Microcanonical Thermostatistical Investigation
of the Blackbody Radiation

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In this work is presented the microcanonical analysis of the blackbody radiation. In our model the electromagnetic radiation is confined in an isolated container with volume $V$ in which the radiation can not escape, conserving this way its total energy, $E$. Our goal is to precise the meaning of the Thermodynamic Limit for this system as well as the description of the nonextensive effects of the generalized Planck’s formula for the spectral density of energy. Our analysis shows the sterility of the intents of finding nonextensive effects in normal conditions, the traditional description of the blackbody radiation is extraordinarily exact. The nonextensive effects only appear in the low temperature region, however, they are extremely difficult to detect.

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I. INTRODUCTION

In the present paper we recover the famous problem of the macroscopic description of the blackbody radiation. In the last years some investigators have been reconsidered some aspects of this theory due to the coming of new ideas and conceptions during the development of a new thermodynamics: that which allows the study of the nonextensive systems.

In this frame, this problem have been approached by mean of the very well known nonextensive thermodynamic of Tsallis \cite{1-4}. In general way this conception constitutes a generalization of the Boltzmann-Gibbs formalism. It is based mainly in the introduction of nonextensive entropy, $S_q$, which is a generalization of the Shannon-Boltzmann-Gibbs extensive entropy \cite{2-3}:

$$ S_q = - \sum_k p_k^q \ln_q p_k $$ \hfill (1)

where $\ln_q x$ is the $q$-generalization of the logarithmic function:

$$ \ln_q x = \frac{x^{1-q} - 1}{1 - q} $$ \hfill (2)

Among the fundamental aspects of this new conception we find the possibility to deal the study of systems with potential distributions, that is, systems with fractal characteristics. In spite of the attractiveness of this formulation, the same one suffers of a logical internal inconsistence due to dependence in all the theory of a phantasmagorical parameter, the entropic index, $q$. This parameter represents the measure of the degree of nonextensivity of the systems, an intrinsic characteristic of the same, however it can not be inferred from the general principles. In fact, it acts as a adjustment parameter of the theory, we need the experiment or the computational simulation in order to precise it.

In the present system there are no long-range correlations due to the presence of long-range interactions. In fact, this is a typical extensive system. The only possible nonextensive effects in the theory of the blackbody radiation could be found due to the non realization of the Thermodynamic Limit, that is, due to the finite nature of the system.

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It is very interesting to specify the meaning of the thermodynamic limit for this system, as well as which are the main peculiarities of the physical quantities product to the finite nature of the same one.

To reach these objectives we will make use of the main results of the microcanonical thermostatistic of D.H.E. Gross [7,8]. This theory returns to the pre-gibbsian times assuming the Boltzmann’s definition of entropy:

\[
S_B = \ln W (E; a)
\]  

(3)

his famous gravestone’s epitaph. The thermodynamic formalism of this formulation has been conceived in order to be equivalent in the thermodynamic limit with the formalism of the classical thermodynamic for those systems that become in this limit in a ordinary extensive systems [9–11], so that, it could be applied to this kind of finite systems. Our problem is in fact a typical example for the application of this theory.

II. MICROCANONICAL ANALYSIS OF THE BLACKBODY RADIATION.

In the present section we will perform the microcanonical analysis of the blackbody radiation. Firstly, we will expose some general aspects of the microcanonical approaching of the quantum systems of identical noninteracting particles. After, we will begin the analysis remembering some results of the traditional thermodynamic. Subsequently the density of accessible states will be calculated, being determined this way the Boltzmann’s entropy of the system. It will be carried out a comparison between the microcanonical temperature derived of this entropy with the absolute temperature of the traditional description. This analysis will help to specify the meaning of the thermodynamic limit for the blackbody radiation.

The generalization of the Planck’s formula for the spectral density of energy will be obtained analyzing its more significant particularities due to the finite nature of the system as well as their behavior during the step to the thermodynamic limit.

A. Microcanonical description of a quantum system of identical noninteracting particles.

In analogy to the classical case, the macroscopic state of the system is determined by the knowledge of certain set of integrals of movement, \{I\}. From the general theory it is known that macroscopic observables are obtained by the relation:

\[
\mathcal{O} = \frac{Sp (\hat{O} \hat{\rho})}{Sp (\hat{\rho})}
\]

(4)

where \(\hat{\rho}\) is matrix density operator. For the microcanonical description:

\[
\hat{\rho}_M (I; a) \equiv \delta \left( I - \hat{I} (a) \right)
\]

(5)

where \(a\) represent the external parameters of the systems. Let us pay attention to the accessible states density of the system:

\[
\Omega (I; a) = Sp \left( \delta \left( I - \hat{I} (a) \right) \right)
\]

(6)

For the special case of the systems of noninteracting particles is convenient to make use of the integral representation of the Dirac delta function:

\[
\Omega (I; a) = \int d^n \xi \exp \left( i \xi \cdot I \right) Sp \left( \exp \left( -i \xi \cdot \hat{I} (a) \right) \right)
\]

(7)

where \(d^n \xi = \frac{d^n \xi}{(2\pi)^n}\) and \(\xi = \xi - i\beta\) with \(\beta \in \mathbb{R}^n\). We recognize immediately the partition function of the canonical description, this time with complex argument:

\[
\mathcal{Z} (z; a) = Sp \left( \exp \left( -z \cdot \hat{I} (a) \right) \right) \quad (z = \beta + i\xi)
\]

(8)

Introducing the Planck’s potential, \(\mathcal{P} (z; a)\):
the Eq. is rewritten as:

\[ \Omega (I; a) = \int d^3 \xi \exp (i \xi \cdot I - P (i \xi; a)) \]  

(10)

It is very well known that the Planck’s potential of a system of identical noninteracting particles is expressed as:

\[ P (z; a) = -\eta \sum_k g_k (a) \ln [1 + \eta \exp (-z \cdot i_k (a))] \]  

(11)

where the index \( k \) represent the state with eingenvalues \( i_k (a) \) for the integrals of movement and degeneracy \( g_k (a) \). The constant \( \eta \) is equal to \(-1\) for the bosons or \(+1\) for fermions. When the thermodynamic limit takes place in the system the microcanonical and the canonical description are equivalent if under the scaling transformation the integrals of movement and the Planck’s potential are scaled homogeneously:

\[ I \rightarrow \alpha I; \ a \rightarrow \alpha^s a \ (s = 0, 1) \Rightarrow P (z; a) \rightarrow \alpha P (z; a) \]  

(12)

In this case will be valid the Legendre transformation between the fundamental potentials of the ensembles:

\[ S_B (I; a) \approx \min_{\beta} \{ \beta \cdot I - P (\beta; a) \} \]  

(13)

When the minimum requirement is not satisfied it is due to the occurrence of critical phenomena in the systems i.e., phase transitions. Outside the thermodynamic limit or during the phase transitions the only way to describe the system is microcanonically. Ordinarily we have to deal with finite systems, so that, we need to precise when we could consider that the thermodynamic limit takes place in it.

B. The model

Let us consider an isolated container with volume \( V \) in which the electromagnetic radiation has been confined and it can not scape. Let us suppose too that this container is the sufficiently large in order to be valid the continuum approximation for the occupational states density:

\[ g_p (V) = 2 \frac{V 4 \pi p^2 dp}{(2\pi \hbar)^3} = \frac{V}{\pi^2 \hbar^3} p^2 dp \]  

(14)

The only one integral of movement that we consider here is the total energy, \( E \). For the state with momentum \( p \) the correspondent energy eingenvalue is \( \varepsilon_p = pc \), where \( c \) is the speed of light in vacuum. The Planck’s potential in this case is given by:

\[ P (z; V) = \frac{V}{\pi^2 \hbar^3} \int_0^{+\infty} p^2 \ln [1 - \exp (-zpc)] dp \]  

(15)

\[ = -\frac{V}{3\pi^2 \hbar^3} \frac{z^2 c}{\exp (zpc) - 1} \]  

\[ = -\frac{\tilde{\sigma} V}{3z^3} \left( \text{with } \tilde{\sigma} = \frac{\pi^2}{15\hbar^3 c^3} \right) \]

From the above relation is inferred that the thermodynamic limit is carried out when \( \{ V \rightarrow \infty \mid \frac{\tilde{\sigma} V}{3z^3} \sim \text{const} \} \). Substituting \( z \) by \( \beta \), in correspondence with the canonical description, the total energy of the system is given by:

\[ E = \frac{\partial}{\partial \beta} P (\beta; V) = \tilde{\sigma} V \beta^{-4} \]  

(16)

arriving this way to the Stephan-Boltzmann’s law:

\[ \rho = \frac{E}{V} = \frac{4}{3} \sigma T^4 \]  

(17)
where $\sigma = \frac{2 \hbar c}{\pi m c^3}$ is the Stephan-Boltzmann constant. On other hand, the pressure is expressed as:

$$p = -kT \frac{\partial}{\partial V} \mathcal{P} (\beta; V) = \frac{1}{3} \bar{\sigma} V \beta^{-4} \equiv \frac{1}{3} \bar{\sigma}$$

(18)

Applying the Legendre transformation we can find the information entropy of the system, $S_c (E; V)$:

$$S_c (E; V) = \beta E - \mathcal{P} (\beta; V) = \frac{4}{3} \bar{\sigma} V \beta^{-3}$$

(19)

$$S_c (E; V) = \frac{4}{3} (\bar{\sigma} V E^3)^{\frac{4}{3}}$$

(20)

According to the Eq.[10] the states density is obtained by the calculation of the following integral:

$$\Omega (E; V) = \int_{-\infty}^{+\infty} d\xi \exp \left( i \xi E + \frac{1}{3} \bar{\sigma} V \frac{1}{(i \xi)^3} \right)$$

(21)

whose integration yields:

$$\Omega (E; V) = \frac{1}{E} \sum_{n=1}^{\infty} \frac{1}{n! (3n-1)!} \left( \frac{1}{3} \bar{\sigma} V E^3 \right)^n$$

(22)

The function $W (E; V)$ in the definition of the Boltzmann`s entropy (the Eq.[3]) is obtained multiplying the states density by a suitable energy constant, $\varepsilon_0$. A reasonable choice of $\varepsilon_0$ for the blackbody radiation is:

$$\varepsilon_0 = \frac{\pi \hbar c}{V^{\frac{4}{3}}} = p_0 c$$

(23)

where $p_0$ is the length of the elemental cells in the momentum space for the present problem. This way, the function $W (E; V)$ is given by:

$$W (E; V) = \left( \frac{256 \pi^5}{1215} \right)^{\frac{4}{3}} \frac{1}{(S_c)^{\frac{4}{3}}} H (S_c)$$

(24)

where $S_c$ is the canonical entropy of the Eq.[20] and the function $H (x)$ is given by the expression:

$$H (x) = \sum_{n=1}^{\infty} \frac{1}{n! (3n-1)!} \left( 3^4 \left( \frac{x}{4} \right)^4 \right)^n$$

(25)

and therefore, the Boltzmann`s entropy is expressed by:

$$S_B (E; V) = \ln H (S_c (E; V)) - \frac{4}{3} \ln S_c (E; V) + const$$

(26)

In order to analyze the realization of the thermodynamic limit let us to find the microcanonical temperature, $T_m$:

$$\beta_m = \frac{1}{kT_m} = \frac{\partial}{\partial E} S_B = \left( \frac{\partial S_B}{\partial S_c} \right) \frac{\partial}{\partial E} S_c$$

(27)

Defining the function $\gamma (S_c)$ by:

$$\gamma (S_c) = \left( \frac{\partial S_B}{\partial S_c} \right)$$

(28)

we can express the microcanonical temperature as:

$$T_m = \gamma^{-1} (S_c) T_c = \gamma^{-1} (S_c (E; V)) \left( \frac{c E}{4 \sigma V} \right)^{\frac{4}{3}}$$

(29)
The Eq. 29 constitutes the microcanonical generalization of the Stephan-Boltzmann law (see fig.1). The function \( \gamma^{-1}(S_c) \) allows us to evaluate the convergency of the microcanonical temperature, \( T_m \), to the canonical one, \( T_c \) (see fig.2). In the thermodynamic limit we expect that it approaches to the unity. This means that the function \( H(x) \) should have an exponential behavior for \( x \gg 1 \). In effect, for large values of \( x \) the \( n \)-th term in the serie Eq. 25 is comparable with:

\[
\frac{1}{n!(3n-1)!} \left( \frac{3}{4} \right)^n \sim \frac{1}{(4n)^{3n}} x^{4n} \sim \frac{x^{4n}}{(4n)!}
\]

and therefore, \( H(x) \propto \exp(x) \). A better analysis shows that \( H(x) \) converges asymptotically for large values of \( x \) to:

\[
H(x) \approx C x^\theta \exp(x)
\]

where \( C = 0.171(2) \) and \( \theta = 0.501(4) \) (this asymptotical behavior is almost exact with great accuracy for \( x > 80 \)). From here, we find the asymptotic behavior for the Boltzmann’s entropy:

\[
S_B \approx S_c - \left( \frac{4}{3} - \theta \right) \ln S_c
\]

and the function \( \gamma(S_c) \):

\[
\gamma(S_c) \approx 1 - \left( \frac{4}{3} - \theta \right) \frac{1}{S_c}
\]

We can say with a precision of \( 10^{-3} \) that the thermodynamic limit is established in the system when \( S_c > S^* = 800 \). On other hand, the total energy of the system must satisfy the exigency:

\[
E > \left( \frac{1215}{256 \pi^2} S^* \right)^{\frac{3}{4}} \epsilon_0 \approx 1.8 \times 10^3 \epsilon_0
\]

This numerical result shows the high degree of accuracy that the traditional methodology offers to the macroscopic description of the blackbody radiation in ordinary conditions. In terms of the temperature the above condition is rewritten as:

\[
kT_c > \left( \frac{45}{4 \pi^5} S^* \right)^{\frac{3}{4}} \epsilon_0 \approx 3.1 \epsilon_0
\]

It is very interesting to evaluate numerically all the above conditions. For the elemental energy constant, \( \epsilon_0 \), we have:

\[
\epsilon_0 = \frac{6.2406 \times 10^{-25}}{V^\frac{1}{3}} \text{ J (V in m}^3\text{)}
\]

and therefore:

\[
T_c > \frac{0.13952}{V^\frac{1}{3}} \text{ K (V in m}^3\text{)}
\]

This is a very important result because it shows that the predictions of our model could be corroborated in low temperature experiments. Let us now to study the microcanonical modifications to the Planck’s formula for the spectral density of energy. From the definition Eq. 3 we can obtain the particles density by mean of the integral:

\[
\Omega(E; V) n_p(E; V) = \int_{-\infty}^{+\infty} d\tilde{\xi} \exp \left( i\tilde{\xi} E + \frac{1}{3} (\tilde{\xi})^3 \right) \frac{g_p(V)}{\exp(i\tilde{\xi} \epsilon_p) - 1}
\]

Easily is obtained the following result:

\[
n_p(E; V) = g_p(V) \sum_{n=1}^{N_p} \frac{W(E - n \epsilon_p; V)}{W(E; V)} \text{ with } N_p = \left[ \frac{E}{\epsilon_p} \right]
\]
Multiplying the particles density by the energy eigenvalue of the state, rewriting it again in terms of the frequencies making use of the Planck’s relation:

\[ \varepsilon_p = pc = \hbar \omega \]  

we have finally the microcanonical spectral energy density, \( u_m(\omega; E, V) \):

\[ u_m(\omega; E, V) \, d\omega = \frac{\omega^2}{\pi^2 c^3} \hbar \omega \sum_{n=1}^{N_\omega} \frac{W(E - n\hbar \omega; V)}{W(E; V)} \, d\omega \quad \text{with} \quad N_\omega = \left[ \frac{E}{\hbar \omega} \right] \]  

Among the most interesting peculiarities of the above expression we found a cut off at the frequency \( \omega_c \):

\[ \omega_c = \frac{E}{\hbar} \]  

around the same one the function \( u_m(\omega; E, V) \) behaves as:

\[ u_m(\omega \approx \omega_c; E, V) \, d\omega \approx \frac{1}{6\pi^2 c^3} \frac{\tilde{\sigma} V E^3}{W(E; V)} (\omega_c - \omega)^2 \, d\omega \]  

On other hand, at the low frequency limit we have:

\[ u_m(\omega \approx 0; E, V) \, d\omega \approx \frac{\omega^2}{\pi^2 c^3} kT_m d\omega \]  

This is the microcanonical generalization of the Raleigh-Jean’s formula, where \( T_m \) is the microcanonical temperature. Finally, during the thermodynamic limit:

\[ u_c(\omega; E, V) \, d\omega = \frac{\omega^2}{\pi^2 c^3} \hbar \omega \lim_{V \to \infty} \lim_{(T-\text{const})} \left( \sum_{n=1}^{N_\omega} \exp \left[ \ln W(E - n\hbar \omega; V) - \ln W(E; V) \right] \right) \, d\omega \]  

\[ u_c(\omega; T_c) \, d\omega = \frac{\omega^2}{\pi^2 c^3} \frac{\hbar \omega d\omega}{\exp (\beta_c \hbar \omega) - 1} \]  

we arrive to the celebrated Planck’s formula.

### III. CONCLUSIONS

In the present work we have found the conditions in which can be considered that the thermodynamic limit has been established for the electromagnetic radiation confined in an isolated container with volume \( V \). Our calculations showed that the total energy, \( E \), only needs to be some thousands of times the elementary energy constant, \( \varepsilon_0 \). In normal conditions the traditional description of the blackbody radiation is extraordinary exact. This fact clarifies the sterility of the intents of finding nonextensive effects under these conditions. This conclusion is supported by direct experiments trying to find nonextensive effects in the cosmic microwave background radiation from the data obtained via Cosmic Background Explorer Satellite by Mather et al \cite{12}. The analysis of these data through the Statistic of Tsallis showed the great accuracy of the traditional description: with 95% of confidence \(|q - 1| < 3.6 \times 10^{-5}\) (the Tsallis’s Thermodynamic becomes in the traditional thermodynamic at \( q = 1 \)).

However, for the blackbody radiation confined at low temperatures in a finite and isolated container, our results predicts deviations with respect to the traditional analysis if those temperatures do not satisfy the condition in Eq.\[37\]. This nonextensive effects could be detected through of measurements of the spectral density of energy at low energies. According to the Eq.\[41\], at high frequencies must be detected the cut off frequency \( \omega_c \) which is related with the total energy via the Eq.\[42\]. The microcanonical temperature could be found analyzing the low frequency region of the spectral density by mean of the Eq.\[44\]. This way we can check the validity of the Stephan-Boltzmann’s law. Unfortunately there is no a ideal containers, since they are composed by a big number of particles, that is, they are extensive systems. This fact leads to hard difficulties for the practical realization of those experimental measurements.
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