SMOOTH MANIFOLDS WITH PRESCRIBED RATIONAL COHOMOLOGY RING

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Abstract. The Hirzebruch signature formula provides an obstruction to the following realization question: given a rational Poincaré duality algebra \( A \), does there exist a smooth manifold \( M \) such that \( H^\ast(M; \mathbb{Q}) = A \)? This problem is especially interesting for rational truncated polynomial algebras whose corresponding integral algebra is not realizable. For example, there are number theoretic constraints on the dimension \( n \) in which there exists a closed smooth manifold \( M \) with \( H^\ast(M; \mathbb{Q}) = \mathbb{Q}[x]/(x^3) \). We limit the possible existence dimension to \( n = 8(2^a + 2^b) \). For \( n = 32 \), such manifolds are not two-connected. We show that the next smallest possible existence dimension is \( n = 128 \). As there exists no integral \( \mathbb{O}P^m \) for \( m > 2 \), the realization of the truncated polynomial algebra \( \mathbb{Q}[x]/(x^{m+1}), |x| = 8 \) is studied. Similar considerations provide examples of topological manifolds which do not have the rational homotopy type of a smooth closed manifold.

The appendix presents a recursive algorithm for efficiently computing the coefficients of the \( L \)-polynomials, which arise in the signature formula.

1. Introduction

Let \( A \) be a 1-connected graded commutative algebra over \( \mathbb{Q} \) satisfying the Poincare duality. We ask if there exists a simply-connected closed manifold \( M \) such that \( H^\ast(M; \mathbb{Q}) = A \). In general, such realization problem can be studied by rational surgery ([Bar76], [Sul77]). When the dimension of \( A \) is \( n = 4k \), there exists such a manifold \( M \) only if there is a choice of fundamental class \( \mu \in (A^0) \) such that the bilinear form on \( A^{2k} \) defined as \( (\cdot \cdot, \mu) \) has the signature \( \sigma(A, \mu) \) equal to the signature of the manifold \( \sigma(M) \), which by Hirzebruch signature theorem is equal to the \( L \)-genus \( \langle L_k(p(\tau_M)), [M] \rangle \).

The case of \( A = \mathbb{Q}[x]/(x^k), |x| = 2k, k > 8 \) was studied in [Su14]. A closed smooth manifold with such rational cohomology ring is called a rational projective plane. It is shown that a rational projective plane must be of dimension \( n = 8k \) for \( k > 1 \). After 4, 8, and 16 (i.e., the dimensions of \( \mathbb{C}P^2 \), \( \mathbb{H}P^2 \), and \( \mathbb{O}P^2 \) respectively), the next smallest \( n \) for which there is an \( n \)-dimensional rational projective plane is \( n = 32 \).

Main results. In Section 2 by a number theoretic observation on the coefficients of the signature equation, we limit the possible existence dimensions of a projective plane to \( n = 8(2^a + 2^b) \), where \( a, b \) are arbitrary nonnegative integers.

Theorem A. For \( n \geq 8 \), a rational projective plane can only exist in dimension \( n = 8k \) with \( k = 2^a + 2^b \) for some nonnegative integers \( a, b \).

Certain specific dimensions after 32 will be eliminated in Section 4 so that the next smallest \( n \) for which an \( n \)-dimensional rational projective plane could exist is \( n = 128 \).
In Section 6 we study the connectivity of the 32-dimensional rational projective planes from [Su14]. In particular, we show that such a manifold does not admit a Spin structure and thus is not 2-connected. As a consequence, we have the following result.

**Theorem B.** A 32-dimensional rational projective plane is not 2-connected.

The truncated polynomial algebra $\mathbb{Q}[x]/(x^{m+1}), |x| = 8$ will be considered in Section 6. While the construction of $\mathbb{O}P^2$ does not generalize to give any higher dimensional octonionic projective spaces, we show that a rational $\mathbb{O}P^m$ exists for $m$ odd.

**Theorem C.** If $m > 2$ is odd, there exists a closed smooth $8m$-dimensional manifold $M$ with rational cohomology ring

$$H^*(M; \mathbb{Q}) \cong \mathbb{Q}[x]/(x^{m+1}) \quad |x| = 8.$$ 

Finally, in Section 7 we use similar approach to study if a Milnor’s $E_8$ manifold have the rational homotopy type of a smooth manifold.

**Theorem D.** If an $E_8$ manifold $M^{4k}$ ($k > 1$) has the rational homotopy type of a smooth manifold, then $k$ is even and $k$ has no more than 5 nonzero bits in its binary expansion. In particular, $M^8$ has the rational homotopy type of a smooth manifold.

A surprising consequence is that the 504-dimensional $E_8$ manifold is a simply-connected topological manifold which does not have the rational homotopy type of a smooth manifold.

In the Appendix, we give an efficient recursive algorithm for computing the coefficients of the $\bar{L}$-polynomial. Such an algorithm is a useful tool for the study of the signature obstruction.

2. Rational projective planes

The Hopf Invariant One theorem implies that $\mathbb{C}P^2, \mathbb{H}P^2, \mathbb{O}P^2$ are the only closed manifolds whose integral cohomology ring is $H^*(-; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^3)$. Ignoring the torsion, we seek realization of the rational cohomology algebra by closed smooth manifolds in dimension greater than 16. We call a simply-connected smooth closed manifold $M^{4k}$ such that $H^*(M; \mathbb{Q}) \cong \mathbb{Q}[x]/(x^3)$ a rational projective plane. It has been shown that a rational projective plane only exists in dimension $n = 8k$ if $k > 1$ ([Su14]).

If $M^{8k}$ is a rational projective plane, by the Hizebruch signature theorem,

$$\langle \bar{L}_k(p(\tau_M)), [M] \rangle = \sigma(M) = \pm 1$$

Since $H^*(M; \mathbb{Q}) \cong \mathbb{Q}[x]/(x^3), |x| = 4k$, the Pontryagin class $p_i(\tau_M) \in H^{4i}(M; \mathbb{Q})$ is zero for all $i$ except $p_k(\tau_M)$ and $p_{2k}(\tau_M)$, the signature equation then becomes

$$s_{k,k}p_k^2(\tau_M), [M]) + s_{2k}(p_{2k}(\tau_M), [M]) = \pm 1$$

Let $\alpha \in H^{4k}(M; \mathbb{Z})$ be the generator such that $\langle \alpha, [M] \rangle = \pm 1$, then $p_k(\tau_M) = x\alpha$ and $p_{2k}(\tau_M) = y\alpha^2$ for some integers $x$ and $y$. Then we have the diophantine equation

$$s_{k,k}x^2 + s_{2k}y = \pm 1. \tag{2.1}$$
A formulas for $s_{k,k}$ and $s_{2k}$ can be found below in Equation (A.1) and Equation (A.2).

By examining the coefficients $s_{k,k}$ and $s_{2k}$, we will prove the following theorem.

**Theorem A** For $n \geq 8$, a rational projective plane can only exist in dimension $n = 8k$ with $k = 2^a + 2^b$ for some nonnegative integers $a, b$.

**Proof.** We firstly prove the following proposition on the 2-adic valuation of the coefficients $s_{k,k}$ and $s_{k}$.

**Proposition 2.1.** Let $k$ be an integer with $m > 2$ nonzero bits in its binary expansion. Then the numerators of the irreducible fractions of $s_{k,k}$ and $s_{2k}$ are divisible by $2^{m-2}$.

**Proof.** We fix our notation: the Hamming weight function $\nu : \mathbb{Z} \to \mathbb{Z}$ sends an integer $x$ to the number of 1’s in the binary expansion of $x$; the 2-adic valuation, $\nu_2 : \mathbb{Z} \to \mathbb{Z}$, sends $x$ to the largest integer $\nu$ so that $2^\nu$ divides $x$; the extended 2-adic valuation, $\nu_2 : \mathbb{Q} \to \mathbb{Z}$ is defined as $\nu_2(\frac{x}{y}) = \nu_2(x) - \nu_2(y)$. Then the proposition asserts that $\nu_2(s_{k,k})$ and $\nu_2(s_{2k})$ are at least $\nu_2(k) - 2$ when $\nu_2(k) \geq 2$. We will need the following lemma to prove the proposition.

**Lemma 2.2.** For any integer $x$,

$$\nu_2(x) + \nu_2(x!) = x.$$

**Proof.** Let $\lfloor \cdot \rfloor$ and $\{ \cdot \}$ denote the floor and fractional part function, respectively. Then

$$x = \sum_{k=1}^{\infty} \frac{x}{2^k} = \left( \sum_{k=1}^{\infty} \left\lfloor \frac{x}{2^k} \right\rfloor \right) + \left( \sum_{k=1}^{\infty} \frac{x}{2^k} \right) = \nu_2(x) + \nu_2(x!).$$

\[\square\]

Let $B_k = \frac{\text{num}(B_k)}{\text{den}(B_k)}$ denotes the irreducible fraction. It is a fact that

$$\text{den}(B_k) = \prod_{p \mid 2^k} p$$

where the product is over all the primes $p$ such that $p - 1 \mid k$. This implies that when $k > 0$, $\nu_2(B_k) = -1$.

By formula (A.1), the coefficient

$$s_k = \frac{2^{2k}(2^{2k-1} - 1)B_{2k}}{(2k)!},$$

then the 2-adic valuation of $s_k$ is

$$\nu_2(s_k) = \nu_2(2^{2k}) + \nu_2(2^{2k-1}) - 1 + \nu_2(B_{2k}) - \nu_2((2k)!)
= 2k + \nu_2(B_{2k}) - \nu_2((2k)!)$$
$$= (\nu_2(2k) + \nu_2((2k)!)) + \nu_2(B_{2k}) - \nu_2((2k)!)$$
$$= \nu_2((2k)!)) + \nu_2(B_{2k})$$
$$= \nu_2(B_{2k})$$
$$= \nu_2(B_k)$$
$$= \nu_2(k) - 1.$$
We also have \( \nu_2(s_{2k}) = \text{wt}(2k) - 1 = \text{wt}(k) - 1 \). Therefore when \( \text{wt}(k) > 1 \), \( \text{num}(s_{2k}) \) is divisible by \( 2^{\text{wt}(k)-1} \).

On the other hand, the coefficient
\[
s_{k,k} = \frac{s_{k}^2 - s_{2k}}{2},
\]
so when \( k > 0 \), the 2-adic valuation is
\[
\nu_2(s_{k,k}) \geq \min\{\nu_2(s_{k}^2), \nu_2(s_{2k})\} - 1
= \min\{2 \cdot (\text{wt}(k) - 1)), \text{wt}(k) - 1\} - 1
= \text{wt}(k) - 2.
\]

Therefore when \( \text{wt}(k) > 2 \), \( \text{num}(s_{k,k}) \) is divisible by \( 2^{\text{wt}(k)-2} \). This proves Proposition 2.1. □

Now we return to the proof of Theorem A. Recall that a necessary condition for the existence of a rational projective plane in dimension \( 8k \) is that the Diophantine equation
\[
s_{k,k}x^2 + s_{2k}y = \pm 1
\]
has integer solutions \( x \) and \( y \). By Proposition 2.1 when \( \text{wt}(k) > 2 \), we have \( 2 \mid \text{num}(s_{k,k}) \) and \( 2 \mid \text{num}(s_{2k}) \). Therefore the diophantine equation has no solution when \( \text{wt}(k) > 2 \). Hence a rational projective plane can only exist in dimension \( n = 8k \) where \( w(k) = 1 \) or \( 2 \), i.e., when \( k = 2^a + 2^b \) for some nonnegative integers \( a, b \).

□

3. Ruling out some specific dimensions up to 128

It has been shown that after dimension 16 (for which there is the octonionic projective plane), the next smallest dimension where a rational analog of projective plane exists is 32 ([Su14]). By Theorem A the candidate dimensions are \( n = 40, 48, 64, 72, 80, 96, 128, \ldots \). We will rule out all those that are less than 128 and show that the signature equation does have a solution for dimension 128.

Proposition 3.1. There is no rational projective plane in dimension 64.

Proof. It suffices to show that there is no solution to \( s_{8,8}x^2 \pm s_{16}y = \pm 1 \) for integers \( x \) and \( y \). Note that 37 divides the numerator of \( s_{16} \), because 37 divides \( B_{32} \), a well-known fact considering 37’s status as the smallest irregular prime. So it is enough to show there is no solution to \( x^2 \not\equiv \pm 1/s_{8,8} \pmod{37} \). Since \( s_{16} \equiv 0 \pmod{37} \),
\[
 s_{8,8} \equiv \frac{s_{8}^2 - s_{16}}{2} \equiv \frac{s_{8}^2}{2} \pmod{37},
\]
but neither 2 nor \(-2\) is a quadratic residue modulo 37. □

This method of searching for a particular prime—played by 37 to prove nonexistence of a 64-dimensional example—can be applied to rule out many other possible examples.

Lemma 3.2. There is no \( 8n \)-dimensional rational projective plane whenever there is a prime \( p \) so that
\begin{itemize}
  \item 2 and \(-2\) are quadratic nonresidues modulo \( p \),
  \item \( \nu_p(s_{2n}) > 0 \), but
\end{itemize}
Table 1. Primes \( p \) ruling out \( n \)-dimensional rational projective planes via Lemma 3.2

| \( n \) | \( p \) |
|--------|--------|
| 48     | 2294797|
| 64     | 37     |
| 72     | 26315271553053477373|
| 96     | 653    |
| 136    | 101    |
| 160    | 10589  |

- \( \nu_p(s_n) = 0 \).

To ensure 2 is a quadratic nonresidue, it is enough that \( p \equiv \pm 1 \pmod{8} \); to ensure that \(-2\) is also a quadratic nonresidue, we further want \( p \equiv 5 \pmod{8} \). By checking the handful of primes \( p \) which divide the numerator of \( s_{2n} \), we can quickly find witnesses which rule out \( n \)-dimensional projective planes for \( n \in \{48, 64, 72, 96\} \); these witnesses are listed in Table 1.

Between 32 and 128, the only integers \( n \) which have at most two nonzero bits in its binary expansion are 40, 48, 64, 72, 80, and 96. So by Proposition 2.1, these are only possibilities for the dimensions of a rational projective plane when \( 32 < n < 128 \).

In light of Table 1, we can restrict attention to \( n = 40 \) and \( n = 80 \).

**Proposition 3.3.** There is no 40-dimensional rational projective plane.

**Proof.** If there were, there would be integers \( x \) and \( y \) for which

\[
 s_{5,5}x^2 + s_{10}y = \pm 1,
\]

but there are no such integers.

Suppose there were \( x \) and \( y \) so that \( s_{5,5}x^2 + s_{10}y = 1 \). We use the fact that \( 2^{2^{10} - 1} - 1 = 524287 \) is prime and the fact that \( B_{20} = -174611/330 \). Then equation (A.1) yields

\[
 s_{10} = \frac{2^{2^{10}}(2^{2^{10} - 1} - 1)B_{210}}{(2 \cdot 10)!} = \frac{2^{20} \cdot 524287 \cdot 174611}{330 \cdot (20)!} = \frac{2^{2^{10}} \cdot 524287 \cdot 283 \cdot 617}{330 \cdot (20)!}.
\]

Set \( p = 283 \), so \( s_{10} \equiv 0 \pmod{283} \). Then we have a solution to \( x^2 \equiv 1/8_{5,5} \pmod{283} \). But by Equation (A.2),

\[
 1/8_{5,5} = -4593988395871875/527062321 \equiv 146 \pmod{283},
\]

and 146 is not a quadratic residue modulo 283, which is a contradiction.

There is another case to consider: suppose there were \( x \) and \( y \) so that \( s_{5,5}x^2 + s_{10}y = -1 \). In this case, consider \( q = 524287 \). Again, we have \( s_{10} \equiv 0 \pmod{q} \), and we assume there is an integer \( x \) so that \( x^2 \equiv -1/s_{5,5} \pmod{q} \). But arithmetic reveals the contradiction: \(-1/s_{5,5} \equiv 318975 \pmod{q} \), and 318975 is not a quadratic residue modulo \( q \).

\( \square \)
We can boil this proof down into a lemma.

**Lemma 3.4.** There is no $8n$-dimensional rational projective plane whenever there are primes $p$ and $q$ so that

- $2$ and $-2$ are quadratic nonresidues modulo $p$,
- $\nu_p(s_{2n}) > 0$ and $\nu_q(s_{2n}) > 0$,
- $1/s_{n,n}$ is a quadratic nonresidue modulo $p$, and
- $-1/s_{n,n}$ is a quadratic nonresidue modulo $q$.

We can rule out an $80$-dimensional rational projective plane by choosing

\[ p = 1897170067619, \]
\[ q = 79. \]

**Proposition 3.5.** There is no $n$-dimensional rational projective plane for $32 < n < 128$.

The situation is quite different when $n = 128$.

**Proposition 3.6.** The signature equation (Equation (2.1)) has a solution in dimension $128$.

**Proof.** In dimension $128$, the signature equation is

\[ s_{16,16}x^2 + s_{32}y = \pm 1, \]

where, using Equations (A.1) and (A.2), we compute

\[ s_{16,16} = \frac{-157 \cdot 311 \cdot 4759 \cdot 7841 \cdot 483239 \cdot 204435234192497636810897021371}{31 \cdot 5^{16} \cdot 7^{8} \cdot 11^{5} \cdot 13^{3} \cdot 17^{4} \cdot 19^{3} \cdot 23^{2} \cdot 29^{2} \cdot 31^{2} \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61}, \]
\[ s_{32} = \frac{31 \cdot 5^{15} \cdot 7^{8} \cdot 11^{5} \cdot 13^{4} \cdot 17^{4} \cdot 19^{3} \cdot 23^{2} \cdot 29^{2} \cdot 31^{2} \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61}{31 \cdot 5^{15} \cdot 7^{8} \cdot 11^{5} \cdot 13^{4} \cdot 17^{4} \cdot 19^{3} \cdot 23^{2} \cdot 29^{2} \cdot 31^{2} \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61}, \]

where $N = 87057315354522179184989699791727$. Equation (3.1) can be simplified to

\[ x^2 \equiv \pm a \pmod{m}, \]

where

\[ a = 98719348515711444512355076910350678632922916684640405411745, \]
\[ m = 1005007135687838990955550319132617999314780397894537794155 \]
\[ = 5 \cdot 73 \cdot 127 \cdot 337 \cdot 92737 \cdot 649657 \cdot 1226592271 \cdot N. \]

Using the factorization of $m$, it can be verified that $a$ is a quadratic residue for each of the factors. Therefore Equation (3.6) has a solution. \(\square\)

**Remark 3.7.** The signature equation is only a necessary condition for the existence of a rational projective plane. By rational surgery ([Bar76], [Sul77]), there exists a $128$-dimensional rational projective plane if and only if there exists integers $x$ and $y$ such that

(i) The signature equation (A.1) holds;

(ii) The integers $x^2$ and $y$ are the Pontragin numbers of a genuine closed smooth manifold, i.e., $x^2 = \langle \tau_{16}(\tau_N), [N] \rangle$ and $y = \langle \tau_{32}(\tau_N), [N] \rangle$ for a $128$-dimensional closed smooth manifold $N$.

Condition (ii) is equivalent to a set of congruence relations on $x^2$ and $y$ (see [Su14]), which requires a significant amount of calculations in dimension $128$. 
4. Ruling out some very high-dimensional examples

We apply Lemma 3.2 and Lemma 3.4 to rule out some higher dimensional examples.

**Proposition 4.1.** There is no \( n \)-dimensional rational projective plane for \( 128 < n < 256 \).

**Proof.** By Proposition 2.1, we have \( n \in \{136, 144, 160, 192\} \). But \( n \neq 136 \) and \( n \neq 160 \) by the witnesses in Table 1. We can rule out a 144-dimensional rational projective plane by applying Lemma 3.4 with
\[
p = 1872341908760688976794226499636304357567811,
q = 228479.
\]
And we can rule out a 192-dimensional rational projective plane by choosing
\[
p = 415593423131,
q = 191.
\]

We can also direct our attention to even higher dimensions. When is there a \( 2^{k+3} \)-dimensional rational projective plane? The fact that divisors of \( 2^{2^k-1} - 1 \) are rarely (never?) 5 mod 8 tells us not to look for such primes among the divisors of the Mersenne factor. The desiderata of Lemma 3.2 are satisfied by finding a prime \( p \) so that
\[
\begin{align*}
& \cdot \ p \equiv 5 \pmod{8}, \\
& \cdot \ p > 4 \cdot 2^k, \\
& \cdot \ p \text{ divides the numerator of } B_{4 \cdot 2^k}, \\
& \cdot \ p \text{ does not divide the numerator of } B_{2 \cdot 2^k}.
\end{align*}
\]

For example, the prime \( p = 502261 \) is 5 mod 8 and divides the numerator of \( B_{4 \cdot 2^{11}} \), but not \( B_{2 \cdot 2^{11}} \), which rules out a rational projective plane in dimension 2^{14} = 4096; similarly, the prime \( p = 69399493 \) is 5 mod 8, divides the numerator of \( B_{4 \cdot 2^{21}} \) but not \( B_{2 \cdot 2^{21}} \), which rules out a projective plane in dimension 2^{24}. These calculations are possible due to tables of irregular primes produced by Joe P. Buhler and David Harvey [BH11].

5. Connectedness of the 32 dimensional rational projective planes

There exist 32-dimensional rational projective planes. In light of this, we consider the torsion structure of such closed smooth manifolds. We show that a 32-dimensional rational projective plane does not admit a Spin structure.

**Lemma 5.1.** There does not exist a simply-connected closed Spin manifold \( M \) in dimension 32 such that
\[
H^*(M; \mathbb{Q}) \cong \begin{cases} 
\mathbb{Q} & \text{if } s = 0, 16, 32, \text{ and} \\
0 & \text{otherwise.}
\end{cases}
\]

The key application of this theorem is to show that a 32-dimensional rational projective plane cannot be very highly (integrally) connected.

**Theorem 5.** Any 32-dimensional rational projective plane is not 2-connected.
Proof: Suppose $M$ is a 32-dimensional rational projective plane. By Lemma 5.1 $M$ is not Spin. But the Stiefel-Whitney class $w_2(M)$ is the obstruction to a Spin structure, so $w_2(M) \neq 0$, and so $H^2(M, \mathbb{Z}_2) \neq 0$, and so by universal coefficients, $H^2(M, \mathbb{Z}) \neq 0$. \hfill \Box

Proof of Lemma 5.3. If there exists such a simply-connected closed smooth manifold $M^{32}$, the Hizebruch signature theorem implies that

$$s_{k,4}(\nu_8(\tau_M), [M]) + s_N(\nu_8(\tau_M), [M]) = \pm 1.$$  

As in Section 2 if $\alpha \in H^{16}(M; \mathbb{Z})$ is the generator such that $(\alpha \sim \alpha, [M]) = \pm 1$, then $\nu_4(\tau_M) = x\alpha$ and $\nu_8(\tau_M) = y\alpha^2$ for integers $x$ and $y$, then we may write $(\nu_4(\tau_M), [M]) = x^2$ and $(\nu_8(\tau_M), [M]) = y$ to have the diophantine equation

$$s_{k,4}x^2 + s_Ny = \pm 1$$

Moreover, the integers $x^2$ and $y$ must be the Pontryagin numbers of a genuine Spin manifold of dimension 32. The following theorem of Stong characterized the congruence relations that determines the set of all possible Pontryagin numbers of a Spin manifold.

**Theorem 5.2** (Stong). For 8k-dimensional closed Spin manifolds, the stable tangent bundle $\tau_N : N \to B_{Spin}$ induces a homomorphism

$$\tau : \Omega_{8k}^{Spin}/\text{tor} \to H_{8k}(B_{Spin}; \mathbb{Q}).$$

The image of the homomorphism $\tau$ is a lattice consisting exactly the elements $x \in H_{8k}(B_{Spin}; \mathbb{Q})$ such that

$$(\mathbb{Z}[e_1(\gamma), e_2(\gamma), \ldots] \cdot \hat{A}(p_i(\gamma)), x) \in \mathbb{Z}.$$  

We explain the notations in the theorem. The KO-theoretic Pontryagin character $e_i(\gamma)$ is defined as follows: the total Pontryagin class of the universal vector bundle $\gamma$ over $B_{Spin}$ can be formally expressed as $p(\gamma) = \Pi(1 + x_j^2)$ by the splitting principle. The class $e_i(\gamma) \in H^{i}(B_{Spin}; \mathbb{Q})$ is the $i$-th elementary symmetric polynomial of the variables $e_j, e^{−1}_j, e^{−2}_j, \ldots$.

Here, $\hat{A}(p_i(\gamma))$ denotes the total $\hat{A}$-polynomial of the Pontryagin classes $p_i(\gamma)$'s.

Since the Pontryagin class $p_i(\gamma)$ is exactly the $i$-th elementary symmetric polynomial in the variables $x_j^2$'s, each class $e_i(\gamma)$, which can be expanded as a symmetric polynomials of the variables $x_j^2$'s, can be written as a polynomial in the $p_i(\gamma)$'s. Therefore Equation (5.2) can be expressed as a set of congruence relations on the Pontryagin numbers $(p_I(\gamma), x)$ over the partitions $I$ of $2k$.

Since for any closed Spin manifold $N$, $(p_I(\tau_N), [N]) = (p_I(\gamma), \tau_N[N]) = (p_I(\gamma), x)$, the relations on $(p_I(\gamma), x)$ in (5.2) determine a set of integrality conditions on the Pontryagin numbers $(p_I(\tau_N), [N])$. Therefore Theorem 5.2 implies the following Lemma, which characterizes all the possible Pontryagin numbers of a closed Spin manifold.

**Corollary 5.3.** If $M$ is an 8k-dimensional closed Spin manifold, then

$$(\mathbb{Z}[e_1(\tau_M), e_2(\tau_M), \ldots] \cdot \hat{A}(p_i(\tau_M)), [M]) \in \mathbb{Z},$$

where each class $e_i(\tau_M) \in H^*(M; \mathbb{Q})$ is the pull back of $e_i(\gamma) \in H^*(B_{Spin}; \mathbb{Q})$ by the stable tangent bundle $\tau_N : N \to B_{Spin}$.
Table 2. Expression for $e_i$ in terms of $p_4$ and $p_8$ from [Su14].

$$
e_1 = -\frac{1}{5040}p_4 + \frac{1}{2615348736000}p_4^2 - \frac{1}{1307674368000}p_8,$$

$$e_2 = -\frac{1}{25401600}p_4^2 + \frac{1}{435891456000}p_8,$$

$$e_1e_1 = -\frac{1}{25401600}p_4^2,$$

$$e_3 = -\frac{1}{3}p_4 + \frac{19}{39916800}p_4^2 - \frac{31}{28512000}p_8, \quad e_1e_2 = -\frac{1}{201600}p_4^2, \quad e_1^3 = 0,$$

$$e_4 = p_4 + \frac{1}{12096000}p_4^2 + \frac{457}{604800}p_8, \quad e_1e_3 = \frac{1}{15120}p_4^2, \quad e_2e_2 = \frac{1}{1600}p_4^2,$$

$$e_5 = -\frac{43}{25200}p_8, \quad e_1e_4 = -\frac{1}{5040}p_4^2, \quad e_2e_3 = \frac{1}{120}p_4^2,$$

$$e_6 = \frac{29}{180}p_8, \quad e_2e_4 = \frac{1}{40}p_4^2, \quad e_3^2 = \frac{1}{9}p_4^2, \quad e_1e_5 = 0,$$

$$e_7 = -\frac{2}{3}p_8, \quad e_3e_4 = -\frac{1}{3}p_4^2, \quad e_2e_5 = 0, \quad e_1e_6 = 0,$$

$$e_8 = p_8, \quad e_4e_4 = p_4^2, \quad e_3e_5 = 0, \quad e_2e_6 = 0, \quad e_1e_7 = 0.$$

In our case, a rational projective plane $M^{32}$ has only nonzero Pontryagin classes $p_4 \in H^{16}(M; \mathbb{Q})$ and $p_8 \in H^{32}(M; \mathbb{Q})$. For dimension 32, the expressions of the $e_i$ classes in terms of $p_4$ and $p_8$ was calculated in [Su14]. This data is recalled in Table 2.

We are now in a position to calculate the $\hat{A}$-genus. For each partition $I$ of $k$, let $a_I$ denote the coefficient of the $I$-th Pontryagin class $p_I$ in the $k$-th $\hat{A}$-polynomial, then similar to the derivation of the coefficients in the $L$-genus through Equation (A.1) and Equation (A.2), we have corresponding formulas

$$a_k = -\frac{|B_{2k}|}{2(2k)!},$$

$$a_{k,k} = \frac{1}{2}(a_k^2 - a_{2k}).$$

The total $\hat{A}$-polynomial up to dimension 32 in terms of $p_4$ and $p_8$ is

$$(5.4) \quad \hat{A} = 1 - \frac{1}{2419200}p_4 + \frac{14527}{85364982743040000}p_4^2 - \frac{3617}{21341245685760000}p_8.$$

The basis of $\mathbb{Z}[e_1, e_2, \ldots] \cdot \hat{A}$ in dimension 32 is displayed Table 3. Corollary 5.3 says each of the above basis class should satisfy $\langle -, [M] \rangle \in \mathbb{Z}$. Let $\langle p_4^2, [M] \rangle = x^2$ and $\langle p_8, [M] \rangle = y$, the integrality conditions can then be simplified to the following
Table 3. Basis of $\mathbb{Z}[e_1, e_2, \ldots] \cdot \hat{A}$ in dimension 32.

$$
\begin{align*}
1 \cdot \hat{A} &= \frac{14527}{85364982743040000} p_4^2 - \frac{3617}{21341245685760000} p_8 \\
e_1 \cdot \hat{A} &= \frac{431}{5230697472000} p_4^2 - \frac{1}{1307674368000} p_8 \\
e_2 \cdot \hat{A} &= -\frac{7}{871782912000} p_4^2 + \frac{31}{217945728000} p_8, \quad e_1 e_1 \cdot \hat{A} = \frac{1}{25401600} p_4^2 \\
e_3 \cdot \hat{A} &= \frac{1}{11404800} p_4^2 + \frac{457}{604800} p_8, \quad e_1 e_3 \cdot \hat{A} = \frac{1}{15120} p_4^2, \quad e_2 e_2 \cdot \hat{A} = \frac{1}{1600} p_4^2 \\
e_4 \cdot \hat{A} &= -\frac{43}{2520} p_8, \quad e_1 e_4 \cdot \hat{A} = -\frac{1}{5040} p_4^2, \quad e_2 e_3 \cdot \hat{A} = -\frac{1}{120} p_4^2 \\
e_5 \cdot \hat{A} &= \frac{29}{180} p_8, \quad e_2 e_4 \cdot \hat{A} = \frac{1}{40} p_4^2, \quad e_3 e_3 \cdot \hat{A} = \frac{1}{9} p_4^2 \\
e_6 \cdot \hat{A} &= -\frac{2}{3} p_8, \quad e_3 e_4 \cdot \hat{A} = -\frac{1}{3} p_4^2 \\
e_7 \cdot \hat{A} &= p_8, \quad e_4 e_4 \cdot \hat{A} = p_4^2.
\end{align*}
$$

congruence relations

\begin{align*}
85364982743040000 &\mid 14527 x^2 - 14468 y \\
5230697472000 &\mid 431 x^2 - 4 y \\
871782912000 &\mid -2771 x^2 + 21844 y \\
11404800 &\mid 7 x^2 - 124 y \\
2419200 &\mid x^2 + 1828 y \\
25401600 &\mid x^2 \\
2520 &\mid y
\end{align*}

Recall that the signature equation says

$$
-444721 x^2 + 118518239 y = \pm 162820783125.
$$

We will show that if the above integrality conditions are satisfied, the signature equation does not have any solution. Since $2520 \mid y$, the signature equation is equivalent to the following quadratic residue problem

$$(5.5) \quad -444721 x^2 \equiv \pm 162820783125 \pmod{(118518239) \cdot (2520)}
\quad x^2 \equiv \pm (-444721)^{-1} \cdot 162820783125 \pmod{298665962280}
\quad x^2 \equiv \pm 11010868155 \pmod{298665962280}.
$$

The modulus has the prime factorization $298665962280 = 2^3 \cdot 3^2 \cdot 5^1 \cdot 7^2 \cdot 31 \cdot 151 \cdot 3617$, since $11010868155 \equiv 3 \pmod{8}$ and $-11010868155 \equiv 5 \pmod{8}$, both $11010868155$ and $-11010868155$ are quadratic nonresidue mod 2. Therefore, Equation (5.5) has no solution. This shows that there does not exist a simply-connected closed Spin
manifold $M^{32}$ such that $H^*(M;\mathbb{Q}) = \mathbb{Q}[x]/(x^3)$. We conclude that none of the 32-dimensional rational projective planes admit a Spin structure.

6. Rational projective spaces

The construction of the octonionic projective plane $\mathbb{O}P^2$ does not generalize to any projective space $\mathbb{O}P^m$ for $m > 2$. If $H^*(X;\mathbb{Z}) = \mathbb{Z}[x]/(x^{m+1})$ with $m > 2$, then $|x| \in \{2, 4\}$ (Hat02). We study the existence of “rational octonionic spaces.” In other words, we ask whether there exist a closed smooth manifold $M^{8k}$ such that $H^*(M;\mathbb{Q}) = \mathbb{Q}[x]/(x^{m+1}), |x| = 8$ for $m > 2$. We apply rational surgery to show the existence of rational $\mathbb{O}P^m$ for $m$ odd.

Our main technical tool is the following result of Barge and Sullivan.

**Theorem 6.1** (Barge [Bar76], Sullivan [Sul77]). Let $X$ be an $n = 4k$-dimensional simply-connected, $\mathbb{Q}$-local, $\mathbb{Q}$-Poincaré complex, where $k \neq 1$. There exists a simply-connected $4k$-dimensional, closed, smooth manifold $M$, and a $\mathbb{Q}$-homotopy equivalence $f : M \rightarrow X$ if and only if there exist cohomology classes $p_i \in H^{4i}(X;\mathbb{Q})$ for $i = 1, \ldots, k$, and a fundamental class $\mu \in H_{4k}(X;\mathbb{Q}) \cong \mathbb{Q}$ such that

(i) the pairing of the $k$th $L$-polynomials of $p_i$'s and $\mu$ is equal to the signature of $X$, i.e., $(L_k(p_1, \ldots, p_k), \mu) = \sigma(X)$;

(ii) the intersection form $\lambda : H^{2k}(X;\mathbb{Q}) \times H^{2k}(X;\mathbb{Q}) \rightarrow \mathbb{Q}$ defined as $\langle \cdot, \cdot \rangle$ is isomorphic to a direct sum of copies of $(1)$'s and $(-1)$'s; and

(iii) the pairings $(p_1, \mu) = (p_{i_1} \cdots p_{i_r}, \mu)$ over all the partitions $I = (i_1, \ldots, i_r)$ of $k$ form a set of Pontryagin numbers of a genuine closed smooth manifold, i.e., there exists a $4k$-dimensional closed smooth manifold $N$ such that $\langle p_1(\tau_N), [N] \rangle = (p_1, \mu)$

for all partitions $I$ of $k$.

If the choice of $p_i$'s and $\mu$ satisfies all the conditions above, surgery theory will construct a $\mathbb{Q}$-homotopy equivalence $f : M \rightarrow X$ such that $f_*[M] = \mu$ and $f^*(p_i) = p_i(\tau_M)$, where $p_i(\tau_M)$ is the $i$-th Pontryagin class of the tangent bundle of $M$. As a consequence, the Pontryagin numbers $p_I[M] = (p_I, \mu)$ for all partitions $I$ of $k$.

**Theorem C** If $m > 2$ is odd, there exists a closed smooth $8m$-dimensional manifold $M$ with rational cohomology ring $H^*(M;\mathbb{Q}) \cong \mathbb{Q}[x]/(x^{m+1}), |x| = 8$.

**Proof.** If a rational Poincare duality algebra $\mathcal{A}$ is intrinsically formal, it contains a unique rational homotopy type ([FHS2]), i.e., for any two simply-connected spaces $X$ and $Y$ such that $H^*(X;\mathbb{Q}) \cong H^*(Y;\mathbb{Q}) \cong \mathcal{A}$, $X$ and $Y$ are rational homotopy equivalent to each other. Any truncated rational polynomial algebra is intrinsically formal ([FHS2]), so $\mathcal{A} = \mathbb{Q}[x]/(x^4), |x| = 8$ is intrinsically formal. Similar to the approach to study the existence of rational projective planes in [Su14], we construct a $\mathbb{Q}$-local space $X$ such that $H^*(X;\mathbb{Z}) \cong H^*(X;\mathbb{Q}) \cong \mathcal{A}$, and apply the rational surgery realization Theorem 6.1 to determine if there exists a manifold $M$ which is rational homotopy equivalent to $X$, thus $H^*(M;\mathbb{Q}) \cong \mathcal{A}$. Given a rational homotopy type, the theorem provides the sufficient and necessary condition for the existence of a simply-connected closed smooth manifold within the rational homotopy type.

Now, consider Theorem 6.1. In our case, $H^*(X;\mathbb{Q}) \cong \mathbb{Q}[x]/(x^{m+1}), |x| = 8$. Note that when $m$ is odd, 8 does not divide $4m$, therefore the middle dimensional cohomology group $H^{4m}(X;\mathbb{Q}) = 0$, so there is no obstruction from condition (ii).
Moreover, the signature $\sigma(X) = 0$. Then in the rational surgery realization Theorem 6.1, the choice that each cohomology class $p_i = 0$ with any fundamental class $\mu$ would satisfy condition (i),(ii) and (iii). Such a realizing manifold has all the Pontryagin numbers vanish.

Proof. Suppose $k > 1$ is odd and that there exists a closed smooth manifold $N^{4k}$ which is rational homotopy equivalent to $M$, so $H^*(N, \mathbb{Q}) \cong H^*(M; \mathbb{Q})$. Note that $p_k[N]$ is the only nonzero Pontryagin number of $N$, so the signature equation requires

$$\langle \mathcal{L}(p_{\tau_N}), [N] \rangle = s_k p_k[N] = \sigma(N) = \sigma(M) = 8. $$

Moreover, since $N$ is a smooth manifold, $p_k[N]$ is an integer.

But, as we will see, there is no integer $x$ so that $s_k x = 8$ when $k > 1$ is odd. Suppose there is such an integer $x$, then

$$s_k x = \frac{2^{2k} (2^{2k-1} - 1) \text{num}(B_{2k})}{(2k)! \text{denom}(B_{2k})} x = \frac{2^{2k} (2^{2k-1} - 1) \text{num}(\frac{B_{2k}}{2k})}{(2k-1)! \text{denom}(\frac{B_{2k}}{2k})} x = 8. $$

It is known that $\text{num}(\frac{B_{2k}}{2k}) = 1$ only for $2k = 2, 4, 6, 8, 10, 14$, otherwise $\text{num}(\frac{B_{2k}}{2k})$ is a product of powers of irregular primes $p$ such that $p > 2k + 1$.

Suppose $k \notin \{3, 5, 7\}$; in this case, let $p$ be an irregular prime that divides $\text{num}(\frac{B_{2k}}{2k})$, then the signature equation implies $p$ divides $8 \cdot (2k-1)! \cdot \text{denom}(\frac{B_{2k}}{2k})$. But since the odd irregular prime $p > 2k+1$, $p$ does not divide $8 \cdot (2k-1)! \cdot \text{denom}(\frac{B_{2k}}{2k})$. Therefore $s_k x = 8$ has no integer solution $x$.

The other cases can be handled individually. When $k \in \{3, 5, 7\}$, the signature equations are

$$s_3 x = \frac{62}{945} x = 8, \quad s_5 x = \frac{146}{13365} x = 8, \quad s_7 x = \frac{32764}{18243225} x = 8.$$
respectively, and each equation has no integer solution. □

Similar to the proof of Theorem A we use the 2-adic valuation of $s_{k,k}$ and $s_{2k}$ to show the following nonexistence result.

**Proposition 7.2.** The $E_8$ manifold $M^{sk}$ does not have the rational homotopy type of a smooth manifold when the binary expansion of $k$ has more than 5 nonzero bits.

**Proof.** Suppose there exists a closed smooth manifold $N^{sk}$ which is rational homotopy equivalent to $M$. In this case, the signature equation says

$$s_{k,k}(p_{k,k}, \mu) + s_{2k}(p_{2k}, \mu) = 8,$$

where $(p_{k,k}, \mu)$ and $(p_{2k}, \mu)$ are integers. As in Section 2 define $\text{wt}(k)$ to be the number of nonzero bits in the binary expansion of $k$. By Proposition 2.1 the numerators of both $s_{k,k}$ and $s_k$ are divisible by $2^\text{wt}(k) - 2$. When $\text{wt}(k) > 5$, the left hand side of the signature equation is divisible by 16, and therefore the equation has no integer solution. □

A specific instance of Proposition 7.2 is in dimension 504; note that

$$504 = 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8,$$

and so $\text{wt}(504) = 6$. The Milnor $E_8$ manifold $M^{504}$ is a topological manifold which does not have the rational homotopy type of a smooth manifold.

**Proposition 7.3.** The 8-dimensional $E_8$ manifold $M^8$ has the rational homotopy type of a smooth manifold.

**Proof.** Following our approach to the realization of rational projective planes and rational $OP^4$, we again apply the surgery realization Theorem 6.1. Let $X$ be the $\mathbb{Q}$-localization of $M^{8k}$, so $H^*(X; \mathbb{Z}) \cong H^*(M^{8k}; \mathbb{Q})$. Let $a_i \in H^i(X; \mathbb{Q}) = \mathbb{Q}^8$ be a generator of the $i$-th summand of $\mathbb{Q}^8$, and let $\mu \in H_8(X; \mathbb{Q})$ be a fundamental class, such that the intersection form on $H^4(X; \mathbb{Q})$ with respect to the basis $a_i$’s and $\mu$ is exactly the $E_8$ form. We seek a choice of $p_1, p_2 \in H^*(X; \mathbb{Q})$ such that the signature formula

$$(7.1) \quad s_{1,1}(p_1^2, \mu) + s_2(p_2, \mu) = 8$$

is satisfied. Note that $p_1^2[CP^4] = 25$, and $p_2[CP^4] = 10$. The signature formula says $s_{1,1}p_1^2[CP^4] + s_2p_2[CP^4] = 1$. Then we let $p_1 = 10a_1$, $p_2 = 40a_1^2$ so that

$$\langle p_{1,1}, \mu \rangle = 100(a_1^2, \mu) = 200 = 8p_1^2[CP^4],$$

$$\langle p_{2}, \mu \rangle = 80(a_1^2, \mu) = 80 = 8p_2[CP^4].$$

Then the signature formula (7.1) is satisfied. Also, $(p_{1,1}, \mu)$ and $(p_2, \mu)$ are the Pontrajin numbers of a genuine closed smooth manifold. Having met all the conditions of the surgery realization Theorem 6.1, we may conclude that $M^8$ has the rational homotopy of a smooth manifold. □

Propositions 7.1, 7.2, and 7.3 combine to yield Theorem D.
Appendix A. Computing the \( L \)-polynomial

In general, our approach of studying the realization of a rational cohomology ring by smooth manifold requires finding the coefficients of the \( k \)-th \( L \)-polynomial. This is harder than it may seem at first. A na"ıve approach for this calculation is directly express the homogenous part of degree \( k \) in

\[
1 + \mathcal{L}_1(p_1) + \mathcal{L}_2(p_1, p_2) + \ldots + \mathcal{L}_k(p_1, \ldots, p_k) = f(t_1)f(t_2)\cdots f(t_k),
\]

by the elementary symmetric polynomials \( p_i = \sigma_i(t_i) \), where the generating function

\[
f(t) = \frac{\sqrt{t}}{\tanh \sqrt{t}} = \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k} t^k}{(2k)!} = 1 + \frac{t}{3} - \frac{t^2}{45} + \ldots.
\]

This approach involves expanding the product of power series of large degree; it is not nearly efficient enough for our desired applications. Here we give an recursive algorithm for calculating the \( L \)-polynomial.

For each partition \( I = i_1, \ldots, i_r \) of \( k \), let \( s_I \) denote the coefficient of the \( I \)-th Pontryagin class \( p_I = p_{i_1} \cdots p_{i_r} \) in the \( k \)-th \( L \)-polynomial, i.e., we express

\[
\mathcal{L}_k(p_1, p_2, \ldots, p_k) = \sum_{|I| = k} s_I p_I
\]

From \cite{MS74}, \( s_k \) can be calculated by the formula

\[
(A.1) \quad s_k = \frac{2^{2k}(2^{2k-1} - 1)|B_{2k}|}{(2k)!}
\]

And in \cite{And69}, it was derived that

\[
(A.2) \quad s_{k,k} = \frac{1}{2}(s_k^2 - s_{2k})
\]

This is the first case of a recursive formula. In general, for any partition \( I \) of \( k \), one can calculate \( s_I \) in terms of the \( s_J \)'s, where \( I \) is a refinement of \( J \).

**Proposition A.1.** Let \( I = i_1, \ldots, i_j, \ldots, i_s, \ldots, i_n \) be a partition of \( k \), then

\[
(A.3) \quad s_I = \frac{1}{\mu(i_1)! \cdots \mu(i_n)!} \left( s_{\mu(i_1)}^{\mu(i_1)} \cdots s_{\mu(i_n)}^{\mu(i_n)} - \sum_{|J| = k, J \supsetneq I} \mu(J) \mu(J_{\mu}) a_J s_J \right),
\]

where \( J > I \) means that \( J \) is any partition of \( k \) such that \( J \) is a proper refinement of \( I \). For each \( J = j_1, \ldots, j_l, \ldots, j_m, j_{m+1}, \ldots, j_n \), the coefficient \( a_J \) counts the number of partitions \( P = \{A_1, \ldots, A_m\} \) of the multiset \( \{I\} \) such that \( J = \sum_{a \in A_1} a, \ldots, \sum_{a \in A_m} a \)

**Proof.** For any partition \( I = i_1, \ldots, i_j, \ldots, i_s, \ldots, i_n \) of \( |I| = k \), consider the corresponding product of spheres \( X = \prod_{l=1}^{\mu(i_1)} S^{4i_1}_{(l)} \times \cdots \times \prod_{l=1}^{\mu(i_n)} S^{4i_n}_{(l)} \). Let \( \xi_i \) be any real vector bundle over the \( l \)-th 4\( i \)-dimensional sphere \( S^{4i}_{(l)} \) in the product. We denote the total Pontryagin class \( p(\xi_i) = 1 + p_i^{(l)} \), where \( p_i^{(l)} \in H^{4i}(S^{4i}_{(l)}; \mathbb{Z}) \). Consider the product bundle \( \eta = \prod_{i=1}^{\mu(i_1)} \xi_i^{(l)} \times \cdots \times \prod_{i=1}^{\mu(i_n)} \xi_i^{(l)} \), the total Pontryagin class
Having obtained two expressions for the coefficient of \(\frac{L}{p}\) in Example A.2. On the other hand, by the multiplicativity of the \(\mathcal{L}\)-polynomials,

\[
\mathcal{L}(p(n)) = \prod_{i=1}^{\mu(i)} \mathcal{L}(p(\xi_{i1}^{(l)})) \times \cdots \times \prod_{i=1}^{\mu(i_n)} \mathcal{L}(p(\xi_{i_n}^{(l)}))
\]

\[
= \prod_{i=1}^{\mu(i)} (\cdots + s_{i1}p_{i1}^{(l)} + \cdots) \times \cdots \times \prod_{i=1}^{\mu(i_n)} (\cdots + s_{i_n}p_{i_n}^{(l)} + \cdots)
\]

\[
= \cdots + s_{i1}^{(l)} \times s_{i_n}^{(l)} \left( \prod_{i=1}^{\mu(i)} p_{i1}^{(l)} \times \cdots \times \prod_{i=1}^{\mu(i_n)} p_{i_n}^{(l)} \right) + \cdots
\]

Having obtained two expressions for the coefficient of \(\prod_{i=1}^{\mu(i)} p_{i1}^{(l)} \times \cdots \times \prod_{i=1}^{\mu(i_n)} p_{i_n}^{(l)}\) in \(\mathcal{L}(p(n))\), we may equate them to get the equation

\[
s_{i1}^{(l)} \cdots s_{i_n}^{(l)} = \sum_{\vert J \vert = k, J \not\supset I} a_J \mu(j_1)! \cdots \mu(j_m)! s_J
\]

\[
= \mu(i_1)! \cdots \mu(i_n)! s_I + \sum_{\vert J \vert = k, J \not\supset I} a_J \mu(j_1)! \cdots \mu(j_m)! s_J,
\]

from which we solved

\[
s_I = \frac{1}{\mu(i_1)! \cdots \mu(i_n)!} \left( s_{i1}^{(l)} \cdots s_{i_n}^{(l)} - \sum_{\vert J \vert = k, J \not\supset I} a_J \mu(j_1)! \cdots \mu(j_m)! s_J \right)
\]

\(\square\)

**Example A.2.** Consider the case of finding \(s_{1,1,2,2}\). The corresponding product of sphere is \(X = S^4 \times S^4 \times S^8 \times S^8\), over which any vector bundle \(\eta = \xi_1^{(1)} \times \xi_1^{(2)} \times \xi_2^{(1)} \times \xi_2^{(2)}\) has the total Pontryagin class \(p(\eta) = (1 + p_1^{(1)} \times (1 + p_1^{(2)} \times (1 + p_2^{(1)} \times (1 + p_2^{(2)})\). We want to find the coefficient of \(p_1^{(1)} \times p_1^{(2)} \times p_2^{(1)} \times p_2^{(2)}\) in the \(\mathcal{L}\)-polynomial \(\mathcal{L}_6(p(\eta)) = \sum_{J} s_J p_J(\eta)\). Firstly note that \(p_1^{(1)} \times p_1^{(2)} \times p_2^{(1)} \times p_2^{(2)}\) is only contained in a \(p_J(\eta)\) where \(J \supset I\), i.e., partition \(J\) such that \(I = 1, 1, 2, 2\) is a refinement of \(J\).

For starters, consider \(J = (2, 4)\),

\[
p_{2,4}(\eta) = p_2(\eta)p_4(\eta)
\]

\[
= (p_1^{(1)} \times p_1^{(2)} \times 1 \times 1 + 1 \times 1 \times 1 \times 1 \times p_2^{(1)} \times 1 + 1 \times 1 \times 1 \times p_2^{(2)})
\]

\[
( p_1^{(1)} \times p_1^{(2)} \times p_2^{(1)} \times 1 + p_1^{(1)} \times p_2^{(1)} \times 1 \times 1 \times p_2^{(2)} + 1 \times 1 \times p_2^{(1)} \times p_2^{(2)})
\]

\[
= (3)(p_1^{(1)} \times p_1^{(2)} \times p_2^{(1)} \times p_2^{(2)}) + \text{other terms},
\]

where the coefficient \(a_J = 3\) counts the 3 partitions

\[
\{\{1, 1\}, \{2, 2\}\}, \{\{2\}, \{1, 1, 2, 2\}\}, \{\{2\}, \{1, 1, 2, 1\}\}
\]
Table 4. Partitions $J$ (and corresponding $a_J$) which refine to $(1, 1, 2, 2)$ for use in Example A.2.

| $J$ ($J \geq I$) | corresponding partitions of the multiset $\{I\} = \{1_1, 1_2, 2_1, 2_2\}$ | $a_J$ |
|----------------|-------------------------------------------------|-------|
| $(1, 1, 2, 2)$  | $\{\{1_1\}, \{1_2\}, \{2_1\}, \{2_2\}\}$         | 1     |
| $(2, 2, 2)$     | $\{\{1_1, 1_2\}, \{2_1\}, \{2_2\}\}$             | 1     |
| $(1, 2, 3)$     | $\{\{1_1\}, \{2_1\}, \{1_2, 2_2\}\}, \{\{1_1\}, \{2_2\}, \{1_2, 2_1\}\}, \{\{1_2\}, \{2_1\}, \{1_1, 2_2\}\}$ | 4     |
|                | $\{\{1_2\}, \{2_2\}, \{1_1, 2_1\}\}$             |       |
| $(1, 1, 4)$     | $\{\{1_1\}, \{1_2\}, \{2_1, 2_2\}\}$             | 1     |
| $(2, 4)$        | $\{\{1_1, 1_2\}, \{2_1, 2_2\}\}, \{\{2_1\}, \{1_1, 1_2, 2_2\}\}, \{\{2_2\}, \{1_1, 1_2, 2_1\}\}$ | 3     |
| $(3, 3)$        | $\{\{1_1, 2_1\}, \{1_2, 2_2\}\}, \{\{1_1, 2_2\}, \{1_2, 2_1\}\}$ | 2     |
| $(1, 5)$        | $\{\{1_1\}, \{1_2, 2_1, 2_2\}\}, \{\{1_2, 2_1, 2_2\}\}$ | 2     |
| $(6)$           | $\{\{1_1, 1_2, 2_1, 2_2\}\}$                      | 1     |

of the multiset $\{I\} = \{1_1, 1_2, 2_1, 2_2\}$. These are the only partitions $\{A_1, A_2\}$ corresponding to $J = (2, 4)$ in the sense that $\sum_{a \in A_i} a = 2$, and $\sum_{a \in A_i} a = 4$.

Next, consider $J = (2, 2, 2)$,

$$p_{2,2,2}(\eta) = p_2(\eta)p_2(\eta)p_2(\eta)$$

$$= (p_1^{(1)} \times p_1^{(2)}) \times 1 \times 1 \times p_2^{(1)} \times 1 \times 1 \times p_2^{(2)}$$

$$= (1)(3!)(p_1^{(1)} \times p_1^{(2)} \times p_2^{(1)} \times p_2^{(2)}) + \text{other terms},$$

where the coefficient $a_J = 1$ counts the single partition $\{A_1, A_2, A_3\} = \{\{1_1, 1_2\}, \{2_1\}, \{2_2\}\}$ of the multiset $\{I\} = \{1_1, 1_2, 2_1, 2_2\}$ such that $\sum_{a \in A_i} a = 2$ for $i = 1, 2, 3$. The coefficient $3!$ came from the multiplicity of the parts of $J$.

We proceed in this fashion, eventually producing the Table ?? which lists all such partitions $J$ and the corresponding $a_J$.

The coefficient of $p_1^{(1)} x p_1^{(2)} x p_2^{(1)} x p_2^{(2)}$ in the $L$-polynomial $L_0(p(\eta)) = \sum_s s \mu_s(\eta)$ is then $\sum_{J \geq I} a_J \mu(j_1)! \cdots \mu(j_m)! s_J$. On the other hand,

$$L(p(\eta)) = L(p(\xi_1^{(1)})) \times L(p(\xi_1^{(2)})) \times L(p(\xi_2^{(1)})) \times L(p(\xi_2^{(2)}))$$

$$= \cdots + (s_1 s_1 s_2 s_2) p_1^{(1)} \times p_2^{(1)} \times p_2^{(2)} + \cdots.$$

Equate the two formulas for the coefficient. Then we obtain

$$s_1^2 s_2^2 = \sum_{J \geq I} a_J \mu(j_1)! \cdots \mu(j_m)! s_J$$

$$= (1)(2!)(s_{1,1,2,2} + (1)(3!)s_{2,2,2} + (4)s_{1,2,3} + (1)(2!)s_{1,1,4} + (3)s_{2,4} + (2)(2!)s_{3,3} + (2)s_{1,5} + (1)s_6,$$

and from this we conclude

$$s_{1,1,2,2} = \frac{1}{2!} (s_1^2 s_2^2 - 6s_{2,2,2} - 4s_{1,2,3} - 2s_{1,1,4} - 3s_{2,4} - 4s_{3,3} - 2s_{1,5} - s_6).$$

We can also find formulas which generalize Equation [A.2] for $s_{n,n}$.

**Example A.3.** Here we list some explicit examples of recursive formulas for $s_J$.

- If $m \neq n$, then $s_{m,n} = s_m s_n - s_{m+n}$. 
• For any \( n \),
\[
s_{n,n,n} = \frac{1}{3!} (s^3_n - 3s_{2n,n} - s_{3n}).
\]

• For any \( n \),
\[
s_{n,n,2n} = \frac{1}{2} (s_{2n}s^2_n - 2s_{3n,n} - 2s_{2n,2n} - s_{4n})
\]

• If \( m \neq n \) and \( m \neq 2n \), then
\[
s_{n,n,m} = \frac{1}{2} (s_ms^2_n - s_{m,2n} - 2s_{m+n,n} - s_{m+2n}).
\]

• If \( m < n < k \) and \( m + n \neq k \), then
\[
s_{m,n,k} = s_ms_{n}sk - s_{m+n,k} - s_{m+k,n} - s_{n+k,m} - s_{m+n+k}.
\]

Formulas such as these will prove useful in working through more complicated versions of the spaces considered in Section 6.

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