REGULARIZATIONS OF POSITIVE ENTROPY PSEUDO-AUTOMORPHISMS

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ABSTRACT. We study positive entropy birational automorphisms of threefolds. We identify some conditions which imply that such an automorphism is non-regularizable. We show that this criterion applies in the example of a positive entropy birational automorphism of $\mathbb{P}^3$ constructed in [Bla13], thus showing that for a general choice of parameters it is non-regularizable. Additionally, we establish a criterion which proves that the automorphism in this example does not preserve a structure of a fibration over a surface.

1. INTRODUCTION

Let $X$ be any normal projective variety defined over an algebraically closed field $k$ of characteristic 0. We shall say that a birational automorphism $\varphi : X \to X$ is regularizable if there exists a birational map $\alpha : X \to Y$ to a variety $Y$ such that $\psi = \alpha \circ \varphi \circ \alpha^{-1}$ is a regular automorphism of $Y$. Understanding whether a given birational self-map is regularizable or not is a delicate problem.

We note immediately that any birational self-map of a curve, or of a variety of general type is regularizable. Any map of finite order is also regularizable (see, e.g., [PS14, Lemma-Definition 3.1]).

The question whether one can regularize a birational automorphism $\varphi : X \to X$ of infinite order becomes increasingly difficult when dimension of $X$ grows. The case of surfaces is now well-understood and precise criteria have been devised in [DF01] and [BC16] which are all based on the growth of degrees of the iterates of $\varphi$. Since the degree growth also plays a key role in all criteria that are known in higher dimension, we recall some basic properties of such.

Let us fix any ample line bundle $H$ over $X$. The $i$-th degree of $\varphi$ for $0 \leq i \leq d$ is defined as the following number:

$$\deg_i(\varphi) = (\varphi)^*(H^i) \cdot H^{d-i}.$$  

The growth of degrees $\deg_i(\varphi^n)$ is a birational invariant by [DS05]. Moreover one can prove for all $0 \leq i \leq d$, that the sequence $\deg_i(\varphi^n)$ for all $n > 0$ is essentially submultiplicative; thus, we can define the $i$-th dynamical degree of $\varphi$:

$$\lambda_i(\varphi) = \lim_{n \to \infty} (\deg_i(\varphi^n))^{\frac{1}{n}}.$$  

These numbers are birational invariants, they are real and satisfy $\lambda_i(\varphi) \geq 1$ for each $1 \leq i \leq d - 1$. In case of birational automorphisms we have $\lambda_0(\varphi) = \lambda_d(\varphi) = 1$. The sequence of real numbers $\lambda_1(\varphi), \ldots, \lambda_{d-1}(\varphi)$ is log-concave; i.e. we have the following inequalities for all $1 \leq i \leq d - 1$:

$$\lambda_{i-1}(\varphi) \cdot \lambda_{i+1}(\varphi) \leq \lambda_i(\varphi)^2.$$  

In the sequel we shall use the following convenient terminology: $\varphi$ has positive entropy iff $\lambda_1(\varphi) > 1$. Note that by log-concavity, the latter condition is equivalent to $\lambda_i(\varphi) > 1$ for any $i = 1, \ldots, d - 1$. For any birational automorphism $\varphi : X \to X$ of a variety $X$ the growth of degrees $\deg_i(\varphi^n)$ and the dynamical degrees $\lambda_i(\varphi)$ give strong constraints on the possibility for $\varphi$ to be regularizable.

We now list all currently known criteria ensuring regularizability or non-regularizability.

- If $X$ is a surface and $\lambda_1(\varphi) = 1$, then $\varphi$ is is regularizable if and only if $\deg_1(\varphi^n) \approx n^2$ by [DF01]. If $\lambda_1(\varphi) > 1$ is a Salem number, then $\varphi$ is regularizable by [BC16].
- If $\deg_1(\varphi^n)$ is bounded and $X$ is any variety then $\varphi$ can be regualarized by [Wei55].
- If $\deg_1(\varphi^n) \approx n^d$ where $d$ is an odd integer and $X$ is any variety then $\varphi$ can not be regularized by [CDX21] Proposition 1.2).
- If $X$ is any algebraic variety and $\varphi$ is regularizable, then $\lambda_1(\varphi)$ is an algebraic integer. However, if $\lambda_1(\varphi)$ lies in $\mathbb{Z}$ and greater than 1 then the birational automorphism $\varphi$ is not regularizable by [CDX21] Proposition 1.1]. In case when the dimension of $X$ equals 3 there are more
restrictions on the algebraic number $\lambda_1(\varphi)$ for a regularizable automorphism $\varphi$ which are listed in [LB19] Proposition 4.6.7, 4.7.2, 5.0.1.

- If $X$ is a projective space $\mathbb{P}^d$ over a field $k$ of characteristic 0 and $\varphi: \mathbb{P}^d \to \mathbb{P}^d$ is a birational automorphism, then by [CDX21] Corollary 1.7 one can find a linear transformation $A \in \text{PGL}_{d+1}(k)$ such that $A \circ \varphi$ is not regularizable. Moreover, by [CDX21] Theorem 1.5 the set of such $A$ is Zariski dense if $k$ is uncountable. Thus, a very general birational automorphism of $\mathbb{P}^d$ over an uncountable field is not regularizable.

In this paper we explore the problem or regularization in the case when $X$ is a smooth threefold and $\varphi: X \to X$ is a pseudo-automorphism such that $\lambda_1(\varphi) > 1$. Recall that a birational map is a pseudo-automorphism if $\varphi$ and $\varphi^{-1}$ does not contract any divisor in $X$. Note that in case of surfaces any pseudo-automorphism can be extended to a regular automorphism. Thus, pseudo-automorphisms form a class of birational automorphisms which are very close to being regular. One might expect that any pseudo-automorphism can be regularized; however, it turns out to be false. There are several examples of non-regularizable pseudo-automorphisms, e.g. [BK09] and [BCK14].

The easiest construction of an automorphism $\varphi$ of threefold $X$ with $\lambda_1(\varphi) > 1$ is to take $X = S \times C$, where $S$ is a surface admitting a positive entropy automorphism $S$, and any birational automorphism $\varphi$ of $X$ can not be constructed on any sequence of blow-ups with smooth centers of a smooth Fano variety.

Since a birational map is a pseudo-automorphism, then by [CDX21, Corollary 1.7] one can find a linear transformation $A \in \text{PGL}_{d+1}(k)$ such that $A \circ \varphi$ is not regularizable. Moreover, by [CDX21] Theorem 1.5 the set of such $A$ is Zariski dense if $k$ is uncountable. Thus, a very general birational automorphism of $\mathbb{P}^d$ over an uncountable field is not regularizable.

The easiest construction of an automorphism $\varphi$ of threefold $X$ with $\lambda_1(\varphi) > 1$ is to take $X = S \times C$, where $S$ is a surface admitting a positive entropy automorphism $S$, and any birational automorphism $\varphi$ of $X$ can not be constructed on any sequence of blow-ups with smooth centers of a smooth Fano variety.

We know several constructions of positive entropy primitive pseudo-automorphisms of rational threefolds: [OT15], [BK14], [BCK14], [DO88], etc.. The example in [OT15] is the only known example of regular positive entropy primitive automorphism of a rational threefold. For examples [BK14] and [BCK14] there was proved that they are not regularizable. In these papers authors proved that if $\varphi: X \to X$ is a birational automorphism and there is a $\varphi$-invariant surface $S$ in $X$ such that $\varphi|_S$ induces a non-regularizable birational automorphism of $S$, then $\varphi$ is also non-regularizable. Thus, they find an invariant surfaces and applied criteria for positive entropy birational automorphisms for surfaces.

Our result is a new criterion of non-regularizibility of pseudo-automorphisms of threefolds.

**Condition A.** We say that a pseudo-automorphism $\varphi: X \to X$ of a smooth projective threefold $X$ satisfies Condition A if

1. $\lambda_1(\varphi)^2 > \lambda_2(\varphi)$;
2. there exists a curve $C$ such that $\theta_1(\varphi) \cdot [C] < 0$;
3. there exists infinitely many integers $m > 0$ such that $C \not\subseteq \text{Ind}(\varphi^{-m})$.

Some comments are in order. It is a theorem due to [Tru14] that the property (1) implies the existence of a unique class $\theta_1(\varphi)$ in the group of classes of divisors $N^1(X)$ which satisfies the property $\varphi^* \theta_1(\varphi) = \lambda_1(\varphi) \theta_1(\varphi)$. Property (2) implies that the class $\theta_1(\varphi)$ is not nef and the last property is required to avoid transformations obtained after flopping a curve with infinite orbit (see Section 5).

**Theorem 1.1.** Assume that $\varphi: X \to X$ is a pseudo-automorphism of a smooth projective threefold $X$ which satisfies Condition A. Then $\varphi$ is not regularizable.

The main step of the proof of Theorem 1.1 consists in proving that if we have a regularization $\psi: Y \to Y$ of $\varphi$ and if $\alpha: X \to Y$ is a birational map such that $\varphi = \alpha^{-1} \circ \psi \circ \alpha$, then we have $\theta_1(\varphi) = \alpha^* \theta_1(\psi)$. This fact relies on our assumption that $\varphi$ is a pseudo-automorphism, but it can fail in general. Property (2) of Condition A implies that all proper images of $C$ are included in $\text{Ind}(\alpha)$. We then obtain a contradiction with property (3).

Our second result is motivated by searching a criterion of primitivity for birational automorphisms.
Theorem 1.2. Assume that \( \varphi : X \rightarrow X \) is pseudo-automorphism of a smooth projective threefold \( X \) such that \( \lambda_1(\varphi)^2 > \lambda_2(\varphi) \). If \( \varphi \) satisfies Condition [A] then there exists no dominant rational map \( \pi : X \rightarrow S \) to a normal surface \( S \) such that \( \pi \circ \varphi = f \circ \pi \) for some birational map \( f : S \rightarrow S \).

Condition [A] implies only that the pseudo-automorphism \( \varphi \) can not preserve the structure of a fibration over a surface. The case of a fibration over a curve seems to be much harder.

We apply Theorems 1.1 and 1.2 to the family of pseudo-automorphisms introduced by J. Blanc in [Bla13] that we now recall. Let \( Q \subset \mathbb{P}^d \) be a smooth cubic hypersurface for some \( d \geq 2 \). We associate a birational involution \( \sigma_p \) of \( \mathbb{P}^d \) to each point \( p \in Q \). For a general line \( L \) passing through \( p \) and intersecting \( Q \) in three distinct points \( p, q_1 \) and \( q_2 \) we define \( \sigma_p|_L \) as a unique non-trivial involution of \( \mathbb{P}^1 \) fixing both points \( q_1 \) and \( q_2 \). This defines a birational transformation of the projective space.

Now take any general points \( p_1, \ldots, p_k \) on \( Q \) with \( k \geq 3 \). Then \( \sigma_{p_1} \circ \cdots \circ \sigma_{p_k} \) induces a positive entropy birational automorphism of \( \mathbb{P}^d \) by [Bla13] Proposition 2.3]. If \( d = 2 \), then the composition \( \sigma_{p_1} \circ \cdots \circ \sigma_{p_k} \) is a regularizable non-primitive automorphism by [Bla08]. Here is the main result of this paper.

Theorem 1.3. Assume that \( Q \subset \mathbb{P}^3 \) is a very general smooth cubic surface over \( \mathbb{C} \) and \( p_1, p_2, p_3 \) are general points on \( Q \). Then the composition \( \varphi = \sigma_{p_2} \circ \sigma_{p_3} \circ \sigma_{p_1} \) is a positive entropy birational automorphism of \( \mathbb{P}^3 \) which is non-regularizable and does not preserve the structure of a fibration over a surface.

Note that none of the previously known criteria from [BK14] and [CDX21] Proposition 1.1] apply in this situation.

To prove Theorem 1.3 we find a \( \theta_1(\varphi) \)-negative curve \( C \) on a birational model \( X \) of \( \mathbb{P}^3 \) where \( \varphi \) induces an algebraically stable automorphism. The most difficult part in the proof of Theorem 1.3 is to show that \( C \) satisfies property (3) of Condition [A]. We do this by computing the orbit of a well-chosen point in \( C \). Namely, we take the cubic surface \( Q \) such that its coefficients when we write it in some coordinates of \( \mathbb{P}^3 \) are algebraically independent. We fix points \( p_1, p_2 \) and \( p_3 \), then we can write formulas defining \( \sigma_i \) for \( i = 1, 2, 3 \). Then we chose some concrete point in \( \mathbb{P}^3 \) and consider its images under the action of \( \sigma_i \) as a set of four polynomials of coefficients of \( Q \). We show that after several iterations of involutions these polynomials has some form which remains the same after applying new involutions. Thus, we manage to show that its orbit never falls into the indeterminacy locus of \( \varphi \).

Dealing with three involutions makes our computation already quite tricky. We believe that our theorem is valid for any composition of at least three involutions associated to general points on \( Q \).

The paper is organized in the following way. In Section 2 we recall properties of birational maps. In Section 3 we prove Theorems 1.1 and 1.2. In Section 4 we recall the construction of the positive entropy automorphism of \( \mathbb{P}^3 \) introduced in [Bla13], show that Condition [A] is satisfied for it and prove Theorem 1.2. In Section 5 we give an example of a regularizable pseudo-automorphism which satisfies all but third properties of Condition [A].

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2. Preliminaries

2.1. Birational maps acting on the divisor class group. Throughout this paper we consider smooth algebraic varieties over an algebraically closed field of characteristic 0. If \( \alpha : X \rightarrow Y \) is a rational map
between two varieties we denote by \(\text{Ind}(\alpha)\) the complement to the greatest open subset of \(X\) on which \(\alpha\) is regular, we call it the *locus of indeterminacy of \(\alpha\). By \(\text{Exc}(\alpha)\) we denote the union of divisors in \(X\) which are contracted under the action of \(\alpha\) and we call it the *exceptional locus of the map \(\alpha\).

Assume that \(\alpha: X \rightarrow Y\) is a rational map between smooth varieties \(X\) and \(Y\). Consider a smooth variety \(V\) and two regular morphisms \(\delta_X\) and \(\delta_Y\) to \(X\) and \(Y\) respectively such that the diagram commutes and \(\delta_X\) is birational:

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & Y \\
\delta_X & & \delta_Y \\
V & \xleftarrow{\delta_X} & \xrightarrow{\delta_Y} \text{Ind}(\alpha)
\end{array}
\]

Note that such a diagram always exists (take, e.g., the resolution of indeterminacy of the graph of \(\alpha\) in \(X \times Y\)). Moreover, here we can assume that \(\text{Ind}(\alpha)\) coincides with \(\text{Ind}(\delta_X^{-1})\).

By the **total image** of a subset \(W\) in \(X\) we denote the subset \(\alpha(W) = \delta_Y(\delta_X^{-1}(W))\) in \(Y\). By the **proper transform** of a subvariety \(W\) which does not lie in \(\text{Ind}(\alpha)\) we denote the subvariety \(\tilde{\alpha}(W) = \delta_Y(W \setminus \text{Ind}(\alpha))\) of \(Y\). Note that the proper transform of an irreducible subvariety is always irreducible. There are many choices for the smooth variety \(V\) as in the diagram (2.1). However, constructions of the proper transform and total image do not depend on this choice.

If \(X\) is a smooth variety we can define the \(\mathbb{R}\)-vector space \(N^1(X)\) of classes of Cartier divisors modulo numerical equivalence. For any element in the divisor class group \(D \in N^1(Y)\) we can define its *inverse image*:

\[
\alpha^*D = \delta_X_*(\delta_Y^*(D)) \in N^1(X).
\]

This operation also does not depend on the choice of resolution of \(\alpha\). The divisor class group \(N^1(X)\) of a smooth variety \(X\) is a finite-dimensional vector space. If \(S\) is a proper irreducible reduced hypersurface in \(X\), then we denote by \([S]\) the class of \(S\) in \(N^1(X)\).

On a smooth variety \(X\) we consider the group of numerical classes of curves which is the \(\mathbb{R}\)-vector space \(N_1(X)\) generated by classes of irreducible reduced curves modulo numerical equivalence. If \(C\) is an irreducible reduced curve in \(X\), then we denote by \([C]\) its class in \(N_1(X)\). On a smooth variety \(X\) there is a natural perfect pairing between \(N^1(X)\) and \(N_1(X)\).

We say that the class \(D \in N^1(X)\) is nef if \(D \cdot [C] \geq 0\) for any effective curve \(C\) on \(X\). The inverse image of a nef class under a regular map is nef. In case of rational maps it is not true, but we have the following generalization of this fact.

**Lemma 2.2.** Consider a birational map \(\alpha: X \rightarrow Y\) between smooth varieties \(X\) and \(Y\). If \(D\) is a class of a nef divisor on \(Y\) and \(C\) is a curve on \(X\) such that \(\alpha^*D \cdot [C] < 0\), then \(C\) lies in \(\text{Ind}(\alpha)\). In particular, in case \(\dim(X) = 3\) the set of \(\alpha^*D\)-negative curves is finite for any nef class of divisor \(D\).

**Proof.** Consider a diagram as in (2.1). If \(D\) is a nef divisor in \(Y\), then its pullback \(\tilde{D} = \delta_Y^*D\) is also nef.

Consider the class \(\delta_X_*(\delta_Y^*(D))\); the difference \(E\) between this class and \(\tilde{D}\) is supported on the exceptional locus of \(\delta_X\):

\[
\tilde{D} = \delta_X_*(\delta_Y^*(D)) - E.
\]

By Lemma 2.3 we get that \(E\) is the class of an effective divisor.

Take any irreducible curve \(C\) on \(X\) outside \(\delta_X(\text{Exc}(\delta_X))\). Denote by \(\tilde{C}\) the proper transform of \(C\) in \(V\) so that \(\delta_X*[\tilde{C}] = [C]\). Then we have

\[
\delta_X_*(\tilde{D}) \cdot [C] = \delta_X_*(\tilde{D}) \cdot \delta_X*[\tilde{C}] = \delta_X_*(\delta_X_*(\tilde{D})) \cdot [\tilde{C}] = (\tilde{D} + E) \cdot [\tilde{C}] \geq 0.
\]

The last inequality is true since \(\tilde{D}\) is nef, \(E\) is effective and \(\tilde{C}\) lies outside the support of \(E\). Thus, we obtain that the product of \(\delta_X_*(\tilde{D})\) and any curve outside \(\delta_X(\text{Exc}(\delta_X))\) is not negative and this implies the result.

By the above definition a rational map \(\alpha: Y \rightarrow X\) between smooth varieties \(X\) and \(Y\) defines a map between their groups of numerical classes of divisors \(\alpha^*: N^1(X) \rightarrow N^1(Y)\). If \(\alpha\) is regular, then \(\alpha^*\) is the standard functor of the inverse image; in particular, the composition of the inverse images of two regular
maps is equal to the inverse image of the composition of maps. In case of rational maps the situation is more complicated.

**Lemma 2.3.** Let $X, Y$ and $Z$ be smooth varieties and $\alpha: Y \dasharrow X$ and $\beta: Z \dasharrow Y$ be rational maps such that $\beta(Z)$ does not lie in $\text{Ind}(\alpha)$. Then the composition $\alpha \circ \beta: Z \dasharrow X$ is well-defined and for each class $D \in N^1(X)$ we have the following equality:

$$
\beta^*(\alpha^*(D)) - (\alpha \circ \beta)^*(D) = E,
$$

where $E$ is a divisor on $Z$ supported in $\beta^{-1}(\text{Ind}(\alpha))$. If $D$ is nef, then $E$ is effective.

**Proof.** Denote by $\Gamma_\alpha$ and $\Gamma_\beta$ resolutions of singularities of graphs of maps $\alpha$ and $\beta$ respectively. Denote by $p_\alpha$, $q_\alpha$ projections to $X$ and $Y$ from $\Gamma_\alpha$ and by $p_\beta$ and $q_\beta$ projection from $\Gamma_\beta$ to $Y$ and $Z$. Both maps $q_\alpha$ and $q_\beta$ are birational and $q_\alpha(\text{Exc}(q_\alpha^{-1})) = \text{Ind}(\alpha)$. The composition $q_\alpha^{-1} \circ p_\beta$ induces the rational map $\gamma: \Gamma_\beta \dasharrow \Gamma_\alpha$. Denote by $\Gamma$ the resolution of the graph of $\gamma$ and by $p$ and $q$ projections to $\Gamma_\alpha$ and $\Gamma_\beta$. The morphism $q$ is birational and the following diagram commutes:

$$
\begin{array}{ccccc}
& & & & \\
& & & & \\
& & & & \\
& & & & \\
\Gamma & \xrightarrow{\xi} & \Gamma & \xrightarrow{\alpha} & \Gamma_\alpha \\
\downarrow p & & \downarrow q & & \downarrow p_\alpha \\
\Gamma_\beta & \xrightarrow{\gamma} & \Gamma & \xrightarrow{\beta} & \Gamma_\beta \\
\downarrow q_\beta & & \downarrow q & & \downarrow p_\beta \\
Z & \xrightarrow{\beta} & Y & \xrightarrow{\alpha} & X \\
\end{array}
$$

By definition $(\alpha \circ \beta)^*D = q_\beta\ast (q_\alpha\ast (p_\alpha^*D)))$ and $\beta^*(\alpha^*D) = q_\beta\ast (q_\alpha\ast (p_\alpha^*D)))$. Denote by $\tilde{D}$ the class $p_\alpha^*D$ in $N^1(\Gamma_\alpha)$ and denote by $\tilde{E}$ the following difference:

$$
\tilde{E} = p_\beta^*(q_\alpha\ast \tilde{D}) - q_\ast (p^* \tilde{D}).
$$

By definition we see that $\beta^*(\alpha^*D) - (\alpha \circ \beta)^*D = q_\beta\ast \tilde{E}$. Consider the following equality:

$$
\tilde{E} = q_\ast (q^* \tilde{E}) = q_\ast (q^* (p_\beta^* (q_\alpha \ast \tilde{D}))) - q_\ast (p^* \tilde{D}) = q_\ast (q^* (p_\beta^* (q_\alpha \ast \tilde{D}))) - q_\ast (p^* \tilde{D}) = q_\ast (p^* (\tilde{D} + E')) - q_\ast (p^* \tilde{D}) = q_\ast (p^* E'),
$$

where $E' = q_\ast (q_\alpha \ast \tilde{D}) - \tilde{D}$ is a class of divisor with the support in $\text{Exc}(q_\alpha)$. Thus, the image of the support of $\tilde{E} = q_\ast (p^* E')$ under $p_\beta$ lies in $\text{Ind}(\alpha)$. This implies that $E$ is a divisor on $Z$ supported in $\beta^{-1}(\text{Ind}(\alpha))$.

Now assume that $D$ is a nef divisor. Then so is $\tilde{D}$; thus, the class $E'$ is $q_\ast$-antinef by construction and $q_\alpha \ast E' = 0$ is an effective divisor. By [KM98, Lemma 3.39] we get that $E'$ is effective. Consider the divisor class $q^* E'$; it is $q$-nef and $E' = q_\ast (q^* E')$ is effective. Thus, by [KM98, Lemma 3.39] we get that $q^* E'$ is effective. Then we is $E$ and this finishes the proof. \qed

Let $\alpha: X \dasharrow Y$ be a rational map between smooth varieties $X$ and $Y$. Then using the natural perfect pairing between $N^1(X)$ and $N_1(X)$ we can define the direct image $\alpha_*: N_1(X) \rightarrow N_1(Y)$ as the dual map to $\alpha^*: N^1(Y) \rightarrow N^1(X)$. If the curve $C$ does not lie in the indeterminacy locus of $\alpha$, then we have the following interpretation of the direct image of a curve:

**Lemma 2.5.** If $\delta: Y \rightarrow X$ is a blow-up of a smooth variety $X$ in smooth center $Z$ and $C \subset Z$ is an irreducible curve in $X$, then for any irreducible curve $T$ such that $\delta(T)$ is a point we have

$$
\delta^*[C] = [\tilde{C}] + \mu[T] \in N_1(X),
$$

where $\tilde{C}$ is the proper transform of $C$ and $\mu$ is a non-negative number.

**Proof.** Take some class of divisor $D$ in $X$. Since

$$
\delta^* D \cdot ([\tilde{C}] - \delta^*[C]) = D \cdot \delta_* [\tilde{C}] - \delta^*(D \cdot [C]) = 0,
$$

we get that $\delta^*[C] = [\tilde{C}] + \xi$, where $\xi \in N_1(X)$ is a class of curve such that $\delta_\ast (\xi) = 0$. Denote by $E$ the exceptional divisor of $\delta$. Since $\delta$ is a blow-up along a smooth center, the class $\xi$ is proportional to the class
of the curve $T$ in $E_1$ such that $δ(T)$ is a point. By the projection formula we have

$$0 = E \cdot δ^*[C] = E \cdot (\tilde{C} + \xi) = E \cdot [\tilde{C}] + E \cdot \xi$$

Since $\tilde{C}$ does not lie inside $E$ the product $E \cdot [\tilde{C}]$ is non-negative. Thus, $E \cdot \xi \leq 0$ then we get that $\xi = μ[T]$ is effective and $μ ≥ 0$.

**Lemma 2.6.** Assume that $α: X \to Y$ is a birational map between smooth varieties $X$ and $Y$ and $C$ is an irreducible curve on $X$ such that $C \not\in \text{Ind}(α)$. If $\tilde{C}$ is the proper image of $C$ under $α$, then there exist effective curves $T_1, \ldots, T_N$ in $α(\text{Ind}(α))$ and non-negative numbers $μ_1, \ldots, μ_N ≥ 0$ such that

$$α_*(C) = [\tilde{C}] + \sum_{i=1}^M μ_i[T_i] \in N_1(X).$$

In particular, if $C$ does not lie in $α(\text{Ind}(α))$, then the class $α_*(C)$ is effective.

**Proof.** Consider the diagram as in (2.1). We can assume that $C$ differs. Nevertheless, in higher dimension there are restrictions on the indeterminacy locus of a pseudo-automorphisms coincides with the one of regular automorphisms, while in higher dimensions the notions differ. Nevertheless, in higher dimension there are restrictions on the indeterminacy locus of a pseudo-automorphism.

Consider the blow-up $δ_i: X_i \to X_{i-1}$ along a smooth center $Z$ and a curve $T$ which does not lie in $Z$. Denote by $T$ the proper preimage of $T$ in $X_i$. Denote by $E$ the exceptional divisor of $δ_i$ and consider the intersection number $n = E \cdot \tilde{T}$. It is positive since $T$ does not lie in $E$. Then by projection formula we get $δ_i^*[T] = [\tilde{T}] + n \cdot F$, where $F$ is the class of the curve on the extremal ray of the contraction $δ$. The class $F$ is effective, its representative lies in $E$ and can be chosen such that it does not lie in a closed set of codimension 2.

By induction we prove that $δ_X^*[C] = δ_X(C) + \sum μ'_i[T'_i]$, in the same way of the curves $T'_i$ lie in $\text{Exc}(δ_X)$ and numbers $μ'_i$ are non-negative.

Now by $[\text{Fn98}]$ the direct image under a regular map $δ_Y: \text{ind} \text{ of the linear combination of classes of effective curves } δ_X^*[C] = δ_X(C) + \sum μ'_i[T'_i] \text { is a linear combination with the same coefficients of images of these curves under } δ_Y$. This finishes the proof.

Recall also that $φ$ is **regularizable**, if there exist a smooth variety $Y$, a birational map $α$ and a regular automorphism $ψ: Y \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc}
X - φ & \to & X \\
α \downarrow & & α \downarrow \\
Y & \xrightarrow{ψ} & Y
\end{array}$$

In this situation we call the triple $(Y, ψ, α)$ a smooth regularization of $φ$. By functorial desingularization (see, for instance, [Kol07, Theorem 3.26]) any birational map which admits a regularization also admits a smooth regularization.

Recall that a birational map $φ: X \to Y$ is called a **pseudo-isomorphism**, if sets $\text{Exc}(φ)$ and $\text{Exc}(φ^{-1})$ are empty. If $Y = X$, then we call such map a **pseudo-automorphism**. On surfaces the notion of pseudo-automorphisms coincides with the one of regular automorphisms, while in higher dimensions the notions differ. Nevertheless, in higher dimension there are restrictions on the indeterminacy locus of a pseudo-automorphism.

**Lemma 2.7 ([BK14]).** If $φ: X \to X$ is a pseudo-automorphism of a smooth variety $X$ of dimension 3 or greater, then $\text{Ind}(φ)$ has no isolated points.

Observe that pseudo-automorphisms induce invertible maps on the group of classes of divisors on the variety:

**Lemma 2.8.** If $φ: X \to X$ is a pseudo-automorphism of a smooth variety $X$, then

$$φ^*: N^1(X) \to N^1(X)$$

is an isomorphism of vector spaces and $(φ^n)^* = (φ^*)^n$.

**Proof.** This follows from Lemma 2.3. □
2.2. A special construction of a flop. In this section we consider the construction of a concrete pseudo-isomorphism and construct the resolution of its graph.

Consider a smooth threefold $X$ and two smooth curves $\Gamma_1$ and $\Gamma_2$ on $X$ such that the intersection $\Gamma_1 \cap \Gamma_2$ is a finite set of points and the union $\Gamma_1 \cup \Gamma_2$ is a nodal curve.

Denote by $\delta_1 : Y_1 \to X$ the blow-up of $X$ along the curve $\Gamma_1$. Let $\delta_2 : Y_1 \to Y_2$ be the proper transform of the curve $\Gamma_2$ under $\delta_1$. Let $\delta_2 : Y_2 \to Y_1$ be the blow-up of $Y_1$ along $\Gamma_2$.

Denote by $\delta_{2+} : Y_{2+} \to X$ the blow-up of $X$ along the curve $\Gamma_{2+}$, by $\hat{\delta}_1$ the proper transform of $\Gamma_1$ to $Y_{2+}$, and by $\delta_{2+} : Y_{21+} \to Y_{2+}$ the blow-up of $Y_{2+}$ in $Y_{12}$. Consider the birational map $\tau : Y_{12} \to Y_{21+}$ induced by the following diagram:

\[
\begin{array}{ccc}
Y_{12} & \xrightarrow{\tau} & Y_{21+} \\
\downarrow \delta_1 & & \downarrow \delta_{2+} \\
Y_1 & \xrightarrow{\delta_2} & Y_{2+} \\
\end{array}
\]

The map $\tau$ is a pseudo-isomorphism and its restriction to $(\delta_1 \circ \delta_{12})^{-1} (X \setminus (\Gamma_1 \cap \Gamma_2)) \subset Y_{12}$ induces an isomorphism to $(\delta_{2+} \circ \delta_{21+})^{-1} (X \setminus (\Gamma_1 \cap \Gamma_2)) \subset Y_{21+}$.

For each point $p_i$ in $\Gamma_1 \cap \Gamma_2$ we consider a curve $C_i$ defined as the irreducible component of $(\delta_1 \circ \delta_{12})^{-1} (p_i)$ which does not lie in the exceptional divisor of $\delta_{12}$. Analogously, we denote by $C_i^+$ the irreducible component of $(\delta_{2+} \circ \delta_{21+})^{-1} (p_i)$ which does not lie in the exceptional divisor of $\delta_{21+}$. These are smooth rational curves and curves $C_i$ and $C_j$ (respectively, $C_i^+$ and $C_j^+$) do not intersect in $Y_{12}$ (respectively $Y_{21+}$) for distinct $i$ and $j$.

Then the map $\tau$ is a pseudo-isomorphism between $Y_{12}$ and $Y_{21+}$; it is a flop, see [HM10, Example 1.12]. Denote by $f_1$ and $f_2$ the curve classes of the fibers of the morphisms $\delta_1$ and $\delta_{12}$ over $\Gamma_1$ and $\hat{\delta}_1$ respectively. Then the following assertion is true:

**Proposition 2.9 ([HM10 Example 1.12]).** The map $\tau$ is a pseudo-isomorphism. The sets of indeterminacy of maps $\tau$ and $\tau^{-1}$ consist of disjoint unions of curves:

\[
\text{Ind}(\tau) = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_m; \quad \text{Ind}(\tau^{-1}) = C_1^+ \sqcup C_2^+ \sqcup \cdots \sqcup C_m^+.
\]

Moreover, the class of each curve $C_i$ and of its direct image under $\tau$ satisfy:

\[
[C_i] = f_1 - f_2 \quad \quad \quad \quad \tau_*[C_i] = -[C_i^+].
\]

**Sketch of proof.** Denote by $\Delta : V \to Y_{12}$ the blow-up of the variety $Y_{12}$ in the smooth curve $C_1 \sqcup \cdots \sqcup C_m$ and by $\Delta_+ : V \to Y_{21+}$ the induced map to $Y_{21+}$.

We use the universal property of blow-ups (see [Har77, Proposition II.7.14]). First we apply this property to the map $\delta_1 \circ \delta_{12} \circ \Delta : V \to X$. Since the preimage of the curve $\Gamma_2$ is of pure dimension 2 and since $V$ is smooth we get that the following map is regular:

\[
(\delta_1^{-1} \circ \delta_1 \circ \delta_{12}) : V \to Y_{12}.
\]

Repeating this argument we can show that there exists a regular map $f : V \to V_+$, where $V_+$ is a blow-up of the variety $Y_{21+}$ in the smooth curve $C_{1+} \sqcup \cdots \sqcup C_{m+}$ such that the following diagram commutes:

\[
\begin{array}{ccc}
V & \xrightarrow{\Delta} & V_+ \\
\downarrow \tau & & \downarrow \Delta_+ \\
Y_{12} & \xrightarrow{\tau} & Y_{21+} \\
\end{array}
\]

Since Picard numbers of $V$ and $V_+$ are same, this implies that the map $f$ is an isomorphism.

Considering the Picard group of $Y_{12}$ we can prove that $[C_i] = f_1 - f_2$. Note that since $\tau$ is a pseudo-isomorphism, groups $N^i(Y_{12})$ and $N^i(Y_{21+})$ are isomorphic for $i = 1$. Since $Y_{12}$ is a threefold and group of classes of curves is dual to the divisor class group, we have also the isomorphism for $i = 2$. Under this identification we get $[C_i^+] = f_2 - f_1$, thus, $\tau_*[C_i^+] = -[C_i^+]$. 

\[\square\]
2.3. **Dynamical degrees.** If \( \varphi \) is any birational automorphism of a smooth projective variety \( X \), then we define its dynamical degrees \( \lambda_i(\varphi) \) for all \( 0 \leq i \leq \dim(X) \) as follows:

\[
\lambda_i(\varphi) = \lim_{n \to \infty} \left( (\varphi^n)^*(H^i) \cdot H^{\dim(X)-i} \right)^{\frac{1}{n}},
\]

where \( H \) is any ample class in \( N^1(X) \). The fact that the limit exists and does not depend on the choice of the ample class is proved in \([\text{DS05}, \text{Tru20}]\). Dynamical degrees are positive real numbers which are greater than or equal to 1 and they are birational invariants of the automorphism \( \varphi \). The projection formula implies:

\[
\lambda_i(\varphi) = \lambda_{\dim(X) - i}(\varphi^{-1}).
\]

Dynamical degrees are log-concave, see \([\text{DN11}]\): i.e. for all indices \( 0 \leq i \leq \dim(X) \) one has:

\[
\lambda_i(\varphi)^2 \geq \lambda_{i+1}(\varphi)\lambda_{i-1}(\varphi).
\]

Note that by log-concavity we have \( \lambda_1(\varphi) = 1 \) if and only if \( \lambda_i(\varphi) = 1 \) for all \( 1 \leq i \leq \dim(X) \). By log-concavity we have also the following inequality \( \lambda_1(\varphi)^2 \geq \lambda_2(\varphi) \) for all birational automorphisms \( \varphi \). If this inequality is strict and \( \varphi \) is a pseudo-automorphism, then the action of \( \varphi^* \) on the group of classes of divisors has the following property:

**Theorem 2.10** ([Tru14, Theorem 1, Corollary 3]). Assume that \( \varphi \colon X \dasharrow X \) is a pseudo-automorphism of a smooth projective variety \( X \) satisfying \( \lambda_1^2(\varphi) > \lambda_2(\varphi) \). Then there exists a non-zero class \( \theta_1(\varphi) \in N^1(X) \) such that:

1. For any ample class \( H \) the limit \( \lim_{n \to \infty} \frac{(\varphi^n)^*(H) \cdot H^{\dim(X) - i}}{\lambda_1(\varphi)^n} \) exists, is non-zero and proportional to \( \theta_1(\varphi) \);
2. \( \varphi^*(\theta_1(\varphi)) = \lambda_1(\varphi)\theta_1(\varphi) \);
3. The eigenvalue \( \lambda_1(\varphi) \) is simple, i.e. there is a \( \varphi^* \)-invariant decomposition \( N_1(X) = \mathbb{R}\theta_1(\varphi) \oplus V \).

Moreover, the absolute value of any eigenvalue of \( \varphi^* \) distinct from \( \lambda_1(\varphi) \) is less than or equal to \( \sqrt{\lambda_2(\varphi)} \).

In \([\text{DF20}]\) was proved a generalization of this theorem in the case of any birational automorphism.

In case when the birational automorphism \( \varphi \) is regular or if \( \dim(X) = 2 \), then the class \( \theta_1(\varphi) \) is nef. However, even in the case of a pseudo-automorphisms of threefolds this class can intersect some curves negatively.

**Lemma 2.11.** Let \( \varphi \colon X \dasharrow X \) be a pseudo-automorphism of a smooth threefold \( X \) such that \( \lambda_1(\varphi)^2 > \lambda_2(\varphi) \). If \( C \) is an irreducible curve and \( \theta_1(\varphi) \cdot [C] < 0 \), then there exists an integer \( N \) such that

\[
C \subset \bigcap_{n > N} \text{Ind}(\varphi^n).
\]

**Proof.** Fix a curve \( C \) such that \( \theta_1(\varphi) \cdot C < 0 \) and an ample divisor \( H \) on \( X \). Then by Theorem 2.10 there exists \( N \) such that for all \( n > N \) we have the following inequality

\[
f^{n*}(H) \cdot C < 0.
\]

By Lemma 2.2 this is possible only if \( C \) lies in \( \text{Ind}(\varphi^n) \) for all \( n > N \). \( \square \)

The set of curves \( C \) such that \( \theta_1(\varphi) \cdot [C] < 0 \) is finite hence \( \theta_1(\varphi) \) is movable in the sense of \([\text{BDP13}]\).

3. **Proofs of Theorems 1.1 and 1.2**

3.1. **Regularizations of pseudo-automorphisms.** Here we consider smooth threefolds \( X \) and \( Y \). Let \( \varphi \) be a pseudo-automorphism of \( X \) and \( \psi \) be a regular automorphism of \( Y \). The birational map \( \alpha \colon X \dasharrow Y \) is such that \( \varphi \circ \alpha = \alpha \circ \psi \).

Consider some resolution of the graph of the map \( \alpha \):

\[
\begin{array}{c}
\delta_X \\
\varphi \swarrow \\
X \longrightarrow Y \swarrow \delta_Y \\
\end{array}
\]

We can choose \( V, \delta_X \) and \( \delta_Y \) in such a way that \( \text{Ind}(\alpha) = \delta_X(\text{Exc}(\delta_X)) \). Moreover, we can assume that \( \delta_X \) is a composition of blow-ups along smooth centers.
We consider classes of divisors $\theta_1(\varphi)$ and $\theta_1(\psi)$ in $N^1(X)$ and $N^1(Y)$ respectively; inverse images of these classes to $V$ are connected in the following way:

**Lemma 3.2.** Let $\varphi$ be a pseudo-automorphism with a regularization $(Y, \psi, \alpha)$ fitting into a diagram of the form (3.1). If $\lambda_1(\varphi)^2 > \lambda_2(\varphi)$, then there exists a class $E$ of an effective divisor in $\text{Exc}(\delta_X)$ such that
$$\delta_X^*\theta_1(\varphi) = \delta_Y^*\theta_1(\psi) + E.$$

**Proof.** First we justify that $\alpha^*\theta_1(\psi) \neq 0$. Suppose by contradiction that $\alpha^*\theta_1(\psi) = \delta_X^*(\delta_Y^*\theta_1(\psi)) = 0$. Since $\delta_X$ is a composition of blow-ups along smooth centers this implies that the divisor class $\delta_Y^*\theta_1(\psi)$ is supported on a subset of $\text{Exc}(\delta_X)$. Then there exists a class $E$ of the divisor on $V$ supported on $\text{Exc}(\delta_X)$ such that
$$\delta_Y^*\theta_1(\psi) = E.$$

The class $\theta_1(\psi)$ is nef, so is $E$. By Mori negativity lemma [KM98, Lemma 3.39] since $\delta_X^*E = 0$ is effective we conclude that $E$ is anti-effective. However, $\delta_Y^*E = \theta_1(\psi)$ is a pseudo-effective class. Thus, we get a contradiction.

Now we can assume that $\alpha^*\theta_1(\psi)$ is a non-zero class. We apply Lemma 2.3

$$\lambda_1(\psi)\alpha^*\theta_1(\psi) = \alpha^*(\psi^*\theta_1(\psi)) = \psi\circ\alpha^*\theta_1(\psi) = (\alpha\circ\varphi)^*\theta_1(\psi) = \varphi^*(\alpha^*\theta_1(\psi)).$$

By [Tn20, Theorem 1.1] we have $\lambda_1(\psi) = \lambda_1(\varphi)$. Since both classes $\theta_1(\psi)$ and $\theta_1(\varphi)$ are pseudo-effective, there exists a number $C > 0$ such that
$$\theta_1(\varphi) = C \cdot \alpha^*\theta_1(\psi).$$

Pulling back this equation by $\delta_X$ we obtain an effective divisor $E$ supported in $\text{Exc}(\delta_X)$ such that
$$\delta_X^*\theta_1(\varphi) = C \cdot \delta_Y^*\theta_1(\psi) + E.$$

Since the class $\theta_1(\psi)$ is nef, the class $-E$ is $\delta_X$-nef. Then by [KM98, Lemma 3.39] we get that $E$ is effective. \hfill $\square$

**Lemma 3.3.** Let $\varphi$ be a pseudo-automorphism with a regularization $(Y, \psi, \alpha)$ fitting into a diagram of the form (3.1) and $\lambda_1(\varphi)^2 > \lambda_2(\varphi)$. Then if $C$ is an irreducible curve such that $\theta_1(\varphi) \cdot [C] < 0$, then $C$ lies in $\text{Ind}(\alpha)$.

**Proof.** Choose an irreducible curve $\tilde{C}$ in $\delta_X^{-1}(C) \subset V$ such that $\delta_X(\tilde{C}) = C$. Then there exists a positive number $m$ such that $\delta_X^*([\tilde{C}]) = m \cdot [C]$.

By the projection formula and Lemma 3.2 we get the following
$$(\delta_Y^*(\theta_1(\psi)) + E) \cdot [\tilde{C}] = \delta_X^*(\theta_1(\varphi)) \cdot [\tilde{C}] = m \cdot \theta_1(\varphi) \cdot [C] < 0.$$

Since $\theta_1(\psi)$ is a nef class this implies that $E \cdot [\tilde{C}] < 0$. Since $E$ is an effective exceptional divisor of $\delta_X$ this is possible only if $\tilde{C}$ is included in the support of $E$. As $E$ is contracted by $\delta_X$, we get
$$C = \delta_X(\tilde{C}) \subset \delta_X(E) \subset \text{Ind}(\alpha).$$

Thus, we get that a $\theta_1(\varphi)$-negative curve lies in the indeterminacy locus of any regularization map. \hfill $\square$

**Lemma 3.4.** Assume that $X$ and $Y$ are smooth varieties, $\dim(X) = 3$, $\varphi: X \dasharrow X$ is a pseudo-automorphism, $\alpha: X \dasharrow Y$ is a rational map and $\psi: Y \dasharrow Y$ is a birational automorphism such that $\alpha \circ \varphi = \varphi \circ \alpha$. Let either $\psi$ be a regular automorphism or $Y$ be a surface. If $D$ is a nef divisor class on $Y$ and $C$ is a curve on $X$ such that $\alpha^*D \cdot [C] < 0$ and $C \subset \bigcap_{t > N} \text{Ind}(\varphi^t)$, then for any $m > 0$ except a finite set we have $C \subset \text{Ind}(\varphi^{-m})$.

**Proof.** Assume that $C$ is an irreducible $\alpha^*D$-negative curve and the following set is infinite:
$$I = \{ m \in \mathbb{Z} \mid C \not\subset \text{Ind}(\varphi^m) \}.$$

By assumption we have that $I$ is included in $\{ m \leq N \}$ for some integer $N > 0$.

For each $-m \in I$ we have that $C$ does not lie in $\text{Ind}(\varphi^{-m})$. Denote by $C_m$ the proper image of the curve $C$ under $\varphi^{-m}$ for each $-m \in I$. Since $\varphi$ is a pseudo-automorphism the curve $C_m$ does not lie in $\text{Ind}(\varphi^m)$ and its proper image under $\varphi^m$ is $C$. \hfill $\square$
By Lemma 3.3 the curve $C$ lies in $\text{Ind}(\alpha)$. Since $\text{Ind}(\psi^n)$ does not contain curves for any $n$ this implies that

$$C \subset \text{Ind}(\psi^{-m} \circ \alpha) = \text{Ind}(\alpha \circ \varphi^{-m}),$$

for all $-m \in I$. Thus, the curve $C_{-m}$ lies in $\text{Ind}(\alpha)$ for all $-m$ in $I$.

Since $\text{Ind}(\alpha)$ contains only finite number of curves there is some $-m \in I$ such that $C_{-m} = C$.

The proper image of $C_{-m}$ under $\varphi^m$ is $C$. Then $C = C_{-m}$ does not lie in $\text{Ind}(\varphi^m)$ and also in $\text{Ind}(\varphi^{km})$ for all $k > 0$. However, this contradicts our assumption and concludes the proof.

Now we are ready to prove the criterion for non-regularizable automorphisms.

Proof of Theorem 1.2 By Lemma 2.11 we only have to prove that if there is a curve $C$ on $X$ such that Condition $A$ is satisfied, then the pseudo-automorphism $\varphi$ can not be regularized.

By contradiction we assume that there exists a regularization of $\varphi$ as on the diagram (3.1). By Lemma 3.2 we get that $\theta_1(\varphi) = \alpha^*\theta_1(\psi)$. Moreover, by [DF01] the class $\theta_1(\psi)$ is nef. Then Lemma 3.4 applied to the divisor $\theta_1(\psi)$ and the curve $C$ from Condition $A$ leads us to the contradiction.

3.2. Proof of Theorem 1.2. We start with considering the situation when $f$ is algebraically stable i.e. when we have an equality $(f^*)^n = (f^n)^*$ of endomorphisms of $N^1(S)$ for all integers $n \in \mathbb{Z}$. In fact, all birational automorphisms of surfaces are conjugated to algebraically stable ones by [DF01, Theorem 0.1].

If the pseudo-automorphism $\varphi$ preserves the fibration $\pi$, then we can construct a good model of $S$.

Lemma 3.5. Assume that $\varphi: X \dashrightarrow S$ is a pseudo-automorphism of a smooth threefold, $\pi: X \dashrightarrow S$ is a dominant rational map to a smooth surface $S$ and $f: S \dashrightarrow \tilde{S}$ is a birational automorphism of $S$ such that $\pi \circ \varphi = f \circ \pi$. Then there exists a birational morphism $\delta: \tilde{S} \dashrightarrow S$ such that the automorphism $\tilde{f} = \delta^{-1} \circ f \circ \delta$ of $\tilde{S}$ is algebraically stable if and only if we denote $\tilde{\pi} = \delta^{-1} \circ \pi$, then

$$\left(\tilde{f} \circ \tilde{\pi}\right)^* = \tilde{\pi}^* \circ \tilde{f}^*.$$

Proof. Denote by $\Gamma$ the resolution of singularities of the graph of $\pi: X \dashrightarrow S$ and by $p: \Gamma \dashrightarrow X$ and $q: \Gamma \dashrightarrow S$ the projections to $X$ and $S$ respectively:

$$\xymatrix{ & \Gamma \\
X \ar@{-->}[r]^{\pi} \ar@{-->}[ur]^{p} & \ar@{-->}[u]^{q} \ar@{-->}[r]_{\pi} & S}
$$

Consider the set of points $Z$ in $S$ such that for any $z \in Z$ the fiber $q^{-1}(z)$ is a divisor in $\Gamma$. Since $\Gamma$ is irreducible $Z$ is finite. Denote by $\delta': S' \dashrightarrow S$ the blow-up of $S$ in $Z$. Then by [DF01, Theorem 0.1] there exists a birational morphism $\delta'': S' \dashrightarrow S'$ such that the automorphism $\tilde{f}: \tilde{S} \dashrightarrow \tilde{S}$ induced by $f$ is algebraically stable.

Denote by $\delta$ the composition $\delta' \circ \delta''$ and by $\tilde{\pi}$ the composition $\delta^{-1} \circ \pi$ and consider the following diagram:

$$\xymatrix{ X \ar[r]^{\varphi} & X \\
Y \ar[r]_{\tilde{\pi}} \ar[u]_{\pi} & \tilde{S} \ar[u]_{\tilde{\pi}} \ar[r]_{\tilde{f}} & \tilde{S} \ar[u]_{\tilde{\pi}} \ar[r]_{\tilde{\pi}} & \tilde{S}}
$$

By construction the proper image of any divisor on $X$ is not a point on $\tilde{S}$. In particular, the proper image of any divisor on $X$ does not lie in $\text{Ind}(\tilde{f})$. Thus, by Lemma 2.8 we get that $\tilde{\pi}^* \circ \tilde{f}^* = (\tilde{f} \circ \tilde{\pi})^*$.

This lemma allows us to prove Theorem 1.2.

Proof of Theorem 1.2. Assume that there exists a surface $S$, a rational map $\pi: X \dashrightarrow S$ and a birational automorphism $f: S \dashrightarrow S$ such that $\pi \circ \varphi = f \circ \pi$. By [Tru20, Theorem 1.1] since the relative dimension of $\pi$ equals 1 we have the following equality:

$$1 < \lambda_1(\varphi) = \lambda_1(f).$$
By Lemma 3.5 we can replace $S$ by its birational model such that $\pi^* \circ f^* = (f \circ \pi)^*$ and $f$ is algebraically stable. Since $\varphi$ is pseudo-automorphism no divisor in $X$ maps under $\varphi$ to a component of $\text{Ind}(\pi)$. Then by Lemmas 2.3 and 3.5 we get the following.

$$\varphi^*(\pi^*(\theta_1(f))) = (\pi \circ \varphi)^*(\theta_1(f)) = (f \circ \pi)^*(\theta_1(f)) = \pi^*(f^*(\theta_1(f))) = \lambda_1(f) \cdot \pi^*(\theta_1(f)) = \lambda_1(\varphi) \cdot \pi^*(\theta_1(f)).$$

Then Theorem 2.10 implies that the class $\theta_1(\varphi)$ is proportional to $\pi^*(\theta_1(f))$ with strictly positive coefficient. Since the divisor $\theta_1(f)$ is a nef divisor we get the contradiction with Condition A by Lemma 3.4.

4. Blanc’s pseudo-automorphism

4.1. Construction. This family of pseudo-automorphisms is described in the paper of Blanc [Bla13]. We recall the construction of the pseudo-automorphism only in dimension 3, though, in other dimensions everything is similar. We consider a smooth cubic hypersurface $Q$ in $\mathbb{P}^3$. To each smooth point $p \in Q$ we associate a birational automorphism of the projective space $\sigma_p : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$.

If $L$ is a general line passing through $p$ and intersecting $Q$ in three distinct points $p$, $q_1$, $q_2$, then $\sigma_p|_L$ is a unique involution of $L \cong \mathbb{P}^1$ such that $\sigma_p|_L(q_i) = q_i$ for $i = 1, 2$. Thus, $\sigma_p$ is a birational involution of $\mathbb{P}^3$, it preserves pointwise an open subset of $Q$ and its indeterminacy locus consists of the point $p$ and an irreducible curve $\Gamma \subset Q$ of degree 6, see [Bla13] Section 2.

Consider now $k$ distinct points $p_1, \ldots, p_k$ on $Q$: denote by $\Gamma_1, \ldots, \Gamma_k$ the curves in the base loci of the associated involutions $\sigma_{i_1}, \ldots, \sigma_{i_k}$. Denote by $\delta : X \to \mathbb{P}^3$ the successive blow-ups of points $p_1, \ldots, p_k$, then the proper transforms of curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$. Also denote by $\tilde{\sigma}_i$ the birational involution of $X$ induced by $\sigma_i$:

$$\tilde{\sigma}_i : X \dashrightarrow X.$$

We introduce the following condition on the points $p_1, \ldots, p_k$ on a smooth cubic surface $Q$ in $\mathbb{P}^3$:

$$p_i \not\in \Gamma_j \quad \forall i \neq j \quad |\Gamma_i \cap \Gamma_j| = 6 \quad \forall i \neq j \quad \Gamma_i \cap \Gamma_j \cap \Gamma_k = \emptyset \quad \forall \text{ distinct indices } i, j, k.$$

The condition $|\Gamma_i \cap \Gamma_j| = 6$ means that $\Gamma_i$ and $\Gamma_j$ intersect transversally on the surface $Q$. It is proved in [Bla13] that Condition (4.1) is satisfied for a general set of points $p_1, \ldots, p_k \in Q$. This gives us a construction of a pseudo-automorphism on a smooth rational threefold with dynamical degree greater than 1.

Theorem 4.2 ([Bla13] Theorem 1.2]). Assume Condition (4.1) is satisfied. Then the composition of involutions $\tilde{\sigma}_1 \circ \cdots \circ \tilde{\sigma}_k$ defines a pseudo-automorphism $\varphi : X \dashrightarrow X$. If $k \geqslant 3$, then we have $\lambda_1(\varphi) = \lambda_2(\varphi) > 1$.

We can represent the pseudo-automorphism $\varphi$ as a composition of simpler birational maps.

We denote by $\delta_i : X_i \to \mathbb{P}^3$ the successive blow-ups of points $p_1, \ldots, p_k$, then the proper transforms of curves $\Gamma_1, \Gamma_1, \ldots, \Gamma_{i-1}, \Gamma_{i+1}, \ldots, \Gamma_k$. Note that $X_1 = X$ and $\delta_1 = \delta$.

By [Bla13] Proposition 2.2] the involution $\sigma_i$ induces a regular automorphism $\tilde{\sigma}_i$ on the variety $X_i$. We denote by $\tau_{i,j} : X_i \to X_j$ the birational map induced by the identity map on $\mathbb{P}^3$. Note that this map is a composition of flops described in Proposition 2.4.

Lemma 4.3. In notations and assumptions of Theorem 4.2 the pseudo-automorphism $\varphi$ is a composition of regular involutions $\tilde{\sigma}_i : X_i \to X_i$ and flops $\tau_{i,i+1} : X_{i+1} \dashrightarrow X_i$ and the following diagram commutes:

$$X_1 \underbrace{\tilde{\sigma}_1}_{\varphi} X_1 \dashrightarrow X_1 \underbrace{\tau_{1,2}}_{\varphi} X_2 \underbrace{\tilde{\sigma}_2}_{\varphi} \cdots \underbrace{\tau_{k-1,k}}_{\varphi} X_{k-1} \underbrace{\tilde{\sigma}_{k-1}}_{\varphi} X_k \underbrace{\tau_{k,k+1}}_{\varphi} X_{k+1} \dashrightarrow X_1.$$

4.2. Action of $\varphi^*$. In this section we study the action of the inverse image of $\varphi$ on groups $N^1(X)$ and $N_1(X) = N^2(X)$. First, let us define the generators of these groups.

We denote by $H$ the pullback of the hyperplane section from $\mathbb{P}^3$ to $X$, by $E_j$ the exceptional divisors over points $p_j$, and by $F_j$ the exceptional divisors over $\Gamma_j$. There exist classes $e_j, f_j \in N^2(X)$ which correspond to extremal rays of the contractions of $E_j$ and $F_j$ respectively. We denote by $h \in N^2(X)$ the image of the class of a line in $\mathbb{P}^3$. 
Lemma 4.5. Under the assumptions of Theorem 4.2, the groups $N^1(X)$ and $N^2(X)$ are generated by the following classes:

$$N^1(X_i) = \langle H, E_1, \ldots, E_k, F_1, \ldots, F_k \rangle;$$
$$N^2(X_i) = \langle h, e_1, \ldots, e_k, f_1, \ldots, f_k \rangle.$$

Moreover, these two bases are “almost dual” in the sense that

$$H \cdot h = 1; \quad E_i \cdot h = 0; \quad F_i \cdot h = 0 \text{ for all } i;$$
$$H \cdot e_i = 0; \quad E_i \cdot e_i = -1; \quad F_i \cdot e_i = 0 \text{ for all } i;$$
$$H \cdot f_i = 0; \quad E_i \cdot f_i = 0; \quad F_i \cdot f_i = -1 \text{ for all } i;$$
$$E_i \cdot e_j = 0; \quad F_i \cdot e_j = 0 \text{ for all } i \neq j;$$
$$E_i \cdot f_j = 0; \quad F_i \cdot f_j = 0 \text{ for all } i \neq j.$$

We also use another set of elements in the group $N^1(X)$ to describe the action of the inverse image $\varphi^*$. Define a class $\nu_j \in N^1(X)$ as follows:

$$\nu_j = 2H - 2E_j - F_j.$$

Lemma 4.6 (Bla13 Proposition 2.3)]. Under the assumptions of Theorem 4.2, the involution $\hat{\sigma}_i$ acts on $N^1(X)$ as follows:

$$\hat{\sigma}_i^*(D) = D + \nu_i, \text{ if } D = H \text{ or } E_i;$$
$$\hat{\sigma}_i^*(F_i) = F_i + 2\nu_i;$$
$$\hat{\sigma}_i^*(D) = D, \text{ if } D = E_j \text{ or } F_j \text{ for } j \neq i;$$
$$\hat{\sigma}_i^*(\nu_i) = -\nu_i;$$
$$\hat{\sigma}_i^*(\nu_j) = \nu_j + 2\nu_i.$$

Since we know the action of involutions on groups of classes of divisors of $X_1$ we are able to compute the first dynamical class of $\varphi$.

Lemma 4.7 (Bla13 Proof of Proposition 2.3)]. Under the assumptions of Theorem 4.2, the first dynamical class of $\varphi$ is as follows:

$$(4.8) \quad \theta_1(\varphi) = \sum_{i=1}^{k} \alpha_i \nu_i \in N^1(X),$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_k$ are strictly positive numbers and $\alpha_1 > \alpha_2 + 2 \left( \sum_{i=3}^{k} \alpha_i \right)$.

This implies in particular that the class $\theta_1(\varphi)$ is not nef. By Condition (4.1), curves $\Gamma_1$ and $\Gamma_2$ lie on $Q$ and their intersection is transversal in $Q$. Choose a point $q \in \Gamma_1 \cap \Gamma_2$ and denote by $L$ the proper transform in $X$ of the line in $\mathbb{P}^3$ passing through $p_1$ and $q$.

Corollary 4.9. Under the assumptions of Theorem 4.2, we have $\theta_1(\varphi) \cdot [L] < 0$.

Proof. By formulas in (Bla13 Section 2), we see that $\tilde{\sigma}_1(L)$ is the fiber of $F_1$ over the point $q$ on $\Gamma_1$. Thus, by Proposition 2.10, we get that $\sigma_1(L)$ is a curve of indeterminacy of $\tau_{12}$. Then by Proposition 2.20, we see

$$\tilde{\sigma}_{1*}[L] = f_1 - f_2.$$

Since $\tilde{\sigma}_1$ is an involution, we get $\tilde{\sigma}_1^*[L] = \tilde{\sigma}_{1*}[L]$. Then by Lemma 4.6, we have the following equalities:

$$H \cdot [L] = \tilde{\sigma}_1^*(H) \cdot \tilde{\sigma}_1^*[L] = (3H - 2E_1 - F_1) \cdot (f_1 - f_2) = 1;$$
$$E_1 \cdot [L] = \tilde{\sigma}_1^*(E_1) \cdot \tilde{\sigma}_1^*[L] = (2H - E_1 - F_1) \cdot (f_1 - f_2) = 1;$$
$$F_1 \cdot [L] = \tilde{\sigma}_1^*(F_1) \cdot \tilde{\sigma}_1^*[L] = (4H - 4E_1 - F_1) \cdot (f_1 - f_2) = 1;$$
$$F_2 \cdot [L] = \tilde{\sigma}_1^*(F_2) \cdot \tilde{\sigma}_1^*[L] = F_2 \cdot (f_1 - f_2) = 1.$$
By Lemma 4.5 this implies that \([L] = h - e_1 - f_1 - f_2\). Then by Lemma 4.7 we get
\[
\theta_1(\varphi) \cdot [L] = \left( \sum_{i=1}^{k} \alpha_i v_i \right) \cdot (h - e_1 - f_1 - f_2) = \sum_{i=1}^{k} \alpha_i (2H - 2E_j - E_j) \cdot (h - e_1 - f_1 - f_2) = 2 \sum_{i=1}^{k} \alpha_i - 3\alpha_1 - \alpha_2 < 0.
\]
Thus, \(L\) is an effective curve which intersects the class \(\theta_1(\varphi)\) negatively.

4.3. Composition of three involutions. In these section we consider the Blanc’s pseudo-automorphism for \(k = 3\). We denote \(F = \sigma_3 \circ \sigma_2 \circ \sigma_1\), it is a birational automorphism of \(\mathbb{P}^3\):

(4.10)
\[
\begin{array}{ccccccc}
X_1 & \overset{\tilde{\tau}_1}{\longrightarrow} & X_1 & \overset{\tau_1}{\longrightarrow} & X_2 & \overset{\tilde{\sigma}_2}{\longrightarrow} & X_2 & \overset{\tau_2}{\longrightarrow} & X_3 & \overset{\tilde{\sigma}_3}{\longrightarrow} & X_3 & \overset{\tau_3}{\longrightarrow} & X_1 \\
\Delta & \downarrow & \Delta & \downarrow & \Delta & \downarrow & \Delta \in \mathbb{P}^3 & \overset{F}{\longrightarrow} & \mathbb{P}^3
\end{array}
\]

Choose homogeneous coordinates \(x_0, x_1, x_2\) and \(x_3\) on \(\mathbb{P}^3\). Denote by \(f\) the equation of the cubic \(Q\):
\[
\frac{\partial f}{\partial x_i} = f(x_0, x_1, x_2, x_3) = \sum_{|I|=3} a_I x_I.
\]

We fix the centers of three involutions, these are points \(p_1, p_2\) and \(p_3\). Fix a point \(q\) in the intersection of curves \(\Gamma_1\) and \(\Gamma_2\).

Lemma 4.11. If the cubic surface \(Q\) and points \(p_1, p_2\) and \(p_3\) are sufficiently general, then Condition (1.1) is satisfied and up to conjugating \(\mathbb{P}^3\) by an element in \(\text{PGL}(4, \mathbb{C})\) we have
\[
q = (1 : 0 : 0 : 0); \quad p_1 = (0 : 1 : 0 : 0); \quad p_2 = (0 : 0 : 1 : 0); \quad p_3 = (0 : 0 : 0 : 1).
\]
In this situation we get the following conditions on coefficients of \(f\):
\[
a_{3000} = a_{0300} = a_{0030} = a_{0003} = a_{2100} = a_{2010} = 0.
\]

**Proof.** First, let us prove that curves \(\Gamma_1\) and \(\Gamma_2\) intersect transversally in all points for a general choice of the cubic surface \(Q\) and points \(p_1\) and \(p_2\).

The first generality condition is that \(p_2\) does not lie on \(\Gamma_1\). Then any point \(q\) in \(\Gamma_1 \cap \Gamma_2\). Points \(p_1, p_2\) and \(q\) are not colinear since the line passing through \(p_1\) and \(q\) does not intersect \(Q\) in any other points and \(p_2 \neq q\) by our condition.

Then there exist an automorphism of the projective space \(\mathbb{P}^3\) which maps points \(q, p_1\) and \(p_2\) to points \((1 : 0 : 0 : 0), (0 : 1 : 0 : 0)\) and \((0 : 0 : 1 : 0)\). Since points \(q, p_1, p_2\) lie on the cubic surface \(Q\) this implies that in some coordinates we have \(a_{3000} = a_{0300} = a_{0030} = 0\).

Consider the affine chart \(U_0 = \{x_0 \neq 0\}\) of \(\mathbb{P}^3\). This chart is isomorphic to \(\mathbb{A}^3\) with coordinates \(t_1, t_2, t_3\) where \(t_i = \frac{x_i}{x_0}\) for all \(i = 1, 2, 3\). By [Bla13] the curve \(\Gamma_i\) is the intersection of the surfaces \(\{f = 0\}\) and \(\{\frac{\partial f}{\partial x_0} = 0\}\) for \(i = 1, 2\). Denote by \(g(t_1, t_2, t_3)\) the equation induced by \(f\) on the chart \(U_0\). Then the tangent line \(L_i\) to \(\Gamma_i\) in the point \(q = (0, 0, 0)\) in \(U_0\) is as follows:
\[
T_{L_i,q} = \left\{ \sum_{j=1}^{3} \frac{\partial g}{\partial t_j}(0,0,0)dt_j = \sum_{i=1}^{3} \frac{\partial^2 g}{\partial t_i \partial t_j}(0,0,0)dt_i = 0 \right\} \subset T_{U_0,q} = \langle dt_1, dt_2, dt_3 \rangle.
\]

The intersection of curves \(\Gamma_1\) and \(\Gamma_2\) in the point \(q\) is not transversal if the tangent lines \(T_{L_1,q}\) and \(T_{L_2,q}\) coincide. Computing derivatives of \(g\) we get that lines \(T_{L_1,q}\) and \(T_{L_2,q}\) coincide if the following polynomial vanishes:
\[
\det \begin{pmatrix}
\frac{\partial g}{\partial t_1}(0,0,0) & \frac{\partial g}{\partial t_2}(0,0,0) & \frac{\partial g}{\partial t_3}(0,0,0) \\
\frac{\partial^2 g}{\partial t_1 \partial t_1}(0,0,0) & \frac{\partial^2 g}{\partial t_2 \partial t_2}(0,0,0) & \frac{\partial^2 g}{\partial t_3 \partial t_3}(0,0,0) \\
\frac{\partial^2 g}{\partial t_1 \partial t_2}(0,0,0) & \frac{\partial^2 g}{\partial t_2 \partial t_3}(0,0,0) & \frac{\partial^2 g}{\partial t_3 \partial t_1}(0,0,0)
\end{pmatrix} = a_{2001}(4a_{1200}a_{1020} - a^2_{1110}).
\]

This implies that if \(Q\) is sufficiently general and \(p_2\) does not lie on \(\Gamma_1\), then curves \(\Gamma_1\) and \(\Gamma_2\) intersect transversally.
Now take three points $p_1, p_2$ and $p_3$ on a cubic $Q$ such that they are not collinear, $p_i$ does not lie on $\Gamma_j$ for $i \neq j$ and curves $\Gamma_i$ and $\Gamma_j$ intersect transversally for all $1 \leq i < j \leq 3$. Choose any point $q$ in $\Gamma_1 \cap \Gamma_2$, then points $q, p_1, p_2$ and $p_3$ does not lie on one plane in $\mathbb{P}^3$. Thus, we can choose coordinates of $\mathbb{P}^3$ such that our four points are $(1 : 0 : 0 : 0), (0 : 1 : 0 : 0), (0 : 0 : 1 : 0)$ and $(0 : 0 : 0 : 1)$ respectively.

Since points $q, p_1, p_2$ and $p_3$ lie on $Q$ we get $a_{0300} = a_{0030} = a_{0003} = 0$. Moreover, since the line passing through $p_i$ and $q$ is tangent to $Q$ in $q$ for $i = 1$ and 2 we get that $a_{2100} = a_{0210} = 0$.

**Lemma 4.12.** [Bla13 Section 2] Let $Q = \{ f = 0 \}$ be an equation of a cubic surface in $\mathbb{P}^3$ which contains points $q = (1 : 0 : 0 : 0), p_1 = (0 : 1 : 0 : 0), p_2 = (0 : 0 : 1 : 0)$ and $p_3 = (0 : 0 : 0 : 1)$; lines passing through $p_i$ and $q$ are tangent to $\mathcal{D}_f$ in $q$ for $i = 1$ and 2. Then the involutions associated with points $p_1, p_2$ and $p_3$ are given by the following formulas:

\[
\begin{align*}
\sigma_1(x_0, x_1, x_2, x_3) &= \left( x_0 \frac{\partial f}{\partial x_1}(x) : x_1 \frac{\partial f}{\partial x_1}(x) - 2f(x) : x_2 \frac{\partial f}{\partial x_1}(x) : x_3 \frac{\partial f}{\partial x_1}(x) \right); \\
\sigma_2(x_0, x_1, x_2, x_3) &= \left( x_0 \frac{\partial f}{\partial x_2}(x) : x_1 \frac{\partial f}{\partial x_2}(x) - 2f(x) : x_2 \frac{\partial f}{\partial x_2}(x) : x_3 \frac{\partial f}{\partial x_2}(x) \right); \\
\sigma_3(x_0, x_1, x_2, x_3) &= \left( x_0 \frac{\partial f}{\partial x_3}(x) : x_1 \frac{\partial f}{\partial x_3}(x) - 2f(x) : x_2 \frac{\partial f}{\partial x_3}(x) : x_3 \frac{\partial f}{\partial x_3}(x) \right).
\end{align*}
\]

**Proof.** It suffices to prove that $\sigma_1$ is the necessary involution. Let us consider a line $l$ passing through the point $p_1 = (0 : 1 : 0 : 0)$. By Bezout theorem this line intersects the plane $\{x_1 = 0\}$ in an only one point $(a : 0 : b : c)$. Then the line $l$ is the following set of points:

\[
l = \{(as : t : bs : cs) | (s : t) \in \mathbb{P}^1\}.
\]

If we compute equations $f$ and $\frac{\partial f}{\partial x_1}$ in any point of the line $l$ we get the following:

\[
f(as, t, bs, cs) = \mu_1st^2 + \mu_2s^2t + \mu_3s^3
\]

\[
\frac{\partial f}{\partial x_1}(as, t, bs, cs) = 2\mu_1st + \mu_2s^2.
\]

Here $\mu_1, \mu_2$ and $\mu_3$ are some numbers in $\mathbb{C}$ depending only on the line $l$ and the cubic $Q$. Then if we apply the formula of $\sigma_1$ to the point $(as : t : bs : cs)$, we get the following point on $\mathbb{P}^3$:

\[
\sigma_1(as, t, bs, cs) = (as^2(2\mu_1t + \mu_2s) : st(2\mu_1t + \mu_2s) - 2(\mu_1st^2 + \mu_2s^2t + \mu_3s^3) : bs^2(2\mu_1t + \mu_2s) : cs^2(2\mu_1t + \mu_2s) = (a(2\mu_1t + \mu_2s) : -\mu_2t - 2\mu_3s : b(2\mu_1t + \mu_2s) : c(2\mu_1t + \mu_2s)).
\]

Then the map $\sigma_1$ induces an automorphism of the line $l$; in coordinates $s, t$ it can be written as the following matrix in $\text{PGL}(2, \mathbb{C})$:

\[
\sigma_1|_l = \begin{pmatrix} \mu_2 & -2\mu_3 \\
2\mu_1 & -\mu_2 \end{pmatrix}
\]

It is easy to check that this is an involution and its eigenvectors correspond to points $(s : t)$ such that $f(as, t, bs, cs) = 0$ and $(s : t) \neq (0 : 1)$. \qed

Set $B = \{(2001), (1200), (1110), (1101), (0210)\}$. Denote by $R$ the following ring:

\[
R = \mathbb{Z}[a_1]_{l \in B} = \mathbb{Z}[a_{2001}, a_{1200}, a_{1110}, a_{1101}, a_{0210}].
\]

Consider a free commutative ring $R[X, Y]$. Set $w = a_{2001}a_{0210} + a_{1110}a_{1101}$ and denote by $I$ the ideal $(4, 2w)$ in $R[X, Y]$, i.e., the ideal of polynomials whose coefficients lie in $(4, 2w) \subset R$. If $P$ is an element in $R[X, Y]$ we consider it as a polynomial of $X$ and $Y$ with coefficients in $R$.

By $\text{deg}_Y(P)$ we denote the degree of the polynomial $P$ in $Y$. By the leading term in $Y$ of the polynomial $P$ we call the polynomial $r(X)$ such that $\text{deg}(P - r(X)Y^{\text{deg}(P)})$ is strictly less than $\text{deg}(P)$. We need the following useful property of degrees of polynomials:

**Lemma 4.13.** Assume that $P_1$ and $P_2$ are two polynomials:

\[
P_i = r_i(X)Y^{d_i} + Q_i(X, Y);
\]
Assume also that the degree in $Y$ of all monomials in $Q_1$ is strictly less that $d_i$. If $d_1 > d_2$ and $r_1(X)$ is a non-zero polynomial which does not lie in $\mathcal{I}$, then $P_1 + P_2$ also does not lie in $\mathcal{I}$ and $\deg_Y(P_1 + P_2) = d_1$.

**Proof.** Since $X$ and $Y$ are free variables, then the polynomial $\sum_{i,j} r_{i,j} X^i Y^j$ lies in $\mathcal{I}$ only if all coefficients $r_{i,j}$ lie in $\mathcal{I}$. Since $\deg(P_1) > \deg(P_2)$, then the leading term in $Y$ of $P_1 + P_2$ coincides with the leading term of $P_1$. \hfill $\square$

Consider involutions $\sigma_{R1}, \sigma_{R2}$ and $\sigma_{R3}$ of $\mathbb{P}^3_{R[X,Y]}$ defined by formulas from Lemma [1.12] for a cubic surface $Q$ given by an equation $\sum_{i \in B} a_i x_i^i = 0$. Denote by $F_R$ the composition of these involutions:

$$F_R = \sigma_{R3} \circ \sigma_{R2} \circ \sigma_{R1} : \mathbb{P}^3_{R[X,Y]} \rightarrow \mathbb{P}^3_{R[X,Y]}$$

Our goal is to compute the orbit of a point under the action of $F_R^{-1}$. We will consider points of the following form.

**Notation 4.15.** Let $p = (M_0 + 2N_0 : M_1 + 2N_1 : \tilde{g}_2 + 2g_2 : \tilde{g}_3 + 3g_3)$ be a point in $\mathbb{P}^3_{R[X,Y]}$, such that $M_0, N_0, M_1, N_1, g_2, \tilde{g}_2, g_3$ and $\tilde{g}_3$ are elements of $R[X,Y]$ of the following form:

- (B1) Polynomials $\tilde{g}_2$ and $\tilde{g}_3$ are elements of $\mathcal{I}$; also $g_2$ and $g_3$ lie in $2R[X,Y]$ and the leading term in $Y$ of $\tilde{g}_3$ do not lie in $4R[X,Y]$;
- (B2) The leading terms of polynomials $M_i(X, Y)$ do not lie in the ideal $2R[X,Y]$ for $i = 0$ and $i = 1$.
- (B3) We have the following conditions on degrees of polynomials:

$$\deg_Y(\tilde{g}_2) \leq \deg_Y(M_0) < \deg_Y(\tilde{g}_3) < \deg_Y(M_1).$$

Now we are going to show that the image of the point in $\mathbb{P}^3_{R[X,Y]}$ which satisfies conditions (B1 – B3) under $F_R^{-1}$ still satisfies these conditions.

**Lemma 4.16.** Assume that $p = (M_0 + 2N_0 : M_1 + 2N_1 : \tilde{g}_2 + 2g_2 : \tilde{g}_3 + 3g_3)$ is a point in the projective space $\mathbb{P}^3_{R[X,Y]}$ such that polynomials $M_0, N_0, M_1, N_1, g_2, \tilde{g}_2, g_3$ and $\tilde{g}_3$ satisfy conditions (B1 – B3). Then the point $p$ does not lie in $\text{Ind}(F_R^{-1})$. Moreover, $F_R^{-1}(p) = (M_0' + 2N_0' : M_1' + 2N_1' : \tilde{g}_2' + 2g_2' : \tilde{g}_3' + 3g_3')$ where polynomials $M_0', N_0', M_1', N_1', g_2', \tilde{g}_2', g_3'$ and $\tilde{g}_3'$ satisfy conditions (B1 – B3).

**Proof.** In view of condition (B3) one has $\deg_Y(M_0) = d_0$, $\deg_Y(M_1) = d_1$, $\deg_Y(\tilde{g}_2) = d_2$, $\deg_Y(\tilde{g}_3) = d_3$ and we have the following inequality:

$$d_2 < d_0 < d_3 < d_1. \tag{4.17}$$

In order to compute $F_R^{-1}(p)$ we use the formulas from Lemma [1.13] associated with the cubic surface given by the equation $\sum_{i \in B} a_i x_i^i$. Then $F_R^{-1}(p) = (q_0 : q_1 : q_2 : q_3)$, where the expressions $q_i$ can be computed in Sage explicitly.

First thing we note is that all polynomials $q_0, q_1, q_2$ and $q_3$ lie in $2R[X,Y]$. Our goal is to construct polynomials $M_0', N_0', M_1', N_1', g_2', \tilde{g}_2', g_3'$ and $\tilde{g}_3'$ such that

- $q_0 = 2(M_0' + 2N_0')$;
- $q_1 = 2(M_1' + 2N_1')$;
- $q_2 = 2(\tilde{g}_2' + 2g_2')$;
- $q_3 = 2(\tilde{g}_3' + 3g_3')$.

We are going to do it considering all components $q_0, q_1, q_2$ and $q_3$ one by one.

Consider the component $q_0$. By formulas in Lemma [1.12] we can see that this is a homogeneous polynomial of $M_0, N_0, M_1, N_1, g_2, \tilde{g}_2, g_3$ and $\tilde{g}_3$ of degree 27. We define $M_0'$ to be the sum of all monomials in $\frac{1}{2}q_0$ except those divisible by $2, g_2, g_2, g_3$ or $\frac{1}{2}m$ where $m$ is a degree 2 monomial of $g_2, g_3, g_2, \tilde{g}_3$. Note that by construction $\frac{1}{2}q_0 - M_0'$ lies in $2R[X,Y]$ since so do $g_2, g_2, g_3$ and $\tilde{g}_3$. Denote $N_0' = \frac{1}{2}(\frac{1}{2}q_0 - M_0')$. The computation in Sage shows:

$$M_0' = a_{1,101}^8 a_{0,210}^2 M_0^{11} M_1^{15} (wa_{120} M_0 + \frac{1}{2} a_{1,110} a_{1,101} a_{0,210} g_2 + \frac{1}{2} a_{1,101} a_{0,210} g_3) + h M_0', \tag{4.18}$$
where \( h_{M'_0} \) is a homogeneous polynomial of \( M_0, N_0, M_1, N_1, g_2, \bar{g}_2, g_3 \) and \( \bar{g}_3 \) of degree 27 and the degree in \( M_1 \) of all monomials is strictly less than 15. By Lemma 4.13 and (4.17) we get that the leading term in \( Y \) of \( M'_0 \) does not lie in \( 2R[X, Y] \)

\[
\deg_Y(M'_0) = 11d_0 + 15d_1 + d_3.
\]  

Consider the component \( g_1 \). This is a homogeneous polynomial of \( M_0, N_0, M_1, N_1, g_2, \bar{g}_2, g_3 \) and \( \bar{g}_3 \) of degree 27. We define \( M'_1 \) to be the sum of all monomials in \( \frac{1}{2}q_1 \) except those divisible by \( 2, g_2, \bar{g}_2, g_3, \bar{g}_3 \) or \( \frac{1}{2}m \) where \( m \) is a degree 2 monomial of \( g_2, \bar{g}_2, g_3 \) and \( \bar{g}_3 \). Note that by construction \( \frac{1}{2}q_1 - M'_1 \) lies in \( 2R[X, Y] \). Denote \( N'_1 = \frac{1}{2}(\frac{1}{2}q_1 - M'_1) \). The computation in Sage shows:

\[
M'_1 = a_{12000}a_{11101}^3 a_{0210}^2 M_0^10 M_1^17 + h_{M'_1},
\]

where \( h_{M'_1} \) is a homogeneous polynomial of \( M_0, N_0, M_1, N_1, g_2, \bar{g}_2, g_3 \) and \( \bar{g}_3 \) of degree 27 and the degree in \( M_1 \) of all monomials is strictly less than 17. By Lemma 4.13 and (4.17) we get that the leading term in \( Y \) of \( M'_1 \) does not lie in \( 2R[X, Y] \)

\[
\deg_Y(M'_1) = 10d_0 + 17d_1.
\]

Consider the component \( g_2 \). This is a homogeneous polynomial of \( M_0, N_0, M_1, N_1, p_2 = \bar{g}_2 + 2g_2 \) and \( p_3 = g_3 + 2g_3 \) of degree 27. We define \( g'_2 \) to be the sum of all monomials of \( g_2 \) except those divisible by \( 8, 2g_2, 2p_3 \) or a degree 2 monomial of \( p_2 \) and \( p_3 \). Note that by construction \( g_2 - g'_2 \) lies in \( 2I \). The computation in Sage shows:

\[
g'_2 = 4wa_1200^2 M_0^8 M_1^12 M_0^0 (a_{1110} M_0 + a_{0210} M_1)(a_{2001} M_0^0 + a_{1101} M_1^8)
\]

Since \( g'_2 \) is divisible by \( 4w \) it lies in \( 2I \). Therefore, by construction \( g_2 \) lies in \( 2I \). Define \( \tilde{g}'_2 \) to be the sum of all monomials of \( \frac{1}{2}g_2 \) except those divisible by \( 4, 2g_2, 2p_3, 2g_3, \bar{g}_3 \) or a degree 2 monomial of \( g_2, \bar{g}_2, g_3 \) and \( \bar{g}_3 \). Note that by construction \( g_2 = \frac{1}{2}(g_2 - \tilde{g}'_2) \) lies in \( 2R[X, Y] \). The computation in Sage shows:

\[
\tilde{g}'_2 = a_{0210} a_{1101}^8 M_0^10 M_1^15 (2wa_1200^2 M_0^2 + a_{2001} a_{1200} a_{0210}^2 M_0 g_2 +
+ \frac{1}{2} a_{1110} a_{1101} a_{0210}^2 g_2 + a_{1200} a_{0210} a_{0210} M_0 g_3 + a_{2001} a_{1200} a_{0210} \tilde{g}_3)) + h_{\tilde{g}'_2},
\]

where \( h_{\tilde{g}'_2} \) is a homogeneous polynomial of \( M_0, N_0, M_1, N_1, g_2, \bar{g}_2, g_3 \) and \( \bar{g}_3 \) of degree 27 and the degree in \( M_1 \) of all monomials is strictly less than 15. By (4.17) and we get that

\[
\deg_Y(\tilde{g}'_2) \leq 11d_0 + 15d_1 + d_3.
\]

Consider the component \( g_3 \). This is a homogeneous polynomial of \( M_0, N_0, M_1, N_1, p_2 = \bar{g}_2 + 2g_2 \) and \( p_3 = g_3 + 2g_3 \) of degree 27. We define \( g'_3 \) to be the sum of all monomials of \( g_3 \) except those divisible by \( 8, 2g_2, 2p_3 \) or a degree 2 monomial of \( p_2 \) and \( p_3 \). Note that by construction \( g_3 - g'_3 \) lies in \( 2I \). The computation in Sage shows:

\[
g'_3 = 4wa_{1200}^2 M_0^{11} M_1^7 (a_{2001} M_0 + a_{1101} M_1)(a_{1101} M_0^0 + a_{0210} M_1^7)
+ a_{2001} a_{0210}^2 M_1^7 (a_{2001} M_0^0 + a_{1101} M_1^7)(a_{2001} M_0^0 + a_{1101} M_1^7).
\]

Since \( g'_3 \) is divisible by \( 4w \) it lies in \( 2I \). Therefore, by construction \( g_3 \) lies in \( 2I \).

Define \( \tilde{g}'_3 \) to be the sum of all monomials of \( \frac{1}{2}g_3 \) except those divisible by \( 4, 2g_2, 2\bar{g}_2, 2g_3, \bar{g}_3 \) or degree 2 monomial of \( g_2, \bar{g}_2, g_3 \) and \( \bar{g}_3 \). Note that by construction \( g_3 = \frac{1}{2}(g_3 - \tilde{g}'_3) \) lies in \( 2R[X, Y] \). Moreover, the Sage computation shows:

\[
\tilde{g}'_3 = a_{1200} a_{0210}^2 a_{1101}^7 M_0^10 M_1^16 (2wa_{1200} M_0 + a_{1110} a_{1101} a_{0210} \tilde{g}_2 + a_{1200} a_{0210} \tilde{g}_3) + h_{\tilde{g}'_3}.
\]

Here \( h_{\tilde{g}'_3} \) is a homogeneous polynomial of \( M_0, N_0, M_1, N_1, g_2, \bar{g}_2, g_3 \) and \( \bar{g}_3 \) of degree 27 and the degree in \( M_1 \) of all monomials is strictly less than 16. By Lemma 4.13 and (4.17) we get that the leading term in \( Y \) of \( \tilde{g}'_3 \) does not lie in \( 4R[X, Y] \)

\[
\deg_Y(\tilde{g}'_3) = 10d_0 + 16d_1 + d_3.
\]

We have constructed polynomials \( M'_0, N'_0, M'_1, N'_1, g'_2, \bar{g}'_2, g'_3 \) and \( \bar{g}'_3 \), it remains to show conditions (B1–B3) for these polynomials.
Polynomials $\tilde{g}_2 + 2g_2'$ and $\tilde{g}_3 + 2g_3'$ lie in $\mathcal{I}$ by equations (4.22) and (4.25). Polynomials $g_2'$ and $g_3'$ lie in $2R[X,Y]$ by the construction. Also by (4.23) and (4.26) we get that the leading terms of $\tilde{g}_2$ and $\tilde{g}_3$ are not divisible by 4. Thus, the condition (B1) is satisfied.

By (4.18) and (4.20) we get that the leading monomials of $M_0'$ and $M_1'$ do not lie in $2R[X,Y]$. Thus, the condition (B2) is satisfied.

Finally the condition (B3) follows from the degree computation (4.19), (4.21), (4.24), (4.27) and the assumption (4.17).

Since by our computation polynomials $q_0, q_1, q_2$ and $q_3$ are non-zero, we get that the point $p$ does not lie in $\text{Ind}(F_R^{-1})$.

\textbf{Corollary 4.28.} Let $p = (X : Y : 0 : 0)$ be a point in $\mathbb{P}^3_{\mathbb{R}(X,Y)}$ and $F_R$ be a birational automorphism as in (4.14). Then $p$ does not lie in $\text{Ind}(F_R^{-n})$ for all $n > 0$.

\textbf{Proof.} Consider $F_R^{-1}(p) = q = (q_0 : q_1 : q_2 : q_3)$. Below we construct polynomials $M_0', N_0', M_1', N_1', \tilde{g}_2', g_2', \tilde{g}_3'$ and $g_3'$ such that

\begin{equation}
\begin{aligned}
q_0 &= 2(M_0' + 2N_0'); \\
q_1 &= 2(M_1' + 2N_1'); \\
q_2 &= 2(\tilde{g}_2' + 2g_2'); \\
q_3 &= 2(\tilde{g}_3' + 2g_3').
\end{aligned}
\end{equation}

Consider the component $q_0$, it is a homogeneous polynomial in $X$ and $Y$ of degree 27. Define $M_0'$ to be the sum of all monomials of $\frac{1}{2}q_0$, which coefficients are not divisible by 4. Set $N_0' = \frac{1}{2}(\frac{1}{2}q_0 - M_0')$. Using the formula (4.18) and substituting there $M_0 = X, M_1 = Y, \tilde{g}_2 = \tilde{g}_3 = 0$ we can see that the leading term in $Y$ of $M_0'$ is equal to $w_{1200}a_{1101}^2a_{210}^0X^{12}Y^{15}$. It does not lie in $2R[X,Y]$ and the degree in $Y$ of $M_0'$ is $\text{deg}_Y(M_0') = 15$.

Consider the component $q_1$, it is a homogeneous polynomial in $X$ and $Y$ of degree 27. Define $M_1'$ to be the sum of all monomials of $\frac{1}{2}q_1$, which coefficients are not divisible by 4. Set $N_1' = \frac{1}{2}(\frac{1}{2}q_1 - M_1')$. Using the formula (4.20) and substituting there $M_0 = X, M_1 = Y, \tilde{g}_2 = \tilde{g}_3 = 0$ we can see that the leading term in $Y$ of $M_1'$ is equal to $a_{1200}a_{1101}^3a_{210}^0X^{10}Y^{17}$. It does not lie in $2R[X,Y]$ and the degree in $Y$ of $M_1'$ is $\text{deg}_Y(M_1') = 17$.

Consider the component $q_2$. By the formula (4.22) we see that $q_2$ lies in the ideal $\mathcal{I}$. Define $\tilde{g}_2'$ to be the sum of all monomials of $\frac{1}{2}q_2$ which are not divisible by 4. Set $g_2' = \frac{1}{2}(\frac{1}{2}q_2 - \tilde{g}_2')$. Using the formula (4.23) and substituting there $M_0 = X, M_1 = Y, \tilde{g}_2 = \tilde{g}_3 = 0$ we can see that the leading term in $Y$ of $\tilde{g}_2'$ is equal to $2wa_{210}a_{210}^2a_{1101}^3X^{12}Y^{15}$. Then the degree in $Y$ of $\tilde{g}_2'$ is $\text{deg}_Y(\tilde{g}_2') = 15$.

Consider the component $q_3$. By the formula (4.25) we see that $q_3$ lies in the ideal $\mathcal{I}$. Define $\tilde{g}_3'$ to be the sum of all monomials of $\frac{1}{2}q_3$ which are not divisible by 4. Set $g_3' = \frac{1}{2}(\frac{1}{2}q_3 - \tilde{g}_3')$. Using the formula (4.26) and substituting there $M_0 = X, M_1 = Y, \tilde{g}_2 = \tilde{g}_3 = 0$ we can see that the leading term in $Y$ of $\tilde{g}_3'$ is equal to $2wa_{210}a_{210}^2a_{1101}^3X^{11}Y^{16}$. It does not lie in $4R[X,Y]$ and the degree in $Y$ of $\tilde{g}_3'$ is $\text{deg}_Y(\tilde{g}_3') = 16$.

Thus, one get that $q$ has the form (4.29) and by the construction polynomials $M_0, N_0, M_1, N_1, \tilde{g}_2, g_2, \tilde{g}_3$ and $g_3'$ satisfy conditions (B1 − B3).

Thus, the point $p$ does not lie in the indeterminacy locus of $F_R^{-1}$. Moreover, by Lemma 4.16 the point $q = F_R^{-1}(p)$ does not lie in $\text{Ind}(F_R^{-n})$ for all $n > 0$. This finishes the proof. □

\textbf{Corollary 4.30.} Let $Q$ be a very general cubic surface in $\mathbb{P}^3_\mathbb{C}$ and let points $p_1, p_2$ and $p_3$ be general points on $Q$. Assume that $F$ is the birational automorphism of $\mathbb{P}^3_\mathbb{C}$ as in (4.10). Then the line $L \subset \mathbb{P}^3_\mathbb{C}$ passing through $p_1$ and some point in $\text{Ind}(\sigma_1) \cap \text{Ind}(\sigma_2)$ does not lie in $\text{Ind}(F^{-n})$ for all $n > 0$. □
Proof. Consider a 4-dimensional complex vector space $V$ and associated projective space $\mathbb{P}(V) = \mathbb{P}^3$. A cubic surface in $\mathbb{P}(V)$ is given by an equation in $\mathbb{P}(S^3V^*)$. The coordinates of the space $\mathbb{P}(S^3V^*)$ are coefficients $a_I$ of the cubic equation $\sum a_I x^I$. Denote by $W$ the following subspace of $S^3V^*$:

$$W = \{ a_I \mid I \neq (3000), (0300), (0030), (0003), (2100), (2010) \} \subset S^3V^*.$$ 

Consider a universal cubic surface $\mathcal{S} \subset \mathbb{P}(V) \times \mathbb{P}(W)$. Denote by $\Pi: \mathcal{S} \rightarrow \mathbb{P}(W)$ the projection. The fiber of $\Pi$ over a point $f \in \mathbb{P}(W)$ is a cubic surface $\mathcal{S}_f = \{ f = 0 \}$. Moreover, by construction $\mathcal{S}_f$ contains points $q = (1 : 0 : 0 : 0)$, $p_1 = (0 : 1 : 0 : 0)$, $p_2 = (0 : 0 : 1 : 0)$ and $p_3 = (0 : 0 : 0 : 1)$; lines passing through $p_i$ and $q$ are tangent to $\mathcal{S}_f$ in $q$ for $i = 1$ and $2$. Note that for any point $f \in \mathbb{P}(W)$ the point $q$ lies in $\text{Ind}(\sigma_1) \cap \text{Ind}(\sigma_2)$ where $\sigma_1$ and $\sigma_2$ are involutions associated with the cubic equation $f$ and points $p_1$ and $p_2$.

Then we can define involutions $\sigma_{21}, \sigma_{22}$ and $\sigma_{23}$ of $\mathbb{P}(V) \times \mathbb{P}(W)$ by formulas in Lemma 14.12. Denote by $F_{\sigma}$ the composition $\sigma_{21} \circ \sigma_{22} \circ \sigma_{23}$.

Consider the following cubic equation $f_0$:

$$f_0 = a_{2001}x_0^2x_3 + a_{1200}x_0x_1^2 + a_{1110}x_0x_1x_2 + a_{1101}x_0x_1x_3 + a_{0210}x_1^2x_2,$$

here coefficients $a_{2001}, a_{1200}, a_{1110}, a_{1101}, a_{0210}$ are non-zero complex numbers. By Corollary 14.28 we get that if $X$ and $Y$ are very general complex numbers then the point $(X : Y : 0 : 0)$ does not lie in the indeterminacy locus of $F^n$ for all $n > 0$.

This implies that the point $((X : Y : 0 : 0), f_0)$ does not lie in $\text{Ind}(F^n)$ for all $n > 0$. Denote by $L$ the line connecting $q$ and $p_1$. Then the point $(X : Y : 0 : 0)$ lies on $L$ in $\mathbb{P}^3$. Therefore, the subvariety $L \times \mathbb{P}(W)$ does not lie in $\text{Ind}(F^n)$ for all $n > 0$.

Since $\bigcup_{n>0}\text{Ind}(F^n)$ is a countable union of closed codimension 2 subvarieties of $\mathbb{P}(V) \times \mathbb{P}(W)$, then for a very general point $f$ in $\mathbb{P}(W)$ the line $L$ passing through $p_1$ and $q$ does not lie in $\text{Ind}(F^n)$ for all $n > 0$.

Finally, since by Lemma 14.11 for a set of a cubic surface $Q$ and three general points $p_1, p_2$ and $p_3$ there exists a choice of coordinates such that the equation of $Q$ lies in $W$ and $p_1 = (0 : 1 : 0 : 0), p_2 = (0 : 0 : 1 : 0)$ and $p_3 = (0 : 0 : 0 : 1)$ we get the result. \qed

4.4. Proof of Theorem 1.3. We consider the composition of three birational involutions $\sigma_1, \sigma_2$ and $\sigma_3$ on $\mathbb{P}^3$ associated with points $p_1, p_2$ and $p_3$ on a smooth cubic surface $Q$. Without loss of generality we can assume that they are as in Lemma 14.11. Then involutions are defined by formulas in Lemma 14.12.

If we prove that for some cubic surface and 3 points on it the composition $F = \sigma_3 \circ \sigma_2 \circ \sigma_1$ is not birationalizable, then the same is true for any very general birational automorphism of this type. Thus, we assume that coefficients $a_I$ of the equation $f$ of cubic $Q$ are transcendental and algebraically independent over $\mathbb{Q}$.

We denote by $\delta: X \rightarrow \mathbb{P}^3$ the consequent blow-up of the proper preimages of curves $\Gamma_1, \Gamma_2$ and $\Gamma_3$. By Theorem 14.2 the birational automorphism $F$ induces a pseudo-automorphism $\varphi$ on $X$. By Theorems 14.2 and 2.10 the class $\theta_1(\varphi)$ is well-defined and we can compute it by Lemma 14.4.

Consider the curve $L$ on which is the proper preimage under $\delta$ of the line $\delta(L)$ on $\mathbb{P}^3$ passing through points $p_1$ and $q$. By Corollary 14.9 we get that $\theta_1(\varphi) \cdot [L] < 0$. However, by Corollary 14.30 we get that the curve $\delta(L)$ does not lie in $\text{Ind}(F^-N)$ for all $N > 0$.

This implies that the curve $L$ does not lie in $\text{Ind}(\varphi^-N)$ for infinitely many numbers $N > 0$. Thus, Condition $[A]$ is satisfied for the pseudo-automorphism $\varphi$. Then by Theorem 14.1 we get that there is no birational model of $X$ on which $\varphi$ defines a regular automorphism. Moreover, by Theorem 14.2 this also implies that $\varphi$ does not preserve any fibrations over a surface.

5. Example of a regularizable pseudo-automorphism with non-nef class $\theta_1$

We recall here the construction from [OT15] and [Les18] Section 7 in order to give an example of a regularizable (not regular) pseudo-automorphism $\varphi_+$ such that the first dynamical class $\theta_1(\varphi_+)$ is not nef.

First let us recall the construction of the regular primitive automorphism of a rational threefold with the dynamical degree greater than 1 described in [OT15].

In order to construct it we consider the lattice of Eisenstein integers $\mathbb{Z}[j]$ in $\mathbb{C}$, here $j$ is a primitive cubic root of unity. Denote by $E$ the elliptic curve $\mathbb{C}/\mathbb{Z}[j]$. The group $G = \langle -j \rangle$ isomorphic to $\mathbb{Z}/6\mathbb{Z}$ acts on $E$ and this action induces a diagonal action of $G$ on the abelian variety $A = E \times E \times E$. 
We denote by $q: A \to A/G$ the quotient map. Since the action of $G$ was not free, $A/G$ is a singular variety; there are 27 non-terminal isolated quotient singularities on $A/G$. Denote by $\delta: X \to A/G$ the blow-up of all these points. Then $X$ is a smooth rational threefold by [OT15, Theorem 1.4].

Any matrix $M$ in $\text{SL}(3, \mathbb{Z})$ defines a regular automorphism of $A$. Since the action of $\text{SL}(3, \mathbb{Z})$ and $G$ commutes, this action extends to a regular action on $A/G$ and on $X$. We fix a matrix $M$ as in [OT15, Lemma 4.3]. It is an integer invertible matrix such that all roots of its characteristic polynomial are distinct real numbers. Then $M$ induces primitive regular automorphisms $\varphi_A$ and $\varphi$ of $A$ and $X$ respectively and

$$\lambda_1(\varphi)^2 \geq \lambda_2(\varphi).$$

Denote by $C$ a proper image of a curve $E \times \{0\} \times \{0\}$ on $A$ under the finite rational map $\delta \circ q$. Then $C$ is a smooth rational curve on $X$. By [Les18, Section 7] there is a standard Atiyah flop in the curve $C$; i.e. there exists a pseudo-isomorphism $\alpha: X \dashrightarrow X_+$ such that $\text{Ind}(\alpha) = C$ and the total image of $C$ under $\alpha$ is a smooth rational curve $C_+$. Moreover, if we denote by $p: W \to X$ the blow-up of $X$ in $C$, then the exceptional divisor of $p$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and the blow-down of another ruling induces a regular birational morphism $p_+: W \to X_+$.

By the choice of the matrix $M$ the orbit of the curve $C$ is infinite and the intersection number $\theta_1(\varphi) \cdot [C]$ is strictly positive. Then we can apply the following lemma.

**Lemma 5.1.** Let $\alpha: X \dashrightarrow X_+$ be a flop in a curve $C$. Then for any $D \in N^1(X)$ such that $D \cdot [C] > 0$ we have $(\alpha^*D) \cdot [C^+] < 0$.

In the case of the standard Atiyah flop this lemma is an easy computation. Since $\theta_1(\varphi_+) = (\alpha^{-1})^*\theta_1(\varphi)$ by Lemma 4.2 we get the following inequality:

$$\theta_1(\varphi_+) \cdot [C_+] < 0.$$ 

By the construction the curve $C_+$ lies in the indeterminacy locus $\text{Ind}((\varphi^n))$ for all non-zero integers $n$. Thus, the pseudo-automorphism $\varphi_+$ satisfies all but last properties of Condition A.

**APPENDIX A. COMPUTATIONS FOR LEMMA 4.16**

In the proof of Lemma 4.16 we need to compute the preimage of the point in the projective space $\mathbb{P}^3_{R[X,Y]}$ described in Notation 4.15. Here is the Sage code we used to perform the computations. We define birational involutions $\sigma_j = 1_j$ for $j = 1, 2$ and 3 and their composition $F_{R}^{-1} = F_{\text{inverse}}$. Variables $M_0, N_0, M_1, N_1, g_2, gg_2, g_3$ and $gg_3$ play roles of $M_0, N_0, M_1, N_1, g_2, gg_2, g_3$ and $gg_3$ respectively. Variables $p_2$ and $p_3$ represents elements $\tilde{g}_2 + 2g_2$ and $\tilde{g}_3 + 2g_3$ of the ideal $\mathcal{I}$. Then we apply $F_{\text{inverse}}$ to the point $p = (N_0 + 2N, M_0 + 2M, g_2^2 + 2gg_2, g_3^2 + 2gg_3)$ and get set of for polynomials $q_1$. Also we apply $F_{\text{inverse}}$ to the point $p = (N_0 + 2N, M_0 + 2M, p_2, p_3)$ and get $q_2$.

```sage
K.<a_2001, a_1200, a_1110, a_1101, a_0210, M_0, M_1, N_0, N_1, g_2, gg_2, g_3, gg_3, x_0, x_1, x_2, x_3, p_2, p_3>=ZZ[]

f = a_2001*x_0^-2*x_3 + a_1200*x_0*x_1^-2 + a_1110*x_0*x_1*x_2 + a_1101*x_0*x_1*x_3 + a_0210*x_1^-2*x_2
dfdx1 = f.derivative(x_1)
dfdx2 = f.derivative(x_2)
dfdx3 = f.derivative(x_3)

def I1(x_0,x_1,x_2,x_3):
```
return x0*dfdx1(x0=x0, x1=x1, x2=x2, x3=x3), x1*dfdx1(x0=x0, x1=x1, x2=x2, x3=x3) -
2*f(x0=x0, x1=x1, x2=x2, x3=x3), x2*dfdx1(x0=x0, x1=x1, x2=x2, x3=x3),
 x3*dfdx1(x0=x0, x1=x1, x2=x2, x3=x3)
def I2(x0, x1, x2, x3):
 return x0*dfdx2(x0=x0, x1=x1, x2=x2, x3=x3), x1*dfdx2(x0=x0, x1=x1, x2=x2, x3=x3),
 x2*dfdx2(x0=x0, x1=x1, x2=x2, x3=x3) - 2*f(x0=x0, x1=x1, x2=x2, x3=x3),
 x3*dfdx2(x0=x0, x1=x1, x2=x2, x3=x3)
def I3(x0, x1, x2, x3):
 return x0*dfdx3(x0=x0, x1=x1, x2=x2, x3=x3), x1*dfdx3(x0=x0, x1=x1, x2=x2, x3=x3),
 x2*dfdx3(x0=x0, x1=x1, x2=x2, x3=x3), x3*dfdx3(x0=x0, x1=x1, x2=x2, x3=x3) -
2*f(x0=x0, x1=x1, x2=x2, x3=x3)
def F_inverse(x0, x1, x2, x3):
 return (I1(*I2(*I3(x0, x1, x2, x3))))
Q1 = F_inverse(M0+2* N0, M1 +2* N1, g2 + 2* gg2, g3 +2* gg3)
Q2 = F_inverse(M0+2* N0, M1 +2* N1, p2, p3)

Set of polynomials Q is exactly the set of polynomials (q0, q1, q2, q3) from the proof of Lemma 4.16. To get the
result we need several additional functions. The function mod_2I deletes monomials in a polynomial which lie
in the ideal 2 \mathcal{I}.

def mod_2I(P):
 Q = 0
 for m in P.monomials():
 c = P.monomial_coefficient(m)
 d2 = m.degree(p2)
 d3 = m.degree(p3)
 if d2+d3==2:
     # here we factor mod (pi*pj) \subset 2I
 Q = Q + c
 if d2+d3==1:
     # here we factor mod (2pi) \subset 2I
 c2 = c % 2
 Q = Q + c2*m
 if d2+d3==0:
     # here we factor mod (8) \subset 2I
 c8 = c % 8
 Q = Q + c8*m
 return Q

The function mod_4 deletes all monomials in a polynomial which lie in the ideal 4\mathcal{R}[X, Y].

def mod_4(P):
 Q = 0
 for m in P.monomials():
 c = P.monomial_coefficient(m)
 d2 = m.degree(g2)
 d3 = m.degree(g3)
 dd2 = m.degree(gg2)
 dd3 = m.degree(gg3)
 if d2+d3+dd2+dd3 ==1:
     # here we factor mod (2gi) \subset 4I
 c2 = c % 2
 Q = Q + c2*m
 if d2+d3+dd2+dd3 ==0:
     # here we factor mod (4) \subset 4I
 c4 = c % 4
 Q = Q + c4*m
 return Q

The function mod_8 deletes all monomials in a polynomial which lie in the ideal 8\mathcal{R}[X, Y].
def mod_8(P):
    Q = 0
    for m in P.monomials:
        c = P.monomial_coefficient(m)
        d2 = m.degree(g2)
        d3 = m.degree(g3)
        dd2 = m.degree(gg2)
        dd3 = m.degree(gg3)
        if d2 + d3 + dd2 + dd3 == 2:
            # here we factor mod (2^2 * gi * gj) \subset (8)
            c2 = c % 2
            Q = Q + c2 * m
        if d2 + d3 + dd2 + dd3 == 1:
            # here we factor mod (4 * gi) \subset (8)
            c4 = c % 4
            Q = Q + c4 * m
        if d2 + d3 + dd2 + dd3 == 0:
            # here we factor mod (8) \subset (8)
            c8 = c % 8
            Q = Q + c8 * m
    return Q

The function leading_term_M1 returns the leading term in $M_1$ of the polynomial.

def leading_term_M1(P):
    Q = 0
    d = P.degree(M1)
    for m in P.monomials:
        if d == m.degree(M1):
            c = P.monomial_coefficient(m)
            Q = Q + c * m
    return Q

The last thing we need is a function factorization which factors polynomials into a product of irreducible polynomials. Unfortunately the computation above results in so-called symbolic functions, so to factor them we transform the symbolic function into a polynomial and then use the factorization in polynomials.

b_2001, b_1200, b_1110, b_0210, Z0, Z1, W0, W1, f0, f1, f2, f3, ff2, ff3, q2, q3 =
PolynomialRing(RationalField(), 17, ['a_2001', 'a_1200', 'a_1110', 'a_0210',
'M0', 'M1', 'N0', 'N1', 'g0', 'g1', 'g2', 'g3', 'gg2', 'gg3', 'p2', 'p3']).gens()

def factorization(P):
    Q = 0
    for m in P.monomials:
        c = P.monomial_coefficient(m)
        d2001 = m.degree(a_2001)
        d1200 = m.degree(a_1200)
        d1110 = m.degree(a_1110)
        d1101 = m.degree(a_1101)
        d0210 = m.degree(a_0210)
        dx0 = m.degree(M0)
        dx1 = m.degree(M1)
        dy0 = m.degree(N0)
        dy1 = m.degree(N1)
        d2 = m.degree(g2)
        d3 = m.degree(g3)
        dd2 = m.degree(gg2)
        dd3 = m.degree(gg3)
        dp2 = m.degree(p2)
        dp3 = m.degree(p3)
Q = q * c * b_2001^d2001 * b_1200^d1200 * b_1110^d1110 * b_1101^d1101 * b_0210^d0210 * f2^d2 * f3^d3 * ff2^d2 * ff3^d3 * Z0^dx0 * Z1^dx1 * W0^dy0 * W1^dy1 * q2^dp2 * q3^dp3

return Q.factor()

Now we are ready to get results:

M0new = mod_4(Q1[0])
M1new = mod_4(Q1[1])
Q2_new = mod_2I(Q2[2])
Q3_new = mod_2I(Q2[3])
G2new = mod_8(Q1[2])
G3new = mod_8(Q1[3])

print(' Leading term of 2* M_0_new = ', factorization(leading_term_M1(M0new)))
print(' Leading term of 2* M_1_new = ', factorization(leading_term_M1(M1new)))
print(' Q2_new modulo 2*I = ', factorization(Q2_new))
print(' Q3_new modulo 2*I = ', factorization(Q3_new))
print(' Leading term of 2* G2new = ', factorization(leading_term_M1(G2new)))
print(' Leading term of 2* G3new = ', factorization(leading_term_M1(G3new)))

We get six lines of results. The first line produces the leading term in $M_1$ for $2M_0'$, we use this formula in (1.18). The second line produces the leading term in $M_1$ for $2M_1'$, we use this formula in (1.20). The third line produces the expression for $q_2'$, we use it in (1.22). The fourth line produces the expression for $q_3'$, we use it in (1.25). The fifth and sixth lines produce the leading terms in $M_1$ of $\tilde{g}_2'$ and $\tilde{g}_3'$ respectively, we use them in (1.23) and (1.26).

References

[BC16] J. Blanc and S. Cantat. Dynamical degrees of birational transformations of projective surfaces. J. Amer. Math. Soc., 29(2):415–471, 2016.
[BCK14] E. Bedford, S. Cantat, and K. Kim. Pseudo-automorphisms with no invariant foliation. J. Mod. Dyn., 8(2):221–250, 2014.
[BDPP13] S. Boucksom, J.-P. Demailly, M. Păun, and T. Peternell. The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension. J. Algebraic Geom., 22(2):201–248, 2013.
[BK09] E. Bedford and K. Kim. Dynamics of rational surface automorphisms: linear fractional recurrences. J. Geom. Anal., 19(3):553–583, 2009.
[BK14] E. Bedford and K. Kim. Dynamics of (pseudo) automorphisms of 3-space: periodicity versus positive entropy. Publ. Mat., 58(1):65–119, 2014.
[Bla08] J. Blanc. On the inertia group of elliptic curves in the Cremona group of the plane. Michigan Math. J., 56(2):315–330, 2008.
[Bla13] J. Blanc. Dynamical degrees of (pseudo)-automorphisms fixing cubic hypersurfaces. Indiana Univ. Math. J., 62(4):1143–1164, 2013.
[CDX21] S. Cantat, J. Déserti, and J. Xie. Three chapters on Cremona groups. Indiana Univ. Math. J., 70(5):2011–2064, 2021.
[DF01] J. Diller and C. Favre. Dynamics of bimeromorphic maps of surfaces. Amer. J. Math., 123(6):1135–1169, 2001.
[DF20] N.-B. Dang and C. Favre. Spectral interpretations of dynamical degrees and applications. arXiv:2006.10262, 2020.
[DN11] T.-C. Dinh and V.-A. Nguyên. Comparison of dynamical degrees for semi-conjugate meromorphic maps. Comment. Math. Helv., 86(4):817–840, 2011.
[DO88] I. Dolgachev and D. Ortland. Point sets in projective spaces and theta functions. Astérisque, (165):210 pp. (1989), 1988.
[DS05] T.-C. Dinh and N. Sibony. Une borne supérieure pour l’entropie topologique d’une application rationnelle. Ann. of Math. (2), 161(3):1637–1644, 2005.
[Ful98] W. Fulton. Intersection theory, volume 2 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, second edition, 1998.
[Har77] R. Hartshorne. Algebraic geometry. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.
[HM10] C. D. Hacon and J. McKernan. Flips and flops. In Proceedings of the International Congress of Mathematicians. Volume II, pages 513–539. Hindustan Book Agency, New Delhi, 2010.
[KM98] J. Kollár and S. Mori. Birational geometry of algebraic varieties, volume 134 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.
[Kol07] J. Kollár. Lectures on resolution of singularities, volume 166 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2007.

[LB19] F. Lo Bianco. On the cohomological action of automorphisms of compact Kähler threefolds. Bull. Soc. Math. France, 147(3):469–514, 2019.

[Les18] J. Lesieutre. Some constraints of positive entropy automorphisms of smooth threefolds. Ann. Sci. Éc. Norm. Supér. (4), 51(6):1507–1547, 2018.

[OT15] K. Oguiso and T. T. Truong. Explicit examples of rational and Calabi-Yau threefolds with primitive automorphisms of positive entropy. J. Math. Sci. Univ. Tokyo, 22(1):361–385, 2015.

[PS14] Yu. Prokhorov and C. Shramov. Jordan property for groups of birational selfmaps. Compos. Math., 150(12):2054–2072, 2014.

[Tru14] T. T. Truong. The simplicity of the first spectral radius of a meromorphic map. Michigan Math. J., 63(3):623–633, 2014.

[Tru20] T. T. Truong. Relative dynamical degrees of correspondences over a field of arbitrary characteristic. J. Reine Angew. Math., 758:139–182, 2020.

[Wei55] A. Weil. On algebraic groups of transformations. Amer. J. Math., 77:355–391, 1955.

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