Non-homogeneous Lorentz-Herz Spaces: Interpolation and Applications

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ABSTRACT

Herz spaces with special exponents initially appeared in the work of C. S. Herz on mapping properties of Fourier transform on a Lipschitz class. After S. Lu and D. Yang introduced these spaces with general indices, they have gained a considerable interest in harmonic analysis. The purpose of this paper is to study a Lorentz space variant of the classical Herz spaces called the Lorentz-Herz spaces $HL_{p,q}^{a,r}$, in which the underlying Lebesgue space metric is replaced with a Lorentz space metric. Besides their fundamental properties, we obtain their Banach function space characterization. Using the real interpolation method, we determine their interpolation spaces. In the non-diagonal case ($q \neq p$), it turns out that the scale of non-homogeneous Lorentz-Herz spaces is closed under interpolation. As an application, we prove the boundedness of a class of operators on these spaces, which include Maximal and Caledron-Zygmund singular operators.

keywords: Herz spaces; Lorentz spaces; Banach function spaces; Interpolation.

1 Introduction

The classical Herz spaces originated from the work of C. S. Herz [3], who introduced a class of functions as a suitable environment for the range of Fourier transform acting on a class of Lipschitz spaces. However, the space of functions introduced by Herz is just prototype of Herz spaces. R. Johnson [8] characterized the norm of these spaces in terms of Lebesgue space norms over annuli. Following this characterization, Lu and Yang introduced two types of Herz spaces [6], called the homogeneous Herz spaces and the non-homogeneous Herz spaces. Let $S_k = \{x \in \mathbb{R}^N : |x| < 2^k \}$ and $R_k = S_k \setminus S_{k-1}$ for $k \in \mathbb{Z}$. Denote $\tilde{\chi}_{R_k} = \chi_{R_k}$ for $k \in \mathbb{Z}^+$ and $\tilde{\chi}_{R_{-1}} = \chi_{S_{-1}}$.

Definition 1.1. For $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$. The homogeneous Herz space $\dot{K}_{p,q}^\alpha (\mathbb{R}^N)$ is defined by

$$\dot{K}_{p,q}^\alpha (\mathbb{R}^N) = \{ f \in L_{\text{loc}}^p (\mathbb{R}^N \setminus \{0\}) : \| f \|_{K_{p,q}^\alpha (\mathbb{R}^N)} < \infty \},$$

where

$$\| f \|_{K_{p,q}^\alpha (\mathbb{R}^N)} = \left( \sum_{k \in \mathbb{Z}} 2^{kap} \| f \chi_{R_k} \|_{L^q}^p \right)^{\frac{1}{p}}.$$
The non-homogeneous Herz space $K^\alpha_{p,q}(\mathbb{R}^N)$ is defined by

$$K^\alpha_{p,q}(\mathbb{R}^N) = \{ f \in L^q_{\text{loc}}(\mathbb{R}^N) : \| f \|_{K^\alpha_{p,q}(\mathbb{R}^N)} < \infty \},$$

where

$$\| f \|_{K^\alpha_{p,q}(\mathbb{R}^N)} = \left( \sum_{k=-1}^{\infty} 2^{k \alpha p} \| f \chi_{R_k} \|_{L^q}^p \right)^{\frac{1}{p}}.$$

The usual modifications are made if $p$ or $q$ is infinite.

In last few decades, the theory of these spaces had a remarkable development due to several applications. For instance, they appear to be good substitutes of ordinary Hardy spaces when considering boundedness of non-translation invariant singular integral operators [26, 7], they are useful in characterizing multipliers on Hardy spaces and in regularity theory for elliptic and parabolic equations [5, 27].

It was observed in the first half of last century that the scale of classical Lebesgue spaces is not sufficiently rich to describe some fine properties of functions and operators. As a result, several other classes of measurable functions were developed and studied. In particular, G. G. Lorentz introduced Lorentz spaces [23, 24], which is more general scale of function spaces than the classical Lebesgue spaces. These spaces include the classical Lebesgue spaces as a special case and play very important role in analysis, for example, in the interpolation theory and Fourier analysis [1, 2, 25].

The scale of Herz spaces can be naturally improved by replacing the underlying Lebesgue norm with a Lorentz norm. The Lorentz version of Herz spaces, which we call Lorentz-Herz spaces, can be seen as refinements of both Herz spaces and weak Herz spaces. One of the main objectives of this paper is to isolate all those Herz-Lorentz spaces which fall in Banach function space category. Such a characterization can be fruitful as Banach function spaces have a well-developed theoretical foundation. Also, this study investigates the interpolation properties of these spaces and the behavior of some well-known classical operators acting on them. The Lorentz-Herz spaces appear to be useful in certain aspects, for example, in the description of interpolation spaces of the Herz spaces.

The paper is organized as follows: In Section 2, we collect some background material and fix some notations required in this paper. The Lorentz-Herz spaces are introduced in Section 3, where we provide sufficient conditions for them to be Banach function spaces. In the same section, we also obtain their completeness and give some embedding results involving these spaces. Section 4 deals with the description of associate spaces and a few topological properties of Lorentz-Herz spaces. Therein, a Banach function space characterization of these spaces is given. The intermediate spaces using the real interpolation method are obtained in Section 5. Finally, in Section 6 we establish the boundedness of a class of operators on the aforementioned spaces from the corresponding boundedness on Lorentz spaces.

2 Preliminaries

This section quotes some notations, definitions and results that will be followed throughout the paper. Let $(\Omega, \Sigma, \mu)$ denotes a $\sigma$-finite measure space, where $\Omega \subseteq \mathbb{R}^N$ (for some $N \in \mathbb{N}$) and let $\mathcal{M}$ be the set of $\mu$-measurable functions on $\Omega$. Also, let $\mathcal{M}^+$ be the set of non-negative elements of $\mathcal{M}$ and $\chi_E$ be the indicator function of $E \subseteq \Omega$. Moreover, unless specified, $C$ denotes a constant irrespective of its value. We start with the definition of a function norm.
Definition 2.1. \((\mathcal{M})\) A map \(\eta : \mathcal{M}^+ \to [0, \infty]\) is said to be a (Banach) function norm over \(\Omega\), if the following properties hold for all \(f, g, f_n (n = 1, 2, 3, \ldots)\) in \(\mathcal{M}^+\) and for all \(\mu\)-measurable subsets \(E \subseteq \Omega\):

(P1) \(\eta(f) = 0 \iff f = 0 \text{ } \mu\text{-a.e.} ; \quad \eta(\alpha f) = \alpha \eta(f) \quad (\alpha \geq 0) ; \quad \eta(f + g) \leq \eta(f) + \eta(g)\).

(P2) If \(0 \leq f \leq g \text{ } \mu\text{-a.e.}, \text{ then } \eta(f) \leq \eta(g)\).

(P3) If \(0 \leq f_n \uparrow f \text{ } \mu\text{-a.e.}, \text{ then } \eta(f_n) \uparrow \eta(f)\).

(P4) If \(\mu(E) < \infty\), then \(\eta(1_E) < \infty\).

(P5) If \(\mu(E) < \infty\), then \(\int_E f \, d\mu \leq C_E \eta(f)\) for some finite positive constant \(C_E\), independent of \(f\).

If \(\eta\) is a function norm, then a Banach function space \(X\) is the collection \(X = \{f \in \mathcal{M} : \eta(|f|) < \infty\}\) equipped with the norm \(\|f\|_X := \eta(|f|)\). Moreover, given a function norm \(\eta\), there is a function norm \(\eta'\), defined by

\[
\eta'(g) = \sup \left\{ \int_\Omega f g \, d\mu : f \in \mathcal{M}^+, \eta(f) \leq 1 \right\}, \quad g \in \mathcal{M}^+.
\]

\(\eta'\) is called the associate function norm. It determines another Banach function space known as the associate space of \(X\), denoted by \(X'\). The norm of a function \(g \in X'\) is given by

\[
\|g\|_{X'} = \sup \left\{ \int_\Omega |fg| \, d\mu : f \in X, \|f\|_X \leq 1 \right\}.
\]

Let \(X\) be a Banach function space. A function \(f \in X\) is said to have absolutely continuous norm if \(\|f_{1_{EN}}\|_X \to 0\) for every sequence \(\{E_N\}_{N=1}^\infty\) of \(\mu\)-measurable subsets of \(\Omega\) satisfying \(1_{E_N} \to 0\) pointwise \(\mu\)-a.e.. Suppose \(X_a\) denotes the collection of all functions in \(X\) having absolutely continuous norm. Then \(X\) is said to have absolutely continuous norm if \(X_a = X\). Also, let \(X_b\) denotes the closure of the set of simple functions in \(X\). The following results taken from [1] are useful in the sequel.

Theorem 2.2. Let \(X^*\) be the dual space of \(X\). Then \(X^*\) is canonically isometrically isomorphic to the associate space \(X'\) if and only if \(X\) has absolutely continuous norm.

Theorem 2.3. The following inclusions hold:

\[X_a \subseteq X_b \subseteq X.\]

Theorem 2.4. The space \(X\) is separable if and only if \(X\) has absolutely continuous norm and the underlying measure \(\mu\) is separable.

Next, for \(f \in \mathcal{M}\), the function \(\mu_f : [0, \infty) \to [0, \infty]\) defined by

\[
\mu_f(\alpha) = \mu\{t \in \Omega : |f(t)| > \alpha\}, \quad \alpha \geq 0
\]

is called the distribution function of \(f\). Further, the non-increasing rearrangement of a function \(f\) is the map \(f^* : [0, \infty) \to [0, \infty]\) given by

\[
f^*(s) = \inf\{\alpha \geq 0 : \mu_f(\alpha) \leq s\}, \quad s \in [0, \infty).
\]

Additionally, the average function \(f^{**} : (0, \infty) \to [0, \infty]\) is defined by

\[
f^{**}(s) = \frac{1}{s} \int_0^s f^*(t) \, dt.
\]

We have the following useful estimate for the non-increasing rearrangement of sums.
Theorem 2.5. ([9]) Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence in \( \mathcal{M}^+ \) and \( c_n > 0 \) be such that \( \sum_n c_n = 1 \). Then we have

\[
\left( \sum_{n \in \mathbb{N}} f_n \right)^* (3t) \leq \sum_{n \in \mathbb{N}} \left( f_n^{**}(t) + \frac{1}{t} \int_{c_n t}^t f_n^*(s) ds \right).
\]

The following corollary is evident from the above theorem.

Corollary 2.6. For a sequence \( \{f_n\}_{n \in \mathbb{N}} \) in \( \mathcal{M}^+ \), we have the following inequality

\[
\left( \sum_{n \in \mathbb{N}} f_n \right)^* (t) \leq 2 \sum_{n \in \mathbb{N}} f_n^{**} \left( \frac{t}{3} \right).
\]

Now we define two parameter Lorentz spaces.

Definition 2.7. The Lorentz space \( L^{p,r} = L^{p,r}(\Omega) \) consists of all \( \mu \)-measurable functions on \( \Omega \) for which the functional \( \|f\|_{L^{p,r}} \) is finite, where

\[
\|f\|_{L^{p,r}} = \begin{cases} 
\left( \int_0^\infty \left( t^\frac{1}{p} f^*(t) \right)^r \frac{dt}{t} \right)^{\frac{1}{r}} & \text{if } 0 < p < \infty, 0 < r < \infty \\
\sup_{t>0} t^\frac{1}{p} f^*(t) & \text{if } 0 < p < \infty, r = \infty.
\end{cases}
\]  

(1)

Note that \( \|f\|_{L^{p,r}} \) is a quasi-norm but not necessarily a norm [1][10]. For \( 0 < q \leq p \leq r \leq \infty \), we have continuous embedding.

\[
L^{p,q} \subset L^{p,p} \subset L^{p,r} \subset L^{p,\infty}.
\]  

(2)

Let \( \|f\|_{L^{p,r}}^* \) be the functional obtained from equation (1) by replacing \( f^* \) with \( f^{**} \). Then \( L_r^p = (L_{p,r}^*, \| \cdot \|_{L_{p,r}^*}) \) is a normed space for \( 1 < p < \infty \), \( 1 \leq r \leq \infty \), \( p = r = 1 \) or \( p = r = \infty \). The following relation between the quasi-norm \( \|f\|_{L^{p,r}}^* \) and the norm \( \|f\|_{L^{p,r}} \) is known.

Theorem 2.8. ([10]) If \( 1 < p < \infty \), \( 1 \leq r \leq \infty \) or \( p = r = \infty \), then

\[
\|f\|_{L^{p,r}} \leq \|f\|_{L^{p,r}}^* \leq \left( \frac{p}{p-1} \right) \|f\|_{L^{p,r}},
\]

with the convention that \( \frac{p}{p-1} = 1 \) for \( p = \infty \).

Moreover, it is easy to observe that \( \|f\|_{L^{1,1}} = \|f\|_{L^{1,\infty}}^* \). For more details on these topics, readers can refer to [1][2][10].

Now we give a quick review of the real interpolation method. As mentioned several times in [11] that the definitions and facts about the real interpolation of Banach spaces are equally applicable for quasi-Banach spaces without any essential modifications, we proceed with the real interpolation method for quasi-Banach spaces. Let \( Z = (Z_0, Z_1) \) be a pair of quasi-Banach spaces with each of them continuously embedded in a Hausdorff topological vector space \( H \), we call \( (Z_0, Z_1) \) a compatible couple. For \( z \in Z_0 + Z_1 \), the vector sum of \( Z_0 \) and \( Z_1 \), and \( 0 < t < \infty \), the K-functional is defined as

\[
K(t, z; Z_0, Z_1) = \inf_{z_0 + z_1} \{ \|z_0\|_{Z_0} + t\|z_1\|_{Z_1} : z_i \in Z_i, i = 0, 1 \}.
\]
Let $0 < \theta < 1, 0 < q \leq \infty$ or $0 \leq \theta \leq 1, q = \infty$, the space $(Z_0, Z_1)_{\theta,q} = Z_{\theta,q}$ consists of all $z \in Z_0 + Z_1$ for which the functional $\|z\|_{Z_{\theta,q}}$ is finite, where

$$\|z\|_{Z_{\theta,q}} = \begin{cases} \left( \int_0^\infty \left( \frac{K(t,z;Z_0,Z_1)}{p^r} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} & \text{if } 0 < \theta < 1, 0 < q < \infty \\ \sup_{0 < t < \infty} \frac{K(t,z;Z_0,Z_1)}{p^r} & \text{if } 0 \leq \theta \leq 1, q = \infty. \end{cases}$$

$Z_{\theta,q}$ is a quasi-Banach space with respect to the quasi-norm $\|\cdot\|_{Z_{\theta,q}}$. For more details on these topics, reader may consult [1, 11]. Let $B(B_1, B_2)$ be the set of all bounded linear operators from a quasi-Banach space $B_1$ to a quasi-Banach space $B_2$.

**Definition 2.9.** (11) An operator $L \in B(B_1, B_2)$ is said to be a retraction if there exists $M \in B(B_2, B_1)$ such that $LM = I$, the identity operator on $B_2$. In this case, $M$ is called as coretraction (belonging to $L$) and $B_2$ is called retract of $B_1$.

**Theorem 2.10.** (11) Let $X = (X_0, X_1)$ and $Y = (Y_0, Y_1)$ be two compatible couples of quasi-Banach spaces. Assume that $L \in B(X_0 + X_1, Y_0 + Y_1)$ and $M \in B(Y_0 + Y_1, X_0 + X_1)$ such that the restriction $M_i = M|_{Y^i_i}$, is a coretraction of $B(Y^i_i, X^i_i)$, and the restriction $L_i = L|_{X^i_i}$ is a retraction belonging to $B(X^i_i, Y^i_i)$, $i = 0, 1$. (If $i$ belongs to $L_i$ in the sense of above definition.) Then

$$\|y\|_{Y_{a,r}} \approx \|M y\|_{X_{a,r}},$$

where $'\approx'$ represents the isomorphism.

### 3 Lorentz-Herz Spaces

For any integer $a \geq 0$, set $A_u = \{x \in \Omega : 2^{u-1} \leq |x| < 2^u\}$ and $A_{-1} = \{x \in \Omega : |x| < \frac{1}{2}\}$. It is easy to see that $\{A_u : -1 \leq u < \infty\}$ is a partition of $\Omega$. We call a function $f$ in $M$ to be $L^{p,r}$ locally integrable on $\Omega$ (for some $0 < p, r \leq \infty$) if $f_{\chi_A} \in L^{p,r}$ for every compact subset $A \subset \Omega$. We denote such a class of functions by $L^{p,r}_{loc}(\Omega)$.

**Definition 3.1.** For $0 < p, q, r \leq \infty$ and $a \in \mathbb{R}$, define $HL^{a,r}_{p,q} = HL^{a,r}_{p,q}(\Omega)$ by

$$HL^{a,r}_{p,q} := \{ f \in L_{loc}^{p,r}(\Omega) : \|f\|_{HL^{a,r}_{p,q}} < \infty \},$$

where

$$\|f\|_{HL^{a,r}_{p,q}} = \begin{cases} \left[ \sum_{u \geq -1} 2^{u a q} \|f_{\chi_{A_u}}\|_{L_{p,r}}^q \right]^{\frac{1}{q}} & \text{if } 0 < q < \infty \\ \sup_{u \geq -1} 2^{u a q} \|f_{\chi_{A_u}}\|_{L^{p,r}} & \text{if } q = \infty. \end{cases}$$

with usual modifications for the cases $p = \infty$ and/or $r = \infty$.

It is easy to see that $HL^{0,p}_{p,q}(\Omega) = L^p(\Omega)$ and $HL^{p,0}_{p,q}(\Omega) = K^a_{p,q}(\Omega)$. Therefore, this space can be considered as a natural generalization of the Herz space. Moreover, it must be noted that for $a \in \mathbb{R}$, $p = \infty$, $0 < q \leq \infty$ and $0 < r < \infty$, the space $HL^{a,p}_{p,q}(\Omega)$ contains only those functions which are zero $\mu$-a.e. (see p.41 [10]). Hence, such a case is not of interest and will not be paid
any further attention. By similar considerations, we can also define homogeneous Lorentz-Herz spaces. However, in this manuscript we discuss only the non-homogeneous case. Moreover, it is clear that if $0 < r_1 \leq p \leq r_2 \leq \infty$, then $H^a_{L^{p,q}} \subseteq H^a_{L^{p,q}} = K^a_{p,q} \subseteq H^a_{L^{p,q}} = W^a_{K_{p,q}}$, where $W_{K_{p,q}}$ is a weak non-homogeneous Herz space. Thus, Lorentz-Herz spaces can be seen as refinements of both Herz spaces and weak Herz spaces. In the following theorem, we describe the linearity of the space $H^a_{L^{p,q}}$ and the nature of the functional $\|f\|_{H^a_{L^{p,q}}}$.

**Theorem 3.2.** Let $a \in \mathbb{R}$ and $0 < p, q, r \leq \infty$. The space $H^a_{L^{p,q}}$ is a linear space and the functional $\| \cdot \|_{H^a_{L^{p,q}}}$ is a quasi-norm.

**Proof.** Let $f, g \in H^a_{L^{p,q}}$ and $\alpha$ be any scalar. We begin with the case $0 < p, r \leq \infty$ and $0 < q < \infty$. Using the fact that $\| \cdot \|_{L^{p,r}}$ is a quasi-norm (see [10]), we have

$$
\|f + g\|_{H^a_{L^{p,q}}} = \left[ \sum_{u \geq 1} 2^u a q \|f + g\|_A y \|L^{p,r}\| \right]^{\frac{1}{q}}
\leq C \left[ \sum_{u \geq 1} 2^u a q \left( \|f \chi_{A_u}\|_{L^{p,r}} + \|g \chi_{A_u}\|_{L^{p,r}} \right)^q \right]^{\frac{1}{q}}
\leq C \left[ \sum_{u \geq 1} 2^u a q \left( \max \left\{ \|f \chi_{A_u}\|_{L^{p,r}}^q, \|g \chi_{A_u}\|_{L^{p,r}}^q \right\} \right) \right]^{\frac{1}{q}}
\leq C \left[ \sum_{u \geq 1} 2^u a q \left( \|f \chi_{A_u}\|_{L^{p,r}}^q + \|g \chi_{A_u}\|_{L^{p,r}}^q \right) \right]^{\frac{1}{q}}
\leq C \left[ 2 \max \left\{ \sum_{u \geq 1} 2^u a q \|f \chi_{A_u}\|_{L^{p,r}}^q, \sum_{u \geq 1} 2^u a q \|g \chi_{A_u}\|_{L^{p,r}}^q \right\} \right]^{\frac{1}{q}}
\leq C \max \left\{ \left( \sum_{u \geq 1} 2^u a q \|f \chi_{A_u}\|_{L^{p,r}}^q \right)^\frac{1}{q}, \left( \sum_{u \geq 1} 2^u a q \|g \chi_{A_u}\|_{L^{p,r}}^q \right)^\frac{1}{q} \right\}
= C \max \left\{ \|f\|_{H^a_{L^{p,q}}}^q, \|g\|_{H^a_{L^{p,q}}}^q \right\}
\leq C \left( \|f\|_{H^a_{L^{p,q}}}^q + \|g\|_{H^a_{L^{p,q}}}^q \right).
$$

In addition to this, it is easy to observe that $\|\alpha f\|_{H^a_{L^{p,q}}} = |\alpha| \|f\|_{H^a_{L^{p,q}}}$. Now let $0 < p, r \leq \infty$ and $q = \infty$. Then again by the fact that $\| \cdot \|_{L^{p,r}}$ is a quasi-norm and the subadditivity of supremum, we have $\|f + g\|_{H^a_{L^{p,q}}} \leq C \left( \|f\|_{H^a_{L^{p,q}}} + \|g\|_{H^a_{L^{p,q}}} \right)$. Again, it is easy to see that $\|\alpha f\|_{H^a_{L^{p,q}}} = |\alpha| \|f\|_{H^a_{L^{p,q}}}$. This completes the proof. $\square$

To discuss the Banach space structure on the non-homogeneous Lorentz-Herz space, we must introduce $\|f\|_{H^a_{L^{p,q}}}^*$. Let $\|f\|_{H^a_{L^{p,q}}}^*$ be the functional obtained from $\|f\|_{H^a_{L^{p,q}}}$ by replacing $\|f \chi_{A_u}\|_{L^{p,r}}$ with $\|f \chi_{A_u}\|_{L^{p,r}}^q$. Then $\|f\|_{H^a_{L^{p,q}}}^* = \|f\|_{H^a_{L^{p,q}}}^{\frac{1}{q}}$. In addition to this, it is easy to observe that $\|\alpha f\|_{H^a_{L^{p,q}}}^* = |\alpha| \|f\|_{H^a_{L^{p,q}}}^*$. Now let $0 < p, r \leq \infty$ and $q = \infty$. Then again by the fact that $\| \cdot \|_{L^{p,r}}$ is a quasi-norm and the subadditivity of supremum, we have $\|f + g\|_{H^a_{L^{p,q}}} \leq C \left( \|f\|_{H^a_{L^{p,q}}} + \|g\|_{H^a_{L^{p,q}}} \right)$. Again, it is easy to see that $\|\alpha f\|_{H^a_{L^{p,q}}} = |\alpha| \|f\|_{H^a_{L^{p,q}}}$. This completes the proof. $\square$
Therefore, for any $u \in \mathbb{R}$, $0 < p < \infty$, $0 < q, r \leq \infty$, the space $HL_{p,q}^{a,r}$ with the quasi-norm $\| \cdot \|_{HL_{p,q}^{a,r}}$ is complete. Furthermore, the space \( \left( HL_{p,q}^{a,r}, \| \cdot \|_{HL_{p,q}^{a,r}}^* \right) \) is a Banach space, provided $p, q$ and $r$ satisfy one of the following:

(i) $1 < p < \infty$, $1 \leq q, r \leq \infty$.

(ii) $p = r = 1$, $1 \leq q \leq \infty$ and

(iii) $p = r = \infty$, $1 \leq q \leq \infty$.

**Proof.** We first assume that $0 < q < \infty$. Let \((f_n)\) be a Cauchy sequence in \( (HL_{p,q}^{a,r}, \| \cdot \|_{HL_{p,q}^{a,r}}^*) \). Then for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

\[
\left[ \sum_{u \geq -1} 2^{uaq} \|(f_n - f_m)\chi_{A_u}\|_{L_{p,r}}^q \right]^{\frac{1}{q}} < \frac{\epsilon}{2} \quad \forall n, m \geq N.
\]

Thus for any $u \geq -1$,

\[
\|(f_n - f_m)\chi_{A_u}\|_{L_{p,r}} < \frac{\epsilon}{2} := \epsilon'.
\]

Therefore, for any $u \geq -1$, \((f_n\chi_{A_u})\) is a Cauchy sequence in \( (L_{p,r}, \| \cdot \|_{L_{p,r}}) \). As the space \( (L_{p,r}, \| \cdot \|_{L_{p,r}}) \) is complete \([10]\), there exists \( f\chi_{A_u} \in L_{p,r} \) such that

\[
\|(f_n - f)\chi_{A_u}\|_{L_{p,r}} \to 0 \text{ as } n \to \infty.
\]

(3)

Since $\| \cdot \|_{L_{p,r}}$ is a quasi-norm,

\[
\|(f_n - f)\chi_{A_u}\|_{L_{p,r}} \leq C \left( \|(f_n - f)\chi_{A_u}\|_{L_{p,r}} + \|(f_n - f_N)\chi_{A_u}\|_{L_{p,r}} \right).
\]

Letting $n \to \infty$ and using (5), we get

\[
\|(f_n - f)\chi_{A_u}\|_{L_{p,r}} \leq C \lim_{n \to \infty} \|(f_n - f_N)\chi_{A_u}\|_{L_{p,r}}.
\]

(4)

Now using (4) and Fatou’s lemma, we obtain

\[
\|f_N - f\|_{HL_{p,q}^{a,r}}^q \leq C^q \lim_{n \to \infty} \|f_n - f_N\|_{HL_{p,q}^{a,r}}^q.
\]

As \((f_n)\) is Cauchy, we get

\[
\|f_N - f\|_{HL_{p,q}^{a,r}} < \epsilon C.
\]

Therefore, \( f = (f - f_N) + f_N \in HL_{p,q}^{a,r} \). Also, by (3) and the dominated convergence theorem, \( \lim_{n \to \infty} \|f_n - f\|_{HL_{p,q}^{a,r}}^q = 0 \). The case $q = \infty$ can be proved by somewhat different but easy arguments. Thus, \( \left( HL_{p,q}^{a,r}, \| \cdot \|_{HL_{p,q}^{a,r}}^* \right) \) is complete. Further, it is not difficult to verify that $\| \cdot \|_{HL_{p,q}^{a,r}}^*$ defines a norm on $HL_{p,q}^{a,r}$ for the values of $p, q$ and $r$ that satisfy one of the assumptions (i), (ii) or (iii). Moreover, in view of Theorem 2.8 for such a case, $\| \cdot \|_{HL_{p,q}^{a,r}}^*$ and $\| \cdot \|_{HL_{p,q}^{a,r}}$ are equivalent.

Therefore, \( \left( HL_{p,q}^{a,r}, \| \cdot \|_{HL_{p,q}^{a,r}}^* \right) \) is a Banach space. \( \square \)
Example 3.4. Let $0 < p, q, r < \infty$, $B_{-1} = B_0 = \emptyset$, $B_u = \{ x \in \mathbb{R} : 2^u - \frac{1}{u^2} \leq |x| < 2^u \}$ for $u \in \mathbb{N}$ and $E = \bigcup_{u=-1}^{\infty} B_u$. Then $\mu(E) = \sum_{u \geq -1} \mu(B_u) = \sum_{u \geq -1} \frac{2}{u^2} < \infty$ and

$$
\| \chi_E \|^{a,r}_{H^{a,r}_{p,q}} = \left[ \sum_{u \geq -1} 2^{uaq} \| \chi_{E \cap A_u} \|_{L^{p,r}}^q \right]^{\frac{1}{q}}
$$

$$
= \left[ \sum_{u \geq 1} 2^{uaq} \| \chi_{B_u} \|_{L^{p,r}}^q \right]^{\frac{1}{q}}
$$

$$
= \left[ \sum_{u \geq 1} 2^{uaq} \left( \frac{p}{r} \right)^{\frac{q}{r}} \mu(B_u)^{\frac{q}{p}} \right]^{\frac{1}{q}}
$$

$$
= \left[ \sum_{u \geq 1} 2^{uaq} \left( \frac{p}{r} \right)^{\frac{q}{r}} \left( \frac{2}{u^2} \right)^{\frac{q}{p}} \right]^{\frac{1}{q}}.
$$

In particular, if $a > 0$, we see that $\| \chi_E \|^{a,r}_{H^{a,r}_{p,q}} = \infty$.

The above example shows that the Lorentz-Herz spaces are not Banach function spaces in general. We shall want to isolate all those spaces from Lorentz-Herz spaces which fall in Banach function space category. In the next theorem, we provide sufficient conditions for the non-homogeneous Lorentz-Herz space to be a Banach function space.

Theorem 3.5. Let $a \in \mathbb{R}$, $1 < p < \infty$ and $1 \leq q, r \leq \infty$. Then the space $(H^{a,r}_{p,q}, \| f \|_{H^{a,r}_{p,q}}^*)$ is a Banach function space if for every $\mu$-measurable subset $E \subseteq \Omega$ with $\mu(E) < \infty$, the following conditions hold:

(a) $\sum_{u \geq -1} 2^{uaq} \mu(A_u \cap E)^{\frac{q}{p}} < \infty$,

(b) $\sum_{u \geq 1} 2^{-uaq'} \mu(A_u \cap E)^{\frac{q'}{p'}} < \infty$.

Where $p'$ and $q'$ are, respectively, the conjugate exponents of $p$ and $q$. Usual modifications can be made if $q = \infty$ or $q' = \infty$.

Proof. First, suppose that $1 \leq q < \infty$, we only show that the properties $(P4)$ and $(P5)$ of Definition 2.1 are satisfied. The remaining properties are easy to verify. So assume that $E \subseteq \Omega$ with $\mu(E) < \infty$. Using Theorem 2.8 we obtain

$$
\| \chi_E \|_{H^{a,r}_{p,q}}^* = \left[ \sum_{u \geq -1} 2^{uaq} \| \chi_{(A_u \cap E)} \|_{L^{p,r}}^q \right]^{\frac{1}{q}}
$$

$$
\leq \left( \frac{p}{p-1} \right) \left[ \sum_{u \geq -1} 2^{uaq} \| \chi_{(A_u \cap E)} \|_{L^{p,r}}^q \right]^{\frac{1}{q}}
$$

$$
\leq C \left[ \sum_{u \geq -1} 2^{uaq} \mu(A_u \cap E)^{\frac{q}{p}} \right]^{\frac{1}{q}} < \infty.
$$
This verifies (P4). Furthermore, using the Hölder inequality for Lorentz spaces [12] as well as the Hölder inequality for sum, we get

\[
\int_E f \, d\mu = \int_\Omega f \chi_E \, d\mu = \sum_{u \geq -1} \int_{A_u} f \chi_{A_u \cap E} \, d\mu \\
\leq \sum_{u \geq -1} \|\chi_{A_u \cap E}\|_{L^{p',r'}} \|f \chi_{A_u}\|_{L^{p,r}} \\
\leq C \sum_{u \geq -1} (A_u \cap E)^{\frac{1}{p'}} \|f \chi_{A_u}\|_{L^{p,r}} \\
= C \sum_{u \geq -1} 2^{-ua} (A_u \cap E)^{\frac{1}{p'}} \cdot 2^{ua} \|f \chi_{A_u}\|_{L^{p,r}}^{*} \\
\leq C \left[ \sum_{u \geq -1} 2^{-uaq'} (A_u \cap E)^{\frac{q'}{p'}} \left[ \sum_{u \geq -1} 2^{uaq} \|f \chi_{A_u}\|_{L^{p,r}}^{*q} \right] \right]^{\frac{1}{q'}} \\
= C_E \sum_{u \geq -1} \|f \chi_{A_u}\|_{L^{p,r}}^{*q} \\
= C_E \|f\|_{H^{a,r}_{p,q}}^{*},
\]

where $r'$ is the conjugate exponent $r$. This shows that the property (P5) holds. Now it remains to consider the case when $q = \infty$. In this case, (P1), (P2), (P4) and (P5) are easily verifiable. For (P3), let $0 \leq f_n \uparrow f \, \mu$ a.e., therefore, for all $u \geq -1$, $\|f_n \chi_{A_u}\|_{L^{p,r}} \uparrow \|f \chi_{A_u}\|_{L^{p,r}}$. This gives $\sup_{u \geq -1} 2^{ua} \|f_n \chi_{A_u}\|_{L^{p,r}} \uparrow \|f \chi_{A_u}\|_{L^{p,r}}$. Thus, $\|f_n\|_{H^{a,r}_{p,q}} \uparrow \|f\|_{H^{a,r}_{p,q}}$ and this completes the proof. \hfill \Box

\textbf{Remark 3.6.} \hspace{1em} 1. If $\Omega \subseteq \mathbb{R}^N$ is bounded a.e., then both the conditions of the above theorem are satisfied.

2. If $\Omega \subseteq \mathbb{R}^N$, then $\sum_{u \geq -1} \mu(A_u \cap E) = \mu \left( \bigcup_{u \geq -1} (A_u \cap E) \right) = \mu(\Omega \cap E) = \mu(E) < \infty$.

Consequently, the space $HL^{0,r}_{p,p}$ is a Banach function space for $1 < p < \infty$ and $1 \leq r \leq \infty$. In particular, we get the well-known fact that $L^p(\Omega)$ is a Banach function space for $1 < p < \infty$.

The Banach function space nature of non-homogeneous Lorentz-Herz spaces enables us to quickly determine some of their topological properties (see Section 4).

Our next step is to state a few embeddings among non-homogeneous Lorentz-Herz spaces. Since there are several parameters involved, we have to consider each one of them separately.

\textbf{Theorem 3.7.} The following embedding results are valid:
(A) If \( a \in \mathbb{R}, 0 < p < \infty, 0 < r_1 \leq r_2 \leq \infty \) and \( 0 < q \leq \infty \), then
\[
HL_{p,q}^{a,r_1} \hookrightarrow HL_{p,q}^{a,r_2}.
\]

(B) If \( a_2 \leq a_1 \) and \( 0 < p, q, r \leq \infty \), then
\[
HL_{p,q}^{a_1,r} \hookrightarrow HL_{p,q}^{a_2,r}.
\]

(C) If \( a \in \mathbb{R}, 0 < r_1, r_2 \leq \infty, 0 < q \leq \infty \) and \( 0 < p_1 \leq p_2 < \infty \), then
\[
HL_{p_2,q}^{a,r_2} \hookrightarrow HL_{p_1,q}^{a,r_1}.
\]

(D) If \( a \in \mathbb{R}, 0 < p, r \leq \infty \) and \( 0 < q_2 \leq q_1 \leq \infty \), then
\[
HL_{p,q_2}^{a,r} \hookrightarrow HL_{p,q_1}^{a,r}.
\]

Proof. The items (A), (B) and (D) can be proved by fairly simple arguments. In fact, (A) is the direct consequence of the corresponding result in Lorentz spaces (p.52, Theorem 4.4 [10]), (B) follows from straightforward calculations and (D) is a result of the inequality
\[
\sum_{k \geq -1} |a_k|^s \leq \left( \sum_{k \geq -1} |a_k| \right)^s \forall s \geq 1.
\]

To prove item (C), we first observe that
\[
\| f \chi_{A_u} \|_{L^{p_1,r_1}} = \left[ \int_0^{\mu(A_u)} t^{\frac{r_1}{p_1} - 1} (f \chi_{A_u})^{r_1} dt \right]^\frac{1}{r_1} \\
\leq \left[ \int_0^{\mu(A_u)} t^{\frac{r_1}{p_1} - \frac{r_2}{p_2} - 1} \left( \frac{1}{r_2} (f \chi_{A_u})^r(t) \right)^{r_1} dt \right]^\frac{1}{r_1} \\
\leq \left[ \int_0^{\mu(A_u)} t^{\frac{r_1}{p_1} - \frac{r_2}{p_2} - 1} \left( \sup_{s \geq 0} s^{\frac{1}{r_2}} (f \chi_{A_u})^r(s) \right)^{r_1} dt \right]^\frac{1}{r_1} \\
= \| f \chi_{A_u} \|_{L^{p_2,\infty}} \left[ \int_0^{\mu(A_u)} t^{\frac{r_1}{p_1} - \frac{r_2}{p_2} - 1} dt \right]^\frac{1}{r_1} \\
= \frac{\mu(A_u)^{\frac{r_1}{p_1} - \frac{r_2}{p_2}}}{(\frac{r_1}{p_1} - \frac{r_2}{p_2})^{1/r_1}} \| f \chi_{A_u} \|_{L^{p_2,\infty}}.
\]

Therefore, we have \( HL_{p_2,q}^{a,r_2} \hookrightarrow HL_{p_1,q_1}^{a,r_1} \). Using item (A) we are done. \( \square \)
4 Associate Space

We commence this section by proving the absolute continuity of the norm to achieve the associate space of $H^{a,r}_{p,q}$.

**Proposition 4.1.** Let $a \in \mathbb{R}$, $1 < p < \infty$ and $1 \leq q, r < \infty$. The Banach function space $(H^{a,r}_{p,q}, \| \cdot \|_{H^{a,r}_{p,q}}^{*})$ has absolutely continuous norm.

**Proof.** Let $f \in H^{a,r}_{p,q}$ and $\{E_n\}_{n=1}^{\infty}$ be a sequence of $\mu$-measurable subsets of $\Omega$ with $\chi_{E_n} \to 0 \mu$-a.e.. By using the dominated convergence theorem, Theorem 2.8 and the fact that $(f\chi_{A})^{*}(t) \leq f^{*}(t)\chi_{[0,\mu(A))}(t)$, we get

$$\lim_{n \to \infty} \|f\chi_{E_n}\|^{*}_{H^{a,r}_{p,q}} = \lim_{n \to \infty} \left[ \sum_{u \geq -1} 2^{uaq} \|f\chi_{A_u} \cdot \chi_{E_n}\|^{q}_{L^{p,r}} \right]^{\frac{1}{q}}$$

$$\leq \left( \frac{p}{p-1} \right) \left[ \sum_{u \geq -1} 2^{uaq} \lim_{n \to \infty} \|f\chi_{A_u} \cdot \chi_{E_n}\|^{q}_{L^{p,r}} \right]^{\frac{1}{q}}$$

$$\leq \left( \frac{p}{p-1} \right) \left[ \sum_{u \geq -1} 2^{uaq} \lim_{n \to \infty} \left( \int_{0}^{\infty} t^{\frac{r}{p}-1}(f\chi_{A_u})^{*r}(t) \cdot \chi_{[0,\mu(E_n))}(t) dt \right) \right]^{\frac{2}{q}}$$

$$= 0. \quad \square$$

We have the following corollaries of Proposition 4.1 in light of Theorem 2.3 and Theorem 2.4 respectively.

**Corollary 4.2.** For $a \in \mathbb{R}$, $1 < p < \infty$ and $1 \leq q, r < \infty$, the simple functions are dense in the Banach function space $(H^{a,r}_{p,q}, \| \cdot \|_{H^{a,r}_{p,q}}^{*})$.

**Corollary 4.3.** For $a \in \mathbb{R}$, $1 < p < \infty$ and $1 \leq q, r < \infty$, the Banach function space $(H^{a,r}_{p,q}, \| \cdot \|_{H^{a,r}_{p,q}}^{*})$ is separable if and only if the measure $\mu$ is separable.

Next we prove the following Hölder type inequality for non-homogeneous Lorentz-Herz spaces.

**Theorem 4.4.** Let $a \in \mathbb{R}$, $1 < p < \infty$ and $1 \leq q, r \leq \infty$. Then for all $\mu$-measurable functions $f$ and $g$ on $\Omega$,

$$\int_{\Omega} |fg|d\mu \leq \|f\|_{H^{a,r}_{p,q}} \|g\|_{H^{-a,r'}_{p',q'}},$$

where $p', q'$ and $r'$ are the conjugate exponents of $p, q$ and $r$ respectively.
Proof. For $\mu$-measurable functions $f$ and $g$ on $\Omega$ and $a \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q, r \leq \infty$, we have

$$
\int_{\Omega} |fg|d\mu = \sum_{u \geq -1} \int_{A_u} |fg|d\mu
$$

$$
= \sum_{u \geq -1} \int_{\Omega} |f\chi_{A_u}| \cdot |g\chi_{A_u}|d\mu
$$

$$
\leq \sum_{u \geq -1} \|f\chi_{A_u}\|_{L^{p,r}} \|g\chi_{A_u}\|_{L^{p',r'}}
$$

$$
= \sum_{u \geq -1} 2^{ua} \|f\chi_{A_u}\|_{L^{p,r}} \cdot 2^{-ua} \|g\chi_{A_u}\|_{L^{p',r'}}
$$

$$
\leq \left[ \sum_{u \geq -1} 2^{ua} \|f\chi_{A_u}\|^q_{L^{p,r}} \right]^{\frac{1}{q}} \left[ \sum_{u \geq -1} 2^{-ua} \|g\chi_{A_u}\|^q_{L^{p',r'}} \right]^{\frac{1}{q}} \quad \text{(Assuming } q < \infty)\]

$$
= \left\| f \right\| _{HL^{a,r}_{p,q}} \left\| f \right\| _{HL^{-a,r'}_{p',q'}}.
$$

Similar arguments can be used for $q = \infty$. □

Now let $B = \{\beta_u\}_{u \geq -1}$, where $\beta_u$’s are Banach spaces. For $a \in \mathbb{R}$ and $0 < q \leq \infty$, define

$$
\ell^a_q(B) := \left\{ \alpha : \alpha = (\alpha_u)_{u \geq -1}, \alpha_u \in \beta_u \text{ and } \left[ \sum_{u \geq -1} 2^{ua} \|\alpha_u\|_{\beta_u} \right]^{\frac{1}{q}} < \infty \right\},
$$

with usual modification for $q = \infty$. Suppose that $X^*$ denotes the dual space of a Banach space $X$ and $B^* := \{\beta^*_u\}_{u \geq -1}$. Now it is easy to observe that for $1 < p < \infty$ and $1 \leq r \leq \infty$, if $\beta_u = (L^{p,r}, \|f\|^r_{L^{p,r}})$ and $\alpha_u = f\chi_{A_u}$ for all $u \geq -1$, then clearly $\ell^a_q(B) = (HL^{a,r}_{p,q}, \|\cdot\|_{HL^a_{p,q}}^r)$. Therefore, we have the following result (see Theorem 2.1, [4]).

**Theorem 4.5.** Let $a \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q < \infty$ and $1 \leq r \leq \infty$. The Banach space dual of $(HL^{a,r}_{p,q}, \|\cdot\|_{HL^a_{p,q}}^r)$ is the space $(HL^{-a,r'}_{p',q'}, \|\cdot\|_{HL^{-a,r'}_{p,q}}^r)$. 

**Corollary 4.6.** For $a \in \mathbb{R}$ and $1 < p, q, r < \infty$, the space $(HL^{a,r}_{p,q}, \|f\|_{HL^a_{p,q}}^r)$ is reflexive.

**Theorem 4.7.** If $a \in \mathbb{R}$, $1 < p < \infty$ and $1 \leq q, r < \infty$, then the associate space $(HL^{a,r}_{p,q})'$ can be identified with the space $HL^{-a,r'}_{p',q'}$.

**Proof.** In view of Theorem 2.2 and Proposition 4.1, the proof is immediate. □

Theorem 3.5 provides the sufficient conditions for $(HL^{a,r}_{p,q}, \|f\|_{HL^a_{p,q}}^r)$ to be a Banach function space. On the other hand, assume that the conditions (a) and (b) of Theorem 3.5 are satisfied for
$a \in \mathbb{R}$ and $1 < p, q, r < \infty$. Then by property $(P4)$, $\|\chi_E\|_{HL^{a,r}_{p,q}}^* < \infty$ for every $\mu$-measurable subset $E \subseteq \Omega$ with $\mu(E) < \infty$. Now

$$
\|\chi_E\|_{HL^{a,r}_{p,q}}^* = \left[ \sum_{u \geq -1} 2^{u a q} \|\chi_{(A_u \cap E)}\|_{L^p,r}^* \right]^{\frac{1}{q}}
$$

$$
= \left[ \sum_{u \geq -1} 2^{u a q} \left( \int_0^{\mu(\Omega)} t^{r-1} (\chi_{(A_u \cap E)})^{**r}(t) dt \right)^{\frac{2}{r}} \right]^{\frac{1}{q}}
$$

$$
= \left[ \sum_{u \geq -1} 2^{u a q} \left( \int_0^{\infty} t^{r-1} \min \left\{ 1, \frac{\mu(A_u \cap E)}{t} \right\} dt \right)^{\frac{2}{r}} \right]^{\frac{1}{q}}
$$

$$
= \left[ \sum_{u \geq -1} 2^{u a q} \min \left( \int_0^{\mu(A_u \cap E)} t^{r-1} dt, \int_0^{\infty} t^{r-1} \mu(A_u \cap E)\, dt \right)^{\frac{2}{r}} \right]^{\frac{1}{q}}
$$

$$
= \left[ \sum_{u \geq -1} 2^{u a q} \min \left( \frac{p}{r} \mu(A_u \cap E)\, t^{\frac{q}{p}} \mu(A_u \cap E)^{\frac{2}{p}}, \left[ \frac{p}{r(p-1)} \right] \mu(A_u \cap E)^{\frac{2}{p}} \right) \right]^{\frac{1}{q}}
$$

Thus, it follows that $\sum_{u \geq -1} 2^{u a q} \mu(A_u \cap E)^{\frac{2}{p}} < \infty$. Similarly, using the property $(P4)$ of the associate space \($HL^{a,r}_{p',q'}$, $\|\cdot\|_{HL^{a,r}_{p',q'}}^*$\), we get $\sum_{u \geq -1} 2^{-u a q'} \mu(A_u \cap E)^{\frac{2}{p'}} < \infty$. Therefore, we have the following characterization.

**Theorem 4.8.** For $1 < p, q, r < \infty$ and $a \in \mathbb{R}$, the space \(HL^{a,r}_{p,q}\) is a Banach function space if and only if for every subset, $E \subseteq \Omega$ with, $\mu(E) < \infty$, the following conditions hold:

(a) $\sum_{u \geq -1} 2^{u a q} \mu(A_u \cap E)^{\frac{2}{p}} < \infty$,

(b) $\sum_{u \geq -1} 2^{-u a q'} \mu(A_u \cap E)^{\frac{2}{p'}} < \infty$.

The above characterization can lead to several fruitful outcomes because Banach function spaces have a well-developed theory. However, in this paper, we stick to our stated objectives rather than seeking such outcomes.

5 Real Interpolation spaces

In this section we apply real interpolation method to non-homogeneous Lorentz-Herz spaces and characterize their intermediate spaces. In particular, we get interpolation spaces for non-homogeneous
Theorem 5.1. Let $\ell$ be a quasi-Banach space, set $B$ is a quasi-Banach space, set $\beta_u = B$ for all $u \geq -1$ and $\ell^{q}_{\ell}(B) = \ell^{q}_{\ell}(B)$ (see Section 4). Then it is easy to see that $\ell^{q}_{\ell}(B)$ is a quasi-Banach space. We recall the following interpolation result for $\ell^{a}_{\ell}(B)$ (see [21], Section 5.6 and [11], Section 1.18.1).

**Theorem 5.2.** Assume that $0 < \theta < 1$, $0 < q_0, q_1, q \leq \infty$ and $\alpha_0, \alpha_1, \alpha \in \mathbb{R}$.

1. If $\alpha_0 \neq \alpha_1$ and $\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1$, then
   \[
   (\ell^{\alpha_0}_{\ell}(B), \ell^{\alpha_1}_{\ell}(B))_{\theta,q} = \ell^{\alpha}_{\ell}(B).
   \]

2. If $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, then
   \[
   (\ell^{\alpha_0}_{\ell}(B), \ell^{\alpha_1}_{\ell}(B))_{\theta,q} = \ell^{\alpha}_{\ell}(B).
   \]

3. If $0 < q_0, q_1 < \infty$ and $(B_0, B_1)$ is a compatible couple of quasi-Banach spaces, then
   \[
   (\ell^{\alpha_0}_{\ell}(B_0), \ell^{\alpha_1}_{\ell}(B_1))_{\theta,q} = \ell^{\alpha}_{\ell}((B_0, B_1)_{\theta,q})
   \]
   provided that $\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

We also recall an interpolation result for Lorentz spaces ([21], Section 5.3 and [11], Section 1.18.6).

**Theorem 5.2.** Assume that $0 < \theta < 1$, $1 < p_0 \neq p_1 < \infty$, $1 \leq r_0, r_1, r \leq \infty$. Then

\[
(L^{p_0,r_0}, L^{p_1,r_1})_{\theta,r} = L^{p,r},
\]

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. This formula is still valid if we replace $L^{p_0,r_0}$ by $L^1$ (then $p_0 = 1$) and $L^{p_1,r_1}$ by $L^\infty$ (then $p_1 = \infty$). In fact, if $1 \leq p_0 < p < p_1 \leq \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, then (up to equivalence of norms),

\[
(L^{p_0}, L^{p_1})_{\theta,r} = L^{p,r}.
\]

Let $L(f) = \{L(f)_u\}_{u \geq 1} = \{f \chi_{A_u}\}_{u \geq 1}$ and $M (\{f_j\}) = \sum_{j \geq 1} f_j \chi_{A_j}$ whenever the later is meaningful. The following lemma is essential in proving our interpolation results.

**Lemma 5.3.** $L : HL^{a,r}_{p,q} \rightarrow \ell^{a}_{q}(L^{p,r})$ and $M : \ell^{a}_{q}(L^{p,r}) \rightarrow HL^{a,s}_{p,q}$ are isometries. Moreover, $M \circ L = I$, the identity map.

**Proof.** Suppose $f \in (HL^{a,r}_{p,q}, \| \cdot \|_{HL^{a,r}_{p,q}})$, then

\[
\|f\|_{HL^{a,r}_{p,q}} = \left[ \sum_{u \geq 1} 2^{uaq} \|f \chi_{A_u}\|_{L^{p,r}}^{q} \right]^{\frac{1}{q}}
= \left[ \sum_{u \geq 1} 2^{uaq} \|L(f)_u\|_{L^{p,r}}^{q} \right]^{\frac{1}{q}}
= \|L(f)\|_{\ell^{a}_{q}(L^{p,r})}.
\]
Moreover, for \( f = (f_j \chi_{A_j})_{j \geq -1} \in \ell^a_q \left( L^{p,r} \right) \), we have

\[
\| M(f) \|_{H L^{a,r}_{p,q}} = \left[ \sum_{u \geq -1} 2^{uaq} \left( \sum_{j \geq -1} f_j \chi_{A_j} \| A_u \|_{L^{p,r}} \right)^q \right]^{\frac{1}{q}} = \left[ \sum_{u \geq -1} 2^{uaq} \| f_u \chi_{A_u} \|_{L^{p,r}}^q \right]^{\frac{1}{q}} = \| f \|_{\ell^a_q \left( L^{p,r} \right)}. 
\]

This completes the proof. \( \square \)

Now we are in a position to formulate interpolation results for Lorentz-Herz spaces.

**Theorem 5.4.** Assume that \( 0 < \theta < 1, 0 < p < \infty, \quad 0 < q_0, q_1, q, r \leq \infty \) and \( a_0, a_1, a \in \mathbb{R} \). Then the following interpolation formulae are valid:

1. If \( a_0 \neq a_1 \) and \( a = (1 - \theta)a_0 + \theta a_1 \), then

\[
(H L^{a_0, r}_{p, q_0}, H L^{a_1, r}_{p, q_1})_{\theta, q} = H L^{a, r}_{p, q},
\]

2. If \( \frac{1}{q} = \frac{1}{q_0} + \frac{\theta}{q_1} \), then

\[
(H L^{a, r}_{p, q_0}, H L^{a, r}_{p, q_1})_{\theta, q} = H L^{a, r}_{p, q},
\]

3. If \( 0 < q_0, q_1 < \infty, 1 \leq p_0 \neq p_1 < \infty \) and \( 1 \leq r_0, r_1, q \leq \infty \), then

\[
(H L^{a_0, r_0}_{p_0, q_0}, H L^{a_1, r_1}_{p_1, q_1})_{\theta, q} = H L^{a_0, q}_{p_0, q},
\]

provided that \( a = (1 - \theta)a_0 + \theta a_1, \frac{1}{p} = \frac{1}{p_0} + \frac{\theta}{p_1} \) and \( \frac{1}{q} = \frac{1}{q_0} + \frac{\theta}{q_1} \). In addition, if \( p = q \), then

\[
(H L^{a_0, r_0}_{p_0, q_0}, H L^{a_1, r_1}_{p_0, q_1})_{\theta, q} = K^{a_0}_{p_0, q_0}.
\]

4. If \( 0 < q_0, q_1 < \infty, a = (1 - \theta)a_0 + \theta a_1 \) and \( \frac{1}{q} = \frac{1}{q_0} + \frac{\theta}{q_1} \), then

\[
(K^{a_0}_{1, q_0}, K^{a_1}_{1, q_1})_{\theta, q} = H L^{a, q}_{1, q}. \]

In addition, if \( 1 \leq p_0 < p < p_1 \leq \infty \) and \( \frac{1}{p} = \frac{1}{p_0} + \frac{\theta}{p_1} \), then (up to equivalence of norms),

\[
(K^{a_0}_{p_0, q_0}, K^{a_1}_{p_1, q_1})_{\theta, q} = H L^{a_0, q}_{p_0, q}. \]

**Proof.** In view of Theorem 2.10, Lemma 5.3 and Theorem 5.1, we have

\[
\| f \|_{(H L^{a_0, r}_{p, q_0}, H L^{a_1, r}_{p, q_1})_{\theta, q}} \approx \| L(f) \|_{(\ell^{a_0}_{q_0}(L^{p,r}), \ell^{a_1}_{q_1}(L^{p,r}))_{\theta, q}} = \| f \chi_{A_u} \|_{\ell^{a}_q (L^{p,r})},
\]

i.e.,

\[
(H L^{a_0, r}_{p, q_0}, H L^{a_1, r}_{p, q_1})_{\theta, q} = H L^{a, r}_{p, q}.
\]

The proof of Item (2) is similar. Item (3) and Item (4) can be easily proved by using the above procedure along with Theorem 5.2. \( \square \)
Theorem 5.4 shows that, with appropriately chosen parameters, the interpolation spaces of non-homogeneous Lorentz-Herz spaces are again non-homogeneous Lorentz-Herz spaces. Thus, they are closed under interpolation. In addition, the non-homogeneous Lorentz-Herz space $HL_{p,q}^{a,r}$ appears as an interpolation space between the non-homogeneous Herz spaces. Furthermore, if $p = r$ and $p_i = r_i$, $i = 0, 1$, we get real interpolation spaces of the non-homogeneous Herz spaces.

6 Applications

In this section, we prove boundedness of a wide class of sublinear operators on non-homogeneous Lorentz-Herz spaces. We also strengthen the conclusion of our result (for $q = r$) in case linear operators by applying the above interpolation results. The class of operators whose kernels satisfy the standard size condition $|k(x, y)| \leq \frac{C}{|x-y|^N}$, have been extensively studied due to its wide range of applications in Harmonic analysis (e.g., see [4, 13, 14, 15, 16]). In particular, the authors in [4, 17] proved that if a sublinear operator $T$ is bounded on $L^p(\mathbb{R}^N)$ and is satisfying the size condition

$$|T f(x)| \leq C \int_{\mathbb{R}^N} \frac{|f(y)|}{|x-y|^N} d\mu(y), \quad x \notin \text{supp}(f)$$  \hspace{1cm} (5)$$

for all $f \in L^1(\mathbb{R}^N)$ with compact support, where $C$ is independent of $f$ and $x$. Then it is bounded on $K^a_{p,q}(\mathbb{R}^N)$. We prove a version of this result for the spaces $HL_{p,q}^{a,r}(\mathbb{R}^N)$. Our result is more general and reduces to the corresponding result on Herz spaces when $r = p$ (see [4]). It is even slightly stronger, because unlike in [4], we do not assume $T$ to be linear for $a = 0$. Moreover, if $r = \infty$, we get the corresponding result for weak Herz spaces, which we could not find in the literature. We first need the following technical lemma. As the proof involves only elementary calculations, we leave it to the reader as an exercise.

**Lemma 6.1.** Let $u, v \geq -1$ be integers. For $1 < p < \infty$ and $1 \leq r \leq \infty$,

$$2^{-uN} \|\chi_{A_u}\|_{L^p,r} \|\chi_{A_v}\|_{L^{p',r'}} \leq C 2^{\frac{N}{p}(v-u)}.$$  

**Theorem 6.2.** Let $T$ be a sublinear operator satisfying the size condition (5) for all $f \in L^1(\mathbb{R}^N)$ with compact support. Suppose that $1 < p < \infty$, $1 \leq q, r \leq \infty$ and $T$ is bounded on $L^{p,r}(\mathbb{R}^N)$. Then for $\frac{-N}{p} < a < \frac{N}{p}$, $T$ is bounded on $HL_{p,q}^{a,r}(\mathbb{R}^N)$.
Proof. Let \( f \in H^{a,r}_{p,q}(\mathbb{R}^n) \). Since \( T \) is sublinear, we have

\[
\|Tf\|_{H^{a,r}_{p,q}} = \left[ \sum_{u \geq 1} 2^{uaq} \|\chi_{A_u} \cdot Tf\|_{L^{p,r}}^q \right]^{\frac{1}{q}}
\]

\[
\leq \left[ \sum_{u \geq 1} 2^{uaq} \|\chi_{A_u} \cdot \sum_{v \geq 1} T(f\chi_{A_v})\|_{L^{p,r}}^q \right]^{\frac{1}{q}}
\]

\[
= \left[ \sum_{u \geq 1} 2^{uaq} \left( \int_0^\infty t^{\frac{p}{p-1}} \left( \sum_{v \geq 1} \chi_{A_u} T(f\chi_{A_v}) \right)^r (t) dt \right) \right]^{\frac{1}{q}}
\]

Using Corollary 2.6 and Tonelli’s Theorem, we get

\[
\|Tf\|_{H^{a,r}_{p,q}} \leq C \left[ \sum_{u \geq 1} 2^{uaq} \left( \int_0^\infty t^{\frac{p}{p-1}} (\sum_{v \geq 1} \chi_{A_u} T(f\chi_{A_v}))^{*r} (t) dt \right) \right]^{\frac{1}{q}}
\]

\[
\leq C \left[ \sum_{u \geq 1} 2^{uaq} \left( \sum_{v \geq 1} \left( \int_0^\infty t^{\frac{p}{p-1}} (\chi_{A_u} T(f\chi_{A_v}))^{*r} (t) dt \right)^{\frac{1}{q}} \right)^q \right]^{\frac{1}{q}}
\]

\[
= C \left[ \sum_{u \geq 1} 2^{uaq} \left( \sum_{v \geq 1} (\chi_{A_u} T(f\chi_{A_v}))^{*r} \right) \right]^{\frac{1}{q}}
\]

Therefore,

\[
\|Tf\|_{H^{a,r}_{p,q}} \leq C \left[ \sum_{u \geq 1} 2^{uaq} \left( \sum_{v \geq 1} (\chi_{A_u} T(f\chi_{A_v}))^{*r} \right)^{\frac{1}{q}} \right]^{\frac{1}{q}}
\]

Using Minkowski’s inequality, we get

\[
\|Tf\|_{H^{a,r}_{p,q}} \leq C \left[ S_1 + S_2 + S_3 \right],
\]

where

\[
S_1 = \left[ \sum_{u \geq 1} 2^{uaq} \left( \sum_{v \geq u-1} (\chi_{A_u} T(f\chi_{A_v}))^{*r} \right)^{\frac{1}{q}} \right],
\]

\[
S_2 = \left[ \sum_{u \geq 1} 2^{uaq} \left( \sum_{v \geq u+1} (\chi_{A_u} T(f\chi_{A_v}))^{*r} \right)^{\frac{1}{q}} \right],
\]

and

\[
S_3 = \left[ \sum_{u \geq 1} 2^{uaq} \left( \sum_{v \geq u+2} (\chi_{A_u} T(f\chi_{A_v}))^{*r} \right)^{\frac{1}{q}} \right].
\]
Now observe that for \( v \leq u - 2 \) and a.e. \( x \in A_u \), the size condition (5) and the Hölder inequality for \( L^{p,r} \) spaces imply that

\[
|T(f \chi_{A_u})(x)| \leq C 2^{-uN} \|f \chi_{A_u}\|_{L^{p,r}} \|\chi_{A_u}\|_{L^{p',r'}}.
\]

Therefore,

\[
S_1 \leq C \left[ \sum_{u \geq -1} 2^{ua} \left( \sum_{v = -1}^{u-2} 2^{-uN} \|\chi_{A_u}\|_{L^{p,r}} \|f \chi_{A_v}\|_{L^{p,r}} \|\chi_{A_v}\|_{L^{p',r'}} \right) \right]^{q \frac{1}{q}}.
\]

Using Lemma 6.1, we get

\[
S_1 \leq C \left[ \sum_{u \geq -1} 2^{ua} \left( \sum_{v = -1}^{u-2} 2^{va} \|f \chi_{A_v}\|_{L^{p,r}} \cdot 2^{\frac{N}{p}(v-u)-va} \right) \right]^{q \frac{1}{q}}
\]

\[
= C \left[ \sum_{u \geq -1} \left( \sum_{v = -1}^{u-2} 2^{va} \|f \chi_{A_v}\|_{L^{p,r}} \cdot 2^{\beta(v-u)} \right) \right]^{q \frac{1}{q}}.
\]

Where \( \beta = \frac{N}{p} - a > 0 \). Using Hölder’s inequality for the inner sum, we get

\[
S_1 \leq C \left[ \sum_{u \geq -1} \left( \sum_{v = -1}^{u-2} 2^{va} \|f \chi_{A_v}\|_{L^{p,r}} \cdot 2^{\beta(v-u)/2} \right) \right]^{q \frac{1}{q}}
\]

\[
\leq C \left[ \sum_{u \geq -1} \left( \sum_{v = -1}^{u-2} 2^{va} \|f \chi_{A_v}\|_{L^{p,r}} \cdot 2^{\beta(v-u)/2} \right) \right]^{q \frac{1}{q}}.
\]

Interchanging the order of summations, we get

\[
S_1 \leq C \left[ \sum_{v \geq -1} \left( \sum_{u \geq v+2} 2^{va} \|f \chi_{A_v}\|_{L^{p,r}} \cdot 2^{\beta(v-u)/2} \right) \right]^{q \frac{1}{q}}.
\]

Therefore,

\[
S_1 \leq C \|f\|_{H^L_{p,q,r}} \quad (7)
\]

Now \( S_2 = \left[ \sum_{u \geq -1} 2^{ua} \left( \sum_{v = u-1}^{u+1} \|\chi_{A_v} T f(\chi_{A_v})\|_{L^{p,r}} \right) \right]^{q \frac{1}{q}} \leq \left[ \sum_{u \geq -1} 2^{ua} \left( \sum_{v = u-1}^{u+1} \|T f(\chi_{A_v})\|_{L^{p,r}} \right) \right]^{q \frac{1}{q}} \).
Using the boundedness of $T$ on $L^{p,r}$ and Minkowski’s inequality, we get

\[ S_2 \leq \left[ \sum_{u \geq -1} 2^{ua} \left( \sum_{v=u-1}^{u+1} \| f(\chi_{A_v}) \|_{L^{p,r}} \right) \right]^{\frac{1}{q}} \]

\[ \leq C \left[ \sum_{u \geq -1} 2^{ua} \| f(\chi_{A_u-1}) \|_{L^{p,r}} \right]^{\frac{1}{q}} + \left[ \sum_{u \geq -1} 2^{ua} \| f(\chi_{A_u}) \|_{L^{p,r}} \right]^{\frac{1}{q}} + \left[ \sum_{u \geq -1} 2^{ua} \| f(\chi_{A_{u+1}}) \|_{L^{p,r}} \right]^{\frac{1}{q}} \]

\[ \leq C \left[ \sum_{u \geq -1} 2^{ua} \| f(\chi_{A_u}) \|_{L^{p,r}} \right]^{\frac{1}{q}}. \]

Thus,

\[ S_2 \leq C \| f \|_{HL^{a,r}_{p,q}}. \quad (8) \]

Finally, for $v \geq u + 2$ and a.e. $x$ in $A_u$, the size condition (5) and Hölder’s inequality for Lorentz spaces imply that

\[ |T(f \chi_{A_u})(x)| \leq C 2^{-vN} \| f \chi_{A_u} \|_{L^{p,r}} \| \chi_{A_u} \|_{L^{p',r'}}. \]

Therefore, by similar calculations as in $S_1$, we get

\[ S_3 \leq C \left[ \sum_{u \geq -1} 2^{ua} \left( \sum_{v \geq u+2} 2^{va} \| f \chi_{A_v} \|_{L^{p,r}} \cdot 2^{N(u-v)-va} \right) \right]^{\frac{1}{q}} \]

\[ = C \left[ \sum_{u \geq -1} \left( \sum_{v \geq u+2} 2^{va} \| f \chi_{A_v} \|_{L^{p,r}} \cdot 2^{\beta'(u-v)} \right) \right]^{\frac{1}{q}}. \]

Where $\beta' = \frac{N}{p} + a > 0$. Using Hölder’s inequality for the inner sum, we get

\[ S_3 \leq C \left[ \sum_{u \geq -1} \left( \sum_{v \geq u+2} 2^{va} \| f \chi_{A_v} \|_{L^{p,r}} \cdot 2^{\beta'(u-v)/2} \right) \right]^{\frac{1}{q}} \]

\[ \leq C \left[ \sum_{u \geq -1} \left( \sum_{v \geq u+2} 2^{va} \| f \chi_{A_v} \|_{L^{p,r}} \cdot 2^{\beta'(u-v)/2} \right) \right]^{\frac{1}{q}}. \]

Interchanging the order of summations, we get

\[ S_3 \leq C \left[ \sum_{v \geq -1} \left( \sum_{u \geq v-2} 2^{va} \| f \chi_{A_v} \|_{L^{p,r}} \cdot 2^{\beta'(u-v)/2} \right) \right]^{\frac{1}{q}} \leq C \| f \|_{HL^{a,r}_{p,q}}. \quad (9) \]

Combining the estimates (6), (7), (8) and (9), we obtain $\| Tf \|_{HL^{a,r}_{p,q}} \leq C \| f \|_{HL^{a,r}_{p,q}}$. This completes the proof.
Remark 6.3. The condition (5) was first considered in [13]. This condition is satisfied by several operators of critical importance in harmonic analysis. Some examples are Caledrón-Zygmund operators, Hardy-Littlewood maximal operator, Bochner-Riesz means, Carleson’s maximal operator, and certain singular integrals (see [13, 14, 18]).

Since the Hardy-Littlewood maximal operator is bounded on $L^{p,r}(\mathbb{R}^N)$ for $1 < p < \infty$, $1 \leq r \leq \infty$ (see [19], Corollary 4) and the Caledrón-Zygmund operators are bounded on $L^{p,r}(\mathbb{R}^N)$ for $1 < p < \infty$, $1 \leq r < \infty$ (see [20], Theorem 1.2), we have the following corollary of Theorem 6.2.

Corollary 6.4. Let $1 < p < \infty$ and $-\frac{N}{p} < a < \frac{N}{p'}$.

1. If $1 \leq q, r \leq \infty$, then the Hardy-Littlewood maximal operator is bounded on $HL^{a,r}_{p,q}(\mathbb{R}^N)$.

2. If $1 \leq r < \infty$ and $1 \leq q \leq \infty$, then the Caledrón-Zygmund operators are bounded on $HL^{a,r}_{p,q}(\mathbb{R}^N)$.

In some particular cases, we can relax bounds and strengthen the conclusion for linear operators by using interpolation results.

Theorem 6.5. Let $T$ be a linear operator satisfying the size condition (5) for all $f \in L^1(\mathbb{R}^N)$ with compact support. Suppose that $1 < p < \infty$, $0 < q < \infty$ and $T$ is bounded on $L^p(\mathbb{R}^N)$. Then for $-\frac{N}{p} < a < \frac{N}{p'}$, $T$ is bounded on $HL^{a,q}_{p,q}(\mathbb{R}^N)$.

Proof. Putting $q_0 = q_1 = q$, $a_0 = a_1 = a$ and $\theta = \frac{1}{p'}$ in Item (4) of Theorem 5.4, we get

$$(K^a_{1,q}, K^a_{\infty,q})_{\theta,q} = HL^{a,q}_{p,q}.\)$$

Since $T$ is bounded on both $K^a_{1,q}$ and $K^a_{\infty,q}$ (see [4], Corollary 5.4). Therefore, by interpolation theorem ([1], Chapter 5, Theorem 1.12), we deduce that

$$\|Tf\|_{HL^{a,q}_{p,q}} \leq C\|f\|_{HL^{a,q}_{p,q}}.\)$$

Notice that, in this case, we need $T$ to be bounded on $L^p(\mathbb{R}^N)$ and not on $L^{p,q}(\mathbb{R}^N)$. Thus, this result is stronger than Theorem 6.2 applied to linear operators with $r = q$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest.
Funding

This research work did not receive any external funding.

Acknowledgements

The first author (M. Ashraf Bhat) is thankful to Prime Minister’s Research Fellowship (PMRF) scheme for fellowship (Appl. No. PMRF-192002-1745).

Funding

Not applicable

Text for this section . . .

Availability of data and materials

Not applicable

Competing interests

The authors declare that they have no competing interests.

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