The Dirac operator and gamma matrices for quantum Minkowski spaces

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Abstract

Gamma matrices for quantum Minkowski spaces are found. The invariance of the corresponding Dirac operator is proven. We introduce momenta for spin 1/2 particles and get (in certain cases) formal solutions of the Dirac equation.

0 Introduction

It is widely recognized that geometry of space–time should drastically change at very small distances, comparable with Planck’s length. On the other hand there is no satisfactory physical theory which would describe such a change. A lot of effort was devoted to simple physical models describing possible changes of geometry which can occur. One of possibilities is provided by the theory of quantum groups: the related examples of quantum space–times have a (quantum) group of symmetries which can be as big as the classical Poincaré group. If we want to have a quantum space–time which has exactly the same properties as the classical Minkowski space endowed with the action

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of (spinorial) Poincaré group, we get the classification of quantum Minkowski spaces and quantum Poincaré groups given in [7]. The related differential structure on quantum Minkowski spaces was determined in [8]. Therefore we are able to write Klein–Gordon equation for spin 0 particles and solve it (at least formally) in many cases. The same holds for Dirac equation for spin 1/2 particles provided we are able to find the gamma matrices (and certain other objects). This remaining task is undertaken in the present paper. In Section 1 we recall the theory of quantum Minkowski spaces and quantum Poincaré groups. In Section 2 we prove that the requirement of invariance of the Dirac operator determines all the gamma matrices up to two constants \( a, b \in \mathbb{C} \). The square of the Dirac operator is equal to the Laplacian (as in the classical case) if and only if \( ab = 1 \). The normalization \( a = b = 1 \) is chosen. We study certain expressions like the deformed Lagrangian. In Section 3 we get (in certain cases) formal solutions of the Dirac equation and introduce the momenta for spin 1/2 particles.

The gamma matrices for the example of the standard \( q \)-Lorentz group and \( q \)-Minkowski space ([9], [3]) were considered in [3] (cf. also [12], [11], [1]). This case however doesn’t fall into our scheme (the corresponding \( q \)-Poincaré group contains dilatations) and involves essentially only one 2-dimensional \( R \)-matrix (while in our considerations we have two independent 2-dimensional \( R \)-matrices: \( L \) and \( X \)).

The gamma matrices (given by (2.25) and (2.9)) and relations among them (the condition 2. of Theorem 2.2 and (2.7)) were announced in [9]. Recently, when the present paper was essentially completed, some gamma matrices (satisfying a condition like the condition 2. of Theorem 2.2) appeared also in [2]. That paper contains explicit formulae for the gamma matrices and metric tensor in some cases. Also the transformation properties are discussed there.

We sum over repeated indices (Einstein’s convention). If \( V, W \) are vector spaces then \( \tau : V \otimes W \to W \otimes V \) is given by \( \tau(x \otimes y) = y \otimes x, x \in V, y \in W. \) We denote the unit \( N \times N \) matrix by \( \mathbf{1}_N \) or \( \mathbf{1} \). If \( \mathcal{A} \) is an algebra, \( v \in M_N(\mathcal{A}), w \in M_K(\mathcal{A}) \), then the tensor product \( v \otimes w \in M_{NK}(\mathcal{A}) \) is defined by

\[
(v \otimes w)^{ij}_{kl} = v^i_k w^j_l, \quad i, k = 1, \ldots, N, \quad j, l = 1, \ldots, K.
\]

We set \( \dim v = N \). If \( \mathcal{A} \) is a \( * \)-algebra then the conjugate of \( v \) is defined as \( \bar{v} \in M_N(\mathcal{A}) \) where \( \bar{v}^i_j = (v^i_j)^* \). We also set \( v^* = \bar{v}^T \) (\( v^T \) denotes the
transpose of $v$, i.e. $(v^T)_{ij} = v_{ji})$. We write $a \sim b$ if $a, b$ are proportional, i.e. $a = kb$ for $k \in \mathbb{R}^\times$.

Throughout the paper quantum groups $H$ are abstract objects described by the corresponding Hopf $\ast$-algebras $\text{Poly}(H) = (\mathcal{A}, \Delta)$. We denote by $\Delta, \varepsilon, S$ the comultiplication, the counit and the coinverse of $\text{Poly}(H)$. In particular, $S$ is invertible ($S^{-1} = \ast \circ S \circ \ast$). We say that $v$ is a representation of $H$ ($v \in \text{Rep } H$) if $v \in M_N(\mathcal{A}), N \in \mathbb{N}$, and

$$\Delta v_{ij} = v_{ik} \otimes v_{kj}, \varepsilon(v_{ij}) = \delta_{ij}, i, j = 1, \ldots, N,$$

in which case $S(v_{ij}) = (v^{-1})_{ij}$. The conjugate of a representation and tensor products of representations are also representations. If $v, w \in \text{Rep } H$, $\dim v = N, \dim w = M$, then we say that $A \in M_{M \times N}(\mathbb{C})$ intertwines $v$ with $w$ if $Av = wA$. We say that $v, w$ are equivalent ($v \simeq w$) if such $A$ can be chosen as invertible. For $\rho \in \mathcal{A}'$ (the dual vector space of $\mathcal{A}$), $a \in \mathcal{A}$, we set $\rho \ast a = (\text{id} \otimes \rho)\Delta a$, $a \ast \rho = (\rho \otimes \text{id})\Delta a$.

1 Quantum Lorentz and Poincaré groups

In this section we recall the definitions and properties of quantum Lorentz and Poincaré groups as well as quantum Minkowski spaces. These objects are the natural generalizations of the standard objects known from the relativistic physics.

Quantum Lorentz groups are defined as quantum groups with the same properties as the classical Lorentz group $SL(2, \mathbb{C})$ [13]. The classification of quantum Lorentz groups is given as follows [13]. The set $\mathcal{A}$ of polynomials on a quantum Lorentz group is the universal unital $\ast$-algebra generated by $w_{AB}$, $A, B = 1, 2$, satisfying

$$w^A_B w^C_D E^{BD} = E^{AC},$$

$$E'_{AC} w^A_B w^C_D = E'_{BD},$$

$$X^{AB} C D w^C_K w^{D_L} = w^A_C w^B_D X^{CD}_{KL},$$

$A, B, C, D, K, L = 1, 2$, where the matrices $E \in M_{4 \times 1}(\mathbb{C}), E' \in M_{1 \times 4}(\mathbb{C}), X \in M_{4 \times 4}(\mathbb{C})$ are given (up to a nonzero factor) in [13]. The set $\mathcal{A}$ becomes
a Hopf \(\ast\)-algebra if we define the comultiplication and the counit in such a way that \(w\) becomes a representation, i.e.

\[
\Delta w^A_B = w^A_C \otimes w^C_B, \quad \varepsilon(w^A_B) = \delta^A_B, \quad A, B = 1, 2. \tag{1.4}
\]

The equations (1.1)-(1.3) can be also written as

\[
(w \otimes w)E = E, \quad E'(w \otimes w) = E', \quad X(w \otimes \bar{w}) = (\bar{w} \otimes w)X. \tag{1.5}
\]

In particular, setting \(E_{11} = E_{22} = 0, E_{12} = 1, E'_{AB} = -E^A_B, X^{AB}_{CD} = \delta^A_D \delta^B_C\), we get the classical Lorentz group \(SL(2, \mathbb{C})\). Then \(w^A_B\) are the matrix elements of the fundamental representation of \(SL(2, \mathbb{C})\), i.e. \(w^A_B(h) = h^A_B \in \mathbb{C}, h \in SL(2, \mathbb{C})\). Moreover, \(f^\ast(g) = \overline{f(g)} \in \mathbb{C}\) for \(f \in \mathcal{A}, g \in SL(2, \mathbb{C})\).

For any quantum Lorentz group we define its representation \(\Lambda\) as

\[
\Lambda = V^{-1}(w \otimes \bar{w})V \tag{1.6}
\]

where

\[
V^{AB}_{i} = (\sigma_i)_{AB}, \quad (V^{-1})^i_{AB} = \frac{1}{2}(\sigma_i)_{AB} = \frac{1}{2}(\sigma_i)_{BA}, \tag{1.7}
\]

and \(\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\)

are the Pauli matrices. Then \(\bar{\Lambda} = \Lambda\) is irreducible.

In the next step we introduce the quantum Poincaré groups, i.e. the quantum groups with the properties of the (spinorial) Poincaré group. Their definition and (almost complete) classification are given in [4]. It turns out that each quantum Poincaré group is related to one of quantum Lorentz groups described by \(E, E'\) and \(X = \tau Q'\) as follows:

1) \(E = e_1 \otimes e_2 - e_2 \otimes e_1, E' = -e^1 \otimes e^2 + e^2 \otimes e^1, Q' = \begin{pmatrix} t^{-1} & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t^{-1} \end{pmatrix}, \quad 0 < t \leq 1, \quad \text{or} \)
2) 

\[ E, E' \text{ as above} , \quad Q' = \begin{pmatrix} 1 & 0 & 0 & c^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{or} \]

3) \( E = e_1 \otimes e_2 - e_2 \otimes e_1 + ce_1 \otimes e_1, \quad E' = -e_1 \otimes e^2 + e^2 \otimes e^1 + ce^2 \otimes e^2, \)

\[ Q' = \begin{pmatrix} 1 & 0 & 0 & rc^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad r \geq 0, \quad \text{or} \]

4) 

\[ E, E' \text{ as above} , \quad Q' = \begin{pmatrix} 1 & c & c & 0 \\ 0 & 1 & 0 & -c \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{or} \]

5) \( E = e_1 \otimes e_2 + e_2 \otimes e_1, \quad E' = e^1 \otimes e^2 + e^2 \otimes e^1, \)

\[ Q' = i \begin{pmatrix} t^{-1} & 0 & 0 & 0 \\ 0 & -t & 0 & 0 \\ 0 & 0 & t^{-1} & 0 \\ 0 & 0 & 0 & t^{-1} \end{pmatrix}, \quad 0 < t \leq 1, \quad \text{or} \]

6) 

\[ E, E' \text{ as above} , Q' = i \begin{pmatrix} 1 & 0 & 0 & c^2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{or} \]

7) 

\[ E, E' \text{ as above} , \quad Q' = i \begin{pmatrix} r & 0 & 0 & \varsigma \\ 0 & -r & \varsigma & 0 \\ 0 & \varsigma & -r & 0 \\ \varsigma & 0 & 0 & r \end{pmatrix}, \quad \text{or} \]

\[ r = (t + t^{-1})/2, \quad \varsigma = (t - t^{-1})/2, \quad 0 < t < 1, \]
$e_1, e_2$ form the standard basis of $\textbf{C}^2$, $e^1, e^2$ the corresponding dual basis of $(\textbf{C}^2)^*$, $c \in \textbf{R}_*$ (cf. Remark below). One has $q = 1$ in the cases 1)-4) and $q = -1$ for 5)-7). We set $q^{1/2} = 1$ for $q = 1$ and $q^{1/2} = i$ for $q = -1$.

The set $\mathcal{B}$ of polynomials on a quantum Poincaré group is the universal unital *-algebra generated by $A$ and $y^i$, $i = 0, 1, 2, 3$, satisfying $I_B = I_A$,

\[
(R - 1)^{ij} k_{kl}(y^k y^l - Z^{kl} m y^m + T^{kl} - \Lambda^k m \Lambda^l m T^{mn}) = 0, \quad (1.8)
\]

\[
y^i w^i_A = G^{iA}_{Cj} w^j_B y^j + (H_V)^{iA}_{Cj} w^j_B - \Lambda^j_{k} w^A_{C} (H_V)^{jC}_B, \quad (1.9)
\]

\[
(y^i)^* = y^i, \quad (1.10)
\]

where $R = (V^{-1} \otimes V^{-1})(1_2 \otimes X \otimes 1_2)(L \otimes L)(1_2 \otimes X^{-1} \otimes 1_2)(V \otimes V)$, $G = (V^{-1} \otimes 1_2)(1_2 \otimes X)(L \otimes 1_2)(1_2 \otimes V)$, $Z = (1_4 \otimes V^{-1})[H_V \otimes 1_2 + (G \otimes 1_2)(1_2 \otimes H_V)]V$, $H_V = -q^{-1/2}(1_4 + qEE')$, $L = q^\pm L_\tau$, $s = \pm 1$, $(H_V)^{iC}_{D} = (V^{-1})^i_{AB} H^{ABC} D$, $T^{ij} = (V^{-1})^i_{AB} (V^{-1})^j_{CD} T^{ABCD}$, the possible $H^{ABC} D, T^{ABCD} \in \textbf{C}$ (for given quantum Lorentz group and $s$), $A, B, C, D = 1, 2, 3$, are provided in [7] (for $c, k = 1$). Then $\mathcal{B}$ becomes a Hopf *-algebra if we define the comultiplication and counit in such a way that

\[
\Delta y^i = \Lambda^i_{j} \otimes y^j + y^i \otimes I, \quad \varepsilon(y^i) = 0, \quad i = 0, 1, 2, 3, \quad (1.11)
\]

and [1,4] holds.

**Remark.** We set $c = k = 1$ for the objects considered in [1]. The objects with $c, k \in \textbf{R}_*$ ($c = 1$ in cases 1)-5)-7)) are isomorphic to those with $c = k = 1$ in the following way: $N = \text{diag}(c, 1)$, $N' = V^{-1}(N \otimes N)V$, $w(c) = NW^{-1}$, $X(c) = (N \otimes N)X(N^{-1} \otimes N^{-1})$, $E(c) \sim (N \otimes N)E$, $E'(c) \sim E'(N^{-1} \otimes N^{-1})$, $\Lambda(c) = NN^{-1}$, $y^i(c, k) = kN^i_j y^j$, $H(c, k) = k(N \otimes N \otimes N)HN^{-1}$, $T(c, k) = k^2(N \otimes N \otimes N)T$ etc. One should mention that formally taking $k \rightarrow 0$, next $c \rightarrow 0$ (in cases 2)-3)-4)) or $t \rightarrow 1$ (in case 1)), we can deform all the objects related to the cases 1)-4) with $s = 1$ (and any allowed $H, T$) to the classical case [1], $s = t = 1$, $H = T = 0$. Therefore $k, c$ and $1 - t$ can serve as small deformation parameters.

In particular, $H = T = 0$ is always allowed. If in addition we consider the classical Lorentz group and $s = 1$, we get the classical (spinorial) Poincaré group. It is defined as

\[
P = SL(2, \textbf{C}) \ltimes \textbf{R}^4 = \{(g, a) : g \in SL(2, \textbf{C}), a \in \textbf{R}^4\}
\]
with multiplication \((g,a) \cdot (g',a') = (gg',a + \lambda_g(a'))\) where the double covering \(SL(2, \mathbb{C}) \ni g \rightarrow \lambda_g \in SO_0(1,3)\) is given by \(\lambda_g(x)^i \sigma_i = g(x^j \sigma_j) g^*, \ g \in SL(2, \mathbb{C}), \ x \in \mathbb{R}^4\) (we treat \(\lambda_g\) as mapping from \(\mathbb{R}^4\) into \(\mathbb{R}^4\)). The group \(P\) is the double covering of the (connected component of) vectorial Poincaré group

\[
P = SO_0(1,3) \ltimes \mathbb{R}^4 = \{(M,a) : M \in SO_0(1,3), a \in \mathbb{R}^4\}
\]

with multiplication \((M,a) \cdot (M',a') = (MM',a + Ma')\). This covering \(\pi : P \rightarrow \tilde{P}\) is defined by \(\pi(g,a) = (\lambda_g,a)\). We should mention that \(P\) is more important in quantum field theory than \(\tilde{P}\). In these notations \(f(g,a) = f(g)\) for \(f \in A\) (in particular, \(w_{AB}(g,a) = w_{AB}(g) = g^A B \in \mathbb{C}\) and \(y^i(g,a) = a^i = i\)-th coordinate of \(a \in \mathbb{R}^4\). Then the relations (1.1)–(1.3) and (1.8)–(1.9) express the commutativity of our algebra \((R = \tau, G = \tau, Z = 0\) in this case). The case 1), \(s = 1, 0 < t < 1, H = T = 0\), corresponds to the quantum Poincaré group of [4].

Finally, we pass to quantum Minkowski spaces, i.e. the quantum analogues of the Minkowski space. Their definition and properties are also given in [7]. According to [7], each quantum Poincaré group admits exactly one quantum Minkowski space. The set \(C\) of polynomials on a quantum Minkowski space is introduced as the universal unital \(*\)-algebra generated by \(x^i, i = 0, 1, 2, 3\) satisfying the relations

\[
(R - 1_4)^{ij}{}_{kl}(x^k x^l - Z^{kl} m x^m + T^{kl}) = 0, \tag{1.12}
\]

\[
(x^i)^* = x^i, \tag{1.13}
\]

\(i, j = 0, 1, 2, 3\). The action of the quantum Poincaré group on the quantum Minkowski space is given by the unital \(*\)-homomorphism \(\Psi : C \rightarrow B \otimes C\) satisfying \((id \otimes \Psi)\Psi = (\Delta \otimes id)\Psi, (\varepsilon \otimes id)\Psi = id\) and

\[
\Psi(x^i) = \Lambda^i{}_{j} x^j + y^i \otimes I, \quad i = 0, 1, 2, 3. \tag{1.14}
\]

In particular, for the classical Poincaré group we get the classical Minkowski space \(M = \mathbb{R}^4\). Then \(x^i\) are the coordinates on \(M\): \(x^i(v) = v^i = i\)-th coordinate of \(v \in M\). Moreover, \(\Psi\) corresponds to the action \(\sigma : P \times M \rightarrow M\) of \(P\) on \(M\) given by \(\sigma(p,v) = (g,a)v = \lambda_g(v) + a, \ p = (g,a) \in P, \ v \in M\). One has

\[
\Psi(f) = f \circ \sigma \tag{1.15}
\]
for $f \in \mathcal{C}$.

Let us recall [8] that 4-dimensional covariant differential calculus on a quantum Minkowski space exists if and only if

$$\tilde{F} \equiv [(R-1) \otimes 1_4]\{(1_4 \otimes Z)Z-(Z \otimes 1_4)Z+T \otimes 1_4-(1_4 \otimes R)(R \otimes 1_4)(1_4 \otimes T)\} = 0.$$ 

This requirement singles out some quantum Poincaré groups which are described after the proof of Theorem 1.1 of [8] (in particular, the trivial choice $H = T = 0$ is always allowed). From now on we limit ourselves to quantum Poincaré groups and quantum Minkowski spaces with $\tilde{F} = 0$. Then for given quantum Minkowski space the 4-dimensional covariant differential calculus exists and is unique. It is described by partial derivatives $\partial_i : \mathcal{C} \rightarrow \mathcal{C}$, $i = 0, 1, 2, 3$, which are determined by the following properties:

$$\partial_i(I) = 0, \quad \partial_i(x^k f) = \delta^k_i f + (R^{kl} m x^n + Z^{kl}_i)(\partial_l f), \quad f \in \mathcal{C}. \quad (1.16)$$

The $\partial_i$ satisfy the following covariance property

$$(id \otimes \partial_j)(\Psi f) = (\Lambda^i_j \otimes I)[\Psi(\partial_i f)], \quad f \in \mathcal{C}. \quad (1.18)$$

According to Proposition 3.1.2 of [8], its proof and [7], $\partial_i$ can be also obtained as follows. We set $\tilde{G} = (V^{-1} \otimes 1_2)(1_2 \otimes \tilde{L})(X^{-1} \otimes 1_2)(1_2 \otimes V)$ and define a unital homomorphism $f = (f^i_j)_{i,j=0}^3 : \mathcal{A} \rightarrow M_4(\mathcal{C})$ by $f^i_j(w^C_D) = G^{iC}D_j$, $f^i_j(w^C_D^*) = \tilde{G}^{iC}D_j$. Then there exists a unital homomorphism $X : \mathcal{B} \rightarrow M_5(\mathcal{C})$ given by

$$\text{X}(a) = \left( \begin{array}{ccc} (f^i_j(S(a)))_{i,j=0}^3 & 0 & 0 \\ 0 & \varepsilon(a) & 0 \end{array} \right), \quad a \in \mathcal{A}, \quad (1.19)$$

$$\text{X}(y^i) = \left( \begin{array}{cc} (Z^i_0)_{j=0}^3 & (\delta^i_0)_{j=0}^3 \\ 0 & 0 \end{array} \right), \quad i = 0, 1, 2, 3. \quad (1.20)$$

In practical computations one can use the formula $f^i_j(S(a)) = f^{i,j}(a^*)$. It turns out that

$$\text{X} = \left( \begin{array}{cc} (X^i_j)_{j,\ell=0}^3 & (Y^i_j)_{j=0}^3 \\ 0 & \varepsilon \end{array} \right), \quad (1.21)$$

with $X^i_j, Y^i_j \in \mathcal{B}'$. One has

$$\partial_j = (Y_j \otimes \text{id})\Psi, \quad \rho^k_j = (X^k_j \otimes \text{id})\Psi \quad (1.22)$$
where $\rho_{j}^{k} : \mathcal{C} \rightarrow \mathcal{C}$ also appear in the proof of Theorem 1.1 of [8].

The metric tensor is defined [8] as an invertible matrix $g = (g^{ij})_{i,j=0}^{3} \in M_{4 \times 4}(\mathbb{C})$ such that its matrix entries $g^{ij}$ satisfy the invariance and self–adjointness conditions:

$$
\Lambda_{j}^{i} \Lambda_{k}^{l} g^{jl} = g^{ik},
$$

$$
\overline{g^{ik}} = g^{ki},
$$
i, k = 0, 1, 2, 3. Such $g$ is unique up to a nonzero real multiplicative factor. It satisfies $Rg = g$. Here we choose it as

$$
g = -2q^{1/2} (V^{-1} \otimes V^{-1}) (1_{2} \otimes X \otimes 1_{2}) (E \otimes \tau E) \quad (1.24)
$$

(cf. [8] and Remark below). Matrix elements of $g^{-1}$ are denoted by $g_{ij}$. The Laplacian is given [8] as

$$
\Box = g^{ij} \partial_{j} \partial_{i}. \quad (1.25)
$$

It is invariant and commutes with partial derivatives:

$$
(id \otimes \Box) [\Psi(f)] = \Psi(\Box f), \quad f \in \mathcal{C}, \quad (1.26)
$$

$$
\Box \partial_{i} = \partial_{i} \Box, \quad i = 0, 1, 2, 3. \quad (1.27)
$$

According to [4] and [8], the “sizes” of all our constructions are the same as for the classical Poincaré group, the classical Minkowski space and the standard differential calculus on it (we consider only the polynomial functions).

**Remark.** The factor $(-2)$ in (1.24) gives the standard metric tensor $g = \text{diag}(1, -1, -1, -1)$ for the classical Poincaré group and was taken into account in the expression for a propagator near the end of Section 4 of [8]. However, it was not fixed in the considerations after (3.8) of [8].

In particular cases the metric tensor $g = (g^{ij})_{i,j=0}^{3}$ equals

1) $g = q \text{diag}(t, -t^{-1}, -t^{-1}, -t)$,

2) $g = \frac{1}{2} q \left( \begin{array}{cccc}
2 - c^{2} & 0 & 0 & -c^{2} \\
0 & -2 & 0 & 0 \\
0 & 0 & -2 & 0 \\
-c^{2} & 0 & 0 & -2 - c^{2} \\
\end{array} \right)$,
In this section we prove that the requirement of invariance of the Dirac operator \( \partial / \partial x \) determines all the gamma matrices up to two constants. Then using the condition \( \partial / \partial x = \Box \) we provide the exact form of the gamma matrices. We also study certain expressions like the deformed Lagrangian.

We shall consider gamma matrices \( \gamma^i \in M_{4 \times 4}(\mathbb{C}) \), \( i = 0, 1, 2, 3 \). At the moment they are not determined yet. The Dirac operator has form \( \partial / \partial x = \gamma^i \otimes \partial_i \) (cf. [8]). It acts on bispinor functions \( \phi \in \tilde{\mathcal{C}} \equiv \mathbb{C}^4 \otimes \mathbb{C} \) (in a more advanced approach we should consider square integrable functions \( \phi \)).

In the classical case the Poincaré group \( P \) acts on \( \tilde{\mathcal{C}} \) as follows:

\[
\phi'(x') = S(g)\phi(x)
\]

where \( \phi \in \tilde{\mathcal{C}} \), \( x \in M \), \( \phi(x) \in \mathbb{C}^4 \), \( \phi' \) is \( \phi \) transformed by \( p = (g,a) \in P \), \( x' = p \cdot x \), \( \phi'(x') \in \mathbb{C}^4 \), \( S \) is a representation of the Lorentz group \( SL(2, \mathbb{C}) \) acting in the space \( \mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2 \) of bispinors, \( S \simeq w \oplus \bar{w} \) (undotted and dotted spinors). Writing \( \phi = \varepsilon_a \otimes \phi^a \) where \( \varepsilon_a \), \( a = 1, 2, 3, 4 \), form the standard basis of \( \mathbb{C}^4 \), replacing \( x \) by \( p^{-1}x \) and then \( p \) by \( p^{-1} \), using (1.15) and setting

\[
[\tilde{\Psi}(\phi)](p,x) = \phi'(x), \quad (2.1)
\]

\( \Psi(\phi^a) = \phi^{a(1)} \otimes \phi^{a(2)} \) (Sweedler’s notation, exception of Einstein’s convention), one obtains

\[
\tilde{\Psi}(\varepsilon_a \otimes \phi^a) = G_a^l \phi^{a(1)} \otimes \varepsilon_l \otimes \phi^{a(2)} \quad (2.2)
\]
where $G = (S^{-1})^T \simeq w \oplus \bar{w}$ (cf. remarks after (2.3)).

For the general quantum Minkowski space we define the action $\tilde{\Psi} : \tilde{\mathcal{C}} \rightarrow \mathcal{B} \otimes \tilde{\mathcal{C}}$ of quantum Poincaré group on $\tilde{\mathcal{C}}$ by (2.2) where $G$ is a representation equivalent to $w \oplus \bar{w}$. For the future convenience we choose

$$G = \epsilon w \oplus \bar{w} \quad (2.3)$$

where $\epsilon w = (w^T)^{-1}$. We notice that (1.3) implies (here we treat $E$ as $2 \times 2$ matrix and then $E^T = E^{-1}$)

$$wEw^T = E. \quad (2.4)$$

Therefore $\epsilon w \simeq w$ :

$$\epsilon w = E^{-1}wE. \quad (2.5)$$

In the classical case one has the invariance of the Dirac operator:

$$[\partial \phi] = \partial \phi'. \quad (2.6)$$

Then assuming that $\phi$ satisfies the Dirac equation $(i\partial - m)\phi = 0$, we get

$$0 = [(i\partial - m)\phi] = i(\partial \phi)' - m\phi' = (i\partial - m)\phi', \quad i.e. \phi' also satisfies the Dirac equation. \ It means that the Dirac equation is invariant. \ On the other hand using (2.1), the invariance condition (2.6) is equivalent to

$$\tilde{\Psi}(\phi) = (\text{id} \otimes \partial)[\tilde{\Psi}(\phi)], \quad \phi \in \tilde{\mathcal{C}}. \quad (2.7)$$

We set (2.7) as the condition of invariance of the Dirac operator for the general quantum Minkowski space. The main result of the present Section is contained in

**Theorem 2.1** *The following are equivalent:*

1. *The Dirac operator is invariant (i.e. (2.7) is satisfied)*

2. 

$$\gamma^i = \begin{pmatrix} 0 & bA_i \\ a\sigma_i & 0 \end{pmatrix} \quad (2.8)$$
where
\[ A_i = q^{-1/2} E^T (\sigma_i \circ D) E, \]  
(2.9)

\[(\sigma_i \circ D)_{KL} = (\sigma_i)_{AB} D^{AB}_{KL}, \quad D = \tau X^{-1} \tau, \quad a, b \in \mathbb{C} \quad (E \text{ is regarded here as } 2 \times 2 \text{ matrix}).\]

**Proof.** Setting \( \phi = \varepsilon_a \otimes \phi^a \) and using (2.2), one easily checks that the LHS of (2.7) equals
\[ G_m^s (\gamma^i)^m_a (\partial_j \phi^a)(1) \otimes \varepsilon_s \otimes (\partial_j \phi^a)(2). \]  
(2.10)

Similarly, using (2.2) and (1.18), we obtain that the RHS of (2.7) is equal to
\[ G^l_a \phi^a(1) \otimes (\gamma^i)_l \varepsilon_s \otimes \partial_i [\phi^a(2)] = G^l_a \Lambda^j_i (\partial_j \phi^a)(1)(\gamma^i)_l \otimes \varepsilon_s \otimes (\partial_j \phi^a)(2). \]  
(2.11)

Choosing \( \phi = x^r \varepsilon_t \), one gets \( \phi^a = x^r \delta^a_t \), \( \partial_j \phi^a = \delta^r_j \delta^a_t I \). In this case (2.7) (i.e. the equality of (2.10) and (2.11)) yields
\[ G_m^s (\gamma^i)^m_t = G^l_a \Lambda^j_i (\gamma^i)_l. \]  
(2.12)

On the other hand (2.12) implies that (2.10) equals (2.11) and (2.7) follows (for general \( \phi \)). Therefore (2.7) and (2.12) are equivalent.

Setting
\[ W_{l, i}^s = (\gamma^i)_l, \]  
(2.13)

we can write (2.12) as
\[ WG = (G \otimes \Lambda) W. \]  
(2.14)

Defining
\[ N = (1_4 \otimes V) W, \]  
(2.15)

using (1.6) and (2.3), (2.14) can be translated as
\[ N(\varepsilon w \oplus \bar{w}) = (\varepsilon w \otimes w \otimes \bar{w} \oplus w \otimes \bar{w} \oplus \bar{w} \otimes \bar{w}) N. \]  
(2.16)

According to (2.3) and Proposition 2.1 of [7] (cf. also [13]),
\[ \varepsilon w \otimes w \otimes \bar{w} \simeq (w^1 \otimes \bar{w}) \oplus \bar{w}, \quad \bar{w} \otimes w \otimes \bar{w} \simeq (w \otimes \bar{w}^T) \oplus w \]  
(decompositions into irreducible components, we use the notation of Proposition 2.1 of [7]). Thus
\[ N = \begin{pmatrix} 0 & N_1 \\ N_2 & 0 \end{pmatrix}, \]  
(2.17)
\( (2.18) \) means that

\[
N_1 \bar{w} = (c w \otimes w \otimes \bar{w}) N_1, \tag{2.18}
\]

\[
N_2 c w = (\bar{w} \otimes w \otimes \bar{w}) N_2 \tag{2.19}
\]

and \( N_1, N_2 \) are fixed up to multiplicative constants. Using the definition of \( c w, (1.3) \) and conjugated \((1.1)\), one can check that solutions of \((2.18), (2.19)\) are given by

\[
(N_1)^{ABC}_D = 2a \delta^A_B \delta^C_D,
\]

\[
(N_2)^{ABC}_D = 2b q^{-1/2} (X^{-1})^{BC}_{KLE} E^{KA} E^{LD},
\]

\( a, b \in \mathbb{C} \), where additional scalar factors \( 2, q^{-1/2} \) are added for future convenience. Using \((2.13), (2.15)\) and \((1.7)\), we finally get \((2.8)\).

In the standard Dirac theory \( \partial^2 = \square \). We set this as the additional condition for our gamma matrices. Then the Dirac equation \((i \partial - m) \phi = 0\) implies (formally) the Klein–Gordon equation \((\square + m^2) \phi = 0\).

**Theorem 2.2** Assume \((2.8)\). The following are equivalent:

1. \( \partial^2 = \square \).
2. \( \gamma^i \gamma^j + R^{ji}_{lk} \gamma^k \gamma^l = 2 g^{ji} 1, \quad i, j = 0, 1, 2, 3. \)
3. \( ab = 1. \)

**Remark.** The condition 2 was considered in \([5]\) (cf. \([10], [11]\)). For the classical Poincaré group it gives the standard relation \( \gamma^i \gamma^j + \gamma^j \gamma^i = 2 g^{ji} 1. \)

**Proof.** According to \([5]\), the condition 2 implies the condition 1. Conversely, assume that the condition 1 holds. Applying its both sides to \( x^m x^n \), using \((1.13), (1.17)\) and \( R g = g \), we get the condition 2. It remains to prove the equivalence of conditions 2 and 3.

Using \((2.13)\), it is easy to check that the condition 2 is equivalent to

\[
[1_4 \otimes (R + 1)](W \otimes 1_4)W = 2 \cdot 1_4 \otimes g. \tag{2.20}
\]

In virtue of \((2.15), (2.20)\) can be translated as

\[
[1_4 \otimes (R_L + 1)](N \otimes 1_4)N = 2 \cdot 1_4 \otimes g_L, \tag{2.21}
\]
where
\[ R_L = (1_2 \otimes X \otimes 1_2)(L \otimes \bar{L})(1_2 \otimes X^{-1} \otimes 1_2), \]
\[ g_L = -2q^{1/2}(1_2 \otimes X \otimes 1_2)(E \otimes \tau E). \]

This in turn means that
\[ [1_2 \otimes (R_L + 1_{16})](N_1 \otimes 1_4)N_2 = 2 \cdot 1_2 \otimes g_L, \quad (2.22) \]
\[ [1_2 \otimes (R_L + 1_{16})](N_2 \otimes 1_4)N_1 = 2 \cdot 1_2 \otimes g_L. \quad (2.23) \]

But we can write
\[ N_1 = 2a(M \otimes 1_2), \quad N_2 = 2bq^{-1/2}(1_2 \otimes X^{-1} \otimes T)(\tau E \otimes E \otimes 1_2) \]
where \( M : C \rightarrow C^2 \otimes C^2 \) and \( T : C^2 \otimes C^2 \rightarrow C \) are such that \( M^{AB} = T_{AB} = \delta^{AB} \). Using this,
\[ (1_2 \otimes T)(M \otimes 1_2) = (T \otimes 1_2)(1_2 \otimes M) = 1_2 \]
and the 16 relations (2.1), (2.3)–(2.9), (2.17)–(2.20) and (2.35)–(2.38) of [7], after some computations we get that the left hand sides of (2.22)–(2.23) are both equal to \( 2ab1_2 \otimes g_L \). Therefore the condition 2 is equivalent to \( ab = 1 \). \( \blacksquare \).

**Remark.** Here we explain why each of conditions (2.22)–(2.23) (expressing equalities of 32 \times 2 matrices) leads to only one numerical condition. Let us begin with (2.22). In virtue of (2.18), (2.19) \( (N_1 \otimes 1_4)N_2 \) intertwines \( c w \) with \( z = c w \otimes L \otimes L \) where \( L = w \otimes \bar{w} \). Using Proposition 2.1 of [7],
\[ L \otimes L \simeq I \oplus w^1 \oplus \bar{w}^1 \oplus (w^1 \otimes \bar{w}^1). \quad (2.24) \]
Thus
\[ z \simeq w \oplus (w \oplus w^{3/2}) \oplus (w \otimes \bar{w}^1) \oplus (w \otimes \bar{w}^1 \oplus w^{3/2} \otimes \bar{w}^1). \]

Hence \( (N_1 \otimes 1_4)N_2 \) can be nonzero only in first two components of (2.24). Moreover, using the remarks after (2.25) of [7], \( R_L + 1_{16} \) kills the second component of (2.24) and the LHS of (2.22) intertwines \( c w \) with \( c w \otimes I \) (here \( I \) denotes the trivial subrepresentation of \( L \otimes L \)). The same is true for the RHS (cf. (1.23)). Such intertwiners are represented by numbers and (2.22) is equivalent to one numerical condition. We treat (2.23) in a similar way.
Our remarks are valid in particular in the classical case.

The transformation \( a \mapsto x^{-1}a, \ b \mapsto xb \ (x \in \mathbb{C}\setminus\{0\}) \) corresponds to \( \gamma^i \mapsto D\gamma^iD^{-1} \) where \( D = xI_2 \oplus 1_2 \), which is equivalent to scaling of the undotted spinor. Therefore we may set \( a = b = 1 \) and obtain

\[
\gamma^i = \begin{pmatrix} 0 & A_i \\ \sigma_i & 0 \end{pmatrix}
\] (2.25)

where \( A_i \) are given by (2.9).

In particular cases one gets:

1) \( A_0 = qt\sigma_0, \ A_1 = -qt^{-1}\sigma_1, \ A_2 = -qt^{-1}\sigma_2, \ A_3 = -qt\sigma_3, \)

2) \( A_0 = q\begin{pmatrix} 1 - c^2 & 0 \\ 0 & 1 \end{pmatrix}, \ A_1 = -q\sigma_1, \ A_2 = -q\sigma_2, \ A_3 = q\begin{pmatrix} -1 - c^2 & 0 \\ 0 & 1 \end{pmatrix}, \)

3)

\[
A_0 = \begin{pmatrix} c^2(1 - r) + 1 & c \\ c & 1 \end{pmatrix}, \ A_1 = \begin{pmatrix} -2c & -1 \\ -1 & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ A_3 = \begin{pmatrix} c^2(1 - r) - 1 & c \\ c & 1 \end{pmatrix},
\]

4)

\[
A_0 = \begin{pmatrix} c^2 + 1 & 2c \\ 2c & 1 \end{pmatrix}, \ A_1 = -\sigma_1, \ A_2 = -\sigma_2, \ A_3 = \begin{pmatrix} c^2 - 1 & 2c \\ 2c & 1 \end{pmatrix}, \)

7) \( A_0 = (\varsigma - r)\sigma_0, \ A_1 = (r + \varsigma)\sigma_1, \ A_2 = (r - \varsigma)\sigma_2, \ A_3 = (r + \varsigma)\sigma_3 \)

\((q = 1 \text{ for } 1) - 4)\) and \( q = -1 \text{ for } 5) - 7)\).

In the following we shall study the analogues of certain sesquilinear expressions which appear in the standard Dirac theory. They involve the gamma matrix \( \gamma^0 \). But in the deformed case the corresponding matrix will be in general different from \( \gamma^0 \) and denoted by \( A \). We set

\[
A = \begin{pmatrix} 0 & K^T \\ K & 0 \end{pmatrix}, \quad \text{(2.26)}
\]
where
\[ K = -E(E^{-1})^T \]  
(2.27)
(as 2 × 2 matrices). For \( \phi = \varepsilon_a \otimes \phi^a \in \tilde{C} \) we put \( \phi^\dagger = \lambda_a \otimes \phi^{a*} \), \( \bar{\phi} = \phi^\dagger (A \otimes I) \) where \( \lambda_a \in (\mathbb{C}^4)^* \) form a basis dual to the standard basis \( \varepsilon_a \) of \( \mathbb{C}^4 \). We shall prove that expressions like \( \bar{\phi} \phi \) and deformed Lagrangian \( \mathcal{L} = \bar{\phi}(i\bar{\gamma} - m)\phi \) transform themselves in the same way as in the standard theory:

**Proposition 2.3** For \( \phi = \varepsilon_a \otimes \phi^a \), \( \chi = \varepsilon_a \otimes \chi^a \in \tilde{C} \) one has
\[
(\bar{\Psi} \phi)^\dagger (I \otimes A \otimes I) (\bar{\Psi} \chi) = \Psi[\phi^\dagger (A \otimes I) \chi],
\]
(2.28)
\[
(\bar{\Psi} \phi)^\dagger (I \otimes A \otimes I) [I \otimes (i\bar{\gamma} - m)] (\bar{\Psi} \chi) = \Psi[\phi^\dagger (A \otimes I) (i\bar{\gamma} - m) \chi],
\]
(2.29)
where \( (b_a \otimes c_a)^\dagger = b^*_a \otimes c^*_a \) for \( b_a \in \mathcal{B} \), \( c_a \in \tilde{C} \).

**Proof.** Using (2.5) and (2.4), we notice that
\[
wK(c_w)^T = wKE^T w^T (E^{-1})^T = -wEw^T (E^{-1})^T = -E(E^{-1})^T = K.
\]
Applying the hermitian conjugation, one obtains \( \bar{w}K^Tw^T = K^T \). These two equations and (2.3) give
\[
\bar{G}AG^T = A.
\]
(2.30)
Therefore
\[
(\bar{\Psi} \phi)^\dagger (I \otimes A \otimes I) (\bar{\Psi} \chi) = \left[(G_a^I \phi^{a(1)})^* \otimes \lambda_l \otimes \phi^{a(2)*}\right] [(I \otimes A \otimes I) [G_b^* \chi^{b(1)} \otimes \varepsilon_s \otimes \chi^{b(2)}]]
\]
\[
= \phi^{a(1)*} G_a^I \lambda_l G_b^* \chi^{b(1)} \otimes \phi^{a(2)*} \chi^{b(2)} = \phi^{a(1)*} A_{ab} \chi^{b(1)} \otimes \phi^{a(2)*} \chi^{b(2)}
\]
\[
= A_{ab} \Psi(\phi^{a*}) \Psi(\chi^b) = \Psi(\lambda_a A \varepsilon_b \otimes \phi^{a*} \chi^b) = \Psi(\phi^\dagger (A \otimes I) \chi)
\]
and (2.28) follows. Replacing \( \chi \) by \( (i\bar{\gamma} - m) \chi \) and using (2.7), one gets (2.29).
\( \square \).

**Remark.** We were not able to get a similar fact for \( \bar{\phi} \gamma^i \phi \).
In particular cases \( K \) equals:

1)2) \( K = 1_2 \), 3)4) \( K = \begin{pmatrix} 1 & -2c \\ 0 & 1 \end{pmatrix} \), 5)6)7) \( K = -1_2 \).
3 Solutions of the Dirac equation and momenta

In this section we use [8] to get (in certain cases) formal solutions of the Dirac equation. We introduce the momenta for spin $1/2$ particles. They have good transformation properties, commute with the Dirac operator, are selfadjoint w.r.t. the inner product introduced in Section 2 and in general differ from momenta for spin 0 particles. Unfortunately, only in some cases they are diagonalizable with real eigenvalues. Considerations in this section are largely formal (except of Proposition 3.1).

In the case when the Lorentz group is classical (case 1), $t = 1$, we get $R = \tau$. Then we obtain formal solutions of the Dirac equation $(i\partial - m)\phi = 0$ ($m \geq 0$) as in the case 2. of Section 4 of [8]. In these cases metric tensor $g = \text{diag}(1, -1, -1, -1)$ and the gamma matrices

$$
\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3,
$$

are classical. Solutions have form $\phi = v \otimes e^{-ix^a p_a}$ (according to the conventions of our paper, we use a different order of tensor product than in [8]) where $p_a$ are real numbers and $v$ is a solution of

$$
P_j \gamma^j v = mv \tag{3.1}
$$

with

$$
m = \sqrt{P_0^2 - P_1^2 - P_2^2 - P_3^2}
$$

and $P_j \in \mathbb{R}$ obtained from $p_i$ as in [8]. We solve (3.1) as in the standard theory:

for $m > 0$ we get

$$
v = \begin{pmatrix} \varphi \\ m^{-1} \sigma_k \varphi \end{pmatrix} \quad (\varphi \in \mathbb{C}^2),
$$

for $m = 0$ one has

$$
v \in \text{span}\left\{ \begin{pmatrix} P_1 - iP_2 \\ -P_0 - P_3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ P_0 + P_3 \\ P_1 + iP_2 \end{pmatrix} \right\} \quad (P_0 \neq -P_3),
$$
\[ v \in \text{span}\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \} \quad (\mathcal{P}_0 = -\mathcal{P}_3 \neq 0) \]

(or \( v \in \mathbb{C}^4 \) if all \( \mathcal{P}_i = 0 \) – unphysical case).

Now let us consider the case 1. of Section 4 of [8] (for \( Z = 0 \)). In addition to gamma matrices we need also *-algebra \( \mathcal{F} \) and its *-representations \( \pi \) in Hilbert spaces \( H \) with bases \( e_k, k \in K \). We recall that \( \mathcal{F} \) is generated by \( p^a, a = 0, 1, 2, 3, \) satisfying \((p^a)^* = p^a, p^k p^l = R^{kl}_{ji} p^j p^i \). Then solutions \( \varphi \) of the Dirac equation \( i\partial \varphi = m\varphi \) \((m \geq 0)\) are provided in terms of \( \varphi_{sl} = (\text{id} \otimes \pi_{sl})(e^{-ix \otimes p}) \) where \( x \otimes p = x^a \otimes g_{ab} p^b, s, l \in K, \pi_{sl}(a) = (e_s | \pi(a)e_l) \) for \( a \in \mathcal{F} \). Namely, \( \varphi = \varphi_{cl} = \varepsilon_i \otimes \nu^s \varphi_{sl} \) \((i = 1, 2, 3, 4)\) for \( v \in H \otimes \mathbb{C}^4 \) such that \( Uv = mv \). Here \( U^{si}n_k = \pi_{ks}(p^i)g_{al}(\gamma^a)^i_n, m = \pi(g_{ij}p^j p^i)^{1/2} \) (we assume that it is a number, which holds e.g. for \( \pi \) irreducible as in [8]).

In the following we limit ourselves to the cases 1)-2) of Section 1. Then \( g_{ab} \) are real, \( x \otimes p \) is selfadjoint, \( e^{-ix \otimes p} \) unitary and the components of \( \varphi \) are bounded (since the summation over \( s \) will be finite). Set \( A = p^0 + p^3, B = p^1 - ip^2, B^* = p^1 + ip^2, D = p^0 - p^3 \). We have found the following admissible \( \pi \):

1a) \( \pi_{abd} \) in \( l^2(\mathbb{Z}) \) with orthonormal basis \( e_n, n \in \mathbb{Z} \), defined by

\[
\pi_{abd}(A)e_n = t^{-2n}ae_n, \\
\pi_{abd}(B)e_n = be_{n-1}, \\
\pi_{abd}(B^*)e_n = be_{n+1}, \\
\pi_{abd}(D)e_n = t^{2n}de_n,
\]

\( a, d \in \mathbb{R}, (a, d) \neq (0, 0), b > 0 \).

b) \( \bar{\pi}_{abd} \) in \( \mathbb{C} \) defined by

\[
\bar{\pi}_{abd}(A) = a, \quad \bar{\pi}_{abd}(B) = b, \quad \bar{\pi}_{abd}(B^*) = \bar{b}, \quad \bar{\pi}_{abd}(D) = d,
\]

\( a, d \in \mathbb{R}, b = 0 \) or \( a = d = 0, b \in \mathbb{C}_s \).

2a) \( \pi_{ad} \) in \( l^2(\mathbb{N}) \) with orthonormal basis \( e_n, n \in \mathbb{N} \), defined by

\[
\pi_{ad}(A)e_n = c^2(a + nd)e_n,
\]

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\[ \pi_{ad}(B)e_n = cdn^{1/2}e_{n-1}, \]
\[ \pi_{ad}(B^*)e_n = cd(n + 1)^{1/2}e_{n+1}, \]
\[ \pi_{ad}(D)e_n = de_n, \]
\[ d \in \mathbb{R}, a \in \mathbb{R}, e_{-1} = 0. \]

b) \( \tilde{\pi}_{ab} \) in \( \mathbb{C} \) defined by
\[ \tilde{\pi}_{ab}(A) = a, \quad \tilde{\pi}_{ab}(B) = b, \quad \tilde{\pi}_{ab}(B^*) = \bar{b}, \quad \tilde{\pi}_{ab}(D) = 0, \]
\[ a \in \mathbb{R}, b \in \mathbb{C}. \]

After some computations one finds \( U, v \) and finally solutions \( \varphi \) of the Dirac equation \( i\partial \varphi = m\varphi \) \( (m \geq 0) \). They are (up to linear combinations) as follows:

1) \( m = (t^{-1}ad - t | b |^2)^{1/2}, \)
a) \( m > 0: \)
\[ \varphi = \begin{pmatrix} m\varphi_{nl} \\ 0 \\ t^{2n-1}d\varphi_{nl} \\ -tb\varphi_{n-1,l} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ m\varphi_{nl} \\ -tb\varphi_{n+1,l} \\ t^{-2n-1}a\varphi_{nl} \end{pmatrix}; \]
\( m = 0: \)
\[ \varphi = \begin{pmatrix} b\varphi_{nl} \\ dt^{2n-2}\varphi_{n-1,l} \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ b\varphi_{nl} \\ -at^{-2n}\varphi_{n-1,l} \end{pmatrix}. \]
b) \( m > 0 (b = 0, ad > 0): \)
\[ \varphi = \begin{pmatrix} m\varphi_{11} \\ 0 \\ t^{-1}d\varphi_{11} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ m\varphi_{11} \\ 0 \\ t^{-1}a\varphi_{11} \end{pmatrix}; \]
\( m = 0: \)
\[ \varphi = \begin{pmatrix} \varphi_{11} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ \varphi_{11} \\ 0 \end{pmatrix} \quad (a = b = 0), \]
2) $m = c(ad)^{1/2}$, $m > 0$:

\[
\varphi = \begin{pmatrix}
\varphi_{11} \\
0 \\
0 \\
0 \\
\varphi_{11}
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
\varphi_{11}
\end{pmatrix} \quad (d = b = 0).
\]

m = 0:

\[
\varphi = \begin{pmatrix}
\frac{m}{n} \varphi_{n+1,t} \\
0 \\
d \varphi_{n,t} \\
- c d n^{1/2} \varphi_{n-1,t}
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
\varphi_{n} \\
- c n^{1/2} \varphi_{n-1,t}
\end{pmatrix}.
\]

\[m = 0:\]

\[
\varphi = \begin{pmatrix}
\frac{c(n + 1)^{1/2}}{n} \varphi_{n+1,t} \\
\varphi_{n} \\
0 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
\varphi_{n} \\
0
\end{pmatrix}.
\]

b) $m = 0$ for $b = 0$, $a \neq 0$:

\[
\varphi = \begin{pmatrix}
\varphi_{11} \\
0 \\
0 \\
0 \\
\varphi_{11}
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
\varphi_{11}
\end{pmatrix}
\]

(we have omitted unphysical case $a = b = 0$ when $\varphi = \rho \otimes \varphi_{11}$ with any $\rho \in \mathbb{C}^4$).

Now we shall pass to the momenta for spin 1/2 particles (in general case). Let us recall that for spin 0 particles the momenta were defined as $P_j = i \partial_j$ and partial derivatives $\partial_j$ can be also obtained by $\partial_j = (Y_j \otimes \text{id}) \Psi$ (see (1.22)). It suggests to define the momenta for spin 1/2 particles as

\[
\tilde{P}_j = i \tilde{\partial}_j, \quad (3.2)
\]

\[
\tilde{\partial}_j = (Y_j \otimes \text{id}) \tilde{\Psi} : \tilde{C} \rightarrow \tilde{C}. \quad (3.3)
\]

This choice is justified by the following

**Proposition 3.1** Let us define $\tilde{\partial}_j$ as in (3.3). Then

\[
\tilde{\partial}_j \varphi = \varphi \tilde{\partial}_j, \quad (3.4)
\]

\[
(id \otimes \tilde{\partial}_j) \tilde{\Psi}(a) = (\Lambda_j \otimes I) \tilde{\Psi}(\tilde{\partial}_l(a)), \quad a \in \tilde{C}. \quad (3.5)
\]
Remark. Proposition 3.1 is valid for any homogeneous quantum space endowed with the action of inhomogeneous quantum group in the sense of \([8]\) (with \(\tilde{F} = 0\)). Equation (3.4) implies that (up to technical difficulties) the deformed momenta are well defined in the space of solutions of the Dirac equation \((i\partial - m)\phi = 0\) \((m \geq 0)\). Equation (3.5) means that the momenta for spin 1/2 particles transform themselves in the same way as the momenta for spin 0 particles (cf. (1.18)).

Proof. We set (cf. (1.22))
\[
\tilde{\rho}^j_k = (X^j_k \otimes \text{id}) \tilde{\Psi} : \tilde{C} \longrightarrow \tilde{C}.
\] (3.6)
Applying \(Y_j \otimes \text{id}\) or \(X^j_k \otimes \text{id}\) to (2.7), we get (3.4) and
\[
\tilde{\rho}^j_k \partial = \partial \tilde{\rho}^j_k.
\] (3.7)

Next we are going to prove
\[
Y_j * a = \Lambda^i_j \{a * Y_i\},
\] (3.8)
\[
\{X^j_k * a\} \Lambda^i_j = \Lambda^j_k \{a * X^i_j\},
\] (3.9)
a \(\in \mathcal{B}\). For \(a \in \mathcal{A}\) (3.8) is trivial while (3.9) follows from (1.19), (1.5) of \([8]\) and invertibility of \(S\). For \(a = y^s\) (3.8) is trivial while (3.9) follows from (1.19), (1.20) and (3.60) of \([8]\) \((p_i\) are called \(y^i\) now). But the set of \(a \in \mathcal{B}\) satisfying (3.8), (3.9) is an algebra (use the homomorphism property of \(X\) of (1.24)). Therefore (3.8), (3.9) follow for all \(a \in \mathcal{B}\).

Moreover, using (2.2), one obtains
\[
(id \otimes \tilde{\Psi}) \tilde{\Psi} = (\Delta \otimes \text{id}) \tilde{\Psi}.
\] (3.10)
Now tensoring (3.8), (3.9) from right by \(b \in \tilde{C}\), replacing \(a \otimes b\) by \(\tilde{\Psi}(x)\), \(x \in \tilde{C}\), using (3.10) and (3.3), (3.6), one obtains (3.5) and
\[
(id \otimes \tilde{\rho}^j_i) \tilde{\Psi}(a) (\Lambda^i_j \otimes I) = (\Lambda^i_j \otimes I) \tilde{\Psi}(\tilde{\rho}^j_i(a)).
\] (3.11)
In the same way, using \((id \otimes \Psi) \Psi = (\Delta \otimes \text{id}) \Psi\), one can also get (1.11)–(1.12) of \([8]\). \(\square\)

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Using the properties of $\mathbf{X}$ and (2.2), one gets

$$\tilde{\partial}_m = X^j_m (\mathcal{G}_a^l) E^a_i \otimes \partial_j, \quad \tilde{\rho}_m^k = X^j_m (\mathcal{G}_a^l) E^a_i \otimes \rho_j^k$$

where $E^a_i$ are matrix units ($E^a_i \varepsilon_b = \delta^a_b \varepsilon_i$). First consider the case when the Lorentz group is classical $[1], t = 1$ and $s = 1$. Then $f_{ij} = \delta_{ij} \varepsilon$. Using (1.19), one gets

$$\tilde{\partial}_m = \text{id} \otimes \partial_m, \quad \tilde{\rho}_m^k = \text{id} \otimes \rho_m^k.$$

Taking the solution $\varphi = v \otimes e^{-ix^a p_a}$ as in the beginning of the present section, one obtains

$$\tilde{P}_m \varphi = v \otimes i \partial_m e^{-ix^a p_a} = P_m \varphi$$

(cf. the case 2. of Section 4 of [8]). Thus in this case the momenta are just like the momenta for spin 0 particles. For all other cases it is easy to check that $f^i_m (w^A B) \neq \delta^i_m \delta^A_B$, $X^i_m (\mathcal{G}_a^l) \neq \delta^i_m \delta^l_a$ and

$$\tilde{\partial}_i \neq \text{id} \otimes \partial_i, \quad \tilde{\rho}_i^k \neq \text{id} \otimes \rho_i^k$$

(acting on $\phi = \varepsilon_a \otimes x^b$ or $\phi = \varepsilon_a \otimes \text{I}$). It means that in general momenta depend on spin.

Let us return to the general case. Set

$$F^t_r = g^{tm} X^j_m (\mathcal{G}_a^l) g_{jr} E^a_i,$$

$$\tilde{P}_t = ig^m \tilde{\partial}_m, \quad P^r = ig^r \partial_j. \text{ Then}$$

$$\tilde{P}^t = F^t_r \otimes P^r.$$

The transformation property of the momenta $\tilde{P}^t$ easily follows from (3.3):

$$(\text{id} \otimes \tilde{P}^t) \tilde{\Psi}(a) = [(g^{-1T} \Lambda g^T)_m^t \otimes \text{I}] \tilde{\Psi}(\tilde{P}^m(a)).$$

Due to (3.4), $\tilde{P}^t$ commute with $\tilde{\theta}$. One can check that $F^t_r$ are selfadjoint w.r.t. the inner product defined in Section 2 ($\tilde{\phi} F^t_r \tilde{\psi} = F^t_r \phi \psi$ for $\phi, \psi \in \mathbb{C}^4$) while $P^r$ are selfadjoint according to [8]. Therefore $\tilde{P}^t$ are selfadjoint. However, the inner product of Section 2 is not positively defined and selfadjointness doesn’t guarantee diagonalizability or reality of the spectrum as we shall see. Namely, after long computations one gets the following form for $\tilde{P}^t$ in cases 1)-2), $Z = 0$ (case 1. of Section 4 of [8] as considered above):
identifying \( \varphi_{nl} \) with \( e_n \) one can represent \( \tilde{P}^t \) as operators in \( C^4 \otimes H = (C^2 \otimes H) \oplus (C^2 \otimes H) \) such that

\[
\tilde{P}^t = \tilde{R}^t \oplus (\tilde{R}^t)^*
\]

(this is related to the selfadjointness of \( \tilde{P}^t \), * is given by the standard hermitian structures in \( C^2 \) and \( H \)) where

1)

\[
\tilde{R}^0 = \frac{s}{2} \begin{pmatrix}
  t\pi(A) + t^{-1}\pi(D) & 0 \\
  0 & t^{-1}\pi(A) + t\pi(D)
\end{pmatrix},
\]

\[
\tilde{R}^1 = \frac{s}{2} \begin{pmatrix}
  t^{-1}\pi(B)^T + t\pi(B^*)^T & 0 \\
  0 & t^{-1}\pi(B^*)^T + t\pi(B)^T
\end{pmatrix},
\]

\[
\tilde{R}^2 = i\frac{s}{2} \begin{pmatrix}
  t^{-1}\pi(B)^T - t\pi(B^*)^T & 0 \\
  0 & t\pi(B)^T - t^{-1}\pi(B^*)^T
\end{pmatrix},
\]

\[
\tilde{R}^3 = \frac{s}{2} \begin{pmatrix}
  t\pi(A) - t^{-1}\pi(D) & 0 \\
  0 & t^{-1}\pi(A) - t\pi(D)
\end{pmatrix},
\]

2)

\[
\tilde{R}^0 = \frac{s}{2} \begin{pmatrix}
  \pi(A) + \pi(D) & -c^2\pi(B)^T \\
  0 & \pi(A) + \pi(D)
\end{pmatrix},
\]

\[
\tilde{R}^1 = \frac{s}{2} \begin{pmatrix}
  \pi(B^*)^T + \pi(B)^T & -c^2\pi(D) \\
  0 & \pi(B^*)^T + \pi(B)^T
\end{pmatrix},
\]

\[
\tilde{R}^2 = i\frac{s}{2} \begin{pmatrix}
  \pi(B)^T - \pi(B^*)^T & c^2\pi(D) \\
  0 & \pi(B)^T - \pi(B^*)^T
\end{pmatrix},
\]

\[
\tilde{R}^3 = \frac{s}{2} \begin{pmatrix}
  \pi(A) - \pi(D) & -c^2\pi(B)^T \\
  0 & \pi(A) - \pi(D)
\end{pmatrix}.
\]

According to (3.4), \( \tilde{P}^t \) act in the subspace of solutions of the Dirac equation. However, considering \( \tilde{P}^1, \tilde{P}^2 \) w.r.t. the found solutions, there appears complex spectrum (case 1) or nondiagonalizability (case 2), \( m > 0 \). But in the case 2), \( m = 0 \) all \( \tilde{P}^r \) are diagonalizable with real spectrum (in the subspace of found solutions). One can try to overcome this difficulty by
modifying the definition of $\tilde{\partial}_m$. However, one checks that it is not possible to get always (in all cases 1)2)) diagonalizability with real spectrum (without spoiling the other properties of $\tilde{P}^m$) by choosing another ansatz of the form $\tilde{\partial}_m = M_m^j \otimes \partial_j$ ($M_m^j \in \mathbb{C}$, $M_m^j \neq 0$).

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