Existence and Nonexistence for Elliptic Equation with Cylindrical Potentials, Subcritical Exponent and Concave Term

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Abstract

In this paper, we consider the existence and nonexistence of nontrivial solutions to elliptic equations with cylindrical potentials, concave term and subcritical exponent. First, we shall obtain a local minimizer by using the Ekeland’s variational principle. Secondly, we deduce a Pohozaev-type identity and obtain a nonexistence result.

Keywords: Existence; Nonexistence; Elliptic equation; Nontrivial solutions

Introduction

In this paper we study the existence, multiplicity and nonexistence of nontrivial solutions of the following problem

\[
(\mathcal{P}_{\beta, \lambda, \mu}) \quad \{ -\Delta u - \mu |u|^{q-2} u = |u|^{\beta} g(x) |u|^{\gamma} \sin x, \quad x \neq 0 \}
\]

where \( k \) and \( N \) be integers such that \( N \geq 3 \) and \( k \) belongs to \( \mathbb{N} \setminus \{0\} \), \( 2^* = 2N/(N-2) \) is the critical Sobolev exponent, \( \mu > 0 \), \( \gamma \leq 2^* \), \( 0 \leq a < 1 \), \( 1 < q < 2 \), \( g \) is a bounded function on \( \mathbb{R}^N \), \( \lambda \) and \( \beta \) are parameters which we will specify later.

We denote point \( x \) in \( \mathbb{R}^N \) by the pair \( (\nu, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k} \), \( D_{x}^{2} = D_{x}^{2}\left((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}\right) \) and \( \mathcal{H}_{\nu} = \mathcal{H}_{\nu}\left((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}\right) \), the closure of \( C_{0}^{1}\left((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}\right) \) with respect to the norms

\[
|h| = \left( \int_{\mathbb{R}^N} |F(x)|^{1/2} \right)^{1/2} \quad \text{and} \quad |h|_{s} = \left( \int_{\mathbb{R}^N} |F(x)|^{-1/2} \right)^{1/2}
\]

We define the weighted Sobolev space \( D = \mathcal{H}_{\nu} \cap C^{0}\left(|x|^{-\alpha}\right) \times C^1\left(|x|^{-\alpha}\right) \) with \( b = a+1 \), which is a Banach space with respect to the norm defined by \( \|u\|_{s} = \left( \int_{\mathbb{R}^N} |F(x)|^{-1/2} \right)^{1/2} \).

My motivation of this study is the fact that such equations arise in search for solitary waves of nonlinear evolution equations of the Schrödinger or Klein-Gordon type [1-3]. Roughly speaking, a solitary wave is a nonsingular solution which travels as a localized packet in the spatial variable. Accordingly, a solitary wave preserves intrinsic properties of particles such as the energy, the angular momentum and the charge, whose finiteness is strictly related to the finiteness of the \( L^{2} \)- norm. Owing to their particle-like behavior, solitary waves can be regarded as a model for extended particles and they arise in many problems of mathematical physics, such as classical and quantum field theory, nonlinear optics, fluid mechanics and plasma physics [4].

Several existence and nonexistence result are available in the case \( k = N \), we quote for example [5,6] and the reference therein. When \( \mu = 0 \) \( g(x) = 1 \), problem \((\mathcal{P}_{\beta, \lambda, \mu})\) has been studied in the famous paper by Brezis and Nirenberg [7] and B. Xuan [8] which consider the existence and nonexistence of nontrivial solutions to quasilinear Brezis-Nirenberg-type problems with singular weights.

Concerning existence result in the case \( k < N \) we cite [9,10], and the reference therein. As noticed in [11], for \( a = 0 \) and \( \lambda = 0 \), M. Badiale et al. has considered the problem \((\mathcal{P}_{\beta, \lambda, \mu})\). She established the nonexistence of nonzero classical solutions when \( k < N \) and the pair \((\beta, \gamma)\) belongs to the region, i.e. \( (\beta, \lambda) \in A = A \cup A \cup A \) where \( A := \{ (\beta, \lambda) \in \mathbb{R}^2 : \beta \in (0, 2), \lambda \in (2^*, 2^* \right) \} \), \( A := \{ (\beta, \lambda) \in \mathbb{R}^2 : \beta \in (2, N), \lambda \in (2^*, 2^* \right) \} \), \( A := \{ (\beta, \lambda) \in \mathbb{R}^2 : \beta \in [N, +\infty), \lambda \in [2^*, 2^* \right) \} \).

Since our approach is variational, we define the functional \( I_{\beta, \lambda, \mu} \) on \( D \) by

\[
I_{\beta, \lambda, \mu}(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 dx - \lambda \int_{\mathbb{R}^N} g(x) |u|^\gamma dx.
\]

We say that \( u \in D \) is a weak solution of the problem \((\mathcal{P}_{\beta, \lambda, \mu})\) if it is a nontrivial nonnegative function and satisfies

\[
\int_{\mathbb{R}^N} \{ \nabla u \nabla v - \mu |u|^{q-2} uv \} dx = \int_{\mathbb{R}^N} (g(x) |u|^\gamma uv - \lambda |u|^\gamma uv - \lambda g(x) |u|^\gamma uv) dx = 0,
\]

Concerning the perturbation \( g \) we assume

\[
(G) g \in L^{1}\left(\mathbb{R}^N\right) \quad \text{and} \quad g(\cdot) > 0 \quad \text{for all} \quad \nu \in \mathbb{R}^N.
\]

In our work, we prove the existence of at least one critical points of \( I_{\beta, \lambda, \mu} \) by the Ekeland’s variational in [12]. By the Pohozaev type identities in [12], we show the nonexistence of positive solution for our problem.

We shall state our main result

Theorem 1 Assume \( 2 < k \leq N \), \( \mu \leq \mu_{k} = (k-2)/2 \), \( \beta = 2 \), \( 0 < a < q < 2 \) and \( (G) \) hold.

If \( \gamma \in (2^*, 2^* \right) \) then there exist \( \Lambda_0 \) and \( \Lambda^* \) such that the problem \((\mathcal{P}_{\beta, \lambda, \mu})\) has at least one nontrivial solution for any \( \lambda \in (\Lambda^*, \Lambda_0) \).

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Theorem 2 Let $2 < k \leq N$, $0 < a < 1$ and (G) hold.

If $\beta \in (2,3)$, $\gamma \in (2,2')$, with $2p_{\beta,2'} - 2N/(N - 2(\beta - 2a)) = \lambda < 0$ and $1 < q_S < 2$, then $(P_{\lambda,q_S})$ has no positive solutions.

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to the proof of Theorem 1. Finally in the last section, we give a nonexistence result by the proof of Theorem 2.

Preliminaries

We list here a few integrals inequalities. The first inequality that we need is the weighted Hardy inequality [13]

$$\pi \int_\mathbb{R} \left| \partial_x^\alpha f(x) \right|^2 \, dx \leq \mu \int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \, dx, \forall \mu \in \mathcal{H},$$

The starting point for studying $(P_{\lambda,q_S})$ in the Hardy-Sobolev-Maz'ya inequality that is peculiar to the cylindrical case $k < N$ and that was proved by Maz'ya in [14]. It state that there exists positive constant $C_k$ such that

$$C_k \left( \int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \, dx \right)^{\frac{1}{2}} \leq \mu \int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \, dx,$$

for $\mu > 0$ of equation $(P_{\lambda,q_S})$ is related to a family of inequalities given by Caffarelli, Kohn and Nirenberg [15], for any $\nu \in C^1_c(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^{2\gamma}$. The embedding $\mathcal{H} \to L^p(\mathbb{R}^2, |\nu|^p \, dx)$ is compact where $\nu = ay$ and $L^p(\mathbb{R}^2, |\nu|^p \, dx)$ is the weighted $L^p$ space with respect to the norm

$$|\nu|_p = \int \left| \partial_x^\alpha \sqrt{|\nu(x)|} \right|^2 \, dx.$$

Definition 1 Assume $2 \leq k < N$, $0 < \mu \leq \mu_k$ and $2 < \gamma < 2'$. Then the infimum $S_{\mu,k}$ defined by

$$S_{\mu} = S_{\mu,k}(\kappa, \gamma) := \inf_{\nu \in C^1_c(\mathbb{R}^2 \setminus \{0\})} \frac{\int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \, dx}{\int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \, dx},$$

is achieved on $\mathcal{H}$.

Lemma 1 Let $(u_n) \subset D$ be a Palais-Smale sequence $(PS)_\delta$ for short of $(I_{\lambda,q_S} )$, such that

$$I_{\lambda,q_S}(u_n) \to \delta \text{ and } I'_{\lambda,q_S}(u_n) \to 0 \text{ in } D \text{ (dual of }D) \text{ as } n \to \infty,$$

for some $\delta \in \mathbb{R}$. Then if $\lambda < \lambda_0 = \frac{q(2 - \gamma)}{2}\gamma$, $u_n \to u$ in $D$ and

$$I_{\lambda,q_S}(u_n) \to 0.$$

Proof. From (6), we have

$$\left( 1/2 \right) \left[ \int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \right] dx - \frac{\lambda}{\gamma} \int_\mathbb{R} g(x) \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 dx = \delta + o_{\delta}(1)$$

and

$$\left[ \int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \right] dx - \int_\mathbb{R} g(x) \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 dx = o_{\delta}(1),$$

then $(u_n)$ is bounded in $D$. Going if necessary to a subsequence, we can assume that there exists $u \in D$ such that

$$u_n \to u \text{ strongly in } L^p(\mathbb{R}^2, |\nu|^p \, dx)$$

and

$$u_n \to u \text{ strongly in } L^\infty(\mathbb{R}^2).$$

Consequently, we get for all $\nu \in C^1_c((\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^{2\gamma})$,

$$\int_\mathbb{R} \left( \nu \partial_x^\alpha g + \frac{1}{\gamma} \nu \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \right) dx + \lambda \int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \nu \, dx = 0,$$

which means that

$$I_{\lambda,q_S}(u) = 0.$$

Existence Result

Firstly, we require following Lemmas

Lemma 2 Let $(u_n) \subset D$ be a $(PS)_\delta$ sequence of $(I_{\lambda,q_S})$ for some $\delta \in \mathbb{R}$.

Then,

$$u_n \to u \text{ in } D$$

and either

$$u_n \to u \text{ a.e.} \in \mathbb{R}^2.$$

Denote $\nu = u_n - u$, then $\nu_n \to 0$. As in Brézis and Lieb [16], we have

$$\int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \left| \nu_n \right|^2 dx - \int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \left| u_n \right|^2 dx - \nu_n \to 0$$

and

$$\int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \left| \nu_n \right|^2 dx \leq \left( \int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \left| \nu \right|^2 dx \right)^{\frac{1}{2}} \left( \int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \left| \nu_n \right|^2 dx \right)^{\frac{1}{2}}.$$

From Lebesgue theorem and by using the assumption $(G)$, we obtain

$$\lim_{n \to \infty} \int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \left| \nu_n \right|^2 dx = \lim_{n \to \infty} \int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \left| u_n \right|^2 dx.$$

Then, we deduce that

$$I_{\lambda,q_S}(u_n) - I_{\lambda,q_S}(u) + \frac{1}{2} \left[ \int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \right] dx + o_{\delta}(1)$$

and

$$I_{\lambda,q_S}(u_n) = \left[ \int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \right] dx + o_{\delta}(1).$$

From the fact that $\nu_n \to 0$ in $D$, we can assume that

$$\lim_{n \to \infty} \left[ \int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \right] dx = \alpha \geq 0.$$

Assume $\alpha > 0$, we have by definition of $S_{\mu,k}$

$$\alpha \geq S_{\mu,k}^{(\gamma)}(\gamma),$$

and so

$$\alpha \geq S_{\mu,k}^{(\gamma)}(\gamma).$$

Then, we get

$$\delta \geq I_{\lambda,q_S}(u) + \frac{1}{2} \left[ \int_\mathbb{R} \left| \partial_x^\alpha \sqrt{f(x)} \right|^2 \right] dx + o_{\delta}(1).$$

Therefore, if not we obtain $\alpha = 0$, i.e $u_n \to u$ in $D$.

Lemma 3 Suppose $2 \leq k \leq N$, $0 < \mu_k$, and (G) hold. There exist $\lambda' > 0$ such that if $\lambda > \lambda'$, then there exist $\rho$ and $\nu$ positive constants such that,
there exist $\omega \in \mathbb{R}^n$ such that $I_{t_i,\varphi}(\omega) < 0$,

ii) we have $I_{t_i,\varphi}(u) \geq v > 0$ for $\|u\|_{L^\infty} = \rho_i$.

Proof. i) Let $t_i > 0$, $t_i$ small and $\varphi \in C^0_c((\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^+)$ such that $\varphi \neq 0$. Choosing $\Lambda' = [\varphi \theta ]^\tau < \Lambda_\varphi = \frac{q(y-2)}{2(y-q)}$ then, if $\lambda \in (\Lambda', \Lambda_\varphi)$

$$I_{t_i,\varphi}(\theta) := \int_{\mathbb{R}^n} -\left( \frac{\gamma'}{2} + \mathcal{L} \right)[\varphi]^\gamma \left[ \int_{\mathbb{R}^n} |\nabla \varphi|^{-q} |\varphi|^{q-1} \right] \mathcal{L} \varphi(x) dx \\
- \left( \frac{\gamma'}{2} + \mathcal{L} \right)[\varphi]^\gamma \left[ \int_{\mathbb{R}^n} |\nabla \varphi|^{-q} |\varphi|^{q-1} \right] \mathcal{L} \varphi(x) dx$$

$$\leq 0$$

Thus, if $\omega = t_i \theta$, we obtain that $I_{t_i,\varphi}(\omega) < 0$.

ii) By the Holder inequality and the definition of $S_{\varphi,\gamma}$, and since $\gamma > 2$, we get for all $u \in \mathcal{D}$:

$$I_{t_i,\varphi}(u) := \frac{1}{(1/2)} \left[ \int_{\mathbb{R}^n} \left( \frac{\gamma'}{2} + \mathcal{L} \right)[\varphi]^\gamma \left[ \int_{\mathbb{R}^n} |\nabla \varphi|^{-q} |\varphi|^{q-1} \right] \mathcal{L} \varphi(x) dx \\
- \left( \frac{\gamma'}{2} + \mathcal{L} \right)[\varphi]^\gamma \left[ \int_{\mathbb{R}^n} |\nabla \varphi|^{-q} |\varphi|^{q-1} \right] \mathcal{L} \varphi(x) dx \right]$$

$$\geq \frac{1}{(1/2)} \left[ \int_{\mathbb{R}^n} \left( \frac{\gamma'}{2} + \mathcal{L} \right)[\varphi]^\gamma \left[ \int_{\mathbb{R}^n} |\nabla \varphi|^{-q} |\varphi|^{q-1} \right] \mathcal{L} \varphi(x) dx \\
- \left( \frac{\gamma'}{2} + \mathcal{L} \right)[\varphi]^\gamma \left[ \int_{\mathbb{R}^n} |\nabla \varphi|^{-q} |\varphi|^{q-1} \right] \mathcal{L} \varphi(x) dx \right]$$

If $\lambda > 0$, then there exist $v > 0$ and $\rho_i > 0$ small such that $I_{t_i,\varphi}(u) \geq v > 0$ for $\|u\|_{L^\infty} = \rho_i$.

We also assume that $t_i$ is so small enough such that $\|\theta \|_{L^\infty} < \rho_i$.

Thus, we have $c_i = \inf \{ I_{t_i,\varphi}(u) : u \in B_{\rho_i} \} > 0$, where $\mathcal{B}_{\rho_i} = \{ u \in \mathcal{D} : \|u\|_{N^*} < \rho_i \}$.

Using the Ekeland’s variational principle, for the complete metric space $\mathcal{B}_{\rho_i}$ with respect to the norm of $\mathcal{D}$, we can prove that there exists a $(PC)$ sequence $(u_n) \subset \mathcal{B}_{\rho_i}$ such that $u_n \to u_i$ for some $u_i$ with $\|u_i\|_{N^*} \leq \rho_i$.

Now, we claim that $u_n \to u_i$. If not, by Lemma 2, we have $c_i \geq I_{t_i,\varphi}(u_n) + (\gamma'(2) + \gamma') \|S_{\varphi,\gamma}\|^{\gamma'(2)}$

$$\geq c_i + (\gamma'(2) \|S_{\varphi,\gamma}\|^{\gamma'(2)}$$

which is a contradiction.

Then we obtain a critical point $u_i$ of $I_{t_i,\varphi}$ for all $\lambda \in (\Lambda', \Lambda_\varphi)$.

Proof of Theorem 1

Proof. From Lemmas 2 and 3, we can deduce that there exists at least a nontrivial solution $u_i$ for our problem $(P_{t_i,\varphi})$ with positive energy [17-19].

Nonexistence Result

By a Pohozaev type identity we show the nonexistence of positive solution of $(P_{t_i,\varphi})$ when $\beta \in (2\mathbb{Z}, 2)$, $\gamma \in (2\mathbb{Z}, 2)$ with $2\mathbb{Z} - 2 N / (N - 2(\beta - 2a))$, $\Lambda_0 < 1$, $\Lambda < 2$ and $(G)$ hold with $0 < a < 1$.

First, we need the following lemma

Lemma 4 Let $\mathcal{U}$ be a positive solution of $(P_{t_i,\varphi})$ and. Then the following identity holds

$$\int_{\mathbb{R}^n} \|u\|^{\gamma'} dx - \int_{\mathbb{R}^n} \|u\|^{\gamma'} dx$$

where $\lambda \left( \frac{N}{q} - \frac{N - \beta}{2} \right) \int_{\mathbb{R}^n} |\nabla \varphi|^{q-2} |\varphi| \int_{\mathbb{R}^n} |\nabla \varphi|^{q-2} |\varphi| dx$.

Proof: we shall state the similar proof of proposition 30 and Lemma 31 in [11].

1) Multiplying the equation of $(P_{t_i,\varphi})$ by the inner product $(u \nabla u)$ and integrating on $\mathbb{R}^n$, we obtain

$$\int_{\mathbb{R}^n} |\nabla u|^{q} dx + \lambda \left( \frac{N}{q} - \frac{N - \beta}{2} \right) \int_{\mathbb{R}^n} |\nabla \varphi|^{q-2} |\varphi| \int_{\mathbb{R}^n} |\nabla \varphi|^{q-2} |\varphi| dx$$

2) By multiplying the equation of $(P_{t_i,\varphi})$ by $u$, using the identity

$$u \Delta u \ dx (u \nabla u) - |\nabla u|^q dx$$

in $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^+$ and applying the divergence theorem on $\mathbb{R}^n$, we obtain

$$\int_{\mathbb{R}^n} |\nabla u|^{q} dx + \lambda \left( \frac{N}{q} - \frac{N - \beta}{2} \right) \int_{\mathbb{R}^n} |\nabla \varphi|^{q-2} |\varphi| \int_{\mathbb{R}^n} |\nabla \varphi|^{q-2} |\varphi| dx$$

From (3), we have

$$\int_{\mathbb{R}^n} |\nabla u|^{q} dx + \lambda \left( \frac{N}{q} - \frac{N - \beta}{2} \right) \int_{\mathbb{R}^n} |\nabla \varphi|^{q-2} |\varphi| \int_{\mathbb{R}^n} |\nabla \varphi|^{q-2} |\varphi| dx$$

Combining (??) and (??), we obtain

$$\int_{\mathbb{R}^n} |\nabla u|^{q} dx + \lambda \left( \frac{N}{q} - \frac{N - \beta}{2} \right) \int_{\mathbb{R}^n} |\nabla \varphi|^{q-2} |\varphi| \int_{\mathbb{R}^n} |\nabla \varphi|^{q-2} |\varphi| dx$$

Proof of Theorem 2. We proceed by contradictions.

From Lemma 4, since $(G)$ hold and $1 < q < 2$ therefore, if $\beta \in (2\mathbb{Z}, 2)$, $\gamma \in (2\mathbb{Z}, 2)$ with $2\mathbb{Z} - 2 N / (N - 2(\beta - 2a))$ we obtain that $\Lambda > 0$ which contradicts the fact that $\Lambda > 0$.

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