Cost of Guessing: Applications to Distributed Data Storage and Repair

Suayb S. Arslan, Member, IEEE, and Elif Haytaoglu, Member, IEEE

Abstract

In this paper, we introduce the notion of cost of guessing and provide an optimal strategy for guessing a random variable taking values on a finite set whereby each choice may be associated with a positive finite cost value. Moreover, we drive asymptotically tight upper and lower bounds on the moments of cost of guessing problem. Similar to previous studies on the standard guesswork, established bounds on the moments quantify the accumulated cost of guesses required for correctly identifying the unknown choice and are expressed in terms of the Rényi’s entropy. A new random variable is introduced to bridge between cost of guessing and the standard guesswork. Finally, we establish the guessing cost exponent on the moments of the optimal guessing by considering a sequence of random variables. Furthermore, these bounds are shown to serve quite useful for bounding the overall repair latency cost (data repair complexity) for distributed data storage systems in which sparse graph codes may be utilized.

Index Terms

Guessing, entropy, moments, bounds, sparse graph codes, data repair.

I. INTRODUCTION

The classical question of guessing involves finding the value of a realization of a random variable \( X \) from a finite set \( \mathcal{X} \) by asking a sequence of questions "Is \( X \) equal to \( x \in \mathcal{X} \)?" until the answer becomes “Yes”. In association with these questions, an optimal guessing strategy i.e., a bijective function from \( \mathcal{X} \) to \([|\mathcal{X}|]=\{1,\ldots,|\mathcal{X}|\}\) is adapted to minimize the average number of guesses. In [1], this problem is named as Guesswork and Massey established a lower bound on guessing number in terms of Shannon’s entropy for the first time [2]. Later, asymptotically tight bounds are derived on the moments of the expected number of guesses for a typical guesswork [3]. This study has related the asymptotic exponent of the best achievable guessing moment to the Rényi’s entropy. Such findings are successfully applied to various recent applications of data compression [4], channel coding [5], networking and data storage security [6] by tweaking the original problem so that it fits within the requirements of the application at hand.

Making a guess about the unknown value of a random variable (even in presence of a side information [3]) leads to a certain amount of cost. Therefore, making a choice among multiple possibilities may lead to different types and amounts of costs overall. In fact, these costs may dynamically be changing after making subsequent guesses about a series of random variables \( X_i \) distributed identically but not independently. Independent and identically distributed random variable case is thoroughly studied and some extensions to Markovian dependencies are also considered [7]. To our best of knowledge, the cost of guessing is only mentioned recently in [4] in a limited context whereby the guesser is allowed to stop guessing and declare an error.

Applications of such a problem are abundant. In a distributed system for instance, cost of data communication depends on the link loads, node availabilities and current traffic at the time of communication etc. These costs can be expressed in terms of latency, bandwidth used to transfer information or computation complexity. Inspired from these observations in this study, we proposed a generalization of the guessing framework and derive asymptotically tight bounds by using a quantity related to the Rényi’s entropy. Furthermore, we consider a distributed data storage scenario in which nodes are repaired in case of failures or unexpected departures from the network using graph codes such as low density parity check (LDPC) codes [8].

II. PROBLEM STATEMENT AND GUESSING STRATEGY

Let us use \( C_G(x) \) to denote the total cost of guessing required by a particular guessing strategy \( G \) when \( X = x \). If the cost of making each guess \( X = x \) is independent of other guesses and amounts to 1 (unity), then this problem would be the same as the characterization of the average number guesses (average guessing number) and is identical to Massey’s original guessing problem [2]. Later, bounds on the moments of optimal guessing are derived [3] and improved [9], [10]. Particularly, the relationship between Rényi’s entropy and expected guessing number is interesting and useful in different engineering contexts.

Let us assume that the random variable \( X \) can take on values from a finite set \( \mathcal{X} = \{x_1, \ldots, x_M\} \) according to a distribution \( P_X(x) \) with costs \( C = \{c_{x_1}, \ldots, c_{x_M}\} \). Sets have cardinalities \( |\mathcal{X}| = |C| = M \) in which using a particular guessing strategy \( G \),
the probability that a randomly selected element of $X$ can be found in the $i$-th guess is $p_i = P_X(G^{-1}(i))$ with cost $c_i = c_{G^{-1}(i)}$, independently of already made guesses. Then, the average cost of guessing can be expressed as follows

$$E[C_G(X)] = \sum_{i=1}^{M} \sum_{j=1}^{M} c_{j}p_{i} = \sum_{i=1}^{M} f_{i}p_{i},$$  
\hspace{1cm} (1)

where $f_{i} = \sum_{j=1}^{i} c_{j}$ and $g_{i} = \sum_{j=1}^{i} p_{j}$. The minimization of this value is a function of both guessing strategy and the probability distribution of $X$.

One of the questions is the best guessing strategy that would minimize $E[C_G(X)]$. In case of $c_i = 1, \forall i \in [M]$, the strategy is simple i.e., guess the possible values of $X$ in the order of non-increasing probabilities [2]. In other words, without loss of generality, we can assume $p = (p_1, p_2, \ldots, p_M)$ with $p_1 \geq p_2 \geq \cdots \geq p_M$ are the probabilities of choosing values from $[M]$ and $G(x_i) = i$. Then with this choice, $\sum_i ip_i$ would be minimized. However, the same conclusion could not be drawn where an arbitrary vector of costs $c = (c_1, \ldots, c_M)$ is present. Let us consider two possible scenarios.

A. Configurable Costs Determined A Posteriori

In this particular scenario, although the cost vector is given i.e., $\{c_i\}$, the assignments are not made i.e., costs can be associated with each choice as it fits in the beginning. In this case, the best strategy is to guess the possible values of $X$ in the order of non-increasing probabilities and associate the more probable choice with the least cost value. In other words, for the assignment (permutation of $c$) $\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_M)$ with $\tilde{c}_1 \leq \tilde{c}_2 \leq \cdots \leq \tilde{c}_M$ and $\tilde{c}_i \in c$, it is easy to see that

$$\sum_{i=1}^{M} \sum_{j=1}^{i} c_jp_i \geq \sum_{i=1}^{M} \sum_{j=1}^{i} \tilde{c}_jp_i.$$  
\hspace{1cm} (3)

In case the cumulative costs are given by the moments of the guess number i.e., $f_{i} = i^\rho$ for any $\rho \geq 1$, then it is easy to see that $c_i = i^\rho - (i - 1)^\rho$ which implies that $c_1 \leq c_2 \leq \cdots \leq c_M$ is satisfied. Thus, the best strategy would again be to guess the possible values of $X$ in the order of non-increasing probabilities as argued in [3].

B. Non-configurable Costs Determined A Priori

In this case, cost associated with each choice is determined externally. In other words, costs and choices are bound and determined prior to guessing. In this case, the best strategy would not necessarily be guessing the possible values of $X$ in the order of non-increasing probabilities. Consider for instance three choices i.e., $M = 3$ with $(1, p_1 = 0.5, c_1 = 20)$, $(2, p_2 = 0.4, c_2 = 2)$ and $(3, p_3 = 0.1, c_3 = 1)$. In that case the guessing order $(2, 3, 1)$ would be preferable than $(1, 2, 3)$ with average costs 12.6 and 21.1, respectively, where the latter choice, based on the order of non-increasing probabilities, is clearly not optimal. The following proposition establishes a necessary condition for the optimal guessing strategy $G^*$.

**Proposition 2.1.** Using the optimal guessing strategy, namely $G^*$, we always have $c_i p_j \leq c_j p_i$ for all $i, j \in \{1, \ldots, M\}$ satisfying $i \leq j$.

**Proof.** Consider swapping the $i$-th and $(i+1)$-th guessed values. Let $G_{i,i+1}$ be the original guessing strategy and $G_{i+1,i}$ be the swapped version. Then it is straightforward to show that the difference is

$$E[C_{G_{i,i+1}}(x)] - E[C_{G_{i+1,i}}(x)] = c_i(1 - g_{i-1})$$  
$$+ c_{i+1}(1 - g_{i-1} - p_i)$$  
$$- c_{i+1}(1 - g_{i-1})$$  
$$- c_i(1 - g_{i-1} - p_{i+1})$$  
$$= c_i p_{i+1} - c_{i+1} p_i$$  
\hspace{1cm} (4)

which implies that if $c_i p_{i+1} > c_{i+1} p_i$, then we swap $i$-th and $(i+1)$-th guessed values in order to reduce the average cost of guessing, otherwise no swapping is performed.

Since each swapping leads to lower cost, for any $i, j \in \{1, \ldots, M\}$ with $i \leq j$, the optimal guessing strategy $G^*$ would satisfy

$$c_i p_{i+1} \leq c_{i+1} p_i$$  
$$c_{i+1} p_{i+2} \leq c_{i+2} p_{i+1}$$  
$$\vdots$$  
$$c_{j-1} p_{j} \leq c_{j} p_{j-1}$$  
\hspace{1cm} (9)

(10)
Algorithm 1

1: function OptimalCostGuess(p, c)
2: \( M \leftarrow |p| \) \( \triangleright \) Selection Order
3: \( I \leftarrow \{1, 2, 3, \ldots, M\} \)
4: swapped \( \leftarrow \text{true} \)
5: \( i \leftarrow 1 \)
6: while swapped do
7: \( \text{swapped} \leftarrow \text{false} \)
8: for \( j = 1 : M - i \) do
9: if \( c_{j}p_{j+1} > c_{j+1}p_{j} \) then \( \triangleright \) If condition holds
10: swap(c_{j}, c_{j+1}), swap(p_{j}, p_{j+1}), swap(I_{j}, I_{j+1})
11: swapped \( \leftarrow \text{true} \)
12: end if
13: end for
14: \( i \leftarrow i + 1 \)
15: end while
16: return \( I \)

where multiplying left-hand terms and right-hand terms individually would give us the desired result since all \( p_{i} \)s and \( c_{i} \)s are non-negative.

In observation of above, let us provide an algorithmic solution to optimal guessing. We notice that if the swapping in proposition is executed within a Bubble-sort\(^1\) style for the given strategy, the convergence is guaranteed and we can find the optimal strategy with the best and the worst time complexities of \( \Omega(M) \) and \( \Theta(M^2) \), respectively. The naive algorithm for finding the optimal cost of guessing is provided in Algorithm 1 where swap(.,.) function swaps the entries of a given array in the argument.

In the next section, we focus on the moments of the cost of guessing whereby the average cost would a special case. Furthermore, lower and upper bounds are derived in terms of a popular information theoretic measure, namely Rényi’s entropy.

III. Bounds on Moments of the Cost of Guessing

Throughout this section, we assume static costs determined a priori and focus on moments of guessing as the average cost of guessing would be a special case.

A. Lower and Upper Bounds

Let \( P_{X}(x) \) to denote the probability distribution of \( X \) and define the moments of the cost of guessing using a particular guessing function \( G \) as

\[
\mathbb{E}[C_{G}(X)^{\rho}] = \sum_{i} P_{X}(G^{-1}(i)) \left[ \sum_{j} c_{G^{-1}(j)} \right]^{\rho}
\]  

(13)

where the costs are not necessarily integers. Let us use the previous notation \( c_{i} = c_{G^{-1}(i)} \) and define \( e^{*} = \{c_{1}^{*}, c_{2}^{*}, \ldots, c_{M}^{*}\} \) which is the order obtained by running Algorithm 1 for optimal guessing strategy \( G^{*} \). This shall be useful in expressing the lower and upper bounds in the following.

Theorem 3.1. For any guessing function \( G \), \( \rho > 0 \) and costs \( c_{j} > 1 \), \( \rho \)-th moment of the cost of guessing is lower bounded by

\[
\mathbb{E}[C_{G}(X)^{\rho}] \geq \mathbb{E}[C_{G^{*}}(X)^{\rho}] \\
\geq \left( \frac{1 + \gamma^{*}}{M} \right)^{\rho} \exp \left\{ \rho H_{1} \left( \frac{1}{\gamma^{*}} \right) \right\} 
\]  

(14)

where \( \gamma^{*} \) is the harmonic mean of \( \{\sum_{j} c_{j}^{*} - 1\} \)'s for \( i = \{1, 2, \ldots, M\} \) and \( H_{\alpha}(X) \) is Rényi’s entropy of order \( \alpha \).

Proof. Let us start with an inequality. Let \( a_{i} \) be a positive real number for all \( i \), \( M \) be a natural number, and \( \gamma \) be the harmonic mean of \( \{a_{1}, \ldots, a_{n}\} \), then we have

\[
\sum_{i=1}^{M} \frac{1}{1 + a_{i}} \leq \frac{M}{1 + \gamma}
\]  

(15)

\(^{1}\)Bubble-sort is a sorting algorithm that works by repeatedly swapping the adjacent elements in a given list based on a condition.
which can easily be proved using Radon’s inequality \[11\]. Now, let us express the lower bound of the moments of the cost of guessing as follows,

\[
\mathbb{E}[C_G(X)^\rho] \geq \mathbb{E}[C_{G^*}(X)^\rho]
\]

\[
\geq \left( \sum_i \frac{1}{\sum_j c_j^{\alpha}} \right)^{-\rho} \left( \sum_i P_X(G^{-1}(i))^{\frac{1}{1+\rho}} \right)^{1+\rho}
\]

which easily follows from Arikan’s work \[3\]. In fact this can be shown by a direct application of Hölder’s inequality. Let us remember Hölder inequality stated as follows.

**Lemma 3.2 (Hölder’s inequality).** Let \(a_i\) and \(b_i\) for \((i = 1, \ldots, n)\) be positive real sequences. If \(q > 1\) and \(1/q + 1/r = 1\), then

\[
\left( \sum_i a_i^q \right)^{1/q} \left( \sum_i b_i^r \right)^{1/r} \geq \sum_i a_i b_i
\]

Let us set \(r = 1 + \rho\), \(q = (1 + \rho)/\rho\) so that \(1/q + 1/r = 1\) is satisfied for \(\rho > 0\). We also let

\[
a_i = \left( \sum_j c_j^{-1}(j) \right)^{-\rho/(1+\rho)}
\]

\[
b_i = \left( \sum_j c_j^{-1}(j) \right)^{\rho/(1+\rho)} P_X(G^{-1}(i))^{1/(1+\rho)}
\]

Now, using Hölder’s inequality, we have

\[
\left[ \sum_i \frac{1}{\sum_j c_j^{-1}(j)} \right]^{\rho/(1+\rho)} (\mathbb{E}[C_G(X)^\rho])^{1/(1+\rho)} \geq \sum_i P_X(G^{-1}(i))^{1/(1+\rho)}
\]

from which inequality \[17\] follows for the optimal strategy \(G^*\).

Now, considering the ordering of costs that minimizes the right hand side, we shall have,

\[
\mathbb{E}[C_G(X)^\rho] \geq \left( \frac{M}{1+\gamma^*} \right)^{-\rho} \left[ \sum_i P_X(x_i) \right]^{1+\rho}
\]

\[
= \left( \frac{1+\gamma^*}{M} \right)^\rho \exp \left\{ \rho H_{1+\gamma^*}(X) \right\}
\]

where \(\gamma^*\) is the harmonic mean of \(\{\sum_j c_j^{-1}\}\)’s for \(i = 1, 2, \ldots, M\) and \(H_\alpha(X)\) is Rényi’s entropy of order \(\alpha (\alpha > 0, \alpha \neq 1)\) for random variable \(X\) defined as,

\[
H_\alpha(X) = \frac{\alpha}{1-\alpha} \ln \left( \sum_x P_X(X)^\alpha \right)^{1/\alpha}
\]

Note that inequality \[22\] followed from the inequality \[15\].

Let us demonstrate that the bound given in Theorem 3.3 is tight within a factor of \((M/(1+\gamma^*))^\rho\).

**Theorem 3.3.** For the optimal guessing function \(G^*\), and \(\rho \geq 0\), \(\rho\)-th moment of the cost of guessing is upper bounded by

\[
\mathbb{E}[C_{G^*}(X)^\rho] \leq \exp \left\{ \rho H_{1+\gamma^*}(Y) \right\}
\]

where the random variable \(Y\) is defined to take on values from a finite set \(\mathcal{Y} = \{y_1, y_2, \ldots, y_{\sum x_i e_x^*} \}\) with probabilities \(P_Y(y) = P_X(x)/[e_x]\) for all \(y\) satisfying

\[
\frac{x-1}{x'} e_{x'} \leq y \leq \frac{x}{x'} e_{x'}
\]
and $H_\alpha(X)$ is Rényi’s entropy of order $\alpha$.

**Proof.** Let us first observe that with the optimal guessing strategy $G^*$ that minimizes the expected cost of guessing $x$,

$$C_{G^*}(x) = \sum_{x' : C_{G^*}(x') \leq C_{G^*}(x)} \sum_{x''} 1 \quad (27)$$

$$\leq \sum_{x' : C_{G^*}(x') \leq C_{G^*}(x)} \sum_{x''} c_{x'} P_X(x'') / c_{x'} P_X(x) \quad (28)$$

$$\leq \sum_{x' : C_{G^*}(x') \leq C_{G^*}(x)} \sum_{x''} \left( c_{x'} P_X(x'') / c_{x'} P_X(x) \right) \frac{1}{1+\rho} \quad (29)$$

$$= \sum_{x'} \left( c_{x'} P_X(x') / P_X(x) \right) \frac{1}{1+\rho} \quad (30)$$

where the inequality (29) follows from the necessary condition $c_{x'} P_X(x') \leq c_x P_X(x)$ for all $\{x' : C_{G^*}(x') \leq C_{G^*}(x)\}$ that needs to hold for the optimal strategy $G^*$. Also, although the exponent $1/(1+\rho)$ decreases the value, it is still greater than 1 due to $c_{x'} P_X(x') / c_x P_X(x) \geq 1$. Using the inequality given in (31) in equation (13), we get

$$E[C_{G^*}(X)^\rho] = \sum_x P_X(x) C_{G^*}(x)^\rho \quad (32)$$

$$\leq \sum_x P_X(x) \left( \sum_{x'} c_{x'}^{1/\rho} \left( c_{x'} P_X(x') / P_X(x) \right) \frac{1}{1+\rho} \right)^\rho = \left[ \sum_x c_{x'}^{1/\rho} P_X(x) \right]^{1+\rho} \quad (33)$$

$$= \left[ \sum_x c_x (P_X(x)/c_x)^{1/\rho} \right]^{1+\rho} \quad (34)$$

On the other hand, we notice that

$$\frac{P_X(x)}{c_x} = c_x P_X(x) / \sum_x c_x \geq \frac{P_X(x)}{c_x} \left( \frac{c_x}{\sum_x c_x} \right)^{1+\rho} \quad (35)$$

from which the following inequality follows for $\rho \geq 0$,

$$\left[ c_x \right] (P_X(x)/c_x)^{1/\rho} \geq c_x (P_X(x)/c_x)^{1/\rho} \quad (36)$$

Thus, using the inequality (36) and the defined random variable $Y$ earlier, we finally express the upper bound in a more compact form

$$E[C_{G^*}(X)^\rho] \leq \left[ \sum_x [c_x] (P_X(x)/c_x)^{1/\rho} \right]^{1+\rho} \quad (37)$$

$$\leq \left[ \sum_x [c_x] (P_X(x)/c_x)^{1/\rho} \right]^{1+\rho} = \left[ \sum_y P_Y(y)^{1/\rho} \right]^{1+\rho} \quad (38)$$

$$= \exp\{\rho H_{1/\rho}(Y)\} \quad (39)$$

Notice that this upper bound will reduce to Arikan’s $\exp(\rho H_{1/\rho}(X))$ with all costs set to unity.

**B. Relation to Guesswork and Guessing Cost Exponent**

Introduction of the random variable $Y$ is useful for establishing a relationship with the standard guesswork. From the earlier discussions on the random variable $Y$, we can express a looser lower bound (compared to (14)) for any guessing function $G(.)$ by observing the following,

$$E[C_Y(X)^\rho] \geq E[C_H(Y)^\rho] \geq \left( 1 + \ln \left( \sum_x [c_x] \right) \right)^{-\rho} \exp\{\rho H_{1/\rho}(Y)\} \quad (40)$$
which follows directly from the standard Guesswork [3] and the definition of the random variable $Y$. Better lower bounds can be given, however this loose lower bound is enough to prove the following asymptotically tight result. Here the guessing function $H(Y)$ for the random variable $Y$ defined earlier is directly induced from $G(X)$. The guessing cost exponent is given by the following theorem.

**Theorem 3.4.** Let $X = (X_1, \ldots, X_n)$ be a sequence of i.i.d. random variables over $X$ where each is defined with the same cost distribution $C$. Let $G^*(X_1, \ldots, X_n)$ be an optimal guessing function for $X$. Then, for any $\rho > 0$, we have

$$
\lim_{n \to \infty} \frac{1}{n} \ln \left( E[C_{G^*}(X_1, X_2, \ldots, X_n)^{\rho}] \right)^{1/\rho} = H_{1/\rho}(Y)
$$

\[(41)\]

**Proof.** Let us define random variables $Y_i \sim Y$ as in Theorem 3.3 for the corresponding random variable $X_i$ for $i = 1, \ldots, n$. Now consider the upper bound for i.i.d. random variables and observe

$$
C_{G^*}(x_1, \ldots, x_n) \leq \sum_{x_1^i, x_2^i, \ldots, x_n^i : x^i_1 \leq \cdots \leq c_{x_n^i}} \left( \prod_{i} c_{x_i} P_{X}(x_i) \right)^{1/\rho} \leq \left[ \sum_{x_1^i} c_{x_1} \frac{P_{X}(x_1)}{c_{x_1}} \right]^{n(1+\rho)}
$$

\[(42)\]

Due to independence and series of inequalities $c_{x_1} P_{X}(x_1) \leq c_{x_2} P_{X}(x_1), c_{x_2} P_{X}(x_2) \leq c_{x_3} P_{X}(x_2), \ldots, c_{x_n} P_{X}(x_n)$ for all $\{x_i : G^*(x_i) \leq c_{G^*}(x_i)\}$ where $i = 1, \ldots, n$ that needs to hold for the optimal strategy $G^*$ required by the necessary condition. Finally, we can upper bound the expected guessing cost for a sequence of i.i.d. random variables as

$$
E[C_{G^*}(X_1, \ldots, X_n)^{\rho}] = \sum_{x} P_{X}(x_1, \ldots, x_n) C_{G^*}(x_1, \ldots, x_n)^{\rho}
$$

\[(44)\]

$$
\leq \left[ \sum_{x_1} c_{x_1} P_{X}(x_1)/c_{x_1} \right]^{n(1+\rho)}
$$

\[(45)\]

$$
\leq \left[ \sum_{y} P_{Y_1}(y_1) \right]^{n(1+\rho)} = \exp \left\{ \rho n H_{1/\rho}(Y_1) \right\}
$$

\[(46)\]

where the last inequality follows due to inequalities similar to (36) for each random variable $X_i$. In addition we can extend the lower bound given in (40) for a sequence of random variables as

$$
E[C_{G^*}(X_1, \ldots, X_n)^{\rho}] \geq E[C_{H^*}(Y_1, \ldots, Y_n)^{\rho}] \geq \left( 1 + n \ln \left( \sum_{x} \left| c_{x} \right| \right) \right)^{-1/n} \exp \left\{ \frac{\rho n H_{1/\rho}(Y_1)}{1+\rho} \right\}
$$

\[(47)\]

where the first inequality can be shown to be true through induction and the second inequality directly follows from [3]. Note that $H^*$ is the optimal induced strategy from $G^*$. As a consequence, using (47) we have

$$
\lim_{n \to \infty} \frac{1}{n} \ln \left( E[C_{G^*}(X_1, X_2, \ldots, X_n)^{\rho}] \right)^{1/\rho} \geq \lim_{n \to \infty} \ln \left( 1 + n \ln \left( \sum_{x} \left| c_{x} \right| \right) \right)^{-1/n} + H_{1/\rho}(Y_1)
$$

\[(48)\]

Combining equations (47) with (40), the intended result follows.

The above result indicates that the complexity of guessing cost of a random variable $X$ with strategy $G$ can be tied to the complexity of guessing another random variable $Y$ with strategy $H$ as defined earlier.

**IV. AN APPLICATION: LONG BLOCK LENGTH SPARSE GRAPH CODES WITH A BACK-UP MASTER**

Let us consider a master-slave configuration for a distributed data storage scenario in which the data protection is provided by a long block length $(n, k)$ sparse graph code whereby each slave node stores a single symbol. In addition, a master node constitutes a back-up system and keeps the copy of all symbols. In case of a slave node failure, there would be multiple options of repair. To be able to maintain instantaneous reliability, it may not be possible to get all failure information quickly within the same network (due to other failures or network link breakages) or else it may be too time and bandwidth costly to contact the master for that information. Thus, the failed node needs to adapt the best guessing strategy and choose among the multiple repair options to complete the repair process as quickly as possible.

Let us suppose one of the symbols shown as gray-colored node in Fig. 1 is to be repaired the degree of which is assumed to be $d_v$. Suppose it is connected to check nodes of degrees $d_{c_1}, d_{c_2}, \ldots, d_{c_{d_v}}$, as shown. We define the costs associated with

\[\text{Fig. 1: Master-slave configuration.}\]

"..."
approximated by \( n \) block length (number of nodes \( X \) random variables). Let \( G \) right check node for a successful repair. For instance \( X \) where \( c \) guessing strategy. Note that such a constraint naturally places a lower bound on \( c \) i.e., the cost of downloading the lost symbol from the master is large enough so that it is contacted at the end in the optimal downloads). On the other hand, it is possible that none of the check relations would be able to help with the repair process make sure that the condition in Proposition 2.1 is satisfied and hence we minimize the average cost (the number of symbol the example irregular LDPC code, using optimal guessing strategy, it can be shown that (using Proposition 2.1)

\[
p_j = (1 - q)^{c_j} \prod_{i=1}^{j-1} (1 - (1 - q)^{c_i}).
\]

(50)

Using these probabilities, it can be shown that if the guesses are made based on costs ordered in ascending order, we make sure that the condition in Proposition 2.1 is satisfied and hence we minimize the average cost (the number of symbol downloads). On the other hand, it is possible that none of the check relations would be able to help with the repair process where the back-up master may complete it with success probability \( p_M \triangleq 1 - \sum_{j=1}^{d_v} p_j \) where \( M = d_v + 1 \). Finally, \( c_M \geq c_j \) i.e., the cost of downloading the lost symbol from the master is large enough so that it is contacted at the end in the optimal guessing strategy. Note that such a constraint naturally places a lower bound on \( c_M \) in terms of \( q \) and costs. For instance for the example irregular LDPC code, using optimal guessing strategy, it can be shown that (using Proposition 2.1)

\[
c_M \geq c_{\max} \left( (1 - q)^{c_{\max}} - 1 \right)
\]

(51)

where \( c_{\max} = \max\{c_1, \ldots, c_{d_v}\} \). Now, for any code symbol we associate a random variable \( X_v \) which will identify the right check node for a successful repair. For instance \( X_v = 3 \) indicates that the \( 3^{rd} \) check relation is the first possibility for a successful recovery. Let \( G^*(X_1, X_2, \ldots, X_n) \) denote the guessing function for the value of a joint realization of i.i.d. random variables \( X_1, X_2, \ldots, X_n \) where each represents one of the code symbols. Then due to Theorem 3.4 for large enough block length (number of nodes \( n \) tends large), the moments of repair latency (cost) using the optimal guessing can be well approximated by

\[
E[C_{G^*}(X_1, X_2, \ldots, X_n)^{\rho}] \approx \exp\{n\rho H_{\frac{1}{1+\rho}}(Y)\}
\]

(52)

where the random variable \( Y \) is defined to take on values from a finite set \( \mathcal{Y} = \{y_1, y_2, \ldots, y_M\} \) with probabilities \( P_Y(y) = p_{y_j}/c_j \) for all \( y \) satisfying \( \sum_{i=1}^{j-1} c_i < y \leq \sum_{i=1}^{j} c_i \) and \( (p_{y_j}, c_j) \) are sorted version in ascending order of costs as argued before. Note that this formulation directly applies to both regular and irregular LDPC codes. Further comparisons between short block length regular and irregular LDPC constructions on the basis of moments of guessing cost for different interpretations of the cost function is one of our future works.

\footnote{Here, due to large block length assumption, it is assumed that subsequent guesses cannot help each other. In addition, other cost metrics can be used.}

\footnote{In a more general version of the problem, the costs of the check nodes may take values independent of the degrees (e.g., the communication cost required for obtaining a variable node may be different).}

| \( \frac{1}{2} \ln E[C_{G^*}(X)^2] \) | \( \frac{1}{3} \ln E[C_{G^*}(X)^3] \) |
|-----------------|-----------------|
| LB (Eq. (10))  | 4.4782          | 4.5128          |
| UB (Eq. (21))  | 6.0195          | 6.0806          |
| UB (Eq. (40))  | 5.4735          | 5.6213          |
| Exact Value    | 5.1697          | 5.3693          |

TABLE I

LOWER AND UPPER BOUNDS FOR \( \frac{1}{2} \ln E[C_{G^*}(X)^2] \) AND \( \frac{1}{3} \ln E[C_{G^*}(X)^3] \): UB: UPPER BOUND, LB: LOWER BOUND.

Fig. 1. An example repair process using an LDPC code Tanner graph. \( d_v \) represents the degree number of the lost symbol whereas the \( d_{c_1}, \ldots, d_{c_v} \) are the degrees of the potential repair check relations.

Each choice to be the number of downloaded symbols, i.e., \( c_j \triangleq d_{c_j} - 3 \) for \( 1 \leq j \leq M - 1 \). On the other hand, assume that each slave node is unavailable/failed with probability \( q > 0 \). The probability that \( j \)-th check node will successfully repair the gray-colored node can be shown to be of the form\(^3\)

\[
p_j = (1 - q)^{c_j} \prod_{i=1}^{j-1} (1 - (1 - q)^{c_i}).
\]

(50)

Using these probabilities, it can be shown that if the guesses are made based on costs ordered in ascending order, we make sure that the condition in Proposition 2.1 is satisfied and hence we minimize the average cost (the number of symbol downloads). On the other hand, it is possible that none of the check relations would be able to help with the repair process where the back-up master may complete it with success probability \( p_M \defeq 1 - \sum_{j=1}^{d_v} p_j \) where \( M = d_v + 1 \). Finally, \( c_M \geq c_j \) i.e., the cost of downloading the lost symbol from the master is large enough so that it is contacted at the end in the optimal guessing strategy. Note that such a constraint naturally places a lower bound on \( c_M \) in terms of \( q \) and costs. For instance for the example irregular LDPC code, using optimal guessing strategy, it can be shown that (using Proposition 2.1)

\[
c_M \geq c_{\max} \left( (1 - q)^{c_{\max}} - 1 \right)
\]

(51)
V. CONCLUSIONS AND FUTURE WORK

Before we conclude, let us provide several numerical results to be able to illustrate how close the provided bounds are for finite values of costs, \( \rho \) and \( M \). The exact moments for the optimal guessing strategy are calculated using Algorithm 1. The results are provided in Table 1 and these results indicate that lower and upper bounds approximate the actual results well. More specifically, we consider the second and third moments where the exact values of \( \frac{1}{2} \ln \mathbb{E}[C_{G^*}(X)^2] \) and \( \frac{1}{3} \ln \mathbb{E}[C_{G^*}(X)^3] \) and their lower and upper bounds are calculated and compared. The probability of each choice is generated using geometric distribution as assumed in [10] with the restricted probability distribution \( P_X(x) = (1-a)\frac{a^{x-1}}{(1-a^M)} \) with \( M = 32 \) and the parameter \( a = 0.9 \). The non-integer cost values are generated based on a normal distribution with mean and variance \( \mu = \sigma^2 = 16 \).

One other observation from the numerical results is that the provided bounds have the potential for improvement particularly in the non-asymptotic regime similar in spirit to works such as [9] and [10]. For instance, we make the following conjecture for the upper bound,

\[
\mathbb{E}[C_{G^*}(X)^\rho] \leq \frac{1}{1+\rho} \left[ \exp \left\{ \rho H_{\frac{1+\rho}{\rho+1}}(Y) \right\} - 1 \right] 
+ \exp \left\{ (\rho - 1)^+ H_{1/\rho}(Y) \right\}
\]

where \((z)^+ \triangleq \max\{z, 0\}\) for \(z \in \mathbb{R}\). This result is also included in Table 1 for comparison purposes. As can be seen, we can tighten up the upper bound with this conjecture, particularly for small \( \rho \).

Finally, it is of interest in a distributed storage protocol design to consider giving up on the guessing the next value based on a condition such as the total accumulated cost. Characterization of the cost of guessing in that case would have to be expressed in terms of smooth Rényi’s entropy. Recent studies such as [4] considered similar constraints for the standard guesswork within the context of source coding.

REFERENCES

[1] J. O. Pliam. (1999) The Disparity Between Work and Entropy in Cryptology. Available Online: [http://philby.ucsd.edu/cryptolib/1998/98-24.html](http://philby.ucsd.edu/cryptolib/1998/98-24.html)

[2] J. L. Massey, “Guessing and entropy,” in Proc. IEEE Int. Symp. on Information Theory Trondheim, Norway, 1994, pp. 204.

[3] E. Arikan, “An inequality on guessing and its application to sequential decoding,” IEEE Trans. Inform. Theory, vol. 42, pp. 99–105, Jan. 1996.

[4] S. Kuzuoka, “On the Conditional Smooth Rényi Entropy and Its Application in Guessing,” IEEE International Symposium on Information Theory (ISIT), Paris, France, 2019, pp. 647-651.

[5] K. R. Duffy, J. Li and M. Médard, “Capacity-Achieving Guessing Random Additive Noise Decoding,” in IEEE Transactions on Information Theory, vol. 65, no. 7, pp. 4023-4040, July 2019.

[6] Bracher, Annina, Eran Hof, and Amos Lapidoth. “Guessing attacks on distributed-storage systems.” IEEE Transactions on Information Theory 65.11 (2019): pp. 6975-6998.

[7] D. Malone and W. G. Sullivan, “Guesswork and entropy,” in IEEE Transactions on Information Theory, vol. 50, no. 3, pp. 525-526, March 2004.

[8] R. Gallager, “Low-density parity-check codes,” in IRE Transactions on Information Theory, vol. 8, no. 1, pp. 21-28, January 1962.

[9] S. Boztas, “Comments on “An inequality on guessing and its application to sequential decoding”,” in IEEE Transactions on Information Theory, vol. 43, no. 6, pp. 2062-2063, Nov. 1997.

[10] I. Sason and S. Verdú, “Improved Bounds on Guessing Moments via Rényi Measures,” IEEE International Symposium on Information Theory (ISIT), Vail, CO, 2018, pp. 566-570.

[11] W. K. Lai and E. Kim, “Some inequalities involving geometric and harmonic means,” In International Mathematical Forum, (2016), Vol. 11, No. 4, pp. 163-169.