INVARIANCE PROPERTY FOR PARTIAL MEANS

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Abstract. We study the properties of the mean-type mappings \( M : I^p \rightarrow I^p \) of the form
\[
M(x_1, \ldots, x_p) := (M_1(x_{\alpha_1,1}, \ldots, x_{\alpha_1,d_1}), \ldots, M_p(x_{\alpha_p,1}, \ldots, x_{\alpha_p,d_p})),
\]
where \( p \) and \( d_i \)-s are positive integers, each \( M_i \) is a \( d_i \)-variable mean on an interval \( I \subset \mathbb{R} \), and \( \alpha_{i,j} \)-s are elements from \( \{1, \ldots, p\} \).

We show that, under some natural assumption on \( M_i \)-s, the problem of existing the unique \( M \)-invariant mean can be reduced to the ergodicity of the directed graph with vertexes \( \{1, \ldots, p\} \) and edges \( \{(\alpha_{i,j}, i): i, j \text{ admissible}\} \).

1. Introduction

Invariance property is a very important aspect in the theory of means. There are two classical studies, Lagrange \cite{16} and Gauss \cite{11}, which could be considered as a beginning of this field. It has been extensively studied by many authors since then. For example J. M. Borwein and P. B. Borwein \cite{5} extended some earlier ideas \cite{10, 17, 28} and generalized the original iteration to a vector of continuous, strict means of an arbitrary length. For several recent results about Gaussian product of means see the papers by Baják–Páles \cite{1–4}, by Daróczzy–Páles \cite{6–8}, by Głazowska \cite{13, 14}, by Jarczyk–Jarczyk \cite{15}, by Matkowski \cite{18–21}, by Matkowski–Páles \cite{24}, and by the author \cite{25}. In the vast majority of these studies authors implicitly assumed that the invariant mean is uniquely determined. There are also few results without uniqueness assumption; see Deregowska–Pasteczka \cite{9}, Matkowski–Pasteczka \cite{22, 23}, and Pasteczka \cite{26, 27}. The main outcome of these papers is that the uniqueness of invariant means is deeply related with strictness and continuity.

Basic definition and notions. Before we proceed further recall that, for a given \( p \in \mathbb{N} \) and an interval \( I \subset \mathbb{R} \), a \textit{\( p \)-variable mean on \( I \)} is an arbitrary function \( M : I^p \rightarrow I \) satisfying the inequality
\[
\min(x) \leq M(x) \leq \max(x) \text{ for all } x \in I^p.
\]
Property (1.1) is referred as a *mean property*. If the inequalities in (1.1) are strict for every nonconstant vector $x$ then we say that a mean $M$ is *strict*. Moreover, for such objects we define natural properties like continuity, symmetry (when the value of mean does not depend on the order of its arguments), monotonicity (which states that $M$ is nondecreasing in each of its variables), etc.

A mean-type mapping is a selfmapping of $I^p$ which has a $p$-variable mean on each of its coordinates. More precisely $M: I^p \to I^p$ is called a *mean-type mapping* if $M = (M_1, \ldots, M_p)$ for some $p$-variable means $M_1, \ldots, M_p$ on $I$. In this framework a function $K: I^p \to \mathbb{R}$ is called $M$-invariant if it solves the functional equation $K \circ M = K$. Usually we restrict solutions of this equation to the family of means and say about $M$-invariant means.

**Posting the problem.** There is the natural issue to give a condition to $M$ that warranty the uniqueness of $M$-invariant mean. There are three natural conditions which are proposed in the literature. Namely, if the continuous mean-type mapping $M = (M_1, \ldots, M_p): I^p \to I^p$ satisfies one of the following three conditions:

- each $M_i$ is a strict mean;
- $M$ is *contractive*, that is $\max M(x) - \min M(x) < \max(x) - \min(x)$ for every nonconstant vector $x \in I^p$;
- $M$ is *weakly contractive*, which states that for every nonconstant vector $x \in I^p$ there exists a natural number $n(x)$ such that

\begin{equation}
\max M^{n(x)}(x) - \min M^{n(x)}(x) < \max(x) - \min(x),
\end{equation}

then there exists exactly one $M$-invariant mean (cf. [5], [21], and [23], respectively). Obviously the last condition is the most general, however it is also the most difficult to verify.

We try to check the last condition in the example.

**Example 1.** Let $M: \mathbb{R}_+^4 \to \mathbb{R}_+^4$ be a mean-type mapping given by

\begin{equation}
M(x, y, z, t) := \left( \frac{2xy}{x+y}, \frac{z+t}{2}, \sqrt{\frac{t^2 + x^2}{2}}, \sqrt{\frac{t^2 + y^2}{2}} \right).
\end{equation}

Then each coordinate of $M$ is a bivariate mean. Moreover $M$ is not contractive, since this condition voids for all vectors of the form $(a, a, b, b)$. On the other hand, one can prove that each coordinate of

\[ M^2(x, y, z, t) = \left( \frac{4\sqrt{yz}xy}{2xy + (x+y)\sqrt{yz}}, \frac{1}{2} \sqrt{\frac{1}{2}yz(t+z)^2}, \frac{t+z + \sqrt{2t^2 + 2z^2}}{4} \frac{1}{2} \sqrt{t^2 + x^2 + \frac{8x^2y^2}{(x+y)^2}} \right) \]
is a trivariate, strict mean on $I$. Thus $M^2$ is a contractive mean-type mapping. Consequently (1.2) holds with $n(x) := 2$, $M$ is weakly contractive, and there exists the unique $M$-invariant mean. □

Observe that $M$ has quite an interesting structure. Namely, in each coordinate we take one of classical means (harmonic, geometric, arithmetic, and quadratic) but we omit some arguments. The aim of this paper it to deliver a robust framework and to prove some natural properties for these sort of mean-type mappings.

**Partial means.** Now we introduce the essential definition from the point of view of this manuscript. Namely, a $p$-variable mean $M: I^p \to I$ is called a *partial mean* if it satisfies a mean property (1.1) and it is independent on some variable. More precisely, there exists $k \in \{1, \ldots, p\}$ such that for all $x, x' \in I^p$ satisfying the equality $x_i = x'_i$ for all $i \in \{1, \ldots, p\} \setminus \{k\}$ we have $M(x) = M(x')$.

For a given $d, p \in \mathbb{N}$, a sequence $\alpha := (\alpha_1, \ldots, \alpha_d) \in \{1, \ldots, p\}^d$, and a $d$-variable mean $M: I^d \to I$ we define the mean $M^{(p;\alpha)}: I^p \to I$ by

$$M^{(p;\alpha)}(x_1, \ldots, x_p) := M(x_{\alpha_1}, \ldots, x_{\alpha_d}),$$

for all $(x_1, \ldots, x_p) \in I^p$.

In the case $d < p$, mean $M^{(p;\alpha)}$ is a $p$-variable partial mean on $I$.

For example if $A: \mathbb{R}^2 \to \mathbb{R}$ is a bivariate arithmetic mean, $p \geq 3$ and $\alpha = (2, 3)$ then $A^{(p;\alpha)}: I^p \to I$ is given by

$$A^{(p;\alpha)}(x_1, \ldots, x_p) = A^{(p;2,3)}(x_1, \ldots, x_p) = \frac{x_2 + x_3}{2}$$

for all $(x_1, \ldots, x_p) \in I^p$.

The opposite statement is also valid in some sense, indeed for every partial mean $M: I^p \to I$, there exists $d < p$, a sequence $\alpha \in \{1, \ldots, p\}^d$, and a $d$-variable mean $M_*: I^d \to I$ such that $M = M_*^{(p;\alpha)}$. We are going to study the invariance of mean-type mappings which contain partial means.

2. **Uniformly weak contractive mappings**

It turns out that in this setup it is natural to define a property which is between the contractivity and weak contractivity. We say that a mean-type mapping $M: I^p \to I^p$ is *uniformly weak contractive* if there exists a natural number $n_0 \in \mathbb{N}$ (which does not depend on $x$) such that

$$\max M^{n_0}(x) - \min M^{n_0}(x) < \max(x) - \min(x),$$

for every nonconstant vector $x \in I^p$. Obviously every contractive mean-type mapping is uniformly weak contractive and every uniformly weak contractive mean-type mapping is weakly contractive.
2.1. Invariance property. In this section we show a counterpart of the result contained in [21, Theorem 1]. The main difference is that we generalize the original setting to the family of uniformly weakly contractive mean-type mappings.

**Theorem 1.** Given an interval $I \subset \mathbb{R}$, $p \in \mathbb{N}$, and the uniformly weak contractive, continuous mean-type mapping $M : I^p \to I^p$. Then

(i) for every $n \in \mathbb{N}$, the mapping $M^n$ is a mean-type mapping;
(ii) there is a continuous mean $K : I^p \to I$ such that the sequence of iterates $(M^n)_{n=0}^{\infty}$ converges, uniformly on compact subsets of $I^p$, to the mean-type mapping $K : I^p \to I^p$, $K = (K_1, \ldots, K_p)$ such that

\[ K_1 = \cdots = K_p = K; \]

(iii) $K : I^p \to I^p$ is $M$-invariant, that is, $K = K \circ M$ or, equivalently, the mean $K$ is $M$-invariant;
(iv) $M$-invariant mean (mean-type mapping) is unique;
(v) if $M = (M_1, \ldots, M_p)$ and all $M_i$-s are strict means, then so is $K$ (and $K$);
(vi) if $M = (M_1, \ldots, M_p)$ and all $M_i$-s are nondecreasing with respect to each variable then so is $K$;
(vii) if $I = (0, +\infty)$ and $M$ is positively homogeneous, then every iterate of $M$ and $K$ are positively homogeneous.

**Proof.** The case when $M$ is contractive is due to [21, Theorem 1]. Moreover parts (i), and (iv) are due to [23, Theorem 2], where they were proved for all continuous, weakly contractive mean-type mappings.

Now assume that $M : I^p \to I^p$ is uniformly weak contractive, and take $k \in \mathbb{N}$ such that $M^k$ is contractive.

To show (ii) observe that for every compact subset $X$ of $I^p$, there exists a compact interval $J \subset I$ such that $X \subset J^p \subset I^p$. Consequently one may assume that $I$ is compact. Moreover, by the contractive part, assertion (i) holds for the subsequence $(M^k)_{n=0}^{\infty}$. More precisely, there exists $K : I^p \to I^p$ with required properties such that for every $\varepsilon > 0$ there exists $n_\varepsilon$ such that

\[ \|M^{kn}(x) - K(x)\|_\infty < \varepsilon \text{ for all } x \in I^p \text{ and } n \geq n_\varepsilon. \]

Equivalently, the property

(2.1) \[ K(x) - \varepsilon < \min M^s(x) \leq \max M^s(x) < K(x) + \varepsilon \]

holds for $s = kn_\varepsilon$. But by the mean value property we know that the mapping $s \mapsto \min M^s(x)$ is nondecreasing and $s \mapsto \max M^s(x)$ is non-increasing. Thus (2.1) holds for all $s \geq kn_\varepsilon$ which completes the proof of (ii).

Having this proved, for all $x \in I^p$, we obtain

\[ K(M(x)) = \lim_{n \to \infty} M^n(M(x)) = \lim_{n \to \infty} M^n(x) = K(x), \]

and
and therefore \( K \circ M = K \), which is (iii).

To prove (vi) note that, under the assumption that all \( M_i \)-s are strict means, for an arbitrary nonconstant vector \( x \in I^p \) we have

\[
K(x) = K \circ M(x) \leq \max(M(x)) < \max x.
\]

Similarly we can show the inequality \( K(x) > \min(x) \).

Finally we proceed to the proof of (vi). First, let us define the partial ordering \( \preceq \) on \( I^p \) by

\[
(x_1, \ldots, x_p) \preceq (y_1, \ldots, y_p) \quad \text{if and only if} \quad x_i \leq y_i \quad \text{for all} \quad i \in \{1, \ldots p\}.
\]

Then, since each \( M_i \) are nondecreasing, for every \( x, y \in I^p \) with \( x \preceq y \) we get \( M(x) \preceq M(y) \). Thus, by a simple induction, we also have \( M^n(x) \preceq M^n(y) \) for all \( n \in \mathbb{N} \). In the limit case when \( n \to \infty \), in view of (ii), we obtain that \( K \) is monotone with respect to \( \preceq \), which is (vi).

To show the last assertion take \( c > 0 \) and \( x \in (0, +\infty)^p \) arbitrarily. Then, since \( M \) is homogeneous, by (ii), we have

\[
K(cx) = \lim_{n \to \infty} M^n(cx) = \lim_{n \to \infty} cM^n(x) = cK(x),
\]

and thus \( K(cx) = cK(x) \). \( \square \)

3. \( d \)-AVERAGING MAPPINGS

In this section we introduce an important subfamily of mean-type mappings and study their properties within this class. Our aim is to reduce the properties of a subclass mean-type mappings which appeared in the previous section to certain properties of directed graphs.

For the sake of completeness, let us introduce formally \( \mathbb{N} := \{1, \ldots \} \), and \( \mathbb{N}_p := \{1, \ldots, p\} \) (where \( p \in \mathbb{N} \)). Then, for \( p \in \mathbb{N} \) and a vector \( d = (d_1, \ldots, d_p) \in \mathbb{N}_p \), let \( \mathbb{N}^d_p := \mathbb{N}^d_1 \times \cdots \times \mathbb{N}^d_p \). Using this notations, a sequence of means \( M = (M_1, \ldots, M_p) \) is called \( d \)-averaging mapping on \( I \) if each \( M_i \) is a \( d_i \)-variable mean on \( I \).

For a \( d \)-averaging mapping \( M \) and a vector of indexes \( \alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{N}^{d_1}_p \times \cdots \times \mathbb{N}^{d_p}_p = \mathbb{N}^d_p \) define a mean-type mapping \( M_\alpha : I^p \to I^p \) by

\[
M_\alpha := (M_1^{(p;\alpha_1)}, \ldots, M_p^{(p;\alpha_p)});
\]

recall that \( M_i^{(p;\alpha_i)} \)-s were defined in (1.4).

Few examples of this sort of mean-types mapping are presented in the last section. We aim to study the family of means which are \( M_\alpha \)-invariant. The important part of our consideration will use some facts from the graph theory.

Remark 1. Observe that for each element on \( \alpha \in \mathbb{N}^d_p \) we can recover the value of \( p \) (based on the vector of \( d \)) but we cannot do the same for the
single element of $\alpha_i \in \mathbb{N}_p^{d_i}$ (in this example $i \in \{1, \ldots, k\}$). Therefore it is natural to use notations $M_\alpha$ and $M_{i(p,\alpha)}$.

3.1. Graphs of averaging mappings. Now we recall some elementary facts concerning graphs. For the details let us refer the reader to the classical book [12].

A digraph is a pair $G = (V, E)$, where $V$ is a finite set of vertexes, and $E \subset V \times V$ is a set of edges. For each $v \in V$ we denote by $N^-_G(v)$ and $N^+_G(v)$ sets of in- and out-neighbors, respectively. More precisely $N^-_G(v) = \{ w \in V : (w, v) \in E \}$ and $N^+_G(v) = \{ w \in V : (v, w) \in E \}$. Edges of the form $(v, v)$ for $v \in V$ are called loops.

A sequence $(v_0, \ldots, v_n)$ of elements in $V$ such that $(v_{i-1}, v_i) \in E$ for all $i \in \{1, \ldots, n\}$ is called a walk from $v_0$ to $v_n$. Number $n$ is a length of the walk. If for all $v, w \in V$ there exists a walk from $v$ to $w$, then $G$ is called irreducible.

A cycle in a graph is a non-empty walk in which only the first and last vertices are equal. A directed graph is said to be aperiodic if there is no integer $k > 1$ that divides the length of every cycle of the graph. A graph which is simultaneously irreducible and aperiodic is called ergodic.

After this extensive introduction, for a given $p \in \mathbb{N}$, $d = (d_1, \ldots, d_p) \in \mathbb{N}^p$, and $\alpha \in \mathbb{N}_p^d$, we define the $\alpha$-incidence graph $G_\alpha = (V_\alpha, E_\alpha)$ as follows: $V_\alpha := \mathbb{N}_p$ and $E_\alpha := \{ (\alpha_i, j) : i \in \mathbb{N}_p \text{ and } j \in \mathbb{N}_{d_i} \}$.

Going back to the mean-type mapping $M_\alpha$ we can easily see that $x_k$ appears in $[M_\alpha]_i$, as an argument if and only if $G_\alpha$ contains the edge from $k$ to $i$. It turns out that the natural assumption to warranty that $M_\alpha$-invariant mean is uniquely determined, is that $G_\alpha$ is ergodic. Therefore, for $p \in \mathbb{N}$ and $d \in \mathbb{N}^p$, we set

\[
\text{Erg}(d) := \{ \alpha \in \mathbb{N}_p^d : G_\alpha \text{ is ergodic} \}.
\]

Lemma 1. For an ergodic digraph $G = (V, E)$ define $T_G : \{-1, 0, 1\}^V \to \{-1, 0, 1\}^V$ as follows: for an arbitrary $c : V \to \{-1, 0, 1\}$ and $v \in V$ we set

\[
T_G(c)(v) := \begin{cases} 
1 & \text{if } c(w) = 1 \text{ for all } w \in N^-_G(v); \\
-1 & \text{if } c(w) = -1 \text{ for all } w \in N^-_G(v); \\
0 & \text{otherwise}.
\end{cases}
\]

(3.1)

Then for every function $c_0 : V \to \{-1, 0, 1\}$ there there exist a number $\bar{c} \in \{-1, 0, 1\}$ such that $T_G^n(c_0) \equiv \bar{c}$ for all $n \geq 3^{|V|}$.

Moreover $\bar{c} = 0$ unless $c_0 \equiv 1$ or $c_0 \equiv -1$.

Proof. For the sake of brevity, for all $n \in \mathbb{N}$, we denote briefly $c_n := T_G^n(c_0)$. First observe that if $c_{n_0}$ is constant for some $n_0 \in \mathbb{N}$ then, since constant functions are fixed points of $T_G$, we obtain $c_n = c_{n_0}$ for
all $n \geq n_0$. Thus it is sufficient to show that $c_{n_0}$ is constant for some $n_0 \in \{0, \ldots, 3^{|V|}\}$.

By $c_{n+1} = T_G(c_n)$, since the range of $T$ has $3^{|V|}$ elements, we know that there exist $\alpha, n_0 \in \{1, \ldots, 3^{|V|}\}$ such that

$$(3.2) \quad c_{n+\alpha} = c_n \text{ for all } n \geq n_0.$$ 

Next we show that $c_{n_0}$ is a constant function. First, since $\text{Rg}(c_{n_0})$ is a nonempty subset of $\{-1, 0, 1\}$, we know that at least one of the following conditions is valid:

A. $\text{Rg}(c_{n_0}) = \{0\}$,  \quad B. $1 \in \text{Rg}(c_{n_0})$,  \quad C. $-1 \in \text{Rg}(c_{n_0})$.

It splits our proof to three (possibly overlapping) cases.

**Case A.** Once we have $\text{Rg}(c_{n_0}) = \{0\}$, that is $c_{n_0} \equiv 0$, then $T_G^0(c_0) = T^{n-n_0}_G(c_{n_0}) \equiv 0$ for all $n \geq n_0$. Therefore, since $n_0 \leq 3^{|V|}$, the proof in this simple case is completed.

**Case B.** If $1 \in \text{Rg}(c_{n_0})$ then take $v_0 \in V$ such that $c_{n_0}(v_0) = 1$. By $(3.2)$ we have

$$(3.3) \quad c_{n_0+k\alpha}(v_0) = 1 \text{ for all } k \in \mathbb{N}.$$ 

Next, define a sequence $(V_n)_{n=0}^{\infty}$ of vertexes by $V_0 := \{v_0\}$, and

$V_p := \{v \in V : \text{there exists a walk from } v \text{ to } v_0 \text{ of length } p\}, \quad p \geq 1.$ 

Observe that $w \in V_{p+1}$ if and only if there exists an edge from $w$ to some vertex in $V_p$, that is $V_{p+1} = N_G(V_p)$ ($p \geq 0$). Thus $c_n(v) = 1$ for all $v \in V_p$ implies $c_{n-1}(v) = 1$ for all $v \in V_{p+1}$. By the simple induction, in view of $(3.3)$, one gets

$$(3.4) \quad c_0(v) = 1 \text{ for all } k \in \mathbb{N} \text{ and } v \in V_{n_0+k\alpha}.$$ 

Next, since $G$ is ergodic, it is a well-known fact that there exists $q_0$ such that for all $v \in V$ and $q \geq q_0$ there exists a walk from $v$ to $v_0$ of length exactly $q$. In particular $V_q = V$ for all $q \geq q_0$.

Now we take $k_0$ such that $n_0 + k_0\alpha \geq q_0$. By $(3.4)$ we have $c_0(v) = 1$ for all $v \in V_{n_0+k_0\alpha} = V$. Whence, $c_n = T_G^n(c_0) \equiv 1$ for every nonnegative integer $n$.

**Case C.** Whenever $-1 \in \text{Rg}(c_{n_0})$ then, analogously to the previous case, one gets $c_n = T_G^n(c_0) \equiv -1$ for every nonnegative integer $n$.

Binding all latter cases, we have proved that $c_{n_0}$ is a constant function, that is $c_{n_0} \equiv \bar{c}$ for some $\bar{c} \in \{-1, 0, 1\}$. Since $n_0 \leq 3^{|V|}$, and $T_G(c_{n_0}) = c_{n_0}$, we obtain

$$T_G^n(c_0) = T_G^{n-n_0}(T_G^{n_0}(c_0)) = T_G^{n-n_0}(c_{n_0}) = c_{n_0} \equiv \bar{c} \text{ for all } n \geq 3^{|V|}.$$ 

Moreover if $\bar{c} \neq 0$ then we are not in the case A, and therefore $c_0 \equiv 1$ or $c_0 \equiv -1$ as it has been proved in cases B and C, respectively. □
Now we show the first nontrivial results referring directly to $d$-averaging mappings.

**Proposition 1.** Given $p \in \mathbb{N}$, $d \in \mathbb{N}^p$, $\alpha \in \text{Erg}(d)$, and a $d$-averaging mapping $M = (M_1, \ldots, M_p)$. If all $M_i$-s are strict means then $M_\alpha$ is uniformly weak contractive.

Moreover, for all $x \in I^p$, either $M^{3p}_\alpha(x)$ is a constant vector or

$$\min(x) < \min M^{3p}_\alpha(x) < \max M^{3p}_\alpha(x) < \max x.$$  \hfill (3.5)

**Proof.** Let $\Gamma$ be a family of all nonconstant vectors in $I^p$. For $n \in \{0, 1, \ldots \}$ and $x = (x_1, \ldots, x_p) \in \Gamma$ define $c_{x,n} : \mathbb{N}_p \to \{-1, 0, 1\}$ by

$$c_{x,0}(i) := \begin{cases} 1 & \text{if } x_i = \max(x); \\ -1 & \text{if } x_i = \min(x); \\ 0 & \text{otherwise}; \end{cases}$$

$$c_{x,n}(i) := \begin{cases} 1 & \text{if } [M^n_\alpha(x)]_i = \max(x); \\ -1 & \text{if } [M^n_\alpha(x)]_i = \min(x); \\ 0 & \text{otherwise} \end{cases}$$

for $n \geq 1$. Then $c_{x,n}(i) = 1$ for some $x \in \Gamma$ and $n \geq 1$ yields $[M^n_\alpha(x)]_i = \max(x)$, thus

$$M_i([M^n_\alpha(x)]_{\alpha_1,1}, \ldots, [M^n_\alpha(x)]_{\alpha_{i-1},i}) = \max(x).$$

Since $M_i$ is a strict mean, one has

$$[M^n_\alpha(x)]_{\alpha_{i,j}} = \max(x) \text{ for all } j \in \mathbb{N}_p.$$  In the other words, $c_{x,n-1}(\alpha_{i,j}) = 1$ for all $j \in \mathbb{N}_p$. Therefore if $c_{x,n}(i) = 1$ for some $x \in \Gamma$ and $n \in \mathbb{N}$, then $c_{x,n-1}(v) = 1$ whenever $v \in N_{Ga}(i)$.

The converse implication is also valid. Indeed, if $c_{x,n-1}(v) = 1$ for all $v \in N_{Ga}(i)$ then $[M^n_\alpha(x)]_k = \max(x)$ for all $k \in \{\alpha_{i,1}, \ldots, \alpha_{i,d_i}\}$ which yields $[M^n_\alpha(x)]_i = \max(x)$. Similarly $c_{x,n}(i) = -1$ if and only if $c_{x,n-1}(v) = -1$ for all $v \in N_{Ga}(i)$.

Consequently for every $x \in \Gamma$ and $n \geq 0$ we have $c_{x,n+1} = T_{Ga}(c_{x,n})$, where $T_{Ga}$ is defined by (3.1). Thus $c_{x,n} = T^n_{Ga}(c_{x,0})$. By Lemma 1

$c_x := c_{x,3p}$ is a constant function for every $x \in \Gamma$. Now we have three cases.

First, if $c_x \equiv -1$ or $c_x \equiv 1$ then $M^{3p}_\alpha(x)$ is a constant vector and therefore, since $x$ is nonconstant, one has

$$\max M^{3p}_\alpha(x) - \min M^{3p}_\alpha(x) < \max x - \min x.$$  \hfill (3.6)

Next, if $c_x \equiv 0$ then $\min(x) < [M^{3p}_\alpha(x)]_i < \max(x)$ for all $i \in \mathbb{N}_p$, which also yields (3.6). Thus, by the mean-value property, one gets

$$\max M^n_\alpha(x) - \min M^n_\alpha(x) < \max x - \min x$$  \hfill (3.7)$$\text{for all } x \in \Gamma \text{ and } n \geq 3p,$$
which implies the uniform weak concontractivity of $M_\alpha$. Finally note that, based on the proof above, we can easily deduce the moreover part. □

3.2. Invariance problem. Following the convention which is used by several authors (for example it was used in [21]) and Theorem 1 above, we are going to bind several results in a single theorem. The idea beyond this result (and simultaneously the sketch of its proof) is to bind Proposition 1 (which states that, under certain conditions, $M_\alpha$ is uniformly weak contractive) and Theorem 1 (which provides a number of properties of such mappings). Before we formulate this result, which should be considered as the most important outcome of this paper, let us underline two issues. First, we shuffle the order of items in these results (in our opinion they are more natural). Second, in some parts we need to give some effort in the proof, since we cannot apply Theorem 1 directly.

Theorem 2. Given an interval $I \subset \mathbb{R}$, a parameters $p \in \mathbb{N}$, $d \in \mathbb{N}$, and a $d$-averaging mapping $M = (M_1, \ldots, M_p)$ on $I$ such that all $M_i$-s are strict. Then, for all $\alpha \in \text{Erg}(d)$,

(a) there exists the unique $M_\alpha$-invariant mean $K_\alpha : I^p \to I$;
(b) $K_\alpha$ is continuous;
(c) $K_\alpha$ is strict;
(d) $M_\alpha^n$ converges, uniformly on compact subsets of $I^p$, to the mean-type mapping $K_\alpha : I^p \to I^p$, $K_\alpha = (K_\alpha, \ldots, K_\alpha)$;
(e) $K_\alpha : I^p \to I^p$ is $M_\alpha$-invariant, that is $K_\alpha = K_\alpha \circ M_\alpha$;
(f) if $M_1, \ldots, M_p$ are nondecreasing with respect to each variable then so is $K_\alpha$;
(g) if $I = (0, +\infty)$ and $M_1, \ldots, M_p$ are positively homogeneous, then every iterate of $M_\alpha$ and $K_\alpha$ are positively homogeneous.

Proof. By Proposition 1 we know that $M_\alpha$ is uniformly weakly contractive. Then a vast majority of the proof is based on Theorem 1; more precisely: (iv) yields (a), (ii) implies (b) and (d), and (iii) implies (c).

Moreover, for all $i \in \mathbb{N}_p$, if $M_i$ is nondecreasing then the mapping

$I^p \ni (x_1, \ldots, x_p) \mapsto M_i(x_{\alpha_{i,1}}, \ldots, x_{\alpha_{i,d_i}})$

is also nondecreasing (in each of its variable), and therefore $[M_\alpha]_i$ is nondecreasing. Since $i \in \mathbb{N}_p$ is arbitrary, in view of Theorem 1 part (vi), we get (f). Analogously, using part (vii) of the same result, one can prove (g).

At this stage (c) is the only remaining part to be proved, whence we need to prove that $K_\alpha$ is a strict mean on $I$. To this end, take a nonconstant vector $x \in I^p$. We show that $K_\alpha(x) < \max(x)$ (the proof of the second inequality is analogous).

Due to the moreover part of Proposition 1 we have that either (3.5) holds or $M_\alpha^{3p}(x)$ is a constant vector. In the first case, since $K_\alpha$ is
\( M_\alpha \)-invariant, using the inequality \( K_\alpha = K_\alpha \circ M_\alpha^{3p} \) and the mean-value property of \( K_\alpha \), we obtain

\[
K_\alpha(x) = K_\alpha \circ M_\alpha^{3p}(x) < \max\left( M_\alpha^{3p}(x) \right) < \max(x).
\]

If \( M_\alpha^{3p}(x) \) is a constant vector (that is \( M_\alpha^{3p}(x) = K_\alpha(x) \)) then let \( n_0 \in \{0, \ldots, 3^p\} \) be the smallest number such that \( M_\alpha^{n_0}(x) = K_\alpha(x) \).

Obviously \( n_0 > 0 \) since \( x \) is nonconstant. Moreover, \( y := M_\alpha^{n_0-1}(x) \) is a nonconstant vector with \( K_\alpha(x) = M_\alpha(y) \).

Then \( y_k = \min(y) \) for some \( k \in \mathbb{N}_p \). Since \( G_\alpha \) is irreducible, we have \((k, i) \in E \) for some \( i \in \mathbb{N}_p \). Therefore, by the definition of \( G_\alpha \), one gets \( \alpha_{i,j} = k \) for some \( j \in \mathbb{N}_d \). Then \( \max(y_{\alpha_{i,1}}, \ldots, y_{\alpha_{i,d_j}}) \leq \max(y) \) and \( y_{\alpha_{i,j}} = y_k = \min(y) < \max(y) \). Therefore, since \( M_i \) is strict,

\[
K_\alpha(x) = [K_\alpha(x)]_i = [M_\alpha(y)]_i = M_i(y_{\alpha_{i,1}}, \ldots, y_{\alpha_{i,d_j}}) < \max(y).
\]

However, since \( M_\alpha \) is the mean-type mapping, we obtain

\[
\max(y) = \max(M_\alpha^{n_0-1}(x)) \leq \max(x),
\]

and thus we get \( K_\alpha(x) < \max(x) \).

Similarly one can prove the inequality \( K_\alpha(x) > \min(x) \). Since \( x \) is an arbitrary nonconstant vector in \( I^p \), we obtain that \( K_\alpha \) is a strict mean, which was the last unproved part of this statement. \( \square \)

4. Applications and examples

4.1. An application to functional equations. As a first application we solve the functional equation \( F \circ M_\alpha = F \). Obviously, under standard conditions, it has a unique solution in the family of means. We show that if we extend the considered family to all functions which are continuous on the diagonal, we are able to follow the pattern of invariant means.

**Theorem 3.** Given an interval \( I \subset \mathbb{R} \), a parameters \( p \in \mathbb{N} \), \( d \in \mathbb{N}_p \), \( \alpha \in \text{Erg}(d) \), and a \( d \)-averaging mapping \( M = (M_1, \ldots, M_p) \) such that all \( M_i \)-s are continuous and strict.

A function \( F: I^p \to \mathbb{R} \) which is continuous on the diagonal \( \Delta(I^p) := \{(u_1, \ldots, u_p) \in I^p: u_1 = \cdots = u_p\} \) is invariant with respect to the mean-type mapping \( M_\alpha \), i.e. \( F \) satisfies the functional equation

\[
F \circ M_\alpha = F
\]

if, and only if, there is a continuous function \( \varphi: I \to \mathbb{R} \) such that \( F = \varphi \circ K_\alpha \), where \( K_\alpha: I^p \to I \) is the unique \( M_\alpha \) invariant mean.

**Proof.** Take an \( M_\alpha \)-invariant function \( F: I^p \to \mathbb{R} \) which is continuous of the diagonal. Then, for all \( n \in \mathbb{N} \), we have \( F \circ M_\alpha^n = F \). In the limit case, by Theorem 2 part (d), since \( F \) is continuous on the diagonal, we get \( F = F \circ K_\alpha \). Thus \( F = \varphi \circ K_\alpha \) for \( \varphi(x) := F(x, \ldots, x) \).
Conversely, if $F = \varphi \circ K_\alpha$ then, since $K_\alpha$ in $M_\alpha$-invariant, for all $x \in I^p$ we get

$$F \circ M_\alpha(x) = \varphi \circ K_\alpha \circ M_\alpha(x) = \varphi \circ K_\alpha(x) = F(x),$$

which completes the proof. 

4.2. Classical application of Theorem 2. In this section we apply Theorem 2 to show that the mean-type mapping given by (1.3) has the unique invariant mean. This was one of motivations to writing this paper. In this and the subsequent section all mean-type mapping contains only power means (since the mean-type mapping $M$ does not play any important role in the assumptions of Theorem 2). Recall that the $n$-variable power mean of order $s$ is defined by

$$P_s(x_1, \ldots, x_n) = \begin{cases} (x_1^s + \cdots + x_n^s)^{1/s} & \text{if } s \in \mathbb{R} \setminus \{0\}, \\ \sqrt[n]{x_1 \cdots x_n} & \text{if } s = 0, \end{cases}$$

where $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{R}_+$. For simplicity, let us assume that all means in this section are on $\mathbb{R}_+$.

Example 2. Let $M: \mathbb{R}_+^4 \to \mathbb{R}_+^4$ be given by (1.3). We show that there exists the unique $M$-invariant mean $K: \mathbb{R}_+^4 \to \mathbb{R}_+$. Additionally, $K$ is continuous and strict.

Indeed, in the framework of $d$-averaging mappings, we express $M$ defined in (1.3) as $M_\alpha$, where $M$ consists of bivariate power means, that is

$$M = (P_{-1}, P_0, P_1, P_2), \quad \text{and } \alpha = ((1, 2), (2, 3), (3, 4), (4, 1)).$$

The vector $d$ contains lengths of elements in $\alpha$ (since $\alpha \in \mathbb{N}_1^4$), thus $d = (2, 2, 2, 2)$. Obviously all means in $M$, being power means, are continuous and strict. Moreover the $\alpha$-incidence graph is aperiodic (since every vertex has a loop) and irreducible (as (4321) is its Hamiltonian cycle). Consequently the $\alpha$-incidence graph is ergodic.
Thus, in view of Theorem 2, there exists exactly one \( M \)-invariant mean \( K: \mathbb{R}_+^4 \to \mathbb{R}_+ \). Moreover, by the same theorem, we know that it is continuous and strict.

4.3. Mean-type mappings without ergodic incidence graph. In the last section we show few difficulties which comes up in this setting. Moreover in each example we present the incidence graph, which would help to understand the problems appearing when it comes to deal with the invariance problem.

In the first example we show we show what happens if the incidence graph is disconnected.

**Example 3** (Disconnected incidence graph). Let \( p = 4 \),
\[
d = (2, 2, 2, 2),
\]
\[
\alpha = ((1, 2), (1, 2), (3, 4), (3, 4)) \in \mathbb{N}^d,
\]
\[
M = (\mathbb{P}^{-1}, \mathbb{P}_1, \mathbb{P}^{-1}, \mathbb{P}_1).
\]
Then the mean-type mapping \( M_\alpha: \mathbb{R}_+^4 \to \mathbb{R}_+^4 \) is of the form
\[
M_\alpha(x, y, z, t) = \left( \frac{2xy}{x+y}, \frac{x+y}{2}, \frac{2zt}{z+t}, \frac{z+t}{2} \right).
\]

![Figure 2. Graph \( G_\alpha \) related to Example 3.](image)

Observe that, in this case, \( M_\alpha \) is not weakly contractive, for example
\[
M^n_\alpha(1, 1, 2, 2) = (1, 1, 2, 2) \text{ for all } n \in \mathbb{N}.
\]
As a matter of fact, we can split the mapping \( M_\alpha \) into two bivariate mappings. Then, using the classical result saying that the arithmetic-harmonic mean coincide with the geometric mean, we obtain
\[
\lim_{n \to \infty} M^n_\alpha(x, y, z, t) = (\sqrt{xy}, \sqrt{xy}, \sqrt{zt}, \sqrt{zt}).
\]
As a result we have two natural \( M_\alpha \)-invariant means. Namely
\[
K_1(x, y, z, t) = \sqrt{xy} \text{ and } K_2(x, y, z, t) = \sqrt{zt}.
\]
Therefore \( K(x, y, z, t) = S(\sqrt{xy}, \sqrt{zt}) \) is \( M_\alpha \)-invariant for every mean \( S: \mathbb{R}_+^2 \to \mathbb{R}_+ \). Note that we cannot exclude that there are other \( M_\alpha \)-invariant means, however all continuous solutions are of this form. □
In next two examples we deal with the weakly connected graphs which are not irreducible. We do believe that it is possible to generalize these examples to a result which covers weakly connected graphs. However at this stage it is the conjecture.

Example 4 (Weakly connected incidence graph I). Let $p = 4$,
\[
\begin{align*}
d & = (3, 3, 3, 3), \\
\alpha & = ((1, 2, 3), (1, 2, 3), (1, 2, 3), (1, 2, 3)) \in \mathbb{N}_4^d, \\
M & = (\mathcal{P}_{-1}, \mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2).
\end{align*}
\]

Then the mean-type mapping $M_\alpha : \mathbb{R}_+^4 \to \mathbb{R}_+^4$ is of the form
\[
M_\alpha(x, y, z, t) = \left( \frac{3xyz}{xy+yz+zx}, \sqrt[3]{xyz}, \frac{x+y+z}{3}, \sqrt{\frac{x^2+y^2+z^2}{3}} \right).
\]
Thus $M_\alpha$ does not depend on the last coordinate. As a result we cannot suppose that the $M_\alpha$-invariant mean is, for example, monotone or strict. In this case, however, $M_\alpha$-invariant mean is uniquely determined, since $M_\alpha$ restricted to first three variables (both is domain and values) admit the unique invariant mean. □

Example 5 (Weakly connected incidence graph II). Let $p = 4$,
\[
\begin{align*}
d & = (2, 2, 2, 2), \\
\alpha & = ((1, 2), (1, 2), (2, 4), (3, 4)) \in \mathbb{N}_4^d, \\
M & = (\mathcal{P}_{-1}, \mathcal{P}_1, \mathcal{P}_{-1}, \mathcal{P}_1).
\end{align*}
\]
Then $M_\alpha$ is of the form
\[
M_\alpha(x, y, z, t) = \left( \frac{2xy}{x+y}, \frac{x+y}{2}, \frac{2yt}{y+t}, \frac{z+t}{2} \right).
\]
Figure 4. Graph $G_\alpha$ related to Example 5.

Analogously to Example 3, we have

$$\lim_{n \to \infty} M^n_\alpha(x, y, z, t) = (\sqrt{xy}, \sqrt{xy}, \sqrt{xy}, \sqrt{xy}).$$

Therefore $K(x, y, z, t) = \sqrt{xy}$ is the only $M$-invariant mean (see [23, Theorem 1] for the detailed proof). Remarkably, it depends on neither $z$ nor $t$. □

Finally we show an example with periodic incidence graph, with the conjecture as in the case of weakly connected graphs.

Example 6 (Periodic incidence graph). Let $p = 4$,

$$\mathbf{d} = (2, 2, 2, 2),$$
$$\alpha = ((3, 4), (3, 4), (1, 2), (1, 2)) \in \mathbb{N}^4_d,$$
$$M = (\mathcal{P}_1, \mathcal{P}_1, \mathcal{P}_1, \mathcal{P}_1).$$

Then $M_\alpha$ is of the form

$$M_\alpha(x, y, z, t) = \left(\frac{2zt}{z + t}, \frac{z + t}{2}, \frac{2xy}{x + y}, \frac{x + y}{2}\right).$$

Figure 5. Graph $G_\alpha$ related to Example 6.

Then, after some computations, we have

$$M^2_\alpha(x, y, z, t) = \left(\frac{4xy(x + y)}{x^2 + 6xy + y^2}, \frac{x^2 + 6xy + y^2}{4(x + y)}, \frac{4zt(z + t)}{z^2 + 6zt + t^2}, \frac{z^2 + 6zt + t^2}{4(z + t)}\right).$$
Analogously Example 3 and the previous example we have
\[ \lim_{n \to \infty} \mathbf{M}_\alpha^{2n}(x, y, z, t) = (\sqrt{xy}, \sqrt{xy}, \sqrt{zt}, \sqrt{zt}), \]
consequently
\[ \lim_{n \to \infty} \mathbf{M}_\alpha^{2n+1}(x, y, z, t) = \mathbf{M}_\alpha \left( \lim_{n \to \infty} \mathbf{M}_\alpha^{2n}(x, y, z, t) \right) = (\sqrt{zt}, \sqrt{zt}, \sqrt{xy}, \sqrt{xy}). \]
Therefore \( K(x, y, z, t) = S(\sqrt{xy}, \sqrt{zt}) \) is \( \mathbf{M}_\alpha \)-invariant for every symmetric mean \( S : \mathbb{R}_+^2 \to \mathbb{R}_+ \). Moreover, using ideas from Example 3, one can show that all continuous \( \mathbf{M}_\alpha \)-invariant means are of this form. \( \square \)

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