Numerical Optimization of Rectangular Mufflers with Multi-channel Splitters Using the Simulated Annealing Method

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Abstract. Rectangular mufflers internally hybridized with splitters have been extensively applied in industrial noise abatement. However, there has been a palpable lack of academic work directed toward space-constrained mufflers conjugated with multi-channel splitters that disperse venting fluid and reduce secondary noise. That being so, an analysis of the Sound Transmission Loss (STL) of rectangular mufflers internally hybridized with multiple parallel splitters that are optimally designed to perform within a limited space will be considered, here. By using an acoustical lumped method, a four-pole system matrix for evaluating the acoustic performance (STL) emerges. During the optimization process, the simulated annealing (SA) method, which is a robust scheme utilized to search for the global optimum by imitating a physical annealing process, is used. Before dealing with a broadband noise, the STL’s maximization relative to a one-tone noise (250Hz) is offered to confirm the SA method’s reliability. Moreover, the mathematical model is also checked for accuracy. To appreciate the influence of acoustical efficiency with respect to the design parameters, the sensitivity analysis of six design parameters (dh: the diameter of a perforated hole; W: the width of air channel; R: acoustic flow resistivity of the acoustic fiber; σ: the porosity of the perforated plate; Df: the thickness of the acoustic fiber; L1: the horizontal length of the splitter) is performed. Subsequently, to bring into focus the acoustical interaction with respect to the number of air-channels (between the splitters), three types of mufflers (mufflers A–C) hybridized with one, two, and four air-channels (with parallel splitters) have been surveyed. Results divulge that for a rectangular muffler internally conjugated with a one-channel splitter, the maximal STL is located at the desired tone. The acoustical performance of a rectangular muffler will increase and the induced back pressure will decrease simultaneously if the number of the splitters internally conjugated within the rectangular muffler increases. Consequently, optimally designed rectangular mufflers with multiple parallel splitters that avoid secondary noise induced by high speed flow while simultaneously maximizing acoustical performance within a constrained space are preferable.

1. Nomenclature

This paper is constructed on the basis of the following notations:

c₀: sound speed (m s⁻¹)

dh: the diameter of a perforated hole on the front perforated plate (m)
Df: the thickness of the acoustic fiber
L1: diameter of the i-th horizontal segment of the muffler (m)
d1,2: diameter of the inlet/outer tube (m)
f: cyclic frequency (Hz)
H: the thickness of the splitter (m)
iter: maximum iteration
j: imaginary unit
k: wave number (=ω/c0)
k1: real part of complex kfiber-i
k2: image part of complex kfiber-i
kfiber: complex propagation constant of the acoustic fiber
kk: cooling rate in SA
L0: total length of the muffler (m)
L1: the horizontal length of the splitter (m)
L2: the height of the splitter (m)
M: mean flow Mach number
OBJ: objective function (dB)
p0: acoustic pressure at the i-th node (Pa)
ph(T): transition probability
Q: volume flow rate of venting gas (m³ s⁻¹)
R: acoustic flow resistivity of the acoustic fiber (MKS raysl m⁻¹)
Rfiber: real part of the complex Zfiber
Si: section area at the i-th node(m²)
STL: sound transmission loss (dB)
SWLO: unsilenced sound power level inside the muffler’s inlet (dB)
SWLT: overall sound power level inside the muffler’s output (dB)
qu: the thickness of the front perforated plate in the splitter (m)
TCij: components of four-pole transfer matrices for an acoustical mechanism with sudden-contracted ducts
TEij: components of four-pole transfer matrices for an acoustical mechanism with sudden-expanded ducts
TSij: components of four-pole transfer matrices for an acoustical mechanism with straight ducts
TSPij: components of a four-pole transfer matrix for an acoustical mechanism with a one-channel splitter
T: current temperature (°C)
Tij: components of a four-pole transfer system matrix
To: initial temperature (°C)
u0: acoustic particle velocity at the i-th node (m s⁻¹)
W: the width of air channel (m)
Zi: specific normal impedance at i.
Zfiber: characteristic impedance of the acoustic fiber
Zp: characteristic impedance of the perforated front plate
Xfiber: image part of the complex Zfiber
ρa: air density (kg m⁻³)
ρf: acoustical density at the i-th node (kg m⁻³)
v: kinematic viscosity of air (=15*10⁻⁶ m²/s)
σ: the porosity of the perforated plate (m)
δ: viscous boundary layer thickness of the i-th layer of the perforated plate (m)
ω: angular velocity (=2πf)
2. Introduction
Morse, in 1939 [1], began research on mufflers that reduced high frequency noise using a dissipative duct (a duct lined with sound absorbing material). Later, Scott [2] analyzed the sound transmission of circular and rectangular mufflers internally lined with porous material using bulk reactive model. Ko [3], in 1975, investigated the sound transmission loss in acoustically lined flow ducts separated by porous splitters. The research mentioned above [1-3] focused on the sound attenuation of an infinite duct. Cummings and Chang [4] then, in 1988, analyzed a finite length dissipative flow duct silencer with an internal mean flow in the absorbent using a modal method. Peat [5], addressing the volume modulus, adopted a transfer matrix for evaluating the acoustical performance of an absorption silencer element in 1991. Subsequently, using a one-dimensional analytical method and a three-dimensional boundary element method (BEM), Selamet et al. [6, 7] assessed the acoustical attenuation for perforated concentric absorbing silencers and hybrid silencers. In 2003, Munjal [8] analyzed pod silencers using a four-pole transfer matrix. Then, in 2004, Xu et al. [9] proposed a characteristic equation to calculate the sound attenuation in dissipative expansion chambers. However, an assessment of a muffler’s optimal design within a limited space was seldom addressed. Therefore, Chiu [10] has examined shaped optimization of a circular muffler internally lined with sound absorbing material within a constrained space. Later, Chiu [11] assessed circular mufflers internally equipped with multi-channel splitters using neural networks in conjunction the boundary element method and a genetic algorithm. However, the study is limited in the optimization assessment for pure tone noise. In order to develop a space-constrained rectangular muffler that is internally equipped with multi-channel splitters used in reducing broadband noise, a mathematical mode of a rectangular muffler internally equipped with multiple splitters is proposed using an acoustical lump analysis. Three types of rectangular mufflers linked with multiple splitters (muffler A: a muffler equipped with a one-channel splitter; muffler B: a muffler equipped with two-channel splitters; muffler C: a muffler equipped with four-channel splitters) are presented. It should also be noted that the acoustical lump method used to form a four-pole system matrix is compatible with the simulated annealing method.

3. Theoretical background
Three types of rectangular mufflers internally connected with multiple parallel splitters have been adopted for noise elimination in the steam turbine system shown in Fig. 1. Before the acoustical fields of the mufflers were analyzed, the acoustical elements had been identified. As shown in Fig. 2, four types of muffler components that included four straight ducts (points 1–2, points 3–4, points 7–8, and points 9–10), two sudden expanded ducts (points 2–3, points 6–7), two sudden contracted ducts (points 4–5, points 8–9), and one multiple parallel splitter element (points 5–6) are identified. Additionally, the acoustical field within the muffler is represented by ten points. As derived in previous work [12, 13, 14] and shown in appendices A–C, individual transfer matrices with respect to straight ducts, dissipative ducts, and sudden expanded/contracted ducts are described below.

Figure 1. Noise elimination on a steam turbine within a limited space.
Figure 2. Acoustical elements in three types of rectangular mufflers internally equipped with parallel splitters (muffler A ~ muffler C).
3.1. Muffler A (a rectangular muffler internally connected with a one-channel splitter)

As indicated in Fig. 2, for the acoustical straight element, the four-pole matrix between nodes 1 and 2 is [12, 13, 14]

\[
\begin{bmatrix}
    p_1 \\
    \rho_c u_1
\end{bmatrix} = f_1(L_i, d_i, M_i) \begin{bmatrix}
    TS_{1,1} \\
    TS_{1,2}
\end{bmatrix} \begin{bmatrix}
    p_2 \\
    \rho_c u_2
\end{bmatrix}
\]

(1)

For the acoustical sudden-expanded element, the four-pole matrix between nodes 2 and 3 is [12, 13, 14]

\[
\begin{bmatrix}
    p_2 \\
    \rho_c u_2
\end{bmatrix} = \begin{bmatrix}
    TE_{1,1} \\
    TE_{1,2}
\end{bmatrix} \begin{bmatrix}
    \rho_c u_3 \\
    p_3
\end{bmatrix}
\]

(2)

Similarly, the four-pole matrix between nodes 3 and 4 is

\[
\begin{bmatrix}
    p_3 \\
    \rho_c u_3
\end{bmatrix} = f_3(L_i, L_j, M_j) \begin{bmatrix}
    TS_{2,1} \\
    TS_{2,2}
\end{bmatrix} \begin{bmatrix}
    \rho_c u_4 \\
    p_4
\end{bmatrix}
\]

(3)

For the acoustical sudden-contracted element, the four-pole matrix between nodes 4 and 5 is [12, 13, 14]

\[
\begin{bmatrix}
    p_4 \\
    \rho_c u_4
\end{bmatrix} = \begin{bmatrix}
    TC_{1,1} \\
    TC_{1,2}
\end{bmatrix} \begin{bmatrix}
    \rho_c u_5 \\
    p_5
\end{bmatrix}
\]

(4)

As derived in Appendix A, for a one-channel splitter, the four-pole matrix between nodes 5 and 6 yields

\[
\begin{bmatrix}
    p_5 \\
    \rho_c u_5
\end{bmatrix} = \begin{bmatrix}
    TS_{P1,1} \\
    TS_{P1,2}
\end{bmatrix} \begin{bmatrix}
    \rho_c u_6 \\
    p_6
\end{bmatrix}
\]

(5)

Likewise, the four-pole matrix between nodes 6 and 7 for a sudden-expanded duct is

\[
\begin{bmatrix}
    p_6 \\
    \rho_c u_6
\end{bmatrix} = \begin{bmatrix}
    TE_{2,1} \\
    TE_{2,2}
\end{bmatrix} \begin{bmatrix}
    \rho_c u_7 \\
    p_7
\end{bmatrix}
\]

(6)

The four-pole matrix between nodes 7 and 8 in a straight duct is

\[
\begin{bmatrix}
    p_7 \\
    \rho_c u_7
\end{bmatrix} = f_3(L_i, L_j, M_j) \begin{bmatrix}
    TS_{3,1} \\
    TS_{3,2}
\end{bmatrix} \begin{bmatrix}
    \rho_c u_8 \\
    p_8
\end{bmatrix}
\]

(7)

The four-pole matrix between nodes 8 and 9 for a sudden-contracted duct is

\[
\begin{bmatrix}
    p_8 \\
    \rho_c u_8
\end{bmatrix} = \begin{bmatrix}
    TC_{2,1} \\
    TC_{2,2}
\end{bmatrix} \begin{bmatrix}
    \rho_c u_9 \\
    p_9
\end{bmatrix}
\]

(8)

Moreover, the four-pole matrix between nodes 9 and 10 in a straight duct is

\[
\begin{bmatrix}
    p_9 \\
    \rho_c u_9
\end{bmatrix} = f_3(L_i, L_j, M_j) \begin{bmatrix}
    TS_{4,1} \\
    TS_{4,2}
\end{bmatrix} \begin{bmatrix}
    \rho_c u_{10} \\
    p_{10}
\end{bmatrix}
\]

(9)

The total transfer matrix assembled by multiplication is

\[
\begin{bmatrix}
    p_1 \\
    \rho_c u_1
\end{bmatrix} = \begin{bmatrix}
    f_1(L_i, d_i, M_i) f_2(L_i, L_3, M_3) f_3(L_i, L_4, M_4) & f_4(L_i, d_i, M_i)
\end{bmatrix} \begin{bmatrix}
    TS_{1,1} \\
    TS_{1,2}
\end{bmatrix} \begin{bmatrix}
    TE_{1,1} \\
    TE_{1,2}
\end{bmatrix} \begin{bmatrix}
    \rho_c u_{10}
\end{bmatrix}
\]

(10)

\[
\begin{bmatrix}
    p_1 \\
    \rho_c u_1
\end{bmatrix} = \begin{bmatrix}
    T_{11} & T_{12}
\end{bmatrix} \begin{bmatrix}
    p_{10}
\end{bmatrix}
\]

(11)
3.2. Muffler B (a rectangular muffler internally connected with two-channel splitters)

Similarly, the acoustical four-pole matrix between points 1–2, points 2–3, points 3–4, points 4–5, points 6–7, points 7–8, points 8–9, and points 9–10 is the same as Eqs.(1)–(4) and Eqs.(6)–(9).

As derived in Appendix B, for two parallel air-channel splitters, the four-pole matrices between nodes 5a–6a and nodes 5b–6b can be combined into an equivalent matrix

$$\begin{pmatrix}
p_5 \\
\rho_c, u_{10}
\end{pmatrix}
=\begin{bmatrix}
TSP1_{1,1} & \frac{1}{2} \cdot TSP1_{1,2} \\
2 \cdot TSP1_{2,2} & \frac{1}{2} \cdot TSP1_{2,1}
\end{bmatrix}
\begin{pmatrix}
p_6 \\
\rho_c, u_{10}
\end{pmatrix}
$$

(12)

where \([TSP1_{ij}]\) is the four-pole matrix for a one-channel splitter. Consequently, the total transfer matrix assembled by multiplication is

$$\begin{pmatrix}
p_1 \\
\rho_c, u_{10}
\end{pmatrix}
=\begin{bmatrix}
T1_{1,1} & T1_{1,2} \\
T1_{2,1} & T1_{2,2}
\end{bmatrix}
\begin{pmatrix}
p_{10} \\
\rho_c, u_{10}
\end{pmatrix}
$$

(14)

3.3. Muffler C (a rectangular muffler internally connected with three-channel splitters)

Likewise, the acoustical four-pole matrix between points 1–2, points 2–3, points 3–4, points 4–5, points 6–7, points 7–8, points 8–9, and points 9–10 is the same as Eqs.(1)–(4) and Eqs.(6)–(9).

As derived in Appendix C, for four parallel air-channel splitters, the four-pole matrices between nodes 5a–6a, 5b–6b, 5c–6c, and 5d–6d nodes can be combined into an equivalent matrix

$$\begin{pmatrix}
p_5 \\
\rho_c, u_{10}
\end{pmatrix}
=\begin{bmatrix}
TSP1_{1,1} & \frac{1}{4} \cdot TSP1_{1,2} \\
4 \cdot TSP1_{2,2} & \frac{1}{4} \cdot TSP1_{2,1}
\end{bmatrix}
\begin{pmatrix}
p_6 \\
\rho_c, u_{10}
\end{pmatrix}
$$

(15)

where \([TSP1_{ij}]\) is the four-pole matrix for a one-channel splitter. Consequently, the total transfer matrix assembled by multiplication is

$$\begin{pmatrix}
p_1 \\
\rho_c, u_{10}
\end{pmatrix}
=\begin{bmatrix}
T1_{1,1} & T1_{1,2} \\
T1_{2,1} & T1_{2,2}
\end{bmatrix}
\begin{pmatrix}
p_{10} \\
\rho_c, u_{10}
\end{pmatrix}
$$

(17)
3.4. Overall sound power level

The sound transmission loss (STL) of mufflers A is defined as

\[
STL_i(Q, f, RT_1, RT_2, RT_3, RT_4, RT_5, RT_6) = 20 \log \left( \frac{T_{i1}^* + T_{i2}^* + T_{i3}^* + T_{i4}^*}{2} \right) + 10 \log \left( \frac{S_i}{S_{10}} \right)
\]

\[
STL_2(Q, f, RT_1, RT_2, RT_3, RT_4, RT_5, RT_6) = 20 \log \left( \frac{T_{i1}^{**} + T_{i2}^{**} + T_{i3}^{**} + T_{i4}^{**}}{2} \right) + 10 \log \left( \frac{S_i}{S_{10}} \right)
\]

\[
STL_3(Q, f, RT_1, RT_2, RT_3, RT_4, RT_5, RT_6) = 20 \log \left( \frac{T_{i1}^{***} + T_{i2}^{***} + T_{i3}^{***} + T_{i4}^{***}}{2} \right) + 10 \log \left( \frac{S_i}{S_{10}} \right)
\]

where

\[
R_{T1} = R; \ R_{T2} = \sigma; \ R_{T3} = d; \ R_{T4} = D_f / H; \ R_{T5} = W; \ R_{T6} = L_i;
\]

The silenced octave sound power level emitted from a muffler’s outlet is

\[
SWL = SWLO(f_i) - STL(f_i)
\]

(18d)

where (1) \(SWLO(f_i)\) is the original SWL at the inlet of a muffler (or pipe outlet), and \(f_i\) is the relative octave band frequency.

(2) \(STL(f_i)\) is the muffler’s STL with respect to the relative octave band frequency \(f_i\).

(3) \(SWL\) is the silenced SWL at the outlet of a muffler with respect to the relative octave band frequency.

Finally, the overall SWLT silenced by a muffler at the outlet is

\[
SWL_{f-K} = 10^* \log \left( \sum_{i=1}^{10} \frac{SWL}{10} \right)
\]

\[
= 10^* \log \left( 10^{\frac{SWLO(f_i)-10}{SWL(f_i)-10}} + 10^{\frac{SWLO(f_i)-10}{SWL(f_i)-10}} + \ldots + 10^{\frac{SWLO(f_i)-10}{SWL(f_i)-10}} \right)
\]

4. Objective function

4.1 STL maximization for a tone (F) noise

For muffler A (a two-chamber muffler hybridized with one dissipative duct), the objective function in maximizing the STL at a pure tone \(f\) is

\[
OBJ_1 = STL_i(Q, f, RT_1, RT_2, RT_3, RT_4, RT_5, RT_6)
\]

(21a)

The related ranges of the parameters are

\[
Q = 0.001 (m^3/s); \ L_0 = 1.2 (m); \ L_s = 0.3 (m); \ L_c = 0.1 (m); \ L_e = 0.1 (m); \ d_1 = 0.1 (m); \ d_2 = 0.1 (m); \ q = 0.008 (m); \ RT1 = [3000, 20000]; \ RT2 = [5, 30]; \ RT3 = [0.003, 0.015]; \ RT4 = [0, 0.95]; \ RT5 = [0.25, 0.15]; \ RT6 = [0.3, 1.0]
\]

(21b)

4.2 SWL minimization for a broadband noise

To minimize the overall SWLT, the objective functions for mufflers A–C are

\[
OBJ_2 = SWL_{f-T}(Q, RT_1, RT_2, RT_3, RT_4, RT_5, RT_6)
\]

\[
OBJ_3 = SWL_{f-T}(Q, RT_1, RT_2, RT_3, RT_4, RT_5, RT_6)
\]

\[
OBJ_3 = SWL_{f-T}(Q, RT_1, RT_2, RT_3, RT_4, RT_5, RT_6)
\]

(22a)
5. Model check
Before performing the SA optimal simulation on the mufflers, an accuracy check of the mathematical model on the acoustical elements of a one-channel splitter is performed using the experimental data. As depicted in Fig. 3, the theoretical and experimental data are in agreement. Therefore, the proposed fundamental mathematical model for the dissipative tube is acceptable. Consequently, the model linked with the numerical method is applied to the shape optimization in the following section.

![Figure 3](image)

**Figure 3.** Performance of a rectangular muffler with a one-chamber splitter (A=0.6 M, B=0.05 M, C=0.2 M, p=20%, d=0.005 M, glass fiber (32kg/m³)).

5.1 Case Study
The noise reduction of a steam turbine system within a space-constrained room is introduced and shown in Fig. 1. The sound power level (SWL) inside the venting outlet is shown in Table 1 where the overall SWL reaches 127.4 dB. To efficiently reduce the venting noise emitted from the steam turbine system, a rectangular muffler having splitter acoustical elements is necessary. Here, three kinds of rectangular mufflers internally connected with multiple parallel splitters (mufflers A~C) that disperse venting fluid and decrease the secondary flowing noise are considered.

To obtain the best acoustical performance within a fixed space, numerical assessments linked to a SA optimizer are applied. Before the minimization of a broadband noise is performed, a reliability check of the SA method by maximization of the STL at a targeted tone (250 Hz) in muffler A is performed. As shown in Fig. 1, the available space for a muffler is 0.3 m in width (per one channel) and 1.2 m in length. The flow rate (Q) is preset at 0.001 (m³/s). The corresponding OBJ functions, space constraints, and ranges of the design parameters are summarized in Eqs.(21)–(22).

| Table 1. Unsilenced SWL of a fan inside a duct outlet. |
|-----------------|----------------|----------------|----------------|----------------|----------------|
| f(Hz)           | 125            | 250            | 500            | 1k             | 2k             | overall        |
| SWL – dBA       | 95             | 110            | 118            | 125            | 122            | 127.4          |
5.2. Simulated Annealing Method [13]

Because of the needs of the classical gradient methods EPFM, IPFM and FDM as an appropriate starting point (design data) before optimization is performed, accuracy is limited [15]. Simulated Annealing (SA), one of the best stochastic search methods, was first introduced by Metropolis et al. [16] and later developed further by Kirkpatrick et al. [17]. Because no starting point is required before the optimization is performed, SA is then adopted as the optimizer used in the muffler’s shape optimization. Because annealing is the process of heating while simultaneously maintaining a metal at a stabilized temperature as it cools, it allows particles to remain close to the minimal energy state. Generating a random initial solution will start the algorithm. The scheme of SA is a variation of the hill-climbing algorithm where all downhill movements for improvement are accepted for the decrement of the system’s energy. In order to emulate the SA’s evolution, a new random solution (X’) is selected from the neighborhood of the current solution (X). If it happens that there is a negative change in the objective function, or energy, (\( \Delta F \leq 0 \)), then the resulting solution will be recognized as the new current solution with the transition property \( pb(X') \) of 1. However, if there is not a negative change (\( \Delta F > 0 \)), the probability of transitioning to the new state X’ will be the function \( pb(\Delta F/CT) \). As shown in Eq. (23), the new transition property \( pb(X') \) varied from 0~1 will be calculated using the Boltzmann factor \( pb(X') = \exp(\Delta F / CT) \) where \( C \) and \( T \) are the Boltzmann constant and the current temperature.

\[
pb(X') = \begin{cases} 
1, & \Delta F \leq 0 \\
\exp(-\Delta F/CT), & \Delta F > 0 
\end{cases} 
\quad (23a)
\]

\[
\Delta F = F(X') - F(X) \quad (23b)
\]

Each successful substitution of the new current solution will conduct to the decay of the current temperature as

\[
T_{\text{new}} = kk * T_{\text{old}} 
\quad (24)
\]

where \( kk \) is the cooling rate. The process is repeated until the predetermined number (iter) of the outer loop is reached.

6. Results and discussion

6.1. Results

The accuracy of the SA optimization depends on two types of SA parameters that include \( kk \) (cooling rate) and \( iter \) (maximum iteration). To achieve good optimization, the following parameters are varied step by step:

\( kk(0.91, 0.93, 0.95, 0.97, 0.99); \) iter \((50, 100, 500, 1000, 2000, 5000)\).

Two results of optimization (one, pure tone noises used for SA’s accuracy check; and the other, a broadband noise occurring in a venting system) are described below.

6.2. Pure tone noise optimization

Before dealing with a broadband noise, the STL’s maximization with respect to a one-tone noise (250Hz) of muffler A is introduced for a reliability check on the SA method. By using Eq. (21), the maximization of the STL with respect to muffler A (a rectangular muffler internally connected with a one-channel splitter) at the specified pure tone (250Hz) was performed first. As indicated in Table 2, ten sets of SA parameters are tried in the muffler’s optimization. Obviously, the optimal design data can be obtained from the last set of SA parameters at \((kk, iter) = (0.99, 5000)\). Using the optimal design in a theoretical calculation, the optimal STL curves with respect to various SA parameters \((kk, iter)\) are plotted and depicted in Figs. 4 and 5. As revealed in Figs. 4 and 5, the STL is precisely maximized at the desired frequency (250Hz). Consequently, the SA optimizer is reliable in the optimization process.
Table 2. Optimal STL for splitter muffler A (equipped with one air channel) at various SA parameters (targeted tone of 250 Hz).

| SA parameter | Design parameters | Result |
|--------------|-------------------|--------|
| kk           | iter              | RT1    | RT2    | RT3    | RT4    | RT5    | RT6    | STL_{250Hz} dB |
| 0.91         | 50                | 10060  | 15.38  | 0.00798 | 0.3943 | 0.07688 | 0.5905 | 6.8  |
| 0.93         | 50                | 5969   | 9.367  | 0.005096 | 0.1659 | 0.04683 | 0.4224 | 8.3  |
| 0.95         | 50                | 5505   | 8.685  | 0.004769 | 0.14   | 0.04342 | 0.4032 | 9.6  |
| 0.97         | 50                | 11530  | 17.54  | 0.009018 | 0.4764 | 0.08769 | 0.6511 | 10.7 |
| 0.99         | 50                | 4739   | 7.558  | 0.004228 | 0.0972 | 0.03779 | 0.3716 | 10.9 |
| 0.99         | 100               | 12530  | 19.01  | 0.009727 | 0.5326 | 0.09507 | 0.6924 | 12.4 |
| 0.99         | 500               | 19340  | 29.03  | 0.01454  | 0.9133 | 0.1452  | 0.973  | 13.4 |
| 0.99         | 1000              | 19780  | 29.67  | 0.01484  | 0.9374 | 0.1483  | 0.9908 | 13.5 |
| 0.99         | 2000              | 13950  | 21.11  | 0.01073  | 0.6120 | 0.1055  | 0.7509 | 13.8 |
| 0.99         | 5000              | 15950  | 24.04  | 0.01214  | 0.7235 | 0.1202  | 0.8331 | 14.5 |

Figure 4. STL with respect to various kk [muffler A: target tone=250 Hz].

Figure 5. STL with respect to various iter [muffler A: target tone=250 Hz].

6.3. Broadband noise optimization

Similarly, considering Eq. (22a-c) and using the same SA parameters in the broadband optimization process, the minimization of the $SWL_{T-1}$, $SWL_{T-2}$, and $SWL_{T-3}$ with respect to mufflers A–C was performed and shown in Table 3. As illustrated in Table 3, the resultant sound power levels with
respect to three types of mufflers have been reduced from 127.4 dB(A) to 113.1 dB(A), 106.5 dB(A), and 99.5 dB dB(A). Using this optimal design in a theoretical calculation, the optimal STL curves with respect to various mufflers are plotted and compared with the original SWL depicted in Fig. 6.

Table 3. Comparison of the minimized SWL$_T$ of three kinds of mufflers (mufflers A–C) [broadband noise].

| Muffler Type   | Design parameters | Result |
|----------------|-------------------|--------|
|                | RT1               | RT2    | RT3    | RT4    | RT5    | RT6    | SWL$_T$ |
| Muffler A (one channel) | 13820             | 20.91  | 0.01064 | 0.6045 | 0.1045 | 0.7454 | 113.1   |
| Muffler B (two channel)   | 15100             | 22.8   | 0.01154 | 0.6765 | 0.1140 | 0.7984 | 106.5   |
| Muffler C (four channel)  | 15170             | 22.9   | 0.01159 | 0.6801 | 0.1145 | 0.8012 | 99.5    |

Fig. 6 Comparison of the optimal STLs of three kinds of mufflers (mufflers A, B, and C) and the original SWL

6.4. Discussion

In order to decrease the secondary flowing noise generated from the higher speed flow, new muffler designs with multiple parallel splitters used to disperse the venting fluid are presented. To achieve a sufficient optimization, the selection of the appropriate SA parameter set is essential. As indicated in Table 2, the best SA set of muffler A at the targeted pure tone noise of 250 Hz has been shown. The related STL curves with respect to various SA parameters are plotted in Figs. 4 and 5. Figs. 4 and 5 reveal the predicted maximal value of the STL is located at the desired frequency.

In dealing with the broadband noise, the acoustical performance among three types of rectangular mufflers connected with multiple splitters (mufflers A, B, and C) are shown in Table 3 and Fig. 6. As can be observed in Table 3, the overall sound transmission losses with respect to mufflers A–C are 14.3 dB, 20.9 dB, and 27.9 dB. Results shown in Table 3 and Fig. 6 indicate that the rectangular muffler hybridized with more parallel splitters is superior to the other mufflers equipped with fewer splitters. It also can be seen that the passing velocity within the air way will decrease if the number of air-channels increase. With this, the related back pressure will decrease; therefore, the overall back pressure of the mufflers with more parallel splitters will decrease.

To appreciate the influence of the acoustical effect with respect to the design parameters, a sensitivity analysis of six design parameters ($dh$: the diameter of the perforated holes; $W$: the width of the air channel; $R$: acoustic flow resistivity of the acoustic fiber; $\sigma$: the porosity of the perforated plate; $D_f$: the thickness of the acoustic fiber; $L_1$: the horizontal length of the splitter) is performed. The influence of acoustical performance with respect to $R$ (acoustic flow resistivity of the acoustic fiber) is shown in Fig. 7. As indicated in Fig. 7, the acoustical performance will increase when $R$ decreases. In addition,
the influence of the acoustical performance with respect to $\sigma$ (the porosity of the perforated plate) is depicted in Fig. 8. Fig. 8 indicates that the acoustical performance will increase if $\sigma$ increases. However, as indicated in Fig. 9, there is no influence on the acoustical performance when the $dh$ (the diameter of a perforated hole) varies. Also, the influence of acoustical performance with respect to $D_f$ (the thickness of the acoustic fiber) is illustrated in Fig. 10. As indicated in Fig. 10, the acoustical performance at low frequencies will increase when $D_f$ decreases (i.e., the increment of the air layer inside the splitter). This means that the resonating effect at the low frequencies will be remarkable due to the increment of the air layer inside the splitter. Subsequently, the influence of the acoustical performance with respect to $W$ (the width of air channel) is shown in Fig. 11. As indicated in Fig. 11, the acoustical performance at low frequencies (below 600 Hz) will increase when $W$ increases; However, the acoustical performance at higher frequencies (beyond than 1300 Hz) will decrease when $W$ increases. Consequently, the influence of the acoustical performance with respect to $L_1$ (the horizontal length of the splitter) is depicted in Fig. 12. As can be seen in Fig. 12, the acoustical performance at the frequencies of 0~1000 Hz will increase when $L_1$ increases.

Figure 7. The influence of the acoustical performance with respect to $R$.

Figure 8. The influence of the acoustical performance with respect to $\sigma$. 
Figure 9. The influence of the acoustical performance with respect to $d_h$.

Figure 10. The influence of the acoustical performance with respect to $D_f$.

Figure 11. The influence of the acoustical performance with respect to $W$. 
7. Conclusion

It has been shown that rectangular mufflers internally connected with multiple parallel splitters can be easily and efficiently optimized within a limited space by using an acoustical lumped technique, a plane wave theory, a four-pole transfer matrix, and a SA optimizer. As indicated in Table 2 and Figs. 4–5, two kinds of SA parameters (\( k_k \) and \( \text{iter} \)) play essential roles in the solution’s accuracy during SA optimization. Figs. 4–5 indicate that the tuning ability established by adjusting design parameters of muffler A is reliable. Additionally, the appropriate acoustical performance curve of three types of rectangular mufflers internally connected with multiple parallel splitters (mufflers A–C) has been assessed. As indicated in Table 3 and Fig. 6, the resultant \( \text{SWL}_T \) with respect to these mufflers is 113.1 dB(A), 106.5 dB(A), and 99.5 dB(A). Obviously, the muffler hybridized with more parallel splitters is superior to the other mufflers equipped with fewer parallel splitters. It can be seen that more splitters installed inside the rectangular muffler will disperse the venting fluid, avoid the secondary flowing noise, and reduce the back pressure of the muffler. Moreover, as investigated in Section 7.2, the influence of the acoustical performance with respect to five design parameters (\( W \): the width of air channel; \( R \): acoustic flow resistivity of the acoustic fiber; \( \sigma \): the porosity of the perforated plate; \( D_f \): the thickness of the acoustic fiber; \( L_1 \): the horizontal length of the splitter) is enormous. Consequently, an appropriate selection of the design parameters will result in a more efficient noise reduction for the mufflers.

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APPENDIX A

Transfer Matrix of a One-Channel Splitter Element

A one-layer sound absorber shown in Fig.13(A) is partitioned with two rigid stiffeners of L_1 and L_2. Based on the plane wave theory, \( f_c < \frac{c_0}{2L_{\text{max}}} \) where \( L_{\text{max}} \) is the maximum of \((L_1, L_2)\). The acoustic impedance on the perforated front plate is obtained from the bottom wall where the value of the impedance is infinity. As indicated in Fig. 13(B), there exists four points representing the absorbing impedance within the absorbers. The absorber is composed of a “rigid-backing plate +L, thickness of air + D_{fi} thickness of the acoustic fiber + q_1 thickness of the perforated front plate.” As derived in a previous study [15], for a wave propagating normally in a quiescent medium symbolized by “m,” the general matrix form between node \( i \) and node \( i+1 \) is expressed as
Therefore, for the upper sound absorber, the relationship of the acoustic pressure $p$ and the acoustic particle velocity $u$ between node 00 and node 01 is expressed as the transfer matrix and shown below.

$$
\begin{pmatrix}
  p_{01} \\
  u_{01}
\end{pmatrix} =
\begin{bmatrix}
  \cos(k_x L) & jZ_a \sin(k_x L) \\
  j Z_a \sin(k_x L) & \cos(k_x L)
\end{bmatrix}
\begin{pmatrix}
  p_0 \\
  u_0
\end{pmatrix} 
$$

(A1)

The relationship of the acoustic pressure $p$ and the acoustic particle velocity $u$ with respect to node 01 and node 02 is expressed in the transfer matrices below.

$$
\begin{pmatrix}
  p_{02} \\
  u_{02}
\end{pmatrix} =
\begin{bmatrix}
  \cos(k_{\text{fiber}} D_f) & jZ_{\text{fiber}} \sin(k_{\text{fiber}} D_f) \\
  j Z_{\text{fiber}} \sin(k_{\text{fiber}} D_f) & \cos(k_{\text{fiber}} D_f)
\end{bmatrix}
\begin{pmatrix}
  p_{01} \\
  u_{01}
\end{pmatrix} 
$$

(A4)

By developing Eq. (A4), an alternative form of Eq.(A4) yields

$$
Z_{02} = -j \rho_a c_a \cos\left(\frac{\omega(H - D_f)}{c_a}\right) 
$$

(A3)

By adopting the formula of the specific normal impedance and wave number [18], Eq. (A5) is written as

$$
Z_{02} = \begin{bmatrix}
  \sinh(k_{\text{fiber}} D_f) \cos(k_{\text{fiber}} D_f) - \\
  j \sin(k_{\text{fiber}} D_f) \cosh(k_{\text{fiber}} D_f)
\end{bmatrix}
\begin{bmatrix}
  \sinh(k_{\text{fiber}} D_f) \cosh(k_{\text{fiber}} D_f) \\
  j \sinh(k_{\text{fiber}} D_f) \sin(k_{\text{fiber}} D_f)
\end{bmatrix}
$$

(A6)

For sound flowing into the perforated plate, it is assumed that the incident sound passes through the holes of the perforated plate and is immediately transmitted to the porous material behind the perforated plate, the particle velocity is hardly reduced [19, 20].  The continuity of the particle velocity is then applied and expressed as

$$
u_{02} = u_2 
$$

(A7)

Acoustic impedance yields

$$
p_2 = Z_p u_{02} + p_{02} 
$$

(A8)

Combining Eqs. (A7)–(A8), the transfer matrix between node 02 and node 2 yields

$$
\begin{pmatrix}
  p_2 \\
  u_2
\end{pmatrix} =
\begin{bmatrix}
  1 & Z_p \\
  0 & 1
\end{bmatrix}
\begin{pmatrix}
  p_{02} \\
  u_{02}
\end{pmatrix} 
$$

(A9)

Developing Eq. (A9) and substituting Eq. (A7), the specific normal impedance at node 2 is

$$
Z_2 = Z_{02} + Z_p 
$$

(A10)

Adopting the formula of the specific normal impedance and the wave number of the perforated plate [21] yields

$$
Z_p = \frac{\rho_a}{\sigma} \sqrt{8 \sigma \omega (1 + \frac{q}{2dh})} + j \frac{\omega \rho_a}{\sigma} \sqrt{8 \omega \left(1 + \frac{q}{2dh}\right) + q + \delta} 
$$

(A11a)

$$
\delta = 0.85(2dh)\left(1 + 1.47\sqrt{\sigma} + 0.47\sqrt{\sigma}\right) 
$$

(A11b)

Similarly, for the lower sound absorber, the transfer matrix between node 02 and node 3 yields
Developing Eq. (A12), the specific normal impedance at node 3 is

$$Z_3 = Z_{i0} + Z_p$$  \hspace{1cm} (A13)

A sound wave propagates from the left side to the right side. Considering the acoustical mass conservation and boundary condition yields

$$p_1 = p_4; p_2 = p_4; p_3 = p_4; \hspace{1cm} (A14a)$$

$$\nu_1 = \nu_2 + \nu_3 + \nu_4; \hspace{1cm} (A14b)$$

$$S_1 = S_4 = S_p; \hspace{1cm} (A14c)$$

$$S_2 = S_1 = L_1 \ast L_2 \ast \sigma; \hspace{1cm} (A14d)$$

Defining the acoustical impedance of node 2 and node 3, the impedance can be expressed as

$$Z_{m=1,n=1,k=1} = Z_2 = \frac{P_2}{u_2}; Z_{m=1,n=2,k=1} = Z_3 = \frac{P_3}{u_3} \hspace{1cm} (A15)$$

Eq. (A15) can be expressed as

$$u_2 = \frac{P_2}{Z_2}; u_3 = \frac{P_3}{Z_3} \hspace{1cm} (A16)$$

Rearranging Eqs. (A14)-(A16) yields

$$\begin{pmatrix}
    p_1 \\
    \rho_v \nu \cdot u_1
  \end{pmatrix} = \begin{pmatrix}
    1 & 1 & 1 \\
    Z_{m=1,n=1,k=1} & Z_{m=1,n=2,k=1} & 1
  \end{pmatrix} \begin{pmatrix}
    p_4 \\
    \rho_v \nu \cdot u_4
  \end{pmatrix} \hspace{1cm} (A17)$$

The matrix form is expressed as

$$\begin{pmatrix}
    p_1 \\
    \rho_v \nu \cdot u_1
  \end{pmatrix} = \begin{pmatrix}
    \frac{1}{Z_{m=1,n=1,k=1}} & \frac{1}{Z_{m=1,n=2,k=1}} & 0 \\
    1
  \end{pmatrix} \begin{pmatrix}
    p_4 \\
    \rho_v \nu \cdot u_4
  \end{pmatrix} \hspace{1cm} (A18a)$$

An alternative form is

$$\begin{pmatrix}
    p_1 \\
    \rho_v \nu \cdot u_1
  \end{pmatrix} = \begin{pmatrix}
    TSPI_{11} & TSPI_{12} \\
    TSPI_{21} & TSPI_{22}
  \end{pmatrix} \begin{pmatrix}
    p_4 \\
    \rho_v \nu \cdot u_4
  \end{pmatrix} \hspace{1cm} (A18b)$$

(A) Three-dimensional view of a one-channel splitter muffler composed of two absorbers and an air-channel.
APPENDIX B

Transfer Matrix of a Two-Channel Splitter Element

As indicated in Fig. 14, two identical one-channel splitters are parallel and combined to form a two-channel splitter element. As derived in Appendix A, the acoustical four-pole matrix between node 5a and 6a for the first one-channel splitter type element is

\[
\begin{bmatrix}
p_{5a} \\
p_\rho c_\rho u_{5a}
\end{bmatrix} = \begin{bmatrix} TSP_{1,1} & TSP_{1,2} \\ TSP_{1,2} & TSP_{1,2} \end{bmatrix} \begin{bmatrix} p_{6a} \\
p_\rho c_\rho u_{6a}
\end{bmatrix}
\]  

(B1)

Developing Eq. (B1) yields

\[p_{5a} = TSP_{1,1} \cdot p_{6a} + TSP_{1,2} \cdot p_\rho c_\rho u_{6a}\]  

(B2a)

\[p_\rho c_\rho u_{5a} = TSP_{1,2} \cdot p_{6a} + TSP_{1,2} \cdot p_\rho c_\rho u_{6a}\]  

(B2b)

Similarly, the acoustical four-pole matrix for the second one-channel splitter element between node 5b and 6b is

\[
\begin{bmatrix}
p_{5b} \\
p_\rho c_\rho u_{5b}
\end{bmatrix} = \begin{bmatrix} TSP_{2,1} & TSP_{2,2} \\ TSP_{2,2} & TSP_{2,2} \end{bmatrix} \begin{bmatrix} p_{6b} \\
p_\rho c_\rho u_{6b}
\end{bmatrix}
\]  

(B3)

Developing Eq. (B3) yields

\[p_{5b} = TSP_{2,1} \cdot p_{6b} + TSP_{2,2} \cdot p_\rho c_\rho u_{6b}\]  

(B4a)

\[p_\rho c_\rho u_{5b} = TSP_{2,2} \cdot p_{6b} + TSP_{2,2} \cdot p_\rho c_\rho u_{6b}\]  

(B4b)

Combining Eq. (B2a) and Eq. (B4a) yields

\[p_{5a} + p_{5b} = TSP_{1,1} \cdot [p_{6a} + p_{6b}] + TSP_{1,2} \cdot p_\rho c_\rho u_{6a} + u_{6b}\]  

(B5)

Likewise, combining Eq. (B2b) and Eq. (B4b) yields

\[p_\rho c_\rho u_{5a} + u_{5b} = TSP_{2,1} \cdot [p_{6a} + p_{6b}] + TSP_{2,2} \cdot p_\rho c_\rho u_{6a} + u_{6b}\]  

(B6)

where

\[p_5 = p_{5a} = p_{5b}, \quad p_6 = p_{6a} = p_{6b}, \quad u_5 = u_{6a} + u_{6b}, \quad u_6 = u_{6a} + u_{6b}\]  

(B7)

Plugging Eq. (B7) into Eqs. (B5) and (B6) yields

\[2 \cdot p_5 = 2 \cdot TSP_{1,1} \cdot p_6 + TSP_{1,2} \cdot p_\rho c_\rho u_6\]  

(B8a)

\[p_\rho c_\rho u_5 = 2 \cdot TSP_{2,1} \cdot p_6 + TSP_{2,2} \cdot p_\rho c_\rho u_6\]  

(B8b)

Rearranging Eq. (B8) in a matrix form, the equivalent four-pole matrix between nodes 5 and 6 shown in Fig. 4 is

\[
\begin{bmatrix}
p_5 \\
p_\rho c_\rho u_5
\end{bmatrix} = \begin{bmatrix} TSP_{1,1} & \frac{1}{2} \cdot TSP_{1,2} \\ 2 \cdot TSP_{2,3} & TSP_{2,2} \end{bmatrix} \begin{bmatrix} p_6 \\
p_\rho c_\rho u_6
\end{bmatrix}
\]  

(B9)
APPENDIX C

Transfer Matrix of a Four-Channel Splitter Element

As indicated in Fig. 15, four identical one-channel splitters are parallel and combined to form a four-channel splitter element. As derived in Appendix A, the acoustical four-pole matrix between node 5a and 6a for the first one-channel splitter type element is

\[
\begin{bmatrix}
p_{5a} \\
\rho_c u_{5a}
\end{bmatrix} =
\begin{bmatrix}
T_{SPI1,1} & T_{SPI1,2} \\
T_{SPI2,1} & T_{SPI2,2}
\end{bmatrix}
\begin{bmatrix}
p_{6a} \\
\rho_c u_{6a}
\end{bmatrix}
\] (C1)

Developing Eq.(C1) yields

\[
p_{5a} = T_{SPI1,1} \cdot p_{6a} + T_{SPI1,2} \cdot \rho_c u_{6a}
\] (C2a)

\[
\rho_c u_{5a} = T_{SPI2,1} \cdot p_{6a} + T_{SPI2,2} \cdot \rho_c u_{6a}
\] (C2b)

Similarly, the acoustical four-pole matrices between node 5b and 6b, node 5c and 6c, and node 5d and 6d are

\[
\begin{bmatrix}
p_{5b} \\
\rho_c u_{5b}
\end{bmatrix} =
\begin{bmatrix}
T_{SPI1,1} & T_{SPI1,2} \\
T_{SPI2,1} & T_{SPI2,2}
\end{bmatrix}
\begin{bmatrix}
p_{6b} \\
\rho_c u_{6b}
\end{bmatrix}
\] (C3a)

\[
\begin{bmatrix}
p_{5c} \\
\rho_c u_{5c}
\end{bmatrix} =
\begin{bmatrix}
T_{SPI1,1} & T_{SPI1,2} \\
T_{SPI2,1} & T_{SPI2,2}
\end{bmatrix}
\begin{bmatrix}
p_{6c} \\
\rho_c u_{6c}
\end{bmatrix}
\] (C3b)

\[
\begin{bmatrix}
p_{5d} \\
\rho_c u_{5d}
\end{bmatrix} =
\begin{bmatrix}
T_{SPI1,1} & T_{SPI1,2} \\
T_{SPI2,1} & T_{SPI2,2}
\end{bmatrix}
\begin{bmatrix}
p_{6d} \\
\rho_c u_{6d}
\end{bmatrix}
\] (C3c)

Developing Eqs.(C3a)–(C3c) yields

\[
p_{5b} = T_{SPI1,1} \cdot p_{6b} + T_{SPI1,2} \cdot \rho_c u_{6b}
\] (C4a)

\[
\rho_c u_{5b} = T_{SPI2,1} \cdot p_{6b} + T_{SPI2,2} \cdot \rho_c u_{6b}
\] (C4b)

\[
p_{5c} = T_{SPI1,1} \cdot p_{6c} + T_{SPI1,2} \cdot \rho_c u_{6c}
\] (C4c)

\[
\rho_c u_{5c} = T_{SPI2,1} \cdot p_{6c} + T_{SPI2,2} \cdot \rho_c u_{6c}
\] (C4d)
Combining Eq.(C2a) and Eq.(C4a), (C4c), (C4e) yields
\[ p_{5d} = TSP1_{11} \cdot p_{ad} + TSP1_{1,2} \cdot \rho \cdot \rho \cdot u_{ad} \]  \hfill (C4e)
\[ \rho \cdot \rho \cdot u_{sd} = TSP1_{21} \cdot p_{ad} + TSP1_{2,2} \cdot \rho \cdot \rho \cdot u_{ad} \]  \hfill (C4f)

Likewise, combining Eq.(C2b) and Eq.(C4b), (C4d), and (C4f) yields
\[ p_{6a} + p_{b} + p_{c} + p_{sd} = TSP1_{11} \cdot \{ p_{6a} + p_{b} + p_{c} + p_{sd} \} + TSP1_{1,2} \cdot \rho \cdot \rho \cdot \{ u_{6a} + u_{b} + u_{c} + u_{sd} \} \]  \hfill (C5)
\[ \rho \cdot \rho \cdot u_{sd} = TSP1_{21} \cdot \{ p_{6a} + p_{b} + p_{c} + p_{sd} \} + TSP1_{2,2} \cdot \rho \cdot \rho \cdot \{ u_{6a} + u_{b} + u_{c} + u_{sd} \} \]  \hfill (C6)

where
\[ \rho = p_{6a} = p_{b} = p_{c} = p_{sd} = p_{6b} = p_{6c} = p_{sd} = p_{6a} \]
\[ u_{5} = u_{6a} + u_{b} + u_{c} + u_{sd} \]
\[ u_{6} = u_{6a} + u_{b} + u_{c} + u_{sd} \]  \hfill (C7)

Plugging Eq.(C7) into Eqs.(C5) and (C6) yields
\[ 4 \cdot p_{5} = 4 \cdot TSP1_{11} \cdot p_{b} + TSP1_{1,2} \cdot \rho \cdot \rho \cdot u_{b} \]  \hfill (C8a)
\[ \rho \cdot \rho \cdot u_{5} = 4 \cdot TSP1_{1,2} \cdot p_{b} + TSP1_{2,2} \cdot \rho \cdot \rho \cdot u_{6} \]  \hfill (C8b)

Rearranging Eq.(C8) in a matrix form, the equivalent four-pole matrix between nodes 5 and 6 shown in Fig. 4 is
\[ \begin{bmatrix}
p_{5} \\
\rho \cdot \rho \cdot u_{5}
\end{bmatrix} = \begin{bmatrix}
TSP1_{11} & \frac{1}{4} \cdot TSP1_{1,2} \\
4 \cdot TSP1_{2,2} & \frac{1}{2} \cdot TSP1_{2,2}
\end{bmatrix} \begin{bmatrix}
p_{b} \\
\rho \cdot \rho \cdot u_{6}
\end{bmatrix} \]  \hfill (C9)

Figure 15. A four-channel splitter element.