Analytic (non)integrability of Arutyunov-Bassi-Lacroix model

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Abstract

We use the notion of the gauge/string duality and discuss the Liouvillian (non)integrability criteria for string sigma models in the context of recently proposed Arutyunov-Bassi-Lacroix (ABL) model [JHEP 03 (2021), 062]. Our analysis complements those previous results due to numerical analysis as well as Lax pair formulation. We consider a winding string ansatz for the deformed torus $T_k^{(\lambda_1, \lambda_2, \lambda)}$ which can be interpreted as a system of coupled pendulums. Our analysis reveals the Liouvillian nonintegrability of the associated sigma model. We also obtain the generalized decoupling limit and confirm the analytic integrability for the decoupled sector.

1 Introduction and motivation

Integrable systems are very special - they posses an infinite tower of conserved charges those are in involution. The notion of integrability was first introduced in the seminal work of Bethe in an attempt to solve the Heisenberg spin chain model [1]. Subsequently, these techniques were further developed and applied to models like QCD in [2]. In recent years, following the pioneering work of [3], there has been a surge in the study of integrable systems within the framework of gauge/string duality [4]-[27].

In this work, we use the methodology of the AdS/CFT duality [28, 29, 30] in order to study the dynamics of string solitons over Arutyunov-Bassi-Lacroix (ABL) background $R \times T_k^{(\lambda_1, \lambda_2, \lambda)}$ [31, 32]. Along the way, we explore the classical (non)integrability of the associated phase space. This ABL background can be considered as a generalization of the Einsteinian $T^{1, 1}$ manifold [33] and is given by [31, 32]

\begin{align*}
    ds^2 &= -dt^2 + \sum_{i=1}^2 \lambda_i^2 (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) + \lambda^2 (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2, \\
    B_2 &= k (d\psi + \cos \theta_1 d\phi_1) \wedge (d\psi + \cos \theta_2 d\phi_2).
\end{align*}

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Here \( \lambda_1, \lambda_2, \lambda \) and \( k \) are real parameters. Interestingly, the classical string dynamics is integrable only when \( k = \lambda^2 \). When \( k \neq \lambda^2 \), the integrability of the system is lost and it becomes chaotic. This has been confirmed very recently using numerical techniques [32].

In this paper, we complement all those previous results [31, 32] by taking a third path which is based on the notion of Kovacic’s algorithm [34, 35]. The algorithm essentially offers a set of rules in order to verify the Liouvillian non-integrability criteria for a classical 2d sigma model over general backgrounds. As our analysis reveals, it establishes a clear compatibility between the Liouvillian (non)integrability and the Lax pair formulation of 2d sigma model over ABL background.

In our analysis, we consider winding string ansatz where a string is wrapped around all the isometry directions \((\phi_1, \phi_2, \psi)\). We observe that when the modes of the string along the \((\phi_1, \phi_2)\) directions are coupled, the dynamics of the system can be modelled as a system of coupled pendulums. Subsequently, the application of the Kovacic’s algorithm ensures non-integrability of this coupled pendulum model. We also observe a generalized special limit when these pendulums decouple and become integrable. Interestingly, the decoupling limit studied in [31, 32] can be seen as a special case of our analysis. While achieving this later limit, we observe that the string must satisfy certain conditions that have been discussed in detail in the section 3.

The structure of the paper is as follows: In section 2 we study the string sigma model in the ABL background (1). Using winding string ansatz, we explicitly compute the set of differential equations that represents two coupled harmonic oscillators. We explicitly determined the conditions that make the Virasoro constraints to satisfy. Furthermore, we apply the Kovacic’s algorithm to show analytically the non-integrability of this system of coupled pendulums. In section 3 we analyze the decoupling limits of these coupled pendulum system and checked the integrability in individual system. Finally, we conclude in section 4. Additionally, we give a brief account of the Kovacic’s algorithm in Appendix A in order to make the article self-contained. Appendix B contains lengthy expressions of certain constant coefficients.

2 Classical sigma model on \( \mathcal{R} \times T^\lambda_1 \lambda_2 \lambda \)

The classical 2d string sigma model can be expressed in the conformal gauge as [36]

\[
S_P = \frac{1}{4\pi\alpha'} \int \mathrm{d}\tau \mathrm{d}\sigma (\eta^{\alpha\beta} G_{MN} + \epsilon^{\alpha\beta} B_{MN}) \partial_\alpha X^M \partial_\beta X^N = \frac{1}{4\pi\alpha'} \int \mathrm{d}\tau \mathrm{d}\sigma L_P. \tag{3}
\]

Here \( \alpha, \beta \) are string worldsheet coordinates with metric \( \eta_{\alpha\beta} = \text{diag}(-1, 1) \), and the target space coordinates are labelled as \( X^\mu = (t, \theta_1, \theta_2, \phi_1, \phi_2, \psi) \).

Using (1) and (2), the Lagrangian density \( L_P \) appearing in (3) can be written as

\[
L_P = (\dot{t}^2 - \dot{t}^2) + \lambda_1^2 \left( \dot{\theta}_1^2 - \dot{\theta}_1^2 \right) + \lambda_2^2 \left( \dot{\theta}_2^2 - \dot{\theta}_2^2 \right) + \lambda^2 \left( \dot{\psi}^2 - \dot{\psi}^2 \right) + (\lambda_1^2 \sin^2 \theta_1 + \lambda^2 \cos^2 \theta_1) \left( \dot{\phi}_1^2 - \dot{\phi}_1^2 \right) + (\lambda_2^2 \sin^2 \theta_2 + \lambda^2 \cos^2 \theta_2) \left( \dot{\phi}_2^2 - \dot{\phi}_2^2 \right) + 2\lambda^2 \cos \theta_1 \left( \dot{\psi} \dot{\phi}_1' - \dot{\psi} \dot{\phi}_1' \right) + 2k \cos \theta_2 \left( \dot{\psi} \dot{\phi}_2' - \dot{\psi} \dot{\phi}_2' \right) + \left( \dot{\phi}_1 \dot{\phi}_2' - \dot{\phi}_2 \dot{\phi}_1' \right).
\tag{4}
\]
The conserved charges associated with the string $\sigma$-model (3) are simply derived from the above density (4) and are written as

$$E = \frac{\partial L_P}{\partial \dot{t}} = 2i,$$  \hspace{1cm} (5)

$$P_\psi = \frac{\partial L_P}{\partial \dot{\psi}} = -2 \left[ \lambda^2 \dot{\psi} + (k \dot{\phi}_1 + \lambda^2 \dot{\phi}_1) \cos \theta_1 - (k \dot{\phi}'_2 - \lambda^2 \dot{\phi}_2) \cos \theta_2 \right],$$  \hspace{1cm} (6)

$$P_{\phi_1} = \frac{\partial L_P}{\partial \dot{\phi}_1} = -2 \left[ (\lambda_1^2 \sin^2 \theta_1 + \lambda^2 \cos^2 \theta_1) \dot{\phi}_1 - \left( k \psi' - \lambda^2 \dot{\psi} \right) \cos \theta_1 \right.$$  
$$- \left( k \dot{\phi}'_2 - \lambda^2 \dot{\phi}_2 \right) \cos \theta_1 \cos \theta_2 \left], \right.$$  \hspace{1cm} (7)

$$P_{\phi_2} = \frac{\partial L_P}{\partial \dot{\phi}_2} = -2 \left[ (\lambda_2^2 \sin^2 \theta_2 + \lambda^2 \cos^2 \theta_2) \dot{\phi}_2 + \left( k \psi' + \lambda^2 \dot{\psi} \right) \cos \theta_2 \right.$$  
$$+ \left( k \dot{\phi}'_1 + \lambda^2 \dot{\phi}_1 \right) \cos \theta_1 \cos \theta_2 \left]. \right.$$  \hspace{1cm} (8)

Next, we consider the winding string ansatz of the following form [32]

$$t = \tau, \hspace{1cm} \theta_1 = \theta_1(\tau), \hspace{1cm} \theta_2 = \theta_2(\tau),$$  \hspace{1cm} (9)

$$\phi_1 = \omega_1 \tau + \ell_1 \sigma, \hspace{1cm} \phi_2 = \omega_2 \tau + \ell_2 \sigma, \hspace{1cm} \psi = \alpha \tau + \ell_3 \sigma,$$  \hspace{1cm} (10)

where $\ell_i$s are the respective winding numbers along the isometries of the target space.

Substituting (9) and (10) into (4) we find

$$\ddot{\ell}_P = 1 - \lambda_1^2 \dot{\theta}_1^2 - \lambda_2^2 \dot{\theta}_2^2 + \lambda^2 (\ell_3^2 - \alpha^2) + \left( \lambda_1^2 \sin^2 \theta_1 + \lambda^2 \cos^2 \theta_1 \right) \left( \ell_1^2 - \omega_1^2 \right)$$  
$$+ \left( \lambda_2^2 \sin^2 \theta_2 + \lambda^2 \cos^2 \theta_2 \right) \left( \ell_2^2 - \omega_2^2 \right) + 2 \lambda^2 \cos \theta_1 (\ell_1 \ell_3 - \alpha \omega_1)$$  
$$+ 2 \lambda^2 \cos \theta_2 (\ell_2 \ell_3 - \alpha \omega_2) + \lambda \cos \theta_1 \cos \theta_2 (\ell_1 \ell_2 - \omega_1 \omega_2)$$  
$$+ 2 k \cos \theta_1 (\omega_1 \ell_3 - \alpha \ell_1) + 2 k \cos \theta_2 (\alpha \ell_2 - \omega_2 \ell_3) + 2 k \cos \theta_1 \cos \theta_2 (\omega_1 \ell_2 - \omega_2 \ell_1).$$  \hspace{1cm} (11)

The equations of motion corresponding to $\theta_i$ ($i = 1, 2$) can be derived from the above Lagrangian density (11) as

$$\ddot{\theta}_1 + \left( 1 - \frac{\lambda_1^2}{\lambda_1^2} \right) (\ell_1^2 - \omega_1^2) \sin \theta_1 \cos \theta_1 - \frac{\sin \theta_1}{\lambda_1^2} \left[ k(\omega_1 \ell_3 - \alpha \ell_1) + \lambda^2 (\ell_1 \ell_3 - \alpha \omega_1) \right]$$  
$$- \frac{\sin \theta_1}{\lambda_1^2} \left[ k(\omega_1 \ell_2 - \omega_2 \ell_1) + \lambda^2 (\ell_1 \ell_2 - \omega_1 \omega_2) \right] = 0,$$  \hspace{1cm} (12)

$$\ddot{\theta}_2 + \left( 1 - \frac{\lambda_2^2}{\lambda_2^2} \right) (\ell_2^2 - \omega_2^2) \sin \theta_2 \cos \theta_2 - \frac{\sin \theta_2}{\lambda_2^2} \left[ k(\alpha \ell_2 - \omega_2 \ell_3) + \lambda^2 (\ell_2 \ell_3 - \alpha \omega_2) \right]$$  
$$- \frac{\sin \theta_2}{\lambda_2^2} \left[ k(\omega_1 \ell_2 - \omega_2 \ell_1) + \lambda^2 (\ell_1 \ell_2 - \omega_1 \omega_2) \right] = 0.$$  \hspace{1cm} (13)

This looks like an interacting oscillator model. In other words, the sigma model could be thought of as a model of coupled oscillators which are mutually interacting.

In addition, the corresponding Virasoro constraints can be written as

$$T_{\tau\tau} = T_{\sigma\sigma} = \frac{1}{2} \left[ -1 + \lambda_1^2 \dot{\theta}_1^2 + \lambda_2^2 \dot{\theta}_2^2 + \lambda^2 (\ell_3^2 + \alpha^2) + \left( \lambda_1^2 \sin^2 \theta_1 + \lambda^2 \cos^2 \theta_1 \right) (\ell_1^2 + \omega_1^2)$$  
$$+ \left( \lambda_2^2 \sin^2 \theta_2 + \lambda^2 \cos^2 \theta_2 \right) (\ell_2^2 + \omega_2^2) + 2 \lambda^2 \cos \theta_1 (\ell_1 \ell_3 + \alpha \omega_1)$$  
$$+ 2 \lambda^2 \cos \theta_2 (\ell_2 \ell_3 + \alpha \omega_2) + 2 \lambda^2 \cos \theta_1 \cos \theta_2 (\ell_1 \ell_2 + \omega_1 \omega_2) \right] ,$$  \hspace{1cm} (14)
\[ T_{\tau\sigma} = T_{\pi\tau} = \lambda^2 \alpha \ell_3 + (\lambda_1^2 \sin^2 \theta_1 + \lambda^2 \cos^2 \theta_1) \omega_1 \ell_1 + (\lambda_1^2 \sin^2 \theta_2 + \lambda^2 \cos^2 \theta_2) \omega_2 \ell_2 \\
+ \lambda^2 \cos \theta_1 (\alpha \ell_1 + \omega_1 \ell_3) + \lambda^2 \cos \theta_2 (\alpha \ell_2 + \omega_2 \ell_3) \\
+ \lambda^2 \cos \theta_1 \cos \theta_2 (\omega_1 \ell_2 + \ell_1 \omega_2) \]  

(15)

### 2.1 Consistency requirements

Using the equations of motion (12), (13) we may write

\[ \partial_\tau T_{\tau\tau} \]

\[ = \omega_1^2 (\lambda_1^2 - \lambda^2) \sin 2\theta_1 \dot{\theta}_1 + \omega_2^2 (\lambda_2^2 - \lambda^2) \sin 2\theta_2 \dot{\theta}_2 - 2\alpha \lambda^2 \left( \omega_1 \sin \theta_1 \dot{\theta}_1 + \omega_2 \sin \theta_2 \dot{\theta}_2 \right) \\
- 2\lambda^2 \omega_1 \omega_2 \left( \sin \theta_1 \cos \theta_2 \dot{\theta}_1 + \sin \theta_2 \cos \theta_1 \dot{\theta}_2 \right) + k \sin \theta_1 \dot{\theta}_1 \left[ \omega_1 (\ell_3 + \ell_2 \cos \theta_2) \right. \\
- \ell_1 (\alpha + \omega_2 \cos \theta_2) \bigg]\right] + k \sin \theta_2 \dot{\theta}_2 \left[ \ell_2 (\alpha + \omega_1 \cos \theta_1) - \omega_2 (\ell_3 + \ell_1 \cos \theta_1) \right], \\
\]  

(16)

\[ \partial_\tau T_{\tau\sigma} = 2\omega_1 \ell_1 \sin \theta_1 \cos \theta_1 \cdot \dot{\theta}_1 \left( \lambda_1^2 - \lambda^2 \right) + 2\omega_2 \ell_2 \sin \theta_2 \cos \theta_2 \cdot \dot{\theta}_2 \left( \lambda_2^2 - \lambda^2 \right) \\
- \lambda^2 (\alpha \ell_1 + \omega_1 \ell_3) \sin \theta_1 \cdot \dot{\theta}_1 - \lambda^2 (\alpha \ell_2 + \omega_2 \ell_3) \sin \theta_2 \cdot \dot{\theta}_2 \\
- \lambda^2 (\omega_1 \ell_2 + \omega_2 \ell_1) \left( \sin \theta_1 \cos \theta_2 \cdot \dot{\theta}_1 + \sin \theta_2 \cos \theta_1 \cdot \dot{\theta}_2 \right). \]

(17)

In the next step we demand the invariance of the conserved charges \( Q_i \) associated with the reduced sigma model (11). This allows us to compute the following constraint equations:

\[ \partial_\tau J_{\psi} = \left( k \ell_1 + \lambda^2 \omega_1 \right) \sin \theta_1 \cdot \dot{\theta}_1 - \left( k \ell_2 + \lambda^2 \omega_2 \right) \sin \theta_2 \cdot \dot{\theta}_2 = 0, \]

(18)

\[ \partial_\tau J_{\phi_1} = \left[ 2\omega_1 \left( \lambda_1^2 - \lambda^2 \right) \sin \theta_1 \cos \theta_1 + \sin \theta_1 \left( k \ell_3 \right) + \alpha \lambda_1^2 \right) + \sin \theta_1 \cos \theta_2 \left( k \ell_2 - \omega_2 \lambda_2^2 \right) \right] \cdot \dot{\theta}_1 \\
+ \left( k \ell_2 - \omega_2 \lambda_2^2 \right) \sin \theta_2 \cos \theta_1 \cdot \dot{\theta}_2 = 0, \]

(19)

\[ \partial_\tau J_{\phi_2} = \left[ 2\omega_2 \left( \lambda_2^2 - \lambda^2 \right) \sin \theta_2 \cos \theta_2 - \sin \theta_2 \left( k \ell_3 + \alpha \lambda_2^2 \right) - \sin \theta_2 \cos \theta_1 \left( k \ell_1 + \omega_1 \lambda_1^2 \right) \right] \cdot \dot{\theta}_2 \\
- \left( k \ell_1 + \omega_1 \lambda_1^2 \right) \sin \theta_1 \cos \theta_2 \cdot \dot{\theta}_1 = 0. \]

(20)

The above set of equations (18)-(20) implies

\[ \Pi_{\theta_1} = \dot{\theta}_1 = 0, \quad \Pi_{\theta_2} = \dot{\theta}_2 = 0, \]

(21)

where \( \Pi_{\theta_i} \) are the momenta conjugate to \( \theta_i \).

Using (21), it is now trivial to check that

\[ \partial_\tau T_{\tau\tau} = 0 = \partial_\tau T_{\tau\sigma}, \]

(22)

which satisfy the consistency requirements for the Virasoro constraints.

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1Here we have used the following standard definition of the conserved charge: \( J_i = \frac{1}{2\pi \alpha'} \int_0^{2\pi} d\sigma P_i, \quad i = (\psi, \phi_1, \phi_2) \) and the \( P_i \)'s are given in (6)-(9).
2.2 The coupled pendulum model

Next, in order to study the interacting two (gravitational) pendulum system described by (12), (13) we choose the following invariant plane in the phase space [9]:

\[ \theta_2 \sim 0, \quad \Pi_{\theta_2} \sim \dot{\theta}_2 \sim 0. \] (23)

Using (23) the equation of motion for \( \theta_1 \) can be written as

\[ \ddot{\theta}_1 + A \sin \theta_1 \cos \theta_1 - (B + D) \sin \theta_1 = 0, \] (24)

where

\[ A = \left(1 - \frac{\lambda^2}{\lambda_1^2}\right) (\ell_1^2 - \omega_1^2); \] (25)

\[ B = \frac{1}{\lambda_1^2} \left[k(\omega_1 \ell_3 - \alpha \ell_1) + \lambda^2(\ell_1 \ell_3 - \alpha \omega_1)\right]; \] (26)

\[ D = \frac{1}{\lambda_1^2} \left[k(\omega_1 \ell_2 - \omega_2 \ell_1) + \lambda^2(\ell_1 \ell_2 - \omega_1 \omega_2)\right]. \] (27)

Let \( \bar{\theta}_1 \) be the solution to (24). In the next step, we shall consider small fluctuations (\( \eta \)) about the invariant plane \( \theta_2 \sim 0, \Pi_{\theta_2} \sim 0 \) which results in the normal variational equation (NVE) of the following form:

\[ \ddot{\eta} + (F - G)\eta - \tilde{D} \cos \bar{\theta}_1 \cdot \eta = 0, \] (28)

where

\[ F = \left(1 - \frac{\lambda^2}{\lambda_1^2}\right) (\ell_2^2 - \omega_2^2); \] (29)

\[ G = \frac{1}{\lambda_1^2} \left[k(\alpha \ell_2 - \omega_2 \ell_3) + \lambda^2(\ell_2 \ell_3 - \alpha \omega_2)\right]; \] (30)

\[ \tilde{D} = \frac{1}{\lambda_2^2} \left[k(\omega_1 \ell_2 - \omega_2 \ell_1) + \lambda^2(\ell_1 \ell_2 - \omega_1 \omega_2)\right]. \] (31)

In order to study the NVE (28) it will be convenient to introduce the variable \( z \) such that

\[ \cos \bar{\theta}_1 = z. \] (32)

With this change in variable, (28) can be expressed as

\[ \eta''(z) + \frac{f'(z)\eta'(z)}{2f(z)} + \left(F - G - \tilde{D} \frac{\eta(z)}{f(z)}\right) = 0. \] (33)

The above equation (33) is similar to a second order linear homogeneous differential equation, known as the Lamé equation [9], with

\[ f(z) = \hat{\theta}_1^2 \sin^2 \hat{\theta}_1 = [2E^2 - A(1 - z^2) - 2(B + D)z] (1 - z^2). \] (34)

It is difficult to solve the NVE (33) exactly. However, we can expand the coefficients of (33) for small values of the variable \( |z| \). The resulting solution can be expressed in terms of special
functions and can be written as

\[
\eta(z) = \exp \left[ \frac{\tilde{D}z}{\Upsilon} \left( \frac{A}{2} - 1 \right) + \frac{z}{\Upsilon} (B + D)(F - G) \right] \left\{ C_1 \text{ Hermite} \left[ \frac{\chi_1}{8\Upsilon^3} \frac{\chi_2}{2\Upsilon^2} + \frac{\Upsilon^2}{A - 2} \right] \right. \\
+ C_2 \ {}_1F_1 \left[ -\frac{\chi_1}{16\Upsilon^3} \frac{1}{2} \left( \frac{\chi_2}{2\Upsilon^2} + \frac{\Upsilon^2}{A - 2} \right) \right] \right\},
\]

where \( C_1, C_2 \) are constants of integration,

\[
\Upsilon = A^2 + (B + D)^2 - 3A + 2,
\]

and the detail expressions for \( \chi_1 \) and \( \chi_2 \) are provided in appendix B. Here \text{ Hermite}[n, z] \text{ and } \text{ } {}_1F_1[a; b; z] \text{ are the Hermite polynomial function and Kummer confluent hypergeometric function, respectively. Clearly, this form of the solution is not Liouvillian indicating the non-integrability of the system \cite{9, 20, 21, 22, 25, 27}.

Furthermore, using the change of variable \cite{20, 21, 22, 25, 27}

\[
\eta(z) = \exp \left[ \int \left( w(z) - \frac{B(z)}{2} \right) dz \right]
\]

we can recast (33) in the Schrödinger form

\[
w'(z) + w^2(z) = \mathcal{V}(z) \equiv \frac{2B'(z) + B^2(z) - 4A(z)}{4},
\]

where

\[
A(z) = \frac{F - G - \tilde{D}z}{f(z)}, \quad B(z) = \frac{f'(z)}{2f(z)}.
\]

Using (39) in (38) we can compute the function \( \mathcal{V}(z) \) as

\[
\mathcal{V}(z) = -\frac{3}{4(z^2 - 1)^2} + \frac{(\beta_1 + z\tilde{\beta}_1)}{4(z^2 - 1)} - \frac{3\beta_2}{4(4Az^2 - 2(B + D)z - (A - 2))^2}
\]

\[
+ \frac{(\beta_3 + z\tilde{\beta}_3)}{4(4Az^2 - 2(B + D)z - (A - 2))},
\]

where

\[
\beta_1 = \frac{1}{\beta_0} \left( 1 - A + 2\tilde{D}(B + D) - 2(F - G) \right); \quad \beta_0 = (B + D)^2 - 1,
\]

\[
\tilde{\beta}_1 = \frac{1}{\beta_0} \left( (1 - A)(B + D) - 2(B + D)(F - G) \right),
\]

\[
\beta_2 = A(A - 2) + (B + D)^2,
\]

\[
\beta_3 = \frac{1}{\beta_0} \left( A^2 - 2AB^2 - 4ABD - 2AB\tilde{D} - 2AD^2 - 2AD\tilde{D} + 2AF - 2AG - A - 4B^2F + 4B^2G + 2B^2 - 8BDF + 8BDG + 4BD + 4B\tilde{D} - 4D^2F + 4D^2G + 2D^2 + 4D\tilde{D} \right),
\]

\[
\tilde{\beta}_3 = \frac{1}{\beta_0} \left( A(A - 1)(B + D) - 2AD + 2A(F - G)(B + D) \right).
\]
Notice that, in writing (40) we have set the constant of integration \( E = 1 \) in (34).
Clearly, the function \( V(z) \) has poles of order 2 at
\[
z = \pm 1, \quad z = \frac{1}{A} \left[ (B + D) \pm \sqrt{A(A - 2) + (B + D)^2} \right],
\]
and the order of \( V(z) \) at infinity can be computed to be 1. Hence, \( V(z) \) does not belong to any of the three cases discussed in the Appendix A. Therefore, the form of the solution to equation (38) must be non-Liouvillian.

In fact, for small values of \(|z|\) we can series expand \( V(z) \) in (40) and can find the solution to (38) as
\[
w(z) = \frac{\chi_3}{\Pi_2} \cdot \frac{\text{Bi}'(\Pi_1) + C_3 \text{Ai}'(\Pi_1)}{\text{Bi}(\Pi_1) + C_3 \text{Ai}(\Pi_1)},
\]
where \( C_3 \) is a constant of integration,
\[
\Pi_1 = \frac{\chi_3 z + \chi_4}{\Pi_2}, \quad \Pi_2 = 2^{\frac{7}{4}}(A - 2)\chi_3^\frac{3}{2},
\]
\[
\chi_3 = 8A^2B + 8A^2D - 4A^2\hat{D} - 8ABF + 8ABG - 12AB - 8ADF + 8ADG - 12AD
\quad + 16A\hat{D} + 12B^3 + 36B^2D + 36BD^2 + 16BF - 16BG - 8B + 12D^3 + 16DF
\quad - 16DG - 8D - 16\hat{D},
\]
\[
\chi_4 = -4A^3 + 4A^2F - 4A^2G + 20A^2 - 3AB^2 - 6ABD - 3AD^2 - 16AF + 16AG - 32A
\quad + 6B^2 + 12BD + 6D^2 + 16F - 16G + 16.
\]

As it is clear from the above expression (47), the solution to (38) is written in terms of Airy functions \( \text{Ai}(z), \text{Bi}(z) \) and their derivatives \( \text{Ai}'(z), \text{Bi}'(z) \) thereby making it non-Liouvillian. This reassures the non-integrability of the system.

### 3 Analytic integrability of the decoupled systems

In this section we shall discuss the analytic integrability of the two pendulum system in the decoupling limit. It is easy to check from (12), (13) that the two pendula decouples from each other in the limit
\[
\frac{k}{\lambda^2} = \frac{(\omega_1\omega_2 - \ell_1\ell_2)}{(\omega_1\ell_2 - \omega_2\ell_1)},
\]
and the equations of motion (12) and (13) simplify to
\[
\ddot{\theta}_1 + A \sin \theta_1 \cos \theta_1 - B \sin \theta_1 = 0, \quad (52)
\]
\[
\ddot{\theta}_2 + F \sin \theta_2 \cos \theta_2 - G \sin \theta_2 = 0, \quad (53)
\]
where \( A, B, F \) and \( G \) have been defined earlier.

We notice that, when
\[
\omega_1 \leftrightarrow -\ell_1, \quad \text{or} \quad \omega_2 \leftrightarrow \ell_2, \quad (54)
\]
the decoupling limit (51) reduces to what was studied in [31, 32]; namely, \( k = \lambda^2 \). The constraint (54) implies that either the string must wind anti-clockwise along the isometry direction \( \phi_1 \),
or it winds in the clockwise direction along $\phi_2$. Thus we may consider the limit $k = \lambda^2$ as a special case of (51).

Next, in order to check the integrability of the individual systems, we study their dynamics in the corresponding phase spaces $\{\theta_i, \Pi_{\theta_i}\}$ ($i = 1, 2$). For the first pendulum we choose the invariant plane given by

$$\theta_1 = \dot{\theta}_1 = 0,$$

and consider the fluctuation $\delta \theta_1 \sim \sigma(\tau)$ around this plane to compute the NVE as

$$\ddot{\sigma} + (A - B)\sigma = 0,$$

which is nothing but simple harmonic motion with frequency $\sim \sqrt{A - B}$, hence trivially integrable as per differential Galois theory [9].

In a similar manner we can choose an invariant plane $\theta_2 = \dot{\theta}_2 = 0$ in the phase space of the second pendulum. Subsequently, the Kovacic’s algorithm ensures the integrability of this system as well.

4 Conclusions

In this paper, we have established classical Liouvillian non-integrability of string sigma model in the recently proposed Arutyunov-Bassi-Lacroix (ABL) background (1) which is a generalization of the Einsteinian $T^{1,1}$ manifold [31]. We use the Kovacic’s algorithm [34, 35] in order to perform our computations. Our analysis complements the claim made in [31, 32] and is compatible with the Lax pair formulation of [31].

We observe that, if we consider a string that winds around the deformed torus $T_k^{(\lambda_1, \lambda_2, \lambda)}$, the system can be described by two coupled harmonic oscillators. We analyse the corresponding coupled differential equation. By appropriately choosing an invariant plane in the phase space of the system, we analyse the corresponding normal variational equation (NVE). The solution to this equation turns out to be non-Liouvillian which therefore establishes the non-integrability. We further recast the NVE in the Schrödinger form (38) and carefully analyze the polynomial function (40) arising from it. We observe incompatibility of this function with the Kovacic’s classification (see Appendix A), thereby establishing the non-integrability.

Next, we proceed to study the limit in which the coupled oscillators decouple and turn out to be integrable. As a matter of fact, for the particular example of winding string, we find a generalized decoupling limit and, under special conditions (54), it gives the decoupling limit studied in [31, 32]. This analysis reveals that, in transiting to this later limit in [31, 32], the string winds (anti-)clockwise along the isometry direction $(\phi_1 \phi_2)$.

The Kovacic’s algorithm has been proven to be an excellent mathematical tool to (dis)prove analytic integrability over general backgrounds [8], [9], [20]-[27]. As has been shown in the present work, one can reliably use this formalism as it complements the existing methodologies in the literature [31, 32]. It will be interesting to apply this formulation, along with the other analytic methods, to explore integrable structures of other systems in future.

Acknowledgments

J.P., A.M. and D.R. is indebted to the authorities of IIT Roorkee for their unconditional support towards researches in basic sciences. A.M. acknowledges The Science and Engineering Research Board (SERB), India for financial support. D.R. would also like to acknowledge The Royal Society, UK for financial assistance, and acknowledges the Grant (No. SRG/2020/000088) received
from The Science and Engineering Research Board (SERB), India. The work of A.L. is supported by the Chilean National Agency for Research and Development (ANID)/ FONDECYT / POSTDOCTORADO BECAS CHILE / Project No. 3190021.

A The Kovacic’s algorithm

The Kovacic’s algorithm is a systematic procedure to check the Liouvillian non-integrability of a dynamical system [34, 35]. This algorithm is implemented in order to realize whether the second order linear homogeneous differential equation of the type

$$
\eta''(z) + M(z)\eta'(z) + N(z)\eta(z) = 0
$$

(A1)

with polynomial coefficients $M(z), N(z)$ are integrable in the Liouvillian sense. We look for solutions of (A1) in the Liouvillian form; namely, those solutions which can be expressed in terms of simple algebraic polynomial functions, exponentials, or trigonometric functions. As a matter of fact, this algorithm is a variant of the Galois theory of differential equations (the Piccard-Vessoit theory [37]) and utilizes the $SL(2, \mathbb{C})$ group of symmetries of the differential equation (A1) in its analysis.

Without going into detailed mathematical analysis which is rather involved, we limit ourselves to find the relation between the coefficients $M(z), M'(z)$ and $N(z)$ that makes (A1) integrable. In order to do so, we perform the following change of variable:

$$
\eta(z) = \exp \left[ \int \left( w(z) - \frac{M(z)}{2} \right) dz \right]. \quad (A2)
$$

This allows us to rewrite (A1) as

$$
w'(z) + w^2(z) = V(z) \equiv \frac{M'(z) + M^2(z) - 4N(z)}{4}, \quad (A3)
$$

where $''$ denotes differentiation w.r.to $z$.

The group of symmetry transformations of the solutions of (A1) is in fact a subgroup $G$ of $SL(2, \mathbb{C})$ which can be classified as follows: (i) For any $a, b \in \mathbb{C}$, if $G$ is generated by the matrix $M = \{\{a, 0\}, \{b, 1/a\}\}$ then $w(z)$ is a rational polynomial function of degree 1; (ii) When $M = \{\{a, 0\}, \{0, 1/a\}\}, M = \{\{0, a\}, \{-1/a, 0\}\}$ $w(z)$ is a function of degree 2; (iii) When $G$ is a finite group and not generated by a matrix of the form mentioned in (i) and (ii) above, $w(z)$ is a function of degree 4, 6, or 12; (iv) When $G = SL(2, \mathbb{C})$, any non-vanishing solution $w(z)$ is not Liouvillian.

The Kovacic’s algorithm also proposes a set of three necessary but not sufficient conditions for the rational polynomial function $V(z)$ that must be compatible with the aforementioned group theoretical analysis [34]. These are as follows: (i) $V(z)$ has pole of order 1, or $2n$ ($n \in \mathbb{Z}^+$). Also, the order of $V(z)$ at infinity, defined as the highest power of the denominator minus that of the numerator, is either $2n$ or greater than 2; (ii) $V(z)$ either has pole of order 2, or poles of order $2n + 1$ greater than 2; (iii) $V(z)$ has poles not greater than 2 and the order of $V(z)$ at infinity is at least 2. If none of these conditions are met, the solution to (A1) is non-Liouvillian and ensures the non-integrability of (A1). On the other hand, fulfillment of any one of the above conditions makes us eligible to apply the Kovacic’s algorithm to the ODE (A1). It is then necessary to determine whether $w(z)$ is a polynomial function of degree 1, 2, 4, 6, or 12 in which case (A1) is integrable.
B Expressions for the constants $\chi_1$ and $\chi_2$ in (35)

\[
\chi_1 = -4FA^5 + 4GA^5 + \tilde{D}^2 A^4 + 2B\tilde{D}A^4 + 2D\tilde{D}A^4 + 32FA^4 - 32GA^4 - 8\tilde{D}^2 A^3 - 14B\tilde{D}A^3 - 14D\tilde{D}A^3 - 4B^2 FA^3 - 4D^2 FA^3 - 8BDF A^3 + 4B\tilde{D}FA^3 + 4D\tilde{D}FA^3 - 100FA^3 + 4B^2 GA^3 + 4D^2 GA^3 + 8BDGA^3 - 4B\tilde{D}GA^3 + 100GA^3 + 24D^2 A^2 + 4B^2 F^2 A^2 + 4D^2 F^2 A^2 + 8BDF^2 A^2 + 4B^2 G^2 A^2 + 4D^2 G^2 A^2 + 8BDG^2 A^2 + 2B^3 \tilde{D}A^2 + 2D^3 \tilde{D}A^2 + 6BD^2 \tilde{D}A^2 + 36B\tilde{D}A^2 + 6B^2 D\tilde{D}A^2 + 36D\tilde{D}A^2 + 20B^2 FA^2 + 20D^2 FA^2 + 40BDFA^2 - 24B\tilde{D}FA^2 - 24D\tilde{D}FA^2 + 152FA^2 - 20B^2 GA^2 - 20D^2 GA^2 - 40BDGA^2 + 24B\tilde{D}GA^2 + 24D\tilde{D}GA^2 - 8B^2 GA^2 - 8D^2 GA^2 - 16BDFGA^2 - 152GA^2 - 32\tilde{D}^2 A - 16B^2 F^2 A - 16D^2 F^2 A - 32BDF^2 A - 16B^2 G^2 A - 16D^2 G^2 A - 32BDG^2 A - 8B^3 \tilde{D}A - 8D^3 \tilde{D}A - 24BD^2 \tilde{D}A - 4B\tilde{D}A - 40B\tilde{D}A - 42B^2 FA - 32D^2 FA - 64BDFA + 48B\tilde{D}FA + 48B\tilde{D}FA - 112FA + 32B^2 GA + 32D^2 GA + 64BDGA - 48B\tilde{D}GA - 48D\tilde{D}GA + 32B^2 FGA + 32D^2 FGA + 64BDFGA + 112GA + 16\tilde{D}^2 + 16B^2 F^2 + 16D^2 F^2 + 32BDFA^2 + 16B^2 G^2 + 16D^2 G^2 + 32BDG^2 + 8B^3 + 32BDG^2 + 8D^3 \tilde{D} + 24BD\tilde{D}A + 24D^2 \tilde{D} + 16BD\tilde{D} + 16B^2 F + 16D^2 F + 32BDF - 32B\tilde{D}F - 32D\tilde{D}F + 32F - 16B^2 G - 16D^2 G - 32BDG + 32B\tilde{D}G + 32D\tilde{D}G - 32B^2 FG - 32D^2 FG - 64BDFG - 32G.
\]

(B1)

\[
\chi_2 = -A^2 B - A^2 D - A^2 \tilde{D} - 2ABF + 2ABG + 3AB - 2ADF + 2ADG + 3AD + 4A\tilde{D} - B^3 - 3B^2 D - 3BD^2 + 4BF - 4BG - 2B - D^3 + 4DF - 4DG - 2D - 4\tilde{D}.
\]

(B2)

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