ESTIMATING THE GREATEST COMMON DIVISOR
OF THE VALUE OF TWO POLYNOMIALS

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Abstract. Let \( p \) be a fixed prime, and let \( v(a) \) stand for the exponent of \( p \) in the prime factorization of the integer \( a \). Let \( f \) and \( g \) be two monic polynomials with integer coefficients and nonzero resultant \( r \). Write \( S \) for the maximum of \( v(\gcd(f(n), g(n))) \) over all integers \( n \). It is known that \( S \leq v(r) \). We give various lower and upper bounds for the least possible value of \( v(r) - S \) provided that a given power \( p^s \) divides both \( f(n) \) and \( g(n) \) for all \( n \). In particular, the least possible value is \( ps^2 - s \) for \( s \leq p \) and is asymptotically \( (p - 1)s^2 \) for large \( s \).

Let \( f, g \in \mathbb{Z}[x] \) be monic polynomials with nonzero resultant \( r \). Our interest is in the range of the greatest common divisor of \( f(n) \) and \( g(n) \) as \( n \) varies in \( \mathbb{Z} \). In the recent paper \[1\] by J. Pelikán and the first author, it was shown\[1\] that

1. \( \gcd(f(n), g(n)) \) divides \( r \) for all \( n \); moreover,
2. for square-free \( r \), its range is the set of all (positive) divisors of \( r \);
3. If \( r \) is allowed to have square divisors, then \( |r| \) need not be in the range. For example, \( f(x) = x^2 + 1 \) and \( g(x) = x^2 - 1 \) have resultant 4 but never have gcd 4.
4. If \( r \) has no divisors of the form \( p^2 \) with \( p \) prime, then 1 appears in the range.

For statement (3), there is an even worse example with resultant 4: \( f(x) = x^2 + x + 1 \) and \( g(x) = x^2 + x - 1 \) have \( f(n) \) and \( g(n) \) coprime for all \( n \). For statement (4) with the condition on \( r \) removed, there again is a counterexample with resultant 4: \( f(x) = x^2 + x + 2 \) and \( g(x) = x^2 + x \) have \( \gcd(f(n), g(n)) = 2 \) for all \( n \). On the other hand, it will turn out that if \( r \) is in the range, then so are all its divisors; see Theorem \[6\] below.

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\[1\]Statement \[4\] was essentially known before, cf. \[2, 6\]
In the present paper, we undertake a refined study of the case when \( r \) can have prime power divisors with high exponents. Fix a prime \( p \), and let \( v(a) \) stand for the exponent of \( p \) in the prime factorization of the integer \( a \). It suffices to study the range of \( v(\gcd(f(n), g(n))) \), since if we understand this for all \( p \), then the Chinese remainder theorem allows us to read off the range of \( \gcd(f(n), g(n)) \).

Write \( S \) for the maximum of \( v(\gcd(f(n), g(n))) \) as \( n \) varies in \( \mathbb{Z} \). By \([1,\text{Proposition 2(a)}]\), we have \( S \leq v(r) \). Our main goal is to estimate the least possible value of \( v(r) - S \) provided that \( v(\gcd(f(n), g(n))) \geq s \) for all \( n \). We develop two different methods. Up to Theorem 3, we use the definition of the resultant in terms of the coefficients of \( f \) and \( g \), while from Construction 4 on, we use the equivalent definition in terms of the roots of \( f \) and \( g \).

Let
\[
(1) \quad f(x) = a_0x^k + a_1x^{k-1} + \cdots + a_k
\]
and
\[
(2) \quad g(x) = b_0x^l + b_1x^{l-1} + \cdots + b_l,
\]
where \( a_0 = b_0 = 1 \). Recall that, by definition, \( r \) is the determinant of the Sylvester matrix
\[
(3) \quad M = \begin{pmatrix}
a_0 & a_1 & \cdots & a_k \\
a_0 & a_1 & \cdots & \cdots & a_k \\
\vdots & \vdots & \ddots & \cdots & \cdots \\
b_0 & b_1 & \cdots & b_l \\
b_0 & b_1 & \cdots & \cdots & b_l
\end{pmatrix}
\]
of the two polynomials. Note that \( M \) is an \((l + k)\)-square matrix; the first \( l \) rows are built from the coefficients of \( f \), and the last \( k \) rows are built from the coefficients of \( g \), padded with zeros.

We shall need the following interpretation of the resultant.

**Lemma 1.** If \( f \) and \( g \) are monic polynomials with integer coefficients and nonzero resultant \( r \), then \( |r| = |\mathbb{Z}[x]/(f, g)| \), where \((f, g)\) stands for the ideal generated by \( f \) and \( g \).

Note that for \( r = 0 \) (which is excluded throughout this paper), we would have \( |\mathbb{Z}/(f, g)| = \infty \) because \( f \) and \( g \) would have a nonconstant common divisor in \( \mathbb{Z}[x] \).

Note also that Lemma \([1]\) implies \([1,\text{Proposition 2(a)}]\): the greatest common divisor \((f(n), g(n))\) divides the resultant \( r \). Indeed, there is a surjective ring homomorphism from \( \mathbb{Z}[x]/(f, g) \) onto \( \mathbb{Z}/(f(n), g(n)) \).

The statement and proof of Lemma \([1]\) are reminiscent of \([3,\text{Theorem 1.19}]\), which was reproved as \([1,\text{Theorem 5}]\). In that theorem, the
coefficients come from a field $F$, and the claim is that the corank of the Sylvester matrix $M$ is the dimension over $F$ of the quotient ring $F[x]/(f, g)$, i.e., the degree of the polynomial $\gcd(f, g)$.

**Proof.** Let us identify the free Abelian group $\mathbb{Z}^{k+l}$ with the additive group $\mathbb{Z}[x]_{<k+l}$ of polynomials of degree less than $k + l$ with integer coefficients. Let any such polynomial correspond to the list of its coefficients, starting with the coefficient of $x^k$ and ending with the constant term.

Under this correspondence, the subgroup generated by the rows of the Sylvester matrix $M$ is identified with the set of polynomials of the form $\phi f + \psi g$, where $\phi, \psi \in \mathbb{Z}[x]$ have degree less than $l$ and $k$, respectively. Any polynomial of this form is in $(f, g)$. Conversely, any element of $(f, g)$ of degree less than $k + l$ is an integral linear combination of the rows. To see this, we first write such a polynomial as $\phi_0 f + \psi_0 g$, where we know nothing about the degree of $\phi_0, \psi_0 \in \mathbb{Z}[x]$, but then we write $\phi_0 = q g + \phi$ with $\phi$ of degree less than $l$, and we define $\psi = qf + \psi_0$. Then $\phi_0 f + \psi_0 g = \phi f + \psi g$; moreover, this polynomial and $\phi f$ both have degree less than $k + l$, whence so does $\psi g$, showing that $\psi$ has degree less than $k$.

Thus, the subgroup of $\mathbb{Z}^{k+l}$ generated by the rows of $M$ is identified with the degree $< k + l$ part $(f, g)_{<k+l}$ of the ideal $(f, g)$ of $\mathbb{Z}[x]$. The determinant $r$ of $M$ is the signed volume of the parallelootope spanned by the rows, therefore $|r|$ is the volume of this parallelootope, which is the cardinality of the quotient

$$\mathbb{Z}^{k+l}/\langle \text{rows of } M \rangle \simeq \mathbb{Z}[x]_{<k+l}/(f, g)_{<k+l} \simeq (\langle f, g \rangle + \mathbb{Z}[x]_{<k+l})/(f, g) = \mathbb{Z}[x]/(f, g).$$

\[\square\]

For integers $S \geq s \geq 0$, let

$$I_{S, s} = \{ f \in \mathbb{Z}[x] : p^s \mid f(n) \text{ for all } n, \text{ and } p^S \mid f(0) \}.$$ 

This is an ideal of $\mathbb{Z}[x]$. Put $R_{S, s} = \mathbb{Z}[x]/I_{S, s}$. The cardinality of this quotient ring will play a central role in our computations. The cardinality can be expressed in terms of the functions

$$\alpha(j) = v(j!) = \left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{j}{p^2} \right\rfloor + \left\lfloor \frac{j}{p^3} \right\rfloor + \ldots$$

and $\beta(m) = \min\{ j : \alpha(j) \geq m \}$. Put $B(s) = \sum_{m=1}^{\alpha(s)} \beta(m)$.

Note that $\alpha$ is superadditive:

$$\alpha(j_1 + j_2) \geq \alpha(j_1) + \alpha(j_2)$$

for all nonnegative integers $j_1$ and $j_2$. It follows that $\beta$ is subadditive:

$$\beta(m_1 + m_2) \leq \beta(m_1) + \beta(m_2)$$

for all nonnegative integers $m_1$ and $m_2$. 

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Note also that \( \alpha(j) = \lfloor j/p \rfloor \) for \( 0 \leq j < p^2 \), and \( \alpha(p^2) = p+1 \), whence 
\( \beta(m) = pm \) for \( 1 \leq m \leq p \) and \( B(s) = p^{(s+1)/2} \) for \( 1 \leq s \leq p \). On the other hand, \( \alpha(j) \sim j/(p-1) \) for large \( j \), whence \( \beta(m) \sim (p-1)m \) for large \( m \) and \( B(s) \sim (p-1)s^2/2 \) for large \( s \).

**Lemma 2.** We have

\[
|R_{s,s}| = p^{S-s+B(s)}.
\]

*Proof.* For \( S = s \), the ring \( R_{s,s} = R_{s,s} \) is the ring of polynomial functions \( \mathbb{Z}/(p^s) \to \mathbb{Z}/(p^s) \). By a classical result of Kempner \([5]\), reproved by Keller and Olson \([4, Corollary 2.2]\), this ring has cardinality \( p^{B(s)} \).

For \( S \geq s \), observe that \( I_{s,s} \) is the kernel of the map \( I_{s,s} \to \mathbb{Z}/(p^S) \), \( f \mapsto f(0) \). The image of this map is \( (p^s)/(p^S) \), whence \( |I_{s,s}/I_{s,s}| = p^{S-s} \). But \( I_{s,s}/I_{s,s} \) is the kernel of the surjective map \( R_{s,s} \to R_{s,s} \), therefore \( |R_{s,s}/|R_{s,s}| = p^{S-s} \) and the Lemma follows. \( \square \)

The first main result of this paper is the following refinement of \([1, Proposition 8(a)]\).

**Theorem 3.** Let \( f \) and \( g \) be monic polynomials with integer coefficients and nonzero resultant \( r \). Assume that a fixed prime power \( p^s \) divides both \( f(n) \) and \( g(n) \) for all \( n \). Let

\[
S = \max_{n \in \mathbb{Z}} v(\gcd(f(n), g(n))).
\]

Then \( v(r) - S \geq B(s+t) - 2B(t) - s \) for all nonnegative integers \( t \).

*Proof.* The resultant being translation invariant, we may and do assume that \( p^S \) divides \( \gcd(f(0), g(0)) \). Using Lemma 1, we have

\[
v(r) = v\left(\left|\mathbb{Z}[x]/(f, g)\right|\right) \geq v\left(\left|\mathbb{Z}[x]/((f, g) + I_{s+t,s+t})\right|\right) = v\left(\left|R_{s+t,s+t}/(\bar{f}, \bar{g})\right|\right),
\]

where \( \bar{f} \) and \( \bar{g} \) are the natural images in \( R_{s+t,s+t} \) of \( f \) and \( g \), respectively. Now observe that in the \( \mathbb{Z}[x] \)-module \( R_{s+t,s+t} \), both elements \( \bar{f} \) and \( \bar{g} \) are annihilated by the ideal \( I_{t,t} \). Hence \( v\left(\left|\bar{f}\right|\right) \leq v\left(\left|R_{t,t}\right|\right) = B(t) \) by Lemma 2 and similarly for \( \bar{g} \). Now

\[
v\left(\left|\bar{f}, \bar{g}\right|\right) = v\left(\left|\bar{f}\right|\right) + v\left(\left|\bar{g}\right|\right) - v\left(\left|\bar{f} \cap \bar{g}\right|\right) \leq 2B(t),
\]

whence

\[
v(r) \geq v\left(\left|R_{s+t,s+t}\right|\right) - v\left(\left|\bar{f}, \bar{g}\right|\right) \geq (S + t) - (s + t) + B(s + t) - 2B(t)
\]

and the Theorem follows. \( \square \)

For \( s = 1 \), we may choose \( t = 0 \) in Theorem 3, to get \( v(r) \geq S + p - 1 \geq p \), which recovers \([1, Proposition 8(a)]\). For general \( s \geq 0 \), choosing \( t = s \), we get \( v(r) - S \geq B(2s) - 2B(s) - s \). When \( s \leq p/2 \), we have \( B(s) = p^{(s+1)/2} \) and \( B(2s) = p^{(2s+1)/2} \), whence \( v(r) - S \geq ps^2 - s \). It shall follow from Theorem 4 and Construction 8 that this lower bound holds true, and is sharp, even under the weaker assumption
that \( s \leq p \). On the other hand, for large \( s \), we have \( B(s) \sim (p - 1)s^2/2 \) and \( B(2s) \sim 2(p - 1)s^2 \), whence \( v(r) - S \gtrsim (p - 1)s^2 \). We now present a construction showing that this is asymptotically sharp for any fixed \( p \).

**Construction 4.** Consider the polynomials

\[
    f(x) := \prod_{j=0}^{\beta(s)-1} (x - j); \\
    g(x) := p^s + \prod_{i=0}^{p-1} (x - i)^{s+1}
\]

for an integer \( s \geq 0 \). Then \( v(\gcd(f(n), g(n))) = s \) for all integers \( n \). For the resultant \( r \), we have \( v(r) = s\beta(s) \), whence \( v(r) - s = s(\beta(s) - 1) \sim (p - 1)s^2 \) when \( s \gg p \).

**Proof.** Firstly, note that \( f(\beta(s)) = \beta(s)! \) divides \( f(n) \) for any integer \( n \) since the binomial coefficient \( \binom{n}{\beta(s)} = f(n)/\beta(s)! \) is an integer. Therefore, we have \( s \leq \alpha(\beta(s)) = v(\beta(s)! \leq v(f(n)) \). On the other hand, we have \( v(g(n)) = s \) for all \( n \) since \( p^s+1 \) divides \( \prod_{i=0}^{p-1} (n - i)^{s+1} \)

for any integer \( n \). Hence the statement on \( v(\gcd(f(n), g(n))) \). Further, we compute

\[
    v(r) = v \left( \prod_{j=0}^{\beta(s)-1} g(j) \right) = \sum_{j=0}^{\beta(s)-1} v(g(j)) = s\beta(s).
\]

Let us return to the notations and conditions of Theorem 3. In the rest of this paper, our main goal is to obtain a sharp lower bound for \( v(r) - S \) when \( s \leq p \). For this, we recall a bit of \( p \)-adic number theory. Let \( K \) be the splitting field of the product \( f \) over the field \( \mathbb{Q}_p \) of \( p \)-adic numbers for the fixed prime \( p \). So we may write \( f(x) = \prod_{i=1}^{k} (x - \gamma_i) \) and \( g(x) = \prod_{j=1}^{l} (x - \delta_j) \) with \( \gamma_i, \delta_j \in O \) \( (i = 1, \ldots, k; \ j = 1, \ldots, l) \), where \( O \) denotes the valuation ring in \( K \) with uniformizer \( \pi \) and residue field \( F = O/(\pi) \). We put \( e = v_\tau(p) \) for the absolute ramification index of \( K \), where \( v_\tau \) stands for the \( \tau \)-adic valuation. We extend the \( p \)-adic valuation \( v \) to \( K \) by putting \( v = v_\tau/e \). In particular, we have \( v(\pi) = 1/e \), and the \( v \)-value of any element of \( O \) is a nonnegative integer multiple of \( 1/e \). We have \( e \cdot |F : F_p| = |K : \mathbb{Q}_p| \), but this will not be used in the sequel.

For integers \( n \in \mathbb{Z} \) and \( 0 \leq s \in \mathbb{Z} \), the value \( f(n) \in \mathbb{Z} \) is divisible by \( p^s \) if and only if \( \sum_{i=1}^{k} v(n - \gamma_i) \geq s \). On the other hand, the resultant of \( f \) and \( g \) equals

\[
    r = \prod_{i,j} (\gamma_i - \delta_j) \in \mathbb{Z}.
\]
For any fixed \( n \in \mathbb{Z} \), we have the following trivial estimate for the \( p \)-adic valuation of \( r \):

\[
(4) \quad v(r) = \sum_{i,j} v(\gamma_i - \delta_j) \geq \sum_{i,j} \min(v(n - \gamma_i), v(n - \delta_j)).
\]

Note that the above trivial estimate again implies [1, Proposition 2(a)]: the greatest common divisor \((f(n), g(n))\) divides the resultant \( r \). Indeed, it suffices to check this locally, i.e.,

\[
v(\gcd(f(n), g(n))) = \min(v(f(n)), v(g(n))) = \min\left(\sum_i v(n - \gamma_i), \sum_j v(n - \delta_j)\right) \leq v(r)
\]

for all primes \( p \). The latter inequality follows easily from [4] by choosing a maximum among the multiset

\[
\{v(n - \gamma_i), v(n - \delta_j) \mid 1 \leq i \leq k, 1 \leq j \leq l\}.
\]

In order to estimate this further from below, we need the following lemma stating (in the special case of \( I = \emptyset \)) that whenever \( s \leq p \) and \( f(n) \) is divisible by \( p^s \) for all \( n \), then there are at least \( s \) roots of \( f \) in \( \mathbb{Q}_p \) congruent to each integer modulo \( p \).

**Lemma 5.** Let \( m \in \mathbb{Z} \) be a fixed integer, and let \( I \subseteq \{1, \ldots, k\} \) be an arbitrary subset such that for all \( i \in I \) we have \( v(m - \gamma_i) \notin \mathbb{Z} \). Further, let \( 0 \leq t_I < p \) be the number of indices \( i \in \{1, \ldots, k\} \setminus I \) with \( v(m - \gamma_i) > 0 \). Then there exists an integer \( n \in \mathbb{Z} \) such that \( n \equiv m \pmod{p} \) and \( v(f(n)) \leq \sum_{i \in I} v(m - \gamma_i) + t_I \).

**Proof.** First of all, note that

\[
v(f(n)) = \sum_{i=1}^{k} v(n - \gamma_i) = \sum_{i \in I} v(n - \gamma_i) + \sum_{i \in \{1, \ldots, k\} \setminus I} v(n - \gamma_i).
\]

On the one hand, for any integer \( n \in \mathbb{Z} \) and \( i \in I \), we have \( v(n-m) \in \mathbb{Z} \), whence \( v(n-m) \neq v(m - \gamma_i) \), as the latter is not an integer by assumption. So we compute

\[
v(n-\gamma_i) = v((n-m)+(m-\gamma_i)) = \min(v(n-m), v(m-\gamma_i)) \leq v(m-\gamma_i).
\]

On the other hand, we want to pick \( n \in \mathbb{Z} \) in such a way that we can estimate

\[
\sum_{i \in \{1, \ldots, k\} \setminus I} v(n - \gamma_i)
\]
efficiently. We have to have \( n \equiv m \pmod{p} \), and we choose \( n \) modulo \( p^2 \) so that all indices \( i \in \{1, \ldots, k\} \setminus I \) satisfy \( v(n-\gamma_i) \leq 1 \) (equivalently, \( < 1 + 1/e \)). Indeed, we can achieve this by the pigeonhole principle: there are \( p \) choices for \( n \) mod \( p^2 \) and these are pairwise incongruent mod \( p^{e+1} \), so any element \( \gamma \in \mathcal{O} \) can only be congruent to one of these choices modulo \( p^{e+1} \).
This way we obtain an integer \( n \equiv m \pmod{p} \) such that
\[
v(f(n)) = \sum_{i=1}^{k} v(n - \gamma_i) \leq \\
\leq \sum_{i \in I} v(m - \gamma_i) + \sum_{i \notin I, v(m - \gamma_i) > 0} 1 = \sum_{i \in I} v(m - \gamma_i) + t_1
\]
as desired. \( \square \)

The second main result of this paper is the following refinement of Proposition 8(a).

**Theorem 6.** Let \( f \) and \( g \) be monic polynomials with integer coefficients and nonzero resultant \( r \). Assume that \( s \leq p \) and that the power \( p^s \) divides both \( f(n) \) and \( g(n) \) for all \( n \). Let
\[ S = \max_{n \in \mathbb{Z}} v(\gcd(f(n), g(n))). \]

(a) We have
\[ v(r) - S \geq ps^2 - s. \]

(b) If equality holds here, then \( v(\gcd(f(n), g(n))) \) takes all the integer values in the interval \([s, S]\).

**Proof.** We may assume without loss of generality that
\[ v(\gcd(f(0), g(0))) = S. \]

Fix an integer \( m \in \mathbb{Z} \), and set \( a_i = v(m - \gamma_i) \) and \( b_j = v(m - \delta_j) \) \((i = 1, \ldots, k; j = 1, \ldots, l)\). By assumption, \( p^s \) divides \( \gcd(f(m), g(m)) \), so we have \( \sum_{i=1}^{k} a_i \geq s \) and \( \sum_{j=1}^{l} b_j \geq s \).

We may assume without loss of generality (possibly swapping \( f \) and \( g \) and permuting their roots) that the maximum of
\[ \{a_i, b_j \mid 1 \leq i \leq k, 1 \leq j \leq l\} \]
is achieved at \( b_l \).

**Lemma 7.** (a) We have
\[
\sum_{i,j: \gamma_i \equiv m \equiv \delta_j \pmod{\pi}} v(\gamma_i - \delta_j) \geq \begin{cases} 
 s^2 & (m \in \mathbb{Z}) \\
 s^2 - s + S & (m = 0). 
\end{cases}
\]

(b) If equality holds for \( m = 0 \), then either \( S = s \), or all of the following hold:
\[ b_l \geq S - s + \text{sgn } s, \]
\[ b_j \leq \text{sgn } s \quad \text{for all } j < l, \]
and
\[ \sum_{j=1}^{l-1} b_j = s - \text{sgn } s. \]
Here \( \text{sgn} \, 0 = 0 \) and \( \text{sgn} \, s = 1 \) for \( s \geq 1 \).

**Proof.** (a) We have \( v(\gamma_i - \delta_j) \geq \min(a_i, b_j) \) as before. Note that whenever \( m \not\equiv \gamma_i \mod \pi \) or \( m \not\equiv \delta_j \mod \pi \), then \( \min(a_i, b_j) \) vanishes. Hence we obtain

\[
\sum_{i,j: \gamma_i \equiv m \equiv \delta_j \mod \pi} v(\gamma_i - \delta_j) \geq \sum_{j=1}^{l} \sum_{i=1}^{k} \min(a_i, b_j)
\]

by adding these together. Fix \( j \in \{1, \ldots, l\} \) for now, and put

\[
I_j := \{ i \in \{1, \ldots, k\} \mid a_i \leq \min(1, b_j) \text{ and } a_i \notin \mathbb{Z} \}.
\]

Let \( t_j \) be the number of indices \( i \in \{1, \ldots, k\} \setminus I_j \) such that \( a_i \neq 0 \). Applying Lemma 5 to the subset \( I := I_j \), we find

\[
s \leq \sum_{i \in I_j} a_i + t_j.
\]

On the other hand, for any \( i \in \{1, \ldots, k\} \setminus I_j \) with \( a_i \neq 0 \), we have \( a_i \geq \min(1, b_j) \), so

\[
\sum_{i=1}^{k} \min(a_i, b_j) \geq \sum_{i \in I_j} a_i + t_j \min(1, b_j) \geq \\
\left( \sum_{i \in I_j} a_i + t_j \right) \min(1, b_j) \geq s \min(1, b_j).
\]

Now Lemma 5 applied to the polynomial \( g \) and to the subset

\[
I := \{ j \in \{1, \ldots, n\} \mid 0 < b_j < 1 \}
\]

yields

\[
s \leq \sum_{j \in I} b_j + t_I \leq \sum_{j=1}^{l} \min(1, b_j).
\]

The first statement in (a) is a combination of (5), (6), and (7).

Let \( m = 0 \). By the maximality of \( b_l \), we have

\[
\sum_{i=1}^{k} \min(a_i, b_l) = \sum_{i=1}^{k} a_i = v(f(0)) \geq S.
\]

Also,

\[
1 + \sum_{j=1}^{l-1} \min(1, b_j) \geq \sum_{j=1}^{l} \min(1, b_j) \geq s.
\]
This yields
\[ \sum_{j=1}^{l} \sum_{i=1}^{k} \min(a_i, b_j) = \sum_{j=1}^{l-1} \sum_{i=1}^{k} \min(a_i, b_j) + \sum_{i=1}^{k} \min(a_i, b_l) \geq \sum_{j=1}^{l-1} \min(1, b_j) + S \geq s(s - 1) + S \]
as desired.

(b) Fix \( j < l \). To have equality in the last chain of inequalities, we must have equality in (6), whence \( \min(a_i, b_j) = \min(1, b_j) \) for all \( i \) such that \( i \not\in I_j \) and \( a_i > 0 \). We must also have \( \sum_{i=1}^{k} a_i = S \) and, in case \( s \geq 1 \), we must have \( b_l \geq 1 \) and \( \sum_{j=1}^{l} \min(1, b_j) = s \).

If \( b_l > 1 \) for some \( j < l \), then \( a_i = 1 \) for all \( i \) such that \( i \not\in I_j \) and \( a_i > 0 \), which means that \( a_i \leq 1 \) for all \( i \). But (6) holds with equality, so we have \( \sum_{i=1}^{k} a_i = s \), whence \( S = s \).

If \( b_l \leq 1 \) for all \( j < l \), then \( \min(1, b_j) = b_j \) for all \( j < l \), hence
\[ S \leq v(g(0)) = \sum_{j=1}^{l} b_j = b_l + \sum_{j=1}^{l-1} \min(1, b_j). \]
If \( s \geq 1 \), then this is \( b_l - 1 + s \), and \( b_l \geq S - s + 1 \) follows. If \( s = 0 \), then, since (6) holds with equality, we deduce either \( a_1 = \cdots = a_k = 0 \) and therefore \( S = 0 = s \), or \( b_l = \cdots = b_{l-1} = 0 \) and therefore \( b_l \geq S \).

Adding up the estimates of Lemma (7a) for \( m = 0, 1, \ldots, p - 1 \), we deduce Theorem (6a). For (b), observe that the value \( S \) is obviously taken. Observe also that if \( v(r) - S = ps^2 - s \), then the value \( s \) is also taken, for otherwise Theorem (6a) yields
\[ v(r) - S \geq p(s + 1)^2 - (s + 1), \]
a contradiction. Moreover, equality holds in Lemma (7a) for all \( m \), in particular, for \( m = 0 \). Thus, Lemma (7b) applies. If \( S = s \), then Theorem (6b) obviously holds. We treat the other case given in Lemma (7b). Let \( \text{sgn} s < u < S - s \). sgn \( s \).

We have \( v(p^u - \delta_i) = u \) and \( v(p^u - \delta_j) = b_j \) for all \( 1 \leq j \leq l - 1 \). So we compute
\[ v(g(p^u)) = \sum_{j=1}^{l} v(p^u - \delta_j) = u + \sum_{j=1}^{l-1} b_j = u + s - \text{sgn} s. \]

We have \( v(f(p^u)) \geq u \), but also \( v(f(p^u)) \geq s + u - 1 \). To prove the latter, we distinguish two cases. If \( a_i \leq u \) for all \( 1 \leq i \leq k \), then \( v(p^u - \gamma_i) \geq a_i \), which yields
\[ v(f(p^u)) = \sum_{i=1}^{k} v(p^u - \gamma_i) \geq \sum_{i=1}^{k} a_i \geq S \geq s + u - 1. \]
So assume that there exists an index $1 \leq i \leq k$ with $a_i > u$, say $a_k > u$. Put

$$I := \{1 \leq i \leq k \mid 0 < a_i < 1\}$$

and let $t_I$ be the number of indices $i$ with $a_i \geq 1$. By Lemma 5 we find $s \leq \sum_{i \in I} a_i + t_I$. On the other hand, we have $v(p^u - \gamma_i) = a_i$ for all $i \in I$. Summing yields

$$v(f(p^u)) = \sum_{i=1}^k v(p^u - \gamma_i) = u + \sum_{i=1}^{k-1} v(p^u - \gamma_i) =$$

$$= u + \sum_{i \in I} a_i + \sum_{i \in \{1,\ldots,k-1\}\backslash I} v(p^u - \gamma_i) \geq u + \sum_{i \in I} a_i + t_I - 1 \geq u + s - 1.$$

We deduce that

$$v(\gcd(f(p^u), g(p^u))) = u + s - \text{sgn } s,$$

which takes all integer values in the open interval $(s, S)$ when $u$ runs over integers in $(\text{sgn } s, S - s + \text{sgn } s)$. \qed

**Remark.** Assuming $s \geq 1$ and noting $S \geq s$ in Theorem 6(a) yields $v(r) \geq p$, which is the statement of Proposition 8(a)].

**Remark.** The above proof shows that one can weaken the assumption in Theorem 6(b): it suffices to assume that the estimate in case $m = 0$ of Lemma 7(a) is sharp for the choice of $f$ and $g$.

**Construction 8.** Let $p$ be a prime and assume that $0 \leq s \leq S$ and, in case $p = 2 \leq s$, also that $2s + 1 \leq S$. Then there exists a pair $f, g \in \mathbb{Z}[x]$ of monic polynomials such that $\min_{n \in \mathbb{Z}} v(\gcd(f(n), g(n))) = s$, $\max_{n \in \mathbb{Z}} v(\gcd(f(n), g(n))) = S$, and $v(r) - S = ps^2 - s$ holds for the resultant $r$. In particular, the estimate in Theorem 6(a) is sharp for any prime $p \geq 2$ and any $0 \leq s \leq p$.

**Proof.** If $s = S = 0$ we simply take $f(x) = 1$ and $g(x)$ arbitrary. In case $s = S = 0$ (resp. $s = 1 \leq S$) we pick $f(x) = x$ (resp. $f(x) = x(x-1)$) and $g(x) = x - p^S$ (resp. $g(x) = (x - p^S)(x-1) - p$).

For $s \geq 2$ and $p$ odd, the example is

$$f(x) = x(x-2p)^{s-1} \prod_{j=1}^{p-1} (x-j)^s$$

and

$$g(x) = (x - p^{s-s+1})(x-p)^{s-1} \prod_{j=1}^{p-1} (x-j-p)^s.$$

Under this choice, we clearly have $s = \min_{n \in \mathbb{Z}} v(\gcd(f(n), g(n)))$. On the other hand, $f(0) = 0$ and $v(g(0)) = S$, whence

$$\max_{n \in \mathbb{Z}} v(\gcd(f(n), g(n))) \leq S.$$
Moreover, if $n \equiv j \neq 0 \pmod{p}$ ($j = 1, \ldots, p - 1$), then $n$ cannot be congruent to both $j$ and $j + p$ modulo $p^2$, whence
\[ v(\gcd(f(n), g(n))) = s. \]

Further, if $p | n$, then we distinguish three cases:

(i) $n \equiv 0 \pmod{p^{s - s + 2}}$. Then
\[ v(n - p^{s - s + 1}) = S - s + 1 \text{ and } v(n - p) = 1, \]
whence $v(g(n)) = S$.

(ii) $n \equiv p^{s - s + 1} \pmod{p^{s - s + 2}}$. Then
\[ v(n) = S - s + 1 \text{ and } v(n - 2p) = 1, \]
showing that $v(f(n)) = S$.

(iii) $0 \not\equiv n \not\equiv p^{s - s + 1} \pmod{p^{s - s + 2}}$. In this case, we have
\[ v(n) = v(n - p^{s - s + 1}) \leq S - s + 1, \]
and $n$ cannot be congruent to both $p$ and $2p$ modulo $p^2$, showing that $v(\gcd(f(n), g(n))) \leq S$.

In all cases, we obtained $v(\gcd(f(n), g(n))) \leq S$, showing that $S$ is the maximum. Finally, we compute
\[
\begin{align*}
v(r) &= v \left( g(0)g(2p)^{s-1}\prod_{j=1}^{p-1} g(j)^s \right) \\
&= v(g(0)) + (s - 1)v(g(2p)) + s \sum_{j=1}^{p-1} v(g(j)) = S + (s - 1)s + s(p - 1)s = ps^2 - s + S
\end{align*}
\]
as claimed.

Finally, if $p = 2 \leq s \leq (S - 1)/2$, then we take
\[ f(x) = x(x - 2)^{s-1}(x - 1)^s \]
and
\[ g(x) = (x - 2^{s-2}x^2)(x - 4)^{s-1}(x - 3)^s. \]
A simple computation similar to the one above shows the statement.

\[\Box\]

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