Refinements and Sharpening of some Huygens and Wilker Type Inequalities

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Abstract In the article, some Huygens and Wilker type inequalities involving trigonometric and hyperbolic functions are refined and sharpened.

Keywords: refinement, sharpening, Huygens inequality, Wilker inequality, trigonometric function, hyperbolic function

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1. Introduction

The famous Huygens inequality for the sine and tangent functions states that for \( x \in \left( 0, \frac{\pi}{2} \right) \)

\[
2 \sin x + \tan x > 3x. \tag{1.1}
\]

The hyperbolic counterpart of (1.1) was established in [10] as follows: For \( x > 0 \)

\[
2 \sinh x + \tanh x > 3x. \tag{1.2}
\]

The inequalities (1.1) and (1.2) were respectively refined in [10], Theorem 2.6 as

\[
2 \sin x + \tan x > 2 \frac{x}{\sin x} + \frac{x}{\tan x} > 3 \tag{1.3}
\]

for \( 0 < x < \frac{\pi}{2} \) and

\[
2 \sinh x + \tanh x > 2 \frac{x}{\sinh x} + \frac{x}{\tanh x} > 3, x \neq 0. \tag{1.4}
\]

In [8], the inequality (1.2) was improved as

\[
2 \sinh x + \tanh x > 3 + \frac{3}{20} x^4 - \frac{3}{56} x^6, x > 0. \tag{1.5}
\]

In [10], the following inequality is given

\[
\frac{3}{\sin x} + \cos x > 4. \tag{1.6}
\]

For more information in this area, please refer to [13,16], [14], Section 1.7 and Section 7.3] and closely related references therein.

In [18], Wilker proved

\[
\left( \sin \frac{x}{x} \right)^2 + \tan \frac{x}{x} > 2 \tag{1.7}
\]

and proposed that there exists a largest constant \( c \) such that

\[
\left( \sin \frac{x}{x} \right)^2 + \tan \frac{x}{x} > 2 + cx^3 \tan x \tag{1.8}
\]

for \( 0 < x < \frac{\pi}{2} \). In [17], the best constant \( c \) in (1.7) was found and it was proved that

\[
2 + \frac{8}{45} x^3 \tan x > \left( \sin \frac{x}{x} \right)^2 + \tan \frac{x}{x} > 2 + \left( \frac{2}{\pi} \right)^4 x^3 \tan x
\]

for \( 0 < x < \frac{\pi}{2} \). The constants \( \frac{8}{45} \) and \( \left( \frac{2}{\pi} \right)^4 \) in the above inequality are the best possible. For more information on this topic, please see [4,5,19], [[14], pp. 38-40, Section 8] and closely related references therein.

Recently the inequalities (1.3) and (1.6) were respectively refined in [9] as

\[
2 + \sin x + \tan x > \sin \frac{x}{x} + 2 \frac{\tan \left( x/2 \right)}{x/2} > \frac{x}{\sin x} + \frac{x}{\tan x} > 3
\]

and

\[
\left( \sin \frac{x}{x} \right)^2 + \tan \frac{x}{x} > \left( \frac{x}{\sin x} \right)^2 + \frac{x}{\tan x} > \frac{\sin \left( x+ x/2 \right)}{\sin x} + \left( \frac{x}{\tan \left( x/2 \right)} \right)^2 > 2. \tag{1.8}
\]
The hyperbolic counterparts of the last two inequalities in (1.8) were also given in [9] as follows:

\[
\frac{\sinh x}{x} + \frac{\tanh(x/2)^2}{x/2} > x \frac{\sinh x}{\sinh x} + \frac{x/2}{\tanh(x/2)^2} > 2.
\]

The aim of this paper is to refine and sharpen some of the above-mentioned Huygens and Wilker type inequalities.

2. Some Lemmas

In order to attain our aim, we need several lemmas below.

**Lemma 2.1.** The Bernoulli numbers \(B_{2n}\) for \(n \in \mathbb{N}\) have the property

\[
(-1)^{n-1}B_{2n} = [B_{2n}], \tag{2.1}
\]

where the Bernoulli numbers \(B_i\) for \(i \geq 0\) are defined by

\[
\frac{x}{e^x - 1} = \sum_{i=0}^{\infty} \frac{B_i}{i!} x^i = \frac{1}{2} x + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!} x^{2i} < 2 \pi.
\]

**Proof.** In [[2], p. 16 and p. 56], it is listed that for \(q \geq 1\)

\[
\zeta(2q) = (-1)^{q-1} \frac{\pi^{2q}}{2^{2q}(2q)!} B_{2q}, \tag{2.2}
\]

where \(\zeta\) is the Riemann zeta function defined by

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2.3}
\]

From (2.2), the formula (2.1) follows.

**Lemma 2.2.** For \(0 < |x| < \pi\), we have

\[
\frac{x}{\sin x} = 1 + \sum_{n=1}^{\infty} \frac{2(2^{2n-1}-1)[B_{2n}]}{(2n)!} x^{2n}. \tag{2.4}
\]

**Proof.** This is an easy consequence of combining the equality

\[
\frac{1}{\sin x} = \csc x = \frac{1}{\sin x} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2 (2^{2n-1}-1) B_{2n}}{(2n)!} x^{2n-1}, \tag{2.5}
\]

see [[1], p. 75, 4.3.68], with Lemma 2.1.

**Lemma 2.3** ([[1], p. 75, 4.3.70]). For \(0 < |x| < \pi\),

\[
\cot x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} x^{2n-1}. \tag{2.6}
\]

**Lemma 2.4.** For \(0 < |x| < \pi\),

\[
\frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n} (2n-1) B_{2n}}{(2n)!} x^{2(n-1)}. \tag{2.7}
\]

**Proof.** Since

\[
\frac{1}{\sin^2 x} = \csc^2 x = - \frac{d}{dx} (\cot x),
\]

the formula (2.7) follows from differentiating (2.6).

**Lemma 2.5.** For \(0 < |x| < \pi\),

\[
\frac{\cos x}{\sin^2 x} = \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{2(2n-1)(2^{2n-1}-1) B_{2n}}{(2n)!} x^{2(n-1)}. \tag{2.8}
\]

**Proof.** This follows from differentiating on both sides of (2.5) and using (2.1).

**Lemma 2.6.** For \(0 < |x| < \pi\),

\[
\frac{1}{\sin^3 x} = \frac{1}{2x^3} + \sum_{n=1}^{\infty} \frac{1}{2^n(2n-1)!} \left[ \frac{(2^{2n+1}-1) B_{2n+2}}{n+1} \right] x^{2n-1} + \sum_{n=2}^{\infty} \frac{1}{2^n n!} x^{2n-3}. \tag{2.9}
\]

**Proof.** Combining

\[
\frac{1}{\sin^3 x} = \frac{1}{2x^3} - \frac{1}{2(\sin^2 x)^{3/2}}
\]

with Lemma 2.5, the identity (2.5), and Lemma 2.1 gives (2.9).

The equality (2.10) follows from combination of

\[
\frac{\cos x}{\sin x} = \frac{1}{x^2} \left( 1 - \frac{1}{(\sin^2 x)^{1/2}} \right)
\]

with Lemma 2.4.

**Lemma 2.7.** Let \(f\) and \(g\) be continuous on \([a,b]\) and differentiable in \((a,b)\) such that \(g'(x) \neq 0\) in \((a,b)\). If \(f'/g'(x)\) is increasing (or decreasing) in \((a,b)\), then the functions \(f(x)/g(x)\) and \(f(x)-g(x)\) are also increasing (or decreasing) in \((a,b)\).

The above Lemma 2.7 can be found, for examples, in [[3], p. 292, Lemma 1], [[6], p. 57, Lemma 2.3], [[11], p. 92, Lemma 1], and [[12], p. 161, Lemma 2.3].

**Lemma 2.8.** Let \(a_k\) and \(b_k\) for \(k \in \mathbb{N}\) be real numbers and the power series

\[
A(x) = \sum_{k=1}^{\infty} a_k x^k \quad \text{and} \quad B(x) = \sum_{k=1}^{\infty} b_k x^k \tag{2.11}
\]

be convergent on \((-R, R)\) for some \(R > 0\). If \(b_k > 0\) and the ratio \(\frac{a_k}{b_k}\) is strictly increasing for \(k \in \mathbb{N}\), then the function \(\frac{A(x)}{B(x)}\) is also strictly increasing on \((0, R)\).

The above Lemma 2.8 can be found, for examples, in [[3], p. 292, Lemma 2], [[15], p. 71, Lemma 1], and [[20], Lemma 2.2].
3. Main Results

Now we are in a position to state and prove our main results, refinements and sharpening of some Huygens and Wilker type inequalities mentioned in the first section.

**Theorem 3.1.** For \( |x| \in \left(0, \frac{\pi}{2}\right]\), we have
\[
3 + \frac{1}{60} x^3 \sin x < 2 - \frac{x}{\tan x} < 3 + \frac{8\pi - 24}{\pi^3} x^3 \sin x. \quad (3.1)
\]
The scalars \( \frac{1}{60} \) and \( \frac{8\pi - 24}{\pi^3} \) in (3.1) are the best possible.

**Proof.** Let
\[
f(x) = \frac{2x}{\sin x} - \frac{x}{\tan x} - \frac{2x}{\sin^2 x} + \frac{x \cos x}{\sin^2 x} - \frac{3}{x^3}
\]
for \( x \in \left(0, \frac{\pi}{2}\right] \). By virtue of (2.4), (2.7), and (2.8), we have
\[
\begin{align*}
f(x) &= \frac{1}{3} \left( \sum_{n=1}^{\infty} \frac{2^{2n+1} (2n-1)}{(2n)!} \left| B_{2n} \right| x^{2n-1} - \sum_{n=1}^{\infty} \frac{2^{2n-2}}{(2n)!} \left| B_{2n} \right| x^{2n-1} - \sum_{n=1}^{\infty} \frac{(2n-2)(2n-1)}{(2n)!} \left| B_{2n} \right| x^{2n-1} \right) \\
&= \sum_{n=2}^{\infty} \frac{(n-1)2^{2n+1} + 4(n+1)}{(2n)!} \left| B_{2n} \right| x^{2n-4}. \\
\end{align*}
\]
So the function \( f(x) \) is strictly increasing on \( \left(0, \frac{\pi}{2}\right] \).

Moreover, it is easy to obtain
\[
\lim_{x \to 0^+} f(x) = \frac{1}{60} \quad \text{and} \quad \lim_{x \to (\pi/2)^-} f(x) = \frac{8\pi - 24}{\pi^3}.
\]
The proof of Theorem 3.1 is complete.

**Theorem 3.2.** For \( 0 < |x| < \pi/2 \),
\[
\begin{align*}
2 + \frac{17}{270} x^3 \sin x < \frac{x}{\sin x} + \left[ \frac{x/2}{\tan (x/2)} \right]^2 < 2 + \frac{\pi^2 + 8\pi - 32}{2\pi^3} x^3 \sin x. \\
\end{align*}
\]
The constants \( \frac{17}{270} \) and \( \frac{\pi^2 + 8\pi - 32}{2\pi^3} \) in (3.2) are the best possible.

**Proof.** By using (2.4), (2.7), (2.9), and (2.10), the function
\[
g(x) = \frac{x + \left[ \frac{x/2}{\tan (x/2)} \right]^2}{x^3 \sin x}
\]
may be expanded as
\[
\begin{align*}
g(x) &= 1 + \sum_{n=1}^{\infty} \frac{2^{2n+1} (2n-1)! \left| B_{2n} \right| x^{2n-1}}{(2n)!} \\
&\quad - \sum_{n=1}^{\infty} \frac{2^{2n-2} (2n-1)(2n-1)! \left| B_{2n} \right| x^{2n-1}}{(2n)!} \\
&\quad + \sum_{n=1}^{\infty} \frac{(2n-2)(2n-1)(2n-1)! \left| B_{2n} \right| x^{2n-1}}{(2n)!} \\
&\quad - \sum_{n=1}^{\infty} \frac{8(2n-2)(2n-2)! \left| B_{2n} \right| x^{2n-1}}{(2n)!} \\
&\quad + \sum_{n=1}^{\infty} \frac{3n-2n! \left| B_{2n} \right| x^{2n-4}}{(2n)!} \\
&\quad \triangleq \sum_{n=2}^{\infty} \frac{b_n}{(2n)!} \left| B_{2n} \right| x^{2n-4}.
\end{align*}
\]
Since \( b_2 = 17 \) and
\[
b_{n+1} - b_n = 4^n (6n-1) - 4n + 1 = (1+3)^n (6n-1) - 4n + 1 > 3n (6n-1) - 4n + 1 = 18n (n-2) + 29(n-2) + 59 > 0
\]
for \( n \geq 2 \), the sequence \( b_n \) is increasing and \( b_n \geq b_2 = 17 > 0 \). Thus, the function \( g(x) \) is increasing on \( \left(0, \frac{\pi}{2}\right) \). Moreover,
\[
\lim_{x \to 0^+} g(x) = \frac{17}{720}
\]
and
\[
\lim_{x \to (\pi/2)^-} g(x) = \frac{\pi^2 + 8\pi - 32}{2\pi^3}.
\]
The proof of Theorem 3.2 is complete.

**Theorem 3.3.** For \( x > 0 \), we have
\[
2 \sinh x \frac{x}{x} + \tanh x > 3 + \frac{3}{20} x^3 \tanh x. \quad (3.3)
\]
The constant \( \frac{3}{20} \) is the best possible.

**Proof.** Let

\[
F(x) = \frac{2 \sinh x + \tan x}{x^3 \tan x} = \frac{\sin 2x + \sinh x - 3x \cosh x}{x^4 \sinh x}
\]

and let

\[
f(x) = \sinh 2x + \sin 3x \cosh x
\]

and

\[
g(x) = x^4 \sinh x.
\]

From the power series expansions

\[
\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}
\]

and

\[
\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}
\]

it follows that

\[
f'(x) = 2 \cosh 2x - 2 \cosh x - 2x \sinh x
\]

\[
= \sum_{n=0}^{\infty} \frac{x^{2n+4}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{4x^{2n+4}}{(2n)!} - \sum_{n=0}^{\infty} \frac{3x^{2n+2}}{(2n+1)!}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{x^{2n+4}}{(2n+1)!} + \frac{4x^{2n+4}}{(2n)!} - \frac{3x^{2n+2}}{(2n+1)!} \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{(2n+5)x^{2n+4}}{(2n+1)!}
\]

\[
= \sum_{n=0}^{\infty} \frac{(n+1)(n-1)x^{2n}}{(2n)!}
\]

\[
= \sum_{n=0}^{\infty} \frac{4(n-1)(4n^2-1)x^{2n}}{(2n)!}
\]

and

\[
g'(x) = 4x^3 \sinh x + x^4 \cosh x
\]

\[
= \sum_{n=0}^{\infty} \frac{4x^{2n+4}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{4x^{2n+4}}{(2n)!}
\]

\[
= \sum_{n=0}^{\infty} \frac{(2n+5)x^{2n+4}}{(2n+1)!}
\]

\[
= \sum_{n=0}^{\infty} \frac{4(n-1)(4n^2-1)x^{2n}}{(2n)!}
\]

It is easy to see that the quotient

\[
c_n = \frac{a_n}{b_n} = \frac{2^{2n+1} - 6n - 2}{4n(1)(4n^2 - 1)}
\]

satisfies

\[
c_{n+1} - c_n = \left( \frac{6n^2 - 17n + 1}{2n^2 (2n + 3)} \right) \left( 4n^2 - 1 \right) > 0
\]

for \( n \geq 0 \). This means that the sequence \( c_n \) is increasing.

By Lemma 2.8, the function \( G(x) = \frac{f'(x)}{g(x)} \) is increasing on \((0, \infty)\), and so, by Lemma 2.7, the function

\[
F(x) = \frac{f(x)}{g(x)} = \frac{f(x) - f(0)}{g(x) - g(0)}
\]

is increasing on \((0, \infty)\).

Moreover, it is not difficult to obtain

\[
\lim_{x \to 0^+} F(x) = c_2 = \frac{3}{20}.
\]

Theorem 3.3 is thus proved.

**Theorem 3.4.** For \( x > 0 \),

\[
\frac{\sinh x}{x} + \left[ \frac{\tanh (x/2)}{x/2} \right]^2 > 2 + \frac{23}{720} x^3 \tanh x.
\]

The number \( \frac{23}{720} \) in (3.6) is the best possible.

**Proof.** Let

\[
F(x) = \frac{\sinh x + \left[ \frac{\tanh (x/2)}{x/2} \right]^2 - 2}{x^3 \tanh x}
\]

and let

\[
f(x) = \cosh x \left[ x \sinh x \cosh x + x \sinh x \right] + 4 \cosh x - 2x^2 \cosh x - 4 - 2x^2
\]

and

\[
g(x) = x^5 \sinh x (1 + \cosh x).
\]

By the power series expansions in (3.4), we obtain

\[
f(x) = x \sinh x + \frac{1}{4} \sinh x - \frac{3}{4} x \sinh x
\]

\[
+ \frac{1}{2} \sinh (2x) - 4 \cosh x + 2 \cosh (2x)
\]

\[
- x^2 \cosh (2x) - 2x^2 \cosh x + 2 - x^2
\]

\[
= \sum_{n=0}^{\infty} \frac{3^{2n+1}}{4} - n^{2n+1} - 4n - \frac{7}{4} x^{2n+2}
\]

\[
+ \sum_{n=0}^{\infty} \frac{2^{2n+1} - 4}{2n+1} x^{2n+2}
\]

\[
= \sum_{n=0}^{\infty} \frac{3^{2n+1}}{4} - n^{2n+1} - 4n - \frac{7}{4} x^{2n+2}
\]

\[
+ \sum_{n=2}^{\infty} \frac{2^{2n+1} - 4}{2n+1} x^{2n+2}
\]
\begin{align*}
&= \sum_{n=2}^{\infty} \frac{3^{2n-1} - (n-1)2^{2n-1} - 4n + \frac{9}{4} x^{2n}}{4(2n-1)!} \\
&\quad + \sum_{n=2}^{\infty} \frac{2^{2n+1} - 4}{2n!} x^{2n} \\
&= \sum_{n=3}^{\infty} \frac{1}{(2n)!} \left( \frac{3^{2n-1} - (n-1)2^{2n-1} - 8n + \frac{9}{2}}{2} \right) x^{2n} \\
&\quad + \sum_{n=3}^{\infty} a_n x^{2n}
\end{align*}

and

\begin{align*}
g(x) &= x^5 \left( \frac{1}{2} \sinh(2x) + \sinh x \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left( (1+2^{-n})2n-4 \right) \left( 2n-3 \right) x^{2n} \\
&\quad + \sum_{n=0}^{\infty} b_n x^{2n}
\end{align*}

The ratio

\[ c_n = \frac{a_n}{b_n} \]

\begin{align*}
n \left[ \frac{3^{2n-1} - (n-1)2^{2n-1} - 8n + \frac{9}{2}}{2} \right] + 2^{2n+1} - 4 \\
= \frac{(1+2^{-n})(2n-4)(2n-3)(2n-2)(2n-1)2n}{(2n-2)(2n-3)(2n-2)(2n-1)2n}
\end{align*}

satisfies

\[ c_3 = \frac{23}{720} \approx 0.031, \quad c_4 = \frac{17}{336} \approx 0.050, \quad \text{and} \]

\[ c_5 = \frac{5099}{85680} \approx 0.059 \ldots \]

Furthermore, when \( n \geq 6 \) by a simple computation, we have

\[ c_{n+1} - c_n = \frac{\left[ f_1(n) + f_2(n) \right]}{\left[ f_1(n) + f_4(n) \right]} + \frac{\left[ 3n \left( 16 + 4^n \right) \right]}{\left[ n - 2 \right] \left( 2n - 3 \right) \left( 2n - 1 \right) \left( 4n^2 - 1 \right) \left( n^2 - 1 \right)} , \]

where

\[ f_1(n) = 16^n \left( 144n^3 - 24n^2 - 648n + 240 \right) \]
\[ = 16^n \left( 144n - 6 \right)^2 + 1704n(n - 6) + 4392(n - 6) + 26532 \]
\[ > 0, \]

\[ f_2(n) = 9^n \left( 1024n^3 - 3072n^2 - 640n + 3456 \right) \]
\[ = 9^n \left( 1024n - 6 \right)^2 + 9216n(n - 6) + 128(129n + 27) \]
\[ > 0, \]

\[ f_3(n) = 4^n \left[ 9^n (10n^3 - 57n^2 - 13n + 54) \right] \]
\[ = 4^n \left[ -2016n^4 + 8622n^3 + 5541n^2 - 33327n + 17490 \right] \]
\[ > 4^n \left[ 84(1 + 8)^n - 2016n^4 + 8622n^3 + 5541n^2 - 33327n + 17490 \right] \]

\[ = 17574 - 107023n + 144421n^2 - 70226n^3 + 12320n^4 \]
\[ = 12320(n - 6)^4 + 225454(n - 6)^3 + 1541473(n - 6)^2 + 4686101(n - 6) + 5372496 \]
\[ > 0 \]

for \( n \geq 6 \). Hence, the sequence \( c_n \) is increasing. By Lemma 2.8, the function \( F(x) \) is increasing. Finally, it is easy to see that \( \lim_{x \to 0^+} F(x) = c_3 = \frac{23}{720} \). The proof of Theorem 3.4 is complete.

**Theorem 3.5.** For \( 0 < |x| < \pi/2 \), we have

\[
4 + \frac{1}{10} x^3 \sin x < \frac{3x}{\sin x} + \cos x < 4 + \frac{12\pi - 32}{\pi^3} x^3 \sin x .
\]

The numbers \( \frac{1}{10} \) and \( \frac{12\pi - 32}{\pi^3} \) are the best possible.

**Proof.** Let

\[
f(x) = \frac{3 - x + \cos x - 4}{x^3 \sin x} = \frac{1}{x^3} \left( \frac{3x}{\sin^2 x} + \cot x - \frac{4}{\sin x} \right).
\]
By (2.4), (2.6), and (2.7), we have

\[
f(x) = \frac{1}{x^3} \left[ \sum_{n=1}^{\infty} \frac{3(2n-1)2^{2n}}{(2n)!} B_{2n} x^{2n-1} + \frac{1}{x} \right]
\]

\[
= \sum_{n=1}^{\infty} \frac{3(2n-1)2^{2n} - 22n - 4(2n-1)}{(2n)!} B_{2n} x^{2n-4}
\]

This shows that the function \( f(x) \) is increasing on \( \left( 0, \frac{\pi}{2} \right) \). Moreover, it is straightforward to obtain

\[
\lim_{x \to 0^+} f(x) = a_2 = \frac{1}{10}
\]

and

\[
\lim_{x \to \pi/2^-} f(x) = \frac{12\pi - 32}{\pi^3}.
\]

The proof of Theorem 3.5 is complete.

**Remark 3.1.** This paper is a slightly revised version of the preprint [7].

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