Diffusion limit of the Vlasov equation in the weak turbulent regime

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Abstract

In this paper we study the Hamiltonian dynamics of charged particles subject to a non-self-consistent stochastic electric field, when the plasma is in the so-called weak turbulent regime. We show that the asymptotic limit of the Vlasov equation is a diffusion equation in the velocity space, but homogeneous in the physical space. We obtain a diffusion matrix, quadratic with respect to the electric field, which can be related to the diffusion matrix of the resonance broadening theory and of the quasilinear theory, depending on whether the typical autocorrelation time of particles is finite or not. In the self-consistent deterministic case, we show that the asymptotic distribution function is homogenized in the space variables, while the electric field converges weakly to zero. We also show that the lack of compactness in time for the electric field is necessary to obtain a genuine diffusion limit. By contrast, the time compactness property leads to a “cheap” version of the Landau damping: the electric field converges strongly to zero, implying the vanishing of the diffusion matrix, while the distribution function relaxes, in a weak topology, towards a spatially homogeneous stationary solution of the Vlasov-Poisson system.

1 Introduction

Here, we are interested in a problem of particle diffusion which is produced by the wave-particle interaction. In plasma physics, the wave-particle interaction is an important phenomenon, which stands at the root of Landau damping, of wave heating, of numerous instabilities, and of some regimes of anomalous transport in magnetically confined plasmas. This work is closely related to the so-called quasilinear (QL) theory, which describes the nonlinear relaxation of the weak warm beam-plasma instability through the derivation of a diffusion equation in the velocity variable conjugated with the prediction of an associated diffusion coefficient. This topic has led to a longstanding controversy that is not solved yet [57, 26, 27, 52, 28, 2, 36, 14, 10, 11, 42, 18, 56, 19, 45, 46, 38, 25, 54, 43, 52, 33, 31, 10, 29, 30, 34, 44]. References cited above are not exhaustive but testify to the
huge literature on this subject. For a brief history on the development of the QL theory, we refer the reader to references [10, 44]. Furthermore, the QL diffusion coefficient is quite frequently used for modelling particle transport in different branches of plasma physics, such as laser-plasma interaction or magnetized plasma turbulence. Since the QL approximation is ubiquitous, particularly in kinetic modelling, it is then important to assess, in the most rigorous possible way, whether the QL theory is valid or not. A complete and rigorous proof of the QL theory goes beyond the purpose of this paper, which aims at taking stock of what can or cannot be rigorously proven at this time.

We now sketch this problem in dimension one (see, e.g., [34] for an intuitive introduction and Chapters 8-9 of [21] or Chapter 7 of [31] for a more exhaustive one). We consider a two-dimensional distribution function of particles in the two-dimensional phase-space \((x, v)\). This distribution function is initially given by a one-dimensional (in \(v\)) spatially uniform (in \(x\)) beam-plasma system. This beam corresponds to a gentle and small bump on the tail of the electronic plasma velocity distribution function (see, e.g., Section 9.4 of [39]). The study of the bump-on-the-tail instability dates back to the pioneering work of Buneman [16] on the two-stream instability, where each stream is considered as a mono-kinetic beam. Using the Nyquist method (see, e.g., Section 9.6 of [39]) Penrose [49] derived a criterion, the so-called Penrose criterion for instability, under which the beam-plasma system distribution function (spatially uniform and one-dimensional in the velocity variable) constitutes an unstable equilibrium (i.e., an unstable stationary solution of the Vlasov–Poisson equations). Then, any initial small perturbations of the beam-plasma system are destabilized by the inversion of the electron population corresponding to the positive slope interval of the velocity distribution. This gives rise to electrostatic waves, which first grow linearly until the beginning of a saturation stage, where the amplitude of waves reaches a non-negligible value. In this resulting wave spectrum, the particle dynamics becomes chaotic enough in their range of phase velocities, so that the bump is eroded with eventually a plateau formation in the distribution function. Simultaneously, there is a transfer of momentum from particles to electric waves, generating a turbulent spectrum of waves. This scenario was first predicted on a theoretical basis [57, 26] by considering the wave-particle interaction as perturbative and neglecting all nonlinear wave-wave interactions in the Vlasov–Poisson equation, except for their effect on the space-averaged distribution function \(f = f(t, v)\). This led to the set of QL equations coupling the distribution function \(f\) and the Fourier modes \(E(t, k)\) of the electric field,

\[
\frac{\partial_t f(t, v)}{\partial_v(D_{QL}(t, v) \partial_v f(t, v))} = 0, \quad (1)
\]

\[
\frac{\partial_t |E(t, k)|^2}{\partial_v} = 2\gamma(t, k)|E(t, k)|^2, \quad (2)
\]

where the QL diffusion coefficient is given by

\[
D_{QL}(t, v) = \pi \sum_{k \in \mathbb{Z}} |E(t, k)|^2 \delta(\omega(t, k) - kv), \quad (3)
\]

The real functions \((t, k) \mapsto \omega(t, k)\) and \((t, k) \mapsto \gamma(t, k)\) satisfy the following dispersion equation,

\[
\mathbb{D}(k, \omega(t, k) + i\gamma(t, k)) = 0, \quad \text{with}
\]

\[
\mathbb{D}(k, \omega(t, k) + i\gamma(t, k)) := 1 + \frac{\omega_p^2}{k^2} \int_{\mathbb{R}} dv \frac{k \partial_v f(t, v)}{\omega(t, k) - kv + i\gamma(t, k)}, \quad (4)
\]

where \(\omega_p\) is the plasma frequency [57, 26, 21, 39]. The system (1)-(4) is a closed and self-consistent system of equations. We recall that in the case of the gentle-bump-on-the-tail instability an approximate solution of the dispersion equation is given (see, e.g., [39]) by the so-called Bohm–Gross approximation

\[
D_{QL}(t, v) = \sum_{k \in \mathbb{Z}} E(t, k)|^2 \frac{\omega(t, k) - kv}{(\omega(t, k) - kv)^2 + \gamma(t, k)} \quad \text{in the resonant region of velocity space that corresponds to particle velocities satisfying } \omega(t, k) - kv = 0.
\]

1In fact, this diffusion coefficient is a formal approximation for small \(\gamma\) of the quasilinear diffusion coefficient \(D_{QL}(t, v) = \sum_{k \in \mathbb{Z}} E(t, k)|^2 \frac{\omega(t, k) - kv}{(\omega(t, k) - kv)^2 + \gamma(t, k)}\) in the resonant region of velocity space that corresponds to particle velocities satisfying \(\omega(t, k) - kv = 0\).
relation,
\[ \omega^2(t, k) \simeq \omega^2(k) := \omega_p^2 \left(1 + 3k^2 \lambda_D^2 \right), \quad (5) \]

with the Debye length \( \lambda_D := v_{th} / \omega_p \) and the thermal velocity squared \( \nu_{th}^2 := \int v^2 dv / \int f dv \). The approximate growth rate is given by

\[ \gamma(t, k) \simeq \frac{\pi}{2} \frac{\omega_p^2}{k^2} \omega(k) \int_\mathbb{R} dv \delta(\omega(k) - kv) k \partial_v f(t, v). \quad (6) \]

Therefore the system constituted by (1)-(2) and (5)-(6), is also a closed and self-consistent system. This approximate solution relies on the following assumptions: \( v \rightarrow f(t, v) \) is even, \( \gamma / \omega \ll 1 \) (weak instability) and \( k \lambda_D \ll 1 \) (long wavelength approximation) (see, e.g., [30] for more details). Let us note that the dispersion equation and its approximate solution are the same as for the Landau damping case, where the damping rate \( \gamma \) given by (6) is negative because the slope of \( f \) so is.

We must emphasize that, even from a physical and physicists point of view, the derivation of quasilinear theory from either a deterministic or a probabilistic approach is actually not clear. Indeed the original 1962 derivation [26, 57], briefly exposed above, is deterministic. Right after there were many other derivations of the QL theory, most of them (see, e.g., [21, 3, 5] and references therein) appeal to some statistical arguments, like the random phase approximation (RPA), and invoke some time/space decorrelation hypotheses. From a numerical point of view, it has been shown in [10], that a statistical ensemble average of solutions of the Vlasov–Poisson system is required to recover a QL description of the long time behavior of the weak warm beam-plasma system.

In this work, we consider both the self-consistent deterministic case and the non-self-consistent stochastic case. Here, the term “self-consistent” means that the electric field is produced by the particles themselves, through the coupling with the Poisson equation. In the self-consistent deterministic case, we show that the asymptotic distribution function is homogenized in the space variables, while the self-consistent electric field converges weakly to zero. As already observed in related works (e.g. [4]), we show that the lack of time compactness for the electric field is compulsory to obtain a non-trivial and thus a diffusion limit for the Vlasov equation. By contrast, the time compactness property leads to a cheap version of the Landau damping, where the electric field converges strongly to zero (entailing a null diffusion matrix) and the distribution function converges weakly to a spatially homogeneous stationary solution of the Vlasov-Poisson system. Actually the difficult part is to show a non-zero diffusion limit in the presence of fast time oscillations. Using a Duhamel formula, we formally derive a diffusion equation for the asymptotic distribution function, which depends only on the time and velocity variables. Unfortunately we are not able to justify rigorously this diffusion limit. This task requires a new approach, which will be the matter of a future work. It is worthwhile to mention that in the nonlinear regime, the saturation of the weak warm beam-plasma instability generates in phase space a type of turbulence, which has a very close connection to Hamiltonian chaos theory (see, e.g., [8, 35] and references therein). A complete treatment of the self-consistent deterministic case remains an open issue, the proof of which must be based at least (but not only) on the same ingredients than those used for proving the Landau damping [48], and more particularly on the control of nonlinear wave-wave interactions (e.g. plasma echoes). In contrast with Landau damping, the main and not the least difficulty is that perturbations are not arbitrarily small, since wave amplitudes are amplified by the instability. From a mathematical point of view, this makes the nonlinear wave-wave interactions more difficult to control, especially for showing that the latter remains negligible at least at the end of the relaxation process. The proof of QL diffusion for this “inverse landau damping” problem remains a challenge. Nevertheless, studying the non-self-consistent problem remains meaningful. Indeed, it was observed in numerical simulations of the self-consistent problem [11], that when the distribu-
tion function is enough phase-space homogenized (after quite a long time), the problem falls into
the non-self-consistent framework, even in the strong nonlinear regime.

As explained in plasma physics literature (see, e.g., [21, 3]), diffusion in the QL theory comes
from the time decorrelation property of the electric field, which can be considered as a random field.
We then place ourselves in similar modelling hypotheses. This second framework is then closer to
the case of particles evolving in a given bath of (random) waves [18, 9, 31, 29, 30] or particles subject
to a reversible reflection law, which has convenient mixing properties [6]. As a result, we prove that
the asymptotic limit of the Vlasov equation is a diffusion equation in the velocity space, where the
diffusion matrix is given by the space-time autocorrelation function of the stochastic electric field,
the so-called Reynolds electric stress tensor. Hence the diffusion matrix is quadratic with respect
to the electric field. By specializing a little bit more the structure of our electric field, we recover,
at least from a formal point of view, the diffusion matrix predicted by the QL theory. Our diffusion
matrix can also be related to a refinement of the QL theory called the resonance broadening theory
[27, 1, 58, 51, 21]. For the present problem and to our knowledge, our results have not been found in
the literature so far. For the proof of the non-self-consistent stochastic case, we follow the strategy
introduced in the paper [50] which relies on short-time decorrelation properties. These techniques
have been successfully used in various physical contexts [47, 7, 15, 20, 37].

The outline of this paper is as follows. Section 2 describes the weak turbulent regime, which
is characterized by some dimensionless parameters. In Section 3 we deal with the self-consistent
deterministic case. In Section 4, we deal with the non-self-consistent stochastic case. Section 4.1
collects all the hypotheses on the stochastic electric field. Section 4.2 contains our main result
about the diffusion limit of the Vlasov equation, the proof of which is done in Section 4.3. Finally
Section 4.4 connects our result with some kinetic turbulence theories of plasma physics such as the
resonance broadening theory and the quasilinear theory.

2 The weak turbulent regime

2.1 Dimensionless parameters

The Vlasov–Poisson system, describing the self-consistent evolution of the distribution function of
particles \( f = f(t, x, v) \) in an electrostatic plasma, reads

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f + \frac{q}{m} E \cdot \nabla_v f &= 0, \\
E &= -\nabla \Phi, \quad -\Delta \Phi = \frac{q}{\varepsilon_0} \left( \int_{\mathbb{R}^d} dv f - 1 \right).
\end{align*}
\]

Here \( t \in \mathbb{R}, x \in \mathbb{T}^d := (\mathbb{R}/2\pi \mathbb{Z})^d \), and \( v \in \mathbb{R}^d \), represent respectively time, position, and velocity of
particles of charge \( q \) and mass \( m \), which are accelerated by the “turbulent” electric field \( E = E(t, x) \).
Since the plasma is globally neutral we have

\[
\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} dx \int_{\mathbb{R}^d} dv f = 1.
\]

In order to have a well-posed problem, we must add the zero-mean electrostatic condition

\[
\int_{\mathbb{T}^d} dx E = 0.
\]

Indeed the condition (10) is necessary to invert the Laplacian operator \( \Delta \). The phase space is
denoted by \( Q := \mathbb{T}^d \times \mathbb{R}^d \). In order to write the Vlasov–Poisson system (7)-(8) in a dimensionless
form, we need to introduce a time unit \( \hat{t} \), a length unit \( \hat{x} \), a velocity unit \( \hat{v} \), and typical amplitudes
\( \hat{E}, \hat{\Phi} \) and \( \hat{f} \) for the electric field, the electric potential and the distribution function respectively. The dimensionless variables and physical quantities read

\[
t' = \frac{t}{\hat{t}}, \quad x' = \frac{x}{\hat{x}}, \quad v' = \frac{v}{\hat{v}}, \quad E' = \frac{E}{\hat{E}}, \quad \Phi' = \frac{\Phi}{\hat{\Phi}}, \quad f' = \frac{f}{\hat{f}}.
\]

We set

\[
\hat{n} := \hat{f} \hat{v} d,
\]

the typical value of the macroscopic (charge) density of particles. Using the Poisson equation (8), we obtain the following dimensional equation,

\[
\hat{E} = \hat{n} \hat{x} q / \varepsilon_0.
\]

In an electrostatic plasma, the typical length scale is the Debye length \( \lambda_D \), while the typical velocity is the thermal velocity \( v_{th} \). The plasma frequency \( \omega_p \), which is related to a typical fast oscillation time of an electrostatic plasma, is then given by

\[
\omega_p = \frac{v_{th} \lambda_D}{\lambda_D}.
\]

The typical electric and kinetic energies are respectively \( \mathcal{E}_{el} = \varepsilon_0 |\hat{E}|^2 \), and \( \mathcal{E}_{kin} = \hat{n} \hat{m} \hat{v}^2 \). The distribution function \( f \) has a typical evolution/relaxation time \( \tau_{rel} \), while the turbulent electric field has two time scales. A slow time scale \( \tau_L \) is associated with the instantaneous growth or damping rate \( \gamma_L := 1/\tau_L \) of the electric field, while a fast time scale is related to both the wave (electric field) autocorrelation time \( \tau_{ac} \) and the particle autocorrelation time \( \tau_D \). The time \( \tau_{ac} \) is the lapse of time needed for a resonant particle, traveling at the same velocity as the phase velocity of a typical wave, to cross the localized spatial extent of the oscillatory electric field disturbance. This time can also be seen as the time needed for a resonant particle to resolve the finite frequency width of the wave spectrum. In other words the time \( \tau_{ac} \) can be seen as the turnover or the life time of a typical wave measured or felt by a resonant particle traveling at the same velocity as the phase velocity of this wave. Then the synchronization between a wave and a particle occurs in a lapse of time of the order of \( \tau_{ac} \), during which they interact by momentum transfer. The time \( \tau_D \) is the autocorrelation or spreading time of particles, i.e. the lapse of time after which two close particles or orbits are completely separated from each other. In the plasma physics literature, the time \( \tau_D \) is called the Dupree time [27]. A particle distribution function evaluated at two different times separated by a time interval of the order of \( \tau_D \) is then decorrelated. The relaxation time \( \tau_{rel} \) of the distribution function is then of the order of \( \tau_D \). In the self-consistent case, where the Poisson equation is used to compute the electric field from the particle distribution function, this implies that two evaluations in time of the electric field, separated by a time interval of the order of \( \tau_D \), are also decorrelated. We now set

\[
\hat{t} := \tau_L, \quad \hat{x} := \lambda_D, \quad \hat{v} := v_{th}.
\]

Let \( \varepsilon \in (0,1) \) be a small dimensionless parameter and \( \tau \in [0,\infty] \) be a positive dimensionless parameter, which may be finite or infinite. Then, the weak turbulence regime of an electrostatic plasma is defined by (see, e.g., Chapter 7 in [21]),

\[
\frac{\mathcal{E}_{el}}{\mathcal{E}_{kin}} = \varepsilon_0 |\hat{E}|^2 / \hat{n} \hat{m} \hat{v}^2 = \varepsilon, \quad \frac{1}{\omega_p \hat{t}} = \varepsilon^2, \quad \frac{\tau_{ac} \hat{t}}{\hat{t}} = \varepsilon^2, \quad \frac{\tau_D \hat{t}}{\hat{t}} = \tau.
\]
Using (11)-(16) and dropping the prime notation for dimensionless variables and physical quantities, we obtain from (7)-(8), the dimensionless Vlasov–Poisson equations,

\[
\partial_t f^\varepsilon + \frac{v}{\varepsilon^2} \cdot \nabla_x f^\varepsilon + \frac{E^\varepsilon}{\varepsilon} \cdot \nabla_v f^\varepsilon = 0, \quad (t, x, v) \in \mathbb{R}^+ \times \mathbb{T}^d \times \mathbb{R}^d,
\]

(17)

\[
E^\varepsilon = -\nabla \Phi^\varepsilon, \quad -\Delta \Phi^\varepsilon = \int_{\mathbb{R}^d} dv \, f^\varepsilon - 1.
\]

(18)

The global neutrality condition (9) and the zero-mean electrostatic condition (10) keep the same. We just have to substitute \( f^\varepsilon \) to \( f \) in (9) and \( E^\varepsilon \) to \( E \) in (10).

### 2.2 Notation

In the rest of this paper, the notation \( \langle \cdot, \cdot \rangle \) denotes the duality bracket between the space of distributions \( \mathcal{D}'(\mathbb{R}^+ \times Q) \) and the space \( \mathcal{D}(\mathbb{R}^+ \times Q) \) of indefinitely differentiable functions with compact support in \( \mathbb{R}^+ \times Q \). The \( L^2 \)-scalar product on the phase space \( Q = \mathbb{T}^d \times \mathbb{R}^d \) is defined by

\[
\langle f, g \rangle := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} dx \int_{\mathbb{R}^d} dv \, f^g, \quad \forall f, g \in L^2(Q),
\]

(19)

where the notation \( (\cdot)^* \) stands for the complex conjugate. We then have, for \( g \in L^1_{\text{loc}}(\mathbb{R}^+ \times Q) \),

\[
\langle g, \varphi \rangle := \int_{\mathbb{R}^+} dt \, (g, \varphi), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^+ \times Q).
\]

We denote the space average on the torus \( \mathbb{T}^d \) by

\[
\int dx \, g(t, x, v) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} dx \, g(t, x, v).
\]

The one-parameter family of functions \( \{g^\varepsilon\}_{\varepsilon>0} \), which we call sequences (respectively, subsequences) by abuse of language, must be understood as generalized sequences (respectively, subsequences) such as nets (respectively, subnets) in the sense of Moore–Smith or filters (respectively, finer filters) in the sense of Cartan (for more details see, e.g., references [13, 59]). We also use the notation \( g^\varepsilon \) to denote the cluster point, at least in the sense of distributions, of a family of functions \( \{g^\varepsilon\}_{\varepsilon>0} \). We next define the free-flow operator by

\[
\mathcal{L} := v \cdot \nabla_x.
\]

We note \( t \mapsto S_t^\varepsilon \) the group on \( L^p(Q) \), \( 1 \leq p \leq \infty \), generated by the free-flow operator \( \varepsilon^{-2} \mathcal{L} \). Then, an explicit formula for the group \( t \mapsto S_t^\varepsilon \) is given by

\[
(S_t^\varepsilon g)(x, v) = \exp \left( -\frac{t}{\varepsilon^2} \mathcal{L} \right) g(x, v) = g(x - vt/\varepsilon^2, v), \quad \forall g \in L^p(Q).
\]

(20)

Eventually, the symbol \( |\cdot| \) denotes either the modulus or the Euclidean norm depending on whether we deal with complex/real scalars or vectors.

### 3 The self-consistent deterministic case

In this section we deal with the self-consistent deterministic case.
3.1 Ergodic theorem

We observe two orthogonal behaviors for the asymptotic limit of the Vlasov-Poisson system (17)-(18), depending on whether one makes or not the hypothesis of time compactness. The cornerstone of such observations is the ergodic property of the free-flow operator $\mathcal{L}$ on the torus, we recall in

**Lemma 1** (Ergodicity of the free flow on the torus) Let $g \in L^p(Q)$, with $1 \leq p \leq \infty$, satisfy

$$\mathcal{L}g = 0, \quad \text{in } \mathcal{D}'(Q).$$

Then $g = \int dx \, g$ ($g$ is independent of the variable $x$).

**Proof.** We first integrate the free-flow operator by using characteristic curves, and second we use the spatial Fourier transform of the obtained solution. Indeed, the characteristic curves $(X(\tau), V(\tau))$ of the free flow satisfy the ODEs $\dot{X}(\tau) = V(\tau)$, $V(\tau) = 0$, with initial conditions $X(0) = x_0$, and $V(0) = v_0$. Its solution is given by $(X(\tau) = x_0 + v_0 \tau, V(\tau) = v_0)$. Since $(dg/d\tau)(X(\tau), V(\tau)) = (v \cdot \nabla_x g)(X(\tau), V(\tau)) = 0$, we obtain $g(X(\tau), V(\tau)) = g(X(0), V(0))$ or $g(x, v) = g(x - v \tau, v)$, for a.e. $(x, v) \in Q$ and all $\tau \in \mathbb{R}^+$. The Fourier transform in space of this last equation gives

$$(1 - \exp(ik \cdot v \tau)) \hat{g}(k, v) = 0, \quad \forall k \in \mathbb{Z}^d, \quad v \in \mathbb{R}^d, \quad \tau \in \mathbb{R}.$$

This relation and $g \in L^p(Q)$, with $1 \leq p \leq \infty$ ($g$ cannot be a Dirac mass in velocity), imply that the support of $\hat{g}$ is contained in the set $\text{supp}(\hat{g}) := \{(k, v) \in \mathbb{Z}^d \times \mathbb{R}^d \mid k \cdot v \tau \in 2\pi \mathbb{Z}, \forall \tau \in \mathbb{R}\}$. For any $\delta, T, r, R > 0$, such that $\delta < T$, and $r < R$, the Lebesgue measure of the set $\text{supp}(\hat{g})$ for $\delta < \tau < T$ and $r < |v| < R$ is zero. This forces $\hat{g}$ to be equal to zero for all $k \neq 0$ and then $\text{supp}(\hat{g}) := \{(k, v) \mid k = 0, \quad v \in \mathbb{R}^d\}$. Therefore $g = \int dx \, g$ and $g$ is independent of the variable $x$. This ends the proof of Lemma 1.

**Remark 1** The Proof of Lemma 1 is reminiscent of the Proof of Theorem 2.1 in [17] but it is not exactly the same. Indeed, here we do not use the Riemann–Lebesgue lemma (in the Fourier dual variable of $v$) whereas [17] does. More precisely the proof of Theorem 2.1 in [17], which is based on the weak formulation of the equation $g(x, v) = g(x - v \tau, v)$ against continuous compactly supported test functions, uses first the Fourier transform in the phase space $Q$ to switch from real variables to Fourier variables in the weak formulation, and it uses second the Riemann–Lebesgue lemma together with the Lebesgue dominated convergence theorem to pass to the limit.

Based on the ergodicity of the free flow, we state

**Theorem 1** Let $\{f^\varepsilon_0\}_{\varepsilon > 0}$ be a sequence of non-negative initial data and $C_0$ be a positive constant such that

$$\|f^\varepsilon_0\|_L^1(Q) + \|f^\varepsilon_0\|_L^\infty(Q) \leq C_0, \quad \int_Q \! f^\varepsilon_0 |v|^2 \, dx \, dv \leq C_0, \quad \text{and}$$

$$\left\| E^\varepsilon_0 := \nabla \Delta^{-1}\left( \int_{\mathbb{R}^d} f^\varepsilon_0 \, dv - 1 \right) \right\|_{L^2(\mathbb{T}^d)} \leq C_0.$$

Let $(f^\varepsilon, E^\varepsilon)_{\varepsilon > 0}$ be a sequence of weak solutions of the Vlasov–Poisson system (17)-(18), with initial data $f^\varepsilon |_{t=0} = f^\varepsilon_0$, the existence of which has been proved in [4, 22, 23, 12] for all $\varepsilon > 0$. Then,
(i) There exists a function \( f = f(t,v) \), independent of the variable \( x \), such that \( f \in L^\infty(\mathbb{R}^+;L^1 \cap L^\infty(\mathbb{R}^d)) \), and up to subsequences one has,

\[
f^\varepsilon \rightharpoonup f \quad \text{in} \quad L^\infty(\mathbb{R}^+;L^\infty(Q)) \quad \text{weak-*},
\]

\[
\int dx f^\varepsilon \rightharpoonup f \quad \text{in} \quad L^\infty(\mathbb{R}^+;L^\infty(\mathbb{R}^d)) \quad \text{weak-*}.
\]

(ii) The electric field \( E^\varepsilon \) converges weakly to zero as \( \varepsilon \to 0 \), more precisely,

\[
E^\varepsilon \rightharpoonup 0 \quad \text{in} \quad L^\infty(\mathbb{R}^+;W^{1,1+2/d}(\mathbb{T}^d)) \quad \text{weak-*}.
\]

(iii) The expression

\[
\nabla_v \cdot \int dx \frac{E^\varepsilon f^\varepsilon}{\varepsilon},
\]

is uniformly (with respect to \( \varepsilon \)) bounded in \( \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d) \); hence, up to a subsequence, it converges in \( \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d) \) and we obtain

\[
\partial_t f + \nabla_v \cdot \int dx \frac{E^\varepsilon f^\varepsilon}{\varepsilon} = 0, \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d),
\]

\[
f_{t=0} = \int dx f_0.
\]

(iv) Let \( d \leq 4 \). Moreover, if we suppose that there exists a constant \( \kappa \), independent of \( \varepsilon \) such that

\[
\|E^\varepsilon\|_{W^{1,1}^{s}(\mathbb{R}^+;L^1(\mathbb{T}^d))} \leq \kappa, \quad \text{with} \quad s > 0, \quad \text{and} \quad \|\partial_t \Phi^\varepsilon\|_{L^{1}(\mathbb{R}^+;L^1(\mathbb{T}^d))} \leq \kappa,
\]

then,

\[
\int dx \frac{E^\varepsilon f^\varepsilon}{\varepsilon} \rightharpoonup 0 \quad \text{in} \quad \mathcal{D}'([0,T] \times \mathbb{R}^d),
\]

\[
E^\varepsilon \rightharpoonup 0 \quad \text{in} \quad L^1([0,T] \times \mathbb{T}^d) \quad \text{strong},
\]

as \( \varepsilon \to 0 \), and equations \((21)-(22)\) degenerate into the following equations,

\[
\partial_t f = 0 \quad \text{in} \quad \mathcal{D}'([0,T] \times \mathbb{R}^d),
\]

\[
f_{t=0} = \int dx f_0.
\]

**Proof.** Since \( \|f_0\|_{L^1(Q)} + \|f_0\|_{L^\infty(Q)} \leq C_0 < \infty \), by weak compactness arguments there exists a function \( f_0 \in L^1 \cap L^\infty(Q) \) such that \( f^\varepsilon_0 \) (up to a subsequence) converges in \( L^\infty(Q) \) weak-* to \( f_0 \). Indeed from \( \|f^\varepsilon_0\|_{L^1(Q)} + \|f^\varepsilon_0\|_{L^\infty(Q)} \leq C_0 < \infty \), and using standard weak compactness theorems, we obtain that there exists \( f_0 \in \mathcal{M}_b \cap L^\infty(Q) \) such that \( f^\varepsilon_0 \rightharpoonup f_0 \) in \( \mathcal{M}_b \cap L^\infty(Q) \) weak-*.

\( \mathcal{M}_b(Q) \) is the set of bounded measures on \( Q \). Moreover \( \|f^\varepsilon_0\|_{L^1(Q)} + \|f^\varepsilon_0\|_{L^\infty(Q)} \leq C_0 < \infty \) implies that \( \|f^\varepsilon_0\|_{L^p(Q)} \leq C_0 < \infty \), for \( 1 \leq p \leq \infty \). Therefore we have also \( f^\varepsilon_0 \rightharpoonup f_0 \) in \( L^p(Q) \) weak-*, for \( 1 < p \leq \infty \). This and the De La Vallé-Poussin theorem on the criterion for uniform equi-integrability implies that the family \( \{f^\varepsilon_0\}_{\varepsilon>0} \) is uniformly equi-integrable. Finally, using Dunford-Pettis theorem, uniform equi-integrability implies that we also have \( f^\varepsilon_0 \rightharpoonup f_0 \) in \( L^1(Q) \) weak. Therefore, \( f_0 \in L^1 \cap L^\infty(Q) \).

From the standard theory of existence of weak solutions for the Vlasov–Poisson system \([4, 22, 23, 12]\), we obtain for all \( \varepsilon > 0 \),

\[
\|f^\varepsilon(t)\|_{L^p(Q)} \leq \|f^\varepsilon_0\|_{L^p(Q)} \leq C_0 < \infty, \quad 1 \leq p \leq \infty.
\]
Then, by weak compactness arguments there exists a function \( f \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(Q)) \), such that \( f^\varepsilon \) (up to a subsequence) converges in \( L^\infty(\mathbb{R}^+; L^\infty(Q)) \) weak-* to \( f \), and \( \int dx f^\varepsilon \) (up to a subsequence) converges in \( L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^d)) \) weak-* to \( \int dx f \). From properties of weak solutions for the Vlasov–Poisson system, weak solutions of \([17, 18]\) satisfy the following a priori bound:

\[
\mathcal{E}(t) \leq \mathcal{E}(0), \quad \forall t \geq 0, \tag{28}
\]

where the total energy \( \mathcal{E}(t) \) is defined by

\[
\mathcal{E}(t) := \frac{1}{2} \int_Q dx dv |v|^2 f^\varepsilon(t, x, v) + \frac{\varepsilon}{2} \int_{\mathbb{T}^d} dx |E^\varepsilon(t, x)|^2.
\]

From \((28)\) and the initial data assumptions of Theorem \([1]\), we infer that there exists a constant \( K_0 \), depending on \( C_0 \) such that

\[
\int_Q dx dv |v|^2 f^\varepsilon(t, x, v) \leq K_0, \quad \text{and} \quad \|E^\varepsilon\|_{L^\infty(\mathbb{R}^+; L^2(\mathbb{T}^d))} \leq \frac{K_0}{\varepsilon}. \tag{29}
\]

Taking \( \varphi \in D(\mathbb{R}^+ \times Q) \) as a test function and using the \( L^2\)-scalar product \([19]\), the weak formulation of Vlasov equation \((17)\) reads

\[
\varepsilon^2 \langle f^\varepsilon, \partial_t \varphi \rangle + \langle f^\varepsilon, v \cdot \nabla_x \varphi \rangle + \varepsilon \langle f^\varepsilon, E^\varepsilon \cdot \nabla_v \varphi \rangle = 0. \tag{30}
\]

Using the \( L^\infty\)-bound for \( f^\varepsilon \), and a priori estimate \((29)\) for the electric field \( E^\varepsilon \), we obtain

\[
|\varepsilon^2 \langle f^\varepsilon, \partial_t \varphi \rangle| \leq \varepsilon^2 (2\pi)^{-d} \|\partial_t \varphi\|_{L^1(\mathbb{R}^+ \times Q)} \|f^\varepsilon\|_{L^\infty(\mathbb{R}^+ \times Q)} \leq C \varepsilon^2, \tag{31}
\]

and

\[
\varepsilon |\langle f^\varepsilon, E^\varepsilon \cdot \nabla_v \varphi \rangle| \leq \varepsilon \|E^\varepsilon\|_{L^\infty(\mathbb{R}^+; L^2(\mathbb{T}^d))} \|f^\varepsilon\|_{L^\infty(\mathbb{R}^+; L^2(Q))} \|\nabla_v \varphi\|_{L^1(\mathbb{R}^+; L^2(\mathbb{R}^d; L^\infty(\mathbb{T}^d))} \leq C \sqrt{\varepsilon}. \tag{32}
\]

Using \((31)-(32)\) to pass to the limit \( \varepsilon \to 0 \) in \((30)\), we obtain

\[
v \cdot \nabla_x f = 0 \quad \text{in} \quad D'(\mathbb{R}^+ \times Q).
\]

From Lemma \([1]\) we infer that \( f \) is independent of \( x \) and \( \int dx f(t) = f(t), \) for a.e. \( t > 0. \) This proves point i). For proving point ii), we first define the charge density \( \rho^\varepsilon \) by

\[
\rho^\varepsilon = \int_{\mathbb{R}^d} dv f^\varepsilon.
\]

From a standard interpolation inequality (see e.g., \([22, 23, 12]\)), there exists a constant \( C_d \) depending on \( d \) such that

\[
\|\rho^\varepsilon\|_{L^\infty(\mathbb{R}^+; L^{1+2/d}(\mathbb{T}^d))} \leq C_d \|f^\varepsilon\|_{L^\infty(\mathbb{R}^+ \times Q)}^{2/(2+d)} \|f^\varepsilon\|_{L^\infty(\mathbb{R}^+; L^1(Q))}^{d/(d+2)} \leq \kappa_0 < \infty, \tag{33}
\]

where the constant \( \kappa_0 \) depends on \( C_0 \), but is independent of \( \varepsilon \). From the Poisson equation \((18)\), the bound \((33)\) on the charge density \( \rho^\varepsilon \) and standard elliptic regularity estimates, we obtain

\[
\|E^\varepsilon\|_{L^\infty(\mathbb{R}^+; W^{1,1+2/d}(\mathbb{T}^d))} \leq c_0 < \infty, \tag{34}
\]

where the constant \( c_0 \) depends on initial data but is independent of \( \varepsilon \). Then, by weak compactness there exists a function \( E \in L^\infty(\mathbb{R}^+; W^{1,1+2/d}(\mathbb{T}^d)) \) such that \( E^\varepsilon \) (up to a subsequence) converges in
\( L^\infty(\mathbb{R}^+; W^{1,1+2/d}(\mathbb{T}^d)) \) weak-* to \( E \). To determine the limit point \( E \), we use the Poisson equation (18). Observing that
\[
\int_{\mathbb{R}^d} dv \, f = 1,
\]
and passing to the limit \( \varepsilon \to 0 \) in the Poisson equation (18), we obtain
\[
\Delta \Phi = 0, \quad \text{in} \ D'(\mathbb{R}^+ \times \mathbb{T}^d),
\]
which leads to \( \Phi = 0 \) and \( E = 0 \) in \( D'(\mathbb{R}^+ \times \mathbb{T}^d) \). This ends the proof of point \( ii \). For point \( iii \), using the \( L^2 \)-scalar product (19), we first write the following weak formulation of the Vlasov equation (17) being previously averaged in space,
\[
\left\langle f^\varepsilon, \partial_t \varphi \right\rangle + \left\langle \frac{E^\varepsilon f^\varepsilon}{\varepsilon}, \nabla_v \varphi \right\rangle = 0, \quad \forall \varphi \in D(\mathbb{R}^+ \times \mathbb{R}^d).
\tag{35}
\]
Using a priori estimates (27) or point \( i \) of Theorem 1, we obtain from (35),
\[
\left| \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \varphi \cdot \nabla_v \varphi \cdot \int dx \frac{E^\varepsilon f^\varepsilon}{\varepsilon} \right| \leq \| f^\varepsilon \|_{L^\infty([0,T] \times \mathbb{Q})} \| \partial_t \varphi \|_{L^1(\mathbb{R}^+ \times \mathbb{R}^d)} \leq C < \infty,
\]
where \( C \) is independent of \( \varepsilon \). This implies that \( \nabla_v \cdot \int dx E^\varepsilon f^\varepsilon / \varepsilon \) (up to a subsequence) converges in \( D'(\mathbb{R}^+ \times \mathbb{R}^d) \). Then, using point \( i \) of Theorem 1, we can pass to the limit \( \varepsilon \to 0 \) in (35) to obtain equation (21). For proving point \( iv \), we start by establishing some strong convergence properties for the sequences \( E^\varepsilon \) and \( \Phi^\varepsilon \). Using \( \| E^\varepsilon \|_{W^{s,p}(\mathbb{R}^+; L^1(\mathbb{T}^d))} \leq \kappa \), with \( s > r \) (assumption (23) of Theorem 1), and \( E^\varepsilon \in L^\infty(\mathbb{R}^+; W^{1,1+2/d}(\mathbb{T}^d)) \), we obtain, from a Lions–Aubin theorem [55], that the sequence
\[
E^\varepsilon \quad \text{is compact in} \quad L^{1+2/d}([0,T] \times \mathbb{T}^d) \quad \text{strong,} \quad \forall T > 0.
\tag{36}
\]
We next deal with the sequence \( \Phi^\varepsilon \). Using the Poisson equation (18), the bound (33) on the charge density \( \rho^\varepsilon \) and standard elliptic regularity estimates, we obtain that \( \Phi^\varepsilon \in L^\infty(\mathbb{R}^+; W^{2,1+2/d}(\mathbb{T}^d)) \). Using the Sobolev embedding \( W^{s,p}(\mathbb{R}^d) \hookrightarrow W^{r,q}(\mathbb{R}^d) \), with \( s > r \), \( d > (s-r)p \) and \( p \leq q \leq dp/(d - (s-r)p) \), we obtain
\[
L^\infty(\mathbb{R}^+; W^{2,1+2/d}(\mathbb{T}^d)) \hookrightarrow L^\infty(\mathbb{R}^+; W^{s,1+2/d}(\mathbb{T}^d)) \tag{37}
\]
with \( \max\{0, 2 - d/(1 + 2/d)\} < \delta = \max\{2, (-d^2 + 4d + 4)/(d+2)\} \), and \( d \leq 4 \). Using the embedding (37), and the bound \( \| \partial_t \Phi^\varepsilon \|_{L^\infty(\mathbb{R}^+; L^1(\mathbb{T}^d))} \leq \kappa \), (assumption (23) of Theorem 1), we obtain from a Lions–Aubin theorem [55], that the sequence
\[
\Phi^\varepsilon \quad \text{is compact in} \quad L^{1+d/2}([0,T] \times \mathbb{T}^d) \quad \text{strong,} \quad \forall T > 0.
\tag{38}
\]
Multiplying the Vlasov equation (17) by \( \varepsilon \Phi^\varepsilon \), then averaging in space, multiplying the result by a test function \( \varphi \in D([0,T] \times \mathbb{R}^d) \) and finally integrating with respect to the time and velocity variables, we obtain
\[
\varepsilon \int_0^T dt \int_{\mathbb{R}^d} dv \int dx \varphi \Phi^\varepsilon \partial_t f^\varepsilon + \frac{1}{\varepsilon} \int_0^T dt \int_{\mathbb{R}^d} dv \int dx \varphi \Phi^\varepsilon v \cdot \nabla_x f^\varepsilon + \int_0^T dt \int_{\mathbb{R}^d} dv \int dx \varphi \Phi^\varepsilon \nabla_v \cdot (E^\varepsilon f^\varepsilon) = 0. \tag{39}
\]
Using integration by parts, we obtain from (39),
\[ \varepsilon \int_0^T dt \int_{\mathbb{R}^d} dv \int dx f^\varepsilon (\Phi^\varepsilon \partial_t \varphi + \varphi \partial_t \Phi^\varepsilon) - \int_0^T dt \int_{\mathbb{R}^d} dv \varphi v \cdot \int dx \frac{E^\varepsilon f^\varepsilon}{\varepsilon} + \int_0^T dt \int_{\mathbb{R}^d} dv \int dx f^\varepsilon \Phi^\varepsilon E^\varepsilon \cdot \nabla v \varphi = 0. \tag{40} \]

Using the $L^\infty$-bound (27) for $f^\varepsilon$ and assumption (23), we obtain, for the first term of (40),
\[ \varepsilon \left| \int_0^T dt \int_{\mathbb{R}^d} dv \int dx f^\varepsilon (\Phi^\varepsilon \partial_t \varphi + \varphi \partial_t \Phi^\varepsilon) \right| \leq \varepsilon (2\pi)^{-d} \| f^\varepsilon \|_{L^\infty([0,T] \times \mathbb{R}^d)} \left( \| \partial_t \varphi \|_{L^\infty(0,T; L^1(\mathbb{R}^d))} + \| \varphi \|_{L^\infty(0,T; L^1(\mathbb{R}^d))} \right) \leq C \varepsilon. \tag{41} \]

Using (36) and (38), we infer that the product $\Phi^\varepsilon E^\varepsilon$ converges strongly in $L^1([0,T] \times \mathbb{R}^d)$. Using this strong convergence, the weak convergence of $f^\varepsilon$ in $L^\infty([0,T] \times Q)$ weak–∗, and the fact that the limit point of $E^\varepsilon$ vanishes, we obtain for the third term of (40),
\[ \left| \int_0^T dt \int_{\mathbb{R}^d} dv \int dx f^\varepsilon \Phi^\varepsilon E^\varepsilon \cdot \nabla v \varphi \right| \rightarrow 0, \text{ as } \varepsilon \rightarrow 0. \tag{42} \]

Using (40)-(42), we obtain
\[ \int_0^T dt \int_{\mathbb{R}^d} dv \varphi v \cdot \int dx \frac{E^\varepsilon f^\varepsilon}{\varepsilon} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0. \tag{43} \]

Choosing a test function $\varphi$ such that $0_{\mathbb{R}^d} \notin \text{supp}(\varphi)$, and passing to the limit $\varepsilon \rightarrow 0$ in (35), we then obtain from (43),
\[ \int_0^T dt \int_{\mathbb{R}^d} dv f \partial_t \varphi = 0. \tag{44} \]

Since $f \in L^\infty([0,T] \times \mathbb{R}^d)$, equation (41) is valid for any $\varphi \in \mathcal{D}([0,T] \times \mathbb{R}^d)$; hence we obtain (21)-(26), which ends the proof. \qed

Few remarks on Theorem 1 are in order.

**Remark 2** (electric energy) *A priori estimate* (29) for $E^\varepsilon$ is not optimal. In fact, we have $E^\varepsilon \in L^\infty(\mathbb{R}^+; L^2(\mathbb{T}^d))$ for $d \leq 4$, uniformly with respect to $\varepsilon$. Indeed, for $d \leq 2$, it is obvious from estimate (34). For $3 \leq d \leq 4$, it comes from estimate (34) and the Sobolev embedding $W^{1,1+2/d}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$, with $q = d(d+2)/(d-2)/(d+1)$ and $d > 2$.

**Remark 3** (time compactness)

1. To obtain time compactness there are a priori three ways. The first one is to obtain time compactness for the electric field by using standard control on the charge current $j^\varepsilon$ (see, e.g., [22, 23, 12]), and the Ampère equation given by $\varepsilon^2 \partial_t E^\varepsilon + j^\varepsilon = 0$. This method fails because the presence of the factor $\varepsilon^2$ in front the time partial derivative does not give uniform bound (with respect to $\varepsilon$) for $\partial_t E^\varepsilon$. We obtain the same result from the charge conservation law (to obtain time compactness on the charge density $\rho^\varepsilon$, and thus on $E^\varepsilon$ via the Poisson equation), since the latter can be recovered by applying the spatial divergence operator to the
Ampère equation. The second method is to use averaging lemmas [24]. With this method, we only obtain compactness in the space variables but not in the time variable, because in the limit \( \varepsilon \to 0 \), the term \( \varepsilon^2 \partial_t f^\varepsilon \) in the Vlasov equation disappears [24, 53]. A third way is to obtain time compactness for the distribution function instead of the electric field. For this, we can show uniform convergence with respect to time in a weak topology for the phase-space variables. Showing time equi-continuity for the distribution function requires using the Vlasov equation. Here again, the presence of the factor \( \varepsilon^2 \) in front of the term \( \partial_t f^\varepsilon \) in the Vlasov equation makes this method to fail.

2. Point iv) of Theorem 1 shows that the lack of time compactness is in fact a necessary condition for obtaining a genuine or a non-degenerate diffusion equation in the limit \( \varepsilon \to 0 \).

Remark 4 (boundary conditions) As shown in the proof of Theorem 1, the relation \( v \cdot \nabla_x f = 0 \), is a direct consequence of a priori estimates. By a direct computation using Fourier series (see the proof of Lemma 1) it has been proved that the limit point \( f \) is independent of the space variable \( x \). This is the ergodic property of the torus. Obviously the same property is true when the torus is replaced by any domain where the free flow trajectory \( (x_0, v_0) \mapsto (x_0 + v_0 t, v_0) \) with specular reflections at the boundary are dense (this is a definition of ergodicity). Extending the present analysis to this more general case may be very useful.

Remark 5 (“cheap” Landau damping)

1. What we proved for the rescaled Vlasov–Poisson system, given by (17)–(18), is a “cheap” version of the Mouhot–Villani version of the Landau damping [48], i.e. that (under convenient hypotheses of regularity for initial conditions and smallness for initial perturbations) the self-consistent electric field \( E \) of the Vlasov–Poisson system (7)–(8) vanishes strongly when \( t \to +\infty \), while the distribution \( f \) relaxes, in a weak topology, towards a spatially homogeneous stationary solution of the Vlasov-Poisson system. Indeed, if \( \varepsilon \) is the ratio of the electric field \( E(t) \) of (7)–(8) to the electric field \( E^\varepsilon(t) \) of (17)–(18) (implying that \( |E^\varepsilon(t)| > |E(t)| \)), with the change of time scale \( t \to t/\varepsilon^2 \), the Vlasov–Poisson system (7)–(8) becomes the rescaled Vlasov–Poisson system (17)–(18). In other words, the limit \( t \to +\infty \) in (7)–(8) is equivalent to the limit \( \varepsilon \to 0 \) in (17)–(18), and \( \varepsilon = 1/\sqrt{t} \) stands for the smallest rate at which the electric field \( E(t) \) tends to zero when \( t \to +\infty \).

2. By considering the rescaled Vlasov–Poisson system (17)–(18) in the framework of the Landau damping, we observe that under the hypotheses of [48] the electric field converges strongly to zero. From Theorem 1, this strong convergence corresponds to a zero diffusion. The Mouhot–Villani result [48] is obtained for small perturbations (in some analytic norms) of a stable equilibrium profile (in velocity variables). Here, we are interested in unstable equilibrium profiles that lead to a non-zero diffusion in the velocity space.

The velocity diffusion operator should arise when we pass to the limit in the term \( \varepsilon^{-1} \nabla_v \cdot \int dx \ E^\varepsilon f^\varepsilon \). A rigorous proof of this fact remains an open issue and will be the matter of a future work. Nevertheless, we can show, at least formally, what is the structure of this term by using a simple iteration of the Duhamel formula. This is the aim of the next section.

3.2 Duhamel formula and Fick-type law

Here, we derive formally a Fick-type law for the flux term

\[
\int dx \ \frac{E^\varepsilon f^\varepsilon}{\varepsilon},
\]
appearing in \((21)\). Most of developments of this section are formal, but they allow us to point out the difficulties for showing rigorously the diffusion limit. This Fick-type law can be obtained from two ways. The first way is a global in time approach, which involves the initial condition \(f_0^\varepsilon\), while the second one, a local in time approach, does not. Each approach has its advantages (Lemmas 2 and 3) and drawbacks (Remarks 7 and 10). In addition, for both approaches, the absence of time decorrelation properties prevent us to determine the structure and the properties of the diffusion matrix. Nevertheless a formal WKB approximation allows us to obtain a non-negative diffusion matrix in the non-self-consistent case.

### 3.2.1 Global in time approach

Using the Duhamel formula and \((20)\), we obtain from the Vlasov equation \((17)\), the following representation formula for \(f^\varepsilon(t)\), solution to \((17) - (18)\),

\[
f^\varepsilon(t) = S_t^\varepsilon f_0^\varepsilon - \frac{1}{\varepsilon} \int_0^t ds \left( S_{t-s}^\varepsilon E^\varepsilon(s) \cdot \nabla_v f^\varepsilon(s) \right).
\]

Substituting \((45)\) into

\[- \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} dv \varphi \nabla_v \cdot \int dx \frac{E^\varepsilon f^\varepsilon}{\varepsilon} = \frac{1}{\varepsilon} \int_{\mathbb{R}^d} dv \int dx \nabla_v \varphi \cdot E^\varepsilon f^\varepsilon, \quad \forall \varphi \in D(\mathbb{R}^+ \times \mathbb{R}^d),\]

we obtain

\[- \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} dv \varphi \nabla_v \cdot \int dx \frac{E^\varepsilon f^\varepsilon}{\varepsilon} = T_1^\varepsilon(\varphi) + T_2^\varepsilon(\varphi),\]

where

\[T_1^\varepsilon(\varphi) := \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} dv \frac{1}{\varepsilon} \nabla_v \varphi(t, v) \cdot \int dx E^\varepsilon(t, x) f_0^\varepsilon(x - vt/\varepsilon^2, v),\]

and

\[T_2^\varepsilon(\varphi) := - \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} dv \frac{1}{\varepsilon^2} \nabla_v \varphi(t, v) \cdot \int_0^t ds \int dx E^\varepsilon(t, x) E^\varepsilon(s, x - vt/\varepsilon^2) \cdot \left( \nabla_v f^\varepsilon \right)(s, x - vt^2/\varepsilon^2, v).\]

For the term \(T_1^\varepsilon\) we have

**Lemma 2** Assume that \(f_0^\varepsilon\) satisfies the hypotheses of Theorem 7. In addition we suppose that there exists a constant \(C_0\), independent of \(\varepsilon\), such that for \(|\alpha| \leq 1,\)

\[
\sum_{k \in \mathbb{Z}^d} \left( ||k||^{-1} \left\| \partial_v^\alpha f_0^\varepsilon(k) \right\|_{L^1(\mathbb{R}^d)} \right)^2 \leq C_0, \quad \text{if } d = 1, \quad \text{and,}
\]

\[
\sum_{k \in \mathbb{Z}^d} \left( ||k||^{-2} \left\| \partial_v^\alpha f_0^\varepsilon(k) \right\|_{L^1(\mathbb{R}^d)} \right)^{1+2/d} \leq C_0, \quad \text{if } d \geq 2.
\]

Then

\[T_1^\varepsilon \rightharpoonup 0 \quad \text{in} \quad D'(\mathbb{R}^+ \times \mathbb{R}^d).\]

**Proof.** Using Fourier series and the zero-mean electrostatic condition \((10)\), we rewrite the term \(T_1^\varepsilon\) as

\[T_1^\varepsilon(\varphi) = \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} dv \sum_{k \in \mathbb{Z}^d} \frac{1}{\varepsilon} \nabla_v \varphi(t, v) \cdot \tilde{E}^\varepsilon(t, -k) \tilde{f}_0^\varepsilon(k, v) \exp(-ik \cdot vt/\varepsilon^2).\]
Using a velocity integration by parts, we obtain

\[
T_1^\varepsilon(\varphi) = -\varepsilon \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}^d} \nabla v \cdot \left( \frac{k}{|k|^2} \nabla \varphi(t, v) \cdot \hat{E}^\varepsilon(t, -k) \hat{f}_0^\varepsilon(k, v) \right) \exp(-i k \cdot v t / \varepsilon^2),
\]

which leads to

\[
|T_1^\varepsilon(\varphi)| \leq \varepsilon \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}^d} |\hat{E}^\varepsilon(t, k)||k|^{-1}
\]

\[
\left( \|\nabla^2 \varphi(t)\|_{L^\infty(\mathbb{R}^d)} \|\hat{f}_0^\varepsilon(k)\|_{L^1(\mathbb{R}^d)} + \|\nabla \varphi(t)\|_{L^\infty(\mathbb{R}^d)} \|\nabla \hat{f}_0^\varepsilon(k)\|_{L^1(\mathbb{R}^d)} \right). \tag{49}
\]

Using the bound (34) and the Hausdorff-Young inequality, we obtain for \( d \geq 2, \)

\[
\|k|\hat{E}^\varepsilon\|_{L^\infty(\mathbb{R}^+; L^{1+4/d}(\mathbb{Z}^d))} \leq \|E^\varepsilon\|_{L^\infty(\mathbb{R}^+; W^{1,1+4/d}(\mathbb{T}^d))} \leq c_0. \tag{50}
\]

Using Hölder’s inequality, (50) and the assumption (47), we obtain from (49),

\[
|T_1^\varepsilon(\varphi)| \leq \varepsilon 2c_0c_0^{d/(d+2)} \|\varphi/t\|_{L^1(\mathbb{R}^+; W^{2,\infty}(\mathbb{R}^d))}.
\]

In the same way, using Remark 2 and the Cauchy-Schwarz inequality, we obtain for \( d = 1, \)

\[
|T_1^\varepsilon(\varphi)| \leq \varepsilon 2c_0^{1/2} \|E^\varepsilon\|_{L^\infty(\mathbb{R}^+; L^2(\mathbb{T}^d))} \|\varphi/t\|_{L^1(\mathbb{R}^+; W^{2,\infty}(\mathbb{R}^d))},
\]

which ends the proof of Lemma 2. \( \square \)

**Remark 6** In Lemma 2, the regularity assumption for \( f_0^\varepsilon \) might be refined but with the presence of the factor \( \varepsilon^{-1} \) in the term \( T_1^\varepsilon \), some mixing-type hypotheses seem compulsory.

We now deal with the term \( T_2^\varepsilon \). Performing the change of time variable \( s = t - \varepsilon^2 \sigma \), followed by the change of space variable \( x' = x - \sigma v \), and using a velocity integration by parts and \( x \)-periodicity, we obtain

\[
T_2^\varepsilon(\varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int dx \int_0^{t/\varepsilon^2} d\sigma \int f^\varepsilon(t - \varepsilon^2 \sigma, x, v) \nabla \cdot \left( E^\varepsilon(t - \varepsilon^2 \sigma, x, v) \otimes E^\varepsilon(t, x + \sigma v) \nabla \varphi(t, v) \right). \tag{51}
\]

Using the time characteristic function \( \chi_{[0,t/\varepsilon^2]}(\sigma) \), equation (51) can be recast as

\[
T_2^\varepsilon(\varphi) = J^\varepsilon(\varphi) + M^\varepsilon(\varphi), \tag{52}
\]

where

\[
J^\varepsilon(\varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv f(t, v) \nabla \cdot \left( \int_{\mathbb{R}^+} d\sigma \int_{\mathbb{R}^d} dx \chi_{[0,t/\varepsilon^2]}(\sigma) E^\varepsilon(t - \varepsilon^2 \sigma, x, v) \otimes E^\varepsilon(t, x + \sigma v) \nabla \varphi(t, v) \right),
\]

and

\[
M^\varepsilon(\varphi) := \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_{\mathbb{R}^+} d\sigma
\int dx \chi_{[0,t/\varepsilon^2]}(\sigma) \left( f^\varepsilon(t - \varepsilon^2 \sigma, x, v) - f(t, v) \right) \nabla \cdot \left( E^\varepsilon(t - \varepsilon^2 \sigma, x, v) \otimes E^\varepsilon(t, x + \sigma v) \nabla \varphi(t, v) \right).\]
If we assume that
\[ \lim_{\varepsilon \to 0} J^\varepsilon(\varphi) \quad \text{exists,} \tag{53} \]
and
\[ \lim_{\varepsilon \to 0} M^\varepsilon(\varphi) = 0, \tag{54} \]
then we obtain from (52),
\[ \lim_{\varepsilon \to 0} T_2^\varepsilon(\varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \, f(t,v) \nabla_v \cdot (\mathcal{D}(t,v)^T \nabla_v \varphi(t,v)), \tag{55} \]
with
\[ \mathcal{D}(t,v) := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^+} d\sigma \int_{\mathbb{R}^d} dx \, \chi_{[0,t/\varepsilon^2]}(\sigma) E^\varepsilon(t,x + v\sigma) \otimes E^\varepsilon(t - \varepsilon^2 \sigma, x). \tag{56} \]
Using (16), (48) and (55) to pass to the limit \( \varepsilon \to 0 \) in (55), we obtain the following diffusion equation,
\[ \partial_t f(t,v) - \nabla_v \cdot (\mathcal{D}(t,v) \nabla_v f(t,v)) = 0, \quad \text{in } \mathcal{D}'([0,T] \times \mathbb{R}^d). \tag{57} \]
Few remarks are now in order.

**Remark 7 (open issues)**

1. All computations involving the term \( T_2^\varepsilon \) are formal and must be justified in a convenient functional framework. In order to justify (53), we have to show that
\[ R^\varepsilon(t,\sigma,x,v) := \chi_{[0,t/\varepsilon^2]}(\sigma) \nabla_v \cdot (E^\varepsilon(t - \varepsilon^2 \sigma, x) \otimes E^\varepsilon(t, x + v\sigma) \nabla_v \varphi(t,v)) \]
converges weakly in \( L^1(\mathbb{R}^+_t \times \mathbb{R}^+_x \times Q) \). In order to prove (53) and justify (55), we have to show that \( R^\varepsilon \) converges strongly in \( L^1(\mathbb{R}^+_t \times \mathbb{R}^+_x \times Q) \), since \( f^\varepsilon(t - \varepsilon^2 \sigma, x,v) - f(t,v) \to 0 \) in \( L^\infty(\mathbb{R}^+_t \times \mathbb{R}^+_x \times Q) \) weak-*.

2. As already observed, the bound \( \| E^\varepsilon \|_{L^\infty([0,T],W^{1,1+2d/4}(\mathbb{R}^d))} \leq c_0 < \infty \), does not imply strong convergence (because of the lack of time control or compactness) for the electric field \( E^\varepsilon \), which would help to justify the above formal computations for the term \( T_2^\varepsilon \). However, this lack of time compactness is in fact necessary if we do not want to obtain a trivial equation, as stated in point iv) of Theorem 7. Indeed, from point iv) of Theorem 7, time compactness entails a strong convergence to zero of the electric field \( E^\varepsilon \). This implies the vanishing of the diffusion matrix \( \mathcal{D} \) given by (56). Without time compactness, the electric field \( E^\varepsilon \) always converges weakly to zero, but not the quadratic electric tensor \( E^\varepsilon \otimes E^\varepsilon \) (this is a property of weak convergence), which implies a non-trivial diffusion matrix \( \mathcal{D} \). Therefore, weak convergence seems mandatory to obtain a diffusion limit.

3. Instead of time compactness, time decorrelation properties could help to justified rigorously above computations. In the presence of a non-self-consistent but stochastic electric field, with convenient hypotheses, some time decorrelation properties allow us to justify rigorously the limit of the Vlasov equation (17) towards diffusion equations (55) - (57). This is the object of Section 4.
Remark 8 (periodic or quasi-periodic time oscillations) Since the defect of time compactness means that the system contains fast oscillations in time, it would be tempting to apply the analysis of this section to the case of a non-self-consistent deterministic electric field (satisfying convenient regularity assumptions) with two time scales, one being slow and not periodic, and the other being fast and (quasi-)periodic. Such a standard homogenization problem would lead to solve a hierarchy of equations where the free-streaming operator $v \cdot \nabla_x$, with periodic boundary condition, must be inverted at each stage of the hierarchy. Since the free-flow operator is not a Fredholm operator, the Fredholm alternative does not hold for such transport equation. On the contrary, when considering a non-self-consistent stochastic electric field the situation is completely different and we can obtain a diffusion behavior for the statistical average of the distribution function. This is what is done in Section 4.

An explicit form of the diffusion matrix in the non-self-consistent deterministic case.

Here we pursue a little bit further the above formal analysis to explicit the structure of the diffusion matrix (56) by constructing a well-suited non-self-consistent deterministic electric field. This gives an example of diffusion matrix (56), which is not zero and non-negative. Introducing the Fourier series decomposition of $E_\varepsilon$,

$$E_\varepsilon(t, x) = \sum_{k \in \mathbb{Z}^d} e^{i k \cdot x} \widehat{E}_\varepsilon(t, k),$$

we assume the formal WKB expansion for the Fourier mode $\widehat{E}_\varepsilon(t, k)$,

$$\widehat{E}_\varepsilon(t, k) = \sum_{j \geq 0} \varepsilon^j \widehat{E}_j(t, k, \Omega(t, k)/\varepsilon^2),$$

where complex vector-valued functions $(k, \tau) \mapsto \widehat{E}_j(t, k, \tau)$ are $2\pi$-periodic with respect to the variable $\tau$. Here, functions $\widehat{E}_j(t, k, \Omega(t, k)/\varepsilon^2)$ are Hermitian, i.e.

$$\widehat{E}_j^*(t, k, \Omega(t, k)/\varepsilon^2) = \widehat{E}_j(t, -k, \Omega(t, -k)/\varepsilon^2),$$

and the real-valued function $k \mapsto \Omega(t, k)$ is odd with respect to the variable $k$. As a first approximation of (58), we obtain

$$\widehat{E}_\varepsilon(t, k) = \widehat{E}_0(t, k) \exp \left( -i \frac{\Omega(t, k)}{\varepsilon^2} \right) + O(\varepsilon),$$

where the real vector-valued function $k \mapsto \widehat{E}_0(t, k)$ is even with respect to the variable $k$. Using (59) and time Taylor expansions, we obtain from the definition of the diffusion matrix (56),

$$\mathcal{D}(t, v) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^+} d\sigma \sum_{k \in \mathbb{Z}^d} \chi_{[0, t/\varepsilon^2]}(\sigma) \exp (i k \cdot v \sigma) \widehat{E}_\varepsilon(t, k) \otimes \widehat{E}_\varepsilon(t - \varepsilon^2 \sigma, -k)$$

$$= \lim_{\varepsilon \to 0} \left( \int_{\mathbb{R}^+} d\sigma \sum_{k \in \mathbb{Z}^d} \chi_{[0, t/\varepsilon^2]}(\sigma) \exp \left( i [k \cdot v \sigma + \Omega(t - \varepsilon^2 \sigma, k)/\varepsilon^2 - \Omega(t, k)/\varepsilon^2] \right) \right)$$

$$\widehat{E}_0(t, k) \otimes \widehat{E}_0(t - \varepsilon^2 \sigma, -k) + O(\varepsilon)$$

$$= \lim_{\varepsilon \to 0} \left( \int_{\mathbb{R}^+} d\sigma \sum_{k \in \mathbb{Z}^d} \chi_{[0, t/\varepsilon^2]}(\sigma) \exp \left( -i \sigma [\partial_t \Omega(t, k) - k \cdot v] \right) \widehat{E}_0(t, k) \otimes \widehat{E}_0(t, k) + O(\varepsilon) \right).$$

(60)
Using
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^+} d\sigma x[0, t_\varepsilon^2](\sigma) e^{-i\sigma T} = \pi \delta(T) - i \text{p.v.} \left(\frac{1}{T}\right) \quad \text{in } \mathcal{D}'(\mathbb{R}),
\]
and parity of functions \(\Omega(t, k)\) and \(\hat{E}_0(t, k)\), we obtain from (60),
\[
\mathcal{D}(t, v) = \pi \sum_{k \in \mathbb{Z}^d} \hat{E}_0(t, k) \otimes \hat{E}_0(t, k) \delta(\partial_t \Omega(t, k) - k \cdot v).
\] (61)

If we assume \(\Omega(t, k) = \omega(k)t\) then (61) is the diffusion matrix given by the quasilinear theory [39, 21], i.e.,
\[
\mathcal{D}(t, v) = \pi \sum_{k \in \mathbb{Z}^d} |\hat{E}_0(t, k)|^2 \frac{k \otimes k}{|k|^2} \delta(\omega(k) - k \cdot v).
\] (62)

**Remark 9** If the electric field \(E^\varepsilon\) derives from a potential \(\Phi^\varepsilon\), i.e., \(E^\varepsilon(t, x) = -\nabla \Phi^\varepsilon(t, x)\), then, following the above computations, a WKB expansion of \(\Phi^\varepsilon(t, x)\) similar to (58) leads to the diffusion matrix
\[
\mathcal{D}(t, v) = \pi \sum_{k \in \mathbb{Z}^d} |\hat{E}_0(t, k)|^2 \frac{k \otimes k}{|k|^2} \delta(\omega(k) - k \cdot v).
\]

### 3.2.2 Local in time approach

We first integrate, with respect to the time variable, the space-averaged Vlasov equation
\[
\partial_t \int dt \int dx f^\varepsilon + \nabla_v \cdot \int dx \frac{E^\varepsilon f^\varepsilon}{\varepsilon} = 0,
\]
between the time \(t\) and \(t + \theta\), with \(\theta > 0\). Then, we multiply the result by \(\varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)\) and we perform an integration with respect to the time and velocity variables. Finally, using the \(L^2\)-scalar product (19) and a velocity integration by parts, we obtain
\[
\left< \frac{f^\varepsilon(t + \theta) - f^\varepsilon(t)}{\theta}, \varphi \right> = \frac{1}{\varepsilon \theta} \int_{\mathbb{R}^+} d\sigma \int_{\mathbb{R}^d} d\tau \int_t^{t+\theta} \int_t^s ds \int dx f^\varepsilon(s) E^\varepsilon(s) \cdot \nabla_v \varphi(t, v).
\] (63)

Using the Duhamel formula and the notation (20), we obtain from the Vlasov equation (17), the following representation formula for \(f^\varepsilon(s)\),
\[
f^\varepsilon(s) = S^{\varepsilon}_{s-t+\hat{\theta}} f^\varepsilon(t - \hat{\theta}) - \frac{1}{\varepsilon} \int_{t-\hat{\theta}}^{s} d\sigma S^{\varepsilon}_{s-\sigma} E^\varepsilon(\sigma) \cdot \nabla_v f^\varepsilon(\sigma),
\] (64)
with \(\hat{\theta}\) being an arbitrary non-negative time. Substituting (61) into (63), we obtain
\[
\left< \frac{f^\varepsilon(t + \theta) - f^\varepsilon(t)}{\theta}, \varphi \right> = T_1^\varepsilon(\varphi) + T_2^\varepsilon(\varphi),
\] (65)
where
\[
T_1^\varepsilon(\varphi) := \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_t^{t+\theta} ds \int dx \frac{1}{\varepsilon \theta} E^\varepsilon(s) \cdot \nabla_v \varphi(t, v) S^{\varepsilon}_{s-t+\hat{\theta}} f^\varepsilon(t - \hat{\theta}),
\] (66)
and
\[
T_2^\varepsilon(\varphi) := \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_t^{t+\theta} ds \int_{t-\theta}^{s} d\sigma \int dx \frac{1}{\varepsilon \theta} S^{\varepsilon}_{s-\sigma} f^\varepsilon(\sigma) \nabla_v \left( S^{\varepsilon}_{s-\sigma} E^\varepsilon(\sigma, x) \otimes E^\varepsilon(s, x) \nabla_v \varphi(t, v) \right).
\]
For the term $T_1^\varepsilon$, we assume that

$$T_1^\varepsilon \rightarrow 0 \text{ in } D'(\mathbb{R}^+ \times \mathbb{R}^d).$$

We now deal with the term $T_2^\varepsilon$. Using $x$-periodicity and the change of time variable $s' = s - t$, the term $T_2^\varepsilon$ becomes

$$T_2^\varepsilon (\varphi) = \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} dv \int_0^\theta ds \int_{t-\hat{\theta}}^{t+s} d\sigma \int dx \frac{1}{\varepsilon^2} f^\varepsilon (\varphi, x, v) \nabla_v \cdot \left( E^\varepsilon (\varphi, x) \otimes E^\varepsilon (t + s, x + v(t + s - \varepsilon^2)/\varepsilon^2) \nabla_v \varphi(t, v) \right).$$

(68)

Using the change of time variable $\sigma' = (t + s - \varepsilon^2)/\varepsilon^2$, equation (68) becomes

$$T_2^\varepsilon (\varphi) = \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} dv \int_0^\theta ds \int_0^{(s+\hat{\theta})/\varepsilon^2} d\sigma \int dx \frac{1}{\varepsilon^2} f^\varepsilon (t + s - \varepsilon^2, x, v) \nabla_v \cdot \left( E^\varepsilon (t + s - \varepsilon^2, x) \otimes E^\varepsilon (t + s, x + v\varepsilon^2) \nabla_v \varphi(t, v) \right).$$

(69)

Taking $\theta = \varepsilon^2 \tau$ and $\hat{\theta} = \varepsilon^2 \eta$, and using the change of time variable $s' = s/\varepsilon^2$, followed by the change of time variable $\sigma' = \sigma - \eta$, equation (69) becomes

$$T_2^\varepsilon (\varphi) = \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} dv \frac{1}{\varepsilon^2} \int_0^\tau ds \int_{-\eta}^{s} d\sigma \int dx f^\varepsilon (t + (s - \eta - \sigma)\varepsilon^2, x, v) \nabla_v \cdot \left( E^\varepsilon (t + (s - \eta - \sigma)\varepsilon^2, x) \otimes E^\varepsilon (t + s\varepsilon^2, x + v\eta) \nabla_v \varphi(t, v) \right).$$

This equation can be recast as

$$T_2^\varepsilon (\varphi) = \mathcal{J}^\varepsilon (\varphi) + \mathcal{M}^\varepsilon (\varphi),$$

(70)

where

$$\mathcal{J}^\varepsilon (\varphi) = \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} dv f(t, v) \nabla_v \cdot \left( \frac{1}{\varepsilon^2} \int_0^\tau ds \int_{-\eta}^{s} d\sigma \int dx E^\varepsilon (t + (s - \eta - \sigma)\varepsilon^2, x) \otimes E^\varepsilon (t + s\varepsilon^2, x + v\eta) \nabla_v \varphi(t, v) \right),$$

and

$$\mathcal{M}^\varepsilon (\varphi) := \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} dv \frac{1}{\varepsilon^2} \int_0^\tau ds \int_{-\eta}^{s} d\sigma \int dx \left( f^\varepsilon (t + (s - \eta - \sigma)\varepsilon^2, x, v) - f(t, v) \right) \nabla_v \cdot \left( E^\varepsilon (t + (s - \eta - \sigma)\varepsilon^2, x) \otimes E^\varepsilon (t + s\varepsilon^2, x + v\eta) \nabla_v \varphi(t, v) \right).$$

The next lemma justifies that in the case where $\tau$ and $\eta$ are finite, the term $\mathcal{J}^\varepsilon (\varphi)$ has a limit as $\varepsilon \rightarrow 0$. Defining

$$\mathcal{D}^\varepsilon (t, v) := \frac{1}{\tau} \int_0^\tau ds \int_{-\eta}^{s} d\sigma \int dx E^\varepsilon (t + s\varepsilon^2, x + v\eta) \otimes E^\varepsilon (t + (s - \eta - \sigma)\varepsilon^2, x),$$

(71)

we have
Lemma 3 Let $\tau$ and $\eta$ be finite. Then, $\mathcal{J}^\varepsilon$ has a limit in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$ such that
\[
\lim_{\varepsilon \to 0} \mathcal{J}^\varepsilon(\varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \nabla_v \cdot (\mathcal{D}^T \nabla_v \varphi), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d),
\] (72)
where $\mathcal{D}$ is the weak limit of $\mathcal{D}^\varepsilon$ (up to a subsequence) in the following sense,
\[
\mathcal{D}^\varepsilon \rightharpoonup \mathcal{D} \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}^+; W^{1,1}_{\text{loc}}(\mathbb{R}^d)) \quad \text{weak},
\] (73)
and
\[
\mathcal{D}^\varepsilon \rightharpoonup \mathcal{D} \quad \text{in} \quad L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^d)) \quad \text{weak}^*.
\] (74)

Proof. Since $f \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$, to prove (72)-(73) it is sufficient to show that the term
\[
g^\varepsilon(t, v) := \nabla_v \cdot (\mathcal{D}^\varepsilon(t, v)^T \nabla_v \varphi(t, v))
\]
converges weakly in $L^1(\mathbb{R}^+ \times \mathbb{R}^d)$ as $\varepsilon \to 0$, i.e. for the weak topology $\sigma(L^1, L^\infty)$. For this, we appeal to the Dunford–Pettis theorem. First we show that $g^\varepsilon \in L^1(\mathbb{R}^+ \times \mathbb{R}^d)$ uniformly in $\varepsilon$. For this, we use the fact that weak solutions of (L7-13) are such that $E^\varepsilon \in L^\infty(\mathbb{R}^+; W^{1,1+2/d}(\mathbb{T}^d))$ uniformly with respect to $\varepsilon$. For $d = 2$, we use the $L^2$ duality and $E^\varepsilon \in L^\infty(\mathbb{R}^+; L^2(\mathbb{T}^d))$ uniformly in $\varepsilon$ (see Remark 2). For $d = 3$, we use the $L^p-L^q$ duality with $(p, q) = (1 + 2/d, 1 + d/2)$ and the Sobolev embedding $W^{1,1+2/d}(\mathbb{T}^d) \hookrightarrow L^{1+d/2}(\mathbb{T}^d)$. Since the case $d = 1$ is simpler (using the regularity $E^\varepsilon \in L^\infty(\mathbb{R}^+ \times \mathbb{T}^d)$ uniformly in $\varepsilon$), we only give the proof for $d \geq 2$. Then, for $d \geq 2$, using Hölder’s inequality, we obtain
\[
\|g^\varepsilon\|_{L^1(\mathbb{R}^+ \times \mathbb{R}^d)} \leq \frac{1}{\tau} \sum_{i,j} \int_{-\tau}^{\tau} ds \int_{-\eta}^{\eta} d\sigma \int dx (\sigma + \eta) \left[ E^\varepsilon_i(t + (s - \eta - \sigma)e^2, x) (\partial_{x_i} E^\varepsilon)(t + s\varepsilon^2, x + v(\sigma + \eta)) \right]_{L^1(\mathbb{R}^+ \times \mathbb{R}^d)}
\]
\[
+ \left[ \sum_{i,j} \frac{1}{\tau} \int_{-\tau}^{\tau} ds \int_{-\eta}^{\eta} d\sigma \int dx (\sigma + \eta) \right]_{L^1(\mathbb{R}^+ \times \mathbb{R}^d)} \left( E^\varepsilon_j(t + (s - \eta - \sigma)e^2, x) E^\varepsilon_j(t + s\varepsilon^2, x + v(\sigma + \eta)) \partial_{x_i}^\varepsilon \varphi(t, v) \right)_{L^1(\mathbb{R}^+ \times \mathbb{R}^d)}
\]
\[
\leq (\tau^2 + 3\tau\eta + 3\eta^2)(2\pi)^{-d} \|E^\varepsilon\|_{L^\infty(\mathbb{R}^+; L^{1+2/d}(\mathbb{T}^d))}
\]
\[
\left\{ \|\nabla_x E^\varepsilon\|_{L^\infty(\mathbb{R}^+; L^{1+2/d}(\mathbb{T}^d))} \|\nabla_v \varphi\|_{L^1(\mathbb{R}^+ \times \mathbb{R}^d)} + \|E^\varepsilon\|_{L^\infty(\mathbb{R}^+; L^{1+2/d}(\mathbb{T}^d))} \|\nabla_v \varphi\|_{L^1(\mathbb{R}^+ \times \mathbb{R}^d)} \right\} < \infty.
\]
For showing uniform equi-integrability of the family $g^\varepsilon$, we take $A \subset \mathbb{R}^+ \times \mathbb{R}^d$ such that $|A| \leq \delta$, with $\delta$ small. Following the above computations we obtain
\[
\|g^\varepsilon\|_{L^1(A)} \leq (\tau^2 + 3\tau\eta + 3\eta^2)(2\pi)^{-d} \|E^\varepsilon\|_{L^\infty(\mathbb{R}^+; L^{1+2/d}(\mathbb{T}^d))}
\]
\[
\left\{ \|\nabla_x E^\varepsilon\|_{L^\infty(\mathbb{R}^+; L^{1+2/d}(\mathbb{T}^d))} \|\nabla_v \varphi\|_{L^1(A)} + \|E^\varepsilon\|_{L^\infty(\mathbb{R}^+; L^{1+2/d}(\mathbb{T}^d))} \|\nabla_v \varphi\|_{L^1(A)} \right\}
\]
\[
\leq (\tau^2 + 3\tau\eta + 3\eta^2) |A|(2\pi)^{-d} \|E^\varepsilon\|_{L^\infty(\mathbb{R}^+; L^{1+2/d}(\mathbb{T}^d))}
\]
\[
\left\{ \|\nabla_x E^\varepsilon\|_{L^\infty(\mathbb{R}^+; L^{1+2/d}(\mathbb{T}^d))} \|\nabla_v \varphi\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)} + \|E^\varepsilon\|_{L^\infty(\mathbb{R}^+; L^{1+2/d}(\mathbb{T}^d))} \|\nabla_v \varphi\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)} \right\}
\]
\[
\lesssim \delta,
\]
which shows uniform equi-integrability of the family \( g^\varepsilon \). It remains to prove (74). Using Fourier series in the space variables, the matrix (71) rewrites as

\[
D^\varepsilon(t,v) = \frac{1}{\tau} \int_0^\tau ds \int_{-\eta}^{s} d\sigma \sum_{k \in \mathbb{Z}^d} e^{ik \cdot v(\sigma + \eta)} \hat{E}^\varepsilon(t + s\varepsilon^2, k) \otimes \hat{E}^\varepsilon(t + (s - \eta - \sigma)\varepsilon^2, -k).
\]

From (50), there exists \( \lambda > 0 \) such that

\[
|\hat{E}^\varepsilon(t,k)| \leq c_0 (1 + |k|)^{- (1 + \lambda)}.
\]

Using (75)-(76) and weak compactness, we obtain (74), which ends the proof of Lemma 3.

\[\Box\]

If we now assume

\[
\lim_{\varepsilon \to 0} \mathcal{M}^\varepsilon(\varphi) = 0,
\]

then, using Lemma 3 and (67), we obtain from (65) and (70) the diffusion equation (57) with

\[
\mathcal{D}(t,v) := \lim_{\varepsilon \to 0} \frac{1}{\tau} \int_0^\tau ds \int_{-\eta}^{s} d\sigma \int dx E^\varepsilon(t + s\varepsilon^2, x) \otimes E^\varepsilon(t + (s - \eta - \sigma)\varepsilon^2, x - v(\sigma + \eta))
\]

Few remarks are now in order.

**Remark 10** (open issues)

1. In order to justify (77), we have to prove that the term

\[
\nabla_v \cdot \left( E^\varepsilon(t + (s - \eta - \sigma)\varepsilon^2, x) \otimes E^\varepsilon(t + s\varepsilon^2, x + v(\sigma + \eta)) \nabla_v \varphi(t,v) \right),
\]

converges strongly in \( L^1(\mathbb{R}_t^+ \times \mathbb{R}_x^+ \times \mathbb{R}_{\xi,\eta}^+ \times Q) \) as \( \varepsilon \to 0 \).

2. Show that \( T_1^\varepsilon \to 0 \) in \( \mathcal{D}'(\mathbb{R}_t^+ \times \mathbb{R}^d) \) remains an open issue. Nevertheless, we may expect that there exist some mixing-type hypotheses, which could justify such limit.

3. Taking \( \hat{\theta} = \varepsilon^2 \tau \) in equation (66), the term \( T_1^\varepsilon \) is reminiscent of the term (103) appearing in the case of the non-self-consistent stochastic electric field (see Section 4). Let us note that in the stochastic case, without additional regularity assumptions on weak solutions, this term vanishes by using a time decorrelation property between \( E^\varepsilon \) and \( f^\varepsilon \) and by using the fact that the stochastic average of \( E^\varepsilon \) vanishes.

4. The parameter \( \tau \) is reminiscent of the autocorrelation time of particles \( \tau \), which is introduced in case of the non-self-consistent stochastic electric field (see Section 4).

**An explicit form of the diffusion matrix in the non-self-consistent deterministic case.**

Without being able to prove some time decorrelation properties for the electric field \( E^\varepsilon \), it is difficult to deduce the structure of the diffusion matrix \( \mathcal{D} \). However, as in Section 3.2.1, we can design a non-self-consistent deterministic electric field by using the WKB expansion (58) to obtain an explicit diffusion matrix. This gives a formal example for which the diffusion matrix (78) is not zero and non-negative. Using the WKB approximation (59), the notation

\[
\Delta \Omega := \partial_t \Omega(t,k) - k \cdot v,
\]
and the parity of functions $\Omega(t, k)$ and $\hat{E}_0(t, k)$ in the variable $k$, we obtain from similar computations leading to (60),

$$
\mathcal{D}(t, v) = \lim_{\varepsilon \to 0} \frac{1}{\tau} \int_0^\tau ds \sum_{k \in \mathbb{Z}^d} e^{i k \cdot v (\tau + s)} \hat{E}^\varepsilon (t + s, k) \otimes \hat{E}^\varepsilon (t + (\tau - s - \sigma)\varepsilon^2, k)^*
$$

$$
= \sum_{k \in \mathbb{Z}^d} \frac{1}{\tau} \int_0^\tau ds \int_0^{s+\eta} ds' e^{-i \sigma \Delta \Omega(t, k)} \hat{E}_0(t, k) \otimes \hat{E}_0(t, k)
$$

$$
= \sum_{k \in \mathbb{Z}^d} \sin(\tau \Delta \Omega/2) \sin((\tau/2 + \eta)\Delta \Omega) \hat{E}_0(t, k) \otimes \hat{E}_0(t, k).
$$

At this point we obtain some important limits with respect to the parameters $\eta$ and $\tau$. The parameter $\tau$ is the same as the one that we have defined in Section 2 and used in Section 4. Hence parameters $\tau$ and $\eta$ can be seen as normalized particle autocorrelation times.

The first significant limit is $\eta \to +\infty$. Indeed using the limit $\lim_{\eta \to +\infty} \sin(\eta \pi) / \tau = \pi \delta(\tau)$ in $\mathcal{D}'(\mathbb{R})$, and taking the limit $\eta \to +\infty$ in (79), we recover the same diffusion matrix (61) that we have obtained with the global-in-time approach of Section 3.2.1. Therefore, we recover the quasilinear diffusion matrix (62) too. In a sense, the limit $\eta \to +\infty$ corresponds to take into account all the past of the distribution function and particularly the initial condition as it was done in the global-in-time approach of Section 3.2.1. Therefore it is consistent to obtain the result of the global-in-time approach by taking the limit $\eta \to +\infty$ in the local-in-time approach.

The second significant limit is $\eta \to 0$. Indeed for $\eta = 0$, we obtain from (79), the following non-negative diffusion matrix,

$$
\mathcal{D}(t, v) = \frac{\tau}{2} \sum_{k \in \mathbb{Z}^d} \left( \sin \left( \frac{\pi}{2} (\partial_t \Omega(t, k) - k \cdot v) \right) \right)^2 \hat{E}_0(t, k) \otimes \hat{E}_0(t, k).
$$

This diffusion matrix seems more regular in velocity than the quasilinear diffusion matrix (62). This regularity improvement in velocity is reminiscent of the finite-$\tau$ effect that we observe in the framework of the non-self-consistent stochastic electric field (see Section 4) and also in the resonance broadening like theory (see Section 4.4.1).

The last significant limit is to keep $\eta$ finite and to pass to the limit $\tau \to +\infty$ in (79) or (80). In this limit, we also recover the quasilinear diffusion matrix (61) or (62), and Remark 9 still holds true. This result is consistent with the developments of Section 4.4.2 for the non-self-consistent stochastic electric field.

Remark 11 As in Remark 2, if the electric field $E^\varepsilon$ derives from a potential $\Phi^\varepsilon$, then in the diffusion matrices (79) and (80) we should replace the matrix $\hat{E}_0(t, k) \otimes \hat{E}_0(t, k)$ by the matrix $|\hat{E}_0(t, k)|^2 k \otimes k / |k|^2$.

### 4 The non-self-consistent stochastic case

In this section we deal with the non-self-consistent stochastic case. Before stating our main Theorem 2 in Section 4.2, we start by describing the features of the stochastic electric field in the following section.
4.1 The turbulent electric field

Here, electrostatic turbulence is modeled through the random vector field $E^\varepsilon$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, with $\mathbb{P}$ being a $\sigma$-finite measure. A random vector $F$ is real vector-valued function defined on $\Omega$. When $F : \Omega \to \mathbb{R}^d$ is an integrable random vector, its expectation is given by

$$E[F] = \int_\Omega \mathrm{d}\mathbb{P}(\omega) F(\omega).$$

From considerations of Section 2.1, the turbulent electric field $E^\varepsilon$ has two time scales, one slow and the other fast. We then choose a turbulent electric field $E^\varepsilon$ given by

$$E^\varepsilon(t, x) = E(t, t/\varepsilon^2, x; \omega),$$

where, the integrable random vector field $E$ satisfies the following “stochastic” assumptions:

(H1): The random vector field $E$ is centered, i.e.

$$\mathbb{E}[E(t, \tau, x)] = 0, \quad \forall (t, \tau, x) \in \mathbb{R}^+ \times \mathbb{R}^+ \times T^d.$$

(H2): There exists a constant $\tau > 0$ such that for every $x, y \in \mathbb{R}^d$ and for every $\tau, \sigma \in \mathbb{R}^+$ the electric fields $E(t, \tau, x)$ and $E(s, \sigma, y)$ are independent random vector fields as soon as $|\tau - \sigma| \geq \tau$. The autocorrelation time $\tau$ is supposed fixed and finite, hence independent of $\varepsilon$.

(H3): There exists a matrix-valued function $R_\tau : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times T^d \to \mathbb{R}^{2d}$, called the autocorrelation matrix or the Reynolds electric stress tensor, such that

$$\mathbb{E}[E(t, \tau, x) \otimes E(s, \sigma, y)] = R_\tau(t, s, \tau - \sigma, x - y).$$

Hypothesis (H1) sets the stochastic average of $E^\varepsilon$ to zero, which is standard and not restrictive. Assumption (H2) means that the turbulent electric field $E^\varepsilon$ is time decorrelated on a time scale $\varepsilon^2 \tau$. Assumption (H2) can be seen as a hypothesis of propagation of “stochasticity” or propagation of independence of random vector fields. Therefore, two evaluations in time of the electric field, separated by a lapse of time larger than $\varepsilon^2 \tau$, are independent random vector fields. Hypothesis (H3) is the standard spatio-temporal homogeneity property of the turbulence, i.e. the spatio-temporal autocorrelation of the electric field $E^\varepsilon$ is invariant under space and time translations. These assumptions are similar to the ones of [50].

**Remark 12** In the nonlinear regime, the property of time decorrelation seems to be the cornerstone of the diffusion process, for both the self-consistent and the non-self-consistent setting [21, 3, 5]. An open and very difficult problem remains to show mathematically such time decorrelation property from only the deterministic Vlasov-Poisson system (7)-(8) and random initial data $f_0$. In a sense, this is what has been shown numerically in [10, 11]. This property of propagation of “stochasticity” is reminiscent of the property of propagation of chaos in statistical mechanics.

In order to justify rigorously the diffusion limit, the stochastic electric field requires the following regularity assumptions.

(H4): The regularity of $E$ is such that

$$E \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+; W^{2,\infty}(T^d)), \quad \text{and} \quad \mathbb{E} \left[ \|E\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^+; W^{2,\infty}(T^d))}^3 \right] =: C_E < \infty.$$

Assumption (H4) imposes the regularity (especially in space) of the random vector field $E$. It is worthwhile to end this section by giving an explicit example of a random field $E$, which satisfies assumptions (H1)-(H4). Following the spirit of Example 2 in [50], we construct in Appendix A a random vector field $E$ satisfying these requirements.
4.2 Main theorem

Concerning the non-self-consistent stochastic case, we establish

**Theorem 2** Let $E$ be an integrable random vector field satisfying assumptions (H1)-(H4), and let $E^\varepsilon$ be given by (81). Let $\{f^\varepsilon\}_{\varepsilon>0}$ be a sequence of independent random non-negative initial data and $C_0$ be a positive constant such that for a.e. $\omega \in \Omega$, $\|f^\varepsilon_0\|_{L^1(Q)} + \|f^\varepsilon_0\|_{L^\infty(Q)} \leq C_0 < \infty$. Let $\mathcal{D}_\tau = \mathcal{D}_\tau(t,v)$ be the matrix-valued function defined by

$$\mathcal{D}_\tau(t,v) = \int_0^t d\sigma \mathcal{R}_\tau(t,t,\sigma,v), \quad (83)$$

the properties of which, are described in Proposition 1. Let $f^\varepsilon$ be the unique weak solution of the Vlasov equation (14), with initial data $f^\varepsilon|_{t=0} = f^\varepsilon_0$. Then up to extraction of a subsequence, $\mathbf{E}[f^\varepsilon]$ converges in $L^\infty(Q)$ weak-* to a function $f_0 \in L^1 \cap L^\infty(Q)$, $\mathbf{E}[f^\varepsilon]$ converges in $L^\infty(\mathbb{R}^+;L^1 \cap L^\infty(\mathbb{R}^d))$ weak-* to a function $f \in L^\infty(\mathbb{R}^+;L^1 \cap L^\infty(\mathbb{R}^d))$, and $\mathbf{E}[f \mathbf{dx} f^\varepsilon]$ converges in $L^\infty(\mathbb{R}^+;L^\infty(\mathbb{R}^d))$ weak-* to $f$. Moreover $\mathbf{E}[f \mathbf{dx} f^\varepsilon]$ converges in $\mathcal{C}(0,T;L^p(\mathbb{R}^d) - \text{weak})$ to $f$, for $1 < p < \infty$ and for all $T > 0$. The limit point $f = f(t,v)$ is solution of the following diffusion equation in the sense of distributions:

$$\partial_t f - \nabla_v \cdot (\mathcal{D}_\tau \nabla_v f) = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d),$$

$$f|_{t=0} = \int dx f_0.$$ 

The Proof of Theorem 2 is postponed to Section 4.3. General properties of the diffusion matrix $\mathcal{D}_\tau$ and the autocorrelation matrix $\mathcal{R}_\tau$ of Theorem 2 are stated in

**Proposition 1** (properties of the diffusion matrix $\mathcal{D}_\tau$) Under assumptions (H1)-(H4), the matrix-valued function $\mathcal{R}_\tau : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{2d}$, and the diffusion matrix $\mathcal{D}_\tau : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{2d}$ satisfy the following properties:

i) $\mathcal{R}_\tau(t,t,\tau,x) = \mathcal{R}_\tau^T(t,t,-\tau,-x)$, and $\mathcal{R}_\tau(t,t,\tau,x + 2\pi k) = \mathcal{R}_\tau(t,t,\tau,x)$, $\forall k \in \mathbb{Z}$.

ii) $\mathcal{R}_\tau \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R};W^2,\infty(\mathbb{R}^d))$, and $\text{supp}(\mathcal{R}_\tau) \subset \mathbb{R}^+ \times \mathbb{R}^+ \times [-\tau,\tau] \times \mathbb{R}^d$.

iii) $\mathcal{D}_\tau \in L^\infty(\mathbb{R}^+;W^2,\infty(\mathbb{R}^d))$, and $\text{supp}(\mathcal{D}_\tau) \subset \mathbb{R}^+ \times \mathbb{R}^d$.

iv) The symmetric part of $\mathcal{D}_\tau$ is non-negative, i.e. $X^T \mathcal{D}_\tau X \geq 0$, $\forall X \in \mathbb{R}^d$.

**Proof.** We start with property i). Using (H3), we obtain

$$\mathcal{R}_{\tau ij}(t,t,\tau-\sigma,x-y) = \mathbf{E}[E_i(t,\tau,x)E_j(t,\sigma,y)] = \mathbf{E}[E_j(t,\sigma,y)E_i(t,\tau,x)] = \mathcal{R}_{\tau ji}(t,t,-(\tau-\sigma),-(x-y)),$$

and for all $k,k' \in \mathbb{Z}$,

$$\mathcal{R}_\tau(t,t,\tau-\sigma,x-y) = \mathbf{E}[E(t,\tau,x) \otimes E(t,\sigma,y)] = \mathbf{E}[E(t,\tau,x + 2\pi k) \otimes E(t,\sigma,y + 2\pi k')]
= \mathcal{R}_{\tau ij}(t,t,\tau-\sigma,x-y + 2\pi (k-k')).$$

The regularity property ii) for $\mathcal{R}_\tau$ comes immediately from the regularity assumption (H4) for $E$ and the definition (H3) for $\mathcal{R}_\tau$. The regularity property iii) for $\mathcal{D}_\tau$ is the straight consequence of the definition (83) for $\mathcal{D}_\tau$ and the regularity property ii) for $\mathcal{R}_\tau$. The support of $\mathcal{D}_\tau$ is obvious, while the support of $\mathcal{R}_\tau$ results from assumptions (H1)-(H3). Indeed, if $|\tau-\sigma| > \tau$, then (H1)-(H3)
imply that \( \mathcal{R}_\tau(t, t - \tau, \sigma, x - y) = \mathbb{E}[E(t, \tau, x) \otimes E(t, \sigma, y)] = \mathbb{E}[E(t, \tau, x)] \otimes \mathbb{E}[E(t, \sigma, y)] = 0. \) We end with the property iv). Using properties i) and ii), we obtain

\[
X^T \mathcal{D}_\tau X = \sum_{i,j} X_i X_j \int_0^\tau d\sigma \mathcal{R}_{ij}(t, t, \sigma, \sigma v) = \frac{1}{2} \sum_{i,j} X_i X_j \int_{-\tau}^{\tau} d\sigma \mathcal{R}_{ij}(t, t, \sigma, \sigma v)
\]

\[
= \frac{1}{2} \sum_{i,j} X_i X_j \int_{-\tau}^{\tau} d\sigma \mathcal{R}_{ij}(t, t, \sigma, \sigma v).
\]  \hfill (85)

Using Lemma 3.1 of [50], which states that for all \( g \in L^1(\mathbb{R}) \) we have

\[
\int_{-\infty}^{+\infty} g(s) ds = \lim_{R \to +\infty} \frac{1}{2R} \int_{-R}^{R} ds \int_{-R}^{R} dt g(s-t),
\]

we obtain from (85),

\[
X^T \mathcal{D}_\tau X = \lim_{R \to +\infty} \frac{1}{4R} \sum_{i,j} X_i X_j \int_{-R}^{R} d\sigma \int_{-R}^{R} d\theta \mathbb{E}[E_i(t, \sigma, \sigma v)E_j(t, \theta, \theta v)]
\]

\[
= \lim_{R \to +\infty} \frac{1}{4R} \int_{-R}^{R} d\sigma \int_{-R}^{R} d\theta \mathbb{E}[X \cdot E(t, \sigma, \sigma v)X \cdot E(t, \theta, \theta v)]
\]

\[
= \lim_{R \to +\infty} \frac{1}{4R} \mathbb{E} \left[ \left( \int_{-R}^{R} d\sigma X \cdot E(t, \sigma, \sigma v) \right)^2 \right] \geq 0,
\]

which ends the Proof of Proposition 1. \( \square \)

To end this section, it is worthwhile to compare the diffusion matrix obtained here for the stochastic case and the one obtained for the deterministic case in Section 3.2. If \( \varepsilon \) is small enough such that \( t/\varepsilon^2 > \tau \), then using (H3), (81), and the compact support of \( \mathcal{R}_\tau(\cdot, \cdot, \sigma, \cdot) \) in the variable \( \sigma \) (included in \([-\tau, \tau]\); see Proposition 1), the stochastic diffusion matrix \( \mathcal{D}_\tau \), defined by (83), rewrites as

\[
\mathcal{D}_\tau(t, v) = \lim_{\varepsilon \to 0} \int_{-\tau}^{\tau} d\sigma \mathbb{E}[E^\varepsilon(t, x) \otimes E^\varepsilon(t - \varepsilon^2 \sigma, x - \sigma v)]
\]

\[
= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^+} d\sigma \chi_{[0,t/\varepsilon^2]}(\sigma) \mathbb{E}[E^\varepsilon(t, x) \otimes E^\varepsilon(t - \varepsilon^2 \sigma, x - \sigma v)],
\]  \hfill (86)

Comparing the deterministic diffusion matrix (56) (obtained for the global-in-time approach in Section 3.2.1) and stochastic diffusion matrix (86), we observe that they are the same except that the space average is replaced by the statistical average. If we suppose that in definition (56) of the deterministic diffusion matrix, the electric field \( E^\varepsilon \) is a random vector field, and if we take the expectation value of (56), then the homogeneity property (H3) implies that the space average is trivially the identity. Therefore we recover the stochastic diffusion matrix (86) or (83) from the “deterministic” one (56) by a statistical average and the homogeneity property. The link between the deterministic and stochastic diffusion matrices is also reinforced by

**Corollary 1** Let \( (t, k) \mapsto \Omega(t, k) := \omega(k) t \) be a real-valued function, where the given real-valued function \( k \mapsto \omega(k) \) is odd in the variable \( k \). Let \( (t, k) \mapsto \tilde{E}_0(t, k) \) be a given real vector-valued function, which is even in the variable \( k \) and such that \( |k| \tilde{E}_0 | \in L^\infty(\mathbb{R}^+; \ell^2(\mathbb{Z}^d)) \). Then there exists
a matrix-valued function $\mathcal{R}_t^0 : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{T}^d \to \mathbb{R}^{2d}$ such that the associated diffusion matrix, defined by formula (83) of Theorem 2, is

$$\mathcal{R}_t(t, v) = \frac{\tau}{2} \sum_{k \in \mathbb{Z}^d} \left( \sin \left( \frac{\tau}{2} (\partial_k \omega(t, k) - k \cdot v) \right) \right)^2 \widehat{E}_0(t, k) \otimes \widehat{E}_0(t, k). \tag{87}$$

Moreover the autocorrelation matrix $\mathcal{R}_t^0$ and its associated diffusion matrix satisfy properties i)–iv) of Proposition.

Before giving the proof of Corollary 11 we observe that the diffusion matrix is the same as the diffusion matrix obtained, for the local-in-time approach of the non-self-consistent deterministic case in Section 3.2.2.

Proof of Corollary 1. We consider the Fourier series decomposition of a given smooth electric field $E$,

$$E(t, \tau, x) = \sum_{k \in \mathbb{Z}^d} e^{i k \cdot x} \widehat{E}(t, \tau, k),$$

where, without loss of generality, we take

$$\widehat{E}(t, \tau, k) = \widehat{E}_0(t, \tau, k) e^{-i \Omega(\tau, k)} = \widehat{E}_0(t, \tau, k) e^{-i \omega(k) \tau},$$

with the function $\Omega(t, k)$ (respectively, $\omega(k)$) chosen like in the statement of Corollary 11. Since $\widehat{E}(t, \tau, k)$ is Hermitian (i.e. $\widehat{E}^*(t, \tau, k) = \widehat{E}(t, \tau, -k)$), and because the function $k \mapsto \omega(k)$ is odd, we could choose either $\widehat{E}_0(t, \tau, k)$ Hermitian, or real and even in $k$. We restrict ourselves to the case where $\widehat{E}_0(t, \tau, k)$ is real and even in $k$. In order to be consistent with hypothesis (H3), using the real vector-valued function $\widehat{E}_0(t, k)$ like in the statement of Corollary 11 we can choose $\widehat{E}_0(t, \tau, k)$ such that

$$\mathbb{E} \left[ \widehat{E}_0(t, \tau, k) \otimes \widehat{E}_0(t, \sigma, k') \right] = \widehat{E}_0(t, k) \otimes \widehat{E}_0(t, k) A_\tau(\tau - \sigma) \delta(k + k'), \tag{88}$$

where $s \mapsto A_\tau(s) : \mathbb{R} \to \mathbb{R}^+$ is a real non-negative even bounded function. From (88) we obtain

$$\mathbb{E} [E(t, \tau, x) \otimes E(t, \sigma, y)] = \sum_{k \in \mathbb{Z}^d} \widehat{E}_0(t, k) \otimes \widehat{E}_0(t, k) A_\tau(\tau - \sigma) e^{i k \cdot (x - y)} e^{-i \omega(k)(\tau - \sigma)},$$

and then from (82) we obtain

$$\mathcal{R}_t^0(t, \tau, x) = \sum_{k \in \mathbb{Z}^d} \widehat{E}_0(t, k) \otimes \widehat{E}_0(t, k) A_\tau(\tau - \sigma) e^{-i \omega(k)(\tau - \sigma - x)}. \tag{89}$$

The autocorrelation matrix $\mathcal{R}_t^0$ given by (89) satisfies the property i) of Proposition 11. Setting the resonance function

$$R_\tau(\xi) := \frac{\tau}{2} \left( \frac{\sin(\tau \xi/2)}{\tau \xi/2} \right)^2,$$

we define the time autocorrelation function $A_\tau$ as the inverse Fourier transform of the function $2R_\tau$, i.e.

$$A_\tau(\sigma) := \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{i \sigma \xi} R_\tau(\xi) d\xi = \Lambda(\sigma/\tau), \tag{90}$$

where the function $s \mapsto \Lambda(s)$ is the triangular function of support $]-1,1[$. As a consequence $\|A_\tau\|_{L^\infty(\mathbb{R})} \leq 1$, and using $|k| |\widehat{E}_0| \in L^\infty(\mathbb{R}^+; l^2(\mathbb{Z}^d))$, we obtain that $\mathcal{R}_t^0 \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}; W^{2,\infty}(\mathbb{T}^d))$. In addition from (90), we deduce that $\text{supp}(\mathcal{R}_t^0) \subset \mathbb{R}^+ \times \mathbb{R}^+ \times [-\tau, \tau] \times \mathbb{T}^d$. 

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Therefore the autocorrelation matrix $R^0_{\tau}$ given by (87)-(90) satisfies the property $\text{ii}$) of Proposition 1. Finally, let us compute the diffusion matrix $D_{\tau}$ from (89) and (91). Using parity properties of the functions $\omega$, $A_{\tau}$ and $\hat{E}_0$, we obtain

$$D_{\tau}(t,v) = \int_0^\tau d\sigma R^0_{\tau}(t,\tau,\sigma v)$$

$$= \sum_{k \in \mathbb{Z}^d} \hat{E}_0(t,k) \otimes \hat{E}_0(t,k) \int_0^\tau d\sigma e^{i(\omega(k) - k \cdot v)\sigma} A_{\tau}(\sigma)$$

$$= \sum_{k \in \mathbb{Z}^d} \hat{E}_0(t,k) \otimes \hat{E}_0(t,k) \int_{\mathbb{R}^+} d\sigma e^{i(\omega(k) - k \cdot v)\sigma} \Lambda(\sigma/\tau)$$

$$= \frac{1}{2} \sum_{k \in \mathbb{Z}^d} \hat{E}_0(t,k) \otimes \hat{E}_0(t,k) \int_{\mathbb{R}} d\sigma e^{i(\omega(k) - k \cdot v)\sigma} \Lambda(\sigma/\tau)$$

$$= \frac{1}{2} \sum_{k \in \mathbb{Z}^d} \hat{E}_0(t,k) \otimes \hat{E}_0(t,k) \left( \frac{\sin \left( \frac{\tau}{2} (\omega(k) - k \cdot v) \right)}{\omega(k) - k \cdot v} \right)^2,$$

(91)

which is (87). By a direct differentiation of (91) with respect to $v$, we verify easily the property $\text{iii)}$ of Proposition 1 while the property $\text{iv)}$ of Proposition 1 is obvious from the structure of (91). □

4.3 Proof of Theorem 2

Let us rewrite the Vlasov equation (17) in the following form,

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon^2} \mathcal{L} f^\varepsilon = \mathcal{N}_t^\varepsilon f^\varepsilon,$$  \hspace{1cm} (92)

$$f^\varepsilon|_{t=0} = f^\varepsilon_0,$$  \hspace{1cm} (93)

where the linear operators $\mathcal{L}$ and $\mathcal{N}_t^\varepsilon$ are defined by

$$\mathcal{L} = v \cdot \nabla_x, \quad \mathcal{N}_t^\varepsilon = -\frac{1}{\varepsilon} E^\varepsilon(t,x) \cdot \nabla_v = -\frac{1}{\varepsilon} E(t,t/\varepsilon^2,x) \cdot \nabla_v.$$  \hspace{1cm} (94)

Obviously the operators $\mathcal{L}$ and $\mathcal{N}_t^\varepsilon$ are skew-adjoint for the scalar product of $L^2(Q)$, while the operator $\mathcal{L}$ and the deterministic group $S_t^\varepsilon$, generated by $\varepsilon^{-2} \mathcal{L}$ (see Section 2.2), commute with $E$. Of course space and statistical averages commute. From hypothesis (H2), the random operators $\mathcal{N}_t^\varepsilon$ and $\mathcal{N}_s^\varepsilon$ are independent as soon as $|t-s| > \varepsilon^2 \tau$.

The next useful proposition states that time decorrelation of the stochastic electric field also entails time decorrelation between the distribution function and the electric field.

**Proposition 2** (time decorrelation property between $f^\varepsilon$ and $E^\varepsilon$) Assume (H2). Suppose that the random initial data $f^\varepsilon_0$ and the electric field $E^\varepsilon$ are independent. Then $\mathcal{N}_s^\varepsilon$ is independent of $f^\varepsilon(t)$ as soon as $s \geq t + \varepsilon^2 \tau$.

**Proof.** From the Duhamel formula

$$f^\varepsilon(t) = S_t^\varepsilon f^\varepsilon_0 + \int_0^t d\sigma S_t^\varepsilon S_{t-\sigma}^\varepsilon \mathcal{N}_\sigma^\varepsilon f^\varepsilon(\sigma),$$

where $t \mapsto S_t^\varepsilon$ is the deterministic group generated by $\varepsilon^{-2} \mathcal{L}$ (see Section 2.2), we observe that $f^\varepsilon(t)$ depends only on $f^\varepsilon_0$ and $\mathcal{N}_\sigma^\varepsilon$ (or $E^\varepsilon(\sigma, \cdot)$) for $\sigma \leq t$. Since $f^\varepsilon_0$ is independent of $E^\varepsilon(t, \cdot)$, $\forall t \in \mathbb{R}$, and
The solution of the limit point $f$ and for a.e. $\omega \in \Omega$, $\|f^\varepsilon_0\|_{L^1(Q)} + \|f^\varepsilon_0\|_{L^\infty(Q)} \leq C_0 < \infty$.

Then, for any $\varepsilon > 0$, the Cauchy’s problem (92)-(93) has a unique non-negative solution $f^\varepsilon \in C(\mathbb{R}^+; L^1 \cap L^\infty(Q))$, which is given by

$$f^\varepsilon(t,x,v) = f^\varepsilon_0(X^\varepsilon(0; t, x, v), V^\varepsilon(0; t, x, v)),$$  \hspace{1cm} (95)

where the characteristic curves $(X^\varepsilon, V^\varepsilon)$ are solutions to the ODEs,

$$\frac{dX^\varepsilon}{dt}(t) = \frac{1}{\varepsilon} V^\varepsilon(t), \quad \frac{dV^\varepsilon}{dt}(t) = \frac{1}{\varepsilon} E^\varepsilon(t, X(t)), \quad X^\varepsilon(0; 0, x, v) = x, \quad V^\varepsilon(0; 0, x, v) = v.$$  \hspace{1cm} (96)

Moreover we have the a priori estimates,

$$\|f^\varepsilon(t)\|_{L_p(Q)} = \|f^\varepsilon_0\|_{L_p(Q)}, \quad 1 \leq p \leq \infty.$$  

In addition, there exist a function $f_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$, and a function $f \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^d))$, such as, up to subsequences,

$$\mathbb{E}[f^\varepsilon_0] \rightharpoonup f_0 \text{ in } L^\infty(Q) \text{ weak-\ast}, \quad \text{and } \mathbb{E}[f^\varepsilon] \rightharpoonup f \text{ in } L^\infty(\mathbb{R}^+; L^\infty(Q)) \text{ weak-\ast}.$$  

The limit point $f$ is such that $\int dx f = f \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^d))$. The function $\mathbb{E}[\int dx f^\varepsilon]$ is the solution of

$$\partial_t \mathbb{E} \left[ \int dx f^\varepsilon \right] + \nabla_v \cdot \mathbb{E} \left[ \int dx \frac{E^\varepsilon f^\varepsilon}{\varepsilon} \right] = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d),$$  \hspace{1cm} (97)

$$\mathbb{E} \left[ \int dx f^\varepsilon \right] \big|_{t=0} = \mathbb{E} \left[ \int dx f^\varepsilon_0 \right].$$  \hspace{1cm} (98)

**Proof.** Since $\|\mathbb{E}[f^\varepsilon_0]\|_{L^1(Q)} + \|\mathbb{E}[f^\varepsilon_0]\|_{L^\infty(Q)} \leq \mathbb{E}[\|f^\varepsilon_0\|_{L^1(Q)}] + \mathbb{E}[\|f^\varepsilon_0\|_{L^\infty(Q)}] \leq C_0 < \infty$, by weak compactness arguments there exists a function $f_0 \in L^1 \cap L^\infty(Q)$ such that $\mathbb{E}[f^\varepsilon_0]$ (up to a subsequence) converges in $L^\infty(Q)$ weak-\ast to $f_0$. Using the regularity hypothesis (H4) for the electric field $E$, the Cauchy–Lipschitz–Picard theorem for ODEs gives existence and uniqueness of a regular Lagrangian flow $(X^\varepsilon, V^\varepsilon)$, which is a solution of (96). It follows from standard results on first-order transport equations (see, e.g., [12]) that the Lagrangian solution to (92)-(93) is given by (95). From (95), we obtain $\|\mathbb{E}[f^\varepsilon]\|_{L^\infty(Q)} = \|\mathbb{E}[f^\varepsilon_0]\|_{L^\infty(Q)} \leq C_0 < \infty$. Moreover using skew-adjointness of $L$ and $N^\varepsilon_t$, we can prove, following standard lines, that $\partial_t\|f^\varepsilon(t)\|_{L^p(Q)} = 0$, for $1 \leq p \leq \infty$. This leads to $\mathbb{E}[\|f^\varepsilon\|_{L^\infty(\mathbb{R}^+; L^p(Q))}] = \mathbb{E}[\|f^\varepsilon_0\|_{L^p(Q)}] \leq C_0 < \infty$ and $\|\mathbb{E}[f^\varepsilon]\|_{L^\infty(\mathbb{R}^+; L^p(Q))} \leq \|\mathbb{E}[f^\varepsilon]\|_{L^\infty(\mathbb{R}^+; L^p(Q))} \leq C_0 < \infty$. By weak compactness arguments there exists a function $f \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(Q))$, such that $\mathbb{E}[f^\varepsilon]$ (up to a subsequence) converges in $L^\infty(\mathbb{R}^+; L^\infty(Q))$ weak-\ast to $f$. Now we claim that $\varepsilon^2 N^\varepsilon_t f^\varepsilon \rightarrow 0$, in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$ as $\varepsilon$ tends to zero. Indeed, we have for all $\varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$,

$$|\varepsilon^2 N^\varepsilon_t f^\varepsilon, \varphi| = \varepsilon^2 |(f^\varepsilon, N^\varepsilon_t * \varphi)| \leq \varepsilon \|f^\varepsilon_0\|_{L^\infty(Q)} \|\varphi\|_{L^\infty(\mathbb{R}^+; W^{1,1}(Q))} \|E\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^d)} \rightarrow 0,$$  

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as $\varepsilon \to 0$. Then multiplying the Vlasov equation (92) by $\varepsilon^2$, taking its expectation value, and letting $\varepsilon$ go to zero, we find

$$\mathcal{L} f = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times Q),$$

(99)

where we have used the commutation property between $E$ and $\mathcal{L}$. From Lemma 1 and (99), we infer that $f$ is independent of $x$ and $\int dx f = f \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^d))$. Finally, the Vlasov equation (92) is averaged in space and then rewritten in a weak form. Taking the expectation value of the result, we obtain (97)-(98).

The rest of the proof is devoted to pass to the limit in equation (97). For this, we can first start with a simple iteration of the Duhamel formula as it was done in Section 3.2.2 for the deterministic case. As explained in Section 4.3.1 this method fails. To solve this problem we use the method of Section 3.2.2 for the deterministic case. Letting $\varepsilon$ go to zero, we find $\varepsilon f(\theta)$, which consists to apply a double iteration of the Duhamel formula for $f$. This is described in Section 4.3.2.

### 4.3.1 Simple iteration of the Duhamel formula

Following what we have done in Section 3.2.2 where a simple iteration of the Duhamel formula is used, we obtain, for all $\varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$,

$$\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \varphi(t,v) \frac{\mathbb{E}[ \int dx f^\varepsilon(t+\theta) ] - \mathbb{E}[ \int dx f^\varepsilon(t) ]}{\theta} = T_1^\varepsilon(\varphi) + T_2^\varepsilon(\varphi),$$

(100)

where

$$T_1^\varepsilon(\varphi) := \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_t^{t+\theta} ds \int dx \frac{1}{\varepsilon \theta} \mathbb{E}[ E^\varepsilon(s) \cdot \nabla_v \varphi(t,v) S^\varepsilon_{s-t-\theta} f^\varepsilon(t - \theta) ],$$

(101)

and

$$T_2^\varepsilon(\varphi) := \mathcal{J}^\varepsilon(\varphi) + \mathcal{M}^\varepsilon(\varphi),$$

with

$$\mathcal{J}^\varepsilon(\varphi) := \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_t^{t+\theta} ds \int_{t-\theta}^s d\sigma \frac{1}{\varepsilon \theta} f(t,v) \nabla_v \cdot \left( \int dx \mathbb{E}[ S^\varepsilon_{s-\sigma} E^\varepsilon(\sigma,x) \otimes E^\varepsilon(s,x) \nabla_v \varphi(t,v) ] \right),$$

(102)

and

$$\mathcal{M}^\varepsilon(\varphi) := \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_t^{t+\theta} ds \int_{t-\theta}^s d\sigma \frac{1}{\varepsilon \theta} \int dx \mathbb{E}[ (S^\varepsilon_{s-\sigma} f^\varepsilon(\sigma) - f(t,v)) \nabla_v \cdot (S^\varepsilon_{s-\sigma} E^\varepsilon(\sigma,x) \otimes E^\varepsilon(s,x) \nabla_v \varphi(t,v) ) ].$$

By choosing $\hat{\theta} \geq \varepsilon^2 \tau$, we can use the time decorrelation hypothesis (H2), Proposition 2 and assumption (H1) to show that for equation (102) we have $T_1^\varepsilon(\varphi) = 0$. The term $\mathcal{J}^\varepsilon(\varphi)$, defined by equation (101), can be treated as it was done in Section 3.2 and it gives the diffusion term. It remains to deal with the error term $\mathcal{M}^\varepsilon(\varphi)$, which is of order zero with respect to $\varepsilon$, i.e. $\mathcal{M}^\varepsilon(\varphi) = O(\varepsilon^0)$. For this, we observe that we have to evaluate the expectation of a cubic product between $f^\varepsilon(\sigma) - f(t)$, $E^\varepsilon(\sigma)$ and $E^\varepsilon(s)$, i.e. schematically $\mathbb{E}[ f^\varepsilon(\sigma) - f(t) ] E^\varepsilon(\sigma) E^\varepsilon(s)$. Using hypotheses (H2)-(H3) and Proposition 2, we would like to replace an expression of the form $\mathbb{E}[ (f^\varepsilon(\sigma) - f(t)) E^\varepsilon(\sigma) E^\varepsilon(s) ]$ by an expression of the form $\mathbb{E}[ f^\varepsilon - f ] \mathbb{E}[ E^\varepsilon E^\varepsilon ]$, because from Proposition 3 we know that $\mathbb{E}[ f^\varepsilon - f ] \to 0$.
in \( L^\infty(\mathbb{R}^+; L^\infty(Q)) \) weak-*.
Unfortunately it is not possible, since \( f^\varepsilon(\sigma) \) and \( E^\varepsilon(\sigma) \) are evaluated at the same time \( \sigma \).
To remedy this problem we follow the procedure of [50], which consists of using a double iteration of the Duhamel formula for \( f \).
A double instead of a simple iteration of the Duhamel formula is used to go back in time far enough in order to use the time decorrelation property \((H2)\). The price of this procedure is the introduction of a second error term, namely \( \mu^\varepsilon_t \) (defined by (108)).
For the error term \( \mu^\varepsilon_t \), we face the same problem as for the term \( \mathcal{M}^\varepsilon \), i.e. we cannot use the time decorrelation hypotheses \((H2)-(H3)\) and Proposition 2. Nevertheless, the error term \( \mu^\varepsilon_t \) is of the order \( \mathcal{O}(\varepsilon) \). Therefore, to show that \( \lim_{\varepsilon \rightarrow 0} \mu^\varepsilon_t = 0 \) in the distributional sense, we do not use time decorrelation hypotheses (which are useless), but we appeal to the regularity hypotheses on the electric field \( E^\varepsilon \), namely \((H4)\).

### 4.3.2 Double iteration of the Duhamel formula

First we recall that \( t \mapsto S^\varepsilon_t \) is the (deterministic) group on \( L^p(Q) \), \( 1 \leq p \leq \infty \), generated by \( \varepsilon^{-2}L \) (see equation (20)).
Using the group \( S^\varepsilon_t \) and the Duhamel formula, the formal solution to (92) is given by

\[
f^\varepsilon(t) = S^\varepsilon_{t-s}f^\varepsilon(s) + \int_s^t d\tau S^\varepsilon_{t-\tau}N^\varepsilon_{\tau}f^\varepsilon(\tau). \tag{103}
\]

Taking \( s = t - \varepsilon^2 \tau \) in (103), and making the change of variable \( \tau = t - \sigma \), we obtain from (103),

\[
f^\varepsilon(t) = S^\varepsilon_{\varepsilon^2\tau}f^\varepsilon(t - \varepsilon^2 \tau) + \int_0^{2\varepsilon^2\tau} d\sigma S^\varepsilon_{\sigma}N^\varepsilon_{\tau-s}f^\varepsilon(t - \sigma). \tag{104}
\]

In the integral term of (104), we observe that the electric field and the distribution function are evaluated at the same time \( t - \sigma \). As a consequence, if we substitute (104) to \( f^\varepsilon \) in the right hand side of (97) (like it was done in Section 4.3.1), we obtain a quadratic term with respect to the electric field that we cannot decorrelate in time from the distribution function.
For this reason and following [50], we iterate a second time the Duhamel formula. In the same way that we obtained (103), we obtain

\[
f^\varepsilon(t - \sigma) = S^\varepsilon_{2\varepsilon^2\tau-s}f^\varepsilon(t - 2\varepsilon^2 \tau) + \int_0^{2\varepsilon^2\tau-s} d\sigma S^\varepsilon_{\sigma}N^\varepsilon_{\tau-s}f^\varepsilon(t - \sigma - s). \tag{105}
\]

Substituting the right-hand side of (105) to \( f^\varepsilon(t - \sigma) \) in the right-hand side of (104), and using the properties of the group \( S^\varepsilon_t \), we obtain

\[
f^\varepsilon(t) = S^\varepsilon_{\varepsilon^2\tau}f^\varepsilon(t - \varepsilon^2 \tau) + \int_0^{2\varepsilon^2\tau} d\sigma S^\varepsilon_{\sigma}N^\varepsilon_{\tau-s}S^\varepsilon_{\sigma}S^\varepsilon_{2\varepsilon^2\tau}f^\varepsilon(t - 2\varepsilon^2 \tau)
\]
\[
+ \int_0^{2\varepsilon^2\tau-s} d\sigma \int_0^{2\varepsilon^2\tau-s} d\sigma S^\varepsilon_{\sigma}N^\varepsilon_{\tau-s}S^\varepsilon_{\sigma}N^\varepsilon_{\tau-s}f^\varepsilon(t - \sigma - s). \tag{106}
\]

Applying the operator \( N^\varepsilon_t \) to (106), and then applying successively the average in space and the expectation value, we obtain

\[
- \nabla_v \cdot \mathbf{E} \left[ \int dx \frac{E^\varepsilon(t)f^\varepsilon(t)}{\varepsilon} \right] = \int dx \mathbf{E} \left[ N^\varepsilon_t S^\varepsilon_{\varepsilon^2\tau}f^\varepsilon(t - \varepsilon^2 \tau) \right]
\]
\[
+ \int_0^{2\varepsilon^2\tau} d\sigma \int dx \mathbf{E} \left[ N^\varepsilon_t S^\varepsilon_{\sigma}N^\varepsilon_{\tau-s}S^\varepsilon_{\sigma}S^\varepsilon_{2\varepsilon^2\tau}f^\varepsilon(t - 2\varepsilon^2 \tau) \right] + \mu^\varepsilon_t, \tag{107}
\]
with
\[ \mu^ε_1 = \int_0^{\varepsilon^2 \tau} d\sigma \int_0^{2\varepsilon^2 \tau - \sigma} ds \int dx E \left[ N^ε_{i1} S^ε_{\sigma} N^ε_{i\sigma} S^ε_{\tau} N^ε_{i\tau - \sigma} f^ε(t - \sigma - s) \right]. \] (108)

Using Proposition 2, we obtain that \( f^ε(t) \) is independent of \( N^ε_{i\sigma} \) as soon as \( s \geq t + \varepsilon^2 \tau \). Then, using hypothesis (H1), we obtain
\[ E \left[ N^ε_{i1} S^ε_{\sigma} f^ε(t - \varepsilon^2 \tau) \right] = E \left[ N^ε_{i1} \right] S^ε_{\sigma} E \left[ f^ε(t - \varepsilon^2 \tau) \right] = 0. \] (109)

We note that the analysis of the term \((109)\) is the same as the one done for the term \((101)\). Therefore the first term of the right-hand side of \((107)\) vanishes. Since Proposition 2 implies that \( N^ε_{i\sigma} \) and \( N^ε_{i\tau - \sigma} \) are independent of \( f^ε(t - 2\varepsilon^2 \tau) \), for \( 0 \leq \sigma \leq \varepsilon^2 \tau \), we obtain from \((107)\),
\[ -\nabla_v \cdot E \left( \int dx \frac{E^ε(t) f^ε(t)}{\varepsilon} \right) = \int_0^{\varepsilon^2 \tau} d\sigma \int dx E \left[ N^ε_{i1} S^ε_{\sigma} N^ε_{i\sigma} S^ε_{\tau} \right] E \left[ S^ε_{2\varepsilon^2 \tau} f^ε(t - 2\varepsilon^2 \tau) + \mu^ε_1 \right] + \mu^ε_1. \] (110)

In fact, we have to consider a weak form of \((110)\), which is given by

**Proposition 4** We define the differential operator \( \Theta^ε_i \) as
\[ \Theta^ε_i \varphi = \int_0^\tau d\sigma (\sigma \nabla_x \cdot + \nabla_v \cdot) E \left[ E^ε(t - \varepsilon^2 \sigma, x - \sigma v) \otimes E^ε(t, x) \right] \nabla_v \varphi, \quad \varphi \in D(\mathbb{R}^+ \times \mathbb{R}^d), \] (111)
and the bilinear form \( \nu^ε_i \) as
\[ \nu^ε_i(\psi, \varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int dx \psi \Theta^ε_i \varphi, \quad \forall \psi \in L^\infty(\mathbb{R}^+ \times Q), \quad \forall \varphi \in D(\mathbb{R}^+ \times \mathbb{R}^d). \] (112)

Then, the weak formulation of \((110)\) reads: \( \forall \varphi \in D(\mathbb{R}^+ \times \mathbb{R}^d), \)
\[ \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \nabla_v \cdot E \left[ \int dx \frac{f^ε(t) E^ε(t)}{\varepsilon} \right] = \nu^ε_i \left( E \left[ f^ε(t) \right], \varphi \right) + \nu^ε_i \left( E \left[ S^ε_{2\varepsilon^2 \tau} f^ε(t - 2\varepsilon^2 \tau) - f^ε(t) \right], \varphi \right) + \mu^ε_1(\varphi), \] (113)
where the remainder term \( \mu^ε_1(\varphi) \) is given by
\[ \mu^ε_1(\varphi) = -\varepsilon^4 \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_0^\tau d\sigma \int_0^{2\varepsilon^2 \tau - \sigma} ds \int dx E \left[ f^ε(t - \varepsilon^2 (\sigma + s)) N^ε_{i\varepsilon^2 (\sigma + s)} S^ε_{\sigma} N^ε_{i\sigma} S^ε_{\tau} \right]. \] (114)

**Proof.** We have to show that the weak formulation of \((110)\) is given by \((113)\). The left hand side of \((113)\) is obtained straightforwardly from the left hand side of \((110)\). The first two terms of the right-hand side of \((113)\) can be obtained in a similar way from the first two terms of the
right-hand side of (110) respectively. Indeed, we consider a non-random function \( \psi = \psi(t, x, v) \), which can be either \( \mathbb{E}[f^\varepsilon(t)] \) or \( \mathbb{E}\left[ S_{2\varepsilon^2\tau}^\varepsilon f^\varepsilon(t - 2\varepsilon^2\tau) - f^\varepsilon(t) \right] \). Then the quantity of interest is

\[
\int_0^{\varepsilon^2\tau} d\sigma \int dx \mathbb{E}\left[ N_t^\varepsilon S_\sigma^\varepsilon N_{t-\sigma}^\varepsilon S_{-\sigma}^\varepsilon \right] \psi.
\] (115)

Multiplying (115) by a test function \( \varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d) \), and integrating with respect to the time and velocity variables, we obtain, after expanding all operators,

\[
\frac{1}{\varepsilon^2} \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \varphi(t, v) \int dx \int_0^{\varepsilon^2\tau} d\sigma \nabla_v \cdot \left( \mathbb{E}\left[ E^\varepsilon(t, x) \otimes E^\varepsilon(t - \sigma, x - \sigma v/\varepsilon^2) \right] \left( \frac{\sigma}{\varepsilon^2} \nabla_x + \nabla_v \right) \psi(t, x, v) \right).
\] (116)

Using integrations by parts with respect to the variables \( x \) and \( v \), and making the change of time variable \( \sigma' = \sigma/\varepsilon^2 \), we obtain from (116),

\[
\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int dx \psi(t, x, v) \int_0^\tau d\sigma \left( \sigma \nabla_x \cdot + \nabla_v \cdot \right) \left( \mathbb{E}\left[ E^\varepsilon(t - \varepsilon^2\sigma, x - \sigma v) \otimes E^\varepsilon(t, x) \right] \nabla_v \varphi(t, v) \right) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int dx \psi \Theta_t^\varepsilon \varphi = \nu_t^\varepsilon(\psi, \varphi).
\] (117)

The first two terms of the right-hand side of (113) follow by replacing respectively \( \psi \) by \( \mathbb{E}[f^\varepsilon(t)] \) and \( \mathbb{E}\left[ S_{2\varepsilon^2\tau}^\varepsilon f^\varepsilon(t - 2\varepsilon^2\tau) - f^\varepsilon(t) \right] \) in (117). To obtain the weak formulation of the remainder term \( \mu_t^\varepsilon \), we use the skew-adjointness of \( N_t^\varepsilon \) and the dual of \( S_t^\varepsilon \) given by \( S_t^\varepsilon = S_{-t}^\varepsilon \). Using multiple velocity integrations by parts, several changes of variables in space, and the changes of time variables \( \sigma' = \sigma/\varepsilon^2 \) and \( s' = s/\varepsilon^2 \), we obtain from the third term of the right-hand side of (113),

\[
\mu_t^\varepsilon(\varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \varphi \mu_t^\varepsilon
\]

\[
= \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \varphi \int_0^{\varepsilon^2\tau} \sigma \int_0^{2\varepsilon^2\tau-\sigma} ds \int dx \mathbb{E}\left[ N_t^\varepsilon S_\sigma^\varepsilon N_{t-\sigma}^\varepsilon S_{-\sigma}^\varepsilon \right] f^\varepsilon(t - \sigma - s)
\]

\[
= \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \varphi \int_0^{\varepsilon^2\tau} \sigma \int_0^{2\varepsilon^2\tau-\sigma} ds \mathbb{E}\left[ \int_{\mathbb{R}^d} dv f^\varepsilon(t - \sigma - s) S_{-\sigma}^\varepsilon N_{t-\sigma}^\varepsilon S_{-\sigma}^\varepsilon N_t^\varepsilon \varphi \right]
\]

\[
= -\varepsilon^4 \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_0^\tau ds \int_0^{2\tau-\sigma} dx \mathbb{E}\left[ f^\varepsilon(t - \varepsilon^2(\sigma + s)) S_{-\varepsilon^2\sigma}^\varepsilon N_{t-\varepsilon^2\sigma}^\varepsilon S_{-\varepsilon^2\sigma}^\varepsilon N_t^\varepsilon \varphi \right],
\]

which is (114). This ends the Proof of Proposition 4.

The next lemma states the limit of the operator \( \Theta_t^\varepsilon \) as \( \varepsilon \to 0 \).

**Lemma 4** Under hypothesis (H3), for all \( \varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d) \), the operator \( \Theta_t^\varepsilon \) defined by (111) becomes

\[
\Theta_t^\varepsilon \varphi = \nabla_v \cdot \left( \int_0^\tau d\sigma R_\tau(t - \varepsilon^2\sigma, t - \sigma, -\sigma \varphi) \right) \nabla_v \varphi,
\] (118)
and we obtain

\[ \Theta^0_t \varphi \rightarrow \Theta^0_t \varphi \quad \text{in} \quad L^1(\mathbb{R}^+ \times \mathbb{R}^d), \]

where the operator \( \Theta^0_t \) is defined by

\[ \Theta^0_t \varphi = \nabla_v \cdot \left( \left( \int_0^\tau d\sigma \mathcal{R}_\tau(t, t, -\sigma, -\sigma v) \right) \nabla_v \varphi \right). \]  

(120)

Proof. Using assumption (H3), i.e. \( \mathbb{E}[E(t, \tau, x) \otimes E(s, \sigma, y)] = \mathcal{R}_\tau(t, s, \tau - \sigma, x - y) \), for all \( \varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d) \), we obtain from (111),

\[ \Theta^\varepsilon_t \varphi = \int_0^\tau d\sigma (\sigma \nabla_x \cdot + \nabla_v \cdot) \mathbb{E} \left[ E^\varepsilon(t - \varepsilon^2 \sigma, x - \sigma v) \otimes E^\varepsilon(t, x) \right] \nabla_v \varphi \]

\[ = \int_0^\tau d\sigma (\sigma \nabla_x \cdot + \nabla_v \cdot) \mathcal{R}_\tau(t - \varepsilon^2 \sigma, t, -\sigma, -\sigma v) \nabla_v \varphi \]

\[ = \int_0^\tau d\sigma \nabla_v \cdot \left( \mathcal{R}_\tau(t - \varepsilon^2 \sigma, t, -\sigma, -\sigma v) \nabla_v \varphi \right) \]

\[ = - \int_0^\tau d\sigma \left( \nabla_x \cdot \mathcal{R}_\tau(t - \varepsilon^2 \sigma, t, -\sigma, -\sigma v) \right) \nabla_v \varphi + \int_0^\tau d\sigma \mathcal{R}_\tau(t - \varepsilon^2 \sigma, t, -\sigma, -\sigma v) \nabla_v \varphi. \]

Since \( \mathcal{R}_\tau \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}; W^{2,\infty}(T^d)) \), and because \( \tau \) is bounded, we obtain

\[ \int_0^\tau d\sigma \left( \nabla_x \cdot \mathcal{R}_\tau(t - \varepsilon^2 \sigma, t, -\sigma, -\sigma v) \right) \nabla_v \varphi \rightarrow \int_0^\tau d\sigma \left( \nabla_x \cdot \mathcal{R}_\tau(t, t, -\sigma, -\sigma v) \right) \nabla_v \varphi, \]

for almost every \((t, v) \in \mathbb{R}^+ \times \mathbb{R}^d\), as \( \varepsilon \rightarrow 0 \), and

\[ \int_0^\tau d\sigma \mathcal{R}_\tau(t - \varepsilon^2 \sigma, t, -\sigma, -\sigma v) \nabla_v \varphi \rightarrow \int_0^\tau d\sigma \mathcal{R}_\tau(t, t, -\sigma, -\sigma v) \nabla_v \varphi, \]

for almost every \((t, v) \in \mathbb{R}^+ \times \mathbb{R}^d\), as \( \varepsilon \rightarrow 0 \). Moreover using the regularity of \( \mathcal{R}_\tau \) we obtain

\[ |\Theta^\varepsilon_t \varphi| \leq \|\sigma \nabla_x \mathcal{R}_\tau\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}; L^\infty([-\tau, \tau]; L^\infty(T^d)))} \|\nabla_v \varphi\| + \|\mathcal{R}_\tau\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}; L^1([-\tau, \tau]; L^\infty(T^d)))} \|\nabla_v \varphi\|, \]

(121)

where the right-hand side of (121) defines a function in \( L^1(\mathbb{R}^+ \times \mathbb{R}^d) \) independent of \( \varepsilon \). Therefore, using the Lebesgue dominated convergence theorem, we obtain the limit (119), where the limit point is given by (120). This ends the proof. \( \square \)

To deal with the second term of right-hand side of (113), we use

Lemma 5 Under hypothesis (H3), for all \( \varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d) \), we obtain

\[ \lim_{\varepsilon \rightarrow 0} \nu^\varepsilon_t \left( \mathbb{E}[S_{2\varepsilon^2\tau} f^\varepsilon(t - 2\varepsilon^2 \tau) - f^\varepsilon(t)], \varphi \right) = 0. \]

Proof. Using definitions (112) and (118) of respectively the bilinear form \( \nu^\varepsilon_t \) and the operator \( \Theta^\varepsilon_t \), we obtain

\[ \nu^\varepsilon_t \left( \mathbb{E}[S_{2\varepsilon^2\tau} f^\varepsilon(t - 2\varepsilon^2 \tau) - f^\varepsilon(t)], \varphi \right) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} dx \mathbb{E}[S_{2\varepsilon^2\tau} f^\varepsilon(t - 2\varepsilon^2 \tau) - f^\varepsilon(t)] \Theta^\varepsilon_t \varphi \]

\[ = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \Theta^\varepsilon_t \varphi \mathbb{E} \left[ \int_{\mathbb{R}^d} dx f^\varepsilon(t - 2\varepsilon^2 \tau) - \int_{\mathbb{R}^d} dx f^\varepsilon(t) \right]. \]

(122)
On the one hand, from Proposition 3 we obtain for all \( \psi \in \mathcal{D} \), we get

\[ \mathbb{E}[f dx f^\varepsilon] = f dx \mathbb{E}[f^\varepsilon] \rightarrow f dx f = f \text{ in } L^\infty(\mathbb{R}^+ \times \mathbb{R}^d) \text{ weak-}* \text{ as } \varepsilon \to 0 \]. Then, \( \mathbb{E}[f dx f^\varepsilon(t - 2\varepsilon^2\tau - f dx f^\varepsilon(t)] \rightarrow 0 \text{ in } L^\infty(\mathbb{R}^+ \times \mathbb{R}^d) \text{ weak-}* \text{ as } \varepsilon \to 0 \). On the other hand, from Lemma 4 we obtain \( \Theta^0_t \varphi \rightarrow \Theta^0_t \varphi \) in \( L^1(\mathbb{R}^+ \times \mathbb{R}^d) \) strongly as \( \varepsilon \to 0 \). Therefore, we can pass to the limit \( \varepsilon \to 0 \text{ in } (122) \), and we obtain the desired result.

The asymptotic behavior of the term \( \mu^\varepsilon_t \) is given by

Lemma 6 Under hypothesis (H4), for all \( \varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d) \), we obtain

\[ |\mu^\varepsilon_t(\varphi)| \leq \varepsilon^4 C_0 C_E \|\varphi\|_{L^\infty(\mathbb{R}^+; W^{3,1}(\mathbb{R}^d))} \].

Proof. Introducing the operator \( \Gamma^\varepsilon_{t,\sigma} \), defined by

\[ \Gamma^\varepsilon_{t,\sigma} = N^\varepsilon_{t,-\varepsilon^2 \sigma} S^\varepsilon_{-\varepsilon^2 \sigma}, \]

we obtain from definition (114) of \( \mu^\varepsilon_t(\varphi) \),

\[ |\mu^\varepsilon_t(\varphi)| \leq \varepsilon^4 Q^2 \|f^\varepsilon(t - \varepsilon^2(\sigma + s))\|_{L^\infty(\mathbb{R}^+ \times [0,\tau]\times [0,2\tau]\times \mathbb{R}^d)} \mathbb{E} \left[ \|\Gamma^\varepsilon_{t,-\varepsilon^2 \sigma,s} \Gamma^\varepsilon_{t,\sigma} N^\varepsilon_t \varphi\|_{L^\infty(\mathbb{R}^+ \times [0,\tau]\times [0,2\tau]\times L^1(\mathbb{R}^d))} \right] \]

\[ \leq C_0 \varepsilon^4 Q^2 \|\Gamma^\varepsilon_{t,-\varepsilon^2 \sigma,s} \Gamma^\varepsilon_{t,\sigma} N^\varepsilon_t \varphi\|_{L^\infty(\mathbb{R}^+ \times [0,\tau]\times [0,2\tau]\times L^1(\mathbb{R}^d))}. \]

Introducing the translation operator \( T_{-\sigma \nu} \) in the \( x \)-direction, defined by

\[ T_{-\sigma \nu} \psi(t, x, v) = \psi(t, x + \sigma v, v), \]

we obtain for any smooth function \( \psi = \psi(t, x, v) \),

\[ \Gamma^\varepsilon_{t,\sigma} \psi = -\varepsilon^{-1} E^\varepsilon(t - \varepsilon^2 \sigma, x) \cdot \nabla_v (T_{-\sigma \nu} \psi) = -\varepsilon^{-1} E(t - \varepsilon^2 \sigma, t/\varepsilon^2 - \sigma, x) \cdot (\sigma T_{-\sigma \nu} \nabla_x \psi + T_{-\sigma \nu} \nabla_v \psi). \]

Using (124) and (126), we obtain for any smooth function \( \psi = \psi(t, x, v) \),

\[ \Gamma^\varepsilon_{t,-\varepsilon^2 \sigma,s} \Gamma^\varepsilon_{t,\sigma} \psi = N^\varepsilon_{t,-\varepsilon^2 (\sigma + s)} S^\varepsilon_{-\varepsilon^2 s} \Gamma^\varepsilon_{t,\sigma} \psi \]

\[ = -\varepsilon^{-2} \mathbb{E}[T_{-\sigma \nu} E^\varepsilon(t - \varepsilon^2 \sigma, x)] \cdot T_{-\sigma \nu} \sigma \nabla_x \psi + \nabla_v \psi \]

\[ = \varepsilon^{-2} \sum_{i,j} E^\varepsilon_i(t - \varepsilon^2 (\sigma + s), x) \left\{ s T_{-\sigma \nu} \partial_{x_i} E^\varepsilon_j(t - \varepsilon^2 \sigma, x) \left[ \sigma T_{-\sigma \nu} \partial_{x_j} \psi \right. \right. \]

\[ + T_{-(\sigma + s) \nu} \partial_{v_j} \psi \right\} + \left. T_{-\sigma \nu} E^\varepsilon_j(t - \varepsilon^2 \sigma, x) \left[ \sigma (\sigma + s) \sigma \nabla_v \psi \right. \right. \]

\[ + \sigma T_{-(\sigma + s) \nu} \partial_{v_{i,j}} \psi \left. \right\} + \left. T_{-\sigma \nu} \partial_{v_{i,j}} \psi \right\} \right\}. \]

Using (127), we obtain for all \( \psi \in L^\infty(\mathbb{R}^+; W^{2,1}(\mathbb{R}^d)) \),

\[ \|\Gamma^\varepsilon_{t,-\varepsilon^2 \sigma,s} \Gamma^\varepsilon_{t,\sigma} \psi\|_{L^\infty(\mathbb{R}^+ \times [0,\tau]\times [0,2\tau]\times L^1(\mathbb{R}^d))} \leq \frac{\tau^2}{\varepsilon^2} \|E\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^+; W^{2,1}(\mathbb{R}^d))} \|\psi\|_{L^\infty(\mathbb{R}^+; W^{2,1}(\mathbb{R}^d))}. \]

Using (128) we obtain for all \( \varphi \in L^\infty(\mathbb{R}^+; W^{3,1}(\mathbb{R}^d)) \),

\[ \|\Gamma^\varepsilon_{t,-\varepsilon^2 \sigma,s} \Gamma^\varepsilon_{t,\sigma} N^\varepsilon_t \varphi\|_{L^\infty(\mathbb{R}^+ \times [0,\tau]\times [0,2\tau]\times L^1(\mathbb{R}^d))} \leq \frac{\tau^2}{\varepsilon^3} \|E\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^+; W^{2,1}(\mathbb{R}^d))} \|\varphi\|_{L^\infty(\mathbb{R}^+; W^{3,1}(\mathbb{R}^d))}. \]
Combining (125) and (129), we obtain from hypothesis (H4) the final estimate (123), which ends the proof of Lemma 6. □

We are now able to conclude the proof of Theorem 2 by showing that we can pass to the limit $\varepsilon \to 0$ in (97). From (97) and (113), we have for all $\varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$,

$$
\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \mathbb{E} \left[ \int dx f^{\varepsilon} \right] \partial_t \varphi - \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \nabla_v \varphi \cdot \mathbb{E} \left[ \int dx \frac{f^{\varepsilon} E^{\varepsilon}}{\varepsilon} \right] = 0,
$$

(130)

with

$$
\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \nabla_v \varphi \cdot \mathbb{E} \left[ \int dx \frac{f^{\varepsilon} E^{\varepsilon}}{\varepsilon} \right] = \nu^\varepsilon_t(\mathbb{E}[f^{\varepsilon}], \varphi) + \nu^\varepsilon_t(\mathbb{E}[S^{\varepsilon}_{2\varepsilon^2\tau}f^{\varepsilon}(t-2\varepsilon^2\tau) - f^{\varepsilon}(t)], \varphi) + \mu^\varepsilon_t(\varphi). \tag{131}
$$

From Proposition 3, we have $\mathbb{E}[f^{\varepsilon}] \to f$ in $L^\infty(\mathbb{R}^+ \times Q)$ weak–*, as $\varepsilon \to 0$. Then, we obtain $\mathbb{E}[f dx f^{\varepsilon}] = \int dx \mathbb{E}[f^{\varepsilon}] \to \int dx f = f$ in $L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ weak–*, as $\varepsilon \to 0$. Using this weak limit, we obtain for the first term of (130),

$$
\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \mathbb{E} \left[ \int dx f^{\varepsilon} \right] \partial_t \varphi \to \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \varphi, \quad \text{as } \varepsilon \to 0.
$$

(132)

Using definitions (112) and (113) of respectively the bilinear form $\nu^\varepsilon_t$ and the operator $\Theta^\varepsilon_t$, we obtain

$$
\nu^\varepsilon_t(\mathbb{E}[f^{\varepsilon}], \varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int dx \mathbb{E}[f^{\varepsilon}] \Theta^\varepsilon_t \varphi
$$

$$
= \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \mathbb{E} \left[ \int dx f^{\varepsilon} \right] \Theta^\varepsilon_t \varphi.
$$

(133)

Moreover, from Lemma 4, we obtain $\Theta^\varepsilon_t \varphi \to \Theta^0_t \varphi$ in $L^1(\mathbb{R}^+ \times \mathbb{R}^d)$ strong, as $\varepsilon \to 0$. Using the weak limit $\mathbb{E}[f dx f^{\varepsilon}] \to f$ in $L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ weak–*, as $\varepsilon \to 0$, we can then pass to the limit in (133) or in the first term of the right-hand side of (131) and we obtain

$$
\nu^\varepsilon_t(\mathbb{E}[f^{\varepsilon}], \varphi) \to \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv f \Theta^0_t \varphi, \quad \text{as } \varepsilon \to 0.
$$

(134)

Using Lemma 5, the second term of the right-hand side of (131) vanishes as $\varepsilon \to 0$. Finally, for the third term of the right-hand side of (131), we obtain from Lemma 6 that $\mu^\varepsilon_t \to 0$ in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$, as $\varepsilon \to 0$. Therefore, we obtain from (131) and (134),

$$
\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \nabla_v \varphi \cdot \mathbb{E} \left[ \int dx \frac{f^{\varepsilon} E^{\varepsilon}}{\varepsilon} \right] \to \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv f \Theta^0_t \varphi, \quad \text{as } \varepsilon \to 0,
$$

(135)

where

$$
\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv f \Theta^0_t \varphi = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \varphi \nabla_v \cdot \left( \int_0^\tau d\sigma \mathcal{R}_\tau(t, t, \sigma, \sigma v) \nabla_v f \right).
$$

(136)

Passing to the limit $\varepsilon \to 0$ in (130), and using (132)-(135), we obtain

$$
\partial_t f - \nabla_v \cdot \left( \int_0^\tau d\sigma \mathcal{R}_\tau(t, t, \sigma, \sigma v) \nabla_v f \right) = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d),
$$

which shows (84) with (83).
It remains to show time continuity of the limit point \( f \). From (37) and the convergence result (135), we have for all \( \varphi \in D(\mathbb{R}^d) \),
\[
\frac{d}{dt} \int_{\mathbb{R}^d} dv \varphi \mathbb{E} \left[ \int dx f^\varepsilon \right] = \int_{\mathbb{R}^d} dv \nabla_v \varphi \cdot \mathbb{E} \left[ \int dx f^\varepsilon E^\varepsilon \frac{f^\varepsilon}{\varepsilon} \right] \leq C(\varphi), \quad \forall t \in \mathbb{R}^+,
\]
where the constant \( C(\varphi) > 0 \) depends on \( \varphi \), but not on \( t \) and \( \varepsilon \). Therefore we obtain
\[
\left\| \int_{\mathbb{R}^d} dv \left( \mathbb{E} \left[ \int dx f^\varepsilon(t) \right] - \mathbb{E} \left[ \int dx f^\varepsilon(t - 2\varepsilon^2 \tau) \right] \right) \varphi \right\| \leq \left| \int_{t-2\varepsilon^2 \tau}^t ds \int_{\mathbb{R}^d} dv \int dx \varepsilon^{-1} \mathbb{E} [E^\varepsilon(s, x) \cdot \nabla_v f^\varepsilon(s, x, v)] \varphi(v) \right|
\leq \varepsilon^{-1} \left\| f_0^\varepsilon \right\|_{L^\infty(Q)} \int_{t-2\varepsilon^2 \tau}^t ds \int_{\mathbb{R}^d} dv \int dx |\nabla_v \varphi(v)| \mathbb{E} [|E^\varepsilon(s, x)|]
\leq 2\varepsilon \tau C_0 \mathbb{E} \| \varphi \|_{W^{1,1}(\mathbb{R}^d)} \to 0, \quad \text{as} \ \varepsilon \to 0.
\]
Therefore the set \( \{ \int_{\mathbb{R}^d} dv \mathbb{E} [\int dx f^\varepsilon(t)] \mid \varphi \}_{\varepsilon > 0} \) is uniformly equi-continuous in time, and by Ascoli-Arzelà theorem this set is relatively compact in \( \mathscr{C}([0, T]) \). Then, there exists a subsequence, still labeled by \( \varepsilon \) such that \( \int_{\mathbb{R}^d} dv \mathbb{E} [\int dx f^\varepsilon(t)] \varphi \) converges uniformly in time to \( \int_{\mathbb{R}^d} f(t) \varphi dv \), with in particular \( f|_{t=0} = \int dx f_0 \). Finally the bound
\[
\left\| \mathbb{E} \left[ \int dx f^\varepsilon \right] \right\|_{L^\infty(0, T; L^p(\mathbb{R}^d))} \leq (2\pi)^{-d/p} \mathbb{E} \left[ \| f^\varepsilon \|_{L^\infty(0, T; L^p(Q))} \right]
= (2\pi)^{-d/p} \mathbb{E} \left[ \| f_0^\varepsilon \|_{L^p(Q)} \right] \leq (2\pi)^{-d/p} C_0 < \infty,
\]
( obtained by using Hölder’s inequality) and the density of \( D(\mathbb{R}^d) \) in \( L^q(\mathbb{R}^d) \) with \( 1/p + 1/q = 1 \) and \( 1 < q < \infty \), imply by standard arguments that \( \mathbb{E} [\int dx f^\varepsilon(t)] \to f \) in \( \mathscr{C}([0, T]; L^p(\mathbb{R}^d) - \text{weak}) \), for all \( T > 0 \) and \( 1 < p < \infty \). This completes the proof of Theorem 2.

4.4 Links with some kinetic turbulence theories of plasma physics

In this section we relate the results of Section 4.2 to two kinetic turbulence theories of plasma physics. First, when the autocorrelation time of particles \( \tau \) is finite, our result leads to a diffusion matrix, which is reminiscent of the diffusion matrix of the resonance broadening theory \[27, 1, 58, 51, 21\], a refinement of the quasilinear theory. This diffusion matrix falls into the framework of Theorem 2. Second, even if Theorem 2 and in particular hypothesis (H2) hold only for \( \tau \) finite, it is worthwhile to pass, at least formally, to the limit \( \tau \to +\infty \). When the autocorrelation time of particles \( \tau \) tends to infinity, we recover formally the structure of the diffusion matrix of the quasilinear theory \[26, 57\] from the diffusion matrix \[83\].

4.4.1 Resonance broadening like theory: Finite \( \tau \)

Here, we show how to relate the diffusion matrix \[83\] to the diffusion matrix obtained in the resonance broadening theory \[27, 1, 58, 51, 21\]. The diffusion matrix \[83\] obtained in Theorem 2 can be recast as
\[
\mathscr{D}_\tau(t, v) = \int_0^\tau d\sigma \mathbb{E} [E(t, 0, x) \otimes E(t, -\sigma, x - \sigma v)].
\]
Introducing the Fourier series decomposition of \( E \),
\[
E(t, \tau, x) = \sum_{k \in \mathbb{Z}^d} e^{ik \cdot x} \widehat{E}(t, \tau, k),
\]

35
we can suppose without loss of generality that \( \hat{E}(t, \tau, k) = e^{-i\omega(k)\tau} \tilde{E}(t, \tau, k) \) where the real-valued function \( \mathbb{Z}^d \ni k \mapsto \omega(k) \in \mathbb{R} \) is odd, i.e. \( \omega(-k) = -\omega(k) \) for all \( k \in \mathbb{Z}^d \). This transformation, which can be seen as a WKB ansatz, is just a change of unknown functions. In dimensional variables, \( \omega(k) \) should scale as \( \omega_p \), which means that \( \tau \) (respectively, \( t \)) represents the fast (respectively, slow) time variable. In the framework of the resonance broadening theory of plasma physics, which is self-consistent, frequencies \( \omega(k) \) are given by the resolution of the dispersion relation \( (\mathbf{1}) \) or by its approximation \( (\mathbf{3}) \). Since here we work in a non-self-consistent frame, we suppose that frequencies \( \omega(k) \) are simply given by a suitable function of \( k \), which is regular and bounded with respect to \( k \).

In the same spirit as assumption \( (\mathbf{H3}) \), we now make the following assumption:

\( (\mathbf{H3'}) \): There exist a non-negative real-valued function \( \mathcal{E}(t, k) : \mathbb{R}^+ \times \mathbb{Z}^d \to \mathbb{R}^+ \), with \( \mathcal{E}(t, k) = \mathcal{E}(t, -k) \) and \( |k|^2 |\mathcal{E}(t, k)|^{1/2} \in L^\infty(\mathbb{R}^+; \ell^1(\mathbb{Z}^d)) \), and a bounded function \( A_t(\tau, k) : [-\tau, \tau] \times \mathbb{Z}^d \to \mathbb{R}^+ \), even and compactly supported in \( \tau \), such that

\[
\mathbb{E}[\tilde{E}(t, \tau, k) \otimes \tilde{E}(t, \sigma, k')] = A_t(\tau - \sigma, k) \mathcal{E}(t, k) \frac{k \otimes k}{|k|^2} \delta(k + k').
\]

The term \( \delta(k + k') \), which provides spatial homogeneity, is reminiscent of what plasma physicists call the random phase approximation. Indeed the random phase approximation assumes that \( \tilde{E}(t, \tau, k) \propto \exp(i\phi_k) \), where \( (\phi_k)_{k \in \mathbb{Z}^d} \) are independent random variables equidistributed on \([0, 2\pi]\) such that \( \phi_{-k} = -\phi_k \). The matrix \( k \otimes k/|k|^2 \) means that we choose an electric field which is the gradient of an electric potential. The function \( A_t \) corresponds to a time autocorrelation function. We suppose that the function \( \tau \mapsto A_t(\tau, k) \) is bounded, even, and with compact support included in \([-\tau, \tau]\), for all \( k \in \mathbb{Z}^d \). The function \( \mathcal{E}(t, k) \) corresponds to the energy of the \( k \)-th mode of the electric field, whose time scale of evolution in dimensional variables is \( \tau_L \), i.e. a slow time scale in comparison to \( 1/\omega_p \) (remember that \( 1/(\omega_p \tau_L) = \varepsilon^2 \)). In the self-consistent framework of resonance broadening theory, the function \( \mathcal{E}(t, k) \) are given by equation \( (\mathbf{2}) \) in which the energy of the \( k \)-th mode of the electric field, \(|E_t(k)|^2\), is replaced by \( \mathcal{E}(t, k) \) and the grow rate \( \gamma(t, k) \) is given by a \( d \)-dimensional version of \( (\mathbf{3}) \) (or its approximation \( (\mathbf{3}) \)). Since here we work in a non-self-consistent frame, we also suppose that energy amplitudes \( \mathcal{E}(t, k) \) are simply given by a suitable bounded function, which decreases fast enough in the variable \( k \) to satisfy hypothesis \( (\mathbf{H4}) \). Hypothesis \( (\mathbf{H3'}) \), which is a particular case of hypothesis \( (\mathbf{H3}) \), is consistent with the spatio-temporal homogeneity property of the turbulence. Actually, the property \( (\mathbf{H3'}) \) implies the property \( (\mathbf{H3}) \), in other words the property \( (\mathbf{H3'}) \) is less general than the property \( (\mathbf{H3}) \). Indeed, we obtain from \( (\mathbf{H3'}) \),

\[
\mathbb{E}[E(t, \tau, x) \otimes E(t, \sigma, y)] = \sum_{k \in \mathbb{Z}^d} A_t(\tau - \sigma, k) \mathcal{E}(t, k) \frac{k \otimes k}{|k|^2} e^{ik(x-y)} e^{-i\omega(k)(\tau-\sigma)},
\]

from which we easily observe that the spatio-temporal autocorrelation function \( \mathbb{E}[E(t, \tau, x) \otimes E(t, \sigma, y)] \) is invariant under time and space translations. In terms of the autocorrelation matrix \( \mathcal{R}_t \), hypothesis \( (\mathbf{H3'}) \) is equivalent to

\[
\mathcal{R}_t(t, \tau, x, y) = \sum_{k \in \mathbb{Z}^d} e^{-i\omega(k)\sigma} A_t(\sigma, k) \mathcal{E}(t, k) \frac{k \otimes k}{|k|^2} e^{ik \cdot x}.
\]

Using assumption \( (\mathbf{H3'}) \) in \( (\mathbf{137}) \), where the electric field \( E \) is written in terms of its Fourier series decomposition, we obtain

\[
\mathcal{D}_t(t, v) = \sum_{k \in \mathbb{Z}^d} \mathcal{E}(t, k) \frac{k \otimes k}{|k|^2} \int_0^\tau d\sigma e^{-i\omega(k)(\sigma-v)} A_t(\sigma, k).
\]
The diffusion matrix (139) can be rewritten as

\[ \mathcal{D}_\tau(t, v) = \sum_{k \in \mathbb{Z}^d} E(t, k) \frac{k \otimes k}{|k|^2} R_\tau(\omega(k) - k \cdot v, k), \] (140)

where the resonance function \( R_\tau \) is given by

\[ R_\tau(\omega(k) - k \cdot v, k) = \Re e^{\int_0^\tau d\sigma e^{-i(\omega(k) - k \cdot v)\sigma} A_\tau(\sigma, k)} = \frac{1}{2} \int_{-\infty}^{+\infty} d\sigma e^{-i(\omega(k) - k \cdot v)\sigma} A_\tau(\sigma, k). \] (141)

In (141) we have used the even parity and the compact support (included in \([-\tau, \tau]\)) of the real function \( \tau \mapsto A_\tau(\tau, k) \) to obtain the last equality.

Substituting hypothesis (H3') for hypothesis (H3) in Theorem 2, we obtain

**Corollary 2** Let \( E \) be an integrable random vector field satisfying the assumptions (H1)-(H2) and (H3')-(H4), and let \( E^\varepsilon \) be given by (81). Let \( \{f^\varepsilon_0\}_{\varepsilon > 0} \) be a sequence of independent random non-negative initial data and \( C_0 \) be a positive constant such that for a.e. \( \omega \in \Omega, \|f^\varepsilon_0\|_L^1(Q) + \|f^\varepsilon_0\|_{L^\infty(Q)} \leq C_0 \leq \infty. \) Let \( \mathcal{D}_\tau \) be the matrix-valued function defined by (140)-(141). Let \( f^\varepsilon \) be the unique weak solution of the Vlasov equation (17), with initial data \( f^\varepsilon_{|t=0} = f^\varepsilon_0. \) Then

1. Up to extraction of a subsequence, \( \mathbb{E}[f^\varepsilon_0] \) converges in \( L^\infty(Q) \) weak-* to a function \( f_0 \in L^1 \cap L^\infty(Q), \) \( \mathbb{E}[f^\varepsilon] \) converges in \( L^\infty(\mathbb{R}^+; L^1(Q)) \) weak-* to a function \( f \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^d)), \) and \( \mathbb{E}[f^\varepsilon] \) converges in \( L^{\infty}(\mathbb{R}^+; L^\infty(\mathbb{R}^d)) \) weak-* to \( f. \) Moreover \( \mathbb{E}[f^\varepsilon dx f^\varepsilon] \) converges in \( \mathcal{C}(0, T; L^p(\mathbb{R}^d) - \text{weak}) \) to \( f, \) for \( 1 < p < \infty \) and for all \( T > 0. \) The limit point \( f = f(t, v) \) is solution of the following diffusion equation in the sense of distributions:

\[ \partial_t f - \nabla_v \cdot (\mathcal{D}_\tau \nabla_v f) = 0, \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d), \]

\[ f_{|t=0} = \int dx f_0. \]

2. The diffusion matrix \( \mathcal{D}_\tau \) is symmetric, non-negative and analytic in the velocity variables.

**Proof.** The proof of point 1 of Corollary 2 is the same as the proof of Theorem 2. It remains to deal with the proof of point 2. Symmetry of the diffusion matrix \( \mathcal{D}_\tau \) is obvious, while reality of the function \( R_\tau \) defined by (141) follows from the parity (even) of the function \( \sigma \mapsto A_\tau(\sigma, k). \) Non-negativity of the diffusion matrix \( \mathcal{D}_\tau \) comes from the non-negativity of the function \( R_\tau, \) which results from both the Bochner theorem (see, e.g., Theorem 2 p. 346 in [60]) and the fact that the function \( \sigma \mapsto A_\tau(\sigma, k) \) is positive definite in the following sense:

\[ \int_{\mathbb{R}} d\tau \int_{\mathbb{R}} d\sigma A_\tau(\tau - \sigma, k) \varphi(\tau) \varphi(\sigma) \geq 0, \] (142)

for every continuous function \( \varphi \) with compact support. Indeed, from assumption (H3') and using \( \overline{E}(t, \tau, -k) = \overline{E}^*(t, \tau, k), \) for all vector \( X \in \mathbb{R}^d \setminus \{0, k^+\} \) and \( \varphi \in \mathcal{C}_c^0(\mathbb{R}; \mathbb{R}) \) (the set of continuous and compactly supported functions from \( \mathbb{R} \) to \( \mathbb{R} \)), we obtain

\[ \frac{|k \cdot X|^2}{|k|^2} E(t, k) \int_{\mathbb{R}} d\tau \int_{\mathbb{R}} d\sigma A_\tau(\tau - \sigma, k) \varphi(\tau) \varphi(\sigma) \]

\[ = \mathbb{E} \left[ \left. \left( \int_{\mathbb{R}} d\tau \overline{E}(t, \tau, k) \cdot X \varphi(\tau) \right) \left( \int_{\mathbb{R}} d\sigma \overline{E}(t, \sigma, k) \cdot X \varphi(\sigma) \right) \right|^* \right] \geq 0, \]
which implies (142). Finally, \( \mathcal{D}_\tau(t, v) \), defined by (140)-(141), is an analytic function in \( v \) because the function \( R_\tau \) is the Fourier transform of the compactly supported function \( A_\tau \).

The diffusion matrix (140) is reminiscent of the diffusion matrix of resonance broadening theory [27, 58, 21]. In Remark 13, we present very briefly the result of the resonance broadening theory to point out the similarities and the differences between the resonance broadening effect of Corollary 2 and that of the resonance broadening theory. As already mentioned in Section 3.2.2 from a mathematical point of view, the consequence of the finite-\( \tau \) or resonance broadening effect of Corollary 2 is a regularity improvement of the diffusion matrix in the velocity variables (by comparison with the diffusion matrix of the quasilinear theory, see Section 4.4.2). From a physical point of view, our broadening resonance effect corresponds to a broadening in time frequency of a width \( \Delta \omega = 1/\tau \) for the resonance \( \omega(k) - k \cdot v = 0 \) (\( \Delta \omega = 0 \) for the quasilinear theory, see Section 4.4.2). Here, up to hypothesis (H3’), we can choose freely the time autocorrelation function \( A_\tau \). From (141), we observe that the resonance function \( R_\tau \) is obtained as the Fourier transform of \( A_\tau/2 \). In particular, we can recover diffusion matrices (80) and (87) obtained respectively in Section 3.2.2 and Corollary 2.

**Remark 13 (resonance broadening theory)** The finite-\( \tau \) or resonance broadening effect obtained in Corollary 2 is reminiscent of the resonance broadening theory [27, 1, 58, 54, 21], but it is not exactly the same since the derivation of resonance broadening theory is quite different and more involved. The resonance broadening theory is a correction to the quasilinear theory, which takes into account the particle diffusion coming from \( \mathcal{D}_\tau \), for calculating \( \mathcal{D}_\tau \) itself. Indeed, in the original derivation of the quasilinear theory, the diffusion matrix of the quasilinear is obtained by assuming that the perturbed dynamics of particles (called fluctuations) can be approximated by the ballistic motion or the free flow because fluctuations are small enough. This approximation is consistent with the limit \( \tau \to +\infty \) (or \( \tau_D \to +\infty \) in dimensional variables) since for such limit, particles follow almost straight lines. If the fluctuation amplitude is sufficiently large and/or the wave spectrum is sufficiently narrow in \( k \)-space, the diffusion of particle trajectories can produce an appreciable broadening of wave-particle resonances \( \omega(k) - k \cdot v \approx 0 \), even when \( \mathcal{E}_{el}/\mathcal{E}_{kin} \ll 1 \). To take these effects into account, the resonance broadening theory aims at modifying appropriately the diffusion matrix from the particle dynamics in a self-consistent way, i.e. by incorporating the diffusion of particle trajectories. Using a statistical approach [27, 58, 21], the broadening resonance theory states that the autocorrelation function should be given by

\[
A_\tau(\sigma, k) = \exp\left(-\frac{\sigma}{\tau^3} \right)^3, \quad \text{with} \quad \tau = (k \otimes k : \mathcal{D}_{\tau_3}(\sigma, v))^{-1/3}, \quad \text{i.e.} \quad A_\tau(\sigma, k) = A_{\tau_3}(\sigma, k, \mathcal{D}_{\tau_3}(\sigma, v)) := \exp\left(-\frac{3}{2} k \otimes k : \mathcal{D}_{\tau_3}(\sigma, v) \right)^3.
\]

We refer the reader to Remark 14 below for a rough but short explanation of this kind of result in one dimension (see [27, 58] for an original and detailed derivation). We now observe that the autocorrelation time \( \tau \) is no more a free parameter but is determined by the diffusion matrix \( \mathcal{D}_{\tau_3} \). Likewise, the time autocorrelation function \( A_{\tau_3} \) can no more be chosen freely but is an explicit function of the diffusion matrix \( \mathcal{D}_{\tau_3} \). According to the resonance broadening theory, the diffusion matrix \( \mathcal{D}_{\tau_3} \) is now a solution to the nonlinear functional integral equation given by

\[
\mathcal{D}_{\tau_3}(t, v) = \sum_{k \in \mathbb{Z}^d} \mathcal{E}(t, k) \frac{k \otimes k}{|k|^2} \Re \int_0^\infty d\sigma e^{-i(\omega(k) - k \cdot v)\sigma} A_{\tau_3}(\sigma, k, \mathcal{D}_{\tau_3}(\sigma, v)) = \sum_{k \in \mathbb{Z}^d} \mathcal{E}(t, k) \frac{k \otimes k}{|k|^2} R_{\tau_3}(\omega(k) - k \cdot v, k, \mathcal{D}_{\tau_3}).
\]
The resonance function $R_{rb}$ can no more be seen as the Fourier transform (problem of convergence for negative time) of the time autocorrelation function $A_{rb}$, but as the Laplace transform of it. The broadening of wave-particle resonance is produced by the term \( k \otimes k : \mathcal{D}_{rb}(\sigma, v)\sigma^3/3 \), which can be seen an approximation of the ensemble-average mean square deviation from the mean of particle trajectories in the turbulent electric field (see Remark 14 for a rough but short explanation, and [27, 58] for a more rigorous and exhaustive one). Resonance broadening theory would certainly deserve a rigorous mathematical treatment, but it is out of the scope of this paper. This remark just aims at enlightening similarities and differences with the resonance broadening effect obtained in Corollary 2.

**Remark 14** In this remark we roughly explain where typical expression (143) comes from. Here, we give a short and simplified derivation of (143) just to give a flavor. A more involved and detailed derivation can be found in [27, 58]. In addition, without loss of generality, we consider the dimension \( d = 1 \) to simplify the calculation. Indeed, we observe that

\[
\exp(ik\sigma) = \mathbb{E}[\exp(ik\Delta X(\sigma))],
\]

with \( (\Delta X(\sigma), \Delta V(\sigma)) = (X(\sigma) - X(0), V(\sigma) - V(0)) = (\sigma, v) \). Considering a higher approximation of the characteristic curves (in particular an approximation that takes into account the diffusion of particles), we should add to the free-flow approximation a correction \( (\delta X(\sigma), \delta V(\sigma)) \), such that we have, \( X(\sigma; 0, x, v) \simeq x + v\sigma + \delta X(\sigma) \) and \( V(\sigma; 0, x, v) = v + \delta V(\sigma) \). The autocorrelation function \( \sigma \mapsto A_{rb}(\sigma, k) \) is then defined by

\[
A_{rb}(\sigma, k) := \mathbb{E}[\exp(ik\delta X(\sigma))] = \mathbb{E}\left[ \exp \left( ik \int_0^\sigma ds \delta V(s) \right) \right]. \tag{145}
\]

We now suppose that \( \delta V(\sigma) \) is a Gaussian random function, with a probability distribution \( \mathbb{P} \) given by

\[
\mathbb{P}(\delta V(\sigma)) = \frac{1}{\sqrt{2\pi\mathcal{D}(v)s}} \exp \left( -\frac{(\delta V(\sigma))^2}{2\mathcal{D}(v)s} \right),
\]

where the standard deviation or the diffusion coefficient \( \mathcal{D}(v) \) is constant in time. Therefore we have,

\[
\mathbb{E}[\delta V(\sigma)] = 0, \quad \mathbb{E}[(\delta V(\sigma))^2] = \mathcal{D}(v)s, \quad \text{and} \quad \mathbb{E}[\delta V(\sigma)\delta V(\sigma')] = \mathcal{D}(v)\min(s, s'), \tag{146}
\]

which means that particle velocity follows a diffusion process characterized by the diffusion coefficient \( \mathcal{D} \). Since \( \delta V(\sigma) \) is a Gaussian random function, using (146), we obtain from (145),

\[
A_{rb}(\sigma, k) = \mathbb{E}\left[ \exp \left( ik \int_0^\sigma \delta V(s)ds \right) \right]
= \exp \left( -\frac{1}{2}k^2 \int_0^\sigma ds \int_0^\sigma ds' \mathbb{E}[\delta V(s)\delta V(s')] \right)
= \exp \left( -\frac{1}{2}k^2 \int_0^\sigma ds \int_0^\sigma ds' \mathcal{D}(v)\min(s, s') \right)
= \exp \left( -\frac{1}{6}k^2 \mathcal{D}(v)s^3 \right),
\]

which gives, up to a multiplicative constant in the exponential, the same result as (143).
4.4.2 Quasilinear theory: Infinite $\tau$

Here, we show that we can retrieve formally the diffusion matrix of the quasilinear theory \cite{26, 57, 21} from the diffusion matrix (83) or (140) by taking the formal limit $\tau \to +\infty$. For this, we choose an autocorrelation function $A_\tau$, such that $A_\tau \to 1$ a.e. as $\tau \to +\infty$ (e.g. $A_\tau(\sigma, k) = 1_{[-\tau, \tau]}(\sigma)$). Then, letting $\tau$ go to infinity we obtain in the sense of distributions,

$$\lim_{\tau \to +\infty} \int_0^\tau d\sigma e^{-i(\omega(k) - k \cdot v)\sigma} A_\tau(\sigma, k) = \pi \delta(\omega(k) - k \cdot v) - i \text{p.v.} \left( \frac{1}{\omega(k) - k \cdot v} \right),$$

and

$$\lim_{\tau \to +\infty} R_\tau(\omega(k) - k \cdot v, k) = \pi \delta(\omega(k) - k \cdot v). \quad (147)$$

Therefore, the diffusion matrix (140) becomes the quasilinear diffusion matrix of plasma physics literature (see, e.g., \cite{39, 21}),

$$\mathcal{D}_\infty(t, v) = \pi \sum_{k \in \mathbb{Z}^d} \mathcal{E}(t, k) \left( k \otimes \frac{k}{|k|^2} \right) \delta(\omega(k) - k \cdot v). \quad (148)$$

The diffusion matrix (148) seems not very regular, since it involves a sum of Dirac masses, namely resonances $\delta(\omega(k) - k \cdot v)$. Then, well-posedness of the diffusion equation (84) with such a diffusion matrix remains an open issue at least for non-smooth distribution functions. We note that we can also recover the quasilinear diffusion matrix (148) from the diffusion matrix $\mathcal{D}_{rb}$ of the resonance broadening theory (see Remark 13) by observing that $R_{rb}(\omega(k) - k \cdot v, k, 0) = \delta(\omega(k) - k \cdot v)$.

Finally, we note that the resonance broadening theory \cite{27, 58} is actually a statistical (probabilistic) theory of the Vlasov equation and does not have a deterministic counterpart in the plasma physics literature. Nevertheless, in Section 5.2.2 for the deterministic case, we have been able to introduce a finite autocorrelation time of particles $\tau$, and to derive formally a diffusion matrix that is consistent with the quasilinear one in the limit $\tau \to \infty$.  

When $\tau \to \infty$, the autocorrelation time of particles tends to infinity and the time decorrelation of the electric field defined in Section 4.1 occurs at infinite time. This can be interpreted as follows. The electric field becomes deterministic and particles trajectories are almost straight lines. This seems consistent with the fact that the original derivation of the QL theory performed by physicists \cite{26, 57} is deterministic. Indeed this deterministic derivation is based on two main ingredients. First the wave-particle interaction is assumed perturbative, and the perturbed dynamics of particles is approximated by the free flow or the ballistic motion. Second, all nonlinear wave-wave interactions, except for their effect on the space-averaged distribution function, are neglected. After the original 1962 derivation which is deterministic, other derivations of the QL theory (see, e.g., \cite{21, 6, 5}) appeal to some statistical arguments and decorrelation hypotheses, to establish the QL diffusion. Therefore, considering quasilinear theory as a probabilistic or deterministic theory remains an open question.
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Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

A An example of a random field $E$

In this appendix we construct an example of random electric field $E = E(t, \tau, x)$ that satisfies hypotheses (H1)-(H4) of Section 4.1. For this, we are inspired by the Example 2 in [50]. Let $r$ and $\rho$ be positive real numbers such that $\rho \geq 1/r$ and $r \geq 1$. We set $\tau = 1/r + \rho$. With this decomposition of the autocorrelation time $\tau$, we can choose $\tau$ as small as we want by taking finite but large $r$, and choose also $\tau$ as large as we want by taking any fixed $r$ and $\rho$ large enough. Let $(T^n_k, X^n_k) \in \mathbb{R}^{1+d}$ with $(n, k) \in \mathbb{Z} \times \mathbb{Z}^d$ be independent random variables equidistributed in

$$\left\{ \frac{n}{r} + \left[ -\frac{1}{2r}, \frac{1}{2r} \right] \right\} \times \left\{ k + [-1/2, 1/2]^d \right\}.$$

We consider also other independent random variables $\alpha^n_k$, with $(n, k) \in \mathbb{Z} \times \mathbb{Z}^d$, such that $\mathbb{E}[\alpha^n_k] = 0$ and $\mathbb{E}[\alpha^n_k]^2 = 1$. Let $\eta \in \mathcal{E}_c^\infty(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d; \mathbb{R})$ be a real scalar function that is compactly supported in $\mathbb{R}^+ \times [-\rho/2, \rho/2] \times \mathbb{R}^d$. We define the random function $\mathcal{E} = \mathcal{E}(t, \tau, x)$ by

$$\mathcal{E}(t, \tau, x) = \sum_{(n,k) \in \mathbb{Z}^{1+d}} \alpha^n_k \eta(t, \tau - T^n_k, x - X^n_k).$$

Since $\eta$ has a compact support, the sum defining $\mathcal{E}$ in this equation is finite. Obviously, $\mathcal{E} \in \mathcal{E}_c^\infty(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d; \mathbb{R})$ and $\mathbb{E}[\mathcal{E}] = 0$, which means that hypotheses (H1) and (H4) are satisfied. For a fixed $\tau \in \mathbb{R}$, the function $\mathcal{E}(t, \tau, x)$ depends only on $(T^n_l, X^n_l, \alpha^n_l)$ with

$$n \in \left[ \tau - \frac{1}{2r} - \frac{\rho}{2}, \tau + \frac{1}{2r} + \frac{\rho}{2} \right].$$

Similarly, we define

$$\mathcal{E}'(s, \sigma, y) = \sum_{(n,k) \in \mathbb{Z}^{1+d}} \alpha^n_k \eta'(s, \sigma - T^n_k, y - X^n_k),$$

with $\eta' \in \mathcal{E}_c^\infty(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d; \mathbb{R})$ being a real scalar function which is also compactly supported in $\mathbb{R}^+ \times [-\rho/2, \rho/2] \times \mathbb{R}^d$. For a fixed $\sigma \in \mathbb{R}$, the function $\mathcal{E}'(s, \sigma, x)$ depends only on $(T^n_k, X^n_k, \alpha^n_k)$ with

$$m \in \left[ \sigma - \frac{1}{2r} - \frac{\rho}{2}, \sigma + \frac{1}{2r} + \frac{\rho}{2} \right].$$

Then, as soon as $|\tau - \sigma| \geq 1/r + \rho =: \tau$, the random functions $\mathcal{E}(t, \tau, x)$ and $\mathcal{E}'(s, \sigma, y)$ are independent, which means that hypothesis (H2) is satisfied. It remains to show hypothesis (H3). For this,
we estimate \( \mathbb{E}[\mathcal{E}(t, \tau, x) \mathcal{E}'(s, \sigma, y)] \) as follows. Using the independence of random variables \((T^n_k, X^n_k)\) and \(\alpha^n_k\), and using \(\mathbb{E}[\alpha^n_k \alpha'^n_l] = \delta_{mn} \delta_{kl}\), we obtain

\[
\mathbb{E}[\mathcal{E}(t, \tau, x) \mathcal{E}'(s, \sigma, y)] = \sum_{(n,k) \in \mathbb{Z}^{1+d}} \sum_{(m,l) \in \mathbb{Z}^{1+d}} \mathbb{E}[\alpha^n_k \alpha'^n_l] \mathbb{E}[\eta(t, \tau - T^n_k, x - X^n_k) \eta'(s, \sigma - T'^m_l, y - X'^m_l)]
\]

\[
= \sum_{(n,k) \in \mathbb{Z}^{1+d}} \mathbb{E}[(\alpha^n_k)^2] \mathbb{E}[\eta(t, \tau - T^n_k, x - X^n_k) \eta'(s, \sigma - T'^m_l, y - X'^m_l)]
\]

\[
= \sum_{(n,k) \in \mathbb{Z}^{1+d}} \int_{[-1/2,1/2]^d} dz \int_{-\pi}^{\pi} d\phi \int_{(n+1)/r}^{(n-1)/r} \eta(t, \tau - \theta, x - k - z) \eta'(s, \sigma - n/r - \theta, y - k - z)
\]

\[
= \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} \eta(t, \tau - \theta, x - z) \eta'(s, \sigma - \theta, y - z)
\]

\[
= \tilde{\mathcal{R}}_{\eta \eta'}(t, s, \tau - \sigma, x - y).
\]

We then obtain \(\tilde{\mathcal{R}}_{\eta \eta'} \in \mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d; \mathbb{R})\). To construct the autocorrelation matrix \((\mathcal{R}_{\tau ij})_{(i,j)}\), we first choose real scalar functions \(\eta_i \in \mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d; \mathbb{R})\), for \(i \in \{1, \ldots, d\}\), which are compactly supported in \(\mathbb{R}^+ \times [-\theta/2, \theta/2] \times \mathbb{R}^d\). Second, we define the i-th component of the random electric field \(E\) by

\[
E_i(t, \tau, x) = \sum_{(n,k) \in \mathbb{Z}^{1+d}} \alpha^n_k \eta_i(t, \tau - T^n_k, x - X^n_k).
\]

Finally, the autocorrelation matrix \(\mathcal{R}_\tau\) is defined by

\[
\mathcal{R}_{\tau ij}(t, s, \tau - \sigma, x - y) := \tilde{\mathcal{R}}_{\eta_i \eta_j}(t, s, \tau - \sigma, x - y).
\]

Another way to obtain the autocorrelation matrix \(\mathcal{R}_\tau\) is to take the gradient of \(\tilde{\mathcal{R}}_{\eta \eta'}(t, s, \tau - \sigma, x - y)\) with respect to the variables \(x\) and \(y\), i.e.

\[
\mathcal{R}_{\tau ij}(t, s, \tau - \sigma, x - y) := \partial_x \partial_y \tilde{\mathcal{R}}_{\eta \eta'}(t, s, \tau - \sigma, x - y) = \tilde{\mathcal{R}}_{\partial_x \eta \partial_y \eta'}(t, s, \tau - \sigma, x - y)
\]

\[
= \mathbb{E}[\partial_x \mathcal{E}(t, \tau, x) \partial_y \mathcal{E}'(s, \sigma, y)].
\]

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