Two Dimensional Conformal Field Theory on Open and Unoriented Surfaces *

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Abstract

Introduction to two dimensional conformal field theory on open and unoriented surfaces. The construction is illustrated in detail on the example of SU(2) WZW models.

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1 Introduction

Two dimensional Conformal Field Theory (CFT) on open and unoriented surfaces is not a recent discovery. Its systematic study began in two seemingly different developments. On the one hand the implications of the presence of a boundary in two dimensional systems and the corresponding boundary conditions and boundary fields were first analyzed by Cardy [1] and further in [2, 3]. On the other hand a general prescription for the systematic construction of open and unoriented string models from a given closed oriented string model was proposed by Sagnotti [4] and further elaborated in [5, 6]. However it was only after the discovery of D-branes [7] that the topic attracted so much attention and a huge number of different models have been explicitly constructed (any list will be incomplete). A parallel development was the study of the general consistency conditions for the models, and in particular of the compatibility conditions between the Klein bottle projection and the annulus partition function embodied by the Möbius strip projection. As often in two dimensional conformal theories a rational completely solved model like the SU(2) Wess–Zumino–Witten model provided a good playground for such an analysis and exhibited three interesting properties

- for the diagonal models there is a standard solution which extends the Cardy ansatz for the annulus to the unoriented case [8];
- there may be several different Klein bottle projections corresponding to different spectra in the unoriented sector;
- the annulus partition function satisfies a completeness condition (satisfies the chiral fusion algebra) [9].

The last property extends also to all other explicitly solved examples, but a better understanding of the physical principle underlying the completeness condition in the general case, in particular in the framework of string theory where so far open and closed string completeness conditions appear rather asymmetrically, is still absent. Another important open problem is whether there will be new constraints on the unoriented sector coming from higher genus surfaces.

Two dimensional conformal field theory on surfaces with boundaries and crosscaps is a large and rapidly developing subject. The aim of these lectures is to give an introduction to the topic, hence we have chosen to present a self-contained exposition based on one relatively simple and completely solved example, namely the SU(2) Wess–Zumino–Witten (WZW) model. Even so some aspects like the explicit realization of the models in terms of D-branes
and orientifolds \[10\] and their geometry are not covered. Other important developments which have to be mentioned are the relations of boundary conformal theory to graph theory (for a review see \[11\]) and to topological field theory \[12\].

The material is organized as follows. In section \(\underline{2}\) we review some general properties of two dimensional CFT. In section \(\underline{3}\) we derive explicit expressions for the 4-point functions in the \(SU(2)\) WZW model, the corresponding exchange operators and fusion matrix. Section \(\underline{4}\) is devoted to the derivation of the sewing constraints for the correlation functions on open and unoriented surfaces. In section \(\underline{5}\) we analyze the partition functions and the consistency conditions they satisfy.

\section{General properties of two dimensional CFT}

\subsection{The stress energy tensor in two dimensions}

Let us begin by recalling the particular properties of the stress energy tensor in two dimensional conformal field theory. It is useful to introduce together with the flat Minkowski space light cone coordinates \(x_\pm = x^0 \pm x^1\) also the coordinates on the cylindric space \(S^1 \times \mathbb{R}^1\) (on which the conformal transformations are well defined globally \[13\]) \(t_\pm = \xi^0 \pm \xi^1\). Here \(\xi^0\) is the non-compact time variable on the cylinder, while \(\xi^1\) is the compact space variable \((\xi^1 + 2\pi\) is identified with \(\xi^1)\). We shall use also the analytic picture on the compact space \(S^1 \times S^1\) with coordinates

\[ z = e^{it}, \quad \bar{z} = e^{i\bar{t}} \quad , \tag{1} \]

where the complex variables \(z\) and \(\bar{z}\) are obtained from the Minkowski light cone coordinates by a Cayley transform

\[ z = \frac{1 + \frac{1}{2}x_-}{1 - \frac{1}{2}x_-}, \quad \bar{z} = \frac{1 + \frac{1}{2}x_+}{1 - \frac{1}{2}x_+}. \tag{2} \]

Note that \(z\) and \(\bar{z}\) are complex conjugate only if one starts from the Euclidean picture where \(\xi^0\) is purely imaginary, while \(\xi^1\) is real. Nonlinear transformations of the coordinates, like \(\underline{14}\), require nontrivial accompanying changes of the field variables. To find the transformation law for the stress energy
tensor let us first write its components in the light cone basis $x_\pm$

\[ \Theta_{\mu\nu} dx^\mu dx^\nu = \Theta^{++} dx_+^2 + \Theta^{+-} dx_+ dx_- + \Theta^{--} dx_-^2 , \quad (3) \]

where

\[ \Theta^{++} = \frac{1}{4} (\Theta_{00} + \Theta_{10} + \Theta_{01} + \Theta_{11}) \]

\[ \Theta^{--} = \frac{1}{4} (\Theta_{00} - \Theta_{10} - \Theta_{01} + \Theta_{11}) \]

\[ \Theta^{+-} = \Theta^{-+} = \frac{1}{4} (\Theta_{00} - \Theta_{11}) . \]

The energy density with our choice of metric

\[ \eta_{\mu\nu} = \text{diag}(-,+) \quad (4) \]

is given by $\Theta^0_0 = -\Theta_{00}$, so let us choose the three independent components of $\Theta_{\mu\nu}$ as

\[ \Theta = -\Theta^{--}, \quad \tilde{\Theta} = -\Theta^{++}, \quad \Theta_0 = -\Theta^{+-} = \frac{1}{4} \text{Tr} \Theta . \quad (5) \]

The conservation of the stress energy tensor $\partial_{\mu} \Theta^{\mu\nu} = 0$ then implies

\[ \partial_+ \Theta = -\partial_- \Theta_0 \quad \partial_- \tilde{\Theta} = -\partial_+ \Theta_0 , \quad (6) \]

where $\partial_\pm = 1/2(\partial_0 \pm \partial_1)$. The corresponding fields in the analytic picture are

\[ T(z, \bar{z}) = 2\pi \left( \frac{1}{i \partial z} \right)^2 \Theta(x_+(\bar{z}), x_-(z)) \]

\[ \tilde{T}(z, \bar{z}) = 2\pi \left( \frac{1}{i \partial \bar{z}} \right)^2 \tilde{\Theta}(x_+(\bar{z}), x_-(z)) \quad (7) \]

\[ T_0(z, \bar{z}) = 2\pi \left( \frac{1}{i \partial z} \right) \left( \frac{1}{i \partial \bar{z}} \right) \Theta_0(x_+(\bar{z}), x_-(z)) . \]

The conservation of $\Theta$ leads to the equations

\[ \partial T = -\partial T_0 \quad \partial \tilde{T} = -\partial \tilde{T}_0 \quad \left( \partial = \frac{\partial}{\partial z}, \tilde{\partial} = \frac{\partial}{\partial \bar{z}} \right) . \quad (8) \]
Thus if the stress energy tensor is traceless ($T_0 = 0$) each of the two components $T$ and $\bar{T}$ depends on a single variable $T = T(z)$ and $\bar{T} = \bar{T}(\bar{z})$.

A similar separation in chiral and antichiral components is valid also for an abelian current $j_\mu$ that is conserved together with its dual

$$\partial_\mu j^\mu = 0 = \partial^\mu \epsilon_{\mu\nu} j^\nu.$$  \hfill (9)

We shall call such fields which split into chiral and antichiral components local observables. In other words, one can define the two dimensional conformal field theory as a quantum field theory in which the observable algebra is a tensor product of two algebras

$$\mathcal{A} \otimes \bar{\mathcal{A}}.$$  \hfill (10)

The chiral (or analytic) algebra $\mathcal{A}$ and the antichiral (or antianalytic) algebra $\bar{\mathcal{A}}$ are related by space reflection. For the rest of these lectures we shall assume that $\mathcal{A}$ and $\bar{\mathcal{A}}$ are isomorphic. The algebra $\mathcal{A}$ is generated by a finite number of local fields $O_n(z)$. It should be stressed that this condition does not lead necessarily to a finite number of fields in the theory. Locality implies that all $O_n(z)$ mutually commute for different arguments, more precisely for any given $n$ and $m$ there exists an integer $N_0(n,m)$ such that for all $N \geq N_0$

$$(z_1 - z_2)^N [O_n(z_1), O_m(z_2)] = 0.$$  \hfill (11)

The general solution of this equation is given by a linear combination of the $\delta$ function and its derivatives

$$[O_n(z_1), O_m(z_2)] = \sum_{\ell=0}^{N_0-1} C_\ell(z_2) \delta^{(\ell)}(z_{12}),$$  \hfill (12)

where $\delta$ on the unit circle can be defined as

$$\delta(z_{12}) = \frac{1}{z_1} \sum_n \left( \frac{z_2}{z_1} \right)^n = \frac{1}{z_1} \sum_{n=0}^{\infty} \left( \frac{z_2}{z_1} \right)^n + \frac{1}{z_2} \sum_{n=0}^{\infty} \left( \frac{z_1}{z_2} \right)^n$$  \hfill (13)

and satisfies

$$\oint \delta(z_{12}) f(z_2) \frac{dz_2}{2\pi i} = f(z_1).$$  \hfill (14)
For the currents (of scale dimension 1) and for the stress energy tensor (of scale dimension 2) this leaves undetermined only one constant. In particular

$$[T(z_1), T(z_2)] = -\frac{c}{12} \delta'''(z_{12}) - \delta'(z_{12})(T(z_1) + T(z_2)) \ ,$$

where the constant $c$ is called central charge. The same relation holds also for the antichiral component $\bar{T}$ with central charge $\bar{c}$ (due to the assumption that $\mathcal{A}$ and $\bar{\mathcal{A}}$ are isomorphic). All fields from $\mathcal{A}$ commute with all fields from $\bar{\mathcal{A}}$, hence $T(z)$ and $\bar{T}(\bar{z})$ commute. Under a general analytic reparametrization $z \to w(z)$ the stress energy tensor transforms according to

$$T(z) \to T(w) = \left(\frac{\partial z}{\partial w}\right)^2 T(z(w)) + \frac{c}{12} \{w, z\} \ ,$$

where $\{w, z\}$ is the Schwartz derivative

$$\{w, z\} = \frac{w'''}{w} - \frac{3}{2} \left(\frac{w''}{w}\right)^2 .$$

The central term in (15), (16) is related to the conformal anomaly. $T(z)$ has a Laurent expansion of the form

$$T(z) = \sum_n \frac{L_n}{z^{n+2}} \ ,$$

where the modes $L_n$ are given by

$$L_n = \frac{1}{2\pi i} \oint_{S^1} dz \ T(z) \ z^{n+1} .$$

The commutator (13) for the chiral components of the stress energy tensor implies for the modes $L_n$ the commutation relations of the Virasoro algebra $Vir$

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n(n^2 - 1)\delta_{n+m} \ ,$$

where $\delta_\ell$ denotes the Kronecker symbol $\delta_{\ell,0}$. The central term in (20) vanishes for $n = 0, \pm 1$. The corresponding subalgebra generated by $L_{-1}$, $L_0$ and $L_1$ is $SL(2, \mathbb{R})$. The unique vacuum vector $|0\rangle$ is annihilated by $L_{-1}$, $L_0$ and $L_1$ (and by their antichiral counterparts)

$$L_{0,\pm 1}|0\rangle = 0 = \bar{L}_{0,\pm 1}|0\rangle .$$
The Hermiticity of the stress energy tensor gives for the modes

\[ L_n^1 = L_{-n} \quad . \] (22)

Not all the fields in the theory split into chiral and antichiral parts. In particular there exist “primary” conformal fields \([14, 16]\), of conformal weights \(\Delta\) and \(\bar{\Delta}\), which under reparametrizations \(z \to w(z)\), \(\bar{z} \to \bar{w}(\bar{z})\) transform as

\[ \phi_{\Delta\bar{\Delta}}(z, \bar{z}) \to \phi_{\Delta\bar{\Delta}}(w, \bar{w}) = \left( \frac{\partial z}{\partial w} \right)^{\Delta} \left( \frac{\partial \bar{z}}{\partial \bar{w}} \right)^{\bar{\Delta}} \phi_{\Delta\bar{\Delta}}(z(w), \bar{z}(\bar{w})) \quad . \] (23)

This transformation law implies the following commutation relations between the primary fields and the generators of the Virasoro algebra \(L_n\)

\[
\begin{align*}
[L_n, \phi_{\Delta\bar{\Delta}}(z, \bar{z})] & = z^n \left( z \partial_z + (n + 1) \Delta \right) \phi_{\Delta\bar{\Delta}}(z, \bar{z}) \quad (24) \\
[\bar{L}_n, \phi_{\Delta\bar{\Delta}}(z, \bar{z})] & = \bar{z}^n \left( \bar{z} \partial_{\bar{z}} + (n + 1) \bar{\Delta} \right) \phi_{\Delta\bar{\Delta}}(z, \bar{z}) \quad . \end{align*}
\]

The corresponding states obtained by acting with the primary fields on the vacuum are also called primary

\[ |\Delta, \bar{\Delta}\rangle = \phi_{\Delta\bar{\Delta}}(0, 0) |0\rangle \quad . \] (26)

They are annihilated by all the generators \(L_n\) with \(n > 0\)

\[ L_n |\Delta, \bar{\Delta}\rangle = \bar{L}_n |\Delta, \bar{\Delta}\rangle = 0 \quad \text{for} \quad n > 0 \quad . \] (27)

The conformal dimension of a primary field is equal to the sum of its two conformal weights, while its spin (or helicity) is equal to their difference

\[ d = \Delta + \bar{\Delta} \quad , \quad s = \Delta - \bar{\Delta} \quad . \] (28)

There exist also fields that satisfy (24,25) only for \(n = 0, \pm 1\). Such fields are called quasiprimary (or conformal descendants). The corresponding quasiprimary states are obtained from the primary states (26) by the action of polynomials in \(L_n\) with negative \(n\). All the properties of the quasiprimary fields follow from those of the underlying primary one.
2.2 Rational conformal field theories

One important class of theories are the Rational Conformal Field Theories (RCFT). In a RCFT there are only a finite number of primary fields. For example, in the unitary minimal models \[14, 16\] corresponding to central charge of the Virasoro algebra

\[ c = 1 - \frac{6}{m(m+1)} \quad m \geq 3 \] (29)

the primary fields have weights

\[ \Delta_{r,s} = \frac{[r(m+1) - sm]^2 - 1}{4m(m+1)} \quad 1 \leq r \leq m-1 \quad 1 \leq s \leq m . \] (30)

Another important example are the superconformal models. The supersymmetry generator \( G(z) \) has conformal weight \( 3/2 \) and hence a Laurent expansion

\[ G(z) = \sum_r \frac{G_r}{z^{r+\frac{3}{2}}} . \] (31)

Since \( G(z) \) has half integer spin, it can be chosen either periodic (Ramond sector) or antiperiodic (Neveu-Schwarz sector) \[17\]. In the Ramond sector the sum in (31) is over \( r \) integer, while in the Neveu-Schwarz sector it is over \( r \) half-integer. The (anti)commutation relations between \( L_n \) and \( G_r \) are

\[ [L_n, G_r] = \left( \frac{n}{2} - r \right) G_{n+r} \] (32)

\[ \{G_r, G_s\} = 2L_{r+s} + \frac{c}{3} \left( r^2 - \frac{1}{4} \right) \delta_{r+s} . \] (33)

The unitary \( N = 1 \) superconformal models have central charge

\[ c = \frac{3}{2} \left[ 1 - \frac{8}{m(m+2)} \right] \quad m \geq 3 , \] (34)

while the conformal weights of the primary fields are \[18, 19\]

\[ \Delta_{r,s} = \frac{[r(m+2) - sm]^2 - 4}{8m(m+2)} + \frac{1}{32} \left[ 1 - (-1)^{r-s} \right] , \] (35)

where \( 1 \leq r \leq m-1 \) and \( 1 \leq s \leq m \). The Neveu-Schwarz sector contains the fields with \( r - s \) even, while the Ramond sector contains the fields with \( r - s \) odd.
In order to describe the $N = 2$ superconformal models \[20\] it is convenient to study first the simplest example of a conformal current algebra, namely the abelian $U(1)$ case. The chiral part of the $U(1)$ current satisfying \[9\] has the following expansion in Laurent modes
\[
J(z) = \sum_n \frac{J_n}{z^{n+1}}, \quad J_n^\dagger = J_{-n}.
\]
(36)

Since the $U(1)$ current is a primary field of the Virasoro algebra of weight one, its commutation relations with the modes of the stress energy tensor are
\[
[L_n, J_m] = -m J_{m+n}.
\]
(37)
The locality condition \[12\] determines completely also the commutation relations between two currents
\[
[J(z_1), J(z_2)] = -\delta'(z_{12}) \quad \text{or} \quad [J_n, J_m] = n\delta_{n+m}.
\]
(38)
where for convenience we have chosen to normalize the central term to one. The same relations hold also for the antichiral components. The primary fields of the $U(1)$ conformal current algebra are characterized by their charges $q$ and $\bar{q}$ and satisfy the following commutation relations with the current components
\[
[J(z_1), \phi_{qq}(z_2, \bar{z}_2)] = -q\phi_{qq}(z_2, \bar{z}_2) \delta(z_{12})
\]
(39)
\[
[J(z_1), \phi_{q\bar{q}}(z_2, \bar{z}_2)] = -\bar{q}\phi_{q\bar{q}}(z_2, \bar{z}_2) \delta(\bar{z}_{12})
\]
(40)
The stress energy tensor can be expressed in terms of the currents by the Sugawara formula \[21\] and the central charge of the Virasoro algebra is equal to one
\[
T(z) = \frac{1}{2} : J^2(z) : \quad \Rightarrow \quad c(u(1)) = 1,
\]
(41)
which for the Laurent modes gives
\[
L_n = \frac{1}{2} \left( \sum_{m \geq 1} + \sum_{m \geq -n} \right) J_{-m} J_{m+n}.
\]
(42)
The consistency of equations \[39\], \[42\] and \[24\] implies a relation between the $U(1)$ charges and the conformal weights
\[
\Delta = \frac{1}{2} q^2, \quad \bar{\Delta} = \frac{1}{2} \bar{q}^2,
\]
(43)
as well as the following equations for the primary fields

$$\begin{align*}
\partial_z \phi_{q\bar{q}}(z, \bar{z}) + q : J(z) \phi_{q\bar{q}}(z, \bar{z}) : &= 0 \\
\partial_{\bar{z}} \phi_{q\bar{q}}(z, \bar{z}) + \bar{q} : \tilde{J}(\bar{z}) \phi_{q\bar{q}}(z, \bar{z}) : &= 0 .
\end{align*}$$

(44) (45)

The $N = 2$ superconformal algebra contains two supersymmetry generators $G^\alpha(z)$, $\alpha = 1, 2$, with Laurent expansions (31) and a $U(1)$ current $J(z)$ with expansion (36). The new (anti)commutation relations are

$$\begin{align*}
\{ G_r^\alpha, G_s^\beta \} &= 2\delta^{\alpha\beta} L_{r+s} \\
i(r-s)e^{\alpha\beta} J_{r+s} + \frac{c}{3} \left( r^2 - \frac{1}{4} \right) \delta^{\alpha\beta} \delta_{r+s} \\
[J_m, G_r^\alpha] &= i\epsilon^{\alpha\beta} G_r^\beta ,
\end{align*}$$

(46) (47)

where $\epsilon^{\alpha\beta}$ is antisymmetric and $\epsilon^{12} = 1$. There are three sectors: in the Neveu–Schwarz and Ramond sectors the $U(1)$ current has integer modes, while in the twisted sector the $U(1)$ current has half integer modes [24]. The unitary minimal $N = 2$ superconformal models correspond to central charges

$$c = 3 \left( 1 - \frac{2}{m} \right) \quad m \geq 3 .$$

(48)

2.3 Nonabelian conformal current algebras

The nonabelian generalization of the $U(1)$ conformal current algebra (38) known also as Wess-Zumino-Witten (WZW) model is one of the few cases of two dimensional CFT for which one can write also an explicit action [25]. Alternatively one can use the following definition. Let $G$ be a compact semi-simple Lie group and $\mathcal{G}$ be its Lie algebra of dimension $d_G$. The chiral conformal current algebra $\mathcal{A}(\mathcal{G})$ is the algebra generated by the $d_G$ chiral currents in the adjoint representation of $\mathcal{G}$. The currents are primary fields of the Virasoro algebra of conformal weight one and have the Laurent expansion

$$J^a(z) = \sum_n \frac{J^a_n}{z^{n+1}} \quad J^a_n = J^a_{-n} .$$

(49)

The commutation relations for their modes are

$$[J^a_n, J^b_m] = i \sum_c f_{abc} J^c_{n+m} + \frac{k}{2n} \delta_{ab} \delta_{n+m} ,$$

(50)
where $f_{abc}$ are the structure constants of $\mathcal{G}$ and the level $k$ is a nonnegative integer. These relations define an affine Kac-Moody algebra [26].

The stress energy tensor can be expressed in terms of the currents (49) by the Sugawara formula

$$2h \ T(z) = \sum_{a=1}^{d_G} : J^2_a(z) : ,$$  \hspace{1cm} (51)

where the height $h$ is the sum of the level $k$ and the dual Coxeter number of $\mathcal{G}$, $h = k + g_\hat{c}$ ($= k + N$ for $SU(N)$). In terms of the Laurent modes (51) becomes

$$2hL_n = \left( \sum_{\ell=1}^{\infty} + \sum_{\ell=-n}^{\infty} \right) \sum_{a=1}^{d_G} J^a_{-\ell} J^a_{n+\ell} ,$$  \hspace{1cm} (52)

while the central charge of the Virasoro algebra is

$$c = \frac{k}{h} \ d_G .$$  \hspace{1cm} (53)

The primary fields of $\mathcal{A}(\mathcal{G})$ are in one-to-one correspondence with the irreducible representations of $\mathcal{G}$, hence we can label them by highest weight vectors $\Lambda = (\lambda_1, \ldots, \lambda_r)$ of $\mathcal{G}$. We shall denote the primary fields by $V_\Lambda(z)$. They satisfy the following commutation relations with the currents (for brevity we omit the dependence on $\bar{z}$ and write only the relations in the chiral sector)

$$[J^a(z_1), V_\Lambda(z_2)] = \delta(z_{12}) \ V_\Lambda(z_2) \ t^a_\Lambda$$  \hspace{1cm} (54)

or in terms of the modes (19)

$$[J^a_n, V_\Lambda(z)] = z^n \ V_\Lambda(z) \ t^a_\Lambda ,$$  \hspace{1cm} (55)

where $t^a_\Lambda$ are the matrices of $J^a_0$ in the representation $\Lambda$. The consistency of equations (52) and (55) with (24) implies the relation

$$2h \ \Delta_\Lambda = C_2(\Lambda)$$  \hspace{1cm} (56)

between the conformal weight of the primary field and the eigenvalue of the second order Casimir operator in the representation $\Lambda$, as well as the operator form of the Knizhnik–Zamolodchikov (KZ) equation [22, 23]

$$h \ \frac{d}{dz} \ V_\Lambda(z) = \sum_{a=1}^{d_G} : V_\Lambda(z) \ t^a_\Lambda J^a(z) : .$$  \hspace{1cm} (57)
The primary fields in a two dimensional conformal theory transforming as in (23) in general do not split in a sum of chiral and antichiral components. Rather they are given by a (finite in the case of a rational conformal theory) sum of products of chiral and antichiral vertex operators [27, 28]. In order to properly define a chiral vertex operator we have to specify a triple of weights \( (\Lambda_f, \Lambda_i) \) where \( \Lambda_i \) is the weight on which \( V_{\Lambda} \) acts, while \( \Lambda_f \) is the weight to which \( V_{\Lambda} \) maps. In other words, the chiral vertex operators can be represented as

\[
V_{\Lambda}^{\Lambda_f, \Lambda_i}(z) = \Pi_{\Lambda_f} V_{\Lambda} (z) \Pi_{\Lambda_i},
\]

where \( \Pi_{\Lambda} \) are orthogonal projectors, and in general are multivalued functions of \( z \)

\[
V_{\Lambda}^{\Lambda_f, \Lambda_i}(e^{2\pi i} z) = e^{2\pi i(\Delta_{\Lambda_f}-\Delta_{\Lambda_i})} V_{\Lambda}^{\Lambda_f, \Lambda_i}(z).
\]

The correlation functions of the chiral vertex operators are called chiral conformal blocks and due to (59) are also multivalued functions of the coordinates. The two dimensional primary fields \( \phi(z, \bar{z}) \) can be written in terms of the chiral vertex operators (58) as

\[
\phi_{\Lambda \bar{\Lambda}}(z, \bar{z}) = \sum_{\Lambda_f, \Lambda_i} V_{\Lambda}^{\Lambda_f, \Lambda_i}(z) \bar{V}_{\Lambda}^{\bar{\Lambda}_f, \bar{\Lambda}_i}(\bar{z}).
\]

Locality and (52) imply that the spin of all fields \( \Delta_{\Lambda} - \Delta_{\bar{\Lambda}} \) has to be integer. Note that this selection rule must be respected also by the pairs of weights (\( \Lambda_i, \bar{\Lambda}_i \)) and (\( \Lambda_f, \bar{\Lambda}_f \)). One large class of theories which satisfy trivially this requirement are the diagonal theories with \( \Lambda = \bar{\Lambda} \).

### 2.4 Partition function, modular invariance

Due to the factorization of the observable algebra (10) we can analyze independently the chiral and antichiral sectors, but in order to reconstruct the whole two dimensional theory we need also the pairings between the fields from the two sectors. They can be found requiring the modular invariance of the partition function on the torus. From the viewpoint of string theory the modular invariance condition is very natural, since it ensures that one can define the theory on surfaces of arbitrary genus [30, 31]. In Statistical Mechanics models its physical meaning is more subtle, since the modular transformations relate the low and the high temperature behaviour of the theory [32].
Let us briefly recall the construction of the partition function. To every primary field \( \varphi_i \) of \( A \) corresponds a character of the Virasoro algebra \( [26] \)

\[
\chi_i(\tau) = \text{Tr}_\mathcal{H}_i e^{2\pi i r(L_0 - \frac{c}{24})},
\]

(61)

where the trace is over the space of all quasiprimary descendants of \( \varphi_i \). Note that the energy operator \( L_0 \) on the torus is modified according to (16). In this notation the torus partition function

\[
Z_T = \text{Tr} \left( e^{2\pi i r(L_0 - \frac{c}{24})} e^{2\pi i r(L_0 - \frac{c}{24})} \right)
\]

(62)

can be rewritten as (we recall that \( \bar{c} = c \))

\[
Z_T = \sum_{i,j} \chi_i X_{ij} \bar{\chi}_j,
\]

(63)

where \( X_{ij} \) are non-negative integers which give the multiplicities of the two dimensional fields. For the rational theories the sum in (63) is over a finite set of characters.

Not all values of \( \tau \) in (62) correspond to inequivalent tori. In particular the transformations

\[
S : \tau \rightarrow -\frac{1}{\tau} \quad (64)
\]

\[
T : \tau \rightarrow \tau + 1 \quad (65)
\]

are just redefinitions of the fundamental cell of the torus. They generate the modular group \( PSL(2, \mathbb{Z}) \) under which \( \tau \) transforms as

\[
\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d} \quad ad - bc = 1 \quad (66)
\]

with integer \( a, b, c \) and \( d \). These transformations act linearly on the characters \( \chi_i \)

\[
\chi_i \left( -\frac{1}{\tau} \right) = \sum_j S_{ij} \chi_j(\tau) \quad \chi_i(\tau + 1) = \sum_j T_{ij} \chi_j(\tau),
\]

(67)

where \( T \) is a diagonal matrix, while \( S \) is a symmetric matrix. Both \( S \) and \( T \) are unitary and satisfy \( S^2 = (ST)^3 = C \), where the matrix \( C \) is called charge conjugation matrix and satisfies \( C^2 = 1 \).
The modular invariance of the torus partition function implies

\[ SXS^\dagger = X \quad \text{TXT} = X . \] (68)

The solutions to these equations are of two distinct types \[^3\]. The first one are called permutation (or automorphism) invariants, for which

\[ X_{ij} = \delta_{i\sigma(j)} . \] (69)

where \(\sigma(j)\) is a permutation of the labels \(j\). The second one correspond to extensions of the observable algebra and can always be rewritten as a permutation invariant (69) in terms of the characters of the maximally extended observable algebra (that are linear combinations of the characters of the unextended one).

Let us denote by \([\varphi_i]\) the conformal family of the primary field \(\varphi_i\) i.e. the collection of all the conformal descendants of \(\varphi_i\). The product of two conformal families is determined by the fusion algebra

\[ [\varphi_i] \times [\varphi_j] = \sum_k N_{ij}^k [\varphi_k] . \] (70)

The non negative integers \(N_{ij}^k\), called fusion rules can be expressed in terms of the modular matrix \(S\) by the Verlinde formula

\[ N_{ij}^k = \sum_\ell S_{i\ell} S_{j\ell} S_{k\ell}^\dagger S_{1\ell} \] (71)

and as matrices \((N_i)_{j}^k\) satisfy the commutative and associative fusion algebra \[^3\]

\[ (N_i) (N_j) = \sum_k N_{ij}^k (N_k) . \] (72)

There are several known classifications of modular invariant partition functions, e.g.\[^35\], \[^36\], \[^37\], \[^38\], but the problem is still not solved in general. We shall often refer to the \(A - D - E\) classification of Cappelli, Itzykson and Zuber \[^35\] of the modular invariants of the \(SU(2)\) conformal current algebra. In this classification, the diagonal \(A\) and the \(D_{\text{odd}}\) series are permutation invariants, the \(D_{\text{even}}\) series, \(E_6\) and \(E_8\) are diagonal invariants of an extended algebra, while \(E_7\) is a nontrivial permutation invariant of an extended algebra.
There is also an alternative method to compute the allowed pairings between the fields of the two sectors that makes no use of higher genus partition functions. In two dimensional conformal field theory the product of two primary fields can be expressed as a sum of primary fields and their conformal descendants using the Operator Product Expansion (OPE)

\[
\phi_{\Delta_i, \bar{\Delta}_i}(z, \bar{z}) \phi_{\Delta_j, \bar{\Delta}_j}(w, \bar{w}) = \sum_{k, \bar{k}} C^{(k, \bar{k})}_{(i, \bar{i})(j, \bar{j})} \phi_{\Delta_k, \bar{\Delta}_k}(w, \bar{w}) \ldots \tag{73}
\]

where the dots stand for the descendants. The two dimensional structure constants \(C^{(k, \bar{k})}_{(i, \bar{i})(j, \bar{j})}\) vanish whenever the corresponding fusion rules \(N_{ij}^k\) or \(N_{\bar{i} \bar{j}}^\bar{k}\) are zero and completely define the theory. In particular they determine also the allowed pairings between the fields of the two sectors. Moreover they permit to reconstruct all the Green functions of the two dimensional fields. In rational conformal field theories the structure constants can in principle be computed imposing the locality (or crossing symmetry) of the 4-point Green functions. Indeed for a generic 4-point function

\[
\langle \phi_{\Delta_1, \bar{\Delta}_1}(z_1, \bar{z}_1) \phi_{\Delta_2, \bar{\Delta}_2}(z_2, \bar{z}_2) \phi_{\Delta_3, \bar{\Delta}_3}(z_3, \bar{z}_3) \phi_{\Delta_4, \bar{\Delta}_4}(z_4, \bar{z}_4) \rangle \tag{74}
\]

we can apply the OPE (73) in three different ways which schematically can be denoted as \((12)(34)\), \((13)(24)\) and \((14)(23)\). This gives two duality relations between the structure constants and determines them up to global rescalings of the two dimensional fields. In practice this procedure is very complicated and the closed expressions for the two dimensional structure constants are known only in a very limited number of cases (in particular for the \(SU(2)\) current algebra models and for the unitary minimal models \([39, 40]\)).

Let us stress that while the crossing symmetry relations are satisfied also for any subset of primary fields closed under OPE, e.g. for the identity operator alone to give a trivial example, the modular invariance condition is satisfied only by the maximal (or complete) set of fields.

In fact these two approaches are complementary, since as demonstrated in \([1, 28]\) both the condition of crossing symmetry of the 4-point functions and the modular invariance of the torus partition function are necessary and sufficient for the consistency of the theory on a surface of arbitrary genus.
3 Correlation functions in current algebra models

In the conformal current algebra models the operator Knizhnik-Zamolodchikov equation (57) implies a system of first order partial differential equations for the \( n \)-point chiral conformal blocks. This allows one to reformulate all the properties of the primary conformal fields as conditions on their chiral correlators. Moreover, for the \( SU(2) \) models that we shall review in some detail this also allows to obtain explicit expressions for the chiral conformal blocks and to compute the structure constants that enter the two dimensional operator product expansion (73).

3.1 Properties of the chiral conformal blocks

Let \( G \) be a simply connected compact Lie group with Lie algebra \( \mathcal{G} \) and let \( V_i = V(\Lambda_i), \ i = 1, 2, \ldots, n \) be chiral vertex operators of highest weight \( \Lambda_i \) such that the space \( \mathcal{J}_n = \mathcal{J}(\Lambda_1, \ldots, \Lambda_n) \) of \( G \) invariant tensors is non trivial (\( d_\mathcal{J} = \dim \mathcal{J}_n > 0 \)). Consider the \( d_\mathcal{J} \) dimensional vector space \( \mathcal{L}_n \) of holomorphic functions \( w_n = w(z_1, \Lambda_1; \ldots; z_n, \Lambda_n) \) called chiral conformal blocks [14] with values in \( \mathcal{J}_n \).

Möbius invariance of the vacuum implies that the functions \( w_n \) are covariant under local Möbius transformations. In particular they are translation invariant (hence depend only on the differences \( z_{ij} \)), they transform covariantly under uniform dilations \( z_i \to \rho z_i, \ \rho > 0 \)

\[
\rho^{\Delta_1 + \ldots + \Delta_n} w(\rho z_1, \Lambda_1; \ldots; \rho z_n, \Lambda_n) = w(z_1, \Lambda_1; \ldots; z_n, \Lambda_n) ,
\]

where \( \Delta_i = \Delta(\Lambda_i) \) are the conformal weights (56). Finally, \( w_n \) are covariant under infinitesimal special conformal transformations \( z \to z/(1 + \varepsilon z) \) with \( \varepsilon \to 0 \), thus satisfy the differential equation

\[
\sum_{i=1}^{n} z_i \left( z_i \frac{\partial}{\partial z_i} + 2\Delta_i \right) w_n = 0 .
\]

The operator form of the Knizhnik-Zamolodchikov equation (77) implies that all elements in \( \mathcal{L}_n \) satisfy the system of partial differential equations

\[
\left( \frac{\partial}{\partial z_i} + \frac{1}{\hbar} \sum_{j=1}^{n} \sum_{j \neq i} t^a t^a_{\Lambda_i \Lambda_j} \frac{1}{z_{ij}} \right) w_n = 0
\]
for $i = 1, \ldots, n$, where $h$ is the height defined after equation (51).

Every function $w_n$ of $L_n$ admits a path dependent multivalued analytic continuation in the product of complex planes minus the diagonal $\{ z_i \in \mathbb{C}, z_i \neq z_j \text{ for } i \neq j \}$. Let us choose a basis $\{ w_n^\nu, \nu = 1, \ldots, d_J \}$ in $L_n$ and consider the analytic continuation of $w_n^\nu$ along a pair of paths $C_i^\pm$ that exchange two neighbouring arguments $z_i, z_{i+1}$ in positive/negative directions

$$C_i^\pm : \begin{pmatrix} z_i \\ z_{i+1} \end{pmatrix} \to \frac{1}{2} (z_i + z_{i+1}) + \frac{1}{2} \left( \frac{z_{i+1} - z_{i}}{-z_{i}} \right) e^{\pm i \pi t}, \quad (78)$$

where $0 \leq t \leq 1$. This operation followed by the permutation of the two weights $\Lambda_i$ a $\Lambda_{i+1}$ defines the action of two exchange operators $B_i$ and $B_i$ [27, 28, 42]. The exchange operator $B_i$ transforms the basis $\{ w_n^\nu \}$ in $\mathcal{L}(\Lambda_1, \ldots, \Lambda_i, \Lambda_{i+1}, \ldots, \Lambda_n)$ in a basis $\{ w_n^\mu \}$ in $\mathcal{L}(\Lambda_1, \ldots, \Lambda_{i+1}, \Lambda_i, \ldots, \Lambda_n)$.

$$B_i = B_i^{\Lambda_1 \ldots \Lambda_n} : \mathcal{L}(\Lambda_1, \ldots, \Lambda_i, \Lambda_{i+1}, \ldots, \Lambda_n) \to \mathcal{L}(\Lambda_1, \ldots, \Lambda_{i+1}, \Lambda_i, \ldots, \Lambda_n). \quad (79)$$

The exchange operator $\bar{B}_i$ is the inverse to $B_i$. More precisely

$$\bar{B}_i^{\Lambda_1 \ldots \Lambda_i + 1 \ldots \Lambda_n} B_i^{\Lambda_1 \ldots \Lambda_i \Lambda_{i+1} \ldots \Lambda_n} = 1. \quad (80)$$

For real analytic $w_n^\nu$ the matrix $\bar{B}_i$ is complex conjugate to $B_i$. The operators $B_i, i = 1, \ldots, n - 1$ with various order of the weights $(\Lambda_1, \ldots, \Lambda_n)$ generate a representation of the exchange (called also braid [43]) algebra $B_n$.

The two dimensional $n$-point Green functions $G_n$ can be written as a finite sum of products of $n$-point chiral and antichiral blocks

$$G_n = \langle 0 | \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) | 0 \rangle = \bar{w}_n^{\mu} Q^{\Lambda_1 \ldots \Lambda_n} w_n^\nu. \quad (81)$$

Local commutativity of the two dimensional fields is equivalent to the invariance of the Green functions $G_n$ under the combined action of the two exchange algebras which implies a braid invariance condition for the matrices $Q^{\Lambda_1 \ldots \Lambda_n}$ [12]

$$(B_i^{\Lambda_1 \ldots \Lambda_i \Lambda_{i+1} \ldots \Lambda_n})^\dagger Q^{\Lambda_1 \ldots \Lambda_i \Lambda_{i+1} \ldots \Lambda_n} B_i^{\Lambda_1 \ldots \Lambda_i \Lambda_{i+1} \ldots \Lambda_n} = Q^{\Lambda_1 \ldots \Lambda_i + 1 \Lambda_i \ldots \Lambda_n}. \quad (82)$$

The relative normalization of $G_n$ for different $n$ and different sets of weights are constrained by the factorization properties implied by the two dimensional operator product expansion [73].
3.2 Regular basis of 4-point functions in the $SU(2)$ model

We shall consider in some detail only the simplest non-trivial case of 4-point functions for $G = SU(2)$. Note that there is an infinite series of such models corresponding to integer height $h = k + 2$ and Virasoro central charge $c = \frac{3k}{k + 2}$. The primary fields can be labelled by their isospin $I$ which has to satisfy the integrability condition $I \leq k/2$ and have conformal dimension $\Delta(I) = \frac{I(I+1)}{(k+2)}$. The fusion rules can be computed from the Verlinde formula and in terms of the isospins of the fields are

$$[I_1] \times [I_2] = \sum_{I = |I_1 - I_2|}^{\min(I_1 + I_2, k - I_1 - I_2)} [I]. \quad (83)$$

Exploiting Möbius invariance one can reduce the KZ equation to a system of ordinary differential equations. In order to write more compact formulae we shall make use of the polynomial realization of the irreducible $SU(2)$ modules and introduce an auxiliary variable $\zeta$ to keep track of the third isospin projection $m$ of the operators. In particular we shall set

$$V^I(z, \zeta) = \sum_{m=-I}^{m=I} \frac{\zeta^{I+m}}{(I+m)!} V^I_m(z). \quad (84)$$

The $SU(2)$ generators act on $V^I(z, \zeta)$ as first order differential operators in $\zeta$, while the correlation functions are polynomials in $\zeta$. We shall also assume that the isospins of the four fields satisfy the inequalities ($I_{ij} = I_i - I_j$)

$$I(= \min I_i) = I_4 \quad |I_{12}| \leq I_{34} \quad |I_{23}| \leq I_{14}. \quad (85)$$

The other cases can be treated in exactly the same way.

Möbius and $SU(2)$ invariance imply that the 4-point chiral conformal blocks have the form

$$w(z_1, \zeta_1, I_1; \ldots; z_4, \zeta_4, I_4) = g(z_{ij}, \Delta) p(\zeta_{ij}, I_{ij}) F(\eta, \xi_1, \xi_2). \quad (86)$$

Here $g(z_{ij}, \Delta)$ is a scale prefactor

$$g(z_{ij}, \Delta) = \frac{z_1^{\Delta_2+\Delta_3} z_2^{\Delta_1+\Delta_4} z_3^{\Delta_2+\Delta_4} z_4^{\Delta_1+\Delta_3} (1 - \eta)^{\Delta_w}}{z_1^{\Delta_{12}} z_2^{\Delta_{23}} z_3^{\Delta_{24}} z_4^{\Delta_{14}}}, \quad (87)$$

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\( \eta \) is the Möbius invariant crossratio
\[
\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}} = 1 - \frac{z_{14}z_{23}}{z_{13}z_{24}},
\]  
while \( \Delta_s \) and \( \Delta_u \) are the threshold dimensions in the \( s- (12)(34) \) and \( u- (23)(14) \) channels. For isospins constrained by (85) they are given by
\[
\Delta_s = \Delta(I_{34}) = \frac{1}{h} I_{34}(I_{34} + 1) \quad \Delta_u = \Delta(I_{14}) = \frac{1}{h} I_{14}(I_{14} + 1).
\]  
The polynomial \( p(\zeta_{ij}, I_{ij}) \) is
\[
p(\zeta_{ij}, I_{ij}) = \zeta_{14} I_{34} + \zeta_{23} I_{12} - \zeta_{13} I_{24} + \zeta_{12} I_{34}.
\]  
Finally, the Möbius invariant function \( F \) is a homogeneous polynomial
\[
F(\eta; \xi_1, \xi_2) = \sum_{\ell=0}^{2I} (\xi_2 \eta)^\ell [\xi_1 (1 - \eta)]^{2I-\ell} f_\ell(\eta)
\]  
in the combinations
\[
\xi_1 = \zeta_{12} \zeta_{34}, \quad \xi_2 = \zeta_{14} \zeta_{23}, \quad (\xi_1 + \xi_2 = \zeta_{13} \zeta_{24}).
\]  
Inserting these formulae into the KZ equation (77) for \( n=4 \) after some algebra we obtain a system of first order ordinary differential equations for the functions \( f_\ell(\eta) \)
\[
\frac{df_\ell}{d\eta} = \left\{ \frac{\ell}{\eta} [\alpha + \gamma - 1 + (\ell - 1)\delta] - \frac{2I - \ell}{1 - \eta} [\beta + \gamma - 1} + (2I - \ell - 1)\delta \right\} f_\ell + \frac{\ell + 1}{1 - \eta} (\alpha + \ell \delta) f_{\ell+1} - \frac{2I - \ell + 1}{\eta} [\beta + (2I - \ell)\delta] f_{\ell-1},
\]  
where
\[
h\alpha = 1 + I_{34} - I_{12} \quad , \quad h\beta = 1 + I_{14} + I_{23} \quad , \quad h\gamma = 1 + I_{34} + I_{12} \quad , \quad h\delta = 1 \quad , \quad (h = k + 2).
\]
The system \([93]\) has \(2I + 1\) linearly independent solutions \(f_\lambda\), \(\lambda = 0, 1, \ldots, 2I\) which for \(0 < \eta < 1\) are given by the integral representations

\[
f_\lambda(\eta) = \int_0^\eta dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{\lambda-1}} dt_\lambda \int_\eta^1 dt_{\lambda+1}\]
\[
\times \int_{t_{\lambda+1}}^1 dt_{\lambda+2} \cdots \int_{t_{2I-1}}^1 dt_{2I} P_\lambda(t; \eta; \alpha, \beta, \gamma, \delta) ,
\]
where

\[
P_\lambda = \prod_{i=1}^{2I} t_i^\alpha (1 - t_i)^\beta \prod_{i=1}^\lambda (\eta - t_i)^{\gamma-1} \prod_{i<j}^{2I} (t_j - \eta)^{\delta-1} \Pi_{i<j} (\varepsilon_{\lambda j} t_{ij})^{2\delta}
\]

\[
\varepsilon_{\lambda j} = \begin{cases} 1 & \text{for } \lambda \geq j \\ -1 & \text{for } \lambda < j \end{cases}, \quad t_{ij} = t_i - t_j .
\]

The sum in \([96]\) extends over all \((2I)!\) permutations \(\sigma : (1, \ldots, 2I) \rightarrow (i_1, \ldots, i_{2I})\). Note that the integration contours in \([95]\) never go to infinity. This is an important difference with respect to the commonly used integral representations \([45,39,40]\) which correspond to tree expansions. Our choice has the advantage that the solutions are linearly independent and non-singular (if all four external isospins satisfy the integrability condition \(I_i \leq k/2\)). In particular the exchange operators are also well defined.

### 3.3 Matrix representation of the exchange algebra

Each basis of solutions \(\{w^\lambda, \lambda = 0, \ldots, 2I\}\) of the (4-point) KZ equation gives rise to a matrix representation of the algebra of exchange operators \(B_1, B_2\) and \(B_3\) \([12,18]\). We shall work out only the action of \(B_1\) and \(B_2\) on the 4-point blocks \([86]\) since \(B_3\) is proportional to \(B_1\) (see equation \([109]\) below). According to \([78]\) \(B_i\) act on the cross ratio \(\eta\) \([88]\) as follows

\[
B_1 : \eta \rightarrow \frac{\eta e^{i\pi}}{1 - \eta} \left(= \lim_{t \rightarrow 1} \frac{\eta e^{i\pi t}}{1 + i\eta e^{i\pi t} \sin \frac{\pi}{2} t}\right)
\]

\[
B_2 : \eta \rightarrow \frac{1}{\eta} \left(= \lim_{t \rightarrow 1} \frac{\eta \cos \frac{\pi}{2} t - i \sin \frac{\pi}{2} t}{\cos \frac{\pi}{2} t - i\eta \sin \frac{\pi}{2} t}\right).
\]
The expressions within parentheses indicate the analytic continuation path in the $\eta$ plane, hence $B_1$ carries $\eta$ around 0 from above, while $B_2$ carries $\eta$ around 1 from below. Note that in order to specify the domain and the target space of the exchange operators $B_i$ one actually has to indicate all four isospins. We shall use the notation

$$B^I_1 I_2 I_3 I_4 : \mathcal{L}(I_1 I_2 I_3 I_4) \rightarrow \mathcal{L}(I_2 I_1 I_3 I_4) \quad (99)$$

$$B^I_2 I_2 I_3 I_4 : \mathcal{L}(I_1 I_2 I_3 I_4) \rightarrow \mathcal{L}(I_1 I_3 I_2 I_4) . \quad (100)$$

The action of the exchange operators on the basis constructed in the previous subsection, $B_i : w^\lambda \rightarrow (B_i)^{\lambda}_{\mu} w^\mu$, can be obtained by analytic continuation of the integral representations (95). Note that $B_i$ not only transform the integrand (96), but also reorder the integration contours in (95). The explicit expressions can be written in a more compact form, if one introduces $q$-deformed numbers

$$[\lambda] = \frac{q^{\lambda} - q^{-\lambda}}{q - q^{-1}} \quad (101)$$

where

$$q = e^{i\pi \delta} = e^{i\pi h} (\Rightarrow q^h = -1) \quad \bar{q} = q^{-1} \quad (102)$$

and $q$-deformed binomial coefficients

$$\left[ \begin{array}{c} \mu \\ \lambda \end{array} \right] = \frac{[\mu]!}{[\lambda]![\mu - \lambda]!} \quad [\lambda]! = [\lambda][\lambda - 1]! \quad [0]! = 1 . \quad (103)$$

The exchange matrix $B_1$ is upper triangular in our basis

$$(B^I_1 I_2 I_3 I_4)^{\lambda}_{\mu} = (-1)^{I_1 + I_2 - I_3 - I_4 - \mu}$$

$$\times q^{(I_3 + \mu)(I_3 + \lambda + 1) + I_2 (\mu - \lambda) - I_1 (I_1 + 1) - I_2 (I_2 + 1)} \left[ \begin{array}{c} \mu \\ \lambda \end{array} \right] , \quad (104)$$

while the exchange matrix $B_2$ is lower triangular and is related to $B_1$ by a similarity transformation,

$$B^I_2 I_2 I_3 I_4 = F^{I_2 I_3 I_4} I_2 I_1 I_3 I_4 B^I_1 I_2 I_3 I_4 . \quad (105)$$

The matrix $F^{I_1 I_2 I_3 I_4} : \mathcal{L}(I_1 I_2 I_3 I_4) \rightarrow \mathcal{L}(I_3 I_2 I_1 I_4)$, called fusion matrix [28], is involutive

$$F^{I_3 I_2 I_1 I_4} F^{I_1 I_2 I_3 I_4} = 1 \quad (106)$$
and in the basis (95) is represented by an antidiagonal matrix whose elements are independent of the order of the isospins

$$(F^{I_1\ldots I_4})^\lambda_{\mu} = \delta_{\mu}^{2I-\lambda}.$$  \hfill (107)

Using the expressions (104) and (105) one can verify that the exchange operators $B_i$ satisfy the parameter free Yang–Baxter equation \[49\]

$$B_{I_2I_3I_4} B_{I_1I_4} = B_{I_1I_2I_4} B_{I_3I_4} = (−1)^{I_1+I_2+I_3+I_4} q^{I_5(I_4+1)-I_1(I_1+1)-I_2(I_2+1)-I_3(I_3+1)} B_{I_1I_2I_3I_4}.$$ \hfill (108)

Let us note also that for the 4-point functions $B_{3}^{I_1I_2I_3I_4}$ and $B_{4}^{I_1I_2I_3I_4}$ are proportional

$$B_{3}^{I_1I_2I_3I_4} = (−1)^{I_3+I_4-I_1-I_2} q^{I_1(I_1+1)+I_2(I_2+1)-I_3(I_3+1)-I_4(I_4+1)} B_{4}^{I_1I_2I_3I_4}.$$ \hfill (109)

The exchange operators for the 3-point functions are just phases, since the space of $SU(2)$ invariants is one dimensional in this case. They can be obtained as a special case (for $I_4 = 0$) from the general expressions (104), (105)

$$B_{1}^{I_1I_2I_3} = (−1)^{I_1+I_2-I_3} q^{I_4(I_3+1)-I_1(I_1+1)-I_2(I_2+1)}, \hfill (110)$$

$$B_{2}^{I_1I_2I_3} = (−1)^{I_2+I_3-I_1} q^{I_1(I_1+1)-I_2(I_2+1)-I_3(I_3+1)}.$$ \hfill (111)

The exchange operator for the 2-point function which exists only for $I_2 = I_1$ is given by

$$B_{I_1I_2} = (−1)^{2I_1} q^{-2I_1(I_1+1)}.$$ \hfill (112)

### 3.4 Two dimensional braid invariant Green functions

So far we have computed only the exchange operators in the chiral sector of the theory. To compute the two dimensional Green functions (81) we need also the expressions for the antichiral sector. To derive them let us recall that the corresponding current algebras are isomorphic, while the orientation of the analytic continuation contours (78) are opposite in the two sectors. Thus the exchange operators in the antichiral sector are complex conjugate of the corresponding chiral ones and can be obtained from them by the substitution $q \to q^{-1} (= \bar{q})$ (see equation (102)).
The locality condition for the two dimensional Green functions (81) implies the braid invariance constraints (82) for the matrices \(Q\). For generic value of \(q\) on the unit circle, (or equivalently for a generic value of the level \(k\)) the solution of (82) is unique and corresponds to a diagonal pairing of the two sectors (a diagonal modular invariant). For special values of the level \(k\) there exist also other solutions which correspond to non-diagonal modular invariants. Let us first consider the generic diagonal case. The solution of the braid invariance condition (82) is

\[
Q_{\mu \nu} (I_1, I_2, I_3, I_4) = (-1)^{\mu + \nu} \frac{[\mu]! [\mu - I_{12} + I_{34}]! [\nu - I_{12} + I_{34}]!}{[2I_1]! [2I_2]! [2I_3]! [2I_4]!} \times \sum_{\rho = 0}^{\min(\mu, \nu, k(I))} T_{\rho} (\mu, \nu; I_i),
\]

(113)

where \(\mu, \nu = 0, \ldots, 2I_4\),

\[
k(I) = k - I_1 - I_2 - I_3 + I_4
\]

and

\[
T_{\rho} (\mu, \nu; I_i) = \frac{[2I_{34} + 2\rho + 1]}{[I_1 + I_2 + I_{34} + \rho + 1]! [I_1 + I_2 - I_{34} - \rho]! [2I_4 - \rho]!} \times \frac{[I_3 + \mu + \rho + 1]! [I_3 + \nu + \rho + 1]!}{[\mu - \rho]! [\nu - \rho]! [I_{34} - I_{12} + \rho]!}
\]

It is straightforward but rather lengthy to check that (113) satisfies (82).

In order to study the factorization properties of the two dimensional Green functions let us rewrite the 4-point chiral conformal blocks in the tree bases. In the s-channel, which exhibits the singularities of the solutions for small \(z_{12}\) (hence small \(\eta\)) we find

\[
S_{I_{34}+\lambda}^{(I_1, I_2, I_3, I_4)} (z, \zeta) = \sum_{\nu=0}^{2I_4} u_{\nu}^{(I_1, I_2, I_3, I_4)} (z, \zeta) ~ \sigma_{\nu \lambda}^{-1} (I_1, I_2, I_3, I_4)
\]

\[
= \sum_{\nu = \lambda}^{2I_4} (-1)^{\nu - \lambda} \frac{[\nu]! [\nu - I_{12} + I_{34}]! [2I_{34} + 2\lambda + 1]!}{[\nu - \lambda]! [\lambda]! [\lambda - I_{12} + I_{34}]! [2I_{34} + \nu + \lambda + 1]!} \times u_{\nu}^{(I_1, I_2, I_3, I_4)} (z, \zeta).
\]

(114)

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Let us stress that for \( q \) a root of unity (note that \( q^{k+2} = -1 \)) the matrix elements of the matrix \( \sigma^{-1} \) are well defined only if
\[
I_1 + I_2 + I_{34} + \lambda \leq k. \tag{115}
\]
In other words the s-channel conformal blocks (114) are well defined only for intermediate fields that respect the fusion rules (83). In the rest of the paper, we shall use (114) and all other tree bases formulae only for such intermediate fields. Having this in mind we can introduce also the matrix formally inverse to
\[
\sigma_{\lambda \mu}^{-1}(I_1, I_2, I_3, I_4) = \begin{bmatrix} \sigma_{I_1} & \sigma_{I_2} & \sigma_{I_3} & \sigma_{I_4} \end{bmatrix}.
\]
\[
\sigma_{I_1}(I_1, I_2, I_3, I_4) = \frac{[2I_{34} + \lambda + \mu]![I_{34} - I_{12} + \lambda]![\lambda]}{[2I_{34} + 2\lambda]![I_{34} - I_{12} + \mu]![\mu]}.
\]
In the s-channel basis (114) the exchange operator \( B_1 \) has a simple diagonal form
\[
((B_1^s)^{I_1 I_2 I_3 I_4})_{\lambda \mu} = \delta_\mu^\lambda (-1)^{I_1 + I_2 - I_{34} - \mu} q^{I_{34} + \mu}(I_{34} + \mu + 1 - I_{12} - I_{14} + I_{23}) \tag{117}
\]
while the exchange operator \( B_2 \) and the fusion matrix \( F \) are given by complicated expressions. In particular for \( F \) one finds
\[
((F^s)^{I_1 I_2 I_3 I_4})_{\lambda \mu} = \sum_{\lambda=0}^{2I} \sigma_{\nu \lambda}(I_1, I_2, I_3, I_4) \sigma_{I_1}^{-1}(I_3, I_2, I_1, I_4), \tag{118}
\]
where \( I = I_4 \) (see equation (85)). Inserting the expressions for \( \sigma \) and \( \sigma^{-1} \) we obtain
\[
((F^s)^{I_1 I_2 I_3 I_4})_{\lambda \mu} = \sum_{\lambda=0}^{\min(\nu, 2I - \mu)} \sum_{\lambda=0}^{\min(2I - \lambda - \mu)} \frac{(-1)^{2I - \lambda - \mu} [2I_{34} + \nu + \lambda]![I_{34} - I_{12} + \nu]![\nu]!}{[2I_{34} + 2\nu]![I_{34} - I_{12} + \lambda]![\nu - \lambda]![\lambda]!}
\]
\[
\times \frac{[2I - \lambda]![2I - \lambda - I_{32} + I_{14}]![2I_{14} + 2\mu + 1]!}{[2I - \lambda - \mu]![\mu]![\mu - I_{32} + I_{14}]![2I_{14} + 2I - \lambda + \mu + 1]!}. \tag{119}
\]
The other tree basis, the u-channel, exhibits the singularities of the solutions for small \( z_{23} \) (hence small \( 1 - \eta \)). To construct it let us note that the KZ equation as differential equation in \( 1 - \eta \) for the conformal blocks with isospin order \( I_3 I_2 I_1 I_4 \) coincides with the KZ equation in \( \eta \) for the conformal
blocks with isospin order \( I_1 I_2 I_3 I_4 \). Thus we can define the u-channel blocks which diagonalize the exchange operator \( B_2 \) as

\[
U_{I_{14}+\lambda}^{(I_1, I_2, I_3, I_4)}(\eta) = S_{I_{14}+\lambda}^{(I_2, I_3, I_4, I_1)}(1 - \eta) = (-1)^{I_2 + I_3 - I_1 - I_4} \\
\times q^{I_1(I_1+1) + I_4(I_4+1) - I_2(I_2+1) - I_3(I_3+1)} S_{I_{14}+\lambda}^{(I_2, I_3, I_1, I_4)}(1 - \eta),
\]

where the second equation follows from the diagonal form of the exchange operator \( B_1 \) (117), (and hence also of \( B_3 \) and \( B_1 B_3^{-1} \)) in the s-channel basis.

The u-channel blocks (120) are related to the s-channel blocks (114) by the fusion matrix \( F^s \) (119)

\[
U_{I_{14}+\mu}^{(I_1, I_2, I_3, I_4)}(\eta) = \sum_{\nu} \left((F^s)^{(I_1, I_2, I_3, I_4)}\right)_{\mu\nu} S_{I_{34}+\nu}^{(I_1, I_2, I_3, I_4)}(\eta).
\]

The two dimensional Green functions can be expressed in terms of the tree conformal blocks as

\[
G_4^{(I)}(z, \bar{z}; \zeta, \bar{\zeta}) = \sum_{\nu=0}^{\min(2I, k(I))} [g_s]_{\nu}^{(I_1, I_2, I_3, I_4)}(\bar{z}, \bar{\zeta}) S_{I_{34}+\nu}(z, \zeta)
\]

\[
= \sum_{\mu=0}^{\min(2I, k(I))} [g_u]_{\mu}^{(I_1, I_2, I_3, I_4)}(z, \zeta) U_{I_{14}+\mu}(z, \zeta)
\]

The normalization constants \([g_s]\) and \([g_u]\) can be written as

\[
[g_s]_{\nu}^{(I_1, I_2, I_3, I_4)} = \frac{C_{I_1 I_2 I_3 + \nu} C_{I_4 I_3 + \nu}}{N_{I_{34}+\nu}}
\]

\[
[g_u]_{\mu}^{(I_1, I_2, I_3, I_4)} = \frac{C_{I_3 I_4 + \mu} C_{I_1 I_2 + \mu}}{N_{I_{14}+\mu}}
\]

where \( C_{I_1 I_2 I_3} \) are

\[
C_{I_1 I_2 I_3} = [I_1 + I_2 + I_3 + 1]! \\
\times \frac{[I_1 + I_2 - I_3]![I_2 + I_3 - I_1]![I_1 + I_3 - I_2]!}{[2I_1]![2I_2]![2I_3]!},
\]

while \( N_I \) is equal to

\[
N_I = C_{I I 0} = [2I + 1].
\]
Now we can impose the factorization property in both the s- and the u-channels. Comparison of (123) and (124) with the two dimensional OPE (73) shows that the two dimensional structure constants in the diagonal model (in which the two dimensional fields have equal chiral and antichiral labels) are given by

\[ C^{(KK)}_{(II)(JJ)} = \frac{C_{IJK}}{N_K}. \]  

(127)

Moreover, if we choose the normalizations of the 2-point functions to be equal to \( N_I \) (126), the normalizations of the 3-point functions are equal to \( C_{I_1I_2I_3} \) (125).

This construction can be extended also to the non-diagonal SU(2) current algebra models. For the \( D_{odd} \) series of models, which exist for values of the level \( k = 4p - 2 \), the structure constants are

\[ C^{(KK)}_{(II)(JJ)} = \epsilon_{(II)(JJ)(KK)} \sqrt{\frac{C_{IJK}C_{IJK}}{N_KN_K}}, \]  

(128)

where the signs \( \epsilon \) are symmetric in all three pairs of indices and differ from +1 only if two pairs of the isospins, say \( I, \bar{I}, J, \bar{J} \), are half integers, in which case they are equal to \((-1)^K \ (= (-1)^K\).

For the other SU(2) current algebra models denoted by \( D_{even} \), \( E_6 \), \( E_7 \), \( E_8 \) one can also compute the structure constants [50]. The resulting expressions are not as simply related to the diagonal ones. This can be explained by the fact that these models correspond to extensions of the observable algebra, so their structure is determined by this extension, rather than by the underlying SU(2) current algebra.

4 CFT on surfaces with holes and crosscaps

Conformal field theories in presence of boundaries have been introduced by Cardy to describe critical phenomena in Statistical Mechanics and solid state physics [1, 2, 29]. An alternative approach, called open and unoriented descendants construction, was proposed by Sagnotti in the framework of string theory to unify in a consistent way open strings with closed oriented and unoriented strings [4]. In this section we shall review some general properties of boundary CFT. The SU(2) conformal current algebra models will be again used as an example. On one hand, they are relatively simple and all the
necessary data (chiral conformal blocks, structure constants, exchange operators) are explicitly known. On the other hand the $SU(2)$ models exhibit many features of the general case (like infinite series of nondiagonal models and non abelian fusion rules).

4.1 Open sector, sewing constraints

The presence of a boundary breaks the two dimensional conformal symmetry, since the boundary cannot be invariant under all the transformations of $Vir \otimes \overline{Vir}$. If the central charges of the two chiral algebras are equal $\overline{c} = c$, it is possible to introduce boundaries which are preserved at most by the diagonal subalgebra $Vir_{\text{diag}}$. We shall call such boundaries conformal boundaries. In the rest we shall assume that all boundaries are conformal. The introduction of non conformal boundaries is also possible, but one cannot use anymore the tools of conformal field theory for their study.

Assume that the conformal boundary coincides with the line $x^1 = 0$. The conformal invariance condition means that there is no energy transfer across the boundary, hence the stress energy tensor satisfies

$$\Theta(x_-) = \overline{\Theta}(x_+) \quad \text{for} \quad x_- = x_+ \iff x^1 = 0,$$

since $x_{\pm} = x^0 \pm x^1$. So one can define the stress energy tensor in the theory with conformal boundaries as

$$\Theta_d(x) = \begin{cases} \Theta(x_-) & \text{for} \ x^1 \geq 0 \\ \overline{\Theta}(x_+) & \text{for} \ x^1 < 0 \end{cases}$$

(130)

In a similar way, if the two dimensional theory is invariant under the product of two isomorphic conformal current algebras $\mathcal{A} \otimes \overline{\mathcal{A}}$ with equal levels $\overline{k} = k$, the boundary can be preserved at most by the diagonal subalgebra $\mathcal{A}_{\text{diag}}$. Such boundaries are called symmetry preserving, the currents in this case are defined as

$$j^a_{\alpha}(x) = \begin{cases} j^a(x_-) & \text{for} \ x^1 \geq 0 \\ \overline{j}^a(x_+) & \text{for} \ x^1 < 0 \end{cases}$$

(131)

One can introduce also conformal boundaries that are preserved only by a proper subalgebra $\mathcal{A}' \subset \mathcal{A}_{\text{diag}}$ (such that the boundary is still invariant under $Vir_{\text{diag}}$). Such boundaries are called symmetry breaking (or symmetry non preserving) boundaries and have been also studied [51]. In these lectures we
shall restrict our attention only to the simpler case of symmetry preserving boundaries.

We can pass to the analytic picture by mapping the boundary onto the unit circle by a Cayley transform (2). The stress energy tensor becomes

\[
T_d(z) = \begin{cases} 
T(z) \text{ for } |z| \leq 1 \\
\frac{1}{z^2}T\left(\frac{1}{z}\right) \text{ for } |z| > 1 
\end{cases}
\]  

(132)

(where we used \(\bar{z} \leftrightarrow 1/z\) in this picture), while the currents are

\[
J_d^a(z) = \begin{cases} 
J^a(z) \text{ for } |z| \leq 1 \\
-\frac{1}{z^2}J^a\left(\frac{1}{z}\right) \text{ for } |z| > 1 
\end{cases}
\]  

(133)

The sign change with respect to (132) comes from the prefactor in the Cayley transform.

Let us introduce also the following combinations of the Laurent modes of the stress energy tensor \(T\) and the currents \(J^a\):

\[
\mathcal{L}_n = L_n - \bar{L}_{-n}
\]  

(134)

and

\[
\mathcal{J}_n^a = J_n^a + \bar{J}_{-n}^a.
\]  

(135)

Since the left and right central charges and levels are equal (\(\bar{c} = c, \bar{k} = k\)) the modes (134) satisfy the commutation relations of the Virasoro algebra with central charge equal to zero

\[
[\mathcal{L}_n, \mathcal{L}_m] = (n - m)\mathcal{L}_{n+m},
\]  

(136)

while the modes (135) satisfy the commutation relations of the current algebra with level equal to zero

\[
[\mathcal{J}_n^a, \mathcal{J}_m^b] = i f^{abc} \mathcal{J}_{n+m}^c.
\]  

(137)

These two algebras have no nontrivial representations, hence the modes (134), (135) annihilate all the boundary states \(|B\rangle\) in the theory

\[
\mathcal{L}_n|B\rangle = (L_n - \bar{L}_{-n})|B\rangle = 0
\]  

(138)

and

\[
\mathcal{J}_n^a|B\rangle = (J_n^a + \bar{J}_{-n}^a)|B\rangle = 0.
\]  

(139)
For rational models a basis of states that satisfy (138) called Ishibashi states has been constructed in [52] as infinite sums of products of left and right states
\[ |\mathcal{I}_\Lambda\rangle = \sum_m |\Lambda, m\rangle \otimes |\Lambda, m\rangle, \tag{140} \]
where the sum extends over all the quasiprimary descendants of the primary state $|\Lambda\rangle$. Note that the Ishibashi states are not eigenvalues of the energy $L_0 + \bar{L}_0$ and are not normalizable in the usual sense.

One important consequence of equations (132), (133) is that in the presence of boundaries the $n$-point functions of two dimensional primary fields $\phi_{\Lambda \bar{\Lambda}}(z, \bar{z})$ and the chiral conformal blocks of $2n$-chiral vertex operators with the same weights satisfy the same equations as functions of the $2n$ variables $(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n)$ [1]. Indeed, since the chiral and antichiral parts of the stress energy tensor and of the currents act independently on the chiral and antichiral vertex operators in the decomposition of the two dimensional primary fields (see also equation (60))
\[ \phi_{\Lambda \bar{\Lambda}}(z, \bar{z}) = \sum_{\Lambda_i, \bar{\Lambda}_i} V_{\Lambda_i}^{\Lambda_{\bar{\Lambda}_i}}(z) \bar{V}_{\bar{\Lambda}_i}^{\bar{\Lambda}_{\Lambda_i}}(\bar{z}) \, n_{\Lambda_i}^{\bar{\Lambda}_i}, \tag{141} \]
the $n$-point functions of the two dimensional fields in the theory with boundaries are linear combinations of $2n$-point chiral conformal blocks. Note the difference with respect to the case without boundaries reviewed in the previous section, where the two dimensional functions are sesquilinear combinations of $n$-point conformal blocks.

Another important property of the boundary is the existence of one dimensional fields $\psi$ called boundary fields [1]. They are defined only on the boundary (on the unit circle in the analytic picture). Equations (132), (133) imply that the Virasoro algebra and the conformal current algebra which act on the boundary have the same central charge and the same level as the chiral ones. Hence the primary boundary fields can be labelled by the same set of weights $\Lambda$. There can be different boundary conditions on different portions of the boundary, which we denote by labels $a, b, c$. The boundary fields carry two boundary condition labels $\psi_{\Lambda}^{ab}(x)$ and change the boundary condition from $b$ to $a$. In general also a degeneracy label accounting for the multiplicity of the boundary fields may be necessary. For simplicity we shall omit the degeneracy labels. For a more accurate analysis of this point see e.g.[53]. We shall denote the argument of the boundary fields by $x$ which
takes values only on the unit circle, to distinguish it from \( z (\bar{z}) \) which take values inside (outside) the unit circle.

In general the boundary fields do not locally commute, rather they behave much like the chiral vertex operators under the exchange algebra. In other words in correlation functions the ordering of their arguments on the circle cannot be changed arbitrarily. In particular, this implies that the 4-point functions of boundary fields satisfy only one crossing symmetry relation, called planar duality, in contrast with the two dimensional case where crossing symmetry of the 4-point functions implies two duality relations.

To simplify the notation in this section we shall consider only the \( SU(2) \) current algebra models and label the fields by \( i = 2I_i + 1 \) and \( \bar{i} = 2\bar{I}_i + 1 \) rather than by their weights \( \Lambda_i \) and \( \bar{\Lambda}_i \). Hence the identity operators carry the label 1. When this is not ambiguous, we shall also omit the space-time \((z \text{ and } x)\) and \( SU(2) (\zeta) \) variables.

The operator product expansion for the boundary fields schematically has the form (note the continuity of the boundary indices)

\[
\psi_i^{ab} \psi_j^{bc} \sim \sum_l C_{ijl}^{abc} \psi_l^{ac}
\]

(142)

where the sum is over all the values allowed by the fusion rules \((83)\). The boundary structure constants \( C_{ijl}^{abc} \) are in general not symmetric. Other important data are the normalizations of the 2-point functions of the boundary fields, since they cannot be chosen arbitrarily \([4]\). To define them one has to specify also the order of the arguments, since the boundary fields do not commute. Both variables are on the unit circle so we can order them by their phase

\[
\langle \psi_i^{ab}(x_1; \zeta_1) \psi_i^{ba}(x_2; \zeta_2) \rangle = \frac{\alpha_i^{ab}(\zeta_{12})^{2I_i}}{(x_{12})^{2\Delta_i}}
\]

(143)

for \( \text{Arg}(x_2) < \text{Arg}(x_1) \),

where \( I_i \) is the isospin of \( \psi_i \). The normalizations of the fields with exchanged boundary labels are related. For example, for the \( SU(2) \) current algebra models one finds

\[
\alpha_i^{ab} = \alpha_i^{ba} (-1)^{2I_i}.
\]

(144)

Let us stress that even if we consider in detail only the \( SU(2) \) conformal current algebra case, most of the formulae are valid also in more general cases.
(with minor modifications in the numerical factors). For instance, in the unitary minimal models case one just has to omit all the isospin dependence.

Using the boundary OPE (142) we can compute in two different ways the three-point functions of the boundary fields \( \langle \psi_i^{ab} \psi_j^{bc} \psi_l^{ca} \rangle \) and \( \langle \psi_j^{bc} \psi_l^{ca} \psi_i^{ab} \rangle \). This gives the following consistency conditions

\[
C_{ijl}^{abc} \alpha_i^{ac} = C_{jli}^{bec} \alpha_i^{ab} \quad \text{and} \quad C_{jli}^{bec} \alpha_i^{ba} = C_{lij}^{cab} \alpha_j^{bc} ,
\]

that together with (144) imply also

\[
C_{ijl}^{abc} \alpha_i^{ac} = (-1)^{2I_i} C_{lij}^{cab} \alpha_j^{bc} .
\]

The natural normalization of the boundary identity operator is

\[
C_{iII}^{abb} = 1 \quad \langle 1^{aa} \rangle = \alpha_1^{aa} ,
\]

while all other one-point functions of the boundary fields vanish.

The planar duality constraint for the 4-point functions \( \langle \psi_i^{ab} \psi_j^{bc} \psi_k^{cd} \psi_l^{da} \rangle \) reads

\[
\sum_p C_{ijp}^{abc} C_{klp}^{cda} \alpha_p^{ac} S_p(i,j,k,l) = \sum_q C_{jkq}^{bed} C_{liq}^{dab} \alpha_q^{bd} U_q(i,j,k,l)
\]

and after expressing the \( u \)-channel blocks (120) in terms of the \( s \)-channel blocks (114) by the fusion matrix \( F^s \) (119) as (see also (121))

\[
U_q(i,j,k,l) = \sum_p F_{qp}(i,j,k,l) S_p(i,j,k,l)
\]

we obtain a quadratic relation for the boundary structure constants \( C_{ijk}^{abc} \) and the 2-point normalizations \( \alpha_i^{ab} \)

\[
C_{ijp}^{abc} C_{klp}^{cda} \alpha_p^{ac} = \sum_q C_{jkq}^{bed} C_{liq}^{dab} \alpha_q^{bd} F_{qp}(i,j,k,l) .
\]

These relations do not determine completely the boundary structure constants. In other words, the boundary theory cannot be considered independently, but only as a part of the two dimensional conformal theory.

The relation between the bulk and boundary fields is encoded into the bulk-to-boundary expansion

\[
\phi_i^a \bigg|_a \sim \sum_j C_{(i,j)j}^a \psi_j^{aa} ,
\]
that expresses the two dimensional fields in front of a portion of boundary with given boundary condition \(a\) in terms of the corresponding boundary fields. The sum is again over all the values allowed by the fusion rules. The proper normalization of the identity operator gives

\[
C^a_{(1,1)1} = 1
\]  

for all boundary conditions \(a\).

The consistency of the operator product expansions \((73), (142)\) and \((151)\) have been studied by Lewellen \[3\], who has shown that the complete set of relations (called also sewing constraints) which guarantee the consistency of the theory includes two more equations, the first one involving 4-point functions and the second one involving 5-point functions. The first relation arises from the correlation functions of one two dimensional bulk field and two boundary fields. As already stressed the boundary fields have a fixed order of the arguments, but the two dimensional fields have to be local also in presence of boundary fields, which implies

\[
\langle \phi_{(i,\bar{i})} \psi^b_{j} \psi^a_{k} \rangle = \langle \psi^b_{j} \phi_{(i,\bar{i})} \psi^a_{k} \rangle .
\]  

Note that the bulk field is expanded in front of portions of the boundary with different boundary conditions in the left and in the right hand sides of this equation. Using also \((142), (151)\) one obtains

\[
\sum_l C^b_{(i,\bar{i})l} C^{ba}_{ljk} \alpha^b_k S_l(i, \bar{i}, j, k) = \sum_n C^a_{(i,\bar{i})n} C^{baa}_{jnk} \alpha^a_k U_n(j, i, \bar{i}, j, k) .
\]  

To derive the constraint on the structure constants we have to relate the \(U\)- and the \(S\)- blocks. A convenient way to do this is to use repeatedly the fusion matrix (and its inverse) in such a way that the exchange operators act always diagonally (see \((117)\)). In other words before applying \(B_1\) or \(B_3\) we change to the s-channel basis, while before applying \(B_2\) we change to the u-channel basis. The resulting composite exchange operator is

\[
U_n(j, i, \bar{i}, j, k) = \sum_{m,r,s,p,l} F_{nm}(j, i, \bar{i}, j, k) (B_1)_{mr}(i, j, \bar{i}, k) F_{rs}^{-1}(i, j, \bar{i}, k)
\times (B_2)^{-1}_{sp}(i, \bar{i}, j, k) F_{pl}(i, \bar{i}, j, k) S_l(i, \bar{i}, j, k) .
\]  

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Inserting this in (154) and using the explicit expressions for the exchange operators $B_1$ and $B_2$ in the $SU(2)$ model, we obtain the constraint

$$C^{ab}_{(i,\bar{i})l} C^{bab}_{jk\bar{l}} \alpha_{l}^{bb} = \sum_{m,n,p} (-1)^{(I_m-I_j+2I_p-I_l-I_m)} \ e^{-i\pi(D_1-D_3-D_m+D_p)}$$

$$\times C^{aa}_{(i,\bar{i})m} C^{aaa}_{knj} \alpha_n^{aa} F_{nm}(j, i, \bar{i}, k) F^{-1}_{mp}(i, \bar{i}, j, k) F_{pl}(i, \bar{i}, j, k). \quad (156)$$

The other independent relation can be derived from the 5-point functions of two bulk fields and one boundary field of the form

$$\langle \phi(i,\bar{i}) \phi(j,\bar{j}) \psi_k^{aa} \rangle. \quad (157)$$

This function can again be computed in two different ways. We can first use the two dimensional OPE (73) followed by a bulk to boundary OPE (151), alternatively we can use twice the bulk to boundary OPE (151) followed by a boundary OPE (142). Before proceeding we have to define a basis in the space of 5-point functions. We shall use a tree representation which decomposes the 5-point functions into products of a 4-point function and a 3-point function (denoted by $g_{1,2,3}^{(1,2,3,4,5)} = S_p(1,2,q,5) \ g(q,3,4)$ (158)

In this notation the equivalence of the two ways of computing the function (157) implies

$$\sum_{p,q} C^{(p,q)}_{(i,\bar{i})j} C^{aa}_{(p,q)k} \ alpha_k^{aa} X_{pq}(j, i, \bar{i}, j, k) = \sum_{p,q} C^{aa}_{(i,\bar{i})p} C^{aaa}_{(j,\bar{j})q} \ alpha_k^{aa} X_{pq}(i, \bar{i}, j, \bar{j}, k). \quad (159)$$

The two expressions can again be related by the exchange operators, for example by $F_2^{-1} B_2^{-1} B_1 F_2^{-1}$, where the label in brackets indicates on which 4-point subtree acts the fusion matrix $F$. We can use the Yang-Baxter equations (108) to express $B_2 B_1 B_2$ in terms of $F_{1}$ obtaining (if $\alpha_k^{aa} \neq 0$)

$$C^{(p,q)}_{(i,\bar{i})j} C^{aa}_{(p,q)k} = \sum_{r,s,t} (-1)^{(I_j-I_t+2I_r-I_p-I_m)} \ e^{-i\pi(D_1-D_3-D_m+D_p)} \ C^{aaa}_{rsk} \ C^{aa}_{(i,\bar{i})r} F_{st}(r, j, \bar{j}, k) F_{rp}(j, i, \bar{i}, t) F^{-1}_{pq}(i, \bar{i}, j, k). \quad (160)$$
Both equations (156) and (160) can be written in several different equivalent forms, since the exchange operators satisfy duality relations (like the Yang–Baxter equation (108)) [28]. Our derivation follows [9]. For alternative ones see also [3, 53, 54]. In particular [54] contains the general solution of the sewing constraints for the unitary minimal models with a detailed analysis of the residual normalization freedom. Here we shall address only a simpler problem, namely we shall try to count the allowed boundary conditions. Note that in all the sewing constraints the boundary fields enter as external insertions, so one can always start with only one type of boundary labels, say $a$, and solve only the corresponding subsystem. There is however a systematic way to determine also the whole set of allowed boundary conditions. In other words, only by analyzing the sewing constraints one can find all boundary states $|a\rangle$. To illustrate this point, let us consider one particular case of the function (157), namely $\langle \phi(i,\bar{i}) \phi(j,\bar{j}) 1^{aa}\rangle$. Then the condition (160) becomes
\begin{equation}
C^{(q,\bar{q})}_{(i,\bar{i})(j,\bar{j})} C^{aa}_{(q,\bar{q})1} \alpha_{1}^{aa} = \sum_{p} (-1)^{(I_{j}-I_{\bar{j}}+I_{p})} e^{-i\pi(\Delta_{j}-\Delta_{\bar{j}}+\Delta_{p})} \times \alpha_{p}^{aa} C^{a}_{(i,\bar{i})p} C^{a}_{(j,\bar{j})p} F_{pq}(j,\bar{i},i,\bar{j}) . \tag{161}
\end{equation}
Multiplying by $F_{qr}^{-1}(j,\bar{i},i,\bar{j})$, summing on $q$ and keeping only the equation corresponding to $r = 1$ we find a system of equations for the bulk-to-boundary coefficients in front of the boundary identity operators
\begin{equation}
B_{i}^{a} = C^{a}_{(i,\bar{i})1} , \tag{162}
\end{equation}
where to simplify notation we used that for a permutation modular invariant the antichiral label of a field $\bar{i}$ is determined by its chiral label $i$. The resulting relation has the form
\begin{equation}
B_{i}^{a} B_{j}^{a} = \sum_{l} X_{ij}^{l} B_{l}^{a} , \tag{163}
\end{equation}
for all $a$ with $a$-independent structure constants $X_{ij}^{l}$ that vanish if the fusion rules $N_{ij}^{l}$ are zero. The number of different solutions of these equations determines also the number of allowed boundary conditions. In order to compute the values of the structure constants $X_{ij}^{l}$ one needs to know the two dimensional structure constants and the expressions for the fusion matrix in the model. As already stressed, these data are known only in a very restricted number of cases. In order to bypass this difficulty, in [53] an alternative
approach was proposed. One can postulate that (163) holds and that the structure constants $X_{ij}^l$ form a commutative and associative algebra, called Classifying algebra. Then the reflection coefficients $B^a_i$ are given by the representations of this algebra which in some cases can be explicitly found.

For the $SU(2)$ case from the explicit expressions of the fusion matrix (119) and the two dimensional structure constants (128), we can compute the values of $X_{ij}^l$ both in the diagonal $A$ models and in the non-diagonal $D_{odd}$ models, obtaining

$$B^a_i B^a_j = \sum_l \epsilon_{ijl} N_{ij}^l B^a_l,$$

where the signs $\epsilon_{ijl}$, present only for the $D_{odd}$ models, are defined after equation (128) (they are symmetric in all three indices and are equal to $(-1)$ only if two of the isospins are half integer, while the third isospin is an odd integer).

As an illustration we shall write down the solutions of the system (164) in the two $SU(2)$ models of level $k = 6$. In the diagonal $A$ model there are seven different solutions for the reflection coefficients $B_i$, which are reported in table 1. Note that in the diagonal models the number of boundary conditions is always equal to the number of two dimensional fields.

Table 1: Reflection coefficients for the diagonal $SU(2)$ level $k = 6$ model

|   | $B_1$ | $B_3$ | $B_5$ | $B_7$ | $B_2$ | $B_4$ | $B_6$ |
|---|-------|-------|-------|-------|-------|-------|-------|
| 1 | 1     | 1 + $\sqrt{2}$ | 1     | 1     | $\sqrt{2} + \sqrt{2}$ | $\sqrt{2} (2 + \sqrt{2})$ | $\sqrt{2} + \sqrt{2}$ |
| 2 | 1     | 1     | -1    | -1    | $\sqrt{2}$     | 0     | $-\sqrt{2}$    |
| 3 | 1     | 1 - $\sqrt{2}$ | 1     | 1     | $\sqrt{2} - \sqrt{2}$ | $-\sqrt{2} (2 - \sqrt{2})$ | $2 - \sqrt{2}$ |
| 4 | 1     | -1    | 1     | -1    | 0     | 0     | 0     |
| 5 | 1     | 1 - $\sqrt{2}$ | 1     | -1    | $-\sqrt{2} - \sqrt{2}$ | $\sqrt{2} (2 - \sqrt{2})$ | $-\sqrt{2} - \sqrt{2}$ |
| 6 | 1     | 1     | -1    | -1    | $-\sqrt{2}$    | 0     | $\sqrt{2}$    |
| 7 | 1     | 1 + $\sqrt{2}$ | 1     | 1     | $-\sqrt{2} + \sqrt{2}$ | $-\sqrt{2} (2 + \sqrt{2})$ | $-\sqrt{2} + \sqrt{2}$ |
Table 2: Reflection coefficients for the non-diagonal $SU(2)$ level $k = 6$ model

| a | $B_1$ | $B_3$ | $B_5$ | $B_7$ | $B_4$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | -1 | -1 | 0 |
| 2 | 1 | $1 + \sqrt{2}$ | $1 + \sqrt{2}$ | 1 | 0 |
| 3 | 1 | -1 | 1 | -1 | 2 |
| 4 | 1 | -1 | 1 | -1 | -2 |
| 5 | 1 | $1 - \sqrt{2}$ | $1 - \sqrt{2}$ | 1 | 0 |

In the non-diagonal $D_5$ model, with torus partition function

$$Z_{D_5}^T = |\chi_1|^2 + |\chi_3|^2 + |\chi_5|^2 + |\chi_7|^2 + |\chi_4|^2 + \chi_2 \bar{\chi}_6 + \chi_6 \bar{\chi}_2$$

(165)
two of the coefficients ($B_2$ and $B_6$) vanish, since the corresponding two dimensional fields are non-diagonal, while the presence of the signs $\epsilon_{ijl}$ modifies the equations for $B_4$ as follows

$$B_4 B_{2I+1} = (-1)^I B_4,$$

$$B_4 B_4 = B_1 - B_3 + B_5 - B_7.$$  

(166)

Hence there are only five different solutions for the reflection coefficients $B_i$, which are reported in table 2.

Note that again the number of different boundary conditions is equal to the number of two dimensional fields with charge conjugate chiral and anti-chiral labels (or equivalently to the number of different Ishibashi states (140)). This in fact is a general property of the two dimensional conformal theories with boundaries [53, 56].

4.2 Closed unoriented sector, crosscap constraint

To study the behaviour of the two dimensional fields on non-oriented surfaces let us first introduce the crosscap. The crosscap is the projective plane and can be represented as a unit disc with diametrically opposite points identified. Two dimensional surfaces with crosscaps cannot be oriented. For example the Klein bottle is topologically equivalent to a cylinder terminating at two crosscaps.
Our analysis will follow closely the one in the boundary case. Like the boundaries, the crosscap breaks the two dimensional conformal symmetry since it is not invariant under all transformations of $\text{Vir} \otimes \overline{\text{Vir}}$. If the central charges of the two algebras are equal ($\bar{c} = c$) there exist crosscaps that are preserved at most by the diagonal subalgebra $\text{Vir}_{\text{diag}}$. Let us again pass to the analytic picture mapping the boundary of the crosscap onto the unit circle. Then the crosscap implies the identification $\bar{z} \leftrightarrow -1/z$. Similarly to the case of a boundary, the absence of energy flux through the crosscap allows to define the stress energy tensor as

$$T_d(z) = \begin{cases} T(z) & \text{for } |z| \leq 1 \\ \frac{1}{z^2} \bar{T}(-\frac{1}{z}) & \text{for } |z| > 1 \end{cases}$$

while the currents are

$$J^a_d(z) = \begin{cases} J^a(z) & \text{for } |z| \leq 1 \\ -\frac{1}{z} J^a(-\frac{1}{z}) & \text{for } |z| > 1 \end{cases}$$

The combinations of the Laurent modes of the stress energy tensor and of the currents that satisfy the Virasoro algebra with vanishing central charge (136) and the current algebra of zero level (137) are in this case

$$L_n = L_n - (-1)^n \bar{L}_{-n}$$

and

$$J^a_n = J^a_n + (-1)^n \bar{J}^a_{-n}.$$ 

The crosscap states $|C\rangle$ [57] in the theory are annihilated by the modes (169) and (170)

$$L_n |C\rangle = (L_n - (-1)^n \bar{L}_{-n}) |C\rangle = 0$$

and

$$J^a_n |C\rangle = (J^a_n + (-1)^n \bar{J}^a_{-n}) |C\rangle = 0.$$ 

These equation have the same number of solutions as the corresponding equations (138) and (139) for the boundary states and one can explicitly construct the Ishibashi-type crosscap states like in (140). There is however an important difference with the boundary case, since the consistency conditions imply the crosscap constraint [58] [59], which singles out one crosscap state $|C\rangle$. Let us stress that in general there may be several different crosscap states corresponding to different actions of the involution $\Omega : z \leftrightarrow -1/z$ on the fields.
The crosscap constraint tells us only that two different crosscap states cannot exist simultaneously in the same theory.

Just like in the boundary case, the presence of a crosscap implies that the \(n\)-point functions of the two dimensional primary fields are linear combinations of the \(2n\)-point chiral conformal blocks. However, in contrast with the boundary case one cannot introduce non-trivial crosscap operators, since the involution \(\Omega : z \leftrightarrow -1/z\) has no fixed points. In particular only the identity operator (which has no \(z\) dependence and hence is the only invariant under \(\Omega\) one) can contribute to the expansion of a two dimensional primary field in front of a crosscap

\[
\phi_{\Lambda \bar{\Lambda}}(z, \bar{z})\big|_{\text{crosscap}} \sim \Gamma_{\Lambda \bar{\Lambda}} \delta_{\Lambda \bar{\Lambda} C} 1 .
\]  

(173)

Here \(\Gamma_{\Lambda \bar{\Lambda}}\) is a normalization constant and \(\bar{\Lambda}^C\) is the charge conjugate of \(\bar{\Lambda}\). Let us stress that the expansion \(173\) can be used only for the computation of the one point functions of the fields in front of a crosscap. The reason is that the operator product expansions are valid only if the arguments can be connected without encountering other singularities, but in all \(n \geq 2\) point functions in front of a crosscap \(z\) and \(\bar{z} = -1/z\) are always separated by the arguments of the other fields.

The involution \(\Omega\) acts on the two dimensional primary fields \(141\) transforming the chiral vertex operators into antichiral ones and vice versa, and thus relating the two dimensional field \(141\) to the field with weights and arguments exchanged

\[
\phi_{\Lambda \bar{\Lambda}}(\bar{z}, z) = \sum_{\Lambda_i \Lambda_f} V_{\bar{\Lambda}_i}^{\Lambda_i} (\bar{z}) \bar{V}^{\Lambda_f}_{\Lambda_i}(z) n_{i f} .
\]  

(174)

To simplify the notation let us denote the two weights of the field by a single label (this is unambiguous for a permutation modular invariant) setting \(\phi_i = \phi_{\Lambda_i \bar{\Lambda}_i}\) and \(\phi_i = \phi_{\bar{\Lambda}_i \Lambda_i}\). The action of \(\Omega\) is

\[
\Omega \phi_i(z, \bar{z}) = \epsilon_i \phi_i(\bar{z}, z) .
\]  

(175)

Since \(\Omega\) is an involution, the \(\epsilon_i\) are just signs

\[
\epsilon_i = \epsilon_i = \pm 1
\]  

(176)

which have to respect the fusion rules \(71\), hence

\[
\epsilon_i \epsilon_j \epsilon_k = 1 \text{ if } N_{ijk} \neq 0 .
\]  

(177)
As an example let us again take the $SU(2)$ current algebra. In this case the equations (177) have only two different solutions: $\epsilon_i = +1$ for all integer isospin fields, and $\epsilon_i = \epsilon = \pm 1$ for all half integer isospin fields.

One convenient way to compute the $n$-point functions of the two dimensional fields in the front of a crosscap is to introduce the crosscap operator

$$\hat{C} = \sum_l \Gamma_l \frac{|\Delta_l\rangle \langle \bar{\Delta}_l|}{\sqrt{N_l}}, \quad (178)$$

where $N_l$ is the normalization constant of the two dimensional 2-point function (126). The operator $\hat{C}$ allows to explicitly correlate the $n$-point functions of the two dimensional fields in presence of a crosscap with the $2n$-point chiral conformal blocks

$$\langle \phi_{1,1}..\phi_{n,n}\rangle_C = \langle 0|\hat{C}\phi_{1,1}..\phi_{n,n}|0 \rangle = \sum_l \frac{\Gamma_l}{\sqrt{N_l}} \langle 0|V_{\Delta_1}(z_1)..V_{\Delta_n}(z_n)|\Delta_l\rangle \langle \bar{\Delta}_l|\bar{V}_{\bar{\Delta}_1}(\bar{z}_1)..\bar{V}_{\bar{\Delta}_n}(\bar{z}_n)|0 \rangle \quad (179)$$

The relation (175) for the two dimensional fields implies for their functions in presence of a crosscap

$$\langle \phi_{i,i}(z_i, \bar{z}_i) X \rangle_C = \epsilon_{(i,i)} \langle \phi_{\bar{i},\bar{i}}(\bar{z}_i, z_i) X \rangle_C \quad (180)$$

where $X$ is an arbitrary polynomial in the fields. These equations determine the coefficients $\Gamma_n$. In particular for the one point functions which satisfy

$$\langle \phi_{i,i}(z, \bar{z})\rangle_C = \sum_l \frac{\Gamma_l}{\sqrt{N_l}} \langle 0|V_i(z)|\Delta_l\rangle \langle \bar{\Delta}_l|\bar{V}_i(\bar{z})|0 \rangle = \frac{\Gamma_i}{\sqrt{N_i}} \delta_{ii} \langle 0|V_i(z)\bar{V}_i(\bar{z})|0 \rangle = \langle \phi_{i,i}(z, \bar{z})\rangle_C \quad (181)$$

equation (180) implies the vanishing of $\Gamma_{\ell}$ for all fields on which $\Omega$ acts nontrivially ($\epsilon_{\ell} = -1$). Note that the factor $\sqrt{N_i}$ in (181) is compensated by the normalization of the chiral function $\langle 0|V_i V_i|0 \rangle$ in accord with (173).

To derive the crosscap constraint let us apply (180) for the 2-point functions in presence of a crosscap. The left hand side is

$$\langle \phi_{i,i}(z_1, \bar{z}_1) \phi_{j,j}(z_2, \bar{z}_2)\rangle_C$$
\[
\sum_l \frac{\Gamma_l}{\sqrt{N_l}} \langle 0 | V_i(z_1) \ V_j(z_2) | \Delta_l \rangle \langle \bar{\Delta}_l | \bar{V}_i(\bar{z}_1) \ \bar{V}_j(\bar{z}_2) | 0 \rangle \\
= \sum_l \Gamma_l \ \tilde{C}^{(l,l)}_{(i,i)(j,j)} \ S_l(z_1, z_2, \bar{z}_1, \bar{z}_2) ,
\]

(182)

where \( S_l \) are the normalized \( s \)-channel chiral conformal blocks \(^{114}\) (note the order of the arguments \( z_i \)). The constants \( \tilde{C} \) are proportional to the two dimensional structure constants \(^{128}\)

\[
\tilde{C}^{(l,l)}_{(i,i)(j,j)} = \sqrt{N_l} \ C^{(l,l)}_{(i,i)(j,j)} .
\]

(183)

In the same way for the right hand side we obtain

\[
\langle \phi_{i,i}(\bar{z}_1, z_1) \ \phi_{j,j}(z_2, \bar{z}_2) \rangle_C \\
= \sum_l \frac{\Gamma_l}{\sqrt{N_l}} \langle 0 | V_i(\bar{z}_1) \ V_j(z_2) | \Delta_l \rangle \langle \bar{\Delta}_l | \bar{V}_i(z_1) \ \bar{V}_j(\bar{z}_2) | 0 \rangle \\
= \sum_l \Gamma_l \ \tilde{C}^{(l,l)}_{(i,i)(j,j)} \ S_l(\bar{z}_1, z_2, z_1, \bar{z}_2) .
\]

(184)

The \( s \)-channel blocks \( S_l(\bar{z}_1, z_2, z_1, \bar{z}_2) \) are proportional to the \( u \)-channel blocks \( U_l(z_1, z_2, \bar{z}_1, \bar{z}_2) \) (see equation \(^{120}\)) and can be related to the conformal blocks in \(^{182}\) by the exchange operator \( B_1 B_3 \)^{-1} F. Using also the explicit form of \( B_1 \) and \( B_3 \) \(^{117}\) we find

\[
S_l(i,j,i,j) = (-1)^{\Delta_i - \Delta_j} \ \sum_n \ F_{ln}(i,j,i,j) \ S_n(i,j,i,j) .
\]

(185)

Inserting \(^{182,184,185}\) into equation \(^{180}\) we obtain the final form of the crosscap constraint \(^{39}\)

\[
\epsilon_{(i,\bar{i})} (-1)^{\Delta_i - \Delta_j + \Delta_j - \Delta_j} \ \Gamma_n \ \tilde{C}^{(n,n)}_{(i,\bar{i})(j,j)} \\
= \sum_l \ \Gamma_l \ \tilde{C}^{(l,l)}_{(i,i)(j,j)} \ F_{ln}(i,j,i,j) \quad (186)
\]

for all \( n \). Applying \( \Omega \) to the second field in the two point function leads to the same equation. Note that the crosscap constraint is linear in \( \Gamma \), hence it determines only the ratios \( \Gamma_l / \Gamma_1 \). The remaining freedom is only in the
normalization of the two dimensional identity operator in front of the crosscap $\Gamma_1$. The simplest way to determine $\Gamma_1$ is to impose the integrality condition on the partition functions which we shall describe in the next section. An alternative approach would be to use the topological equivalence of three crosscaps to a handle and one crosscap that is expected to give a nonlinear relation for $\Gamma_l$. The explicit form of this relation is however still not known.

5 Partition functions

The two dimensional structure constants are explicitly known only in a very limited number of cases. This does not allow in general to compute the $n$-point functions in the presence of boundaries or crosscaps and to solve the sewing constraints. Here we shall describe an alternative approach, proposed in [4] in the framework of string theory. It gives less detailed information about the theory, but is applicable in all cases when the modular matrices $S$ and $T$ (67) are known. Just like modular invariant torus partition functions are classified in many cases when the structure constants are not known, the partition function on the annulus and the Klein bottle and Möbius strip projections can be explicitly computed in many cases when we cannot obtain detailed information about the corresponding $n$-point functions in presence of boundaries or crosscaps. The method is particularly powerful if the completeness of the boundary conditions [9] is used. Note that modular invariance of the torus partition function plays also the role of completeness condition for the two dimensional fields.

One starts with a general (not necessary rational) two dimensional theory with isomorphic chiral and antichiral observable algebras $\mathcal{A}$ and $\mathcal{\bar{A}}$, corresponding to a symmetric $X_{ij} = X_{ji}$ torus modular invariant (63).

To simplify the formulae we shall assume that the theory is rational and that the modular invariant is of the permutation type (69). This has the advantage that one can write all expressions using only chiral labels, while in the general case additional degeneracy labels may be needed to distinguish fields with multiplicities larger than one.

5.1 Klein bottle projection

Let us first construct the non-oriented sector. The simplest non orientable surface, the Klein bottle, can be represented as a cylinder terminating at two
crosscaps. The Klein bottle contribution to the partition function is a linear combination of the Virasoro characters \([4, 5, 6]\), hence in general it is not a modular invariant. In fact there are two distinct expressions for the Klein bottle contribution called direct and transverse channel which are related by the modular \(S\) transformation \((64)\). In the string theory language they correspond to inequivalent choices of time on the worldsheet. In the direct channel the Klein bottle contribution is a projection of the torus partition function that describes the (anti)symmetry properties of the two dimensional fields under the involution \(\Omega\) \((175)\)

\[
K = \sum_i \chi_i K^i ,
\]

where the integers \(K^i\) satisfy

\[
|K^i| \leq X_{ii} \quad K^i = X_{ii} \pmod{2} .
\]

Hence for permutation invariants \(K^i\) can take only the values 0 or \(\pm 1\). The \(K_i\) are related (but not necessarily equal) to the signs \(\epsilon_i\) in \((175)\).

The modular \(S\) transformation turns \((187)\) into the transverse channel, which describes the reflection of the two dimensional fields from the two crosscaps at the ends of the cylinder. It has the form

\[
\tilde{K} = \sum_i \chi_i \Gamma_i^2 ,
\]

where the reflection coefficients \(\Gamma_i\) are the normalizations of the one point functions of the two dimensional fields in front of the crosscap (see equation \((173)\)), so they vanish if \(X_{iiC} = 0\).

The complete partition function in the unoriented case is given by the half sum of the torus and direct channel Klein bottle contributions

\[
Z_{\text{unoriented}} = \frac{1}{2}(Z_T + K) .
\]

The multiplicity of a field \(\phi_{ij}(= \phi_{ji})\) can be read of the partition function \((190)\) as follows (for a permutation invariant, if there are multiplicities the argument applies for each copy of the fields)

- if \(i \neq j\) it is equal to \(1/2(X_{ij} + X_{ji})\) and is non-negative integer due to the assumption that the torus invariant is symmetric. Only one combination
of the two fields $\phi_{ij}$ and $\phi_{ji}$ remains in the spectrum, the other is projected out.

- if $i = j$ it is equal to $1/2(X_{ii} + K_i)$ and is non-negative integer since $K_i$ satisfy (188). In particular if $X_{ii} = 1$ the fields with $K_i = 1$ remain in the spectrum, while the ones with $K_i = -1$ are projected out.

If the ground state is degenerate, the Klein bottle projects out the part antisymmetric under the left-right exchange rather that the whole field.

We shall illustrate the construction on the example of the non-diagonal $D_5$ model of the $SU(2)$ current algebra with level $k = 6$ with torus partition function (165). There are two different Klein bottle projections, corresponding to the two choices of the signs $\epsilon_i$ in (175) [59]. For reasons that will become clear in the next subsection, we shall distinguish them by the subscripts $r$ and $c$ (for "real" and "complex")

$$K_{r}^{D_5} = \chi_1 + \chi_3 + \chi_5 + \chi_7 - \chi_4$$
$$K_{c}^{D_5} = \chi_1 + \chi_3 + \chi_5 + \chi_7 + \chi_4$$  \hspace{1cm} (191)

Comparison with the values of $\epsilon_i$ given by

- $\epsilon_i = 1$ for all $i$ in the real case
- $\epsilon_i = (-1)^{i-1}$ in the complex case

shows that there is a relative factor $(-1)^{2I}$ between $K_i$ and the signs $\epsilon_i$ which comes from the $SU(2)$ structure of the fields. Indeed the singlet is in the symmetric (antisymmetric) part of the tensor product of integer (half integer) isospins. So the real Klein bottle projection correspond to keeping all singlets, while the complex one projects out the singlet corresponding to $\chi_4$.

As an application of these ideas to string theory, let us mention that in [60] by a non-standard Klein bottle projection has been constructed the first tachyon free non-supersymmetric open string model.

### 5.2 Annulus partition function

The spectrum of the boundary fields is described by the annulus (or cylinder) partition function with all possible boundary conditions at the two ends. Again the partition function is linear in the characters, hence not a modular invariant, so there are two distinct expressions for the annulus contribution [1]. They are called direct and transverse channel and are related by the modular $S$ transformation [54].
In the direct channel the annulus partition function counts the number of operators that intertwine the boundary conditions at the two ends and can be represented as

\[ A = \sum_{i,a,b} \chi_i A_{ab}^i n_a n_b, \]  

(193)

where the non-negative integers \( A_{ab}^i \) give the multiplicities of the boundary fields \( \psi_{ab}^i \). The auxiliary multiplicities \( n^a \) associated with the boundaries in open string models correspond to the introduction of Chan-Paton gauge groups \([61]\), which can be \( U(n) \), \( O(n) \) or \( USp(2n) \) \([62]\). In the case of \( U(n) \) groups the boundaries can be oriented, since there are two inequivalent choices of the fundamental representation, hence the Chan-Paton charges come in numerically equal pairs \( \bar{n} = n \). We shall call such charges complex. The other two cases, \( USp(2n) \) and \( O(2n) \), do not lead to similar identifications and we shall call the corresponding charges real. The labels \( r \) and \( c \) on the partition functions originate from this interpretation. In applications to Statistical Mechanics one may regard (193) as a generating function for the multiplicities of the allowed boundary fields.

The transverse channel, related to (193) by a modular \( S \) transformation, has a very different interpretation. It describes the reflection of a two dimensional field from the two boundaries and can be represented as

\[ \tilde{A} = \sum_{i} \chi_i \left[ \sum_{a} B_{ia} n^a \right]^2. \]  

(194)

Since only fields with charge conjugate chiral and antichiral labels can couple to the boundaries it is again sufficient to specify only the chiral label. The reflection coefficient \( B_{ia} \) for the field \( i \) (\( \bar{i} \)) from a boundary \( a \) is proportional to the coefficient of the identity operator in the bulk-to-boundary expansion of the two dimensional field in front of the boundary (151)

\[ B_{ia} = \frac{C_{i \bar{i}, i \bar{i}}^{\alpha \bar{\alpha}}}{\sqrt{N_{i \bar{i}}}}. \]  

(195)

One can define charge conjugation on the boundary labels. It is non-trivial only if the boundaries are oriented (that corresponds to complex charges) and is given by the involutive matrix \( (A_1)_{ab} = (A_1)^{ab} \), such that

\[ A_{ia} b = \sum_{c} A_{1ac} A_{i}^{cb}, \]  

(196)
hence $(A_1)^b_a = \delta^b_a$. Let us also assume that the boundaries are a complete set. To justify this assumption let us recall that the modular invariance condition of the torus partition function plays also the role of completeness condition for the two dimensional fields. The completeness condition for the boundaries has two equivalent formulations. The first one [4] is to require that the coefficients $A_{ia}^b$ satisfy the fusion algebra

$$
\sum_b A_{ia}^b A_{jb}^c = \sum_k N_{ij}^k A_{ka}^c.
$$

(197)

Intuitively this relation corresponds to two different ways of counting the boundary fields. The second one [56] is to require that the boundary states are related to the Ishibashi states (140) by a unitary transformation, which in particular implies that they are the same number.

Equation (197) contains only chiral information, so it cannot determine completely the multiplicities $A_{ia}^b$. The two dimensional input is provided by the torus modular invariant (62). In particular if for some $j$ the torus coefficient $X_{jj}^j = 0$ (where $j^c$ is the charge conjugate of $j$) then there is no two dimensional field with these labels, so the coefficients $C_a^{(jj^c)}$ are zero for all $a$. Hence, due to (195) vanish also all $B_{ja}$ and $\chi_j$ will not contribute to (194). After a modular transformation this implies

$$
\sum_i A_{ia}^b \delta^i_j = 0
$$

(198)

for all $a$ and $b$ and this particular $j$.

Hence we can reformulate the problem of finding the annulus partition function in the following way: solve over the non-negative integers the two equations (197) and (198). In general this system may have several solutions, but in all known cases fixing also the boundary charge conjugation matrix $(A_1)_{ab}$ determines completely all $A_{ia}^b$, and thus the only freedom is in choosing the orientation on pairs of boundaries. The proof of this fact in the general case is however still a challenging open problem.

As an illustration let us again consider the $D_5$ model with torus partition function (163). In the real charge case $(A_1)_{ab} = \delta_{ab}$ and the solution is (the labels of the charges correspond to the first column in table 2)

$$
A_{r}^{D_5} = \chi_1 (n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2)
+ (\chi_2 + \chi_6)(2n_1n_2 + 2n_1n_5 + 2n_3n_5 + 2n_4n_5)
$$

45
\[ A_{c}^{D5} = \chi_{1}(n_{1}^{2} + n_{2}^{2} + 2n\bar{n} + n_{3}^{2}) + (\chi_{2} + \chi_{6})(2n_{1}n_{2} + 2n_{1}n_{5} + 2nn_{5} + 2\bar{n}n_{5}) + \chi_{3}(n_{1}^{2} + n^{2} + \bar{n}^{2} + 2n_{1}n + 2n_{1}\bar{n} + 2n_{2}n_{5} + 2n_{5}^{2}) + \chi_{4}(4n_{1}n_{5} + 2n_{2}n + 2n_{2}\bar{n} + 2nn_{5} + 2\bar{n}n_{5}) + \chi_{5}(n_{1}^{2} + 2n_{2}^{2} + 2n_{1}n + 2n_{1}\bar{n} + 2n_{2}n_{5} + 2\bar{n}n) + \chi_{7}(n_{1}^{2} + n_{2}^{2} + n_{5}^{2} + n^{2} + \bar{n}^{2}) \]  

(199)

In the complex case the two charges \( n_{3} \) and \( n_{4} \) become a complex pair \( \bar{n} = n \) and the solution is

\[ A_{c}^{D5} = \chi_{1}(n_{1}^{2} + n_{2}^{2} + 2n\bar{n} + n_{3}^{2}) + (\chi_{2} + \chi_{6})(2n_{1}n_{2} + 2n_{1}n_{5} + 2nn_{5} + 2\bar{n}n_{5}) + \chi_{3}(n_{1}^{2} + n^{2} + \bar{n}^{2} + 2n_{1}n + 2n_{1}\bar{n} + 2n_{2}n_{5} + 2n_{5}^{2}) + \chi_{4}(4n_{1}n_{5} + 2n_{2}n + 2n_{2}\bar{n} + 2nn_{5} + 2\bar{n}n_{5}) + \chi_{5}(n_{1}^{2} + 2n_{2}^{2} + 2n_{1}n + 2n_{1}\bar{n} + 2n_{2}n_{5} + 2\bar{n}n) + \chi_{7}(n_{1}^{2} + n_{2}^{2} + n_{5}^{2} + n^{2} + \bar{n}^{2}) \]  

(200)

Note that in both cases some boundary fields (corresponding to the \( n_{5} \) charge) have multiplicities equal to two.

### 5.3 Möbius strip projection

The consistency of the theory in presence of both boundaries and crosscaps is determined by the Möbius strip contribution [4, 5, 6]. The Möbius strip can be represented as a cylinder terminating at one boundary and at one crosscap. Hence in the transverse channel the two dimensional field reflects from the boundary and the crosscap with the same reflection coefficients \( B_{ia} \) and \( \Gamma_{i} \), which enter equations (189), (194).

\[ \tilde{M} = \sum_{i} \hat{\chi}^{i} \Gamma_{i} \left[ \sum_{a} B_{ia} n^{a} \right] \]  

(201)

As we have seen there are in general more than one solutions for both \( B_{ia} \) and \( \Gamma_{i} \), so we have to specify also which of these solutions we shall use in equation (201). To determine this we can pass to the direct channel (by a \( P \) transformation, see equation (205) below)

\[ M = \sum_{i} \hat{\chi}^{i} M_{i}^{a} n_{a} \]  

(202)
and compare this expression with the annulus partition function (193). The integer coefficients $M_i^a$ can be interpreted as twists (or projections) of the open spectrum and thus have to satisfy

$$M_i^a = A_i^{aa} \pmod{2} \quad |M_i^a| \leq A_i^{aa}.$$  \hfill (203)

These equations choose consistent pairs of annulus and Klein bottle partition functions.

The natural modular parameter in the direct channel for the Möbius strip is $(i\tau + 1)/2$, while in the transverse channel it is $(i+\tau)/2\tau$. The non-vanishing real part of the direct channel modular parameter implies that the natural basis of characters for the Möbius strip is

$$\hat{\chi}_j = e^{-i\pi(\Delta_j - c/24)} \chi_j \left( \frac{i\tau + 1}{2} \right),$$  \hfill (204)

hence the transformation which relates the direct and the transverse channel is given by [5]

$$P = T^{1/2} S T^2 S T^{1/2},$$  \hfill (205)

and satisfies $P^2 = C$. The square root of $T$ in (205) denotes the diagonal matrix whose eigenvalues are square roots of the eigenvalues of $T$.

By a formula similar to the Verlinde formula (71) one can define the coefficients $Y_{ijk}^k$ [8]

$$Y_{ij}^k = \sum_{\ell} S_{i\ell} P_{j\ell} P_{k\ell}^{\dagger} S_{1\ell},$$  \hfill (206)

which are integers [63, 64] and satisfy the fusion algebra

$$\sum_i Y_{im}^l Y_{jl}^n = \sum_{\ell} N_{ij}^\ell Y_{\ell m}^n \hfill (207)$$

$$\sum_i Y_{ijk} Y_{il}^m = \sum_i Y_{ijm} Y_{ik}^l.$$  \hfill (208)

The complete partition function in the unoriented open sector is

$$Z_{\text{open}} = \frac{1}{2} (A \pm M).$$  \hfill (209)

Its integrality is guaranteed by the conditions (203). Note that the overall sign of the Möbius strip projection is not determined by the conformal theory.
In open string models this sign is fixed by the tadpole cancellation conditions \cite{98} and determines the gauge group.

The completeness condition \cite{197} implies two relations between the integer coefficients in the direct channel partition functions $A$, $M$ and $K$ and the $Y$ tensor \cite{206}:

\[
\begin{align*}
\sum_b A_i^{ab} M_{jb} &= \sum_l Y_{ij}^l M_i^a \\
\sum_b M_i^{a} M_{ib} &= \sum_l Y_{ij}^l K_l
\end{align*}
\] (210)

that put very strong constraints on $K_i$ and $M_i^a$ for given $A_i^{ab}$ (in all known cases they completely determine them).

Coming back to our example, in the non diagonal $D_5$ model there are two consistent choices for the annulus and Klein bottle partition functions, namely the pairs with the same subscript ($r$ or $c$). The two Möbius strip projections are correspondingly

\[
M_{D_5}^r = \hat{\chi}_1(n_1 - n_2 + n_3 + n_4 - n_5) + \hat{\chi}_3(-n_1 + 2n_5) + \hat{\chi}_5(n_1 + n_3 + n_4) + \hat{\chi}_7(n_1 + n_2 + n_5)
\] (212)

and

\[
M_{D_5}^c = \hat{\chi}_1(-n_1 + n_2 + n_5) + \hat{\chi}_3(n_1 + n + \bar{n}) + \hat{\chi}_5(n_1 + 2n_5) + \hat{\chi}_7(n_1 + n_2 + n + \bar{n} + n_5)
\] (213)

It is instructive to verify that these indeed satisfy the polynomial equations and to determine the open spectrum of the models. Note that when the annulus coefficient is equal to $2n_5^2$, there are two possibilities for the Möbius strip coefficient. It can be either $2n_5 = n_5 + n_5$ or $0 = n_5 - n_5$. This corresponds to two operators with equal or opposite symmetrization properties.
5.4 Solutions for the partition functions

If the torus modular invariant is given by the charge conjugation matrix $X = C$ then the number of boundary conditions coincides with the number of chiral representations, so we can label both by the same label. In this case the standard solution for the annulus was found in [1], while the expressions for the Klein bottle and Möbius strip were found in [8]

$$A_{ijk} = N_{ijk} \quad (214)$$

$$M_{ij} = Y_{ji1} \quad (215)$$

$$K_i = Y_{i11} \quad . \quad (216)$$

Using the properties of $N_{ijk}$ and $Y_{ijk}$ it is straightforward to verify that these solutions satisfy all the consistency requirements. Moreover, the standard Klein bottle projection (216) is equal to the Frobenius-Schur indicator [64] and corresponds to keeping all the singlets in the spectrum. A modular transformation to the transverse channel gives

$$\tilde{A} = \sum_i \left( \sum_j S_{ij} n^j \right)^2 \chi_i \quad (217)$$

$$\tilde{M} = \sum_i \left( \sum_j P_{i1} S_{ij} n^j \right) \hat{\chi}_i \quad (218)$$

$$\tilde{K} = \sum_i \left( \frac{P_{i1}}{\sqrt{S_{ii}}} \right)^2 \chi_i \quad . \quad (219)$$

Before these general formulae were known, in [66] the standard Klein bottle and Möbius partition functions for the diagonal case of the unitary minimal models had been explicitly constructed.

As a simple example of a non standard solution we shall list also the expressions for the second possible solution in the diagonal $SU(2)$ current algebra models of level $k$ denoted by $A_{k+1}$. The modular matrices $S$ and $T$ (we label the fields by $j = 2I + 1$) are

$$S_{jl} = \sqrt{\frac{2}{k+2}} \sin \left( \frac{\pi j l}{k+2} \right) \quad (220)$$

$$T_{jl} = \delta_{jl} e^{i \pi \left( \frac{j^2}{\pi (k+2)} - \frac{1}{4} \right)} \quad . \quad (221)$$
The charge conjugation matrix is identity \( C = S^2 = (ST)^3 = 1 \). The modular matrix \( P = T^{1/2}ST^2ST^{1/2} \) which satisfies \( P^2 = C = 1 \) is

\[
P_{jl} = \frac{2}{\sqrt{k+2}} \sin \left( \frac{\pi jl}{2(k+2)} \right) \left( E_k E_{j+l} + O_k O_{j+l} \right),
\]

where \( E_n \) and \( O_n \) are projectors on \( n \) even and \( n \) odd correspondingly.

The standard solution in the diagonal model has \( k + 1 \) real charges and is given by (214, 219). The explicit expression for the direct channel Klein bottle projection is

\[
K_c^{(A_{k+1})} = \sum_{j=1}^{k+1} Y_{j,1}^1 \chi_j = \sum_{j=1}^{k+1} ( -1 )^{j-1} \chi_j
\]

(223)

hence indeed all singlets are kept in the unoriented spectrum.

The second solution has also \( k + 1 \) charges (most are in complex pairs) and in the direct channel is given by (8)

\[
K_c^{(A_{k+1})} = \sum_{j=1}^{k+1} Y_{j,k+1}^1 \chi_j = \sum_{j=1}^{k+1} \chi_j
\]

(224)

\[
A_c^{(A_{k+1})} = \sum_{j,l,m=1}^{k+1} N_{lm}^j \chi_{k+2-j} n_l^m
\]

(225)

\[
M_c^{(A_{k+1})} = \sum_{j,l=1}^{k+1} Y_{l,k+1}^1 \bar{\chi}_j n_l.
\]

(226)

Note that the Klein bottle projects out the singlets for all the half integer isospin fields, so they cannot couple to the identity on the boundaries or the crosscap, hence the corresponding reflection coefficient should vanish. After a modular transformation we find in the transverse channel

\[
\tilde{K}_c^{(A_{k+1})} = \sum_i \left( \frac{P_{k+1,i}}{\sqrt{S_{1i}}} \right)^2 \chi_i
\]

(227)

\[
\tilde{A}_c^{(A_{k+1})} = \sum_i ( -1 )^{i-1} \left( \sum_j S_{ij} n_j \right)^2 \chi_i
\]

(228)

\[
\tilde{M}_c^{(A_{k+1})} = \sum_i \left( \sum_j P_{k+1,i} S_{ij} n_j \right) \bar{\chi}_i.
\]

(229)
The vanishing of the reflection coefficients of the fields with half integer isospin in \((228)\) implies the complex charge identifications \(n_{k+2-i} = \bar{n}_i = n_i\) for all \(i\).

In the \(D_{odd}\) models there are again two different choices for the Klein bottle projection (which generalize equations \((191,192)\) for \(D_5\)). Both lead to \(k/2 + 2\) charges. The corresponding annulus and Möbius strip partition functions are rather involved \([5, 59]\). The solutions for \(D_{even}, E_6\) and \(E_8\) (with charge conjugation modular invariants if considered as models with extended symmetry) are given by the general formulae \((214,219)\). The solution for the exceptional case \(E_7\) is given in \([59]\). In the \(D_{even}\) and \(E\) models one can study also boundary conditions that do not respect the extended symmetry of the bulk model, but only the \(SU(2)\) symmetry. The corresponding solutions are given in \([53]\).

Many other solutions have been found. Let us note only the general formulae in \([67]\) where the partition functions for all simple currents modular invariants are given.

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