Interaction of photons traversing a slowly varying electromagnetic background

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(Dated: June 19, 2014)

When two electromagnetic fields counterpropagate, they are modified due to mutual interaction via the polarised virtual electron-positron states of the vacuum. By studying how photon-photon scattering effects such as birefringence and four-wave mixing evolve as the fields pass through one another, we find a significant increase during overlap. The results have particular relevance for calculations based on a constant field background.

I. INTRODUCTION

That electromagnetic fields can polarise virtual electron-positron pairs of the vacuum has been known since the early pioneering calculations of Sauter [1], Halpern [2], Weisskopf [3], and Heisenberg and Euler [4], later being rederived in the language of quantum electrodynamics by Schwinger [5]. The polarised pairs facilitate the process of photon-photon scattering, which can be broadly split into inelastic processes such as vacuum pair-creation and elastic processes where fermion states do not persist on the mass shell. There are many predicted manifestations of elastic effects. The polarisation of scattered photons could be used to verify this phenomenon through birefringence, polarisation rotation [6–12] and helicity flipping [9, 13]. The propagation direction of scattered photons could also be used, through diffraction [14–17] and reflection [18–20] signals have been calculated. Also in the frequency of scattered photons, signals can occur through the process of four-wave mixing [17, 21], photon-splitting [8, 22–25] and photon-merging [26–28].

These phenomena are of interest in astrophysics, for example to describe the behaviour of magnetised neutron stars [29–33] and in particular astrophysical electromagnetic shocks [31–36], and also in high-intensity laser physics [37–39] being searched for in terrestrial experiments [40–43].

When photon wavelengths are much shorter than the length on which pair creation occurs, photon-photon scattering can be described using an effective theory for interacting electromagnetic fields given by the Heisenberg-Euler Lagrangian. Typically one considers the effect on some weak “probe” field, which can be a single photon, as it passes through a “strong” field. In applications to potential laser experiments, it is the asymptotic state of the probe field which is of primary interest as detection apparatus is necessarily far removed from the interaction region. In simulations of astrophysics in the magnetospheres of neutron stars, one typically calculates the effect on propagating photons in a classical magnetic field, which is taken to vary adiabatically, with the constant-field solution being integrated over macroscopic regions in kinetic equations [29, 44].

In the current paper, we focus on the evolution of an oscillating probe field that scatters in a slowly-varying strong background, with both fields being described as plane waves. Using the Heisenberg-Euler Lagrangian, we will identify a signal of elastic photon-photon scattering that increases with the overlap of the fields and disappears when the overlap tends to zero. Moreover, we will find that this scattered “overlap field” can be much larger than the “asymptotic” scattered field which persists after the probe has passed through the background, particularly for parameters considered in high-intensity laser experiments. The presence of the overlap field implies a difference in the predicted physics when one calculates effects in a forever-constant background compared to in a constant background evolved adiabatically from the infinite past. Furthermore, the overlap field is neglected whenever an approximation to elastic photon scattering in inhomogeneous fields is made by integrating over forever-constant background scattering rates.

II. ANALYTICAL METHOD

Let us consider the electromagnetic field to be the sum of a weak probe and strong background field

\[ F^{\mu\nu} = F_p^{\mu\nu} + F_s^{\mu\nu}, \]  

(1)
where $F$ is the Faraday tensor and the subscripts $p$ and $s$ pertain to the probe and strong fields respectively. If one defines dimensionless electromagnetic and secular invariants,

$$
\mathcal{F} = -\frac{F^2}{4E_c^2}, \quad \mathcal{G} = -\frac{F^*F}{4E_c^2},
$$

$$
a = \left[ \sqrt{F^2 + G^2 + F} \right]^{1/2}, \quad b = \left[ \sqrt{F^2 + G^2 - F} \right]^{1/2}
$$

(2)

where $F^2 = F^\mu\nu F_{\mu\nu}$, $F F^* = F^\mu\nu F_{\mu\nu}$, giving $\mathcal{G} = E \cdot B$ and $\mathcal{F} = (E^2 - B^2)/2$, in which $E^2 = E \cdot E$ and electric and magnetic fields $E$, $B$ are dimensionless, having been normalised by the critical field strength $E_c = m^2/e$. We set here and throughout $\hbar = c = 1$. The Heisenberg-Euler Lagrangian is

$$
\mathcal{L}_{HE} = -\frac{\alpha m^4}{8\pi^2} \int_0^\infty ds \frac{e^{-s}}{s^3} \left[ s^2 ab \cot as \coth bs - 1 + \frac{s^2}{3} (a^2 - b^2) \right].
$$

This low-frequency limit is a good approximation for describing the physics of spacetime-dependent fields for photon frequencies $\omega E/m \ll 1$, or equivalently wavelengths much less than the pair-creation length scale $\lambda/E$ for reduced Compton wavelength $\lambda = 1/m$. As we are interested in the effects on electromagnetic fields and wish to avoid a discussion on pair creation, we perform a weak-field expansion of Eq. (4) for when $E \ll 1$:

$$
\mathcal{L}_{HE} = \sum_{n=1}^\infty \mathcal{L}_n, \quad \mathcal{L}_1 = \frac{\mu_1}{4\pi \alpha} \left[ (E^2 - B^2)^2 + 7(E \cdot B)^2 \right],
$$

$$
\mathcal{L}_2 = \frac{\mu_2}{4\pi \alpha} \left[ 2(E^2 - B^2)^2 + 13(E \cdot B)^2 \right],
$$

$$
\mathcal{L}_3 = \frac{\mu_3}{4\pi \alpha} \left[ 3(E^2 - B^2)^4 + 22(E^2 - B^2)^2(E \cdot B)^2 + 19(E \cdot B)^4 \right],
$$

(5)

where $\mu_1 = \alpha m^4/90\pi$, $\mu_2 = \alpha m^4/315\pi$, $\mu_3 = 4\alpha m^4/945\pi$ (the fine-structure constant occurs in the denominator in the Lagrange densities Eq. (4) due to rewriting fields in terms of the critical field). The term $\mathcal{L}_n$ describes the effective scattering of $2n$ photons and we will restrict ourselves to the leading-order effects of $\mathcal{L}_1$ corresponding to effective four-photon scattering or the "box diagram", and $\mathcal{L}_2$ corresponding to effective six-photon scattering or the "hexagon diagram", as demonstrated in Fig. 1. Applying the Euler-Lagrange equations to $\mathcal{L} = \mathcal{L}_{MW} + \mathcal{L}_{HE}$, where $\mathcal{L}_{MW} = m^4(E^2 - B^2)/8\pi \alpha$ leads to the classical Maxwell equations, one arrives at a wave equation modified by vacuum polarisation:

$$
\Box \mathbf{E} = \mathbf{T}[\mathbf{E}, \mathbf{B}],
$$

(7)

where we have defined a source term:

$$
\mathbf{T} = 4\pi [\nabla \times \partial_t \mathbf{M} + \partial_t^2 \mathbf{P} - \nabla (\nabla \cdot \mathbf{P})]
$$

(8)

for magnetisation $\mathbf{M} = \partial \mathcal{L}_{HE}/\partial \mathbf{B}$ and polarisation $\mathbf{P} = \partial \mathcal{L}_{HE}/\partial \mathbf{E}$. To simplify the discussion, let us consider the probe and strong fields to counterpropagate with normalised wavevectors $\mathbf{k}_p = (0, 0, 1)$, $\mathbf{k}_s = (0, 0, -1)$ and calculate scattering along the axis of symmetry. This effectively reduces the system to one spatial and one temporal dimension and neglects the final term in Eq. (8). Assuming the change in the fields due to scattering is small, we then solve:

$$
(\partial_t^2 - \partial_z^2) \Delta \mathbf{E} = \mathbf{T}[\mathbf{E}^{(0)}],
$$

(9)

where $\Box \mathbf{E}^{(0)} = \Box \mathbf{B}^{(0)} = 0$ are vacuum solutions to the wave equation, and $\mathbf{B}_j^{(0)} = \mathbf{k}_j \cdot \mathbf{E}^{(0)}$ for $j \in \{s, p\}$. The solution to Eq. (9) is acquired using:

$$
\mathbf{E}(t, z) = \mathbf{E}^{(0)}(t, z) + \Delta \mathbf{E}(t, z)
$$

(10)

$$
\Delta \mathbf{E}(t, z) = \int dt' dz' G_R(t - t', z - z') \mathbf{T}[\mathbf{E}(t', z')],
$$

(11)

where $G_R$ is the retarded Green's function for the wave equation in one spatial and one temporal dimension:

$$
G_R(t, z) = \frac{1}{2} \theta(t) \theta \left( \frac{t}{v} - |z| \right),
$$

(12)

for propagation speed $v$, which we will assume to be equal to the speed of light $v = 1$ in all calculations in this paper. Applying this method to Eq. (8), we have:

$$
(\partial_t^2 - \partial_z^2) \Delta \mathbf{E} = \mathbf{T}[\mathbf{E}^{(0)}].
$$

(13)

Since the source $\mathbf{T}$ contains the cube of electromagnetic fields, the lowest order neglected term is $\sim E^2 \partial \mathbf{E}$, which if $\Delta \mathbf{E} \sim E^3$, is much smaller than the included term when we note that truncating the weak-field expansion we already assume $E^2 \ll 1$. Through partial integration in $t$, the scattered field becomes a sum of forward- (positive $z$-direction) and backward-propagating scattered fields

$$
\Delta \mathbf{E}(t, z) = \Delta \mathbf{\overline{E}}(t, z) + \Delta \mathbf{\overline{E}}(t, z),
$$

(14)

where boundary terms can be neglected when the initial overlap of fields is zero, and

$$
\Delta \mathbf{\overline{E}}(t, z) = \int_0^z \frac{dz'}{2} \mathbf{J}(x^+ - z', z'),
$$

(15)

$$
\Delta \mathbf{\overline{E}}(t, z) = \int_z^\infty \frac{dz'}{2} \mathbf{J}(x^+ - z', z'),
$$

(16)

FIG. 1. The Heisenberg-Euler Lagrangian contains effective vertices for classical electromagnetic fields interacting via the quantum effects described by the four-photon scattering box diagram and six-photon scattering hexagon diagram.
where \( x^\pm = t \pm z \) and \( \mathbf{J} \) is the current occurring in Maxwell's equations

\[
\mathbf{J}(t, z) = \mathbf{k}_p \wedge \partial_t \mathbf{M}(t, z) + \partial_z \mathbf{P}(t, z).
\] (17)

The forward-scattered field is proportional to an integration over the backward-travelling current, and the backward-scattered field to an integration over the forward-travelling current, where the current is proportional to the derivative of a combination of the strong and probe fields.

### A. Overlap and asymptotic field

To make clearer what is happening, we calculate by way of example, part of the forward-scattered field Eq. [15] arising from the second term in the current Eq. [17].

The polarisation is

\[
\mathbf{P}|\mathbf{E}| = \frac{\mu_1}{2\pi} \left[ 2 (E^2 - B^2) \mathbf{E} + 7 (\mathbf{E} \cdot \mathbf{B}) \mathbf{B} \right],
\] (18)

where we recall we consider \( \mathbf{P}|\mathbf{E}|(0) \) and since \( \mathbf{E}(0)(t, z) = \mathbf{E}_s(x^+) + \mathbf{E}_p(x^-) \) and similarly for the magnetic field, which are both plane waves, we see different combinations of powers of \( E_s \) and \( E_p \) will occur in \( \mathbf{P} \).

For brevity, let us focus on terms proportional to the probe field squared. Then the corresponding part of the scattered field is

\[
\epsilon \int_{z_0}^{z} \frac{dz'}{2} \partial_{z'} \left( E_p^2 E_s \right)
\] (19)

where \( \epsilon \) is the polarisation vector that absorbs all other constants in this example, \( z_0 \leq z \) is the \( z \)-coordinate from which the effect is calculated and the derivative is evaluated at \( t' = x^- + z' \), which with the chain rule becomes

\[
-\epsilon \int_{z_0}^{z} \frac{dz'}{2} \partial_{z'} \left( E_p^2 E_s + \epsilon \int_{z_0}^{z} \frac{dz'}{2} E_p^2 \partial_{z'} (E_s) \right).
\] (20)

Eq. [20] is the asymptotic plus the overlap field respectively. To elaborate these labels, we can use that the derivatives are evaluated on the lightcone of the probe field so that Eq. [20] becomes:

\[
\frac{1}{2} \epsilon E_p(x^-) E_p(x^-) \int_{-2(x-z_0)}^{0} dy \ E_s(x^+ + y)
+ \frac{1}{2} \epsilon E_p(x^-) \left[ E_s(x^+) - E_s(x_0^+) \right],
\] (21)

where ' indicates the derivative and \( x_0^+ = t + z_0 \). For the first term, we see that on the probe lightcone (e.g. \( x^- = 0 \)), long after the collision in the asymptotic limit \( t, z \rightarrow \infty \), the term remains (assuming the integration over the strong field is non-vanishing). Therefore we label this the asymptotic scattered field. The second term corresponds to a surface term and the strong and probe fields are evaluated on their respective light cones.

When the overlap of the fields tends to zero, so does this term and therefore we label this the overlap scattered field. We note that if a constant field is adiabatically evolved from the infinite past, the overlap field does occur. This should be contrasted with the case of an ever-present constant field, in which the overlap field vanishes identically.

In this example we considered the probe field squared, corresponding to a generation of a second harmonic (the frequency of the strong field is taken to be much smaller than that of the probe), also referred to as "photon-merging". A division into overlap and asymptotic scattered fields can be made in each combination of powers of strong and probe fields that occur in the interaction.

To investigate these ideas, we will choose the probe and strong fields in this paper to be of the form

\[
\mathbf{E}_p(\varphi_p) = \epsilon_p \mathbf{E}_s \ e^{-\left( \frac{x_0^2}{2} \right)} \cos \varphi_p
\] (22)

\[
\mathbf{E}_s(\varphi_s) = \epsilon_s \mathbf{E}_s \ e^{-\left( \frac{x_0^2}{2} \right)},
\] (23)

where \( \Phi_j = \omega_j t_j \) and \( \varphi_j = k_j^p x \mu \) for \( j \in \{ s, p \}, \epsilon_p \cdot \epsilon_p = \epsilon_s \cdot \epsilon_s = 1 \) and we are interested in the case \( \omega_p t_s \gg 1 \).

### III. NUMERICAL METHOD

The numerical solution of the nonlinear Maxwell equations following from the sum of classical and Heisenberg-Euler Lagrangians is based on the PCMOL (Pseudo Characteristic Method of Lines) [46], matrix inversion and the CVODE ODE-Solver from SUNDIALS (SUite of Nonlinear and DIfferential/ALgebraic equation Solvers) [47]. Since we assume propagation only in the \( z \)-direction and only transverse polarisations, the resulting equations of motion can be written in matrix form:

\[
(1 + \mathbf{A}) \partial_t \mathbf{f} + (\mathbf{Q} + \mathbf{B}) \partial_z \mathbf{f} = 0,
\] (24)

where \( \mathbf{f} = (E_s, E_y, B_z, B_y)^T \), \( 1 + \mathbf{A} = \text{diag}(1, 1, 1, 1) \) is the identity matrix in four dimensions, \( \mathbf{Q} = \text{adiag}(1, -1, -1, 1) \) is an anti-diagonal matrix and \( \mathbf{A} = (a_{ij}) \) and \( \mathbf{B} = (b_{ij}) \) are the nonlinear corrections resulting from Eq. [15] with \( a_{ij} = b_{ij} = 0 \) for \( i > 2 \). For the weak field expansion \( \mathcal{L}_1 (\mathcal{L}_2) \), the components are quadratic (fourth order) polynomials of the field components. Let us first consider the linear case with \( \mathbf{A} = \mathbf{B} = 0 \). In the PCMOL, one uses the diagonalisability of the matrix \( \mathbf{Q} \), which means one can find a basis \( \mathbf{u} := \mathbf{P} \mathbf{f} \) such that \( \Lambda = \mathbf{PQP}^{-1} = \text{diag}(-1, -1, 1, 1) \) is diagonal with real eigenvalues:

\[
\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix}
-1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
\end{pmatrix}, \quad \mathbf{u} := \mathbf{P} \mathbf{f} = \frac{1}{\sqrt{2}} \begin{pmatrix}
B_s - B_x \\
E_s + B_z \\
E_s + B_z \\
B_s - B_y \\
\end{pmatrix}
\] (25)

The new set of equations is given by:

\[
\partial_t \mathbf{u} + \mathbf{A} \partial_z \mathbf{u} = 0.
\]
The eigenvalues are called the characteristic speeds and the positive (negative) sign corresponds to a component propagating in the positive (negative) z-direction. The system, which is taken to be of a length of 320 μm is discretised in space using \( N = 2 \cdot 10^3 \) points. The four-dimensional vector \( \mathbf{u} \) can then be mapped onto a 4N-dimensional one, \( \mathbf{u} = (\ldots u_{1}^{t-1}u_1u_2u_3u_1^{t+1} \ldots) \), where \( 0 < i \leq N \) labels the grid point. The spatial derivatives of the components \( u^i \) are approximated with upwind-biased finite differences determined by the sign of the characteristic speed. This is done using fourth-order stencils [48], where the values of the derivative near the boundary are also approximated with fourth-order accuracy using grid points only inside the simulation box. Since the derivatives at one point are calculated with the field values at the specific and surrounding points, the action of the derivative can be written as a matrix multiplication: \( \partial_t \mathbf{u} \approx \mathbf{D} \mathbf{u} \), where \( \mathbf{D} \) is a 4N × 4N matrix. In the PCMOL, the equations are now transformed back to the original basis \( \mathbf{f} \), but the system is solved in \( \mathbf{u} \), which is completely equivalent. This has the advantage of automatically implementing open boundary conditions due to the upwind character in the single components. The electric and magnetic fields are then obtained by applying \( \mathbf{P}^{-1} \) for output at each grid point.

We now consider the nonlinear case. To bring the system to an ODE form \( \mathbf{u}'(t) = f(\mathbf{u},t) \) (\( f \) is called “right-hand-side function”), the \( \tau \) denotes the time derivative), we need to invert the matrix \((\mathbb{I}_4 + \mathbf{A})\). Since \( \mathbf{A} \) is a local operator of the field components, it is only necessary to consider the inversion for each single grid point. We rewrite \( \mathbf{A} \) as \( \mathbf{A} = \mathbf{MN} \) with

\[
\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}. \tag{26}
\]

and apply the Woodbury Formula [49],

\[
(\mathbb{I}_4 + \mathbf{A})^{-1} = \mathbb{I}_4 - \mathbf{M}(\mathbb{I}_2 + \mathbf{N})^{-1}\mathbf{N} \tag{27}
\]

to reduce the inversion of the 4 × 4-matrix \((\mathbb{I}_4 + \mathbf{A})\) to one of the 2 × 2 matrix

\[
(\mathbb{I}_2 + \mathbf{N}) = \begin{pmatrix} 1 + a_{11} & a_{12} \\ a_{21} & 1 + a_{22} \end{pmatrix}, \tag{28}
\]

which is performed at each evaluation of the right-hand-side function \( f \) via an LU-factorisation. Since our method employs a weak field expansion, we expect only small corrections from the nonlinearities \( \mathbf{A} \) and \( \mathbf{B} \), such that the the matrix \((\mathbb{I}_4 + \mathbf{A})^{-1}(\mathbf{Q} + \mathbf{B})\) has similar spectral properties (i.e. the same signs of the eigenvalues) as \( \mathbf{Q} \). Therefore we use the same biased differencing as in the linear case. The system is now solved using the parallel, extended-precision version of CVODE, where the nonlinear right-hand-side function is given by

\[
f(\mathbf{u},t) = -\mathbf{P}(\mathbb{I}_4 + \mathbf{A})^{-1}(\mathbf{Q} + \mathbf{B})\mathbf{P}^{-1} \partial_t \mathbf{u}. \tag{29}
\]

\( \mathbf{P}(\mathbb{I}_4 + \mathbf{A})^{-1}(\mathbf{Q} + \mathbf{B})\mathbf{P}^{-1} \) is now a block-diagonal 4N × 4N matrix with 4 × 4 blocks acting on each grid point, as explained above. We use the provided Adams-Moulton methods and the functional iteration to solve the corresponding linear system of equations. The signals are analysed using a spatial Fourier Transform in Wolfram Mathematica [50] and the frequency components are filtered under the assumption \( \omega = |\mathbf{k}| \) and transformed back to spatial co-ordinates. To analyse the DC-component, we subtract the analytical expression of the strong pulse from the signal.

IV. COMPETING VACUUM PROCESSES

For clarity, we consider each frequency component of the scattered field separately and neglect the change in frequency due to the background frequency scale \( \delta \omega \ll \omega_p \). To leading order in \( \mathcal{E}_s, \mathcal{E}_p \ll 1 \), the scattered field can be written as

\[
\Delta \mathbf{E} = \sum_{l=1}^{\infty} \mathcal{E}_p \mathcal{E}_s \omega_p e^{-i(\frac{\omega_p}{\tau})^2} \left[ C_l \sin l \varphi_p + \bar{C}_l \cos l \varphi_p \right] + \mathcal{E}_p^2 \left[ C_0 + \bar{C}_0 \right]. \tag{30}
\]

In Eq. (30) we note that the scattered field is written as a sum over harmonics of the probe field. Each higher harmonic involves a higher power of \( \mathcal{E}_p \ll 1 \) so in general higher-harmonics are less likely in this regime. For each harmonic we then note two spacetime-dependent vector terms with coefficients \( C_l \) and \( \bar{C}_l \). The \( C_l \) terms are out of phase with the probe field and form the asymptotic field whereas the \( \bar{C}_l \) terms are in phase with the probe field and correspond to the overlap field. Although we neglect processes of a higher order than four- and six-photon scattering in the current analysis, they can be calculated straightforwardly using the method used here. In the following we comment on the first few harmonics.

A. Fundamental harmonic

If the scattered field is much weaker than the probe, it can be described by analogy with a modified refractive index, \( 1 + \delta n, \delta n \ll 1 \). The probe field light-cone then becomes \( \varphi_p = \omega_p t / (1 + \delta n) - z \). Expanding \( \cos \varphi_p \) in \( \delta n \), the leading-order scattered field is in antiphase with the probe field, so this effect should be entirely covered by the asymptotic field in our analysis. For the current scenario we find:

\[
C_1 = -\mu_1 \mathcal{E}_s \mathcal{E}_s \omega_p \pi \sqrt{\pi} \frac{1}{2} \text{erf}(\sqrt{2} \varphi_s / \Phi_s) \tag{31}
\]

\[
\bar{C}_1 = -\mu_1 \mathcal{E}_s \mathcal{E}_s (\varphi_s), \tag{32}
\]

where the polarisation of the scattered field is given by

\[
\mathcal{E}_{s,1} = c_{1,1} \mathcal{E}_s + c_{1,2} \mathbf{K}_s \wedge \mathcal{E}_s. \tag{33}
\]
with \( c_{1,1} = 8 \varepsilon_p \cdot \varepsilon_p, \ c_{1,2} = 14 \varepsilon_x \cdot \hat{k}_p \wedge \varepsilon_p \). We note that the well-known modified refractive index in a constant background is given by \([5]\):

\[
\delta n(\varphi_s) = \frac{2\alpha E_s^2(\varphi_s)}{45\pi} \left[ 4 (\varepsilon_p \cdot \varepsilon_s)^2 + 7 (\varepsilon_p \wedge \varepsilon_s)^2 \right]. \tag{34}
\]

If the corresponding phase difference \( \delta \varphi_p(\varphi_s) \) is calculated by integrating Eq. \((34)\) over the shape of \( E_s(\varphi_s) \) in probe field light-cone co-ordinates:

\[
\delta \varphi_p(\varphi_s) = -\int_{-\infty}^{\varphi_s-\varphi_p} d\varphi_p' \delta n(\varphi_s') \bigg| \varphi_s' = 2\varphi_s + \varphi_p \tag{35},
\]

then the asymptotic field and Eq. \((31)\) can be recovered exactly.

The presence of the overlap field in the fundamental harmonic cannot be described by a modified index of refraction. If the background is wider than several probe wavelengths \( (\omega_p \tau_s \gg 1) \), then the amplitude of the overlap field is much smaller than of the asymptotic. Both parts of the scattered field have the same polarisation as the probe.

**B. Second harmonic**

![Graphs showing electric field and angular distribution](image)

**FIG. 2.** The smaller panels plot the state of the total electric field above the larger panels showing the corresponding state of the scattered overlap (red dashed) and asymptotic (blue solid) second harmonic field. The electric fields are in units of the probe field amplitude and the abscissae are spatial co-ordinate in units of \( \tau_s \).

The strongest contribution to the scattered field with double the frequency of the probe originates from four- and six-photon scattering. We find

\[
\mathcal{C}_2 = -\frac{\mu_2}{2} \varepsilon_p, 2\varepsilon_s^2 \omega_p \sqrt{\tau} (\varphi_s) \text{erf}(\sqrt{3} \varphi_s / \varphi_s) \tag{36}
\]

\[
\tilde{\mathcal{C}}_2 = -\frac{\mu_1}{2} \varepsilon_{p,1} E_s(\varphi_s) - \frac{\mu_2}{2} \varepsilon_{s,2} E_s^3(\varphi_s) \tag{37}
\]

where

\[
\varepsilon_{p,1} = c_{1,1} \varepsilon_p + c_{1,2} \hat{k}_p \wedge \varepsilon_p \tag{38}
\]

\[
\varepsilon_{s,2} = c_{2,1} \varepsilon_s + c_{2,2} \hat{k}_s \wedge \varepsilon_s \tag{39}
\]

with \( c_{2,1} = 3 c_{1,1}^2 / 2 + 13 c_{1,2}^2 / 49 \) and \( c_{2,2} = 13 c_{1,1} c_{1,2} / 14 \). In one temporal and one spatial dimension, four-photon scattering in a strictly constant background is suppressed for kinematical reasons. However, when the background contains some inhomogeneity, the second harmonic can be generated. This is also the case when a constant background is adiabatically evolved from the infinite past.

Since the second-harmonic overlap field is of order \( \alpha^2 \) and the asymptotic field of order \( \alpha^3 \), there is a range of parameters for which the overlap field dominates. Let us define the gauge- and relativistically-invariant parameter \( \zeta \):

\[
\zeta = \int_{-\infty}^{\infty} d\varphi_s \, \zeta(\varphi_s) \tag{40}
\]

where \( \zeta(\varphi_s) = [\chi(\varphi_s)]^2 / \eta \), \( \chi = \sqrt{k_p F_s^2} / m \) is the so-called quantum non-linearity parameter \([5]\) and \( \eta = k_p k_s / m^2 \). For the current scenario, \( \zeta = \varepsilon_p^2 \omega_p \tau_s \sqrt{2} \pi \) and by comparing Eqs. \((36)\) and \((37)\), we note that when \( \zeta \ll 1 \), the overlap field can dominate. The evolution of the scattered field is illustrated in Fig. 2 and directly compared with the position of the probe and strong fields. In Fig. 2 the maximum of the amplitude of the simulated second-harmonic signal is plotted and the evolution for the asymptotic and overlap fields compared. In the second harmonic, the rate of change of the overlap field is proportional to the gradient of the background. In Fig. 2 we observe that the maximum of the amplitude of the overlap field initially increases to an overall maximum when the probe and strong fields most overlap, after which the second harmonic is further generated field but phase-shifted by \( \pi \) and destructively interferes with the already present second harmonic field.

**C. Higher harmonics**

In the presence of some background inhomogeneity, a given harmonic is generated in the overlap field at one order in \( \alpha \) lower than in the scattered field. For example, in one spatial and one temporal dimension, no asymptotic signal is generated from the hexagon diagram, but an overlap signal is permitted. We find

\[
\tilde{\mathcal{C}}_3 = -\frac{\mu_2}{4} \varepsilon_p, 2 E_s^2(\varphi_s) \tag{41}
\]
where \( \varepsilon_{p,2} = \varepsilon_p + \varepsilon_{2,2} \mathbf{k}_p \wedge \varepsilon_p \) and the leading order term in \( C_3 \) can be found by calculating octagon diagram in the weak-field Heisenberg-Euler expansion.

Although the overlap and asymptotic fields have different spacetime dependencies, we find that the polarisation selection rules for higher harmonic generation are identical. In particular

\[
I_{\gamma_1} \rightarrow \gamma'_1; \quad 2I_{\gamma_\perp} \rightarrow \gamma'_1; \quad (2I - 1)_{\gamma_\perp} \rightarrow \gamma'_\perp \tag{42}
\]

where \( \gamma_1 \) corresponds to a probe photon obeying \( \varepsilon_p \wedge \varepsilon_s = 0 \), \( \gamma_\perp \) to \( \varepsilon_p \cdot \varepsilon_s = 0 \), \( l \in \mathbb{N}^+ \) and \( \gamma' \) is the photon generated through scattering. Therefore odd harmonics exhibit a slightly different polarisation behaviour and in particular admit a photon-merging cascade in the \( \perp \) component. However, since this requires a minimum of three photons to merge, it is presumably only of relevance when the probe photon density is very high or path length very long. Another feature of this mechanism is that probe photons that are in a superposition of linear polarisations can access the \( \gamma_\perp + \gamma_\perp \rightarrow \gamma'_\perp \) channel, but only once. This can be seen by the coefficient of the outgoing \( \perp \) channel depending on the overlap of probe photon \( \perp \) and \( \parallel \) components (e.g. \( c_{2,2} \) in Eq. (39)). Once scattered, the merged photons are then confined to residing in a polarisation eigenmode thereafter.

The lowest-order non-trivial effect of the polarised vacuum on probe photons is a modification of the index of refraction (Eq. (34)) leading to \( k^2 \neq 0 \). We note that taking this non-trivial dispersion into account, more harmonics can be generated for a given-order diagram in which photons are no longer described by null fields as \( F^2 \neq 0 \). In particular, the signal must no longer be in harmonics of the incoming field. We will postpone analysis of this particular problem, which requires longer path lengths, for a future publication.

### D. Zeroth harmonic

With the zeroth harmonic, DC component, or rectification, we are referring to a signal with the low frequency \( \approx 1/\tau_s \ll \omega_p \) of the background. One probe photon is absorbed by and one photon emitted from the polarised vacuum pairs, leaving a photon of the frequency associated with the background. From momentum conservation, the scattered field has a momentum vector in the backwards direction. We find

\[
C_0 = -\mu_1 \varepsilon_{p,1} \varepsilon_s \sqrt{\frac{\pi}{2 \tau_s \Phi_s}} \left( 1 + \text{erf} \left( \sqrt{2} \frac{\varepsilon_s}{\Phi_p} \right) \right) e^{-\left( \frac{\varepsilon_s}{\Phi_p} \right)^2}.
\]

Emission in the backwards-direction is demonstrated in Fig. 4 and contrasts with the photon-merging behaviour shown in Fig. 3. Moreover, when the background varies,
Frequency down-conversion can also produce non-DC components, for example in $\gamma + \gamma \rightarrow \gamma + \gamma'$, for probe photons $\gamma$, the scattered photon $\gamma'$ is at the fundamental frequency. However, the more photons that participate, the smaller the effect when $\mathcal{E}_p \ll 1$. We stress the difference of frequency down-conversion from photon splitting, as in the current case, no photon quantum is being split into quanta of lower energy. For that to occur, dispersion has to be taken into account, the effects of which have been neglected in the current work.

V. DISCUSSION

Our results suggest that when one approximates photon-photon scattering in spacetime-varying fields by assuming that scattering is at each instant equivalent to in a strictly constant background, important physics is missed. We have seen how this even arises when a photon propagates through a quasi-constant background, due to the photon-photon interaction current involving derivatives of combinations of fields. When this current is integrated over, part of the scattered field is generated by a surface term that depends on the state of the background in the photon’s past. Therefore it can occur that changes in the background strength, even when over distances much larger than the photon wavelength, can lead to a significant contribution to the rate of photon-photon scattering. In particular, the predicted evolution of the total scattered field is different for a strictly constant background compared to a constant background that has been adiabatically evolved from the infinite past. This could have potential implications for effects calculated in the overlap of probe and strong fields that rely on the instantaneously-constant background approximation such as the so-called “vacuum resonance” in strongly-magnetised pulsars, which could be searched for in a programme similar to the GEMS mission.

The intense magnetic field of certain neutron stars offers an excellent possibility to study strong-field quantum electrodynamical effects using polarisation measurements of emitted photons. The process of photon-splitting has been hypothesised to be of particular importance in the magnetospheres of neutron stars. Here we compare the density of photons (number per unit volume) that split $\rho_{\gamma \rightarrow 2\gamma'}$ with those that merge $\rho_{2\gamma \rightarrow \gamma'}$ in a quasi-constant magnetic field of strength $B$. We repeated the calculation leading to Eqs. (40) and (41), first taking the limit $\tau_{\gamma \rightarrow e}$, $\tau_{e \rightarrow e'} \rightarrow \infty$ and setting the background electric field to zero but allowing a background field strength difference $\Delta B$ over the seed photons’ history, defining $\Delta = \Delta B/B$. If a photon is generated in the background, then this presumably occurs over some formation region, at the start of which, the overlap of photon and background fields is negligible, which allows us to again neglect surface terms in Eq. (14). We then find:

$$\rho_{2\gamma \rightarrow \gamma'}^\text{as} = 2\alpha^3 \left[ \frac{11 \pm 3}{180\pi} \right]^2 B^2 \Delta^2 \frac{\omega}{m} (\rho \pi \lambda^3) \rho \quad (44)$$

$$\rho_{2\gamma \rightarrow \gamma'}^\infty = 8\alpha^3 \left[ \frac{37 \mp 11}{315\pi} \right]^2 B^2 \Delta^2 \frac{\omega}{m} (\rho \pi \lambda^3) \rho \quad (45)$$

$$\rho_{\gamma \rightarrow 2\gamma'} = \frac{\alpha^3}{10} \left[ \frac{19}{315\pi} \right]^2 L \frac{L_\omega}{\omega} \frac{L}{T} \rho \quad (46)$$

where $\rho$ is the density of seed photons with frequency $\omega$, $L$ is propagation distance of the seed photon, $\zeta = B^2 / \omega L$ and $\pm$ refer to seed photon polarisation being perpendicular or parallel to that of the external field and we have adapted the rate for photon-splitting from [56]. Photon-splitting requires dispersion to be taken into account and has a strong dependence on the frequency being split $\sim (\omega/m)^3$, whereas photon merging requires a high density of photons such that the number of seeds in a cylindrical volume of radius $\lambda$ around the photon’s trajectory is not too small. Although a full comparison is beyond the scope of this paper, if one notes that in a photon gas at temperature $T$ the density of photons with energies $\in [\omega, \omega + \delta \omega]$, $\delta \omega/\omega \ll 1$ is of the order $\rho \sim \omega^2 \delta \omega \exp(\omega/T - 1)^{-1}$, then the ratio of second harmonic generation to photon splitting is of the dependency

$$\frac{\rho_{2\gamma \rightarrow \gamma'}}{\rho_{\gamma \rightarrow 2\gamma'}} \sim \frac{L}{\omega} \frac{\delta \omega}{\omega} - 1$$

$$\frac{\rho_{2\gamma \rightarrow \gamma'}}{\rho_{\gamma \rightarrow 2\gamma'}} \sim \left( \frac{m \Delta}{\omega L} \right)^2 \frac{\Delta B}{\Delta} \frac{\delta \omega}{\omega} - 1 \quad (47)$$

When is harmonic generation more prevalent than electron-positron pair creation in a strongly-magnetised thermal photon gas? If the number density of pairs created in photon-photon collisions is $\rho_{2\gamma}$, and pairs created through photon decay in a background constant magnetic field $\rho_{\gamma}$, then

$$\rho_{\gamma \rightarrow e^+e^-} \sim \frac{2}{\lambda} \frac{L}{\lambda} \frac{\alpha}{2\pi} \left( \frac{T}{m} \right)^3 e^{-\frac{2\pi}{\lambda}} \quad (48)$$

$$\rho_{\gamma \rightarrow e^+e^-} \sim \frac{3^{3/4} \alpha}{4 \sqrt{2\pi} \epsilon} \left( \frac{T}{m} \right)^2 \frac{1}{\lambda} \frac{T}{\lambda} \frac{2^{1/4} e^{-\frac{1}{\sqrt{2}}}} { \quad (49)$$

for $T/m \ll 1$ and $\Delta = TB/2m \ll 1$ where the pair-creation densities were adapted from [57, 58] for a constant magnetic background. In order to calculate the total density of merged photons created in a photon gas, we would have to extend our calculation to include merging of photons with different wavevectors and integrate the double-photon rate over a double Bose-Einstein distribution. However, from Eqs. (49) and (50) we already note that for $T/m \ll 1$, pair-creation is exponentially suppressed whereas photon merging (and splitting) are perturbative in $T/m$. Since $T/m \sim 10^{-4}$
for strongly-magnetised neutron stars \cite{33}, one could pose the question whether harmonic generation, along with photon splitting, can be an important factor in the evolution of these stellar objects.

We close by noting that only the asymptotic photon merging signal is of relevance to laser physics, and then only when the laser background occurs to an even power and hence contains a slowly-varying component. This occurs in six-photon scattering if pulses collide at an angle which is proportional to $\mathcal{E}_p^2 \mathcal{E}_s^2$, or in eight-photon scattering which is proportional to $\mathcal{E}_p^3 \mathcal{E}_s^3$ and considering that $\mathcal{E}_{p,s} \ll 1$, these signals are greatly suppressed. This suppression can be potentially overcome by using an ultra-short strong laser pulse and looking off-axis for emitted photons \cite{17} using more than two laser frequencies and off-axis beams \cite{21}, or using a charged projectile such as a proton \cite{59, 60}.

VI. CONCLUSION

We have shown that when an oscillating probe field propagates through a background field with some inhomogeneity, a source of photon-photon scattering appears when the two fields overlap. This “overlap field” disappears when the overlap of the two fields tends to zero and is distinct from the “asymptotic field” that persists after scattering has taken place. Moreover, the overlap field permits high harmonic generation for a specific harmonic at an order of the fine structure constant lower than in the asymptotic field. By integrating the weak-field expansion of the Heisenberg-Euler Lagrangian using the Green’s function for the wave equation in one spatial and one temporal dimension, we compared the nature of the overlap and asymptotic fields and identified a suitable scaling parameter. We have highlighted the potential importance of this effect in astrophysical environments by calculating the density of merged photons and contrasted this with the density of photons split and density of photons seeding pair creation.

VII. ACKNOWLEDGMENTS

B. K. acknowledges the hospitality of H. R. and the Arnold Sommerfeld Center for Theoretical Physics at the Ludwig Maximilians University as well as useful editorial suggestions from T. Heinzl. P. B. acknowledges the very useful advice of A. Hindmarsh during development of the computational simulation. This work was supported by Grant No. DFG, FOR1048, RU633/1-1, by SFB TR18 project B12 and by the Cluster-of-Excellence “Munich-Centre for Advanced Photonics” (MAP). Plots were generated with Matplotlib \cite{61}.

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