Lifshitz cosmology: quantum vacuum and Hubble tension

Dror Berechya and Ulf Leonhardt

Weizmann Institute of Science, Rehovot 7610001, Israel

Accepted 2021 August 9. Received 2021 June 15; in original form 2020 November 24

ABSTRACT

Dark energy is one of the greatest scientific mysteries of today. The idea that dark energy originates from quantum vacuum fluctuations has circulated since the late ’60s, but theoretical estimations of vacuum energy have disagreed with the measured value by many orders of magnitude, until recently. Lifshitz theory applied to cosmology has produced the correct order of magnitude for dark energy. Furthermore, the theory is based on well-established and experimentally well-tested grounds in atomic, molecular and optical physics. In this paper, we confront Lifshitz cosmology with astronomical data. We find that the dark–energy dynamics predicted by the theory is able to resolve the Hubble tension, the discrepancy between the observed and predicted Hubble constant within the standard cosmological model. The theory is consistent with supernovae data, Baryon Acoustic Oscillations and the Cosmic Microwave Background. Our findings indicate that Lifshitz cosmology is a serious candidate for explaining dark energy.

Key words: dark energy.

1 INTRODUCTION

The cosmological standard model, the Λ Cold Dark Matter (ΛCDM) model, has been spectacularly successful. With a few basic principles, it explains a vast range of phenomena over an enormous range of time scales. With only six free parameters, it fits the complex and detailed fluctuation spectra of the cosmic microwave background (CMB). Nevertheless, the ΛCDM model lacks an explanation of the underlying nature of three of its pillars, known as the dark sector – inflation, dark matter, and dark energy.

In recent years, the cosmology community has been actively looking for cracks in the ΛCDM model in the form of tensions between several independent phenomena (Verde, Treu & Riess 2019). Presently, the most severe such tension is known as the Hubble tension: the discrepancy between the Hubble constant (the present–day expansion rate) inferred from early–universe phenomena and the value obtained by local probes of cosmic expansion (Verde et al. 2019; Riess 2020). Not everyone agrees that these tensions are real (Efstathiou 2020) but by revealing cracks in the ΛCDM model they may shed light on the dark sector.

There have been numerous attempts to explain the Hubble tension (Di Valentino et al. 2021). Without exception, they either require significant changes to general relativity, the cosmological principle, or modifications to the standard model of particle physics that have not been experimentally tested elsewhere.

Here enters the Lifshitz theory in cosmology (Leonhardt 2019). This theory is based on solid foundations in atomic, molecular, and optical (AMO) physics that have been experimentally tested with percent-level precision (Decca 2014). The connection to cosmology is the analogy between curved space-times and dielectric media (Plebanski 1960; Leonhardt 2010) which is also the foundation of the well-developed field of transformation optics (Service & Cho 2010).

A homogeneous and isotropic, expanding universe with scale factor $a(t)$ is perceived by the electromagnetic field as a medium with an homogeneous and isotropic but evolving refractive index $n(t) \propto a(t)$. Then, calculating the vacuum energy in the universe should be done as if it were a dielectric medium with an evolving refractive index in what is known as Lifshitz theory (Lifshitz 1954; Landau, Lifshitz & Pitaevskii 1980). Applied to cosmology, the Lifshitz vacuum energy turns out to have the same order of magnitude as the measured cosmological constant $\Lambda$ (Leonhardt 2019).

In this paper, we compare the predictions of Lifshitz theory with astronomical data. We also formulate the theory such that it can be taken up by astronomers. Lifshitz theory in cosmology has not been designed to alleviate the Hubble tension, but we show that the most naive choice of its coupling parameter fits the SH0ES value (Riess et al. 2021) with perfect precision. We also find that the theory is consistent with the Pantheon type Ia supernova (SN Ia) data at the same level or slightly better than the ΛCDM model, that it agrees with the measured baryon acoustic oscillations (BAO) and does not lead to deviations from the measured CMB spectra within the accuracy of the cosmic parameters. There are still many opportunities for further analysis, but the findings reported here already show that Lifshitz cosmology is a serious contender for a realistic explanation of dark energy, rooted in established physics.

2 LIFSHITZ THEORY IN COSMOLOGY

2.1 Background

Most of our universe is empty space. Yet, this ‘emptiness’ is far from being ‘nothingness.’ According to the modern view of quantum field theory (QFT), the universe is filled with quantum fields in at least their ground state – also known as the vacuum state. Since the early days...
of QFT, it is known that the vacuum state of a quantum field contains non-vanishing energy density, and due to Casimir in the late ’40s, we know that this energy density may even exert measurable forces (Casimir 1948; Casimir & Polder 1948). The physics of the quantum vacuum has been well tested (Munday, Capasso & Parsegian 2009; Rodriguez, Capasso & Johnson 2011; Decca 2014; Zhao et al. 2019) and explains a vast set of phenomena, from the adhesion of geckos to walls (Autumn & Gravish 2008) to the limit trees can grow (Koch et al. 2004).

So, the state of affairs is as follows. We know the universe is filled with quantum fields at their ground state, we know that this ground state exhibits non-vanishing energy density and may exert forces, and finally, we know that the universe is also filled with a mysterious energy density we call dark energy. It is therefore tempting to combine the physics of the quantum vacuum and dark energy.

Zel’dovich was the first to suggest, in 1968, that the cosmological constant \( \Lambda \) comes from the physics of the quantum vacuum (Zel’dovich 1968). By calculating the bare energy density of the vacuum, with a cut-off at the Planck scale where presumably GR breaks, one gets the correct structure of the cosmological constant. So, have we found an explanation of dark energy? Not quite yet. The problem is that the quantitative prediction of the vacuum energy density is off by about 120 order of magnitude (Weinberg 1989). Furthermore, if the theory is made to agree with the observed value of the vacuum energy density by choosing a sufficiently low cut-off for the vacuum fluctuations, it severely disagrees with measurements of vacuum forces (Mahajan, Sarkar & Padmanabhan 2006).

This situation does not seem very encouraging. However, the case for a Casimir cosmology is not closed yet (Leonhardt 2019, 2020); the idea that dark energy stems from vacuum fluctuations (Sakharov 1967; Zel’dovich 1968; Weinberg 1989) may still be valid. One encouraging insight is that curved space-times are the same as dielectric media in the eyes of the electromagnetic field: Maxwell’s equations in curved space-time are equivalent to Maxwell’s equations in dielectric media (Plebanski 1960; Leonhardt 2010), and our spatially flat, expanding universe is just another curved space-time. It would be a far more unreasonable assumption that the universe is one particular space-time with different rules or that vacuum physics is different in the lab and the universe. Therefore, we assume that we can calculate vacuum energy in the universe as if it were the corresponding dielectric medium.

Now, since Zel’dovich, substantial progress has been made in understanding the quantum vacuum forces such that formal arguments can be replaced by empirically tested theory (Rodriguez et al. 2011; Scheel 2014; Simpson & Leonhardt 2015). Without exception, the empirical evidence for forces of the quantum vacuum and the comparison with theory comes from AMO physics. There the quantum vacuum produces attractive or repulsive forces (Munday et al. 2009; Zhao et al. 2019) between dielectric objects and inside inhomogeneous media. For example, in the Casimir effect (Casimir 1948), vacuum fluctuations cause two dielectric plates to attract each other. Here the spatial variation of the refractive index from free space to the material of the plates generates a vacuum force on the surface of each plate. This effect is a general phenomenon: variations of the refractive index create variations in the electromagnetic energy density and stress \( \sigma \) in media (Lifshitz 1954; Dzyaloshinskii, Lifshitz & Pitaevskii 1961; Landau et al. 1980; Scheel 2014), which gives the force density \( \mathbf{V} \cdot \mathbf{\sigma} \). This fact means that Casimir forces do not only act between dielectric bodies such as mirrors but also inside inhomogeneous bodies. Inhomogeneous dielectric media do exert local vacuum forces (Landau, Lifshitz & Pitaevskii 1995; Griniasty & Leonhardt 2017).

The theory that agrees with most measurements (Decca 2014) of Casimir forces is the Lifshitz theory (Lifshitz 1954; Dzyaloshinskii et al. 1961; Landau et al. 1980; Rodriguez et al. 2011; Scheel 2014). Due to the analogy mentioned above between space-times and media, Lifshitz theory can be applied to cosmology; in that case, the electromagnetic field and its fluctuations perceive the (spatially-flat, homogeneous, and isotropic) expanding universe as a spatially-uniform but time-dependent medium with a refractive index that is proportional to the scale factor \( a \) (Leonhardt 2019, 2020). Admittedly, when applying Lifshitz theory to that specific kind of medium, we extrapolate the theory outside its well-tested zone and introduce some new ideas. Nevertheless, the application of Lifshitz theory to the expanding universe was shown to produce the correct order of magnitude for the dark energy density (Leonhardt 2019).

In time-dependent media, the vacuum energy turns out to be time-dependent and responding to the evolution of the refractive index, or in the case of cosmology – to the evolution of the universe. Then, by the Friedmann equations, the universe is reacting to the vacuum energy. In the following, we present the resulting self-consistent dynamics.

### 2.2 Equations of motion

In the framework of the flat–\( \Lambda \)CDM model, the background (homogeneous and isotropic) universe evolves by the Friedmann equations, which can be summarized into one equation as

\[
H^2(a) = H_0^2(\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_\Lambda) \quad (\Lambda\text{CDM})
\]

where \( H(a) \) is the Hubble parameter, \( H_0 \) the Hubble constant (Hubble parameter at the present–day), \( a \) is the scale factor, and \( \Omega \), with \( x = r, m, \Lambda \) are the density parameters for radiation, matter, and the cosmological constant \( \Lambda \).

Let us now see how this equation changes for the Lifshitz theory in cosmology (hereafter, ‘Lifshitz cosmology,’ LC). For a given cosmic expansion, i.e. for a given \( a(t) \), Lifshitz theory predicts for a medium with \( n(t) \) the energy–momentum tensor of the quantum vacuum in that medium (Leonhardt 2019), in our case, in the universe. In turn, the vacuum energy and stress react back on the cosmic evolution through the Friedmann equation, influencing \( a(t) \). This mutual interaction between the vacuum energy and the background universe results in self-consistent dynamics (Leonhardt 2019), which we express here as\(^1\)

\[
\begin{align*}
H^2(a) & = H_0^2(\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_{LC}) \\
H_0^2\Omega_{LC} & = 8\pi a_\Lambda H_0^4 H^{-1} \quad \text{(Lifshitz cosmology)}
\end{align*}
\]

where \( \Omega_{LC} \) is the density parameter for dark energy in the Lifshitz cosmology. \( a_\Lambda \) is a dimensionless coupling parameter that depends on the cut–off, assumed near the Planck scale, and on the possible contributions of other fields in the standard model of particle physics (Leonhardt 2019). As these influences are not known within the present theory, \( a_\Lambda \) is a free parameter that must be fitted against observations. Taking only the electromagnetic field into account and assuming a sharp cut–off at exactly the Planck scale, we get (Leonhardt 2019) \( a_\Lambda H = (9\pi)^{-1} \) (the ‘TH’ superscript indicates that this value is a theoretical prediction under the above-mentioned

\(^1\)The dynamics that would result from the original calculations in Leonhardt (2019) are somewhat different from the dynamics we bring here; the reason for this difference is a different definition of the vacuum state. See Appendix A for more details.
The integration constant. In this way, we get the first-order correction to the dynamics of the universe.

In Section 2.2, we saw that this theory also gives a testable prediction: a modified expansion history, embodied in equations (2). In this section, we will find an approximate solution for the dynamics predicted by Lifshitz cosmology. Later, we will use this approximate solution to analyse the dynamics and demonstrate the theory’s plausibility. Here we present the approximate solution along general lines; for further details, see Appendix B.

The contribution of the cosmological constant in the ΛCDM model is negligible at last-scattering as ΩΛ/Ωm(1+z)3 ≈ 1.7·10−9 with values provided by Planck Collaboration (2020), and it is even smaller before that time. We assume that in Lifshitz cosmology the vacuum contribution is negligible before last-scattering as well and verify this later. The right-hand side of the second equation in equations (2) is zero for linear H−1; this means that ΩLe is constant in both radiation- and matter-domination eras where H−1 is linear. Lifshitz cosmology may intervene only in the transition period around aeq (as we will see in detail in Section 5). Hence, if we start with a negligible vacuum contribution during radiation domination, then the vacuum contribution will remain negligible at the beginning of matter-domination if the effects of Lifshitz cosmology around aeq are small. Here, we assume that this is the case; in Section 5 we check the validity of this assumption (see Fig. 6). Thus, we adopt ΛCDM’s dynamics at the early universe and focus our attention on the late universe. Hence we drop the radiation term in our calculations (as it turns out, see Appendix B, this is a crucial simplification for our calculations).

Even without the radiation term, finding a closed analytical solution for equation (2) remains a real challenge. Moreover, finding a numerical solution is no less challenging, mainly for the following two reasons. First, equations (2) are ‘stiff equations,’ causing havoc with step size and accuracy, and second, the equation for ΩLe depends on high derivatives of a (up to fourth-order), which is problematic since the highest derivatives take the lead in differential equation solvers. In reality, the dynamics of ΩLe are a mere correction to the dynamics of the universe.

Therefore, we solve for the dynamics after last-scattering perturbatively. αM is presumably small (recall that the theoretical prediction is α2TH = (9π)−1 ≈ 0.035), so we calculate ΩLe up to first-order in αM. We plug the zeroth-order Hubble parameter (the ΛCDM’s Hubble parameter, equation (1) without the radiation term) into the equation for ΩLe (the second equation in equations 2), and we integrate it (analytically, see Appendix B) with Ω∞ ≡ lim a→∞ ΩLe as the integration constant. In this way, we get the first-order correction to ΩLe, that we substitute into the equation for H(a) (the first equation in equations 2). Thus, we get

\[ H^2 = H_0^2 (\Omega_m a^{-3} + \Omega_{Le}). \]

\[ \Omega_{Le} = \Omega_{\infty} \left[ 1 + 18 \alpha_M \left( \ln \left( \frac{\Omega_m}{\Omega_{\infty}} a^{-3} + 1 \right) - \frac{3}{\Omega_m} \frac{\dot{a}}{a} \right) \right] \]

(for aeq ≪ a).

The next step is to fit the theory’s parameters with cosmological data sets, such as CMB power spectra, SN Ia, and BAO. The complete way of fitting the parameters is to include the modified equation for the background dynamics, i.e. the new equation for H(a), in the relevant computer codes and perform statistical analysis (such as likelihood-based MCMC or Fisher information). In this paper, however, we aim to explore the Lifshitz dynamics for the first time to test whether this theory can plausibly resolve the Hubble tension at all, which would then justify further research. For this, we take the ΛCDM’s value for the sound horizon, rs, as we assume that the deviations of Lifshitz cosmology from the ΛCDM model are negligible in the early universe. The following section shows that the resulting dynamics are consistent with BAO and SN Ia measurements. To preserve the acoustic angular scale of the CMB fluctuations (θs = rs / Dm), we must demand that the angular diameter distance to the surface of last-scattering, DM, is unchanged as well:

\[ D_M = c \int_0^{\theta_s} \frac{dz}{H(z)} = D_M^{\Lambda\text{CDM}} \]

where \( D_M^{\Lambda\text{CDM}} \) is calculated with the ΛCDM model. In effect, this demand gives us a relationship between αM and the combination \( H_0/\Omega_{\infty} \) for the following reason. Since \( H_0/\Omega_{\infty} \) is proportional to the physical matter density, it should be a model-independent quantity; therefore, we may use ΛCDM’s value for this combination. The Planck collaboration2 determined \( \Omega_m h^2 = 0.1430 \pm 0.0011 \) (Planck Collaboration 2020) via the relative heights of the CMB acoustic peaks (approximately) model-independently (Planck Collaboration et al. 2014). Here and throughout this paper h ≡ H0 / 100[km s−1 Mpc−1] and the ‘p’ superscript, hereafter, denotes that the value is determined by Planck. Fig. 1 shows the resulting relationship. The \( \Omega_m h^2 \) errors presented in this figure are estimates based solely on propagating the ΛCDM errors in determining \( D_M^{\Lambda\text{CDM}} \); that is, for any other ΛCDM’s quantity, we take the mean value given by Planck’s TT,TE,EE+LowE+lensing analysis (Planck Collaboration et al. 2020) without errors (see Appendix B for more details).

Thus, we are left only with αM as a free parameter. The value of αM will determine H(a), and hence, will fix the value of H0, Fig. 2, as well as the values of the other parameters in Table 1.

Fig. 2 shows that whatever the actual value of H0 may be, Lifshitz cosmology may reproduce it (at least nominally, see the discussion in Section 4.1). The theoretical prediction under the assumptions of only electromagnetic contributions and a sharp cut-off at exactly the Planck length is more or less at the middle of the local measurements, and remarkably, it is right on the latest measurement by the SH0ES team (Riess et al. 2021).

To study the influence of different values of αM and hence of different sets of parameters, we will explore the resulting dynamics of two representative realizations of Lifshitz cosmology. The first one, which we call ‘M1,’ is the theoretical prediction, for which we have \( \alpha_M^{M1} = \alpha_M = (9\pi)^{-1} \). For the second realization, which we

2Throughout the paper, we will use italic letters to designate the Planck collaboration and distinguish it from Planck the person or other contexts in which this name might appear.
show the yellow point), the theoretical prediction for the electromagnetic contribution of Lifshitz cosmology considered in this paper are also shown: $M_1$ (black and red point). The theoretical prediction for the electromagnetic contribution of Lifshitz cosmology considered in this paper are also shown: $M_1$ (black and red point). This relationship results when demanding that Lifshitz cosmology preserves $\Omega_{\infty} h^2$ equal to the intercept of the magnitude–redshift relation. $H_0$ in units of km s$^{-1}$ Mpc$^{-1}$ as a function of $a_\Lambda$ in the range of interest. The two realizations of Lifshitz cosmology considered in this paper are also shown: $M_1$ (black and yellow point), the theoretical prediction for the electromagnetic contribution along with a sharp cut-off at exactly the Planck length, $a_\Lambda^M = (9\pi)^{-1}$ and $M_2$ (black point), $a_\Lambda^M = 0.0225$.

4 LOW REDSHIFT PROBES

At this point, we have in our hands a theory explaining the physical origin of dark energy, one that stems from well-known physics, an approximate solution for the theory’s dynamics assuming unmodified early evolution, and two sets of parameters (in Table 1) for two realizations of the theory: $M_1$ ($\alpha^M_1 = (9\pi)^{-1}$, a theoretical prediction) and $M_2$ ($\alpha^M_2 = 0.0225$).

Now, we are ready to compare the resulting dynamics with low redshift probes of cosmic expansion, viz. SNe Ia and BAOs. We shall see that $M_1$ fits better the SN data with SH0ES calibration of the absolute magnitude $M_B$, which might be crucial regarding the Hubble tension (Efstathiou 2021; Benevento, Hu & Ravera 2020); however, $M_1$’s fit to BAO data is somewhat lesser than ΛCDM’s. On the other hand, $M_2$ fits BAO data better than $M_1$ and seemingly falls from ΛCDM’s fit only by a small margin; yet, $M_2$ fits SN data with a lower value (more negative) of $M_B$ and can only relieve the tension (see the discussion in the following subsection).

We will compare the resulted dynamics of $M_1$ and $M_2$ with observational findings, refraining from a more complex statistical analysis for the time being. Our analysis already indicates the viability of Lifshitz cosmology. However, only a complete statistical analysis will determine the actual set of values for the theory’s parameters, instead of $M_1$ and $M_2$, which are demonstrations obtained by choosing $a_\Lambda$ and imposing equation (4), and will enable us to decide which is the better theory. This further analysis poses an opportunity for future research.

4.1 The Hubble diagram and distance ladders

We start with SNe Ia observations. Ultimately, each SN Ia measures the luminosity distance via the relation

$$\mu \equiv m_B - M_B + \delta \mu = 5 \log_{10} \frac{D_L(z)}{\text{Mpc}} + 25,$$

where

$$D_L(z) = c (1+z) \int_0^z \frac{dz'}{H(z')}.$$

$\mu$ is the distance modulus, $M_B$ is the absolute magnitude (in the B-band), $m_B$ is the apparent magnitude, and $\delta \mu$ summarizes corrections due to effects such as colour, light-curve’s shape, and host-galaxy mass; these effects can be either measured or fitted using SNe Ia data alone, independently of cosmology (Riess et al. 2018a). Roughly speaking, in each measurement, we measure $m_B$ and $z$, and we wish to infer $D_L(z)$. $M_B$ is thus a nuisance parameter that must be determined or marginalized over. This nuisance parameter is degenerate with $H_0$ in the SN Ia data: as a prefactor in $H(z)$, $H_0$ would shift $M_B$ by $5 \log_{10} H_0$ in equation (5); thus, both $M_B$ and $H_0$ get swallowed into the intercept of the magnitude–redshift relation.

One way to break that degeneracy is to use a distance ladder to infer $M_B$ by calibrating SN Ia. Generally, there are two approaches to measuring $H_0$ using distance ladders. One is to use geometrical measurements to anchor local probes of distance (first rung), such as Cepheids (e.g. SH0ES team, Riess et al. 2021) or TRGB (e.g. Soltis et al. 2021 and Freedman et al. 2020), then use these probes...
Table 1. Two realizations of Lifshitz cosmology. For each realization, we choose $\alpha_i$. Then $\Omega_{\infty} h^2$ is fixed by equation (4) (taking $\alpha_i^0 = 0.1430$ and $\alpha_i^* = 0.3107$ (Planck Collaboration 2020), see Appendix B for details of the calculations), and the rest of the parameters of Lifshitz cosmology follow. The errors in the parameters result from propagating the errors in $\Omega_{\infty} h^2$ that are estimated by the lowest RMSD using the (unbinned) Pantheon data. For comparison, we also calculated the RMSD with the reproduced SH0ES absolute magnitude ($M_B = -19.244$) (Efstathiou 2021) and with absolute magnitude obtained by fitting a late dark energy model with Planck, BAO, and Pantheon data (retaining $M_B$ in the Pantheon likelihood without the SH0ES constraint), $M_B = -19.415$ (Benevento et al. 2020). The results are presented in Table 2. The bottom panel of Fig. 3 shows $\Delta \mu$ for the binned Pantheon data with the best $M_B$ of M1 and M2, together with the theoretical curves. We find $M_B^{*} = -19.421$ for $\Lambda CDM$, $M_B^{*} = -19.330$ for M1, and $M_B^{*} = -19.388$ for M2 (the ‘*’ superscript indicates a value corresponding to the lowest RMSD). The three RMSD values that correspond to the three $M_B^{*}$’s are comparable to one another, so it seems that the three models fit the unbinned Pantheon data comparably well.

\[
M_B = -19.244 \pm 0.042 \text{ mag.}
\]

We have adopted this value and used it with the Pantheon data set (which is given by Scolnic et al. 2018) and publicly available in doi: 10.17909/T95Q4X) to extract the $\mu$’s of observed SNe. We also calculated $\mu$ (equation 5) for $\Lambda CDM$ and the two Lifshitz cosmologies (M1 and M2). Fig. 3 shows $\Delta \mu \equiv \mu - \mu_{\Lambda CDM}$ for the theories and the observed data points with $M_B = -19.244$ (top panel). This figure also shows binned data from Scolnic et al. (2018).

It has been noted (Efstathiou 2021; Benevento et al. 2020) that, in principle, SH0ES does not measure $H_0$ directly but measures $M_B$ instead; $H_0$ is inferred from the low redshift ($z < 0.15$ (Efstathiou 2021; Benevento et al. 2020)) SNe in the Pantheon sample with the measured $M_B$. According to this view, the Hubble tension is really an $M_B$ tension: a significant gap of about $\Delta M_B \approx 0.2$ between the SH0ES $M_B$ and the one inferred from the Pantheon data without including the SH0ES constraint (retaining $M_B$ in the likelihood) (Benevento et al. 2020) or the one obtained by inverse distance ladder (Efstathiou 2021). For theories that modify the dynamics above $z \approx 0.15$, these two viewpoints should be equivalent; however, for theories that modify the dynamics below that redshift, only the latter viewpoint ($M_B$ tension) should be considered, as in this case, $H_0$ is not constrained by the Pantheon data (see figure 1 in Benevento et al. 2020)) and SH0ES analysis would be oblivious to this modification (Efstathiou 2021; Benevento et al. 2020). That is, if our universe would evolve according to a theory that modifies the dynamics below $z \approx 0.15$, it will not appear in the SH0ES analysis, and they would approximately measure the same $\Lambda CDM$ value for $H_0$ as inferred from the CMB.

Even though Lifshitz cosmology starts to modify the dynamics at $z > 0.15$ (see Fig. 5), we would like to estimate how the theory will perform regarding the $M_B$ tension. To give some quantitative measure, for each model, we calculate the root–mean–square deviation (RMSD) given by

\[
\text{RMSD} = \sqrt{\frac{\sum_{i=1}^{N}(\mu_i(z_i) - \mu_i)^2}{N}}
\]

where $z_i$ and $\mu_i$ are, respectively, the measured redshift and distance modulus (with a given $M_B$) of measurement $i$, $\mu(z_i)$ is the theoretical prediction, and $N$ is the number of data points (for the Pantheon sample, $N = 1,048$). For each model, we have found $M_B$ that gives the lowest RMSD using the (unbinned) Pantheon data. For comparison, we also calculated the RMSD with the reproduced SH0ES absolute magnitude ($M_B = -19.244$) (Efstathiou 2021) and with absolute magnitude obtained by fitting a late dark energy model with Planck, BAO, and Pantheon data (retaining $M_B$ in the Pantheon likelihood without the SH0ES constraint), $M_B = -19.415$ (Benevento et al. 2020). The results are presented in Table 2. The bottom panel of Fig. 3 shows $\Delta \mu$ for the binned Pantheon data with the best $M_B$ of M1 and M2, together with the theoretical curves. We find $M_B^{*} = -19.421$ for $\Lambda CDM$, $M_B^{*} = -19.330$ for M1, and $M_B^{*} = -19.388$ for M2 (the ‘*’ superscript indicates a value corresponding to the lowest RMSD). The three RMSD values that correspond to the three $M_B^{*}$’s are comparable to one another, so it seems that the three models fit the unbinned Pantheon data comparably well.

These results show that while M2 can only mitigate the $M_B$ tension by about 19 per cent at mean value, as $\Delta M_B^{*} = M_B^{\text{SH0ES}} - M_B^{*} = 0.144$, M1 can relax it considerably by about 51 per cent at mean value, as $\Delta M_B^{*} = 0.086$ (for $\Lambda CDM$, one gets $\Delta M_B^{*} = 0.177$). On the other hand, M2 fits the shape of the binned Pantheon data exceedingly well, as shown in Fig. 3, while M1’s fit to the shape is only moderate. The shape of the binned data might depend on the model, e.g. via model dependence of the redshift weights of the surveys (Benevento et al. 2020); in addition, the RMSD($M_B$) profile (for the unbinned data) is shallow around the minimum, so the two Lifshitz cosmologies might do even better (this seem to be especially true for M1). Only a rigorous statistical analysis would be able to tell. Nevertheless, based on our current analysis, we conclude that the two Lifshitz cosmologies fit the Pantheon sample as well as $\Lambda CDM$ does, and they both reduce the $M_B$ tension (M1 might even, hopefully, resolve this tension). As we will see later, while both cosmologies (M1 and M2) seem to fit the BAO data comparably to $\Lambda CDM$, M2 is a better fit there.

4.2 Distance-ladder-independent analysis

Before we turn to BAO data, let us compare our dynamics with measurements of $E(z) = H(z) / H_0$ that are independent of any distance ladder, going around the $M_B$ dilemma. The quantity $E(z)$ is independent of $H_0$ and thus avoids the $H_0-M_B$ degeneracy; therefore, it can be measured using SN Ia data alone (Riess et al. 2018a).

To extract $E(z)$ from the SN Ia data, Riess et al. (2018a) parametrize it by its value at several specific redshifts and interpolate
Figure 3. $\Delta \mu \equiv \mu - \mu_{\Lambda CDM}$ as a function of $z$. Top panel: Pantheon data (Scolnic et al. 2018) (yellow (unbinned data) and blue (binned data) points) with the reproduced SH0ES absolute magnitude $M_B = -19.244$ from Efstathiou (2021). The theoretical predictions are also shown: unbroken black curve represents M1, and dashed black curve represents M2. The grey band around each curve shows the $\pm 1\sigma$ errors in $\Delta \mu$ obtained by propagating the errors in $\Omega_\infty h^2$. Bottom panel: The binned Pantheon data are shown with $M_B = -19.330$ (Green) and with $M_B = -19.388$ (Purple), the best $M_B$ in terms of RMSD (equation 8) for M1 and M2, respectively.

Table 2. Root–mean–square deviation (RMSD) calculated with equation (8) for the (unbinned) Pantheon data (Scolnic et al. 2018). The top block shows the best (in terms of RMSD) $M_B$ and the corresponding RMSD for each model. The bottom block shows results with the reproduced SH0ES absolute magnitude ($M_B = -19.244$) (Efstathiou 2021) and the absolute magnitude obtained by fitting a late dark energy model with Planck, BAO, and Pantheon data ($M_B = -19.415$) (Benevento et al. 2020).

| $M_B$     | $\Delta \mu_{\Lambda CDM}$ | M1  | M2  |
|-----------|-------------------------------|-----|-----|
| -19.421   | 0.1449                        | -   | -   |
| -19.330   | -                             | 0.1511 |   |
| -19.388   | -                             | -   | 0.1453 |
| -19.244   | 0.2291                        | 0.1736 | 0.2046 |
| -19.415   | 0.1450                        | 0.1736 | 0.1477 |

to define the complete $E(z)$ function, which can then be used to compute the luminosity distance and compare to the data while fully marginalizing over the absolute magnitude. This way, they constrained the value of $E^{-1}(z)$ at six different redshifts model–independently (except for assuming a spatially–flat universe). Fig. 4 shows $\Delta [E^{-1}] \equiv E^{-1} - E^{-1}_{\Lambda CDM}$ for these six data points together with the two Lifshitz cosmologies. It can be seen that among the six

Figure 4. $\Delta [E^{-1}] \equiv E^{-1} - E^{-1}_{\Lambda CDM}$ as a function of $z$. The six grey points are model–independent measurements of $E^{-1}$ performed by Riess et al. (2018a) based on SN Ia data alone. The theoretical predictions are also shown: unbroken black curve represents M1, and dashed black curve represents M2. The grey band around each curve shows the $\pm 1\sigma$ errors in $\Delta [E^{-1}]$ obtained by propagating the errors in $\Omega_\infty h^2$. Among the six data points, three (at $z = 0.07, 0.35$, and 0.9) are situated closer to M1’s curve, and the remaining three (at $z = 0.2, 0.55$, and 1.5) are situated closer to the $\Lambda CDM$ baseline. M2 lies in between, and it seems to agree with all data points. All in all, Lifshitz cosmology appears to fit the data comparably to $\Lambda CDM$. 

MNRAS 507, 3473–3485 (2021)
data points, three (at $z = 0.07, 0.35$, and 0.9) are situated closer to M1’s curve, and the remaining three (at $z = 0.2, 0.55$, and 1.5) are closer to the $\Lambda$CDM’s baseline. M2 lies in between, and it seems to agree with all data points. All in all, we conclude that Lifshitz cosmology does fit the $E^{-1}$ data to a degree comparable to $\Lambda$CDM.

4.3 BAO measurements

Now we turn to BAOs, the second low redshift probe we consider in this paper. BAO measurements can be used for measuring $H(z)$, as these measurements constrain the product $H(z)r_d$ (Di Valentinio et al. 2021; Arendse, Agnello & Wojtak 2019), where $r_d$ is the sound horizon at the end of the baryon–drag epoch ($z_d = 1, 059.94$ (Planck Collaboration et al. 2020)).

The BAO constraint (at $z \geq 0.38$) on $H(z)r_d$ can be extrapolated to $z = 0$ using a lower redshift probe, such as SN Ia, to obtain a constraint on $H_0r_d$ (Arendse et al. 2019). This procedure of extrapolating the BAO measurements is model–dependent (Arendse et al. 2019). Nonetheless, the extrapolation can be performed using various cosmographic techniques, such as cosmology–agnostic expansions of the Hubble parameter or distances, so that the final measurement might be considered as independent of a cosmological model (Arendse et al. 2019). Therefore, a point has been made that due to the extrapolated BAO constraint on $H_0r_d$, one cannot rise $H_0$ without reducing $r_d$ since this would introduce tension with the extrapolated BAO measurements (Arendse et al. 2019).

However, while the cosmographic techniques are agnostic to cosmology, they are still models, and it is not clear how well they may capture the Lifshitz cosmology. Di Valentinio et al. (2021) has noted that the BAO data are extracted under the assumption of a $\Lambda$CDM scenario, so one should be careful in excluding all the ‘Late Time solutions’ only using this argument. Moreover, the use of SN Ia to extrapolate the BAO measurements in this inverse distance ladder procedure might be problematic, as also noted by Di Valentinio et al. (2021), which recommended not to use this approach. They write, ‘the fiducial absolute magnitudes’... value depends on the method used to produce a light curve fit, which bands are included, the light curve age where it is defined, and the fiducial reference point chosen. Errors would arise from unintended mismatches between SN analyses and missing covariance data’ (Di Valentinio et al. 2021).

Lastly, Arendse et al. (2020) used strong gravitational lensing to break the degeneracy between $r_d$ and $H_0$ in the extrapolated BAO constraints; they found a small trend in the measured $r_d$ when using each lens (at different redshift) separately (see fig. 5 there). While statistically insignificant (1.6σ) at the moment, this trend might signal residual systematics, either in the lenses themselves or in the procedure used to extrapolate the BAO measurements (Arendse et al. 2020). Arendse et al. (2020) has noted that a recent ($z \approx 0.4$) change in dark energy might produce this behaviour and re–absorb this trend.

Only a more thorough analysis of Lifshitz cosmology would be able to answer the question posed by extrapolating the BAO measurements; at the moment, we may take Planck’s value for $r_d$ (147.09 Mpc (Planck Collaboration et al. 2020)) to see whether the Lifshitz cosmologies (M1 and M2) are consistent with the BAO measurements of $H(z)$ at $z \geq 0.38$. We take $\Lambda$CDM’s value for the sound horizon at the end of the baryon–drag epoch to approximate the Lifshitz cosmology’s value since $z_d$ shortly follows last–scattering ($z_d = 1, 089.92$ (Planck Collaboration 2020)). Fig. 5 shows $H(z)/(1 + z)$ at $z = 0.38, 0.51, 0.61, 1.48, 2.34$, and 2.35 from BAO measurements (with $r_d = 147.09$ Mpc). In this figure, the data point at $z = 0.61$ already disagrees with $\Lambda$CDM, but more severely so with $M2$ and even more with $M1$; the data point at $z = 0.38$ agrees with Lifshitz cosmology (M1 and M2) slightly better, and so does the point at $z = 1.48$.

5 EARLY UNIVERSE

Up to this point, we have ignored the early universe and solved the theory assuming only late-universe modifications. Now, we shall turn our attention to this point. We will estimate the expected early-universe modifications due to Lifshitz cosmology to assess the validity of the assumption that led us to drop the radiation term. Specifically, we will verify that our Lifshitz cosmology’s dynamics are consistent with negligible dark energy contribution at the early universe.

We return to the two coupled equations, equation (2), that describe the mutual interaction between the expanding universe and dark energy according to Lifshitz cosmology. This time, we do not drop the radiation term, and we need to find a new solution that includes this term. In addition, we can no longer assume that the sound horizon at last-scattering $r_s$ is unchanged. Therefore, if we wish to proceed in the spirit of Section 3 and find a set of values for the theory’s parameters by imposing a relationship between $\alpha_s$ and $\Omega_\infty h^2$; then, instead of demanding that $D_M$ is kept unchanged (equation 4), we should demand that the CMB’s angular acoustic scale $\theta_*$ is
where $\omega_x$ the equation for $H/\alpha/Lambda_1$ considering only late modifications (Table 1), and the new plug (as matter domination at $/Omega_1LC$).

Now, we can use the so-obtained and vacuum-matter equality and becomes at the present $f_{de}(z = 0) \approx 0.733$ for M1 or $\approx 0.707$ for M2. Far in the future, it approaches 1.}

unchanged. That is, we should, in principle, demand

$$\theta_x = \frac{r_x}{D_M} = \theta_x^{ACDM}. \tag{9}$$

As it turns out (see Appendix B2), it is not straightforward to generalize our solution (equation 3) to accommodate $\Omega_x a^{-4}$ and then solve equation (9). Therefore, we will approximate by taking a detour: we take $\alpha_x$ and $\Omega_\infty \, h^2$ that we found for M1 and M2 when considering only late modifications (Table 1), and then we plug (as the zeroth-order solution) $H^2 = H^2_0 (\Omega_0 a^{-4} + \Omega_x a^{-3} + \Omega_\infty)$ into the equation for $H^2_0 \Omega_L (\alpha_1 = \Omega_L h^2)^2$ (the second equation in equations 2), and we integrate it (numerically, see Appendix B2) with $\Omega_\infty \, h^2$ as the integration constant. This way, we get an approximation for the first-order (in $\alpha_x$) $\Omega_{L,C} \, h^2$ that includes early-universe effects. Now, we can use the so-obtained $\Omega_{L,C} \, h^2$ to calculate the relative contribution of the vacuum energy, $f_{de}$, throughout the entire cosmic evolution

$$f_{de} \equiv \frac{\Omega_{L,C}}{\Omega_x a^{-4} + \Omega_x a^{-3} + \Omega_{L,C}} \tag{10}$$

where $\alpha_x \equiv \Omega_x \, h^2$ for $x = r, \, m, \, LC$.

The results are shown in Fig. 6. As expected, the dark energy’s dynamics kick in around the transition period from radiation to matter domination at $z_{eq} = 3, 402$ (Planck Collaboration et al. 2020). The early-universe evolution of $f_{de}$, according to Lifshitz cosmology, takes place roughly at the range $223 \leq z \leq 30, 350$ for M1 or $388 \leq z \leq 17, 950$ for M2 where $f_{de} \leq -0.005$, and it peaks around $z_{eq}$ with $f_{de} \approx -0.019$ at the peak for M1 or $\approx 0.012$ for M2 (inset). At late times, $f_{de}$ rises drastically and becomes at the present $f_{de}(z = 0) \approx 0.733$ for M1 or $\approx 0.707$ for M2. Far in the future, it approaches 1.

$3Recall that Lifshitz cosmology predicts dark energy evolution only in transition periods.

6 DISCUSSION

Viewing the universe as one giant ‘dielectric medium’ with time-dependent refractive index and applying Lifshitz theory for calculating the vacuum energy inside the medium, one can find a physical description of dark energy (Leonhardt 2019). This description is based on well-established and well-tested physics (Landau et al. 1980; Leonhardt 2020) which makes it unique among all other models of dark energy. The theory comes with two free parameters, $\alpha_x$ and $\Omega_\infty \, h^2$ (replacing $\Omega_\Lambda \, h^2$ of $\Lambda$CDM, such we have a total of seven parameters). We call this theory Lifshitz cosmology.

In this paper, we have investigated two realizations of Lifshitz cosmology; for each realization, we choose a value for the coupling parameter $\alpha_x$, and then, by demanding equation (4), $\Omega_\infty \, h^2$ is fixed together with the predicted dynamics.

Our first considered realization (M1) is $\alpha_M^{M1} = \frac{\alpha_\Lambda T^H}{(9\pi)^{-1}}$ based on the assumption of only electromagnetic contribution to the vacuum energy with a sharp cut-off at exactly the Planck length. Amazingly, this naive theoretical prediction gives the SHOES value for $H_0 (73.2 \pm 1 \text{[km s}^{-1} \text{Mpc}^{-1}])$ at mean value. We may (and in some instances, we should) (Benevento et al. 2020; Efstathiou 2021)) view the Hubble tension as a tension between the SHOES value for $H_0$ and the one obtained by calibrating the Pantheon data with $\Lambda$CDM or using inverse distance ladders (Benevento et al. 2020; Efstathiou 2021). Table 2 shows that M1 can considerably relieve the tension by 51 per cent at the best $M_B$ value; the relatively small difference in RMSD between the best $M_B$ and the SHOES value suggests that M1 might even resolve this $M_B$ tension completely. M1 also seems to fit $E^{-1}$ measurements based on distance-ladder-independent SN Ia data (Fig. 4). On the other hand, while M1 appears to fit the shape of the binned SN Ia data (Fig. 3) at the lower redshift region ($z \approx 0.2$), it does not fit the shape at the higher redshift region (to the extent that this shape does not depend on the model). M1 also seems to fit BAO measurements of $H(z)$ only moderately (Fig. 5).

Our second considered realization (M2) is $\alpha_M^{M2} = 0.0225$. This model seems to be the middle ground between M1 and $\Lambda$CDM; it gives a nominal value of $H_0 = 69.9 \pm 0.7 \text{[km s}^{-1} \text{Mpc}^{-1}]$, and it shrinks $\Delta M_B$ by only $\approx 19$ per cent. On the other hand, M2 fits all the $E^{-1}$ data points (Fig. 4), it perfectly fits the shape of the binned SN Ia data over the entire redshift range (Fig. 3), and its fit to BAO measurements of $H(z)$ is comparable to $\Lambda$CDM’s and only slightly worse (Fig. 5).
We have compared Lifshitz cosmology with astronomical data for the first time. There is certainly room for improvement and there are opportunities for further research. One problem with our analysis is that it treats $\alpha_1$ and $\Omega_{m0}\bar{h}^2$ as related via equation (4), whereas they should be regarded as two independent parameters. By treating these parameters independently, a future analysis may yield better results, as we demonstrate in Appendix C with a toy model. One could also determine the best among the models by implementing the approximate solution we found here (equation 3) in numerical codes to perform a parameter fitting and likelihood analysis of the CMB together with other key cosmological data. In any case, our analysis already proves that Lifshitz cosmology deserves serious consideration.

**ACKNOWLEDGEMENTS**

Our research was supported by the Israel Science Foundation, the Murray B. Koffler Professorial Chair and Aalto University.

**DATA AVAILABILITY**

All data are incorporated into the article.

**REFERENCES**

Agathe V. d. S. et al., 2019, A&A, 629, A85
Alam S. et al., 2017, MNRAS, 470, 2617
Arendse N., Agnello A., Wojtak R. J., 2019, A&A, 632, A91
Arendse N. et al., 2020, A&A, 639, A57
Autumn K., Gravish N., 2008, Phil. Trans. R. Soc. A, 366, 1575
Benedict G. F. et al., 2007, AJ, 133, 1810
Benevento G., Hu W., Raveri M., 2020, Phys. Rev. D, 101, 103517
Birrell N. D., Davies P. C. W., 1982, Quantum Fields in Curved Space. Cambridge Monographs on Mathematical Physics. Cambridge Univ. Press, Cambridge,
Birrer S. et al., 2020, A&A, 643, A165
Blakeslee J. P., Jensen J. B., Ma C.-P., Milne P. A., Greene J. E., 2021, ApJ, 911, 65
Blomqvist M. et al., 2019, A&A, 629, A86
Casimir H. B. G., 1948, Koninkl. Ned. Akad. Wetenschap., 51, 793
Casimir H. B. G., 1948, Phys. Rev., 73, 360
Davi P. C. W., 1975, J. Phys. A, 8, 609
de Jaeger T., Stahl B. E., Zheng W., Filippenko A. V., Riess A. G., Galbany L., 2020, MNRAS, 496, 3402
de Jaeger T., Stahl B. E., Zheng W., Filippenko A. V., Riess A. G., Galbany L., 2020, MNRAS, 496, 3402
Decca R. S., 2014, in Forces of the Quantum Vacuum. World Scientific, Singapore, p. 195
Di Valentino E., 2021, MNRAS, 502, 2065
Di Valentino E. et al., 2021, Class. Quantum Gravity
Dzyaloshinskii I. E., Lifshitz E. M., Pitaevskii L. P., 1961, Advan. Phys., 10, 165
Efthasious G., 2020, A Lockdown Perspective on the Hubble Tension (with comments from the SH0ES team). preprint (arXiv:2007.10716)
Efthasious G., 2021, MNRAS, 505, 3866
Fixsen D. J., 2009, ApJ, 707, 916
Freedman W. L. et al., 2020, ApJ, 891, 57
Fulling S. A., 1973, Physical Review D, 7, 2850
Grinastiy I., Leonhardt U., 2017, Physical Review A, 96, 032123
Hou J. et al., 2021, MNRAS, 500, 1201
Khetan N. et al., 2021, A&A, 647, A72
Koch G. W., Sillert S. C., Jennings G. M., Davis S. D., 2004, Nature, 428, 851
Korukchi E., Tully R. B., Anand G. S., Courtois H. M., Dupuy A., Neill J. D., Rizzi L., Seibert M., 2020, ApJ, 896, 3
Landau L., Lifshitz E., Pitaevskii L., 1980, Statistical Physics, Part 2. Course of Theoretical Physics, Vol. 9. Pergamon Press, Oxford
Landau L., Lifshitz E., Pitaevskii L., 1995, Electrodynamics of Continuous Media. Course of Theoretical Physics, Vol. 8. Elsevier Science, Amsterdam
Leonhardt U., 2010, Essential Quantum Optics: From Quantum Measurements to Black Holes. Cambridge Univ. Press, Cambridge
Leonhardt U., 2019, Annal. Phys., 411, 167973
Leonhardt U., 2020, Phil. Trans. R. Soc. A, 378, 20190229
Lifshitz E. M., 1954, J. Exper. Theoret. Phys. USSR, 29, 94
Mahajan G., Sarkar S., Padmanabhan T., 2006, Phys. Lett. B, 641, 6
Munday J. N., Capasso F., Parsegian V. A., 2009, Nature, 457, 170
Philcox O. H. E., Sherwin B. D., Farren G. S., Baxter E. J., 2021, Phys. Rev. D, 103, 023538
Petryński G. et al., 2019, Nature, 567, 200
Planck Collaboration et al., 2014, A&A, 571, A16
Planck Collaboration et al., 2020, A&A, 641, A6
Plebanski J., 1960, Phys. Rev., 118, 1396
Reid M. J., Pesce D. W., Riess A. G., 2019, ApJ, 886, L27
Riess A. G., 2020, Nat. Rev. Phys., 2, 10
Riess A. G. et al., 2018a, ApJ, 853, 126
Riess A. G. et al., 2018b, ApJ, 855, 136
Riess A. G., Casertano S., Yuan W., Bowers J. B., Macri L., Zinn J. C., Scolson D., 2021, ApJ, 908, L6
Rodriguez A. W., Capasso F., Johnson S. G., 2011, Nat. Photonics, 5, 211
Sakharov A. D., 1967, Doklady Akademii Nauk SSSR, 177, 70
Scheel S., 2014, in Forces of the Quantum Vacuum. World Scientific, Singapore, p. 107
Scolson D. M. et al., 2018, ApJ, 859, 101
Service R. F., Cho A., 2010, Science, 330, 1622
Simpson W. M. R., Leonhardt U., eds, 2015, Forces of the Quantum Vacuum. World Scientific, Singapore
Solís J., Casertano S., Riess A. G., 2021, ApJ, 908, L5
Unruh W. G., 1976, Phys. Rev. D, 14, 870
Van Leeuwen F., Feast M. W., Whitelock P. A., Laney C. D., 2007, MNRAS, 379, 723
Verde L., Treu T., Riess A. G., 2019, Nat. Astron., 3, 891
Weinberg S., 1989, Rev. Modern Phys., 61, 1
Zeldovich Y. B., 1968, Soviet Physics Uspekhi (SPU), 11, 381
Zhao R., Li L., Yang S., Bao W., Xia Y., Ashby P., Wang Y., Zhang X., 2019, Science, 364, 984

**APPENDIX A: THE VACUUM STATE**

The equation governing the evolution of $\Omega_{LC}$, the second equation in equations (2), which we introduced in Section 2.2 as

$$H^2 \Omega_{LC} = 8\pi G H^3 \rho^{-1}.$$  \hspace{1cm} (A1)

was introduced in Leonhardt (2019) in a different form (See Eq. (21) there):

$$H^2 \Omega_{LC} = 8\pi G H \left( \dot{\rho}^{-1} + H \rho \dot{H}^{-1} \right).$$  \hspace{1cm} (A2)

The difference between these two forms of $\Omega_{LC}$ stems from different definitions of the cosmologically relevant vacuum state.

The vacuum state of a quantum field is defined as the state that gets annihilated by all of the annihilation operators. Each set of creation and annihilation operators is defined in a specific coordinate system (Fulling 1973), and as a result, the definition of the vacuum state also depends on that coordinate system (Fulling 1973). Consider a state that gets annihilated by all the annihilation operators in one frame of reference and thus appears as a vacuum there; this same state might not be annihilated by all the annihilation operators in another frame and hence appear as an excited state there (Fulling 1973). This fact sometimes goes unappreciated or misunderstood, but it is known for a long time now (Fulling 1973). The best-known example is the Unruh-Fulling-Davies effect (Fulling 1973; Davies 1975; Unruh...
B1 Late universe
To obtain the late-universe evolution, we start from equation (2), which we write again here:
\[
\begin{aligned}
H^2(a) &= \frac{\Omega_m a^{-4} + \Omega_m a^{-3} + \Omega_{\Lambda C}}{\Omega_m a^{-4} + \Omega_m a^{-3} + \Omega_{\Lambda C}}, \\
H^2_0 \Omega_{\Lambda C} &= 8\Omega_m H_0^2 H^{-1}.
\end{aligned}
\] (B1)

These two coupled equations describe the mutual interaction between the cosmic expansion and the evolution of dark energy. By solving these equations, we can find the evolution of the background universe (homogeneous and isotropic) according to Lifshitz cosmology. The problem is that these equations are not easy to solve. As mentioned in Section 3, even obtaining a reliable numerical solution is considerably hard for the following two reasons. First, the equation for the dark energy’s dynamics, the second equation in equation (B1), depends on high derivatives of the scale–factor \( a \) (up to fourth-order derivative); this is a problem because, in differential equation solvers, the highest derivatives take the lead, whereas in reality, for most of the period of interest, the dynamics of \( \Omega_{\Lambda C} \) are a mere correction to the dynamics of the universe. Second, equation (B1) constitute what is known as ‘stiff equations,’ causing havoc with step size and accuracy. Therefore, we will approximate and solve perturbatively, where \( \alpha_A \) will be our small parameter. As also discussed in Section 3, our first simplification will be dropping the radiation term \( H^2_0 \Omega_{\Lambda C} a^{-4} \).

Next, the combination \( H_0^2 \Omega_{\Lambda C} \), that appears in the first equation of equation (B1) is proportional to the present-day physical density of matter \( \rho_0 \); as this density is a physical entity, \( H_0^2 \Omega_{\Lambda C} \) should be a model-independent combination. Indeed, \( H_0^2 \Omega_{\Lambda C} \) can be determined model–independently by the relative heights of the CMB acoustic peaks (Planck Collaboration 2014). Therefore, we may replace \( \omega_m \equiv \Omega_m h^2 \) (where \( h \equiv H_0 / 100 [\text{km s}^{-1} \text{Mpc}^{-1}] \)) in equation (B1) by its \( \Lambda \text{CDM} \)’s equivalent
\[
\omega_m^p \equiv \frac{\Omega_m h^2}{\rho_c} = 0.1430 \pm 0.0011
\]
(Planck, TT,TE,EE+lowE+lensing),
\] (B2)
where \( [\Omega_m h^2] \) is obtained by Planck’s TT,TE,EE+lowE+lensing \( \Lambda \text{CDM} \) analysis (Planck Collaboration 2020) (the ‘p’ superscript denotes that we use Planck’s \( \Lambda \text{CDM} \) value). In the following, we will also use Planck’s value (Planck Collaboration 2020) of
\[
\omega_\Lambda^p \equiv \frac{\Omega_\Lambda h^2}{\rho_c} = 0.3107 \pm 0.0082
\]
(Planck, TT,TE,EE+lowE+lensing).
\] (B3)
Now, for mathematical convenience, we re–scale and re–define the variables
\[
v \equiv \ln \left[ (\omega_m^p / \omega_\Lambda^p)^{1/3} a \right], \quad \xi \equiv \sqrt{\omega_m^p} t, \quad \eta \equiv \frac{\omega_L C}{\omega_\Lambda^p},
\] (B4)
where \( \omega_L C \equiv \Omega_{\Lambda C} h^2 \). We also regard \( \xi \) (time) as a function of \( v \) (scale–factor) and define \( \Theta \) as the derivative of \( \xi \) with respect to \( v \).

It is easy to show that \( \Theta = \sqrt{\omega_\Lambda^p} H^{-1} \) (expressed in terms of \( a \) and \( t \)):
\[
\Theta \equiv \frac{d\xi}{dv} = \left( \frac{dv}{d\xi} \right)^{-1} = \left( \frac{da}{d\xi} \frac{dv}{da} \right)^{-1} = \sqrt{\omega_\Lambda^p} \left( \frac{da}{dt} \right)^{-1} \equiv \sqrt{\omega_\Lambda^p} H^{-1}.
\] (B5)

Then, equation (B1) become (dropping the radiation term and replacing \( \omega_m \) with \( \omega_m^p \))
\[
\theta = (e^{-3v} + \eta^{-1/2}), \quad \eta = 8 \alpha_A \frac{1}{\theta^2} \left( \frac{\theta'}{\theta} - 1 \right) \frac{\theta'}{\theta^2}
\] (B6)
\] (B7)
where $\theta = 100[\text{km s}^{-1}\text{Mpc}^{-1}]\Theta$ and the primes indicate differentiation with respect to $v$.

At this point, we notice that given a solution $\{\theta(v), \eta(v)\}$ of equations (B6) and (B7), it is easy to show that $\{\theta_0(v), \eta_0(v)\}$ is also a solution, where

$$\theta_0(v) = \theta_0(v - \delta)e^{(3/2)\delta}, \quad (B8)$$

$$\eta_0(v) = \eta_0(v - \delta)e^{-3\delta} \quad (B9)$$

for some constant $\delta$. One immediate result is that from any solution $\{\theta_0(v), \eta_0(v)\}$ with asymptotic behaviour $\eta_0 \to 1$ for $v \to +\infty$, we may construct a solution $\{\theta(v), \eta(v)\}$ with any other constant asymptotic behaviour by choosing an appropriate value for $\delta$, as may be seen from

$$\lim_{v \to +\infty} \eta_0(v) = \lim_{v \to +\infty} \eta_0(v - \delta)e^{-3\delta} = e^{-3\delta}.$$  

(B10)

This relation translates $\delta$ to the integration constant in equation (B7) when integrating $\eta$ from far in the future ($v \to +\infty$) to some other value $v'$; thus, we can solve the equations with a convenient integration constant and later find the physically relevant one by adjusting $\delta$. This symmetry also translates $\delta$ to a value of $\omega$ in the far future

$$\omega_\infty = \omega_\infty^P e^{-3\delta}, \quad (B11)$$

where $\omega_\infty = \lim_{v \to +\infty} \omega_\infty L(v)$. Thus, any solution $\theta(v; \alpha_\Lambda, \delta)$ is characterized by two parameters $\alpha_\Lambda$ and $\delta$.

Now, we proceed in two steps. The first one is to find $\eta$ up to first-order in $\alpha_\Lambda$ with $\eta_\infty = 1$ (equivalent to $\delta = 0$) from equation (B6):

$$\theta(0) = (e^{-3\delta} + 1)^{-1/2}.$$  

(B12)

Then, we plug $\theta(0)$ into equation (B7) and integrate it with $\eta(\infty) = 1$ to find $\alpha_\Lambda$:

$$\eta(1) = 1 + 18\alpha_\Lambda \left[ \ln(e^{-3\delta} + 1) - \frac{3}{e^{3\delta} + 1} \right].$$  

(B13)

Finally, we plug $\eta(1)$ into equation (B6) to find $\alpha_\Lambda$ and then use $\theta(1)$ in equation (B8) to find a general and adjustable (first-order) solution:

$$\theta(v; \alpha_\Lambda, \delta) = e^{(3/2)\delta} \sqrt{\frac{1}{e^{-3\delta} + 1} + 18\alpha_\Lambda \ln(e^{-3\delta} + 1) - \frac{3}{e^{3\delta} + 1}}.$$  

(B14)

We do not go beyond the first-order since, as it turns out, the expansion in $\alpha_\Lambda$ is a divergent series. By undoing the re-scalings and re-definitions of equation (B4), taking $\theta = 100[\text{km s}^{-1}\text{Mpc}^{-1}]\sqrt{\omega_\Lambda H}^{-1}$ and reinstating the original $\omega_\Lambda$, we get equation (3).

For the second step, let us express $D_M$ with our terminology here:

$$D_M = \frac{c}{100[\text{km s}^{-1}\text{Mpc}^{-1}]} \int_{v_0}^{v_0} \theta(v; \alpha_\Lambda, \delta)e^{-3\delta} dv' = \int_{v_0}^{v_0} \theta(v; 0, \delta)e^{-3\delta} dv'.$$  

(B15)

where $v_0 = \frac{1}{2} \ln \left[ \frac{\omega_\Lambda}{\omega_\Lambda + \omega_\Omega} \right]$, $v_s = v_0 - \ln(1 + z_s)$. Hence, the demand $D_M = D_M^{(\Lambda CDM)}$ becomes

$$\int_{v_s}^{v_0} \theta(v; \alpha_\Lambda, \delta)e^{-3\delta} dv' = \int_{v_s}^{0} \theta(v; 0, 0)e^{-3\delta} dv'.$$  

(B16)

as $\alpha_\Lambda = 0$ and $\delta = 0$ gives the $\Lambda CDM$ solution. Lastly, since for times before last-scattering Lifshitz cosmology and $\Lambda CDM$ (approximately) coincide, we may take, for numerical simplicity, the lower limit of the integration on both sides of equation (B16) to minus infinity:

$$\int_{-\infty}^{v_0} \theta(v; \alpha_\Lambda, \delta)e^{-3\delta} dv' = \int_{-\infty}^{0} e^{-3\delta} dv' = 2 \left[ \frac{\alpha_\Lambda}{\omega_\Lambda} \right]^{1/6} \zeta F_1 \left( \frac{1}{6}, \frac{1}{6} \cdot \frac{1}{6}, \frac{1}{6}, \frac{\omega_\Lambda}{\omega_\infty^P} \right).$$

(B17)

where $\theta(v; \alpha_\Lambda, \delta)$ is given by equation (B14) and $\zeta$ is Gauss’ hypergeometric function. We can numerically solve equation (B17) for a given $\theta < \alpha_\Lambda < 1$ to obtain the corresponding value of $\delta$. (For a given $\theta < \alpha_\Lambda < 1$, $\delta$ is a monotonic function of $D_M$.) Thus, we get the relationship presented in Fig. 1 (where we translated $\delta$ into $\omega$).

To estimate the errors in $\delta$ (or equivalently, $\sigma_\delta$) that we obtain with the procedure above, we estimate the errors in $D_M^{(\Lambda CDM)}$ as

$$\sigma_\delta = \sqrt{\left( \frac{\partial D_M^{(\Lambda CDM)}}{\partial \lambda^P} \right)^2 + \left( \frac{\partial D_M^{(\Lambda CDM)}}{\partial \sigma_\delta} \right)^2}.$$  

(B18)

And according to equation (B15) we get

$$s_D M = \left( \frac{c (\omega_\Lambda^P)^{-1} (\sigma_\Lambda^P)^{-1}}{100[\text{km s}^{-1}\text{Mpc}^{-1}]} \right)^{-1} \sigma_\delta M \approx 0.00363,$$

(B19)

where we use the values from Planck Collaboration et al. (2020). Then, we calculate the errors in $\delta$ as $\sigma_\delta = \left| \delta - \delta_0 \right|$, where $\delta_0$ is calculated by adding $s_D M$ to the right-hand side of equation (B17) and solving for $\delta$, and $\sigma_\delta$ is calculated by subtracting $s_D M$ from the right-hand side. This estimation procedure makes sense as $\delta$ is a monotonic function of $D_M$. These errors in $\delta$ (or $\sigma_\delta$) are estimates solely based on Planck’s errors in $D_M$ and are probably underestimated.

### B2 Early universe

To see how Lifshitz cosmology modifies the early dynamics, we must go back to equation (B1) and begin all over again without dropping the radiation term. This term complicates matters. For example, the scaling and shifting symmetry of equations (B8) and (B9) does not hold anymore, and adjusting the physical value of the integration constant becomes difficult. Another issue is that we can no longer assume that the sound horizon at last-scattering $r_s$ is unchanged. Therefore, to find the integration constant, keeping the spirit of our analysis of the late universe, we should demand that the CMB’s angular acoustic scale $\theta_\Lambda = r_s / D_M$ is unchanged: $\theta_\Lambda = \theta_\Lambda^{(\Lambda CDM)}$ (instead of demanding $D_M = D_M^{(\Lambda CDM)}$ as before).

To go around these issues, we will approximate by building upon our late-universe result. This time, we remain with $t$ and $\omega_L$ without re-scaling, and instead of $v$, we take

$$\bar{v} \equiv \ln a,$$

(B20)

and define $\bar{\theta} = dt / d\bar{v} = H^{-1}$ and $\bar{\theta} = 100[\text{km s}^{-1}\text{Mpc}^{-1}] \bar{\theta}$, then we get

$$\bar{\theta} = (\omega_\Lambda e^{-3\delta} + \omega_\Lambda P e^{-3\delta} + \omega_\Lambda L) C^{-1/2},$$

(B21)

$$\omega_\Lambda L = 8 \alpha_\Lambda \frac{1}{\bar{\theta}} \frac{1}{\bar{\theta}} \left( \frac{\bar{\theta}'}{\bar{\theta}'} + \frac{1}{2} \frac{\bar{\theta}''}{\bar{\theta}'} \right),$$

(B22)
where now primes indicate differentiation with respect to $\nu$, and $\omega_r$ can be determined (model-independently) from the Stefan–Boltzmann law of black–body radiation with the present–day average CMB temperature ($T_0 = 2.7255$ K (Fixsen 2009)):

$$\omega_r = \frac{8\pi G}{(100[km/s/Mpc])^2} \left[ 1 + \frac{3}{8} \left( \frac{4}{11} \right)^{4/3} \right] \frac{\pi^2(k_B T_0)^4}{15c^5 \hbar^3} = 4.15 \cdot 10^{-5}. \quad \text{(B23)}$$

At this point, we use our late–universe result. For the zeroth-order solution ($\alpha/\Lambda_1 = 0$), we take $\theta(0) = (\omega_r e^{-4\nu} + \omega_p e^{-3\nu} + \omega_\infty)^{1/2}$ with $\omega_\infty = \omega_p e^{-3\nu}$ (equation B11), where we use the value of $\delta$ we found in the late-universe solution. Then, we find $\omega_{LC}$ up to first–order in $\alpha/\Lambda_1$ by integrating equation (B22) with $\theta(0)$ using $\omega_\infty$ as the integration constant:

$$\omega_{LC} = 8 \alpha_\Lambda \int_{\infty}^{\ln a} \frac{1}{(\theta(0))^2} \left( \frac{\partial}{\partial \theta} \frac{(\theta(0))^2}{\theta(0)} - \frac{1}{2} \left( \frac{\theta(0)}{\theta(0)} \right)^2 \right) d\theta + \omega_\infty. \quad \text{(B24)}$$

Then, we may use $\omega_{LC}$ to calculate the relative contribution of the vacuum energy, $f_{de}$, throughout the entire evolution of the cosmos:

$$f_{de} = \frac{\omega_{LC}}{\omega_r a^{-2} + \omega_p a^{-3} + \omega_{lc}}. \quad \text{(B25)}$$

The results are shown in Fig. 6.

**APPENDIX C: TOY MODEL**

In this paper, we have introduced and analyzed for the first time Lifshitz cosmology with the specific goals of solving for the theory’s dynamics and testing the viability of this theory. Therefore, we have avoided the more complex statistical analysis that must be eventually done to draw firm conclusions; this analysis is left open for future research. As we mentioned in the discussion section (Section 6), our analysis treats $\alpha_\Lambda$ and $\Omega_\infty h^2$ as two mutually dependent parameters related via equation (4), whereas in reality, they are two independent parameters. By treating these parameters independently, a future analysis may yield better results. To demonstrate this idea, we arbitrarily choose the combination $\alpha_\Lambda = 0.025$ and $\Omega_\infty h^2 = 0.4625$, which produces $D_M$ that is (statistically insignificant) $1.7\sigma$ away from its Planck value; we call this realization ‘toy model.’ This toy model gives a nominal value of $H_0 = 71.7 [km/s/Mpc^{-1}]$, and it shrinks $\Delta M_B$ by about 43 per cent ($M_B^* = -19.346$) with RMSD = 0.1466, and thus it performs better than our M2. In addition, the toy model seems to fit the shape of the binned SN Ia data, the distance–ladder–independent $E - 1$ measurements from SN Ia, and the BAO measurements of $H(z)$ better than M1 (and comparably or better than M2), see Figs C1–C3. Overall, the toy model seems to work better than M1 and M2 and might be closer to the truth. Rigorous statistical analysis will probably produce an even better realization.

**Figure C1.** Same as Fig. 3; showing the prediction by the toy model ($\alpha_\Lambda = 0.025$ and $\Omega_\infty h^2 = 0.4625$), see Appendix C. The orange points at the bottom panel show the binned Pantheon data with $M_B = -19.346$, the best $M_B$ in terms of RMSD (equation 8) for the Toy model.
Figure C2. Same as Fig. 4; showing the prediction by the toy model ($\alpha_\Lambda = 0.025$ and $\Omega_\infty h^2 = 0.4625$), see Appendix C.

Figure C3. Same as Fig. 5; showing the prediction by the toy model ($\alpha_\Lambda = 0.025$ and $\Omega_\infty h^2 = 0.4625$), see Appendix C.