Towards A Supersymmetric $w_{1+\infty}$ Symmetry in the Celestial Conformal Field Theory

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Abstract

The $w_{1+\infty}$ symmetry algebra appears in the Einstein-Yang-Mills theory, initiated by Strominger recently. In this paper, by using the known results on the operator product expansions (OPEs) between the graviton, gravitino, gluon and gluino in the supersymmetric version of above theory, we describe the supersymmetric $w_{1+\infty}$ symmetry. We calculate the four additional commutator relations between the soft currents explicitly. By analyzing the previous works by Odake et al. and Pope et al. thirty years ago, and introducing the additional symmetry current which corresponds to celestial gluino operator, the whole seven commutator relations can be identified with the ones in the supersymmetric $w_{1+\infty}$ algebra with $SU(N)$ symmetry under the restrictions of wedge modes.


1 Introduction

In the tree level Einstein-Yang-Mills theory, the leading operator product expansions (OPEs) on the celestial sphere of conformal primary gravitons and gluons are determined \[1\]. The structure constants in the right hand side of these OPEs are given by the Euler beta function of arbitrary weights for above operators. By analyzing the singular behavior of the Euler beta function with an appropriate limiting procedure, the OPEs between the conformally soft positive-helicity gluon and graviton operators are obtained and the corresponding three commutator relations are determined \[2\]. It turns out that the structure constants in these commutators are functions of weights and modes of the operators in complicated way. By absorbing the various gamma functions which depend on the weights and modes into each
current, they become very simple form [3]. The commutator between the gravitons can be interpreted as the wedge subalgebra of \(w_{1+\infty}\) algebra [4]. The wedge means that the mode can vary between one minus the weight and the weight minus one. Note that the mode of the graviton contains also the half integers as well as integers while the one in [4] is an arbitrary integer. The operators in the commutators [2, 3] do depend on the complex coordinate \(z\) of celestial sphere. In [5], the mode expansion for the graviton in the holomorphic sector is performed further and this leads to the additional contour integral during the calculation of the commutator relation. Then we are left with the commutator which does not depend on the above \(z\) coordinate. See also the relevant papers in [6, 7, 8] and the review papers in [9, 10, 11] on the celestial holography.

The \(w_\infty\) algebra [4], as an extension of the Virasoro algebra, is the Lie algebra where the currents have the conformal weight (or spin) \(s = 2, 3, 4, \ldots, \infty\). This algebra admits the usual central extension in the Virasoro sector [12] and can be obtained also from the contraction [13] of the \(W_\infty\) algebra [12, 14] which admits the central extensions for all sectors of arbitrary conformal weights. By adding the conformal weight \(s = 1\) further into the \(W_\infty\) algebra, the \(W_{1+\infty}\) algebra is found in [15] and the complete expression is also given by [13]. The structure constants in the \(W_\infty\) algebra are different from the ones in the \(W_{1+\infty}\) algebra. After taking the zero limit of a parameter, we obtain the Lie algebra between the currents of weights \(s = 1, 2, \ldots, \infty\) having the central extension in the Virasoro sector. It is natural to denote this algebra by \(w_{1+\infty}\) algebra which did not appear in any literatures as far as I know. See also [16]. In this paper, we are considering the \(w_{1+\infty}\) algebra with a vanishing central term.

In [17], the additional adjoint currents of weights \(s = 1, 2, \ldots, \infty\) under the \(SU(N)\) are added to the above the \(W_{1+\infty}\) algebra. By requiring that the weight 1 adjoint current should produce the affine \(SU(N)\) algebra with a level \(k\) and the adjoint currents should transform under the zero mode of the weight 1 adjoint current and after using the Jacobi identities between the currents, two additional commutator relations in addition to the known commutator relation in the \(W_{1+\infty}\) algebra are completely fixed. The central charge of the Virasoro current is given by \(c = Nk\). In the commutator between the adjoint currents, there are also the symmetric tensor \(d\) symbol dependent terms as well as the (antisymmetric) structure constant dependent terms. We expect to have the \(w_{1+\infty}\) algebra with \(SU(N)\) symmetry after taking the proper limit of a parameter [18]. In general, there is a central term from the commutator between the weight 1 adjoint currents as well as a central term of Virasoro current.

The \(\mathcal{N} = 2\) supersymmetric extension of \(w_\infty\) algebra is studied in [19] by generalizing the \(\mathcal{N} = 2\) superconformal algebra and can be written in terms of graded Poisson bracket.

\[^1\text{See also Strominger’s talk in strings 2021.}\]
along the line of [4]. There are no central terms. On the other hands, there exists the $\mathcal{N} = 2$ supersymmetric extension of $W_\infty$ algebra [20] where the bosonic sector is given by the sum of $W_\infty$ algebra and $W_{1+\infty}$ algebra. By taking the proper limit of the parameter, the corresponding $\mathcal{N} = 2 w_\infty$ algebra is obtained in [21] where there are (anti)commutator relations with the central terms, compared with the ones in [19]. Based on the construction of the twisted $\mathcal{N} = 2$ superconformal algebra [22, 23], the topological $W_\infty$ algebra [24] is obtained by twisting the $\mathcal{N} = 2$ supersymmetric extension of $W_\infty$ algebra. It turns out that there is no central term and the structure constants appearing two non trivial commutator relations are the same. According to the contraction procedure (by introducing new currents with a parameter and taking this parameter to be zero), the right hand sides of two commutators become very simple. Of course, the anticommutator between the fermionic currents vanishes. See also [25].

In this paper, we generalize the results of [2] to the supersymmetric Einstein-Yang-Mills theory studied in [26] where the additional nontrivial OPEs are given by the ones between bosonic and fermionic operators. Note that the work of [6] deals with the similar studies in different context. The OPEs between the fermionic operators are regular. This is rather unusual because in the conventional conformal field theory they have nontrivial singular behaviors. There exists some paper [27] which is an supersymmetric extension of the $w_\infty$ algebra, by generalizing the $\mathcal{N} = 1$ superconformal algebra. On the other hands, we can reduce the supersymmetry to lower one by using the twisting procedure done in [24] where the anticommutator relation between the fermionic currents vanishes.

By using the known results on the operator product expansions(OPEs) between the graviton, gravitino, gluon and gluino in the above theory, we would like to describe the supersymmetric $w_{1+\infty}$ symmetry in the celestial conformal field theory. The four additional commutator relations between the soft currents are calculated explicitly. One of them can be extracted from the work of [24]. By analyzing the previous works in [17] and [24] and introducing the additional symmetry current which corresponds to celestial gluino operator, the remaining three can be determined. Eventually we obtain that the whole seven commutator relations can be identified with the ones in the supersymmetric $w_{1+\infty}$ algebra with wedge modes having the $SU(N)$ symmetry.

In section 2, we calculate four commutators for the soft currents. In section 3, we present the supersymmetric $w_{1+\infty}$ algebra corresponding to seven commutators from the soft currents. In section 4, we summarize the main result of this paper and collect some future directions. In Appendices A and B, we repeat the result of [2] and four commutator relations are described respectively.
There are some works \[28, 29, 30, 31, 32, 33, 34, 35\] in different contexts.

2 A supersymmetric Einstein-Yang-Mills theory

We will consider the OPEs in \[26\] and obtain the commutator relations for soft current algebra.

2.1 A soft current algebra between the graviton and the gluon: A review

From the positive-helicity (conformally primary) graviton operator \(G^\Delta(z, \bar{z})\) with two-dimensional conformal weight \(\Delta\), a family of (conformally) soft positive-helicity graviton current is defined as \[2\]

\[
H^k(z, \bar{z}) = \lim_{\varepsilon \to 0} \varepsilon G^{\Delta+\varepsilon}_{k+\varepsilon}(z, \bar{z}), \quad k = 2, 1, 0, -1, -2, \ldots, \tag{2.1}
\]

where the (celestial) left and right conformal weights are given by

\[
(h, \bar{h}) = \left(\frac{k + 2}{2}, \frac{k - 2}{2}\right). \tag{2.2}
\]

Note that the additional factor of \(\varepsilon\) in (2.1) is necessary to cancel the pole of beta function appearing in the original OPE coefficient.

Let us take the holomorphic and antiholomorphic expansions for the above soft graviton current as follows:

\[
H^k(z, \bar{z}) = \sum_{n=k/2}^{2-k} \frac{H^k_n(z)}{z^{n+k/2}} = \sum_m \sum_{n=k/2}^{2-k} \frac{H^k_{m,n}}{z^{m+k/2} \bar{z}^{n+k/2}}. \tag{2.3}
\]

Instead of using the mode \(H^k_n(z)\) which depends on the holomorphic coordinate \(z\), we further expand this with respect to the holomorphic mode \(m\) in order to obtain the closed algebra with the \(SL(2, R)_R\) generators \[5\]. Then the operator \(H^k_{m,n}\) does not depend on \(z\) and \(\bar{z}\). Among them, we focus on the case where the mode \(m\) is equal to \((1 - h)\) together with (2.2)

\[
\hat{H}^k_n \equiv H^k_{m=1-h, n}. \tag{2.4}
\]

This will lead to \(1/z\) dependence in that particular terms of (2.3). As usual, the mode in (2.4) can be written in terms of the following contour integral

\[
\hat{H}^k_n = \oint \frac{dz}{2\pi i} z^{1-k/2-k/2-1} \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{n+k/2-1} H^k(z, \bar{z}), \tag{2.5}
\]
where we intentionally write down the power of $z$ explicitly in the integrand. It is straightforward to check (2.5) by substituting the relation (2.3) into the right hand side of (2.5).

We can repeat the computation done in [2]. For the calculation of $[\hat{H}^p_m, \hat{H}^q_n]$, we should perform the contour integrals over the $z_1, \tilde{z}_1, z_2$ and $\tilde{z}_2$ with the OPE $H^k(z_1, \tilde{z}_1)H^l(z_2, \tilde{z}_2)$ in addition to some powers of $z_1$ and $\tilde{z}_2$. The three contour integrals except the coordinate $z_2$ can be done exactly without any modification and we are left with the contour integral over $z_2$ acting on the $\sum_p \frac{H^{k+l}_{p,m+n}}{z_2^{p+q+2}}$ as well as the mode and weight dependent terms. This leads to $\hat{H}^{k+l}_{m+n}$ coming from $\frac{1}{z_2}$ factor. We present the explicit commutator relation between the soft graviton currents in Appendix A. Similarly, the analysis for the commutator between the soft gluon currents and the one between the soft graviton current and the soft gluon current can be done. They are presented in Appendix (A.1) also for convenience.

The main observation of [3] is to consider the absorption of the mode and weight dependent terms appearing in the denominator of the right hand side of the commutator relation properly. Then the corresponding factors in the numerator can be absorbed in the soft current of the right side. In other words, we have [3, 5]

$$\hat{w}_n^p \equiv \frac{1}{\kappa} (p - n - 1)! (p + n - 1)! \hat{H}^{-2p+4}_n, \quad \hat{j}_m^q \equiv (q - m - 1)! (q + m - 1)! \hat{R}^{3-2q,a}_m, \quad (2.6)$$

where $\kappa$ is the gravitational coupling constant and the index $a$ is an adjoint index of $SU(N)$.

Therefore, with the help of (2.6), the equations (2.7), (3.6) and (3.8) of [3] are rewritten in terms of

$$[\hat{w}_m^p, \hat{w}_n^q] = \left[ m(q - 1) - n(p - 1) \right] \hat{w}_{m+n}^{p+q-2},$$
$$[\hat{j}_m^q, \hat{j}_n^a] = -i \hat{f}^{ab} \hat{j}_{m+n}^{p+q-1,c},$$
$$[\hat{w}_m^p, \hat{j}_n^a] = \left[ m(q - 1) - n(p - 1) \right] \hat{j}_{m+n}^{p+q-2,a}. \quad (2.7)$$

As observed in [3], there exists a finite number of modes $1 - q \leq n \leq q - 1$ for each $\hat{w}^q$ which provides $(2q - 1)$ dimensional closed algebra and plays the role of $SL(2, R)_R$ generators. Here $q$ is the positive half integer value $q = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$. When we fix $p = 2$ in the first equation of (2.7), this implies that the $n$-th mode of a weight $q$ transforms as a primary under the $SL(2, R)_R$ generator $\hat{w}_m^2$. From the third equation of (2.7), it is easy to check that the $n$-th mode of a weight $q$ transforms as a primary under the $\hat{w}_m^2$ and $q$ runs over $q = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$.

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2 We present two identities on the finite sums $\sum_{s=-(p+1)}^{1-k}\frac{(-1)^s(s+k-l)!}{(s+m+l+n-1)!}$ and $\sum_{s=-m+\frac{l-1}{2}}^{1-k}\frac{(-1)^s(s+k-l)!}{(s+m+\frac{l+1}{2}+n-1)!}$, which are used in [2].
as before. The first equation of (2.7) consists of the wedge subalgebra of $w_{1+\infty}$ algebra [4]. We will see the corresponding description in the conventional conformal field theory in next section.

### 2.2 Further soft current algebra in the presence of graviton and gluino currents

We continue to calculate the soft current algebra when there are fermionic currents.

#### 2.2.1 The commutator between the graviton and the gravitino

The OPE between the soft positive-helicity graviton and the soft positive-helicity gravitino (denoted by $I^l(z_2, \bar{z}_2)$) can be described by

$$H^k(z_1, \bar{z}_1) I^l(z_2, \bar{z}_2) = -\kappa \frac{1}{2} \sum_{n=0}^{1-k} \left( 1 - n + \frac{1}{2} - k - l \right) \frac{z_{12}^{n+1}}{n!} \bar{\partial}^n I^{k+l}(z_2, \bar{z}_2). \quad (2.8)$$

The bracket stands for a binomial coefficient. Note that the right hand side of (2.8) looks very similar to the equation (3.5) of [26] in the sense that after we replace $l$ with $l + \frac{1}{2}$ we obtain (2.8). Note that the maximum value for the dummy variable $n$ in the summation of (2.8) from the infinite sum in (2.9) can be understood from the fact that the highest power of $\bar{z}_1$ in (2.8) is $(2 - k)$ and then we have the relation $\bar{\partial}^{(3-k)} H^k(z_1, \bar{z}_1) = 0$ and in the right hand side of (2.8), the highest power of $\bar{z}_{12}$ should be $(2 - k)$ also.

Then the corresponding commutator relation is given by

$$[\hat{H}^k_{m}, \hat{I}^l_{n}] = -\kappa \frac{1}{2} \oint \frac{d\bar{z}_1}{2\pi i} \hat{z}_1^{m+k-1} \oint \frac{d\bar{z}_2}{2\pi i} \hat{z}_2^{n-1} \oint \frac{dz_1}{2\pi i} \oint \frac{dz_2}{2\pi i} H^k(z_1, \bar{z}_1) I^l(z_2, \bar{z}_2). \quad (2.10)$$

After inserting the relation (2.8) into the relation (2.10), then there are four contour integrals. The contour integral over the $z_1$ coordinate with a factor $\frac{1}{2\pi i}$, where the corresponding OPE of conformal primary graviton and conformal primary gravitino of arbitrary weights is given by [26]

$$O_{\Delta_1, +} (z_1, \bar{z}_1) O_{\Delta_2, +} (z_2, \bar{z}_2) = -\frac{\kappa}{z_{12}} \sum_{n=0}^{\infty} B(\Delta_1 - 1 + n, \Delta_2) \frac{z_{12}^{n+1}}{n!} \bar{\partial}^n O_{\Delta_1, +} (z_2, \bar{z}_2). \quad (2.9)$$

In the corresponding expression of [26], there is also $\bar{z}_{12}$ in the numerator which is moved into the inside of the summation in (2.9). Intentionally, we write down the OPE where the bosonic operators are located at the position of $(z_1, \bar{z}_1)$ rather than $(z_2, \bar{z}_2)$ in this subsection in order to use the previous relations on the finite sum.

The mode expansion is given by $I^l(z, \bar{z}) = \sum_{n=-l}^{\infty} \frac{\bar{z}^{l-n}}{z^{n-l}} \hat{I}^l_{m,n}$ and $\hat{I}^l_{m,n} = \sum_{n=-l}^{\infty} \frac{\bar{z}^{l-n}}{z^{n-l}} \hat{I}^l_{m,n}$.
The integrand is \( \frac{1}{z_{1z}} \) is simply one. The contour integral over the \( z_1 \) coordinate can be obtained by using Appendix (A.7) identity of [2]. The next contour integral over the \( \bar{z}_2 \) coordinate can be done by following the identity of Appendix (A.9) of [2]. Finally, the contour integral over the coordinate \( z_2 \) selects \( \hat{I}_{m+n}^{k+l} \). Therefore we are left with

\[
[\hat{H}_{m}^{k}, \hat{I}_{n}^{l}] = -\frac{\kappa}{2} \frac{(-1)^m \frac{b}{2} \left(-m - n - k + l - \frac{3}{2}\right)!}{(\frac{1}{2} - l)! (\frac{3}{2} - m)!} \times \sum_{s=-m-k+\frac{1}{2}}^{1-k} \frac{(-1)^s (s + 1) \left(\frac{3}{2} - s - k - l\right)!}{(1 - s - k)! (s + m + \frac{k}{2})! (m - n - \frac{k + l - 3}{2})!} \hat{I}_{m+n}^{l}. \tag{2.11}
\]

As we expect, this intermediate result is the same as the one in [2] when we replace \( l \) with \( (l + \frac{1}{2}) \). See also the first identity appearing in the footnote [2]. At the moment, it is not clear how to write down the finite sum over the dummy variable \( s \) in above in terms of closed form. However, we can check by assuming the expression obtained from [2] (or above first identity) and substituting several values for the modes and weights into this, the above finite sum can be written in terms of gamma functions.

It turns out that the above result (2.11) can be reduced to

\[
[\hat{H}_{m}^{k}, \hat{I}_{n}^{l}] = \frac{\kappa}{2} \left[m \left(\frac{3}{2} - l\right) - n \left(2 - k\right)\right] \times \frac{(\frac{2-k}{2} - m + \frac{3-l}{2} - n - 1)! (\frac{2-k}{2} + m + \frac{3-l}{2} + n - 1)!}{(\frac{2-k}{2} - m)! (\frac{3-l}{2} - n)! (\frac{2-k}{2} + m)! (\frac{3-l}{2} + n)!} \hat{I}_{m+n}^{l}. \tag{2.12}
\]

We can easily observe that the numerical mode and weight dependent factor in the right hand side of (2.12) is the same as the one in \([\hat{H}_{m}^{k}, \hat{I}_{n}^{l}]\) where the weight \( l \) is replaced by \((l + \frac{1}{2})\). See also Appendix (B.1) we present for convenience.

As done in (2.6), we would like to absorb the denominator of (2.12) into the currents by changing the weights with mode dependent parts. Together with the first relation of (2.6), we introduce the following similar quantity

\[
\hat{G}_{n}^{q} \equiv (q - n - 1)! (q + n - 1)! \hat{I}_{n}^{q-2q}. \tag{2.13}
\]

Then the above commutator can be summarized by

\[
[\hat{w}_{m}^{p}, \hat{G}_{n}^{q}] = \left[m (q - 1) - n (p - 1)\right] \hat{G}_{m+n}^{p+q-2}. \tag{2.14}
\]

The \( n \)-th mode of a weight \( q \) in (2.14) transforms as a primary under the \( \hat{w}_{m}^{2} \) and \( q \) runs over \( q = 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots \) and its mode \( n \) varies as \( 1 - q \leq n \leq q - 1 \). Note that the \( m, n, p \) and \( q \) dependence here is the same as the ones in (2.7).
2.2.2 The commutator between the gluon and the gluino

The regular term \( \delta_{ab} \) where

\[
\frac{\partial}{\partial z_1^2} \sum_{n=0}^{\infty} \left( 1 - n + \frac{1}{2} - k - l \right) \frac{z_1^n}{n!} \partial^n L^{k+l-1,c}(z_2, \bar{z}_2). \tag{2.16}
\]

The structure constant \( f^{ab}_c \) is for the \( SU(N) \). Note that the infinite sum in (2.15) is reduced to the finite sum due to the fact that \( \partial^{(2-k)}_{z_1} R^{k,a}(z_1, \bar{z}_1) = 0 \). As observed in previous subsection, this OPE (2.16) looks very similar to the equation (2.7) of [2] and the binomial coefficient where \( l \) is replaced by \( (l + \frac{1}{2}) \) becomes the above one. Then the corresponding commutator relation \( 3 \) can be written in terms of

\[
[\hat{R}^{k,a}_{m}, \hat{L}^{l,b}_n] = \oint \frac{d\bar{z}_1}{2\pi i} \frac{z_1^{m+\frac{m}{2}+1}}{z_1^{m+\frac{m}{2}+1}} \oint \frac{d\bar{z}_2}{2\pi i} \frac{z_2^{n+\frac{n}{2}+1}}{z_2^{n+\frac{n}{2}+1}} \oint \frac{dz_1}{2\pi i} \oint \frac{dz_2}{2\pi i} R^{k,a}(z_1, \bar{z}_1) L^{l,b}(z_2, \bar{z}_2). \tag{2.17}
\]

After we calculate the contour integrals over \( z_1, \bar{z}_1, \bar{z}_2 \) and \( z_2 \) successively, we obtain the following intermediate result from (2.17)

\[
[\hat{R}^{k,a}_{m}, \hat{L}^{l,b}_n] = -i f^{ab}_c \left( -1 \right)^{m+\frac{m}{2}+1} \frac{1-k-m+\frac{1}{2}l-n}{\left( \frac{1}{2} - l \right) \left( \frac{1-k}{2} - m \right)!} \times \sum_{s=-m+\frac{k}{2}}^{1-k} \frac{\left( -1 \right)^s \left( \frac{3}{2} - s - k - l \right)!}{\left( 1 - s - k \right)! \left( s + m + \frac{k-1}{2} \right)! \left( \frac{1-k}{2} - m + \frac{1}{2}l - n - s \right)!} \hat{L}^{k+l-1,c}_{m+n}. \tag{2.18}
\]

As mentioned before, the above finite sum in (2.18) can be read off from the analysis of the equation of Appendix (A.8) of [2] in the replacement of \( l \) and \( (l + \frac{1}{2}) \). See also the second identity appearing in the footnote [2]. Although it is not obvious to observe the closed form in terms of gamma functions, we can check the corresponding identity by applying several values for the modes and weights. It turns out that there is

\[
[\hat{R}^{k,a}_{m}, \hat{L}^{l,b}_n] = -i f^{ab}_c \frac{1-k-m+\frac{1}{2}l-n}{\left( \frac{1}{2} - k \right) \left( \frac{1}{2} - m \right)! \left( \frac{1}{2} - n \right)! \left( \frac{1}{2} + l \right)!} \hat{L}^{k+l-1,c}_{m+n}. \tag{2.19}
\]

\( \delta_{ab} \) of [26] is not included here.

5The OPE of (conformal primary) gluon and (conformal primary) gluino of arbitrary weights is given by

\[
\mathcal{O}^a_{\Delta_1, +1}(z_1, \bar{z}_1) \mathcal{O}^b_{\Delta_2, +\frac{1}{2}}(z_2, \bar{z}_2) = -i f^{ab}_c \sum_{n=0}^{\infty} B(\Delta_1 - 1 + n, \Delta_2 - \frac{1}{2}) \frac{z_1^n}{n!} \partial^n \mathcal{O}^c_{\Delta_1 + \Delta_2 - 1, +\frac{1}{2}}(z_2, \bar{z}_2). \tag{2.15}
\]

The mode expansion is given by \( L^{l,b}(z, \bar{z}) = \sum_{n\text{ odd}} \frac{\hat{L}^{l,b}_n(z)}{z^{n+\frac{1}{2}l}} = \sum_{m\text{ even}} \frac{\hat{L}^{l,b}_n(z)}{z^{n+\frac{1}{4}l}}, \hat{L}^{l,b}_n = \hat{L}^{l,b}_n(z), \text{ and } \hat{L}^{l,b}_n \text{ is not included here.}

6The mode expansion is given by \( L^{l,b}(z, \bar{z}) = \sum_{n=0}^{\infty} \frac{\hat{L}^{l,b}_n(z)}{z^{n+\frac{1}{2}l}} = \sum_{m=0}^{\infty} \frac{\hat{L}^{l,b}_n(z)}{z^{n+\frac{1}{4}l}}, \text{ and } \hat{L}^{l,b}_n \text{ is not included here.} \)
In order to simplify the commutation relation of (2.19), we introduce
\[ \hat{\psi}_m^a \equiv (q - n - 1)! (q + n - 1)! \hat{L}_m^{\bar{z} - 2q}, \] (2.20)
together with the second relation of (2.7). Then we obtain the final commutator relation as follows:
\[ [\hat{J}_m^{p,a}, \hat{\psi}_n^b] = -i f^{a b}_{c} \hat{\psi}^{p+q-1,c}_{m+n}. \] (2.21)
We will observe, in next subsection, that the \( n \)-th mode of a weight \( q \) in (2.21) transforms as a primary under the \( \hat{w}_m^2 \) and \( q \) runs over \( q = 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots \) and its mode \( n \) varies as \( 1 - q \leq n \leq q - 1 \) as before. Note that the weight of the right hand side of (2.21) is given by \( (p + q - 1) \) as in the second case of (2.7).

### 2.2.3 The commutator between the gluon and the gravitino

The OPE between the soft positive-helicity gluon and the soft positive-helicity gravitino can be obtained by \[ [26] \]
\[ R_{m}^{k,a}(z_1, \bar{z}_1) I_{n}^{l}(z_2, \bar{z}_2) = -\frac{1}{z_{12}} \sum_{n=0}^{1} \sum_{l=0}^{1} \left( -n + \frac{1}{2} - k - l \right) \frac{z_{12}^{n+1}}{n!} \bar{\partial}^{n} L_{m}^{k+l,a}(z_2, \bar{z}_2). \] (2.23)

The corresponding commutator relation can be obtained from (2.23) with various contour integrals. In the expression of (2.22), there is an additional factor \( z_{12} \) which is joined in the inside of the summation of (2.23). As before, by realizing the power of \( z_1 \) and \( z_2 \) in the integrand via the conformal weights of currents, we determine the following expression
\[ \left[ \hat{R}_{m}^{k,a}, \hat{L}_{n}^{l} \right] = \oint d\bar{z}_1 \oint d\bar{z}_2 \frac{z_1^{m+k-1}}{2\pi i} \frac{z_2^{n+l-1}}{2\pi i} \oint d\bar{z}_1 \oint d\bar{z}_2 \frac{z_1^{m+k}}{2\pi i} \frac{z_2^{n+l}}{2\pi i} R_{m+n}^{k,a}(z_1, \bar{z}_1) I_{n}^{l}(z_2, \bar{z}_2). \] (2.24)

By substituting the OPE in (2.23) into (2.24) and performing each contour integral successively, we have the following intermediate result, which consists of the finite sum with the mode and weight dependent overall factor,
\[ \left[ \hat{R}_{m}^{k,a}, \hat{L}_{n}^{l} \right] = -\frac{(-1)^{1+m+k-l}}{(1/2 - l)! (1/2 - m)!} \times \sum_{s=-1-m+n}^{-1} \left( -1 \right)^{s} \frac{(-s - k - l)!}{(1 + s + m + k-1/2)! (-m - n - k + l - s)!} \hat{L}_{m+n}^{k+l,a}. \] (2.25)

\[ \text{Note that there exists a gluino in the right hand side of this OPE.} \]
Note that the above finite sum with \( l \) replaced by \( l - \frac{1}{2} \) in (2.25) is calculated in [2] for the commutator between the soft gluon and soft graviton. See also the first identity appearing in the footnote [2]. Then we obtain the final result, by using the explicit form in terms of various gamma functions in the fractional form

\[
[H^k_m, \tilde{H}^l_n] = \left[ m\left(\frac{3}{2} - l\right) - n(1-k) \right] \times \frac{(\frac{1}{2} - m + \frac{3}{2} - n - 1)! (\frac{1}{2} + m + \frac{3}{2} + n - 1)!}{(\frac{1}{2} - m)!(\frac{3}{2} - n)!(\frac{1}{2} + m)!(\frac{3}{2} + n)!} \tilde{H}^{k+l,n}_m. \tag{2.26}
\]

By using the equations, (2.6), (2.13) and (2.20), the above commutation relation (2.26) is

\[
[H^k_m, \tilde{H}^l_n] = \left[ 2m(q-1) - 2n(p-1) \right] \tilde{H}^{p+q-2,n}_m. \tag{2.27}
\]

Note that the \( m, n, p \) and \( q \) dependence here is the same as the ones in (2.7).

### 2.2.4 The commutator between the graviton and the gluino

The OPE between the soft positive-helicity graviton and the soft positive-helicity gluino can be obtained by [3]

\[
H^k(z_1, \bar{z}_1) L^l,z_2, \bar{z}_2) = -\frac{\kappa}{2} \frac{1}{z^{1-k}_{12}} \sum_{n=0}^{1-k} \left( 1 - n - \frac{1}{2} - k - l \right) \frac{z^{n+1}_{12}}{n!} \partial^n H^k(z_1, \bar{z}_1) L^l,z_2, \bar{z}_2). \tag{2.29}
\]

Again, there exist the finite terms from (2.28). It is easy to observe that from the equation (4.2) of [2] by replacing the \( l \) with \( l + \frac{1}{2} \), the binomial coefficient there becomes the above one in (2.29). Again the commutator between the two currents can be determined by the following expression

\[
[H^k_m, \tilde{L}^l_n] = \oint \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \left[ -1 \right]^{m+n} \left[ -m - n - \frac{k+l-\frac{3}{2}}{2} \right]! \left( \frac{3}{2} - l \right)! \partial^n H^k(z_1, \bar{z}_1) L^l,z_2, \bar{z}_2). \tag{2.30}
\]

We arrive at the intermediate result for the commutator, from (2.29) and (2.30),

\[
[H^k_m, \tilde{L}^l_n] = -\left( -1 \right)^{m+n} \left[ -m - n - \frac{k+l-\frac{3}{2}}{2} \right]! \left( \frac{3}{2} - l \right)! \partial^n H^k(z_1, \bar{z}_1) L^l,z_2, \bar{z}_2). \tag{2.31}
\]
We describe the supersymmetric \( \mathcal{W} \) in the supersymmetric Einstein-Yang-Mills theory discussed in previous section. We realize that the finite sum in (2.31) appears in (2.11) and by replacing \( l \) with \( (l + 1) \) the latter becomes the former. See also the first identity of the footnote \( 2 \). We obtain

\[
[\hat{H}_m, \hat{L}_n] = \frac{K}{2} \left[ m \left( \frac{1}{2} - l \right) - n(2 - k) \right] \times \frac{(\frac{2-k}{2} - m + \frac{k-1}{2} - n - 1)! \left( \frac{2-k}{2} + m + \frac{k-1}{2} + n - 1 \right)!}{(\frac{2-k}{2} - m)! (\frac{2-k}{2} + m)! (\frac{2-k}{2} + n)!} \hat{I}^{k+l,a}_{m+n}.
\]

(2.32)

By using the equations (2.6) and (2.20), the above commutation relation (2.32) becomes

\[
[\hat{w}_m^p, \hat{\psi}_n^{q,a}] = \left[ m(q - 1) - n(p - 1) \right] \hat{\psi}^{p+q-2,a}_{m+n}.
\]

(2.33)

The \( n \)-th mode of a weight \( q \) in (2.33) transforms as a primary under the \( \hat{w}_m^2 \) and \( q \) runs over \( q = 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots \) and its mode \( n \) varies as \( 1 - q \leq n \leq q - 1 \) as before.

### 2.2.5 Summary of this subsection

We collect the previous four commutator relations, (2.14), (2.21), (2.27) and (2.33) as follows:

\[
[\hat{J}_m^{p,a}, \hat{\psi}_n^{q,b}] = -i f^{ab}_{\phantom{ab}c} \hat{\psi}^{p+q-1,c}_{m+n},
\]

\[
[\hat{w}_m^p, \hat{G}_n^q] = \left[ m(q - 1) - n(p - 1) \right] \hat{G}^{p+q-2}_{m+n},
\]

\[
[\hat{J}_m^{p,a}, \hat{G}_n^q] = 2 \left[ m(q - 1) - n(p - 1) \right] \hat{\psi}^{p+q-2,a}_{m+n},
\]

\[
[\hat{w}_m^p, \hat{\psi}_n^{q,a}] = \left[ m(q - 1) - n(p - 1) \right] \hat{\psi}^{p+q-2,a}_{m+n}.
\]

(2.34)

Therefore, we obtain the whole seven commutator relations given by (2.7) and (2.34). We have checked that the graded Jacobi identities containing commutator or anticommutator between four currents are satisfied by using the definition of \( (-1)^{AC} [X^A, [X^B, X^C]] + \text{cycl. perm.} = 0 \) where \( X^A \) stands for any currents. The factor \( (-1)^{AC} \) gives us \( -1 \) for the fermionic currents \( X^A \) and \( X^C \) and \( 1 \) for other three cases. In next section, we will describe the corresponding supersymmetric \( w_{1+\infty} \) algebra in the conventional conformal field theory\(^{10}\).

### 3 A supersymmetric \( w_{1+\infty} \) symmetry

We describe the supersymmetric \( w_{1+\infty} \) algebra in order to understand the symmetry behind in the supersymmetric Einstein-Yang-Mills theory discussed in previous section.

\(^{10}\)There are also three vanishing anticommutator relations, \( \{ \hat{\psi}_m^{p,a}, \hat{\psi}_n^{q,b} \} = 0 \), \( \{ \hat{\psi}_m^{p,a}, \hat{G}_n^q \} = 0 \) and \( \{ \hat{G}_m^p, \hat{G}_n^q \} = 0 \) from the corresponding regular OPEs in [20].
3.1 A $w_{1+\infty}$ algebra with $SU(N)$ symmetry

By introducing the affine current $J^{q,a}$ with level $k$ which has weight $q = 1, 2, \cdots$ and an adjoint index $a = 1, 2, \cdots, (N^2 - 1)$ of $SU(N)$ in addition to the current $w^p$ where $p = 1, 2, \cdots$, in the $W_{1+\infty}$ algebra \[15\], Odake and Sano \[17\] have found that the commutators between them are determined as follows \[11\] :

\[
[W_m^p, W_n^q] = \sum_{r \geq 2, \text{even}}^{p+q-1} \lambda^{-2} g_{r-2}^{-q-2}(m, n) W_{m+n}^{p+q-r} + \delta^{pq} \delta_{m+n,0} \lambda^{2(p-2)} c_{p-2}(m),
\]

\[
[W_m^p, J_n^{q,a}] = \sum_{r \geq 2, \text{even}}^{p+q-1} \lambda^{-2} g_{r-2}^{-q-2}(m, n) J_{m+n}^{p+q-r,a},
\]

\[
[J_m^{p,a}, J_n^{q,b}] = \frac{i}{2} f_{c}^{ab} \sum_{r \geq 1, \text{odd}}^{p+q-1} \lambda^{-2} g_{r-2}^{-q-2}(m, n) J_{m+n}^{p+q-r,c} + \delta^{pq} \delta_{m+n,0} \lambda^{2(p-2)} k_{p-2}(m)
\]

\[+ \sum_{r \geq 2, \text{even}}^{p+q-1} \lambda^{-2} g_{r-2}^{-q-2}(m, n) \left( d_{c}^{ab} J_{m+n}^{p+q-r,c} + \frac{1}{N} \delta^{ab} W_{m+n}^{p+q-r} \right), \tag{3.1} \]

which corresponds to the equations (5), (6) and (7) of \[17\] \[12\]. This is called as the $\hat{SU}(N)_k$ $W_{1+\infty}$ algebra. Note that the dummy variable $r$ is even or odd. What they obtain is to take the first relation in (3.1) from \[15\] and to make an ansatz for the remaining two with arbitrary coefficients which can be determined by various Jacobi identities.

By taking the new currents as

\[
W_m^p \rightarrow w_m^p, \quad J_m^{p,a} \rightarrow \lambda J_m^{p,a}, \tag{3.2} \]

and taking the limit $\lambda \rightarrow 0$, the resulting algebra from (3.1) can be described as

\[
[w_m^p, w_n^q] = g_0^{-q-2}(m, n) w_{m+n}^{p+q-2} + \delta^{pq} \delta_{m+n,0} c_0(m),
\]

\[
[w_m^p, J_n^{q,a}] = g_0^{-q-2}(m, n) J_{m+n}^{p+q-2,a},
\]

\[
[J_m^{p,a}, J_n^{q,b}] = \frac{i}{2} f_{c}^{ab} g_{-1}^{-q-2}(m, n) J_{m+n}^{p+q-1,c} + \delta^{pq} \delta_{m+n,0} k_{-1}(m), \tag{3.3} \]

where the structure constants are $g_0^{-q-2}(m, n) = m(q - 1) - n(p - 1)$ and $g_{-1}^{-q-2}(m, n) = \frac{1}{2}$, and the central terms are given by $c_0(m) = \frac{1}{12} m(m^2 - 1) c$ and $k_{-1}(m) = \frac{m}{16} k$. Note that the central charge $c$ is given by $c = Nk$. See also \[18\] for relevant description. By taking $k = 0$ and rescaling the current $J^{q,a}$ with (assuming that the structure constants are the same as

\[11\] In old days, usually the weight is presented by $p + 2$ (or $p + \frac{3}{2}$) rather than an arbitrary $p$ and we take into account of this shift properly everywhere.

\[12\] Their $V_{-1,2}$ and $W_{j-2,a}$ correspond to our $W^i$ and $J^{i,a}$ respectively in this paper.
the ones in previous section) \(-\frac{1}{4}\) factor, then the above algebra \((3.3)\) with wedge modes

\[
[w^p_m, w^q_n] = \left[m(q - 1) - n(p - 1)\right] w^{p+q-2}_{m+n},
\]

\[
[J^p_m, J^q_n] = -i f^{ab}_c J^{p+q-1,c}_{m+n},
\]

\[
[w^p_m, J^q_{n+a}] = \left[m(q - 1) - n(p - 1)\right] J^{p+q-2,a}_{m+n}, \tag{3.4}
\]

coincides with the one in \((2.7)\) by putting the hats. In general, there are no restrictions in the modes in \((3.4)\) and the weights \(p\) and \(q\) are positive integers \(p, q = 1, 2, \ldots\). Note that in \((2.7)\), the modes can be half integers \(^{13}\)

### 3.2 A supersymmetric topological \(w_\infty\) algebra

It is known that the \(\mathcal{N} = 2\) supersymmetric \(W_\infty\) algebra \([20]\) can be twisted to provide a topological \(W_\infty\) algebra. By taking one of the fermionic generators as the nilpotent BRST charge, the corresponding nontrivial commutator relations are given by

\[
[V^p_m, V^q_n] = \sum_{l \geq 2} g^{p+q-2}_{l-2} (m, n) V^{p+q-l}_{m+n},
\]

\[
[V^p_m, G^q_{n+\frac{1}{2}}] = \sum_{l \geq 2} g^{p+q-\frac{3}{2}}_{l-2} (m, n) G^{p+q-l}_{m+n+\frac{1}{2}}. \tag{3.6}
\]

Here there are no central terms. The weight of \(G^q\) is given by \(q = \frac{3}{2}, \frac{5}{2}, \ldots\) while the weight of \(V^p\) is given by \(p = 2, 3, \ldots\) where \(p = 1\) is not included. The bosonic currents \(\hat{V}^p_m\) in \((3.6)\) are given by two kinds of bosonic ones in \([20]\) and the explicit relation can be found in the equation (8) of \([24]\). By construction, the bosonic current \(\hat{V}^p\) is obtained by linear combination of the bosonic current \(V^p\) of \(W_\infty\) algebra and the bosonic current \(\hat{V}^p\) of \(W_{1+\infty}\) algebra. Furthermore the structure constants \(g^{p+q-2}_{l-2} (m, n)\) appearing in \((3.6)\) are given by the equation (11) of \([24]\). Note that the structure constants in two commutators are the same.

\(^{13}\)The three commutators in \((3.4)\) are equivalent to the following OPEs in the antiholomorphic sector (by decomposing the usual mode expansions)

\[
w^p(\bar{z}_1) \bar{w}^q(\bar{z}_2) = \frac{(p + q - 2)}{\bar{(z}_1 - \bar{z}_2)^2} w^{p+q-2}(\bar{z}_2) + \frac{(p - 1)}{\bar{(z}_1 - \bar{z}_2^2}) \bar{\partial} w^{p+q-2}(\bar{z}_2) + \cdots,
\]

\[
J^p(\bar{z}_1) J^q(\bar{z}_2) = -i f^{ab}_c J^{p+q-1,c}(\bar{z}_2) + \cdots,
\]

\[
w^p(\bar{z}_1) J^q(\bar{z}_2) = \frac{(p + q - 2)}{\bar{(z}_1 - \bar{z}_2)^2} J^{p+q-2,a}(\bar{z}_2) + \frac{(p - 1)}{\bar{(z}_1 - \bar{z}_2^2}) \bar{\partial} J^{p+q-2,a}(\bar{z}_2) + \cdots. \tag{3.5}
\]

According to \([36]\), by field redefinitions of the currents, the weight 1 current in the \(W_{1+\infty}\) algebra is decoupled and the remaining currents generate the \(W_\infty\) algebra. Moreover, the so-called \(w_N\) algebra, which is a truncation of the \(w_\infty\) algebra, is introduced in \([37]\).
As done in [24], the relevant contraction is described by introducing $v^p_m \to \lambda^{p-2} \hat{V}^p_m$ and $G^p_m \to \lambda^{p-2} G^p_m$ and taking the limit $\lambda \to 0$ along the line of (3.3)\(^{14}\). Then it is easy to observe that

$$[v^p_m, v^n_q] = \hat{g}_{0}^{p-2, q-2}(m, n) v^{p+q-2}_{m+n} = \left[ m(q - 1) - n(p - 1) \right] v^{p+q-2}_{m+n},$$
$$[v^p_m, G^q_{n+\frac{1}{2}}] = \hat{g}_{0}^{p-2, q-2}(m, n) G^{p+q-2}_{m+n+\frac{1}{2}} = \left[ m(q - 1) - n(p - 1) \right] G^{p+q-2}_{m+n+\frac{1}{2}},$$

(3.7)

where the nontrivial structure constant in (3.7) can be obtained by calculating the four terms in the equation (11) of [24] with the help of some formulas in [20].

$$\hat{g}_{0}^{p-2, q-2}(m, n) = \frac{(2pq - 3p + 2q^2 - 8q + 8)(-2pn - p + 2qm - 3m + 2n + 1)}{2(2q - 3)(2p + 2q - 5)} + \frac{(2pq - 3p + 2q^2 - 8q + 7)(-2pn - p + 2qm - 3m + 2n + 1)}{2(2q - 3)(2p + 2q - 5)} - \frac{2(p-2)(p+m-1)(-\frac{1}{4}) + 2(p-1)(p+m-1)\left(\frac{1}{2}\right)}{(2p-3)} = m(q - 1) - n(p - 1),$$

(3.8)

which is present in the equation of (42) in [24]. Under the $\nu^2$, the weights of $\nu^p$ and $G^q$ are $p = 2, 3, 4, \cdots$ and $q = 2, 3, 4, \cdots$ respectively. Under the twisting procedure, the original weights of $q$ in the current $G^q$ is shifted by $\frac{1}{2}$. In (3.7), the mode of fermionic current is given by the half integers (NS sector). This can be seen from the relation (2.13) by taking $(n - \frac{1}{2})$ rather than $n$ in the left hand side. See also [18] for relevant discussion\(^{15}\).

### 3.3 A supersymmetric $\mathfrak{u}_{1+∞}$ algebra with $SU(N)$ symmetry

By examining the construction of two previous subsections, we realize that the currents $w^p_m$ in (3.3) is the same as the current $v^p_m$ in (3.7). The currents $\hat{V}^p_m$ in (3.6) consists of four parts

\(^{14}\)Their $\nu^i$ and $G^j$ correspond to our $\nu^{i-2}$ and $G^{j-2}$ in this paper.

\(^{15}\)In terms of the OPEs, the following relations corresponding to (3.7) and (3.8) are satisfied, similar to (3.5),

$$\nu^p(\bar{z}_1) \nu^q(\bar{z}_2) = \frac{(p + q - 2)}{(\bar{z}_1 - \bar{z}_2)^2} \nu^{p+q-2}(\bar{z}_2) + \frac{(p-1)}{(\bar{z}_1 - \bar{z}_2)} \nu^{p+q-2}(\bar{z}_2) + \cdots,$$
$$\nu^p(\bar{z}_1) G^q(\bar{z}_2) = \frac{(p + q - 2)}{(\bar{z}_1 - \bar{z}_2)^2} G^{p+q-2}(\bar{z}_2) + \frac{(p-1)}{(\bar{z}_1 - \bar{z}_2)} G^{p+q-2}(\bar{z}_2) + \cdots.$$

(3.9)

From the first equation, we can check the corresponding commutator relation in (3.7). That is, in the expression of $[v^p_m, v^n_q] = \oint \frac{dz}{2\pi i} \frac{1}{(z_1 - z_2)^2} \nu^{p+q-2}(z_2)$, this leads to $(m + p - 1)(p + q - 2) - (p - 1)(m + n + p + q - 2) = m(q - 1) - n(p - 1)$. For the second relation of (3.9) corresponding to the second relation in (3.7), there exists a shift in the dummy variable in the mode expansion of fermionic current, contrary to the bosonic current.
and two of them do not have any singular OPEs with $J^q_a$ in (3.1) and the remaining two terms have nontrivial OPEs with $J^q_a$. One of them is exactly the same as the $W^p_m$ and the other is a term of $W^{p-1}_m$. The latter provides no contribution after taking the above similar contraction procedure ($W^p_m \rightarrow \lambda^{p-2} w^p_m$ and $J^p_a \rightarrow \lambda^p$).

In this subsection we would like to construct a supersymmetric $w_{1+\infty}$ algebra with $SU(N)$ symmetry which contains the previous algebras (3.4) and (3.7). We are looking for this extended algebra in a minimal way. In other words, we introduce the extra currents minimally. So far, we have the currents $w^p_m$, $G^p_m$ and $J^p_a$. At least we should have the superpartner of $J^p_a$ in the sense that the first two are superpartners each other.

Let us construct how we obtain the OPE between $J^p_a$ and $G^q$. Then we expect that the right hand side of this should contain a fermionic current with an adjoint index $a$ of $SU(N)$. This implies that the right hand side of this OPE should contain the superpartner of $J^p_a$ having an adjoint index $a$. We have some experience on the OPE structure in the context of $w_{\infty}$ algebra. That is, the OPE between the primary current with weight $p$ and the current with weight $q$ consists of both the second order pole and the first order pole. The relative coefficients between these are completely fixed by the formula in [38, 39]. Then we are left with one unknown structure constant appearing in the second order pole.

Then we can try to write down the following ansatz, in the antiholomorphic sector,

$J^p_a(z_1) G^q(z_2) = \frac{(p + q - 2)}{(z_1 - z_2)^2} \psi^{p+q-2,a}(z_2) + \frac{(p - 1)}{(z_1 - z_2)^2} \bar \partial \psi^{p+q-2,a}(z_2) + \ldots. \quad (3.10)$

Here the coefficient appearing in the second order pole can be taken from the normalizations in [14, 10]. Furthermore, we can observe that the relative coefficient $\frac{(p - 1)}{(p+q-2)}$ can be understood from the formula $\frac{\tilde h_p - \tilde h_q + \tilde h_{p+q-2}}{2\tilde h_{p+q-2}}$ [38, 39] with each weight $\tilde h_p = p$, $\tilde h_q = q$ and $\tilde h_{p+q-2} = (p+q-2)$. If we assume more currents, then they will appear in other singular terms up to the central term at the $(p+q)$-th order pole. Then we can write down the corresponding commutator relation from (3.10), by using the procedure appearing in (3.9), as follows:

$[J^p_a, G^q] = \left[ m(q - 1) - n(p - 1) \right] \psi^{p+q-2,a}_{m+n}. \quad (3.11)$

Of course, the overall sign of the right hand side of (3.11) can be related to the redefinition of the fermionic current $\psi^{p+q-2,a}$.

Then we should obtain the OPEs between $\psi^q_b$ and its superpartner $J^p_a$ and the currents $w^p_m$. From the OPE between the affine currents, we generalize this to the following OPE where there is no central term

$J^p_a(z_1) \psi^q_b(z_2) = \frac{-i \int f^{ab}_{cd}}{(z_1 - z_2)} \psi^{p+q-1,c}(z_2) + \ldots. \quad (3.12)$
In terms of the commutator, we obtain

$$\left[ J^{p,a}_m, \psi_{q,b}^c_n \right] = -i f^{ab}_c \psi_{m+n}^{p+q-1,c}. \quad (3.13)$$

As usual, from the Jacobi identities between the generalized affine (bosonic and fermionic) currents, the sign of the right hand side of \((3.13)\) can be fixed also by using the Jacobi identity between the structure constants.

For the final OPE we would like to construct, we expect to obtain the similar OPE with \((3.10)\). The right hand side of the OPE between \(w^p(\bar{z}_1) \psi^{q,a}(\bar{z}_2)\) should contain the fermionic current having an adjoint index \(a\). Then we have the following OPE

$$w^p(\bar{z}_1) \psi^{q,a}(\bar{z}_2) = \frac{(p + q - 2)}{(\bar{z}_1 - \bar{z}_2)^2} \psi^{p+q-2,a}(\bar{z}_2) + \frac{(p - 1)}{(\bar{z}_1 - \bar{z}_2)} \bar{\partial} \psi^{p+q-2,a}(\bar{z}_2) + \cdots. \quad (3.14)$$

Its commutator relation, by following the procedure done in \((3.9)\), can be written in terms of

$$\left[ w^p_m, \psi_{q,a}^c_n \right] = \left[ m(q - 1) - n(p - 1) \right] \psi_{m+n}^{p+q-2,a}. \quad (3.15)$$

This is a natural generalization in the sense that the fermionic current \(\psi_{q,a}^c\) is primary weight \(q\) under the stress energy tensor \(w^2\).

Therefore, there are four additional OPEs, \((3.10)\), \((3.12)\), \((3.14)\) and the second OPE in \((3.9)\) corresponding to \((3.11)\), \((3.13)\), \((3.15)\) and the second relation of \((3.7)\). They, under the wedge modes, correspond to the ones in \((2.34)\) by putting the hats with some rescaling in the \(\psi^{p,a}\). In this correspondence, we should assume that the weight of the current \(w^p\) should be generalized to include the \(p = 1\) case. We can check that the graded Jacobi identities between four currents are satisfied\(^{16}\).

### 3.4 Appearance of a supersymmetric \(w_{1+\infty}\) symmetry in the celestial conformal field theory

Then we have a match between the symmetry present in the supersymmetric Einstein-Yang-Mills theory and the symmetry we described in the context of the conventional conform field theory, by the following field identifications

\[
\hat{w}^p \leftrightarrow w^p, \quad \hat{J}^{p,a} \leftrightarrow J^{p,a}, \quad \hat{G}^p \leftrightarrow G^p, \quad \hat{\psi}^{p,a} \leftrightarrow \psi^{p,a}. \quad (3.16)
\]

\(^{16}\)There are also three vanishing anticommutator relations, \(\{ \psi^{p,a}_m, \psi^{q,b}_n \} = 0\) from the expectation of vanishing of the central term in the generalization of the affine current algebra, \(\{ G^p_m, G^q_n \} = 0\) from the result of topological \(w_{\infty}\) algebra, and \(\{ \psi^{p,a}_m, G^q_n \} = 0\). One realization for the algebra in this paper is given by the following expressions \(w^p_m = iy^{p-2} e^{imx} \left[ (p - 1) \frac{\partial}{\partial x} - i m y \frac{\partial}{\partial y} \right], J^{p,a}_m = -i t^a y^{q-1} e^{imx}, G^p_m = \theta w^p_m\) and \(\psi^{p,a}_m = \theta J^{p,a}_m\) where there are no \(\frac{\partial}{\partial \theta}\) terms from the analysis in \([18]\).
In (3.16), the hatted currents have the $\frac{1}{2}$ terms in the holomorphic and antiholomorphic mode decomposition as in (2.3) and (2.4) while the unhatted ones have the standard antiholomorphic mode decomposition with unrestricted modes. Then only after we impose the wedge modes into the unhatted currents, we observe the above matching between two conformal field theories. The weights in the hatted currents are given by positive integers or half integers while the weights in the unhatted currents are given by positive integers.

4 Conclusions and outlook

We have identified the soft current algebra in the supersymmetric Einstein-Yang-Mills theory with the supersymmetric $w_{1+\infty}$ algebra where the subalgebra of this consists of i) the contraction of $\hat{SU}(N)_k$ $W_{1+\infty}$ algebra and ii) the contraction of the topological $W_{\infty}$ algebra. Moreover, there exist three additional OPEs (or commutator relations) in order to realize the above soft algebra.

In this paper, we have considered the supersymmetric $w_{1+\infty}$ algebra in abstract way without presenting any (free field) realization. As we expect that the bosonic $w_{1+\infty}$ algebra can be generalized to the $W_{1+\infty}$ algebra at the quantum level, it is natural to ask what is the quantum version of the supersymmetric $w_{1+\infty}$ algebra we have obtained in this paper. In other words, there are two subalgebras in our findings whose quantum versions are known before we are taking any contractions. It is an open problem whether the full quantum version of the (classical) supersymmetric $w_{1+\infty}$ algebra we have obtained in this paper. In other words, there are two subalgebras in our findings whose quantum versions are known before we are taking any contractions. It is an open problem whether the full quantum version of the (classical) supersymmetric $w_{1+\infty}$ algebra exists or not. Maybe it will not be obvious to observe them in the $\mathcal{N} = 1$ supersymmetric theory and it is natural to consider the $\mathcal{N} = 2$ supersymmetric $W_{\infty}$ algebra and to add the bosonic and fermionic affine currents appropriately with the help of twisting procedure. The final check will use the Jacobi identity to fix the unknown structure constants.

In general, the OPE between the currents in the $W_{\infty}$ algebra contains other higher spin currents as long as the conformal weights of the right hand side of the OPE are less than or equal to the sum of the conformal weights of the left hand side of the OPE. In other words, what we have found in this paper is the OPEs where the nontrivial singular terms are given by both the second and the first order poles in the antiholomorphic sector. The question is how we can identify all the currents appearing in the singular terms more than the third order pole in the corresponding (supersymmetric) Einstein-Yang-Mills theory?

So far, we have considered the $\mathcal{N} = 1$ supersymmetric Einstein-Yang-Mills theory. Maybe it is also an open problem to examine the $\mathcal{N} = 2$ version of this theory. Are there any OPEs between the $\mathcal{N} = 2$ soft currents in the celestial conformal field theory which correspond
to the known $\mathcal{N} = 2$ supersymmetric $W_\infty$ algebra (and its variants)? On the one hand, it is known that there exists the $\mathcal{N} = 4$ supersymmetric higher spin algebra mentioned in the introduction and it is an open problem to observe whether this algebra occurs in the corresponding celestial conformal field theory or not.

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A A soft current algebra I

For convenience, we present the result of [2] as follows:

\[
\begin{align*}
[\hat{R}_{m}, \hat{R}_{n}] &= -i f^{ab}_{c} \frac{(1-k - m + \frac{1-l}{2} - n)! (1-k + m + \frac{1-l}{2} + n)!}{(\frac{1-k}{2} - m)! (\frac{1-l}{2} - n)! (\frac{1-k}{2} + m)! (\frac{1-l}{2} + n)!} \hat{R}_{m+n-1,c}, \\
[\hat{H}_{m}, \hat{H}_{n}] &= \frac{\kappa}{2} \left[ m(2-l) - n(2-k) \right] \\
& \times \frac{(2-k - m + \frac{1-l}{2} - n - 1)! (2-k + m + \frac{1-l}{2} + n - 1)!}{(\frac{2-k}{2} - m)! (\frac{2-l}{2} - n)! (\frac{2-k}{2} + m)! (\frac{2-l}{2} + n)!} \hat{H}_{m+n}, \\
[\hat{H}_{m}, \hat{R}_{n}] &= \frac{\kappa}{2} \left[ m(1-l) - n(2-k) \right] \\
& \times \frac{(2-k - m + \frac{1-l}{2} - n - 1)! (2-k + m + \frac{1-l}{2} + n - 1)!}{(\frac{2-k}{2} - m)! (\frac{2-l}{2} - n)! (\frac{2-k}{2} + m)! (\frac{2-l}{2} + n)!} \hat{R}_{m+n}. & (A.1)
\end{align*}
\]

B A soft current algebra II

The four additional commutator relations appearing in (2.19), (2.12), (2.26), and (2.32), are summarized by

\[
\begin{align*}
[\hat{R}_{m}, \hat{L}_{n}] &= -i f^{ab}_{c} \frac{(1-k - m + \frac{1-l}{2} - n)! (1-k + m + \frac{3-l}{2} + n)!}{(\frac{1-k}{2} - m)! (\frac{1-l}{2} - n)! (\frac{1-k}{2} + m)! (\frac{3-l}{2} + n)!} \hat{L}_{m+n-1,c}, \\
[\hat{H}_{m}, \hat{L}_{n}] &= \frac{\kappa}{2} \left[ m(3-l) - n(2-k) \right] \\
& \times \frac{(2-k - m + \frac{3-l}{2} - n - 1)! (2-k + m + \frac{3-l}{2} + n - 1)!}{(\frac{2-k}{2} - m)! (\frac{3-l}{2} - n)! (\frac{2-k}{2} + m)! (\frac{3-l}{2} + n)!} \hat{L}_{m+n}, \\
[\hat{R}_{m}, \hat{L}_{n}] &= \frac{\kappa}{2} \left[ m(3-l) - n(1-k) \right] \\
& \times \frac{(1-k - m + \frac{3-l}{2} - n - 1)! (1-k + m + \frac{3-l}{2} + n - 1)!}{(\frac{1-k}{2} - m)! (\frac{3-l}{2} - n)! (\frac{1-k}{2} + m)! (\frac{3-l}{2} + n)!} \hat{L}_{m+n}, \\
[\hat{H}_{m}, \hat{L}_{n}] &= \frac{\kappa}{2} \left[ m(1-l) - n(2-k) \right] \\
& \times \frac{(2-k - m + \frac{1-l}{2} - n - 1)! (2-k + m + \frac{1-l}{2} + n - 1)!}{(\frac{2-k}{2} - m)! (\frac{1-l}{2} - n)! (\frac{2-k}{2} + m)! (\frac{1-l}{2} + n)!} \hat{L}_{m+n}. & (B.1)
\end{align*}
\]

Here we do not simplify these expressions by factorizing them.
References

[1] M. Pate, A. M. Raclariu, A. Strominger and E. Y. Yuan, “Celestial operator products of gluons and gravitons,” Rev. Math. Phys. 33, no.09, 2140003 (2021) doi:10.1142/S0129055X21400031 [arXiv:1910.07424 [hep-th]].

[2] A. Guevara, E. Himwich, M. Pate and A. Strominger, “Holographic Symmetry Algebras for Gauge Theory and Gravity,” [arXiv:2103.03961 [hep-th]].

[3] A. Strominger, “w(1+infinity) and the Celestial Sphere,” [arXiv:2105.14346 [hep-th]].

[4] I. Bakas, “The Large n Limit of Extended Conformal Symmetries,” Phys. Lett. B 228, 57 (1989) doi:10.1016/0370-2693(89)90525-X

[5] E. Himwich, M. Pate and K. Singh, “Celestial Operator Product Expansions and w_{1+\infty} Symmetry for All Spins,” [arXiv:2108.07763 [hep-th]].

[6] H. Jiang, “Holographic Chiral Algebra: Supersymmetry, Infinite Ward Identities, and EFTs,” [arXiv:2108.08799 [hep-th]].

[7] H. Jiang, “Celestial OPEs and w_{1+\infty} algebra from worldsheet in string theory,” [arXiv:2110.04255 [hep-th]].

[8] T. Adamo, L. Mason and A. Sharma, “Celestial w_{1+\infty} symmetries from twistor space,” [arXiv:2110.06066 [hep-th]].

[9] A. M. Raclariu, “Lectures on Celestial Holography,” [arXiv:2107.02075 [hep-th]].

[10] S. Pasterski, “Lectures on Celestial Amplitudes,” [arXiv:2108.04801 [hep-th]].

[11] P. B. Aneesh, G. Compère, L. P. de Gioia, I. Mol and B. Swidler, “Celestial Holography: Lectures on Asymptotic Symmetries,” [arXiv:2109.00997 [hep-th]].

[12] C. N. Pope, L. J. Romans and X. Shen, “The Complete Structure of W(Infinity),” Phys. Lett. B 236, 173-178 (1990) doi:10.1016/0370-2693(90)90822-N

[13] C. N. Pope, “Lectures on W algebras and W gravity,” [arXiv:hep-th/9112076 [hep-th]].

[14] C. N. Pope, L. J. Romans and X. Shen, “W(infinity) and the Racah-wigner Algebra,” Nucl. Phys. B 339, 191-221 (1990) doi:10.1016/0550-3213(90)90539-P
[15] C. N. Pope, L. J. Romans and X. Shen, “A New Higher Spin Algebra and the Lone Star Product,” Phys. Lett. B 242, 401-406 (1990) doi:10.1016/0370-2693(90)91782-7

[16] S. Chaudhuri and J. D. Lykken, “String theory, black holes, and SL(2,R) current algebra,” Nucl. Phys. B 396, 270-302 (1993) doi:10.1016/0550-3213(93)90267-S [arXiv:hep-th/9206107 [hep-th]].

[17] S. Odake and T. Sano, “W(1) + infinity and superW(infinity) algebras with SU(N) symmetry,” Phys. Lett. B 258, 369-374 (1991) doi:10.1016/0370-2693(91)91101-Z

[18] E. Sezgin, “Area preserving diffeomorphisms, w(infinity) algebras and w(infinity) gravity,” [arXiv:hep-th/9202086 [hep-th]].

[19] C. N. Pope and X. Shen, “Higher Spin Theories, W(infinity) Algebras and Their Superextensions,” Phys. Lett. B 236, 21-26 (1990) doi:10.1016/0370-2693(90)90588-W

[20] E. Bergshoeff, C. N. Pope, L. J. Romans, E. Sezgin and X. Shen, “The Super W(infinity) Algebra,” Phys. Lett. B 245, 447-452 (1990) doi:10.1016/0370-2693(90)90672-S

[21] E. Sezgin, “Aspects of W(infinity) symmetry,” [arXiv:hep-th/9112025 [hep-th]].

[22] E. Witten, “Topological Quantum Field Theory,” Commun. Math. Phys. 117, 353 (1988) doi:10.1007/BF01223371

[23] T. Eguchi and S. K. Yang, “N=2 superconformal models as topological field theories,” Mod. Phys. Lett. A 5, 1693-1701 (1990) doi:10.1142/S0217732390001943

[24] C. N. Pope, L. J. Romans, E. Sezgin and X. Shen, “W topological matter and gravity,” Phys. Lett. B 256, 191-198 (1991) doi:10.1016/0370-2693(91)90672-D

[25] P. Horava, “Space-time diffeomorphisms and topological w(infinity) symmetry in two-dimensional topological string theory,” Nucl. Phys. B 414, 485-516 (1994) doi:10.1016/0550-3213(94)90438-3 [arXiv:hep-th/9302020 [hep-th]].

[26] A. Fotopoulos, S. Stieberger, T. R. Taylor and B. Zhu, “Extended Super BMS Algebra of Celestial CFT,” JHEP 09, 198 (2020) doi:10.1007/JHEP09(2020)198 [arXiv:2007.03785 [hep-th]].

[27] E. Sezgin, “GAUGE THEORIES OF INFINITE DIMENSIONAL HAMILTONIAN SUPERALGEBRAS,” IC/89/108.
[28] C. Ahn, “Adding Complex Fermions to the Grassmannian-like Coset Model,” arXiv:2107.01781 [hep-th].

[29] C. Ahn, “The Grassmannian-like Coset Model and the Higher Spin Currents,” JHEP 03, 037 (2021) doi:10.1007/JHEP03(2021)037 arXiv:2011.11240 [hep-th].

[30] C. Ahn and M. H. Kim, “The $\mathcal{N} = 4$ higher spin algebra for generic $\mu$ parameter,” JHEP 02, 123 (2021) doi:10.1007/JHEP02(2021)123 arXiv:2009.04852 [hep-th].

[31] C. Ahn, D. g. Kim and M. H. Kim, “The $\mathcal{N} = 4$ coset model and the higher spin algebra,” Int. J. Mod. Phys. A 35, no.11n12, 2050046 (2020) doi:10.1142/S0217751X20500463 arXiv:1910.02183 [hep-th].

[32] L. Eberhardt and T. Prochážka, “The matrix-extended $W_{1+\infty}$ algebra,” JHEP 12, 175 (2019) doi:10.1007/JHEP12(2019)175 arXiv:1910.00041 [hep-th].

[33] T. Creutzig, Y. Hikida and T. Uetoko, “Rectangular $W$-algebras of types $so(M)$ and $sp(2M)$ and dual coset CFTs,” JHEP 10, 023 (2019) doi:10.1007/JHEP10(2019)023 arXiv:1906.05872 [hep-th].

[34] T. Creutzig and Y. Hikida, “Rectangular $W$ algebras and superalgebras and their representations,” Phys. Rev. D 100, no.8, 086008 (2019) doi:10.1103/PhysRevD.100.086008 arXiv:1906.05868 [hep-th].

[35] T. Creutzig and Y. Hikida, “Rectangular $W$-algebras, extended higher spin gravity and dual coset CFTs,” JHEP 02, 147 (2019) doi:10.1007/JHEP02(2019)147 arXiv:1812.07149 [hep-th].

[36] C. N. Pope, L. J. Romans and X. Shen, “Ideals of Kac-Moody Algebras and Realizations of $W$($\infty$),” Phys. Lett. B 245, 72-78 (1990) doi:10.1016/0370-2693(90)90167-5

[37] K. Li, “Linear $W$(N) gravity,” CALT-68-1724.

[38] R. Blumenhagen, M. Flohr, A. Kliem, W. Nahm, A. Recknagel and R. Varnhagen, “$W$ algebras with two and three generators,” Nucl. Phys. B 361, 255-289 (1991) doi:10.1016/0550-3213(91)90624-7

[39] C. Ahn, “The Higher Spin Currents in the N=1 Stringy Coset Minimal Model,” JHEP 04, 033 (2013) doi:10.1007/JHEP04(2013)033 arXiv:1211.2589 [hep-th].
[40] I. Bakas and E. Kiritsis, “Bosonic Realization of a Universal $W$-Algebra and $Z_\infty$ Parafermions,” Nucl. Phys. B 343, 185-204 (1990) [erratum: Nucl. Phys. B 350, 512-512 (1991)] doi:10.1016/0550-3213(90)90600-I