Extending Prolog for Quantified Boolean Horn Formulas

Anish Mallick\textsuperscript{1} and Anil Shukla\textsuperscript{2}

\textsuperscript{1} Pontifica Universidad Catolica de Chile, Chile.
\texttt{anish.mallick@mat.uc.cl}

\textsuperscript{2} Department of Computer Science and Engineering, IIT Ropar, India.
\texttt{anilshukla@iitrpr.ac.in}

\textbf{Abstract.} Prolog is a well known declarative programming language based on propositional Horn formulas. It is useful in various areas, including artificial intelligence, automated theorem proving, mathematical logic and so on. An active research area for many years is to extend Prolog to larger classes of logic. Some important extensions of it includes the constraint logic programming, and the object oriented logic programming. However, it cannot solve problems having arbitrary quantified Horn formulas.

To be precise, the facts, rules and queries in Prolog are not allowed to have arbitrary quantified variables. The paper overcomes this major limitations of Prolog by extending it for the quantified Boolean Horn formulas. We achieved this by extending the SLD-resolution (SLD-Res) proof system for quantified Boolean Horn formulas, followed by proposing an efficient model for implementation. The paper shows that the proposed implementation also supports the first-order predicate Horn logic with arbitrary quantified variables.

The paper also introduces for the first time, a declarative programming for the quantified Boolean Horn formulas.

1 Introduction

Prolog is a declarative programming language developed by Alain Colmerauer, Bob Pasero and Philippe Roussel in 1972 \cite{8,31}. It was first designed for solving problems related to natural language processing. Very soon, arithmetic aspects of logic was added to it, using the ideas from Robert Kowalski SL-resolution prover \cite{24}. Today, Prolog is useful not only in the areas of language processing and automated theorem proving, but also in the areas of artificial intelligence, relational database, and design automation to name a few.

Prolog is, in a way an interactive system, which supports a kind of man-machine conversation. That is, a user is allowed to specify some relevant knowledge and subsequently ask it valid questions regarding the same. For example, one may specify several facts about a family, say family members and their relationships, along with some specific rules, for example a rule about siblings. Then one may ask if two elements in the space are siblings?

In Prolog, facts and rules are defined first then query follows. Though all facts and rules have to be predicate horn formulas with quantifiers, any rules defined have to be univerally quantified. For example,

\[
\forall x, y \text{ female}(x) \land \text{parent}(x, y) \rightarrow \text{mother}(x, y)
\]

is a valid rule but

\[
\forall x, y \exists z \text{ parent}(z, x) \land \text{parent}(z, y) \rightarrow \text{ sibling}(z, y)
\]

is not.

On the other hand for queries we are allowed logical Horn clauses with existential quantifiers only. For example,

\[
\exists x. \text{parent}(x, \text{bob})
\]

is a valid query but

\[
\exists x \forall y. \text{ancestor}(x, y)
\]
is not. The primary focus of this work is to remove these limitations (see, e.g., [33, Chapter 14] for reference) and extend the scope of Prolog to handle larger class of problems. We also propose an efficient algorithm to the problem

\[ \text{Is } \mathcal{F} = \Pi. \phi \implies \Pi. (\phi \land C) ? \]

where, \( \mathcal{F} \) is a first-order predicate Horn formula, and \( C \) a first-order predicate Horn clause. Note that the above problem is in general undecidable [15, 37]. The efficiency lies in the fact that the algorithm will detect loop or infinite branching in linear time and stop. It should be clear that we are not claiming that any SLD-Q-resolution (Definition 13) will not halt if this algorithm outputs loop, just that it will determine possibility of not halting following the current course of action. So this should not be viewed as an attempt towards halting problem but a contingency. These are explained further in the following section.

1.1 Our Contributions

In this section we summarize the contribution of this article.

1. **Extending Prolog for quantified Boolean Horn formulas.** A Prolog algorithm can be expressed as logical part plus control part. Logical part is expressed as facts and rules. The facts and rules are represented as (propositional) Horn formulas (Section 2.5). The control part is based on the SLD-Res proof system (Definition 10). In particular, once a query clause \( C \) is presented by the user, Prolog control tries to find an SLD-Res refutation of \( \neg C \), and accordingly return the answer.

The article extended the SLD-Res proof system for the quantified Boolean Horn formulas and defined the SLD-Q-Res proof system (Section 3). Quantified Boolean Horn formulas are all QBFs in closed prefix form with CNFs in which every clause has at most one positive literal (Definition 12). The article shows that SLD-Q-Res is sound and complete for the quantified Boolean Horn formulas and also proposes an efficient implementation for the same. That is, given a quantified Boolean Horn formula \( F = \Pi. \phi \) as facts and rules, and a quantified Horn query clause \( C \), the paper proposes an efficient algorithm (Algorithm 2) to answer the following:

\[ \text{Is } \mathcal{F} = \Pi. \phi \implies \Pi. (\phi \land C) ? \]

Obviously, one may encode verification problems as a quantified Boolean Horn formulas [2, 5, 30], which in general is undecidable. Our proposed algorithm is efficient in the sense that it can detect a possible loop during evaluation efficiently.

As a result, the article allows us to specify facts and rules as a quantified Boolean Horn formulas with quantified query clause. Refer Section 6.1 for an application.

The article shows how our efficient algorithm (Algorithm 2) handles the first-order predicate Horn logic as well. To be precise, the article shows that given facts, rules and queries as first-order predicate Horn clauses, Algorithm 2 solves efficiently the above implication problem.

2. **Overcomes the weaknesses of existing Prolog.** As already discussed, Prolog does not allow arbitrary quantified variables in facts, rules and queries. All variables appearing in facts and rules are universally quantified and all variables appearing in the query are existentially quantified. These limitations arise due to the Prolog control (engine) part, which is based on SLD-Res proof systems for Horn formulas.

This article overcomes the limitations of the existing Prolog by allowing arbitrary quantified variables in facts, rules and queries. As a result several problems which were impossible to be solved via existing Prolog can now be attempted. We present few examples below after few comments.

The encoding of the problem statement of the Example 1 and Example 2 (below) uses only universally quantified variables and can be specified in the existing Prolog. However, the query of the corresponding problem cannot be specified in the existing Prolog as it requires universally quantified variables.

The encoding of the problem statement and the query of the Example 3 (below) uses both the existentially and universally quantified variables, and so cannot be specified in the existing Prolog. However, the problem is easily handled by the extended Prolog (QBF-Prolog, Section 4) of this article.
**Example 1.** **Tree-bipartite-problem:** Consider the problem related to trees and bipartite graphs. Recall, a graph $G = (V, E)$ is said to be bipartite iff its vertex set $V$ can be partitioned into two parts $U$ and $W$ such that every edge in $E$ connects a vertex in $U$ to a vertex in $W$. Consider the following: undirected graphs without cycle are bipartite, and trees are acyclic. These facts can be encoded as rules in the existing Prolog. However, consider the following query: are trees bipartite? The query required universal variables, and hence cannot be handled by the existing Prolog. However, the extended Prolog of the paper is capable to handle the problem. For the detailed encoding and solution of the problem, refer Section 6.2.

**Example 2.** **Bipartite problem:** Consider a Prolog program which specifies the set of all bipartite graphs $G$. One can easily encode bipartite graphs in first-order predicate Horn logic using only universally quantified variables. Hence, this can be encoded in the existing Prolog. But, consider the following query regarding a bipartite graph $G = (\{U,W\}, E)$: pick a vertex $x$ belonging to $U$ and move to some vertex $y$ using an edge $\{x,y\}$ and from $y$ again move to a vertex $z$ using the edge $\{y,z\}$. Is the vertex $z \in U$? Obviously, the above query is valid and needed universal variables in the encoding. Hence, the existing Prolog does not support such queries. However, the present paper extends Prolog to handle such queries as well. For the detailed encoding of the bipartite problem, refer Section 6.3.

**Example 3.** **Simple relations:** The problem uses the following predicates:

$$P(h,k) /* \text{the predicate is 1 iff } k = 2h */$$

$$R(h,k) /* \text{the predicate is 1 iff } h < k */$$

We may give several interpretations to these predicates. For example, if $k$ and $h$ represent graphs, then the predicate $R(k,h)$ is 1 iff the graph $h$ is a subgraph of the graph $k$. The predicate $P(k,h)$ is 1 iff the graph $k$ is a superset of the graph $h$. That is, the graph $k$ is constructed from $h$ by say adding a vertex.

Let us now consider the following rules:

$$\forall h,k.P(h,k) \rightarrow R(h,k) /* \text{rule 1} */$$

$$\forall h_1,h_2,h_3.[R(h_1,h_2) \land R(h_2,h_3)] \rightarrow R(h_1,h_3) /* \text{rule 2} */$$

Now consider the following query:

$$\forall h \exists g,k.[P(h,g) \land P(g,k)] \rightarrow R(h,k) /* \text{query} */$$

Clearly the query is correct according to our interpretation. We show that this problem can be easily solved by the QBF-Prolog of this article. For the detailed encoding and solution of the problem, refer Section 6.3.

Finally, we show an example (Example 4) which cannot be handled by the existing Prolog, but also cannot be solved via the extended Prolog. The proofs of such problems are induction based, however, our algorithm is unable to mimic the inductive proof and just return a loop as the output. This is not surprising, even the pigeonhole principle has a short inductive proof, but is hard for resolution [14]. Consider the example below.

**Example 4.** Consider a Prolog program which specifies the set of all trees, with a unique root and such that every child has a unique parent. Consider the following query: is the tree connected? Existing prolog unables to handle such programs. QBF-Prolog, proposed in this article just returns a loop for this problem. For the detailed encoding of the tree and query, refer Section 6.5.

3. **An interactive QBF-solver for the quantified Boolean Horn formulas (QBF-Prolog, Section 4).** There exists several QBF-solver in the literature (Section 2.2), then why QBF-Prolog? First of all, QBF-Prolog, deals with the quantified Horn formulas. It is well known that several problems of practical importance, such as program verifications [2, 5, 30], can be encoded as quantified Horn formulas. It is important to design an efficient QBF solver for the same. QBF-Prolog is such a solver. It uses the structure of the quantified Boolean Horn formulas and solves
the satisfiability problem in linear time. (Of course, the QBF-Prolog may return a loop as well, but it detects the same in linear time).

QBF-Prolog can also be used for the quantified renamable Horn formulas. We say that a QBF \( \mathcal{F} \) is a quantified renamable Horn formula, if \( \mathcal{F} \) can be transformed into a quantified Horn formula by negating every instance of one or more of its variables. For example, the QBF \( \exists x_1 \forall x_2 \exists x_3. (x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor x_2 \lor \neg x_3) \) is a quantified renamable Horn formula, since by complementing \( x_1 \) and \( x_3 \) we get the following quantified Boolean formula: \( \exists x_1 \forall x_2 \exists x_3. (\neg x_1 \lor \neg x_2 \lor x_3) \land (x_1 \lor \neg x_2 \lor \neg x_3) \). This operation of choosing a subset of variables and replacing each positive literal of such a variable by the corresponding negative literal and vice versa, is called renaming. It is easy to observe that renaming preserves satisfiability. That is, a QBF \( \mathcal{F} \) is satisfiable if and only if the QBF \( \mathcal{F}' \) obtained via renaming is satisfiable.

Given a QBF formula \( \mathcal{F} \) it is possible to determine in linear time whether \( \mathcal{F} \) is quantified renamable Horn formula [16]. The algorithm also gives the set of variables which needs to be complemented in order to get a quantified Horn formula. Thus, using the algorithm from [16], and via the QBF-Prolog, one can solve the satisfiable problem for a quantified renamable Horn formula efficiently. Note that, [16] solves the problem for the propositional case, however, the same algorithm works for the quantified formulas as well, we just need to ignore the quantifiers.

Another advantage of QBF-Prolog is its interactive nature: As it is based on Prolog, it supports man-machine conversation. To the best of our knowledge, all existing QBF solvers are imperative in nature. Also, QBF satisfiability problem is well suited to be solved via backtracking and hence is very natural to pick Prolog and extend it for the QBFs.

1.2 Organisation of the paper

The remainder of the paper is organized as follows. We review the basic notations and preliminaries in Section 2. In Section 3, we extend the SLD-Res proof systems for the quantified Boolean Horn formulas and proof the completeness and soundness of the new proof systems for the same. We extend Prolog for the quantified Boolean formulas, with the restrictions that the query clause contains no new variables, in Section 4, and also propose an efficient implementation model for the same. We extend Prolog for quantified Boolean Horn formulas without any restriction and for first-order predicate Horn formulas in Section 5. In Section 6, we present some applications of the QBF-Prolog. Finally, we present conclusions in Section 7.

2 Notations and Preliminaries

Quantified Boolean formulas (QBFs) are extensions of propositional formulas, in which each variable is quantified by universal or existential quantifiers. QBF does not increase the expressive power of propositional logic, it simply offers an exponentially more succinct encoding of problems. As a result, problems from various important industrial fields, such as formal verification and model checking can be encoded succinctly as QBFs. This leads to the desire of building efficient QBF-SAT solvers. Let us define now QBFs more formally.

**Quantified Boolean Formulas (QBFs):** It extend propositional logic with Boolean quantifiers with the standard semantics that \( \forall x. F \) is satisfied by the same truth assignments as \( F|_{x=0} \land F|_{x=1} \) and \( \exists x. F \) as \( F|_{x=0} \lor F|_{x=1} \).

We say that a QBF is in **closed prenex form** with a CNF (Conjunctive Normal Form) matrix, if the QBF instance is of the form \( \Pi. \phi \), where \( \Pi = Q_1 X_1 Q_2 X_2 \ldots Q_n X_n \) is called the quantifier prefix, with \( Q_i \in \{\exists, \forall\} \), and \( \phi \) is a quantifier-free CNF formula in the variables \( X_1 \cup \ldots \cup X_n \). We have \( X_i \cap X_j = \emptyset \), \( Q_i \neq Q_{i+1} \), and \( Q_1, Q_n = \exists \).

A literal is either a variable or its complement. For a clause \( C \), \( \text{var}(C) \) is a set containing all the variables of \( C \). For a QBF \( \mathcal{F} \), \( \text{var}(\mathcal{F}) \) is a set containing all the variables of \( F \) (i.e., \( \text{var}(\mathcal{F}) = \cup_{C \in \mathcal{F}} \{\text{var}(C)\} \)). A quantifier \( Q(\Pi, \ell) \) of a literal \( \ell \) is \( Q_1 \) if the variable \( \text{var}(\ell) \) of \( \ell \) is in \( X_i \). A literal \( \ell \) is existential if \( Q(\Pi, \ell) = \exists \) and universal if \( Q(\Pi, \ell) = \forall \). For literals \( \ell \) and \( k \), with \( Q(\Pi, \ell) = Q_i \) and \( Q(\Pi, k) = Q_j \), we say that the literal \( \ell \) is on the left of literal \( k \) (\( \ell \leq k \)) if and only if \( i < j \). Within a clause, we order literals according to their ordering in the quantifier prefix.

We assume that QBF is in this form, unless noted otherwise. The order \( \leq_H \) is arbitrary extended to variables withing each block \( X_i \). For a literal \( \ell \), \( \text{level}(\Pi, \ell) = i \) if \( \text{var}(\ell) \in X_i \).

A QBF \( Q_1 x_1 \ldots Q_k x_k . \phi \) can be seen as a game between two players: universal (\( \forall \)) and existential (\( \exists \)). In the \( i^{th} \) step of the game, the player \( Q_i \) assigns a value to the variable \( x_i \). The existential
player wins if $\phi$ evaluates to 1 under the assignment constructed in the game. The universal player wins if $\phi$ evaluates to 0.

An assignment tree \[ T \] of a QBF $F$ is a complete binary tree of depth $|\text{var}(F)| + 1$. Internal nodes correspond to the variables of $F$. The order of the variables in $T$ respect the orders in the prefix in $F$. Internal nodes which corresponds to an existential variables act as an OR-nodes, whereas universal nodes corresponds to an AND-nodes. Every internal nodes $x$ has two children, one for literal $x$ (true) and one for literal $\neg x$ (false). An existential node is labelled with $\top$ (true) if at least one of its children is labelled with $\top$ (true). A universal node is labelled with $\top$ if both of its children are labelled with $\top$. A path in $T$ is a sequence of literals. A path $\tau$ from root to a leave in $T$ is a complete assignment and the leave is labelled with the value of QBF under $\tau$. The QBF $F$ is true if the root of $T$ is labelled with $\top$, and $F$ is false if the root is labelled with $\bot$.

A subtree $T'$ of the assignment tree $T$ is a pre-model \[ \{T' : |T'| \leq |T| \} \] of the QBF $F$ if the root of $T$ is also the root of $T'$, both children of every universal node must be present in $T'$, and exactly one child of every existential node are in $T'$. We say that a pre-model $T'$ of a QBF $F$ is a model of $F$, denoted $T' \models F$ if every node in $T'$ is labelled with $\top$. Similarly, a false QBF has countermodels which can be defined dually.

A QBF $F$ is satisfiable if it has at least one model. For QBFs $F$ and $H$, we say that $H$ is implied by $F$, denoted $F \models H$ if, for all $T$, if $T \models F$ then $T \models H$.

We say that two QBFs $F$ and $H$ are (logically) equivalent denoted $(F \equiv H)$ if $F \models H$ and $H \models F$, and satisfiability equivalent, denoted $F \equiv_{\text{sat}} H$, iff $F$ is satisfiable whenever $H$ is satisfiable.

### 2.1 QBF proof complexity

Since past few decades, QBFs solving is an active area of research. As a result, several proof systems for QBFs have been developed. For example, Kleine Büning et al. in \[ 21 \] introduced Q-resolution (Q-Res), which is an extension of the propositional resolution proof system for QBFs. Before defining Q-Res, let us quickly define the Resolution (Res) proof system.

**Resolution (Res)**: (Res) is well studied propositional proof system introduced by Blake in \[ 6 \] and proposed by Robinson in \[ 32 \] as automated theorem proving. The lines in the resolution proofs are clauses. Given a CNF formula $F$, Res can infer new clauses according to the following inference (resolution) rule:

\[
\frac{C \lor x \land D \lor \neg x}{C \lor D},
\]

where $C$ and $D$ are clauses and $x$ is a variable being resolved, called as pivot variable. The clause $C \lor D$ is called the resolvent. Let $F$ be an unsatisfiable CNF formula. A resolution proof (refutation) $\pi$ of $F$ is a sequence of clauses $D_1, \ldots, D_l$ with $D_l = \Box$ and each clause in the sequence is either from $F$ or is derived from previous clauses in the sequence using the above resolution rule.

We can also view $\pi$ as a directed acyclic graph $G_{\pi}$, where the source nodes are the clauses from $F$, internal nodes are the derived clauses and the empty node is the unique sink. Edges in $G_{\pi}$ are from the hypotheses to the conclusion for each resolution step. In $G_{\pi}$, we say that a clause $C$ is descendant to a clause $D$ if there is a directed path from $C$ to $D$.

**Q-resolution**: Q-Res uses the resolution rule, but with a side condition that the pivot variable must be existential, and the resolvent does not have simultaneously a $z$ and $\neg z$. In addition, Q-Res has a universal reduction rule ($\forall$-Red) which allows dropping a universal literal $u$ from a clause provided the clause has no existential literal $\ell$ with $u \leq_{\forall} \ell$. That is, there exists no existential literal to the right of the reduced literal $u$. We say in this case, that $u$ is not blocked.

**QU-resolution** \[ 12 \], removes the restriction from Q-Res that the resolved variable must be existential variable and allows resolution on universal variables as well. Thus QU-resolution (QU-Res) is classical resolution augmented with a $\forall$-Red rule. Interested readers are referred to \[ 35 \], for more details on QBF proof complexity.

### 2.2 QBFs solvers

Since, the propositional satisfiability (SAT) problem is NP-complete, and the QBF-SAT problem is PSPACE-complete \[ 1 \], only algorithms with exponential worst-case time complexity are known for these problems. Despite of this, several efficient SAT solvers, capable of solving the satisfiability
instances with thousands of variables and clauses, have been designed. For example, the GRASP search algorithm for the satisfiability problem \(30\), implemented by Marques-Silva and Sakallah. GRASP is based on conflict-driven clause learning (CDCL) algorithms \(34\).

Motivated by the success of propositional SAT solvers, many QBF solvers have also been developed. We would like to mention some important ones:

- **DepQBF**: DepQBF, developed by Lonsing and Egly \(26\), is based on QCDCL \(40\), which is an extension of CDCL for QBFs. DepQBF, implements a variant of QCDCL, which is based on a generalization of Q-Res. This generalization is due to some additional axioms. Q-Res proofs can be extracted from instances of the QCDCL algorithm. This generalization, in DepQBF implementation, helps to produce exponential shorter refutation, than the traditional QCDCL-based solvers.

- **RAREQS**: Recursive Abstraction Refinement QBF Solver \(18\), is a recursive algorithm for QBF solving, based on the Counter-example Guided Abstraction Refinement (CEGAR) technique \(7\). Initially, CEGAR was used to solve QBFs with two level of quantifiers \(19\). RAREQS, extends the technique to QBFs with an arbitrary number of quantifiers using recursion.

  CEGAR technique was designed to tackle problems whose implicit representation is infeasible to solve, and an abstract instance is tackled instead. Thus, in CEGAR-based algorithm, we have a concrete (an original) representation, and an abstract representation of a problem. Both the representations are connected as follows: if the abstract problem has no solution, then concrete problem also does not have a solution. If a solution to the abstract problem is not a solution to the concrete problem, then a counterexample can be constructed for this fact. And finally, if there are no counterexamples for a solution to the abstract problem, then the solution, is also a solution to the concrete problem.

  Based on the above properties one can compute a solution to the original problem as follows: compute a solution to the abstract problem, if no such solution exists, return: no solution to the original problem. Otherwise, check if it is also a solution to the original problem: if no counterexample exists, return: the same solution is a solution to the original problem. Otherwise, use the counterexample to refine the abstract representation and repeat. This gives the name: CounterExample Guided Abstraction Refinement technique.

- **GhostQ**: GhostQ solver \(23\) is originally implemented to support the QBF solving for instances which are not in prenex form. GhostQ is a DPLL-based QBF solver which uses ghost variables. Ghost literal, first introduced in \(13\), is a powerful propagation technique for QBFs. Interested readers are referred to \(23\).

### 2.3 First-order predicate logic

First-order predicate logic extends the propositional logic by adding both the predicates and the quantification. A Predicate \(P(x_1, x_2, \ldots, x_n)\) is an \(n\)-ary function whose range is only 0 (false) or 1 (true). Unlike propositional logic, where variable can take only 0-1 values, in predicate logic variables can take any values from some universal set \(U\). Predicates with arity 0 are just the propositional variables. Some examples of predicates are as follows. Assume the set of all humans as the universal domain set \(U\) for the below examples.

**Example 5.** “\(x\) is a male \(\equiv\) male(\(x\))”: here variable \(x\) ranges over the universal set \(U\). The predicate male(\(x\)) is true iff \(x\) is male.

**Example 6.** “\(x\) is the mother of \(y \equiv\) mother(\(x, y\))”: again \(x, y\) ranges over the universal set \(U\), and the predicate mother(\(x, y\)) is true iff the variables \(x\) and \(y\) are initialized with some values \(a, b \in U\) respectively, such that \(a\) is the mother of \(b\).

Clearly, as compared to the propositional logic, in the first-order predicate logic, it is possible to express knowledge much easier, for example the statement “for every \(x\), if \(x\) is an indian then \(x\) is a human” can be expressed as:

\[
\forall x \text{ indian}(x) \rightarrow \text{human}(x).
\]

As another example, consider the following statement: If \(x\) is a female, and parents of \(x\) and \(y\) are same, then \(x\) is a sister of \(y\). This statement can be expressed as:

\[
\forall x, y, m, f \quad \text{Female}(x) \land \text{Parent}(x, m, f) \land \text{Parent}(y, m, f) \rightarrow \text{Sister}(x, y).
\]
Observe that the above example is equivalent to the following:
\[ \forall x, y, m, f \left( \neg \text{Female}(x) \lor \neg \text{Parent}(x, m, f) \lor \neg \text{Parent}(y, m, f) \lor \text{Sister}(x, y) \right). \]

This rule has just one positive predicate, and hence is a Horn clause. A **first-order predicate Horn logic** consists of Horn clauses only. However, each literal on a clause can be replaced by an arbitrary predicates of any arity. Also, the variables can be quantified arbitrary.

We skip the definition of first-order predicate logic more formally here. Interested readers are referred to the book by Robert S. Wolf [38].

**Substitution and Unification**: Given a first-order predicate formula \( F(x_1, \ldots, x_n) \), over distinct variables, \( x_1, \ldots, x_n \), a substitution is a process of binding each variable \( x_i \) with a predicate \( P_i \) (denoted, \( x_i/P_i \)). The arity of \( P_i \) is allowed to be zero.

Recall, in Res, the resolution rules are performed over clauses with complementary literals \( x \) and \( \neg x \). Consider two literals \( P(a) \) and \( \neg P(x) \) appearing in two distinct clauses in a first-order predicate logic. The literals are almost complementary: the first contains a constant, whereas the second contains a variables. The substitution can be applied here two make the literals complementary: substitute \( x/a \) in the second. Thus, the literal \( P(a) \) and \( \neg P(x) \) becomes complementary after the substitution. One may now perform resolution step on the same. Thus substitution can be applied to make two first-order predicate formulas syntactically equal.

The **unification algorithm** is a general method for comparing first-order predicate formulas. The algorithms also computes the substitution which is needed to make the formulas syntactically equal. Interested readers are referred to [29].

### 2.4 Prolog

Prolog stands for PROgramming in LOGic. Prolog is a declarative programming language. Unlike, the procedural programming languages, where we have to tell the computer what to do when, and how to achieve a certain goal, in Prolog on the other hand, we only need to tell what is true, and then querying it to draw conclusions. To be precise, in Prolog the description of a problem and the procedure for solving it are separated from each other. According to Robert Kowalski, a Prolog algorithm can be expressed as:

\[
\text{algorithm} = \text{logic} + \text{control}
\]

In the above equation, the logic part gives the description of the problem, that is, what the algorithm should do, and the control part indicates how it should be done. In particular, the program logic is expressed as facts and rules, and a computation is initiated by running a query over these facts and rules.

The facts and rules are represented as **Horn formulas**. Horn formulas are a special case of CNF formulas, where each clause is allowed to have at most one positive literals.

Once a query as a goal clause is provided, Prolog tries to find a resolution refutation of the negated query. If the refutation is found, the query is said to be successful. This part constitute the control part.

To be precise, the Prolog control part is based on SLD-resolution, which is an important refinement of Linear Resolution [9]. A linear resolution derivation of a clause \( C \) from a CNF formula \( F \) is a sequence of clauses \( C_1, C_2, \ldots, C_m \), such that \( C_1 \in F, C_m = C \), and for every \( i < m \), \( C_{i+1} \) is the resolvent of \( C_i \) either with a clause \( D \in F \) or with a clause \( C_k \) for \( k < i \). In order to define the basic principle used in processing a logic program in Prolog, we need to first define the SLD-resolution (Definition [10]).

### 2.5 Proof system for Prolog: SLD-Res

In order to cut down the search space for SAT solvers, several refinements of resolution proof system have been introduced. One of the most important refinement is the linear resolution. Linear resolution is known to be complete and sound for unsatisfiable CNF formulas [20]. To even cut down the search space, several refinements of linear resolution have been proposed, for example, s-linear Res [28][39], t-linear Res [24], SL-res [24], and SLD-Res [25]. All the above mentioned refinements are known to be complete and sound for unsatisfiable CNF formulas, except the SLD-Res. As Prolog is based on SLD-Res we define it next.
**SLD-Res** SLD-Res, a refinement of linear resolution, was introduced by Robert Kowalski [25]. The name SLD-Resolution stands for SL resolution with definite clauses. SLD-Res is complete and sound for propositional Horn formulas. Before introducing SLD-Res, let us quickly re-visit propositional Horn formulas in detail.

**Propositional Horn Formulas** Horn formulas are a special case of CNF formulas, named after the logician Alfred Horn. A clause $C$ is a Horn clause if and only if $C$ contains at most 1 positive literal. A Horn formula is a conjunction of Horn clauses. A definite Horn clause is a clause with exactly one positive literal and zero or more negative literals. A conjunction of definite Horn clauses is called a definite Horn formula. We usually denote a definite Horn clause $(x \lor \neg x_1 \lor \ldots \lor \neg x_k)$ as $x \leftarrow x_1, \ldots, x_k$, which is equivalent to the following:

$$\text{IF } x_1 \land x_2 \land \cdots \land x_k \text{ then } x$$

Here, $x, x_1, \ldots, x_k$ are variables. The variable $x$ is called the head of the clause, and $x_1, \ldots, x_k$ is called the body of the clause. We denote the head of a Horn clause $C$ by $C^+$ (or head$(C)$) and the body by $C^-$. A clause with only negative literals are referred as negative clause. Clearly a negative clause is also a Horn clause. We usually refer a negative Horn clause as a goal clause. We denote a goal clause $(\neg x_1 \lor \neg x_2 \lor \cdots \lor \neg x_k)$ as $x \leftarrow x_1, x_2, \ldots, x_k$. The empty clause $\Box$ is also considered as a goal clause. We quickly list some important facts about Horn formulas.

**Lemma 7.** [22] For any Horn formula (other than the empty clause), the following holds:

1. A definite Horn formula is satisfiable.
2. A Horn formula $F$ is satisfiable if $F$ contains no positive unit clause (clauses with only one literal are called unit clauses).

**Proof.** Every clause in a definite Horn formula has exactly one positive literal. Thus the assignment which assigns each variable the value true, satisfies the formula.

If the Horn formula has no positive unit clause, then each clause contains at least one negative literals. Again the assignment which assigns each variable the value false, satisfies the formula. □

Therefore, for a Horn formula to be unsatisfiable, it must contain at least one negative clause. In fact the following also holds:

**Lemma 8.** [22] Let $F = F_1 \cup F_2$ be an unsatisfiable Horn formula, where $F_1$ contains only definite Horn clauses and $F_2$ contains only negative clauses. Then

$$F \text{ is unsatisfiable } \iff \exists C \in F_2 : F_1 \cup \{C\} \text{ is unsatisfiable}$$

**Proof.** Suppose not. That is, let $F$ be unsatisfiable and for all $C \in F_2$, $F_1 \cup \{C\}$ is satisfiable. Assuming this we show that $F$ is satisfiable as well, which is clearly a contradiction.

Let variables $x_1, x_2, \ldots, x_m$ are consequences of $F_1$, that is $F_1 \implies x_i$ for $i \in [m]$. Clearly, var$(C) \nsubseteq \{x_1, \ldots, x_m\}$ for every $C \in F_2$, otherwise $F$ would have been satisfiable: $F_1$ is satisfiable from Lemma [7] and if var$(C) \subseteq \{x_1, \ldots, x_m\}$ for all $C$ in $F_2$, then $F_2$ is also satisfiable.

Let $I$ be the following assignment: $I(x_j) = 1$ for $j \in [m]$, and $I(y) = 0$ for all other variables $y$. Clearly, $I(C) = 1$ for all $C \in F_2$: as $C$ is negative clause and at least one variable $y$ must belong to $C$. Therefore we have $I(F_2) = 1$.

On the other hand, observe that $I(F_1) = 1$ as well: since $I(x_j) = 1$ and $F_1 \implies x_j$, for $j \in [m]$, therefore $I(F_1)$ must be 1. □

Since we are discussing Horn formulas, we quickly introduce an important refinement of resolution proof system which is incomplete for CNF formulas but complete for Horn formulas: unit resolution (unit-Res). A resolution proof system is called a unit resolution proof systems iff for every resolution step one of its parent clause is a unit-clause. We have the following:

**Lemma 9.** [22] Let $F$ be a Horn formula, then

$$F \text{ is unsatisfiable } \iff F_{\text{unit-Res}}$$
Definition 10 (SLD-resolution). Let $F$ be a Horn formula. Let $D, G \subseteq F$ be disjoint subsets of clauses from $F$ such that $D$ is the set of all definite Horn clauses, and $G = \{G_1, \ldots, G_q\}$ be the set of all goal (negative) clauses. Let $C$ be a goal clause. An SLD-Res derivation of $C$ from $F$ is a sequence of negative clauses $\pi = N_0, N_1, \ldots, N_m$ such that:

1. $N_0 = G_j$, where $G_j$ is some goal from $F$.
2. Each $N_i$ is a resolvent of $N_{i-1}$ and a definite input clause $C_i \in D$, resolved over the head $x$ of $C_i$. The variable $x$ (i.e., the literal $\neg x$) of $N_{i-1}$ is the selected variable in the body of $N_{i-1}$.
   We call, the clause $N_{i-1}$ a centre clause, and the clause $C_i$, a side clause.
3. $N_m = C$.

$N_0$ is the top clause, and the $C_i$ are the input clauses of this SLD-Res derivation. If an SLD-Res derivation of $C$ from $F$ exists, we write $F \vdash_{SLD-Res} C$. If $N_m = \square$, we call $\pi$ an SLD-Res refutation of the Horn formula $F$.

Observe that, an SLD-Res refutation is not just a linear resolution but also an input and a negative resolution (every resolution step has at least one negative clause as parent). In fact, all the clauses $N_i$ is the SLD-Res derivation are goal clauses. This is because the top clause $N_0$ is a goal clause. If we pick $N_0$ to be a definite clause from $F$, then all clauses in the SLD-Res derivation will be definite clauses. Note that the input clauses of any SLD-Res derivations are always the definite clauses.

We know that SLD-Res proof system is sound and complete for Horn formulas.

Theorem 11. SLD-Res proof system is sound and complete for Horn formulas.

Prolog treats clauses as multisets of literals (it retains all the repeated literals in a resolvent). For Prolog not only the order of literals within a clause matters, but also the order of clauses within a Horn formula matters. The Horn formulas in which both the clauses and the literals within the clauses are given a fixed ordering are called a Prolog program. Given a Prolog program $\pi$ and a query $\gamma$ one can easily define the Prolog mechanism via a recursive algorithm. Interested readers are referred to [22].

3 SLD-resolution for QBFs (SLD-Q-Res)

If we allow universal variables, along with existential variables, to be used as a pivot variable, for the resolution step (as in QU-Res), then SLD-Res can be easily extended for QBFs: just add the $\forall$-Red rules as in [3]. This proof system is interesting, and we plan to study this in future. However, as per our knowledge, there exists no QBF solver based on QU-Res. Therefore, extending SLD-Res to QBFs with a restriction that only existential variables are allowed, as a pivot variable in the resolution step (as in Q-resolution), is more important. We work on this direction now. As, SLD-Res proof system is complete only for Horn formulas, let us define quantified Horn formulas first:

Definition 12. A quantified Boolean formula $F \equiv Q_1X_1Q_2X_2\ldots Q_nX_n(C_1 \land \cdots \land C_m)$ is a quantified Horn formula iff each clause $C_i$ in the matrix has at most one positive literals (existential or universal). We use notations as in the propositional case.

It is well known that the satisfiability problem for the quantified Horn formula can be solvable in polynomial time [22], however the equivalence problem for them are coNP complete [22].

It is clear that every clause in any quantified Horn formula $F$ belongs to one of the following set:
– \( F_3 \): set of all clauses \( C \) with exactly one positive existential literal \( \ell \).
– \( F_\emptyset \): set of all clauses with exactly one positive universal literal.
– \( F_{\text{goal}} \): set of all goal clauses with only negative (existential or universal) literals.

**Observation 1.** [22] A quantified Horn formula \( F \) is false if and only if there exists a clause \( C' \in F_\emptyset \cup F_{\text{goal}} \) such that \( Q_1 X_1 Q_2 X_2 \ldots Q_n X_n (F_3 \land C') \) is false.

**Observation 2.** Let \( F \) be a false quantified Horn formula. Let us assume, that in Observation \ref{observation:qbf} we have \( C' \in F_{\text{goal}} \). That is, assume \( C' \in F_{\text{goal}} \), and \( Q_1 X_1 Q_2 X_2 \ldots Q_n X_n (F_3 \land C') \) is false. Then for every negated existential literal \( q \) in \( C' \) (i.e., \( \neg q \in C' \)), we have a clause \( C \in F_3 \), with the variable \( q \) (i.e., literal \( q \)) as its head.

**Proof.** We know that the initial QBF \( F \) is false. Therefore, there does not exists any winning strategy for the existential player.

Consider the following strategy for the existential player: assign 1 to all the heads of the clauses from \( F_3 \). Let us call this assignment \( \alpha \). For sure, \( \alpha \) satisfies all the clauses of \( F_3 \). However, as the original formula is false, this assignment does not satisfy the clause \( C' \). In fact \( C'|_{\alpha} \) is either 0, or is left with a bunch of negated universal literals. Otherwise \( \alpha \) can be extended as a winning strategy for the existential player.

This tells us that, every negated existential literal (\( \neg \ell \)) of \( C' \) has a corresponding head \( \ell \) in some clause of \( F_3 \). If not, one can extend \( \alpha \) to winning strategy for the existential player. \( \square \)

**Observation 3.** Let us assume, that in Observation \ref{observation:qbf} we have \( C' \in F_\emptyset \). Then, for every negated existential literal \( q \) in \( C' \) (i.e., \( \neg q \in C' \)), we have a clause \( C \in F_3 \), with the variable \( q \) (i.e., literal \( q \)) as its head.

**Proof.** The proof is exactly as in Observation \ref{observation:qbf} \( \square \)

Let us define SLD-Res for QBFs (SLD-Q-Res). An SLD-Q-Res derivation of a clause \( C \) from a quantified Horn formula \( F \) is exactly as in the propositional case, except that a clause \( N_i \) in the sequence may also be derived via a \( \forall \)-Red rule from \( N_{i-1} \). That is, by deleting a universal variable of \( N_{i-1} \) which has not been blocked in \( N_{i-1} \). To be precise,

**Definition 13 (SLD-Q-Res).** Let \( F \) be a false quantified Horn formula. Let \( C \) be a goal clause. An SLD-Q-Res derivation of \( C \) from \( F \) is a sequence of negative clauses \( \pi = N_0, N_1, \ldots, N_m \) such that:

1. \( N_0 \in \mathcal{F} \), is an initial goal clause.
2. Each \( N_i \) is a resolvent of \( N_{i-1} \) and a definite input clause \( C_i \), resolved over the head \( x \) of \( C_i \). The variable \( x \) (i.e., the literal \( \neg x \)) of \( N_{i-1} \) is the selected variable in the body of \( N_{i-1} \).
   (Observe that \( x \) is an existential literal by definition). Resolution is only allowed on existential variables.
3. Each \( N_i \) is derived from \( N_{i-1} \) via a \( \forall \)-Red step. That is, by deleting a universal variable \( u \) from \( N_{i-1} \) such that \( u \) is not blocked in \( N_{i-1} \).
4. \( N_m = C \).

If \( N_m = \square \) then \( \pi \) is an SLD-Q-Res refutation of the quantified Horn formula \( F \). We denote by \( \mathcal{F} \vdash_{\text{SLD-Q-Res}} \square \) the fact that \( \mathcal{F} \) has an SLD-Q-Res refutation.

**Theorem 14.** SLD-Q-Res is sound and complete for false quantified Horn formulas.

**Proof.** As Q-Res is sound and can simulate SLD-Q-Res, we only need to prove completeness. Let \( F \) be a false quantified Horn formula. Then, by Observation \ref{observation:qbf} \( \exists C' \in F_{\text{goal}} \cup F_\emptyset \) such that \( Q_1 X_1 Q_2 X_2 \ldots Q_n X_n (F_3 \land C') \) is false. So we have two cases:

**Case 1:** When \( C' \in F_{\text{goal}} \): If \( C' \) has only negated universal literals, just apply the \( \forall \)-Red rules and derive the empty clause. Clearly, this is an SLD-Q-Res refutation. Otherwise, let \( C' \) have the following existential negated literals (with some negated universal literals as well): \( \neg x_1, \neg x_2, \ldots, \neg x_k \).

Let \( C_i \in F_3 \) be the clause with head \( x_i \). By Observation \ref{observation:qbf} we must have a \( C_i \in F_3 \) for every \( x_i \).

We start constructing an SLD-Q-Res refutation with the top clause \( C' \) as follow: resolve \( C' \) and the clause \( C_1 \) with pivot \( x_1 \). Let the resolvent be \( C'_1 \). Resolve \( C'_1 \) with the clause \( C_2 \) with pivot \( x_2 \), and so on. Call the resolvent after \( k \) step as \( C'_k \).
We claim that \( C'_i \) is either an empty clause, or consists of only negated universal literal. If this claim is true, we are done. We have an SLD-Q-Res refutation of \( F \).

Suppose not. Then, \( C'_i \) has an existential variable say \( q \). Clearly, \( q \) is a negated literal. Because, each \( C'_i \) is in fact a negated clause only. Observe that \( \neg q \notin C' \), otherwise, it would have been resolved via an input clause containing \( q \) as head. So, it must be the case, that \( \neg q \) belongs to some \( C_{i-1} \in F_3 \) and introduced in the resolvent \( C'_i \).

Assuming this, we now give a winning strategy for the existential player for the QBF \( F \): assign 1 to all the heads of the clauses from \( F_3 \), except the clause \( C_{i-1} \). In \( C_{i-1} \) assign 0 to the head \( x_{i-1} \) and a 0 to the negated literal \( q \). Clearly, the assignment satisfies all clauses from \( F_3 \). Also, it satisfies the clause \( C' \): as \( \neg x_{i-1} \in C' \) and \( x_{i-1} = 0 \). Note that \( \neg x_{i-1} \) surely is present in \( C' \), that is why we resolved \( C'_i \) with \( C_{i-1} \). This proves our claim.

**Note:** as \( F \) is a QBF Horn formula, we do not have any universal variables \( u \) and \( \neg u \) together in any resolvent \( C'_i \).

**Case 2:** When \( C' \in F'_4 \): This case is similar to the Case 1. Again, begin with clause \( C' \) and start resolving the negated existential literals of \( C' \) with the corresponding definite clauses. This is possible by Observation 3. At the end, we are left with the empty clause or with only negated universal literals.

### 4 Extending Prolog to QBFs (QBF-Prolog)

In this Section, we develop theory for the QBF-Prolog. We follow the ideas from [22,27]. Given facts and rules as a quantified Horn formulas \( F = \Pi \phi \), and a query Horn clause \( C = x \leftarrow x_1, x_2, \ldots, x_n \). We need to answer whether

\[
\Pi \phi \models \Pi \phi \wedge \{C\}
\]

We note that \( C \) may have new variables as well, and there are no restrictions on how these variables are quantified and where they are put within the prefix.

Let us first develop theory for the case when the query clause \( C \) has no new variables. We show that the other case is trivial. That is, we show that the case in which the query clause contains new literals have a straightforward answer: if the new literal in the query clause is existential, then the answer is yes. If the new literal \( u \) is universal, then one may drop \( u \) from the query clause without effecting the answer.

#### 4.1 Theory for QBF-Prolog when the query Horn clause has no new variables

Let \( F = \Pi \phi \) be a quantified Boolean Horn formula. Let \( C \) be a Horn clause, such that \( \text{var}(C) \subseteq \text{var}(F) \). We need to answer whether

\[
\Pi \phi \models \Pi \phi \wedge \{C\}
\]

We use the ideas from [27].

**Abstractions:** Given a QBF \( F = \Pi \phi \) with prefix \( \Pi = Q_1X_1 \cdots Q_iX_iQ_{i+1}X_{i+1} \cdots Q_nX_n \), and an \( i \) with \( 0 \leq i \leq n \). An abstraction QBF is defined as \( F_i = \Pi_i \phi \) where

\[
\Pi_i = \exists (X_1 \cup \cdots \cup X_i) Q_{i+1}X_{i+1} \cdots Q_nX_n
\]

Note that \( F_0 \equiv F \) and \( F_n \) is just the CNF formula \( \phi \) over all existential variables \( X_1 \cup \cdots \cup X_n \). We state the following Lemma from [27] regarding abstractions without proof:

**Lemma 15 (27).** Let \( F = \Pi \phi \) and \( F' = \Pi \phi' \) be two QBFs with same prefix. Then for all \( i \), if \( \Pi_i \phi \equiv \Pi_i \phi' \) then \( F \equiv F' \).

Let \( i = \max_{\ell \in C}\{\text{level}(\Pi, \ell)\} \), that is, \( i \) be the maximum level of literals in the query Horn clause \( C \). For a clause \( C \) (say, \( (\ell_1 \vee \cdots \vee \ell_k) \)), we define \( \overline{C} \) as the complement of \( C \), that is, conjunction of all the negated literals of \( C \) (that is, \( (\neg \ell_1 \wedge \cdots \wedge \neg \ell_k) \)). We have the following:

**Lemma 16 (27).** Let \( F = \Pi \phi \) be a Horn QBF, \( C \) be a Horn clause with \( \text{var}(C) \subseteq \text{var}(F) \) and \( i = \max_{\ell \in C}\{\text{level}(\Pi, \ell)\} \). If \( \Pi_i (\phi \wedge \overline{C}) \models_{\Pi_i \phi \wedge C} \square \) then \( \Pi \phi \equiv \Pi_i(\phi \wedge C) \)
Note that in \[27\], they proved the result via quantified unit propagation \[27\] Definition 9]. However, here we show that the result is valid for SLD-Q-Res as well.

Lemma 17 \(27\). Let \(\Pi,\phi\) be a Horn QBF, with \(\Pi = Q_1X_1 \ldots Q_nX_n\) and \(C\) be a Horn query clause with \(\text{var}(C) \subseteq X_1\), and \(i = \max_{\ell \in C}\{\text{level}(\Pi, \ell)\}\). If \(\Pi_1.(\phi \land C)\) \(\not\vdash_{\text{SLD-Q-Res}}\) then \(\Pi_1.\phi \equiv \Pi_1.(\phi \land C)\).

As \(\Pi_1.\phi\) is equal to \(\Pi.\phi\). We have that if \(\Pi.(\phi \land C)\) \(\not\vdash_{\text{SLD-Q-Res}}\) then \(\Pi.\phi \equiv \Pi.(\phi \land C)\).

Proof. Suppose not, and there exists a \(T\) (assignment tree) with \(T \models \Pi.\phi\) but \(T \not\models \Pi.(\phi \land C)\). It follows that there exists a path \(\tau\) in \(T\) with \(\tau(C) = \bot\). As \(\Pi_1.(\phi \land C)\) \(\not\vdash_{\text{SLD-Q-Res}}\), the Horn QBF \(\Pi.(\phi \land C)\) is unsatisfiable, and so \(T \not\models \Pi.(\phi \land C)\). Since \(\tau(C) = \bot\), we have \(\tau(C) = \top\) and hence \(T \models \Pi.(\phi \land C)\), a contradiction.

Lemma 18 \(27\). Let \(\Pi,\phi\) be a Horn QBF, \(C\) be a Horn query clause and \(i = \max_{\ell \in C}\{\text{level}(\Pi, \ell)\}\). If \(\Pi_i.(\phi \land C)\) \(\not\vdash_{\text{SLD-Q-Res}}\) then \(\Pi_i.\phi \equiv \Pi_i.(\phi \land C)\).

Proof. Since all variables from \(C\) are existentially quantified in the abstraction \(\Pi_i.(\phi \land C)\), they all belong to the first quantifier block and the Lemma follows from Lemma 17.

Using the ideas from \[22\] and Lemma 16, we next develop an efficient implementation model. We start by presenting an exponential time algorithm for the same.

### 4.2 Exponential time algorithm for QBF-Prolog when the query clause has no new variables

In this Section we develop an efficient implementation for the QBF-Prolog when the query clause has no new variables. In particular, we need to develop an efficient implementation details for the Lemma 16. However, Prolog does not work with clauses, but with multi-clauses as a list. Therefore, we need to do more.

As in the propositional case, QBF-Prolog treats clauses as multisets, let us call this proof system SLD-Q_{multi}-resolution. That is, SLD-Q_{multi}-resolution proof system is just an SLD-Q-Res proof system which retains every multiple copies of any literals within a resolvent clause. For example: \((x \leftarrow y), (y \leftarrow x, z)\) \(\vdash_{\text{SLD-Q-Res}}\) \(x, x, z\), and not \(x, z\). Also, QBF-Prolog treats clauses as list of literals. Therefore, the order of literals in any clause matters for QBF-Prolog. To make it clear, let us define the proof system precisely for QBF-Prolog.

**Definition 19 (SLD-Q_{Prolog}-resolution)**. \[22\] Chapter 5 An SLD-Q_{Prolog}-resolution derivation of a quantified Horn formula is an SLD_{multi}-resolution derivation in which clauses are regarded and processed as lists. In every resolution step, the pivot variable \(x\) which is resolved upon must occur as a negative literal at the start of the list for the centre clause \(\neg x, \neg y_1, \neg y_2, \ldots, \neg y_r\) and as a positive literal at the start of the list for the side clause \(x, \neg s_1, \neg s_2, \ldots, \neg s_p\). The resolvent list is formed by concatenating the remainder of the centre clause to the remainder of the side clause \(\neg s_1, \neg s_2, \ldots, \neg s_p, \neg y_1, \neg y_2, \ldots, \neg y_r\). There are no additional constraints for the \(\forall\)-Red rules.

In QBF-Prolog, not only the order of literals within a clause matters, but also the order of clauses within a quantified Horn formula matters. A QBF-Prolog program \(P\) is a quantified Horn formula in which both the clauses and the literals within the clauses are given a fixed ordering. Positive literals always appear in the front of the lists for any definite clause.

Now, let us restate the problems mentioned at the beginning of Section 4.1, with the above discussed modifications.

Instead of a quantified definite Horn formula \(F = \Pi.\phi\), we have a QBF-Prolog program \(P = \Pi.\phi\) in which both the clauses and the literals within a clause have fixed ordering. In addition, clauses in \(\phi\) are given as a list, with the positive literal in the front. We also have given the query definite Horn clause \(C = (x \lor \neg x_1 \lor \neg x_2 \lor \cdots \lor \neg x_n)\) as a list of literals, with \(\text{var}(C) \subseteq \text{var}(P)\). We need to answer whether \(P = \Pi.\phi\) \(\equiv \Pi.(\phi \land C)\)?

Let \(i = \max_{\ell \in C}\{\text{level}(\Pi, \ell)\}\). Lemma 16 says that our Prolog Algorithm should return a \textit{yes}, for the above problem, if we can search an SLD-Q-Res refutation of the following QBF-Prolog program:

\[
P' = \Pi_i.(\neg x \land x_1 \land x_2 \cdots \land x_n \land \phi)
\]
Note that all the literals of \( C \) becomes existential literals in the above QBF-Prolog program. To be precise, we need to search an SLD-Q_{Prolog} resolution refutation. We first present an exponential time recursive algorithm (Algorithm 1) to search a required SLD-Q_{Prolog} resolution refutation. The Algorithm search a refutation with \( \neg \alpha \) as the top clause. That is, it start searching the refutation with negation of the head of the query clause \( C \). We call the Algorithm 1 with the following input parameters: the QBF-Prolog program \( P' \), and \( \neg \alpha \) (\( \neg \alpha \) will copy in the list \( \gamma \)).

**QBF-Prolog-Recursive-Algo**(*quantified cnf-formula *P, list of quantified cnf-variables * \( \gamma \))

**Input:** QBF-Prolog program \( P \) as a list of definite quantified Horn clauses, and a goal list \( \gamma \) as a list of quantified (negated) literals. For our problem, \( \gamma \) is the negation of the head of the query definite clause \( C \) and \( P \) is equalt to \( P' \) (Refer Equation 1).

**Output:** true; if the literals of the goal list are derivable from \( P \) using the prolog procedure, false (or possibly a loop); otherwise

```plaintext
if is-empty(\( \gamma \)) then
    return true
else
    foreach clause \( C \in P \) do
        if first-pos-lit(\( C \)) == first(\( \gamma \)) then
            /* first-pos-lit(\( C \)) returns the only positive literal of \( C \), similarly, first(\( \gamma \)) returns the first existential literal of \( \gamma \). If they are same, call recursively the main program with \( P \) as the first argument and the goal list \( \gamma' \) as the second argument. \( \gamma' \) is obtained as follows: Let rest(\( C \)) outputs exactly the same list except the first positive existential literal. Similarly, we have rest(\( \gamma \)) returns the same list except the first existential literal of \( \gamma \). And \( \gamma' = \text{append(rest(}C\), rest(\( \gamma \))) \), where append concatenates the rest of \( \gamma \) list to the rest of \( C \) and then removes all the universe literals which are not blocked in the concatenated sequence. */
            if QBF-Prolog-Recursive-Algo(\( P, \gamma' \)) then
                /* When the recursive call return sucess, we return success */
                return true
            end
            /* For \( \gamma \), we considered each clause from \( P \) in sequence, but unable to proccess \( \gamma \) completely, so return a failure */
            return false
        end
    end
end
```

**Algorithm 1:** Recursive algorithm for Prolog search mechanism for QBFs inspired from [22].

QBF-Prolog-Recursive-Algo, uses SLD-Q_{Prolog}-resolution with a combination of depth first search from left to right with backtracking. Observe that Algorithm 1 may enters an infinite loop, even for the propositional Prolog program.

Infact, due to the deterministic approach, the completeness of the SLD-Q_{Prolog}-resolution has lost.

The above QBF-Prolog mechanism can be well explained with the concept of a refutation tree [22]. Let \( P \) be a QBF-Prolog program and \( \gamma \) be the goal query. The refutation tree \( T_P(\gamma) \) is a tree such that its root is labelled with the query clause \( \gamma \). Every node labelled with a goal clause \( \leftarrow x_1, x_2, \ldots, x_n \) has a successor corresponding to every definite Horn clause \( x_1 \leftarrow y_1, \ldots, y_k \in P \). The successors are labelled with the SLD-Q_{Prolog}-resolution resolvent \( \leftarrow y_1 \ldots y_k, x_2 \ldots x_n \) in this order.

For example, Figure 1 shows a refutation tree for the QBF-Prolog program

\[
P = \exists a, b, c, d, e \forall f \exists g. (a \leftarrow e, c, g) \land (a \leftarrow d, b) \land (d \leftarrow b, f) \land (e \leftarrow f) \land (b) \land (g)
\]

and a query clause \( \leftarrow a \).

While searching for a refutation, given a goal query \( \gamma \) and \( P \), three outcomes are possible:
Empty clause $\square$ is found. We say that the algorithm found the refutation. The result is a yes.

- The entire tree is searched without finding a node with $\square$. The result is a no, which means that the derivation of the empty clause is not possible.
- During the search algorithm follow an infinite branch in the refutation tree. The result is a loop.

For a QBF-Prolog program $P$ and a Horn query clause $C$, let us define the following function:

$$\text{output}(P, C) = \begin{cases} 
    \text{yes} & \text{if the algorithm stops with a success} \\
    \text{no} & \text{if the algorithm stops with a failure} \\
    \text{loop} & \text{if the algorithm enters a loop}
\end{cases}$$

We next give a linear time algorithm for computing the function $\text{output}(P, C)$.

### 4.3 Linear time algorithm for computing the output function

Given a QBF-Prolog program $P$ and a query Horn clause $C$, we describe an algorithm which computes the $\text{output}(P, C)$ function in linear time, in the length of the inputs. The challenge is to detect whether the algorithm enters into loop in linear time.

As in the propositional case, we develop QBF-Prolog which is not only interested in finding a refutation, but all possible refutations. Following the ideas from [22], our proposed algorithm maintains five distinct states for each existential variable of the QBF Prolog program. During the course of processing, each existential variable $x$ is in one of the following states:

- $\text{state}(x) = \text{new}$: the goal $\leftarrow x$ (i.e., $\neg x$) has not yet processed.
- $\text{state}(x) = \text{yes}$: the goal $\leftarrow x$ can be refuted in finite number of ways and the refutation tree $T_P(\leftarrow x)$ does not contain an infinite branch.
- $\text{state}(x) = \text{no}$: refutation of the goal $\leftarrow x$ is not possible.
- $\text{state}(x) = \text{loop}$: algorithm follows an infinite branch in the refutation tree $T_P(\leftarrow x)$ while searching for the refutation. That is, algorithm goes into a loop.
- $\text{state}(x) = \text{inf}$: there exists infinitely many ways to refute the goal $\leftarrow x$. Or, after finitely many refutations, the algorithm goes into a loop (follows an infinite branch in the refutation tree).

Any universal variable $u$ of the QBF-Prolog program can be in any one of the following two states. These state depends on the clause in which variable $u$ belongs to.

- $\text{blocked}(u) = \text{yes}$: let $C$ be a clause and $u \in C$, and one cannot apply the $\forall$-red step to $u$ in the clause $C$. To be precise, there exists an existential literal $\ell \in C$ with $u \leq_H \ell$, and $(\text{state}(\ell) = \text{no} \text{ or } \text{state}(\ell) = \text{loop})$. 

---

**Fig. 1.** Refutation tree for QBF-Prolog program $P$ and query $\leftarrow a$
- $\text{blocked}(u) = no$: let $C$ be a clause and $u \in C$. There does not exist an existential literal $\ell$ with $u \leq_{\Pi} \ell$. That is, one can apply a $\forall$-red step to $u$ in the clause $C$.

Or, for any existential literal $\ell \in C$ with $u \leq_{\Pi} \ell$ we have $\text{state}(\ell) = yes$. That is, one can perform the $\forall$-red step on $u$ in future after refuting all the existential literals blocking $u$.

Before proceeding further, we present a simple example to clarify the definition of loop and inf.

Consider the following slightly modified QBF-Prolog program from [22]:

$$P = \exists a \forall d \exists b, c. (a \leftarrow d, b, c) \land (b \leftarrow b)$$

We have $\text{state}(b) = inf$: since we have a successful refutation of $\leftarrow b$ followed by an infinite branch (loop) in $T_P(\leftarrow b)$. Refer, Figure 2.

![Fig. 2. Refutation tree $T_P(\leftarrow a)$ showing that $\text{state}(b) = inf$](image)

However, we have $\text{state}(a) = loop$: since in the refutation tree $T_P(\leftarrow a)$, there exists no refutation for $\leftarrow d, c$ and also has an infinite branch. Refer, Figure 3.

![Fig. 3. Refutation tree $T_P(\leftarrow a)$ showing that $\text{state}(a) = loop$](image)

Observe that given the state of all existential literals of a clause, computing the states of a universal variable is simple. Denote it by $\text{comp-block}(u, C)$.

So, consider the problem of computing $\text{state}(x)$ for an existential literal $x$, given a QBF-Prolog program $P$. That is, we need to refute $\leftarrow x$. Clearly, for computing the SLD-QProlog-resolution refutation of $\leftarrow x$, one has to perform resolution of the clause $\leftarrow x$ with a clause having head $x$. There can be several clauses in $P$ with head $x$. For computing $\text{state}(x)$, we need to consider each of them in the order in which they appear in $P$. However, in order to make discussion simple, let us focus on just one clause $C$ with head $x$ and compute the intermediate result $\text{state}(x, C)$. We use this result for computing the final one. Let $C = x \leftarrow x_1, x_2, \ldots, x_n$ be a clause with head $x$. Note that some of the $x_i$’s may be universal, but $x$ is existential.

- $\text{state}(x, C) = yes$: if $n = 0$,
- $\text{state}(x, C) = inf$: if for all existential $x_i$, $1 \leq i \leq n$, $\text{state}(x_i) = yes$ and for all universal variables $x_i$, $1 \leq i \leq n$, $\text{blocked}(x_i) = no$.
- $\text{state}(x, C) = inf$: if for all existential $x_i$, $1 \leq i \leq n$, $\text{state}(x_i) \in \{yes, inf\}$, and for all universal $x_i$, $1 \leq i \leq n$, $\text{blocked}(x_i) = no$, and there exist an $x_j$, $1 \leq j \leq n$, $\text{state}(x_j) = inf$. **
\( state(x, C) = loop \): there exists an existential \( x_i, 1 \leq i \leq n, state(x_i) = loop \), and for all existential \( x_k, 1 \leq k < i, state(x_k) = \{yes, inf\} \), and for all universal \( x_k, 1 \leq k < i, blocked(x_k) = no \).

Or, there exists an existential \( x_i, 1 \leq i \leq n \), such that while refuting \( \leftarrow x_i \), the refutation of \( \leftarrow x \) is called recursively, and for all existential \( x_k, 1 \leq k < i, state(x_k) = \{yes, inf\} \), and for all universal \( x_k, 1 \leq k < i, blocked(x_k) = no \).

Or, there exists \( x_i, 2 \leq i \leq n, state(x_i) = no \), and there exists an existential \( x_k, 1 \leq k < i, state(x_k) = inf \), and for all existential \( x_k, 1 \leq k < i, state(x_k) = \{yes, inf\} \), and for all universal \( x_k, 1 \leq k < i, blocked(x_k) = no \).

\( state(x, C) = no \): there exists an existential \( x_i, 1 \leq i < n, state(x_i) = no \), and for all existential \( x_k, 1 \leq k < i, state(x_k) = yes \), and for all universal \( x_k, 1 \leq k < i, blocked(x_k) = no \).

Or, there exists a universal variable \( x_k, 1 \leq k \leq n, blocked(x_k) = yes \).

Using these intermediate results, we are ready to describe the procedure of refuting \( \leftarrow x \) with a QBF-Prolog program \( P \). That is, procedure to compute \( state(x) \), for any existential variable \( x \).

Let the definite Horn clauses \( C^1_x, C^2_x, \ldots, C^m_x \) be the clauses of \( P \) with head \( x \).

\( state(x) = yes \): there exists an \( i, 1 \leq i \leq m, state(x, C^i_x) = yes \), and for all \( k, 1 \leq k < i, state(x, C^k_x) = no \), and for all \( j, i \leq j \leq n, state(x, C^j_x) \notin \{inf, loop\} \).

\( state(x) = inf \): there exists an \( i, 1 \leq i \leq m, state(x, C^i_x) = inf \), and for all \( k, 1 \leq k < i, state(x, C^k_x) = no \)

Or there exists an \( i, 1 \leq i \leq m, state(x, C^i_x) = yes \), and for all \( k, 1 \leq k < i, state(x, C^k_x) = no \), and for all \( j, i < j \leq m, state(x, C^j_x) \in \{loop, inf\} \).

\( state(x) = loop \): there exists an \( i, 1 \leq i \leq m, state(x, C^i_x) = loop \), and for all \( k, 1 \leq k < i, state(x, C^k_x) = no \).

\( state(x) = no \): for all \( i, 1 \leq i \leq m, state(x, C^i_x) = no \).

Now, we are ready to present our linear time algorithm for computing the function \( output(P, C) \). Our algorithm is inspired and a slight modification of the Algorithm from [22, Algorithm 5.16].

**QBF-Prolog-linear: linear time algorithm**

Recall the problem: given a QBF-Prolog program \( P = \Pi.\phi \) and a query definite Horn clause \( C = \langle x \leftarrow x_1, \ldots, x_n \rangle \), with \( \text{var}(C) \subseteq \text{var}(P) \), answer the following:

\[ \text{Is } \Pi.\phi \models \Pi.(\phi \land C) ? \]

Following Lemma [16] we modify \( P \) as \( P' = \Pi_i. (\neg x \land x_1 \land \cdots \land x_n \land \phi) \) and want to find an SLD-Q\(_{\text{Prolog}}\)-resolution of \( \leftarrow x \) from \( P' \) efficiently.

In other words, we need to compute \( state(x) \). We design the following function for the same:

\( \text{Refutation}(x, \text{loc-h}(x)) \): the function computes \( state(x) \). It takes two parameters: variable \( x \) and \( \text{loc-h}(x) \): list of clauses from \( P' \) with head \( x \). It output one of the following: \( no, yes, loop, inf \).

We maintain these outputs as enum data structure: enum status = \( \{ \text{no}, \text{yes}, \text{loop}, \text{inf} \} \). (Thus, here enumeration variable ‘no’ represents 0). Finally we set the function \( output(P, C) \) as follows:

\[
\text{output}(P, C) = \begin{cases} 
\text{yes} & \text{if } \text{Refutation}(x, \text{loc-h}(x)) \in \{\text{yes}, \text{inf}\} \\
\text{Refutation}(x, \text{loc-h}(x)) & \text{otherwise}
\end{cases}
\]
Here is our efficient pseudocode:

```
status QBF-Prolog-linear(quantified cnf-formula *P, cnf-clause * C)
Input: QBF-Prolog program P = \Pi.\phi as a list of definite quantified Horn clauses, and
a definite query clause C as a list
Output: output(P,C): yes, if Refutation return \{yes, inf\}, otherwise return the value
of the Refutation function; that is, either loop or no. The return type is the
enumeration status
/* C = x ← x_1, x_2, . . . , x_n, let i represent the maximum level of the
literals of C */
P' = \Pi_i. (\neg x \land x_1 \land \cdots \land x_n \land \phi)
foreach existential variable x ∈ var(P') do
  state(x) = new
end
result = Refutation(x,loc-h-x)
/* Call the Refutation function [Algorithm 3], which computes
state(x) and store the output in the variable result. Refutation takes the variable x, and a list of clauses
from P' with head x as its parameters. */
if result ∈ \{yes, inf\} then
  return yes
else
  return (result)
/* result can be no or loop */
end
End Function
```

**Algorithm 2:** Linear time algorithm for QBF-Prolog inspired from [22].

We only need to present the linear time function Refutation. Before presenting the pseudocode
for the same, we mention some preliminary remarks:

Observe that, for computing \textit{state}(x) for an existential variable \(x\), we need to consider all
definite clauses \(C_x^i\) of the QBF-Prolog program \(P\) with head \(x\) in sequence. If \textit{state}(x, \(C_x^i\)) = \textit{no} for
all of them then we need to return a \textit{no}. Otherwise, if we found a \textit{loop} or an \textit{inf} at any moment,
then we must return immediately a \textit{loop} or an \textit{inf} respectively. Only when we encounter a \textit{yes}
for some definite Horn clause \(C_x^i\), rest of the clause of \(P'\) with head \(x\) need to be considered.

During the course of execution, state of variables are going to change. At the begining, we
initialize \textit{state}(x) = \textit{new} for all existential variables \(x\) (refer, Algorithm 2). When we try to find
a state of a variable \(x\), we replace it’s state from new to loop, as the first initialization. This will
make sure that the case of a recursive call for determining the state of \(x\) is handled correctly.
When a first refutation for \(\leftarrow x\) is found, we changed the state from loop to inf. This is the second
initialization in our Refutation function. Finally, we check that the correct answer is a \textit{yes} (finitely
many refutations) or an \textit{inf} (infinitely many refutations).

To make task easy, Refutation function uses another function **TestClause**. **TestClause:** the function takes a definite Horn clause from \(P'\), and finds the state of the positive literal of this clause based on the status of it’s negative literals. In other words, it helps the
Refutation function for computing \textit{state}(x) by computing \textit{state}(x, \(C_x^i\)), where \(C_x^i\) is a clause from
\(P'\) with head \(x\). TestClause uses Refutation function for the same.
We now present the Refutation function in Algorithm 3.

Algorithm 3: Linear time algorithm for QBF-Prolog inspired from [22].
Now we finish this Section by presenting the function TestClause:

\[
\text{status TestClause}(\text{CNF clause } *C)
\]

**Input:** Clause \( C = x \leftarrow x_1, x_2, \ldots, x_n \) with existential head

**Output:** \( \text{state}(\text{head}(C)) \) with respect to the tail of \( C \)

\[
\begin{align*}
&\text{/* literals of } C \text{ are arranged as per } II, \text{ Initialize as } n = 0 \quad */ \\
&\text{success} = \text{true}; \text{ infflag} = \text{false}; \text{ result} = \text{yes}; i = 1; \\
&\text{/* compute state of all universal literals of } C \quad */ \\
&\forall i, \text{ with } x_i \text{ universal, result-array}[i] = \text{com-block}(x_i, C') \text{, where } C' = C \setminus x \\
\end{align*}
\]

while (success) and (not is-empty(C)) do

\[
\begin{align*}
&\text{if } \text{sign(first}(C)) = \text{positive} \text{ then} \\
&\quad /* \text{first}(C) \text{ returns first literal of } C, \text{ i.e., ignore the head of } C \quad */ \\
&\quad C = C \setminus \text{first}(C) \\
&\text{else} \\
&\quad /* \text{consider next literal from the body of } C \quad */ \\
&\quad \text{if first}(C) = \forall \text{ then} \\
&\quad \quad /* \text{answer found, } x_i \text{ is blocked} \quad */ \\
&\quad \quad \text{result} = \text{no} \\
&\quad \quad \text{break} /* \text{ break the while loop} \quad */ \\
&\quad \quad i++ /* x_i \text{ not blocked} \quad */ \\
&\quad \text{else} \\
&\quad \quad \text{result-array}[i] = \text{Refutation(first}(C), \text{loc-h(first}(C))) \\
&\quad \quad \text{result} = \text{result-array}[i]; i++; \\
&\quad \quad \text{success} = (\text{result} == \text{yes}) \text{ or } (\text{result} == \text{inf}) \\
&\quad \quad /* \text{when result ==loop, success = false and while loop ends} \quad */ \\
&\quad \text{if result == inf then} \\
&\quad \quad \text{infflag} = \text{true} \\
&\quad \quad C = C \setminus \text{first}(C) \\
&\end{align*}
\]

end

switch result do

\[
\begin{align*}
\text{case } \text{result} \in \{\text{yes, inf}\} \text{ do} \\
&\text{if infflag then} \\
&\quad \text{return inf} \\
&\text{else} \\
&\quad \text{return yes} \\
&\end{align*}
\]

end

\[
\begin{align*}
\text{case } \text{result} == \text{loop} \text{ do} \\
&\text{return loop} \\
\end{align*}
\]

end

\[
\begin{align*}
\text{case } \text{return} == \text{no} \text{ do} \\
&\text{if infflag then} \\
&\quad \text{return loop} \\
&\text{else} \\
&\quad \text{return no} \\
&\end{align*}
\]

end

end

Algorithm 4: TestClause Function computing \( \text{state}(x, C_x) \) inspired from [22].

**Time Complexity Analysis:**
The time complexity of Refutation is linear in the length of the QBF-Prolog program \( P' \), as every clause of \( P' \) is processed at most once. Only we need to be careful in the function TestClause, as we need to compute the state of universal variables as well. For this, we just maintained and array ‘result-array’ and computed the required information at the very beginning. Thus, this increases the complexity additively. This proves the following.
Theorem 20. For a QBF-Prolog program $P$ and a query clause $C$, with $\text{var}(C) \subseteq \text{var}(P)$, it is possible to compute the function $\text{output}(P, C)$ in linear time in the length of $P$ and $C$.

5 Theory for QBF-Prolog with no restrictions on the query clause

In this Section, we develop theory for the QBF-Prolog without any restrictions on the query Horn clause. That is, the query clause may contains new variables as well. To be precise, given a QBF-Prolog program $P = \Pi.\phi$ and a query Horn clause $C$, is the following true:

$$P = \Pi.\phi \models \Pi'.(\phi \land C)?$$

where, $C$ may contains new variables which do not occur in $P$. Here, $\Pi'$ is obtained by extending $\Pi$ by the new variables of $C$. There is no restriction on how these new variables are placed in the prefix $\Pi$.

We show in this section, that this case is not interesting for the QBFs. That is, if the query clause $C$ contains a new existential variable, then $C$ will be implied by $P$ for sure, and if $C$ has a new universal variable $u$, one can drop $u$ and still preserves the answer.

We show this using the property $\text{QIOR}^+$ from [27]. We show that if the query clause contains a new existential variable, then $C$ will satisfy the $\text{QIOR}^+$ property for sure and hence $C$ will be redundant for the QBF-Prolog program $P$. We need the following definitions:

Definition 21 (Outer clause [17]). The outer clause of a clause $C$ on literal $\ell \in C$ with respect to the prefix $\Pi$ is the clause $\text{OC}(\Pi, C, \ell) = \{k \mid k \in C, k \leq_H \ell, k \neq \ell\}$.

Clearly, $\text{OC}(\Pi, C, \ell)$ of clause $C$ on literal $\ell \in C$ contains all literals of $C$, excluding $\ell$, which are smaller than or equal to $\ell$ in the variable ordering of prefix $\Pi$.

Definition 22 (Outer resolvent [17]). Let $C$ be a clause with $\ell \in C$ and $D$ a clause occurring in QBF $\Pi.\phi$ with $\neg \ell \in D$. The outer resolvent of $C$ with $D$ on literal $\ell$ with respect to the quantifier prefix $\Pi$, denoted $\text{OR}(\Pi, C, D, \ell)$ is the following:

$$\text{OR}(\Pi, C, D, \ell) = C \setminus \{\ell\} \cup \text{OC}(\Pi, D, \neg \ell)$$

The following Definition is the extension of the property Quantified Implied Outer Resolvent ($\text{QIOR}$) from [17].

Definition 23 ($\text{QIOR}^+$ [27]). A clause $C$ has property $\text{QIOR}^+$ with respect to QBF $\Pi.\phi$ on literal $\ell \in C$ iff

$$\Pi.\phi \equiv \Pi.(\phi \land \text{OR}(\Pi, C, D, \ell))$$

for each clause $D \in \phi$ with $\neg \ell \in D$.

Recall, that by, $\Pi.\phi \equiv \Pi.(\phi \land \text{OR}(\Pi, C, D, \ell))$, we mean that both QBFs are logical equivalent, that is, every model of one QBF is also the model of another. It has been proved in [27], that a clause with $\text{QIOR}^+$ property on some existential literal $\ell$ with respect to a QBF $\mathcal{F} = \Pi.\phi$ is redundant for the QBF $\mathcal{F}$. That is, we may add or remove the clause $C$ from $\mathcal{F}$ without effecting the satisfiability of $\mathcal{F}$. We below state the corresponding Theorem from [27].

Theorem 24 ([27]). Given a QBF $\mathcal{F} = \Pi.\phi$ and a clause $C$ with $\text{QIOR}^+$ on an existential literal $\ell \in C$ with respect to QBF $\mathcal{F}' = \Pi'.\phi'$, where $\phi' = \phi \setminus \{C\}$ and $\Pi'$ is same as $\Pi$ with variables and respective quantifiers removed that no longer appear in $\phi'$. Then $\mathcal{F} \equiv_{\text{sat}} \mathcal{F}'$.

Thus, Theorem [24] states that the clause $C$ with $\text{QIOR}^+$ property on some existential literal is redundant for the QBF, and can be added or removed. We also have universal elimination theorem from [27].

Theorem 25 ([27]). Given a QBF $\mathcal{F}_0 = \Pi.\phi$ and $\mathcal{F} = \Pi.(\phi \cup \{C\})$ where $C$ has $\text{QIOR}^+$ on a universal literal $\ell \in C$ with respect to the QBF $\mathcal{F}_0$. Let $\mathcal{F}' = \Pi.(\phi \cup \{C'\})$, with $C' = C \setminus \{\ell\}$. Then $\mathcal{F} \equiv_{\text{sat}} \mathcal{F}'$. 
Remarks: Although, QIOR$^+$ is extremely powerful in terms of redundancy detection, checking for the QIOR$^+$ property in practice is too costly. Since, even for the propositional case checking whether an outer resolvent is implied by a propositional formula is co-NP hard. So, one needs a redundancy property that can be checked in polynomial time. One such redundancy property is the QRAT [17]. Following QRAT, QORAT$^+$ [27] have been introduced, which is a polynomial time redundancy property for a clause, based on QUP (quantified unit propagation). It has been proved in [27], that QORAT$^+$ is more powerful than QRAT in terms of redundancy detection.

However, we do not need QORAT$^+$ to prove our results, QIOR$^+$ will suffice.

**Observation 4.** Let $P = \Pi.\phi$ be a QBF Prolog program. And, let $C$ be a definite Horn query clause. If there exists an existential literal say, $\ell \in C$ which is new. That is, $\ell \in \text{var}(C)$ but $\ell \notin \text{var}(\phi)$. Then, we know that $C$ for sure has QIOR$^+$ property with respect to the QBF Prolog program $P$: since there exists no clause $D$ in $\phi$ with $\neg\ell \in D$. Therefore, $C$ is redundant by Thereom 24. That is, the answer is yes for our problem.

**Observation 5.** Let $P = \Pi.\phi$ be a QBF Prolog program. And, let $C$ be a definite Horn query clause. If there exists a universal literal $\ell \in C$ which is a new literal, that is universal $\ell \in \text{var}(C)$ but $\ell \notin \text{var}(\phi)$. Then by Thereom 24 we can drop universal literal $\ell$ from $C$ safely and then ask the problem for the remaining clause.

**QBF-Prolog for First-order Predicate Horn Logic:** Recall the Definition of first-order predicate Horn logic from Section 2. In this logic, we may replace each literal in a Horn clause by arbitrary predicate of any bounded arity. The QBF-Prolog defined in the previous section is capable of solving the first-order Horn logic. That is, given facts, rules and query as first-order predicate Horn formulas, the QBF-Prolog computes the function output$(P,C)$ in linear time. Note that a predicate may contains arbitrary quantified variables, which is not allowed in the existing Prolog. The algorithm for first-order predicate Horn logic is exactly as the Algorithm 2. The only difference is that now the algorithm treats each predicate of the query Horn clause as a literal. For the resolution step of literals corresponding to the predicates, the algorithm may use unification to make them syntactically equal.

To be precise, let $C$ be a first-order query Horn clause. Every predicates $P(x_1, \cdots, x_k)$ appearing in clause $C$ is considered as a literal $\ell$. Also, the literal $\ell = P(x_1, \cdots, x_k)$ is considered as an existential literal (recall the definition of abstraction). We can resolve the literal $P(x_1, \cdots, x_k)$ with the same predicate $P(y_1, \cdots, y_k)$ in rule clause $C$ only if for each index $i$,

1. the quantifier of $y_i$ is $\forall$, or,
2. if quantifier of $y_i$ is $\exists$ then the quantifier of $x_i$ is $\exists$.

Recall that the variables occur in the same order as in the quantifier prefix. As the quantification of variables stays same in the resolvent, it can restricts the set of clause with which one can resolve a predicate $P$.

Note that for any literals $P(x)$ and $P(y)$ with $x \neq y$, separate states will be maintained by the algorithm.

**6 Applications of QBF-Prolog**

In this Section, we present some applications of the QBF-Prolog. Observe that the existing Prolog is a subset of QBF-Prolog. Hence, QBF-Prolog can handle problems that can be handled by the existing Prolog. Now we present some examples which can be handled by QBF-Prolog but not by the existing Prolog.

**6.1 Application 1: QBFs**

Recall the limitations of Prolog: it is not possible to define rules having an existentially quantified head, and having body with arbitrary quantifications.

Below example illustrates, that the QBF-Prolog has no such restrictions. Let us consider the following simple QBF-Prolog program [4]:

$$P_n = \Pi.\phi = \exists e_0 \forall u_1 \exists e_1 \cdots u_n \exists e_n.$$
For $i \in [n], D_i : (e_{i-1} \lor \neg u_i \lor \neg e_i) \land
D_{n+1} : (e_n)\\

Clearly, in this QBF-Prolog program, the head $(e_{i-1})$ of the rules are existential, which is not possible in Prolog. Also, the body of the Horn clauses have both existentially (i.e., $e_i$’s) and universally (i.e., $u_i$’s) quantified variables.

Consider a simple query clause $C = (e_0)$. Observe that $\text{var}(C) \subseteq \text{var}(P_n)$. As, the variable $e_0$ belongs to the first level in the quantifier prefix $\Pi$, we have the abstraction $\Pi_1.\phi$ is same as $\Pi.\phi$. Since, we have simple SLD-Q-Res refutation of $\Pi_1.(-e_0 \land \phi)$ (Figure 3), the QBP-Prolog is capable to show that the following: $\Pi.\phi \models (\neg e_0 \land \phi)$ (Figure 4), the QBP-Prolog is capable to show that the following:

To be precise, in order to solve the above problem, we invoke the Algorithm 2 with parameters $P_n$ and $(e_0)$. The Algorithm in turn calls the Algorithm 3 with parameters $e_0$ (precisely, $-e_0$) and $(e_0 \lor -u_1 \lor -e_1)$. Algorithm 3 in turn call the Algorithm 4, with the parameter $(e_0 \lor -u_1 \lor -e_1)$. The recursion will end when the TestCase($(e_n)$) function returns a yes, which eventually reaches to the first level of recursion and Algorithm 2 also returns a yes.

\[ \neg C \quad \neg e_0 \quad e_0 \lor \neg u_1 \lor \neg e_1 \quad D_1 \]
\[ \neg u_1 \lor \neg e_1 \quad e_1 \lor \neg u_2 \lor \neg e_2 \quad D_2 \]
\[ \neg u_1 \lor \neg u_2 \lor \neg e_2 \quad e_2 \lor \neg u_3 \lor \neg e_3 \quad D_3 \]
\[ \neg u_1 \lor \neg u_2 \lor \neg u_3 \lor \neg e_3 \]
\[ \neg u_1 \lor \cdots \lor \neg u_n \lor \neg e_n \quad e_n \quad D_{n+1} \]
\[ \neg u_1 \lor \cdots \lor \neg u_n \quad n \forall-\text{Red steps} \]

\[ \square \]

Fig. 4. An SLD-Q-Res refutation of $\Pi.(-e_0 \land \phi)$ \[4\] for Application 6.1

6.2 Application 2: Tree-bipartite-problem

Consider the problem tree-bipartite from Example 1 of Section 1.1. The problem has the following rules: undirected graphs without cycles are bipartite, and trees are acyclic. Such rules can be easily encoded in the Existing Prolog. Let us present an encoding in detail. The encoding uses the following predicates with the following interpretations:

\[ \text{Nocycle}(g) \] / * the predicate is 1 iff \( g \) has no cycles * /
\[ \text{Bipartite}(g) \] / * the predicate is 1 iff \( g \) is bipartite * /
\[ \text{Tree}(g) \] / * the predicate is 1 iff \( g \) is a tree * /

Now, using these predicates, we have the following QBF-Prolog program for the tree-bipartite problem.

\[ \forall g \text{.Nocycle}(g) \rightarrow \text{Bipartite}(g) \] / * rule 1 * /
\[ \forall g \text{.Tree}(g) \rightarrow \text{Nocycle}(g) \] / * rule 2 * /
Consider the following query which cannot be handled by the existing Prolog as they required universally quantified variables: are all trees bipartite? Below is the query clause for the same.

\[
\forall g. \text{Tree}(g) \rightarrow \text{Bipartite}(g) \quad / \ast \text{ query } / \ast
\]

Our algorithm solves the problem as follows: It first adds the negation of the query clause in the QBF-Prolog program. That is we have,

\[
P' = \Pi_i, \neg \text{Bipartite}(g) \land \text{Tree}(g) \land \phi
\]

Certainly, the predicates corresponding to the query clause becomes existential. The algorithm picks the \(\neg \text{Bipartite}(g)\) as the top clause and derives the empty clause using rule 1, and rule 2, as follows:

\[
\begin{array}{c}
\neg \text{Bipartite}(g) \\
\neg \text{Nocycle}(g) \lor \text{Bipartite}(g) \\
\neg \text{Nocycle}(g) \\
\neg \text{Tree}(g) \lor \text{Nocycle}(g)
\end{array}
\]

\[
\text{Tree}(g)
\]

6.3 Application 3: Bipartite problem

Consider the problem related to the simple bipartite graphs \(G = (V, E)\) with vertex partitions \(U\) and \(W\), from Section 6.1 (refer Example 2). Assume that \(G\) has no loops and multiple edges. Consider a QBF-Prolog program which specifies the class of bipartite graphs. This can be encoded in first-order predicate Horn logic with only universally quantified variables. Let us present below one of the encodings in detail. The encoding uses the following predicates with the following interpretations:

\(E(x,y)\) / * the predicate is 1 iff there exists and edge between vertices \(x\) and \(y\) / *

\(\text{first}(x)\) / * the predicate is 1 iff \(x\) belongs to the first part \(U\) of the bipartite graph / *

\(\text{second}(x)\) / * the predicate is 1 iff \(x\) belongs to the second part \(W\) of the bipartite graph / *

Now, using these predicates, we have the following QBF-Prolog program for the bipartite graphs.

\[
\forall x, y, [\text{first}(x) \land E(x,y) \rightarrow \text{second}(y)] \quad / \ast \text{ rule 1 } / \ast
\]

\[
\forall x, y, [E(x,y) \land \text{second}(x)] \rightarrow \text{first}(y) \quad / \ast \text{ rule 2 } / \ast
\]

\[
\forall x, \text{first}(x) \rightarrow \neg \text{second}(x) \quad / \ast \text{ rule 3 } / \ast
\]

\[
\forall x, E(x,y) \rightarrow E(y,x) \quad / \ast \text{ rule 4 } / \ast
\]

We quickly explain the above rules: Rule 1 says that if an edge \(\{x,y\}\) is present in the bipartite graph \(G\) and \(x \in U\) then we have \(y \in W\). Similarly, rule 2 can be explained. Rule 3 says that if a vertex belong to \(U\) then it does not belong to \(W\). Rule 4 says that the predicate \(E(x,y)\) is symmetric in nature.

Now, consider the following query regarding a bipartite graph \(G = (\{U,W\}, E)\) which uses universal variables and hence cannot be supported by the existing Prolog: starting from any vertex \(x \in U\), and after jumping two hops via edges in \(G\), do we again reach the set \(U\)? Below is the quantified query clause \(C\) for the same:

\[
\forall x, y, z, [\text{first}(x) \land E(x,y) \land E(y,z)] \rightarrow \text{first}(z) \quad / \ast \text{ query Horn clause for QBF-Prolog } / \ast
\]

Certainly, such queries are valid for the QBF-Prolog. The algorithm solves the problem as follows: It first adds the negation of the query clause \(C\) in the QBF-Prolog program. That is, we have

\[
P' = \Pi_i, \neg \text{first}(z) \land \text{first}(x) \land E(x,y) \land E(y,z) \land \phi
\]

Clearly, all the literal in \(P'\) corresponding to the predicate of \(C\) becomes existential. The algorithm picks \(\neg \text{first}(z)\) and resolves it with rule 2 after substituting \(y/z\) in the same. Substituting \(y/z\) in rule 2 we get: \(\neg E(x,z) \lor \neg \text{second}(x) \lor \text{first}(z)\). That is, we have

\[
\neg E(x,z) \lor \neg \text{second}(x) \lor \text{first}(z)
\]
\[
\neg \text{first}(z) \quad \neg E(x, z) \lor \neg \text{second}(x) \lor \text{first}(z) \\
\neg E(x, z/y) \lor \neg \text{second}(x) \\
\neg \text{second}(x)
\]

We derived \(\neg \text{second}(x)\) at this moment. We proceed as follows:

\[
\neg \text{second}(x/y) \quad \neg \text{first}(x) \lor \neg E(x, y) \lor \neg \text{second}(y) \\
\neg \text{first}(x) \lor \neg E(x, z) \\
\neg E(x, z/y) \lor \neg \text{second}(x) \\
\neg \text{second}(x)
\]

Thus, the algorithm returns true for this query.

### 6.4 Application 4: Simple relations

Consider the problem of Example 3 of the Section 1.1. The problem uses the following predicates:

\[
P(h, k) \quad \text{/* the predicate is 1 iff } k = 2h */
\]

\[
R(h, k) \quad \text{/* the predicate is 1 iff } h < k */
\]

We may give several interpretations to these predicates. For example, if \(k\) and \(h\) represent graphs, then the predicate \(R(k, h)\) is 1 iff the graph \(h\) is a subgraph of the graph \(k\). The predicate \(P(k, h)\) is 1 iff the graph \(k\) is a superset of the graph \(h\). That is, the graph \(k\) is constructed from \(h\) by say adding a vertex.

Let us now consider the following facts and rules:

\[
\forall h \exists k. P(h, k) \quad \text{/* fact 1 */}
\]

\[
\forall h, k. P(h, k) \rightarrow R(h, k) \quad \text{/* rule 1 */}
\]

\[
\forall h_1, h_2, h_3. [R(h_1, h_2) \land R(h_2, h_3)] \rightarrow R(h_1, h_3) \quad \text{/* rule 2 */}
\]

Now consider the following query:

\[
\forall h \exists g, k. [P(h, g) \land P(g, k)] \rightarrow R(h, k) \quad \text{/* query */}
\]

Clearly the query is correct according to our interpretation. Our algorithm solves the problem as follows: It first adds the negation of the query clause in the QBF-Prolog program. That is, we have,

\[
P' = \Pi_i \neg R(h, k) \land P(h, g) \land P(g, k) \land \phi
\]

Recall, all predicates of the query clause becomes existential. The algorithm picks \(\neg R(h, k)\) as the top clause and derive the empty clause as follows:

\[
\neg R(h, k) \quad \neg P(h, k) \lor R(h, k) \quad \text{/* rule 1 */}
\]

\[
\neg P(h, k) \quad \text{/* rule 1 */}
\]

\[
P(h, g/k) \quad \text{/* added clause from } P'\text{ */}
\]

### 6.5 Application 5

Consider the Prolog program mentioned in Example 4. The problem cannot be handled by existing Prolog, as it requires variables with both quantifiers. Also, the proof for the problem is based on induction, as a result, instead of solving the problem, the QBF-Prolog return a loop.

Let us now encode the problem: given trees with unique root and such that every node has only one parent node. The program uses the following predicates:

\[
E(x, y) \quad \text{/* the predicate is 1 iff there exists and edge between vertices } x \text{ and } y */
\]

\[
\text{Notroot}(x) \quad \text{/* the predicate is 1 iff } x \text{ is not the root */}
\]
Using the above predicates we now present a QBF-Prolog program specifying the trees.

\[
\begin{align*}
\forall x, y. & E(x, y) \rightarrow \neg E(y, x) \quad \text{/* rule 1: tree is directed */} \\
\forall x \exists y. & \text{Notroot}(x) \rightarrow E(y, x) \quad \text{/* rule 2: if x is not root then it has a parent y */} \\
\forall x \forall y \forall z. & E(y, x) \land \text{Notroot}(x) \land \neg (z = y) \rightarrow \neg E(z, x) \quad \text{/* rule 3: every non root node has unique parent */} \\
\forall x. & \text{Root}(x) \\
\forall x. & \text{Notroot}(x) \\
\forall x \exists y \forall z. & E(y, x) \land \text{Root}(x) \land \neg (z = y) \rightarrow \neg E(z, x) \quad \text{/* rule 4: if x is a root then x is not a Nonroot */} \\
\exists x \forall y \forall z. & E(y, x) \land \text{Root}(x) \land \neg (z = y) \rightarrow \neg E(z, x) \quad \text{/* rule 5: there is a unique root */} \\
\exists x. & \text{Root}(x) \\
\forall x \exists y \forall z. & E(y, x) \land \text{Root}(x) \land \neg (z = y) \rightarrow \neg E(z, x) \quad \text{/* rule 6: if x is a root then the indicator P(x) is true */} \\
\forall x, y. & P(x) \land E(y, x) \rightarrow P(y) \quad \text{/* rule 7: this rule is for connectivity */} \\
\end{align*}
\]

Now, consider the following query:

\[
\forall x. P(x) \quad \text{/* is tree connected? */}
\]

Observe that one may easily prove this query via induction. However, QBF-Prolog will detect a loop since it unables to mimic the inductive proofs. To be precise, the algorithm adds the negation of the predicate of the query clause in the QBF-Prolog program. That is, we have

\[
P' = \Pi_i. \neg P(x) \land \phi
\]

So we have

\[
\begin{align*}
\neg P(x) & \quad \neg P(x) \lor \neg E(x, y/x) \lor P(y/x) \quad \text{rule 7: substitute y by x} \\
\neg P(x) & \lor \neg E(x, y/x)
\end{align*}
\]

The algorithm again needs to refute \( \leftarrow P(x) \) hence outputs a loop.

7 Conclusion

The paper overcomes one of the major logical limitations of Prolog of not allowing arbitrary quantified variables in the rules, facts and queries. The paper achieves this by extending the SLD-Res proof systems to quantified Boolean Horn formulas, followed by proposing an efficient implementation for the following problem: given a quantified Boolean Horn formula \( \mathcal{F} = \Pi_i. \phi \) and a clause \( C \), is \( \Pi_2. \phi \Rightarrow \Pi_2. (\phi \land C) \)? The paper shows that the implementation can also handles the first-order predicate Horn logic.

We also saw that the proposed algorithms unables to solve problems with inductive proofs. The next step is to extend the algorithm for simple problems based on induction. For example, extending the algorithm to handle the tree problem of Application 6.5.

The satisfiability problem for the first-order predicate formulas is known as the Satisfiability Modulo Theories (SMT) problem. Designing efficient SMT solvers for formulas over arbitrary quantified variables is challenging. Our implementation behaves as an SMT-solver for the first-order predicate Horn formulas, over arbitrary quantified variables.

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