Partially Observable Total-Cost Markov Decision Processes with Weakly Continuous Transition Probabilities

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Abstract

This paper describes sufficient conditions for the existence of optimal policies for Partially Observable Markov Decision Processes (POMDPs) with Borel state, observation, and action sets and with the expected total costs. Action sets may not be compact and one-step cost functions may be unbounded. The introduced conditions are also sufficient for the validity of optimality equations, semi-continuity of value functions, and convergence of value iterations to optimal values. Since POMDPs can be reduced to Completely Observable Markov Decision Processes (COMDPs), whose states are posterior state distributions, this paper focuses on the validity of the above mentioned optimality properties for COMDPs. The central question is whether transition probabilities for a COMDP are weakly continuous. We introduce sufficient conditions for this and show that the transition probabilities for a COMDP are weakly continuous, if transition probabilities of the underlying Markov Decision Process are weakly continuous and observation probabilities for the POMDP are continuous in the total variation. Moreover, the continuity in the total variation of the observation probabilities cannot be weakened to setwise continuity. The results are illustrated with counterexamples and examples.

1 Introduction

Partially Observable Markov Decision Processes (POMDPs) play an important role in operations research, electrical engineering, and computer science. They have a broad range of applications to various areas including sensor networks, artificial intelligence, target tracking, control and maintenance of complex systems, finance, and medical decision making. In principle, it is known how to solve POMDPs. A POMDP can be reduced to a Completely Observable Markov Decision Process (COMDP) with the state space being the set of belief (posterior state) probabilities for the POMDP; see Hinderer [21, Section 7.1] and Sawarigi and

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Yoshikawa [26] for countable state spaces and Rhenius [24], Yushkevich [33], Dynkin and Yushkevich [12, Chapter 8], Bertsekas and Shreve [8, Chapter 10], and Hernández-Lerma [18, Chapter 4] for Borel state spaces. After an optimal policy for the COMDP is found, it can be used to compute an optimal policy for the POMDP. However, except finite state and action POMDPs (Sondik [30]), problems with a continuous filtering transition probability $H$ (Hernández-Lerma [18, Chapter 4], Hernández-Lerma and Romera [20]), and a large variety of particular problems considered in the literature, little is known regarding the existence and properties of optimal policies for COMDPs and POMDPs with expected total costs.

This paper investigates the existence of optimal policies for COMDPs and therefore for POMDPs with the expected total discounted costs and, if the one-step costs are nonnegative, with the expected total costs. We provide conditions for the existence of optimal policies and for the validity of other properties of optimal values and optimal policies: they satisfy optimality equations, optimal values are lower semi-continuous functions, and value iterations converge to optimal infinite-horizon values.

Since a COMDP is a Markov Decision Process (MDP) with Borel state and action sets, it is natural to apply results on the existence of optimal policies for MDPs to COMDPs. Feinberg et al. [15] introduced a mild assumption, called Assumption ($W^*$), for the existence of stationary optimal policies for infinite horizon MDPs, lower semi-continuity of value functions, characterization of the sets of optimal actions via optimality equations, and convergence of value iterations to optimal values for the expected total discounted costs, if one-step costs are bounded below, and for the expected total costs, if the one-step costs are nonnegative (according to the main result in [15], if another mild assumption is added to Assumption ($W^*$), then there exist stationary optimal policies for average costs per unit time). Assumption ($W^*$) consists of two conditions: transition probabilities are weakly continuous and one-step cost functions are $K$-inf-compact. The notion of $K$-inf-compactness (see the definition below) was introduced in Feinberg et al. [14], and it is slightly stronger than the lower semi-continuity of the cost function and its inf-compactness in the action parameter. In operations research applications, one-step cost functions are usually $K$-inf-compact. Here we consider a POMDP whose underlying MDP satisfies Assumption ($W^*$). According to Theorem 3.4, this implies the $K$-inf-compactness of the cost function for the COMDP and it remains to prove the weak continuity of transition probabilities for the COMDP to verify Assumption ($W^*$) for the COMDP and therefore the existence of optimal policies and the validity of additional optimality properties for the COMDP and POMDP.

For problems with incomplete information, the filtering equation $z_{t+1} = H(z_t, a_t, y_{t+1})$ presented in equation (3.4) below, that links the posterior state probabilities $z_t, z_{t+1}$, the selected action $a_t$, and the observation $y_{t+1}$, plays an important role. This equation presents a general form of Bayes’s rule. Hernández-Lerma [18, Chapter 4] showed that the weak continuity of $H$ in all three variables and the weak continuity of transition and observation probabilities imply the weak continuity of transition probabilities for the COMDP. In this paper we introduce another condition, Assumption (H), which is weaker than the weak continuity of the filtering kernel $H$ in $(z_t, a_t, y_{t+1})$. We prove that this condition and setwise continuity of the stochastic kernel on the observation set, given a posterior state probability and prior action, imply the weak continuity of the transition probability for the COMDP; Theorem 3.5. In particular, if either these assumptions or the weak continuity of $H$ and observation probabilities are added to Assumption ($W^*$) for the underlying MDP of the POMDP, the COMDP satisfies Assumption ($W^*$) and therefore various optimality properties, includ-
ing the existence of stationary optimal policies and convergence of value iterations hold; see Theorem 3.2.

By using Assumption (H) this paper answers the following question: which conditions on transition and observation probabilities are sufficient for weak continuity of transition probabilities for the COMDP? Theorem 3.7 states that weak continuity of transition probabilities and continuity of observation probabilities in the total variation imply weak continuity of transition probabilities for COMDPs. Thus, Assumption \((W^*)\) and continuity of observation probabilities in the total variation imply that the COMDP satisfies Assumption \((W^*)\) and therefore optimal policies exist for the COMDP and for the POMDP, value iterations converge to the optimal value, and other optimality properties hold; Theorem 3.6. Example 4.1 demonstrates that continuity of observation probabilities in the total variation cannot be relaxed to setwise continuity.

If the observation set is countable and it is endowed with the discrete topology, convergence in the total variation and weak convergence are equivalent. Thus, Theorem 3.7 implies weak continuity of the transition probability for the COMDP with a countable observation set endowed with the discrete topology and with weakly continuous transition and observation kernels; Hernández-Lerma [18, p. 93]. However, as Example 4.2 demonstrates, under these conditions the filtering transition probability \(H\) may not be continuous. This example motivated us to introduce Assumption (H).

The main results of this paper are presented in Section 3. Section 4 contains three counterexamples. In addition to the two examples described above, Example 4.3 demonstrates that setwise continuity of the stochastic kernel on the observation set, given a posterior state probability and prior action, is essential to ensure that Assumption (H) implies continuity of the transition probability for the COMDP. Section 5 describes properties of stochastic kernels used in the proofs of main results presented in Section 6. Section 7 introduces a sufficient condition for the weak continuity of transition probabilities for the COMDP that combines Assumption (H) and the weak continuity of \(H\). Combining these properties together is important because Assumption (H) may hold for some observations and continuity of \(H\) may hold for others. Section 8 contains three illustrative examples: (i) a model defined by stochastic equations including Kalman’s filter, (ii) a model for inventory control with incomplete records (for particular inventory control problems of such type see Bensoussan et al. [4–7] and references therein), and (iii) the classic Markov decision model with incomplete information studied by Aoki [1], Dynkin [11], Shiryaev [29], Hinderer [21] Section 7.1, Sawaragi and Yoshikawa [26], Rhenius [24], Yushkevich [33], Dynkin and Yushkevich [12] Chapter 8, for which we provide a sufficient condition for the existence of optimal policies, convergence of value iterations to optimal values, and other optimality properties formulated in Theorems 3.2.

2 Model Description

For a metric space \(S\), let \(B(S)\) be its Borel \(\sigma\)-field, that is, the \(\sigma\)-field generated by all open subsets of the metric space \(S\). For a Borel set \(E \in B(S)\), we denote by \(B(E)\) the \(\sigma\)-field whose elements are intersections of \(E\) with elements of \(B(S)\). Observe that \(E\) is a metric space with the same metric as on \(S\), and \(B(E)\) is its Borel \(\sigma\)-field. For a metric space \(S\), we denote by \(P(S)\) the set of probability measures on \((S, B(S))\). A sequence of probability measures \(\{\mu^{(n)}\}_{n=1,2,...}\) from \(P(S)\) converges weakly (setwise) to \(\mu \in P(S)\) if for
any bounded continuous (bounded Borel-measurable) function \( f \) on \( \mathbb{S} \)

\[
\int_{\mathbb{S}} f(s) \mu^{(n)}(ds) \to \int_{\mathbb{S}} f(s) \mu(ds) \quad \text{as} \quad n \to \infty.
\]

A sequence of probability measures \( \{\mu^{(n)}\}_{n=1,2,...} \) from \( \mathbb{P}(\mathbb{S}) \) converges in the total variation to \( \mu \in \mathbb{P}(\mathbb{S}) \) if

\[
\sup \left\{ \int_{\mathbb{S}} f(s) \mu^{(n)}(ds) - \int_{\mathbb{S}} f(s) \mu(ds) : f : \mathbb{S} \to [-1,1] \text{ is Borel-measurable} \right\} \to 0 \quad \text{as} \quad n \to \infty.
\]

Weak convergence, setwise convergence, and convergence in the total variation are used in Yüksel and Linder [32] to describe convergence of observation channels. Note that \( \mathbb{P}(\mathbb{S}) \) is a separable metric space with respect to the weak convergence topology for probability measures, when \( \mathbb{S} \) is a separable metric space; Parthasarathy [23] Chapter II. For separable metric spaces \( \mathbb{S}_1 \) and \( \mathbb{S}_2 \), a (Borel-measurable) stochastic kernel \( R(ds_1|s_2) \) on \( \mathbb{S}_1 \) given \( \mathbb{S}_2 \) is a mapping \( R(\cdot | \cdot) : \mathcal{B}(\mathbb{S}_1) \times \mathbb{S}_2 \to [0,1] \), such that \( R(\cdot | s_2) \) is a probability measure on \( \mathbb{S}_1 \) for any \( s_2 \in \mathbb{S}_2 \), and \( R(B|\cdot) \) is a Borel-measurable function on \( \mathbb{S}_2 \) for any Borel set \( B \in \mathcal{B}(\mathbb{S}_1) \). A stochastic kernel \( R(ds_1|s_2) \) on \( \mathbb{S}_1 \) given \( \mathbb{S}_2 \) defines a Borel measurable mapping \( s_2 \to R(\cdot | s_2) \) of \( \mathbb{S}_2 \) to the metric space \( \mathbb{P}(\mathbb{S}_1) \) endowed with the topology of weak convergence. A stochastic kernel \( R(ds_1|s_2) \) on \( \mathbb{S}_1 \) given \( \mathbb{S}_2 \) is called weakly continuous (setwise continuous, continuous in the total variation), if \( R(\cdot | x^{(n)}) \) converges weakly (setwise, in the total variation) to \( R(\cdot | x) \) whenever \( x^{(n)} \) converges to \( x \) in \( \mathbb{S}_2 \). For one-point sets \( \{s_1\} \subset \mathbb{S}_1 \), we sometimes write \( R(s_1|s_2) \) instead of \( R(\{s_1\}|s_2) \).

For a Borel subset \( S \) of a metric space \( (\mathbb{S}, \rho) \), where \( \rho \) is a metric, consider the metric space \( (S, \rho) \). A set \( B \) is called open (closed, compact) in \( S \) if \( B \subseteq S \) and \( B \) is open (closed, compact, respectively) in \( (S, \rho) \). Of course, if \( S = \mathbb{S} \), we omit “in \( \mathbb{S} \)”. Observe that, in general, an open (closed, compact) set in \( S \) may not be open (closed, compact, respectively).

Let \( \mathbb{X} \), \( \mathbb{Y} \), and \( \mathbb{A} \) be Borel subsets of Polish spaces (a Polish space is a complete separable metric space), \( P(dx'|x,a) \) be a stochastic kernel on \( \mathbb{X} \) given \( \mathbb{X} \times \mathbb{A} \), \( Q(dy|a,x) \) be a stochastic kernel on \( \mathbb{Y} \) given \( \mathbb{A} \times \mathbb{X} \), \( Q_0(dy|x) \) be a stochastic kernel on \( \mathbb{Y} \) given \( \mathbb{X} \), \( p \) be a probability distribution on \( \mathbb{X} \), \( c : \mathbb{X} \times \mathbb{A} \to \mathbb{R} = \mathbb{R} \cup \{+\infty\} \) be a bounded below Borel function on \( \mathbb{X} \times \mathbb{A} \), where \( \mathbb{R} \) is a real line.

A POMDP is specified by a tuple \((\mathbb{X}, \mathbb{Y}, \mathbb{A}, P, Q, c)\), where \( \mathbb{X} \) is the state space, \( \mathbb{Y} \) is the observation set, \( \mathbb{A} \) is the action set, \( P(dx'|x,a) \) is the state transition law, \( Q(dy|a,x) \) is the observation kernel, \( c : \mathbb{X} \times \mathbb{A} \to \mathbb{R} \) is the one-step cost.

The partially observable Markov decision process evolves as follows:

- at time \( t = 0 \), the initial unobservable state \( x_0 \) has a given prior distribution \( p \);
- the initial observation \( y_0 \) is generated according to the initial observation kernel \( Q_0(\cdot|x_0) \);
- at each time epoch \( t = 0,1, \ldots \), if the state of the system is \( x_t \in \mathbb{X} \) and the decision-maker chooses an action \( a_t \in \mathbb{A} \), then the cost \( c(x_t,a_t) \) is incurred;
- the system moves to a state \( x_{t+1} \) according to the transition law \( P(\cdot|x_t,a_t), t = 0,1, \ldots \);
- an observation \( y_{t+1} \in \mathbb{Y} \) is generated by the observation kernel \( Q(\cdot|a_t,x_{t+1}), t = 0,1, \ldots \).
Define the observable histories: \( h_0 := (p, y_0) \in \mathbb{H}_0 \) and \( h_t := (p, y_0, a_0, \ldots, y_{t-1}, a_{t-1}, y_t) \in \mathbb{H}_t \) for all \( t = 1, 2, \ldots \), where \( \mathbb{H}_0 := P(\mathcal{X}) \times \mathcal{Y} \) and \( \mathbb{H}_t := \mathbb{H}_{t-1} \times \mathcal{A} \times \mathcal{Y} \) if \( t = 1, 2, \ldots \). A policy \( \pi \) for the POMDP is defined as a sequence \( \pi = \{\pi_t\}_{t=0,1,\ldots} \) of stochastic kernels \( \pi_t \) on \( \mathcal{A} \) given \( \mathbb{H}_t \). A policy \( \pi \) is called nonrandomized, if each probability measure \( \pi_t(\cdot | h_t) \) is concentrated at one point. A nonrandomized policy is called Markov, if all decisions depend only on the current state and time. A Markov policy is called stationary, if all decisions depend only on current states. The set of all policies is denoted by \( \Pi \).

The Ionescu Tulcea theorem (Bertsekas and Shreve [8, pp. 140-141] or Hernández-Lerma and Lasserre [19, p.178]) implies that a policy \( \pi \in \Pi \) and an initial distribution \( p \in \mathcal{P}(\mathcal{X}) \), together with the stochastic kernels \( P, Q \), determine a unique probability measure \( P^\pi \) on the set of all trajectories \( (\mathcal{X} \times \mathcal{Y} \times \mathcal{A})^\infty \) endowed with the \( \sigma \)-field defined by the products of Borel \( \sigma \)-fields \( \mathcal{B}(\mathcal{X}), \mathcal{B}(\mathcal{Y}), \) and \( \mathcal{B}(\mathcal{A}) \). The expectation with respect to this probability measure is denoted by \( E^\pi_p \).

For a finite horizon \( T = 0, 1, \ldots \), the expected total discounted costs are

\[
V^\pi_{T, \alpha}(p) := \mathbb{E}^\pi_p \sum_{t=0}^{T-1} \alpha^t c(x_t, a_t), \quad p \in \mathcal{P}(\mathcal{X}), \pi \in \Pi, \tag{2.1}
\]

where \( \alpha \geq 0 \) is the discount factor, \( V^\pi_{0, \alpha}(p) = 0 \). Consider the following assumptions.

**Assumption (D).** \( c \) is bounded below on \( \mathcal{X} \times \mathcal{A} \) and \( \alpha \in [0, 1) \).

**Assumption (P).** \( c \) is nonnegative on \( \mathcal{X} \times \mathcal{A} \) and \( \alpha \in [0, 1] \).

When \( T = \infty \), formula (2.1) defines the infinite horizon expected total discounted cost, and we denote it by \( V^\pi_\alpha(p) \). We use the notations (D) and (P) following Bertsekas and Shreve [8, p. 214], where cases (D), (N), and (P) are considered. However, Assumption (D) here is weaker than the conditions assumed in case (D) in [8, p. 214], where one-step costs are assumed to be bounded.

Since the function \( c \) is bounded below on \( \mathcal{X} \times \mathcal{A} \), a discounted model can be converted into a model with nonnegative costs by shifting the cost function. In particular, let \( c(x, a) \geq -K \) for all \( (x, a) \in \mathcal{X} \times \mathcal{A} \). Consider a new cost function \( \hat{c}(x, a) := c(x, a) + K \) for all \( (x, a) \in \mathcal{X} \times \mathcal{A} \). Then the corresponding total discounted reward is equal to

\[
\hat{V}^\pi_\alpha(p) := V^\pi_\alpha(p) + \frac{K}{1 - \alpha}, \quad \pi \in \Pi, \ p \in \mathcal{P}(\mathcal{X}).
\]

Thus, optimizing \( V^\pi_\alpha \) and \( \hat{V}^\pi_\alpha \) are equivalent problems, and \( V^\pi_\alpha \) is the objective function for a model with nonnegative costs. Though Assumption (P) is more general, Assumption (D) is met in a wide range of applications. Thus we formulate the results for either of these Assumptions.

For any function \( g^\pi(p) \), including \( g^\pi(p) = V^\pi_{T, \alpha}(p) \) and \( g^\pi(p) = V^\pi_\alpha(p) \), define the optimal values

\[
g(p) := \inf_{\pi \in \Pi} g^\pi(p), \quad p \in \mathcal{P}(\mathcal{X}).
\]

A policy \( \pi \) is called optimal for the respective criterion, if \( g^\pi(p) = g(p) \) for all \( p \in \mathcal{P}(\mathcal{X}) \). For \( g^\pi = V^\pi_{T, \alpha} \), the optimal policy is called \( T \)-horizon discount-optimal; for \( g^\pi = V^\pi_\alpha \), it is called discount-optimal.

In this paper, for the expected total costs and objective values we use similar notations for POMDPs, MDPs, and COMDPs. However, we reserve the symbol \( V \) for POMDPs, the symbol \( v \) for MDPs, and the
notation \( \bar{v} \) for COMDPs. So, in addition to the notations \( V^\pi_{T, \alpha}, V^\alpha_{T, \alpha}, V_T, V_\alpha \), and \( V_\alpha \) introduced for POMDPs, we shall use the notations \( v^\pi_{T, \alpha}, v^\alpha_{T, \alpha}, v_T, v_\alpha \) and \( \bar{v}^\pi_{T, \alpha}, \bar{v}^\alpha_{T, \alpha}, \bar{v}_T, \bar{v}_\alpha \) for the similar objects for MDPs and COMDPs, respectively.

We recall that a function \( c \) defined on \( \mathbb{X} \times \mathbb{A} \) with values in \( \mathbb{R} \) is inf-compact if the set \( \{(x, a) \in \mathbb{X} \times \mathbb{A} : c(x, a) \leq \lambda \} \) is compact for any finite number \( \lambda \). A function \( c \) defined on \( \mathbb{X} \times \mathbb{A} \) with values in \( \mathbb{R} \) is called \( \mathbb{K} \)-inf-compact on \( \mathbb{X} \times \mathbb{A} \), if for any compact set \( K \subseteq \mathbb{X} \), the function \( c : K \times \mathbb{A} \to \mathbb{R} \) defined on \( K \times \mathbb{A} \) is inf-compact; Feinberg et al. [13, 14 Definition 1.1]. According to Feinberg et al. [14] Lemma 2.5], a bounded below function \( c \) is \( \mathbb{K} \)-inf-compact on the product of metric spaces \( \mathbb{X} \) and \( \mathbb{A} \) if and only if it satisfies the following two conditions:

(a) \( c \) is lower semi-continuous;

(b) if a sequence \( \{x^{(n)}\}_{n=1,2,...} \) with values in \( \mathbb{X} \) converges and its limit \( x \) belongs to \( \mathbb{X} \) then any sequence \( \{a^{(n)}\}_{n=1,2,...} \) with \( a^{(n)} \in \mathbb{A}, n = 1, 2, \ldots \), satisfying the condition that the sequence \( \{c(x^{(n)}, a^{(n)})\}_{n=1,2,...} \) is bounded above, has a limit point \( a \in \mathbb{A} \).

For a POMDP \( (\mathbb{X}, \mathbb{Y}, \mathbb{A}, P, Q, c) \), consider the MDP \( (\mathbb{X}, \mathbb{A}, P, c) \), in which all the states are observable. An MDP can be viewed as a particular POMDP with \( \mathbb{Y} = \mathbb{X} \) and \( Q(B|a, x) = Q(B|x) = I\{x \in B\} \) for all \( x \in \mathbb{X}, a \in \mathbb{A}, \) and \( B \in \mathcal{B} (\mathbb{X}) \). In addition, for an MDP an initial state is observable. Thus for an MDP an initial state \( x \) is considered instead of the initial distribution \( p \). In fact, this MDP possesses a special property that action sets at all the states are equal. For MDPs, Feinberg et al. [15] provides general conditions for the existence of optimal policies, validity of optimality equations, and convergence of value iterations. Here we formulate these conditions for an MDP whose action sets in all states are equal.

**Assumption (W\(^*\))** (cf. Feinberg et al. [15] and Lemma 2.5 in [14]).

(i) the function \( c \) is \( \mathbb{K} \)-inf-compact on \( \mathbb{X} \times \mathbb{A} \);

(ii) the transition probability \( P(\cdot | x, a) \) is weakly continuous in \((x,a) \in \mathbb{X} \times \mathbb{A} \).

**Theorem 2.1.** (cf. Feinberg et al. [15] Theorem 2). Let MDP \((\mathbb{X}, \mathbb{A}, P, c)\) satisfy Assumption (W\(^*\)). Let either Assumption (P) or Assumption (D) hold. Then:

(i) the functions \( v_{t, \alpha}, t = 0, 1, \ldots \), and \( v_\alpha \) are lower semi-continuous on \( \mathbb{X} \), and \( v_{t, \alpha}(x) \to v_\alpha(x) \) as \( t \to \infty \) for all \( x \in \mathbb{X} \);

(ii) for each \( x \in \mathbb{X} \) and \( t = 0, 1, \ldots \),

\[
v_{t+1, \alpha}(x) = \min_{a \in \mathbb{A}} \left\{ c(x, a) + \alpha \int_{\mathbb{X}} v_{t, \alpha}(y) P(dy|x, a) \right\}, \tag{2.2}
\]

where \( v_{0, \alpha}(x) = 0 \) for all \( x \in \mathbb{X} \), and the nonempty sets

\[
A_{t, \alpha}(x) := \left\{ a \in \mathbb{A} : v_{t+1, \alpha}(x) = c(x, a) + \alpha \int_{\mathbb{X}} v_{t, \alpha}(y) P(dy|x, a) \right\}, \quad x \in \mathbb{X}, t = 0, 1, \ldots ,
\]

satisfy the following properties: (a) the graph \( \text{Gr}(A_{t, \alpha}) = \{(x, a) : x \in \mathbb{X}, a \in A_{t, \alpha}(x)\}, t = 0, 1, \ldots , \) is a Borel subset of \( \mathbb{X} \times \mathbb{A} \), and (b) if \( v_{t+1, \alpha}(x) = +\infty \), then \( A_{t, \alpha}(x) = \mathbb{A} \) and, if \( v_{t+1, \alpha}(x) < +\infty \), then \( A_{t, \alpha}(x) \) is compact;

(iii) for each \( T = 1, 2, \ldots \), there exists an optimal Markov \( T \)-horizon policy \((\phi_0, \ldots , \phi_{T-1})\), and if for a \( T \)-horizon Markov policy \((\phi_0, \ldots , \phi_{T-1})\) the inclusions \( \phi_{T-1}(x) \in A_{t, \alpha}(x), x \in \mathbb{X}, t = 0, \ldots , T - 1 \), hold, then this policy is \( T \)-horizon optimal;
(iv) for each \( x \in \mathcal{X} \)
\[
 v_\alpha(x) = \min_{a \in \mathcal{A}} \left\{ c(x, a) + \alpha \int_{\mathcal{X}} v_\alpha(y) P(dy|x, a) \right\},
\]  
and the nonempty sets
\[
 A_\alpha(x) := \left\{ a \in \mathcal{A} : v_\alpha(x) = c(x, a) + \alpha \int_{\mathcal{X}} v_\alpha(y) P(dy|x, a) \right\}, \quad x \in \mathcal{X},
\]
satisfy the following properties: (a) the graph \( \text{Gr}(A_\alpha) = \{(x, a) : x \in \mathcal{X}, a \in A_\alpha(x)\} \) is a Borel subset of \( \mathcal{X} \times \mathcal{A} \), and (b) if \( v_\alpha(x) = +\infty \), then \( A_\alpha(x) = \emptyset \) and, if \( v_\alpha(x) < +\infty \), then \( A_\alpha(x) \) is compact;

(v) for infinite-horizon problems there exists a stationary discount-optimal policy \( \phi_\alpha \), and a stationary policy \( \phi_\alpha^* \) is optimal if and only if \( \phi_\alpha^*(x) \in A_\alpha(x) \) for all \( x \in \mathcal{X} \);

(vi) (Feinberg and Lewis \cite{Feinberg:1987} Proposition 3.1(iv)) if \( c \) is inf-compact on \( \mathcal{X} \times \mathcal{A} \), then the functions \( v_{t, \alpha} \), \( t = 1, 2, \ldots \), and \( v_\alpha \) are inf-compact on \( \mathcal{X} \).

### 3 Reduction of POMDPs to COMDPs and Main Results

In this section we formulate the main results of the paper, Theorems \ref{thm:3.2} \ref{thm:3.6}, and the relevant statements. These theorems provide sufficient conditions for the existence of optimal policies for COMDPs and therefore for POMDPs with expected total costs, as well as optimality equations and convergence of value iterations for COMDPs. These conditions consist of two major components: the conditions for the existence of optimal policies for MDPs and additional conditions on the POMDP. Theorem \ref{thm:3.2} states that the continuity of the observation kernel \( Q \) in the total variation is the additional sufficient condition under which there is a stationary optimal policy for the COMDP, and this policy satisfies the optimality equations and can be found by value iterations. In particular, the continuity of \( Q \) in the total variation and the weak continuity of \( P \) imply the setwise continuity of the stochastic kernel \( R \) defined in \ref{eq:3.2} and the validity of Assumption (H) introduced in this section; Theorem \ref{thm:3.7}. These two additional properties imply the weak continuity of the transition probability \( q \) for the COMDP (Theorem \ref{thm:3.5}) and eventually the desired properties of the COMDP; Theorem \ref{thm:3.2}.

This section starts with the description of known results on the general reduction of a POMDP to the COMDP; Bertsekas and Shreve \cite{Bertsekas:1995} Section 10.3, Dynkin and Yushkevich \cite{Dynkin:1973} Chapter 8, Hernández-Lerma \cite{Hernandez-Lerma:1989} Chapter 4, Rhenius \cite{Rhenius:1977}, and Yushkevich \cite{Yushkevich:1972}. To simplify notations, we sometimes drop the time parameter. Given a posterior distribution \( z \) of the state \( x \) at time epoch \( t = 0, 1, \ldots \) and given an action \( a \) selected at epoch \( t \), denote by \( R(B \times C|z, a) \) the joint probability that the state at time \( t + 1 \) belongs to the set \( B \in \mathcal{B}(\mathcal{X}) \) and the observation at time \( t + 1 \) belongs to the set \( C \in \mathcal{B}(\mathcal{Y}) \),

\[
 R(B \times C|z, a) := \int_{\mathcal{X}} \int_{B} Q(C|a, x') P(dx'|x, a) z(dx), \quad B \in \mathcal{B}(\mathcal{X}), \ C \in \mathcal{B}(\mathcal{Y}), \ z \in \mathcal{P}(\mathcal{X}), \ a \in \mathcal{A}. \quad (3.1)
\]

Observe that \( R \) is a stochastic kernel on \( \mathcal{X} \times \mathcal{Y} \) given \( \mathcal{P}(\mathcal{X}) \times \mathcal{A} \); see Bertsekas and Shreve \cite{Bertsekas:1995} Section 10.3, Dynkin and Yushkevich \cite{Dynkin:1973} Chapter 8, Hernández-Lerma \cite{Hernandez-Lerma:1989} p. 87, Yushkevich \cite{Yushkevich:1972} or Rhenius \cite{Rhenius:1977} for details. The probability that the observation \( y \) at time \( t + 1 \) belongs to the set \( C \in \mathcal{B}(\mathcal{Y}) \), given that at time
Observe that \( R' \) is a stochastic kernel on \( \mathbb{Y} \) given \( \mathbb{P}(\mathbb{X}) \times \mathbb{A} \). By Bertsekas and Shreve [8, Proposition 7.27], there exist a stochastic kernel \( H \) on \( \mathbb{X} \) given \( \mathbb{P}(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y} \) such that

\[
R(B \times C | z, a) = \int_C H(B | z, a, y) R'(dy | z, a), \quad B \in \mathcal{B}(\mathbb{X}), \ C \in \mathcal{B}(\mathbb{Y}), \ z \in \mathbb{P}(\mathbb{X}), \ a \in \mathbb{A}. \tag{3.3}
\]

The stochastic kernel \( H( \cdot | z, a, y) \) defines a measurable mapping \( H : \mathbb{P}(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y} \rightarrow \mathbb{P}(\mathbb{X}) \), where \( H(z, a, y)(\cdot) = H(\cdot | z, a, y) \). For each pair \((z, a) \in \mathbb{P}(\mathbb{X}) \times \mathbb{A}\), the mapping \( H(z, a, \cdot) : \mathbb{Y} \rightarrow \mathbb{P}(\mathbb{X}) \) is defined \( R'( \cdot | z, a) \)-almost surely uniquely in \( y \in \mathbb{Y} \); Bertsekas and Shreve [8, Corollary 7.27.1] or Dynkin and Yushkevich [12, Appendix 4.4]. For a posterior distribution \( z_t \in \mathbb{P}(\mathbb{X}) \), action \( a_t \in \mathbb{A} \), and an observation \( y_{t+1} \in \mathbb{Y} \), the posterior distribution \( z_{t+1} \in \mathbb{P}(\mathbb{X}) \) is

\[
z_{t+1} = H(z_t, a_t, y_{t+1}). \tag{3.4}
\]

However, the observation \( y_{t+1} \) is not available in the COMDP model, and therefore \( y_{t+1} \) is a random variable with the distribution \( R'( \cdot | z_t, a_t) \), and the right-hand side of (3.4) maps \((z_t, a_t) \in \mathbb{P}(\mathbb{X}) \times \mathbb{A}\) to \( \mathbb{P}(\mathbb{P}(\mathbb{X})) \). Thus, \( z_{t+1} \) is a random variable with values in \( \mathbb{P}(\mathbb{X}) \) whose distribution is defined uniquely by the stochastic kernel

\[
q(D | z, a) := \int_{\mathbb{Y}} 1_{\{H(z, a, y) \in D\}} R'(dy | z, a), \quad D \in \mathcal{B}(\mathbb{P}(\mathbb{X})), \ z \in \mathbb{P}(\mathbb{X}), \ a \in \mathbb{A}; \tag{3.5}
\]

Hernández-Lerma [18, p. 87]. The particular choice of a stochastic kernel \( H \) satisfying (3.3) does not effect the definition of \( q \) from (3.5), since for each pair \((z, a) \in \mathbb{P}(\mathbb{X}) \times \mathbb{A}\), the mapping \( H(z, a, \cdot) : \mathbb{Y} \rightarrow \mathbb{P}(\mathbb{X}) \) is defined \( R'( \cdot | z, a) \)-almost surely uniquely in \( y \in \mathbb{Y} \); Bertsekas and Shreve [8, Corollary 7.27.1], Dynkin and Yushkevich [12, Appendix 4.4].

Similar to the stochastic kernel \( R \), consider a stochastic kernel \( R_0 \) on \( \mathbb{X} \times \mathbb{Y} \) given \( \mathbb{P}(\mathbb{X}) \) defined by

\[
R_0(B \times C | p) := \int_B Q_0(C | x) p(dx), \quad B \in \mathcal{B}(\mathbb{X}), \ C \in \mathcal{B}(\mathbb{Y}), \ p \in \mathbb{P}(\mathbb{X}).
\]

This kernel can be decomposed as

\[
R_0(B \times C | p) = \int_C H_0(B | p, y) R'_0(dy | p), \quad B \in \mathcal{B}(\mathbb{X}), \ C \in \mathcal{B}(\mathbb{Y}), \ p \in \mathbb{P}(\mathbb{X}), \tag{3.6}
\]

where \( R'_0(C | p) = R_0(\mathbb{X} \times C | p), \ C \in \mathcal{B}(\mathbb{Y}), \ p \in \mathbb{P}(\mathbb{X}) \), is a stochastic kernel on \( \mathbb{Y} \) given \( \mathbb{P}(\mathbb{X}) \) and \( H_0(dx | p, y) \) is a stochastic kernel on \( \mathbb{X} \) given \( \mathbb{P}(\mathbb{X}) \times \mathbb{Y} \). Any initial prior distribution \( p \in \mathbb{P}(\mathbb{X}) \) and any initial observation \( y_0 \) define the initial posterior distribution \( z_0 = H_0(p, y_0) \) on \( (\mathbb{X}, \mathcal{B}(\mathbb{X})) \). Similar to (3.4), the observation \( y_0 \) is not available in the COMDP and this equation is stochastic. In addition, \( H_0(p, y) \) is defined \( R'_0(dy | p) \)-almost surely uniquely in \( y \in \mathbb{Y} \) for each \( p \in \mathbb{P}(\mathbb{X}) \).

Similar to (3.5), the stochastic kernel

\[
q_0(D | p) := \int_{\mathbb{Y}} 1\{H_0(p, y) \in D\} R'_0(dy | p), \quad D \in \mathcal{B}(\mathbb{P}(\mathbb{X})) \text{ and } p \in \mathbb{P}(\mathbb{X}), \tag{3.7}
\]

for each \( p \in \mathbb{P}(\mathbb{X}) \) and selected action is \( a \), is

\[
R'(C | z, a) := \int_{\mathbb{X}} \int_{\mathbb{X}} Q(C | a, x') P(dx' | x, a) z(dx), \quad C \in \mathcal{B}(\mathbb{Y}), \ z \in \mathbb{P}(\mathbb{X}), \ a \in \mathbb{A}. \tag{3.2}
\]
on \( \mathbb{P}(X) \) given \( \mathbb{P}(X) \) defines the initial distribution on the set of posterior probabilities. Define \( q_0(p)(D) = q_0(D|p) \), where \( D \in \mathcal{B}(\mathbb{P}(X)) \). Then \( q_0(p) \) is the initial distribution of \( z_0 = H_0(p, y_0) \) corresponding to the initial state distribution \( p \).

The COMDP is defined as an MDP with parameters \((\mathbb{P}(X), A, q, \tilde{c})\), where

(i) \( \mathbb{P}(X) \) is the state space;

(ii) \( A \) is the action set available at all states \( z \in \mathbb{P}(X) \);

(iii) the one-step cost function \( \tilde{c} : \mathbb{P}(X) \times A \to \mathbb{R} \), defined as

\[
\tilde{c}(z, a) := \int_X c(x, a)z(dx), \quad z \in \mathbb{P}(X), a \in A;
\]  

(iv) transition probabilities \( q \) on \( \mathbb{P}(X) \) given \( \mathbb{P}(X) \times A \) defined in (3.3).

Denote by \( i_t, t = 0, 1, \ldots \), a \( t \)-horizon history for the COMDP, also called an information vector,

\[
i_t := (z_0, a_0, \ldots, z_{t-1}, a_{t-1}, z_t) \in I_t, \quad t = 0, 1, \ldots,
\]

where \( z_0 \) is the initial posterior distribution and \( z_t \in \mathbb{P}(X) \) are recursively defined by equation (3.4), \( I_t := \mathbb{P}(X) \times (A \times \mathbb{P}(X))^t \) for all \( t = 0, 1, \ldots \), with \( I_0 := \mathbb{P}(X) \). An information policy (I-policy) is a policy in a COMDP, i.e. \( I \)-policy is a sequence \( \delta = \{ \delta_t : t = 0, 1, \ldots \} \) such that \( \delta_t(\cdot | i_t) \) is a stochastic kernel on \( A \) given \( I_t \) for all \( t = 0, 1, \ldots \); Bertsekas and Shreve [8, Chapter 10], Hernández-Lerma [18, p. 88]. Denote by \( \bigtriangleup \) the set of all \( I \)-policies. We also consider Markov \( I \)-policies and stationary \( I \)-policies.

For an \( I \)-policy \( \delta = \{ \delta_t : t = 0, 1, \ldots \} \), define a policy \( \pi^\delta = \{ \pi^\delta_t : t = 0, 1, \ldots \} \) in \( \Pi \) as

\[
\pi^\delta_t(\cdot | h_t) := \delta_t(\cdot | i_t(h_t)) \quad \text{for all } h_t \in H_t \text{ and } t = 0, 1, \ldots,
\]  

(3.9)

where \( i_t(h_t) \in I_t \) is the information vector determined by the observable history \( h_t \) via (3.4). Thus \( \delta \) and \( \pi^\delta \) are equivalent in the sense that \( \pi^\delta_t \) assigns the same conditional probability on \( A \) given the observed history \( h_t \) as \( \delta_t \) for the history \( i_t(h_t) \). If \( \delta \) is an optimal policy for the COMDP then \( \pi^\delta \) is an optimal policy for the POMDP. This follows from the facts that \( V_{t,\alpha}(p) = \bar{v}_{t,\alpha}(q_0(p)) \), \( t = 0, 1, \ldots \), and \( V_{\alpha}(p) = \bar{v}_{\alpha}(q_0(p)) \); Hernández-Lerma [18, p. 89] and references therein. Let \( z_t(h_t) \) be the last element of the information vector \( i_t(h_t) \). With a slight abuse of notations, by using the same notations for a measure concentrated at a point and a function at this point, if \( \delta \) is Markov then (3.9) becomes \( \pi^\delta_t(h_t) = \delta_t(z_t(h_t)) \) and if \( \delta \) is stationary then \( \pi^\delta_t(h_t) = \delta(z_t(h_t)) \), \( t = 0, 1, \ldots \).

Thus, an optimal policy for a COMDP defines an optimal policy for the POMDP. However, very little is known for the conditions on POMDPs that lead to the existence of optimal policies for the corresponding COMDPs. For the COMDP, Assumption (W\(^\ast\)) has the following form:

(i) \( \tilde{c} \) is \( X \)-inf-compact on \( \mathbb{P}(X) \times A \);

(ii) the transition probability \( q(\cdot | z, a) \) is weakly continuous in \((z, a) \in \mathbb{P}(X) \times A \).

Recall that the notation \( \bar{v} \) has been reserved for the expected total costs for COMDPs. The following theorem follows directly from Theorem 2.1 applied to the COMDP \((\mathbb{P}(X), A, q, \tilde{c})\).
Theorem 3.1. Let either Assumption (D) or Assumption (P) hold. If the COMDP $\{P(X), A, q, c\}$ satisfies Assumption (W*), then:

(i) the functions $\bar{v}_{t,\alpha}$, $t = 0, 1, \ldots$, and $\bar{v}_{\alpha}$ are lower semi-continuous on $P(X)$, and $\bar{v}_{t,\alpha}(z) \to \bar{v}_{\alpha}(z)$ as $t \to \infty$ for all $z \in P(X)$;

(ii) for each $z \in P(X)$ and $t = 0, 1, \ldots$,

\[
\bar{v}_{t+1,\alpha}(z) = \min_{a \in A} \left\{ \bar{c}(z, a) + \alpha \int_{P(X)} \bar{v}_{t,\alpha}(z') q(dz'|z, a) \right\} = \\
\min_{a \in A} \left\{ \int_X c(x, a) z(dx) + \alpha \int_X \int_X \int_Y \bar{v}_{t,\alpha}(H(z, a, y)) Q(dy|x, x') P(dx'|x, a) z(dx) \right\},
\]

(3.10)

where $\bar{v}_{0,\alpha}(z) = 0$ for all $z \in P(X)$, and the nonempty sets

\[A_{t,\alpha}(z) := \left\{ a \in A : \bar{v}_{t+1,\alpha}(z) = \bar{c}(z, a) + \alpha \int_{P(X)} \bar{v}_{t,\alpha}(z') q(dz'|z, a) \right\}, \quad z \in P(X), \quad t = 0, 1, \ldots,
\]

satisfy the following properties: (a) the graph $Gr(A_{t,\alpha}) = \{(z, a) : z \in P(X), a \in A_{t,\alpha}(z)\}$, $t = 0, 1, \ldots$, is a Borel subset of $P(X) \times A$, and (b) if $\bar{v}_{t+1,\alpha}(z) = +\infty$, then $A_{t,\alpha}(z) = \emptyset$ and, if $\bar{v}_{t+1,\alpha}(z) < +\infty$, then $A_{t,\alpha}(z)$ is compact;

(iii) for each $T = 1, 2, \ldots$, there exists an optimal Markov $T$-horizon I-policy $(\phi_0, \ldots, \phi_{T-1})$, and if for a $T$-horizon Markov I-policy $(\phi_0, \ldots, \phi_{T-1})$ the inclusions $\phi_{T-1-t}(z) \in A_{t,\alpha}(z)$, $z \in P(X)$, $t = 0, \ldots, T-1$, hold, then this I-policy is $T$-horizon optimal;

(iv) for each $z \in P(X)$

\[
\bar{v}_{\alpha}(z) = \min_{a \in A} \left\{ \bar{c}(z, a) + \alpha \int_{P(X)} \bar{v}_{\alpha}(z') q(dz'|z, a) \right\} = \\
\min_{a \in A} \left\{ \int_X c(x, a) z(dx) + \alpha \int_X \int_X \int_Y \bar{v}_{\alpha}(H(z, a, y)) Q(dy|x, x') P(dx'|x, a) z(dx) \right\},
\]

(3.11)

and the nonempty sets

\[A_{\alpha}(z) := \left\{ a \in A : \bar{v}_{\alpha}(z) = \bar{c}(z, a) + \alpha \int_{P(X)} \bar{v}_{\alpha}(z') q(dz'|z, a) \right\}, \quad z \in P(X),
\]

satisfy the following properties: (a) the graph $Gr(A_{\alpha}) = \{(z, a) : z \in P(X), a \in A_{\alpha}(z)\}$ is a Borel subset of $P(X) \times A$, and (b) if $\bar{v}_{\alpha}(z) = +\infty$, then $A_{\alpha}(z) = \emptyset$ and, if $\bar{v}_{\alpha}(z) < +\infty$, then $A_{\alpha}(z)$ is compact.

(v) for infinite horizon problems there exists a stationary discount-optimal I-policy $\phi_{\alpha}$, and a stationary $I$-policy $\phi_{\alpha}$ is optimal if and only if $\phi_{\alpha}(z) \in A_{\alpha}(z)$ for all $z \in P(X)$.

(vi) if $c$ is inf-compact on $P(X) \times A$, then the functions $\bar{v}_{t,\alpha}$, $t = 1, 2, \ldots$, and $\bar{v}_{\alpha}$ are inf-compact on $P(X)$.

Thus, in view of Theorem 3.1, the important question is under which conditions on the original POMDP, the COMDP satisfies the conditions under which there are optimal policies for MDPs. Hernández-Lerma [18, p. 90] provides the following conditions for this: (a) $A$ is compact, (b) the cost function $c$ is bounded and
continuous, (c) the transition probability \( P(dx'|x, a) \) and the observation kernel \( Q(dy|a, x) \) are weakly continuous stochastic kernels; (d) there exists a weakly continuous stochastic kernel \( H \) on \( \mathbb{X} \) given \( \mathbb{P}(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y} \) satisfying (3.3). Consider the following relaxed version of assumption (d) that does not require that \( H \) is continuous in \( y \). We introduce this assumption, called Assumption (H), because it holds in many important situations when a weakly continuous stochastic kernel \( H \) satisfying (3.3) does not exist; see Example 4.2 and Theorem 3.7.

**Assumption (H).** There exists a stochastic kernel \( H \) on \( \mathbb{X} \) given \( \mathbb{P}(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y} \) satisfying (3.3) such that: if a sequence \( \{z^{(n)}\}_{n=1,2,...} \subseteq \mathbb{P}(\mathbb{X}) \) converges weakly to \( z \in \mathbb{P}(\mathbb{X}) \), and a sequence \( \{a^{(n)}\}_{n=1,2,...} \subseteq \mathbb{A} \) converges to \( a \in \mathbb{A} \) as \( n \to \infty \), then there exists a subsequence \( \{(z^{(nk)}, a^{(nk)})\}_{k=1,2,...} \subseteq \{(z^{(n)}, a^{(n)})\}_{n=1,2,...} \) and a measurable subset \( C \) of \( \mathbb{Y} \) such that \( R'(C|z,a) = 1 \) and for all \( y \in C \)

\[
H(z^{(nk)}, a^{(nk)}, y) \text{ converges weakly to } H(z, a, y). \tag{3.12}
\]

In other words, (3.12) holds \( R'(.|z,a) \)-almost surely

**Theorem 3.2.** If the following assumptions hold:

(a) either Assumption (D) or Assumption (P) holds;

(b) the function \( c \) is \( \mathbb{K} \)-inf-compact on \( \mathbb{X} \times \mathbb{A} \);

(c) either

(i) the stochastic kernel \( R'(dy|z,a) \) on \( \mathbb{Y} \) given \( \mathbb{P}(\mathbb{X}) \times \mathbb{A} \) is setwise continuous and Assumption (H) holds,

or

(ii) the stochastic kernels \( P(dx'|x, a) \) on \( \mathbb{X} \) given \( \mathbb{X} \times \mathbb{A} \) and \( Q(dy|a, x) \) on \( \mathbb{Y} \) given \( \mathbb{A} \times \mathbb{X} \) are weakly continuous and there exists a weakly continuous stochastic kernel \( H(dx|z,a,y) \) on \( \mathbb{X} \) given \( \mathbb{P}(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y} \) satisfying (3.3),

then the COMDP \( (\mathbb{P}(\mathbb{X}), \mathbb{A}, q, \tilde{c}) \) satisfies Assumption (W*) and therefore statements (i)–(vi) of Theorem 3.7 hold.

**Remark 3.3.** Throughout this paper we follow the terminology according to which finite sets are countable. If \( \mathbb{Y} \) is countable, then equation (5.10) transforms into

\[
\tilde{v}_{t+1,\alpha}(z) = \min_{a \in \mathbb{A}} \left\{ \int_{\mathbb{X}} c(x, a) z(dx) + \alpha \sum_{y \in \mathbb{Y}} \tilde{v}_{t,\alpha}(H(z,a,y)) R'(y|z,a) \right\}, \quad z \in \mathbb{P}(\mathbb{X}), \; t = 0, 1, ..., \tag{5.10}
\]

and equation (5.11) transforms into

\[
\tilde{v}_{\alpha}(z) = \min_{a \in \mathbb{A}} \left\{ \int_{\mathbb{X}} c(x, a) z(dx) + \alpha \sum_{y \in \mathbb{Y}} \tilde{v}_{\alpha}(H(z,a,y)) R'(y|z,a) \right\}, \quad z \in \mathbb{P}(\mathbb{X}). \tag{5.11}
\]

11
Theorem 3.4 follows from Theorems 3.1, 3.4, and 3.8. In particular, Theorem 3.4 implies that if Assumption (D) or (P) holds for a POMDP, then it also holds for the corresponding COMDP.

**Theorem 3.4.** If the function \( c : X \times A \to \mathbb{R} \) is bounded below and \( K \)-inf-compact on \( X \times A \), then the cost function \( \tilde{c} : P(X) \times A \to \mathbb{R} \) defined for the COMDP in (3.3) is bounded from below by the same constant as \( c \) and \( K \)-inf-compact on \( P(X) \times A \).

**Theorem 3.5.** The stochastic kernel \( q(dz'|z,a) \) on \( P(X) \) given \( P(X) \times A \) is weakly continuous if condition (c) from Theorem 3.2 holds.

The following theorem provides sufficient conditions for the existence of optimal policies for the COMDP and therefore for the POMDP in terms of the initial parameters of the POMDP.

**Theorem 3.6.** Let assumptions (a) and (b) of Theorem 3.2 hold, the stochastic kernel \( P(dx'|x,a) \) on \( X \) given \( X \times A \) be weakly continuous, and the stochastic kernel \( Q(dy|a,x) \) on \( Y \) given \( A \times X \) be continuous in the total variation. Then the COMDP \( (P(X), A, q, \tilde{c}) \) satisfies Assumption (W') and therefore statements (i)–(vi) of Theorem 3.1 hold.

Theorem 3.6 follows from Theorem 3.4 and from the following statement.

**Theorem 3.7.** The weak continuity of the stochastic kernel \( P(dx'|x,a) \) on \( X \) given \( X \times A \) and continuity in the total variation of the stochastic kernel \( Q(dy|a,x) \) on \( Y \) given \( A \times X \) imply that condition (i) from Theorem 3.2 holds (that is, \( R' \) is setwise continuous and Assumption (H) holds) and therefore the stochastic kernel \( q(dz'|z,a) \) on \( P(X) \) given \( P(X) \times A \) is weakly continuous.

Example 4.1 demonstrates that, if the stochastic kernel \( Q(dy|a,x) \) on \( Y \) given \( A \times X \) is setwise continuous, then the transition probability \( q \) for the COMDP may not be weakly continuous in \((z,a) \in P(X) \times A\).

In this example the state set consists of two points. Therefore, if the stochastic kernel \( P(dx'|x,a) \) on \( X \) given \( X \times A \) is setwise continuous (even if it is continuous in the total variation) in \((x,a) \in X \times A\) then the setwise continuity of the stochastic kernel \( Q(dy|a,x) \) on \( Y \) given \( A \times X \) is not sufficient for the weak continuity of \( q \).

**Corollary 3.8.** (cp. Hernández-Lerma [18] p. 93) If the stochastic kernel \( P(dx'|x,a) \) on \( X \) given \( X \times A \) is weakly continuous, \( Y \) is countable, and for each \( y \in Y \) the function \( Q(y|a,x) \) is continuous on \( A \times X \), then the following statements hold:

(a) for each \( y \in Y \) the function \( R'(y|z,a) \) is continuous on \( P(X) \times A \) with respect to the topology of weak convergence on \( P(X) \), and Assumption (H) holds;

(b) the stochastic kernel \( q(dz'|z,a) \) on \( P(X) \) given \( P(X) \times A \) is weakly continuous;

(c) if, in addition to the above conditions, assumptions (a) and (b) from Theorem 3.2 hold, then the COMDP \( (P(X), A, q, \tilde{c}) \) satisfies Assumption (W') and therefore statements (i)–(vi) of Theorem 3.1 hold.

**Proof.** For a countable \( Y \), the continuity in the total variation of the stochastic kernel \( Q(dy|x,a) \) on \( Y \) given \( A \times X \) follows from the continuity of \( Q(y|a,x) \) for each \( y \in Y \) in \((a,x) \in A \times X \) and from \( Q(Y|a,x) = 1 \) for all \((a,x) \in A \times X \). Indeed, let \((a^n), x^n \to (a,x) \) as \( n \to \infty \). Since \( Q(y|a,x) \) is continuous in \((a,x) \in ...
This and the continuity of $Q$ imply
\[
\sum_{y \in Y} |Q(y|a^{(n)}, x^{(n)}) - Q(y|a, x)| \to 0 \quad \text{as} \quad n \to \infty.
\]
Thus, $\sup_{C \in B(Y)} |Q(C|a^{(n)}, x^{(n)}) - Q(C|a, x)| \leq \sum_{y \in Y} |Q(y|a^{(n)}, x^{(n)}) - Q(y|a, x)| \to 0$ as $n \to \infty$, and

continuity in the total variation takes place. Statements (a) and (b) follow from Theorem 3.7 and statement (c) follows from Theorem 3.6.

4 Counterexamples

In this section we provide three counterexamples. Example 4.1 demonstrates that the assumption in Theorems 3.6 and 3.7 that the stochastic kernel $Q$ is continuous in the total variation, cannot be weakened to the assumption that $Q$ is setwise continuous. Example 4.2 shows that, under conditions of Corollary 3.8, a weakly continuous mapping $H$ satisfying (3.4) may not exist. The existence of such a mapping is mentioned in Hernández-Lerma [18, p. 93]. Example 4.3 illustrates that the setwise continuity of the the stochastic kernel $R'(dy|z, a)$ on $\mathbb{Y}$ given $P(X) \times A$ is essential in condition (i) of Theorem 3.2. Without this assumption, Assumption (H) alone is not sufficient for the weak continuity of the stochastic kernel $q(dx'|z, a)$ on $P(X)$ given $P(X) \times A$ and therefore for the correctness of Theorems 3.2 and 3.5.

We would like to mention that before the authors constructed Example 4.1 Huijzen Janey Yu provided them with an example when the weak continuity of the observation kernel $Q$ is not sufficient for the weak continuity of the stochastic probability $q(\cdot | z, a)$. In her example, $X = \{1, 2\}$, the system does not move, $Y = A = [0, 1]$, at state 1 the observation is 0 for any action $a$ and in state 2, under an action $a \in A$, the observation is uniformly distributed on $[0, a]$. The initial belief distribution is $z = (0.5, 0.5)$.

Example 4.1. Continuity of $Q$ in the total variation cannot be relaxed to setwise continuity in Theorems 3.6 and 3.7. Let $X = \{1, 2\}$, $Y = [0, 1]$, and $A = \{0\} \cup \left\{ \frac{1}{n} : n = 1, 2, \ldots \right\}$. The system does not move. This means that $P(x|x, a) = 1$ for all $x = 1, 2$ and $a \in A$. This stochastic kernel $P$ is weakly continuous and, since $X$ is finite, it is setwise continuous and continuous in the total variation. The observation kernel $Q$ is $Q(dy|a, 1) = Q(dy|0, 2) = m(dy)$, $a \in A$, where $m$ is the Lebesgue measure on $Y = [0, 1]$ and $Q(dy|1/n, 2) = m^{(n)}(dy)$, $n = 1, 2, \ldots$, where $m^{(n)}$ is the absolutely continuous measure on $Y = [0, 1]$ with the density $f^{(n)}$,

\[
f^{(n)}(y) = \begin{cases} 0, & \text{if } 2k/2^n < y < (2k + 1)/2^n \text{ for } k = 0, 1, \ldots, 2^{n-1} - 1; \\
2, & \text{otherwise.}
\end{cases}
\]

First we show that $Q(dy|a, x)$ on $\mathbb{Y}$ given $A \times \mathbb{X}$ is setwise continuous in $(a, x)$. In our case, this means that the probability distributions $Q(dy|1/n, i)$ converge setwise to $Q(dy|0, i)$ as $n \to \infty$, where $i = 1, 2$. For $i = 1$ this statement is trivial, because $Q(dy|a, 1) = m(dy)$ for all $a \in A$. For $i = 2$ we need to verify that $m^{(n)}$ converge setwise to $m$ as $n \to \infty$. According to Bogachev [10, Theorem 8.10.56], which is Pflanzagl’s
Thus the posterior probability distribution \( X \) at the current step then nothing new can be learned about the state during the next step. Thus the prior probability of the event \( y \) at the current step. The new posterior state probabilities depend on the event \( y \) given \( X \) at the current step, since the observation kernel \( q \) is \( 0.5 \), because \( z(2) = 0.5 \). Now let an action \( 1/n \) be selected at the current step. The new posterior state probabilities depend on the event \( A = \{ f(n)(y) = 2 \} \). If the event \( D \) takes place (the state is 2), then the probability of the event \( A \) is 1 and the probability of the event \( \bar{A} = \{ f(n)(y) = 0 \} \) is 0. If the event \( D \) takes place (the new state is 1), then the probabilities of the events \( A \) and \( \bar{A} \) are 0.5. Bayes’s formula implies that the posterior probabilities are \( (1/3, 2/3) \), if \( f(n)(y) = 2 \), and \( (1, 0) \), if \( f(n)(y) = 0 \). Since \( f(n)(2) = 2 \) with probability \( 3/4 \) and \( f(n)(y) = 0 \) with probability \( 1/4 \), then \( q((1/3, 2/3)|z, 1/n) = 3/4 \) and \( q((1,0)|z, 1/n) = 1/4 \). So, all the measures \( q(\cdot|z, 1/n) \) are constants and they are not equal to the measure \( q(\cdot|z, 0) \), which is concentrated at the point \( z = (0.5, 0.5) \). Thus the transition kernel \( q \) on \( \mathbb{P}(X) \) given \( \mathbb{P}(X) \times A \) is not weakly continuous. \( \square \)

**Example 4.2.** Under conditions of Corollary 3.8 there is no weakly continuous stochastic kernel \( H(\cdot|z, a, y) \) on \( X \) given \( \mathbb{P}(X) \times A \times Y \) satisfying (3.4). Let the state and observation spaces \( X = Y = \{1, 2\} \); the action space \( A = [-1, 1] \); the system do not move, that is \( P(1|1, a) = P(2|2, a) = 1 \) for all \( a \in A \); for each \( y \in Y \) the observation kernel \( Q(y|a, x) \) be continuous in \( a \in A \),

\[
Q(1|a, 1) = \begin{cases} |a|, & a \in [-1, 0), \\ a^2, & a \in [0, 1], \end{cases} \quad Q(1|a, 2) = \begin{cases} a^2, & a \in [-1, 0), \\ |a|, & a \in [0, 1], \end{cases}
\]

\[
Q(2|a, 1) = \begin{cases} 1 - |a|, & a \in [-1, 0), \\ 1 - a^2, & a \in [0, 1], \end{cases} \quad Q(2|a, 2) = \begin{cases} 1 - a^2, & a \in [-1, 0), \\ 1 - |a|, & a \in [0, 1]; \end{cases}
\]

and \( z = (z(1), z(2)) = (\frac{1}{2}, \frac{1}{2}) \) be the probability measure on \( X = \{1, 2\} \).

Formula (3.1) with \( B = \{1\} \) and \( C = \{1\} \) implies

\[
R((1,1)|z,a) = \frac{1}{2} \cdot \frac{1}{2} Q(1|a, 1) = \begin{cases} \frac{|a|}{2}, & a \in [-1, 0), \\ \frac{a^2}{2}, & a \in [0, 1]. \end{cases}
\] (4.2)
Setting $C = \{1\}$ in (3.2) we obtain

$$R'(1|z, a) = \frac{1}{2}Q(1|a, 1) + \frac{1}{2}Q(1|a, 2) = \frac{|a| + a^2}{2}, \quad a \in [-1, 1]. \tag{4.3}$$

Formulas (4.2) and (4.3) imply that, if $H$ satisfies (3.3), then

$$H(1|z, a, 1) = \frac{R((1, 1)|z, a)}{R'(1|z, a)} \begin{cases} \frac{|a|}{|a| + a^2}, & a \in [-1, 0), \\ \frac{a^2}{|a| + a^2}, & a \in (0, 1]. \end{cases}$$

Therefore,

$$\lim_{a \uparrow 0} H(1|z, a, 1) = 1 \quad \text{and} \quad \lim_{a \downarrow 0} H(1|z, a, 1) = 0.$$

Thus, the stochastic kernel $H$ on $\mathcal{X}$ given $\mathbb{P}(\mathcal{X}) \times \mathcal{A} \times \mathcal{Y}$ is not weakly continuous in $a$, that is, $H : \mathbb{P}(\mathcal{X}) \times \mathcal{A} \times \mathcal{Y} \rightarrow \mathbb{P}(\mathcal{X})$ is not a continuous mapping. In view of Corollary 3.8 Assumption (H) holds. \hfill $\square$

**Example 4.3.** Stochastic kernels $P$ on $\mathcal{X}$ given $\mathcal{X} \times \mathcal{A}$ and $Q$ on $\mathcal{Y}$ given $\mathcal{A} \times \mathcal{X}$ are weakly continuous, the stochastic kernel $R'$ on $\mathcal{Y}$ given $\mathbb{P}(\mathcal{X}) \times \mathcal{A}$, defined by formula (3.2), is weakly continuous, but it is not setwise continuous. Though Assumption (H) holds, the stochastic kernel $q$ on $\mathbb{P}(\mathcal{X})$ given $\mathbb{P}(\mathcal{X}) \times \mathcal{A}$, defined by formula (3.3), is not weakly continuous.

Let $\mathcal{X} = \{1, 2\}$, $\mathcal{Y} = \mathcal{A} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots \} \cup \{0\}$ with the metric $d(a, b) = |a - b|$, $a, b \in \mathbb{R}$, and $P(x|a, a) = 1$, $x \in \mathcal{X}$, $a \in \mathcal{A}$. Let also $Q(0|0, x) = 1$, $Q(b|0, x) = Q(b|\frac{1}{2}, x) = 0$, $x \in \mathcal{X}$, and $Q(\frac{1}{n}, 1, n) = a_{m, n} \sin^2(\frac{\pi}{2m})$, $Q(\frac{1}{2}, 1, 2) = a_{m, n} \cos^2(\frac{\pi}{2m})$, $m, n = 1, 2, \ldots$, where $a_{m, 2mk+\ell} = \frac{1}{2^{k+1}m}$ for $k = 0, 1, \ldots, \ell = 1, 2, \ldots, 2m$. Since $\sum_{\ell=1}^{2m} \sin^2(\frac{\pi}{2m}) = \sum_{\ell=1}^{2m} \cos^2(\frac{\pi}{2m}) = \sum_{m} (\sin^2(\frac{\pi}{2m}) + \cos^2(\frac{\pi}{2m})) = m$, then $\sum_{n=1}^{\infty} Q(\frac{1}{n}, \frac{1}{m}, x) = \sum_{k=0}^{m} \sum_{\ell=1}^{2m} \sin^2(\frac{\pi}{2m}) = \sum_{k=1}^{\infty} \frac{1}{2^{k+1}m} = 1$, $x \in \mathcal{X}$, and $Q$ is a stochastic kernel on $\mathcal{Y}$ given $\mathcal{A} \times \mathcal{X}$. The stochastic kernels $P$ on $\mathcal{X}$ given $\mathcal{X} \times \mathcal{A}$ and $Q$ on $\mathcal{Y}$ given $\mathcal{A} \times \mathcal{X}$ are weakly continuous. The former is true because of the same reasons as in Example 4.1. The latter is true because of the reason that $\limsup_{m \rightarrow \infty} Q(C|a, x) = Q(C|0, x)$ for any closed set $C$ in $\mathcal{Y}$. Indeed, a set $C$ is closed in $\mathcal{Y}$ if and only if either (i) $0 \in C$ or (ii) $0 \notin C$ and $C$ is finite. Let $C \subseteq \mathcal{Y}$ be closed. In case (i), $\limsup_{m \rightarrow \infty} Q(C|a, x) = 1 = Q(C|0, x)$ as $a \rightarrow 0$, $x \in \mathcal{X}$. In case (ii), $\limsup_{m \rightarrow \infty} Q(C|a, x) = 0 = Q(C|0, x)$ as $a \rightarrow 0$, since $\lim_{m \rightarrow \infty} Q(\frac{1}{n}|a, x) = 0 = Q(\frac{1}{n}|0, x)$ for $n = 1, 2, \ldots$ and for $x \in \mathcal{X}$.

Formula (3.1) implies that $R(1, \frac{1}{n}, \frac{1}{m}, \frac{1}{m}) = z(1)a_{m, n} \sin^2(\frac{\pi}{2m})$, $R(2, \frac{1}{n}, \frac{1}{m}, \frac{1}{m}) = z(2)a_{m, n} \cos^2(\frac{\pi}{2m})$, $R(0, \frac{1}{n}, \frac{1}{m}, \frac{1}{m}) = 0$, $R(2, 0, \frac{1}{n}, \frac{1}{m}, \frac{1}{m}) = 0$, and $R(1, \frac{1}{n}, \frac{1}{n}, 0) = 0$, $R(2, \frac{1}{n}, \frac{1}{n}, 0) = 0$, $R(1, 0, \frac{1}{n}, 0) = (z(1) - 1, z(2) - 2) \in \mathbb{P}(\mathcal{X})$. Formula (3.2) yields $R'(0|z, \frac{1}{m}, \frac{1}{m}) = 0$, $R'(0, \frac{1}{m}, \frac{1}{m}) = z(1)a_{m, n} \sin^2(\frac{\pi}{2m}) + z(2)a_{m, n} \cos^2(\frac{\pi}{2m})$, and $R'(0, z, 0) = 1$, $R'(0, 0, 0) = 0$ for $m, n = 1, 2, \ldots$, $z = (z(1), z(2)) \in \mathbb{P}(\mathcal{X})$. Therefore, $R'(0|z, \frac{1}{m}, \frac{1}{m}) \neq R'(0, z, 0)$ as $m \rightarrow \infty$. Thus the stochastic kernel $R'$ on $\mathcal{Y}$ given $\mathbb{P}(\mathcal{X}) \times \mathcal{A}$ is not setwise continuous. However, stochastic kernel $R'$ on $\mathcal{Y}$ given $\mathbb{P}(\mathcal{X}) \times \mathcal{A}$ is weakly continuous.

Observe that $\mathbb{P}(\mathcal{X}) = \{(z(1), z(2)) : z(1), z(2) \geq 0, z(1) + z(2) = 1\} \subseteq \mathbb{R}^2$. Let $z = (z(1), z(2)) \in \mathbb{P}(\mathcal{X})$. If $R'(y|z, a) > 0$, in view of (3.3), $H(x', z, a, y) = R((x', y)|z, a)/R'(y|z, a)$ for all $x' \in \mathcal{X}$, $a \in \mathcal{A}$,
and $y \in \mathbb{Y}$. Thus, if $R'(y|z,a) > 0$ then

$$H(z,a,y) = \begin{cases} \frac{z(1) \sin^2(\frac{\pi m}{2}) + z(2) \cos^2(\frac{\pi m}{2})}{(z(1) \sin^2(\frac{\pi m}{2}) + z(2) \cos^2(\frac{\pi m}{2}))}, & \text{if } a = \frac{1}{m}, \ y = \frac{1}{n}, \ m, \ n = 1, 2, \ldots, \\ (z(1), z(2)), & \text{if } a = y = 0. \end{cases}$$

If $R'(y|z,a) = 0$, we set $H(z,a,y) = z = (z(1), z(2))$. In particular, $H(z, \frac{1}{m}, 0) = z$ for all $m = 1, 2, \ldots$.

Observe that Assumption (H) holds, because, if $R'(y|z,a) > 0$ and if sequences $\{z^{(N)}\}_{N=1,2,\ldots} \subseteq \mathbb{P}(X)$ and $\{a^{(N)}\}_{N=1,2,\ldots} \subseteq \Lambda$ converge to $z \in \mathbb{P}(X)$ and $a \in \Lambda$ respectively as $N \to \infty$, then $H(z^{(N)}, a^{(N)}, y) \to H(z, a, y)$ as $N \to \infty$. Indeed, it is sufficient to verify this property only for the following two cases: (i) $y = \frac{1}{n}, \ a = \frac{1}{m}$, and $R'(\frac{1}{n}|z, \frac{1}{m}) > 0$, where $m, n = 1, 2, \ldots$, and (ii) $y = a = 0$. In case (i), $a^{(N)} = \frac{1}{m}$, when $N$ is large enough, and $H(z^{(N)}, \frac{1}{m}, \frac{1}{m}) \to H(z, \frac{1}{m}, \frac{1}{m})$ as $N \to \infty$ because the function $H(z, \frac{1}{m}, \frac{1}{m})$ is continuous in $z$, when $R'(\frac{1}{n}|z, \frac{1}{m}) > 0$. For case (ii), $H(z^{(N)}, a^{(N)}, 0) = z^{(N)} \to z$ as $N \to \infty$.

Fix $z = (\frac{1}{2}, \frac{1}{2})$. According to the above formulæ, $H(z, \frac{1}{m}, \frac{1}{n}) = (\sin^2(\frac{\pi m}{2mn}), \cos^2(\frac{\pi n}{2mn}))$ and $R'(\frac{1}{n}|z, \frac{1}{m}) = \frac{a_{m,n}}{2}$. Consider a closed subset $D = \{(z', z''): z = (\frac{1}{2}, \frac{1}{2}) \in \mathbb{P}(X) : z'(1) \geq \frac{3}{4}\}$ in $\mathbb{P}(X)$. Then

$$q(D|z, \frac{1}{m}) = \sum_{n=1,2,\ldots} \mathbb{I}\{\sin^2(\frac{\pi m}{2mn}) \geq \frac{3}{4}\} \frac{a_{m,n}}{2} = \sum_{k=0}^{2m} \mathbb{I}\{\sin^2(\frac{\pi k}{2m}) \geq \frac{3}{4}\} \frac{a_{m,2m+k}}{2} = \sum_{k=0}^{2m} \mathbb{I}\{\sin^2(\frac{\pi k}{2m}) \geq \frac{3}{4}\} \frac{1}{2m} \sum_{k=0}^{\infty} \frac{\sin^2(\frac{\pi k}{2m})}{\pi k+1} \to \frac{3}{4} \geq 0 \text{ as } m \to \infty,$$

where the limit takes place because $\int \frac{1}{2m} \sum_{k=0}^{\infty} \frac{\sin^2(\frac{\pi k}{2m})}{\pi k+1} \leq \frac{1}{4}$, where $[\cdot]$ is an integer part of a number, and $\sum_{k=0}^{\infty} \frac{1}{\pi k+1} = 1$. In addition, $q(D|z, 0) = 0$ since $z \notin D$ and $q(z|z, 0) = \mathbb{I}\{H(z, 0, 0) = z\}R'(0|z, 0) = 1$. Thus, $\lim_{m \to \infty} q(D|z, \frac{1}{m}) = \frac{3}{4} > 0 = q(D|z, 0)$ for a closed set $D \in \mathbb{P}(X)$. This implies that the stochastic kernel $q$ on $\mathbb{P}(X)$ given $\mathbb{P}(X) \times \Lambda$ is not weakly continuous.

## 5 Continuity of Transition Kernels for Posterior Probabilities

This section contains the proofs of Theorems 5.5 and 5.7. The following two versions of Fatou’s lemma for a sequence of measures $\{\mu^{(n)}\}_{n=1,2,\ldots}$ are used in the proofs provided below.

**Lemma 5.1.** (Generalized Fatou’s Lemma). Let $\mathbb{S}$ be an arbitrary metric space, $\{\mu^{(n)}\}_{n=1,2,\ldots} \subseteq \mathbb{P}(\mathbb{S})$, and $\{f^{(n)}\}_{n=1,2,\ldots}$ be a sequence of measurable nonnegative $\mathbb{R}$-valued functions on $\mathbb{S}$. Then:

(i) (Royden [25], p. 231) if $\{\mu^{(n)}\}_{n=1,2,\ldots} \subseteq \mathbb{P}(\mathbb{S})$ converges setwise to $\mu \in \mathbb{P}(\mathbb{S})$, then

$$\int_{\mathbb{S}} \liminf_{n \to \infty} f^{(n)}(s) \mu(ds) \leq \liminf_{n \to \infty} \int_{\mathbb{S}} f^{(n)}(s) \mu^{(n)}(ds); \quad (5.1)$$

(ii) (Schü. [27], Lemma 2.3(ii), Jaskiewicz and Nowak [22], Lemma 3.2, Feinberg et al. [15], Lemma 4], [16], Theorem 1.1) if $\{\mu^{(n)}\}_{n=1,2,\ldots} \subseteq \mathbb{P}(\mathbb{S})$ converges weakly to $\mu \in \mathbb{P}(\mathbb{S})$, then

$$\int_{\mathbb{S}} \liminf_{n \to \infty} f^{(n)}(s') \mu(ds) \leq \liminf_{n \to \infty} \int_{\mathbb{S}} f^{(n)}(s) \mu^{(n)}(ds). \quad (5.2)$$
Proof of Theorem 3.3. According to Parthasarathy [23, Theorem 6.1, p. 40], Shiryaev [29, p. 311], Billingsley [9, Theorem 2.1], the stochastic kernel \( q(dz'|z,a) \) on \( \mathbb{P}(X) \) given \( \mathbb{P}(X) \times A \) is weakly continuous if and only if \( q(D|z,a) \) is lower semi-continuous in \((z,a) \in (\mathbb{P}(X) \times X)\) for every open set \( D \) in \( \mathbb{P}(X) \), that is,

\[
\liminf_{n \to \infty} q(D|z^{(n)},a^{(n)}) \geq q(D|z,a),
\]

(5.3)

for all \( z, z^{(n)} \in \mathbb{P}(X) \), and \( a, a^{(n)} \in A, n = 1, 2, \ldots \), such that \( z^{(n)} \to z \) weakly and \( a^{(n)} \to a \).

To prove (5.3), suppose that

\[
\liminf_{n \to \infty} q(D|z^{(n)},a^{(n)}) < q(D|z,a).
\]

Then there exists \( \varepsilon^* > 0 \) and a subsequence \( \{z^{(n,1)}, a^{(n,1)}\}_{n=1,2,\ldots} \subseteq \{z^{(n)}, a^{(n)}\}_{n=1,2,\ldots} \), such that

\[
q(D|z^{(n,1)},a^{(n,1)}) \leq q(D|z,a) - \varepsilon^*, \quad n = 1, 2, \ldots .
\]

(5.4)

If condition (ii) of Theorem 3.2 holds, then formula (3.5), the weak continuity of the stochastic kernel \( R' \) on \( Y \) given \( \mathbb{P}(X) \times A \) (this weak continuity is proved in Hernández-Lerma [18, p. 92]), and Lemma 5.1(ii) contradict (5.4). If condition (i) of Theorem 3.2 holds, then there exists a subsequence \( \{z^{(n,2)}, a^{(n,2)}\}_{n=1,2,\ldots} \subseteq \{z^{(n)}, a^{(n)}\}_{n=1,2,\ldots} \) such that \( H(z^{(n,2)}, a^{(n,2)}, y) \to H(z,a,y) \) weakly as \( n \to \infty \), \( R'(|\cdot| z,a) \)-almost surely in \( y \in Y \). Therefore, since \( D \) is an open set in \( \mathbb{P}(X) \),

\[
\liminf_{n \to \infty} I\{H(z^{(n,2)}, a^{(n,2)}, y) \in D\} \geq I\{H(z,a,y) \in D\}, \quad R'(|\cdot| z,a) \)-almost surely in \( y \in Y \). \quad (5.5)

Formulas (3.5), (5.5), the setwise continuity of the stochastic kernel \( R' \) on \( Y \) given \( \mathbb{P}(X) \times A \), and Lemma 5.1(i) imply \( \liminf_{n \to \infty} q(D|z^{(n,2)},a^{(n,2)}) \geq q(D|z,a) \), which contradicts (5.4). Thus (5.3) holds. \( \square \)

In order to prove Theorem 3.7, we need to formulate and prove several auxiliary facts. Let \( S \) be a metric space, \( F(S) \) and \( C(S) \) be respectively the spaces of all real-valued functions and all bounded continuous functions defined on \( S \). A subset \( A_0 \subseteq F(S) \) is said to be equicontinuous at a point \( s \in S \), if \( \sup_{f \in A_0} |f(s') - f(s)| \to 0 \) as \( s' \to s \). A subset \( A_0 \subseteq F(S) \) is said to be uniformly bounded, if there exists a constant constant \( M < +\infty \) such that \( |f(s)| \leq M \), for all \( s \in S \) and for all \( f \in A_0 \). Obviously, if a subset \( A_0 \subseteq F(S) \) is equicontinuous at all the points \( s \in S \) and uniformly bounded, then \( A_0 \subseteq C(S) \).

Theorem 5.2. Let \( S_1, S_2, \) and \( S_3 \) be arbitrary metric spaces, \( \Psi(d\mathcal{S}_2|s_1) \) be a weakly continuous stochastic kernel on \( S_2 \) given \( S_1 \), and a subset \( A_0 \subseteq C(S_2 \times S_3) \) be equicontinuous at all the points \((s_2, s_3) \in S_2 \times S_3\) and uniformly bounded. If \( S_2 \) is separable, then for every open set \( O \) in \( S_2 \) the family of functions defined on \( S_1 \times S_3 \),

\[
\mathcal{A}_O = \left\{(s_1, s_3) \to \int_{O} f(s_2,s_3) \Psi(d\mathcal{S}_2|s_1) : f \in A_0\right\},
\]

is equicontinuous at all the points \((s_1, s_3) \in S_1 \times S_3\) and uniformly bounded.

Proof. The family \( \mathcal{A}_O \) consists of a single function which is identically equal to 0. Thus, the statement of the theorem holds when \( O = \emptyset \). Let \( O \neq \emptyset \). Since \( A_0 \subseteq C(S_2 \times S_3) \) is uniformly bounded, then

\[
\mathcal{M} = \sup_{f \in A_0} \sup_{s_2 \in S_2} \sup_{s_3 \in S_3} |f(s_2,s_3)| < \infty.
\]

(5.6)
and, since $\Psi(ds_2|s_1)$ is a stochastic kernel, the family of functions $A_\mathcal{O}$ is uniformly bounded by $\mathcal{M}$.

Let us fix an arbitrary nonempty open set $\Omega \in \mathcal{S}_2$ and an arbitrary point $(s_1, s_3) \in \mathcal{S}_1 \times \mathcal{S}_3$. We shall prove that $A_\mathcal{O} \subset \mathcal{F}(\mathcal{S}_1 \times \mathcal{S}_3)$ is equicontinuous at the point $(s_1, s_3)$. For any $s \in \mathcal{S}_2$ and $\delta > 0$ denote by $B_\delta(s)$ and $\overline{B}_\delta(s)$ respectively the open and closed balls in the metric space $\mathcal{S}_2$ of the radius $\delta$ with the center $s$ and by $S_\delta(s)$ the sphere in $\mathcal{S}_2$ of the radius $\delta$ with the center $s$. Note that $S_\delta(s) = B_\delta(s) \setminus \overline{B}_\delta(s)$ is the boundary of $B_\delta(s)$. Every ball $B_\delta(s)$ contains a ball $B_{\delta'}(s), 0 < \delta' \leq \delta$, such that

$$
\Psi(B_{\delta'}(s) \setminus B_{\delta'}(s)|s_1) = \Psi(S_{\delta'}(s)|s_1) = 0,
$$

that is, $B_{\delta'}(s)$ is a continuity set for the probability measure $\Psi(\cdot | s_1)$; Parthasarathy [23, p. 50]. Since $\Omega$ is an open set in $\mathcal{S}_2$, for any $s \in \Omega$ there exists $\delta_s > 0$ such that $B_{\delta_s}(s)$ is a continuity set for a probability measure $\Psi(s|s_1)$ and $B_{\delta_s}(s) \subset \Omega$. The family $\{B_{\delta_s}(s) : s \in \Omega\}$ is a cover of $\Omega$. Since $\mathcal{S}_2$ is a separable metric space, by Lindelöf’s lemma, there exists a sequence $\{s^{(j)}\}_{j=1,2,\ldots} \subset \Omega$ such that $\{B_{\delta_s(s^{(j)})}(s^{(j)}) : j = 1, 2, \ldots \}$ is a cover of $\Omega$. The sets

$$
A^{(1)} := B_{\delta_{s^{(1)}}}(s^{(1)}), \quad A^{(2)} := B_{\delta_{s^{(2)}}}(s^{(2)}) \setminus B_{\delta_{s^{(1)}}}(s^{(1)}), \ldots, \quad A^{(j)} := B_{\delta_{s^{(j)}}}(s^{(j)}) \setminus \left( \bigcup_{i=1}^{j-1} B_{\delta_{s^{(j)}}}(s^{(i)}) \right), \ldots
$$

are continuity sets for the probability measure $\Psi(s|s_1)$. In view of Parthasarathy [23, Theorem 6.1(e), p. 40],

$$
\Psi(A^{(j)}|s_1') \to \Psi(A^{(j)}|s_1) \quad \text{as} \quad s_1' \to s_1, \quad j = 1, 2, \ldots.
$$

Moreover,

$$
\cup_{j=1,2,\ldots} A^{(j)} = \Omega \quad \text{and} \quad A^{(i)} \cap A^{(j)} = \emptyset \quad \text{for all} \quad i \neq j.
$$

The next step of the proof is to show that for each $j = 1, 2, \ldots$

$$
\sup_{f \in A_0} \left| \int_{A^{(j)}} f(s_2, s_3) \Psi(ds_2|s_1') - \int_{A^{(j)}} f(s_2, s_3) \Psi(ds_2|s_1) \right| \to 0 \quad \text{as} \quad (s_1', s_3) \to (s_1, s_3).
$$

Fix an arbitrary $j = 1, 2, \ldots$. If $\Psi(A^{(j)}|s_1) = 0$, then formula (5.9) directly follows from (5.7) and (5.6). Now let $\Psi(A^{(j)}|s_1) > 0$. Formula (5.7) implies the existence of such $\delta > 0$ that $\Psi(A^{(j)}|s_1') > 0$ for all $s_1' \in B_{\delta}(s_1)$. We endow $A^{(j)}$ with the induced topology from $\mathcal{S}_2$ and set

$$
\Psi_j(C|s_1') := \frac{\Psi(C|s_1')}{\Psi(A^{(j)}|s_1')}, \quad s_1' \in B_{\delta}(s_1), \quad C \in \mathcal{B}(A^{(j)}).
$$

Formula (5.7) yields

$$
\Psi_j(ds_2|s_1') \text{ converges weakly to } \Psi_j(ds_2|s_1) \text{ in } \mathbb{P}(A^{(j)}) \text{ as } s_1' \to s_1.
$$

According to Parthasarathy [23, Theorem 6.8, p. 51],

$$
\sup_{f \in A_0} \left| \int_{A^{(j)}} f(s_2, s_3) \Psi(ds_2|s_1') - \int_{A^{(j)}} f(s_2, s_3) \Psi(ds_2|s_1) \right| \to 0 \quad \text{as} \quad s_1' \to s_1.
$$

Equicontinuity of $A_0 \subset \mathcal{C}(\mathcal{S}_2 \times \mathcal{S}_3)$ at all the points $(s_2, s_3) \in \mathcal{S}_2 \times \mathcal{S}_3$ and the inequality $|f(s_2', s_3') - f(s_2', s_3)| \leq |f(s_2', s_3) - f(s_2', s_3)| + |f(s_2', s_3) - f(s_2, s_3)|$ imply

$$
\limsup_{(s_2', s_3') \to (s_2, s_3)} \sup_{f \in A_0} |f(s_2', s_3') - f(s_2', s_3)| = 0 \quad \text{for all} \quad s_2 \in \mathcal{S}_2.
$$
Thus, formulas (5.12), (5.10) and Lemma 5.1(ii) imply
\[
\limsup_{(s_1',s_3') \to (s_1,s_3)} \sup_{f \in A_0} \left| \int_{A(j)} \left( f(s_2,s_3') - f(s_2,s_3) \right) \Psi(ds_2|s_1') \right| \\
\leq \int_{A(j)} \limsup_{(s_1',s_3') \to (s_1,s_3)} \sup_{f \in A_0} \left| f(s_2,s_3') - f(s_2,s_3) \right| \Psi(ds_2|s_1) = 0.
\] (5.13)

Formula (5.9) follows from (5.11) and (5.13).

Since, for all \( j = 1, 2, \ldots \) and for all \((s_1',s_3') \in S_1 \times S_3\),
\[
\sup_{f \in A_0} \left| \int_{A(j)} f(s_2,s_3')\Psi(ds_2|s_1') - \int_{A(j)} f(s_2,s_3)\Psi(ds_2|s_1) \right| \leq 2M\Psi(A(j)|s_1),
\]
and \( \sum_{j=1}^{\infty} \Psi(A(j)|s_1) = \Psi(O|s_1) \leq 1 \), then equicontinuity of \( A_O \) at the point \((s_1,s_3)\) follows from (5.8) and (5.9). Indeed,
\[
\sup_{f \in A_0} \left| \int_{O} f(s_2,s_3')\Psi(ds_2|s_1') - \int_{O} f(s_2,s_3)\Psi(ds_2|s_1) \right| \leq \sum_{j=1,2,\ldots} \sup_{f \in A_0} \left| \int_{A(j)} f(s_2,s_3')\Psi(ds_2|s_1') - \int_{A(j)} f(s_2,s_3)\Psi(ds_2|s_1) \right| \to 0 \quad \text{as } (s_1',s_3') \to (s_1,s_3).
\]

As \((s_1,s_3) \in S_1 \times S_3\) is arbitrary, the above inequality implies that \( A_O \) is equicontinuous at all the points \((s_1,s_3) \in S_1 \times S_3\). In particular, \( A_O \subseteq C(S_1 \times S_3) \). \( \blacksquare \)

For a set \( B \in \mathcal{B}(\mathbb{X}) \), let \( \mathcal{R}_B \) be the following family of functions defined on \( \mathbb{P}(\mathbb{X}) \times \mathbb{A} \):
\[
\mathcal{R}_B = \{(z,a) \to R(B \times C|z,a) : C \in \mathcal{B}(\mathbb{Y})\}.
\] (5.14)

**Lemma 5.3.** Let the stochastic kernel \( P(dx'|x,a) \) on \( \mathbb{X} \) given \( \mathbb{X} \times \mathbb{A} \) be weakly continuous and the stochastic kernel \( Q(dy|x,a) \) on \( \mathbb{Y} \) given \( \mathbb{X} \times \mathbb{A} \) be continuous in the total variation. Consider the stochastic kernel \( R(\cdot|z,a) \) on \( \mathbb{X} \times \mathbb{Y} \) given \( \mathbb{A} \times \mathbb{X} \), defined in formula (5.7). Then, for every pair of open sets \( O_1 \) and \( O_2 \) in \( \mathbb{X} \), the family of functions \( \mathcal{R}_{O_1 \setminus O_2} \) defined on \( \mathbb{P}(\mathbb{X}) \times \mathbb{A} \) is uniformly bounded and is equicontinuous at all the points \((z,a) \in \mathbb{P}(\mathbb{X}) \times \mathbb{A}\), that is, for all \((z_{(n)}) \in \mathbb{P}(\mathbb{X})\), \( a, a^{(n)} \in \mathbb{A} \), \( n = 1, 2, \ldots \), such that \( z^{(n)} \to z \) weakly and \( a^{(n)} \to a \),
\[
\sup_{C \in \mathcal{B}(\mathbb{Y})} \left| R((O_1 \setminus O_2) \times C|z^{(n)},a^{(n)}) - R((O_1 \setminus O_2) \times C|z,a) \right| \to 0.
\] (5.15)

**Proof.** Since \( R \) is a stochastic kernel, all the functions in the family \( \mathcal{R}_{O_1 \setminus O_2} \) are nonnegative and bounded above by 1. Thus, this family is uniformly bounded. The remaining proof establishes the equicontinuity of \( \mathcal{R}_{O_1 \setminus O_2} \) at all the points \((z,a) \in \mathbb{P}(\mathbb{X}) \times \mathbb{A}\). First we show that \( \mathcal{R}_{O_1} \) is equicontinuous at all the points \((z,a) \in \mathbb{X} \times \mathbb{A}\) when \( O_1 \) is an open set in \( \mathbb{X} \). Theorem 5.2 with \( S_1 = \mathbb{X} \times \mathbb{A}, S_2 = \mathbb{X}, S_3 = \mathbb{A}, O = O, \Psi = P, \) and \( A_0 = \{(a,x) \to Q(C|a,x) : C \in \mathcal{B}(\mathbb{Y})\} \subseteq C(\mathbb{A} \times \mathbb{X}) \), implies that the family of functions \( A_0^O = \{(x,a) \to \int_{O} Q(C|a,x')P(dx'|x,a) : C \in \mathcal{B}(\mathbb{Y})\} \) is equicontinuous at all the points \((x,a) \in \mathbb{X} \times \mathbb{A}\). In particular, \( A_0^O \subseteq C(\mathbb{A} \times \mathbb{X}) \). Thus, Theorem 5.2 with \( S_1 = \mathbb{P}(\mathbb{X}), S_2 = \mathbb{X}, S_3 = \mathbb{A}, O = \mathbb{X}, \Psi(\mathbb{B}|z) = z(B), B \in \mathcal{B}(\mathbb{X}), z \in \mathbb{P}(\mathbb{X}) \), and \( A_0 = A_0^O \), implies that the family \( \mathcal{R}_{O_1} \) is equicontinuous at
all the points \((z, a) \in \mathbb{P}(X) 	imes A\). Second, let \(O_1\) and \(O_2\) be arbitrary open sets in \(X\). Then the families of functions \(R_{O_1}, R_{O_2},\) and \(R_{O_1 \cup O_2}\) are equicontinuous at all the points \((z, a) \in \mathbb{P}(X) \times A\). Thus, for all \(z, z^{(n)} \in \mathbb{P}(X), a, a^{(n)} \in A, n = 1, 2, \ldots\), such that \(z^{(n)} \to z\) weakly and \(a^{(n)} \to a\),

\[
\sup_{C \in \mathcal{B}(Y)} |R((O_1 \setminus O_2) \times C|z^{(n)}, a^{(n)}) - R((O_1 \setminus O_2) \times C|z, a)| \\
\leq \sup_{C \in \mathcal{B}(Y)} |R((O_1 \cup O_2) \times C|z^{(n)}, a^{(n)}) - R((O_1 \cup O_2) \times C|z, a)| \\
+ \sup_{C \in \mathcal{B}(Y)} |R(O_2 \times C|z^{(n)}, a^{(n)}) - R(O_2 \times C|z, a)| \to 0, \tag{5.16}
\]

that is, the family of functions \(R_{O_1 \setminus O_2}\) is equicontinuous at all the points \((z, a) \in \mathbb{P}(X) \times A\). \(\square\)

**Corollary 5.4.** Let assumptions of Lemma 5.3 hold. Then the stochastic kernel \(R'(dy|z, a)\) on \(Y\) given \(\mathbb{P}(X) \times A\), defined in formula (3.2), is continuous in the total variation.

**Proof.** This corollary follows from Lemma 5.3 applied to \(O_1 = X\) and \(O_2 = \emptyset\). \(\square\)

**Theorem 5.5.** Let \(S\) be an arbitrary metric space, \(\{h, h^{(n)}\}_{n=1,2,\ldots}\) be Borel-measurable uniformly bounded real-valued functions on \(S\), \(\{\mu^{(n)}\}_{n=1,2,\ldots} \subset \mathbb{P}(S)\) converges in the total variation to \(\mu \in \mathbb{P}(S)\), and

\[
\sup_{S \in \mathcal{B}(S)} \left| \int_S h^{(n)}(s) \mu^{(n)}(ds) - \int_S h(s) \mu(ds) \right| \to 0 \quad \text{as} \quad n \to \infty. \tag{5.17}
\]

Then \(\{h^{(n)}\}_{n=1,2,\ldots}\) converges in probability \(\mu\) to \(h\), and therefore there is a subsequence \(\{n_k\}_{k=1,2,\ldots}\) such that \(\{h^{(n_k)}\}_{k=1,2,\ldots}\) converges \(\mu\)-almost surely to \(h\).

**Proof.** Fix an arbitrary \(\varepsilon > 0\) and set

\[
S^{(n,+)} := \{s \in S : h^{(n)}(s) - h(s) \geq \varepsilon\}, \quad S^{(n,-)} := \{s \in S : h(s) - h^{(n)}(s) \geq \varepsilon\}, \quad S^{(n)} := \{s \in S : |h^{(n)}(s) - h(s)| \geq \varepsilon\} = S^{(n,+)} \cup S^{(n,-)}, \quad n = 1, 2, \ldots
\]

Note that for all \(n = 1, 2, \ldots\),

\[
\varepsilon \mu^{(n)}(S^{(n,+)}) \leq \int_{S^{(n,+)}} h^{(n)}(s) \mu^{(n)}(ds) - \int_{S^{(n,+)}} h(s) \mu^{(n)}(ds) \\
\leq \left| \int_{S^{(n,+)}} h^{(n)}(s) \mu^{(n)}(ds) - \int_{S^{(n,+)}} h(s) \mu(ds) \right| + \left| \int_{S^{(n,+)}} h(s) \mu^{(n)}(ds) - \int_{S^{(n,+)}} h(s) \mu(ds) \right|. \tag{5.18}
\]

The convergence in the total variation of \(\mu^{(n)}\) to \(\mu \in \mathbb{P}(S)\) implies that

\[
\left| \int_{S^{(n,+)}} h(s) \mu^{(n)}(ds) - \int_{S^{(n,+)}} h(s) \mu(ds) \right| \to 0 \quad \text{and} \quad |\mu^{(n)}(S^{(n,+)}) - \mu(S^{(n,+)})| \to 0 \quad \text{as} \quad n \to \infty. \tag{5.19}
\]

Formulas (5.17)–(5.19) yield

\[
\left| \int_{S^{(n,+)}} h^{(n)}(s) \mu^{(n)}(ds) - \int_{S^{(n,+)}} h(s) \mu(ds) \right| \to 0 \quad \text{as} \quad n \to \infty, \quad \text{and, in view of (5.18),}
\]

\[
\mu^{(n)}(S^{(n,+)}) \to 0 \quad \text{as} \quad n \to \infty. \tag{5.20}
\]
Since \( C \in \mathcal{B} \), \( (ii) \) any open set \( C \in \mathcal{B} \) can be represented as \( \bigcup_{i=1}^{\infty} O_i \) of the sequence \( \{O_i\}_{i=1}^{\infty} \). However, in this paper we use the intersection assumption.

Lemma 5.6. If the topology on \( X \) has a countable base \( \mathcal{T}_b = \{O^{(j)}\}_{j=1,2,\ldots} \) such that, for each finite intersection \( \mathcal{O} = \bigcap_{i=1}^{N} O^{(j_i)} \) of its elements \( O^{(j_i)} \in \mathcal{T}_b, i = 1, 2, \ldots, N \), the family of functions \( \mathcal{R}_O \) defined in (5.14) is equicontinuous at all the points \( (z,a) \in \mathbb{P}(X) \times A \), then for any sequence \( \{(z^{(n)}, a^{(n)})\}_{n=1,2,\ldots} \) such that \( \{z^{(n)}\}_{n=1,2,\ldots} \in \mathbb{P}(X) \) converges weakly to \( z \in \mathbb{P}(X) \) and \( \{a^{(n)}\}_{n=1,2,\ldots} \in A \) converges to \( a \), there exists a subsequence \( \{z^{(n_k)}, a^{(n_k)}\}_{k=1,2,\ldots} \) and a set \( C^* \in \mathcal{B}(Y) \) such that

\[
R'(C^* | z,a) = 1 \quad \text{and} \quad H(\cdot | z^{(n_k)}, a^{(n_k)}, y) \text{ converges setwise to } H(\cdot | z,a,y) \text{ for all } y \in C^*,
\]

and, therefore, Assumption (H) holds.

As clear from the proof of Lemma 5.6, the intersection assumption is equivalent to the same assumption for finite unions. However, in this paper we use the intersection assumption.

Proof: According to Bogachev [10, Theorem 8.10.56], \( (5.21) \) holds if there exists a subsequence \( \{(z^{(n_k)}, a^{(n_k)})\}_{k=1,2,\ldots} \) of the sequence \( \{(z^{(n)}, a^{(n)})\}_{n=1,2,\ldots} \) and a set \( C^* \in \mathcal{B}(Y) \) such that

\[
R'(C^* | z,a) = 1 \quad \text{and} \quad H(\mathcal{O}| z^{(n_k)}, a^{(n_k)}, y) \rightarrow H(\mathcal{O}| z,a,y) \quad \text{as } k \rightarrow \infty \quad \text{for all } y \in C^*,
\]

(5.22)

for all open sets \( \mathcal{O} \in X \). The rest of the proof establishes the existence of a subsequence \( \{(z^{(n_k)}, a^{(n_k)})\}_{k=1,2,\ldots} \) of the sequence \( \{(z^{(n)}, a^{(n)})\}_{n=1,2,\ldots} \) and a set \( C^* \in \mathcal{B}(Y) \) such that (5.22) holds for all open sets \( \mathcal{O} \in X \).

Let \( A_1 \) be a family of all the subsets of \( X \) that are finite unions of sets from \( \mathcal{T}_b \), and let \( A_2 \) be a family of all subsets \( B \) of \( X \) such that \( B = \hat{\mathcal{O}} \setminus \mathcal{O}' \) with \( \mathcal{O} \in \mathcal{T}_b \) and \( \mathcal{O}' \in A_1 \). Observe that: (i) both \( A_1 \) and \( A_2 \) are countable, (ii) any open set \( \mathcal{O} \in X \) can be represented as

\[
\mathcal{O} = \bigcup_{j=1,2,\ldots}^{\infty} \mathcal{O}^{(j,1)} = \bigcup_{j=1,2,\ldots}^{\infty} B^{(j,1)} \quad \text{for some } \mathcal{O}^{(j,1)} \in \mathcal{T}_b, \quad j = 1, 2, \ldots,
\]

(5.23)

where \( B^{(j,1)} = \mathcal{O}^{(j,1)} \setminus \left( \bigcup_{i=1}^{j-1} \mathcal{O}^{(i,1)} \right) \) are disjoint elements of \( A_2 \) (it is allowed that \( \mathcal{O}^{(j,1)} = \emptyset \) or \( B^{(j,1)} = \emptyset \) for some \( j = 1, 2, \ldots \)).

To prove (5.22) for all open sets \( \mathcal{O} \in X \), we first show that there exists a subsequence \( \{(z^{(n_k)}, a^{(n_k)})\}_{k=1,2,\ldots} \) of the sequence \( \{(z^{(n)}, a^{(n)})\}_{n=1,2,\ldots} \) and a set \( C^* \in \mathcal{B}(Y) \) such that (5.22) holds for all \( \mathcal{O} \in A_2 \).

Consider an arbitrary \( \mathcal{O}^* \in A_1 \). Then \( \mathcal{O}^* = \bigcup_{i=1}^{n} \mathcal{O}^{(j_i)} \) for some \( n = 1, 2, \ldots \), where \( \mathcal{O}^{(j_i)} \in \mathcal{T}_b, i = 1, 2, \ldots, n \). Let \( A^{(n)} = \{\bigcap_{m=1}^{k} \mathcal{O}^{i_m} : \{i_1, i_2, \ldots, i_k\} \subseteq \{j_1, j_2, \ldots, j_n\}\} \) be the finite set of possible intersections of \( \mathcal{O}^{(j_1)}, \ldots, \mathcal{O}^{(j_n)} \). The principle of inclusion-exclusion implies that for \( \mathcal{O}^* = \bigcup_{i=1}^{n} \mathcal{O}^{(j_i)} \), \( C \in \mathcal{B}(Y), z, z' \in \mathbb{P}(X), \) and \( a, a' \in A \),

\[
|R(\mathcal{O}^* \times C| z,a) - R(\mathcal{O}^* \times C| z',a')| \leq \sum_{B \in A^{(n)}} |R(B \times C| z,a) - R(B \times C| z',a')|.
\]

(5.24)
In view of the assumption of the lemma regarding finite intersections of the elements of the base \( \tau_0 \), for each \( O^* \in A_1 \) the family \( \mathcal{R}_{O^*} \) is equicontinuous at all the points \( (z,a) \in \mathbb{P}(X) \times A \). Inequality (5.16) implies that for each \( B \in A_2 \) the family \( \mathcal{R}_B \) is equicontinuous at all the points \( (z,a) \in \mathbb{P}(X) \times A \), that is, (5.15) holds with arbitrary \( O_1 \in \tau_0 \) and \( O_2 \in A_1 \). This fact along with the definition of \( H \) (see (5.23)) means that

\[
\lim_{n \to \infty} \sup_{C \in \mathcal{B}(\mathcal{Y})} \left| \int_C H(B | z^{(n)}, a^{(n)}, y) R'(dy | z^{(n)}, a^{(n)}) - \int_C H(B | z, a, y) R'(dy | z, a) \right| = 0, \tag{5.25}
\]

for any \( B \in A_2 \).

Since the set \( A_2 \) is countable, let \( A_2 := \{B(j) : j = 1, 2, \ldots\} \). Denote \( z^{(n,0)} = z^{(n)} \), \( a^{(n,0)} = a^{(n)} \) for all \( n = 1, 2, \ldots \). For \( j = 1, 2, \ldots \), from (5.25) and Theorem 5.5 with \( S = \mathcal{Y} \), \( s = y \), \( h^{(n)}(s) = H(B(j) | z^{(n,j-1)}, a^{(n,j-1)}, y) \), \( \mu^{(n)}(\cdot) = R'(\cdot | z^{(n,j-1)}, a^{(n,j-1)}) \), \( h(s) = H(B(j) | z, a, y) \), and \( \mu(\cdot) = R'(\cdot | z, a) \), there exists a subsequence \( \{(z^{(n,j)}, a^{(n,j)})\}_{n=1,2,\ldots} \) of the sequence \( \{(z^{(n,j)}, a^{(n,j)})\}_{n=1,2,\ldots} \) and a set \( C^{(s,j)} \in \mathcal{B}(\mathcal{Y}) \) such that

\[
R'(C^{(s,j)} | z, a) = 1 \quad \text{and} \quad \lim_{n \to \infty} H(B(j) | z^{(n,j)}, a^{(n,j)}, y) = H(B(j) | z, a, y) \quad \text{for all} \ y \in C^{(s,j)}. \tag{5.26}
\]

Let \( C^* := \bigcap_{j=1}^{\infty} C^{(s,j)} \). Observe that \( R'(C^* | z, a) = 1 \). Let \( z^{(n_k)} = z^{(k,k)} \) and \( a^{(n_k)} = a^{(k,k)} \), \( k = 1, 2, \ldots \). As follows from Cantor’s diagonal argument, (5.22) holds with \( O = B(j) \) for all \( j = 1, 2, \ldots \). In other words, (5.22) holds for all \( O \in A_2 \).

Let \( O \) be an arbitrary open set in \( X \) and \( B^{(1,1)}, B^{(2,1)}, \ldots \) be disjoint elements of \( A_2 \) satisfying (5.23). Countable additivity of probability measures implies that

\[
\lim_{k \to \infty} H(O | z^{(n_k)}, a^{(n_k)}, y) = \lim_{k \to \infty} \sum_{j=1,2,\ldots} H(B^{(j,1)} | z^{(n_k)}, a^{(n_k)}, y) = \sum_{j=1,2,\ldots} \lim_{k \to \infty} H(B^{(j,1)} | z, a, y) = H(O | z, a, y),
\]

for all \( y \in C^* \). Since \( R'(C^* | z, a) = 1 \), (5.22) holds.

**Proof of Theorem 3.7** The setwise continuity of the stochastic kernel \( R' \) follows from Corollary 5.4 that states the continuity of \( R' \) in the total variation. The validity of Assumption (H) follows from Lemma 5.3 and Lemma 5.6. In particular, \( \tau_0 \) is any countable base of the state space \( X \), and, in view of Lemma 5.3 the family \( \mathcal{R}_O \) is equicontinuous at all the points \( (z,a) \in \mathbb{P}(X) \times A \) for each open set \( O \) in \( X \), which implies that the assumptions of Lemma 5.6 hold.

**Remark 5.7.** The proof of Theorem 3.7 is based on Lemma 5.6 that states that the measures \( H(z^{(n_k)}, a^{(n_k)}, y) \) converge setwise to \( H(z, a, y) \) as \( k \to \infty \). This is a stronger property than the weak convergence required in Assumption (H). This implies two conclusions: (i) stronger properties than Assumption (H) take place under the assumptions of Theorem 3.7 and (ii) it is possible that the assumptions of Theorem 3.7 can be relaxed. However, regarding (ii), Example 4.1 indicates that relaxing the continuity in the total variation of \( Q \) to the setwise continuity does not lead to the weak continuity of \( q \) even if the weak continuity of \( P \) is strengthened to the continuity in the total variation.
6 Preservation of Properties of One-Step Costs and Proof of Theorem 3.4

As shown in this section, the reduction of a POMDP to the COMDP preserves properties of one-step cost functions that are needed for the existence of optimal policies. These properties include inf-compactness and \( K \)-inf-compactness. In particular, in this section we prove Theorem 3.4 and thus complete the proof of Theorem 3.2.

We recall that an \( \mathbb{R} \)-valued function \( f \), defined on a nonempty subset \( U \) of a metric space \( U \), is called inf-compact on \( U \) if all the level sets \( \{ y \in U : f(y) \leq \lambda \} \), \( \lambda \in \mathbb{R} \), are compact. A function \( f \) is called lower semi-continuous, if all the level sets are closed.

The notion of a \( K \)-inf-compact function \( c(x,a) \), defined in Section 2 for a function \( c : X \times A \to \mathbb{R} \), is also applicable to a function \( f : S_1 \times S_2 \to \mathbb{R} \), where \( S_1 \) and \( S_2 \) are metric spaces, or certain more general topological spaces; see Feinberg et al. [14, 13] for details, where the properties of \( K \)-inf-compact functions are described. In particular, according to Feinberg et al. [14 Lemma 2.1], if a function \( f \) is inf-compact on \( S_1 \times S_2 \) then it is \( K \)-inf-compact on \( S_1 \times S_2 \). According to Feinberg et al. [14 Lemmas 2.2, 2.3], a \( K \)-inf-compact function \( f \) on \( S_1 \times S_2 \) is lower semi-continuous on \( S_1 \times S_2 \), and, in addition, for each \( s_1 \in S_1 \), the function \( f(s_1, \cdot) \) is inf-compact on \( S_2 \).

**Lemma 6.1.** If the function \( c : X \times A \to \mathbb{R} \) is bounded below and lower semi-continuous on \( X \times A \), then the function \( \bar{c} : \mathbb{P}(X) \times A \to \mathbb{R} \) defined in (3.8) is bounded below and lower semi-continuous on \( \mathbb{P}(X) \times A \).

**Proof.** The statement of this lemma directly follows from generalized Fatou’s Lemma 5.1(ii). \( \square \)

The inf-compactness of \( c \) on \( X \times A \) implies the inf-compactness of \( \bar{c} \) on \( \mathbb{P}(X) \times A \). We recall that an inf-compact function on \( X \times A \) with values in \( \mathbb{R} = \mathbb{R} \cup \{ +\infty \} \) is bounded below on \( X \times A \).

**Theorem 6.2.** If \( c : X \times A \to \mathbb{R} \) is an inf-compact function on \( X \times A \), then the function \( \bar{c} : \mathbb{P}(X) \times A \to \mathbb{R} \) defined in (3.8) is inf-compact on \( \mathbb{P}(X) \times A \).

**Proof.** Let \( c : X \times A \to \mathbb{R} \) be an inf-compact function on \( X \times A \). Fix an arbitrary \( \lambda \in \mathbb{R} \). To prove that the level set \( D_c(\lambda; \mathbb{P}(X) \times A) = \{ (z,a) \in \mathbb{P}(X) \times A : \bar{c}(z,a) \leq \lambda \} \) is compact, consider an arbitrary sequence \( \{ z^{(n)} , a^{(n)} \}_{n=1,2,...} \subset D_c(\lambda; \mathbb{P}(X) \times A) \). It is enough to show that \( \{ z^{(n)}, a^{(n)} \}_{n=1,2,...} \) has a limit point \( (z,a) \in D_c(\lambda; \mathbb{P}(X) \times A) \).

Let us show that the sequence of probability measures \( \{ z^{(n)} \}_{n=1,2,...} \) has a limit point \( z \in \mathbb{P}(X) \). Define \( X_{<+\infty} := X \setminus X_{+\infty} \), where \( X_{+\infty} := \{ x \in X : c(x,a) = +\infty \text{ for all } a \in A \} \).

The inequalities
\[
\int_X c(x,a^{(n)}) z^{(n)}(dx) \leq \lambda, \quad n = 1,2,\ldots,
\]
(6.1)

imply that \( z^{(n)}(X_{+\infty}) = 0 \) for any \( n = 1,2,\ldots \). Thus (6.1) transforms into
\[
\int_{X_{<+\infty}} c(x,a^{(n)}) z^{(n)}(dx) \leq \lambda, \quad n = 1,2,\ldots.
\]
(6.2)

By definition of inf-compactness, the function \( c : X \times A \to \mathbb{R} \) is inf-compact on \( X_{<+\infty} \times A \). According to Feinberg et al. [15 Corollary 3.2], the real-valued function \( \psi(x) = \inf_{a \in A} c(x,a) \), \( x \in X_{<+\infty} \), with values
in $\mathbb{R}$, is inf-compact on $X_{<+\infty}$. Furthermore, (6.2) implies that $\int_{X_{<+\infty}} \psi(x) z^{(n)}(dx) \leq \lambda$, $n = 1, 2, \ldots$. Thus, Hernández-Lerma and Lasserre [19] Proposition E.8 and Prohorov’s Theorem [19] Theorem E.7, yield relative compactness of the sequence $\{z^{(n)}\}_{n=1,2,\ldots}$ in $P(X_{<+\infty})$. Thus there exists a subsequence $\{z^{(nk)}\}_{k=1,2,\ldots} \subseteq \{z^{(n)}\}_{n=1,2,\ldots}$ and a probability measure $z \in P(X_{<+\infty})$ such that $z^{(nk)}$ converges to $z$ in $P(X_{<+\infty})$. Let us set $z(X_{<+\infty}) = 0$. As $z^{(n)}(X_{<+\infty}) = 0$ for all $n = 1, 2, \ldots$, then the sequence of probability measures $\{z^{(nk)}\}_{k=1,2,\ldots}$ converges weakly and its limit point $z$ belongs to $P(X)$.

The sequence $\{a^{(nk)}\}_{k=1,2,\ldots}$ has a limit point $a \in A$. Indeed, inequality (6.1) implies that for any $k = 1, 2, \ldots$ there exists at least one $x^{(k)}(\in X$ such that $c(x^{(k)}, a^{(nk)}) \leq \lambda$. Inf-compactness of $c : X \times A \to \mathbb{R}$ on $X \times A$ implies that $\{a^{(k)}\}_{k=1,2,\ldots}$ has a limit point $a \in A$. To finish the proof note that Lemma [6.1] generalized Fatou’s Lemma (5.1(ii)), and (6.1) imply that $\int_X c(x,a) z(dx) \leq \lambda$. ■

Proof of Theorem 3.4. If $c$ is bounded below on $X \times A$, then formula (3.8) implies that $\bar{\chi}$ is bounded below on $P(X) \times A$ by the same lower bound as $c$. Thus, it is enough to prove $K$-inf-compactness of $\bar{\chi}$ on $P(X) \times A$.

Let a sequence of probability measures $\{z^{(n)}\}_{n=1,2,\ldots}$ on $X$ weakly converges to $z \in P(X)$. Consider an arbitrary sequence $\{a^{(n)}\}_{n=1,2,\ldots} \subseteq A$ satisfying the condition that the sequence $\{\bar{\chi}(z^{(n)}, a^{(n)})\}_{n=1,2,\ldots}$ is bounded above. Observe that $\{a^{(n)}\}_{n=1,2,\ldots}$ has a limit point $a \in A$. Indeed, boundedness below of the $\mathbb{R}$-valued function $c$ on $X \times A$ and generalized Fatou’s Lemma (5.1(ii)) imply that for some $\lambda < +\infty$

$$\int_X \lambda(x) z(dx) \leq \liminf_{n \to \infty} \int_X c(x, a^{(n)}) z^{(n)}(dx) \leq \lambda,$$

where

$$\lambda(x) := \liminf_{y \to x, n \to \infty} c(y, a^{(n)}).$$

Inequality (6.3) implies the existence of $x^{(0)}(\in X$ such that $\lambda(x^{(0)}) \leq \lambda$. Therefore, formula (6.4) implies the existence of a subsequence $\{a^{(nk)}\}_{k=1,2,\ldots} \subseteq \{a^{(n)}\}_{n=1,2,\ldots}$ and a sequence $\{y^{(k)}\}_{k=1,2,\ldots} \subseteq X$ such that $y^{(k)} \to x^{(0)}$ as $k \to \infty$ and $c(y^{(k)}, a^{(nk)}) \leq \lambda + 1$ for $k = 1, 2, \ldots$. Since $c : X \times A \to \mathbb{R}$ is $K$-inf-compact on $X \times A$, the sequence $\{a^{(nk)}\}_{k=1,2,\ldots}$ has a limit point $a \in A$, which is the limit point of the initial sequence $\{a^{(n)}\}_{n=1,2,\ldots}$. Thus, the function $\bar{\chi}$ is $K$-inf-compact on $P(X) \times A$. ■

Arguments similar to the proof of Theorem 3.4 imply the inf-compactness of $\bar{\chi}(z, a)$ in $A$ for any $z \in P(X)$, if $c(x, a)$ is inf-compact in $a \in A$ for any $x \in X$.

Theorem 6.3. If the function $c(x, a)$ is inf-compact in $a \in A$ for each $x \in X$ and bounded below on $X \times A$, then the function $\bar{\chi}(z, a)$ is inf-compact in $a \in A$ for each $z \in P(X)$ and bounded below on $P(X) \times A$.

Proof. Fix $z \in P(X)$ and consider a sequence $\{a^{(n)}\}_{n=1,2,\ldots}$ in $A$ such that $c(z, a^{(n)}) \leq \lambda$ for some $\lambda < \infty$, $n = 1, 2, \ldots$. The classic Fatou’s lemma implies that (6.3) holds with $z^{(n)} = z$, $n = 1, 2, \ldots$, and $\lambda(x) = \liminf_{n \to \infty} c(x, a^{(n)})$, $x \in X$. Thus, there exists $x^{(0)} \in X$ such that $\liminf_{n \to \infty} c(x^{(0)}, a^{(n)}) \leq \lambda$. This together with the inf-compactness of $c(x^{(0)}, a)$ in $a \in A$ implies that the sequence $\{a^{(n)}\}_{n=1,2,\ldots}$ has a limit point in $A$. ■

Proof of Theorem 3.2 Theorem 3.2 follows from Theorems 3.1, 3.4 and 3.5.
7 Combining Assumption (H) and the Weak Continuity of $H$

Theorem 3.2 assumes either the weak continuity of $H$ or Assumption (H) together with the setwise continuity of $R'$. For some applications, see e.g., Subsection 3.2 that deals with inventory control, the filtering kernel $H$ satisfies Assumption (H) for some observations and it is weakly continuous for other observations. The following theorem is applicable to such situations.

**Theorem 7.1.** Let the observation space $\mathbb{Y}$ be partitioned into two disjoint subsets $\mathbb{Y}_1$ and $\mathbb{Y}_2$ such that $\mathbb{Y}_1$ is open in $\mathbb{Y}$. If the following assumptions hold:

(a) the stochastic kernels $P$ on $\mathbb{X}$ given $\mathbb{X} \times \mathbb{A}$ and $Q$ on $\mathbb{Y}$ given $\mathbb{A} \times \mathbb{X}$ are weakly continuous;

(b) the measure $R'(\cdot|z,a)$ on $(\mathbb{Y}_2, B(\mathbb{Y}_2))$ is setwise continuous in $(z,a) \in P(\mathbb{X}) \times \mathbb{A}$, that is, for every sequence $\{(z^{(n)}, a^{(n)})\}_{n=1,2,\ldots}$ in $P(\mathbb{X}) \times \mathbb{A}$ converging to $(z,a) \in P(\mathbb{X}) \times \mathbb{A}$ and for every $C \in B(\mathbb{Y}_2)$, we have $R'(C|z^{(n)}, a^{(n)}) \to R'(C|z,a)$;

(c) there exists a stochastic kernel $H$ on $\mathbb{X}$ given $P(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y}$ satisfying (3.3) such that:

(i) the stochastic kernel $H$ on $\mathbb{X}$ given $P(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y}_1$ is weakly continuous;

(ii) Assumption (H) holds on $\mathbb{Y}_2$, that is, if a sequence $\{(z^{(n)}, a^{(n)})\}_{n=1,2,\ldots} \subseteq P(\mathbb{X})$ converges weakly to $z \in P(\mathbb{X})$ and a sequence $\{a^{(n)}\}_{n=1,2,\ldots} \subseteq A$ converges to $a \in A$, then there exists a subsequence $\{(z^{(n_k)}, a^{(n_k)})\}_{k=1,2,\ldots} \subseteq \{(z^{(n)}, a^{(n)})\}_{n=1,2,\ldots}$ and a measurable subset $C$ of $\mathbb{Y}_2$ such that $R'(\mathbb{Y}_2 \setminus C|z,a) = 0$ and $H(z^{(n_k)}, a^{(n_k)}, y)$ converges weakly to $H(z,a,y)$ for all $y \in C$;

then the stochastic kernel $q$ on $P(\mathbb{X})$ given $P(\mathbb{X}) \times \mathbb{A}$ is weakly continuous. If, in addition to the above conditions, assumptions (a) and (b) from Theorem 3.2 hold, then the COMDP $(P(\mathbb{X}), A, q, \bar{c})$ satisfies Assumption (W*) and therefore statements (i)–(vi) of Theorem 3.1 hold.

**Proof.** The stochastic kernel $q(dz'|z,a)$ on $P(\mathbb{X})$ given $P(\mathbb{X}) \times \mathbb{A}$ is weakly continuous if and only if for every open set $D$ in $P(\mathbb{X})$ the function $q(D|z,a)$ is lower semi-continuous in $(z,a) \in P(\mathbb{X}) \times \mathbb{A}$; Billingsley [9, Theorem 2.1]. Thus, if $q$ is not weakly continuous, there exist an open set $D$ in $P(\mathbb{X})$ and sequences $z^{(n)} \to z$ weakly and $a^{(n)} \to a$, where $z^{(n)} \in P(\mathbb{X})$ and $a^{(n)} \in A$, $n = 1, 2, \ldots$, such that

$$\liminf_{n \to \infty} q(D|z^{(n)}, a^{(n)}) < q(D|z,a).$$

Then there exists $\varepsilon^* > 0$ and a subsequence $\{z^{(n_1)}, a^{(n_1)}\}_{n=1,2,\ldots} \subseteq \{z^{(n)}, a^{(n)}\}_{n=1,2,\ldots}$ such that for all $n = 1, 2, \ldots$

$$\int_{\mathbb{Y}_1} \mathbb{I}\{H(z^{(n_1)},a^{(n_1)}, y) \in D\} R'(dy|z^{(n_1)}, a^{(n_1)}) + \int_{\mathbb{Y}_2} \mathbb{I}\{H(z^{(n_1)},a^{(n_1)}, y) \in D\} R'(dy|z^{(n_1)}, a^{(n_1)}) = q(D|z^{(n_1)}, a^{(n_1)}) \leq q(D|z,a) - \varepsilon^* \quad (7.1)$$

$$= \int_{\mathbb{Y}_1} \mathbb{I}\{H(z,a, y) \in D\} R'(dy|z,a) + \int_{\mathbb{Y}_2} \mathbb{I}\{H(z,a, y) \in D\} R'(dy|z,a) - \varepsilon^*,$$

where the stochastic kernel $H$ on $\mathbb{X}$ given $P(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y}$ satisfies (3.3) and assumption (c) of Theorem 7.1.
Since $\mathcal{Y}_1$ is an open set in $\mathcal{Y}$ and the stochastic kernel $H$ on $\mathcal{X}$ given $\mathbb{P}(\mathcal{X}) \times A \times \mathcal{Y}_1$ is weakly continuous, for all $y \in \mathcal{Y}_1$

$$\liminf_{n \to \infty} I\{H(z^{(n,1)}, a^{(n,1)}, y') \in D\} = \liminf_{y' \to y, y \in \mathcal{Y}_1} I\{H(z^{(n,1)}, a^{(n,1)}, y') \in D\} \geq I\{H(z, a, y) \in D\}. \quad (7.2)$$

The weak continuity of the stochastic kernels $P$ and $Q$ on $\mathcal{X}$ given $\mathcal{X} \times A$ and on $\mathcal{Y}$ given $A \times \mathcal{X}$ respectively imply the weak continuity of the stochastic kernel $R'$ on $\mathcal{Y}$ given $\mathbb{P}(\mathcal{X}) \times A$; Hernández-Lerma [18, p. 92]. Therefore,

$$\liminf_{n \to \infty} \int_{\mathcal{Y}_1} I\{H(z^{(n,1)}, a^{(n,1)}, y) \in D\} R'(dy|z^{(n,1)}, a^{(n,1)}) \geq \int_{\mathcal{Y}_1} \liminf_{n \to \infty} I\{H(z^{(n,1)}, a^{(n,1)}, y') \in D\} R'(dy|z^{(n,1)}, a^{(n,1)}) \geq \int_{\mathcal{Y}_1} I\{H(z, a, y) \in D\} R'(dy|z, a), \quad (7.3)$$

where the first inequality follows from Lemma 5.1(ii) and the second one follows from formula (7.2).

The inequality

$$\limsup_{n \to \infty} \int_{\mathcal{Y}_2} I\{H(z^{(n,1)}, a^{(n,1)}, y) \in D\} R'(dy|z^{(n,1)}, a^{(n,1)}) \geq \int_{\mathcal{Y}_2} I\{H(z, a, y) \in D\} R'(dy|z, a) \quad (7.4)$$

together with (7.3) contradicts (7.1). This contradiction implies that $q(\cdot | z, a)$ is a weakly continuous stochastic kernel on $\mathbb{P}(\mathcal{X})$ given $\mathbb{P}(\mathcal{X}) \times A$.

To complete the proof of Theorem 7.1 we prove inequality (7.4). If $R'(\mathcal{Y}_2| z, a) = 0$, then inequality (7.4) holds. Now let $R'(\mathcal{Y}_2| z, a) > 0$. Since $R'(\mathcal{Y}_2| z^{(n,1)}, a^{(n,1)}) \to R'(\mathcal{Y}_2| z, a)$ as $n \to \infty$, there exists $N = 1, 2, \ldots$ such that $R'(\mathcal{Y}_2| z^{(n,1)}, a^{(n,1)}) > 0$ for any $n \geq N$. We endow $\mathcal{Y}_2$ with the same metric as in $\mathcal{Y}$ and set

$$R'_1(C| z', a') := \frac{R'(C| z', a')}{R'(\mathcal{Y}_2| z', a')}, \quad z' = z, z^{(n,1)}, a' = a, a^{(n,1)}, n \geq N, C \in \mathcal{B}(\mathcal{Y}_2).$$

Assumption (b) of Theorem 7.1 means that the stochastic kernel $R'_1(dy| z, a)$ on $\mathcal{Y}_2$ given $\mathbb{P}(\mathcal{X}) \times A$ is setwise continuous. Assumption (ii) of Theorem 7.1 implies the existence of a subsequence $(z^{(n,2)}, a^{(n,2)})_{n=1,2,\ldots} \subset (z^{(n,1)}, a^{(n,1)})_{n=1,2,\ldots}$ and a measurable subset $C$ of $\mathcal{Y}_2$ such that $R'_1(\mathcal{Y}_2 \setminus C| z, a) = 0$ and $H(z^{(n,2)}, a^{(n,2)}, y)$ converges weakly to $H(z, a, y)$ as $n \to \infty$ for all $y \in C$. Therefore, since $D$ is an open set in $\mathbb{P}(\mathcal{X})$, we have

$$\liminf_{k \to \infty} I\{H(z^{(n,2)}, a^{(n,2)}, y) \in D\} \geq I\{H(z, a, y) \in D\}, \quad y \in C. \quad (7.5)$$

Formulas (3.5), (7.5), the setwise continuity of the stochastic kernel $R'_1$ on $\mathcal{Y}_2$ given $\mathbb{P}(\mathcal{X}) \times A$, and Lemma 5.1(ii) imply

$$\frac{1}{R'(\mathcal{Y}_2| z, a)} \liminf_{k \to \infty} \int_{\mathcal{Y}_2} I\{H(z^{(n,2)}, a^{(n,2)}, y) \in D\} R'(dy| z^{(n,2)}, a^{(n,2)}) \geq \liminf_{k \to \infty} \int_{\mathcal{Y}_2} \frac{I\{H(z^{(n,2)}, a^{(n,2)}, y) \in D\}}{R'(\mathcal{Y}_2| z^{(n,2)}, a^{(n,2)})} \geq \frac{\int_{\mathcal{Y}_2} I\{H(z, a, y) \in D\} R'(dy| z, a)}{R'(\mathcal{Y}_2| z, a)} \quad (7.4),$$

and thus (7.4) holds.
Corollary 7.2. Let the observation space $\mathcal{Y}$ be partitioned into two disjoint subsets $\mathcal{Y}_1$ and $\mathcal{Y}_2$ such that $\mathcal{Y}_1$ is open in $\mathcal{Y}$ and $\mathcal{Y}_2$ is countable. If the following assumptions hold:

(a) the stochastic kernels $P$ on $\mathcal{X}$ given $\mathcal{X} \times \mathcal{A}$ and $Q$ on $\mathcal{X}$ given $\mathcal{A} \times \mathcal{X}$ are weakly continuous;

(b) $Q(y|a, x)$ is a continuous function on $\mathcal{A} \times \mathcal{X}$ for each $y \in \mathcal{Y}_2$;

(c) there exists a stochastic kernel $H$ on $\mathcal{X}$ given $\mathbb{P}(\mathcal{X}) \times \mathcal{A} \times \mathcal{Y}$ satisfying (3.3) such that the stochastic kernel $H$ on $\mathcal{X}$ given $\mathbb{P}(\mathcal{X}) \times \mathcal{A} \times \mathcal{Y}_1$ is weakly continuous;

then assumptions (b) and (ii) of Theorem 7.1 hold, and the stochastic kernel $q$ on $\mathbb{P}(\mathcal{X})$ given $\mathbb{P}(\mathcal{X}) \times \mathcal{A}$ is weakly continuous. If, in addition to the above conditions, assumptions (a) and (b) from Theorem 3.2 hold, then the COMDP $(\mathbb{P}(\mathcal{X}), \mathcal{A}, q, c)$ satisfies Assumption (W*) and therefore statements (i)–(vi) of Theorem 3.1 hold.

Proof. To prove the corollary, it is sufficient to verify conditions (b) and (ii) of Theorem 7.1. For each $B \in \mathcal{B}(\mathcal{X})$ and for each $y \in \mathcal{Y}_2$, Hernández-Lerma [18] Proposition C.2(b), Appendix C], being repeatedly applied to formula (3.1) with $C = \{y\}$, implies the continuity of $R(B \times \{y\}|z, a)$ in $(z, a) \in \mathbb{P}(\mathcal{X}) \times \mathcal{A}$. In particular, the function $R'(y|\cdot, \cdot)$ is continuous on $\mathbb{P}(\mathcal{X}) \times \mathcal{A}$. If $R'(y|z, a) > 0$ then, in view of (3.3), $H(B|z, a, y) = R(B \times \{y\}|z, a)/R'(y|z, a)$, and, if $y$ is fixed, this function is continuous at the point $(z, a)$. Thus, condition (ii) of Theorem 7.1 holds. Since the set $\mathcal{Y}_2$ is closed in $\mathcal{Y}$, the function $Q(\mathcal{Y}_2|a, x)$ is upper semi-continuous in $(a, x) \in \mathcal{A} \times \mathcal{X}$. Generalized Fatou’s Lemma [5.1] being repeatedly applied to (3.2) with $C = \mathcal{Y}_2$, implies that $R'(\mathcal{Y}_2|z, a)$ is upper semi-continuous in $(z, a) \in \mathbb{P}(\mathcal{X}) \times \mathcal{A}$. This implies that, for every $Y \subseteq \mathcal{Y}_2$ and for every sequence $\{z^{(n)}, a^{(n)}\}_{n=1, 2, \ldots} \subset \mathbb{P}(\mathcal{X}) \times \mathcal{A}$ converging to $(z, a) \in \mathbb{P}(\mathcal{X}) \times \mathcal{A}$,

$$|R'(Y|z^{(n)}, a^{(n)}) - R'(Y|z, a)| \leq \sum_{y \in \mathcal{Y}_2} |R'(y|z^{(n)}, a^{(n)}) - R'(y|z, a)| \to 0 \quad \text{as} \quad n \to \infty,$$

where the convergence takes place because of the same arguments as in the proof of Corollary 3.8. Thus, condition (b) of Theorem 7.1 holds.

\[\Box\]

8 Examples of Applications

To illustrate theoretical results, they are applied in this section to three particular models: (i) problems defined by stochastic equations; see Striebel [31], Bensoussan [3], and Hernández-Lerma [18, p. 83], (ii) inventory control, and (iii) Markov Decision Model (MDM).

8.1 Problems Defined by Stochastic Equations

Let $\{\xi_t\}_{t=0, 1, \ldots}$ be a sequence of identically distributed finite random variables with values in $\mathbb{R}$ and with the distribution $\mu$. Let $\{\eta_t\}_{t=0, 1, \ldots}$ be a sequence of random variables uniformly distributed on $(0, 1)$. An initial state $x_0$ is a random variable with values in $\mathbb{R}$. It is assumed that the random variables $x_0, \xi_0, \eta_0, \xi_1, \eta_1, \ldots$ are defined on the same probability space and mutually independent.
Consider a stochastic partially observable control system

\[ x_{t+1} = F(x_t, a_t, \xi_t), \ t = 0, 1, \ldots \]

\[ y_{t+1} = G(a_t, x_{t+1}, \eta_{t+1}), \ t = 0, 1, \ldots \]

where \( F \) and \( G \) are given measurable functions from \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) to \( \mathbb{R} \) and from \( \mathbb{R} \times \mathbb{R} \times (0, 1) \) to \( \mathbb{R} \) respectively. The initial observation is \( y_0 = G_0(x_0, \eta_0) \), where \( G_0 \) is a measurable function from \( \mathbb{R} \times (0, 1) \) to \( \mathbb{R} \). The states \( x_t \) are not observable, while the states \( y_t \) are observable. The goal is to minimize the expected total discounted costs.

Instead of presenting formal definitions of functions \( a_t \), we describe the above problem as a POMDP with the state space \( X = \mathbb{R} \), observation space \( Y = \mathbb{R} \), and action space \( A = \mathbb{R} \). The transition law is

\[ P(B | x, a) = \int_{\mathbb{R}} \mathbf{1}_\{F(x, a, s) \in B \} \mu(ds), \quad B \in \mathcal{B}(\mathbb{R}), \ x \in \mathbb{R}, \ a \in \mathbb{R}. \]  

(8.3)

The observation kernel is

\[ Q(C | a, x) = \int_{(0,1)} \mathbf{1}_\{G(a, x, s) \in C \} \lambda(ds), \quad C \in \mathcal{B}(\mathbb{R}), \ a \in \mathbb{R}, \ x \in \mathbb{R}, \]

where \( \lambda \in \mathcal{P}((0,1)) \) is the Lebesgue measure on \( (0,1) \). The initial state distribution \( p \) is the distribution of the random variable \( x_0 \), and the initial observation kernel \( \lambda_0(C | x) = \int_{(0,1)} \mathbf{1}_\{G_0(x, s) \in C \} \lambda(ds) \) for all \( C \in \mathcal{B}(\mathbb{R}) \) and for all \( x \in X \).

Assume that \( (x, a) \to F(x, a, s) \) is a continuous mapping on \( \mathbb{R} \times \mathbb{R} \) for \( \mu \)-a.e. \( s \in \mathbb{R} \). Then the stochastic kernel \( P(dx' | x, a) \) on \( \mathbb{R} \) given \( \mathbb{R} \times \mathbb{R} \) is weakly continuous: Hernández-Lerma [18, p. 92].

Assume that: (i) \( G \) is a continuous mapping on \( \mathbb{R} \times \mathbb{R} \times (0,1) \), (ii) the partial derivative \( g(x, y, s) = \frac{\partial G(x, y, s)}{\partial s} \) exists everywhere and is continuous, and (iii) there exists a constant \( \beta > 0 \) such that \( |g(a, x, s)| \geq \beta \) for all \( a \in \mathbb{R}, x \in \mathbb{R}, \) and \( s \in (0,1) \). Denote by \( \tilde{G} \) the inverse function for \( G \) with respect the last variable. Assume that \( \tilde{G} \) is continuous.

Let us prove that under these assumptions the observation kernel \( Q \) on \( \mathbb{R} \) given \( \mathbb{R} \times \mathbb{R} \) is continuous in the total variation. For each \( \varepsilon \in (0, \frac{1}{4}) \), for each Borel set \( C \in \mathcal{B}(\mathbb{R}) \), and for all \( (a', x'), (a, x) \in \mathbb{R} \times \mathbb{R} \)

\[ |Q(C | a', x') - Q(C | a, x)| = \left| \int_0^1 \mathbf{1}_{\{G(a', x', s) \in C \}} \lambda(ds) - \int_0^1 \mathbf{1}_{\{G(a, x, s) \in C \}} \lambda(ds) \right| \]

\[ \leq 4\varepsilon + \left| \int_\varepsilon^{1-\varepsilon} \mathbf{1}_{\{G(a', x', s) \in C \}} \lambda(ds) - \int_\varepsilon^{1-\varepsilon} \mathbf{1}_{\{G(a, x, s) \in C \}} \lambda(ds) \right| \]

\[ = 4\varepsilon + \left| \int_{G(a', x', \varepsilon, 1-\varepsilon)} \mathbf{1}_{\{\tilde{s} \in C \}} \lambda(d\tilde{s}) - \int_{G(a, x, \varepsilon, 1-\varepsilon)} \mathbf{1}_{\{\tilde{s} \in C \}} \lambda(d\tilde{s}) \right| \]

\[ \leq 4\varepsilon + \frac{1}{\beta} \int_{G(a, x, \varepsilon, 1-\varepsilon)} |g(a', x', G(a', x', \tilde{s})) - g(a, x, G(a, x, \tilde{s}))| \lambda(d\tilde{s}), \]

where \( \lambda \) is the Lebesgue measure on \( \mathbb{R} \), the second equality holds because of the changes \( \tilde{s} = G(a', x', s) \) and \( \tilde{s} = G(a, x, s) \) in the corresponding integrals, and the second inequality follows from direct estimations.
Since, the function $G$ is continuous, $G(a', x', \varepsilon) \to G(a, x, \varepsilon)$ and $G(a', x', 1 - \varepsilon) \to G(a, x, 1 - \varepsilon)$ as $(a', x') \to (a, x)$, for any $(a, x, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times (0, \frac{1}{2})$. Thus, if
\[
\int_{\mathbb{R}} D(a, x, a', x', \varepsilon, \tilde{s}) \lambda(d\tilde{s}) \to 0 \quad \text{as} \quad (a', x') \to (a, x),
\]
where $D(a, x, a', x', \varepsilon, \tilde{s}) := |g(a', x', G(a', x', \tilde{s})) - g(a, x, G(a, x, \tilde{s}))|$, when $\tilde{s} \in G(a, x, [\varepsilon, 1 - \varepsilon]) \cap G(a', x', [\varepsilon, 1 - \varepsilon])$, and $D(a, x, a', x', \varepsilon, \tilde{s}) = 0$ otherwise, then
\[
\lim_{(a', x') \to (a, x)} \sup_{C \in \mathcal{B}(\mathbb{R})} \left| Q(C|a', x') - Q(C|a, x) \right| = 0.
\]

So, to complete the proof of the continuity in the total variation of the observation kernel $Q$ on $\mathbb{R}$ given $\mathbb{R} \times \mathbb{R}$, it is sufficient to verify (8.4). We fix an arbitrary vector $(a, x, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times (0, \frac{1}{2})$ and consider arbitrary converging sequences $a^{(n)} \to a$ and $x^{(n)} \to x$. Let $(a', x') = (a^{(n)}, x^{(n)})$, $n = 1, 2, \ldots$. Since the sets $K := \{(a^{(n)}, x^{(n)}) : n = 1, 2, \ldots\} \cup \{(a, x)\}$ and $[\varepsilon, 1 - \varepsilon]$ are compact and the function $g$ is continuous on $\mathbb{R} \times \mathbb{R} \times (0, 1)$, the function $|g|$ is bounded above on the compact set $K \times [\varepsilon, 1 - \varepsilon]$ by a positive constant $M$. Thus, the integrand in (8.4) is bounded above by $2M$ on the compact set $G(K \times [\varepsilon, 1 - \varepsilon])$ and is equal to 0 on its complement. Since $G$, $g$, and $\mathcal{G}$ are continuous functions, for each $\tilde{s} \in \mathbb{R}$ the integrand in (8.4) converges to 0 as $(a', x') \to (a, x)$. Therefore, (8.4) follows from the dominated convergence theorem, because the Lebesgue measure of the set $G(K \times [\varepsilon, 1 - \varepsilon])$ is finite since this set is compact.

Finally, we assume that the one-period cost function $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is bounded below and $\mathbb{K}$-inf-compact. Thus, the assumptions of Theorem 3.6 are satisfied. Therefore, for this COMDP there exists a stationary optimal policy, the optimality equations hold, and value iterations converge to the optimal value.

We remark that the one-dimensional Kalman filter in discrete time satisfies the above assumptions. In this case, $F(x_t, a_t, \xi_t) = d^* x_t + b^* a_t + \xi_t$ and $G(a_t, x_{t+1}, \eta_{t+1}) = h^* x_{t+1} + c^* \Phi^{-1}(\eta_{t+1})$, where $c^* \neq 0$ and $\Phi^{-1}$ is the inverse to the cumulative distribution function of a standard normal distribution ($\Phi^{-1}(\eta_{t+1})$ is a standard normal random variable). In particular, $|g(a, x, s)| = |c^*| (2\pi)^{\frac{1}{2}} e^{s^2/2} \geq |c^*| (2\pi)^{\frac{1}{2}} > 0$ for all $s \in [0, 1]$. Thus, if the cost function $c(x, a)$ is $\mathbb{K}$-inf-compact, then the conclusions of Theorem 3.6 hold for the Kalman filter. In particular, the quadratic cost function $c(x, a) = c_1 x^2 + c_2 a^2$ is $\mathbb{K}$-inf-compact if $c_1 \geq 0$ and $c_2 > 0$. Thus, the linear quadratic Gaussian control problem is a particular case of this model. The one-step cost functions $c(x, a) = (a - x)^2$ and $c(x, a) = |x - a|$, which are typically used for identification problems, are also $\mathbb{K}$-inf-compact. However, these two functions are not inf-compact. This illustrates the usefulness of the notion of $\mathbb{K}$-inf-compactness.

### 8.2 Inventory Control with Incomplete Records

This example is motivated by Bensoussan et al. [4–7], where several inventory control problems for periodic review systems, when the Inventory Manager (IM) may not have complete information about inventory levels, are studied. In Bensoussan et al. [4, 7], a problem with backorders is considered. In the model considered in [4], the IM does not know the inventory level, if it is nonnegative, and the IM knows the inventory level, if it is negative. In the model considered in [7], the IM only knows whether the inventory level is negative or nonnegative. In [5] a problem with lost sales is studied, when the IM only knows whether
a lost sale happened or not. The underlying mathematical analysis is summarized in [6], where additional references can be found. The analysis includes transformations of density functions of demand distributions.

The current example studies periodic review systems with backorders and lost sales, when some inventory levels are observable and some are not. The goal is to minimize the expected total costs. Demand distribution may not have densities.

In the case of full observations, we model the problem as an MDP with a state space $X = \mathbb{R}$ (the current inventory level), action space $\mathcal{A} = \mathbb{R}$ (the ordered amount of inventory), and action sets $\mathcal{A}(x) = \mathcal{A}$ available at states $x \in X$. In a particular case, when we do not assume that demands are nonnegative, this model also covers cash balancing problems. Since we do not assume that demands are nonnegative, this model also covers cash balancing problems. However, for problems with lost sales it is assumed that the initial state distribution $p$ is concentrated on $[0, \infty)$, and this implies that states $x < 0$ will never be visited. We assume that the distribution function $F_D$ is atomless (an equivalent assumption is that the function $F_D$ is continuous). The state transition law $P$ on $X$ given $X \times \mathcal{A}$ is

$$P(B|x, a) = \int_{\mathbb{R}} \mathbb{I}\{F(x, a, s) \in B\} dF_D(s), \quad B \in \mathcal{B}(X), \ x \in X, \ a \in \mathcal{A}. \quad (8.5)$$

Since we do not assume that demands are nonnegative, this model also covers cash balancing problems and problems with returns; see Feinberg and Lewis [17] and references therein. In a particular case, when $C(a) = +\infty$ for $a < 0$, orders with negative sizes are infeasible, and, if an order is placed, the ordered amount of inventory should be positive.

As mentioned above, some states (inventory levels) $x \in X = \mathbb{R}$ are observable and some are not. Let inventory be stored in containers. From a mathematical prospective, containers are elements of a finite or countably infinite partition of $X = \mathbb{R}$ into disjoint convex sets, and each of these sets is not a singleton.

In other words, each container $B_{i+1}$ is an interval (possibly open, closed, or semi-open) with ends $d_i$ and $d_{i+1}$ such that $-\infty \leq d_i < d_{i+1} \leq +\infty$, and the union of these disjoint intervals is $\mathbb{R}$. In addition, we assume that $d_{i+1} - d_i \geq \gamma$ for some constant $\gamma > 0$ for all containers, that is, the sizes of all the containers are uniformly bounded below by a positive number. We also follow an agreement that the 0-inventory level belongs to a container with end points $d_0$ and $d_1$, and a container with end points $d_i$ and $d_{i+1}$ is labeled as the $(i+1)$-th container $B_{i+1}$. Thus, container $B_1$ is the interval in the partition containing point 0. Containers’ labels can be nonpositive. If there is a container with the smallest (or largest) finite label $n$ then $d_{n-1} = -\infty$ (or $d_n = +\infty$, respectively). If there are containers with labels $i$ and $j$ then there are containers with all the labels between $i$ and $j$. In addition each container is either transparent or nontransparent. If the
inventory level $x_t$ belongs to a nontransparent container, the IM only knows which container the inventory level belongs to. If an inventory level $x_t$ belongs to a transparent container, the IM knows that the amount of inventory is exactly $x_t$.

For each nontransparent container with end points $d_i$ and $d_{i+1}$, we fix an arbitrary point $b_{i+1}$ satisfying $d_i < b_{i+1} < d_{i+1}$. For example, it is possible to set $b_{i+1} = 0.5d_i + 0.5d_{i+1}$, when $\max\{|d_i|, |d_{i+1}|\} < \infty$. If an inventory level belongs to a nontransparent container $B_i$, the IM observes $y_t = b_i$. Let $L$ be the set of labels of the nontransparent containers. We set $Y_L = \{b_i : i \in L\}$ and define the observation set $\mathbb{Y} = \mathbb{T} \cup Y_L$, where $\mathbb{T}$ is the union of all transparent containers $B_i$ (transparent elements of the partition). If the observation $y_t$ belongs to a transparent container (in this case, $y_t \in \mathbb{T}$), then the IM knows that the inventory level belongs to the container $B_i$, and this container is nontransparent. Of course, the distribution of this level can be computed.

Let $\rho$ be the Euclidean distance on $\mathbb{R} : \rho(a, b) = |a - b|$ for $a, b \in \mathbb{Y}$. On the state space $\mathbb{X} = \mathbb{R}$ we consider the metric $\rho_\mathbb{X}(a, b) = |a - b|$, if $a$ and $b$ belong to the same container, and $\rho_\mathbb{X}(a, b) = |a - b| + 1$ otherwise, where $a, b \in \mathbb{X}$. The space $(\mathbb{X}, \rho_\mathbb{X})$ is a Borel subset of a Polish space (consisting of closed containers, that is, each finite point $d_i$ is represented by two points: one belonging to the container $B_i$ and another one to the container $B_{i+1}$). We notice that $\rho_\mathbb{X}(x^{(n)}, x) \to 0$ as $n \to \infty$ if and only if $|x^{(n)} - x| \to 0$ as $n \to \infty$ and the sequence $\{x^{(n)}\}_{n=N,N+1,\ldots}$ belongs to the same container as $x$ for a sufficiently large $N$. Thus, convergence on $\mathbb{X}$ in the metric $\rho_\mathbb{X}$ implies convergence in the Euclidean metric. In addition, if $x \neq d_i$ for all containers $i$, then $\rho_\mathbb{X}(x^{(n)}, x) \to 0$ as $n \to \infty$ if and only if $|x^{(n)} - x| \to 0$ as $n \to \infty$. Therefore, for any open set $B$ in $(\mathbb{X}, \rho_\mathbb{X})$, the set $B \setminus \cup_i \{d_i\}$ is open in $(\mathbb{X}, \rho)$. We notice that each container $B_i$ is an open and closed set in $(\mathbb{X}, \rho_\mathbb{X})$.

Observe that the state transition law $P$ given by (8.5) is weakly continuous in $(x, a) \in \mathbb{X} \times \mathbb{A}$. Indeed, let $B$ be an open set in $(\mathbb{X}, \rho_\mathbb{X})$ and $\rho_\mathbb{X}(x^{(n)}, x) \to 0$ and $|a^{(n)} - a| \to 0$ as $n \to \infty$. The set $B^o := B \setminus \cup_i \{d_i\} = B \cap \{x^{(n)}, x \in \mathbb{X} \mid |x^{(n)} - x| \to 0\}$ is open in $(\mathbb{X}, \rho)$. Since $F$ (as a function from $(\mathbb{X}, \rho_\mathbb{X}) \times (\mathbb{A}, \rho) \times (\mathbb{R}, \rho)$ into $(\mathbb{X}, \rho_\mathbb{X})$) is a continuous function in the both models, with backorders and lost sales, Fatou’s lemma yields

$$
\liminf_{n \to \infty} P(B^o|x^{(n)}, a^{(n)}) \geq \liminf_{n \to \infty} \int_\mathbb{R} I\{F(x^{(n)}, a^{(n)}, s) \in B^o\} dF_D(s) \geq \int_\mathbb{R} \liminf_{n \to \infty} I\{F(x, a, s) \in B^o\} dF_D(s) = P(B^o|x, a).
$$

Therefore, $\liminf_{n \to \infty} P(B|x^{(n)}, a^{(n)}) \geq P(B|x, a)$ because for the model with backorders $P(x^*|x', a') = 0$ for all $x^*, x', a' \in \mathbb{R}$ in view of the continuity of the distribution function $F_D$, and, for the model with lost sales, $P(x^*|x', a') = 0$ for any $x', a' \in \mathbb{R}$ and $x^* \neq 0$, and $P(0|x', a') = 1 - F_D(x' + a')$ is continuous in $(x', a') \in \mathbb{X} \times \mathbb{A}$. Since $B$ is an arbitrary open set in $(\mathbb{X}, \rho_\mathbb{X})$, the stochastic kernel $P$ on $\mathbb{X}$ given $\mathbb{X} \times \mathbb{A}$ is weakly continuous. Therefore, $\limsup_{n \to \infty} P(B|x^{(n)}, a^{(n)}) \leq P(B|x, a)$, for any closed set $B$ in $(\mathbb{X}, \rho_\mathbb{X})$. Since any container $B_i$ is simultaneously open and closed in $(\mathbb{X}, \rho_\mathbb{X})$, we have $P(B_i|x^{(n)}, a^{(n)}) \to P(B_i|x, a)$ as $n \to \infty$.

Set $\Psi(x) = x$, if the inventory level $x$ belongs to a transparent container, and $\Psi(x) = b_i$, if the inventory level belongs to a nontransparent container $B_i$ with a label $i$. As follows from the definition of the metric
\( \rho \), the function \( \Psi : (X, \rho) \to (Y, \rho) \) is continuous. Therefore, the observation kernels \( Q_0 \) and \( Q \) on \( Y \) given \( X \times X \), \( Q_0(c|x) := Q(C|a, x) := I\{\Psi(x) \in C\}, C \in B(Y), a \in A, x \in X \), are weakly continuous.

If all the containers are nontransparent, the observation set \( Y = Y_L \) is countable, and conditions of Corollary 3.8 hold. In particular, the function \( Q(b|x) = I\{x \in B_i\} \) is continuous, if the metric \( \rho \) is considered on \( X \). If some containers are transparent and some are not, the conditions of Corollary 7.2 hold. To verify this, we set \( Y_1 := T \) and \( Y_2 := Y_L \) and note that \( Y_2 \) is countable and the function \( Q(b|x) = I\{x \in B_i\} \) is continuous for each \( b_i \in Y_L \) because \( B_i \) is open and closed in \( (X, \rho) \). Note that \( H(B|z, a, y) = P(B|y, a) \) for any \( B \in B(X), C \in B(Y), z \in P(X), a \in A, y \in T \). The kernel \( H \) is weakly continuous on \( P(X) \times A \times Y_L \). In addition, \( T = \bigcup_i B_i^t \), where \( B_i^t \) are transparent containers, is an open set in \( (X, \rho) \). Thus, if either Assumption (D) or Assumption (P) holds, then POMDP \( (X, Y, A, P, Q, c) \) satisfies the assumptions of Corollary 7.2. Thus, for the corresponding COMDP, there are stationary optimal policies, optimal policies satisfy the optimality equations, and value iterations converge to the optimal value.

The models studied in Bensoussan et al. [4, 5, 7] correspond to the partition \( B_1 = (-\infty, 0] \) and \( B_2 = (0, +\infty) \) with the container \( B_2 \) being nontransparent and with the container \( B_1 \) being either nontransparent (backordered amounts are not known [7]) or transparent (models with lost sales [5], backorders are observable [4]). Note that, since \( F_D \) is atomless, the probability that \( x_t + a_t - D_t = 0 \) is 0, \( t = 1, 2, \ldots \).

The model provided in this subsection is applicable to other inventory control problems, and the conclusions of Corollary 7.2 hold for them too. For example, for problems with backorders, a nontransparent container \( B_0 = (-\infty, 0) \) and a transparent container \( B_1 = [0, +\infty) \) model a periodic review inventory control system for which nonnegative inventory levels are known, and, when the inventory level is negative, it is known that they are backorders, but their values are unknown.

### 8.3 Markov Decision Model with Incomplete Information

Consider a Markov decision model (MDM) with incomplete information; Dynkin and Yushkevich [12, Chapter 8], Rhenius [24], Yushkevich [33] (see also Bäuerle and Rieder [2] for a version of this model with transition probabilities having densities). This model is defined by an observed state space \( Y \), an unobserved state space \( W \), an action space \( A \), nonempty sets of available actions \( A(y) \), where \( y \in Y \), a stochastic kernel \( P \) on \( Y \times W \) given \( Y \times W \times A \), and a one-step cost function \( c : G \to \mathbb{R} \), where \( G = \{(y, w, a) : y \in Y, w \in W, a \in A(y)\} \) is the graph of the mapping \( A(y, w) = A(y), (y, w) \in Y \times W \). Assume that:

(i) \( Y, W \) and \( A \) are Borel subsets of Polish spaces. For all \( y \in Y \) a nonempty Borel subset \( A(y) \) of \( A \) represents the set of actions available at \( y \);

(ii) the graph of the mapping \( A : Y \to 2^A \), defined as \( \text{Gr}(A) = \{(y, a) : y \in Y, a \in A(y)\} \) is measurable, that is, \( \text{Gr}(A) \in B(Y \times A) \), and this graph allows a measurable selection, that is, there exists a measurable mapping \( \phi : Y \to A \) such that \( \phi(y) \in A(y) \) for all \( y \in Y \);

(iii) the transition kernel \( P \) on \( X \) given \( Y \times W \times A \) is weakly continuous in \( (y, w, a) \in Y \times W \times A \);

(iv) the one-step cost \( c \) is \( K \)-inf-compact on \( G \), that is, for each compact set \( K \subseteq Y \times W \) and for each
\( \lambda \in \mathbb{R} \), the set \( \mathcal{D}_{K,c}(\lambda) = \{(y, w, a) \in G : c(y, w, a) \leq \lambda \} \) is compact.

Let us define \( \mathcal{X} = \mathcal{Y} \times \mathcal{W} \), and for \( x = (y, w) \in \mathcal{X} \) let us define \( Q(C|x) = I\{y \in C\} \) for all \( C \in \mathcal{B}(\mathcal{Y}) \). Observe that this \( Q \) corresponds to the continuous function \( y = F(x) \), where \( F(y, w) = y \) for all \( x = (y, w) \in \mathcal{X} \) (here \( F \) is a projection of \( \mathcal{X} = \mathcal{Y} \times \mathcal{W} \) on \( \mathcal{Y} \)). Thus, as explained in Example 4.1 the stochastic kernel \( Q(dy|x) \) is weakly continuous in \( x \in \mathcal{X} \). Then by definition, an MDM is a POMDP with the state space \( \mathcal{X} \), observation set \( \mathcal{Y} \), action space \( \mathcal{A} \), available action sets \( A(y) \), transition kernel \( P \), observation kernel \( Q(dy|a, x) := Q(dy|x) \), and one-step cost function \( c \). However, this model differs from our basic definition of a POMDP because action sets \( A(y) \) depend on observations and one-step costs \( c(x, a) = c(y, w, a) \) are not defined when \( a \notin A(y) \). To avoid this difficulty, we set \( c(y, w, a) = +\infty \) when \( a \notin A(y) \). The extended function \( c \) is \( \mathcal{K} \)-inf-compact on \( \mathcal{X} \times \mathcal{A} \) because the set \( \mathcal{D}_{K,c}(\lambda) \) remains unchanged for each \( K \subseteq \mathcal{Y} \times \mathcal{W} \) and for each \( \lambda \in \mathbb{R} \).

Thus, an MDM is a special case of a POMDP \((\mathcal{X}, \mathcal{Y}, \mathcal{A}, P, Q, c)\), when \( \mathcal{X} = \mathcal{Y} \times \mathcal{W} \) and observation kernels \( Q \) and \( Q_0 \) are defined by the projection of \( \mathcal{X} \) on \( \mathcal{Y} \). The observation kernel \( Q(\cdot|\cdot|x) \) is weakly continuous in \( x \in \mathcal{X} \). As Example 4.1 demonstrates, in general this is not sufficient for the weak continuity of \( q \) and therefore for the existence of optimal policies. The following example confirms this conclusion for MDMs by demonstrating even the stronger assumption, that \( P \) is setwise continuous, is not sufficient for the weak continuity of the transition probability \( q \).

**Example 8.1.** Setwise continuity of a transition probability \( P \) on \( \mathcal{X} \) given \( \mathcal{X} \times \mathcal{A} \) for an MDM is not sufficient for the weak continuity of the transition probability \( q \) for the corresponding COMDP. Set \( \mathcal{W} = \{1, 2\}, \mathcal{Y} = [0, 1], \mathcal{X} = \mathcal{Y} \times \mathcal{W} \), and \( \mathcal{A} = \{0\} \cup \{\frac{1}{n} : n = 1, 2, \ldots\} \). Let \( m \) be the Lebesgue measure on \( \mathcal{Y} = [0, 1] \) and \( m^{(n)} \) be an absolutely continuous measure on \( \mathcal{Y} = [0, 1] \) with the density \( f^{(n)} \) defined in (4.1). As shown in Example 4.1, the sequence of probability measures \( \{m^{(n)}\}_{n=1,2,...} \) converges setwise to the Lebesgue measure \( m \) on \( \mathcal{Y} = [0, 1] \). Recall that \( Q(C|a, y, w) = I\{y \in C\} \) for \( C \in \mathcal{B}(\mathcal{Y}) \). In this example, the setwise continuous transition probability \( P \) is chosen to satisfy the following properties: \( P(B|y, w, a) = P(B|w, a) \) for all \( B \in \mathcal{B}(\mathcal{X}), y \in \mathcal{Y}, w \in \mathcal{W}, a \in \mathcal{A} \), that is, the transition probabilities do not depend on observable states, and \( P(\mathcal{X} \times \{w^1\}|w, a) = 0 \), when \( w^1 \neq w \) for all \( w, w^1 \in \mathcal{W}, a \in \mathcal{A} \), that is, the unobservable states do not change. For \( C \in \mathcal{B}(\mathcal{Y}), w \in \mathcal{W}, \) and \( a \in \mathcal{A} \), we set

\[
P(C \times \{w\}|w, a) = \begin{cases} 
m^{(n)}(C), & w = 2, a = \frac{1}{n}, n = 1, 2, \ldots; 
m(C), & \text{otherwise.} \end{cases}
\]

Fix \( z \in P(\mathcal{X}) \) defined by

\[
z(C \times \{w\}) = 0.5(I\{w = 1\} + I\{w = 2\})m(C), \quad w \in \mathcal{W}, \quad C \in \mathcal{B}(\mathcal{Y}).
\]

Direct calculations according to formulas (3.1)–(3.3) imply that for \( C, C' \in \mathcal{B}(\mathcal{Y}) \) and \( w \in \mathcal{W} \)

\[
R(C \times \{w\} \times C'|z, a) = \begin{cases} 
0.5m^{(n)}(C \cap C'), & \text{if } w = 2 \text{ and } a = \frac{1}{n}; 
0.5m(C \cap C'), & \text{otherwise}; 
\end{cases}
\]

which implies \( R'(C'|z, \frac{1}{n}) = 0.5(m(C') + m^{(n)}(C')) \), \( R'(C'|z, 0) = m(C') \), and therefore we can choose
For any nonempty open sets $O_1$ in $Y$ and $O_2$ in $W$ respectively, Theorem 5.2, with $S_1 = \mathbb{P}(X)$, $S_2 = X$, $S_3 = \mathbb{A}$, $\mathcal{O} = X$, $\Psi(B|z) = z(B)$, and $A_0 = \{ (x,a) \rightarrow P((O_1 \cap C) \times O_2)|x,a) : C \in \mathcal{B}(Y) \}$, implies the equicontinuity of the family of functions

$$R_{O_1 \times O_2} = \{ (z,a) \rightarrow R(O_1 \times O_2 \times C|z,a) : C \in \mathcal{B}(Y) \},$$
defined on $\mathbb{P}(X) \times A$, at all the points $(z, a) \in \mathbb{P}(X) \times A$. Being applied to $O_1 = \mathbb{Y}$ and $O_2 = \mathbb{W}$, this fact implies that the stochastic kernel $R'$ on $\mathbb{Y}$ given $\mathbb{P}(X) \times A$ is continuous in the total variation. In particular, the stochastic kernel $R'$ is setwise continuous.

Now, we show that Assumption (H) holds. Since the metric spaces $\mathbb{Y}$ and $\mathbb{W}$ are separable, there exist countable bases $\tau_b^Y$ and $\tau_b^W$ of the topologies for the separable metric spaces $\mathbb{Y}$ and $\mathbb{W}$, respectively. Then $\tau_b = \{O^Y \times O^W : O^Y \in \tau_b^Y, O^W \in \tau_b^W\}$ is a countable base of the topology of the separable metric space $X = \mathbb{Y} \times \mathbb{W}$. Therefore, Assumption (H) follows from Lemma 5.6, the equicontinuity of the family of functions $R_{O_1 \times O_2}$ for any open sets $O_1$ in $\mathbb{Y}$ and $O_2$ in $\mathbb{W}$, and the property that for any finite subset $N$ of $\{1, 2, \ldots\}$,

$$\bigcap_{j \in N} (O^Y_j \times O^W_j) = \left( \bigcap_{j \in N} O^Y_j \right) \times \left( \bigcap_{j \in N} O^W_j \right) \in \tau_b, \quad O^Y_j \in \tau_b^Y, O^W_j \in \tau_b^W \text{ for all } j \in N.$$  

\[\square\]

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