Three Dimensional de Sitter Gravity and the Correspondence

Bruno Carneiro da Cunha
Enrico Fermi Inst. and Dept. of Physics
The University of Chicago
5640 S. Ellis Ave., Chicago IL 60637, USA

Abstract

Certain aspects of three dimensional asymptotically de Sitter spaces are studied, with emphasis on the mapping between gravity observables and the representation of the conformal symmetry of the boundary. In particular, we show that non-real conformal weights for the boundary theory correspond to space-times that have non-zero angular momentum. Some miscellaneous results on the role of the holonomies and isometry groups are also presented.
1 Introduction

In the recent effort to understand quantum gravity in de Sitter (dS) spaces \[1, 2, 3, 4, 5, 6\], the role of asymptotic symmetries was particularly stressed. This is the group which naturally acts on the conserved charges in classical theories of gravitation \[7\] and then on states of the Hilbert states of the quantum theory. On the other hand, in cases where we perturb space-time away from some simple vacuum, – for instance, Minkowski, de Sitter or the usual anti-de Sitter (AdS) space-times – holography states that basically all the information about the whole of space-time is encoded in the conformal boundary. The asymptotic symmetry group would then take for quantum gravity a similar role as the Poincaré group has for the usual quantum field theory.

The particular case of dS contrasts sharply with AdS in that there is no model of a more fundamental description, no model derived from string theory\[4\]. On the other hand, dS not only presents itself as perhaps a more realistic background but also displays some puzzles that have to be addressed before a background independent formulation can even be dreamed of.

Among these is the finite observable entropy of dS spaces \[1, 8, 9, 10, 11\]. The puzzle relies on how a natural cut-off could be implemented that reduces the degrees of freedom so drastically to a finite number – and perhaps worse, a finite number large enough to be of any phenomenological interest. Another puzzle, perhaps related, is the apparent instability of dS with respect to quantum fluctuations \[12\]. For instance, dS can “decay” by a topology change process – hence differently from the usual Minkowski instability – into many copies of itself. These questions have just recently begun to be addressed, and they certainly possess some keys to the understanding of quantum gravity in general.

Noteworthy progress in this direction was recently achieved by Strominger \[2\], who proposed that the conjugacy between scalar fields in the bulk and boundary operators worked much the same way as in the AdS-CFT correspondence. Specifically, the conformal weight of operators in the boundary was given by the asymptotic form of the field profile as one approaches the conformal boundary – in the dS case, past and

\[\text{despite some recent efforts in this direction \[4\].} \]
future infinity. A curious bound appears: for fields whose mass was significantly larger than the de Sitter mass, the corresponding operator would have complex dimension. Although it poses no serious problem for the definition of the boundary theory, which is after all Euclidean, the question of what is the exact space-time interpretation of those operators still remains. The clarification of this particular question in three dimensions will be the aim of this short note.

The article is organized as follows: after a short introduction to the Chern-Simons description of three dimensional gravity, emphasizing the de Sitter case, we will present the general solution of three dimensional de Sitter gravity and show that it can be seen as an identification of global de Sitter under the action of a finite group. We will close by discussing how these results can be used to clarify some of the points raised above.

2 Chern-Simons and Asymptotic Data

In three dimensions, the spin connection $\omega_{ab}$ is dual to a vector field $\omega^a = \varepsilon^{abc} \omega_{ab}$. It was found in [15] that the Einstein-Hilbert Lagrangian in three dimensions – with a cosmological constant term $\Lambda$ – can be recast in terms of the vector-valued one forms $A^a_\pm = \omega^a \pm \sqrt{-\Lambda} e^a$, with $e^a$ being the triad. In the $\Lambda < 0$ case, both $A^a_\pm$ are real and independent. In the $\Lambda > 0$ case (which will be dwelled in from now on), they are the complex conjugate of each other, and we will drop the subscript and call $A_-$ by $A^*$. The action that results from the change of variables has the Chern-Simons form:

$$S = \frac{1}{8\pi G} \text{Im} \left[ \int \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \right]$$

(1)

where Im denotes the imaginary part. We will set $\Lambda = 1$ and $4G = 1$ from now on. The equations of motion then simply state that $A$ is a locally flat connection:

$$dA^a + f^a_{\ bc} A^c \wedge A^b = 0$$

(2)

with $f^a_{\ bc}$ being the structure constants of the $SL(2, C)$ group. Being locally flat, the connection itself doesn’t carry too much information about the space-time. In parallel to the AdS case, one has to provide some set of boundary conditions that introduce

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2In [10] or [13] the reader will find a much more intelligible discussion.
some notion of “asymptotic triviality” and still allow for non-trivial holonomies. From
the fact that dS$_3$ has two disconnected conformal boundaries at future and past
infinity $t = \pm \infty$, one is naturally led to the following boundary conditions:

$$ds^2\big|_{\mathcal{I}^-} = -dt^2 + e^{-2t}dzd\bar{z} + \ldots$$  \hspace{1cm} (3)

and

$$ds^2\big|_{\mathcal{I}^+} = -dt^2 + e^{2t}dzd\bar{z} + \ldots$$  \hspace{1cm} (4)

As we will see this turns out to be too restrictive. In a holographic context, one
argues that quantum gravity in dS consists of a transfer matrix which relates the
Hilbert spaces in the asymptotic past and future. By requiring the two conditions
above we are really determining a particular state in both Hilbert spaces, and hence
the whole matrix element. In other words, if we restrict ourselves to the gravity sector
alone, the only space satisfying both of the conditions above is global dS$_3$. We should
then drop at least one of the conditions. It became customary to drop (4), and then
the form of the connection can be derived much the same way as in the AdS case
\cite{16}:

$$A = -i \begin{pmatrix}
-\frac{1}{2}dt & e^{-t}dz \\
\mathcal{O}(e^{t}) & \frac{1}{2}dt
\end{pmatrix}.$$  \hspace{1cm} (5)

where we represented $SL(2, C)$ with Pauli matrices. Following the argumentation of
section 4 in \cite{13} one can see that the general solution of the equations of motion with
the boundary conditions above is, up to a diffeomorphism:

$$ds^2 = -dt^2 + (e^{-2t} + L(z)\tilde{L}(\bar{z})e^{2t})dzd\bar{z} + L(z)dz^2 + \tilde{L}(\bar{z})d\bar{z}^2$$  \hspace{1cm} (6)

with $L(z)$ an arbitrary holomorphic function and $\tilde{L}(\bar{z}) = [L(z)]^*$. The actual proof
is entirely analogous to the AdS case (actually for AdS with Euclidean time, which
is what is done in \cite{13}) and will not be shown here. The metric (6) gives rise to the
potential:

$$A = -i \begin{pmatrix}
-\frac{1}{2}dt & e^{-t}dz \\
e^{t}L(z)dz & \frac{1}{2}dt
\end{pmatrix}.$$  \hspace{1cm} (7)

\footnote{The calculation was indeed carried out for dS$_3$ recently \cite{17}.}
It is perhaps worth noting that the analogue of the Brown-Henneaux symmetry is particularly clear in these coordinates, and not particularly surprisingly so. Setting $\bar{L}(\bar{z}) = 0$ for the sake of clarity, one sees that the transformation:

$$
\delta t = \frac{1}{2} \partial \epsilon, \quad \delta z = \epsilon, \quad \delta \bar{z} = \frac{1}{2} e^{2\epsilon} \partial^2 \epsilon
$$

(8)
is a symmetry if we allow $L(z)$ to transform as:

$$
\delta L = - (\epsilon \partial L + 2 \partial \epsilon L + \frac{1}{2} \partial^3 \epsilon).
$$

(9)

From which we can rederive the central charge [2, 18, 19] by recovering the dependence on $l$ and $G$. The point of view naturally taken is this case is that the transformation (8) is a diffeomorphism that changes the asymptotic charges and then is not a gauge transformation but rather a global transformation.

So now we turn to the problem of relating the function $L(z)$ to physical quantities of the space-time metric such as mass and angular momentum.

### 3 Cartography

We will now turn to the relation of the metric (6) with the three-dimensional Kerr-de Sitter solution [11]:

$$
ds^2 = - \left( M - r^2 + \frac{J^2}{4r^2} \right) dt^2 + \frac{dr^2}{\left( M - r^2 + \frac{J^2}{4r^2} \right)} + r^2 \left( d\varphi - \frac{J}{2r^2} dt \right)^2.
$$

(10)

For that consider (6) with constant $L(z) = L_0$. The mapping

$$
t' = t - \frac{1}{4} \ln(L_0 \bar{L}_0), \quad x = \sqrt{L_0 z} + \sqrt{\bar{L}_0 \bar{z}}, \quad iy = \sqrt{L_0 z} - \sqrt{\bar{L}_0 \bar{z}}
$$

(11)

turns the metric into

$$
ds^2 = - dt'^2 + \cosh^2 t' dx^2 + \sinh^2 t' dy^2.
$$

(12)
The Kerr-de Sitter metric can also be turned into (12) by the following change of coordinates:

$$
sinh t' = \sqrt{\frac{r^2 - r_+^2}{r_+^2 + r_-^2}}, \quad y = r_+ \bar{t} + r_- \varphi, \quad x = r_- \bar{t} - r_+ \varphi.
$$

(13)
where $M = r_+^2 - r_-^2$ and $J = 2r_+r_-$. Note that only the region $r > r_+$, $t' > 0$, and thus lying outside the cosmological horizon can be mapped. In this coordinate change we find

$$L_0 = (r_+ - ir_-)^2 = M - iJ \equiv \lambda^2$$

(14)

and also that $r_- y - r_+ x$ is identified with $2\pi(r_+^2 + r_-^2)$ translations. For (6) that means $z + \bar{z} = \varphi \in [0, 2\pi]$. Note that $L_0$ is complex for non-zero $J$. Spatial slices have then the topology of a cylinder, whose non-contractible loop is indeed necessary for the existence of a non-trivial holonomy for $A$. Also, it fits in the usual description of mass sources as conical defects in global dS where the spatial slices are spheres.

It is also interesting to study the global structure of the metric (6), or, alternatively, (12). We will restrict ourselves to the case of zero angular momentum, in which $x$ is an angular variable with a conical defect $x \in [0, 2\pi \lambda]$ but $y$ is not restricted. By defining:

$$\tan \hat{t} = \sinh t' \cosh y, \quad \cos \theta = \frac{\sinh t' \tanh y}{\sqrt{\cosh^2 t' - \tanh^2 y}}, \quad \phi = x,$$

(15)

one turns (12) into the familiar form of global dS [20]:

$$ds^2 = \sec^2 \hat{t}(-dt'^2 + d\theta^2 + \sin^2 \theta d\phi^2)$$

(16)

The patch covered by $\{t', x, y\}$ can be seen in Figure 1. Note that it covers both $I^+$ and $I^-$ which also happen to be the holographic screens of dS.

It may not be clear that the null boundary of the hourglass is at $y \to \infty$. To see that one just needs to consider the inverse transformation:

$$\tanh y = \frac{\cos \theta}{\sin \hat{t}}$$

(17)

which defines the boundary for $\sin \hat{t} = \pm \cos \theta$. Also, the point at $\hat{t} = 0$ is merely a coordinate singularity, just like in the AdS version of (6).

We now turn to the study of the observables and the identifications that introduce non-trivial holonomies from global dS$_3$.

### 4 Holonomies and Identifications

Since $A$ is a flat connection, it can be written locally by $A = i\mathcal{G}^{-1} d\mathcal{G}$, where $\mathcal{G}$ is an element of $SL(2, C)$. For $A$ as in (6), $\mathcal{G}$ can be written as:
Figure 1: The patch of de Sitter space covered by the coordinates (6) for real $L_0$ is the central hourglass-like region. The horizontal lines and vertical lines correspond to $t'$ and $y$ being constant respectively. The null boundary is at $y = \pm \infty$ in (12).

\[ G = \begin{pmatrix} \Psi_1'(z) & \Psi_1(z) \\ \Psi_2'(z) & \Psi_2(z) \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \] (18)

with $\Psi_{1,2}(z)$ the solutions of the Schrödinger equation $(\partial_z^2 - L(z))\Psi(z) = 0$, set to have the Wronskian equal to one.

For $L(z) = L_0 = \lambda^2$ constant it is useful to define a $2 \times 2$ coefficient matrix $B$ as:

\[ g(z) = \begin{pmatrix} \Psi_1'(z) & \Psi_1(z) \\ \Psi_2'(z) & \Psi_2(z) \end{pmatrix} \equiv B \begin{pmatrix} \lambda e^{\lambda z} & e^{\lambda z} \\ -\lambda e^{-\lambda z} & e^{-\lambda z} \end{pmatrix} \] (19)

Note that the actual elements of $B$ are not so important, since they can be gauged away by a global $SL(2, C)$ transformation, but we have to require $\det B = \frac{1}{2\lambda}$. Using the fact that the $z$ plane has the topology of a cylinder, the monodromy can be easily computed:

\[ g(z + \pi) = Be^{2\pi \lambda \sigma_3} B^{-1} g(z), \] (20)

and then the holonomy is $W = 2 \cosh(\pi \lambda)$.

Finally, we will show that (11) can be obtained from global dS$_3$ via a proper identification. One way of accomplish that is by embedding the hourglass region (12) in
$SL(2,\mathbb{C})$:  
\[ g = e^{i(x+iy)\sigma_2} e^{2i\sigma_3} e^{i(x-iy)\sigma_2} \]  
and the action of the $SL(2,\mathbb{C})$ group is given by $g \rightarrow hgh^*$. The particular group element that implements the isometry that brings $x + iy$ to $x + iy + 2\pi \lambda$ is  
\[ h = e^{2\pi i \sigma_2 \lambda}, \]  
and then the space (8) can be thought of as the quotient of global $dS_3$ under the action of the group generated by $h$.

5 Discussion

In this short note we have collected some results about asymptotic $dS_3$ spaces which may help clarify the role of holography in the quantum theory with that background. Specifically, we have mapped the conformal weights of operators of the boundary theory into space-time quantities like the mass and the angular momentum of the space. It was shown that the weights are complex in general, in accordance with the results of [2].

Given the form of the general metric solution (10), one might be tempted to work in cosmological patch where time is a Killing vector field. The point of view taken here is different in which we stress the role of the asymptotic symmetries. This is in consonance with holographic arguments which infer that the states of quantum gravity in de Sitter can be encoded in the conformal boundary of space-time. It is particularly clear in this language the space-time role of the Virasoro generators of the boundary conformal field theory.

Of course, there are a few caveats. It is unclear how exactly the description given in terms of the asymptotic data relates to what a given observer sees. Recovering the observer’s point of view may be more than a technicality given that Susskind et al. argued on generic grounds [21] that a quantum description of gravity based on local degrees of freedom cannot see beyond a horizon. Being more specific, the local formulations of observables for each side of the horizon are not compatible with one another. Also, one would like to verify which mechanism in the boundary conformal theory is responsible for the finiteness of the $dS$ states. One expects that
the correlation between different states in the boundary theory to be large, thus reducing its entropy. Note that here the same scenario envisioned by [22] happens: there are patches of space-time which have alternative projections on either past or future infinity screens. Following that argument, the language best suited for coping with these issues, especially if one’s goal is to describe local states in the bulk of space, may be that of a correlated or entangled state (see also [23].)

As this work was in its final states of preparation, we learned of a similar work [24] which has overlaps with the results listed here. After this work was made publicly available we became aware of the work by Park [25] in which he also derives the space-time interpretation of the asymptotic conformal theory. Unlike that work, we argue, based on the causal structure of de Sitter, that the boundary conformal theory is defined on a pair of two-spheres instead of a cylinder. We also gave a detailed account on how the Virasoro symmetry acts on the space-time metric, and discussed the role of identifications and holonomies. We have also found a real central charge for the Virasoro symmetry, corroborating the results of [4, 5, 14, 18, 19, 24].

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