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To cite this version:
Giacomo Monari, Benoit Famaey, Arnaud Siebert. Modelling the Galactic disc: perturbed distribution functions in the presence of spiral arms. Monthly Notices of the Royal Astronomical Society, Oxford University Press (OUP): Policy P - Oxford Open Option A, 2016, 457 (3), pp.2569-2582. 10.1093/mnras/stw171. hal-03156672

HAL Id: hal-03156672
https://hal.archives-ouvertes.fr/hal-03156672
Submitted on 2 Mar 2021

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Modelling the Galactic disc: perturbed distribution functions in the presence of spiral arms

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Accepted 2016 January 18. Received 2016 January 11; in original form 2015 December 2

ABSTRACT

Starting from an axisymmetric equilibrium distribution function (DF) in action space, representing a Milky Way thin disc stellar population, we use the linearized Boltzmann equation to explicitly compute the response to a three-dimensional spiral potential in terms of the perturbed DF. This DF, valid away from the main resonances, allows us to investigate a snapshot of the velocity distribution at any given point in three-dimensional configuration space. Moreover, the first-order moments of the DF give rise to non-zero radial and vertical bulk flows – namely breathing modes – qualitatively similar to those recently observed in the extended solar neighbourhood. We show that these analytically predicted mean stellar motions are in agreement with the outcome of test-particle simulations. Moreover, we estimate for the first time the reduction factor for the vertical bulk motions of a stellar population compared to the case of a cold fluid. Such an explicit expression for the full perturbed DF of a thin disc stellar population in the presence of spiral arms will be helpful in order to dynamically interpret the detailed information on the Milky Way disc stellar kinematics that will be provided by upcoming large astrometric and spectroscopic surveys of the Galaxy.

Key words: Galaxy: disc – Galaxy: evolution – Galaxy: kinematics and dynamics – solar neighbourhood – Galaxy: structure – galaxies: spiral.

1 INTRODUCTION

The primary objective of the current and future large spectroscopic and astrometric surveys of the Milky Way, culminating with the Gaia mission (Prusti 2012), will be to provide a detailed dynamical model of the Galaxy, including all of its components, and giving us insight into its structure, its formation and its evolutionary history.

The top-down dynamical approach consists in producing ab initio simulations of Milky Way-like galaxies in a cosmological context. This approach can be useful to understand some general features of galaxy formation (e.g. Minchev et al. 2014). However, it is not flexible enough to produce an acceptable model for the wide range of extremely detailed data soon to be available for our own Galaxy. On the other side, the bottom-up approach for dynamical modelling consists in starting from the actual Galactic data, rather than from simulations, in order to construct a model of the Galaxy. To avoid the redundancy and computational waste of representing the orbits of every single particle in the model, one can use a phase-space distribution function (DF) to represent each population of constituent particles (typically, various stellar populations and dark matter; see e.g. Binney & Piffl 2015; Piffl, Penoyre & Binney 2015). The model-building generally starts from the assumptions of dynamical equilibrium and axisymmetry. These assumptions allow us to make use of Jeans’ theorem constraining the DF to depend only on three integrals of motion, which can typically be chosen to be the radial, azimuthal, and vertical action variables. However, one should remember, especially when modelling the stellar populations of the Galactic disc, that the Galaxy is obviously not axisymmetric, as it harbours a central bar as well as spiral arms. Such perturbations can obviously be treated through perturbation theory, whose foundations in the case of flat 2D discs have been laid down by Kalnajs (1971). For instance, following up on the work of Binney & Lacey (1988) who derived the orbit-averaged Fokker–Planck equation for a 2D stellar disc, recent investigations (e.g. Fouvry, Binney & Pichon 2015) have focused on the long-term secular evolution of such a flat disc by means of diffusion through action space at resonances, producing ridges in action space. Here, we are rather interested in the present-day perturbed DF in the action-angle space of the unperturbed Hamiltonian, in the presence of a 3D spiral arm perturber, which could be fitted to a snapshot of the Galaxy taken by current and upcoming large surveys. Our philosophy is thus closer to that of McMillan (2013), except that the shape of the perturbed DF will be computed directly from the linearized Boltzmann equation. Moreover, in this paper, we will first concentrate only on the response away from the main resonances, the extremely interesting effects expected at resonances, as well as the effect of resonance overlaps of multiple perturbers (e.g. Quillen 2003;
There are in principle an infinity of sets of isolating integrals of the motion to choose from.

On the other hand, if one of the configuration space variables of a dynamical system is absent from the Hamiltonian, then its conjugate momentum is itself an integral of the motion, as is evident from Hamilton’s equations. Conversely, if an integral of the motion has a canonically conjugate variable, the Hamiltonian does not depend on that variable. Hence by choosing three isolating integrals of the motion having canonically conjugate variables, the Hamiltonian can be written in its simplest form, purely as a function of the three integrals of the motion. This makes such a choice of integrals particularly appealing. Such integrals are called ‘action variables’ \( J \), and correspond to new generalized momenta. Their canonically conjugate variables are called the ‘angle variables’ \( \theta \), because they can be normalized such that the position in phase-space is \( 2\pi \)-periodic in them.

The equations of motion (Hamilton’s equations) are conveniently expressed as

\[
\theta = \frac{\partial H_0}{\partial J} = \omega(J), \quad J = -\frac{\partial H_0}{\partial \theta} = 0. \tag{2}
\]

For a star in an axisymmetric disc galaxy, for which the usual phase-space coordinates are the cylindrical coordinates \((R, \phi, z)\) and their associated velocities \((v_R, v_\phi, v_z)\) \(\equiv (R, R\dot{\phi}, z)\), \(J = (J_R, J_\phi, J_z)\) are the actions, \(\theta = (\theta_R, \theta_\phi, \theta_z)\) the angles, and \(H_0(J)\) is the Hamiltonian corresponding to the axisymmetric time-independent potential \(\Phi_0\). The motion is as simple as one can imagine, since the actions \(J\) are constant in time, and define orbital tori on which the angles just evolve linearly with time, i.e. \(\theta(t) = \omega t + \theta_0\), where \(\omega(J) \equiv \partial H_0/\partial J\) are the orbital fundamental frequencies.

One of the drawbacks is that we can write analytical relations between the action-angles \((J, \theta)\) and the usual phase-space coordinates \((x, v)\) only in some rare cases\(^1\) of potentials \(\Phi_0\). But the advantages are numerous. First of all, in an equilibrium configuration for the Galaxy, the phase of the stars, \(\theta\), is uniformly distributed (phase mixed) on orbital tori specified by \(J\) alone, and the phase-space density of stars \(f_0(J)d^3J\) is just the number of stars \(dN\) in a given infinitesimal action range divided by a factor \((2\pi)^3\). Secondly, the actions are adiabatically invariant for a slow secular evolution of the Galactic potential. And, finally, they are very natural coordinates for perturbation theory: the linearized collisionless Boltzmann equation takes a rather simple form with these variables (see Section 3).

For simplicity, we are going to work here in the epicyclic and adiabatic approximations (for various more rigorous ways of evaluating the actions, see e.g. McGill & Binney 1990; McMillan & Binney 2008; Binney & McMillan 2011; Binney 2012; Bovy & Rix 2013; Sanders & Binney 2014), assuming separable motion in the vertical and horizontal directions. The epicyclic approximation is roughly valid for the thin disc we want to model here, i.e. for not too eccentric orbits and close enough to the Galactic plane. It consists in locally approximating the radial and vertical motions of an orbit

\[\frac{df_0}{df} = 0. \tag{1}\]

\(^1\) Note that, for any choice of integrals, the third integral cannot, in general, be written analytically for a disc galaxy, apart when the vertical motion is considered separable from the horizontal one as assumed here, or if the potential is of Stäckel form (e.g. Famaey & Dejonghe 2003). Bienaymé, Robin & Famaey (2015) provide typical analytic approximations for the third integral in more realistic potentials, based on the Stäckel approximation, but the corresponding actions are not analytic either.
of angular momentum $L_c \equiv R v_R$ with harmonic motions, i.e. with an effective potential in the meridional plane of the form

$$\Phi_{0,\text{eff}} = \Phi_0 + \frac{L_c^2}{2R^2} \simeq E_c + \Phi_{0,R} + \Phi_{0,\theta},$$  

(3)

where $\Phi_{0,R} \equiv \kappa^2 (R - R_0)^2/2$, $\Phi_{0,\theta} \equiv v_L^2 \zeta_0^2/2$, and the radial and vertical epicyclic frequencies, $\kappa$ and $v_L$, are evaluated at $R_0$, the radius of a circular orbit of angular momentum $L_c$, whose energy is $E_c$. The techniques and results developed in this paper are nevertheless in principle generalizable to more precise and general estimates of the actions for a wider range of orbits, which will be the topic of further papers. Within the adiabatic and epicyclic approximations, the actions $(J_R, J_\theta, J_z)$ are approximated by the following explicit analytic form:

$$J_\theta = \frac{1}{2\pi} \int_0^{2\pi} d\phi L_z = L_z,$$

$$J_z \simeq \frac{1}{\pi} \int_{z_{max}}^{z_{min}} dz \sqrt{2[E_z - \Phi_{0,\theta}] - E_z \frac{v_L}{v_z}},$$

$$J_R \simeq \frac{1}{\pi} \int_{R_{max}}^{R_{min}} dR \sqrt{2[E_R - \Phi_{0,R}] - E_R \frac{v_R}{\kappa}}$$  

(4)

where $E_R = v_R^2/2 + \kappa^2 (R - R_0)^2/2$ is the radial epicyclic energy and $E_z = v_L^2/2 + \zeta_0^2 v_z^2/2$ is the vertical energy. The canonically conjugate angle variables can then also be expressed explicitly (Dehnen 1999; Binney & Tremaine 2008) as:

$$\theta_\theta \simeq \phi + \Delta \phi,$$

$$\theta_z \simeq \tan^{-1} \left( -\frac{v_L}{v_z} \right),$$

$$\theta_R \simeq \tan^{-1} \left( -\frac{v_R}{\kappa (R - R_0)} \right),$$  

(5)

where

$$\Delta \phi = -\frac{\gamma}{\kappa} \sqrt{\frac{2R}{\kappa}} \sin \theta_R - \frac{J_R}{\kappa} \frac{d \ln \kappa}{dJ_\theta} \sin (2\theta_R),$$

(6)

with

$$\gamma \equiv 2\Omega/\kappa,$$

(7)

and $\Omega$ the angular circular frequency evaluated at $R_0(J_\theta)$. Finally, the orbital frequencies are approximated by

$$\omega_\phi \simeq \Omega + (d\phi/dJ_\theta) J_R,$$

$$\omega_z \simeq v_z,$$

$$\omega_R \simeq \kappa.$$  

(8)

The possible choices of realistic DFs to represent the different components of the Galactic disc are again numerous (see e.g. Binney 2010; Binney et al. 2014). Here we will make the simplest assumption, i.e. that the axisymmetric thin disc is well represented by a Schwarzschild DF (Binney & Tremaine 2008), i.e.

$$f_0(J_R, J_\theta, J_z) = \frac{\gamma \Sigma_0 \exp(-R_0/h_0)}{4(2\pi)^{(2/3)} \Sigma_0^2 \delta \kappa \zeta_0} \exp \left(-\frac{J_R \kappa}{\Sigma_0^2} - \frac{J_z v_z}{\Sigma_0^2} \right),$$

(9)

where $\Sigma_0$, $\delta \kappa$, $\zeta_0$, $\kappa$, $v_z$, and $\gamma$ are all functions of $J_z$ through a chosen dependence on $R_0(J_\theta)$. Note, however, that most results in Section 3 will be fully independent of this particular choice for $f_0$.

2 Rigorously speaking $\theta_\phi$, the canonical conjugate of $J_\phi$, should also include a term dependent on the vertical motion $-J_z (d \ln v_z/dJ_\phi) \sin \theta_\phi$, which in typical thin disc situations is tiny, much smaller than the already small $-J_\phi (d \ln \kappa/dJ_\phi) \sin (2\theta_R)$, and which therefore we omit.

3 LINEARIZED COLLISIONLESS BOLTZMANN EQUATION

3.1 General solution

In this section, we will consider a small perturbation to the potential, denoted as $\epsilon \Phi_1$, where $\epsilon \ll 1$, $\Phi_1$ has the same order of magnitude as the axisymmetric background potential $\Phi_0$, and the total potential is $\Phi = \Phi_0 + \epsilon \Phi_1$. Instead of searching for new action-angle variables for the perturbed Hamiltonian $H_1 = H_0 + \epsilon \Phi_1$, we will continue here to work with the variables defined within the unperturbed Hamiltonian $H_0$. These are obviously no longer action-angle variables within $H_1$, but they remain canonical. The following calculations in this section are fully independent from the specific action-angle estimate and choice of DF mentioned at the end of Section 2. We will move to specific predictions involving our specific choice of variables only in Section 3.2.

With such a perturbation, the DF becomes, to first order in $\epsilon$, $f = f_0 + \epsilon f_1$, which should still be a solution of the collisionless Boltzmann equation, equation (1). To first order in $\epsilon$ (i.e. dropping higher-order terms), this leads to the linearized collisionless Boltzmann equation, which reads (equations 5.13 and 5.14 of Binney & Tremaine 2008):

$$\frac{df_1}{dt} + [f_0, \Phi_1] = 0,$$  

(10)

where the time-derivative of $f_1$ is a total derivative and $[f_0, \Phi_1]$ is the Poisson bracket estimated along the unperturbed orbits. It thus appears immediately that for a given axisymmetric equilibrium DF, $f_0$, and a given perturbing potential, $\Phi_1$, the response $f_1$ can be computed.

Integrating equation (10) within angle-action coordinates leads to

$$f_1(J, \theta, t) = \int_{-\infty}^{t} dt' \frac{\partial f_0}{\partial J'} (J', \theta', t') \frac{\partial \Phi_1}{\partial \theta'} (J', \theta', t'),$$

(11)

where the coordinates $(J', \theta')$ correspond to the orbits in the unperturbed potential. Note that the perturbing potential $\Phi_1$ is assumed to have an explicit dependence on time.

Since the angle variables are defined such that the position in phase-space is $2\pi$-periodic in them, we consider only cases where $\Phi_1$ is cyclic in the angle coordinates, i.e.

$$\Phi_1 |_{\theta_i} = \Phi_1 |_{\theta_i + 2\pi},$$

(12)

where $\theta_i$ is any angle of the angle coordinates and the vertical line means that $\Phi_1$ is evaluated keeping constant all the other variables. Then, $\Phi_1$ can be expanded in a Fourier series as

$$\Phi_1(J, \theta, t) = \Re \left\{ G(t) \sum_n c_n(J) e^{in\theta} \right\},$$

(13)

where $G(t)$ controls the strength of the perturbation as a function of time. It is convenient to factorize this function into two factors, $G(t) = g(t) h(t)$, where $g(t)$ is a well behaved function controlling the general amplitude of the perturbation, and $h(t)$ is a periodic sinusoidal function of frequency $\omega_\phi$, which can account for a perturbing potential rotating with a fixed pattern speed. Here, $n$ is a triple of indexes running from $-\infty$ to $\infty$. Then equation (11) becomes

$$f_1(J, \theta, t) = \Re \left\{ \frac{i}{\omega_\phi} \frac{\partial f_0}{\partial J} (J) \sum_n n c_n(J) \right\} \times \int_{-\infty}^{t} dt' g(t') h(t') e^{i\omega_\phi(t-t')}.$$  

(14)
Integrating by parts, the solution of the integral in equation (14) is

$$
\int_{-\infty}^{t} dt' g(t') h(t') e^{i\omega t'} = \sum_{k=0}^{\infty} \left[ (-1)^k \frac{h(t') e^{i\omega t'} g^{(k)}(t')}{(in \cdot \omega + i\omega_0)^{k+1}} \right]_{-\infty}^{t}.
$$

We assume that the perturbation and its time derivatives are null far back in time, i.e. \( g^{(k)}(-\infty) = 0 \). Moreover, we assume in the following that the amplitude of the perturbation is constant at the present time \( t \), hence \( g^{(k)}(t) = 1 \), and \( g^{(k)}(t) = 0 \), for \( k = 1, \ldots, \infty \). This finally leads to

$$
f_i(t, \theta, t) = \Re \left\{ \frac{\partial f_0}{\partial J}(J) \cdot \sum_n n c_n(J) \frac{h(t) e^{i\omega \theta}}{n \cdot \omega + i\omega_0} \right\}.
$$

Within the above assumption of a currently non-varying amplitude of the perturbation, this solution is completely general and independent of any choice of action-angle coordinates and of any choice of a particular form of the axisymmetric equilibrium DF \( f_0 \). Note that Carlberg & Sellwood (1985) and Carlberg (1987) have on their side investigated the lasting changes in the DF after a transient spiral has come and gone. While similar in spirit to the present work, the goal was very different and needed to consider the second-order response, since to first-order, after the spiral has vanished, the DF goes back to its initial state through phase-mixing. Our approach is rather approximating what happens when the amplitude of the spiral wave reaches a plateau at its maximum.

### 3.2 Fourier modes perturbing potential within the epicyclic approximation

To be more specific, we now consider a perturbing potential of the form

$$
\Phi_1(R, \phi, z, t) = \Re \left\{ \Phi_1(R, z) e^{i\omega t - \Omega_0 t} \right\},
$$

i.e. a pure Fourier mode in \( \phi \), which is a good approximation for the potential of a given spiral arm mode, or the bar (at least away from the centre of the Galaxy). Note that we only consider hereafter plane-symmetric potentials \( \Phi_1(R, z) \), thereby not addressing perturbations such as corrugations. Here, \( \Omega_0 \) is simply the pattern speed, while \( m \) is the azimuthal wavenumber (e.g. \( m = 2 \) for the bar or a two-armed spiral, \( m = 4 \) for a four-armed spiral).

At this point, in order to specify the above solution \( f_i \) (equation 16) within that perturbing potential, we have to rewrite \( \Phi_1 \) as in equation (13). To do so, we approximate \( \Phi_1(R, z) \) close to the plane as

$$
\Phi_1(R, z) \approx \Phi_1(R, 0) + \frac{1}{2} \frac{\partial^2 \Phi_1(R, 0)}{\partial z^2} z^2,
$$

which is valid in the same range of \( z \) as the epicyclic approximation. So, \( \Phi_1 \) becomes

$$
\Phi_1 \approx \Phi_{1,R}(R, \phi) + \Phi_{1,z}(R, \phi, z),
$$

where

$$
\Phi_{1,R} \equiv \Re \left\{ \Phi_1(R, 0) e^{i\omega t - \Omega_0 t} \right\},
$$

$$
\Phi_{1,z} \equiv \Re \left\{ \frac{\partial^2 \Phi_1(R, 0)}{\partial z^2} \frac{z^2}{2} e^{i\omega t - \Omega_0 t} \right\}.
$$

We start with \( \Phi_{1,R} \). The radial motion in the epicyclic approximation is written as

$$
R = R_\Sigma (1 - e) \cos \theta_k,
$$

where

$$
e(J_k, J_\phi) \equiv \sqrt{2J_\phi / (\kappa R_\Sigma^2)}
$$

is the eccentricity of the orbit. We consider orbits with low \( e \), for which the epicyclic approximation holds. Using the definition of \( e \) and the mapping of equations (4) and (5), we can rewrite \( \Phi_{1,R} \) and expand it in powers of \( e \), dropping all the terms that are \( O(e^2) \), to obtain (e.g. Weinberg 1994)

$$
\Phi_{1,R} = \Re \left\{ \Phi_1(R, 0) e^{i\omega t - \Omega_0 t} \right\} \approx \Re \left\{ \left[ \delta_{i0} + \delta_{i1} \frac{\kappa}{2} \text{sgn}(j) \gamma e \right] \Phi_1(R, 0) \right\},
$$

where \( \gamma \) is defined as in equation (7). Note that the function \( h(t) \) in equation (13) is just \( h(t) = \exp(-i\Omega_0 t) \) in this case, and the frequency \( \omega_0 \) in equations (15)–(16) is thus \( \omega_0 = -m \Omega_0 \). We can now evaluate the Fourier coefficients for \( \Phi_{1,R} \) in the traditional way

$$
c_{jk}^R(J_k, J_\phi, J_z) = \frac{1}{(2\pi)^3} \int_{0}^{2\pi} d\theta_k \int_{0}^{2\pi} d\phi_0 \int_{0}^{2\pi} \int_{0}^{2\pi} \frac{d\theta_0}{\pi} e^{i(k \theta_0 + j \phi_0 + \delta_{i0})} \delta_{i0} \left\{ \delta_{i0} + \delta_{i1} \frac{\kappa}{2} \text{sgn}(j) \right\} \Phi_1(R, \phi, z),
$$

where \( \delta \) is the Kronecker delta. We can now also treat \( \Phi_{1,z} \) in the same way, replacing \( \Phi_1(R, 0) \) by \( \frac{\partial^2 \Phi_1(R, 0)}{\partial z^2} \). From equation (5), we note that \( \gamma^2 \) can be expressed as

$$
\gamma^2 = \frac{2J_\phi}{v} \cos^2 \theta_k = \frac{J_\phi}{v} \sum_{l=1}^{\infty} \frac{c_{l0}^2}{2l^2}.
$$

The Fourier coefficients for \( \Phi_{1,z} \) are then

$$
c_{jk}^z(J_k, J_\phi, J_z) = \frac{1}{(2\pi)^3} \int_{0}^{2\pi} d\theta_k \int_{0}^{2\pi} d\phi_0 \int_{0}^{2\pi} \int_{0}^{2\pi} \frac{d\theta_0}{\pi} e^{i(k \theta_0 + j \phi_0 + \delta_{i0})} \frac{\partial^2 \Phi_1(R, 0)}{\partial z^2} \frac{z^2}{2} e^{i\omega t - \Omega_0 t} \delta_{i0} \left\{ \delta_{i0} + \delta_{i1} \frac{\kappa}{2} \text{sgn}(j) \gamma e \right\} \Phi_1(R, \phi, z),
$$

where

$$
f_{1} = f_{1,R} + f_{1,z},
$$

$$
f_{1,R} \equiv \Re \left\{ \sum_{j=1}^{\infty} c_{j0}^R f_{j0} e^{i(j \theta_k + m(\phi_0 - \Omega_0 t))} \right\},
$$

$$
f_{1,z} \equiv \Re \left\{ \sum_{j,f=1}^{\infty} c_{j2}^z f_{j2} e^{i[j \theta_k + m(\phi_0 - \Omega_0 t) + 2 \phi_0]} \right\}.
$$
where the Fourier coefficients $c^{R}_{jkl}$ and $c^{z}_{jkl}$ are given by equations (24) and (26), and
\begin{equation}
F_{jkl}(J_R, J_\phi, J_z) \equiv \frac{j \delta \partial_{R} + k \delta \partial_{\phi} + l \delta \partial_{z}}{jk + k (\omega_0 - \Omega_\psi) + j l}.
\end{equation}

### 3.3 Moments of the distribution function

One of the main motivations of the present work is to understand the present response of a disc stellar population, represented by a DF $f_0$ in an axisymmetric potential, to a quasi-static perturbing non-axisymmetric potential in terms of radial and vertical mean motions (e.g. Faure, Siebert & Famaey 2014). Such mean motions can be computed through the zeroth- and first-order moments of the perturbed DF $f = f_0 + \epsilon f_1$. Here, we will assume a given form for $f_0$, namely the Schwarzschild DF of equation (9).

We will focus on the mean motions projected on the plane (for the radial motion) and on both sides of the plane (for the vertical motion). Indeed, it was already shown numerically (Faure, Siebert & Famaey 2014; Monari, Famaey & Siebert 2015) that spiral or bar perturbations typically lead to a *breathing mode* response of the disc, i.e. a density response that has even parity with respect to the Galactic plane (i.e. plane-symmetric), and a mean vertical velocity field that has odd parity. Hence, we will concentrate hereafter on the projected surface density $\Sigma(R, \phi)$, the projected mean radial velocity field $(v_R(R, \phi, \theta))$, and the difference between the mean vertical velocity field above and below the plane
\begin{equation}
\Delta(v_z)(R, \phi) \equiv \langle v_z(z > 0) \rangle - \langle v_z(z < 0) \rangle.
\end{equation}

These can be computed by integrating the perturbed DF over all $z$ (or half of them in the case of the vertical motion) and all velocities, i.e.
\begin{equation}
\Sigma(R, \phi) = \int_{-\infty}^{\infty} dz \int d^3v (f_0 + \epsilon f_1),
\end{equation}
\begin{equation}
\Sigma(R, \phi)(v_k)(R, \phi) = \int_{-\infty}^{\infty} dz \int d^3v v_k (f_0 + \epsilon f_1),
\end{equation}
\begin{equation}
\Sigma(R, \phi)\Delta(v_z)(R, \phi) = 4 \int_{0}^{\infty} dz \int d^3v v_z (f_0 + \epsilon f_1),
\end{equation}
where
\begin{equation}
d^3v = dv_\phi dv_\theta dv_z.
\end{equation}

Note that, by integrating over half of the $z$ for $\Delta(v_z)$, we get only half of the surface density, and have to multiply by two again to get the subtraction between the mean vertical velocities above and below the plane, hence the factor of 4. Now, using the parity of the functions, equation (32) simplifies to
\begin{equation}
\Sigma(R, \phi) = \int_{-\infty}^{\infty} dz \int d^3v (f_0 + \epsilon f_1),
\end{equation}
\begin{equation}
\Sigma(R, \phi)(v_k)(R, \phi) = \epsilon \int_{-\infty}^{\infty} dz \int d^3v v_k f_1,
\end{equation}
\begin{equation}
\Sigma(R, \phi)\Delta(v_z)(R, \phi) = 4\epsilon \int_{0}^{\infty} dz \int d^3v v_z f_{1,z}.
\end{equation}

These integrals have to be solved at constant $(R, \phi, t)$. To compute the integrals over all velocities, we pass from the integration coordinates $(v_R, v_\phi, v_z)$ (where $v_R$ and $v_z$ range from $-\infty$ to $\infty$, and $v_\phi$ from 0 to $\infty$) to $(\theta_R, \theta_z, J_R)$ (where $\theta_R$ and $\theta_z$ range from $-\pi/2$ to $\pi/2$, and $J_R$ from 0 to $\infty$) via the transformations
\begin{equation}
v_R = -\kappa (R - R_g) \tan \theta_R,
\end{equation}
\begin{equation}
v_\phi = J_R / R,
\end{equation}
\begin{equation}v_z = -v_\phi \tan \theta_z,
\end{equation}
and
\begin{equation}
J_R = \frac{(R - R_g)^2 \kappa}{2 \cos^2 \theta_R},
\end{equation}
\begin{equation}J_z = \frac{z^2 \kappa}{2 \cos^2 \theta_z},\end{equation}
\begin{equation}\theta_\phi = \phi + \Delta \phi(\theta_R).
\end{equation}
The Jacobian of the transformation is
\begin{equation}
dv_R dv_\phi dv_z = \kappa \nu (R - R_g) r \cos \theta_R \cos \theta_\phi.
\end{equation}

Using these transformations, as well as the approximations $\omega_0 \approx \Omega_\psi$, $\exp (i m \Delta \phi) \approx (1 + i m \Delta \phi)$, $\Delta \phi \approx -\gamma / R_g \sqrt{2} J_R / \kappa \sin \theta_R$ (i.e. up to the first order in $\epsilon$), we compute the integrals of equation (34), and the DF $f_0$ of equation (9), to obtain
\begin{equation}
\Sigma = \Sigma_0 + \epsilon \Sigma_1,
\end{equation}
where
\begin{equation}
\Sigma_0 = \frac{(2\pi)^{3/2}}{R} \int_{0}^{\infty} dJ_R \frac{\bar{\sigma}_R \bar{\sigma}_z^2}{v} \nu f_0(0, J_R, 0),
\end{equation}
\begin{equation}\Sigma_1 = \text{Re} \left\{ \frac{(2\pi)^{3/2}\kappa}{R} \int_{0}^{\infty} dJ_R \frac{\bar{\sigma}_R \bar{\sigma}_z^2}{v} \nu \left( \bar{\Sigma}_j + \bar{\Sigma}_{j1} \right) \right\}.
\end{equation}

For the mean radial velocity, we get
\begin{equation}
\Sigma(v_R) = \text{Re} \left\{ -i \frac{(2\pi)^{3/2}\kappa}{R} \int_{0}^{\infty} dJ_R \frac{\bar{\sigma}_R \bar{\sigma}_z^2}{v} \nu \left( \bar{\Sigma}_j + \bar{\Sigma}_{j1} \right) \right\} \left( \delta_{j0} m \gamma - \delta_{j1} (j + m \gamma \delta_R) \right) \right\}.
\end{equation}

Finally, for the difference of mean vertical velocities above and below the plane, we get
\begin{equation}
\Sigma \Delta(v_z) = \text{Re} \left\{ -i \frac{8\pi \kappa}{R} \int_{0}^{\infty} dJ_R \frac{\bar{\sigma}_R \bar{\sigma}_z^2}{v} \nu \left( \delta_k - j m \gamma \delta_k \right) \right\} \left( \delta_{k1} m \gamma - \delta_{k1} (j + m \gamma \delta_R) \right) \right\}.
\end{equation}

where
\begin{equation}\delta_R = \frac{R - R_g}{R_g},
\end{equation}
\begin{equation}\hat{\phi} \equiv m \left( \phi - \Omega_\psi t \right),
\end{equation}
\footnote{Actually, the explicit results of equations (38)-(40) are valid not only for the Schwarzschild DF of equation (9) but also for any DF that has a dependence on $J_R$ and $J_z$ of the form $\exp(-\frac{\kappa R}{2\sigma_z^2} - \frac{\nu R}{2\sigma_\phi}).$}
For the length and height parameters of this spiral potential, we choose \( R_0 = 1 \) kpc and \( h_s = 0.1 \) kpc. We also fix a phase \( \phi_s = -26^\circ \) and consider our following results at present time \( t = 0 \). The spiral is chosen to be tightly wound with \( p = -9.9^\circ \), and the amplitude parameter is chosen to be \( A = 683.7 \) km\(^2\) s\(^{-2}\). Finally, we choose to consider a two-armed spiral with \( \Omega_p = 18.9 \) km s\(^{-1}\) kpc\(^{-1}\), so that the main resonances are relatively away from the solar neighbourhood. The inner Lindblad resonance would be hidden in the central bar region of the Galaxy (ILR = 1.56 kpc) and the corotation is in the outer galaxy (CR = 11.49 kpc). These parameters have been partly inspired by the 2D spiral potential considered in Siebert et al. (2012), and produce at \((R, z) = (8 \) kpc, 0) a maximum radial force of the spiral that is 1 per cent of the force due to the axisymmetric background.

With this form of the background potential, axisymmetric equilibrium DF, and spiral potential, we can now compute the Fourier coefficients of equations (24) and (26), as well as the perturbed DF of equation (27) and its moments of equations (38)–(40).

4.2 Moments of the distribution function

4.2.1 Radial velocity gradient

We first consider the integrals in equations (38) and (39), which have to be computed numerically. In practice, for a given \( R \), the integral on \( J_2 \) is computed in the interval of angular momenta corresponding to circular orbits at the radii where the circular velocity is \( v_c \pm 2\delta R \) (we tested that the results obtained in this way are stable on larger integration ranges.). The moments are actually Fourier modes themselves, i.e. they have the form \( q(R, \phi) = \text{Re}(q_l(R) \exp(i\phi)) \) where \( \phi \) is defined as in equation (41b). We evaluate \( q_l(R) \) numerically on a grid of \( R \) values between 1 and 10 kpc with a step 0.25 kpc, and use a third-order polynomial interpolation on this grid to obtain the value \( q_l(R) \) at a generic \( R \) point.

In Fig. 1 we plot \( \Sigma/\Sigma_0 \) and \( \Sigma(v_R)/\Sigma_0 \) as obtained from equations (38)–(40). As we see, the maxima of the response density wake \( \Sigma/\Sigma_0 \) closely follow the loci of the spiral arm potential (dashed red curves), as one expects. On the other hand, stars on the arms tend to move towards the centre of the Galaxy \((v_R < 0)\), while those in the interarm regions tend to move outside \((v_R > 0)\).

In order to illustrate how our analytic calculations allow us to physically interpret the outcome of simulations, we compare the moments induced by the perturbation derived analytically with those computed with a numerical test-particle simulation. The initial conditions are drawn from \( f_0 \) and with the same potential \( \Phi_0 + \epsilon \Phi_1 \) (where \( \Phi_1 \) grows slowly with time, until it reaches the final amplitude used for the analytical predictions). The details of this simulation can be found in Monari et al. (2015), where the only difference with the present simulations is that, in that previous work, \( \Phi_1 \) was a bar potential instead of the spiral arms that we use here. The results of this simulation are depicted in Fig. 1. We find a very good agreement between the position of the maxima and minima of the moments and the loci of the spiral arms. Moreover, the amplitude of the perturbed density and motions appear to be similar to the analytical predictions. A closer look to this comparison with the simulation is presented in Fig. 2. Here the comparison is made at three different radii, in the neighbourhood of \( R_0 = 8 \) kpc: \( R = 7 \) kpc, \( R = 8 \) kpc, and \( R = 9 \) kpc. These plots confirm the agreement between the simulation and the analytical predictions. Some small discrepancies are of course present, and are due to a combination of different effects. One of them is the discrete nature of the simulations, and the fact that they never reach complete phase-mixing. The
second is that, although the area covered is away from the ILR and CR, there still are non-linear effects due to the resonances of higher order than the Lindblad resonances that our analytical method does not describe (e.g. due to the 4:1 inner ultra-harmonic resonance between $\Omega - \Omega_p$ and $\kappa$, which in our case falls at $R = 7.61$ kpc).

The third is the presence of very eccentric orbits, especially in the inner regions of the Galaxy, while equation (38) and equation (39) are valid only for moderate eccentricities.

All this is especially interesting in view of the large-scale radial velocity gradient first observed in the Galaxy by Siebert et al. (2011) with the RAdial Velocity Experiment (RAVE) survey. This was interpreted as the possible effect of either a $m = 2$ spiral (Siebert et al. 2012) or the Galactic bar (Monari et al. 2014). In this respect, it is interesting to note that the amplitude of the radial velocity fluctuations generated by our spiral potential here is of the same order of magnitude as those observed. It should, however, be noted that subsequently observed large-scale line-of-sight velocity fluctuations with a few red clump stars from the APOGEE survey seem to be more compatible with the effect of the bar (Bovy et al. 2015; Grand et al. 2015).

In Siebert et al. (2012), a comparison between the RAVE data and various spiral models was made by using the traditional reduction factor $F$ of Lin & Shu (1964, 1966), Lin et al. (1969) – see also Binney & Tremaine (2008). In the case of a cold, pressureless fluid it can indeed be shown that the linear response to a non-axisymmetric rotating density perturbation $\epsilon \Phi_1$ in the radial velocity on the Galactic plane is

$$
\epsilon u_{R,1}(R, \phi) = \epsilon \text{Re} \left\{ u_R^\delta(R) e^{i(\phi - \Omega_p t)} \right\},
$$

where

$$
u_R^\delta(R) = \frac{i}{\Delta(R)} \left\{ [\Omega_p - \Omega(R)] \frac{d\Phi_1}{dR}(R, 0) - \frac{2\Omega(R)\Phi_1(R, 0)}{R} \right\},
$$

Figure 1. Moments induced by the potential perturbation equation (43) on the Binney & Tremaine (2008) Model I potential. Top left: density wake $\Sigma_1/\Sigma_0$ obtained from equation (38). Top right: average radial speed $\langle v_R \rangle/\Sigma_0$ obtained from equation (39). Bottom left: density wake $\Sigma_1/\Sigma_0$ obtained from the simulation. Bottom right: average radial speed $\langle v_R \rangle$ obtained from the simulation. $\Sigma_0$ is computed in the simulation averaging $\Sigma$ over $\phi$ at a certain $R$. The dashed red curves represent the loci of the arms.
and \( \Delta(R) \equiv \kappa(R)^2 - m^2(\Omega_\phi - \Omega(R))^2 \). When the perturbing potential is a tightly wound spiral, the second term in the r.h.s. of equation (46) is much smaller than the first term, and can be omitted, so that equation (46) simplifies to

\[
u_R^t(R) \approx \frac{m}{\Delta(R)} \frac{\Omega_\phi - \Omega(R)}{\Delta(R)} \frac{\mathrm{d} \Phi_1}{\mathrm{d} R}(R, 0).
\] (47)

Lin & Shu (1964, 1966) and Lin et al. (1969) offer a way to rewrite equation (47) in the case of a stellar disc, i.e. by multiplying it by a reduction factor \( \mathcal{F} \) whose derivation is reported in appendix K of Binney & Tremaine (2008). In Fig. 3 we compare all these predictions with equation (39) of the present work and the outcome of our numerical simulations at \( R = R_0 \). Since \( \mathcal{F} \) was derived for tightly wound spirals only, we use the best fit tightly wound spiral potential with the same pitch angle \( p \) to \( \Phi_1 \) of this work in the range of \( R, 6 < R < 8 \) kpc (left panel), \( 7 < R < 9 \) kpc (central panel), and \( 8 < R < 10 \) kpc (right panel). We notice that there is a noticeable difference in the amplitude predicted by the Lin–Shu approximation, even with the reduction factor (a factor of \( \sim 2 \) or more), and the results obtained using equation (39) of the present work: the latter case actually describes much better the numerical simulation, calling for a re-investigation of non-axisymmetric kinematic features in future surveys with our present DF-based method rather than a simple reduction factor. There are several likely reasons for this difference. First of all, our approach is three-dimensional, and takes explicitly into account the vertical velocity dispersion of stars in the response to the perturbation. Secondly, we do not neglect the tangential force term which is usually neglected for tightly wound spirals. Thirdly, we use the guiding radius to evaluate our quantities instead of the present position which is used as a proxy in the Lin–Shu approach. Finally, the Lin–Shu approach assumes for the time-variation of the azimuthal angle that of a circular orbit, which is a good approximation only for very small eccentricities. It is a combination of these effects which leads to the present difference with the Lin–Shu reduction factor.

4.2.2 Vertical bulk motions: breathing mode of the disc

One of the immense advantages of working with a 3D spiral model is that it allows us to investigate the effect of the spiral on mean stellar vertical motions. This is especially interesting given that recent Milky Way large spectroscopic surveys have consistently indicated that the mean vertical motion of stars above and below the plane was typically non-zero (Widrow et al. 2012; Carlin et al. 2013; Williams et al. 2013). Such a behaviour was originally associated uniquely with external excitations of the disc by a passing satellite galaxy or a dark matter substructure (Widrow et al. 2012; Gómez et al. 2013; Yanny & Gardner 2013; Feldmann & Spolyar 2015; D’Onghia et al. 2015). It is, however, useful to separate such stellar bulk motions into two types of vertical oscillations. If the density perturbation has odd parity with respect to the Galactic plane, and the vertical velocity field has even parity, the disc itself is subject to a corrugation pattern which is called a ‘bending mode’. These are indeed mostly caused by external perturbers (de la Vega et al. 2015; Gómez et al. 2016; Xu et al. 2015). On the other hand, if the density wake has even parity while the vertical velocity field has odd parity (i.e. a rarefaction–compression pattern), the oscillation is called a ‘breathing mode’. Such breathing modes have been shown through test-particle simulations and approximate analytical considerations to be natural consequences of internal non-axisymmetries such as the bar and spiral arms (Faure et al. 2014; Monari et al. 2015). The same effect was also found in self-consistent simulations of isolated galaxies developing spiral instabilities (Debattista 2014). It was even shown that the breathing mode present in the simulation of a Milky Way like galaxy bombarded by satellites, which was analysed by Widrow et al. (2014), was actually most probably linked to the bar formation rather than induced by the satellites themselves (Monari et al. 2015).

Our present analytic calculations allow for the first time a rigorous and fully dynamical understanding of spiral-induced breathing modes away from the main resonances (and in the absence of resonance overlaps of multiple patterns, which will be the topic of further work). For this, it suffices to integrate equation (40) in a similar manner as equation (39). The resulting \( \Sigma \Delta(v_z) / \Sigma_0 \) is plotted in Fig. 4. As can be seen, stars tend to vertically move away from the Galactic plane at the outer edge of spiral arms \( \Delta(v_z) > 0 \) and towards the plane at the inner edge \( \Delta(v_z) < 0 \), with a clear phase-shift w.r.t. the mean radial motion, already noted in Faure et al. (2014). Again, we compare this to the results of our test-particle simulation (Figs 4 and 5) and find a good agreement.
4.3 Distribution function at a point in configuration space

Our computation of the exact form of the perturbed DF away from the main resonances also allows us to study the detailed behaviour of \( f = f_0 + e f_1 \) at a given point in configuration space \((R, \phi, z)\), in terms of the actions and angles, and compare it with the unperturbed version of the DF, \( f_0 \). First, let us note that the dimensions of phase-space, given the constraint of a fixed point in configuration space, \((R, \phi, z) = \text{constant}\), decrease from 6 to 3, even when the DF depends both on actions and angles. We focus on the case \((R, \phi, z) = (8 \text{ kpc}, 0, 0)\) (i.e. the typical position of the Sun in our model) and we add the constraint \( f = 0 \), additionally decreasing the dimensionality of phase-space to two dimensions.

The two variables that we choose to display are \((\theta_R, J_R)\). The other angles and actions are constrained by \( R = R(J_\phi) = \sqrt{2J_\psi/k(J_\psi)} \cos \theta_R, J_z = 0, \theta_\phi = \phi + \Delta \phi(J_\phi, J_z, \theta_R) \), and \( \theta_z = \pi/2 \) (because \( z = 0 \)). In practice we solve numerically the first of this constraints for each pair \((\theta_R, J_R)\) to get \( J_\phi \), and it is then trivial to get \( \theta_\phi \). In the case of the unperturbed DF, \( f_0 \), the true dependence is obviously on \( J_R \) and \( J_\phi \), but we can translate it in terms of \((\theta_R, J_R)\) in terms of the above constraints at a fixed point in configuration space.

The comparison between \( f_0(\theta_R, J_R) \) and \( f(\theta_R, J_R) \) is shown in Fig. 7. As is apparent from this figure, both \( f_0 \) and \( f \) decrease with \( J_R \), but \( f_0 \) is symmetric with respect to \( \theta_R = \pi \) while \( f \) not, which is due to the \( \exp(\pm i \theta_R) \) terms. The asymmetries in Fig. 7 can be directly translated to features in the \((v_R, v_\phi)\) velocity space. To visualize this transformation, we are helped by the map in Fig. 1 (products of multiple perturbers will affect these vertical motions (Monari et al., in preparation).
In Fig. 4 (top panel) we show the perturbed DF \( \Delta (v_z) \) obtained from equation (40). Right panel: north–south difference between the average vertical speed \( \bar{v}_z \approx 9 \) and \( \bar{v}_\varphi > 0 \). The consequence of the perturbation is to deform the density contours so that the stars are not anymore distributed symmetrically between positive and negative \( v_R \). In particular, there is an excess of stars slightly lagging rotation and moving outwards around \( v_R \approx 30 \text{ km s}^{-1} \) and \( v_\varphi = 210 \text{ km s}^{-1} \). This particular configuration of the density contours is reminiscent of that created by the Hyades moving group in the solar neighbourhood.\(^4\) These features can be easily interpreted in light of Figs 7 and 8. For example, fixing \( J_\varphi = 30 \text{ km s}^{-1} \text{ kpc} \) and moving clockwise from \( \theta_R = 0 \), we first encounter in Fig. 7 an underdensity at \( \theta_R \approx \pi/5 \). Then the density increases again at \( \theta_R = \pi/2 \), forming in Fig. 9 (top) the Hyades-like distortion. At \( \theta_R \approx \pi \) the density is almost constant, to slightly decrease again for \( \theta_R > \pi \). The general velocity distribution is slightly skewed towards negative radial velocities. In the bottom panel of Fig. 9, we then also show \( f(v_R, v_\varphi) \) at \( (R, \phi, z) = (6 \text{ kpc}, 0, 0) \), and \( v_\varphi = 0 \). Here we find more stars than in the previous case at \( v_R < 220 \text{ km s}^{-1} \). Moreover, the two configurations in the DFs of Fig. 9 explain why there is a net \( \langle v_y \rangle < 0 \) motion at \( (R, \phi, z) = (8 \text{ kpc}, 0, 0) \) in the Galaxy, while \( \langle v_y \rangle > 0 \) at \( (R, \phi) = (6 \text{ kpc}, 0, 0) \), due to the asymmetry of the general velocity distribution.

In Fig. 10 we now show \( f(v_R, v_\varphi) \) fixing \( v_R = v_\varphi = 7.5 \text{ kpc} \) and \( (R, \phi, z) = (7.5 \text{ kpc}, 0, 0.3 \text{ kpc}) \) (top panel) and \( v_R = v_\varphi = 9.5 \text{ kpc} \) and \( (R, \phi, z) = (9.5 \text{ kpc}, 0, 0.3 \text{ kpc}) \) (bottom panel), hence at \( z = 0.3 \text{ kpc} \) height from the Galactic plane.\(^5\) The former case has \( \Delta \langle v_z \rangle < 0 \) and \( \langle v_y \rangle > 0 \), while the latter \( \Delta \langle v_z \rangle > 0 \) and \( \langle v_y \rangle < 0 \). The consequence of the perturbation is a tilt of the velocity ellipsoid in the \( v_R - v_\varphi \) space, which has opposite sign in the two points. Such a tilt would be impossible, by construction, with the unperturbed \( f_0 \) DF which is plane-parallel, and has a similar amplitude of that found by studies of stars in the solar neighbourhood (e.g. Pasetto et al. 2012). The velocity ellipsoid is thus clearly influenced by the spiral potential, and this intuitively explains why there is a transition from positive to negative mean vertical motion precisely in between the arms and in

\(^4\) However, it is likely that the Hyades moving group is a resonant feature (Sellwood 2010; Hahn, Sellwood & Pryor 2011; McMillan 2011, 2013), thereby not precisely reproduced by the present model.

\(^5\) To obtain the DFs at \( z = -0.3 \text{ kpc} \) it is sufficient to flip \( v_z \) with \( -v_z \).
Perturbed distribution functions

Figure 6. Several predictions for the response of mean $v_z$ of a stellar system or a cold fluid to the potential model used in this work [red solid line equation (49), green line method by Monari et al. (2015) for a cold stellar disc, blue dashed line equation (40), blue solid line simulation]. Top panel: $R = 7$ kpc. Central panel: $R = 8$ kpc. Bottom panel: $R = 9$ kpc.

Figure 7. Isocontours of the DFs in the $(\theta R, J_R)$ space at the point $(R, \phi, z) = (8 \text{ kpc}, 0, 0)$ of the Galactic plane. Top panel: $f_0(\theta_R, J_R)$. Bottom panel: $f(\theta_R, J_R)$. The contours enclose (from bottom to top) 12, 21, 33, 50, 68, 80, 90, 95, and 99 per cent of the stars.

the middle of the arms (where the mean radial motion is maximal), because the ellipsoid becomes plane-parallel again. Nevertheless, a tilt of the ellipsoid alone cannot cause a net vertical motion, as the average $v_z$ would still be 0. But this tilt is actually accompanied by a lopsidedness of the $v_z$ distribution, which is maximal when the tilt is maximal.

5 CONCLUSION AND PERSPECTIVES

This work presents a general way to calculate the effects of a non-axisymmetric gravitational disturbance on an axisymmetric DF, $f_0$, describing the phase-space density of stars in a collisionless stellar system (i.e. governed by the collisionless Boltzmann equation). We assume that the axisymmetric $f_0$ alone solves the collisionless Boltzmann equation in an axisymmetric potential $\Phi_0$ where the relationship between the ordinary positions and velocities and the action and angle variables are known (Section 2).

We apply this method to construct a 3D model of the Milky Way’s thin disc, where the non-axisymmetric gravitational disturbance $\epsilon \Phi_1$ is a Fourier mode in azimuth (Section 3). In particular, we chose to describe bisymmetric spiral arms with a $\sim \text{sech}^2$ vertical falloff (Section 4.1). As a result, we obtain formulas for the DF and its zeroth- and first-order moments (density and mean motions) that are shown to be in agreement with a numerical test-particle
Figure 8. Curves of constant $\theta_R$ and $J_R$ in velocity space at $(R, \phi, z) = (8 \, \text{kpc}, 0, 0)$ for the Binney & Tremaine (2008) Model I potential. See also McMillan (2011).

A simulation representing the effect of the same bisymmetric spiral arms on the Milky Way’s thin disc (Section 4.2). In particular, we estimate for the first time the reduction factor for the vertical bulk motions of a stellar population compared to the case of a cold fluid.

An inspection of the DF at given points in 3D configuration space (Section 4.3) also helps to interpret these macroscopic properties of the stellar system. One interesting result is that the spiral arms induce a tilt and a lopsidedness in the $v_R - v_z$ velocity ellipsoid that changes of sign and magnitude as a function of the position of the point where it is calculated w.r.t. the spiral arms. In addition, it is shown that distortions typical of moving groups such as the Hyades are naturally reproduced in velocity space.

We nevertheless point out that our results here are only valid away from the main resonances. Indeed, our method consists in a linear treatment of the collisionless Boltzmann equation, i.e. it assumes that the non-axisymmetric gravitational disturbance $\epsilon/\Phi_1$ and DF response $\epsilon f_1$ are small. In particular, $f_0$ should always be larger than $\epsilon f_1$ in order to preserve physical meaning. While most of the non-axisymmetric gravitational disturbances of the Milky Way are indeed much smaller than its background axisymmetric gravitational potential, certain regions of phase-space are particularly affected by the perturbations. These are the resonances, where the rotational, radial, and vertical frequencies and the perturbation pattern speed are commensurable. The linear regime breaks down at the resonances, as is evident from equation (30), whose denominator vanish at the resonances. Even if there is an infinite number of resonances, those that affect a significant portion of phase-space are rare. In our treatment they appear, for example, at the corotation and Lindblad resonances that, in the case of the spiral arms we chose in this paper, are all quite far from the solar neighbourhood. The same cannot be stated in the case of the bar, where the outer Lindblad resonance is probably close to the Sun. One way to treat the resonances that we will explore in forthcoming work is to pass, in their vicinity, to another system of angle-action variables (fast and slow variables), that allows us to focus on the librations around the resonant orbits, neglecting all the high-frequency motions (see Binney & Tremaine 2008).

Another future issue, even more complex to treat, is related to the non-linear effects due to the presence of more than one perturber. In the linear perturbation theory presented here, the effect of more than one perturber would simply be the linear combination of the single responses. However, from numerical studies (Monari et al., in preparation), it can be shown that non-linear effects arise simply by superposing different perturbers, as the bar and spiral arms (see also Vera-Ciro & D’Onghia 2015). This is especially important in terms of the amplitude of the vertical breathing mode generated by the spirals in this work, which is qualitatively similar to observations (Williams et al. 2013), but not quantitatively. The effect of
multiple perturbers could be especially important in that case. Future analytic calculations should investigate this question. Also, in the present work, we concentrated on the response of a given disc stellar population in equilibrium to a perturbing three-dimensional spiral potential, but we did not investigate yet the conditions for self-consistency, which, especially in 3D, is a more complex problem than the present one, to be treated in the future too.

Finally, we note that, while we used the adiabatic and epicyclic approximations to estimate the angle and action variables in this work, the method to obtain the DF that we present at the beginning of the paper is completely general (Section 3.1). Our choice of using a Schwarzschild DF to represent the axisymmetric equilibrium configuration can trivially be generalized to other forms of the DF. Moreover, our results can also, in principle, be used with more sophisticated approximations of the angles and actions in the Milky Way potential already present in the literature. For this reason, the method presented in this paper will be helpful in the future to dynamically characterize the Milky Way disc stellar kinematic information that will be provided by upcoming large astrometric and spectroscopic surveys of the Galaxy, as it offers the possibility to interpret the latter in the dynamical sense (rather than just subtracting the residuals from a fiducial axisymmetric model), using a rather low number of free parameters.

ACKNOWLEDGEMENTS

This work has been supported by a postdoctoral grant from the Centre National d’Études Spatiales (CNES) for GM.

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