Pencils and nets of small degree on curves on smooth, projective surfaces of Picard rank 1 and very ample generator

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Abstract

Let $S$ be a smooth, projective surface of Picard rank 1 and very ample generator embedding $S$ into $\mathbb{P}^n$. Let $C \in |\mathcal{O}_S(m)|$ for $m \geq 5$ be a smooth curve. We prove that any base-point free, complete $g^r_d$ on $C$ for $r \in \{1, 2\}$ and $d$ small enough is cut out by a hyperplane section restricted to a multisecant $(n-r-1)$-plane.

1 Introduction

Let $C$ be a smooth curve embedded in $\mathbb{P}^n$, and let $|A|$ be a complete, base-point free linear series $g^r_d$ on $C$, where $r \in \{1, 2\}$. Our aim in this paper is to find instances for when $|A|$ is cut out by hyperplane sections restricted to an $(n-r-1)$-plane secant on $C$.

The question in focus has previously been much studied, in particular for the case when $C \subseteq \mathbb{P}^3$. In [3], it was proven that when $C$ lies on a surface of degree $> 4$ in $\mathbb{P}^3$, the linear system of plane sections is the only $g^3_{\deg(C)}$ that exists. In [1], it was proven that when $C$ is a smooth complete intersection curve in $\mathbb{P}^3$, any pencil computing the gonality is given by a plane section minus a multisecant of maximal degree. In [4], Hartshorne and Schlesinger proved that the gonality of a general ACM curve in $\mathbb{P}^3$ is also computed by multisecants. A further overview of previous research done on gonality and multisecants is given in the introduction of [4].

This paper is an attempt to prove a slightly more general result for the case where $C$ lies on a smooth, projective surface $S$ with Picard rank 1 and very ample generator. We will here study $g^1_d$’s of not necessarily minimal degree, in addition to $g^2_d$’s. The methods we use are taken from [4, Proposition 5.5]. The idea is to create a map from $\mathcal{O}_S(1)$ to the $g^1_d A$ on $C$, and make sure that the map is surjective on global sections.

In what follows, $S$ will denote a smooth, projective surface with Picard rank 1 and very ample generator. We will here study $g^1_d$’s of not necessarily minimal degree, in addition to $g^2_d$’s. The methods we use are taken from [4, Proposition 5.5]. The idea is to create a map from $\mathcal{O}_S(1)$ to the $g^1_d A$ on $C$, and make sure that the map is surjective on global sections.

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Theorem 1.1. Let $S$ and $\mathcal{O}_S(H)$ be as above. Let $C \in |\mathcal{O}_S(m)|$ for $m \geq 5$ be a smooth curve on $S$. Then any base-point free, complete $g^r_d$ on $C$, with $r \in \{1, 2\}$ and $d \leq mH^2$, is given by restricting hyperplane sections to an $(n-r-1)$-plane secant on $C$.

An immediate consequence of Theorem 1.1 is the following:

Corollary 1.2. Let $S$ be a general complete intersection surface in $\mathbb{P}^4$ of type $(2, 3)$ (hence a K3 surface), with Picard group generated by the class of hyperplane sections $H$. Then, for $m \geq 5$, the minimal degree of a $g^2_3$ on a smooth curve $C \in |\mathcal{O}_S(m)|$ is $6m - 3$.

Proof. Let $T$ be the Fano scheme of lines contained in the quadric hypersurface containing $S$. This scheme corresponds precisely to the set of 3-secant lines on $S$. The dimension of $T$ is 3.

Let $\mathcal{I}$ be the incidence variety given by pairs $(C, \Gamma)$ such that $C$ is smooth in $|\mathcal{O}_S(m)|$ and $\Gamma \in T$ is a 3-secant line on $C$. Denoting smooth curves in $|\mathcal{O}_S(m)|$ by $|\mathcal{O}_S(m)|_s$, we have the following diagram, where $p$ and $q$ denote the natural projections:

\[
\begin{array}{c}
\mathcal{I} \\
\downarrow q \\
|\mathcal{O}_S(m)|_s \\
\downarrow p \\
T
\end{array}
\]

For any $\Gamma \in T$, $\Gamma \cap S$ will impose 3 independent conditions on the elements in $|\mathcal{O}_S(m)|_s$. Since the elements in $|\mathcal{O}_S(m)|_s$ are general in $|\mathcal{O}_S(m)|$, giving us that $\dim |\mathcal{O}_S(m)|_s = \dim |\mathcal{O}_S(m)| = 3m^2 + 1$, it is therefore clear that $q$ is surjective, and that each fibre corresponds to a codimension 3 subscheme of $|\mathcal{O}_S(m)|_s$. Since $\dim(T) = 3$, it follows that each fibre of $p$ must have dimension 0, and so each curve in $|\mathcal{O}_S(m)|_s$ has a finite number of 3-secant lines.

Since $S$ can’t have any 4-secant lines, it follows that the minimal degree of a $g^2_3$ on any $C \in |\mathcal{O}_S(m)|_s$ is $C.H - 3 = mH^2 - 3 = 6m - 3$. \hfill \Box

2 Proof of the theorem

Following the work of Lazarsfeld and Tyurin [3, 8], given a smooth curve $C$ of genus $g$ on $S$ and a base-point free, complete $g^r_d[A]$ on $C$, one defines a vector-bundle $\mathcal{F}_{C,A}$ on $S$ as the kernel of the evaluation morphism $H^0(C, A) \otimes \mathcal{O}_S \rightarrow A \rightarrow 0$. The bundle $\mathcal{F}_{C,A}$ (or, more frequently, its dual) is often referred to as the associated Lazarsfeld–Mukai bundle of $C$ and $A$. The bundle has the following properties:

- $\text{rk}(\mathcal{F}_{C,A}) = r + 1$.
- $\text{det}(\mathcal{F}_{C,A}) = \mathcal{O}_S(-C)$.
- $e_2(\mathcal{F}_{C,A}) = d$.
- $h^0(S, \mathcal{F}_{C,A}) = 0$.
- The dual, $\mathcal{F}^n_{C,A}$, is globally generated away from a finite set.

In the cases we are interested in, where $C \in |\mathcal{O}_S(m)|$ with $m \geq 5$, and $d \leq mH^2$, it follows from [2, Theorem] that $\mathcal{F}_{C,A}$ is non-$\mathcal{O}_S(H)$-stable. There thus exists a maximal destabilising sequence,

\[
0 \rightarrow M \rightarrow \mathcal{F}_{C,A} \rightarrow N \rightarrow 0,
\]
where $M$ is a vector-bundle with $\text{rk}(M) \leq r$ satisfying $\frac{1}{\text{rk}(M)}c_1(M).H > -\frac{1}{r+1}mH^2$, and $N$ is torsion-free and $\mathcal{O}_S(H)$-stable.

Before presenting the proof of Theorem 1.1, we prove the following lemma:

**Lemma 2.1.** Let $\mathcal{F}_{C,A}$ be as above, with $C \in |\mathcal{O}_S(m)|$, $m \geq 5$, $d \leq mH^2$ and $r \leq 2$. In the exact sequence (1), we have the following:

- $c_1(M) = \mathcal{O}_S(-H)$,
- $c_2(M) \geq 0$, and
- $c_2(N) \geq 0$.

As a consequence of the above, $\text{rk}(M) = r$ and $\text{rk}(N) = 1$.

**Proof.** We begin by proving that $c_1(M) = \mathcal{O}_S(-a)$ for some $a > 0$. In the case where $\text{rk}(M) = 1$, this is clear since $h^0(S, \mathcal{F}_{C,A}) = 0$. In the case where $\text{rk}(M) = 2$, we dualise (1) and get

$$0 \to N^\vee \to \mathcal{F}_{C,A}^\vee \to \tilde{M} \to 0,$$

where $\tilde{M}$ satisfies $\tilde{M}^\vee = M$. Since $\mathcal{F}_{C,A}^\vee$ is globally generated away from a finite set, then the same must be the case for $\tilde{M}$, and so $c_1(\tilde{M}) = \mathcal{O}_S(a)$ with $a \geq 0$ (proof: there exists a saturated sub-linebundle $\mathcal{O}_S(t)$ with $t \geq 0$ which we can inject into $\tilde{M}$, and the cokernel must be globally generated away from a finite set). If $a = 0$, then $\tilde{M} = \mathcal{O}_S^2$. However, it then follows that $c_2(\mathcal{F}_{C,A}) = 0$, which is a contradiction. It follows that $a > 0$.

We now prove that $c_2(M)$ and $c_2(N)$ are both nonnegative. Since $N$ is torsion-free and $\mathcal{O}_S(H)$-stable, then by [2] Theorem, $c_2(N) \geq 0$. The inequality $c_2(M) \geq 0$ is clear for the $\text{rk}(M) = 1$ case. For the $\text{rk}(M) = 2$ case, suppose $c_2(M) < 0$, and note that by [2] Theorem, $M$ must then be non-$\mathcal{O}_S(H)$-stable. This gives us a maximal destabilising sequence

$$0 \to \mathcal{O}_S(-b) \to M \to \mathcal{O}_S(b-a) \otimes \mathcal{I}_\eta \to 0,$$

where $b > 0$ since $\mathcal{F}_{C,A}$ and hence also $M$ cannot have any global sections; where $\mathcal{I}_\eta$ is the ideal sheaf of a finite subscheme $\eta$ (possibly empty); and where $-bH^2 > -\frac{1}{4}aH^2$, implying that $2b < a$. Since $c_2(M) < 0$, we have $(-b)(b-a)H^2 < 0$, implying that $b-a > 0$, giving us the desired contradiction.

We now prove that $a = 1$: If $c_1(M) = \mathcal{O}_S(-a)$ and $c_1(N) = \mathcal{O}_S(a-m)$ for $a \geq 2$ satisfying $\frac{1}{\text{rk}(M)}c_1(M).H > -\frac{1}{r+1}mH^2$, this gives us $c_2(\mathcal{F}_{C,A}) \geq c_1(M).c_1(N) = a(m-a)H^2 > mH^2$, which contradicts the assumption we have for $c_2(\mathcal{F}_{C,A})$. (As long as $m \geq 3$, we cannot have $c_1(M) = \mathcal{O}_S(1-m)$ at the same time as $\frac{1}{\text{rk}(M)}c_1(M).H > -\frac{1}{r+1}mH^2$. The condition $m \geq 5$ is to ensure that the inequality $a(m-a)H^2 > mH^2$ is strict.)

It now follows that $\text{rk}(M) = r$, because of the following: If $r = 2$ with $\text{rk}(M) = 1$, then by [2] Theorem, the stability of $N$ implies that $\frac{1}{4}c_1(N)^2 - c_2(N) \leq 0$, so that $c_2(N) \geq \frac{1}{4}(m-1)^2H^2$. This gives us $c_2(\mathcal{F}_{C,A}) \geq (m-1)H^2 + \frac{1}{4}(m-1)^2H^2 > mH^2$, a contradiction.

We now prove the main theorem.
Proof of Theorem 1.1. Let $C \in |\mathcal{O}_S(m)|$ be smooth, where $m \geq 5$, let $A$ be a base-point free, complete $g^d_0$ on $C$ with $r \in \{1, 2\}$ and $d \leq 4mH^2$, and let $\mathcal{F}_{C,A}$ be the associated Lazarsfeld–Mukai vector bundle. By Lemma 2.1, there exists a rank $r$ vector bundle $M$ with $c_1(M) = \mathcal{O}_S(\mathcal{H})$ and $c_2(M), c_2(N) \geq 0$ such that

$$0 \to M \to \mathcal{F}_{C,A} \to N \to 0.$$ 

Because $M$ injects into $\mathcal{F}_{C,A}$, we can compose with the map from $\mathcal{F}_{C,A}$ into $\mathcal{O}^\oplus_{S^r+1}$ and get the following commutative diagram, where $G$ is the cokernel:

$$
\begin{array}{ccc}
0 & \to & M \\
\downarrow & & \downarrow \phi \\
0 & \to & \mathcal{F}_{C,A} \\
\downarrow & & \downarrow \phi \\
0 & \to & \mathcal{O}^\oplus_{S^r+1} \\
\end{array}
$$

We see that $G$ is torsion-free, since any possible torsion element of $G$ would map to 0 in $A$, and by the snake lemma, $\ker(\phi) \cong N$, which is torsion-free. We have $\text{rk}(G) = 1$ and $c_1(G) = \mathcal{O}_S(1).$

Note that $h^0(N) = 0$, since $N$ is torsion-free of rank 1 and has negative $c_1$. This implies that the image of each global section under the map $\phi$ must be nonzero in $A$. Furthermore, since $\text{ev} : \mathcal{O}^\oplus_{S^r+1} \to A$ is surjective on global sections, the same must apply for $\phi : G \to A$. It follows that $h^0(S, G) = h^0(C, A) = r + 1.$

Since $\text{rk}(G) = 1$, then $G = \mathcal{O}_S(1) \otimes I_{\xi}$, where $I_{\xi}$ is the ideal sheaf of a finite subscheme $\xi$ (nonempty because $h^0(S, \mathcal{O}_S(1)) > r + 1$). Let $A' \in |A|$. We prove that $A' + \xi' = H'_C$, where $\xi' := \xi \cap C$ and $H'$ is a hyperplane section. The map $\phi$ corresponds to a nonzero element in $\text{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(1) \otimes I_{\xi}, A) = \text{Hom}_{\mathcal{O}_S}(I_{\xi}, A \otimes \mathcal{O}_S(-1)).$ This implies that $h^0(S - \xi, A \otimes \mathcal{O}_S(-1)) > 0$, i.e., $A' > H'_C - \xi'$ for some hyperplane section $H'$. Since $A$ is base-point free, then this means that either $A' = H'_C - \xi'$ or $h^0(C, A) > h^0(C, \mathcal{O}_C(1) \otimes I_{\xi'}).$ However, since $h^0(C, A) = h^0(S, \mathcal{O}_S(1) \otimes I_{\xi})$, the latter inequality would imply that $\mathcal{O}_S(1) \to \mathcal{O}_C(1)$ has a nonzero kernel on global sections, which cannot be the case since we are assuming that $C \in \mathcal{O}_S(m)$ with $m \geq 5$. It thus follows that $A' = H'_C - \xi'$. Since $|A|$ is a $g^d_0$, then the base-locus of the hyperplane sections containing $\xi'$ must be an $(n - r - 1)$-plane multisecant. \hfill \Box

Remark 2.2. It would be interesting to see if this result can be extended to $3 \leq r \leq H^0(\mathcal{O}_S(1)) - 2$. Most of the methods used in Lemma 2.1 seem to work, but with the exception that we lose control over $c_2(M)$. However, if we manage to prove that $c_2(M) \geq 0$ in some (or all) of these cases, then Theorem 1.1 would also apply for these values of $r$.

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References

[1] B. Basili. Indice de Clifford des intersections complètes de l’espace. Bull. Soc. Math. France, 124(1):61–96, 1996.
[2] F.A. Bogomolov. Stable vector bundles on projective surfaces. *Sb. Math.*, 81(2):397–419, 1995.

[3] C. Ciliberto and R. Lazarsfeld. On the uniqueness of certain linear series on some classes of curves. In *Complete Intersections*, pages 198–213. Springer, 1984.

[4] R. Hartshorne and E. Schlesinger. Gonality of a general ACM curve in $\mathbb{P}^3$. *Pacific J. Math.*, 251(2):269–313, 2011.

[5] S.O. Kim. Noether–Lefschetz locus for surfaces. *Trans. Amer. Math. Soc.*, 324(1):369–384, 1991.

[6] R. Lazarsfeld. Brill–Noether–Petri without degenerations. *J. Diff. Geom.*, 23:299–307, 1986.

[7] S. Lefschetz. On certain numerical invariants of algebraic varieties with application to abelian varieties. *Trans. Amer. Math. Soc.*, 22(3):327–406, 1921.

[8] A.N. Tyurin. Cycles, curves and vector bundles on an algebraic surface. *Duke Math. J.*, 54(1):1–26, 1987.

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