Geometric Scattering Monodromy

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Abstract
In this paper we give geometric conditions so that the integral mapping of a Liouville integrable Hamiltonian system with a focus-focus equilibrium point has scattering monodromy. Using a complex version of the Morse lemma, we show that scattering monodromy is the same as the scattering monodromy of the standard focus-focus system.

Keywords Scattering monodromy · Complex Morse lemma

Mathematics Subject Classification 70H06

Introduction
In [1] the hyperbolic oscillator integrable Hamiltonian system $(u, v, \mathbb{R}^4, \Omega = d\xi_1 \wedge d\xi_2 - d\eta_1 \wedge d\eta_2)$, where

$$u : \mathbb{R}^4 \to \mathbb{R} : (\xi, \eta) \mapsto \xi_1 \eta_1 + \xi_2 \eta_2 = h$$

and

$$v : \mathbb{R}^4 \to \mathbb{R} : (\xi, \eta) \mapsto \frac{1}{2} (\xi_1^2 + \xi_2^2 - \eta_1^2 - \eta_2^2) = \ell$$

was shown to have scattering monodromy. Geometrically this means that a motion in $\mathbb{R}^4$ of the hyperbolic oscillator of energy $h$ and angular momentum $\ell$ projects onto a branch of a hyperbola in the $(\xi_1, \xi_2)$ plane, whose outgoing asymptote forms an angle $\tan^{-1} \frac{h}{\ell}$ with its incoming asymptote. This angle is called the scattering angle of the hyperbolic motion. As $(h, \ell)$ traverses a circle in the energy-momentum plane
centered at the origin, the scattering angle of a hyperbolic motion, which starts at a point in the image of a section of the bundle formed by the integral map, increases by $2\pi$. This is the scattering monodromy of the hyperbolic oscillator system.

In [3] it was shown that the quantum Kepler problem has scattering monodromy. The references in [5] list all the contributors to the proof of the toral geometric monodromy theorem. In [4] the relation of scattering monodromy to the geometric monodromy of a toral fibration was treated using rotation forms. However, a geometric scattering monodromy theorem was not formulated. This paper remedies this omission.

Our formulation of the geometric scattering monodromy theorem follows that of the geometric (toral) monodromy theorem given in [5]. Our proof follows the line of argument for the proof of the toral geometric monodromy theorem given in [2] with all reasoning involving compactness being avoided. We use a complex version of the Morse lemma, inspired by [6], to reduce the proof of the geometric scattering monodromy theorem to the computation of the scattering monodromy of the complexified standard focus-focus system $(q_1, q_2, \mathbb{R}^4, \omega = dx \wedge dp_x + dy \wedge dp_y)$, where

\[ q_1 : \mathbb{R}^4 \to \mathbb{R} : (x, y, p_x, p_y) \mapsto xp_x + yp_y \]  
\[ \text{and} \]  
\[ q_2 : \mathbb{R}^4 \to \mathbb{R} : (x, y, p_x, p_y) \mapsto xp_y - yp_x. \]

We now state the geometric scattering monodromy theorem.

The origin 0 of $\mathbb{R}^4$ is a focus-focus equilibrium point of the Liouville integrable system $(h_1, h_2, \mathbb{R}^4, \omega = dx \wedge dp_x + dy \wedge dp_y)$ if and only if

1. The complete vector fields $X_{h_1}$ and $X_{h_2}$ vanish at 0, that is, 0 is an equilibrium point of $X_{h_1}$ and $X_{h_2}$.

2. The space spanned by the linearized Hamiltonian vector fields $DX_{h_1}(0)$ and $DX_{h_2}(0)$ is conjugate by a real linear symplectic mapping of $(\mathbb{R}^4, \omega)$ into itself to the Cartan subalgebra of $\mathfrak{sp}(4, \mathbb{R})$ spanned by $X_{q_1}$ and $X_{q_2}$, where $q_1 = xp_x + yp_y$ and $q_2 = xp_y - yp_x$.

From point 2 we may assume that $h_i = q_i + r_i$ for $i = 1, 2$, where $r_i$ is a smooth function on $\mathbb{R}^4$, which is flat to 2nd order at 0, that is, $r_i \in O(2)$.

The remainder of this paper is devoted to proving

**Theorem (Geometric Scattering Monodromy).** Let $(h_1, h_2, \mathbb{R}^4, \omega)$ be a Liouville integrable system with a focus-focus equilibrium point at $0 \in \mathbb{R}^4$. Consider the integral map

\[ F : \mathbb{R}^4 \to \mathbb{R}^2 : z \mapsto (h_1(z), h_2(z)) = (c_1, c_2), \]  

where $F(0) = (0, 0)$. Suppose that $F$ has the following properties.

1. There is an open neighborhood $U$ of the origin $(0, 0)$ in $\mathbb{R}^2$ such that $(0, 0)$ is the only critical value of the integral map $F$ in $U$. 

2. For every \( c \in U^x = U \setminus \{(0, 0)\} \) the fiber \( F^{-1}(c) \) is noncompact and connected. The fibration \( \rho = F | F^{-1}(U^x) : F^{-1}(U^x) \to U^x \) is trivial.

3. The singular fiber \( F^{-1}(0, 0) \) is noncompact and connected. For every \( z \in F^{-1}(0, 0) \setminus \{0\} \) the rank of \( DF(z) \) is 2. Then the fibration \( \hat{\rho} = F | F^{-1}(C) : F^{-1}(C) \to C \) over the smooth circle \( C \) in \( U^x \) encircling \( (0, 0) \) is trivial. So \( F^{-1}(C) = C \times (S^1 \times \mathbb{R}) \). There is an integral \( I \) of the vector field \( X_{h_1} \) whose flow \( \varphi^I_t \) is periodic on \( F^{-1}(C) \). Also there is a connection 1-form \( \theta \) on \( F^{-1}(C) \) which is invariant under the flow \( \varphi^I_t \) of \( X_I \). For each \( c \in C \subseteq U^x \), the curve \( t \mapsto \Gamma_{\sigma(c)}(t) = \varphi^h_{t1}(\sigma(c)), \) where \( \sigma \) is a global section of the bundle \( \hat{\rho} \), has scattering phase \( \Theta(c) = \arg c + \int_{\Gamma_{\sigma(c)}} \theta. \) The degree of the map \( \Theta : U^x \to S^1 : c \mapsto \Theta(c) \) is 1. This is the scattering monodromy of the focus-focus system.

The main idea of the proof is to construct a smooth local isotopy from the focus-focus system to the standard focus-focus system, which pulls back the connection 1-form to a connection 1-form so that we can compute the scattering phase of the curve \( \Gamma_{\sigma(c)} \).

1 The Singular Fiber

In this section we show that the singular fiber \( F^{-1}(0, 0) \) of the integral mapping \( F(3) \) is homeomorphic to a once pinched cylinder.

Let \( \varphi^h_{t1} \) and \( \varphi^h_{u2} \) be the flows of the vector fields \( X_{h_1} \) and \( X_{h_2} \), respectively. The hyperbolicity of \( X_{h_1} \) at 0 implies that there is an open ball \( B \) in \( \mathbb{R}^4 \) (with the Euclidean inner product) centered at 0 having radius \( r \) such that the local stable \( W_s^B(0) \) and unstable \( W_u^B(0) \) manifolds of 0 in \( B \) are smooth connected manifolds, whose tangent space at 0 is the \( \mp 1 \) eigenspace of the linear mapping \( X_{\theta_1} \), respectively. The global stable manifold \( W_s(0) \) is \( \bigcup_{t>0} \varphi^h_{t1}(W_s^B(0)) \); while the global unstable manifold \( W_u(0) \) is \( \bigcup_{t>0} \varphi^h_{t2}(W_u^B(0)) \). When \( w \in W_{s,u}(0) \) as \( t \to \infty, -\infty \) we have \( F(w) = F(\varphi^h_{t1}(w)) \to F(0) = (0, 0) \). Thus \( W_{s,u}(0) \subseteq F^{-1}(0, 0) \).

**Claim 1.1** \( F^{-1}(0, 0) \setminus \{0\} = (W_s(0) \setminus \{0\}) \coprod (W_u(0) \setminus \{0\}) \).

**Proof** Since \( F^{-1}(0, 0) \) is locally invariant under the flow \( \varphi^h_{t1} \), it is globally invariant. Thus \( \varphi^h_{t1} | F^{-1}(0, 0) \) is defined for every \( t \in \mathbb{R} \). Because of hypothesis 3, the set \( F^{-1}(0, 0)^x = F^{-1}(0, 0) \setminus \{0\} \) is a smooth 2-dimensional submanifold of \( \mathbb{R}^4 \). So \( \varphi^h_{t1}(W_{s,u}^B(0)) \setminus \{0\} \) is an open subset of \( F^{-1}(0, 0)^x \). Thus \( W_{s,u}(0)^x = W_{s,u}(0) \setminus \{0\} = \bigcup_{t\ge0} \varphi^h_{t1}(W_{s,u}^B(0) \setminus \{0\}) \). The set \( F^{-1}(0, 0) \) is invariant under the flow \( \varphi^h_{t2} \). Because \( (h_1, h_2) = 0 \), the flows \( \varphi^h_{t1} \) and \( \varphi^h_{t2} \) commute. Thus \( F^{-1}(0, 0) \) is invariant under the \( \mathbb{R}^2 \)-action

\[
\Xi : \mathbb{R}^2 \times \mathbb{R}^4 \to \mathbb{R}^4 : (t, v, w) \mapsto (\varphi^h_{t1} \circ \varphi^h_{t2})(w).
\]

So the \( \mathbb{R}^2 \)-action \( \Psi_{(t, v)} = \Xi_{(t, v)} | F^{-1}(0, 0) \) on \( F^{-1}(0, 0) \) is defined. Because \( (0, 0) \in \mathbb{R}^2 \) is an isolated critical value of \( F \) by hypothesis 1, it follows that \( 0 \in \mathbb{R}^4 \) is an isolated
equilibrium point of $X_{h_1}$ and $X_{h_2}$. Thus 0 is an isolated fixed point of the $\mathbb{R}^2$-action $\Psi_{(t,v)}$ on $F^{-1}(0,0)$. If $w \in W_{s,u}(0)$, then $\varphi_t^{h_1}(\varphi_v^{h_2}(w)) = \varphi_v^{h_2}(\varphi_t^{h_1}(w)) \to \varphi_v^{h_2}(0) = 0$ when $t \to \infty$, $-\infty$. So $W_{s,u}(0)$ is invariant under the flow $\varphi_v^{h_2}$. Because 0 is a fixed point of the $\mathbb{R}^2$-action $\Psi_{(t,v)}$ on $F^{-1}(0,0)$, it follows that $W_{s,u}(0) \times = W_{s,u}(0) \setminus \{0\}$ is invariant under both flows $\varphi_t^{h_1}$ and $\varphi_v^{h_2}$. By hypothesis 2 the vector fields $X_{h_1}$ and $X_{h_2}$ are linearly independent at each point of $F^{-1}(0,0)^\times$. Consequently, every orbit $O$ of the $\mathbb{R}^2$-action $\Psi_{(t,v)}$ on $F^{-1}(0,0)^\times$ is open. Because the complement of $O$ in a connected component of $F^{-1}(0,0)^\times$ is the union of other $\mathbb{R}^2$ orbits of $\Psi_{(t,v)}$, it is also open. Thus $O$ is a connected component of $F^{-1}(0,0)^\times$. A similar argument shows that $W_{s,u}(0)^\times$ is an $\mathbb{R}^2$-orbit in $F^{-1}(0,0)^\times$. The orbit $O$ is open and closed in $F^{-1}(0,0)^\times$, which implies that it is open in $F^{-1}(0,0) = F^{-1}(0,0)^\times \cup \{0\}$. If 0 is not in the closure of $O$ in $\mathbb{R}^4$, then $O$ is closed in $F^{-1}(0,0)$. Hence $O$ is a connected component of $F^{-1}(0,0)$. But $0 \in F^{-1}(0,0)$, which is a contradiction. So $O \cup \{0\}$ is a closed subset of $\mathbb{R}^4$.

Thus 0 is the unique limit point in $\mathbb{R}^4 \setminus O$ of the $\mathbb{R}^2$-orbit $O$. For any $O$ in $F^{-1}(0,0)^\times$, we know that the closure of $O$ in $\mathbb{R}^4$ is $O \cup \{0\}$. In particular, this holds when $O = W_{s,u}(0)^\times$. If $O$ is an orbit of the $\mathbb{R}^2$-action $\Psi_{(t,v)}$ on $F^{-1}(0,0)$ and $w \in O$, then the mapping $(t,v) \mapsto \varphi_t^{h_1}(\varphi_v^{h_2}(w))$ induces a diffeomorphism of $\mathbb{R}^2/J_w$ onto $O$, where $J_w = \{(t,v) \in \mathbb{R}^2 \mid \varphi_t^{h_1}(\varphi_v^{h_2}(w)) = w\}$ is the isotropy group of $w$. $J_w$ is an additive subgroup of $\mathbb{R}^2$, which does not depend on $w$, because $O$ is connected. Therefore we will write $J_O$ instead of $J_w$. Suppose that $O$ is an $\mathbb{R}^2$-orbit of the action $\Psi_{(t,v)}$ on $F^{-1}(0,0)^\times$ and that $J_O \cap (\mathbb{R} \times \{0\}) \neq \emptyset$. Then the flow $\varphi_t^{h_1}$ of $X_{h_1}$ would be periodic with period $T > 0$. Because periodic integral curves of $X_{h_1}$, which lie in $O$ and start near 0 leave a fixed neighborhood of 0, have an arbitrarily large period, we deduce that a periodic solution of $X_{h_1}$, which starts near 0, must stay close to 0. Because $X_{h_1}$ is hyperbolic at 0 it does not have any periodic solutions which remain close to 0 other than 0. This is a contradiction. So $J_O \cap (\mathbb{R} \times \{0\}) = \emptyset$.

Combined with the fact that 0 is the only limit point in $\mathbb{R}^4 \setminus O$ of $O$ and that $O$ is contained in a connected component of $F^{-1}(0,0)$ of $\mathbb{R}^4$, it follows that for every $w \in O$ we have $\varphi_t^{h_1}(w) \to 0$ as $t \to \infty$ or as $t \to -\infty$. In other words, $w \in W_s(0)^\times$ or $W_u(0)^\times$. Because $W_{s,u}(0)^\times$ are $\mathbb{R}^2$-orbits of the action $\Psi_{(t,v)}$ on $F^{-1}(0,0)^\times$, it follows that $F^{-1}(0,0)^\times = W_s(0)^\times \sqcup W_u(0)^\times$. Here the notation $\sqcup$ means disjoint union. So $F^{-1}(0,0) = W_s(0) \cup W_u(0)$ and the connected components of $F^{-1}(0,0)^\times$ are $W_{s,u}(0)^\times$.

We now prove

**Claim 1.2** In a suitable open neighborhood of 0 in $(\mathbb{R}^4, \omega = -d\alpha = -d(p_x dx + p_y dy))$ there is a Hamiltonian function $I$, which equals $h_2 + O(2)$, whose associated Hamiltonian vector field $X_I$ has a periodic flow $\varphi_t^I$, and $I$ Poisson commutes with the Hamiltonians $h_1$ and $h_2$.

To construct the function $I$ we prove

**Lemma 1.3** There is a local diffeomorphism $\tilde{\Phi}$ of $\mathbb{R}^4$, which fixes the origin, is near the identity, and is isotopic to the identity map, such that $\tilde{\Phi}^* h_i = q_i$ for $i = 1, 2$. 


Proof We will use a complex version of the Morse lemma, which is proved in the appendix, to construct the desired local diffeomorphism. Introduce complex coordinates \((z_1, z_2) = (x - iy, px + ipy)\) on \(\mathbb{R}^4\). Then

\[ t : \mathbb{R}^4 \to \mathbb{C}^2 : (x, y, px, py) \mapsto (z_1, z_2) \]

is an invertible real linear mapping. The integral map

\[ F : \mathbb{R}^4 \to \mathbb{R}^2 : (x, y, px, py) \mapsto (xp_x + ypy + r_1(x, y, px, py), xp_y - ypx + r_2(x, y, px, py)) \]

becomes the differentiable function

\[ \mathcal{H} : \mathbb{C}^2 \to \mathbb{C} : (z_1, z_2) \mapsto z_1z_2 + R(z_1, z_2), \tag{5} \]

where \(R(z_1, z_2) = (r_1 + ir_2)(t^{-1}(z_1, z_2))\). Because \(r_i\) is flat to second order at 0 for \(i = 1, 2\), the function \(R\) is flat to second order at \((0, 0)\). We have \(\mathcal{H} = j \circ F \circ t^{-1}\), where \(j : \mathbb{R}^2 \to \mathbb{C} : (x, y) \mapsto x + iy\). Check: \((F \circ t^{-1})(z_1, z_2) = F(x, y, px, py)\).

So

\[ j((F \circ t^{-1})(z_1, z_2)) = \left((xp_x + ypy) + i(xp_y - ypx) + (r_1 + ir_2)(x, y, px, py)\right) = z_1z_2 + R(z_1, z_2). \]

Since \(D^2\mathcal{H}(0, 0) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\) is invertible, \((0, 0)\) is a nondegenerate critical point of \(\mathcal{H}\). By the complex Morse lemma, there is an open neighborhood \(U\) of \((0, 0)\) in \(\mathbb{C}^2\) and a complex diffeomorphism \(\varphi^U_1 : U \to U\), which fixes \((0, 0)\), is near \(id_U\), and is isotopic to \(id_U\), such that for every \((z_1, z_2) \in U\)

\[ (\mathcal{H} \circ \varphi^U_1)(z_1, z_2) = \frac{1}{2} D^2\mathcal{H}(0, 0)(z_1, z_2) = \tilde{\mathcal{H}}(z_1, z_2), \]

where \(\tilde{\mathcal{H}} : U \subseteq \mathbb{C}^2 \to \mathbb{C} : (z_1, z_2) \mapsto z_1z_2\). In real terms \(\tilde{\mathcal{H}}\) is the integral map

\[ \tilde{F} = j^{-1} \circ \tilde{\mathcal{H}} \circ t, \]

where

\[ \tilde{F} : \tilde{U} = t^{-1}(U) \subseteq \mathbb{R}^4 \to \mathbb{R}^2 : (x, y, px, py) \mapsto (xp_x + ypy, xp_y - ypx) = (q_1, q_2), \tag{6} \]

is the integral map of the standard focus-focus system \((q_1, q_2, \mathbb{R}^4, \omega)\). Also in real terms, the complex local diffeomorphism \(\varphi^U_1\) corresponds to the real local diffeomorphism \(\Phi = t^{-1} \circ \varphi^U_1 \circ t\) of \(\tilde{U}\) into itself, which fixes \(0\), is near the identity, that is, \(\Phi = id_{\tilde{U}} + O(2)\), and is isotopic to the identity. Since \(\tilde{F} = j^{-1} \circ (\mathcal{H} \circ \varphi^U_1) \circ t = F \circ \Phi\), we obtain \(\Phi^*h_i = q_i\) for \(i = 1, 2\). Warning: \(\Phi\) is not a symplectic diffeomorphism of \((\tilde{U}, \omega|\tilde{U})\) into itself. \(\square\)

We now begin the construction of the function \(I\) in claim 1.2. Let \(B\) be a ball of radius \(r\) in \(\mathbb{R}^4\) centered at 0, which is contained in the open set \(\tilde{U}\). Let \(Y\) be the vector
field \( \tilde{\Phi}^* X_{q_2} \) whose flow is \( \psi_s = \tilde{\Phi}^{-1} \circ \varphi_{q_2}^s \circ \tilde{\Phi} \). Hence the integral curve \( \Gamma_w \) of \( Y \) starting at \( w = \tilde{\Phi}^{-1}(w') \in B \setminus \{0\} \) is periodic of period \( 2\pi \), because

\[
\Gamma_w(s) = \psi_s(w) = \tilde{\Phi}^{-1}(\varphi_{q_2}^s(\tilde{\Phi}(w))) = \tilde{\Phi}^{-1}(\gamma_{w'}(s)),
\]

where \( \gamma_{w'} \) is an integral curve of \( X_{q_2} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} - p_y \frac{\partial}{\partial p_x} + p_x \frac{\partial}{\partial p_y} \) starting at \( w' \neq 0 \), which is periodic of period \( 2\pi \). Since

\[
L_y h_i = L_{\tilde{\Phi}^* X_{q_2}} h_i = \tilde{\Phi}^* (L_{X_{q_2}} (\tilde{\Phi}^{-1})^* h_i) = \tilde{\Phi}^* (L_{X_{q_2}} q_i) = 0,
\]

the flow \( \psi_s \) of \( Y \) preserves the level sets of the integral map \( F(3) \).

For \( w \in B \setminus \{0\} \) let

\[
I : B \setminus \{0\} \subseteq \mathbb{R}^4 \to \mathbb{R} : w \mapsto I(w) = \frac{1}{2\pi} \int_{\Gamma_w} \alpha.
\]  

(7)

Then \( I = \tilde{\Phi}^* K \), where

\[
K : \tilde{\Phi}(B) \setminus \{0\} \subseteq \mathbb{R}^4 \to \mathbb{R} : w' \mapsto \frac{1}{2\pi} \int_{\gamma_{w'}} (\tilde{\Phi}^{-1})^* \alpha.
\]  

(8)

**Proof** We compute

\[
I(w) = \frac{1}{2\pi} \int_{\Gamma_w} \alpha = \frac{1}{2\pi} \int_0^{2\pi} \left( \alpha \bigg| \frac{d\psi_t}{dt} \right)(\Gamma_w(t)) \, dt
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \langle \alpha(\psi_t(w)) \big| Y(\psi_t(w)) \rangle \, dt
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \left( \alpha(\tilde{\Phi}^{-1}(\varphi_{q_2}^t(w'))) \big| T\tilde{\Phi} X_{q_2}(\varphi_{q_2}^t(w')) \right) \, dt, \text{ since } Y = \tilde{\Phi}^* X_{q_2}
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} (\tilde{\Phi}^{-1})^* \alpha \big| X_{q_2}(\varphi_{q_2}^t(w'))) \, dt
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \left( (\tilde{\Phi}^{-1})^* \alpha \bigg| \frac{d\varphi_{q_2}^t}{dt} \right)(w') \, dt
\]

\[
= \frac{1}{2\pi} \int_{\gamma_{w'}} (\tilde{\Phi}^{-1})^* \alpha = K(w') = K(\tilde{\Phi}(w)).
\]

\[ \square \]

Next we show that \( I \) (7) is smooth near 0.

**Proof** Let \( z_1 = x - iy \) and \( z_2 = p_x + ip_y \). Then \( q_2 = \text{Im} \, z_1 z_2 \) and the flow \( \varphi_{q_2}^t \) of \( X_{q_2} \) is \((t, (z_1, z_2)) \mapsto (e^{it} z_1, e^{-it} z_2) \). Let \( D = \{ \zeta \in \mathbb{C} \mid |\zeta| \leq 1 \} \) with boundary \( \partial D = S^1 = \{ \zeta \in \mathbb{C} \mid |\zeta| = 1 \} \). Let

\[
k : D \times \mathbb{C}^2 \to \mathbb{C}^2 : (\zeta, (z_1, z_2)) \mapsto (\zeta z_1, \zeta^{-1} z_2)
\]
with \( k_z : D \to \mathbb{C}^2 : \zeta \mapsto (\zeta z_1, \zeta^{-1} z_2) \). Using Stokes’ theorem we have

\[
K(\zeta^{-1} z) = \int_{\partial D} k_z^*((\tilde{\Phi}^{-1})^*\alpha) = -\int_D k_z^*((\tilde{\Phi}^{-1}(\omega))).
\]  (9)

Since \( D \) is compact and \( \omega, k_z \) are smooth, it follows that \( K \) is smooth near 0. Thus \( I = \tilde{\Phi}^*K \) is smooth near 0. \( \square \)

**Claim 1.4** The function \( I (7) \) has the following properties.

1. The function \( I \) Poisson commutes with \( h_i \) for \( i = 1, 2 \) on \( F^{-1}(c) \), where \( c \) is a regular value of the integral map \( F (3) \).
2. For all values of \( c \) close to but not equal to 0 and for all \( w \in F^{-1}(c) \), the tangent vectors \( X_I(w) \) and \( X_{h_1}(w) \) to \( F^{-1}(c) \) at \( w \) are linearly independent.
3. For all \( c \) close to but not equal to 0, the flow \( \varphi^t_I \) of \( X_I \) on \( F^{-1}(c) \) is periodic of period \( T_c \).

**Proof** 1. We compute. \( \{ I, h_i \} = L_{X_{h_i}} I = \int_{\Gamma_w} L_{X_{h_i}} \alpha \), because we can move \( \Gamma_w \) by a homotopy in \( F^{-1}(c) \) without changing the integral, the new integral does not depend on \( w \). Thus we can take the Lie derivative under the integral sign. But

\[
\int_{\Gamma_w} L_{X_{h_i}} \alpha = \int_{\Gamma_w} X_{h_i} \underbrace{\alpha}_{\underbrace{\text{d} \alpha}_{\underbrace{1}} + \underbrace{\text{d}(X_{h_i} \alpha)}_{\underbrace{\text{d}(-h_i + X_{h_i} \alpha)}_{\underbrace{0}}}} = 0, \text{ since } \Gamma_w \text{is a closed curve.}
\]  (10)

2. Since \( \tilde{\Phi} = \text{id}_U + \mathcal{O}(1)^2 \), we get \( \tilde{\Phi}^{-1} = \text{id}_U + \mathcal{O}(1)^2 \). So \((\tilde{\Phi}^{-1})^*\alpha = \alpha + \mathcal{O}(1)\), which gives

\[
K(w) = \frac{1}{2\pi} \int_{\gamma_w} (\tilde{\Phi}^{-1})^*\alpha = \left( \frac{1}{2\pi} \int_{\gamma_w} \alpha \right) + \mathcal{O}(2) = q_2 + \mathcal{O}(2).
\]  (11)

Therefore

\[
I = \tilde{\Phi}^* K = \tilde{\Phi}^* q_2 + \mathcal{O}(2) = h_2 + \mathcal{O}(2).
\]  (12)

Since the vector fields \( X_{h_1} \) and \( X_{h_2} \) are linearly independent on \( F^{-1}(c) \) for \( c \) near, but not at the origin, the vector fields \( X_I \) and \( X_{h_1} \) are also.
3. For all \( c \in \mathbb{R}^2 \) near but not at 0, there are smooth functions \( a(c) \) and \( b(c) \) such that

\[
X_I = a(c) X_{h_1} + b(c) X_{h_2}
\]
on \( F^{-1}(c) \). Thus \( \varphi^t_I = \varphi^t_{a(c)} \circ \varphi^t_{b(c)} \) is defined for all \( t \in \mathbb{R} \), since the vector fields \( X_{h_i} \) for \( i = 1, 2 \) are complete. Hence the vector field \( X_I \) on \( F^{-1}(c) \) is complete. Since \( \tilde{\Phi}^* I = K \), the vector field \( \tilde{X}_K \) on \( (\mathbb{R}^4, \tilde{w} = \tilde{\Phi}^* \omega) \), which equals \( \tilde{\Phi}^* X_I \), is complete. From \( 0 = \{ I, h_i \} = \{ \tilde{\Phi}^* K, \tilde{\Phi}^* q_i \} = \tilde{\Phi}^* \{ \{ K, q_i \} \} \), where \( \{ \{ , \} \} \) is the Poisson
bracket on \((R^4, \omega)\), it follows that the vector fields \(\tilde{X}_K\) and \(\tilde{X}_{q_1}\) on \((R^4, \omega)\) commute. So their flows \(\tilde{\varphi}_u^K\) and \(\tilde{\varphi}_u^{q_1}\) commute. Recall that \(\tilde{F} : R^4 \to R^2 : z \mapsto (q_1(z), q_2(z))\). The flow \(\tilde{\varphi}_u^K\) on \(\tilde{F}^{-1}(c)\) is periodic for every \(c \in R^2\) close to but not at 0. To see this we argue as follows. We have an \(R^2\) action
\[
\Lambda : R^2 \times \tilde{F}^{-1}(c) \to \tilde{F}^{-1}(c) : ((u, t), w) \mapsto (\tilde{\varphi}_u^K \circ \tilde{\varphi}_t^{q_1})(w).
\]
For any \(\tilde{w} \in \tilde{F}^{-1}(c)\) the isotropy group \(\Lambda_{\tilde{w}} = \{(u, t) \in R^2 | \Lambda(u, t, \tilde{w}) = \tilde{w}\}\) is a rank 1 lattice, since \(R^2 / \Lambda_{\tilde{w}} = \tilde{F}^{-1}(c) = S^1 \times R\). Let \((u_0, t_0) \in \Lambda_{\tilde{w}}\). Then
\[
(\tilde{\varphi}_{u_0}^K \circ \tilde{\varphi}_{t_0}^{q_1})(\tilde{w}) = \tilde{w}.
\]
Suppose that there is \(t' > 0\) such that \((u_0, t_0 + t') \in \Lambda_{\tilde{w}}\). Then
\[
(\tilde{\varphi}_{u_0}^K \circ \tilde{\varphi}_{t_0 + t'}^{q_1})(\tilde{w}) = \tilde{w}.
\]
So \(\tilde{\varphi}_{t'}(\tilde{w}) = \tilde{\varphi}_u^K(\tilde{w})\), since \(\tilde{\varphi}_u^K\) and \(\tilde{\varphi}_t^{q_1}\) commute. Thus \(\tilde{\varphi}_t^{q_1}(\tilde{\varphi}_{t'}^{q_1}(\tilde{w})) = \tilde{\varphi}_t^{q_1}(\tilde{w})\), which implies that the integral curve \(t \mapsto \tilde{\varphi}_t^{q_1}(\tilde{w})\) of \(\tilde{X}_{q_1}\) is periodic of period \(-t'\). Since the diffeomorphism \(\Phi\) is isotopic to the identity map, each integral curve of \(\tilde{X}_{q_1}\) is homotopic to an integral curve of \(X_{q_1}\), because the symplectic form \(\omega\) is homotopic to the symplectic form \(\tilde{\omega}\). See lemma 4.3. But \(X_{q_1}\) has no periodic integral curves. Thus \(t' = 0\). Since the rank of \(\Lambda_{\tilde{w}}\) is 1, there is a \(u' > 0\) such that \((u_0 + u', t_0) \in \Lambda_{\tilde{w}}\), that is, \((\tilde{\varphi}_{u_0}^K \circ \tilde{\varphi}_u^K)(\tilde{w}) = \tilde{w}\). So \(\tilde{\varphi}_{u'}(\tilde{w}) = \tilde{w}\), since \(\tilde{\varphi}_u^K\) and \(\tilde{\varphi}_t^{q_1}\) commute. Thus the integral curve \(u \mapsto \tilde{\varphi}_u^K(\tilde{w})\) of \(\tilde{X}_K\) is periodic of period \(u\). Since \(\tilde{w}\) is an arbitrary point of \(\tilde{F}^{-1}(c)\), the flow \(\tilde{\varphi}_u^K\) of \(\tilde{X}_K\) on \(\tilde{F}^{-1}(c)\) is periodic of period \(T_\epsilon = u\). Thus the flow of \(X_I\) on \(F^{-1}(c)\) is periodic of period \(T_\epsilon\), because \(I = \Phi^* K\). This proves 3 and completes the proof of claim 1.4.

\[\square\]

This completes the proof of claim 1.2.

\[\square\]

**Claim 1.5** \(F^{-1}(0, 0)\) is homeomorphic to a pinched cylinder, that is, a cylinder \(S^1 \times R\) with one of its generating circles pinched to the origin 0. The singular fiber \(F^{-1}(0, 0)\) has two transverse tangent planes at 0.

**Proof** Since the action \(I\) Poisson commutes with the integrals \(h_i\) for \(i = 1, 2\), the flow \(\varphi_u^I\) of \(X_I\) leaves the fiber \(F^{-1}(0, 0) \cap V\) invariant. Here \(V \subseteq B\) is an open neighborhood of 0, which is invariant under the \(S^1\)-action generated by \(\varphi_u^I\). Note that \(W_{s,u}(0) \cap V\) is invariant under the flow \(\varphi_u^I\).

We now extend the \(S^1\)-action \(\varphi_u^I(V_{s,u}(0) \cap V)\) to all of \(W_{s,u}(0)\). Let \(p \in W_s(0)\) or \(W_u(0)\). Then there is an open neighborhood \(V_p\) of 0 in \(R^4\) and a time \(t_p > 0\) such that \(\varphi_{t_p}^I(V_p) \subseteq V\). For every \(\tilde{p} \in V_p\) let \(\tilde{\varphi}_u^I(\tilde{p}) = (\varphi_{t_p}^I \circ \varphi_u^I)(\tilde{p})\), where \(\{t_p \in W_s(0): t_p \in W_u(0)\}\). Then \(\tilde{\varphi}_u^I\) defines an \(S^1\)-action on an open neighborhood \(\tilde{V} = \bigcup_{p \in W_s(0)} V_p \cup \bigcup_{p \in W_u(0)} V_p\) of \(F^{-1}(0, 0) = W_s(0) \cup W_u(0)\). Therefore we have an \(R^2\)-action on \(\tilde{V}\) defined by
\[
\tilde{\Lambda} : R^2 \times \tilde{V} \to \tilde{V} : ((u, t), \tilde{w}) \mapsto (\tilde{\varphi}_u^K \circ \tilde{\varphi}_t^{q_1})(w).
\]

Note that \(F^{-1}(0, 0)\) is invariant under the action \(\tilde{\Lambda}\) and that 0 is the only fixed point of the flow \(\varphi_u^I\) on \(F^{-1}(0, 0)\). So we have an \(R^2\)-action \(\tilde{\Lambda}(u, t) = (u, t)\). Let \(J_O\) be the isotropy group for the \(R^2\)-orbit \(O = W_s(0)\) or \(W_u(0)\). Because \(F^{-1}(0, 0)\) is not compact, the rank of the lattice \(J_O\) can not be equal to 2. But \(J_O \neq \emptyset\). So
is a smooth surjective submersion, which defines a trivial fibration whose fiber
of the

In this section we study the fibers of the integral map $F$ (3), which are close to the singular fiber $F^{-1}(0, 0)$. We prove

Claim 2.1 There is an open neighborhood $W$ of $F^{-1}(0, 0)$ in $\mathbb{R}^4$, which is invariant under the flows $\phi_t^{h_1}$ and $\phi_t^{I}$, and an open neighborhood $U$ of $(0, 0)$ in $\mathbb{R}^2$ such that

$$F\lfloor (W \setminus F^{-1}(0, 0)) : W \setminus F^{-1}(0, 0) \to U \setminus \{(0, 0)\} : p \mapsto (h_1(p), I(p))$$

is a smooth surjective submersion, which defines a trivial fibration whose fiber $F^{-1}(c) \cap W$ for each $c \in U \setminus \{(0, 0)\}$ is a smooth cylinder $S^1 \times \mathbb{R}$, that is, an orbit of the action

$$\hat{\Sigma} : \mathbb{R}^2 \times W \to W : ((u, t), p) \mapsto (\phi_t^{I} \circ \phi_t^{h_1})(p).$$

Proof Let $B$ be an open ball in $\mathbb{R}^4$ centered at 0 whose radius is small enough that $\partial B$ intersects $F^{-1}(0, 0)$ in two circles $W_{s,u}^B(0) \cap \partial B$. We can arrange that all of the orbits of the $\mathbb{R}^2$ action

$$\tilde{\Sigma} : \mathbb{R}^2 \times \mathcal{V} \to \mathcal{V} : ((u, t), w) \mapsto (\phi_t^{I} \circ \phi_t^{h_1})(w),$$

where $\mathcal{V}$ is an open neighborhood of $F^{-1}(0, 0)$ in $\mathbb{R}^4$, which is invariant under the flow $\phi_t^{I}$, are 2 dimensional, and all of its orbits near 0 intersect $\partial B$ in two circles, which are close to the circles $W_{s,u}(0) \cap \partial B$.

Consider the local $\mathbb{R}^2$ action $\tilde{\Sigma}_{(u,t)}|B$. Let $W$ be the union of $\mathbb{R}^2$-orbits which intersect $\mathcal{V} \cap \partial B$ or $\{0\}$. The union of $\tilde{\Sigma}$ orbits in $\overline{B}$ which intersect of $\mathcal{V} \cap \partial B$, is an open subset of $\overline{B}$, which contains a small neighborhood $V$ of 0 in $\mathbb{R}^4$. This follows because by hyperbolicity of the integral curves of $X_{h_1}$ at a distance $\delta > 0$ from 0 enter and leave $B$ at points on $\partial B$, which are at a distance $O(\delta)$ from $W_{s,u}(0) \cap \partial B$. Thus $W$ is an open neighborhood of $F^{-1}(0, 0)$ in $\mathbb{R}^4$ such that $F^{-1}(c) \cap W$ is equal to the $\tilde{\Sigma}$ orbit in $\overline{B}$ through $F^{-1}(c) \cap \partial B$ for every $c \in U \setminus \{(0, 0)\}$. Hence the smooth mapping $F\lfloor W : W \to U \setminus \{(0, 0)\}$ is surjective with connected fibers. The invariance of $F$ under the local $\mathbb{R}^2$ action $\tilde{\Sigma}_{(u,t)}|\overline{B}$, together with the fact that at every $z \in \mathcal{V} \cap \partial B$
the rank of $DF(z)$ is 2, implies that at each point $w$ on an $\mathbb{R}^2$-orbit which intersects $V \cap \partial B$, the rank of $DF(w)$ is 2. Thus the map (14) is a surjective submersion, which defines a fibration with connected fibers. By hypothesis 2 this fibration is trivial.

We now show that each fiber of the fibration (14) is a smooth cylinder. Since the flows $\varphi_i^{p_i}$ and $\varphi_u$ leave the fibers $F^{-1}(c) \cap W$ invariant, they define an $\mathbb{R}^2$-action

$$\Xi : \mathbb{R}^2 \times (W \setminus F^{-1}(0, 0)) \to W \setminus F^{-1}(0, 0) : ((u, t), w) \mapsto (\varphi_u \circ \varphi_i^{p_i})(w)$$

on $W \setminus F^{-1}(0, 0)$. Because the vector fields $X_{h_1}$ and $X_I$ are linearly independent at each point of $W \setminus F^{-1}(0, 0)$, an $\mathbb{R}^2$ orbit $O$ is an open subset of $W \setminus F^{-1}(0, 0)$. Since $W \setminus F^{-1}(0, 0)$ is connected, it follows that $O = W \setminus F^{-1}(0, 0)$. Now $O$ is diffeomorphic to $\mathbb{R}^2/J_O$ and $O$ is not compact. Thus $J_O$ is a rank 1 lattice in $\mathbb{R}^2$. Hence $O$ is a smooth cylinder $S^1 \times \mathbb{R}$.

Let $\Sigma_{\xi}$ be the image of a smooth local section $\sigma : U \setminus \{(0, 0)\} \to W \setminus F^{-1}(0, 0)$ of the fibration (14) at $\xi \in W \setminus F^{-1}(0, 0)$, which is invariant under the flow $\varphi_u$ of $X_I$. Because $T_c$ is the period of the flow $\varphi_u$, it is the period of every integral curve $u \mapsto \varphi_u^I(\xi')$ for every $\xi' \in \Sigma_{\xi} \cap F^{-1}(c)$ with $c \in U \setminus \{(0, 0)\}$. Thus for every $c \in U \setminus \{(0, 0)\}$ we have $(T_c, 0) \in J_{\xi'}$.

Let $\xi \in W_s^B(0)$ and $\eta \in W_u^B(0)$. Let $\Sigma_{\xi, \eta}$ be the image in $B$ of a smooth local section of the fibration (14) at $\xi, \eta$, which is invariant under the $S^1$ action $\varphi_u$ on $W$. For each $c \in U \setminus \{(0, 0)\}$, it follows that $F^{-1}(c) \cap (W \cap \Sigma_{\xi, \eta})$ is diffeomorphic to a circle $C_{\xi, \eta}(c)$, which is an orbit of the $S^1$ action $\varphi_u$. For each $\xi(c) \in C_{\xi}(c)$ there is a smallest positive time $\tau(c)$ such that the integral curve $t \mapsto \varphi_u^{h_1}(\xi(c))$ lies in $C_\eta(c)$. In other words, $\eta(c) = \varphi_{\tau(c)}(\xi(c)) \in C_\eta(c)$. As $\xi(c)$ traces out $C_{\xi}(c)$ once, $\eta(c)$ traces out $C_{\eta}(c)$ once. Thus the circles $C_{\xi, \eta}(c)$ bound a subset of $F^{-1}(c)$, which is diffeomorphic to a compact cylinder $S^1 \times [0, 1]$.

3 Holomorphic Focus-Focus System

In this section we give a complex variables treatment of the standard focus-focus system $(q_1, q_2, \mathbb{R}^4, \omega)$, see (1) and (2).

Let $(z_1, z_2) = (x - iy, p_x + i p_y)$ be coordinates on $\mathbb{C}^2$. Let $\sigma = dz_1 \wedge dz_2$ be a complex symplectic form on $\mathbb{C}^2$. Consider the holomorphic Hamiltonian

$$\mathcal{H} : \mathbb{C}^2 \to \mathbb{C} : (z_1, z_2) \mapsto z_1 z_2 = (xp_x + y p_y) + i(x p_y - y p_x) = q_1 + i q_2.$$

The complex Hamiltonian vector field on $(\mathbb{C}^2, \sigma)$ associated to $\mathcal{H}$ is

$$X_{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial z_2} \frac{\partial}{\partial z_1} - \frac{\partial \mathcal{H}}{\partial z_1} \frac{\partial}{\partial z_2} = z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2},$$

since

$$X_{\mathcal{H}}(dz_1 \wedge dz_2) = z_1 dz_1 + z_2 dz_2 = d(z_1 z_2) = d\mathcal{H}.$$
The complex integral curves of $X_{\mathcal{H}}$ satisfy
\[ \frac{dz_1}{d\tau} = z_1 \quad \text{and} \quad \frac{dz_2}{d\tau} = -z_2. \]

Here $\tau$ is complex time parameter. The complex flow of $X_{\mathcal{H}}$ on $\mathbb{C}^2$ is
\[ \varphi_{\mathcal{H}} : \mathbb{C} \times \mathbb{C}^2 \to \mathbb{C}^2 : (\tau, (z_1, z_2)) \mapsto (e^{\tau} z_1, e^{-\tau} z_2). \]

Let
\[ \tilde{\Sigma} : \mathbb{C}^\times = \mathbb{C} \setminus \{0\} \to \mathbb{C}^2 : c \mapsto (c, 1). \quad (15) \]

Since $\mathcal{H}(\tilde{\Sigma}(c)) = c$ for every $c \in \mathbb{C}^\times$, it follows that $\tilde{\Sigma}(c) \in \mathcal{H}^{-1}(c)$ for every $c \in \mathbb{C}^\times$. Thus $\tilde{\Sigma}$ is a global section of the bundle
\[ \widehat{\rho} = \mathcal{H} \vert_{\mathcal{H}^{-1}(\mathbb{C}^\times)} : \mathcal{H}^{-1}(\mathbb{C}^\times) \to \mathbb{C}^\times : (z_1, z_2) \mapsto \mathcal{H}(z_1, z_2). \quad (16) \]

**Lemma 3.1** The bundle $\widehat{\rho}$ is trivial.

**Proof** Consider the map
\[ \tau : \mathcal{H}^{-1}(\mathbb{C}^\times) \subseteq \mathbb{C}^2 \to \mathbb{C}^\times \times \mathcal{H}^{-1}(1) : (z_1, z_2) \mapsto (z_1 z_2, (z_1, z_2(z_1 z_2)^{-1})), \]
whose inverse is
\[ \tau^{-1} : \mathbb{C}^\times \times \mathcal{H}^{-1}(1) \to \mathcal{H}^{-1}(\mathbb{C}^\times) : (c, (w, w^{-1})) \mapsto (w, cw^{-1}). \]

Check:
\[ (\tau \circ \tau^{-1})(c, (w, w^{-1})) = \tau(w, cw^{-1}) = (c, (w, cw^{-1})(wcw^{-1})^{-1}) = (c, (w, w^{-1})) \]
and
\[ (\tau^{-1} \circ \tau)(z_1, z_2) = \tau^{-1}(z_1 z_2, (z_1, z_2(z_1 z_2)^{-1})) = (z_1, (z_1 z_2)z_2(z_1 z_2)^{-1}) = (z_1, z_2). \]

Thus the map $\tau$ trivializes the bundle $\widehat{\rho}$. \qed

In real terms, the function $\mathcal{H} : \mathbb{C}^2 \to \mathbb{C}$ is the energy momentum map
\[ \tilde{F} : \mathbb{R}^4 \to \mathbb{R}^2 : w = (x, y, p_x, p_y) \mapsto (q_1(w), q_2(w)) \]
of the standard focus-focus system. Here $\tilde{F} = j \circ \mathcal{H} \circ t^{-1}$, where $j : \mathbb{C} \to \mathbb{R}^2 : z \mapsto (\text{Re } z, \text{Im } z)$. 
In real terms the section $\tilde{\Sigma} : \mathbb{C}^\times \to \mathbb{C}^2 : c \mapsto (c, 1)$ (15) of the bundle $\hat{\rho}$ is the map

$$\Sigma : (\mathbb{R}^2)^\times = \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^4 : (c_1, c_2) \mapsto t^{-1} \circ \tilde{\Sigma} \circ j^{-1} = (c_1, -c_2, 1, 0). \quad (17)$$

Using the fact that $\Sigma$ is a global section we obtain

**Corollary 3.1A** The fibration

$$\tilde{F}^{-1}(R) : \tilde{F}^{-1}(R) \subseteq \mathbb{R}^4 \to R = (\mathbb{R}^2)^\times \subseteq \mathbb{R}^2 : w \mapsto (q_1(w), q_2(w))$$

is trivial.

In real terms the complex flow $\varphi^H_t$ of the Hamiltonian vector field $X^H_t$ is $t^{-1} \circ \varphi^H_t \circ t$, where $t = t + is$. We compute.

$$
(t^{-1} \circ \varphi^H_t \circ t)(x, y, p_x, p_y) \\
= t^{-1} \left( e^{t + is}(x - iy), e^{-t - is}(p_x + ip_y) \right) \\
= t^{-1} \left( e^t \left[ (x \cos s + y \sin s) + i(x \sin s - y \cos s) \right], \right.

\begin{align*}
& e^{-t} \left[ (p_x \cos s + p_y \sin s) + i(-p_x \sin s + p_y \cos s) \right] \\
& = (e^t (x \cos s + y \sin s), e^{-t} (-x \sin s + y \cos s), \\
& e^{-t} (p_x \cos s + p_y \sin s), e^{-t} (-p_x \sin s + p_y \cos s) \\
& = \varphi^{q_1}_{t} \left( \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} \right) \\
& = (\varphi^{q_1}_{t} \circ \varphi^{q_2}_{-s})(x, y, p_x, p_y).
\end{align*}
$$

Fix $\varepsilon > 0$ and let $c \in \mathbb{C}$. Look at the circles

$$\xi_s : \mathbb{R} \to \mathcal{H}^{-1}(c) : s \mapsto \left( \frac{c}{e^{-is} \varepsilon}, e^{-is} \varepsilon \right) \quad (18a)$$

and

$$\eta_s : \mathbb{R} \to \mathcal{H}^{-1}(c) : s \mapsto \left( e^{is} \varepsilon, \frac{c}{e^{is} \varepsilon} \right). \quad (18b)$$

**Lemma 3.2** For every $s \in \mathbb{R}$ the point $\xi_s(0) = (0, e^{-is} \varepsilon)$ lies on the stable manifold $W_s(0)$ of the vector field $X^H_q$ on $\mathbb{R}^4$, while the point $\eta_s(0) = (e^{is} \varepsilon, 0)$ lies on the unstable manifold $W_u(0)$ of the vector field $X^H_q$ for every $s \in \mathbb{R}$.

**Proof** We compute.

$$\varphi^{H}_{t+iu}(0, e^{-is} \varepsilon) = (0, e^{-(t+iu)}(e^{-is} \varepsilon)) = (0, e^{-t}(e^{-i(s+u)} \varepsilon)).$$

So $\lim_{t \to \infty} \varphi^{H}_{t+iu}(0, e^{-is} \varepsilon) = (0, 0)$, that is, $(0, e^{-is} \varepsilon) \in W_s(0)$, since $\varphi^{H}_{t+iu}$ is $\varphi^{q_1}_{t} \circ \varphi^{q_2}_{-s}$ in real terms. Similarly, $(e^{is} \varepsilon, 0) \in W_u(0)$.

$\Box$
Lemma 3.3 With \( c \neq 0 \) the circle \( C_u = \{ \eta_s(c) \mid s \in \mathbb{R} \} \) in \( \mathcal{H}^{-1}(c) \) is a cross section for the flow \( \varphi_t^H(c_1, z_2) \) on \( \mathcal{H}^{-1}(c) \). In other words, every complex integral curve of \( X_H \) on \( \mathcal{H}^{-1}(c) \) intersects \( C_u \).

Proof To show that the complex integral curve \( \tau \mapsto \varphi_t^H(c_1, z_2) \) intersects \( C_u \) we need to show that there is \( (e^{iv}, \frac{c}{e^{iv}}) \in C_u \) and a \( \tau = t + is \) such that

\[
(e^{iv}, \frac{c}{e^{iv}}) = \varphi_t^H(c_1, z_2) = (e^{t+is}, e^{-t-is} z_2).
\]

It is enough to solve

\[
e^{t+is} z_1 = e^{iv}, \tag{19}\]

because equation (19) implies \( e^{-t-is} z_2 = \frac{c}{e^{iv}} \) since \( z_1^0z_2^0 = c \) and \( c \neq 0 \). Write \( z_1^0 = |z_1^0| e^{i\arg z_1^0} \). Then (19) reads

\[
e^{t+is+\arg z_1^0}|z_1^0| = e^{iv}. \tag{20}\]

So \( s = v - \arg z_1^0 \) and \( t = \ln \frac{s}{|z_1^0|} \) solves (20). Here \( v \) may be chosen freely.

\[\Box\]

Lemma 3.4 Fix \( c \neq 0 \). Let \( C_s \) be the circle \( \{ \xi_s(c) \mid s \in \mathbb{R} \} \) in \( \mathcal{H}^{-1}(c) \). The points \( \xi_s(c) \in C_s \) and \( \eta_s(c) \in C_u \) lie on the same complex integral curve of \( X_H \).

Proof Set \( \tau = -\ln \frac{\xi}{\varepsilon} \). Then

\[
\varphi^H_\tau(\xi_s(c)) = \left( e^{-\ln \frac{\xi}{\varepsilon}} \left( \frac{c}{e^{-iv}} \right), e^{\ln \frac{\xi}{\varepsilon}} \left( e^{-iv} \right) \right) = (e^{iv}, \frac{c}{e^{iv}}) = \eta_s(c).
\]

\[\Box\]

Since Re \( \tau = 2 \ln \varepsilon - \text{Re} \ln c \), we get \( -\text{Im} \tau = \text{Im} \ln c \). Thus \( -\text{Im} \tau \) depends only on \( c \neq 0 \) and not on the choice of the circles \( C_s, u \) on \( \mathcal{H}^{-1}(c) \) or on the choice of points \( \xi_s(c) \in C_s \) or \( \eta_s(c) \in C_u \). Hence \( -\text{Im} \tau \) is an intrinsic property of the flow of \( \varphi^H_t \) of the vector field \( X_H \) on \( \mathcal{H}^{-1}(c) \).

We look at the map

\[ \Theta : \mathbb{C}^\times \to S^1 : \xi \mapsto \Theta(\xi) = -\text{Im} \tau(\xi) = \text{Im} \ln c. \]

Claim 3.5 The winding number of the map \( \Theta \) is 1, which is the scattering monodromy of the standard focus-focus system.

Proof We now verify this last assertion. On \( \mathbb{C}^\times \) with coordinate \( z_1 \) we have a real 1-form \( \vartheta = \text{Im} \frac{dz_1}{z_1} = \text{Im} d \ln z_1 \). So

\[
\vartheta = \text{Im} \frac{dz_1}{z_1} = \text{Im} \left( \frac{dx - i\,d y}{x - iy} \right) = (x^2 + y^2)^{-1} \text{Im}[(x + i y)(d x - i\,d y)] = (x^2 + y^2)^{-1}(y\,d x - x \,d y) = d \left( \tan^{-1} \frac{x}{y} \right).
\]
Let $\pi : \mathbb{C}^\times \times \mathbb{C} \to \mathbb{C}^\times : (z_1, z_2) \mapsto z_1$ be the projection map on the first factor. Set $\theta = \pi^*(-\theta)|\mathcal{H}^{-1}(\mathbb{C}^\times)$. Then $\theta$ is a closed 1-form on $\mathcal{H}^{-1}(\mathbb{C}^\times)$ since $d\theta = d\pi^*(-\theta)|\mathcal{H}^{-1}(\mathbb{C}^\times) = (\pi^*(-d\theta))|\mathcal{H}^{-1}(\mathbb{C}^\times) = 0$. Because
\[
X_{\Im \mathcal{H}} \theta = X_{q_2} \theta = \left( -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} - p_y \frac{\partial}{\partial p_x} + p_x \frac{\partial}{\partial p_y} \right) \theta = (x^2 + y^2)^{-1}(y^2 + x^2) = 1,
\]
$\theta$ is a connection 1-form on the bundle $\mathcal{H}^{-1}(\mathbb{C}^\times) \mapsto \mathbb{C}^\times$ (16). Since $-\Im \left[ \frac{de^{i\theta z_1}}{e^{iz_1}} \right] = -\Im \frac{dz_1}{z_1}$, the connection 1-form $\theta$ is invariant under the flow of $X_{\Im \mathcal{H}}$ on $\mathcal{H}^{-1}(\mathbb{C}^\times)$.

Let
\[
\Sigma^\vee = \tilde{\Sigma}|S^1_r \subseteq \mathbb{C}^\times \to \mathcal{H}^{-1}(S^1_r) : s \mapsto (s, 1)
\]
be a smooth section of the bundle $\tilde{\rho}$. Consider the real curve $\Gamma_{\Sigma^\vee(s)}$ on $\mathcal{H}^{-1}(S^1_r)$, where $\Gamma_{\Sigma^\vee(s)}(t) = \varphi^\Re(t)\mathcal{H}(\Sigma^\vee(s))$. We have
\[
(z_1(t), z_2(t)) = \Gamma_{\Sigma^\vee(s)}(t) = \varphi^q_t(s, 1) = (e^t s, e^{-t}).
\]

Thus for each $s \in S^1_r$ the scattering phase $\Theta(s)$ of $\Gamma_{\Sigma^\vee(s)}$ with respect to the connection 1-form $\theta$ is given by $\Theta(s) = \Im \ln s + \int_{\Gamma_{\Sigma^\vee(s)}} \theta$. We have
\[
\int_{\Gamma_{\Sigma^\vee(s)}} \theta = \int_{\Gamma_{\Sigma^\vee(s)}} \theta \frac{d\Gamma_{\Sigma^\vee(s)}}{dt} dt = \int_{-\infty}^{\infty} \langle \theta | X_{q_1} \rangle dt
\]
\[
= \int_{-\infty}^{\infty} d\theta = \langle \Gamma_{\Sigma^\vee(s)}(t) \rangle dt, \quad \text{by definition of infinitesimal elevation, see [1, p.436]}
\]
\[
= -\int_{-\infty}^{\infty} \frac{d}{dt} \left( \Im \ln z_1(t) \right) dt = -\Im \ln z_1(t) |_{-\infty}^{\infty} = 0.
\]

Thus $\Theta(s) = \Im \ln s$. This verifies the assertion.

\begin{center}
\textit{4 Connection 1-Form}
\end{center}

In this section we construct a connection 1-form on $(\mathbb{R}^4, \tilde{\omega} = \tilde{\Phi}^*\omega)$, which is invariant under the flow of $\tilde{X}_K$, where $K$ is given by (8).

\begin{lemma}
Let $\gamma_w$ be an integral curve of the vector field $X_{q_2} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} - p_y \frac{\partial}{\partial p_x} + p_x \frac{\partial}{\partial p_y}$ on $(\mathbb{R}^4, \omega = -d\alpha = -d(p_x dx + p_y dy))$ starting at $w = (x, y, p_x, p_y)$. Then
\[
q_2(w) = \frac{1}{2\pi} \int_{\gamma_w} \alpha.
\]
\end{lemma}
Proof. We compute. By definition
\[
\frac{1}{2\pi} \int_{\gamma_w} \alpha = \frac{1}{2\pi} \int_{0}^{2\pi} \langle \alpha | \frac{d\varphi_t^{q_2}}{ds}(w) \rangle \, ds = \frac{1}{2\pi} \int_{0}^{2\pi} \langle \alpha | X_{q_2} \rangle \left( \varphi_t^{q_2}(w) \right) \, ds.
\]
But \( \langle \alpha | X_{q_2} \rangle = X_{q_2} \cdot \alpha = -yp_x + xp_y = q_2 \). So
\[
\frac{1}{2\pi} \int_{\gamma_w} \alpha = \frac{1}{2\pi} \int_{0}^{2\pi} q_2(\varphi_t^{q_2}(w)) \, ds = q_2(w),
\]
since \( q_2 \) is an integral of \( X_{q_2} \).

Let \( \Phi_t = \iota^{-1} \circ \varphi_t^X \circ \iota \), where \( \varphi_t^X \) for \( t \in [0, 1] \) is the flow of the vector field \( X \) constructed in the proof of the complex Morse lemma for the complex function \( \hat{f} \)
(28) on \([0, 1] \times \mathbb{C}^2\).

Lemma 4.2 For \( t \in [0, 1] \) let
\[
K_t(w) = \frac{1}{2\pi} \int_{\gamma_w} \alpha_t = \frac{1}{2\pi} \int_{\gamma_w} (\Phi_t^{-1})^* \alpha.
\]
Then \( K_t = (\Phi_t^{-1})^* q_2 \). Since \( \Phi_1 = \tilde{\Phi} \), it follows that \( K_1 = K \).

Proof. We compute. From lemma 4.1 we obtain
\[
((\Phi_t^{-1})^* q_2)(w) = \frac{1}{2\pi} \int_{\gamma_w} \alpha = \frac{1}{2\pi} \int_{\gamma_w} (\Phi_t^{-1})^* \alpha = \frac{1}{2\pi} \int_{\gamma_w} \alpha_t = K_t(w).
\]

Lemma 4.3 For \( t \in [0, 1] \) let \( \omega_t = (\Phi_t^{-1})^* \omega \). Since \( \Phi_1 = \tilde{\Phi} \), it follows that \( \omega_1 = (\tilde{\Phi})^* \omega = \tilde{\omega} \). Then \( X_{K_t} = (\Phi_t^{-1})^* X_{q_2} \).

Proof. We compute. By definition
\[
X_{K_t} \cdot \omega_t = dK_t = d((\Phi_t^{-1})^* q_2), \text{ by lemma 4.2}
\]
\[
= (\Phi_t^{-1})^* dq_2 = (\Phi_t^{-1})^* (X_{q_2} \cdot \omega)
\]
\[
= (\Phi_t^{-1})^* X_{q_2} \cdot ((\Phi_t^{-1})^* \omega) = (\Phi_t^{-1})^* X_{q_2} \cdot \omega_t.
\]
So \( X_{K_t} = (\Phi_t^{-1})^* X_{q_2} \), since \( \omega_t \) is nondegenerate.

Let \( \theta = \pi^* \left( -d \tan^{-1} \frac{w_2}{w_1} \right) \). Then \( \theta \) is a connection 1-form on \((\mathbb{R}^4, \omega)\), because \( \theta \) is closed and \( X_{q_2} \cdot \theta = 1 \). \( \theta \) is \( \varphi_t^{q_2} \)-invariant.

Claim 4.4 For \( t \in [0, 1] \) let \( \theta_t = (\Phi_t^{-1})^* \theta \). Then \( \theta_t \) is a connection 1-form on \((\mathbb{R}^4, \omega_{1-t}) = (\Phi_{1-t}^{-1})^* \omega \). In particular, since \( \Phi_0 = \text{id}_{\mathbb{R}^4} \), we obtain \( \theta_1 = \theta \), which is a connection 1-form on \((\mathbb{R}^4, \omega)\). Also \( \theta_0 = (\Phi_1^{-1})^* \theta \) is a connection 1-form on \((\mathbb{R}^4, \omega_1 = \tilde{\omega})\).
We are able to prove the geometric scattering monodromy theorem.

5 Geometric Monodromy Theorem

We are able to prove the geometric scattering monodromy theorem.

For each $t \in [0, 1]$ consider the integrable Hamiltonian system $\left(\Phi_t^{-1}\right)^* q_1, K_t, \mathbb{R}^4, \left(\Phi_t^{-1}\right)^* \omega)$, where $K_t = \left(\Phi_t^{-1}\right)^* q_2$. For $t = 0$ this system is $(q_1, q_2, \mathbb{R}^4, \tilde{\omega})$; while for $t = 1$ the system is the focus-focus system $(h_1, h_2, \mathbb{R}^4, \omega)$, since $\omega = \tilde{\Phi}_s \omega = (\Phi_1)_s \omega$ and $\tilde{\Phi}_s h_j = (\Phi_1)_s h_j = q_j$ for $j = 1, 2$.

Let $S_r^1$ be a circle of radius $r$ in $U \setminus \{(0, 0)\} \subseteq \mathbb{R}^2$. Consider the bundle

$$\widehat{\rho}_t = \widehat{F}_t|\widehat{F}_t^{-1}(S_r^1) : \widehat{F}_t^{-1}(S_r^1) \subseteq \mathbb{R}^4 \to S_r^1 \subseteq U \setminus \{(0, 0)\} : w \mapsto \left(\left(\Phi_t^{-1}\right)^* q_1(w), K_t(w)\right).$$

Note that $\widehat{F}_1 = F$ (3) and $\widehat{F}_0 = \tilde{F}$ (6). The bundle $\widehat{\rho}_t$ has a section

$$\Sigma_t : S_r^1 \subseteq \mathbb{R}^2 \to F_t^{-1}(S_r^1) : s \mapsto (\Phi_0 \circ \tilde{\Sigma})(s),$$

where

$$\tilde{\Sigma} = t^{-1} \circ \Sigma \circ j : S_r^1 \subseteq \mathbb{R}^2 \to \tilde{F}_0^{-1}(S_r^1) \subseteq \mathbb{R}^4$$

and $\tilde{\Sigma}$ is given by (15). Note that $\Sigma_0 = \tilde{\Sigma}$. For each $s \in S_r^1$ consider the curve

$$\widehat{\Gamma}_{\Sigma_t(s)} : \mathbb{R} \to \widehat{F}_t^{-1}(S_r^1) \subseteq \mathbb{R}^4 : v \mapsto \varphi_v^{K_t}(\Sigma_t(s)),$$
Claim 5.1 The degree of the scattering phase map \( s \mapsto \Theta_0(s) \) associated to the curve \( \tilde{\Gamma}_{\Sigma_0(s)} \) on \( F^{-1}(S^1_r) \) contained in \( (\mathbb{R}^4, \omega) \) is equal to the degree of the scattering phase map \( s \mapsto \Theta_1(s) \) associated to the curve \( \tilde{\Gamma}_{\Sigma_1(s)} \) on \( F^{-1}(S^1_r) \) contained in \( (\mathbb{R}^4, \tilde{\omega}) \).

**Proof** We compute.

\[
\int_{\tilde{\Gamma}_{\Sigma_1^{-1}\Theta_1^{-1}}(s)} \theta_1 = \int_{\tilde{\Gamma}_{\Sigma_1^{-1}\Theta_1^{-1}}(s)} \Phi_t^{-1}*\theta = \int_{\Phi_t^{-1}\tilde{\Gamma}_{\Sigma_1^{-1}\Theta_1^{-1}}(s)} \theta = \int_{\tilde{\Gamma}_{\Sigma_1}(s)} \theta. \tag{23}
\]

The second to last equality above follows because

\[
\tilde{\Gamma}_{\Sigma_1^{-1}\Theta_1^{-1}}(v) = \varphi_u^K(\Sigma_t(s)) = (\Phi_t \circ \varphi_u^q \circ \Phi_t^{-1})(\Sigma_t(s)), \quad \text{because } X_{K_1} = (\Phi_t^{-1})^* X_{q_2}
\]

\[
= \Phi_t(\varphi_u^q(\Sigma_t(s))), \quad \text{since } \Sigma_t = \Phi_t \circ \Sigma
\]

\[
= \Phi_t(\tilde{\Gamma}_{\Sigma_0(s)}(v)), \quad \text{since } \Sigma_0 = \tilde{\Sigma}.
\]

Let \( \arg \Sigma_t(s) = \arg \pi_1(\Phi_t \circ \tilde{\Gamma}_{\Sigma_0(s)}) \). Here we have

\[
\pi_1 : \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 : (x, y, p_x, p_y) \mapsto (x, y),
\]

where \( \pi = t^{-1} \circ \pi_1 \circ t^{-1} : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \) is projection on the first factor. So the scattering phase map \( s \mapsto \Theta_t(s) \) associated to the curve \( \tilde{\Gamma}_{\Sigma_1^{-1}\Theta_1^{-1}}(s) \) is

\[
\Theta_t(s) = \arg \Sigma_t(s) + \int_{\tilde{\Gamma}_{\Sigma_1^{-1}\Theta_1^{-1}}(s)} \theta_1 = \arg \Sigma_t(s) + \int_{\tilde{\Gamma}_{\Sigma_1}(s)} \theta, \quad \text{using equation (23)}
\]

\[
= \arg \Sigma_t(s) - \arg \Sigma_1(s) + \Theta_1(s).
\]

Since \( \Phi_t \) with \( t \in [0, 1] \) is an isotopy of \( \Phi_1 \) to the identity map, the degree of the map \( s \to \Theta_1(s) \) is equal to the degree of \( s \mapsto \Theta_0(s) \).

To finish the proof of the geometric scattering theorem we show that

Claim 5.2 The degree of the scattering phase map \( s \mapsto \Theta_0(s) = \arg \tilde{\Gamma}_{\Sigma_0(s)} + \int_{\tilde{\Gamma}_{\Sigma_0(s)}} \theta \) associated to the curve

\[
\tilde{\Gamma}_{\Sigma_0(s)} : \mathbb{R} \to F^{-1}(S^1_r) : v \mapsto \varphi_u^q(\Sigma_0(s)) \tag{24}
\]

in \( (\mathbb{R}^4, \tilde{\omega}) \) is equal to the degree of the scattering phase map \( s \mapsto \Theta(s) = \arg \Gamma_{\Sigma_1(s)} = \text{Im } \ln s \) associated to the curve

\[
\Gamma_{\Sigma(s)} : \mathbb{R} \to F^{-1}(S^1_r) : v \mapsto \varphi_u^q(\tilde{\Sigma}(s)). \tag{25}
\]

in \( (\mathbb{R}^4, \omega) \).
We have \( \tilde{\omega} = \tilde{\Phi}_* \omega \), where \( \tilde{\Phi} \) is a near identity diffeomorphism. More precisely, \( \tilde{\Phi} = \text{id}_{\tilde{U}} + \Psi \), where \( \Psi \) is flat to order 2. Shrinking \( \tilde{U} \) if necessary the map \( \tilde{\Phi}^\epsilon = \text{id}_{\tilde{U}} + (1 - \epsilon) \Psi \) on \([0, 1]\) is an isotropy of \( \tilde{\Phi} \) to \( \text{id}_{\tilde{U}} \). Let

\[
\Sigma^\epsilon : S^1_r \to \tilde{F}^{-1}_\epsilon(S^1_r) : s \mapsto \tilde{\Phi}^\epsilon \circ \tilde{\Sigma}(s).
\]

Then \( \Sigma^\epsilon \) is a section of the bundle

\[
\tilde{F}^\epsilon \big| (\tilde{F}^\epsilon)^{-1}(S^1_r) : (\tilde{F}^\epsilon)^{-1}(S^1_r) \to S^1_r,
\]

where

\[
\tilde{F}^\epsilon : \mathbb{R}^4 \to \mathbb{R}^2 : w \mapsto \big((\tilde{\Phi}^\epsilon)_* q_1(w), (\tilde{\Phi}^\epsilon)_* q_2(w)\big).
\]

Let

\[
\Gamma_{\Sigma^\epsilon(s)} : \mathbb{R} \to (\tilde{F}^\epsilon)^{-1}(S^1_r) : w \mapsto \varphi^K_\epsilon \big(\Sigma^\epsilon(s)\big),
\]

where \( K^\epsilon = (\tilde{\Phi}^\epsilon)_* q_2 \) on \((\mathbb{R}^4, \omega^\epsilon = (\tilde{\Phi}^\epsilon)_* \omega)\). Let \( \theta^\epsilon = (\tilde{\Phi}^\epsilon)_* \theta \). A calculation similar to the one which verified equation (23) shows that

\[
\int_{\Gamma_{\Sigma^\epsilon(s)}} \theta^\epsilon = \int_{(\tilde{F}^\epsilon)^{-1}(S^1_r)} \theta = \int_{\Gamma_{\Sigma^1(s)}} \theta,
\]

since \( \Gamma_{\Sigma^\epsilon(s)} = \tilde{\Phi}^\epsilon \circ \Gamma_{\Sigma^1(s)} \). Here \( \Sigma^1(s) = \tilde{\Sigma}(s) \). Next define \( \arg \Sigma^\epsilon(s) = \pi_1(\tilde{\Phi}^\epsilon \circ \Gamma_{\Sigma^1(s)}) \). Then the scattering phase map \( \Theta^\epsilon \) associated to the curve \( \Gamma_{\Sigma^\epsilon(s)} \) is

\[
\Theta^\epsilon(s) = \arg \Sigma^\epsilon(s) + \int_{\Sigma^\epsilon(s)} \theta^\epsilon
= \arg \Sigma^\epsilon(s) + \int_{\Gamma_{\Sigma^1(s)}} \theta, \text{ using equation (26)}
= \arg \Sigma^\epsilon(s) - \arg \Sigma^1(s) + \Theta^1(s).
\]

Since \( \tilde{\Phi}^\epsilon \) is an isotopy of \( \tilde{\Phi} \) to \( \text{id}_{\tilde{U}} \), the degree of the mapping \( s \mapsto \Theta^0(s) = \Theta^1(s) \) is equal to the degree of the mapping \( s \mapsto \Theta^1(s) = \Theta(s) = \text{Im} \ln s \) by the proof of claim 3.5. Hence the degree of \( \Theta^0(s) \) is 1.

Using claim 5.1 we have proved

**Theorem 5.3** (Geometric scattering) The focus-focus integrable system \((h_1, h_2, \mathbb{R}^4, \omega)\) has scattering monodromy.

**Proof** The scattering phase map \( \Theta : s \to \Theta^1(s) \) associated to the curve \( \tilde{\Sigma}(s) \) in \( F^{-1}(S^1_r) \) contained in \((\mathbb{R}^4, \omega)\) has degree 1. \(\square\)
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Declarations

Human and animal rights  This research involved no animals or humans and neither generated nor used any computer programs or data.

6 Appendix

In this appendix we prove a complex version of the Morse lemma. Our proof was inspired by the proof of the focus-focus Morse lemma in [6].

Lemma A1  (Morse lemma). Let
\[ \hat{\mathcal{H}} : [0, 2] \times \mathbb{C}^2 \to \mathbb{C} : (t, z) \mapsto \mathcal{H}_t(z) = Q(z) + tR(z), \tag{27} \]
where \( Q \) is a nondegenerate homogeneous quadratic polynomial and \( R \) is a smooth function, which is flat to second order at the origin \((0, 0)\). Then there is an open neighborhood \( U \) of \((0, 0)\) in \( \mathbb{C}^2 \) and a diffeomorphism \( \Phi \) of \( U \) into itself with \( \Phi(0, 0) = (0, 0) \) such that \( \Phi^* \mathcal{H}_1 = Q \) on \( U \). Moreover, \( \Phi \) is isotopic to \( \text{id}_U \).

Proof  By a complex linear change of coordinates we may assume that \( Q(z) = z_1z_2 \). We want to find a time dependent vector field \( X_t = X_t + \frac{\partial}{\partial t} \) on \((0, 2) \times \mathbb{C}^2 \) whose flow \( \varphi^X_t \) satisfies
\[ (\varphi^X_t)^* \mathcal{H}_t = Q, \text{ for every } t \in [0, 1]. \tag{28} \]

Differentiating (28) gives \( 0 = (\varphi^X_t)^* (\frac{\partial \hat{\mathcal{H}}}{\partial t} + L_{X_t} \mathcal{H}_t) \). Since \( \frac{\partial \mathcal{H}}{\partial t} = R \), we need to find a vector field \( X_t \) on \( \mathbb{C}^2 \) such that
\[ d\mathcal{H}_t(z)X_t(z) = -R(z) \text{ for all } t \in [0, 1]. \tag{29} \]

Now \( d\mathcal{H}_t(z) = (z_2 + t \frac{\partial R}{\partial z_1}(z)) \, dz_1 + (z_1 + t \frac{\partial R}{\partial z_2}(z)) \, dz_2 \). For some smooth functions \( A \) and \( B \) on \((0, 2) \times \mathbb{C}^2 \)
\[ X_t(z) = A(t, z) \frac{\partial}{\partial z_1} + B(t, z) \frac{\partial}{\partial z_2}. \tag{30} \]

Since \( R(0) = 0 \), by the integral form of Taylor’s theorem \( R(z) = G_1(z)z_1 + G_2(z)z_2 \), where \( G_j \) are smooth functions with \( G_j(0) = \frac{\partial R}{\partial z_j}(0) \) for \( j = 1, 2 \). Since \( R \) is flat to second order at 0, we get \( G_j(0) = 0 \). Thus (29) can be written as
\[ -G_1(z)z_1 - G_2(z)z_2 = A(t, z) \left( z_2 + t \frac{\partial R}{\partial z_1}(z) \right) + B(t, z) \left( z_1 + \frac{\partial R}{\partial z_2}(z) \right). \tag{31} \]
Again by Taylor’s theorem, \( \frac{\partial R}{\partial z_j}(z) = F_j(z)z_1 + E_j(z)z_2 \) for \( j = 1, 2 \), where \( F_j(0) = \frac{\partial^2 R}{\partial z_1 \partial z_j}(0) \) and \( E_j(0) = \frac{\partial R}{\partial z_2 \partial z_j}(0) \). Since \( R \) is flat to second order at 0, it follows that \( F_j(0) = 0 \) and \( E_j(0) = 0 \). Thus equation (31) becomes

\[
-G_1(z)z_1 - G_2(z)z_2 = A(t, z)\left(z_2 + t(F_1(z)z_1 + E_1(z)z_2)\right) + B(t, z)\left(z_1 + t(F_2(z)z_1 + E_2(z)z_2)\right).
\]

Equating the coefficients of \( z_1 \) and \( z_2 \) in the equation above, we get

\[
\begin{pmatrix} G_1(z) \\ G_2(z) \end{pmatrix} = \begin{pmatrix} tF_1(z) & 1 + tF_2(z) \\ 1 + tE_1(z) & tE_2(z) \end{pmatrix} \begin{pmatrix} A(t, z) \\ B(t, z) \end{pmatrix} = \mathcal{A}(t, z)\begin{pmatrix} A(t, z) \\ B(t, z) \end{pmatrix}.
\]

So

\[
|\det \mathcal{A}(t, z)| = |t + t(E_1(z) + F_2(z)) + t^2(E_1(z)F_2(z) - E_2(z)F_2(z))| \\
\geq 1 - |t| \cdot |(E_1(z) + F_2(z)) + t^2(E_1(z)F_2(z) - E_2(z)F_2(z))|.
\]

Let \( U \) be an open neighborhood of \( 0 \in \mathbb{C}^2 \) such that for \( i = 1, 2 \)

\[
|E_i(z)| < \frac{1}{16} \quad \text{and} \quad |F_i(z)| < \frac{1}{16}.
\]

Then

\[
|t| \cdot |(E_1(z) + F_2(z)) + t^2(E_1(z)F_2(z) - E_2(z)F_2(z))| \\
\leq |t| \cdot |E_1(z)| + |F_2(z)| + |t| \cdot \left(|E_1(z)||F_2(z)| + |E_2(z)||F_1(z)|\right) \\
< 2 \left[ \frac{1}{16} + \frac{1}{16} + 2 \left( \frac{1}{16} \cdot \frac{1}{16} + \frac{1}{16} \cdot \frac{1}{16} \right) \right], \quad \text{using (32) and } t \in [0, 2]
\]

\[
= \frac{17}{64}.
\]

Thus the matrix \( \mathcal{A}(t, z) \) is invertible for all \( z \in U \) and all \( t \in [0, 2] \). With \( \begin{pmatrix} A(t, z) \\ B(t, z) \end{pmatrix}^{-1} \begin{pmatrix} G_1(z) \\ G_2(z) \end{pmatrix} \) we have determined the vector field \( X_t \) (30) on \( (0, 2) \times \mathbb{C}^2 \) which solves equation (29).

Because \( X_t(0, 0) = (0, 0) \) we can shrink \( U \) if necessary so that the flow \( \varphi_t^X \) of the vector field \( X \) sends \([0, 1] \times U \) to \( U \). Set \( \Phi = \varphi_t^X \). Then \( \Phi^* \mathcal{H}_t = (\varphi_t^X)^* \mathcal{H}_t = Q \) on \( U \). The diffeomorphism \( \Phi \) is isotopic to \( \text{id}_U \), since it is the time 1 map of a flow. \( \square \)

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