VISUAL BOUNDARIES OF DIESTEL-LEADER GRAPHS

KEITH JONES, GREGORY A. KELSEY

Abstract. Diestel-Leader graphs are neither hyperbolic nor CAT(0), so their visual boundaries may be pathological. Indeed, we show that for \(d > 2\), \(\partial \text{DL}_d(q)\) carries the indiscrete topology. On the other hand, \(\partial \text{DL}_2(q)\), while not Hausdorff, is \(T_1\), totally disconnected, and compact. Since \(\text{DL}_2(q)\) is a Cayley graph of the lamplighter group \(L_q\), we also obtain a nice description of \(\partial \text{DL}_2(q)\) in terms of the lamp stand model of \(L_q\) and discuss the dynamics of the action.

1. Introduction

The visual boundary \(\partial M\) of a complete CAT(0) metric space \(M\) is the topological space obtained by giving the set of asymptotic equivalence classes of geodesic rays in \(M\) the compact-open topology [2, Ch. II.8]. For any base point \(p \in M\), one can simply take \(\partial_p M\) to be the set of geodesic rays emanating from \(p\), and \(\partial_p M\) and \(\partial M\) are homeomorphic. In this setting, the visual boundary has nice properties: for instance, \(M \cup \partial M\) is contractible, and if \(M\) is proper, then \(\partial M\) provides a compactification of \(M\) under the “Cone topology”. An action of a group \(G\) by isometries on \(M\) can be extended to an action by homeomorphisms on \(\partial M\), and studying the dynamics of this action can prove quite fruitful. One can define the visual boundary more generally (i.e., outside the context of CAT(0) spaces), and ask whether these nice properties still arise or whether the study of the action on the boundary is still fruitful.

When a group \(G\) acts geometrically on a space, one may take the boundary of the space as a boundary of the group. For word hyperbolic groups, the visual boundary is unique, and has proven very useful [7]. Outside this class of groups, the situation is not so nice. Croke and Kleiner have shown that even CAT(0) groups may not have unique visual boundaries [4]. Even worse, outside this context we may run into pathological situations. In [8], it is shown that the visual boundary of the Cayley graph of \(\mathbb{Z}^2\) with respect to the standard generating set is uncountable, yet it has the indiscrete (a.k.a. trivial) topology. In short, this occurs because one is able to play the asymptotic classes (two rays are equivalent if they are close in the long term) against the compact-open topology (two rays are close if they agree in the short term) to obtain a sequence of asymptotic rays representing an arbitrary point of the boundary and whose limit is another arbitrary point of the boundary.

This paper investigates whether visual boundaries for non-hyperbolic, non-CAT(0) groups may carry interesting topologies, and if so what this might tell us about...
those groups. In particular, we study the family of lamplighter groups \( L_q = \mathbb{Z}_q \wr \mathbb{Z} \), \( q \geq 2 \) an integer. Using the appropriate generating set, one obtains a particularly nice Cayley graph for \( L_q \), called the Diestel-Leader graph \( DL_2(q) \) \([1], [13], \S 2\). The boundary \( \partial DL_2(q) \) is not a canonical boundary for \( L_q \), since using a different Cayley graph might give rise to a different boundary. However, \( \partial DL_2(q) \) is appealing in that it can be well understood using the standard “lamp stand” model for the lamplighter group. This model and the (well-known) geometry of Diestel-Leader graphs provide ample tools for studying the visual boundary.

More generally, the Diestel-Leader graph \( DL(q_1, q_2, \ldots, q_d) \) can be realized as a subspace of the product of \( d \) trees, having respective valence \( q_1 + 1, q_2 + 1, \ldots, q_d + 1 \) \([1]\). The notation \( DL_d(q) \) is used when each tree has valence \( q + 1 \). While we will state our results only for the cases when the degrees are equal, allowing the degrees to vary between trees has no affect on our analysis. Recently, Stein and Taback have described the metric on these graphs \([11]\) and Duchin, Leli`evre, and Mooney have discussed geodesics in \( DL_d(q) \) in their work on sprawl \([5]\). While not all Diestel-Leader graphs are Cayley graphs \([6, \text{Theorem 1.4}], [1, \text{Corollary 2.15}]\), this paper discusses the relation to \( L_2 \) when applicable and also has results that apply in the non-Cayley graph case. Because Diestel-Leader graphs inherit much of their structure from trees, which are prototypical CAT(0) and hyperbolic spaces, it seems natural to ask whether the boundaries of such graphs inherit any nice properties from the boundaries of trees.

In Section 2 we provide some background on visual boundaries, lamplighter groups, and Diestel-Leader graphs. In Section 3 we collect some basic facts about geodesic rays in Diestel-Leader graphs and prove:

**Theorem (A - Corollary 3.7).** As a set, \( \partial DL_2(q) \) is a disjoint union of two punctured Cantor sets.

In Section 4 we prove:

**Theorem (B).** \( \partial DL_2(q) \) is not Hausdorff, but it is \( T_1 \), compact, and totally disconnected.

The proof of this Theorem is collected in Observations 4.5, 4.7, Proposition 4.8 and Observation 4.9. Additionally, we discuss the dynamics of the action by \( L_2 \) on \( \partial DL_2(q) \) in Theorem 4.10 and Corollary 4.11. In Section 5 we prove:

**Theorem (C - Theorem 5.7).** For \( d > 2 \), the topology of \( \partial DL_d(q) \) is indiscrete.

Roughly speaking, this is a consequence the additional degree of freedom a third tree provides, which allows one to play asymptotic classes against the compact-open topology as described above.

One feature of CAT(0) and hyperbolic spaces is that the horofunction\(^2\) boundary \( \partial_h X \) is naturally homeomorphic to \( \partial X \) \([2, \text{II.8.13}]\), since all horofunctions are Busemann functions (i.e. they come from geodesic rays). However, outside this setting, one may find horofunctions which are not Busemann functions \([12]\). An investigation of the horofunction boundary will appear in a forthcoming work, where we will

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2In short, for a metric space \( X \), a horofunction is a point in \( C(X) \) (the space of continuous functions on \( X \) with the topology of compact convergence on bounded subsets) which is a limit of a sequence of functions \( d_y(x) = d_X(x, y), y \in X \). Horofunctions represent points of the horofunction boundary \( \partial_h(X) \). Busemann functions are those horofunctions obtained as a limit of points along a geodesic ray in \( X \). See \([2, \text{II.8.12-14}]\) for details.
show that while $\partial X$ embeds in $\partial_h X$, there are many horofunctions which are not Busemann functions.

2. Background

2.1. The Visual Boundary. Let $X$ be a geodesic space with base point $x_0$. Two geodesic rays $\gamma, \gamma' : [0, \infty) \to X$ are said to be asymptotic if there is a $\lambda \geq 0$ such that for all $t \geq 0$, $d(\gamma(t), \gamma'(t)) \leq \lambda$. The visual boundary of $X$ is the space $\partial X$ consisting of all asymptotic equivalence classes of geodesic rays in $X$, endowed with the quotient topology from the topology of uniform convergence on compact sets. The based visual boundary of $X$ with base point $x_0$, denoted $\partial(X, x_0)$, is the same topology restricted to the subset of geodesic rays emanating from $x_0$. In general the based and unbased visual boundaries need not agree.

In this paper we consider (based) Diestel-Leader graphs, which are known to be vertex transitive [1 Proposition 2.4], so the based visual boundary is independent of base point. In Proposition 3.10 we show that in $DL_2(q)$, the based and unbased boundaries are the same, so throughout Sections 3 and 4 we abuse notation and use $\partial DL_2(q)$ to refer to the based visual boundary of $DL_2(q)$. In Section 2.2 we still abuse notation and use $\partial DL_d(q)$ to refer to the based visual boundary, even though we do not consider whether it is the same as the unbased visual boundary.

2.2. The Diestel-Leader Graph $DL_d(q)$. For an integer $q$, let $T$ be the regular $q + 1$-valent simplicial tree. Following [13 §2] and [11 §2], we orient the edges of $T$ so that each vertex $v$ has exactly one predecessor $v^-$ and $q$ successors. This induces a partial ordering on the set of vertices of $T$, under which any two vertices $v$ and $w$ have a greatest common ancestor $v \wedge w$. Choosing a base vertex $o$ in $T$ allows us to define a height function $h(v) = d_T(v, v \wedge o) - d_T(o, v \wedge o)$, where the function $d_T$ measures distance in $T$ when each edge is given length 1. The partial ordering provides a chosen endpoint $\omega$ of $T$, obtained by any geodesic ray that always follows predecessors, and this height function is the Busemann function for $\omega$ corresponding to the ray emanating from $o$. For a vertex $v$ in the horocycle $H_k = \{ v \in T | h(v) = k \}$, its unique predecessor $v^-$ is in $H_{k-1}$, and each of its $d$ successors is in $H_{k+1}$ (see Figure 1). In particular, for a given initial vertex $v$, there is a unique “downward” path of length $k$, for each $k$, and a unique downward ray: that which leads to $\omega$.

We now define $DL_d(q)$. Let $T_0, T_1, \ldots, T_{d-1}$ be copies of $T$, with base points $o_i \in T_i$. The Diestel-Leader graph $DL_d(q)$ is the graph whose vertex set consists of $d$-tuples $v = (x_0, x_1, \ldots, x_{d-1})$, $x_i \in T_i$ a vertex, such that $\sum_{i=0}^{d-1} h(x_i) = 0$. Let $h_i(v)$ denote the height function $h(x_i)$ on $T_i$. There is a natural basepoint $(o_0, o_1, \ldots, o_{d-1})$ for $DL_d(q)$.

The edges of $DL_d(q)$ correspond to pairs $(v, w) = ((x_0, \ldots, x_{d-1}), (y_0, \ldots, y_{d-1}))$ such that there are $i$ and $j$, $i \neq j$ with an edge joining $x_i$ to $y_i$ in $T_i$, an edge joining $x_j$ to $y_j$ in $T_j$, and $x_k = y_k$ for all $k \neq i, j$. The relation
\[
h(y_i) - h(x_i) = \pm 1 = h(x_j) - h(y_j)
\]
follows from the definition of vertices of $DL_d(q)$. Thus, moving along an edge in $DL_d(q)$ means simultaneously choosing one tree in which to increase height, and another tree in which to decrease height, while holding the position constant in every other tree.
Convention. We adopt the convention that all geodesic rays in $DL_d(q)$ under discussion emanate from $o$, unless otherwise stated.

2.3. Lamplighter groups. The Diestel-Leader graph $DL_2(q)$ is the Cayley graph of the lamplighter group $L_q = \mathbb{Z}_q \wr \mathbb{Z}$ with generating set \{t, at, a^2t, ..., a^{q-1}t\} (t is the generator of $\mathbb{Z}$ in the wreath product and $a$ is the generator of $\mathbb{Z}_q$). Each element of the lamplighter group (and thus each vertex of $DL_2(q)$) is associated with a “lamp stand.” In the case $q = 2$, the lamp stand consists of a row of lamps in bijective correspondence with $\mathbb{Z}$, a finite number of which are lit, and a lamplighter positioned at one of the lamps. If $q > 2$, then the lamps have $q$ settings: these can be interpreted as off and $q - 1$ levels of brightness while lit or $q - 1$ different colors [10]. The Diestel-Leader graph $DL_3(q)$ has a similar interpretation, except that there is a rhombic grid of lamps [3].

We should also note that for $d > 3$, if $q$ has a prime factor $p$ such that $p < d - 1$, it is open whether $DL_d(q)$ is a Cayley graph of some group. If $q$ has no such prime factor, then $DL_d(q)$ is a Cayley graph [11, Corollary 3.17]. If $q$ does have such a prime factor, it is only known that $DL_d(q)$ is quasi-isometric to a Cayley graph [1, Corollary 3.21].

In the $d = 2$ case, the base vertex $(o_0, o_1)$ corresponds to the lamp stand with no lit lamps and the lamplighter at position 0. In general, the lamp stand corresponding to vertex $(x_0, x_1)$ has the lamplighter at position $h(x_0)$. An edge in $DL_2(q)$ corresponds to the lamplighter stepping between adjacent lamps. If the edge is associated with generator $t$, then the lamplighter moves without switching any bulbs. If the edges is associated with generator $at$, then the lamplighter switches the bulb before he leaves (if he is moving in the positive direction) or after he arrives (if he is moving in the negative direction) [13].

So, for example, the word $t^3(at)t^{-2}(at)^{-2}t^{-1}$ corresponds with the lamplighter starting at position 0, moving three to the right, lighting lamp 3 and stepping to position 4, stepping back to position 2, then stepping back to light lamps 1 and 0, and finally stepping back to lamp $-1$. The end result is the lamp stand pictured in Figure 2. Notice that this lamp stand is also obtainable by the word $(at)^2t(at)t^{-5}$, and so these words represent the same element of $L_2$.

Multiplication of elements corresponds to “composition” of lamp stands. To compute the lamp stand for the element $g \cdot h$, take the lamp stand for $g$, then have...
the lamplighter perform the same switching of lamps as in \( h \), but starting from the lamplighter’s end position in \( g \) instead of at 0. So, for example, for \( g \) the lamp stand in Figure 2 \( g \cdot t \) would have the same set of lit lamps, but the lamplighter would be at 0 instead of \(-1\). The lamp stand for \( g \cdot t a t \) would have the lamplighter at position 1 and lit lamps only at positions 1 and 3.

3. The boundary of \( DL_2(q) \)

3.1. Using projections to bound distance. We begin with two observations that apply in the general case. Since a path in \( DL_d(q) \) projects to a path in a tree \( T_i \), we have a lower bound on the distance between two vertices:

**Observation 3.1.** Let \( v = (v_0, v_1, \ldots, v_{d-1}) \) and \( w = (w_0, w_1, \ldots, w_{d-1}) \) be two vertices in \( DL_d(q) \). Then \( d(v, w) \geq \max\{d_{T_i}(v_i, w_i) \ | \ 0 \leq i \leq d-1\} \).

Moreover, there is a simple upper bound on the distance as well:

**Lemma 3.2.** Let \( v = (v_0, v_1, \ldots, v_d) \) and \( w = (w_0, w_1, \ldots, w_d) \) be two vertices in \( DL_d(q) \). Then

\[
d(v, w) \leq \sum_{i=0}^{d-1} d_{T_i}(v_i, w_i)
\]

**Proof.** We may assume \( v \) is the origin \( o \), since any path between vertices may be translated via isometry to a path from the origin.

Let \( k \) be the index of a tree such that \( h_k(w) \) is minimal. Then \( h_k(w) \leq 0 \). For each tree \( T_i \) other than \( T_k \), in turn, follow the path from \( o_i \) to \( v_i \), always compensating in tree \( T_k \) (i.e. moving up in \( T_k \) when moving down in \( T_i \), and vice versa), following the rule that when the current vertex in \( T_k \) has negative height and we must move up in \( T_k \), we choose to stay on the ray from \( o_k \) to \( \omega_k \). After all trees other than \( T_k \) have been so traversed, a total distance of \( \sum_{i=1, i \neq k}^{d-1} d_{T_i}(o_i, w_i) \) has been traveled. At this point, the current vertex \( x \in T_k \) lies at height \( h_k(w) \) and along the ray from \( o_k \) to \( \omega_k \), implying that \( x \) lies on the geodesic from \( o_k \) to \( w_k \); and so \( d_{T_k}(x, w_k) \leq d_{T_k}(o_k, w_k) \). Thus we can move from \( o \) to \( w \) taking no more than \( \sum_{i=0}^{d-1} d_{T_i}(o_i, w_i) \) steps.

3.2. Asymptotic equivalence classes. Let \( p \) be a path between vertices \( v \) and \( w \) in \( DL_d(q) \). Thinking of \( p \) as a sequence \((v = v_0, v_1, \ldots, v_n = w)\) of vertices of \( DL_d(q) \), we refer to a subsequence \( v_i, v_{i+1}, v_{i+2} \) such that \( h_j(v_i) = h_j(v_{i+1}) + 1 = h_j(v_{i+2}) \) as a turn in \( p \); a bottoming out in the \( j^{th} \) tree. In the lamp stand interpretation for \( d = 2 \), bottoming out corresponds with the lamplighter turning around and moving in the opposite direction along the row of lamps: if the path
bottoms out in \( T_0 \), then the lamplighter stops moving to the left (towards \(-\infty\)) along the lamp stand and begins moving to the right, and vice versa for \( T_1 \).

**Definition 3.1.** We can project a path \( \gamma = (v_0, v_1, \ldots, v_n) \) through \( DL_2(q) \) to a tree \( T_j \). Let \( v_i = (x_{i,0}, x_{i,1}, \ldots, x_{i,d-1}) \) be the \( i^{th} \) vertex along \( \gamma \). Then \( \gamma^{(j)} \) is the corresponding path \( (x_{0,j}, x_{1,j}, \ldots, x_{n,j}) \) through \( T_j \). We will refer to \( \gamma^{(j)} \) as the projection of \( \gamma \) to tree \( T_j \).

For the rest of this section, we restrict our attention to \( DL_2(q) \), though the notation and terminology we use will extend to the general case. In [1 Figure 1] it is shown that a geodesic in \( DL_2(q) \) has at most two turns. However, the case of geodesic rays is simpler.

**Lemma 3.3.** There are no 2-turn geodesic rays in \( DL_2(q) \).

**Proof.** Let \( \gamma \) be a ray in \( DL_2(q) \) emanating from \( o \) with two turns and no backtracking. Then \( \gamma \) “bottoms out” once in \( T_i \), \( i \in \{0,1\} \), and then once in \( T_{1-i} \). Let \( k \in \mathbb{Z} \), \( k > 0 \) be the distance traveled before the first turn, so that \( \gamma \) bottoms out in \( T_i \) at \( \gamma(k) \) with heights \( h_i(\gamma(k)) = -k \) and \( h_{1-i}(\gamma(k)) = k \). Let \( l \in \mathbb{Z} \), \( l > 0 \), be the distance traveled before the second turn, so that \( \gamma \) bottoms out in \( T_{1-i} \) at \( \gamma(k+l) \) with heights \( h_i(\gamma(k+l)) = l - k \) and \( h_{1-i}(\gamma(k+l)) = k - l \). Because \( \gamma \) has exactly two turns, \( \gamma \) then proceeds to descend eternally in \( T_i \) while ascending in \( T_{1-i} \). Consider the vertex \( \gamma(k+2l) \). In \( T_i \), \( \gamma \) descends to \( \gamma^{(i)}(k) \), then ascends for \( l \) edges followed by a descent of the same distance, so that \( \gamma^{(i)}(k) = \gamma^{(i)}(k+2l) \).

We now show there is a path through \( DL_2(q) \) from \( o \) that arrives at \( \gamma(k+2l) \) in shorter time. It may be helpful to refer to Figure 3 which provides an example. Let \( d \) be the distance in \( T_{1-i} \) from \( o_{1-i} \) to \( \gamma^{(1-i)}(k+2l) \). Then either \( d = k \) if \( k \geq l \) or \( d = 2l - k \) if \( l > k \). Choose a path \( \gamma' \) from \( o \) to \( \gamma(k+2l) \) so that \( \gamma^{(1-i)} \) is the geodesic in \( T_{1-i} \) from \( o_{1-i} \) to \( \gamma^{(1-i)}(k+2l) \), and so that \( \gamma^{(i)} \) changes height appropriately with each edge in \( \gamma^{(1-i)} \). The actual choice of \( \gamma^{(i)} \) is irrelevant: regardless of how it ascends, it must then descend to \( \gamma^{(i)}(k+2l) = \gamma^{(i)}(k) \). Thus \( \gamma'(d) = \gamma(k+2l) \) and \( d < k+2l \), so \( \gamma \) cannot be a geodesic ray.

It is worth noting that because there are 2-turn geodesic paths in \( DL_2(q) \), Lemma 3.3 implies that \( DL_2(q) \) is not geodesically complete, i.e. there are geodesics which cannot be extended to geodesic rays. This fact is proved in [11 Theorem 12], where the authors demonstrate that \( DL_2(q) \) has dead-end elements.

Let \( \gamma \) be a geodesic ray based at the origin vertex \( o \in DL_2(q) \), and suppose \( \gamma \) first descends to height \(-h \leq 0 \) in \( T_i \) before ascending eternally in \( T_i \). Then \( \gamma^{(i)}|_{[0,h]} \) is fixed, while \( \gamma^{(1-i)}|_{[0,h]} \) follows one of \( q^h \) paths. On the other hand, \( \gamma^{(i)}|_{[h,\infty]} \) chooses some endpoint other than \( o_1 \) of \( T_i \), while \( \gamma^{(1-i)}|_{[h,\infty]} \) must approach \( o_{1-i} \).

**Lemma 3.4.** If \( \gamma \) and \( \gamma' \) are geodesic rays in \( DL_2(q) \) emanating from \( o \) which both descend to height \(-h \leq 0 \) in \( T_i \) before turning, then they are asymptotic if and only if \( \gamma^{(i)} = \gamma'^{(i)} \). In this case, they eventually merge and upon merging, never split.

**Proof.** First, assume that \( \gamma^{(i)} = \gamma'^{(i)} \). Because \( h_{1-i}(\gamma^{(1-i)}(h)) = h_{1-i}(\gamma'^{(1-i)}(h)) = h \), and both of these vertices have \( o_{1-i} \) as an ancestor, we have \( \gamma^{(1-i)}(2h) = \gamma'^{(1-i)}(2h) = o_{1-i} \). From this point on, since both \( \gamma^{(1-i)} \) and \( \gamma'^{(1-i)} \) are descending in \( T_{1-i} \), they are the same. Hence, \( \gamma|_{[2h,\infty]} = \gamma'|_{[2h,\infty]} \). So \( \gamma \) and \( \gamma' \) merge at or before distance \( 2h \), and they are asymptotic.
Figure 3. Two paths in $DL(2, 2)$ having the same endpoints. A 1-turn path which is geodesic, and a 2-turn path which is minimally non-geodesic.

Now, assume that there exists $t$ such that $\gamma(t) \neq \gamma'(t)$. Since both rays descend to height $-h$, we must have $t > h$. Since $\gamma$ and $\gamma'$ are geodesic rays bottoming out at height $h$ in $T_i$, it follows that $\gamma|_{[h, \infty)}$ and $\gamma'(t)|_{[h, \infty)}$ are geodesic rays in $T_i$. By assumption they are not equal, so since $T_i$ is a tree, they are not asymptotic in $T_i$. From Observation 3.1, $\gamma$ and $\gamma'$ are not asymptotic. □

Furthermore, Observation 3.1 also ensures the following:

**Observation 3.5.** Let $\gamma$ and $\gamma'$ be geodesic rays in $DL_2(q)$.
1. If $\gamma$ begins by descending in $T_i$, while $\gamma'$ begins by ascending in $T_i$, or vice-versa, then $\gamma$ and $\gamma'$ are not asymptotic.
2. If $\gamma$ and $\gamma'$ are geodesic rays descending to heights $h$ and $h'$ respectively before turning, with $h \neq h'$, then $\gamma$ and $\gamma'$ are not asymptotic.

Combining Lemmas 3.3, 3.4 and Observation 3.5, we obtain the following description of asymptotic equivalence classes of geodesic rays in $\partial DL_2(q)$:

**Theorem 3.6.** Two geodesic rays $\gamma$ and $\gamma'$ in $DL_2(q)$ are asymptotic if and only if their projections $\gamma(0)$, $\gamma'(0)$ approach the same end of $T_0$ and their projections $\gamma(1)$, $\gamma'(1)$ approach the same end of $T_1$.

**Corollary 3.7.** The family of rays whose projections do not approach $\omega_i \in \partial T_i$, $i \in \{0, 1\}$, is in one-to-one correspondence with a Cantor set minus the point corresponding to $\omega_i$. Hence, as a set, $\partial DL_2(q)$ is a disjoint union of two deleted Cantor sets:

$$\partial DL_2(q) = ((\partial T_0 - \omega_0) \times \{\omega_1\}) \coprod ((\partial T_1 - \omega_1) \times \{\omega_0\})$$

It is perhaps not surprising that $\partial DL_2(q)$ should be so closely related to a Cantor set, given that $DL_2(q)$ is a one dimensional subset of a product of trees. Lemma 3.4 leads to the picture in Figure 3.2 of a typical element of $\partial DL_2(q)$.

**3.3. Lamp stand interpretation of $\partial DL_2(2)$.** We can understand the visual boundary using the lamp stand. For each geodesic ray starting at the base point, the lamplighter starts at position 0 on an unlit row of lamps. He starts moving in a direction (always the same direction as the projection of the ray to $T_0$), perhaps...
lighting lamps along the way. If the ray “bottoms out” in one tree, then the lamplighter will turn around and proceed in the other direction, again possibly switching lamps along the way. So each element of the visual boundary corresponds with a lamp stand with the lamplighter standing at either \(+\infty\) or \(-\infty\). If the lamplighter is at \(+\infty\), then the set of lit lamps (if it is non-empty) has a minimum. If the lamplighter is at \(-\infty\), then the set of lit lamps (if it is non-empty) has a maximum.

Notice that if the lamplighter turns, he can reset the lamps he has already passed to undo any lighting that he has done or to light any lamps that he missed the first time. In this way, we can see how the “pre-turn” segment of the ray does not affect the asymptotic equivalence class.

Since the height of the associated vertex in tree \(T_0\) is the position of the lamplighter, this means that the points in \((\partial T_0 - \omega_0) \times \omega_1\) have the lamplighter at \(+\infty\) and the points in \((\partial T_1 - \omega_1) \times \omega_0\) have the lamplighter at \(-\infty\).

The lamp stand interpretation for \(\partial\mathbb{DL}_2(2)\) is essentially the same, except that the lamps can take on \(q\) different states, instead of simply on and off.

### 3.4 Action of \(L_2\) on \(\partial\mathbb{DL}_2(2)\)

We can compute the action of the lamplighter group \(L_2\) on the visual boundary \(\partial\mathbb{DL}_2(q)\) by using the lamp stand interpretation in Section 3.3. For \(\gamma\) a geodesic ray in \(\mathbb{DL}_2(q)\), we write \([\gamma]\) for its asymptotic equivalence class in \(\partial\mathbb{DL}_2(q)\). For \(g \in L_2\) and \([\gamma]\) \(\in\) \(\partial\mathbb{DL}_2(2)\), to compute the lamp stand for \(g \cdot [\gamma]\), start with the lamp stand for \(g\). Then, have the lamplighter perform the lighting proscribed by \([\gamma]\), but starting from the lamp lighter’s end position in \(g\) instead of at position \(0\). See Figure 5 for an example.

Notice that for any \([\gamma]\) \(\in\) \((\partial T_0 - \omega_0) \times \omega_1\) and any \(g \in L_2\), we will have \(g \cdot [\gamma]\) \(\in\) \((\partial T_0 - \omega_0) \times \omega_1\). Similarly, \((\partial T_1 - \omega_1) \times \omega_0\) is also invariant under the action of \(L_2\).

**Observation 3.8.** The action of the generators \(t\) and \(at\) on the lamp stand model for \(\partial\mathbb{DL}_2(2)\) is as follows:

- \(t\) shifts the lit lamps one spot to the right (i.e. towards \(+\infty\))
- \(t^{-1}\) shifts the lit lamps one spot to the left (i.e. towards \(-\infty\))
- \(at\) shifts the lamps one spot to the right and then switches the lamp located at \(0\).
- \((at)^{-1}\) switches the lamp located at \(0\) and then shifts the lamps one spot to the left.
For $k \in \mathbb{Z}$, let $a_k$ represent the element $t^k a t^{-k} \in L_2$. Notice that in the lamps model for $L_2$, this is the element associated with only lamp $k$ lit and the lamplighter at position 0.

**Observation 3.9.** The action of $a_k$ on the lamps model of $\partial DL_2(2)$ is to switch the lamp at position $k$.

In Section 4.6, we use the lamp stand interpretation of $\partial DL_2(2)$ to compute the dynamics of this action.

3.5. $\partial DL_2(q)$ **without a basepoint.** In Section 2.1 we introduced the based and unbased visual boundaries. When $X$ is CAT(0) or $\delta$-hyperbolic, these agree. The following shows that the same is true for $DL_2(q)$.

**Proposition 3.10.** Let $\gamma$ be a geodesic ray in $DL_2(q)$. Then there exists a geodesic ray $\tau$ emanating from the origin which is asymptotic to $\gamma$.

**Proof.** In one tree $T_i$, $i \in \{0, 1\}$, $\gamma$ chooses a non-distinquished end $e \neq \omega_i$, and in $T_{1-i}$, $\gamma$ approaches $\omega_{1-i}$. Let $\tau$ be any geodesic ray emanating from $o$ that approaches $e \in T_i$ and $\omega_{1-i} \in T_{1-i}$. Because the projections $\gamma^{(i)}$ and $\tau^{(i)}$ approach the same end of $T_i$, they must merge since $T_i$ is a tree. I.e., there exist $r_1, r_2 \in \mathbb{Z}$ such that $\gamma^{(i)}(r_1) = \tau^{(i)}(r_2)$. Similarly, there are $s_1, s_2 \in \mathbb{Z}$ such that $\gamma^{(1-i)}(s_1) = \tau^{(1-i)}(s_2)$. Setting $n = \max\{r_1, r_2, s_1, s_2\}$, one of $\gamma^{(i)}(k)$ and $\tau^{(i)}(k)$ is an ancestor of the other for all $k \geq n$, and the opposite relation holds for $\gamma^{(1-i)}(k)$ and $\tau^{(1-i)}(k)$. The distance from $\gamma(k)$ to $\tau(k)$ is constant, regardless of $k$, and so the rays are asymptotic. \(\square\)

4. **Topology of $\partial DL_2(q)$**

4.1. **Some important sets.** The natural topology on the visual boundary of a space is the topology of uniform convergence on compact sets. Informally, this means that two asymptotic equivalence classes are close if there are representatives of those classes that share a long initial segment. More formally, given a ray $\gamma$, a compact subset $K$ of $[0, \infty)$ and $\epsilon > 0$, define the set

$$B_K(\gamma, \epsilon) = \{\gamma' \mid \sup\{d(\gamma(x), \gamma'(x)) \mid x \in K\} < \epsilon\}.$$
The sets $B_K(\gamma, \epsilon)$ form a basis for the topology on the set of geodesic rays $U_7.4$]. The corresponding sets in the quotient space form a basis for the topology on the visual boundary (the set of equivalence classes of rays). Often in our proofs, we will work with representatives in the space of rays, rather than the equivalence classes themselves. We will denote the equivalence class of a ray $\gamma$ by $[\gamma]$. Abusing notation, we will write $B_K([\gamma], \epsilon)$ for the image of $B_K(\gamma, \epsilon)$ in the quotient space.

**Definition 4.1.** For $i \in \{0, 1\}$ and $n \in \mathbb{N}$, we define $C_n^i$ to be the set of equivalence classes of geodesic rays that “bottom out” in the $T_i$ after descending for exactly $n$ edges. We define $C_0^i$ to be the set of equivalence classes of rays that ascend forever in the $T_i$ without ever turning. Notice that when equipped with the subspace topology, the sets $C_n^i$ are homeomorphic to the Cantor set.

In terms of the lamp stand, elements of $C_n^0$ (for $n > 0$) have a lit lamp at position $-n$, no lit lamps below that position, and the lamplighter at $+\infty$. Similarly, elements of $C_n^1$ (for $n > 0$) have a lit lamp at position $n - 1$, no lit lamps above that position, and the lamp lighter at $-\infty$. The lamp stand for an element of $C_0^i$ has the lamplighter at $+\infty$ and no lamps lit below 0. The lamp stand for an element of $C_0^0$ has the lamplighter at $-\infty$ and no lit lamps above 0.

**Definition 4.2.** For $k \in \mathbb{N}$, we define the set $C^i_{k,\infty} = \bigcup_{n=k}^{\infty} C_n^i$, which is the set of equivalence classes of geodesic rays that descend at least $k$ edges in $T_i$ before turning and ascending in $T_i$ forever. When equipped with the subspace topology, these sets are homeomorphic to the punctured Cantor set.

We now prove some of the important properties of these sets.

**Lemma 4.1.** For $n > 0$, the set $C_n^i$ is open.

**Proof.** Fix $n > 0$. For each $j \in \{1, 2, \ldots, q\}$, let $[\gamma_j] \in C_n^i$ such that if $j \neq j'$ then $\gamma_j^{(i)}(n + 1) \neq \gamma_{j'}^{(i)}(n + 1)$ (note that $\gamma_j^{(i)}|[0,n] = \gamma_{j'}^{(i)}|[0,n]$).

For $0 < \epsilon < 1$, notice that

$$C_n^i = \bigcup_{j=1}^{q} B_{[0,n+1]}([\gamma_j], \epsilon).$$

Thus, $C_n^i$ is open. $\square$

Lemma 4.1 does not apply when $n = 0$ because in this case the open sets $B_{[0,1]}([\gamma_j], \epsilon)$ include all elements of $C_{1,\infty}^i$ (i.e. every class that bottoms out in the opposite tree). Hence $C_0^i$ cannot be formed as a union in the same way.

**Lemma 4.2.** The set $C_0^i$ is not open.

**Proof.** Fix $[\gamma] \in C_0^i$. For each $n > 0$, let $\gamma_n$ be a ray that agrees with $\gamma$ on the first $n$ edges, but then bottoms out in $T_{1-i}$ and ascends in $T_{1-i}$ forever. In other words, $\gamma(x) = \gamma_n(x)$ for all $x \in [0,n]$ and $[\gamma_n] \in C_n^{1-i}$. Notice that $[\gamma_n] \notin C_0^i$.

Consider a basis element of the form $B_K(\gamma, \epsilon)$. If $n > \text{sup } K$, then $\gamma_n \in B_K(\gamma, \epsilon)$ and thus $[\gamma_n] \in B_K([\gamma], \epsilon)$. Thus, $[\gamma]$ is a limit point of $\{[\gamma_n]\}$, and so the complement of $C_0^i$ is not closed. Hence, $C_0^i$ is not open. $\square$

**Observation 4.3.** For any $k \in \mathbb{N}$, the set $C_0^i \cup C_{k,\infty}^i$ is open.
Proof. Fix $k \in \mathbb{N}$. Let $\gamma$ be a ray that descends in $T_{1-i}$ for $k$ edges before turning (and therefore ascends in $T_i$ for $k$ edges). For $0 < \epsilon < 1$, notice that passing to the quotient:

$$C^i_0 \cup C^{1-i}_{k,\infty} = B_{[0,k]}([\gamma],\epsilon).$$

\[\Box\]

**Observation 4.4.** For $n \geq 0$, the set $C^i_n$ is closed.

**Proof.** The complement is open by the previous observations.

Recall that the induced subspace topology on each $C^i_n$ is a Cantor set and on each $C^i_{k,\infty}$ is a punctured Cantor set. We use these sets to understand the topology of $\partial DL_2(q)$. See Figure 6 for some examples of nested open sets.

![Figure 6](image)

**Figure 6.** Some nested open neighborhoods of an element of $\partial DL_2(2)$

### 4.2. Separability

The boundary $\partial DL_2(q)$ has some interesting separability properties that distinguish it from visual boundaries of hyperbolic or CAT(0) spaces.

**Definition 4.3.** [9, §2.6]

A topological space $X$ is $T_1$ if for every pair of points $x, y \in X$, there exist open sets $O_x, O_y$ such that $x \in O_x, y \notin O_x$ and $y \in O_y, x \notin O_y$.

This is a weaker form of separability than the Hausdorff condition (also known as $T_2$), which requires that the open sets $O_x, O_y$ be disjoint.

**Observation 4.5.** The visual boundary $\partial DL_2(q)$ is not Hausdorff.

**Proof.** Let $\gamma$ and $\gamma'$ be distinct geodesic rays that ascend forever in $T_i$ with no turns (i.e. $[\gamma], [\gamma'] \in C^i_0$). Notice that $[\gamma] \neq [\gamma']$. For each $n > 0$, let $[\gamma_n] \in C^{1-i}_n$ be as in the proof of Lemma 4.2, that is, $\gamma_n$ agrees with $\gamma$ on the first $n$ edges before bottoming out in $T_{1-i}$. Notice that in the asymptotic equivalence class of $\gamma_n$, there is an element $\gamma'_n$ that agrees with $\gamma'$ on the first $n$ edges before bottoming out in tree $T_{1-i}$.

Thus, $[\gamma]$ and $[\gamma']$ are distinct limit points of the sequence $\{[\gamma_n]\} = \{[\gamma'_n]\}$, and so the topology is not Hausdorff.

\[\Box\]

We could also prove that the topology is not Hausdorff using the following observation:

**Observation 4.6.** Any open set containing an element of $C^i_0$ necessarily contains $C^{1-i}_{k,\infty}$ for some $k$. 
Observation 4.7. The visual boundary \( \partial DL_2(q) \) is \( T_1 \).

Proof. Let \( \gamma \) and \( \gamma' \) be geodesic rays that are not asymptotic to each other. So there exists some \( k \in \mathbb{N} \) such that \( \gamma(n) \neq \gamma'(n) \) for all \( n \geq k \). For \( 0 < \epsilon < 1 \), consider the basis elements \( B_{0,k}([\gamma], \epsilon) \) and \( B_{0,k}([\gamma'], \epsilon) \). For any ray \( \tilde{\gamma} \in [\gamma] \), notice that \( \tilde{\gamma}(k) \neq \gamma'(k) \), so \( d(\gamma(k), \gamma'(k)) \geq 1 > \epsilon \), and \( [\gamma] \notin B_{0,k}([\gamma'], \epsilon) \). By symmetry, the reverse holds as well. \( \square \)

4.3. Compactness. For \( X \) non-positively curved, \( \partial X \) is homeomorphic to the horofunction boundary of \( X \) and is also an inverse limit of compact sets \([2] \S 1.8\), both of which imply compactness. Since \( DL_2(q) \) is not CAT(0) or unique geodesic, we have to prove compactness directly.

Proposition 4.8. \( \partial DL_2(q) \) is compact.

Proof. Let \( A = \{A_i\}_{i \in I} \) be some index set \( I \) be an open cover of \( \partial DL_2(q) \).

As sets, \( \partial T_0 \cup \partial T_1 = \partial DL_2(q) \cup \{\omega_0, \omega_1\} \). We extend \( A \) to a cover \( \tilde{A} \) of \( \partial T_0 \cup \partial T_1 \) by defining

\[
\tilde{A}_i = A_i \cup \{x \mid x = \omega_j \text{ for } j = 0, 1 \text{ and } C_{k,\infty}^i \subseteq A_i \text{ for } k > 0\}.
\]

Since \( A \) covers \( DL_2(q) \), Observation 4.6 ensures \( \tilde{A} \) covers \( \partial T_0 \cup \partial T_1 \).

We now define covers \( \tilde{A}^0 \) and \( \tilde{A}^1 \) of \( \partial T_0 \) and \( \partial T_1 \), respectively by \( \tilde{A}^0_i = \tilde{A}_i \cap \partial T_0 \) and \( \tilde{A}^1_i = \tilde{A}_i \cap \partial T_1 \). We claim that these projections are open sets in the boundaries of the trees.

Proof of the claim: Suppose \( A \in A \) and let \( [\gamma] \in \tilde{A}^0 \) (the proof for \( \tilde{A}^1 \) is similar). If \( [\gamma] = \omega_0 \), then \( \tilde{A}^0 \) contains an open neighborhood of \( [\gamma] \) by construction. So we may assume \( [\gamma] \in C_n^0 \) for some \( n \). If \( n > 0 \), then there is an open neighborhood \( N \) of \( [\gamma] \) such that \( N \subseteq A \cap C_n^0 \) (\( A \) is open by assumption and \( C_n^0 \) is open by Lemma [4.1]) and so \( N \subseteq \tilde{A}^0 \). If \( n = 0 \), then there is an open neighborhood \( N \subseteq A \cap (C_0^0 \cup C_{1,\infty}^0) \) for some \( k \). Set \( N' = N \cap C_0^0 \), Note that \( N' \) is not open in \( \partial DL_2(q) \), but it is open in the induced subspace topology on \( C_0^0 \) and thus is open in \( \partial T_0 \). Since \( [\gamma] \in N' \subseteq \tilde{A}^0 \), this completes the proof of our claim.

Thus, the covers \( \tilde{A}^0, \tilde{A}^1 \) are open. Since \( \partial T_0 \) and \( \partial T_1 \) are compact, there exists a finite \( F \subseteq I \) such that \( \{\tilde{A}^0_i\}_{i \in F} \) covers \( \partial T_0 \) and \( \{\tilde{A}^1_i\}_{i \in F} \) covers \( \partial T_1 \). Then \( \{A_i\}_{i \in F} \) is a finite subcover of \( \partial DL_2(q) \).

4.4. Connectedness. We have been considering the visual boundary through the Cantor sets \( C_n^0 \) and punctured Cantor sets \( C_{k,\infty}^0 \), so it is reasonable to expect that the visual boundary is disconnected in a similar manner to a Cantor set.

Observation 4.9. \( \partial DL_2(q) \) is totally disconnected.

Proof. Let \( S \) be a subset of \( \partial DL_2(q) \) containing at least two elements.

Suppose that \( S \cap C_n^0 \neq \emptyset \) for some \( n > 0 \) and some \( i \in \{0, 1\} \). If \( S \subseteq C_n^0 \), then \( S \) is disconnected since \( C_n^0 \) is a Cantor set. Else, since \( C_i^0 \) is both open and closed, it and its complement form a separation of \( S \).
If \( S \cap C_i^k = \emptyset \) for all \( n > 0 \) and \( i \in \{0, 1\} \), then \( S \subseteq (C_0^0 \cup C_1^1) \). So \( S \) is a subset of a Cantor set, and thus is disconnected.

4.5. **Intuitive picture of the topology of \( \partial DL_2(q) \).** Intuitively, the visual boundary \( \partial DL_2(q) \) can be viewed as a pair of punctured Cantor sets in which the punctures are “filled” by a portion of the other Cantor set. Specifically, every open neighborhood of \( \omega_i \) becomes an open neighborhood of \( C_{1-i}^0 \). Figure 7 illustrates this notion.

Clearly, \( \partial DL_2(q) \) is not homogeneous.

![Figure 7. An informal visualization of \( \partial DL_2(2) \)](image)

4.6. **Dynamics of the action of \( L_2 \) on \( \partial DL_2(2) \).** Notice that the exponent sum of \( t \) in a word representing an element \( g \) of \( L_2 \) is equal to the position of the lamplighter in the lamp stand representation of \( g \). Thus, the exponent sum is an invariant of the group element. We will use \( \exp_t(g) \) to denote the exponent sum of \( t \) for \( g \).

Notice that for an element \( g \in L_2 \), if \( \exp_t(g) = 0 \), then \( g^2 \) is trivial (since the second application of \( g \) will switch off all the lights that the first application of \( g \) switched on). Notice also that if \( \exp_t(g) \neq 0 \), then \( g \) will have infinite order since the lamplighter for \( g^n \) with \( n \neq 0 \) is at position \( n \cdot \exp_t(g) \neq 0 \). In other words, for \( g \in L_2 \) non-trivial, then the order of \( g \) is either 2 (when \( \exp_t(g) = 0 \)) or infinite (when \( \exp_t(g) \neq 0 \)).

**Definition 4.4.** Let \( g \in L_2 \) with \( \exp_t(g) > 0 \), so its lamp stand has no lit lamps below some position \( m \). Consider the lamp stand for \( g^n \) for \( n \in \mathbb{N} \). The lamplighter for \( g^n \) is at position \( n \cdot \exp_t(g) \) and no matter how many more times we multiply by \( g \), the lamps below position \( n \cdot \exp_t(g) + m \) will not be switched again. Thus, since \( n \cdot \exp_t(g) + m \to \infty \) as \( n \to \infty \), we have a well-defined lamp stand for \( g^\infty \). This lamp stand can be realized by a geodesic ray in \( DL_2(2) \) (since it is the Cayley graph of \( L_2 \)) by starting with \( t^m(at) \) and then multiplying by \( t \) or \( at \) for each successive lamp, depending on whether the lamp is lit or unlit in \( g^\infty \). Thus, \( g^\infty \) is an element of \( \partial DL_2(2) \).

We can similarly define \( g^\infty \) for \( g \) with \( \exp_t(g) < 0 \) (except that it will have no lit lamps above \( m \)).

Intuitively, \( g^\infty \) is the “lamp stand limit” of \( g^n \). For example, \( t^\infty \) is the lamp stand with no lit lamps and the lamplighter at \(+\infty\).
Definition 4.5. For \( g \in L_2 \) with \( \exp_t(g) \neq 0 \), we define \( g^{-\infty} \) to be \( (g^{-1})^{\infty} \).

Theorem 4.10. If a non-trivial element \( g \) of \( L_2 \) has \( \exp_t(g) = 0 \), then its action on \( \partial DL_2(2) \) will be periodic of order 2. Otherwise, \( g \) will act with north-south dynamics on the boundary, with the attractor in \( (\partial T_0 - \omega_0) \times \omega_1 \) and the repeller in \( (\partial T_1 - \omega_1) \times \omega_0 \) if \( \exp_t(g) > 0 \) and vice versa if \( \exp_t(g) < 0 \).

Proof. If \( \exp_t(g) = 0 \), then the action of \( g \) on an element of \( \partial DL_2(q) \) will simply be to switch a finite set of lamps (the ones that are lit in the lamp stand interpretation of \( g \)).

For \( g \in L_2 \) with \( \exp_t(g) > 0 \) and \([\gamma] \in \partial DL_2(q), \) define \([\gamma_n] \in \partial DL_2(q) \) to be \( g^n \cdot [\gamma] \). Assume \([\gamma] \) (and thus \([\gamma_n] \) for all \( n \)) has the lamplighter at \( +\infty \) (i.e. \([\gamma] \in (\partial T_0 - \omega_0) \times \omega_1 \)). Thus, there is a minimum lit lamp, say at position \( m \), in the lamp stand for \([\gamma] \). Then in the representation for \( g^n \cdot [\gamma] \), all lamps at positions below \( n \cdot \exp_t(g) + m \) will be lit or unlit according to \( g^n \)’s lamp stand. For any \( K \subseteq [0, \infty) \) compact and any \( \epsilon > 0 \), let \( n \) be large enough so that \( n \cdot \exp_t(g) + m > \sup K \). Then notice that \([\gamma_n] \in B_K(g^\infty, \epsilon) \). Thus, \([\gamma_n] \to g^\infty \).

Similar arguments show the rest of the result.

\[ \square \]

Corollary 4.11. The action on \( \partial DL_2(2) \) of a non-torsion element of \( L_2 \) is hyperbolic.

5. \( \partial DL_d(q) \) for \( d > 2 \)

5.1. Differences from the \( d = 2 \) case. We will show that \( \partial DL_d(q) \), \( d > 2 \), has the indiscrete topology.

Definition 5.1. We say that two geodesic rays have the same ends if whenever one of them has a projection to a tree that has infinitely many edges, so does the other, and the two projections go to the same end of that tree.

The visual boundary of \( DL_d(q) \), \( d > 2 \), will be significantly larger than that of \( DL_2(q) \), as set, not just because additional punctured Cantor sets will be added for the trees, but also because it is no longer guaranteed that two geodesic rays having the same ends will be asymptotic, due to the additional degree of freedom offered by a third tree. However, since we aim to show that when \( d > 2 \) the boundary has the indiscrete topology, we will not delve into this. We will show that any point of \( \partial DL_d(q) \) is topologically indistinguishable from a point that approaches a distinguished end \( \omega_i \) in some \( T_i \), a nondistinguished end \( e_j, j \neq i \), in some \( T_j \), and is trivial in every other tree, so that this issue can be avoided in proving that \( \partial DL_d(q) \) has the indiscrete topology.

Observation 5.1. If \( \tau_n \) are geodesic rays in \( DL_d(q) \) that are asymptotic to another geodesic ray \( \gamma \) for all \( n \in \mathbb{N} \), and \( \tau \) is a geodesic ray that is a limit point of \( \{\tau_n\} \), then \([\gamma]\) and \([\tau]\) are topologically indistinguishable elements of \( \partial DL_d(q) \).

Proof. Clearly every neighborhood of \([\tau]\) contains \([\gamma]\) and since the basis definition of the topology is symmetric, every neighborhood of \([\gamma]\) contains \([\tau]\).
5.2. Finite projections.

**Lemma 5.2.** Let \( \gamma \) be a geodesic ray in \( DL_d(q) \) for \( d > 2 \). Partition the set \( \{ T_0, T_1, T_2, \ldots, T_{d-1} \} \) into sets \( \mathcal{I} \) and \( \mathcal{F} \), where the projection of \( \gamma \) to any tree in \( \mathcal{F} \) is eventually constant, and the height of the projection to any tree in \( \mathcal{I} \) approaches \( \pm \infty \). Then we can construct a geodesic ray \( \tau \) that is asymptotic to \( \gamma \), has the same ends as \( \gamma \), and the projection of \( \tau \) to any tree in \( \mathcal{F} \) is trivial. (Here trivial means the image is constant at the origin.)

**Proof.** Let \( N \) be large enough that for each \( T_i \in \mathcal{F} \), all edges of \( \gamma \) that project onto \( T_i \) come before \( N \), and for each tree in \( \mathcal{I} \) in which \( \gamma \) bottoms out, \( \gamma \) does so before \( N \).

Let \( \rho \) be a geodesic ray such that for each tree \( T_i \) in \( \mathcal{I} \), the projection \( \rho^{(i)} \) approaches the same end as the projection \( \gamma^{(i)} \), chosen so that the projection of \( \rho \) to any tree in \( \mathcal{F} \) is trivial, and \( \rho \) satisfies the same condition on edges after \( N \) that \( \gamma \) does (i.e., all turns occur before \( N \); clearly this is achievable, since \( \gamma \) does it while possibly additionally moving through trees in \( \mathcal{F} \)).

Let \( \tau \) be defined by \( \tau_{[0,N]} = \rho_{[0,N]} \), and for \( n > N \), the \( n \)th edge of \( \tau \) simply “tracks” the \( n \)th edge of \( \gamma \). That is, when the \( n \)th edge of \( \gamma \) moves upward in some \( T_{i_n} \) and downward in some \( T_{j_n} \), \( \tau \) does the same, choosing the upward branch that takes it toward the same point of \( \partial T_{i_n} \) that \( \gamma^{(i_n)} \) approaches. Since \( \rho_{[0,N]} \) is a geodesic and all turns in \( \rho \) occur before \( N \), \( \tau \) is a geodesic ray.

The ray \( \tau \) has been chosen so that for each tree \( T_i \), for \( n > N \), \( d_{T_i}(\tau^{(i)}(n), \gamma^{(i)}(n)) \) is constant. Lemma 3.2 then ensures \( \tau \) and \( \gamma \) are asymptotic. □

5.3. Infinite projections.

**Lemma 5.3.** For \( d > 2 \), let \( \gamma \) be a geodesic ray in \( DL_d(q) \) with empty projection to \( T_i \). Let \( \tau \) be another geodesic ray whose projections in trees other than \( T_i \) have the same ends as \( \gamma \) and whose projection to \( T_i \) is infinite. Then, \( [\gamma] \) and \( [\tau] \) are topologically indistinguishable.

**Proof.** Let \( N \) be large enough so that all turns and finite projections of \( \gamma \) and \( \tau \) come before \( N \). For \( n > N \), define \( \tau_n \) to be the ray that matches \( \tau \) up through \( n \) edges in \( T_i \) and then “tracks” \( \gamma \) by going up and down in the same trees for each edge as in the proof of Lemma 5.2. By the same argument as in the proof of Lemma 5.2, each \( \tau_n \) is asymptotic to \( \gamma \). But clearly \( \tau_n \to \tau \), so by Observation 5.1 we are done. □

**Corollary 5.4.** For \( d > 2 \), an element of \( \partial DL_d(q) \) is topologically indistinguishable from at least one other element that only has non-empty projections in two trees: one that eventually ascends in height without bound, and the other which eventually descends in height without bound.

**Proof.** This follows immediately from Lemmas 5.2 and 5.3. □

**Lemma 5.5.** Suppose that \( \gamma \) is a geodesic ray in \( DL_d(q) \) for \( d > 2 \) that has no projection to \( T_i \) and infinite projection to \( T_j \). Then we can construct a geodesic ray \( \tau \) such that \( \tau \) has no projection to \( T_j \), \( [\tau] \) is in every open set that contains \( [\gamma] \), and \( \gamma \) and \( \tau \)'s edges are exactly the same except that whenever \( \gamma \) has an edge that would project to \( T_j \), \( \tau \) projects to the same exact edge in \( T_i \).
In other words, $\gamma$ and $\tau$ are the same ray, just swapping the projections in $T_i$ and $T_j$ (one of which is empty), and the asymptotic equivalence classes of $\gamma$ and $\tau$ are topologically indistinguishable in $\partial \text{DL}_d(q)$.

**Proof.** We begin by assuming that $\gamma^{(j)}$ eventually increases without bound in height. The descending case is analogous. Let $\ell$ be the number of edges that $\gamma^{(j)}$ descends before increasing forever.

In the obvious way, we can construct a geodesic ray $\tau$ that exactly matches $\gamma$, except that the infinite projection to $T_j$ and the empty projection to $T_i$ are swapped. We will now construct a sequence of geodesic rays $\tau_n$ such that $\tau_n \in \gamma$ and $\tau_n \to \tau$, which will show topological indistinguishability.

Let $N$ be large enough so that every turn and finite projection of $\gamma$ (and thus of $\tau$ also) occurs before $N$. For $n > N$, we construct $\tau_n$ as follows:

The first $n$ edges of $\tau_n$ exactly match the first $n$ edges of $\tau$. By choice of $N$, we have partitioned the trees into “up,” “down,” and “empty” for projections of $\tau$. That is, any edge after $N$ has its up projection in one of the “up” trees and its down projection in one of the “down” trees (after $N$ there are no edge projections in the “empty” trees). Notice that $T_j$ is “empty” for $\tau$, but is “up” for $\gamma$.

So for the next $\ell$ edges of $\tau_n$, continue to copy $\tau$, except that the “down” projections of the edges should all be in $T_j$ instead. By the definition of $\ell$, these down edges will exactly reach the point where $\gamma^{(j)}$ turns.

For all subsequent edges, $\tau_n$ “mimics” $\gamma$ by going up and down in the same trees as $\gamma$. As a result, for $x \geq n + \ell$, the distance between $\tau_n(x)$ and $\gamma(x)$ will be equal to the distance between $\tau_n(n + \ell)$ and $\gamma(n + \ell)$, so the two rays are in the same asymptotic equivalence class. Since $\tau_n \to \tau$, by Observation 5.1 we are done.

**Corollary 5.6.** For $d > 2$, an element of $\partial \text{DL}_d(q)$ is topologically indistinguishable from at least one other element that only has non-empty projections in trees $T_0$ and $T_1$ such that the projection to $T_0$ eventually ascends in height without bound, and the projection to $T_1$ eventually descends in height without bound.

**Theorem 5.7.** For $d > 2$, $\partial \text{DL}_d(q)$ has the indiscrete topology.

**Proof.** Let $\gamma$ and $\gamma'$ be geodesic rays in $\text{DL}_d(q)$. By Corollary 5.6, we may assume that the only non-empty projections of $\gamma$ and $\gamma'$ are in trees $T_0$ and $T_1$, that $\gamma^{(0)}, \gamma'^{(0)}$ both eventually ascend in height without bound, and $\gamma^{(1)}, \gamma'^{(1)}$ both eventually descend in height without bound. Let $\ell$ (and $\ell'$) be the number of down edges in $\gamma^{(0)}$ (respectively, $\gamma'^{(0)}$) before turning.

For $N$ sufficiently large so that all the turns in $\gamma$ and $\gamma'$ occur after $N$ and for $n > N$, define $\tau_n$ so that the first $n$ edges go up always taking the leftmost edge in $T_2$ (recall $d > 2$) and down in $T_1$. For the next $\ell$ edges, $\tau_n$ goes down in $T_0$ and up in $T_2$ (again, always taking the leftmost edge). After that, $\tau_n$ goes up in $T_0$ and down in $T_1$ towards the ends of $\gamma$. We define $\tau'_n$ similarly, but using $\ell'$ and $\gamma'$.

Notice that $\tau_n|_{[0,n]} = \tau'_n|_{[0,n]}$.

The ray $\tau_n$ is asymptotic to $\gamma$, since the two rays are never further apart than their distance at $\tau_n(n + \ell)$ and $\gamma(n + \ell)$. Similarly, $\tau'_n$ is asymptotic to $\gamma'$.

But notice that for any $K \subseteq [0, \infty)$ compact and any $\epsilon > 0$, we have $\tau_n \in B_K(\tau_n', \epsilon)$ for any $n \geq \max K$. Thus, $[\gamma]$ and $[\gamma']$ are topologically indistinguishable. □
References

[1] Laurent Bartholdi, Markus Neuhauser, and Wolfgang Woess. Horocyclic products of trees. *J. Eur. Math. Soc. (JEMS)*, 10(3):771–816, 2008.

[2] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[3] Sean Cleary and Tim R. Riley. Erratum to: “A finitely presented group with unbounded dead-end depth” [Proc. Amer. Math. Soc. 134 (2006), no. 2, 343–349; see MR2176000]. *Proc. Amer. Math. Soc.*, 136(7):2641–2645, 2008.

[4] Christopher B. Croke and Bruce Kleiner. Spaces with nonpositive curvature and their ideal boundaries. *Topology*, 39(3):549–556, 2000.

[5] Moon Duchin, Samuel Lelièvre, and Christopher Mooney. Statistical hyperbolicity in groups. *Algebr. Geom. Topol.*, 12(1):1–18, 2012.

[6] Alex Eskin, David Fisher, and Kevin Whyte. Coarse differentiation of quasi-isometries I: Spaces not quasi-isometric to Cayley graphs. *Ann. of Math. (2)*, 176(1):221–260, 2012.

[7] Ilya Kapovich and Nadia Benakli. Boundaries of hyperbolic groups. In *Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001)*, volume 296 of *Contemp. Math.*, pages 39–93. Amer. Math. Soc., Providence, RI, 2002.

[8] Kyle Kitzmiller and Matt Rathbun. The visual boundary of \( \mathbb{Z}^2 \). *Involve*, 4(2):103–116, 2011.

[9] James R. Munkres. *Topology: a first course*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.

[10] Walter Parry. Growth series of some wreath products. *Trans. Amer. Math. Soc.*, 331(2):751–759, 1992.

[11] M. Stein and J. Taback. Metric Properties of Diestel-Leader Groups. *Michigan Math. J.*, 62(2):365–286, 2013.

[12] Corran Webster and Adam Winchester. Busemann points of infinite graphs. *Trans. Amer. Math. Soc.*, 358(9):4209–4224 (electronic), 2006.

[13] Wolfgang Woess. Lamplighters, Diestel-Leader graphs, random walks, and harmonic functions. *Comb. Probab. Comput.*, 14(3):415–433, May 2005.