On the Integrable Deformations of the Maximally Superintegrable Systems

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Abstract: In this paper, we present the integrable deformations method for a maximally superintegrable system. We alter the constants of motion, and using these new functions, we construct a new system which is an integrable deformation of the initial system. In this manner, new maximally superintegrable systems are obtained. We also consider the particular case of Hamiltonian mechanical systems. In addition, we use this method to construct some deformations of an arbitrary system of first-order autonomous differential equations.

Keywords: superintegrable systems; integrable deformations; Poisson bracket; Nambu bracket

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1. Introduction

The construction of integrable deformations of a given integrable system was the subject of some recent articles. In [1], taking into account the fact that Poisson algebras endowed with a co-product map give rise to a systematic way of constructing integrable systems, integrable deformations of a class of three-dimensional Lotka–Volterra equations were given. In [2], considering Poisson–Lie groups as deformations of Lie–Poisson (co)algebras, integrable deformations of certain integrable types of Rössler and Lorenz systems were presented. Based on the construction of a family of compatible Poisson structures, in [3], a family of integrable deformations of the Bogoyavlenskij–Itoh systems was constructed. In [4], altering the constants of motion, integrable deformations of the Euler top were constructed. In the same manner, in [5–9], integrable deformations of some three-dimensional systems were obtained. Moreover, in [10], the integrable deformations method for a three-dimensional system of differential equations was presented. The property that allows to construct integrable deformations of the above-mentioned three-dimensional systems is their maximal superintegrability.

Both in classical mechanics and in quantum mechanics, the superintegrable systems have been widely investigated (see, e.g., [11–13] and references therein). In classical mechanics, a superintegrable system on a $2n$-dimensional phase space is a completely integrable Hamiltonian system which possesses more functionally independent first integrals than degrees of freedom. Moreover, such a system is called maximally superintegrable if the number of the independent first integrals is $2n – 1$ (see, e.g., [14,15]). Similarly, for an arbitrary natural number $n$, a system of first-order differential equations on $\mathbb{R}^n$ which has $n – 1$ functionally independent constants of motion is called a maximally superintegrable system.

In this paper, we present some integrable deformations of maximally superintegrable systems. Firstly, using $n – 1$ functionally independent functions, we construct a family of maximally superintegrable systems for which the considered functions are constants of motion, and we point out a Hamilton–Poisson realization of such a system. Secondly, the Hamilton–Poisson realization allows us to give integrable deformations of the obtained system, which are also maximally superintegrable Hamilton–Poisson systems. Moreover,
under certain conditions, the obtained system and its integrable deformations are generalized Nambu–Hamilton systems [16]. In Section 3 we analyze the particular case of Hamiltonian mechanical systems. In Section 4 we extend the method given in [10] to the case of \( n \)-dimensional systems of first-order differential equations.

### 2. The Integrable Deformations Method for Maximally Superintegrable Systems

In this section, using the idea given in [4], we give integrable deformations of a maximally superintegrable system. For this purpose, we present a construction of a maximally superintegrable system considering \( n - 1 \) functionally independent functions as its constants of motion.

Let \( \Omega \subseteq \mathbb{R}^n \) be an open set. Consider the functions \( C_1, C_2, \ldots, C_{n-1} \in C^\infty(\Omega, \mathbb{R}) \) such that \( \Delta \neq 0 \) on \( \Omega \), where

\[
\Delta := \frac{\partial(C_1, C_2, \ldots, C_{n-1})}{\partial(x_1, x_2, \ldots, x_{n-1})} = \begin{vmatrix}
\partial x_1 C_1 & \cdots & \partial x_{n-1} C_1 \\
\vdots & \ddots & \vdots \\
\partial x_1 C_{n-1} & \cdots & \partial x_{n-1} C_{n-1}
\end{vmatrix},
\]

(1)

with \( \partial x_i C_k = \frac{\partial C_k}{\partial x_i} \). Thus, \( C_1, C_2, \ldots, C_{n-1} \) are \( n - 1 \) functionally independent functions on \( \Omega \).

Considering that these functions are constants of motion, that is, \( \dot{C}_k = \frac{\partial C_k}{\partial t} = 0 \) for every \( k \in \{1, 2, \ldots, n-1\} \), it follows that:

\[
\begin{align*}
\dot{x}_1 C_1 x_1 + \dot{x}_2 C_2 x_2 + \ldots + \dot{x}_{n-1} C_{n-1} x_{n-1} &= -\partial x_n C_1 x_n \\
\vdots \\
\dot{x}_1 C_{n-1} x_1 + \dot{x}_2 C_{n-2} x_2 + \ldots + \dot{x}_{n-1} C_{n-1} x_{n-1} &= -\partial x_n C_{n-1} x_n.
\end{align*}
\]

This linear system has a unique solution given by

\[
\begin{align*}
\dot{x}_k &= \frac{\Delta_k}{\Delta} \cdot \dot{x}_n, \quad k \in \{1, 2, \ldots, n-1\},
\end{align*}
\]

(3)

where

\[
\Delta_k := \frac{\partial(C_1, C_2, \ldots, C_{n-1}, x_k)}{\partial(x_1, x_2, \ldots, x_{n-1}, x_n)}. \quad (4)
\]

From (1) and (4), it is easy to see that \( \Delta_n = \Delta \).

Set \( x_n = \Delta \). Therefore, the functions \( C_1, \ldots, C_{n-1} \) give rise to the following differential system on \( \mathbb{R}^n \)

\[
\dot{x}_k = \Delta_k, \quad k \in \{1, 2, \ldots, n\}. \quad (5)
\]

Considering the Poisson bracket on \( \mathbb{R}^n \) generated by \( C_1, C_2, \ldots, C_{n-2} \) (see, e.g., [17]), namely,

\[
\{f, g\} c_1, \ldots, c_{n-2} := \frac{\partial(C_1, C_2, \ldots, C_{n-2}, f, g)}{\partial(x_1, x_2, \ldots, x_{n-1}, x_n)},
\]

(6)

for every \( f, g \in C^\infty(\mathbb{R}^n, \mathbb{R}) \), system (5) reads

\[
\dot{x}_k = \{H, x_k\} c_1, \ldots, c_{n-2}, \quad k \in \{1, 2, \ldots, n\}. \quad (7)
\]

Thus, it is a Hamilton–Poisson system with the Hamiltonian \( H := C_{n-1} \) and Casimirs \( C_1, C_2, \ldots, C_{n-2} \). We notice that \( \{C_i, C_j\} c_1, \ldots, c_{n-2} = 0 \) for every \( i, j \in \{1, 2, \ldots, n-1\} \).

Now, we set \( \dot{x}_n = \nu \Delta = \nu \Delta_n \), where \( \nu \in C^1(\Omega, \mathbb{R}) \) is an arbitrary function. Then system (3) becomes

\[
\dot{x}_k = \nu \cdot \frac{\partial(C_1, C_2, \ldots, C_{n-1}, x_k)}{\partial(x_1, x_2, \ldots, x_{n-1}, x_n)}, \quad k \in \{1, 2, \ldots, n\}, \quad (8)
\]
and it is obvious that it has the same constants of motion $C_1, \ldots, C_{n-1}$. Furthermore, System (8) is the Hamilton–Poisson system

$$
\dot{x}_k = \{H, x_k\}_I^{C_1, \ldots, C_{n-2}}, \; k \in \{1, 2, \ldots, n\},
$$

(9)

with the Hamiltonian $H := C_{n-1}$ and the Poisson bracket (see, e.g., [18]) given by

$$
\{f, g\}^{C_1, \ldots, C_{n-2}} := v \cdot \frac{\partial (C_1, C_2, \ldots, C_{n-2}, f, g)}{\partial (x_1, x_2, \ldots, x_{n-1}, x_n)}, \; f, g \in C^\infty(\mathbb{R}^n, \mathbb{R}).
$$

(10)

In this manner, a family of maximally superintegrable systems with the same constants of motion was constructed. On the other hand, for a given maximally superintegrable system, there is a unique function $v$ such that the system takes the form (9) [17,18] (see the next Remark). The function $v$ is called the rescaling function [18] and it is usually defined on an open and dense subset of $\Omega$.

**Remark 1.** In the papers [17,18], a system $\dot{x} = X(x)$ on $\mathbb{R}^n$ is considered. For this system, it is assumed there are $n - 1$ functionally independent constants of motion $H_1, \ldots, H_{n-1}$. In [17] it was proven that $X = (X_1, \ldots, X_n)$ is orthogonal to the vectors $\nabla H_k, k \in \{1, 2, \ldots, n - 1\}$, and consequently there is a function $u$ such that $X_k = u : \frac{\partial (H_1, H_2, \ldots, H_{n-1})}{\partial (x_1, x_2, \ldots, x_n)}$, $\forall k \in \{1, 2, \ldots, n\}$. In [18], using the orthogonality of the vector fields $X$ and $\nabla H_k, k \in \{1, 2, \ldots, n - 1\}$, it was obtained that $X$ is given as the vector field $\ast(\nabla H_I \wedge \ldots \wedge \nabla H_{n-1})$ multiplied by a $C^1$ real function, where $\ast$ denotes the Hodge star operator for multivector fields.

In the final part of this section, we give integrable deformations of system (9).

By a deformation of a system $\dot{x} = f(x)$, where $x = (x_1, x_2, \ldots, x_n)$, $f = (f_1, f_2, \ldots, f_n)$, we understand a system $\dot{x} = f(x) + u(x)$, where $u = (u_1, u_2, \ldots, u_n)$ and each function $u_i, i \in \{1, 2, \ldots, n\}$ depends on some real parameters such that when these parameters vanish, then $u$ vanishes. When the initial system and its deformation are maximally superintegrable, then we talk about an integrable deformation.

In the next result, we give a family of integrable deformations of a maximally superintegrable system.

**Theorem 1.** Consider system (9) generated by the constants of motion $C_1, \ldots, C_n$, a rescaling function $v$, and the Poisson bracket (10).

Let $a_1, a_2, \ldots, a_{n-1} \in C^1(\Omega, \mathbb{R})$ be arbitrary functions such that the functions

$$
I_k = C_k + g_k a_k, \; k \in \{1, 2, \ldots, n - 1\}
$$

(11)

are functionally independent on an open set $\Omega \subset \mathbb{R}^n$, where $g_1, g_2, \ldots, g_{n-1} \in \mathbb{R}$. Then an integrable deformation of system (9) is given by

$$
x_k = \{C_{n-1}, x_k\}_I^{C_1, \ldots, C_{n-2}} + g_1 \{C_{n-1}, x_k\}^{C_1, C_2, \ldots, C_{n-2}} + \sum_{i=1}^{n-3} g_i \{C_{n-1}, x_k\}^{C_1, \ldots, I_i, a_i, C_i, \ldots, C_{n-2}}
+ g_{n-2} \{C_{n-1}, x_k\}^{I_1, \ldots, I_{n-1}, C_1, \ldots, C_{n-2}} + g_{n-1} \{a_{n-1}, x_k\}^{I_1, \ldots, I_{n-2}}, \; k \in \{1, 2, \ldots, n\}.
$$

(12)

**Proof.** The functions $I_1, \ldots, I_{n-1}$ and the rescaling function $v$ generate the system

$$
x_k = \{I_{n-1}, x_k\}^{I_1, \ldots, I_{n-2}}, \; k \in \{1, 2, \ldots, n\},
$$

(13)

where the Poisson bracket $\{., .\}^{I_1, \ldots, I_{n-2}}$ is given by (10). Using the properties of determinants, for every $k \in \{1, 2, \ldots, n\}$ equation (13) successively becomes

$$
\dot{x}_k = \{C_{n-1}, x_k\}^{I_1, \ldots, I_{n-2}} + g_{n-1} \{a_{n-1}, x_k\}^{I_1, \ldots, I_{n-2}},
$$

(14)

$$
\dot{x}_k = \{C_{n-1}, x_k\}^{I_1, \ldots, I_{n-3}, C_{n-2}} + g_{n-2} \{C_{n-1}, x_k\}^{I_1, \ldots, I_{n-3}, a_{n-2}} + g_{n-1} \{a_{n-1}, x_k\}^{I_1, \ldots, I_{n-2}},
$$

(15)
and finally, system (13) takes the form (12).

It is easy to see that if the deformation parameters \( g_1, g_2, \ldots, g_{n-1} \) vanish, then system (12) is the initial system (9). Moreover, functions \( f_1, \ldots, f_{n-1} \) are a constant of motions of (12); therefore, system (12) is an integrable deformation of system (9). \( \Box \)

**Remark 2.** In order to apply the above Theorem to a maximally superintegrable system \( \dot{x} = f(x) \), where \( x = (x_1, x_2, \ldots, x_n) \), \( f = (f_1, f_2, \ldots, f_n) \), first we determine the rescaling function \( \nu \), as in Remark 1. Secondly, we alter the constants of motion and then apply (12). The new system (12) can be considered as the controlled initial system, where deformation parameters become control parameters. By choosing appropriate deformation functions \( \alpha_1, \ldots, \alpha_{n-1} \), the dynamics of the initial system may be controlled in a desired way. In addition, the methods of geometric mechanics are applicable to such systems.

**Remark 3.** Every three-dimensional Hamilton–Poisson system is a maximally superintegrable system; therefore, Theorem 1 gives integrable deformations for such systems.

**Remark 4.** The integrable deformation (12) of system (9) is a Hamilton–Poisson system with the Hamiltonian (12) becomes

\[
\dot{q} = \{ f_1, f_2, \ldots, f_n \}_N := \frac{\partial (f_1, f_2, \ldots, f_n)}{\partial (x_1, x_2, \ldots, x_n)}, \quad f_1, f_2, \ldots, f_n \in C^\infty(\mathbb{R}^n, \mathbb{R}).
\]

Under some supplementary conditions, for example \( \dot{\nu} = 0 \) (see, e.g., [20]), the bracket

\[
\{ f_1, f_2, \ldots, f_n \}_N := \nu \cdot \{ f_1, f_2, \ldots, f_n \}_N
\]

is also a Nambu bracket, therefore system (9) is a generalized Nambu–Hamilton system. Consequently, we have the following integrable deformation of this Nambus system:

\[
\dot{x}_k = \{ C_1, \ldots, C_{n-1}, x_k \}_N + g_1 \{ x_1, C_2, \ldots, C_{n-2}, C_{n-1}, x_k \}_N + \sum_{i=2}^{n-3} g_i \{ 1, \ldots, i-1, \alpha_i, C_{i+1}, \ldots, C_{n-2}, C_{n-1}, x_k \}_N + g_{n-2} \{ 1, \ldots, i-3, \alpha_{n-2}, C_{n-1}, x_k \}_N + g_{n-1} \{ 1, \ldots, i-2, \alpha_{n-1}, x_k \}_N, \quad k \in \{ 1, 2, \ldots, n \}.
\]

**Remark 5.** In order to preserve the Poisson bracket (10) generated by \( C_1, \ldots, C_{n-2} \), we alter only the Hamiltonian \( C_{n-1} \), namely, \( I_{n-1} = C_{n-1} + p \alpha_{n-1} \). In this case, the integrable deformation (12) becomes

\[
\dot{x}_k = \{ C_{n-1}, x_k \}_N + p \{ \alpha_{n-1}, x_k \}_N, \quad k \in \{ 1, 2, \ldots, n \},
\]

where \( p \) is a deformation parameter.

**Example 1.** Consider the periodic Volterra system (see, e.g., [21])

\[
\dot{x}_1 = x_1(x_2 - x_4), \quad \dot{x}_2 = x_2(x_3 - x_1), \quad \dot{x}_3 = x_3(x_4 - x_2), \quad \dot{x}_4 = x_4(x_1 - x_3)
\]

that describes an interaction between only the close neighboring variables. This system has the Hamiltonian function \( H = x_1 + x_2 + x_3 + x_4 \) and the constants of motion \( C_1 = x_2 x_4 \) and \( C_2 = x_1 x_3 \). Therefore, it is maximally superintegrable. Furthermore, this system is the Nambu system

\[
\dot{x}_k = \{ C_1, C_2, H, x_k \}_N, \quad k \in \{ 1, 2, 3, 4 \},
\]
where the Nambu bracket is given by (17)

\[
\{ f_1, f_2, f_3, f_4 \}_N := \frac{\partial (f_1, f_2, f_3, f_4)}{\partial (x_1, x_2, x_3, x_4)}, \quad f_1, f_2, f_3, f_4 \in C^\infty (\mathbb{R}^4, \mathbb{R}).
\]

Consider the deformation parameters \( g_i \in \mathbb{R} \) and deformation functions \( \alpha_i, i \in \{1, 2, 3\} \). Denote \( I_1 = C_1 + g_1 \alpha_1, I_2 = C_2 + g_2 \alpha_2, \hat{H} = H + g_3 \alpha_3 \). Therefore, system

\[
\begin{align*}
\dot{x}_1 &= C_1, C_2, H, x_k \} + g_1 \{ \alpha_1, C_2, H, x_k \} + g_2 \{ I_1, I_2, H, x_k \} + g_3 \{ I_1, I_2, \alpha_3, x_k \} N, \\
\end{align*}
\]

\( k \in \{1, 2, 3, 4\} \),
gives an integrable deformation of the considered system.

Considering particular deformation functions, one can obtain various new Nambu systems. For example, when we choose \( \alpha_1 = -x_1^2, \alpha_2 = -x_3^2, \alpha_3 = 0 \), then we obtain the following integrable deformation:

\[
\begin{align*}
\dot{x}_1 &= x_1(x_2 - x_4 + 2g_1 x_2) \\
\dot{x}_2 &= x_2(x_3 - x_1 - 2g_2 x_1) \\
\dot{x}_3 &= (x_3 - 2g_2 x_1)(x_4 - x_2 - 2g_1 x_2) \\
\dot{x}_4 &= (x_4 - 2g_1 x_2)(x_1 - x_3 + 2g_2 x_1).
\end{align*}
\]

3. Integrable Deformations of a Maximally Superintegrable Hamiltonian System

In this section, we discuss some integrable deformations of a Hamiltonian system on a \( 2n \)-dimensional phase space in the case when it possesses \( 2n - 1 \) functionally independent integrals of motion.

Let \( M \subset \mathbb{R}^{2n} \) be a smooth manifold and let \( (M, \omega, H) \) be a Hamiltonian mechanical system, where \( \omega = \sum_{i=1}^n dp_i \wedge dq_i \) and \( H \) is the Hamiltonian. The equations of motion are given by

\[
\begin{align*}
\dot{q}_i &= \frac{\partial H}{\partial p_i} = \{ H, q_i \}_\omega, \\
\dot{p}_i &= -\frac{\partial H}{\partial q_i} = \{ H, p_i \}_\omega, \quad i \in \{1, 2, \ldots, n\},
\end{align*}
\]

where \( \{ \cdot, \cdot \}_\omega \) is the Poisson bracket on \( \mathbb{R}^{2n} \), that is, \( \{ f, g \}_\omega = \sum_{i=1}^n \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) \) (see, e.g., [22,23]).

Consider that \( C_1, C_2, \ldots, C_{2n-2}, H = C_{2n-1} \) are functionally independent integrals of motion of system (20), thus, \( C_j = \{ H, C_j \}_\omega = 0, \quad j \in \{1, 2, \ldots, 2n - 2\} \). System (20) has the form \( \dot{x} = f(x) \), where \( x = (x_1, x_2, \ldots, x_{2n}), \quad f = (f_1, f_2, \ldots, f_{2n}) \), and the procedure used in the previous section gives integrable deformations of system (20). It is natural to ask whether such a deformation is a Hamiltonian mechanical system. In addition, if the Hamiltonian mechanical system (20) is completely integrable in Arnold–Liouville’s sense, then it is maximally superintegrable [15] and, consequently, the following question arises: is the above-mentioned deformation a maximally superintegrable Hamiltonian mechanical system? To give a partial answer to this questions, we consider the case when only the Hamiltonian is deformed.

First, we perturb the Hamiltonian function \( H \); that is, we consider the new Hamiltonian function \( \hat{H} \) given by \( \hat{H} = H + k\alpha \), where \( k \in \mathbb{R} \) and \( \alpha \) is an arbitrary differentiable function on \( M \) such that \( C_1, C_2, \ldots, C_{2n-2}, H \) are functionally independent. Then we obtain the following Hamiltonian mechanical system

\[
\begin{align*}
\dot{q}_i &= \frac{\partial H}{\partial p_i} + k \cdot \frac{\partial \alpha}{\partial p_i} = \{ H, q_i \}_\omega + k \{ \alpha, q_i \}_\omega \\
\dot{p}_i &= -\frac{\partial H}{\partial q_i} - k \cdot \frac{\partial \alpha}{\partial q_i} = \{ H, p_i \}_\omega + k \{ \alpha, p_i \}_\omega, \quad i \in \{1, 2, \ldots, n\},
\end{align*}
\]
which is a deformation of system (20). In general, we notice that the equality \( \{ \tilde{H}, C_j \}_{\omega} = 0 \) does not hold.

On the other hand, taking into account Remark 1, there is a function \( \nu \) such that
\[
\nu \cdot \frac{\partial (C_1, \ldots, C_{2n-2}, H, q_i)}{\partial (p_1, \ldots, p_n, q_1, \ldots, q_n)} = \frac{\partial H}{\partial p_i}, \quad \nu \cdot \frac{\partial (C_1, \ldots, C_{2n-2}, H, p_i)}{\partial (p_1, \ldots, p_n, q_1, \ldots, q_n)} = -\frac{\partial H}{\partial q_i}, \quad i \in \{1, 2, \ldots, n\}.
\]

Using the Poisson bracket (10)
\[
\{f, g\}_x^{C_1, \ldots, C_{2n-2}} = \nu \cdot \frac{\partial (C_1, \ldots, C_{2n-2}, f, g)}{\partial (p_1, \ldots, p_n, q_1, \ldots, q_n)}, \quad f, g \in C^\infty(M, \mathbb{R}),
\]
system (20) reads
\[
q_i = \{H, q_i\}_x^{C_1, \ldots, C_{2n-2}}, \quad p_i = \{H, p_i\}_x^{C_1, \ldots, C_{2n-2}}, \quad i \in \{1, 2, \ldots, n\}. \tag{22}
\]

Therefore, via Theorem 1, an integrable deformation of system (22) in the case \( \tilde{H} = H + ka \) is given by
\[
\begin{cases}
q_i = \{H, q_i\}_x^{C_1, \ldots, C_{2n-2}} + k\{a, q_i\}_x^{C_1, \ldots, C_{2n-2}} & , i \in \{1, 2, \ldots, n\}.
\end{cases} \tag{23}
\]

It is clear that \( \{\tilde{H}, C_j\}_x^{C_1, \ldots, C_{2n-2}} = \{C_i, C_j\}_x^{C_1, \ldots, C_{2n-2}} = 0, i, j \in \{1, 2, \ldots, 2n - 2\} \).

**Remark 6.** Systems (21) and (23) represent two deformations of the same Hamiltonian system (20) obtained by a perturbation of the Hamiltonian \( H \). In the first case, system (21) is still a Hamiltonian mechanical system, but it does not preserve the other constants of motion. In the second case, system (23) preserves the integrals of motion, and it is a Hamilton–Poisson system; however, generally, it is not a Hamiltonian mechanical system with the Hamiltonian function \( \tilde{H} \). We notice that if the deformation function \( a \) is chosen such that
\[
\{a, q_i\}_x^{C_1, \ldots, C_{2n-2}} = \frac{\partial a}{\partial p_i}, \quad \{a, p_i\}_x^{C_1, \ldots, C_{2n-2}} = -\frac{\partial a}{\partial q_i}, \quad i \in \{1, 2, \ldots, n\},
\]
then systems (21) and (23) are identical. Therefore, system (21) preserves the first integrals \( C_1, \ldots, C_{2n-2} \) and (23) is also a Hamiltonian mechanical system. Moreover, if the Hamiltonian mechanical system (20) is maximally superintegrable, then (21) is also a maximally superintegrable Hamiltonian mechanical system.

**Example 2.** Consider the Hamiltonian function \( H(q_1, q_2, p_1, p_2) = U(q_1 - q_2) + V(p_1 - p_2), \) where \( U, V \) are smooth functions. The equations of motion are given by
\[
q_1 = V'(p_1 - p_2), \quad q_2 = -V'(p_1 - p_2), \quad p_1 = -U'(q_1 - q_2), \quad p_2 = U'(q_1 - q_2).
\]

It immediately follows that the functions \( C_1 = \frac{q_1 + q_2}{2} \) and \( C_2 = p_1 + p_2 \) are first integrals of the above system. This Hamiltonian mechanical system is maximally superintegrable. In addition, it can be written in the form (22) with \( \nu = 1 \).

Taking \( l_1 = C_1 + q_1a_1, l_2 = C_2 + q_2a_2, l_3 = H + q_3a_3, \) Theorem 1 furnishes the following integrable deformation of the considered system
\[
q_1 = V'(p_1 - p_2) + q_1\{H, q_1\}_{a_1, C_2} + q_2\{H, q_2\}_{a_2, C_2} + q_3\{a_3, q_1\}_{a_3, l_1},
q_2 = -V'(p_1 - p_2) + q_1\{H, q_2\}_{a_1, C_2} + q_2\{H, q_2\}_{a_2, C_2} + q_3\{a_3, q_2\}_{a_3, l_1},
\]
\[
p_1 = -U'(q_1 - q_2) + q_1\{H, p_1\}_{a_1, C_2} + q_2\{H, p_1\}_{a_2, C_2} + q_3\{a_3, p_1\}_{a_3, l_1},
\]
\[
p_2 = U'(q_1 - q_2) + q_1\{H, p_2\}_{a_1, C_2} + q_2\{H, p_2\}_{a_2, C_2} + q_3\{a_3, p_2\}_{a_3, l_1}.
\]
Particularly, if we alter only the Hamiltonian function, that is, \(g_1 = g_2 = 0\), then the above integrable deformation becomes
\[
q_1 = V'(p_1 - p_2) + \frac{g_3}{2} \left( \frac{\partial \alpha_3}{\partial p_1} - \frac{\partial \alpha_3}{\partial p_2} \right), \quad q_2 = -V'(p_1 - p_2) + \frac{g_3}{2} \left( \frac{\partial \alpha_3}{\partial p_2} - \frac{\partial \alpha_3}{\partial p_1} \right),
\]
\[
p_1 = -U'(q_1 - q_2) - \frac{g_3}{2} \left( \frac{\partial \alpha_3}{\partial q_1} - \frac{\partial \alpha_3}{\partial q_2} \right), \quad p_2 = U'(q_1 - q_2) - \frac{g_3}{2} \left( \frac{\partial \alpha_3}{\partial q_2} - \frac{\partial \alpha_3}{\partial q_1} \right).
\]

By Remark 6, we deduce that this particular integrable deformation is in fact a Hamiltonian mechanical system with the Hamiltonian function \(\tilde{H} = I_3 = H + g_3\alpha_3\) if, and only if \(\alpha_3(q_1, q_2, p_1, p_2) = W(q_1 - q_2, p_1 - p_2)\), where \(W\) is a smooth function. In this case, the above integrable deformation is a maximally superintegrable Hamiltonian mechanical system.

**Remark 7.** While the deformation (21) of system (20) can be obtained in one way, namely by a perturbation of the Hamiltonian function, integrable deformations of the same system, written in the form (22), can be obtained in many more ways by altering one or more of the functions \(C_1, C_2, \ldots, C_{2n-1}\).

**4. Some Deformations of an Arbitrary System of First-Order Autonomous Differential Equations**

In [10], some deformations of three-dimensional systems of differential equations were obtained. First, a Hamilton–Poisson part of the considered system is identified. Then an integrable deformation of this part is constructed by the alteration of its constants of motion. Lastly, the non-Hamilton–Poisson part and the obtained integrable deformation are put together to obtain a new system, that is, a deformation of the initial system. In this section we extend this method to systems of differential equations on \(\mathbb{R}^n, n > 3\). More precisely, the same method works if we can identify a maximally superintegrable part of the considered system.

Consider a dynamical system of the form
\[
x = f(x),
\]  
where \(x = (x_1, x_2, \ldots, x_n), f = (f_1, f_2, \ldots, f_n)\). We assume that system (24) can be written in the form
\[
x = s(x) + h(x),
\]  
where the system
\[
x = s(x)
\]  
is maximally superintegrable. Then, using the notations from Section 2, system (26) takes the form (9) and one of its integrable deformations is given by Theorem 1. We denote this deformation by
\[
x = s(x) + \tilde{s}(x).
\]  
Adding \(h(x)\) to the right side, we obtain the following deformation of system (24)
\[
x = f(x) + \tilde{s}(x).
\]

Note that in the case of maximally superintegrable systems, some constants of motions can be preserved. If a system has a constant of motion, but it is not maximally superintegrable, its above-mentioned deformation does not generally preserve that constant of motion. It is natural to ask whether there are deformations that preserve a constant of motion.

**Example 3.** Another Lotka–Volterra type system (see, e.g., [21]) is given by
\[
\dot{x}_1 = x_1 x_2, \quad \dot{x}_2 = x_2 (x_3 - x_1), \quad \dot{x}_3 = x_3 (x_4 - x_2), \quad \dot{x}_4 = -x_3 x_4.
\]
The function $H = x_1 + x_2 + x_3 + x_4$ is a constant of motion of this system. Apparently, this system is not maximally superintegrable. However, we can consider a maximally superintegrable part of it, namely,

\[ \dot{x}_1 = 0, \quad \dot{x}_2 = x_2 x_3, \quad \dot{x}_3 = -x_2 x_3, \quad \dot{x}_4 = -x_3 x_4. \]

Indeed, the functions $C_1 = x_1, C_2 = x_2 + x_3, \text{ and } C_3 = x_2 x_4$ are constants of motion. Moreover, the above system takes the form

\[ \dot{x}_k = \{C_3, x_k\}_{C_1, C_2}, \quad k \in \{1, 2, 3, 4\}, \]

where $\nu = x_3$ and the bracket is given by (10). If we consider $I_1 = C_1 + g_1 \alpha_1, I_2 = C_2 + g_2 \alpha_2$, $I_3 = C_3 + g_3 \alpha_3$, then an integrable deformation of the maximally superintegrable part of the considered system is given by Theorem 1

\[ \dot{x}_k = \{C_3, x_k\}_{C_1, C_2} + g_1 \{C_3, x_k\}_{I_1, I_2} + g_2 \{C_3, x_k\}_{I_1, I_2} + g_3 \{\alpha_3, x_k\}_{I_1, I_2}, \quad k \in \{1, 2, 3, 4\}. \]

Now we bring back the unused terms into the system and obtain the following deformation of the considered system

\[
\begin{align*}
\dot{x}_1 &= x_1 x_2 + g_1 \{C_3, x_1\}_{I_1, I_2} + g_2 \{C_3, x_1\}_{I_1, I_2} + g_3 \{\alpha_3, x_1\}_{I_1, I_2}, \\
\dot{x}_2 &= x_2 (x_3 - x_1) + g_1 \{C_3, x_2\}_{I_1, I_2} + g_2 \{C_3, x_2\}_{I_1, I_2} + \{\alpha_3, x_2\}_{I_1, I_2}, \\
\dot{x}_3 &= x_3 (x_4 - x_2) + g_1 \{C_3, x_3\}_{I_1, I_2} + g_2 \{C_3, x_3\}_{I_1, I_2} + g_3 \{\alpha_3, x_3\}_{I_1, I_2}, \\
\dot{x}_4 &= x_3 x_4 + g_1 \{C_3, x_4\}_{I_1, I_2} + g_2 \{C_3, x_4\}_{I_1, I_2} + g_3 \{\alpha_3, x_4\}_{I_1, I_2}.
\end{align*}
\]

A particular deformation is obtained if we consider particular deformation functions. For example, if we take $g_1 = g_3 = 0$, and $\alpha_2(x_1, x_2, x_3, x_4) = \beta(x_2)$, then we get

\[ \dot{x}_1 = x_1 x_2, \quad \dot{x}_2 = x_2 (x_3 - x_1), \quad \dot{x}_3 = x_3 (x_4 - x_2) - g_2 x_2 x_3 \beta'(x_2), \quad \dot{x}_4 = -x_3 x_4. \]

It is easy to see that $H$ is not a constant of motion of the above system. Nevertheless, we can find some deformation functions such that $H$ will be a constant of motion. For example, if we choose $g_1 = g_2 = g, g_3 = 0$ and $\alpha_1(x_1, x_2, x_3, x_4) = \alpha(x_4), \alpha_2(x_1, x_2, x_3, x_4) = \beta(x_4)$, then we obtain

\[ \dot{x}_1 = x_1 x_2 + g x_3 x_4 \alpha'(x_4), \quad \dot{x}_2 = x_2 (x_3 - x_1), \quad \dot{x}_3 = x_3 (x_4 - x_2) + g x_3 x_4 \beta'(x_4), \quad \dot{x}_4 = -x_3 x_4. \]

We obtain

\[ H = g x_3 x_4 (\alpha'(x_4) + \beta'(x_4)). \]

Therefore, if $\alpha(x_4) + \beta(x_4)$ = constant, then $H$ is a constant of motion of the deformed system.

5. Conclusions

Integrable deformations of a maximally superintegrable system were obtained by the alteration of their constants of motion. The new systems are also maximally superintegrable. Moreover, this method can be used to perturb a maximally superintegrable part of an arbitrary system of first-order differential equations, which gives a deformation of the considered system.

Note that the systems obtained by deformations depend on more parameters than the initial system. Therefore, they can exhibit rich dynamics and can display some types of bifurcations, such as Hopf, zero-Hopf, Bogdanov–Takens, Bautin, or even three co-dimension bifurcations. Moreover, the functions added to a system by deformation can be viewed as control functions, and as a consequence, some natural queries arise, such as: how the stability changes, whether they can be stabilized by some states/orbits, how the periodic motion is affected, and whether new chaotic systems can be obtained or whether some chaotic trajectories can be stabilized, and so forth.
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