Primes between consecutive powers

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Abstract
This paper updates the explicit interval estimate for primes between consecutive powers. It is shown that there is least one prime between $n^{155}$ and $(n + 1)^{155}$ for all $n \geq 1$. This result is in part obtained with a new explicit version of Goldston’s 1983 estimate for the error in the truncated Riemann–von Mangoldt explicit formula.

1 Primes in intervals

There are a variety of results on primes in intervals of the form $(x, x + f(x)]$, for some $f(x) < x$. These results typically hold for sufficiently large $x$, and some explicitly calculate the range of $x$ for which they hold. The latter is particularly useful for bounding gaps between large primes. We have computation on gaps between primes up to $4 \cdot 10^{18}$ [30], so explicit interval estimates are instrumental above this. One of the smallest explicit interval results is from Dudek [13], with primes between consecutive cubes for all $n \geq \exp(\exp(33.3))$. It appears difficult to substantially extend this range, so we can instead look at results for higher powers. The purpose of this paper is to reduce the $m$ for which we know $(n^m, (n+1)^m)$ contains a prime for all $n \geq 1$. Dudek [13] showed that we can take $m = 5 \cdot 10^9$, and Mattner [27] lowered this to $m = 1.5 \cdot 10^6$. This is improved to the following.

Theorem 1. There exists at least one prime in the interval $(n^{155}, (n + 1)^{155})$ for all $n \geq 1$.

This result can also be used in work on prime-representing functions, e.g. see [15]. To give more context for Theorem 1, the following gives a short summary on the different forms of $f(x)$ which allow $(x, x + f(x)]$ to contain a prime.

\footnote{Le projet TME-EMT from Olivier Ramaré is a very useful resource for these and related results.}
Interval estimates date back to Bertrand’s postulate of 1845. He proposed that there should be at least one prime in \((x, 2x)\) for all integers \(x > 1\). This was proved by Chebyshev in 1852. Intervals with \(f(x) = Cx\) for \(1 < C < 2\) are the largest in the long-run, but can be the smallest for sufficiently small \(x\). These results have been refined in \([40], [37], [24]\), and most recently by the author and Lee in \([9]\), with corrections in \([10]\). From \([10]\), we know there is at least one prime in \((x(1 - \Delta^{-1}), x)\) for all \(x \geq x_0\) with \((x_0, \Delta) = (4 \cdot 10^{18}, 3.9 \cdot 10^7)\) or \((e^{600}, 2.5 \cdot 10^{11})\), among others.

For sufficiently large \(x\), the next smallest intervals have \(f(x) = C_k x(\log x)^{-k}\), with some integer \(k \geq 2\) and constant \(C_k\). These intervals can be deduced from a certain type of error estimate for the prime number theorem (PNT), as was done by Trudgian \([43, \text{Cor. 2}]\) and Dusart \([14, \text{Prop. 5.4}]\), among others. For example, Corollary 2 of \([43]\) states that for \(k = 2\) we can take \(C_k = 1/111\) for all \(x \geq 2898242\).

The smallest intervals in the long-run have \(f(x) = C x^a\), with \(C > 0\) and \(a \in (1/2, 1)\). Most of these results use estimates for the Riemann zeta-function \(\zeta(s)\) and its zeros. For example, Ingham \([21]\) found that we can take \(f(x) = x^{\theta+\epsilon}\) with \(\theta = (1 + 4c)/(2 + 4c)\) if we have \(|\zeta(1/2 + it)| \leq At^c\) as \(t \to \infty\) with constants \(c, A > 0\) and sufficiently large \(x\). Bourgain \([8]\) showed that we can take \(c = 13/84 + \epsilon\), the smallest to date, which gives \(\theta = 34/55\) in Ingham’s method. More recently, the best results have incorporated sieve methods. Iwaniec and Jutila \([22]\) first used a sieving argument to prove \(\theta = 5/9\) for sufficiently large \(x\). At present, the smallest interval is from Baker, Harman, and Pintz \([2]\), of \([x, x^{0.525}]\).

The result of \([2]\) is considered particularly strong because of how close it comes to results which assume the Riemann hypothesis (RH). Cramér \([7]\) showed that assuming RH gives us primes in \((x, x + C \sqrt{x} \log x)\) for some \(C\) and sufficiently large \(x\). Carneiro, Milinovich, and Soundararajan \([3, \text{Thm. 5}]\) give the best explicit version, of \(C = 22/25\) for \(x \geq 4\).

As mentioned, the best explicit result in the long run is Dudek’s consecutive cubes \([13]\). This was proved following Ingham’s method in \([21]\) with \(f(x) = 3x^{4/7}\). It would be similarly possible to take \(f(x) = mx^{1-\frac{1}{m}}\), and seek primes between consecutive \(m^{th}\) powers. This is done in Section \([10]\) we refine Dudek’s method, and utilise the interval results in \([10]\), to arrive at Theorem \([1]\). Improvements come from using corrected and/or more recent estimates for the zeros of \(\zeta(s)\), the PNT, and the Riemann–von Mangoldt explicit formula — detailed in Section \([2]\). We also carry out further optimisation, using additional parameters and numerical optimisation functions in Python. In Section \([8]\) we make explicit an asymptotically better estimate for the error in the truncated Riemann–von Mangoldt explicit formula from Goldston \([18]\). Section \([5]\) discusses the relative impact of each of these results on Theorem \([1]\) and the room for improvement.
2 Estimates to be used

The proof of Theorem 1 begins with Chebyshev’s functions
\[ \theta(x) = \sum_{p \leq x} \log p \quad \text{and} \quad \psi(x) = \sum_{n \leq x} \Lambda(n), \]
where \( \Lambda(n) \) is the von Mangoldt function. Results on the zeros of \( \zeta(s) \) are useful for estimating these functions. We will use the most recent estimates for the location and density of the non-trivial zeros (in the “critical strip” \( 0 < \text{Re}(s) < 1 \)), as well as computational verification of RH. RH states that all non-trivial zeros of \( \zeta(\sigma + it) \) have \( \sigma = 1/2 \), and has most recently been verified over \( |t| \leq 3,000,175,332,800 \) by Platt and Trudgian \([32]\). The previous computation from Platt \([33]\) up to \( 3.06 \times 10^{10} \) was used in many of the following results, so will be marked with \( H_p \). Otherwise, the largest known “Riemann height” will be denoted \( H_0 \).

Above \( H_0 \), the non-trivial zeros are known to lie outside zero-free regions. The classical region is deduced from Hadamard and de la Vallée Poussin’s proof of the PNT, and has been made explicit and refined in a number of papers, including \([17]\) and \([29]\). Depending on the range of \( t \) for which the result is used, some of these estimates are better than others. We will use Ford’s, in Theorem 3 of \([17]\). It states that for \( |t| \geq 2 \times 10^{14} \) there are no zeros with \( \sigma \geq 1 - \nu_1(t) \), where
\[ \nu_1(t) = \frac{1}{R(|t|) \log |t|}, \quad R(t) = \frac{J(t) + 0.685 + 0.155 \log \log t}{\log t \left( 0.04962 - \frac{0.0196}{J(t)+1.15} \right)}, \tag{1} \]
and \( J(t) = \log(t)/6 + \log \log t + \log(0.77) \). For larger \( t \), there is an asymptotically wider region from Korobov \([26]\) and Vinogradov \([44]\). Ford \([17]\) also made this explicit, proving that for \( c = 57.54 \) and \( |t| \geq 3 \) there are no zeros with \( \sigma \geq 1 - \nu_2(t) \) for
\[ \nu_2(t) = \frac{1}{c \log^{2/3} t (\log \log t)^{1/3}}. \tag{2} \]

Outside the zero-free region, but within the critical strip, there are estimates on the number of zeros up to some \( T > 0 \), denoted \( N(T) \). Backlund \([1]\) proved that
\[ N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8} + S(T) + O \left( \frac{1}{T} \right), \]
where \( S(T) = O(\log T) \). This was made explicit by Rosser \([38]\), and most recently by Hasanalizade, Shen, and Wong \([19, \text{Cor. 1.2}]\). It was shown that for \( T \geq e \),
\[ \left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{7}{8} \right| \leq a_1 \log T + a_2 \log \log T + a_3 \tag{3} \]

\[^2\text{This definition of } J(t) \text{ has been improved by Hiary } [20] \text{ and corrected as per the comments in Section 2 and footnote 3 of } [31] .\]
with $a_1 = 0.1038$, $a_2 = 0.2573$, and $a_3 = 9.3675$. Sharper estimates are possible for smaller areas of the critical strip: we can estimate
\[
N(\sigma, T) = |\{\rho = \beta + i\gamma : \zeta(\rho) = 0, 0 < \gamma < T \text{ and } \sigma < \beta < 1\}|.
\]

One of the best explicit estimates is from Kadiri, Lumley, and Ng [25]. Their result builds on Ramaré’s explicit version of Ingham’s zero-density estimate in [21], and is valid for any $\sigma > \frac{1}{2} + \frac{d}{\log H}$, with $d > 0$ and $H \in [1002, H_0)$. For any $T \geq H_0$, Kadiri et al. give
\[
N(\sigma, T) \leq N_1(\sigma, T) = C_1(\sigma) (\log(kT))^{2\sigma} (\log T)^{5-4\sigma} T^\frac{4(1-\sigma)}{\sigma} + C_2(\sigma) \log^2 T,
\]
for any $k \in [10^9 H^{-1}, 1]$. Using $H_0 = H_p$, Table 1 of [25] lists values of $C_1$ and $C_2$ for specific $\sigma$, after optimising over several parameters. Another zero-density estimate was given by Simonić [42], of an explicit version of Selberg’s zero-density estimate [41]. For $1/2 \leq \sigma \leq \sigma_0 = \frac{1}{2} + \frac{8}{\log T_0}$, $T \geq 2T_0 \geq 2H_0$, and given constant $C(\sigma_0, T_0)$, we can take
\[
N(\sigma, T) \leq C(\sigma_0, T_0) T^{1-\frac{4\sigma-4}{\sigma}} \log \frac{T}{2}.
\]

For $\sigma \in [1/2, 37/58]$, (5) will be better than (4) for sufficiently large $T$.

Zero-density estimates are commonly used in estimates for the PNT. The PNT in terms of $\psi(x)$ can be deduced from the Riemann–von Mangoldt formula: for non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$, and any $x > 1$ not a prime power,
\[
\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2} \log \left(1 - \frac{1}{x^2}\right).
\]

The sum over $\rho$ is divergent for unordered $\rho$, but can be truncated to write
\[
\psi(x) = x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} + E(x, T),
\]
with $E(x, T)$ decreasing in $T$. Dudek [13] showed that for half odd integers $x > e^{60}$, we have
\[
|E(x, T)| \leq \frac{2x \log^2 x}{T}
\]
with any $T \in (50, x)$. Goldston [18] proved an asymptotically smaller estimate of
\[
E(x, T) = O \left(\frac{x \log x \log \log x}{T}\right),
\]
which can be made explicit. This is done in the following section — see Theorem 2.

3Values for $C_1$ and $C_2$ have been re-calculated in [10] and [28].
The PNT estimates for $\psi(x)$ can be translated into those for $\theta(x)$ if needed. Costa Pereira [6, Thm. 5] (see also Dusart [14]) gives the best lower bound on their difference,

$$\psi(x) - \theta(x) > 0.999x^{\frac{1}{2}} + x^{\frac{1}{4}},$$

which holds for $x \geq e^{38}$, and Broadbent et al. [4, Cor. 5.1] give the most recent explicit upper bounds. Of these, we will use the version which holds for all $x \geq e^{1000}$,

$$\psi(x) - \theta(x) < a_1x^{\frac{1}{2}} + a_2x^{\frac{1}{4}},$$

with $a_1 = 1 + 1.99986 \cdot 10^{-12}$ and $a_2 = 1 + 1.936 \cdot 10^{-8}$.

### 3 An explicit version of Goldston’s result

Davenport’s exposition in Chapter 17 of [12] (see, in particular, equation (3)) shows that for $T > 0$ and $x \geq 2$ which is not a prime power,

$$\psi(x) = \left(1 - \frac{\zeta'(s)}{\zeta(s)}\right) \frac{x^s}{s} ds + O^\ast\left(\frac{1}{\pi T} \sum_{n=1}^{\infty} \Lambda(n) \left(x^s - \frac{1}{n}\right) \left|\log \frac{x}{n}\right|^{-1}\right),$$

where, here and hereafter, $O^\ast$ denotes a constant of absolute value not exceeding 1. The integral gives the exact main term and smaller-order terms in the truncated Riemann–von Mangoldt explicit formula (see (6)). The dominant error terms come from the sum. Goldston [18] showed that $E(x, T)$ of (6) can be reduced to

$$E(x, T) = O\left(\frac{x \log x \log \log x}{T} + \frac{x \log T}{T} + \log x\right),$$

for sufficiently large $x \geq 3$ and $T \geq 3$. This is made explicit in Lemma 1 and will be combined with Dudek’s estimate [13] for the integral to prove Theorem 2.

**Lemma 1.** For half odd integers $x \geq x_K$, and $c = 1 + 1/\log x$, we have

$$\sum_{n=1}^{\infty} \Lambda(n) \left(x^s - \frac{1}{n}\right) \left|\log \frac{x}{n}\right|^{-1} < Mx \log x \log \log x,$$

where pairs of $x_K$ and $M$ are given in Table 1.

**Theorem 2.** For $50 < T < x$ and half odd integers $x \geq x_K$ we have

$$\psi(x) = x - \sum_{|\gamma| < T} \frac{x^\rho}{\rho} + O^\ast\left(\frac{Kx \log x \log \log x}{T}\right),$$

where pairs of $x_K$ and $K$ are given in Table 2.
Proof of Lemma 1. Let \( x \) be half an odd integer, and
\[
S(x) = \sum_{n=1}^{\infty} \Lambda(n) \left( \frac{x}{n} \right)^c \left| \log \frac{x}{n} \right|^{-1},
\]
with \( \alpha > 1 \) a parameter. The sum can be split into five parts,
\[
\sum_{n=1}^{\infty} = \sum_{n=1}^{\lfloor x/\alpha \rfloor} + \sum_{n=\lfloor x/\alpha \rfloor + 1}^{\lfloor x \rfloor} + \sum_{n=\lfloor x \rfloor + 1}^{\lfloor \alpha x \rfloor} + \sum_{n=\lfloor \alpha x \rfloor + 1}^{\infty},
\]
denoting each partial sum with \( S_i \), consecutively from \( i = 1 \) to \( 5 \). The resulting bound on \( S(x) \) will be optimised over \( \alpha \). Estimates for the first and last sums can be taken straight from [13], as they are both relatively small compared to the overall bound, with
\[
S_1 + S_5 = \frac{e}{\log \alpha} x \log x.
\]
The estimate for \( S_3 \) can be just as small: using \( \Lambda(n) \leq \log n \) and \( \lfloor x \rfloor = x - \frac{1}{2} \) we have
\[
S_3 \leq \left( \frac{x}{x - \frac{1}{2}} \right)^c \log \left( x - \frac{1}{2} \right) \left| \log \left( \frac{x}{x - \frac{1}{2}} \right) \right|^{-1} + \left( \frac{x}{x + \frac{1}{2}} \right)^c \log \left( x + \frac{1}{2} \right) \left| \log \left( \frac{x}{x + \frac{1}{2}} \right) \right|^{-1}.
\]
To bound the first term we can use
\[
\left| \log \left( \frac{x}{x - \frac{1}{2}} \right) \right|^{-1} < \left| \log \left( \frac{x}{x + \frac{1}{2}} \right) \right|^{-1} = \left( \log \left( \frac{x + \frac{1}{2}}{x} \right) \right)^{-1},
\]
where
\[
\log \left( \frac{x + \frac{1}{2}}{x} \right) = \int_{x}^{x+1/2} \frac{1}{t} dt > \frac{1}{2x + 1},
\]
and for \( x \geq e^{100} \) we have
\[
\left( \frac{x}{x - \frac{1}{2}} \right)^c < 1 + 10^{-43}.
\]
Combining these gives
\[
S_3 < 2(1 + 10^{-43})(2x + 1) \log (x + 1/2) < (4 + 10^{-20})x \log x.
\]
The estimates for \( S_2 \) and \( S_4 \) utilise Goldston’s method. The improvement largely comes from incorporating the Brun–Titchmarsh theorem for primes in intervals. For \( S_2 \), the inner sum is over \( p^k > x/\alpha \), and as we later need \( \alpha < 2 \) we can write
\[
S_2 = x^c \sum_{1 \leq k \leq \log x/\log 2} \sum_{p^k = \lfloor x/\alpha \rfloor + 1}^{\lfloor x \rfloor - 1} \frac{\log p}{p^k} \left| \log \frac{x}{p^k} \right|^{-1} < \alpha^c \log x \frac{1}{\alpha} \sum_{1 \leq k \leq \log x/\log 2} \sum_{p^k = \lfloor x/\alpha \rfloor + 1}^{\lfloor x \rfloor - 1} \left| \log \frac{x}{p^k} \right|^{-1}.
\]
Using the Taylor series for $\log(1-u)$ with $|u| < 1$,

$$
\left| \log \frac{x}{p^k} \right| = - \log \left( 1 - \frac{x - p^k}{x} \right) > \frac{x - p^k}{x} \left( 1 + \frac{x - p^k}{2x} \right),
$$

which results in

$$
S_2 < \alpha^c x \log \frac{x}{\alpha} \sum_{1 \leq k \leq \log x} \sum_{p^k = [x/\alpha]+1}^{[x]-1} \frac{1}{x - p^k} \left( 1 + \frac{x - p^k}{2x} \right)^{-1}.
$$

To estimate the double sum, we will consider the cases $k = 1$ and $k \geq 2$ separately. As $x$ is half an odd integer, it can always be placed in $(2m, 2m + 2)$ for some $m \in \mathbb{Z}^+$. When $k = 1$, we are summing over primes in $\left[ \left\lfloor x/\alpha \right\rfloor + 1, \lfloor x \rfloor - 1 \right]$. Since we have $\lfloor x \rfloor \leq 2m + 1$ and $\left\lfloor x/\alpha \right\rfloor + 1 \geq m + 1$ for $1 < \alpha < 2$, this interval is contained within $[m + 1, 2m]$ if we choose $\alpha < 2$. In this case,

$$
\sum_{p = \left\lfloor x/\alpha \right\rfloor + 1}^{\left\lfloor x \right\rfloor - 1} \frac{1}{x - p} \leq \sum_{m < p \leq 2m - 1} \frac{1}{2m - p}.
$$

Let $P(x, y)$ denote the number of primes in $(x - y, x]$, so that

$$
\sum_{m \leq p \leq 2m - 1} \frac{1}{2m - p} = \sum_{n = 1}^{m} \frac{1}{n} \left[ P(2m, n + 1) - P(2m, n) \right] \\
\leq \sum_{2 \leq n \leq x/\alpha} \frac{1}{n(n - 1)} P(2m, n) + \frac{\alpha}{x} P(2m, x/\alpha + 1).
$$

Montgomery and Vaughan’s [28] version of the Brun–Titchmarsh theorem implies

$$
P(x, y) \leq \frac{2y}{\log y}
$$

for $1 < y < x$. With this, and the Euler–Maclaurin formula for the resulting sum, we have

$$
\sum_{p = \left\lfloor x/\alpha \right\rfloor + 1}^{\left\lfloor x \right\rfloor - 1} \frac{1}{x - p} \leq \sum_{2 \leq n \leq x/\alpha} \frac{2}{(n - 1) \log n} + 2 \left( 1 + \frac{\alpha}{x} \right) \frac{1}{\log (\frac{x}{\alpha} + 1)} \\
\leq \sum_{3 \leq n \leq x/\alpha} \frac{2}{n \log n} + \frac{2}{\log 2} + \frac{1}{\log 3} + \frac{2}{\log (\frac{x}{\alpha} + 1)} \\
\leq 2 \log \log \frac{x}{\alpha} + 3.92 + \frac{1}{\log \frac{xK}{\alpha}} \left( \frac{\alpha}{x_{K}} + 2 \right),
$$

with the last line valid for $x \geq x_{K}$. Note that it would be possible to reduce the constant term by evaluating more terms in the sum over $n$. For example, if the sum were directly evaluated
for $2 \leq n \leq 30$, the constant term would drop to 3.69. The constant and lower-order term could also be reduced by using more terms in the Euler–Maclaurin expansion.

For $k \geq 2$, the inner sum can be estimated using an integral, with

$$\sum_{\frac{x}{p} \leq p^k \leq x} \frac{1}{x - p^k} \leq \int_{\frac{x}{2}}^{x-1} \frac{1}{x - y} d[y^{1/k}]$$

$$\leq \frac{1}{k} \int_{\frac{x}{2}}^{x-1} \frac{y^{-\frac{1}{k}}}{x - y} dy = \frac{1}{k} \int_{x-1}^{x} \frac{1}{y^{\frac{1}{k}}} \left( \log \left( \frac{1 + \sqrt{1 - 1/x}}{1 - \sqrt{1 - 1/x}} \right) - 2 \log(1 + \sqrt{2}) \right).$$

Summing over $k$,

$$\log \frac{x}{\alpha} \sum_{2 \leq k \leq \log x} \frac{\sum_{p^k = [x/\alpha] + 1}^{[x] - 1}}{x - p^k} < \frac{\log \frac{x}{\alpha}}{\sqrt{x}} \left( \log \left( \frac{1 + \sqrt{1 - 1/x}}{1 - \sqrt{1 - 1/x}} \right) - 2 \log(1 + \sqrt{2}) \right) \sum_{2 \leq k \leq \log x} \frac{1}{k}$$

$$< \frac{\log^2 x}{\sqrt{x}} \left( \log \left( \frac{\log x}{\log 2} \right) + \gamma + \frac{\log 2}{2\log x} - 1 \right).$$

Not only is this bound decreasing with $x$, it is practically negligible for sufficiently large $x$: less than $10^{-200}$ for $x \geq e^{1000}$. Combining the estimates for $k = 1$ and $k \geq 2$ gives

$$S_2 < 2\alpha^c x \log \frac{x}{\alpha} \left( \log \log \frac{x}{\alpha} + 1.96 + \frac{1}{2\log \frac{x}{\alpha}} \left( \frac{\alpha}{x} + 2 \right) \right)$$

$$< 2M_1 \alpha^c x \log \frac{x}{\alpha},$$

with

$$M_1 = 1 + \frac{1.96}{\log \log \frac{x}{\alpha}} + \frac{1}{2\log \frac{x}{\alpha} \log \log \frac{x}{\alpha}} \left( \frac{\alpha}{x} + 2 \right).$$

A similar method can be used for $S_4$, but with $x/p^k < 1$ and

$$\left| \log \frac{x}{p^k} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{p^k - x}{p^k} \right)^n > \frac{p^k - x}{p^k}.$$

We thus have

$$S_4 < x^c \sum_{1 \leq k \leq \log x} \frac{\sum_{p^k = [x]+2}^{[x]}}{p^k (p^k - x)}$$

$$< x \log x \sum_{1 \leq k \leq \log x} \frac{1}{k} \sum_{p^k = [x]+2}^{[x]} \frac{1}{p^k - x},$$
as the factor $p^{-k(c-1) \log p}$ is decreasing for $p \geq e^{\frac{\log x}{k}}$. For $k = 1$, the second sum at most covers primes in $(x + 1, \alpha x]$. It can be bounded by a sum over integers $n$, where the $n$ term is included if there is a prime in $(x + n, x + n + 1]$. Using (11), we find

$$\sum_{x+1<p \leq \alpha x} \frac{1}{p - x} \leq \sum_{1 \leq n \leq (\alpha - 1)x - 1} \frac{1}{n} \left[ P(x + n + 1, n + 1) - P(x + n, n) \right]$$

$$\leq \sum_{2 \leq n \leq (\alpha - 1)x - 1} \left( \frac{1}{n - 1} - \frac{1}{n} \right) P(x + n, n) + \frac{2(\alpha - 1)x}{((\alpha - 1)x - 1) \log((\alpha - 1)x)},$$

so that for $x \geq x_K$ we have

$$\sum_{x+1<p \leq \alpha x} \frac{1}{p - x} \leq \sum_{1 \leq n \leq (\alpha - 1)x - 2} \frac{2}{n \log(n + 1)} + \frac{2(\alpha - 1)x}{((\alpha - 1)x - 1) \log((\alpha - 1)x)}$$

$$\leq 2 \log \log((\alpha - 1)x) + 3.92 + \frac{2(\alpha - 1)x_K + 3}{(\alpha - 1)x_K \log((\alpha - 1)x_K)},$$

For $k \geq 2$, the inner sum can be estimated with

$$\sum_{x+1<p \leq \alpha x} \frac{1}{p^k - x} \leq \int_{x+1}^{\alpha x} \frac{1}{y - x} d[y^{1/k}]$$

$$\leq \frac{1}{k} \int_{x+1}^{\alpha x} \frac{y^{-\frac{1}{k}}}{y - x} dy = \frac{1}{k \sqrt{x}} \left( 2 \log \left( \sqrt{x + 1} + \sqrt{x} \right) + \log \left( \frac{\sqrt{\alpha - 1}}{\sqrt{\alpha + 1}} \right) \right)$$

$$\leq \frac{2 \log \left( \sqrt{x + 1} + \sqrt{x} \right)}{k \sqrt{x}}.$$

Thus giving

$$\sum_{2 \leq k \leq \log \frac{\alpha x}{\log 2}} \frac{1}{k} \sum_{x+1<p \leq \alpha x} \frac{1}{p^k - x} \leq \frac{2 \log \left( \sqrt{x + 1} + \sqrt{x} \right)}{\sqrt{x}} \sum_{2 \leq k \leq \log \frac{\alpha x}{\log 2}} \frac{1}{k^2},$$

where we can use

$$\sum_{2 \leq k \leq \frac{\log \alpha x}{\log 2}} \frac{1}{k^2} \leq \frac{\pi^2}{6} - 1 - \frac{\log 2}{\log 2 \alpha x}.$$

The overall bound on $S_4$, for $x \geq x_K$, is

$$S_4 < 2x \log x \left( \log \log((\alpha - 1)x) + 1.96 + \frac{(\alpha - 1)x_K + 3/2}{(\alpha - 1)x_K \log((\alpha - 1)x_K)} \right)$$

$$+ 2 \left( \frac{\pi^2}{6} - 1 \right) \sqrt{x} \log x \log(2\sqrt{x + 1})$$

$$< 2M_2x \log x \log \log((\alpha - 1)x)$$

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where
\[ M_2 = 1 + 1.96 \sqrt{x_K} + \left( \frac{\pi^2}{6} - 1 \right) \log(2\sqrt{x_K} + 1) + \frac{2x_K + 3(\alpha - 1)^{-1}}{2x_K \log((\alpha - 1)x_K) \log \log((\alpha - 1)x_K)}. \]

The lower bound on \( \alpha \) is also needed here, to ensure the last line is true for sufficiently large \( x \). It is worth noting that the constant 2 cannot be reduced by increasing the smallest \( x \) for which the result holds, as it comes directly from the constant in (11).

Combining the estimates for each \( S_i \) gives
\[
S(x) < 2M_1 \alpha^c x \log \frac{x}{\alpha} \log \log \frac{x}{\alpha} + 2M_2 x \log x \log((\alpha - 1)x) + \left( \frac{e}{\log \alpha} + 4 + 10^{-20} \right) x \log x 
< Mx \log x \log \log x,
\]
with
\[
M = \frac{2M_1 \alpha^c \log \frac{x}{\alpha} \log \log \frac{x}{\alpha}}{\log x_K \log \log x_K} + \frac{2M_2 \log((\alpha - 1)x_K)}{\log \log x_K} + \left( \frac{e}{\log \alpha} + 4 + 10^{-20} \right) \frac{1}{\log \log x_K}
\]
for all \( x \geq x_K \). As \( x_K \to \infty \), \( M \to 2(1 + \alpha) \), implying that smaller \( \alpha \) is preferable for larger \( x \). However, for intermediate values of \( x_K \), there will be an optimal value of \( \alpha \in (1, 2) \). Table 1 lists the smallest possible values of \( M \) for each \( x_K \) after optimising over \( \alpha \).

| \( \log x_K \) | 10³ | 10⁴ | 10⁵ | 10⁶ |
| --- | --- | --- | --- | --- |
| \( \alpha \) | 1.3933 | 1.3501 | 1.3186 | 1.2943 |
| \( M \) | 7.9074 | 7.1157 | 6.6260 | 6.2904 |

Table 1: Admissible values of \( M \), with optimised \( \alpha \), in Lemma 1.

Lemma 1 implies that for \( x \geq x_K \) we have
\[
\psi(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left( -\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^s}{s} ds + O\left( \frac{Mx \log x \log \log x}{\pi T} \right).
\]

Cauchy’s theorem can be used to evaluate the integral. From [13], we have
\[
|E(x, T)| < \frac{\zeta'(0)}{\zeta(0)} + \frac{1}{2} \log(1 - x^{-2}) + 2|I_2| + |I_3| + \frac{Mx \log x \log \log x}{\pi T},
\]
\[\text{There is a small typo in the bound for } |I_7| \text{ in [13], which has been corrected here.}\]
with, for some positive odd integer $U$,

$$|I_2| < \frac{2x \log T}{T-1} + \frac{9 + \log \sqrt{U^2 + (T+1)^2}}{\pi T x^U} + \frac{9 + \log \sqrt{U^2 + (T+1)^2}}{2\pi x(T-1)}$$

$$+ \frac{ex \left(\log^2 T + 20 \log T\right)}{2\pi(T-1) \log x} + \frac{e x \log x}{\pi (T-1)} = J_2(x, T, U)$$

and

$$|I_3| < \frac{9 + \log \sqrt{U^2 + T^2}}{\pi x^U} = J_3(x, T, U).$$

With these estimates, the error term can be simplified to

$$|E(x, T)| \leq \frac{K x \log x \log \log x}{T},$$

where

$$K = \frac{T}{x_K \log x_K \log \log x_K} \left(\frac{\zeta'(0)}{\zeta(0)} + \frac{\log(1 - x_K^{-2})}{2} + 2J_2(x_K, x_K, U) + J_3(x_K, x_K, U)\right) + \frac{M}{\pi}. $$

Although we are free to choose $U$, the size of $x_K$ is such that there is no apparent difference in $K$ as $U$ varies. We expect $J_2$ and $J_3$ to be minimised for small $U$ however, so we chose $U = 1$. The following table gives values of $K$ for specific $x_K$.

| $\log x_K$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|------------|--------|--------|--------|--------|
| $K$        | 3.4747 | 2.9814 | 2.6821 | 2.4798 |

Table 2: Admissible values of $K$ in (12) and Theorem 2.

The limiting value of $K$ is $M/\pi$. So, as $x \to \infty$, $K$ approaches $2(1 + \alpha)/\pi$. This could be reduced by refining the estimates for $I_2$ and $I_3$. However, even if $I_2$ and $I_3$ were zero, we find that $K$ only drops to around 2.5 for the estimate over $\log x \geq 1000$.

4 Primes between consecutive powers

There will be at least one prime in $(n^m, (n+1)^m)$ for all $n \geq n_0$ if there is a prime in $(x, x + mx^{1-\frac{1}{m}} + \ldots + mx^{\frac{1}{m}} + 1)$ for all $x \geq n_0^m$. Therefore, it will be sufficient to show there is at least one prime in $(x, x + mx^{1-\frac{1}{m}})$ for sufficiently large $x$. Discarding the smaller-order terms in the upper endpoint does not actually affect the final result, as they become relatively negligible at the values of $x$ we are interested in.
There will be a prime in $[x,x+h]$ if
\[ \theta(x+h) - \theta(x) = \sum_{x<p \leq x+h} \log p \] (13)
is positive. This can be translated to $\psi(x)$ with (8) and (9), in that we have
\[ \theta(x+h) - \theta(x) > \psi(x+h) - \psi(x) + 0.999 \sqrt{x + x^{1/3}} - a_1 \sqrt{x+h} - a_2(x+h)^{1/3} \] (14)
for $x \geq e^{1000}$, where $a_1$ and $a_2$ are given after (9). Theorem 2 gives us the estimate for $\psi(x)$, so for half odd integers $x \geq x_K$ and $50 < T < x$ we have
\[ \psi(x+h) - \psi(x) \geq h - \sum_{|\gamma|<T} \frac{(x+h)^{\rho} - x^{\rho}}{\rho} - K \frac{G(x,h)}{T} \] (15)
where $G(x,h) = (x+h) \log(x+h) \log(x+h) + x \log x \log \log x$, and values for $K$ and $x_K$ are in Table 2. Between the sum and last term, there should be some optimal value of $T$ which maximises the overall bound. To bound the sum in (15),
\[ \left| \sum_{|\gamma|<T} \frac{(x+h)^{\rho} - x^{\rho}}{\rho} \right| \leq \int_{x}^{x+h} u^{\rho-1} du \leq \int_{x}^{x+h} u^{\beta-1} du \leq hx^{\beta-1}. \]
More terms are possible in this bound, but it would not affect the final result. Hence we have
\[ \left| \sum_{|\gamma|<T} \frac{(x+h)^{\rho} - x^{\rho}}{\rho} \right| \leq h \sum_{|\gamma|<T} x^{\beta-1}. \] (16)
The sum can be estimated by writing
\[ \sum_{|\gamma|<T} (x^{\beta-1} - x^{-1}) = \sum_{|\gamma|<T} \int_{0}^{\beta} x^{\sigma-1} \log x d\sigma = \int_{0}^{1} \sum_{|\gamma|<T} x^{\sigma-1} \log x d\sigma, \]
which re-arranges to
\[ \sum_{|\gamma|<T} x^{\beta-1} = 2 \int_{0}^{1} x^{\sigma-1} \log x \sum_{0<\gamma<T} 1 d\sigma + 2x^{-1} \sum_{0<\gamma<T} 1. \]
Estimates for $N(T)$ and $N(\sigma,T)$ can be used for the two sums. We can also incorporate zero-free regions. For $\nu(T) = \max\{\nu_1(T), \nu_2(T)\}$, defined in (11) and (2), we have
\[ \sum_{|\gamma|<T} x^{\beta-1} = 2x^{-1} N(T) + \frac{2 \log x}{x} \left( \int_{0}^{1/2} N(T) x^{\sigma} d\sigma + \int_{1/2}^{1-\nu(T)} N(\sigma,T) x^{\sigma} d\sigma \right). \] (17)
The estimate in (11) for \( N(\sigma, T) \) is asymptotically smaller than (3) for \( \sigma > 5/8 \) and sufficiently large \( T \), so it may be useful to use \( N(\sigma) \) for some range of \( \sigma \in [1/2, 1) \). As such, we will re-write the two integrals in (17) to be split at some \( \sigma_1 \in [1/2, 1 - \nu(T)] \), i.e.

\[
\int_0^{\sigma_1} N(T)x^{\sigma}d\sigma + \int_{\sigma_1}^{1-\nu(T)} N_1(\sigma, T)x^{\sigma}d\sigma \leq \frac{T \log T}{2\pi} \left( \frac{x^{\sigma_1} - 1}{\log x} \right) + C_1 T^{8/3} \log^5 T \left( \frac{W^{1-\nu(T)} - W^{\sigma_1}}{\log W} \right) + C_2 \log^2 T \left( \frac{x^{1-\nu(T)} - x^{\sigma_1}}{\log x} \right),
\]

(18)

where \( C_1 = C_1(1) \), \( C_2 = C_2(\sigma_1) \), and \( W = x(T^{4\over 5} \log T)^{-2} \). Note that the choice of \( C_1 \) and \( C_2 \) is because \( C_1(\sigma) \) is increasing and \( C_2(\sigma) \) is decreasing as per (4.72) and (4.73) in [25].

To simplify the bound on (17), let \( T = x^\alpha \) for some \( \alpha \in (0,1) \). With (18) we have

\[
\sum_{|\gamma|<T} x^{\beta-1} < \frac{\alpha \log x}{\pi x^{1-\alpha-\sigma_1}} + 2C_1 \alpha^3 \left( \frac{W^{-\nu(x^\alpha)} - W^{\sigma_1-1}}{\log W} \right) \log^4 x + 2C_2 \alpha^2 \log^2 x \left( x^{-\nu(x^\alpha)} - x^{\sigma_1-1} \right).
\]

The negative term tending to zero as \( x \to \infty \) can be discarded, so that

\[
\sum_{|\gamma|<T} x^{\beta-1} < F(x) = \frac{\alpha \log x}{\pi x^{1-\alpha-\sigma_1}} + 2C_1 \alpha^3 \left( \frac{W^{-\nu(x^\alpha)} - W^{\sigma_1-1}}{\log W} \right) \log^4 x + 2C_2 \alpha^2 \log^2 x \left( x^{-\nu(x^\alpha)} \right),
\]

(19)

Returning to (14) with the bounds in (15) and (19),

\[
\theta(x + h) - \theta(x) > h \left[ 1 - F(x) - K \left( \frac{G(x, h)}{x^{\alpha}h} + \frac{E(x)}{h} \right) \right],
\]

(20)

where \( E(x) = 0.999x^{1/2} + x^{3/4} - a_1(x + h)^{1/2} - a_2(x + h)^{1/3} \).

It remains to optimise over \( \alpha \) and \( \sigma_1 \) to find the smallest \( m \) satisfying

\[
1 - F(x) - K \left( \frac{G(x, h)}{x^{\alpha}h} + \frac{E(x)}{h} \right) > 0
\]

(21)

for \( x \geq x_0 \). The LHS of (21) will only be positive and increasing if we take \( \alpha > 1/m \) and \( \sigma_1 < 1 - \alpha \). Of these two parameters, \( \alpha \) is far more influential. In fact, the size of \( \sigma_1 \) has a negligible effect on \( F(x) \) for large \( x \), as long as the second condition is true. We will use the computation in [30] and intervals in [10] to verify the \( m \)th powers interval for small \( x < x_0 \), so there is little need to optimise over \( \sigma_1 \). Hence, we will set \( \sigma_1 = 0.6 \), to consider any \( m \geq 3 \).

Values for \( C_1 \) and \( C_2 \) can be taken directly from Table 1 of [25]. However, this table can be updated with the latest Riemann height and a new explicit estimate for the squared divisor function: replacing (3.13) of [25] with Theorem 2 of [11]. A correction also needs to be
made to Lemma 3.2 of [25] — see Remark 1.4 in [16] for more detail. Making these changes, we can take \( C_1 = 17.418 \) (computed as in Lemma 2.6 of [23]) and \( C_2 = 5.272 \).

For interest, the smallest feasible interval result from this method is consecutive cubes. Taking \( m = 3 \) and \( \alpha = 1/3 + 10^{-10} \), (21) holds for \( x \geq \exp(\exp(33.990)) \). This result does hold for all such \( x \), despite having only considered half odd integer \( x \). The result for those \( x \) would imply primes in \((x, X + 3X^{2/3}]\) for \( X = [x + 0.5] + 0.5 \) and any \( x \) in the admissible range. The upper endpoint is always less than \( x + 3x^{2/3} + 3x^{1/3} + 1 \), therefore, we have primes between \( n^3 \) and \((n + 1)^3\) for all \( n \geq \exp(\exp(32.892)) \).

An interesting aspect of the result for cubes is that it only uses \( \nu_2(T) \) in the zero-free region. We find that \( \nu_1(T) \) is a better estimate than \( \nu_2(T) \) over \( T \leq e^{54594.17} = \lambda \), and thus becomes useful when we consider larger \( m \), as (21) will hold over smaller \( x, \alpha \), and hence \( T \). Because of how \( \nu(T) \) is defined, the LHS of (21) decreases to a local minimum at \( \lambda^{1/\alpha} \) as \( x \) decreases, then increases as the zero-free region switches from using \( \nu_2(T) \) to \( \nu_1(T) \). If this minimum is still positive, the smallest \( x \) for which (21) holds will be a function of \( \nu_1(T) \). Therefore, to solve (21) for large \( m \), we need an \( \alpha > 1/m \) for which (21) is positive at \( x = \lambda^{1/\alpha} \). Moreover, we want the largest such \( \alpha \), as this will maximise the LHS of (21).

We could now find the smallest \( m \) for which (21) holds for all \( x \geq 1 \). However, the intervals in [10] can be smaller than an \( m^{th} \)-powers interval for small \( x \). This means that (21) need only hold for \( x \geq x_0 \) if the intervals in [10] verify the \( m^{th} \)-powers interval for \( x < x_0 \). The smallest \( m \) for which this was found to work was \( m = 155 \). From Table 2 we can take \( K = 3.4747 \) over \( \log x \geq 1000 \), and with \( \alpha = 0.0080146 \) we find that (21) holds for all \( \log x \geq e^{4810} \). The intervals in [10] for \( x \geq 4 \cdot 10^{18} \) and \( x \geq e^{1200} \) are smaller than that of consecutive 155\(^{th}\) powers for \( 4 \cdot 10^{18} \leq x \leq e^{4850} \), by solving \( x(1 - \Delta^{-1})^{-1} \leq x + mx^{1-1/m} \) for \( x \). The computation in [30] verifies the interval for the remaining \( x \leq 4 \cdot 10^{18} \). Thus, we can say there is at least one prime in \((n^{155}, (n + 1)^{155})\) for all \( n \geq 1 \).

5 Discussion

Theorem 1 is largely determined by the estimates for the zero-free region, zero-counting function, zero-density function, and error term in the truncated Riemann–von Mangoldt explicit formula. Although, some of these estimates are more influential than others. Most notably, a smaller constant in the zero-free region is more likely to affect the results than feasible asymptotic improvements in the zero-density estimate.

Reducing the constants in Kadiri, Lumley, and Ng’s zero-density estimate (4) does affect Theorem 1, albeit less so for smaller powers. However, to widen the range for which we have primes between consecutive cubes, a smaller power of \( T \) in (4) would be needed. Simonić’s estimate in (5) has just this, so it would have been possible to split the second integral in

\[ C_2 \text{ was computed using } \{k, \mu, \alpha, \delta, d\} = \{1, 1.2362, 0.2419, 0.3025, 0.3485\} \text{ in [25, Thm. 1.1].} \]
to incorporate (5) over some range of $\sigma \in [1/2, 37/58]$. However, not even $N(\sigma, T) = 0$ in this range would have made a difference to Theorem 1. This is owing to the choice of $\sigma_1$ in (21); we took $\sigma_1 < 1 - 1/m$ to make $F(x)$ of (19) decreasing in the long-run. This makes the terms from the trivial estimate and Simonič’s estimate negligible compared to the main term. Therefore, a better zero-density estimate will only be useful for this method if it can be used for $\sigma$ up to the zero-free region.

The zero-free regions are arguably the most influential ingredients in the proof. A smaller constant in either would affect Theorem 1. For example, if it were possible to take $c = 50$ in (2), it would give primes between consecutive $150^{th}$ powers.

Using the explicit version of Goldston’s estimate in place of (7) affected both Theorem 1 and the range for consecutive cubes. This suggests it would be worth refining this estimate. The author and Johnston [8] recently worked on an explicit version of an estimate from Wolke [43] and Ramaré [35], which looks likely to improve Theorem 1. Theorem 1.2 of [8] would allow us to take $E(x, T) = O(x/T)$ in (6) for $\log x \leq T \leq \sqrt{x}$.

The tactics to cover small $x$ are another important aspect of the result. The intervals in [10] were used to verify the $m^{th}$ powers interval over small $x$, and allowed a much smaller value for $m$ than if only the condition of (21) were used. If the results of [10] were not as strong, a combination of intervals could have been used, to differentially cover smaller values of $x$. A good option for some mid-range of $x$ would be intervals of the form

$$\left(x, x + \frac{C_k x}{\log^k x}\right),$$

which contain a prime for all $x \geq x_1$, given any positive integer $k$, and $C_k > 0$ determined by $k$ and $x_1$. These are implied by PNT estimates of the form

$$\frac{|\psi(x) - x|}{x} \leq A \left(\frac{\log x}{R}\right)^B \exp\left(-C \sqrt{\frac{\log x}{R}}\right),$$

with positive constants $A$, $B$, $C$, and $R$, given explicitly in [39, Thm. 11] and [34, Thm. 1], among others. See, for example, Corollary 5.5 in [14]. Say (21) held for all $x \geq x_0$. Then, adjusting $k$ as needed and starting at $x = x_0$, intervals of the form (22) could incrementally verify an $m^{th}$-powers interval for $x_1 \leq x \leq x_0$. This method was not needed in the present work, however, because the range covered by the intervals deduced from (23) did not extend past that of the intervals of [10].

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