Finite-time trajectory control for a quadrotor aircraft using disturbance observer

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Abstract
In this article, we investigate the problem of finite-time trajectory tracking control for a quadrotor aircraft with unknown external disturbances. To improve convergence rate and disturbance rejection performance, a new composite controller is proposed by integrating finite-time control design and disturbance estimation attenuation technique. Explicit Lyapunov function is given to ensure the finite-time stability of the closed-loop control system. Numerical simulations also show the effectiveness of the proposed method.

Keywords
Finite-time control, quadrotor aircraft, trajectory tracking control, external disturbances, disturbance observer

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Introduction
In recent years, the small unmanned aerial vehicle (UAV) is drawing an increasing interest due to the large demand in the civil and military fields.\(^1\)\(^-\)\(^3\) Quadrotor aircraft is one of the most important vehicles in the UAVs for its simple structure, low cost, and the feature of vertical takeoff and landing.\(^4\)\(^,\)\(^5\) Because of the powerful thrust generated by the four rotors, quadrotor aircraft also performs stronger maneuver ability to achieve various flight tasks through individual vehicle and multiple vehicles team, for example, air surveillance, search, rescue, sensor networks, delivery of small goods, and pesticide spraying.\(^6\)\(^-\)\(^8\) From the viewpoint of system control, quadrotor aircraft is a typical underactuated system which has nonlinear, coupling, multiple input–multiple output, how to design controller and analysis stability still is a challenging problem.

In the literature, various effective nonlinear control methods have been adopted to develop high-performance controller to meet the increasing demands on the quadrotor aircraft, such as backstepping control,\(^9\) adaptive control,\(^10\)\(^,\)\(^11\) sliding mode control,\(^12\)\(^-\)\(^14\) robust control,\(^15\)\(^,\)\(^16\) and fuzzy control.\(^17\) In these works, the double-loop control strategy is a common approach to deal with the complex dynamics of quadrotor aircraft, that is to say, the system model of quadrotor aircraft is divided into two control loop, the outer-loop is the position subsystem and the inter-loop is the attitude subsystem. To ensure the feasibility of the double-loop control, an essential requirement is that the convergence rate of the interloop is faster than the outer-loop. Therefore, the problem of the attitude control has also been studied by a large number of researchers, some attitude control algorithms were presented in the literature.\(^18\)\(^-\)\(^22\) However, most of the existing control algorithms belong to the field of the smooth

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control, which only ensures that the closed-loop system is asymptotically stable, that is, the settling time of system states that converge to equilibrium point is infinite time. To improve convergence rate, a novel nonlinear control method named finite-time control has been developed in the study by Bhat and Bernstein and attracted the attention of many researchers. The main feature of finite-time control lies in that the system state converges to the equilibrium point in finite time. Based on this, the problem of finite-time hovering control for a quadrotor aircraft was studied by Zhu et al. The attitude control algorithm was given using finite-time control technology. Besides, the external disturbance is inevitable in the flight process of a quadrotor aircraft, gust wind, and so on. Through theoretical analysis and experimental verification, it is shown that finite-time control has better anti-disturbance ability.

The finite-time control method with the above advantages is suitable for the control of quadrotor aircraft. A finite-time controller with disturbance observer is designed for a quadrotor aircraft system without external disturbance. Different from most existing results, this article studies the problem of finite-time control for quadrotor aircraft with external disturbances. The main difficulty is to achieve finite-time stabilization, meanwhile the external disturbance is estimated and feedforward compensated. Based on homogeneous systems theory, the finite-time position and attitude controllers are presented to improve the system performance, and it can also simplify the design process of the controller. Meanwhile, considering the external disturbance in position subsystem and attitude subsystem, two finite-time disturbance observers are designed, then the external disturbance can be estimated in a finite time. Finally, by combing the finite-time disturbance observers, a finite-time control strategy with disturbance observer is proposed, whose validity is proved through Lyapunov theory and numerical simulations.

**Preliminaries**

**Basic description of quadrotor system**

As shown in the literature, the construction of quadrotor aircraft system is given in Figure 1, which consists of a frame with cross structure, four propellers, and four motors. In general, two transformation coordinates are introduced to describe the motion of quadrotor aircraft in three-dimensional space. One is the inertial coordinate \(C^i_n = \{x_n, y_n, z_n\}\) which is determined by the orientation of the earth, the other one is the body coordinate \(C^b_b = \{x_b, y_b, z_b\}\) which is fixed with the frame of quadrotor aircraft. As shown in Figure 1, the full information for the motion of quadrotor aircraft is described by \(\chi = (x, y, z)^T\) in \(R^3\) denoted in the inertial coordinate system, the attitude information is represented by the three Euler angles \(\Phi = (\phi, \theta, \psi)^T\) in \(R^3\) with respect to the inertial coordinate. The lift forces are denoted as \(F_i, i = 1, 2, 3, 4\) and the counter torques are represented by \(Q_i, i = 1, 2, 3, 4\), which are generated by four rotating propellers. For the quadrotor aircraft, the total thrust produced by four propellers is defined as \(T = F_1 + F_2 + F_3 + F_4\).
Dynamics model of quadrotor aircraft

Based on the difference in position control and attitude control, the dynamics model of quadrotor aircraft is viewed as two control subsystems, one is the position dynamical model and the other is the attitude dynamical model.

Position part. According to the literature, the model for the position part of quadrotor aircraft is obtained as follows

\[ \ddot{x} = - \frac{K_1}{m} \dot{x} + \frac{T}{m} (\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi) + \Delta_1(t) \]
\[ \ddot{y} = - \frac{K_2}{m} \dot{y} + \frac{T}{m} (\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi) + \Delta_2(t) \]
\[ \ddot{z} = - \frac{K_3}{m} \dot{z} + \frac{T}{m} \cos \phi \cos \theta - g + \Delta_3(t) \]  

(1)

where \( T \in R \) is the total thrust generated by the four rotors, \( m \in R \) denotes the mass of the quadrotor aircraft, \( g \in R \) represents the acceleration of gravity which is a positive constant, \( K_i \in R \) and \( \Delta_i(t) \in R \), \( i = 1, 2, 3 \) represent the aerodynamic damping coefficient and the unknown external disturbance, respectively.

Attitude part. From the literature, it follows from the description method of Euler angle that the model for the attitude part of quadrotor aircraft is given by

\[ J_1 \dot{\phi} = -K_4 \dot{\phi} + l \tau_1 + d_1(t) \]
\[ J_2 \dot{\theta} = -K_5 \dot{\theta} + l \tau_2 + d_2(t) \]
\[ J_3 \dot{\psi} = -K_6 \dot{\psi} + c \tau_3 + d_3(t) \]  

(2)

where \( J_i \in R \), \( i = 1, 2, 3 \), represent the moment of inertia, \( \tau_i \), \( i = 1, 2, 3 \), denote three attitude torques generated by the four motors, \( K_i \in R \), \( i = 4, 5, 6 \), denote the damping coefficients in aerodynamics, \( c \in R \) and \( l \in R \) are the related scale factors which describe the relationship between the attitude torque with force generated by rotors, and \( d_i(t) \in R \), \( i = 1, 2, 3 \), are the unknown external disturbance in attitude control channel, such as wind and friction force.

Remark 1. Note that in the quadrotor aircraft model (1)–(2), the control inputs are the total thrust \( T \) and the attitude control torque \( \tau = (\tau_1, \tau_2, \tau_3) \). However, in practice, the directly/real control inputs for the quadrotor aircraft are the four motors’ speed. There is a transform relationship between motors’ speed and motors’ torque which can be found in the study by Abdelhamid and Stephen. In this article, for the sake of statement, only the control inputs with \((T, \tau_1, \tau_2, \tau_3)\) are considered.

Control objective

For the quadrotor aircraft system in this article, the control objective is to develop a finite-time trajectory tracking control law \( \tau \) so that the position of quadrotor aircraft can track the reference position trajectory in finite time even if there exist the time-varying external disturbance \( \Delta_i(t) \), \( d_i(t) \in R \), \( i = 1, 2, 3 \). To this end, some assumptions are given as follows.

Assumption 1. Assumed that the reference position trajectory is denoted as \( x_d = (x_d, y_d, z_d) \), its derivatives \( \dot{x}_d \) and \( \ddot{x}_d \) are continuous and bounded signals.

Assumption 2. Assumed that the external disturbances \( \Delta_i(t), d_i(t), i = 1, 2, 3 \) and its derivatives \( \Delta_i, d_i(t), i = 1, 2, 3 \) are bounded.

Relevant definitions and lemmas

Definition 1. Consider the following nonlinear system

\[ \dot{x} = f(x), \quad f(0) = 0, \quad x \in R^n \]  

(3)

where \( f(\cdot) \) is a continuous function refers to variable \( x \). If the system is Lyapunov stable and the system state can converge to the equilibrium point in a finite time, it is also called finite-time stable. The finite-time convergence is defined as that there is a finite time \( T(x_0) \) such that \( \lim_{t \rightarrow T(x_0)} x(t, x_0) = 0 \) and \( x(t, x_0) = 0 \) for all \( t \geq T(x_0) \).

Definition 2. A class nonlinear function is defined as \( \text{sig}^\alpha(x) = \text{sign}(x)[\text{sign}(x)]^\alpha, \quad \alpha \geq 0, \quad x \in R \), where \( \text{sign}(\cdot) \) is a signal function.

Definition 3. For the system (3) and any \( \varepsilon > 0 \), if there exists \( (r_1, \ldots, r_n) \) with \( r_i > 0, i = 1, \ldots, n \), such that

\[ f_i(\varepsilon x_1, \ldots, \varepsilon x_n) = \varepsilon^{k+r_i} f_i(x), \quad i = 1, \ldots, n, \]  

where \( k > -\min\{r_i, i = 1, \ldots, n\} \), then \( f(x) \) is a homogeneous vector field whose degree is \( k \) refers to the dilation \( (r_1, \ldots, r_n) \).

Lemma 1. For the following system:

\[ \dot{x} = f(x) + \dot{f}(x), f(0) = 0, \dot{f}(0) = 0, \quad x \in R^n \]  

(4)

where \( f(x) \) is a continuous homogeneous vector field and its degree \( k < 0 \) refers to the dilation \( (r_1, \ldots, r_n) \) and \( f(x) \) is a continuous higher order term with respect to the nominal part \( f(x) \). If the system \( \dot{x} = f(x) \) is asymptotically stable and the equilibrium point is \( x = 0 \) and for all \( \forall x \neq 0 \), \( \lim_{t \rightarrow -\infty} \frac{\int_{-\infty}^{t} f(x_1(x_0, x, \varepsilon x_0))}{x_1} = 0 \), \( i = 1, 2, \ldots, n \), then system (4) is local finite-time stable. Besides, it can be concluded that the system (4) is global finite-time stable when it is local finite-time stable and globally asymptotically stable.

Main results

The main idea of the controller design is based on backstepping approach. To be specific, the detailed procedure can be viewed as two steps.

- Step 1: For the position control subsystem, by treating the attitude \( \Phi = (\phi, \theta, \psi) \) as the known information, the nonlinear dynamics involving attitude is viewed as virtual controller to guarantee that the
quadrotor position can track the reference position in a finite time.

- Step 2: For the attitude control subsystem, the desired attitude is generated by previous position controller, then a tracking control law \( \tau_i \) is constructed to ensure that the desired attitude can be tracked by the real attitude in a finite time.

Based on the separation principle, the finite-time observer is designed independently to compensate the external disturbances in the position subsystem or attitude subsystem.

**Design of position controller**

For the sake of simple expression, define

\[
\begin{align*}
      u_x &= \frac{T}{m} (\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi) \\
      u_y &= \frac{T}{m} (\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi) \\
      u_z &= \frac{T}{m} \cos \phi \cos \theta - g
\end{align*}
\]

under which the position equation of system (1) is rewritten as

\[
\begin{align*}
      \dot{x} &= -\frac{K_1}{m} \dot{x} + u_x + \Delta_1(t) \\
      \dot{y} &= -\frac{K_2}{m} \dot{y} + u_y + \Delta_2(t) \\
      \dot{z} &= -\frac{K_3}{m} \dot{z} + u_z + \Delta_3(t)
\end{align*}
\]

**Lemma 2.** For quadrotor’s position model (6) without the external disturbance, that is, \( \Delta_i(t) = 0, i = 1, 2, 3 \), if the controller is designed as

\[
\begin{align*}
      u_x &= \ddot{x}_d + \frac{K_1}{m} \dot{x}_d + k_p \text{sgn}^\alpha_1 (x_d - x) + k_d \text{sgn}^\alpha_2 (\ddot{x}_d - \dot{x}) \\
      u_y &= \ddot{y}_d + \frac{K_2}{m} \dot{y}_d + k_p \text{sgn}^\alpha_1 (y_d - y) + k_d \text{sgn}^\alpha_2 (\ddot{y}_d - \dot{y}) \\
      u_z &= \ddot{z}_d + \frac{K_3}{m} \dot{z}_d + k_p \text{sgn}^\alpha_1 (z_d - z) + k_d \text{sgn}^\alpha_2 (\ddot{z}_d - \dot{z})
\end{align*}
\]

where \( 0 < \alpha_1 < 1, \alpha_2 = 2\alpha_1/(1 + \alpha_1), k_p > 0, k_d > 0 \), then the reference position trajectory can be tracked in a finite time, that is, \((x, y, z)^T \to (x_d, y_d, z_d)^T\) in a finite time.

**Proof.** Because of the stability proofs for the three position components are similar, without loss of generality, we will only give the proof about the \( x \)-axis.

Let \( e_x = x_d - x \) be about the position tracking error of the \( x \)-axis. Under the proposed controller (7), it follows from equation (6) that the error equation of position tracking is

\[
\begin{align*}
      \dot{e}_x &= \ddot{x}_d - \dot{x} \\
      \dot{e}_x &= -\frac{K_1}{m} e_x - k_p \text{sgn}^\alpha_1 (e_x) - k_d \text{sgn}^\alpha_2 (\dot{e}_x)
\end{align*}
\]

In the sequel, it is proved that the closed-loop system (8) is globally finite-time stable. The entire process is based on Lemma 1, which is divided into two parts, that is, global asymptotical stability and local finite-time stability.

**Step 1: Proof of global asymptotical stability.** In the first step, it will be proved that system (8) is globally asymptotically stable. We can choose the following Lyapunov function

\[
V = k_p \int_0^\varepsilon \text{sgn}^\alpha_1 (\rho) \, d\rho + \frac{1}{2} \dot{e}_x^2
\]

whose derivative regarding for system (8) is

\[
\begin{align*}
V|_8 &= k_p \text{sgn}^\alpha_2 (e_x) \dot{e}_x + \dot{e}_x \ddot{e}_x \\
&= -\frac{K_1}{m} \dot{e}_x^2 - k_d |\dot{e}_x|^{\alpha_2+1} \leq 0
\end{align*}
\]

Clearly, it can be concluded from equation (10) that the error and its derivative always are zero when the derivative of Lyapunov function equals to zero, that is, \( \dot{e}_x \equiv \ddot{e}_x \equiv 0 \) when \( V \equiv 0 \). That means that the elements in the invariant set \( \mathcal{V} = \{ (e_x, \dot{e}_x)| V \equiv 0 \} \) are \( \dot{e}_x \equiv 0 \) and \( \ddot{e}_x \equiv 0 \). Thus, according to LaSalle’s invariant principle, the system (8) is globally asymptotically stable.

**Step 2: Proof of local finite-time stability.** In this step, it is proved that the system (8) is also locally finite-time stable based on homogeneous systems theory. The system (8) can be decomposed into the following form

\[
\begin{align*}
      \dot{e}_x &= \ddot{e}_x - \dot{x} \\
      \dot{e}_x &= -k_p \text{sgn}^\alpha_1 (e_x) - k_d \text{sgn}^\alpha_2 (\dot{e}_x) + h(\dot{e}_x)
\end{align*}
\]

with higher order term \( h(\dot{e}_x) = -\frac{k_d}{k_p} \dot{e}_x \). For the part of equation (11) without higher order term

\[
\dot{e}_x = \ddot{e}_x - \dot{x}
\]

by similar proof in step 1, Lyapunov function is chosen as

\[
U = k_p \int_0^\varepsilon \text{sgn}^\alpha_1 (\rho) \, d\rho + \frac{1}{2} \dot{e}_x^2 , \quad \text{we have}
\]

\[
\dot{U}|_{12} = -k_d |\dot{e}_x|^{\alpha_2+1} \leq 0
\]

According to LaSalle’s invariant principle, the system (12) is also globally asymptotically stable. Meanwhile, it follows from Definition 3 that the system (12) is homogeneous vector field whose degree is \( k = (\alpha_1 - 1)/2 < 0 \) refers to the dilation \((r_1, r_2) = (1, (\alpha_1 + 1)/2)\).
Note that the higher order term \( h(\dot{\varepsilon}_s) = -\frac{K_s}{m} \dot{\varepsilon}_s \) satisfies, for any \( \dot{\varepsilon}_s \neq 0 \)
\[
\lim_{\varepsilon \to 0} h(\varepsilon^T \dot{\varepsilon}_s) = \lim_{\varepsilon \to 0} \left( -\frac{K_s}{m} \varepsilon^T \dot{\varepsilon}_s \right) = 0
\] (14)

Using Lemma 1, it can be found that system (11) is locally finite-time stable. Actually, the system (8) is globally finite-time stable by combining the globally asymptotically stable in step 1 with the local finite-time stability in step 2.

In practice, the influence of external disturbance on quadrotor aircraft cannot be ignored. For the convenience of explanation, the following assumption is given according to Assumption 2.

**Assumption 3.** Supposed that there are two known positive constants \( M_1 \) and \( M_2 \), so that \( |\Delta(t)| \leq M_1 \), \( |\Delta_i(t)| \leq M_2 \), \( i = 1, 2, 3 \), are satisfied.

**Lemma 3.** For the system (6) under Assumption 3, the disturbance observers are designed as
\[
\begin{align*}
\frac{\partial \dot{x}_1}{\partial t} &= -\frac{K_1}{m} \dot{\varepsilon}_1 + u_x + \dot{\Delta}_1 + \lambda_1 \text{sign}(\dot{\varepsilon}_1 - \dot{x}_1) \\
\dot{\Delta}_1 &= \lambda_2 \text{sign}(\dot{\varepsilon}_1 - \dot{x}_1) \\
\frac{\partial \dot{y}_2}{\partial t} &= -\frac{K_2}{m} \dot{\varepsilon}_2 + u_y + \dot{\Delta}_2 + \lambda_1 \text{sign}(\dot{\varepsilon}_2 - \dot{y}_2) \\
\dot{\Delta}_2 &= \lambda_2 \text{sign}(\dot{\varepsilon}_2 - \dot{y}_2) \\
\frac{\partial \dot{z}_3}{\partial t} &= -\frac{K_3}{m} \dot{\varepsilon}_3 + u_z + \dot{\Delta}_3 + \lambda_1 \text{sign}(\dot{\varepsilon}_3 - \dot{z}_3) \\
\dot{\Delta}_3 &= \lambda_2 \text{sign}(\dot{\varepsilon}_3 - \dot{z}_3)
\end{align*}
\] (15)

then the estimation disturbances \( (\dot{\Delta}_1, \dot{\Delta}_2, \dot{\Delta}_3) \) will converge to the real disturbances \( (\Delta_1, \Delta_2, \Delta_3) \) in a finite time, where the selection of observer gains satisfies
\[
\lambda_2 > M_2, \quad \lambda_1 > \sqrt{\frac{2}{\lambda_2 - M_2} \cdot \left( \lambda_2 + M_2 \right)^2 (1 + q) - \frac{1}{1 - q}}
\] (16)

with \( 0 < q < 1 \).

**Proof.** Similarly, without loss of generality, only the proof about the x-axis is given here. Define the error as \( \delta_1 = \dot{x} - \dot{x} \), \( \delta_2 = \Delta_1 - \dot{\Delta}_1 \), under which the error equation of disturbance observer can be obtained from equations (6) and (15) that
\[
\begin{align*}
\dot{\delta}_1 &= \delta_2 - \lambda_1 \text{sign}(\delta_1) + \Delta_1 \\
\dot{\delta}_2 &= -\lambda_2 \text{sign}(\delta_1) + \dot{\Delta}_1
\end{align*}
\] (17)

By Assumption 3, we have \( -M_2 < \dot{\Delta}_1 < M_2 \). As a consequence, it can be concluded from the study by Jorge et al.\textsuperscript{36} that system (17) is finite-time stable under the condition (16), which means that the observation \( \dot{\Delta}_1 \) can converge to the disturbance \( \Delta_1 \) in a finite time. The proof is completed.

Based on this, a finite-time controller with the compensation of external disturbance is given to achieve the position tracking control of quadrotor aircraft.

**Theorem 1.** For the quadrotor’s position model (6), if the control law is designed as
\[
\begin{align*}
\dot{u}_x &= \ddot{x} + \frac{K_1}{m} \dot{x} + k_p \text{sign}(\dot{x}_d - x) + k_d \text{sign}(\dot{x}_d - \dot{x}) - \dot{\Delta}_1 \\
\dot{u}_y &= \ddot{y} + \frac{K_2}{m} \dot{y} + k_p \text{sign}(\dot{y}_d - y) + k_d \text{sign}(\dot{y}_d - \dot{y}) - \dot{\Delta}_2 \\
\dot{u}_z &= \ddot{z} + \frac{K_3}{m} \dot{z} + k_p \text{sign}(\dot{z}_d - z) + k_d \text{sign}(\dot{z}_d - \dot{z}) - \dot{\Delta}_3
\end{align*}
\] (18)

where \( 0 < \alpha_1 < 1, \alpha_2 = 2\alpha_1/(1 + \alpha_1), \quad k_p > 0, k_d > 0 \), then the reference position trajectory can be tracked in a finite time, that is, \( (x, y, z)^T \rightarrow (x_d, y_d, z_d)^T \) in a finite time.

**Proof:** Because the disturbance observer (15) is finite-time stable, which means that there exists a finite time \( T_o \), such that the observation equals to the real disturbance, that is, \( \dot{\Delta}_1 = \Delta_1, \dot{\Delta}_2 = \Delta_2, \dot{\Delta}_3 = \Delta_3, \forall t > T_o \). After the time \( T_o \), the closed-loop control system (6)–(18) is
\[
\begin{align*}
\ddot{x} &= \ddot{x}_d + \frac{K_1}{m} (\dot{x}_d - \dot{x}) + k_p \text{sign}(\dot{x}_d - x) + k_d \text{sign}(\dot{x}_d - \dot{x}) \\
\ddot{y} &= \ddot{y}_d + \frac{K_2}{m} (\dot{y}_d - \dot{y}) + k_p \text{sign}(\dot{y}_d - y) + k_d \text{sign}(\dot{y}_d - \dot{y}) \\
\ddot{z} &= \ddot{z}_d + \frac{K_3}{m} (\dot{z}_d - \dot{z}) + k_p \text{sign}(\dot{z}_d - z) + k_d \text{sign}(\dot{z}_d - \dot{z})
\end{align*}
\] (19)

By Lemma 2, the proof is straightforwardly obtained.

**Design of attitude controller.** In the previous section, the nonlinear dynamics involving attitude information \( \Phi = (\phi, \theta, \psi) \) is taken as a virtual control input and is designed. Next, the attitude control of the aircraft system will be considered. According to the relationship (5), the desired attitude can be obtained from the virtual control input \( u_x, u_y, u_z \). Expected attitude of aircraft is expressed by \( \Phi_d = (\phi_d, \theta_d, \psi_d) \), then it follows from equation (5) that
\[
\begin{align*}
T &= m \sqrt{u_x^2 + u_y^2 + (u_z + g)^2} \\
\phi_d &= \arcsin\left( \frac{m}{T} (u_x \sin \psi_d - u_z \cos \psi_d) \right) \\
\theta_d &= \arctan\left( \frac{1}{u_y + g} (u_z \cos \psi_d + u_y \sin \psi_d) \right)
\end{align*}
\] (20)
Due to the free variable \( \psi_d \), for the convenience of analysis, we set the desired yaw angle to zero, that is, \( \psi_d = 0 \). Similar to the design process for position controller in the section, the absence of external interference will be considered first.

**Lemma 4.** For the quadrotor’s attitude model (2) without external disturbance, that is, \( d_i(t) = 0, i = 1, 2, 3 \), if the control law is designed as

\[
\begin{align*}
\tau_1 &= \frac{J_1}{I} \left( \dot{\psi}_d + \frac{K_{a1}}{J_1} \dot{\theta}_d + a_{p1} \text{sign}^\beta_1 (\phi_d - \phi) + a_{d1} \text{sign}^\beta_2 (\dot{\phi}_d - \dot{\phi}) \right), \\
\tau_2 &= \frac{J_2}{I} \left( \dot{\theta}_d + \frac{K_{a1}}{J_2} \dot{\theta}_d + a_{p1} \text{sign}^\beta_1 (\theta_d - \theta) + a_{d1} \text{sign}^\beta_2 (\dot{\theta}_d - \dot{\theta}) \right), \\
\tau_3 &= \frac{J_3}{c} \left( \dot{\psi}_d + \frac{K_{a6}}{J_3} \dot{\psi}_d + a_{p1} \text{sign}^\beta_1 (\psi_d - \psi) + a_{d1} \text{sign}^\beta_2 (\dot{\psi}_d - \dot{\psi}) \right)
\end{align*}
\]

(21)

where \( 0 < \beta_1 < 1, \beta_2 = 2\beta_1/(1 + \beta_1) \), \( a_{p1} > 0, a_{d1} > 0 \), then the desired attitude can be achieved in a finite time, that is, \( (\phi, \theta, \psi)^T \rightarrow (\phi_d, \theta_d, \psi_d)^T \) in a finite time.

**Proof.** Similarly, without loss of generality, only the stability proof of roll angle \( \varphi \) is provided. Define the error as \( e_\phi = \phi_d - \phi \), then it follows from equations (2) and (21) that the error equation is

\[
\begin{align*}
\dot{e}_\phi &= \dot{\phi}_d - \dot{\phi} \\
\ddot{e}_\phi &= \frac{K_{a1}}{J_1} \dot{\theta}_d + a_{p1} \text{sign}^\beta_1 (e_\varphi) - a_{d1} \text{sign}^\beta_2 (\dot{e}_\phi)
\end{align*}
\]

(22)

Under the action of the proposed attitude controller (21), the closed-loop system is

\[
\begin{align*}
\dot{e}_\phi &= \dot{\phi}_d - \dot{\phi} \\
\ddot{e}_\phi &= -\frac{K_{a1}}{J_1} \dot{e}_\varphi - a_{p1} \text{sign}^\beta_1 (e_\varphi) - a_{d1} \text{sign}^\beta_2 (\dot{e}_\phi)
\end{align*}
\]

(23)

It can be found that system (23) is similarly like the system (8). Therefore, it is easy to prove that the system (23) is globally finite-time stable by a similar process.

Considering the external disturbance, for the sake of simple writing, the attitude system (2) can be rewritten as

\[
\begin{align*}
\ddot{\phi} &= -\frac{K_{a1}}{J_1} \dot{\phi} + \frac{l}{J_1} \tau_1 + d_\phi \\
\ddot{\theta} &= -\frac{K_{a1}}{J_2} \dot{\theta} + \frac{l}{J_2} \tau_2 + d_\theta \\
\ddot{\psi} &= -\frac{K_{a6}}{J_3} \dot{\psi} + \frac{c}{J_3} \tau_3 + d_\psi
\end{align*}
\]

(24)

where \( d_\phi = \frac{d_\phi(t)}{J_1}, d_\theta = \frac{d_\theta(t)}{J_2}, d_\psi = \frac{d_\psi(t)}{J_3} \).

For the external interference in system (24), a disturbance estimation and compensation method is adopted. As that in the literature,\textsuperscript{37-40} for external disturbances, the following assumptions are given.

**Assumption 4.** Assumed that there are known positive constants \( L_1, L_2 \) such that \( |d_\phi(t)| \leq L_1, |d_\theta(t)| \leq L_1, |d_\psi(t)| \leq L_1, |d_\phi(t)| \leq L_2, |d_\theta(t)| \leq L_2, |d_\psi(t)| \leq L_2 \).

**Lemma 5.** For the system (24) under Assumption 4, the states \( (\dot{\phi}, \dot{\theta}, \dot{\psi}) \) of the observers

\[
\begin{align*}
\dot{\hat{\phi}} &= -\frac{K_{a1}}{J_1} \dot{\hat{\phi}} + \frac{l}{J_1} \tau_1 + \dot{\hat{\phi}} + \rho_1 \text{sign}^{1/2} (\hat{\phi} - \dot{\phi}) \\
\dot{\hat{\theta}} &= -\frac{K_{a1}}{J_2} \dot{\hat{\theta}} + \frac{l}{J_2} \tau_2 + \dot{\hat{\theta}} + \rho_1 \text{sign}^{1/2} (\hat{\theta} - \dot{\theta}) \\
\dot{\hat{\psi}} &= -\frac{K_{a6}}{J_3} \dot{\hat{\psi}} + \frac{c}{J_3} \tau_3 + \dot{\hat{\psi}} + \rho_1 \text{sign}^{1/2} (\hat{\psi} - \dot{\psi})
\end{align*}
\]

(25)

will converge to the states \( (\hat{\phi}, \hat{\theta}, \hat{\psi}) \) in a finite time, where

\[
\rho_2 > L_2, \quad \rho_1 > \sqrt{\frac{2}{\rho_2 - L_2} \frac{(\rho_2 + L_2)(1 + p)}{1 - p}}
\]

(26)

with \( 0 < p < 1 \).

**Proof.** The proof process is the same as that of the position loop disturbance observer, that is, Lemma 3. The proof is straightforward.

Based on the accurate estimation of external disturbance, an attitude tracking control method based on the observer is proposed to realize finite-time attitude tracking control.

**Theorem 2.** For the quadrotor’s attitude model (2), if the control law is designed as

\[
\begin{align*}
\tau_1 &= \frac{J_1}{I} \left( \dot{\psi}_d + \frac{K_{a1}}{J_1} \dot{\psi}_d + a_{p1} \text{sign}^{1/2}_1 (\phi_d - \phi) \\
&\quad + a_{d1} \text{sign}^{1/2}_2 (\dot{\phi}_d - \dot{\phi}) \right) \\
\tau_2 &= \frac{J_2}{I} \left( \dot{\theta}_d + \frac{K_{a1}}{J_2} \dot{\theta}_d + a_{p1} \text{sign}^{1/2}_1 (\theta_d - \theta) \\
&\quad + a_{d1} \text{sign}^{1/2}_2 (\dot{\theta}_d - \dot{\theta}) \right) \\
\tau_3 &= \frac{J_3}{c} \left( \dot{\psi}_d + \frac{K_{a6}}{J_3} \dot{\psi}_d + a_{p1} \text{sign}^{1/2}_1 (\psi_d - \psi) \\
&\quad + a_{d1} \text{sign}^{1/2}_2 (\dot{\psi}_d - \dot{\psi}) \right)
\end{align*}
\]

(27)
where \( 0 < \beta_1 < 1, \beta_2 = 2\beta_1/(1 + \beta_1), \ a_p > 0, a_d > 0, \) then the desired attitude can be tracked in a finite time, that is, \((\phi, \theta, \psi)^T \to (\phi_d, \theta_d, \psi_d)^T\) in a finite time.

**Proof.** Because the disturbance observer (25) is finite-time stable, which means that there exists a finite time \( T_o \) such that the observation equals to the real disturbance, that is, \( \hat{d}_\phi = d_\phi, \hat{d}_\theta = d_\theta, \hat{d}_\psi = d_\psi, \forall t \geq T_o. \) After the time \( T_o, \) the closed-loop control system (24)–(27) is

\[
\ddot{\phi} = \frac{K_s}{J_1} (\dot{\phi}_d - \dot{\phi}) + a_p \sin^{\beta_1} (\phi_d - \phi) + a_d \sin^{\beta_2} (\dot{\phi}_d - \dot{\phi}) \\
\ddot{\theta} = \frac{K_s}{J_2} (\dot{\theta}_d - \dot{\theta}) + a_p \sin^{\beta_1} (\theta_d - \theta) + a_d \sin^{\beta_2} (\dot{\theta}_d - \dot{\theta}) \\
\ddot{\psi} = \frac{K_s}{J_3} (\dot{\psi}_d - \dot{\psi}) + a_p \sin^{\beta_1} (\psi_d - \psi) + a_d \sin^{\beta_2} (\dot{\psi}_d - \dot{\psi})
\]

(28)

With the result of Lemma 4, the proof is straightforward.

**Remark 2.** Note that the whole controller consists of position controller (18) and attitude controller (27). In addition, though the considered dynamics of roll, pitch, and yaw angles are linear and decoupled, the some results, such as in the study by Abdessameud and Tayebi, the physical experiment is given to support the feasibility when the controller design is based on such simplified model.

### Numerical simulations

To verify the previous theoretical results, some simulation cases are given in this section. Firstly, considering the existence of external disturbances, three kinds of control methods are employed, that is, Proportional-Differentiation (PD) control method, finite-time control method, and finite-time control method with disturbance observer.

**Model parameters and controllers gains**

The system’s parameter values of the quadrotor aircraft are given as follows: The mass of quadrotor aircraft is \( m = 1.5kg, \) the frame length is \( l = 0.4m, \) the scale factor of aerodynamics is \( c = 2, \) the damping coefficients of aerodynamics are \( K_i = 0.01N\cdot s/m, i = 1, \ldots, 6, \) the gravitational acceleration is \( g = 9.81m/s^2, \) the inertia matrix of quadrotor aircraft is \( J = \text{diag}(0.0049, 0.0049, 0.0088) kg\cdot m^2, \) the external disturbances of position loop are \( \Delta_1(t) = 0.5\sin (t + 2), \Delta_2(t) = 0.6\cos (3t + 1), \Delta_3(t) = 0.4\sin (2t + 6), \) and the external disturbances of attitude loop are \( d_1(t) = 0.002\sin (t + 5), d_2(t) = 0.002\cos (2t + 3), d_3(t) = 0.003\sin (3t + 2). \)

The parameters of three different controllers are chosen as follows. For the proposed finite-time controller with observer, the control gains are \( \alpha_1 = \beta_1 = 3/4, k_p = 4.2, k_d = 3.3, a_p = 3.0, a_d = 4.5, \rho_1 = \lambda_1 = 4, \rho_2 = \lambda_2 = 2. \) For the classical finite-time controller, the control gains are \( \alpha_1 = \beta_1 = 3/4, k_p = 4.2, k_d = 3.3, a_p = 3.0, a_d = 4.5. \) For
Figure 3. The curves of attitude tracking error.

Figure 4. The curves of the position-loop’s disturbance observations $\hat{\Delta}_i$, $i = 1, 2, 3$. 
the traditional PD controller, the control gains are $k_p = 4.2, k_d = 3.3, a_p = 3.0, a_d = 4.5$.

**Simulation results**

To verify the effectiveness of the proposed control law, some simulation parameters are set as follows: the reference position trajectory $x_d = (x_d, y_d, z_d)^T = (2\sin(0.5t), 2\cos(0.5t), 1)^T$, the desired roll angle and pitch angle are calculated using equation (20), and the yaw angle $\psi_d$ is set to 0. The initial conditions of quadrotor aircraft are chosen as $(x, y, z, \phi, \theta, \psi)^T = (0, 0, 0, 0, 0, 0)^T$.

To handle external interference, three different control laws are adopted, that is, the classical PD control, the general finite-time control, and the proposed finite-time control with disturbance observer. The comparison results for position and attitude tracking under three control laws are shown in Figures 2 and 3. The results show that the proposed control method has good robustness in dealing with external interference. Moreover, the disturbance observation $\hat{d}_i, i = 1, 2, 3$, and $d_i, i = \phi, \theta, \psi$ are shown in Figures 4 and 5, respectively, which the effectiveness of the proposed finite-time perturbation estimation method is verified.

**Conclusion**

In this article, a finite-time control method with disturbance observer was proposed to realize the finite-time control of a quadrotor aircraft in the presence of the unknown external disturbances. According to the model characteristics of quadrotor aircraft, the controller design has two steps, that is, the design of finite-time position tracking controller and the design of finite-time attitude tracking controller. By combing the estimation of external disturbance, a finite-time control law with the compensation of disturbance was proposed. For position loop and attitude loop of aircraft with external disturbance, the corresponding controllers were designed to improve the performance. Through rigorous stability analysis and simulation results, the effectiveness of the controllers was verified. Future work includes conducting experimental research to verify the proposed control algorithm.

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