Observable Sets, Potentials and Schrödinger Equations

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Abstract: Consider the Schrödinger equation: \( i\frac{\partial}{\partial t} u = Hu \) over \( \mathbb{R}^n \), where \( H \) is a self-adjoint operator on \( L^2(\mathbb{R}^n) \) which is the sum of \( -\Delta \) and some potential. This paper aims to study the observability for the above equation, including observable sets and the observable time. We mention that a measurable subset \( E \subset \mathbb{R}^n \) is called an observable set at time \( T > 0 \) for the above equation, if there is a constant \( C > 0 \) (depending on \( T \) and \( E \)) such that
\[
\int_{\mathbb{R}^n} |u_0(x)|^2 \, dx \leq C \int_0^T \int_E |e^{-itH} u_0|^2 \, dx \, dt \quad \text{for all } u_0 \in L^2(\mathbb{R}^n).
\]

First, we characterize observable sets for the 1-dim case where \( H = -\frac{\partial^2}{\partial x^2} + x^{2m} \) (with \( m \in \mathbb{N} := \{0, 1, \ldots\} \)). More precisely, we obtain what follows: (i) When \( m = 0 \), \( E \subset \mathbb{R} \) is an observable set at some time if and only if it is thick, namely, there are constants \( \gamma, L > 0 \) so that
\[
|E \cap [x, x + L]| \geq \gamma L \quad \text{for each } x \in \mathbb{R};
\]
(ii) When \( m = 1 \) (\( m \geq 2 \) resp.), \( E \) is an observable set at some time (at any time resp.) if and only if it is weakly thick, namely
\[
\lim_{x \to +\infty} \frac{|E \cap [-x, x]|}{x} > 0.
\]

These reveal how potentials \( x^{2m} \) affect the observability. Second, we obtain what follows for the \( n \)-dim case where \( H = -\Delta + |x|^2 \) (the Harmonic oscillator): (i) For each \( r > 0 \), the exterior domain \( B^C(0, r) \) is an observable set at any time; (ii) Let \( E_1 \) be a half of \( B^C(0, r) \) bisected by a hyperplane across the origin. Then \( E_1 \) is an observable set at time \( T > 0 \) if and only if \( T > \frac{\pi}{2} \).
1. Introduction

The equations under study can be put into the framework:

\[ i\partial_t u(t, x) = Hu(t, x), \quad t \in \mathbb{R}^+(0, \infty), \quad x \in \mathbb{R}^n; \quad u(0, \cdot) \in L^2(\mathbb{R}^n), \]  

(1.1)

where \( n \in \mathbb{N}^+ := \{1, 2, \ldots\} \), \( H \) is a self-adjoint operator on \( L^2(\mathbb{R}^n) \) which is the sum of \(-\Delta\) and some potential, \( L^2(\mathbb{R}^n) := L^2(\mathbb{R}^n; \mathbb{C}) \). We are going to study the observability (including the geometric structures of observable sets and the minimal observable time) for the equation (1.1). To this end, we introduce the following definitions:

(D)_1 A measurable set \( E \subset \mathbb{R}^n \) is called an observable set at any time for (1.1), if for any \( T > 0 \), there is a constant \( C_{obs} = C_{obs}(T, E) > 0 \) so that

\[ \int_{\mathbb{R}^n} |u(0, x)|^2 \, dx \leq C_{obs} \int_0^T \int_E |u(t, x)|^2 \, dx \, dt, \quad \text{when } u \text{ solves (1.1).} \]  

(1.2)

(Here and in what follows, \( C(\cdots) \) stands for a positive constant depending on what are enclosed in the brackets.)

(D)_2 A measurable set \( E \subset \mathbb{R}^n \) is called an observable set at any time for (1.1), if there is \( T > 0 \) and \( C_{obs} = C_{obs}(T, E) > 0 \) so that (1.2) holds. Similarly, the set \( E \) is called an observable set at time \( T > 0 \) for (1.1), if there is \( C_{obs} = C_{obs}(T, E) > 0 \) so that (1.2) holds.

The inequality (1.2) is the standard observability inequality for (1.1). Thus, \( E \) is an observable set at some time for (1.1) if and only if (1.1), with controls restricted in \( E \), is exactly controllable over \((0, T)\) for some \( T > 0 \), while \( E \) is an observable set at any time for (1.1) if and only if (1.1), with controls restricted in \( E \), is exactly controllable over \((0, T)\) for any \( T > 0 \).

Motivation The goal of this paper is to try to understand how potentials affect the observability for the equation (1.1). We are partially motivated by the existing fact for heat equations: different potentials may have different effects on the geometric structure of observable sets for heat equations in \( \mathbb{R}^n, n \in \mathbb{N}^+ \). More precisely, consider the following heat equations:

\[ \partial_t u(t, x) + Hu(t, x) = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \quad u(0, x) \in L^2(\mathbb{R}^n), \]  

(1.3)

where \( H = -\Delta + |x|^{2m}, m \in \mathbb{N} := \{0, 1, 2, \ldots\} \). Recall that a set \( E \) is said to be an observable set (at any time) for (1.3), if for any \( T > 0 \), there is a constant \( C_{obs} = C_{obs}(T, E) > 0 \) so that

\[ \int_{\mathbb{R}} |u(T, x)|^2 \, dx \leq C_{obs} \int_0^T \int_E |u(t, x)|^2 \, dx \, dt, \quad \text{when } u \text{ solves (1.3).} \]

First, when \( m = 0 \), it was shown independently in [18,50] that \( E \) is an observable set for (1.3) if and only if \( E \) is thick in \( \mathbb{R}^n \), i.e., for some \( \gamma > 0 \) and \( L > 0 \),

\[ |E \cap Q_L(x)| \geq \gamma L^n \quad \text{for each } x \in \mathbb{R}^n. \]  

(1.4)

(Here and in what follows, when \( E \) is a measurable set in \( \mathbb{R}^n, |E| \) stands for the Lebesgue measure of \( E \) in \( \mathbb{R}^n \), \( Q_L(x) \) is the closed cube in \( \mathbb{R}^n \), centered at \( x \) and of the length \( L \).)

Second, when \( m \geq 2 \), it was obtained in [14,37] that the cone \( E = \{x \in \mathbb{R}^n : |x| \geq \} \)
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where \( r_0 > 0 \) and \( \Theta_0 \) is a nonempty and open subset of \( \mathbb{S}^{n-1} \) is an observable set for (1.3), while when \( m = 1 \), the above cone is no longer an observable set for (1.3). Third, when \( m = 1 \), it was proved in [3] that if \( E \) is thick, then \( E \) is an observable set for (1.3).

Our study is divided into two cases: 1-dim and \( n \)-dim. The main results in 1-dim are more delicate than those in the \( n \)-dim case.

1.1. Characterizations of observable sets for Schrödinger equations in \( \mathbb{R} \). We consider two Schrödinger equations in \( \mathbb{R} \). The first one is as:

\[
i \partial_t u(t, x) = (-\partial_x^2 + c)u(t, x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}; \quad u(0, \cdot) \in L^2(\mathbb{R}),
\]

where \( c \) is a real number, while the second one reads as:

\[
i \partial_t u(t, x) = Hu(t, x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}; \quad u(0, \cdot) \in L^2(\mathbb{R}),
\]

where

\[
H := -\partial_x^2 + x^{2m}, \quad m \in \mathbb{N}^*.
\]

Equation (1.5) with \( c = 0 \) is referred to as the free Schrödinger equation. Equation (1.6) is known as Schrödinger equation associated with harmonic \((m = 1)\) and anharmonic \((m \geq 2)\) oscillators [22, 41], which is also a basic model in quantum mechanics. Equation (1.6) with \( m = 1 \) is also called the Hermite Schrödinger equation. Both (1.5) and (1.6) are well-posed in \( L^2(\mathbb{R}) \).

Our aim is to characterize the observable sets for the above two equations. For this purpose, we need two definitions as follows:

(D3) A measurable set \( E \subset \mathbb{R} \) is said to be thick if it satisfies (1.4), i.e., there is \( \gamma > 0 \) and \( L > 0 \) so that

\[
|E \cap [x, x + L]| \geq \gamma L \quad \text{for each} \quad x \in \mathbb{R}.
\]

(D4) A measurable set \( E \subset \mathbb{R} \) is said to be weakly thick, if

\[
\lim_{x \to +\infty} \frac{|E \cap [-x, x]|}{x} > 0.
\]

These two classes of sets have appeared in various studies. The concept of thick sets arose from studies of the uncertainty principle, see e.g. [5, p.5], [23, p.113]. It has been applied to the controllability of nonlinear wave equations [7, Corollary 6.5] and heat equations [18, 50] on the whole space. In the recent work [3], the authors have used thick and weakly thick sets in \( \mathbb{R}^n \) to study spectral inequalities for Hermite functions. We point out that, the class of thick sets is strictly contained in the class of weakly thick sets, see Proposition 4.7 and 4.8 for details.

There are two main theorems for the 1-dim case. We will state and explain them one by one.

**Theorem 1.1.** Let \( E \subset \mathbb{R} \) be a measurable set. The following statements are equivalent:

(i) The set \( E \) is thick.

(ii) The set \( E \) is an observable set at some time for the equation (1.5) with \( c \in \mathbb{R} \).
We make the following remarks related to Theorem 1.1:

(a1) As far as we are aware, Theorem 1.1 seems to be the first work to characterize the observable set at some time for the free Schrödinger equation (1.5). There have been works showing sufficient conditions on the observable sets for the Schrödinger equation on \( \mathbb{R}^n \). For instance, it is shown in [51, Remark (a6)] that, if \( E \) contains \( B^c(x_0, r) \) for some \( r > 0 \), then \( E \) is an observable set at any time for the free Schrödinger equation. The same conclusion holds true [42, Theorem 3.1] for the Schrödinger equation with Schwartz class potentials. It deserves mentioning that we now are not able to either extend Theorem 1.1 to higher dimensions or prove the minimal observable time for some technical reasons, see Remark 2.2 for details.

(a2) For the Schrödinger equation on compact Riemannian manifolds, the observability has been extensively studied: It was shown in [34] (see also [39]) that any open set with geometric control condition (GCC) is an observable set at any time. It was further proved in [35] that the GCC is also necessary in the manifolds with periodic geodesic flows (or in Zoll manifolds). It was obtained that on the flat torus \( \mathbb{T}^n := (\mathbb{R}/2\pi \mathbb{Z})^n \), every non-empty open set \( E \subset \mathbb{T}^n \) is an observable set at any time (see, for instance, [24, 26, 31, 46] for the free Schrödinger equation and [1, 6, 10] for the Schrödinger equations with potentials). Moreover, it was verified in [11] that on \( \mathbb{T}^2 \), each measurable set \( E \), with a positive measure, is an observable set at any time. For the observability of Schrödinger equations on negatively curved manifolds, we refer the readers to [2, 16, 17, 28]. It also deserves mentioning the works [16, 17] where the authors obtained a quantitative resolvent inequality of Hautus type. One of the key ingredients is the use of the fractal uncertainty principle, which was first formulated in [15]. As a comparison, the uncertainty principle also plays an important role in our paper: a generalization of the Logvinenko-Sereda uncertainty principle (see [32]) was used in the proof of Theorem 1.1, while a Nazarov’s uncertainty principle (see [27]) is crucial in proving the following Theorem 1.3.

Theorem 1.2. Let \( E \subset \mathbb{R} \) be a measurable set. The following statements are equivalent:

(i) The set \( E \) is weakly thick.
(ii) The set \( E \) is an observable set at some time for the equation (1.6) with \( m = 1 \).
(iii) The set \( E \) is an observable set at any time for the equation (1.6) with \( m \geq 2 \).

The following remarks are in order.

(b1) Theorem 1.2 characterizes observable sets for the equation (1.6). In this direction, we would like to mention [14] which shows that a half line: \((-\infty, x_0)\) (or \((x_0, \infty)\), with \( x_0 \in \mathbb{R} \), is an observable set at some time for the equation (1.6) with \( m = 1 \) (see [14, Proposition 3]). Compared to this, our Theorem 1.2 shows that the observable set can be much smaller. For example, one can take \( E = \bigcup_{k=1}^{\infty} [kx_0, (k + 1/2)x_0] \) with \( x_0 > 0 \). With regard to observable sets for (1.6) with \( m \geq 2 \), we do not find any result in the existing literatures.

(b2) Since every thick set is a weakly thick set, and the reverse may be not true, it follows from Theorem 1.1 and Theorem 1.2 that the appearance of the potentials \( x^{2m}(m \geq 1) \) enlarges the class of observable sets. The reason is as follows: Differing from the operator \((-\partial_x^2 + c)\) (which has the continuous spectrum),

---

1 Here and in what follows, \( B(x_0, r) \) denotes the closed ball in \( \mathbb{R}^n \), centered at \( x_0 \in \mathbb{R}^n \) and of radius \( r > 0 \), while \( B^c(x_0, r) \) denotes its complementary set.
the operator $H = -\frac{d^2}{dx^2} + x^{2m}$ has purely discrete spectrum consisting of all simple and real eigenvalues $\{\lambda_k\}_{k=1}^\infty$, with a gap condition (see (3.8)). This gap condition ensures that $E$ is an observable set at some time if and only if $\|\varphi_k\|_{L^2(E)}$ has a uniform positive lower bound for all $k \in \mathbb{N}^+$, where $\varphi_k$ is the $L^2$ normalized eigenfunction corresponding to $\lambda_k$ (see Proposition 3.3). Furthermore, we observed that each eigenfunction $\varphi_k$ is either even or odd (see Key Observation in Sect. 4.1). This fact suggests that the observable set can be chosen only on a half line, which is not a thick set clearly.

(b3) Theorem 1.2, together with Theorem 1.4 below, also reveals the essential difference between the equation (1.6) with $m = 1$ and $m \geq 2$ respectively, from the perspective of the minimal observable time. In fact, on one hand, the half line $(a, \infty)$ (with $a \in \mathbb{R}$) is clearly a weakly thick set (see Example 4.9), while on the other hand, $(a, \infty)$ is an observable set at time $T$ for the equation (1.6) with $m = 1$ if and only if $T > \frac{\pi^2}{2}$ (see Theorem 1.4). The reason behind this difference is closely related to the different asymptotic behaviours of the spectral gaps for $H = -\frac{d^2}{dx^2} + x^{2m}$ between the case $m = 1$ and $m \geq 2$. More precisely, by (4.11) and (5.1), we have

$$\begin{align*}
\lambda_{k+1} - \lambda_k &\rightarrow \infty, \quad k \rightarrow \infty, \quad m \geq 2, \\
\lambda_{k+1} - \lambda_k &\geq 2, \quad k \in \mathbb{N}^+, \quad m = 1.
\end{align*}$$

We refer to Proposition 3.3 for general results in this direction.

1.2. Observable sets for the Hermite Schrödinger equation in $\mathbb{R}^n$. We now turn to introduce our results on the observable sets in higher dimensions in which the problem become much more difficult. In fact, the proofs of Theorem 1.1 and Theorem 1.2 mainly rely on a uniform resolvent inequality of Hautus type (see Proposition 2.1), while as mentioned in the note (a1), there are essential difficulties in extending the inequality to higher dimensions (see Remark 2.2 and Remark 4.6).

Fortunately, for the Hermite Schrödinger equation

$$i\partial_t u(t, x) = (-\Delta + |x|^2)u(t, x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n; \quad u(0, \cdot) = u_0(\cdot) \in L^2(\mathbb{R}^n),$$

the solution can be written explicitly via Mehler’s formula and the solution satisfies some periodic property (see Sect. 5). These facts allow us to use a more direct approach in all dimensions. Hence, we only consider the observable sets for the Hermite Schrödinger equation in higher dimensions.

We present two sufficient conditions on observable sets at any/some time for (1.10) and also give a necessary condition on observable sets at some time for (1.10).

**Theorem 1.3.** Given $x_0 \in \mathbb{R}^n$ and $r > 0$. Then for any $T > 0$, there is $C = C(n) > 0$ so that when $u$ solves (1.10),

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \, dx \leq C \left(1 + \frac{1}{T}\right) e^{Cr^2/(1+\tau)} \int_0^T \int_{B^c(x_0, r)} |u(t, x)|^2 \, dx \, dt. \quad (1.11)$$

Thus, the exterior domain $B^c(x_0, r)$ is an observable set at any time for (1.10).

**Theorem 1.4.** Let $E = B^c(0, r) \cap \{x \in \mathbb{R}^n : a \cdot x \geq 0\}$ with some $r > 0$ and $a \in \mathbb{S}^{n-1}$. Then the following assertions are equivalent:
(i) The set $E$ is an observable set at time $T > 0$ for (1.10).

(ii) It holds that $T > \frac{\pi}{2}$.

**Theorem 1.5.** Suppose that $E \subset \mathbb{R}^n$ is an observable set at some time for (1.10). Then there are constants $L > 0$ and $c > 0$ so that

$$\left| E \cap B(y, L\rho(y)) \right| \geq ce^{-|y|^2} \text{ for all } y \in \mathbb{R}^n,$$

(1.12)

where $\rho(y) := \max(1, |y|)$.

Several remarks on Theorem 1.3–1.5 are as follows:

(e1) From Theorem 1.3 and Theorem 1.4, we see that for the Hermite Schrödinger equation in $\mathbb{R}^n$, different kinds of $E$ lead to different types of observability: when $E$ is the complement of any closed ball, it is an observable set at any time for (1.10), while when $E$ is half of the complement of any closed ball centered at the origin, it is an observable set at time $T > 0$ for (1.10) if and only if $T > \frac{\pi}{2}$. (These can be viewed as supplements of Theorem 1.2.) In this direction, we mention that Theorem 1.4 where $n = 1$ has been stated in [14, Prop. 5.1] without proof. It also deserves mentioning the recent paper [8] dealing with the time optimal observability for the two-dim Grushin Schrödinger equation on the finite cylinder: $\Omega = (-1, 1) \times \mathbb{T}_y$.

(e2) The proofs of Theorem 1.3 and Theorem 1.4 rely on an observability inequality for (1.10) at two points in time, given by Theorem 5.1 which is based on the Nazarov’s uncertainty principle (see [27]) and is of independent interest. With regard to the observability inequality at two points in time for Schrödinger equations, we mention papers [51] and [25].

(e3) Theorem 1.5 gives a necessary condition on observable sets at some time for the $n$-dim Hermite Schrödinger equation. The condition (1.12) is another kind of density condition. In the 1-dim case, it is strictly weaker than the weakly thick condition (1.9) (see Remark 5.2).

1.3. Plan of the paper. The rest of the paper is organized as follows: Sect. 2 proves Theorem 1.1; Sect. 3 shows a sufficient condition (see Theorem 3.1) on the observable sets at some time for Schrödinger equations with more general potentials; Sect. 4 gives the proof of Theorem 1.2, as well as the comparison of thicknesses for two kinds of observable sets; Sect. 5 presents the proofs of Theorem 1.3–Theorem 1.5; Finally, Appendix A-E provide the detailed proofs of several intermediate and technical results, respectively.

2. Proof of Theorem 1.1

We start with introducing a resolvent condition on the observability for some evolution equation, i.e., the next Proposition 2.1, which is another version of [36, Theorem 5.1] (see also [9,33,40] and [49]). It will be used in the proof of Theorem 1.1, as well as in the proof of Theorem 1.2. To state it, we consider the equation:

$$i\partial_t u(t, x) = \mathcal{A} u(t, x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}; \quad u(0, x) \in L^2(\mathbb{R}),$$

(2.1)

where $\mathcal{A}$ is a self-adjoint operator on $L^2(\mathbb{R})$. 

Proposition 2.1. Let $E \subset \mathbb{R}$ be a measurable set. Then the following statements are equivalent:

(i) The set $E$ is an observable set at some time for (2.1).

(ii) There is $M > 0$ and $m > 0$ so that

$$
\|u\|_{L^2_{x}(\mathbb{R})}^2 \leq M\|(-\partial_x^2 - \lambda)u\|_{L^2_{x}(\mathbb{R})}^2 + m\|u\|_{L^2_{x}(E)}^2 \quad \text{for all } u \in D(\mathcal{A}) \text{ and } \lambda \in \mathbb{R}.
$$

(2.2)

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Note that if $u$ is a solution of $i \partial_t u = -\partial_x^2 u$ and $c \in \mathbb{R}$, then $u_c(t, x) := e^{-ict}u(t, x)$ is a solution of $i \partial_t u = (-\partial_x^2 + c)u$. Thus it suffices to prove Theorem 1.1 in the case $c = 0$. Fix an arbitrary measurable subset $E \subset \mathbb{R}$. We organize the proof by two steps.

Step 1. We show that (i) of Theorem 1.1 implies (ii) of Theorem 1.1 with $c = 0$.

Suppose that $E$ is thick. Arbitrarily fix $\lambda \in \mathbb{R}$ and $u \in H^2(\mathbb{R})$. In the case that $\lambda \geq 0$, we deduce from [21, Proposition 1] that for some $m = m(E) > 0$ and $C = C(E) > 0$,

$$
\|u\|_{L^2_{x}(\mathbb{R})}^2 \leq C \left( \|(-\partial_x^2 - \lambda)u\|_{L^2_{x}(\mathbb{R})}^2 + m\|u\|_{L^2_{x}(E)}^2 \right).
$$

(2.3)

In the case that $\lambda < 0$, we have trivially $\|(-\partial_x^2 - \lambda)u\|_{L^2_{x}(\mathbb{R})}^2 \leq \|(-\partial_x^2 - \lambda)u\|_{L^2_{x}(\mathbb{R})}^2$, This, along with (2.3) where $\lambda = 0$, shows that (2.3) also holds for all $\lambda < 0$.

Combining the above two cases, we know that (2.2) holds with $\mathcal{A} = -\partial_x^2$. According to Proposition 2.1, we conclude that $E$ is an observable set at some time.

Step 2. We show that (ii) of Theorem 1.1 with $c = 0$ implies (i) of Theorem 1.1.

We borrow some ideas from [50] in this step. Recall that the kernel of the Schrödinger equation (1.5) with $c = 0$ is

$$
K(t, x) = (4\pi it)^{-1/2}e^{-|x|^2/4it}, \quad t > 0, x \in \mathbb{R}.
$$

Thus, given $u_0 \in \mathcal{S}(\mathbb{R})$ (the Schwartz class), the function defined by

$$
(t, x) \rightarrow \int_{\mathbb{R}} K(t, x - y)u_0(y) \, dy, \quad (t, x) \in (0, \infty) \times \mathbb{R},
$$

(2.4)

is a solution to the equation (1.5) (where $c = 0$), with the initial condition $u(0, x) = u_0(x), x \in \mathbb{R}$. Arbitrarily fix $x_0 \in \mathbb{R}$. By taking

$$
u(t, x) = (4\pi)^{-1/2}e^{-|x-x_0|^2/4}, \quad x \in \mathbb{R},$$

in (2.4), we get the following solution to the equation (1.5) (where $c = 0$):

$$
v(t, x) = (4\pi (it + 1))^{-1/2}e^{-|x-x_0|^2/4(it+1)}, \quad t \geq 0, x \in \mathbb{R}.
$$

(2.5)

The key ingredient in the proof of this result relies on a generalization of the Logvinenko-Sereda uncertainty principle: If $E$ is thick, then for all $f$ in $L^2(\mathbb{R})$ with Fourier support contained in finite disjoint intervals, one has $\|f\|_{L^2_{x}(\mathbb{R})} \leq C\|f\|_{L^2_{x}(E)}$. This is due to Kovrijkine, see [32, Theorem 2].
We now suppose that \( E \) satisfies (ii) of Theorem 1.1 with \( c = 0 \). Then there is \( T > 0 \) and \( C = C_{ob3}(T, E) > 0 \) so that any solution \( u \) to (1.5) (where \( c = 0 \)) satisfies (1.2), from which, it follows that

\[
\int_{\mathbb{R}} |v(0, x)|^2 \, dx \leq C \int_0^T \int_E |v(t, x)|^2 \, dx \, dt. \tag{2.6}
\]

Besides, we have the following two observations: First, a direct computation gives

\[
\int_{\mathbb{R}} |v(0, x)|^2 \, dx = \frac{1}{2\sqrt{2\pi}}. \tag{2.7}
\]

Second, for arbitrarily fixed \( L > 0 \), we have

\[
\int_0^T \int_E |v(t, x)|^2 \, dx \, dt \\
= \int_0^T \int_E \frac{1}{4\pi(1 + t^2)} e^{-\frac{(x-x_0)^2}{2(1+t^2)}} \, dx \, dt \\
\leq \int_0^T \int_E \frac{1}{2\pi} e^{-\frac{(x-x_0)^2}{2(1+t^2)}} \, dx \, dt \\
= \frac{T}{2\pi} \left( \int_E \left| \int_{[x_0-L/2, x_0+L/2]} e^{-\frac{(x-x_0)^2}{2(1+t^2)}} \, dx \right| + \int_E \left| \int_{[x_0-L/2, x_0+L/2]^c} e^{-\frac{(x-x_0)^2}{2(1+t^2)}} \, dx \right| \right). \tag{2.8}
\]

Since

\[
\int_E \left| \int_{[x_0-L/2, x_0+L/2]} e^{-\frac{(x-x_0)^2}{2(1+t^2)}} \, dx \right| \leq \left| E \bigcap [x_0 - L/2, x_0 + L/2] \right|
\]

and

\[
\int_E \left| \int_{[x_0-L/2, x_0+L/2]^c} e^{-\frac{(x-x_0)^2}{2(1+t^2)}} \, dx \right| \leq e^{-\frac{(L/2)^2}{4(1+t^2)}} \int_E \left| \int_{[x_0-L/2, x_0+L/2]^c} e^{-\frac{(x-x_0)^2}{4(1+t^2)}} \, dx \right| \leq \sqrt{4\pi(1 + T^2)} e^{-\frac{L^2}{16(1+T^2)}},
\]

we deduce from (2.8) that

\[
\int_0^T \int_E |v(t, x)|^2 \, dx \, dt \\
\leq \frac{T}{2\pi} \left( \left| E \bigcap [x_0 - L/2, x_0 + L/2] \right| + \sqrt{4\pi(1 + T^2)} e^{-\frac{L^2}{16(1+T^2)}} \right). \tag{2.9}
\]

Now, it follows from (2.6), (2.7) and (2.9) that

\[
\frac{1}{2\sqrt{2\pi}} \leq \frac{CT}{2\pi} \left( \left| E \bigcap [x_0 - L/2, x_0 + L/2] \right| + \sqrt{4\pi(1 + T^2)} e^{-\frac{L^2}{16(1+T^2)}} \right).
\]

In the above, by taking \( L = L_0 > 0 \) so that

\[
\frac{CT}{2\pi} \sqrt{4\pi(1 + T^2)} e^{-\frac{L_0^2}{16(1+T^2)}} \leq \frac{1}{4\sqrt{2\pi}},
\]
we find that
\[
\frac{1}{4\sqrt{2\pi}} \leq \frac{CT}{2\pi} |E\cap[x_0 - L_0/2, x_0 + L_0/2]|,
\]
from which, it follows that
\[
|E\cap[x_0 - L_0/2, x_0 + L_0/2]| \geq \gamma L_0 \text{ with } \gamma = \frac{\sqrt{2\pi}}{4CTL_0}.
\]
Since \(x_0\) was arbitrarily taken from \(\mathbb{R}\), we obtain from the above that
\[
|E\cap[x - L_0/2, x + L_0/2]| \geq \gamma L_0 \text{ for all } x \in \mathbb{R}.
\]
This implies that
\[
|E\cap[x, x + L_0]| \geq \gamma L_0 \text{ for all } x \in \mathbb{R},
\]
i.e., \(E\) is a thick set.

Thus, we end the proof of Theorem 1.1. □

Remark 2.2. (a) For the implication \((i) \Rightarrow (ii)\) in Theorem 1.1, we used the resolvent inequality of Hautus type (2.2). This, together with the result in [21], gives the sharp condition on the observable set. But such method does not give sharp range of the exact observability times in general (for instance, it follows from [49, Theorem 6.6.1] that observability inequality (1.2) holds for all \(T > \pi \sqrt{M}\)). Indeed, in the special case \(E = B^c(0, r)\), it is known that (1.2) holds for all \(T > 0\). This has been proved in [42] by applying the strategy of compactness uniqueness and also in [51] based on Nazarov uncertainty principle. However, both techniques in [42,51] do not seem to apply directly to the more general case of thick sets.

(b) Let us mention the situation in higher dimensions. The argument we presented in \((ii) \Rightarrow (i)\) above does not depend on the dimension, thus thick set is a necessary condition for the observable set of the free Schrödinger equation for all \(n \geq 1\).

The main difficulty comes from the sufficiency part. Precisely, if we still use the resolvent approach to prove \((i) \Rightarrow (ii)\), we have to confirm the following version of the uncertainty principle: When \(E\) is thick (in \(\mathbb{R}^n\)), then there exist some \(C, \delta > 0\), such that for all \(\lambda \geq 1\), one has
\[
\int_E |g|^2 dx \geq C \int_{\mathbb{R}^n} |g|^2 dx, \text{ supp } \hat{g} \subset \{\xi \in \mathbb{R}^n, |\xi| - \lambda \leq \frac{\delta}{\lambda}\},
\]
which, to our best knowledge, is not known when \(n \geq 2\).

3. Observable Sets for General Potentials

In this section, we will give a sufficient condition on observable sets at some time for the Schrödinger equation:
\[
i\partial_t u(t, x) = (-\partial_x^2 + V(x))u(t, x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}; \quad u(0, x) \in L^2(\mathbb{R}),
\]
where the potential \(V\) satisfies the following condition:

**Condition (H).** The real-valued function \(V\) belongs to the space \(C^3(\mathbb{R})\) and there is \(c \geq 1\) so that
(i) for some compact interval $0 \in K \subset \mathbb{R}$, $V''(x) > 0$ and $x V'(x) \geq 2cV(x) > 0$, when $x \in \mathbb{R} \setminus K$;
(ii) when $j = 1, 2, 3$, $V^{(j)}(x) = O(x^{-j}|x|^{2c})$ as $x \to \infty$.

Here and in what follows, given two functions $f$ and $g$, by $f(x) = O(g(x))$ as $x \to \infty$, we mean that there is $C > 0$ and $M > 0$ so that $|f(x)| \leq C|g(x)|$, when $|x| \geq M$.

Several notes on Condition (H) are given in order.

(d1) Condition (H) is a variant of the condition given in [52], where the smoothness of the fundamental solution of Schrödinger equations with similar perturbations was studied. Typical examples of potentials satisfying this condition are as:

$$V_1(x) = C(1 + |x|^2)^c, \quad x \in \mathbb{R}, \quad \text{with} \quad C > 0, \quad c \geq 1;$$

and

$$V_2(x) = \sum_{j=0}^{2m} a_j x^j, \quad x \in \mathbb{R}, \quad \text{with} \quad a_{2m} > 0, \quad a_j \in \mathbb{R}, \quad m \in \mathbb{N}^+.$$

(d2) By Condition (H), we have that for some $x_0 > 0$,

$$V'(x) > 0, \quad \text{when} \quad x \geq x_0; \quad (3.2)$$

we also have two constants $D > D' > 0$ so that

$$D'x^{2c} \leq V(x) \leq Dx^{2c}, \quad \text{when} \quad |x| \geq x_0. \quad (3.3)$$

The later shows that $V(x) \to +\infty$ as $|x| \to +\infty$. Hence, Condition (H) implies the following weaker condition:

$V$ is real-valued, locally bounded and $V(x) \to +\infty$, as $|x| \to \infty$. \quad (3.4)

(d3) Suppose that $V$ satisfies (3.4). Then, according to [4, Theorem 1.1, p.50], the operator $H = -\partial^2_x + V$ has the properties: it is essentially self-adjoint (i.e., its closure is self-adjoint); its resolvent is compact. Thus, we have $\sigma(H) = \{\lambda_k\}_{k=1}^{\infty}$, with

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k \to +\infty, \quad (3.5)$$

where $\lambda_k$, $k \in \mathbb{N}^+$, are all eigenvalues of $H$. We further have that each $\lambda_k$ is simple. (See Proposition 3.2.)

(d4) Throughout this section, we write $\{\lambda_k\}_{k=1}^{\infty}$, with (3.5), for the family of all eigenvalues of $H$, and $\{\psi_k\}_{k=1}^{\infty}$ for the family of corresponding normalized eigenfunctions.

The main result of this section is as:

**Theorem 3.1.** Suppose that Condition (H) holds. If $E \subset \mathbb{R}$ is a measurable set satisfying

$$\lim_{x \to +\infty} \frac{|E \cap [0, x]|}{x} > 0, \quad (3.6)$$

then $E$ is an observable set at some time for the equation (3.1).
Before proving Theorem 3.1, we outline our strategy: First, by the following Proposition 3.3, we can transform the desired result to an problem of uniform lower bound on the mass of eigenfunctions on the subdomain \( E \). Then we divide our analysis into the low frequency case and the high frequency case. To deal with the low frequency case, we only need the weaker assumption (3.4); while for the high frequency case, we shall use \textbf{Condition (H)} to obtain more explicit information on the asymptotic behavior of the high frequency eigenfunctions, moreover, the geometric assumption (3.6) on \( E \) is required to obtain a uniform lower bound.

3.1. \textit{Low frequency part—the operator \( H \) with the weak condition (3.4)}. In this subsection, we will study some properties of eigenvalues and eigenfunctions of \( H \) under the assumption (3.4). The conclusions (i) and (ii) of the following Proposition 3.2 are given by [4, Corollary, p.64] and [4, Proposition 3.3, p.65], respectively.

\textbf{Proposition 3.2.} Assume that (3.4) holds. Then the following statements are true:

(i) Every eigenfunction \( \varphi_k \) of \( H \) has at most finite zero points.

(ii) Every eigenvalue \( \lambda_k \) of \( H \) is simple, i.e., each eigenspace has dimension one.

Since \( H \) is essentially self-adjoint (which follows from (3.4)), we have the following facts: First, we can put the equation:

\[
\frac{i}{\partial t} u(t, x) = (-\partial_x^2 + V(x))u(t, x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}; \quad u(0, x) \in L^2(\mathbb{R}) \tag{3.7}
\]

into the framework of (2.1). Second, the \textit{observable sets at some time} for (3.7) can be defined in the same manner as that in the definition (D1) in Sect. 1. Third, \(-iH\) generates a unitary group \( e^{-iHt} \) in \( L^2(\mathbb{R}) \). Thus, the solution to equation (3.7) is as: \( u(t, \cdot) = e^{-iHt}u(0, \cdot) \) for each \( t \in \mathbb{R}^+ \).

The next proposition gives connections among \textit{observable sets}, eigenfunctions and eigenvalues of \( H \).

\textbf{Proposition 3.3.} Suppose that (3.4) is true. Further assume that eigenvalues of \( H \) satisfy that for some \( \varepsilon_0 > 0 \) (independent of \( k \),

\[
\lambda_{k+1} - \lambda_k \geq \varepsilon_0 > 0 \quad \text{for all } k \geq 1. \tag{3.8}
\]

Then for any measurable set \( E \subset \mathbb{R} \), the following statements are equivalent:

(i) The set \( E \) is an observable set at some time for (3.7).

(ii) The \( L^2 \)-normalized eigenfunctions of \( H \) satisfy that for some \( C > 0 \),

\[
\int_E |\varphi_k(x)|^2 dx \geq C > 0 \quad \text{for all } k \geq 1. \tag{3.9}
\]

In addition, if (ii) holds and \( H \) satisfies the following stronger spectral gap condition:

\[
\lambda_{k+1} - \lambda_k \rightarrow \infty, \quad \text{as } k \rightarrow \infty, \tag{3.10}
\]

then \( E \) is an observable set at any time for (3.7).

\textbf{Proof.} The proof is given in Appendix A. \( \Box \)
Lemma 3.4 (Lower bound for low frequency). Suppose that (3.4) is true. Then for each subset $E \subset \mathbb{R}$ of positive measure and each $\ell \in \mathbb{N}^+$, there exists $C = C(E, \ell) > 0$ so that
\[
\int_E |\varphi_k(x)|^2 \, dx \geq C, \quad \text{when } 1 \leq k \leq \ell.
\]

Proof. Arbitrarily fix a subset $E \subset \mathbb{R}$ of positive measure and $\ell \in \mathbb{N}^+$. First, by (i) of Proposition 3.2, we can easily see that for each $1 \leq k \leq \ell$, there is $C_k = C(k, E) > 0$ so that
\[
\int_E |\varphi_k(x)|^2 \, dx \geq C_k.
\]
(3.12)

Next, by setting $C := \min_{1 \leq k \leq \ell} C_k > 0$, we get (3.11) from (3.12) at once. This ends the proof of Lemma 3.4. \qed

3.2. High frequency part—the operator $H$ with Condition (H). In this subsection, we will study some properties of eigenvalues and eigenfunctions of $H$ under Condition (H). In particular, we shall give a uniform lower bound of high frequency eigenfunctions $\varphi_k$ for all $k > \ell$. The ideas in [20,48] (where the WKB method was applied to study the asymptotic behaviors of eigenfunctions of $H$) will be used.

We start with several facts. Fact One: Each $\varphi_k$ satisfies
\[
-\varphi''_k(x) + V(x)\varphi_k(x) = \lambda_k \varphi_k(x), \quad x \in \mathbb{R}.
\]
(3.13)

Fact Two: By Condition (H) and by (3.5), there is $k_0 \in \mathbb{N}^+$ so that $\lambda_k \geq 1$, when $k \geq k_0$ and so that
\[
0 \in \Omega_k, \quad \text{when } k \geq k_0,
\]
(3.14)

where
\[
\Omega_k := \{x \in \mathbb{R}, \ V(x) \leq \lambda_k/2\}, \quad k \geq k_0.
\]
(3.15)

Moreover, we find from (3.3) that $\Omega_k$ is a bounded set and that for some $C > 0$ (independent of $k$),
\[
|\Omega_k| \leq C\lambda_k^{1/4} \quad \text{for all } k \geq k_0.
\]
(3.16)

Fact Three: Recall the following Liouville transform (see e.g. [48, p. 119]):
\[
\begin{align*}
y &= S(x) = \int_0^x \sqrt{\lambda_k - V(s)} \, ds, \quad x \in \Omega_k \\
w &= w(y) = \left(\lambda_k - V(S^{-1}(y))\right)^{1/4} \varphi_k\left(S^{-1}(y)\right), \quad y \in S(\Omega_k).
\end{align*}
\]
(3.17)

By (3.17), we see that
\[
S'(x) > 0 \text{ over } \Omega_k; \quad S^{-1}(\cdot) \text{ exists over } S(\Omega_k).
\]
(3.18)

By (3.17) and (3.14), we obtain
\[
0 \in S(\Omega_k), \quad \text{as } k \geq k_0.
\]
For each $k \geq k_0$, by making the above Liouville transform to (3.13), which is restricted over $\Omega_k$, we obtain

$$\frac{d^2 w(y)}{dy^2} + w(y) + q(y)w(y) = 0, \quad y \in S(\Omega_k),$$

(3.20)

where

$$q(y) = \frac{V''(x)}{4(\lambda_k - V(x))^2} + \frac{5(V'(x))^2}{16(\lambda_k - V(x))^3}, \quad \text{with} \quad x = S^{-1}(y) \in \Omega_k.$$

Fact Four: The function $w$ (given by (3.17)) depends on $k$. By (3.17) and (3.19), we see that when $k \geq k_0$, 0 is in the domain of $w$. Hence, $w(0)$ and $w'(0)$ make sense.

With the above notations, we present an asymptotic expansion of the eigenfunction $\varphi_k(x)$ when $k$ is large enough in next lemma. The expansion includes a main contribution term and an error term.

**Lemma 3.5.** Suppose that Condition (H) holds for some $c \geq 1$. Let $\Omega_k$ be given by (3.15). Let $S$ and $w$ be given by (3.17). Then there exist constants $C, C' > 0$ and $k_0' \in \mathbb{N}^+$ so that for all $k \geq k_0'$,

$$\varphi_k(x) = (\lambda_k - V(x))^{-\frac{1}{4}} \cdot \Re (C_{\lambda_k} e^{iS(x)}) + R_k(x) \quad \text{for each} \quad x \in \Omega_k,$$

(3.21)

where $R_k$ is a function satisfying

$$|R_k(x)| \leq C(\lambda_k - V(x))^{-\frac{1}{4}} \cdot |C_{\lambda_k}| \cdot \lambda_k^{-\frac{1}{2}} \quad \text{for each} \quad x \in \Omega_k,$$

(3.22)

with

$$C_{\lambda_k} = w(0) - iw'(0)$$

(3.23)

satisfying the following two-sided bounds

$$C'\lambda_k^{\frac{1}{2} - \frac{1}{4c}} \leq |C_{\lambda_k}| \leq C\lambda_k^{\frac{1}{2} - \frac{1}{4c}}.$$  

(3.24)

**Proof.** The proof of the upper bound in (3.24) is given in Appendix B, while all other results have been proved in [52, Lemma 3.1&3.2]. □

With Lemma 3.5 in hand, we show that the eigenfunction $\varphi_k(x)$ can not be concentrated on small sets. More precisely, we shall give a uniform positive lower bound on $L^2$ norms of high-frequency eigenfunctions over sets satisfying the density condition (3.6).

This is the most important part in the proof of Theorem 3.1.

**Lemma 3.6 (Lower bound for high frequency).** Assume that Condition (H) holds and $E \subset \mathbb{R}$ is a measurable set satisfying (3.6). Then there exist $k''_0 \in \mathbb{N}^+$ and $C > 0$ so that

$$\int_E |\varphi_k(x)|^2 dx \geq C \quad \text{for all} \quad k \geq k''_0.$$  

(3.25)

**Proof.** The proof is given in Appendix C. □
3.3. Proof of Theorem 3.1. Arbitrarily fix a subset $E \subset \mathbb{R}$ satisfying (3.6). Then we have $|E| > 0$. We first claim that there exists $\varepsilon_0 > 0$ so that

$$\lambda_{k+1} - \lambda_k \geq \varepsilon_0 \quad \text{for all} \quad k \in \mathbb{N}^+. \quad (3.26)$$

In fact, because of Condition (H), we can apply [52, Lemma 3.3] to find $k_0 \in \mathbb{N}^+$ and $C > 0$, which are independent of $k$, so that

$$\lambda_{k+1} - \lambda_k \geq C\lambda_k^{\frac{1}{m}} - \frac{1}{m^2} \quad \text{for all} \quad k \geq k_0,$$

which, along with the conclusion (ii) in Proposition 3.2, leads to (3.26).

Next, we claim that there exists $C > 0$, independent of $k$, so that

$$\int_E |\varphi_k(x)|^2dx \geq C \quad \text{for all} \quad k \in \mathbb{N}^+. \quad (3.27)$$

Indeed, by (3.6), we can apply Lemma 3.6 to find $k''_0 \in \mathbb{N}^+$ and $C > 0$, independent of $k$, so that (3.27) holds for all $k \geq k''_0$. This, together with Lemma 3.4, leads to (3.27).

Finally, by (3.26) and (3.27), we can use Proposition 3.3 to see that $E$ is an observable set at some time for the equation (3.1). This ends the proof of Theorem 3.1. \qed

4. Proof of Theorem 1.2

In this section, we mainly prove Theorem 1.2, besides, we give the difference between thick sets and weakly thick sets. The proof of Theorem 1.2 is based on the following two theorems:

Theorem 4.1. Let $E \subset \mathbb{R}$ be weakly thick. Then the following conclusions are true:

(i) The set $E$ is an observable set at some time for the equation (1.6) with $m = 1$.

(ii) The set $E$ is an observable set at any time for the equation (1.6) with $m \geq 2$.

Theorem 4.2. If $E$ is an observable set at some time for the equation (1.6) with $m \in \mathbb{N}^+$, then it is weakly thick.

Proof of Theorem 1.2. The equivalence of (i) and (ii) in Theorem 1.2 follows from (i) of Theorem 4.1 and Theorem 4.2. The equivalence of (i) and (iii) in Theorem 1.2 follows from (ii) of Theorem 4.1 and Theorem 4.2. This ends the proof of Theorem 1.2. \qed

The rest of this section is organized as follows: Sects. 4.1 and 4.2 prove Theorem 4.1 and Theorem 4.2, respectively; Sect. 4.3 presents the difference between thick sets and weakly thick sets.

4.1. Proof of Theorem 4.1. First of all, we outline our main idea. To show that the weakly thick condition (1.9) is sufficient for observable sets at some time to the equation (1.6), we first use Theorem 3.1 (where $V(x) = x^{2m}$) to get the sufficient condition (3.6) on observable sets at some time for the equation (1.6), (Notice that the potential $V(x) = x^{2m}$ satisfies Condition (H).) then we find connections between (3.6) and the weakly thick condition (1.9), through using the property: each eigenfunction is either even or odd.
We now start our proof. Arbitrarily fix a weakly thick set $E \subset \mathbb{R}$. Then we have
\[
\lim_{x \to +\infty} \frac{|E \cap [-x, x]|}{x} > 0.
\] (4.1)

Arbitrarily fix $m \in \mathbb{N}^+$. First of all, we claim that the eigenfunctions $\{\varphi_k\}_{k \in \mathbb{N}^+}$ (to the operator $H = -\frac{d^2}{dx^2} + x^{2m}$) satisfy
\[
\int_E |\varphi_k(x)|^2 \, dx \geq C \text{ for all } k \in \mathbb{N}^+,
\] (4.2)
where $C > 0$ is independent of $k$. To this end, we introduce the following sets:
\[
E_+ := E \cap [0, \infty); \quad E_- := E \cap (-\infty, 0); \quad E_* := \{-x, x \in E_-\}.
\] (4.3)

It is clear that $E = E_+ \cup E_-$ and $E_+ \cap E_- = \emptyset$ and that
\[
E \cap [-x, x] = \left(E_+ \cap [0, x]\right) \cup \left(E_- \cap [-x, 0]\right) \text{ for each } x > 0,
\]
which, together with (4.3), implies that
\[
|E \cap [-x, x]| = |E_+ \cap [0, x]| + |E_- \cap [-x, 0]| = |E_+ \cap [0, x]| + |E_* \cap [0, x]| \leq 2 \left|\left(E_+ \cup E_*\right) \cap [0, x]\right| = 2 \left|\tilde{E} \cap [0, x]\right|,
\] (4.4)
where
\[
\tilde{E} := E_+ \cup E_* \subset [0, \infty).
\] (4.5)

Now it follows from (4.1) and (4.4) that
\[
\lim_{x \to +\infty} \frac{|\tilde{E} \cap [0, x]|}{x} > 0.
\] (4.6)

Because of (4.6), we can apply Theorem 3.1 to conclude that $\tilde{E}$ is an observable set at some time for (1.6). Further, according to Proposition 3.3, there is $C > 0$ (independent of $k$) so that
\[
\int_{\tilde{E}} |\varphi_k(x)|^2 \, dx \geq C \text{ for all } k \in \mathbb{N}^+.
\] (4.7)

To proceed, we need the following Key Observation: Each eigenfunction of $H$ is either even or odd. Indeed, we have
\[
\left(-\frac{d^2}{dx^2} + x^{2m}\right)\varphi_k(x) = \lambda_k \varphi_k(x), \quad x \in \mathbb{R}.
\] (4.8)

Let $\tilde{\varphi}_k(x) := \varphi_k(-x), x \in \mathbb{R}$. One can easily check that $\tilde{\varphi}_k$ also satisfies (4.8) and that $\|\varphi_k\|_{L^2(\mathbb{R})} = \|\tilde{\varphi}_k\|_{L^2(\mathbb{R})}$. These, along with the conclusion $(ii)$ Proposition 3.2, give immediately that either $\varphi_k = \tilde{\varphi}_k$ or $\varphi = -\tilde{\varphi}_k$, which leads to Key Observation.

By Key Observation and (4.3), we infer that for all $k \in \mathbb{N}^+$,
\[
\int_{E_-} |\varphi_k(x)|^2 \, dx = \int_{E_*} |\varphi_k(x)|^2 \, dx.
\] (4.9)
It follows from (4.3), (4.9) and (4.5) that
\[
\int_E |\varphi_k(x)|^2 dx = \int_{E_+} |\varphi_k(x)|^2 dx + \int_{E_-} |\varphi_k(x)|^2 dx \geq \int_E |\varphi_k(x)|^2 dx. \tag{4.10}
\]
Combining (4.7) and (4.10), we find that
\[
\int_E |\varphi_k(x)|^2 dx \geq C \quad \text{for all } k \in \mathbb{N}^+,
\]
which leads to (4.2).

Moreover, we shall use the following explicit asymptotic expression of eigenvalues \(\lambda_k\) of \(-\partial_x^2 + x^{2m}\): (See e.g. [19,43].)
\[
\lambda_k = \left(\frac{\pi}{B(3/2, 1/(2m))}\right)^{\frac{2m}{m+1}} k \cdot (1 + r_k), \quad k \in \mathbb{N}^+; \quad \text{and } \lim_{k \to +\infty} r_k = 0, \tag{4.11}
\]
where \(B(\cdot, \cdot)\) is the Beta function. We can verify directly from (4.11) that for some \(\varepsilon_0 > 0,^3\)
\[
\lambda_{k+1} - \lambda_k \geq \varepsilon_0 > 0 \quad \text{for all } k \geq 1. \tag{4.12}
\]

Next we prove (i) of Theorem 4.1. Indeed, by (4.2) and (4.12), we can use Proposition 3.3 to conclude that \(E\) is an observable set at some time for (1.6) with \(m \in \mathbb{N}^+\), which leads to (i) clearly.

Finally, we prove (ii) of Theorem 4.1. Arbitrarily fix \(m \geq 2\). By (4.11) again and some calculations, we can find \(C > 0\) (independent of \(k\)) so that
\[
\lambda_{k+1} - \lambda_k \geq C k^{\frac{m-1}{m+1}} \to \infty, \quad \text{as } k \to \infty.
\]
Then because of (4.2) and (4.12), we apply the last statement in Proposition 3.3 to conclude that \(E\) is an observable set at any time for (1.6). This completes the proof of Theorem 4.1. \(\square\)

4.2. Proof of Theorem 4.2. Now suppose that \(E\) is an observable set at some time \(T_0 > 0\) for (1.6), with an arbitrarily fixed \(m \in \mathbb{N}^+\). Then there is \(C_0 = C_0(T_0, E) > 0\) so that
\[
\|u_0\|_{L^2}^2 \leq C_0 \int_0^{T_0} \int_E |e^{-itH}u_0|^2 dx dt \quad \text{for all } u_0 \in L^2(\mathbb{R}). \tag{4.13}
\]
Let \(u_0 = \varphi_k\) in (4.13), where \(\varphi_k\) is the \(L^2\) normalized eigenfunction of \(H\), we find
\[
\int_E |\varphi_k|^2 dx \geq C_1 \quad \text{for all } k \in \mathbb{N}^+, \tag{4.14}
\]
where \(C_1 = 1/(T_0C_0)\). (Notice that (4.14) can also be obtained by Proposition 3.3.)

In order to show that \(E\) is weakly thick from the uniform inequality (4.14), the asymptotic expression (3.21) for general potentials with Condition (H) does not seem

\(^3\) It deserves mentioning that since Condition (H) holds for \(V(x) = x^{2m}\), (4.12) has been proved in the proof of Theorem 3.1 (see (3.26)).
to be enough. We need a finer asymptotic expression of \( \varphi_k \) for the case (1.7). This will be given by the next Lemma 4.3. To state it, we write

\[
\mu_k := \frac{1}{\lambda_k} \quad \text{for each } k \gg 1 \text{ (so that } \lambda_k > 0) .
\]

(4.15)

For each \( x \in \mathbb{R} \) and each \( k \gg 1 \), we define

\[
S_k^{-}(x) := \int_0^x \sqrt{|\mu_k^{2m} - t^{2m}|} \, dt \quad \text{and} \quad S_k^{+}(x) := \int_{\mu_k}^{\infty} \sqrt{|t^{2m} - \mu_k^{2m}|} \, dt .
\]

(4.16)

We also notice that \( \varphi_k \) satisfies

\[
\begin{cases}
-\varphi''_k(x) + x^{2m} \varphi_k(x) = \mu_k^{2m} \varphi_k(x), & x \in \mathbb{R}; \\
\|\varphi_k\|_{L^2(\mathbb{R})} = 1 .
\end{cases}
\]

(4.17)

The next Lemma 4.3 is the key in our proof.

**Lemma 4.3.** With notations in (4.15) and (4.16), when \( k \to +\infty \), either

\[
\varphi_k(x) = \begin{cases}
ad_k^{-}(\mu_k^{2m} - |x|^{2m})^{-\frac{1}{4}} (\cos S_k^{-}(x) + R_k(x)), & |x| < \mu_k - \delta \cdot \mu_k^{\frac{2m-1}{3}}, \\
o(\mu_k^{2m} - |x|^{2m})^{-\frac{1}{4}}, & \mu_k - \delta \cdot \mu_k^{\frac{2m-1}{3}} \leq |x| \leq \mu_k + \delta \cdot \mu_k^{\frac{2m-1}{3}}, \\
ad_k^{+}(\mu_k^{2m} - |x|^{2m})^{-\frac{1}{4}} e^{-S_k^{+}(x)} (1 + R_k(x)), & |x| > \mu_k + \delta \cdot \mu_k^{\frac{2m-1}{3}},
\end{cases}
\]

or

\[
\varphi_k(x) = \begin{cases}
ad_k^{-+}(\mu_k^{2m} - |x|^{2m})^{-\frac{1}{4}} (\sin S_k^{-}(x) + R_k(x)), & |x| < \mu_k - \delta \cdot \mu_k^{\frac{2m-1}{3}}, \\
o(\mu_k^{2m} - |x|^{2m})^{-\frac{1}{4}}, & \mu_k - \delta \cdot \mu_k^{\frac{2m-1}{3}} \leq |x| \leq \mu_k + \delta \cdot \mu_k^{\frac{2m-1}{3}}, \\
ad_k^{++}(\mu_k^{2m} - |x|^{2m})^{-\frac{1}{4}} e^{-S_k^{+}(x)} (1 + R_k(x)), & |x| > \mu_k + \delta \cdot \mu_k^{\frac{2m-1}{3}},
\end{cases}
\]

(4.18)

where \( \delta > 0 \) is independent of \( k \) and \( x \) and

\[
|a_k^\pm| \sim \mu_k^{\frac{m-1}{2}}, \quad R_k(x) = O \left( |x^{2m} - \mu_k^{2m}|^{-\frac{1}{2}} \cdot |x - \mu_k|^{-1} \right) .
\]

(4.20)

**Proof.** The proof is given in Appendix D. \( \square \)

Here and in what follows, given sequences of numbers \( \{\alpha_k\} \) and \( \{\gamma_k\} \), by \( \alpha_k \sim \gamma_k \), we mean that there is \( C_1 > 0 \) and \( C_2 > 0 \) so that \( C_1 |\gamma_k| \leq |\alpha_k| \leq C_2 |\gamma_k| \) for all \( k \), while by \( \alpha_k = O(\gamma_k) \), we mean that there is \( C_3 > 0 \) so that \( |\alpha_k| \leq C_3 |\gamma_k| \) for all \( k \).

We now back to the proof of Theorem 4.2. In order to apply Lemma 4.3, we make the following decomposition:

\[
\int_E |\varphi_k|^2 \, dx = I_1 + I_2 + I_3 ,
\]

(4.21)

where

\[
I_1 := \int_E \left\{ x : |x| < \mu_k - \delta \mu_k^{\frac{2m-1}{3}} \right\} |\varphi_k(x)|^2 \, dx ,
\]
\[
I_2 := \int_E \left\{ x : \mu_k - \frac{2m-1}{3} \leq |x| \leq \mu_k + \frac{2m-1}{3} \right\} |\varphi_k(x)|^2 \, dx,
\]
\[
I_3 := \int_E \left\{ x : |x| > \mu_k + \frac{2m-1}{3} \right\} |\varphi_k(x)|^2 \, dx.
\]

To deal with the term \(I_1\), we observe that by (4.20), there is \(C > 0\) (independent of \(k\) and \(x\)) so that
\[
|R_k(x)| \leq C \left( \mu_k^{2m-1} - \frac{2m-1}{3} \mu_k \right)^{-\frac{1}{2}} \mu_k^{\frac{2m-1}{3}} \leq C,
\]
when \(x\) satisfies
\[
|\varphi_k(x)|^2 \leq C.
\]
when \(x\) satisfies
\[
|\varphi_k(x)|^2 \leq C.
\]
Now, we write
\[
I_1 = \int_E \left\{ x : |x| < \rho \mu_k \right\} |\varphi_k(x)|^2 \, dx + \int_E \left\{ x : \rho \mu_k < |x| < \mu_k - \frac{2m-1}{3} \mu_k \right\} |\varphi_k(x)|^2 \, dx
\]
\[
:= I_{1,1} + I_{1,2}, \tag{4.23}
\]
where \(\rho \in (0, 1)\) is some constant to be chosen later. Notice that for any given \(\rho \in (0, 1)\), we have
\[
\rho \mu_k < \mu_k - \frac{2m-1}{3} \mu_k, \quad k \to \infty.
\]
Thus we can use (4.22), as well as Lemma 4.3, to find \(k_1 \in \mathbb{N}^+\) so that when \(k \geq k_1\),
\[
I_{1,1} \leq C_2 \int_E \mu_k^{m-1} \left( \mu_k^{2m} - x^{2m} \right)^{-\frac{1}{2}} \, dx
\]
\[
\leq C_2 \left( 1 - \rho^{2m} \right)^{-\frac{1}{2}} \frac{|E \cap \{ |x| < \rho \mu_k \}|}{\mu_k} \tag{4.24}
\]
and
\[
I_{1,2} \leq C_3 \int_E \mu_k^{m-1} \left( \mu_k^{2m} - x^{2m} \right)^{-\frac{1}{2}} \, dx
\]
\[
\leq C_3 \int_{\rho}^{1} (1 - x^{2m})^{-\frac{1}{2}} \, dx, \tag{4.25}
\]
where \(C_2, C_3 > 0\) are two absolute constants. Since \(\int_{\rho}^{1} (1 - x^{2m})^{-\frac{1}{2}} \, dx \to 0\) as \(\rho \to 1^-\), we can choose \(\rho = \rho_0 \in (0, 1)\) so that
\[
\int_{\rho_0}^{1} (1 - x^{2m})^{-\frac{1}{2}} \, dx < \frac{C_1}{100C_3}. \tag{4.26}
\]
Then it follows from (4.23)-(4.26) that
\[
I_1 \leq C_2 \left( 1 - \rho_0^{2m} \right)^{-\frac{1}{2}} \frac{|E \cap \{ |x| < \rho_0 \mu_k \}|}{\mu_k} + \frac{C_1}{100}. \tag{4.27}
\]
For the term $I_2$, we can use Lemma 4.3 again to find $k_2 \in \mathbb{N}^+$ and $C_4 > 0$ so that
\[
I_2 \leq C_4 \mu_k^\frac{m-2}{3} \mu_k^{-\frac{2m-1}{3}} = C_4 \mu_k^{-\frac{m+1}{3}} \leq \frac{C_1}{100} \text{ for all } k \geq k_2. \tag{4.28}
\]

We next deal with the term $I_3$. First we claim that for large $k$,
\[
S_k^+(x) > \frac{\sqrt{3}}{3} \mu_k^{-\frac{2m-1}{3}} (|x| - \mu_k), \text{ when } |x| > \mu_k + \delta \mu_k^{-\frac{2m-1}{3}}. \tag{4.29}
\]
(Here, $S_k^+(x)$ is given by (4.16).) Indeed, by (4.16), one can directly check that
\[
S_k^+(\mu_k + \delta \mu_k^{-\frac{2m-1}{3}}) \geq \frac{2}{3} \delta^\frac{3}{2} \text{ for all } k >> 1. \tag{4.30}
\]

We define the function:
\[
F(x) = S_k^+(x) - \frac{\sqrt{3}}{3} \mu_k^{-\frac{2m-1}{3}} (|x| - \mu_k), \quad x \in \mathbb{R}. \tag{4.31}
\]
From (4.31) and (4.30), we have that
\[
F\left(\mu_k + \delta \mu_k^{-\frac{2m-1}{3}}\right) > 0 \text{ for all } k >> 1 \tag{4.32}
\]
and that when $k \in \mathbb{N}^*$,
\[
F'(x) = \sqrt{|x^{2m} - \mu_k^{2m}| - \frac{\sqrt{3}}{3} \mu_k^{-\frac{2m-1}{3}}} \geq \left(\left(\mu_k + \delta \mu_k^{-\frac{2m-1}{3}}\right)^{2m} - \mu_k^{2m}\right)^{\frac{1}{2}} - \frac{\sqrt{3}}{3} \mu_k^{-\frac{2m-1}{3}} \geq \left(\sqrt{2m} - \frac{1}{3}\right) \sqrt{\delta \mu_k^{2m}} > 0, \text{ when } |x| \geq \mu_k + \delta \mu_k^{-\frac{2m-1}{3}}. \tag{4.33}
\]
From (4.33), (4.32) and (4.31), we obtain (4.29). Now by Lemma 4.3, (4.22) and (4.29), there is some constant $C_5 > 0$ (independent of $k$) so that
\[
I_3 \leq C_5 \mu_k^{m-1} \int_{\{x:|x|>\mu_k+\delta \mu_k^{-\frac{2m-1}{3}}\}} (x^{2m} - \mu_k^{2m})^{-\frac{1}{2}} \exp\left\{-\frac{\sqrt{3}}{3} \mu_k^{-\frac{2m-1}{3}} (|x| - \mu_k)\right\} \, dx
\]
\[
\leq C_5 \mu_k^{\frac{1}{2}} \int_{\mu_k+\delta \mu_k^{-\frac{2m-1}{3}}}^{\infty} (x - \mu_k)^{-\frac{1}{2}} \exp\left\{-\frac{\sqrt{3}}{3} \mu_k^{-\frac{2m-1}{3}} (x - \mu_k)\right\} \, dx.
\]
By changing variable $\mu_k^{-\frac{2m-1}{3}} (x - \mu_k) = y$ in the second integral above, we can find $k_3 \in \mathbb{N}^+$ and $C_6 > 0$ so that
\[
I_3 \leq C_6 \mu_k^{-\frac{1}{2}} \mu_k^{m-1} \leq \frac{C_1}{100} \text{ for all } k \geq k_3. \tag{4.34}
\]
Finally, it follows from (4.14), (4.21), (4.27), (4.28) and (4.34) that there is $C > 0$, independent of $k$, so that

$$|E \cap \{ x : |x| < \rho_0 \mu_k \}| \geq C, \text{ when } k \geq k_4 := \max\{k_1, k_2, k_3\}. \quad (4.35)$$

By (4.35) and the following Lemma 4.4, we conclude that $E$ is weakly thick. Therefore we have completed the proof of Theorem 4.2. □

**Lemma 4.4.** (a) Let $E \subset \mathbb{R}$ be a measurable subset. Then for each $a > 0$,

$$\lim_{N \ni k \to \infty} \frac{|E \cap [-a\lambda_k^{\frac{1}{2m}}, a\lambda_k^{\frac{1}{2m}}]|}{a\lambda_k^{\frac{1}{2m}}} = \lim_{N \ni k \to \infty} \frac{|E \cap [-abk^{\frac{1}{m+1}}, abk^{\frac{1}{m+1}}]|}{abk^{\frac{1}{m+1}}}, \quad (4.36)$$

where

$$b = \left( \frac{\pi}{B(3/2, 1/2m)} \right)^{\frac{1}{m+1}}. \quad (4.37)$$

(b) Let $E \subset \mathbb{R}$ be a measurable set. Then the following statements are equivalent:

(i) The set $E$ is weakly thick, i.e.,

$$\lim_{x \to +\infty} \frac{|E \cap [-x, x]|}{x} > 0.$$

(ii) For some $a > 0$ and $l > 0$,

$$\lim_{N \ni k \to \infty} \frac{|E \cap [-ak^l, ak^l]|}{ak^l} > 0.$$

**Proof.** The proof is given in Appendix E. □

**Remark 4.5.** It follows immediately from Theorem 4.2 that any bounded set in $\mathbb{R}$ is not an observable set at any time for (1.6) (with $m \in \mathbb{N}^+$). Indeed, if $E \subset [-R, R]$ for some $R > 0$, then

$$\frac{|E \cap [-x, x]|}{|x|} \leq \frac{2R}{|x|} \to 0, \text{ as } x \to +\infty. \quad (4.38)$$

Thus $E$ is not weakly thick, consequently, it is not an observable set for (1.6).

As a comparison, it is natural to ask if a bounded measurable subset $E \subset \mathbb{R}$ is an observable set for the heat equation: $\partial_t u + Hu = 0$ where $H = -\Delta + |x|^{2m}$ with $m > 1$. This seems to be open (see [37]).

For example, let

$$E := \bigcup_{j=1}^{\infty} E_j, \text{ with } E_j := [j, j + (j + 1)^{-\varepsilon}], \ \varepsilon > 0. \quad (4.39)$$

It is clear that $E \subset [0, \infty)$; $E$ is unbounded; $|E| = \infty$ if $0 < \varepsilon \leq 1$. Let $x > 2$ and let $j_0$ be the unique positive integer so that $j_0 \leq x < j_0 + 1$. Then by (4.39) we have

$$\frac{|E \cap [-x, x]|}{|x|} \leq \frac{1}{j_0} \sum_{j=1}^{j_0} (j + 1)^{-\varepsilon} \leq Cx^{-\varepsilon}. \quad (4.40)$$

Since the right hand side of (4.40) tends to 0 as $x \to +\infty$, $E$ is not weakly thick, therefore it is not an observable set for (1.6) for all $m \in \mathbb{N}^+$. 


Remark 4.6. The situation in higher dimensions is quite different and let us consider the Hermite operator \( H = -\Delta + |x|^2 \) for simplicity. Note that the eigenvalue is no longer simple when \( n \geq 2 \) (except for the smallest eigenvalue, see (5.4)), in particular, the concentration property of the corresponding (high frequency) eigenfunctions becomes very complicated (see [30, Sect. 5]). This, in view of (4.14), makes the characterization of observable sets in higher dimension much more difficult. Nevertheless, we can still obtain some interesting results (including the minimal observable time) based on a more direct approach (see Sect. 5 below).

4.3. Comparison of thicknesses for two kinds of observable sets. In this subsection, we shall show that the class of thick sets is strictly included in the class of weakly thick sets.

Proposition 4.7. Every thick set is weakly thick.

Proof. Let \( E \) be a thick set. According to the definition (D3), there is \( L > 0 \) and \( \gamma > 0 \) so that
\[
\frac{|E \cap [x, x + L]|}{L} \geq \gamma \quad \text{for all} \quad x \in \mathbb{R}.
\] (4.41)

Given \( x \geq L \), there is \( n_x \in \mathbb{N}^+ \) so that \( n_x L \leq x < (n_x + 1)L \). Then we find
\[
[-x, x] \supset [-n_x L, n_x L] = \bigcup_{-n_x \leq j \leq n_x - 1} [jL, (j + 1)L].
\]

This, along with (4.41), yields
\[
|E \cap [-x, x]| \geq \sum_{-n_x \leq j \leq n_x - 1} |E \cap [jL, (j + 1)L]| \geq \sum_{-n_x \leq j \leq n_x - 1} \gamma L
\]
\[
= 2 \gamma n_x L \geq \gamma (n_x + 1) L > \gamma x.
\]

Since \( x \geq L \) is arbitrarily given, the above leads to
\[
\lim_{x \to +\infty} \frac{|E \cap [-x, x]|}{x} \geq \gamma > 0.
\]

Hence, \( E \) is weakly thick. This ends the proof of Proposition 4.7. \( \square \)

Proposition 4.8. A weakly thick set may not be thick.

This can be seen from the next two examples.

Example 4.9. Let \( E = [a, \infty) \) with \( a \in \mathbb{R} \). Then \( E \) is weakly thick but not thick.

It follows directly from definitions (1.8) and (1.9), so we omit the details.

Example 4.10. Let
\[
E = \bigcup_{j=1}^{\infty} \left( [2^j, 2^{j+1} - j] \cup [-2^{j+1} + j, -2^j] \right).
\] (4.42)

Then \( E \) is weakly thick but not thick.
Here is the proof: We first show that $E$ is weakly thick. To this end, we first claim that

$$\lim_{x \to +\infty} \frac{|E \cap [0, x]|}{x} > 0. \quad (4.43)$$

For this purpose, we arbitrarily fix $x \geq 32$. Then there is a unique integer $n \geq 5$ so that

$$2^n \leq x < 2^{n+1}. \quad (4.44)$$

From $(4.44)$, we find that $[0, x] \supset [0, 2^n] \supset [2^{n-1}, 2^n - n + 1]$. This, along with $(4.42)$, yields that

$$E \cap [0, x] \supset [2^{n-1}, 2^n - n + 1].$$

The above, together with $(4.44)$, implies that

$$\left| E \cap [0, x] \right| \geq 2^{n-1} - n + 1 > x/8,$$

which leads to $(4.43)$. By $(4.43)$, we find that $E$ is weakly thick.

We next show that $E$ is not thick. To this end, we arbitrarily fix $L > 0$. Let $k \in \mathbb{N}^+$ so that $k > 3L$. Using $(4.42)$, we deduce that

$$E \cap [2^k - 3L, 2^k - L] = \emptyset. \quad (4.45)$$

But clearly one has

$$[x, x + L] \subset [2^k - 3L, 2^k - L], \quad \text{when} \quad x \in [2^k - 3L, 2^k - 2L]. \quad (4.46)$$

Combining $(4.45)$ and $(4.46)$, we find that

$$E \cap [x, x + L] = \emptyset, \quad \text{when} \quad x \in [2^k - 3L, 2^k - 2L].$$

This shows that $E$ is not thick.

**Remark 4.11.** Given a sequence $\{a_j\}_{j=1}^\infty$, with $0 < a_j < 2^{j-1}$, let

$$\tilde{E} = \bigcup_{j=1}^\infty \left( [2^j, 2^{j+1} - a_j] \bigcup [-2^{j+1} + a_j, -2^{j+1}] \right). \quad (4.47)$$

Then $\tilde{E}$ is thick if and only if $\{a_j\}_{j=1}^\infty$ is bounded, i.e., there is $L_0 > 0$ (independent of $j$) so that

$$a_j \leq L_0 \quad \text{for all} \quad j \geq 1. \quad (4.48)$$

Indeed, if $\tilde{E}$ is thick, then by $(1.8)$ (where $E = \tilde{E}$ and $x = 2^{j+1} - a_j$), we see that for all $j$ large enough,

$$\left| [2^{j+1}, 2^{j+1} - a_j + L] \right| \geq \left| \tilde{E} \cap [x, x + L] \right| \geq L_0 > 0.$$

Since $a_j < 2^{j-1}$, the above leads to $(4.48)$. Conversely, we suppose that $(4.48)$ is true. Set $g(x) := |\tilde{E} \cap [x, x + 2L_0]|/(2L_0)$. By $(4.47)$ and $(4.48)$, we can find $M > 0$ so that $g(x) \geq \frac{1}{2}$, when $|x| > M$. Since $g(\cdot)$ is continuous and positive over $[-M, M]$, we can choose $\gamma_0 > 0$ so that $g(x) \geq \gamma_0$ for all $x \in \mathbb{R}$. So $\tilde{E}$ is thick.

Intuitively speaking, the sequence $\{a_j\}_{j=1}^\infty$ describes the gaps of the set $\tilde{E}$. From $(4.48)$, we see that a thick set must have uniformly bounded gaps, while from $(4.42)$ ($a_j = j$), we find that a weakly thick set can contain increasing gaps of arbitrarily large size.
5. Proofs of Main Theorems for the n-Dim Case

We start with recalling several known facts related to the spectral theory of the Hermite operator \( H = -\Delta + |x|^2 \) (in \( L^2(\mathbb{R}^n) \)), which can be found in [47,48]. The first one is about eigenvalues:

\[
\sigma(H) = \{ n + 2k, \ k = 0, 1, 2, \ldots \}. \tag{5.1}
\]

The second one is about eigenfunctions of \( H \): For each \( k \in \mathbb{N} \), let

\[
\varphi_k(x) = (2^k k! \sqrt{\pi})^{-\frac{1}{2}} H_k(x) e^{-\frac{x^2}{2}}, \ x \in \mathbb{R}, \tag{5.2}
\]

where \( H_k \) is the Hermite polynomial given by

\[
H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} (e^{-x^2}), \ x \in \mathbb{R}. \tag{5.3}
\]

Notice that \( \| \varphi_k(x) \|_{L^2(\mathbb{R})} = 1 \) for all \( k \in \mathbb{N} \). Now for each multi-index \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) (\( \alpha_i \in \mathbb{N} \)), we define the following \( n \)-dimensional Hermite function by tensor product:

\[
\Phi_\alpha(x) = \prod_{i=1}^n \varphi_{\alpha_i}(x_i), \ x = (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{5.4}
\]

Then for each \( k \in \mathbb{N} \), \( \Phi_\alpha \) (with \( |\alpha| = k \)) is an eigenfunction of \( H \) corresponding to the eigenvalue \( n + 2k \), and \( \{ \Phi_\alpha : \alpha \in \mathbb{N}^n \} \) forms a complete orthonormal basis in \( L^2(\mathbb{R}^n) \). The third one is about the solution \( u \) of the Hermite-Schrödinger equation (1.10) with the initial condition \( u(0, \cdot) = f(\cdot) \in L^2(\mathbb{R}^n) \):

\[
u(t, x) = e^{-itH} f = \sum_{\alpha \in \mathbb{N}^n} e^{-it(n+2|\alpha|)} a_\alpha \Phi_\alpha(x), \ t \geq 0, \ x \in \mathbb{R}^n, \tag{5.5}\]

where \( a_\alpha = \int_{\mathbb{R}^n} f(x) \Phi_\alpha(x) \, dx \) is the Fourier-Hermite coefficient. Let \( K(t, x, y) \) be the kernel associated to the operator \( e^{-itH} \). Then by Mehler’s formula (see e.g. in [45,47]), we have

\[
e^{-itH} f = \int_{\mathbb{R}^n} K(t, x, y) f(y) \, dy, \ t \in \mathbb{R}^+ \setminus \frac{\pi}{2} \mathbb{N}, \tag{5.6}\]

where

\[
K(t, x, y) = \frac{e^{-i\pi n/4}}{(2\pi \sin 2t)^{n/2}} \exp \left( \frac{i}{2} (|x|^2 + |y|^2) \cot 2t - \frac{ix \cdot y}{\sin 2t} \right), \ x, y \in \mathbb{R}^n. \tag{5.7}\]

Meanwhile, it follows by (5.5) that

\[
\| e^{-itH} f \|_{L^2(\mathbb{R}^n)} = \| f \|_{L^2(\mathbb{R}^n)} \text{ for all } t \geq 0 \tag{5.8}\]

and that

\[
e^{-i(t+\pi)H} f = e^{-i\pi n} e^{-itH} f \text{ for all } t \geq 0. \tag{5.9}\]
5.1. Proof of Theorem 1.3. As already mentioned in Remark (c) in the introduction, we first build up the observability inequality at two points in time for the equation (1.10), then by using it, obtain the observability inequality (1.11) for any \( T > 0 \). Since we are in the general case where \( n \geq 1 \), the spectral approach used to prove Theorem 1.1 and Theorem 1.2 seem not work. (At least, we do not know how to use it.) Fortunately, the kernel, associated with \( e^{-itH} \), has an explicit expression given by (5.7). This expression can help us to look at the problem from a new perspective. In particular, we realize some connections between uncertainty principles in harmonic analysis and observability inequalities. It deserves mentioning what follows: (i) The aforementioned observability inequality at two time points was obtained in [51] for the free Schrödinger equation; (ii) In [25], the authors considered a class of decaying potentials \( V \) and established observability inequality at two time points for \( H = -\Delta + V \). To our best knowledge, no such kind of results have been proved for potentials that are increasing to infinity when \( |x| \to \infty \).

**Theorem 5.1.** The following conclusions are true:

(i) If \( T < S \geq 0 \), with \( T < S = \frac{k\pi}{2} \) for all \( k \in \mathbb{N}^+ \), then there is \( C := C(n) \) so that

\[
\int_{\mathbb{R}^n} |u(0, x)|^2 \, dx \leq C e^{\frac{C_1 r_2}{2} \pi} \left( \int_{B^c(x_1, r_1)} |u(S, x)|^2 \, dx + \int_{B^c(x_2, r_2)} |u(T, x)|^2 \, dx \right)
\]

(5.10)

holds for any closed balls \( B(x_1, r_1) \) and \( B(x_2, r_2) \) in \( \mathbb{R}^n \) and any solution \( u \) to (1.10).

(ii) If \( T < S \geq 0 \), with \( T < S = \frac{k\pi}{2} \) for some \( k \in \mathbb{N}^+ \), then for any closed balls \( B(x_1, r_1) \) and \( B(x_2, r_2) \) in \( \mathbb{R}^n \), there is no \( C > 0 \) so that

\[
\int_{\mathbb{R}^n} |u(0, x)|^2 \, dx \leq C \left( \int_{B^c(x_1, r_1)} |u(S, x)|^2 \, dx + \int_{B^c(x_2, r_2)} |u(T, x)|^2 \, dx \right)
\]

(5.11)

holds for all solutions \( u \) to (1.10).

**Proof.** Let \( K(t, x, y) \) be the kernel associated to \( e^{-itH} \). The following two facts are needed. First, when \( t \in \mathbb{R}^+ \setminus \frac{\pi}{2} \mathbb{N} \), \( K(t, x, y) \) is given by (5.7). (Notice that the structure of the above kernel breaks down and becomes singular at resonant times \( t = \frac{\pi}{2} \cdot k \), \( k = 0, 1, 2, \ldots \)) Second, one has (see e.g. in [29])

\[
K \left( \frac{k\pi}{2}, x, y \right) = e^{-\frac{ik\pi n}{2} \delta} \left( x - (-1)^k y \right), \quad x, y \in \mathbb{R}^n.
\]

(5.12)

where \( \delta \) is the Dirac function.

We will use (5.7) to show the conclusion (i). Arbitrarily fix a solution \( u \) to (1.10) and two balls \( B(x_1, r_1) \) and \( B(x_2, r_2) \). By (5.8) and (5.9), we can assume, without loss of generality, that \( S = 0 \) and \( 0 < T \leq \pi \). In this case, we have \( 0 < T < \pi \) and \( T \neq \frac{\pi}{2} \), which implies that \( \sin 2T \neq 0 \). The key observation is as:

\[
u(T, x) = \frac{e^{-i\pi n/4}}{(2\pi \sin 2T)^{n/2}} \int_{\mathbb{R}^n} \exp \left( \frac{i}{2} (|x|^2 + |y|^2) \cot 2T - \frac{i}{\sin 2T} x \cdot y \right) u(0, y) \, dy
\]

\[
= \frac{e^{-i\pi n/4}}{(2\pi \sin 2T)^{n/2}} e^{\frac{|y|^2}{2} \cot 2T} \int_{\mathbb{R}^n} e^{-i\frac{x}{\sin 2T} \cdot y} u(0, y) e^{i\frac{|y|^2}{2} \cot 2T} \, dy
\]
where $\mathcal{F}$ stands for the Fourier transform. Recall that the uncertainty principle built up in [27] says: for any $S$, $\Sigma \subset \mathbb{R}^n$ with $|S| < \infty$ and $|\Sigma| < \infty$, there is a positive constant

$$C(n, S, \Sigma) := C e^{C \min \{|S|, |\Sigma|\}^{1/n} \omega(S), |\Sigma|^{1/n} \omega(S)},$$

(5.14)

with $C = C(n)$, so that for any $g \in L^2(\mathbb{R}^n),

$$\int_{\mathbb{R}^n} |g|^2 \, dx \leq C(n, S, \Sigma) \left( \int_{S^c} |g|^2 \, dx + \int_{\Sigma^c} |\hat{g}|^2 \, dx \right).$$

(5.15)

(Here $\omega(S)$ denotes the mean width of $S$, we refer the readers to [27] for its detailed definition. In particular, when $S$ is a ball in $\mathbb{R}^n$, $\omega(S)$ is the diameter of the ball.)

By (5.15), where $g(x) = e^{i|x|^2/2} \cot 2T u(0, x)$ and $(S^c, \Sigma^c)$ is replaced by $(B^c(x_1, r_1), B^c(x_2, r_2))$ (here we have used the notation $kE := \{kx, x \in E\}$) and then by (5.13), we find

$$\int_{\mathbb{R}^n} |u(0, x)|^2 \, dx \leq C \left( \int_{B^c(x_1, r_1)} |u(0, x)|^2 \, dx + \int_{B^c(x_2, r_2)} |u(T, x)|^2 \, dx \right),$$

(5.16)

where $C := C \left( n, B(x_1, r_1), B(x_2, r_2) \right)$ is given by (5.14). In view of (5.14), we find that

$$C \left( n, B(x_1, r_1), B(x_2, r_2) \right) \leq C e^{C r_1 r_2},$$

(5.17)

which, along with (5.16), leads to (5.10).

Next, we will use (5.12) to prove the conclusion (iii). Without loss of generality, we can assume that $(S, T) = (0, \pi/2)$ or $(S, T) = (0, \pi)$. In the case when $(S, T) = (0, \pi)$, we see from (5.12) that $K(\pi, x, y) = e^{-i\pi n} \delta(x - y)$, which implies that for any solution $u$ to (1.10),

$$|u(0, x)| = |u(\pi, x)|, \quad x \in \mathbb{R}^n.$$ 

(5.18)

To simplify matters, we set $x_1 = x_2 = 0$. Let $f$ be a nonzero function in $C_0^\infty(B(0, r))$, with $r = \min \{r_1, r_2\}$. Let $v$ be the solution to (1.10) with the initial condition: $v(\cdot, 0) = f(\cdot)$. Then, for this solution $v$, the left hand side of the inequality (5.11) is strictly positive, but the right hand side of (5.11) is zero since both integrals vanish. (Here we used (5.18).) This shows that for this solution $v$, (5.11) is not true in the case that $(S, T) = (0, \pi)$.

We now consider the case that $(S, T) = (0, \pi/2)$. To simplify matters, we again set $x_1 = x_2 = 0$. Let $u_0(\cdot) \in C_0^\infty((-r/\sqrt{n}, r/\sqrt{n}))$ (with $r = \min \{r_1, r_2\}$) be a nonzero real-valued even function (i.e., $u_0(x) = u_0(-x)$ for all $x \in \mathbb{R}$). Then define a function by

$$g(x) := \prod_{i=1}^n u_0(x_i), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$ 

(5.19)
Let \( w \) be the solution to (1.10) with the initial condition: \( w(0, \cdot) = g(\cdot) \), where \( g \) is given by (5.19). It is clear that \( \text{supp} \ w(0, x) \subset B(0, r) \). We claim
\[
|w(0, x)| = |w(\pi/2, x)|, \quad x \in \mathbb{R}^n.
\]
Indeed, since \( \{\Phi_\alpha : \alpha \in \mathbb{N}^n\} \) forms a complete orthonormal basis in \( L^2(\mathbb{R}^n) \), we have
\[
w(0, x) = \sum_{\alpha \in \mathbb{N}^n} \langle w(0, \cdot), \Phi_\alpha(\cdot) \rangle_{L^2(\mathbb{R}^n)} \Phi_\alpha.
\]
Meanwhile, by (5.2), (5.3) and (5.4), we see that when \( \alpha_i \) is odd/even, \( \varphi_{\alpha_i} \) is odd/even. Thus, when \( |\alpha| \) is odd, there exists some \( j, 1 \leq j \leq n \), such that \( \alpha_j \) is odd, consequently, \( \varphi_{\alpha_j} \) is an odd function, which implies that \( \langle u_0(x_j), \varphi_{\alpha_j}(x_j) \rangle_{L^2(\mathbb{R})} = 0 \). This, along with (5.19), yields
\[
\langle w(0, \cdot), \Phi_\alpha(\cdot) \rangle_{L^2(\mathbb{R}^n)} = \prod_{i=1}^{\frac{n}{2}} \langle u_0(\cdot), \varphi_{\alpha_i}(\cdot) \rangle_{L^2(\mathbb{R})} = 0, \quad \text{if } |\alpha| \text{ is odd}.
\]
From (5.21) and (5.22), we see
\[
w(0, \cdot) = \sum_{|\alpha| \text{ is even}} \langle w(0, \cdot), \Phi_\alpha(\cdot) \rangle_{L^2(\mathbb{R}^n)} \Phi_\alpha(x), \quad x \in \mathbb{R}^n.
\]
On the other hand, we obtain from (5.5) that
\[
w(\pi/2, x) = \sum_{\alpha \in \mathbb{N}^n} e^{-i\pi(\alpha_1 + \cdots + \alpha_n)} \langle w(0, \cdot), \Phi_\alpha(\cdot) \rangle_{L^2(\mathbb{R}^n)} \Phi_\alpha(x)
\]
\[
= e^{-\frac{i\pi}{2}} \sum_{\alpha \in \mathbb{N}^n} (-1)^{|\alpha|} \langle w(0, \cdot), \Phi_\alpha(\cdot) \rangle_{L^2(\mathbb{R}^n)} \Phi_\alpha(x)
\]
\[
= e^{-\frac{i\pi}{2}} \sum_{|\alpha| \text{ is even}} \langle w(0, \cdot), \Phi_\alpha(\cdot) \rangle_{L^2(\mathbb{R}^n)} \Phi_\alpha(x), \quad x \in \mathbb{R}^n,
\]
which, together with (5.23), leads to (5.20).
Hence, for the above solution \( w \), the left hand side of the inequality (5.11) is strictly positive, but the right hand side of (5.11) is zero. (Here we used (5.20).) This ends the proof of Theorem 5.1. \( \square \)

Based on Theorem 5.1, we now can show show Theorem 1.3.

**Proof of Theorem 1.3.** Arbitrarily fix a ball \( B(x_0, r) \) and a solution \( u \) to (1.10). We first consider the case when \( 0 < T \leq \pi/4 \). According to (i) of Theorem 5.1 (with \( r_1 = r_2 = r \) and \( x_1 = x_2 = x_0 \)), there exists \( C = C(n) \) so that when \( 0 \leq s < t < T \),
\[
\int_{\mathbb{R}^n} |u(0, x)|^2 \, dx \leq Ce^{\frac{C^2}{\sin^2(2(\pi - r))}} \left( \int_{B^c(x_0, r)} |u(s, x)|^2 \, dx + \int_{B^c(x_0, r)} |u(t, x)|^2 \, dx \right).
\]
(5.24)
Since \( 0 \leq s < t < T \) and \( T < \pi/4 \), we have \( 0 < 2(t - s) < \pi/2 \). Thus \( \sin(2(t - s)) \geq 4(t - s)/\pi \). Moreover, if \( (s, t) \in [0, T/3] \times [2T/3, T] \), we have \( 2(t - s) \geq 2T/3 \). These show that
\[
\sin(2(t - s)) \geq 4T/(3\pi), \quad (s, t) \in [0, T/3] \times [2T/3, T].
\]
(5.25)
Integrating (5.24) with \( s \) over \( s \in [0, T/3] \) and \( t \) over \( t \in [2T/3, T] \), using (5.25), we obtain that
\[
\left( \frac{T}{3} \right)^2 \int_{\mathbb{R}^n} |u(0, x)|^2 \, dx 
\leq C \int_0^{T/3} \int_0^{2T/3} e^{\frac{3\pi Cr^2}{4T}} \left( \int_{B^c(x_0, r)} |u(s, x)|^2 \, dx + \int_{B^c(x_0, r)} |u(t, x)|^2 \, dx \right) \, dt \, ds
\leq CT \frac{e^{3\pi Cr^2}}{3} \left( \int_0^{T/3} \int_0^{2T/3} |u(s, x)|^2 \, dx \, ds + \int_0^{T/3} \int_{B^c(x_0, r)} |u(t, x)|^2 \, dx \, dt \right)
\leq CT \frac{e^{3\pi Cr^2}}{3} \int_0^{T/3} \int_{B^c(x_0, r)} |u(t, x)|^2 \, dx \, dt.
\]

From the above, we obtain
\[
\int_{\mathbb{R}^n} |u(0, x)|^2 \, dx \leq \frac{3C}{T} e^{3\pi Cr^2/4T} \int_0^{T/3} \int_{B^c(x_0, r)} |u(t, x)|^2 \, dx \, dt,
\]
which leads to (1.11) for the case that \( 0 < T \leq \pi/4 \).

We next consider the case when \( T > \pi/4 \). By (5.26) with \( T = \pi/4 \), we find
\[
\int_{\mathbb{R}^n} |u(0, x)|^2 \, dx \leq \frac{12C}{\pi} e^{3\pi Cr^2/4T} \int_0^{T/3} \int_{B^c(x_0, r)} |u(t, x)|^2 \, dx \, dt,
\]
from which, it follows that when \( T > \pi/4 \),
\[
\int_{\mathbb{R}^n} |u(0, x)|^2 \, dx \leq \frac{12C}{\pi} e^{3\pi Cr^2/4T} \int_0^{T/3} \int_{B^c(x_0, r)} |u(t, x)|^2 \, dx \, dt.
\]
The above leads to (1.11) for the case where \( T > \pi/4 \). This ends the proof of Theorem 1.3. \( \Box \)

5.2. Proof of Theorem 1.4. Since the equation (1.10) is rotation invariant, we can assume, without loss of generality, that \( a = (1, \ldots, 0) \in \mathbb{R}^n \). In what follows, \( E := B^c(0, r) \cap \{ x \in \mathbb{R}^n : x_1 \geq 0 \} \).

**Step 1.** We prove that \((i) \Rightarrow (ii)\).

Given \( k \in \mathbb{N}^+ \), write \( \vec{k} := (k, \ldots, k) \in \mathbb{R}^n \); let
\[
u_{0,k}(x) := \pi^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} - ik \cdot x, \quad x \in \mathbb{R}^n,
\]
and write \( u_k(t, x) \) for the solution of (1.10), with the initial condition: \( u_k(0, x) = u_{0,k}(x) \). We claim that
\[
\|u_{0,k}\|_{L^2(\mathbb{R}^n)} = 1 \quad \text{for all } \ k \in \mathbb{N}^+ \tag{5.28}
\]
and that
\[
\lim_{k \to \infty} \int_0^T \int_E |u_k(t, x)|^2 \, dx \, dt = 0 \quad \text{for each } \ T \in \left(0, \frac{\pi}{2}\right]. \tag{5.29}
\]
When this is done, “(i) ⇒ (ii)” follows from (5.28) and (5.29) at once.

The equality (5.28) follows from (5.27) and the direct calculation:
\[
\int_{\mathbb{R}^n} |u_{0,k}|^2 \, dx = \pi^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-|x|^2} \, dx = 1 \quad \text{for each} \quad k \in \mathbb{N}^+.
\]

We next show (5.29). By (5.6) and (5.27), we have
\[
\begin{align*}
 u_k(t, x) &= \frac{e^{-i\pi/4}}{(2\pi \sin 2t)^{n/2}} \int_{\mathbb{R}^n} \exp \left( \frac{i}{2} \left( |x|^2 + |y|^2 \right) \cot 2t - \frac{i}{\sin 2t} x \cdot y \right) u_{0,k}(y) \, dy \\
&= \frac{e^{-i\pi/4}}{(2\pi \sin 2t)^{n/2}} e^{i\frac{\pi/2}{2} \cdot \cot 2t} \int_{\mathbb{R}^n} \exp \left( -\frac{|x|^2}{2} \cot 2t \right) u_{0,k}(y) e^{i\frac{\pi/2}{2} \cdot \cot 2t} \, dy, \quad t \in (0, \pi/2), x \in \mathbb{R}^n.
\end{align*}
\]

By changing variables in the above, we find
\[
\begin{align*}
 u_k(t, x) &= \frac{e^{-i\pi/4}}{(2\pi \sin 2t \cdot A_t)^{n/2}} \exp \left\{ -\frac{|x|^2}{2} \cdot \cot 2t - \frac{|x + \vec{k}|^2}{2A_t} \right\}, \quad t \in (0, \pi/2), x \in \mathbb{R}^n,
\end{align*}
\]

where \( A_t := 1 - i \cot 2t \). This implies that when \( 0 < t < \frac{\pi}{2} \) and \( x \in \mathbb{R}^n \),
\[
|u_k(t, x)| \leq C \exp \left\{ -\frac{|x + \tilde{k}|^2}{2(1 + \cot^2 2t)} \right\} = C \exp \left\{ -\frac{|x + \tilde{k} \sin 2t|^2}{2} \right\}. \quad (5.30)
\]

Now we arbitrarily fix \( 0 < \epsilon < \frac{\pi}{4} \). Several facts are given in order. Fact One:
\[
\begin{align*}
\int_0^{\frac{\pi}{2}} \int_E |u_k(t, x)|^2 \, dx \, dt &\leq \int_0^{\frac{\epsilon}{2}} \int_{\mathbb{R}^n} |u_k(t, x)|^2 \, dx \, dt + \int_{\frac{\epsilon}{2}}^{\frac{\pi}{2} - \frac{\epsilon}{2}} \int_E |u_k(t, x)|^2 \, dx \, dt \\
&\quad + \int_{\frac{\pi}{2} - \frac{\epsilon}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}^n} |u_k(t, x)|^2 \, dx \, dt.
\end{align*} \quad (5.31)
\]

Fact Two: It follows directly from (5.28) that
\[
\int_0^{\frac{\epsilon}{2}} \int_{\mathbb{R}^n} |u_k(t, x)|^2 \, dx \, dt + \int_{\frac{\pi}{2} - \frac{\epsilon}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}^n} |u_k(t, x)|^2 \, dx \, dt = \frac{2\epsilon}{3}. \quad (5.32)
\]

Fact Three: Since \( \sin 2t \geq \sin (2\epsilon/3) > 0 \), when \( \frac{\epsilon}{3} \leq t \leq \frac{\pi}{2} - \frac{\epsilon}{3} \), we deduce from (5.30) and the definition of \( E \) that
\[
\begin{align*}
\int_E |u_k(t, x)|^2 \, dx &\leq C \int_E e^{-|x + \tilde{k} \sin 2t|^2} \, dx \\
&\leq C \int_{k \sin (2\epsilon/3)}^{\infty} e^{-x_1^2} \, dx_1 \prod_{j=2}^{n} \int_{\mathbb{R}} e^{-x_j^2} \, dx_j \longrightarrow 0, \quad \text{as} \quad k \to +\infty,
\end{align*}
\]

from which, we can find \( K > 0 \) so that
\[
\int_{\frac{\pi}{2} - \frac{\epsilon}{3}}^{\frac{\pi}{2}} \int_E |u_k(t, x)|^2 \, dx \, dt \leq \frac{\epsilon}{3} \quad \text{for all} \quad k > K. \quad (5.33)
\]
Because $\epsilon > 0$ can be arbitrarily small, (5.29) follows from (5.31), (5.32) and (5.33) immediately.

Hence, we have proved “(i) \Rightarrow (ii)”. 

**Step 2. We prove that “(ii) \Rightarrow (i)”**.

Arbitrarily fix $T > \frac{\pi}{2}$. By contradiction, we suppose that (ii) is not true for the aforementioned $T$. Then there exists a sequence of functions $\{v_0,k\}_{k=1}^\infty$ in $L^2(\mathbb{R}^n)$ so that for each $k \in \mathbb{N}^+$,

\[\|v_0,k\|_2 = 1 \quad \text{and} \quad \int_0^T \int_E |e^{-itH}v_0,k|^2 \, dx \, dt \leq \frac{\|v_0,k\|^2_{L^2}}{k} = \frac{1}{k}. \tag{5.34}\]

Several observations are given in order. First, from (5.6), we see that when $t \in \mathbb{R}^+ \setminus \frac{\pi}{2} \mathbb{N}$ and $x \in \mathbb{R}^n$,

\[
\left( e^{-i(t+\pi/2)H}v_0,k \right)(x) = \frac{e^{-i\pi n/2}}{(2\pi \sin 2(t + \pi/2))^{n/2}} \int_{\mathbb{R}^n} \exp \left( \frac{i}{2}(|x|^2 + |y|^2) \cot 2(t + \pi/2) - \frac{i}{\sin 2(t + \pi/2)} x \cdot y \right) v_0,k(y) \, dy
\]

\[
= \frac{e^{-i\pi n/4}}{(2\pi \sin 2t)^{n/2}} \int_{\mathbb{R}^n} \exp \left( \frac{i}{2}(|x|^2 + |y|^2) \cot 2t - \frac{i}{\sin 2t} (-x) \cdot y \right) v_0,k(y) \, dy
\]

\[
= e^{-i\pi n/2} \left( e^{-iH}v_0,k \right)(-x). \tag{5.35}\]

Second, since $T > \frac{\pi}{2}$, we have

\[
\int_0^T \int_E |e^{-itH}v_0,k|^2 \, dx \, dt = \int_0^T \int_{\mathbb{R}^n} |e^{-itH}v_0,k|^2 \, dx \, dt + \int_{\frac{\pi}{2}}^T \int_E |e^{-itH}v_0,k|^2 \, dx \, dt. \tag{5.36}\]

Third, by the change of variable $t \to t + \frac{\pi}{2}$ and the fact (5.35), we find

\[
\int_{\frac{\pi}{2}}^T \int_E |e^{-itH}v_0,k(x)|^2 \, dx \, dt = \int_0^{T-\frac{\pi}{2}} \int_E |e^{-i(t+\pi/2)H}v_0,k(x)|^2 \, dx \, dt
\]

\[
= \int_0^{T-\frac{\pi}{2}} \int_{-E} |e^{-itH}v_0,k(x)|^2 \, dx \, dt, \tag{5.37}\]

where $-E := B^c(0, r) \cap \{x \in \mathbb{R}^n : x_1 \leq 0\}$. Last, it is clear that

\[E \cup (-E) = B^c(0, r). \tag{5.38}\]

Set $T_0 = \min\{T - \frac{\pi}{2}, \frac{\pi}{2}\}$. Then by the second inequality in (5.34) and by (5.36), (5.37) and (5.38), we have

\[
\int_0^{T_0} \int_{B^c(0, r)} |e^{-itH}v_0,k|^2 \, dx \, dt \leq \frac{1}{k}. \tag{5.39}\]

Now by Theorem 1.3 (with $x_0 = 0$), the first equality in (5.34) and (5.39), we find

\[1 \leq C(T_0) \int_0^{T_0} \int_{B^c(0, r)} |e^{-itH}v_0,k|^2 \, dx \, dt \leq \frac{C(T_0)}{k} \to 0, \quad \text{as} \quad k \to \infty, \tag{5.40}\]

which leads to a contradiction. So (i) is true.

Thus we end the proof of Theorem 1.4. \qed
5.3. Proof of Theorem 1.5. In its proof, we borrow some ideas from [14] and [50]. Suppose that \( E \subset \mathbb{R}^n \) is an observable set at time \( T > 0 \) for the equation (1.10), i.e., there is \( C = C(T, E) > 0 \) so that when \( u \) solves (1.10),

\[
\int_{\mathbb{R}^n} |u(0, x)|^2 \, dx \leq C \int_0^T \int_E |u(t, x)|^2 \, dx \, dt.
\]

(5.41)

Notice that the semigroup \( \{e^{-tH}\}_{t \geq 0} \), generated by the Hermite operator \(-H\), can be extended over \( \mathbb{C}^+ := \{ z \in \mathbb{C} : \Re z > 0 \} \). Furthermore, the kernel associated with \( e^{-zH} \) (over \( \mathbb{C}^+ \)) can be written in the form:

\[
K_z(x, y) = \sum_{\alpha \in \mathbb{N}^n} e^{-(n+2|\alpha|)z} \Phi_\alpha(x) \Phi_\alpha(y), \quad z \in \mathbb{C}^+, \ x, y \in \mathbb{R}^n,
\]

(5.42)

where \( \Phi_\alpha(x) \) is given by (5.4). Thanks to the Mehler’s formula (see e.g. in [47, p. 85]),

the series in (5.42) can be summed explicitly. More precisely, we have that when \( z \in \mathbb{C}^+ \) and \( x, y \in \mathbb{R}^n \),

\[
K_z(x, y) = (2\pi \sinh 2z)^{-\frac{n}{2}} \exp \left( -\frac{\cosh 2z}{2}(|x|^2 + |y|^2) + \frac{2}{\sinh 2z} x \cdot y \right).
\]

(5.43)

where \( \cosh \) and \( \sinh \) are hyperbolic trigonometric functions. From (5.43), we see that for each fixed \( (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \), the kernel is an analytic function of \( z \) over \( \mathbb{C}^+ \).

We claim

\[
|K_{s+it}(x, y)| \leq C s^{-\frac{n}{2}} \exp \left( -\frac{s}{4(s^2 + t^2)} |x - y|^2 \right) \quad \text{for all } x, y \in \mathbb{R}^n, s > 0, t \in [0, T].
\]

(5.44)

To prove (5.44), we first recall the following result on analytic function: (It can be found in [13, Lemma 9].) Let \( F \) be an analytic function on \( \mathbb{C}^+ \). Suppose that there is \( a_1 > 0, a_2 > 0, \beta \geq 0 \) and \( \alpha \in (0, 1] \) so that

\[
|F(re^{i\theta})| \leq a_1 (r \cos \theta)^{-\beta}, \quad |F(r)| \leq a_1 r^{-\beta} \exp (-a_2 r^{-\alpha}) \quad \text{for all } r > 0 \text{ and } |\theta| < \pi/2.
\]

Then

\[
|F(re^{i\theta})| \leq a_1 2^\beta (r \cos \theta)^{-\beta} \exp \left( -\frac{a_2}{2} r^{-\alpha} \cos \theta \right) \quad \text{for all } r > 0 \text{ and } |\theta| < \pi/2.
\]

(5.45)

Now we are in a position to prove (5.44). On one hand, when \( z = s > 0 \), one can use the inequalities: \( e^{2s} + e^{-2s} \geq 2 \) and \( \sinh 2s \geq 2s \), to find some \( C > 0 \) so that

\[
K_s(x, y) \leq C s^{-\frac{n}{2}} \exp \left( -\frac{|x - y|^2}{2s} \right) \quad \text{for all } x, y \in \mathbb{R}^n \text{ and } s > 0.
\]

(5.46)

(One can also use the fact: \( -\Delta + |x|^2 \geq -\Delta \), to get (5.46), see e.g. in [44].) On the other hand, by (5.43), it follows that there is some \( C > 0 \) so that

\[
|K_{s+it}(x, y)| \leq C s^{-\frac{n}{2}} \quad \text{for all } x, y \in \mathbb{R}^n, s > 0 \text{ and } t \geq 0.
\]

(5.47)
In view of (5.46) and (5.47), the desired estimate (5.44) follows by applying (5.45) with $a_2 = |x - y|^2/2, \alpha = 1, \beta = n/2$ and $\cos \theta = s/(s^2 + r^2)$.

Next, we arbitrarily fix $y_0 \in \mathbb{R}^n$. Let

$$u_0(x) := K_1(x, y_0), \quad x \in \mathbb{R}^n.$$ (It is clear that $u_0 \in L^2(\mathbb{R}^n)$.) By the group property, the solution $v(t, x) := e^{-itH}u_0(x)$ ($(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$) is as:

$$v(t, x) = \int K_{it}(x, y)K_1(y, y_0) \, dy = K_{1+it}(x, y_0), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n. \quad (5.48)$$

Since the Hermite function $\Phi_0(x) = \pi^{-n/4}e^{-|x|^2/2}$ ($x \in \mathbb{R}^n$), given by (5.4), is the normalized eigenfunction of $H = -\Delta + |x|^2$ corresponding to the smallest eigenvalue $\lambda_0 = n$, it follows from the eigenfunction expansion of $u_0(x)$ that

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \, dx = \int |\sum_{\alpha \in \mathbb{N}^n} e^{-(n+2)|\alpha|} \Phi_\alpha(x) \Phi_\alpha(y_0)|^2 \, dx \geq e^{-2n}|\Phi_0(y_0)|^2 = Ce^{-|y_0|^2}. \quad (5.49)$$

Meanwhile, by (5.48) and (5.44), we see that for any $L > 0$,

$$|v(t, x)| \leq Ce^{\frac{\gamma|x-y_0|^2}{8}} e^{\frac{\gamma L^2|\rho(y_0)|^2}{8}}, \quad \text{when } |x - y_0| \geq L\rho(y_0) \text{ and } t \in [0, T]. \quad (5.50)$$

where $\gamma := \frac{1}{1+T^2}$ and $\rho(y_0)$ is given by (1.12).

Finally, by (5.49), (5.41) (where $u$ is replaced by the above $v$) and (5.50), we see that for any $L > 0$,

$$e^{-|y_0|^2} \leq C \int_0^T \int_{E \bigcap B(y_0, L\rho(y_0))} |v(t, x)|^2 \, dx \, dt + Ce^{-\frac{\gamma L^2|\rho(y_0)|^2}{4}} \int_0^T \int_{B(y_0, L\rho(y_0)))} e^{-\frac{\gamma|x-y_0|^2}{4}} \, dx \, dt \leq C \int_0^T \int_{E \bigcap B(y_0, L\rho(y_0))} |v(t, x)|^2 \, dx \, dt + CT e^{-\frac{\gamma L^2|\rho(y_0)|^2}{4}}. \quad (5.51)$$

In the above, by choosing $L > 0$ so that $CT e^{-\frac{\gamma L^2|\rho(y_0)|^2}{4}} < \frac{e^{-|y_0|^2}}{2}$, we obtain, with the help of (5.44), that

$$\frac{e^{-|y_0|^2}}{2} \leq C \int_0^T \int_{E \bigcap B(y_0, L\rho(y_0))} e^{-\frac{\gamma|x-y_0|^2}{2}} \, dx \, dt \leq CT |E \bigcap B(y_0, L\rho(y_0))|, \quad (5.52)$$

which leads to (1.12) with $c = \frac{1}{2CT}$. This ends the proof of Theorem 1.5. \(\square\)
Remark 5.2. In the 1-dim case, the necessary condition (1.12) is strictly weaker than the weakly thick property (1.9). This can be seen by three facts as follows: First, if \( E \subset \mathbb{R} \) satisfies (1.9), then \( E \) must satisfy (1.12). Second, the set \( E \), defined by (4.39), holds (1.12). Third, the set \( E \), defined by (4.39), does not satisfy (1.9). The first fact follows from Theorem 1.2 and Theorem 1.5. The second fact can be proved in the following way: Since

\[
[y - 3\rho(y), \, y + 3\rho(y)] \supseteq [0, 2\rho(y)] \quad \text{for all } y \in \mathbb{R},
\]

here \( \rho(y) \) is given by (1.12). It follows from (4.39) that for some \( c > 0 \),

\[
\left| E \cap [y - 3\rho(y), \, y + 3\rho(y)] \right| \geq \rho(y)^{-\varepsilon} \geq ce^{-|y|^2} \quad \text{for all } y \in \mathbb{R},
\]

which shows that \( E \) satisfies (1.12). The third fact follows from (4.40) in Remark 4.5.

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Appendix A. Proof of Proposition 3.3

First, from the proof of [4, Theorem 3.1, p.57], we see that \( H \) has a compact resolvent. Then according to [40, Theorem 1.3, p.195], we have the fact: \( E \) is an observable set at some time for (3.7) if and only if there exists \( \varepsilon > 0 \) and \( C > 0 \) such that

\[
\int_E |\varphi(x)|^2 \, dx \geq C \int_{\mathbb{R}} |\varphi(x)|^2 \, dx \quad \text{for all } \varphi \in \bigcup_{\lambda \in \mathbb{R}} \text{span} \left\{ \varphi_m : m \in J_\varepsilon(\lambda) \right\}, \quad (A.1)
\]

where

\[
J_\varepsilon(\lambda) := \{ m \in \mathbb{N}^+ : |\lambda_m - \lambda| < \varepsilon \}. \quad (A.2)
\]

From this, we in particular have what follows:

The conclusion (i) of Proposition 3.3 \( \iff \quad (A.1) \quad \text{with } \varepsilon = \varepsilon_0/2. \quad (A.3) \)

(Here, \( \varepsilon_0 \) is given by (3.8)).

Next, by (3.8), (A.2) (where \( \varepsilon = \varepsilon_0/2 \)) and (ii) of Proposition 3.2, one can directly check that

\[
\bigcup_{\lambda \in \mathbb{R}} \text{span} \left\{ \varphi_m : m \in J_\varepsilon(\lambda) \right\} = \bigcup_{k \in \mathbb{N}^+} \{ a\varphi_k : a \in \mathbb{C} \}, \quad \text{with } \varepsilon = \varepsilon_0/2.
\]

This yields that

(A.1) with \( \varepsilon = \varepsilon_0/2 \iff \int_E |\varphi_k(x)|^2 \, dx \geq C \int_{\mathbb{R}} |\varphi_k(x)|^2 \, dx \quad \text{for all } k \in \mathbb{N}^+. \quad (A.4)
Now, since \( \int_{\mathbb{R}} |\varphi_k(x)|^2 \, dx = 1 \) for all \( k \in \mathbb{N}^+ \), it follows from (A.3) and (A.4) that \((i) \Leftrightarrow (ii)\).

Finally, if (3.10) holds, then we can apply [49, Corollary 6.9.6] to see directly that \( E \) is an observable set at any time for (3.7). This ends the proof of Proposition 3.2.

Appendix B. Proof of the upper bound in (3.24) in Lemma 3.5

By Condition (H), we have the notes (d_2) (see (3.2) and (3.3)) and (d_3) (see (3.5)). Let \( c \) and \( x_0 \) be given by Condition (H) and the note (d_2) respectively. According to [52, Lemma 3.1&3.2] and the fact (3.5), there is \( \hat{k} \in \mathbb{N}^+ \) so that when \( k \geq \hat{k} \), (3.21)-(3.23) hold and

\[
\lambda_k \geq \max \left\{ 1, 2D(2(x_0 + 1))^{2c}, \left( \frac{6\sqrt{2\pi}}{2} \right)^{2c/(c+1)} \left( 2D \right)^{1/(c+1)}, \left( \frac{4C}{\sqrt{3}} \right)^2 \right\},
\]

where \( C \) and \( D \) are given by (3.22) and (3.3) respectively. Arbitrarily fix \( k \geq \hat{k} \). We divide the rest of the proof into several steps.

Step 1. Define the following interval:

\[
I_k := [x_k/2, x_k], \quad \text{with} \quad x_k := \alpha \lambda_k^{1/2}, \quad \alpha := \left( \frac{1}{2D} \right)^{1/2}.
\]

We claim that

\[
I_k \subset \Omega_k \bigcap [x_0 + 1, \infty).
\]

We first show that \( I_k \subset [x_0 + 1, \infty) \). Indeed, by (B.1), we have \( \lambda_k \geq 2D(2(x_0 + 1))^{2c} \), which, along with (B.2), yields \( x_k \geq 2(x_0 + 1) \). This, together with (B.2), leads to \( I_k \subset [x_0 + 1, \infty) \).

We next show that \( I_k \subset \Omega_k \). In fact, since \( I_k \subset [x_0, \infty) \), it follows from (3.2) that

\[
V'(x) > 0 \quad \text{for all} \quad x \in I_k.
\]

Meanwhile, by the definitions of \( x_k \) and \( \alpha \) (see (B.2)), and by (3.3), we find

\[
V(x_k) = V(\alpha \lambda_k^{1/2}) \leq \lambda_k/2.
\]

Combining (B.4) and (B.5) gives that

\[
V(x) \leq \lambda_k/2 \quad \text{for all} \quad x \in I_k.
\]

This, along with (3.15), leads to \( I_k \subset \Omega_k \). Hence, (B.3) has been proved.

Step 2. Define the following set:

\[
J := \{ j \in \mathbb{N} : S(x_k/2) + \pi \leq j \pi + \theta_0 \leq S(x_k) - \pi \},
\]

where \( S(\cdot) \) is given by (3.17) and \( \theta_0 \) is defined as:

\[
\theta_0 := \arctan w'(0)/w(0), \quad \text{when} \quad w(0) \neq 0; \quad \theta_0 := \pi/2, \quad \text{when} \quad w(0) = 0,
\]

where
where \( w(\cdot) \) is given by (3.17). We claim

\[
\sharp J \geq \frac{\alpha}{6\pi \sqrt{2}} \lambda_k^{\frac{1}{2} + \frac{1}{c}} \geq 1. \tag{B.8}
\]

Here and below, \( \sharp J \) denotes the cardinality of \( J \).

Two facts deserve to be mentioned: First, (B.8) holds for all \( \theta_0 \in [-\pi/2, \pi/2] \) and that we choose it satisfying (B.7) for later use. Second, when (B.8) is proved, we have that \( J \neq \emptyset \).

To prove (B.8), we see from (B.3), (3.3) and (3.15) that

\[
0 \leq V(x) \leq \lambda_k / 2, \quad x \in I_k. \tag{B.9}
\]

Then, by the definition of \( S(x) \) (see (3.17)) and by (B.9), we have

\[
S(x_k) - S(x_k / 2) = \int_{x_k / 2}^{x_k} \sqrt{\lambda_k - V(x)} \, dx \geq \frac{x_k \sqrt{\lambda_k}}{2\sqrt{2}}.
\]

Since \( \lambda_k \geq \left(\frac{6\sqrt{2\pi}}{2\pi} \left(2D\right)^{1/(c+1)} \right) \) (see (B.1)) and because \( \theta_0 \in (-\pi/2, \pi/2) \) (see (B.7)), the above, along with (B.6) and the definitions of \( x_k \) and \( \alpha \) (see (B.2)), shows

\[
\sharp J \geq \frac{S(x_k) - S(x_k / 2) - 2\pi}{\pi} \geq \frac{x_k \sqrt{\lambda_k}}{2\pi \sqrt{2}} - 2 = \frac{\alpha \lambda_k^{1+\frac{1}{c}}}{2\pi \sqrt{2}} - 2 \geq \frac{\alpha}{6\pi \sqrt{2}} \lambda_k^{\frac{1}{2} + \frac{1}{c}} \geq 1,
\]

which leads to (B.8).

Step 3. Define, for each \( j \in J \), the following set:

\[
E_{k, j} := [x_{k, j} - \mu x_{k, j}^{-c}, x_{k, j}], \quad \text{with} \quad \mu := \frac{\pi}{6 \cdot 2^c \sqrt{D}}. \tag{B.10}
\]

where \( x_{k, j} \) is the unique solution to the equation:

\[
S(x) = j \pi + \theta_0, \quad x \in I_k. \tag{B.11}
\]

We claim

\[
E_{k, j} \subset I_k, \quad j \in J; \tag{B.12}
\]

\[
E_{k, j} \cap E_{k, j'} = \emptyset \quad \text{for all} \quad j, j' \in J \quad \text{with} \quad j \neq j'; \tag{B.13}
\]

\[
| \cos \left( S(x) - \theta_0 \right) | \geq \sqrt{3}/2 \quad \text{for all} \quad x \in E_{k, j} \quad \text{and} \quad j \in J. \tag{B.14}
\]

First of all, by (3.18) and (B.3), we infer that the function \( S(\cdot) \) is strictly increasing over \( I_k \). This fact and the definition (B.6) imply that the equation (B.11) has a unique solution \( x_{k, j} \), which satisfies that \( x_k / 2 \leq x_{k, j} \leq x_k \) for all \( j \in J \).

We claim that for each \( j \in J \),

\[
0 \leq V(x) \leq \lambda_k / 2, \quad \text{when} \quad x \in E_{k, j}. \tag{B.15}
\]

To this end, we arbitrarily fix \( j \in J \). Since \( x_k / 2 \leq x_{k, j} \); \( x_k / 2 \geq x_0 + 1 \) (see (B.3)); \( \mu(x_k / 2)^{-c} \leq 1 \) (see (B.1) and (B.10)), we have

\[
x_{k, j} - \mu x_{k, j}^{-c} \geq (x_k / 2) - \mu(x_k / 2)^{-c} \geq x_0. \tag{B.16}
\]
Since \(x_{k,j} \leq x_k\), it follows by (B.16) that
\[
E_{k,j} \subset [x_0, x_k].
\] (B.17)

Notice that \(V'(x) > 0\) and \(V(x) > 0\) over \([x_0, +\infty)\) (see (3.2) and (3.3)). From these, (B.17) and (B.5), we are led to (B.15).

We now show (B.14). Since \(x_{k,j}\) satisfies (B.11), we have that when \(x \in E_{k,j}\),
\[
|S(x) - \theta_0 - j\pi| = |S(x) - S(x_{k,j})| \leq |x - x_{k,j}| \cdot \sup_{x \in E_{k,j}} |V'(x)|
\]
\[
= |x - x_{k,j}| \cdot \sup_{x \in E_{k,j}} \sqrt{\lambda_k - V(x)}
\]
\[
\leq \mu x_{k,j}^{-c} \sqrt{\frac{\lambda_k}{2}} \leq \frac{\mu}{\sqrt{2}} 2^c \sqrt{2D} = \frac{\pi}{6}.
\] (B.18)

(On the last line of (B.18), we used (B.15) and the fact \(x_{k,j} \geq x_k/2 = 2^{-1}(2D)^{-1/(2c)} \lambda_k^{1/(2c)}\), as well as the definition of \(\mu\) in (B.10).) From (B.18), we are led to (B.14) at once.

We next show (B.12). Indeed, by the monotonicity of \(S^{-1}\) on \(I_k\) (see (3.18) and (B.3)) and by (B.6), we see
\[
x_k/2 \leq S^{-1}(j\pi + \theta_0 - \frac{\pi}{6}) \leq S^{-1}(j\pi + \theta_0 + \frac{\pi}{6}) \leq x_k \text{ for all } j \in J.
\] (B.19)

Meanwhile, by (B.18), we have
\[
E_{k,j} \subset S^{-1}([j\pi + \theta_0 - \frac{\pi}{6}, j\pi + \theta_0 + \frac{\pi}{6}]) \text{ for all } j \in J.
\] (B.20)

From (B.20) and (B.19), we obtain \(E_{k,j} \subset [x_k/2, x_k]\), i.e., (B.12) holds.

We finally show (B.13). Indeed, it follows from (B.20) that for all \(j, j' \in J\) with \(j \neq j'\),
\[
E_{k,j} \bigcap E_{k,j'} \subset S^{-1}(K_{k,j}) \bigcap S^{-1}(K_{k,j'}),
\] (B.21)

where
\[
K_{k,j} := [j\pi + \theta_0 - \frac{\pi}{6}, j\pi + \theta_0 + \frac{\pi}{6}],
\]
\[
K_{k,j'} := [j'\pi + \theta_0 - \frac{\pi}{6}, j'\pi + \theta_0 + \frac{\pi}{6}].
\]

Since \(S^{-1}\) is strictly monotonic on \(I_k\) (see (3.18) and (B.3)), the set on the right hand side of (B.21) is empty. This leads to (B.13).

**Step 4. Define the following subset:**
\[
G_k := \left\{ x \in I_k : |\cos(S(x) - \theta_0)| \geq \frac{\sqrt{3}}{2} \right\}. 
\] (B.22)

We claim
\[
|\varphi_k(x)| \geq \frac{\sqrt{3}}{4} \left( \frac{\lambda_k}{2} \right)^{-\frac{1}{4}} |C_{\lambda_k}|, \ x \in G_k,
\] (B.23)
and
\[ G_k \supset \bigcup_{j \in J} E_{k,j}. \]  

We start with proving (B.23). Because \( k \geq \hat{k}_0 \), we can use (3.23) to get
\[
\Re (C_{\lambda_k} e^{iS(x)}) = w(0) \cos S(x) + w'(0) \sin S(x) = |C_{\lambda_k}| \cos (S(x) - \theta_0), \ x \in \Omega_k. 
\]

From (3.21), (3.22), (B.25), (B.22) and (B.9), we see that
\[
|\varphi_k(x)| \geq (\lambda_k - V(x))^{-\frac{1}{4}} (|\Re (C_{\lambda_k} e^{iS(x)})| - C|C_{\lambda_k}|^{\frac{1}{2}}) \geq \left( \frac{\lambda_k}{2} \right)^{-\frac{1}{4}} |C_{\lambda_k}| \left( \frac{\sqrt{3}}{2} - C\lambda_k^{-\frac{1}{2}} \right) \geq \frac{\sqrt{3}}{4} \left( \frac{\lambda_k}{2} \right)^{-\frac{1}{4}} |C_{\lambda_k}|, \text{ when } x \in G_k. 
\]

(On the last step above, we used the fact that \( \lambda_k \geq \left( \frac{4C}{\sqrt{3}} \right)^2 \) which follows from (B.1). This leads to (B.23).

Now (B.24) follows from (B.12), (B.14) and (B.22) directly.

**Step 5. We complete the proof.**
We have
\[
1 \geq \int_{G_k} |\varphi_k|^2 \, dx 
\]
(by (B.23)) \[
\geq \int_{G_k} \frac{3}{16} \left( \frac{\lambda_k}{2} \right)^{-\frac{1}{2}} |C_{\lambda_k}|^2 \, dx 
\]
(by (B.24) and (B.13)) \[
\geq \sum_{j \in J} \int_{E_{k,j}} \frac{3}{16} \left( \frac{\lambda_k}{2} \right)^{-\frac{1}{2}} |C_{\lambda_k}|^2 \, dx 
\]
\[= \frac{3}{16} \left( \frac{\lambda_k}{2} \right)^{-\frac{1}{2}} |C_{\lambda_k}|^2 \cdot \# J \cdot |E_{k,j}| 
\]
(by (B.8) and (B.10)) \[
\geq \frac{3\alpha \mu}{96\pi} |C_{\lambda_k}|^2 \lambda_k^{-\frac{1}{2}} x_{k,j}^{-c} 
\]
\[\geq \frac{3\mu}{96\pi} \alpha^{-1-c} \lambda_k^{-\frac{1}{2} + \frac{1}{2}} |C_{\lambda_k}|^2. \]  

(On the last inequality in (B.26), we used the fact that \( x_{k,j} \leq x_k \) and the definition of \( x_k \) in (B.2).) By (B.26), we see
\[
|C_{\lambda_k}| \leq C(D, C, c) \lambda_k^{-\frac{1}{4} - \frac{1}{4}}. 
\]

Since the above holds for any \( k \geq \hat{k}_0 \), we obtain the upper bound in (3.24). This ends the proof of Lemma 3.5.
Appendix C. Proof of Lemma 3.6

Suppose that $E$ satisfies (3.6). Then there is $\gamma \in (0, 1]$ and $L > 0$ so that
\[
\frac{|E \cap [0, x]|}{x} \geq \gamma \quad \text{for all } x \geq L. \tag{C.1}
\]
Let $\delta := \gamma/2 \in (0, 1/2]$. It follows from (C.1) that
\[
\frac{|E \cap [\delta x, x]|}{x} \geq \gamma/2 \quad \text{for all } x \geq L. \tag{C.2}
\]
We will use (C.2) later. Now we let $k_1 = k'_0$ be the constant given by Lemma 3.5. Set
\[
I_k^\delta := [\delta x_k, x_k], \quad k \geq k_1, \tag{C.3}
\]
where $x_k$ is given by (B.2). Similar to (B.3), we can find $k_2 > k_1$ so that when $k \geq k_2$,
\[
I_k^\delta \subset (\Omega_k \cap [x_0 + 1, \infty) \cap [L, \infty)) , \tag{C.4}
\]
where $\Omega_k$ and $x_0$ are given by (3.15) and (3.2), respectively. The rest of the proof is organized by several steps.

Step 1. Given $k \geq k_2$ and $\varepsilon \in (0, 1)$, we define
\[
F_{k, \varepsilon} := \left\{ x \in I_k^\delta : \cos^2 (S(x) - \theta_0) \leq \varepsilon \right\}, \tag{C.5}
\]
where $S(\cdot)$ and $\theta_0$ are given by (3.17) and (B.7) respectively. We claim that there is a constant $C_1 > 0$, independent of $\varepsilon$ and $k$, so that
\[
|F_{k, \varepsilon}| \leq C_1 \sqrt{\frac{\lambda_k}{\varepsilon}} \quad \text{for all } k \geq k_2 \text{ and } \varepsilon \in (0, 1). \tag{C.6}
\]
For this purpose, we arbitrarily fix $k \geq k_2$ and $\varepsilon \in (0, 1)$. Define the following set:
\[
J' := \left\{ j \in \mathbb{N} : S(\delta x_k) - \pi \leq j \pi + \frac{\pi}{2} + \theta_0 \leq S(x_k) + \pi \right\}. \tag{C.7}
\]
Several observations are given in order. First, by a very similar way to that used in the proof of (B.8), we can obtain
\[
\#J' \leq \frac{S(x_k) - S(\delta x_k)}{\pi} + 2 \leq \frac{1}{2} \alpha \lambda_k^{\frac{1}{2} + \frac{1}{\pi}} + 2. \tag{C.8}
\]
Second, by (C.5) and (C.7), we have
\[
F_{k, \varepsilon} \subset \bigcup_{j \in J'} \left\{ x \in I_k^\delta : j \pi + \frac{\pi}{2} - \arcsin \sqrt{\varepsilon} \leq S(x) - \theta_0 \leq j \pi + \frac{\pi}{2} + \arcsin \sqrt{\varepsilon} \right\}. \tag{C.9}
\]
Third, there is $C_2 > 0$ (independent of $k$ and $\varepsilon$) so that when $j \in J'$,
\[
\left| \left\{ x \in I_k^\delta : j \pi + \frac{\pi}{2} - \arcsin \sqrt{\varepsilon} \leq S(x) - \theta_0 \leq j \pi + \frac{\pi}{2} + \arcsin \sqrt{\varepsilon} \right\} \right| = S^{-1} (j \pi + \frac{\pi}{2} + \theta_0 + \arcsin \sqrt{\varepsilon}) - S^{-1} (j \pi + \frac{\pi}{2} + \theta_0 - \arcsin \sqrt{\varepsilon})
\]
\[ \leq 2(\arcsin \sqrt{\varepsilon}) \cdot \sup_{x \in I_k^\delta} \frac{1}{\sqrt{\lambda_k - V(x)}} \]

\[ \leq C_2 \sqrt{\varepsilon} \lambda_k^{-\frac{1}{2}}. \]  \hspace{1cm} (C.10)

In (C.10), for the first equality, Line 2, we used the fact that \( S(\cdot) \) is continuous and strictly increasing on \( I_k \) (which follows from (C.4) and (3.2)); for the first inequality, Line 3, we used the rule of the derivative of inverse function and the fact that \( S'(x) = \sqrt{\lambda_k - V(x)} \) (which follows from (3.17)); for the last inequality, Line 4, we used the fact \( V(x) \leq \lambda_k/2 \) for \( x \in I_k^\delta \) (which follows from (C.4) and (3.15)) and \( \arcsin \sqrt{\varepsilon} \sim \sqrt{\varepsilon} \).

According to (C.8)-(C.10), there is \( C_1 > 0 \) (independent of \( k \) and \( \varepsilon \)) so that

\[ |F_{k, \varepsilon}| \leq \frac{\gamma}{2} \alpha \lambda_k^{\frac{1}{2} \frac{1}{\varepsilon}} \leq C_1 \sqrt{\varepsilon} \lambda_k^{\frac{1}{2} \frac{1}{C}} \] for all \( k \geq k_2 \) and \( \varepsilon \in (0, 1) \),

which leads to (C.6).

**Step 2.** We prove (3.25).

Several observations are given in order. First, by (C.2) (where \( x = x_k \)), (C.3) and (C.4), we find

\[ |E \cap I_k^\delta| \geq \gamma \frac{\alpha}{2} \lambda_k^{\frac{1}{2} \frac{1}{\varepsilon}} , \] \hspace{1cm} when \( k \geq k_2 \).  \hspace{1cm} (C.11)

Write

\[ \varepsilon_0 := \left( \frac{1}{1 + 4C_1/\gamma \alpha} \right)^2 (\in (0, 1)). \]

Then it follows from (C.6) that

\[ |F_{k, \varepsilon_0}| \leq C_1 \sqrt{\varepsilon_0} \lambda_k^{\frac{1}{2} \frac{1}{\varepsilon}} \leq \gamma \frac{\alpha}{4} \lambda_k^{\frac{1}{2} \frac{1}{\varepsilon}} \] for all \( k \geq k_2 \).  \hspace{1cm} (C.12)

Combining (C.11) and (C.12), we get

\[ \left| (E \cap I_k^\delta) \backslash F_{k, \varepsilon_0} \right| \geq \gamma \frac{\alpha}{4} \lambda_k^{\frac{1}{2} \frac{1}{\varepsilon}} , \] \hspace{1cm} when \( k \geq k_2 \).  \hspace{1cm} (C.13)

Second, by (C.5) (where \( \varepsilon = \varepsilon_0 \)), we have that when \( k \geq k_2 \),

\[ \cos^2(S(x) - \theta_0) \geq \varepsilon_0 \] for all \( x \in (E \cap I_k^\delta) \backslash F_{k, \varepsilon_0} \).  \hspace{1cm} (C.14)

Third, noting that when \( k \geq k_2 \), we have \( I_k^\delta \subset \Omega_k \) (see (C.4)), then using (3.21), (3.22) and (B.25), we obtain that when \( k \geq k_2 \),

\[ \int_E |\varphi_k(x)|^2 \, dx \geq \int_{E \cap I_k^\delta} |\varphi_k(x)|^2 \, dx \]

\[ \geq \frac{1}{2} \int_{E \cap I_k^\delta} (\lambda_k - V)^{-\frac{1}{2}} |C_{\lambda_k}|^2 \cos^2(S(x) - \theta_0) \, dx - 3C \int_{E \cap I_k^\delta} (\lambda_k - V)^{-\frac{1}{2}} \cdot |C_{\lambda_k}|^2 \cdot \lambda_k^{-1} \, dx, \]  \hspace{1cm} (C.15)

where \( C \) is given by (3.22).
Next, we will estimate two terms on the right hand side of (C.15), with the aid of Lemma 3.5 and the first two facts above-mentioned. First, since $V \geq 0$ over $I^\delta_k$ (see (3.3) and (4.4)), it follows from (C.14), (C.13) and the lower bound (3.24) that when $k \geq k_2$,

$$
\int_{E \cap I^\delta_k} (\lambda_k - V(x))^{-\frac{1}{2}} |C_{\lambda_k}|^2 \cos^2(S(x) - \theta_0) \, dx \\
\geq \varepsilon_0 \int_{(E \cap I^\delta_k) \setminus F_0} (\lambda_k - V(x))^{-\frac{1}{2}} |C_{\lambda_k}|^2 \, dx \\
\geq \varepsilon_0 \lambda_k^{-\frac{1}{2}} |C_{\lambda_k}|^2 \left| (E \cap I^\delta_k) \setminus F_{k, \varepsilon_0} \right| \\
\geq C_3 \varepsilon_0 \gamma \alpha
$$

for some $C_3 > 0$ (independent of $k$). Second, because $V \leq \lambda_k/2$ over $I^\delta_k$ (see (4.4) and (3.15)), it follows from (3.16) and the upper bound in (3.24) that when $k \geq k_2$,

$$
\int_{E \cap I^\delta_k} (\lambda_k - V(x))^{-\frac{1}{2}} |C_{\lambda_k}|^2 \lambda_k^{-1} \, dx \leq |I^\delta_k| (\lambda_k/2)^{-\frac{1}{2}} |C_{\lambda_k}|^2 \lambda_k^{-1} \\
\leq C_4 \lambda_k^{-\frac{1}{2}} \lambda_k^{-1} \lambda_k^{-\frac{1}{2}} \lambda_k^{-1} = C_4 \lambda_k^{-1}
$$

for some $C_4 > 0$ (independent of $k$).

Finally, inserting (C.16) and (C.17) into (C.15), we find that when $k \geq k_2$,

$$
\int_E |\varphi_k(x)|^2 \, dx \geq \frac{1}{2} C_3 \varepsilon_0 \gamma \alpha - 3 C_4 \lambda_k^{-1}.
$$

Since $\lambda_k^{-1} \to 0$ as $k \to \infty$, we can find $k''_0 \geq k_2$ so that when $k \geq k''_0$,

$$
\frac{1}{2} C_3 \varepsilon_0 \gamma \alpha - 3 C_4 \lambda_k^{-1} \geq \frac{1}{4} C_3 \varepsilon_0 \gamma \alpha,
$$

which, together with (C.18), leads to (3.25).

Hence, we end the proof of Lemma 3.6.

**Appendix D. Proof of Lemma 4.3—WKB approximate solutions**

One can use the standard WKB method (see e.g. in [4,20,48]) to obtain asymptotic expressions of the form $\varphi_k = f(x) e^{iS(x)}$ for certain amplitude $f$ and phase function $S$. The corresponding result for the case $m = 1$ was stated in [30, Lemma 5.1] without proof. Since Lemma 4.3 will play an important role in our proof, we will give its detailed proof here for the sake of completeness of the paper.

According to **Key Observation** in the proof of Theorem 4.1, each $\varphi_k$ is either even or odd. We claim: in the case that $\varphi_k$ is even, (4.18), as well as (4.20), holds, while in the case that $\varphi_k$ is odd, (4.19), as well as (4.20), holds. We only give the proof for the case that $\varphi_k$ is even, while the proof for the second case is very similar. Thus we will assume, in what follows, that $\varphi_k$ is even.

Notice that the equation (4.17) has two turning points $x = \pm \mu_k$ (with $\mu_k = \lambda_k^{1/2m}$). Since $\varphi_k$ is even, we need only focus our studies on $[0, \infty)$ and the turning point $x = \mu_k$. Since $\varphi_k$ has different behaviors for the cases that $x$ is small (compared to $\mu_k$), $x$ is close to $\mu_k$; and $x$ is large, it has three different expressions in (4.20).
Case 1. Asymptotic behavior of $\varphi_k$ when $0 \leq x < \mu_k - \delta \frac{2m-1}{3} (\mu_k > 0)$ (Here, $\delta > 0$ is arbitrarily fixed. Notice that for large $k$, we have $\mu_k - \delta \frac{2m-1}{3} > 0$.)

This case is corresponding to the classical allowed region. Consider the following standard Liouville transform (see e.g. [48, p. 119]):

$$
\left\{
\begin{array}{l}
y = S^-(x) = \int_0^x \sqrt{\mu_k^{2m} - s^{2m}} \, ds, \quad x \in \hat{\Omega}_k := \{ x : |x| \leq \mu_k - \delta \frac{2m-1}{3} \} \\
w = w(y) = (\mu_k^{2m} - x^{2m})^{\frac{1}{2}} \varphi_k(x), \quad \text{with } y = S^-(x) \in S^-(\hat{\Omega}_k).
\end{array}
\right.
$$ (D.1)

Applying the above transform to the equation (4.17) (restricted over $\hat{\Omega}_k$), we find

$$
\frac{d^2 w(y)}{dy^2} + w(y) + q_1(\mu_k, y)w(y) = 0, \quad y \in S^-(\hat{\Omega}_k),
$$ (D.2)

where

$$q_1(\mu_k, y) = \frac{m(2m - 1)x^{2m-2}}{2(\mu_k^{2m} - x^{2m})} + \frac{5m^2 x^{2(2m-1)}}{4(\mu_k^{2m} - x^{2m})^3}, \quad \text{with } y = S^-(x) \in S^-(\hat{\Omega}_k).$$ (D.3)

Using Duhamel’s formula to (D.2), we have

$$w(y) = w(0) \cos y - \int_0^y \sin(y - z)q_1(\mu_k, z)w(z) \, dz, \quad y \in S^-(\hat{\Omega}_k).$$ (D.4)

By (D.3), after some computations, we see that when $0 \leq x < \mu_k - \delta \frac{2m-1}{3}$,

$$
\int_0^y q_1(\mu_k, z) \, dz = \int_0^y \left( \frac{m(2m - 1)x^{2m-2}}{2(\mu_k^{2m} - t^{2m})^{\frac{3}{2}}} + \frac{5m^2 x^{2(2m-1)}}{4(\mu_k^{2m} - t^{2m})^3} \right) \, dt
$$

(by changing variable: $z = S^-(t)$)

$$\leq C \left( \int_0^{\mu_k - \delta \frac{2m-1}{3}} \frac{t^{2m-2}}{(\mu_k^{2m} - t^{2m})^{\frac{3}{2}}} \, dt + \int_0^{\mu_k - \delta \frac{2m-1}{3}} \frac{t^{2(2m-1)}}{(\mu_k^{2m} - t^{2m})^3} \, dt \right)
$$

$$\leq C \mu_k^{-\frac{2}{3}(m+1)} + C \leq C. \quad (D.5)
$$

Here and in what follows, $C$ stands for a positive constant (independent of $k$) which may vary in different contexts. Moreover, when $0 \leq x < \mu_k$, we have

$$
\int_0^x \frac{t^{2m-2}}{(\mu_k^{2m} - t^{2m})^{\frac{3}{2}}} \, dt + \int_0^x \frac{t^{2(2m-1)}}{(\mu_k^{2m} - t^{2m})^3} \, dt \leq C(\mu_k^{2m} - x^{2m})^{-\frac{1}{2}} (\mu_k - x)^{-1}. \quad (D.6)
$$

By (D.5), we can apply Gronwall’s inequality in (D.4) to see

$$|w(y)| \leq C \cdot |w(0)|, \quad \text{when } y \in S^-(\hat{\Omega}_k). \quad (D.7)
$$

Inserting (D.7) into (D.4), using (D.6) and (D.1), we obtain

$$\varphi_k(x) = w(0)(\mu_k^{2m} - |x|^{2m})^{-\frac{1}{2}} \left( \cos S^-(x) + R_k(x) \right), \quad |x| < \mu_k - \delta \frac{2m-1}{3}. \quad (D.8)$$
where the error term $R_k(x)$ satisfies

$$|R_k(x)| \leq C(\mu_k^{2m} - x^{2m})^{-\frac{1}{2}}(\mu_k - x)^{-1}, \quad 0 \leq x < \mu_k. \quad (D.9)$$

Comparing (D.8) (as well as (D.9)) with (4.18) (as well as (4.20)), we see that the remainder in Case 1 is to show that

$$w(0) \sim \mu_k^{\frac{m-1}{2}}.$$

But this is a direct consequence of Lemma 3.5 and $C_{\lambda_k} := w(0)$. The latter follows from the fact that $w$ is even and the definition $C_{\lambda_k} := w(0) - i w'(0)$.

**Case 2. Asymptotic behavior of $\phi_k$ when $\mu_k - \delta \mu_k^{\frac{2m-1}{3}} \leq x \leq \mu_k + \delta \mu_k^{\frac{2m-1}{3}}$ (Here, $\delta > 0$ will be given later.)**

Notice that near the turning point $\mu_k$, the approximation in Case 1 breaks down since the factor $|x^{2m} - \mu_k^{2m}|^{-1/4}$ in (D.8) goes to infinity when $|x - \mu_k| \to 0$. The way to pass this barrier is to linearize the potential $x^{2m}$ near the turning point. Indeed, plugging the Taylor’s expansion:

$$x^{2m} = \mu_k^{2m} + 2m\mu_k^{2m-1}(x - \mu_k) + \mu_k^{2m-2}(x - \mu_k)^2 \cdot T(x - \mu_k),$$

(where $T(t) = c_0 + c_1t + \cdots + c_{2m-2}t^{2m-2}$ is some polynomial of degree $2m - 2$) into (4.17) yields

$$-\phi_k''(x) + 2m\mu_k^{2m-1}(x - \mu_k)\phi_k(x) + \mu_k^{2m-2}(x - \mu_k)^2T(x - \mu_k) \cdot \phi_k(x) = 0. \quad (D.10)$$

Using the transform:

$$y = (2m\mu_k^{2m-1})^{1/2}(x - \mu_k), \quad (D.11)$$

and choosing $\delta$ so that

$$0 < \delta < \frac{(2m)^{-1/3}}{10},$$

we change the equation (D.10) into

$$\phi_k''(y) - y \cdot \phi_k(y) - q_2(\mu_k, y)\phi_k(y) = 0, \quad -1/10 < y < 1/10, \quad (D.12)$$

where

$$q_2(\mu_k, y) = \mu_k^{\frac{2(m+1)}{3}} \cdot y^2 \cdot T((2m\mu_k^{2m-1})^{-\frac{1}{2}}y). \quad (D.13)$$

The idea to deal with (D.12) is as: the term $q_2(\mu_k, y)$ is small as $k$ is large; when it is ignored, the above equation becomes the standard Airy equation. Thus, it can be solved explicitly in terms of $Ai(\cdot)$ and $Bi(\cdot)$ (which are two linear independent solutions of the Airy equation: $\psi''(y) - y\psi(y) = 0, y \in \mathbb{R}$) by the method of variation of parameters (see [38]).

With the above idea, we write the solution of (D.12) as:

$$\phi_k(y) = c_{\mu_k} \cdot (Ai(y) + r(y)), \quad -1/10 < y < 1/10. \quad (D.14)$$
Since $Ai''(y) - y \cdot Ai(y) = 0$, the error term $r(\cdot)$ satisfies
\[
r''(y) - y \cdot r(y) - q_2(\mu_k, y)(Ai(y) + r(y)) = 0, \quad -1/10 < y < 1/10. \tag{D.15}\]
By the variation of parameters and by using of the identity:
\[
W(Ai(y), Bi(y)) := Ai(y)Bi'(y) - Bi(y)Ai'(y) = 1/\pi, \quad y \in \mathbb{R},
\]
we can get from (D.15) that (see e.g. [38, p.400])
\[
r(y) = \pi \int_{-1/10}^{y} (Bi(y)Ai(v) - Ai(y)Bi(v))q_2(\mu_k, v)(Ai(v) + r(v)) \, dv, \tag{D.16}
\]
when $-1/10 < y < 1/10$. Since the functions $Ai(\cdot)$ and $Bi(\cdot)$ don’t vanish over $[-1/10, 1/10]$ ([38, p.395]), we can find some absolute constant $C > 0$ so that
\[
\frac{1}{C} \leq |Ai(y)|, \quad |Bi(y)| \leq C \quad \text{for all} \quad -1/10 < y < 1/10. \tag{D.17}
\]
This, along with (D.13), yields
\[
\int_{-1/10}^{y} |(Bi(y)Ai(v) - Ai(y)Bi(v))q_2(\mu_k, v)| \, dv \leq C\mu_k^{-\frac{2(m+1)}{3}}, \quad -1/10 < y < 1/10. \tag{D.18}
\]
By (D.18), we can apply Gronwall’s inequality to (D.16) to see
\[
|r(y)| \leq C\mu_k^{-\frac{2(m+1)}{3}}, \quad -1/10 < y < 1/10. \tag{D.19}
\]
Now, it follows from (D.14) and (D.11) that
\[
\varphi_k(x) = c_{\mu_k} \cdot \left( Ai\left( (2m\mu_k^{2m-1})^{\frac{1}{2}}(x - \mu_k) \right) + r\left( (2m\mu_k^{2m-1})^{\frac{1}{3}}(x - \mu_k) \right) \right), \tag{D.20}
\]
where $r$ satisfies the estimate (D.19). The remainder is to estimate $c_{\mu_k}$ in (D.20). By (D.17), one has $Ai(x) \sim 1$, when $|x| \leq 1/10$. Therefore we have
\[
\varphi_k(x) \sim c_{\mu_k}, \quad \text{when} \quad \mu_k - \delta \mu_k^{-\frac{2m-1}{3}} \leq x \leq \mu_k + \delta \mu_k^{-\frac{2m-1}{3}}. \tag{D.21}
\]
Meanwhile, if $w$ is given by (D.1), then by results in Case 1 and by the fact $|w(0)| \sim \mu_k^{-\frac{m-1}{2}}$, we see that $x \not< \mu_k - \delta \mu_k^{-\frac{2m-1}{3}}$,
\[
\varphi_k(x) \rightarrow w(0) \left( \mu_k^{2m} - (\mu_k - \delta \mu_k^{-\frac{2m-1}{3}})^{2m} \right)^{-1/4} \sim \mu_k^{-\frac{m-2}{6}}.
\]
This, together with the continuity of the function $\varphi_k$, yields that
\[
c_{\mu_k} \sim \mu_k^{\frac{m-2}{6}}. \tag{D.22}
\]
From (D.21) and (D.22), we get (4.18) for Case 2.

Case 3. Asymptotic behavior of $\varphi_k$ when $x > \mu_k + \delta \mu_k^{-\frac{2m-1}{3}}$. 

This case is corresponding to the classically forbidden region since the potential energy $V = x^{2m}$ is greater than total energy $\lambda_k = \mu_k^{2m}$. So we use the following transform (instead of (D.1)):

\[
\begin{cases}
y = S^+(x) = \int_{\mu_k}^x \sqrt{s^{2m} - \mu_k^{2m}} \, ds, \\
w = w(y) = (x^{2m} - \mu_k^{2m})^{\frac{1}{2}} \varphi_k(x), \text{ with } y = S^+(x).
\end{cases}
\]  

(D.23)

Under this transform, the equation (4.17) is as:

\[
\frac{d^2 w(y)}{dy^2} - w(y) + q_3(\mu_k, y)w(y) = 0,
\]

where

\[
q_3(\mu_k, y) = \frac{m(2m - 1)x^{2m-2}}{2(x^{2m} - \mu_k^{2m})^2} + \frac{5m^2x^{2(2m-1)}}{4(x^{2m} - \mu_k^{2m})^3}, \quad y = S^+(x).
\]  

(D.25)

Then by (D.25) and a direct computation, one has

\[
\int_{S^+(\mu_k + \delta \mu_k^{-(2m-1)/3})}^{\infty} q_3(\mu_k, y) \, dy 
\leq C \int_{\mu_k + \delta \mu_k^{-(2m-1)/3}}^{\infty} \left( \frac{x^{2m-2}}{(x^{2m} - \mu_k^{2m})^2} + \frac{x^{2(2m-1)}}{(x^{2m} - \mu_k^{2m})^3} \right) \, dx \leq C.
\]

This shows that the unbounded potential $x^{2m}$ is reduced to an integrable one after the transform (D.23). Then, according to some fundamental results in ODE (see e.g. [4, Theorem 4.1]), the solution satisfies

\[
w(y) = C_{\mu_k} e^{-y} (1 + o(1)), \quad \text{as } y \to +\infty,
\]

(D.26)

and it is uniquely defined by

\[
w(y) = C_{\mu_k} e^{-y} - \int_y^{\infty} \sinh(t - y)q_3(\mu_k, t)w(t) \, dt.
\]

(D.27)

To proceed, observe that by (D.26), one has

\[
| \sinh(t - y) \cdot w(t) | \leq C_{\mu_k} \left( e^{t-y} - e^{-(t-y)} \right) e^{-t} \leq C_{\mu_k} e^{-y}, \quad \text{when } t > y.
\]

Plugging this into the integral in (D.27) and making change of variable $t = S^+(s)$ (notice that $y = S^+(x)$), one has

\[
\left| \int_y^{\infty} \sinh(t - y)q_3(\mu_k, t)w(t) \, dt \right| \leq C_{\mu_k} e^{-y} \cdot \int_y^{\infty} q_3(\mu_k, t) \, dt 
\leq C_{\mu_k} e^{-y} \cdot \int_x^{\infty} \left( \frac{s^{2m-2}}{(s^{2m} - \mu_k^{2m})^2} + \frac{s^{2(2m-1)}}{(s^{2m} - \mu_k^{2m})^3} \right) \, ds
\]

\[
\leq C_{\mu_k} e^{-y} \cdot (x^{2m} - \mu_k^{2m})^{-\frac{1}{2}} (x - \mu_k)^{-1}, \quad x > \mu_k.
\]

(D.28)
Therefore we obtain from (D.27), (D.28) and (D.23) that
\[
\varphi_k(x) = C_{\mu_k} (x^{2m} - \mu_k^{2m})^{-\frac{1}{4}} e^{-S^+(x)} (1 + R_k(x)), \quad \text{when } x > \mu_k + \delta \mu_k^{\frac{2m-1}{3}},
\] (D.29)
and the reminder term \( R_k(x) \) satisfies
\[
|R_k(x)| \leq C (x^{2m} - \mu_k^{2m})^{-\frac{1}{2}} (x - \mu_k)^{-1}, \quad \text{when } x > \mu_k + \delta \mu_k^{\frac{2m-1}{3}}. \quad (D.30)
\]
We next claim that
\[
C_{\mu_k} \sim \mu_k^{\frac{m-1}{2}}. \quad (D.31)
\]
Indeed, according to (D.23) and (D.30), there is an absolute constant \( C > 0 \) such that
\[
0 < S^+(\mu_k + \delta \mu_k^{\frac{2m-1}{3}}) \leq \mu_k^{\frac{2m-1}{3}} \cdot \sqrt{(\mu_k + \delta \mu_k^{\frac{2m-1}{3}})^{2m} - \mu_k^{2m}} \leq C,
\]
and
\[
R_k(\mu_k + \delta \mu_k^{\frac{2m-1}{3}}) \leq C.
\]
By these, we can use (D.29) and the continuity of \( \varphi_k \) near the point \( \mu_k + \delta \mu_k^{\frac{2m-1}{3}} \) to find that when \( x \searrow \mu_k + \delta \mu_k^{\frac{2m-1}{3}} \),
\[
\varphi_k(x) \to C_{\mu_k} \left( (\mu_k + \delta \mu_k^{\frac{2m-1}{3}})^{2m} - \mu_k^{2m} \right)^{-1/4} \sim \mu_k^{\frac{m-2}{6}},
\]
which leads to (D.31).

Finally, by (D.29), (D.30) and (D.31), we get (4.18) and (4.20) for Case 3. This ends the proof of Lemma 4.3.  \( \square \)

Appendix E. Proof of Lemma 4.4

First we prove statement (a). By (4.11), we have
\[
bk^{\frac{1}{m+1}} (1 - |r_k|)^{\frac{1}{2m}} \leq \lambda_k^{\frac{1}{2m}} \leq bk^{\frac{1}{m+1}} (1 + |r_k|)^{\frac{1}{2m}} \quad \text{for all } k \in \mathbb{N}^+,
\] (E.1)
where \( b \) is given by (4.37). Arbitrarily fix \( a > 0 \). Then by (E.1), we find that for all \( k \in \mathbb{N}^+ \),
\[
\left( E \cap [-a\lambda_k^{\frac{1}{2m}}, a\lambda_k^{\frac{1}{2m}}] \right) \subset \left( E \cap [-abk^{\frac{1}{m+1}} (1 + |r_k|)^{\frac{1}{2m}}, abk^{\frac{1}{m+1}} (1 + |r_k|)^{\frac{1}{2m}}] \right).
\] (E.2)

From (E.2), we see
\[
\lim_{N \to \infty} \frac{|E \cap [-a\lambda_k^{\frac{1}{2m}}, a\lambda_k^{\frac{1}{2m}}]|}{a\lambda_k^{\frac{1}{2m}}} \leq \lim_{N \to \infty} \frac{|E \cap [-abk^{\frac{1}{m+1}} (1 + |r_k|)^{\frac{1}{2m}}, abk^{\frac{1}{m+1}} (1 + |r_k|)^{\frac{1}{2m}}]|}{abk^{\frac{1}{m+1}} (1 - |r_k|)^{\frac{1}{2m}}}
\]
\[
\leq \lim_{N \ni k \to \infty} \frac{|E \cap [-abk^{1/n}, abk^{1/n}]|}{abk^{1/n}(1 - |r_k|)^{1/m}} + \lim_{N \ni k \to \infty} \frac{2((1 + |r_k|)^{1/n} - 1)}{(1 - |r_k|)^{1/m}}. \tag{E.3}
\]

Since \(\lim_{N \ni k \to \infty} r_k = 0 \) (see (4.11)), the second term on the right hand side of (E.3) vanishes. So we obtain from (E.3) that

\[
\lim_{N \ni k \to \infty} \frac{|E \cap [-a^{1/n}, a^{1/n}]|}{a^{1/n}} \leq \lim_{N \ni k \to \infty} \frac{|E \cap [-ab^{1/n}, ab^{1/n}]|}{ab^{1/n}}. \tag{E.4}
\]

On the other hand, it follows from (E.1) that when \(k\) is large enough so that \(|r_k| < 1\),

\[
\left( E \cap [-a^{1/n}, a^{1/n}] \right) \supset \left( E \cap [-ab^{1/n}, ab^{1/n}(1 - |r_k|)^{1/m}, ab^{1/n}(1 - |r_k|)^{1/m}] \right). \tag{E.5}
\]

Similar to (E.4), one can deduce from (E.5) that

\[
\lim_{N \ni k \to \infty} \frac{|E \cap [-a^{1/n}, a^{1/n}]|}{a^{1/n}} \geq \lim_{N \ni k \to \infty} \frac{|E \cap [-ab^{1/n}, ab^{1/n}]|}{ab^{1/n}}. \tag{E.6}
\]

Therefore, (4.36) follows from (E.4) and (E.6) at once.

Now we prove statement (b). We will only prove the part \((ii) \implies (i)\). The reverse \((i) \implies (ii)\) can be proved by almost the same argument.

By (ii), there is \(k_0 \in \mathbb{N}^+\) and \(\gamma > 0\) so that

\[
|E \cap [-ak, ak]| \geq \gamma\text{ for all } k \geq k_0. \tag{E.7}
\]

Arbitrarily fix \(x \in \mathbb{R}\) so that

\[
x \geq c_2 := a(k_0 + 1)^l. \tag{E.8}
\]

Then, there exists a unique \(n \in \mathbb{N}\) so that

\[
an^l \leq x < a(n + 1)^l. \tag{E.9}
\]

The fact (E.8), together with (E.9), implies that \(n > k_0\). Meanwhile, from (E.9), we also have

\[
E \cap [-x, x] \supset E \cap [-an^l, an^l],
\]

which, along with (E.7), leads to

\[
|E \cap [-x, x]| \geq |E \cap [-an^l, an^l]| \geq \gamma an^l. \tag{E.10}
\]

Now, from (E.10) and (E.9), we find that for all \(x \in [c_2, +\infty)\),

\[
|E \cap [-x, x]| \geq \gamma\left(\frac{n}{n + 1}\right)^l x \geq \gamma 2^{-l} x. \tag{E.11}
\]

It follows from (E.11) that

\[
\lim_{x \to +\infty} \frac{|E \cap [-x, x]|}{x} \geq \gamma 2^{-l} > 0,
\]

which leads to (i). This ends the proof of Lemma 4.4.
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