Every super-polynomial proof in $M\rightarrow$ has a polynomially sized proof in $K\rightarrow$

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Abstract
In this note we show how any formula $\delta$ with a proof of super-polynomial size in $M\rightarrow$ has a poly-sized proof in $K\rightarrow$. This fact entails that any propositional classical tautology has short proofs, i.e., $NP=CoNP$.

1 Every classical tautology has short proofs

This article reports the results on the investigation on the existence of short proofs for every Classical Propositional Logic tautology. Firstly, the functional completeness of the fragment $\{\rightarrow, \perp\}$ of the propositional language says that any tautology in the full fragment can be expressed in this implicational based fragment. Let $t(\alpha)$ be the translation of the propositional formula $\alpha$ in terms of $\{\rightarrow, \perp\}$. We know that, from any proof $\Pi$ of $\alpha$, in the system of Natural Deduction for CPL, there is a proof $\Pi'$ of $t(\alpha)$, such that, $|\Pi'| \leq p(|\Pi|)$, where $p(x)$ is a polynomial. Thus, if there is a super-polynomially sized proof of $\alpha$ then, there is a super-polynomially sized proof of $t(\alpha)$. Proving that every formula in $\{\rightarrow, \perp\}$ has short (polynomially sized proofs on the length ($||\alpha||$) entails that every formula in the full language has short proofs too. Without loss of generality we focus the discussion on the existence of short proofs in the fragment $\{\rightarrow, \perp\}$. We remember that the negation $\neg\beta$ can be seen as a shorthand for $\beta \rightarrow \perp$ whenever we find it convenient.

The length of a formula $\alpha$, $||\alpha||$, is the number of occurrences of symbols of the alphabet of $\alpha$ in the string $\alpha$. The size of a proof $\Pi$, $|\Pi|$, is the number of occurrences of symbols in the string that represents a linearization of $\Pi$ (viewed as a tree). In what follows we use $\vdash_{Cla} \alpha$, $\vdash_{Int} \alpha$ and $\vdash_{Min} \alpha$ to denote Classical, Intuitionistic and Minimal validity, respectively. Our proof-theoretical analysis is based on the systems of Natural Deduction for Classical, Intuitionistic and
1 EVERY CLASSICAL TAUTOLOGY HAS SHORT PROOFS

\[ \exists \Pi \vdash \Pi_{\text{Cla}} \delta \]

Figure 1: From \( \vdash_{\text{Cla}} \delta \) to \( \vdash_{\text{K}} \delta \)

Minimal logics, as they appear in section 3. The notation \( \vdash_{\mathcal{L}} \alpha \) denotes the existence of a proof of \( \alpha \) in the Natural Deduction system for \( \mathcal{L} \). In this article the purely implicational minimal logic (\( \mathcal{M}_{\rightarrow} \)) has a central role. It is the system of Natural Deduction for minimal logic only with two rules: \( \rightarrow\text{-intro} \) and \( \rightarrow\text{-elim} \) (cf. section 3). The Natural Deduction for the CPL in the fragment \( \{\rightarrow, \bot\} \) is denoted by \( \mathcal{K}_{\rightarrow} \), cf. section 3.

Let \( \delta \) be a classical propositional tautology. Consider Glyvenko’s theorem, namely, \( \vdash_{\text{Cla}} \delta \) if and only if, \( \vdash_{\text{Int}} \neg\neg \delta \). It is easy to see that from a polynomial, on the size of \( \delta \), proof of \( \neg\neg \delta \) in IPL we obtain a polynomial proof of \( \delta \) in CPL. In [1] there is a translation \( J \), such that, \( \vdash_{\text{Int}} \beta \), if and only if, \( \vdash_{\mathcal{M}_{\rightarrow}} J(\beta) \). Besides that, if \( \Pi \) is a proof of \( \beta \) in Intuitionistic logic, then there is a proof \( \Pi' \) of \( J(\beta) \) in \( \mathcal{M}_{\rightarrow} \), such that, \( p(\| \Pi \|) \leq \| \Pi' \| \), where \( p(x) \) is a polynomial. In this article we show, in section 7, that for any formula \( \alpha \) in \( \mathcal{M}_{\rightarrow} \), with a super-polynomially sized proof \( \Pi \), on the \( |\alpha| \), there is a polynomially sized proof \( \Pi^* \) of \( \alpha \), on \( |\alpha| \) too, in \( \mathcal{K}_{\rightarrow} \). Note that \( \Pi^* \) is a classical proof. Putting together what was said above, we have that from any super-polynomially on \( |\delta| \) sized proof of \( \delta \) in \( \mathcal{K}_{\rightarrow} \), there is super-polynomially sized proof of \( \neg\neg \delta \) in \( \mathcal{M}_{\rightarrow} \). By theorem 1 there is a polynomially sized proof of \( \neg\neg J(\delta) \) in \( \mathcal{K}_{\rightarrow} \). It happens, that this proof is of a special kind, expressed by the fact that the rules for implication and intuitionistic absurd are of a restricted form. We call this Natural Deduction system \( \mathcal{M}_{\rightarrow} \).

Proposition 1 states that from any proof \( \Pi \) of \( J(\gamma) \) in \( \mathcal{M}_{\rightarrow} \) there is a proof \( \Pi^* \) of \( \gamma \) polynomially bounded by \( |\Pi| \) in \( \mathcal{K}_{\rightarrow} \). Thus, from any polynomially sized proof of \( \neg\neg J(\delta) \) in \( \mathcal{M}_{\rightarrow} \) we obtain a polynomially sized proof of \( \neg\neg \delta \) in \( \mathcal{K}_{\rightarrow} \). From this last proof we obtain a polynomially sized proof of \( |\delta| \) proof of \( \delta \) in \( \mathcal{K}_{\rightarrow} \). Hence, every classical propositional tautology in the fragment \( \{\rightarrow, \bot\} \) has a polynomially sized proof, so has every classical propositional tautology in the full language. We have proved that \( NP = \text{CoNP} \). The schema of this argument is depicted in figure 1.

The main result in the above argument is theorem 1. Next section provides
2 Roadmap to obtain polynomially sized proofs in $\mathbf{K}_{\rightarrow}$

The strategy that we will apply is the following:

1. Consider a normal proof $\Pi$ of a $\mathbf{M}_{\rightarrow}$ tautology $\delta$ that is super-polynomially sized on $||\delta||$. This means that no polynomial on $||\delta||$ is greater than the size of $\Pi$, for sufficiently big formulas $\delta$.

2. Since $\Pi$ is built with (possibly all) polynomially many sub-formulas of $\delta$, for the sub-formula principle holds in $\mathbf{M}_{\rightarrow}$, then there must exists a formula $\xi$ that occurs super-polynomially many-times on $||\delta||$ in $\Pi$. We use lemma 1 to draw this conclusion.

3. From the existence of $\xi$, using lemmas 6 and 7 we conclude that a super-polynomial sub-derivation $\Pi^{\rightarrow}$ of $\Pi$, of the form shown in figure 8 exists. The sub-derivation $\Pi_{k+1}$ (where $k$ is super-polynomial in $||\delta||$) is important in the sequel.

4. Using lemma 9 we shown a way to reduce $\Pi^{\rightarrow}$ in a derivation of the same conclusion, $\gamma$, from the same set of hypothesis that is polynomial on $|\Pi_{k+1}|$.

5. We argument that it is always possible to have such sub-derivation with $\Pi_{k+1}$ of size polynomial on $||\delta||$. The reader will note that in general there are many sub-derivations of the form of $\Pi^{\rightarrow}$, but one having $\Pi_{k+1}$ of polynomial size is always possible.

6. The possible number of derivations of the form of $\Pi^{\rightarrow}$ is related to the number of R-connected chains present in $\Pi$, that is bounded by $||\delta||$, according to lemma 4.

7. Thus, by iterating the reductions shown in the proof of lemma 9 we obtain a proof of $\delta$ that is polynomial on $||\delta||$.

8. Some care is needed each time the reduction is applied resulting in a polynomially sized sub-derivation of $\gamma$ from $\Gamma$. Thus, in order to isolate
the classical rules present in it from the rest of the $M\rightarrow$ proof, we cut the derivation by replacing it by a rule application that has one premise for each hypothesis in $\Gamma$ and conclusion $\gamma$. Thus, the resulting derivation of $\delta$ is a normal derivation in $M\rightarrow$ extended with these new rule application. These applications happens only as the topmost applications in of a rule in any branch.

9. After performing all polynomially many reductions we obtain a polynomial derivation of $\delta$ that is polynomial on $||\delta||$ even after replacing the new rules mentioned in the previous item by their corresponding polynomial $K\rightarrow$ sub-derivations.

We notice again, that the proof that this procedure that is a central step in proving that CoNP = NP is explained in the introduction of the article.

3 Natural Deduction: Minimal, Intuitionistic and Classical logics

This section shows the Natural Deduction systems that we use in this article. Figure 2 defines the set of elimination and introduction rules for the minimal logic, while figure 3 and 4 defines Natural Deduction for Intuitionistic and Classical logics, respectively.

The purely implicational minimal logic $M\rightarrow$ is defined by the set of rules that includes only $\rightarrow$-i and $\rightarrow$-e. The classical counterpart of $M\rightarrow$ is $K\rightarrow$, it includes $\rightarrow$-i, $\rightarrow$-e and $\bot$-Cla. $\neg$-e and $\neg$-i are taken as particular cases of $\rightarrow$-e and $\rightarrow$-i for both systems $M\rightarrow$ and $K\rightarrow$.

The system $MK\rightarrow$ is defined by restricting some rule applications of $K\rightarrow$. The derivations obtained from the super-polynomial $M\rightarrow$ derivations, by applying the reduction shown in lemma 9 are in $MK\rightarrow$. In fact $MK\rightarrow$ is defined here only because it is the (Classical) system that is used to represent the derivations obtained by this very reduction in lemma 9 $MK\rightarrow$ is defined as $M\rightarrow$, extended with $\neg$-i, $\bot$-Cla, $\bot$-Int and the restricted $\neg$-e rule, called $\neg$-e$^R$, where the major is derived from no formula, but itself. We write it overlined in order to indicate this restriction.

$$\frac{\alpha}{\bot \alpha} \neg-e^R$$

4 Main definitions and terminology

Proof Theory is a main tool in this work. In this section we only write down the definitions and results, without proof, that are used in this article in order to have a self-contained presentation. However, this does not substitute in any level a more detailed consult and reading of the literature in proof theory. There are very good references in this subject, among them we cite [2] that has terminology and main concepts closest to our presentation.
Figure 2: Natural Deduction for Minimal Logic $M$

Minimal Logic rules $+ \quad \frac{-A}{\bot}$ -Int

Figure 3: Natural Deduction for Intuitionistic Logic $Int$

Minimal Logic $+ \quad \frac{-A}{\bot}$ -Cla

or ....

Intuitionistic Logic $+ \quad \frac{-A}{\bot}$ -Cla

Figure 4: Classical logic rules on top of $M$ and $Int$
A derivation in Natural Deduction is any tree of N.D. rule applications having a conclusion $\alpha$ and $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ as top-formulas (not discharged). The elements of $\Gamma$ are said to be open assumptions in $\Pi$. A proof is a derivation with no open assumption.

**Definition 1 (Maximal Formula)** In a proof $\Pi$ in $M\rightarrow$, a maximal formula is a formula occurrence in $\Pi$ that is conclusion of an $\rightarrow$-i and major premise of an $\rightarrow$-e application.

Below it is shown a typical maximal formula.

$$
\begin{array}{cc}
\Pi_1 & \Pi_2 \\
A & B \\
A & A \rightarrow B \\
B
\end{array}
$$

A maximal formula can be eliminated from a proof without changing the conclusion of the proof. It is enough to apply the following reduction to the maximal formula occurrence.

$$
\begin{array}{cc}
\Pi_1 & \Pi_2 \\
A & B \\
A & A \rightarrow B \\
\end{array}
$$

**Definition 2 (Normal Proof)** A derivation $\Pi$ is a normal derivation in $M\rightarrow$, iff, there is no maximal formula in $\Pi$.

It is possible to obtain normal derivations from any derivation, by applying the reduction shown above in a special order, such that, each time a maximal formula is eliminated, there is a measure that indicates that the process of elimination goes down in a finite number of applications of the already mentioned reduction.

**Definition 3 (Normalization)** For every derivation $\Pi$ of $\alpha$ from $\Gamma$ there is a normal derivation of $\alpha$ from $\Gamma' \subseteq \Gamma$.

**Definition 4 (Branch)** A branch in $\Pi$ is any sequence $\langle \alpha_1, \ldots, \alpha_k \rangle$ of formulas, such that, $\alpha_k$ is the conclusion of $\Pi$ or a minor premise of a $\rightarrow$-elim, $\alpha_1$ is premise of a rule application and it is not conclusion of any rule application. For each $i = 1, k - 1$, $\alpha_i$ is a major premise of a $\rightarrow$-elim or $\rightarrow$-intro rule applications and $\alpha_{i+1}$ is the conclusion of this rule.

**Definition 5 (Level of a branch)** Any Branch that the last formula is the conclusion of the derivation is a level 0 branch or the principal branch of the proof. Any branch that the last formula is minor premise of a $\rightarrow$-e application of a branch of level $n$ is a branch of level $n + 1$. 
We use \(\langle \ldots \rangle\) to denote lists of formulas, as it is the case regarding branches.

**Definition 6 (Sub-sequence)** Let \(\Delta = \langle \alpha_1, \ldots, \alpha_k \rangle\) be a sequence of formulas. A sub-sequence of \(\Delta\) is any sequence \(\Gamma = \langle \alpha_i, \ldots, \alpha_j \rangle\), where \(1 \leq i, j \leq k\), and for every formula \(n\), \(i < n < j\), \(\alpha_n\) occurs in \(\Gamma\), immediately after \(\alpha_{n-1}\) and before \(\alpha_{n+1}\).

**Definition 7 (Sub-branch)** Let \(\Pi\) be a proof in \(\mathcal{M}\rightarrow\) and \(\Delta\) be a branch in \(\Pi\). A sub-branch of \(\Delta\) is any sub-sequence of \(\Delta\).

**Proposition 1 (Form of Normal proofs)** Let \(\Pi\) be a normal proof in \(\mathcal{M}\rightarrow\), then each branch in \(\Pi\) has the following form:

- The branch has a formula occurrence \(\alpha\), that is conclusion of an \(\rightarrow\)-e and premise of an \(\rightarrow\)-i rule application in \(\Pi\), namely, the minimal formula.
- Every formula in this branch occurring above \(\alpha\) is premise of a \(\rightarrow\)-e rule, this sub-sequence is called the E-part of the branch;
- Every formula in the branch occurring below \(\alpha\) is premise of a \(\rightarrow\)-i rule, this sub-sequence is called the I-part of the branch;

**Proposition 2 (Sub-formula principle)** Let \(\Pi\) be a normal derivation of \(\alpha\) from \(\Gamma\) in \(\mathcal{M}\rightarrow\). Every formula occurrence \(\beta\) belonging to a branch \(b\) of \(\Pi\) is either sub-formula of \(\alpha\) or of some formula \(\gamma_i\) in \(b\).

## 5 Auxiliary Lemmata

Given a graph \(\mathcal{G} = \langle G, E \rangle\), the cardinality of \(\mathcal{G}\), \(\text{card}(\mathcal{G})\), is defined as the number of nodes in \(\mathcal{G}\), i.e., \(\text{card}(G)\).

**Lemma 1** Let \(\mathcal{G} = \langle G, E \rangle\) be a finite graph, \(L\) be a finite and non-empty set that labels nodes of \(G\) and \(L : \text{nodes}(G) \rightarrow L\) be the labeling function. Assume that \(\text{card}(L) \leq n^k\) for some \(n\), \(2 \leq n\), and \(k > 1\), \(k, n \in \mathbb{N}\), and, suppose that \(\text{card}(\mathcal{G})\) is super-polynomial on \(n^k\). Then, there is at least one \(l \in L\), such that, \(\text{card}(\{v/\mathcal{L}(v) = l\})\) is super-polynomial on \(n^k\).

**bf Proof of lemma** Let us define \(S_l = \{v/\mathcal{L}(v) = l\}\). Since \(\mathcal{L}\) is a total function, we have that \(G = \bigcup_{l \in L} S_l\), and \(S_l \neq S_{l'}\), for every \(l \neq l' \in L\). Hence, \(\text{card}(\mathcal{G}) = \sum_{l \in L} \text{card}(S_l)\), that is upper bounded by \(n^k \times \max_{l \in L} (\text{card}(S_l))\). Since multiplication and addition of polynomials is a polynomial too, there is at least one \(l \in L\), such that \(S_l\) is super-polynomial on \(n^k\), that is, \(\text{card}(S_l)\) upper bounds any polynomial on \(n^k\), in the same way that \(\text{card}(\mathcal{G})\) does.

\(\square\)
Lemma 2 Let $\Pi$ be a derivation, in $\text{M}_{\rightarrow}$, $\text{MK}_{\rightarrow}$ or $\text{K}_{\rightarrow}$ of $\alpha$ from a set $\Gamma$ of assumptions. Let $\Pi'$ be a sub-derivation of $\Pi$ depending on a set $\Delta$ of assumptions and having $\beta$ as conclusion. Let $\Sigma$ be a derivation of $\beta$ depending on $\Delta' \subseteq \Delta$. Denote by $\Pi[\Pi' \leftarrow \Sigma]$ the derivation obtained by substituting the sub-derivation $\Pi'$ by $\Sigma$ in $\Pi$. $\Pi[\Pi' \leftarrow \Sigma]$ is a derivation of $\alpha$ from $\Gamma' \subseteq \Gamma$. As a corollary, if $\Pi$ is a proof of $\alpha$, then $\Pi[\Pi' \leftarrow \Sigma]$ so is.

Proof of lemma 2 We only have to note that any $\rightarrow$-introduction rule application in $\Pi$ that discharges a formula $\epsilon$ occurring in $\Pi'$ that does not occur in $\Sigma$ as assumption may discharge vacuously $\epsilon$ if there is no other occurrence of $\epsilon$ in $\Pi$ as assumption. In this last case, the derivation is a valid derivation in the respective logic system yet.

\[\square\]

Definition 8 Let $\Pi$ be a derivation of $\alpha$ from $\Gamma$ in $\text{M}_{\rightarrow}$ and $\Pi'$ a sub-derivation of $\Pi$ that derives $\beta$ from $\{\gamma'_1, \ldots, \gamma'_n\} \subseteq \Gamma$. Consider the rule $R_{\Pi'}$ of the following form:

\[
\frac{\gamma'_1, \ldots, \gamma'_n}{\beta}
\]

The result of substituting $\Pi'$ by a $R$-application in $\Pi$ is denoted by $\Pi[\Pi' \leftarrow R]$.

Definition 9 Let $\Pi$ be a derivation of $\delta$, in $\text{M}_{\rightarrow}$, having a sub-derivation $\Pi'$ that has $\beta$ as conclusion. Consider the system $\text{M}_{\rightarrow} + R_{\Pi'}$, $\Pi[\Pi' \leftarrow R_{\Pi'}]$ is normal in $\text{M}_{\rightarrow} + R_{\Pi'}$, if, $\Pi[\Pi' \leftarrow \beta]$ is a normal derivation in $\text{M}_{\rightarrow}$.

By examining the definitions above we that the following proposition holds.

Proposition 3 Let $\Pi$ be a derivation in $\text{M}_{\rightarrow}$, $\text{K}_{\rightarrow}$ or $\text{MK}_{\rightarrow}$. Let $\Pi'$ be a sub-derivation of $\Pi$ and $R_{\Pi'}$ its associated rule, as defined in definition above. If $\Pi$ is a normal derivation, then $\Pi[\Pi' \leftarrow R_{\Pi'}]$ is normal too.

Definition 10 $R$-Connected-occurrences. Let $\Pi$ be a derivation in $\text{M}_{\rightarrow}$. Let $\delta_1$ and $\delta_2$ be two different occurrences of the same formula $\delta$ in $\Pi$. We say that $\delta_1$ and $\delta_2$ are $R$-connected$\Box$ in $\Pi$, if and only if, one of the following conditions holds:

1. $\delta_1$ occurs in the $E$-part of a branch and $\delta_2$ occurs in the $I$-part of the same branch, or vice-versa (cf. fig $\Box$ where $\delta_1 = \delta_2 = \alpha \rightarrow \beta$);

2. $\delta_1$ occur in a branch as major premisse of an $\rightarrow$-Elim that has its minor premise occurring below $\delta_2$ in the respective secondary branch, or vice-versa, i.e., $\delta_2$ is the major premise and $\delta_1$ occurs above its corresponding minor premise in the respective secondary branch in $\Pi$ (cf. fig $\Box$);

3. $\delta_1$ is minor premise of an $\rightarrow$-Elim and $\delta_2$ occurs below the major premise of this $\rightarrow$-Elim rule, in the branch determined by it, or vice-versa (cf. fig $\Box$ where $\delta_1 = \delta_2 = \delta$).
Figure 5: R-connected formulas, first case (see \[1\]) with $\delta_1 = \delta_2 = \alpha \to \beta$

Figure 6: R-connected formulas, second case (see \[2\]) with $\delta_1 = \delta_2 = \alpha \to \beta$

Figure 7: R-connected formulas, second case (see \[3\])
Definition 11 R-connected-chain Let $\Pi$ be a derivation in $M\rightarrow$, and, $\langle \delta_1, \ldots, \delta_n \rangle$ be occurrences of the formula $\delta$ in $\Pi$. $\langle \delta_1, \ldots, \delta_n \rangle$ is an R-chain, iff, for each $i = 1, n - 1$, $\delta_i$ is R-connected to $\delta_{i+1}$.

We say that an R-connected-chain $C$ in a derivation $\Pi$ is maximal whenever there is no formula occurrence in $\Pi$ that can be added to $C$ such that the result is an R-connected-chain. Besides that, we observe that the sequence of formulas occurrences in any R-connected-chain are such that no more than two consecutive formulas in the $R$-chain belong to a same branch.

Lemma 3 order-in-chains Let $\Pi$ be a normal derivation in $M\rightarrow$, and $C = \langle \delta_1, \ldots, \delta_n \rangle$ be an R-connected-chain of $\delta$ occurrences in $\Pi$. The following properties hold on $C$:

1. $\delta$ is of the form $\alpha \rightarrow \beta$;
2. Each branch in $\Pi$ contains at most two $\delta$ occurrences that belong to $C$;
3. Either, for each $i = 1, n - 1$, whenever $\delta_i$ belongs to a branch of level $k$, if $\delta_{i+1}$ does not belong to the same branch, then it belongs to a branch of level $k - 1$, or, for each $i = 1, n - 1$, whenever $\delta_i$ belongs to a branch of level $k$, if $\delta_{i+1}$ does not belong to the same branch, then it belongs to a branch of level $k + 1$.
4. Either $\delta_1$ belongs to (occurs in) the lowest level or to the highest level branch, among the branches that each $\delta_i$ belongs, $i = 1, n$;

Proof of lemma 3

1. Items 2 and 3 assume that $\delta$ is major premise of an $\rightarrow$-elim and item 1 suppose that $\delta$ is conclusion of $\rightarrow$-intro rule applications;
2. By definition of $R$-chain, only item 1 allows two $\delta$ occurrences from a same branch, since one occurrence is in the I-part and the other in the E-part, then in order to a third $\delta$ occurrence be in the same branch it is not a branch in a normal derivation, for it would have $\rightarrow$-elim applications below $\rightarrow$-intro rule applications in this branch.
3. By items 2 and 3 each two consecutive formulas $\delta_i$ and $\delta_{i+1}$ in $C$, either are in the same branch, or one is major premise of a $\rightarrow$-elim and the other occurs in the branch whose minor premise is.
4. By the argument above, $C$ is a linear order, so that, $\delta_1$ is the last or the first in it. So, we have it either occurring in the highest branch or in the lowest one.

1R-connected means Repeatition-connected
Lemma 4 Let $\Pi$ be a normal derivation in $M_\rightarrow$. Let $\delta$ be any formula that occurs in $\Pi$. There are at most $||\delta||$ maximal $R$-connected-chains in $\Pi$ and every occurrence of $\delta$ in $\Pi$ belongs to one and only one maximal $R$-connected chain.

Proof of lemma 4 Let $C = \langle \delta_1, \ldots, \delta_k \rangle$ be a maximal $R$-connected chain in $\Pi$. By lemma 3, first item, we can consider $\delta$ being of the form $\alpha \rightarrow \beta$. If $C$ contains every occurrence of $\delta$ in $\Pi$, there is nothing to prove. Thus, we assume that there is a $\delta$ occurrence in $\Pi$ that does not belong to $C$. In this case we have the following sub-derivation in $\Pi$, where $\alpha'$ is not $\alpha \rightarrow \beta$.

\[
\begin{array}{c|c|c|c|c}
\Pi_1^l & \Pi_2^l & \Pi_1^r & \Pi_2^r \\
\hline
\alpha & \alpha \rightarrow \beta & \alpha & \alpha \rightarrow \beta \\
\hline
\beta & \beta & \Sigma^l & \Sigma^r \\
\hline
\alpha' & \alpha' \rightarrow \beta' & \beta' \\
\end{array}
\]

We must observe that $\alpha' \rightarrow \beta'$ cannot be $\alpha \rightarrow \beta$, since the main branch of $\Sigma^r$, ending in $\alpha' \rightarrow \beta'$, cannot have $\rightarrow$-intro applications below the $\beta$ showed as conclusion of an $\rightarrow$-elim. Neither $\alpha \rightarrow \beta$ can be subformula of $\beta$ either. The main branch of $\Sigma^r$ only has applications of elimination rules. Let us analyse what are the possible cases concerning $\Sigma^l$. Let us investigate whether $\Sigma^l$ may have $\rightarrow$-intro application in its main branch, that has $\alpha'$ as conclusion. We have the following cases, where $b$ is the branch under consideration. If the $\beta$ occurrence pointed out in $b$ starts the $I$-part of $b$ it is premise of an $\rightarrow$-intro application. By the sub-formula property, the conclusion of this rule, namely $\gamma$, has $\beta$ as proper sub-formula. On the other hand, the last formula of $b$, $\alpha'$ is sub-formula of $\alpha' \rightarrow \beta'$ that is also proper sub-formula of $\beta$, since it is below $\beta$ in an $E$-part of a branch. Summing up, we have that $\beta \prec \gamma \prec \beta'$ and hence $\beta \prec \beta'$, which is an absurd. The same conclusion is drawn if we consider $\gamma$ as being conclusion of any $\rightarrow$-Intro rule in $b$. Thus, $b$ has only elimination rules. As $b$ has only elimination rules, there is at most one possible secondary branch, like $b$, to each sub-formula of $\beta$. Hence the possible $R$-connected chains of $\alpha \rightarrow \beta$ occurrences is limited to the set of sub-formulas of the conclusion of $\Pi$. Finally, every $\alpha \rightarrow \beta$ occurrence in $\Pi$ is in one and only one $R$-connected chain.

$\square$

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2 We use $A \prec B$ to denote that $A$ is a proper sub-formula of $B$.
Using $M_\rightarrow$ to simulate derivations in $K_\rightarrow$

In this section we show how to translate any $K_\rightarrow$ provable formula into a $M_\rightarrow$ provable formula. Firstly we use Glyvenko’s theorem that establishes that:

$$\vdash_{K_\rightarrow} \alpha, \iff, \vdash_{\text{Int}} \neg \neg \alpha$$

Since the proof in $K_\rightarrow$ only proves $\alpha$ from $\neg \neg \alpha$, then $| \Pi_1 | \leq | \Pi_2 | + k$, where $k$ is linear on the size of $\alpha$ number of rules used to prove $\alpha$ from $\neg \neg \alpha$.

Using the derivation below:

$$\frac{\neg \alpha}{\neg \neg \alpha} \frac{\neg \neg \alpha}{\alpha}$$

$k$ is $4 \times || \alpha ||$.

The next step is to simulate de intuitionistic absurd rule. As suggested by Johansson in [3] and implemented by Statman in [1], we choose a new propositional letter (variable) that does not occur in $\alpha$ to serve this task. We consider the set of formulas $A_q = \{ q \rightarrow p/p \in \text{letters}(\alpha) \}$.

**Definition 12** Johansson-Statman translation We define the formula $J(q, \alpha)$, as defined as following:

$$J(q, p) = p \quad (1)$$

$$J(q, \bot) = q \quad (2)$$

$$J(q, \alpha_1 \rightarrow \alpha_2) = J(q, \alpha_1) \rightarrow J(q, \alpha_2) \quad (3)$$

Using the definition above, with a propositional variable that does not occur in $\alpha$ we have the following proposition, that is a particular case of what is obtained in [4] regarding polynomial simulation of provability for propositional systems satisfying the principle of sub-formula. The translation of $\alpha$ that is provable in $M_\rightarrow$, iff, $\alpha$ is provable in Int is:

$$J^*(q, \alpha) = \beta_1 \rightarrow (\beta_2 \rightarrow (\ldots \rightarrow (\beta_{n-1} \rightarrow (\beta_n \rightarrow J(q, \alpha))))))$$

Where $A_q = \{ \beta_1, \ldots, \beta_n \}$ We have the following proposition ensuring equi-provability of $\alpha$ and $J^*(q, \alpha)$

**Proposition 4** Let $\Pi$ be a proof of $\alpha$ in Int and $q$ a propositional letter that does not occur in $\alpha$, then there is a proof $\Pi'$ of $J^*(q, \alpha)$, such that, $| \Pi | \leq k \times | \Pi' |$, where $k$ is linear on the number of inference rules in $\Pi$.

**Proof of proposition** [4] Firstly, we can consider, w.g.l, each application of $\bot$-Int in $\Pi$ having only atomic conclusions. Since $\alpha$ and $J(q, \alpha)$ are in the fragment $\{ \rightarrow, \bot \}$, the only rule that occurs in $\Pi$ and cannot be in $\Pi'$ is the intuitionistic absurd. Thus, each application of $\bot$-Int in $\Pi$ as below:

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3 Remember that in $K_\rightarrow$ all formulas are in the fragment $\{ \rightarrow, \bot \}$ of the propositional language.
\[
\Pi_1 \\
\overline{p}
\]

Is replaced by:

\[
\Pi'_1 \\
q \qrightarrow p
\]

Where \(\Pi'_1\) is the result of this replacing schema when applied to \(\Pi_1\). The final step is to use \(n\)-times \(\rightarrow\)-i applications to obtain \(J^*(q, \alpha)\) from the derivation of \(J(q, \alpha)\) from \(A_q\).

\[\square\]

7 Reducing the size of proofs in \(M\rightarrow\) via classical reasoning

In this section we show how to obtain a polynomially sized proof of a formula \(\alpha\) in classical logic, from a super-polynomially sized proof of \(\alpha\) in \(M\rightarrow\). The system of classical propositional logic is represented by a restricted subsystem of \(K\rightarrow\) in Natural Deduction.

**Lemma 5** Let \(\Pi\) be a normal proof of \(\delta\) in \(M\rightarrow\) that is super-polynomially sized on \(||\delta||\). So there is a formula \(\chi\) that occurs super-polynomially many times on \(||\delta||\) in \(\Pi\).

**Proof of lemma 5** Every proof can be viewed as a graph, in fact a tree, labeled with formula occurrences. A normal proof in \(M\rightarrow\) satisfies the sub-formula principle, namely, every formula occurrence in the proof is a sub-formula of the conclusion of the proof. The number of sub-formulas of any formula is linear on the length of the formula. Thus, we have that a normal proof of \(\alpha\) is a graph labeled with a set of sub-formulas that has cardinality polynomial on \(||\alpha||\). By lemma 1 there must be a sub-formula \(\chi\) of \(\alpha\) that occurs super-polynomially many times in any super-polynomially sized on \(||\alpha||\) normal proof of \(\alpha\).

\[\square\]

**Lemma 6** Let \(\Pi\) be a normal proof of \(\delta\) in \(M\rightarrow\) that has a formula occurring super-polynomially many times on \(||\delta||\) in it, then there is a formula \(\alpha \rightarrow \beta\) that occurs super-polynomially many times in \(\Pi\) as major premise of \(\rightarrow\)-Elim rules applications in \(\Pi\) and it is also top-formula, discharged by some \(\rightarrow\)-intro rule application in \(\Pi\).

**Proof of lemma 6** Let \(\chi\) be a formula that occurs super-polynomially many times in \(\Pi\). Any of its occurrences must be in the I-part or in the E-part.
of super-polynomially many branches in \( \Pi \). We firstly prove that, without loss of generality, we can consider that its occurrence is in the E-part. If \( \chi \) occurs in the I-part, then, as there are polynomially many possible sub-formulas of \( \delta \) and super-formulas of \( \chi \), there must be one that happens super-polynomially many times as conclusion of the respective branches where \( \chi \) occurs in the I-part, by lemma 7. These conclusions are minor premises of \( \rightarrow \)-elim, so the respective major premises occurs super-polynomially many times too. Since there are only polynomially many possible major premises, by lemma 7 at least one major premise occurs super-polynomially many times in E-parts of branches. Considering, without lost of generality, that \( \chi \) occurs super-polynomially many times in E-parts, and hence, that it occurs super-polynomially many times as major premise of \( \rightarrow \)-elim. Each branch having \( \chi \) in its E-part has a top-formula that is major premise of an \( \rightarrow \)-Elim rule. Since there are polynomially many possible top-formulas, by the sub-formula property, there must be, by lemma 7 again, one formula of the form \( \alpha \rightarrow \beta \) as top-formula occurring super-polynomially many times in \( \Pi \). Since \( \Pi \) is a proof, each one of these occurrences is discharged by some \( \rightarrow \)-intro application in \( \Pi \). □

Lemma 7 If in a normal proof \( \Pi \) of \( \delta \) there is formula \( \alpha \rightarrow \beta \) that occurs super-polynomially many times in \( \Pi \) as major premise of \( \rightarrow \)-elim rule applications and discharged assumption, and hence is a top-formula, then there is a super-polynomial sub-derivation \( \Pi^* \) of \( \Pi \) that has a sub-formula \( \gamma \) of \( \delta \) as conclusion, and it is of the form depicted in figure 8, with \( k \) super-polynomial on \( || \delta || \).

Proof of lemma 7 Each top-formula occurrence \( \alpha \rightarrow \beta \) determines one branch and only one branch, by branches definition. Take into account the lowest level branch among those having an occurrence of \( \alpha \rightarrow \beta \) as top-formula. It is \( \langle [\alpha \rightarrow \beta], \beta, \ldots, \gamma \rangle \), where \( \gamma \) is minor premise of an \( \rightarrow \)-elim or the conclusion \( \delta \) of \( \Pi \) and \( \alpha \rightarrow \beta \) is discharged by some \( \rightarrow \)-intro rule application in \( \Pi \). Since this is the lowest branch that has an occurrence of \( \alpha \rightarrow \beta \) as major premise of an \( \rightarrow \)-rule application there is no \( \alpha \) as minor premise of an \( \rightarrow \)-elim below it, with \( \alpha \rightarrow \beta \) as major premise. \( \gamma \) can be \( \alpha \), the proof of this lemma proceeds equally if \( \gamma \) is \( \alpha \). We call \( \Pi_1 \) the derivation determined by this lowest branch and each of its secondary branches and higher branches, but the branch having \( \alpha \) as conclusion and minor premise of the major premise \( \alpha \rightarrow \beta \) of the pointed out \( \rightarrow \)-Elim.

Consider \( \Pi_2 \) the derivation that has the minor premise \( \alpha \) of the \( \rightarrow \)-elim rule application with \( \alpha \rightarrow \beta \) as the major premise belonging to the main branch of \( \Pi_1 \). Since, there is a formula \( \alpha \rightarrow \beta \) above the conclusion of \( \Pi_2 \), we can iterate this identification of \( \alpha \rightarrow \beta \) occurrences from bottom up. Thus, \( \Pi_i \) is determined by the reference to the \( i \)-th lowest occurrence of \( \alpha \rightarrow \beta \) in \( \Pi \). We know that \( i \) ranges from 1 to a super-polynomial value \( k \) on \( || \delta || \), by hypothesis.

Since this sequence of occurrences of \( \alpha \rightarrow \beta \) is not infinite, there must be an occurrence of \( \alpha \) above which there is no occurrence of \( \alpha \rightarrow \beta \). This occurrence determines the sub-derivation \( \Pi_{k+1} \) that has two sets of hypothesis: (1)
the formulas $\gamma_1, \ldots, \gamma_n$, discharged by the $\rightarrow$-intro part of each corresponding derivation $\Pi_{d_i}$, $i = 1, n$, and; (2) the formulas $\epsilon_1, \ldots, \epsilon_m$ that are discharged by $\rightarrow$-intro rules applications that occur below formula $\gamma$ in $\Pi$. The formulas $\epsilon_1, \ldots, \epsilon_m$ appear as assumptions in the derivation of $\gamma$ shown in figure 8, for they are discharged only when this derivation is considered inside the whole $\Pi$.

□

The following lemma helps us understanding the internal structure of each sub-derivation $\Pi_{d_i}$, $i = 2, k$, when related to the whole proof $\Pi$. Remember that $d_i$ is the index of the sub-derivation $\Pi_{d_i}$ that discharges $\gamma_i$ by means of a $\rightarrow$-intro rule application.

Lemma 8 Let $\Pi$ be a normal derivation as depicted in figure 8. For each $i = 1, \ldots, k$, let $\Pi^i$ be the sub-derivation of $\Pi$ determined by the conclusion of $\Pi_i$, namely $\alpha$. Taking into account that $\gamma_i$ is discharged in derivation $\Pi_{d_i}$, it is discharged by an $\rightarrow$-intro rule application that occurs in the $\rightarrow$-intro part of $\Pi_{d_i}$ as shown in figure 9, where $\Sigma_{d_i}$ has only $\rightarrow$-intro rules and $\Sigma^*_d$ has only $\rightarrow$-elim rules in its main branch, and hence, $\beta$ is the minimal formula of the main branch of $\Pi_{d_i}$.

Proof of lemma 8 In order to prove this lemma we only checkout that the statement in the lemma is a direct consequence of the form of normal proofs in $M_\rightarrow$, stated in proposition 1.

□

Lemma 9 Consider a super-polynomial normal proof of $\delta$ in $M_\rightarrow$ that have a sub-derivation of $\gamma$ of the form shown in figure 8. Assume that each $d_i$, $i = 1, \ldots, n$, denotes that the corresponding formula is discharged by an intro-$\rightarrow$ rule applied in the main branch of the subderivation $\Pi_{d_i}$. The formulas $\epsilon_i$, $i = 1, \ldots, m$, are discharged by $\rightarrow$-intro rules below the last conclusion $\beta$ in the (whole) proof of $\delta$. Then, there is a derivation of $\beta$ from $\epsilon_i$, $i = 1, \ldots, n$, in $K_\rightarrow$, that is polynomially (upper) bounded by $|\Pi_{k+1}|$.

Proof of lemma 9 Consider the detailed view of the sub-derivations $\Pi_i$, $i = 2, k$ shown in figure 9 according the proof of lemma 8. Consider each of the formulas in $\gamma_1, \ldots, \gamma_n$ that we know that are discharged by rules in the introduction part of the main branch, namely the $\Sigma_{d_i}$ shown in figure 9 of $\Pi_{d_i}$, for $i = 1, n$. Let each formula $\gamma_i$ in $\Pi_{k+1}$ be discharged by a $\rightarrow$-intro application $d_i$ occurring in the derivation $\Pi_{d_i}$. We denote by $\Sigma_{d_i}$ the sub-derivation of $\Pi_i^k$, analogous to the derivation shown in figure 9 for $i$, corresponding to this $\rightarrow$-intro part. The derivation shown in figure 9 shows that $\gamma$ can be proved in $K_\rightarrow$, by means of a polynomially sized proof. Note that some of the formulas $\epsilon_1, \ldots, \epsilon_m$ that are assumptions of $\Pi_{k+1}$ maybe discharged by $\rightarrow$-intro rule applications in

---

4 this is the sub-derivation that starts with $\Pi_{k+1}$ and $\alpha \rightarrow \beta$ and ends with the conclusion of $\Pi_i$, passing through $\Pi_k, \ldots, \Pi_{i+1}$ until $\Pi_i$
Figure 8: Derivation of $\gamma$ with super-polynomially many occurrences of $\alpha \rightarrow \beta$ as major premises, lemma $[7]$
7 REDUCING THE SIZE OF PROOFS IN $\mathbf{M} \rightarrow$ VIA CLASSICAL REASONING

\[
\begin{array}{c}
\gamma_1 d_1 \ldots \gamma_i d_i, \gamma_{i+1} \ldots \gamma_n \\
\Theta_1 \\
\Pi^{d_i-1} \\
\alpha \rightarrow \beta \\
\beta \equiv \beta_1 \rightarrow \beta_2 \\
\beta_2 \equiv \beta_1 \rightarrow \beta_2 \\
\beta_1 \rightarrow \beta_2 \\
\beta_2 \equiv \beta_1 \rightarrow \beta_2 \\
\beta_3 \rightarrow \beta_3 \\
\beta_3 \rightarrow \beta_3 \\
\beta_3 \rightarrow \beta_3 \\
\Sigma d_i \\
\Sigma d_i \\
\Sigma d_i \\
\Sigma d_i \\
\alpha
\end{array}
\]

Figure 9: Detailed view of the pattern that follows the rules application in each sub-derivation $\Pi^i$, $i = 2, \ldots, k$, lemma 8

$\Pi_1$. These discharges are the same that happens in the original derivation $\Pi$. Thus, no additional assumption is present in the polynomial derivation shown in figure 10. Besides that, lemma 2 shows that the $\rightarrow$-introduction rules that discharge formulas formulas that are in some $\Pi_s$ that are not any more in the reduced derivation, shown in figure 10, do hold.

$\square$

**Theorem 1** Let $\Pi$ be any normal proof of $\delta$ in $\mathbf{M} \rightarrow$ that is super-polynomially sized on $|| \delta ||$ in $\Pi$. There is proof $\Pi'$ of $\delta$ that is polynomially bounded on $|| \delta ||$ in $\mathbf{K} \rightarrow$.

**Proof of theorem 1** Applying lemma 5, lemma 6 and lemma 7 we conclude that there is a sub-derivation $\Pi^\gamma$ of $\gamma$ of the form shown in picture 8. We can prove that it is possible to have this derivation $\Pi^\gamma$ of $\gamma$, such that, its sub-derivation $\Pi_{k+1}$ is polynomially bounded by $|| \delta ||$. Suppose that a first application of lemmas 5, 6 and 7 does not provide us with a $\Pi^\gamma$ having such polynomially bounded sub-derivation $\Pi_{k+1}$, then $\Pi_{k+1}$ itself is super-polynomially sized on $|| \delta ||$. We apply the sequence of lemmas, 5, 6 and 7 to $\Pi_{k+1}$, so we obtain a new sub-derivation $\Pi^\gamma$, that is sub-derivation of $\Pi_{k+1}$, this turn. If this new derivation does not contain a (new) sub-derivation $\Pi_{k+1}$ of some $\alpha$ that it is polynomially bounded on $|| \delta ||$, then we can proceed repeating the argument until finding such polynomially bounded derivation. Each time the argument is applied the sized of the corresponding $\Pi^\gamma$ is a super-polynomial that it is a summand of a bigger super-polynomial. This regression goes down to a polynomial summand, otherwise, every sub-derivation of a super-polynomially sized derivation would be super-polynomial, what is not the case. Once we have determined
Figure 10: Derivation of $\beta$, polynomial on $|\Pi_{k+1}|$ built with sub-derivations of $\Pi$, lemma 9
a sub-derivation $\Pi_k$ containing a polynomially bounded sub-derivation $\Pi_{k+1}$ we apply lemma 9 to obtain a polynomially bounded derivation of $\gamma$ in $K_{\rightarrow}$. Thus, after replacing $\Pi_k$ by the polynomial derivation of $\gamma$ from the same set of open hypothesis than $\Pi_k$ we have a smaller derivation of $\delta$. If this derivation is not yet polynomially bounded on $|| \delta ||$, then we repeat the above steps in the derivation that is the result of the substitution of the classical (in $M_{\rightarrow}$) polynomially bounded derivation of $\gamma$ from the same set of open hypothesis than $\Pi_k$ we have a smaller derivation of $\delta$. If this derivation is not yet polynomially bounded on $|| \delta ||$, then we repeat the above steps in the derivation that is the result of the substitution of the classical (in $M_{\rightarrow}$) polynomially bounded derivation of $\gamma$ by $R_{\Pi_{\rightarrow}}$ as stated in proposition 13. This proposition ensures that the resulting derivation is structurally normal in $M_{\rightarrow}$, i.e., the application of the $R_{\Pi_{\rightarrow}}$ rule does not affect the correct application of lemmas 5, lemma 6 and lemma 7. Finally, we can observe that there are at most $|| \delta ||$ turns of application of lemma 9 to each R-connected maximal chain present in the original $\Pi$. As the bound on the number of R-connected chain is limited polynomially by $|| \delta ||$, according to lemma 4. Hence, the size of the proof obtained at the end of this procedure is a polynomial, on $|| \delta ||$, sum of polynomials on $|| \delta ||$. This is polynomially bounded by $|| \delta ||$. We observe that after all possible reductions are done, we replace back the polynomial bounded derivations $\Pi_k$, in $K_{\rightarrow}$, by the respective rules $R_{\Pi_{\rightarrow}}$. The final proof is a proof of $\delta$, in $K_{\rightarrow}$, that is polynomially bounded on $|| \delta ||$.

□

8 Translating proofs of $J(q, \alpha)$ in $MK_{\rightarrow}$ back to proofs of $\alpha$ in $K_{\rightarrow}$

Lemma 10 Consider a formula $\alpha$ in the fragment $\{ \rightarrow, \perp \}$. Let $q$ be a propositional variable not occurring in $\alpha \cup \Delta$, $S_\alpha = \{ q \rightarrow p/p \in PropVars(\alpha) \}$ and $J^*(q, \beta) = \beta[\perp/q]$, for every $\beta$ a formula in the fragment $\{ \rightarrow, \perp \}$. If $\Pi$ is a derivation of either $J^*(q, \alpha)$ or $\perp$, from $J^*(q, \Delta) \cup S_\alpha$ in $MK_{\rightarrow}$, then there is a derivation $\Pi^*$ of $\alpha$ or $\perp$, respectively, from $\Delta$ in $K_{\rightarrow}$ and a polynomial $p(x)$, such that $|| \Pi^* || \leq p(|| \Pi ||)$.

Proof of lemma 10: Without loss of generality, we consider $\Pi$ as having atomic $\perp$-Int conclusions only. We now prove by induction the desired statement. We use induction on the number of inference rules in $\Pi$. The basis is when $\Pi$ is $J^*(q, \alpha)$, so $J^*(q, \alpha) \in \Delta$, and hence $\Pi^*$ is a deduction of $\alpha$ from $\alpha$. We note that, since $\perp \notin J^*(q, \Delta) \cup S_\alpha$, for any $\Delta$ and $\alpha$, a derivation of $\perp$ of length 1 is not possible. For the inductive cases, we take into account the last inference rule applied in $\Pi$.

→Intro In this case, $\Pi$ is of one of the following forms:

1. The conclusion is of the form $J^*(q, \alpha_1 \rightarrow \alpha_2)$

   \[
   \frac{[J^*(q, \alpha_1)] \quad \Pi_1}{J^*(q, \alpha_2) \quad J^*(q, \alpha_1 \rightarrow \alpha_2)}
   \]

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where we used the fact that \( J^*(q, \alpha_1 \rightarrow \alpha_2) \) is \( J^*(q, \alpha_1) \rightarrow J^*(q, \alpha_2) \). Thus, by inductive hypothesis we know that there is a derivation \( \Pi^*_1 \) of \( \alpha_2 \) from \( \alpha_1 \), polynomial on \( \Pi_1 \), with the same assumptions already used in \( \Pi \). \( \Pi^* \) is then the polynomial, on \( \Pi \), derivation below:

\[
\begin{array}{c}
\alpha_2 \\
\alpha_1 \rightarrow \alpha_2
\end{array}
\]

2. The conclusion is \( J^*(q, \alpha_1) \rightarrow \bot \)

\[
\begin{array}{c}
[J^*(q, \alpha_1)] \\
\Pi_1 \\
\bot
\end{array}
\]

\[
J^*(q, \alpha_1) \rightarrow \bot
\]

Thus, by inductive hypothesis we know that there is a derivation \( \Pi^*_1 \) of \( \bot \) from \( \alpha_1 \), polynomial on \( \Pi_1 \), with the same assumptions already used in \( \Pi \). \( \Pi^* \) is then the polynomial, on \( \Pi \), derivation below:

\[
\begin{array}{c}
\alpha_1 \\
\Pi^*_1 \\
\bot
\end{array}
\]

\[
\alpha_1 \rightarrow \bot
\]

\(-\text{Elim} \) In this case, \( \Pi \) can be only of the following forms, since \( J^*(q, \alpha) \) is always different from \( \bot \). We note that in \( \text{MK}_{\rightarrow} \), \( \neg \alpha \) is not considered as an abbreviation for \( \alpha \rightarrow \bot \). Hence, a derivation of \( \bot \) is possible only when the \( \neg\)-Restricted-Elim rule is the last rule applied. This is treated in the sequel.

\[
\begin{array}{c}
\Pi_1 \\
J^*(q, \beta) \\
\Pi_2
\end{array}
\]

\[
J^*(q, \alpha)
\]

\[
\begin{array}{c}
\Pi_1 \\
q \\
q \rightarrow B
\end{array}
\]

\[
B
\]

In the first case we have that \( \Pi^* \) is as follows:

\[
\begin{array}{c}
\Pi^*_1 \\
\beta \\
\Pi^*_2
\end{array}
\]

\[
\beta \rightarrow \alpha
\]

where \( \Pi^*_1 \) and \( \Pi^*_2 \) are given by applications of the inductive hypothesis. The second case is provided by the following derivation, and the observation that \( q \) is \( J^*(q, \bot) \), and hence \( \Pi^*_1 \) is a derivation of \( \bot \). Observe also that \( J^*(q, B) \) is \( B \) itself.
\[ \Pi_1^* \]
\[ \perp \]
\[ \Gamma \]

\textbf{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{-Restricted-Elim}}}}}}}}}}}}}}\text{ In this case we have that } \Pi \text{ is of the form:}
\[ \begin{array}{c}
\Pi_1 \\
J^*(q, \beta) & \neg J^*(q, \beta) \\
\hline
\perp
\end{array} \]

By inductive hypothesis, \( \Pi_1^* \) is a derivation of \( \beta \) from \( \Delta \) and hence \( \Pi^* \) is as following. Observe that \( \neg J^*(q, \beta) \) is \( J^*(q, \neg \beta) \).
\[ \begin{array}{c}
\Pi_1^* \\
\beta & \neg \beta \\
\hline
\perp
\end{array} \]

\textbf{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{-Int}}}}}}}}}}}}}}\text{ In this case } \Pi \text{ is of the form:}
\[ \Pi_1 \\
\perp \\
J^*(q, \alpha) \]

By inductive hypothesis, there is a derivation of \( \Pi^* \) of \( \perp \) from \( \Delta \) that is polynomial on \( \Pi \), and hence the following polynomial derivation \( \Pi^* \) is also polynomial on \( \Pi \).
\[ \begin{array}{c}
\Pi_1^* \\
\perp \\
\alpha
\end{array} \]

\textbf{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{\texttt{-Cla}}}}}}}}}}}}}}\text{ The derivation is of the form:}
\[ \begin{array}{c}
[\neg J^*(q, \alpha)] \\
\Pi_1 \\
\hline
\perp \\
J^*(q, \alpha)
\end{array} \]

By the hypothesis on the general form of \( \Pi \), we know that all occurrences of \( J^*(q, \neg \alpha) \) are major premisses of \( \neg\)-Restricted-Elim rule applications. Thus, taking into account that \( J^*(q, \neg \alpha) \) is \( \neg J^*(q, \alpha) \), we have, by inductive hypothesis, that there is a derivation \( \Pi_1^* \) of \( \perp \) from \( \Delta \) and \( \neg \alpha \) polynomial on \( \Pi_1 \), hence the following derivation \( \Pi^* \) is polynomial on \( \Pi \).
\[ \begin{array}{c}
[\neg \alpha] \\
\Pi_1^* \\
\hline
\perp \\
\alpha
\end{array} \]
Lemma 11 Let $J$ be Johansson’s translation as it is seen in definition 12. Consider a formula $\alpha$ and $q$ a variable not occurring in it. If $\vdash_{\text{MK} \rightarrow} J(q, \alpha)$, then $\vdash_{\text{K} \rightarrow} \alpha$. Besides that, if $\Pi$ is a proof of $J(q, \alpha)$ in $\text{MK} \rightarrow$, then there is a proof $\Pi^*$ of $\alpha$ in $\text{K} \rightarrow$, such that $|\Pi^*| \leq p(|\Pi|)$, where $p(x)$ is a polynomial on $x$.

Proof of lemma 11 Consider a proof $\Pi$ of $J(q, \alpha)$ in $\text{MK} \rightarrow$. Remember that if $S_{\alpha} = \{q \rightarrow p/p \in \text{PropVars}(\alpha)\} = \{\beta_1, \beta_2, \ldots, \beta_k\}$ and $J^*(q, \alpha) = \alpha[\bot/q]$, with $\alpha$ in the fragment $\{\rightarrow, \bot\}$, then $J(q, \alpha) = \beta_1 \rightarrow (\beta_2 \rightarrow (\ldots \rightarrow (\beta_k \rightarrow J^*(q, \alpha)) \ldots)$. By the restrictions on derivations in $\text{MK} \rightarrow$ a proof of this formula $J(q, \alpha)$ is a derivation of $J^*(q, \alpha)$ from $\{\beta_1, \ldots, \beta_k\}$ in $\text{MK} \rightarrow$. By lemma 10 there is a derivation $\Pi^*$ of $\alpha$ from no assumptions, polynomial on $\Pi$, in $\text{K} \rightarrow$.

$\square$


9 Related work

Richard Statman [1] proved that the purely implicational fragment of Intuitionistic Logic (\(M\rightarrow\)) is PSPACE-complete. He showed a polynomially bounded translation from full Intuitionistic Propositional Logic into its implicational fragment. In [4] we extended Statman’s result in order to show that \(M\rightarrow\) is able to polynomially simulate any propositional logic for which the sub-formula principle holds. More precisely, \(M\rightarrow\) is able to simulate any propositional logic with a complete and sound set \(R\) of Natural Deduction rules, such that, any provable formula \(\alpha\) has a proof in \(R\) built only with sub-formulas of \(\alpha\). That is, for any propositional logic \(L\) satisfying the conditions just mentioned, there is a polynomial translation \(T_L\), such that:

\[ \vdash_L \alpha, \text{ if and only if, } \vdash_{M\rightarrow} T_L(\alpha) \]

By inspecting the proof of the polynomial simulation of a logic satisfying sub-formula principle in \(M\rightarrow\), proposition 2 in [4], we can also conclude that the presented polynomial simulation can be extended to proofs, such that:

If \(\Pi\) is a proof of \(\alpha\) in \(L\) then \(T_L(\Pi)\) is a proof of \(T_L(\alpha)\) in \(M\rightarrow\),

\[ |\Pi| < p(||\Pi||) \]

where \(p(x)\) is a polynomial. Thus, considering any classical propositional logic proof \(\Pi\) of a formula \(\alpha\), restricted to the language \(\{\bot, \rightarrow\}\), i.e., \(K\rightarrow\), if \(\Pi\) is super-polynomially sized on \(\alpha\), then the translation \(T_{K\rightarrow}(\Pi)\) is super-polynomially sized on the conclusion \(T_{K\rightarrow}(\alpha)\) too. Hence, any efficient prover for \(M\rightarrow\), namely one able to produce short proofs for all tautologies, is an efficient prover for propositional logics with the sub-formula property holding. The results proved in the article provide a weaker result, i.e., we have proved that classical propositional logic can have efficient provers, the next step is to obtain this result for \(M\rightarrow\) itself. This seems to be much harder, if not impossible. For example, in [5] it is shown a class of \(M\rightarrow\) tautologies that have least normal proofs of exponential size on the conclusions. These formulas have linear normal proofs if classical absurd rule is allowed to be used. In order to have smaller proofs, without using classical reasoning, the most popular idea is to introduce maximal formulas, that can be obatained by means of interpolants. However, in \(M\rightarrow\), such interpolants seems to be trivial, that is, are either the conclusion or some of the hypothesis.

10 Conclusion

In this article we show that for any implicational formula \(\delta\) with a super-polynomially sized (on \(||\delta||\)) \(M\rightarrow\) normal proof, there is a \(K\rightarrow\) polynomially-

\[ \text{By the PSPACE-completeness of S4, proved by Ladner, and the Godel translation from S4 into Intuitionistic Logic, the PSPACE-completeness of } M\rightarrow \text{ is drawn} \]

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sized proof, on $|| \delta ||$, of $\delta$. Using this fact, we can prove that every classical propositional tautology has a least proof that is polynomially-sized, on its conclusion, in the system of Natural Deduction described in section 4. In section 1 we showed how theorem 1 is used to prove that every Classical Propositional tautology in the fragment $\{ \rightarrow, \bot \}$ has short proofs. As $\{ \bot, \rightarrow \}$ is a functionally complete set of propositional connectives, we know that for each formula $\alpha$ in the full propositional language of classical propositional logic there is a polynomially sized implicational formula $\alpha^*$, on $|| \alpha ||$, such that, $\alpha^*$ is a tautology, if and only if, $\alpha$ is a tautology. Thus, $\alpha^*$ has a short proof, if and only if, $\alpha$ has a short proof too. From what is discussed in this paragraph, we can conclude that $NP = CoNP$.

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