Vibrational control in $H_\infty$ problems

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We consider the application of the theory of vibrational control to $H_\infty$-problems. We study the possibility of introduction of high-frequency parametric vibrations in order to decrease the minimal attainable value of the $H_\infty$-norm. We prove the existence of the stabilizing solution of the Riccati equation with quickly oscillating coefficients. This solution is found using the averaging technique as a series of the small parameter.

One of the goals of control in various dynamical systems is their stabilization with respect to all or to a part of phase variables. Traditionally, the main tool of automatic control is feedback control, which is connected with the necessity to measure the state of the system. However, in some systems the measurement of the state is impossible.

As an alternative to feedback control the principle of vibrational control was proposed by S. M. Meerkov [1]. The essence of vibrational control is in introduction of parametric vibrations into the system in order to change its dynamical properties. By means of vibrational control it is sometimes possible to achieve the desired change without measuring the system’s state. As an example of such system we can mention the inverted pendulum with the oscillating suspension point [1]. The stabilization of the unstable equilibrium of the pendulum in this case occurs for sufficiently high frequency of vibrations.

In the present article we consider the application of the vibrational control theory [1-4] to $H_\infty$ problems [5-8].

The general $H_\infty$-problem in the linear system can be formulated as follows. Consider the linear system

$$\begin{cases}
\dot{x} = A(t)x + B_1(t)u + B_2(t)w, \\
z = Lx,
\end{cases}$$

(1)

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $u \in \mathbb{R}^p$, $w \in \mathbb{R}^q$, $A(t)$, $B_1(t)$, $B_2(t)$ are continuous bounded on $[0, \infty)$ matrix functions. We assume that the pair $(A(t), B_1(t))$ is internally stabilizable, i.e. there exists a such continuous matrix function $K(t)$ that the system

$$\dot{x} = [A(t) + B_1(t)K(t)]x$$

is asymptotically stable.
Consider the space
\[ \mathcal{L}^m_2[0,\infty) = \{ f : [0,\infty) \to \mathbb{R}^m : \int_0^\infty \langle f(t), f(t) \rangle dt < \infty \} \]
and similarly defined spaces \( \mathcal{L}^p_2[0,\infty), \mathcal{L}^q_2[0,\infty) \). In the sequel we shall denote these spaces as \( \mathcal{L}^m_2, \mathcal{L}^p_2, \mathcal{L}^q_2 \).

Let \( U \subset \mathcal{L}^p_2 \) denote the set of controls of the form \( u = K(t)x \) internally stabilizing the pair \( (A(t), B_1(t)) \). We shall call the quantity
\[ || \cdot ||_\infty = \sup_{w \in \mathcal{L}^q_2} \| z \|_{\mathcal{L}^m_2}^2 + \| u \|_{\mathcal{L}^p_2}^2 \]
the \( H_\infty \)-norm of the system for the given \( u \).

Consider the functional
\[ J(u, w) = || z ||_{\mathcal{L}^m_2}^2 + || u ||_{\mathcal{L}^p_2}^2 - \gamma^2 || w ||_{\mathcal{L}^q_2}^2. \] (2)

It is known that (e.g. [8]) that if the Riccati equation
\[ \dot{R} = -A^T R - R A + R \left( B_1 B_1^T - \frac{1}{\gamma^2} B_2 B_2^T \right) R - LL^T \] (3)
has the solution \( R_*(t) = R_0^T(t) > 0 \) for \( t \in [0,\infty) \) such that the system
\[ \dot{x} = \left[ A - \left( B_1 B_1^T - \frac{1}{\gamma^2} B_2 B_2^T \right) R_* \right] x \]
is asymptotically stable, then

1. The system \( \dot{x} = [A - B_1 B_1^T R_*] x \) is asymptotically stable,
2. The pair \( u^* = -B_1 R_* x, w^* = \frac{1}{\gamma^2} B_2 R_* x \) is such that
\[ J(u^*, w) = \min_{u \in U} J(u, w), \quad J(u^*, w^*) = \max_{w \in \mathcal{L}^q_2} J(u, w), \quad J(u^*, w^*) = 0. \]

Moreover, if the solution \( R_* \) with the above mentioned properties exists, it constant if functions \( A, B_1, B_2 \) are constant and is periodic if those functions are periodic.

From 2 obviously follows that if such solution \( R_*(t) \) exists, then \( J(u^*, w) \leq 0 \) for all \( w \in \mathcal{L}^q_2 \) i.e.
\[ || z ||_{\mathcal{L}^m_2} \leq \gamma || w ||_{\mathcal{L}^p_2} \]
and
\[ || \cdot ||_{\infty} \leq 0. \]

It is also known [3] that there exists \( \gamma^*(J) \) - the smallest of \( \gamma \) for which equation (3) has a solution.
In the sequel we will show that $\gamma^*(J)$ can be decreased by the introduction of vibrations into the system. Assume that the system (11) has the form
\[
\begin{aligned}
\dot{x} &= Ax + \frac{1}{\varepsilon} \sin \frac{t}{\varepsilon} K x + B_1 u + B_2 w, \\
z &= L x,
\end{aligned}
\] (4)
where $A$, $B_1$, $B_2$, $K$ are constant matrices and $\varepsilon$ is a small parameter.
As was mentioned before, the existence of a solution of the formulated above problem is related to the existence of positive definite periodic solution of the Riccati equation
\[
\frac{dR}{dt} = - \left( A + \frac{1}{\varepsilon} \sin \frac{t}{\varepsilon} K \right)^T R - R \left( A + \frac{1}{\varepsilon} \sin \frac{t}{\varepsilon} K \right) + \frac{1}{\gamma^2} R \left( B_1 B_1^T - B_2 B_2^T \right) R - LL^T.
\] (5)

Next we prove that such a solution exists for small $\varepsilon$ if there exists a positive definite solution of the averaged algebraic Riccati equation.

We introduce notation
\[D = B_1 B_1^T - \frac{1}{\gamma^2} B_2 B_2^T, \quad C = LL^T.\]

First, we consider the differential Riccati equation
\[
\frac{dR}{dt} = \varepsilon \left[ - A^T(t) R - RA(t) + RD(t) R - C(t) \right],
\] (6)
where $A$, $D$, $C$ are continuous $T$-periodic functions, $D, C$ are symmetric and $\varepsilon$ is a small parameter. We shall look for a solution $R_*(t) = R_*^T(t) > 0$ of the equation (6) such that the system
\[
\dot{x} = [A(t) - D(t) R_*(t)]
\]

is asymptotically stable.

Consider the following spaces
- \(RS^{n\times n}\) - the space of symmetric real $n \times n$ matrices,
- \(CS_T^{n\times n}\) - the space of continuous symmetric $T$-periodic functions,
- \(CSZ_T^{n\times n}\) - subspace of \(CS_T^{n\times n}\), consisting of functions with the zero average.

Obviously,
\[
CS_T^{n\times n} = RS^{n\times n} \oplus CSZ_T^{n\times n},
\]
if the elements of \(RS^{n\times n}\) are considered as constant functions.

Now we introduce the averaging operator $M_0 : CS_T^{n\times n} \to RS^{n\times n}$ defined by the equality
\[
M_0 F = \frac{1}{T} \int_0^T F(t) dt.
\]
Clearly, \( M_0 \) is a projector to the subspace \( \mathbb{R}^n \times \mathbb{R}^n \). We also define the operator \( T_0 \) on \( \mathbb{CSZ}^n \times \mathbb{CSZ}^n \)

\[ T_0 F = F - M_0 F. \]

For brevity we will denote \( M_0 F \) as \( \bar{F} \).

Consider the algebraic Riccati equation

\[ -\bar{A}^T R - R\bar{A} + R\bar{D}R - \bar{C} = 0. \tag{7} \]

**Theorem 1** If equation (7) has such solution \( R_0 = R_0^T > 0 \) that the matrix \( \bar{A} - \bar{D}R_0 \) is Hurwitz, then there exists such \( \varepsilon_0 > 0 \) that for any \( \varepsilon < \varepsilon_0 \) equation (8) has such a \( T \)-periodic symmetric positive definite solution \( R_*(t) \), that the system

\[ \dot{x} = [A(t) - D(t)R_*(t)]x \]

is asymptotically stable.

This solution can be approximately found as

\[ R_{(N)}(t) = R_0 + \varepsilon (R_1 + \Pi_1(t)) + \ldots + \varepsilon^N (R_N + \Pi_N(t)) + \varepsilon^{N+1} \Pi_{N+1}(t), \]

where \( R_0, \ldots, R_N \in \mathbb{RS}^n \times \mathbb{RS}^n, \Pi_1(t), \ldots, \Pi_{N+1}(t) \in \mathbb{CSZ}_T^n \times \mathbb{CSZ}_T^n \), and for some \( C > 0 \) independent of \( \varepsilon \)

\[ \sup_{t \in [0,T]} ||R_*(t) - R_{(N)}(t)|| < C\varepsilon^{N+1}. \tag{8} \]

**Proof.** Formally substituting \( R_{(N)} \) into Eq. (8), we have

\[ \varepsilon \dot{\Pi}_1 + \varepsilon^2 \dot{\Pi}_2 + \ldots + \varepsilon^{N+1} \dot{\Pi}_{N+1} = \]

\[ \varepsilon \left[ -A^T(t) (R_0 + \varepsilon (R_1 + \Pi_1(t)) + \ldots) - (R_0 + \varepsilon (R_1 + \Pi_1(t)) + \ldots) A(t) + (R_0 + \varepsilon (R_1 + \Pi_1(t)) + \ldots) D(t) (R_0 + \varepsilon (R_1 + \Pi_1(t)) + \ldots) - C(t) \right]. \tag{9} \]

Equating in (9) the terms of order \( \varepsilon^1 \) we have

\[ \frac{d\Pi_1}{dt} = -A^T(t)R_0 - R_0 A(t) + R_0 D(t) R_0 - C(t). \]

Averaging the latter equation yields

\[ -\bar{A}^T R_0 - R_0 \bar{A} + R_0 \bar{D}R_0 - \bar{C} = 0 \tag{10} \]

\[ \Pi_1(t) = \int_0^t T_0 \left[ -A^T(s) R_0 - R_0 A(s) + R_0 D(s) R_0 - C(s) \right] ds. \]

According to the assumptions of the theorem, equation (10) has a positive definite solution \( R_0 \in \mathbb{RS}^n \times \mathbb{RS}^n \), for which the matrix \( \bar{A} - \bar{D}R_0 \) is Hurwitz.
Equating the terms of order \( \varepsilon^2 \), we have
\[
\frac{d\Pi_2}{dt} = -A^T(t)(R_1 + \Pi_1(t)) - (R_1 + \Pi_1(t))A(t) +
(R_1 + \Pi_1(t))D(t)R_0 + R_0D(t)(R_1 + \Pi_1(t)).
\]
Averaging of the latter equality yields
\[
\frac{1}{T} \int_0^T (\bar{A}^T - \bar{D}R_0)^T R_1 + R_1(\bar{A}^T - \bar{D}R_0) + M_0[-A^T(t)\Pi_1(t) - \Pi_1(t)A(t) + \Pi_1(t)D(t)R_0 + R_0D(t)\Pi_1(t)] = 0.
\]
Because the matrix \( \bar{A} - \bar{D}R_0 \) is Hurwitz, the Lyapunov equation (11) has the unique solution \( R_1 \).

The matrix \( \Pi_2(t) \) is defined by
\[
\Pi_2(t) = \int_0^t T_0 \left[ A^T(s)(R_1 + \Pi_1(s)) - (R_1 + \Pi_1(s))A(s) +
(R_1 + \Pi_1(s))D(s)R_0 + R_0D(s)(R_1 + \Pi_1(s)) \right] ds.
\]
Similarly, we can find \( R_2, \ldots, R_N \) and \( \Pi_3(t), \ldots, \Pi_{N+1}(t) \).

Next we prove the existence of solution \( R_s(t) \), satisfying the conditions of the present theorem. Obviously,
\[
\frac{dR_{(N)}}{dt} = \varepsilon \left[ -A^T(t)R_{(N)} - R_{(N)}A(t) + R_{(N)}D(t)R_{(N)} - C(t) +
\varepsilon^{N+1}g(t, \varepsilon) \right],
\]
where \( g(t, \varepsilon) \) - \( T \) is a periodic in \( t \) and uniformly bounded in \( \varepsilon \in (0, \varepsilon_0] \) function.

First, we prove that the equation
\[
\frac{d\Theta}{dt} = \varepsilon \left[ -(A - DR_{(N)})^T\Theta - \Theta(A - DR_{(N)}) + \Theta D\Theta - \varepsilon^{N+1}g(t, \varepsilon) \right]
\]
has a \( T \)-periodic solution. Note that the function \( H(\Theta) = \Theta D\Theta \) is such that for any \( \sigma > 0 \) there exists \( \delta > 0 \) such that if \( ||\Theta_1|| < \delta \) and \( ||\Theta_1|| < \delta \), than
\[
||H(\Theta_1 - \Theta_2)|| < \sigma ||\Theta_1 - \Theta_2||.
\]
Let \( \Gamma = M_0\Theta \) and \( \Delta = T_0\Theta \). Then applying operators \( M_0 \) and \( T_0 \) to the left and right hand sides of Eq. (12), we have
\[
0 = \int_0^T \left[ -(A - DR_{(N)})^T(\Gamma + \Delta) - (\Gamma + \Delta)(A - DR_{(N)}) +
(A - DR_{(N)})D(A - DR_{(N)}) - \varepsilon^{N+1}g(t, \varepsilon) \right] dt,
\]
\[
\dot{\Delta} = \varepsilon T_0 \left[ -(A - DR_{(N)})^T(\Gamma + \Delta) - (\Gamma + \Delta)(A - DR_{(N)}) +
(A - DR_{(N)})D(A - DR_{(N)}) - \varepsilon^{N+1}g(t, \varepsilon) \right].
\]
From Eq. (13) we have
\[ A\Gamma + B\Delta + G_1(\Gamma, \Delta, \varepsilon) = 0, \tag{15} \]
where the operators \( A : \mathbb{RS}^{n \times n} \rightarrow \mathbb{RS}^{n \times n} \) and \( B : \mathbb{CSZ}_T^{n \times n} \rightarrow \mathbb{RS}^{n \times n} \) are defined by the equality
\[
A\Gamma = -(\bar{A} - \bar{D}R_0)^T\Gamma - \Gamma(\bar{A} - \bar{D}R_0),
\]
\[
B\Delta = -M_0 \left[ (A(t) - D(t)R_0)^T\Delta + \Delta(A(t) - D(t)R_0) \right],
\]
and
\[
G_1(\Gamma, \Delta, \varepsilon) = M_0 [\varepsilon D\rho_1(t, \varepsilon)(\Gamma + \Delta) + \varepsilon(\Gamma + \Delta)D\rho_1(t, \varepsilon) + (\Gamma + \Delta)D(\Gamma + \Delta) - \varepsilon^{N+1}g(t, \varepsilon)],
\]
where \( \rho_1(t, \varepsilon) = (R(N)(t) - R_0)/\varepsilon. \)

Obviously, for any \( \sigma_1 > 0 \) there exist \( \varepsilon_0 \) and \( \delta \) such that if \( ||\Gamma_1||_{\mathbb{RS}} < \delta, ||\Gamma_2||_{\mathbb{RS}} < \delta, ||\Delta_1||_{\mathbb{CSZ}} < \delta, ||\Delta_2||_{\mathbb{CSZ}} < \delta \) and \( \varepsilon < \varepsilon_0 \) than
\[
||G_1(\Gamma_1, \Delta_1, \varepsilon) - G_1(\Gamma_2, \Delta_2, \varepsilon)||_{\mathbb{CSZ}} < \sigma(||\Gamma_1 - \Gamma_2||_{\mathbb{RS}} + ||\Delta_1 - \Delta_2||_{\mathbb{CSZ}}). \tag{16} \]

Since the matrix \( \bar{A} - \bar{D}R_0 \) is Hurwitz, the operator \( A \) is invertible and from (13) follows that
\[ \Gamma = -A^{-1}B\Delta - A^{-1}G_1(\Gamma, \Delta, \varepsilon). \tag{17} \]

From Eq. (14) we have
\[ \Delta = \varepsilon G_2(\Gamma, \Delta, \varepsilon), \tag{18} \]
where
\[
G_2(\Gamma, \Delta, \varepsilon) = \int_0^t \left[ -(A(s) - D(s)R(N)(s))^T(\Gamma + \Delta(s)) - (\Gamma + \Delta(s))(A(s) - D(s)R(N)(s)) + (\Gamma + \Delta(s))D(\Gamma + \Delta(s)) - \varepsilon^{N+1}g(s, \varepsilon) \right] ds.
\]

\( G_2(\Gamma, \Delta, \varepsilon) \) satisfy a condition similar to (16), namely that for any \( \sigma > 0 \) there exist such \( \varepsilon_0 \) and \( \delta \) that if \( ||\Gamma_1||_{\mathbb{RS}} < \delta, ||\Gamma_2||_{\mathbb{RS}} < \delta, ||\Delta_1||_{\mathbb{CSZ}} < \delta, ||\Delta_2||_{\mathbb{CSZ}} < \delta \) and \( \varepsilon < \varepsilon_0 \) than
\[
||G_2(\Gamma_1, \Delta_1, \varepsilon) - G_2(\Gamma_2, \Delta_2, \varepsilon)||_{\mathbb{CSZ}} < \sigma(||\Gamma_1 - \Gamma_2||_{\mathbb{RS}} + ||\Delta_1 - \Delta_2||_{\mathbb{CSZ}}). \tag{19} \]

System (17), (18) can be rewritten as
\[
\begin{pmatrix}
\Gamma \\
\Delta
\end{pmatrix}
=egin{pmatrix}
0 & -A^{-1}B \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\Gamma \\
\Delta
\end{pmatrix}
+egin{pmatrix}
-A^{-1}G_1(\Gamma, \Delta, \varepsilon) \\
\varepsilon G_2(\Gamma, \Delta, \varepsilon)
\end{pmatrix}
\tag{20}
\]
or
\[ (\Gamma, \Delta) = \varphi(\Gamma, \Delta), \]

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where
\[
\varphi(\Gamma, \Delta) = \begin{pmatrix}
0 & -A^{-1}B \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\Gamma \\
\Delta
\end{pmatrix}
+ \begin{pmatrix}
-A^{-1}G_1(\Gamma, \Delta, \varepsilon) \\
\varepsilon G_2(\Gamma, \Delta, \varepsilon)
\end{pmatrix}.
\]

The map \(\varphi\) acts in the space
\[
X = RS^{n \times n} \times CSZ_T^{n \times n},
\]
where we introduce the vector norm
\[
\| (\Gamma, \Delta) \|_X = (\| \Gamma \|_{RS}, \| \Delta \|_{CSZ})^T.
\]
Next we show that there exists a ball in \(X\), within which the map \(\varphi\) satisfies the generalized contraction principle \([11]\), i.e. there exists the matrix \(K \in \mathbb{R}^{2 \times 2}\) with non-negative elements and the spectral radius \(\rho(K) < 1\) such that
\[
\| \varphi(\Gamma_1, \Delta_1) - \varphi(\Gamma_2, \Delta_2) \|_X \leq K \| (\Gamma_1, \Delta_1) - (\Gamma_2, \Delta_2) \|_X,
\]
where the relation of inequality is understood in the following sense:
- it is said that \(x \leq y\) \((x, y \in \mathbb{R}^2)\) if \(x_i \leq y_i\) for \(i = 1, 2\).

From (20) and estimates (16), (19) follows that if \(\| \Gamma_1 \|_{RS} < \delta\), \(\| \Gamma_2 \|_{RS} < \delta\), \(\| \Delta_1 \|_{CSZ} < \delta\), \(\| \Delta_2 \|_{CSZ} < \delta\), then
\[
\| \varphi(\Gamma_1, \Delta_1) - \varphi(\Gamma_2, \Delta_2) \|_X \leq \left( \| A^{-1} \| \sigma_1 + \| A^{-1} B \| + \| A^{-1} \| \sigma_1 \right) \| (\Gamma_1, \Delta_1) - (\Gamma_2, \Delta_2) \|_X,
\]
where \(\sigma_1, \sigma_2 \to 0\) for \(\delta \to 0\).

Therefore, the map \(\varphi\) satisfies the generalized contraction principle for sufficiently small \(\delta\). Let us now find the value of \(\delta\) for which \(\varphi\) maps the ball of the radius \(\delta\) in the space \(X\) into itself.

From (16) and (19) follows that
\[
\| G_1(\Gamma, \Delta, \varepsilon) - G_1(0, 0, \varepsilon) \| \leq \sigma_1(\| \Gamma \| + \| \Delta \|),
\]
\[
\| G_2(\Gamma, \Delta, \varepsilon) - G_2(0, 0, \varepsilon) \| \leq \sigma_2(\| \Gamma \| + \| \Delta \|),
\]
for \(\| \Gamma \| < \delta\) \(\| \Delta \| < \delta\). This implies that
\[
\| G_1(\Gamma, \Delta, \varepsilon) \| < C_1 \varepsilon^{N+1} + \sigma_1(\| \Gamma \| + \| \Delta \|),
\]
\[
\| G_2(\Gamma, \Delta, \varepsilon) \| < C_2 \varepsilon^{N+1} + \sigma_2(\| \Gamma \| + \| \Delta \|),
\]
where \(C_1, C_2\) are some constants.

From (21) follows that
\[
\| \varphi(\Gamma, \Delta) \|_X \leq
\]
\[
\left( \frac{||A^{-1}||\sigma_1}{\varepsilon\sigma_2}, \frac{||A^{-1}B|| + ||A^{-1}||\sigma_1}{\varepsilon\sigma_2} \right) ||(\Gamma, \Delta)|| x + \left( \frac{C_1\varepsilon^{N+1}}{C_2\varepsilon^{N+1}} \right),
\]
i.e. for some \( C^* \varphi \) maps the ball of the radius \( \rho = C^*\varepsilon^{N+1} \) in the space \( X \) into itself and is contracting, and hence has in this ball the unique fixed point \((\Gamma_*, \Delta_*)(t)\) such that \( \Theta_*(t) = \Gamma_* + \Delta_*(t) \) is the solution of Eq. (12). It is easy to verify that \( R_* = R_{(N)}(t) + \Theta(t) \) is a solution of equation (13).

Next we prove the asymptotic stability of the system
\[
\dot{x} = [A(t) - D(t)R_*(t)]x.
\] (22)

Substituting \( R_*(t) \) in the form \( R_0 + \varepsilon\rho_1(t) + \Theta(t) \) into (22) and taking into account that the matrix \( \bar{A} - \bar{D}R_0 \) is Hurwitz, we have that (22) is asymptotically stable for small \( \varepsilon \).

Positive definiteness of \( R_*(t) \) for small \( \varepsilon \) follows from the positive definiteness of \( R_0 \).

Theorem 1 is proved.

Consider Eq. (5). Introduce the fast time \( \tau = t/\varepsilon \). Then
\[
\frac{dR}{d\tau} = -\sin \tau (K^TR + RK) + \varepsilon( -A^TR - RA + RD - C). \] (23)

Putting \( \varepsilon = 0 \), we have
\[
\frac{dR}{d\tau} = -\sin \tau (K^TR + RK). \] (24)

Equation (24) has the general solution
\[
R = \Psi(\tau)P\Psi^T(\tau), \] (25)
where \( P \) is an arbitrary constant, \( \Psi(\tau) = \exp(K^T(\cos \tau - 1)) \).

Introducing \( P \), defined by (25), as the new variable in (23), we have
\[
\frac{dP}{d\tau} = \varepsilon \left(-\bar{A}^T(\tau)P - PA(\tau) + PD(\tau)P - C(\tau) \right), \] (26)
where
\[
\bar{A}(\tau) = \Psi^T(\tau)A(\Psi^T)^{-1}(\tau), \quad D(\tau) = \Psi^T(\tau)D(\tau), \quad C(\tau) = \Psi^{-1}(\tau)C(\Psi^T)^{-1}(\tau).
\]
Consider the algebraic Riccati equation
\[
-\bar{A}^T P - P\bar{A} + P\bar{D}P - \bar{C} = 0, \] (27)
which is the averaging of the right hand side of (26).
Theorem 2 If equation (27) has a solution $P_0 = P_0^T > 0$ such that the matrix $ar{A} - ar{D}P_0$ is Hurwitz, then there exists a $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ equation (28) has a $2\pi\varepsilon$-periodic symmetric positive definite solution $R_*(\frac{t}{\varepsilon})$ for which the zero solution of the system

$$\frac{dx}{dt} = \left[ A \left( \frac{t}{\varepsilon} \right) - D \left( \frac{t}{\varepsilon} \right) R_*(\frac{t}{\varepsilon}) \right] x$$

is exponentially stable.

For any $N \geq 0$ this solution can be found in the form

$$R_{(N)} \left( \frac{t}{\varepsilon} \right) = \Psi \left( \frac{t}{\varepsilon} \right) \left[ P_0 + \varepsilon \left( P_1 + \Pi_1 \left( \frac{t}{\varepsilon} \right) \right) + \ldots + \varepsilon^N \left( P_N + \Pi_N \left( \frac{t}{\varepsilon} \right) \right) \right] \Psi^T \left( \frac{t}{\varepsilon} \right),$$

where $P_1, \ldots, P_N \in \mathbb{R}S^{n \times n}$, $\Pi_1, \ldots, \Pi_{N+1} \in \mathbb{C}SZ_T^{n \times n}$, and for some $C > 0$

$$\sup_{\tau \in [0,T_\varepsilon]} \left\| R_*(\tau) - R_{(N)} \left( \frac{\tau}{\varepsilon} \right) \right\| < C\varepsilon^{N+1}.$$ 

Proof. According to Theorem 1, equation (26) has a positive definite stabilizing solution $P_*(\tau)$ such that for

$$P_{(N)}(\tau) = P_0 + \varepsilon (P_1 + \Pi_1(\tau)) + \ldots + \varepsilon^N (P_N + \Pi_N(\tau)) + \varepsilon^{N+1} \Pi_{N+1}(\tau)$$

the estimate (29) is satisfied. It is obvious that $R(t/\varepsilon) = \Psi(t/\varepsilon)P_*(t/\varepsilon)\Psi^T(t/\varepsilon)$ is a solution of (28) and for $R_{(N)}(t/\varepsilon)$ and $R_*(t/\varepsilon)$ a similar estimate is satisfied, and that the system

$$\dot{x} = [A(\tau) - D(\tau)R_*(\tau)]x$$

is exponentially stable.

Example. Let matrices $A, B, C$ and $K$ have the form

$$A = \begin{pmatrix} 0 & 1 \\ -0.27 & -2.8 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}.$$

The averaged Riccati equation (27) in this case has the form

$$\begin{pmatrix} 0 & -0.27 - k^2/2 \\ 1 & -2.8 \end{pmatrix} P + P \begin{pmatrix} 0 & 1 \\ -0.27 - k^2/2 & -2.8 \end{pmatrix} + \frac{1}{\gamma^2} P \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} P + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0.$$
The table below shows the values $\gamma^*_K$ for different $k$. These values are found numerically using the conditions for the existence of solutions of Riccati equations, given in [12].

| $k$ | $\gamma^*_K$ |
|-----|--------------|
| 0   | 3,704        |
| 0.25| 3,320        |
| 0.5 | 2,532        |
| 0.75| 1,815        |
| 1   | 1,300        |
| 1.25| 0,925        |
| 1.5 | 0,717        |
| 1.75| 0,556        |

This example shows the possibility of introduction of vibrations into a linear system, which decrease $\gamma^*_K$, i.e. decrease the influence of disturbances on the output of the system.

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