Precise single-qubit control of the reflection phase of a photon mediated by a strongly-coupled ancilla–cavity system

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Abstract

We propose to use the interaction between a single qubit atom and a surrounding ensemble of three level atoms to control the phase of light reflected by an optical cavity. Our scheme employs an ensemble dark resonance that is perturbed by the qubit atom to yield a single-atom single photon gate. We show here that off-resonant excitation towards Rydberg states with strong dipolar interactions offers experimentally-viable regimes of operations with low errors (in the $10^{-3}$ range) as required for fault-tolerant optical-photonic, gate-based quantum computation. We also propose and analyze an implementation within microwave circuit-QED, where a strongly-coupled ancilla superconducting qubit can be used in the place of the atomic ensemble to provide high-fidelity coupling to microwave photons.

1. Introduction

The transmission of quantum information between remote quantum systems represents one of the main technical bottlenecks to the scalability of quantum networks for distributed quantum computing, cryptography, metrology and sensing [1]. Proposals to use light to interlink quantum degrees of freedom of spatially separated nodes fall broadly in two categories. The first engages direct transmission of non-classical states of light [1–8], while the second heralds non-local quantum correlations by joint measurements on signals that are emitted from or have sequentially interacted with spatially separated quantum systems [9–18].

In all cases it is pertinent to have an efficient coupling of the matter and light degrees of freedom, which can be achieved when high Q cavities are used to enhance the coupling of even a single-atom with quantum light [19, 20], and when photons interact with the collective quantum degrees of freedom of large ensembles of atoms. In the former case, the matter–light interaction has enabled quantum gates on the atomic and photonic qubits [4], while in the latter, long-range interactions between the atoms have been used to establish effective optical nonlinearities, e.g., when a delocalized single Rydberg excitation disrupts the propagation of slow light by electromagnetically induced transparency (EIT) through the surrounding medium [18, 21–23].

Ensembles of interacting atoms can be employed to create and manipulate non-classical states of light [24–27] to enable qubit interactions between separate single photon wave packets [5, 8]. It is also possible to address collective qubit degrees of freedom via different internal atomic states [28, 29], but effective coupling of single photons to single atomic qubits that form part of a local register with computing, memory or sensing capabilities remains a challenge. Thus far, experimental single-atom cavity QED [41] has achieved errors in the 10% range with high Q cavities, while proposals based on strongly-coupled Rydberg ensembles, see figure 1(A), suggest this can be brought down to the few % level [30]. However, these proposals involve a conditional transfer (shelving) of population into a qubit atom Rydberg state, which is subject to much shorter lifetimes than the hyperfine ground states and to leakage to nearby Rydberg levels. This has caused Rydberg excitation gates to be limited to the 80% fidelity range [31, 32].

In this work, we build on earlier proposals that use Rydberg blockade to alter the EIT condition for a large ensemble of atoms interacting with a cavity field, and thus alter the reflection property of the cavity surrounding the atoms [5, 8, 30]. While retaining the advantages of the collectively enhanced matter–light interaction, we
instead explore different control and interaction mechanisms and we show that qubit–photon high-fidelity gates can be achieved with significantly reduced excitation of the Rydberg levels. We also generalize the ideas presented to other physical systems such as superconducting circuit-QED with microwave optics, see figure 1(B), showing that these techniques can be applied to a variety of different matter systems that couple strongly to single photons.

The article is organized as follows. In section 2, we present the physical system and we review the cavity EIT mechanism introduced in [8, 30]. In section 3, we present a complete input–output quantum analysis of the general problem, which includes population transfer within the qubit degrees of freedom. We derive a solution of the Schrödinger equation for the combined discrete atomic and cavity degrees of freedom and continuous incident and reflected field modes. In section 4, we present results showing the scaling of the fidelity and identify novel interaction mechanisms for the single-atom single photon phase gate while scanning over different physical parameters. Section 5 discusses implementation of the main physical mechanism of our proposal in circuit-QED taking advantage of the strong coupling available in such systems. Section 6 summarizes the results of the article.

2. The physical light–matter system and previous work

The physical system and atomic level schemes are depicted in figures 1(A) and 2, respectively. The (orange) qubit atom is selectively excited towards a Rydberg state $|2_a\rangle$ from the $|1_s\rangle$ (hyperfine ground) state by a classical laser field with Rabi frequency $\Omega$ and detuning $\Delta$. The ensemble of $N$ (blue) ancilla atoms interacts with the cavity field, exciting the ground product state $|0_a\rangle = |0, 0, \ldots, 0\rangle_a$ to the collectively excited state, $|1_a\rangle = (|1, 0, \ldots, 0\rangle_a + |0, 1, 0, \ldots, 0\rangle_a + |0, 0, 1, \ldots, 0\rangle_a + \cdots + |0, \ldots, 1\rangle_a)/\sqrt{N}$, with the $\sqrt{N}$ times enhanced collective amplitude $G$ for the absorption of a single cavity photon. In the collectively excited state $|1_a\rangle$, the ensemble is subject to strong classical driving with the Rabi frequency $\Omega$ to the collectively excited Rydberg state $|2_a\rangle = (|2, 0, \ldots, 0\rangle_a + |0, 2, 0, \ldots, 0\rangle_a + |0, 0, 2, \ldots, 0\rangle_a + \cdots + |0, \ldots, 2\rangle_a)/\sqrt{N}$. The system has a collective dark state with either a single cavity photon or a shared excitation in the atoms, $|\psi_0\rangle = (G|0\rangle - \Omega |1\rangle |0\rangle)/\sqrt{G^2 + \Omega^2}$, which does not coupling to the intermediate short lived state of the atoms. The energy of the dark state equals the energy of a bare cavity photon, and if the cavity mode and all atoms are initially in the ground state, an incident photon resonant with the bare cavity may resonantly excite the dark state. In contrast, when the qubit atom is excited to its Rydberg state and the strong dipolar interactions effectively block excitation of the nearby ancilla Rydberg states, a large number of ancilla atoms behave as two level atoms and their strong coupling to the cavity mode shifts the cavity resonance and prevents excitation by the incident photon. It was shown in [8, 30], that due to the transfer into the dark state and back into the propagating field, the phase of the reflected photon differs by $\pi$ from the case where the photon is excluded from entering the cavity. In free space, the disruption of EIT leads to absorption, and e.g., to anti-bunched transmission [22, 23, 33], while, by splitting the cavity frequency, it yields a dispersive effect for the photon scattering process (see [34] for recent dispersive variants of free space Rydberg EIT).

Note that either the photon never enters the cavity, or it enters into a mostly photonic state (if $\Omega \gg G$), where the ancilla atoms are only little excited. The qubit atom may have no optical transition in the frequency range of the photons, offering possibilities to separately optimize the qubit lifetime and transmission wavelengths of the quantum network.

A complete input–output theory was employed in [30] to assess the fidelity of the phase gate due to decay and losses and the finite bandwidth of the cavity and the dark state mechanism. A simple estimate of the loss of fidelity due to atomic decay showed that the excitation of the qubit Rydberg state during the reflection process may, indeed, be the dominant error. This is the motivation for the present study, where we retain the qubit atom...
wavefunction mostly within its hyperfine ground states, with a correspondingly reduced Rydberg decay probability.

3. Input–output theory in the Schrödinger picture

As the photon may not only be reflected but its waveform may be entangled with the atoms, we need to account for its continuum wavefunction along with the amplitudes on the various discrete collective states of the cavity field and the atoms. To this end it is useful to apply input–output theory in the Schrödinger picture, rather than in the usually employed Heisenberg picture [35]. In the same way as the input–output theory simplifies when the coupling terms and Hamiltonians are second order in oscillator quadrature operators and the coupled Heisenberg picture equations of motion are linear, the Schrödinger picture equations simplify considerably, when only a single quantum of excitation is introduced and shared between the different components of the system.

3.1. Empty cavity

In this section, we review the equations of motion for the amplitude on the state with an empty cavity illuminated by a one photon continuum wave packet. We expand the state of the system containing a single photon as

\[ |\Psi(t)\rangle = C(t) |1_c, 0_{ph}\rangle + \int \! \! d\omega \phi(\omega, t) |0_c, 1_{ph}(\omega)\rangle, \]

with \( |1_c, 0_{ph}\rangle = \hat{b}^\dagger |0_c, 0_{ph}\rangle \) denoting a photon in the cavity mode and \( |0_c, 1_{ph}(\omega)\rangle = a^\dagger(\omega) |0_c, 0_{ph}\rangle \) in the field eigenmode with frequency \( \omega \) (we assume a definite polarization of the photons and atomic transitions in accordance with the polarization selection rules).

We apply a rotating frame with respect to the cavity mode, so that \( \omega \) denotes the field detuning with respect to the cavity, and the fields are described by the free field Hamiltonian

\[ H_f = \hbar \int \! \! d\omega \, \omega a^\dagger(\omega)a(\omega), \]

and by the coupling due to the input mirror

\[ H_M = i\hbar \int \! \! d\omega \, g(\omega) [a^\dagger(\omega)b - b^\dagger a(\omega)]. \]

Figure 2. The level diagram is plotted for the atom cloud distribution in figure 1(A) as a function of distance between the qubit and ancilla atoms in (A) and (B) for red and blue detuned dressing, respectively. In the red-detuned case, the ancilla EIT resonance is merely shifted, while for blue detuning, the interaction energy \( B(r) \) and detuning \( \Delta \) match and the resonant coupling to \( |1_q, 2_a\rangle \) with both atoms in Rydberg excited states splits the EIT resonance.
The Schrödinger equation yields for \( \phi(\omega, t) \),

\[
\dot{\phi}(\omega, t) = -i\omega \phi(\omega, t) + g(\omega) C(t),
\]

which can be integrated directly, assuming the expansion of the incident photon wave packet on the field eigenmodes at time \( t = 0 \) prior to the reflection process,

\[
\phi(\omega, t) = e^{-i\omega t} \phi(\omega, 0) + g(\omega) \int_0^t ds e^{-i\omega(t-s)} C(s).
\]

Inserting this expression in the equation for \( C(t) \) and employing the Born–Markov approximation yields

\[
\dot{C}(t) = -\frac{\kappa}{2} C(t) - \sqrt{\kappa} \beta_{in}(t),
\]

where we have introduced \( \kappa = 2\pi |g(\omega)|^2 \) evaluated at the cavity eigenfrequency, and

\[
\beta_{in}(t) = \frac{1}{2\pi} \int d\omega e^{-i\omega t} \phi(\omega, 0),
\]

which represents the time dependent arrival of the incident photon wave packet on the input mirror. Note that we only need to solve a single equation for the intracavity photon amplitude subject to a driving term \( \beta_{in}(t) \), given by the shape of the incident wave packet.

Integrating (6) backwards in time from time \( t = T \), long after the reflection process, yields

\[
\phi(\omega, t) = e^{-i\omega(t-T)} \phi(\omega, T) - g(\omega) \int_T^T ds e^{-i\omega(t-s)} C(s).
\]

Inserting this expression in the equation for \( C(t) \) yields

\[
\dot{C}(t) = \frac{\kappa}{2} C(t) - \sqrt{\kappa} \beta_{out}(t),
\]

where \( \beta_{out}(t) = \frac{1}{2\pi} \int d\omega e^{-i\omega(t-T)} \phi(\omega, T) \) is the time dependent shape of the photon wave packet as it propagates away from the cavity mirror.

Subtracting the two equations for \( \dot{C}(t) \) yields the input–output relation \( \beta_{out}(t) = \beta_{in}(t) + \sqrt{\kappa} C(t) \). I.e., after having solved equation (6) for \( C(t) \), we obtain the waveform of the reflected photon. Passing to the frequency domain by a Fourier transformation, equation (6) becomes an algebraic equation,

\[
-\omega C(\omega) = -i\frac{\omega}{2} \dot{C}(\omega) - i\sqrt{\kappa} \beta_{in}(\omega),
\]

and we readily find the frequency dependent reflection coefficient of the cavity,

\[
R(\omega) = \frac{\beta_{out}(\omega)}{\beta_{in}(\omega)} = 1 - \frac{\kappa}{\omega^2 - \kappa/2}.
\]

This coefficient is equal to unity for \( |\omega| \gg \kappa \) and changes sign close to resonance \( |\omega| \ll \kappa \).

### 3.2. Cavity field and atomic system

When the cavity field interacts with atoms inside the cavity, we must identify and solve the corresponding coupled equations for the amplitudes of the states occupied by the system, with equation (6) providing the coupling to the input field.

#### 3.2.1. Hamiltonians

The qubit atom has three states: two low-lying qubit states \( |0_q\rangle, |1_q\rangle \) and a Rydberg excited state \( |2_q\rangle \). We assume that prior to the arrival of the light pulse, an adiabatic or suitably tailored pulse drives the \( |1_q\rangle \) qubit level into a dressed eigenstate of the Hamiltonian,

\[
\hat{H}_{qub} = \Delta |2_q\rangle \langle 2_q| + \epsilon (|1_q\rangle \langle 2_q| + |2_q\rangle \langle 1_q|),
\]

while the uncoupled qubit state \( |0_q\rangle \) is left unchanged. The laser field with detuning \( \Delta \) and Rabi frequency \( \epsilon \) is left on during the entire reflection process, and it is useful to introduce the dressed eigenstates,

\[
|1_q\rangle = \cos(\Theta)|1_q\rangle - \sin(\Theta)|2_q\rangle, \quad |2_q\rangle = \cos(\Theta)|2_q\rangle + \sin(\Theta)|1_q\rangle
\]

with \( \Theta = \tan^{-1}(2\epsilon/\Delta)/2 \) and the energy separation given by \( \Delta = E_2 - E_1 = \sqrt{\Delta^2 + 4\epsilon^2} \).

Each ancilla atom, labeled here by an index \( m \), couples to a classical field with Rabi frequency \( \Omega \) and to the cavity field with coupling strength \( g_m \)

\[
\hat{H}_{anc}^m = \Omega (|1_m^m\rangle \langle 2_m^m| + \text{h.c.}) - \frac{\Gamma}{2} (|1_m^m\rangle \langle 1_m^m| - |2_m^m\rangle \langle 2_m^m|),
\]

\[
\hat{H}_{anc}^b = g_m (|0_m^m\rangle \langle 1_m^m| + b^\dagger |1_m^m\rangle \langle 2_m^m|),
\]

where we have included damping terms, representing the decay of the excited ancilla states. Such decay will cause a loss of norm, ultimately reflected in a reduction of the output single photon field amplitude, and it will constitute part of the gate error of our protocol. Note that we did not incorporate similar decay terms in the qubit Hamiltonian. We shall treat the dressed states as being populated throughout the photon reflection.
process, and estimate the qubit decay error from the excited state population in the dressed state and the duration of the gate.

Finally, we represent the Rydberg interaction between the excited qubit and an ancilla atom by the blockade strength $B_m$,

$$H_{\text{block}} = B_m |\tilde{2}_q q \rangle \langle \tilde{2}_q q|,$$

where different values of $B_m$ are due to different distances $r$ between the ancilla and qubit atoms.

Resonance conditions and polarization selection rules determine which Rydberg excited states are laser coupled to the ground states. The atoms may be exposed to electric fields, polarizing the Rydberg states and causing direct dipolar interactions, $B_m = C_m/r_m^3$, or coupling of different Rydberg product states may perturb the energy levels, giving also rise to energy shifts on the form $B_m = C_m/r_m^3$ near (Förster) resonances [30, 37]. We note that the dipolar interactions are generally anisotropic [30, 37], but as this will only change our results by a numerical factor of order unity, we shall for simplicity consider isotropic interactions in our numerical examples. Mutual interaction between excited ancilla atoms does not occur, since the incident single photon field allows only the excitation of a single ancilla atom.

The blockade interaction can be written in the dressed state basis of the qubit

$$H_{\text{block}}^m = \cos^2(\Theta) B_m |\tilde{2}_q q \rangle \langle \tilde{2}_q q| + \sin^2(\Theta) B_m |\tilde{1}_q q \rangle \langle \tilde{1}_q q|$$

$$+ B_m \cos(\Theta) \sin(\Theta) (|\tilde{2}_q q \rangle \langle \tilde{1}_q q| + |\tilde{1}_q q \rangle \langle \tilde{2}_q q|).$$

Transitions into the dressed states $|\tilde{2}_q q\rangle$ are suppressed by the energy separation $\Delta$ between the dressed states.

3.2.2. Schrödinger equation

We solve the Schrödinger equation by expanding the state of the entire system excited by the incident photon wave packet on a complete basis,

$$|\psi(t)\rangle = C_1(t)|\tilde{1}_q q\rangle + C_2(t)|\tilde{2}_q q\rangle + \sum_m A_m^m(t)|0_q, \tilde{1}_q l_m^m\rangle + A_{\bar{m}}^m(t)|0_q, \tilde{2}_q l_m^m\rangle$$

$$+ \sum_m B_m^m(t)|0_q, \tilde{1}_q l_m^m\rangle + B_{\bar{m}}^m(t)|0_q, \tilde{2}_q l_m^m\rangle,$$

where the collective ancilla states with the $m$th atom excited is denoted $|12_q l_m^m\rangle = |0_q, 0_q, ..., 12_q, ..., 0_q\rangle$.

If the qubit atom is initially in the state $|\tilde{1}_q q\rangle$ and, thus, transferred to the dressed state $|\tilde{1}_q q\rangle$, the Schrödinger equation for the amplitudes, incorporating Hamiltonians (10)–(12), read ($\hbar = 1$)

$$i\dot{C}_1 = \sum_m g_m A_m^m(t) - i\frac{\kappa}{2}C_1 - i\sqrt{\kappa} \beta_m(t), \quad i\dot{C}_2 = \sum_m g_m A_{\bar{m}}^m(t) + \left( \Delta - \frac{i\kappa}{2} \right) C_2(t),$$

$$i\dot{A}_1^m = \Omega B_m^m + g_m C_1 - \frac{\Gamma}{2} A_1^m, \quad i\dot{A}_{\bar{m}}^m = \Omega B_{\bar{m}}^m + g_m C_2 + \left( \Delta - \frac{\Gamma}{2} \right) A_{\bar{m}}^m(t),$$

$$i\dot{B}_1^m = \Omega A_1^m + \left( \sin^2(\Theta) B_m - \frac{i\kappa}{2} \right) B_m^m(t) + \cos(\Theta) \sin(\Theta) B_m B_{\bar{m}}^m(t),$$

$$i\dot{B}_{\bar{m}}^m = \Omega A_{\bar{m}}^m + \left( \cos^2(\Theta) B_m + \Delta - \frac{i\kappa}{2} \right) B_m^m(t) + \cos(\Theta) \sin(\Theta) B_m B_{\bar{m}}^m(t).$$

Note that the input field enters as the inhomogeneous term in the first equation, pertaining to the initial dressed state of the qubit atom, as derived in equation (6). The equations (14) can be converted to algebraic equations in frequency domain by a Fourier transform,

$$-\omega \tilde{C}_1(\omega) = \sum_m g_m \tilde{A}_1^m(\omega) - i\frac{\kappa}{2} \tilde{C}_1(\omega) - i\sqrt{\kappa} \tilde{C}_m(\omega), \quad -\omega \tilde{C}_2(\omega) = \sum_m g_m \tilde{A}_{\bar{m}}^m(\omega) + \left( \Delta - \frac{i\kappa}{2} \right) \tilde{C}_2(\omega),$$

$$-\omega \tilde{A}_1^m(\omega) = \Omega \tilde{B}_m^m(\omega) + g_m \tilde{C}_1(\omega) - \frac{\Gamma}{2} \tilde{A}_1^m(\omega), \quad -\omega \tilde{A}_{\bar{m}}^m(\omega) = \Omega \tilde{B}_{\bar{m}}^m(\omega) + g_m \tilde{C}_2(\omega) + \left( \Delta - \frac{\Gamma}{2} \right) \tilde{A}_{\bar{m}}^m(\omega),$$

$$-\omega \tilde{B}_1^m(\omega) = \Omega \tilde{A}_1^m(\omega) + \left( \sin^2(\Theta) B_m - \frac{i\kappa}{2} \right) \tilde{B}_m^m(\omega) + \cos(\Theta) \sin(\Theta) \tilde{B}_m \tilde{B}_{\bar{m}}^m(\omega),$$

$$-\omega \tilde{B}_{\bar{m}}^m(\omega) = \Omega \tilde{A}_{\bar{m}}^m(\omega) + \left( \cos^2(\Theta) B_m + \Delta - \frac{i\kappa}{2} \right) \tilde{B}_m^m(\omega) + \cos(\Theta) \sin(\Theta) \tilde{B}_m \tilde{B}_{\bar{m}}^m(\omega).$$

These equations can be readily solved in a sequence of analytical steps: first, the lower pair of equations is solved and the resulting $\tilde{B}_m^m, \tilde{B}_{\bar{m}}^m$ are inserted in the middle pair of equations. This allows solution for the pair of variables $\tilde{A}_1^m, \tilde{A}_{\bar{m}}^m$ in terms of $\tilde{C}_1$ and $\tilde{C}_2$. The resulting closed pair of equations for the two variables $\tilde{C}_1(\omega), \tilde{C}_2(\omega)$ is solved by inverting a $2 \times 2$ matrix with $\omega$-dependent coefficients where we may, however, have to evaluate the sums of terms representing the coupling to different ancilla atoms numerically. Even for several
thousand ancilla atoms, these sums are readily evaluated, while for much larger ensembles we have recourse to a binning of the atoms according to their interaction strengths $B_m$ with the qubit atom. The sums over $m$ in the equations for $C_1$ and $C_2$ can then be evaluated as population weighted sums or integrals.

3.2.3. Gate fidelity from reflection coefficient

The output field is given by the solution to the set of coupled equations (14) for the single photon amplitudes $C_1(t)$, $C_2(t)$. We are interested in the field amplitude, correlated with the qubit dressed state $|\tilde{T}_q\rangle$,

$$|\tilde{\beta}_{\text{out}}(\omega)\rangle = \tilde{\beta}_q(\omega) - \sqrt{\kappa C_1(\omega)},$$

while an (undesired) photon wave packet $\tilde{\xi}_{\text{out}}(\omega) = -\sqrt{\kappa C_2(\omega)}$ is correlated with transfer of the qubit to the dressed state $|\tilde{2}_q\rangle$. The outcome of the calculation is the complex reflection coefficient

$$R_m(\omega) = \frac{\tilde{\beta}_{\text{out}}(\omega)}{\tilde{\beta}_{\text{in}}(\omega)},$$

which determines the modification of the photon wave packet by the reflection process and thus forms the basis for the fidelity analysis of the gate. The reflection coefficient $R_m(\omega)$ pertaining to the qubit state $|0_q\rangle$ attains the same values as in [30],

$$R_0(\omega) = 1 - \kappa \left( \frac{\kappa}{2} - i\omega + \frac{G^2}{\Gamma - i\omega + |\Omega|^2/(\Gamma - i\omega)} \right)^{-1},$$  

(16)

where we have introduced the collectively enhanced cavity coupling, $G^2 = \sum_m |g_m|^2$. As we aim to produce a phase gate, the reflection coefficient should have a sign, $R_m(\omega) = \pm 1$, that depends on the qubit state and which is constant over the spectral components of the incident photon. The calculations include decay of the ancilla atoms through the damping rates $\Gamma, \gamma$, but decay of the qubit atom Rydberg level has so far been disregarded. The application of the Fourier transform to the equations (15) assume time independent coupling coefficients, and hence that the qubit atom is only excited before and becomes de-excited only after the reflections process. To estimate the loss of fidelity, due to decay of the qubit atom, we assume that this occurs independently of the evolution of the ancilla atoms and the field, and that it merely amounts to the decay probability from the dressed state $|1\rangle$ during the reflection of the field, yielding a reduction of the excited state amplitudes by the factor

$$\eta = \exp(-\gamma T \sin^2(\theta)/2).$$

In our numerical analysis we consider photon wave packets of the form

$$\phi(t) = \max\{0, \tilde{A}(\exp(-(t-T/2)^2/2\sigma^2) - \exp(-(T/2/8\sigma^2))\},$$

while the frequency dependence of the reflection coefficient favors narrow bandwidth pulses, the decay of the ancilla and qubit atoms during the reflection favors short, and hence broadband, pulses.

The reflection phase associated with the qubit states leads to an atom–photon phase gate, with the photonic qubit basis $|0(1)_{\text{ph}}\rangle$ represented by zero and one photons states, incident on the cavity. Due to the physics of the reflection process, the non-zero phase $\pi$ appears on the $|0(1)_{\text{ph}}\rangle$ state component, and thus yields a CPHASE gate that is controlled by the $|0_q\rangle$ qubit state. The average fidelity and corresponding gate is influenced by the effect of damping and decay as well as the frequency dependence of the reflection coefficient over the pulse bandwidth. Ideally, the reflected photon should populate the same temporal or spectral modes function for both atomic qubit states as quantified by the overlap integrals $T_q = \int_{-\infty}^{\infty} d\omega |\tilde{\beta}_{\text{in}}(\omega)|^2 R_m^2(\omega)$ between the incident and reflected photon wave packets for the 0 and 1 qubit states. For a desired phase rotation $\varphi$ on the $|0_q,1_{\text{ph}}\rangle$ state, we obtain the gate error averaged over all input atomic and photonic qubit states [30, 38],

$$E = 1 - F_{\text{at-ph}} = 1 - \frac{1}{20} \left(1 + |T_q|^2 + \eta^2 + |T_q|^2 \eta^2 + |1 + e^{i\varphi}T_0 + \eta + T_\eta|^2\right).$$

(17)

4. Atom–photon gate

4.1. Rydberg ensemble geometry

We assume that the ensemble of ancilla atoms is arranged in a Gaussian distribution around the central qubit atom,

$$\rho(r) = \rho_0 \exp(-r^2/2R_c^2) \text{d}V,$$

(18)

parametrized by the width parameter $R_c \approx 10 \mu m$ and the peak density $\rho_0 \approx 10^{14}-10^{15} \text{cm}^{-3}$ [39, 40]. As shown in figure 2, both the red and blue detuned Rydberg dressing of the qubit atom perturb the EIT mechanism depending on the distance between the qubit and the ancilla atoms. Solving equation (15) with the appropriate parameters yields the frequency dependent reflection coefficient and the fidelity of the phase gate. We first consider the red-detuned case of dispersive blockade to compare to previous work using shelved Rydberg states, followed by a novel photon interaction mechanism relevant in the blue detuned case.

4.2. Red-detuned dressing

The first main result of the paper is that off-resonant, red-detuned excitation of the qubit atom into a Rydberg dressed state drastically reduces decay and population loss from the excited state while still allowing significant
dispersive blockade of Rydberg excitation of a large number of ensemble atoms. The dispersive interaction of these effective two-level atoms with the cavity mode suffices to shift the cavity resonance and reflect the incident photon.

To demonstrate this improved coupling mechanism we show how the error scales as a function of the dressing (Rydberg state population). In figure 3(A) the three lines correspond to three photon bandwidths of 10, 1 and 0.1 MHz. The stated bandwidths of the pulses are calculated by Fourier transform and numerical extraction of the standard deviation. For each pulse, the average error (equation (17)) is minimized numerically over $\kappa > 20$ MHz, while other parameters are set to realistic values of $\Omega = 30$ MHz, $\Delta = 300$ MHz, $R_c = 5 \mu m$, $C_s = -30$ GHz $\mu m^{-3}$, $\Gamma = 3$ MHz, $\gamma = 1$ kHz, $g_m = 0.5$ MHz, $\rho_0 = 10^{13}$ cm$^{-3}$, $\epsilon$ is calculated for given $\Delta$ to yield the desired dressing fraction. A qubit Rydberg population greater than 0.5 corresponds to a dressed state that is mostly composed of the Rydberg state, and therefore a different initial state is assumed in our calculation (for comparison purposes).

We see that the optimal dressing depends on the photon bandwidth because as the pulses are made shorter a stronger coupling is required to block the photon. The global optimal regime is very low dressing strength where the excited qubit state has greatly enhanced lifetime and a narrow photon bandwidth that can still be fully blockaded.

In figure 3(B), the same physical system is used, but now the Rydberg level rather than the dressing is varied. The same parameters are used but the optimal value of dressing (minimum in figure 3(A)) is assumed. We apply an approximate scaling of the atomic properties with principal quantum number $n$ whereby $C_s \sim -300n^3$ Hz $\mu m^{-3}$ and $\gamma \sim n^{-3}$ GHz [37].

### 4.3. Blue-detuned dressing

The case of off-resonant, blue-detuned excitation of the qubit atom into a Rydberg dressed state offers an entirely different interaction mechanism. Rather than causing further detuning of the state with excited ancilla atoms as in figure 2(A), the blockade interaction here provides a resonant excitation pathway to the fully Rydberg excited qubit state and a Rydberg excited ancilla atom at a distance obeying $B_m = C_s / r_m^3 \approx \Delta$ (see figure 2(B)). This resonant coupling, however, does not lead to population of the two-atom excited state: instead, it splits the dark state by $\pm \epsilon$ and breaks the EIT mechanism, causing reflection of the incident photon. The reflection coefficient $R_i(\omega)$ is given by $\tilde{C}_i(\omega)$, which is found by solving the full set of equations (15). The first line in equation (15), however, indicates how the different ancilla atoms contribute to the reflection coefficient,

$$R_i(\omega) = \frac{\beta_{\text{out}}(\omega)}{\beta_{\text{in}}(\omega)} = 1 - \frac{\kappa}{i\omega - \kappa/2 + \sum_m g_m \tilde{A}_m^{\text{in}}(\omega)/\tilde{C}_i(\omega)}.$$

Figure 4 shows the values of $\tilde{A}_m^{\text{in}}(\omega)/\tilde{C}_i(\omega)$ as function of the distance between the qubit and ancilla. A thin shell of ancilla atoms within the peak in the blue curve obey the two-atom resonance condition and can cause $R_i(\omega)$ to
change from $-1$ to $1$. In comparison, much smaller absolute values pertain to the dispersive mechanism of \cite{8,30} (red lines).

The gate error is plotted in figure 4(A), as a function of the width of the distribution of atoms included from the Gaussian distribution \cite{18}, near the two-atom resonance. In our calculations we assume the (more conservative than figure 3) physical parameters $\Omega = 30$ MHz, $R_\epsilon = 10$ $\mu$m, $C_0 = -18$ GHz $\mu$m$^{-3}$, $\kappa > 20$ MHz, $\Gamma = 3$ MHz, $\gamma = 1$ kHz, $g_{\mu m} = 0.5$ MHz, $\rho_0 = 10^{13}$ cm$^{-3}$, a Rydberg dressing fraction of 0.15, and an incident photon bandwidth of 0.1 MHz. Allowing the Rydberg state population and the photon bandwidth to vary around these values does not significantly improve the fidelity. The benefits of using the two-atom resonance instead of dispersive interaction compounds with the advantages of using dressing (see figure 3(A) for dressing fraction 1), reaching errors in the few $10^{-3}$ range, satisfying fault-tolerance thresholds for several error correcting codes. Compared to the strong coupling to single atoms \cite{4}, the errors here are close to two orders of magnitude lower.

In both the red and the blue detuned case, the qubit–photon interaction is mediated by the ancilla atoms, and it is the weighted number of ancilla atoms contributing to this effective coupling rather than their precise distribution that governs the reflection coefficient for the incident photon. Our results therefore vary only little if the qubit is not exactly centered in the ensemble, or if the blockaded volume or resonant shell are not spherically symmetric due to angular dependence of the dipolar interactions. Further improvement in the gate fidelity may be obtained by spatial or spectral tailoring of the ancilla coupling parameters. Rather than relying on the $C_0/g_{\mu m}$ dependence one may, e.g., exploit long-range local minima in more complex energy spectra \cite{41,42} to increase the number of ancilla atoms experiencing the two-atom resonance.

4.4. Arbitrary controlled-phase

In quantum computing, a key requirement is a universal gate set that spans all possible operations. Our CPHASE gate in combination with local qubit operations provide such a gate set, but being able to achieve other controlled phases than $\pi$ can be of significance in the efficient composition of more complicated operations. As illustrated by the empty cavity (9), the reflection phase varies between 0 and $\pi$ as the frequency of the photon approaches cavity resonance. As the complex phase of our reflection coefficients $R_0(\omega)$ and $R_1(\omega)$ have different frequency dependencies, we may hence investigate the ability to generate arbitrary controlled phases.

In figure 5, we plot the error associated with achieving different phases $\varphi$ in equation (17). The simulation parameters are the same as for figure 4 with a shell width taken as 0.3 $\mu$m and the central frequency of the incoming photon pulse relative to the cavity mode being the crucial optimization parameter. The most difficult to achieve phase is not surprisingly the largest phase difference $\pi$ between the reflection coefficients, which is the phase assumed in the rest of the figures.
5. Implementation with superconducting circuits

The use of an ancilla system as mediator of the interaction between a stationary and a flying qubit is not restricted to atomic systems. Another prominent candidate for quantum information technologies is the circuit-QED architecture, that may ultimately rely on effective microwave communication between separate chips with superconducting qubits for highly parallelized processing within a single or multiple dilution refrigerators.

The arrangement is depicted in figure 1(B), corresponding to a microwave transmission line cavity with two superconducting transmons. The left qubit transmon is restricted to its two lowest energy eigenstates $|0\rangle_1$ and $|1\rangle_1$ while the right ancilla transmon is restricted to its four lowest levels $|0\rangle_a, i=0..3$. As in the atomic case, we assume that the incident microwave photon is resonant with the resonator, which is in turn resonantly coupled with strength $g$ with the $|0\rangle_a \leftrightarrow |1\rangle_a$ ancilla transmon transition, and we assume a classical microwave drive is applied with strength $\Omega$ to the $|1\rangle_a \leftrightarrow |2\rangle_a$ ancilla transmon transition. In addition we assume degeneracy of the product states $|21\rangle_{aq}$ and $|30\rangle_{aq}$ and a coupling between them with strength $\epsilon$. This energy structure is depicted in figure 6(A).

In the limit of low charge dispersion, the transmon qubits can be accurately modeled by Duffing oscillators with low anharmonicity [43, 44], and with Hamiltonian

$$H = \omega a^\dagger a + \omega_1 b^\dagger b + \omega_2 c^\dagger c + \alpha_1 (b^\dagger b)^2 + \alpha_2 (c^\dagger c)^2 + g (b^\dagger a + a^\dagger b) + \frac{\epsilon}{\sqrt{3}} (b^\dagger c + c^\dagger b) + \frac{\Omega}{\sqrt{2}} (b^\dagger + b),$$

where $b$ and $c$ are the respective annihilation operators for the ancilla and qubit transmons, and $\alpha_1$ and $\alpha_2$ their respective anharmonicities. The rotating frame Hamiltonian where we keep only the relevant levels then reads

$$H = g (|1\rangle \langle 1| a^\dagger a + a|0\rangle \langle 0| 1_a) + \Omega (|1\rangle \langle 2|_{1_a} + |2\rangle \langle 1|_{1_a} + |1\rangle \langle 3|_{1_a} + |3\rangle \langle 2|_{1_a} |0\rangle \langle 1_a).$$

Figure 5. Gate error (17) as a function of the desired phase acquired on the $|0\rangle_{1_{aq}}$ state versus the other 3 qubit product states. The red dashed line corresponds to the full cloud geometry with red-detuned dressing, while the blue is for a shell of width $0.3 \mu m$ at a distance of $5 \mu m$ for blue-detuned dressing. Other parameters are the same as in figure 4.

Figure 6. (A) Energy level diagram and couplings for implementing the protocol with superconducting transmon qubits and a microwave cavity. The eigenenergies of the ladder system are shown as black lines, with the dark state split due to the extra coupling. (B) Gate error (17) for the circuit-QED setup from figure 1(B) as a function of the coupling from the cavity to the ancilla transmon qubit. The dotted line is for a lifetime of the qubit of $5 \mu s$, while the solid purple is for $33 \mu s$. The ancilla lifetime is assumed to be half the register qubit’s. Other parameters are described in the text.
We see from figure 6(A) that, for the microwave photon incident on the initial transmon qubit \( |0_q\rangle \) state, the same EIT configuration (left three level ladder) appears as in the previous section, while for the qubit \( |1_q\rangle \) state, the coupling to the state \( |2_a, k_e\rangle \) splits the zero energy eigenstates and prevents photons from entering the cavity. Note that the EIT mechanism has been previously demonstrated [45–47] for superconducting qubits, but here we show it can be applied for a high-fidelity quantum switch.

Solving the Schrödinger equation for the different state amplitudes in the input–output theory yields the complex reflection coefficients,

\[
R_q(\omega) = 1 + \frac{\kappa}{2} - i\omega + |g|^2 \left[ \frac{\Gamma_i}{2} - i\omega + \frac{2|\Omega|^2}{\frac{\Gamma_i}{2} - i\omega + d} \right]^{-\frac{1}{2}},
\]

where \( \Gamma_i \) is the decay from the ith level of the ancilla atom. The reflection coefficient \( R_q(\omega) \) is effectively controlled by the qubit state if \( \epsilon > |\Omega| \gg \Gamma_i \kappa \). Note that we must also assume sufficient anharmonicity of the transmon energy levels to avoid excitation of the qubit transmon to higher states [48, 49] and undesired couplings among ancilla levels due to the strong classical microwave drive \( \Omega \).

This puts requirements on the anharmonicities \( \alpha_i \gg |\Omega|, \epsilon \) of the transmons. These requirements can be fulfilled in experiments, where we may have transmon state lifetimes of tens of microseconds and couplings \( \epsilon \) and \( |\Omega| \) in the tens to low hundreds of MHz can be obtained, while anharmonicities are typically in the few hundreds of MHz. This corresponds relatively well to the parameter regimes we have chosen for the neutral atom Rydberg systems, where Rabi frequencies can be significantly lower but hyperfine and Rydberg lifetimes significantly longer.

We use the same theoretical expression for state overlaps and fidelities as in the previous section to obtain the CPHASE gate error, and we plot this quantity as a function of the cavity vacuum Rabi frequency coupling to the ancilla transmon in figure 6(B). The dashed brown curve is simulated with a qubit lifetime of 5 \( \mu s \) while the solid purple lines is for 33 \( \mu s \). The higher transmon level lifetimes decrease with Fock number. The coupling between transmons is taken as \( \epsilon = 150 \) MHz and the other parameters are optimized numerically as before, specifically the bandwidth and EIT microwave strength, which are in the 10 MHz and 100 MHz range, respectively, but increase with \( g \). Error rates are limited to interaction times of hundreds of ns, similar to circuit-QED two-qubit gates which have similar limitations. Thus, qubit–photon gates can achieve similar intrinsic fidelities as qubit–photon gates [49], with photon traveling losses likely to be the most severe bottleneck for scalable technology in the foreseeable future.

6. Conclusion

We have shown that it is possible to strongly couple single photons to single stationary qubits by means of ancilla systems, which may be optimized for communication while the qubit may be optimized for storage or, e.g., for local interaction within a register of qubits. We presented quantitative analyses, improving on earlier proposals [8, 30] for neutral qubit atoms and we presented an original implementation for superconducting circuits with engineered resonant interactions between the qubit and ancilla transmons within wave guide resonators. For both implementations, the proposal meets a need to achieve high-performance or fault-tolerant scalable quantum operations, and we believe that the main ideas may be applied to other systems, e.g., hybrid atom and solid state systems, where a mediator is needed for the coupling to, and communication by, phonons, plasmons, spin waves and other extended quantum degrees of freedom.

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References

[1] Kimble H J 2008 Nature 453 1023
[2] Duan L M and Kimble H J 2004 Phys. Rev. Lett. 92 127902
[3] Reiserer A, Ritter S and Rempe G 2013 Science 342 1349
[4] Reiserer A, Kall N, Rempe G and Ritter S 2014 Nature 508 237
[5] Hao Y M, Lin G W, Xia K, Lin X M, Niu Y P and Gong S Q 2015 Sci. Rep. 5 10005
[6] Clark S, Peng A, Gu M and Parkins S 2003 Phys. Rev. Lett. 91 177701
[7] Motzoi F, Halperin E, Wang X, Whaley K B and Schirmer S 2016 Phys. Rev. A 94 032313
[8] Das S, Grankin A, Lakoupov I, Brion E, Borregaard J, Boddeda R, Usmani I, Ourjoumtsev A, Grangier P and Sørensen A S 2016 Phys. Rev. A 93 040303
[9] Cabrillo C, Cirac J I, García-Fernández P and Zoller P 1999 Phys. Rev. A 59 1025
[10] Duan L-M, Cirac J I, Zoller P and Polzik E S 2000 Phys. Rev. Lett. 85 5643
[11] Duan L-M, Lukin M D, Cirac J I and Zoller P 2001 Nature 414 413
[12] Moehring D L, Maunz P, Olmschenk S, Younge K M, Duan L-M and Monroe C 2007 Nature 449 68
[13] Kerckhoff J, Bouten L, Silberfarb A and Mabuchi H 2009 Phys. Rev. A 79 023405
[14] Horn N, Schwartz M E, Motzoi F, Macklin C, Vijay R, Eddins A W, Korotkov A N, Whaley K B, Sarovar M and Siddiqi I 2014 Phys. Rev. Lett. 112 170501
[15] Motzoi F, Whaley K B and Sarovar M 2015 Phys. Rev. A 92 032308
[16] Martin L, Motzoi F, Li H, Sarovar M and Whaley K B 2015 Phys. Rev. A 92 062321
[17] Hensen B et al 2015 Nature 526 682
[18] Firstenberg O, Adams C S and Hofferberth S 2016 J. Phys. B: At. Mol. Opt. Phys. 49 152003
[19] Pellizzari T 1997 Phys. Rev. Lett. 79 5242
[20] Wilk T, Webster S C, Kuhn A and Rempe G 2007 Science 317 488
[21] Ates C, Sevinçli S and Pohl T 2011 Phys. Rev. A 83 041802
[22] Chang D E, Vuletić V and Lukin M D 2014 Nat. Photon. 8 685
[23] Tiecke T G, Thompson J D, de Leon N P, Liu L R, Vuletić V and Lukin M D 2014 Nature 508 241
[24] Lukin M D, Fleischhauer M, Cote R, Duan L M, Jaksh D, Cirac J I and Zoller P 2001 Phys. Rev. Lett. 87 037901
[25] Saffman M and Walker T G 2002 Phys. Rev. A 66 055403
[26] Ningyuan J, Georgakopoulos A, Ryou A, Schine N, Sommer A and Simon J 2016 Phys. Rev. A 93 041802
[27] Ding D-S and Shi B-S 2016 2016 Conf. on Lasers and Electro-Optics (CLEO) (Piscataway, NJ: IEEE) pp 1–2
[28] Pedersen I H and Mølmer K 2009 Phys. Rev. A 79 012320
[29] Grankin A, Brion E, Bimbard E, Boddeda R, Usmani I, Ourjoumtsev A and Grangier P 2015 Phys. Rev. A 92 043841
[30] Wade A C J, Mattiodi M and Mølmer K 2016 Phys. Rev. A 94 053830
[31] Mølmer K, Lichtman M T, Xia T, Sun Y, Piotrowicz M J, Carr A W, Isenhower L and Saffman M 2015 Phys. Rev. A 92 022336
[32] Jau Y-Y, Hankin A M, Keating T and Deutsch I H and Biedermann G W 2016 Nat. Phys. 12 71
[33] Firstenberg O, Peyronel T, Liang Q-Y, Gorshkov A V, Lukin M D and Vuletić V 2013 Nature 502 71
[34] Gorshkov A V, Nath R and Pohl T 2013 Phys. Rev. Lett. 110 153601
[35] Gardiner C W 1993 Phys. Rev. Lett. 70 2269
[36] Walker T G and Saffman M 2005 J. Phys. B: At. Mol. Opt. Phys. 38 S309
[37] Beterov I I, Ryabtsev I L, Tretyakov D B and Eintz V M 2009 Phys. Rev. A 79 052504
[38] Pedersen I H, Mølmer N M and Mølmer K 2007 Phys. Lett. A 367 47
[39] Brennecke F, Donner T, Ritter S, Bourdel T, Köhl M and Esslinger T 2007 Nature 450 268
[40] Murch K W, Moore K L, Gupta S and Stamper-Kurn D M 2008 Nat. Phys. 4 561
[41] Greene C H, Dickinson A S and Sadeghpour H R 2000 Phys. Rev. Lett. 85 2458
[42] Boisseau C, Simbotin I and Cote R 2002 Phys. Rev. Lett. 88 133004
[43] Khani B, Gambetta J M, Motzoi F and Wilhelm F K 2009 Phys. Scr. T137 014021
[44] Khani B, Merkel S T, Motzoi F, Gambetta J M and Wilhelm F K 2013 Phys. Rev. A 85 022306
[45] Hoi I-C, Wilson C M, Johansson G, Palomaki T, Peropadre B and Delsing P 2011 Phys. Rev. Lett. 107 073601
[46] Abdumalikov A A, Astafiev O, Zagoskin A M, Pashkin Y A, Nakamura Y and Tsai J S 2010 Phys. Rev. Lett. 104 193601
[47] Murali K, Dutton Z, Oliver W, Crankshaw D and Orlando T 2004 Phys. Rev. Lett. 93 087003
[48] Theis I S, Motzoi F and Wilhelm F K 2016 Phys. Rev. A 93 012324
[49] Goerz M H, Motzoi F, Whaley K B and Koch C P 2017 npj Quantum Inf. 3 37