Explicit expressions for joint moments of $n$-dimensional elliptical distributions

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Abstract

Inspired by Stein’s lemma, we consider the joint moments of $n$-dimensional elliptical distribution. We use two different methods to calculate the expectations $E[X^2 f(X)]$ for any measurable function $f$ satisfying some regular conditions. Applying this result, we obtain recursion formulae for product moments of multivariate Gaussian random variables and multivariate elliptical random variables.

Keywords:
Multivariate elliptical distributions; Multivariate normal distributions; Joint moments; Stein’s Lemma

1. Introduction and Motivation

Stein’s lemma was proposed to study the covariance $\text{Cov}(X, h(Y))$ for a bivariate normal random vector $(X, Y)$, where $h$ is any differentiable function such that $E|h'(Y)|$ exists. Inspired by the original work of Stein, several generalizations appeared in the literature. Stein (1981) and then Liu (1994) generalized the lemma to a multivariate normal context. Further, Landsman (2006) showed that Stein’s type lemma also holds when $(X, Y)$ is bivariate elliptically distributed, and Landsman and Neslehova (2008) extended this result to multivariate elliptical vectors. Landsman et al. (2013) rederive
their results in a more straightforward way. Recently, Shushi (2018) derive
the multivariate Stein’s lemma for truncated elliptical random vectors.

Inspired by Stein’s lemma and its generalizations above, we derive the
expressions of the expectations $E[X_1^2f(X)]$ for any measurable function $f$
satisfying some conditions. In particular, we obtain new formulae for the
expectation of the product of normally distributed random variables.

The rest of the paper is organized as follows. Section 2 reviews the def-
initions and properties of the family of elliptical distributions. In Section 3,
we provide two explicit expressions of joint moments for $n$-dimensional ellip-
tical distributions, and in Section 4 we give the proof for the equivalence of
two expressions under the condition that the scale matrix is positive definite.
Examples are presented in Section 5.

2. The family of elliptical distributions

Elliptical distributions are generalizations of the multivariate normal dis-
tributions and share many of its tractable properties. Moreover, the elliptical
family of distributions can deal with fat tails. This class of distributions was
introduced by Kelker (1970) and was widely discussed in Fang et al. (1990).
An $n \times 1$ random vector $X = (X_1, X_2, \ldots, X_n)^T$ is said to have an elliptically
symmetric distribution if its characteristic function has the form

$$E[\exp(it^TX)] = e^{it^T\mu \phi \left( \frac{1}{2}t^T\Sigma t \right)}$$

for all $t \in \mathbb{R}^n$, denoted $X \sim E_n(\mu, \Sigma, \phi)$, where $\phi$ is called the characteristic
generator satisfying $\phi(0) = 1$, $\mu$ ($n$-dimensional vector) is its location pa-
rameter and $\Sigma$ ($n \times n$ matrix with $\Sigma \geq 0$) is its dispersion matrix (or scale
matrix). The mean vector $E(X)$ (if it exists) coincides with the location
vector and the covariance matrix $\text{Cov}(X)$ (if it exists), being $-\phi'(0)\Sigma$. In
particular, the generator of the multivariate normal distribution is given by 
\( \phi(u) = \exp(-u) \).

In general, the elliptical vector \( \mathbf{X} \sim E_n(\mathbf{\mu}, \Sigma, \phi) \) may not have a density. However, if the density \( f_X(x) \) exists then it is of the form

\[
f_X(x) = \frac{c_n}{\sqrt{|\Sigma|}} g_n \left( \frac{1}{2}(x - \mathbf{\mu})^T \Sigma^{-1} (x - \mathbf{\mu}) \right), \quad x \in \mathbb{R}^n, \tag{1}\]

where \( \mathbf{\mu} \) is an \( n \times 1 \) location vector, \( \Sigma \) is an \( n \times n \) positive definite scale matrix, and \( g_n(u), \ u \geq 0, \) is the density generator of \( \mathbf{X} \). This density generator satisfies the condition

\[
\int_0^\infty t^{n/2-1} g_n(t) dt < \infty.
\]

The normalizing constant \( c_n \) is given by

\[
c_n = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[ \int_0^\infty t^{n/2-1} g_n(t) dt \right]^{-1}. \tag{2}\]

Two important families of elliptical distributions are: multivariate normal family, \( g_n(u) = e^{-u} \), and multivariate generalized Student-\( t \) family, \( g_n(u) = (1 + \frac{u}{k_{n,p}})^{-p} \), where the parameter \( p > \frac{n}{2} \) and \( k_{n,p} \) is some constant that may depend on \( n \) and \( p \).

To calculate the mixed moments for \( n \)-dimensional elliptical distributions we use the cumulative generators \( \overline{G}_n(u) \) and \( \overline{G}_n(u) \). These take the forms

\[
\overline{G}_n(u) = \int_u^\infty g_n(v) dv, \tag{3}\]

and

\[
\overline{G}_n(u) = \int_u^\infty \overline{G}_n(v) dv, \tag{4}\]

respectively (see Landsman et al. (2018)), and the corresponding normalizing constants are

\[
c_n^* = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[ \int_0^\infty t^{n/2-1} \overline{G}_n(t) dt \right]^{-1}, \tag{5}\]
and
\[ c_n^{**} = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[ \int_0^\infty t^{n/2-1} \overline{\mathcal{G}}_n(t) dt \right]^{-1}. \tag{6} \]

Throughout, \( x \in \mathbb{R}^n \) will denote an \( n \)-dimensional vector and \( x^T = (x_1, \cdots, x_n) \) its transpose. For an \( n \times n \) matrix \( \Sigma \in \mathbb{R}^{n \times n} \), \( |\Sigma| \) is the determinant of \( \Sigma \). Note that if \( \Sigma \) is positive definite, the Cholesky decomposition is unique.

3. Main result

To calculate the mixed moments for \( n \)-dimensional elliptical distributions we will fix the following notation. Consider a random vector \( X \sim E_n(\mu, \Sigma, g_n) \) with mean vector \( \mu = (\mu_1, \cdots, \mu_n)^T \) and positive definite matrix \( \Sigma = (\sigma_{ij})_{i,j=1}^n \). Partition \( y \in \mathbb{R}^n \) into two parts \( y = (y_1, y_{(2)})^T \) each with 1, \( n-1 \) components respectively. By Cholesky decomposition (see Golub and Van Loan (2012)), there exists a unique lower triangular matrix \( A = (a_{ij})_{i,j=1}^n \) such that \( AA^T = \Sigma \). In components, one obtains
\[ a_{11} = \sqrt{\sigma_{11}}, \ a_{i1} = \frac{\sigma_{i1}}{a_{11}}, \ i = 2, 3, \cdots, n, \ a_{ik} = 0, i < k, \tag{7} \]
\[ a_{kk} = \sqrt{\sigma_{kk} - \sum_{i=1}^{k-1} a_{ki}^2}, \ a_{ik} = \frac{\sigma_{ik} - \sum_{j=1}^{k-1} a_{ij}a_{kj}}{a_{kk}}, \ i = k + 1, \cdots, n. \tag{8} \]

Let \( f : \mathbb{R}^n \to \mathbb{R} \) be twice continuously differentiable function, we write as usual \( (\nabla_{i,j}f(x))_{i,j=1}^n \) for the Hessian matrix of \( f \). In addition,
\[ \nabla_{i,j}f(x) = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \ i, j = 1, 2, \cdots, n, \]
\[ \nabla_i f(x) = \frac{\partial f(x)}{\partial y_i}, \ i = 1, 2, \cdots, n, \]
∇f(x) = \left( \frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n} \right)^T.

Let \( X^* \sim E_n(\mu, \Sigma, \mathcal{G}_n) \) and \( X^{**} \sim E_n(\mu, \Sigma, \mathcal{G}_n) \) be two elliptical random vectors with generators \( \mathcal{G}_n(u) \) and \( \mathcal{G}_n(u) \), respectively.

The following theorem gives one of the expression of joint moments.

**Theorem 1.** Let \( X \sim E_n(\mu, \Sigma, g_n) \) be an \( n \)-dimensional elliptical vector with density generator \( g_n \), positive definite matrix \( \Sigma = (\sigma_{i,j})_{i,j=1}^n \) and finite expectation \( \mu \) and let \( f : \mathbb{R}^n \to \mathbb{R} \) be twice continuously differentiable function satisfying \( E[\nabla_{i,j} f(X^{**})] < \infty \) and \( E[\nabla_i f(X^*)] < \infty \). Furthermore, we suppose that

\[
\lim_{|x_1| \to \infty} x_1 f(Ax + \mu) \mathcal{G}_n \left( \frac{1}{2}x^T x \right) = 0,
\]

and

\[
\lim_{|x_1| \to \infty} [\nabla_1 f(Ax + \mu)] \mathcal{G}_n \left( \frac{1}{2}x^T x \right) = 0.
\]

Then

\[
E[X_1^2 f(X)] = \sigma_{11} b_n^* E[f(X^*)] + b_{11}^* \sum_{i=1}^n \sum_{j=1}^n \sigma_{i1} \sigma_{j1} E[\nabla_{i,j} f(X^{**})]
+ 2\mu_1 b_n^* \sum_{i=1}^n \sigma_{i1} E[\nabla_i f(X^*)] + \mu_1^2 E[f(X)],
\]

where \( b_n^* = \frac{c_n}{\sqrt{n}} \), \( b_{11}^* = \frac{c_n}{\sqrt{n}} \).

Proof. By definition,

\[
E[X_1^2 f(X)] = \frac{c_n}{\sqrt{|\Sigma|}} \int_{\mathbb{R}^n} x_1^2 f(x) g_n \left\{ \frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu) \right\} dx
= \frac{c_n}{\sqrt{|\Sigma|}} \int_{\mathbb{R}^n} x_1^2 f(x) g_n \left\{ \frac{1}{2}(x - \mu)^T (AA^T)^{-1} (x - \mu) \right\} dx.
\]
Setting \( y = A^{-1}(x - \mu) \), we obtain

\[
E[X^T f(X)] = c_n \frac{|A|}{\sqrt{|\Sigma|}} \int_{\mathbb{R}^n} (a_1 y_1 + \mu_1)^2 f(Ay + \mu) g_n \left\{ \frac{1}{2} y^T y \right\} dy
\]

\[
= c_n \int_{\mathbb{R}^n} (a_1 y_1 + \mu_1)^2 f(Ay + \mu) g_n \left\{ \frac{1}{2} y^T y \right\} dy
\]

\[
= a^2_{1,1} c_n \int_{\mathbb{R}^n} y_1^2 f(Ay + \mu) g_n \left\{ \frac{1}{2} y^T y \right\} dy
\]

\[
+ 2a_{1,1} \mu_1 c_n \int_{\mathbb{R}^n} y_1 f(Ay + \mu) g_n \left\{ \frac{1}{2} y^T y \right\} dy
\]

\[
+ \mu^2_1 c_n \int_{\mathbb{R}^n} f(Ay + \mu) g_n \left\{ \frac{1}{2} y^T y \right\} dy
\]

\[
= a^2_{1,1} c_n \int_{\mathbb{R}^n-1} I_1 dy_1 + 2a_{1,1} \mu_1 c_n \int_{\mathbb{R}^n} I_2 dy_1
\]

\[
+ \mu^2_1 c_n \int_{\mathbb{R}^n} f(Ay + \mu) g_n \left\{ \frac{1}{2} y^T y \right\} dy,
\]

where

\[
I_1 = \int_{\mathbb{R}} y_1^2 f(Ay + \mu) g_n \left\{ \frac{1}{2} y^T y \right\} dy_1
\]

\[
= \int_{\mathbb{R}} y_1 f(Ay + \mu) \frac{\partial}{\partial y_1} \overline{G}_n \left\{ \frac{1}{2} y^T y \right\} dy_1
\]

\[
= \int_{\mathbb{R}} [f(Ay + \mu) + y_1 \nabla_1 f(Ay + \mu)] \overline{G}_n \left\{ \frac{1}{2} y^T y \right\} dy_1
\]

\[
= \int_{\mathbb{R}} f(Ay + \mu) \overline{G}_n \left\{ \frac{1}{2} y^T y \right\} dy_1 - \int_{\mathbb{R}} \nabla_1 f(Ay + \mu) \frac{\partial}{\partial y_1} \overline{G}_n \left\{ \frac{1}{2} y^T y \right\} dy_1
\]

\[
= \int_{\mathbb{R}} f(Ay + \mu) \overline{G}_n \left\{ \frac{1}{2} y^T y \right\} dy_1 + \int_{\mathbb{R}} \nabla_1 f(Ay + \mu) \overline{G}_n \left\{ \frac{1}{2} y^T y \right\} dy_1,
\]

and

\[
I_2 = \int_{\mathbb{R}} y_1 f(Ay + \mu) g_n \left\{ \frac{1}{2} y^T y \right\} dy_1
\]

\[
= -\int_{\mathbb{R}} f(Ay + \mu) \frac{\partial}{\partial y_1} \overline{G}_n \left\{ \frac{1}{2} y^T y \right\} dy_1
\]

\[
= \int_{\mathbb{R}} \nabla_1 f(Ay + \mu) \overline{G}_n \left\{ \frac{1}{2} y^T y \right\} dy_1.
\]
So that we obtain
\[
E[X_1^2 f(X)] = a_{11}^2 \{b_n^* E[f(X^*)] + b_n^{**} A_1^T E(\nabla_{i,j} f(X^{**}))_{i,j=1}^n A_1 \} + 2a_{11} \mu_1 b_n^* A_1^T E[\nabla f(X^*)] + \mu_1^2 E[f(X)],
\]
where \( A = (a_{ij})_{i,j=1}^n = (A_1, A_2, \cdots, A_n) \). Using relations (7) and (8) we get (11). This completes the proof of Theorem 1.

The next theorem gives other expression for \( E[X_1^2 f(X)] \), which can be obtained by Stein’s Lemma. Note that the positive definiteness of \( \Sigma \) is not necessarily.

**Theorem 2.** Assume that \( X \sim \mathcal{N}(\mu, \Sigma, \phi) \), the other conditions are the same as that in Theorem 1, we have
\[
E[X_1^2 f(X)] = \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(X_1, X_i) \text{Cov}(X_i^*, X_j^*) E[\nabla_{i,j} f(X^{**})]
\]
\[
+ 2\mu_1 \sum_{i=1}^n \text{Cov}(X_1, X_i) E[\nabla_i f(X^*)]
\]
\[
+ \text{Cov}(X_1, X_1) E[f(X^*)] + \mu_1^2 E[f(X)].
\]
(12)

**Proof.** Using Lemma 2 in Landsman et al. (2013) we find that
\[
\text{Cov}(X_1, f(X)) = \sum_{i=1}^n \text{Cov}(X_1, X_i) E[\nabla_i f(X^*)],
\]
so that
\[
E[X_1 f(X)] = \text{Cov}(X_1, f(X)) + E[X_1] E[f(X)]
\]
\[
= \sum_{i=1}^n \text{Cov}(X_1, X_i) E[\nabla_i f(X^*)] + E[X_1] E[f(X)].
\]
(13)
Replacing $f(X)$ with $X_1 f(X)$ in formula (13), we obtain

$$
E[X_1^2 f(X)] = \sum_{i=1}^{n} \text{Cov}(X_1, X_i)E[\nabla_i f(X^{*})] + E[X_1]E[X_1 f(X)]
$$

$$
= \sum_{i=1}^{n} \text{Cov}(X_1, X_i)E[X_1^2 \nabla_i f(X^{*})] + \text{Cov}(X_1, X_1)E[f(X^{*})] + E[X_1]E[X_1 f(X)]
$$

$$
= \sum_{i=1}^{n} \text{Cov}(X_1, X_i) \left\{ \sum_{j=1}^{n} \text{Cov}(X_1^*, X_j^*)E[\nabla_{i,j} f(X^{**})] + E[X_1^*]E[\nabla_i f(X^{*})] \right\}
$$

$$
+ \text{Cov}(X_1, X_1)E[f(X^{*})] + E[X_1] \left\{ \sum_{i=1}^{n} \text{Cov}(X_1, X_i)E[\nabla_i f(X^{*})] + E[X_1]E[f(X)] \right\}
$$

$$
= \sum_{i=1}^{n} \sum_{j=1}^{n} \text{Cov}(X_1, X_i)\text{Cov}(X_1^*, X_j^*)E[\nabla_{i,j} f(X^{**})] + 2E[X_1] \sum_{i=1}^{n} \text{Cov}(X_1, X_i)E[\nabla_i f(X^{*})] + \text{Cov}(X_1, X_1)E[f(X^{*})] + (E[X_1])^2 E[f(X)],
$$

which completes the proof Theorem 2.

4. Equivalence of two expressions

In this section we shall prove the equivalence of two expressions (11) and (12) under the condition that $\Sigma$ is positive definite. We will use the following result, see e.g. Fang et al. (1990).

Lemma 1. For every non-negative measurable function $f : \mathbb{R} \to \mathbb{R}^+$, we have

$$
\int_{\mathbb{R}^n} f \left( \frac{1}{2} y^T y \right) dy = \frac{(2\pi)^{n/2}}{\Gamma(n/2)} \int_0^{\infty} u^{n/2-1} f(u) du.
$$
Proposition 1. Under the same conditions as in the Theorem 1, we have
\[
\frac{c_n}{c_n^*} = -\phi'(0), \quad \frac{c_n^*}{c_n} = -\phi^*(0),
\]
where \( \phi \) and \( \phi^* \) are the characteristic generating functions corresponding to the density generators \( g_n \) and \( \overline{G}_n \), respectively.

Proof Define \( Y = \sqrt{-\phi'(0)} \Sigma^{-1}(X - E(X)) \), then \( X \sim E_n(\mu, \Sigma, g_n) \) implies that \( Y \sim E_n(0, I_n, g_n) \) and \( \text{Cov}(Y) = -\phi'(0)I_n \). It follows that
\[
E(Y^T Y) = -n\phi'(0)
= c_n \int_{\mathbb{R}^n} y^T y g_n \left( \frac{1}{2} y^T y \right) dy
= 2c_n \frac{(2\pi)^{n/2}}{\Gamma(n/2)} \int_0^\infty t^{n/2} g_n(t) dt,
\]
where in the last line we have used Lemma 1. Hence
\[
\frac{c_n}{c_n^*} = \frac{-n\phi'(0)}{\frac{(2\pi)^{n/2}}{\Gamma(n/2)} \int_0^\infty t^{n/2} g_n(t) dt}. \tag{14}
\]
Using (9) in Landsman et al. (2013) we have
\[
1 = c_n^* \int_{\mathbb{R}^n} \overline{G}_n \left( \frac{1}{2} y^T y \right) dy = c_n^* \frac{(2\pi)^{n/2}}{\Gamma(n/2 + 1)} \int_0^\infty t^{n/2} g_n(t) dt,
\]
and thus
\[
\frac{c_n^*}{c_n} = \frac{1}{\frac{(2\pi)^{n/2}}{\Gamma(n/2 + 1)} \int_0^\infty t^{n/2} g_n(t) dt}. \tag{15}
\]
The result \( \frac{c_n}{c_n^*} = -\phi'(0) \) follows from (14) and (15).

For \( X^* \sim E_n(\mu, \Sigma, \overline{G}_n) \), define \( Z = \sqrt{-\phi^*(0)} \Sigma^{-1}(X^* - E(X^*)) \), then \( Z \sim E_n(0, I_n, \overline{G}_n) \) and \( \text{Cov}(Z) = -\phi^*(0)I_n \). Using the same arguments as
above we get

\[
E(Z^T Z) = -n\phi''(0)
\]

\[
= c_n^* \int_{\mathbb{R}^n} z^T z \overline{G}_n \left( \frac{1}{2} z^T z \right) \, dz
\]

\[
= 2c_n^* \frac{(2\pi)^{n/2}}{\Gamma(n/2)} \int_0^\infty t^{n/2} \overline{G}_n(t) \, dt
\]

\[
= nc_n^* \frac{(2\pi)^{n/2}}{\Gamma(n/2)} \int_0^\infty t^{n/2-1} \overline{G}_n(t) \, dt
\]

\[
= \frac{nc_n^*}{c_n^{**}},
\]

and hence

\[
\frac{c_n^*}{c_n^{**}} = -\phi''(0).
\]

This ends the proof of Proposition 1.

**Remark 1.** Using Proposition 1 we find that, if \( \Sigma \) is positive definite, the expressions (11) and (12) are equivalent. Moreover, for positive semidefinite matrix \( \Sigma \) we can rewrite (12) in terms of the characteristic generators as follows:

\[
E[X_i^2 f(X)] = \phi'(0) \phi''(0) \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} \sigma_{ij} E[\nabla_{i,j} f(X^{**})]
\]

\[
- 2\mu_1 \phi'(0) \sum_{i=1}^n \sigma_{ii} E[\nabla_i f(X^*)]
\]

\[
- \phi'(0) \sigma_{11} E[f(X^*)] + \mu_2 E[f(X)],
\]

where \( \phi \) and \( \phi^* \) are the characteristic generators corresponding to the density generators \( g_n \) and \( \overline{G}_n \), respectively.

5. Examples

**Example 1.** Suppose we have an \( n \)-variate normal random vector \( X \sim E_n(\mu, \Sigma, \phi) \) with the density generator \( g(u) = \exp\{-u\} \) and the character-
istic generator $\phi(t) = \exp\{-t\}$. Note that $g(u) = \bar{G}(u) = \tilde{G}(u) = \exp\{-u\}$, $\phi(t) = \phi^*(t)$ and $b_n^* = b_n^{**} = 1$. Then

$$E[X_1^2 f(X)] = \sum_{i=1}^n \sum_{j=1}^n \sigma_{1i} \sigma_{1j} E[\nabla_{i,j} f(X)]$$

$$+ 2\mu_1 \sum_{i=1}^n \sigma_{1i} E[\nabla_i f(X)]$$

$$+ \sigma_{11} E[f(X)] + \mu_2 E[f(X)].$$

In general, for any $p_1 \geq 2$, we have the following recursion formula:

$$E[X_1^{p_1} f(X)] = \sum_{i=1}^n \sum_{j=1}^n \sigma_{1i} \sigma_{1j} E[\nabla_{i,j} (X_1^{p_1-2} f(X))]$$

$$+ 2\mu_1 \sum_{i=1}^n \sigma_{1i} E[\nabla_i (X_1^{p_1-2} f(X))]$$

$$+ \sigma_{11} E[X_1^{p_1-2} f(X)] + \mu_2 E[X_1^{p_1-2} f(X)]$$

$$= \sum_{i=1}^n \sum_{j=1}^n \sigma_{1i} \sigma_{1j} E[X_1^{p_1-2} \nabla_{i,j} f(X)] + \sigma_{11}^2 (p_1 - 2) (p_1 - 3) E[X_1^{p_1-4} f(X)]$$

$$+ 2\mu_1 \sum_{i=1}^n \sigma_{1i} E[X_1^{p_1-2} \nabla_i f(X)] + 2\mu_1 \sigma_{11} (p_1 - 2) E[X_1^{p_1-3} f(X)]$$

$$+ \sigma_{11} E[X_1^{p_1-2} f(X)] + \mu_2 E[X_1^{p_1-2} f(X)].$$

This formula can be seen as the development and supplement of the following multivariate normal version of Stein’s identity (see Stein (1981) and Liu (1994)):

$$\text{Cov}(X_1, f(X)) = \sum_{i=1}^n \text{Cov}(X_1, X_i) E(\nabla_i f(X)),$$

or, equivalently,

$$E(X_1 f(X)) = \sum_{i=1}^n \sigma_{1i} E(\nabla_i f(X)) + E(X_1) E(f(X)).$$
Stein’s identity for multivariate elliptical distributions also has a similar result, which can be found in Landsman et al. (2013).

In Kan (2008) and Song and Lee (2015), explicit formulae for the product moments of multivariate Gaussian random variables were derived. The next example considers the product moments of multivariate elliptical random variables.

**Example 2.** Assume that $X \sim E_n(\mu, \Sigma, \phi)$, $X^* \sim E_n(\mu, \Sigma, \phi^*)$ and $X^{**} \sim E_n(\mu, \Sigma, \phi^{**})$. Let $f(x) = x_1^{p_1-2} \prod_{i=2}^{n} x_i^{p_i}$ in Theorem 2 where $p_1$ to $p_n$ are non-negative integers and $p_1 \geq 2$, then we get the following recursion formula:

$$E \left[ \prod_{i=1}^{n} X_i^{p_i} \right] = -\phi'(0)\sigma_{11} E \left[ (X_1^*)^{p_1-2} \prod_{i=2}^{n} (X_i^*)^{p_k} \right]$$

$$+ \phi'(0)\phi''(0) \left\{ \sigma_{11}^2 (p_1 - 2)(p_1 - 3) E \left[ (X_1^{**})^{p_1-4} \prod_{k=2}^{n} (X_k^{**})^{p_k} \right] \right. + 2 \sum_{j=2}^{n} \sigma_{11} \sigma_{j1} (p_1 - 2)p_j E \left[ (X_1^{**})^{p_1-3} (X_j^{**})^{p_j-1} \prod_{k=2, k\neq j}^{n} (X_k^{**})^{p_k} \right]$$

$$+ \sum_{j=2}^{n} \sigma_{j1}^2 p_j (p_j - 1) E \left[ (X_1^{**})^{p_1-2} (X_j^{**})^{p_j-2} \prod_{k=2, k\neq j}^{n} (X_k^{**})^{p_k} \right]$$

$$+ \sum_{j=2}^{n} \sum_{i=2, i\neq j}^{n} \sigma_{j1} \sigma_{i1} p_j p_i E \left[ (X_1^{**})^{p_1-1} (X_j^{**})^{p_j-1} (X_1^{**})^{p_1-2} \prod_{k=2, k\neq i, j}^{n} (X_k^{**})^{p_k} \right] \right\}$$

$$- 2\phi'(0)\mu_{1} \left\{ \sigma_{11} (p_1 - 2) E \left[ (X_1^{**})^{p_1-3} \prod_{k=2}^{n} (X_k^{**})^{p_k} \right] \right. + \sum_{j=2}^{n} \sigma_{j1} p_j E \left[ (X_j^{**})^{p_j-1} (X_1^{**})^{p_1-2} \prod_{k=2, k\neq j}^{n} (X_k^{**})^{p_k} \right]$$

$$\left. \right\} + \mu_1^2 E \left[ X_1^{p_1-2} \prod_{k=2}^{n} X_k^{p_k} \right].$$

In particular, when $\phi(t) = \exp\{-t\}$, then $X \sim N_n(\mu, \Sigma)$, $X^* \sim N_n(\mu, \Sigma)$ and $X^{**} \sim N_n(\mu, \Sigma)$. We obtain the following recursion formula:
\[ E \left[ \prod_{i=1}^{n} X_{i}^{p_{i}} \right] = \sigma_{11} E \left[ (X_1)^{p_1-2} \prod_{i=2}^{n} (X_k)^{p_k} \right] \\
+ \sigma_{11}^2 (p_1 - 2)(p_1 - 3) E \left[ (X_1)^{p_1-4} \prod_{k=2}^{n} (X_k)^{p_k} \right] \\
+ 2 \sum_{j=2}^{n} \sigma_{11} \sigma_{j1}(p_1 - 2)p_j E \left[ (X_1)^{p_1-3}(X_j)^{p_j-1} \prod_{k=2, k\neq j}^{n} (X_k)^{p_k} \right] \\
+ \sum_{j=2}^{n} \sigma_{j1}^2 p_j(p_j - 1) E \left[ (X_1)^{p_1-2}(X_j)^{p_j-2} \prod_{k=2, k\neq j}^{n} (X_k)^{p_k} \right] \\
+ \sum_{j=2}^{n} \sum_{i=2, i\neq j}^{n} \sigma_{j1} \sigma_{i1} p_j p_i E \left[ (X_i)^{p_i-1}(X_j)^{p_j-1}(X_1)^{p_1-2} \prod_{k=2, k\neq i, j}^{n} (X_k)^{p_k} \right] \\
+ 2\mu_1 \left\{ \sigma_{11}(p_1 - 2) E \left[ (X_1)^{p_1-3} \prod_{k=2}^{n} (X_k)^{p_k} \right] \right\} \\
+ \sum_{j=2}^{n} \sigma_{j1} p_j E \left[ (X_j)^{p_j-1}(X_1)^{p_1-2} \prod_{k=2, k\neq j}^{n} (X_k)^{p_k} \right] } \right\} \\
+ \mu_1^2 E \left[ X_1^{p_1-2} \prod_{k=2}^{n} X_k^{p_k} \right].

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