Measurability of Electromagnetic Field: Model and Path Integral Methods*

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Abstract

The problem of the measurability of the electromagnetic field is investigated 1) in the framework of the abstract restricted-path-integral method, and 2) by explicitly accounting the action of the field onto the meter and its back reaction. The meaning of the previously obtained results as well as of the classical results of Bohr and Rosenfeld are made clear. The restricted-path-integral method with integration over field configurations is shown to give an estimation on the measurability of the field by any device not disturbing the measured field (in the process of measurement) more than by the measurement error. Such method of measurement is necessary for the control of the field in electronic devices.

1 Introduction

The problem of measurability of the electromagnetic field was considered by Landau and Peierls (LP) in 1931 [1]. Then the results of LP were revised by Bohr and Rosenfeld (BR) in 1933 [4]. In short, the difference between the two

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works is in that LP tried to prove the existence of an absolute restriction on the measurability of the field, while BR argued that no absolute restriction exists. According to LP, no construction of the measuring device can provide a measurability beyond this absolute limit. BR argued that the measurability may be made arbitrarily small by choosing a sufficiently large charge of the test body.

In the papers [4, 5] and the book [6] the restricted-path-integral (RPI) method has been applied to the problem of the measurability of the electromagnetic field. The analysis in the framework of this method gave some estimation on the uncertainty $\delta E$ of the measurement output that does not always coincide with the previously known results. In the present paper we analyse this estimation in more detail applying a more explicit model of measurement.

Our final conclusion is that the uncertainty $\delta E$ obtained by the RPI method gives the limits in which the measurement may be considered as undisturbing. Besides, more strong limitations for undiscturbing measurements are obtained in the case when the space and time dimensions of the measurement region satisfy the inequality $l < ct/137$ or $l > ct$.

### 2 Restricted-Path-Integral Method

The restricted-path-integral method [6] has been applied to the measurement of the electromagnetic field in [4, 5] (see also [4]). We shall give here a short exposition of the method in its field-theoretic version appropriate for our aim.

The starting point is a functional integral on field configurations (which is often called also path integral). Let a field $\Phi$ be considered, and the action functional for this field be of the form

$$S[\Phi] = \int_{\Omega} d^4x \, L(\Phi, \partial \Phi)$$

where the integral is taken over a space-time region $\Omega$. Then the dynamics of the quantum field is described by the amplitude equal to the integral over field configurations. For the dynamics in the space-time region $\Omega$ the integral must be taken over field configurations in this region:

$$U = \int d[\Phi] \exp(iS[\Phi]),$$
Measuring the field has influence on its dynamics that may be expressed by restricting the path integral. The restriction is determined by the information supplied by the measurement. Let the measurement give an output \( \alpha \). Then the corresponding information may be expressed by some weight functional \( w_\alpha[\Phi] \), small (or equal to zero) for all field configurations \([\Phi]\) incompatible with the information given by the measurement. In this case field configurations in the path integral should be weighted by the functional:

\[
U_\alpha = \int d[\Phi] w_\alpha[\Phi] \exp(iS[\Phi]).
\] (3)

The square modulus of the corresponding amplitude

\[
P_\alpha = |U_\alpha|^2
\]
gives a probability density for different measurement outputs \( \alpha \).

Applying this scheme of consideration to the measurement of the electromagnetic field strength one should take the action for this field and the path integral over its configurations \([4, 5, 6]\). The unrestricted functional integral \([2]\) has in this case the form \([7]\):

\[
U = \int d[A] \delta(\partial_\mu A^\mu) \exp \left[-\frac{i}{4} \int d^4 x (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\right].
\] (4)

It is seen from this equation that the symbol \( d[\Phi] \) in Eq. \([2]\) must be specified as \( d[A] \delta(\partial_\mu A^\mu) \) for the electromagnetic field. Expressing the field action through the field strength, one has

\[
U = \int d[A] \delta(\partial_\mu A^\mu) \exp \left(-\frac{i}{2} \int d^4 x (\mathbf{H}_A^2 - \mathbf{E}_{A}^2)\right).
\] (5)

The preceding formulas are valid for the field in the absence of any measurement. Consider now the situation when the measurement of the field strength in the region \( \Omega \) is performed to give the result presented by the configurations \( \vec{H}(x), \vec{E}(x) \). This supplies an information about the field which can be characterized by the weight functional

\[
w_{[\vec{H}, \vec{E}]}[A] = \exp \left[-\frac{1}{\Omega} \int d^4 x \left(\frac{(\mathbf{H}_A - \vec{H})^2}{\Delta H^2} + \frac{(\mathbf{E}_A - \vec{E})^2}{\Delta E^2}\right)\right]
\] (6)
where Ω is the (four-dimensional) volume of the space-time region Ω where the measurement is arranged.

Indeed, this functional is almost equal to unity for the field configurations close to \( \vec{H}(x), \vec{E}(x) \) and it is almost zero otherwise. “Being close” is understood here in the sense of the square average. The functional decreases in \( e \) times when the square average deviation of the magnetic field becomes larger than \( \Delta H \) or/and the deviation of the electric field becomes larger than \( \Delta E \). Therefore, the functional (6) describes the packet of configurations, which corresponds to the field measurement giving the output \([\vec{H}, \vec{E}]\).

The corresponding restricted (weighted) path integral has the form

\[
U_{[\vec{H}, \vec{E}]} = \int d[A] \frac{1}{\Omega} \delta(\partial_t A^\mu) \exp \left[ -\frac{i}{2} \int d^4x \left( \frac{(H_A - \vec{H})^2}{\Delta H^2} + \frac{(E_A - \vec{E})^2}{\Delta E^2} \right) \right].
\] (7)

This integral gives an amplitude describing the measurement of electromagnetic field. The probability distribution over all possible measurement outputs (all possible field configurations \( \vec{H}(x), \vec{E}(x) \)) is provided by the square modulus of the amplitude:

\[
P_{[\vec{H}, \vec{E}]} = |U_{[\vec{H}, \vec{E}]}|^2
\]

The calculation shows [4, 5, 6] that this probability distribution has the form

\[
P_{[\vec{H}, \vec{E}]} = \exp \left[ -\frac{2}{\Omega} \int d^4x \left( \frac{(\vec{H} - \vec{H}_{\text{class}})^2}{\Delta H^2 + \frac{4}{\Omega^2 \Delta H^2}} + \frac{(\vec{E} - \vec{E}_{\text{class}})^2}{\Delta E^2 + \frac{4}{\Omega^2 \Delta E^2}} \right) \right].
\] (8)

It follows from this distribution [4, 5, 6] that the output of measurement \([\vec{H}, \vec{E}]\) may differ from the classical configuration in such a way that the square average deviations

\[
\| \vec{H} - \vec{H}_{\text{class}} \|^2 = \frac{1}{\Omega} \int d^4x (\vec{H} - \vec{H}_{\text{class}})^2,
\]

\[
\| \vec{E} - \vec{E}_{\text{class}} \|^2 = \frac{1}{\Omega} \int d^4x (\vec{E} - \vec{E}_{\text{class}})^2
\] (9)

be not too large, namely satisfy the following inequalities:

\[
\| \vec{H} - \vec{H}_{\text{class}} \|^2 \lesssim \delta H^2, \quad \| \vec{E} - \vec{E}_{\text{class}} \|^2 \lesssim \delta E^2,
\] (10)

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with
\[ \delta H^2 = \Delta H^2 + \frac{4}{\Omega^2 \Delta H^2}, \quad \delta E^2 = \Delta E^2 + \frac{4}{\Omega^2 \Delta E^2}. \quad (11) \]

Two qualitatively different regimes of the measurement may be considered. If
\[ \Delta H \gg \sqrt{\frac{2}{\Omega}}, \quad \Delta E \gg \sqrt{\frac{2}{\Omega}}, \]
then the second term in the right-hand side of each of the formulas (11) is negligible, so that (11) takes form
\[ \delta H = \Delta H, \quad \delta E = \Delta E. \quad (12) \]

If, on the contrary,
\[ \Delta H \ll \sqrt{\frac{2}{\Omega}}, \quad \Delta E \ll \sqrt{\frac{2}{\Omega}}, \]
then the first terms in (11) become small, so that
\[ \delta H = \frac{2}{\Omega \Delta H}, \quad \delta E = \frac{2}{\Omega \Delta E}. \quad (13) \]
The first regime is a classical one because it completely corresponds to the classical theory of measurement. The second regime is essentially quantum.

Returning to the general formula (11) one sees that the variance of the measurement results has the minimal value:
\[ \delta H_{\text{min}} = \delta E_{\text{min}} = \frac{2}{\sqrt{\Omega}}. \quad (14) \]

This corresponds to an optimal regime of measurement lying on the border between the classical and quantum ones. The existence of the minimum means that there is an absolute restriction on the measurability of the field. The limiting measurability is determined by the variance
\[ \delta H_{\text{min}} = \delta E_{\text{min}} = \frac{2}{\sqrt{\tau_3}}. \quad (15) \]

In all preceding consideration natural units were used in which \( \hbar = c = 1 \). In ordinary units one has for the absolute limit
\[ \delta H_{\text{min}} = \delta E_{\text{min}} = 2 \sqrt{\frac{\hbar}{\tau_3}}. \quad (16) \]
3 Explicit Account of Back Reaction

The restricted-path-integral method used in Sect. 2 gives an absolute limit for the measurability of an electromagnetic field for a certain definition of the measurement (we shall discuss this definition in detail later, in Sect. 4). This means that the quantum measurement noise does not allow one to have information more precise than is expressed by Eq. (16) about the value of the field strength. It is not clear from the phenomenological restricted-path-integral method what the origin of the quantum noise is and what factors do contribute to it. The rest of the paper is devoted to an analysis of these questions.

We shall show that the measurement noise consists of two characteristic parts, the mechanical uncertainty of the probe body and a proper field of this body. It is the first part that has been taken into account in the well-known paper of Bohr and Rosenfeld [2]. In the book [3] this part has been found with the help of the restricted-path-integral method (but with integration over trajectories of a mechanical test body, not over field configurations). We shall remind the results of these works in the present section.

Bohr and Rosenfeld derived their estimation for the measurability of an electric field,

$$\delta E_{BR} = \frac{\hbar c}{\Delta x c \tau Q}, \quad (17)$$

by taking Heisenberg’s uncertainty principle into account. For the measurement of the field the momentum of a probe charge must be found from observing its movement. However a precise localization of the charge in the observation prevents a precise determination of its momentum due to the uncertainty principle. The minimal possible error (17) in the estimation of the field results from optimization of the process.

The same conclusion can be drawn [3] if the observation of the measuring charge is considered with the help of the restricted-path-integral method.

Let, for example, this charge be an oscillator with the frequency\footnote{Free charge can be considered as a special case $\omega = 0$.} $\omega$ and the characteristic frequency of the measured motion is $\Omega$. Then an optimal value of the measurement error may be shown [3] to be

$$\Delta x = \left( \frac{\hbar}{m \tau (\Omega^2 - \omega^2)} \right)^{\frac{1}{2}}, \quad (18)$$
and the precision with which the force acting on the oscillator can be estimated is

$$\delta F = \left( \frac{m\hbar |\Omega^2 - \omega^2|}{\tau} \right)^{\frac{1}{2}}. \quad (19)$$

Taking into account that $F = QE$ and multiplying the preceding formulas one has the same estimation for the uncertainty of the measured value of the field as in the paper of Bohr and Rosenfeld:

$$\delta E_{\text{mech}} = \delta E_{\text{BR}} = \frac{\hbar c}{\Delta x c \tau Q} \quad (20)$$

This formula has a very simple and characteristic structure. Indeed, if one takes into account that $Q\Delta E$ is an uncertainty in the force and $\Delta p = Q\Delta E \tau$ is an uncertainty in the momentum acting on the measuring body then Eq. (20) reduces to the uncertainty relation for this body $\Delta p \Delta x = \hbar$.

Notice that the uncertainty discussed in this section is a consequence of the quantum uncertainties of the mechanical meter used for the measurement of the field. If one takes into account that this mechanical device must have a charge to interact with the field, the question arises about proper fields of the measuring body.

The question about proper fields was discussed in literature and particularly in [2]. It was concluded that proper fields do not prevent measurement because they may be calculated (i.e. they are systematic errors that can be taken into account). We shall however discuss this question in a new light. Namely, we shall consider the situation when these systematic errors are an obstacle for the aim of the measurement. This is, for example, the case when the measurement aims at controlling the field so that the observer would like to know what is the real value of the field, where the influence of the measurement is also taken into account. We shall see that it is just this setup of the problem that is characteristic for the restricted-path-integral approach.

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\[\text{We use an equality instead of an inequality meaning a limiting (optimal in quantum sense) regime of measurement.}\]
4 Accuracy with which the Measurement is Undisturbing

In the paper of Bohr and Rosenfeld [2] the uncertainty (20) is considered as the only obstacle for measurability, and from some point of view it is. However, one could introduce a further characteristic for the measurement, $\delta E$, including an additional field arising during the measurement, into this entity.

The motivation for this is evident. In some situations the aim of the measurement is not the estimation of what the field could be if the measurement is not actually performed, but the estimation of what the field really was, with all actual circumstances accounting, among them the measurement itself. If one knows the entity $\delta E$ one can be sure that the field was actually equal to the measurement outcome, $E$, with the precision $\Delta E$.

We have for this new characteristic the evident formula

$$\delta E = \delta E_{\text{mech}} + E_{\text{meas}} \quad (21)$$

where $E_{\text{meas}}$ is the complete field created by measuring bodies.

Let our measuring body has a charge $Q$ and the measurement is arranged in the region of the size $l$. Accept the Coulomb formula

$$E_{\text{meas}} = \frac{Q}{l^2} \quad (22)$$

as the simplest estimation for the field of such a body in a typical point of the measurement region. Then

$$\delta E = \frac{\hbar c}{\Delta x \, c \tau} + \frac{Q}{l^2} \quad (23)$$

We see from Eq. (23) that the complete (with the additional fields) measurement uncertainty $\delta E$ depends (for given dimensions of the measurement region, $l$ and $\tau$) on the value of the charge $Q$ of the measuring body and the measurement error of the mechanical meter, $\Delta x$. We should choose these parameter in such a way that the uncertainty $\delta E$ be minimal. This will give an absolute limit on the measurability of the field in an undisturbing regime.

Let us consider first the choice of the charge $Q$ for a fixed error $\Delta x$. It is evident that the expression (23) has a minimum achieved for the charge
equal to

\[ Q_{\text{opt}}^2 = \hbar c \frac{l^2}{\Delta x c \tau}. \]  

(24)

For this value of the charge we have the uncertainty equal to

\[ \delta E_{\text{opt}} = 2\sqrt{\frac{\hbar c}{\Delta x c \tau l^2}}. \]  

(25)

It is seen now that one should increase \( \Delta x \) to diminish the uncertainty \( \delta E_{\text{opt}} \). However the error \( \Delta x \) in measurement of the position of the measuring body cannot be more than the size \( l \) of the measuring region:

\[ \Delta x \leq l. \]  

(26)

Taking\(^3\) \( \Delta x = l \) we have for the minimum possible uncertainty the following estimation:

\[ \delta E_{\text{abs}} = 2\sqrt{\frac{\hbar c}{c \tau l^3}}. \]  

(27)

It is evident that this estimation coincides with the estimation (16) found by the restricted-path-integral method.

The formula (27) and the analysis leading to this formula makes more clear the sense of the estimation (16) and, more generally, of the estimations found by the restricted-path-integral method. We see that this method gives a restriction for undisturbing measurement. If the output of the measurement is \( E \) and the uncertainty of the output found by the restricted-path-integral method is \( \delta E \), then we know that the field really was in the limits of the interval \( [E - \delta E, E + \delta E] \), even with the fields of measuring bodies taken into account.

5 **Accounting quantization of charge**

Let us consider now more attentively the case when the choice \( Q = Q_{\text{opt}} \) and \( \Delta x = l \) is possible so that one is led to the above estimation. The problem is that, due to the relation (24), enlarging of \( \Delta x \) leads to diminishing of \( Q_{\text{opt}} \). Choosing \( \Delta x = l \) we determine some value for the charge \( Q_{\text{opt}} \) and

\(^3\)We shall consider in the next sections the situations when this is impossible.
we should be sure that this charge is feasible, i.e., is larger than the charge of an electron,  
\[ Q_{\text{opt}} > e. \]

Otherwise we should take the value for \( \Delta x \) less than \( l \).

The value of \( l/\Delta x \) leading to the value \( e \) for \( Q_{\text{opt}} \) is

\[
\frac{l}{\Delta x} = \frac{1}{137} \frac{c\tau}{l}.
\]

where \( 1/137 \) is the (approximate) value of the fine structure constant, \( e^2/\hbar c \).

Therefore we should in fact choose for \( l/\Delta x \) not the value 1 as was done above, but

\[
\frac{l}{\Delta x} = \max \left( 1, \frac{1}{137} \frac{c\tau}{l} \right).
\]

We see therefore that the above consideration leading to Eq. (27) is valid only for \( l/c\tau \) greater than \( 1/137 \). If the opposite inequality

\[
\frac{l}{c\tau} \leq \frac{1}{137}.
\]

is valid, we are led to the estimation

\[
\delta E_{\text{abs}} = \frac{2e}{l^2} = \frac{1}{6} \frac{\sqrt{\hbar c}}{l^2}
\]

where \( 1/6 \) is accepted as an approximate value for \( 2/\sqrt{137} \). This estimation is valid for the case when an optimal value of the measuring charge is \( e \).

In the latter formula the discrete structure of matter is taken into account which is impossible to do in the framework of the phenomenological restricted-path-integral approach.

6 Accounting causal relations

Let us consider a measurement arranged in the space region of the size \( l \) during the time \( \tau \) but with the inequality

\[
l > c\tau
\]
valid for these parameters. In this case the measurement region is “acausal” in the following sense. In the space region of the size \( l \) some smaller subregions (of the size say \( \lambda \)) exist that, during the time of measurement, \( \tau \), cannot be connected by a sublight signal. Any two events which occurred in two such subregions cannot have causal influence on each other. Such subregions are causally disconnected. Therefore, the whole space-time region where the measurement is arranged is in this sense “acausal”.

Bohr and Rosenfeld considered such acausal measurement regions. This is why we shall also consider them. We shall see that the estimations (27), (16) are incorrect in this case too.\(^4\)

The formula (20) is valid also in the case of acausal region. However Eqs. (27), (16) cannot be used for such a region. Instead, \( l = c\tau \) should be substituted in these formulas. The reason is the following.

The entity \( \Omega \) in Eq. (6) and the subsequent formulas is the four-volume of a region where an “integral” measurement is performed that cannot be reduced to measurements arranged in smaller regions. If an “elementary” measurement is arranged in the region of the volume \( \omega = \lambda^3 c\tau \) then just this volume should be substituted in a denominator of the exponent (6) but the integration should be performed over the whole region \( \Omega \). The resulting estimation for the variety of the measurement outputs is

\[
\Delta E_{\text{opt}}^2 = \Delta H_{\text{opt}}^2 = \frac{2}{\omega} \tag{30}
\]

\[
\delta E_{\text{min}} = \delta H_{\text{min}} = 2\sqrt{\frac{\hbar}{\tau \lambda^3}}. \tag{31}
\]

In the case \( l < c\tau \) an “integral” measurement is possible. If however \( l > c\tau \), the measurement procedure decomposes into a series of independent procedures arranged in causal parts of the whole region. The maximal size of such a part is \( l = c\tau \). This is why we should substitute \( \omega = \lambda^3 c\tau = (c\tau)^4 \) in this case. This leads to

\[
\delta E_{\text{min}} = 2\sqrt{\frac{\hbar c}{(c\tau)^2}}. \tag{32}
\]

The argument concerning the incorporation of the field of measuring bodies should also be changed because the field in each of the causal components

\(^4\)Notice however that we consider another characteristic of the measurement here than in the paper of BR so that the direct comparison of our results with theirs is impossible.
of an acausal region must be considered separately. Let us denote the charge of the measuring bodies in such a subregion by \( q \). Then instead of Eq. (23) we have

\[
\delta E = \frac{\hbar c}{\Delta x c \tau} q + \frac{q}{\lambda^2} = \frac{\hbar c}{\Delta x c \tau} q + \frac{q}{(c \tau)^2}.
\]

(33)

The optimization of this formula (with accounting of \( \Delta x < \lambda = c \tau \)) gives

\[
\delta E_{\text{abs}} = 2 \sqrt{\frac{\hbar c}{(c \tau)^2}}.
\]

(34)

Thus both methods give the same estimation of \( \delta E \) in an acausal case too, but this estimation differs from that obtained for a causal region.

7 Discussion

In the present work we considered the measurement of the electric field in the framework of two different approaches and compared the corresponding conclusions. The first approach based on restricting path integrals is phenomenological. The second approach includes an explicit consideration of mechanical properties of a measuring body as well as its electrical charge. When applying the second approach, we took into consideration both 1) the uncertainty relation for the measuring body as a mechanical system and 2) its proper electric field as an obstacle for undisturbing measurement of an external field.

The comparison of the results of both approaches allows us to conclude the following:

- The measurement uncertainty \( \delta E \) obtained from the restricted-path-integral (RPI) approach determines the precision with which the measurement may be considered undisturbing.

- The value \((16), (27)\) for \( \delta E \) which is obtained by the RPI method is an absolute restriction for the undisturbing measurement arranged in the region with the space dimension of the order of \( l \) during time of the order of \( \tau \).

- The RPI method does not take into account the discrete structure of matter, in particular the fact that any charge is a sum of charges equal
to the electron charge $e$. Taking into account this circumstance one obtains in the case $l < c\tau/137$ the more strong restriction (28) for a undisturbing measurement (than by the RPI method).

- In the case $l > c\tau$, i.e., in an acausal measurement region, the RPI method should be applied to each causal subregion separately. This leads to a more strong limitation (24) than in a generic case.

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