Kähler Potential of Moduli Space
in Large Radius Region of Calabi-Yau Manifold

Katsuyuki Sugiyama†

Department of Fundamental Sciences
Faculty of Integrated Human Studies, Kyoto University
Yoshida-Nihon-Matsu cho, Sakyo-ku, Kyoto 606-8501, Japan

ABSTRACT

We study a Kähler potential $K$ in the large radius region of a Calabi-Yau $d$-fold $M$ embedded in $CP^{d+1}$. It has a Kähler parameter $t$ that describes a deformation of the A-model moduli. Also the metric, curvature and hermitian two-point functions in the large volume region are analyzed. We use a result of our previous paper in the B-model of the mirror. We perform an analytic continuation of a parameter to the large complex structure region. By translating the result in the A-model side of $M$, we determine the $K$. The method is not restricted to this specific model and we apply the recipe to complete intersection Calabi-Yau cases.

† E-mail: sugiyama@phys.h.kyoto-u.ac.jp
1 Introduction

D-branes play important roles to describe the solitonic modes in string theory and could make clear dynamics in strong coupling regions. The physical observables of D-brane’s effective theories have dependences on moduli of compactified strings or wrapped D-branes. We expect that properties of compactified internal spaces essentially control non-perturbative effects in low energy theories. In this paper, we focus on the type II superstring compactified on Calabi-Yau manifold and study its topological sector from the point of view of topological sigma models (A- and B-models)\[1\] to investigate properties of moduli spaces. Because both the A- and B-models are topological theories, they are characterized by their two-point and three point functions that play important roles as the constituent blocks in these models\[2\]-\[10\]. Topological metrics are two-point functions and receive no quantum corrections. On the other hand, the three-point functions of the A-model have information about the fusion structure of observables. The remaining fundamental blocks are a Kähler potential $K$ and associated hermitian two-point functions. They are hermitian and describe correlations of topological ant anti-topological sectors\[11\],\[12\],\[8\],\[10\]. Also the $K$ has information about intersections of homology cycles with even dimensions in the A-model case.

The aim of this paper is to develop a concrete method to construct Kähler potentials applicable in the large radius regions of Calabi-Yau $d$-folds and to investigate their properties in order to understand structures of the moduli spaces. We present formulae of Kähler potentials for $d$ dimensional Calabi-Yau manifolds explicitly.

The paper is organized as follows. In section 2, we explain a mirror manifold paired with a Calabi-Yau $d$-fold embedded in $CP^{d+1}$. We also explain the results in \[10\] about a Kähler potential $K$ in the small complex structure region of the B-model in order to fix notations. In section 3, we introduce a set of periods valid in the large complex structure region. By relating two sets of periods in the large and small complex structure regions, we construct a formula of the $K$ applicable in the large complex structure region of the B-model. In sections 4 and 5, we construct a mirror map and a Kähler potential. The scalar curvature of the Kähler moduli is investigated. The set of correlation functions associated with the Kähler moduli are calculated in the large radius region of the A-model. A concrete application of our result is explained in the quintic case in section 6. Also the result there is generalized to propose a formula of the Kähler potential of a Calabi-Yau $d$-fold in the complete intersection type in section 7. Section 8 is devoted to conclusions and comments. In appendix A, we summarize several examples of the expansion coefficients of a function $\hat{K}$ in lower dimensional cases.
2 Small Complex Structure Region

We take a one-parameter family of Calabi-Yau $d$-fold $M$ realized as a zero locus of a hypersurface embedded in a $CP^{d+1}$

$$M; \; p = X_1^N + X_2^N + \cdots + X_N^N - N\psi X_1 X_2 \cdots X_N = 0.$$ 

The $N$ is related with the complex dimension $d$ of $M$, $N = d + 2$. A mirror manifold $W$ paired with this $M$ is constructed as a orbifold divided by some maximally discrete group $G = Z^\times_N^{(N-1)}$

$$W; \{p = 0\}/G.$$ 

When one thinks about Hodge structure of the $G$-invariant parts of the cohomology group $H^d(W)$, related Hodge numbers[13] are written as

$$h^{d,0} = h^{d-1,1} = \cdots = h^{1,d-1} = h^{0,d} = 1.$$ 

In our previous paper[10], we study the formula of the Kähler potential of the Calabi-Yau $d$-fold $W$ with a moduli parameter $\psi$ of the complex structure.

The $K$ is constructed by combining a set of periods $\tilde{\omega}_k$ quadratically

$$e^{-K} = \sum_{k=1}^{N-1} I_k \tilde{\omega}_k^\dagger \tilde{\omega}_k,$$ (1)

$$I_k = \frac{1}{\pi^N \cdot N^{N+2}} (-1)^{k-1} \left( \sin \frac{\pi k}{N} \right)^N,$$

$$\tilde{\omega}_k(\psi) = \left[ \Gamma \left( \frac{k}{N} \right) \right]^N \left( \frac{(N\psi)^k}{\Gamma(k)} \right) \times \sum_{n=0}^{\infty} \left[ \frac{\Gamma \left( \frac{k}{N} + n \right)}{\Gamma \left( \frac{k}{N} \right)} \right]^N \frac{\Gamma(k)}{\Gamma(Nn + k)} (N\psi)^{Nn}.$$ 

We determined the coefficients $I_k$ in the [10] by requiring consistency conditions with the results of the CFT at the Gepner point. This formula is valid in the small $\psi$ region because of the convergence of the series expansion.

In the following section, we investigate the large complex structure region of the $W$. Its formula is important for the large radius analyses of the manifold $M$.

3 Large Complex Structure Region

We try to rewrite the $K$ in a formula that is valid in the large $\psi$ by an analytic continuation. First we choose a set of periods $\{\Omega_m\} (m = 0, 1, \cdots, N - 2)$ appropriate to describe the
large complex structure region of the mirror W. A generating function of the $\Omega_m$ is defined by using a formal parameter $\rho$ with $\rho^{N-1} = 0$

$$\sum_{m=0}^{N-2} \Omega_m \rho^m = \sqrt{K(\rho)} \cdot \varpi \left( \frac{\rho}{2\pi i} ; z \right),$$

$$\varpi(v) = z^v \cdot \sum_{n=0}^{\infty} \frac{a(n+v)}{a(v)} z^n, \quad z = (N\psi)^{-N},$$

$$a(v) = \frac{\Gamma(Nv+1)}{[\Gamma(v+1)]^N}.$$

Here we introduce a function $\hat{K}(\rho)$.

$$\hat{K}(\rho) := \frac{a \left( +\frac{\rho}{2\pi i} \right)}{a \left( -\frac{\rho}{2\pi i} \right)} = \exp \left[ 2 \sum_{m=1}^{N-1} \frac{N-N^{2m+1}}{2m+1} \zeta(2m+1) \left( \frac{\rho}{2\pi i} \right)^{2m+1} \right]$$

$$= 1 + 2\zeta(3) \frac{c_3}{N} \left( \frac{\rho}{2\pi i} \right)^3 + \mathcal{O}(\rho^5).$$

The leading term in the $\hat{K}$ of the variety W is a 3rd Chern class of the M. Generally the coefficients of $\hat{K}$ contain topological information of M. In fact, Chern classes of $c_\ell$ ($\ell = 1, 2, \cdots, N-2$) of the manifold $M$ are generated by a function $c(\rho)$

$$c(\rho) = \frac{(1+\rho)^N}{1+N\rho} = 1 + \sum_{\ell \geq 1} \rho^\ell \frac{c_\ell}{N}.$$

Typical coefficients $X_\ell = N - N^\ell$ in the $\hat{K}$ are some combinations of Chern classes $c_\ell$

$$c(\rho) = 1 + \sum_{\ell \geq 1} \rho^\ell \frac{c_\ell}{N} = \exp \left( \sum_{\ell \geq 1} (-1)^{\ell-1} \rho^\ell \cdot \frac{X_\ell}{\ell} \right).$$

For examples, we list several $X_\ell$ for the Calabi-Yau case

$$X_1 = 0, \quad X_2 = -\frac{2}{N} c_2, \quad X_3 = \frac{3}{N} c_3,$$

$$X_4 = -\frac{4}{N} c_4 + \frac{2}{N^2} c_2^2, \quad X_5 = \frac{5}{N} c_5 - \frac{5}{N^2} c_3 c_2.$$

The series expansion Eq.(2) converges around $z \sim 0$, that is, large complex structure point of the W. For the purpose of an analytic continuation into the large complex structure region, we find that the two sets of the periods are related by a transformation matrix $\tilde{M}$ with components $\tilde{M}_{k\ell}$

$$\tilde{\varpi}_k = \sum_{\ell=0}^{N-2} \tilde{M}_{k\ell} \Omega_\ell \quad (k = 1, 2, \cdots, N-1),$$

3
\[ \tilde{M}_{k\ell} = (-N) \cdot (2\pi i)^{N-1} \times \left[ \sqrt{A(\rho)} \cdot \frac{\alpha^k}{e^\rho - \alpha^k} \cdot (-\rho)^\ell \right] \bigg|_{\rho = N-2} \]
\[ = (-N) \cdot (2\pi i)^{N-1} \cdot \sum_{m=0}^{N-2} G_{k,m} V_{m,\ell}, \]
\[ G_{k,m} = \frac{-\alpha^k}{(\alpha^k - 1)^{m+1}} \quad (1 \leq k \leq N - 1, \ 0 \leq m \leq N - 2), \]
\[ \alpha = e^{2\pi i / N}, \]
\[ V_{m,\ell} = \left[ \sqrt{A(\rho)} \cdot (e^\rho - 1)^m \cdot (-\rho)^\ell \right] \bigg|_{\rho = N-2} \quad (0 \leq m \leq N - 2, \ 0 \leq \ell \leq N - 2). \]

Here the transformation matrix \( V \) contains a square root of a topological invariant “A-roof” of the Calabi-Yau space \( M \)

\[ \hat{A}(\rho) = \left( \frac{\rho}{\sinh \frac{\rho}{2}} \right)^N \cdot \left( \frac{\sinh \frac{N\rho}{2}}{\frac{N\rho}{2}} \right) = \frac{1}{a(-\rho / 2\pi i)a(+\rho / 2\pi i)} \]
\[ = \exp \left[ + \sum_{m=1}^{\infty} (-1)^m B_m N - N^{2m} \frac{2m^{2m}}{2m!} \rho^{2m} \right] \]
\[ = 1 + \frac{1}{12} N \rho^2 + \mathcal{O}(\rho^4). \]

The \( B_m \)s are Bernoulli numbers and are defined in our convention as
\[ \frac{x}{e^x - 1} = 1 - \frac{x}{2} - \sum_{n=1}^{\infty} (-1)^n \cdot B_{2n} \frac{x^{2n}}{(2n)!}, \]
\[ B_1 = \frac{1}{6}, \ B_2 = \frac{1}{30}, \ B_3 = \frac{1}{42}, \ B_4 = \frac{1}{30}, \cdots. \]

Now we return to the Kähler potential \( K \). By performing the analytic continuation of the Eq. (1), we can obtain a formula of the \( K \) applicable in the large \( \psi \) region.

\[ e^{-K} = (-1)^N \left( \frac{2\pi i}{N} \right)^{N-2} \cdot \frac{1}{N^2} \sum_{\ell,\ell'=0}^{N-2} (V^\dagger \mathcal{I} V)_{\ell,\ell'} (-1)^{\ell+\ell'} \cdot \hat{\Omega}_{\ell} \hat{\Omega}_{\ell'}, \]

Here the \( \mathcal{I} \) is a triangular matrix that combines \( \bar{\Omega} \) and \( \Omega \) quadratically

\[ \mathcal{I}_{m,m'} = 2^N \cdot i^N \cdot \sum_{k=1}^{N-1} \frac{(-1)^k \left( \frac{\sin \frac{\pi k}{N} \right)^N}{(\alpha^k - 1)^{m+1}(\alpha^k - 1)^{m'+1}} = (-1)^m \delta_{m+m',N-2} + \cdots. \]

The \( \mathcal{I}_{m,m'} \) has non-vanishing components only at \( m + m' \geq N - 2 \) (\( m, m' = 0, 1, \cdots, N - 2 \)) and we find an expression for the matrix \( V^\dagger \mathcal{I} V \)

\[ (V^\dagger \mathcal{I} V)_{\ell,\ell'} = (-1)^\ell \delta_{\ell+\ell',N-2}. \quad (3) \]
We check validity of this equation Eq.(3) concretely for \( N \leq 27 \) and propose this formula for arbitrary \( N(\geq 3) \) cases as a conjecture.

Finally we obtain a formula of the \( K \) of \( W \) with this equation

\[
e^{-K} = (-1)^d \left( \frac{2\pi i}{N} \right)^d \cdot \frac{1}{N^2} \left( \Omega^i \Sigma \Omega \right),
\]

\[
\Sigma_{\ell,\ell'} = (-1)^\ell \delta_{\ell+\ell',N-2}.
\]

The \( e^{-K} \) is constructed by combining a holomorphic \( d \) form and an anti-holomorphic one quadratically. Both parts are decomposed by a dual basis of (real) homology cycles and their coefficients are realized as periods. Then we can understand that a matrix which combines the periods and their complex conjugates is an intersection matrix of the cycles. In our case, the \( \Sigma \) is an intersection matrix of homology cycles associated with the set of periods \( \Omega_m \) of \( W \). The result means that the cycles we used here are combined into a symplectic USp\((d+1)\) or an SO\((d+1, d+1)\) invariant bases for respectively \( d = \text{odd} \) or \( d = \text{even} \) cases. But the basis is not an integral one and we have to perform an appropriate linear transformation with fractional rational numbers to construct a canonical basis of a central charge. More details will appear in our next paper.

Let us study behaviors of a metric, a curvature in the large \( \psi \) region of \( W \). Powers of logarithm of \( \psi \) appear in the Kähler potential \( K \) in the leading expansion

\[
e^{-K} = \frac{1}{d!N^2} \{2 \log(N|\psi|) \}^d \times \left[ 1 + \sum_{n=1}^{d} \frac{d!}{(d-n)!} \cdot \frac{(-1)^n \cdot \hat{K}_n}{(2N \log(N|\psi|))^n} \right] + \cdots
\]

\[
= \frac{1}{d!N^2} \{2 \log(N|\psi|) \}^d \times \left[ 1 + 4 \left( \frac{d}{3} \right) (N^3 - N) \zeta(3) \cdot (2N \log(N|\psi|))^{-3}
\]

\[
+ 48 \left( \frac{d}{5} \right) (N^5 - N) \zeta(5) \cdot (2N \log(N|\psi|))^{-5} + \cdots \right].
\]

Here the \( \hat{K}_n \)s are coefficients of the series expansion of the \( \hat{K} \)

\[
\hat{K}(\rho) = 1 + \sum_{n=3} \hat{K}_n \left( \frac{\rho}{2\pi i} \right)^n,
\]

\[
\hat{K}_3 = -\frac{2}{3} (N^3 - N) \zeta(3) = \frac{2}{N} c_3 \zeta(3),
\]

\[
\hat{K}_5 = -\frac{2}{5} (N^5 - N) \zeta(5) = \frac{2}{N} (c_5 - \frac{1}{N} c_3 \cdot c_2) \zeta(5),
\]

They are related to the Chern classes of the \( d \)-fold \( M \). Also there appear \( \zeta(2m+1) \)s \( (m = 1, 2, \cdots \text{ with } 2m+1 \leq d) \), which might be transcendental numbers, in this formula. We summarize several concrete examples of the \( \hat{K}_n \) in the appendix. The exponent of power of the logarithm is at most \( d \) for the \( d \)-fold. In addition, there are parts of infinite series with respect to the \( \psi \) in the \( e^{-K} \). They are omitted as an abbreviated symbol “\( \cdots \)”. 


Next we calculate the Kähler metric of this B-model associated with $W$ 

\[
g_{\psi\bar{\psi}} = \frac{d}{\{2|\psi| \log(N|\psi|)\}^2} \times \left[ 1 - 16 \left( \frac{d-1}{2} \right) (N^3 - N)\zeta(3) \cdot \{2N \log(N|\psi|)\}^{-3} \right.
\]

\[ -288 \left( \frac{d-1}{4} \right) (N^5 - N)\zeta(5) \cdot \{2N \log(N|\psi|)\}^{-5} + \cdots \].

Also we obtain the scalar curvature in this large $\psi$ region

\[
R = -\frac{4}{d} \times \left[ 1 - 80 \left( \frac{d-1}{2} \right) (N^3 - N)\zeta(3) \cdot \{2N \log(N|\psi|)\}^{-3} \right.
\]

\[ -4032 \left( \frac{d-1}{4} \right) (N^5 - N)\zeta(5) \cdot \{2N \log(N|\psi|)\}^{-5} + \cdots \].

In the $|\psi| = \infty$ limit, the curvature is a negative constant and its absolute value is inversely proportional to the dimension $d$. The $|R|$ decreases apart from the point $|\psi| = \infty$ for $N \geq 5$ cases. The leading term of the corrections in the brackets is inversely proportional to the $\{\log(N\psi)\}^{-3}$ and contains $\zeta(3)$ as its coefficient. For the $N = 3, 4$ cases, the associated curvatures $R_s$ are constants except for points in the $\psi$-plane with $\psi^N = 1$.

Next we discuss an invariant coupling. The Kähler potential is not a function but a section of a line bundle and there is an arbitrariness of multiplications by holomorphic and anti-holomorphic functions. Also a $d$-point correlation function in the B-model is a section with a weight 2

\[
K_{\psi\bar{\psi}...\psi} = \frac{1}{N^d} \cdot \frac{N\psi^2}{1 - \psi^N}.
\]

But there is an invariant $d$-point function $K$ that is constructed by combining the metric, the $K$ and $K_{\psi\bar{\psi}...\psi}$

\[
K = (g_{\psi\bar{\psi}})^{-d/2} e^K |K_{\psi\bar{\psi}...\psi}|.
\]

In our normalization, it is expressed in the large $|\psi|$ limit as

\[
K = \frac{d!}{d^{d/2} N^{d-3}} \times \left[ 1 + 20 \left( \frac{d}{3} \right)(N^3 - N)\zeta(3) \cdot \{2N \log(N|\psi|)\}^{-3} \right.
\]

\[ +672 \left( \frac{d}{5} \right)(N^5 - N)\zeta(5) \cdot \{2N \log(N|\psi|)\}^{-5} + \cdots \].

Up to coefficients, the corrections have the same structures for the metric, the $R$ and the $K$.

In the small complex structure limit (at $\psi = 0$), this invariant coupling is evaluated as

\[
K = \left[ \frac{\Gamma\left( \frac{1}{N} \right)}{\Gamma\left( 1 - \frac{1}{N} \right)} \right]^{1/2} \cdot \left[ \frac{\Gamma\left( 1 - \frac{2}{N} \right)}{\Gamma\left( \frac{2}{N} \right)} \right]^{1/2} \times \frac{1}{N^{d-3}}.
\]
For an example, we write down the result in the $d = 3$ case

\[
\kappa = \frac{2}{\sqrt{3}} \left[ 1 - 12c_3\zeta(3) \cdot \left\{ 10 \log(5|\psi|) \right\}^{-3} + \cdots \right], \quad c_3 = -200.
\]

This shows that the coupling increases apart from the $|\psi| = \infty$.

Also we compare the leading value of this $\kappa$ in the small $|\psi|$ with that in the large $|\psi|$ limits

\[
\text{small } \psi; \quad \kappa = \left( \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{7}{3})} \right)^{5/2} \left( \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{5}{3})} \right)^{15/2} = 1.5553189899632389724994495854237822 \cdots
\]

\[
\text{large } \psi; \quad \kappa = \frac{2}{\sqrt{3}} = 1.154700538379251529018297561003914911 \cdots.
\]

The $\kappa$ in the small $\psi$ case can be analyzed by using our previous result in [10]. The coupling in the small $\psi$ region is stronger than that in the large $\psi$ region in an amount of 34.7%.

### 4 Large Radius Region

In the previous section, we study properties of physical quantities in the large complex structure point of the $d$-fold $W$. It is known that the large complex structure point is related to the large radius point of the partner $M$. It is possible to translate the results of $W$ to those of the $M$. For the purpose of this program, we introduce a mirror map $t$

\[
2\pi i t = \log z + \frac{\sum_{n=1}^{\infty} (Nn)!}{(n!)^N} \left( \sum_{m=n+1}^{Nn} \frac{N}{m} \right) z^n, \quad q = e^{2\pi i t},
\]

\[
x_n = \left. \frac{1}{(2\pi i)^n n!} \partial_{\rho}^n \log \left[ \sum_{m=0}^{\infty} \frac{\Gamma(N(m+\rho)+1)}{\Gamma(N\rho+1)} \left( \frac{\Gamma(\rho+1)}{\Gamma(m+\rho+1)} \right)^N z^m \right] \right|_{\rho=0}
\]

\[
= \sum_{m=1}^{\infty} a_{n,m} q^m \quad (n \geq 2),
\]

\[
a_{n,1} = N! \cdot S_n(\beta_1(1), \cdots, \beta_n(1)),
\]

\[
a_{n,2} = \frac{(2N)!}{2^N} \cdot S_n(\beta_1(2), \cdots, \beta_n(2))
\]

\[
- (N!)^2 \left( \sum_{m=2}^{N} \frac{N}{m} \right) \cdot S_n(\beta_1(1), \cdots, \beta_n(1))
\]

\[
- \frac{1}{2} (N!)^2 \cdot S_n(2\beta_1(1), \cdots, 2\beta_n(1)),
\]

\[
\beta_m(n) = N^m \cdot \frac{\sum_{k=1}^{n} (-1)^{k-1}(k - 1)!}{k^m} - N \cdot \frac{\sum_{k=1}^{n} (-1)^{k-1}(k - 1)!}{k^m}.
\]
This $t$ is a coordinate of the Kähler moduli space or the coefficient of the complexified Kähler form

$$B + iJ = t[D], \quad [D] \in H^2(M).$$

The $[D]$ is a Poincaré dual of a divisor "$D$" of the $M$. The $S_n(x_1, x_2, \cdots, x_n)$s are Schur polynomials and are defined as

$$\exp\left(\sum_{m=1} x_m u^m\right) = \sum_{n=0} S_n(x_1, x_2, \cdots, x_n) u^n.$$

Next we rewrite the $K$ in this coordinate

$$e^{-K} = (-1)^d \left(\frac{2\pi i}{N}\right)^d \frac{1}{N^2} (\Omega^! \Sigma \Omega),$$

$$\left(\Omega^! \Sigma \Omega\right) = \sum_{\ell=0}^{N-2} \hat{\Omega}_\ell \cdot (-1)^\ell \Omega_{N-2-\ell} = \left[\hat{K}(\rho) \cdot \varpi \left(\frac{-\rho}{2\pi i}; z\right) \cdot \varpi \left(\frac{\rho}{2\pi i}; z\right)\right]_\rho^{N-2}$$

$$= |\varpi_0|^2 \times \left[\hat{K}(\rho) e^{\rho(t-\bar{t})} \exp\left(\sum_{n \geq 2} \rho^n (x_n + (-1)^n \bar{x}_n)\right)\right]_\rho^{N-2}. \tag{4}$$

Here the first term in the brackets contains characteristic classes of the $M$. Its coefficients in the series expansion are represented as some combinations of Chern classes of the $M$. Also there appear Riemann’s zeta functions evaluated at positive odd integers. These are irrational numbers and might be transcendental numbers. It implies some arithmetic properties of this model. Next the second term is associated to the imaginary part of the $t$. When we translate the formal parameter $\rho$ into a divisor "$D$" of the hyperplane, the $\rho(t - \bar{t})$ is identified with the Kähler form $J$ of $M$

$$\rho(t - \bar{t}) = 2i\rho\text{Im}(t) \leftrightarrow 2i\text{Im}(t)[D] = 2iJ.$$

This second term contains only imaginary part of $t$ and is invariant under an arbitrary shift of $\text{Re}(t) \rightarrow \text{Re}(t) + a$ for $a \in \mathbb{R}$. This symmetry is a classical one and is broken at the quantum level. In fact, the third term in Eq.(4) is not invariant under this arbitrary shift because the $x_n$s are expressed as series expansions of the variable $q = e^{2\pi it}$. This 3rd term is invariant under only integral shifts of $t \rightarrow t + n$ with $n \in \mathbb{Z}$. This term contains information about non-perturbative effects of the worldsheet instantons. They break the Peccei-Quinn symmetry into a symmetry under the integral shift of the $\text{Re}(t)$. The real part of the $t$ is related to the 2nd rank antisymmetric field $B$ in the NS-NS sector

$$\text{Re}(t)[D] = B,$$

and the shift of the $\text{Re}(t) \rightarrow \text{Re}(t) + n$ is equivalent to a shift of the $B$ field, $B \rightarrow B + [D]$. 

8
Finally we will write down our result for the $K$

$$e^{-K} = (-1)^d \left( \frac{2\pi i}{N} \right)^d \frac{1}{N^2} |\varpi_0|^2 \cdot S_d(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_d)$$

$$= (-1)^d \left( \frac{2\pi i}{N} \right)^d \frac{1}{N^2} |\varpi_0|^2 \cdot \sum_{\ell=2}^{d} \frac{d!}{\ell^\ell} \frac{\ell! S_\ell(0, \bar{y}_2, \ldots, \bar{y}_\ell)}{(t - \bar{t})^\ell}$$

$$\hat{k}_{2n+1} := 2 \cdot \frac{N - N^{2n+1}}{2n + 1} \cdot \frac{(2n + 1)}{(2\pi i)^{2n+1}} \ (n \geq 1),$$

$$\hat{K}(\rho) = \exp \left( \sum_{m \geq 1} \hat{k}_{2m+1} \cdot \rho^{2m+1} \right) = 1 + \sum_{n \geq 1} \hat{K}_n \cdot \left( \frac{\rho}{2\pi i} \right)^n,$$

$$y_n := x_n + (-1)^n \bar{x}_n \ (n \geq 2),$$

$$\bar{y}_n = \begin{cases} \bar{t} & (n = 1) \\
\bar{y}_{2m} & (n = 2m; \ m \geq 1) \\
\bar{y}_{2m+1} + \hat{k}_{2m+1} & (n = 2m + 1; \ m \geq 1) \end{cases}.$$ 

The Schur functions contain “loop”\(^1\) and non-perturbative effects of the model

$$S_2 = 2 \text{Re}(x_2), \ S_3 = 2i \text{Im}(x_3) + \hat{k}_3,$$

$$S_4 = 2[\text{Re}(x_4) + (\text{Re}(x_2))^2],$$

$$S_5 = 2i \text{Im}(y_5) + 4i \text{Re}(x_2) \text{Im}(x_3) + \hat{k}_5 + 2 \text{Re}(x_2) \cdot \hat{k}_3.$$ 

Especially we obtain an asymptotic formula of the $y_n$s in the range of the large radius volume

$$\begin{align*}
\bar{y}_{2m} &\to 0 \quad (m \geq 1) \\
\bar{y}_{2m+1} &\to \hat{k}_{2m+1} \quad (m \geq 1) \\
S_\ell(0, \bar{y}_2, \ldots, \bar{y}_\ell) &\to \hat{K}_\ell \quad (\ell \geq 2).
\end{align*}$$

In this limit, we can neglect non-perturbative corrections and write down the $K$, a metric and a scalar curvature of the moduli space of $M$ as power series of the $(t - \bar{t})$

$$e^{-K} = (-1)^d \left( \frac{2\pi i}{N} \right)^d \frac{1}{N^2} \frac{1}{d!} (t - \bar{t})^d \cdot \left[ 1 + \sum_{\ell=3}^{d} \frac{d!}{\ell^\ell} \frac{\ell! \hat{K}_\ell}{(2\pi i)^\ell \cdot (t - \bar{t})^\ell} \right],$$

$$g_{t\bar{t}} = \frac{-d}{(t - \bar{t})^2} \cdot \left[ 1 - \frac{72}{(2\pi i)^3} \cdot \frac{d}{d^d} \cdot (t - \bar{t})^d \right] - \frac{3600}{(2\pi i)^5} \cdot \frac{d}{d^d} \cdot (t - \bar{t})^d + \cdots,$$

$$R_{t\bar{t}} = -\partial_t \partial_{\bar{t}} \log g_{t\bar{t}} = \frac{2}{(t - \bar{t})^2} \cdot \left[ 1 - \frac{432}{(2\pi i)^3} \cdot \frac{d}{d^d} \cdot (t - \bar{t})^d - \frac{54000}{(2\pi i)^5} \cdot \frac{d}{d^d} \cdot (t - \bar{t})^d + \cdots \right].$$

\(^1\)The $\frac{\text{Re}(\hat{k}_n)}{(2\pi i)^n}$ is interpreted as an effect of loop corrections at the 4-loop perturbative calculation of the sigma model with 3 dimensional Calabi-Yau target spaces. We do not know that the other terms $\hat{K}_n$s in the $\hat{K}$ can be interpreted directly as perturbative loop corrections of the sigma model at higher loop calculations.
The line element of the A-model moduli space is given as 

\[ ds^2 = g_{\bar{t}t} dt d\bar{t} \].

The large \( \text{Im}(t) \) limit, the metric and the Ricci tensor are inversely proportional to \( (t - \bar{t}) \). But apart from the point \( \text{Im}(t) = \infty \), there appear terms \( \text{Im}(t)^n (n > 2) \) and also non-perturbative corrections with \( e^{2\pi n t} \) or \( e^{-2\pi n t} (n \geq 1) \) for \( d \geq 3 \) cases. The large \( \text{Im}(t) \) point corresponds to the large \( \psi \) point. On the other hand, a related value of the “\( t \)” at the \( \psi = 0 \) point does not vanish, but it is finite 

\[ t(\psi = 0) = -\frac{1}{2} + \frac{i}{2\tan \frac{\pi}{N}}. \]

At the \( \psi = 0 \) point, the \( R, R_{\bar{t}t} \) and \( g_{\bar{t}t} \) do not vanish and are evaluated as

\[ g_{\bar{t}t} = \left( 2 \sin \frac{2\pi}{N} \right)^2 \left( 2 \cos \frac{\pi}{N} \right)^{-N+2}, \]

\[ R = -4 + 2 \left[ \frac{\Gamma \left( \frac{1}{N} \right) \Gamma \left( \frac{3}{N} \right)}{\Gamma \left( 1 - \frac{1}{N} \right) \Gamma \left( 1 - \frac{3}{N} \right)} \right]^N \left[ \frac{\Gamma \left( 1 - \frac{2}{N} \right)}{\Gamma \left( \frac{2}{N} \right)} \right]^{2N}, \]

\[ R_{\bar{t}t} = \frac{1}{2} g_{\bar{t}t} R. \]

The scalar curvature is positive around the Gepner point for \( d \geq 3 \). When we increase the \( \psi \) from zero to one, the \( R \) increases monotonically with the \( \psi \) for each \( N \). The point \( \psi = 1 \) is a singular point where associated scalar curvatures blow up. When one passes through the \( \psi = 1 \) and increases the value of the \( \psi \), the associated \( R \) decreases monotonically and vanishes at some point \( \psi = \psi_0 \) for each \( N \). In the range \( \psi > \psi_0 \), the \( R \) is negative and its asymptotic value is \(-4/d\). That is to say, it is negative in the large \( \psi \) region for \( d \geq 3 \). The behaviors of the curvatures for \( N = 5, 6, 7 \) are shown in Fig.1.

For the torus case, the curvature is \( R = -4 \) both in the small \( \psi \) and in the large \( \psi \) regions. Similarly, for the K3 case, the \( R \) is \((-2)\) in the two limits. Associated Ricci tensors at the \( \psi = 0 \) are obtained for \( N = 3, 4 \) respectively

\[ R_{\psi\bar{\psi}} = -18 \cdot \left[ \frac{\Gamma \left( \frac{3}{4} \right)}{\Gamma \left( \frac{1}{4} \right)} \right]^6 = -0.30021677774546778674 \cdots, \ \text{\( N = 3 \) case}, \]

\[ R_{\psi\bar{\psi}} = -16 \cdot \left[ \frac{\Gamma \left( \frac{3}{4} \right)}{\Gamma \left( \frac{1}{4} \right)} \right]^4 = -0.20880017792822456419 \cdots, \ \text{\( N = 4 \) case}. \]

When we increase the \( \psi \) from zero to one, the \( R_{\psi\bar{\psi}} \) decreases monotonically with the \( \psi \) for each \( N \). The point \( \psi = 1 \) is a singular point where the \( R_{\psi\bar{\psi}} \) blow up. When one
Figure 1: Scalar Curvature $R$ for $N = 5, 6, 7$ cases. The axis of abscissa represents the value of the $\psi \in \mathbb{R}$. The axis of ordinate corresponds to the scalar curvature $R$. The point $\psi = 1$ is a singular point where associated scalar curvatures blow up. The $R$ vanishes at some point $\psi = \psi_0$ for each $N$. In the range $\psi > \psi_0$, the $R$ is negative and its asymptotic value is $-4/d$. The point $\psi = 1$ passes through the $\psi = 1$ and increases the value of the $\psi$, the associated $R_{\psi\bar{\psi}}$ decreases monotonically for each $N$ as shown in Fig.2. But the scalar curvatures are constants except for a point $\psi = 1$.

Figure 2: Ricci Tensor $R_{\psi\bar{\psi}}$ for torus ($N = 3$) and K3 ($N = 4$) cases. The axis of abscissa represents the value of the $\psi \in \mathbb{R}$. The axis of ordinate corresponds to the Ricci Tensor $R_{\psi\bar{\psi}}$. The point $\psi = 1$ is a singular point where the $R_{\psi\bar{\psi}}$s blow up. But the scalar curvatures are constants except for a point $\psi = 1$.

5 Two-Point Functions

The other constituent blocks of the topological model are three-point couplings $\{\kappa_\ell\}$ and two-point functions. Each A-model operator $O^{(\ell)}$ is associated with a cohomology element
$e_\ell \in H^{2\ell}(M)$. The $K_\ell$ is a fusion coupling of $O^{(1)}$ and $O^{(\ell)}$

\[
O^{(1)} \cdot O^{(\ell)} = K_\ell O^{(\ell+1)} \quad (\ell = 0, 1, \ldots, d - 1) .
\]

$O^{(1)} \cdot O^{(d)} = 0 ,$

$K_0 = 1 ,$

$K_\ell = \partial \frac{1}{K_{\ell-1}} \partial \frac{1}{K_{\ell-2}} \cdots \partial \frac{1}{K_1} \partial \frac{1}{K_0} \partial S_{\ell+1}(t, x_2, x_3, \ldots, x_{\ell+1})$

\[
= 1 + O(q) , \quad (1 \leq \ell \leq d - 1) .
\]

Also the topological metric $\eta_{\ell,m}$ is given as

\[
\eta_{\ell,m} = \langle O^{(\ell)} O^{(m)} \rangle = N \delta_{\ell+m,d} .
\]

On the other hand, the hermitian two-point functions $\langle \bar{O}^{(\ell)} | O^{(m)} \rangle$ are calculated by using a method of the $tt^*$-fusion. In our case, the correlators have diagonal forms and are given as

\[
\langle \bar{O}^{(\ell)} | O^{(m)} \rangle = e^{\tilde{q}_\ell} \delta_{\ell,m} \quad (0 \leq \ell \leq d , 0 \leq m \leq m) ,
\]

$\tilde{q}_0 = q_0 ,$

$\tilde{q}_\ell = q_\ell + \sum_{n=0}^{\ell-1} \log |K_n|^2 \quad (\ell \geq 1) ,$

$\partial \partial \tilde{q}_0 + e^{\tilde{q}_1 - \tilde{q}_0} = 0 ,$

$\partial \partial \tilde{q}_\ell + e^{\tilde{q}_{\ell+1} - \tilde{q}_\ell} - e^{\tilde{q}_\ell - \tilde{q}_{\ell-1}} = 0 \quad (1 \leq \ell \leq d - 1) ,$

$\partial \partial \tilde{q}_d - e^{\tilde{q}_d - \tilde{q}_{d-1}} = 0 .$

In the large radius limit, the $\tilde{q}_\ell$s behave as

\[
e^{\tilde{q}_0} = e^{-K} = (-1)^d \left( \frac{2\pi i}{N} \right)^d \cdot \frac{1}{N^2} \cdot \frac{(t - \ell)^d}{d!} + \cdots ,
\]

$e^{\tilde{q}_{\ell+1} - \tilde{q}_\ell} = - \frac{(\ell + 1)(d - \ell)}{(t - t)^2} + \cdots \quad (0 \leq \ell \leq d - 1) .
\]

In this case, any corrections are suppressed and normalized two-point functions are inversely proportional to the $\text{Im}(t)^{2\ell}$ $(\ell = 0, 1, 2, \ldots, d)$

\[
\frac{\langle \bar{O}^{(\ell)} | O^{(\ell)} \rangle}{\langle \bar{O}^{(0)} | O^{(0)} \rangle} = e^{\tilde{q}_\ell - \tilde{q}_0} = (-1)^\ell (\ell!)^2 \left( \frac{d}{\ell} \right) \frac{1}{(t - t)^{2\ell}} + \cdots \quad (0 \leq \ell \leq d) .
\]

In the finite $\text{Im}(t)$ case, there appear corrections in power series and non-perturbative corrections of exponential types. Those are abbreviated as a symbol “...” in the above formula.
Even in that generic case, these $q$s are described by combining the curvature $R$, the metric $g_{\bar{t}t}$ and the $K$

\[
e^{-\tilde{q}_0} = e^{-K}, \quad e^{\tilde{q}_1-\tilde{q}_0} = g_{\bar{t}t}, \quad e^{\tilde{q}_2-\tilde{q}_1} = g_{\bar{t}t} \left( \frac{R}{2} + 2 \right), \]
\[
e^{\tilde{q}_3-\tilde{q}_2} = g_{\bar{t}t} \left[ 3 \left( \frac{R}{2} + 1 \right) - g^{\bar{t}t} \partial_\bar{t} \partial_t \log \left( \frac{R}{2} + 2 \right) \right], \]
\[
e^{\tilde{q}_4-\tilde{q}_3} = g_{\bar{t}t} \left[ 4 + 3R - 2g^{\bar{t}t} \partial_\bar{t} \partial_t \log \left( \frac{R}{2} + 2 \right) - g^{\bar{t}t} \partial_\bar{t} \partial_t \log \left( \frac{R}{2} + 2 \right) \right] ,
\]
\[
\ldots .
\]

We know the formula of the $K$, $R$, and $g_{\bar{t}t}$ and can evaluate moduli dependences of these correlators.

6 Quintic

In this section, we investigate the $K$ for a quintic $M$ case and compare our results with those by Candelas et al[3]. The set of periods $\{\Omega_\ell\}$ of an associated mirror $W$ is expressed by using the $\varpi$

\[
\varpi \left( \frac{\rho}{2\pi i} \right) \sqrt{K(\rho)} =: \sum_{\ell \geq 0} \rho^\ell \Omega_\ell ,
\]
\[
\sqrt{A(\rho)} = 1 + \frac{5}{12} \rho^2, \quad \sqrt{\tilde{K}(\rho)} = 1 - \frac{40}{(2\pi i)^3} \zeta(3) \rho^3 ,
\]
\[
\hat{c} = \frac{40}{(2\pi i)^3} \zeta(3),
\]
\[
\Omega = \begin{pmatrix}
\Omega_0 \\
\Omega_1 \\
\Omega_2 \\
\Omega_3 
\end{pmatrix} = \begin{pmatrix}
1 \\
t \\
\frac{1}{2}t^2 + S_2(0, x_2) \\
\frac{1}{6}t^3 - \hat{c} + t S_2(0, x_2) + S_3(0, x_2, x_3)
\end{pmatrix} .
\] (5)

The prepotential $F$ of $M$ is expressed as a sum of a polynomial part of $t$ and a non-perturbative part $f$

\[
F = -\frac{\kappa}{6} t^3 + \frac{1}{2} a t^2 + b t + \frac{1}{2} c + f,
\]
\[
a = \frac{-11}{2}, \quad b = \frac{25}{12}, \quad c = \frac{c_3 \zeta(3)}{(2\pi i)^3} = -5 \hat{c}, \quad c_3 = -200 .
\]
The effects of the instantons are encoded in this function $f$. The $F$ leads to a a canonical set of basis $\{\Pi_\ell\}$. The $\Pi_\ell$s are represented as some linear combinations of periods

\[
\Pi = \begin{pmatrix} \Pi_0 \\ \Pi_1 \\ \Pi_2 \\ \Pi_4 \end{pmatrix} = \begin{pmatrix} 1 \\ t \\ \partial_t F \\ t\partial_t F - 2F \end{pmatrix} = \mathcal{N}\Omega,
\]

(6)

\[
\mathcal{N} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b & a & -\kappa & 0 \\ 0 & -b & 0 & -\kappa \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{25}{12} & -\frac{11}{2} & -5 & 0 \\ 0 & -\frac{25}{12} & 0 & -5 \end{pmatrix},
\]

$\kappa = 5$, $a = -11/2$, $b = 25/12$.

By comparing these two approaches Eqs. (5), (6), we obtain relations for the $c$ and the $f$

\[
c = -\kappa\hat{c} = \frac{c_3\zeta(3)}{(2\pi i)^3},
\]

\[
f = \frac{\kappa}{2}S_3(0, x_2, x_3),
\]

\[
\partial_t f = -\kappa S_2(0, x_2).
\]

Then the Kähler potential $K$ in the A-model is evaluated by using the $F$ up to a overall normalization factor with a suitable choice of a section of an associated line bundle

\[
e^{-K} = (t - \bar{t})(\partial F + \bar{\partial} F) - 2(F - \bar{F})
\]

\[
= -\frac{\kappa}{6}(t - \bar{t})^3 + (\bar{a} - a)t\bar{t} - (b - \bar{b})(t - \bar{t}) - (c - \bar{c}) + (t - \bar{t})(\partial f + \bar{\partial} f) - 2(f - \bar{f})
\]

\[
= -\frac{\kappa}{6}(t - \bar{t})^3 - 2c + (t - \bar{t})(\partial f + \bar{\partial} f) - 2(f - \bar{f}),
\]

\[
= -\kappa \left[ \frac{1}{6}(t - \bar{t})^3 - 2\hat{c} + (t - \bar{t})(S_2 + \bar{S}_2) + (S_3 - \bar{S}_3) \right]
\]

\[
= (-1)^3 \int_{C_{Y_3}} \left[ \hat{K}(\rho) \cdot \overline{\varphi(-\frac{\rho}{2\pi i})} \cdot \varphi(+\frac{\rho}{2\pi i}) \right] \bigg|_{\rho=[D]} ,
\]

\[
\int_{C_{Y_3}} [D] : [D] : [D] = \kappa.
\]

Here we used the facts the $a$ and $b$ are real numbers and the $c$ is a pure imaginary number. This formula coincides with our formula Eq.(4). Now we make a remark here: The basis $\Omega$ does not coincide with the $\Pi$, but it is related to the $\Pi$ by a kind of a symplectic transformation, that is represented as a matrix $\mathcal{N}$

\[
\Pi = \mathcal{N}\Omega , \quad \mathcal{N}^\dagger \Sigma \mathcal{N} = (-5) \cdot \Sigma.
\]

It leads to the same $K$ up to a multiplicative factor because the following relation is satisfied

\[
e^{-K} \sim \Pi^\dagger \Sigma \Pi = \Omega^\dagger \mathcal{N}^\dagger \Sigma \mathcal{N} \Omega = (-5) \cdot \Omega^\dagger \Sigma \Omega.
\]
7 Generalization

In this section, we propose a formula of the $K$ by generalizing the previous result of the Fermat type. We consider complete intersections $M$ of $\ell$ hypersurfaces $\{p_j = 0\}$ in products of $k$ projective spaces $M$

$$M := \left( \begin{array}{c} \mathbb{P}^{n_1}(w_1^{(1)}, \ldots, w_{n_1+1}^{(1)}) \vline d_1^{(1)} \cdots d_\ell^{(1)} \\ \vdots \vline \vdots \\ \mathbb{P}^{n_k}(w_1^{(k)}, \ldots, w_{n_{k+1}}^{(k)}) \vline d_1^{(k)} \cdots d_\ell^{(k)} \end{array} \right).$$

The $d_j^{(i)}$ are degrees of the coordinates of $\mathbb{P}^{n_i}(w_1^{(i)}, \ldots, w_{n_{i+1}}^{(i)})$ in the $j$-th polynomial $p_j$ $(i = 1, 2, \ldots, k; j = 1, 2, \ldots, \ell)$. We propose a formula of the Kähler potential of the A-model up to some overall normalization factors

$$e^{-K} = (-1)^d \int_M \left[ \tilde{K}(\lambda) \cdot \frac{\lambda[D_i]}{2\pi i} ; z_k \right] \cdot \frac{\lambda[D_i]}{2\pi i} ; z_k \right] \times |\omega_0|^{-2}$$

$$= (-1)^d \int_M \left[ \tilde{K}(\lambda) \cdot \frac{\lambda[D_i]}{2\pi i} ; z_k \right] \cdot \frac{\lambda[D_i]}{2\pi i} ; z_k \right] \cdot [H] \times |\omega_0|^{-2},$$

$$\omega(v; z) := \sum_n \frac{a(n + v)}{a(v)} z^{n+v}, \quad z^{n+v} := \prod_{k=1}^p z_k^{n_k+v_k},$$

$$a(n + v) := \frac{\prod_{j=1}^\ell \Gamma \left( 1 + \sum_{i=1}^k d_j^{(i)}(n_i + v_i) \right)}{\prod_{i=1}^k \prod_{j'=1}^\ell \Gamma \left( 1 + w_j^{(i)}(n_i + v_i) \right)},$$

$$\tilde{K}(\lambda) := \frac{a \left( + \frac{\lambda[D_i]}{2\pi i} \right)}{a \left( - \frac{\lambda[D_i]}{2\pi i} \right)} = \exp \left[ +2 \sum_{m=1}^{\ell} \frac{\zeta(2m+1)}{2m+1} \cdot \left( \frac{\lambda}{2\pi i} \right)^{2m+1} \cdot X_{2m+1} \right],$$

$$X_n := \sum_{i=1}^k \sum_{j'=1}^{n_i+1} (w_j^{(i)}[D_i])^n - \sum_{j=1}^\ell \left( \sum_{i=1}^k d_j^{(i)}[D_i] \right)^n, \quad [H] = \frac{\prod_{j=1}^\ell \left( \sum_{i=1}^k d_j^{(i)}[D_i] \right)}{\prod_{i=1}^k \prod_{j'=1}^{n_i+1} w_j^{(i)}},$$

$$d = -\ell + \sum_{i=1}^k n_i, \quad \text{ (dimension)}.$$

The $[D_i]$s $(i = 1, 2, \ldots, k)$ are Poincaré duals of divisors "$D_i$" of the model. We will rewrite these in order to interpret them in the A-model language. First mirror maps are given as

$$2\pi i t^i = \log z^i + \partial_{z_i} \log \tilde{\omega},$$

$$\tilde{\omega} = \sum_{n=0}^{\infty} \frac{a(n + v)}{a(v)} z^n.$$
Then the normalized $\omega$ is expressed as
\[
\omega_0^{-1} \omega \left( \frac{\lambda[D_i]}{2\pi i} \right) = \exp \left( \lambda[D_i] \cdot t^i + \sum_{\ell=2}^{\infty} \lambda^\ell x_\ell \right),
\]
where
\[
x_\ell = \frac{1}{(2\pi i)^\ell \ell!} ([D] \cdot \partial)^\ell \log \omega,
\]
\[
[D] \cdot \partial := \sum_i [D_i] \frac{\partial}{\partial v_i},
\]
\[
\hat{K}(\lambda) = \exp \left( \sum_{m=1}^{\infty} \lambda^{2m+1} \hat{k}_{2m+1} \right),
\]
\[
\hat{k}_{2m+1} = 2 \cdot \frac{X_{2m+1}}{2m+1} \zeta(2m+1) \frac{(2\pi i)^{2m+1}}{2m+1}.
\]

By using the above relations, we can express the Kähler potential in a compact form
\[
e^{-K} = (-1)^d \int_M S_d(\tilde{y}_1, \tilde{y}_2, \cdots, \tilde{y}_d) \cdot [H],
\]
\[
\tilde{y}_1 = \sum_i [D_i](t^i - \bar{t}^i),
\]
\[
\tilde{y}_{2n} = x_{2n} + \bar{x}_{2n}, \quad (n \geq 1),
\]
\[
\tilde{y}_{2n+1} = x_{2n+1} - \bar{x}_{2n+1} + \hat{k}_{2n+1}, \quad (n \geq 1).
\]

To confirm the validity of this formula, we restrict ourselves to the 3-fold case and consider the $K$. First let us consider a general type of Calabi-Yau 3-fold $M$ with Kähler parameters $t^i$ ($i = 1, 2, \cdots, h^{1,1}$). Its prepotential $F$ is described as
\[
F = -\frac{1}{6} K_{ijk} t^i t^j t^k + \frac{1}{2} a_{ij} t^i t^j + b_i t^i + \frac{1}{2} c + f,
\]
\[
e^{-K} = (t^i - \bar{t}^i)(\partial_i F + \bar{\partial}_i \bar{F}) - 2(F - \bar{F})
\]
\[
= -\frac{1}{6} K_{ijk} (t^i - \bar{t}^i)(t^j - \bar{t}^j)(t^k - \bar{t}^k) - (a_{ij} - \bar{a}_{ji}) t^i \bar{t}^j
\]
\[
- (b_i - \bar{b}_i)(t^i - \bar{t}^i) - (c - \bar{c}) + (t^i - \bar{t}^i)(\partial_i f + \bar{\partial}_i \bar{f}) - 2(f - \bar{f}).
\]

When we impose a condition that the $e^{-K}$ is invariant under a constant shift of $t^i \rightarrow t^i + 1$ for an arbitrary $i$, the term $(a_{ij} - \bar{a}_{ji}) t^i \bar{t}^j$ must vanish. It means that the $a_{ij}$s must be hermitian as components of a matrix
\[
\bar{a}_{ij} = a_{ji}.
\]

Also the $b_i$s are related to a second Chern class $c_2$ of the $M$
\[
b_i = \frac{1}{24} \int_M c_2 \wedge J_i = \int_M \sqrt{A} \wedge J_i,
\]
and they are real numbers. On the other hand, the $c$ is a pure imaginary number and is associated with a 3rd Chern class
\[
c = \frac{c_3 \zeta(3)}{(2\pi i)^3}.
\]
Collecting all these facts, we can rewrite the $K$

$$e^{-K} = -\frac{1}{6} \kappa_{ijk} (t^i - \bar{t}^i)(t^j - \bar{t}^j)(t^k - \bar{t}^k) - 2c$$

$$+ (t^i - \bar{t}^i)(\partial_i f + \partial_i \bar{f}) - 2(f - \bar{f}).$$

The Eq. (8) is obtained by using a prepotential $F$ for a 3-fold. In contrast, we can obtain our result for the formula $K$ without requiring an existence of the prepotential. Let us write down our result Eq. (7) in the 3 dimensional case. All we have to do is to evaluate the $S_3$ in this case

$$S_3 = \frac{1}{6} [D_i][D_j][D_k](t - \bar{t})^i(t - \bar{t})^j(t - \bar{t})^k + [D_i](t - \bar{t})^i(x_2 + \bar{x}_2) + (x_3 - \bar{x}_3) + \hat{k}_3.$$  

Then the $K$ is obtained

$$e^{-K} = -\frac{1}{6} \kappa_{ijk} (t - \bar{t})^i(t - \bar{t})^j(t - \bar{t})^k$$

$$+ (t - \bar{t})^i(\partial_i f + \partial_i \bar{f}) - 2(f - \bar{f}) - 2\frac{c_3 \zeta(3)}{(2\pi i)^3},$$

(9)

$$K_{ijk} = \int_M [D_i][D_j][D_k] \cdot [H],$$

$$\partial_i f = -\frac{1}{2} \frac{1}{(2\pi i)^2} K_{ijk} \frac{\partial}{\partial v_j} \frac{\partial}{\partial v_k} \log \hat{\omega},$$

$$f = \frac{1}{12} \frac{1}{(2\pi i)^3} K_{ijk} \frac{\partial}{\partial v_i} \frac{\partial}{\partial v_j} \frac{\partial}{\partial v_k} \log \hat{\omega},$$

$$c_3 = \frac{1}{3} \left[ \sum_i \sum_{j'} (w_{j'}^{(i)})^3 K_{iii} - \sum_{i,j,k} \sum_{j'} d_{j'}^{(i)} d_{j'}^{(j)} d_{j'}^{(k)} K_{ijk} \right].$$

This formula Eq. (9) coincides with that of the 3-fold case, Eq. (8). But Eq. (7) is not restricted to this 3 dimensional case. In fact, Eq. (7) is applicable to smooth $d$-folds cases because the periods $\sqrt{K} \cdot \omega$ have appropriate intersection forms

$$\hat{K}(\lambda) \cdot \omega \left( -\frac{\lambda [D_i]}{(2\pi i)} \right) \cdot \omega \left( +\frac{\lambda [D_i]}{(2\pi i)} \right) \bigg|_{\lambda^d} = \sum_{n,n'} \bar{\Omega}_n \cdot \Sigma_{n,n'} \cdot \Omega_{n'},$$

$$\Omega_m = \frac{1}{m!} \frac{1}{(2\pi i)^m} ([D] \cdot \partial)^m \left[ \sqrt{K} \cdot \omega \right], \Sigma_{n,n'} = (-1)^n \delta_{n+n',d}.$$ 

For singular $d$-fold case, we might have to modify some parts which have information about intersection numbers associated with the divisors.

8 Conclusions and Discussions

In this article, we develop a method to calculate the Kähler potential in the topological A-model. Generally the Kähler potential $K$ is represented as $\sim (t - \bar{t})^d$ when no quantum
corrections exist. But it is known that there are loop corrections in the two dimensional
$N = 2$ non-linear sigma models with Calabi-Yau target spaces. A term $\sim (t - \bar{t})^{d-3} \times \frac{c_3 \zeta(3)}{(2\pi i)^3}$
in the $K$ reflects a perturbative correction at the four loop calculation. The $c_3$ is the 3rd
Chern class of the CY. In our case, the $\hat{K}$ seems to describe loop corrections to the sigma
model. In general, there might be more corrections and they could be controlled by one
function $\hat{K}$. It is interesting to give a field theoretical interpretation to these $\hat{K}_n$s from the
point of view of direct higher loop calculations.

The basis we pick here is a symplectic (or an SO-invariant) basis with intersection matrix
$\Sigma$. But it is not integrable but rational basis. In order to obtain a set of canonical basis, we
have to do some linear transformation on the $\Omega$. It is needed to discuss the D-branes charges
or central charges in BPS mass formulae. We will study these topics in the next paper.

**Acknowledgment**

This work is supported by the Grant-in-Aid for Scientific Research from the Ministry of
Education, Science and Culture 10740117.
A Examples of $\hat{K}$

We write down several concrete examples of the $\hat{K}_n$ for the $d$-fold $M$. First the generating function $\hat{K}(\rho)$ is defined by using Riemann’s zeta functions

$$\hat{K}(\rho) = \exp \left( 2 \sum_{m=1}^{\infty} \frac{N - N^{2m+1}}{2m+1} \zeta(2m+1) \cdot \left( \frac{\rho}{2\pi i} \right)^{2m+1} \right)$$

$$= 1 + \sum_{n=1}^{d} \hat{K}_n \cdot \left( \frac{\rho}{2\pi i} \right)^n, \quad N = d + 2.$$ 

The coefficients in the series expansion are represented as some combinations of Chern classes of the $M$

$$\hat{K}_1 = \hat{K}_2 = 0, \quad \hat{K}_3 = \frac{2}{N} c_3 \zeta(3), \quad \hat{K}_4 = 0,$$

$$\hat{K}_5 = \left( \frac{2}{N} c_5 - \frac{2}{N^2} c_3 c_2 \right) \zeta(5), \quad \hat{K}_6 = \frac{2}{N^2} c_3^2 \zeta(3)^2,$$

$$\hat{K}_7 = \left( \frac{2}{N} c_7 - \frac{2}{N^2} c_5 c_2 - \frac{2}{N^2} c_4 c_3 + \frac{2}{N^3} c_3 c_2 \right) \zeta(7),$$

$$\hat{K}_8 = \left( \frac{4}{N^2} c_5 c_3 - \frac{4}{N^3} c_3^2 c_2 \right) \zeta(3) \zeta(5).$$

A finite number of $\hat{K}_n$s ($n \leq d$) appear for the $d$-fold case. The $\hat{K}_1$ and $\hat{K}_2$ always vanish and the $\hat{K}(\rho)$ is identity for the torus and K3 cases ($N = 3, 4$ respectively). We will summarize several examples for lower dimensional cases

$$\hat{K}(N = 3) = 1, \quad \hat{K}(N = 4) = 1,$$

$$\hat{K}(N = 5) = 1 - 80 \zeta(3) \left( \frac{\rho}{2\pi i} \right)^3, \quad \hat{K}(N = 6) = 1 - 140 \zeta(3) \left( \frac{\rho}{2\pi i} \right)^3,$$

$$\hat{K}(N = 7) = 1 - 224 \zeta(3) \left( \frac{\rho}{2\pi i} \right)^3 - 6720 \zeta(5) \left( \frac{\rho}{2\pi i} \right)^5,$$

$$\hat{K}(N = 8) = 1 - 336 \zeta(3) \left( \frac{\rho}{2\pi i} \right)^3 - 13104 \zeta(5) \left( \frac{\rho}{2\pi i} \right)^5 + 56448 \zeta(3)^2 \left( \frac{\rho}{2\pi i} \right)^6,$$

$$\hat{K}(N = 9) = 1 - 480 \zeta(3) \left( \frac{\rho}{2\pi i} \right)^3 - 23616 \zeta(5) \left( \frac{\rho}{2\pi i} \right)^5 + 115200 \zeta(3)^2 \left( \frac{\rho}{2\pi i} \right)^6 - 1366560 \zeta(7) \left( \frac{\rho}{2\pi i} \right)^7,$$

$$\hat{K}(N = 10) = 1 - 660 \zeta(3) \left( \frac{\rho}{2\pi i} \right)^3 - 39996 \zeta(5) \left( \frac{\rho}{2\pi i} \right)^5 + 217800 \zeta(3)^2 \left( \frac{\rho}{2\pi i} \right)^6 - 2857140 \zeta(7) \left( \frac{\rho}{2\pi i} \right)^7 + 26397360 \zeta(3) \zeta(5) \left( \frac{\rho}{2\pi i} \right)^8.$$
References

[1] E. Witten, *Mirror Manifolds and Topological Field Theory*, in *Essays on Mirror Manifolds*, ed. S.-T. Yau, (Int. Press, Hong Kong, 1992), pp.120-180.

P. Aspinwall and D. Morrison, Commun. Math. Phys. **151** (1993) 245.

[2] B. Greene and M. Plesser, Nucl. Phys. **B338** (1990) 15.

P. Candelas, M. Lynker and R. Schimmrigk, Nucl. Phys. **B341** (1990) 383.

[3] P. Candelas, X. de la Ossa, P. Green and L. Parkes, Phys. Lett. **258B** (1991) 118;

Nucl. Phys. **B359** (1991) 21.

[4] B. Greene, D. Morrison and M. Plesser, Commun. Math. Phys. **173** (1995) 559, hep-th/9402119.

[5] M. Nagura and K. Sugiyama, Int. J. Mod. Phys. **A10** (1995) 233, hep-th/9312159.

[6] K. Sugiyama, Ann. Phys. **247** (1996) 106, hep-th/9410184.

[7] K. Sugiyama, Int. J. Mod. Phys. **A11** (1996) 229, hep-th/9504114.

[8] K. Sugiyama, Nucl. Phys. **B459** (1996) 693, hep-th/9504115.

[9] K. Sugiyama, Nucl. Phys. **B537** (1999) 599, hep-th/9805058.

[10] K. Sugiyama, “Kähler Potential of Moduli Space of Calabi-Yau $d$-fold embedded in $CP^{d+1}$”.

[11] S. Cecotti and C. Vafa, Nucl. Phys. **B367** (1991) 359.

[12] S. Cecotti, Int. J. Mod. Phys. **A6** (1991) 1749,

S. Cecotti, Nucl. Phys. **B355** (1991) 755.

[13] V. Batyrev and D. Dais, Topology **35** (1996) 901.