Features of librational motions around $L_4$

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Abstract. Trojan bodies are present in the Solar system in great number, as Trojan asteroids and also as Trojan moons. Thus it is possible that their presence is similar in extrasolar planetary systems too. We investigated features of librational motions around $L_4$ with numerical methods on the mass parameter - eccentricity plane in the elliptic restricted three-body problem. We determined the lifetime of Trojan bodies, until they remained in the librational domain around $L_4$, also illustrated the distribution of the three possible endgames of trojan motion. Finally we determined the frequencies and resonances of the librational motion in the stable region.

1. Introduction

It is not rare in our Solar system, that in a three-body system the third body moves nearly in the same orbit as one of the others, by 60° behind or in front of it. They are in 1:1 mean motion resonance. Trojan asteroids, which are moving co-orbital with Jupiter, are the most known group of configuration of this kind. The first Trojan asteroid, Achilles was discovered in 1906 by Max Wolf. To date astronomers discovered more than 3000 companions of Achilles. We know asteroids at the $L_4$ and $L_5$ Lagrangian points of Jupiter nearly 100 years back, while Trojan followers of other planets were detected only in the recent past. By now we know 4 Trojan asteroids of Mars and 6 of Neptune. Trojan bodies can also be moons, namely Saturn has a few such moons. After all, it would not be astonishing, if observations would report about detections of Trojan bodies in extrasolar planetary systems. Configurations of this kind can be studied by using the circular or elliptic restricted three-body problem.

2. The elliptic restricted three-body problem

The recent study investigates a special case of the three-body problem, namely the elliptic restricted three-body problem, when the mass of one body (an asteroid) is negligible, compared to the others, which are moving in elliptic orbits around their common centre of mass. This case can be applied for Trojan asteroids or trojan moons as well.

The mathematical difficulty of this problem is that it can be analysed only by numerical methods. First we investigated the stability time of fictitious trojan bodies by integrating the
Figure 1. Left panel: The logarithm of the lifetime of orbits with given mass parameter and eccentricity (Érdi et al., 2009). Right panel: Variation of lifetimes as a function of the mass parameter in the unstable region at \( e = 0.394 \).

equations of motion of the elliptic restricted three-body problem, putting the trojan body in the \( L_4 \) point and computing how long it remained there depending on the parameters of the system. \( P_1 \) and \( P_2 \) marks the two heavier bodies, and \( P_3 \) the asteroid, whose mass is negligible. Let \( P_1 \) be the centre of the coordinate system, and suppose that \( P_3 \) moves in the same plane as the others, so movements in the \( z \) direction is neglected. The masses of the components are \( 1 - \mu, \mu, \) and 0, respectively, in units of the total mass of the primaries. The initial positions and velocities of \( P_3 \) and \( P_2 \) were taken such that both begin their motion at the apocentre of their respective elliptic orbit, \( P_3 \) being by 60° before \( P_2 \).

In this case the equations of motion contain two free parameters: \( \mu \) (the mass parameter of the system), and the eccentricity \( e \) of the elliptic orbit of \( P_2 \). Thus we studied the properties of the system on a grid covering the \( \mu - e \) plane. We changed the mass parameter \( \mu \) from 0.0001 to 0.1 with stepsize 0.0001, and the eccentricity from 0 to 1 with stepsize 0.002. This means about 500 000 \((\mu, e)\) points.

We also investigated the frequencies of motion around \( L_4 \) depending on \( \mu \) and \( e \). Analysis of the frequencies is based on the definitions of Érdi et al. (2007). To determine the frequencies of the librational motions we used a numerical library for C programs, Gnu Scientific Library. This provides a wide range of mathematical routines, so we used a Fast Fourier Transform (FFT). The description of the algorithm is given in Temperton (1983).

3. Lifetime of the system

We examined the lifetime of the systems with numerical integration through 5000 periods of \( P_2 \). We considered the motion of \( P_3 \) stable, if it remained in the surroundings of \( L_4 \) during the entire time of integration. Also considered it as unstable, if it moved away from \( P_2 \) farther than 180° or they became impacted\(^4\). On the left panel of Figure 1 one can see the stability time of the system at given mass parameter and eccentricity. The plotted value is the logarithm of the lifetime. In the stable region \( P_3 \) stays during the entire integration time near \( L_4 \) (the region with the bright grey colour, where the logarithm of the lifetime is 3.70), while in the unstable region it remains there only for a few dozen periods (the logarithm of the lifetime is less than 1.40).

The border of the stable region is quite clear, while lifetimes in the unstable region shows a decreasing tendency with some local prominences. This is shown in the right panel of Figure 1

\(^4\) Collision was defined so, that the separation of \( P_2 \) and \( P_3 \) is smaller, than the Hill radius of \( P_2 \).
Figure 2. Left panel: Possible motions of $P_3$. Black: stable motion. Grey: $P_3$ moved away from $P_2$ farther than 180˚. White: collision occurred between $P_2$ and $P_3$. Black square: The position of the right panel. Right panel: A part of the left panel, integrated with higher resolution: $\mu \in [0.04; 0.06]$ with stepsize 0.00001, and $e \in [0.6; 0.8]$ with stepsize 0.0001.

for $e = 0.394$. At $\mu = 0.05$ the lifetime increases with a few periods, that can be due to some resonances, but this needs to be investigated more in the future. At higher eccentricities the system dissolves sooner, lifetimes take here just a few periods.

4. The dissolution of the system
The motion of $P_3$ has the three above-mentioned endgames: a) librational motion through the whole time; b) it moves farther away from $P_2$ than 180˚; c) collision occurs.

On the left panel of Figure 2 is the distribution of the three possible ending of the system. The integration was done here for 5000 periods of $P_2$ also. The black region shows the systems, in which the motion of $P_3$ is stable during the whole integration time, grey colour is for ejection from the librational domain, and the white fine fibrous structure is where $P_2$ and $P_3$ collide. It can be seen, that at smaller eccentricities the system dissolves usually because of collision, and at higher eccentricities ejection from the librational domain dominates the $\mu - e$ plane. Near to the border of the stable region, the third body with the very small mass usually moves farther away than 180˚ from $P_2$, opposite to the $\mu > 0.05$ domain, where colliding systems form fine fibrous structures, usually by constant eccentricities. This can look like integrational error, but at higher resolution they become thicker, and beside them several new thin structures are also appearing (Figure 2, right panel).

5. Frequencies and resonances
To determine the frequencies of the librational motion of $P_3$ we used Fourier transform, and integrated for 1250 periods of $P_2$ to get practicable Fourier spectra. But this method could not map the unstable region, because at ejection from the librational domain the spectrum has exponential shape, no peak could be identified, and in the case of collision the stability time was too small to get a practicable Fourier spectrum.

Figure 3 shows the variation of the four frequencies of the librational motion ($n_s$, $1 - n_l$, $n_l$, $1 - n_s$, corresponding to the elliptic restricted three-body problem, $n_s$, $n_l$ being the normalized frequencies of short and long period libration) as the function of the mass parameter. The lack of data around $\mu = 0.03$ is due to the instabil region, where we could not get the frequencies with FFT.

Knowing the frequencies of the librational motion, we can determine resonances. Resonances can appear when the ratios of frequencies are rational numbers, so they can be given as the
Figures 3. Variation of the frequencies at $e = 0.01$ as the function of the mass parameter. Key: cross: $n_s$, star: $1 - n_l$, triangle: $n_l$, diamond: $1 - n_s$ frequencies.

Figures 4. The $(1 - n_l) : n_l$ resonances. Black thick lines: resonances derived numerically, from left to right: 5:1, 4:1, 3:1, 2:1, 4:3, 5:4, 2:3. White dashed lines are the same resonances derived with analytical methods.

ratios of integer numbers. We used this fact to find resonances on the $\mu - e$ plane. For the four frequencies of the elliptic restricted three-body problem 6 types of resonances exist (Érdi et al. 2007, 2009). Figure 4 shows one of them: $(1 - n_l) : n_l$. The grey region is where the system is stable, black thick lines denotes the resonances obtained by numerical integration, and the dashed lines presents resonances determined by analytical methods. The latter was computed by Érdi et al. (2007), using Rabe’s equation (Rabe, 1973). It can be seen that resonances, determined by different methods, fit very well at smaller eccentricities. Of course the method of Rabe can be used only for small and moderate values of $e$, that is why the curves differ at higher eccentricities. The turning points can denote the validity limit of Rabe’s equation.

6. Summary
We investigated the lifetime, until the asteroid with negligible mass remained in the three-body system, for the $\mu - e$ plane (Figure 1), and found, that this plane can be divided into a stable and unstable region. For the previous case $P_3$ remained in the surroundings of $L_4$, and the latter led to dissolution. Lifetime of the system decreases quickly in the unstable region, as the eccentricity increases, or far from the boundary of the stable domain.

Distribution of the three possible outcomes were determined, this is shown in Figure 2. We found that at smaller eccentricities dissolution is due to collision, and as $e$ increases, ejection from librational domain becomes dominant.

Furthermore we determined frequencies and resonances for the stable librational motions, and found that resonance, obtained with numerical integration fit in with that, obtained with analytical methods (Figure 4). Future work could be to determine the characteristic roots of the motions in the whole $\mu - e$ plane, thus get the possible frequencies and resonances in the unstable region.

7. References
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