Pairing of Solitons in Two-Dimensional $S = 1$ Magnets

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We discuss the structure of topological solitons in a general non-Heisenberg model of isotropic two-dimensional magnet with spin $S = 1$, in the vicinity of a special point where the model symmetry is enhanced to $SU(3)$. It is shown that upon perturbing the $SU(3)$ symmetry, solitons with odd topological charge become unstable and bind into pairs.

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Introduction.– Many condensed matter systems can be successfully described with the help of effective continuum field models. In systems with reduced spatial dimensionality, topologically nontrivial field configurations are known to play an important role. Magnetic systems are usually modeled with the help of the Heisenberg exchange interaction. In many instances the fluctuations of the length of the local magnetic moment occur at a large energy scale and can be neglected; the continuum field theory in that case is the so-called $O(3)$ nonlinear sigma model (NLSM) describing the dynamics of a three-component real unit vector field, and the topological excitations of this model are well understood.

However, for spin $S > 1/2$ the general isotropic exchange goes beyond the purely Heisenberg interaction bilinear in spin operators $S_i$, and may include higher-order terms of the type $(S_i S_j)^n$ with $n$ up to $2S$. Particularly, a general $S = 1$ model with the isotropic nearest-neighbor exchange on a two-dimensional (2d) square lattice is described by the Hamiltonian

$$
\mathcal{H} = -\sum_{\langle ij \rangle} h_{ij} - J (S_i S_j) + K (S_i S_j)^2,
$$

where $\langle ij \rangle$ denotes the sum over nearest neighbors, and $J$ and $K > 0$ are respectively the bilinear (Heisenberg) and biquadratic exchange constants. The model has been discussed recently in connection with $S = 1$ bosonic gases in optical lattices and in the context of the deconfined quantum criticality. The effective field theory for the above model is generally more complicated than NLSM: the order parameter belongs to the 2d complex projective space $CP^2$, and at two points, $J = K$ and $J = 0$, the model symmetry is enlarged to $SU(3)$.

Continuum field description.– The spin-1 state $|\psi\rangle_j$ at a given site $j$ is a linear superposition of three basis states $|\sigma\rangle_j$ with $S_j^z|\sigma\rangle_j = \sigma|\sigma\rangle_j$, $\sigma = 0, \pm 1$. It is convenient to write down the spin-1 state at site $j$ as

$$
|\psi\rangle_j = \sum_{a=x,y,z} t_{j,a}|\alpha\rangle_j,
$$

using the “cartesian” states $|\alpha\rangle = |0\rangle, |x\rangle = (|1\rangle - |+1\rangle)/\sqrt{2}, |y\rangle = i(|1\rangle - |+1\rangle)/\sqrt{2}$, then the three numbers $t_{j,a}$ transform under rotations as the components of a complex vector $t_j$. The normalization condition brings the constraint $t_j^* \cdot t_j = 1$. The states can be viewed as $SU(3)$ coherent states corresponding to the bosonic operators $t_{j,a}$, and the $S = 1$ operator can be represented as $S_j^z = -i\epsilon_{abc} t_{j,b} t_{j,c}$. Taking into account that the state may contain an arbitrary overall phase factor, one concludes that the order parameter space $M$ of the problem is four-dimensional and isomorphic to $CP^2$.

The lattice Lagrangian of the model expressed in terms of the complex unit vector $t$ takes the form

$$
\mathcal{L} = \sum_j i(t_j^* \cdot \partial t_j) - W,
$$

where the local Hamiltonian average $\langle \hat{h}_{i,j} \rangle$ is given by

$$
\langle \hat{h}_{i,j} \rangle = J(t_i^* \cdot t_j)(t_j^* \cdot t_i) + (J - K)(t_i^* \cdot t_j^*)(t_i \cdot t_j).
$$

This makes obvious that the system is always invariant under global rotations $t_{j,a} \mapsto R_{ab} t_{j,b}$, with an arbitrary $O(3)$ rotation matrix $R$, as well as under local “gauge” transformation $t_j \mapsto t_j e^{i\chi_j}$. At $J = K$ the symmetry becomes higher as there is an invariance under a global transformation $t_j \mapsto U t_j$ with $U \in SU(3)$. Moreover, if the lattice is bipartite, at $J = 0$ the energy is invariant under making an arbitrary $SU(3)$ rotation on the sites belonging to one sublattice if this is accompanied by a conjugate transformation $t_j \mapsto U^* t_j$ at the other sublattice, so the point $J = 0$ is $SU(3)$-invariant as well.

 Breaking up the complex vector $t = u + iv$ into two real vectors representing its real and imaginary parts, one can write on the site spin and quadrupole averages as

$$
\langle S \rangle = 2(u \times v),
$$

$$
S_{ab} \equiv \langle S_a S_b + S_b S_a \rangle = 2(\delta_{ab} - u_a u_b - v_a v_b).
$$

(5)
One can use a different parametrization, directly connected to the physical averages, by introducing the eight-component vector \( n \),

\[
n_\alpha = \text{tr}(t^* \cdot \hat{\lambda}_\alpha t),
\]

where \( \hat{\lambda}_\alpha, \alpha = 1, \ldots 8 \) are the well-known Gell-Mann matrices which form, together with a unit matrix \( \mathbb{1} \), a basis in the \( SU(3) \) matrix space. The vector \( n \) is subject to the following two constraints:

\[
n^2 = 4/3, \quad n \cdot (n \times n) = 8/(3\sqrt{3}),
\]

where the \( \times \)-product of any two vectors \( n \) and \( n' \) is defined as \( (n \times n')_\alpha = \sqrt{3}d_{\alpha\beta\gamma}n_\beta n'_\gamma \), and \( d_{\alpha\beta\gamma} \) are the structure constants defined by the anticommutation properties of the Gell-Mann matrices, \( \{\lambda_\alpha, \lambda_\beta\} = \frac{4}{3}\delta_{\alpha\beta}\mathbb{1} + 2d_{\alpha\beta\gamma}\lambda_\gamma \).

One can show that the constraints (7) in fact reduce the dimension of the \( n \)-space to four. The quantities \( n_\alpha \) correspond to the following on-site averages:

\[
\begin{align*}
n_2 &= \langle S_z \rangle, & n_5 &= -\langle S_y \rangle, & n_7 &= \langle S_x \rangle, \\
n_4 &= S_{xz}, & n_6 &= S_{yz}, & n_1 &= S_{xy}, \\
n_3 &= (S_{xx} - S_{yy})/2, & n_8 &= \sqrt{3}S_{zz}/(2 - 2/3),
\end{align*}
\]

which can be split into the vector of spin averages \( m \) and the vector of quadrupolar averages \( d \),

\[
m = (n_7, -n_5, n_2), \quad d = (n_1, n_3, n_4, n_6, n_8).
\]

In these variables, the Hamiltonian takes the simple form

\[
\langle h_{ij} \rangle = -\frac{K}{3} - \frac{K}{2} (d_i \cdot d_j) + \frac{1}{2}(K - 2J)(m_i \cdot m_j),
\]

which explicitly shows that \( J > K \) corresponds to a ferromagnet (FM), \( J < 0 \) to an antiferromagnet (AFM), and \( 0 < J < K \) to a quadrupolar (spin nematic) order (hereafter we assume that \( K > 0 \) and will not discuss the so-called orthogonal spin nematic present at \( K < 0 \)).

In terms of \( n \), the lattice Lagrangian can be written as

\[
\mathcal{L} = \sum_j \Phi(n_j) - \sum_{(ij)} \langle h_{ij} \rangle,
\]

with the dynamic part

\[
\Phi(n) = \frac{3n_0 \cdot (n \wedge \partial_t n)}{4 + \frac{1}{2}n_0 \cdot n}.
\]

Here the \( SU(3) \)-crossproduct is defined as \( (n \wedge n')_\alpha = f_{\alpha\beta\gamma}n_\beta n'_\gamma \), where \( f_{\alpha\beta\gamma} \) is another set of structure constants defined by commutators of the group generators \( \{\lambda_\alpha, \lambda_\beta\} = 2if_{\alpha\beta\gamma}\lambda_\gamma \), and \( n_0 \) is an arbitrary vector satisfying the constraints (7).

Topological analysis. – To describe topological solitons, one needs to pass to the continuum description first. The continuum Lagrangian of the model (3) can be obtained by the gradient expansion of the discrete energy \( W \) retaining the leading terms, that gives \( W = \int d^2x \) with

\[
w = J\{[\partial_t t^2] - [t^* \cdot \partial_t t^2] + (J - K)|t^2|^2 \\
- (J - K)\{t \cdot \partial_t t^2 + \frac{1}{2}[t^* (\partial_t t^2)^2 + c.c.]\},
\]

where \( \mu \) runs over space coordinates \( (x, y) \). The above form is valid for the region \( J > K/2 \), where the short-range spin-spin correlations are of the ferromagnetic type, as can be seen from (10).

To classify the topological excitations, one needs to know the so-called degeneracy space \( \mathbb{M}_D \) that includes all values of the order parameter field corresponding to the ground state of the system. For the model (1) the space \( \mathbb{M}_D \) is continuous and depends on the type of the ground state: for FM or AFM it coincides with the unit sphere \( S^2 \), for the nematic case it is a 2d real projective space \( RP^2 = S^2/Z^2 \) (a unit sphere with the opposite points identified), and at \( J = K \) the degeneracy space is enlarged to \( CP^2 \). For all the above spaces, the second homotopy group is nontrivial, \( \pi_2(\mathbb{M}_D) = \mathbb{Z} \) which makes possible the existence of so-called localized topological solitons, whose order parameter distribution becomes uniform away from some point.

If the order parameter lies completely in \( \mathbb{M}_D \), the energy contains only terms with gradients, so there is no natural space scale. If corresponding soliton solutions exist, they have a finite energy which does not depend on their size, and are stable against collapse. Another possibility is to allow the order parameter to leave \( \mathbb{M}_D \), which breaks the scale invariance. Static solitons of that type are unstable against collapse due to the Hobart-Derrick theorem, but they can be stabilized by some internal dynamics (1-2). We will study the structure of both types of solitons for the model (1).

For the sake of analyzing static soliton solutions the Lagrangian (3) with the energy (12) is equivalent to the 2d \( CP^2 \) model (6) with an additional “anisotropy term” proportional to \( (J - K) \). Let us start from the \( SU(3) \)-symmetric point \( J = K \). In that case a localized topological soliton corresponds to the field configuration with nonzero topological charge (3):

\[
q = -\frac{i}{2\pi} \int d^2x \epsilon_{\mu\nu} (\partial_\mu t^* \cdot \partial_\nu t),
\]

where the indices \( \mu, \nu \) run over \( (x, y) \). The invariant (13) takes only integer values and corresponds to the mapping of the compactified 2d space \( S^2 \) onto \( CP^2 \). The exact \( q = 1 \) soliton solution is well known (6):

\[
t = (\xi a + z b)/\sqrt{|z|^2 + \xi^2},
\]

where \( z = x + iy \) is the complex coordinate (the soliton center is assumed to be at the origin), \( a \) and \( b \) are two mutually orthonormal complex vectors, and \( \xi \) has the meaning of the soliton size. The energy of such excitation according to (12) is \( E = 2\pi K \). For an arbitrary value of \( q \), the general soliton solution can be written as

\[
t_a = \frac{f_a}{(\sum_a |f_a|^2)^{1/2}}, \quad f_a = c_a \prod_{k=1}^a (z - z_{k,a}), \quad a = x, y, z,
\]

(15)
and the corresponding energy is \( E = 2\pi K |q| \).

**Ferromagnetic solitons.**— On the ferromagnetic side \( J > K \) the minimum of energy is achieved for

\[
t = (e_1 + ie_2)/\sqrt{2}
\]

with \( e_{1,2} \) being a pair of orthogonal real unit vectors. In that case on the degeneracy space \( \mathbb{M}_D \) the order parameter is equivalent to the unit vector \( \mathbf{m} = (e_1 \times e_2) \) (a rotation around \( \mathbf{m} \) corresponds to a change of the overall phase factor \( t \to te^{i\phi} \) and thus does not change the physical state). Thus, localized topological solitons for \( J > K \) correspond to the mapping \( S^2 \to S^2 \) and are characterized by another topological charge

\[
Q_m = \frac{1}{8\pi} \int d^2 x \varepsilon_{\mu\nu} \mathbf{m} \cdot (\partial_\mu \mathbf{m} \times \partial_\nu \mathbf{m}).
\]  

(17)

It is easy to calculate the topological charge [13] for a restricted field configuration satisfying (16): a general pair of orthonormal vectors \( e_{x,y} \) by an arbitrary rotation \( \mathbf{R}(\theta, \varphi, \psi) \), where \( \theta \) and \( \varphi \) are respectively the polar and azimuthal angles characterizing the direction of the unit magnetization vector \( \mathbf{m} \), and the third angle \( \psi \) corresponds to the rotation around \( \mathbf{m} \). A straightforward calculation yields

\[
q = \frac{1}{2\pi} \int d^2 x \sin \theta \epsilon_{\mu\nu}(\partial_\mu \theta)(\partial_\nu \varphi) = 2Q_m.
\]  

(18)

One is led to conclude that solitons of the \( CP^2 \) model tend to pair upon perturbing the \( SU(3) \) symmetry, which constitutes the central observation of the present paper.

The above result can be also obtained by noticing that for the configurations (10) the energy takes the form \( W = (J/2) \int d^2 x (\partial_\mu m_\mu)^2 \). This is exactly the energy of the \( O(3) \) NLSP, and the well-known Belavin-Polyakov (BP) soliton solution [7] with the topological charge \( Q_m = 1 \) will have the energy \( E = 4\pi J \), which in the limit \( J \to K \) is twice the energy of the \( q = 1 \) soliton of the \( CP^2 \) model. In fact, one can explicitly check that the ferromagnetic BP soliton is a particular solution of the general solution (15) with \( q = 2 \).

**Solitons for spin nematic.**— On the nematic side \( J < K \) the minimum of energy is reached for \( t = u e^{i\chi} \), where \( u \) is a real unit vector and \( \chi \) is an arbitrary phase. The degeneracy space is thus \( \mathbb{M}_D = RP^2 \). The energy then takes the form \( W = K \int d^2 x (\partial_\mu u)^2 \), where \( u \) must be understood as a director, i.e., \( u \) and \( -u \) are physically identical. It is worth noting that in contrast to the other phases the spin nematic allows for a nontrivial \( \pi_1 \)-topological charge as well, \( \pi_1(RP^2) = \mathbb{Z}_2 \).

If one defines the topological charge \( Q_u \) according to (17), simply replacing \( \mathbf{m} \) by \( u \), then in the BP soliton with \( Q_u = 1 \) the director \( u \) goes over \( \mathbb{M}_D \) twice; the energy of such a solution is \( E_{BP} = 8\pi J \). However, the director property of \( u \) allows one to construct a solution [8] with \( u \) going over \( \mathbb{M}_D \) just once, which has \( Q_u = 1/2 \) and the energy \( \tilde{E}_{BP} = 4\pi J \). In the limit \( J \to K \) this is again twice as much as the energy of the \( q = 1 \) solution (14), which suggests that this soliton is a descendant of the \( q = 2 \) solution of the \( CP^2 \) model. This indicates that the tendency to pairing exists on the nematic side as well.

The fate of solitons with \( q = 1 \).— Up to now we have considered only static solitons with the order parameter lying completely inside \( \mathbb{M}_D \). We found that for \( J \neq K \) the lowest energy solutions of that type are descendants of \( q = 2 \) solitons of the \( CP^2 \) model, while the \( q = 1 \) solution seems to exist only at \( J = K \). To get further understanding of what happens in the vicinity of the \( SU(3) \)-symmetric point \( J = K \), let us discuss the \( CP^2 \)-soliton with \( q = 1 \) for small but finite \( J - K \). One can easily see that at \( J \neq K \) any solutions with \( q = 1 \) must involve a deviation of the order parameter from the degeneracy space \( \mathbb{M}_D \). Due to the Hobbart-Derrick theorem, this means instability of static solitons with \( q = 1 \) against collapse. However, \( q = 1 \) solitons can be stabilized by internal dynamics in presence of additional integrals of motion, e.g., stable solitons with the magnetization vector precessing around the easy axis exist in the uniaxial ferromagnet [2]. In our case, it is also possible to construct such a solution. In terms of the complex vector \( t = u + iv \) this is a planar configuration, where \( u \) and \( v \) are parallel to the plane (1, 2) orthogonal to some axis \( e_3 \), for definiteness let it be the z axis (a more general solution can be obtained by an arbitrary rotation). It is convenient to use the 8-vector notation (6): only four components of \( \mathbf{n} \) are nonzero and it takes the form

\[
\mathbf{n} = (R_x, R_z, R_y, 0, 0, 0, 0, 1/\sqrt{3}),
\]  

(19)

where \( \mathbf{R} \) is a unit vector combining one spin average \( R_z = m_3 \) and two quadrupolar variables \( R_x = d_1, R_y = d_2 \) (cf. (8)). Using (10) and (11), one obtains the effective Lagrangian for the chosen subspace,

\[
\mathcal{L}_R = \frac{1}{2} \sum_{ij} \frac{R_0 \cdot (R_j \times \partial_j R_j)}{1 + R_0 \cdot R_j} - W_R
\]  

(20)

\[
W_R = -\sum_{i<j} \left[ \frac{K}{2} R_i R_j + (J - K) R_{iz} R_{zj} \right].
\]

where \( R_0 = (0, 0, -1) \), and in (11) we have used \( n_0 = (0, 0, -1, 0, 0, 0, 0, 0, 1/\sqrt{3}) \). The Lagrangian (20) describes the dynamics of a classical anisotropic ferromagnet with the unit magnetization vector \( \mathbf{R} \); the anisotropy constant is proportional to \( J - K \). At the isotropic point \( J = K \) the energy \( W_R = (K/2) \int (\nabla \mathbf{R})^2 d^2 x \), and there exists a BP-type soliton that has the energy \( E_1 = 2\pi K \) and is a special case of the \( q = 1 \) \( CP^2 \) solution (14). The \( CP^2 \) charge \( q \) given by (14) is obviously equal to the Pontryagin index \( Q_R \) defined by (17) with \( \mathbf{m} \to \mathbf{R} \); the BP solution corresponds to the mapping of \( S^2 \) onto the subspace \( CP^1 \) embedded into \( CP^2 \) and has \( Q_R = q = 1 \).
For a finite “anisotropy” \((J - K)\) the BP soliton becomes unstable against collapse, but the situation is different for the spin-nematic and FM regions. In the FM case \((J > K)\) the anisotropy is of the easy-axis type, and there exist \(Q_R = q = 1\) dynamic solutions, with \(R\) precessing around the \(z\) axis \([2]\), which are smoothly connected to the BP solitons in the \(J \to K\) limit. A detailed analysis \([3]\) shows that the minimal energy of such dynamic solitons exhibits a nonanalytical behavior of the type \(E_{\text{min}} = E_{J=K}(1 + 3.74 \sqrt{J/K - 1})\), as shown in Fig. 1. For small \(J - K \leq 0.1K\), when the above expression is valid, the energy of a static \(q = 2\) \((Q_m = 1)\) soliton considered above stays higher than the energy of the \(q = 1\) \((Q_R = 1)\) dynamical soliton, but at the same time, it remains smaller that the energy of two dynamical \(q = 1\) solitons, which indicates that it is energetically favorable to bind two \(Q_R = 1\) solitons into a single \(Q_m = 1\) one.

In the nematic case \((J < K)\) we effectively have a ferromagnet with the easy-plane anisotropy. For such case, delocalized \(\pi_1\)-solitons (vortices) exist. Vortices in \(R\)-field correspond to spin-nematic disclinations considered in Ref. \([10]\). The energy of a single vortex diverges logarithmically with the system size, so a static vortex-antivortex pair is unstable against collapse. The BP soliton can be considered as a pair of “merons” carrying topological charge \(Q_R = \frac{1}{2}\) each \([11]\). For small \((K - J)\) those “merons” can be viewed as a vortex and antivortex with a finite out-of-plane component of the vector \(R\), they are subject to a gyroforce \([10]\), and there may exist stable dynamic solutions (rotational pairs of vortices) similar to those studied in Ref. \([12, 13]\). Their energy will tend to \(2\pi K\) in the limit \(J \to K\); similarly to the FM case, in the vicinity of the \(J = K\) point the \(Q_R = 1\) topological solitons will be unstable against pairing into “nematic” Belavin-Polyakov solitons with \(Q_u = \frac{1}{2}\).

Finally, a few words are to be said about the other, antiferromagnetic \(SU(3)\)-symmetric point \(J = 0\). From \([1]\) one can see that on any bipartite lattice the transformation \(t_j \mapsto t_j^*\) for all \(j\) belonging to one sublattice maps the points \(J = 0\) and \(J = K\) onto each other. As can be seen from \([10]\), and is especially clear from the “spin analogy” \([20]\), for \(J < K/2\) the short-range correlations are antiferromagnetic, and the proper transition to the continuum description becomes more complicated; however, one can show that the difference concerns only dynamics and does not affect the static properties. The arguments leading to \([18]\) and thus the conclusion on soliton pairing equally apply to the vicinity of \(J = 0\) point.

**Summary.**—We have studied the structure of topologically nontrivial solitons in a general non-Heisenberg model of the 2d isotropic \(S = 1\) magnet. In the vicinity of special points with \(SU(3)\) symmetry the system can be described with the help of the \(CP^2\) model. It is shown that when the \(SU(3)\) symmetry is broken down to \(SU(2)\), solitons of the \(CP^2\) model with odd topological charge become unstable and bind into pairs.

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