On energy dissipation in a friction-controlled slide of a body excited by random motions of a foundation

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We study a friction controlled slide of a body excited by random motions of a foundation and show that this problem can be treated in an analytic manner. Assuming the random excitation is switched off at some time, we derive the moments of the body displacement and of the total distance traveled, and calculate an average energy loss due to friction. To accomplish that we use the Pugachev–Sveshnikov equation for the characteristic function of a continuous random process, which is solved by reduction to the parametric Riemann boundary value problem of complex analysis.

I. INTRODUCTION

In the present paper, we address a problem that concerns dynamical behavior of a solid body sliding with dry friction over the surface of foundation that moves randomly. This problem is known to be extremely difficult, yet very important for various physical applications.

Interest in such a topic has been increasingly growing over the last few decades. In the 60s, that was engineers who were first to start studying influence of random fluctuations on the dry friction phenomenon in order to model behavior of buildings during earthquakes [1, 2]. More recently, physicists got involved and began to study similar problems, but from a “more microscopic” perspective. Their studies relate to nanofrictional systems [3], particles separation [4], ratchets [5–8], granular motors [9, 10], and dynamics of droplets on moving surfaces [11–13]. The quality all these studies share is that, in a way, they are all connected with forces similar to dry friction.

The typical mechanical problem statement we deal with is given in [14]. The author of that paper studies motion of an object, a solid body, sliding with friction over the surface of a horizontal uniformly rough foundation, which is vibrating laterally being subjected to an external Gaussian white noise excitation. The author uses the simplest description possible for dry friction, namely, he takes into account only kinetic part of it, ignoring static one. In this case, if we take the lower body as a frame of reference, relative velocity of the sliding body will satisfy a simple stochastic differential (Langevin) equation with “sign (x).” The latter means the resistant force has a constant absolute value, only its direction is always opposite to the one of velocity. This equation was a starting point for numerous mathematical studies on the topic, in turn leading to solution of different practical problems.

So far, authors of physical papers have restricted themselves to calculating only characteristics of the object’s velocity. The main approach they use is the Fokker–Planck equation [15], the path integral [16], or a weak-noise limit [17]. At the same time, engineers and specialists in related fields have always wanted to have a more detailed description of the problem. For instance in the mid 70s, S. H. Crandall proposed to study displacements of the object. Soon, the corresponding problem was named after Crandall, but neither he nor his co-authors found an exact solution: they had to be satisfied with approximate one given by statistical linearization.

It is worth noticing there exist further mechanical studies of classic Crandall’s problem and of more general ones. For instance, some authors improved the statistical linearization technique [18], others changed the type of excitation used [19] or studied the case of two sliding bodies [20]. However, all of these works are, at the end of the day, based on the concept of linearization and give an approximate solution only.

In the present paper we suggest an alternative explicit method based on the Pugachev–Sveshnikov equation. This equation is not wide known, yet it is very effective for the problems considered. Unlike the Fokker–Planck equation, the Pugachev–Sveshnikov equation describes behavior of the random process in terms of the characteristic function. This equation allows us to get an exact expression not only for the object’s velocity, but also for the displacement and for the total distance traveled. This distance is proportional to the amount of energy dissipated during the slide, and it turns out there is a certain natural conservation law for it.

The Pugachev–Sveshnikov equation method is thoroughly described in [21, 22], also some preliminary study of the present topic can be found in [23].

II. PROBLEM STATEMENT

We consider a rigid body of mass $m$ placed on a massless foundation. This foundation is subjected to random Gaussian white noise excitation $\xi(t)$ of intensity $h$, switched off at time $t_0$; the corresponding covariance function is $K_\xi(t_1, t_2) = \delta(t_2 - t_1)$. That excitation causes body to move with velocity $V(t)$ relative to the foundation. The resistant force $F_{res}(t)$ between the foundation and the body is assumed to obey Coulomb’s friction law, namely, $F_{res}(t) = -\mu mg \text{sign}(V(t))$, where $\mu$ is the coefficient of dry (kinetic) friction between the two surfaces.
in contact (see Fig. 1), and \( g \) is the acceleration of gravity.

![Crandall's problem](image)

**FIG. 1.** Crandall’s problem

The body’s velocity \( V(t) \), displacement \( U(t) \), and the distance \( S(t) \) body has traveled satisfy equations that read as follows:

\[
\begin{align*}
\dot{V}(t) &= -\mu g \text{sign}(V(t)) + \eta(t_0 - t) h \xi(t), \\
\dot{U}(t) &= V(t), \\
\dot{S}(t) &= |V(t)|,
\end{align*}
\]

(1)

where \( \eta(t) \) is the Heaviside step function, that is in charge of switching excitation off. The first component \( V(t) \) of the process (1) is the so-called Caughey–Dienes process \([1]\), the second component \( U(t) \) is the Crandall process \([2]\), and the third component \( S(t) \) is a new one, that has not been thoroughly studied to date.

After scaling equations (1) to dimensionless form with a transform

\[
\begin{align*}
U_1 &= \frac{\mu g}{h^2} V, \\
U_2 &= \frac{\mu^2 g^3}{2h^4} U, \\
U_3 &= \frac{\mu^2 g^3}{2h^4} S, \\
\tau &= \frac{t}{h^2},
\end{align*}
\]

(2)

we can rewrite (1) in the following form:

\[
\begin{align*}
\dot{U}_1(\tau) &= -2 \text{sign}(U_1(\tau)) + \eta(\tau_0 - \tau) \sqrt{2} \xi(\tau), \\
\dot{U}_2(\tau) &= U_1(\tau), \\
\dot{U}_3(\tau) &= |U_1(\tau)|.
\end{align*}
\]

(3)

The process \( \dot{\xi}(\tau) \) is another Gaussian white noise process given by

\[
\dot{\xi}(\tau) = \frac{\sqrt{2} h}{\mu g} \xi \left( \frac{2h^2}{\mu^2 g^2} \tau \right).
\]

(4)

Suppose that the system had been at rest before the moment \( t = 0 \) when excitation suddenly occurred, and this is when we started measuring \( U(t) \) and \( S(t) \). In such a situation initial conditions become homogeneous \( U_1(0) = U_2(0) = U_3(0) = 0 \).

System (3) includes two nonlinearities

\[
\Psi_1(U) = \text{sign}(U), \quad \Psi_2(U) = |U|,
\]

(5)

both of which are piecewise linear having two domains of linearity. Therefore, this system can be treated with methods from \([21]\).

### III. AVERAGE ENERGY CONSERVATION LAW

From now on, we denote specific kinetic energy (per unit mass) of the body by \( Q(t) \) and specific energy dissipated due to friction by \( W(t) \):

\[
Q(t) = \frac{1}{2} V^2(t),
\]

\[
W(t) = \mu g \int_0^t V(s) \text{sign}(V(s)) \, ds = \mu g S(t).
\]

(6)

As long as external excitation is switched off, the body stops at some random moment \( t_s > t_0 \) because of friction, and thus \( V(t_s) = 0 \). Let us write the Itô’s formula for \( Q(t) \), using (3), in the following form:

\[
dQ(t) = \left( -\mu g |V(t)| + \frac{1}{2} \eta(t_0 - t) h^2 + \right. \\
+ \eta(t_0 - t) h V(t) \xi(t) \left. \right) dt.
\]

(7)

Having taken the mathematical expectation of (7) we will get

\[
d\bar{q}(t) = -d\bar{w}(t) + \frac{1}{2} \eta(t_0 - t) h^2 dt,
\]

(8)

where \( \bar{q}(t) = M[Q(t)] \), \( \bar{w}(t) = M[W(t)] \), and \( \bar{w}_s = M[W(t_s)] \).

Now, if we consider two time intervals \((0, t_0)\) and \((t_0, t_s)\), and integrate (8) over each of them, taking into account initial condition, we will get the average energy conservation law

\[
\bar{q}(t_0) = -\bar{w}(t_0) + \frac{1}{2} h^2 t_0,
\]

(9)

\[
-\bar{q}(t_0) = -(\bar{w}_s - \bar{w}(t_0)).
\]

(10)

The specific kinetic energy of the foundation which moves with velocity \( V_f(t) \) is given by

\[
K_f(t) = \frac{1}{2} V_f^2(t).
\]

(11)

Since the excitation in (3) is a Gaussian white noise process of intensity \( h \), the process \( V_f(t) \) will be the Wiener one. Therefore, the average kinetic energy of the foundation has form

\[
\bar{k}_f(t) = \frac{1}{2} h^2 t.
\]

(12)

That allows us to rewrite (9) as

\[
\bar{q}(t_0) = -\bar{w}(t_0) + \bar{k}_f(t_0),
\]

(13)

\[
-\bar{q}(t_0) = - (\bar{w}_s - \bar{w}(t_0)).
\]

(14)

Summing up expressions (13) and (14), we will finally get to the formula

\[
\bar{w}_s = \bar{k}_f(t_0),
\]

(15)
which means the average specific energy spent on friction during the motion equals the average specific kinetic energy of the foundation at the time the excitation is switched off. This result gives us an opportunity to calculate the average energy due to friction at the final moment $t_f$. To investigate transient behavior we are going to use the Pugachev–Sveshnikov equation approach. The latter let us find the moments of velocity $V(t)$, displacement $U(t)$, distance $S(t)$, and, therefore, these of dissipated energy $W(t)$.

IV. PUGACHEV–SVESHNIKOV EQUATION FORMALISM

As we pointed out earlier, system (3) is piecewise linear, and it has two domains of linearity. This allows us to use the Pugachev–Sveshnikov equation approach. Assuming $t < t_0$, the singular integral-differential equation for the characteristic function $E(z_1, z_2, z_3; \tau)$ of the Markov process $(U_1(\tau), U_2(\tau), U_3(\tau))$ will take form

$$
\frac{\partial E}{\partial \tau} - z_2 \frac{\partial E}{\partial z_1} + \frac{2z_2}{\pi} E + \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{E|_{z_1= \xi}}{s - z_1} ds + \int_{-\infty}^{\infty} \frac{E|_{z_1= \xi}}{s - z_1} ds = \frac{i z_3}{\pi} \frac{\partial E}{\partial z_1} + \int_{-\infty}^{\infty} \frac{E|_{z_1= \xi}}{s - z_1} ds$$

with $E|_{\tau = 0} = 1$. It is important to underline that the integral in (16) is improper, the principle value one. If we apply the technique developed in (21), we can find the characteristics of $(U_1(\tau), U_2(\tau), U_3(\tau))$, and, thus, these of $V(t), U(t), S(t)$, and $W(t)$.

Let us introduce the Cauchy type integral

$$F(\zeta; z_2, z_3, \tau) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{E(z_1, z_2, z_3; \tau)}{z_1 - \zeta} ds,$$ (17)

which is known to be a complex analytic function for $\text{Im}(\zeta) \geq 0$, and let us denote its limit values when $\zeta \to z_1 \pm 0$ by $F^{\pm}(z_1, z_2, z_3, \tau)$. The latter functions satisfy the well-known Sokhotski–Plemelj formulas (21)

$$E = F^{+} - F^{-}, \quad \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{E|_{z_1= \xi}}{s - z_1} ds = F^{+} + F^{-}.$$ (18)

Then, applying Liouville’s theorem from complex analysis, after having substituted (18) into (16) we easily get the so-called master equation

$$\frac{\partial F^{\pm}}{\partial \tau} - (z_2 \pm z_3) \frac{\partial F^{\pm}}{\partial z_1} + z_1 (z_1 \pm 2i) F^{\pm} = G_0 + z_1 G_1, \quad F^{\pm}|_{\tau = 0} = \pm \frac{1}{2}.$$ (19)

Here, $G_0(z_2, z_3; \tau)$ and $G_1(z_2, z_3; \tau)$ are new intermediary entire complex variable functions we need to find. This can be done using analytic properties of $F^{\pm}$, that is, $F^{+}$ is analytic for $\text{Im}(z_1) > 0$, and $F^{-}$ is analytic for $\text{Im}(z_1) < 0$. After such the functions are found, we solve equations (19) and use (18) to get the desired characteristic function $E$ back.

V. METHOD OF MOMENTS

For the sake of simplicity and to fulfill practical needs, we find moments of $(U_1(\tau), U_2(\tau), U_3(\tau))$ only. Detailed description of this method is given in (21). Let us start by applying the Laplace transform to (19) with respect to $\tau$. The corresponding series decomposition reads

$$\hat{F}^{\pm}(z_1; z_2, z_3, p) = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \hat{F}^{\pm}_{\alpha\beta}(z_1; p) \frac{z_2}{\alpha!\beta!}.$$ (21)

$$\hat{G}_{\alpha\beta}(z_2, z_3, p) = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \frac{\hat{G}_{\alpha\beta}(z_2, z_3; p) z_2 z_3}{\alpha!\beta!}.$$ (22)

Inserting (21) and (22) into (20) and comparing coefficients in front of $z_2^0 z_3^0$, we will have an infinite system of equations

$$(z_1^2 + 2i z_1 + p) \hat{F}^{\pm}_{kl} = \hat{G}_{0kl} + z_1 \hat{G}_{1kl} + \frac{1}{2} \delta_{0k} \delta_{0l} + k \frac{\partial \hat{F}^{\pm}_{(k-1)l}}{\partial z_1} \pm \frac{\partial \hat{F}^{\pm}_{kl}}{\partial z_1}, \quad k, l \geq 0.$$ (23)

where $\delta$ is the Kronecker delta, and $\hat{F}^{\pm}_{kl} = 0$ if $k < 0$ or $l < 0$.

To get mixed moments of $(U_1(\tau), U_2(\tau), U_3(\tau))$ we introduce the functions

$$m_{jkl}(\tau) = M[U^j_1(\tau) U^k_2(\tau) U^l_3(\tau)].$$ (24)

Formulas (18) and (21) lead us to the expression

$$\hat{m}_{jkl}(p) = \left. \frac{1}{\mu^j + k + l} \left( \hat{F}^{+}_{kl}(0; p) - \hat{F}^{-}_{kl}(0; p) \right) \right).$$ (25)

We will calculate $\hat{F}^{\pm}_{kl}$ successively by grouping them with respect to $k$ and $l$, so that $k + l = 0, 1, \ldots$. There is the one term for $k + l = 0$, the term $\hat{F}^{\pm}_{00}$, which can be easily found from (23).

$$\hat{F}^{\pm}_{00} = \frac{\hat{G}_{00} + z_1 \hat{G}_{10} \pm \frac{1}{2}}{z_1^2 + 2i z_1 + p}.$$ (26)
These rational functions have simple poles in the complex plane, and in order to turn them into analytic functions for \( \text{Im}(z_1) \geq 0 \), respectively, we need to vanish numerators of the fractions at \( \pm i\mu \), where \( \mu = \sqrt{p+1} - 1 \). That gives us a system of linear equations to find \( \tilde{G}_{0,00} \) and \( \tilde{G}_{1,00} \):

\[
\tilde{G}_{0,00} + i\mu \tilde{G}_{1,00} + \frac{1}{2} = 0, \quad (27)
\]

\[
\tilde{G}_{0,00} - i\mu \tilde{G}_{1,00} - \frac{1}{2} = 0. \quad (28)
\]

And finally we get

\[
\tilde{F}^\pm_{00} = \frac{i}{2\mu(z_1 \pm i(\sqrt{p+1} + 1))}. \quad (29)
\]

For \( k+l > 0 \) we will have the following recurrent formulas

\[
\tilde{F}^\pm_{kl} = \frac{\tilde{G}_{0,kl} + z_1\tilde{G}_{1,kl} + k\frac{\partial \tilde{F}^\pm_{(k-1)l}}{\partial z_1} \pm l\frac{\partial \tilde{F}^\pm_{kl-1}}{\partial z_1}}{z_1^2 \pm 2iz_1 + p}. \quad (30)
\]

In the similar way as earlier, to make these complex valued functions analytic, we need to vanish numerators at \( i\mu^\pm \). That gives us two equations

\[
\tilde{G}_{0,kl} + i\mu \tilde{G}_{1,kl} + k\frac{\partial \tilde{F}^\pm_{(k-1)l}}{\partial z_1} \pm l\frac{\partial \tilde{F}^\pm_{kl-1}}{\partial z_1} = 0, \quad (31)
\]

where we can take either + or − for all of ± simultaneously.

Using the procedure described, we will get Laplace transforms of the first moments:

\[
\hat{m}_{100} = \hat{m}_{010} = 0, \quad \hat{m}_{001} = \frac{1}{p^2(\sqrt{p+1} + 1)}, \quad (32)
\]

\[
\hat{m}_{200} = \frac{2}{p(\sqrt{p+1} + 1)^2}, \quad (33)
\]

\[
\hat{m}_{020} = \frac{4\sqrt{p+1} + 1}{p^2(p+1)(\sqrt{p+1} + 1)^3}, \quad (34)
\]

\[
\hat{m}_{002} = \frac{9p - 4\sqrt{p+1} + 1}{2p^3(p+1)(\sqrt{p+1} + 1)^2}. \quad (35)
\]

All the rest moments can be found in the similar manner.

### VI. RESULTS

Applying the inverse Laplace transform to (32) and (34) we will have

\[
\bar{u}_1(\tau) = 0, \quad \bar{u}_2(\tau) = 0, \quad (36)
\]

\[
\bar{u}_3(\tau) = \frac{1}{8} \left[ 2(1 + 2\tau) \sqrt{\frac{\tau}{\pi}} e^{-\tau} - 4\tau^2 \text{Erf}(\sqrt{\tau}) + (4\tau - 1) \text{Erf}(\sqrt{\tau}) \right], \quad (37)
\]

\[
\sigma^2_{\bar{u}_1}(\tau) = \frac{1}{2} \left[ \text{Erf}(\sqrt{\tau}) + 4\tau(\tau + 1) \text{Erf}(\sqrt{\tau}) - 2(1 + 2\tau) \sqrt{\frac{\tau}{\pi}} e^{-\tau} \right], \quad (38)
\]

\[
\sigma^2_{\bar{u}_2}(\tau) = \frac{5}{8} \tau - \frac{27}{32} + e^{-\tau} \left[ 1 - \sqrt{\frac{\tau}{\pi}} \left( \frac{7}{2} \tau^3 + 7 \frac{12}{12} \tau^2 - 13 \frac{12}{12} \tau + 5 \frac{16}{16} \right) \right] + \text{Erf}(\sqrt{\tau}) \left( \frac{1}{2} \tau^4 + \frac{5}{6} \tau^3 - \frac{1}{2} \tau^2 + \frac{3}{8} \tau - \frac{5}{32} \right), \quad (39)
\]

\[
\sigma^2_{\bar{u}_3}(\tau) = \frac{1}{4} \tau^2 + \frac{1}{8} \tau - \frac{11}{32} + e^{-\tau} \left[ \frac{1}{2} + \tau \sqrt{\frac{\tau}{\pi}} \left( \frac{13}{24} - \frac{7}{12} \tau - \frac{1}{2} \tau^2 \right) \right] + \text{Erf}(\sqrt{\tau}) \left( \frac{1}{2} \tau^4 + \frac{5}{6} \tau^3 - \frac{1}{2} \tau^2 + \frac{3}{8} \tau - \frac{5}{32} \right) - \bar{u}_3^2, \quad (40)
\]

for \( \tau \leq \tau_0 = \frac{\mu^2 g^2}{2k} t_0 \), where \( \text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \) and \( \text{Erfc}(x) = 1 - \text{Erf}(x) \).

After the external excitation is turned off, the body will remain moving until the moment \( t_s \), or \( \tau_s = \frac{\mu^2 g^2}{2k} t_s \), in the dimensionless formulation. The expression (29) gives a possibility to derive the probability density function of \( \tau_s \). Inverting (29) by Fourier with respect to \( z_1 \) and
by Laplace with respect to $p$, one can get

$$f_{\tau_s}(y, \tau) = \frac{1}{2} \left( e^{-2|y|} \text{Erfc} \left( \frac{|y| - 2\tau}{2\tau} \right) + \frac{1}{\sqrt{\pi} \tau} e^{-\frac{(|y| + 2\tau)^2}{4\tau}} \right).$$

(41)

Since for $\tau > \tau_0$ the equations (3) will have no stochastic component, it is easy to solve them. Then, we will have

$$U_1(\tau) = U_1(\tau_0) - 2\text{sign}(U_1(\tau_0))(\tau - \tau_0),$$

(42)

$$U_2(\tau) = U_2(\tau_0) + U_1(\tau_0)(\tau - \tau_0) - \text{sign}(U_1(\tau_0))(\tau - \tau_0)^2,$$

(43)

$$U_3(\tau) = U_3(\tau_0) + |U_1(\tau_0)|(\tau - \tau_0) - (\tau - \tau_0)^2,$$

(44)

where $\tau \leq \tau_s$, and at time $\tau_s$ the body stops.

From (42) it follows that $\tau_s$ is given by the formula

$$\tau_s = \tau_0 + \frac{1}{2} |U_1(t_0)|.$$

(46)

If we substitute (46) into expression (45), go back to units with (2), and use (6), we will get to the formula $\bar{w}_s = \bar{w}(t_0) + \bar{q}(t_0)$. Thus, by use of (37) and (38) after units conversion (2), we will have $\bar{w}_s = \frac{k^2 t_0}{2} = \bar{k}_f(t_0)$, which is in perfect agreement with (19).

The formulas (36)–(46) allow us to plot some meaningful pictures (see Fig. 2, 3, 4, and 5).

FIG. 2. Average scaled traveled distance $\bar{u}_3(\tau)$

FIG. 3. Variance of scaled velocity $\sigma^2_{U_1}(\tau)$

FIG. 4. Variance of scaled displacement $\sigma^2_{U_2}(\tau)$

FIG. 5. Variance of scaled traveled distance $\sigma^2_{U_3}(\tau)$

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