QUASI-PLURISUBHARMONIC ENVELOPES 3: SOLVING
MONGE-AMPÈRE EQUATIONS ON HERMITIAN MANIFOLDS

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Abstract. We develop a new approach to $L^\infty$-a priori estimates for degenerate complex Monge-Ampère equations on complex manifolds. It only relies on compactness and envelopes properties of quasi-plurisubharmonic functions. In a prequel [GL21a] we have shown how this method allows one to obtain new and efficient proofs of several fundamental results in Kähler geometry. In [GL21b] we have studied the behavior of Monge-Ampère volumes on hermitian manifolds. We extend here the techniques of [GL21a] to the hermitian setting and use the bounds established in [GL21b], producing new relative a priori estimates, as well as several existence results for degenerate complex Monge-Ampère equations on compact hermitian manifolds.

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Introduction

Complex Monge-Ampère equations have been one of the most powerful tools in Kähler geometry since Yau’s solution to the Calabi conjecture [Yau78]. In recent years degenerate complex Monge-Ampère equations have been intensively studied by many authors, in relation to the Minimal Model Program (see [GZ, Don18, BBEGZ, BBJ21] and the references therein). The main analytical input came here from pluripotential theory which allowed Kołodziej [Kol98] to establish uniform a priori estimates in quite degenerate settings.

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The study of complex Monge-Ampère equations on compact hermitian manifolds was undertaken by Cherrier [Cher87] and Hanani [Han96]. It has gained considerable interest in the last decade, after Tosatti and Weinkove solved an appropriate version of Yau’s theorem in [TW10b], following important progress by Guan-Li [GL10] and geometric motivation for constructing special hermitian metrics (see e.g. [FLY12]). The smooth Gauduchon-Calabi-Yau conjecture has been solved by Székelyhidi-Tosatti-Weinkove [STW17], while the pluripotential theory has been partially extended by Dinew, Kołodziej, and Nguyen [DK12, KN15, Din16, KN19], allowing these authors to establish various uniform a priori estimates in a non Kähler setting.

In [GL21a] we have developed a new approach for establishing uniform a priori estimates, restricting to the context of Kähler manifolds for simplicity. While the pluripotential approach consists in measuring the Monge-Ampère capacity of sublevel sets ($\varphi < -t$), we directly measure the volume of the latter, avoiding delicate integration by parts. Our approach thus applies in the hermitian setting, providing several new results that we now describe more precisely.

We let $X$ denote a compact complex manifold of complex dimension $n$, equipped with a hermitian metric $\omega_X$. We fix $\omega$ a semi-positive $(1,1)$-form and set

$$v_-(\omega) := \inf \left\{ \int_X (\omega + dd^c u)^n : u \in \text{PSH}(X, \omega) \cap L^\infty(X) \right\},$$

where $d = \partial + \overline{\partial}$, $dd^c = i(\overline{\partial} - \partial)$, and PSH$(X, \omega)$ is the set of $\omega$-plurisubharmonic functions: these are functions $u$ which are locally the sum of a smooth and a plurisubharmonic function, and such that $\omega + dd^c u \geq 0$ is a positive current.

When $\omega$ is closed, simple integration by parts reveal that $v_-(\omega) = \int_X \omega^n$ is positive as soon as the differential form $\omega$ is positive at some point. Bounding from below $v_-(\omega)$ is a much more delicate issue in general which we discuss at length in [GL21b]. Our first main result is the following uniform a priori estimate when $v_-(\omega)$ is positive (Theorem 2.2):

**Theorem A.** Let $\omega$ be semi-positive with $v_-(\omega) > 0$. Let $\mu$ be a probability measure such that PSH$(X, \omega) \subset L^m(\mu)$ for some $m > n$. Any solution $\varphi \in \text{PSH}(X, \omega) \cap L^\infty(X)$ to $(\omega + dd^c \varphi)^n = c \mu$, where $c > 0$, satisfies

$$\text{Osc}_X(\varphi) \leq T$$

for some uniform constant $T$ which depends on an upper bound on $\frac{c}{v_-(\omega)}$ and $A_m(\mu) := \sup \left\{ \left( \int_X (-\psi)^m d\mu \right)^{1/m} : \psi \in \text{PSH}(X, \omega) \text{ with } \sup_X \psi = 0 \right\}.$

This result covers the case when $\mu = fdV_X$ is absolutely continuous with respect to Lebesgue measure, with density $f$ belonging to $L^p$, $p > 1$, or to an appropriate Orlicz class $L^w$ (for some convex weight $w$ with "fast growth" at infinity), thus partially extending the case of hermitian forms treated by Dinew-Kołodziej [DK12] and Kołodziej-Nguyen [KN15, KN21].

We also provide a new and direct alternative proof of this a priori estimate when $\omega$ is hermitian, relying only on the local resolution of the classical Dirichlet problem for the complex Monge-Ampère equation, and twisting the right hand side with an exponential (see Theorem 2.1).

As far as solutions to such equations are concerned, we obtain the following:
Theorem B. Let $\omega$ be a semi-positive $(1,1)$ form which is either big or such that $\nu_-(\omega)>0$. Fix $0 \leq f \in L^p(dV_X)$, where $p > 1$ and $\int_X f dV_X = 1$. Then

- there exists a unique constant $c(\omega, f) > 0$ and a bounded $\omega$-psh function $\varphi$ such that $(\omega + dd^c \varphi)^n = c(\omega, f) f dV_X$;
- for any $\lambda > 0$ there exists a unique $\varphi_\lambda \in \text{PSH}(X, \omega) \cap L^\infty (X)$ such that $(\omega + dd^c \varphi_\lambda)^n = e^{\lambda \varphi_\lambda} f dV_X$.

By analogy with the Kähler setting, we say here that $\omega$ is big if there exists an $\omega$-psh function with analytic singularities $\rho$ such that $\omega + dd^c \rho \geq \delta \omega_X$ for some $\delta > 0$. A celebrated result of Demailly-Păun [DP04, Theorem 0.5] ensures, when $X$ is Kähler, that the existence of $\rho$ is a consequence of the condition $\nu_-(\omega) > 0$. This result has been partially extended to the hermitian setting in [GL21b].

A slight refinement of our technique allows one to establish important stability estimates (see Theorems 2.3 and 3.5). Similar results have been obtained by quite different methods over the last decade (see [Blo11, DK12, Szek18, KN19, LPT20]).

There are several geometric situations when one cannot expect the Monge-Ampère potential $\varphi$ to be globally bounded. These corresponds to probability measures $\mu = f dV_X$ whose density $f$ belongs to an Orlicz class $L^w$, for some convex weight $w$ with slow growth (see section 2.2). Our next main result provides the following a priori estimate, which extends to the hermitian setting a result proved by DiNezza-Lu [DuL17] in the context of quasi-projective varieties:

Theorem C. Let $\omega$ be a semi-positive $(1,1)$ form which is big and such that $\nu_-(\omega) > 0$. Let $\mu = f dV_X$ be a probability measure, where $0 \leq f = g e^{-A\psi} \in L^w$ with $g \in L^p(dV_X)$ for some $p > 1$, $A > 0$ and $\psi \in \text{PSH}(X, \omega)$. Assume $\varphi$ is a bounded $\omega$-psh function such that $(\omega + dd^c \varphi)^n = cf dV_X$ and $\sup_X \varphi = 0$. Then

$$\alpha \psi - \beta \leq \varphi \leq 0$$

for any $0 < \alpha \leq 1$, where $\beta > 0$ is a uniform constant that depends on $p$, the weight $w$, and upper bounds for $\|g\|_L^p(A \frac{c}{\alpha \nu_-(\omega)}$, and the Luxembourg norm $\|f\|_w$.

The proof of DiNezza-Lu uses generalized Monge-Ampère capacities, hence relies on Bedford-Taylor’s comparison principle. It has been shown by Chiose [Chi16] that this comparison principle holds only under the restrictive condition $dd^c \omega = dd^c \omega^2 = 0$. Our approach is completely different: we use a quasi-psh envelope construction to replace $f$ by $e^{-2A\varphi}$ (a similar idea has been recently used in [LN20, DDL19]) and then use Theorem A, whose proof also heavily relies on quasi-psh envelopes.

We then move on to show the existence of solutions to such degenerate complex Monge-Ampère equations. In this context we prove the following:

Theorem D. Let $\omega$ be a semi-positive $(1,1)$ form which is big and such that $\nu_-(\omega) > 0$. Fix $\rho \in \text{PSH}(X, \omega)$ with analytic singularities along a divisor $E$, such that $\omega + dd^c \rho$ dominates a hermitian form. Let $\mu = f dV_X$ be a probability measure, where $0 \leq f$ is smooth and positive in $X \setminus D$, and $f = e^{\psi+\psi^\pm}$ for some quasi-plurisubharmonic functions $\psi^\pm$.

Then there exist $c > 0$ and $\varphi \in \text{PSH}(X, \omega)$ such that

- $\varphi$ is smooth in the open set $X \setminus (D \cup E)$;
- $\alpha (\psi^+ + \rho) - \beta(\alpha) \leq \varphi$ in $X$ and $\sup_X \varphi = 0$;
- $(\omega + dd^c \varphi)^n = cf dV_X$ in $X \setminus (D \cup E)$;

where $0 < \alpha \leq 1$ is arbitrarily small, and $\beta(\alpha)$ is a uniform constant.
This result can be seen as a generalization of the main result of [TW10b]. It encompasses the case of smooth Monge-Ampère equations on mildly singular compact hermitian varieties, as well as more degenerate settings, hermitian analogues of the main results of [DnL17, Theorems 1 and 3]. It is obtained as a combination of Theorems 3.7 and 4.2. When moreover $f \in L^p(dV_X)$, $p > 1$, then $\varphi$ is globally bounded (one can take $\alpha = 0$) and it suffices to assume that $\omega$ is big (see Theorem 4.1).

We finally apply our results to solve a singular version of the hermitian Calabi-Yau theorem. We work over a compact complex variety $V$ which has log terminal singularities (see Section 4.2 for a precise definition). If the first Bott-Chern class $c_{BC}^1(V)$ vanishes, one says that $V$ is a $\mathbb{Q}$-Calabi-Yau variety. In that context we construct many Ricci-flat hermitian metrics:

**Theorem E.** Let $V$ be a $\mathbb{Q}$-Calabi-Yau variety and $\omega_V$ a hermitian form. There exists a function $\varphi \in \text{PSH}(V, \omega_V) \cap L^\infty(V)$ such that
- $\varphi$ is smooth in $V_{\text{reg}}$;
- $\omega_V + dd^c \varphi$ is a hermitian form and $\text{Ric}(\omega_V + dd^c \varphi) = 0$ in $V_{\text{reg}}$.

We actually prove a singular hermitian analogue of Yau’s celebrated solution to the Calabi conjecture, see Theorem 4.5. We expect many further geometric implications of the present work, but leave this for a future project.

**Comparison with other works.** Yau’s proof of his famous $L^\infty$-a priori estimate [Yau78] goes through a delicate Moser iteration process. A PDE proof of the $L^\infty$-estimate has been recently provided by Guo-Phong-Tong [GPT21] using an auxiliary Monge-Ampère equation, inspired by the recent breakthrough by Chen-Cheng on constant scalar curvature metrics [CC21]. An important generalization of Yau’s estimate has been provided by Kolodziej [Kol98] using pluripotential techniques. This technique has been further generalized in [EGZ09, EGZ08, DP10, BEGZ10] in order to deal with less positive or collapsing classes. All these works require the underlying manifold to be Kähler.

Blocki has provided a different approach [Blo05, Blo11] which is based on the Alexandroff-Bakelman-Pucci maximum principle and a stability estimate due to Cheng-Yau ($L^2$-case) and Kolodziej ($L^p$-case). Blocki’s method works in the hermitian case and has been generalized to various settings by Székelyhidi, Tosatti and Weinkove [STW17, Szek18, TW18]. It requires the reference form to be strictly positive, but applies to a large family of equations (see also [TW15] for a slightly weaker $L^\infty$ estimate based on Moser iteration process).

The first steps of pluripotential theory have been developed in the hermitian setting by Dinew, Kolodziej and Nguyen [DK12, KN15, Din16, KN19]. The presence of torsion requires the reference form to be positive in order to control error terms in delicate integration by parts.

Our approach consists in showing that the volume of the sublevel sets ($\varphi < -t$) goes down to zero in finite time by directly measuring their $\mu$-size. It relies on weak compactness of normalized $\omega$-plurisubharmonic functions and basic properties of quasi-psh envelopes, allowing us to deal with semi-positive forms.

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1. Preliminaries

In the whole article we let $X$ denote a compact complex manifold of complex dimension $n \geq 1$, equipped with a hermitian form $\omega_X$, $dV_X$ a smooth probability measure, and $\omega$ a smooth semi-positive $(1,1)$-form on $X$ such that $\int_X \omega^n > 0$.

1.1. Positivity properties and envelopes.

1.1.1. Monge-Ampère measure. A function is quasi-plurisubharmonic if it is locally given as the sum of a smooth and a psh function.

**Definition 1.1.** Quasi-plurisubharmonic functions $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ satisfying $\omega_\varphi := \omega + dd^c \varphi \geq 0$ in the weak sense of currents are called $\omega$-psh functions.

We let $\text{PSH}(X, \omega)$ denote the set of all $\omega$-plurisubharmonic ($\omega$-psh) functions which are not identically $-\infty$.

We refer the reader to [Dem, GZ, Din16] for basic properties of $\omega$-psh functions. Recall that:

- The set $\text{PSH}(X, \omega)$ is a closed subset of $L^1(X)$, for the $L^1$-topology.
- $\text{PSH}(X, \omega) \subset L^r(X)$ for $r \geq 1$; the induced $L^r$-topologies are equivalent;
- the subset $\text{PSH}_A(X, \omega) := \{ u \in \text{PSH}(X, \omega) : -A \leq \sup X u \leq 0 \}$ is compact in $L^r(X)$ for any $r \geq 1$ and all $A > 0$.

The complex Monge-Ampère measure $(\omega + dd^c u)^n = \omega^n_u$ is well-defined for any $\omega$-psh function $u$ which is bounded, as follows from Bedford-Taylor theory.

The mixed Monge-Ampère measures $(\omega + dd^c u)^j \wedge (\omega + dd^c v)^{n-j}$ are also well defined for any $0 \leq j \leq n$, and any bounded $\omega$-psh functions $u, v$. We recall the following classical inequality (see [GL21a, Lemma 1.2]):

**Lemma 1.2.** Let $\varphi, \psi$ be bounded $\omega$-psh functions such that $\varphi \leq \psi$, then

\[ 1_{\{\psi = \varphi\}} (\omega + dd^c \varphi)^j \wedge (\omega + dd^c \psi)^{n-j} \leq 1_{\{\psi = \varphi\}} (\omega + dd^c \psi)^n, \]

for all $1 \leq j \leq n$.

We shall also need the following generalization of the inequality of arithmetic and geometric means (see [N16, Lemma 1.9]):

**Lemma 1.3.** Let $\varphi_1, \ldots, \varphi_n$ be bounded $\omega$-psh functions such that $(\omega + dd^c \varphi_i)^n \geq f_i dV_X$. Then $(\omega + dd^c \varphi_1) \wedge \cdots \wedge (\omega + dd^c \varphi_n) \geq (\Pi_{i=1}^n f_i)^{\frac{1}{n}} dV_X$.

1.1.2. Positivity assumptions. On a few occasions we will need to assume slightly stronger positivity properties of the form $\omega$:

**Definition 1.4.** We say $\omega$ satisfies condition (B) if there exists $B \geq 0$ such that

$$-B \omega^2 \leq dd^c \omega \leq B \omega \quad \text{and} \quad -B \omega^3 \leq d\omega \wedge dd^c \omega \leq B \omega^3.$$ 

Here are three different contexts where this condition is satisfied:

- any hermitian metric $\omega > 0$ satisfies condition (B);
- if $\pi : X \to Y$ is a desingularization of a singular compact complex variety $Y$ and $\omega_Y$ is a hermitian metric, then $\omega = \pi^* \omega_Y$ satisfies condition (B);
- if $\omega$ is semi-positive and closed, then it satisfies condition (B).

Combining these one obtains further settings where condition (B) is satisfied.

**Definition 1.5.** We say that

- $\omega$ is non-collapsing if for any bounded $\omega$-psh function, the complex Monge-Ampère measure $(\omega + dd^c u)^n$ has positive mass: $\int_X (\omega + dd^c u)^n > 0$;
• \( \omega \) is uniformly non-collapsing if \( v_-(\omega) > 0 \), where

\[
v_-(\omega) := \inf \left\{ \int_X (\omega + dd^c u)^n : u \in \text{PSH}(X, \omega) \cap L^\infty(X) \right\}.
\]

These positivity notions are studied at length in [GL21b]. It is shown there that condition (B) implies non-collapsing, we further expect it implies uniform non-collapsing (at least in the case if \( X \) belongs to the Fujiki class).

**Definition 1.6.** We say \( \omega \) is big if there exists an \( \omega \)-psh function \( \rho \) with analytic singularities such that \( \omega + dd^c \rho \geq \delta \omega_X \) dominates a hermitian form.

If \( V \) is a compact complex space endowed with a hermitian form \( \omega_V \), and \( \pi : X \rightarrow V \) is a resolution of singularities, then \( \omega = \pi^* \omega_V \) is big. This follows from classical arguments (see e.g. [FT09, Proposition 3.2]).

It is expected that \( \omega \) is big if and only if \( v_-(\omega) > 0 \). This is a generalization of a conjecture of Demailly-Păun [DP04, Conjecture 0.8], which has been addressed in [GL21b]: it is in particular shown in [GL21b, Theorem 4.6] that if \( v_+(\omega_X) < +\infty \) and \( v_-(\omega) > 0 \) then \( \omega \) is big, where

\[
v_+(\omega_X) := \sup \left\{ \int_X (\omega_X + dd^c u)^n : u \in \text{PSH}(X, \omega_X) \cap L^\infty(X) \right\}.
\]

1.1.3. Envelopes.

**Definition 1.7.** Given a Lebesgue measurable function \( h : X \rightarrow \mathbb{R} \), we define the \( \omega \)-psh envelope of \( h \) by

\[
P_\omega(h) := (\sup\{u \in \text{PSH}(X, \omega) : u \leq h \text{ on } X\}^*),
\]

where the star means that we take the upper semi-continuous regularization.

The following has been established in [GL21b, Theorem 2.3]:

**Theorem 1.8.** If \( h \) is bounded from below, quasi-l.s.c., and \( P_\omega(h) < +\infty \), then

- \( P_\omega(h) \) is a bounded \( \omega \)-plurisubharmonic function;
- \( P_\omega(h) \leq h \) in \( X \setminus P \), where \( P \) is pluripolar;
- \( (\omega + dd^c P_\omega(h))^n \) is concentrated on the contact set \( \{P_\omega(h) = h\} \).

A useful consequence is the following (see [GL21b, Lemma 2.5]):

**Lemma 1.9.** Fix \( \lambda \geq 0 \) and let \( u, v \) be bounded \( \omega \)-psh functions. Fix two smooth semi-positive \((1,1)\)-forms \( \omega_1, \omega_2 \) such that \( \omega_1 \geq \omega, \omega_2 \geq \omega \).

(i) If \( (\omega_1 + dd^c u)^n \leq e^{\lambda u} fdV_X \) and \( (\omega_2 + dd^c v)^n \leq e^{\lambda v} gdV_X \), then

\[
(\omega + dd^c P_\omega(\min(u, v)))^n \leq e^{\lambda P_\omega(\min(u, v))} \max(f, g)dV_X.
\]

(ii) If \( (\omega + dd^c u)^n \geq e^{\lambda u} fdV_X \) and \( (\omega + dd^c v)^n \geq e^{\lambda v} gdV_X \), then

\[
(\omega + dd^c \max(u, v))^n \geq e^{\lambda \max(u, v)} \min(f, g)dV_X.
\]

**Proof.** The second statement follows from [GL21b, Lemma 2.5]. We prove the first one. Setting \( \varphi := P_\omega(\min(u, v)) \), by Theorem 1.8 and Lemma 1.2 we have

\[
(\omega + dd^c \varphi)^n \leq 1_{\{\varphi = u < v\}} (\omega + dd^c \varphi)^n + 1_{\{\varphi = v\}} (\omega + dd^c \varphi)^n
\leq 1_{\{\varphi = u < v\}} (\omega_1 + dd^c \varphi)^n + 1_{\{\varphi = v\}} (\omega_2 + dd^c \varphi)^n
\leq 1_{\{\varphi = u < v\}} (\omega_1 + dd^c u)^n + 1_{\{\varphi = v\}} (\omega_2 + dd^c v)^n
\leq 1_{\{\varphi = u < v\}} e^{\lambda \varphi} \max(f, g)dV_X + 1_{\{\varphi = v\}} e^{\lambda \varphi} \max(f, g)dV_X
\leq e^{\lambda \varphi} \max(f, g)dV_X.
\]

\(\square\)
The following is a key tool to our new approach for uniform estimates:

**Lemma 1.10.** Fix χ : ℝ⁻ → ℝ⁻ a concave increasing function such that χ(0) ≥ 1. Let ϕ, φ be bounded ω-psh functions with ϕ ≤ φ, and set ψ = ϕ + χ(φ − ϕ). Then

\[(ω + dd^c P_ω(ψ))^n ≤ 1_{\{P_ω(ψ)=0\}}(χ'(φ−φ))^n (ω + dd^c φ)^n.\]

The proof is identical to that of [GL21a, Lemma 1.6], a consequence of Theorem 1.8 and Lemma 1.2.

1.2. Comparison and domination principles.

1.2.1. **Comparison principle.** The comparison principle plays a central role in Kähler pluripotential theory. A "modified comparison principle" has been established by Kłodziej-Nguyen [KN15, Theorem 0.2] when ω is hermitian. We extend the latter in this section, assuming that ω is merely big: we fix

- an ω-plurisubharmonic function ρ with analytic singularities such that ω + dd^c ρ ≥ δω_X for some δ > 0; we set Ω := \{ρ > −∞\};
- a constant B_1 > 0 such that for all x ∈ Ω,

\[-B_1 ω^2 ≤ dd^c ω ≤ B_1 ω^2, \quad \text{and} \quad -B_1 ω^3 ≤ dω ∧ dd^c ω ≤ B_1 ω^3.\]

The existence of B_1 is clear since \(-B_1 ω_X^2 ≤ dd^c ω ≤ B_1 ω_X^2\) for some B > 0, and

\(-B_1 ω_X^3 ≤ dω ∧ dd^c ω ≤ B_1 ω_X^3.\)

**Theorem 1.11.** Assume ω is big and let ρ, B_1 be as above. Let u be a bounded ω-psh function, and set m := inf_X(u − ρ). Then for s > 0 small enough we have

\[(1 - 4B_1 s(n - 1)^2)^n \int_{\{u < ρ + m + s\}} ω^n ≤ \int_{\{u < ρ + m + s\}} ω^n.\]

In particular, ω is non-collapsing.

**Proof.** We set φ = max(u, ρ + m + s) and U := \{u < φ\} = \{u < ρ + m + s\}. Observe that U is relatively compact in the open set Ω. For each k ≥ 0 we set T_k := ω_X^k ∧ φ^{n-k} and T_l = 0 for l ≤ 0. Set α = B_1 s(n-1)^2. We prove by induction on k = 0, 1, ..., n − 1 that

\[(1 - 4α) \int_U T_k ≤ \int_U T_{k+1}.\]

The conclusion follows since ω^n_φ = ω^n_φ on the plurifine open set U = \{u < φ\}.

We first prove (1.1) for k = 0. Since u ≤ φ Lemma 1.2 ensures that

\[1_{\{u=φ\}}φ^n ≥ 1_{\{u=φ\}}ω_u ∧ φ^{n-1}.\]

Noting that X \ U = \{u = φ\} we can write

\[\int_U (T_0 - T_1) = \int_U (φ^n - ω_u ∧ φ^{n-1}) ≤ \int_X (φ^n - ω_u ∧ φ^{n-1}) = \int_X dd^c (φ - u) ∧ φ^{n-1}.\]

Observe that

\[dd^c φ^{n-1} = (n - 1)dd^c ω ∧ φ^{n-2} + n(n - 1)dω ∧ dd^c ω ∧ φ^{n-3}\]

\[≤ B_1(n - 1)ω^2 φ ∧ φ^{n-2} + (n - 1)(n - 2)B_1 ω^3 φ ∧ φ^{n-3}.\]
Since $0 \leq \phi - u \leq s$ and $U = \{u < \phi\}$, it follows from Stokes’ theorem that
\[
\int_X dd^c(\phi - u) \wedge \omega^{n-1}_\phi = \int_X (\phi - u) dd^c \omega^{n-1}_\phi \\
\leq sB_1(n-1) \int_U (\omega^2_\rho \wedge \omega^{n-2}_\phi + (n-2)\omega^3_\rho \wedge \omega^{n-3}_\phi) \\
\leq sB_1(n-1)^2 \int_U \omega^n_\phi,
\]
using that $\omega^k_u \wedge \omega^{n-k}_\phi = \omega^n_\phi$ on the plurifine open set $U$. We thus get $\int_U (T_0 - T_1) \leq sB_1(n-1)^2 \int_U T_0$, proving (1.1) for $k = 0$.

We now assume that (1.1) holds for $j \leq k - 1$, and we check that it still holds for $k$. Observe that
\[
 dd^c \left( \omega^k_u \wedge \omega^{n-[k+1]}_\phi \right) \\
= kdd^c \omega \wedge \omega^k_u \wedge \omega^{n-[k+1]}_\phi + (n-[k+1])dd^c \omega \wedge \omega^k_u \wedge \omega^{n-[k+2]}_\phi \\
+ 2(k-[k+1])d\omega \wedge dd^c \omega \wedge \omega^k_u \wedge \omega^{n-[k+2]}_\phi \\
+ k(k-1)d\omega \wedge dd^c \omega \wedge \omega^{k-2}_u \wedge \omega^{n-[k+1]}_\phi \\
+ (n-[k+1]) (n-[k+2])d\omega \wedge dd^c \omega \wedge \omega^k_u \wedge \omega^{n-[k+3]}_\phi.
\]
The same arguments as above therefore show that
\[
\int_U (T_k - T_{k+1}) \leq \int_X (T_k - T_{k+1}) = \int_X (\phi - u) dd^c (\omega^k_u \wedge \omega^{n-[k+1]}_\phi) \\
\leq B_1s \int_U (k(k-1)T_{k-2} + 2k[n-k]T_{k-1} + (n-[k+1])^2T_k) \\
\leq a \left( \frac{1}{(1-4a)^2} + \frac{1}{1-4a} + 1 \right) \int_U T_k \\
\leq 4a \int_U T_k,
\]
where the third inequality uses the induction hypothesis, while the fourth follows from the upper bound $4a < 1/8$. From this we obtain (1.1) for $k$.

We finally prove that $\omega$ is non-collapsing. If $u \in \text{PSH}(X, \omega)$ is bounded and $\omega^n_u = 0$ then the first statement of the proposition implies, since $\omega^n_\rho \geq \delta^n \omega^n_X$, that $\omega^n_X(u < \rho + m + s) = 0$ for $s > 0$ small enough. Since $u$ and $\rho$ are $\omega$-psh and $\omega_X > 0$, this implies $u \geq \rho + m + s$, contradicting the definition of $m$. \hfill $\square$

1.2.2. Domination principle. Several versions of the domination principle have been established in [LPT20, Proposition 2.2], [GL21b, Proposition 2.8]. We shall need the following generalization, valid for mildly unbounded $\omega$-psh functions:

**Proposition 1.12.** Fix $\rho$ an $\omega$-psh function with analytic singularities such that $\omega + dd^c \rho \geq \delta \omega_X$, with $\delta > 0$. Let $u, v$ be $\omega$-psh functions such that, for all $\varepsilon > 0$,
\[
\inf_X (\min(u, v) - \varepsilon \rho) > -\infty.
\]
If $\omega^n_u \leq c\omega^n_v$ on $\{u < v\} \cap \{\rho > -\infty\}$ for some $c \in [0, 1)$, then $u \geq v$.

The condition $\inf_X (\min(u, v) - \varepsilon \rho) > -\infty$ can be equivalently formulated as follows: for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $u, v \geq \varepsilon \rho - C_\varepsilon$. In particular
u and v are locally bounded in the Zariski open set $\Omega = \{ \rho > -\infty \}$, hence the Monge-Ampère measures $\omega^n_u$ and $\omega^n_v$ are well-defined in $\Omega$.

**Proof.** We fix a constant $a > 0$ so small that $\omega^n_u \leq c_1 \omega^n_\phi$ on $\{ u < v \}$, where $0 < c_1 < 1$ and $\phi = (1 - a)v + a\rho$. By adding a constant we can assume that $v \geq \rho$ so that $\omega^n_u \leq c_1 \omega^n_\phi$ on $\{ u < \phi \}$.

We now fix $b > 1$ large and consider $\varphi := P_\omega(bu - (b - 1)\phi)$. It follows from Theorem 1.18 that $\varphi$ is bounded on $X$ and $\omega^n_\varphi$ is supported on the contact set $C := \{ \varphi = bu - (b - 1)\phi \}$. Since $b^{-1}\varphi + (1 - b^{-1})\phi \leq u$ with equality on the contact set, Lemma 1.2 yields

$$1_C b^{-n} \omega^n_\varphi + 1_C (1 - b^{-1})^n \omega^n_\phi \leq 1_C \omega^n_u.$$ 

Thus for $b > 1$ large enough $\omega^n_\varphi$ vanishes in $C \cap \{ u < \phi \} = C \cap \{ \varphi < \phi \}$, hence on $\{ \varphi < \max(\varphi, \phi) \}$ since $\omega^n_\varphi$ is supported on $C$. The domination principle ensures that $\varphi \geq \phi$. We infer $u \geq \phi$ since

$$\varphi = P_\omega(bu - (b - 1)\phi)) \leq bu - (b - 1)\phi.$$ 

Thus $u \geq (1 - a)v + a\rho$ and letting $a \to 0^+$ yields the conclusion. \hfill $\Box$

Here is a useful consequence of the domination principle.

**Corollary 1.13.** Assume $\omega$ is non-collapsing and let $u, v$ be bounded $\omega$-psh functions. If

$$(\omega + dd^c u)^n \leq \tau (\omega + dd^c v)^n$$

for some constant $\tau > 0$, then $\tau \geq 1$.

In particular if $(\omega + dd^c u)^n = c\mu$ and $(\omega + dd^c v)^n = c'\mu$ for the same measure $\mu$, then $c = c'$.

**Proof.** If $\tau < 1$ the domination principle yields $u \geq v + C$ for any constant $C$, a contradiction. \hfill $\Box$

The same result holds when $u, v$ are mildly singular:

**Corollary 1.14.** Assume $\omega$ is big and fix $\rho$ an $\omega$-psh function with analytic singularities such that $\omega + dd^c \rho \geq \delta \omega_X$, with $\delta > 0$. Let $u, v$ be $\omega$-psh functions such that for all $\varepsilon > 0$,

$$\inf_X (\min(u, v) - \varepsilon \rho) > -\infty.$$ 

(1) If $(\omega + dd^c u)^n = \tau (\omega + dd^c v)^n$ then $\tau = 1$.

(2) If $e^{-\lambda u}(\omega + dd^c v)^n \geq e^{-\lambda u}(\omega + dd^c u)^n$ for some $\lambda > 0$, then $v \leq u$.

**Proof.** The proof of (1) is similar to that of Corollary 1.13, so we focus on (2) whose proof follows again from Proposition 1.12: fix $\delta > 0$ and observe that in $\{ u < v - \delta \} \cap \{ \rho > -\infty \}$ we have

$$(\omega + dd^c u)^n \leq e^{\lambda (u - v)}(\omega + dd^c v)^n \leq e^{-\lambda \delta} (\omega + dd^c (v - \delta))^n,$$

with $c = e^{-\lambda \delta} < 1$. Thus $v - \delta \leq u$ and the conclusion follows by letting $\delta \to 0$. \hfill $\Box$

2. Uniform a priori estimates

2.1. Global $L^\infty$-bounds.
2.1.1. Hermitian forms. When $\omega$ is a hermitian form and $\mu$ is a smooth volume form, it has been shown by Tosatti-Weinkove [TW10b] that there exists a unique $c > 0$ and a unique smooth sup-normalized $\omega$-psh function $\varphi$ such that

$$(\omega + dd^c\varphi)^n = c\mu.$$  

This landmark result relies on several previous attempts to generalize Yau’s result to the hermitian setting, notably by Cherrier [Cher87], Hanani [Han96] and Guan-Li [GL10]. The key result that was missing and provided by [TW10b] is an a priori $L^\infty$-estimate for the solution. An alternative a priori estimate using pluripotential techniques has been provided by Dinew-Kołodziej in [DK12], who treated the case when $\mu = f dV_X$ with $f \in L^p$, $p > 1$.

**Theorem 2.1** (Dinew-Kołodziej). Assume $\omega$ is a hermitian form, $p > 1$ and $f \in L^p(dV_X)$ is such that $A^{-1} \leq \left( \int_X \frac{1}{f^p} dV_X \right)^n \leq \left( \int_X f^p dV_X \right)^{\frac{1}{p}} \leq A$ for some $A > 1$. If $c > 0$ and $u \in \text{PSH}(X, \omega) \cap L^\infty(X)$ are such that $(\omega + dd^c u)^n = c f dV_X$, then

$$c + c^{-1} + \text{Osc}_X(u) \leq T,$$

where the constant $T$ only depends on $p, n, \omega$ and an upper bound for $A$.

The proof by Dinew-Kołodziej is a non trivial extension of the pluripotential approach developed by Kołodziej in the Kähler case [Kol98], bypassing extra difficulties coming from the non closedness of $\omega$. We provide a direct proof of this result here, that only relies on local resolutions of Monge-Ampère equations.

**Proof.** Step 1. Constructing a bounded subsolution. We claim that there exist uniform constants $0 < m = m(p, \omega) < M = M(p, \omega)$ such that for any $0 \leq g \in L^p$ with $\int_X g^p dV_X \leq 1$, we can find $v \in \text{PSH}(X, \omega) \cap L^\infty(X)$ such that

$$(\omega + dd^c v)^n \geq mgdV_X \quad \text{and} \quad \text{Osc}_X v \leq M.$$  

Consider indeed a finite double cover of $X$ by small ”balls” $B_j, B'_j = \{ \rho_j < 0 \}$, with $B_j \subset \subset B'_j$ which are bounded in a local holomorphic chart. Here $\rho_j : X \to \mathbb{R}$ denotes a smooth function which is strictly plurisubharmonic in a neighborhood of $B'_j$. We solve $(dd^c v_j)^n = gdV_X$ in $B'_j$ with $-1$ boundary values. It follows from [Kol98] that the plurisubharmonic solution $v_j$ is uniformly bounded in $B'_j$. Considering max$(v_j, \lambda_j \rho_j)$ we can choose $\lambda_j > 1$ and obtain a uniformly bounded function $w_j$ with the following properties:

- $w_j$ coincides with $v_j$ in $B_j$ where it satisfies $(dd^c w_j)^n = gdV_X$;
- $w_j$ is plurisubharmonic in $B'_j$ and uniformly bounded;
- $w_j$ coincides with $\lambda_j \rho_j$ in $X \setminus B'_j$ and in a neighborhood of $\partial B'_j$.

As $w_j$ is smooth where it is not plurisubharmonic, its curvature $dd^c w_j$ is bounded below by $-\delta^{-1} \omega$ for some uniform $\delta > 0$. Thus $\delta w_j$ is $\omega$-psh and $v := \frac{\delta}{N} \sum_{j=1}^N w_j$ is the bounded subsolution we are looking for, since in $B_j$ we obtain

$$(\omega + dd^c v)^n \geq \frac{\delta}{N^n}(\omega + dd^c w_j)^n \geq \frac{\delta^n}{N^n}(dd^c w_j)^n = \frac{\delta^n}{N^n}gdV_X.$$  

**Step 2. Uniform a priori bounds.** We can normalize $u$ by sup$_X u = 0$. It follows from Skoda’s uniform integrability (see [GZ, Theorem 8.11]) that one can find $\varepsilon > 0$, $p' = p'(\varepsilon, p) \in (1, p)$, and $C = C(\varepsilon, p) > 0$ independent of $u$ such that $g = e^{-\varepsilon u} f \in L^{p'}$ with

$$||g||_{p'} \leq ||f||_p \cdot ||e^{-\varepsilon^{\frac{p'}{p'}} u}||_{p^{\frac{p'}{p'}}} \leq C(\varepsilon, p) A.$$
Let $v$ be the bounded subsolution provided by Step 1 for the density $\frac{g}{||g||_{p'}}$, i.e.
\[(\omega + dd^c v)^n \geq m' \frac{g}{||g||_{p'}} dV_X \geq \frac{m'}{A} f dV_X = \frac{m'}{Ac} (\omega + dd^c u)^n,\]
using that $u \leq 0$ hence $g \geq f$. It follows from Corollary 1.13 that $c \geq m'/A$. Note that the upper bound for $c$ follows easily from Lemma 1.3.

We finally observe that the uniform bound on $v$ also provides a uniform bound for $u$. Indeed since $v$ is bounded we obtain
\[(\omega + dd^c v)^n \geq m'' e^{-\varepsilon u} f dV_X \geq e^{\varepsilon(v-u-C)} c f dV_X,\]
hence Corollary 1.14 ensures that $u \geq v - C$. \hfill \Box

### 2.1.2. Semi-positive forms

We now extend this key $L^\infty$-estimate to the case when the form $\omega$ is not necessarily positive, assuming instead that $v_-(\omega) > 0$:

**Theorem 2.2.** Let $\omega$ be semi-positive with $v_-(\omega) > 0$. Let $\mu$ be a probability measure such that $\text{PSH}(X, \omega) \subset L^m(\mu)$ for some $m > n$. Any solution $\varphi \in \text{PSH}(X, \omega) \cap L^\infty(X)$ to $(\omega + dd^c \varphi)^n = c \mu$, where $c > 0$, satisfies
\[\text{Osc}_X(\varphi) \leq T\]
for some uniform constant $T$ which depends on upper bounds for $\frac{c}{v_-(\omega)}$ and
\[A_m(\mu) := \sup \left\{ \left( \int_X (-\psi)^m d\mu \right)^{\frac{1}{m}} : \psi \in \text{PSH}(X, \omega) \text{ with } \sup_X \psi = 0 \right\}.\]

Since any quasi-psh function belongs to $L^r(dV_X)$ for all $r > 1$, this theorem applies to measures $\mu = f dV_X$, where $f \in L^p$ with $p > 1$, as follows from Hölder inequality. As in [Kol98, Theorem 2.5.2] our technique also covers the case of more general densities. We refer the reader to [GL21a, Section 2.2] for more details.

**Proof.** The proof is very similar to that of [GL21a, Theorem 2.1]. Shifting by an additive constant we can assume $\sup_X \varphi = 0$. Set
\[T_{\text{max}} := \sup \{ t > 0 : \mu(\varphi < -t) > 0 \}.\]

Our goal is to establish a precise bound on $T_{\text{max}}$. By definition, $-T_{\text{max}} \leq \varphi$ almost everywhere with respect to $\mu$, hence everywhere by the domination principle, providing the desired a priori bound $\text{Osc}_X(\varphi) \leq T_{\text{max}}$.

We let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ denote a concave increasing function such that $\chi(0) = 0$ and $\chi'(0) = 1$. We set $\psi = \chi \circ \varphi$, $u = P(\psi)$ and observe that
\[\omega + dd^c \psi = \chi' \circ \varphi \omega + [1 - \chi' \circ \varphi] \omega + \chi'' \circ \varphi d\varphi \wedge dd^c \varphi \leq \chi' \circ \varphi \omega \varphi.\]

It follows from Lemma 1.10 that
\[\frac{1}{c}(\omega + dd^c u)^n \leq 1_{\{u=\psi\}}(\chi' \circ \varphi)^n \mu.\]

**Controlling the energy of $u$.** We fix $\varepsilon > 0$ so small that $n < n + 3\varepsilon = m$. The concavity of $\chi$ and the normalization $\chi(0) = 0$ yields $|\chi(t)| \leq |t|\chi'(t)$. Since
$u = \chi \circ \varphi$ on the support of $(\omega + dd^c u)^n$, Hölder’s inequality yields
\[
\int_X (u)_+^\varepsilon (\omega + dd^c u)^n \leq \int_X (-\chi \circ \varphi)^\varepsilon (\chi' \circ \varphi)^n d\mu \leq \int_X (-\varphi)^\varepsilon (\chi' \circ \varphi)^{n+\varepsilon} d\mu \\
\leq \left( \int_X (-\varphi)^{n+2\varepsilon} d\mu \right)^\frac{\varepsilon}{n+2\varepsilon} \left( \int_X (\chi' \circ \varphi)^{n+2\varepsilon} d\mu \right)^\frac{n+\varepsilon}{n+2\varepsilon} \\
\leq A_m(\mu)^\varepsilon \left( \int_X (\chi' \circ \varphi)^{n+2\varepsilon} d\mu \right)^\frac{n+\varepsilon}{n+2\varepsilon}
\]

using that $\varphi$ belongs to the set of $\omega$-psh functions $v$ normalized by $\sup_X v = 0$ which is compact in $L^{n+2\varepsilon}(\mu)$.

Controlling the norms $||u||_{L^m}$. We are going to choose below the weight $\chi$ in such a way that $\int_X (\chi' \circ \varphi)^{n+2\varepsilon} d\mu = B \leq 2$ is a finite constant under control. This provides a uniform lower bound on $\sup_X u$ as we now explain: indeed
\[
0 \leq (-\sup_X u)_+^\varepsilon \frac{v_- (\omega)}{c} \leq (-\sup_X u)_+^\varepsilon \int_X \frac{(\omega + dd^c u)^n}{c} \\
\leq \int_X (-u)_+^\varepsilon (\omega + dd^c u)^n \leq 2A_m(\mu)^\varepsilon
\]
yields
\[
- \left( \frac{2c}{v_- (\omega)} \right)^{1/\varepsilon} A_m(\mu) \leq \sup_X u \leq 0.
\]

We infer that $u$ belongs to a compact set of $\omega$-psh functions, hence
\[
||u||_{L^m(\mu)} \leq \left[ 1 + \left( \frac{2c}{v_- (\omega)} \right)^{1/\varepsilon} \right] A_m(\mu) =: \tilde{A}^m.
\]

From $u \leq \chi \circ \varphi \leq 0$ we get $||\chi \circ \varphi||_{L^m} \leq ||u||_{L^m} \leq \tilde{A}^m$. Using Chebyshev inequality we thus obtain
\[
(2.1) \quad \mu(\varphi < -t) \leq \frac{\tilde{A}}{|\chi|^m(-t)}.
\]

Choice of $\chi$. Recall that if $g : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing with $g(0) = 1$, then
\[
\int_X g \circ (-\varphi) d\mu = \mu(X) + \int_0^{T_{\max}} g'(t) \mu(\varphi < -t) dt.
\]
Setting $g(t) = [\chi'(-t)]^{n+2\varepsilon}$ we define $\chi$ by imposing $\chi(0) = 0$, $\chi'(0) = 1$, and
\[
g'(t) = \frac{1}{(1 + t)^2} \frac{\mu(\varphi < -t)}{\mu(\varphi < -t)} \quad \text{if } t < T_{\max}.
\]
This choice guarantees that $\chi$ is concave increasing with $\chi' \geq 1$ on $\varphi(X)$, and
\[
\int_X (\chi' \circ \varphi)^{n+2\varepsilon} d\mu \leq \mu(X) + \int_0^{+\infty} \frac{dt}{(1 + t)^2} \leq 2.
\]
A slight technical point is that one should first work on a compact subinterval $[0, T']$ with $0 < T' < T_{\max}$ and then let $T'$ tend to $T_{\max}$. We refer the reader to the proof of [GL21a, Theorem 2.1] for more details on this twist.
Conclusion. We set \( h(t) = -\chi(-t) \) and work with the positive counterpart of \( \chi \). Note that \( h(0) = 0 \) and \( h'(t) = |g(t)|^{\frac{1}{1+2\varepsilon}} \) is positive increasing, hence \( h \) is convex. Observe also that \( g(t) \geq g(0) = 1 \) hence \( h'(t) = |g(t)|^{\frac{1}{1+2\varepsilon}} \geq 1 \) yields

\[
(2.2) \quad h(1) = \int_0^1 h'(s)ds \geq 1.
\]

Together with (2.1) our choice of \( \chi \) yields, for all \( t \in [0,T_{\text{max}}) \),
\[
\frac{1}{(1+t)^2 g'(t)} = \mu(\varphi < -t) \leq \frac{\tilde{A}}{h^m(t)}.
\]

This reads
\[
h^m(t) \leq \tilde{A}(1+t)^2 g'(t) = (n+2\varepsilon)\tilde{A}(1+t)^2 h''(t)(h')^{n+2\varepsilon-1}(t).
\]

Multiplying by \( h' \), integrating between 0 and \( t \), we infer that for all \( t \in [0,T_{\text{max}}) \),
\[
\frac{h^{m+1}(t)}{m+1} \leq (n+2\varepsilon)\tilde{A} \int_0^t (1+s)^2 h''(s)(h')^{n+2\varepsilon}(s)ds
\]
\[
\leq (n+2\varepsilon)\tilde{A}(t+1)^2 \frac{((h')^{n+2\varepsilon+1}(t) - 1)}{n+2\varepsilon + 1}
\]
\[
\leq \tilde{A}(1+t)^2(h')^{n+2\varepsilon+1}(t).
\]

Recall that we have set \( m = n + 3\varepsilon \) so that
\[
\alpha := m + 1 = (n + 2\varepsilon + 1) + \varepsilon > \beta := n + 2\varepsilon + 1 > 2.
\]

The previous inequality then reads \( (1+t)^{-\frac{\varepsilon}{\beta}} \leq C h'(t)h(t)^{-\frac{\varepsilon}{\beta}} \), for some uniform constant \( C \) depending on \( n, m, \tilde{A} \). Since \( \alpha > \beta > 2 \) and \( h(1) \geq 1 \) (by (2.2)), integrating the above inequality between 1 and \( T_{\text{max}} \) we obtain \( T_{\text{max}} \leq C' \), for some uniform constant \( C' \) depending on \( C, \alpha, \beta \). \( \square \)

2.1.3. Stability estimate. Adapting similarly the proof of [GL21a, Theorem 2.4], we also obtain:

**Theorem 2.3.** Let \( \omega, \mu \) be as in Theorem 2.2. Let \( \varphi \in \text{PSH}(X, \omega) \cap L^\infty(X) \) be such that \( \sup_X \varphi = 0 \) and \( (\omega + dd^c \varphi)^n = c\mu \) for some \( c > 0 \). Then
\[
\sup_X (\phi - \varphi)_+ \leq T \left( \int_X (\phi - \varphi)_+ d\mu \right)^\gamma,
\]
for any \( \phi \in \text{PSH}(X, \omega) \cap L^\infty(X) \), where \( \gamma = \gamma(n,m) > 0 \) and \( T \) is a uniform constant which depends on upper bounds for \( \frac{\tilde{\varphi}}{v_{-\omega}} \), \( ||\phi||_{L^\infty} \), and
\[
A_m(\mu) := \sup \left\{ \left( \int_X (-\psi)^m d\mu \right)^{\frac{1}{m}} : \psi \in \text{PSH}(X, \omega) \text{ with } \sup_X \psi = 0 \right\}.
\]

If \( \phi \in \text{PSH}(X, \omega) \cap L^\infty(X) \) also satisfies \( (\omega + dd^c \phi)^n = c'\mu' \) for some \( c' > 0 \) and \( \mu' \) having similar properties as that of \( \mu \), the above result yields an \( L^1 - L^\infty \)-stability estimate,
\[
||\phi - \varphi||_{L^\infty(X)} \leq M ||\phi - \varphi||_{L^1(\mu + \mu')} \gamma.
\]

Thus if a sequence of such solutions \( \varphi_j \) converges in \( L^1(\mu) \), this should allow one to conclude that it actually uniformly converges. We shall use this strong information on several occasions in the sequel.
2.2. Relative $L^\infty$-bounds. We consider in this section the degenerate complex Monge-Ampère equation

\[(\omega + dd^c \varphi)^n = cf dV_X,\]

where $0 \leq f \in L^1(X)$. It follows from abstract measure theoretic arguments that $f$ belongs to an Orlicz class $L^\mu(dV_X)$ for some convex increasing weight $w : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\frac{w(t)}{t} \to +\infty$ as $t \to +\infty$. The Luxemburg norm

$$||f||_w := \inf \left\{ r > 0 : \int_X w \left( \frac{f(x)}{r} \right) dx < 1 \right\}$$

is finite if and only if $\int_X w \circ f \ dV_X < +\infty$.

If the weight $w$ grows fast enough at infinity (e.g. $w(t) = t^p$ with $p > 1$, or $w(t) = t(\log t)^m$ with $m > n$), then one can expect the solution to be uniformly bounded (see [Kol98, Theorem 2.5.2]). We assume here this is not the case.

When $f \leq e^{-\psi}$ for some quasi-psh function $\psi$, and $X$ is Kähler, it has been shown by DiNezza-Lu [DnL17, Theorem 2] that the normalized solution $\varphi$ to (2.3) is locally bounded in the complement of the pluripolar set $\{\psi = -\infty\}$. We extend this important result here to the hermitian setting:

**Theorem 2.4.** Assume $v_-(\omega) > 0$ and fix $A > 0$. Fix $\mu = f dV_X$ a probability measure with $f \in L^\mu(dV_X)$ and assume $f \leq ge^{-A\psi}$, where $g \in L^p(dV_X)$ for some $p > 1$ and $\psi \in \text{PSH}(X, \omega)$. If $\varphi$ is a bounded $\omega$-psh function such that $(\omega + dd^c \varphi)^n = cf dV_X$ and $\sup_X \varphi = 0$, then

$$\psi - \beta \leq \varphi \leq 0$$

where $\beta > 0$ depends on $p$, $w$ and upper bounds for $c, v_-((\omega)^{-1}); ||g||_{L^p}, ||f||_w, A$.

Proceeding by approximation, we are going to use in the sequel this lower bound to ensure that some approximating solutions $\varphi_j$ are uniformly bounded from below by a fixed function $\psi - \beta$. In particular any cluster point $\varphi$ of the $\varphi_j$’s will be locally bounded in $X \setminus \{\psi = -\infty\}$. Replacing $A$ and $\psi$ by $A/\alpha$ and $\alpha \psi$, with $0 < \alpha \leq 1$, and increasing $\beta$ in the statement, one obtains

$$\alpha \psi - C_\alpha \leq \varphi \leq 0.$$

An interesting consequence of this improved a priori estimate is that it produces limiting functions $\varphi$ having zero Lelong number at all points.

The proof of DiNezza-Lu makes use of a theory of generalized Monge-Ampère capacities further developed in [DnL15]. It seems quite delicate to extend their arguments to the hermitian setting, as they rely on very precise integration by parts which produce several extra terms when $\omega$ is not closed. Our new approach allows us to bypass these difficulties.

**Proof.** Uniform integrability of $\varphi$. Set $\chi_1(t) := -(w^*)^{-1}(-t)$, where $w^*$ denotes the Legendre transform of $w$. Thus $\chi_1 : \mathbb{R}^- \to \mathbb{R}^-$ is a convex increasing, such that $\chi_1(-\infty) = -\infty$ and

$$\int_X (\chi_1 \circ \varphi)(\omega + dd^c \varphi)^n \leq c \int_X w \circ f dV_X + c \int_X (-\varphi)dV_X \leq C_0,$$

as follows from the additive version of Hölder-Young inequality and the compactness of sup-normalized $\omega$-psh functions. We set, for $t \leq 0$,

$$\chi(t) = -\int_t^0 \sqrt{(-\chi_1)(s)} \, ds \quad \text{and} \quad \chi_2(t) = -\sqrt{(-\chi_1) \circ \chi^{-1}(t)}.$$
The weight $\chi$ is concave increasing with $\chi(-\infty) = -\infty$ and $\chi'(\infty) = +\infty$, while $
abla$ is convex increasing with $\chi(\infty) = -\infty$. Set $\Phi = P_\omega(\chi \circ \varphi) \leq \chi \circ \varphi \leq 0$. It follows from Lemma 1.10 that

$$\int_X (\chi_2 \circ \Phi) (\omega + dd^c \Phi)^n \leq \int_X (\chi_2 \circ \chi \circ \varphi) \chi' \circ \varphi (\omega + dd^c \varphi)^n = \int_X (\chi_1 \circ \varphi) (\omega + dd^c \varphi)^n \leq C_0.$$ 

Since $v_+(\omega) > 0$ this provides a uniform bound on $\sup_X \Phi$ depending on $C_0$, $v_+(\omega)$ and $\chi_2$, so that $\Phi$ belongs to a compact family of $\omega$-psh functions. It follows from Skoda’s uniform integrability result that there exist uniform constants $\alpha, C > 0$ such that

$$\int_X \exp(-\alpha \chi \circ \varphi) dV_X \leq \int_X \exp(-\alpha \Phi) dV_X \leq C < +\infty.$$ 

Finally since $\chi(t)$ grows faster than $t$ at infinity, we infer that for any $\lambda > 0$,

$$\int_X \exp(-\lambda \varphi) dV_X \leq M_\lambda$$

for a uniform constant $M_\lambda$ which depends on upper bounds for $||f||_w$ and $v_+(\omega)^{-1}$.

**Quasi-psh envelope.** Consider $u := P_\omega(2\varphi - \psi)$. One can show that $\sup_X u \leq C_0$ is uniformly bounded from above by generalizing [GZ, Theorem 9.17.1] to the hermitian setting. The previous argument also shows that $\sup_X P_\omega(2\varphi)$ is uniformly bounded from below, hence the same holds for $P_\omega(2\varphi - \psi)$ as we can assume without loss of generality that $\psi \leq 0$.

Since $\varphi$ is bounded, we have $u = P_\omega(2\varphi - \max(\psi, -t))$ for $t > 0$ large enough. We can thus assume that $\psi$ is also bounded. Since $2\varphi - \psi$ is bounded and quasi continuous, it follows from Theorem 1.8 that the Monge-Ampère measure $(\omega + dd^c u)^n$ is concentrated on the contact set $C = \{ u = 2\varphi - \psi \}$. We claim that

$$(\omega + dd^c u)^n \leq 1c2^n (\omega + dd^c \varphi)^n.$$ 

Indeed consider $v = u + \psi$. This is a $(2\omega)$-psh function such that $v \leq 2\varphi$; it follows therefore from Lemma 1.2 that

$$1_{\{v = 2\varphi\}} (2\omega + dd^c v)^n \leq 1_{\{v = 2\varphi\}} (2\omega + dd^c (2\varphi))^n$$

The claim follows since $\{ v = 2\varphi \} = C$ and $(\omega + dd^c u)^n \leq (2\omega + dd^c v)^n$. Here the boundedness of $\psi$ ensures that the Monge-Ampère measure $(\omega + dd^c u)^n$ is well-defined. Now $(\omega + dd^c \varphi)^n = cf dV_X \leq cge^{-A\psi} dV_X$, with

$$1cge^{-A\psi} = 1cge^{Au}e^{-2A\varphi} \leq e^{A^\omega} uge^{-2A\varphi}.$$ 

Using Hölder inequality and (2.4), we infer that $g' = g e^{-2A\varphi}$ is in $L^p'$, for some $p' > 1$ arbitrarily close to $p$, and with an upper bound on $||g'||_{L^p'}$ that only depends on $w$ and upper bounds for $A, ||g||_{L^p}$, and $v_-(\omega)^{-1}$.

Since $(\omega + dd^c u)^n \leq 2^n c e^{A\varphi} g'dV_X$, we can invoke Theorem 2.2 to ensure that the oscillation of $u$ is uniformly bounded. The desired lower bound follows since $2\varphi = (2\varphi - \psi) + \psi \geq u + \psi$. 

### 3. Existence of locally bounded solutions

We now show the existence of solutions to degenerate complex Monge-Ampère equations under various positivity assumptions on $\omega$. We first establish a general
existence result for bounded solutions in section 3.1, then treat the case of solutions that are merely locally bounded in a Zariski open set in section 3.3. Higher regularity of these solutions will be studied in section 4.1.

3.1. Bounded solutions.

3.1.1. The normalizing constant. Fix $p > 1$ and $0 \leq f \in L^p(X, dV_X)$ with $\int_X f dV > 0$. It follows from [KN15, Theorem 0.1] and [N16, Corollary 0.2] that for each $\varepsilon > 0$, there exists a unique constant $c_\varepsilon = c(\omega + \varepsilon \omega_X, f) > 0$ such that one can find $u \in \text{PSH}(X, \omega + \varepsilon \omega_X) \cap L^\infty(X)$ satisfying

$$(\omega + \varepsilon \omega_X + dd^c u)^n = c_\varepsilon f dV.$$ 

Observe that $\varepsilon \mapsto c_\varepsilon$ is increasing. Indeed assume $\varepsilon > \varepsilon'$ and $u_{\varepsilon}, u_{\varepsilon'}$ are bounded solutions to the corresponding equations. Then

$$(\omega + \varepsilon \omega_X + dd^c u_{\varepsilon'})^n \geq (\omega + \varepsilon' \omega_X + dd^c u_{\varepsilon'})^n = c_{\varepsilon'} f dV = c_{\varepsilon}(\omega + \varepsilon \omega_X + dd^c u_{\varepsilon})^n.$$ 

It follows therefore from Corollary 1.13 that $c_\varepsilon \geq c_{\varepsilon'}$. We can thus consider:

**Definition 3.1.** We set $c(\omega, f) := \lim_{\varepsilon \to 0} c_\varepsilon$.

This definition is independent of the choice of the hermitian metric $\omega_X$:

**Proposition 3.2.** The constant $c(\omega, f)$ does not depend on $\omega_X$. Moreover $c(\omega, f) = \inf \{c(\omega', f) : \omega'$ hermitian form such that $\omega' > \omega\}$.

**Proof.** Fix $\omega_1, \omega_2$ two hermitian metrics on $X$. Then $A^{-1} \omega_1 \leq \omega_2 \leq A \omega_1$ for some positive constant $A$. It follows that, for all $\varepsilon > 0$,

$$(\omega + A^{-1} \varepsilon \omega_1) \leq (\omega + \varepsilon \omega_2) \leq (\omega + A \varepsilon \omega_1).$$

Corollary 1.13 ensures that

$$c(\omega + A^{-1} \varepsilon \omega_1, f) \leq c(\omega + \varepsilon \omega_2, f) \leq c(\omega + A \varepsilon \omega_1, f).$$

Letting $\varepsilon \to 0$ yields the conclusion. The last assertion follows similarly. \(\square\)

3.1.2. Existence of solutions.

**Lemma 3.3.** Fix $p > 1$ and assume either $\omega$ is big or $\nu(\omega) > 0$. Then there exists $0 < c = c(p, \omega)$ such that for any $f \in L^p(dV_X)$ with $\int_X f^p dV_X \leq 1$, we can find $u \in \text{PSH}(X, \omega) \cap L^\infty(X)$ such that

$$(\omega + dd^c u)^n \geq c f dV_X \quad \text{and} \quad -1 \leq u \leq 0.$$ 

**Proof.** We first assume $\omega$ is big. We fix $\rho \in \text{PSH}(X, \omega)$ with analytic singularities such that $\omega + dd^c \rho \geq \delta X$ for some $\delta > 0$, with $\sup_X \rho = -1$. Since $\rho$ belongs to $L^r$ for any $r > 1$, Hölder inequality ensures that $|\rho|^{2n} f \in L^q(dV_X)$ for some $q > 1$. It follows from the first step of the proof of Theorem 2.1 that there exist a uniform $c_1 > 0$ and $v \in \text{PSH}(X, \omega_X) \cap L^\infty(X)$ such that $\sup_X v = -1$ and

$$(\omega_X + dd^c v)^n \geq c_1 |\rho|^{2n} f dV_X.$$ 

Observe that the function $\delta v + \rho$ is $\omega$-psh with $\omega + dd^c (\delta v + \rho) \geq \delta (\omega_X + dd^c v)$. Set $u := - (\delta v + \rho)^{-1} = \chi \circ (\delta v + \rho)$ with $\chi(t) = -t^{-1}$ convex increasing on $R^-$. Our normalizations ensure $-1 \leq u \leq 0$ and a direct computation yields

$$\omega + dd^c u \geq \delta \chi' \circ (\delta v + \rho)(\omega_X + dd^c v) = \frac{\delta}{(\delta v + \rho)^2} (\omega_X + dd^c v).$$
We infer \( \omega^n_u \geq |dv + \rho|^{-2n} \delta^n c_1 |\rho|^{2n} f dV_X \). Since \( v \leq -1 \) is bounded and \( \rho \leq -1 \), it follows that \( \omega^n_u \geq a f dV_X \) for some uniform constant \( a > 0 \).

We now assume \( v_-(\omega) > 0 \). Set \( \omega_j = \omega + 2^{-j} \omega_X \). Rescaling \( \omega_X \) if necessary we can assume that \( \omega \leq \omega_j \leq \omega_X \). Using [KN15, Theorem 0.1] we solve
\[
(\omega_j + dd^c u_j)^n = c_j (1 + f) dV_X, \sup_{X} u_j = 0
\]
for any fixed \( j > 0 \), where \( u_j \in \text{PSH}(X, \omega_j) \cap L^\infty(X) \). Fix \( a > 0 \) such that \( \omega^n_X \geq a dV_X \), and observe that \( (\omega_X + dd^c u_j)^n \geq (\omega_j + dd^c u_j)^n \geq c_j dV_X \), hence
\[
(\omega_X + dd^c u_j)^n \wedge \omega^n_X - 1 \geq c_j^2 a^{n-1} dV_X,
\]
as follows from Lemma 1.3.

We infer that \( c_j \) is uniformly bounded from above. Since \( \| (1 + f) \|_p \leq 2 \) and \( v_-(\omega_j) \geq v_-(\omega) > 0 \), it follows from Theorem 2.2 that \( u_j \) is uniformly bounded.

Extracting and relabelling we can assume that \( u_j \to u \in \text{PSH}(X, \omega) \). Theorem 2.2 yields \(-T \leq u \leq 0\) for some uniform constant \( T > 0 \). Set
\[
\Phi_j := \left( \sup_{\ell \geq j} u_\ell \right)^* .
\]
By Lemma 1.9, we have
\[
(\omega + 2^{-j} \omega_X + dd^c \Phi_j)^n \geq v_-(\omega) f dV_X .
\]
Since \( \Phi_j \searrow u \in \text{PSH}(X, \omega) \cap L^\infty(X) \), we obtain \( (\omega + dd^c u)^n \geq v_-(\omega) f dV_X \).

If \( 0 \leq T \leq 1 \) we are done, otherwise we consider \( \tilde{u} = u/T \in \text{PSH}(X, \omega) \) and observe that \(-1 \leq \tilde{u} \leq 0\) with \( (\omega + dd^c \tilde{u})^n \geq T^{-n} v_-(\omega) f dV_X \).

**Theorem 3.4.** Let \( \mu = f dV_X \) be a probability measure, where \( 0 \leq f \in L^p(dV_X) \) for some \( p > 1 \). Assume either \( \omega \) is big or \( v_-(\omega) > 0 \).

1. For any \( \lambda > 0 \) there exists \( \varphi_\lambda \in \text{PSH}(X, \omega) \cap L^\infty(X) \) such that \( (\omega + dd^c \varphi_\lambda)^n = e^{\lambda \varphi_\lambda} f dV_X \).

Moreover \( \| \varphi_\lambda \|_\infty \leq C \), where \( C \) depends on upper bounds on \( \| f \|_p, \| f \|^{-1}_{1/n}, \lambda^{-1} \).

2. One has \( c(\omega, f) > 0 \) and there exists \( \varphi \in \text{PSH}(X, \omega) \cap L^\infty(X) \) such that \( (\omega + dd^c \varphi)^n = c(\omega, f) \mu \).

Here \( c(\omega, f) \) denotes the normalizing constant from Definition 3.1. It is possible to study Hölder regularity of \( \varphi \) following [DDGKPZ], [LPT20], [KN19], but we leave it for a future project.

**Proof.** We first prove (1). For simplicity we assume \( \lambda = 1 \). By [N16, Theorem 0.1], for each \( \varepsilon > 0 \) there exists \( \varphi_\varepsilon \in \text{PSH}(X, \omega + \varepsilon \omega_X) \cap L^\infty(X) \) such that
\[
(\omega + \varepsilon \omega_X + dd^c \varphi_\varepsilon)^n = e^{\varphi_\varepsilon} \mu .
\]
The bounded subsolution \( u \) obtained in Lemma 3.3 satisfies
\[
(\omega + \varepsilon \omega_X + dd^c u)^n \geq (\omega + dd^c u)^n \geq e^{u_0 + \ln c} \mu ,
\]
since \( u \leq 0 \). The domination principle now yields \( u + \ln c \leq \varphi_\varepsilon \), for all \( \varepsilon > 0 \).

It follows again from the domination principle that \( \varphi_\varepsilon \) decreases to some function \( \varphi \in \text{PSH}(X, \omega) \) which is bounded below by the previous estimate, and bounded above by \( \varphi_1 \), hence it is uniformly bounded.

The uniqueness is a simple consequence of Corollary 1.14.
We now prove (2). It follows from [KN15, KN19, Theorem 0.1] that there exists $c_j > 0$ and $\varphi_j \in \text{PSH}(X, \omega) \cap L^\infty(X)$ such that $\sup_X \varphi_j = -1$ and

$$ (\omega + dd^c \varphi_j)^n = c_j \mu, $$

where $\omega_j := \omega + 2^{-j} \omega_X$. By Lemma 3.3 we have a lower bound $c_j \geq a$, while the domination principle ensures that $j \mapsto c_j$ is decreasing, so $c_j \to c(\omega, f) \in [a, c_1]$.

Fix $\gamma > 0$ so small that $g_j := e^{-\gamma \varphi_j} f$ is uniformly bounded in $L^q(X)$ for some $q > 1$. This is possible thanks to the Skoda integrability theorem [Sk072] and Hölder inequality. By the previous step the solutions $u_j \in \text{PSH}(X, \omega)$ to

$$ (\omega + dd^c u_j)^n = e^{\gamma u_j} g_j dV_X $$

are uniformly bounded. Since $u_j$ satisfies

$$ (\omega + 2^{-j} \omega_X + dd^c u_j)^n \geq e^{\gamma (u_j - \gamma^{-1} \ln c_1)} c_1 g_j dV_X, $$

while $\varphi_j$ satisfies

$$ (\omega + 2^{-j} \omega_X + dd^c \varphi_j)^n \leq e^{\gamma \varphi_j} c_1 g_j dV_X, $$

the domination principle ensures that $\varphi_j \geq u_j - \gamma^{-1} \ln c_1 \geq -C$.

Extracting a subsequence we can assume that $\varphi_j$ converges in $L^1$ and a.e. to $\varphi \in \text{PSH}(X, \omega) \cap L^\infty(X)$. Set

$$ \Phi^+_j := \left( \sup_{\ell \geq j} \varphi_\ell \right) \quad \text{and} \quad \Phi^-_j := P_\omega \left( \inf_{\ell \geq j} \varphi_\ell \right). $$

The function $\Phi^+_j$ (resp. $\Phi^-_j$) is $\omega_j$-psh (resp. $\omega$-psh) and uniformly bounded on $X$. It follows from Lemma 1.9 that

$$ (\omega_j + dd^c \Phi^+_j)^n \geq c_j^- f dV_X \quad \text{with} \quad c_j^- := \inf_{\ell \geq j} c_\ell. $$

Since $\Phi^+_j$ decreases to $\varphi$, while $c_j^-$ increases to $c$, it follows from the continuity of the complex Monge-Ampère along monotonic sequences (see [BT76, BT82]) that

$$ (\omega + dd^c \varphi)^n \geq c f dV_X \quad \text{on} \quad X. $$

Writing $(\omega + dd^c \varphi)^n = e^{\varphi_j - \varphi} c_j f dV_X$, it follows from Lemma 1.9 that

$$ (\omega + dd^c \Phi^-_j)^n \leq e^{\Phi^-_{j, \inf_{\ell \geq j} \varphi_\ell} c_j^+} f dV_X \quad \text{with} \quad c_j^+ := \sup_{\ell \geq j} c_\ell. $$

The functions $\Phi^-_j$ increase to some $\varphi' \in \text{PSH}(X, \omega) \cap L^\infty(X)$ which satisfies

$$ (\omega + dd^c \varphi')^n \leq e^{\varphi' - \varphi} c f dV_X \quad \text{on} \quad X. $$

The domination principle then yields $\varphi' \geq \varphi$ but we also have by construction that $\varphi' \leq \varphi$, hence $\varphi = \varphi'$ is a solution. \hfill \Box

3.2. Uniqueness of solutions. We now study the dependence of the solution with respect to the density, dealing as well with the case when the RHS depends on the solution in a convex way.

**Theorem 3.5.** Assume $\omega$ is either big or such that $v_-(\omega) > 0$. Fix $f_1, f_2 \in L^p(dV_X)$ with $p > 1$ and $A^{-1} \leq \left( \int_X f_1^p dV_X \right)^n \leq \left( \int_X f_2^p dV_X \right)^n \leq A$, for some constant $A > 1$. Assume $\varphi_1, \varphi_2 \in \text{PSH}(X, \omega) \cap L^\infty(X)$ satisfy

$$ (\omega + dd^c \varphi_i)^n = e^{\lambda \varphi_i} f_i dV_X. $$
(1) If $\lambda > 0$, then $||\varphi_1 - \varphi_2||_{\infty} \leq T||f_1 - f_2||_p^{1/\lambda}$, where $T$ is a constant which depends on $n, p$ and upper bounds for $A, \lambda^{-1}, \lambda$.

(2) If $\lambda = 0$ and $f_1 \geq c_0 > 0$, then the same estimate holds with $T$ depending on $n, p$ and upper bounds for $A, c_0^{-1}$.

In particular, there is at most one bounded $\omega$-psh solution $\varphi$ to the equation $(\omega + dd^c \varphi)^n = e^{\lambda \varphi} f \, dV_X$ in case $\lambda > 0$, or when $\lambda = 0$ and $f \geq c_0 > 0$.

When $\omega$ is hermitian such a stability estimate has been provided by Kołodziej-Nguyen for $\lambda = 0$ [KN19]. We adapt some arguments of Lu-Plung-To [LPT20], [GLZ18], who treated the case $\lambda > 0$, and obtained refined estimates in the case $\lambda = 0$.

**Proof.** (1) We first consider the simpler case when $\lambda > 0$. Rescaling we can assume without loss of generality that $\lambda = 1$. We first observe that by Theorem 3.4 $\varphi_1, \varphi_2$ are uniformly bounded. Since $\varphi_1, \varphi_2$ play the same role, it suffices to show that $\varphi_1 - \varphi_2 \leq T||f_1 - f_2||_p^{1/\lambda}$. Using Theorem 3.4 we solve

$$(\omega + dd^c u)^n = c \left( \frac{|f_1 - f_2|}{||f_1 - f_2||_p^{1/\lambda}} + 1 \right) dV_X,$$

with $\sup_X u = 0$ and $|u|, c, c^{-1}$ uniformly bounded. We can assume that

$$\varepsilon := e^{(\sup_X u - \log c)/n} ||f_1 - f_2||_p^{1/\lambda}$$

is small enough. We next consider

$$\phi := (1 - \varepsilon)\varphi_1 + \varepsilon u - C\varepsilon,$$

and proceed exactly the same as in [LPT20, Theorem 1.1] to prove that

$$(\omega + dd^c \phi)^n \geq e^{C\varepsilon} f_2 dV_X.$$

The domination principle thus gives $\phi \leq \varphi_2$, yielding the desired estimate.

(2) We now take care of the case $\lambda = 0$. As usual we use $C$ to denote various uniform constants. By Theorem 3.4 we have that $\sup_X (||\varphi_1| + |\varphi_2|) \leq C$.

**Step 1.** We first assume that $0 < f_1, f_2$ are smooth, and

$$2^{-\varepsilon} f_1 \leq f_2 \leq 2^\varepsilon f_1,$$

for some small constant $\varepsilon > 0$. Our goal is to show that $|\varphi_1 - \varphi_2| \leq C\varepsilon$.

Let $m = \inf_X (\varphi_1 - \varphi_2)$ and $M = \sup_X (\varphi_1 - \varphi_2)$. To simplify the notation we set $\mu = \int f_1 dV_X$. Recall that by Lemma 3.3 there exists a constant $c = c(\omega, p) > 0$ such that if $0 \leq f \in L^p(X, dV_X)$ with $||f||_p \leq 1$ then we can find $u \in \text{PSH}(X, \omega)$ such that $(\omega + dd^c u)^n \geq 4cf dV_X$ with $-1 \leq u \leq 0$. We define

$$t_0 := \sup \{ t \in [m, M] : \mu(\varphi_1 < \varphi_2 + t) \leq e^t \},$$

$$T_0 := \inf \{ t \in [m, M] : \mu(\varphi_1 > \varphi_2 + t) \leq e^t \}.$$

**Step 1.1.** We claim that $t_0 \leq m + C\varepsilon$ and $T_0 \geq M - C\varepsilon$.

Assume $t \leq t_0$ and $\mu(u < v + t) \leq e^t$. Set

$$\hat{f} := \frac{1}{2c} 1_{\{\varphi_1 < \varphi_2 + t\}} f_1 + \frac{1}{2||f_1||_p} 1_{\{\varphi_1 \geq \varphi_2 + t\}} f_1.$$

It follows from Lemma 3.3 that $c(\omega, \hat{f}) \geq 4c$, since

$$\int_X \hat{f}^p dV_X \leq \frac{1}{(2c)^p} \int_{\{\varphi_1 < \varphi_2 + t\}} f_1^p dV + 2^{-p} \leq 1.$$
Using Theorem 3.4 we find $\phi \in \text{PSH}(X, \omega) \cap L^\infty(X)$ such that $\sup_X (\phi - \varphi_2) = 0$ and

$$(\omega + dd^c \phi)^n = c(\omega, \hat{f}) \hat{f} dV_X.$$ 

Since $\hat{f} \geq C^{-1} f_1$, it follows that $4c \leq c(\omega, \hat{f}) \leq C$, hence $\phi$ is uniformly bounded.

We next consider $\psi := (1 - \varepsilon)\varphi_2 + \varepsilon \phi + t$. Then $\{\varphi_1 < \psi\} \subset \{\varphi_1 < \varphi_2 + t\}$ and on the latter set we have

$$\omega^n_\psi \geq ((1 - \varepsilon)2^{-\varepsilon/n} + 2^{1/n}\varepsilon)^n f_1 dV \geq (1 + \gamma)f_1 dV = (1 + \gamma)\omega^n_{\varphi_1},$$

for some $\gamma > 0$. The domination principle ensures that $\varphi_1 \geq \psi \geq \varphi_2 + t - C\varepsilon$, hence $t \leq \varphi_1 - \varphi_2 + C\varepsilon$. Since this inequality is true for all $x \in X$, we can take $x$ such that $\varphi_1(x) - \varphi_2(x) = \inf_X (\varphi_1 - \varphi_2)$ and obtain

(3.2) \hspace{1cm} t_0 \leq m + C\varepsilon.

Using a similar argument we obtain

(3.3) \hspace{1cm} T_0 \geq M - C\varepsilon.

**Step 1.2.** Fixing $t_0 < a < b < T_0$, we claim that

(3.4) \hspace{1cm} I_{a,b} := \int_{E_{a,b}} d(\varphi_1 - \varphi_2) \wedge dd^c(\varphi_1 - \varphi_2) \wedge \omega^{n-1}_{\varphi_1} \leq C(b - a)(\varepsilon + b - t_0),

and

(3.5) \hspace{1cm} I_{a,b} \int_{E_{a,b}} \omega_{\varphi_1} \wedge \omega^{n-1}_X \geq C^{-1} \left( \int_{E_{a,b}} |\partial(\varphi_1 - \varphi_2)| dV_X \right)^2,

where $E_{a,b} := \{\varphi_2 + a < \varphi_1 < \varphi_2 + b\}$.

To prove (3.4) we set $\varphi := \max(\varphi_1, \varphi_2 + a)$, $\psi := \max(\varphi_1, \varphi_2 + b)$, and consider

$$J := \int_X (\varphi - \psi)(\omega^n_\varphi - \omega^n_\psi) = \int_X (\varphi - \psi) dd^c(\psi - \varphi) \wedge T,$$

where $T = \sum_{j=0}^{n-1} \omega^n_\varphi \wedge \omega^{n-1-j}_\psi \geq \omega^{n-1}_\varphi$. Integrating by parts we obtain

$$J = \int_X d(\varphi - \psi) \wedge dd^c(\varphi - \psi) \wedge T - \frac{1}{2} \int_X (\varphi - \psi)^2 dd^c T$$

$$\geq \int_{E_{a,b}} d(\varphi_1 - \varphi_2) \wedge dd^c(\varphi_1 - \varphi_2) \wedge \omega^{n-1}_{\varphi_1} - C(b - a)^2.$$

We have used here $0 \leq \psi - \varphi \leq b - a$, $\omega^{n-1}_{\varphi_1} = \omega^{n-1}_\varphi$ on $\{\varphi_1 > \varphi_2 + a\}$, and

$$dd^c T \geq -C(\omega^2 \wedge (\omega_{\varphi_1} + \omega_{\varphi_2})^{n-2} + \omega^3 \wedge (\omega_{\varphi_1} + \omega_{\varphi_2})^{n-3}),$$

the integral over $X$ of the latter form being uniformly bounded. We have also used that $\partial \psi = 0$ a.e. on $\{\varphi_1 \geq \varphi_2 + b\} \cup \{\varphi_1 \leq \varphi_2 + a\}$.

On the other hand, $(\varphi - \psi)(\omega^n_\varphi - \omega^n_\psi) = 0$ on $\{\varphi_1 < \varphi_2 + a\} \cup \{\varphi_1 \geq \varphi_2 + b\}$, and

$$\left| \int_{E_{a,b}} (\varphi - \psi)(\omega^n_\varphi - \omega^n_\psi) \right| \leq (b - a) \int_X |f - g| dV \leq C(b - a)\varepsilon.$$
Now
\[\left| \int_{\{\varphi_1 = \varphi_2 + a\}} (\varphi - \psi)(\omega^n_\varphi - \omega^n_\psi) \right| = (b-a) \left| \int_{\{\varphi_1 = \varphi_2 + a\}} (\omega^n_\varphi - \omega^n_{\varphi_2}) \right| \]
\[= (b-a) \left| \int_{\{\varphi_1 \leq \varphi_2 + a\}} (\omega^n_\varphi - \omega^n_{\varphi_2}) \right| \]
\[\leq C(b-a)\varepsilon + (b-a) \left| \int_{\{\varphi_1 \leq \varphi_2 + a\}} (\omega^n_\varphi - \omega^n_{\varphi_1}) \right| \]
\[= C(b-a)\varepsilon + (b-a) \int_{X} (\omega^n_\varphi - \omega^n_{\varphi_1}) \]
\[\leq C(b-a)\varepsilon + (b-a)C \sup X (\varphi - \varphi_1) \]
\[\leq C(b-a)(\varepsilon + a + t_0). \]

These inequalities are justified as follows:

- in the second line we used the identity \(\omega^n_\varphi = \omega^n_{\varphi_2}\) on \(\{\varphi_1 < \varphi_2 + a\}\).
- in the third line we used \(\left| \int_{\{\varphi_1 \leq \varphi_2 + a\}} (\omega^n_\varphi - \omega^n_{\varphi_2}) \right| \leq \int_{X} |f - g|dV_X \leq C\varepsilon.\)
- in the fourth line we used the identity \(\omega^n_\varphi = \omega^n_{\varphi_1}\) on \(\{\varphi_1 > \varphi_2 + a\}\).
- the fifth line consists in the following integration by part,

\[\left| \int_{X} (\omega^n_\varphi - \omega^n_{\varphi_1}) \right| = \left| \int_{X} (\varphi - \varphi_1) \frac{d^n}{dz}\sum_{j=0}^{n-1} \omega^n_\varphi \wedge \omega^{n-1-j}_{\varphi_1} \right| \leq C \sup X (\varphi - \varphi_1). \]

Altogether this yields (3.4), as
\[I_{a,b} \leq C(b-a)(\varepsilon + a + t_0 + b - a) \leq C_1(b-a)(b-t_0 + \varepsilon). \]

We next prove (3.5). The following pointwise inequality is a reformulation of [Pop16, Lemma 3.1]: if \(\alpha \geq 0\) is a \((1,1)\)-form and \(\omega_1, \omega_2\) are Hermitian forms then
\[\frac{\alpha \wedge \omega_1^{n-1} \wedge \omega_2^{n-1}}{\omega_1^n \wedge \omega_2^n} \geq \frac{1}{n} \frac{\omega^n_1 \alpha \wedge \omega^{n-1}_2}{\omega^n_2 \wedge \omega^n_2}. \]

Applying this for \(\alpha = d(\varphi_1 - \varphi_2) \wedge d^c(\varphi_1 - \varphi_2),\ \omega_1 = \omega_{\varphi_1}, \ \omega_2 = \omega_X\) we obtain
\[d(\varphi_1 - \varphi_2) \wedge d^c(\varphi_1 - \varphi_2) \wedge \omega_{\varphi_1}^{n-1} \wedge \omega_X^{n-1} \geq \frac{1}{n} \frac{\omega^n_{\varphi_1} |\partial(\varphi_1 - \varphi_2)|^2}{\omega^n_X} \geq \frac{c_0}{nC} |\partial(\varphi_1 - \varphi_2)|^2. \]

Applying Cauchy-Schwarz inequality we obtain (3.5). Let us stress that this is the only place where the extra assumption \(f_1 \geq c_0\) is used.

**Step 1.3.** We finally set \(t_1 = t_0 + \varepsilon, \ t_k := t_0 + 2^{k-1}(t_1 - t_0), \) for \(k \geq 2,\) so that \(t_{k+1} - t_k = 2^{k-1}\varepsilon.\) Fix \(k\) such that \(t_0 \leq t_k < t_{k+1} \leq T_0.\) Then
\[\mu(\varphi_1 < \varphi_2 + t_k) \geq c^p\text{ and } \mu(\varphi_1 > \varphi_2 + t_{k+1}) \geq c^p, \]
hence Lemma 3.6 below (with \(h = \varphi_1 - \varphi_2 - t_k\) and \(\delta = t_{k+1} - t_k\)) yields
\[\int_{\{\varphi_2 + t_k < \varphi_1 \leq \varphi_2 + t_{k+1}\}} |\partial(\varphi_1 - \varphi_2)|dV_X \geq C^{-1}(t_{k+1} - t_k) = C^{-1}2^{k-1}\varepsilon. \]
Combining this with (3.4) and (3.5) for \( a = t_k \) and \( b = t_{k+1} \) we obtain
\[
C 2^{2k} \varepsilon^2 \int_{E_{a,b}} \omega_{\varphi_1} \wedge \omega_X^{n-1} \geq I_{a,b} \int_{E_{a,b}} \omega_{\varphi_1} \wedge \omega_X^{n-1} \geq C^{-1} 2^{2k} \varepsilon^2,
\]
hence
\[
\int_{\{ \varphi_2 + t_k < \varphi_1 \leq \varphi_2 + t_{k+1} \}} \omega_{\varphi_1} \wedge \omega_X^{n-1} \geq C^{-1}.
\]
We take \( N \) such that \( t_N < T_0 \leq t_{N+1} \). Summing up for \( k \) from 1 to \( N \) we obtain
\[
\int_{\{ \varphi_2 + t_k < \varphi_1 \leq \varphi_2 + t_N \}} \omega_{\varphi_1} \wedge \omega_X^{n-1} \geq NC^{-1}.
\]
Since the integral \( \int_X \omega_{\varphi_1} \wedge \omega_X^{n-1} \) is uniformly bounded, this yields a uniform upper bound for \( N \). Using (3.2) and (3.3) we thus obtain the desired bound
\[
M - m = M - T_0 + T_0 - t_{N+1} + \sum_{k=1}^{N+1} (t_k - t_{k-1}) + t_0 - m \leq C \varepsilon.
\]

**Step 2.** To remove the assumption (3.1) on \( f_1, f_2 \) from Step 1 one can proceed as in [LPT20] to which we refer the reader (this requires the case \( \lambda = 1 \) which has been treated independently above).

We finally remove the smoothness assumption. Let \( f_{1,j}, f_{2,j} \) be sequences of smooth positive densities converging to \( f_1, f_2 \) respectively in \( L^p(X, dV_X) \). Let also \( \varphi_{1,j}, \varphi_{2,j} \) be smooth functions decreasing to \( \varphi_1, \varphi_2 \) respectively. Using [Cher87] we solve, for \( i = 1, 2 \),
\[
(\omega_j + dd^c u_{i,j})^n = e^{u_{i,j} - \varphi_{i,j}} f_{i,j} dV_X,
\]
where \( \omega_j = \omega + 2^{-j} \omega_X \). Theorem 3.4 ensures that the functions \( u_{i,j} \) are uniformly bounded. Note that the constants \( c(p, \omega_j) \) from Lemma 3.3 satisfy \( c(p, \omega_j) \geq c(p, \omega) > 0 \). By Step 1 we have
\[
|u_{1,j} - u_{2,j}| \leq T \|g_{1,j} - g_{2,j}\|_p^{1/n},
\]
where \( g_{i,j} = e^{u_{i,j} - \varphi_{i,j}} f_{i,j} \). As \( j \to +\infty \), arguing as in the proof of Theorem 3.4 we can show that \( u_{i,j} \) converge to \( u_i \) solving
\[
(\omega + dd^c u_i)^n = e^{u_i - \varphi_i} f_i dV_X.
\]
The domination principle ensures that \( u_i = \varphi_i \), hence \( g_{1,j} - g_{2,j} \rightarrow f_1 - f_2 \) finishing the proof. \( \square \)

We have used the following elementary result (see [KN19, Lemma 2.6]):

**Lemma 3.6.** Let \( h \) be a real-valued function in \( W^{1,1}(X) \) such that
\[
Vol(h \leq 0) \geq \gamma \quad \text{and} \quad Vol(h \geq \delta) \geq \gamma
\]
where \( \delta > 0 \) and \( \gamma > 0 \) are constants. Then
\[
\int_{\{0 < h < \delta\}} |\partial h| dV_X \geq C^{-1} \gamma^{\frac{2n-1}{2n}} \delta.
\]
Here \( W^{1,1}(X) \) denotes the set of \( L^1 \) functions whose gradient belongs to \( L^1 \).
3.3. Locally bounded solutions. We assume in this section that \( \nu_-(\omega) > 0 \) and \( \omega \) is big. We fix \( \rho \) an \( \omega \)-psh function such that

- \( \rho \) has analytic singularities and \( \sup_X \rho = 0 \);
- \( \omega + dd^c \rho \geq \delta \omega_X \) is a hermitian current;

Given \( \psi \) a quasi-plurisubharmonic function on \( X \) and \( c > 0 \), we set

\[
E_c(\psi) := \{ x \in X : \nu(\psi, x) \geq c \},
\]

where \( \nu(\psi, x) \) denotes the Lelong number of \( \psi \) at \( x \). A celebrated theorem of Siu ensures that for any \( c > 0 \), the set \( E_c(\psi) \) is a closed analytic subset of \( X \).

**Theorem 3.7.** Assume \( \omega \) is a big form and \( \nu_-(\omega) > 0 \). Fix \( p > 1 \) and \( q = \frac{p}{p-1} \).

Let \( \mu = f dV_X \) be a probability measure, where \( f = e^{-\psi} \) with \( 0 \leq g \leq L^p(dV_X) \), and \( \psi \) is quasi-plurisubharmonic. Assume \( f \) belongs to a fixed Orlicz class \( L^w \).

Then there exist a unique constant \( c > 0 \) and a function \( \varphi \in \text{PSH}(X, \omega) \) such that

- \( \varphi \) is locally bounded and continuous in \( \Omega := X \setminus \{ \rho = -\infty \} \cup E_{q-1}(\psi) \);
- \( (\omega + dd^c \varphi)^n = cf dV_X \) in \( \Omega \);
- for any \( \alpha > 0 \), \( \alpha(A^{-1} \psi + \rho) - \beta \leq \varphi \leq 0 \);

where \( A > 0 \) is a constant such that \( dd^c \psi \geq -A \omega_p \) and \( \beta > 0 \) depends on \( p, w \), upper bounds for \( \alpha^{-1}, ||g||_{L^p}, ||f||_{L^w}, \nu_-(\omega)^{-1} \).

**Proof.** Reduction to analytic singularities. We let \( q \) denote the conjugate exponent of \( p \), set \( r = \frac{2p}{p+q} \), and note that \( 1 < r < p \). If the Lelong numbers of \( \psi \) are all less than \( \frac{1}{q} \), it follows from Hölder inequality that \( f \in L^r(dV_X) \), since

\[
\int_X g^r dV_X = \int_X g^{r} e^{-r \psi} dV_X \leq \left( \int_X g^p dV_X \right)^{\frac{r}{p}} \cdot \left( \int_X e^{-pr \psi} dV_X \right)^{\frac{p-r}{p}},
\]

where the last integral is finite by Skoda’s integrability theorem [GZ, Theorem 8.11] if \( \frac{p-r}{p} \nu(\psi, x) < 2 \) for all \( x \in X \), which is equivalent to \( \nu(\psi, x) < \frac{1}{q} \).

It is thus natural to expect that the solution \( \varphi \) will be locally bounded in the complement of the closed analytic set \( E_{q-1}(\psi) \). It follows from Demailly’s equisingular approximation technique (see [Dem15]) that there exists a sequence \((\psi_m)\) of quasi-psh functions on \( X \) such that

- \( \psi_m \geq \psi \) and \( \psi_m \rightarrow \psi \) (pointwise and in \( L^1 \));
- \( \psi_m \) has analytic singularities, it is smooth in \( X \setminus E_{m-1}(\psi) \);
- \( dd^c \psi_m \geq -K \omega_X \), for some uniform constant \( K > 0 \);
- \( \int_X e^{2m(\psi_m - \psi)} dV_X < +\infty \) for all \( m \).

We choose \( m = [q] \), set \( g_m := e^{\psi_m - \psi} \), and observe that

\[
\int_X g_m^r \leq \left( \int_X e^{2m(\psi_m - \psi)} dV_X \right)^{\frac{2m}{2m-r}} \cdot \left( \int_X g_m^{2m-r} dV_X \right)^{\frac{2m-r}{2m}} < +\infty
\]

if we choose \( r^{-1} = p^{-1} + (2m)^{-1} < 1 \) so that \( \frac{2m}{2m-r} = p \). By replacing \( \psi \) by \( \psi_{[q]} \geq \psi \) and \( g \) by \( g_m \in L^r \) in the sequel, we can thus assume that \( \psi \) has analytic singularities and is smooth in \( X \setminus E_{q-1}(\psi) \).

**Bounded approximation.** The densities \( f_j := e^{-\text{max}(\psi, -j)} \) belong to \( L^p \) and increase towards \( f \). By Theorem 3.4 there exists \( \varphi_j \in \text{PSH}(X, \omega) \cap L^\infty(X) \) such
that
\[(\omega + dd^c \varphi_j)^n = c_j f_j dV_X,\]
where \(c_j > 0\) and \(\sup_X \varphi_j = 0\).

Observe that \(dV_X \geq \delta_1 \omega^n_X\) for some \(\delta_1 > 0\). It follows therefore from the arithmetico-geometric means inequality that
\[c_j^{1/n} f_j^{1/n} \delta_1^{1/n} \omega^{n/n}_X \leq (\omega + dd^c \varphi_j) \vee \omega^{n-1}_X.\]

As \(j\) increases to \(+\infty\), we have \(\int_X f_j^{1/n} \omega^n_X \to \int_X f^{1/n} \omega_X > 0\), while
\[\int_X (\omega + dd^c \varphi_j) \vee \omega^{n-1}_X \leq \int_X \omega \vee \omega^{n-1}_X + B \int_X (\varphi_j) \omega^n_X,\]
using that \(dd^c \omega^{n-1}_X \leq B \omega^n_X\) and \(\varphi_j \leq 0\). Thus the positive constants \(c_j\) are uniformly bounded from above (the functions \(\varphi_j\) are \(\omega\)-psh and normalized, hence they are relatively compact in \(L^1(X)\)).

They are also bounded away from zero, since
\[c_j \int_X f_j dV_X = \int_X (\omega + dd^c \varphi_j)^n \geq v_-(\omega),\]
and \(\int_X f_j dV_X \to \int_X f dV_X = 1\).

\(L^\infty\)-estimates. Fix \(A > 0\) large enough so that \(\omega + dd^c \psi \geq -A \omega\). Since \(c \leq c_j \leq C\), \(f_j \leq f\) where \(f\) belongs to a fixed Orlicz class \(L^w\), \(f_j \leq ge^{-A \tilde{\psi}}\) where \(\tilde{\psi} = \frac{\psi}{A} + \rho \in \text{PSH}(X,\omega)\), it follows from Theorem 2.4 that
\[\alpha(A^{-1} \psi + \rho) - \beta \leq \varphi_j \leq 0.\]

Thus the functions \(\varphi_j\) are locally uniformly bounded in \(\Omega := X \setminus \{\rho + \psi = -\infty\}\).

We can thus extract a subsequence such that \(c_j \to c > 0\) and

- \(\varphi_j \to \varphi\) a.e. and in \(L^1\), with \(\varphi \in \text{PSH}(X,\omega)\);
- \(\sup_X \varphi = 0\), as follows from Hartogs lemma (see [GZ, Theorem 1.46]);
- \(\alpha(A^{-1} \psi + \rho) - \beta \leq \varphi \leq 0\) in \(\Omega\),

hence the Monge-Ampère measure \((\omega + dd^c \varphi)^n\) is well-defined in \(\Omega\).

Convergence of the approximants. Set
\[\Phi_j^+ := \left(\sup_{\ell \geq j} \varphi_\ell\right)^* \quad \text{and} \quad \Phi_j^- := P_\omega \left(\inf_{\ell \geq j} \varphi_\ell\right).\]

The function \(\Phi_j^+\) (resp. \(\Phi_j^-\)) is \(\omega_j\)-psh (resp. \(\omega\)-psh) and locally bounded in \(\Omega\). Note that \(\Phi_j^+\) decreases to \(\varphi\), while \(\Phi_j^-\) increases to some \(\omega\)-psh function \(\Phi' \leq \varphi\).

We are going to show that \(\Phi' = \varphi\).

We also set \(f_j^+ := \sup_{\ell \geq j} f_\ell\,\), \(f_j^- := \inf_{\ell \geq j} f_\ell\,\), \(c_j^+ := \sup_{\ell \geq j} c_\ell\,\), \(c_j^- := \inf_{\ell \geq j} c_\ell\,\). It follows from Lemma 1.9 that
\[(\omega + dd^c \Phi_j^+)^n \geq c_j^- f_j^- dV_X.\]

Since \(\Phi_j^+\) decreases to \(\varphi\), while \(c_j^- f_j^-\) increases to \(cf\), it follows from the continuity of the complex Monge-Ampère along monotonic sequences that
\[(\omega + dd^c \varphi)^n \geq cf dV_X \quad \text{in} \quad \Omega.\]

It follows from Lemma 1.9 that
\[(\omega + dd^c \Phi_j^+)^n \geq c_j^- f_j^- dV_X.\]
Since \( \Phi_j^+ \) decreases to \( \varphi \), while \( c_j^{-} f_j^{-} \) increases to \( cf \), it follows from the continuity of the complex Monge-Ampère along monotonic sequences that
\[
(\omega + dd^c \varphi)^n \geq cf dV_X \quad \text{in} \quad \Omega.
\]
It also follows from Lemma 1.9 that
\[
(\omega + dd^c \Phi_j^-)^n \leq e^{\Phi_j^- - \inf_{\Omega} (\varphi + c_j^+ f_j^+)} dV_X,
\]
so \( \Phi_j^- \) increase to some \( \Phi' \in \text{PSH}(X, \omega) \) such that
\[
(\omega + dd^c \Phi')^n \leq e^{\Phi' - \varphi} cf dV_X \leq e^{\Phi' - \varphi} (\omega + dd^c \varphi)^n
\]
in \( \Omega \). It thus follows from Proposition 1.12, that \( \Phi' \geq \varphi \), hence \( \Phi' \leq \varphi \). Altogether this shows that \( \varphi \) solves the desired equation.

The uniqueness of the constant \( c > 0 \) follows from Corollary 1.14.

Continuity of the solutions. We finally prove that any solution \( \phi \in \text{PSH}(X, \omega) \) satisfying \( \text{inf}_X (\phi - \varepsilon \rho) > -\infty \), for all \( \varepsilon > 0 \), is continuous in \( \Omega \). We can assume without loss of generality that \( \phi \leq -1 \). Note that \( \phi \) has zero Lelong number everywhere on \( X \). Using Demailly’s regularization we can thus find \( \phi_j \) smooth functions such that \( \omega + dd^c \phi_j \to -2^{-j} \omega_X \), and \( \phi_j \searrow \phi \). Fix \( \varepsilon > 0 \) and solve
\[
(\omega + dd^c u_{j, \varepsilon})^n = e^{u_{j, \varepsilon}} (g_{j, \varepsilon} e^{-2\varepsilon^{-1} \phi} + h) dV_X,
\]
where \( h = \omega^n / dV_X \), \( g_{j, \varepsilon} = \varepsilon^{-n} \mathbb{1}_{\{\phi < \phi_j + \varepsilon\}} \). Observe that \( g_{j, \varepsilon} e^{-\varepsilon^{-1} \phi} \to 0 \) in \( L^p(X) \) as \( j \to +\infty \). It follows from Theorem 3.5 that the functions \( u_{j, \varepsilon} \) uniformly converge to \( 0 \) as \( j \to +\infty \). Consider
\[
v_{j, \varepsilon} := (1 - \varepsilon) \phi_j + \frac{\varepsilon}{2}(\psi + A \rho + u_{j, \varepsilon}) - C \varepsilon,
\]
where \( A > 0 \) is a constant ensuring that \( \omega + dd^c (\psi + A \rho) \geq 0 \), and \( C > 0 \) is a constant chosen below. A direct computation shows that
\[
(\omega + dd^c v_{j, \varepsilon})^n \geq 2^{-n} e^{u_{j, \varepsilon}} \mathbb{1}_{\{\phi < \phi_j + \varepsilon\}} g e^{-\varepsilon^{-1} \phi} dV_X.
\]
Recall that, by Theorem 2.4, \( \phi_j \geq \phi \geq \frac{1}{2} (\psi + A \rho) - C_1 \), for some positive constant \( C_1 \). On the set \( \{ \phi < v_{j, \varepsilon} \} \) we have
\[
\phi < \phi_j - \varepsilon \left( \phi_j - \frac{1}{2} (\psi + A \rho) \right) + \frac{\varepsilon}{2} \sup_X |u_{j, \varepsilon}| - C \varepsilon < \phi_j - \varepsilon,
\]
if we choose \( C > 1 + \sup_X |u_{j, \varepsilon}| + C_1 \). On this set we also have
\[
2^{-n} e^{u_{j, \varepsilon}} g e^{-\phi} \geq 2^{-n} e^{-\sup_X |u_{j, \varepsilon}|} e^{C - \psi} \geq 2 ge^{-\psi},
\]
if we choose \( C > 2^n + \sup_X |u_{j, \varepsilon}| + \ln(2) \). Thus
\[
(\omega + dd^c \phi)^n \leq 2^{-1} (\omega + dd^c v_{j, \varepsilon})^n = 2^{-1} (\omega + dd^c \max(\phi, v_{j, \varepsilon}))^n
\]
on the set \( \{ \phi < v_{j, \varepsilon} \} \). Thus \( \phi \geq \max(\phi, v_{j, \varepsilon}) \geq v_{j, \varepsilon} \) by Proposition 1.12. Fixing a compact set \( K \Subset \Omega \) and letting \( j \to +\infty \) we obtain
\[
\lim_{j \to +\infty} \inf_K (\phi - \phi_j) \geq -C' \varepsilon.
\]
Since \( \phi_j \to \phi \) uniformly on \( K \) as \( \varepsilon \to 0 \), we conclude that \( \phi \in C^0(\Omega) \). \( \square \)

Remark 3.8. This relative \( L^{\infty} \)-bound requires one to assume both that \( \omega \) is big and \( v_-(\omega) > 0 \). It is natural to expect that it suffices to assume that \( \omega \) is big, however such an estimate is an open problem even in the simplest case when \( \omega \) is a hermitian form! On the other hand these two conditions are expected to be equivalent; such is the case e.g. when \( X \) belongs to the Fujiki class (see [GL21b]).
4. Geometric applications

In this section we apply the previous analysis on compact complex varieties $V$ with mild singularities. We refer the reader to [EGZ09, Section 5] for an introduction to complex analysis on singular varieties. Roughly speaking one can consider local embeddings $\mathcal{O}_{\alpha} \rightarrow \mathbb{C}^N$ and consider objects (quasi-psh functions, forms, etc) that are restrictions of global ones.

One can also consider $\pi : X \rightarrow V$ a resolution of singularities and pull-back these objects to $X$ in order to study the corresponding equations in a smooth environment. The draw-back is a loss of positivity along some divisor $E = \pi^{-1}(V_{\text{sing}})$ which lies above the singular locus of $V$. Considering a hermitian form $\omega_V$ on $V$, we are thus lead to work with $\omega = \pi^* \omega_V$, which is semi-positive and big but no longer hermitian. We fix a $\omega$-psh function $\rho$ with analytic singularities along a divisor $E$ such that $\omega + dd^c \rho \geq 0$. Moreover the functions $\rho$ are quasi-psh with $\rho_{\pm}$ and $\omega = \rho_e^+ - \rho_e^-$. Then there exist $c > 0$ such that $\sup_{X} \varphi = 0$ and

- $\varphi$ is smooth in the open set $X \setminus (D \cup E)$;
- $-T \leq \varphi \leq 0$ in $X$ for some uniform $T > 0$;
- $(\omega + dd^c \varphi)^n = c f dV_X$ on $X$.

We also establish higher regularity under less restrictive assumptions on the density $f$, but we then need to assume that $\omega$ is both big and satisfies $v_-(\omega) > 0$, in order to have a good relative $L^\infty$-bound:

Theorem 4.2. Assume $\omega$ is big and $v_-(\omega) > 0$. Assume $f = e^{\psi^+ - \psi^-} \in L^p(dV_X)$, $p > 1$, and $f$ is smooth and positive in $X \setminus D$, with $\psi^\pm$ quasi-plurisubharmonic functions. Then there exist $c > 0$ and $\varphi \in \text{PSH}(X, \omega)$ such that $\sup_{X} \varphi = 0$ and

- $\varphi$ is smooth in the open set $X \setminus (D \cup E)$;
- for any $\alpha > 0$ there is $\beta(\alpha) > 0$ with $\alpha(\delta \psi^- + \rho) - \beta(\alpha) \leq \varphi \leq 0$ in $X$;
- $(\omega + dd^c \varphi)^n = c f dV_X$ in $X \setminus (D \cup E)$.

One can keep track of the dependence of the constants on the data, as was done in Theorem 3.7. This result can be seen as a generalization of the main result of [TW10b] which dealt with the case when $\omega > 0$ is hermitian on $X$ and $f$ is globally smooth and positive.

Proof. We prove Theorem 4.1 and Theorem 4.2 at once.

Smooth approximation. We assume without loss of generality that $\psi^\pm \leq 0$. For a Borel function $g$ we let $g_\varepsilon := \rho_\varepsilon(g)$ denote the Demailly regularization of $g$ (see [Dem94, (3.1)]):

$$
\rho_\varepsilon(g)(x) := \frac{1}{\varepsilon^{2n}} \int_{\zeta \in T_x X} g(\text{exp}_{x}(\zeta)) \chi \left( \frac{|\zeta|^2}{\varepsilon^2} \right) d\lambda(\zeta).
$$

Since $\psi^-$ is quasi-psh, the corresponding regularization $\psi^-_\varepsilon$ satisfies $\psi^- \leq \psi^-_\varepsilon + A \varepsilon^2$, while $\psi^+_\varepsilon \leq 0$. Moreover the functions $\psi^\pm_\varepsilon$ are quasi-psh with $dd^c \psi^\pm_\varepsilon \geq -K^\pm \omega_X$ for uniform constants $K^\pm \geq 0$. In particular

$$
K^- \omega + dd^c (\delta \psi^-_\varepsilon + K^- \rho) \geq -K^- \delta \omega_X + K^- (\omega + dd^c \rho) \geq 0,
$$

where $\delta \omega_X$ is decided by $\omega_X$. Then $K^- \omega + dd^c (\delta \psi^-_\varepsilon + K^- \rho) \geq -K^- \delta \omega_X + K^- (\omega + dd^c \rho) \geq 0$,
so \( \alpha \psi_\varepsilon^- + \rho \in \text{PSH}(X, \omega) \) for all \( 0 < \alpha < \frac{\delta}{K} \). Up to replacing \( \delta \) with \( \frac{\delta}{K} \), we assume in the sequel that \( K = 1 \).

We fix \( 0 < \varepsilon \leq 1 \) and set \( \omega_\varepsilon := \omega + \varepsilon \omega_X \). It follows from [TW10b] that there exist unique constants \( c_\varepsilon > 0 \) and smooth \( \omega_\varepsilon \)-psh functions \( \varphi_\varepsilon \) such that

\[
(\omega + \varepsilon \omega_X + dd^c \varphi_\varepsilon)^n = c_\varepsilon e^{\psi^+_\varepsilon - \psi^-_\varepsilon} dV_X,
\]

with \( \sup_X \varphi_\varepsilon = 0 \). Note that by Jensen inequality we have

\[
e^{\psi^+_\varepsilon - \psi^-_\varepsilon} \leq f_\varepsilon.
\]

Since \( f_\varepsilon \) converges to \( f \) in \( L^1(X, dV_X) \), extracting a subsequence we can assume that \( e^{\psi^+_\varepsilon - \psi^-_\varepsilon} \leq F \), for some \( F \in L^1(dV_X) \) which belongs to an Orlicz class \( L^w \), for some convex increasing weight \( w : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( \frac{\omega(t)}{t} \to +\infty \) as \( t \to +\infty \).

\( C^0 \)-estimates. This is the only place where the proofs of Theorem 4.1 and Theorem 4.2 slightly differ, and also the reason why the assumptions on \( \omega \) look slightly stronger in the second case.

Arguing as in the proof of Theorem 3.7, one shows that the constants \( c_\varepsilon \) are uniformly bounded away from zero and \( +\infty \). If \( f \) belongs to \( L^p(dV_X) \) for some \( p > 1 \), it follows from Theorem 2.2 (if \( v_-(\omega) > 0 \)) and Theorem 3.4 (if \( \omega \) is big) that \(-T \leq \varphi_\varepsilon \leq 0 \), for some uniform constant \( T \) independent of \( \varepsilon > 0 \).

If we rather assume that \( f \in L^1(dV_X) \) with \( \omega \) big and \( v_-(\omega) > 0 \), note that

\[
(\omega + \varepsilon \omega_X + dd^c \varphi_\varepsilon)^n \leq Ce^{-A(\delta \psi^-_\varepsilon + \rho)} dV_X,
\]

with \( A = \delta^{-1} \). It then follows from Theorem 2.4 that for any \( \alpha > 0 \) there exists \( \beta(\alpha) \) such that

\[
(\varphi_\varepsilon) \text{ is locally uniformly bounded in } \Omega := X \setminus (E \cup D).
\]

\( C^2 \)-estimates. In the sequel we fix \( \alpha = 1 \) and we establish uniform bounds on \( \Delta_{\omega_X} \varphi_\varepsilon \) on compact subsets of \( \Omega \). We follow the computations of [TW10a, Proof of Theorem 2.1] and [Tô18] with a twist in order to deal with unbounded functions. We use \( C \) to denote various uniform constants which may be different. Consider

\[
H := \log \text{Tr}_{\omega_X}(\tilde{\omega}) - \gamma(u)
\]

where

\[
\tilde{\omega} = \omega_\varepsilon + dd^c \varphi_\varepsilon, \quad u = (\varphi_\varepsilon - \rho - 2\delta \psi^-_\varepsilon + \beta + 1) > 1,
\]

and \( \gamma : \mathbb{R} \to \mathbb{R} \) is a smooth concave increasing function such that \( \gamma(\infty) = +\infty \). We are going to show that \( H \) is uniformly bounded from above for an appropriate choice of \( \gamma \). Since \( u \) is uniformly bounded on compact subsets of \( \Omega \), this will yield for each \( \varepsilon > 0 \) and \( K \subset \subset \Omega \) a uniform bound

\[
\Delta_{\omega_X}(\varphi_\varepsilon) = \text{Tr}_{\omega_X}(\omega_\varepsilon + dd^c \varphi_\varepsilon) - \text{Tr}_{\omega_X}(\omega_\varepsilon) \leq C_K.
\]

We let \( g \) denote the Riemannian metric associated to \( \omega_X \) and \( \tilde{g} \) the one associated to \( \tilde{\omega} = \omega_\varepsilon + dd^c \varphi_\varepsilon \). To simplify notations we will omit the subscript \( \varepsilon \) in the sequel. Since \( \psi^\pm \) are quasi-psh, up to multiplying \( \omega_X \) with a large constant, we can assume that \( \omega_X + dd^c \psi^\pm \geq 0 \).

Since \( \rho \to -\infty \) on \( E \) the maximum of \( H \) is attained at some point \( x_0 \in X \setminus E \). We use special coordinates at this point, as defined by Guan-Li in [GL10]:

\[
g_{ij} = \delta_{ij}, \quad \frac{\partial g_{ij}}{\partial z_j} = 0 \quad \text{and} \quad \tilde{g}_{ij} \text{ is diagonal}.
\]
To achieve this we use a linear change of coordinates so that \( g_{\tilde{j}\tilde{i}} = \delta_{\tilde{ij}} \) and \( \tilde{g}_{\tilde{j}\tilde{i}} \) is diagonal at \( x_0 \), and we then make a change of coordinates as in \([GL10, \, (2.19)]\).

We first compute

\[
\Delta_\bar{\omega} \text{Tr}_{\omega_X}(\bar{\omega}) = \sum_{i,j,k,l} \bar{g}^{i\tilde{j}} \partial_i \partial_j (g^{k\tilde{l}} \bar{g}_{k\tilde{l}})
\]

\[
= \sum_{i,k} \bar{g}^{i\tilde{j}} \bar{g}_{k\tilde{n}} - 2 \Re \left( \sum_{i,j,k} \bar{g}^{i\tilde{j}} g_{j\tilde{k}i} \bar{g}_{k\tilde{j}} \right) + \sum_{i,j,k} \bar{g}^{i\tilde{j}} g_{j\tilde{k}i} g_{k\tilde{n}j} \bar{g}_{k\tilde{k}} \\
+ \sum_{i,j,k} \bar{g}^{i\tilde{j}} g_{j\tilde{k}i} g_{k\tilde{n}j} - \sum_{i} \bar{g}^{i\tilde{j}} g_{k\tilde{n}i} \bar{g}_{k\tilde{k}}
\]

\[
\geq \sum_{i,k} \bar{g}^{i\tilde{j}} \bar{g}_{k\tilde{n}} - 2 \Re \left( \sum_{i,j,k} \bar{g}^{i\tilde{j}} g_{j\tilde{k}i} \bar{g}_{k\tilde{j}} \right) - C \text{Tr}_{\omega_X}(\bar{\omega}) \text{Tr}_{\bar{\omega}}(\omega_X).
\]

Using this and

\[
\text{Tr}_{\omega_X} \text{Ric}(\bar{\omega}) = \sum_{i,k} \bar{g}^{i\tilde{j}} \left( - \bar{g}_{i\tilde{k}k} + \sum_{j} \bar{g}^{i\tilde{j}} \bar{g}_{j\tilde{k}i} \bar{g}_{j\tilde{k}} \right)
\]

we obtain

\[
(4.2) \quad \Delta_\bar{\omega} \text{Tr}_{\omega_X}(\bar{\omega}) \geq \sum_{i,j,k} \bar{g}^{i\tilde{j}} g_{j\tilde{k}i} \bar{g}_{j\tilde{k}i} - \text{Tr}_{\omega_X} \text{Ric}(\bar{\omega}) - C \text{Tr}_{\omega_X}(\omega_X)
\]

\[
- C \text{Tr}_{\omega_X}(\bar{\omega}) \text{Tr}_{\bar{\omega}}(\omega_X) - 2 \Re \left( \sum_{i,j,k} \bar{g}^{i\tilde{j}} g_{j\tilde{k}i} \bar{g}_{k\tilde{j}} \right),
\]

noting that \( |\bar{g}_{i\tilde{k}k} - \bar{g}_{i\tilde{k}k}| \leq C \). Here \( \text{Ric}(\bar{\omega}) \) is the Chern-Ricci form of \( \bar{\omega} \). From \( \text{Tr}_{\omega_X}(\bar{\omega}) \text{Tr}_{\bar{\omega}}(\omega_X) \geq n, \) (4.2) and

\[
\text{Ric}(\bar{\omega}) = \text{Ric}(\omega_X) - dd^c (\psi^+ - \psi^-) \leq C \omega_X + dd^c \psi^-,
\]

we obtain

\[
(4.3) \quad \Delta_\bar{\omega} \text{Tr}_{\omega_X}(\bar{\omega}) \geq \sum_{i,j,k} \bar{g}^{i\tilde{j}} g_{j\tilde{k}i} \bar{g}_{j\tilde{k}i} - \text{Tr}_{\omega_X}(\omega_X + dd^c \psi^-) - C \text{Tr}_{\bar{\omega}}(\omega_X)
\]

\[
- 2 \Re \left( \sum_{i,j,k} \bar{g}^{i\tilde{j}} g_{j\tilde{k}i} \bar{g}_{k\tilde{j}} \right) - C \text{Tr}_{\omega_X}(\bar{\omega}) \text{Tr}_{\bar{\omega}}(\omega_X).
\]

Our special choice of coordinates at \( x_0 \) ensures that \( g_{\tilde{j}\tilde{i}} = 0 \). Using Cauchy-Schwarz inequality and \( |\bar{g}_{k\tilde{j}i} - \bar{g}_{i\tilde{j}k}| \leq C \), we therefore obtain

\[
2 \Re \left( \sum_{i,j,k} \bar{g}^{i\tilde{j}} g_{j\tilde{k}i} \bar{g}_{k\tilde{j}} \right) \leq 2 \Re \left( \sum_i \sum_{j \neq k} \bar{g}^{i\tilde{j}} g_{j\tilde{k}i} \bar{g}_{j\tilde{j}k} \right) + C \text{Tr}_{\omega_X}(\omega_X)
\]

\[
\leq \sum_i \sum_{j \neq k} \left( \bar{g}^{i\tilde{j}} g_{j\tilde{k}i} \bar{g}_{k\tilde{j}} + \bar{g}^{i\tilde{j}} g_{j\tilde{j}k} \bar{g}_{k\tilde{k}i} \right) + C \text{Tr}_{\omega_X}(\omega_X)
\]

\[
\leq \sum_i \sum_{j \neq k} \bar{g}^{i\tilde{j}} g_{j\tilde{k}i} \bar{g}_{j\tilde{j}k} + C \text{Tr}_{\omega_X}(\bar{\omega}) \text{Tr}_{\bar{\omega}}(\omega_X) + C \text{Tr}_{\omega_X}(\omega_X).
\]

Together with (4.3) this yields

\[
(4.4) \quad \Delta_\bar{\omega} \text{Tr}_{\omega_X}(\bar{\omega}) \geq I - C \text{Tr}_{\omega_X}(\omega_X + dd^c \psi^-) - C \text{Tr}_{\omega_X}(\bar{\omega}) \text{Tr}_{\bar{\omega}}(\omega_X) - C \text{Tr}_{\omega_X}(\omega_X)
\]
with \( I := \sum_{i,j} \bar{g}^i \bar{g}^{ij} \bar{g}^{ij} \bar{g}^{i,j,j} \). We next compute
\[
|\partial \text{Tr}_{\omega X}(\bar{\omega})|^2_{\bar{\omega}} = \sum_{i,j,k} g^i \bar{g}^{ij} \bar{g}^{j} \bar{g}^{k} = \sum_{i,j,k} g^i (T_{ij} + \bar{g}^{ijj}) (T_{ik} + \bar{g}^{ik}) = \sum_{i,j,k} g^i \bar{g}^{ij} \bar{g}^{j} \bar{g}^{k} + \sum_{i,j,k} g^i T_{ij} \bar{g}^{j} \bar{g}^{k} + 2 \Re (\bar{g}^i T_{ijj} \bar{g}^{k}) .
\]
where \( T_{ij} = \bar{g}^{ijj} - \bar{g}^{ijj} \) is the torsion term corresponding to \( \omega + \varepsilon \omega X \) which is under control: \( |T_{ij}| \leq C \). We bound the first term by Cauchy-Schwarz inequality
\[
\sum_{i,j,k} g^i \bar{g}^{ij} \bar{g}^{j} \bar{g}^{k} = \sum_{i} \left| \sum_{j} g^i \bar{g}^{ij} \right|^2 \geq \left( \sum_{i} g^i \bar{g}^{ij} \right) \left( \sum_{j} \bar{g}^{ijj} \right) = I \text{Tr}_{\omega X}(\bar{\omega}) .
\]
We thus get
\[
(4.5) \quad \frac{|\partial \text{Tr}_{\omega X}(\bar{\omega})|^2_{\bar{\omega}}}{\left( \text{Tr}_{\omega X}(\bar{\omega}) \right)^2} \leq \frac{I}{\text{Tr}_{\omega X}(\bar{\omega})} + C \frac{\text{Tr}_{\omega X}(\omega X)}{\left( \text{Tr}_{\omega X}(\bar{\omega}) \right)^2} + \frac{2 \Re (\sum g^i T_{ij} \bar{g}^{j})}{\left( \text{Tr}_{\omega X}(\bar{\omega}) \right)^2} ,
\]
Since \( \partial_t H = 0 \) at the point \( x_0 \), we obtain by differentiating \( H \) once
\[
g_{k\bar{k}} = \text{Tr}_{\omega X}(\bar{\omega}) \gamma'(u) u_T .
\]
Cauchy-Schwarz inequality yields
\[
\left| \frac{2}{\left( \text{Tr}_{\omega X}(\bar{\omega}) \right)^2} \Re \left( \sum g^i T_{ij} \bar{g}^{j} \bar{g}^{k} \right) \right| \leq C \frac{\gamma'(u)^2}{(-\gamma''(u))} \frac{\text{Tr}_{\omega X}(\omega X)}{\left( \text{Tr}_{\omega X}(\bar{\omega}) \right)^2} + (-\gamma''(u)) |\partial u|_{\bar{\omega}}^2 .
\]
Noting that \( |g_{k\bar{k}} - \bar{g}_{k\bar{k}}| \leq C \) we infer
\[
\left| \frac{2}{\left( \text{Tr}_{\omega X}(\bar{\omega}) \right)^2} \Re \left( \sum g^i T_{ij} \bar{g}^{j} \bar{g}^{k} \right) \right| \leq C \left( \frac{\gamma'(u)^2}{(-\gamma''(u))} + 1 \right) \frac{\text{Tr}_{\omega X}(\omega X)}{\left( \text{Tr}_{\omega X}(\bar{\omega}) \right)^2} + (-\gamma''(u)) |\partial u|_{\bar{\omega}}^2 .
\]
Since \( 0 \geq \Delta_{\bar{\omega}} H \) at \( x_0 \), it follows from (4.4), (4.5) that
\[
(4.6) \quad 0 \geq \Delta_{\bar{\omega}} H = \Delta_{\bar{\omega}} \text{Tr}_{\omega X}(\bar{\omega}) - \frac{|\partial \text{Tr}_{\omega X}(\bar{\omega})|^2_{\bar{\omega}}}{\left( \text{Tr}_{\omega X}(\bar{\omega}) \right)^2} - \gamma'(u) \Delta_{\bar{\omega}}(u) - \gamma''(u) |\partial u|_{\bar{\omega}}^2 \\
\geq - C \text{Tr}_{\omega X}(\omega X + dd^c \psi^-) - \gamma'(u) (n - \delta \text{Tr}_{\omega X}(3\omega X + 2dd^c \psi^-)) \\
- C \left( \frac{\gamma'(u)^2}{(-\gamma''(u))} + 1 \right) \frac{\text{Tr}_{\omega X}(\omega X)}{\left( \text{Tr}_{\omega X}(\bar{\omega}) \right)^2} - C \frac{\text{Tr}_{\omega X}(\omega X)}{\text{Tr}_{\omega X}(\bar{\omega})} - C \text{Tr}_{\omega X}(\omega X) .
\]
We now choose the function \( \gamma \) so as to obtain a simplified information. We set
\[
\gamma(u) := \frac{C + 1}{\min(\delta, 1)} u + \ln(u) .
\]
Since \( u \geq 1 \), we observe that
\[
\frac{C + 1}{\min(\delta, 1)} \leq \gamma'(u) \leq 1 + \frac{C + 1}{\min(\delta, 1)} \quad \text{and} \quad \frac{\gamma'(u)^2}{|\gamma''(u)|} + 1 \leq C_1 u^2 .
\]
By incorporating this into (4.6) we obtain
\[ 0 \geq -\frac{C \text{Tr}_{\omega_X}(\omega_X + dd^c \psi^-)}{\text{Tr}_{\omega_X}(\tilde{\omega})} - C_2 + (C + 1)(\text{Tr}_{\tilde{\omega}}(\omega_X) + \text{Tr}_{\tilde{\omega}}(\omega_X + dd^c \psi^-)) \]
\[ - C_2(u^2 + 1)\frac{\text{Tr}_{\omega_X}(\omega_X)(\text{Tr}_{\omega_X}(\tilde{\omega}))^2}{(\text{Tr}_{\omega_X}(\tilde{\omega}))^2} - C \frac{\text{Tr}_{\tilde{\omega}}(\omega_X)}{\text{Tr}_{\omega_X}(\tilde{\omega})} - C \text{Tr}_{\tilde{\omega}}(\omega_X). \]

Using \( \text{Tr}_{\omega_X}(\omega_X + dd^c \psi^-) \leq \text{Tr}_{\tilde{\omega}}(\omega_X + dd^c \psi^-) \text{Tr}_{\omega_X}(\tilde{\omega}) \) we thus arrive at
\[ (4.7) \quad 0 \geq \text{Tr}_{\tilde{\omega}}(\omega_X) - C_2(u^2 + 1)\frac{\text{Tr}_{\tilde{\omega}}(\omega_X)}{(\text{Tr}_{\omega_X}(\tilde{\omega}))^2} - C \frac{\text{Tr}_{\tilde{\omega}}(\omega_X)}{\text{Tr}_{\omega_X}(\tilde{\omega})} - C_2. \]

At the point \( x_0 \) we have the following alternative:

- If \( \text{Tr}_{\omega_X}(\tilde{\omega})^2 \geq 4C_2(u^2 + 1) + (4C)^2 \) then
  \[ C_2(u^2 + 1)\frac{\text{Tr}_{\tilde{\omega}}(\omega_X)}{(\text{Tr}_{\omega_X}(\tilde{\omega}))^2} \leq \frac{\text{Tr}_{\tilde{\omega}}(\omega_X)}{4} \text{ and } C \frac{\text{Tr}_{\tilde{\omega}}(\omega_X)}{\text{Tr}_{\omega_X}(\tilde{\omega})} \leq \frac{\text{Tr}_{\tilde{\omega}}(\omega_X)}{4}, \]
  hence from (4.7) we get \( \text{Tr}_{\tilde{\omega}}(\omega_X) \leq 2C_2. \) Now
  \[ \text{Tr}_{\omega_X}(\tilde{\omega}) \leq n\frac{\tilde{\omega}_n n}{\omega_X}(\text{Tr}_{\tilde{\omega}}(\omega_X))^{n-1} \leq n(2C_2)^{n-1}C e^{\psi^+ - \psi^-} \]
  yields \( \text{Tr}_{\omega_X}(\tilde{\omega}) \leq C e^{\psi^+ - \psi^-}. \) It follows therefore from (4.1) that
  \[ H(x_0) \leq \log(2C_2) - \psi^- - \frac{C + 1}{\min(\delta, 1)}(\varphi - \rho - 2\delta \psi^-) \]
  \[ \leq \log(2C_2) - (C + 1)(\varphi - \rho - \delta \psi^-) \leq C_3. \]

- If \( \text{Tr}_{\omega_X}(\tilde{\omega})^2 \leq 4C_2(u^2 + 1) + (4C)^2 \) then
  \[ H(x_0) \leq \log \sqrt{4C_2(u^2 + 1) + (4C)^2} - \gamma(u) \leq C_4. \]

Thus \( H(x_0) \) is uniformly bounded from above, yielding the desired estimate.

**Higher order estimates.** With uniform bounds on \( \|\Delta_{\omega_X} \varphi_\varepsilon\|_{L^\infty(K)} \) in hands, we can use a complex version of Evans-Krylov-Trudinger estimate (see [TW10a, Section 4] in this context) and eventually differentiate the equation to obtain -using Schauder estimates- uniform bounds, for each \( K \subset \subset \Omega, \ 0 < \beta < 1, \ j \geq 0, \)
\[ \sup_{\varepsilon > 0} \|\varphi_\varepsilon\|_{C_{j,\beta}(K)} = C_{j,\beta}(K) < +\infty, \]
which guarantee that \( \varphi_\varepsilon \) is relatively compact in \( C^\infty(\Omega). \)

We now extract a subsequence \( \varepsilon_j \to 0 \) such that

- \( c_{\varepsilon_j} \to c > 0; \)
- \( \varphi_{\varepsilon_j} \to \varphi \) in \( L^1 \) with \( \varphi \in \text{PSH}(X, \omega) \) and \( \sup_X \varphi = 0 \) (Hartogs lemma);
- \( \varphi \in C^\infty(\Omega) \) with \( (\omega + dd^c \varphi)^n = c f dV_X \) in \( \Omega; \)
- \( \varphi \geq \alpha(\delta \psi^- + \rho) - \beta \) in \( X, \)

where \( \alpha > 0 \) is arbitrarily small, as follows from (4.1). When \( f \in L^p(dV_X), \ p > 1, \) the solution \( \varphi \) is even uniformly bounded on \( X. \)

**Remark 4.3.** By comparison with the Kähler setting, it is not clear how to make sense of the Monge-Ampère measure \( (\omega + dd^c \varphi)^n \) across the singularity divisor \( D \cup E \) if \( \varphi \) is unbounded, and it is delicate to establish uniqueness of the solution, even if the solution is globally bounded (see however Theorem 3.5).
4.2. Singular hermitian Calabi conjecture. Let $V$ be a compact complex variety with log-terminal singularities, i.e. $V$ is a normal complex space such that the canonical bundle $K_V$ is $\mathbb{Q}$-Cartier and for some (equivalently any) resolution of singularities $\pi: X \to V$, we have

$$K_X = \pi^* K_V + \sum_i a_i E_i,$$

where the $E_i$’s are exceptional divisors with simple normal crossings, and the rational coefficients $a_i$ (the discrepancies) satisfy $a_i > -1$.

Given $\phi$ a smooth metric of $K_V$ and $\sigma$ a non vanishing local holomorphic section of $K_V$, we consider the "adapted volume form"

$$\mu_\phi := \left( i^{rn^2} \sigma \wedge \overline{\sigma} \right)^{\frac{1}{2}} |\sigma|_{r_{\phi}}^{-2}.$$

This measure is independent of the choice of $\sigma$, and it has finite mass on $V$, since the singularities are log-terminal. Given $\omega_V$ a hermitian form on $V$, there exists a unique metric $\phi = \phi(\omega_V)$ of $K_V$ such that

$$\omega_V^\phi = \mu_\phi.$$

**Definition 4.4.** The Ricci curvature form of $\omega_V$ is $\text{Ric}(\omega_V) := -dd^c \phi$.

Recall that the Bott-Chern space $H_{BC}^{1,1}(V, \mathbb{R})$ is the space of closed real $(1, 1)$-forms modulo the image of $dd^c$ acting on real functions. The form $\text{Ric}(\omega_V)$ determines a class $c_1^{BC}(V)$ which maps to the usual Chern class $c_1(V)$ under the natural surjection $H_{BC}^{1,1}(V, \mathbb{R}) \to H^{1,1}(V, \mathbb{R})$.

By analogy with the Calabi conjecture from Kähler geometry, it is natural to wonder conversely, whether any representative $\eta \in c_1^{BC}(V)$ can be realised as the Ricci curvature form of a hermitian metric $\omega_V$. We provide a positive answer, as a consequence of Theorem 3.4 and Theorem 4.1:

**Theorem 4.5.** Let $V$ be a compact hermitian variety with log terminal singularities equipped with a hermitian form $\omega_V$. For every smooth closed real $(1, 1)$-form $\eta$ in $c_1^{BC}(V)$, there exists a function $\varphi \in \text{PSH}(V, \omega_V)$ such that

* $\varphi$ is globally bounded on $V$ and smooth in $V_{\text{reg}}$;
* $\omega_V + dd^c \varphi$ is a hermitian form and $\text{Ric}(\omega_V + dd^c \varphi) = \eta$ in $V_{\text{reg}}$.

In particular if $c_1^{BC}(V) = 0$, any hermitian form $\omega_V$ is "$dd^c$-cohomologous" to a Ricci flat hermitian current. Understanding the asymptotic behavior of these singular Ricci flat currents near the singularities of $V$ is, as in the Kähler case, an important open problem.

**Proof.** It is classical that solving the (singular) Calabi conjecture is equivalent to solving a complex Monge-Ampère equation. We let $\pi: X \to V$ denote a log resolution of singularities and observe that

$$\pi^* \mu_\phi = fdV, \quad \text{where} \quad f = \prod_{i=1}^k |s_i|^{2a_i},$$

has poles (corresponding to $a_i < 0$) or zeroes (corresponding to $a_i > 0$) along the exceptional divisors $E_i = (s_i = 0)$ and $dV$ is a smooth volume form on $X$.

We set $\psi^+ = \sum_{a_i > 0} 2a_i \log |s_i|$, $\psi^- = \sum_{a_i < 0} 2|a_i| \log |s_i|$, and fix $\phi$ a smooth metric of $K_V$ such that $\eta = -dd^c \phi$. Finding $\omega_V + dd^c \varphi$ such that $\text{Ric}(\omega_V + dd^c \varphi) =$
\( \eta \) is thus equivalent to solving the Monge-Ampère equation \((\omega_V + dd^c \varphi)^n = c \mu_\phi \).

Passing to the resolution this boils down to solve
\[
(\omega + dd^c \tilde{\varphi})^n = ce^{\psi^+ - \psi^-} dV
\]
on \( X \), where \( \omega = \pi^* \omega_V \) and \( \tilde{\varphi} = \varphi \circ \pi \in \text{PSH}(X, \omega) \).

Since \( \omega \) is semi-positive and big, and since \( \psi^\pm \) are quasi-plurisubharmonic functions which are smooth in \( X_0 = \pi^{-1}(V_{\text{reg}}) \), it follows from Theorem 3.4 and Theorem 4.1 that there exists a solution \( \tilde{\varphi} \) with all the required properties. The function \( \varphi = \pi^* \tilde{\varphi} \) is the potential we were looking for. \( \square \)

**References**

[BT76] E. Bedford, B.A. Taylor, *The Dirichlet problem for a complex Monge-Ampère equation*. Invent. Math. 37 (1976), no. 1, 1–44.

[BT82] E. Bedford, B. A. Taylor, *A new capacity for plurisubharmonic functions*. Acta Math. 149 (1982), no. 1-2, 1–40.

[BEGZ] R.J. Berman, S.Boucksom, P.Eyssidieux, V.Guedj, A.Zeriahi, *Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties*, J. Reine Ang. Math. 751 (2019), 27–89.

[BBJ21] R. J. Berman, S.Boucksom, M.Jonsson, *A variational approach to the Yau-Tian-Donaldson conjecture*, J. Amer. Math. Soc. (2021).

[Blo05] Z. Błocki, *On uniform estimate in Calabi-Yau theorem*. Sci. China Ser. A 48 (2005), suppl., 244–247.

[Blo11] Z. Błocki, *On uniform estimate in Calabi-Yau theorem II*. Sci. China Math. 54 (2011), no. 7, 1375–1377.

[BEGZ10] S. Boucksom, P. Eyssidieux, V. Guedj, A. Zeriahi, *Monge-Ampère equations in big cohomology classes*, Acta Math. 205 (2010), no. 2, 199–262.

[Cher87] P. Cherrier, *Équations de Monge-Ampère sur les variétés hermitiennes compactes*, Bull. Sci. Math. (2) 111 (1987), no. 4, 343–385.

[Chi16] I. Chiose, *On the invariance of the total Monge-Ampère volume of Hermitian metrics*, Preprint arXiv:1609.05945.

[CC21] X.X. Chen, J. Cheng, *On the constant scalar curvature Kähler metrics*, J. Amer. Math. Soc. (2021).

[Dem94] J.P. Demailly, *Regularization of closed positive currents of type \((1,1)\) by the flow of a Chern connection*, Contributions to complex analysis and analytic geometry, 105–126, Friedr. Vieweg, Braunschweig (1994).

[Dem15] J.P. Demailly, *On the cohomology of pseudoeffective line bundles*. Complex geometry and dynamics, Abel Symp., vol. 10, Springer, Cham, 2015, pp. 51–99.

[Dem] J.P. Demailly, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, 1, International Press; Higher Education Press, Beijing, 2012. viii+231 pp.

[DDGKPZ] J.P. Demailly, S. Dinew, V. Guedj, H.H. Pham, S. Kołodziej, A. Zeriahi, *Hölder continuous solutions to Monge-Ampère equations*. J. Eur. Math. Soc. 16 (2014), 619–647.

[DP10] J.P. Demailly, N. Pali, *Degenerate complex Monge-Ampère equations over compact Kähler manifolds*. Internat. J. Math. 21 (2010), no. 3, 357–405.

[DP04] J.P. Demailly, M. Paun, *Numerical characterization of the Kähler cone of a compact Kähler manifold*. Ann. of Math. (2) 159 (2004), no. 3, 1247–1274.

[Din16] S. Dinew, *Pluripotential theory on compact Hermitian manifolds*, Ann. Fac. Sci. Toulouse Math. (6) 25 (2016), no. 1, 91–139.

[DK12] S. Dinew, S.Kołodziej, *Pluripotential estimates on compact Hermitian manifolds*. Advances in geometric analysis, 69–86, Adv. Lect. Math. (ALM), 21, Int. Press, 2012.

[DnL15] E. Di Nezza, C. H. Lu, *Generalized Monge-Ampère capacities*. Int. Math. Res. Not. IMRN 2015, no. 16, 7287–7322.

[DnL17] E. Di Nezza, C. H. Lu, *Complex Monge-Ampère equations on quasi-projective varieties*, J. Reine Angew. Math. 727 (2017), 145–167.

[DGL19] T. Darvas, E. Di Nezza, C.H. Lu, *The metric geometry of singularity types*, J. Reine Angew. Math. 771 (2021), 137–170.

[Don18] S. Donaldson, *Some recent developments in Kähler geometry and exceptional holonomy*, Proc. Int. Cong. Math., Rio de Janeiro 2018. Vol. I. 425–451, World Sci. Publ., NJ, 2018.
[EGZ08] P. Eyssidieux, V. Guedj, A. Zeriahi, A priori $L^\infty$-estimates for degenerate complex Monge-Ampère equations, I.M.R.N., Vol. 2008, Article ID rnm070, 8 pages.

[EGZ09] P. Eyssidieux, V. Guedj, A. Zeriahi, Singular Kähler-Einstein metrics, J. Amer. Math. Soc. 22 (2009), no. 3, 607–639.

[FT09] A. Fino, A. Tomassini, Blow-ups and resolutions of strong Kähler with torsion metrics. Adv. Math. 221 (2009), no. 3, 914–935.

[FLY12] J. Fu, J. Li, S.-T. Yau, Balanced metrics on non-Kähler Calabi-Yau threefolds. J. Differential Geom. 90 (2012), no. 1, 81–129.

[GL10] B. Guan, Q. Li, Complex Monge-Ampère equations and totally real submanifolds. Adv. Math. 225 (2010), no. 3, 1185–1223.

[GLZ18] V. Guedj, C.H. Lu, Stability of solutions to complex Monge-Ampère flows. Ann. Inst. Fourier (Grenoble) 68 (2018), no. 7, 2819–2836.

[GL21a] V. Guedj, C.H. Lu, Quasi-plurisubharmonic envelopes 1: Uniform estimates on Kähler manifolds, Preprint arXiv:2106.04273 (2021).

[GL21b] V. Guedj, C.H. Lu, Quasi-plurisubharmonic envelopes 2: Bounds on Monge-Ampère volumes, Preprint arXiv:2106.04272 (2021).

[GZ] V. Guedj, A. Zeriahi, Degenerate complex Monge-Ampère equations, EMS Tracts in Mathematics, vol. 26, European Mathematical Society (EMS), Zürich, 2017.

[GPT21] B. Guo, D.H. Phong, F. Tong, On $L^\infty$-estimates for complex Monge-Ampère equations. Preprint arXiv:2106.02224.

[Han96] A. Hanani, Équations du type de Monge-Ampère sur les variétés hermitiennes compactes, J. Funct. Anal. 137 (1996), no. 1, 49–75.

[KS] H. Skoda, Sous-ensembles analytiques d’ordre fini ou infini dans $\mathbb{C}^n$, Bull. Soc. Math. France, 100 (1972), 353–408.

[Szek18] G. Székelyhidi, Fully non-linear elliptic equations on compact Hermitian manifolds. J. Differential Geom. 109 (2018), no. 2, 337–378.

[STW17] G. Székelyhidi, V. Tosatti, B. Weinkove, Gauduchon metrics with prescribed volume form, Acta Math. 219 (2017), no. 1, 181–211.

[Tö18] T.D. Tô, Regularizing properties of complex Monge-Ampère flows II: Hermitian manifolds. Math. Ann. 372, 699–741 (2018).

[TW18] V. Tosatti, B. Weinkove, The Aleksandrov-Bakelman-Pucci estimate and the Calabi-Yau equation. Nonlinear analysis in geometry and applied mathematics. Part 2, 147–158, Int. Press, Somerville, MA, 2018.

[Yau78] S.T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. Comm. Pure Appl. Math. 31 (1978), no. 3, 339–411.
