Real rectifiable currents and algebraic cycles

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Abstract

We study some fundamental properties of real rectifiable currents and give a generalization of King’s theorem to characterize currents defined by holomorphic chains with real coefficients. A consequence of this result is a sufficient condition for the Hodge conjecture.

1 Introduction

Since the publication of the foundational paper “Normal and integral currents” ([9]) by Federer and Fleming, geometric measure theory becomes important tool in many areas of mathematics ([7]). One particular fascinating question to us is to characterize currents defined by analytic varieties, or more generally, by holomorphic chains, which are some formal linear combination of analytic varieties. The first major progress was made by King in his marvelous paper [16] where he proved that holomorphic chains with positive integral coefficients are those $d$-closed rectifiable positive currents. Three years later Harvey and Shiffman improved King’s result ([15]) to show that $d$-closed rectifiable currents of type $(k, k)$ with $(2k+1)$-Hausdorff measure 0 support are integral holomorphic chains. They also conjectured that the assumption on support is not necessary. This conjecture was resolved about twenty years later by Alexander ([1]). So for holomorphic chains with integral coefficients, the characterization is complete. For holomorphic chains with real coefficients, as far as we know, not much is known. A significant difference lies in the fact that for a rectifiable current $T$, its density $\Theta(||T||, x)$ is a nonnegative integer, but for a real rectifiable current $R$, its density $\Theta(||R||, x)$, is a nonnegative real number ([8]). For example consider $R := \sum_{n=1}^{\infty} \frac{1}{2^n} [\frac{1}{n}]$ where $[\frac{1}{n}]$ is the current defined by the point $\frac{1}{n}$. The current $R$ is obviously a $d$-closed, positive, type $(0, 0)$ real rectifiable current but it can not be a holomorphic 0-chain on $\mathbb{C}$ since the sequence $\{\frac{1}{n}\}$ has a limit point 0. This means that the conditions $d$-closedness, real rectifiability and positivity are not sufficient to characterize positive real holomorphic chains. The main point of this paper is to show that in addition to the three conditions mentioned above, we need to add that $N := \{x|\Theta(||R||, x) > 0\}$ is of $\mathcal{H}^{2k}$-locally finite. Restriction on support also appears in studying plurisubharmonic positive currents ([11], [12], [13], [14]). We then show that currents with these four properties are positive real holomorphic chains. This generalizes King’s result. Since the last condition is automatically satisfied by rectifiable currents, our result also largely simplified King’s proof. This simplification is possible because of our use
of Siu’s famous semicontinuity theorem ([21]). Techniques from geometric measure theory are already important tools in studying algebraic cycles([19] [10] [11] [18]). In our opinion, our characterization of real holomorphic chains may find important applications in studying the Hodge conjecture. We give a sufficient condition for homology classes that can be represented by algebraic cycles with rational coefficients on complex projective manifolds.

This paper is organized as follows. Section 2 studies some fundamental properties of real rectifiable currents. This includes an integral representation for real rectifiable currents which plays an important role in later development. We show that a locally normal current with $H^{2k}$-locally finite support is actually real rectifiable. Section 3 gives a generalization and at the same time, a new proof of King’s theorem. In section 4 we apply our result to give a sufficient condition for homology classes to be represented by algebraic cycles with rational coefficients.

2 Real rectifiable currents

In this article, $M$ will denote an oriented smooth real manifold and $X$ will be a complex manifold. We first fix some notations that will be used throughout this paper. Let $A^r(M)$ be the space of complex valued smooth $r$-forms on $M$ and let $A^r_c(M)$ be the space of complex-valued $r$-forms with compact support on $M$. Dually, $D'_r(M)$ is the space of currents of dimension $r$ and $E'_r(M)$ is the space of currents with compact supports.

**Definition 2.1.** Let $K \subset M$ be a compact set. An element $T \in RR^r, K(M)$ is called a real rectifiable $r$-current on $M$ with support in $K$ is defined as follows: $T \in E'_r(M)$ and for every $\varepsilon > 0$, there is an open subset $U$ of some $\mathbb{R}^n$, a Lipschitz map $f : U \to M$ and a finite real polyhedral $r$-chain $P$ (in this article, we assume that simplices are nonoverlapping) with $f(\text{supp} P) \subset K$ such that the mass

$$M(T - f_*(P)) < \varepsilon$$

We define $RR_r(M)$, the real rectifiable $r$-currents on $M$ as $\bigcup RR^r, K(M)$, where the union is taken over all compacta $K \subset M$. Locally real rectifiable $r$-currents on $M$ are elements of the set

$$RR^r_{\text{loc}}(M) := \{ T \in D'_r(M) : \text{for } x \in M, \text{ there is } T_x \in RR_r(M) \text{ such that } x \notin \text{supp}(T - T_x) \}.$$ 

We recall some definitions and results that we need later.

**Definition 2.2.** Suppose that $N$ is a $\mathcal{H}^n$-measurable subset of $\mathbb{R}^{n+k}$ and $\theta$ is a positive locally $\mathcal{H}^n$-integrable function on $N$. We say that a given $n$-dimensional vector subspace $P$ of $\mathbb{R}^{n+k}$ is the approximate tangent space for $N$ at $x$ with respect to $\theta$ if for all $f \in A^0_c(\mathbb{R}^{n+k})$,

$$\lim_{\lambda \to 0} \lambda^{-n} \int_N f(\lambda^{-1}(z - x))\theta(z)d\mathcal{H}^n(z) = \theta(x) \int_P f(y)d\mathcal{H}^n(y)$$
The following result is basically [17, Proposition 5.4.3]. Since later we need some more information about $T_j$ which is contained in the construction [17, Lemma 5.4.2], we include it in the statement of the theorem.

**Theorem 2.3.** Suppose $N \subset \mathbb{R}^{n+k}$ is $\mathcal{H}^n$-measurable and is countably $n$-rectifiable. Then $N = \bigcup_{j=0}^{\infty} S_j$ where

1. $\mathcal{H}^n(S_0) = 0$;
2. $S_i \cap S_j = \emptyset$ if $i \neq j$;
3. for $j \geq 1$, $S_j \subseteq T_j$, and $T_j$ is an $n$-dimensional, embedded $C^1$ submanifold of $\mathbb{R}^{n+k}$.
   Furthermore, there is an open subset $U_j \subset \mathbb{R}^n$ and a $C^1$-diffeomorphism $f_j : U_j \to T_j$, and $T_x N = T_x T_j$ for almost all $x \in S_j$.

The following result is from [20, Theorem 11.6].

**Theorem 2.4.** Suppose that $N$ is $\mathcal{H}^n$-measurable. Then $N$ is countably $n$-rectifiable if and only if there is a positive locally $\mathcal{H}^n$-integrable function $\theta$ on $N$ with respect to which the approximate tangent space $T_x N$ exists for $\mathcal{H}^n$-almost every $x \in N$.

In the following, we generalize the integral representation theorem ([9, Theorem 8.16]) for integral currents to real rectifiable currents. This result plays a fundamental role in later development.

**Theorem 2.5.** If $T \in \mathcal{R}R_k(\mathbb{R}^n)$, then for all $\varphi \in A^k_c(\mathbb{R}^n)$,

$$T(\varphi) = \int_W \langle \varphi(x), \overrightarrow{T}(x) \rangle \theta(x) \, d\mathcal{H}^k$$

where $W$ is countably $k$-rectifiable, $\mathcal{H}^k$-measurable, $\theta$ is a positive $\mathcal{H}^k$-integrable function on $W$, $\|\overrightarrow{T}(x)\| = 1$ and $\overrightarrow{T}(x)$ orients the approximate tangent space $T_x W$ for $\mathcal{H}^k$-almost every $x \in W$.

**Proof.** Let $C$ be the class of all $T \in \mathcal{E}'_k(\mathbb{R}^n)$ which has the integral representation as stated. The result will be proved in six steps as in [9, Theorem 8.16], but we only need to check the second step since the proof of other statements are the same.

The statement we want to check is:

if $T_i \in C$ for $i \in \mathbb{N}$, $\sum_{i=1}^{\infty} T_i = T \in \mathcal{E}'_k(\mathbb{R}^n)$ and $\sum_{i=1}^{\infty} M(T_i) < \infty$, then $T \in C$

Let $\varphi \in A^k_c(\mathbb{R}^n)$. By assumption

$$T_i(\varphi) = \int_{W_i} \langle \varphi(x), \overrightarrow{T}_i(x) \rangle \theta_i(x) \, d\mathcal{H}^k$$
and
\[ T(\varphi) = \sum_{i=1}^{\infty} \int_{\cup_i W_i} \langle \varphi(x), \vec{T}_i(x) \rangle \theta_i(x) \, d\mathcal{H}^k. \]

By the Lebesgue’s dominated convergence theorem, we have
\[ \int_{\cup_i W_i} \sum_{i=1}^{\infty} \theta_i(x) d\mathcal{H}^k = \sum_{i=1}^{\infty} \int_{W_i} \theta_i(x) d\mathcal{H}^k = \sum_{i=1}^{\infty} M(T_i) < \infty \]
which implies
\[ \sum_{i=1}^{\infty} \theta_i \in L^1(\mathcal{H}^k) \]

Since \( \varphi \) has compact support, we may assume \( \|\varphi\| \leq 1 \). Note that \( \|\vec{T}_i(x)\| = 1 \) for almost every \( x \in W_i \). So for any \( n \in \mathbb{N} \),
\[ \sum_{i=1}^{n} | < \varphi(x), \vec{T}_i(x)\theta(x) > | \leq \sum_{i=1}^{\infty} \theta_i(x) \]

By the Lebesgue dominated convergence theorem again,
\[ T(\varphi) = \int_{\cup_i W_i} < \varphi(x), \nu(x) > \, d\mathcal{H}^k \]
where \( \nu(x) = \sum_{i=1}^{\infty} \theta_i(x) \vec{T}_i(x) \in \wedge^k(T_x \mathbb{R}^n) \) is convergent.

Next, by hypothesis, for each \( i \), there is a subset \( Y_i \subset W_i \) such that for every \( x \in Y_i \), \( \vec{T}_i(x) \) exists, \( \|\vec{T}_i(x)\| = 1 \), \( \vec{T}_i(x) \) orients the approximate tangent space of \( W_i \) at \( x \), and \( \mathcal{H}^k(W_i \setminus Y_i) = 0 \). Let \( W = \bigcup_{i=1}^{\infty} W_i \) and \( Y = \bigcup_{i=1}^{\infty} Y_i \).

Then we use the notation in Theorem 2.3 to express \( W \) as \( W = \bigcup_{j=0}^{\infty} S_j \) and \( S_j \subseteq T_j \) for some \( C^1 \)-manifold \( T_j \).

Let \( Z = S_0 \bigcup (W \setminus Y) \). Then \( \mathcal{H}^k(Z) = 0 \). Fix \( x \in W \setminus Z \). Then \( x \) lies in some \( T_j \). If \( x \in W_i \cap W_j \), then both \( \vec{T}_i(x) \) and \( \vec{T}_j(x) \) orient \( T_i T_j \) which means \( \vec{T}_i(x) = \pm \vec{T}_j(x) \). Since \( \sum_{i=1}^{\infty} \theta_i(x) \vec{T}_i(x) \) converges in \( \wedge^k(T_x \mathbb{R}^n) \), so \( \sum_{i=1}^{\infty} \theta_i(x) \vec{T}_i(x) \) is of the form \( \theta(x) \vec{T}(x) \) where \( \vec{T}(x) \) orients \( T_x T_j \) with unit norm and \( \theta(x) \geq 0 \). Since \( \theta = \sum_{i=1}^{\infty} a_i \theta_i \) where \( a_i = 1 \) or \( -1 \) and \( \sum_{i=1}^{\infty} \theta_i \in L^1(\mathcal{H}^k) \), by the Lebesgue dominated convergence theorem, \( \theta \) is in \( L^1(\mathcal{H}^k) \).

Then for all \( \varphi \in A^k_c(\mathbb{R}^n) \),
\[ T(\varphi) = \int_W \langle \varphi(x), \vec{T}(x) \rangle \theta(x) \, d\mathcal{H}^k \]
which means that \( T \in C \).

The converse of the above result is also true.
Theorem 2.6. If $T \in \mathcal{E}'_t(\mathbb{R}^n)$ and 
\[ T(\varphi) = \int_W \langle \varphi(x), \overrightarrow{T}(x) \rangle \theta(x) \, d\mathcal{H}^k \]
for all $\varphi \in A^k_c(\mathbb{R}^n)$ where $W$ is a countably $k$-rectifiable and $\mathcal{H}^k$-measurable set, $\theta$ is a positive $\mathcal{H}^k$-integrable function on $\mathbb{R}^n$, $\| \overrightarrow{T}(x) \| = 1$ and $\overrightarrow{T}(x)$ orients the approximate tangent space $T_xW$ for $\mathcal{H}^k$-almost every $x \in W$, then $T \in RR_k(\mathbb{R}^n)$.

Before we prove Theorem 2.6, we need a simple result whose validity is rather clear.

Lemma 2.7. Let $A$ be a bounded $\mathcal{L}^n$-measurable subset of $\mathbb{R}^n$. For any given $\varepsilon > 0$, there is a finite set of disjoint $n$-simplices which coincide with $A$ except for a set of measure less then $\varepsilon$.

Now we can prove Theorem 2.6.

Proof. Since $T$ has compact support, we may assume that $W$ is bounded. By Theorem 2.3, we may write $W = \bigcup_{j=0}^{\infty} S_j$ where all $S_j \subseteq T_j$ have properties as stated in Theorem 2.3. We have 
\[ M(T) = \int_W \theta(x) d\mathcal{H}^k(x) = \sum_{i=1}^{\infty} \int_{S_i} \theta(x) d\mathcal{H}^k(x) = \sum_{i=1}^{\infty} M(T|S_i) < \infty. \]
Given $\varepsilon > 0$. Choose $m \in \mathbb{N}$ such that 
\[ \sum_{i=m+1}^{\infty} M(T|S_i) < \varepsilon. \]

Fix $i$ with $1 \leq i \leq m$. Suppose that the $C^1$-manifold $T_i$ is parametrized by the $C^1$-diffeomorphism $f_i : U \to T_i$ for some open subset $U \subset \mathbb{R}^k$, and let $U$ be oriented by the natural orientation inherited from $\mathbb{R}^k$. For $x \in S_i$, there is $y \in U$ such that $f_i(y) = x$ and we define 
\[ \tilde{\theta}_i(x) = \begin{cases} \theta(x), & \text{if } \langle \bigwedge^k(D_y f_i), e_1 \wedge \cdots \wedge e_k \rangle \text{ and } \overrightarrow{T}(x) \text{ determine the same orientation}. \\
-\theta(x), & \text{otherwise.} \end{cases} \]

Let 
\[ \tilde{T}_i(x) = \frac{\langle \bigwedge^k(D_y f_i), e_1 \wedge \cdots \wedge e_k \rangle}{\| \langle \bigwedge^k(D_y f_i), e_1 \wedge \cdots \wedge e_k \rangle \|}. \]

Then 
\[ (T|S_i)(\varphi) = \int_{S_i} \langle \varphi(x), \tilde{T}_i(x) \rangle \tilde{\theta}_i(x) d\mathcal{H}^k(x) \]

Let $\tilde{\theta}_i = \tilde{\theta}_i \circ f_i$. Then $\tilde{\theta}_i$ is Lebesgue integrable. By the change of variable formula, we have $T|S_i = f_i^*(U \wedge \tilde{\theta}_i)$. Given $\lambda_i > 0$. Choose a simple function $\sum_{j=1}^{N} a^i_j \chi_{E^i_j}$ that is close to $\tilde{\theta}_i$ in $L^1$-norm where $a^i_j \in \mathbb{R}$ and all $E^i_j \subset \mathbb{R}^n$ are Lebesgue measurable such that 
\[ M(U \wedge \tilde{\theta}_i - U \wedge \sum_{j=1}^{N} a^i_j \chi_{E^i_j}) \leq \| \tilde{\theta}_i - \sum_{j=1}^{N} a^i_j \chi_{E^i_j} \|_{L^1(U)} < \lambda_i \]
For each $j \in \{1, \ldots, N\}$, by Lemma 2.7 we can find finitely many disjoint polyhedrals $\Delta_{j,l}^i$ in $\mathbb{R}^n$ for $l = 1, \ldots, q_j$ such that

$$\mathcal{H}^k((E_j^i \setminus \bigcup_{l=1}^{q_j} \Delta_{j,l}^i) \cup \bigcup_{l=1}^{q_j} \Delta_{j,l}^i \setminus E_j^i) < \frac{\lambda_i}{N|a_j^i|}.$$  

Let $P_i = \sum_{j=1}^{N} \sum_{l=1}^{q_j} a_j^i \Delta_{j,l}^i$. Then

$$M(U \wedge \sum_{j=1}^{N} a_j^i \chi_{E_j} - P_i) \leq \sum_{j=1}^{N} \int_{(E_j^i \setminus \bigcup_{l=1}^{q_j} \Delta_{j,l}^i) \cup \bigcup_{l=1}^{q_j} \Delta_{j,l}^i \setminus E_j^i} |a_j^i| d\mathcal{H}^k(x) < \lambda_i$$

This implies that

$$M(T[S_i - f_i, P_i]) \leq \text{Lip}(f_i)^k [M(U \wedge \hat{\theta}_i - U \wedge \sum_{j=1}^{N} a_j^i \chi_{E_j}) + M(U \wedge \sum_{j=1}^{N} a_j^i \chi_{E_j} - P_i)] < 2\text{Lip}(f_i)^k \lambda_i.$$  

Now take $C = m(\max_{i=1, \ldots, m}\text{Lip}(f_i)^k)$ and $\lambda_i = \frac{\varepsilon}{2C}$. Then we have

$$M(T - \sum_{i=1}^{m} f_i, P_i) \leq M(\bigcup_{i=m+1}^{\infty} T[S_i]) + \sum_{i=1}^{m} M(T[S_i - f_i, P_i]) < 2\varepsilon$$

This completes the proof.  

**Definition 2.8.**  
1. A triple $(W, \theta, \overrightarrow{T})$ is called an oriented real $k$-rectifiof if $W$ is a countably $k$-rectifiable and $\mathcal{H}^k$-measurable set, $\theta$ is a positive locally $\mathcal{H}^k$-integrable function on $W$, $\overrightarrow{T}(x)$ orients the approximate tangent space $T_xW$ and $\|\overrightarrow{T}(x)\| = 1$ for $\mathcal{H}^k$-almost every $x \in W$.

2. The real rectifiable current $T \in RR^\text{loc}_k(\mathbb{R}^n)$ associated to an oriented real $k$-rectifiof $(W, \theta, \overrightarrow{T})$ is the current

$$T(\varphi) = \int_W \langle \varphi(x), \overrightarrow{T}(x) \rangle \theta(x) \, d\mathcal{H}^k$$

for $\varphi \in A^k_c(\mathbb{R}^n)$.

**Definition 2.9.** Let $U \subset \mathbb{R}^n$ be an open set. We say that a subset $A \subset U$ is of $\mathcal{H}^k$-locally finite if for any $u \in U$, there is $r > 0$ such that the Hausdorff measure $\mathcal{H}^k(A \cap B_r(u)) < \infty$.

**Proposition 2.10.** Suppose that $T \in RR^\text{loc}_k(\mathbb{R}^n)$ is the real rectifiable $k$-current associated to an oriented real $k$-rectifiof $(W, \theta, \overrightarrow{T})$. Let $N = \{x \in \mathbb{R}^n : \Theta^k(\|T\|, x) > 0\}$. Then

$$T(\varphi) = \int_N \langle \varphi(x), \overrightarrow{T}(x) \rangle \Theta^k(\|T\|, x) \, d\mathcal{H}^k$$

for all $\varphi \in A^k_c(\mathbb{R}^n)$, and $W$ is $\mathcal{H}^k$-locally finite if and only if $N$ is $\mathcal{H}^k$-locally finite.
Proof. Let \( \mu = \mathcal{H}^k|\theta = ||T|| \). The existence of the approximate tangent plane (Theorem 2.4) of \( W \) implies that (see [20, pg 63])

\[
0 < \theta(x) = \lim_{r \to 0^+} \frac{\int_{W \cap B_r(x)} \theta(y) d\mathcal{H}^k(y)}{\Omega(k)r^k} = \Theta^k(\mu, x)
\]

for \( \mu \)-almost every \( x \in W \), where \( \Omega(k) \) is the volume of the unit ball in \( \mathbb{R}^k \). This implies that \( \mathcal{H}^k(W - N) = 0 \).

Let

\[
\bar{\theta}(x) = \begin{cases} 
\theta(x), & \text{if } x \in W \\
1, & \text{otherwise.}
\end{cases}
\]

and \( \bar{\mu} = \mathcal{H}^k|\bar{\theta} \).

Since \( \theta \) is locally \( \mathcal{H}^k \)-integrable, for each \( x \in W \), there is a neighborhood \( B_x \) of \( x \) such that \( M(T|B_x) = \int_{W \cap B_x} \theta d\mathcal{H}^k < \infty \). Since \( \bar{\mu} \) is Borel regular, \( W \subset \mathbb{R}^n \) is \( \bar{\mu} \)-measurable and

\[
\bar{\mu}(W \cap B_x) = \int_{W \cap B_x} \bar{\theta}(x) d\mathcal{H}^k(x) = \int_{W \cap B_x} \theta(x) d\mathcal{H}^k(x) = M(T|B_x) < \infty
\]

by [20, Theorem 3.5],

\[
\Theta^k(\bar{\mu}, W, y) = \Theta^k(\bar{\mu}, W \cap B_x, y) = 0
\]

for \( \bar{\mu} \)-almost every \( y \in B_x - W \). Let \( V = \bigcup_{x \in W} B_x \). Then

\[
\Theta^k(\bar{\mu}, W, y) = 0
\]

for \( \bar{\mu} \)-almost every \( y \in V - W \). Clearly, \( \Theta^k(\bar{\mu}, W, y) = 0 \) for all \( y \in \mathbb{R}^n - U \). Therefore \( \Theta^k(||T||, x) = 0 \) for \( \mathcal{H}^k \)-almost every \( x \in \mathbb{R}^n - W \). This implies \( \mathcal{H}^k(N - W) = 0 \).

From the equality

\[
\mathcal{H}^k[(W \setminus N) \cup (N \setminus W)] = 0
\]

we may rewrite

\[
T(\varphi) = \int_W < \varphi(x), \vec{T}(x) > d\mathcal{H}^k|\theta = \int_N < \varphi(x), \vec{T}(x) > \Theta^k(||T||, x) \ d\mathcal{H}^k
\]

for all \( \varphi \in A^k_c(\mathbb{R}^n) \) and we see that \( W \) is \( \mathcal{H}^k \)-locally finite if and only if \( N \) is \( \mathcal{H}^k \)-locally finite. \( \square \)

**Theorem 2.11.** If \( T \in N^\text{loc}_k(\mathbb{R}^n) \) has \( \mathcal{H}^k \)-locally finite support, then \( T \) is real rectifiable.

**Proof.** Let \( S = \text{spt}(T) \). Since \( S \) is \( \mathcal{H}^k \)-locally finite, by [9, pg 494 (4)],

\[
\Theta^k(\mathcal{H}^k, S, x) = \lim_{r \to 0^+} \frac{\mathcal{H}^k(S \cap B_r(x))}{\Omega(k)r^k} \geq 2^{-k}
\]
for $\mathcal{H}^k$-almost every $x \in S$. By [20, Page 63]
\[
\theta(x) = \lim_{r \to 0^+} \frac{||T|| \cdot (B_r(x))}{\mathcal{H}^k(S \setminus B_r(x))} = \lim_{r \to 0^+} \frac{||T|| \cdot (B_r(x))}{\Omega(k) r^k} \cdot \frac{\Omega(k) r^k}{\mathcal{H}^k(S \setminus B_r(x))} \\
\leq 2^k \lim_{r \to 0^+} \frac{||T|| \cdot (B_r(x))}{\Omega(k) r^k} = 2^k \Theta^k(||T||, x)
\]
for $||T||$-almost every $x \in S$. Thus $\Theta^k(||T||, x) > 0$ for $||T||$-almost every $x \in S$. Since $||T||((\mathbb{R}^n - S)) = 0$, $\Theta^k(||T||, x) > 0$ for $||T||$-almost every $x \in \mathbb{R}^n$. By [20, Theorem 32.1], $T$ is real rectifiable. 

3 A generalization of King’s theorem

From now on, we let $U$ be an open subset of $\mathbb{C}^n$.

**Definition 3.1.** Let $\omega$ be the Euclidean-Kähler form in $\mathbb{C}^n$, $\omega_k = \omega_{k,1}$ and let $T \in \mathcal{P}'_{k,k}(U)$ be a positive closed current. The Lelong number $n(T, a)$ of $T$ at a point $a \in U$ is defined to be $\Theta^{2k}(T \wedge \omega_k, a)$.

**Lemma 3.2.** If $T \in RR_{k,k}^{loc}(U)$ is positive and closed, then
\[n(T, a) = \Theta^{2k}(||T||, a)\]
for all $a \in U$. In particular, $\Theta^{2k}(||T||, a)$ exists for all $a \in U$.

**Proof.** By Wirtinger’s inequality ([6, Theorem 4.1]) and the integral representation of $T$ (Theorem 2.5), we have
\[(T \cap B) \wedge \omega_k = \int_B < \omega_k, \overrightarrow{T} > \theta d\mathcal{H}^{2k} = \int_B \theta d\mathcal{H}^{2k} = ||T||(B)\]
for any Borel set $B \subset U$. In particular, $||T||(B_r(a)) = (T \cap B_r(a)) \wedge \omega_k = (T \wedge \omega_k) \cap B_r(a)$. Hence $||T|| = T \wedge \omega_k$, and therefore $n(T, a) = \Theta^{2k}(||T||, a)$. Since Lelong number $n(T, a)$ exists for all $a \in U$, so does $\Theta^{2k}(T \wedge \omega_k, a)$. 

Since we have integral representation for locally real rectifiable currents. The following result is a simple modification of [15, Lemma 1.12].

**Lemma 3.3.** Suppose that $T \in RR_{k,k}^{loc}(U)$ is associated to the oriented real $2k$-rectifolds $(W, \theta(x), \overrightarrow{T}(x))$. Then $T \in RR_{k,k}^{loc}(U)$ if and only if $\overrightarrow{T}(x)$ is complex (i.e. $\overrightarrow{T}(x)$ represents a complex subspace of $\mathbb{C}^n$) for $||T||$-almost every $x \in U$. Furthermore, $T \in RR_{k,k}^{loc}(U)$ is positive if and only if $\overrightarrow{T}(x)$ is complex and positive for $||T||$-almost every $x \in U$. 

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Definition 3.4. A current $T \in \mathcal{D}_k(U)$ is said to be a real holomorphic $k$–chain on $U$ if $T$ can be written in the form $T = \sum r_j[V_j]$ where $r_j \in \mathbb{R}$ and $V = \bigcup V_j$ is a purely $k$-dimensional holomorphic subvariety of $U$ with irreducible components $\{V_j\}$. The vector space of real holomorphic $k$-chains on $U$ is denoted by $\mathcal{R} \mathcal{Z}_k(U)$. Also let $\mathcal{R} \mathcal{Z}^+(U)$ denote the set of positive real holomorphic $k$-chains on $U$, i.e., those real holomorphic $k$-chains with nonnegative coefficients.

We recall the following semicontinuity theorem of Siu’s (see [2, 21]).

Theorem 3.5. (Siu’s semicontinuity theorem) If $T$ is a closed positive current of bidimension $(k, k)$ on a complex manifold $X$, then the upperlevel sets

$$E_r(T) = \{ x \in X : n(T, x) \geq c \}$$

are holomorphic subvarieties of dimension $\leq k$.

Proposition 3.6. Let $U$ be an open subset $\mathbb{C}^n$, and let $A_i, i = 1, 2, \ldots$, be an irreducible holomorphic subvariety of dimension $k$ in $U$. If $A = \bigcup_{i=1}^\infty A_i$ is $\mathcal{H}^{2k}$-locally finite, then $A$ is a holomorphic subvariety of $U$.

Proof. Suppose that there is a point $a \in U$ such that each neighborhood of $a$ meets infinitely many $A_i$. Fix any $r > 0$ with $B_{4r}(a) \subset U$. Then $B_r(a)$ meets infinitely many $A_i$. Assume that $B_r(a)$ meets $A_{ij}, j = 1, 2, \ldots$. For each $j$, there is a point $a_j \in A_{ij}$ such that $|a - a_j| < r$. Hence $B_r(a_j) \subset B_{4r}(a)$. By [22, Theorem B],

$$\mathcal{H}^{2k}(B_{4r}(a) \cap A_{ij}) \geq \Omega(2k)r^{2k}$$

Thus,

$$\mathcal{H}^{2k}(B_{4r}(a) \cap A) \geq \sum_{j=1}^\infty \Omega(2k)r^{2k}.$$ 

Since $r$ is arbitrary, we find that $A$ is not $\mathcal{H}^{2k}$-locally finite which is a contradiction.

Proposition 3.7. If $T \in RR_k^{loc}(U)$ and $N = \{ x \in U : \Theta^k(\|T\|, x) > 0 \}$, then $\text{spt}(T) = \bar{N}$.

Proof. By Proposition [2,10]

$$T(\varphi) = \int_N \langle \varphi(x), \bar{T}(x) \rangle \Theta^k(\|T\|, x) \, d\mathcal{H}^k$$

for all $\varphi \in A^k_c(U)$. If $a \notin \bar{N}$, there is a neighborhood $V$ of $a$ such that $V \cap \bar{N} = \emptyset$. Hence for any $w \in A^k_c(U)$ with $\text{spt}(w) \subset V$, we have $T(w) = 0$. Therefore $V \subset (\text{spt}(T))^c$, and this shows that $\bar{N}^c \subset (\text{spt}(T))^c$, equivalently, $\text{spt}(T) \subset \bar{N}$. On the other hand, for $a \in N$, $\Theta^k(\|T\|, a) > 0$ implies that there are infinitely many $r > 0$ such that

$$\frac{\|T\|(B_r(a))}{\Omega(k)r^k} > 0$$
For each such \( r > 0 \), since

\[
\|T\|(B_r(a)) = \sup \{T(w) : w \in A^k_c(U) \text{ with } \|w\| \leq 1 \text{ and } \text{spt}(w) \subset B_r(a)\}
\]

we can find at least one \( w \in A^k_c(U) \) with \( \text{spt}(w) \subset B_r(a) \) such that \( T(w) > 0 \). This shows that \( a \in \text{spt}(T) \), and hence \( \bar{N} \subset \text{spt}(T) \) since \( \text{spt}(T) \) is closed in \( U \).

We need the following result from [16, Proposition 3.1.3].

**Proposition 3.8.** If \( V \) is a \( k \)-dimensional holomorphic subvariety of a complex manifold \( X \), then for any closed flat chain \( T \in F^\text{loc}(X) \) with \( \text{spt}(T) \subset X \), \( T \) is of the form \( \sum a_j[V_j] \), where the \( V_j \) are the global irreducible components of \( V = \bigcup V_j \) and \( a_j \in \mathbb{C} \).

Now we give a generalization of King’s theorem to realistic currents.

**Theorem 3.9.** If \( T \in RR^\text{loc}_{k,k}(U) \) is positive, closed and \( N = \{x \in U : \Theta^k(\|T\|, x) > 0\} \) is \( \mathcal{H}^{2k} \)-locally finite, then \( T \in \mathcal{R} \mathcal{Z}^+_k(U) \).

**Proof.** By Proposition [2,10] \( N = \{x \in U : n(T, x) > 0\} \). Write

\[
N = \bigcup_{n=1}^\infty E_n \text{ where } E_n = \{x \in U : n(T, x) \geq \frac{1}{n}\}
\]

Then by Siu’s semicontinuity theorem, \( E_n \) is a holomorphic subvariety of \( U \) with dimension \( \leq k \) for all \( n \in \mathbb{N} \). By the Measure Support Theorem ([16, Theorem 2.4.2]), we may assume that each \( E_n \) is of purely \( k \)-dimensional. By assumption, \( N \) is of \( \mathcal{H}^{2k} \)-locally finite and hence by Proposition [3,6] \( N \) is a holomorphic subvariety and hence closed in \( U \). By Proposition [3,7] \( \text{spt}(T) = \bar{N} = N \). By King’s theorem ([16, Theorem 4.2.2]), \( n(T) \) is a positive integer. This proves that if \( n(T, x) \in \mathbb{Z} \) (respectively \( \mathbb{Q} \) for all \( x \in U \), then \( T \) is a holomorphic chain with positive integral (respectively rational) coefficients.

**Corollary 3.10.**

1. Suppose that \( T \in RR^\text{loc}_{k,k}(U) \) is positive, closed and \( N = \{x \in U : \Theta^k(\|T\|, x) > 0\} \) is \( \mathcal{H}^{2k} \)-locally finite. If \( n(T, x) \in \mathbb{Z} \) (respectively \( \mathbb{Q} \)) for all \( x \in U \), then \( T \) is a holomorphic chain with integral coefficients.

2. (King’s theorem) Suppose that \( T \in R^\text{loc}_{k,k}(U) \) is positive and \( d \)-closed, then \( T \) is a holomorphic chain with integral coefficients.

**Proof.**

1. By Theorem [3,9] \( T = \sum_{j=1}^m a_j[V_j] \) is a holomorphic chain for some positive real numbers \( a_j \) and irreducible \( k \)-dimensional holomorphic subvarieties \( V_j \) of \( U \). For each \( j \), choose \( x_j \in V_j - \bigcap_{i \neq j} V_i \). Then \( n(T, x_j) = a_j n([V_j], x_j) \) and by Thie’s theorem ([16, Theorem 4.2.2]), \( n([V_j], x_j) \) is a positive integer. This implies that if \( n(T, x_j) \in \mathbb{Z} \) (respectively \( \mathbb{Q} \)), then \( a_j \in \mathbb{Z} \) (respectively \( \mathbb{Q} \)).

2. By Lemma [15, Lemma 1.14], \( \text{spt}(T) \) has \( \mathcal{H}^{2k} \)-locally finite measure, so King’s theorem follows from (1).
Corollary 3.11. Let $U \subset \mathbb{C}^n$ be an open set. If $T \in RR^\text{loc}_{k,k}(U)$ is a closed, positive real rectifiable current and $n(T,a)$ is either 0 or larger than $b$, where $b > 0$ for all $a \in U$, then $T$ is a holomorphic chain with real coefficients.

Proof. Because the result is local, we may restrict $U$ to a smaller open subset, also denoted by $U$, such that $M(T) < \infty$. Let $N = \{a \in U : n(T,a) > 0\}$. Then

$$T(\varphi) = \int_N \langle \varphi(x), \overrightarrow{T}(x) \rangle \ n(T,x) dH^{2k}(x)$$

Hence

$$bH^{2k}(N) \leq M(T) < \infty.$$ 

Therefore by Theorem 3.9, $T$ is a holomorphic chain with real coefficients.

4 Applications

Let $X$ be a compact complex manifold. Recall that a $k$-current $T \in \mathcal{D}'_k(X)$ is said to be real if $\overline{T}(\varphi) = T(\overline{\varphi})$ for all $\varphi \in A^k(X)$ where $\overline{\varphi}$ is the conjugation of $\varphi$. Let $\mathcal{D}'(X;\mathbb{R})$ be the space of all real currents on $X$. It follows from a well known result of Federer and Fleming [9, 7] that the homology $H_*(\mathcal{D}'_\bullet(X;\mathbb{R}))$ of the chain complex $(\mathcal{D}'_\bullet(X;\mathbb{R}), d)$ is isomorphic to the singular homology $H_*(X;\mathbb{R})$ with real coefficients. In the following, $H_*(X;\mathbb{R})$ denotes the homology of the chain complex $(\mathcal{D}'_\bullet(X;\mathbb{R}), d)$.

Proposition 4.1. Let $X$ be a complex projective manifold of complex dimension $n$ and $e \in A^{n-k,n-k}(X)$ is a $d$-closed form. If $e$ considered as a current has the following property:

$$e = R + dd^cb$$

where $R$ is a current such that the $(k,k)$-part $R_{k,k}$ of $R$ is positive and $R_{k,k}$ has $H^{2k}$-locally finite support, then $e$ is homologous to some algebraic cycles with real coefficients.

Proof. Since $R_{k,k}$ is positive, it is normal. The support of $R_{k,k}$ is of $H^{2k}$-locally finite and $dR_{k,k} = 0$, by Theorem 2.11 $R_{k,k} \in RR_{k,k}(X)$, and by Theorem 3.9 and Chow’s Theorem, it is an algebraic cycle with real coefficients. The result follows from the fact that $e = R_{k,k} + dd^cb_{k+1,k+1}$ where $b_{k+1,k+1}$ is $(k+1,k+1)$-part of $b$.

Lemma 4.2. Let $X$ be a compact Kähler manifold and $P \in \mathcal{D}'_k(X)$ be a $d$-closed $k$-current. If $\gamma = -G(d^*P)$, then

1. $\text{spt}(\gamma) \subset \text{spt}(d^*P) \subset \text{spt}(P)$;
2. $P + d\gamma$ is $d$- and $d^c$-closed, and $\text{spt}(P + d\gamma) \subset \text{spt}(P)$.
Proof. Since $X$ is compact Kähler, by the Hodge decomposition, we have
\[ d^c P = H(d^c P) + \Delta G(d^c P) = dd^* G(d^c P) = dd^c(-G(d^* P)) \]

Let $\gamma = -G(d^* P)$. Note that $P + d\gamma$ is clearly $d$-closed and
\[ d^c(P + d\gamma) = d^c P - dd^c \gamma = dd^c \gamma - dd^c \gamma = 0 \]
By the Hodge theorem again, we have
\[ d^* P = \Delta G(d^* P) \]
If there is $x \in \text{spt}(G(d^* P)) - \text{spt}(d^* P)$, there exists an open ball $B_\epsilon(x)$ lying in some local chart centered at $x$ such that for any $k$-forms $\phi$ support in $B_\epsilon(x)$, $d^* P(\phi) = 0$. Since $x \in \text{spt}(G(d^* P))$, there exists a $k$-form $\alpha$ support in $B_\epsilon(x)$ such that $G(d^* P)(\alpha) \neq 0$. Let $\beta$ be the solution of the Poisson’s equation ([5, page 41, Theorem 15])
\[ \begin{cases} \Delta \beta = \alpha, & \text{in } B_\epsilon(x); \\ \beta = 0, & \text{on the boundary of } B_\epsilon(x) \end{cases} \]
and we extend $\beta$ by zeros to whole $X$, then $\beta \in A^k_\epsilon(X)$ and support in $B_\epsilon(x)$. Therefore
\[ 0 = (d^* P)(\beta) = (\Delta G(d^* P))(\beta) = G(d^* P)(\alpha) \neq 0 \]
which is a contradiction. It implies that
\[ \text{spt}(G(d^* P)) \subset \text{spt}(d^* P) \subset \text{spt}(P), \quad \text{spt}(d\gamma) \subset \text{spt}(P) \]
and hence $\text{spt}(P + d\gamma) \subset \text{spt}(P)$. \hfill \qed

**Theorem 4.3.** Let $X$ be a complex projective manifold of complex dimension $n$. Fix a smooth triangulation of $X$. Let $e \in A^{n-k,n-k}(X)$ be a $d$-closed form. If $e$ considered as a current is homologous to a simplicial $k$-cycle $P$ with rational coefficients and the $(k,k)$-part of $P - dG(d^* P)$ is positive, then $[e] \in H_{2k}(X; \mathbb{Q}) \subset H_{2k}(X; \mathbb{C})$ is represented by some algebraic $k$-cycle with rational coefficients on $X$.

**Proof.** By assumption we have
\[ e = P + da \]
for some $a \in \mathcal{D}'_{2(k+1)}(X)$. By Lemma [4.2], there is $\gamma \in \mathcal{D}'_{2(k+1)}(X)$ such that $P + d\gamma$ is $d$- and $d^c$-closed, and $\text{spt}(P + d\gamma) \subset \text{spt}(P)$. So we may rewrite
\[ e = P + d\gamma + d(a - \gamma) = P + d\gamma + dd^c b \]
for some $b$ by the $dd^c$-lemma.

Since $H^{2k}(\text{spt}(P)) < \infty$, by Lemma [4.2], $\text{spt}(P + d\gamma)$ has $H^{2k}$-finite support, and by assumption, its $(k,k)$-part is positive, hence $P + d\gamma$ full fill the hypothesis of Proposition [4.1] and therefore $e$ is homologous to an algebraic cycle with real coefficients.
Note that by assumption, \([e] \in H_{2k}(X; \mathbb{Q})\). Let \(C_k(X; \mathbb{Q}) \subset H_{2k}(X; \mathbb{Q}), C_k(X; \mathbb{R}) \subset H_{2k}(X; \mathbb{R})\) be the subspaces generated by algebraic cycles with rational and real coefficients respectively. Since we may find a basis for \(C_k(X; \mathbb{R})\) from algebraic cycles with integral coefficients, these algebraic cycles also form a basis for \(C_k(X; \mathbb{R})\), then we have the equality

\[
C_k(X; \mathbb{Q}) = H_{2k}(X; \mathbb{Q}) \cap C_k(X; \mathbb{R})
\]

Applying the above observation to our case, we have \([e] \in H_{2k}(X; \mathbb{Q}) \cap C_k(X; \mathbb{R})\), and hence in \(C_k(X; \mathbb{Q})\).

\[\square\]

**Corollary 4.4.** Let \(X\) be a complex projective manifold of dimension \(n\). Given a smooth triangulation of \(X\) and \(e \in A^{n-k,n-k}(X)\). If \(e\) is homologous to a simplicial 2k-cycle \(P\) with rational coefficients which is \(d^c\)-closed and the \((k,k)\)-part of \(P\) is positive, then \(e\) is homologous to an algebraic cycle with rational coefficients.

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