Detection of multipartite entanglement based on Heisenberg-Weyl representation of density matrices

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Abstract  We study entanglement and genuine entanglement of tripartite and four-partite quantum states by using Heisenberg-Weyl (HW) representation of density matrices. Based on the correlation tensors in HW representation, we present criteria to detect entanglement and genuine tripartite and four-partite entanglement. Detailed examples show that our method can detect more entangled states than previous criteria.

Keywords  Heisenberg-Weyl representation · Genuine entanglement · Correlation tensor

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1 Introduction

One of the most distinguished features of quantum mechanics is the quantum entanglement which has wide applications in various areas of quantum information processing such as quantum cryptography [1], quantum secure communication [2] and quantum channel protocols [3]. Many criteria have been given to detect quantum entanglement, such as positive partial transpose criterion [4], entanglement witness [5] and realignment of density matrices [6–9]. Genuine multipartite entanglement based on the lower bounds of concurrence was studied in [10]. In [11] inequalities for detecting quantum entanglement of bipartite quantum systems and necessary conditions of separability for mixed states were given. A witness test to detect genuine multipartite entanglement of bound entangled states was presented in [12]. New separability criteria for bipartite and multipartite quantum states based on the Bloch representation of density matrices have been derived in [13]. Separability of bipartite quantum systems was discussed in terms of Bloch representation in [14]. In [15], separability for different classes of quantum states in multipartite systems was studied.

While the Bloch representation of density matrices uses three kinds of generators of special unitary Lie group \( SU(d) \), the Heisenberg-Weyl (HW) representation has only one type of generators. The obvious conciseness of the HW representation suggests possibility to simplify some of the calculations related to density matrices. In [16], generalized Pauli matrices based on the HW operators were introduced and a criterion to detect entanglement by the bounds of the sum of expectation values of any set of anti-commuting observables was given. Moreover, separability criteria in terms of the HW observable basis for multipartite quantum systems were presented in [17].

In this paper, we study the genuine multipartite entanglement (GME) of tripartite and four-partite quantum states by exploiting the HW representation of density matrices. This paper is organized as follows. In Section 2, we use the correlation tensor to construct a special matrix to give a criterion of genuine tripartite entanglement according to the HW representation. An example is given to show that our result can detect more entanglement. In Section 3, a new criterion of genuine four-partite entanglement is presented. By a detailed example, our results are seen to outperform some previously available results. In Section 4, conclusions are given and results are summarized.

2 Detection of Genuine Tripartite Entanglement

We first consider the GME for tripartite quantum states. Let \( d \) be the dimension of the underlying space. The HW operators are defined as follows [15]:

\[
D(l, m) = \sum_{k=0}^{d-1} e^{\frac{2\pi i kl}{d}} |k\rangle \langle (k + m) \text{ mod } d|, \quad l, m = 0, 1, ..., d - 1.
\]

(1)

The HW observable basis is given by

\[
Q(l, m) = \mathcal{X} D(l, m) + \mathcal{X}^* D(l, m), \quad \mathcal{X} = \frac{(1 \pm i)}{2},
\]

(2)

which satisfies the following relations,

\[
\text{Tr}\{Q(l, m)Q(l', m')\} = d \delta_{l, l'} \delta_{m, m'}.
\]

(3)

When \( d = 2 \) the HW observable basis \( Q(0, 1) \), \( Q(1, 0) \) and \( Q(1, 1) \) give rise to the standard Pauli matrices.
Let $H^d_n$ ($n = 1, 2, 3$) be $d_n$-dimensional Hilbert spaces and $I$ the $d_n \times d_n$ identity matrix. Denote $\| \cdot \|$ the Hilbert-Schmidt norm (Frobenius norm) and $\| \cdot \|_r$ the trace norm defined by $\|A\|_r = \sum_1 \sigma_i = \text{Tr} \sqrt{A^t A}$, $A \in \mathbb{R}^{m \times n}$, where $\sigma_i$ ($i = 1, 2, \cdots, \min\{m, n\}$) are the singular values of the matrix $A$ arranged in descending order. A state $\rho \in H^d_1$ can be expressed as $\rho = \frac{1}{d_1} I + \frac{1}{2} \sum t_{lm} Q(l, m)$, where $t_{lm} = \frac{2}{d_1} \text{Tr} \{ \rho Q(l, m) \}$, $l, m = 0, ..., d_1 - 1$, $(l, m) \neq (0, 0)$. Let $T^{(1)}$ be the column vector with entries $t_{lm}$. For any pure state, it is known that $\|T^{(1)}\|^2 = \frac{4(d_1 - 1)}{d_1^2}$ [17].

A bipartite state $\rho \in \text{End}(H^d_1 \otimes H^d_2)$ can be expressed as

$$
\rho = \frac{1}{d_1 d_2} I_{d_1} \otimes I_{d_2} + \sum_{f=1}^2 \frac{d_f}{2d_1 d_2} \sum_{(l, m_f) \neq (0, 0)} t^{(f)}_{l m_f} Q^{(f)}(l_f, m_f) + \frac{1}{4} \sum_{(l, m_1), (l, m_2) \neq (0, 0)} t^{12}_{l_1 m_1 l_2 m_2} Q^1(l_1, m_1) Q^2(l_2, m_2),
$$

(4)

where $Q^1(l_f, m_f) = Q(l_f, m_f) \otimes I$, $Q^2(l_f, m_f) = I \otimes Q(l_f, m_f)$, $t^{(f)}_{l m_f} = \frac{2}{d_f} \text{Tr} \{ \rho Q^{(f)}(l_f, m_f) \}$. We have $\|T^{(f)}\|^2 = \sum_{(l, m_f) \neq (0, 0)} (t^{(f)}_{l m_f})^2$ and $\|T^{(12)}\|^2 = \sum_{(l, m_1), (l, m_2) \neq (0, 0)} (t^{12}_{l_1 m_1 l_2 m_2})^2$.

For any tripartite state $\rho \in \text{End}(H^d_1 \otimes H^d_2 \otimes H^d_3)$, we have

$$
\rho = \frac{1}{d_1 d_2 d_3} I_{d_1} \otimes I_{d_2} \otimes I_{d_3} + \sum_{f=1}^3 \frac{d_f}{2d_1 d_2 d_3} \sum_{(l, m_f, l_g, m_g) \neq (0, 0)} t^{(f g)}_{l_f m_f l_g m_g} Q^{(f g)}(l_f, m_f) Q^g(l_g, m_g) + \frac{1}{8} \sum_{(l, m_1), (l, m_2), (l, m_3) \neq (0, 0)} t^{123}_{l_1 m_1 l_2 m_2 l_3 m_3} Q^1(l_1, m_1) Q^2(l_2, m_2) Q^3(l_3, m_3),
$$

(5)

where $Q^{(f g)}(l_f, m_f)$ stands for that the operator $Q(l_f, m_f)$ acts on the $f$th space and identity acts on the remaining spaces, the coefficients $t^{(f g)}_{l_f m_f l_g m_g}$ and $t^{123}_{l_1 m_1 l_2 m_2 l_3 m_3}$ are defined similarly. Let $T^{(f)}$, $T^{(f g)}$ and $T^{(123)}$ be the column vectors with entries $t^{(f g)}_{l_f m_f l_g m_g}$ and $t^{123}_{l_1 m_1 l_2 m_2 l_3 m_3}$, respectively. We have

$$
\|T^{(f)}\|^2 = \sum_{(l, m_f) \neq (0, 0)} (t^{(f)}_{l m_f})^2, \quad \|T^{(f g)}\|^2 = \sum_{(l, m_f), (l, m_g) \neq (0, 0)} (t^{(f g)}_{l_f m_f l_g m_g})^2, \quad \|T^{(123)}\|^2 = \sum_{(l, m_1), (l, m_2), (l, m_3) \neq (0, 0)} (t^{123}_{l_1 m_1 l_2 m_2 l_3 m_3})^2.
$$

(6)

**Lemma 1** For any pure state $\rho = |\phi\rangle \langle \phi| \in H^d_1 \otimes H^d_2$ we have

$$
\|T^{(12)}\|^2 \leq \frac{16(d_2^2 - 1)}{d_1 d_2^2}.
$$
Proof Since $\rho$ is pure, we have $\frac{d_4}{4} \|T^{(1)}\|^2 - \frac{d_4}{4} \|T^{(2)}\|^2 = \frac{1}{d_2} - \frac{1}{d_3}$ by using $\text{Tr} \rho_1^2 = \text{Tr} \rho_2^2$, where $\rho_1$ and $\rho_2$ are the reduced density operators on $H_1^{d_1}$ and $H_2^{d_2}$, respectively. From (4) we have

$$
\text{Tr} \rho_2^2 = \frac{1}{d_1 d_2} + \frac{d_1}{4d_2} \sum_{(i_1, m_1) \neq (0,0)} t_{i_1, m_1} + \frac{d_2}{4d_1} \sum_{(i_2, m_2) \neq (0,0)} t_{i_2, m_2} \\
+ \frac{d_1 d_2}{16} \sum_{(i_1, m_1), (i_2, m_2) \neq (0,0)} t_{i_1, m_1} t_{i_2, m_2}
$$

(7)

Therefore,

$$
\|T^{(12)}\|^2 = \sum_{(i_1, m_1), (i_2, m_2) \neq (0,0)} t_{i_1, m_1} t_{i_2, m_2} \\
= \frac{16(d_2^2 - 1)}{d_1 d_2} - \frac{4(d_1 + d_2)}{d_1^2 d_2} \sum_{(i_2, m_2) \neq (0,0)} t_{i_2, m_2}^2
$$

(8)

Lemma 2 Let $\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes H_3^{d_3}$ be a pure state such that $d_1 d_j \geq d_k, i, j, k \in \{123, 132, 231\}$. Then

$$
\|T^{(123)}\|^2 \leq \frac{64[d_1 d_2 d_3 (d_1 d_2 d_3 + 2) - (d_1^2 d_2^2 + d_1^2 d_3^2 + d_2^2 d_3^2)]}{d_1^2 d_2^2 d_3^2}.
$$

(9)

Proof Since $\rho$ is a pure state, we have $\text{Tr}(\rho^2) = 1$ and $\text{Tr}(\rho_1^2) = \text{Tr}(\rho_2^2)$, where $\rho_i$, $(i_1 = 1, 2, 3)$ and $\rho_{23i}$ ($1 \leq i_2 < i_3 \leq 3$) are the reduced density operators of $\rho$ with respect to the subsystems $H_{d_1}$ and $H_{d_2} \otimes d_3$, respectively. Therefore,

$$
\|T^{(123)}\|^2 = \frac{64(d_1 d_2 d_3 - 1)}{d_1^2 d_2^2 d_3^2} - \frac{16}{d_1^2 d_2^2 d_3^2} \|T^{(1)}\|^2 + \frac{1}{d_1^2 d_2^2 d_3^2} \|T^{(2)}\|^2 + \frac{1}{d_1^2 d_2^2 d_3^2} \|T^{(3)}\|^2 \\
- \frac{4}{d_1^2} \|T^{(12)}\|^2 + \frac{1}{d_2^2} \|T^{(13)}\|^2 + \frac{1}{d_2^2} \|T^{(23)}\|^2 \\
\leq \frac{64[d_1 d_2 d_3 (d_1 d_2 d_3 + 2) - (d_1^2 d_2^2 + d_1^2 d_3^2 + d_2^2 d_3^2)]}{d_1^2 d_2^2 d_3^2} \\
- \frac{16}{d_1^2 d_2^2 d_3^2} [(d_2 d_3 - d_1) \cdot \|T^{(1)}\|^2 + (d_1 d_3 - d_2) \cdot \|T^{(2)}\|^2 \\
+ (d_1 d_2 - d_3) \cdot \|T^{(3)}\|^2] \\
\leq \frac{64[d_1 d_2 d_3 (d_1 d_2 d_3 + 2) - (d_1^2 d_2^2 + d_1^2 d_3^2 + d_2^2 d_3^2)]}{d_1^2 d_2^2 d_3^2}.
$$

(10)
If a tripartite state $\rho \in H_{d_1} \otimes H_{d_2} \otimes H_{d_3}$ is separable under the bipartition $f|gh$ ($f \neq g \neq h \in \{1, 2, 3\}$), $\rho$ can be written as
\[
\rho = \sum_l p_l \rho_l^f \otimes \rho_l^{gh}, \quad p_l > 0, \quad \sum_l p_l = 1,
\]
where
\[
\rho_l^f = \frac{1}{d_f} I_{d_f} + \frac{1}{2} \sum_{(l_f, m_f) \neq (0,0)} t_{l_f m_f}^f \mathcal{Q}^f(l_f, m_f),
\]
\[
\rho_l^{gh} = \frac{1}{d_g d_h} I_{d_g} \otimes I_{d_h} + \frac{1}{2 d_h} \sum_{(l_g, m_g) \neq (0,0)} t_{l_g m_g}^{gh} \mathcal{Q}^g(l_g, m_g) \otimes I_{d_h}
\]
\[
+ \frac{1}{2 d_g} \sum_{(l_g, m_g) \neq (0,0)} t_{l_g m_g}^{gh} I_{d_g} \otimes \mathcal{Q}^h(l_g, m_g)
\]
\[
+ \frac{1}{4} \sum_{(l_g, m_g), (l_h, m_h) \neq (0,0)} t_{l_g m_g l_h m_h}^{gh} \mathcal{Q}^g(l_g, m_g) \otimes \mathcal{Q}^h(l_h, m_h).
\]

If $\rho$ is separable under the tripartition $f|gh$, then $\rho$ is of the form
\[
\rho = \sum_l p_l \rho_l^f \otimes \rho_l^{gh} \otimes \rho_l^h, \quad p_l > 0, \quad \sum_l p_l = 1,
\]
where $\rho_l^f$ is of the form (12) and
\[
\rho_l^g = \frac{1}{d_g} I_{d_g} + \frac{1}{2} \sum_{(l_g, m_g) \neq (0,0)} t_{l_g m_g}^g \mathcal{Q}^g(l_g, m_g),
\]
\[
\rho_l^h = \frac{1}{d_h} I_{d_h} + \frac{1}{2} \sum_{(l_h, m_h) \neq (0,0)} t_{l_h m_h}^h \mathcal{Q}^h(l_h, m_h).
\]

We first consider the separability of $\rho$ under bipartition $f|gh$. Set
\[
S(\rho_{f|gh}) = \begin{pmatrix} 2 T(g)^t & (T^{(gh)})^t \\ 2 T(f) & T(fg) \end{pmatrix}.
\]

We have the following result.

**Theorem 1** If a tripartite quantum state $\rho \in H_{d_1} \otimes H_{d_2} \otimes H_{d_3}$ is separable under the bipartition $f|gh$, then
\[
\|S(\rho_{f|gh})\|_{tr} \leq \frac{2}{d_f d_g d_h} \sqrt{(d_f^2 + 4d_f - 4)(d_g^2 d_h^2 + d_g^2 d_h^3 - d_g d_h^3 + 4d_g^2 d_h^2 - 4d_g^2 d_h^3) \over d_g d_h}.
\]
Proof. By (11), (12) and (13), we have
\[
T(f) = \sum_ipiv_i^f, \quad T(g) = \sum_ipiv_i^g, \quad T(h) = \sum_ipiv_i^h,
\]
\[
T(fg) = \sum_ipiv_i^f(v_i^g)^t, \quad T(fh) = \sum_ipiv_i^f(v_i^h)^t.
\] (19)

Applying the relations \(\|A + B\|_r \leq \|A\|_r + \|B\|_r\) and \(\|a\rangle\langle b\|_r = \||a\rangle\langle b\|\), we get
\[
\|S(\rho_{fg|h})\|_r = \|\sum_ipi\left(\frac{1}{v_i^f}\right) (2(v_i^g)^t (v_i^h)^t)\|_r
\]
\[
\leq \sum_ipi\left(\frac{1}{v_i^f}\right) (2(v_i^g)^t (v_i^h)^t) \|_{tr}
\]
\[
= \sum_ipi\left(\frac{1}{v_i^f}\right) \| (2(v_i^g)^t (v_i^h)^t) \|
\]
\[
= \sum_ipi\sqrt{1 + \|v_i^f\|^2} \sqrt{4 + \|v_i^g\|^2 + \|v_i^h\|^2}
\]
\[
\leq \frac{2}{d_fd_gd_h} \sqrt{(d_f^2 + 4d_f - 4)(d_g^2d_h^2 + d_gd_h + d_f^2 - 2d_f - 2)},
\]
where the second inequality is due to the Lemma 1 in [17] and the above Lemma 1.

Theorem 2. If a tripartite quantum state \(\rho \in H_1^d \otimes H_2^d \otimes H_3^d\) is (fully) separable under the tripartition \(f\mid g\mid h\), then
\[
\|S(\rho_{fg|h})\|_r \leq \frac{2}{d_fd_gd_h} \sqrt{(d_f^2 + 4d_f - 4)(d_g^2d_h^2 + d_gd_h + d_f^2 - 2d_f - 2)},
\] (20)

where
\[
S(\rho_{fg|h}) = \left(\frac{2}{2T(f)} \frac{T(g)}{T(fg)} \frac{T(h)}{T(fh)}\right).
\] (21)

Proof. By (12), (14), (15) and (16), we have
\[
T(f) = \sum_ipiv_i^f, \quad T(g) = \sum_ipiv_i^g, \quad T(h) = \sum_ipiv_i^h,
\]
\[
T(fg) = \sum_ipiv_i^f(v_i^g)^t, \quad T(fh) = \sum_ipiv_i^f(v_i^h)^t.
\] (22)

Then \(S(\rho_{fg|h})\) can be expressed as follows:
\[
S(\rho_{fg|h}) = \sum_ipi\left(\frac{1}{v_i^f}\right) (2(v_i^g)^t (v_i^h)^t)
\]
\[
= \sum_ipi\left(\frac{1}{v_i^f}\right) (2(v_i^g)^t (v_i^h)^t),
\] (23)
We have the following theorem.

\[ \|S(\rho_{fgh})\|_{tr} \leq \sum_{i} p_i \| \left( \frac{1}{v_i^f} \right) (2(v_i^q)^t (v_i^h)^t) \|_{tr} \]
\[ = \sum_{i} p_i \left( \frac{1}{v_i^f} \right) \| (2(v_i^q)^t (v_i^h)^t) \| \]
\[ = \sum_{i} p_i \sqrt{1 + \|v_i^f\|^2} \sqrt{4 + \|v_i^q\|^2 + \|v_i^h\|^2} \]
\[ = \frac{2}{d_fd_gd_h} \sqrt{(d_f^2 + 4d_f - 4)S_{d_f^2d_g^2d_h^2}} \]
from which we obtain
\[ \|S(\rho_{fgh})\|_{tr} \leq \sum_{i} p_i \| \left( \frac{1}{v_i^f} \right) (2(v_i^q)^t (v_i^h)^t) \|_{tr} \]
\[ = \sum_{i} p_i \left( \frac{1}{v_i^f} \right) \| (2(v_i^q)^t (v_i^h)^t) \| \]
\[ = \sum_{i} p_i \sqrt{1 + \|v_i^f\|^2} \sqrt{4 + \|v_i^q\|^2 + \|v_i^h\|^2} \]
\[ = \frac{2}{d_fd_gd_h} \sqrt{(d_f^2 + 4d_f - 4)S_{d_f^2d_g^2d_h^2}} \]
where we have used formulae \( A + B \leq \|A\|_{tr} + \|B\|_{tr} \) and \( \|a\|_{tr} ||b||_{tr} = \|a\| \cdot ||b|| \).

Next we consider the genuine tripartite entanglement. For any biseparable mixture state can be written as \( \rho = \sum q_i \rho_i^1 \otimes \rho_i^2 \otimes \rho_i^3 + s_i \rho_i^1 \otimes \rho_i^2 \otimes \rho_i^3, \) \( q_i \geq 0, r_i \geq 0, s_i \geq 0. \) A state is said to be genuine multipartite entangled if it can not be expressed as the convex combination of biseparable states. Let \( M_1(\rho) = \frac{1}{3}[\|S(\rho_{123})\|_{tr} + \|S(\rho_{213})\|_{tr} + \|S(\rho_{312})\|_{tr}] \) and

\[ M_1 = \max \left\{ \frac{2}{d_fd_gd_h} \sqrt{(d_f^2 + 4d_f - 4)S_{d_f^2d_g^2d_h^2}}, \frac{2}{d_fd_gd_h} \sqrt{(d_g^2 + 4d_g - 4)S_{d_g^2d_f^2d_h^2}}, \frac{2}{d_fd_gd_h} \sqrt{(d_h^2 + 4d_h - 4)S_{d_h^2d_f^2d_g^2}} \right\}. \]

We have the following theorem.

**Theorem 3** A mixed state \( \rho \in H_1^d \otimes H_2^d \otimes H_3^d \) is genuine tripartite entangled if \( M_1(\rho) > M_1. \)

**Proof**

\[ M_1(\rho) = \frac{1}{3}[\|S(\rho_{123})\|_{tr} + \|S(\rho_{213})\|_{tr} + \|S(\rho_{312})\|_{tr}] \]
\[ \leq \frac{1}{3} \left( \sum q_i\|S(\rho_i)_{123}\|_{tr} + \sum r_i\|S(\rho_i)_{213}\|_{tr} + \sum s_i\|S(\rho_i)_{312}\|_{tr} \right) \]
\[ \leq \frac{1}{3} (M_1 + M_1 + M_1) \]
\[ = M_1. \]

Consequently, if \( M_1(\rho) > M_1, \) \( \rho \) is genuine tripartite entangled.

We consider a special quantum state, if a density matrix is permutational invariant, i.e. \( \rho^p = ppm^t, \) where the \( p \) denotes the any permutation of the qudits. Then any biseparable permutational invariant state can be written as \( \rho = \sum q_i \rho_i^1 \otimes \rho_i^2 \otimes \rho_i^3 + r_i \rho_i^2 \otimes \rho_i^1 \otimes \rho_i^3 + s_i \rho_i^3 \otimes \rho_i^2, \) where \( q_i, r_i, s_i \) are all non-zero. We can get the following corollary:
Corollary 1 If a density matrix is permutational invariant, then
\[
M_1(\rho) = \frac{1}{3}\left[\|S(\rho_{123})\|_{tr} + \|S(\rho_{213})\|_{tr} + \|S(\rho_{312})\|_{tr}\right] \leq J_1.
\]
Thus, if \(M_1(\rho) > J_1\), \(\rho\) is a genuinely entangled tripartite state, where
\[
J_1 = \frac{2}{d_1 d_2 d_3} \sqrt{\frac{(d_1^2 + 4d_1 - 4)(d_2^2 d_3^2 + d_2^2 d_3^2 - d_2^2 d_3^2 + 4d_2^2 d_3^2 - 4d_2^2)}{d_2 d_3}}
\]
\[
+ \frac{2}{d_2 d_1 d_3} \sqrt{\frac{(d_2^2 + 4d_2 - 4)(d_1^2 d_3^2 + d_1^2 d_3^2 - d_1^2 d_3^2 + 4d_1^2 d_3^2 - 4d_1^2)}{d_1 d_3}}
\]
\[
+ \frac{2}{d_3 d_1 d_2} \sqrt{\frac{(d_3^2 + 4d_3 - 4)(d_1^2 d_2^2 + d_1^2 d_2^2 - d_1^2 d_2^2 + 4d_1^2 d_2^2 - 4d_1^2)}{d_1 d_2}}.
\]

Example 1 Consider \(2 \times 2 \times 2 \) quantum state
\[
\rho = \frac{x}{8} I_8 + (1 - x)|\text{GHZ}\rangle \langle \text{GHZ}|, \quad 0 \leq x \leq 1,
\]
where \(|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\) and \(I_8\) is the \(8 \times 8\) identity matrix. By Theorem 1 when \(d = 2\) we have \(f_1(x) = \|S(\rho f_{gh})\|_{tr} - 4 = (2\sqrt{2} + 1)(1 - x) + \sqrt{(x - 1)^2 + 4} - 4\). When \(f_1(x) > 0\), \(\rho\) is not separable under the bipartition \(f_{gh}\) for \(0 \leq x < 0.4941\). According to the Theorem 1 in [18], when \(f_2(x) = -3x + 4 - 2\sqrt{3} > 0\), \(\rho\) is not separable under bipartition \(f_{gh}\) for \(0 \leq x < 0.179\). This shows that our theorem detects more entanglement in such partition, see Fig. 1.

From Corollary 1, we have that if \(f_3(x) = (2\sqrt{2} + 1)(1 - x) + \sqrt{(x - 1)^2 + 4} - 4 > 0\), then \(\rho\) is genuine tripartite entangled for \(0 \leq x < 0.4941\). From the lower bound of concurrence the authors in [19] show that \(\rho\) is genuine tripartite entangled for \(0 \leq x < 0.08349\). Our theorem can detect more genuine multipartite entanglement of the state.
3 Detection of Genuine Four-party Entanglement

Next we consider genuine entanglement of four-party quantum states. A four-party state \( \rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes H_3^{d_3} \otimes H_4^{d_4} \) can be written as

\[
\rho = \frac{1}{d_1 d_2 d_3 d_4} I \otimes I \otimes I + \sum_{f=1}^{4} \frac{d_f}{2d_1 d_2 d_3 d_4} \sum_{l_f m_f} t_{l_f m_f}^f Q^f(l_f, m_f) \\
+ \sum_{1 \leq f < g \leq 4} \frac{d_f d_g}{4d_1 d_2 d_3 d_4} \sum_{l_f m_f, l_g m_g} t_{l_f m_f l_g m_g}^{fg} Q^f(l_f, m_f) Q^g(l_g, m_g) \\
+ \sum_{1 \leq f < g < h \leq 4} \frac{d_f d_g d_h}{8d_1 d_2 d_3 d_4} \sum_{l_f m_f, l_g m_g, l_h m_h} t_{l_f m_f l_g m_g l_h m_h}^{fg h} Q^f(l_f, m_f) Q^g(l_g, m_g) Q^h(l_h, m_h) \\
+ \frac{1}{16} \sum_{l_1 m_1 l_2 m_2 l_3 m_3 l_4 m_4} t_{l_1 m_1 l_2 m_2 l_3 m_3 l_4 m_4}^{1234} Q^1(l_1, m_1) Q^2(l_2, m_2) Q^3(l_3, m_3) Q^4(l_4, m_4),
\]

where \((l_a, m_a) \neq (0, 0)\), \(a \in \{1, 2, 3, 4\}\), \(Q^f(l_f, m_f)\), \(Q^g(l_g, m_g)\) and \(Q^h(l_h, m_h)\) are operators acting on the spaces \(H_{f_{a}}\), \(H_{g_{a}}\) and \(H_{h_{a}}\), respectively. The coefficients are given by \(t_{l_f m_f}^f = \frac{1}{d_f} \text{Tr}(\rho Q^f(l_f, m_f))\), \(t_{l_f m_f l_g m_g}^{fg} = \frac{4}{d_f d_g} \text{Tr}(\rho Q^f(l_f, m_f) Q^g(l_g, m_g))\), \(t_{l_f m_f l_g m_g l_h m_h}^{fg h} = \frac{8}{d_f d_g d_h} \text{Tr}(\rho Q^f(l_f, m_f) Q^g(l_g, m_g) Q^h(l_h, m_h))\), \(t_{l_1 m_1 l_2 m_2 l_3 m_3 l_4 m_4}^{1234} = \frac{6}{d_1 d_2 d_3 d_4} \text{Tr}(\rho Q^1(l_1, m_1) Q^2(l_2, m_2) Q^3(l_3, m_3) Q^4(l_4, m_4))\). Denote \(T(f)\), \(T(fg)\), \(T(fgh)\) and \(T(1234)\) the column vectors given by \(t_{l_f m_f}^f\), \(t_{l_f m_f l_g m_g}^{fg}\), \(t_{l_f m_f l_g m_g l_h m_h}^{fg h}\) and \(t_{l_1 m_1 l_2 m_2 l_3 m_3 l_4 m_4}^{1234}\), respectively. We have

\[
\|T(f)\|^2 = \sum (t_{l_f m_f}^f)^2,
\]

\[
\|T(fg)\|^2 = \sum (t_{l_f m_f l_g m_g}^{fg})^2,
\]

\[
\|T(fgh)\|^2 = \sum (t_{l_f m_f l_g m_g l_h m_h}^{fg h})^2,
\]

\[
\|T(1234)\|^2 = \sum (t_{l_1 m_1 l_2 m_2 l_3 m_3 l_4 m_4}^{1234})^2,
\]

where \((l_a, m_a) \neq (0, 0)\) and \(a \in \{1, 2, 3, 4\}\).

For the four-party quantum state \( \rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes H_3^{d_3} \otimes H_4^{d_4} \), we denote the bipartitions and tripartitions as follows: \( f \mid ghe \), \( fg \mid he \) and \( f \mid ghe \), where \( f, g, h \) and \( e \) are not equal to each other, and take from the set \( \{1, 2, 3, 4\}\). If \( \rho \) is separable under the bipartition \( f \mid ghe \), we have

\[
\rho_{f \mid ghe} = \sum_{l_f} p_l \rho_l^f \otimes \rho_l^{(ghe)}, \quad p_l > 0, \quad \sum_l p_l = 1,
\]

where

\[
\rho_l^f = \frac{1}{d_f} I_{d_f} + \frac{1}{2} \sum_{(l_f, m_f) \neq (0, 0)} t_{l_f m_f}^f Q^f(l_f, m_f),
\]
\begin{align*}
\rho^{\text{he}} = & \frac{1}{d_g d_h d_e} I_{d_g} \otimes I_{d_h} \otimes I_{d_e} + \frac{3}{2d_g d_h d_e} \sum_{g=1}^3 \frac{d_g}{(l_g, m_g) \neq (0,0)} t_{l_g m_g}^q Q^q(l_g, m_g) \otimes I_{d_h} \otimes I_{d_e} \\
& + \sum_{1 \leq g < h \leq 3} \frac{d_g d_h}{4d_g d_h d_e} \sum_{(l_g, m_g), (l_h, m_h) \neq (0,0)} t_{l_g m_g l_h m_h}^q Q^q(l_g, m_g) \otimes Q^q(l_h, m_h) \otimes I_{d_e} \\
& + \frac{1}{8} \sum_{(l_g, m_g), (l_h, m_h), (l_e, m_e) \neq (0,0)} t_{l_g m_g l_h m_h l_e m_e}^q Q^q(l_g, m_g) \otimes Q^q(l_h, m_h) \otimes Q^e(l_e, m_e). \\
\end{align*}

If \( \rho \) is separable under the bipartition \( fg|he \), we have

\[ \rho_{fg|he} = \sum_i p_i \rho_i^{(fg)} \otimes \rho_i^{(he)}, \quad p_i > 0, \quad \sum_i p_i = 1, \tag{31} \]

where

\begin{align*}
\rho_i^{(fg)} &= \frac{1}{d_f d_g} I_{d_f} \otimes I_{d_g} + \frac{1}{2d_f} \sum_{(l_f, m_f) \neq (0,0)} t_{l_f m_f}^l Q^l(l_f, m_f) \otimes I_{d_g} \\
& + \frac{1}{2d_f} \sum_{(l_g, m_g) \neq (0,0)} t_{l_g m_g}^l I_{d_g} \otimes Q^l(l_g, m_g) \\
& + \frac{1}{4} \sum_{(l_f, m_f), (l_g, m_g) \neq (0,0)} t_{l_f m_f l_g m_g}^l Q^l(l_f, m_f) \otimes Q^l(l_g, m_g), \\
\rho_i^{(he)} &= \frac{1}{d_h d_e} I_{d_h} \otimes I_{d_e} + \frac{1}{2d_e} \sum_{(l_h, m_h) \neq (0,0)} t_{l_h m_h}^h Q^h(l_h, m_h) \otimes I_{d_e} \\
& + \frac{1}{2d_h} \sum_{(l_e, m_e) \neq (0,0)} t_{l_e m_e}^h I_{d_e} \otimes Q^h(l_e, m_e) \\
& + \frac{1}{4} \sum_{(l_h, m_h), (l_e, m_e) \neq (0,0)} t_{l_h m_h l_e m_e}^h Q^h(l_h, m_h) \otimes Q^e(l_e, m_e). \\
\end{align*}

If \( \rho \) is separable under the tripartitions \( fg|he \), we have

\[ \rho_{fg|he} = \sum_i p_i \rho_i^{(f)} \otimes \rho_i^{(g)} \otimes \rho_i^{(he)}, \quad p_i > 0, \quad \sum_i p_i = 1, \tag{34} \]

where \( \rho_i^{(f)} \) and \( \rho_i^{(he)} \) are given in (29) and (33), respectively,

\[ \rho_i^q = \frac{1}{d_g} I_{d_g} + \frac{1}{2} \sum_{(l_g, m_g) \neq (0,0)} t_{l_g m_g}^q Q^q(l_g, m_g). \tag{35} \]

**Theorem 4** If \( \rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes H_3^{d_3} \otimes H_4^{d_4} \) is separable under the bipartition \( fg|he \) such that \( d_i d_j \geq d_k, ijk \in \{123, 132, 231\} \), then

\[ \| S(\rho_{fg|he}) \|_\tau \leq \sqrt{\frac{4d_2^2 + 4d_3 - 4}{d_2^2} \cdot \frac{4d_1^2 + 4d_2 + 4}{d_1^2} \cdot \frac{4d_3^2 + 4d_4 + 4}{d_3^2} \cdot \frac{4d_4^2 + 4d_1 + 4}{d_4^2}} + \frac{64[d_g d_h d_c (d_g d_h d_c + 2) - (d_g^2 d_h^2 + d_h^2 d_c^2 + d_c^2 d_g^2)]}{d_g^2 d_h^2 d_c^2}. \tag{36} \]
where

\[ S(\rho_{fg|he}) = \left( \frac{2}{2T_{(f)}} \right)^{(T^{(h)})^t} \left( \frac{2}{T_{(f)}} \right)^{(T^{(he)})^t} \cdot (37) \]

**Proof** If a four-partite mixed state \( \rho \) is separable under bipartition \( fg|he \), by (28), (29), and (30) we have

\[ T^{(f)} = \sum_l p_l v^f_l, \quad T^{(h)} = \sum_l p_l v^h_l, \quad T^{(he)} = \sum_l p_l v^{he}_l, \quad T^{(fghe)} = \sum_l p_l v^{fghe}_l. \]

Using Lemma 1 in [17] and our Lemma 2 we have

\[ ||S(\rho_{fg|he})||_{tr} = || \sum_l p_l \left( \frac{2}{2T_{(f)}} (v^h_l)^t (v^{he}_l)^t \right) ||_{tr} \]

\[ = || \sum_l p_l \left( \frac{1}{v^f_l} \right) (2 (v^h_l)^t (v^{he}_l)^t) ||_{tr} \]

\[ \leq \sum_l p_l \left( \frac{1}{v^f_l} \right) (2 (v^h_l)^t (v^{he}_l)^t) ||_{tr} \]

\[ = \sum_l p_l \left( \frac{1}{v^f_l} \right) \cdot || (2 (v^h_l)^t (v^{he}_l)^t) || \]

\[ = \sum_l p_l \sqrt{1 + \frac{||v^f_l||^2}{4 + ||v^h_l||^2 + ||v^{he}_l||^2}} \]

\[ \leq \sqrt{\frac{4d^2_h + 4d_h - 4}{d^2_h}} + \frac{64[d_g d_h d_c (d_g d_h d_c + 2) - (d^2_g d^2_h d^2_c + d^2_g d^2_h d^2_c + d^2_g d^2_h d^2_c)]}{d^3_g d^3_h d^3_c} \cdot \sqrt{\frac{d^2_f + 4d_f - 4}{d^2_f}}. \]

\[ \square \]

**Theorem 5** If the quantum state \( \rho \in H^d_1 \otimes H^d_2 \otimes H^d_3 \otimes H^d_4 \) is separable under partition \( fg|he \), then

\[ ||S(\rho_{fg|he})||_{tr} \leq \sqrt{1 + \frac{16(d^2_g - 1)}{d^2_f d^2_g}} \sqrt{4 + \frac{4(d_h - 1)}{d^2_h}} + \frac{16(d^2_c - 1)}{d^2_n d^2_c}. \]

(40)

**Proof** If \( \rho \) is separable under partition \( fg|he \), according (31), (32), and (33) we have

\[ T^{(fg)} = \sum_l p_l v^{fg}_l, \quad T^{(h)} = \sum_l p_l v^h_l, \quad T^{(he)} = \sum_l p_l v^{he}_l, \]

\[ T^{(fghe)} = \sum_l p_l v^{fghe}_l. \]

(42)
Thus, $S(\rho_{fg|he})$ can be written as,

$$
S(\rho_{fg|he}) = \sum_l p_l \left( \frac{2}{2v_l^g} \left( v_l^h \right)^t \left( v_l^{he} \right)^t \right)
= \sum_l p_l \left( \frac{1}{v_l^g} \left( 2 \left( v_l^h \right)^t \left( v_l^{he} \right)^t \right) \right).
$$

(43)

Thus,

$$
\|S(\rho_{fg|he})\|_{tr} = \| \sum_l p_l \left( \frac{2}{2v_l^g} \left( v_l^h \right)^t \left( v_l^{he} \right)^t \right) \|_{tr}
\leq \sum_l p_l \left( \frac{1}{v_l^g} \left( 2 \left( v_l^h \right)^t \left( v_l^{he} \right)^t \right) \right) \|_{tr}
= \sum_l p_l \left( \frac{1}{v_l^g} \| \cdot \| \left( 2 \left( v_l^h \right)^t \left( v_l^{he} \right)^t \right) \right) \|
= \sum_l p_l \sqrt{1 + \|v_l^g\|^2} \sqrt{1 + \|v_l^h\|^2 + \|v_l^{he}\|^2}
\leq \sqrt{1 + \frac{16(d^2_3 - 1)}{d^2_f d^2_g}} \sqrt{1 + \frac{4(d_h - 1)}{d^2_h}} + \frac{16(d^2_3 - 1)}{d_h d^2_e}.
$$

(44)

For the tripartition $f|g|he$, we introduce

$$
S(\rho_{f|g|he}) = \begin{pmatrix}
(T^{(g)})^t & (T^{(gh)})^t & (T^{(ghe)})^t \\
T^{(fs)} & T^{(fgh)} & T^{(f|g|he)}
\end{pmatrix}.
$$

(45)

**Theorem 6** If the quantum state $\rho \in H^d_1 \otimes H^d_2 \otimes H^d_3 \otimes H^d_4$ is separable under tripartition $f|g|he$, then

$$
\|S(\rho_{f|g|he})\|_{tr} \leq \sqrt{1 + \frac{4(d_f - 1)}{d^2_f}} \sqrt{1 + \frac{4(d_h - 1)}{d^2_h}} + \frac{16(d^2_3 - 1)}{d_h d^2_e} \sqrt{\frac{4(d_g - 1)}{d^2_g}}.
$$

(46)

**Proof** If a four-partite mixed state $\rho$ is separable under tripartition $f|g|he$, by (29), (33), (34 and 35) we have

$$
T^{(g)} = \sum_l p_l v_l^g, \quad T^{(gh)} = \sum_l p_l v_l^g \otimes v_l^h, \quad T^{(ghe)} = \sum_l p_l v_l^g \otimes (v_l^{he})^t, \quad T^{(fs)} = \sum_l p_l v_l^f (v_l^g)^t,
T^{(fgh)} = \sum_l p_l v_l^f \otimes (v_l^g \otimes v_l^h)^t, \quad T^{(f|g|he)} = \sum_l p_l v_l^f \otimes (v_l^g \otimes v_l^{he})^t.
$$

(47)
Thus,
\[
\|S(\rho_{f|ghe})\|_{tr} = \| \sum_{i} p_i \left( \left( v_i^g \right)^t v_i^f \left( v_i^g \otimes v_i^h \right)^t v_i^f \left( v_i^g \otimes v_i^h \right)^t \right) \|_{tr}
\]
\[
\leq \sum_{i} p_i \left( \left( \frac{1}{v_i^f} \right)^t (v_i^h)^t (v_i^g)^t \otimes (v_i^g)^t \right) \|_{tr}
\]
\[
= \sum_{i} p_i \left( \left( \frac{1}{v_i^f} \right)^t \| (v_i^h)^t (v_i^g)^t \| \cdot \| v_i^g \|
\]
\[
= \sum_{i} p_i \sqrt{1 + \| v_i^f \|^2 \sqrt{1 + \| v_i^h \|^2 + \| v_i^{he} \|^2}} \sqrt{\frac{4(d_f - 1)}{d_f^2} + \frac{4(d_h - 1)}{d_h^2} + \frac{16(d_e^2 - 1)}{d_e^2} \frac{4(d_g - 1)}{d_g^2}}.
\]

To deal with the genuine entanglement of four-partite quantum states, we consider the trace norm of \( S \) summing over all possible bipartitions \( f|ghe \) and \( fg|he \), define \( M_2(\rho) = \frac{1}{16} \|S(\rho_{1|234})\|_{tr} + \cdots + \|S(\rho_{12|34})\|_{tr} \). Set

\[
M_2 = \text{Max}\left\{ \sqrt{\frac{d_f^2 + 4d_f - 4}{d_f^2}} \sqrt{\frac{4d_h^2 + 4d_h - 4}{d_h^2}} + \frac{64[d_g d_h d_e(d_g d_h d_e + 2) - (d_g^2 d_h^2 + d_h^2 d_e^2 + d_e^2 d_g^2)]}{d_g^2 d_h^2 d_e^2}, \right. \\
\left. \sqrt{1 + \frac{16(d_g^2 - 1)}{d_f d_g^3}} \sqrt{4 + \frac{4(d_h - 1)}{d_h^2} + \frac{16(d_e^2 - 1)}{d_e^2}} \right\},
\]

(48)

where \( f, g, h \) and \( e \) are not equal to each other, and take from the set \( \{1, 2, 3, 4\} \). Using the similar method to Theorem 3, we have

Theorem 7 For any four-partite state \( \rho \in H_1^d \otimes H_2^d \otimes H_3^d \otimes H_4^d \) such that \( d_id_j \geq d_e, \ ij \in \{123, 132, 231\} \), if \( M_2(\rho) > M_2 \) then \( \rho \) is genuine four-partite entangled.

If the density matrix is permutational invariant, we can get the following corollary:

Corollary 2 If a density matrix is permutational invariant, then \( M_2(\rho) \leq J_2 \), if \( M_2(\rho) > J_2 \), \( \rho \) is a genuinely entangled tripartite state, where

\[
J_2 = \frac{1}{16} \sum \sqrt{\frac{d_f^2 + 4d_f - 4}{d_f^2}} \sqrt{\frac{4d_h^2 + 4d_h - 4}{d_h^2}} + \frac{64[d_g d_h d_e(d_g d_h d_e + 2) - (d_g^2 d_h^2 + d_h^2 d_e^2 + d_e^2 d_g^2)]}{d_g^2 d_h^2 d_e^2}
\]
\[
+ \sum \left( \sqrt{1 + \frac{16(d_g^2 - 1)}{d_f d_g^3}} \sqrt{4 + \frac{4(d_h - 1)}{d_h^2} + \frac{16(d_e^2 - 1)}{d_e^2}} \right),
\]

the first \( \sum \) sums over all possible bipartitions \( f|ghe \), the second \( \sum \) sums over all possible bipartitions \( fg|he \).
Example 2 Consider the four-qubit quantum state $\rho \in H_1^2 \otimes H_2^2 \otimes H_3^2 \otimes H_4^2$, 

$$\rho = x|\psi\rangle\langle\psi| + \frac{1-x}{16}I_{16}, \quad (49)$$

where $|\psi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$ and $I_{16}$ is the $16 \times 16$ identity matrix.

For bipartition $f|ghe$, we have $f_4(x) = \|S(\rho_f|ghe)\|_{tr} - \sqrt{18} = (4+\sqrt{2})x+2-\sqrt{18} > 0$ by Theorem 4. We also get $f_5(x) = \|S(\rho_f|ghe)\|_{tr} - \sqrt{10} = (\sqrt{2}+5)x-\sqrt{10}$ under tripartition $f|ghe$ by Theorem 6. By Theorem 3 of Ref. [20], $f_6(x) = 9x^2 - 4$ and $f_7(x) = 9x^2 - 3$ for $f|ghe$ and $f|ghe$ can be obtained. The entanglement $\rho$ in different partitions is shown in Table 1. Fig. 2 shows that our method can detect more entangled states. Moreover, from our Corollary 2, $M_2(\rho) = \frac{1}{4}(4+\sqrt{2})x+2 + \frac{1}{4}|\psi\rangle\langle\psi| + \frac{1-x}{16}I_{16}$.

Table 1 The entanglement range of $\rho$ in different partition

| the range of entanglement for $f|ghe$ | the range of entanglement for $f|ghe$ |
|----------------------------------------|----------------------------------------|
| $f_4(x) > 0$, $0.4142 < x \leq 1$     | $f_5(x) > 0$, $0.493 < x \leq 1$     |
| $f_6(x) > 0$, $0.6667 < x \leq 1$     | $f_7(x) > 0$, $0.5774 < x \leq 0.6667$ |

The authors [17] studied the separability for bipartite quantum systems by all correlation tensors and only gave a necessary condition of fully separable for multipartite quantum states. While in our current approach, we study the entanglement under different partition and present fully separable, bi-separable and tri-separable necessary conditions in tripartite and four-partite quantum systems. Moreover, we derive the genuine multipartite entanglement criteria of tripartite and four-partite quantum systems. Our main structural matrices in the algorithm differ from the
Ref. [17] both in structure and in form. We only use a part of correlation tensors in the HW representation to study the entanglement and genuine multipartite entanglement. Detailed examples show that our criteria can detect more entanglement than previous studies.

4 Conclusion

We have studied the entanglement and genuine multipartite entanglement in tripartite and four-partite quantum systems. Taking the advantage of the HW representation, we have derived the upper bounds on the norms of correlation tensors and the separability criteria under any partitions. Detailed examples show that our criteria are able to detect genuine multipartite entanglement more effectively than some existing criteria. Our approach can be also applied to more general multipartite systems.

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