On improved Cepheid distance estimators

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Abstract. The use of Cepheids as distance indicators on Galactic and extragalactic distance scales is based upon the Cepheid period - luminosity (PL) and period - luminosity - colour (PLC) relations. These relations are usually derived in terms of the properties of Cepheids at mean light i.e. averaged over their pulsation cycle. In this paper, we derive a physical argument for the existence of PL and PLC relations at maximum light. We examine in detail a sample of Cepheids in the Large Magellanic Cloud, and compare the variance of some PL and PLC type distance indicators based on mean and maximum light.

We show that for the LMC data considered, a PLC relation based on maximum light leads to a distance estimator with a dispersion about 10\% smaller than its counterpart using mean light. We also show that for the LMC, a PLC type relation constructed using observations at both maximum and mean light has a significantly (\(\geq 50\%\)) smaller dispersion than a PLC relation using either maximum or mean light alone. A comparable (\(\geq 30\%\)) reduction in the dispersion of the corresponding distance estimator, however, in this case requires the relation be applied to a large \((n > 30)\) group of equidistant Cepheids in, e.g., a distant galaxy. Recent HST observations of IC4182, M81 and M100 already provide suitable candidate data sets for this relation. The use of maximum light in constructing PLC type relations for galactic and extragalactic Cepheids is, therefore, shown to be an interesting topic for further study. These investigations are under way.

Key words: Stars: oscillations, Stars: fundamental parameters, Cosmology: distance scale

1. Introduction

Cepheids are high luminosity radially pulsating variable stars. Their intrinsic brightness ranges from \(-2 > M_V > -6\) and makes them suitable candidates for distance indicators on Galactic and extragalactic distance scales. This is based on the Cepheid period - luminosity (PL) and period - luminosity - colour (PLC) relations. Examples of such PL and PLC relations for the Magellanic Clouds are given in Figure 4 of Madore and Freedman (1991) and in Figure 3 of Caldwell and Coulson (1986; hereafter CC) respectively. In these relations the luminosity, measured by the magnitude, and colour are mean quantities taken over the pulsational cycle. Motivated by the work of Simon, Kanbur and Mihalas (1993; hereafter SKM), in this paper we derive a physical argument for the existence of PL and PLC relations for Cepheids based on their properties at maximum light. This argument thus provides a physical justification for the work of Sandage and Tammann (1968), who introduced a PL relation at maximum light. However, we have extended their work by the introduction of a colour term and the simultaneous use of maximum and mean light.

We then examine in detail a sample of Cepheids in the Large Magellanic Cloud (LMC), using multicolour photometric data originally presented in Martin and Warren (1979) and discussed in Martin, Warren and Feast (1979; hereafter MWF). These data have (B-V) colours and were used purely as an illustration since this was the only data available to us at the time. The principles of our analysis are applicable in any wavelength range, however, and the extension of our analysis to other data sets will form part of our future work. Following the statistical formalism described in Hendry and Simmons (1990, 1994; hereafter HS90, HS94), we obtain ‘optimal’ (in the sense of unbiased and minimum variance) distance estimators corresponding to several different PL and PLC-type relations derived from this calibrating sample. We show that distance estimators based on the properties of the Cepheid at maximum light can have significantly smaller variance than those derived for Cepheids at mean light.

The paper is organised as follows: Sect. 2 describes the theoretical basis for the PL and PLC relations, originally introduced by Sandage (1958). Sect. 3 explains the physical reasoning behind our reformulation and extension of these relations in terms of Cepheid maximum light and colour. Sect. 4 outlines briefly how one may use our new relations to derive the corresponding Cepheid distance indicators. In the appendix we discuss the statistical model with which we derive these distance indicators, and describe the significance tests which we used to compare the variance of distance estimators defined for Cepheids at mean and maximum light. In Sect. 5 we describe the LMC Cepheid data upon which our analysis is based, and test the validity of the assumptions made in the appendix concerning the statistical properties of this sample. Sect. 6 presents our results and discussions. In Sect. 7 and 8 we report our conclusions and point out further work in this area.
2. Cepheid relations at mean light

The theoretical basis for the empirical Cepheid PL and PLC relations was first outlined by Sandage (1958) as a consequence of the period mean density relation for pulsating variable stars, the Stefan Boltzmann law and the existence of a linear mass luminosity relation for Cepheids. We outline this argument below.

The period mean density relation states that

\[ P^{3/2} = Q \] (1)

where \( P \) is the Cepheid period, \( Q \) is a slowly varying function of stellar parameters and \( \overline{\rho} \), the mean density, satisfies

\[ \overline{\rho} \propto M R^{-3} \] (2)

where \( M \) is the total mass and \( R \) is the radius of the star. The Stefan Boltzmann law, however, states that

\[ L = \alpha R_{eq}^2 T^4 \] (3)

where \( L \) and \( R_{eq} \) are the equilibrium luminosity and radius respectively, and \( T \) is the effective temperature. For Cepheids, we assume that the equilibrium \( L \) and \( R_{eq} \) are close to their average values over a pulsational cycle. It then follows that

\[ R^{3/2} \propto L^{3/4} T^{-3} \] (4)

Substituting equations (2) and (4) into equation (1), the period mean density relation, and taking logarithms we obtain

\[ \log P + \frac{1}{2} \log M - \frac{3}{4} \log L + 3 \log T = \log Q \] (5)

If we assume that there exists a power law mass-luminosity relation for Cepheids,

\[ \log L = \alpha \log M + \beta \] (6)

where \( \alpha \) and \( \beta \) are constants, then equation (5) may be reduced to

\[ \log P + \left(\frac{1}{2\alpha} - \frac{3}{4}\right) \log L + 3 \log T = \log Q + \frac{\beta}{2\alpha} \] (7)

Equation (7) is the theoretical basis for the Cepheid PLC relation, and its projection onto the plane of mean magnitude and period gives the PL relation – examples of which are given in Figure 4 of Madore and Freedman (1991). It can be seen that there is a scatter about the regression line of mean magnitude on period. That is, for a given period there is a range of mean magnitudes. This dispersion results from a combination of several different factors – including the effect of reddening, which may not be properly corrected, and observational errors in the apparent visual magnitudes. Another factor contributing to the dispersion is intrinsic, however, and is caused by the finite width of the instability strip in the HR diagram.

A Cepheid can be brighter than that luminosity given by the regression line by having a hotter surface temperature than a Cepheid of the same period but with a mean magnitude given exactly by the regression line. Similarly, a Cepheid of given period can be dimmer than the regression line by having a cooler surface temperature. This is completely in accord with equation (7). Hence the possible range of Cepheid surface temperatures (ie. the range of \( T \) in equation (7) at a given period), as well as the other factors mentioned above, leads to the scatter in the Cepheid PL relation (Sandage 1958). The possible range of Cepheid surface temperatures is determined by the width of the instability strip - the locus of \( L \) and \( T \) points in the HR diagram in which pulsation occurs. It is also clear that a smaller range of \( T \) in equation (7) will lead to a tighter correlation between \( P \), \( \log L \), and \( \log T \). We discuss instances in which this is the case in the next section.

There has been considerable debate in the literature over the exact cause of the scatter in the Cepheid PL relation. Some authors claim that observational errors and an incorrect allowance for reddening combined with a narrow, but non zero, intrinsic width of the instability strip mean that it is difficult to disentangle the effects of reddening and intrinsic temperature variations (Madore and Freedman 1991; Clube and Dawe 1980; Stift 1982, 1990). Others maintain that observations and estimates of reddening and colour excess are accurate enough to imply that the scatter in the PL relation is due to intrinsic temperature variations (MWF; CC; Feast and Walker 1987). The arguments presented in these papers and the work of Laney and Stobie (1986) who showed the existence of a significant colour term in the infrared PLC relation for LMC Cepheids provide ample support for this latter view.

3. The Cepheid at Maximum light

At maximum light, the period mean density law is still valid. We assume, moreover, that the Stefan Boltzmann law still applies,

\[ L_{max} \propto R_{max}^2 T_{max}^4 \] (8)

where \( L_{max} \) is the maximum luminosity, \( R_{max} \) is the radius at light maximum and \( T_{max} \) is the photospheric temperature at maximum light. In the optical and ultraviolet most of the light variations in Cepheids are caused by temperature fluctuations, not radius variations (c.f. Cox 1974, McGonegal et al 1982) and in any case maximum light occurs when the star is expanding through its equilibrium radius (Cox 1974; SKM), so we can also assume,

\[ R_{max} \approx R \] (9)

where \( R \) is the equilibrium radius of the star. However, even at longer wavelengths, the relative radius fluctuations are no more than five to ten percent (Cox 1974) and equation (9) will still be approximately valid. Thus our physical derivation would still be applicable in the infrared. Therefore, using equation (8) and (9) we obtain,

\[ R^{3/2} \propto L_{max}^{3/4} T_{max}^{-3} \] (10)

Substituting equation (10) into the period mean density law, taking logarithms and assuming equation (6), yields

\[ \log P + \frac{1}{2\alpha} \log L - \beta - \frac{3}{4} \log L_{max} + 3 \log T_{max} = \log Q \] (11)

Equation (11) suggests the existence of a relation between period, mean luminosity, maximum luminosity and maximum temperature. The appearance of the \( T_{max} \) term in equation (11) is important because at maximum light, the range of Cepheid
photospheric temperatures is smaller (about 600 K) than at mean light (about 1000 K), as was shown in SKM. This is because at maximum light the photosphere occurs at the base of the hydrogen ionization zone, independent of the period of the Cepheid. For typical Cepheid densities and temperatures, the hydrogen ionization zone occurs at about 6200 K, although very long period Cepheids \( (P > 40 \text{ days}) \) have such extended envelopes that the photosphere is further out in the envelope. Consequently these longer period Cepheids have at maximum light a range of photospheric temperatures similar to that at mean light (SKM). If one considers only Cepheids of shorter period \( (P < 40 \text{ days}) \), however, then one would expect the range of \( T_{\text{max}} \) in equation (11) to be smaller than the range of \( T \) in equation (7). Consequently one can expect that PL and PLC relations based on maximum light may have smaller scatter than those based on mean light.

Another advantage of this approach is the following. The Stefan Boltzmann law applied at mean and maximum light yields,

\[
L^{-3/4} T_0^3 = L_{\text{max}}^{-3/4} T_{\text{max}}^3
\]

which implies,

\[
-\frac{3}{4} \log L + 3 \log T_0 = -\frac{3}{4} \log L_{\text{max}} + 3 \log T_{\text{max}}
\]

Substituting for \( \log T_0 \) from equation (7) into the above equation yields equation (11). We are constructing a relationship using the properties of the pulsation at two phase points (mean and maximum) to obtain equation (11), rather than using just the properties at mean light as in equation (7). Hence Cepheid relations based on equation (11) incorporate more information about the pulsation than their counterparts based on equation (7).

We can rewrite equation (11) in the following form:

\[
\log P + \frac{1}{2 \alpha} (\log L - \log L_{\text{max}} + \log L_{\text{max}} - \beta) - \frac{3}{4} \log L_{\text{max}} + 3 \log T_{\text{max}} = \log Q
\]

which implies,

\[
\log P + \frac{1}{2 \alpha} (\log L - \log L_{\text{max}}) + (\frac{1}{2 \alpha} - \frac{3}{4} \log L_{\text{max}} + 3 \log T_{\text{max}} = \log Q
\]

Equation (15) suggests the possibility of a period, maximum light, semi-amplitude relationship.

We can convert equation (11) into a form which can be more easily compared with observations by writing,

\[
\log L = 0.4 (c - M_{\text{bol}})
\]

and

\[
\log L_{\text{max}} = 0.4 (c - M_{\text{bol}})_{\text{max}}
\]

where \( M_{\text{bol}} \) denotes the absolute bolometric magnitude, which is related to the absolute visual magnitude, \( M_v \), by

\[
M_v = M_{\text{bol}} + \text{BC}
\]

We can convert the bolometric correction, BC, and the temperature to an observed colour using relations of the form,

\[
\text{BC} = a + b \log (B - V)_0
\]

and

\[
\log T = x + y(B - V)_0
\]

where \( a, b, x \) and \( y \) are constants and \( (B - V)_0 \) is the dereddened colour.

Combining equations (15) - (20) we have

\[
\log P - \frac{0.2}{\alpha} M_v + 0.3 M_{\text{max}} + (\frac{0.2}{\alpha} b + 0.3 b + 3 y)(B - V)_0 = \text{constant}
\]

Clearly one can also convert the period, maximum light, semi-amplitude relation suggested by equation (15) into a similar form involving magnitudes and colours. Much as equation (7) is the theoretical justification for the Cepheid PLC and PL relations at mean light \( (\text{Sandage 1958; Sandage and Tammann 1968}) \), equation (11) is the theoretical justification for Cepheid PLC and PL relations at maximum light. Although Sandage and Tammann (1968) constructed a period maximum light relation for Cepheids in the Large and Small Magellanic Clouds, M31 and NGC 6822, to our knowledge the above argument has never been given.

4. Cepheids as distance indicators

As we indicated in Sect. 1, our main motivation in studying and extending PL and PLC relations for Cepheids is their usefulness as primary distance indicators. In this section we illustrate how one would apply our new relations, calibrated with a sample of Cepheids of known distance, to infer the distance of other Cepheids grouped in a more distant galaxy or cluster. We follow the statistical formalism and notation adopted in HS90 and HS94 in discussing optimal galaxy distance indicators such as the Tully–Fisher or \( D_n - \sigma \) relations. In particular, we adopt the standard statistical convention of denoting an estimator of a quantity by a caret. In order to avoid a surfeit of confusing subscripts we also drop the subscript ‘v’ from the absolute and apparent visual magnitude.

The basic relationship between the absolute magnitude, \( M \), and apparent magnitude, \( m \), (assumed corrected for absorption) for an object at distance \( D \) Mpc is

\[
m = M + 5 \log D + 25
\]

For Cepheids, these magnitudes are usually taken to be the mean value over their pulsation cycle. Clearly, however, for any Cepheid we also have

\[
m_{\text{max}} = M_{\text{max}} + 5 \log D + 25
\]

An obvious estimate of the distance, \( D \), (or more correctly the log distance) of the Cepheid is then

\[
\hat{\log D} = 0.2 (m - \hat{M} - 25)
\]

or

\[
\hat{\log D} = 0.2 (m_{\text{max}} - \hat{M}_{\text{max}} - 25)
\]

Here \( \hat{M} \) and \( \hat{M}_{\text{max}} \) denote estimators of the mean and maximum absolute magnitude respectively. Examples of such estimators include
\[ \hat{M} = a_1 + b_1 \log P \]  
(26)

or

\[ \hat{M} = a_2 + b_2 \log P + c_2(B - V)_0 \]  
(27)

where \( P \) and \( (B - V)_0 \) take their usual meanings and \( a_1, b_1, a_2 \) and \( b_2 \) are constants. The motivation for considering estimators of this form is clearly the PL and PLC relations – the theoretical basis for which has already been described in the preceding sections.

The constant coefficients in equations (26) and (27) are obtained by fitting such relations to the relevant observations – i.e. to the apparent visual magnitude, period and colour – for Cepheids of known distance. For example, if the calibrating Cepheids are all at distance, \( D_{\text{cal}} \) Mpc, then we may rewrite equation (26) as

\[ \hat{m} - 5 \log D_{\text{cal}} - 25 = a_1 + b_1 \log P \]  
(28)

In other words the estimator of absolute visual magnitude given by equation (26) is equivalent to the estimator of apparent visual magnitude given by equation (28). That is, in order to best fit the intrinsic relations involving mean and maximum absolute magnitudes, we need only consider the corresponding relations involving apparent magnitudes, provided all the calibrating stars are assumed equidistant. Similar remarks apply to all the PLC-type relations discussed in this paper.

The fitting procedure mentioned above is usually a maximum likelihood or linear regression analysis, and requires a statistical model for the relationship between the relevant variables. A suitable model for the PL relation, for example, might be

\[ M = a + b \log P + \epsilon \]  
(29)

where the errors, \( \epsilon \), are taken to be normally distributed with mean 0 and variance \( \sigma^2 \). Given such an error model, the estimator, \( \hat{M} \), given by equation (26) with \( a_1 = a \) and \( b_1 = b \) corresponds to the expected absolute magnitude conditional upon log period, and the values of \( a \) and \( b \) are in this case equal to the slope and zero point respectively of the direct linear regression of \( M \) on \( \log P \). An example of a more sophisticated model for the PLC relation is given in the appendix of this paper. In the appendix of this paper we describe the statistical model adopted in the present analysis.

Given an appropriate statistical model it is straightforward to determine the distribution, \( p(\log \hat{D} | D_T) \), of \( \log \hat{D} \) conditional upon the true Cepheid distance, \( D_T \). One can then use the moments of this distribution as a suitable criterion for comparing the properties of different estimators. For example, a desirable property is that \( \log \hat{D} \) be unbiased. This requires that \( \log \hat{D} \) satisfies the relation

\[ E(\log \hat{D} | D_T) = \int \log D \ p(\log D | D_T) \ d\log D = \log D_T \]  
(30)

(c.f. eqn. [4] of HS94).

In other words \( \log \hat{D} \) will, on average, yield the true log distance of the Cepheid, whatever that true distance is. In a similar manner the risk, \( \mathcal{R} \), of an estimator is defined as the expected value of the second moment of \( p(\log \hat{D} | D_T) \), i.e.

\[ \mathcal{R} = \int (\log D - \log D_T)^2 \ p(\log D | D_T) \ d\log D \]  
(31)

(c.f. eqn. [5] of HS94). Note that for an unbiased estimator the risk is identically equal to the variance. In this study we will consider as optimal log distance estimators which are unbiased and which have minimum risk or variance – a criterion which we discuss more fully in the appendix.

Generally, better physics should translate into better statistics. For example, equation (27) reflects a more complete physical model for \( \hat{M} \) than that given by equation (26), and we would therefore expect that the distance estimator of equation (24) would have a smaller variance if \( \hat{M} \) is given by equation (27) rather than equation (26). In fact, previous studies such as MWF and CC have indeed shown that for LMC Cepheids, the introduction of a colour term offers a significant reduction in scatter over a PL relation.

In this study we investigate the variance of log distance estimators taking the form of equations (24) and (25), where our models for \( \hat{M} \) and \( M_{\text{max}} \) are prompted by equations (11) and (15) respectively.

It follows trivially from equations (22) - (25) that for an unbiased estimator \( \mathcal{R} \) is given by

\[ \mathcal{R} = 0.04 \sigma^2 \]  
(32)

where an asterisk denotes ‘mean’ or ‘maximum’ magnitude as appropriate. The rms percentage distance error, \( \Delta \), of log \( \hat{D} \) is given by

\[ \Delta \simeq 20 \ln 10 \sigma_{\hat{M}} \]  
(33)

Specifically, in this paper we examine the following estimators of mean or maximum absolute magnitude.

\[ \hat{M} = a + b \log P \]  
(34)

\[ \hat{M}_{\text{max}} = a + b \log P \]  
(35)

\[ \hat{M} = a + b \log P + c(B - V) \]  
(36)

\[ \hat{M}_{\text{max}} = a + b \log P + c(B - V)_{\text{max}} \]  
(37)

\[ \hat{M}_{\text{max}} = a + b \log P + c(B - V)_{\text{max}} + d(M_{\text{mean}} - M_{\text{max}}) \]  
(38)

\[ \hat{M}_{\text{max}} = a + b \log P + c(B - V)_{\text{max}} + dM_{\text{mean}} \]  
(39)

Of course the constants \( a, b, c \) and \( d \) are different in each case. Again, to avoid a surfeit of subscripts we have dropped the subscript ‘0’ from \( (B - V) \) and \( (B - V)_{\text{max}} \). Henceforth, unless we state otherwise, all colours will be assumed to be corrected for reddening. Equations (34) and (36) are the standard Cepheid PL and PLC relations at mean light. Equation (35) is a Cepheid PL(max) relation, as described in Sandage (1968). Equations (37), (38) and (39) are new and are based upon equations (11) and (15).

Note that the semi-amplitude term on the right hand side of equation (38) involves the true value of both \( M_{\text{mean}} \) and \( M_{\text{max}} \), neither of which is directly observable (otherwise we would have no need to estimate them). This presents no difficulty, since \( M_{\text{mean}} - M_{\text{max}} \) may be re-written as \( m_{\text{mean}} - m_{\text{max}} \), i.e. the difference between the mean and maximum apparent magnitude, a quantity which is readily observable. Thus,
\[ M_{\text{max}} = a + b \log P + c(B - V)_{\text{max}} + d(m_{\text{mean}} - m_{\text{max}}) \] (40)

The presence of \( m_{\text{mean}} \) on the right hand side of equation (39), however, cannot be dealt with in this way. In order to overcome the obvious problem that \( M_{\text{mean}} \) is not directly observable, we replace equation (39) by

\[ \hat{M}_{\text{max}}' = a + b \log P + c(B - V)_{\text{max}} + d(m_{\text{mean}} - < M_{\text{mean}}>) \] (41)

where \( < M_{\text{mean}} > \) denotes the sample averaged absolute magnitude at mean light of a group of equidistant Cepheids (e.g. in a distant galaxy or cluster, the distance of which we wish to estimate). Of course the zero point, \( a \), of equation (41) will differ from that in equation (39). Note that \( M_{\text{mean}} - < M_{\text{mean}} > \) is a distance independent quantity (as is \( M_{\text{mean}} \)) but is directly observable, being equal simply to \( m_{\text{mean}} - < m_{\text{mean}} > \). Note also, however, that the distribution of \( < M_{\text{mean}} > \) depends upon the sample size of Cepheids, and in particular, that one could not sensibly apply equation (41) to estimate the maximum absolute magnitude of an individual field Cepheid.

Some algebra easily establishes that the variance of \( \hat{M}_{\text{max}}' \) is given by the sum of the variance of \( M_{\text{max}} \) in equation (39) and the variance of \( < M_{\text{mean}} > \), i.e.

\[ \sigma_{\hat{M}_{\text{max}}'}^2 = \sigma_{\hat{M}_{\text{max}}}^2 + \sigma_{< M_{\text{mean}}>}^2/n \] (42)

where \( \sigma_M \) is the dispersion of the intrinsic distribution of magnitudes at mean light (which of course is not known a priori, but which is estimated from the LMC calibrating Cepheids) and \( n \) is the number of observed Cepheids in the more distant galaxy or cluster. The risk, \( \mathcal{R} \), of the corresponding log distance estimator is given by \( \mathcal{R} = 0.04\sigma_{\hat{M}_{\text{max}}'} \), in accordance with equation (32) above.

Finally in this section we should note that the above analysis takes no account of the metallicity dependence of our fitted relations. Although the Cepheid PL relations (34) and (35) are insensitive to composition differences, this will not be the case for the remaining PLC-type relations. One can overcome this problem as follows. One fits equations (36) - (39) to the LMC data and then adjusts the coefficients of the fit (using eg. Table (b1) in CC) to give the PLC relation of a normal metal abundance Cepheid. The zero point of this corrected relation can then be then calibrated using Galactic Cepheids of known distance. CC use the results of Iben and Tuggle (1975), Becker, Iben and Tuggle (1977) and Bell and Gustafsson (1978) to obtain their Table (b1). The composition dependence of the Cepheid mass luminosity law is part of the reason for the metallicity sensitivity of the PLC. It is not known exactly how this will affect e.g. the PLC(max) relation. Since Feast and Walker (1987) suggest that the metal deficiency of LMC Cepheids is slight, however, (1.4 times less than solar), it seems likely that the difference between the values of \( b \) and \( c \) in equation (37) and those appropriate for a metal normal Cepheid will be small. Because our primary intention in this paper is to establish how the use of Cepheids at mean and maximum light affects the dispersion of our distance estimators, we leave a more precise zero point calibration of relations (36) to (39) – accounting for metallicity dependence as described above – for subsequent papers.

5. LMC Data

In order to test our assertion of the existence of Cepheid relations of the form suggested by equations (7), (11) and (15), we use multicolour photoelectric observations of LMC Cepheids taken by Martin and Warren (1979) and discussed in MWF. These observations were taken in the BVI system. The list of stars used in this analysis is given in Table 1, together with their period in days. Note that some of the stars in the original data set presented in Martin, Warren and Feast (1979) were omitted because there were not enough observed points to obtain accurate mean and maximum magnitudes. These excluded stars are shown in Table 2. Rejecting these stars resulted in the total sample of 39 stars shown in Table 1. MWF and Feast (1984) have discussed the small dispersion in reddenings for their LMC Cepheids: hence we adopt a constant value of \( E(B - V) = 0.1 \) given by Madore and Freedman (1991). We take \( R = A_V/E(B-V) \) to be 3.3, again following MWF. Since the only data set available to us was in magnitudes and not intensity fluxes, our PL and PLC relations are determined in terms of magnitudes.

Figure 1 shows a plot of period vs. semi-amplitude for all 39 stars in our sample. It clearly shows a group of stars at long period which are well separated from the other stars in the sample. Rather than postulate why this is so, we will present results with and without this set of stars. These stars are presented in Table 3, marked by an asterix in the third column. Recall that in Sect. 3 and in SKM it was suggested that at maximum light Cepheids have a smaller range of surface temperatures than at mean light, but that for periods \( P \geq 40 \) days this distinction is not found. Since the range of surface temperatures also introduces scatter, we will consider a third subset of the data consisting only of those stars with periods \( P < 40 \) days. Stars with periods \( P > 40 \) days are denoted by a ‘+’ in Table 3. Moreover, note that the disjoint group of stars listed in Table 3 all have periods \( P > 40 \), and so are also rejected from this third sample. We therefore consider three data sets, with 39, 35 and 31 stars respectively. In the appendix, we introduce a multivariate normal model to describe the joint distribution of the six relevant variables in the Cepheid data: maximum magnitude, mean magnitude, maximum colour, mean colour, log period and semi-amplitude. We require this joint distribution...
to be normal in order to make a strictly valid application in Sect. 6 of our hypothesis tests that the variance of distance estimators based on maximum light is significantly smaller than for those based upon mean light.

We can test the normality of the distribution of each of these variables in several ways. First, we plot a sample cumulative distribution function (CDF) and compare this with the theoretical CDF of a normal distribution with mean and dispersion equal to the sample mean and dispersion of that variable. Figures 2, 3 and 4 show selected results obtained for the samples of 31, 35 and 39 stars respectively. The agreement with the theoretical normal curves is generally quite good, although somewhat worse for the distribution of semi-amplitude than for the other variables. To quantify this agreement we apply the Kolmogorov–Smirnov (KS) test (c.f. Kendall & Stuart, 1963), for which the test statistic, \( D_n \), is defined as the maximum absolute deviation between the sample and model CDF curves. Table 4 lists \( D_{\text{obs}} \), the observed value of the test statistic, for each of the six relevant observables and using each of the three data sets under consideration, together with the corresponding significance of the KS test – i.e. the probability that \( D_n > D_{\text{obs}} \) under the null hypothesis that the sample CDF is drawn from the modelled theoretical distribution.

The main advantage of the KS test is its robustness, however, and it would seem prudent to apply more powerful tests of normality to confirm the validity of our assumptions. We next calculated the sample skewness, \( S \), and kurtosis, \( K \), for each variable, \( x \), defined as

\[
S = \frac{1}{N} \sum_{j=1}^{N} \left( \frac{x_j - \overline{x}}{\sigma} \right)^3
\]

(43)

and

\[
K = \frac{1}{N} \sum_{j=1}^{N} \left( \frac{x_j - \overline{x}}{\sigma} \right)^4 - 3
\]

(44)

where \( \overline{x} \) and \( \sigma \) denote the sample mean and standard deviation of the sample of size, \( N \), respectively.

The intrinsic skewness and kurtosis should both be identically zero for a normal distribution. The sampled value of each statistic will fluctuate around zero, however, and the variance of the sampling distribution depends upon the sample size, \( N \). Under the null hypothesis of a normal variable, the variance
of the sample skewness and kurtosis is approximately equal to 15/N and 96/N respectively (c.f. Kendall and Stuart, 1963).

Table 5 lists the sample skewness and kurtosis, expressed in terms of the number of standard deviations under the null hypothesis of a normal distribution, for each of our six Cepheid observables and for the three different sample sizes considered. From these results we see that the sample of 31 stars shows no significant evidence of non-zero skewness and kurtosis – the largest sampled skewness being $\sim 0.72\sigma$ for mean apparent magnitude. In the larger samples, however, there is some indication of significant skewness above the one $\sigma$ level for mean magnitude and maximum colour.

The final test which we apply is based upon Shapiro and Wilks’ W statistic (Royston, 1982). This is a considerably more powerful test and uses the properties of the order statistics of a normal distribution – i.e. the properties of a sample arranged in ascending or descending order. (c.f. David, 1981; Hendry, O’Dell and Collier Cameron, 1993). The test statistic is given by

$$W = \frac{\sum a_j x_j^2}{\sum (x_j - \overline{x})^2}$$  \hspace{1cm} (45)

where $\overline{x}$ denotes the sample mean and the set of normalising weights, $a_j$, depend upon the sample size, $N$.

The mean value of the statistic under the null hypothesis of a sample drawn from a normal distribution is unity.

Table 6 shows the values of the $W$ statistic and the calculated significance of the test in each relevant case. From these results we see that our sample of 31 stars satisfies well the assumption of normality: only for the observed semi-amplitudes is there any evidence, at the $\sim 5\%$ level, for significant deviation from a normal distribution. In the larger samples, however, the validity of the normal assumption is more marginal. In fact for our full sample of 39 stars only for the mean colour would the null hypothesis be accepted at the 15% level, and for the maximum colour it would be quite strongly rejected, at a level of $\sim 0.01\%$. Clearly, then, we can safely apply hypothesis tests based on normality to the restricted sample of 31 stars, but must be somewhat more cautious in drawing conclusions from their application to the larger samples. We will comment further upon this point in Sect. 6 and Sect. 7.

Finally, it is important to note that – in modelling the joint distribution of the six relevant physical variables as multivariate normal – we do not make any explicit assumptions concerning the nature of the scatter in their observed distribution. In particular we do not attempt to separate intrinsic assumptions concerning the nature of the scatter in their observed distribution. Undoubtedly some component of the variance in our fitted distance relations will be due to observational errors and related effects such as line of sight spread in the true distance of the calibrating stars – both effects which are, at least in principle, removable. Our point is that the relative contribution of intrinsic scatter and observational errors is unlikely to differ dramatically in each of our distance estimators. Hence our conclusions concerning the relative reduction in variance resulting from the use of magnitudes and colours at maximum light will not be significantly changed – even if the absolute value of the variance could be reduced in both cases by the use of more precise observations (observations which – notwithstanding these remarks – are now available from HST and 8m class terrestrial telescopes, and from the use of observations at redder wavelengths).

6. Results and Discussion

In this section we present the results of fitting the coefficients of the relations defined in equations (34) - (39) by a multilinear regression model as described in the appendix. Each relation has been fitted using the three LMC samples – of 31, 35 and 39 stars respectively. The relations have been converted to absolute magnitudes at mean and maximum light assuming an LMC distance modulus of 18.5 mag for the LMC (c.f. Madore and Freedman, 1991; Pierce et al., 1994). Tables 7 and 8 present fitted PL and PLC relations respectively at mean and maximum light, and also a period, luminosity, semi-amplitude (PLA) relation - again fitted at both mean and maximum light. Table 9 presents the results of fitting a period, maximum luminosity, mean luminosity, maximum colour (PLL'C) and period, maximum luminosity, semi-amplitude, maximum colour (PLAC) relation.

The table headings indicate the form of the fitted relation and the variables used in the fit. The next rows indicate the values of the regression coefficients obtained. The final two columns of each table give the variance of $M_\star$ derived from the linear regression (where, as before, a star indicates mean or maximum light as appropriate) and the percentage rms error of the corresponding distance estimator, calculated from equa-
tation (33) - with the exception of the PLL'C relation in table 9, the distance error dispersion of which we discuss separately. Next to the fitted regression coefficients for each relation we give their computed standard errors. For the coefficients of the relevant physical variables in each fit, we also indicate the computed probability of obtaining a sample regression coefficient larger in modulus than the estimated value, under the null hypothesis that the true regression coefficient is identically zero, applying the t test described in the appendix. This provides a clear and useful indication of the relative importance of the different independent variables in each relation. Finally, for the relations with two or more independent variables, we also present beneath each table the results of a second significance test involving the partial sample multiple correlation coefficient (SMCC) of each regression fit, as described in the appendix. This test provides a direct measure of the significance of including the final independent variable in reducing the overall dispersion of the fit. We list the partial SMCC for each fit and the probability that the value of the test statistic W be greater than its observed value under the null hypothesis that the true partial SMCC be identically zero: i.e. that the final independent variable makes no contribution to the reduction of the scatter in the relation.

6.1. PL relations

Table 7 presents our results for equations (34) and (35), the PL relations at mean and maximum light. Note that there is hardly any difference in the variance of the fit, or the dispersion of the corresponding distance estimator, between mean and maximum light for each of the three samples. As one would naturally expect, the regression coefficient of log P is in all cases highly significant: cepheid magnitudes at mean or maximum light are not well described by a constant. The coefficient of log P in each case varies from about -2.0 to -2.5, and is larger in modulus for the maximum light relation for each sample size, with comparable standard error - although the difference is always within 2 σ. This range is somewhat different from existing PL relations for the LMC – for example -2.88 (MF), -2.69 (CC) and a range from -2.59 to -2.90 (MWF). Table 3 of MWF lists the data set used in their work: as we noted in Sect. 5, our data set is a subset of this sample. The results of table 7 indicate, therefore, that the coefficient of log P in a linear least squares fit to a subset of the MWF data set differs from the value obtained when the entire data set is used. This does reinforce the fact that PL relations are inherently statistical in nature and care must be taken in comparing the results of studies which sample different period ranges – a point also recently addressed in Pierce et al., 1994. In the present context, this means that we can usefully compare the dispersion of different distance relations obtained for the same sample – of 31, 35 or 39 stars – but must be more cautious in comparing results for a given relation across the three samples, particularly in the light of the normality test results described in Sect. 5.

6.2. PLC and PLA relations

Table 8 presents our results for PLC relations, based on equations (36) and (37), at mean and maximum light. We can see that there is an appreciable reduction in the dispersion of each relation, and for all three samples, compared with the corresponding PL relations. This conclusion is borne out in a quantitative manner in several ways. First note that the t test applied to the regression coefficients strongly rejects the null hypothesis of a zero regression coefficient for colour in all cases, although it is interesting to note that the significance of a non-zero regression coefficient of log P is still considerably higher than that for colour, suggesting that the greater contribution to the reduction of scatter is coming from the PL relation. Notwithstanding this, the results of the W test applied to the partial SMCCs for each relation do confirm that the addition of colour significantly reduces the scatter compared with the PL relation. In all cases the null hypothesis of no reduction in scatter is rejected strongly, with a probability of acceptance of 4.9 × 10^-4 or less. Hence, our application of this hypothesis test reconfirm the results of MWF and CC who established the existence of a colour term for LMC Cepheids. The coefficients of log P and (B − V)_{mean} which we have obtained are in all cases within one standard error of the values given by MWF and CC.

We can see from table 8 that the PLC relation at maximum light leads to a distance estimator with about 10% less dispersion than its counterpart at mean light for the three samples. This result is further supported by the fact the partial SMCC test more strongly rejects the null hypothesis for the PLC relation at maximum light than at mean light. Applying an F test to determine the significance of this reduction in dispersion, we find a significance of 0.27, 0.24 and 0.34, for the samples of 31, 35 and 39 stars respectively. Thus, our sample of calibrating Cepheids is still too small to confirm at a high significance level the improvement of the PLC relation at maximum light. Figures 5 and 6 show plots of period against (B − V) colour at mean and maximum light respectively. From these plots one can conclude that there is only a very small reduction in the width of the instability strip at maximum light for our LMC sample: a fact which may well account for the inferred reduction in scatter of only 10% in the maximum light PLC relation. Moreover, it can be seen from figure 3 that the maximum (B − V) colour is essentially independent of period for log P < 1.5 (c.f. SKM). Figure 7 of SKM, however, illustrates that the temperature range of Galactic Cepheids at maximum light is only about 600 K. This suggests that a PLC relation at maximum light constructed for Galactic Cepheids may lead to bigger reduction in scatter compared to that at mean light.

Fig. 5. log period plotted against unreddened B − V colour at mean light, for the full sample of 39 LMC stars
Fig. 6. log period plotted against unreddened $B - V$ colour at maximum light, for the full sample of 39 LMC stars

Table 8 also presents the results of fitting PLA relations to the LMC data, which represent a projection of the PLAC relation suggested by equation (38) and considered below. In all cases it was found that the addition of a semi-amplitude term did not significantly reduce the dispersion of the relation. The partial SMCC test accepted the null hypothesis at a level of more than 10% and the regression coefficient of semi-amplitude was not significantly different from zero in any case. What is interesting to note, however, is that the PLA relation appeared to fare best when applied to the full sample of 39 stars in the sense that, in this case, the null hypothesis of the partial SMCC test is accepted least strongly. This is perhaps not too surprising since the 39 star sample includes the group of four stars which, from figure 1, clearly have unusual amplitudes - and one might expect that a relation involving an amplitude term would be most readily able to accommodate such stars. It would be unwise to attach too much importance to this result, however, since we have shown in Sect. 5 that the semi-amplitudes satisfy the assumption of normality least to this result, however, since we have shown in Sect. 5 that such stars. It would be unwise to attach too much importance to introducing the sample averaged absolute magnitude at mean light in a group of distant Cepheids -- effectively turning the $M_{\text{mean}}$ term in equation (39) into a quantity which is both distance independent and directly observable -- but at the cost of increasing the dispersion of our distance estimator due to the sampling variance of $< M_{\text{mean}} >$, and thus restricting the application of our PLL'C relation to a sufficiently large group of distant Cepheids.

In figure 7 we plot the rms percentage error dispersion, $\Delta^\ast$, of the corresponding distance estimator, divided by the percentage distance error of the PLL'C relation at maximum light, as a function of $n$, the number of distant Cepheids observed. We use the values of $\sigma_n^2 M_{\text{mean}}$ and $\sigma_n^2$ derived from the LMC sample of 31 stars, and compute $\Delta^\ast$ using equations (33) and (42). We can see from figure 8 that $\Delta^\ast$ falls off quite slowly with $n$. Moreover, we see that for small samples of 10 or fewer Cepheids $\Delta^\ast$ is greater than unity -- i.e. the resultant distance error dispersion of the PLL'C relation is in fact larger than that for the PLC relation. This is because the reduction in the variance of $M_{\text{mean}}$ is more than cancelled out by the large sample variance of $< M_{\text{mean}} >$. For larger values of $n$, however, we do see an appreciable reduction in the dispersion of the distance estimator, falling to $\sim 63\%$ for $n = 30$ and to $\sim 53\%$ for $n = 40$. It seems, therefore, that our modified PLL'C can significantly improve upon the PLC relation for fairly large but realistic sample sizes of distant Cepheids. Similar results are obtained for the samples of 35 and 39 Cepheids.

Of course we should note here that we are neglecting the effect of sampling variance on the determination of the distribution parameters in our LMC calibrating data: clearly with a calibrating sample of less than 40 stars this effect cannot be considered negligible. It will have no bearing upon our comparison of the relative dispersion of any of our estimators, however, since it will affect every relation in precisely the same way. Moreover, its effect on the calibrating sample is in principle removable as the number of calibrating Cepheids is increased.

Finally, we observe from table 9 the important fact that the regression coefficient of log $P$ for all three samples is significantly different from the value predicted from equation (21), i.e. $-3.33$. In fact, if we force the coefficient of log $P$ to be $-3.33$ in our regression fit, we find that the coefficient of $M_{\text{mean}} - < M_{\text{mean}} >$ is fitted to be essentially zero. In other words the fit reverts to the PLC relation at maximum light, with correspondingly increased dispersion. A similar sensitivity is, in fact, found with the PLC relation at mean light, for which the theoretically expected coefficient of log $P$ is about when stars of unusual amplitude for their period are considered.

The PLL'C relation, on the other hand, is seen to have a considerably reduced dispersion -- by a factor of two or more -- for all three samples. The partial SMCC null hypothesis is now very strongly rejected and the regression coefficient of $M_{\text{mean}} - < M_{\text{mean}} >$ differs from zero with a very high significance and small standard error.

The outstanding difficulty with the PLL'C relation lies with how one can convert it into a useful distance estimator. The basic problem is that absolute magnitude at mean and maximum light are related to their apparent magnitude counterparts via an identical function of distance, so that equation (39) may be distance-degenerate: i.e. it yields essentially no distance information if the coefficient of $M_{\text{mean}}$ is close to unity. As we saw in Sect. 4, we have attempted to overcome this problem by introducing the sample averaged absolute magnitude at mean light in a group of distant Cepheids -- effectively turning the $M_{\text{mean}}$ term in equation (39) into a quantity which is both distance independent and directly observable -- but at the cost of increasing the dispersion of our distance estimator due to the sampling variance of $< M_{\text{mean}} >$, and thus restricting the application of our PLL'C relation to a sufficiently large group of distant Cepheids.

6.3. PLL'C and PLAC relations

Table 9 presents the results of our fits to the PLL'C relation, in the modified form of equation (41), and PLAC relation given by equation (38). Firstly note that the PLAC relation does not offer a significant reduction in dispersion over the PLC relation at mean or maximum light in the samples of 31 or 35 stars, consistent with the results of the previous section. There is a significant improvement in the relation for the 39 star sample, however. The partial SMCC null hypothesis is rejected at the 3% level and the regression coefficient of semi-amplitude is consistent with zero at a similarly small probability. This would again seem to be due to the inclusion of the group of stars of unusual amplitude in our sample. Notwithstanding our earlier note of caution about the consistency of the semi-amplitude distribution with a normal distribution, we nevertheless believe this result to indicate that a semi-amplitude term has an important role to play in improving Cepheid PLC relations -- not as a replacement for, but in addition to, a colour term --
4. Forcing the coefficient of of log $P$ to be equal to this value results in a significant change in the $(B - V)_{\text{mean}}$ coefficient.

The change in the coefficient of log $P$ in the PLL'C relation would appear to be due to the very strong correlation between absolute magnitude at mean and maximum light (which, as we have discussed, renders the relation nearly distance degenerate). Thus, while the measured values of log $P$ and maximum colour continue to provide useful physical information to constrain the inferred absolute magnitude at maximum light, the main contribution to $M_{\text{max}}$ comes from the measured value of $M_{\text{mean}} - < M_{\text{mean}} >$ and it is the relationship between these two variables which dominates the form of the fitted relation. In a sense, therefore, the numerical values of the coefficients in our PLL'C relation are determined as much by the fact that we are combining observations of Cepheids at two phase points as by the underlying physical relationship between period, luminosity and colour.

7. Conclusions

Using the period-mean density relation, the Stefan Boltzmann law and the existence of a linear mass luminosity law for Cepheids, we have derived a new linear relation for Cepheids, connecting the maximum and mean luminosity, the period and effective temperature. This new equation suggests the possibility of deriving PLC relations using observations of Cepheids at maximum light, and applying these relations to estimate the distance of Cepheids. We have fitted these new relations— together with standard PL and PLC relations at mean light—to a sample of Cepheids in the LMC. We have adopted a general statistical model which allows for non-zero correlation between all pairs of the observables, and we have obtained distance estimators which are ‘optimal’ in the sense of being unbiased and having minimum variance.

More specifically, our conclusions are the following:

1. Our PL and PLC results at mean light are similar to those given in MWF and CC. Moreover, we also confirm that the introduction of a colour term for LMC Cepheids produces a significant reduction in scatter over a PL relation.

2. We have derived a PLC relation for LMC Cepheids at light maximum. The introduction of the colour term at maximum light again offers a significant reduction in scatter compared with the corresponding PL(max) relation. Moreover, for these data we find that our PLC(max) relation has around 10 percent less scatter than a PLC relation at mean light.

3. The maximum light, maximum colour, period, semi-amplitude relation generally has a comparable dispersion to that obtained for PLC relations at mean or maximum light, but has a significantly smaller dispersion when stars with unusual amplitudes for their period are included in the sample. This result provides evidence to support the arguments leading to equation (15).

4. The maximum light, maximum colour, period and mean magnitude relation given by equation (39) offers a highly significant reduction in the dispersion of PLC relations at maximum or mean light. In converting the relation into a form which is not close to distance degenerate, we find that the corresponding distance estimator has a dispersion nearly 40% smaller than that derived from a PLC relation at maximum light provided the relation is applied to a sufficiently large ($n = 30$) group of equidistant Cepheids in, e.g., a distant galaxy: an observational constraint which does not seem too unreasonable. This relation appears to derive both from the underlying physical relationship between luminosity, colour and period, and from the fact that Cepheid observations at two different phase points are used in its construction. Our results would seem to offer support for the derivation leading to equation (14), although further analysis of other data sets would prove very useful in order to more fully understand this relation.

8. Further Work

We present below some topics which we feel deserve further investigation based on the results presented in this paper.

1. MWF present individual reddening corrections for many of the stars in table 1. We propose to repeat the analysis using these reddenings instead of using $E(B - V) = 0.1$ for all stars. The use of accurate reddening corrections would be important in the practical implementation of this method for distance determinations. It would also be important to model the geometry of the LMC, as is carried out in CC.

2. CC present data for Cepheids in the SMC. We have started to further test our results on SMC Cepheids and other data sets such as those described in Table (1) of Jacoby et. al (1992).

3. We plan to investigate in more detail our 4 variable estimators using maximum and mean light, period and maximum colour. We would welcome receiving new Cepheid data sets, such as those available from the MACHO project and Hubble Telescope observations, in order to further test and extend the results described here. We also plan to investigate the construction of PLL'C estimators using apparent magnitude data at two arbitrary phase points, not just at mean and maximum light as studied in this paper.

4. We have begun a reexamination of calibrating Galactic Cepheids, as presented in e.g. table 2 of Feast and Walker (1987). Although Fernie and McGonegal (1983) found that the introduction of a colour term did not significantly reduce the scatter of the PL relation at mean light, it would
be interesting to apply the results of this paper to Galactic Cepheids and establish if a significant reduction in dispersion can be achieved in this case at maximum light.

5. Successful completion of (1), (2), (3) and (4) will enable us to properly calibrate our new relations and compute a distance modulus to the LMC and SMC.

6. Madore and Freedman (1991) show that the dispersion in Cepheid PL relations at mean light decreases as wavelength increases. In this work we have used $B-V$ observations. We plan to extend this by using observations at redder wavelengths in the hope that this will further reduce the scatter and in addition serve to reduce the effects of reddening and metallicity. Thus our method might serve to further improve the large body of work carried out in the infrared in the past decade, which has already offered a very significant improvement in Cepheid distance estimators.

7. Given that our new estimators using maximum light hold up to the tests mentioned in (1) and (2) – and given that they depend upon the line of sight dispersion in the true distance of the observed Cepheids being small, it seems appropriate to apply our estimator to Cepheids in galaxies at cosmological distances. Prime candidates for such analysis would seem to be IC 4182 (Saha et. al, 1994) and M100 (Freedman et al, 1994). It should be noted that the former authors adopted a Galactic metallicity for IC 4182. It might be argued that the extra work needed in obtaining light curves with sufficient phase coverage to define an accurate maximum is not justified by the relatively small reduction in dispersion when using maximum light relations. At least in the case of the LMC, however, the Cepheid data already obtained from the MACHO (Alcock et. al 1994) project mean that such observations are already in place.

Much of the above work is well under way and will be reported upon in subsequent papers.

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10. Appendix: the statistical model

The distance estimators introduced in Sect. 4 depend upon various linear combinations of the following six observable quantities: maximum apparent magnitude, mean apparent magnitude, maximum colour, mean colour, log period and semi-amplitude. More specifically, the relations set down in equations (34) - (41) involve using a linear combination of a subset of these observables to infer an estimate of the absolute magnitude at mean or maximum light, which is then combined with the apparent magnitude to obtain a distance estimator following equation (24) or (25). In order to compare the variance of these distance estimators, we therefore require to adopt an appropriate statistical model for the joint distribution of the six intrinsic physical quantities: maximum absolute magnitude, mean absolute magnitude, maximum colour, mean colour, log period and semi-amplitude. (Of course, since the calibrating Cepheids in the LMC are assumed to be equidistant, any model for the distribution of mean and maximum absolute magnitudes will apply equivalently to apparent magnitudes corrected for absorption).

We adopt a model which is general, realistic and non-restrictive, while remaining analytically tractable: an obvious candidate is the multivariate normal distribution. Thus, we as-
sume that the above six physical quantities are drawn from a distribution denoted by,
\[ \mathbf{X} \sim N(\mu, \Sigma) \]  
with probability density function given by,
\[ \left( \frac{1}{(2\pi)^{3/2}|\Sigma|^{1/2}} \right) \exp\left[ -\frac{1}{2}(\mathbf{X} - \mu)^T \Sigma^{-1} (\mathbf{X} - \mu) \right] \]  
where \( \mathbf{X} \) is the vector of variables, \( \mu \) is the vector of their mean values and \( \Sigma \) is the 6×6 covariance matrix describing their mutual correlation, \( \Sigma \) is assumed to be positive definite, real, symmetric but is otherwise arbitrary. In particular, we do not assume \( \Sigma \) to be diagonal; i.e. we allow for all six physical variables to be correlated with each other – as clearly may be the case in reality.

Let the elements of \( \mathbf{X}, \mu \) and \( \Sigma \) be denoted \((X_1, X_2, ..., X_6)\), \((\mu_1, \mu_2, ..., \mu_6)\) and \(\sigma_{ij}\) respectively. The statistical problem with which we are dealing concerns the optimal estimation, or prediction, of \( X_j \) for some \( j \) (denoting the mean or maximum absolute magnitude) from a linear combination of the measured values of some or all of the other variables, \( X_i \). This is an example of a linear prediction problem - a topic which is treated extensively in the general statistics literature (c.f. Kendall and Stuart 1963; Graybill, 1976)

Following the standard notation, we define the best prediction function \( g_1(X_2, ..., X_6) \) of, say, \( X_1 \) based upon the measured values of \( X_2, ..., X_6 \) to be the function such that,
\[ E[(X_1 - g_1(X_2, ..., X_6))^2] \leq E[(X_1 - g(X_2, ..., X_6))^2] \]  
for all other such functions, \( g(X_2, ..., X_6) \), of \((X_2, ..., X_6)\) (c.f. eqn (12.2.2) of Graybill, 1976). That is, the best prediction function is 'best' in the sense of minimum variance and unbiased - i.e. minimum risk, as we define in Sect. 4.

It can be shown (c.f. Graybill 1976; Kendall and Stuart 1963) that when \( \mathbf{X} \) has a multivariate normal distribution, the best linear prediction function of \( X_1 \) based upon the measured values of \( X_2, ..., X_6 \) is
\[ E(X_1|X_2 = x_2, ..., X_6 = x_6) = \beta_1 + \sum_{k=2}^{6} \beta_k x_k \]  
where \( x_j \) denotes the measured value of the variable \( X_j \), and the regression coefficient, \( \beta_j \), is defined by,
\[ \beta_j = \frac{\text{cov}[(X_1, X_j|X_2, ..., X_{j-1}, X_{j+1}, ..., X_6)]}{\text{var}[(X_j|X_2, ..., X_{j-1}, X_{j+1}, ..., X_6)]} \]  

In other words, the best linear prediction function of \( X_1 \) is given simply by the linear regression of \( X_1 \) on \( X_2, ..., X_6 \). Obviously the equivalent result holds for the best linear predictor of the other \( X_j \). Moreover, the result also holds in the case where the covariance matrix, \( \Sigma \) is not known a priori but must be estimated from observations - as is obviously the case in the current study. In this event, the covariance matrix - and the regression coefficients in equation (A.5) calculated from it - are simply replaced by their sample estimates.

Our procedure for finding the optimum set of prediction variables for \( M_{\text{max}} \) or \( M_{\text{mean}} \) is based upon the forward selection procedure, as described in e.g. Chapter 12 of Graybill (1976). Again, we outline this procedure where one is predicting \( X_1 \) from measurements of the other variables, with the obvious equivalent procedure for predicting some other \( X_j \).

First, we determine the sample multiple correlation coefficient, \( \rho_{1(2)} \), of \( X_1 \) with \( X_k \) for \( k = 2, ..., 6 \), and identify the largest of these in modulus. Suppose, without loss of generality, that this is \( \rho_{1(2)} \). Then \( X_2 \) is the best single predictor of \( X_1 \). We then compute all multiple correlation coefficients with \( X_1 \) of pairs of variables which include the best single predictor, \( X_2 \); i.e. \( \rho_{1(2,3)}; \rho_{1(2,4)}; ..., \rho_{1(2,6)} \). Again, we select the largest - say \( \rho_{1(2,3)} \). It then follows that \( X_2 \) and \( X_3 \) are the best two predictors of \( X_1 \) which include the best single predictor \( X_2 \).

We can continue this procedure until all the remaining variables have been included in the prediction function or, more usefully, until the addition of a new variable does not appreciably improve the estimator of \( X_1 \). We can test the significance of adding a given variable - and hence quantify what we mean by 'appreciably improve' the fit - in two different ways. First, we can test the null hypothesis that the regression coefficient of a given variable, \( X_j \), is equal to zero, i.e.,
\[ H_0 : \beta_j = 0 \]  
If the null hypothesis is strongly rejected, then one should include \( X_j \) in the best linear prediction function. If the null hypothesis is accepted, then \( X_j \) has little effect on the prediction of \( X_1 \) and could be considered superfluous. It can be shown (c.f. Graybill, 1976) that under the null hypothesis the transformed variable \( t_j = \beta_j / \hat{\sigma} (\beta_j) \) has a \( t \) distribution with \( n - \alpha \) degrees of freedom. Here \( \hat{\sigma} \) denotes the standard error of the estimated regression coefficient and \( n \) and \( \alpha \) are the number of data points in the linear regression fit and the number of independent variables in the fit (i.e. the number of prediction variables) respectively. Testing the null hypothesis that \( \beta_j = 0 \) is therefore equivalent to the test,
\[ H_0 : t_j = 0 \]  
the results of which for each of our estimators are described in detail in Sect. 5.

We can also test the significance of adding a given variable in terms of the computed sample multiple correlation coefficients of the different relations. Suppose we have applied the forward selection procedure described above to identify the best linear prediction function of \( X_1 \) as depending on the variables \( X_2, ..., X_q \), and we wish to test whether the new variable, \( X_{q+1} \) significantly improves the estimation of \( X_1 \). If \( X_{q+1} \) has no effect on the prediction then the multiple correlation coefficient of \( X_1 \) with \( X_2, ..., X_q \) will be identically equal to that of \( X_1 \) with \( X_{2}, ..., X_{q+1} \), i.e.,
\[ \hat{\rho}_{1(2,...,q+1)}^2 = \rho_{1(2,...,q)}^2 \]  
One can then use the sampled values of the multiple correlation coefficients to construct a suitable hypothesis test based upon this property, i.e. to test the null hypothesis,
\[ H_0 : \rho_{1(2,...,q+1)}^2 = \hat{\rho}_{1(2,...,q)}^2 \]  
where the caret denotes that the multiple correlation coefficients are sample values estimated from the calibration data.

In fact, the test in equation (A.9) is equivalent to testing the null hypothesis,
\[ H_0 : \rho_{1,q+1}^2(2,...,q) = 0 \]
where $\rho_{1,q+1|(2,...q)}$ is the partial sample multiple correlation coefficient of $X_1$ and $X_{q+1}$ given $X_2,...X_q$.

When $H_0$ is true, equation (A.10) is distributed as a central $F$ random variable with 1 and $n - q - 2$ degrees of freedom. It is this hypothesis test which we have applied to each of the relations introduced in Sect. 4.

Since the assumption of a multivariate normal distribution is required in most of the above, we have devoted Sect. 6 to the task of testing carefully how well our LMC data satisfy this normality assumption.
Table Captions

Table 1: LMC stars
Table 2: Excluded LMC stars
Table 3: Unusual LMC stars
Table 4: Results of KS test applied to LMC data
Table 5: Skewness and kurtosis results
Table 6: Shapiro and Wilks normality test results
Table 7: Fitted PL relations
Table 8: Fitted PLC and PLA relations
Table 9: Fitted PLL'C and PLAC relations
| Star   | Period (days) |
|--------|--------------|
| HV2353 | 3.1080       |
| HV12765| 3.4290       |
| HV12700| 8.1530       |
| HV12823| 8.3020       |
| HV2854 | 8.6350       |
| HV2733 | 8.7220       |
| HV12816| 9.1140       |
| HV971  | 9.2970       |
| HV2301 | 9.4990       |
| HV6105 | 10.4400      |
| HV2864 | 10.9840      |
| HV874  | 12.6820      |
| HV2260 | 12.9360      |
| HV2527 | 12.9480      |
| HV997  | 13.1470      |
| HV2579 | 13.4310      |
| HV2352 | 13.6260      |
| HV955  | 13.7320      |
| HV2324 | 14.4660      |
| HV2549 | 16.1970      |
| HV2580 | 16.9450      |
| HV2836 | 17.5260      |
| HV1005 | 18.7100      |
| HV2793 | 19.1840      |
| HV1013 | 24.1264      |
| HV12815| 26.1690      |
| HV1023 | 26.5880      |
| HV1002 | 30.4700      |
| HV899  | 31.0270      |
| HV2294 | 36.5270      |
| HV2294 | 36.5270      |
| HV879  | 36.7820      |
| HV2338 | 42.1669      |
| HV877  | 45.1853      |
| HV2369 | 48.3190      |
| HV2827 | 78.8582      |
| HV5497 | 98.7802      |
| HV2883 | 109.0000     |
| HV2447 | 119.4400     |
| HV883  | 134.0000     |
### TABLE 2

**EXCLUDED LMC STARS**

| Star   | Period (days) |
|--------|---------------|
| HV5665 | 14.2110       |
| HV2262 | 15.8460       |
| HV909  | 37.5700       |
| HV2257 | 42.1669       |
| HV900  | 47.5330       |
| HV953  | 47.890        |

### TABLE 3

**UNUSUAL LMC STARS**

| Star   | Period (days) |
|--------|---------------|
| HV877  | 45.1853       |
| HV2827 | 78.8582       |
| HV5497 | 98.7802       |
| HV2883 | 109.000       |
| HV2447 | 119.440       |
| HV883  | 134.000       |
**TABLE 4**

RESULTS OF KS TEST APPLIED TO LMC DATA

| Variable      | $D_{obs}$ | Prob($D_n > D_{obs}$) |
|---------------|-----------|-----------------------|
| **31 stars**  |           |                       |
| $V_{max}$     | 0.099     | 0.905                 |
| $V_{mean}$    | 0.103     | 0.880                 |
| $(B - V)_{max}$ | 0.096    | 0.927                 |
| $(B - V)_{mean}$ | 0.099   | 0.907                 |
| log $P$       | 0.125     | 0.688                 |
| semi-amplitude | 0.144    | 0.504                 |
| **35 stars**  |           |                       |
| $V_{max}$     | 0.100     | 0.858                 |
| $V_{mean}$    | 0.120     | 0.663                 |
| $(B - V)_{max}$ | 0.120    | 0.663                 |
| $(B - V)_{mean}$ | 0.098   | 0.875                 |
| log $P$       | 0.121     | 0.651                 |
| semi-amplitude | 0.151    | 0.372                 |
| **39 stars**  |           |                       |
| $V_{max}$     | 0.111     | 0.697                 |
| $V_{mean}$    | 0.152     | 0.304                 |
| $(B - V)_{max}$ | 0.172    | 0.179                 |
| $(B - V)_{mean}$ | 0.100   | 0.798                 |
| log $P$       | 0.124     | 0.562                 |
| semi-amplitude | 0.150    | 0.315                 |
| Variable | Skewness (no. of $\sigma$) | Kurtosis (no. of $\sigma$) |
|----------|---------------------------|-----------------------------|
| 31 stars |                           |                             |
| $V_{\text{max}}$ | 0.56 | 0.04 |
| $V_{\text{mean}}$ | 0.72 | 0.01 |
| $(B - V)_{\text{max}}$ | 0.57 | 0.38 |
| $(B - V)_{\text{mean}}$ | 0.25 | 0.42 |
| $\log P$ | 0.72 | 0.29 |
| semi-amplitude | 0.52 | 0.58 |
| 35 stars |                           |                             |
| $V_{\text{max}}$ | 0.77 | 0.18 |
| $V_{\text{mean}}$ | 0.98 | 0.11 |
| $(B - V)_{\text{max}}$ | 1.51 | 0.63 |
| $(B - V)_{\text{mean}}$ | 0.32 | 0.43 |
| $\log P$ | 0.46 | 0.51 |
| semi-amplitude | 0.61 | 0.60 |
| 39 stars |                           |                             |
| $V_{\text{max}}$ | 0.60 | 0.56 |
| $V_{\text{mean}}$ | 1.02 | 0.40 |
| $(B - V)_{\text{max}}$ | 1.68 | 0.10 |
| $(B - V)_{\text{mean}}$ | 0.15 | 0.65 |
| $\log P$ | 0.66 | 0.09 |
| semi-amplitude | 0.46 | 0.74 |
| Variable          | $W_{obs}$ | Significance |
|-------------------|-----------|--------------|
| 31 stars $V_{max}$| 0.977     | 0.7546       |
|                  | $V_{mean}$| 0.961        |
|                  | $(B - V)_{max}$ | 0.954       |
|                  | $(B - V)_{mean}$ | 0.959       |
|                  | log $P$    | 0.944        |
|                  | semi-amplitude | 0.927       |
| 35 stars $V_{max}$| 0.960     | 0.3010       |
|                  | $V_{mean}$| 0.948        |
|                  | $(B - V)_{max}$ | 0.928       |
|                  | $(B - V)_{mean}$ | 0.972       |
|                  | log $P$    | 0.965        |
|                  | semi-amplitude | 0.927       |
| 39 stars $V_{max}$| 0.940     | 0.0508       |
|                  | $V_{mean}$| 0.919        |
|                  | $(B - V)_{max}$ | 0.865       |
|                  | $(B - V)_{mean}$ | 0.953       |
|                  | log $P$    | 0.948        |
|                  | semi-amplitude | 0.928       |
TABLE 7

FITTED PL RELATIONS

31 stars

\[ \dot{M} = a + b \log P \]

|         | \( \hat{\beta} \) | \( \sigma_{\hat{\beta}} \) | \( \log P(\beta = 0) \) | \( \sigma_{\dot{M}_P} \) | \( \Delta \) |
|---------|--------------------|------------------|----------------|----------------|--------|
| a       | -1.90              | 0.27             |                 | 0.324          | 14.9   |
| b       | -2.05              | 0.23             | -0.36          |                |        |

\[ \dot{M}_{\text{max}} = a + b \log P \]

|         | \( \hat{\beta} \) | \( \sigma_{\hat{\beta}} \) | \( \log P(\beta = 0) \) | \( \sigma_{\dot{M}_{\text{max}}} \) | \( \Delta \) |
|---------|--------------------|------------------|----------------|----------------|--------|
| a       | -1.88              | 0.26             |                 | 0.316          | 14.5   |
| b       | -2.48              | 0.22             | -10            |                |        |

35 stars

\[ \dot{M} = a + b \log P \]

|         | \( \hat{\beta} \) | \( \sigma_{\hat{\beta}} \) | \( \log P(\beta = 0) \) | \( \sigma_{\dot{M}_P} \) | \( \Delta \) |
|---------|--------------------|------------------|----------------|----------------|--------|
| a       | -1.80              | 0.21             |                 | 0.323          | 14.9   |
| b       | -2.16              | 0.17             | -10            |                |        |

\[ \dot{M}_{\text{max}} = a + b \log P \]

|         | \( \hat{\beta} \) | \( \sigma_{\hat{\beta}} \) | \( \log P(\beta = 0) \) | \( \sigma_{\dot{M}_{\text{max}}} \) | \( \Delta \) |
|---------|--------------------|------------------|----------------|----------------|--------|
| a       | -1.93              | 0.22             |                 | 0.326          | 15.0   |
| b       | -2.44              | 0.17             | -10            |                |        |

39 stars

\[ \dot{M} = a + b \log P \]

|         | \( \hat{\beta} \) | \( \sigma_{\hat{\beta}} \) | \( \log P(\beta = 0) \) | \( \sigma_{\dot{M}_P} \) | \( \Delta \) |
|---------|--------------------|------------------|----------------|----------------|--------|
| a       | -1.64              | 0.18             |                 | 0.317          | 14.6   |
| b       | -2.30              | 0.13             | -10            |                |        |

\[ \dot{M}_{\text{max}} = a + b \log P \]

|         | \( \hat{\beta} \) | \( \sigma_{\hat{\beta}} \) | \( \log P(\beta = 0) \) | \( \sigma_{\dot{M}_{\text{max}}} \) | \( \Delta \) |
|---------|--------------------|------------------|----------------|----------------|--------|
| a       | -2.04              | 0.18             |                 | 0.316          | 14.5   |
| b       | -2.34              | 0.13             | -10            |                |        |
### TABLE 8

**FITTED PLC AND PLA RELATIONS**

**31 stars**

\[
\dot{M} = a + b \log P + c (B - V)
\]

| \(\hat{\beta} \) | \(\sigma_{\hat{\beta}} \) | \(\log P(\beta = 0) \) | \(\sigma_{\dot{M},a} \) | \(\Delta \) |
|-----|-----|----------------|-----------------|-----|
| a   | -2.11 | 0.23 |                      | 0.265 | 12.2 |
| b   | -3.03 | 0.31 | < -10               |        |      |
| c   | 1.75  | 0.44 | -3.61               |        |      |

partial SMCC = 0.597 \quad \text{P}(H_0) = 4.9 \times 10^{-4}

\[
\dot{M}_{\text{max}} = a + b \log P + c (B - V)_{\text{max}}
\]

| \(\hat{\beta} \) | \(\sigma_{\hat{\beta}} \) | \(\log P(\beta = 0) \) | \(\sigma_{\dot{M},a} \) | \(\Delta \) |
|-----|-----|----------------|-----------------|-----|
| a   | -2.39 | 0.22 |                      | 0.235 | 10.8 |
| b   | -2.87 | 0.18 | < -10               |        |      |
| c   | 2.05  | 0.41 | -4.80               |        |      |

partial SMCC = 0.683 \quad \text{P}(H_0) = 3.2 \times 10^{-5}

**35 stars**

\[
\dot{M} = a + b \log P + c (B - V)
\]

| \(\hat{\beta} \) | \(\sigma_{\hat{\beta}} \) | \(\log P(\beta = 0) \) | \(\sigma_{\dot{M},a} \) | \(\Delta \) |
|-----|-----|----------------|-----------------|-----|
| a   | -2.09 | 0.18 |                      | 0.257 | 11.8 |
| b   | -3.11 | 0.25 | < -10               |        |      |
| c   | 1.83  | 0.41 | -4.37               |        |      |

partial SMCC = 0.622 \quad \text{P}(H_0) = 8.6 \times 10^{-5}

\[
\dot{M}_{\text{max}} = a + b \log P + c (B - V)_{\text{max}}
\]

| \(\hat{\beta} \) | \(\sigma_{\hat{\beta}} \) | \(\log P(\beta = 0) \) | \(\sigma_{\dot{M},a} \) | \(\Delta \) |
|-----|-----|----------------|-----------------|-----|
| a   | -2.32 | 0.16 |                      | 0.227 | 10.5 |
| b   | -2.92 | 0.14 | < -10               |        |      |
| c   | 2.00  | 0.33 | -6.25               |        |      |

partial SMCC = 0.727 \quad \text{P}(H_0) = 1.1 \times 10^{-6}
TABLE 8 continued

39 stars
\[ \hat{M} = a + b \log P + c(B - V) \]

|   | \( \hat{\beta} \) | \( \sigma_{\hat{\beta}} \) | \( \log P(\beta = 0) \) | \( \sigma_{\hat{M}_j} \) | \( \Delta \) |
|---|------------------|------------------|------------------|------------------|-------|
| a | -2.01            | 0.16             |                  | 0.251            | 11.6  |
| b | -3.22            | 0.22             |                  | < -10            |       |
| c | 1.88             | 0.39             |                  | 4.83             |       |

partial SMCC = 0.623 \[ P(H_0) = 3.0 \times 10^{-5} \]

\[ \hat{M}_{\text{max}} = a + b \log P + c(B - V)_{\text{max}} \]

|   | \( \hat{\beta} \) | \( \sigma_{\hat{\beta}} \) | \( \log P(\beta = 0) \) | \( \sigma_{\hat{M}_j} \) | \( \Delta \) |
|---|------------------|------------------|------------------|------------------|-------|
| a | -2.11            | 0.13             |                  | 0.234            | 10.8  |
| b | -2.96            | 0.15             |                  | < -10            |       |
| c | 1.64             | 0.29             |                  | -5.92            |       |

partial SMCC = 0.682 \[ P(H_0) = 2.4 \times 10^{-6} \]

31 stars
\[ \hat{M} = a + b \log P + c(M_{\text{mean}} - M_{\text{max}}) \]

|   | \( \hat{\beta} \) | \( \sigma_{\hat{\beta}} \) | \( \log P(\beta = 0) \) | \( \sigma_{\hat{M}_j} \) | \( \Delta \) |
|---|------------------|------------------|------------------|------------------|-------|
| a | -1.88            | 0.27             |                  | 0.321            | 14.8  |
| b | -2.40            | 0.36             |                  | -6.77            |       |
| c | -0.19            | 0.65             |                  | -0.41            |       |

partial SMCC = 0.230 \[ P(H_0) = 0.222 \]

\[ \hat{M}_{\text{max}} = a + b \log P + c(M_{\text{mean}} - M_{\text{max}}) \]

|   | \( \hat{\beta} \) | \( \sigma_{\hat{\beta}} \) | \( \log P(\beta = 0) \) | \( \sigma_{\hat{M}_j} \) | \( \Delta \) |
|---|------------------|------------------|------------------|------------------|-------|
| a | -1.88            | 0.27             |                  | 0.321            | 14.8  |
| b | -2.40            | 0.36             |                  | -6.77            |       |
| c | 0.81             | 0.65             |                  | -0.96            |       |

partial SMCC = -0.054 \[ P(H_0) = 0.776 \]
### TABLE 8 continued

35 stars

\[ \dot{M} = a + b \log P + c(M_{\text{mean}} - M_{\text{max}}) \]

| \( \beta \) | \( \sigma_{\beta} \) | \( \log P(\beta = 0) \) | \( \sigma_{\dot{M}} \) | \( \Delta \) |
|---------|---------|-----------------|--------|------|
| a       | -1.85   | 0.23            | 0.325  | 15.0 |
| b       | -2.28   | 0.22            | < -10  |      |
| c       | 0.44    | 0.51            | -0.70  |      |

partial SMCC = 0.150 \hspace{1cm} P(H_0) = 0.4 \times 10^{-5}

\[ \dot{M}_{\text{max}} = a + b \log P + c(M_{\text{mean}} - M_{\text{max}}) \]

| \( \beta \) | \( \sigma_{\beta} \) | \( \log P(\beta = 0) \) | \( \sigma_{\dot{M}} \) | \( \Delta \) |
|---------|---------|-----------------|--------|------|
| a       | -1.85   | 0.23            | 0.325  | 15.0 |
| b       | -2.28   | 0.22            | < -10  |      |
| c       | -0.56   | 0.51            | -0.85  |      |

partial SMCC = -0.190 \hspace{1cm} P(H_0) = 0.283

39 stars

\[ \dot{M} = a + b \log P + c(M_{\text{mean}} - M_{\text{max}}) \]

| \( \beta \) | \( \sigma_{\beta} \) | \( \log P(\beta = 0) \) | \( \sigma_{\dot{M}} \) | \( \Delta \) |
|---------|---------|-----------------|--------|------|
| a       | -1.84   | 0.21            | 0.310  | 14.3 |
| b       | -2.32   | 0.13            | < -10  |      |
| c       | 0.51    | 0.32            | -1.21  |      |

partial SMCC = 0.254 \hspace{1cm} P(H_0) = 0.124

\[ \dot{M}_{\text{max}} = a + b \log P + c(M_{\text{max}} - M_{\text{mean}}) \]

| \( \beta \) | \( \sigma_{\beta} \) | \( \log P(\beta = 0) \) | \( \sigma_{\dot{M}} \) | \( \Delta \) |
|---------|---------|-----------------|--------|------|
| a       | -1.84   | 0.21            | 0.310  | 14.3 |
| b       | -2.32   | 0.13            | < -10  |      |
| c       | -0.49   | 0.32            | -1.52  |      |

partial SMCC = -0.245 \hspace{1cm} P(H_0) = 0.138
### TABLE 9

**FITTED PLL’C AND PLAC RELATIONS**

#### 31 stars

\[
\hat{M}_{\text{max}} = a + b \log P + c (B - V) + d (M_{\text{mean}} - < M_{\text{mean}} >)
\]

| \(\hat{\beta}\) | \(\sigma_{\hat{\beta}}\) | \(\log P(\beta = 0)\) | \(\sigma_{\hat{M}_{\text{max}}}\) | \(\Delta\) |
|---|---|---|---|---|
| a | -3.95 | 0.89 | -6.19 | 0.078 |
| b | -0.89 | 0.14 | -10 | |
| c | 0.57 | 0.17 | -2.94 | |
| d | 0.83 | 0.05 | < -10 | |

Partial SMCC = 0.946 \(P(H_0) = 9 \times 10^{-15}\)

\[\hat{M}_{\text{max}} = a + b \log P + c (B - V)_{\text{max}} + d (M_{\text{mean}} - M_{\text{max}})\]

| \(\hat{\beta}\) | \(\sigma_{\hat{\beta}}\) | \(\log P(\beta = 0)\) | \(\sigma_{\hat{M}_{\text{max}}}\) | \(\Delta\) |
|---|---|---|---|---|
| a | -2.41 | 0.22 | -10 | 0.234 | 10.7 |
| b | -3.14 | 0.30 | < -10 | |
| c | 2.19 | 0.43 | -10 | |
| d | 0.56 | 0.50 | -0.87 | |

Partial SMCC = 0.213 \(P(H_0) = 0.267\)

#### 35 stars

\[
\hat{M}_{\text{max}} = a + b \log P + c (B - V) + d (M_{\text{mean}} - < M_{\text{mean}} >)
\]

| \(\hat{\beta}\) | \(\sigma_{\hat{\beta}}\) | \(\log P(\beta = 0)\) | \(\sigma_{\hat{M}_{\text{max}}}\) | \(\Delta\) |
|---|---|---|---|---|
| a | -4.13 | 0.90 | -10 | 0.083 |
| b | -0.92 | 0.15 | -6.40 | |
| c | 0.76 | 0.15 | -5.03 | |
| d | 0.79 | 0.05 | < -10 | |

Partial SMCC = 0.932 \(P(H_0) = 3.1 \times 10^{-15}\)

\[\hat{M}_{\text{max}} = a + b \log P + c (B - V)_{\text{max}} + d (M_{\text{mean}} - M_{\text{max}})\]

| \(\hat{\beta}\) | \(\sigma_{\hat{\beta}}\) | \(\log P(\beta = 0)\) | \(\sigma_{\hat{M}_{\text{max}}}\) | \(\Delta\) |
|---|---|---|---|---|
| a | -2.43 | 0.18 | -10 | 0.224 | 10.33 |
| b | -3.12 | 0.21 | < -10 | |
| c | 2.23 | 0.37 | -6.22 | |
| d | 0.53 | 0.40 | -1.01 | |

Partial SMCC = 0.232 \(P(H_0) = 0.193\)
TABLE 9 continued

39 stars

}\[M_{\text{max}} = a + b \log P + c(B - V) + d(M_{\text{mean}} - < M_{\text{mean}}>)\]

\[
\begin{array}{cccc}
\hat{\beta} & \sigma_{\hat{\beta}} & \log P(\beta = 0) & \sigma_{\hat{\beta}} \\
a & -4.22 & 0.85 & 0.090 \\
b & -1.07 & 0.14 & -8.26 \\
c & 1.02 & 0.12 & -9.47 \\
d & 0.72 & 0.05 & < -10 \\
\end{array}
\]

\text{partial SMCC} = 0.925 \quad P(H_0) = 3.0 \times 10^{-16}

\[M_{\text{max}} = a + b \log P + c(B - V)_{\text{max}} + d(M_{\text{mean}} - M_{\text{max}})\]

\[
\begin{array}{cccc}
\hat{\beta} & \sigma_{\hat{\beta}} & \log P(\beta = 0) & \sigma_{\hat{\beta}} \\
a & -2.40 & 0.18 & 0.22 & 10.2 \\
b & -3.19 & 0.17 & < -10 \\
c & 2.16 & 0.36 & -6.33 \\
d & 0.68 & 0.30 & -1.79 \\
\end{array}
\]

\text{partial SMCC} = 0.352 \quad P(H_0) = 3.2 \times 10^{-2}