On a vector-host epidemic model with spatial structure

Pierre Magal1, Glenn Webb2 and Yixiang Wu2

1 Mathematics Department, University of Bordeaux, Bordeaux, France
2 Mathematics Department, Vanderbilt University, Nashville, TN, United States of America

E-mail: yixiang.wu@vanderbilt.edu

Received 19 April 2018, revised 23 August 2018
Accepted for publication 17 September 2018
Published 15 November 2018

Recommended by Professor Michael Jeffrey Ward

Abstract
In this paper, we study a reaction–diffusion vector-host epidemic model. We define the basic reproduction number \( R_0 \) and show that \( R_0 \) is a threshold parameter: if \( R_0 \leq 1 \) the disease free equilibrium is globally stable; if \( R_0 > 1 \) the model has a unique globally stable positive equilibrium. Our proof combines arguments from monotone dynamical system theory, persistence theory, and the theory of asymptotically autonomous semiflows.

Keywords: reaction–diffusion, epidemic models, global stability, basic reproduction number
Mathematics Subject Classification numbers: 35B40, 35P05, 35Q92

1. Introduction
In recent years, many authors (e.g. [1, 5–8, 9–14, 16, 18, 20, 22, 23, 29, 34, 35, 37, 38, 41]) have proposed reaction–diffusion models to study the transmission of diseases in spatial settings. Among them, Fitzgibbon et al [11, 12] applied a reaction–diffusion system on non-coincident domains to describe the circulation of diseases between two locations; Lou and Zhao [22] proposed a reaction–diffusion model with delay and nonlocal terms to study the spatial spread of malaria; and Vaidya, Wang and Zou [34] studied the transmission of avian influenza in wild birds with a reaction–diffusion model with spatial heterogeneous coefficients.

New formulations of diffusive epidemic models have been used recently to study epidemics in spatial contexts. In [23] the spatial spread of influenza in Puerto Rico was analyzed using a diffusive SIR model based on geographical population data. In [14] the effectiveness of a diffusive vector-host epidemic model was demonstrated in understanding the recent Zika outbreak in Rio De Janeiro. In these works it was shown that the beginning location and
magnitude of an epidemic can have significant impact on the spatial development and final size of the epidemic. The simulations in these works highlighted the limitations of incomplete spatial epidemic data in the applications of diffusive models to real world situations. Despite these limitations, spatial epidemic models offer the possibility of better understanding of the evolution of epidemic outbreaks in regions, and the possibility of mitigating their greater regional impact with intervention measures. Our objective in this manuscript is to provide an extended analysis of the reaction–diffusion spatial epidemic model proposed in [14]. A more complete understanding of the model in [14] can help to predict the possibility that current Zika epidemics will become regionally endemic.

Suppose that individuals are living in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial \Omega$. Let $H_i(x,t), V_u(x,t)$ and $V_i(x,t)$ be the densities of infected hosts, uninfected vectors, and infected vectors at position $x$ and time $t$, respectively. Then the model proposed in [14] to study the outbreak of Zika in Rio De Janerio is the following reaction–diffusion system

$$
\begin{cases}
\frac{\partial}{\partial t} H_i - \nabla \cdot \delta_1(x) \nabla H_i = -\lambda(x) H_i + \sigma_1(x) H_u(x) V_i, & x \in \Omega, t > 0, \\
\frac{\partial}{\partial t} V_u - \nabla \cdot \delta_2(x) \nabla V_u = -\sigma_2(x) V_u H_i + \beta(x)(V_u + V_i) - \mu(x)(V_u + V_i) V_u, & x \in \Omega, t > 0, \\
\frac{\partial}{\partial t} V_i - \nabla \cdot \delta_2(x) \nabla V_i = \sigma_2(x) V_u H_i - \mu(x)(V_u + V_i) V_i, & x \in \Omega, t > 0,
\end{cases}
$$

with homogeneous Neumann boundary condition

$$
\frac{\partial}{\partial n} H_i = \frac{\partial}{\partial n} V_u = \frac{\partial}{\partial n} V_i = 0, \quad x \in \partial \Omega, t > 0,
$$

and initial condition

$$
(H_i(\cdot,0), V_u(\cdot,0), V_i(\cdot,0)) = (H_{0i}, V_{0u}, V_{0i}) \in C(\bar{\Omega}; \mathbb{R}^3),
$$

where $\delta_1, \delta_2 \in C^{1+\alpha}(\bar{\Omega}; \mathbb{R})$ are strictly positive, the functions $\lambda, \beta, \sigma_1, \sigma_2$ and $\mu$ are strictly positive and belong to $C^\alpha(\bar{\Omega}; \mathbb{R})$, and the function $H_u \in C^\alpha(\bar{\Omega}; \mathbb{R})$ is nonnegative and non-trivial. The flux of new infected humans is given by $\sigma_1(x) H_u(x) V_i(t,x)$ in which $H_u(x)$ is the density of susceptible population depending on the spatial location $x$. The main idea of this model is to assume that the susceptible human population is (almost) not affected by the epidemic during a relatively short period of time and therefore the flux of new infected is (almost) constant. Such a functional response mainly permits to take care of realistic density of population distributed in space. For Zika in Rio De Janerio the number of infected is fairly small in comparison with the number of the total population (less than 1% according to [3]). Therefore the density of susceptibles can be considered to be constant without being altered by the epidemic.

In section 2, we define the basic reproductive number $R_0$ as the spectral radius of $-CB^{-1}$, i.e. $R_0 = r(-CB^{-1})$, where $B : D(B) \subset C(\bar{\Omega}; \mathbb{R}^2) \to C(\bar{\Omega}; \mathbb{R}^2)$ and $C : C(\bar{\Omega}; \mathbb{R}^2) \to C(\bar{\Omega}; \mathbb{R}^2)$ are linear operators on $C(\bar{\Omega}; \mathbb{R}^2)$ with

$$
B = \begin{pmatrix}
\nabla \cdot \delta_1 \nabla & 0 \\
0 & \nabla \cdot \delta_2 \nabla
\end{pmatrix}
+ \begin{pmatrix}
-\lambda & \sigma_1 H_u \\
\sigma_2 V & -\mu V
\end{pmatrix}
\quad \text{and} \quad
C = \begin{pmatrix}
0 & 0 \\
0 & \sigma_2 V
\end{pmatrix},
$$

with the suitable domain $D(B)$ (see [25, 30]).

The equilibria of (1.1)–(1.3) are solutions of the following elliptic system:
\[
\begin{align*}
-\nabla \cdot \delta_1(x)\nabla H_i &= -\lambda(x)H_i + \sigma_1(x)\nabla \nabla V_i, \quad x \in \Omega, \\
-\nabla \cdot \delta_2(x)\nabla V_a &= -\sigma_2(x)V_a H_i + \beta(x)(V_a + V_i) - \mu(x)(V_a + V_i) V_a, \quad x \in \Omega, \\
-\nabla \cdot \delta_2(x)\nabla V_i &= \sigma_2(x)V_a H_i - \mu(x)(V_a + V_i) V_i, \quad x \in \Omega, \\
\frac{\partial}{\partial n} H_i &= \frac{\partial}{\partial n} V_a = \frac{\partial}{\partial n} V_i = 0, \quad x \in \partial \Omega. 
\end{align*}
\]

The system always has one trivial equilibrium \(E_0\) and a unique semi-trivial equilibrium \(E_1 = (0, 0, 0)\). In section 2, we prove that \(E_1\) is globally asymptotically stable if \(R_0 < 1\) in theorem 2.4.

Our main result is in section 3, where we show that (1.1)–(1.3) has a unique globally asymptotically stable positive equilibrium \(E_2 = (H_i, V_a, V_i)\) if \(R_0 > 1\) (see theorem 3.12). We remark that it is usually not an easy task to prove the global stability of the positive equilibrium for a three-equation parabolic system when there is no clear Lyapunov type functional. Our proof combines arguments from monotone dynamical system theory, persistence theory, and the theory of asymptotically autonomous semiflows.

We briefly summarize our idea of proof here. Adding up the second and third equations in (1.1) and letting \(V := V_a + V_i\), \(V\) satisfies the diffusive logistic equation \(\partial_t V - \nabla \cdot \delta_2\nabla V = \beta V - \mu V^2\). Since this equation has a globally stable positive equilibrium \(\hat{V}\), it is tempting to assume that the dynamics of (1.1)–(1.3) is determined by the limit system

\[
\begin{align*}
\frac{\partial}{\partial t} \hat{H}_i - \nabla \cdot \delta_1 \nabla \hat{H}_i &= -\lambda \hat{H}_i + \sigma_1 \hat{H}_i \hat{V}_i, \quad x \in \Omega, t > 0, \\
\frac{\partial}{\partial t} \hat{V}_i - \nabla \cdot \delta_2 \nabla \hat{V}_i &= \sigma_2(\hat{V} - \hat{V}_i)^+ \hat{H}_i - \mu \hat{V}_i, \quad x \in \Omega, t > 0.
\end{align*}
\]

However even for ordinary differential equation (ODE) systems, Thieme [33] gives many examples where the dynamics of the limit and original systems are quite different. A remedy to this is the theory of asymptotically autonomous semiflows (see [31, theorem 4.1]), which is generalized from the well-known theory by Markus on asymptotically autonomous ODE systems. Applying this theory, to prove the convergence of \((H_i(\cdot, t), V_i(\cdot, t))\), it suffices to show: (a) system (1.5) has a unique positive equilibrium \((\hat{H}_i, \hat{V}_i)\); (b) The equilibrium \((\hat{H}_i, \hat{V}_i)\) of (1.5) is globally stable in \(W := \{(H_{i0}, V_{i0}) \in C(\Omega; \mathbb{R}_+^2) : H_{i0} + V_{i0} \neq 0\}\); (c) The \(\omega\)-limit set of \((H_i(\cdot, t), V_i(\cdot, t))\) intersects \(W\). The proof of (a) is given in section 3.1.1. The proof of (b) is provided in section 3.1.2, where we take advantage of the monotonicity of (1.5). To show (c), we use the uniform persistence theory in [15] to obtain \(\lim \inf_{t \to \infty} \| H_i(\cdot, t) \|_{\infty} + \| V_i(\cdot, t) \|_{\infty} \geq \epsilon\) for some \(\epsilon > 0\) (see lemma 3.11). Interested readers may read the appendix on the ODE system for the idea of the proof first.

In section 4, we prove the global stability of \(E_3\) for the critical case \(R_0 = 1\). Here the main difficulty is to prove the local stability of \(E_3\) as the linearized system at \(E_3\) has principal eigenvalue equaling zero. In section 5, we give some concluding remarks. In particular, we summarize our results on the basic reproduction number \(R_0\), which will be presented in a forthcoming paper. We also remark that our idea is applicable to other models (e.g. [18, 19, 26, 28]).

2. Disease free equilibria

The objective of this section is to define the basic reproduction number and investigate the stability of the trivial and semi-trivial equilibria. The existence, uniqueness, and positivity of global classical solutions of (1.1)–(1.3) have been shown in [14]. Let \(V = V_a + V_i\). Then \(V(x, t)\) satisfies
\[\begin{aligned}
V_t - \nabla \cdot \delta_2(x) \nabla V &= \beta(x)V - \mu(x)V^2, & x \in \Omega, t > 0, \\
\frac{\partial}{\partial n} V &= 0, & x \in \partial \Omega, t > 0, \\
V(\cdot, 0) &= V_0 \in C(\bar{\Omega}; \mathbb{R}_+). 
\end{aligned}\]

The following result about (2.1) is well-known (see, e.g. [4]).

**Lemma 2.1.** For any nonnegative nontrivial initial data \(V_0 \in C(\bar{\Omega}; \mathbb{R})\), (2.1) has a unique global classic solution \(V(x, t)\). Moreover, \(V(x, t) > 0\) for all \((x, t) \in \bar{\Omega} \times (0, \infty)\) and

\[\lim_{t \to \infty} \|V(\cdot, t) - \bar{V}\|_\infty = 0,\]

where \(\bar{V}\) is the unique positive solution of the elliptic problem

\[\begin{aligned}
-\nabla \cdot \delta_2(x) \nabla \bar{V} &= \beta(x)\bar{V} - \mu(x)\bar{V}^2, & x \in \Omega, \\
\frac{\partial}{\partial n} \bar{V} &= 0, & x \in \partial \Omega. 
\end{aligned}\]

By lemma 2.1, \(V_0(x, t) + V_1(x, t) \to \bar{V}(x)\) uniformly for \(x \in \Omega\) as \(t \to \infty\) if \(V_{00} + V_{01} \neq 0\). As usual, we consider two types of equilibria for (1.1)–(1.2): disease free equilibrium (DFE) and endemic equilibrium (EE). A nonnegative solution \((\bar{H}_t, \bar{V}_n, \bar{V}_i)\) of (1.4) is a DFE if \(\bar{H}_t = \bar{V}_i = 0\), and otherwise it is an EE. By lemma 2.1, we must have \(\bar{V}_n + \bar{V}_i = \bar{V}\) or \(\bar{V}_n + \bar{V}_i = 0\). It is then not hard to show that (1.1) and (1.2) has two DFE: trivial equilibrium \(E_0 = (0, 0, 0)\) and semi-trivial equilibrium \(E_1 = (0, \bar{V}, 0)\). We denote the EE by \(E_2 = (\bar{H}, \bar{V}_n, \bar{V}_i)\), which will be proven to be unique if exists.

It is not hard to show that \(E_0\) is always unstable. Linearizing (1.1) around \(E_1\), we arrive at the following eigenvalue problem:

\[\begin{aligned}
\kappa \phi &= \nabla \cdot \delta_1 \nabla \varphi - \lambda \varphi + \sigma_1 H_n \psi, & x \in \Omega, \\
\kappa \psi &= \nabla \cdot \delta_2 \nabla \varphi - \sigma_2 V \varphi + \beta(\varphi + \psi) - 2\mu \bar{V} \varphi - \mu \bar{V} \psi, & x \in \Omega, \\
\kappa \varphi &= \nabla \cdot \delta_2 \nabla \psi + \sigma_2 \bar{V} \varphi - \mu \bar{V} \psi, & x \in \Omega, \\
\frac{\partial}{\partial n} \varphi &= \frac{\partial}{\partial n} \psi = 0, & x \in \partial \Omega. 
\end{aligned}\]

Since the second equation of (2.4) is decoupled from the system, we consider the problem

\[\begin{aligned}
\kappa \varphi &= \nabla \cdot \delta_1 \nabla \varphi - \lambda \varphi + \sigma_1 H_n \psi, & x \in \Omega, \\
\kappa \psi &= \nabla \cdot \delta_2 \nabla \varphi + \sigma_2 \bar{V} \varphi - \mu \bar{V} \psi, & x \in \Omega, \\
\frac{\partial}{\partial n} \varphi &= \frac{\partial}{\partial n} \psi = 0, & x \in \partial \Omega. 
\end{aligned}\]

Problem (2.5) is cooperative, so it has a principal eigenvalue \(\kappa_0\) associated with a positive eigenvector \((\varphi_0, \psi_0)\) (e.g. see [17]).

For \(\delta \in C^1(\Omega; \mathbb{R})\) being strictly positive on \(\bar{\Omega}\) and \(f \in C(\bar{\Omega}; \mathbb{R})\), let \(\kappa_1(\delta, f)\) be the principal eigenvalue of

\[\begin{aligned}
\kappa \phi &= \nabla \cdot \delta(x) \nabla \phi + f \phi, & x \in \Omega, \\
\frac{\partial}{\partial n} \phi &= 0, & x \in \partial \Omega. 
\end{aligned}\]

It is well known that \(\kappa_1(\delta, f)\) is the only eigenvalue associated with a positive eigenvector, and it is monotone in the sense that if \(f_1 \geq f_2\) then \(\kappa_1(\delta, f_1) > \kappa_2(\delta, f_2)\).
Lemma 2.2. \( E_1 \) is locally asymptotically stable if \( \kappa_0 < 0 \) and unstable if \( \kappa_0 > 0 \).

Proof. Since \( \dot{V} \) is a positive solution of (2.3), we have \( \kappa_1(\delta_2, \beta - \mu V) = 0 \). Therefore, \( \kappa_1(\delta_2, \beta - 2\mu V) < 0 \).

Suppose \( \kappa_0 < 0 \). Let \( \kappa \) be an eigenvalue of (2.4). Then \( \kappa \) is an eigenvalue of either (2.5) or the following eigenvalue problem:

\[
\begin{aligned}
\kappa \phi &= \nabla \cdot \delta_2 \nabla \phi + \beta \phi - 2\mu \dot{V} \phi, & x \in \Omega, \\
\frac{\partial}{\partial n} \phi &= 0, & x \in \partial \Omega.
\end{aligned}
\]

Since \( \kappa_0 < 0 \) and \( \kappa_1(\delta_2, \beta - 2\mu \dot{V}) < 0 \), the real part of \( \kappa \) is less than zero. Since \( \kappa \) is arbitrary, \( E_1 \) is linearly stable. By the principle of linearized stability, \( E_1 \) is locally asymptotically stable.

Suppose \( \kappa_0 > 0 \). Let \((\varphi_0, \psi_0)\) be a positive eigenvector associated with \( \kappa_0 \). By \( \kappa_1(\delta_2, \beta - 2\mu \dot{V}) < 0 \) and the Fredholm alternative, the following problem has a unique solution \( \phi_0 \):

\[
\begin{aligned}
\kappa_0 \phi &= \nabla \cdot \delta_2 \nabla \phi - \sigma_2 \dot{V} \varphi_0 + \beta(\varphi + \psi_0) - 2\mu \dot{V} \varphi - \mu \dot{V} \psi_0, & x \in \Omega, \\
\frac{\partial}{\partial n} \phi &= 0, & x \in \partial \Omega.
\end{aligned}
\]

Hence (2.4) has an eigenvector \((\varphi_0, \phi_0, \psi_0)\) corresponding to eigenvalue \( \kappa_0 > 0 \). So \( E_1 \) is linearly unstable. By the principle of linearized instability, \( E_1 \) is unstable. \( \blacksquare \)

We adopt the approach of [32, 36] to define the basic reproduction number of (1.1). Let \( B : C(\Omega; \mathbb{R}^2) \to C(\Omega; \mathbb{R}^2) \) be the operator such that

\[
D(B) := \left\{ (\varphi, \psi) \in \bigcap_{\rho \geq 1} W^{2,p}(\Omega; \mathbb{R}^2) : \frac{\partial}{\partial n} \varphi = \frac{\partial}{\partial n} \psi = 0 \text{ on } \partial \Omega \text{ and } B(\varphi, \psi) \in C(\Omega; \mathbb{R}^2) \right\}
\]

and

\[
B(\varphi, \psi) = \left( \begin{array}{c}
\nabla \cdot \delta_1 \nabla \varphi \\
\nabla \cdot \delta_2 \nabla \psi
\end{array} \right) + \left( \begin{array}{cc}
-\lambda & \sigma_1 H_x \\
0 & -\mu \dot{V}
\end{array} \right) \left( \begin{array}{c}
\varphi \\
\psi
\end{array} \right), \quad (\varphi, \psi) \in D(B).
\]

Define

\[
C = \left( \begin{array}{cc}
0 & 0 \\
\sigma_2 \dot{V} & 0
\end{array} \right).
\]

Let \( A = B + C \). Then \( A \) and \( B \) are resolvent positive (see [32] for the definition), and \( A \) is a positive perturbation of \( B \). It is easy to check that the spectral bound of \( B \) is negative, i.e. \( s(B) < 0 \). By [32, theorem 3.5], \( \kappa_0 = s(A) \) has the same sign with \( r(-CB^{-1}) \), where \( r(-CB^{-1}) \) is the spectral radius of \(-CB^{-1}\). Then we define the basic reproduction number \( R_0 \) by

\[
R_0 = r(-CB^{-1}).
\]

We immediately have the following result:

Lemma 2.3. \( R_0 - 1 \) and \( \kappa_0 \) have the same sign. Moreover, \( E_1 \) is locally asymptotically stable if \( R_0 < 1 \) and unstable if \( R_0 > 1 \).
We then consider the global dynamics of the model when \( R_0 < 1 \).

**Theorem 2.4.** If \( R_0 < 1 \), then \( E_1 \) is globally asymptotically stable, i.e. \( E_1 \) is locally stable and, for any initial data \((H_0, V_0, V_0) \in C(\bar{\Omega}; \mathbb{R}^3_+)\) with \( V_{a0} + V_{i0} \neq 0 \), we have

\[
\lim_{t \to \infty} \|(H_i(t), V_u(t), V_i(t)) - E_1\|_\infty = 0.
\]

**Proof.** By lemma 2.3, \( E_1 \) is locally asymptotically stable and \( \kappa_0 < 0 \). Then we can choose \( \epsilon > 0 \) small such that the following eigenvalue problem

\[
\begin{align*}
\kappa \varphi &= \nabla \cdot \sigma (\nabla \varphi - \lambda \varphi) + \sigma_1 H_u \psi, \\
\kappa \psi &= \nabla \cdot \sigma_2 (\nabla \psi + \epsilon) \varphi - \mu (\hat{V} - \epsilon) \psi, \\
\frac{\partial}{\partial n} \varphi &= \frac{\partial}{\partial n} \psi = 0,
\end{align*}
\]

has a principal eigenvalue \( \kappa_\epsilon < 0 \) with a corresponding positive eigenvector \((\varphi_\epsilon, \psi_\epsilon)\). By \( V_{a0} + V_{i0} \neq 0 \) and lemma 2.1, we know that \( V_u(x, t) + V_i(x, t) \to \hat{V}(x) \) uniformly on \( \Omega \) as \( t \to \infty \). Hence there exists \( t_0 > 0 \) such that \( V(x) - \epsilon < V_u(x, t) + V_i(x, t) < V(x) + \epsilon \) for \( x \in \Omega \) and \( t > t_0 \). It then follows that

\[
\begin{align*}
\frac{\partial}{\partial t} H_i - \nabla \cdot \sigma_1 \nabla H_i &= -\lambda H_i + \sigma_1 H_u V_i, \\
\frac{\partial}{\partial t} V_i - \nabla \cdot \sigma_2 \nabla V_i &\leq \sigma_2 (\hat{V} + \epsilon) H_i - \mu (\hat{V} - \epsilon) V_i, \\
\frac{\partial}{\partial t} V_i(x, t_0) &= M \varphi_\epsilon(x), \\
\hat{V}(x, t_0) &= M \psi_\epsilon(x),
\end{align*}
\]

So \((H_i, V_i)\) is a lower solution of the following problem

\[
\begin{align*}
\frac{\partial}{\partial t} \hat{H}_i - \nabla \cdot \sigma_1 \nabla \hat{H}_i &= -\lambda \hat{H}_i + \sigma_1 \hat{H}_u \hat{V}_i, \\
\frac{\partial}{\partial t} \hat{V}_i - \nabla \cdot \sigma_2 \nabla \hat{V}_i &= \sigma_2 (\hat{V} + \epsilon) \hat{H}_i - \mu (\hat{V} - \epsilon) \hat{V}_i, \\
\frac{\partial}{\partial t} \hat{H}_i(x, t_0) &= \frac{\partial}{\partial n} \hat{V}_i(x, t_0) = 0, \\
\hat{H}_i(x, t_0) &= M \varphi_\epsilon(x), \\
\hat{V}_i(x, t_0) &= M \psi_\epsilon(x),
\end{align*}
\]

where \( M \) is large such that \( H_i(x, t_0) \lesssim \hat{H}_i(x, t_0) \) and \( V_i(x, t_0) \lesssim \hat{V}_i(x, t_0) \). By the comparison principle for cooperative systems (e.g. [27]), \( H_i(x, t) \lesssim \hat{H}_i(x, t) \) and \( V_i(x, t) \lesssim \hat{V}_i(x, t) \) for all \( x \in \Omega \) and \( t \geq t_0 \). It is easy to check that the unique solution of the linear problem (2.8) is \((\hat{H}_i(x, t), \hat{V}_i(x, t)) = (M \varphi_\epsilon(x) e^{\kappa_\epsilon (t-t_0)}, M \psi_\epsilon(x) e^{\kappa_\epsilon (t-t_0)})\). Since \( \kappa_\epsilon < 0 \), we have \( \hat{H}_i(x, t) \to 0 \) and \( \hat{V}_i(x, t) \to 0 \) uniformly for \( x \in \Omega \) as \( t \to \infty \). Hence \( H_i(x, t) \to 0 \) and \( V_i(x, t) \to 0 \) uniformly for \( x \in \Omega \) as \( t \to \infty \). By \( V_u(., t) + V_i(., t) \to \hat{V} \) in \( C(\Omega; \mathbb{R}) \), we have \( V_u(x, t) \to \hat{V}(x) \) uniformly for \( x \in \Omega \) as \( t \to \infty \). [1]

**3. Global dynamics when \( R_0 > 1 \)**

The objective in this section is to prove the convergence of solutions of (1.1)–(1.3) to the unique positive equilibrium when \( R_0 > 1 \).
3.1. The limit problem

By lemma 2.1, we have \( V_a(\cdot, t) + V_i(\cdot, t) \to \hat{V} \) in \( C(\overline{\Omega}; \mathbb{R}) \) as \( t \to \infty \) if \( V_{a0} + V_{i0} \neq 0 \). This suggests us to study the following limit problem of (1.1)–(1.3):

\[
\begin{align*}
\frac{\partial}{\partial n} H_i - \nabla \cdot \delta_1 \nabla H_i &= -\lambda H_i + \sigma_1 H_a V_i, \quad x \in \Omega, t > 0, \\
\frac{\partial}{\partial n} V_i - \nabla \cdot \delta_2 \nabla V_i &= \sigma_2 (\hat{V} - V_i) + H_i - \mu \hat{V} V_i, \quad x \in \Omega, t > 0, \\
\frac{\partial}{\partial n} H_i &= \frac{\partial}{\partial n} V_i = 0, \quad x \in \partial \Omega, t > 0, \\
H_i(x, 0) &= H_{i0}(x), \quad V_i(x, 0) = V_{i0}(x), \quad x \in \Omega.
\end{align*}
\]

The equilibria of (3.1) are nonnegative solutions of the problem:

\[
\begin{align*}
-\nabla \cdot \delta_1 \nabla H_i &= -\lambda H_i + \sigma_1 H_a V_i, \quad x \in \Omega, \\
-\nabla \cdot \delta_2 \nabla V_i &= \sigma_2 (\hat{V} - V_i) + H_i - \mu \hat{V} V_i, \quad x \in \Omega, \\
\frac{\partial}{\partial n} H_i &= \frac{\partial}{\partial n} V_i = 0, \quad x \in \partial \Omega.
\end{align*}
\]

Clearly \((0, 0)\) is an equilibrium. In this section, we prove that if a positive equilibrium of (3.1) exists, it is globally stable in \( \{ (H_{i0}, V_{i0}) \in C(\Omega; \mathbb{R}^2_+) : H_{i0} + V_{i0} \neq 0 \} \).

3.1.1. Uniqueness of positive equilibrium. In the following lemmas, we prove that the positive equilibrium of (3.1) is unique if it exists. We are essentially using the fact that (3.2) is cooperative and sublinear, and similar ideas can be found in [2, 42].

**Lemma 3.1.** If \((\hat{H}_i, \hat{V}_i)\) is a nontrivial nonnegative equilibrium of (3.1), then \( \hat{H}_i(x), \hat{V}_i(x) > 0 \) for all \( x \in \Omega \) and \( V_i(x_0) < \hat{V}(x_0) \) for some \( x_0 \in \overline{\Omega} \).

**Proof.** Since \((\hat{H}_i, \hat{V}_i)\) is nontrivial, \( \hat{H}_i \neq 0 \) or \( \hat{V}_i \neq 0 \). Since \((\lambda - \nabla \cdot \delta_1 \nabla) \hat{H}_i = \sigma_1 H_a \hat{V}_i\), we must have \( \hat{H}_i \neq 0 \) and \( \hat{V}_i \neq 0 \). By the maximum principle, we have \( \hat{H}_i(x), \hat{V}_i(x) > 0 \) for all \( x \in \Omega \). Assume to the contrary that \( \hat{V}_i(x) \geq \hat{V}(x) \) for all \( x \in \Omega \), then

\[-\nabla \cdot \delta_2 \nabla \hat{V}_i = \sigma_2 (\hat{V} - \hat{V}_i) + \hat{H}_i - \mu \hat{V} \hat{V}_i = -\mu \hat{V} \hat{V}_i.
\]

This implies \( \hat{V}_i = 0 \), which is a contradiction. \( \blacksquare \)

By the previous lemma, any nontrivial nonnegative equilibrium must be positive. For any \( C_1, C_2 > 0 \), define

\[
S = \{ V_i \in C(\Omega; \mathbb{R}^+) : \|V_i\|_\infty \leq C_1 \text{ and } V_i(x_0) < \hat{V}(x_0) \text{ for some } x_0 \in \overline{\Omega} \},
\]

and \( f : S \subset C(\Omega) \to C(\Omega) \) by

\[
f(V_i) = (C_2 - \nabla \cdot \delta_2 \nabla)^{-1} \left[ \sigma_2 (\hat{V} - V_i) + (\lambda - \nabla \cdot \delta_1 \nabla)^{-1} \sigma_1 H_a V_i + (C_2 - \mu \hat{V}) V_i \right], \quad V_i \in S.
\]

**Lemma 3.2.** If \((\hat{H}_i, \hat{V}_i)\) is a positive equilibrium, then there exists \( C_1^* > 0 \) such that \( V_i \) is a nontrivial fixed point of \( f \) for all \( C_1 > C_1^* \) and \( C_2 > 0 \).

**Proof.** By the first equation of (3.2), \( \hat{H}_i = (\lambda - \nabla \cdot \delta_1 \nabla)^{-1} \sigma_1 H_a \hat{V}_i \). Substituting it into the second equation, we obtain
\[-\nabla \cdot \delta_2 \nabla \tilde{V}_i = \sigma_2 (\lambda - \nabla \cdot \delta_1 \nabla)^{-1} \sigma_1 H_u \tilde{V}_i - \mu \tilde{V}_i.\]

By lemma 3.1, \(V_i\) is a nontrivial fixed point of \(f\) if \(C_1\) is large. \(\blacksquare\)

**Lemma 3.3.** For any \(C_1 > 0\), there exists \(C_2 > 0\) such that \(f\) is monotone for all \(C_2 > C_2^*\) in the sense that \(f(V_i) \leq f(\tilde{V}_i)\) for all \(V_i, \tilde{V}_i \in S\) with \(V_i \leq \tilde{V}_i\).

**Proof.** It suffices to prove that \(f(V_i) \leq f(V_i + h)\) for any \(V_i \in S\) and \(0 \leq h \leq \tilde{V}_i - V_i\). Define
\[
\tilde{f}(V_i) = \sigma_2 (\lambda - \nabla \cdot \delta_1 \nabla)^{-1} \sigma_1 H_u V_i + (C_2 - \mu \tilde{V}_i) V_i.
\]

Then, we have
\[
\tilde{f}(V_i + h) - \tilde{f}(V_i) = \sigma_2 ((\tilde{V}_i - V_i - h)^+ - (\tilde{V}_i - V_i)^+)(\lambda - \nabla \cdot \delta_1 \nabla)^{-1} \sigma_1 H_u V_i \\
+ \sigma_2 (\tilde{V}_i - V_i - h)^+ (\lambda - \nabla \cdot \delta_1 \nabla)^{-1} \sigma_1 H_u h + (C_2 - \mu \tilde{V}_i) h \\
\geq h[\sigma_2 (\lambda - \nabla \cdot \delta_1 \nabla)^{-1} \sigma_1 H_u V_i + C_2 - \mu \tilde{V}_i],
\]

where we have used
\[
|((\tilde{V}_i - V_i - h)^+ - (\tilde{V}_i - V_i)^+)| \leq h.
\]

By the elliptic estimate, the following set is bounded:
\[
\{(\lambda - \nabla \cdot \delta_1 \nabla)^{-1} \sigma_1 H_u V_i, \ \ V_i \in S\}.
\]

Hence, \(\tilde{f}(V_i + h) - \tilde{f}(V_i) \geq 0\) if \(C_2\) is large. Therefore, \(f(V_i + h) - f(V_i) \geq 0\), and \(f\) is monotone if \(C_2\) is large. \(\blacksquare\)

For any \(f_1, f_2 \in C(\bar{\Omega}; \mathbb{R})\), we say \(f_1 \ll f_2\) if \(f_1(x) < f_2(x)\) for all \(x \in \bar{\Omega}\).

**Lemma 3.4.** For any \(k \in (0, 1)\) and \(V_i \in S\) with \(V_i \gg 0\), \(k f(V_i) \ll f(k V_i)\).

**Proof.** By the definition of \(S\), there exists \(x_0 \in \bar{\Omega}\) such that \(\tilde{V}(x_0) > V_i(x_0)\). So \((V(x) - V_i(x_0))^+ < (V(x) - k V_i(x_0))^+\) and \((V(x) - V_i(x))^+ \leq (V(x) - k V_i(x))^+\) for all \(x \in \bar{\Omega}\). It then follows that \(k f(V_i(x_0)) < f(k V_i)(x_0)\) and \(k f(V_i) \leq f(k V_i)\). The assertion now just follows from the fact that \((C_2 - \nabla \cdot \delta_2 \nabla)^{-1}\) is strongly positive (i.e., if \(g \in C(\bar{\Omega}; \mathbb{R})\) such that \(g \geq 0\) and \(g(x_0) > 0\) for some \(x_0 \in \bar{\Omega}\), then \((C_2 - \nabla \cdot \delta_2 \nabla)^{-1} g \gg 0\)). \(\blacksquare\)

**Lemma 3.5.** The positive equilibrium of (3.1), if exists, is unique.

**Proof.** Suppose to the contrary that \((H^1_i, V^1_i)\) and \((H^2_i, V^2_i)\) are two distinct positive equilibria. Then \(V^1_i \neq V^2_i\) by the first equation of (3.2). Without loss of generality, we may assume \(V^1_i \leq V^2_i\). Define
\[
k = \max \{\bar{k} \geq 0 : \bar{k} V^1_i \leq V^2_i\}.
\]

Then \(k \in (0, 1)\). By the definition of \(k\), \(k V^1_i \leq V^2_i\) and \(k V^1_i(x_0) = V^2_i(x_0)\) for some \(x_0 \in \Omega\). We can choose \(C_1\) and \(C_2\) such that \(V^1_i\) and \(V^2_i\) are fixed points of \(f\), i.e., \(f(V^1_i) = V^1_i\) and \(f(V^2_i) = V^2_i\). By the previous lemmas, we have
Thus $kV_i^t \ll V_i^2$, which contradicts $kV_i^t(x_0) = V_i^2(x_0)$. \hfill \Box

**Remark 3.6.** It is possible to improve lemma 3.1 by proving $\tilde{V}_i(x) < \tilde{V}(x)$ for all $x \in \tilde{\Omega}$, which means that a positive equilibrium of (3.1) is always a positive equilibrium of (1.1). To see this, let $\tilde{V}_i = \tilde{V} - V_i$. Since $\tilde{V}$ satisfies $-\nabla \cdot \delta_2 \nabla \tilde{V} = \beta \tilde{V} - \mu \tilde{V}^2$ and $V_i$ satisfies $-\nabla \cdot \delta_2 \nabla V_i = \sigma_2 (V - V_i)^+ H_i - \mu \tilde{V} V_i$, we have

$$
\begin{cases}
-\nabla \cdot \delta_2 \nabla \tilde{V}_u = \beta \tilde{V} - \sigma_2 H_i \tilde{V}_u^+ - \mu \tilde{V} \tilde{V}_u, & x \in \Omega, \\
\frac{\partial \tilde{V}_u}{\partial n} = 0, & x \in \partial \Omega.
\end{cases}
$$

Let $x_0 \in \Omega$ such that $\tilde{V}_u(x_0) = \min_{x \in \Omega} \tilde{V}_u(x)$. Assume to the contrary that $\tilde{V}_u(x_0) \leq 0$. By a comparison principle due to Lou and Ni [21], we have $\tilde{V}(x_0) - \sigma_2 H_i(x_0) \tilde{V}_u(x_0) \leq 0$, which implies $\tilde{V}_u(x_0) \tilde{V}(x_0) \leq 0$, which contradicts the assumption $\tilde{V}_u(x_0) < 0$. Therefore, $\tilde{V}_i(x) < \tilde{V}(x)$ for all $x \in \Omega$.

### 3.1.2. Global stability of positive equilibrium.

Let $F_1(H_i, V_i) = -\lambda H_i + \sigma_1 H_i V_i$ and $F_2(H_i, V_i) = \sigma_2 (V - V_i)^+ H_i - \mu \tilde{V} V_i$. Since $\partial F_1/\partial V_i \geq 0$ and $\partial F_2/\partial H_i \geq 0$, system (3.1) is cooperative. Let $\Phi(t): C(\Omega; \mathbb{R}^2) \rightarrow C(\Omega; \mathbb{R}^2)$ be the semiflow induced by the solution of (3.1), i.e. $\Phi(t)(H_0, V_0) = (H_i(t), V_i(t))$ for all $t \geq 0$. Then $\Phi(t)$ is monotone (e.g. see [27]).

**Lemma 3.7.** For any nonnegative nontrivial initial data $(H_0, V_0)$, the solution of (3.1) satisfies that $H_i(x, t) > 0$ and $V_i(x, t) > 0$ for all $x \in \Omega$ and $t > 0$.

**Proof.** By the comparison principle for cooperative systems, $H_i(x, t) \geq 0$ and $V_i(x, t) \geq 0$ for all $x \in \Omega$ and $t \geq 0$. Suppose $V_0 \not= 0$. Noticing

$$
\frac{\partial}{\partial t} V_i - \nabla \cdot \delta_1 \nabla V_i \geq -\mu \tilde{V} V_i
$$

and by the comparison principle, we have $V_i(x, t) > 0$ for all $x \in \Omega$ and $t > 0$. Then,

$$
\frac{\partial}{\partial t} H_i - \nabla \cdot \delta_1 \nabla H_i \geq -\lambda H_i,
$$

where the inequality is strict for some $x \in \tilde{\Omega}$ as $H_0$ is nontrivial. So by the comparison principle, $H_i(x, t) > 0$ for all $x \in \tilde{\Omega}$ and $t > 0$.

Suppose $V_0 = 0$. Since $(H_0, V_0)$ is nontrivial, we have $H_0 \not= 0$. By

$$
\frac{\partial}{\partial t} H_i - \nabla \cdot \delta_1 \nabla H_i \geq -\lambda H_i,
$$

and the comparison principle, we have $H_i(x, t) > 0$ for all $x \in \tilde{\Omega}$ and $t > 0$. By the continuity of $V_i(x, t)$ and $V_i(x, 0) = 0$, $(V - V_i(x, t))^+ > 0$ for all $(x, t) \in \tilde{\Omega} \times (0, t_0]$ for some $t_0 > 0$. Then by

$$
\frac{\partial}{\partial t} V_i - \nabla \cdot \delta_1 \nabla V_i \geq -\mu \tilde{V} V_i, \quad x \in \tilde{\Omega}, t \in (0, t_0]
$$
and the comparison principle, we have $V_i(x, t) > 0$ for all $(x, t) \in \bar{\Omega} \times (0, t_0]$. Finally by (3.3), we have $V_i(x, t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$.

**Lemma 3.8.** For any nonnegative initial data $(H_0, V_0)$, there exists $M > 0$ such that the solution of (3.1) satisfies

$$0 \leq H_i(x, t), V_i(x, t) \leq M, \quad \text{for all } x \in \bar{\Omega}, t > 0.$$ 

**Proof.** Let $M_1 = \max\{\|\bar{V}\|_\infty, \|V_0\|_\infty\}$. By the second equation of (3.1) and the comparison principle, we have $V_i(x, t) \leq M_1$ for all $x \in \bar{\Omega}$ and $t > 0$. Then by the first equation of (3.1), we have

$$\frac{\partial}{\partial t} H_i - \nabla \cdot \delta_i(x) \nabla H_i \leq -\lambda(x) H_i + \sigma_1(x) H_u(x) M_1, \quad x \in \Omega, t > 0.$$ 

So $H_i$ is a lower solution of the problem:

$$\begin{align*}
\frac{\partial}{\partial t} w - \nabla \cdot \delta_i(x) \nabla w &= -\lambda(x) w + \sigma_1(x) H_u(x) M_1, \quad x \in \Omega, t > 0, \\
\frac{\partial}{\partial n} w &= 0, \quad x \in \partial \Omega, t > 0, \\
w(x, 0) &= H_{i0}(x), \quad x \in \Omega.
\end{align*}$$ 

Let $M_2 = \max\{\|\sigma_1\|_\infty\|H_u\|_\infty M_1/\lambda_m, \|H_{i0}\|_\infty\}$, where $\lambda_m = \min\{\lambda(x) : x \in \Omega\}$. Then we have $0 \leq w(x, t) \leq M_2$ for all $(x, t) \in \bar{\Omega} \times (0, \infty)$. Hence by the comparison principle, we have $0 \leq H_i(x, t) \leq w(x, t) < M_2$. Therefore, the claim holds for $M = \max\{M_1, M_2\}$.

**Lemma 3.9.** If the positive equilibrium $(\hat{H}, \hat{V})$ of (3.1) exists, it is globally asymptotically stable, i.e. it is locally stable and, for any nonnegative nontrivial initial data $(H_{i0}, V_{i0})$,

$$\lim_{t \to \infty} H_i(\cdot, t) = \hat{H}, \quad \lim_{t \to \infty} V_i(\cdot, t) = \hat{V} \in C(\Omega; \mathbb{R}).$$ 

**Proof.** By lemma 3.7, we have $H_i(x, t) > 0$ and $V_i(x, t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$. So without loss of generality, we may assume $H_{i0}(x) > 0$ and $V_{i0}(x) > 0$ for all $x \in \bar{\Omega}$.

Suppose that $(\hat{H}, \hat{V})$ is a positive equilibrium of (3.1), which is unique by lemma 3.5. Let $(\mathbf{H}, \mathbf{V}) = (\epsilon \hat{H}, \epsilon \hat{V})$ for some $\epsilon > 0$. We may choose $\epsilon$ small such that the following is satisfied:

$$\begin{align*}
-\nabla \cdot \delta_i(x) \nabla \mathbf{H} &\leq -\lambda(x) \mathbf{H} + \sigma_1(x) \mathbf{H}_u(x) \mathbf{V}, \quad x \in \Omega, \\
-\nabla \cdot \delta_2(x) \nabla \mathbf{V} &\leq \sigma_2(x) (\hat{V} - \mathbf{V})^+ \mathbf{H} - \mu(x) \mathbf{V} \mathbf{V}, \quad x \in \Omega, \\
\frac{\partial}{\partial n} \mathbf{H} &= \frac{\partial}{\partial n} \mathbf{V} = 0, \quad x \in \partial \Omega, \\
\mathbf{H}(x) &\leq H_{i0}(x), \quad \mathbf{V}(x) \leq V_{i0}(x), \quad x \in \Omega.
\end{align*}$$ 

Hence by [27, corollary 7.3.6], $\Phi(t)(\mathbf{H}, \mathbf{V})$ is monotone increasing in $t$ and converges to a positive equilibrium of (3.1). Since $(\hat{H}, \hat{V})$ is the unique positive equilibrium of (3.1), we must have $\Phi(t)(\mathbf{H}, \mathbf{V}) \to (\hat{H}, \hat{V})$ in $C(\Omega; \mathbb{R}^2)$ as $t \to \infty$. 


Similarly, we may define \((\bar{H}_i, \bar{V}_i) = (k\bar{H}_i, k\bar{V}_i)\) with \(k\) large such that (3.4) is satisfied with inverse inequalities, and then \(\Phi(t)(\bar{H}_i, \bar{V}_i) \to (\bar{H}_i, \bar{V}_i)\) in \(C(\Omega; \mathbb{R}^2)\) as \(t \to \infty\). Since \((\bar{H}_0, \bar{V}_0) \leq (H_0, V_0) \leq (\bar{H}_i, \bar{V}_i)\) and \(\Phi(t)\) is monotone, we have \(\Phi(t)(\bar{H}_i, \bar{V}_i) \leq (H_0, V_0) \leq (1 + \varepsilon')(\bar{H}_i, \bar{V}_i)\) for all \(t \geq 0\). Therefore, \(\Phi(t)(H_0, V_0) \to (\bar{H}_i, \bar{V}_i)\) in \(C(\Omega; \mathbb{R}^2)\) as \(t \to \infty\).

For any \(\varepsilon' > 0\) and initial data \((H_0, V_0)\) satisfying \((1 - \varepsilon')(\bar{H}_i, \bar{V}_i) \leq (H_0, V_0) \leq (1 + \varepsilon')(\bar{H}_i, \bar{V}_i)\), similar to the previous arguments, we can show \((1 - \varepsilon')(\bar{H}_i, \bar{V}_i) \leq (H(t, \cdot), V(t, \cdot)) \leq (1 + \varepsilon')(\bar{H}_i, \bar{V}_i)\) for all \(t \geq 0\). Therefore, \((\bar{H}_i, \bar{V}_i)\) is locally stable. This proves the lemma.

### 3.2. Global stability of \(E_2\)

In this section, we prove the convergence of solutions of (1.1)–(1.3) to the unique positive equilibrium \(E_2\) when \(R_0 > 1\). We begin by proving the ultimate boundedness of the solutions.

**Lemma 3.10.** There exists \(M > 0\), independent of initial data, such that any solution \((\bar{H}_i, \bar{V}_i, V_i)\) of (1.1)–(1.3) satisfies

\[
0 \leq \bar{H}_i(x, t), V_u(x, t), V_i(x, t) \leq M, \quad x \in \bar{\Omega}, t \geq t_0,
\]

where \(t_0\) is dependent on initial data.

**Proof.** By Lemma 2.1, we have \(V_u(x, t) + V_i(x, t) \to \bar{V}(x)\) uniformly on \(\bar{\Omega}\) as \(t \to \infty\) if \(V_0 \neq 0\). Hence there exists \(t_1 > 0\) depending on initial data such that \(V_u(x, t) + V_i(x, t) \leq \|\bar{V}\|\_\infty + 1\) for \(t > t_1\) and \(x \in \bar{\Omega}\). By the first equation of (1.1) and the comparison principle, we have \(\bar{H}_i(x, t) \leq \bar{H}_i(x)\) on \(\bar{\Omega} \times [t_1, \infty)\), where \(\bar{H}_i\) is the solution of the problem

\[
\begin{align*}
\frac{\partial}{\partial t} \bar{H}_i - \nabla \cdot \delta_1(x) \nabla \bar{H}_i &= -\lambda(x)\bar{H}_i + \sigma_1(x)H_u(x)(\|\bar{V}\|\_\infty + 1), \quad x \in \bar{\Omega}, t > t_1, \\
\frac{\partial}{\partial n} \bar{H}_i &= 0, \quad x \in \partial \bar{\Omega}, t > t_1, \\
\bar{H}_i(x, t_1) &= H_i(x, t_1), \quad x \in \bar{\Omega}.
\end{align*}
\]

We know that \(\bar{H}_i(x, t) \to \bar{H}'_i(x)\) uniformly on \(\bar{\Omega}\) as \(t \to \infty\), where \(\bar{H}'_i\) is the unique solution of the problem

\[
\begin{align*}
-\nabla \cdot \delta_1(x) \nabla \bar{H}_i &= -\lambda(x)\bar{H}_i + \sigma_1(x)H_u(x)(\|\bar{V}\|\_\infty + 1), \quad x \in \bar{\Omega}, \\
\frac{\partial}{\partial n} \bar{H}_i &= 0, \quad x \in \partial \bar{\Omega}.
\end{align*}
\]

Therefore there exists \(t_0 > t_1\) such that \(\bar{H}_i(x, t) \leq \bar{H}'_i(x, t) < \|\bar{V}\|\_\infty + 1\) for all \(x \in \bar{\Omega}\) and \(t \geq t_0\). Therefore, the claim holds with \(M = \max\{\|\bar{V}\|\_\infty + 1, \|\bar{H}'_i\|\_\infty + 1\}\).

Let \((X, d)\) be a complete metric space and \(\Phi(t): X \to X\) be a continuous semiflow. The distance from a point \(z \in X\) to a subset \(A\) of \(X\) is defined as \(d(z, A) := \inf_{x \in A} d(z, x)\). Suppose that \(X = X_0\), where \(X_0\) is an open subset of \(X\). Then \(X = X_0 \cup \partial X_0\) with the boundary \(\partial X_0 = X - X_0\) being closed in \(X\). The semiflow \(\Phi(t)\) is said to be uniformly persistent with respect to \((X_0, \partial X_0)\) if there is an \(\varepsilon > 0\) such that \(\liminf_{t \to \infty} d(T(t)x, \partial X_0) \geq \varepsilon\) for all \(x \in X_0\).
In the following of this section, let $X = C(\bar{\Omega}; \mathbb{R}_+^3)$ with the metric induced by the supremum norm $\| \cdot \|_\infty$. Define
\[
\partial X_0 := \{(H_i, V_a, V_i) \in X : H_i + V_i = 0 \text{ or } V_a + V_i = 0\}
\]
and
\[
X_0 := \{(H_i, V_a, V_i) \in X : H_i + V_i > 0 \text{ and } V_a + V_i > 0\}.
\]
Then $X = X_0 \cup \partial X_0$, $X_0$ is relatively open with $X_0 = X$, and $\partial X_0$ is relatively closed in $X$. Let $w(x, t) = (H_i(x, t), V_a(x, t), V_i(x, t))$ be the solution of (1.1)–(1.3) with initial data $w_0 = (H_{00}, V_{a0}, V_{i0}) \in X$. Let $\Phi(t) : X \to X$ be the semiflow induced by the solution of (1.1)–(1.3), i.e. $\Phi(t)w_0 = w(\cdot, t)$ for $t \geq 0$. Then $\Phi(t)$ is point dissipative by Lemma 3.10 (see, e.g. [15] for the definition). Moreover, $\Phi(t)$ is compact for any $t > 0$, since (1.1)–(1.3) is a standard reaction–diffusion system.

We prove the following persistence result when $R_0 > 1$, which is necessary for proving the convergence of solutions to the positive equilibrium.

**Lemma 3.11.** If $R_0 > 1$, then (1.1)–(1.3) is uniformly persistent in the sense that there exists $\epsilon > 0$ such that, for any initial data $(H_{i0}, V_{a0}, V_{i0}) \in X_0$, \[
\liminf_{t \to \infty} \inf_{w \in \partial X_0} \| (H_i(\cdot, t), V_a(\cdot, t), V_i(\cdot, t)) - w \|_\infty \geq \epsilon.
\]

Moreover, (1.1)–(1.3) has at least one EE.

**Proof.** We prove this result in several steps.

**Step 1.** $X_0$ is invariant under $\Phi(t)$.

Let $U_0 = (H_{i0}, V_{a0}, V_{i0}) \in X_0$. Then $H_{i0} + V_{i0} > 0$ and $V_{a0} + V_{i0} > 0$. Suppose $V_{a0} = 0$. Then $H_{i0} \neq 0$ and $V_{i0} \neq 0$. By the first equation of (1.1), we have
\[
\frac{\partial}{\partial t} H_i - \nabla \cdot \delta_1 \nabla H_i \geq -\lambda H_i.
\]

Then by $H_{i0} \neq 0$ and the maximum principle, we have $H_i(x, t) > 0$ for $x \in \bar{\Omega}$ and $t > 0$. By the second equation of (1.1), we have
\[
\frac{\partial}{\partial t} V_a - \nabla \cdot \delta_2 \nabla V_a \geq V_a(-\sigma_2 H_i + \beta - \mu(V_a + V_i)).
\]

Then by $V_{a0} \neq 0$ and the maximum principle, we have $V_a(x, t) > 0$ for $x \in \bar{\Omega}$ and $t > 0$. Noticing the third equation of (1.1), we have
\[
\frac{\partial}{\partial t} V_i - \nabla \cdot \delta_3 \nabla V_i \geq -\mu(V_a + V_i)V_i, \quad x \in \Omega, t > 0.
\]

Then by the maximum principle, we have $V_i(x, t) > 0$ for $x \in \bar{\Omega}$ and $t > 0$.

Suppose $V_{i0} \neq 0$. Noticing
\[
\frac{\partial}{\partial t} V_i - \nabla \cdot \delta_3 \nabla V_i \geq -\mu(V_a + V_i)V_i
\]

...
and by the maximum principle, we have $V_i(x, t) > 0$ for all $x \in \Omega$ and $t > 0$. By the first equation of (1.1), we have

$$\frac{\partial}{\partial t} H_i - \nabla \cdot \delta_1 \nabla H_i \geq -\lambda H_i, \quad x \in \Omega, t > 0,$$

where the inequality is strict for some $x \in \Omega$ as $H_i$ is nontrivial. So by the comparison principle, $H_i(x, t) > 0$ for all $x \in \Omega$ and $t > 0$. By the second equation of (1.1), we have

$$\frac{\partial}{\partial t} V_a - \nabla \cdot \delta_2 \nabla V_a > V_a(-\sigma_2 H_i + \beta - \mu(V_a + V_i)), \quad x \in \Omega, t > 0,$$

which implies $V_a(x, t) > 0$ for all $x \in \Omega$ and $t > 0$. Therefore, we have $\Phi(t)w_0 \in X_0$ for all $t > 0$. Hence $X_0$ is invariant under $\Phi(t)$.

Step 2. $\partial X_0$ is invariant under $\Phi(t)$. For any $w_0 \in \partial X_0$, the $\omega$-limit set $\omega(w_0)$ is either $\{E_0\}$ or $\{E_1\}$.

Suppose $w_0 = (H_0, V_{a0}, V_0) \in \partial X_0$. Then, $H_0 + V_0 = 0$ or $V_{a0} + V_0 = 0$. If $H_0 + V_0 = 0$ and $V_{a0} \neq 0$, then we have $H_i(\cdot, t) = V_i(\cdot, t) = 0$ for all $t \geq 0$ by the first and third equations of (1.1). Then the second equation of (1.1) is

$$\frac{\partial}{\partial t} V_a - \nabla \cdot \delta_2 \nabla V_a = V_a(\beta - \mu V_a).$$

Hence by lemma 2.1, we have $V_a(x, t) > 0$ for $x \in \Omega$ and $t > 0$, and $V_a(\cdot, t) \to \tilde{V}$ uniformly on $\Omega$ as $t \to \infty$. So $\Phi(t)w_0 \in \partial X_0$ with $\omega(w_0) = \{E_1\}$.

If $V_{a0} + V_0 = 0$, then by the second and third equations of (1.1), we have $V_a(\cdot, t) = V_i(\cdot, t) = 0$ for all $t \geq 0$. Then the first equation of (1.1) is

$$\frac{\partial}{\partial t} H_i - \nabla \cdot \delta_1 \nabla H_i = -\lambda H_i,$$

which implies that $H_i(\cdot, t) \to 0$ uniformly on $\Omega$ as $t \to 0$. Therefore, we have $\Phi(t)w_0 \in \partial X_0$ with $\omega(w_0) = \{E_0\}$.

By Step 2, the semiflow $\Phi_\partial(t) := \Phi(t)|_{\partial X_0}$, the restriction of $\Phi(t)$ on $\partial X_0$, admits a compact global attractor $A_\partial$. Moreover, it is clear that

$$\bar{A}_\partial := \cup_{w_0 \in A_\partial} \omega(w_0) = \{E_0, E_1\}.$$

Step 3. $A_\partial$ has an acyclic covering $M = \{E_0\} \cup \{E_1\}$.

It suffices to show that $\{E_1\} \not\rightarrow \{E_0\}$, i.e. $W^a(E_1) \cap W^s(E_0) = \emptyset$. Suppose to the contrary that there exists $w_0 = (H_{a0}, V_{a0}, V_0) \in W^a(E_1) \cap W^s(E_0)$. Let $(H(\cdot, t), V_0(\cdot, t), V(\cdot, t))$ be a complete orbit through $w_0$. By $w_0 \in W^s(E_0)$ and lemma 2.1, we have $V_{a0} = V_0 = 0$, and hence $V_a(\cdot, t) = V_i(\cdot, t) = 0$ for all $t \in (-\infty, \infty)$. Therefore $V_a(\cdot, t) \not\rightarrow \tilde{V}$ as $t \to -\infty$, contradicting $w_0 \in W^a(E_1)$. Therefore $M = \{E_0\} \cup \{E_1\}$ is an acyclic covering of $A_\partial$.

Step 4. $W^s(E_0) \cap X_0 = \emptyset$ and $W^a(E_1) \cap X_0 = \emptyset$.

We will actually show:

$$W^a(E_0) = \{(H_{a0}, V_{a0}, V_0) \in \partial X_0 : \ V_{a0} = V_0 = 0\} \quad (3.6)$$

and

$$W^a(E_1) = \{(H_{a0}, V_{a0}, V_0) \in \partial X_0 : \ H_{a0} = V_{a0} = 0 \text{ and } V_0 \neq 0\}.$$

By the proof of step 2, it suffices to show that there exists $\epsilon > 0$ such that, for any initial data $(H_{a0}, V_{a0}, V_0) \in X_0$, we have
\[
\limsup_{t \to \infty} \| (H_i(\cdot, t), V_u(\cdot, t), V_\ell(\cdot, t)) - E_0 \|_\infty \geq \epsilon \tag{3.7}
\]
and
\[
\limsup_{t \to \infty} \| (H_i(\cdot, t), V_u(\cdot, t), V_\ell(\cdot, t)) - E_1 \|_\infty \geq \epsilon.
\tag{3.8}
\]

We first prove (3.8). By Lemma 2.3 and \( R_0 > 1 \), we have \( \kappa_0 > 0 \). Hence there exists \( \epsilon_0 > 0 \) such that the following problem has a principal eigenvalue \( \kappa_0 > 0 \) corresponding to a positive eigenvector \((\phi_{\epsilon_0}, \psi_{\epsilon_0})\):
\[
\begin{cases}
\kappa \varphi = \nabla \cdot \delta_1 \nabla \varphi - \lambda \varphi + \sigma_1 H_u \psi, & x \in \Omega, \\
\kappa \psi = \nabla \cdot \delta_2 \nabla \psi + \sigma_2 (\hat{V} - \epsilon_0) \varphi - \mu (\hat{V} + 2 \epsilon_0) \psi, & x \in \Omega, \\
\frac{\partial}{\partial n} \varphi = \frac{\partial}{\partial n} \psi = 0, & x \in \partial \Omega.
\end{cases}
\]

Assume to the contrary that (3.8) does not hold. Then there exists some \( w_0 = (H_{0b}, V_{0u}, V_{0\ell}) \in X_0 \) such that the corresponding solution satisfies
\[
\limsup_{t \to \infty} \| (H_i(\cdot, t), V_u(\cdot, t), V_\ell(\cdot, t)) - E_1 \|_\infty < \epsilon_0.
\]

Hence there exists \( t_0 > 0 \) such that \( \hat{V} - \epsilon_0 < V_u(\cdot, t) < \hat{V} + \epsilon_0 \) and \( V_\ell(\cdot, t) < \epsilon_0 \) for all \( t \geq t_0 \). It then follows from the second and third equations of (1.1) that
\[
\begin{cases}
\frac{\partial}{\partial t} H_i - \nabla \cdot \delta_1 \nabla H_i = -\lambda H_i + \sigma_1 H_u V_i, & x \in \Omega, t \geq t_0, \\
\frac{\partial}{\partial t} V_i - \nabla \cdot \delta_2 \nabla V_i \geq \sigma_2 (\hat{V} - \epsilon_0) H_i - \mu (\hat{V} + 2 \epsilon_0) V_i, & x \in \Omega, t \geq t_0.
\end{cases}
\]

In Step 1, we have shown that \( H_i(x, t), V_i(x, t) > 0 \) for all \( x \in \Omega \) and \( t > 0 \). Thus we can choose \( m > 0 \) small such that \( H_i(\cdot, t_0) \geq m \phi_{\epsilon_0} \) and \( V_i(\cdot, t_0) \geq m \psi_{\epsilon_0} \). Hence \((H_i, V_i)\) is an upper solution of the problem
\[
\begin{cases}
\frac{\partial}{\partial t} \tilde{H}_i - \nabla \cdot \delta_1 \nabla \tilde{H}_i = -\lambda \tilde{H}_i + \sigma_1 H_u \tilde{V}_i, & x \in \Omega, t \geq t_0, \\
\frac{\partial}{\partial t} \tilde{V}_i - \nabla \cdot \delta_2 \nabla \tilde{V}_i = \sigma_2 (\hat{V} - \epsilon_0) \tilde{H}_i - \mu (\hat{V} + 2 \epsilon_0) \tilde{V}_i, & x \in \Omega, t \geq t_0, \\
\frac{\partial}{\partial n} \tilde{H}_i = \frac{\partial}{\partial n} \tilde{V}_i = 0, & x \in \partial \Omega, t \geq t_0, \\
H_i(\cdot, t_0) = m \phi_{\epsilon_0}, \quad V_i(\cdot, t_0) = m \psi_{\epsilon_0}.
\end{cases}
\]

We observe that the solution of this problem is \((\tilde{H}_i, \tilde{V}_i) = m e^{\kappa_0(t-t_0)} (\phi_{\epsilon_0}, \psi_{\epsilon_0})\). By the comparison principle of cooperative systems, we have \( H_i(\cdot, t) \geq \tilde{H}_i(\cdot, t) \) and \( V_i(\cdot, t) \geq \tilde{V}_i(\cdot, t) \) for \( t \geq t_0 \). Since \( \kappa_0 > 0 \), we have \( H_i(\cdot, t) \to \infty \) and \( V_i(\cdot, t) \to \infty \) as \( t \to \infty \), which contradicts the boundedness of the solution. This proves (3.8).

We then prove (3.7). Suppose to the contrary that (3.7) does not hold. Then for given small \( \epsilon_1 > 0 \), there exists initial data \((H_{0b}, V_{0u}, V_{0\ell}) \in X_0 \) such that
\[
\limsup_{t \to \infty} \| (H_i(\cdot, t), V_u(\cdot, t), V_\ell(\cdot, t)) - E_0 \|_\infty < \epsilon_1.
\]

Hence there exists \( t_1 > 0 \) such that \( V_u(\cdot, t) < \epsilon_1 \) and \( V_\ell(\cdot, t) < \epsilon_1 \) for all \( t \geq t_1 \). However by Lemma 2.1, we know that \( V_u(\cdot, t) + V_\ell(\cdot, t) \to \hat{V} \) uniformly on \( \Omega \) as \( t \to \infty \), which is a contradiction as \( \epsilon_1 \) is small.
Finally by steps 1–4 and [15, theorem 4.1], there exists $\epsilon > 0$ such that (3.5) holds. Moreover by [42, theorem 1.3.7], (1.1)–(1.3) has an EE.

Combing lemmas 3.9 and 3.11, we can prove the main result in this section.

**Theorem 3.12.** If $R_0 > 1$, then for any initial data $(H_0, V_0) \in X_0$, the solution $(H_t, V_t)$ of (1.1)–(1.3) satisfies that

$$\lim_{t \to \infty} (H_t(x,t), V_t(x,t), V_t(x,t)) = (\hat{H}_t, \bar{V}_t, \bar{V}_t) \quad \text{uniformly on } \bar{\Omega},$$

where $E_2 = (\hat{H}_t, \bar{V}_t, \bar{V}_t)$ is the unique EE of (1.1).

**Proof.** By lemma 3.11, there exists an EE, $E_2 := (\hat{H}_t, \bar{V}_t, \bar{V}_t)$, of (1.1)–(1.3) when $R_0 > 1$. By lemma 2.1, $V_a + \bar{V} = \bar{V}$. So $(\hat{H}_t, V_t)$ is a positive solution of (3.2), which is unique by lemma 3.5. Hence, $E_2$ is the unique EE of (1.1)–(1.3).

Let $(H_0, V_0) \in X_0$. Then $V_0 + \bar{V} = \bar{V}$. By lemma 2.1, we have $V_a(t) + \bar{V}(t) \to \bar{V}$ in $C(\bar{\Omega} : \mathbb{R})$ as $t \to \infty$. By lemma 3.11, there exists $\epsilon > 0$ such that

$$\liminf_{t \to \infty} \|H_a(t)\|_\infty + \|V_a(t)\|_\infty \geq \epsilon.$$

We focus on the first and third equations of (1.1) and rewrite them as:

$$\begin{align*}
\begin{aligned}
\frac{\partial}{\partial t} H_t - \nabla \cdot \delta_1 \nabla H_t & = -\lambda H_t + \sigma_1 H_a V_t, & x \in \Omega, t > 0, \\
\frac{\partial}{\partial t} V_t - \nabla \cdot \delta_2 \nabla V_t & = \sigma_2 (\bar{V} - V_t)^+ H_t - \mu \bar{V} V_t + F(x, t), & x \in \Omega, t > 0, \\
\frac{\partial}{\partial n} H_t & = \frac{\partial}{\partial n} V_t = 0, & x \in \partial \Omega, t > 0, \\
H_t(x, 0) & = H_0(x), \quad V_t(x, 0) = V_0(x), & x \in \bar{\Omega}.
\end{aligned}
\end{align*}$$

(3.10)

where

$$F(t) = \sigma_2 (V_a(t) - (\bar{V} - V_t)^+) H_t - \mu (V_a(t) + V_t(t) - \bar{V}) V_t(t).$$

Since

$$|V_a(t) - (\bar{V} - V_t(t))^+| \leq |V_a(t) + V_t(t) - \bar{V}|,$$

we have $F(t) \to 0$ in $C(\bar{\Omega} : \mathbb{R})$ as $t \to \infty$. Then by [24, proposition 1.1], (3.10) is asymptotically autonomous with limit system (3.1). By (3.9), the $\omega$–limit set of (3.10) is contained in $W := \{(H_t, V_t) \in C(\bar{\Omega} ; \mathbb{R}^2) : H_t + V_t \neq 0\}$. By lemma 3.9, $W$ is the stable set (or basin of attraction) of the equilibrium $(\hat{H}_t, \bar{V}_t)$ of (3.1). Hence by the theory of asymptotically autonomous semiflows (originally due to Markus. See [31, theorem 4.1] for the generalization to asymptotically autonomous semiflows), we have $(H_t(t), V_t(t)) \to (\hat{H}_t, \bar{V}_t)$ in $C(\bar{\Omega} : \mathbb{R}^2)$ as $t \to \infty$. Moreover, by $V_a(t) + V_t(t) \to \bar{V}$ and $V_t + V_a = \bar{V}$, we have $V_a(t) \to V_a$ in $C(\bar{\Omega} : \mathbb{R})$ as $t \to \infty$. This completes the proof. □
4. Global stability when $R_0 = 1$

In this section, we prove the global stability of $E_1$ for the critical case $R_0 = 1$. The following result is well known. Since we cannot locate a reference and for the convenience of readers, we attach a proof.

**Lemma 4.1.** *The positive equilibrium $\hat{V}$ of (2.1) is exponentially asymptotically stable.*

**Proof.** It is easy to see that $\hat{V}$ is locally asymptotically stable. To see this, linearizing (2.1) around $\hat{V}$, we obtain
\[
\left\{ \begin{array}{ll}
\kappa \phi = \nabla \cdot \delta_2 \nabla \phi + \beta \phi - 2\mu \hat{V} \phi, & x \in \Omega, \\
\frac{\partial \phi}{\partial n} = 0, & x \in \partial \Omega.
\end{array} \right.
\]  
\tag{4.1}

Since $\hat{V}$ satisfies (2.3), we have $\kappa_1(\delta_2, \beta - \mu \hat{V}) = 0$. Hence $\alpha := \kappa_1(\delta_2, \beta - 2\mu \hat{V}) < 0$, i.e. the principal eigenvalue of (4.1) is negative. Therefore, $\hat{V}$ is linearly stable. By the principle of linearized stability, it is locally asymptotically stable.

Let $\epsilon > 0$ be given. Since $\hat{V}$ is locally asymptotically stable, there exists $\delta > 0$ such that $\|V(., t) - \hat{V}\|_\infty < \epsilon$ for all $V_0 \in C(\Omega; \mathbb{R}_+)$ with $\|V_0 - \hat{V}\|_\infty < \delta$. Let $w(., t) = V(., t) - \hat{V}$.

Then $w$ satisfies
\[
\left\{ \begin{array}{ll}
w_t = \nabla \cdot \delta_2 \nabla w + (\beta - 2\mu \hat{V})w - 2\mu w^2, & x \in \Omega, t > 0, \\
\frac{\partial w}{\partial n} = 0, & x \in \partial \Omega, t > 0, \\
w(x, 0) = V_0 - \hat{V}, & x \in \Omega.
\end{array} \right.
\]  
\tag{4.2}

Let $S(t)$ be the semigroup generated by $\nabla \cdot \delta_2 \nabla + (\beta - 2\mu \hat{V})$ (associated with Neumann boundary condition) in $C(\Omega; \mathbb{R})$. Then there exists $M_1 > 0$ such that $\|S(t)\| \leq M_1 e^{-at}$ for all $t \geq 0$. Then by (4.2), we have
\[
w(., t) = S(t)w(., 0) - \int_0^t S(t-s)\mu w(., s)^2 \, ds.
\]

It then follows that
\[
\|w(., t)\|_\infty \leq \|S(t)w(., 0)\|_\infty + \int_0^t \|S(t-s)\mu w(., s)^2\|_\infty \, ds \\
\leq M_1 e^{-at} \|w(., 0)\|_\infty + \epsilon M_1 \|\mu\|_\infty \int_0^t e^{-a(t-s)} \|w(., s)\|_\infty \, ds.
\]

By the Gronwall’s inequality, if $\epsilon \leq \alpha / 2 \|\mu\|_\infty M_1$, we have
\[
\|w(., t)\|_\infty \leq M_1 \|V_0 - \hat{V}\|_\infty e^{(M_1 \|\mu\|_\infty e^{-a})t} \leq M_1 \|V_0 - \hat{V}\|_\infty e^{-at/2}.
\]

Therefore, $\hat{V}$ is exponentially asymptotically stable.  

We then prove the local stability of $E_1$ when $R_0 = 1$. 

\[\blacksquare\]
Lemma 4.2. If $R_0 = 1$, then $E_1$ is locally stable.

Proof. Let $\epsilon > 0$ be given. Denote $V = V_u + V_i$. By lemma 4.1, there exist $\delta, M_1, b > 0$ such that, if $\|V_{a0} + V_0 - \bar{V}\|_\infty < 2\delta$, then

$$\|V - \bar{V}\|_\infty \leq M_1 \|V_{a0} + V_0 - \bar{V}\|_\infty e^{-bt}. \quad (4.3)$$

Suppose that $(H_{00}, V_{a0}, V_0)$ satisfies $\|H_{00}\|_\infty \leq \delta, \|V_{a0} - \bar{V}\|_\infty \leq \delta$ and $\|V_0\|_\infty \leq \delta$ such that (4.3) holds.

Since $\kappa_0$ has the same sign with $R_0 - 1$, we have $\kappa_0 = 0$. Let $T(t)$ be the positive semigroup generated by $A = B + C$ in $C(\Omega; \mathbb{R}^2)$. Then there exists $M_2 > 0$ such that $\|T(t)\|_\leq M_2$ for all $t \geq 0$ [39, propositon 4.15]. By (1.1)–(1.3), we have

$$\text{Let } u(t) = \text{max}\{\|H_1(\cdot, t)\|_\infty, \|V_i(\cdot, t)\|_\infty\} \text{ by (4.3), we have}$$

$$u(t) \leq M_2u(0) + 2M_2 \text{max}\{\|\sigma_2\|_\infty, \|\mu\|_\infty\} \int_0^t \|V(\cdot, s) - \bar{V}\|_\infty u(s) ds$$

$$\leq M_2\delta + \delta C \int_0^t e^{-bs}u(s) ds$$

where $C = 4M_1M_2 \text{max}\{\|\sigma_2\|_\infty, \|\mu\|_\infty\}$. Then by Gronwall’s inequality,

$$u(t) = \text{max}\{\|H_1(\cdot, t)\|_\infty, \|V_i(\cdot, t)\|_\infty\} \leq M_2e^{Ct/b}\delta. \quad (4.4)$$

Moreover, by (4.3), we have

$$\|V_u(\cdot, t) - \bar{V}\|_\infty \leq \|V_u(\cdot, t) + V_i(\cdot, t) - \bar{V}\|_\infty + \|V_i(\cdot, t)\|_\infty \leq 2M_1\delta + M_2e^{Ct/b}\delta. \quad (4.5)$$

Combining (4.4) and (4.5), we can find $\delta = \delta(\epsilon) > 0$ such that

$$\|H_1(\cdot, t)\|_\infty \leq \epsilon, \|V_u(\cdot, t) - \bar{V}\|_\infty \leq \epsilon, \text{ and } \|V_i(\cdot, t)\|_\infty \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $E_1$ is locally stable. □

We then prove the global attractivity of $E_1$ when $R_0 = 1$.

Theorem 4.3. If $R_0 = 1$, then $E_1$ is globally stable in the sense that it is locally stable and, for any nonnegative initial data $(H_{00}, V_{a0}, V_0)$ with $V_{a0} + V_0 \neq 0$,

$$\lim_{t \to \infty} \|(H_1(\cdot, t), V_u(\cdot, t), V_i(\cdot, t)) - E_1\|_\infty = 0.$$

Proof. Let

$$\mathcal{M} = \{(H_{00}, V_{a0}, V_0) \in C(\Omega; \mathbb{R}^3_+) : V_{a0} + V_0 = \bar{V}\}.$$
It suffices to show: (a) $E_1$ is a locally stable equilibrium of (1.1)–(1.3); (b) the stable set (or basin of attraction) of $E_1$ contains $M$; (c) the $\omega-$limit set of $(H_0, V_0, V_0)$ with $V_{a0} + V_{b0} \neq 0$ is contained in $M$.

By lemma 4.2, $E_1$ is locally stable, which gives (a). If $V_{a0} + V_{b0} \neq 0$, we have $V_a(\cdot, t) + V_b(\cdot, t) \rightarrow \tilde{V}$ in $C(\bar{\Omega}; \mathbb{R})$ as $t \rightarrow \infty$, which implies (c).

To prove (b), suppose $(H_0, V_0, V_0) \in M$. Then the solution of (1.1)–(1.3) satisfies $V_a(x, t) + V_b(x, t) = \tilde{V}(x)$ for all $x \in \Omega$ and $t \geq 0$. Hence $(H_1(x, t), V_1(x, t))$ is the solution of the limit problem (3.1).

Since $R_0 = 1$, we have $\kappa_0 = 0$. Let $(\tilde{\varphi}_0, \phi_0)$ be a positive eigenvector associated with $\kappa_0$ of the eigenvalue problem (2.5). Motivated by [7, 40], for any $w_0 := (H_0, V_0)$, we define

$$c(t; w_0) := \inf \{ \tilde{c} \in \mathbb{R} : H_1(\cdot, t) \leq \tilde{c} \varphi_0 \text{ and } V_1(\cdot, t) \leq \hat{c} \phi_0 \}.$$ 

Then $c(t; w_0) > 0$ for all $t > 0$. We now claim that $c(t; w_0) > 0$ is strictly decreasing. To see that, fix $t_0 > 0$, and we define $\bar{H}_1(x, t) = c(t_0; w_0) \varphi_0(x)$ and $\bar{V}_1(x, t) = c(t_0; w_0) \phi_0(x)$ for all $t \geq t_0$ and $x \in \Omega$. Then $(\bar{H}_1(x, t), \bar{V}_1(x, t))$ satisfies

$$\begin{align*}
\frac{\partial}{\partial t} H_i - \nabla \cdot \delta_1 \nabla H_i &= -\lambda H_i + \sigma_1 H_a V_i, \\
\frac{\partial}{\partial t} V_i - \nabla \cdot \delta_2 \nabla V_i &> \sigma_2 (V - \bar{V})^+ H_i - \mu \bar{V}_i, \\
\frac{\partial}{\partial t} H_i(\cdot, t_0) &\geq H_i(\cdot, t_0), \quad \bar{V}_1(\cdot, t_0) \geq \bar{V}_1(\cdot, t_0) \quad \text{for all } x \in \bar{\Omega}.
\end{align*}$$

By the comparison principle for cooperative systems, we have $(\bar{H}_1(x, t), \bar{V}_1(x, t)) \geq (H_i(x, t), V_i(x, t))$ for all $x \in \Omega$ and $t \geq t_0$. By the second equation of (4.6), we have

$$\frac{\partial}{\partial t} V_i - \nabla \cdot \delta_2 \nabla V_i > \sigma_2 (V - \bar{V})^+ H_i - \mu \bar{V}_i.$$ 

By the comparison principle, $\bar{V}_1(x, t) > V_i(x, t)$ for all $x \in \bar{\Omega}$ and $t > t_0$. Then by the first equation of (4.6),

$$\frac{\partial}{\partial t} H_i - \nabla \cdot \delta_1 \nabla H_i \geq -\lambda \bar{H}_i + \sigma_1 H_a \bar{V}_i,$$

where the inequality is strict for some $x \in \bar{\Omega}$ as $H_a$ is nontrivial. By the comparison principle, we have $\bar{H}_1(x, t) > H_i(x, t)$ for all $x \in \Omega$ and $t > t_0$. Therefore, $c(t_0; w_0) \varphi_0(x) > H_i(x, t)$ and $c(t_0; w_0) \phi_0(x) > V_i(x, t)$ for all $x \in \Omega$ and $t > t_0$. By the definition of $c(t; w_0)$, $c(t_0; w_0) > c(t; w_0)$ for all $t > t_0$. Since $t_0 > 0$ is arbitrary, $c(t; w_0)$ is strictly decreasing for $t \geq 0$.

Let $\Phi(t)$ be the semiflow induced by the solution of the limit problem (3.1). Let $\omega := \omega(w_0)$ be the omega limit set of $w_0$. We claim that $\omega = \{(0, 0)\}$. Assume to the contrary that there exists a nontrivial $w_1 \in \omega$. Then there exists $\{t_k\}$ with $t_k \rightarrow \infty$ such that $\Phi(t_k)w_0 \rightarrow w_1$. Let $c_* = \lim_{t_k \rightarrow \infty} c(t_k; w_0)$. We have $c(t; w_1) = c_*$ for all $t \geq 0$. Actually this follows from the fact that $\Phi(t)w_1 = \Phi(t) \lim_{t_k \rightarrow \infty} \Phi(t_k)w_0 = \lim_{t_k \rightarrow \infty} \Phi(t + t_k)w_0$. However since $w_1$ is nontrivial, we can repeat the previous arguments to show that $c(t; w_1)$ is strictly decreasing. This is a contradiction. Therefore $\omega = \{(0, 0)\}$, and $(\bar{H}_1(\cdot, t), \bar{V}_1(\cdot, t)) \rightarrow (0, 0)$ in $C(\bar{\Omega}; \mathbb{R}^2)$ as $t \rightarrow \infty$. Since $V_a(\cdot, t) + V_b(\cdot, t) = \tilde{V}$, we have $V_a(\cdot, t) \rightarrow \tilde{V}$ in $C(\bar{\Omega}; \mathbb{R})$ as $t \rightarrow \infty$. This completes the proof. 

\[\blacksquare\]
5. Concluding remarks

In this paper, we define a basic reproduction number $R_0$ for the model (1.1)–(1.3), and show that it serves as the threshold value for the global dynamics of the model: if $R_0 \leq 1$, then disease free equilibrium $E_1$ is globally asymptotically stable; if $R_0 > 1$, the model has a unique endemic equilibrium $E_2$, which is globally attractive.

As shown in theorem A.4, the global dynamics of the corresponding ODE model of (1.1)–(1.3) is determined by the magnitude of $\sigma_1 \sigma_2 H_\mu/\lambda \mu$. This motivates us to define the local basic reproduction number for model (1.1)–(1.3):

$$ R(x) := R_1(x) R_2(x) = \frac{\sigma_1(x) H_\mu(x) \sigma_2(x)}{\lambda(x) \mu(x)}. $$

Since $R_0$ is difficult to visualize, it is natural to ask: are there any connections between $R_0$ and $R^2$? As the global dynamics of both models are determined by the magnitude of the basic reproduction number, this is equivalent to ask: how the diffusion rates change the dynamics of the model (1.1)–(1.3), and what is the relation between the reaction–diffusion model (1.1)–(1.3), and the corresponding reaction system (the model without diffusion)? We will explore these questions in a forthcoming paper. Our main ingredient is the formula:

$$ R_0 = r(L_1 R_1 L_2 R_2) $$

with $L_1 := (\lambda - \nabla \cdot \delta_1 \nabla)^{-1} \lambda$ and $L_2 := (\mu \hat{\nabla} - \nabla \cdot \delta_2 \nabla)^{-1} \mu \hat{\nabla}$. This formula establishes an interesting connection between $R_0$ and $R$ as we can prove

$$ r(L_1 L_2) = r(L_1) = r(L_2) = 1. $$

Consequences of this formula are:

1. If $R_i(x), i = 1, 2$, is constant, then $R_0 = R$;
2. $R_0 > 1$ if $R_i(x) > 1, i = 1, 2$, for all $x \in \Omega$ and $R_0 < 1$ if $R_i(x) < 1, i = 1, 2$, for all $x \in \Omega$.

Furthermore, when the diffusion coefficients $\delta_1$ and $\delta_2$ are constant, we prove

- $\lim_{\delta_1, \delta_2 \to \infty} R_0 = \frac{\int_\Omega \lambda R_1 dx}{\int_\Omega \mu R_2 dx}$,
- $\lim_{\delta_1, \delta_2 \to 0} \lim_{\delta_1, \delta_2 \to 0} R_0 = \max \{ R(x) : x \in \Omega \}$.

Finally, we remark that our approach is applicable to several other reaction–diffusion models (e.g. [18, 19, 26, 28]). For example, the reaction–diffusion within-host model of viral dynamics studied in [26, 28] is

$$
\begin{align*}
\frac{\partial}{\partial t} T - \nabla \cdot \delta_1(x) \nabla T &= \lambda(x) - \mu T - k_1 TV(-k_2 TI), & x \in \Omega, t > 0, \\
\frac{\partial}{\partial t} I - \nabla \cdot \delta_2(x) \nabla I &= k_1 TV(+k_2 TI) - \mu I, & x \in \Omega, t > 0, \\
\frac{\partial}{\partial t} V - \nabla \cdot \delta_3(x) \nabla V &= N(x) I - \mu_e V, & x \in \Omega, t > 0,
\end{align*}
$$

(5.1)

where $T, I$ and $V$ denote the densities of healthy cells, infected cells and virions, respectively. If $\delta_1 = \delta_2$ and $\mu = \mu_e$, then $E := T + I$ satisfies

$$
\frac{\partial}{\partial t} E - \nabla \cdot \delta_1(x) \nabla E = \lambda(x) - \mu E.
$$

This equation has a unique positive equilibrium $\hat{E}$ and $E(\cdot, t) \to \hat{E}$ in $C(\bar{\Omega})$ as $t \to \infty$. Therefore (5.1) also has a limit system which is monotone:
\[\begin{aligned}
\frac{\partial}{\partial t} I = k_1 (\hat{E} - I)\n + k_2 (\hat{E} - I)^+ I - \mu_1 I, & \quad x \in \Omega, t > 0, \\
\frac{\partial}{\partial t} V = \nabla \cdot \delta_2 (x) \nabla V = N(x) I - \mu V, & \quad x \in \Omega, t > 0.
\end{aligned}\]

For the models in [18, 19], our method is applicable when there are no chemotaxis. The analysis of the basic reproduction number of all these models can also be done similarly.

**Acknowledgment**

The authors would like to thank the referees and editor for many helpful comments.

**Appendix**

Let \(H_i(t), V_u(t)\) and \(V_i(t)\) be the densities of infected hosts, uninfected vectors, and infected vectors at time \(t\) respectively. Then the model is

\[
\begin{align*}
\frac{d}{dt} H_i(t) &= -\lambda H_i(t) + \sigma_1 H_u V_i(t), \quad t > 0, \\
\frac{d}{dt} V_u(t) &= -\sigma_2 V_u(t) H_i(t) + \beta(V_u(t) + V_i(t)) - \mu V_u(t) V_i(t), \quad t > 0, \\
\frac{d}{dt} V_i(t) &= \sigma_2 V_u(t) H_i(t) - \mu (V_u(t) + V_i(t)) V_i(t), \quad t > 0
\end{align*}
\]  

with initial value 

\((H_i(0), V_u(0), V_i(0)) \in M := \mathbb{R}_+^3.\)

The basic reproduction number \(R_0\) is defined as

\[R_0 := \frac{\sigma_1 \sigma_2 H_u}{\lambda \mu} .\]

The equilibria of (A.1) are \(ss_0 = (0,0,0), ss_1 = (0, \beta/\mu, 0),\) and

\[
ss_2 = \left( \frac{\beta (H_u \sigma_1 \sigma_2 - \lambda \mu)}{\lambda \mu \sigma_2}, \frac{\beta \lambda}{H_u \sigma_1 \sigma_2}, \frac{\beta H_u \sigma_1 \sigma_2 - \lambda \mu}{H_u \mu \sigma_1 \sigma_2} \right) = \left( \frac{\beta (R_0 - 1)}{\sigma_2}, \frac{\beta \lambda (R_0 - 1)}{R_0 \mu}, \frac{\beta H_u \sigma_1 \sigma_2}{R_0 \mu} \right) := (\hat{H}_i, \hat{V}_u, \hat{V}_i),
\]

which exists if and only if \(R_0 > 1.\)

If we add the last two equations of (A.1) then \(N(t) := V_u(t) + V_i(t)\) satisfies the logistic equation

\[\frac{d}{dt} N(t) = \beta N(t) - \mu N^2(t).\]  

(A.2)

We decompose the domain \(M := \mathbb{R}_+^3\) into the partition

\[M = \partial M_0 \cup M_0,\]

where

\[\partial M_0 := \{ (H_i, V_u, V_i) \in M : H_i + V_i = 0 \} \cup \{ V_u + V_i = 0 \} .\]
and
\[ M_0 := \{(H_i, V_u, V_i) \in M : H_i + V_i > 0 \text{ and } V_u + V_i > 0\} = M \setminus \partial M_0. \]

Biologically, we can interpret \( \partial M_0 \) as the states without vectors or infected individuals. The subregions \( \partial M_0 \) and \( M_0 \) are both positively invariant with respect to the semiflow generated by (A.1). We can also decompose \( M \) with respect to the subdomain
\[ \partial M_1 := \{(H_i, V_u, V_i) \in M : V_u + V_i = 0\} \]
and
\[ M_1 := \{(H_i, V_u, V_i) \in M : V_u + V_i > 0\}. \]

Since \( N(t) := V_u(t) + V_i(t) \) always satisfies the logistic equation (A.2), the subregions \( \partial M_1 \) and \( M_1 \) are both positively invariant with respect to the semiflow generated by (A.1).

**Lemma A.1.** Both \( \partial M_1 \) and \( M_1 \) are positively invariant by the semiflow generated (A.1). Moreover,
1. if \((H_i(0), V_u(0), V_i(0)) \in \partial M_1\), then
   \[ \lim_{t \to \infty} (H_i(t), V_u(t), V_i(t)) = (0, 0, 0); \]
2. if \((H_i(0), V_u(0), V_i(0)) \in M_1\), then
   \[ \lim_{t \to \infty} V_u(t) + V_i(t) = \frac{\beta}{\mu}. \]

If \((H_i(0), V_u(0), V_i(0)) \in M_1\) the long time behavior of (A.1) is characterized by
\[
\begin{align*}
    \frac{d}{dt} H_i(t) &= -\lambda H_i + \sigma_1 H_u V_i, & t > 0, \\
    \frac{d}{dt} V_u(t) &= \sigma_2 (\beta/\mu - V_i) + H_i - \beta V_i, & t > 0, \\
    H_i(0) &= H_{i0} \geq 0, & V_i(0) = V_{i0} \geq 0. \\
\end{align*}
\]

**Lemma A.2.** Suppose \( R_0 > 1 \). Then (A.3) has a unique positive equilibrium \((\hat{H}_i, \hat{V}_i)\). Moreover, \((\hat{H}_i, \hat{V}_i)\) is locally asymptotically stable, and if \( H_{i0} + V_{i0} \neq 0 \), then the solution \((H_i, V_i)\) of (A.3) satisfies
\[ \lim_{t \to \infty} (H_i(t), V_i(t)) = (\hat{H}_i, \hat{V}_i). \]

**Proof.** The uniqueness of the positive equilibrium \((\hat{H}_i, \hat{V}_i)\) can be checked directly when \( R_0 > 1 \). Let \( D = \mathbb{R}_+^2 \). Then \( D \) is invariant for (A.3). It is not hard to show that the solution of (A.3) is bounded.

Let \( F_1(H_i, V_i) = -\lambda H_i + \sigma_1 H_u V_i \) and \( F_2(H_i, V_i) = \sigma_2 (\beta/\mu - V_i) + H_i - \beta V_i \). Then \( \partial F_1 / \partial V_i \geq 0 \) and \( \partial F_2 / \partial H_i \geq 0 \) on \( D \). So (A.3) is cooperative. Let \( \Phi(t) : D \to D \) be the semiflow generated by the solution of (A.3). Then \( \Phi(t) \) is monotone.

If \( H_{i0} + V_{i0} \neq 0 \), then \( H_i(t) > 0 \) and \( V_i(t) > 0 \) for all \( t > 0 \). So without loss of generality, we may assume \( H_{i0} > 0 \) and \( V_{i0} > 0 \). We can choose \( \delta \) small such that \( F_1(\delta H_i, \delta V_i) \geq 0 \), \( F_2(\delta H_i, \delta V_i) \geq 0 \), \( H_{i0} \geq \delta H_i \), and \( V_{i0} \geq \delta V_i \). By [27, proposition 3.2.1], \( \Phi(t)(\delta H_i, \delta V_i) \) is non-
decreasing for \( t \geq 0 \) and converges to a positive equilibrium as \( t \to \infty \). Since \((\hat{H}, \hat{V})\) is the unique positive equilibrium, we must have \( \Phi(t)(\delta \hat{H}, \delta \hat{V}) \to (\hat{H}, \hat{V}) \) as \( t \to \infty \).

Similarly, we may choose \( k > 0 \) such that \( F_1(k \hat{H}, k \hat{V}) \leq 0, F_2(k \hat{H}, k \hat{V}) \leq 0, H_{i0} \leq k \hat{H}, \) and \( V_{i0} \leq k \hat{V} \). Then \( \Phi(t)(\delta \hat{H}, \delta \hat{V}) \) is non-increasing for \( t \geq 0 \) and \( \Phi(t)(k \hat{H}, k \hat{V}) \to (\hat{H}, \hat{V}) \) as \( t \to \infty \). By the monotonicity of \( \Phi(t) \), we have \( \Phi(t)(\delta \hat{H}, \delta \hat{V}) \leq \Phi(t)(H_{i0}, V_{i0}) \leq \Phi(t)(k \hat{H}, k \hat{V}) \) for \( t \geq 0 \). It then follows that \( \Phi(t)(H_{i0}, V_{i0}) \to (\hat{H}, \hat{V}) \) as \( t \to \infty \).

For any \( \epsilon' > 0 \) and initial data \((H_{i0}, V_{i0})\) satisfying \((1 - \epsilon')(\hat{H}, \hat{V}) \leq (H, V) \leq (1 + \epsilon')(\hat{H}, \hat{V})\), similar to the previous arguments, we can show \((1 - \epsilon')(\hat{H}, \hat{V}) \leq (H(t), V(t)) \leq (1 + \epsilon')(\hat{H}, \hat{V})\) for all \( t \geq 0 \). Therefore, \((\hat{H}, \hat{V})\) is locally stable. This proves the lemma.

We now present a uniform persistence result.

**Lemma A.3.** If \( R_0 > 1 \), then the semiflow generated by (A.1) is uniformly persistent with respect to \((M_0, \partial M_0)\) in the sense that there exists \( \epsilon > 0 \) such that, for any \((H_i(0), V_u(0), V_l(0)) \in M_0\), we have

\[
\liminf_{t \to \infty} \inf_{w \in \partial M_0} |(H_i(t), V_u(t), V_l(t)) - w| \geq \epsilon.
\]

(A.4)

**Proof.** We apply [15, theorem 4.1] to prove this result. Let \( \Phi(t) : \mathbb{R}^3_+ \to \mathbb{R}^3_+ \) be the semiflow generated by (A.1), i.e. \( \Phi(t)w_0 = (H_i(t), V_u(t), V_l(t)) \) for \( t \geq 0 \), where \((H_i(t), V_u(t), V_l(t))\) is the solution of (A.1) with initial condition \( w_0 = (H_i(0), V_u(0), V_l(0)) \in \mathbb{R}^3_+ \).

The semiflow \( \Phi(t) \) is point dissipative in the sense that there exists \( M > 0 \) such that \( \limsup_{t \to \infty} \|\Phi(t)w_0\| \leq M \) for any \( w_0 \in \mathbb{R}^3_+ \). Actually, lemma A.1 implies that \( \limsup_{t \to \infty} V_u(t) \leq \beta/\mu \) and \( \limsup_{t \to \infty} V_l(t) \leq \beta/\mu \). By the first equation of (A.1), we have \( \limsup_{t \to \infty} H_i(t) \leq \sigma_1 \beta H_u/\mu \).

We note that \( M_0 \) and \( \partial M_0 \) are both invariant with respect to \( \Phi(t) \). Moreover, the semiflow \( \Phi_0(t) := \Phi(t)|_{\partial M_0}\), i.e. the restriction of \( \Phi(t) \) on \( \partial M_0 \), admits a compact global attractor \( A_0 \). If \( w_0 = (H_i(0), V_u(0), V_l(0)) \in \partial M_0 \), then the \( \omega \)-limit set of \( w_0 \) is \( \omega(w_0) = \{ss_0\} \) if \( w_0 \in \partial M_l \) and \( \omega(w_0) = \{ss_1\} \) if \( w_0 \in \partial M_u \setminus \partial M_l \). Hence we have

\[
\bar{A}_0 := \cup_{w_0 \in A_0} \omega(w_0) = \{ss_0\} \cup \{ss_1\}.
\]

This covering is acyclic since \( \{ss_1\} \not\to \{ss_0\} \), i.e. \( W^u(ss_1) \cap W^s(ss_0) = \emptyset \). To see this, suppose \( w_0 = (H_i(0), V_u(0), V_l(0)) \in W^u(ss_1) \cap W^s(ss_0) \). Since \( w_0 \in W^u(ss_0) \), we have \( w_0 \in \partial M_l \). Let \((H_i(t), V_u(t), V_l(t))\) be the complete orbit through \( w_0 \), then \( V_u(t) = V_l(t) = 0 \) for \( t \in \mathbb{R} \). So \( w_0 \not\in W^u(ss_1) \), which is a contradiction.

We then show that \( W^p(ss_0) \cap M_0 = \emptyset \) and \( W^s(ss_1) \cap M_0 = \emptyset \). By lemma A.1, \( W^s(ss_0) = \partial M_l \subseteq \partial M_0 \) and hence \( W^p(ss_0) \cap M_0 = \emptyset \). To see \( W^p(ss_1) \cap M_0 = \emptyset \), it suffices to prove that there exists \( \epsilon > 0 \) such that, for any \( w_0 = (H_i(0), V_u(0), V_l(0)) \in M_0 \), the following inequality holds

\[
\limsup_{t \to \infty} |\Phi(t)w_0 - ss_1| \geq \epsilon.
\]

(A.5)

Assume to the contrary that (A.5) does not hold. Let \( \epsilon_0 > 0 \) be given. Then there exists \( w_0 \in M_0 \) such that
\[
\limsup_{t \to \infty} |\Phi(t)w_0 - ss_1| < \epsilon_0.
\]

So there exists \( t_0 > 0 \) such that \( \beta/\mu - \epsilon_0 \leq V_u(t) \leq \beta/\mu + \epsilon_0 \) and \( V_i(t) \leq \epsilon_0 \) for \( t \geq t_0 \).

By (A.1), we have
\[
\begin{align*}
\frac{d}{dt} H_i(t) &= -\lambda H_i + \sigma_1 H_u V_i, \\
\frac{d}{dt} V_i(t) &\geq \sigma_2 \left( \frac{\beta}{\mu} - \epsilon_0 \right) H_i - \mu \left( \frac{\beta}{\mu} + 2\epsilon_0 \right) V_i, \\
&\quad \text{if } t > t_0.
\end{align*}
\]

(A.6)

The matrix associated with the right hand side of (A.6) is
\[
A_{\epsilon_0} := \begin{bmatrix}
-\lambda & \sigma_1 H_u \\
\sigma_2 \left( \frac{\beta}{\mu} - \epsilon_0 \right) & -\mu \left( \frac{\beta}{\mu} + 2\epsilon_0 \right)
\end{bmatrix},
\]

whose eigenvalues \( \lambda_1 \) and \( \lambda_2 \) satisfy that \( \lambda_1 + \lambda_2 = -\lambda - \mu (\beta/\mu + 2\epsilon_0) < 0 \) and \( \lambda_1 \lambda_2 = \lambda \mu (\beta/\mu + 2\epsilon_0) - \sigma_1 \sigma_2 H_u (\beta/\mu - \epsilon_0) \). Since \( R_0 > 1 \), we can choose \( \epsilon_0 \) small such that \( \lambda_1 \lambda_2 < 0 \). Hence either \( \lambda_1 > 0 > \lambda_2 \) or \( \lambda_2 > 0 > \lambda_1 \). Without loss of generality, suppose \( \lambda_1 > 0 > \lambda_2 \). Then by the Perron–Frobenius theorem, there is an eigenvector \( (\phi, \psi) \) associated with \( \lambda_1 \) such that \( \phi > 0 \) and \( \psi > 0 \).

Let \((\tilde{H}_i(t), \tilde{V}_i(t))\) be the solution of the following problem
\[
\begin{align*}
\frac{d}{dt} \tilde{H}_i(t) &= -\lambda \tilde{H}_i(t) + \sigma_1 H_u \tilde{V}_i(t), \\
\frac{d}{dt} \tilde{V}_i(t) &= \sigma_2 \left( \frac{\beta}{\mu} - \epsilon_0 \right) \tilde{H}_i(t) - \mu \left( \frac{\beta}{\mu} + 2\epsilon_0 \right) \tilde{V}_i(t), \\
&\quad \text{if } t > t_0,
\end{align*}
\]

(A.7)

where \( \delta \) is small such that \( H_i(t_0) \geq \tilde{H}_i(t_0) \) and \( V_i(t_0) \geq \tilde{V}_i(t_0) \). By (A.6) and the comparison principle for cooperative systems, we have \((H_i(t), V_i(t)) \geq (\tilde{H}_i(t), \tilde{V}_i(t))\) for \( t \geq t_0 \). We can check that the solution of (A.7) is \((\tilde{H}_i(t), \tilde{V}_i(t)) = (\delta \phi e^{\lambda_1(t-t_0)}, \delta \psi e^{\lambda_2(t-t_0)})\). It then follows from \( \lambda_1 > 0 \) that \( \lim_{t \to \infty} \tilde{H}_i(t) = \infty \) and \( \lim_{t \to \infty} \tilde{V}_i(t) = \infty \), which contradicts the boundedness of the solution.

Our conclusion now just follows from [15, theorem 4.1].

We now present the result about the global dynamics of (A.1).

**Theorem A.4.** The following statements hold.

1. \( ss_0 \) is unstable: If \((H_i(0), V_u(0), V_i(0)) \in \partial M_1\), then
\[
\lim_{t \to \infty} (H_i(t), V_u(t), V_i(t)) = ss_0.
\]

2. Suppose \( R_0 < 1 \). Then \( ss_1 \) is globally asymptotically stable, i.e. \( ss_1 \) is locally asymptotically stable and if \((H_i(0), V_u(0), V_i(0)) \in M_0\), then
\[
\lim_{t \to \infty} (H_i(t), V_u(t), V_i(t)) = ss_1.
\]

3. Suppose \( R_0 > 1 \). Then \( ss_i \) is unstable, and if \((H_i(0), V_u(0), V_i(0)) \in \partial M_0 \setminus \partial M_1\), then
\[
\lim_{t \to \infty} (H(t), V_u(t), V_i(t)) = ss_1.
\]

Moreover, \(ss_2\) is globally asymptotically stable in the sense that \(ss_2\) is locally asymptotically stable and for any \((H_1(0), V_u(0), V_i(0)) \in M_0\),

\[
\lim_{t \to \infty} (H(t), V_u(t), V_i(t)) = ss_2.
\]

**Proof.** We only prove the second convergence result in part 3 (see [14] and lemma A.1 for the other parts). Since the solution of (A.3) is bounded, the omega limit set of the solution of (A.1) exists.

Suppose \((H(0), V_u(0), V_i(0)) \in M_0\). Then the solution \((H(t), V_u(t), V_i(t))\) of (A.1) satisfies that \(H(t), V_u(t), V_i(t) > 0\) for all \(t > 0\). Since \(V_u(0) + V_i(0) \neq 0\), we have \(V_u(t) + V_i(t) \to \beta/\mu\) as \(t \to \infty\). So,

\[
f(t) := \sigma_2 [V_u(t) - (\beta/\mu - V_i(t))^+] H_i(t) + (\beta - \mu(V_u(t) + V_i(t))) V_i(t) \to 0 \text{ as } t \to \infty,
\]

and the limit system of

\[
\begin{align*}
\frac{d}{dt} H_i(t) &= -\lambda H_i + \sigma_1 H_u V_i, \quad t > 0, \\
\frac{d}{dt} V_i(t) &= \sigma_2 V_u H_i - \mu(V_u + V_i)V_i = \sigma_2(\beta/\mu - V_i)^+ H_i - \beta V_i + f(t), \quad t > 0,
\end{align*}
\]

is (A.3). By lemma A.3 and \(V_u(t) + V_i(t) \to \beta/\mu\), there exists \(\epsilon > 0\) such that

\[
\liminf_{t \to \infty} |H_i(t)| + |V_i(t)| \geq \epsilon.
\]

Hence the omega limit set of (A.8) is contained in \(W := \{(H_0, V_0) \in \mathbb{R}_+^2 : H_0 + V_0 \neq 0\}\). By lemma A.2, \(W\) is the stable set of the stable equilibrium \((\dot{H}, \dot{V})\) of (A.3). By the theory of asymptotic autonomous systems, we have \(H_i(t) \to \dot{H}_i\) and \(V_i(t) \to \dot{V}_i\) as \(t \to \infty\). Moreover since \(V_u(t) + V_i(t) \to \beta/\mu = \dot{V}_u + \dot{V}_i\), we have \(V_u(t) \to \dot{V}_u\) as \(t \to \infty\). \(\blacksquare\)

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