DUFLO–SERGANOVA HOMOLOGY FOR EXCEPTIONAL MODULAR LIE SUPERALGEBRAS WITH CARTAN MATRIX

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Abstract. For the exceptional finite-dimensional modular Lie superalgebras $g(A)$ with indecomposable Cartan matrix $A$, and their simple subquotients, we computed non-isomorphic Lie superalgebras constituting the homologies of the odd elements with zero square. These homologies are key ingredients in the Duflo–Serganova approach to the representation theory.

There were two definitions of defect of Lie superalgebras in the literature with different ranges of application. We suggest a third definition and an easy-to-use way to find its value.

In positive characteristic, we found out one more reason to consider the space of roots over reals, unlike the space of weights, which should be considered over the ground field.

We proved that the rank of the homological element (decisive in calculating the defect of a given Lie superalgebra) should be considered in the adjoint module, not the irreducible module of least dimension (although the latter is sometimes possible to consider, e.g., for $p = 0$).

We also computed the above homology for the only case of simple Lie superalgebras with symmetric root system not considered so far over the field of complex numbers, and its modular versions: $\mathfrak{psl}(a|a + pk)$ for $a$ and $k$ small, and $p = 2, 3, 5$.

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1991 Mathematics Subject Classification. Primary 17B50, 17B55, 17B56; Secondary 17B20.

Key words and phrases. Modular Lie superalgebra, Duflo–Serganova homology.
1. Introduction

For notation, see [BGL, BGLd]. All vector spaces, in particular, Lie (super)algebras $\mathfrak{g}$ we consider here, are finite-dimensional over the ground field $K$ of characteristic $p$. Let $p$ denote also the parity of the Lie superalgebra considered. In this paper, $K_{m|n}$ denotes a commutative Lie superalgebra, the one with zero bracket; $\mathfrak{c}$ denotes the center of $\mathfrak{g}$.

1.1. Atypicality and defect. The irreducible finite-dimensional modules over the Lie superalgebra $\mathfrak{g} = \mathfrak{g}(A)$ with Cartan matrix $A$ over $\mathbb{C}$ are of the two types: generic (Kac brightly called them “typical”), which are in one-to-one correspondence with certain $\mathfrak{g}_0$-modules, and “atypical”. It soon became clear that different irreducible modules over simple (and “close” to simple, like $\mathfrak{gl}$ is close to $\mathfrak{psl}$) Lie superalgebras have different levels of atypicality, not exceeding the defined in [KW] defect $\text{df}(\mathfrak{g})$. Kac and Wakimoto defined $\text{df}(\mathfrak{g})$ only for Lie superalgebras.
of the form $g(A)$ over $\mathbb{C}$ with an even non-degenerate invariant symmetric bilinear form (NIS for short).

The typical irreducible $g(A)$-modules were first described analytically by Berezin in several cases ([Bert]). Kac obtained a complete classification of typical irreducible $g(A)$-modules using more adequate algebraic technique, see [Ktyp] with corrections in [Sg, SV].

The atypical irreducible modules were first classified and described in terms of sections of vector bundles on certain Grassmann supervarieties of dimension $0|n$ by J. Bernstein and D. Leites for $\mathfrak{s}(1|n)$ and $\mathfrak{osp}(2|2n)$ whose defect is equal to 1, see [BL, L].

1.2. The Duflo–Serganova functor. Over any ground field, let $g$ be a Lie superalgebra, let $M$ be a $g$-module given by a representation $\rho$. For an odd element $x \in g$ such that $x^2 = 0$ set

\[(1) \quad M_x := \ker \rho_x / \img \rho_x \text{ and } g_x := \ker \ad_x / \img \ad_x.\]

It was known since long time ago that for any dg Lie superalgebra with the odd differential $d$, the space $M_d$ is a $g_d$-module.

This general fact was formulated usually for the non-zero differential $d$ of degree $\pm 1$, see, e.g., [Ge]. In these works, the Maurer-Cartan equation allows one to add only degree $\pm 1$ elements $x \in g_1$ to the differential $d$.

Duflo and Serganova (see [DS]) considered a completely different case where $d = 0$, and the $\mathbb{Z}$-grading of $g$ considered modulo 2 coincides with the parity of $g$. Then, one can add any odd element $x$ such that $x^2 = 0$ to the differential $d$, and still have $(d + x)^2 = 0$.

Duflo and Serganova proved that the Duflo–Serganova functor $DS_x : M \to M_x$ from the category of $g$-modules to the category of $g_x$-modules is a tensor functor; this was a new result.

This functor helped to solve interesting problems, see [EASA, HPS, HR, IRS], where $p = 0$.

For which $g$ was $g_x$ computed? For $p = 0$, Duflo and Serganova ([DS]) considered finite-dimensional Lie superalgebras $g$ of the form $g(A)$ with indecomposable and invertible Cartan matrix $A$, and $g(a|b)$ for any $a, b$ over $\mathbb{C}$, i.e., all simple Lie superalgebras with symmetric root system, except $\mathfrak{psl}(a|a)$.

In this paper, we consider the cases of $\mathfrak{psl}(a|a + kp)$ for $p = 0, 2, 3, 5$, and $k$ and $a$ small.

The Lie superalgebras considered by Duflo and Serganova for $p = 0$, as well as $\mathfrak{psl}(a|a + kp)$ for $p > 0$, and the exceptional simple Lie superalgebras and those of the form $g(A)$ we consider here have an even NIS, see [BKLS, KLLS].

The specifically super analogs of $\mathfrak{gl}(n)$ — Lie superalgebras $g$ of series $q$ their simple subquotients, have an odd NIS and no Cartan matrix. For computation of the DS-homology of these Lie superalgebras, see [KLS]. The case of Poisson Lie superalgebras $\mathfrak{po}(0|n)$ and their simple subquotients $h'(0|n)$ will be considered separately.

1.3. When $g_x \simeq g_y$ for $g = g(A)$ over $\mathbb{C}$. The bulk of [DS] consists of the proof of the fact that, for the finite-dimensional Lie superalgebra $g = g(A)$ with indecomposable and invertible Cartan matrix $A$, the following statement holds:

1.3.1. Theorem ([DS]). We have $g_x \simeq g_y \iff \rank \rho_x = \rank \rho_y$, where

\[(2) \quad \rho = \begin{cases} \id \text{ in } V & \text{for } g \text{ of series } \mathfrak{gl}(V), \text{ or } \mathfrak{sl}(V) \text{ for } \text{sdim } V \neq a|a, \text{ or } \mathfrak{osp}(V), \\ \ad \text{ in } g & \text{for } g = \mathfrak{osp}(4|2; a) \text{ or } \mathfrak{ag}(2) \text{ or } \mathfrak{ab}(3). \end{cases}\]

In cases (2), the module given by $\rho$ is the one of the least dimension, except for $\mathfrak{osp}(4|2; a)$, where $a = 1, 2$ or 3 (or the values of $a$ obtained from these under the action of $S_3$, see...
for the values of $A$, the irreducible modules of the least superdimension are (up to the change of parity) $4[2, 6|4$ and $8|6$, respectively.

These exceptional modules of the least dimension are lucidly (but with a typo) described in [GL2]. For $a = 2$ and $3$, Duflo and Serganova proved Theorem 1.3.1 without considering $\mathfrak{osp}(4|2; a)$-modules of the least dimension. We show that to prove that $\mathfrak{g}_x \simeq \mathfrak{g}_y$ we should compare ranks of the operators in the adjoint module, not in the ones of the least dimension.

To show in which modules should one compute the rank of $\rho(x)$ in order to establish if $\mathfrak{g}_x \simeq \mathfrak{g}_y$, we do consider these modules of least dimension, see Table 7.

For the Lie superalgebra $\mathfrak{g}(A)$ with a symmetrizable indecomposable Cartan matrix $A$, the space spanned by roots $R$ over the ground field inherits the NIS $(-, -)$ given on $\mathfrak{g}(A)$, see [BKLS eq. (4)]. In the modular case, however, we have to use a different definition of roots, most lucidly described in [BLLoS]. (Note that the NIS on $\mathfrak{g}(A)$ induces a non-degenerate symmetric bilinear form on the space spanned by roots over $\mathbb{R}$ as well.)

Observe that there is no NIS on $\mathfrak{sl}(n|n)$, but there is a NIS on $\mathfrak{psl}(n|n)$ which has no $n|n$-dimensional irreducible modules; the $\mathfrak{psl}(n|n)$-module of least dimensional is adjoint.

Recall that a root is called odd if the corresponding root vector is odd; $\beta \in R_1$ is isotropic if $(\beta, \beta) = 0$. Two roots $\alpha, \beta \in R$ are called orthogonal if $(\alpha, \beta) = 0$.

In addition to Theorem 1.3.1, Duflo and Serganova proved the following

1.3.2. Theorem ([DS]: $\mathfrak{g}_x$ for $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $\mathfrak{sl}(a|b)$, where $a \neq b$, and $\mathfrak{osp}(a|2b)$). For $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $\mathfrak{sl}(a|b)$, where $a \neq b$, and $\mathfrak{osp}(a|2b)$ and for any set

\begin{equation}
\{\beta_1, \ldots, \beta_k\} \text{ of linearly independent mutually orthogonal isotropic roots,}
\end{equation}

let $x_k := x_{\beta_1} + \cdots + x_{\beta_k}$, and $\rho = \text{id}$.

Then,

\begin{equation}
\begin{align*}
\text{rank } \rho_{x_k} & = \left\{ \begin{array}{ll}
2k & \text{for } \mathfrak{gl}(m|n) \text{ and } \mathfrak{sl}(m|n), \text{ where } m \neq n, \\
4k & \text{for } \mathfrak{osp}(2m|2n) \text{ and } \mathfrak{osp}(2m + 1|2n),
\end{array} \right.
\end{align*}
\end{equation}

and, respectively,

\begin{equation}
\begin{align*}
\mathfrak{g}_{x_k} & \simeq \left\{ \begin{array}{ll}
\mathfrak{gl}(m - k|n - k) \text{ and } \mathfrak{sl}(m - k|n - k), \\
\mathfrak{osp}(2m - 2k|2n - 2k) \text{ and } \mathfrak{osp}(2m - 2k + 1|2n - 2k).
\end{array} \right.
\end{align*}
\end{equation}

• For $p = 0$, the Lie superalgebras $\mathfrak{gl}(n|n)$ is the only type of Lie superalgebras with non-invertible indecomposable Cartan matrix, see classification [CCLL]. We consider the case of $\mathfrak{psl}(n|n)$, the simple subquotient of $\mathfrak{gl}(n|n)$, in §3 the answer does not depend on $p$.

Kac and Wakimoto define $\text{df}(\mathfrak{g})$ as the maximal cardinality $k$ of the set (3). For $\mathfrak{g} = \mathfrak{g}(A)$ over $\mathbb{C}$, the defect $\text{df}(\mathfrak{g})$ is also equal to

\begin{equation}
\text{the number } \text{nds}(\mathfrak{g}) \text{ of non-isomorphic DS-homology superalgebras } \mathfrak{g}_x.
\end{equation}

Clearly, $\text{nds}(\mathfrak{g})$ (same as def, see eq. (511)) is independent of the existence of NIS on $\mathfrak{g}$.

Let us investigate: is there a relation between the defect and the rank of $\rho_x$ in some irreducible representation $\rho$ apart from the one described in eq. (4)? For example, for the simple Lie superalgebras $\mathfrak{osp}(4|2; a)$, the values of rank of $\rho_{x_{\text{nds}(\mathfrak{g})}}$ are either $\text{ad}$ or the irreducible
representation $\sigma$ of the least dimension, are as follows

| $a$ | rank$(\text{ad}_{\text{nds}(g)})$ | rank$(\sigma_{\text{nds}(g)})$ | nds$(g)$ |
|-----|-------------------------------|--------------------------------|----------|
| 1   | 8                             | 2                              | 1        |
| 2   | 8                             | 4                              | 1        |
| 3   | 8                             | 6                              | 1        |

This table hints to define the defect looking at the adjoint representation instead of the one of the least dimension, but this hint is not as compelling, as the result of table (57).

For the serial Lie superalgebras with Cartan matrix, both choices of $\rho$ (the tautological and adjoint) are OK, the tautological $g_x$-module in eq. (5) is, however, of superdimension $\text{sdim} V - \text{rank} \rho_x | \text{rank} \rho_x,$ where $\rho$ is the representation in tautological $g$-module $V$, not in the adjoint one.

For the series $\text{psl}$, for $\text{osp}(4|2; a)$ for $a$ generic, and for the two exceptional simple Lie superalgebras — $\mathfrak{so}(2)$ and $\mathfrak{ab}(3)$ — the irreducible representation of the least dimension is the adjoint one, so to define the rank of $x$ we have to take ad (and be happy we do not have to consider a module of greater dimension).

1.4. When $g_x \simeq g_y$ for $p > 0$ and $g = g(A)$? We want to answer the following questions:

- Q1) Should we take the space spanned by roots over the ground field $\mathbb{K}$, same as the space of weights, or should we follow the definition suggested by A. Lebedev, see [BLLoS], and consider the inner product in the space spanned by the roots over $\mathbb{R}$, and not the one described in [BKLS]?

- Q2) What should we take for $\rho$ trying to imitate the approach (3)? (Does the answer to the question “under what conditions on $x$ and $y$ we have $g_x \simeq g_y$?” depend on $\rho$?)

The answers prove, by means of SuperLie code, the following hypothesis that guided us.

1.4.1. Hypothesis. Among all Cartan matrices of $g(A)$

(8) with the maximal number of 0’s on its main diagonal

select the one, denote it $A$, whose Dynkin-Kac diagram $D$ has the maximal number $G_{\text{max}}$ of “gray” vertices not connected with each other (in terms of Cartan matrices this means that for all indices $i, j$ such that $A_{ii} = A_{jj} = 0$ there is a maximal number of pairs $A_{ij} = A_{ji} = 0$). Our Hypothesis consists of several parts:

1) For any $g(A)$, except for $\mathfrak{bgl}(4; a)$, the non-isomorphic Lie (super)algebras $g_x$ correspond to $x = x_\beta$ for $\beta = \beta_1 + \cdots + \beta_k$, where $k \leq G_{\text{max}}$ and each $\beta_i$ is one of the roots corresponding to the above “grey” vertices not connected with “grey” vertices corresponding to the other $\beta_j$.

Moreover,

$$\text{df}(g(A)) = \text{nds}(g(A)) = G_{\text{max}}.$$  

1e) For $g = \mathfrak{bgl}(4; a)$, we have $\text{nds} = G_{\text{max}} + 1$ whereas $\text{df}(g(A)) = G_{\text{max}}$.

2) If $A$ is non-invertible, the defect $\text{nds}(h)$ of the simple subquotient $h := g(A)^{(1)}/c$ is given by one of the following formulas (we were unable to find the pattern)

2a) $\text{nd}(h) = G_{\text{max}} - 1$, unless $G_{\text{max}} = 1$ in which case $\text{nds}(h) = G_{\text{max}} = 1$,

2b) $\text{nds}(h) = G_{\text{max}}$, e.g., $\text{nds}(\text{psl}(n|n + pk)) = \text{nds}(\text{gl}(n|n + pk))$ for $k \neq 0$.

Observe that for $g = \text{psl}(n|n)$, we have $\text{def}(g) = \text{nds}(g) \neq \text{df}(g)$ in any characteristic $p$ we verified, see § 3.

1.4.2. Fact (When $g_x \simeq g_y$). For the exceptional simple Lie superalgebra $g$ of the form either $g(A)$, or $g(A)^{(1)}/c$, we have $g_x \simeq g_y$ if and only if $\text{rank} \rho_x = \text{rank} \rho_y$, where $\rho$ is the adjoint
representation of $\mathfrak{g}$, not in the irreducible $\mathfrak{g}$-module $M$ of the least dimension, as examples show, e.g., [55], [57].

1.4.2a. **Our Hypothesis is true over $\mathbb{C}$.** Duflo and Serganova proved the statement equivalent to the claim of our Hypothesis for Lie superalgebras $\mathfrak{g}(A)$ over $\mathbb{C}$ with Cartan matrix $A$, even non-invertible, as for $\mathfrak{gl}(a|a)$, see [DS]. To see where the idea of the answer comes from, consider examples.

Series $\mathfrak{gl}$ and $\mathfrak{osp}$. To select the matrix $A_m$, see eq. (5), we represent the supermatrices — elements of $\mathfrak{gl}(a|b)$ — in the alternating format

$$\text{Par}_{alt} = \begin{cases} (\overline{0}, \overline{1}, \overline{0}, \overline{1}, \ldots, \overline{0}, \overline{1}, \ldots, \overline{1}) & \text{if } a \leq b \\ (\overline{0}, \overline{1}, \overline{0}, \overline{1}, \ldots, \overline{0}, \overline{0}) & \text{if } a > b. \end{cases}$$

Then, the elements $E_{i,i+1}$ (resp. $E_{i+1,i}$) are simple positive (resp. negative) root vectors for all $i$. Since the neighboring root vectors do not commute, their roots are not orthogonal to each other; take every second root. So, $\text{nds}(\mathfrak{gl}(a|b)) = \text{nds}(\mathfrak{sl}(a|b)) = \min(a, b)$.

The embeddings

$$\mathfrak{gl}(a|b) \rightarrow \mathfrak{osp}(2a|2b) \rightarrow \mathfrak{osp}(2a + 1|2b)$$

give the value of $\text{nds}(\mathfrak{g})$ for $\mathfrak{g}$ of the $\mathfrak{osp}$ series, see eq. (6):

$$\mathfrak{gl}(a|b)_{2k} = \mathfrak{gl}(a - k|b - k)$$

and $\mathfrak{osp}(N|2n)_{2k} = \mathfrak{osp}(N - 2k|2n - 2k)$.

Exceptions: $\mathfrak{g} = \mathfrak{osp}(4|2); a$ for a generic, $\mathfrak{ag}(2)$, and $\mathfrak{ab}(3)$. For them, the hypothesis describing when $\mathfrak{g}_x \simeq \mathfrak{g}_y$ is also true; in the hypothesis, $x$ is the adjoint representation, see eq. (12). For $\mathfrak{g} = \mathfrak{osp}(4|2); a$ and $a = 1, 2$ or 3, we can take any of the two representations: either $\text{ad}$, or the irreducible representation $\sigma$ of the least dimension, see eq. (7).

1.5. **Summary of our results.** We suggested a simple method for determining the value of the defect of $\mathfrak{g}(A)$. We computed both $\text{nds}(\mathfrak{g})$ and the Duflo-Serganova homology $\mathfrak{g}_x$ in the two cases:

(i) over $\mathbb{K}$ of characteristic $p > 0$, for each exceptional Lie superalgebra $\mathfrak{g}(A)$ with indecomposable Cartan matrix $A$;

(ii) for simple subquotients of the exceptional modular Lie superalgebras $\mathfrak{g}^{(1)}(A)/\mathfrak{c}$, where $A$ is not invertible, e.g., for $\mathfrak{psl}(n|n + pk)$ the conjecture is verified for $p = 0, 2, 3, 5$, and $n$ and $k$ small.

Hypothesis [1.4.1] is verified directly, by means of the SuperLie package, see [Gr]. Proving Hypothesis and answering Questions in Subsection 1.4 we found out the following.

1.5.1. **Fact** (Where do roots live, which representation determines defect). F1) Over $\mathbb{K}$, the space of roots should be considered over $\mathbb{R}$, as it was defined by A. Lebedev, see [BLLoS], not over the ground field as the space of weights. For motivations, see Subsections 2.3.1 [2.4.3].

F2) To get the correct answer to the question “when $\mathfrak{g}_x \simeq \mathfrak{g}_y$?”, we have to take $x = \text{ad}$, not the irreducible representation $\sigma$ of the least dimension, see, e.g., table (7), and especially table (87): the same rank of $\sigma_x$ and $\sigma_y$ might correspond to $\mathfrak{g}_x \not\simeq \mathfrak{g}_y$, whereas rank $\sigma_x \neq \text{rank } \sigma_y$ might correspond to $\mathfrak{g}_x \simeq \mathfrak{g}_y$.

2. The exceptional cases

By $N\mathfrak{g}(A)$ we denote $\mathfrak{g}(A)$ corresponding to the $N$th Cartan matrix $A$ as listed in [BGL]; recall that $\text{sdim } A/a|B$ means that $\text{sdim } \mathfrak{g}(A) = A|B$ and $\text{sdim } \mathfrak{g}^{(1)}(A)/\mathfrak{c} = a|B$. The odd root vectors are boxed and isotropic roots are underlined. The multiplication tables in $\mathfrak{g}_x$ are
obtained with the aid of SuperLie package. After several cases illustrating our answers, we refer the reader to [BLLoS] for bulky lists of roots and the corresponding root vectors.

2.1. Notation $\mathfrak{A} \oplus_c \mathfrak{B}$, see [BGL]. This notation is needed to describe the following Lie superalgebras or the corresponding DS-homologies

$$g(2, 3), g(2, 6), \text{ and } g(3, 3) \quad \text{for } p = 3,$$

$$\mathfrak{bgl}(4; \alpha), \varepsilon(6, 6), \varepsilon(7, 6), \text{ and } \varepsilon(8, 1) \quad \text{for } p = 2.$$ 

This notation describes the case where $\mathfrak{A}$ and $\mathfrak{B}$ are nontrivial central extensions of the Lie (super)algebras $\mathfrak{a}$ and $\mathfrak{b}$, respectively, and $\mathfrak{A} \oplus_c \mathfrak{B}$ — a nontrivial central extension of $\mathfrak{a} \oplus \mathfrak{b}$ (or, perhaps, a more complicated semidirect sum $\mathfrak{a} \ret \mathfrak{b}$, where $\mathfrak{a}$ is an ideal) with 1-dimensional center spanned by $c$ — is such that the restriction of the extension of $\mathfrak{a} \oplus \mathfrak{b}$ to $\mathfrak{a}$ gives $\mathfrak{A}$ and that to $\mathfrak{b}$ gives $\mathfrak{B}$.

Consider the 4 Lie superalgebras $g(A)$, where $p = 2$, listed in eq. (9) in more details (see [BGL]). Then, $g(A)_0$ is of the form

$$g(B) \oplus_c \mathfrak{hei}(2) \simeq g(B) \oplus \text{Span}(X^+, X^-),$$

where the matrix $B$ is not invertible (so $g(B)$ has a grading element $d$ and a central element $c$), and where $X^+, X^-$ and $c$ span the Heisenberg Lie algebra $\mathfrak{hei}(2)$. The brackets are:

$$[g^{(i)}(B), X^±] = 0;$$

$$[d, X^±] = \begin{cases} X^± & \text{for } \varepsilon(6, 6), \varepsilon(7, 6), \text{ and } \varepsilon(8, 1); \\ \alpha X^± & \text{for } \mathfrak{bgl}(3; \alpha) \\ X^+, X^- & = c. \end{cases}$$

The odd part of $g(A)$ (at least in two of these four cases) consists of two copies of the same $g(B)$-module $N$, the operators $\text{ad}_{X^\pm}$ permute these copies, and $\text{ad}_{X^\pm}^2 = 0$, so each of the operators maps one of the copies to the other, and this other copy to zero.

2.2. Notation $\mathfrak{A} \oplus_c^d \mathfrak{B}$. Recall the definition and examples of double extensions, see [BLS]. Actually, in examples in eq. (11), and $g_x$ for $\varepsilon(7, 6)$, we have double extensions $\mathfrak{A}$ and $\mathfrak{B}$ of the Lie (super)algebras $\mathfrak{a}$ and $\mathfrak{b}$, respectively, and $\mathfrak{A} \oplus_c^d \mathfrak{B}$ is a double extension of $\mathfrak{a} \oplus \mathfrak{b}$ by means of a central element $c$ and an outer derivation $d$ such that the restriction of the extension of $\mathfrak{a} \oplus \mathfrak{b}$ to $\mathfrak{a}$ (resp. $\mathfrak{b}$) gives $\mathfrak{A}$ (resp. $\mathfrak{B}$).

2.3. For $p \geq 5$: $\mathfrak{osp}(4|2; a)$ for $a \neq 0, -1$, $\mathfrak{ag}(2)$, and $\mathfrak{ab}(3)$. The answer is the same as for $p = 0$, namely: $df(g) = 1$ and $g_x$ is given by table (12); verified for $p = 5, 7, 11$:

$$\begin{align*}
g & \mathfrak{osp}(4|2; a) & \mathfrak{ag}(2) & \mathfrak{ab}(3) \\
g_x & \mathbb{K}^{10} & s\mathfrak{l}(2) & s\mathfrak{l}(3) \\
\text{rank } \text{ad}_x & 8 & 14 & 16
\end{align*}$$

Each of the other exceptional Lie superalgebras $g(A)$ with indecomposable Cartan matrix $A$ exists only in characteristics 2, 3 and 5. The two Lie superalgebras $3g(2, 3)$ and $1g(3, 3)$ (indigenous to $p = 3$) have “the same” Cartan matrix as $3g(2)$ and $6\mathfrak{ab}(3)$ (existing for $p = 0$ and any $p > 3$), respectively. Each of the other exceptional superalgebras $g(A)$ has no analogs among other exceptions, except for two pairs $\mathfrak{br}(2; 3) \leftrightarrow \mathfrak{br}(2; 3)$ and $\varepsilon(5, 5) \leftrightarrow \varepsilon(5, 3)$, versions of which we consider one after the other for clarity. Although the algebras in these pairs have “the same” Cartan matrices for $p = 5$ and 3, respectively, their structures are different

\footnote{Pretending that the elements of the Cartan matrix are integers, not elements of $\mathbb{K}$; we do the same describing analogs of Serre relations, see [BGL], [BL].}
(same thing with Lie superalgebras and their desuperizations for \(p = 2\), e.g., \(\mathfrak{sl} \leftrightarrow \mathfrak{w}\mathfrak{l}\)). Let \(\pi_i\) be the \(i\)th fundamental weight of the Lie algebra \(\mathfrak{g}\), let \(R(\sum \pi_i)\) denote the representation of \(\mathfrak{g}\) with highest weight \(\sum \pi_i\) and the corresponding \(\mathfrak{g}\)-module.

2.3.1. \(\mathfrak{brj}(2; 5)\) of \(\text{sdim } 10|12\). We have \(\mathfrak{brj}(2; 5)_0 = \mathfrak{sp}(4)\) and \(\mathfrak{brj}(2; 5)_1 = R(\pi_1 + \pi_2)\) is an irreducible the \(\mathfrak{brj}(2; 5)_0\)-module. We consider the following Cartan matrix and basis elements:

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6 \\
7 & 8 \\
9 & 10
\end{pmatrix}
\]

We get the same answer as for \(x_1\) in eq. (13) for \(x_7, x_8, x_{10}, \) and \(x_1 + x_{10}\).

To find out which of the four homological linearly independent root vectors \(x\)'s in table (13) are orthogonal to one another (to compute the defect) we have to decide do we consider the inner product in the space generated by the set of roots \(R\) over the ground field \(\mathbb{K}\) or over \(\mathbb{R}\): unlike weights, which are considered over \(\mathbb{K}\), the space of roots should be considered over \(\mathbb{R}\) for several reasons as A. Lebedev taught us, see [BGL]; here is one more reason.

Since \((\alpha_1, 2\alpha_1 + 5\alpha_2) \equiv 0 \mod 5\), the only possible pair of mutually orthogonal isotropic candidates: \(x_1\) and \(x_{10}\); but since \(\mathfrak{g}_x\) for \(x = x_1 + x_{10}\) is the same as for \(x = x_1\), see table (13), we conclude that \(nds(\mathfrak{g}) = 1\). Therefore, we have to consider the inner product in the space of roots over \(\mathbb{R}\).

In this case, \((\alpha_1, 2\alpha_1 + 5\alpha_2) = -10 \neq 0\). Thus \(x_1\) and \(x_{10}\) are not orthogonal.

2.3.2. \(\mathfrak{el}(5; 5)\) of \(\text{sdim } 55|32\). We have \(\mathfrak{g}_0 = \mathfrak{o}(11)\) and \(\mathfrak{g}_1 = R(\pi_5)\) as the \(\mathfrak{g}_0\)-module.

\[
\begin{pmatrix}
1 & 2 \\
3 & 15
\end{pmatrix}
\]

We get the same answer as for \(x_3\) in eq. (16) for the other homological elements of the same rank
\(x_4, x_5, x_6, x_7, x_{15}, x_{17}, x_{18}, x_{19}, x_{20}, x_{26}, x_{27}, x_{28}, x_{34}, x_{35}, x_{39}\).

Hypothesis 1.4.1 is confirmed, \(nds(\mathfrak{el}(5; 3)) = 1\).

2.4. For \(p = 3\).

2.4.1. \(\mathfrak{brj}(2; 3)\) of \(\text{sdim } 10|8\). We have \(\mathfrak{g}_0 = \mathfrak{sp}(4)\) and \(\mathfrak{g}_1 = R(2\pi_2)\) is an irreducible \(\mathfrak{g}_0\)-module. We consider the following Cartan matrix and basis elements:

\[
\begin{pmatrix}
0 & -1 \\
-2 & 1
\end{pmatrix}
\]
Hypothesis 2.4.1 is confirmed, nds(\(g\)) = 1. Hypothesis 1.4.1 is confirmed. Since \((\alpha_1, \alpha_1 + 4\alpha_2) = -8\), it follows that nds(\(g\)) = 1, Hypothesis 1.4.1 is confirmed.

2.4.2. \(7\text{sl}(5; 3)\) of \(\text{sdim} = 39\,|\,32\). We have \(g_0 = o(9) \oplus \text{sl}(2)\) and \(g_1 = R(\pi_4) \boxtimes \text{id}\) as the \(g_0\)-module.

Let \(M\) be the irreducible \(g\)-module with the highest weight \((0, 0, 0, 0, 1)\), then dim \(M = 18\,|\,16\); see [BGKL]. The answer in eq. (19) allows to consider \(M\) as well as the adjoint module.

\[
\begin{array}{cccccc}
\text{x} & \ \text{sdim} \ g_x & \ \text{g}_x & \ \text{rank} \ \text{ad}_x & \ \text{rank}_M \ x \\
\hline
\text{x}_1 & 2/0 & K^{2/0} & 8 & \\
\end{array}
\]

We get the same answer as for \(x_1\) in eq. (19) for the other homological elements of the same rank

\[
\begin{array}{cccccc}
\hline
\text{x} & \ \text{sdim} \ g_x & \ \text{g}_x & \ \text{rank} \ \text{ad}_x & \ \text{rank}_M \ x \\
\hline
\text{x}_1 & 15/8 & \text{psl}(1|4) & 24 & 10 \\
\text{x} = \text{x}_4 + \text{x}_5 + \text{x}_6 + \text{x}_7 & 7/0 & \text{psl}(3) & 32 & 16 \\
\end{array}
\]

Hypothesis 1.4.1 is confirmed, nds(\(\text{sl}(5; 3)\)) = 2.

2.4.3. \(1\text{g}(1, 6)\) of \(\text{sdim} 21\,|\,14\). For \(g = g(1, 6)\), we have \(g_0 = \text{sp}(6)\) and \(g_1 = R(\pi_4)\) as the \(g_0\)-module.

\[
\begin{array}{cccccc}
\text{x} & \ \text{sdim} \ g_x & \ \text{g}_x & \ \text{rank} \ \text{ad}_x \\
\hline
\text{x}_3, \ \text{x}_9, \ \text{x}_{12}, \ \text{x}_{15} & 7/0 & \text{psl}(3) & 14 \\
\end{array}
\]

Since \((\alpha_3, 2\alpha_1 + 4\alpha_2 + \alpha_3) = -6 \equiv 0 \mod 3\), the only possible pair of mutually orthogonal isotropic candidates: \(x_3\) and \(x_{15}\); otherwise df(\(g\)) = 1. However, rank \(\text{ad}_{x_3 + x_{15}} = \text{rank} \ \text{ad}_{x_3}\). Therefore, we have to obey the rule (15). Hypothesis 1.4.1 is confirmed.

2.4.4. \(2\text{g}(2, 3)\) of \(\text{sdim} 12\,|\,10\,|\,14\). For \(g = g(2, 3)\), we have \(g_0 = \text{gl}(3) \oplus \text{sl}(2)\) and \(g_1 = \text{psl}(3) \boxtimes \text{id}\) as the \(g_0\)-module. Clearly, \((g(1)(2, 3)/c)_0 = \text{psl}(3) \oplus \text{sl}(2)\).

For \(g = g(2, 3)\), we have

\[
\begin{array}{cccc}
\hline
\text{x} & \ \text{sdim} \ g_x & \ \text{g}_x & \ \text{rank} \ \text{ad}_x \\
\hline
\text{x}_1 & 2/4 & \text{sl}(1|1) \oplus_c \text{sl}(1|1) \oplus K^{1/4} & 10 \\
\text{x}_1 + \text{x}_2 & 1/3 & K^{1/4} & 11 \\
\end{array}
\]

Hypothesis 1.4.1 is confirmed, nds(\(g\)) = 2.

For \(h = g(1)(2, 3)/c\), we have

\[
\begin{array}{cccc}
\text{x} & \ \text{sdim} \ g_x & \ \text{h}_x & \ \text{rank} \ \text{ad}_h \\
\hline
\text{x}_1 & 0/4 & K^{0/4} & 10 \\
\text{x}_1 + \text{x}_2 & 0/4 & K^{0/4} & 10 \\
\end{array}
\]

Hypothesis 1.4.1 is confirmed, nds(\(h\)) = 1.

2.4.5. \(2\text{g}(2, 6)\) of \(\text{sdim} 36\,|\,34\,|\,20\). For \(g = g(2, 6)\), we have \(g_0 = \text{gl}(6)\) and \(g_1 = R(\pi_3)\) as the \(g_0\)-module. Clearly, \((g(1)(2, 6)/c)_0 = \text{psl}(6)\).

\(g = 2g(2, 6)\)

\[
\begin{array}{cccc}
\hline
\text{x} & \ \text{sdim} \ g_x & \ \text{g}_x & \ \text{rank} \ \text{ad}_g \\
\hline
\text{x}_2, \ \text{x}_3, \ \text{x}_4 & 16/0 & \text{gl}(3) \oplus \text{gl}(3) & 20 \\
\end{array}
\]
\[ h = 2g^{(1)}(2, 6)/c \]

\[(24)\]

| \( x \) | sdim \( h_x \) | \( h_x \) | rank \( h_x \) |
|-------|----------------|------|---------|
| \( x_2, x_3, x_4 \) | 14 | 0 | ps(3) + ps(3) | 20 |

We get the same answer for the other homological elements of the same rank. Hypothesis 1.4.1 is confirmed, \( df(g) = nds(h) = 1 \).

2.4.6. 7\( g(3, 3) \) of sdim 23/21/16. Let \( \text{spin}_{\gamma} := R(\pi_3) \). We have

\[ g(3, 3)_0 = (\mathfrak{o}(7) \oplus \mathbb{K} z) \oplus \mathbb{K} d \text{ and } g(3, 3)_1 = (\text{spin}_{\gamma})_+ \oplus (\text{spin}_{\gamma})_-; \]

the action of \( d \) — the outer derivative of \( g(3, 3)^{(1)} \) — separates the identical \( \mathfrak{o}(7) \)-modules \( \text{spin}_{\gamma} \) by acting on these modules as the scalar multiplication by \( \pm 1 \), as indicated by subscripts, \( z \) spans the center of \( g(3, 3) \).

For \( g = g(3, 3) \), we have

\[(25)\]

| \( x \) | sdim \( g_x \) | \( g_x \) | rank \( g_x \) |
|-------|----------------|------|---------|
| \( x_1 \) | 9/2 | gl(3) \oplus R(\pi_3) | 14 |
| \( x_1 + x_3 \) | 7/2 | \( \text{psl}(3) \) | 16 |

Hypothesis 1.4.1 is confirmed, \( nds(g) = 2 \).

For \( h = g^{(1)}(3, 3)/c \), we have

\[(26)\]

| \( x \) | sdim \( h_x \) | \( h_x \) | rank \( h_x \) |
|-------|----------------|------|---------|
| \( x_1 \) | 7/2 | \( \text{psl}(3) \oplus \mathbb{K}^0^2 \) | 14 |
| \( x_1 + x_4 \) | 7/2 | \( \text{psl}(3) \oplus \mathbb{K}^0^2 \) | 14 |

Hypothesis 1.4.1 is confirmed, \( nds(h) = 1 \).

2.4.7. 2\( g(3, 6) \) of sdim 36/40. For \( g = g(3, 6) \), we have \( g_0 = \mathfrak{sp}(8) \) and \( g_1 = R(\pi_3) \) as the \( g_0 \)-module.

\[(27)\]

| \( x \) | sdim \( g_x \) | \( g_x \) | rank \( g_x \) |
|-------|----------------|------|---------|
| \( x_1 \) | 10/14 | \( g^{(1)}(2, 3)/c \) | 26 |
| \( x_1 + x_4 \) | 0/4 | \( \mathbb{K}^0^4 \) | 36 |

We get the same answer for the other homological elements of the same rank. Hypothesis 1.4.1 is confirmed, \( nds(g(3, 6)) = 2 \).

2.4.8. 6\( g(4, 3) \) of sdim 24/26. For \( g = g(4, 3) \), we have \( g_0 = \mathfrak{sp}(6) \oplus \mathfrak{s}(2) \) and \( g_1 = R(\pi_2) \oplus \text{id} \) as the \( g_0 \)-module.

\[(28)\]

| \( x \) | sdim \( g_x \) | \( g_x \) | rank \( g_x \) |
|-------|----------------|------|---------|
| \( x_1, x_2, x_3, x_4, x_8 \) | 6/8 | \( \text{psl}(2)/2 \) | 18 |
| \( x_{12}, x_{17}, x_{20}, x_{22} \) | 12/8 | \( \mathbb{K}^0^2 \) | 24 |

We get the same answer for the other homological elements of the same rank. Hypothesis 1.4.1 is confirmed, \( nds(g(4, 3)) = 2 \).
2.4.9. **13g(8, 3) of sdim 55|50.** We have $g_0 = f(4) \oplus sl(2)$ and $g_1 = R(\pi_4) \boxtimes id$ as the $g_0$-module.

\[
\begin{array}{ccc}
x & \text{sdim } g_x & \text{rank } \text{ad}_x \\
x_2, x_3, x_4, x_5 & 21|16 & g^{(1)}(2, 3)/\mathfrak{c} \\
x_2 + x_5, x_3 + x_5 & 7|2 & \mathfrak{psl}(3) \oplus \mathbb{R}^{1, 2} \\
\end{array}
\]

(29)

We get the same answer for the other homological elements of the same rank. Hypothesis 1.4.1 is confirmed, nds($g$) = 2.

2.4.10. **2g(4, 6) of sdim 66|32.** We have $g_0 = o(12)$ and $g_1 = R(\pi_5)$ as the $g_0$-module.

\[
\begin{array}{ccc}
x & \text{sdim } g_x & \text{rank } \text{ad}_x \\
x_3 & 34|0 & \mathfrak{psl}(6) \\
\end{array}
\]

(30)

We get the same answer for the other homological elements of the same rank. Hypothesis 1.4.1 is confirmed, nds($g$) = 1.

2.4.11. **4g(6, 6) of sdim 78|64.** We have

\[
g(6, 6)_0 = o(13) \text{ and } g(6, 6)_1 = \text{spin}_{13} := R(\pi_6) \text{ as the } g_0 \text{-module.}
\]

\[
\begin{array}{ccc}
x & \text{sdim } g_x & \text{rank } \text{ad}_x \\
x_1 & 34|20 & g^{(1)}(2, 6)/\mathfrak{c} \\
x_1 + x_3, x_1 + x_4 & 14|0 & \mathfrak{psl}(3) \oplus \mathfrak{psl}(3) \\
\end{array}
\]

(31)

We get the same answer for the other homological elements of the same rank. By Hypothesis 1.4.1, nds($g$) = 2, and we are done.

2.4.12. **5g(8, 6) of sdim 133|56.** We have $g_0 = e(7)$ and $g_1 = R(\pi_1)$ as the $g_0$-module.

\[
\begin{array}{ccc}
x & \text{sdim } g_x & \text{rank } \text{ad}_x \\
x_2 & 77|0 & e^{(1)}(6)/\mathfrak{c} \\
\end{array}
\]

(32)

We get the same answer for the other homological elements of the same rank. Hypothesis 1.4.1 is confirmed, nds($g$) = 1. (Recall that dim $e(6) = 79/77|0$ for $p = 3$, whereas dim $e(6) = 78$ for $p \neq 3$.)

2.5. **$p = 2$.** Numbering of vertices of the Dynkin diagram of $e$ type superalgebras goes along the string, starting from the end-vertex of the short branch, the end-vertex connected by an edge with the branching point is the last one.

2.5.1. **bgl(3; $\alpha$), where $\alpha \neq 0, 1$; sdim = 10/8|8.** The roots of $bgl(3; \alpha)$ are the same as those of $osp(4|2; \alpha)$ of sdim = 9|8 with the same division into even and odd ones.

We consider the following Cartan matrix and the corresponding positive root vectors

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & \alpha \\
0 & \alpha & 0
\end{pmatrix}
\]

\[
\begin{array}{c|c|c}
\text{the root vectors} & 0 & 1 \\
\hline
x_1 & 0 & 0 \\
x_2 & 0 & 0 \\
x_3 & 0 & 0 \\
x_4 + [x_1, x_2], x_5 + [x_2, x_3], x_6 & 0 & 0 \\
x_7 & 0 & 0
\end{array}
\]

\[
\begin{array}{c|c|c}
\text{the roots} & 0 & 0 \\
\hline
0_1, 0_2, a_2 & 0 & 0 \\
o_1 + a_2, a_2 + a_3, a_1 + 2a_2 + a_3 & 0 & 0 \\
o_1 + a_2, a_2 + a_3, a_1 + 2a_2 + a_3 & 0 & 0
\end{array}
\]

In what follows, let $M$ be an irreducible highest weight $g$-module of the least dimension.
\( g = \mathfrak{bgl}(3; \alpha) \), let \( M \) be an irreducible module of \( \text{sdim} \ M = 4|4 \), see \cite{BGKL}. We have

\[
\begin{array}{|c|c|c|c|}
\hline
x & \mathfrak{g}_x & \text{rank}_g x & \text{rank}_M x \\
\hline
x_1 & \mathbb{K}^{4|2} & 5 & 2 \\
x_2 & \mathbb{K}^{4|0} & 7 & 3 \\
x_3 & \mathbb{K}^{4|2} & 5 & 2 \\
x_1 + x_3 & \mathbb{K}^{4|0} & 7 & 4 \\
\hline
\end{array}
\]

(34)

For \( \mathfrak{h} = \mathfrak{bgl}^{(1)}(3; \alpha)/\mathfrak{c} \), the highest weight of \( M \) is \((0, 1, 0)\) (with respect to the maximal torus of \( \mathfrak{bgl}(3; \alpha) \)) and \( \text{dim} \ M = 8|6 \), see \cite{BGKL}.

\[
\begin{array}{|c|c|c|c|}
\hline
x & \mathfrak{g}_x & \text{rank}_g x & \text{rank}_M x \\
\hline
x_1 & \mathbb{K}^{2|2} & 6 & 8 \\
x_2 & \mathbb{K}^{0|0} & 8 & 6 \\
x_3 & \mathbb{K}^{2|2} & 6 & 4 \\
x_1 + x_3 & \mathbb{K}^{2|2} & 6 & 6 \\
\hline
\end{array}
\]

(35)

We see that \( \text{rank}_M x \) does not give a conclusive information as to what \( \mathfrak{g}_x \) is, unlike \( \text{rank}_g x \). Hypothesis \[4.1.1\] item 1) is confirmed: \( \text{df}(\mathfrak{g}) = 2 \), item 2) is NOT confirmed \( \text{nds}(\mathfrak{h}) = 2 \).

2.5.2. \( \mathfrak{bgl}(4; \alpha) \), where \( \alpha \neq 0, 1; \text{sdim} = 18|16 \). The roots of \( \mathfrak{bgl}(4; \alpha) \) are the same as those of \( \mathfrak{u}(4; \alpha) \), but divided into even and odd ones. We consider the following Cartan matrix and the corresponding positive root vectors

\[
\begin{pmatrix}
0 & \alpha & 0 & 0 \\
\alpha & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(36)

Let \( M \) be the irreducible \( \mathfrak{g} \)-module with highest weight \((0, 0, 0, 1)\), then \( \text{dim} M = 16|16 \), see \cite{BGKL}. Let \( V \) be the tautological \( \mathfrak{sl}(3) \)-module.

\[
\begin{array}{|c|c|c|c|}
\hline
x & \dim \mathfrak{g}_x & \text{rank}_g x & \text{rank}_M x \\
\hline
x_1 & 8|6 & 10 & 10 \\
x_3, x_4 & 8|6 & 10 & 8 \\
x_{14} & 8|6 & 16 & 8 \\
x_2, x_8, x_9 & 4|2 & 14 & 12 \\
x_{12} & 4|2 & 14 & 14 \\
x_1 + x_3 & 2|0 & 16 & 14 \\
x_1 + x_4 & 2|0 & 16 & 14 \\
x_2 + x_4 & 2|0 & 16 & 14 \\
\hline
\end{array}
\]

(37)

Hypothesis \[4.1.1\] is NOT confirmed: \( \text{nds}(\mathfrak{g}) = 3 \). We see that \( \text{rank}_M x \) does not give a conclusive information as to what \( \mathfrak{g}_x \) is, unlike \( \text{rank}_g x \).

2.5.3. \( \mathfrak{c}(6, 1) \) of \( \text{sdim} = 46|32 \). We have \( \mathfrak{g}_0 \simeq \mathfrak{oc}(2; 10) \oplus \mathbb{K}z \) and \( \mathfrak{g}_1 \) is a reducible module of the form \( R(\pi_4) \oplus R(\pi_5) \) with the two highest weight vectors: \( y_5 \) and

\[
x_{36} = [[x_4, x_5], [x_6, [x_2, x_3]], [[x_3, [x_1, x_2]], [x_6, [x_3, x_4]]]].
\]
Let $M$ be the irreducible $g$-module with highest weight $(1,0,0,0,0,0)$, then \( \dim M = 16|11 \), see [BGKL]. The Cartan matrix of $\mathfrak{e}(6,1)$ we consider is that of $\mathfrak{e}(6)$ with the vector of parities of simple roots being 111100.

| $x$ | $\dim g_x$ | $\mathfrak{g}_x$ | $\mathfrak{r}_x$ | $\mathfrak{r}_{\mathfrak{m}} x$ | $\mathfrak{g}_x$ |
|-----|-------------|------------------|------------------|------------------|------------------|
| $x_1, x_2, x_3, x_4, x_{10}, x_{11}, x_{12}, x_{13}, x_{17}, x_{19}, x_{20}, x_{22}, x_{27}, x_{29}, x_{31}, x_{34}$ | 24|10 | 22 | 6 | $\mathfrak{psl}(5|1)$ |
| $x_1 + x_3, x_2 + x_4$ | 14|0 | 32 | 10 | $\mathfrak{psl}(4)$ |

Hypothesis 1.4.1 is confirmed, \( \mathfrak{nds}(\mathfrak{g}) = 2 \).

2.5.4. $\mathfrak{e}(6,6)$ of $\text{sdim} = 38|40$. In this case, $\mathfrak{g}(B) \simeq \mathfrak{gl}(6)$, see eq. (10). The module $\mathfrak{g}_1$ is irreducible with the highest weight vector

\[
x_{35} = \left[[x_3, x_6], [x_4, [x_2, x_3]], [[x_4, x_5], [x_3, [x_1, x_2]]]\right] \text{ of weight } (0, 0, 1, 0, 1).
\]

Let $M$ be the irreducible $g$-module with highest weight $(0,0,0,0,0,1)$, its dimension is \( \dim M = 15|12 \), see [BGKL]. The Cartan matrix of $\mathfrak{e}(6,6)$ we consider is that of $\mathfrak{e}(6)$ with the vector of parities of simple roots being 111111.

| $x$ | $\dim g_x$ | $\mathfrak{g}_x$ | $\mathfrak{r}_x$ | $\mathfrak{r}_{\mathfrak{m}} x$ | $\mathfrak{g}_x$ |
|-----|-------------|------------------|------------------|------------------|------------------|
| $x_1$ | 16|18 | 22 | 6 | $\mathfrak{psl}(3|3)$ |
| $x_1 + x_3$ | 6|8 | 32 | 10 | $\mathfrak{psl}(2\mathfrak{2})$ |
| $x_1 + x_3 + x_5$ | 0|2 | 38 | 12 | $\mathfrak{K}^{0|2}$ |

We get the same answer for the other elements of the same rank. Hypothesis 1.4.1 is confirmed, \( \mathfrak{nds}(\mathfrak{g}) = 3 \).

2.5.5. $\mathfrak{e}(7,1)$ of $\text{sdim} = 80/78|54$. The Cartan matrix of $\mathfrak{e}(7,1)$ is that of $\mathfrak{e}(7)$ with the parities of simple roots 1111001.

| $x$ | $\dim g_x$ | $\mathfrak{g}_x$ | $\mathfrak{r}_x$ | $\mathfrak{r}_{\mathfrak{m}} x$ | $\mathfrak{g}_x$ |
|-----|-------------|------------------|------------------|------------------|------------------|
| $x_1$ | 46|20 | 34 | $\mathfrak{o}_{10}^{(1)}(2|10)$ |
| $x_1 + x_3$ | 28|2 | 52 | $\mathfrak{o}_{10}^{(1)}(8) \oplus \mathfrak{g}(1|1)$ |
| $x_1 + x_3 + x_7$ | 26|0 | 54 | $\mathfrak{o}_{10}^{(2)}(8)/c$ |

We get the same answer for the other elements of the same rank. Hypothesis 1.4.1 is confirmed, \( \mathfrak{nds}(\mathfrak{g}) = 3 \).

Consider $\mathfrak{h} := (\mathfrak{e}^{(1)}(7,1))/c$. We see that \( \mathfrak{nds}(\mathfrak{h}) = 2 \), hypothesis 1.4.1 is confirmed.

| $x$ | $\dim h_x$ | $\mathfrak{r}_x$ | $\mathfrak{h}_x$ | $\mathfrak{r}_{\mathfrak{h}} x$ | $\mathfrak{h}_x$ |
|-----|-------------|------------------|------------------|------------------|------------------|
| $x_1$ | 44|20 | 34 | $\mathfrak{o}_{10}^{(2)}(2|10)/c$ |
| $x_1 + x_3$ | 26|2 | 52 | $\mathfrak{o}_{10}^{(2)}(8)/c \oplus \mathfrak{K}^{0|2}$ |
| $x_1 + x_3 + x_7$ | 26|2 | 52 | $\mathfrak{o}_{10}^{(2)}(8)/c \oplus \mathfrak{K}^{0|2}$ |
2.5.6. \(\mathfrak{e}(7, 6)\) of \(\text{sdim} = 70/68|64\). The Cartan matrix of \(\mathfrak{e}(7, 6)\) we consider is that of \(\mathfrak{e}(7)\) with the vector of parities of simple roots being 0101010.

| \(x\) | \(\dim \mathfrak{g}_x\) | \(\text{rank}_3 x\) | \(\mathfrak{g}_x\) |
|------|------------|--------|---------|
| \(x_2\) | 36/30 | 34 | \(\mathfrak{pe}^{(1)}(6)\) |
| \(x_2 + x_4\) | 18/12 | 52 | \(\mathfrak{pe}^{(1)}(4) \oplus \mathfrak{gl}(2)\) |
| \(x_2 + x_4 + x_6\) | 8/2 | 62 | solvable, \(\dim \mathfrak{c} = 1|2\), \(\dim \mathfrak{g}_x^{(i)} = \begin{cases} 7|0 & \text{if } i = 1, \\ 1|0 & \text{if } i = 2, \\ 0|0 & \text{if } i = 3. \end{cases}\) |

We get the same answer for the other elements of the same rank. Hypothesis [1.4.1] is confirmed, \(\text{nds}(\mathfrak{g}) = 3\).

Consider \(\mathfrak{h} := (\mathfrak{e}(1)^{(7, 6)})/\mathfrak{c}\). We see that \(\text{nds}(\mathfrak{h}) = 3\), hypothesis [1.4.1] is confirmed.

| \(x\) | \(\dim \mathfrak{h}_x\) | \(\text{rank}_3 x\) | \(\mathfrak{h}_x\) |
|------|------------|--------|---------|
| \(x_2\) | 34/30 | 34 | \(\mathfrak{pe}^{(2)}(6)/\mathfrak{c}\) |
| \(x_2 + x_4\) | 16/12 | 52 | \(\mathfrak{pe}^{(2)}(4)/\mathfrak{c} \oplus \mathbb{K}^{2|0}\) |
| \(x_2 + x_4 + x_6\) | 6/2 | 62 | \(\mathbb{K}^{2|0}\) |

2.5.7. \(\mathfrak{e}(7, 7)\) of \(\text{sdim} = 64/62|70\). The Cartan matrix of \(\mathfrak{e}(7, 7)\) we consider is that of \(\mathfrak{e}(7)\) with the vector of parities of simple roots being 1111111.

| \(x\) | \(\dim \mathfrak{g}_x\) | \(\text{rank}_3 x\) | \(\mathfrak{g}_x\) |
|------|------------|--------|---------|
| \(x_1\) | 30/36 | 34 | \(\mathfrak{oo}_{\text{III}}^{(1)}(6|6)\) |
| \(x_1 + x_3\) | 12/18 | 52 | \(\mathfrak{oo}_{\text{III}}^{(1)}(4|4) \oplus \mathfrak{gl}(1|1)\) |
| \(x_1 + x_3 + x_5\) | 2/8 | 62 | solvable, \(\dim \mathfrak{c} = 1|2\), \(\dim \mathfrak{g}_x^{(i)} = \begin{cases} 1|6 & \text{if } i = 1, \\ 1|0 & \text{if } i = 2, \\ 0|0 & \text{if } i = 3. \end{cases}\) |
| \(x_1 + x_3 + x_5 + x_7\) | 0/6 | 64 | \(\mathbb{K}^{0|6}\) |

We get the same answer for the other elements of the same rank. Hypothesis [1.4.1] is confirmed, \(\text{nds}(\mathfrak{g}) = 3\).

Consider \(\mathfrak{h} := \mathfrak{e}(1)^{(7, 7)}/\mathfrak{c}\). We see that \(\text{nds}(\mathfrak{h}) = 3\), hypothesis [1.4.1] is confirmed.

| \(x\) | \(\dim \mathfrak{h}_x\) | \(\text{rank}_3 x\) | \(\mathfrak{h}_x\) |
|------|------------|--------|---------|
| \(x_1\) | 28/36 | 34 | \(\mathfrak{oo}_{\text{III}}^{(2)}(6|6)/\mathfrak{c}\) |
| \(x_1 + x_3\) | 10/18 | 52 | \(\mathfrak{oo}_{\text{III}}^{(2)}(4|4)/\mathfrak{c} \oplus \mathbb{K}^{0|2}\) |
| \(x_1 + x_3 + x_5\) | 0/8 | 62 | \(\mathbb{K}^{0|8}\) |
| \(x_1 + x_3 + x_5 + x_7\) | 0/8 | 62 | \(\mathbb{K}^{0|8}\) |
2.5.8. \( e(8, 1) \) of \( \text{sdim} = 136|112 \). We have (see eq. (10)) \( \mathfrak{g}(B) \simeq e(7) \). The Cartan matrix of \( e(8, 1) \) we consider is that of \( e(8) \) with the vector of parities of simple roots being 11001111.

\[
\begin{array}{|c|c|c|c|}
\hline
x & \dim \mathfrak{g}_x & \text{rank}_{\mathfrak{g}} x & \mathfrak{g}_x \\
\hline
x_1 & 78|54 & 58 & e^{(1)}(7, 1)/e \\
x_1 + x_5 & 44|20 & 92 & \phi_\mathfrak{m}^{(2)}(2|10)/e \\
x_1 + x_5 + x_7 & 26|2 & 110 & \phi_\mathfrak{m}^{(2)}(8)/e \oplus \mathbb{K}^{0|2} \\
x_1 + x_5 + x_7 + x_8 & 6|2 & 120 & \mathbb{K}^{0|2} \\
\hline
\end{array}
\]

We get the same answer for the other elements of the same rank. By Hypothesis 1.4.1, \( \text{nds}(\mathfrak{g}) = 4 \).

2.5.9. \( e(8, 8) \) of \( \text{sdim} = 120|128 \). In the \( \mathbb{Z} \)-grading with the Cartan matrix with the parities of simple roots 000000001 and with \( \deg e_i^\pm = \pm 1 \) and \( \deg e_i^\pm = 0 \) for \( i \neq 8 \), we have \( \mathfrak{g}_0 = \mathfrak{g}(8) \).

The Lie algebra \( \mathfrak{g}_0 \) is isomorphic to \( \mathfrak{o}_\mathfrak{m}^{(2)}(16) \in \mathbb{K}d \), where \( d = E_6, 6 + \cdots + E_{13,13}, \) and \( \mathfrak{g}_1 \) is an irreducible \( \mathfrak{g}_0 \)-module with the highest weight element \( x_{120} \) of weight \( (1, 0, \ldots, 0) \) with respect to \( h_1, \ldots, h_8; \) \( \mathfrak{g}_1 \) also possesses a lowest weight vector. The Cartan matrix of \( e(8, 8) \) we consider is that of \( e(8) \) with the vector of parities of simple roots being 11111111.

\[
\begin{array}{|c|c|c|c|}
\hline
x & \dim \mathfrak{g}_x & \text{rank}_{\mathfrak{g}} x & \mathfrak{g}_x \\
\hline
x_1 & 62|70 & 58 & e^{(1)}(7, 7)/e \\
x_1 + x_3 & 28|36 & 92 & \phi_\mathfrak{m}^{(2)}(6|6)/e \\
x_1 + x_3 + x_5 & 10|18 & 110 & \phi_\mathfrak{m}^{(2)}(4|4)/e \oplus \mathbb{K}^{0|2} \\
x_1 + x_3 + x_5 + x_7 & 0|8 & 120 & \mathbb{K}^{0|8} \\
\hline
\end{array}
\]

We get the same answer for the other elements of the same rank. By Hypothesis 1.4.1, \( \text{nds}(\mathfrak{g}) = 4 \).

3. \( \mathfrak{psl}(n|n) \) for \( p = 0, 2, 3, 5 \) (checked for \( n = 2, 3, 4 \))

This case was not considered in [DS] [HR]. For it, the Hypothesis on the value of defect \( \text{nds} \) is also true; the form of the answer differs from that in eq. (5). We consider the alternating format (so all Chevalley generators are odd).

For any \( x \in \{x_1, x_3\} \), we have

\[
\begin{align*}
\mathfrak{g} & \quad \text{rank ad}_x, \mathfrak{g}_x \\
\mathfrak{gl}(2|2) & \quad 5, \mathfrak{g}_x = \mathfrak{gl}(1|1) \\
\mathfrak{sl}(2|2) & \quad 6, \mathfrak{g}_x = \mathfrak{sl}(1|1) \simeq \mathfrak{hei}(0|2) \\
\mathfrak{psl}(2|2) & \quad 6, \mathfrak{g}_x = \mathbb{K}^{0|2} \\
\end{align*}
\]

For any \( x \in \{x_1, x_3, x_5\}, \) and \( y = x_1 + x_3, \) and \( z = x_1 + x_3 + x_5, \) we have

\[
\begin{align*}
\mathfrak{g} & \quad \text{rank ad}_y, \mathfrak{g}_y \\
\mathfrak{gl}(3|3) & \quad 9, \mathfrak{g}_y = \mathfrak{gl}(2|2) \\
\mathfrak{sl}(3|3) & \quad 10, \mathfrak{g}_y = \mathfrak{sl}(2|2) \\
\mathfrak{psl}(3|3) & \quad 10, \mathfrak{g}_y = \mathfrak{psl}(2|2) \\
\end{align*}
\]

\[
\begin{align*}
\mathfrak{g} & \quad \text{rank ad}_z, \mathfrak{g}_z \\
\mathfrak{gl}(3|3) & \quad 15, \mathfrak{g}_z = \mathfrak{gl}(1|1) \\
\mathfrak{sl}(3|3) & \quad 16, \mathfrak{g}_z = \mathfrak{hei}(0|2) \\
\mathfrak{psl}(3|3) & \quad 16, \mathfrak{g}_z = \mathbb{K}^{0|2} \\
\end{align*}
\]
For any \( x \in \{x_1, x_3, x_5, x_7\} \), and \( y = x_1 + x_3 \), and \( z = x_1 + x_3 + x_5 \), and \( v = x_1 + x_3 + x_5 + x_7 \), we have

\[
\begin{array}{|c|c|c|c|c|}
\hline
\g & \text{rank ad}_x, \g_x & \text{rank ad}_y, \g_y & \text{rank ad}_z, \g_z & \text{rank ad}_v, \g_v \\
\hline
\gl(4|4) & 13, \g_x = \gl(3|3) & 23, \g_y = \gl(2|2) & 29, \g_z = \gl(1|1) & 31, \g_v = 0 \\
\sl(4|4) & 14, \g_x = \sl(3|3) & 24, \g_y = \sl(2|2) & 30, \g_z = \sl(1|1) & 31, \g_v = \mathbb{K}^{0|1} \\
\psl(4|4) & 14, \g_x = \psl(3|3) & 24, \g_y = \psl(2|2) & 30, \g_z = \mathbb{K}^{0|2} & 30, \g_v = \mathbb{K}^{0|2} \\
\hline
\end{array}
\]

4. \( \psl(n|n + pk) \) for \( p = 2, 3, 5 \) and small non-zero \( n, k \)

4.1. \( p = 2 \). For \( x = x_1 \), and \( y = x_1 + x_3 \), we have

\[
\begin{array}{|c|c|c|c|}
\hline
\g & \text{rank ad}_x, \g_x & \text{rank ad}_y, \g_y \\
\hline
\gl(2|4) & 10, \gl(1|3) & 16, \gl(2) \\
\psl(2|4) & 10, \psl(1|3) & 16, \mathbb{K}^{2|0} \simeq \psl(2) \\
\hline
\gl(2|6) & 14, \gl(1|5) & 24, \gl(5) \\
\psl(2|6) & 14, \psl(1|5) & 24, \psl(4) \\
\hline
\gl(2|8) & 18, \gl(1|7) & 32, \gl(6) \\
\psl(2|8) & 18, \psl(1|7) & 32, \psl(6) \\
\hline
\end{array}
\]

For \( x = x_1, y = x_1 + x_3, \) and \( z = x_1 + x_3 + x_5 \) we have

\[
\begin{array}{|c|c|c|c|}
\hline
\g & \text{rank ad}_x, \g_x & \text{rank ad}_y, \g_y & \text{rank ad}_z, \g_z \\
\hline
\gl(3|5) & 14, \gl(2|4) & 24, \gl(1|3) & 30, \gl(2) \\
\psl(3|5) & 14, \psl(2|4) & 24, \psl(1|3) & 30, \mathbb{K}^{2|0} \simeq \psl(2) \\
\hline
\gl(3|7) & 18, \gl(2|6) & 32, \gl(1|4) & 42, \gl(4) \\
\psl(3|7) & 18, \psl(2|6) & 32, \psl(1|5) & 42, \psl(4) \\
\hline
\gl(3|9) & 22, \gl(2|8) & 50, \gl(1|7) & 54, \gl(6) \\
\psl(3|9) & 22, \psl(2|8) & 50, \psl(1|7) & 54, \psl(6) \\
\hline
\end{array}
\]

For \( x = x_1, y = x_1 + x_3, z = x_1 + x_3 + x_5, \) and \( v = x_1 + x_3 + x_5 + x_7 \) we have

\[
\begin{array}{|c|c|c|c|}
\hline
\g & \text{rank ad}_x, \g_x & \text{rank ad}_y, \g_y & \text{rank ad}_z, \g_z & \text{rank ad}_v, \g_v \\
\hline
\gl(4|6) & 18, \gl(3|5) & 32, \gl(2|4) & 42, \gl(1|3) & 48, \gl(2) \\
\psl(4|6) & 18, \psl(3|5) & 32, \psl(2|4) & 42, \psl(1|3) & 48, \mathbb{K}^{2|0} \simeq \psl(2) \\
\hline
\gl(4|8) & 22, \gl(3|7) & 40, \gl(2|6) & 54, \gl(1|5) & 64, \gl(4) \\
\psl(4|8) & 22, \psl(3|7) & 40, \psl(2|6) & 54, \psl(1|5) & 64, \psl(4) \\
\hline
\gl(4|10) & 26, \gl(3|9) & 48, \gl(2|8) & 66, \gl(1|7) & 80, \gl(6) \\
\psl(4|10) & 26, \psl(3|9) & 48, \psl(2|8) & 66, \psl(1|7) & 80, \psl(6) \\
\hline
\end{array}
\]

4.2. \( p = 3 \). For \( x = x_1, \) and \( y = x_1 + x_3, \) we have

\[
\begin{array}{|c|c|c|}
\hline
\g & \text{rank ad}_x, \g_x & \text{rank ad}_y, \g_y \\
\hline
\gl(2|5) & 12, \gl(1|4) & 20, \gl(3) \\
\psl(2|5) & 12, \psl(1|4) & 20, \psl(3) \\
\hline
\gl(2|8) & 18, \gl(1|7) & 32, \gl(6) \\
\psl(2|8) & 18, \psl(1|7) & 32, \psl(6) \\
\hline
\gl(2|11) & 24, \gl(1|10) & 44, \gl(9) \\
\psl(2|11) & 24, \psl(1|10) & 44, \psl(9) \\
\hline
\end{array}
\]
For \( x = x_1, y = x_1 + x_3, \) and \( z = x_1 + x_3 + x_5, \) we have

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\mathfrak{g} & \text{rank ad}_x, \mathfrak{g}_x & \text{rank ad}_y, \mathfrak{g}_y & \text{rank ad}_z, \mathfrak{g}_z & \text{rank ad}_v, \mathfrak{g}_v \\
\hline
\text{gl}(3|6) & 16, \text{gl}(2|5) & 28, \text{gl}(1|4) & 36, \text{gl}(3) & \\
\text{psl}(3|6) & 16, \text{psl}(2|5) & 28, \text{psl}(1|4) & 36, \text{psl}(3) & \\
\hline
\text{gl}(3|9) & 22, \text{gl}(2|8) & 40, \text{gl}(1|7) & 54, \text{gl}(6) & \\
\text{psl}(3|9) & 22, \text{psl}(2|8) & 40, \text{psl}(1|7) & 54, \text{psl}(6) & \\
\hline
\text{gl}(3|12) & 28, \text{gl}(2|11) & 52, \text{gl}(1|10) & 72, \text{gl}(9) & \\
\text{psl}(3|12) & 28, \text{psl}(2|11) & 52, \text{psl}(1|10) & 72, \text{psl}(9) & \\
\hline
\end{array}
\]

(45)

For \( x = x_1, y = x_1 + x_3, z = x_1 + x_3 + x_5, \) and \( v = x_1 + x_3 + x_5 + x_7, \) we have

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\mathfrak{g} & \text{rank ad}_x, \mathfrak{g}_x & \text{rank ad}_y, \mathfrak{g}_y & \text{rank ad}_z, \mathfrak{g}_z & \text{rank ad}_v, \mathfrak{g}_v \\
\hline
\text{gl}(4|7) & 20, \text{gl}(3|6) & 36, \text{gl}(2|5) & 48, \text{gl}(1|4) & 56, \text{gl}(3) & \\
\text{psl}(4|7) & 20, \text{psl}(3|6) & 36, \text{psl}(2|5) & 48, \text{psl}(1|4) & 56, \text{psl}(3) & \\
\hline
\text{gl}(4|10) & 26, \text{gl}(3|9) & 48, \text{gl}(2|8) & 66, \text{gl}(1|7) & 80, \text{gl}(6) & \\
\text{psl}(4|10) & 26, \text{psl}(3|9) & 48, \text{psl}(2|8) & 66, \text{psl}(1|7) & 80, \text{psl}(6) & \\
\hline
\end{array}
\]

(46)

4.3. \( p = 5. \) For \( x = x_1, \) we have

\[
\begin{array}{|c|c|c|}
\hline
\mathfrak{g} & \text{rank ad}_x, \mathfrak{g}_x \\
\hline
\text{gl}(1|6) & 12, \text{gl}(5) \\
\text{psl}(1|6) & 12, \text{psl}(5) \\
\hline
\text{gl}(1|11) & 22, \text{gl}(10) \\
\text{psl}(1|11) & 22, \text{psl}(10) \\
\hline
\end{array}
\]

(47)

For \( x = x_1, \) and \( y = x_1 + x_3, \) we have

\[
\begin{array}{|c|c|c|}
\hline
\mathfrak{g} & \text{rank ad}_x, \mathfrak{g}_x \\
\hline
\text{gl}(2|7) & 16, \text{gl}(1|6) \\
\text{psl}(2|7) & 16, \text{psl}(1|6) \\
\hline
\text{gl}(2|12) & 26, \text{gl}(1|11) \\
\text{psl}(2|12) & 26, \text{psl}(1|11) \\
\hline
\end{array}
\]

(48)

For \( x = x_1, y = x_1 + x_3, \) and \( z = x_1 + x_3 + x_5, \) we have

\[
\begin{array}{|c|c|c|}
\hline
\mathfrak{g} & \text{rank ad}_x, \mathfrak{g}_x \\
\hline
\text{gl}(3|8) & 20, \text{gl}(2|7) \\
\text{psl}(3|8) & 20, \text{psl}(2|7) \\
\hline
\end{array}
\]

(49)

For \( x = x_1, y = x_1 + x_3, z = x_1 + x_3 + x_5, \) and \( v = x_1 + x_3 + x_5 + x_7, \) we have

\[
\begin{array}{|c|c|c|}
\hline
\mathfrak{g} & \text{rank ad}_x, \mathfrak{g}_x \\
\hline
\text{gl}(4|9) & 24, \text{gl}(3|8) \\
\text{psl}(4|9) & 24, \text{psl}(3|8) \\
\hline
\end{array}
\]

(50)

5. Comments

5.1. Two types of Lie superalgebras. The set of simple \( \mathbb{Z} \)-graded Lie (super)algebras of finite dimension or of polynomial growth and their deformations is a disjoint (at least, if \( p > 3 \)) union of two subsets:

(S) with a symmetric set of roots relative the maximal torus,

i.e., with every root \( \alpha \) there is a root \( -\alpha \) of the same multiplicity;

(N) with a non-symmetric set of roots.
Some (or rather MOST) of the methods used to investigate Lie (super)algebras of type (S) rely on the existence of Casimir elements (in most cases, just one (degree-2) Casimir suffices) and symmetry of the root lattice.

In the study of Lie (super)algebras of type (N) — Lie (super)algebras of vector fields with polynomial or formal coefficients (briefly referred to as vectorial Lie (super)algebras) and modules over them, these Casimirs are completely or partly absent\footnote{Such as the possibility to use the center of the universal enveloping algebra $U(g)$ which is trivial. However, in $\text{Ser}$, Serganova showed that, at least for $g = \mathfrak{pe}(n)$, the superalgebra $U(g)$, whose center is trivial, should be replaced by $\overline{U}(g) := U(g)/\tau(U(g))$, whose center is sufficiently big, where $\tau(A)$ is the radical of the algebra $A$. It is very tempting to investigate applicability of Serganova’s idea to other Lie (super)algebras $g$ with trivial center of $U(g)$. And with non-trivial centers as well.}, and hence can not be used. So, the problems of representation theory of the “non-symmetric” (in particular, vectorial) Lie (super)algebras seem to be much more difficult than for the symmetric Lie (super)algebras.

Fortunately, over $\mathbb{C}$, the description of irreducible continuous (with respect to the natural $(x)$-adic topology) modules over simple vectorial Lie algebras is very simple — modulo representation theory of finite-dimensional simple Lie algebras; superization is similar, see [GLS].

5.2. Two types of homological elements. Another tool for the study of Lie (super)algebras is certain homology relative an element $x$. Here we considered an odd $x$ such that $x^2 = 2$.

For (N)-type simple modular Lie algebras for $p > 2$, one considers the sandwich elements $x$, i.e., such that $(\text{ad}_x)^2 = 0$. S. Kirillov proved that the normalizer of the sandwich subalgebra\footnote{In a totally different setting and over $\mathbb{C}$, the term sandwich algebra is used in [Ch], causing confusion.} is the maximal subalgebra for $p > 3$, see [Kir, KirS].

5.2.1. On inhomogeneous ad-homological elements when $p = 2$. We say that a non-zero $x \in g$ is ad-homological if $(\text{ad}_x)^2 = 0$. For $g = \mathfrak{gl}(2|6)$, inhomogeneous (with respect to parity) elements $x = x_1 + x_3 + x_5$ and $y = x_1 + x_3 + x_5 + x_7$ are ad-homological. We get $g_x = \mathfrak{gl}(2)$ and $g_y = 0$, where $\text{rank}_g \text{ad}_x = 30$, and $\text{rank}_g \text{ad}_y = 32$. The meaning of ad-homological elements and their homology (when $p = 2$) is unknown. Observe that nobody computed the homology corresponding to sandwiches; their meaning is also unknown. Both sandwiches and DS-homology lead to what is called support varieties. There are several non-equivalent definitions of these varieties: compare [DS] with [BaKN, BoKN, DK1, Ba, Bal].

5.3. Other definitions of the defect. In [BoKN], there is given another definition of defect def($g$), equivalent to the above one df($g$) for the simple (relatives of) Lie superalgebras $g$ with Cartan matrix over $\mathbb{C}$, but with a wider range of application:

\begin{equation}
(51) \quad \text{def}(g) := \text{Krull dim}(H^*(g, g_0; \mathbb{C})) \text{ for the trivial } g\text{-module } \mathbb{C}.
\end{equation}

In [BoKN], it is mentioned that this definition is applicable to Lie superalgebras without Cartan matrix, such as $\mathfrak{pe}(n)$, $\mathfrak{q}(n)$ and their (not necessarily simple) relatives, and $\mathfrak{gl}(n|n)$.

We observe that, moreover, the invariant (51) is meaningful for any $\mathbb{Z}/2$-grading of any Lie superalgebra and even for $\mathbb{Z}/2$-graded Lie algebras. A priori, this cohomology is a supercommutative superalgebra whose Krull dimension was recently defined, as on cue, see [MZ].

For $g = \mathfrak{pe}(n)$ and $\mathfrak{q}(n)$, and their relatives, def($g$) was computed in [BoKN], but $g_x$ was not computed for any $x$. For $\mathfrak{q}(n)$, and its simple subquotient, $g_x$ is computed in [KLS].

The third definition of defect — nds — is given by formula (6).

Acknowledgements. For the possibility to conduct difficult computations of this research we are grateful to M. Al Barwani, Director of the High Performance Computing resources at
NYUAD. We thank E. Herscovich for helpful comments. D.L. was partly supported by the grant AD 065 NYUAD. A. K. was partly supported by the QuantiXLie Centre of Excellence, a project cofinanced by the Croatian Government and European Union through the European Regional Development Fund — the Competitiveness and Cohesion Operational Programme (KK.01.1.1.01.0004).

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