ERGODIC THEOREMS
IN BANACH IDEALS OF COMPACT OPERATORS

A. AZIZOV, V. CHILIN, AND S. LITVINOV

Abstract. Let $H$ be an infinite-dimensional Hilbert space, and let $B(H)$ ($K(H)$) be the $C^*$-algebra of bounded (respectively, compact) linear operators in $H$. Let $(E, \| \cdot \|_E)$ be a fully symmetric sequence space. If $\{s_n(x)\}_{n=1}^\infty$ are the singular values of $x \in K(H)$, let $C_E = \{x \in K(H) : \{s_n(x)\} \in E\}$ with $\|x\|_{C_E} = \|\{s_n(x)\}\|_E$, $x \in C_E$, be the Banach ideal of compact operators generated by $E$. We show that the averages $A_n(T)(x) = \frac{1}{n+1} \sum_{k=0}^n T^k(x)$ converge uniformly in $C_E$ for any Dunford-Schwartz operator $T$ and $x \in C_E$. Besides, if $x \in B(H) \setminus K(H)$, there exists a Dunford-Schwartz operator $T$ such that the sequence $\{A_n(T)(x)\}$ does not converge uniformly. We also show that the averages $A_n(T)$ converge strongly in $(C_E, \| \cdot \|_{C_E})$ if and only if $E$ is separable and $E \neq l^1$, as sets.

1. Introduction

Let $B(H)$ be the algebra of bounded linear operators in a Hilbert space $H$, equipped the uniform norm $\| \cdot \|_\infty$. The study of noncommutative individual ergodic theorems in the space of measurable operators affiliated with a semifinite von Neumann algebra $M \subset B(H)$ equipped with a faithful normal semifinite trace $\tau$ was initiated by F. Yeadon. In [23], as a corollary of a noncommutative maximal ergodic inequality in $L^1 = L^1(M,\tau)$, the following individual ergodic theorem was established.

Theorem 1.1. Let $T : L^1 \to L^1$ be a positive $L^1 - L^\infty$-contraction. Then for any $x \in L^1$ there exists $\hat{x} \in L^1$ such that the averages

$$A_n(T)(x) = \frac{1}{n+1} \sum_{k=0}^n T^k(x)$$

converge to $\hat{x}$ bilaterally almost uniformly (in Egorov’s sense), that is, given $\varepsilon > 0$, there exists a projection $e \in M$ such that $\tau(1-e) < \varepsilon$ and

$$\|e(A_n(T)(x) - \hat{x})e\|_\infty \to 0 \quad \text{as} \quad n \to \infty,$$

where $1$ is the unit of $M$.

The study of individual ergodic theorems beyond $L^1(M,\tau)$ started much later with another fundamental paper by M. Junge and Q. Xu [14], where, among other results, individual ergodic theorem was extended to the case with a positive Dunford-Schwartz operator acting in the space $L^p(M,\tau)$, $1 < p < \infty$. In [4]...
(5), utilizing the approach of [17], an individual ergodic theorem was proved for a positive Dunford-Schwartz operator in a noncommutative Lorentz (respectively, Orlicz) space.

Let $H$ be an infinite-dimensional Hilbert space. Let $E \subset c_0$ be a fully symmetric sequence space. Denote by $C_E$ the Banach ideal of compact operators in $H$ associated with $E$. In Section 3 of the article, we obtain the following individual Dunford-Schwartz-type ergodic theorem.

**Theorem 1.4.** (i) Given a Dunford-Schwartz operator $T : C_E \to C_E$ and $x \in C_E$, there exists $\widehat{x} \in C_E$ such that $\|A_n(T)(x) - \widehat{x}\|_\infty \to 0$ as $n \to \infty$;
(ii) if $x \in B(H) \setminus K(H)$, then there exists a Dunford-Schwartz operator $T : B(H) \to B(H)$ such that the averages $A_n(T)(x)$ do not converge uniformly.

Noncommutative mean ergodic theorem can be stated as follows: if $T$ is an $L^1 - L^\infty$-contraction and $1 < p < \infty$, then the averages $A_n(T)$ converge strongly in $L^p = L^p(M, \tau)$, that is, given $x \in L^p$, there exists $\widehat{x} \in L^p$ such that $\|A_n(T)(x) - \widehat{x}\|_p \to 0$ as $n \to \infty$. If $p = 1$ and $\tau(1) = \infty$, this is not true in general. As a consequence, if $\tau(1) = \infty$, mean ergodic theorem may not hold in some noncommutative symmetric spaces. In Yeadon’s paper [24], the following mean ergodic theorem was established.

**Theorem 1.3.** Let $E = (E(M, \tau), \| : \|_E)$ be a noncommutative fully symmetric space such that
(i) $L^1(M, \tau) \cap M$ is dense in $E$;
(ii) $\|e_n\|_E \to 0$ for any sequence of projections $\{e_n\} \subset L^1(M, \tau) \cap M$ with $e_n \downarrow 0$;
(iii) $\|e_n\|_E / \tau(e_n) \to 0$ for any increasing sequence of projections $\{e_n\} \subset M$,
$$0 < \tau(e_n) < \infty, \text{ with } \tau(e_n) \to \infty.$$ Then for any $x \in E$ and a positive $L^1 - L^\infty$-contraction $T : E \to E$ there exists $\widehat{x} \in E$ such that $\|A_n(T)(x) - \widehat{x}\|_E \to 0$.

In [3], a mean ergodic theorem was established for a noncommutative symmetric space $E(M, \tau)$ associated with a fully symmetric function space with nontrivial Boyd indices and order continuous norm.

In Section 4, we give the following criterion for the validity of the mean ergodic theorem in a Banach ideal of compact operators in $H$.

**Theorem 1.4.** The following conditions are equivalent:
(i) For any Dunford-Schwartz operator $T : C_E \to C_E$ the averages $A_n(T)$ converge strongly in $C_E$;
(ii) $(E, \| : \|_E)$ is separable and $E \neq l^1$, as sets.

Commutative counterparts of Theorems 1.2 and 1.4 were established in [8].

In the last section of the article, we present applications of Theorems 1.2 and 1.4 to the well-studied Orlicz, Lorentz, and Marcinkiewicz ideals of compact operators.

2. Preliminaries

Let $l^\infty$ (respectively, $c_0$) be the Banach lattice of bounded (respectively, converging to zero) sequences $\{\xi_n\}_{n=1}^\infty$ of complex numbers equipped with the norm $\|\{\xi_n\}\|_\infty = \text{sup} \{\xi_n\}$, where $N$ is the set of natural numbers. If $2^N$ is the $\sigma$-algebra of subsets of $N$ and $\mu(\{n\}) = 1$ for each $n \in N$, then $(N, 2^N, \mu)$ is a $\sigma$-finite measure.
space such that \( L^\infty(N, 2^N, \mu) = l^\infty \) and
\[
L^1(N, 2^N, \mu) = l^1 = \left\{ \{\xi_n\}_{n=1}^\infty \subset \mathbb{C} : \|\{\xi_n\}\|_1 = \sum_{n=1}^\infty |\xi_n| < \infty \right\} \subset l^\infty,
\]
where \( \mathbb{C} \) is the set of complex numbers.

If \( \xi = \{\xi_n\}_{n=1}^\infty \in l^\infty \), then the non-increasing rearrangement \( \xi^* : (0, \infty) \to (0, \infty) \) of \( \xi \) is defined by
\[
\xi^*(t) = \inf \{\lambda : \mu(\{\xi| > \lambda\}) \leq t\}, \quad t > 0,
\]
(see, for example, [2 Ch. 2, Definition 1.5]). As such, the non-increasing rearrangement of a sequence \( \{\xi_n\}_{n=1}^\infty \in l^\infty \) can be identified with the sequence \( \xi^* = \{\xi^*_n\}_{n=1}^\infty \), where
\[
\xi^*_n = \inf \left\{ \sup_{n\notin F} |\xi_n| : F \subset \mathbb{N}, \ |F| < n \right\}.
\]
If \( \{\xi_n\} \in c_0 \), then \( \xi^*_n \downarrow 0 \); in this case there exists a bijection \( \pi : \mathbb{N} \to \mathbb{N} \) such that \( |\pi(n)| = \xi^*_n, \ n \in \mathbb{N} \).

**Hardy-Littlewood-Polya partial order** in the space \( l^\infty \) is defined as follows:
\[
\xi = \{\xi_n\} \prec \prec \eta = \{\eta_n\} \iff \sum_{n=1}^m \xi_n^* \leq \sum_{n=1}^m \eta_n^* \quad \text{for all} \quad m \in \mathbb{N}.
\]

A non-zero linear subspace \( E \subset l^\infty \) with a Banach norm \( \| \cdot \|_E \) is called a symmetric (fully symmetric) sequence space if
\[
\eta \in E, \ \xi \in l^\infty, \ \xi^* \leq \eta^* \quad \text{(resp.,} \ \xi^* \prec \prec \eta^*) \quad \implies \quad \xi \in E \quad \text{and} \quad \|\xi\|_E \leq \|\eta\|_E.
\]
Every fully symmetric sequence space is a symmetric sequence space. The converse is not true in general. At the same time, any separable symmetric sequence space is a fully symmetric space.

If \( (E, \| \cdot \|_E) \) is a symmetric sequence space, then
\[
\|\xi\|_E = \|\xi^*\|_E = \|\xi^*\|_E \quad \text{for all} \quad \xi \in E.
\]

Besides, if \( E_h = \{\{\xi_n\}_{n=1}^\infty \in E : \ \xi_n \in \mathbb{R} \quad \text{for each} \ n\} \), where \( \mathbb{R} \) is the set of real numbers, then \( (E_h, \| \cdot \|_E) \) is a Banach lattice with respect to the natural partial order
\[
\{\xi_n\} \leq \{\eta_n\} \iff \xi_n \leq \eta_n \quad \text{for all} \quad n \in \mathbb{N}.
\]

Immediate examples of fully symmetric sequence spaces are \( (l^\infty, \| \cdot \|_\infty) \) and \( (c_0, \| \cdot \|_\infty) \) and the Banach lattices
\[
l^p = \left\{ \xi = \{\xi_n\}_{n=1}^\infty \in c_0 : \|\xi\|_p = \left( \sum_{n=1}^\infty |\xi_n|^p \right)^{1/p} < \infty \right\}, \ 1 \leq p < \infty.
\]

For any symmetric sequence space \( (E, \| \cdot \|_E) \) the following continuous embeddings hold [2 Ch. 2, §6, Theorem 6.6]:
\[
(l^1, \| \cdot \|_1) \subset (E, \| \cdot \|_E) \subset (l^\infty, \| \cdot \|_\infty).
\]

Besides, \( \|\xi\|_E \leq \|\xi\|_1 \) for all \( \xi \in l^1 \) and \( \|\xi\|_\infty \leq \|\xi\|_E \) for all \( \xi \in E \).

If there is \( \xi \in E \setminus c_0 \), then \( \xi^* \geq \alpha 1 \) for some \( \alpha > 0 \), where \( 1 = \{1, 1, \ldots\} \).

Consequently, \( 1 \in E \) and \( E = l^\infty \). Therefore, either \( E \subset c_0 \) or \( E = l^\infty \).

Now, let \( (\mathcal{H}, (\cdot, \cdot)) \) be an infinite-dimensional Hilbert space over \( \mathbb{C} \), and let \( (\mathcal{B}(\mathcal{H}), \| \cdot \|_\infty) \) be the \( C^* \)-algebra of bounded linear operators in \( \mathcal{H} \). Denote by
such that $K$-two-sided ideals imply that $y$ is a fully symmetric space. Then, let $\lambda > 0$. This means that $x \in X$, $y \in B(\mathcal{H})$, $\mu_1(y) \leq \mu_1(x)$ for all $t > 0$, which implies that $y \in X$ and $\|y\|_X \leq \|x\|_X$. The spaces $B(\mathcal{H}), \| \cdot \|_X$ and $\mathcal{K}(\mathcal{H}), \| \cdot \|_X$ are examples of fully symmetric spaces.

It should be noted that for every symmetric space $(X, \| \cdot \|_X) \subset B(\mathcal{H})$ and all $x \in X$, $a, b \in B(\mathcal{H})$,

$$\|x\|_X = \|x\|_X = \|Ax\|_X, \ a \in X, \text{ and } \|axb\|_X \leq \|a\|_\infty \|b\|_\infty \|x\|_X.$$ 

**Remark 2.1.** If $X \subset B(\mathcal{H})$ is a symmetric space and there exists a projection $e \in \mathcal{P}(\mathcal{H}) \cap X$ such that $\tau(e) = \infty$, that is, $\dim e(\mathcal{H}) = \infty$, then $\mu_1(e) = \mu_1(1) = 1$ for every $t \in (0, \infty)$. Consequently, $1 \in X$ and $X = B(\mathcal{H})$. If $X \neq B(\mathcal{H})$ and $x \in X$, then $e_\lambda = \{x : \lambda \}

is a finite-dimensional projection, that is, $\dim e_\lambda(\mathcal{H}) < \infty$ for all $\lambda > 0$. This means that $x \in \mathcal{K}(\mathcal{H})$, hence $X \subset \mathcal{K}(\mathcal{H})$. Therefore, either $X = B(\mathcal{H})$ or $X \subset \mathcal{K}(\mathcal{H})$.

Thus, if $\mathcal{H}$ is non-separable, then there exists a proper two-sided ideal $\mathcal{I} \subset B(\mathcal{H})$ such that $\mathcal{K}(\mathcal{H}) \subset \mathcal{I}$ and $(\mathcal{I}, \| \cdot \|_X)$ is a Banach space which is not a symmetric subspace of $B(\mathcal{H})$. 

\[ \tau(x) = \sum_{j \in J} \langle x \varphi_j, \varphi_j \rangle, \quad x \in B(\mathcal{H}), \]

where $\{\varphi_j\}_{j \in J}$ is an orthonormal basis in $\mathcal{H}$ (see, for example, [21, Ch. 7, E. 7.5]).

Let $\mathcal{P}(\mathcal{H})$ be the lattice of projections in $\mathcal{H}$. If $1$ is the identity of $B(\mathcal{H})$ and $e \in \mathcal{P}(\mathcal{H})$, we will write $e^\perp = 1 - e$.

Let $x \in B(\mathcal{H})$, and let $\{e_\lambda\}_{\lambda \geq 0}$ be the spectral family of projections for the absolute value $|x| = (x^*x)^{1/2}$ of $x$, that is, $e_\lambda = \{|x| > \lambda\}$. If $t > 0$, then the $t$-th generalized singular number of $x$, or the non-increasing rearrangement of $x$, is defined as

$$\mu_t(x) = \inf \{\lambda > 0 : \tau(e_\lambda^+) \leq t\}$$

(see [12]).

A non-zero linear subspace $X \subset B(\mathcal{H})$ with a Banach norm $\| \cdot \|_X$ is called symmetric (fully symmetric) if the conditions

$x \in X$, $y \in B(\mathcal{H})$, $\mu_t(y) \leq \mu_t(x)$ for all $t > 0$

(respectively,

$x \in X$, $y \in B(\mathcal{H})$, $\int_0^t \mu_t(y)dt \leq \int_0^t \mu_t(x)dt$ for all $s > 0$ (writing $y \prec \prec x$))

imply that $y \in X$ and $\|y\|_X \leq \|x\|_X$.

The spaces $(B(\mathcal{H}), \| \cdot \|_X)$ and $(\mathcal{K}(\mathcal{H}), \| \cdot \|_X)$ as well as the classical Banach two-sided ideals

$L^p(B(\mathcal{H}), \tau) = C^p = \{x \in \mathcal{K}(\mathcal{H}) : \|x\|_p = \tau(|x|^p)^{1/p} < \infty\}, 1 \leq p < \infty$, are examples of fully symmetric spaces.

It should be noted that for every symmetric space $(X, \| \cdot \|_X) \subset B(\mathcal{H})$ and all $x \in X$, $a, b \in B(\mathcal{H})$,

$$\|x\|_X = \|x\|_X = \|Ax\|_X, \ a \in X, \text{ and } \|axb\|_X \leq \|a\|_\infty \|b\|_\infty \|x\|_X.$$ 

\[ \tau(x) = \sum_{j \in J} \langle x \varphi_j, \varphi_j \rangle, \quad x \in B(\mathcal{H}), \]
If $x \in \mathcal{K}(\mathcal{H})$, then $|x| = \sum_{n=1}^{m(x)} s_n(x)p_n$ (if $m(x) = \infty$, the series converges uniformly), where $\{s_n(x)\}_{n=1}^{m(x)}$ is the set of singular values of $x$, that is, the set of eigenvalues of the compact operator $|x|$ in the decreasing order, and $p_n$ is the projection onto the eigenspace corresponding to $s_n(x)$. Consequently, the non-increasing re-arrangement $\mu(x)$ of $x \in \mathcal{K}(\mathcal{H})$ can be identified with the sequence $\{s_n(x)\}_{n=1}^{\infty}$, $s_n(x) \downarrow 0$ (if $m(x) < \infty$, we set $s_n(x) = 0$ for all $n > m(x)$).

Let $(X, \|\cdot\|_X) \in \mathcal{K}(\mathcal{H})$ be a symmetric space. Fix an orthonormal basis $\{\varphi_j\}_{j \in J}$ in $\mathcal{H}$ and choose a countable subset $\{\varphi_j\}_{j=1}^{\infty}$. Let $p_n$ be the one-dimensional projection on the subspace $\mathbb{C} \varphi_j \subset \mathcal{H}$. It is clear that the set

$$E(X) = \left\{ \xi = \{\xi_n\}_{n=1}^{\infty} \in c_0 : x_{\xi} = \sum_{n=1}^{\infty} \xi_n p_n \in X \right\}$$

(the series converges uniformly), is a symmetric sequence space with respect to the norm $\|\xi\|_{E(X)} = \|x_{\xi}\|_X$. Consequently, each symmetric subspace $(X, \|\cdot\|_X) \subset \mathcal{K}(\mathcal{H})$ uniquely generates a symmetric sequence space $(E(X), \|\cdot\|_{E(X)}) \subset c_0$. The converse is also true: every symmetric sequence space $(E, \|\cdot\|_E) \subset c_0$ uniquely generates a symmetric space $(C_E, \|\cdot\|_{C_E}) \subset \mathcal{K}(\mathcal{H})$ by the following rule (see, for example, [13, Ch. 3, Section 3.5]):

$$C_E = \{x \in \mathcal{K}(\mathcal{H}) : \{s_n(x)\} \in E\}, \quad \|x\|_{C_E} = \|\{s_n(x)\}\|_{E}.$$  

In addition, $E(C_E) = E, \|\cdot\|_{E(C_E)} = \|\cdot\|_E, C_{E(C_E)} = C_E, \|\cdot\|_{C_{E(C_E)}} = \|\cdot\|_{C_E}$.

We will call the pair $(C_E, \|\cdot\|_{C_E})$ a Banach ideal of compact operators (cf. [13, Ch. III]). It is known that $(C^p, \|\cdot\|_{C^p}) = (C^p, \|\cdot\|_{c_0})$ for all $1 \leq p < \infty$ and $(\mathcal{K}(\mathcal{H}), \|\cdot\|_{\infty}) = (C_{c_0}, \|\cdot\|_{c_0})$.

Hardy-Littlewood-Paley partial order in the Banach ideal $\mathcal{K}(\mathcal{H})$ is defined by

$$x \prec y, \ x, y \in \mathcal{K}(\mathcal{H}) \iff \{s_n(x)\} \prec \{s_n(y)\}.$$  

We say that a Banach ideal $(C_E, \|\cdot\|_{C_E})$ is fully symmetric if conditions $y \in C_E$, $x \in \mathcal{K}(\mathcal{H})$, $x \prec y$ entail that $x \in C_E$ and $\|x\|_{C_E} \leq \|y\|_{C_E}$. It is clear that $(C_E, \|\cdot\|_{C_E})$ is a fully symmetric ideal if and only if $(E, \|\cdot\|_E)$ is a fully symmetric sequence space.

Examples of fully symmetric ideals include $(\mathcal{K}(\mathcal{H}), \|\cdot\|_{\infty})$ as well as the Banach ideals $(C^p, \|\cdot\|_{C^p})$ for all $1 \leq p < \infty$. It is clear that $C^1 \subset C_E \subset \mathcal{K}(\mathcal{H})$ for every symmetric sequence space $E \subset c_0$ with $\|x\|_{C_E} \leq \|x\|_1$ and $\|y\|_{\infty} \leq \|y\|_{C_E}$ for all $x \in C^1$ and $y \in C_E$.

We will need the following property of Hardy-Littlewood-Polya partial order.

**Proposition 2.1.** If $x, y, y_k \in \mathcal{K}(\mathcal{H})$ are such that $y_k \prec x$ for all $k \in \mathbb{N}$ and $\|y_k - y\|_{\infty} \to 0$ as $k \to \infty$, then $y \prec x$.

**Proof.** Since $y_k \prec x$, it follows that $\sum_{n=1}^{m} s_n(y_k) \leq \sum_{n=1}^{m} s_n(x)$ for all $m, k \in \mathbb{N}$. By [13, Ch. II, § 2, Sec.3, Corollary 2.3], $|s_n(y_k) - s_n(y)| \leq \|y_k - y\|_{\infty} \to 0$, hence $\sum_{n=1}^{m} s_n(y_k) \to \sum_{n=1}^{m} s_n(y)$ as $k \to \infty$ for every $m \in \mathbb{N}$. Therefore

$$\lim_{k \to \infty} \sum_{n=1}^{m} s_n(y_k) \leq \sum_{n=1}^{m} s_n(x)$$

for all $m$. \qed
Define 
\[ R_\tau = \{ x \in \mathcal{B}(\mathcal{H}) : \mu_t(x) \to 0 \text{ as } t \to \infty \}. \]

By [9] Proposition 2.7, \( R_\tau \) is the closure of \( C^1 \) in \( (\mathcal{B}(\mathcal{H}),\| \cdot \|_\infty) \), implying that 
\( R_\tau = \mathcal{K}(\mathcal{H}) \). Therefore \( \lim_{t \to \infty} \mu_t(x) > 0 \) for every \( x \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H}) \); in particular, there exists \( \lambda > 0 \) such that \( \tau\{|x| > \lambda\} = \infty \).

A linear operator \( T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) is called a Dunford-Schwartz operator if 
\[ \|T(x)\|_1 \leq \|x\|_1 \text{ for all } x \in C^1 \text{ and } \|T(x)\|_\infty \leq \|x\|_\infty \text{ for all } x \in \mathcal{B}(\mathcal{H}). \]

In what follows, we will write \( T \in DS \) (\( T \in DS_+ \)) to indicate that \( T \) is a Dunford-Schwartz operator (respectively, a positive Dunford-Schwartz operator, that is, \( T \in DS \) and \( T(\mathcal{B}_+(\mathcal{H})) \subset \mathcal{B}_+(\mathcal{H}) \)).

Any fully symmetric sequence space \( E \) is an exact interpolation space in the Banach pair \( (l^1,l^\infty) \) (see, for example, [15] Ch. II, §4, Sec. 2). Therefore, for such \( E \), the fully symmetric ideal \( C_E \) is an exact interpolation space in the Banach pair \( (C^1,B(\mathcal{H})) \); see [8] Theorem 2.4. It then follows that \( T(C_E) \subset C_E \) and \( \|T\|_{C_E \to C_E} \leq 1 \) for all \( T \in DS \). In particular, \( T(\mathcal{K}(\mathcal{H})) \subset \mathcal{K}(\mathcal{H}) \) and the restriction of \( T \) on \( \mathcal{K}(\mathcal{H}) \) is a linear contraction (also denoted by \( T \)). We note that if \( T \in DS \), then \( A_n(T) \in DS \); also, \( T(x) \ll x \) and \( A_n(T)(x) \ll x \) for any \( x \in \mathcal{K}(\mathcal{H}) \) and \( n \).

3. Individual ergodic theorem in fully symmetric ideals of compact operators

Let \( \mathcal{H}, \tau : \mathcal{B}_+(\mathcal{H}) \to [0,\infty) \), and \( C^1 \) be as above. Utilizing Theorem 1.1 with \( \mathcal{M} = \mathcal{B}(\mathcal{H}) \) and taking into account that \( \tau(e) \geq 1 \) for every \( 0 \neq e \in \mathcal{P}(\mathcal{H}) \), we arrive at the following.

**Theorem 3.1.** Given \( T \in DS_+ \) and \( x \in C^1 \), there exists \( \hat{x} \in C^1 \) such that 
\[ \|A_n(T)(x) - \hat{x}\|_\infty \to 0 \text{ as } n \to \infty. \]

Theorem 3.1 can be extended to the fully symmetric ideal \( \mathcal{K}(\mathcal{H}) \). In fact, such an extension holds for any \( T \in DS \):

**Theorem 3.2.** Let \( T \in DS \) and \( x \in \mathcal{K}(\mathcal{H}) \). Then there exists \( \hat{x} \in \mathcal{K}(\mathcal{H}) \) such that 
\[ \|A_n(T)(x) - \hat{x}\|_\infty \to 0 \text{ as } n \to \infty. \]

**Proof.** Since \( T(C^2) \subset C^2 \), \( \|T\|_{C^2 \to C^2} \leq 1 \) and the Banach space \( C^2 \) is reflexive, by the mean ergodic theorem [17] Ch. VIII, §5, Corollary 4, the sequence \( \{A_n(T)(x)\} \) converges strongly in \( C^2 \), that is, for every \( x \in C^2 \) there exists \( \hat{x} \in C^2 \) such that \( \|A_n(T)(x) - \hat{x}\|_2 \to 0 \). As \( \|\xi\|_\infty \leq \|\xi\|_2 \) for all \( \xi \in l^2 \), it follows that \( \|x\|_\infty \leq \|x\|_2 \) for all \( x \in C^2 \). Consequently,
\[ \|A_n(T)(x) - \hat{x}\|_\infty \to 0 \quad \text{for every } x \in C^2. \]

Let now \( x \in \mathcal{K}(\mathcal{H}) \) and \( \varepsilon > 0 \). Then there exists \( x_\varepsilon \in \mathcal{F}(\mathcal{H}) \subset C^2 \) such that \( \|x - x_\varepsilon\|_\infty < \varepsilon/4 \). Since the sequence \( A_n(T)(x_\varepsilon) \) converges uniformly, there exists \( N = N(\varepsilon) \) such that
\[ \|A_n(T)(x_\varepsilon) - A_m(T)(x_\varepsilon)\|_\infty < \frac{\varepsilon}{2} \quad \text{whenever } m,n \geq N. \]

Therefore,
\[ \|A_m(T)(x) - A_n(T)(x)\|_\infty \leq \|A_m(T)(x - x_\varepsilon) - A_n(T)(x - x_\varepsilon)\|_\infty \]
\[ + \|A_m(T)(x_\varepsilon) - A_n(T)(x_\varepsilon)\|_\infty \leq 2\|x - x_\varepsilon\|_\infty + \frac{\varepsilon}{2} < \varepsilon. \]
for all \( m, n \geq N \). Thus, since the space \((\mathcal{K}(\mathcal{H}), \| \cdot \|_{\infty})\) is complete, there exists \( \bar{x} \in \mathcal{K}(\mathcal{H}) \) such that \( \| A_n(T)(x) - \bar{x} \|_{\infty} \to 0 \). □

By virtue of Theorem 3.2, we now derive part (i) of Theorem 1.2, an individual ergodic theorem in fully symmetric ideals of compact operators:

**Theorem 3.3.** Let \( \mathcal{C}_E \) be a fully symmetric ideal of compact operators, and let \( T \in DS \). Then, given \( x \in \mathcal{C}_E \), the averages \( A_n(T)(x) \) converge uniformly to some \( \bar{x} \in \mathcal{C}_E \).

**Proof.** As \( \mathcal{C}_E \subset \mathcal{K}(\mathcal{H}) \), it follows from Theorem 3.2 that the sequence \( \{ A_n(T)(x) \} \) converges uniformly to some \( \bar{x} \in \mathcal{K}(\mathcal{H}) \), while Proposition 2.1 implies that \( \bar{x} \prec \prec x \), hence \( \bar{x} \in \mathcal{C}_E \). □

The rest of this section is devoted to proving part (ii) of Theorem 1.2, if \( x \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H}) \), then there exists \( T \in DS \) such that the sequence \( \{ A_n(T)(x) \} \) does not converge uniformly (Theorem 3.3 below).

We begin with a Dunford-Schwartz operator acting in the Banach space \((l^\infty, \| \cdot \|_{\infty})\), that is, when a linear operator \( T : l^\infty \to l^\infty \) is such that \( \| T(\xi) \|_1 \leq \| \xi \|_1 \) for all \( \xi \in l^1 \) and \( \| T(\xi) \|_{\infty} \leq \| \xi \|_{\infty} \) for all \( \xi \in l^\infty \) (writing \( T \in DS \)). In this case, we have a commutative version of Theorem 1.2(ii) (cf. [10, Theorem 3.3]):

**Theorem 3.4.** If \( \xi \in l^\infty \setminus c_0 \), then there exists \( T \in DS \) such that the averages \( A_n(T)(\xi) \) do not converge coordinate-wise, hence uniformly.

**Proof.** Let \( \{ \xi_n \}_{n=1}^{\infty} \subset l^\infty \setminus c_0 \). If \( \xi = \xi_+ - \xi_- \), then either \( \xi_+ = \{ (\xi_+)_{n=1}^{\infty} \in l^\infty \setminus c_0 \) or \( \xi_- \in l^\infty \setminus c_0 \), so let us assume the former. In addition, we may assume that \( \lim_{n \to \infty} (\xi_+)^n = 1 \). It is clear that the set \( \{ n \in \mathbb{N} : (\xi_+)_n \geq 1 \} \) is infinite. In addition, the set

\[ G = \{ m \in \mathbb{N} : 1 \leq (\xi_+)_m \leq 2 \} = \{ m_1 < m_2 < \ldots < m_s < \ldots \} \]

is also infinite.

Let \( 1 = n_0, n_1, n_2, \ldots \) be an increasing sequence of positive integers. Define the function \( \varphi : \mathbb{N} \to \mathbb{R} \) by

\[ \varphi(m) = \chi(m_{n_0}) + \sum_{k=0}^{\infty} \left( \chi(m_{n_{k+1}}, m_{n_{k+2}}, \ldots, m_{n_{k+1}-1})(m) - \chi(m_{n_{k+1}})(m) \right) \] if \( m \in G; \]

\[ \varphi(m) = 0 \] if \( m \notin G \).

Then we have

\[ \varphi(m_1) = 1, \varphi(m_2) = 1, \ldots, \varphi(m_{n_1-1}) = 1, \varphi(m_{n_1}) = -1, \]

\[ \varphi(m_{n_1+1}) = 1, \varphi(m_{n_1+2}) = 1, \ldots, \varphi(m_{n_2-1}) = 1, \varphi(m_{n_2}) = -1, \]

\[ \varphi(m_{n_2+1}) = 1, \varphi(m_{n_2+2}) = 1, \ldots, \varphi(m_{n_3-1}) = 1, \varphi(m_{n_3}) = -1, \ldots \]

Let \( \pi : \mathbb{N} \to \mathbb{N} \) be given by

\[ \pi(m_i) = m_{i+1} \] if \( m_i \in G \) and \( \pi(m) = m \) if \( m \notin G \).

Define a linear operator \( T : l^\infty \to l^\infty \) by

\[ T(\{ \eta_n \}_{n=1}^{\infty}) = \{ \varphi(n) \eta_{\pi(n)} \}_{n=1}^{\infty}, \quad \{ \eta_n \}_{n=1}^{\infty} \in l^\infty. \]

Then, clearly, \( T \in DS \).

Since

\[ T^k(\xi_+) = \varphi(m) \varphi(\pi(m)) \varphi(\pi^2(m)) \ldots \varphi(\pi^{k-1}(m)) (\xi_+)_{\pi^k(m)} \]
for all \( k, m \in \mathbb{N} \), it follows that
\[
A_{n_1-1}(T)(\xi_+)_m = \frac{1}{n_1} \sum_{k=0}^{n_1-1} T^k(\xi_+)_{m_1}
\]
\[
= \frac{1}{n_1} \left( (\xi_+)_m + \sum_{k=1}^{n_1-1} \varphi(m_1)\varphi(m_2)\ldots\varphi(m_k)(\xi_+)_{m_{k+1}} \right)
\]
\[
= \frac{1}{n_1} \sum_{k=0}^{n_1-1} (\xi_+)_{m_{k+1}} \geq 1 > \frac{1}{2}.
\]

Further, since
\[
A_{n_2-1}(T)(\xi_+)_m = \frac{1}{n_2} \left( \sum_{k=0}^{n_2-1} (\xi_+)_{m_{k+1}} - \sum_{k=n_1}^{n_2-1} (\xi_+)_{m_{k+1}} \right)
\]
\[
\leq \frac{1}{n_2-1} \left( 2n_1 - (n_2 - n_1 - 1) \right),
\]

there exists such \( n_2 > n_1 \) that
\[
A_{n_2-1}(T)(\xi_+)_m < -\frac{1}{2}.
\]

As
\[
A_{n_3-1}(T)(\xi_+)_m = \frac{1}{n_3} \left( \sum_{k=0}^{n_3-1} (\xi_+)_{m_{k+1}} - \sum_{k=n_1}^{n_3-1} (\xi_+)_{m_{k+1}} + \sum_{k=n_2}^{n_3-1} (\xi_+)_{m_{k+1}} \right),
\]

one can find \( n_3 > n_2 \) for which
\[
A_{n_3-1}(T)(\xi_+)_m > \frac{1}{2}.
\]

Continuing this procedure, we choose \( n_1 < n_2 < n_3 < \ldots \) to satisfy the inequalities
\[
A_{n_{2k-1}-1}(T)(\xi_+)_m > \frac{1}{2} \quad \text{and} \quad A_{n_{2k-1}}(T)(\xi_+)_m < -\frac{1}{2}, \quad k = 1, 2, \ldots,
\]
implying that \( \{A_n(T)(\xi_+)_m\} \) is a divergent sequence.

Finally, note that \( T(\xi_-) = 0 \), which implies that
\[
A_n(T)(\xi) = A_n(T)(\xi_+) - A_n(T)(\xi_-) = A_n(T)(\xi_+) - \frac{1}{n+1}\xi_-
\]
so, the sequence \( \{A_n(T)(\xi)\} \) does not converge coordinate-wise, hence uniformly.

If \( \xi \in l^\infty \setminus c_0 \), then
\[
\xi = \Re \xi + i \Im \xi, \quad \text{where} \quad \Re \xi = \frac{\xi + \overline{\xi}}{2}, \quad \Im \xi = \frac{\xi - \overline{\xi}}{2i} \in l^\infty.
\]

As shown above, there exists \( T \in DS \) such that the sequence \( \{A_n(T)(\Re \xi)\} \) does not converge coordinate-wise. Then the sequence \( \{A_n(T)(\xi)\} \) also does not converge coordinate-wise, hence uniformly.

Now we need a statement on the existence of conditional expectation in a von Neumann algebra \( B(\mathcal{H}) \) (see, for example, [22]).

**Theorem 3.5.** Let \( \mathcal{N} \) be a von Neumann subalgebra in \( B(\mathcal{H}) \) such that the restriction of the trace \( \tau \) on \( \mathcal{N} \) is a semifinite trace. Then there exists a unique linear map \( U : B(\mathcal{H}) \rightarrow \mathcal{N} \) (conditional expectation on \( \mathcal{N} \)), having the following properties:
(i) \( \tau(x) = \tau(U(x)) \) for all \( x \in \mathcal{C}^1 \);
(ii) \( U(x) = x \) for all \( x \in \mathcal{N} \);
(iii) \( U \in DS_+ \); moreover, \( \|U\|_{\mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})} = 1 \) and \( \|U\|_{\mathcal{C}^1 \rightarrow \mathcal{C}^1} = 1 \).

Assume first that \( (\mathcal{H}, (\cdot, \cdot)) \) is a separable infinite-dimensional complex Hilbert space. Fix an orthonormal basis \( \{\varphi_n\}_{n \in \mathbb{N}} \) in \( \mathcal{H} \). Let \( p_n \) be the one-dimensional projection on the linear subspace \( \mathbb{C} \cdot \varphi_n \subset \mathcal{H} \). It is clear that \( p_m p_n = 0 \) for all \( m, n \in \mathbb{N}, n \neq m \).

For any \( \xi = \{\xi_n\}_{n=1}^\infty \in l^\infty \) and \( h = \sum_{n=1}^{\infty} (h, \varphi_n) \varphi_n \in \mathcal{H} \) we set

\[
x_\xi(h) = \sum_{n=1}^{\infty} \xi_n (h, \varphi_n) \varphi_n = \sum_{n=1}^{\infty} \xi_n p_n(h).
\]

It is clear that \( x_\xi \in \mathcal{B}(\mathcal{H}) \) and \( x_\xi = (u_0) - \sum_{n=1}^{\infty} \xi_n p_n \), where \( (u_0) \) stands for the weak operator topology. In addition,

\[
\mathcal{N} = \{ x_\xi \in \mathcal{B}(\mathcal{H}) : \xi = \{\xi_n\}_{n=1}^\infty \in l^\infty \}
\]

is the smallest commutative von Neumann subalgebra in \( \mathcal{B}(\mathcal{H}) \) containing all projections \( p_n \). Besides, the restriction of the trace \( \tau \) on \( \mathcal{N} \) is a semifinite trace.

Define the linear map \( \Phi : (\mathcal{N}, \| \cdot \|_\infty) \rightarrow (l^\infty, \| \cdot \|_\infty) \) by setting \( \Phi(x_\xi) = \xi \).

**Proposition 3.1.** \( \Phi \) is a positive linear surjective isometry.

**Proof.** By definition of \( \Phi \), we have \( \Phi(\mathcal{N}) = l^\infty \). Using [13, Ch. 1, §1.1, E.1.1.11], we see that

\[
\|x_\xi\|_\infty = \|\xi\|_\infty = \|\Phi(x_\xi)\|_\infty,
\]

that is, \( \Phi \) is a linear surjective isometry.

Since \( \xi = \{\xi_n\}_{n=1}^\infty \geq 0 \) whenever \( x_\xi \in \mathcal{N}_+ \), the map \( \Phi \) is positive. \( \square \)

Let \( (E, \| \cdot \|_E) \subset c_0 \) be a symmetric sequence space, and let \( \mathcal{N}_E = \mathcal{N} \cap C_E \). If \( x_\xi = \sum_{n=1}^{\infty} \xi_n p_n \in \mathcal{N}_E \), then \( \{s_n(x_\xi)\}_{n=1}^\infty = \{\xi_n^*\} \in E \), hence \( \{\xi_n\} \in E \). In addition,

\[
\|x_\xi\|_{C_E} = \|\{\xi_n^*\}\|_E = \|\{\xi_n\}\|_E.
\]

Therefore, we have the following.

**Proposition 3.2.** If \( (E, \| \cdot \|_E) \subset c_0 \) is a symmetric sequence space, then the restriction \( \Phi|_{\mathcal{N}_E} : (\mathcal{N}_E, \| \cdot \|_{C_E}) \rightarrow (E, \| \cdot \|_E) \) is a positive linear surjective isometry (we denote this restriction also by \( \Phi \)).

**Theorem 3.6.** If \( x \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H}) \), then there exists \( T \in DS \) such that the sequence \( \{A_n(T)(x)\} \) does not converge uniformly.

**Proof.** Assume first that \( x \geq 0 \) and \( \mathcal{H} \) is separable. Since \( x \notin \mathcal{K}(\mathcal{H}) \), it follows that there exists a spectral projection \( e_\lambda = \{x > \lambda\}, \lambda > 0 \), such that \( \tau(e_\lambda) = \infty \). Choose an orthonormal basis \( \{\varphi_n\}_{n=1}^\infty \) in \( \mathcal{H} \) such that \( e_\lambda \geq p_n \), for some sequence \( \{n_i\}_{i=1}^\infty \), where \( p_n \) is the one-dimensional projection on the subspace \( \mathbb{C} \cdot \varphi_n \subset \mathcal{H} \).

Let \( \mathcal{N} = \{x_\xi \in \mathcal{B}(\mathcal{H}) : \xi = \{\xi_n\}_{n=1}^\infty \in l^\infty \} \) be the smallest commutative von Neumann subalgebra in \( \mathcal{B}(\mathcal{H}) \) containing all projections \( p_n \). By virtue of Theorem 3.5 there exists a conditional expectation \( U : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{N} \) such that

\[
0 \leq y = U(x) \geq U(\lambda e_\lambda) \geq \lambda U(p_{n_i}) = \lambda p_{n_i}, \quad \text{for all } i \in \mathbb{N}.
\]
Consequently, \( y \notin \mathcal{R}_x \) and \( y = x_\xi \in \mathcal{N} \), where \( 0 \leq \xi = \{ \xi_n \}_{n=1}^{\infty} \in l^\infty \setminus c_0 \). Besides, by definition of \( \Phi \), we have \( \Phi(y) = \xi \).

Next, by Theorem 3.3 there exists an operator \( S : l^\infty \to l^\infty \), \( S \in \mathcal{D} \), such that the sequence \( \{ A_n(S)(\xi) \} \) does not converge uniformly. Consider the operator

\[
T = \Phi^{-1}S\Phi U : \mathcal{B}(\mathcal{H}) \to \mathcal{N} \subset \mathcal{B}(\mathcal{H}).
\]

It is clear that \( T \in \mathcal{D} \). As \( y = U(x) \in \mathcal{N} \), hence \( U(y) = y \) (see Theorem 3.3(ii)), and \( U\Phi^{-1} = \Phi^{-1} \), we have \( T^k(y) = \Phi^{-1}S^k\Phi(y) \) for each \( k \in \mathbb{N} \).

Since \( \Phi^{-1} \) is an isometry and

\[
A_n(T)(y) = \frac{1}{n+1} \sum_{k=0}^{n} T^k(y) = \Phi^{-1} \left( \frac{1}{n+1} \sum_{k=0}^{n} S^k\Phi(y) \right) = \Phi^{-1}(A_n(S)(\xi)),
\]

for all \( n \in \mathbb{N} \), it follows that the sequence \( \{ A_n(T)(y) \}_{n=1}^{\infty} \) does not converge uniformly.

Now, as above, \( y = U(x) \in \mathcal{N} \) entails \( T^k(x) = \Phi^{-1}S^k\Phi(y) = T^k(y) \) for all \( k \in \mathbb{N} \).

Therefore, we have

\[
A_n(T)(x) - A_n(T)(y) = \frac{1}{n+1}(x - y),
\]

and it follows that the sequence \( \{ A_n(T)(x) \}_{n=1}^{\infty} \) also does not converge uniformly.

Let now \( \mathcal{H} \) be non-separable, and let \( 0 \leq x \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H}) \). Since \( x \notin \mathcal{K}(\mathcal{H}) \) it follows that there exists a spectral projection \( e_\lambda = \{ x > \lambda \}, \lambda > 0 \), such that \( \tau(e_\lambda) = \infty \). Choose an orthonormal basis \( \{ \varphi_j \}_{j \in J} \) in \( \mathcal{H} \) such that \( e_\lambda \geq p_{j_n} \) for some sequence \( \{ j_n \}_{n=1}^{\infty} \), where \( p_j \) is the one-dimensional projection on the subspace \( \mathbb{C} \cdot \varphi_j \subset \mathcal{H} \). If \( p = \sup_{n} p_{j_n} \), then \( \mathcal{H}_0 = p(\mathcal{H}) \) is a separable infinite-dimensional Hilbert subspace in \( \mathcal{H} \) such that \( \mathcal{K}(\mathcal{H}_0) = p\mathcal{K}(\mathcal{H})p \).

Since \( z = pxp \in \mathcal{B}(\mathcal{H}_0) \) and \( z \geq \lambda p e \lambda p \geq \lambda p \), it follows that \( z \in \mathcal{B}_+(\mathcal{H}_0) \setminus \mathcal{K}(\mathcal{H}_0) \). In view of the above, there exists a Dunford-Schwartz operator \( D_0 : \mathcal{B}(\mathcal{H}_0) \to \mathcal{B}(\mathcal{H}_0) \) such that the sequence \( \{ A_n(D_0)(z) \}_{n=1}^{\infty} \) does not converge uniformly.

It is clear that \( D(y) = D_0(pyp) \), \( y \in \mathcal{B}(\mathcal{H}) \), is a Dunford-Schwartz operator in \( \mathcal{B}(\mathcal{H}) \) such that \( D^k(x) = D_0^k(z) \) for each \( k \in \mathbb{N} \). Then

\[
A_n(D)(x) - A_n(D_0)(z) = \frac{1}{n+1}(x - z),
\]

and we conclude that the sequence \( \{ A_n(D)(x) \}_{n=1}^{\infty} \) does not converge uniformly.

Further, let \( x \in \mathcal{B}(\mathcal{H})_h \setminus \mathcal{K}(\mathcal{H}) \). Then \( x = x_+ - x_- \) such that \( x_+, x_- \in \mathcal{B}_+(\mathcal{H}) \) and \( x_+ x_- = 0 \). It is clear that either \( x_+ \in \mathcal{B}_+(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H}) \) or \( x_- \in \mathcal{B}_+(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H}) \). Suppose that \( x_+ \in \mathcal{B}_+(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H}) \) and let \( q = s(x_+) \) be the support of \( x_+ \). \( x_- \in \mathcal{B}_+(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H}) \) and \( \mathcal{L} = q(\mathcal{H}) \), then, by the above, there exists a Dunford-Schwartz operator \( S_0 : \mathcal{B}(\mathcal{L}) \to \mathcal{B}(\mathcal{L}) \) such that the sequence \( \{ A_n(S_0)(x_+) \}_{n=1}^{\infty} \) does not converge uniformly. Consider the operator \( S : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) given by \( S(y) = S_0(qyq) \), \( y \in \mathcal{B}(\mathcal{H}) \). It is clear that \( S \in \mathcal{D} \), \( S(x) = S_0(qyq) = S_0(x_+) \), and \( S^k(x) = S_0^k(x_+) \) for all \( k \in \mathbb{N} \). Consequently, the sequence \( \{ A_n(S)(x) \}_{n=1}^{\infty} \) does not converge uniformly.

Finally, if \( x \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H}) \) is arbitrary, then, repeating the ending of the proof of Theorem 3.4 we obtain that there exists \( T \in \mathcal{D} \) such that the sequence \( \{ A_n(T)(x) \} \) does not converge uniformly.

\[\Box\]

Let \( X \subset \mathcal{B}(\mathcal{H}) \) be a fully symmetric space. We will write \( X \in \{ IET \} \) if \( X \) satisfies the following individual ergodic theorem: for any \( x \in X \) and \( T \in \mathcal{D} \) there
exists \( \hat{x} \in X \) such that \( \| A_n(T)(x) - \hat{x} \|_\infty \to 0 \) as \( n \to \infty \). Theorems 3.3 and 3.6 yield the following criterion.

**Theorem 3.7.** Let \( X \subset \mathcal{B}(\mathcal{H}) \) be a fully symmetric space. Then the following conditions are equivalent:

(i) \( X \in (IET) \);

(ii) \( X \subset \mathcal{K}(\mathcal{H}) \).

4. **Mean ergodic theorem in fully symmetric ideals of compact operators**

In this section, our goal is to prove Theorem 4.1. So, let \( (E, \| \cdot \|_E) \subset c_0 \) be a fully symmetric sequence space, and let \( (C_E, \| \cdot \|_{C_E}) \) be a fully symmetric ideal generated by \( (E, \| \cdot \|_E) \). We will write \( C_E \in (MET) \) if the ideal \( (C_E, \| \cdot \|_{C_E}) \) satisfies the mean ergodic theorem, that is, if for any \( x \in C_E \) and \( T \in DS \) there exists \( \hat{x} \in C_E \) such that \( \| A_n(T)(x) - \hat{x} \|_{C_E} \to 0 \) as \( n \to \infty \).

**Proposition 4.1.** \( C^1 \notin (MET) \).

**Proof.** Let \( S : l^\infty \to l^\infty \) be the positive Dunford-Schwartz operator defined by \( S(\{\xi_n\}_{n=1}^\infty) = \{0, \xi_1, \xi_2, \ldots\} \), \( \{\xi_n\}_{n=1}^\infty \in l^\infty \).

If \( \xi = \{1, 0, 0, \ldots \} \in l^1 \), then

\[
\| A_{2n-1}(S)(\xi) - A_{n-1}(S)(\xi) \|_1 = \left\| \frac{1}{2n} \{1, 1, \ldots, 1, 0, 0, \ldots \} - \frac{1}{n} \{1, 1, \ldots, 1, 0, 0, \ldots \} \right\|_1 = 1.
\]

Consequently, the sequence \( \{A_n(S)(\xi)\} \) does not converge in the norm \( \| \cdot \|_1 \).

Let \( \{\varphi_j\}_{j \in J} \) be an orthonormal basis in the Hilbert space \( \mathcal{H} \), and let \( \{\varphi_{j_n}\}_{n=1}^\infty \) be a countable subset of \( \{\varphi_j\}_{j \in J} \). Let \( p_n \) be the one-dimensional projection on the subspace \( \mathbb{C} \cdot \varphi_{j_n} \subset \mathcal{H} \), and let \( p = \sup_n p_n \). It is clear that \( \mathcal{H}_0 = p(\mathcal{H}) \) is a separable infinite-dimensional Hilbert subspace in \( \mathcal{H} \) and \( \mathcal{K}(\mathcal{H}_0) = p(\mathcal{K}(\mathcal{H}))p \).

Let \( \mathcal{N}(\mathcal{H}_0) = \left\{ x_\xi = (\varphi_0) - \sum_{n=1}^{\infty} \xi_n p_n \in \mathcal{B}(\mathcal{H}_0) : \xi = \{\xi_n\}_{n=1}^\infty \in l^\infty \right\} \)

be the smallest commutative von Neumann subalgebra in \( \mathcal{B}(\mathcal{H}_0) \) containing the projections \( p_n, n \in \mathbb{N} \), and let \( \Phi(x_\xi) = \{\xi_n\}_{n=1}^\infty \) be the positive linear surjective isometry from \( (\mathcal{N}(\mathcal{H}_0), \| \cdot \|_\infty) \) onto \( (l^\infty, \| \cdot \|_\infty) \) given in Proposition 3.1. Finally, let \( U : \mathcal{B}(\mathcal{H}_0) \to \mathcal{N}(\mathcal{H}_0) \) be the conditional expectation given in Theorem 3.3.

It is clear that

\[ T = \Phi^{-1} S \Phi U : \mathcal{B}(\mathcal{H}_0) \to \mathcal{N}(\mathcal{H}_0) \subset \mathcal{B}(\mathcal{H}_0) \]

is a positive Dunford-Schwartz operator. If \( \xi = \{1, 0, 0, \ldots\} \in l^1 \) and \( x_\xi = \Phi^{-1}(\xi) \), then \( x_\xi \in \mathcal{N}(\mathcal{H}_0) \cap C^1 \) (see Proposition 3.2), and \( U(x_\xi) = x_\xi \) (see Theorem 3.3 (ii)). Consequently,

\[ T(x_\xi) = \Phi^{-1} S \Phi U(x_\xi) = \Phi^{-1} S \Phi(x_\xi). \]

Now, repeating the proof of Theorem 3.3, we conclude that the averages \( \{A_n(T)(x_\xi)\} \) do not converge in the norm \( \| \cdot \|_1 \), that is, \( C^1 \notin (MET) \). \( \Box \)

Here is another sufficient condition for \( C_E \notin (MET) \):
Proposition 4.2. If \((E, \| \cdot \|_E) \subset c_0\) is non-separable fully symmetric sequence space, then \(C_E \notin \text{(MET)}\).

Proof. If \((E, \| \cdot \|_E) \subset c_0\) is a non-separable fully symmetric sequence space, then there exists \(\xi = \{\xi_n\}_{n=1}^\infty = \{\xi_n^*\}_{n=1}^\infty \in E\), hence \(\xi_n \downarrow 0\), such that

\[
\|\{0, 0, \ldots, 0, \xi_{n+2}, \ldots\}\|_E \downarrow \alpha > 0.
\]

Let the operator \(S \in DS\) be defined as in the proof of Proposition 4.1. Then \(S^k(\xi) = \{0, 0, \ldots, 0, \xi_1, \xi_2, \ldots\}\) and

\[
\sum_{k=0}^n S^k(\xi) = \{n_m^{(n)}\}_{m=1}^\infty,
\]

where

\[
\eta_m^{(n)} = \xi_1 + \xi_2 + \ldots + \xi_m \quad \text{for} \quad 1 \leq m \leq n + 1
\]

and

\[
\eta_m^{(n)} = \xi_{m-n} + \xi_{m-n+1} + \ldots + \xi_m \quad \text{for} \quad m > n + 1.
\]

Since \(\xi_n \downarrow 0\), given \(1 \leq m \leq n + 1\), we have

\[
0 \leq \frac{1}{n+1} \eta_m^{(n)} \leq \frac{1}{n+1} \sum_{k=1}^{n+1} \xi_k \to 0 \quad \text{as} \quad n \to \infty,
\]

implying that \(A_n(S)(\xi) \to 0\) coordinate-wise.

Assume that there exists \(\hat{\xi} \in E\) such that \(\|A_n(S)(\xi) - \hat{\xi}\|_E \to 0\). Then we have \(\|A_n(S)(\xi) - \hat{\xi}\|_E \to 0\); in particular, \(A_n(S)(\xi) \to 0\) coordinate-wise, hence \(\hat{\xi} = 0\).

On the other hand, as \(\xi_n \downarrow 0\), we obtain

\[
A_n(S)(\xi) = \left\{ \frac{\xi_1}{n+1}, \frac{\xi_1 + \xi_2}{n+1}, \ldots, \frac{\xi_1 + \xi_2 + \ldots + \xi_{n+1}}{n+1}, \frac{\xi_2 + \xi_3 + \ldots + \xi_{n+2}}{n+1}, \ldots, \frac{\xi_m}{n+1}, \ldots, \frac{\xi_m + \xi_{m-1} + \ldots + \xi_n}{n+1}, \ldots \right\}
\]

\[
\geq \{0, 0, \ldots, 0, \xi_{n+2}, \ldots\}.
\]

Therefore, in view of [41],

\[
\|A_n(S)(\xi)\|_E \geq \alpha,
\]

implying that the sequence \(\{A_n(S)(\xi)\}\) does not converge in the norm \(\| \cdot \|_E\).

Now, if we define the Dunford-Schwartz operator \(T \in DS\) as in the proof of Proposition 5.1, then repeating its proof for \(x = \Phi^{-1}(\xi)\), we conclude that the sequence \(\{A_n(T)(x)\}\) does not converge in \((C_E, \| \cdot \|_{C_E})\). This means that \(C_E \notin \text{(MET)}\).

Fix \(T \in DS\). By Theorem 5.1, for every \(x \in \mathcal{K}(\mathcal{H})\) there exists \(\tilde{x} \in \mathcal{K}(\mathcal{H})\) such that \(\|A_n(T)(x) - \tilde{x}\|_E \to 0\) as \(n \to \infty\). Therefore, one can define a linear operator \(P_T : \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})\) by setting \(P_T(x) = \tilde{x}\). Then we have

\[
\|P_T(x)\|_E = \lim_{n \to \infty} \|A_n(T)(x)\|_E \leq \|x\|_E,
\]

Besides, since the unit ball in \((C^1, \| \cdot \|_1)\) is closed in measure topology [34, Proposition 3.3] and \(\|A_n(T)(x)\|_1 \leq \|x\|_1\) for all \(x \in C^1\), it follows that \(\|P_T(x)\|_1 \leq \|x\|_1\), \(x \in C^1\).
Consequently, \( \|P_T\|_{c_0 \to c_0} \leq 1 \), and, according to [11, Proposition 1.1], there exists a unique operator \( \hat{P} \in DS \) such that \( \hat{P}(x) = P_T(x) \) whenever \( x \in \mathcal{K}(\mathcal{H}) \). In what follows, we denote \( \hat{P} \) by \( P_T \).

**Lemma 4.1.** If \( T \in DS \) and \( x \in \mathcal{K}(\mathcal{H}) \), then

\[
P_T T(x) = P_T(x) = T P_T(x).
\]

**Proof.** We have

\[
\|(I - T)A_n(T)(x)\|_\infty = \left\| \frac{(I - T^{n+1})(x)}{n+1} \right\|_\infty \to 0 \quad \text{as} \quad n \to \infty.
\]

On the other hand,

\[
TA_n(T)(x) = \frac{1}{n+1} \sum_{k=0}^{n} T^k(Tx) \overset{\| \|}{\longrightarrow} P_T(T(x)),
\]

implying that

\[
(I - T)A_n(T)(x) = A_n(T)(x) - TA_n(T)(x) \overset{\| \|}{\longrightarrow} P_T(x) - P_T(T(x)),
\]

hence \( P_T T(x) = P_T(x) \).

Now, as \( \|A_n(T)(x) - P_T(x)\|_\infty \to 0 \), we have \( T(A_n(T)(x)) - T(P_T(x)) \|_\infty \to 0 \) as \( n \to \infty \), and the result follows.

**Corollary 4.1.** If \( T \in DS \) and \( x \in \mathcal{K}(\mathcal{H}) \), then

\[
T^k(P_T(x)) = P_T(x) \quad \text{and} \quad P_T^2(x) = P_T(x).
\]

We need the following property of separable symmetric sequence spaces [10, Proposition 2.2].

**Proposition 4.3.** Let \( (E, \| \cdot \|_E) \) be a separable symmetric sequence space. If \( C_E \ni y_n \ll x \in C_E \) for every \( n \in \mathbb{N} \) and \( \|y_n\|_\infty \to 0 \) as \( n \to \infty \), then \( \|y_n\|_E \to 0 \) as \( n \to \infty \).

Now we can finalize the proof of Theorem 1.4.

**Proof.** (i) \( \Rightarrow \) (ii): Proposition 4.4 implies that \( E \) is separable. If \( E = l^1 \) as sets, then the norms \( \| \cdot \|_E \) and \( \| \cdot \|_1 \) are equivalent [19, Part II, Ch. 6, §6.1]. Therefore, in view of Proposition 4.1, we would have \( (C_E, \| \cdot \|_E) \notin (MET) \), a contradiction.

(ii) \( \Rightarrow \) (i): Let \( (E, \| \cdot \|_E) \) be separable, \( E \neq l^1 \), and let \( T \in DS \). If \( x \in C_E \) and \( y = x - P_T(x) \), then \( P_T(y) = 0 \), which, by Theorem 4.3, implies \( \|A_n(T)(y)\|_\infty \to 0 \). Since \( E \) is a separable symmetric sequence space, it follows from Proposition 4.3 that

\[
\|A_n(T)(y)\|_E \to 0.
\]

Since \( P_T(z) \ll z \) for all \( z \in \mathcal{K}(\mathcal{H}) \) (see Section 2) and \( T \in DS \), it follows that \( A_n(T)(P_T(x)) \ll P_T(x) \ll x \), hence \( A_n(T)(P_T(x)) - P_T(x) \ll 2x \). Next, as \( A_n(T)(P_T(x)) \overset{\| \|}{\longrightarrow} P_T(x) \), Proposition 4.3 entails

\[
\|A_n(T)(P_T(x)) - P_T(x)\|_E \to 0.
\]

Now, utilizing (4) and (5), we obtain

\[
\|A_n(T)(x) - P_T(x)\|_E = \|A_n(T)(x) - A_n(T)(P_T(x)) + A_n(T)(P_T(x)) - P_T(x)\|_E \\
\leq \|A_n(T)(y)\|_E + \|A_n(T)(P_T(x)) - P_T(x)\|_E \to 0
\]
as \( n \to \infty \). Therefore \( C_E \in (MET) \).

5. Ergodic theorems in Orlicz, Lorentz, and Marcinkiewicz ideals of compact operators

In this section we present applications of Theorems 1.2 and 1.4 to Orlicz, Lorentz and Marcinkiewicz ideals of compact operators.

1. Let \( \Phi \) be an Orlicz function, that is, \( \Phi : [0, \infty) \to [0, \infty) \) is left-continuous, convex, increasing and such that \( \Phi(0) = 0 \) and \( \Phi(u) > 0 \) for some \( u \neq 0 \) (see, for example, [11, Ch. 2, §2.1], [16, Ch. 4]). Let

\[
I^\Phi(N) = \left\{ \xi = \{\xi_n\}_{n=1}^\infty \in l^\infty : \sum_{n=1}^{\infty} \Phi\left(\frac{\xi_n}{a}\right) < \infty \text{ for some } a > 0 \right\}
\]

be the corresponding Orlicz sequence space, and let

\[
\|\xi\|_{\Phi} = \inf \left\{ a > 0 : \sum_{n=1}^{\infty} \Phi\left(\frac{\xi_n}{a}\right) \leq 1 \right\}
\]

be the Luxemburg norm in \( I^\Phi(N) \). It is well-known that \( (I^\Phi(N), \| \cdot \|_{\Phi}) \) is a fully symmetric sequence space.

If \( \Phi(u) > 0 \) for all \( u \neq 0 \), then \( \sum_{n=1}^{\infty} \Phi(a^{-1}) = \infty \) for each \( a > 0 \), hence \( 1 = \{1, 1, ...\} \notin I^\Phi(N) \) and \( I^\Phi(N) \subset c_0 \). If \( \Phi(u) = 0 \) for all \( 0 \leq u < u_0 \), then \( 1 \in I^\Phi \) and \( I^\Phi(N) = l^\infty \).

It is said that an Orlicz function \( \Phi \) satisfies \( (\Delta_2) \)-condition at 0 if there exist \( u_0 \in (0, \infty) \) and \( k > 0 \) such that \( \Phi(2u) < k \Phi(u) \) for all \( 0 < u < u_0 \). It is well known that an Orlicz function \( \Phi \) satisfies \( (\Delta_2) \)-condition at 0 if and only if \( (I^\Phi(N), \| \cdot \|_{\Phi}) \) is separable; see [11, Ch. 2, §2.1, Theorem 2.1.17], [16, Ch. 4, Proposition 4.a.4].

We also note that \( I^\Phi(N) = l^1 \), as sets, if and only if \( \limsup_{u \to 0} \frac{\Phi(u)}{u} > 0 \); see [16, Ch. 4, Proposition 4.a.5], [19, Ch. 16, §16.2].

If \( \Phi(u) > 0 \) for all \( u \neq 0 \), then \( I^\Phi(N) \subset c_0 \), and we can define

\[
C^\Phi = C_{I^\Phi(N)}, \quad \|x\|_{\Phi} = \|x\|_{C^\Phi}, \quad x \in C^\Phi.
\]

Now, Theorems 1.2 and 1.4 yield the following.

**Theorem 5.1.** Let \( \Phi \) be an Orlicz function such that \( \Phi(u) > 0 \) for all \( u \neq 0 \). Then

(i) \( C^\Phi \in (MET) \);

(ii) \( (C^\Phi, \| \cdot \|_{\Phi}) \in (MET) \) if and only if \( \Phi \) satisfies \( (\Delta_2) \)-condition at 0 and \( \lim_{u \to 0} \frac{\Phi(u)}{u} = 0 \).

2. Let \( \psi \) be a concave function on \([0, \infty)\) with \( \psi(0) = 0 \) and \( \psi(t) > 0 \) for all \( t > 0 \), and let

\[
\Lambda_\psi(N) = \left\{ \xi = \{\xi_n\}_{n=1}^\infty \in l^\infty : \|\xi\|_\psi = \sum_{n=1}^{\infty} \xi_n^\psi(n) - \psi(n-1) < \infty \right\},
\]

the corresponding Lorentz sequence space. The pair \((\Lambda_\psi(N), \| \cdot \|_\psi)\) is a fully symmetric sequence space; see, for example, [15, Ch. II, §5], [19, Part III, Ch. 9, §9.1].

Besides, if \( \psi(\infty) = \infty \), then \( 1 \notin \Lambda_\psi(N) \) and \( \Lambda_\psi(N) \subset c_0 \); if \( \psi(\infty) < \infty \), then \( 1 \in \Lambda_\psi(N) \) and \( \Lambda_\psi(N) = l^\infty \).
It is well known that \((\Lambda_\psi(N), \| \cdot \|_\psi)\) is separable if and only if \(\psi(0) = 0\) and \(\psi(\infty) = \infty\); see, for example, [15, Ch. II, §5, Lemma 5.1], [19, Ch. 9, §9.3, Theorem 9.3.1]. It is clear that \(\lim_{t \to \infty} \frac{\psi(t)}{t} > 0\) if and only if the norms \(\| \cdot \|_\psi\) and \(\| \cdot \|_1\) are equivalent on \(\Lambda_\psi(N)\), that is, if \(\Lambda_\psi(N) = L^1\), as sets.

Finally, Theorems 1.2 and 1.4 imply the following.

If \(\psi(\infty) = \infty\), then \(\Lambda_\psi(N) \subset c_0\), and we can define

\[
C_\psi = C_{\Lambda_\psi(N)}, \quad \| x \|_\psi = \| x \|_{C_{\Lambda_\psi(N)}}, \ x \in C_\psi.
\]

Theorems 1.2 and 1.4 imply the following.

**Theorem 5.2.** Let \(\psi\) be a concave function on \([0, \infty)\) with \(\psi(0) = 0\), \(\psi(t) > 0\) for all \(t > 0\), and let \(\psi(\infty) = \infty\). Then

(i) \(C_\psi \in (IET)\);

(ii) \((C_\psi, \| \cdot \|_\psi) \in (MET)\) if and only if \(\psi(0) = 0\) and \(\lim_{t \to \infty} \frac{\psi(t)}{t} = 0\).

3. Let \(\psi\) be as above, and let

\[
M_\psi(N) = \left\{ \xi = \{\xi_n\}_{n=1}^{\infty} \in l^\infty : \| \xi \|_{M_\psi} = \sup_{n \geq 1} \frac{1}{\psi(n)} \sum_{k=1}^{n} \xi_k < \infty \right\},
\]

the corresponding Marcinkiewicz sequence space. The space \((M_\psi(N), \| \cdot \|_{M_\psi})\) is a fully symmetric sequence space; see, for example, [15, Ch. II, §5], [19, Part III, Ch. 9, §9.1]. In addition, \(1 \notin M_\psi(N)\) if and only if \(\lim_{t \to \infty} \frac{\psi(t)}{t} = 0\) ([15, Ch. II, §5]).

Besides, \(M_\psi(N) = L^1\), as sets, if and only if \(\psi(\infty) < \infty\).

If \(\psi(0) = 0\), \(\psi(\infty) = \infty\), and \(\lim_{t \to \infty} \frac{\psi(t)}{t} = \infty\), then \(M_\psi\) is non-separable; see [1, 15, Ch. II §5, Lemma 5.4].

If \(\lim_{t \to \infty} \frac{\psi(t)}{t} = 0\), then \(M_\psi(N) \subset c_0\), and we define

\[
C_{M_\psi} = C_{M_\psi(N)}, \quad \| x \|_{C_{M_\psi}} = \| x \|_{C_{M_\psi(N)}}, \ x \in C_{M_\psi}.
\]

Finally, Theorems 1.2 and 1.4 imply the following.

**Theorem 5.3.** Let \(\psi\) be a concave function on \([0, \infty)\) with \(\psi(0) = 0\), \(\psi(t) > 0\) for all \(t > 0\), and let \(\lim_{t \to \infty} \frac{\psi(t)}{t} = 0\). Then

(i) \((C_{M_\psi}, \| \cdot \|_{C_{M_\psi}}) \in (IET)\);

(ii) if either \(\psi(\infty) = \infty\), \(\psi(0) = 0\), and \(\lim_{t \to \infty} \frac{\psi(t)}{t} = \infty\) or \(\psi(\infty) < \infty\), then \((C_{M_\psi}, \| \cdot \|_{C_{M_\psi}}) \notin (MET)\).

**References**

[1] S.V. Astashkin, F.A. Sukochev, *Banach-Saks property in Marcinkiewicz spaces*, J. Math. Anal. Appl. **336**, 2007, 1231–1258.

[2] C. Bennett, R. Sharpley, *Interpolation of Operators*, Academic Press Inc. (1988).

[3] V. Chilin, A. Azizov, *Ergodic theorems in symmetric sequences spaces*, Colloq. Math. **156** (2), 2018, 57–68. DOI: 10.4064/cm7384-2-2018.

[4] V. Chilin, S. Litvinov, *Ergodic theorems in fully symmetric spaces of \(\tau\)-measurable operators*, Studia Math. **288**(2), 2016, 177–195.

[5] V. Chilin, S. Litvinov, *Individual ergodic theorems in noncommutative Orlicz spaces*, Positivity **21**(1), 2017, 49–59. DOI 10.1007/s11117-016-0402-8.

[6] V. Chilin, S. Litvinov, *The validity space of Dunford-Schwartz pointwise ergodic theorem*, J. Math. Anal. Appl. **461**, 2018, 234–247. DOI 10.1016/j.jmaa.2018.01.001.
[7] Dunford N. and Schwartz J. T. Linear Operators, Part I: General Theory, John Willey and Sons (1988).
[8] P.G. Dodds, T.K. Dodds, and B. Pagter, \textit{Fully symmetric operator spaces}, J. Integr. Equat. Oper. Theory. \textbf{15}, 1992, 942–972.
[9] P. G. Dodds, T. K. Dodds, and B. Pagter, Noncommutative Köthe duality, \textit{Trans. Amer. Math. Soc.}, \textbf{339}(2), 1993, 717-750.
[10] P. G. Dodds, T. K. Dodds, F. A. Sukochev, \textit{Banach-Saks properties in symmetric spaces of measurable operators}, Studia Math. \textbf{178}, 2007, 125–166.
[11] G.A. Edgar, L. Sucheston, \textit{Stopping Times and Directed Processes}, Cambridge University Press (1992).
[12] T. Fack, H. Kosaki, \textit{Generalized s-numbers of τ-measurable operators}, Pacific. J. Math., \textbf{123}, 1986, 269-300.
[13] I.C. Gohberg, M.G. Krein, \textit{Introduction to the theory of linear nonselfadjoint operators}, Translations of Mathematical Monographs \textbf{18}, Amer. Math. Soc., Providence, RI 02904 (1969).
[14] M. Junge, Q. Xu, \textit{Noncommutative maximal ergodic theorems}, J. Amer. Math. Soc. \textbf{20}(2), 2007, 385–439.
[15] S.G. Krein, Ju.I. Petunin, and E.M. Semenov, \textit{Interpolation of Linear Operators}, Translations of Mathematical Monographs, Amer. Math. Soc. \textbf{54}(1982).
[16] J. Lindenstrauss, L. Tzafriri, \textit{Classical Banach spaces I-II}, Springer-Verlag, Berlin Heidelberg New York (1977).
[17] S. Litvinov, \textit{Uniform equicontinuity of sequences of measurable functions and noncommutative ergodic theorems}, Proc. Amer. Math. Sci. \textbf{140}, 2012, 2401–2409.
[18] S. Lord, F. Sukochev, D. Zanin, \textit{Singular Traces}, Walter de Gruyter GmbH, Berlin/Boston (2013).
[19] B.A. Rubshtein, G.Ya. Grabarnik, M.A. Muratov, and Yu.S. Pashkova, \textit{Foundations of Symmetric Spaces of Measurable Functions. Lorentz, Marcinkiewicz and Orlicz Spaces}. Springer International Publishing, Switzerland (2016).
[20] B. Simon, \textit{Trace Ideals and Their Applications} (bf 120 Second edition, Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI (2005).
[21] S. Stratila, L. Zsidó, \textit{Lectures on von Neumann algebras}, Editura Academiei, Bucharest (1979). Revision of the 1975 original, Translated from the Romanian by Silviu Teleman.
[22] M. Takesaki, \textit{Conditional Expectations in von Neumann Algebras}, J. Funct. Analysis. \textbf{9}, 1972, 306–321.
[23] F. J. Yeadon, \textit{Ergodic theorems for semifinite von Neumann algebras. I}, J. London Math. Soc., \textbf{16}(2), 1977, 326–332.
[24] F. J. Yeadon, \textit{Ergodic theorems for semifinite von Neumann algebras: II}, Math. Proc. Camb. Phil. Soc., \textbf{88}, 1980, 135-147.

\textbf{National University of Uzbekistan, Tashkent, 100174, Uzbekistan}
\textit{E-mail address:} azizov.07@mail.ru

\textbf{National University of Uzbekistan, Tashkent, 100174, Uzbekistan}
\textit{E-mail address:} vladimirchil@gmail.com; chilin@ucd.uz

\textbf{Pennsylvania State University, 76 University Drive, Hazleton, PA 18202, USA}
\textit{E-mail address:} snl2@psu.edu