10167 | 10148 | 10163 | 10074 |
| $3^{−}(1D)$ | 10183 | 10155 | 10166 | 10073 |
| $1^{−}(2D)$ | 10467 | 10441 | 10435 | 10423 |
| $2^{−}(2D)$ | 10476 | 10449 | 10443 | 10424 |
| $3^{−}(2D)$ | 10475 | 10450 | 10445 | 10424 |
| $1^{−}(3D)$ | 10742 | 10708 | 10704 | 10731 |
| $2^{−}(3D)$ | 10744 | 10705 | 10711 | 10733 |
| $3^{−}(3D)$ | 10742 | 10706 | 10713 | 10733 |
| $1^{−}(4D)$ | 10987 | 10928 | 10949 | 11013 |
| $2^{−}(4D)$ | 10986 | 10934 | 10957 | 11016 |
| $2^{−}(4D)$ | 10984 | 10935 | 10959 | 11015 |
| $3^{−}(4D)$ | 10981 | 10939 | 10963 | 11015 |

According to the predicted masses by the RFT model and other methods [29, 34, 36], the $4D b\bar{b}$ states should have the masses around the 10950 MeV. The controversial $\Upsilon(11020)$ state might have a significant $4^3D_1$ component since its mass is quite close to the prediction of $4^3D_1$ state. Furthermore, the dielectron width of pure $6S \Upsilon$ state was given about 0.274 KeV [79], which is about two times larger than the experimental measurement of $\Upsilon(11020)$. This result also indicates that the $S$-$D$ mixing effect should be significant for the $\Upsilon(11020)$ state.

### D. High orbital excited states

Up to now, none of the high orbital excited $b\bar{b}$ mesons including $F$-, $G$-, and $H$-wave states have been announced by any experiments. Obviously, it is a challenge for experiments to discover these states. However, the situation may change while the SuperKEKB facility has run last year [26].
the event numbers about $2 \times 10^6 \ \Upsilon(2^3D_1)$ states produced at Belle II in future, the observation of $F$-wave $b\bar{b}$ state could be accessible [26].

Table VII: The masses of high orbital excited $b\bar{b}$ states (in MeV).

| States     | Our  | Ref. [34] | Ref. [35] | Ref. [36] | Ref. [5] |
|------------|------|-----------|-----------|-----------|---------|
| $2^+(1F)$  | 10376| 10350     | 10362     | 10343     | 10315   |
| $3^+(1F)$  | 10391| 10355     | 10366     | 10346     | 10321   |
| $3^+(1F)$  | 10391| 10355     | 10366     | 10347     | 10322   |
| $4^+(1F)$  | 10400| 10358     | 10369     | 10349     | –       |
| $2^+(2F)$  | 10668| 10615     | 10605     | 10610     | 10569   |
| $3^+(2F)$  | 10670| 10619     | 10609     | 10614     | 10573   |
| $3^+(2F)$  | 10668| 10619     | 10609     | 10615     | 10573   |
| $4^+(2F)$  | 10667| 10622     | 10612     | 10617     | –       |
| $2^+(3F)$  | 10920| 10850     | 10809     | –         | 10782   |
| $3^+(3F)$  | 10918| 10853     | 10812     | –         | 10785   |
| $3^+(3F)$  | 10916| 10853     | 10812     | –         | 10785   |
| $4^+(3F)$  | 10912| 10856     | 10815     | –         | –       |
| $3^+(1G)$  | 10588| 10529     | 10533     | 10511     | 10506   |
| $4^+(1G)$  | 10592| 10531     | 10535     | 10512     | –       |
| $4^+(1G)$  | 10591| 10530     | 10534     | 10513     | –       |
| $5^+(1G)$  | 10592| 10532     | 10536     | 10514     | –       |
| $3^+(2G)$  | 10851| 10769     | 10745     | 10712     | 10712   |
| $4^+(2G)$  | 10848| 10770     | 10747     | –         | –       |
| $4^+(2G)$  | 10846| 10770     | 10747     | –         | –       |
| $5^+(2G)$  | 10842| 10772     | 10748     | –         | –       |
| $4^+(1H)$  | 10778| –         | –         | 10670     | –       |
| $5^+(1H)$  | 10776| –         | –         | 10671     | –       |
| $5^+(1H)$  | 10774| –         | –         | 10671     | –       |
| $6^+(1H)$  | 10769| –         | –         | 10672     | –       |

The masses of the $1F \ b\bar{b}$ states are predicted in the region around 10400 MeV, which is comparable with the results given by the lattice nonrelativistic QCD [50]. The $1G \ b\bar{b}$ masses are predicted around 10590 MeV which are slightly above the $B\bar{B}$ threshold at 10.56 GeV. Our predicted masses of $1G \ b\bar{b}$ states seem to be larger than the results obtained by the quark potential models [5, 34–36], but very close to the results from the lattice QCD [50], where the masses of $4^+$ and $4^− \ b\bar{b}$ states were predicted as

$$M(1^G_{D_1}) = 10581 \pm 17 \text{ MeV},$$
$$M(2^G_{D_1}) = 10587 \pm 18 \text{ MeV}.$$  (31)

V. DISCUSSION AND CONCLUSION

We have carried out a systematical study of the bottomonium spectrum for the first time by the relativistic flux tube (RFT) model. We derived a Chew-Frautschi like formula which can give an intuitive description of the spin average mass of the heavy quarkonium systems. With the measured masses of $1S$, $2S$, and $1P \ b\bar{b}$ states, we fixed the three parameters in the Chew-Frautschi like formula, namely the mass of $b$ quark, the string tension $\sigma$, and the dimensionless parameter $A$. Then we tested the mass formula by comparing the predicted the spin average masses of $3S$, $2P$, and $1D$ states with the experimental results. The comparison implied that the Chew-Frautschi like formula could describe the spin average masses of high excited $b\bar{b}$ states well.

Inspired by a good description of the spin average mass, we further incorporate the spin-dependent interactions which include the one gluon exchange (OGE) forces and the longer-ranged inverted spin-orbit term. As shown in the Tables IV and V, the measured masses of the $nS$ ($2 \leq n \leq 6$) and $nP$ ($n = 1$ and 2) states were well reproduced. The predicted masses of $nD$ and other high bottomonium states in Tables VI and VII could be tested in future.

According to our results, the main conclusions are listed as follows:

1. The $\Upsilon(10860)$ could be explained as a predominant $5S$ state since its measured mass is very close to the predictions (see Table IV). The $\Upsilon(10580)$ and $\Upsilon(11020)$ can not be regarded as the pure $4S$ and $6S$ states, respectively, since the predicted masses are much larger than the measurements.

2. The newly discovered $\Upsilon(10750)$ could be regarded as a good candidate of the predominant $3^2D_1$ state since the measured mass is in good agreement with our prediction.

3. The measured masses of $3P \ b\bar{b}$ states seems to be about 20–30 MeV smaller than the theoretical results.

4. Our predicted mass of the $1^2D_2 \ b\bar{b}$ state is consistent with the experimental value. The predicted masses of $1^3D_1$ and $1^2D_1$ states are also comparable with the signals detected by the CLEO [8] and BABAR [9] Collaborations.

In summary, the bottomonium spectrum has been systematically studied by the RFT model, which could be regarded as an important supplement to the available investigations of the bottomonium spectrum. Since the relativistic color flux tube carries both energy and momentum, the RFT model present a different dynamics picture for the heavy quarkonia system. The larger predicted masses of the high orbital excited states by the RFT model can be tested by the experiments in future.

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