Fine Structure of Matrix Darboux-Toda Integrable Mapping

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Abstract

We show here that matrix Darboux-Toda transformation can be written as a product of a number of mappings. Each of these mappings is a symmetry of the matrix nonlinear Shrödinger system of integro–differential equations. We thus introduce a completely new type of discrete transformations for this system. The discrete symmetry of the vector nonlinear Shrödinger system is a particular realization of these mappings.

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1 Introduction

Discrete transformations that leave an integrable system invariant (integrable mappings) provide an important tool in the theory of integrable systems. Furthermore, it has been suggested [1] that the theory of integrable systems is closely related to a representation theory of the group of integrable mappings. In this approach a classification of integrable mappings plays a key role.

Recently, a discrete symmetry of the vector nonlinear Shrödinger system (the $v$-mapping in future references) was discovered by Aratyn [3]. To find it, the author considered transformations that preserve the form of the corresponding Lax operator and equation (a technique that can be applied in the (1+1)d case only).

In the present paper we reveal new discrete symmetries of the (1+2)d matrix nonlinear Shrödinger system (MNLSS) [4, 5] and show that the $v$-mapping (generalized to two spacial dimensions) is a particular case of these symmetries.

The paper is organized as follows. In Section 2 we write down soliton-like solutions of (1+1)d MNLSS. As far as we are aware this form of solutions was not previously known. In Section 3 we show that these solutions are invariant under a certain class of discrete transformations. Moreover, we find that these symmetries are not limited to a particular type of solutions of MNLSS, i.e. these are symmetries of MNLSS itself. These results are generalized to (1+2)d case in Section 4. Finally, in Section 5 we discuss the relationship between different discrete symmetries of MNLSS and show that the $v$-mapping is a particular case of discrete transformations introduced in the present paper.
2 Multi–Soliton Solutions

Here we write down explicit expressions for multi-soliton type solutions of (1+1)d MNLSS. Details and proofs can be found in [5].

MNLSS in (1+1)d is the following system of two coupled nonlinear differential equations:

\[-v_t + v_{xx} + 2vuv = 0\]
\[u_t + u_{xx} + 2uvu = 0\]  \hspace{1cm} (1)

where $u$ and $v$ are $k \times k$ matrix functions of $t$ and $x$. In particular, when the non-zero part of $v$ is a single column and that of $u$ is a single row, system (1) reduces to the vector nonlinear Shrödinger system (VNLSS).

Multi-soliton type solutions of MNLSS (1) can be characterized by a pair of vectors $(\vec{n}, \vec{m})$ with integer valued components $n_i$ and $m_j$ such that $n_i \geq -1$ and $m_j \geq -1$. The solutions can be written symbolically as

\[u_{ij} = \frac{|n_1, \ldots, n_i + 1, \ldots, n_k; m_1, \ldots, m_j - 1, \ldots, m_k|}{|n_1, \ldots, n_k; m_1, \ldots, m_k|}\]
\[v_{ij} = \frac{|n_1, \ldots, n_j - 1, \ldots, n_k; m_1, \ldots, m_i + 1, \ldots, m_k|}{|n_1, \ldots, n_k; m_1, \ldots, m_k|}\]  \hspace{1cm} (2)

where $|n_1, \ldots, n_k; m_1, \ldots, m_k|$ is the determinant of a matrix whose rows are split into segments of lengths $n_1 + 1, \ldots, n_k + 1; m_1 + 1, \ldots, m_k + 1$. Each segment is represented by an integer in equations (2). Segments of the $s$th row that correspond to integers $n_i$ and $m_j$ are

$n_i : e^{2\tau_s}, e^{2\tau_s}\lambda_s, \ldots, e^{2\tau_s}\lambda_s^{n_i}$

$m_j : 1, \lambda_s, \ldots, \lambda_s^{m_j}$

where $\tau_s = \lambda_s x/2 - \lambda_s^2 t/4 + c_s$, and $\lambda_s$ and $c_s$ are arbitrary parameters. For example,

\[|0; 1| = \begin{vmatrix} e^{2\tau_1} & 1 & \lambda_1 \\ e^{2\tau_2} & 1 & \lambda_2 \\ e^{2\tau_3} & 1 & \lambda_3 \end{vmatrix}\]
\[|1; -1| = \begin{vmatrix} e^{2\tau_1} & e^{2\tau_1}\lambda_1 \\ e^{2\tau_2} & e^{2\tau_2}\lambda_2 \end{vmatrix}\]

Using identity (8) from Appendix, one can check that $u$ and $v$ given by (2) are indeed solutions of MNLSS (1). Solutions of VNLSS are obtained from (2) by setting $m_j = -1$ for $j \geq 2$.

3 Discrete Symmetries of MNLSS

In this section we derive mappings between different multi-soliton type solutions (2). Furthermore, we show that these mappings transform any solution of (1+1)d MNLSS (1) into another solution of the same system, i.e. these are symmetries of MNLSS.

First, let us replace a pair of integers in (2), say $n_\alpha$ and $n_\beta$, with $n_\alpha - 1$ and $m_\beta + 1$ respectively and denote the resulting solution by $(\vec{u}, \vec{v})$. It follows from (2) that

\[\tilde{u}_{\alpha\beta} = \frac{1}{v_{\beta\alpha}}\]
Using identity (8) from Appendix, one can also establish the following relations between solutions \((u, v)\) and \((\tilde{u}, \tilde{v})\):

\[
(\tilde{u}_i v_\beta)_{x} = -(uv)_{i\alpha} \quad (\tilde{u}_\alpha j v_\beta)_{x} = -(vu)_{\beta j} \quad \tilde{u}_{\alpha\beta} = \frac{1}{v_{\beta\alpha}}
\]

\[
\begin{align*}
(\frac{v_{ji}}{v_{\beta\alpha}})_{x} &= (\tilde{u}v)_{ai} & (v_{j\alpha})_{x} &= (\tilde{v}u)_{j\beta} \\
\tilde{v}_{ji} &= v_{ji} - \frac{v_{ji}}{v_{\beta\alpha}} v_{\beta\alpha} & \tilde{u}_{ij} &= u_{ij} + \tilde{u}_{ij} v_{\beta\alpha} \tilde{v}_{\beta\alpha}
\end{align*}
\]

\[
\tilde{v}_{\beta\alpha} (\tilde{u}v)_{\alpha\beta} - (vu)_{\beta\alpha} = v_{\beta\alpha} (\ln v_{\beta\alpha})_{xx}
\]

where \(i \neq \alpha\) and \(j \neq \beta\). Note that since \(\alpha\) and \(\beta\) can take \(k\) values each, there are \(k^2\) basic mappings.

Relations (3) connect different solutions of MNLSS (1) of a particular type (multi-soliton). However, it turns out that mappings (3) are not limited to this type of solutions. Indeed, by a direct substitution of (3) into (1) one can check that MNLSS (1) is invariant under transformations (3). A product of any number of mappings (3) is clearly also a discrete symmetry of MNLSS (1).

### 4 2d Case

Here we generalize the result of the previous section (equations (3)) to the case of two spatial dimensions.

In this case, MNLSS [4, 5] (also called 2d matrix Davey–Stewartson system) reads

\[
\begin{align*}
-u_t + av_{xx} + bv_{yy} + 2a \int dy (vu)_x v + 2bv \int dx (uv)_y = 0 \\
u_t + au_{xx} + bu_{yy} + 2au \int dy (vu)_x + 2b \int dx (uv)_y u = 0
\end{align*}
\]

where \(a\) and \(b\) are arbitrary numbers, and \(u\) and \(v\) are \(k \times k\) matrix functions of \(t, x,\) and \(y\). Note that by setting \(x = y\), appropriately choosing constants of integration, and rescaling the time variable \(t\), we can reduce system (4) to its 1d counterpart (1).

System (4) is the third member of the \((1+2)\)d matrix nonlinear Shrödinger hierarchy [6] of integrable systems. This hierarchy is an infinite set of integrable \((1+2)\)d matrix nonlinear integro-differential equations all of which are invariant under the following transformation (matrix Darboux–Toda mapping):

\[
\begin{align*}
\tilde{u} &= v^{-1} & \tilde{v} &= [vu - (v_x v^{-1})_y] v \\
\end{align*}
\]

where \(u\) and \(v\) are assumed to be nonsingular. Mapping (3) can be generalized to the case of two
Plugging $\tilde{u}$ and $\tilde{v}$ instead of $u$ and $v$ into (4) and using equations (6), one can verify that equations (4) are invariant under mapping (6).

In the present paper we do not address the problem of constructing a hierarchy of equations invariant under only a single mapping (6) with a particular choice of $\alpha$ and $\beta$. However, one can show that all equations of (1+2)d matrix nonlinear Shrödinger hierarchy are invariant under all transformations (6). This follows from the fact that matrix Darboux–Toda mapping (5) commutes with all mappings (6) (see Section 5) and from the construction of the hierarchy [6].

## 5 Relations between symmetries of MNLSS

In this section we discuss the relationship between different symmetries of MNLSS, in particular between the $v$-mapping of Ref. [3], matrix Darboux–Toda mapping (5), and transformations (3) and (6) derived in the present paper.

First, the mapping for (1+1)d vector nonlinear Shrödinger system of Ref. [3] can be obtained from symmetries (3) by setting

$$\alpha = \beta = r \quad u_{ir} \equiv u_i \quad v_{ri} \equiv v_i$$

$$u_{ij} = v_{ji} = 0 \quad j \neq r$$

i.e. the $v$-mapping is a particular case of these symmetries.

Next, let us consider matrix Darboux–Toda mapping (5) and transformations (6). Let $T_{\alpha\beta}$ and $M_k$ denote mappings (6) and (5) respectively. One can prove the following relations:

$$T_{11}T_{22} \ldots T_{kk} = M_k$$

$$T_{ij}T_{ji} = T_{ii}T_{jj} \quad T_{ij}T_{jk}T_{ki} = T_{ii}T_{jj}T_{kk} \quad \ldots$$

$$T_{ij}T_{i'j'} = T_{i'j'}T_{ij} \quad T_{ij}M_k = M_kT_{ij}$$

We conclude this section by noting that in (1+1)d case MNLSS possesses an additional discrete symmetry

$$\tilde{u}_x = u - \tilde{u}v\tilde{u} \quad v_x = v\tilde{u}v - \tilde{v} \quad (7)$$

We checked that when $u$ and $v$ are scalar valued functions, this mapping and the 1d counterpart of Darboux-Toda transformation (obtained by setting $x = y$ in (5)) produce the same solutions of (1+1)d nonlinear Shrödinger system. However, presently it is not clear to us whether symmetry (7) has a 2d analogue.
6 Conclusion

We found new discrete symmetries (see equations (3)) of (1+2)d matrix nonlinear Shrödinger system. While Darboux-Toda transformation (5) is limited to nonsingular square matrices, mappings (3) are free from this constraint. In particular, when matrices $u$ and $v$ in equations (4) and (6) reduce to a single row and column respectively, we obtain (1+2)d generalizations of the vector nonlinear Shrödinger system and the corresponding symmetry.

Presently, we do not know how to construct hierarchies of integrable systems that are invariant only under one of mappings (6). The conventional Lax technique is not applicable to the 2d case and an approach similar to that employed in [6] might be required.

7 Appendix

Here we derive an identity

$$|MC_1C_2||MC_3C_4| + |MC_2C_3||MC_1C_4| = |MC_2C_4||MC_1C_3|$$

(8)

where $M$ is an arbitrary $k \times (k-2)$ matrix, $C_1, C_2, C_3$, and $C_4$ are columns of lengths $k$, and $|$ denotes the determinant.

First, consider a $k \times k$ matrix

$$F = \begin{pmatrix} A & a_1 & b_1 \\ a_2 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{pmatrix}$$

where $A$ is a $(k-2) \times (k-2)$ matrix, $a_1$ and $b_1$, and $a_2$ and $b_2$ are columns and rows respectively of length $k$ each, and $c_{1,2}$ and $d_{1,2}$ are scalars. It is simple to show that

$$|F| = |A| \begin{vmatrix} E & A^{-1}a_1 & A^{-1}b_1 \\ a_2 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{vmatrix} = |A| \begin{vmatrix} c_1 - a_2A^{-1}a_1 & d_1 - a_2A^{-1}b_1 \\ c_2 - b_2A^{-1}a_1 & d_2 - b_2A^{-1}a_1 \end{vmatrix}$$

where $E$ is the $k \times k$ unit matrix. Now identity (8) can be proven by applying the above equation to each term in equation (8).

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