A Perturbation Approach to Vector Optimization Problems: Lagrange and Fenchel–Lagrange Duality

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Received: 8 September 2021 / Accepted: 18 May 2022 / Published online: 20 June 2022
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Abstract
In this paper, we study a general minimization vector problem which is expressed in terms of a perturbation mapping defined on a product of locally convex Hausdorff topological vector spaces with values in another locally convex topological vector space. Several representations of the epigraph of the conjugate of the perturbation mapping are given, and then, variants vector Farkas lemmas associated with the system defined by this mapping are established. A dual problem and another so-called loose dual problem of the mentioned problem are defined and stable strong duality results between these pairs of primal–dual problems are established. The results just obtained are then applied to a general class of composed constrained vector optimization problems. For this class of problems, two concrete perturbation mappings are proposed. These perturbation mappings give rise to variants of dual problems including the Lagrange dual problem and several kinds of Fenchel–Lagrange dual problems of the problem under consideration. Stable strong duality results for these pairs of primal–dual problems are derived. Several classes of concrete vector (and scalar) optimization problems are also considered at the end of the paper to illustrate the significance of our approach.

Keywords Vector optimization problems · Perturbation mappings · Perturbation approach · Vector Farkas lemmas · Stable strong duality for vector problems
1 Introduction

We are concerned with the general vector optimization problem

\[(P) \quad \text{WInf}_{x \in X} F(x),\]

where \(X, Y\) are locally convex Hausdorff topological vector spaces (in brief, lcHtvs), \(F: X \to Y \cup \{+\infty, -\infty\}\) and “WInf” stands for the weak infimum w.r.t. an ordering generated by a closed convex cone \(K \subset Y\) with \(\text{int} \ K \neq \emptyset\). We introduce a perturbation mapping \(\Phi: X \times Z \to Y \cup \{+\infty, -\infty\}\) such that \(\Phi(x, 0) = F(x)\) for all \(x \in X\). Here, \(Z\) is a lcHtvs called the perturbation space. Similar to the scalar case, the problem \((P)\) is embedded into a family of perturbed problems

\[(P_z) \quad \text{WInf}_{x \in X} \Phi(x, z),\]

where \(z \in Z\) is the perturbation variable. When \(z = 0\), the problem \((P_0)\) is none other than \((P)\) and it can be expressed as

\[(P) \quad \text{WInf}_{x \in X} \Phi(x, 0Z).\]

The vector problem \((P)\) includes many vector optimization problems in practice as its special cases, such as composite vector problems, convex semi-vector bilevel problems (see [11]), general vector problems with cone constraints [13], the bicriterial optimization problem of a cross current multistage extraction process (see [11], [24, p. 373-375]) and various practical problems in [24, Chapter 13].

The problem \((P)\) with a linear perturbation \(L \in \mathcal{L}(X, Y)\) is

\[(P^L) \quad \text{WInf}_{x \in X} [\Phi(x, 0Z) - L(x)].\]

Conjugate duality has been used to study scalar, vector problems, and also set-valued optimization problems by many authors, see, for instance, [2, 7, 10, 17, 18, 20–22, 25, 33] for scalar and robust (scalar) problems, and [3, 4, 6, 12, 13, 15, 22, 28, 31, 32] for vector and set-valued problems.

For scalar optimization problems, variants of dual problems have been investigated by the perturbation approach (see [2, 9, 21, 22] and references therein) which “is based on the theory of conjugate functions and describes how a dual problem can be assigned to a primal one” (see [9]). Some initial extensions of this approach to vector optimization problems are given in [6] and [23].
For $\emptyset \neq \mathcal{V} \subset \mathcal{L}(X, Y)$ and each $L \in \mathcal{V}$, we introduce the dual problem and the loose dual problem of $(P^L)$, respectively,

$$
(D^L) \quad \text{WSup} \left[ -\Phi^*(L, T) \right], \\
(D^L_\ell) \quad \text{WSup} \left[ -\Phi^*(L, T) \right], \\
$$

where $\mathcal{L}_\Phi := \{ T \in \mathcal{L}(Z, Y) : \exists L \in \mathcal{L}(X, Y) \text{ s.t. } (L, T) \in \text{dom } \Phi^* \}$ and $\mathcal{L}_\Phi^+ := \mathcal{L}_\Phi \cap \mathcal{L}_+(S, K)$, with $\mathcal{L}_+(S, K) = \{ T \in \mathcal{L}(X, Z) : T(S) \subset K \}$, and $S$ is a convex cone in $Z$.

In this paper, the perturbation approach is constructed for establishing (stable) strong duality results for the pairs of primal–dual problems $(P^L)$ - $(D^L)$ and $(P^L_\ell)$ - $(D^L_\ell)$, $L \in \mathcal{V}$.

This paper can be considered as a generalization to the model $(P^L)$ of the work by the same authors [16], where a special cone-constrained vector problem (the problem (VP) below) is considered. To overcome difficulties arising when study the general problem $(P^L)$, various notions (some are just introduced recently in [16]) such as “partition-style subsets of $Y$,” the ordering “$\ll_k$” between such two sets, “extended epigraphs” of conjugate mappings, new operations on the subsets of $Y$, the $\uplus$-sum, and the $\ominus$-sum on the collection of $K$-extended epigraphs of conjugate mappings are used. Also, as a tool, a version of generalized open mapping theorem on a product space is proved.

The novelty of our work is threefold: Firstly, the new tools permit us to establish variants of representations of the epigraph of the conjugate mapping of $\Phi(\cdot, T)$, which pave the way for proving versions of general vector Farkas lemmas and also stable strong duality for the pairs $(P^L)$ - $(D^L)$ and $(P^L_\ell)$ - $(D^L_\ell)$. Such results when specified to some concrete problems (for instance, (VP) and (CCVP) below) will lead to stable strong duality results associated with Fenchel–Lagrange dual problems that probably appear for the first time in the literature for vector optimization problems (Sections 5 and 6). In the case where $Y = \mathbb{R}$, these dual problems go back to the traditional Fenchel–Lagrange dual problems (of scalar problems) as in [2, 7, 18, 19] (Section 6) and this justifies the name “Lagrange and Fenchel–Lagrange dual problems” of (D) and (D_\ell). Secondly, our approach is based on a very general perturbational $K$-convex mapping $\Phi$ (not necessarily continuous) and so it is flexible for applications (i.e., one gets different results with different choices of $\Phi$). Finally, the new results achieved with a general perturbation mapping $\Phi$ give an approach which generalizes and unifies the antecedent works on duality for vector optimization problems and on vector Farkas lemmas such as the ones in [11–14].

To illustrate the meaning of the results obtained, we consider at the end of the paper a class of the composed vector problem with a set and a cone constraint

$$(CCVP) \quad \text{WInf} \{ F(x) + (\kappa \circ H)(x) : x \in C, \ G(x) \in -S \},$$

where $X, Y, Z, W$ are lcHtvs, $S \subset Z$ is a non-empty, convex cone, and $C$ is a closed and convex subset of $X$ while $F : X \to Y \cup \{+\infty_Y, -\infty_Y\}$, $G : X \to
$Z \cup \{+\infty, -\infty\}, \kappa : W \cup \{+\infty, -\infty\} \rightarrow Y \cup \{+\infty, -\infty\}$, $H : X \rightarrow W \cup \{+\infty, -\infty\}$ are proper mappings. A special case of this model is the cone-constrained vector problem

$$(VP) \quad \text{WInf}\{F(x) : x \in C, \ G(x) \in -S\}$$

which is the common model of many practical problems in science and engineering, for instance, the problem of designing of fiber distributed data interface computer networks, the problem of designing of a cross-current multistage extraction process (see, e.g., [11], [24, Chapter 13]). For the problem (CCVP), two perturbation functions $\Phi_1, \Phi_2$ are considered which give rise to generalized vector Farkas lemmas, Lagrange and Fenchel–Lagrange dual problems with corresponding stable strong duality results for the problem under consideration.

The paper is organized as follows: In Section 2, some basic notations and known results, also new notions, and a new version of a generalized open mapping theorem are presented. In Section 3, a representation of the epigraph of the conjugate mapping $\Phi(., 0_Z)^*$ is established under different regularity conditions. Section 4 presents the main results of the paper; necessary and sufficient conditions for $\mathcal{V}$-stable strong duality for the pairs $(\mathcal{P}^L) - (\mathcal{D}^L)$ and $(\mathcal{P}^L) - (\mathcal{D}^L_\ell)$. Sections 5 and 6 are left for applications and illustrative examples of the meaning of the results obtained in Sections 3, 4. Concretely, in Section 5, we consider the composed vector problem (CCVP) while some special cases of this class of problems are considered in Section 6. It is shown that the results obtained when specified to these specific problems lead to variants of results due to different choices of $\Phi$. Some among them cover known ones in the literature and some are new even when coming back to the case when $Y = \mathbb{R}$. Some similar or rather technical proofs of some auxiliary results are left to the “Appendix” at the end of the paper.

2 Preliminaries, First Results, and Generalized Open Mapping Theorem

2.1 Preliminaries

Let $X, Y, Z$ be lcHtvs with their topological dual spaces denoted by $X^*, Y^*$ and $Z^*$, respectively, and endowed with its own corresponding weak*-topology. For a set $U \subset X$, we denote by $\text{int} \ U, \text{cl} \ U, \text{bd} \ U, \text{lin} \ U, \text{aff} \ U, \text{cone} \ U$ the interior, the closure, the boundary, the convex hull, the linear hull, the affine hull, and the conical hull of $U$, respectively. The intrinsic core of $U$, relative interior of $U$ are defined, respectively, as [2, 33]

$$\text{icr} \ U := \{x \in X : \forall x' \in \text{aff} \ U, \exists \delta > 0 \text{ such that } x + \lambda x' \in U, \forall \lambda \in [0, \delta]\},$$

$$\text{ri} \ U := \{x \in \text{aff} \ U : \exists V \text{ neighborhood of } x \text{ such that } (V \cap \text{aff} \ U) \subset U\}.$$

When $X$ is a finitely dimensional space one has $\text{icr} \ U = \text{ri} \ U$. 

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For $\emptyset \neq A \subset X$ and $B \subset A$, $\text{int}_A B$ aims the interior of $B$ w.r.t. the topology induced in $A$.

### 2.1.1 Weak Ordering Generated by a Convex Cone

Let $K$ be a proper closed and convex cone in $Y$ with nonempty interior, i.e., $\text{int} K \neq \emptyset$. It is worth observing that $K + \text{int} K = \text{int} K$, or equivalently,

$$(y \in K \text{ and } y + y' \notin \text{int} K) \implies y' \notin \text{int} K.$$  

We define a **weak ordering** in $Y$ generated by $K$ as follows: for all $y_1, y_2 \in Y$,

$$y_1 <_K y_2 \iff y_1 - y_2 \in -\text{int} K.$$  

In $Y$, we sometimes also consider an usual **ordering** generated by the cone $K$, $\leq_K$, which is defined by $y_1 \leq_K y_2$ if and only if $y_1 - y_2 \in -K$, for $y_1, y_2 \in Y$.

We enlarge $Y$ by attaching a greatest element $+\infty$ and a smallest element $-\infty$ w.r.t. $<_K$, which do not belong to $Y$, and denote $Y^* := Y \cup \{-\infty, +\infty\}$. We understand, by convention, that $-\infty y <_K y <_K +\infty$ for each $y \in Y$ and

$$-(+\infty) = -\infty, \quad -(\infty) = +\infty,$$

$$(+\infty) + y = y + (+\infty) = +\infty, \quad \forall y \in Y \cup \{+\infty\},$$

$$(-\infty) + y = y + (-\infty) = -\infty, \quad \forall y \in Y \cup \{-\infty\}. \quad (1)$$

Moreover, for $M \subset Y$, $M + \{-\infty\} = \{-\infty\} + M = \{-\infty\}, \quad M + [+\infty\} = [+\infty\} + M = [+\infty\}$. The sums $(-\infty) + (+\infty)$ and $(+\infty) + (-\infty)$ are not considered in this paper.

The following notions are the key ones and will be used throughout the paper.

**Definition 2.1** ([6, Definition 7.4.1], [31]) Let $M \subset Y^*$.

(a) An element $\bar{v} \in Y^*$ is said to be a **weakly infimal element** of $M$ if for all $v \in M$ we have $v \not<_K \bar{v}$ and if for any $\bar{v} \in Y^*$ such that $\bar{v} <_K \bar{v}$, then there is some $v \in M$ satisfying $v <_K \bar{v}$. The set of all weakly infimal elements of $M$ is denoted by $\text{WInf} M$ and is called the **weak infimum** of $M$.

(b) An element $\bar{v} \in Y^*$ is said to be a **weakly supral maximal element** of $M$ if for all $v \in M$ we have $\bar{v} \not<_K v$ and if for any $\bar{v} \in Y^*$ with $\bar{v} <_K \bar{v}$, then there exists some $v \in M$ satisfying $\bar{v} <_K v$. The set of all weakly supral maximal elements of $M$ is denoted by $\text{WSup} M$ and is called the **weak supremum** of $M$.

(c) The **weak minimum** of $M$ is the set $\text{WMin} M = M \cap \text{WInf} M$, and its elements are the **weakly minimal elements** of $M$. The definition of the weak maximum of $M$ is similar.

Some similar notions on weakly minimal elements based on quasi interior of the cone, $\text{qi} K$, can be found, for instance, in [22, Definition 3.12]. The next properties related to the sets weak infimum, weak minimum, weak supremum, weak maximum of a subset $M$ of $Y^*$ are traced out from [6, 12, 15] and [16, Proposition 2.1].

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Proposition 2.1 Let $\emptyset \neq M, N \subset Y^\ast$. One has:

(i) $\text{WSup} M \neq \{+\infty_Y\}$ if and only if $Y \setminus (M \setminus \text{int } K) \neq \emptyset$.

(ii) For all $y \in Y$, $\text{WSup}(y + M) = y + \text{WSup} M$.

(iii) $\text{WSup}(\text{WSup} M + \text{WSup} N) = \text{WSup}(M + \text{WSup} N) = \text{WSup}(M + N)$.

Assume further that $M \subset Y$ and $\text{WSup} M \neq \{+\infty_Y\}$, then it holds:

(iv) $\text{WSup} M - \text{int } K = M - \text{int } K$.

(v) The following decomposition of $Y$ holds (the sets in the right-hand side are disjoint)

$$Y = (M - \text{int } K) \cup \text{WSup } M \cup (\text{WSup } M + \text{int } K).$$

(vi) $\text{WSup } M = \text{cl}(M - \text{int } K) \setminus (M - \text{int } K)$.

(vii) If $0_Y \in N \subset -K$ then $\text{WSup}(M + N) = \text{WSup} M$.

In particular, one has $\text{WSup}(M - K) = \text{WSup}(M - \text{bd } K) = \text{WSup } M$.

Remark 2.1 It is clear that $\text{WInf } M = -\text{WSup}(-M)$ for all $M \subset Y^\ast$ and so, Proposition 2.1 holds true also when $\text{WSup}$, $+\infty_Y$, $K$, and $\text{int } K$ are replaced by $\text{WInf}$, $-\infty_Y$, $-K$, and $-\text{int } K$, respectively. For instance, corresponding to (vi), one has:

(vi') $\text{WInf } M = \text{cl}(M + \text{int } K) \setminus (M + \text{int } K)$.

2.1.2 Conjugate Mappings of Vector-Valued Functions

Let $F : X \to Y^\ast$. The domain and the $K$-epigraph of $F$ are defined, respectively, by $\text{dom } F := \{x \in X : F(x) \neq +\infty_Y\}$ and $\text{epi}_K F := \{(x, y) \in X \times Y : y \in F(x) + K\}$. The mapping $F$ is proper if $\text{dom } F \neq \emptyset$ and $-\infty_Y \notin F(X)$. It is $K$-convex if $\text{epi}_K F$ is convex in $X \times Y$, and it is $K$-epi closed if $\text{epi}_K F$ is closed in the product space $X \times Y$ [2], [29, Definition 5.1]. The mapping $F$ is called positively $K$-lower semicontinuous if $y^* \circ F$ is lower semicontinuous (lsc, for brief) for all $y^* \in K^+$ (see [2, 25]) where $K^+ := \{y^* \in Z^\ast : (y^*, k) \geq 0, \forall k \in K\}$ is the (positive) dual cone of $K$. According to [29, Theorem 5.9], every positively $K$-lsc mapping is $K$-epi closed, but the converse is not true. Moreover, when $Y = \mathbb{R}$, the three notions lsc, positively $\mathbb{R}^+_+$-lsc, and $\mathbb{R}^+_+$-epi closed coincide with each other.

For the space $\mathcal{L}(X, Y)$ of all continuous linear mappings from $X$ to $Y$, one equips with the topology of point-wise convergence, i.e., if $(L_i)_{i \in I} \subset \mathcal{L}(X, Y)$ and $L \in \mathcal{L}(X, Y)$, $L_i \to L$ in $\mathcal{L}(X, Y)$ means $L_i(x) \to L(x)$ in $Y$ for all $x \in X$. The zero of $\mathcal{L}(X, Y)$ is $0_{\mathcal{L}}$.

Now, let $S$ be a non-empty convex cone in $Z$. Recall that the cone of positive operators (see [1, 27]) and the cone of weakly positive operators from $Z$ to $Y$ [12] are defined, respectively,

$$\mathcal{L}_+(S, K) := \{T \in \mathcal{L}(Z, Y) : T(S) \subset K\},$$

$$\mathcal{L}_w^+(S, K) := \{T \in \mathcal{L}(Z, Y) : T(S) \cap (-\text{int } K) = \emptyset\}.$$

When $Y = \mathbb{R}$, both these cones collapse to the usual (positive) dual cone $S^+$ of $S$.  

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For $T \in \mathcal{L}(Z, Y)$, $G : X \rightarrow Z \cup \{+\infty\}$, we define the composite function $T \circ G : X \rightarrow Y^*$ as $(T \circ G)(x) = T(G(x))$ if $G(x) \in Z$ and $(T \circ G)(x) = +\infty_Y$ if $G(x) = +\infty_Z$.

The following notion of conjugate mapping is specified from the corresponding one for set-valued mappings in [6, Definition 7.4.2], [31, Definition 3.1].

**Definition 2.2** [6, 31] For $F : X \rightarrow Y^*$, the set-valued mapping $F^* : \mathcal{L}(X, Y) \rightrightarrows Y^*$ defined by $F^*(L) := \text{WSup}\{L(x) - F(x) : x \in X\}$ is called the conjugate mapping of $F$. The $K$-epigraph and the domain of $F^*$ are, respectively, $\text{epi}_K F^* := \{(L, y) \in \mathcal{L}(X, Y) \times Y : y \in F^*(L) + K\}$ and $\text{dom} F^* := \{L \in \mathcal{L}(X, Y) : F^*(L) \neq +\infty_Y\}$.

Throughout this paper, we use the only cone $K$ in $Y$, and so, for the sake of simplicity, from now on we will write $\text{epi} F$ and $\text{epi} F^*$ instead of $\text{epi}_K F$ and $\text{epi}_K F^*$, respectively. Furthermore, by [12, Lemma 3.5], if $F : X \rightarrow Y^*$ is a proper mapping, then $\text{epi} F^*$ is a closed (but not necessarily convex) subset of $\mathcal{L}(X, Y) \times Y$. Moreover, for $(L, y) \in \mathcal{L}(X, Y) \times Y$, one has (see [13, Proposition 2.10])

\[(L, y) \in \text{epi} F^* \iff \left( F(x) - L(x) + y \notin -\text{int} K, \forall x \in X \right) \text{.} \tag{2} \]

Finally, for a subset $D \subset X$, the indicator map $I_D : X \rightarrow Y^*$ is defined by $I_D(x) = 0_Y$ if $x \in D$ and $I_D(x) = +\infty_Y$, otherwise. It is worth noting that [12, Proposition 3.1],

\[\text{dom} I^*_S = \mathcal{L}_S^\pi(S, K). \tag{3}\]

### 2.2 Structure ($\mathcal{P}_p(Y^\infty)$, $\preceq_K$, $\cup$), Extended Epigraphs, and $\boxplus$-Summation

Let $\mathcal{P}_0(Y^*)$ be the collection of all non-empty subsets of $Y^*$. The ordering “$\preceq_K$” on $\mathcal{P}_0(Y^*)$ is defined [15] as, for $M, N \in \mathcal{P}_0(Y^*)$,

\[M \preceq_K N \iff (v \not\in_K u, \forall u \in M, \forall v \in N) . \tag{4}\]

Other orderings on $\mathcal{P}_0(Y^*)$ are also proposed in the literature (see, e.g., [26]).

**Proposition 2.2** The following assertions hold

(i) For all $M, N \in \mathcal{P}_0(Y^*) \setminus \{\{+\infty_Y\}, \{-\infty_Y\}\}$, one has\[M \preceq_K N \iff N \cap (M - \text{int} K) = \emptyset \iff M \cap (N + \text{int} K) = \emptyset.\]

(ii) For all $M \in \mathcal{P}_0(Y^*)$, one has $\text{WInf} M \preceq_K M$ and $M \preceq_K \text{WSup} M$.

(iii) If $M \subset N \subset Y^*$, then $\text{WSup} M \preceq_K \text{WSup} N$.

(iv) For all $M, N \in \mathcal{P}_0(Y^*)$, if $M \preceq_K N$, then $\text{WSup} M \preceq_K \text{WInf} N$.

**Proof** (i) – (iii) are in [16, Proposition 3.1], while (iv) follows from Proposition 3.2 in [15].
Definition 2.3 We say that a subset $U \subset Y$ is a $(Y, K)$-partition style set if the following decomposition of $Y$ holds

$$Y = (U - \text{int} \ K) \cup U \cup (U + \text{int} \ K).$$

The collection of all $(Y, K)$-partition style subsets of $Y$ is denoted by $\mathcal{P}_p(Y)$. Denote also, $\mathcal{P}_p(Y)^\ast := \mathcal{P}_p(Y) \cup \{+\infty_Y, -\infty_Y\}$. Clearly that if $M \subset Y^\ast$, then $\pm \text{WSup} M \in \mathcal{P}_p(Y)^\ast$, $\pm \text{WInf} M \in \mathcal{P}_p(Y)^\ast$ and (by (4)), for any $U \in \mathcal{P}_p(Y)$, one has $U \preceq_K \{+\infty_Y\}$ and $\{\infty_Y\} \preceq_K U$. Moreover, $(\mathcal{P}_p(Y)^\ast, \preceq_K)$ is an partially ordered space [15].

Proposition 2.3 [16, Proposition 3.2, Lemma 3.1] Let $U, V \in \mathcal{P}_p(Y)^\ast$. Then,

(i) If $U \subset V$, then $U = V$.

Moreover, if $U, V \subset Y$ then:

(ii) $U + K = U \cup (U + \text{int} \ K)$ and $U - K = U \cup (U - \text{int} \ K)$.

(iii) $Y = (U - \text{int} \ K) \cup (U + K) = (U - K) \cup (U + \text{int} \ K)$.

(iv) $U \preceq_K V \iff V \subset U + K \iff U \subset V - K$.

(v) $\text{WSup} U = \text{WInf} U = U$.

A new kind of sum of two sets on the collection $\mathcal{P}_p(Y)^\infty := \mathcal{P}_p(Y) \cup \{+\infty_Y\}$ (called WS-sum), introduced recently in [16], will play a role in our further study.

Definition 2.4 [16, Definition 3.1] For $U, V \in \mathcal{P}_p(Y)^\infty$, the WS-sum of $U$ and $V$, denoted by $U \uplus V$, is defined by $U \uplus V := \text{WSup}(U + V)$.

Some properties of the structure $(\mathcal{P}_p(Y)^\infty, \preceq_K, \uplus)$ are given in the next proposition.

Proposition 2.4 [16, Proposition 3.3] Let $U, V, W \in \mathcal{P}_p(Y)^\infty, y \in Y$. One has:

(i) $U \uplus (-\text{bd} K) = U$.

(ii) $U \uplus V = V \uplus U$ (commutative).

(iii) $(U \uplus V) \uplus W = U \uplus (V \uplus W)$ (associative).

(iv) If $U \preceq_K V$, then $U \uplus W \preceq_K V \uplus W$ (compatible of the sum $\uplus$ with $\preceq_K$).

(v) $y \in (U \uplus V) + K$ if and only if there exists $W \in \mathcal{P}_p(Y)$ such that $U \preceq_K W$ and $y \in W \uplus V$.

The next definition originates from [16, Definition 3.2].

Definition 2.5 (i) Let $\mathcal{F} : \mathcal{L}(X, Y) \to \mathcal{P}_p(Y)^\ast$. The $K$-extended epigraph of $\mathcal{F}$ is defined as

$$\mathcal{Epi} \mathcal{F} := \{(L, U) \in \mathcal{L}(X, Y) \times \mathcal{P}_p(Y): \mathcal{F}(L) \subset Y, \mathcal{F}(L) \preceq_K U\}.$$

(ii) For $\mathcal{M}, \mathcal{N} \subset \mathcal{L}(X, Y) \times \mathcal{P}_p(Y)$, the $\Box$-sum of these two sets is defined as follows:

$$\mathcal{M} \Box \mathcal{N} := \{(L_1 + L_2, U_1 \uplus U_2) : (L_1, U_1) \in \mathcal{M}, (L_2, U_2) \in \mathcal{N}\}.$$
It is clear that if \( F : X \to Y^\star \), then the \( K \)-extended epigraph of of the conjugate mapping \( F^\star \) is

\[
\mathcal{E}piF^\star := \{(L, U) \in \mathcal{L}(X, Y) \times \mathcal{P}_p(Y) : L \in \text{dom } F^\star, \ F^\star(L) \preceq_K U\}
\]

and one can understand simply that the \( K \)-extended epigraph \( \mathcal{E}piF^\star \) is the “epigraph” of \( F^\star \) which is considered as a single valued-mapping \( F^\star : \mathcal{L}(X, Y) \to (\mathcal{P}_p(Y) \cup \{+\infty\}, \{-\infty\}) \preceq_K \) and its “epigraph” defined in the same way as the one of a real-valued function. Moreover, if \( F, G : X \to Y^\star \), then the \( \square \)-sum of \( \mathcal{E}piF^\star \) and \( \mathcal{E}piG^\star \) is

\[
\mathcal{E}piF^\star \boxplus \mathcal{E}piG^\star = \{(L_1 + L_2, U_1 \psi U_2) : (L_1, U_1) \in \mathcal{E}piF^\star, \ (L_2, U_2) \in \mathcal{E}piG^\star\}.
\]

It is also worth observing that from the definition of \( \square \)-sum and Proposition 2.4, the \( \square \)-sum is commutative and associative on \( \mathcal{L}(X, Y) \times \mathcal{P}_p(Y) \).

We now consider a set-valued mapping \( \Psi : \mathcal{L}(X, Y) \times \mathcal{P}_p(Y) \rightrightarrows \mathcal{L}(X, Y) \times Y \)

\[
(L, U) \mapsto \Psi(L, U) := \{L\} \times U.
\]

The relation between \( \mathcal{E}pi F^\star \subset \mathcal{L}(X, Y) \times \mathcal{P}_p(Y) \) and \( epi F^\star \subset \mathcal{L}(X, Y) \times Y \), some simple properties of the \( \square \)-sum and the mapping \( \Psi \) will be given in the next proposition.

**Proposition 2.5** Let \( F : X \to Y^\star \) be a proper mapping, and \( \mathcal{M}, \mathcal{N}, \mathcal{Q}, \mathcal{M}_i \subset \mathcal{L}(X, Y) \times \mathcal{P}_p(Y) \), for all \( i \in I \) (I is an arbitrary index set). Then,

(i) \( epi F^\star = \Psi(\mathcal{E}pi F^\star) \).

(ii) \( \mathcal{M} \subset \mathcal{N} \implies \Psi(\mathcal{M}) \subset \Psi(\mathcal{N}) \).

(iii) \( \mathcal{M} \subset \mathcal{N} \implies (\mathcal{M} \boxplus \mathcal{Q}) \subset (\mathcal{N} \boxplus \mathcal{Q}) \).

(iv) \( \bigcup_{i \in I} (\mathcal{M}_i \boxplus \mathcal{N}) = \left(\bigcup_{i \in I} \mathcal{M}_i\right) \boxplus \mathcal{N} \).

(v) \( \Psi\left(\bigcup_{i \in I} \mathcal{M}_i\right) = \bigcup_{i \in I} \Psi(\mathcal{M}_i) \).

**Proof** (i) is [16, Proposition 3.1] while the others are easy from the definitions of \( \Psi \) and \( \square \)-sum. \( \square \)

**Remark 2.2** Noting that when \( Y = \mathbb{R} \) and \( K = \mathbb{R}_+ \), then \( \mathcal{P}_p(Y) = \mathbb{R} \) while the pre-order “\( \preceq_K \)” (see (4)) and “\( \psi \)-sum” (Definition 2.4) become the normal order “\( \preceq \)” and the normal sum “\( + \)” on the set of extended real numbers, respectively. Hence, \( \mathcal{L}(X, Y) \times \mathcal{P}_p(Y) = X^* \times \mathbb{R} \) and the \( \square \)-sum collapses to the usual Minkowski sum of two subsets in \( X^* \times \mathbb{R} \). Both \( K \)-epigraph and \( K \)-extended epigraph of conjugate mappings collapse to their usual epigraphs in the sense of convex analysis. The mapping \( \Psi \) (defined by (5)) then reduces to the identical mapping of \( X^* \times \mathbb{R} \). In other words, if \( \mathcal{M}, \mathcal{N} \subset X^* \times \mathbb{R} \), then one has \( \Psi(\mathcal{M} \boxplus \mathcal{N}) = \Psi(\mathcal{M} + \mathcal{N}) = \mathcal{M} + \mathcal{N} \).
2.3 A Generalized Open Mapping Theorem

Let $\tilde{X}$, $\tilde{Y}$ be lcHtvs and let $\tilde{G}: \tilde{X} \rightrightarrows \tilde{Y}$ be a multifunction. The domain and the graph of $\tilde{G}$ are, respectively, $\text{dom} \tilde{G} := \{ x \in \tilde{X} : \tilde{G}(x) \neq \emptyset \}$ and $\text{gr} \tilde{G} := \{ (x, y) \in \tilde{X} \times \tilde{Y} : x \in \text{dom} \tilde{G} \text{ and } y \in \tilde{G}(x) \}$. The multifunction $\tilde{G}$ is convex (closed, respectively) if $\text{gr} \tilde{G}$ is a convex (closed, respectively) subset of $\tilde{X} \times \tilde{Y}$ [33, p.13].

Lemma 2.1 [33, Lemma 1.3.1] Let $\tilde{G}: \tilde{X} \rightrightarrows \tilde{Y}$ be a closed and convex multifunction and $\tilde{x}_0 \in \tilde{X}$. Assume that $\tilde{X}$ is a first countable and complete space. Then, $\bigcap_{U \in \mathcal{N}(\tilde{x}_0)} \text{int} (\text{cl} \tilde{G}(U)) \subset \bigcap_{U \in \mathcal{N}(\tilde{x}_0)} \text{int} \tilde{G}(U)$, where $\mathcal{N}(\tilde{x}_0)$ is the collection of all neighborhoods of $\tilde{x}_0$ in $\tilde{X}$.

We are now turning back to the lcHtvs $X$, $Y$, $Z$ and the non-empty convex cones $K$, $S$ as in the beginning of this section. Denote by $\pi$ the canonical projection from $X \times Z$ to $Z$, i.e., $\pi(x, z) := z$ for all $(x, z) \in X \times Z$ and set $Z_0 := \text{lin}(\pi(\text{dom } \Phi))$.

For further investigation, we need a generalized open mapping theorem, whose proof is rather technical (based on Lemma 2.1) and is left to the “Appendix”.

Theorem 2.1 (Generalized open mapping theorem) Let $\Phi: X \times Z \to Y^*$ be a proper $K$-convex and $K$-epi closed mapping, $x_0 \in X$ with $(x_0, 0_Z) \in \text{dom } \Phi$. Let further, $U_0$ and $V_0$ be neighborhoods of $x_0$ and $\Phi(x_0, 0_Z)$ in $X$ and $Y$, respectively. Assume that $X$, $Y$ are the first countable and complete spaces, $Z_0$ is a barreled space\(^1\), and that $0_Z \in \text{icr}(\pi(\text{dom } \Phi))$. Then, $0_Z \in \text{int} Z_0 \{ z \in Z : \Phi(x, z) \leq_K y \text{ for some } x \in U_0 \text{ and } y \in V_0 \}$.

Proof (see “Appendix A”) $\square$

3 Conjugate of Perturbation Mapping and Generalized Vector Inequalities

Let $X$, $Y$, $Z$ be lcHtvs and $K$, $S$ be non-empty convex cones in $Y$ and $Z$, respectively, with $\text{int } K \neq \emptyset$, as in Sect. 2. Consider a proper perturbation mapping $\Phi: X \times Z \to Y^*$ associated with the problem $(PL)$ with its conjugate mapping $\Phi^*: L(X, Y) \times L(Z, Y) \rightrightarrows Y^*$. The conjugate mapping of $\Phi(\cdot, 0_Z)^*: L(X, Y) \rightrightarrows Y^*$ will be denoted by $\Phi(\cdot, 0_Z)^*$. Throughout this section, we assume that $0_Z \in \pi(\text{dom } \Phi)$ and hence, $Z_0 = \text{aff}(\pi(\text{dom } \Phi))$.

\(^1\) A lcHtvs $\tilde{X}$ is barreled if every absorbing, convex and closed subset of $\tilde{X}$ is a neighborhood of the origin of $\tilde{X}$ [33, p.9].
3.1 Epigraphs of Conjugate of Perturbation Mappings

In this subsection, we will prove variant representations of epi $\Phi(\cdot, 0_Z)^*$ in terms of epi $\Phi^*(\cdot, T)$ with $T \in \text{proj}_{L(Z,Y)} \text{dom} \Phi^*$, where $\text{proj}_{L(Z,Y)} \text{dom} \Phi^*$ is the projection of dom $\Phi^*$ on the space $L(Z, Y)$. Such representations of epi $\Phi(\cdot, 0_Z)^*$ will play the key role in establishing vector Farkas lemmas below and also in constructing the Lagrange and the Fenchel–Lagrange dual problems in the next section.

For $z_1, z_2 \in Z, z_1 \leq_S z_2$ means that $z_1 - z_2 \in -S$. We also enlarge $Z$ by attaching a greatest element $+\infty_Z$ and a smallest element $-\infty_Z$, which do not belong to $Z$ and define $Z^* := Z \cup \{-\infty_Z, +\infty_Z\}$. In $Z^*$, we adopt the same conventions as in (1). Let us set

$$\mathcal{L}_\Phi := \{ T \in L(Z, Y) : \exists L \in L(X, Y) \text{ s.t. } (L, T) \in \text{dom} \Phi^* \} = \text{proj}_{L(Z,Y)} \text{dom} \Phi^*, \quad (7)$$

$$\mathcal{L}^+_\Phi := \mathcal{L}_\Phi \cap L_+(S, K), \quad (8)$$

$$\mathcal{M} := \bigcup_{T \in \mathcal{L}_\Phi} \text{epi} \Phi^*(\cdot, T), \quad \mathcal{M}_+ := \bigcup_{T \in \mathcal{L}^+_\Phi} \text{epi} \Phi^*(\cdot, T). \quad (9)$$

**Proposition 3.1** It holds that $\text{epi} \Phi(\cdot, T)^* \supset \mathcal{M} \supset \mathcal{M}_+$.

**Proof** The second inclusion $\mathcal{M} \supset \mathcal{M}_+$ is trivial. So, only the first one needs to prove. Take $(L, y) \in \bigcup_{T \in \mathcal{L}_\Phi} \text{epi} \Phi^*(\cdot, T)$. Then, there is $T \in \mathcal{L}_\Phi$ such that $(L, y) \in \text{epi} \Phi^*(\cdot, T)$, or equivalently, $(L, T, y) \in \text{epi} \Phi^*$, and hence, by (2), $\Phi(x, z) - L(x) - T(z) + y \notin \text{int} K$ for all $(x, z) \in X \times Z$. In particular, taking $z = 0_Z$, one has $\Phi(x, 0_Z) - L(x) + y \notin \text{int} K$ for all $x \in X$, which, again by (2), $(L, y) \in \text{epi} \Phi^*(\cdot, T)$ and the proof is complete. \qed

Consider the assumption

\begin{itemize}
  \item [(H1)] The mapping $\Phi$ is $K$-convex
  \item [(C0)] $\exists \tilde{x} \in X : (\tilde{x}, 0_Z) \in \text{dom} \Phi$ and $\Phi(\tilde{x}, z) \leq_K \Phi(\tilde{x}, 0_Z), \forall z \in -S.$
  \item [(C1)] $\forall L \in L(X, Y), \exists y_L \in Y, \exists V_L \in \mathcal{N}(0_Z)$ such that $\forall z \in V_L \cap 0_Z, \exists x \in X : \Phi(x, z) - L(x) \leq_K y_L.$
  \item [(C2)] $\exists (\tilde{x}, \tilde{y}, \tilde{V}) \in X \times Y, \exists V \in \mathcal{N}(0_Z)$ such that $\Phi(\tilde{x}, z) \leq_K \tilde{y}, \forall z \in \tilde{V} \cap 0_Z.$
  \item [(C3)] $\exists \tilde{x} \in X$ such that $\Phi|_{Z_0}(\tilde{x}, \cdot)$ is continuous at $0_Z$.
  \item [(C4)] $\text{dim } Z_0 < \infty$ and $0_Z \in \text{ri}(\pi(\text{dom} \Phi)).$
  \item [(C5)] $X, Y$ are complete and first countable spaces, $Z_0$ is a barreled space, $\Phi$ is $K$-epi closed and $0_Z \in \text{icr}(\pi(\text{dom} \Phi)).$
  \item [(C6)] $\Phi(x, 0_Z) \leq_K \Phi(x, z), \forall (x, z) \in \text{dom} \Phi$ and $\text{int}(\pi(\text{dom} \Phi)) \neq \emptyset.$
  \item [(C7)] $0_Z \in \pi(\text{dom} \Phi) - \text{int} S$, $\Phi(x, 0_Z) \leq_K \Phi(x, z)$ whenever $(x, z) \in \text{dom} \Phi$ and $z \geq_S 0_Z.$
\end{itemize}
Remark 3.1 It is worth noticing that the conditions $(C_1)$-$(C_5)$ extend the ones proposed in [33, Theorem 2.7.1] in the case where $Y = \mathbb{R}$. Concretely, when $Y = \mathbb{R}$ and $\mathcal{L}(X, Y)$ is replaced by $\{x^*\}$, the condition $(C_1)$ collapses to [33, Theorem 2.7.1(i)], the conditions $(C_3)$ and $(C_4)$ are nothing else but [33, Theorem 2.7.1(iii)] and [33, Theorem 2.7.1(viii)], respectively, while $(C_2)$ and $(C_5)$ generalize [33, Theorem 2.7.1(ii)] and [33, Theorem 2.7.1(vi)] to vector problems, respectively.

Theorem 3.1 (The first theorem on representation of epi $\Phi(\cdot, T^*)$) Assume that $(H_1)$ holds. If $(C_1)$ holds, then

$$
epi \Phi(\cdot, T)^* = \mathcal{M}. 
$$

If, in addition that, $(C_0)$ holds, then

$$
epi \Phi(\cdot, T)^* = \mathcal{M} = \mathcal{M}_+. 
$$

Proof Taking Proposition 3.1 into account, to prove (10), it suffices to show that

$$
epi \Phi(\cdot, T)^* \subset \mathcal{M}. 
$$

For this, take $(\tilde{L}, \tilde{y}) \in \nepi \Phi(\cdot, T)^*$. Then, by (2),

$$\tilde{y} \not\in \tilde{L}(x) - \Phi(x, 0_Z) - \text{int } K, \quad \forall x \in X. 
$$

We will show that there exists $\tilde{T} \in \mathcal{L}_\Phi$ such that $(\tilde{L}, \tilde{y}) \in \nepi \Phi^*(., \tilde{T})$. Set

$$\Delta_{\tilde{L}} := \bigcup_{(x, z) \in \text{dom } \Phi} \left( (\tilde{L}(x) - \Phi(x, z) - K) \times \{z\} \right).$$

It is clear that $\Delta_{\tilde{L}} \subset Y \times Z_0$ and it is easy to check that, as $\Phi$ is $K$-convex, $\Delta_{\tilde{L}}$ is a convex subset of $Y \times Z_0$. The proof of (12) is arranged in three steps:

Step 1. We firstly prove that $\text{int}_{Y \times Z_0} \Delta_{\tilde{L}} \neq \emptyset$ and that $(\tilde{y}, 0_Z) \not\in \text{int}_{Y \times Z_0} \Delta_{\tilde{L}}$.

Proof of $\text{int}_{Y \times Z_0} \Delta_{\tilde{L}} \neq \emptyset$. By $(C_1)$, there are $y_{\tilde{L}} \in Y$ and a neighborhood $V_{\tilde{L}}$ of $0_Z$ such that

$$\forall z \in V_{\tilde{L}} \cap Z_0, \exists x \in X : \Phi(x, z) - \tilde{L}(x) \preceq_K y_{\tilde{L}}$$

$$\implies \forall z \in V_{\tilde{L}} \cap Z_0, \exists x \in X : -y_{\tilde{L}} \in \tilde{L}(x) - \Phi(x, z) - K$$

$$\implies \forall (z, k') \in (V_{\tilde{L}} \cap Z_0) \times \text{int } K, \exists x \in X : -y_{\tilde{L}} - k' \in \tilde{L}(x) - \Phi(x, z) - K$$

$$\implies \forall (z, k') \in (V_{\tilde{L}} \cap Z_0) \times \text{int } K, (-y_{\tilde{L}} - k', z) \in \Delta_{\tilde{L}}$$

$$\implies (-y_{\tilde{L}} - \text{int } K) \times (V_{\tilde{L}} \cap Z_0) \subset \Delta_{\tilde{L}}$$

$$\implies (-y_{\tilde{L}} - \text{int } K) \times (V_{\tilde{L}} \cap Z_0) \subset \text{int}_{Y \times Z_0} \Delta_{\tilde{L}}. 
$$

So $\text{int}_{Y \times Z_0} \Delta_{\tilde{L}} \neq \emptyset$ (as $(V_{\tilde{L}} \cap Z_0) \ni 0_Z$ and $\text{int } K \neq \emptyset$).
Proof of \((y, 0_Z) \not\in \text{int}_{Y \times Z_0} \Delta_L\). Assume on the contrary that \((y, 0_Z) \in \text{int}_{Y \times Z_0} \Delta_L\). Then, there exists a neighborhood \(U \times V\) of \((0_Y, 0_Z)\) with \([(\tilde{y} + U) \times V] \cap (Y \times Z_0) \subset \Delta_L\). Take \(k \in U \cap \text{int} K\) then \((\tilde{y} + k, 0_Z) \in \Delta_L\). This yields the existence of \((\tilde{x}, 0_Z) \in \text{dom} \Phi\) such that \(\tilde{y} + k \in \tilde{L}(\tilde{x}) - \Phi(\tilde{x}, 0_Z) - K\), leading to \(\tilde{y} \in \tilde{L}(\tilde{x}) - \Phi(\tilde{x}, 0_Z) - \text{int} K\), which contradicts (13). Thus, \((y, 0_Z) \not\in \text{int}_{Y \times Z_0} \Delta_L\).

Step 2. As \((\tilde{y}, 0_Z) \not\in \text{int}_{Y \times Z_0} \Delta_L\), apply the convex separation theorem ([30, Theorem 3.4]) to the point \((\tilde{y}, 0_Z)\) and the convex set \(\Delta_L\) in \(Y \times Z_0\), there exists \((y_0^*, z_0^*) \in Y^* \times Z_0^*\) such that

\[
y_0^*(\tilde{y}) < y_0^*(y) + z_0^*(z), \quad \forall (y, z) \in \text{int}_{Y \times Z_0} \Delta_L,
\]

and consequently (as \(\Delta_L \subset Y \times Z_0\)),

\[
y_0^*(\tilde{y}) \leq y_0^*(y) + z_0^*(z), \quad \forall (y, z) \in \Delta_L.
\]

Next, we show that

\[
y_0^*(k') < 0, \quad \forall k' \in \text{int} K.
\]

Indeed, take \(k' \in \text{int} K\). With \(-y_L \in Y\) (exists by (C1), used in Step 1) and \(k', \tilde{y} \in Y\), by [11, Lemma 2.1 (i)], there is \(\mu > 0\) such that \(\tilde{y} - \mu k' \in -y_L - \text{int} K\). Hence, \((\tilde{y} - \mu k', 0_Z) \in (\tilde{y} - y_L - \text{int} K) \times (V_L \cap Z_0)\) which, by (14), ensures that \((\tilde{y} - \mu k', 0_Z) \in \text{int}_{Y \times Z_0} \Delta_L\). In turn, (15) leads to \(y_0^*(\tilde{y}) < y_0^*(\tilde{y} - \mu k') + z_0^*(0_Z)\), or \(y_0^*(k') < 0\), and (17) holds.

Step 3. We now build an operator \(\tilde{T} \in L_\Phi\) such that \((\tilde{L}, \tilde{y}) \in \text{epi} \Phi^*(., \tilde{T})\).

Take \(k_0 \in \text{int} K\) such that \(y_0^*(k_0) = -1\) (it is possible by (17)) and \(T: Z_0 \to Y\) defined by \(T(z) = -z_0^*(z)k_0\) for all \(z \in Z_0\) \((z_0^*\text{ exists by the separation theorem in Step 2})\). It is easy to see that \(T \in L(Z_0, Y)\) and for all \(z \in Z_0\), it holds \((y_0^* \circ T)(z) = y_0^*(-z_0^*(z)k_0) = -y_0^*(k_0)z_0^*(z) = z_0^*(z)\). Thus, (16) can be rewritten as \(y_0^*(\tilde{y}) \leq y_0^*(y) + (y_0^* \circ T)(z)\) for all \((y, z) \in \Delta_L\), or equivalently, \(y_0^*(y + T(z) - \tilde{y}) \geq 0\) for all \((y, z) \in \Delta_L\). So, by (17), \(y + T(z) - \tilde{y} \not\in \text{int} K\), yielding

\[
\tilde{y} \notin y + T(z) - \text{int} K, \quad \forall (y, z) \in \Delta_L.
\]

Now, as \((\tilde{L}(x) - \Phi(x, z), z) \in \Delta_L\) for all \((x, z) \in \text{dom} \Phi\), it follows from (18) that

\[
\tilde{y} \notin \tilde{L}(x) - \Phi(x, z) + T(z) - \text{int} K, \quad \forall (x, z) \in \text{dom} \Phi.
\]

On the other hand, using the Hahn–Banach theorem [30, Theorem 3.6], we can extend \(z_0^* \in Z_0^*\) to \(\bar{z}^* \in Z^*\), and hence, \(T\) can be extended to \(\bar{T} = -\bar{z}^*(z)k_0\in L(Z, Y)\). Consequently, (19) holds (for all \((x, z) \in \text{dom} \Phi\)) with \(T\) being replaced by \(\bar{T}\). By (2), \((\tilde{L}, \tilde{T}, \tilde{y}) \in \text{epi} \Phi^*\), showing that \((\tilde{L}, \tilde{T}) \in \text{dom} \Phi^*\) (or, equivalently, \(\bar{T} \in L_\Phi\)) and \((\tilde{L}, \tilde{y}) \in \text{epi} \Phi^*(., \tilde{T})\). Thus, (12) holds and (10) does, too.

We now prove that if, in addition that (C0) holds then (11) holds. For this, we will show that under this extra assumption the operator \(\tilde{T}\) appeared in Step 3 for which (19) holds, must satisfy \(\tilde{T} \in L_+(S, K)\), or equivalently, \(\tilde{T}(S) \subset K\). We claim
firstly that $z_0^* \in -S^+$. For any $s \in S$, $\nu > 0$, one has $-\nu s \in -S$ and hence, by \((C_0)\), $\Phi(\bar{x} , -\nu s) \leq \bar{K} \Phi(\bar{x}, 0Z)$, which implies that $(\bar{x}, -\nu s) \in \text{dom} \Phi$ and that $\bar{L}(\bar{x}) - \Phi(\bar{x}, 0Z) \in \bar{L}(\bar{x}) - \Phi(\bar{x}, -\nu s) - K$. Then, $(\bar{L}(\bar{x}) - \Phi(\bar{x}, 0Z), -\nu s) \in \Delta \bar{L}$. Again, by \((16)\), one has, for any $\nu > 0$, $y_0^*(\bar{y}) \leq y_0^*(\bar{L}(\bar{x}) - \Phi(\bar{x}, 0Z)) + z_0^*(-\nu s)$, or, equivalently,

$$\frac{1}{\nu} y_0^*(\bar{y}) \leq \frac{1}{\nu} [y_0^*(\bar{L}(\bar{x}) - \Phi(\bar{x}, 0Z))] - z_0^*(s), \ \forall \nu > 0.$$  

Letting $\nu$ tends to $+\infty$, one obtains $z_0^*(s) \leq 0$. So, $z_0^* \in -S^+$. We then have, for all $s \in S$, $\bar{T}(s) = T(s) = -z_0^*(s) k_0 \in K$, showing that $\bar{T}(s) \subset K$. Consequently, $\bar{T} \in \mathcal{L}_K^+$, as desired. \hfill \Box

In the next two theorems, Theorems 3.2 and 3.3, we will show that the representations of $\text{epi}_K \Phi(\cdot, T)^*$ in \((11)\) can be derived under different sets of the regularity conditions \((C_2) - (C_7)\). The proofs of these theorems are in the same vein as that of Theorem 3.1 and are left to the “Appendix”.

**Theorem 3.2** (The second theorem on representation of $\text{epi} \Phi(\cdot, T)^*$) Assume that \((H_1)\) holds and that at least one of the conditions \((C_2), (C_3), (C_4), (C_5), \text{or} (C_6)\) holds. Then, \((10)\) holds. Moreover, if, in addition that \((C_0)\) holds, then \((11)\) holds.

**Proof** (See “Appendix B”). \hfill \Box

**Theorem 3.3** (The third theorem on representation of $\text{epi} \Phi(\cdot, T)^*$) Assume that \((H_1), (C_7)\) hold and $\text{int} S \neq \emptyset$. Then, \((11)\) holds.

**Proof** (See “Appendix C”). \hfill \Box

### 3.2 General Vector Inequalities: Vector Farkas Lemmas

We now use the representations of the epigraph $\text{epi} \Phi(\cdot, 0Z)^*$ established in the previous subsection to derive characterizations (i.e., conditions that is equivalent to) of the general vector inequality

$$\Phi(x, 0Z) - L(x) + y \notin -\text{int} \ K, \ \forall x \in X,$$  

\((20)\)

for some $L \in \mathcal{L}(X, Y)$ and $y \in Y$. Each pair of such equivalent conditions is called an extended Farkas lemma for vector mappings, for instance, the pairs \((\alpha) - (\beta)\), \((\alpha) - (\gamma)\) in Theorem 3.4 below are two versions of Farkas lemma. The same observation applies to Corollaries 3.1, 3.2. In some special cases with concrete forms of the mapping $\Phi$ (see Sections 5, 6), these results go back to the vector Farkas lemmas in \([12, 13, 15]\) and even when $Y = \mathbb{R}$, different systems, different functions $\Phi$ will lead to variants of results that cover/extend the known extended Farkas lemmas in the literature (see, e.g., \([2, 18, 20]\), and references therein). We say that a version of Farkas lemma is \(\mathcal{V}\)-stable (or we have a \(\mathcal{V}\)-stable Farkas lemma) if the pair is equivalent for all $L \in \mathcal{V} \subset \mathcal{L}(X, Y)$ and $y \in Y$. It is called stable if it is \(\mathcal{V}\)-stable with $\mathcal{V} = \mathcal{L}(X, Y)$. We are now in a
position to prove characterizations (i.e., necessary and sufficient conditions) of the \( \mathcal{V} \)-stability of Farkas lemmas associated with the vector inequalities (20). However, for the sake of simplicity, we state and prove here only results on stable Farkas lemmas (i.e., with \( \mathcal{V} = \mathcal{L}(X, Y) \)).

**Theorem 3.4** (Characterization of stable vector Farkas lemmas) Consider the statements:

(a) \( \text{epi } \Phi(\cdot, 0_{Z})^{*} = \mathcal{M} \).

(b) \( \text{epi } \Phi(\cdot, 0_{Z})^{*} = \mathcal{M}_{+} \).

(c) For all \((L, y) \in \mathcal{L}(X, Y) \times Y\), two following assertions are equivalent

\[
(\alpha) \ \Phi(x, 0_{Z}) - L(x) + y \notin \text{int } K, \ \forall x \in X,
\]

\[
(\beta) \ \exists T \in \mathcal{L}_{\Phi} : \Phi(x, z) - L(x) - T(z) + y \notin \text{int } K, \ \forall (x, z) \in X \times Z.
\]

(d) For all \((L, y) \in \mathcal{L}(X, Y) \times Y\), two following assertions are equivalent

\[
(\alpha) \ \Phi(x, 0_{Z}) - L(x) + y \notin \text{int } K, \ \forall x \in X,
\]

\[
(\gamma) \ \exists T \in \mathcal{L}_{\Phi}^{+} : \Phi(x, z) - L(x) - T(z) + y \notin \text{int } K, \ \forall (x, z) \in X \times Z.
\]

Then, \([(\alpha) \iff (c)] \text{ and } [(b) \iff (d)].\)

**Proof** Take \((L, y) \in \mathcal{L}(X, Y) \times Y\). Then, by (2), one has

\[
(\alpha) \iff (L, y) \in \text{epi } \Phi(\cdot, T)^{*},
\]

\[
(\beta) \iff \exists T \in \mathcal{L}_{\Phi} \text{ s.t. } (L, T, y) \in \text{epi } \Phi^{*},
\]

\[
\iff \exists T \in \mathcal{L}_{\Phi} \text{ s.t. } (L, y) \in \text{epi } \Phi^{*}(\cdot, T).
\]

The first equivalence \([(\alpha) \iff (c)] \text{ thus follows. The proof of } [(b) \iff (d)] \text{ is similar.} \quad \square

In the case when the perturbation mapping is \( K \)-convex, the regularity conditions \((C_i), i = 0, 1, 2, \ldots, 7\), ensure \( \mathcal{V} \)-stability/stability of vector Farkas lemmas described in (c) and (d) (in Theorem 3.4) as claimed in the next corollaries, whose proofs are based on Theorems 3.1–3.4 and will be omitted.

**Corollary 3.1** (Stable vector Farkas lemma I) Assume that \((H_1)\) and \((C_1)\) hold. Then, (c) in Theorem 3.4 holds. If, in addition, \((C_0)\) fulfills, then (d) in Theorem 3.4 holds.

**Corollary 3.2** (Stable vector Farkas lemma II) Assume that \((H_1)\) holds. The following assertions hold:

(i) If \((C_7)\) holds, then both (c) and (d) of Theorems 3.4 hold.

(ii) If at least one of the conditions \((C_2), (C_3), \ldots, (C_6)\) holds, then (c) of Theorem 3.4 holds. If, in addition that \((C_0)\) holds, then (d) holds as well.

### 4 Perturbation Approach to Duality for Vector Optimization Problems

Let \(X, Y, Z\) and \(K, S\) (with \(\text{int } K \neq \emptyset\)) be the spaces and cones as in Section 3, \(\pi\) be the canonical projection from \(X \times Z\) to \(Z\), and \(\Phi: X \times Z \to Y^{*}\) be a proper...
perturbation mapping. Consider the general vector optimization problem associated with $\Phi$

$$(P) \quad \text{WInf}_x \Phi(x, 0_Z)$$

and assume that $0_Z \in \pi(\text{dom} \Phi)$. This means that the problem (P) is feasible. For (P), we consider weak solutions in the following sense: $\bar{x} \in X$ is a weak solution of (P) if

$$\Phi(\bar{x}, 0_Z) \in \text{WMin} \left\{ \Phi(x, 0_Z) : x \in X \right\}.$$ 

We now denote by $(P_L)$ the problem $(P)$ perturbed by a linear operator $L \in \mathcal{L}(X, Y)$

$$(P_L) \quad \text{WInf}_x \Phi(x, 0_Z - L(x)).$$

The dual problem and loose dual problem of $(P_L)$ are defined as

$$(DL) \quad \text{WSup}_{T \in \mathcal{L}_\Phi} [-\Phi^*(L, T)],$$

$$(DL^\ell) \quad \text{WSup}_{T \in \mathcal{L}^\ell_\Phi} [-\Phi^*(L, T)],$$

where $\mathcal{L}_\Phi$ and $\mathcal{L}^\ell_\Phi$ are the sets defined in (7) and (8), respectively. When $L = 0_{\mathcal{L}_\Phi}$ collapses to (P) and for the dual and loose dual problems, we will write (D) and (D$^\ell$) instead of (D$^0_{\mathcal{L}_\Phi}$) and (D$^0_{\mathcal{L}^\ell_\Phi}$), respectively. It is worth mentioning that such a “loose dual problem” was introduced in [11, Problem (CVD)] for a concrete composite vector problem.

For the sake of simplicity, let us denote the sets of the values of the problems $(P)$, $(P_L)$, $(D_L)$ and $(D_L^\ell)$ by $\text{WInf}(P)$, $\text{WInf}(P_L)$, $\text{WSup}(D_L)$ and $\text{WSup}(D_L^\ell)$, respectively. That means

$$\text{WInf}(P) = \text{WInf}_x \Phi(x, 0_Z); \quad \text{WInf}(P_L) = \text{WInf}_x \Phi(x, 0_Z - L(x));$$

$$\text{WSup}(D_L) = \text{WSup}_{T \in \mathcal{L}_\Phi} [-\Phi^*(L, T)]; \quad \text{WSup}(D_L^\ell) = \text{WSup}_{T \in \mathcal{L}^\ell_\Phi} [-\Phi^*(L, T)].$$

An operator $\hat{T} \in \mathcal{L}_\Phi$ is said to be a solution of the dual problem $(D_L)$ if $[-\Phi^*(L, \hat{T})] \cap \text{WSup}(D_L) \neq \emptyset$. A solution of the loose dual problem $(D_L^\ell)$ is defined similarly.

Now, let $\emptyset \neq \mathcal{V} \subset \mathcal{L}(X, Y)$.

**Definition 4.1** (i) We say that strong duality holds for the pair $(P) - (D)$ if $\text{WInf}(P) = \text{WMax}(D)$.

(ii) We say that $\mathcal{V}$-stable strong duality holds for the pair $(P) - (D)$ (resp. $(P) - (D^\ell)$) if the strong duality holds for the pair $(P_L) - (D_L)$ (resp. $(P_L) - (D_L^\ell)$) for each $L \in \mathcal{V}$. 

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When \( \mathcal{V} = \mathcal{L}(X, Y) \), we simply said that stable strong duality holds for the pair (P) - (D). When \( \mathcal{V} = \{0\} \), \( \{0\} \)-stable strong duality is none other than the usual strong duality. As in Section 3, we consider in detail here only stable strong duality (i.e., when \( \mathcal{V} = \mathcal{L}(X, Y) \)) and strong duality (when \( \mathcal{V} = \{0\} \)).

It can be mentioned here that when \( \text{WInf}(P^L) = \text{WMax}(D^L) \), one has \( \text{WSup}(D^L) = \text{WMax}(D^L) \) (see Proposition 2.3 (i)), which means that for any \( \tilde{y} \in \text{WInf}(P^L) \), one has \( \tilde{y} \in \text{WSup}(D^L) \) and there exists \( \tilde{T} \in \mathcal{L}(Z, Y) \) such that \( \tilde{y} \in [\Phi^*(L, \tilde{T})] \cap \text{WSup}(D^L) = [\Phi^*(L, \tilde{T})] \cap \text{WMax}(D^L) \).

**Example 4.1** Consider the vector optimization problem

\[
(P^1) \quad \text{WInf}\{(x, x^2 + 2x) : x \leq 0\}.
\]

Let \( X = Z = \mathbb{R}, Y = \mathbb{R}^2, S = \mathbb{R}_+ \) and \( K = \mathbb{R}^2_+ \), and set the perturbation mapping \( \Phi: \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2_+ \) defined by \( \Phi(x, z) = (x, x^2 + 2x + z) \) if \( 2x + z \leq 0 \) and \( \Phi(x, z) = +\infty \mathbb{R}^2, \) otherwise.

It is clear that \( \text{WInf}(P^1) = \text{WInf} \Phi(x, 0) \), and as \( \mathcal{L}(X, Y) \cong \mathbb{R}^2, \mathcal{L}(Z, Y) \cong \mathbb{R}^2, \mathcal{L}_+(S, K) \cong \mathbb{R}^2_+ \), we have \( \Phi^*: \mathbb{R}^2 \times \mathbb{R}^2 \Rightarrow \mathbb{R}^2 \) and with \( L := (a, b) \in \mathbb{R}^2, T := (c, d) \in \mathbb{R}^2 \), it is easy to see that

\[
\Phi^*(L, T) = \text{WSup}\{((a-2c-1)x, (b-2d)x - x^2): x \in \mathbb{R}\} + \{(cu, (d-1)u): u \leq 0\}.
\]

The last equality leads to (see Proposition 2.1 (iii))

\[
\Phi^*(L, T) = \text{WSup}\left[\{((a-2c-1)x, (b-2d)x - x^2): x \in \mathbb{R}\} + \text{WSup}\{(cu, (d-1)u): u \leq 0\}\right],
\]

and, the dual and the loose dual problems for (P1) read as

\[
(D^1) \quad \text{WSup}_{c \geq 0 \text{ or } d \geq 1} \text{WInf}\{((2c + 1)x, 2dx + x^2) - (cu, (d-1)u): x \in \mathbb{R}, u \leq 0\},
\]

\[
(D^1_+) \quad \text{WSup}_{c \geq 0 \text{ and } d \geq 0} \text{WInf}\{((2c + 1)x, 2dx + x^2) - (cu, (d-1)u): x \in \mathbb{R}, u \leq 0\}.
\]

Now, for each \( L \in \mathcal{L}(X, Y) \), let us set

\[
N_L := \{\Phi(x, 0) - L(x) : x \in X\},
\]

\[
M_L := \bigcup_{T \in \mathcal{L}_\phi} (-\Phi^*(L, T)) \quad \text{and} \quad M^+_L := \bigcup_{T \in \mathcal{L}^+_\phi} (-\Phi^*(L, T)).
\]

Then, it is clear that \( \text{WInf}(P^L) = \text{WInf} N_L \), \( \text{WSup}(D^L) = \text{WSup} M_L \) and \( \text{WSup}(D^L) = \text{WSup} M^+_L \).

**Theorem 4.1** (Weak duality) For any \( L \in \mathcal{L}(X, Y), T \in \mathcal{L}(Z, Y) \), it holds:

\[
\]
(i) \(-\Phi^*(L, T) \preceq_K W\text{Inf}(P_L)\).
(ii) \(\text{WSup}(D_L^T) \preceq_K \text{WSup}(D_L) \preceq_K W\text{Inf}(P_L)\).

**Proof** (i) As \(\text{dom} \Phi \neq \emptyset\), \(\Phi^*(L, T) \neq \{-\infty\}\). If \(\Phi^*(L, T) = \{+\infty\}\), then (i) holds automatically (recall that \(-\infty\) is the smallest element in \((P_p(Y)^*, \preceq_K)\)). Assume that \(\Phi^*(L, T) \subset Y\). Take \(\tilde{x} \in X\) and \(\tilde{y} = -\Phi^*(L, T)\).

As \(-\tilde{y} \in \Phi^*(L, T)\) then \(W\text{Sup}[L(x) + T(z) - \Phi(x, z) : (x, z) \in X \times Z]\), by Proposition 2.1(v) one gets

\[-\tilde{y} \not\in L(x) + T(z) - \Phi(x, z) - \text{int} \, K, \quad \forall (x, z) \in X \times Z,
\]

and hence, with \(x = \tilde{x}\) and \(z = 0_Z\), \(\tilde{y} \not\in \Phi(\tilde{x}, 0_Z) - L(\tilde{x}) + \text{int} \, K\), or equivalently, \(\Phi(\tilde{x}, 0_Z) - L(\tilde{x}) \not\preceq_K \tilde{y}\). So, \(-\Phi^*(L, T) \preceq_K N_L\). Consequently, by Propositions 2.3(v) and 2.2(iv), \(-\Phi^*(L, T) = W\text{Sup}[-\Phi^*(L, T)] \preceq_K W\text{Inf} N_L = W\text{Inf}(P_L)\).

(ii) As \(L_\Phi^+ \subset L_\Phi\), it follows from Proposition 2.2(iii) that \(W\text{Sup}(D_L^T) \preceq_K W\text{Sup}(D_L)\).

For the last inequality, it follows from (i) that \(M_L \preceq_K W\text{Inf} N_L\), and hence, using Propositions 2.3(v), 2.2(iv) again, one gets \(W\text{Sup}(D_L^T) = W\text{Sup} M_L \preceq_K W\text{Inf}[W\text{Inf}(P_L)] = W\text{Inf}(P_L)\).

\(\square\)

**Theorem 4.2** (Characterizations of stable strong duality for (P)) *Consider the following statements:
(a) \(\epsilon \pi \Phi(\cdot, T)^* = M\).
(b) \(\epsilon \pi \Phi(\cdot, T)^* = M_+\).
(c) Stable strong duality holds for the pair (P)-(D).
(d) Stable strong duality holds for the pair (P)-(D_\ell).

Then, [(a) \(\iff\) (c)] and [(b) \(\iff\) (d)].

**Proof** [(a) \(\Rightarrow\) (c)] Assume that (a) holds and let \(L \in L(X, Y)\). We will show that

\[W\text{Inf}(P_L) = W\text{Max}(D_L^T).\] \hspace{1cm} (22)

Firstly, as \(0_Z \in \pi(\text{dom} \Phi)\), there is \(\tilde{x} \in X\) such that \(\Phi(\tilde{x}, 0_Z) \in Y\), and consequently, \(\Phi(\tilde{x}, 0_Z) - L(\tilde{x}) \in Y\), yielding \(W\text{Inf}(P_L) \neq \{+\infty\}\). If \(W\text{Inf}(P_L) = \{-\infty\}\), then by Theorem 4.1, \(W\text{Sup}(D_L^T) = \{-\infty\}\), and so \(-\Phi^*(L, T) = \{-\infty\}\) for \(T \in L_\Phi\). Consequently, \(W\text{Max}(D_L^T) = \{-\infty\} = W\text{Inf}(P_L)\), and (22) holds. Assume now that \(W\text{Inf}(P_L) \subset Y\) and we will show that

\[W\text{Inf}(P_L) \subset W\text{Max}(D_L^T).\] \hspace{1cm} (23)

Let \(M_L\) and \(N_L\) be the sets in (21). Then, it follows from Theorem 4.1 (i) that

\[M_L \preceq_K W\text{Inf}(P_L) = W\text{Inf} N_L.\] \hspace{1cm} (24)

Take \(y \in W\text{Inf}(P_L) = W\text{Inf} N_L\). Then, by Proposition 2.1(v) (see also, Remark 2.1), \(y \not\in \Phi(x, 0_Z) - L(x) + \text{int} \, K\) for all \(x \in X\), or equivalently,

\[\Phi(x, 0_Z) - L(x) - y \not\in \text{int} \, K, \quad \forall x \in X.\] \hspace{1cm} (25)
Now, as (a) holds, it follows from Theorem 3.4 that (25) is equivalent to the fact that there is \( \bar{T} \in L(Z, Y) \) such that \( \bar{T} \in L_\Phi \) and \( \Phi(x, z) - L(x) - \bar{T}(z) - y \notin -\text{int} \ K \) for all \( (x, z) \in X \times Z \), which is equivalent to (by Proposition 2.1(iv))

\[
- y \notin \{ L(x) + \bar{T}(z) - \Phi(x, z) : (x, z) \in X \times Z \} - \text{int} \ K = \Phi^*(L, \bar{T}) - \text{int} \ K. \tag{26}
\]

As \( \Phi^*(L, \bar{T}) \in \mathcal{P}_p(Y)^4 \), one gets from Proposition 2.3 (iii) that \( Y = (\Phi^*(L, \bar{T}) - \text{int} \ K) \cup (\Phi^*(L, \bar{T}) + K) \), which together with (26) yields

\[
- y \in \Phi^*(L, \bar{T}) + K. \tag{27}
\]

On the other hand, as \( M_L \preceq_K W\inf N_L \) (see (24)) and \( y \in W\inf N_L \), one has \( y \notin K y' \) for all \( y' \in -\Phi^*(L, \bar{T}) \), yielding \( -y \notin \Phi^*(L, \bar{T}) + \text{int} \ K \). From (27) and Proposition 2.3(ii),

\[
- y \in (\Phi^*(L, \bar{T}) + K) \setminus (\Phi^*(L, \bar{T}) + \text{int} \ K) = \Phi^*(L, \bar{T}),
\]

and hence, \( y \in -\Phi^*(L, \bar{T}) \subset M_L \).

We have shown that if \( y \in W\inf (P^L) \), then \( y \in M_L \). Again, as \( M_L \preceq_K W\inf (P^L) \) one has \( y \notin K y' \) for any \( y' \in M_L \). So, by definition of weak maximum, \( y \in W\max M_L = W\max (D^L) \), and by the arbitrariness of \( y \in W\inf (P^L) \), (23) has been proved and then

\[
W\inf (P^L) \subset W\max (D^L) \subset W\sup (D^L). \tag{28}
\]

Now, as \( W\inf (P^L), W\sup (D^L) \in \mathcal{P}_p(Y)^4 \), Proposition 2.3(i) yields \( W\inf (P^L) = W\sup (D^L) \), and we get from (28) that \( W\inf (P^L) = W\max (D^L) \), which is (e).

[(e) \Rightarrow (a)] Assume that (e) holds, i.e., (22) holds for all \( L \in \mathcal{L}(X, Y) \). We will show that (a) holds. For this, taking Proposition 3.1 into account, it suffices to show that

\[
epi \Phi(\cdot, T)^* \subset \mathcal{M}. \tag{29}
\]

Take \((L, y) \in \text{epi } \Phi(\cdot, T)^* \). Then, by (2), \( \Phi(x, 0_Z) - L(x) + y \notin -\text{int} \ K \) for all \( x \in X \), or equivalently, \( -y \notin N_L + \text{int} \ K \) which yields \( W\inf N_L \neq \{-\infty\} \) (see Proposition 2.1(i) and Remark 2.1). So, \( W\inf N_L \subset Y \), and hence, one gets from Proposition 2.1(vi) (see also Remark 2.1) that \( Y = (N_L + \text{int} \ K) \cup W\inf N_L \cup (W\inf N_L - \text{int} \ K) \). As \( -y \notin N_L + \text{int} \ K \) and hence, by Proposition 2.3(ii),

\[
- y \in W\inf N_L \cup (W\inf N_L - \text{int} \ K) = W\inf N_L - K = W\inf (P^L) - K. \tag{30}
\]

Now, as (e) holds, for the given \( L \), one has \( W\inf (P^L) = W\max (D^L) = W\max M_L \subset M_L \) which together with (30) yields the existence of \( \bar{T} \in L_\Phi \) such that \( -y \in -\Phi^*(L, \bar{T}) - K \). The last equation means that \((L, y) \in \text{epi } \Phi^*(\cdot, \bar{T}) \), and hence,
(L, y) ∈ ∪_{T ∈ L} epi Φ∗(·, T) and consequently, (29) holds. The equivalence [(a) ⇔ (e)] has been proved. The proof of [(b) ⇔ (f)] is similar. □

Further on stable strong duality for the pairs (P) - (D), (P) - (Dℓ) are given in the next corollaries.

**Corollary 4.1** (Stable strong duality I) Assume that (H1) and (C1) hold. Then, stable strong duality holds for the pair (P) - (D). Moreover, if, in addition that (C0) holds, then stable strong duality holds for (P) - (Dℓ) as well.

**Proof** As (H1) and (C1) hold, by Theorem 3.1 the condition (a) from Theorem 4.2 holds, and hence, by this theorem stable strong duality holds for the pair (P) - (D). If, in addition that (C0) holds then again, by Theorem 3.1, (b) of Theorem 4.2 holds which ensures stable strong duality for the pair (P) - (Dℓ). □

**Corollary 4.2** (Stable strong duality II) Assume that (H1) holds. The following assertions hold:

(a) If (C7) holds, then stable strong duality holds for the pairs (P) - (D) and (P) - (Dℓ).

(b) If at least one of the conditions (C2), (C3), . . . , (C6) holds, then stable strong duality holds for (P) - (D). If, in addition that (C0) holds then stable strong duality holds for (P) - (Dℓ) as well.

**Proof** Similar to the proof of Theorem 4.1, using Theorems 3.2, 3.3 and 4.2. □

### 5 Composite Vector Problems: Lagrange and Fenchel–Lagrange duality

In this section, we will apply the results obtained in the previous sections to the composed constrained vector optimization problem:

\[(CCVP) \quad \text{WInf} \{F(x) + (κ ∘ H)(x) : x ∈ C, \ G(x) ∈ −S\},\]

where X, Y, Z, W are lcHtvs, P ⊆ W and S ⊆ Z, G : X → Z∞, and C are the cones, the mapping and the set in X, respectively, as in Sections 3 and 4 while F : X → Y∗, κ : W ∪ {+∞W} → Y∗ (with κ(+∞W) = +∞Y), H : X → W∗ are proper mappings. Denote by A := C ∩ G−1(−S) the feasible set of (CCVP) and assume that

\[A ∩ \text{dom} \ F ∩ H^{-1}(\text{dom} \ κ) \neq ∅.\]  

In order to apply the results obtained in the previous sections to (CCVP), we need to define suitable perturbation mappings Φ for (CCVP). Concretely, we will construct some typical perturbation mappings, say Φ1 and Φ2, which give rise to two typical types of dual problems (Lagrange and Fenchel–Lagrange ones) and duality results for (CCVP). Other possible perturbation mappings are introduced in Remark 5.3 together with corresponding dual problems.
**The First Perturbation Mapping $\Phi_1$: Lagrange Duality.** Take $\tilde{Z} := W \times Z$ as the space of perturbation variables and define the perturbation mapping $\Phi_1 : X \times \tilde{Z} \to Y^*$ by

$$\Phi_1(x, w, z) := \begin{cases} F(x) + \kappa(H(x) + w), & \text{if } x \in C \text{ and } G(x) + z \in -S, \\ +\infty_Y, & \text{otherwise}. \end{cases}$$

We consider the cone $\tilde{S} := P \times S$ in $\tilde{Z}$ and observe that $\mathcal{L}_+(\tilde{S}, K) \cong \mathcal{L}_+(P, K) \times \mathcal{L}_+(S, K)$.

Firstly, one has $\Phi_1(x, 0_W, 0_Z) = F(x) + (\kappa \circ H)(x) + I_A(x)$ for all $x \in X$, and

$$\text{dom } \Phi_1 = \{(x, w, z) \in X \times W \times Z : x \in \tilde{C}, \ w \in -H(x) + \text{dom } \kappa, \ z \in -G(x) - S \},$$

and so, $\pi_1(\text{dom } \Phi_1) = -\{(H(x), G(x)) : x \in \tilde{C}\} \cup \{\text{dom } \kappa \times (-S)\}$, where $\pi_1 : X \times \tilde{Z} \to \tilde{Z}$ defined by $\pi_1(x, w, z) = (w, z)$ and $\tilde{C} := C \cap \text{dom } F \cap \text{dom } G \cap \text{dom } H$. As (31) holds, $(0_W, 0_Z) \in \pi_1(\text{dom } \Phi_1)$.

**Lemma 5.1** (i) For all $L \in \mathcal{L}(X, Y)$ and $T := (T_1, T_2) \in \mathcal{L}(W, Y) \times \mathcal{L}(Z, Y)$, it holds

$$\Phi_1^*(L, T) = (F + T_1 \circ H + T_2 \circ G + I_C)^*(L) \cup \kappa^*(T_1) \cup I_S^*(T_2). \tag{32}$$

Moreover, if $T_2 \in \mathcal{L}_+(S, K)$, then $\Phi_1^*(L, T) = (F + T_1 \circ H + T_2 \circ G + I_C)^*(L) \cup \kappa^*(T_1)$.

(ii) It holds:

$$\bigcup_{T \in \mathcal{L}_{\Phi_1}} \text{epi } \Phi_1^*(\cdot, T) = \bigcup_{T \in \mathcal{L}_{\Phi_1}^+} \text{epi } \Phi_1^*(\cdot, T) = \bigcup_{T \in \mathcal{L}_{\Phi_1}} \Psi \left( \text{epi}(F + T_1 \circ H + T_2 \circ G + I_C)^* \boxplus (0_L, \kappa^*(T_1) \cup I_S^*(T_2)) \right),$$

where $\Psi$ is as in (2).

**Proof** (See the Appendix D).

According to Lemma 5.1(ii), the sets $\mathcal{M}$ and $\mathcal{M}_+$ in (9) now become, respectively,

$$\mathcal{M}^1 := \bigcup_{T_1 \in \text{dom } \kappa^*} \Psi \left( \text{epi}(F + T_1 \circ H + T_2 \circ G + I_C)^* \boxplus (0_L, \kappa^*(T_1) \cup I_S^*(T_2)) \right),$$

$\$
\( \mathcal{M}^1_+ := \bigcup_{T_1 \in \text{dom} \kappa^* \cap \mathcal{L}^+(P,K)} \psi \left( \mathcal{E}_{\text{epi}}(F + T_1 \circ H + T_2 \circ G + I_C)^* \oplus (0 \mathcal{L}, \kappa^*(T_1)) \right). \)

By Lemma 5.1(i), for \( T := (T_1, T_2) \in \mathcal{L}(Z_1, Y) \times \mathcal{L}(Z_2, Y) \),

\[ \Phi_1^*(0 \mathcal{L}, T) = (F + T_1 \circ H + T_2 \circ G + I_C)^* \oplus \kappa^*(T_1) \oplus I^*_S(T_2). \]

If \( T_2 \in \mathcal{L}_+(S_2, K) \), then \( \Phi_1^*(0 \mathcal{L}, T) = (F + T_1 \circ H + T_2 \circ G + I_C)^* \oplus \kappa^*(T_1) \).

Note that

\[-(F + T_1 \circ H + T_2 \circ G + I_C)^* \oplus \kappa^*(T_1) \oplus I^*_S(T_2) \]

\[-= - \text{WSup}(F + T_1 \circ H + T_2 \circ G + I_C)^* \oplus \kappa^*(T_1) + I^*_S(T_2)] \]

\[-= - \text{WSup} \left[ \text{WSup}(F - T_1 \circ H - T_2 \circ G - I_C)(X) + \text{WSup}(T_1 - \kappa)(Z) + \text{WSup}(-T_2)(-S) \right] \]

\[= \text{WInf} \left[ (F(x) + (T_1 \circ H)(x) + (T_2 \circ G)(x) + \kappa(z) - T_1(z) + T_2(s)) \right]. \]

and so, the dual problem and the loose dual problem of (CCVP), which are specific cases of (D) and (D\( \ell \)) with \( \Phi = \Phi_1 \), become the Lagrange dual and loose Lagrange dual problems:

\[
\text{(CCVD}_1) \quad \text{WSup}_{T_1 \in \text{dom} \kappa^*} \quad \text{WInf}_{(x,z,s) \in C \times Z \times S} \quad \left[ (F(x) + (T_1 \circ H)(x) + (T_2 \circ G)(x) + \kappa(z) - T_1(z) + T_2(s)) \right],
\]

\[
\text{(CCVD}_1) \quad \text{WSup}_{T_1 \in \text{dom} \kappa^*} \quad \text{WInf}_{(x,z,s) \in C \times Z \times S} \quad \left[ (F(x) + (T_1 \circ H)(x) + (T_2 \circ G)(x) + \kappa(z) - T_1(z)) \right],
\]

When \( \kappa \equiv 0 \), (CCVP) collapses to the vector optimization problem (VP) in Remark 6.2, which leads some known and also new results in duality for (VP). The observation applies to \( \Phi_2 \) below as well.

**Corollary 5.1** (Characterizations of stable strong duality for (CCVP)) Consider the statements:

(a) \( \text{epi}(F + \kappa \circ H + I_A)^* = \mathcal{M}^1. \)

(b) \( \text{epi}(F + \kappa \circ H + I_A)^* = \mathcal{M}^1_+. \)

(e) Stable strong duality holds for the pair (CCVP) - (CCVD\( \ell \)).

(f) Stable strong duality holds for the pair (CCVP) - (CCVD\( \ell \)).

Then, it holds \([a_1] \iff (e_1)]\) and \([b_1] \iff (f_1)]\).

**Proof** It is worth observing that when \( \Phi = \Phi_1 \) one has \( \Phi(., 0_W, 0_Z) = F + (\kappa \circ H) + I_A, \mathcal{M} = \mathcal{M}^1 \) and \( \mathcal{M}_+ = \mathcal{M}^1_+ \). Moreover, the dual problems (CCVD) and (CCVD\( \ell \)) are none other than (D) and (D\( \ell \)) which are specified to this case, respectively. The conclusion now follows from Theorem 4.2.

**Corollary 5.2** (Stable strong Lagrange duality for (CCVP)) Assume that \( F \) (H, G, resp.) is \( K \)-convex (\( P \)-convex, \( S \)-convex, resp.), \( \kappa \) is \( K \)-convex and \( (P, K) \)-nondecreasing and \( C \) is convex. Denote \( Z_0^1 := \text{lin} \left[ \{(H(x), G(x)) : x \in C\} + (-\text{dom} \kappa) \times S \right] \) and assume that \( (C_1^1) \) holds:
\( (C_1^1) \quad \forall L \in \mathcal{L}(X, Y), \exists y_L \in Y, \exists V_L \in \mathcal{N}(0_{Z_1}, 0_{Z_2}) \text{ s.t. } \\forall (w, z) \in V_L \cap Z_0^1, \exists x \in C : G(x) + z \in -S \text{ and } F(x) + \kappa(H(x) + w) - L(x) \leq_K y_L. \)

Then, stable strong duality holds for the pairs (CCVP) - (CCVD\(^1\)) and (CCVP) - (CCVD\(^4\)).

**Proof** Assumptions on cone-convexity of the mappings \( F, H, G \) and \( \kappa \), the convexity of \( C \) and the non-decreasing property of \( \kappa \) entail that \( \Phi_1 \) is a \( K \)-convex mapping. The condition \((C_1^1)\) ensures that \((C_1)\) in previous sections holds with \( \Phi = \Phi_1 \). So, according to Theorem 4.1, stable strong duality holds for the pair (CCVP) - (CCVD\(^1\)).

Moreover, as (31) holds, there is \( \tilde{x} \in X \) such that \( \tilde{x} \in C \) and \( G(\tilde{x}) \in -S \) which yields \( \Phi_1(\tilde{x}, 0_W, 0_Z) = F(\tilde{x}) + (\kappa \circ H)(\tilde{x}) \). Take \( (w, z) \in -(P \times S) \). As \( z \in -S \), \( G(\tilde{x}) + z \in -S \), and hence, \( \Phi_1(\tilde{x}, w, z) = F(\tilde{x}) + \kappa(H(\tilde{x}) + w) \). This, together with the fact that \( w \in -P \) and that \( \kappa \) is \((P, K)\) non-decreasing, yields \( \Phi_1(\tilde{x}, w, z) \leq_K F(\tilde{x}) + (\kappa \circ H)(\tilde{x}) = \Phi_1(\tilde{x}, 0_W, 0_Z) \). We have just proved that \((C_0)\) holds with \( \Phi = \Phi_1 \). So, again by Theorem 4.1, stable strong duality holds for the pair (CCVP) - (CCVD\(^4\)). \( \square \)

**Remark 5.1** Observe that one can specify conditions \((C_2) - (C_7)\) to this setting and use Theorem 4.2 (instead of Theorem 4.1) to get stable strong duality results for pairs (CCVP) - (CCVD\(^1\)) and (CCVP) - (CCVD\(^4\)). However, not to make the paper becomes too long, we omit them. Similar observation also applies to perturbation mapping \( \Phi_2 \) below.

**Remark 5.2** It is worth observing that the dual problem (CCVD\(^4\)) is defined by means of the perturbation mapping \( \Phi_1 \) where both \( \kappa \) and \( G \) are perturbed. When turning back to the scalar case, i.e., when \( Y = \mathbb{R} \), and in some specific circumstance the dual problem (CCVD\(^4\)) goes back to some known dual problem in the literature, for instance, when \( Y = \mathbb{R}, G \equiv 0, C = X \), (CCVP) collapses to the problem (P\(^CC\)) in [2, Page 29] (see also [6, page 100], [8] for other examples) and (CCVD\(^4\)) in this case is exactly the dual problem (D\(^CC\)) (a Lagrange dual problem) in [2], while the other dual problem (D\(^CC\)) of (P\(^CC\)) in [2] is a special case of (CCVD\(^4\)) corresponding to the perturbation mapping \( \Phi_4 \) below (see Remark 5.3).

With \( \Phi = \Phi_1 \), \( \mathcal{V} \)-stability of general vector inequalities in Theorem 3.4 leads to

**Corollary 5.3** (Characterizations of Stable Farkas lemma for composite vector systems)

1. Consider the following statements:
   (a) \( \text{epi}(F + \kappa \circ H + I_A)^* = \mathcal{M}^1_+ \).
   (b) \( \text{epi}(F + \kappa \circ H + I_A)^* = \mathcal{M}^1_+ \).
   (c) For all \( (L, y) \in \mathcal{L}(X, Y) \times Y \), two following assertions are equivalent
      \( (\alpha') \) \( x \in C, G(x) \in -S \implies F(x) + (\kappa \circ H)(x) - L(x) + y \notin - \text{int } K \),
      \( (\beta') \) \( \exists T_1 \in \text{dom } \kappa^*, T_2 \in \mathcal{L}^w_+(S, K) \) such that
      \[ F(x) + (T_1 \circ H)(x) + (T_2 \circ G)(x) - L(x) + y \notin \kappa^*(T_1) + T_2(-S) - \text{int } K, \forall x \in C. \]
For all \((L, y) \in \mathcal{L}(X, Y) \times Y\), two following assertions are equivalent

\((\alpha')\) \(x \in C\), \(G(x) \in -S \implies F(x) + (\kappa \circ H)(x) - L(x) + y \notin - \text{int} \, K\),

\((\gamma')\) \(\exists T_1 \in \text{dom} \, \kappa^* \cap \mathcal{L}_+(P, K)\), \(T_2 \in \mathcal{L}_+(S, K)\) such that

\[
F(x) + (T_1 \circ H)(x) + (T_2 \circ G)(x) - L(x) + y \notin \kappa^*(T_1) - \text{int} \, K, \quad \forall x \in C.
\]

Then, \([a_1] \iff (c_1)]\) and \([b_1] \iff (d_1)]\).

**The Second Perturbation Mapping \(\Phi_2\): Fenchel–Lagrange Duality.** Let \(\tilde{Z} := X \times X \times W \times Z\) be the space of perturbation variables and define the perturbation mapping \(\Phi_2: X \times \tilde{Z} \to Y^*\),

\[
\Phi_2(x, x', x'', w, z) := \begin{cases} F(x + x') + \kappa(H(x) + w), & \text{if } x + x'' \in C, G(x) + z \in -S \\ +\infty, & \text{otherwise}. \end{cases}
\]

We consider in \(\tilde{Z}\) the cone \(\tilde{S} := \{0_X\} \times \{0_X\} \times P \times S\). Then, \(\mathcal{L}_+(\tilde{S}, K) \cong \mathcal{L}(X, Y)^2 \times \mathcal{L}_+(P, K) \times \mathcal{L}_+(S, K)\). We make some quick observations which will be used in the rest of this section.

1. It can be checked that \(\Phi_2(x, 0_X, 0_X, 0_W, 0_Z) = F(x) + (\kappa \circ H)(x) + I_A(x)\) for all \(x \in X\) and \(\pi(\text{dom} \, \Phi_2) = -\left\{ (x, x, H(x), G(x)) : x \in \text{dom} \, H \cap \text{dom} \, G \right\} + \left\{ \text{dom} \, F \cap C \cap \text{dom} \, \kappa \times (-S) \right\}.$

2. For all \(L \in \mathcal{L}(X, Y)\) and \(T := (L', L'', T_1, T_2) \in \mathcal{L}(X, Y)^2 \times \mathcal{L}(W, Y) \times \mathcal{L}(Z, Y)\), it holds \(\Phi_2^* (L, T) = F^* (L') \cup (T_1 \circ H + T_2 \circ G)^* (L - L' - L'') \cup I^*_c (L'') \cup \kappa^*(T_1) \cup I^*_c (L') \cup I^*_c (T_2)\), and if \(T_2 \in \mathcal{L}_+(S, K)\), then \(\Phi_2^* (L, T) = F^* (L') \cup (T_1 \circ H + T_2 \circ G)^* (L - L' - L'') \cup I^*_c (L'') \cup \kappa^*(T_1)\).

3. The sets \(\mathcal{M}\) and \(\mathcal{M}_+\) (when \(\Phi = \Phi_2\)) become, respectively,

\[
\mathcal{M} := \bigcup_{T_1 \in \text{dom} \, \kappa^*} \Psi \left( \text{Epi} \, F^* \mathbin{\biguplus} \text{Epi}(T_1 \circ H + T_2 \circ G)^* \mathbin{\biguplus} \text{Epi} \, I^*_c \mathbin{\biguplus} \mathcal{L}_+(0_L, \kappa^*(T_1) \cup I^*_c (T_2)) \right).
\]
Then, stable strong duality holds for the pairs $(\text{CCVP})$ applying to Theorems 4.2 and 4.1.

Denote $Z_0^2 := \{ (x, x, H(x), G(x)) : x \in \text{dom } H \cap \text{dom } G \} - \{ \text{dom } F \times C \times \text{dom } \kappa \times (-S) \}$.

Corollary 5.4 (Characterizations stable strong duality for (CCVP)) Consider the following statements:

(a) $\text{epi}(F^* + H + I_A) = \mathcal{M}_2^2$.
(b) $\text{epi}(F^* + H + I_A) = \mathcal{M}_2^2$.

(2) Stable strong duality holds for the pair (CCVP) - (CCVD$^2$).

Then, it holds [(a2) $\Leftrightarrow$ (e2)] and [(b2) $\Leftrightarrow$ (f2)].

Corollary 5.5 (Stable strong duality for (CCVP)) Assume that $F$ ($H, G$, resp.) is $K$-convex ($P$-convex, $S$-convex, resp.), $\kappa$ is $K$-convex and ($P, K$) non-decreasing and $C$ is convex. Assume further that the following condition holds:

$$\forall L \in L(X, Y), \exists y_L \in Y, \exists W_L \in N(0_X, 0_X, 0_W, 0_Z) \text{ such that }$$

$$\forall (x', x'', w, z) \in W_L \cap Z_0^2, \exists x \in X : x + x'' \in C, G(x) + z \in -(S),$$

$$F(x + x') + \kappa(H(x) + w) - L(x) \leq y_L.$$
Remark 5.3 It is worth noting that there may have other ways to define perturbation mappings for (CCVP), and then more representations of epi($F + \kappa \circ H + I_A$)$^*$, more corresponding results on duality for (CCVP), vector Farkas lemmas can be derived, for instance:

(i) Take $\tilde{Z} := X \times W \times Z$ as the space of perturbation variables, with $\tilde{S} := \{0_X\} \times P \times S$, and $\mathcal{L}_+(\tilde{S}, K) \cong \mathcal{L}(X, Y) \times \mathcal{L}_+(P, K) \times \mathcal{L}_+(S, K)$. We define the perturbation mapping $\Phi_3 : X \times \tilde{Z} \rightarrow Y^*$,

$$\Phi_3(x, v, w, z) := \begin{cases} F(x + v) + \kappa(H(x) + w), & \text{if } x \in C, \text{ and } G(x) + z \in -S, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, the dual problems (D) and (D$_t$) become different Fenchel–Lagrange dual problems of (CCVP) as follows:

$$(\text{CCVD}^3) \quad \text{WSup}_{L' \in \mathcal{L}(X, Y)} \text{WSup}_{T_1 \in \text{dom } \kappa^*} \text{WSup}_{T_2 \in \mathcal{L}^+_S(S, K)} \left[ -F^*(L') \cup (T_1 \circ H + T_2 \circ G + I_C)^*(-L') \cup \kappa^*(T_1) \cup I_S^-(T_2) \right],$$

$$(\text{CCVD}^4) \quad \text{WSup}_{L', L'', L''' \in \mathcal{L}(X, Y)} \text{WSup}_{T_1 \in \text{dom } \kappa^*} \text{WSup}_{T_2 \in \mathcal{L}^+_S(S, K)} \left[ -F^*(L') \cup (T_1 \circ H)^*(L'') \cup (T_2 \circ G)^*(-L' - L'' - L''') \cup \kappa^*(T_1) \cup I_S^-(T_2) \right].$$

(ii) Take $\tilde{Z} := X^3 \times W \times Z$, $\tilde{S} := \{0_X\}^3 \times P \times S$, $\Phi_4 : X^4 \times W \times Z \rightarrow Y^*$,

$$\Phi_4(x, x', x'', x''', w, z) := \begin{cases} F(x + x') + \kappa(H_1(x + x'') + w), & \text{if } x + x''' \in C, \text{ and } G(x) + z \in -S, \\ +\infty, & \text{otherwise.} \end{cases}$$

This perturbation mapping $\Phi_4$ leads to new types of dual problems of (CCVP):

$$(\text{CCVD}^5) \quad \text{WSup}_{L', L'', L''' \in \mathcal{L}(X, Y)} \text{WSup}_{T_1 \in \text{dom } \kappa^*} \text{WSup}_{T_2 \in \mathcal{L}^+_S(S, K)} \left[ -F^*(L') \cup (T_1 \circ H)^*(L'') \cup (T_2 \circ G)^*(-L' - L'' - L''') \cup \kappa^*(T_1) \cup I_S^-(T_2) \right],$$

$$(\text{CCVD}^6) \quad \text{WSup}_{L', L'', L''' \in \mathcal{L}(X, Y)} \text{WSup}_{T_1 \in \text{dom } \kappa^*} \text{WSup}_{T_2 \in \mathcal{L}^+_S(S, K)} \left[ -F^*(L') \cup (T_1 \circ H)^*(L'') \cup (T_2 \circ G)^*(-L' - L'' - L''') \cup \kappa^*(T_1) \cup I_S^-(T_2) \right].$$

and under suitable conditions, one can get stable strong duality results for pairs (CCVP) - (CCVD$^3$) and (CCVP) - (CCVD$^4$) ($i = 3, 4$), and obviously, some more forms of vector Farkas lemmas as well.
6 Special Cases of (CCVP)

6.1 Lagrange and Fenchel–Lagrange Duality for Composite Vector Problems

We are concerned with the composite vector problem [11]

\[(CVP) \quad \text{WInf} \{ F(x) + (\kappa \circ H)(x) : x \in X \}\]

where \(X, Y, W, P, F, \kappa, H\) are as in Section 5. Assume that \(\text{dom } F \cap H^{-1}(\text{dom } \kappa) \neq \emptyset\). It is clear that (CVP) is a special case of (CCVP) when \(Z = Y, S = \{0_Y\}, C = X, \) and \(G(x) = 0_Y\) for all \(x \in X\), and hence \(L(Z, Y) = L_+(S, K) = L(Y, Y)\) and \(I^*_S(T) = - \text{bd } K\) for all \(T \in L(Y, Y)\). The dual problems (CCVD\(_1\)) and (CCVD\(_2\)) now become

\[(CVD\(_1\)) \quad \text{WSup}_{T \in \text{dom } \kappa^*} \text{WInf}_{(x,z) \in X \times Z} [F(x) + T \circ H(x) + \kappa(z) - T(z)],\]

\[(CVD\(_2\)) \quad \text{WSup}_{T \in \text{dom } \kappa^* \cap L_+(P, K)} \text{WInf}_{(x,z) \in X \times Z} [F(x) + T \circ H(x) + \kappa(z) - T(z)],\]

which are nothing else but the Lagrange and loose Lagrange dual problems (CVD), (CVD\(_2\)) introduced recently in [11] while the problems (CCVD\(_2\)) and (CCVD\(_2\)) lead to new Fenchel–Lagrange dual problems of (CVP):

\[(CVD\(_2\)) \quad \text{WSup}_{L' \in L(X, Y)} \text{WInf}_{T \in \text{dom } \kappa^*} \left[ - F^*(L') \cup (T \circ H)^* (-L') \cup \kappa^*(T) \right],\]

\[(CVD\(_2\)) \quad \text{WSup}_{L' \in L(X, Y)} \text{WInf}_{T \in \text{dom } \kappa^* \cap L_+(P, K)} \left[ - F^*(L') \cup (T \circ H)^* (-L') \cup \kappa^*(T) \right].\]

The sets \(M^1, M^1_+, M^2, \) and \(M^2_+\) reduce to, respectively,

\[A^1 := \bigcup_{T \in \text{dom } \kappa^*} \Psi \left( \text{Epi} (F + T \circ H)^* \boxplus (0_L, \kappa^*(T)) \right),\]

\[A^1_+ := \bigcup_{T \in \text{dom } \kappa^* \cap L_+(P, K)} \Psi \left( \text{Epi} (F + T \circ H)^* \boxplus (0_L, \kappa^*(T)) \right),\]

\[A^2 := \bigcup_{T \in \text{dom } \kappa^*} \Psi \left( \text{Epi } F \boxplus \text{Epi} (T \circ H)^* \boxplus (0_L, \kappa^*(T)) \right),\]

\[A^2_+ := \bigcup_{T \in \text{dom } \kappa^* \cap L_+(P, K)} \Psi \left( \text{Epi } F \boxplus \text{Epi} (T \circ H)^* \boxplus (0_L, \kappa^*(T)) \right).\]

Observe that \(A^1\) and \(A^1_+\) are, respectively, the sets \(A\) and \(B\) in [11]. The next corollary is a direct consequence of Corollary 5.1. Note that Corollary 5.3 also leads to versions of stable Farkas lemma for systems involving composite mapping; however, this will be omitted.
Corollary 6.1 Let \( i \in \{1, 2\} \). Consider the following statements:

(a) \( \text{epi}(F + \kappa \circ H)^* = \mathcal{A}_i^i \).

(b) \( \text{epi}(F + \kappa \circ H)^* = \mathcal{A}_i^i \).

(c) Stable strong duality holds for the pair (CVP) - (CVD\(_i^i\)).

Then, for all \( i \in \{1, 2\} \), it holds \([a_1^i] \Leftrightarrow \text{e}_1^i\] and \([b_2^i] \Leftrightarrow \text{f}_2^i\].

Remark 6.1 (i) The case \( i = 1 \) in Corollary 6.1 goes back to [11, Theorems 3.6, 3.7] while the results corresponding to \( i = 2 \) in Corollary 6.1 up to the best knowledge of the authors, are new.

(ii) When specifying the regularity conditions \((C_0) - (C_7)\) in Section 3 to (CVP), we will get variants of sufficient conditions for \((a_1^i), \(b_1^i\), \(i = 1, 2\), which also means sufficient conditions for stable strong duality for (CVP) in Corollary 6.1.

Remark 6.2 (Lagrange and Fenchel–Lagrange Duality for Cone-Constrained Vector Problems) A special case of (CCVP) is the cone-constrained vector problem (VP):

\[
\text{(VP)} \quad \text{WInf} \{ F(x) : x \in C, \ G(x) \in -S \}.
\]

The Lagrange, Fenchel–Lagrange dual problems of (CCVD\(_i^i\)) and (CCVD\(_i^2\)) (in Section 5) specified to (VP) turn back to the Lagrange and Fenchel–Lagrange dual problems in [16]. Corollaries 5.1 and 5.4 lead to characterizations of stable strong duality for (VP) (similar to Corollary 6.1) which cover [16, Theorem 6.1], [16, Theorem 5.1]. On the other hand, the problems (CCVD\(_i\)), \(i = 1, 2\), collapse to the following ones:

\[
\begin{align*}
\text{(VD\(_1^1\))} & \quad \text{WSup}_{T \in \mathcal{L}_+^{w}(S,K)} \text{WInf}_{(x,s) \in C \times S} \{ F(x) + T \circ G(x) + T(s) \}, \\
\text{(VD\(_2^2\))} & \quad \text{WSup}_{L', L'' \in \mathcal{L}(X,Y)} \text{WInf}_{T \in \mathcal{L}_+^{w}(S,K)} \{ -F^* (L') \cup (T \circ G)^* (-L' - L'') \cup I_C^*(L' - L'') \cup I_{S}^*(T) \}.
\end{align*}
\]

The problem (VD\(_1^1\)) was introduced recently in [15], while (VD\(_2^2\)) is new and so, all the duality results (specified from results in Section 5) associated with the second problem (VD\(_2^2\)) are new.

Remark 6.3 It is worth mentioning here that the problem (VP) in Remark 6.2 shows one of the new and specific feature of our results, which has not yet appeared in the literature. Concretely, for (VP), the use of \( \Phi_2 \) leads to the representation:

\[
\text{epi}(F + I_A)^* = \Psi \left( \text{epi} F^* \boxplus \text{epi} I_C^* \boxplus \bigcup_{T \in \mathcal{L}_+^{w}(S,K)} \text{epi} (T \circ G)^* \right),
\]

(33)

\( A := C \cap G^{-1}(-S) \). When \( Y = \mathbb{R} \), the representation reduces to

\[
\text{epi}(f + i_A)^* = \text{epi} f^* + \text{epi} i_C^* + \bigcup_{z^* \in S^+} \text{epi}(z^* \circ G)^*.
\]
which appeared in many works on scalar optimization (see, e.g., [2, 7, 18–20, 25] and references therein). This shows that the representation of epigraph of conjugate functions as in (33) is extended for the first time to the one for epigraph of conjugate mappings of vector functions.

### 6.2 Scalar Cone-Constrained Composite Problems

Consider the cone-constrained composite (scalar) problem

\[
(\text{CCP}) \quad \inf \{ f(x) + (\kappa \circ H)(x) : x \in C, \ G(x) \in -S \}.
\]

Here, we retain the notions as in Section 5 with \( Y = \mathbb{R} \) and \( K = \mathbb{R}_+ \). Obviously, this is a special case of (CCVP). This model was considered in [5] with the case where \( f \equiv 0 \) or \( G \equiv 0 \) and \( C = X \). Note that, in this case, \( \mathcal{L}_+(P, K) = P^+ \), \( \mathcal{L}^w_+(S, K) = \mathcal{L}_+(S, K) = S^+ \), and so, (CCVD\(_1^1\)) and (CCVD\(_1^1\)), respectively, become the Lagrange dual problems:

\[
(\text{CCD}_1^1) \quad \sup_{(\lambda_1, \lambda_2) \in \text{dom} \ k^* \times S^+} \left[ \inf_{x \in C} \{ f(x) + (\lambda_1 H)(x) + (\lambda_2 G)(x) \} - \kappa^*(\lambda_1) \right],
\]

\[
(\text{CCD}_1^1) \quad \sup_{\lambda_1 \in \text{dom} \ k^* \cap P^+, \ \lambda_2 \in S^+} \left[ \inf_{x \in C} \{ f(x) + (\lambda_1 H)(x) + (\lambda_2 G)(x) \} - \kappa^*(\lambda_1) \right],
\]

and (CCD\(_1^2\)), (CCVD\(_1^2\)) become some forms of Fenchel–Lagrange dual problems:

\[
(\text{CCD}_1^1) \quad \sup_{(x^*, y^*) \in k^* \times S^+, \ (\lambda_1, \lambda_2) \in \text{dom} \ k^* \times S^+} \left[ -f^*(x^*) - (\lambda_1 H + \lambda_2 G)^*(x^* - y^*) \right],
\]

\[
(\text{CCD}_1^1) \quad \sup_{(x^*, y^*, \lambda_2) \in (k^*)^2 \times S^+, \ \lambda_1 \in \text{dom} \ k^* \cap P^+} \left[ -f^*(x^*) - (\lambda_1 H + \lambda_2 G)^*(x^* - y^*) \right].
\]

Corollaries 5.1–5.5 lead to various results on strong duality for (CCP). Some of the important special cases of the problem (CCP) can be listed as: the composite (scalar) problem (CP) (see [2, 5]) and the cone-constrained problem (P1) below:

\[
(\text{CP}) \quad \inf_{x \in X} \{ f(x) + (\kappa \circ H)(x) \},
\]

\[
(\text{P1}) \quad \inf \{ f(x) : x \in C, \ G(x) \in -S \}.
\]

For (CP), the Lagrange dual problems (CCD\(_1^1\)) and (CCD\(_1^1\)) become the usual Lagrange dual problems of (CP) while (CCD\(_1^1\)) and (CCD\(_1^1\)) reduce, respectively, to Fenchel–Lagrange dual problems appeared in [2, page 42] and in [5].

Similarly, for (P1), both the dual problems (CCD\(_1^1\)) and (CCD\(_1^1\)) reduce to the usual Lagrange dual problem while the dual problems (CCD\(_1^1\)) and (CCD\(_1^1\)) coincide with each other and become
\[ (D^2) \sup_{x^*, y^* \in X^*, \lambda \in S^+} \left[ -f^*(x^*) - (\lambda G)^*(-x^* - y^*) - i_C^*(y^*) \right], \]

which are the Fenchel–Lagrange dual problems introduced in [2, 7, 18, 19] and this justifies the name “Fenchel–Lagrange dual problem” for the problems (CCVD^2) and (CCVD^2_ℓ) in Section 5. Corollaries 5.1 and 5.4, in the current setting, imply [16, Corollary 6.2], and hence, they cover the results established in [2, 7, 18, 19] (see also [16, Remark 6.2]).

It is also worth observing that when \( Y = \mathbb{R} \), (C3) and (C4) reduce to \((RC_1^\Phi)\) and \((RC_3^\Phi)\) in [2], respectively, while (C5) is weaker than \((RC_2^\Phi)\) and consequently, Theorem 4.2, specified to the case \( Y = \mathbb{R} \), extends [2, Theorem 1.7] in the sense that we get a stable strong duality results (not only the strong ones) under weaker assumptions.

### 7 Conclusions

In this paper, we consider the general vector optimization problem with linear perturbation

\[ (P^L) \quad \text{WInf}_{x \in X} [\Phi(x, 0_Z) - L(x)], \]

associated with a perturbation mapping \( \Phi : X \times Z \to Y \cup \{+\infty\} \) and a linear operator \( L \in \mathcal{L}(X, Y) \), where X, Y, Z are lcHtv's, “WInf \( M \)” indicates the set of all weak infimum elements of a set \( M \subset Y \) w.r.t. the weak ordering generated by a closed convex cone \( K \subset Y \). The problem \((P^L)\) or \((P)\), when \( L = 0 \in \mathcal{L}(X, Y) \), includes many mathematical models from practical problems (see, e.g., [6, 11, 26]).

In this paper, we firstly establish two representations of the epigraph of the conjugate mapping of \( \Phi(\cdot, T) \) under different regularity conditions. These representations are then used to establish versions of (stable) vector Farkas lemmas corresponding to a system defined by the perturbation mapping \( \Phi \). Secondly, two dual problems of \((P^L)\), namely the dual and the loose dual problems \((D^L)\) and \((D^L_\ell)\), are introduced based on the mentioned representations and stable strong duality results are proved for these pairs of primal–dual problems. Thirdly, the results just obtained are applied to a class of composed constrained vector problems (Section 5) and some specific cases of this class of problems like the unconstrained composite vector problem, or the cone-constrained vector problems (Section 6). For all of these classes of problems, we proposed several choices of perturbation mappings and some of them give rise to the Lagrange dual problems introduced recently in the literature, while some others lead to new kinds of dual problems called “Fenchel–Lagrange” dual problems which, to the best knowledge of the authors, appear for the first time for vector optimization problems. Strong duality results are obtained for these mentioned classes of problems. Particularly, when turning back to scalar cone-constrained composite problems, i.e., when \( Y = \mathbb{R} \), with the perturbation mappings mentioned above specified to this setting, we get back the Lagrange and also Fenchel–Lagrange dual problems known...
in the literature as in [2, 7, 18, 19], and this justifies the name “Fenchel–Lagrange dual problem” that we propose for dual problems in Section 5.

Acknowledgements The authors are very grateful to the anonymous referees for spending so much time in reading carefully our manuscript and for their valuable comments and detailed suggestions which helped us to improve considerably the quality of the paper. This research is funded by the Vietnam National University HoChiMinh city (VNU-HCM) under grant number B2021-28-03.

Appendix A Proof of Theorem 2.1 (Extended open mapping theorem)

Consider the set-valued mapping \( G : X \times Y \Rightarrow Z_0 \) defined by

\[
G(x, y) := \{ z \in Z : \Phi(x, z) \leq_K y \} \subset Z_0.
\]

Then, (6) simply means that \( 0_Z \in \text{int}_{Z_0} G(U_0 \times V_0) \). The proof of (6) proceeds with three steps as follows:

(\( \alpha \)) Let \( U \) and \( V \) be the convex neighborhoods of \( x_0 \) and \( \Phi(x_0, 0_Z) \), respectively. It is easy to check that \( G(U \times V) \) is a convex set (using the \( K \)-convexity of \( \Phi \) and of the sets \( U, V \)). We will show that \( G(U \times V) \) is absorbing in \( Z_0 \). Take \( z \in Z_0 \), we will show that there exists \( \lambda > 0 \) such that \( \lambda z \in G(U \times V) \). Firstly, one has \( 0_Z \in \pi(\text{dom} \Phi) \) (as \( (x_0, 0_Z) \in \text{dom} \Phi \)), so \( \text{aff}(\pi(\text{dom} \Phi)) = \text{lin}(\pi(\text{dom} \Phi)) = Z_0 \). Now, as \( 0_Z \in \text{icr}(\pi(\text{dom} \Phi)) \), there exists \( \delta > 0 \) such that \( \delta z \in \pi(\text{dom} \Phi) \). Then, there is \( x \in X \) such that \( (x, \delta z) \in \text{dom} \Phi \), or equivalently, \( \Phi(x, \delta z) \in Y \). On the other hand, one also has

\[
\Phi(x_0 + \mu(x - x_0), \mu \delta z) = \Phi((1 - \mu)(x_0, 0_Z) + \mu(x, \delta z)) \\
\in (1 - \mu)\Phi(x_0, 0_Z) + \mu\Phi(x, \delta z) - K \text{ (as } \Phi \text{ is } K \text{-convex)} \\
= \Phi(x_0, 0_Z) + \mu(\Phi(x, \delta z) - \Phi(x_0, 0_Z)) - K.
\]

Thus, \( \lambda z \in G(U \times V) \), where \( \lambda = \mu \delta > 0 \), meaning that \( G(U \times V) \) is absorbing in \( Z_0 \).

(\( \beta \)) Now, take an arbitrary neighborhood \( U \times V \) of \( (x_0, \Phi(x_0, 0_Z)) \) and we will prove that \( 0_Z \in \text{int}_{Z_0} \text{cl}(G(U \times V)) \). As \( X \times Y \) is locally convex, replace \( U, V \) by their subsets if necessary, we can suppose that \( U \) and \( V \) are convex. From (\( \alpha \)), \( G(U \times V) \) is convex and absorbing in \( Z_0 \). Consequently, \( \text{cl} G(U \times V) \) is a convex, closed and absorbing subset of \( Z_0 \), and hence, a neighborhood of \( 0_Z \) (as \( Z_0 \) is a barreled space). So, \( 0_Z \in \text{int}_{Z_0} \text{cl}(G(U \times V)) \).

(\( \gamma \)) We now show that (6) follows from Lemma 2.1. From (\( \beta \)), \( 0_Z \) belongs to the intersection of all sets of the form \( \text{int}_{Z_0} \text{cl}(G(U \times V)) \) where \( U \times V \) running from the collection of all neighborhoods of \( (x_0, \Phi(x_0, 0_Z)) \). Moreover, by assumption, \( X \times Y \) is a complete and first countable space, and \( G \) is a closed convex multifunction, since \( \text{gr} G = \text{epi} \Phi \) and \( \Phi \) is \( K \)-convex and \( K \)-epi closed. Then, Lemma 2.1, applying to the multifunction \( G \) with \( X \times Y, Z_0 \), and \( (x_0, \Phi(x_0, 0_Z)) \) playing the roles of \( \tilde{X}, \tilde{Y} \) and
\( \tilde{x}_0 \), respectively, gives

\[
0_Z \in \bigcap_{U \times V \in N(\tilde{x}_0, \Phi(\tilde{x}_0, 0_Z))} \text{int}_Z(\mathcal{G}(U \times V)),
\]

showing that \( 0_Z \in \text{int}_Z \mathcal{G}(U_0 \times V_0) \) and (6) follows. \( \square \)

**Appendix B Proof of Theorem 3.2**

We will show that if one of the conditions \((C_2), (C_3), (C_4), \) or \((C_5)\) holds, then the condition \((C_1)\) in Theorem 3.1 holds, and hence, the conclusion now follows from Theorem 3.1.

(\(\alpha\)) Assume that \((C_2)\) holds. Then, for any \( L \in \mathcal{L}(X, Y) \), take \( y_L := \tilde{y} - L(\tilde{x}) \) and \( V_L := \tilde{V} \), one gets \( \Phi(\tilde{x}, z) - L(\tilde{x}) \leq_K y_L \) for all \( z \in V_L \cap Z_0 \). So, \((C_1)\) holds (with \( \tilde{x} \) playing the role of \( x \)).

(\(\beta\)) Assume that \((C_3)\) holds. Pick \( \tilde{k} \in \text{int } K \). Then, \( \Phi(\tilde{x}, 0_Z) + \tilde{k} - \text{int } K \) is a neighborhood of \( 0 \). So, \( \Phi(\tilde{x}, \tilde{k}) \) belongs to \( 0 \), respectively, gives

Assume that \((\alpha)\) holds. Then, for any \( L \in \mathcal{L}(X, Y) \), take \( y_L := \tilde{y} - L(\tilde{x}) \) and \( V_L := \tilde{V} \), one gets \( \Phi(\tilde{x}, z) - L(\tilde{x}) \leq_K y_L \) for all \( z \in V_L \cap Z_0 \). So, \((C_1)\) holds (with \( \tilde{x} \) playing the role of \( x \)).

Assume that \((\beta)\) holds. Pick \( \tilde{k} \in \text{int } K \). Then, \( \Phi(\tilde{x}, 0_Z) + \tilde{k} - \text{int } K \) is a neighborhood of \( 0 \). So, \( \Phi(\tilde{x}, \tilde{k}) \) belongs to \( 0 \), respectively, gives

showing that \((C_2)\) holds, and consequently, \((C_1)\) holds as well.

(\(\gamma\)) Assume that \((C_4)\) holds. It is worth noting that \( \text{aff} (\pi(\text{dom } \Phi)) = \text{lin}(\pi(\text{dom } \Phi)) \). So, \( \text{ri}(\pi(\text{dom } \Phi)) = \text{int}_{Z_0}\).

Without loss the generality, assume that \( Z_0 = \mathbb{R}^n \) and take \( \{e_i\}_{i=1}^n \) the standard basis of \( \mathbb{R}^n \) (i.e., the \( i^{th} \) coordinate of \( e_i \) is 1 and the others are 0). As \( 0_Z \in \text{int}_{Z_0}(\pi(\text{dom } \Phi)) \), there exists \( \{e_i\}_{i=1}^n \subset \mathbb{R}_+ \setminus \{0\} \) such that \( \{\pm e_i e_i\}_{i=1}^n \subset \pi(\text{dom } \Phi) \). For each \( i = 1, 2, \ldots, n \), as \( \pm e_i e_i \in \pi(\text{dom } \Phi) \), there exists \( x_i, x_i' \in \text{dom } \Phi \) such that \( (x_i, e_i e_i) \) and \( (x_i', -e_i e_i) \) belong to \( \text{dom } \Phi \). Next, for any \( L \in \mathcal{L}(X, Y) \), take \( y_L \in Y \) such that (the existence of \( y_L \) is guaranteed by \([11, \text{Lemma } 2.1 (i))\])

\[
\Phi(x_i, e_i e_i) - L(x_i) < K y_L \quad \text{and} \quad \Phi(x_i', -e_i e_i) - L(x_i') < K y_L, \quad \forall i \in \{1, 2, \ldots, n\}.
\]

(34)

It is easy to see that \( \text{co} \{\pm e_i e_i\}_{i=1}^n \) is a neighborhood of \( 0_Z \) in \( Z_0 \), and hence, there exists the neighborhood \( \tilde{V} \) of \( 0_Z \) such that \( \tilde{V} \cap Z_0 = \text{co} \{\pm e_i e_i\}_{i=1}^n \). Now, for each \( z \in \tilde{V} \cap Z_0 \), there exists \( \{\lambda_i, \lambda_i'\}_{i=1}^n \subset \mathbb{R}_+ \) such that \( \sum_{i=1}^n (\lambda_i + \lambda_i') = 1 \) and \( z = \sum_{i=1}^n (\lambda_i e_i - \lambda_i' e') e_i \). Take \( \tilde{x} = \sum_{i=1}^n (\lambda_i x_i + \lambda_i' x_i') \), by the convexity of \( \Phi \),

\[
\Phi(\tilde{x}, z) \leq_K \sum_{i=1}^n (\lambda_i \Phi(x_i, e_i e_i) + \lambda_i' \Phi(x_i', -e_i e_i)).
\]

This, together with (34), entails \( \Phi(\tilde{x}, z) - L(\tilde{x}) \leq_K y_L \) which means that \((C_1)\) holds.

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(δ) Assume that (C5) holds. Take $L \in \mathcal{L}(X, Y)$. As $0_Z \in \pi(\text{dom } \Phi)$, there exists $x_0 \in X$ such that $(x_0, 0_Z) \in \text{dom } \Phi$. It is worth noting that as $\Phi$ is $K$-convex and $K$-epi closed, $\Phi - L$ is $K$-convex and $K$-epi closed as well. Pick $k_0 \in \text{int } K$. Apply the generalized open mapping theorem, Theorem 2.1, with $\Phi$ being replaced by $\Phi - L$, $U_0 = X$ and $V_0 = \Phi(x_0, 0_Z) - L(x_0) + k_0 - \text{int } K$, one gets that $0_Z \in \text{int}_{Z_0} V_1$, where $V_1$ is the set defined by

$$V_1 := \left\{ z \in Z : \Phi(x, z) - L(x) \leq_K y \text{ for some } x \in X \text{ and } y \in \Phi(x_0, 0_Z) - L(x_0) + k_0 - \text{int } K \right\}.$$ 

Take $y_L = \Phi(x_0, 0_Z) - L(x_0) + k_0$ and $V_L$ the neighborhood of $0_Z$ such that $V_L \cap Z_0 = V_1$. Then, for all $z \in V_L \cap Z_0$, by the definition of the set $V_1$, there exist $x \in X$ and $y \in \Phi(x_0, 0_Z) - L(x_0) + k_0 - \text{int } K$ such that $\Phi(x, z) - L(x) \in y - y_L$. Then, it follows that $\Phi(x, z) - L(x) \in \Phi(x_0, 0_Z) - L(x_0) + k_0 - \text{int } K - K = y_L - \text{int } K \subset y_L - K$, yielding $\Phi(x, z) - L(x) \leq_K y_L$. Consequently, (C1) holds.

(ε) Assume that (C6) holds. By Proposition 3.1, to prove (10), it is sufficient to show that $\text{epi } (\cdot, T)^* \subset \mathcal{M}$. The proof of the last inclusion goes along the line as that of Theorem 3.1.

Firstly, take $(\bar{L}, \bar{y}) \in \text{epi } (\cdot, T)^*$. Then, (13) holds by (2). Let us set

$$\Delta'_L := \bigcup_{(x, z) \in \text{dom } \Phi} \left( \left( \bar{L}(x) - \Phi(x, z) - K \right) \times (z - \pi(\text{dom } \Phi)) \right).$$

As $\Phi$ is $K$-convex, it is easy to check that $\Delta'_L \subset Y \times Z$ is convex. Moreover, as the sets dom $\Phi$, int $K$ and int $\pi(\text{dom } \Phi)$ are nonempty, int $\Delta'_L$ is nonempty as well.

We show that $(\bar{y}, 0_Z) \notin \text{int } \Delta'_L$. Indeed, if $(\bar{y}, 0_Z) \in \text{int } \Delta'_L$ then, by the same argument as in the proof of Theorem 3.1, there exists $k \in \text{int } K$ such that $(\bar{y} + k, 0_Z) \in \Delta'_L$. Hence, there is $(\bar{x}, \bar{z}) \in \text{dom } \Phi$ such that $\bar{y} + k \in \bar{L}(\bar{x}) - \Phi(\bar{x}, \bar{z}) - K$ and $0_Z \leq \bar{z} - \pi(\text{dom } \Phi)$. Taking (C6) into account, one gets $\Phi(\bar{x}, 0_Z) \leq_K \Phi(\bar{x}, \bar{z})$. Consequently, $\bar{y} + k \in \bar{L}(\bar{x}) - \Phi(\bar{x}, 0_Z) - K$, yielding $\bar{y} \in \bar{L}(\bar{x}) - \Phi(\bar{x}, 0_Z) - \text{int } K$, which contradicts (13).

By the separation theorem ([30, Theorem 3.4]) applying to the point $(v, 0_Z)$ and the convex set $\text{int } \Delta'_L$ in $Y \times Z$, there is $(y_0^*, z_0^*) \in Y^* \times Z^* \setminus \{(0y^*, 0z^*)\}$ such that

$$y_0^*(v) < y_0^*(y) + z_0^*(z), \quad \forall (y, z) \in \text{int } \Delta'_L. \quad (35)$$

We now show that

$$y_0^*(k') < 0, \quad \forall k' \in \text{int } K. \quad (36)$$

Take $k' \in \text{int } K$. On the one hand, it follows from the last part of (C6) that there exists $(\hat{x}, \hat{z}) \in \text{dom } \Phi$ satisfying $0_Z \leq \hat{z} - \pi(\text{dom } \Phi)$. On the other hand, according to [11, Lemma 2.1 (i)], there exists $\mu > 0$ such that $v - \mu k' \in \hat{L}(\hat{x}) - \Phi(\hat{x}, \hat{z}) - \text{int } K$. 

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So, the set
\[ W := \left( \bar{L}(\bar{x}) - \Phi(\bar{x}, \bar{z}) - \text{int } K \right) \times \left( \bar{z} - \text{int } \pi(\text{dom } \Phi) \right) \]
is a neighborhood of \((v - \mu k', 0_Z)\). It is easy to see that \(W \subset J'_{\bar{L}}\), yielding \((v - \mu k', 0_Z) \in \text{int } J'_{\bar{L}}\). Applying (35), one has \(y_0^*(v) < y_0^*(v - \mu k') + z_0^*(0, Z)\), which yields (36).

Now, take \(k_0 \in \text{int } K\) such that \(y_0^*(k_0) = -1\) and define \(T \in L(Z, Y)\) by \(T(z) = -z_0^*(z)k_0\) for any \(z \in Z\). Using the same argument as in the proof of Theorem 3.1 (see (18)), we can show that
\[ \bar{y} \notin y + T(z) - \text{int } K, \quad \forall (y, z) \in J'_{\bar{L}} \]
which, together the fact that \((\bar{L}(x) - \Phi(x, z), z) \in J'_{\bar{L}}\) for all \((x, z) \in \text{dom } \Phi\), yields
\[ \bar{y} \notin \bar{L}(x) - \Phi(x, z) + T(z) - \text{int } K, \quad \forall (x, z) \in \text{dom } \Phi. \]

Again, (2) (applying to the mapping \(\Phi\)) yields \((\bar{L}, T, \bar{y}) \in \text{epi } \Phi^*, \text{ or equivalently, } (\bar{L}, \bar{y}) \in \text{epi } \Phi^*(\cdot, T)\). The inclusion \(\Phi(\cdot, T)^* \subset M\) has been proved, and so (10) holds.

The proof of (11) (under extra assumption \((C_0)\)) goes similarly as the last part the proof of Theorem 3.1.

\[ \square \]

**Appendix C Proof of Theorem 3.3**

Take \(k_0 \in \text{int } K\), \((\bar{L}, \bar{y}) \in \text{epi } \Phi(\cdot, T)^*\) and consider the set
\[ J''_{\bar{L}} := \bigcup_{(x, z) \in \text{dom } \Phi} \left[ \left( L(x) - \Phi(x, z) - K \right) \times (z - S) \right]. \]

By a similar argument as in “Appendix B” \((\epsilon)\), using \(J''_{\bar{L}}\) instead of \(J'_{\bar{L}}\), we can establish \(T \in L_\Phi\) such that \((\bar{L}, \bar{y}) \in \text{epi } \Phi^*(\cdot, T)\), and we get (10).

For the proof of (11), take \((\tilde{x}, 0_Z) \in \text{dom } \Phi\). For any \(s \in S\) and \(v > 0\), one has \(-vs \in -S = 0_Z - S\), and hence, \((\bar{L}(\tilde{x}) - \Phi(\tilde{x}), -vz) \in J''_{\bar{L}}\). The same argument as the last part of the proof of Theorem 3.1 leads to \(T \in L_+(S, K)\).

\[ \square \]

**Appendix D Proof of Lemma 5.1**

(i) Take \(L \in \mathcal{L}(X, Y)\) and \(T := (T_1, T_2) \in \mathcal{L}(W, Y) \times \mathcal{L}(Z, Y)\). By a detailed calculation, one has
\[ \Phi^*_1(L, T) = \text{WSup} \left[ (F + T_1 \circ H + T_2 \circ G + I_C)^*(L) + \kappa^*(T_1) + I^-_S(T_2) \right] \]
\[(F + T_1 \circ H + T_2 \circ G + I_C)^*(L) \cup \kappa^*(T_1) \cup I^*_S(T_2).\]

Assume further that \(T_2 \in \mathcal{L}^+(S, K).\) Then, \(I^*_S(T_2) = \text{WSup}[T_2(-S)] = \text{WSup}[0_Y] = -\text{bd} K\) (applying Proposition 2.1 (vii) to \(N = T_2(-S)\) and \(M = \{0_Y\}), and hence,

\[\Phi_1^*(L, T) = (F + T_1 \circ H_1 + T_2 \circ H_2 + I_C)^*(L) \cup \kappa^*(T_1) \cup (-\text{bd} K)\]

\[= (F + T_1 \circ H_1 + T_2 \circ H_2 + I_C)^*(L) \cup \kappa^*(T_1)\] (by Proposition 2.4(i)).

(ii) We prove the first equality, the proof of the second one is similar. Take \(T := (T_1, T_2) \in \mathcal{L}(W, Y) \times \mathcal{L}(Z, Y),\) then

\[(L, y) \in \text{epi} \Phi^*(., T) \iff y \in (F + T_1 \circ H + T_2 \circ G + I_C)^*(L) \cup \kappa^*(T_1) \cup I^*_S(T_2) + K\] (by 32)

\[\iff \exists U \in \mathcal{P}_p(Y) : y \in U \cup \kappa^*(T_1) \cup I^*_S(T_2), \text{ and } (L, U \cup \kappa^*(T_1) \cup I^*_S(T_2)) \]

\[\in (L, U) \boxplus (0_L, \kappa^*(T_1) \cup I^*_S(T_2)) \in \text{Epi}(F + T_1 \circ H + T_2 \circ G + I_C)^* \]

\[\boxplus (0_L, \kappa^*(T_1) \cup I^*_S(T_2))\]

\[\iff (L, y) \in \Psi\left(\text{Epi}(F + T_1 \circ H_1 + T_2 \circ H_2 + I_C)^* \boxplus (0_L, \kappa^*(T_1) \cup I^*_S(T_2))\right),\]

which means,

\[\text{epi} \Phi^*(., T) = \Psi\left(\text{Epi}(F + T_1 \circ H_1 + T_2 \circ H_2 + I_C)^* \boxplus (0_L, \kappa^*(T_1) \cup I^*_S(T_2))\right).\] (37)

If \(T \notin \text{dom} \kappa^* \times \mathcal{L}^w(S, K),\) then \(\kappa^*(T_1) = \{+\infty_Y\}\) or \(I^*_S(T_2) = \{+\infty_Y\}\) (see (3)), with (32), yields \(\Phi_1^*(L, T) = \{+\infty_Y\}, L \in \mathcal{L}(X, Y),\) So, \(\mathcal{L}_{\Phi_1} \subset \text{dom} \kappa^* \times \mathcal{L}^w(S, K),\) and (ii) follows (see (37), epi \(\Phi_1^*(., T) = \emptyset\) if \(T \notin \mathcal{L}_{\Phi_1}\)).

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