Predictive SUSY SO(10) Model with Very Low tanβ

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Abstract

The first fermion family might play a key role in understanding the structure of flavour: a role of the mass unification point. The GUT scale running masses $\bar{m}_{e,u,d}$ are rather close, which may indicate an approximate symmetry limit. Following this observation, we present a new predictive approach based on the SUSY SO(10) theory with $\tan \beta \sim 1$. The inter-family hierarchy is first generated in a sector of hypothetical superheavy fermions and then transferred inversely to ordinary quarks and leptons by means of the universal seesaw mechanism. The Yukawa matrices are simply parametrized by the small complex coefficients $\varepsilon_{u,d,e}$ which are related by the SO(10) symmetry properties. Their values are determined by the ratio of the GUT scale $M_X \approx 10^{16}$ GeV to a higher (possibly string) scale $M \approx 10^{17} - 10^{18}$ GeV. The suggested ansatz correctly reproduces the fermion mass and mixing pattern. By taking as input the masses of leptons and $c$ and $b$ quarks, the ratio $m_s/m_d$ and the value of the Cabibbo angle, the $u,d,s$ quark masses, top mass and $\tan \beta$ are computed. The top quark is naturally in the 100 GeV range, but with upper limit $M_t < 165$ GeV, while the lower bound $M_t > 160$ GeV implies $m_s/m_d > 22$. $\tan \beta$ can vary from 1.4 to 1.7. The proton decaying $d = 5$ operators $qqql$ are naturally suppressed.

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1. Introduction

Understanding the fermion mass spectrum is one of the main issues in particle physics. In the standard model (SM) the Yukawa coupling constants are arbitrary, so one has to think of a more fundamental theory occurring at higher energies. One of the most promising ways beyond the SM is related to supersymmetric grand unified theories (SUSY GUTs) which provide a sound basis for solving the issues of the gauge coupling unification and the weak scale hierarchy. On the other hand, grand unification can also play an important role in understanding the flavour structure, by imposing specific constraints on the fermion mass matrices and thus reducing the number of free parameters. The $SO(10)$ GUT is a very appealing candidate for this purpose. It unifies all quark and lepton states of one family into the irreducible representation 16, providing thereby a possibility to link their masses with certain group-theoretical relations.

In order to understand how the fermion mass spectrum could reflect the grand unification features, it is necessary to compare the quark and lepton running masses $\bar{m}_f$ or their Yukawa constants $\lambda_f$ ($f = u, d, e, \ldots$) at the GUT scale $M_X \simeq 10^{16}$ GeV. In the minimal supersymmetric standard model (MSSM) these are related as $\bar{m}_f = \lambda_f v \sin \beta (\cos \beta)$, where $v$ is the electroweak scale, $\sin \beta$ stands for the case of upper quarks and $\cos \beta$ for the down quarks and charged leptons. One can observe that the vertical mass splitting is small within the first family and is quickly growing with the family number:

$$\frac{\bar{m}_u}{\bar{m}_{d,e}} \sim 1, \quad \frac{\bar{m}_c}{\bar{m}_{s,\mu}} \sim 10, \quad \frac{\bar{m}_t}{\bar{m}_{b,\tau}} \sim 10^2$$

(1)

whereas the splitting between the charged leptons and down quarks remains considerably smaller – the third family is almost unsplit: $\bar{m}_b \approx \bar{m}_\tau$, whereas the first two families are split by a factor of about 3 but $\bar{m}_d \bar{m}_s \simeq \bar{m}_e \bar{m}_\mu$.

The horizontal hierarchy of quarks exhibits the approximate scaling low

$$\frac{1}{\bar{m}_u} : \frac{1}{\bar{m}_c} : \frac{1}{\bar{m}_t} \sim 1 : \varepsilon_u : \varepsilon_u^2, \quad \frac{1}{\bar{m}_d} : \frac{1}{\bar{m}_s} : \frac{1}{\bar{m}_b} \sim 1 : \varepsilon_d : \varepsilon_d^2$$

(2)

with $\varepsilon_u^{-1} = 200 - 300$ and $\varepsilon_d^{-1} = 20 - 30$, while for the charged leptons we have

$$\frac{1}{\bar{m}_e} : \frac{1}{\bar{m}_\mu} : \frac{1}{\bar{m}_\tau} \sim 1 : \varepsilon_e' : \varepsilon_e' \varepsilon_e$$

(3)

with $\varepsilon_e \sim \varepsilon_d$ and $\varepsilon'_e \sim \varepsilon_u$. In addition, the quark mixing angles have the following pattern:

$$s_{12} \sim \varepsilon_d^{1/2}, \quad s_{23} \sim \varepsilon_d, \quad s_{13} \sim \varepsilon_d^2$$

(4)

Moreover, there are intriguing relations between fermion masses and mixing angles, like the well-known formula $s_{12} = (m_d/m_s)^{1/2}$ for the Cabibbo angle.

A popular idea is that the flavour structure is related to the certain restricted form of mass matrices (e.g. so called zero textures of refs. [1]), which can be motivated by specific horizontal symmetry between fermion families. Recently various zero texture ansatzes have been considered on the basis of SUSY $SO(10)$ model and several interesting (and testable) predictions were obtained for the fermion masses and mixing angles. The
key feature of this approach is that the mass generation starts from the third family and proceeds to the lighter ones through the smaller entries in the mass matrices (in fact, this feature is generic for all models \[^{[1]}\]). Namely, the third family is directly coupled to the Higgs 10-plet, so that \( \lambda_{t,b,\tau} \) are equal at the GUT scale. As for the lighter families, their masses are induced by certain higher order operators with the specific \( SO(10) \) structures. The large splitting of the top and bottom masses can be reconciled only at the price of extremely large \( \tan \beta \) of about two orders of magnitude, which can be achieved by certain tuning of parameters in the Higgs potential \[^{[3]}\]. However, then it becomes rather surprising that despite the giant \( \tan \beta \), \( \frac{\tilde{m}_c}{\tilde{m}_u} \) is about 10 times less as compared to \( \frac{\tilde{m}_t}{\tilde{m}_b} \) while \( \tilde{m}_u \) and \( \tilde{m}_d \) are almost unsplit. In order to achieve this, a judicious selection of the \( SO(10) \) Clebsch coefficients is required \[^{[2]}\].

In the present paper we suggest an alternative approach to fermion masses in a SUSY \( SO(10) \) model with \( \tan \beta \sim 1 \). We follow the observation that the masses of the first family exhibit an approximate symmetry limit \( \tilde{m}_e \sim \tilde{m}_u \sim \tilde{m}_d \) (with splitting of about a factor of 2), while the heavier families strongly violate it. In the context of small \( \tan \beta \) this may indicate that the \( SO(10) \) Yukawa unification holds for the constants \( \lambda_{u,d,e} \) rather than for \( \lambda_{t,b,\tau} \). How one could realize such a situation?

Nowadays the idea \[^{[4]}\] becomes popular that quark and lepton masses are induced by the universal seesaw mixing with hypothetical superheavy fermions, just in analogy with the famous seesaw scenario for neutrinos. These are fermion states which have large invariant mass terms or acquire masses after GUT breaking. Thus, their exchanges induce the higher order effective operators cutoff by the scale which can range from the Planck mass to the GUT scale. With such a picture in mind, it is suggestive to think that \( e, u, d \) are ‘unsplit’ since their masses are linked to an energy scale \( M_1 > M_X \) at which \( SO(10) \) symmetry is still good, while the second and third families are split being related to the lower scales \( M_{2,3} \leq M_X \) at which \( SO(10) \) is no longer as good.

In particular, we assume that at the GUT scale the inverse Yukawa matrices have the following form which we call the inverse hierarchy ansatz\[^{[5]}\]

\[
\hat{\lambda}_f^{-1} = \frac{1}{\lambda} \left( \hat{P}_1 + \varepsilon_f \hat{P}_2 + \varepsilon_f^2 \hat{P}_3 \right), \quad f = u, d, e
\]

where the small complex expansion parameters \( \varepsilon_f = \varepsilon_u, \varepsilon_d, \varepsilon_e \) are different for the upper quark, down quark and lepton mass matrices, and \( \hat{P}_{1,2,3} \) are some symmetric rank-1 matrices with \( O(1) \) elements. Without loss of generality, their basis can be chosen as

\[
\hat{P}_1 = (1, 0, 0)^T \cdot (1, 0, 0), \quad \hat{P}_2 = (a, b, 0)^T \cdot (a, b, 0), \quad \hat{P}_3 = (x, y, z)^T \cdot (x, y, z)
\]

so that the inverse Yukawa matrices are alligned and have the form

\[
\hat{\lambda}_f^{-1} = \frac{1}{\lambda} \left( \begin{array}{ccc}
1 + a^2 \varepsilon_f + x^2 \varepsilon_f^2 & ab \varepsilon_f + xy \varepsilon_f^2 & xz \varepsilon_f^2 \\
ab \varepsilon_f + xy \varepsilon_f^2 & b^2 \varepsilon_f + y^2 \varepsilon_f^2 & yz \varepsilon_f^2 \\
xz \varepsilon_f^2 & yz \varepsilon_f^2 & z^2 \varepsilon_f^2
\end{array} \right) \approx \frac{1}{\lambda} \left( \begin{array}{ccc}
1 & ab \varepsilon_f & xz \varepsilon_f^2 \\
ab \varepsilon_f & b^2 \varepsilon_f & yz \varepsilon_f^2 \\
xz \varepsilon_f^2 & yz \varepsilon_f^2 & z^2 \varepsilon_f^2
\end{array} \right)
\]

In lowest order their eigenvalues are given by diagonal entries: \( \lambda_{fi} \approx \lambda \varepsilon_f^{1-i} \) \((i = 1, 2, 3 \text{ is a family index})\). In this way, the quark mass pattern \( \[^{[6]}\],[^{[2]}\] \) is understood by means of

\[\text{[footnote]}\]

\[\text{[footnote]}\] This pattern was first suggested in \[^{[6]}\] in the context of the radiative mass generation scenario.
\[ \varepsilon_u \ll \varepsilon_d: \; \lambda_{u,d} \sim \lambda, \; \lambda_c/\lambda_s \sim (\varepsilon_d/\varepsilon_u) \sim 10 \text{ and } \lambda_l/\lambda_b \sim (\varepsilon_d/\varepsilon_u)^2 \sim 10^2. \]

This also implies that the CKM mixing emerges dominantly from the down quark matrix \( \hat{\lambda}_d \): \( \hat{\lambda}_u \) is much more “stretched” and essentially close to its diagonal form, so that it brings only \( O(\varepsilon_u/\varepsilon_d) \) corrections to the CKM mixing angles. Thus, at lowest order one expects that \( s_{12}, s_{23} \sim \varepsilon_d \) and \( s_{13} \sim \varepsilon_d^2 \), which estimates are indeed good for \( s_{23} \) and \( s_{13} \). For example, from \( (7) \) follows that \( s_{23} \approx |yz/b^2|\varepsilon_d \) and \( \lambda_s/\lambda_b \approx |z^2/b^2|\varepsilon_d \). Then the experimental observation \( s_{23} \sim \lambda_s/\lambda_b \sim \varepsilon_d \) implies that \( z \sim y \sim b \), so that the \( \varepsilon_d^2 y^2 \) term in \( (2,2) \) element can be safely neglected as compared to the leading term \( \varepsilon_d b^2 \). The similar consideration for \( s_{13} \) ensures that \( O(\varepsilon^2) \) terms are negligible also in \((1,1)\) and \((1,2)\) elements.

However, the naive picture with also \( \varepsilon_{d,e} a^2 \ll 1 \) and distinct \( \varepsilon_{e,d} \) immediately encounters the following problems: \( (i) \) the Yukawa couplings \( \lambda_{e,u,d} \) remain unsplit, \( (ii) s_{12} \sim \varepsilon_d \) is too small as compared to the actual value of the Cabibbo angle \( (\sim \sqrt{\varepsilon_d}) \), \( (iii) \) leptons behave as \( \lambda_e : \lambda_u : \lambda_r \sim 1 : \varepsilon^{-1}_e : \varepsilon^{-2}_e \), in contradiction to \( (3) \), \( (iv) \) the grand prix, \( b - \tau \) unification is lost: \( \lambda_b \) and \( \lambda_r \) emerge at \( O(\varepsilon^2) \) level and e.g. the factor of 2 difference among \( \varepsilon_d \) and \( \varepsilon_e \) would cause already factor of 4 splitting between \( \lambda_b \) and \( \lambda_r \).

In this paper we show that all these problems can be naturally solved in the framework of SUSY SO(10) model. As it was argued in \( [3] \), the experimental value of the Cabibbo angle \( s_{12} \approx (m_d/m_s)^{1/2} \) implies that \( |\varepsilon_{a^2}| \approx 1 \). On the other hand, SO(10) symmetry provides the specific relation \( \varepsilon_e = -\varepsilon_d - 2\varepsilon_u \) (see below, eq. \( (12) \)), which ensures that \( \lambda_b \approx \lambda_r \) due to the large \( \lambda_l - \lambda_b \) splitting \( (\varepsilon_u \ll \varepsilon_d) \). In addition, then \( \lambda_d \) and \( \lambda_e \) can split from \( \lambda_u \approx \lambda \) to different sides by about a factor of 2.

The paper is organized as follows. In the next section we demonstrate how the Yukawa matrices of the form \( [3] \) can be obtained in the context of the SUSY SO(10) model \([?]\), and study implications of our scheme for the fermion masses and mixing. Section 4 is devoted to a brief discussion of our results.

### 2. Inverse Hierarchy Picture in SUSY SO(10) Model

Consider a SUSY SO(10) model with three light fermion families \( 16_i^f \) and three families of superheavy fermions \( 16_i^F + \overline{16}_i^F, \; i = 1, 2, 3 \). It is convenient to describe the field content in terms of the Pati-Salam \( G_{PS} = SU(4) \otimes SU(2)_w \otimes SU(2) \) subgroup of \( SO(10) \):

\[
16_i^F = f_i(4, 2, 1) + f_i(4, 1, 2), \\
\overline{16}_i^F = F_i(4, 2, 1) + F_i(4, 1, 2) \\
(8)
\]

(Notice that \( F \)'s are weak isodoublets and \( F \)'s are isosinglets). We introduce also the Higgs 45-plets \( (45=(15,1,1)+(1,3,1)+(1,1,3)+(6,2,2)) \) of the following three types: \( 45_{BL} \) with VEV \( V_{BL} \) on the \((15,1,1)\) fragment, \( 45_R \) with VEV \( V_R \) on the \((1,1,3)\) fragment, and \( 45_X \) having VEV \( V_X \) shared by both \((15,1,1)\) and \((1,1,3)\) components.\(^2\)

\(^2\) The VEV orientation of these 45-plets are determined by their couplings to the Higgs superfields \( 54 \) and \( 16_H + \overline{16}_H \), which are also needed for the \( SO(10) \) symmetry breaking down to \( SU(3) \otimes SU(2)_w \otimes U(1) \) (see e.g. \([4]\)). In particular, \( 45_X \) has VEVs towards both \((15,1,1)\) and \((1,1,3)\) fragments if superpotential includes the terms \( 45^2_{15} 54 \) and \( 45_X 16_H \overline{16}_H \). As for \( 45_{BL} \) and \( 45_R \), in order to ensure the strict ‘zeroes’ in their VEVs, they should couple only to \( 54 \) but not to \( 16_H, \overline{16}_H \). The trilinear term \( 45_{BL} 45_R 45_X \) evades the unwanted Goldstone modes.
For the electroweak symmetry breaking and the quark and lepton mass generation we use a traditional Higgs supermultiplet \( 10 = \phi(1, 2, 2) + T(6, 1, 1) \). In order to maintain the gauge coupling unification, we assume that all VEVs \( V_{BL}, V_R \) and \( V_X \) are of the order of \( M_X \approx 10^{16} \text{ GeV} \), and below this scale SUSY \( SO(10) \) theory reduces to the MSSM with three fermion families \( f_i \) and a couple of the standard Higgs doublets \( h_{1,2} \) contained in \( \phi \). The field \( 45_{BL} \) serves for the solution of the doublet-triplet splitting problem through the “missing VEV” mechanism \[8\]. In this way the Higgs doublets \( h_{1,2} \) are kept light while their colour triplet partners contained in \( T(6,1,1) \) acquire the \( O(M_X) \) mass (otherwise they would cause unacceptably fast proton decay and also would affect the unification of gauge couplings). The VEVs of \( h_{1,2} \) arise radiatively after the SUSY breaking:

\[
\langle \phi \rangle = \begin{pmatrix}
  v_2 & 0 \\
  0 & v_1
\end{pmatrix}; \quad (v_1^2 + v_2^2)^{1/2} = v = 174 \text{ GeV}, \quad \frac{v_2}{v_1} = \tan \beta \quad (9)
\]

Let us assume that the direct Yukawa couplings \( 16^f 16^f 10 \) are forbidden by certain symmetry reasons and the \( 16^f \)'s get mass through the 'seesaw' mixing \[9\] with their heavy partners \( 16_R + \overline{16}_R \). The relevant terms in the superpotential are chosen as \[\]
\[
W_{fF} = \Gamma_{ij} 10 16^f_i 16^f_j + \frac{G_{ij}}{M} (10^F_i 45_{Rj})(45_{Rj} 16^f_j)
\]
\[
W_F = MQ_{ij}^f 16^F_i 16^F_j + Q_{ij}^f 16^F_i 45_X \overline{16}_j + \frac{Q_{ij}^F}{M} (16^F_i 45_X)(45_X \overline{16}_j)
\]

where \( M \gg M_X \) is some large (string?) scale, and \( \hat{\Gamma}, \hat{G}, \hat{Q}_{1,2,3} \) are the coupling constant matrices with \( O(1) \) elements. In what follows we do not specify any concrete texture, assuming only that \( \hat{\Gamma} \) and \( \hat{G} \) are arbitrary nondegenerate matrices and \( \hat{Q}_{1,2,3} \) are some rank-1 matrices. After substituting large VEVs the whole \( 9 \times 9 \) Yukawa matrix for the fermions of different charges gets the form (each entry is \( 3 \times 3 \) matrix in itself):

\[
\begin{pmatrix}
  f^c & F^c & \mathcal{F}^c \\
  0 & \hat{\Gamma}_\phi & 0 \\
  \hat{M}_R & \hat{M}_F & 0
\end{pmatrix}, \quad f = u, d, e, \nu \quad (11)
\]

where the \((2,1)\)-block \( \hat{M}_R = \varepsilon_R^2 \hat{G} M \) is the same for the fermions of all charges. The \((1,2)\) blocks are also the same: the matrix \( \hat{\Gamma} \) stands for the coupling of up-type and down-type fermions with the MSSM Higgses \( h_2 \) and \( h_1 \), respectively. The \((1,3)\)-block is vanishing since the VEV \( \langle 45_R \rangle \) has the \((1,1,3)\) direction, so that the \( \mathcal{F} \)-type fermions are irrelevant for the seesaw mass generation \[10\]. Thus, all information on the flavour structure

\[\]
\[3)\] We do not specify the symmetries leading to this pattern, which question deserves special consideration. The higher order operators cutoff by scale \( M \) can be induced by the exchanges of heavy (with masses \( \sim M \)) fermion superfields in \( 16 + \overline{16} \) representations, so that the combinations in brackets transform as effective 16 or \( \overline{16} \).

\[4)\] This leads to natural suppression of the dangerous \( d = 5 \) operators inducing the proton decay: since the \( f \) and \( \mathcal{F} \) states are unmixed, the colour triplets in \( T(6,1,1) \) can cause transitions of \( f \)’s only into the superheavy \( F \)’s. Thus, the \( LLLL \)-type operators \([qqql]_{\mathcal{F}}\) which bring the dominant contribution to the proton decay automatically vanish. As for the \( RRRR \) type operators \([w^uw^c\overline{d}^c\overline{c}]_F\), they occur due to the \( f^c - F^c \) mixing and have the usual strength. However, these are known to be more safe \[\].
is essentially contained in the matrices of $F$ fermions $\hat{M}_F = M\hat{Q}_F = M(\hat{Q}_1 + \epsilon_f \hat{Q}_2 + \epsilon_f^2 \hat{Q}_3)$, where $\epsilon_f \sim V_X/M$ are the small, generally complex parameters. Since $45_X$ has VEVs both in $(15,1,1)$ and $(1,1,3)$ directions, they have the form $\epsilon_{d,u} = \epsilon_{15} \pm \epsilon_3$, $\epsilon_{e,\nu} = -3\epsilon_{15} \pm \epsilon_3$. Therefore, only two of these four parameters are independent:

$$\epsilon_e = -\epsilon_d - 2\epsilon_u, \quad \epsilon_\nu = 2\epsilon_e + 3\epsilon_u$$ \hspace{1cm} (12)

After decoupling the heavy states in (11) our theory reduces to the MSSM with the Yukawa coupling matrices given by the following expression [5]:

$$(\hat{\lambda}_f \hat{\lambda}_j)^{-1} = (\hat{\Gamma}^\dagger)^{-1} \left[ 1 + \hat{M}_F^\dagger (\hat{M}_R \hat{M}_R^\dagger)^{-1} \hat{M}_F \right] \hat{\Gamma}^{-1} = (\hat{\Gamma} \hat{\Gamma}^\dagger)^{-1} + \frac{1}{\epsilon_R^2} (\hat{G}^{-1} \hat{Q}_R \hat{G}^{-1})^\dagger (\hat{G}^{-1} \hat{Q}_F \hat{G}^{-1})$$ \hspace{1cm} (13)

When $\hat{M}_R \gg \hat{M}_F$, this equation gives the obvious result $\hat{\lambda}_f = \hat{\Gamma}$. On the other hand, for $\hat{M}_R \ll \hat{M}_F$ it reduces to the “seesaw” formula $\hat{\lambda}_f = \hat{\Gamma} \hat{M}_F^{-1} \hat{M}_R$, so that we have

$$\hat{\lambda}_f^{-1} = \frac{1}{\epsilon_R^2} \hat{G}^{-1} \hat{Q}_F \hat{G}^{-1} = \frac{1}{\lambda}(\hat{P}_1 + \epsilon_f \hat{P}_2 + \epsilon_f^2 \hat{P}_3),$$ \hspace{1cm} (14)

where $\lambda = \frac{1}{N^2} \epsilon_R^2$ and $\hat{P}_n = \frac{1}{N} \hat{G}^{-1} \hat{Q}_n \hat{G}^{-1}$ are still rank-1 matrices. For definiteness, the normalization factor $N$ is chosen as the nonzero eigenvalue of the matrix $\hat{G}^{-1} \hat{Q}_1 \hat{G}^{-1}$.

The seesaw limit is certainly very good for all light states apart from $t$ quark: their Yukawa couplings are much smaller than 1, so that the first term in (13) can be safely neglected. However, as far as $\lambda_t \sim 1$ (or, in other words, $\epsilon_R \sim \epsilon_u \ll \epsilon_{d,e}$), for its evaluation one has to use the exact formula (13). Then the top genuine constant $\lambda_t$ is related to the ‘would-be’ Yukawa coupling $\lambda_t$ of the seesaw limit [5] as

$$\tilde{\lambda}_t = \frac{\lambda_t}{\sqrt{1 + (\lambda_t / \Gamma_c)^2}} < \lambda_t$$ \hspace{1cm} (15)

where $\Gamma_c$ is a certain combination of the constants in $\hat{\Gamma}$. For example, in the basis of $\hat{Q}_n$ having a form similar to (5), $\Gamma_i^2 = \sum |\Gamma_{3i}|^2$ ($i = 1, 2, 3$).

In what follows, we assume for simplicity that $\hat{P}_n$ are symmetric. This will not change essentially our results (see comment at the end of this section). Then the inverse Yukawa matrices at the GUT scale have the form (7) which, as was already noted, must be diagonalized by assuming $a^2 \epsilon_f \sim 1$. Thus, the Yukawa eigenvalues are

$$\lambda_{u,d,e} = \frac{\lambda}{|1 + a^2 \epsilon_{u,d,e}|}, \quad \lambda_{e,s,\mu} = \frac{\lambda |1 + a^2 \epsilon_{u,d,e}|}{|b^2 \epsilon_{u,d,e}|}, \quad \lambda_{t,b,\tau} = \frac{\lambda}{|z^2 \epsilon_{u,d,e}|}$$ \hspace{1cm} (16)

and for the CKM angles we obtain

$$s_{12} = \frac{|\epsilon_{dab}|}{|1 + \epsilon_{dab}|} = \sqrt{\frac{\lambda_d}{\lambda_s}} |\epsilon_{dab}|, \quad s_{23} = \frac{\lambda_u}{\lambda_d} \frac{|yz \epsilon_d|}{b^2 \epsilon_d}, \quad s_{13} = \frac{\lambda_d}{\lambda_s} |x z \epsilon_d^2|$$ \hspace{1cm} (17)

These Yukawa constants are linked to the physical fermion masses through the renormalization group (RG) equations. For the heavy quarks $t, b, c$ we take their running masses.
at $\mu = m_{t,b,c}$, while for the light quarks $u, d, s$ at $\mu = 1$ GeV. Then we have (see e.g. 10):

$$
m_u = \lambda_u \eta_u A_u B_t^3 v \sin \beta , \quad m_d = \lambda_d \eta_d A_d v \cos \beta , \quad m_c = \lambda_c \eta_c A_c v \cos \beta
$$

(18)

where the factors $A_f$ account for the gauge coupling induced running from the scale $M_X$ to the SUSY breaking scale $M_S \simeq m_t$, the factors $\eta_f$ encapsulate the QCD+QED running from $M_S$ down to $m_f$ (or to $\mu = 1$ GeV for the light quarks $u, d, s$), and $B_t$ includes the running induced by the top quark Yukawa coupling:

$$
B_t = \exp \left[ -\frac{1}{16\pi^2} \int_{\ln M_S}^{\ln M_X} \frac{\lambda_t^2(\mu)}{\mu} d(\ln \mu) \right]
$$

(19)

From (16) we immediately obtain the relations

$$
\sqrt{\frac{\lambda_b}{\lambda_t}} = \frac{\lambda_d \lambda_s}{\lambda_u \lambda_c} = \left| \frac{\varepsilon_u}{\varepsilon_d} \right| , \quad \sqrt{\frac{\lambda_b}{\lambda_t}} = \frac{\lambda_d \lambda_s}{\lambda_c \lambda_e} = \left| \frac{\varepsilon_e}{\varepsilon_d} \right| = 1 + 2\varepsilon_u \varepsilon_d
$$

(20)

When I get to the bottom I go back to the top: as far as $|\varepsilon_u/\varepsilon_d| = \sqrt{\lambda_b/\lambda_t} \ll 1$, we have the approximate relationships $\lambda_b \approx \lambda_t$ and $\lambda_d \lambda_s \approx \lambda_c \lambda_e$. Therefore, we are not loosing the understanding of $b - \tau$ unification in spite of naive expectation: it is more precise the more $t - b$ are split, and this is granted by the $SO(10)$ symmetry relation (12).

On the other hand, we know that $m_u/m_c = 207$, $m_s/m_d \approx 22$ and $s_{12} \approx \sqrt{22} - c'est \ la\ vie$. Then (14) implies that $|\varepsilon_d a^2| \approx 1$. In addition, the $\lambda_d \lambda_s = \lambda_c \lambda_e$ follows that $\lambda_d/\lambda_e = \lambda_d/\lambda_s = (m_d m_u/m_s m_e)^{1/2} \approx 3$. According to (16), for $|\varepsilon_d a^2| \approx 1$ such a splitting is indeed possible: owing to the relation $\varepsilon_d \approx -\varepsilon_e$, $|1 + \varepsilon_d a^2|$ and $|1 + \varepsilon_e a^2|$ can split to different sides from 1 by about a factor of 2. Then the order of magnitude difference between $\lambda_d/\lambda_e$ and $\lambda_s/\lambda_d$ can be naturally understood. In this way, owing to the numerical coincidence $(\lambda_e/\lambda_d)^2 \sim \varepsilon_u/\varepsilon_d \sim 0.1$, we reproduce the mixed behaviour (3) of leptons. Thus, splitting of the first family and large value of the Cabibbo angle (as compared to other mixing angles which have their natural size $s_{23} \sim \varepsilon_d$ and $s_{13} \sim \varepsilon_2^d$) have the same origin, namely $|\varepsilon_d a^2| \approx 1$ (7) Eqs. (16) also imply

$$
\frac{\lambda_u}{\lambda_d} = \frac{|1 + \varepsilon_d a^2|}{|1 + \varepsilon_u a^2|} \implies \frac{\lambda_u}{\lambda_d} \leq (1 + |\varepsilon_e a^2|) \left( \frac{m_u m_s}{m_d m_e} \right)^{1/2} \approx 0.6 - 0.7
$$

(21)

so that the ratio $m_u/m_d \approx (\lambda_u/\lambda_d) B_t^3 \tan \beta$ is less than 1 for small enough $\tan \beta$.

The detailed numerical study of our ansatz leads to more concrete results. We consider the masses of leptons and heavy quarks $c$ and $b$, the ratio $\zeta = m_s/m_d$ and the Cabibbo

5) One may question how to achieve $\varepsilon_d a^2 \approx 1$, if the Yukawa couplings are $O(1)$ and $\varepsilon_d \sim 1/20 - 1/30$ (see (3)). However, in (14) the Yukawa matrices $Q_n$ are "sandwiched" as $\tilde{P}_n = 1/8 G^{-1} Q_n G^{-1}$ so that $a, b \ldots$ are actually given by the Yukawa constant ratios. Thus, $a \sim 5$ could easily occur due to some spread in the Yukawa couplings, while the latter themselves are small enough to fulfill the perturbativity bound $G^2 N^4 / 4\pi < 1$. As we noted earlier, the mass and mixing pattern of the second and third families suggests that such an accidental enhancement does not happen for other entries in the matrix (3).
angle $s_{12}$ as input, and try to calculate other quantities. For definiteness we take $\alpha_3(M_Z) = 0.11$, $m_b = 4.4$ GeV and $m_c = 1.32$ GeV, and use for the RG running factors the results of ref. [10]. Our computational strategy is the following:

- Substituting the lepton masses in [18], we find $\lambda_c$, $\lambda_u$ and $\lambda_r$ in terms of $\tan\beta$. Analogously, by fixing the values $m_c$ and $m_b$ we find $\lambda_c$ and $\lambda_b$ in terms of $\tan\beta$ and $\tilde{\lambda}_t$. In particular, we have $\lambda_r/\lambda_b = R_{b/\tau}B_t m_r/m_b$, where $R_{b/\tau} = \eta_b A_d/\eta_r A_c \approx 3.1$ for $\alpha_3(M_Z) = 0.11$ [10]. Thus, $\lambda_b = \lambda_r$ is achieved when $\lambda_t \simeq 1.5 (B_t \approx 0.8)$. Then by running the second equation (20) from the GUT scale down to $\mu = 1$ GeV, we readily obtain

$$m_d m_s = R_{d/e}^{-1/2} R_{b/\tau}^{-1/2} B_t^{-1/2} m_e m_\mu \left( \frac{m_b}{m_\tau} \right)^{\frac{1}{2}} = 1100 B_t^{-1/2} \text{ MeV}^2$$

(22)

(notice the very weak dependence on $\tilde{\lambda}_t$), so that for a fixed value of $\zeta$ we get

$$m_d = 7 \cdot (22/\zeta)^{1/2} B_t^{-1/4} \text{ MeV}, \quad m_s = 155 \cdot (\zeta/22)^{1/2} B_t^{-1/4} \text{ MeV}$$

(23)

- For $\Gamma_c$ fixed, the first equation (20) determines $|\varepsilon_u/\varepsilon_d|$ as a function of $\tilde{\lambda}_t$ and $\tan\beta$. Then from the second equation also $\arg(\varepsilon_u/\varepsilon_d)$ can be found in terms of $\tilde{\lambda}_t$ and $\tan\beta$.

- The modulus $|\varepsilon_d a^2|$ is fixed by the value of the Cabibbo angle: $|\varepsilon_d a^2| \approx \zeta s_{12}^2$, whereas $\arg(\varepsilon_d a^2)$ can be found from the equation

$$\frac{\lambda_d}{\lambda_c} = \frac{|1 - \varepsilon_d a^2(1 + 2\varepsilon_u/\varepsilon_d)|}{|1 + \varepsilon_d a^2|} = \left( \frac{\zeta m_\mu}{m_\tau} \right)^{-1/2} \left( \frac{m_\tau}{m_b} R_{b/\tau} B_t \right)^{-1/4}$$

(24)

- In this way, the complex parameters $\varepsilon_{u,d,a}^2$ are all expressed in terms of as yet unknown $\tan\beta$ and $\tilde{\lambda}_t$. Then, using (21) and (18) we find the mass ratio $\rho = m_u/m_d$ as a function of $\tan\beta$ and $\tilde{\lambda}_t$. The isocurves of $\rho$ are shown in Fig. 1 (dotted).

- Once $\varepsilon_{u,d,a}^2$ are known, from (16) we find $\lambda_u = \lambda_c (1 + \varepsilon_d a^2)/(1 + \varepsilon_u a^2)$ and substitute it in the equation $\lambda_t/\lambda_r = (\lambda_u \lambda_c/\lambda_r \lambda_\mu)^2$. Then for fixed $\Gamma_c$ the latter becomes a relation which determines $\tan\beta$ as a function of $\tilde{\lambda}_t$ (see solid curves in Fig. 1). We would like to stress that the chosen values of $\alpha_3(M_Z)$, $m_b$ and $m_c$ are taken at their experimentally allowed extremes. The value of $\tan\beta$ decreases for smaller $m_b$, $m_c$ or larger $\alpha_3(M_Z)$, so that the solid contours actually mark the upper borders of allowed regions.

- Another relation between $\tilde{\lambda}_t$ and $\tan\beta$ emerges by fixing the top pole mass $M_t = m_t [1 + 4\alpha_3(m_t)/3\pi]$ (see dashed curves in Fig. 1). The flat behaviour of these curves for large $\tilde{\lambda}_t$ corresponds to infrared fixed regime when $\tilde{\lambda}_t B_t^6$ is practically independent of $\tilde{\lambda}_t$ and the top pole mass is essentially determined by $\tan\beta$: $M_t = \sin \beta \cdot 190 \text{ GeV}$ [13].

The results of numerical computations are shown in Fig. 1. We see that the constant $\Gamma_c$ which sets the seesaw ‘cutoff’ [13] should be quite close to the perturbativity bound in

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6) The controversy concerning the value of $\alpha_3(M_Z)$ is not resolved yet. The $\gamma$ sum rules analysis implies $\alpha_3(M_Z) = 0.109 \pm 0.001$ [11], whereas the SM global fits based on the LEP/SLD data lead to $\alpha_3(M_Z) = 0.127 \pm 0.005$ [12]. However, as it was argued in [13], the systematic error in the latter value, essentially determined by analyzing $\Gamma(Z \to \text{hadrons})$ can be largely underestimated. The gauge coupling unification, without taking into account the model-dependent threshold corrections, also requires $\alpha_3(M_Z) > 0.12$ [14]. In our SO(10) model, however, $\alpha_3(M_Z) = 0.11$ can be easily adopted due to the threshold corrections emerging e.g. due to large splitting in the third superheavy family $16_F + 16_F$. In particular, the mass of the weak isosinglet upper quark of this family is $\sim \varepsilon_d^2 M$, which is about 2 orders of magnitude below the masses of other fragments ($\sim \varepsilon_u^2 M < M_X$).
order to ensure the sufficiently large $M_t$. Fig. 1 shows that at the perturbativity border ($\Gamma_c = 3.3$), $M_t$ reaches the maximum when $\lambda_t \simeq 1.5$, close to the infrared fix-point \[15\] (we remind that in this case $\lambda_b = \lambda_t$). Namely, for $\Gamma_c = 3.3$ and $\zeta = 22$ the maximal top mass is $M_t^{\text{max}} = 160$ GeV \[16\] which corresponds to $\tan \beta = 1.5$ and $m_u/m_d = 0.5 - 0.6$. The latter values combined with $m_u/m_d = 22$ perfectly fit the famous $\rho - \zeta$ ellipse \[17\]. For smaller $\Gamma_c$ or smaller $\zeta$ the $M_t^{\text{max}}$ sharply decreases (e.g. for $\zeta = 19$ the maximal top mass can be at most 150 GeV). On the other hand, for $\zeta = 25$ we obtain $M_t^{\text{max}} = 165$ GeV, which corresponds to $\tan \beta = 1.7$ and $m_u/m_d = 0.6 - 0.8$ (see Fig. 1B). The latter values seem too large versus $m_u/m_d = 25$ \[18\].

Taking all these into account, we see that the preferable choice of the parameter region corresponds to $m_u/m_d \approx 22$, when $m_u/m_d = 0.5 - 0.6$ and $M_t \approx 160$ GeV ($\tan \beta \approx 1.5$), at the lower edge of the recent CDF result $M_t = 176 \pm 8 \pm 10$ GeV \[17\]. By taking all input parameters at their extremes and also neglecting the perturbativity constraint, the maximal value of $M_t$ in our model can be increased at most up to 170 GeV.

Let us conclude this section with the following comment. The symmetric form of the matrices $\hat{P}_n$ in \[14\] was imposed by hands. In fact, it can be ensured by introducing certain horizontal symmetries, or by extending the gauge symmetry e.g. to $SO(10) \otimes SO(12)$, with 16’s belonging to $SO(10)$ and 16$_F + \overline{16}_F$’s contained in 32-plets of $SO(12)$ \[18\]. The latter case implies that $\hat{Q}_n$ are symmetric and $\hat{G} = \Gamma^T$. On the other hand, for non-symmetric $\hat{P}_n$ instead of \[17\] we obtain the equation $s_{12} s'_{12} = \lambda_d / \lambda_u |\varepsilon_d a^2|$, where $s'_{12}$ stands for the “Cabibbo” mixing of the right-handed states. If $|s_{12}| < |s'_{12}| \approx (m_d/m_s)^{1/2}$, then this equation implies $|\varepsilon_d a^2| < 1$, so that $M_t^{\text{max}}$ becomes smaller. On the contrary, the upper bound on $M_t$ can be lifted if $s'_{12} \gg (m_d/m_s)^{1/2}$. However, such an enhancement of $s'_{12}$ above the naively expected size $O(\varepsilon_d)$ implies substantial fine tuning.

3. Discussion and Outlook

We have considered the SUSY $SO(10)$ model with small $\tan \beta$, in which the flavour structure arises in a mass matrix $\hat{M}_F$ of the superheavy $F$ fermions and is transferred to the light fermions in an inverse way by means of the universal seesaw. The largest eigenvalue of $\hat{M}_F$ is given by the $SO(10)$ invariant mass $M \gg M_t$ and thus is unsplit. The lighter eigenvalues, respectively of the order of $M_X$ and $M_X^2/M$, arise due to the couplings with the Higgs $45_X$ and are thereby split.

As a result, the quark and lepton Yukawa matrices have the inverse hierarchy form \[3\]. The hierarchy of Yukawa constants is described by the approximate scaling low $\lambda_{fi} \sim \lambda_e^{1-i}$, where $i = 1, 2, 3$ is the family number and the expansion parameters $\varepsilon_{e,u,d} \sim M_X/M \sim 10^{-1} - 10^{-2}$ are related by the $SO(10)$ symmetry as $\varepsilon_e = - (\varepsilon_d + 2 \varepsilon_u)$. As far as $M_X \sim 10^{16}$ GeV, the above estimate points to the scale $M \sim 10^{17} - 10^{18}$ GeV, close to the string scale. The $\lambda_b \approx \lambda_h$ unification at the GUT scale implies that $\varepsilon_u \ll \varepsilon_e \approx - \varepsilon_e$. By this reason, the ‘up-down’ splitting is quickly growing with the family number: $\lambda_c/\lambda_s \sim |\varepsilon_d/\varepsilon_u|$ and $\lambda_\tau/\lambda_b \sim |\varepsilon_d/\varepsilon_u|^2$. The first family plays a role of the Yukawa unification

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7) By relaxing the perturbativity bound, $M_t^{\text{max}}$ can increase only by few GeV’s: see solid curve for $\Gamma_c = 6.6$ in Fig. 1A. The flat behaviour for large $\hat{\lambda}_t$ can be easily understood. According to \[15\], for $\Gamma_c > 1$ we have $\hat{\lambda}_t = \lambda_t$, while $\lambda_c \propto m_c / B_t^0$ and the ratio $\lambda_u/\lambda_c \simeq 2$ very weakly depends on $\lambda_t$. Then the equation $\lambda_t/\lambda_c = (\lambda_u/\lambda_c/\varepsilon \lambda_u)^2$ in fact determines the combination $\lambda_t B_t^0 \tan^2 \beta \cos \beta \propto M_t \tan \beta$ which is practically independent of $\lambda_t$ for enough large values of the latter.
point \((\lambda_{e,d,u} \sim \lambda)\), with its splitting understood by the same mechanism that enhances the Cabibbo angle up to the value \(s_{12} \simeq \sqrt{m_d/m_s}\). The other mixing angles stay much smaller (see (17)). For the light quarks we have obtained \(m_s \simeq 150\) MeV, \(m_d \simeq 7\) MeV and \(m_u/m_d \simeq 0.5 - 0.7\). The upper limit on the top mass in our scheme is about 165 GeV, which can be marginally enhanced up to 170 GeV. On the other hand, the lower bound \(M_t \geq 160\) GeV implies \(m_s/m_d \geq 22\). It is worth to mention that small values \(\tan \beta = 1.4 - 1.7\) are of phenomenological interest in testing the MSSM Higgs sector at new colliders [19].

We find it amusing that the inverse hierarchy ansatz implemented in SUSY SO(10) theory reproduces the fermion mass and mixing pattern in a very natural and economical way. Our approach is rather general, with the key assumption that the fermion masses are induced via seesaw mechanism, by means of the couplings (10) with constants \(\tilde{Q}_n\) being rank-1 matrices. We have not specified the concrete symmetry reasons that could support our ansatz. Various possibilities can be envisaged, including normal or R-type discrete and abelian symmetries. (For example, the combination of such a symmetries fixing the proper operator structure for the ansatz [4] have been found recently [2].) Notice, that in contrast to the known predictive frameworks [1, 2] we did not exploit any particular zero texture: except that \(\tilde{Q}'s\) are assumed to be the rank-1 matrices, the Yukawa constants are left completely general. By this reason, the amount of exact predictions in our scheme is less than e.g. in [4]. Clearly, a number of free parameters can be reduced by imposing a proper horizontal symmetry which can restrict the Yukawa matrices at the needed degree and thus enhance predictivity. Last but not least, a clever horizontal symmetry seems to be needed also for evading a potential problem of too large rates for the lepton flavour violating processes [21], which in our scheme should be induced due to the presence of large Yukawa constants above the GUT scale.

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Figure Caption

Fig. 1. Solid curves show the prediction of our ansatz for the $\tilde{\lambda}_t - \tan \beta$ correlation for given $m_u/m_d$ and $\Gamma_c$ ($\zeta = 22 \text{ and } \Gamma_c = 1.5, 2.2, 3.3, 6.6 (\text{Fig. A})$, and $\zeta = 25 \text{ and } \Gamma_c = 3.3 \text{ (Fig. B)})$. Other input parameters are fixed as $\alpha_3(M_Z) = 0.11$, $m_b = 4.4 \text{ GeV}$ and $m_c = 1.32 \text{ GeV}$. Isocurves for fixed top mass are dashed: $M_t = 150, 160, 170$ and $180 \text{ (in GeVs)}$. The isocurves corresponding to different values of $m_u/m_d$ are also shown: $\rho = 0.4, 0.5, 0.6, 0.7$ and $0.8 \text{ (dotted)}$. 
This figure "fig1-1.png" is available in "png" format from:

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