The evolution of the electric field along optical fiber
for the type-2 and 3 PAFs in Minkowski 3-space

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Abstract

In this paper, we introduce the type-2 and the type-3 Positional Adapted Frame (PAF) of spacelike curve and timelike curve in Minkowski 3-space. From these PAFs, we study the evolutions of the electric field vectors of the type-2 and type-3 PAFs. As a result, we also investigate the Fermi-Walker parallel and the Lorentz force equation of the electric field vectors for the type-2 and type-3 PAFs in Minkowski 3-space.

1 Introduction

It is well known that the Frenet frame of a space curve plays an important role in the study of curve and surface theory, and this frame is the most well-known frame along a space curve. However, the Frenet frame is undefined wherever the curvature vanishes, such as at points of inflection or along straight sections of the curve. In order to solve this problem, Bishop [3] introduced a new frame along a space curve which is more suitable for applications, which is called Bishop frame or parallel transport frame. After then, many mathematicians have studying various alternative methods of frame of a space curve. For example, Arbind et al. [1] studied a general 1-dimensional higher-order theory for tubes and rod in terms of the hybrid frame of a space curve. In [15] authors discussed hybrid optical magnetic Lorentz flux by using hybrid frame. Also, Gürbüz et al. [10] presented three formulations associated with the modified nonlinear Schrödinger equation with respect to the hybrid frame in Minkowski 3-space. Recently, in [16] Özen and Tosun introduced the Positional Adapted Frame (PAF) as

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the the another frame for the trajectories with non-vanishing angular momentum in Euclidean 3-space. This frame is used to investigate the kinematics of moving particles.

On the other hand, Berry’s geometric phase [2] is related to the time evolution of a space curve. The geometric phase of linearly polarized light defines its angle of rotation. The evolution of an electric field vector is connected with the geometric phase topic. Also this topic have numerous applications in modern optic. In last years, many mathematicians are studying the geometric phase and the evolution of the electric field vector [4]-[8], [11]-[14] etc.

In this paper, we discuss PAF in Minkowski 3-space. The first author [9] in the present paper studied the type-1 PAF in Minkowski space and obtained the evolution of the electric field vector according to PAF in Minkowski 3-space as natural extensions of Özen and Tosun’s formulation

2 Construction of the type-2 and 3 PAFs

In this section, we construct the new frames in terms of the Frenet frame of the non-null curve in Minkowski 3-space.

The Minkowski 3-space $\mathbb{R}^3_1$ is a real space $\mathbb{R}^3$ with the indefinite inner product $\langle \cdot , \cdot \rangle_L$ defined on each tangent space by

$$\langle x, y \rangle_L = x_1y_1 + x_2y_2 - x_3y_3,$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are vectors in $\mathbb{R}^3_1$.

A nonzero vector $x$ in $\mathbb{R}^3_1$ is said to be spacelike, timelike or null if $\langle x, x \rangle_L > 0$, $\langle x, x \rangle_L < 0$ or $\langle x, x \rangle_L = 0$, respectively.

Let $\beta : I \rightarrow \mathbb{R}^3_1$ be a non-null curve parametrized by the arc-length $s$ in Minkowski 3-space $\mathbb{R}^3_1$. Derivative formulae for the Frenet frame $\{T, N, B\}$ are given by

$$\begin{pmatrix} T_s \\ N_s \\ B_s \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_2 \kappa & 0 \\ -\varepsilon_1 \kappa & 0 & \varepsilon_3 \tau \\ 0 & -\varepsilon_2 \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

where $\langle T, T \rangle_L = \varepsilon_1$, $\langle N, N \rangle_L = \varepsilon_2$, $\langle B, B \rangle_L = \varepsilon_3$, $\varepsilon_i = \pm 1$. Here $\kappa$ and $\tau$ are the curvature and the torsion of the non-null curve $\beta$. On the other hand, the Lorentz cross product implies

$$T \times_L N = \varepsilon_3 B, \quad N \times_L B = \varepsilon_1 T, \quad B \times_L T = \varepsilon_2 N.$$

Suppose that a point particle moves along the non-null curve $\beta$ with the arc-length $s$ and the time $t$ in Minkowski 3-space $\mathbb{R}^3_1$. Then the non-null tangent vector $T$, the velocity vector $v$ and the linear momentum vector $M_t$ are given by

$$T(s) = \frac{dz}{ds}, \quad v(t) = \frac{ds}{dt} T(s), \quad M_t(t) = m \frac{ds}{dt} T(s),$$

where $m$ is a constant mass and $z$ is the position vector of the particle as

$$z = \varepsilon_1 \langle \beta(s), T(s) \rangle_L T(s) + \varepsilon_2 \langle \beta(s), N(s) \rangle_L N(s) + \varepsilon_3 \langle \beta(s), B(s) \rangle_L B(s).$$
The evolution of the electric field

The angular momentum $\mathbf{M}_a$ of the particle is the Lorentz cross product of the position vector $\mathbf{z}$ and the linear momentum vector $\mathbf{M}_t$ at the time $t$, and it is expressed as

$$\mathbf{M}_a = \mathbf{z} \times_L \mathbf{M}_t = -\varepsilon_2 \varepsilon_3 m \frac{ds}{dt} \langle \beta(s), \mathbf{N}(s) \rangle_L \mathbf{B}(s) + \varepsilon_2 \varepsilon_3 m \frac{ds}{dt} \langle \beta(s), \mathbf{B}(s) \rangle_L \mathbf{N}(s).$$

Suppose that the normal component of the angular momentum is not zero, and consider

$$-\mathbf{z} = \varepsilon_1 \langle \beta(s), \mathbf{T}(s) \rangle_L \mathbf{T}(s) + \varepsilon_2 \langle \beta(s), \mathbf{N}(s) \rangle_L \mathbf{N}(s) + \varepsilon_3 \langle \beta(s), \mathbf{B}(s) \rangle_L \mathbf{B}(s). \tag{2.2}$$

Consider the projections $\mathbf{w}_1$ and $\mathbf{w}_2$ on $\text{Span} \{ \mathbf{N}, \mathbf{B} \}$ and $\text{Span} \{ \mathbf{T}, \mathbf{N} \}$ of the vector $-\mathbf{z}$. Then these vectors become

$$\mathbf{w}_1 = \varepsilon_2 \langle \beta(s), \mathbf{N}(s) \rangle_L \mathbf{N}(s) + \varepsilon_3 \langle \beta(s), \mathbf{B}(s) \rangle_L \mathbf{B}(s),$$

$$\mathbf{w}_2 = \varepsilon_1 \langle \beta(s), \mathbf{T}(s) \rangle_L \mathbf{T}(s) + \varepsilon_2 \langle \beta(s), \mathbf{N}(s) \rangle_L \mathbf{N}(s),$$

respectively. It follows that

$$\mathbf{w}_1 - \mathbf{w}_2 = \varepsilon_3 \langle \beta(s), \mathbf{B}(s) \rangle_L \mathbf{B}(s) + \varepsilon_1 \langle \beta(s), \mathbf{T}(s) \rangle_L \mathbf{T}(s). \tag{2.3}$$

We define a new vector $\mathbf{H}$ as follows:

$$\mathbf{H} = \frac{\mathbf{w}_1 - \mathbf{w}_2}{\sqrt{\langle \mathbf{w}_1 - \mathbf{w}_2, \mathbf{w}_1 - \mathbf{w}_2 \rangle_L}} = \varepsilon_1 \frac{\langle \beta(s), \mathbf{T}(s) \rangle_M}{\sqrt{\varepsilon_1 \langle \beta(s), \mathbf{T}(s) \rangle^2_L + \varepsilon_3 \langle \beta(s), \mathbf{B}(s) \rangle^2_L}} \mathbf{T}(s) + \varepsilon_3 \frac{\langle \beta(s), \mathbf{B}(s) \rangle_M}{\sqrt{\varepsilon_1 \langle \beta(s), \mathbf{T}(s) \rangle^2_L + \varepsilon_3 \langle \beta(s), \mathbf{B}(s) \rangle^2_L}} \mathbf{B}(s)$$

and we take another vector $\mathbf{D} = \varepsilon_2 \mathbf{H} \times_L \mathbf{N}$ given by

$$\mathbf{D} = \varepsilon_1 \frac{\langle \beta(s), \mathbf{T}(s) \rangle_L}{\sqrt{\varepsilon_1 \langle \beta(s), \mathbf{T}(s) \rangle^2_L + \varepsilon_3 \langle \beta(s), \mathbf{B}(s) \rangle^2_L}} \mathbf{B}(s) + \varepsilon_3 \frac{\langle \beta(s), \mathbf{B}(s) \rangle_L}{\sqrt{\varepsilon_1 \langle \beta(s), \mathbf{T}(s) \rangle^2_L + \varepsilon_3 \langle \beta(s), \mathbf{B}(s) \rangle^2_L}} \mathbf{T}(s).$$

In this case the moving frame $\{ \mathbf{H}, \mathbf{N}, \mathbf{D} \}$ along the non-null curve $\beta$ is called the type-2 Positional Adapted Frame (type-2 PAF) in Minkowski 3-space.
Now, we give the relationship between the Frenet frame and the type-2 PAF frame.

First of all, if \( \mathbf{N} \) is the timelike vector and \( \mathbf{H}, \mathbf{D} \) are the spacelike vectors (\( \mathbf{T} \) and \( \mathbf{B} \) are the spacelike vectors), then it can be written by

\[
\begin{pmatrix}
\mathbf{H} \\
\mathbf{N} \\
\mathbf{D}
\end{pmatrix}
= 
\begin{pmatrix}
\cos \phi & 0 & -\sin \phi \\
0 & 1 & 0 \\
\sin \phi & 0 & \cos \phi
\end{pmatrix}
\begin{pmatrix}
\mathbf{T} \\
\mathbf{N} \\
\mathbf{B}
\end{pmatrix},
\]

where \( \phi \) is an angle between \( \mathbf{D} \) and \( \mathbf{B} \). On the other hand, the type-2 PAF frame apparatus \( p_1, p_2, p_3 \) are given by

\[
\begin{align*}
p_1 &= \kappa(s) \cos \phi + \tau(s) \sin \phi \\
p_2 &= -\phi' \\
p_3 &= -\kappa(s) \sin \phi + \tau(s) \cos \phi.
\end{align*}
\]

Secondly, if \( \mathbf{D} \) is the timelike vector and \( \mathbf{H}, \mathbf{N} \) are the spacelike vectors, (\( \mathbf{B} \) is the timelike vector), then we have the relationship as follows:

\[
\begin{pmatrix}
\mathbf{H} \\
\mathbf{N} \\
\mathbf{D}
\end{pmatrix}
= 
\begin{pmatrix}
\cosh \phi & 0 & \sinh \phi \\
0 & 1 & 0 \\
\sinh \phi & 0 & \cosh \phi
\end{pmatrix}
\begin{pmatrix}
\mathbf{T} \\
\mathbf{N} \\
\mathbf{B}
\end{pmatrix},
\]

where \( \phi \) is the angle between the vectors \( \mathbf{D} \) and \( \mathbf{B} \). The type-2 PAF frame apparatus \( p_1, p_2, p_3 \) are given by

\[
\begin{align*}
p_1 &= \kappa(s) \cosh \phi(s) - \tau(s) \sinh \phi(s) \\
p_2 &= -\phi'(s) \\
p_3 &= -\kappa(s) \sinh \phi(s) + \tau(s) \cosh \phi(s).
\end{align*}
\]

Also, the derivative formulas of the type-2 PAF frame \( \{\mathbf{H}, \mathbf{N}, \mathbf{D}\} \) of the non-null curve \( \beta \) in Minkowski 3-space are expressed as

\[
\begin{pmatrix}
\mathbf{H}_s \\
\mathbf{N}_s \\
\mathbf{D}_s
\end{pmatrix}
= 
\begin{pmatrix}
0 & \varepsilon_2 p_1 & \varepsilon_3 p_2 \\
-\varepsilon_1 p_1 & 0 & \varepsilon_3 p_3 \\
-\varepsilon_1 p_2 & -\varepsilon_2 p_3 & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{H} \\
\mathbf{N} \\
\mathbf{D}
\end{pmatrix},
\]

where

\[
\langle \mathbf{H}, \mathbf{H} \rangle_L = \varepsilon_1, \quad \langle \mathbf{N}, \mathbf{N} \rangle_L = \varepsilon_2, \quad \langle \mathbf{D}, \mathbf{D} \rangle_L = \varepsilon_3,
\]

\[
\langle \mathbf{H}, \mathbf{N} \rangle_L = \langle \mathbf{N}, \mathbf{D} \rangle_L = \langle \mathbf{H}, \mathbf{D} \rangle_L = 0.
\]

As a similar method, we can define the new frame in terms of the Frenet frame.

Suppose that the binormal component of the angular momentum is not zero. The projections \( \Gamma_1 \) and \( \Gamma_2 \) on \( \text{Span} \{\mathbf{N}, \mathbf{B}\} \) and \( \text{Span} \{\mathbf{T}, \mathbf{B}\} \) of the vector \( -\mathbf{z} \) are given by, respectively

\[
\begin{align*}
\Gamma_1 &= \varepsilon_2 \langle -\beta(s), \mathbf{N}(s) \rangle_L \mathbf{N}(s) + \varepsilon_3 \langle -\beta(s), \mathbf{B}(s) \rangle_L \mathbf{B}(s) \\
\Gamma_2 &= \varepsilon_1 \langle -\beta(s), \mathbf{T}(s) \rangle_L \mathbf{T}(s) + \varepsilon_3 \langle -\beta(s), \mathbf{N}(s) \rangle_L \mathbf{B}(s).
\end{align*}
\]
From this,
\[ \Gamma_1 - \Gamma_2 = \varepsilon_2 \langle -\beta(s), N(s) \rangle_L N(s) + \varepsilon_1 \langle \beta(s), T(s) \rangle_L T(s). \] (2.5)
it follows that we define the new vector \( \mathbf{F} \) as follows:
\[
\mathbf{F} = \frac{\Gamma_1 - \Gamma_2}{\sqrt{|\langle \Gamma_1 - \Gamma_2, \Gamma_1 - \Gamma_2 \rangle_L|}} = \varepsilon_1 \frac{\langle \beta(s), T(s) \rangle_L}{\sqrt{\varepsilon_1 \langle \beta(s), T(s) \rangle^2_L + \varepsilon_2 \langle \beta(s), N(s) \rangle^2_L}} T(s) \\
+ \varepsilon_2 \frac{\langle -\beta(s), N(s) \rangle_L}{\sqrt{\varepsilon_1 \langle \beta(s), T(s) \rangle^2_L + \varepsilon_3 \langle \beta(s), N(s) \rangle^2_L}} N(s).
\]
Also, the binormal vector \( \mathbf{B} \) and the vector \( \mathbf{H} \) lead to the another vector \( \mathbf{P} \):
\[
\mathbf{P} = \varepsilon_1 \mathbf{F} \times_L \mathbf{B} = -\varepsilon_2 \frac{\langle \beta(s), T(s) \rangle_L}{\sqrt{\varepsilon_1 \langle \beta(s), T(s) \rangle^2_L + \varepsilon_3 \langle \beta(s), B(s) \rangle^2_L}} \mathbf{N}(s) \\
- \varepsilon_2 \frac{\langle \beta(s), B(s) \rangle_M}{\sqrt{\varepsilon_1 \langle \beta(s), T(s) \rangle^2_L + \varepsilon_3 \langle \beta(s), B(s) \rangle^2_L}} \mathbf{T}(s).
\]
In this case, the moving frame \( \{ \mathbf{P}, \mathbf{F}, \mathbf{B} \} \) along the non-null curve \( \beta \) is called the type-3 Positional Adapted Frame (type-3 PAF) in Minkowski 3-space.

If \( \mathbf{B} \) is the timelike vector and \( \mathbf{P}, \mathbf{F} \) are the spacelike vectors, then the relationship between the Frenet frame and the type-3 PAF frame are expressed by
\[
\begin{pmatrix} \mathbf{P} \\ \mathbf{F} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix},
\]
where \( \phi \) is the angle between the vectors \( \mathbf{F} \) and \( \mathbf{N} \). Also, the type-3 PAF frame apparatus \( n_1, n_2, n_3 \) are given by
\[
\begin{align*}
n_1 &= \kappa - \phi' \\
n_2 &= -\tau \sin \phi \\
n_3 &= \tau \cos \phi.
\end{align*}
\]
If \( \mathbf{F} \) is the timelike vector (\( \mathbf{N} \) is the timelike) and \( \mathbf{P}, \mathbf{B} \) are the spacelike vectors, then we have
\[
\begin{pmatrix} \mathbf{P} \\ \mathbf{F} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \cosh \phi & \sinh \phi & 0 \\ \sinh \phi & \cosh \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix},
\]
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it follows that the type-3 PAF frame apparatus \( n_1, n_2, n_3 \) are given by

\[
    n_1 = \kappa - \phi' \\
    n_2 = \tau \sinh \phi \\
    n_3 = \tau \cosh \phi.
\]

On the other hand, the derivative formulas of the type-3 PAF frame \( \{P, F, B\} \) of the non-null curve \( \beta \) in Minkowski 3-space become

\[
\begin{pmatrix}
    P_s \\
    F_s \\
    B_s
\end{pmatrix}
= 
\begin{pmatrix}
    0 & \varepsilon_2 n_1 & \varepsilon_3 n_2 \\
    -\varepsilon_1 n_1 & 0 & \varepsilon_3 n_3 \\
    -\varepsilon_1 n_2 & -\varepsilon_2 n_3 & 0
\end{pmatrix}
\begin{pmatrix}
    P \\
    F \\
    B
\end{pmatrix},
\]

where

\[
\langle P, P \rangle_L = \varepsilon_1, \quad \langle F, F \rangle_L = \varepsilon_2, \quad \langle B, B \rangle_L = \varepsilon_3,
\]

\[
\langle P, F \rangle_L = \langle F, B \rangle_L = \langle P, B \rangle_L = 0.
\]

3 The evolution of electric field for the type-2 PAF

In this section, we study the evolution of the electric field vector with respect to the type-2 PAF in Minkowski 3-space. To get results, we split it into three cases according to the type-2 PAF.

Case I. Consider an optical fiber \( \mathcal{O} \) described by the spacelike curve \( \beta \) and the timelike binormal vector of the type-2 PAF in Minkowski 3-space \( \mathbb{R}^3_1 \).

Suppose that the electric field vector \( \mathbf{E}^{(2, PAF)} \) of the type-2 PAF is perpendicular to the spacelike vector \( \mathbf{H} \) with the timelike vector \( \mathbf{D} \), that is,

\[
\langle \mathbf{E}^{(2, PAF)}, \mathbf{H} \rangle_L = 0. \tag{3.1}
\]

On the other hand, the general evolution of the electric field vector \( \mathbf{E}^{(2, PAF)} \) for the type-2 PAF in \( \mathbb{R}^3_1 \) is expressed by

\[
\mathbf{E}^{(2, PAF)}_s = a_1 \mathbf{H} + a_2 \mathbf{N} + a_3 \mathbf{D}, \tag{3.2}
\]

where \( a_1, a_2 \) and \( a_3 \) are the arbitrary smooth functions. Consider that no various loss mechanism along the optic fiber for the electric field vector \( \mathbf{E}^{(2, PAF)} \) of the type-2 PAF in Minkowski 3-space, then it can be written by

\[
\langle \mathbf{E}^{(2, PAF)}, \mathbf{E}^{(2, PAF)} \rangle_L = \text{constant}. \tag{3.3}
\]

Using Eq. (3.1) and Eq. (3.2), we obtain

\[
a_1 = -p_1 \langle \mathbf{E}^{(2, PAF)}, \mathbf{N} \rangle_L + p_2 \langle \mathbf{E}^{(2, PAF)}, \mathbf{D} \rangle_L, \tag{3.4}
\]
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where \( \langle \mathbf{E}^{(2, PAF)}_s, \mathbf{N} \rangle_L \neq 0 \) and \( \langle \mathbf{E}^{(2, PAF)}_s, \mathbf{D} \rangle_L \neq 0 \). Taking the derivative with respect to \( s \) of Eq. (3.3) and using Eq. (3.2), we also have

\[
a_2 = \sigma \langle \mathbf{E}^{(2, PAF)}_s, \mathbf{D} \rangle_L, \quad a_3 = -\sigma \langle \mathbf{E}^{(2, PAF)}_s, \mathbf{N} \rangle_L,
\]

(3.5)

where \( \sigma \) is a parameter. When Eqs. (3.4) and (3.5) are substituted in Eq. (3.2), the evolution of the electric field vector \( \mathbf{E}^{(2, PAF)}_s \) for the type-2 PAF is expressed as

\[
\mathbf{E}^{(2, PAF)}_s = \left( -p_1 \langle \mathbf{E}^{(2, PAF)}_s, \mathbf{N} \rangle_L + p_2 \langle \mathbf{E}^{(2, PAF)}_s, \mathbf{D} \rangle_L \right) \mathbf{H} + \sigma \langle \mathbf{E}^{(2, PAF)}_s, \mathbf{D} \rangle_L \mathbf{N} - \sigma \langle \mathbf{E}^{(2, PAF)}_s, \mathbf{N} \rangle_L \mathbf{D}.
\]

(3.6)

On the other hand, the Fermi-Walker derivative \( \text{FW} \mathbf{E}^{(2, PAF)}_s \) of the electric field \( \mathbf{E}^{(2, PAF)}_s \) with respect to the type-2 PAF in \( \mathbb{R}^3 \) is given by

\[
\text{FW} \mathbf{E}^{(2, PAF)}_s = \mathbf{E}^{(2, PAF)}_s - \langle \mathbf{H}, \mathbf{E}^{(2, PAF)}_s \rangle_L \mathbf{H}_s + \langle \mathbf{H}_s, \mathbf{E}^{(2, PAF)}_s \rangle_L \mathbf{H}.
\]

(3.7)

The electric field \( \mathbf{E}^{(2, PAF)}_s \) is Fermi-Walker parallel of the type-2 PAF if and only if \( \text{FW} \mathbf{E}^{(2, PAF)}_s = 0 \). If the electric field \( \mathbf{E}^{(2, PAF)}_s \) is Fermi-Walker parallel, then Eq. (3.1) and Eq. (3.7) imply

\[
\mathbf{E}^{(2, PAF)}_s = -\langle \mathbf{H}_s, \mathbf{E}^{(2, PAF)}_s \rangle_L \mathbf{H}.
\]

(3.8)

Using Eq. (3.1) the electric field \( \mathbf{E}^{(2, PAF)}_s \) of the type-2 PAF is expressed by

\[
\mathbf{E}^{(2, PAF)}_s = \mathbf{E}^{(2, PAF)}_s \mathbf{N} - \mathbf{E}^{(2, PAF)}_s \mathbf{D},
\]

(3.9)

where

\[
\mathbf{E}^{(2, PAF)}_s \mathbf{N} = \langle \mathbf{E}^{(2, PAF)}_s, \mathbf{N} \rangle_L, \quad \mathbf{E}^{(2, PAF)}_s \mathbf{D} = \langle \mathbf{E}^{(2, PAF)}_s, \mathbf{D} \rangle_L.
\]

Taking derivative of Eq. (3.9) with respect to \( s \), the variation of the electric field vector \( \mathbf{E}^{(2, PAF)}_s \) for the first case is given by

\[
\mathbf{E}^{(2, PAF)}_s = (\mathbf{E}^{(2, PAF)}_s \mathbf{N} + p_3 \mathbf{E}^{(2, PAF)}_s \mathbf{D}) \mathbf{N} - (p_3 \mathbf{E}^{(2, PAF)}_s \mathbf{N} + \mathbf{E}^{(2, PAF)}_s \mathbf{D}) \mathbf{D} + (p_1 \mathbf{E}^{(2, PAF)}_s \mathbf{N} + p_2 \mathbf{E}^{(2, PAF)}_s \mathbf{D}) \mathbf{H}.
\]

(3.10)

Via Eqs. (3.8) and (3.10), one finds

\[
\mathbf{E}^{(2, PAF)}_s \mathbf{N} = -p_3 \mathbf{E}^{(2, PAF)}_s \mathbf{D}, \quad \mathbf{E}^{(2, PAF)}_s \mathbf{D} = -p_3 \mathbf{E}^{(2, PAF)}_s \mathbf{N},
\]

(3.11)

it follows that Eq. (3.11) gives a rotation of the polarization plane by an angle \( p_3 \) for the first case with the type-2 PAF in \( \mathbb{R}^3 \).

A magnetic field vector is given by a closed 2-form \( \mathcal{C} \) in 3-dimensional (pseudo-)Riemannian manifold \( M \). The Lorentz force of a magnetic field \( \mathbf{V} \) is described by skew-symmetric operator \( \Phi \), that is,

\[
\langle \Phi(\mathbf{X}), \mathbf{Y} \rangle = \mathcal{C}(\mathbf{X}, \mathbf{Y})
\]
The evolution of the electric field for all $X, Y \in \chi(M)$ and $\Phi(X) = V \times X$.

The Lorentz force equation $\Phi E^{(2.Paf)}$ of the electric field vector $E^{(2.Paf)}$ with the magnetic vector field $V^{(1)}$ of the type-2 PAF in Minkowski 3-space is

$$\Phi H E^{(2.Paf)} = E^{(2.Paf)} = V^{(1)} \times L E^{(2.Paf)}.$$  \hspace{1cm} (3.12)

The curve $E^{(2.Paf)}V^{(1)}$ moved along the electromagnetic trajectory is called the electromagnetic curve with the type-2 PAF in Minkowski 3-space. The Lorentz force equations $\Phi H$ along the optical fiber via Eqs. (3.6) and (3.12) with the type-2 PAF is obtained by

$$\begin{bmatrix} \Phi H(H) \\ \Phi H(N) \\ \Phi H(D) \end{bmatrix} = \begin{bmatrix} 0 & p_1 & -p_2 \\ -p_1 & 0 & -\sigma \\ -p_2 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} H \\ N \\ D \end{bmatrix}$$ \hspace{1cm} (3.13)

and the magnetic vector field $V^{(1)}$ via Eq. (3.13) with respect to the type-2 PAF becomes

$$V^{(1)} = \sigma H - p_2 N - p_1 D.$$

**Case II.** Suppose that an optical fiber $O$ can be described by a spacelike curve $\beta$ with the timelike normal vector for the type-2 PAF in Minkowski 3-space $\mathbb{R}^3_1$.

Now, we consider the electric field vector $E^{(2.Paf)}$ perpendicular to the timelike vector $N$ according to the type-2 PAF, that is,

$$\langle E^{(2.Paf)}, N \rangle_L = 0.$$ \hspace{1cm} (3.14)

The general variation of the electric field vector $E^{(2.Paf)}$ for the type-2 PAF frame in $\mathbb{R}^3_1$ is given by

$$E^{(2.Paf)}_s = b_1 H + b_2 N + b_3 D,$$ \hspace{1cm} (3.15)

where $b_1$, $b_2$ and $b_3$ are the arbitrary smooth functions. Consider no various loss mechanism along with the optical fiber for the type-2 PAF with $E^{(2.Paf)} \perp N$. Then from Eqs. (3.3), (3.14) and (3.15) we have

$$b_2 = -p_1 \langle E^{(2.Paf)}, H \rangle_L + p_3 \langle E^{(2.Paf)}, D \rangle_L$$ \hspace{1cm} (3.16)

with $\langle E^{(2.Paf)}, H \rangle_M \neq 0$ and $\langle E^{(2.Paf)}, D \rangle_M \neq 0$. By differentiating Eq. (3.3) with respect to $s$ and using Eq. (3.15) we also obtain

$$b_1 = \rho \langle E^{(2.Paf)}, D \rangle_L, \quad b_3 = \rho \langle E^{(2.Paf)}, H \rangle_L,$$ \hspace{1cm} (3.17)

where $\rho$ is a parameter. If Eqs. (3.16) and (3.17) are substituted in Eq. (3.15), thus the evolution of the electric field $E^{(2.Paf)}$ for the type-2 PAF in $\mathbb{R}^3_1$ is derived by

$$E^{(2.Paf)}_s = \rho \langle E^{(2.Paf)}, D \rangle_L H$$

$$+ \left( p_3 \langle E^{(2.Paf)}, D \rangle_L - p_1 \langle E^{(2.Paf)}, H \rangle_L \right) N$$

$$+ \rho \langle E^{(2.Paf)}, H \rangle_L D.$$ \hspace{1cm} (3.18)
The Fermi-Walker derivative of the electric field \( E^{(2,Paf)} \) with respect to the type-2 PAF for the second case in \( \mathbb{R}^3_1 \) is given by

\[
FW E_s^{(2,Paf)} = E_s^{(2,Paf)} + \left\langle N, E^{(2,Paf)} \right\rangle_L N_s - \left\langle N_s, E^{(2,Paf)} \right\rangle_L N. \tag{3.19}
\]

If the electric field \( E^{(2,Paf)} \) of the type-2 PAF is the Fermi-Walker parallel for second case, Eq.(3.14) and Eq.(3.19) imply

\[
E_s^{(2,Paf)} = \left\langle N_s, E^{(2,Paf)} \right\rangle_L N \tag{3.20}
\]

Furthermore, by Eq.(3.14) the electric field \( E^{(2,Paf)} \) of the type-2 PAF for the second case leads to

\[
E^{(2,Paf)} = E^{(2,Paf)H} H + E^{(2,Paf)D} D, \tag{3.21}
\]

where

\[
E^{(2,Paf)H} = \left\langle E^{(2,Paf)}, H \right\rangle_L, \quad E^{(2,Paf)D} = \left\langle E^{(2,Paf)}, D \right\rangle_L.
\]

Taking derivative of Eq.(3.21) with respect to \( s \), the change of the electric field vector \( E^{(2,Paf)} \) in Minkowski 3-space can be expressed as

\[
E_s^{(2,Paf)} = (E_s^{(2,Paf)H} - p_2 E_s^{(2,Paf)D}) H + (p_3 E_s^{(2,Paf)D} - p_1 E_s^{(2,Paf)H}) N + (p_2 E_s^{(2,Paf)H} + E_s^{(2,Paf)D}) D, \tag{3.22}
\]

it follows that from Eq.(3.20) we obtain

\[
E_s^{(2,Paf)H} = p_2 E_s^{(2,Paf)D}, \quad E_s^{(2,Paf)D} = -p_2 E_s^{(2,Paf)H}. \tag{3.23}
\]

Therefore, Eq.(3.23) gives a rotation of the polarization plane by an angle \( p_2 \) for the second class with the type-2 PAF.

On the other hand, the Lorentz force equation \( \Phi \) of the electric field vector \( E^{(2,Paf)} \) of the magnetic vector field \( V^{(2)} \) for the second case is

\[
\Phi^N E^{(2,Paf)} = E_s^{(2,Paf)} = V^{(2)} \times_L E^{(2,Paf)}. \tag{3.24}
\]

The curve \( E^{(2,Paf)} V^{(2)} \) travelled along the electromagnetic trajectory is described by the electromagnetic curve according to the type-2 PAF in \( \mathbb{R}^3_1 \). The Lorentz force equations \( \Phi^N \) of the type-2 PAF along the optical fiber via Eqs.(3.18) and (3.24) with the second case are given by

\[
\begin{bmatrix}
\Phi^N(H) \\
\Phi^N(N) \\
\Phi^N(D)
\end{bmatrix} =
\begin{bmatrix}
0 & p_1 & -\rho \\
p_1 & 0 & -p_3 \\
-\rho & -p_3 & 0
\end{bmatrix}
\begin{bmatrix}
H \\
N \\
D
\end{bmatrix}. \tag{3.25}
\]
Also, using Eq. (3.25) the magnetic vector field \( \mathbf{V}^{(2)} \) with respect to the type-2 PAF for the second case is obtained by
\[
\mathbf{V}^{(2)} = -p_3 \mathbf{H} + \rho \mathbf{N} - p_1 \mathbf{D}.
\]

**Case III.** Let an optical fiber \( \mathcal{O} \) can be described by a spacelike curve \( \beta \) with the timelike binormal vector with respect to the type-2 PAF in the Minkowski 3-space \( \mathbb{R}^3_1 \).

We assume that the electric field vector \( \mathbf{E}^{(2, PAF)} \) is perpendicular to the timelike vector \( \mathbf{D} \), that is,
\[
\left\langle \mathbf{E}^{(2, PAF)}, \mathbf{D} \right\rangle_L = 0,
\]
and we also consider no various loss mechanism along the optical fiber of the electric field vector \( \mathbf{E} \) of the type-2 PAF with the third case in Minkowski 3-space. Then one finds
\[
\left\langle \mathbf{E}^{(2, PAF)}, \mathbf{E}^{(2, PAF)} \right\rangle_L = 0.
\]
The general evolution of the electric field vector \( \mathbf{E}^{(2, PAF)} \) of the type-2 PAF in \( \mathbb{R}^3_1 \) is given by
\[
\mathbf{E}_s^{(2, PAF)} = c_1 \mathbf{H} + c_2 \mathbf{N} + c_3 \mathbf{D},
\]
where \( c_1, c_2 \) and \( c_3 \) are the arbitrary smooth functions. Using Eqs. (3.26), (3.27) and (3.28), we get
\[
c_1 = \xi \left\langle \mathbf{E}^{(2, PAF)}, \mathbf{N} \right\rangle_L, \quad c_2 = -\xi \left\langle \mathbf{E}^{(2, PAF)}, \mathbf{H} \right\rangle_L, \quad (3.29)
\]
\[
c_3 = -p_2 \left\langle \mathbf{E}^{(2, PAF)}, \mathbf{H} \right\rangle_L - p_3 \left\langle \mathbf{E}^{(2, PAF)}, \mathbf{N} \right\rangle_L, \quad (3.30)
\]
with \( \left\langle \mathbf{E}^{(2, PAF)}, \mathbf{H} \right\rangle_L \neq 0 \) and \( \left\langle \mathbf{E}^{(2, PAF)}, \mathbf{N} \right\rangle_L \neq 0 \). Here \( \xi \) is a parameter. When Eqs. (3.29) and (3.30) are written in Eq. (3.28), the evolution of the electric field vector \( \mathbf{E}^{(2, PAF)} \) for the third case with respect to the type-2 PAF in \( \mathbb{R}^3_1 \) is found by
\[
\mathbf{E}_s^{(2, PAF)} = \xi \left\langle \mathbf{E}^{(2, PAF)}, \mathbf{N} \right\rangle_L \mathbf{H} - \xi \left\langle \mathbf{E}^{(2, PAF)}, \mathbf{H} \right\rangle_L \mathbf{N}
\]
\[
- \left( p_3 \left\langle \mathbf{E}^{(2, PAF)}, \mathbf{N} \right\rangle_L + p_2 \left\langle \mathbf{E}^{(2, PAF)}, \mathbf{H} \right\rangle_L \right) \mathbf{D}.
\]
The Fermi-Walker derivative of the electric field \( \mathbf{E}^{(2, PAF)} \) with respect to the type-2 PAF for third case in \( \mathbb{R}^3_1 \) is given by
\[
\text{FW} \mathbf{E}^{(2, PAF)}_s = \mathbf{E}^{(2, PAF)}_s + \left\langle \mathbf{D}, \mathbf{E}^{(2, PAF)} \right\rangle_L \mathbf{D}_s
\]
\[
- \left\langle \mathbf{D}_s, \mathbf{E}^{(2, PAF)} \right\rangle_L \mathbf{D}.
\]
It follows that if the electric field \( \mathbf{E}^{(2, PAF)} \) of the type-2 PAF is Fermi-Walker parallel for third case, then Eqs. (3.26) and (3.32) imply
\[
\mathbf{E}^{(2, PAF)}_s = \left\langle \mathbf{D}_s, \mathbf{E}^{(2, PAF)} \right\rangle_L \mathbf{D}.
\]
Also, the electric field $E^{(2.Paf)}$ for third case with respect to the type-2 PAF becomes

$$E^{(2.Paf)} = E^{(2.Paf)H}H + E^{(2.Paf)N}N,$$  

(3.34)

where

$$E^{(2.Paf)H} = \langle E^{(2.Paf)}, H \rangle_L, \quad E^{(2.Paf)N} = \langle E^{(2.Paf)}, N \rangle_L.$$  

Taking derivative of Eq. (3.34) for $s$, the variation of the electric field vector $E^{(2.Paf)}$ is given by

$$E_s^{(2.Paf)} = (E_s^{(2.Paf)H} - p_1 E_s^{(2.Paf)N})H + (p_1 E_s^{(2.Paf)H} + E_s^{(2.Paf)N})N - (p_2 E_s^{(2.Paf)H} + p_3 E_s^{(2.Paf)N})D.$$  

(3.35)

From this, Eq. (3.35) implies

$$E_s^{(2.Paf)H} = p_1 E_s^{(2.Paf)N}, \quad E_s^{(2.Paf)N} = -p_1 E_s^{(2.Paf)H},$$  

(3.36)

it follows that Eq. (3.36) gives a rotation of the polarization plane by an angle $p_1$ for the third case with the type-2 PAF in $\mathbb{R}_3^1$.

The Lorentz force equation $\Phi^D E^{(2.Paf)}$ of the electric field vector $E^{(2.Paf)}$ with the magnetic vector field $V^{(3)}$ is

$$\Phi^D E^{(2.Paf)} = E_s^{(2.Paf)} = V^{(3)} \times_L E^{(2.Paf)}.$$  

(3.37)

The curve $E^{(2.Paf)}V^{(3)}$ travelled along the electromagnetic trajectory is called the electromagnetic curve with respect to the type-2 PAF in $\mathbb{R}_3^1$. The Lorentz force equations $\Phi^D$ of the type-2 PAF for the optical fiber via Eqs. (3.31) and (3.37) for the third case are given by

$$\begin{bmatrix} \Phi^D(H) \\ \Phi^D(N) \\ \Phi^D(D) \end{bmatrix} = \begin{bmatrix} 0 & -\xi & -p_2 \\ \xi & 0 & -p_3 \\ -p_2 & -p_3 & 0 \end{bmatrix} \begin{bmatrix} H \\ N \\ D \end{bmatrix},$$  

(3.38)

which implies that the magnetic vector field $V^{(3)}$ via Eq. (3.38) for third case with respect to the type-2 PAF is expressed by

$$V^{(3)} = -\xi D - p_2 N + p_3 H.$$  

4 The evolution of electric field for the type-3 PAF

In this section, we consider an optical fiber $O$ described by the type-3 PAF of the timelike curve $\beta$ in Minkowski 3-space $\mathbb{R}_3^1$.

Suppose that the electric field vector $E^{(3.Paf)}$ is perpendicular to the timelike vector $B$ according to the type-2 PAF in $\mathbb{R}_3^1$, that is,

$$\langle E^{(3.Paf)}, B \rangle_L = 0.$$  

(4.1)
Consider that no various loss mechanism along with the optical fiber for the type-3 PAF. Then we have

\[ \langle \mathbf{E}^{(3,PaF)}, \mathbf{E}^{(3,PaF)} \rangle_L = 0. \]  

(4.2)

The general variation of the electric field vector \( \mathbf{E}^{(3,PaF)} \) for the type-3 PAF is given by

\[ \mathbf{E}_s^{(3,PaF)} = d_1 \mathbf{P} + d_2 \mathbf{F} + d_3 \mathbf{B}, \]  

(4.3)

where \( d_1, d_2 \) and \( d_3 \) are the arbitrary smooth functions. Via Eqs. (4.1), (4.2) and (4.3), it is obtained by

\[ d_1 = \lambda \langle \mathbf{E}^{(3,PaF)}, \mathbf{F} \rangle_L, \quad d_2 = -\lambda \langle \mathbf{E}^{(3,PaF)}, \mathbf{P} \rangle_L, \]  

(4.4)

\[ d_3 = n_2 \langle \mathbf{E}^{(3,PaF)}, \mathbf{P} \rangle_L - n_3 \langle \mathbf{E}^{(3,PaF)}, \mathbf{F} \rangle_L, \]  

(4.5)

where \( \langle \mathbf{E}^{(3,PaF)}, \mathbf{P} \rangle_L \neq 0, \langle \mathbf{E}^{(3,PaF)}, \mathbf{F} \rangle_L \neq 0 \) and \( \lambda \) is a parameter. If Eqs. (4.4) and (4.5) are substituted in Eq. (4.3), the change of the electric field vector \( \mathbf{E}^{(3,PaF)} \) for the type-3 PAF is found as the following:

\[ \mathbf{E}_s^{(3,PaF)} = \lambda \langle \mathbf{E}^{(3,PaF)}, \mathbf{F} \rangle_L \mathbf{P} - \lambda \langle \mathbf{E}^{(3,PaF)}, \mathbf{P} \rangle_L \mathbf{F} \]  

\[ + \left( n_3 \langle \mathbf{E}^{(3,PaF)}, \mathbf{F} \rangle_L - n_2 \langle \mathbf{E}^{(3,PaF)}, \mathbf{P} \rangle_L \right) \mathbf{B}. \]  

(4.6)

On the other hand, the Fermi-Walker derivative of the electric field \( \mathbf{E}^{(3,PaF)} \) for the type-3 PAF in \( \mathbb{R}^3 \) is expressed by

\[ ^{FW} \mathbf{E}_s^{(3,PaF)} = \mathbf{E}_s^{(3,PaF)} - \langle \mathbf{B}, \mathbf{E}^{(3,PaF)} \rangle_L \mathbf{B}_s \]  

\[ + \langle \mathbf{B}_s, \mathbf{E}^{(3,PaF)} \rangle_L \mathbf{B}. \]  

(4.7)

If the electric field vector \( \mathbf{E}^{(3,PaF)} \) of the type-3 PAF is Fermi-Walker parallel, then Eqs. (4.1) and (4.7) lead to

\[ \mathbf{E}_s^{(3,PaF)} = -\langle \mathbf{B}_s, \mathbf{E}^{(3,PaF)} \rangle_L \mathbf{B}. \]  

(4.8)

Also, the electric field vector \( \mathbf{E}^{(3,PaF)} \) of the type-3 PAF is given by

\[ \mathbf{E}^{(3,PaF)} = -\mathbf{E}^{(3,PaF)}P + \mathbf{E}^{(3,PaF)}F, \]  

(4.9)

where

\[ \mathbf{E}^{(3,PaF)}P = \langle \mathbf{E}^{(3,PaF)}, \mathbf{P} \rangle_L, \quad \mathbf{E}^{(3,PaF)}F = \langle \mathbf{E}^{(3,PaF)}, \mathbf{F} \rangle_L \]

it follows that the evolution of the electric field vector \( \mathbf{E}^{(3,PaF)} \) is given by

\[ \mathbf{E}_s^{(3,PaF)} = \left(-\mathbf{E}_s^{(3,PaF)}P + n_1 \mathbf{E}^{(3,PaF)}F \right) \mathbf{P} \]  

\[ + \left(-n_1 \mathbf{E}^{(3,PaF)}P + \mathbf{E}_s^{(3,PaF)}F \right) \mathbf{F} \]  

\[ + \left(-n_2 \mathbf{E}^{(3,PaF)}P + n_3 \mathbf{E}^{(3,PaF)}F \right) \mathbf{B}. \]  

(4.10)
The evolution of the electric field  

Furthermore, from Eqs. (4.8) and (4.10) we obtain

\[ E_s^{(3.Paf)}P = n_1 E^{(3.Paf)}F, \quad E_s^{(3.Paf)}F = n_1 E^{(3.Paf)}P, \]  

(4.11)

it gives a rotation of the polarization plane by an angle \( n_1 \) for the type-3 PAF in \( \mathbb{R}_3^1 \).

The Lorentz force equation \( \Phi^B E^{(3.Paf)} \) of the electric field vector \( E^{(3.Paf)} \) with the magnetic vector field \( W \) is

\[ \Phi^B E^{(3.Paf)} = E_s^{(3.Paf)} = W \times L E^{(3.Paf)}. \]  

(4.12)

The curve \( E^{(3.Paf)}W \) moved along the electromagnetic trajectory is called the electromagnetic curve of the type-3 PAF in \( \mathbb{R}_3^1 \). The Lorentz force equations \( \Phi^B \) along the optical fiber via Eqs. (4.6) and (4.12) for the type-3 PAF are given by

\[
\begin{bmatrix}
\Phi^B(P) \\
\Phi^B(F) \\
\Phi^B(B)
\end{bmatrix} =
\begin{bmatrix}
0 & \lambda & -n_2 \\
\lambda & 0 & -n_3 \\
-n_2 & n_3 & 0
\end{bmatrix}
\begin{bmatrix}
P \\
F \\
B
\end{bmatrix}.
\]

(4.13)

The magnetic vector field \( W \) via Eq. (4.13) for the type-3 PAF is found as the following

\[ W = -n_3 P + n_2 F + \lambda B. \]

Conclusion

First of all, we construct the type-2 and type-3 PAFs in Minkowski 3-space. Later, the evolutions of the electric field were presented with respect to the the type-2 and the type-3 PAFs in the Minkowski 3-space. Also the type-2 and the type-3 Lorentz equations, the type-2 and the type-3 PAFs electromagnetic curves were given for the type-2 and the type-3 PAFs in the Minkowski 3-space.

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Declaration of Competing Interest

The authors report no declarations of interest.

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.
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