Radiation Reaction in Classical Field Theory

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Introduction

The aim of this book is to provide a self-contained and systematic introduction to problems of radiation and radiation reaction in classical field theory. The book is not intended to be a textbook in electrodynamics in the usual sense. We shall not attempt to exhibit neither a complete picture of the present state of classical particle electrodynamics nor the historical development of the subject to date. We focus it exclusively on the radiation phenomena in various models of classical field theory and on equations of motions of charged particles where the radiation reaction is taken into account.

Studies of the classical theory of charged particles and their radiation initiated by Lorentz and Abraham have demanded our attention over a century. (The early history of this subject is well presented in textbook [71, Chapter 2], see also Ref. [81].) Abraham and Lorentz intended to develop a theory for the newly discovered electron. The authors designed electron as a tiny charged sphere and summed up all the mutual electromagnetic forces acting between various charge elements. The total force was called the self-force. The first model of electron proposed by Abraham [2] was a rigid sphere with spherically symmetric charge distribution. (For a modern analysis see Ref. [81].) The resulting expression for the self-force acting on Abraham’s sphere with arbitrary charge’s distribution was obtained by Lorentz [48] in form of infinite series in powers of radius $R$ of the sphere. The first term is proportional to electron’s acceleration and inversely proportional to its radius. The second term is proportional to the derivative of acceleration. It is structureless one while the higher terms all depend on the electron’s structure, i.e. its radius $R$ and charge’s distribution. Neglecting the higher order terms the author obtained generalization of the Newton’s second law for a tiny charged sphere where the self-action is taken into account. The first term of the series which diverges as $1/R$ was coupled with particle’s mass which then was proclaimed to be finite (the so-called renormalization of mass). Later Abraham [3] generalized the self-force expression to be valid for relativistic charged particle. Besides the proper time derivative of four-acceleration, his radiation-reaction four-vector contains also the Larmor term for a rate of energy-momentum radiated by oscillating charge. (Liénard and Heaviside generalized Larmor expression (1897) for a rate of radiation of non-relativistic charge on the case of relativistic one.)

Further progress in this field is associated with Dirac [17] who regarded the electron as an elementary particle with no internal structure. The au-
Author produced a proper relativistic derivation of the equation of motion of a point charged particle under the influence of an external force as well as its own electromagnetic field. Its commonly used and widely accepted title is the Lorentz-Dirac equation. To derive the equation Dirac [17] used conservation of energy-momentum. This method is based on the conservation equation \( \text{div} \hat{T} = 0 \) where \( \hat{T} \) is the electromagnetic field energy-momentum tensor density. The differential statement of energy-momentum conservation can immediately be turned into integral statement. Applying Gauss’s theorem the author calculate a flux of Maxwell energy-momentum tensor through a space-like surface enclosing a fragment of particle’s world line. The work of an external force matches the flux of electromagnetic field energy-momentum and the change of particle’s individual four-momentum.

While the verification of energy-momentum conservation is not a trivial matter whenever we treat charges as point particles. Liénard-Wiechert fields are the solutions of wave equations with point-like sources. The fields as well as corresponding stress-energy tensor \( \hat{T} \) get extremely large\(^1\) in the immediate vicinity of particle’s world line. A real challenge is to reveal the part of \( \hat{T} \) that produces finite flows of energy and momentum which detach themselves from the source and lead an independent existence. In his pioneer work [84] Teitelboim splits the stress-energy tensor \( \hat{T} \) of a point source into two parts which are separately conserved off the world line (see also review [85]). Volume integration of the radiative part of the stress-energy tensor gives the integral of the Larmor relativistic rate of emitted energy-momentum over particle’s world line. While the volume integration of the bound part results the term which is of essential value near the particle only. It describes the radiation that never goes far from the source and travels along with it. Angular momentum carried by electromagnetic field is also split into two parts with different properties: the divergent bound angular momentum depends on the instant characteristics of the charged particle while the emitted angular momentum accumulates with time [49].

The bound part of the stress-energy tensor produces the “cloud” of energy and momentum which is permanently attached to the charge and is carried along with it. The bound part contains the self-interaction, which gives rise to the self-energy and self-stress of the source. It contributes to particle’s inertia: four-momentum of charge contains, apart from usual velocity term, also a term which is proportional to the square of charge [84, eq.(4.4)]. A point-like singularity together with surrounding “cloud” constitute new entity: dressed charged particle.

The Larmor relativistic rate of emitted energy-momentum describes the radiation which detaches itself from the source and leads an independent existence. The “long-range” radiation was the subject of active investigation since the early works of Schott and Schwinger [79]. Schott describes the ra-

\(^1\)If a quantity tends to infinity when approaching particle’s position, we call that it has a singularity at the position of the particle.
radiation emitted by circling electron [78]. The radiation loses because of the high radial acceleration experienced by the electrically charged particles in circular accelerators set an upper limit to the attained energy. (Both cyclotron and synchrotron radiation have inherited their names from the devices used to accelerate charged particles in the 1930s and 1940s.) While radiation does more good than harm generally. A sinuous beam of relativistic electrons which passes through a periodic magnetic field produced by arranging magnets with alternating poles is used as the amplification medium in free electron lasers [23]. The high intensity radiation emitted by this device can be tuned over a wide range of wavelengths. It is a very important advantage over conventional lasers.

The phenomenon of the self-force is intimately connected with the radiation; for this reason the “self-force” is also called “radiation-reaction force”. The radiation removes energy, momentum, and angular momentum from the particle, which then undergoes a radiation reaction. On the hypothesis that the total amounts of these quantities do not change we intend “to find a formulation of classical charged particle theory which does not require any reference to, or assumptions about, the particle structure, its charge distribution, and its size.” [71, section 6.2].

The main feature of the present book is that it is exclusively based on symmetries and their associated conservation laws. Behavior of composite particles plus field system is governed by action principle which is invariant under infinitesimal transformations (rotations and translations) which constitute the Poincaré group. According to Noether’s theorem, these symmetry properties yield conservation laws, i.e. these quantities that do not change with time. Conserved quantities place stringent requirements on the dynamics of the system. They demand that the change in field’s energy, momentum, and angular momentum should be balanced by a corresponding change in the momentum and angular momentum of the particles, so that the total particles’ plus field’s momentum and angular momentum are properly conserved. The conservation laws are an immovable fulcrum about which tips the balance of truth regarding renormalization and radiation reaction.

We now give an outline of this book.

Its structure is logical and didactic rather than historical. The necessary mathematical tools are presented in the first two chapters. Chapter 1 contains elementary differential geometry. It is oriented toward physicists who are not expert in this field. This Chapter covers manifolds and their coordinatizations, vector fields and tangent bundles, Cartan’s differential forms and exterior differentiation, Lie groups and Lie algebras etc. A special attention is paid to Stoke’s theorem and integration. Chapter 2 contains some aspects of classical electrodynamics. Section 2.4 presents Maxwell’s equations in coordinate-free form. The equations look the same not only in any coordinates but also in Minkowski space of arbitrary dimensions where the sources and the field “live”. Sections 2.5 and 2.6 review the Lagrangian formalism of classical field theory,
Noether theorem and conservation laws. In Section 2.7 the algebraic properties of the electromagnetic field tensor are analyzed.

Chapter 3 covers solutions to Maxwell’s equation in Minkowski space of various dimensions. We use the Green’s function technique to solve the inhomogeneous wave equation, i.e. equation of motion for the electromagnetic field. In general, a Green’s function consists of “direct” and “tail” parts which are proportional to the Dirac’s delta-function and to the Heaviside step function, respectively. The “direct” part results the electromagnetic field at a point \( x \) which is determined completely by state of motion of a source on the past null cone with vertex at \( x \). It is true for the Liénard-Wiechert potentials in Minkowski space of even dimensions, including four-dimensional spacetime. In odd dimensions the “tail” part of Green’s function arises which allows for additional contribution from source within the past null cone of \( x \) where the field is evaluated.

Chapter 4 treats self-interaction in four-dimensional electrodynamics. The Liénard-Wiechert solutions to wave equations (both the retarded and the advanced) are given in the first two sections. Much attention is paid to their symmetry properties with respect to time inversion. In Section 4.4 we present Dirac’s regularization procedure based on the decomposition of the retarded Liénard-Wiechert field into the singular “mean of the advanced and retarded field” and the finite “radiation” field that reaches a distant sphere. While the main feature of this book is that we prefer the retarded fields as those of true physical meaning. Following our experience, we suppose that point charged particles carry with them fields that behave as outgoing radiation. In Section 4.5 we describe the retarded coordinate systems which are used for calculations of flows of electromagnetic field energy-momentum and angular momentum. We compare Dirac’s retarded coordinates [17] and Bhabha’s coordinates [7] associated with non-inertial reference frame that travels along with an accelerated charge. We find the energy-momentum and angular momentum emitted by the charge as well as their bound counterparts which are not emitted but remain linked to the charge. Much of this Chapter is drawn from the research literature and some of it appears to be new. Particularly noteworthy in Section 4.6 which presents a derivation of the Lorentz-Dirac equation which relies on ten conserved quantities corresponding to Poincaré-invariance of a composite particle plus field system.

Chapter 5 explores further application of the renormalization procedure based on energy-momentum and angular momentum balance equations. It deals with the self-action problem for a point-like charge arbitrarily moving in flat spacetime of six dimensions. Electromagnetic field tensor consists of electric field with five components and magnetic field with ten components. In Section 5.1 we present the solution to wave equation with point-like source. Six-dimensional analog of the Liénard-Wiechert potential depends not only on particle’s position and six-velocity, but also on its six-acceleration. Coulomb electrostatic potential scales as \( |r|^{-3} \) in six dimensions (see Section 5.2 where
static potentials in arbitrary dimensional spacetime are given). Inevitable infinities arising in six-dimensional electrodynamics are stronger than in four dimensions and cannot be removed by the renormalization of mass. A closer look on balance equations explores that the “bare” singularity possesses something like internal angular momentum. Its magnitude is proportional to the norm of six-acceleration. It is the so-called rigid relativistic particle described in Section 5.5. Its motion is governed by the higher derivative Lagrangian depending not only on the arclength of the world line, but also on its curvature. Surrounding electromagnetic “cloud” contributes in six-momentum of dressed charge, so that dynamics of the dressed charged particle is richer than that of “bare” singularity. In Section 5.7 we derive six-dimensional analog of the Lorentz-Dirac equation which was firstly obtained by Kosyakov [42]. Analysis of balance equations is the cornerstone of our treatment of the regularization procedure.

Chapter 6 discusses two-body problem in conventional electrodynamics where interference in the radiation of two point-like charges is taken into account. We determine ten conserved quantities corresponding to Poincaré symmetry of a closed system of two charges and their electromagnetic field. Since the stress-energy tensor is quadratic in field strengths and the field satisfies superposition principle, the tensor contains mixed part which describes interference of outgoing electromagnetic waves from arbitrarily moving point-like sources. The so-called direct particle fields [36] arise due to volume integration of mixed part of two-particle stress-energy tensor. These direct fields (retarded and advanced) are functionals of particles’ world lines; they do not possess degrees of freedom of their own. Therefore, we arrive at the realm of action-at-a-distance electrodynamics [36].

The theory was elaborated by Wheeler and Feynman [90, 91]. It is based on the following assumptions [90, p.160]:

- An accelerated point charge in otherwise charge-free space does not radiate electromagnetic energy.
- The fields which act on a given particle arise only from other particles.
- These fields are represented by one-half the retarded plus one-half the advanced Liénard-Wiechert solutions of Maxwell’s equations. This law is symmetric with respect to past and future.
- Sufficiently many particles are present to absorb completely the radiation given off by the source.

Since the source emanates in all possible directions, all the particles of the universe are required to absorb completely the radiation. They constitute a perfect absorber which possesses a remarkable twofold property: it cancels the (acausal) advanced part of the fields acted on the source particle and doubles the retarded one. Therefore, the complete absorption is the crucial issue of the
theory. For this reason Wheeler and Feynman called it the absorber theory of radiation.

Rigorous calculations performed in Chapter 6 reveal that the combination of retarded Liénard-Wiechert fields forms a resultant field with the desired properties. Then the “perfect absorption” should be replaced by the interference of outgoing waves in Wheeler and Feynman electrodynamics. It allows us to reconcile Wheeler and Feynman theory with the concept of retarded causality.

The Lorentz-Dirac equation governs the motion of a point-like charge in flat spacetime. DeWitt and Brehme [92] generalized Dirac’s work to a curved spacetime. (Their expression for the self-force acting on a point charge radiating electromagnetic waves in a curved space background was later corrected by Hobbs [35].) Background gravitational field “slows down” photons mediating electromagnetic interaction, so that they move with all velocities smaller than or equal to the speed of light. For this reason the charge “fills” its own field, which acts on it just like an external one. In the present book we do not consider electrodynamics in curved spacetime. Instead we analyse in Chapter 7 the self-action problem for an electric charge arbitrarily moving in flat spacetime of three dimensions. An essential feature of 2 + 1 electrodynamics is that the radiation develops a tail, as it is in four-dimensional curved spacetime. This is because the retarded Green’s function associated with D’Alembert operator is supported within the light cone (see Chapter 3, paragraph 3.4.1).

In Section 7.1 we present the Maxwell’s equations in Minkowski space of three dimensions. Electromagnetic field tensor consists of electric field with two components and scalar magnetic field. We outline the remarkable correspondence between the Maxwell’s equations and equations that govern behaviour of superfluid $^4$He film. By this we mean that the dynamics of the low energy quasiparticles and elementary excitations living inside a helium film is governed by Maxwell’s equations in $\mathbb{M}_3$. Therefore, if one study the behavior of electric charges living inside hypothetic spacetime with two space directions, they study the kinetic of vortices and phonon excitations in superfluid $^4$He film.

Computation of electromagnetic field energy, momentum, and angular momentum carried by the tail field is highly non-trivial because outgoing waves emitted by different points of particle’s world line combine with one another. This phenomenon is called the violation of Huygens principle; it is described in Section 7.4. In Section 7.5 we introduce a coordinate system which allows to calculate in Sections 7.6, 7.7, and 7.8 the total flows of (retarded) Noether quantities which flow across the plane $x^0 = t$ with fixed $t$. From the analysis of balance equations we develop the regularization procedure for a theory where Green’s function involves a “tail” part. The renormalization scheme manipulates fields on the world line only. It generalizes Dirac’s scheme of decomposition of the direct retarded field into the singular “mean of the advanced and retarded field” and the finite “radiation” field. In Section 7.9 we derive
integro-differential equation which plays the role the Lorentz-Dirac equation in three dimensions. The radiation reaction is determined by the Lorentz force of point-like charge acting upon itself plus a term that provides finiteness of the self-action. The word “self-force” can be understood literally: the charge in three dimensions is repulsed by itself taken in the past. The integro-differential equation of motion contains path integral over the past motion of the dressed charge.

Chapter 8 explores application of the renormalization procedure for theories with tail fields to a point-like source coupled with neutral massive scalar field. Corresponding equations of motion were first found by Bhabha [7] following a method originally developed by Dirac [17] for the case of electromagnetic field. In this method the finite force and self-force terms in the equations of motion are obtained from the conservation laws for the energy-momentum tensor of the field. It was extended by Bhabha and Harish-Chandra [9] to particles interacting with any tensor field and was applied to the motion of a simple pole of massive scalar field by Harish-Chandra [30].

Recently [68], Quinn has obtained an expression for the self-force on a point-like particle coupled to a massless scalar field arbitrarily moving in a curved spacetime. Quinn establishes that the total work done by the scalar self-force matches the amount of energy radiated away by the particle. To cancel a troublesome part of self-force near the upper limit of path integral, the author averages diverging piece over a small, spatial two-sphere surrounding the particle.

In Chapter 8, we split the energy-momentum and angular momentum carried by massive scalar field into bound and radiative parts. Extracting of radiated portions of Noether quantities is not a trivial matter, since the massive field holds energy and momentum near the source. We apply the procedure elaborated in preceding Chapter, where 2 + 1 electrodynamics has been considered. To substantiate the splitting, we analyze asymptotic behavior of the bound and the radiative pieces as well as the total energy-momentum and angular momentum balance equations. It is of great importance that conservation laws yield the Harish-Chandra equation of motion. This circumstance reinforces our conviction that Dirac’s decomposition scheme can be applied to tail fields. We regard this approach as preferable, because radiative terms are smooth at the location of the particle, so that averaging is not required in computing the self-force.

In Chapters 4-8 the rearrangement of the initial degrees of freedom in classical field theory has been performed which leads to new dynamical entities: dressed charged particles and radiation. A dressed charged particle is a “bare” source surrounded by neighbouring field’s “cloud” while the radiation is the “far” field which detaches the source and leads to independent existence. All the problems considered before are renormalizable: unavoidable divergences are absorbed within the renormalization procedure. The renormalization modifies particles’ individual characteristics. In Chapter 9 we consider an interest-
ing example of non-renormalizable theory: classical electrodynamics of massless charged particles [10, 18]. Unlike the massive case, the charge having zero rest mass is not “dressed”. By this we mean that the photon-like charge does not possess any electromagnetic “cloud” permanently attached to it. It produces the far field which diverges not only on particle’s world line, but also at points on light ray that extends to infinity. The “ray singularity” can not be removed by means of renormalization of particle’s individual characteristics.

Over a sixty figures, tables and diagrams illustrate the book. We hope that it helps to explain the meaning of material, because there are many people for whom geometrical reasoning is easier than purely analytic reasoning. To provide guidance to the book’s contents, section headings within chapters are pointed in two different styles. Fundamental material is marked by **boldface** headings, while supplementary topics are marked by *italics*. 
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