Abstract. Let $M = \tilde{M}/\Gamma$ be a Kähler manifold, where $\Gamma \simeq \pi_1(M)$ and where $\tilde{M}$ is the universal Kähler cover. Let $(L, h) \to M$ be a positive Hermitian holomorphic line bundle. We first prove that the $L^2$ Szegő projector $\tilde{\Pi}_N$ for $L^2$-holomorphic sections on the lifted bundle $\tilde{L}^N \to \tilde{M}$ is related to the Szegő projector for $H^0(M, L^N)$ by $\tilde{\Pi}_N(\gamma \cdot x, y) = \sum_{\gamma \in \Gamma} \tilde{\Pi}_N(x, y)$. We apply this result to give a simple proof of Napier’s theorem on the holomorphic convexity of $\tilde{M}$ with respect to $\tilde{L}^N$ and to surjectivity of Poincaré series.

Let $(M, \omega)$ denote a compact Kähler manifold of dimension $m$, and let $(\tilde{M}, \tilde{\omega})$ denote its universal Kähler cover with deck transformation group $\Gamma = \pi_1(M)$. We assume that $\Gamma$ is an infinite group so that $\tilde{M}$ is complete noncompact. Let $(L, h) \to (M, \omega)$ denote a positive hermitian line bundle and let $(\tilde{L}, \tilde{h})$ be the induced hermitian line bundle over $\tilde{M}$. The first purpose of this note is to prove that for sufficiently large $N \geq N_0(M, L, h)$, the Szegő kernel of the holomorphic projection $\Pi_N : L^2(M, L^N) \to H^0(M, L^N)$ on the quotient is given by the Poincaré series of the Szegő projection for $L^2$ holomorphic sections on the universal cover (Theorem 1). This relation is standard in the theory of the Selberg trace formula on locally symmetric spaces, but seems not to have been proved before in the general setting of positive line bundles over Kähler manifolds. As will be seen, it is a consequence of standard Agmon estimates on off-diagonal decay of the Szegő kernel \cite{Del,L} and of the local structure of the kernel given by the Boutet de Monvel-Sjöstrand parametrix for both Szegő kernels \cite{BouSj,BBSj}. This relation is then used to simplify and unify a number of results on universal covers of compact Kähler manifold. One application is a short proof of the holomorphic convexity with respect to the positive line bundle $(\tilde{L}, \tilde{h})$ (Theorem 2) proved by T. Napier \cite{N}. A second application is a simple proof of surjectivity of Poincaré series (Theorem 3). The problem of determining the least $N_0(M, L, h)$ for which these results are true is not treated in this article.

To state the results, we need to introduce some notations. For any positive hermitian line bundle $(L, h) \to (M, \omega)$ over a Kähler manifold, we denote by $H^0(M, L^N)$ the space of holomorphic sections of the $N$-th power of $L$. We assume throughout that $\omega := -\frac{i}{\pi} \partial \bar{\partial} \log h$ is a Kähler metric. The Hermitian metric $h$ induces the inner
products

\[
\langle s_1, s_2 \rangle_{h^N} = \int_M \langle s_1(z), s_2(z) \rangle_{h^N} dV_M,
\]

where the volume form \( dV_M \) is given by \( dV_M = \omega^m/m! \). The corresponding inner product for \( \tilde{h} \) can be defined in the similar way. We also use \( | \cdot |_{h^N} \) and \( | \cdot |_{\tilde{h}^N} \) to denote the pointwise norms of the metrics \( h \) and \( \tilde{h} \), respectively. We use the notation \( H^0_{L^2}(M, L^N) \) for the space of \( L^2 \) holomorphic sections. More generally, we denote the space of \( L^p \) holomorphic sections by \( H^0_{L^p}(\tilde{M}, \tilde{L}^N) \) for \( p \geq 1 \). We further denote by

\[
\Pi_{h^N} : L^2(M, L^N) \to H^0(M, L^N)
\]

the orthogonal projection or the Szegő kernel with respect to \( \langle \cdot, \cdot \rangle_{h^N} \) and by

\[
\tilde{\Pi}_{h^N} : L^2(\tilde{M}, \tilde{L}^N) \to H^0_{L^2}(\tilde{M}, \tilde{L}^N)
\]

the corresponding orthogonal projection or the Szegő kernel on the universal cover \( \tilde{M} \). Since \( h \) is fixed in this discussion we often simplify the notation by writing \( \Pi_N \) and \( \tilde{\Pi}_N \).

It has been proved by Delin [Del], Lindholm [L] and Christ [Ch] in various settings that the Szegő kernels admit the following Agmon estimate: there exists \( \beta = \beta(\tilde{M}, \tilde{L}, \tilde{h}) > 0 \) such that

\[
|\tilde{\Pi}_N(x, y)|_{\tilde{h}^N} \leq e^{-\beta\sqrt{Nd}(x,y)}
\]

for \( d(x, y) \geq 1 \), where \( d(x, y) \) denotes the distance function of \( \tilde{M} \). We use this relation to study the Poincaré series map

\[
P : H^0_{L^1}(\tilde{M}, \tilde{L}^N) \to H^0(M, L^N), \quad Pf(z) := \sum_{\gamma \in \Gamma} f(\gamma \cdot z)
\]

for any \( N > 0 \).

Each deck transformation \( \gamma \in \Gamma \) determines a displacement function \( d_{\gamma}(x) = d(x, \gamma x) \) on \( \tilde{M} \). Its minimum value is denoted \( L(\gamma) \). The minimum set is the axis of \( \gamma \). When it is of dimension one, it folds up under \( \Gamma_{\gamma} \), the centralizer of \( \gamma \) in \( \Gamma \), to a closed geodesic and \( L(\gamma) \) is its length; in degenerate cases, it equals the common length of the closed geodesics (cf. [EC page 95]).

We denote by \( \ell_1 \) the minimum over \( \gamma \in \Gamma \) of \( L(\gamma) \), i.e. the length of the shortest closed geodesic of \( (M, \omega) \).

The main result of this paper is the following \( \sum_{\gamma} \) relation.

**Theorem 1.** There is an integer \( N_0 = N_0(M, L, h) \), such that if \( N \geq N_0 \), then the degree \( N \) Szegő kernel \( \Pi_N(x, y) \) of \( (L, h) \to M \) and \( \tilde{\Pi}_N(x, y) \) of \( (\tilde{L}, \tilde{h}) \to \tilde{M} \) are related by

\[
\Pi_N(x, y) = \sum_{\gamma \in \Gamma} \tilde{\Pi}_N(\gamma \cdot x, y).
\]

\(^2\)We sometimes also use \( dV_M(y) \) instead of \( dV_M \) if we want to specific a variable \( y \).
There is a classical proof of Theorem 1 (due to Selberg, Godement and Earle \cite{Ear}) for a bounded symmetric domain, which proves the result under the additional assumption on the variation of the Bergman kernel. We also note\footnote{More generally, the relation of solutions of elliptic equations on both $M$ and $\widetilde{M}$ (cf. \cite{A}).} that the analogue of the $\sum_\gamma$ relation of Theorem 1 for heat kernel, or for the wave kernel $\cos t\sqrt{\Delta}$, is simpler and standard. Consider the heat kernel $\tilde{K}(t,x,y)$ for the heat operator $\exp(-t\Box_b)$ generated by the Kohn Laplacian $\Box_b$ associated to $(L,h)$. For each $N$, let $\tilde{K}_N(t,x,y)$ be the component of $\tilde{K}(t,x,y)$ identified with the Kodaira Laplacian of the line bundle $L^N$. We easily see that

$$\tilde{K}_N(t,x,y) = \sum_{\gamma \in \Gamma} \tilde{K}_N(t,\gamma \cdot x, y).$$

To prove (6), we just note that both sides solve the heat equation and that they have the same initial condition, i.e. the delta function. In the case of Szegö kernels, it is also simple to see that both sides are holomorphic projectors. The same argument can be used for the wave kernel, whenever the $\sum_\gamma$ is finite. But it is not apriori clear that the right side is a surjective projection onto $H^0(M,L^N)$. Surjectivity is a kind of replacement for the initial condition in the case of the heat kernel, but it is a more complicated kind of boundary condition. As mentioned above, Theorem 1 is proved by studying the singularity on the diagonal of the Szegö kernel using the parametrix construction in \cite{BouSj, BBSj}. The main idea is that the $\sum_\gamma$ relation must hold because the $L^2$ Szegö kernel has precisely the same local singularity as the quotient Szegö kernel. In principle it is possible to estimate $\beta$ and $\ell_1$ and hence to estimate the minimal power for which the relation is valid.

An alternative approach to Theorem 1 which we do not carry out here, is to use (6) and take the limit $t \to \infty$. One needs to take the limit $t \to \infty$ under the summation sign $\sum_\gamma$ on the right side and show that one obtains the Szegö kernel part of each term. It seems that the limit $t \to \infty$ is monotone decreasing along the diagonal $x = y$ so that the limit may indeed be taken under the $\sum_\gamma$ side. One may then use the existence of a spectral gap for the Kohn Laplacian both upstairs and downstairs to show that the limit of the left side of (6) is the downstairs Szegö kernel and the limit of each term on the right side is the corresponding expression for the upstairs Szegö kernel. However, we opted to work entirely with the Szegö kernel.

Our first application is to give a simple proof of the holomorphic convexity of $\widetilde{M}$ with respect to sufficiently high powers of a positive line bundle. We recall that $\widetilde{M}$ is called holomorphically convex if for each sequence $\{x_n\}$ with no convergent subsequence, there exists a holomorphic function $f$ on $\widetilde{M}$ such that $|f(x_n)|$ is unbounded. It is holomorphically convex with respect to $\tilde{L}^N$ if there exists $s \in H^0(\widetilde{M}, \tilde{L}^N)$ such that $|s(x_n)|^{\tilde{h}_N}$ is unbounded.

In \cite{N}, T. Napier proved a special case of the Shafarevich conjecture, which states the holomorphic convexity of the universal cover of certain complex manifolds, and he also proved holomorphic convexity with respect to high powers of a positive
line bundle. The recent development of the conjecture can be found in [EKPR]. Note that holomorphic convexity is much simpler to prove with the presence of high powers of a positive line bundle. In [3] we give a new proof of the following theorem:

**Theorem 2 ([N]).** Let $M$ be a connected smooth projective variety and $L \to M$ a positive holomorphic line bundle. Then there exists an integer $N_0 = N_0(M, L, h)$ so that for $N \geq N_0$, the universal cover $\tilde{M}$ is holomorphically convex with respect to $\tilde{L}^N$.

Our second application is the surjectivity of the Poincaré map ([3]). In general, the operator $P$ is not surjective. We prove

**Theorem 3.** Suppose that $N$ is large enough so that Theorem 2 holds. Then the Poincaré map is surjective from $H^0_{L^1}(\tilde{M}, \tilde{L}^N) \to H^0(M, L^N)$.

As discussed above, surjectivity is the non-trivial aspect of the $\sum_\gamma$ relation, and as we show in [4] it is an almost immediate consequence of Theorem 1. The original motivation for this article was to simplify the discussion of surjectivity in Kollár's book [K]. We briefly review its approach.

In [K, Theorem 7.12], $P$ is proved to be surjective as long as the Bergman kernel on $L^2$ extends to $L^1$ and $L^\infty$ and is a reproducing kernel on $L^\infty$. Kollár reviews two conditions (7.9) (Condition 1) and (7.11) (Condition 2) under which surjectivity was proved by Earle [Ear]. Condition 1 is that the Bergman projection $\tilde{\Pi}_N$ for $(\tilde{M}, \tilde{h})$ extends to bounded linear maps on $L^1(\tilde{M}, \tilde{L}^N)$ and $L^\infty(\tilde{M}, \tilde{L}^N)$. As verified in [K, Proposition 7.13], it is sufficient that $\tilde{\Pi}_N(\cdot, w) \in L^1(\tilde{M})$ with $\|\tilde{\Pi}_N(\cdot, w)\|_{L^1} \leq C$ for a uniform constant $C$ independent of $w$. Condition 2 is that $\tilde{\Pi}_N$ is a reproducing kernel on all $L^\infty(\tilde{M}, \tilde{L})$. The Agmon estimates are sufficient to ensure Condition 1. In [K, Proposition 7.14], a rather strange condition is used to prove Condition 2: namely that $\frac{\Pi_{2N}(z, w)}{\Pi_N(z, w)}$ is in $L^2$. In particular, that $\Pi_N(z, w)$ is never zero. Kollár writes the conditions in (7.14) are “... quite artificial; it is not clear...how restrictive condition 2 is in reality. Theorem [H] shows that the natural restriction is convergence of the Poincaré series in its statement. We do not study in detail the question of effective estimates, i.e. of finding the minimal value of $N_0$, but the proof of Theorem [H] shows that $N_0$ is determined by balancing the growth rate of $\Gamma$ with optimal off-diagonal estimates ([H] on $\Pi_N(x, y)$. In [Y1, Y2, Y3], Yeung proved some effectiveness results for towers of Galois covers over a Kähler manifold. His techniques may prove to be useful in obtaining an effective estimate of $N_0$. We hope to study this question in a future article.

There exist many additional articles devoted to universal covers of Kähler manifolds and the relations between the complex geometry above and below. See for instance [G1, G2, E, Kai, Ca, Donn, Y1, Y2]. But to our knowledge, they do not use the relation of Theorem 1.

1. **Bergman/Szegő kernels**

In this section, we review the definition of the Bergman/Szegő kernel for a positive Hermitian holomorphic line bundle $(L, h) \to M$. We also go over a basic example where an explicit formula on the universal cover exists.
The Szegő kernel of \((L, h) \rightarrow M\) is the Schwartz kernel of the orthogonal projection \([\mathbb{L}^\infty]\). To obtain a Schwartz kernel we need to introduce a local holomorphic frame \(e_L\) over an open set \(U \subset M\). Then a local holomorphic section may be written \(s = fe_L\) where \(f\) is a local holomorphic function on \(U\). Similarly, \(\epsilon_L^{\otimes N}\) is a local holomorphic frame for \(L^N\). We choose an orthonormal basis \(\{S_j^N\}\) of \(H^0(M, L^N)\) and write \(S_j^N = f_j e_L^{\otimes N} : j = 1, \ldots, d_N\) where \(d_N = \dim H^0(M, L^N)\). Then the Szegő kernel \(\Pi_N(z, w)\) for \((L^N, h^N)\) relative to \(dV_M\) is the section of \(L^N \otimes L^N \rightarrow M \times M\) given by

\[
\Pi_N(z, w) := B_N(z, w) e_L^{\otimes N}(z) \otimes \overline{e_L^{\otimes N}(w)},
\]

where

\[
B_N(z, w) = \sum_{j=1}^{d_N} f_j(z) \overline{f_j(w)}.
\]

In \([\text{BBS}]\), \(B_N(z, w)\) is called the Bergman kernel.

1.1. Szegő kernel for line bundles and the associated disc bundle. Instead of using local frames, one can define scalar kernels if one lifts the Szegő kernels to the unit frame bundle \(X\) associated to the dual Hermitian line bundle \((L^*, h^*) \rightarrow M\) of \((L, h)\). The behavior of the lifts under translations by \(\gamma \in \Gamma\) is somewhat more transparent than for \(\Pi_h(z, w)\) which is a section of \(L^N \otimes L^N\). The choice whether to use the kernels on \(M \times M\) or their lifts to \(X \times X\) is mainly a matter of convenience.

In this section, we review the lift to the unit frame bundle on \(M\) and \(\hat{M}\).

As above, \(L^*\) denotes the dual line bundle to \(L\). The hermitian metric \(h\) on \(L\) induces the dual metric \(h^*(\equiv \bar{h}^{-1})\) on \(L^*\). We define the principal \(S^1\) bundle \(\pi : X \rightarrow M\) by

\[
X = \{ \lambda \in L^* : \|\lambda\|_{h^*} = 1 \} = \partial D, \quad \text{where} \quad D = \{ \lambda \in L^* : \rho(\lambda) > 0 \},
\]

where \(\rho(\lambda) = 1 - \|\lambda\|_{h^*}^2\). We let \(r_\theta x = e^{i\theta} x (x \in X)\) denote the \(S^1\) action on \(X\) and denote its infinitesimal generator by \(\partial/\partial \theta\). The disc bundle \(D\) is strictly pseudoconvex in \(L^*\), since the curvature \(\Theta_h\) of \(h\) is positive, and hence \(X\) inherits the structure of a strictly pseudoconvex CR manifold. Associated to \(X\) is the contact form \(\alpha = -i\partial\bar{\partial}\rho\big|_X = i\partial\bar{\partial}\rho\big|_X\) and the volume form

\[
dV_X = \frac{1}{m!} \alpha \wedge (d\alpha)^m = \pi^m \alpha \wedge \pi^* dV_M.
\]

It induces the \(L^2(X, dV_X)\) inner product

\[
(F_1, F_2) = \frac{1}{2\pi^{m+1}} \int_X F_1 \overline{F_2} dV_X.
\]

A section \(s_N\) of \(L^N\) lifts to an equivariant function \(\hat{s}_N\) on \(L^*\), defined by

\[
\hat{s}_N(\lambda) = (\lambda^{\otimes N}, s_N(z)) , \quad \lambda \in L^*_z, \quad z \in M
\]

We henceforth restrict \(\hat{s}_N\) to \(X\) and then the equivariance property takes the form \(\hat{s}_N(r_\theta x) = e^{i\theta} \hat{s}_N(x)\). We may express the lift in local coordinates \(z\) on \(U \subset M\) and in a local holomorphic frame \(e_L : U \rightarrow M\). They induce local coordinates
(z, θ) on X by the rule \( x = e^{iθ} |e_L(z)|_h e_L^*(z) \in X \). The equivariant lift of a section \( s = f e_L^{⊗N} \in H^0(M, L^N) \) is then given by

\[
\hat{s}(z, θ) = e^{iNθ} |e_L|^N f(z) = e^{N\left[ -\frac{i}{2} \phi(z) + iθ \right]} f(z),
\]

where \( |e_L(z)|_h = e^{-\frac{i}{2} \phi(z)} \) and \( \phi(z) \) is the local Kähler potential. The map \( s \mapsto \hat{s} \) is a unitary equivalence between \( L^2(M, L^N) \) and \( L^2_N(X, dV_X) \), where \( L^2_N(X, dV_X) \subset L^2(X, dV_X) \) is the subspace of equivariant functions transforming by \( e^{iNθ} \) under \( r_θ \).

The Hardy space \( H^2(X) \subset L^2(X, dV_X) \) is by definition the subspace of functions that are annihilated by the Cauchy-Riemann operator \( \bar{∂}_b \). The \( S^1 \) action on \( X \) commutes with \( \bar{∂}_b \) and hence the subspace \( H^2_N(X) \subset H^2(X) \) of equivariant CR functions is the intersection \( H^2(X) \cap L^2_N(X, dV_X) \). The lift of \( s_N \in H^0(M, L^N) \) is then an equivariant CR function \( \hat{s}_N \in H^2(X) \), hence \( H^2(X) = \bigoplus_{N=0}^{∞} H^2_N(X) \). The Szegő kernel \( \Pi \) is the (distribution) kernel of the orthogonal projection \( L^2(X) \to H^2(X) \).

The Szegő kernels \( \Pi_N \) lift to equivariant scalar kernels \( \hat{\Pi}_N \) on \( X \times X \), with \( \hat{\Pi}_N \) the Schwartz kernel of the orthogonal projection \( \hat{\Pi}_N : L^2_N(X, dV_X) \to H^2_N(X) \), defined by

\[
\hat{\Pi}_N F(x) = \int_X \hat{\Pi}_N(x,y) F(y) dV_X(y), \quad F \in L^2(X, dV_X).
\]

Then \( \hat{\Pi}_N \) is the \( N \)th Fourier coefficient of \( \hat{\Pi} \) and in terms of the orthogonal decomposition above, we have

\[
\hat{\Pi} = \sum_N \hat{\Pi}_N
\]
as operators on \( L^2(X) \).

Using (7), the Bergman kernel \( \hat{\Pi}_N \) can be given as

\[
\hat{\Pi}_N(x,y) = \sum_{j=1}^{d_N} \hat{S}^N_j(x) \hat{S}^N_j(y),
\]

where \( \hat{S}^N_1, \ldots, \hat{S}^N_{d_N} \) form an orthonormal basis of \( H^0(M, L^N) \). By (11), the lifted Szegő kernel is given in terms of the Bergman kernel on \( U \times U \) by

\[
\hat{\Pi}_N(z, θ; w, φ) = e^{N\left[ -\frac{i}{2} \phi(z) - \frac{i}{2} \phi(w) + i(θ - φ) \right]} B_N(z, \bar{w}).
\]

Theorem 1 can be restated as follows in terms of the Szegő kernels on the unit circle bundle:

\[
\hat{\Pi}_N(x, y) = \sum_{γ \in Γ} \hat{\Pi}_N(γ \cdot x, y).
\]

In this formulation, translation by \( γ \) acts on a scalar kernel rather than a section of a line bundle. By (13) one has a similar Poincaré series formula for \( \hat{\Pi}_N \).
1.2. Szegő kernels for the hyperbolic disc. To illustrate the notions above, we consider the familiar example of the lifted Szegő kernels on the hyperbolic disc $D$. In this case, the positive line bundle is the canonical bundle $L = T^{(1,0)}D$ equipped with the hyperbolic hermitian metric $h_D$ dual to the hyperbolic metric on $T^{(1,0)}D$. There exists a global holomorphic frame $dz$ for $L \to D$ with Hermitian norm $\|dz\|_{h_D}^2 = (1 - |z|^2)^2$. Hence the Kähler potential is given by $\phi(z) = \log(1 - |z|^2)^{-2}$. Thus for $s_N = f(dz)^N$, one has $\|f(dz)^N\|_{h_D}^2 = |f(z)|^2(1 - |z|^2)^{2N}$. The dual bundle $L^*$ is $T^{(1,0)}D$ with the usual hyperbolic metric, so that $X = \{(z,v) \in T^{(1,0)}D : |v|_z = 1\}$ is the unit tangent bundle of $D$, i.e. equals $PSU(1,1)$. In the local coordinates $(z,\theta)$ on $X$ denote the coordinates of the point $x = e^{i\theta}|\frac{\partial}{\partial z}|^{-1}h_0^{\partial \bar{z}} \in X$, we have

\begin{equation}
\bar{s}(z,\theta) = e^{iN\theta}|dz|^N f(z) = e^{iN\theta}(1 - |z|^2)^N f(z).
\end{equation}

The Bergman kernel for $L^*$ (denoted in $\mathbb{E}ar$ by $k_t$) is explicitly given by

\begin{equation}
2k_t(z,w) = (1 - z\bar{w})^{-2t} = \sum_{j=0}^{\infty}(2t)_j \frac{(z\bar{w})^j}{j!},
\end{equation}

where $t_j = (t-1)t(t+1)\cdots(t+j-1)$. The fact that $k_t = k_t^*$ is a reflection of the homogeneity of $D$. Furthermore, $k_1(z,z)dz = dm = dV_h$ and $k(z,z)^{-1/2} = e^{-\phi/2} = (1 - |z|^2)$ when $L = T^{(1,0)}D$. In the notation above, $k_t(z,w) = F_t^{h_D}(z,w)$, where $F_t$ is the local Bergman kernel in the frame $(dz)^t$. The lifted Bergman kernel is give by

\begin{equation}
\tilde{\Pi}_{h_D}(z,0;w,0) = C_m \left(\frac{(1 - z \cdot \bar{w})}{\sqrt{1 - |z|^2}\sqrt{1 - |w|^2}}\right)^{-2t}.
\end{equation}

for constant $C_m$ depending only on $m$.

We also refer to $\mathbb{E}ar$ for calculations in the general setting of a bounded homogeneous domain $B$ with $L = K$ (the canonical bundle). The Bergman kernel in that setting is induced by the natural inner product on $H^0(B,K)$, i.e. on $(n,0)$ forms and the Bergman kernel $h(z,\zeta)dz \otimes d\zeta$ is naturally an $(n,n)$ form.

1.3. Bergman/Szegő kernel on $\tilde{M}$ and the action of $\Gamma$. We now consider the Kähler cover $\pi : (\tilde{M},\tilde{\omega}) \to (M,\omega)$. By definition, $\tilde{L} = \pi^*L$ and $\tilde{h} = \pi^*h$. We then define the unit circle bundle $\tilde{X} \to \tilde{M}$ similarly.

Because $\tilde{M}$ is simply connected, $\Gamma$ automatically lifts to $\tilde{X}$ as a group of CR holomorphic contact transformations with respect to $\alpha$, and in particular the action of $\Gamma$ linearizes on the spaces $H^0_{\tilde{L}^2}(\tilde{M},\tilde{L}^N)$. We briefly recall the proof: by assumption, $\gamma \in \Gamma$ is an isometry of $\tilde{\omega}$ and thus is a symplectic transformation. We claim that $\gamma$ preserves the holonomy map of the connection 1-form $\alpha$, i.e. the map $H(\beta) = e^{\theta \beta}$ defined by horizontally lifting a loop $\beta : [0,1] \to M$ to $\tilde{\beta} : [0,1] \to X$ with respect to $\alpha$ and expressing $\tilde{\beta}(1) = e^{\theta \beta}\tilde{\beta}(0)$. Then $\gamma$ preserves the holonomy-preserving in the sense that $H(\gamma(\beta)) = H(\beta)$ for all loops $\beta$. Indeed, we may assume that the loop is contained in the domain of a local frame $s : U \to X$, and then $H(\beta) = \exp(2\pi i \int_{\beta} s^*\alpha)$. But $\tilde{M}$ is simply connected so that $\beta = \partial \sigma$ and
\[ \int_\beta s^*\alpha = \int_\sigma \omega. \] Since \( \gamma \) is symplectic, it thus preserves the holonomy around homologically trivial loops and all loops on \( \tilde{M} \) are trivial.

Since \( \Gamma \) acts by holomorphic transformations of \( \tilde{M} \), it lifts to a group of \( CR \) maps on \( \tilde{X} \) which commute with the \( S^1 \) action. It is easy to see that \( \tilde{\Pi}_N \) commutes with the action of \( \Gamma \) on \( H^2_\mathbb{C}(X) \), hence
\[ (19) \quad \tilde{\Pi}_N(\gamma x, \gamma y) = \tilde{\Pi}_N(x, y). \]

This identity is often written as a transformation law for the scalar Szegő kernel \( \hat{\Pi}_N \) of a local frame under \( \Gamma \). In most works such as \([\text{Ear}]\), \( \tilde{M} \) is contractible and \( \tilde{\ell} \to \tilde{M} \) is holomorphically trivial, hence there exists a global frame \( \tilde{e}_L \). Since \( \Gamma \) linearizes on \( H^0(\tilde{M}, \tilde{L}^N) \), there exists a function \( J(\gamma, z) \) (a factor of automorphy) such that \( \gamma^*\tilde{e}_L = J(\gamma, z)\tilde{e}_L \). Then
\[ B_N(\gamma z, \gamma w) = J(\gamma, z)J(\gamma, w)B_N(z, w). \]

2. Proof of Theorem 1

In this section we prove Theorem 1.

2.1. Agmon estimates. We first sketch the proof of the following Agmon estimate for the Szegő kernel, which is almost entirely contained in the previous work of Delin, Lindholm and Berndtsson.

**Theorem 2.1** ([Del] [L] [BBS]) (See Theorem 2 of [Del] and Proposition 9 of [L]). Let \( M \) be a compact Kähler manifold, and let \((L, h) \to M\) be a positive Hermitian line bundle. Then the exists a constant \( \beta = \beta(M, L, h) > 0 \) such that
\[ |\tilde{\Pi}_N(x, y)|_{h^N} \leq e^{-\beta \sqrt{N}d(x, y)}, \quad \text{for } d(x, y) \geq 1. \]
where \( d(x, y) \) is the Riemannian distance with respect to the Kähler metric \( \tilde{\omega} \).

**Review of the proof.** In [L, Proposition 9], the following is proved for a strictly pseudo-convex domain of \( \mathbb{C}^m \). The same argument works on strictly pseudo-convex manifold. In our notation, it was proved that
\[ |\tilde{\Pi}_N(x, y)|_{h^N}^2 \leq C N^{2m} e^{-c \sqrt{N}d(x, y)} \]
for some \( c > 0 \). Since \( d(x, y) \geq 1 \), the polynomial term of \( N \) can be absorbed by the exponential term by shrinking \( \epsilon \).

2.2. \( \bar{\partial} \) estimates and existence theorems on complete Kähler manifolds. The following Hörmander’s \( \bar{\partial} \) estimate is essential in our proof of Theorem 1.

**Theorem 2.2.** Let \((X, \omega)\) be a complete Kähler manifold, and let \( L \to X \) be a hermitian line bundle with the hermitian metric \( h \). Assume that there is an integer \( N_0 \) such that the curvature \( \sqrt{-1}N_0 \Theta(h) + \text{Ric}(\omega) \geq c \omega \) is positive for some \( c > 0 \). Then for any \( N \geq N_0 \), the following is true: for any \( g \in L^2(X, \bigwedge^{0,1} L^N) \) satisfying \( \bar{\partial}g = 0 \), and \( \int_X |g|^2_{h^N} \omega^n < \infty \), there exists \( f \in L^2(X, L^N) \) such that \( \bar{\partial}f = g \) and
\[ \int_X |f|^2_{h^N} \omega^n \leq c^{-1} \int_X |g|^2_{h^N} \omega^n. \]
2.3. Bergman kernels modulo $O(e^{-\delta N})$. We now begin the local analysis of the Bergman-Szegö kernel above and below, following the notation and terminology of [BBS].

Let $B$ be the unit ball in $\mathbb{C}^m$, and let $\chi \in C_c^\infty(B)$ be a smooth cutoff function equal to one on the ball of radius $1/2$. Let $M$ be a Kähler manifold and let $z$ be a fixed point of $M$. Without loss of generality, we assume that the injectivity radius at $z$ is at least 2. We identify $B$ with the unit geodesic ball around $z$ in $M$ and let $\phi$ be a local Kähler potential for $h$ relative to a local frame $e_L$ of $L$. Writing a section $s \in H^0(B, L^N)$ in the form $u_N = u e_L^N$ we identify sections with local holomorphic functions. We define the local $L^2$ norm of the section by

$$\|u_N\|^2_{N\phi} = \frac{1}{m!} \int_B |u|^2 e^{-N\phi} \omega^m.$$  

Let $s$ be a function or a section of a line bundle. We write $s = O(R)$, if there is a constant $C$ such that the norm of $s$ is bounded by $C R$. A family $K_N(x, y)$ of smooth kernels is a reproducing kernel modulo $O(e^{-\delta N})$ for some $\delta > 0$, if there exists an $\epsilon > 0$ such that for any fixed $z$, and any local holomorphic function $u$ on the unit ball $B$, we have

$$u_N(x) = \int_B (K_N(x, y), \chi(y) u_N(y))_{L^N} dV_M(y) + O(e^{-\delta N})\|u_N\|_{N\phi}$$

uniformly in $x \in B_\epsilon = \{ x \mid d(x, z) < \epsilon \}$. Each function $K_N(x, y)$ is called a Bergman kernel modulo $O(e^{-\delta N})$ if it is additionally holomorphic in $x$.

2.4. $\Pi_N(x, y)$ is a Bergman kernel modulo $O(e^{-\delta N})$. In this subsection, we prove the following lemma

**Lemma 4.** There exists a constant $\delta > 0$ such that $\Pi_N(x, y)$ is a Bergman kernel modulo $O(e^{-\delta N})$.

**Proof.** Let $P_N(x, y)$ be the local reproducing kernel constructed in [BBS] and let $P_N$ be the corresponding operator. For any holomorphic function $u$ on the unit disk $B$, write

$$u_N - P_N(\chi u_N) = B_N u_N$$

for operators $B_N$. Then by definition

$$B_N u_N(x) = O(e^{-\delta N})\|u_N\|_{N\phi}$$

for some constant $\delta > 0$ and for $x \in B_\epsilon$.

Let $\chi_1$ be a cut-off function of the unit ball $B$ such that $\chi_1$ is 1 in a fixed neighborhood of the origin. We assume that $\text{supp}(\chi_1) \subset B_\epsilon$. Consider the identity

$$\chi_1(1 - \Pi_N)(\chi u_N) = \chi_1(1 - \Pi_N)(\chi P_N(\chi u_N)) + \chi_1(1 - \Pi_N)(\chi(B_N u_N)).$$

Since $\Pi_N$ is a projection operator, by the definition of $P_N$, we have

$$\|\chi_1(1 - \Pi_N)(\chi(B_N u_N))\|_{N\phi} = O(e^{-\delta N})\|u_N\|_{N\phi}.$$  

(20)

According to [BBS], $P_N(x, y)$ is holomorphic with respect to $x$ when $d(x, y)$ is small. Thus for $y \in B$, we have

$$\|\chi_1(1 - \Pi_N)(\chi P_N(\cdot, y))\|_{N\phi} = O(e^{-\delta N}).$$
The proof is the same as that of [BBS], Theorem 3.1] which we include here for the sake of completeness.

Let \( P_{N,y}(x) = P_N(x, y) \). By the construction of the local reproducing kernel (cf. [BBS], (2.3), (2.7)), we have

\[
\bar{\partial}(\chi_{P_{N,y}}) = O(e^{-\delta N})
\]

for some \( \delta > 0 \). By Hörmander’s \( L^2 \)-estimate (cf. Theorem 2.2), there exists \( v_{N,y} \) such that

\[
\bar{\partial}v_{N,y} = \bar{\partial}(\chi_{P_{N,y}})
\]

with the estimate

\[
\int_{\tilde{M}} |v_{N,y}|^2_{\tilde{h},N} dV_{\tilde{M}} \leq C \int_{\tilde{M}} |\bar{\partial}(\chi_{P_{N,y}})|^2_{\tilde{h},N} dV_{\tilde{M}} = O(e^{-2\delta N}).
\]

By definition,

\[
(1 - \tilde{\Pi}_N)(\chi_{P_N(\cdot, y)}) = (1 - \tilde{\Pi}_N)v_{N,y}.
\]

Therefore for fixed \( y \),

\[
\|\chi_1(1 - \tilde{\Pi}_N)(\chi_{P_N(\cdot, y)})\|_{N_\phi,y} \leq C\|v_{N,y}\|_{N_\phi,y} \leq C e^{-\delta N},
\]

where the norm \( \| \cdot \|_{N_\phi,y} \) is the \( \| \cdot \|_{N_\phi} \) norm for the \( x \) variable and the pointwise norm for the fixed point \( y \). Thus we obtain

\[
\|\chi_1(1 - \tilde{\Pi}_N)(\chi_{P_N(\chi u_N)})\|_{N_\phi} \leq C e^{-\delta N} \|u_N\|_{N_\phi}.
\]

Combining (20) and (21), we conclude that

\[
\|\chi_1(1 - \tilde{\Pi}_N)(\chi u_N)\|_{N_\phi} = O(e^{-\delta N}) \|u_N\|_{N_\phi}.
\]

Note that in a neighborhood of the origin, \( (1 - \Pi_N)(\chi u_N) \) is holomorphic. By working on the ball of radius \( N^{-1} \) and the mean-value inequality, the above \( L^2 \) bound implies the following \( L^\infty \) bound

\[
(1 - \tilde{\Pi}_N)(\chi u_N)(x) = O(N^{2m} e^{-\delta N}) \|u_N\|_{N_\phi}
\]

where \( x \in B_\epsilon \). The \( N^{2m} \) term can be absorbed by the exponential term if we further shrink \( \delta > 0 \), and the lemma is proved.

\[\square\]

2.5. Completing the proof of Theorem [1] Let \( y \) be a fixed point of \( \tilde{M} \). If we use the same local trivialization of \( \tilde{L} \) at each \( \gamma \cdot y \) (that is, these local trivializations are identical via the one at \( \pi(y) \in \tilde{M} \)), then the summation of the right side of the following equation is well-defined

\[
\Pi^\Gamma_N(x, y) := \sum_{\gamma \in \Gamma} \tilde{\Pi}_N(x, \gamma \cdot y).
\]

By Theorem [2.1] the series converges for sufficiently large \( N \).

We first prove that
Lemma 5. There is a constant $\delta > 0$ such that $\Pi^N(x, y)$ is a Bergman kernel mod $O(e^{-\delta \sqrt{N}})$ in the sense that

$$\int_M (\Pi^N(x, y), \chi(y)u_N(y))_{h^N} dV_M(y) = \chi u_N + O(e^{-\delta \sqrt{N}} \|\chi u_N\|_{N\phi})$$

uniformly for any local holomorphic function $u$ on $\tilde{M}$.

Proof. Holomorphic sections $s \in H^0(M, L^N)$ lift to $\tilde{s}$ on $\tilde{M}$ as holomorphic sections of $\tilde{L}^N$, so our integration lifts to the universal cover. Let $z$ be a fixed point of $M$. As assumed before, the injectivity radius at $z$ is at least 2. Let $B$ be the unit ball about $z$. The pre-image of $B$ in the universal cover $\tilde{M}$ are disjoint balls. By abuse of notation, we identify $B$ with one of these balls and the variables $z, x, y$ are used for points on both $M$ and $\tilde{M}$. Let $u$ be a function on $B$. Then $u$ is regarded as local functions on both $M$ and $\tilde{M}$. We regard $u_N$ to be the section of $\tilde{L}^N$ extended by zero outside $B$. Thus we have

$$\int_M (\Pi^N(\cdot, y), \chi(y)u_N(y))_{h^N} dV_M(y) = \sum_{\gamma \in \Gamma} \int_{\tilde{M}} (\tilde{\Pi}_N(\cdot, \gamma \cdot y), \chi(y)u_N(y))_{\tilde{h}^N} dV_{\tilde{M}}(y).$$

Define $d(B, \gamma B)$ to be the distance between $B$ and $\gamma B$. Then the Agmon estimate gives

$$|\tilde{\Pi}_N(x, \gamma \cdot y)|_{\tilde{h}^N} \leq e^{-\beta \sqrt{N} d(B, \gamma B)}$$

for some $\beta = \beta(M, L, h) > 0$ and any $x, y \in B$. Therefore, we have

$$\sum_{\gamma \neq 1} \left| \int_{\tilde{M}} (\tilde{\Pi}_N(\cdot, \gamma \cdot y), \chi(y)u_N(y))_{\tilde{h}^N} dV_{\tilde{M}}(y) \right|_{\tilde{h}^N} \leq C \sum_{\gamma \neq 1} e^{-\beta \sqrt{N} d(B, \gamma B)} \|\chi u_N\|_{N\phi}.$$ 

By compactness, there is a constant $\sigma > 0$ such that

$$\sigma d(x, \gamma y) \leq d(B, \gamma B)$$

for any $x, y \in B, \gamma \neq 1$.

Let

$$\eta = \inf_{\gamma \neq 1} d(B, \gamma B).$$

Then we have

$$\sum_{\gamma \neq 1} e^{-\beta \sqrt{N} d(B, \gamma B)} \leq e^{-\frac{1}{2} \beta \eta \sqrt{N}} \sum_{\gamma \neq 1} e^{-\frac{1}{2} \beta \sqrt{N} d(B, \gamma B)}$$

$$\leq C e^{-\frac{1}{2} \beta \eta \sqrt{N}} \int_M e^{-\frac{1}{2} \beta \sqrt{N} d(z, y)} dV_M(y).$$

By the Bishop volume comparison theorem, since the Ricci curvature has a lower bound, the volume growth of $\tilde{M}$ is at most exponential. Thus the integral is convergent and we have

$$\sum_{\gamma \neq 1} e^{-\beta \sqrt{N} d(B, \gamma B)} \leq C e^{-\frac{1}{2} \beta \eta \sqrt{N}}.$$
Combining the above inequality with lemma 4 we have

$$\int_M (\Pi_N^\Gamma (\cdot, y), \chi(y)u_N(y))_{h^N} dV_M(y) = \chi u_N + O(e^{-\delta_N} + e^{-\frac{1}{2} \beta \eta \sqrt{N}})\|u\|_{N\phi}. $$

Since $\delta$ can be chosen arbitrarily small, the conclusion of the lemma follows.

**Proof of Theorem 1.** Let

$$R_N(z, w) = \Pi_N^\Gamma (z, w) - \Pi_N(z, w).$$

Using the same method as that in Lemma 4 (cf. [BBSJ]), $\Pi_N(z, w)$ is a Bergman kernel of modulo $O(e^{-\delta N})$ for some $\delta > 0$. Combining with Lemma 5, we obtain

$$\int_M (R_N(\cdot, y), \chi(y)u_N(y))_{h^N} dy = O(e^{-\delta \sqrt{N} \|\chi u_N\|_{N\phi}}).$$

Substituting $u_N(y) = \chi(y)e^{-N\phi(x)/2}R_N(y, x)$ into the above equation, we get

$$\int_M \chi(y)|R_N(x, y)|^2_{h^N} dV_M(y) = O(e^{-\delta \sqrt{N}}) \sqrt{\int_M \chi^2 |R_N(x, y)|^2_{h^N} dV_M(y)},$$

which implies

$$\int_M \chi(y)|R_N(x, y)|^2_{h_N} dV_M(y) = O(e^{-2\delta \sqrt{N}}).$$

Note that the above is true uniformly for $x \in B_\epsilon$. Combining this with the Agmon estimate, we obtain

$$\sqrt{\int_{M \times M} |R_N(x, y)|^2_{h_N} dV_M(y) dV_M(x)} = O(e^{-\delta \sqrt{N}}).$$

(22)

The above left hand side bounds the norm of the operator $R_N$ defined by the integral kernel $R_N(z, y)$.

Next we prove that $R_N^2 = R_N$, and hence $R_N = 0$. Recall our convention of identifying points on $M$ with one of their lifts on $\tilde{M}$. Let $x, w \in M$. Then we have

$$\int_M (\Pi_N^\Gamma (x, y), \Pi_N^\Gamma (y, w))_{h^N(y)} e^{-\frac{1}{2} N \phi(w)} dV_M(y)$$

$$= \int_M \sum_{\gamma, \gamma_1 \in \Gamma} (\Pi_N (x, \gamma \cdot y), \Pi_N (y, \gamma_1 \cdot w))_{h^N(y)} e^{-\frac{1}{2} N \phi(w)} dV_M(y).$$
Since $\tilde{\Pi}(x, y)$ is $\Gamma$ invariant (cf. (19)) and since $\Gamma$ acts on $\tilde{M}$ by isometry, we have
\[
\int_M (\Pi_N^\Gamma(x, y), \Pi_N^\Gamma(y, z))h_N(y)e^{-\frac{1}{2}N\phi(z)}dV_M(y)
= \int_M \sum_{\gamma_1 \in \Gamma} (\tilde{\Pi}_N(x, \gamma \cdot y), \tilde{\Pi}_N(\gamma \cdot y, \gamma_1 \cdot w))h_N(y)e^{-\frac{1}{2}N\phi(w)}dV_M(y)
= \int_M \sum_{\gamma_1 \in \Gamma} (\tilde{\Pi}_N(x, \gamma \cdot y), \tilde{\Pi}_N(\gamma \cdot y, \gamma_1 \cdot w))h_N(y)e^{-\frac{1}{2}N\phi(w)}dV_M(y)
= \int_{\tilde{M}} \sum_{\gamma_1 \in \Gamma} (\tilde{\Pi}_N(x, y), \tilde{\Pi}_N(y, \gamma_1 \cdot w))h_N(y)e^{-\frac{1}{2}N\phi(w)}dV_{\tilde{M}}(y).
\]

By the Agmon estimate, any $\tilde{\Pi}_N(y, \gamma_1 \cdot w)e^{-\frac{1}{2}N\phi(w)}$ is an $L^2$ holomorphic section of $\tilde{L}^N$. Thus we have
\[
\int_{\tilde{M}} \sum_{\gamma_1 \in \Gamma} (\tilde{\Pi}_N(x, y), \tilde{\Pi}_N(y, \gamma_1 \cdot w))h_N(y)e^{-\frac{1}{2}N\phi(w)}dV_{\tilde{M}}(y)
= \sum_{\gamma_1} \tilde{\Pi}_N(x, \gamma_1 \cdot w)e^{-\frac{1}{2}N\phi(w)} = \Pi_N^\Gamma(x, w)e^{-\frac{1}{2}N\phi(w)}.
\]

Let $\Pi_N^\Gamma$ be the operator corresponding to the kernel $\Pi_N^\Gamma(x, w)$, then the above computation shows that
\[
(\Pi_N^\Gamma)^2 = \Pi_N^\Gamma.
\]
Since $\Pi_N$ is a projection operator, we have
\[
\Pi_N \Pi_N^\Gamma = \Pi_N^\Gamma.
\]
Since both $\Pi_N$ and $\Pi_N^\Gamma$ are self-adjoint, the above also implies
\[
\Pi_N^\Gamma \Pi_N = \Pi_N^\Gamma.
\]
As a result, we have $R_N^2 = R_N$. Thus $R_N$ is a projection operator. The operator norm of $R_N$ is 1 unless $R_N = 0$. But by (22), the norm is less than one for sufficiently large $N$. Thus $R_N = 0$ and the theorem is proved.

3. Holomorphic convexity: Proof of Theorem 2

We follow the notation of Napier [N] to prove Theorem 2.

Proof. By the proof of Lemma 3, the following result is valid: let $\beta_1 > 0$ be a fixed positive number, then for $N$ sufficiently large,
\[
\sum_{\gamma \in \Gamma} e^{-\beta_1 \sqrt{N}d(x, \gamma y)} \leq C < \infty
\]
for some constant $C$ only depends on the distance $d(x, y)$ of $x, y \in \tilde{M}$. We shall use the fact below repeatedly.
Let \( \{y_j\} \) be a divergent sequence of \( \tilde{M} \). By passing a subsequence if needed, we may assume that \( \pi(y_j) \to x_0 \in M \). By passing a subsequence if needed again, we define the sequence \( \{x_j\} \) inductively by the following conditions:

1. For each \( x_j \), there exists a \( \gamma \in \Gamma \) such that \( x_j = \gamma(x_0) \);
2. \( d(x_j) \geq j \) for all \( j \geq 1 \);
3. \( \inf_{i < j} d(x_i, x_j) \geq \frac{1}{\gamma} \sup_{i < j} d(x_i, x_j) \) for all \( j \geq 1 \);
4. \( d(x_j, y_j) \to 0 \) as \( j \to \infty \).

Define

\[
s(x) = \sum_{j=1}^{\infty} e^{d(x_j)} \tilde{\Pi}_N(x, x_j).\]

Here, as before, we fix a local trivialization of \( L \) at \( x \) so that \( \tilde{\Pi}_N(x, x_j) \) can be identified as a section of \( \tilde{L}^N \) for each \( j \). We claim that the above series is uniformly convergent on compact sets and hence defines a holomorphic section of \( \tilde{L}^N \). To see this, we use the Agmon estimate to obtain

\[
|s(x)|_{\tilde{h}_N} \leq C \sum_{j=1}^{\infty} e^{d(x_j) - \beta \sqrt{N} d(x, x_j)}.
\]

On any compact set, the norm can be estimate by

\[
|s(x)|_{\tilde{h}_N} \leq C \sum_{j=1}^{\infty} e^{d(x_j) - \beta \sqrt{N} d(x, x_j)} \leq C \sum_{j=1}^{\infty} e^{-(\beta \sqrt{N} - 1) d(x_j)} < \infty
\]

for a possibly larger constant \( C \). Thus section \( s \) is well defined.

We verify that \( s(x_k) \to \infty \). In fact, using the Agmon’s estimate and our construction of the sequence \( \{x_j\} \), for any fixed \( k \), we have

\[
\left| \sum_{\ell \neq k} e^{d(x_k)} \tilde{\Pi}_N(y_k, x_{\ell}) \right|_{\tilde{h}_N} \leq \sum_{\ell \neq k} e^{-(\frac{1}{\gamma} \beta \sqrt{N} - 1) d(x_{\ell})} < C < \infty
\]

for a constant \( C \) independent of \( k \). On the other hand, by [SZ] and the Agmon estimate again, we know that

\[
\left| \tilde{\Pi}_N(y_k, x_k) \right|_{\tilde{h}_N} \geq ce^{-d(x_k, y_k)}
\]

for some constant \( c > 0 \). Thus we have

\[
e^{d(x_k)} \tilde{\Pi}_N(y_k, x_k) \to \infty
\]

and this completes the proof. \( \square \)

We remark that such a section \( s \) can never be in \( H^0_{L^2}(\tilde{M}, \tilde{L}^N) \). Indeed, we note that

\[
s(z) = \int_{\tilde{M}} (\tilde{\Pi}_N(z, w), s(w))_{\tilde{h}_N} dV_{\tilde{M}}(w)
\]
so that if $s$ were square integrable, then
\[ |s(z)|^2_{h,N} \leq \int_M |\Pi_N(z, w)|^2 dV(w) \cdot |s|^2_{L^2}. \]
We further note that
\[ \int_M |\Pi_N(z, w)|^2 dV_M(w) = \Pi_N(z, z). \]
But $\Pi_N(z, z)$ is $\Gamma$ invariant and hence bounded. So square integrable holomorphic sections are automatically bounded and we get a contradiction.

4. Application to surjectivity of Poincaré series: Proof of Theorem 3

We now give a simple proof of surjectivity when Theorem 1 and Theorem 2.1 are valid:

Proof. We define the coherent state (or peak section) $\Phi^w_N \in H^0(M, L^N)$ centered at $w$ by
\[ \Phi^w_N(z) = \Pi_N(z, w). \]
By Theorem 1, we have
\[ \Phi^w_N(z) = \sum_{\gamma \in \Gamma} \tilde{\Phi}^w_N(\gamma \cdot z) = P \tilde{\Phi}^w_N(z), \]
where
\[ \tilde{\Phi}^w_N(z) = \tilde{\Pi}_N(z, w). \]
For any $s \in H^0(M, L^N)$,
\[ \langle s, \Phi^w_N(z) \rangle_{h,N} = s(z). \]
Therefore,
\[ s(z) = P(\langle s, \tilde{\Phi}^w_N(z) \rangle_{h,N}) \]
is written as the Poincaré series and the theorem is proved.

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Department of Mathematics, University of California, Irvine, Irvine, CA 92697, USA
E-mail address, Zhiqin Lu: zlu@uci.edu

Department of Mathematics, Northwestern University, Evanston, IL 60208-2370, USA
E-mail address: zelditch@math.northwestern.edu