A NOTE ON THE QUANTIZATION OF TENSOR FIELDS AND QUANTIZATION OF MECHANICAL SYSTEMS

J. MUÑOZ-DÍAZ AND R. J. ALONSO-BLANCO

Abstract. This article continues and completes [10]. We present two methods of quantization associated with a linear connection given on a differentiable manifold. One of these is that presented in [10]. In the final section the equivalence between both methods is demonstrated, as a consequence of a remarkable property of the Riemannian exponential.

Contents

0. Introduction 1
1. Notes on Classical Mechanics and Undulatory Mechanics 2
2. Quantization of contravariant tensors. Dequantization of differential operators 7
3. Quantization by means of Riemannian exponential. Families of quantizations parameterized by \( \hbar \) 11
4. Identity of the two considered rules of quantization 12
References 13

0. Introduction

The purpose of this article is, as that of [10], the research of bridges that link Classical Mechanics with Quantum Mechanics.

In the primitive rules of quantization for conservative mechanical systems, on each solution of the Hamilton-Jacobi equation, the Hamiltonian action supplies us with the phase for the waves. Without leaving the classical path, we can arrive to a wave equation which, by adding some hypotheses, is of “Schrödinger” ([5, 2]). But the Schrödinger equation itself is inaccessible by purely classical means. In Section 1 of the paper we will give a quick review of the notions of Classical Mechanics that we need in the subsequent sections. We will also indicate how, even before any quantization rule, the Broglie waves and the aforementioned classical cases of the Schrödinger equation appear.

In Section 2 we will present a quantization rule of contravariant tensors on a manifold \( M \), which is canonically determined by the datum of a symmetric linear connection \( \nabla \) on \( M \). Such a rule establishes a biunivocal correspondence between contravariant tensors on \( M \) (not necessarily homogeneous) and linear differential operators acting on \( \mathcal{C}^\infty(M) \). The passage differential operator \( \rightarrow \) contravariant tensor, is the “dequantization” determined by \( \nabla \). Since a contravariant tensor on \( M \) is directly interpretable as a Hamiltonian function \( F \in \mathcal{C}^\infty(T^*M) \), we can continue the dequantization with the passage \( F \rightarrow \) Hamiltonian field associated with \( F \) by the symplectic structure of \( T^*M \). In this way, an infinitesimal canonical transformation on the symplectic manifold \( T^*M \) corresponds to each linear differential operator on \( M \). When a Riemannian metric \( T_2 \) (of arbitrary signature) is given on \( M \), makes sense to say whether a tangent field \( D \) on \( T^*M \) is or is not a second order differential equation (that is to say, the field that governs the evolution of a mechanical system with configuration space \( (M, T_2) \)). For the Levi-Civita connection of the metric \( T_2 \) it happens that the necessary and sufficient condition for the field Hamiltonian corresponding to a differential operator \( P \) to be a second order differential equation is that \( P \) is a differential operator “of Schrödinger” \( P = -\hbar^2\Delta + U \) \( (U \in \mathcal{C}^\infty(M), \hbar \) a constant which is fixed when the rule of quantization is given). This fact
1. Notes on Classical Mechanics and Undulatory Mechanics

1.1. Structures previous to the metric. Let $M$ be a smooth manifold of dimension $n$. Let $TM$, $T^*M$ be the tangent and cotangent bundles of $M$, respectively. Let $C^\infty(M)$ be the ring of differentiable functions on $M$ with complex values. We will consider $C^\infty(M)$ as a subring of $C^\infty(TM)$ by means of the injection derived from the canonical projection $\pi: TM \to M$.

The vector fields tangent to $TM$ which (as derivations of the ring $C^\infty(TM)$) kill the subring $C^\infty(M)$, are the vertical tangent fields. The differentiable 1-forms on $TM$ which, by interior product, kill the vertical tangent fields are the horizontal 1-forms on $TM$. The lifting of 1-forms from $M$ to $TM$ by means of $\pi^*$ are horizontal and, locally, any horizontal 1-form on $TM$ is a linear combination of such 1-forms with coefficients in $C^\infty(TM)$.

Each horizontal 1-form $\alpha$ on $TM$ defines on $TM$ a function $\dot{\alpha}$ given by $\dot{\alpha}(u_x) = \langle \alpha, u_x \rangle$ (inner product), for each $u_x \in TM$. In particular, for each fuction $f \in C^\infty(M)$, the function $d\alpha$ will be denoted, for short, $\dot{f}$. Essentially, $\dot{f}$ is $df$: $\dot{f}(u_x) = \langle df, u_x \rangle = u_x(f)$ (derivative of $f$ by $u_x \in TM$). If $(x^1, \ldots, x^n)$ is a system of coordinates on an open subset of $M$, the $(x^1, \ldots, x^n, \dot{x}^1, \ldots, \dot{x}^n)$ are coordinates on the corresponding open subset of $TM$.

Each contravariant tensor field $a$ of degree $r$ on $M$ canonically defines a function $\dot{a}$ on $TM$, polynomial along the fibres: $\dot{a}(u_x) = \langle a, u_x \otimes \cdots \otimes u_x \rangle$. In local coordinates, $\dot{a}$ is obtained by substituting $dx^i$ by $\dot{x}^i$ in the expression of the tensor $a$.

The linear structure of each fibre $T_xM$, allows us to identify the tangent space to $T_xM$ at each one of its points $u_x$ with the very vector space $T_xM$: to the vector $v_x \in T_xM$ it corresponds the vector $V_{u_x} \in T_{u_x}(T_xM)$ that is the “derivative along $v_x$”. We will say that $V_{u_x}$ is the vertical representative of $v_x$ at $u_x$ and that $v_x$ is the geometric representative of $V_{u_x}$.

By going to the definitions it is checked that, for each $f \in C^\infty(M)$, we have $V_{ua}(\dot{f}) = v_x(f)$. In this way, each tangent vector $v_x \in T_xM$ determines on its fibre $T_xM$ a tangent field which is “constant” (= parallel).

Each tangent field on $M$ determines a vertical tangent field on $TM$, constant (=parallel) along each fibre.

As a consequence, each contravariant tensor field $\Phi$ on $M$ determines a vertical contravariant tensor field $\Phi$ on $TM$, constant (=parallel) along each fibre.
In local coordinates, $\Phi$ is obtained from $\Phi$ by substituting each field $\partial/\partial x^j$ by its vertical representative $\partial/\partial \dot{x}^j$.

A symmetric contravariant tensor field $\Phi$ on $M$ determines on $T^*M$ a function $F$, polynomial on the fibres, defined by

$$F(\alpha_x) = \langle \Phi, \alpha_x \otimes \cdots \otimes \alpha_x \rangle$$

(tensor contraction). We will say that $F$ is the Hamiltonian associated with $\Phi$.

If $(x^1, \ldots, x^n)$ are coordinates on an open subset of $M$, the Hamiltonian associated with $\partial/\partial x^j$ is usually denoted by $p_j$:

$$p_j(\alpha_x) = \langle \alpha_x, \partial \partial x^j \rangle.$$

The functions $(x^1, \ldots, x^n, p_1, \ldots, p_n)$ are local coordinates on $T^*M$. For a given contravariant tensor field $\Phi$ on $M$, its associated Hamiltonian is obtained by substituting in the expression of $\Phi$ each $\partial/\partial x^j$ by $p_j$.

Symmetric contravariant tensor fields on $M$, homogeneous or not, canonically corresponds to the functions $\in C^\infty(T^*M)$ that are polynomials along the fibres. We will refer to this particular type of functions as Hamiltonians.

In $T^*M$ it is defined the Liouville 1-form $\theta$ by $\theta_{\alpha_x} = \pi^* \alpha_x$, for each $\alpha_x \in T^*M$ ($\pi: T^*M \to M$ is the canonical projection). We will simplify the notation by putting $\theta_{\alpha_x} = \alpha_x$, understanding that covariant tensors in general rise from $T^*M$ by “pull-back” through $\pi^*$. In local coordinates $(x^1, \ldots, x^n, p_1, \ldots, p_n)$, we have $\theta = p_j \, dx^j$.

The 2-form $\omega_2 := d\theta$ is the symplectic form on $T^*M$. In local coordinates, $\omega_2 = dp_j \wedge dx^j$.

The 2-form $\omega_2$ has no kernel, so establishes an isomorphism between the $C^\infty(T^*M)$-module of tangent fields on $T^*M$ and that of the 1-forms on $T^*M$:

$$D \mapsto \alpha := D \rfloor \omega_2.$$
of $TM$). By acting fibrewise it is obtained a differential operator $\tilde{\Phi}_a$, on $C^\infty(T^*M)$, that kills the subring $C^\infty(M)$.

Therefore, we have the correspondences

$$\hat{a}(x, \dot{x}) \leftrightarrow a(x, dx) \leftrightarrow \Phi_a \leftrightarrow \tilde{\Phi}_a(x, \partial/\partial p).$$

$\Phi_a$ is the polynomial in the $\partial/\partial p_j$ that results by the substitution in the expression of $a$ each $dx^j$ by $\partial/\partial p_j$.

The correspondence $\dot{a} \rightarrow \tilde{\Phi}_a$ is modified by a constant factor $a \rightarrow k^{-r}\tilde{\Phi}_a$ (for tensors of order $r$) if the symplectic form $\omega_2$ is changed to $k\omega_2$ (where $k \in \mathbb{C}$ is arbitrary). In Quantum Mechanics, it is taken $k = i/\hbar$.

The same association \{tensor\} $\rightarrow$ \{vertical differential operator\} is obtained from the Fourier transform, by using the linear duality between fields on $TM$ and $T^*M$. Let us denote by $S(TM)$ the space of complex functions on $TM$ which, when restricted to each fibre $T_xM$, are of class $C^\infty$ and rapidly decreasing they and all of their derivatives. Analogous meaning for $S(T^*M)$. On each fibre $T_xM$ (being a $\mathbb{R}$-linear space) there is a measure invariant by translation $\mu$, univocally determined up to a multiplicative constant ("Haar measure"). Once fixed that factor for each $T_xM$, it is defined the Fourier transform $S(TM) \rightarrow S(T^*M)$ by

$$(\mathcal{F}f)(\alpha_x) := \int_{T_xM} f(v_x)e^{i\langle v_x, \alpha_x \rangle}d\mu(v_x).$$

In local coordinates:

$$(\mathcal{F}f)(x, p) := \int_{T_xM} f(x, \dot{x})e^{i\langle p_jx^j, \dot{x}^1 \cdots \dot{x}^n \rangle}d\dot{x}^1 \cdots d\dot{x}^n.$$ (the constant factor that affects the integral is irrelevant for the following).

By differentiation under the integral sign we get the classical formula

$$\mathcal{F}(\hat{a}(x, \dot{x})f(x, \dot{x})) = a(x, -i\hbar\partial/\partial p)(\mathcal{F}f)(x, p),$$

that is, $\mathcal{F} \circ \hat{a} = \tilde{\Phi}_a \circ \mathcal{F}$, where $\tilde{\Phi}_a$ is the vertical differential operator which results substituting in the tensor $a$ each $dx^j$ by $-i\hbar\partial/\partial p_j$.

This is the correspondence given by the symplectic structure $(i/\hbar)\omega_2$.

For later references, let us write the correspondence between symmetric covariant tensor fields on $M$ and vertical differential operators on $T^*M$, once $\omega_2$ is substituted by $(i/\hbar)\omega_2$:

$$\begin{align*}
\hat{a} &= a(x, \dot{x}) \leftrightarrow a = a(x, dx) \leftrightarrow \tilde{\Phi}_a(x, \partial/\partial p) = a(x, -i\hbar\partial/\partial p).
\end{align*}$$

1.2. Introduction of a metric. Classical mechanical systems. Let $T_2$ be a Riemannian metric (non degenerate of arbitrary signature) on the manifold $M$. Such a metric determines an isomorphism of fibre bundles $TM \simeq T^*M$, that allows us to transport from one to each other all the structures that we have considered. Hence, the Liouville form $\theta$ and the symplectic form $\omega_2$ pass from $T^*M$ to $TM$, where we will denote them in the same way.

If the expression of the metric in local coordinates is $T_2 = g_{jk}(x)dx^jdx^k$, the isomorphism $TM \simeq T^*M$ is expressed by the equations $p_j = g_{jk}\dot{x}^k$. The differential operators $\partial/\partial p_j$ transported to $TM$ become $g^{jk}\partial/\partial x^k$, and in the correspondence (1.1), the operator $\tilde{\Phi}_a$ is $a(x, -i\hbar g^{jk}\partial/\partial x^k)$. For the 1-form $\alpha_j = g_{jk}dx^k$, it holds $\dot{\alpha}_j = p_j$, and the corresponding operator $\tilde{\Phi}_a$ is $g_{jk}(-i\hbar g^{k\ell}\partial/\partial x^\ell) = -i\hbar\partial/\partial \dot{x}^j$:

$$p_j \mapsto -i\hbar\frac{\partial}{\partial \dot{x}^j}$$

in the correspondence of functions on $TM$ linear along the fibres with vertical tangent fields.

In coordinates of $TM$, $\theta = g_{jk}\dot{x}^jdx^k$ and the function associated with $\theta$ on $TM$ is $\tilde{\theta} = g_{jk}\dot{x}^j\dot{x}^k = 2T$ where $T$ is the kinetic energy function.
On $TM$ the Hamiltonian tangent field for the function $-T$ is the geodesic field of $(M, T)$; according its very definition, it holds

\begin{equation}
D_G \omega_2 + dT = 0
\end{equation}

For later references, the well known expression of the geodesic field is

\begin{equation}
D_G = \dot{x}^j \frac{\partial}{\partial x^j} - \Gamma^j_{kl}(x) \dot{x}^k \dot{x}^l \frac{\partial}{\partial x^j}
\end{equation}

where the $\Gamma'$ are the Christoffel symbols of the metric (within our convention, $\Gamma^j_{kl} = \{j_{kl}\}$).

Let us recall that a second order differential equation on $M$ is, by definition, a tangent field $D$ on $TM$ such that, as a derivation, takes each $f \in C^\infty(M)$ to $D f = \dot{f}$. Thereby, $D_G$ in (1.4) is a second order differential equation.

Two second order differential equations on $M$ derive in the same way the subring $C^\infty(M)$ of $C^\infty(TM)$. Thus, any second order differential equation $D$ on $M$ is of the form $D = D_G + V$, where $V$ is a vertical tangent field on $TM$.

The vertical tangent fields are the forces of the Classical Mechanics. A classical-mechanical system is a set comprised by three data $(M, T, V)$ of the space of states $T M$.

Equation (1.5) expresses the biunivocal correspondence between second order differential equations on $M$ and horizontal 1-forms on $TM$. The form $\alpha$ is the work form of the mechanical system.

A mechanical system $(M, T, \alpha)$ is said to be conservative when the $\alpha$ is an exact differential form, $dU$. By taking into account that $\alpha$ is horizontal, $U$ have to be a function $\in C^\infty(M)$. The sum $H = T + U$ is the Hamiltonian of the system, and (1.5) is

\begin{equation}
D \omega_2 + dT + \alpha = 0
\end{equation}

Equation (1.5) expresses the biunivocal correspondence between second order differential equations on $M$ and horizontal 1-forms on $TM$. The form $\alpha$ is the work form of the mechanical system.

Let us highlight the following consequence of (1.5), which will be important in Section 2.

**Proposition 1.1.** The Hamiltonian fields on $T^*M$ that, by means of the metric $T_2$, pass into $TM$ as second order differential equations are exactly those that govern the evolution of conservative mechanical systems on $(M, T_2)$ through (1.4).

No other infinitesimal canonical transformation on $T^*M$ is the law of evolution of a mechanical system on $(M, T_2)$.

Equation (1.6), when is written in coordinates of $T^*M$, is the system of Hamilton canonical equations.

A tangent field $u$ in $M$ is an intermediate integral of the field $D$ when the solution-curves of $u$ in $M$, lifted as curves to $TM$ (each point $x$ of the curve goes to the point $(x, u_x)$ of $TM$) is also a solution of $D$. We can think about a given vector field $u$ as a section of the fibre bundle
When passing to $T^*M$, the section $u$ corresponds to a section $\alpha$ of $T^*M \to M$ where $\alpha = u_j T_2$ or, in other words, $u = \text{grad} \alpha$. If the section $\alpha$ is a lagrangian submanifold of $T^*M$, locally $\alpha = dS$ for a certain function $S$ on $M$ (or on some open subset). In such a way, the necessary and sufficient condition for $u = \text{grad} S$ to be an intermediate integral of the Hamiltonian field $D$ in (1.10) is that

$$H(\text{grad} S) = E, \quad \text{constant},$$

where $H(\text{grad} S)$ is the specialization of $H \in \mathcal{C}^\infty(TM)$ to the section $u = \text{grad} S$. (1.7) is the Hamilton-Jacobi equation (see [2, 3]).

### 1.3. De Broglie waves and Schrödinger equation.

Let us consider a conservative mechanical system with configuration space $(M, T_2)$ and Hamiltonian $H = T + U$. Let $S \in \mathcal{C}^\infty(M)$ be a solution of the Hamilton-Jacobi equation (1.7); $\text{grad} S$ is an intermediate integral of the equations of motion. Let $\text{Grad} S$ be the vertical field on $TM$ whose geometric representative is $\text{grad} S$; we have $\text{Grad} S \cdot \omega_2 = \text{grad} S \cdot T_2 = dS$, whereby $\text{Grad} S$ is the Hamiltonian field whose Hamiltonian function is $S$.

The correspondence (1.1) applied to the tensor $dS$ is

$$\dot{S} \leftrightarrow dS = \frac{\partial S}{\partial x^j} dx^j \leftrightarrow -i\hbar \frac{\partial S}{\partial x^j} \frac{\partial}{\partial p_j} = -i\hbar \text{Grad} S.$$  

For first order differential operators we have a rule, previous to any quantization rule, which assigns a field on $M$ to each vertical vector field constant along the fibres of $TM$: to go from a vertical field to its geometric representative. In this case, $-i\hbar \text{Grad} S \leftrightarrow -i\hbar \text{grad} S$.

The correspondence:

**Classical magnitude (function on $TM$), $\dot{S} \to$ Differential operator on $\mathcal{C}^\infty(M)$, $-i\hbar \text{grad} S$ must remain valid in any quantization law.**

On the section $\text{grad} S$ of $TM$, the function $\dot{S}$ takes the value $\dot{S} \mid_{\text{grad} S} = \langle dS, \text{grad} S \rangle = \|\text{grad} S\|^2 = 2(E - U)$. The functions $\varphi$ on $M$ on which the classical magnitude $\dot{S} \mid_{\text{grad} S}$ and its associated differential operator, $-i\hbar \text{grad} S$, act (the first one by means of multiplying) giving the same result, are those which hold the differential equation:

$$-i\hbar \text{grad} S(\varphi) = 2(E - U) \cdot \varphi.$$  

(1.8)

The parameter $t$ proper for the trajectories of the vector field grad$S$ holds on each trajectory

$$dt = \frac{dS}{\|\text{grad} S\|^2} = \frac{dS}{2(E - U)}.$$  

(1.9)

By changing the parameter $t$ by the parameter $S$ on each trajectory, Equation (1.8) is

$$-i\hbar \frac{d\varphi}{dS} = \varphi,$$

which gives

$$\varphi = \varphi_0 e^{i S / \hbar} = \varphi_0 e^{2\pi i E S / \hbar},$$

(1.10)

where $\varphi_0$ is an arbitrary first integral of the vector field grad$S$.

The wave function $\varphi$ is derived form the condition of that, on it, give the same result the action of the differential operator $-i\hbar \text{grad} S$ and (by multiplication) the classical magnitude $\dot{S}$ (from which that operator proceeds) restricted to the section grad$S$. The generalization of this principle of formation of wave equations is that used in [10].

Note that the advance rate of the wavefronts for $\varphi$ in (1.10) is uniform if the quotient $S/E$ is used as time (or a constant multiple), while the time $t$ that measures the motion of the virtual particles in the mechanical system holds (1.0). These two times are different, except for the geodesic field ($U = 0$).
Going back to (1.10) and taking constant \( \varphi_0 \) an straightforward computation gives the identity

\[
\left(-\frac{\hbar^2}{2} \Delta + U\right) \varphi = \left(\frac{\hbar}{2i} \Delta S + H(\text{grad} S)\right) \varphi.
\]

From that identity it is derived the Proposition 1.2 (2).

Let \((M, T^2, dU)\) be a conservative mechanical system. Let \( S \in C^\infty(M) \).

From the three conditions

A) \( S \) holds the Hamilton-Jacobi equation (1.7)

B) \( S \) is harmonic: \( \Delta S = 0 \)

C) \( \varphi = e^{iS/\hbar} \) holds the Schrödinger equation \((-\frac{\hbar^2}{2} \Delta + U) \varphi = E \varphi\)

each couple of them implies the third one.

When \( M \) is oriented, the metric \( T^2 \) gives a volume form and condition B) can restated as

B') \(-i\hbar \text{grad} S\) is self-adjoint.

By admitting a factor \( \varphi_0 \) a first integral of \( \text{grad} S \), not necessarily constant, we obtained

conditions on \( \varphi_0 \) allowing to generalize the above proposition, but it is not possible to reach a true general Schrödinger equation [5, 2]. There is not a continuous path classical-quantic. The pass requires rules of quantization for tensors of order higher than 1.

2. Quantization of contravariant tensors. Dequantization of differential operators

In this section we will study the way in which a symmetric linear connection \( \nabla \) on the configuration space \( M \) determines a canonical biunivocal correspondence (up to the concrete value of \( \hbar \)) between contravariant tensor fields on \( M \) and linear differential operators acting on \( C^\infty(M) \). The passage tensor \( \rightarrow \) differential operator is the rule of quantization defined by \( \nabla \) and the reverse step differential operator \( \rightarrow \) tensor is the rule of dequantization defined by \( \nabla \); this second passage, once given, can be continued with another one tensor \( \rightarrow \) infinitesimal contact transformation on \( T^*M \), which already only depends on the structure of \( T^*M \).

The quantization rule established with the data \((M, \nabla)\) is an almost obvious generalization of the usual rule of quantization on the flat space \((\mathbb{R}^n, d)\), where we denote by \( d \) the connection canonically associated with the vector structure of \( \mathbb{R}^n \) (the “parallel transport” for \( d \) is the transport by linear parallelism). Let us recall such a rule.

Let \( E \) be a real \( n \)-dimensional vector space. Once fixed a system of vector coordinates \((x^1, \ldots, x^n)\) on \( E \), each real symmetric contravariant tensor of order \( r \) at the origin of \( E \) is written in the form:

\[
\Phi_0 = a^{j_1 \cdots j_r} \left( \frac{\partial}{\partial x^{j_1}} \right)_0 \cdots \left( \frac{\partial}{\partial x^{j_r}} \right)_0,
\]

where the \( a \) are real numbers and by \( \cdots \) we denote the symmetrized tensor product.

The linear structure of \( E \) allows us to propagate “by parallelism” the tensor \( \Phi_0 \) to a tensor field \( \Phi \) on the whole of \( E \), whose expression is the same as that of \( \Phi_0 \), by deleting the subindex \( 0 \). This tensor field \( \Phi \) defines on \( C^\infty(E) \) a differential operator

\[
\hat{\Phi} := (-i\hbar)^r a^{j_1 \cdots j_r} \frac{\partial^r}{\partial x^{j_1} \cdots \partial x^{j_r}}.
\]

The assignation \( \Phi \rightarrow \hat{\Phi} \) is independent of changes of vector coordinates on \( E \). When the numbers \( a^{j_1 \cdots j_r} \) are substituted by functions in \( C^\infty(E) \), the same formula assigns to the tensor field \( \Phi \) a differential operator \( \hat{\Phi} \) independently of the concrete choice of vector coordinates.

In we wish that \( \hat{\Phi} \) to be self-adjoint (for the measure translation invariant of \( E \)) it is sufficient to replace it by \( \frac{1}{2}(\hat{\Phi} + \hat{\Phi}^\dagger) \).
When we work on a concrete problem in curvilinear coordinates, the quantization rule is applied by passing the tensors to vector coordinates, quantizing them according to the above rule and, then, coming back to the given curvilinear coordinates.

This recipe for quantization is \textit{intrinsically determined} by the vector structure of $E$. On each $f \in \mathcal{C}^\infty(E)$ we have

$$\hat{\Phi}(f) = (-i\hbar)^r \langle \Phi, d^r f \rangle,$$

where $\langle , \rangle$ denotes tensor contraction, and $d^r f$ is the $r$-th iterated differential of $f$, that has an intrinsic sense on $E$ because of its vector structure.

The generalization to any smooth manifold $M$ endowed with a symmetric (=torsionless) linear connection $\nabla$ is immediate:

\textbf{Definition 2.1 (Quantization defined by $\nabla$).} For each symmetric covariant tensor field of order $r$, $\Phi$, on $(M, \nabla)$, the quantized of $\Phi$ is the differential operator $\hat{\Phi}$ which, for each $f \in \mathcal{C}^\infty(M)$ gives

$$\hat{\Phi}(f) := (-i\hbar)^r \langle \Phi, \nabla^r_{\text{sym}} f \rangle$$

where $\langle , \rangle$ denotes tensor contraction and $\nabla^r_{\text{sym}} f$ is the symmetrized tensor of the $r$-th covariant iterated differential of $f$ with respect to the connection $\nabla$.

The quantized of a non-homogeneous tensor is the sum of the quantized of its homogeneous components.

\textbf{Remark 2.1.} This definition can be generalized giving a differential operator between sections of fibre bundles for each contravariant tensor $\Phi$ on $M$, once a linear connection is fixed in the first fibre bundle. This generalization does not affect what follows, and we leave it aside.

Let us recall that a differential operator of order $r$ on $M$ (= differential operator of order $r$ on $\mathcal{C}^\infty(M)$) is a $\mathcal{C}$-linear map $P: \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ which holds the following condition: for each point $x \in M$, $P$ takes the ideal $m^r_x$ into $m_x$ ($m_x$ is the ideal of the functions of $\mathcal{C}^\infty(M)$ vanishing at $x$).

It is derived that $P$ takes the quotient $m^r_x/m^{r+1}_x \to \mathcal{C}^\infty(M)/m_x = \mathbb{C}$. By taking into account that $m^r_x/m^{r+1}_x$ is the space of symmetric covariant tensors of order $r$ at the point $x$ (homogeneous polynomials of degree $r$, with coefficients in $\mathbb{C}$, in the $dx^1, \ldots, dx^n$, once taken local coordinates), we see that $P$ determines a symmetric contravariant tensor of order $r$ called symbol of order $r$ of $P$ at $x$, denoted by $\sigma^r_x(P)$,

$$\sigma^r_x(P): m^r_x/m^{r+1}_x = T^r_x M \to \mathbb{R},$$

that is the map canonically associated with $P$ by pass to the quotient.

\textbf{Remark 2.2.} Given $f \in m^r_x$, the differential operator $P$ of order $r$ gives $(P f)(x)$, depending only on the class $[f]_{\text{mod}m^{r+1}_x}$. But the identification of $m^r_x/m^{r+1}_x$ with the space of symmetric covariant tensors of order $r$ at the point $x$ is not unique. In order to fix the tensor $\sigma^r_x(P)$ in such a way that its contraction with the symmetric covariant tensor that represents $[f]_{\text{mod}m^{r+1}_x}$, to be $(P f)(x)$, we take such a covariant tensor as $dx^i f$, computed in any local system of coordinates; the covariant tensor $dx^i f$ so calculated for $f \in m^r_x$, does not depends on the choice of coordinates.

When $x$ runs over $M$, we get the tensor field $\sigma^r(P)$ on $M$ called symbol of order $r$ of $P$. If $\sigma^r(P) = 0$, $P$ is of order $r - 1$.

In the case $M = \mathbb{R}^n$, with vector coordinates $x^1, \ldots, x^n$, let us denote $\partial^a$ the tensor $\partial^a := (\partial/\partial x^1)^{a_1} \cdots (\partial/\partial x^n)^{a_n}$. Its quantized by the rule (2.1) (with the vector connection of $\mathbb{R}^n$) is $\hat{\partial}^a := (-i\hbar)^r \partial^a$, where $\partial^a$ is the differential operator $\partial^{[a}/(\partial x^1)^{a_1} \cdots (\partial x^n)^{a_n}$. It is directly seen that $\sigma^{[a}(D^a) = \partial^a$, so that for any tensor field of order $r = |\sigma|$ on $\mathbb{R}^n$ is obtained, by adding terms,

$$\sigma^r(\hat{\Phi}) = (-i\hbar)^r \Phi$$
Going from $\mathbb{R}^n$ to the general case $(M, \nabla)$ let us observe that, when the iterated differentials of a function $f$ are calculated in local coordinates, the derivatives of order $r$ of $f$ appear in terms which do not contain Christoffel symbols (as in the case of $\mathbb{R}^n$). Since the symbol of an operator of order $r$ depends only on these terms, Formula (2.3) is still valid in general for the quantization rule (2.1) on $(M, \nabla)$.

**Theorem 2.1.** The rule of quantization (2.1) establishes a biunivocal correspondence between linear differential operators $P$ and symmetric contravariant tensor fields (not necessarily homogeneous) on $M$. To the operator $P$ of order $r$ corresponds the tensor $\Phi = \Phi_r + \Phi_{r-1} + \cdots + \Phi_0$ (each $\Phi_j$ denotes the homogeneous component of degree $j$) such that

$$\sigma^r(P) = (\pm i\hbar)^r \Phi_r$$

and, for $k = 1, \ldots, r$:

$$\sigma^{r-k}(P - \hat{\Phi}_r - \cdots - \hat{\Phi}_{r-k+1}) = (\pm i\hbar)^{r-k} \Phi_{r-k}$$

and

(2.4) $$P = \hat{\Phi} = \hat{\Phi}_r + \hat{\Phi}_{r-1} + \cdots + \hat{\Phi}_0.$$ 

**Definition 2.2** (Dequantization). The contravariant tensor $\Phi$ in (2.4) is the dequantized of the differential operator $P$ by the connection $\nabla$.

We have seen in Section 1.1 that symmetric contravariant tensor fields (homogeneous or not) on $M$ canonically correspond with functions $f \in C^\infty(T^*M)$ polynomials along the fibres.

**Definition 2.3.** The function $F \in C^\infty(T^*M)$ corresponding to the tensor $\Phi$ dequantized of the differential operator $P$ will be called Hamiltonian of $P$ with respect to the connection $\nabla$.

The symplectic structure $\omega_2$ of $T^*M$ assign to each $F \in C^\infty(T^*M)$ a Hamiltonian vector field $D_F$, as we have already remembered in section 1.1, by the rule $D_F \cdot \omega_2 = dF$. These hamiltonian fields are the infinitesimal contact transformations of Lie [6, 7]; they are the infinitesimal generators of the 1-parametric groups of automorphisms of $T^*M$ which preserve its symplectic structure.

**Definition 2.4** (Hamiltonian field associated with a differential operator). We will call infinitesimal contact transformation associated with the differential operator $P$ or Hamiltonian field associated with $P$ to the tangent field $D_P$ on $T^*M$ such that

$$D_P \cdot \omega_2 + dF = 0,$$

where $F$ is the Hamiltonian of $P$.

The path $P \rightarrow F \rightarrow D_P$ is univocal. The reverse path $D_P \rightarrow F$ determines $F$ up to an additive constant; then, $F \rightarrow P$ is univocal. Thus, up to an additive constant for $P$, the correspondence $P \leftrightarrow D_P$ is biunivocal.

**Theorem 2.2.** The symmetric linear connection $\nabla$ on $M$ canonically establishes a biunivocal correspondence between linear differential operators $P$ on $C^\infty(M)$ (up to additive constants) and infinitesimal canonical transformations of the simplectic manifold $T^*M$ corresponding to functions polynomial along fibres (Hamiltonians).

Let us assume that $M$ is endowed with a Riemannian metric $T_2$ (of arbitrary signature = pseudoriemannian metric) and $\nabla$ the associated Levi-Civita connection. Under these conditions, it makes sense to say whether or not a tangent field $D$ on $T^*M$ is a second-order differential equation; that is, the tangent field that governs a mechanical system with the configuration space $(M, T_2)$. We have,
Theorem 2.3. The necessary and sufficient condition for a linear differential operator $P$ on $(M, T_2)$ to have as associated infinitesimal contact transformation $D_P$ a second order differential equation is that $P$ is of the form

$$P = -\frac{\hbar^2}{2} \Delta + U$$

where $\Delta$ is the Laplacian operator of the metric and $U \in C^\infty(M)$.

Proof. Let us begin by checking that the tensor $\Phi$, contravariant form of the metric tensor, has as quantized operator $\hat{\Phi} = -\hbar^2 \Delta$. In local coordinates, with $T_2 = g_{jk} dx^j dx^k$, is $\Phi = g^{rs} \frac{\partial}{\partial x^r} \otimes \frac{\partial}{\partial x^s}$. The expression for the second iterated covariant differential is

$$\nabla^2 f = \left( \frac{\partial^2 f}{\partial x^k \partial x^j} - \Gamma^j_{jk} \frac{\partial f}{\partial x^l} \right) dx^j \otimes dx^k,$$

by contracting with $\Phi$,

$$\langle \Phi, \nabla^2 f \rangle = g^{jk} \left( \frac{\partial^2 f}{\partial x^k \partial x^j} - \Gamma^j_{jk} \frac{\partial f}{\partial x^l} \right) = \Delta f.$$

By incorporating to $\phi$ the factor $(-i\hbar)^2$ we see that the quantized of $\Phi$ is $-\hbar^2 \Delta$.

The Hamiltonian function corresponding to the tensor $\Phi$ is $g^{rs} p_r p_s = 2T$ (where $T$ is the kinetic energy function). Finally, for the Hamiltonian $H = T + U$, the corresponding quantum operator is $(-\hbar^2/2) \Delta + U$.

When dequantizing, we go from the operator $(-\hbar^2/2) \Delta + U$ to the Hamiltonian $T + U = H$, and, then to the Hamiltonian field $D_P$ such that $D_P \omega_2 + dH = 0$; $D_P$ is the field of the canonical equations for the mechanical system $(M, T, dU)$.

Conversely, let us assume that $D_P$ is a second order differential equation. Equation (2.5) gives that it holds $D_P \omega_2 + dT + \alpha = 0$, where $\alpha$ is horizontal; since $D_P$ is a contact infinitesimal transformation, $\alpha$ has to be exact, so that of the form $dU$ for some $U \in C^\infty(M)$: $D_P \omega_2 + dH = 0$, for $H = T + U$. Since $T$ is the Hamiltonian function associated with the operator $\frac{1}{2} \Phi$ as before, when quantizing it turns that $P = -\frac{\hbar^2}{2} \Delta + U$. \hfill $\square$

Problem. There is something similar to a Schrödinger equation for non-conservative mechanical systems?

Remarks on $\sigma^r(P)$ for $P$ of order $r$. (1) Let $\Phi_r$ be an homogeneous tensor of order $r$ that corresponds to $P$ by (2.4). Considered as a function on $T^* M$, $\Phi_r$ is $F_r$, homogeneous of degree $r$ on the fibres. The first order partial differential equation $F_r((dS)^r) = 0$ has as solutions the hypersurfaces $S = \text{const}$. characteristic for the differential operator $P$; they are the hypersurfaces of $M$ where the problem of initial conditions cannot be treated by the Cauchy-Kowalevski method (for instance, for $\Delta$, the equation of characteristics is $\|dS\|^2 = 0$, the “Eikonal equation”). The Hamiltonian field of $F_r$ has as solutions the bicharacteristics of $P$. This field does not coincide, in general, with $D_P$. The field which propagates the singularities of $P$ is the hamiltonian field of $F_r$, not the one of the total Hamiltonian of $P$, $D_P$.

(2) The relationship between the Poisson bracket of two Hamiltonians and the commutator of the corresponding quantum operators is:

$$\sigma^{r+s-1}(\hat{\Phi}, \hat{\Psi}) = -\{\sigma^r(\hat{\Phi}), \sigma^s(\hat{\Psi})\}$$

where $\Phi$ is a tensor of order $r$, $\Psi$ of order $s$, $\{,\}$ is the commutator of quantized tensors and $\{,\}$ is the Poisson bracket of $\sigma^r(\hat{\Phi})$, $\sigma^s(\hat{\Psi})$, by identified with the functions they define on $T^* M$ (the minus sign proceeds from the convention taken in (7.3) for the Poisson bracket).

Formula (2.5) is valid for every connection $\nabla$, and its proof can be done as in the case of $\mathbb{R}^n$ with the vector connection, since only highest order terms of the operator intervene and we can use the symplectic form $\omega_2 = dp_j \wedge dx^j$ as in vector coordinates. The checking of (2.5) is a simple calculation.
3. Quantization by means of Riemannian exponential. Families of quantizations parameterized by $h$

In [10] we have presented a quantization rule for the classical system $(M, T_2)$ by means of a linear symmetric connection $\nabla$ on $M$. That rule is defined from the geodesic field $D$ associated with the connection. Instead of use directly $\nabla$, we use the geodesic field $D$ of $\nabla$. The flow of the field $D$ on $TM$ allows us to establish an isomorphism of manifolds between a certain neighborhood $U_x$ of vector 0 in $T_x M$ and a neighborhood $U_x$ of $x$ in $M$, by associating with the vector $v_x \in U_x$ the final point of the geodesic (curve solution of $D$) parameterized by $[0, 1]$ that starts from the point $x$ with initial velocity $v_x$. When $x$ runs over $M$, the union of all the $U_x$ is a neighborhood $U$ of the 0-section in $TM$, and the flow of $D$ defines, in the above described way, a differentiable map $\exp: U \to M$, in which the 0-section of $TM$ is identified (as a part of $U$) with $M$.

For each $f \in C^\infty(M)$, let

$$\hat{f} := \exp^*(f) \in C^\infty(U).$$

Of $\hat{f}$ the only thing we are interested in is its germ at the section 0 of $TM$. If we denote by $\mathcal{O}(M)$ the ring of germs of differentiable functions in neighborhoods of the section 0 of $TM$, we identify $\hat{f}$ with its germ $\in \mathcal{O}(M)$. Thus, we have an injection of rings $C^\infty(M) \hookrightarrow \mathcal{O}(M)$, $f \mapsto \hat{f}$.

In the injection of rings $C^\infty(M) \hookrightarrow C^\infty(TM)$ produced by the natural projection $TM \to M$, the $f \in C^\infty(M)$ give functions constant along the fibres, annihilated by each vertical differential operator on $TM$ (except these of order 0). But in the injection $f \mapsto \hat{f}$, such operators no longer annihilate the $\hat{f}$.

In Section 1.2 we have seen how, with each covariant tensor field $a$ there is a vertical contravariant tensor field $\hat{\Phi}_a$ on $TM$ associated by means of a rule determined by the metric $T_2$ and the symplectic form of $T^*M$ (or, alternatively, the Fourier transform). In local coordinates, $\hat{\Phi}_a$ is obtained by substituting into the expression $a(x, dx)$, each $dx^j$ by the vertical vector field $-ih\partial/\partial x^j$. Or, by considering the coordinates $p_j$ as 1-forms, by substituting each $p_j$ by $-ih\partial/\partial x^j$ (1.2).

The quantization rule given in [10] is

**Definition 3.1** (Quantization by the exponential). Let $(M, T_2)$ a configuration space, $\nabla$ a symmetric linear connection on $M$, $C^\infty(M) \hookrightarrow \mathcal{O}(M)$ ($f \mapsto \hat{f}$) the injection determined by the exponential defined by the geodesic field $D$ of $\nabla$. For each symmetric covariant tensor field $a$ of order $r$, the differential operator $\tilde{a}$ quantized of the function $a$ by $\nabla$ gives, for each $f \in C^\infty(M)$ the value

$$\tilde{a}(f) := \langle \hat{\Phi}_a, df \hat{f} \rangle$$

where $\langle , \rangle$ is the tensor contraction and $df$ is the $r$-th differential of $f$ along each fibre of $TM$, and taking the value at the 0 section.

The “vertical differential” $df$ makes sense due to $T_x M$ is a vector space.

So as not to get lost in technicalities in the discussion that follows, let us suppose that the geodesic field $D$ of $\nabla$ is complete. Let $\{\tau_s\}_{s \in \mathbb{R}}$ the 1-parametric group of automorphisms of the manifold $TM$ generated by $D$. The exponential map is, in this case, the composition

$$TM \xrightarrow{\tau_s} TM \xrightarrow{\exp} M \xrightarrow{\pi}$$

From the mathematical point of view, the restriction of the parameter $s$ to the value 1 is artificial. The natural thing is to consider an arbitrary segment $[0, s]$, $\tau_s$ instead of $\tau_1$,
exp_s = \pi \circ \tau_s \text{ and } \hat{f}_s = \exp_s'(f). The classical magnitude \dot{a} will be quantized as the differential operator \widehat{a}_s:

\begin{equation}
\widehat{a}_s(f) := \langle \hat{\Phi}_a, d^r_0 \hat{f}_s \rangle
\end{equation}

The interesting thing is that the quantization rule \dot{a} \rightarrow \widehat{a}_s changes in such a way that \widehat{a}_s = s^r \hat{a} for tensors of order \( r \). Indeed, since \( D \) is a field of the form \([1,4]\) (it does not matter how are the Christoffel symbols), it holds that \( \pi \circ \tau_s(x, v_x) = \pi \circ \tau_1(x, sv_x) \) (the final point of the geodesic parameterized by \([0, s]\) with initial tangent vector \( v_x \) at \( x \), is the same that the final point of the geodesic parameterized by \([0, 1]\) with initial tangent vector \( sv_x \)). It follows that \( \hat{f}_s(x, v_x) = \hat{f}(x, sv_x) \), that is to say:

\[ \hat{f}_s = \hat{f} \circ \text{(Homothetie of ratio } s \text{ along each fibre of } TM) \]

It is derived that \( d^r_0 \hat{f}_s = s^r d^r_0 \hat{f} \), then \( \widehat{a}_s = s^r \hat{a} \). This means that quantization \( \dot{a} \rightarrow \widehat{a}_s \) is deduced from quantization \( \dot{a} \rightarrow \hat{a} \) by replacing \( h \) by \( sh \). The field \( D \) canonically produces a 1-parametric family of quantizations whose parameter is the Planck “constant”.

4. Identity of the two considered rules of quantization

In this section all the functions are real.

Maintaining the above notation, \( M \) is an smooth manifold of dimension \( n \), \( \nabla \) is a symmetric linear connection on \( M \). The exponential map associated with \( \nabla \) (defined on a neighborhood of \( 0 \) on each fibre \( T_{x_0}M \)) assigns to each vector \( v_{x_0} \in T_{x_0}M \) the point \( \exp(v_{x_0}) \in M \) that is the final point of the geodesic of \( \nabla \) parameterized by \([0, 1]\) which starts from \( x_0 \) with tangent vector \( v_{x_0} \). The local isomorphism \( \exp : T_{x_0}M \rightarrow M \) assigns to each function \( f \in C^\infty(M) \) a differentiable function defined in a neighborhood of \( 0 \) in \( T_{x_0}M \); when \( x_0 \) runs over \( M \), \( f \) gives a function \( \hat{f} \) defined in a neighborhood of the \( 0 \)-section of \( TM \); the map \( f \mapsto \hat{f} \) injects \( C^\infty(M) \) into the ring \( \mathcal{O}(M) \) comprised by germs of smooth functions on neighborhoods of the \( 0 \)-section of \( TM \).

Whatever the local coordinates \( x^1, \ldots, x^n \) in a neighborhood of \( x_0 \), the corresponding \( \dot{x}^1, \ldots, \dot{x}^n \) are linear coordinates on \( T_{x_0}M \). Thus, for each \( g \in C^\infty(T_{x_0}M) \) the following tensor is intrinsically defined

\begin{equation}
d^r_0 g = \sum_{j_1, \ldots, j_r=1}^n \left. \frac{\partial^r g}{\partial \dot{x}^{j_1} \cdots \partial \dot{x}^{j_r}} \right|_{v_{x_0}}(0) \, d\dot{x}^{j_1} \cdots d\dot{x}^{j_r}
\end{equation}

The local isomorphism \( \exp : T_{x_0}M \rightarrow M \) gives an isomorphism of tangent spaces \( T_0(T_{x_0}M) \simeq T_{x_0}M \) (the already known) makes to correspond to each vertical vector in \( TM \) its geometric representative: \( \left( \partial/\partial \dot{x}^j \right|_{x_0} \rightarrow \left( \partial/\partial x^j \right|_{x_0} \). The dual morphism makes to correspond \( d_{x_0}x^j \) to \( d^r_0 \dot{x}^j \). This isomorphism transforms the \( d^r_0 g \) of \([1,1]\) into a tensor at the point \( x_0 \) of \( M \). In particular, for each \( f \in C^\infty(M) \) we define the tensor

\begin{equation}
d^r_{x_0} f := \sum_{j_1, \ldots, j_r=1}^n \left. \frac{\partial f}{\partial \dot{x}^{j_1} \cdots \partial \dot{x}^{j_r}} \right|_{x_0}(0) \, d\dot{x}^{j_1} \cdots d\dot{x}^{j_r}
\end{equation}

which is the tensor at \( x_0 \in M \) that corresponds to \( d^r_0 \hat{f} \) in \([5,2]\) by means the isomorphism \( T_{x_0}M \simeq T_0(T_{x_0}M) \).

**Theorem 4.1.** For each \( f \in C^\infty(M) \) and each \( r \) is

\begin{equation}
d^r_{x_0} f = \nabla^r_{x_0, sym} f
\end{equation}

**Proof.** We begin by considering the case in which \( f \) is one of the coordinates, \( f = x^1 \). The equation of the geodesics of \( \nabla \) gives

\begin{equation}
\frac{d^2 x^1}{ds^2} = -\Gamma^1_{jk} \dot{x}^j \dot{x}^k = \langle \nabla(dx^1), \dot{x} \otimes \dot{x} \rangle
\end{equation}
where \( \langle \quad , \quad \rangle \) is the coupling given by duality and \( \dot{x} \) is the vector tangent to the geodesic in the point corresponding to the value \( s \) of the parameter: \( \dot{x}^j = dx^j/ds \).

Differentiating (4.3) and applying the equation of the geodesics, we obtain

\[
\frac{d^r x^1}{ds^r} = -\left( \frac{\partial \Gamma^j_{rk}}{\partial x^r} - \Gamma^1_{rk} \Gamma^r_{j\ell} - \Gamma^1_{rj} \Gamma^r_{k\ell} \right) \dot{x}^j \dot{x}^k \dot{x}^\ell = \langle \nabla \nabla x^1, \dot{x} \otimes \dot{x} \otimes \dot{x} \rangle
\]

\[
\ldots
\]

By applying at each step the equation of geodesic, we obtain by induction

(4.5)

\[
\frac{d^r x^1}{ds^r} = \langle \nabla^{r-1} \ldots \nabla x^1, \dot{x} \otimes \ldots \otimes \dot{x} \rangle = \langle \nabla^{r} x^1, \dot{x} \otimes \ldots \otimes \dot{x} \rangle
\]

For the geodesic starting from \( x_0 \) with vector velocity \( \xi \) of coordinates \( \dot{x}^j(0) = \xi^j \), we will write its equations in the form \( x^j = x^j(s; \xi^1, \ldots, \xi^n) \). According with definitions, it will be

\[
\hat{x}^j(\xi^1, \ldots, \xi^n) = x^j(1; \xi^1, \ldots, \xi^n).
\]

By applying the well known property of the geodesic (used in Section 3) we will have, for \( |s| < 1 \):

\[
\hat{x}^j(s \xi^1, \ldots, s \xi^n) = x^j(s; \xi^1, \ldots, \xi^n).
\]

Differentiating \( r \) times and, then, taking the value for \( s = 0 \), we obtain, since (4.5) and (4.2),

\[
\langle d^r_0 x^1, \xi \otimes \ldots \otimes \xi \rangle = \langle \nabla^{r}_{x_0, \text{sym}} x^1, \xi \otimes \ldots \otimes \xi \rangle
\]

from which it turns the validity of (4.3) for \( f = x^1 \).

Now, let \( f \in C^\infty(M) \) be such that \( d_{x_0} f \neq 0 \). Such an \( f \) can be taken as coordinate \( x^1 \) in a system of local coordinates \( (x^1, x^2, \ldots, x^n) \) around \( x_0 \). Thus, (4.3) holds for \( f \).

Finally, let \( f \in C^\infty(M) \) arbitrary. We can express \( f \) as the sum \( f = f_1 + f_2 \) where \( d_{x_0} f_1 \neq 0 \) and \( d_{x_0} f_2 \neq 0 \). As (4.3) is linear in \( f \), being true for \( f_1 \) and \( f_2 \), also it is true for \( f \).

**Conclusion.** The quantization defined in Section 3 (by means of the Riemannian exponential) and the one defined in Section 2 (by direct pairing between tensors) are identical. Indeed, the first one is obtained (in addition to the factors \( -i\hbar \)) for each tensor by applying on each \( T_{x_0} M \) the direct quantization described in Section 2 for \( \mathbb{R}^n \), and applying to each \( f \mid_{T_{x_0} M} \) by (4.1), (4.2), (4.3), this is equivalent to directly coupling each \( \nabla^{r}_{x_0, \text{sym}} f \) with the given tensor.

The exponential map reduces the symmetric covariant differential of arbitrary order to the corresponding linear differentials along each fibre.

**References**

[1] Ali, S. T.; Englîss, M., Quantization methods: a guide for physicists and analysts. Rev. Math. Phys. 17 (2005), no. 4, pp. 391-490.

[2] Alonso-Blanco, R.J. and Muñoz Díaz, J., Una nota sobre los fundamentos de la mecánica (spanish), In El legado matemático de Juan Bautista Sancho Guimerà (edited by A. Campillo and D. Hernández-Ruipérez), 111-138. Ediciones Universidad de Salamanca and Real Sociedad Matemática Española, 2015.

[3] Alonso-Blanco, R.J. and Muñoz Díaz, J., A note on the foundation of mechanics. arXiv:1404.1321 [math-ph].

[4] Bongaarts, P., Quantum theory. A mathematical approach. Springer, Cham, 2015.

[5] Holland, P.R., The quantum theory of motion: an account of the de Broglie-Bohm causal interpretation of quantum mechanics, Cambridge University Press, Cambridge, 1993.

[6] Lie, S., Theorie der Transformationsgruppen (german). Written with the help of Friedrich Engel. Teubner, Leipzig, 1888.

[7] Lie, S., Geometrie der Berührungstransformationen (german). Written with the help of Georg Scheffers. B. G. Teubner, Leipzig, 1896.
[8] Lychagin, V., Quantum mechanics on manifolds. Geometrical aspects of nonlinear differential equations. Acta Appl. Math. 56 (1999), no. 2-3, 231-251.

[9] Muñoz Díaz, J., The structure of time and inertial forces in Lagrangian mechanics, Contemporary Mathematics, vol. 549, 2011, pp. 65-94.

[10] Muñoz Díaz, J., and Alonso-Blanco, R.J., Quantization of mechanical systems, J. Phys. Commun. 2 (2018) 025007, https://doi.org/10.1088/2399-6528/aaa850.

[11] Woodhouse, N. M. J., Geometric quantization. Second edition. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1992.

Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1-4, E-37008 Salamanca, Spain.
E-mail address: clint@usal.es, ricardo@usal.es