Braided Lie algebras and bicovariant differential calculi over co-quasitriangular Hopf algebras.

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Abstract

We show that if \( g_\Gamma \) is the quantum tangent space (or quantum Lie algebra in the sense of Woronowicz) of a bicovariant first order differential calculus over a co-quasitriangular Hopf algebra \((A, r)\), then a certain extension of it is a braided Lie algebra in the category of \( A \)-comodules. This is used to show that the Woronowicz quantum universal enveloping algebra \( U(g_\Gamma) \) is a bialgebra in the braided category of \( A \)-comodules. We show that this algebra is quadratic when the calculus is inner. Examples with this unexpected property include finite groups and quantum groups with their standard differential calculi. We also find a quantum Lie functor for co-quasitriangular Hopf algebras, which has properties analogous to the classical one. This functor gives trivial results on standard quantum groups \( O_q(G) \), but reasonable ones on examples closer to the classical case, such as the cotriangular Jordanian deformations. In addition, we show that split braided Lie algebras define ‘generalised-Lie algebras’ in a different sense of deforming the adjoint representation. We construct these and their enveloping algebras for \( O_q(SL(n)) \), recovering the Witten algebra for \( n = 2 \).

Introduction.

In a well-known article [Wor], Woronowicz has given an axiomatic treatment of so called bicovariant first order differential calculi (FODC) over Hopf algebras. It appeared that, given an arbitrary Hopf algebra \( A \), there is no canonical way to associate to it a bicovariant FODC. Nevertheless, each bicovariant FODC \( (\Gamma, d) \) over \( A \) extends to a graded differential algebra \( (\Gamma^\wedge, d) \) - later shown to have a Hopf superalgebra structure [Brz, Schb], and has an associated “quantum Lie algebra” \( g_\Gamma \). A quantum Lie algebra is a vector space \( g \) equipped with a braiding operator \( \sigma \) and a “quantum Lie bracket” satisfying certain identities, which coincide with the usual Lie algebra axioms when the braiding \( \sigma \) on \( g \) is the usual flip. This article is concerned with the universal enveloping algebra \( U(g) \) of a quantum Lie algebra (a certain quotient of the tensor algebra of \( g \)). The main question which we address is whether \( U(g) \) can be equipped with a Hopf algebra or bialgebra structure. To do this we will need additional coassociative structure \( \delta : g \to g \otimes g \), which we axiomatise as a ‘good quantum Lie algebra’ \( (g, \sigma, [\cdot, \cdot], \delta) \) (we then show that in the case of \( g_\Gamma \) associated to an FODC there is a canonical such structure when \( A \) is coquasitriangular.)

Let us recall what are the obstructions for a bialgebra structure on \( U(g_\Gamma) \). Before asking in which sense our required coalgebra structure maps \( \Delta \) and \( \varepsilon \) should be algebra morphisms, the coproduct on \( U(g_\Gamma) \) should be coassociative. Woronowicz [Wor] has shown that bicovariant FODC over \( A \) are parametrized by (in our conventions) \( \text{ad}_L \)-invariant left ideals of \( A \) contained in \( \ker \varepsilon_A \). The “cotangent space” of the FODC \( (\Gamma, d) \) corresponding to such an

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ideal $\mathcal{I}$ is canonically identified with $\ker \varepsilon_A/\mathcal{I}$, and the quantum Lie algebra $\mathfrak{g}_R$ (or tangent space) is the dual of this. When $A$ is commutative, $\ker \varepsilon_A/\mathcal{I}$ has a natural associative algebra structure, say $\mu$ (often without unit). Therefore the dual $\mathfrak{g}_R$ has a natural comultiplication $\delta = \mu^*$, and the space $k \mathfrak{g} \oplus \mathfrak{g}_R$ has a coalgebra structure $(\Delta, \varepsilon)$ such that $\Delta(1) = 1 \otimes 1$, $\Delta(x) = x \otimes 1 + 1 \otimes x + \delta(x)$, $\varepsilon(x) = 0$, for $x \in \mathfrak{g}_R$. Actually, for the standard differential calculus on Lie groups, the ideal $\mathcal{I}$ in question is $(\ker \varepsilon_A)^2$, therefore $\mu = 0$, $\delta = 0$, and we recover the standard coproduct on $U(\mathfrak{g}_R)$. In the non-commutative case, these things do not work anymore. The key idea to solve this problem is the observation that, when $A$ has a co-quasitriangular $\mathfrak{r}$, $\text{ad}_{\mathfrak{r}}$-invariant left ideals of $A$ are two-sided ideals of another algebra, namely $\mathfrak{A}$, the braided version (or ‘braided-group’) of the quantum group $A$ in the category of finite-dimensional objects $\mathcal{L}$ with axioms strong enough to define a braided enveloping algebra $B(\mathcal{L})$ as a quadratic bialgebra in a braided category (it was previously denoted $U(\mathcal{L})$ in [Maj-93]. Moreover, for standard quantum groups $U_q(\mathfrak{g})$ there is an algebra map $B(\mathcal{L}) \rightarrow U_q(\mathfrak{g})$ so that $U_q(\mathfrak{g})$ is essentially generated by $\mathcal{L}$ (the ‘Lie problem for quantum groups’). In this case braided Lie algebra $\mathcal{L}$ is of an $n^2$-dimensional matrix form and can in fact be identified [Maj-98] with the quantum tangent space for the FODC on $O_q(G)$ constructed by the R-matrix method of Jurco [Ju]. However, a general theorem systematically linking the Woronowicz theory and the braided Lie algebra theory has been missing and is what we provide now. Indeed, in our construction $\mathfrak{g}_R = k \mathfrak{g} \oplus \mathfrak{g}_R$ is a braided Lie algebra in $\mathcal{M}^A$ and its braided universal enveloping bialgebra $B(\mathfrak{g}_R)$, therefore provides the homogenization or quadratic central extension of $U(\mathfrak{g}_R)$, with $U(\mathfrak{g}_R)$ a bialgebra quotient of $B(\mathfrak{g}_R)$. Thus we prove the existence of a (braided) bialgebra structure on $U(\mathfrak{g}_R)$ for arbitrary bicovariant FODC over $A$, provided $A$ is co-quasitriangular. The existence of an antipode is more problematic, and actually we can prove that an antipode does not exist in many examples of interest, such as for all finite dimensional bicovariant calculi on standard quantum groups $O_q(G)$, $q$ not a root of unity, and all bicovariant calculi on finite groups. The reason is quite curious: We prove in theorem 4.2 (which is general, i.e., does not use co-quasitriangularity) that when the differential $d$ is implemented by a biinvariant element $\theta$, then $U(\mathfrak{g}_R)$ is a quadratic algebra, in fact a quantum symmetric algebra. In the co-quasitriangular inner case, we find that $U(\mathfrak{g}_R)$ is a quadratic bialgebra. More precisely,

$$U(\mathfrak{g}_R) \simeq B(\mathcal{L}) \quad (\Gamma \text{ inner})$$

is the braided universal enveloping algebra of a braided Lie subalgebra $\mathcal{L} \subset \mathfrak{g}_R$, of the same dimension as $\mathfrak{g}_R$. Since $B(\mathcal{L})$ never has an antipode, $U(\mathfrak{g}_R)$ does not have one as well. Moreover, for the simple bicovariant FODC over standard quantum groups $O_q(G)$ where the braided Lie algebra $\mathcal{L}$ in question is a matrix braided Lie algebra, its braided universal enveloping algebra is an algebra of braided matrices $B(R)$ [Ma]. Therefore $U(\mathfrak{g}_R)$ is also an algebra of braided matrices. The algebra $\mathfrak{A}$, which is a key ingredient of our construction, also proves useful to obtain an analogue of the Lie functor for Lie groups [Maj-94]. Our functor goes from the category of co-quasitriangular Hopf algebras (pairs $(A, r)$) to that of first order differential calculi (triples $(A, r, d)$). We show that it shares many of the properties of the classical one; apparently too many because it sends to zero most of the standard quantizations $O_q(G)$ (an exception is $G = GL(n)$ for which the associated quantum Lie algebra has dimension 1).
Thus it provides another point of view on the non triviality of these quantizations. We show in the case $G = SL(n)$ that for the softer (and less interesting) triangular deformations of $O(G)$, the functor does give results which are close to the classical ones (in particular with a reasonable dimension of the quantum Lie algebra).

The last significant contribution of this paper is to establish a relationship between braided Lie algebras and a third approach to q-deformed Lie algebras defined by representation theory. Here $g_q = g$ as a vector space but with the $q$-deformation of the adjoint representation. For generic $q$ there remains a canonical intertwiner $g_q \otimes g_q \rightarrow g_q$ which could be called ‘Lie bracket’. Even though examples have been computed already for some years\cite{DG}, one does not know a full set of axioms that the obtained $(g_q, [\ , \ ])$ should obey. In this context there is similarly a proposal\cite{LS} for an ‘enveloping algebra’ $U_{LS}(g)$ (say) associated to $g$ semisimple. An open problem here, shown only for $g = sl(2)$, is to find some kind of ‘homogenisation’ of $U_{LS}(g)$ mapping onto (the locally finite part of) $U_q(g)$ and thereby relating these algebras. A corollary of the braided Lie theory is a solution of this problem for $g = sl(n)$, as follows. We consider braided Lie algebras that split as $L = kwL^+$, where $L^+$ is the kernel of the counit of the braided Lie algebra and where we suppose that $[c, ]$ acts as a multiple $\lambda$ of the identity. One can (in principle) axiomatise the inherited properties of $L^+$ and define its enveloping algebra as $B_{red}(L^+) = B(L)/\langle c - \lambda \rangle$. For the standard matrix braided Lie algebra $L = sl_q(n)$ associated to $U_q(sl(n))$ we have $c = tr$ (the quantum trace) and $L^+ = sl_q(n)$ (say) has the classical dimension $n^2 - 1$. The enveloping $B_{red}(sl_q(n))$ by construction has homogenisation $B(sl_q(n))$ mapping to (the locally finite part of) $U_q(sl(n))$. It is clear already from \cite{Maj-94} that $B_{red}(sl_q(2)) = U_{LS}(sl(2))$. We also mention that the latter is isomorphic to the Witten algebra $W_q(sl(2))$ introduced in \cite{Witt}, as was already noticed by L. Le Bruyn in \cite{LeB-95}. This suggests a definition of $W_q(sl(n))$ for any $n$ (although the physical requirements which lead to the definition of $W_q(sl(2))$ are not considered here).

Let us explain the content of the paper in more detail. In the first preliminaries section we recall various well-known facts about (co-)quasitriangular Hopf algebras, crossed modules and Hopf bimodules, and quadratic bialgebras. Then in section 2 we recall the definition of a quantum Lie algebra, following theorems 5.3 and 5.4 in \cite{Wor}. (The notion of quantum Lie algebra is however not left-right symmetric and we take the conventions opposite to \cite{Wor}, the main reason being that we want the quantum Lie bracket to be given by the left adjoint action in the differential calculi setting). We observe that the homogenization of the universal enveloping algebra $U(g)$ of any quantum Lie algebra $g$ is a quantum symmetric Lie algebra. We then investigate the existence of a coproduct on the universal enveloping algebra of a quantum Lie algebra $(g, \sigma)$, not necessarily in the context of differential calculi. For this, we suppose the existence of an underlying braided category $(V, \Psi)$ in which both $g$ and $U(g)$ live (the structure maps $(\sigma, [\ , \ ])$ of $g$ should be morphisms in $V$), and look for a coproduct on $U(g)$ of the form $\Delta(x) = x \otimes 1 + 1 \otimes x + \delta(x)$, $x \in g$, for some “little coproduct” $\delta : g \rightarrow g \otimes g$. Let us stress that not all quantum Lie algebras can be equipped with such a little coproduct. Among those which can, there is a subclass with nice properties, leading to our notion of “good” quantum Lie algebras. An important feature of these good quantum Lie algebras is that their braiding $\sigma$ is not an essential datum: it can be expressed in terms of the other structure maps of $g$. Moreover, $\sigma$ can coincide with the underlying braiding $\Psi_{g, g}$ only in some special cases (which include super Lie algebras). Therefore a generic “good quantum Lie algebra” is equipped with two braidings, the categorical braiding $\Psi_{g, g}$, and the braiding $\sigma$, which should not be confused.

Section 3 about braided Lie algebras, is mainly taken from \cite{Maj-94}, with slight improvements, in particular on some properties of the canonical braiding $\Upsilon$, and on the connection with quantum Lie algebras. Recall that a braided Lie algebra is already a coalgebra $(L, \Delta, \varepsilon)$.
in a braided category, endowed with a “braided Lie bracket” satisfying identities which also mimic usual Lie algebra axioms. One of the differences is that they do not have an antisymmetry axiom, and indeed, such an axiom is impossible to define in general. However, one can consider the subclass of braided Lie algebras $L$ which have a braided Lie algebra imbedding $k \to L$. In this case, the Lie algebra-like object inside $L$ is $L^+$ (the kernel of $\varepsilon$), and there is a natural notion of antisymmetry axiom. We call “good” those braided Lie algebras which meet all these requirements, and show that there is a 1-1 correspondence between good braided Lie algebras and good quantum Lie algebras (given by $L \to L^+$). Good braided Lie algebras are precisely the ones which appear as extensions in the context of FODC over co-quasitriangular Hopf algebras. Not all interesting braided Lie algebras are ‘good’ in this sense, e.g. the matrix braided Lie algebras $L$ above are not (their $L^+$ is not a quantum Lie algebra but a ‘generalised’ one).

In section 4, we first recall how quantum Lie algebras arise in the work of Woronowicz [Won], and make clear what we call the extended (co)tangent spaces of a bicovariant FODC. We work with right invariant 1-forms (and left invariant vector fields), therefore most of our formulas differ from that of [Won]. We then prove the main results of this paper, already mentioned. We give examples of non trivial calculi arising from the quantum Lie functor (this mainly concerns the co-triangular case) and at the far opposite examples of differential calculi over Hopf algebras which are “annihilated” by the quantum Lie functor: finite groups and quantum groups. These examples are well-known [BDM] [Maj-98] [KS] [HS], but they illustrate well the fact that $U(\mathfrak{g})$ is quadratic when $\Gamma$ is inner. Finally, section 4 contains the link between $B(L)$ for such calculi and generalised Lie algebras along the lines of [LS].

1 Preliminaries

Throughout, $k$ is a field, vector spaces, algebras, etc, over $k$. The flip is written $\tau (v \otimes w) = w \otimes v$. We use Sweedler’s notation for coproducts and coactions, omitting the summation sign, and Einstein’s convention for summation over repeated indices.

Crossed modules, Hopf modules, etc. Let $(A, m, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra. The Hopf and full duals of $A$ are $A^* \subset A^\oplus$ respectively. The pairing between $A^*$ and $A$ is written independently $x(a) = (a, x), x \in A^*, a \in A$. We let $\text{ad}_L, \text{ad}_R : A \to A^\otimes$ be the left and right adjoint coaction of $A$ on itself ($\text{ad}_L(a) = a(1)S(a(3)) \otimes a(2)$), and $\text{Ad}_L, \text{Ad}_R$ the left and right adjoint action of $A^\otimes$ on itself ($\text{Ad}_L(x) := x \triangleright y := x(1)yS(y(2))$). Then the left coadjoint action of $A^\otimes$ on $A$ is $\text{Ad}_L^*x(a) = (\langle a(0), x \rangle a(1), a(0) \otimes a(1) = \text{ad}_R(a)$.

We recall the useful lemma (if $\xi : A \otimes A \to k$ is some linear map, we define $\xi_1, \xi_2 : A \to A^*$ by $\xi_1(a)(-\varepsilon) = \xi(a, -\varepsilon)$ and $\xi_2(a)(-) = \xi(-, a)$).

Lemma 1.1. Let $\xi : A \otimes A \to k$ be a linear form satisfying $m^\circ \xi = \xi \circ m$ and $\text{Im} \xi_1 \subset A^\circ$, where $\circ$ is the convolution product. Then $\xi_1$ intertwines the left adjoint and coadjoint actions of $A^\circ$ on $A^\otimes$ and $A$ respectively, i.e. $\text{Ad}_Lh \circ \xi_1 = \xi_1 \circ \text{Ad}_Lh$ for all $h \in A^\circ$. Likewise (if $\text{Im} \xi_2 \subset A^\circ$), $\xi_2$ intertwines the right ones.

We write $A^\otimes M_A^\delta, A^\delta, A^\delta M, A^\delta A^\delta$ the categories of Hopf bimodules, left crossed modules, left modules and left comodules over $A$ respectively. Recall that a Hopf bimodule is a vector space $\Gamma$ which is both a bimodule and a bicomodule (with coactions $\Delta_L : \Gamma \to A \otimes \Gamma$ and $\Delta_R : \Gamma \to \Gamma \otimes A$), both coactions commuting with both actions in the natural way. A left crossed module over $A$ is a vector space $V$ endowed with a left $A$-action (noted $a \otimes \eta \mapsto a \triangleright \eta$) and a left $A$-coaction (noted $\eta \mapsto \delta_L(\eta) = \eta^{(-1)} \triangleright \eta^{(0)}$), such that $\delta_L(a \triangleright \eta) = a^{(1)} \eta^{(-1)}S(a^{(2)}) \otimes a^{(3)} \triangleright \eta^{(0)}$. The category $A^\delta$ is a braided (monoidal) category when the antipode of $A$ is invertible. We shall only need the braiding on $V \otimes W$ which is given by $v \otimes w \mapsto v^{(-1)} \triangleright w \otimes v^{(0)}$. 

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with inverse \( w \otimes v \mapsto v^{(0)} \otimes S^{-1}(v^{(-1)}) \triangleright w \). As in [Wor], the categories \( \mathcal{AM}_A \) and \( \mathcal{AC} \) are equivalent. The left crossed module corresponding to \( \Gamma \) is \( (\Gamma_R, \triangleright, \delta_L) \) where \( \Gamma_R = \{ \eta \in \Gamma : \Delta_R(\eta) = \eta \otimes 1 \} \) is the subspace of right invariants of \( \Gamma \), and \( a \triangleright \eta = a(1)\eta S(a(2)) \), \( \delta_L(\eta) = \Delta_L(\eta) \). There is a canonical projection \( \pi_R : \Gamma \to \Gamma_R \) given by \( \pi_R(v) = v^{(0)} S(v^{(1)}) \), where \( v^{(0)} \otimes v^{(1)} = \Delta_R(v) \). It satisfies \( \pi_R(ab) = a \triangleright \pi_R(b) \varepsilon(b), (a, b \in A, v \in \Gamma) \). Conversely, \( \Gamma \) is recovered from \( (\Gamma_R, \triangleright, \delta_L) \) by

\[
\Gamma \simeq \Gamma_R \otimes A \quad \text{via} \quad \begin{cases} 
  v \mapsto \pi_R(v^{(0)}) \otimes v^{(1)} \\
  \eta a \leftarrow \eta \otimes a
\end{cases}
\]

with tensor product bimodule and bicomodule structure (\( \Gamma_R \) is seen as a trivial right module and comodule, and \( A \) is the regular Hopf bimodule) : writing \( \Gamma = \Gamma_R A \) as free right \( A \)-module (instead of \( \Gamma_R \otimes A \)), the extra structures are \( (\eta \in \Gamma_R, a \in A) \)

\[
a \eta = (a(1) \triangleright \eta), a(2), \quad \Delta_L(\eta, a) = \eta^{(-1)}(a(1)) \otimes (\eta^{(0)}), a(2), \quad \Delta_R(\eta, a) = (\eta, a(1)) \otimes a(2)
\]

A co-quasitriangular structure on a bialgebra \( A \) is a linear map \( r : A \otimes A \to k \) which intertwines the multiplication of \( A \) and its opposite \( (m^{op} \ast r = r \ast m, * \text{ being the convolution product}) \), and satisfies \( r(ab, c) = r(a(2), b) r(a(1), c) \), and \( r(a, bc) = r(a, c(1)) r(b, c(2)) \) for all \( a, b, c \in A \). The maps \( r_1, r_2, r_3 : A \to A^* \) take their values in \( A^0 \) and satisfy:

\[
\begin{align*}
  r_1 : A &\to A^0 \text{ is an algebra/anticoalgebra map}, \\
  r_2 : A &\to A^0 \text{ is an antialgebra/coalgebra map}.
\end{align*}
\]

When \( (A, r) \) is co-quasitriangular, the tensor category \( \mathcal{AM} \) is braided, the braiding on \( V \otimes W \) being

\[
v \otimes w \mapsto r(w^{(-1)}, v^{(-1)}) w^{(0)} \otimes v^{(0)}.
\]

If moreover \( A \) has an antipode \( S \), then \( S^2 \) is inner (hence \( S \) is bijective) and the form \( r \) is convolution invertible with inverse \( \bar{r} \) such that \( \bar{r}(a, b) = r(S(a), b) \), that is \( r_1 = r_1 \circ S = S^{-1} \circ r_1 \), and \( r_2 = r_2 \circ S^{-1} = S \circ r_2 \). Then the braiding on \( V \otimes W \) is invertible with inverse \( w \otimes v \mapsto \bar{r}(w^{(-1)}, v^{(-1)}) v^{(0)} \otimes w^{(0)} \).

In this article, \( (V, \otimes) \) is a monoidal category of the form \( \mathcal{AM}, \mathcal{AC}, \mathcal{AC} \) or variants (switching left and right), \( A \) being a bialgebra or Hopf algebra. Thus, its objects are in particular vector spaces, and \( k \) is the underlying vector space of the unit object. Recall that if a braiding \( \Psi \) exists (eg \( A \) is co-quasitriangular in the case of \( \mathcal{AM} \)), it is a collection of natural (iso)morphisms \( \Psi_{M,N} : M \otimes N \to N \otimes M \) for all pair \((M, N)\) of objects in \( V \). The naturality means that if \( f : M \to M' \) and \( g : N \to N' \) are morphisms, then the equality \( \Psi_{M',N'} \circ (f \otimes g) = (g \otimes f) \circ \Psi_{M,N} \) holds. Also, the structure maps of an algebra, coalgebra, etc are by assumption morphisms in \( V \). Finally, if \( A \) and \( B \) are algebras in \( V \), their tensor product in \( V \) is \( A \otimes B \), with multiplication \( m_A \otimes e_B = (m_A \otimes m_B)(id_A \otimes e_B \otimes e_B) \). The sign \( \otimes \) is to stress the braided structure. Likewise for coalgebras. Thus a bialgebra in \( V \) is both an algebra \((B, m_B, \eta_B)\) and coalgebra \((B, \Delta_B, \varepsilon_B)\), \( \Delta_B : B \to B \otimes B \) and \( \varepsilon_B : B \to k \) being morphisms of algebras in this braided sense.

**Quadratic bialgebras.** Let \((C, \Delta, \varepsilon)\) be a coalgebra in \((V, \otimes, \Psi)\) as above. Its tensor algebra \( T(C) = \bigoplus_{n \geq 0} C^\otimes n \) with \( C^0 = k \), is naturally a bialgebra in \( V \). The coalgebra structure on each summand \( C^\otimes n \) is the (braided) tensor product one. Let \( V \subset C \otimes C \) be a subobject and \( \langle V \rangle \subset T(C) \) be the 2-sided ideal generated by \( V \). Clearly, \( T(C)/\langle V \rangle \) is a bialgebra in \( V \) if and only if \( \Delta_C \otimes_C (V) \subset C^\otimes 2 \otimes V + V \otimes C^\otimes 2 \) and \( \varepsilon_C \otimes_C (V) = 0 \).

**Lemma 1.2.** *The bialgebra \( T(V)/\langle V \rangle \) never has an antipode.*

Proof. Assume that there is a map \( S' : T(C) \to T(C) \) such that the induced map \( S : T(C)/\langle V \rangle \to T(C)/\langle V \rangle, S(a + \langle V \rangle) := S'(a) + \langle V \rangle, \) is an antipode. For all \( c \in C \to T(C) \)
one should have $S'(c_{(1)}) \otimes c_{(2)} - \varepsilon(c)1 = \sum_i a_i \otimes v_i \otimes b_i$ for some $a_i, b_i \in T(C), v_i \in V \subset C \otimes C$. If $\varepsilon(c) = 1$, this would mean that $1 \in \otimes n \geq 1 C \otimes n$, which is false. $\Box$

One can always see $V$ as im$(F)$ for some morphism $F : C \otimes C \rightarrow C \otimes C$ (possibly not of coalgebras). Then (im$(F)$) is a bialgebra ideal if and only if there exist some maps $\Phi, \Phi' : C^\otimes 4 \rightarrow C^\otimes 4$ such that

$$\Delta_{C^\otimes C} \circ F = ((F \otimes \text{id}^\otimes 2) \circ \Phi + (\text{id}^\otimes 2 \otimes F) \circ \Phi') \circ \Delta_{C^\otimes C}, \quad \varepsilon_{C^\otimes C} \circ F = 0 \quad (1.1)$$

(we use implicitly the fact that $\Delta_{C^\otimes C}$ is injective so that any map $C^\otimes 2 \rightarrow C^\otimes 4$ is of the form $\Phi \circ \Delta_{C^\otimes C}$ for some map $\Phi : C^\otimes 4 \rightarrow C^\otimes 4$). We shall use the following special cases.

**Lemma 1.3.** Let $(C, \Delta, \varepsilon)$ be a coalgebra in $\mathcal{V}$. If $\Upsilon : C^\otimes C \rightarrow C^\otimes C$ is a morphism of coalgebras, then $T(C)/(\langle \text{im}(\text{id} - \Upsilon) \rangle)$ is a bialgebra in $\mathcal{V}$, without antipode.

**Proof.** Let $F = \text{id}^\otimes 2 - \Upsilon : C^\otimes C \rightarrow C^\otimes C$. The hypotheses on $\Upsilon$ are exactly that $\sqcup$ is satisfied with $\Phi = \text{id}^\otimes 4$ and $\Phi' = \Upsilon \otimes \text{id} \otimes \text{id}$, (or $\Phi = \text{id} \otimes \text{id} \otimes \Upsilon$ and $\Phi' = \text{id}^\otimes 4$). $\Box$

The following is from $\text{Do}$. Let $(C, \Delta, \varepsilon)$ be a coalgebra in $\mathcal{M}$ (i.e. a usual coalgebra) and $r : C \otimes C \rightarrow k$ some linear map. Define $F_+, F_- : C \otimes C \rightarrow C \otimes C$ by $F_+(a \otimes b) = r(a(1), b(1)) a(2) \otimes b(2)$ and $F_-(a \otimes b) = b(1) \otimes a(1) r(a(2), b(2))$. Then $F = F_+ - F_-$ satisfies (1.1) with $\Phi = \Phi' = \text{id}^\otimes 4$, independently of the choice of the linear map $r$. Thus $A(C, r) := T(C)/(\langle \text{im}(F_+ - F_-) \rangle)$ is a bialgebra, generated $C$ with relations

$$r(a(1), b(1)) a(2) b(2) = b(1) a(1) r(a(2), b(2)) \quad (1.2)$$

Note that if $r$ is convolution invertible, $T(C)/(\langle \text{im}(F_+ - F_-) \rangle) = T(C)/(\langle \text{im}(\text{id} - \Upsilon) \rangle)$, where $\Upsilon(a \otimes b) = \bar{r}(a(1), b(1)) b(2) \otimes a(2) r(a(3), b(3))$ is a morphism of coalgebras.

**Lemma 1.4.** $\text{Do}$ If $r$ is convolution invertible, the following are equivalent:

1. The linear map $r : C \otimes C \rightarrow k$ extends (uniquely) to a co-quasitriangular structure $r$ on $A(C, r)$.
2. The identity $r(a(1), b(1)) r(a(2), c(1)) r(b(2), c(2)) = r(b(1), c(1)) r(a(1), c(2)) r(a(2), b(2))$ holds for all $a, b, c \in C$.
3. The map $\Sigma : C \otimes C \rightarrow C \otimes C, \Sigma(a \otimes b) = r(b(1), a(1)) b(2) \otimes a(2)$, satisfies the braid relation.

**2 Quantum Lie algebras.**

Let $(\mathcal{V}, \otimes)$ be a possibly not braided-monoidal category as in the preliminaries. (Its objects are in particular $k$-vector spaces, $k$ is the underlying vector space of the unit object, and by convention $\gamma$ will always stand for a distinguished basis vector of the unit object).

**Definition 2.1.** A left quantum Lie algebra in $\mathcal{V}$ is a triple $(g, \sigma, [, [,])]$ where $g$ is an object, $\sigma : g \otimes g \rightarrow g \otimes g$ and $[, [,] : g \otimes g \rightarrow g$ are morphisms satisfying the following axioms

1. $\sigma$ satisfies the braid relation.
2. Quantum Jacobi identity : $[x, [y, z]] = [[x, y], z] + \sum_i [y_i, [x_i, z]]$ for all $x, y, z \in g$, where $\sum_i y_i \otimes x_i = \sigma(x \otimes y)$.
3. Writing $\sigma_{12} = (\sigma \otimes \text{id}), \sigma_{23} = \text{id} \otimes \sigma$, and $C(x \otimes y) = [x, y]$

\[
\sigma (\text{id} \otimes C) - (C \otimes \text{id}) \sigma_{23} \sigma_{12} = 0 \quad (2.1) \\
\sigma (C \otimes \text{id}) - (\text{id} \otimes C) \sigma_{12} \sigma_{23} = (C \otimes \text{id}) (\text{id} \otimes \sigma) - \sigma (\text{id} \otimes C) (\sigma \otimes \text{id}) \quad (2.2)
\]
4. Quantum antisymmetry: If \( \sum_i x_i \otimes y_i \in \ker(\text{id} - \sigma) \), then \( \sum_i [x_i, y_i] = 0 \).

The universal enveloping algebra of \( (g, \sigma, [\, , \, ] ) \) is

\[
U(g) = T(g)/(\text{im}(\text{id}^{\otimes 2} - \sigma - [\, , \, ])), \tag{2.3}
\]

the tensor algebra of \( g \) divided by the two-sided ideal generated by all elements of the form \( x \otimes y - \sigma(x \otimes y) - [x, y] \), \( x, y \in g \).

Woronowicz has shown that these axioms appear naturally in the context of bicovariant differential calculi over Hopf algebras \( A \) (theorems 5.3 and 5.4 in \([\text{Wor}]\)). In \([\text{Wor}]\), \( V \) is a \( \mathcal{M} \), but this can be made more precise: it is actually a quantum Lie algebra in the monoidal category \( \mathcal{M} \) (see the comments after proposition \([\text{L}]\)). In the following, “quantum Lie algebra” will mean “left quantum Lie algebra”.

**Lemma 2.2.** Axiom 4 of a quantum Lie algebra is the necessary and sufficient condition for the natural map \( j : g \to T(g) \to U(g) \) to be injective.

**Proof.** Assume injectivity of \( j \) and let \( v \in \ker(\text{id}^{\otimes 2} - \sigma) \subset g \otimes g \). Then, in \( U(g) \),

\[
0 = (\text{id} - \sigma - [\, , \, ])(v) = [-1, [v]] \in j(g).
\]

By the injectivity of \( j \), we get \([v, [v]] = 0 \). Conversely, assume antisymmetry and let \( z \in \ker j \). This means that, as an element of \( T(g) \),

\[
z = \sum_i u_i \otimes (x_i \otimes y_i - \sigma(x_i \otimes y_i) - [x_i, y_i]) \otimes v_i
\]

for some \( u_i, v_i \in T(g), x_i, y_i \in g \). On the r.h.s, terms of degree \( \geq 1 \) must cancel, i.e. one can take \( u_i = v_i = 1 \). Then, terms of degree two must cancel, i.e.

\[
\sum_i (x_i \otimes y_i - \sigma(x_i \otimes y_i)) = 0
\]

so \( z = 0 \).

The other three axioms of a quantum Lie algebra have the following important interpretation. Given an object \( g \) in \( V \), equipped with morphisms \( \sigma : g \otimes g \to g \otimes g \) and \([\, , \, ] : g \otimes g \to g \),

we define its extension \( (\tilde{g}, \tilde{\sigma}) \) as follows. We set \( \tilde{g} = k \gamma \otimes g \) and the morphism \( \tilde{\sigma} : \tilde{g} \otimes \tilde{g} \to \tilde{g} \otimes \tilde{g} \) is defined by

\[
\begin{align*}
\tilde{\sigma}(\gamma \otimes z) &= z \otimes \gamma, \quad \tilde{\sigma}(z \otimes \gamma) = \gamma \otimes z, \quad (z \in \tilde{g}) \\
\tilde{\sigma}(x \otimes y) &= \sigma(x \otimes y) + [x, y] \otimes \gamma \quad (x, y \in g \hookrightarrow \tilde{g})
\end{align*}
\tag{2.4}
\]

**Lemma 2.3.** The following are equivalent:

1. \( \tilde{\sigma} \) satisfies the braid relation;
2. the triple \( (g, \sigma, [\, , \, ] ) \) satisfies axioms 1–3 of a (left) quantum Lie algebra.

**Proof.** Direct calculation. One would obtain the corresponding axioms for a right quantum Lie algebra by defining \( \tilde{\sigma}(x \otimes y) = \sigma(x \otimes y) + \gamma \otimes [x, y] \) for \( x, y \in g \). (2.4)

**Lemma 2.4.** Let \( (g, \sigma, [\, , \, ] ) \) be a quantum Lie algebra. Let \( S_\sigma(g) = T(g)/(\text{im}(\text{id} - \sigma)) \) be the quantum symmetric algebra of \( g \) with respect to the braiding \( \sigma \). Likewise, let \( S_{\tilde{\sigma}}(\tilde{g}) = T(\tilde{g})/(\text{im}(\text{id}^{\otimes 2} - \tilde{\sigma})) \).

(i) There are isomorphisms of algebras

\[
U(g) \simeq S_\sigma(g)/(\gamma - 1) \quad \text{and} \quad S_{\tilde{\sigma}}(\tilde{g}) \simeq S_\sigma(\tilde{g})/(\gamma).
\tag{2.5}
\]

(ii) If there exists a subobject \( L \subset \tilde{g} \) such that \( \tilde{g} = k \gamma \oplus L \) and \( \tilde{\sigma}(L \otimes L) \subset L \otimes L \), then \( S_{\tilde{\sigma}}(\tilde{g}) \simeq k[\gamma] \otimes S_{\tilde{\sigma}_L}(L) \), and

\[
U(g) \simeq S_{\tilde{\sigma}_L}(L) \simeq S_\sigma(g).
\tag{2.6}
\]

**Proof.** (i) is clear from the definition of \( (\tilde{g}, \tilde{\sigma}) \), see (2.4). (ii) If \( L \) has the given properties, then \( S_{\tilde{\sigma}}(\tilde{g}) \) is generated by \( \gamma \) and \( L \) with relations \( x \otimes y = \tilde{\sigma}(x \otimes y) \) and \( \gamma \otimes x = x \otimes \gamma \), \( x, y \in L \) and the first isomorphism follows (since \( \tilde{\sigma}(L \otimes L) \subset L \otimes L \) by hypothesis). Factoring out by
Let $\varphi : \hat{\mathfrak{g}} \to \mathfrak{g}$ be the projection onto $\mathfrak{g}$ with kernel $k\gamma$. Since by hypothesis $\gamma \notin \mathcal{L}$, the map $\varphi$ induces a vector space isomorphism

$$\varphi |_{\mathcal{L}} : \mathcal{L} \cong \mathfrak{g},$$

and satisfies $(\varphi \otimes \varphi) \circ \hat{\sigma} = \sigma \circ (\varphi \otimes \varphi)$. Indeed, checking this on $X \otimes Y$ with either $X$ or $Y$ proportional to $\gamma$ is immediate from $\varphi(\gamma) = 0$ and $[\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g}] = 0$. Otherwise, since $\varphi$ is the identity on $\mathfrak{g}$, and since $\hat{\sigma}(x \otimes y) = \sigma(x \otimes y) + [x, y] \otimes \gamma$ on $\mathfrak{g} \otimes \mathfrak{g}$, we get $(\varphi \otimes \varphi) \circ \hat{\sigma} |_{\mathfrak{g} \otimes \mathfrak{g}} = \sigma = \sigma \circ (\varphi \otimes \varphi) |_{\mathfrak{g} \otimes \mathfrak{g}}$. Therefore, $\varphi |_{\mathcal{L}}$ is a vector space isomorphism which conjugates the braidings $\hat{\sigma}|_{\mathcal{L}}$ on $\mathcal{L}$ and $\sigma$ on $\mathfrak{g}$, hence the last isomorphism.

**Remarks.** When $\mathfrak{g}$ is classical ($\sigma = \tau$ is the flip), the algebra $S_0(\hat{\mathfrak{g}})$ already appears in [LeB-VdB] and called the homogenization of $U(\mathfrak{g})$. We keep our notation to stress that $\mathfrak{g}$ is classical, $\mathcal{L}$ exists only in the case described above ($\mathfrak{g}$ abelian), but more interesting situations do appear in the “quantum case” ($\sigma \neq \tau$).

We give for completeness a third characterization of quantum Lie algebras, by a construction due to D. Bernard [LeB]. It shows that one can associate a co-quasitriangular bialgebra to any quantum Lie algebra $\mathfrak{g}$, in which $U(\mathfrak{g})$ imbeds as an algebra. This bialgebra is not, however, what we are after (one would expect $U(\mathfrak{g})$ to be a quasitriangular bialgebra, not co-

We assume that $\mathcal{V}$ is the category of vector spaces (hence braided).

Let $C$ be a matrix coalgebra, with comultiplication $\lambda \mapsto \lambda(1) \otimes \lambda(2)$. Let $\mathfrak{g}$ be the (unique up to isomorphism) simple left $C$-comodule, with coaction $x \mapsto x(-1) \otimes x(0)$. It can be viewed as a $C$-$k$-bicomodule for the right coaction $x \mapsto x \otimes \gamma$ ($\gamma$ is the grouplike element of the $1$-dimensional coalgebra). One obtains a coalgebra $(\hat{C}, \hat{\Delta}, \hat{\varepsilon})$, where $\hat{C} := C \oplus \mathfrak{g} \oplus k\gamma$ and

$$\hat{\Delta}(\lambda) = \lambda(1) \otimes \lambda(2), \quad \hat{\Delta}(x) = x(-1) \otimes x(0) + x \otimes \gamma, \quad \hat{\Delta}(\gamma) = \gamma \otimes \gamma.$$

(Note that $\hat{C}$ is Morita equivalent to the coalgebra of upper triangular $2 \times 2$ matrices.) Let $\hat{r} : \hat{\mathfrak{g}} \otimes \hat{\mathfrak{g}} \to k$ be a linear map satisfying

$$\hat{r}(\gamma, -) = \hat{r}(-, \gamma) = \hat{\varepsilon}(-), \quad \hat{r}(\mathfrak{g}, -) = 0$$

where “−” stands for “anything”. Thus, $\hat{r}$ is uniquely determined by $r := \hat{r}|_{C \otimes C}$ and $\omega := \hat{r}|_{\mathfrak{g} \otimes \mathfrak{g}}$, which can be arbitrary. We consider the bialgebra $A(\hat{C}, \hat{r})$ as in the preliminaries.

**Proposition 2.5.** Assume $r$ is convolution invertible. $\hat{r}$ extends to a co-quasitriangular structure on $A(\hat{C}, \hat{r})$ if and only if $(\mathfrak{g}, \sigma, [\cdot, \cdot])$ satisfies axioms 1-3 of a (left) quantum Lie algebra, where

$$\sigma(x \otimes y) = r(y(-1), x(-1)) y(0) \otimes x(0), \quad [x, y] = \omega(y(-1), x) y(0).$$

**Proof.** If $r$ is invertible, so is $\hat{r}$ (its inverse also satisfies (2.7), and is given by $\lambda \otimes \mu \mapsto \hat{r}(\lambda, \mu), \lambda \otimes x \mapsto -r(\lambda(1), x(-1)) \omega(\lambda(2), x(0))$ for $\lambda, \mu \in C, x \in \mathfrak{g}$). By lemma [LeB], $\hat{r}$ extends to a co-quasitriangular structure on $A(\hat{C}, \hat{r})$ iff the map $\Sigma : \hat{C} \otimes \hat{C} \to \hat{C} \otimes \hat{C}$, where

$$\Sigma(x \otimes y) = r(y(-1), x(-1)) y(0) \otimes x(0)$$
\[ \Sigma(\alpha \otimes \beta) = \hat{r}(b_{(1)}, a_{(1)}) b_{(2)} \otimes a_{(2)} \], satisfies the braid relation. One easily checks that \( \Sigma \)

 preserves the subspace \( k \gamma \otimes g \), where it takes the form \( (\Sigma, \varepsilon) \), with \( \sigma \) and \( [\cdot, \cdot] \) as stated. Therefore, if \( \Sigma \) satisfies the braid relation, by lemma 2.3, \((g, \sigma, [\cdot, \cdot])\) must satisfy axioms 1-3 of a quantum Lie algebra. The converse is long but straightforward. We omit it.

**Remark.** \( \gamma \) is grouplike central in \( A(\hat{C}, \hat{r}) \), therefore one can consider the quotient \( A(\hat{C}, \hat{r})/(\gamma - 1) \), which is still co-quasitriangular if \( A(\hat{C}, \hat{r}) \) is. The other relations in \( A(\hat{C}, \hat{r}) \) are such that the subalgebra generated by \( C \) is isomorphic is \( A(C, r) \), the subalgebra generated by \( g \) is \( S_{g}(\hat{g}) \), and the crossed relations are given by \((\lambda \in C, x \in g)\):

\[
\lambda x = r(x^{(-1)}, \lambda_{(1)}) x^{(0)} \lambda_{(2)} \\
x \lambda = r(\lambda_{(1)}, x^{(-1)}) \lambda_{(2)} x^{(0)} + \omega(\lambda_{(1)}, x) \lambda_{(2)} x^{(0)} - \lambda_{(1)} x^{(-1)} \lambda_{(2)} \omega(\lambda_{(2)}, x^{(0)})
\]

If \( \omega = 0 \) (i.e. \([\cdot, \cdot] = 0\)), combining the two relations above we get \( x \lambda = r_{21}(x^{(-2)}, \lambda_{(1)}) r(x^{(-1)}, \lambda_{(2)}) x^{(0)} \lambda_{(3)} \). So, if moreover \( r_{21} \ast r = \varepsilon_{C} \otimes \varepsilon_{C} \), the bialgebra \( A(\hat{C}, \hat{r})/(\gamma - 1) \) is just the crossed product of \( A(C, r) \) with the quantum symmetric algebra of its simple comodule. At the far opposite, if \( r_{21} \ast r \) is a non degenerate bilinear form on \( C \), we get \( x \lambda = \lambda x = 0 \) for all \( x \in g, \lambda \in C \). When \( \omega \neq 0 \), the terms involving \( \omega \) are even more unusual.

**“Good” quantum Lie algebras.** In this paragraph we investigate bialgebra structures on \( U(g) \) itself. To give a sense to this, we assume (until the end of the paper) that \((\mathcal{V}, \otimes, \Psi)\) is braided. We stress that \( g \) is now equipped with two braidings, \( \sigma \) and \( \Psi_{g,g} \), which differ in general (indeed, if \( \sigma = \Psi_{g,g} \), one should have \( \sigma(C \otimes \text{id}) = (\text{id} \otimes C) \sigma_{12} \sigma_{23} \) instead of \((\Sigma, \varepsilon)\) by the naturality of \( \Psi \)). The algebra \( U(g) \) has a filtration

\[
U(g)_{(0)} \subset U(g)_{(1)} \subset \ldots \subset U(g)_{(n)} \subset \ldots
\]

induced by the natural \( \mathbb{Z}_{\geq 0} \)-grading of \( T(g) \). By lemma 2.4 one can identify \( U(g)_{(1)} \) with \( k1 \oplus g \). Classically (when \( \sigma = \tau \)), \( U(g) \) is a Hopf algebra in \( \mathcal{V} \) with coalgebra structure \((\Delta, \varepsilon)\) and antipode \( S \) uniquely determined by \( \Delta(x) = x \otimes 1 + 1 \otimes x, \varepsilon(x) = 0 \) and \( S(x) = -x \) for all \( x \in g \). In particular, each term of the above filtration is a subcoalgebra of \( U(g) \), and \( g \subset \ker \varepsilon \). If we require that it is so in the general case, a hypothetical bialgebra structure \((\Delta, \varepsilon)\) on \( U(g) \) is uniquely determined by a coassociative map \( \delta : g \rightarrow g \otimes g \) (which should be a morphism in the category) such that

\[
\Delta(x) = x \otimes 1 + 1 \otimes x + \delta(x), \quad \varepsilon(x) = 0, \quad (x \in g).
\]

We say that \( \delta \) is a compatible coproduct on \((g, \sigma, [\cdot, \cdot])\) if the above formula defines a coalgebra structure on \( U(g) \). Even if there are some similarities with Lie bialgebras and their quantization, the situation is different since \( \delta \) here is coassociative, and in fact, the choice \( \delta = 0 \) is not always possible:

**Lemma 2.6.** Let \((g, \sigma, [\cdot, \cdot])\) be a quantum Lie algebra in \( \mathcal{V} \). Then \( U(g) \) is a Hopf algebra in \( \mathcal{V} \) with coalgebra structure \((\Delta, \varepsilon)\) and antipode \( S \) such that

\[
\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -x
\]

for all \( x \in g \) if and only if \((1 + \Psi_{g,g})(1 - \sigma) = 0 \).

**Proof.** Direct calculation.

In the general case, we shall restrict ourselves to compatible coproducts which satisfy a “nice” criterion (a sufficient but not necessary condition). This criterion is suggested by
A posteriori “motivations” for this choice are given in remark (iii) after theorem 2.6. Let \( \delta : g \to g \otimes g \) be some coassociative morphism. The extension \( \tilde{g} = k\gamma \oplus g \) of \( g \) (see (2.4)) can be seen as a coaugmented coalgebra \((\tilde{g}, \tilde{\delta}, \tilde{\varepsilon})\) by setting
\[
\tilde{\delta}(\gamma) = \gamma \otimes \gamma, \quad \tilde{\delta}(x) = x \otimes \gamma + \gamma \otimes x + \delta(x) \quad (x \in g = \ker \tilde{\varepsilon}).
\]
(2.10)
Note that in fact, \( \tilde{g} \simeq U(g) \) as a coalgebra. Recall that
\[
S_{\tilde{\sigma}}(\tilde{g})/\langle \gamma - 1 \rangle \simeq U(g).
\]
Therefore, since all natural maps \( g \hookrightarrow U(g), g \hookrightarrow \tilde{g} \hookrightarrow S_{\tilde{\sigma}}(\tilde{g}) \) are injective, \( \delta \) is a compatible coproduct on \( g \) iff \( S_{\tilde{\sigma}}(\tilde{g}) \) is a coalgebra in \( V \) with coproduct \( \Delta \) such that \( \Delta(X) = \tilde{\delta}(X) \) for all \( X \in \tilde{g} \).

By lemma 1.3, this is ensured if \( \tilde{\sigma} : \tilde{g} \otimes \tilde{g} \to \tilde{g} \otimes \tilde{g} \) is a morphism of coalgebras, so:

Lemma 2.7. If \( \tilde{\sigma} : \tilde{g} \otimes \tilde{g} \to \tilde{g} \otimes \tilde{g} \) is a morphism of coalgebras, then \( \delta \) is a compatible coproduct on \((g, \sigma, [ , ])\).

Note that what should be a condition on \( \delta \), the maps \( \sigma \) and \([ , ]\) being fixed, is finally better seen as a condition on \( \tilde{\sigma} \) (i.e. \( \sigma \) and \([ , ]\)) with respect to a fixed \( \delta \). Solving this condition leads to the following definition. -We use diagrammatic notations as is conventional; compositions of maps are written from top to bottom, the braiding \( \Psi_{V,W} : V \otimes W \to W \otimes V \) and its inverse \( \Psi_{W \otimes V}^{-1} : W \otimes V \to V \otimes W \) are represented respectively by the symbols:

\[
\Psi = \begin{align*}
&V \\
&W \\
&V \\
&W \\
\end{align*} \\
\Psi^{-1} = \begin{align*}
&W \\
&V \\
&V \\
&W \\
\end{align*}
\]

Definition 2.8. A good quantum Lie algebra in \( V \) is a quadruple \((g, \sigma, [ , ], \delta)\) where \( g \) is an object, \( \sigma : g \otimes g \to g \otimes g, [ , ] : g \otimes g \to g \) and \( \delta : g \to g \otimes g \) morphisms, such that \( \delta \) is coassociative, and obeying the axioms below

(L1') \[
\begin{align*}
[ , ] &= [ , ] + \sigma \\
\end{align*}
\]

(L2'-a) \[
\begin{align*}
\sigma &= \delta[ , ] + \delta[ , ] \\
\end{align*}
\]

(L2'-b) \[
\begin{align*}
\sigma &= \delta[ , ] + \delta[ , ] \\
\end{align*}
\]

(L3') \[
\begin{align*}
[ , ] &= \delta[ , ] + \delta[ , ] \\
\end{align*}
\]

and
\[
\ker(\text{id} \otimes \sigma - [ , ]) \subset \ker([ , ]).
\]
(2.11)

The proof of the following theorem is given after lemma 2.7.
Theorem 2.9. (i) Let \((g, \sigma, [\cdot, \cdot], \delta)\) be a good quantum Lie algebra in \(V\). Then \((g, \sigma, [\cdot, \cdot])\) is a quantum Lie algebra in \(V\), and \(\tilde{\delta}: \tilde{g} \otimes \tilde{g} \rightarrow \tilde{g} \otimes \tilde{g}\) is a morphism of coalgebras. In particular, \(U(g)\) is a bialgebra in \(V\) with coalgebra structure given by

\[
\Delta(x) = x \otimes 1 + 1 \otimes x + \delta(x), \quad \varepsilon(x) = 0, \quad (x \in g).
\]

(ii) Conversely, let \((g, \sigma, [\cdot, \cdot])\) be a quantum Lie algebra in \(V\) equipped with a coassociative morphism \(\delta: g \rightarrow g \otimes g\). If \(\tilde{\delta}: \tilde{g} \otimes \tilde{g} \rightarrow \tilde{g} \otimes \tilde{g}\) is a morphism of coalgebras, then \((g, \sigma, [\cdot, \cdot], \delta)\) is a good quantum Lie algebra.

**Remarks.** (i) The braiding \(\sigma\) of a good quantum Lie algebra \((g, \sigma, [\cdot, \cdot], \delta)\) can be expressed in terms of the maps \([\cdot, \cdot]\), \(\delta\) and the braiding \(\Psi_{g, g}\), therefore it is not an essential datum. Moreover, a good quantum Lie algebra \((g, \sigma, [\cdot, \cdot], \delta)\) satisfies by hypothesis the axioms 2 and 4 of a quantum Lie algebra, therefore the first claim of the theorem is that the axioms 1 and 3 are also satisfied, in particular \(\sigma\) satisfies the braiding relation.

(ii) If either \([\cdot, \cdot] = 0\) or \(\delta = 0\), one must have \(\sigma = \Psi_{g, g} = (\Psi_{g, g})^{-1}\) by axiom (L2'). Said the other way round, if \(\Psi_{g, g}\) is not symmetric, neither \([\cdot, \cdot]\) nor \(\delta\) can be zero (compare with lemma 2.6).

(iii) The axioms of a good quantum Lie algebra are satisfied by the usual Lie algebras \((\sigma = \Psi_{g, g} = \tau, \delta = 0)\). Moreover, they almost characterize the standard coproduct in this case: if \(\sigma = \Psi_{g, g} = \tau\), one easily finds that \((g, \sigma, [\cdot, \cdot], \delta)\) is a good quantum Lie algebra if and only if \((g, [\cdot, \cdot])\) is a usual Lie algebra, \(\delta\) is coassociative, \(\delta g' = 0\) and \(\delta g \subset Z(g) \otimes Z(g)\), where \(g' = [g, g]\) and \(Z(g)\) is the center of \(g\). Therefore, if \(g' = g\) or if \(Z(g) = 0\), one must have \(\delta = 0\).

(iv) There are compatible coproducts which are not “good” : for instance, take \(g = e,(1,1)\) the usual Lie algebra with basis \(e_0, e_+, e_-\) such that \([e_0, e_\pm] = \pm e_\pm,\ [e_+, e_-] = 0\) (if \(k = \mathbb{R}\), \(g\) is the Lie algebra of the pseudo-euclidean plane). Then \(U(g)\) is a Hopf algebra in \(\mathcal{M}\) for all coproducts of the form \((2.8)\) with \(\delta e_\pm = 0, \delta e_0 = \lambda (e_+ \otimes e_- - e_- \otimes e_+)\) for all \(\lambda \in k\); in fact if \(\lambda \neq 0\) it can be rescaled, eg to \(\lambda = 1\), by rescaling \(e_+\) or \(e_-\). The antipode is given by \(S(x) = -x\) for \(x \in g\), independently of \(\lambda\). When \(\lambda \neq 0\), \((g, [\cdot, \cdot], \delta)\) is not good since \(\delta \neq 0\) but \(Z(g) = 0\).

3 Braided Lie algebras and Lie coalgebras

The definition of a good quantum Lie algebra is already coming close to that of a braided Lie algebra [Maj-94]. In this section we recall their definition and main properties, and discuss the connection with good quantum Lie algebras. In particular, we shall see that the axioms of a good quantum Lie algebra can take a much simpler form when expressed in terms of the braided Lie algebra it corresponds to.

3.1 \(L\) and \(B(L)\)

**Definition 3.1.** [Maj-94] A (left) braided Lie algebra in \(V\) is a coalgebra \((L, \Delta, \varepsilon)\) in the category, equipped with a morphism in \(V\) (the braided Lie bracket) \([\cdot, \cdot]: L \otimes L \rightarrow L\) satisfying the axioms pictured below:

\[\text{(L1) } L \overset{L}{\overset{\text{L}}{\underset{\text{L}}{\overset{\Delta}{\text{L}}}}} \text{ = } L \overset{L}{\overset{\text{L}}{\underset{\text{L}}{\overset{\varepsilon}{\text{L}}}}} \quad \text{(L2) } L \overset{L}{\overset{\text{L}}{\underset{\text{L}}{\overset{\Delta}{\text{L}}}}} \text{ = } L \overset{L}{\overset{\text{L}}{\underset{\text{L}}{\overset{\varepsilon}{\text{L}}}}}\]
Lemma 3.2. (i) Let \( \varepsilon \) be a left braided Jacobi identity, (L2) weak braided cocommutativity, and (L3) states that \([ , ] : L \otimes L \to L \) is a morphism of coalgebras.

A braided Lie subalgebra of \( L \) is subcoalgebra \( M \) such that \([ M, M ] \subset M \). A morphism of braided Lie algebras in \( V \) is a morphism of coalgebras \( \phi : L_1 \to L_2 \) such that \([ , ]_2 \circ (\phi \otimes \phi) = \phi \circ [ , ]_1 \).

Remarks. (i) If the braiding on \( L \) is symmetric and if \(( L, \Delta, \varepsilon ) \) is cocommutative \(( \Psi \circ \Delta = \Delta ) \), axiom (L2) is automatically satisfied, independently of \([ , ] \).

(ii) By a braided Lie algebra we shall mean a left one. Axioms for a right braided Lie algebra are obtained from that of a left one by applying a symmetry along the medium vertical axis of each diagram in the definition while keeping the same order of crossing \([ \text{Maj-LN} ] \) (so that the diagrams of axiom (L3) remain unchanged); see for instance \([ \text{Wam} ] \).

It is observed in \([ \text{Wam} ] \) that if \(( L, \Delta, \varepsilon, [ , ] ) \) is a left braided Lie algebra, then \(( L, \Psi^{-1} \circ \Delta, \varepsilon, [ , ] ) \circ \Psi \) is a right one.

(iii) The naturality of the braiding with respect to the morphisms \( \Delta, \varepsilon \) and \([ , ] \) here means that:

\[
(id \circ \Delta) \circ \Psi = \Psi_{12} \circ \Psi_{23} \circ \Delta, \quad (\Delta \circ id) \circ \Psi = \Psi_{23} \circ \Psi_{12} \circ \Delta,
\]

\[
(\varepsilon \otimes id) \circ \Psi = id \otimes \varepsilon, \quad (id \otimes \varepsilon) \circ \Psi = \varepsilon \otimes id,
\]

\[
\Psi \circ ([ , ] \otimes id) = (id \otimes [ , ]) \circ \Psi_{12} \circ \Psi_{23}, \quad \Psi \circ (id \otimes [ , ]) = ([ , ] \otimes id) \circ \Psi_{23} \circ \Psi_{12}.
\]

This identities should be added explicitly to the axioms if one forgets about a background category and consider a vector space \( L \) equipped with maps \(( \Psi, \Delta, \varepsilon, [ , ] ) \). For instance, in dimension 1, there is only one isomorphism class of braided Lie algebra. Indeed, let \( L = k(\gamma) \) with \( \gamma \) grouplike (after scaling). Then there exists scalars \( q, k \in k \) such that \( \Psi(q \gamma \otimes \gamma) = q^2 \gamma \otimes \gamma \) and \([ \gamma, \gamma ] = \lambda \gamma \). Coint and naturality constraints then force \( \lambda = q = 1 \).

Lemma 3.2. (i) Let \(( C, \Delta, \varepsilon ) \) be a coalgebra in \( V \). The map \([ , ]_{\text{triv}} : C \otimes C \to C, [x, y]_{\text{triv}} = \varepsilon(y)x - \varepsilon(x)y \) is a braided Lie bracket on \( C \) if and only if \(( \Psi_{C,C} )^2 = id_{C \otimes C} \).

(ii) Let \(( L_i, \Delta_i, \varepsilon_i, [ , ]_i), i = 1, 2 \) be two braided Lie algebras in \( V \). The direct sum coalgebra \( L = L_1 \oplus L_2 \) equipped with the map \([ , ]_L : L \otimes L \to L \)

\[
[x_1 \oplus x_2, y_1 \oplus y_2]_L = [x_1, y_1]_1 + \varepsilon_2(x_2) y_1 \oplus [x_2, y_2]_2 + \varepsilon_1(x_1) y_2
\]

is a braided Lie algebra if and only if \( \Psi_{L_i, L_j} \circ \Psi_{L_j, L_i} = id_{L_i \oplus L_j} \) when \( i \neq j \). In this case we call \( L \) the direct sum of \( L_1 \) and \( L_2 \).

Proof. (i) Using only the counit axiom of a coalgebra and the naturality of \( \Psi \), one easily checks that this braiding always satisfies axioms (L1) and (L3), but satisfies (L2) if \( \Psi_{C,C} = (\Psi_{C,C})^{-1} \). Note that the trivial bracket on a right braided Lie algebra in case \( \Psi^2 = id \) would be \([ x, y ]_{\text{triv}} = x \varepsilon(y) \). (ii) The reasons are the same as in (i).

Theorem 3.3. Let \(( L, \Delta, \varepsilon ) \) be a coalgebra in \( V \). There is a one-to-one correspondence between

1. morphisms of coalgebras \([ , ] : L \otimes L \to L \) such that \(( L, \Delta, \varepsilon, [ , ] ) \) is a left braided Lie algebra.
2. morphisms of coalgebras \( \Upsilon : L \otimes L \to L \otimes L \) such that
(a) $\Upsilon$ satisfies the braid relation,
(b) $\Upsilon(\ker \varepsilon \otimes \ker \varepsilon) \subseteq \ker \varepsilon \otimes L$,
(c) the following equalities hold (in the box, $Y$ means $\Upsilon$):
\begin{align*}
Y \otimes e & = \Delta Y \\
Y & = \Delta Y \\
\end{align*}
(3.1)

It is given by $\Upsilon := (\cdot, \cdot \otimes \text{id}) \circ (\text{id} \otimes \Psi) \circ (\Delta \otimes \text{id})$, and $\cdot, \cdot := (\text{id} \otimes \varepsilon) \circ \Upsilon$. In diagrammatic form:
\begin{align*}
L L & = L L \\
\end{align*}
(3.2)

We call $\Upsilon$ the canonical braiding of $(L, \Delta, \varepsilon, [\cdot, \cdot])$.

Proof. Let $(L, \Delta, \varepsilon, [\cdot, \cdot])$ be a braided Lie algebra, and define $\Upsilon$ as in (3.2). The fact that it satisfies the braid relation is proved in \cite{Maj-95} \cite{Wam}. By definition of $\Upsilon$ and by the counit axioms, one has
\begin{align*}
(\varepsilon \otimes \text{id}) \circ \Upsilon(x \otimes y) = \varepsilon(y)x, \\
(\text{id} \otimes \varepsilon) \circ \Upsilon(x \otimes y) = [x, y].
\end{align*}
(3.3)
Therefore $\Upsilon$ satisfies (b) and the counit part of the fact that it is a morphism of coalgebras. The coproduct part, i.e. the equality $(\Upsilon \otimes \Upsilon) \circ \Delta_{L \otimes L} = \Delta_{L \otimes L} \circ \Upsilon$, is checked as:
\begin{align*}
\Delta & = \Delta \\
\Delta & = \Delta \\
\Delta & = \Delta \\
\Delta & = \Delta \\
\end{align*}
(3.4)

The first and third equalities use only the naturality and coassociativity axioms, the second axiom (L2) and the fourth axiom (L1). Finally, $\Upsilon$ satisfies the equalities (3.1) since, in view of (3.3), the left one is nothing but its definition, and the right one is an equivalent form of axiom (L2) (multiplied by $\Psi^{-1}$ on the left, i.e. on the bottom).

Conversely, define $\cdot, \cdot := (\text{id} \otimes \varepsilon) \circ \Upsilon : L \otimes L \to L$. Obviously, $\cdot, \cdot$ is a morphism of coalgebras (axiom (L3)), as composition of morphism of coalgebras. Next, axiom (L1) is satisfied since:
\begin{align*}
[\cdot, \cdot] & = [\cdot, \cdot] \\
\end{align*}
(3.4)
The first equality is by definition of $[,] = (\text{id} \otimes \varepsilon) \circ \Upsilon$, the second uses the fact that $\Upsilon$ is a morphism of coalgebras, the third is the braid relation for $\Upsilon$, the fourth is again the definition of $[,]$, the fifth uses the left equality in (3.1), and the sixth is again the definition of $[,]$. Finally, the left equality in (3.4) means that one can reexpress $\Upsilon$ in terms of $[,] := (\text{id} \otimes \varepsilon) \circ \Upsilon$ as $\Upsilon = (([,] \otimes \text{id}) \circ (\Delta \otimes \text{id})$. Therefore the second diagram of (3.4) is again nothing but axiom (L2) for $(L, \Delta, \varepsilon, [,])$.

By definition, the braided enveloping algebra of $(L, \Delta, \varepsilon, [,])$ is the symmetric algebra of $L$ with respect to $\Upsilon$:

$$B(L) := S_\Upsilon(L) = T(L)/\langle \text{id} - \Upsilon \rangle.$$

(Its definition is motivated by the braided Jacobi identity which can be expressed as the equality between the first and fifth diagrams in (3.4)). Since $\Upsilon$ is a morphism of coalgebras, by lemma [1.3] the maps $\Delta: L \to L \otimes L$ and $\varepsilon: L \to k$ extend uniquely to algebra morphisms $B(L) \to B(L) \otimes B(L)$ and $B(L) \to k$ respectively, i.e.:

**Corollary 3.4.** [Maj-94] $B(L)$ is a bialgebra in $\mathcal{V}$.

**Remark.** $B(L)$ is quadratic, and “$\Upsilon$-commutative”, but it lives in $\mathcal{V}$ where the braiding is $\Psi$. Note that $\Upsilon$ need not be invertible (although we do not know any example where it is not), and that there is no way to express the original braiding $\Psi_{L,L}$ in terms of $\Upsilon$, $\Delta$, $\varepsilon$ and $[,]$.

**Proposition 3.5.** (i) The correspondence $L \to B(L)$ is an exact covariant functor.

(ii) Let $L = L_1 \oplus L_2$ be the direct sum of two braided Lie algebras. Then $B(L) \simeq B(L_1) \otimes B(L_2)$ is the tensor product of $B(L_1)$ and $B(L_2)$ in the category.

**Proof.** (i) A morphism of objects $L \to M$ induces a morphism of algebras $T(L) \to T(M)$. By definition, a morphism of braided Lie algebras $f: L \to M$ intertwines all structure maps, in particular $f$ is a morphism of coalgebras and $(f \otimes f) \circ \Upsilon_L = \Upsilon_M \circ (f \otimes f)$. Therefore $f$ induces a bialgebra morphism $f: T(L)/\langle \text{id}^\otimes - \Upsilon_L \rangle \to T(M)/\langle \text{id}^\otimes - \Upsilon_M \rangle$. Clearly $f$ is injective (or surjective) if and only if $f$ is. (ii) By (i), there are bialgebra embeddings $B(L_1) \hookrightarrow B(L) \leftarrow B(L_2)$. The claim follows from the observation that (by definition of a direct sum, see lemma [3.2]), for $x \in L_i$, $y \in L_j$, and $i \neq j$, one has $\Upsilon(x \otimes y) = \Psi(x \otimes y)$.

### 3.2 Good braided Lie algebras.

We shall say that $(L, \Delta, \varepsilon, [,])$ is unital if there exists a morphism of braided Lie algebras $\eta: k \to L$ with

$$[,] \circ (\eta \otimes \text{id}_L) = \text{id}_L, \quad \text{and} \quad [,] \circ (\text{id}_L \otimes \eta) = \eta \circ \varepsilon,$$

or equivalently (by theorem [3.3])

$$\Upsilon \circ (\eta \otimes \text{id}_L) = \text{id}_L \otimes \eta, \quad \text{and} \quad \Upsilon \circ (\text{id}_L \otimes \eta) = \eta \otimes \text{id}_L.$$

In terms of $\gamma = \eta(1)$, $(L, \Delta, \varepsilon, [,], \eta)$ is unital if the span of $\gamma$ is isomorphic to the trivial object (which implies that it is “bosonic”): $\Psi(\gamma \otimes z) = z \otimes \gamma$ and $\Psi(z \otimes \gamma) = \gamma \otimes z$ for all $z \in L$, is grouplike ($\eta$ is a morphism of coalgebras), and $[\gamma, z] = z$, $[z, \gamma] = \varepsilon(z) \gamma$ for all $z \in L$. The last two equalities are equivalent to $\Upsilon(\gamma \otimes z) = z \otimes \gamma$ and $\Upsilon(z \otimes \gamma) = \gamma \otimes z$ for all $z \in L$. In particular $\gamma$ is a central grouplike in $B(L)$. We stress that not all braided Lie algebras are unital and the morphism $\eta: k \to L$ or the element $\gamma$, if it exists, is not unique in general. If $(L, \Delta, \varepsilon, [,], \eta)$ is unital, we define

$$g := \ker \varepsilon, \quad \text{and} \quad \gamma := \eta(1),$$

(3.8)
so that there is a distinguished decomposition $L = k \gamma \oplus \mathfrak{g}$. By the counit axioms and by (3.3), there exists unique morphisms $\sigma : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ and $\tilde{\sigma} : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ such that

$$\Upsilon(x \otimes y) = \sigma(x \otimes y) + [x, y] \otimes \gamma, \quad \Delta(x) = x \otimes \gamma + \gamma \otimes x + \delta(x). \quad (3.9)$$

The axioms of the braided Lie algebra $L$ and the structure of $B(L)$, in the unital case, can be given in terms of $(\mathfrak{g}, \sigma, [\, , \,], \tilde{\sigma})$ as was done in [Maj-94, Fig 11]. Exactly as $\Upsilon$ is expressible in terms of $\Psi_{L,L}$, $\Delta$ and $[\, , \,]$, the map $\sigma$ can be expressed in terms of $\Psi_{\mathfrak{g}, \mathfrak{g}}$, $\tilde{\sigma}$ and $[\, , \,]_{\mathfrak{g} \otimes \mathfrak{g}}$. The braided Jacobi identity -axiom (L1)- for $L$:

$$[X, [Y, Z]] = [[[\Upsilon(X \otimes Y) \otimes Z)], \forall X, Y, Z \in L} \quad (3.10)$$

becomes when restricted to $\mathfrak{g}$:

$$[x, [y, z]] = [[[\sigma(x \otimes y) \otimes z)] + [[x, y] \otimes z], \forall x, y, z \in \mathfrak{g}. \quad (3.11)$$

Similarly the braid relation for $\Upsilon$ implies the braid relation for $\sigma$. Also from the first of (3.3) the relations of $B(L)$ become $\gamma$ central and

$$xy - \cdot \circ \sigma(x \otimes y) = [x, y]\gamma, \quad \forall x, y \in \gamma. \quad (3.12)$$

Note now that (3.11) is axiom 2 of a quantum Lie algebra for $(\mathfrak{g}, \sigma, [\, , \,], \tilde{\sigma})$. In fact if we identify $L = \tilde{\mathfrak{g}}$ then lemma 2.3 with $\Upsilon = \tilde{\sigma}$ ensures that $\mathfrak{g}$ satisfies axioms 1 and 3 of a quantum Lie algebra. The only item missing is an antisymmetry property for the Lie bracket, which can be added at hand:

**Definition 3.6.** A good braided Lie algebra in $\mathcal{V}$ is a unital braided Lie algebra $(L, \Delta, \varepsilon, [\, , \,], \eta)$ such that $\text{ker}(\text{id} \otimes \eta - \sigma) \subseteq \text{ker}([\, , \,]_{\mathfrak{g} \otimes \mathfrak{g}})$, where $(\mathfrak{g}, \sigma, [\, , \,], \tilde{\sigma})$ is defined as above.

From the above discussion, if $(L, \Delta, \varepsilon, \sigma, [\, , \,], \eta)$ is a good braided Lie algebra, then $(\mathfrak{g}, \sigma, [\, , \,])$ is a quantum Lie algebra. Also it is evident from (3.12) that a sufficient condition for a unital braided Lie algebra $L$ to be good is that $\gamma$ is not a zero divisor in $B(L)$. Unsurprisingly from the above discussion, and taking into account the coproduct one has:

**Lemma 3.7.** $(\mathfrak{g}, \sigma, [\, , \,], \tilde{\sigma})$ is a good quantum Lie algebra if and only if its extension $(\tilde{\mathfrak{g}}, \tilde{\sigma}, \tilde{\Delta}, \tilde{\varepsilon})$ is a good braided Lie algebra, with braided Lie bracket $[\, , \,] = (\text{id} \otimes \tilde{\varepsilon}) \circ \tilde{\sigma}$.

**Proof.** Coassociativity of $\tilde{\sigma}$ and $\tilde{\Delta}$ are obviously equivalent, and the antisymmetry axiom is postulated in both definitions. The reader will easily check that axiom (L1), (L2) and (L3) for $\tilde{\mathfrak{g}}$ are equivalent to axioms (L1’), (L2’-b) and (L3’) for $\mathfrak{g}$, while (L2’-a) corresponds to the definition of the canonical braiding $\Upsilon = \tilde{\sigma}$ for $\mathfrak{g}$. We omit the details. (Note that the braided Lie bracket on $\tilde{\mathfrak{g}}$ is given by $[\gamma, \gamma] = \gamma, \gamma, \varepsilon = x, [x, \gamma] = 0, x \in \mathfrak{g}$, and $[\, , \,]_{\mathfrak{g} \otimes \mathfrak{g}} = [\, , \,]$).

**Proof of theorem 2.9.** (i) If $(\mathfrak{g}, \sigma, [\, , \,], \tilde{\sigma})$ is a good quantum Lie algebra, then its extension $(\tilde{\mathfrak{g}}, \tilde{\sigma}, \tilde{\Delta}, \tilde{\varepsilon})$ is a good braided Lie algebra; by the discussion before definition 3.3 this implies that $(\mathfrak{g}, \sigma, [\, , \,], \tilde{\sigma})$ is a quantum Lie algebra. Moreover, $\Upsilon_{\tilde{\mathfrak{g}}} = \tilde{\sigma} : \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}$ is a morphism of coalgebras by theorem 3.3 so that $\tilde{\sigma}$ is a compatible coproduct on $\mathfrak{g}$ by lemma 2.3.

(ii) Let $(\mathfrak{g}, \sigma, [\, , \,], \tilde{\sigma})$ be a quantum Lie algebra with $\tilde{\sigma} : \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}} \otimes \mathfrak{g}$ coassociative, and $(\tilde{\mathfrak{g}}, \tilde{\sigma}, \tilde{\Delta}, \tilde{\varepsilon})$ be the extension of $\tilde{\sigma}$. $\tilde{\sigma}$ is a morphism of coalgebras if $\tilde{\delta} \circ \tilde{\sigma} : (X \otimes Y) = (\tilde{\sigma} \otimes \tilde{\sigma}) \circ \tilde{\delta} \circ \tilde{\sigma} : (X \otimes Y)$, for all $X, Y \in \mathfrak{g}$, where $\tilde{\delta} = (\text{id} \otimes \Psi \otimes \text{id}) \circ (\tilde{\sigma} \otimes \tilde{\sigma})$. One checks that this is trivially satisfied if either $X$ or $Y$ is proportional to $\gamma$. For $X, Y \in \mathfrak{g}$, we obtain an equation in $\tilde{\mathfrak{g}}^{\otimes 4}$. Using the decomposition $\tilde{\mathfrak{g}} = k\gamma \oplus \mathfrak{g}$, this leads to $2^4$ equations, eight of which are trivially satisfied. In the remaining eight, three are the identities corresponding to axioms (L2’-a) and (L3’). Therefore, $(\mathfrak{g}, \sigma, [\, , \,], \tilde{\sigma})$ must be a good quantum Lie algebra. The remaining five identities are automatically satisfied by (i).
Any braided Lie algebra $L$ can be imbedded in a unital one $(L, \eta_L):$ take the direct sum of braided Lie algebras $L = k\gamma \oplus L$ (in the sense of lemma 2.2), with unit $\eta_L(1) = \gamma;$ here $k\gamma$ is the unique one dimensional braided Lie algebra. We call $L$ the trivial extension of $L.$ Not all unital braided Lie algebras are trivial extensions.

**Proposition 3.8.** If $L$ is the trivial extension of some braided Lie algebra $L,$ then $L$ is good, and $(g = \ker \varepsilon_L, \sigma, [, ], \delta)$ is a good quantum Lie algebra (see (3.8), (3.9)). Moreover, $\delta: g \to g \otimes g$ is injective, and there are bialgebra isomorphisms $B(L) \simeq k[\gamma] \otimes B(L)$ and $U(g) \simeq B(L)$.

**Proof.** The bialgebra isomorphism $B(L) \simeq k[\gamma] \otimes B(L)$ (with $\gamma$ grouplike, not primitive) is by proposition 3.3. We note that $\gamma$ is not a zero divisor in $B(L)$ so that $L$ is good. For the other statements, we apply lemma 2.4 to $(g, \delta, \varepsilon) := (L, \Delta_L, \varepsilon_L)$ and $\delta := \Upsilon_L.$ Clearly, $(\Upsilon_L)_L = \Upsilon_L.$ The projection $\varphi: L = g \to g$ with kernel $k\gamma$ (now given by $\varphi(X) = X - \varepsilon(X)\gamma,$ $X \in L$) restricts to an isomorphism $\varphi|_L : L \xrightarrow{\sim} g$ and satisfies $(\varphi \otimes \varphi) \circ \Upsilon_L = \sigma \circ (\varphi \otimes \varphi)$ as already known, and also $(\varphi \otimes \varphi) \circ \Delta_L = \delta \circ \varphi$ as is easily checked. Therefore, when restricted to $L,$ the previous equation tells that $\delta$ is injective (since $\Delta_L|_L = \Delta_L$ is), and that the algebra isomorphism $U(g) \simeq S_{\Upsilon_L}(L) =: B(L)$ of lemma 2.4 is also a bialgebra isomorphism in the present case.

**Remark.** We see that any braided Lie algebra $L$ can help to construct a good quantum Lie algebra $(g, \sigma, [, ], \delta)$ of the same dimension, with $\delta$ injective: take $g = \ker \varepsilon_L$ where $L = k\gamma \oplus L.$ Clearly, not all good quantum Lie algebras are of this form (for instance, the usual Lie algebras). This large class of examples shows that $U(g)$ does not always have an antipode, since in this case $U(g) \simeq B(L)$ cannot have an antipode.

### 3.3 Split braided Lie algebras.

The notion of a unital braided Lie algebra can be generalized as follows. Indeed, while we are interested in unital extensions of braided Lie algebras $L,$ these $L$ themselves are not typically unital. Yet deformation examples should be close to unital ones since the classical model for the entire theory in [Maj-94] is the example $L = k1 \oplus$ a classical Lie algebra.

Thus, we say a braided Lie $L$ is split if there is a morphism $c: k \to L$ in the braided category, or in concrete terms a distinguished element $c \in L$ with span isomorphic to the trivial object, such that

$$\varepsilon(c) = 1, \quad [x, c] = \varepsilon(x)c, \quad \forall x \in L. \quad (3.13)$$

Here $kc$ the trivial object implies $\Psi(c \otimes x) = x \otimes c$ and $\Psi(x \otimes c) = c \otimes x$ for all $x \in L,$ while the second of (3.14) is equivalent (by the counit axioms) to

$$\Upsilon(x \otimes c) = c \otimes x, \quad \forall x \in L. \quad (3.14)$$

Being split is significantly weaker than the unital case. However, (3.14) still ensures that $c$ is still central in $B(L).$ We set $L^+ = \ker \varepsilon_L.$ Then, by again the counit axioms, there exist uniquely determined maps $\omega: (L^+)^{\otimes 2} \to (L^+)^{\otimes 2}, \rho: L^+ \to (L^+)^{\otimes 2}$ and $\Theta: L^+ \to L^+$ such that, for all $x, y \in L^+,$

$$\Upsilon(x \otimes y) = \omega(x \otimes y) + [x, y] \otimes c, \quad (3.15)$$
$$\Upsilon(c \otimes x) = \Theta(x) \otimes c + \rho(x); \quad (3.16)$$
$$\Theta(c) = [c, x]. \quad (3.17)$$

As in the unital case, one may write the axioms of a braided Lie algebra in the split case in terms of $(L^+, \omega, [, ], \rho, \Theta).$ For example, the braided Jacobi identity (3.11) on $L$ gives

$$[x, [y, z]] = [[x, y], z] + [x, [y, z]], \quad \forall x, y, z \in L^+ \quad (3.18)$$
generalising the unital case. Note also that, because of (3.16), lemma 2.3 cannot be applied in general and therefore \( \omega \) (in the place of \( \sigma \)) needs not obey the braid relation. Moreover, \( B(\mathcal{L}) \) in these terms is generated by \( c, \mathcal{L}^+ \) with \( c \) central and the relations

\[
xy - \circ \omega (x \otimes y) = [x, y]c, \quad cx - \Theta (x)c = \circ \rho x, \quad \forall x, y \in \mathcal{L}^+.
\]

(3.19)

Also note that if \( c \) is not a zero divisor in \( B(\mathcal{L}) \) then clearly

\[
\ker (\id - \omega) \subset \ker ([,])
\]

(3.20)

just as for good quantum Lie algebras above.

Let us assume now that \( \mathcal{L}^+ \) is a simple object. Then \( \Theta \) acts as a multiple \( \lambda \) of the identity and we define the reduced enveloping algebra associated to the split braided Lie algebra to be:

\[
B_{\text{red}}(\mathcal{L}^+) = B(\mathcal{L})/\langle c - \lambda \rangle.
\]

It is the tensor algebra \( T(\mathcal{L}^+) \) modulo the ideal generated by the relations for all \( x, y \in \mathcal{L}^+ \):

\[
xy - \circ \omega (x \otimes y) = \lambda [x, y]
\]

(3.21)

\[
\lambda (1 - \lambda) x = \circ \rho (x).
\]

(3.22)

Finally, we suppose that \( \lambda \neq 0 \) and \( c \) is not a zero divisor of \( B(\mathcal{L}) \). If we define

\[
A = \lambda^{-1}(\id - \omega)
\]

then clearly (3.20) and (3.18) appear as

\[
\ker A \subset \ker ([,]), \quad [[,]](A(x \otimes y), z) = [[[x, y], z]], \quad \forall x, y, z \in \mathcal{L}^+
\]

(3.23)

which have been proposed as the axioms of a ‘generalized Lie algebra’ \((\mathcal{L}^+, A, [\cdot, \cdot])\) in [LS]. Here (3.23) is called a generalized Jacobi identity and \( A \) is called a generalized antisymmetrizer (although \( A \) is not required to satisfy any further axioms in this regard). Similarly, according to the definition of [LS], the universal enveloping algebra of \((\mathcal{L}^+, A, [\cdot, \cdot])\) is \( U_{LS}(\mathcal{L}^+) = T(\mathcal{L}^+)/\langle \text{im}(A - [\cdot, \cdot]) \rangle \), i.e. generated by \( \mathcal{L}^+ \) with the relation (3.21). We see that split braided Lie algebras with simple \( \mathcal{L}^+ \) have this general structure but with \( \omega, [,] \) and an additional map \( \rho \) obeying several more axioms inherited from the braided Lie algebra structure. We also see that the natural ‘enveloping algebra’ generated by \( \mathcal{L}^+ \) in this case, namely \( B_{\text{red}}(\mathcal{L}^+) \) has potentially an additional relation (3.22). On the other hand, \( B_{\text{red}}(\mathcal{L}^+) \) comes as a quotient of a quadratic algebra \( B(\mathcal{L}) \) giving its homogenisation and forming a bialgebra (in a braided category), both of them desirable features.

### 3.4 The adjoint action.

Let \( \text{Rep}(L) \) be the category of representations of \( L \) : its objects are pairs \((V, \alpha)\) where \( \alpha : L \otimes V \to V \) is a morphism in \( \mathcal{V} \) satisfying axiom (R1) pictured in (3.24); \( \alpha \) is called the action of \( L \) on \( V \) and we write \( x \triangleright_{\alpha} v = \alpha(x \otimes v) \). A morphism of representations (intertwiner) is a morphism \( f : V \to W \) in \( \mathcal{V} \) satisfying \( f \circ \alpha _V = \alpha _W \circ (\id \otimes f) \). Clearly, \( \text{Rep}(L) \) is the same as \( B(L)_M \). It is a monoidal category with tensor product \((V, \alpha_V) \otimes (W, \alpha_W) = (V \otimes W, \alpha_V \otimes \alpha_W)\) where \( \alpha _{V \otimes W} = (\alpha _V \otimes \alpha _W)(\id _L \otimes \Psi _L, V \otimes \id _W)(\Delta _L \otimes \id _{V \otimes W}) \), and unit object \( k \) (with action afforded by the counit \( \varepsilon \)).

\[
\begin{align*}
\text{(R1)} & \quad \begin{array}{c}
\text{L} \ \
\searrow \ \
\alpha \ \
\downarrow \ \
\text{V}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\text{(R2)} & \quad \begin{array}{c}
\text{L} \\
\downarrow \ \\
\Delta \ \\
\alpha \ \\
\downarrow \ \\
\text{L} \ \
\downarrow \ \
\text{L} \\
\downarrow \ \\
\text{V}
\end{array}
\end{align*}
\]

(3.24)
We also let $\text{Rep}(L)'$ be the subcategory of representations satisfying the property (R2) also pictured in (3.23). Clearly, if $0 \to U \to V \to W \to 0$ is an exact sequence in $\text{Rep}(L)$, $(V, \alpha_V)$ satisfies (R2) if and only if $(U, \alpha_U)$ and $(W, \alpha_W)$ satisfy (R2).

**Proposition 3.9.** $\text{Rep}(L)'$ is a braided monoidal category with braiding $\Psi$, the same braiding as in $\mathcal{V}$.

**Proof.** $\text{Rep}(L)'$ is closed under $\otimes$: Let $(V, \alpha_V)$ and $(W, \alpha_W)$ satisfy (R2). We check that $(V \otimes W, \alpha_V \otimes \alpha_W)$ also satisfies (R2):

$$
\begin{align*}
&= 
\begin{picture}(200,200)
\put(10,10){\includegraphics[width=200px]{diagram.png}}
\end{picture}
\end{align*}
$$

The first, third and fifth equalities use the coassociativity of $\Delta$ and the naturality of $\Psi$, which holds since it already holds in $\mathcal{V}$, the second is (R2) for $(W, \alpha_W)$, the fourth is (R2) for $(V, \alpha_V)$. Next, the braiding $\Psi_{V,W}$ is a morphism in $\text{Rep}(L)'$: we check the equality $\Psi_{V,W} \circ \alpha_V \otimes \alpha_W = \alpha_W \otimes \alpha_V \circ (\text{id} \otimes \Psi_{V,W})$:

$$
\begin{align*}
&= 
\begin{picture}(200,200)
\put(10,10){\includegraphics[width=200px]{diagram.png}}
\end{picture}
\end{align*}
$$

The first and third equalities use the naturality of $\Psi$, the second is property (R2) for $(V, \alpha_V)$. Note that (R2) is used (only for $V$), therefore one cannot conclude anything for $\text{Rep}(L)$ in general.

**Remark.** This is an analogue at the Lie level of the braided category of modules with respect to which a braided group behaves cocommutatively (which in turn was the origin of (L2)), see [Maj-93b] for the general setting of that.

Obviously, $L_\text{Ad} = (L, [\cdot, \cdot])$ is a representation of $L$ and satisfies (R2) by assumption, so $L_\text{Ad} \in \text{Rep}(L)'$, and so does the trivial representation. It is also clear from the axioms of a braided Lie algebra that the maps $\Delta : L_\text{Ad} \to L_\text{Ad} \otimes L_\text{Ad}$, $[\cdot, \cdot] : L_\text{Ad} \otimes L_\text{Ad} \to L_\text{Ad}$ and $\varepsilon : L_\text{Ad} \to k$, are all intertwiners, and so is $\Psi_{L,L} : L_\text{Ad} \otimes L_\text{Ad} \to L_\text{Ad} \otimes L_\text{Ad}$ by the previous proposition. Therefore:

$$
\Upsilon : L_\text{Ad} \otimes L_\text{Ad} \to L_\text{Ad} \otimes L_\text{Ad} \text{ is an intertwiner in } \text{Rep}(L)'.
$$

**Proposition 3.10.** There exists a unique action, the adjoint action noted $x \triangleright_\text{Ad} Y$, of $L$ on $B(L)$ with the properties that $x \triangleright_\text{Ad} y = [x, y]$ for $y \in L \hookrightarrow B(L)$ and that $B(L)$ is an algebra in $\text{Rep}(L)'$ for this action. Under this action, $B(L) = \bigoplus_{n \geq 0} B(L)_n$ is the direct sum of its homogeneous components.

**Proof.** The tensor algebra of the adjoint representation $L_\text{Ad}$ is an algebra in $\text{Rep}(L)$ (as is always the case for the tensor algebra of a representation) and, by the previous proposition, belongs to $\text{Rep}(L)'$ since $L_\text{Ad}$ does. The ideal $(\text{im}(\text{id}^\otimes - \Upsilon))$ is clearly graded, and
a subrepresentation by the above observation (\(\Upsilon\) is an intertwiner). Therefore the quotient 
\(B(L) = T(L_{Ad})/(\text{im}(id \otimes - \circ \Upsilon))\) is an algebra in \(\text{Rep}(L)'\) and has the desired properties. Since \(L\) generates as an algebra, uniqueness is clear.

The adjoint action of \(L\) on \(B(L)\) defines an action of \(B(L)\) on itself, also noted \(X \triangleright \text{Ad} \ Y\). The bracket \([\cdot, \cdot]_{B(L)}: B(L) \otimes B(L) \to B(L)\), \([X, Y]_{B(L)} = X \triangleright \text{Ad} \ Y\), satisfies \((L1)\) and \((L2)\) by the above proposition. It also satisfies \((L3)\), since both \([\cdot, \cdot]\) and \(\Upsilon\) are morphisms of coalgebras. So we have:

**Corollary 3.11.** The bialgebra \(B(L)\) becomes a braided Lie algebra in \(\text{Rep}(L)'\) (also in \(V\)) for the braided Lie product \([X, Y]_{B(L)} = X \triangleright \text{Ad} \ Y\). Each graded summand \(B(L)_n\) is a braided Lie subalgebra.

**Remark.** Assume that \(L\) is a good braided Lie algebra (i.e. \(L = \tilde{\mathfrak{g}}\) is the extension of a good quantum Lie algebra \(\mathfrak{g}\)). Recall that \(U(\mathfrak{g}) \simeq B(\tilde{\mathfrak{g}})/\langle \gamma - 1 \rangle\). By taking appropriate quotients, one easily gets that \(U(\mathfrak{g})\) is a left \(U(\mathfrak{g})\)-module algebra, and a braided Lie algebra in \(U(\mathfrak{g})^*\) (or in \(V\)). However, all statements concerning the grading are lost and should be replaced by "each term \(U(\mathfrak{g})/\langle n \rangle\) of the natural filtration of \(U(\mathfrak{g})\) is a braided Lie subalgebra of \(U(\mathfrak{g})\)". But in turn, if \(L = \tilde{\mathfrak{g}}\) is the trivial extension of some braided Lie algebra \(L\), all the grading properties can be recovered, since \(U(\mathfrak{g}) \simeq B(L)\).

### 3.5 The main example.

The definition of a braided Lie algebra was motivated in [Maj-94] as follows. Let \((H, m, \eta, \Delta, \varepsilon, S)\) be a Hopf algebra in \(V\). Its (left) braided adjoint action is

\[
\text{Ad}_L x(y) = x \triangleright \text{Ad} \ y = m(m \otimes S)(id \otimes \Psi)(\Delta \otimes id)(x \otimes y). \tag{3.25}
\]

We note \([x, y] := x \triangleright \text{Ad} \ y\). \(H\) is a left crossed module over itself (in the braided sense) for the regular coaction \(\Delta\) and the braided adjoint action \(\text{Ad}_L\). One easily checks that the corresponding crossed module braiding \(\Upsilon := ([\cdot, \cdot] \otimes id)(id \otimes \Psi)(\Delta \otimes id)\) satisfies \(m \circ \Upsilon = m\) (this is the equality \(xy = (x(1) \ y S x(2)) x(3)\) when \(\Psi = id_{H \otimes H}\)). Moreover, \((\varepsilon \otimes id) \circ \Upsilon = id \otimes \varepsilon\) and \((id \otimes \varepsilon) \circ \Upsilon = [\cdot, \cdot]\). Therefore there is a unique map \(\sigma : \varepsilon \otimes \varepsilon \to \ker \varepsilon \otimes \varepsilon\) such that \(\Upsilon(x \otimes y) = \sigma(x \otimes y) + [x, y] \otimes 1, x, y \in \ker \varepsilon\). Multiplying this in \(H\), we get

\[
[x, y] = m \circ (\Upsilon - \sigma)(x \otimes y) = m \circ (id - \sigma)(x \otimes y)
\]

for \(x, y \in \ker \varepsilon\). This implies \(\ker (id - \sigma) \subset \ker ([\cdot, \cdot]_{(\ker \varepsilon) \otimes \varepsilon})\). Therefore \((\ker \varepsilon, \sigma, [\cdot, \cdot])\) is always a quantum Lie algebra in \(V\) (all maps are morphisms in \(V\) by assumption, axioms 1-3 come from lemma \[2.3\] applied to \(\mathfrak{g} = \ker \varepsilon\), \(\sigma = \Upsilon\), and axiom 4 (antisymmetry) holds by the above equality). However, \((H, \Delta, \varepsilon, [\cdot, \cdot])\) is not always a braided Lie algebra in \(V\).

**Proposition 3.12.** \(H_L = (H, \Delta, \varepsilon, [\cdot, \cdot], \eta)\) is a good braided Lie algebra in \(V\) if and only if axiom \((L2)\) is satisfied.

**Proof.** As shown in [Maj-94], axiom \((L1)\)-braided Jacobi identity- is always satisfied and, assuming \((L2)\), then \((L3)\) is also satisfied. Thus in this case, \(H_L\) is a braided Lie algebra. The Hopf algebra unit of \(H\) is clearly a unit for \(H_L\) in the sense of \((3.6)\). The good part (antisymmetry axiom) has been checked above.

Consider the case of a usual Hopf algebra \(H\) (i.e. a Hopf algebra in \(\mathcal{M}\)). Then axiom \((L2)\) for \(H_L\) is ensured if \(H\) is cocommutative, but fails to hold in general. However, when the non-cocommutativity of \(H\) is controlled by a quasitriangular structure \(R \in H \otimes H\), then axiom \((L2)\) remains valid, not for \(H\) but for a braided version of \(H\).
Let \((H, R)\) be a usual quasitriangular Hopf algebra \((R \in H \otimes H\) satisfies Drinfeld’s axioms \([Dr]\)) and view \(H\) as an object in \(\mathcal{M}\) by the left adjoint action \(\text{Ad}_L\). We will need the three braidings \(\Upsilon, \Psi_R\) and \(\Xi_{R,R}\) on \(H\) given below:

**Lemma 3.13.** Let \(R, S \in H \otimes H\) be two co-quasitriangular structures on a (usual) Hopf algebra \(H\). The coactions \(\Delta, \lambda_R, \delta_{R,S} : H \to H \otimes H\) below define crossed module structures on \((H, \text{Ad}_L)\), with associated braidings \(\Upsilon, \Psi_R\) and \(\Xi_{R,S}\) as indicated:

\[
\begin{align*}
\Delta(x) &= x(1) \otimes x(2), \\
\lambda_R(x) &= R^{(2)} \otimes R^{(1)} \triangleright \text{Ad} x = R_{21}(1 \otimes x) R_{21}^{-1}, \\
\delta_{R,S}(x) &= R_{21}(1 \otimes x) S, \\
\Upsilon(x \otimes y) &= x(1) \triangleright \text{Ad} y \otimes x(2), \\
\Psi_R(x \otimes y) &= R^{(2)} \triangleright \text{Ad} y \otimes R^{(1)} \triangleright \text{Ad} x, \\
\Xi_{R,S}(x \otimes y) &= (R^{(2)} S^{(1)}) \triangleright \text{Ad} y \otimes R^{(1)} x S^{(2)}
\end{align*}
\]

\(\text{(3.26)}\)

**Proof.** The first one needs no comment. The second one is the image of \((H, \text{Ad}_L)\) under the monoidal functor \(\mathcal{F}_R : \mathcal{M} \to \mathcal{M}'\) which sends arbitrary left module \((M, \cdot)\) to \((M, \cdot \lambda_{R}^{(M)})\) where \(\lambda_{R}^{(M)}(m) = R^{(2)} \otimes R^{(1)}.m\). The braiding of \((M, \cdot)\) calculated in \(\mathcal{M}'\) (thanks to \(R\)) and that of \((M, \cdot \lambda)\) in \(\mathcal{M}\) are equal. The reader will easily check that replacing the factor \(R_{21}^{-1}\) in the definition of \(\lambda_{R}\) by any other co-quasitriangular structure \(S\) does not affect the crossed module properties. Note that \(\lambda_{R} = \delta_{R,R^{-1}}\), and therefore \(\Psi_{R} = \Xi_{R,R^{-1}}\). Moreover, \((H, \text{Ad}_L, \delta_{R,S})\) is in the image of some functor \(\mathcal{F}_R\) (if and) only if \(S = R_{21}^{-1}\), since \(\lambda_T(1) = 1 \otimes 1\) and \(\delta_{R,S}(1) = R_{21}^{-1} S\).

Define the linear maps \(\Delta : H \to H \otimes H, \xi = \varepsilon\) and \(\underline{\xi} : H \to H\) by

\[
\begin{align*}
\Delta(x) &= x(1) S(R^{(2)}) \otimes R^{(1)} \triangleright \text{Ad} x =: x(1) \otimes x(2) \\
\underline{\xi}(x) &= R^{(2)} S(R^{(1)} \triangleright \text{Ad} x)
\end{align*}
\]

\(\text{(3.27, 3.28)}\)

**Proposition 3.14.** View \(H\) as an object in \(\mathcal{M}\) via \(\text{Ad}_L\) (the braiding is \(\Psi_{R}\)).

(i) \(H = (H, m, \eta, \Delta, \xi, S)\) is a Hopf algebra in \(\mathcal{M}\). It is \(\Xi_{R,R^{-1}}\)-cocommutative in the sense that \(\Xi_{R,R^{-1}} \circ \Delta = \Delta\). Its braided adjoint action \(\text{Ad}_L\) coincides with the adjoint action \(\text{Ad}_L\) of \(H\) and the crossed module coactions \(\Upsilon, \Psi_R\) on \((H, \text{Ad}_L, \Delta)\) coincide.

(ii) \(H_L = (H, \Delta, \xi, [\cdot, \cdot])\) is a left braided Lie algebra in \(\mathcal{M}\) for the braided Lie bracket \([x, y] = \text{Ad}_L(x(y)\).

(iii) Let \(L \subset H\) satisfy \([H, L] \subset L\). Then the following are equivalent:

\[
\begin{align*}
(a) \quad & \Delta(L) \subset H \otimes L, \\
(b) \quad & \Delta(L) \subset H \otimes L, \\
(c) \quad & \Delta(L) \subset L \otimes H.
\end{align*}
\]

If one of this condition holds, then \(L\) is a braided Lie subalgebra of \(H_L\).

(We shall need the dual version which is more general, so we omit this proof; see proposition \([Maj-LN]\).)

### 3.6 Braided Lie coalgebras

The dual notion of left ‘braided Lie coalgebras’ is just given in the diagrammatic setting by turning the diagram-axioms of a right braided Lie algebra upside-down, or by reflecting those of a left braided Lie algebra about a horizontal axis and restoring braid crossings, see \([Maj-LN]\).
Definition 3.15. A (left) braided Lie coalgebra in \( \mathcal{V} \) is an algebra \((A, \mu, \eta)\) in the category endowed with a morphism (the braided Lie cobaracket) \( \delta : A \to A \otimes A \) satisfying the axioms below:

\[
\begin{align*}
\text{(C1)} & : \quad \delta = \delta \\
\text{(C2)} & : \quad \delta = \delta \\
\text{(C3)} & : \quad \delta = \delta
\end{align*}
\]

An ideal \( \mathcal{I} \) of \( A \) is an algebra ideal such that \( \delta(\mathcal{I}) \subset A \otimes \mathcal{I} + \mathcal{I} \otimes A \). A counit on \((A, \mu, \eta, \delta)\) is a morphism of braided Lie algebras \( \varepsilon : A \to k \) satisfying

\[
(\varepsilon \otimes \text{id}) \circ \delta = \text{id}, \quad (\text{id} \otimes \varepsilon) \circ \delta = \eta \circ \varepsilon. \tag{3.29}
\]

In the definition, \( k \) is seen as a braided Lie coalgebra with \( \delta(1) = 1 \otimes 1 \). Obviously, \( \mathcal{I} \) is an ideal of the braided Lie coalgebra \( A \) iff the structure maps of \( A \) induce a braided Lie algebra structure on \( A/\mathcal{I} \). By turning proofs upside-down, the following is also clear.

Lemma 3.16. If \((A, \mu_A, \eta_A, \delta_A)\) is a braided Lie coalgebra, the morphism \( \Upsilon_A : A \otimes A \to A \otimes A, \)

\[
\Upsilon_A = (\mu_A \otimes \text{id}) \circ (\text{id} \otimes \Psi) \circ (\delta_A \otimes \text{id}), \tag{3.30}
\]

is a morphism of algebras and satisfies the braid relation.

If \( A \) has a right-dual \( L \) in the categorical sense (in our concrete setting it means if \( A \) is finite-dimensional) there are evaluation and coevaluation maps \( \text{ev} : A \otimes L \to k \) and \( \text{coev} : k \to L \otimes A \), using which it follows by diagrammatic methods that \( A \) is a left braided-Lie coalgebra iff \( L \) is a right braided Lie algebra.

For our purposes we are interested in \( A \in \mathcal{V} = \mathcal{AM} \) for some co-quasitriangular Hopf algebra \( A \) but with \( L \) regarded in the braided category \( \mathcal{MA} \). This is equivalent to the above via the antipode of \( A \), but in an algebraic setting it is more natural to avoid the use of that. Thus we let \( L = A^\ast \) be the usual dual, viewed as a right \( A \)-comodule. We denote by \langle \cdot, \cdot \rangle : A \otimes L \to k \) the evaluation pairing and extend this (as for usual Hopf algebra duality) to \((A \otimes A) \otimes (L \otimes L) \to k \) by setting

\[
\langle a \otimes b, x \otimes y \rangle := \langle a, x \rangle \langle b, y \rangle. \tag{3.31}
\]

Lemma 3.17. (i) Let \( A \in \mathcal{AM} \) be finite-dimensional and \( L = A^\ast \). Then \((L, \Delta_L, \varepsilon_L, [\cdot, \cdot]_L)\) is a braided Lie algebra in \( \mathcal{MA} \) if and only if \((A, \mu_A, \eta_A, \delta_A)\) is a braided Lie coalgebra, with

\[
\langle \mu_A(a \otimes b), x \rangle := \langle a \otimes b, \Delta(x) \rangle, \quad \langle \delta_A(a), x \otimes y \rangle := \langle a, [x, y]_L \rangle, \quad \langle \eta_A(1), x \rangle := \varepsilon_L(x). \tag{3.31}
\]

Moreover, \( \eta_L : k \to L \) is a unit for \( L \) if and only if \( \varepsilon_A : A \to k \) is a counit for \( A \), where \( \varepsilon_A(a) := \langle a, \eta_A(1) \rangle \). (ii) Let \( L = A^\ast \) be unital. Then \( L \) is the trivial extension of some braided Lie algebra \( \mathcal{L} \) if and only if \( \ker \varepsilon_A \) is a unital subalgebra of \( A \) with unit \( \theta \) satisfying \( \delta_A(\theta) = 1 \otimes \theta \). Moreover, \( \mathcal{L} = (1 - \theta)^\perp \).

Proof. (i) This is a straightforward exercise from the definitions. It is important to use compatible conventions for the braidings of \( \mathcal{MA} \) and \( \mathcal{AM} \) as obtained from \( r : A \otimes A \to \}

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\( k \); one may check that they are then adjoint. (ii) A coalgebra decomposition \( L = k \gamma + L \) is equivalent to an algebra decomposition

\[ A = (k \gamma)^\perp + L^\perp = \ker \varepsilon_A \oplus k \xi, \]

for some vector \( \xi \) spanning \( L^\perp \). Since \( A \) is a unital algebra, so must be \( \ker \varepsilon_A \). Since \( (A, \varepsilon_A) \) is a counital braided Lie coalgebra \((3.29)\), one must have \( \delta_A(\xi) = 1 \otimes \xi + w \) for some \( w \in (\ker \varepsilon_A)^\otimes \). The hypothesis \( [L, L] \subset L \) is then equivalent to \( w = 0 \), that is, \( \delta_A \xi = 1 \otimes \xi \).

Finally, if we normalise \( \xi \) so that \( \langle \gamma, \xi \rangle = 1 \), i.e. \( \varepsilon_A(\xi) = 1 \), then \( \xi \) is an idempotent (by definition of \( \xi \) and because \( \Delta(\gamma) = \gamma \otimes \gamma \)). With this normalisation, \( \theta = 1 - \xi \) is the algebra unit of \( \ker \varepsilon_A \) and \( L = \xi^\perp = (1 - \theta)^\perp \).

\section{Link with differential calculi.}

This section contains our main results, namely theorems connecting the above results to bicovariant differential calculi on ordinary Hopf algebras.

\subsection{Extended tangent spaces, inner calculi.}

Following [Wor], a bicovariant first order differential calculus (bicovariant FODC) over a Hopf algebra \( A \) is a pair \( (\Gamma, d) \) where \( \Gamma \) is a Hopf bimodule, with coactions \( \Delta_L : \Gamma \to A \otimes \Gamma \) and \( \Delta_R : \Gamma \to \Gamma \otimes A \), and the linear map \( d : A \to \Gamma \) (the differential) satisfies

\begin{align*}
\langle 1 \rangle & \quad d \text{ is a derivation : } d(ab) = d(a)b + ad(b) \text{ for all } a, b \in A, \\
\langle 2 \rangle & \quad d \text{ is a bicomodule map,} \\
\langle 3 \rangle & \quad \text{ the map } A \otimes A \to \Gamma, a \otimes b \mapsto adb, \text{ is surjective.}
\end{align*}

Let \( \pi_R : \Gamma \to \Gamma_R \) be the canonical projection on right invariants (notations of the preliminaries). \( \Gamma_R \) is called the (right) cotangent space of \( \Gamma \). The differential \( d \) and the (right handed) Maurer-Cartan map \( \omega_R = \pi_R \circ d : A \to \Gamma_R \) are related by

\[ d(a) = \omega_R(a(1))a(2), \quad \omega_R(a) = d(a(1)) S(a(2)) \]  

(4.1)

and therefore are equivalent data. Axioms \( \langle 1 - 3 \rangle \) for \( (\Gamma, d) \) are equivalent to axioms \( \langle 1' - 3' \rangle \) below for the pair \( (\Gamma_R, \omega_R) \):

\begin{align*}
\langle 1' \rangle & \quad \omega_R(ab) = a \triangleright \omega_R(b) + \omega_R(a) \varepsilon(b), \\
\langle 2' \rangle & \quad \Delta_L \omega_R = (\text{id} \otimes \omega_R) \text{ad}_L, \\
\langle 3' \rangle & \quad \omega_R : A \to \Gamma_R \text{ is surjective.}
\end{align*}

\((\triangleright, \Delta_L)\) is the left crossed module structure of \( \Gamma_R \), and \( \text{ad}_L : A \to A \otimes A, \text{ad}_L(a) = a(1) S(a(3)) \otimes a(2), \) is the left adjoint coaction). A calculus \( (\Gamma, d) \) is called inner if \( d \) is an inner derivation, that is, if there exists \( \theta \in \Gamma \) such that for all \( a \in A \)

\[ da = a\theta - \theta a \quad (\text{equivalently : } \omega_R(a) = a(1) \theta S(a(2)) - \varepsilon_A(a) \theta). \quad (4.2) \]

Remark. One can always assume (if necessary by replacing \( \theta \) by \( \pi_R(\theta) \)), that \( \theta \in \Gamma_R \). Indeed, apply \( \pi_R \) to the right equality in \((4.2)\), we get, using the properties of \( \pi_R, \omega_R(a) = \pi_R \omega_R(a) = \pi_R(a(1)) \theta S(a(2)) - \varepsilon_A(a) \pi_R(\theta) = (a - \varepsilon_A(a)) \triangleright \pi_R(\theta) \). Thus \( \theta' = \pi_R(\theta) \) has the same property as \( \theta \) and is right invariant.

Any bicovariant FODC \( (\Gamma, d) \) can be extended to a pair \( (\tilde{\Gamma}, d) \) which satisfies all axioms except \( \langle 3 \rangle \), with the property that it contains \( \Gamma \) as a Hopf sub-bimodule, and that the derivation \( A \xrightarrow{d} \Gamma \xrightarrow{\theta} \tilde{\Gamma} \) is inner; one takes \( \tilde{\Gamma} = \Gamma \oplus \Theta A \) as a right \( A \)-module (\( \Theta \) a free variable), with missing structures fixed by : \( \Theta \) biinvariant and left action \( a \Theta = da + \Theta a \) in
$\tilde{\Gamma}$. $(\tilde{\Gamma}, \tilde{d})$ is called the extended bimodule of $\Gamma$ [Wor], and the crossed module $(\tilde{\Gamma}_R, \triangleright, \tilde{\Delta}_L)$ of right invariants of $\tilde{\Gamma}$ is called the (right) extended cotangent space of $\Gamma$. Let $\bar{\pi}_R : \tilde{\Gamma} \to \Gamma_R$ be the canonical projection and define $\tilde{\omega}_R : A \to \Gamma_R$ by

$$\tilde{\omega}_R(a) = a \triangleright \Theta = \omega_R(a) + \varepsilon_A(a) \Theta$$

Let $(A, m, \text{ad}_L)$ be the left crossed $A$-module, where the left action is the left regular one $(a \triangleright b = m(a \otimes b) = ab)$. Axioms (1’ – 3') can again be restated as the fact that $\tilde{\omega}_R : A \to \Gamma_R$ is a surjective crossed module homomorphism; equivalently (since $\omega_R(1) = 0$), that the restriction $\omega_R : \ker \varepsilon_A \to \Gamma_R$ is a surjective crossed module homomorphism. Therefore, $\tilde{\omega}_R$ and $\omega_R$ respectively induce crossed module isomorphisms

$$(A, m, \text{ad}_L)/\mathcal{I}_R \xrightarrow{\sim} \tilde{\Gamma}_R,$$  

$$\ker \varepsilon_A/\mathcal{I}_R \xrightarrow{\sim} \Gamma_R,$$  

where $\mathcal{I}_R := \ker \tilde{\omega}_R = \ker \omega_R \cap \ker \varepsilon_A$ is called “the left ideal associated to $\Gamma$”. The case $\mathcal{I}_R = 0$ corresponds to the universal extended cotangent space $\Gamma_R\text{univ} = A$, with $\tilde{\omega}_R\text{univ}(a) = \varepsilon_A(a)$ and therefore $\Gamma_R\text{univ} = \ker \varepsilon_A, \omega_R\text{univ}(a) = a - \varepsilon_A(a)$. By definition, the (right) tangent space $\tilde{\mathfrak{g}}_R$ and extended tangent space $\tilde{\mathfrak{g}}_\Gamma$ of $\Gamma$ are

$$\mathfrak{g}_R = \{ x \in A^* : x(\mathcal{I}_R) = 0 \text{ and } x(1) = 0 \},$$  

$$\tilde{\mathfrak{g}}_\Gamma = \{ X \in A^* : X(\mathcal{I}_R) = 0 \},$$  

As subspaces of $A^*$, one has $\tilde{\mathfrak{g}}_\Gamma = k1_{A^*} \oplus \mathfrak{g}_R$ and $\tilde{\mathfrak{g}}_R = \mathfrak{g}_R \cap \ker \varepsilon_A$. Let $[\cdot, \cdot] : A^o \otimes A^o \to A^o$ be defined by

$$[X, Y] := X(1) Y S(X(2)).$$

Recall from [Wor] that there is a unique bilinear form $(\cdot, \cdot) : \Gamma \times \tilde{\mathfrak{g}}_R \to k$ such that $(\omega_R(a), b, x) = \varepsilon(b)(a, x)$ and, if $\dim_k \Gamma_R < \infty$, it allows to identify $\mathfrak{g}_R$ with $(\Gamma_R)^*$ as right crossed modules over $A$. In this case, by axioms (1’, 2’) of a bicovariant FODC, the right action $\triangleleft$-such that $(a \triangleright \omega_R(b), x) = (\omega_R(b), x \triangleleft a)$, coaction $x \mapsto x^{(0)} \otimes x^{(1)}$ on $\tilde{\mathfrak{g}}_\Gamma$, and the corresponding crossed module braiding $\hat{\sigma}$ are defined by:

$$x \triangleleft a = \langle a, x_{(1)} \rangle x_{(2)} - \langle a, x \rangle 1_{A^o}, \quad x^{(0)} \langle x^{(1)} , h \rangle = [h, x]$$

$$\hat{\sigma}(x \otimes y) = [x_{(1)}, y] \otimes x_{(2)} - [x, y] \otimes 1_{A^o}. $$

for all $h \in A^o$. The direct analogue of this is (see for instance [KS]):

**Proposition 4.1.** (i) There is a unique bilinear form $(\cdot, \cdot) : \tilde{\Gamma} \times \tilde{\mathfrak{g}}_R \to k$ such that $(\tilde{\omega}_R(a), b, X) = (a, X) \varepsilon(b)$. (ii) Assume that dim $A/\mathcal{I}_R < \infty$. Then $\tilde{\mathfrak{g}}_\Gamma \subset A^o$ and $\tilde{\mathfrak{g}}_\Gamma$ has the following properties: (a) $1_{A^o} \in \tilde{\mathfrak{g}}_\Gamma$, (b) $\Delta(\tilde{\mathfrak{g}}_\Gamma) \subset A^o \otimes \tilde{\mathfrak{g}}_\Gamma$, (c) $[A^o, \tilde{\mathfrak{g}}_\Gamma] \subset \tilde{\mathfrak{g}}_\Gamma$. Conversely, if $A^o$ separates the elements of $A$, a subspace $\tilde{\mathfrak{g}}_\Gamma$ of $A^o$ satisfying the properties (a), (b) and (c) is the extended tangent space of a unique (up to isomorphism) bicovariant FODC over $A$, with associated ideal $\mathcal{I}_R = \{ a \in A| \forall X \in \tilde{\mathfrak{g}}_\Gamma, X(a) = 0 \}$. Moreover, $\tilde{\mathfrak{g}}_\Gamma \simeq (\Gamma_R)^*$ as right crossed modules over $A$, with right action $\triangleleft$, coaction $X \mapsto X^{(0)} \otimes X^{(1)}$ and crossed module braiding $\hat{\sigma}$ below:

$$X \leftarrow a = \langle a, X_{(1)} \rangle X_{(2)}, \quad X^{(0)} \langle X^{(1)}, h \rangle = [h, X], \quad \hat{\sigma}(X \otimes Y) = [X_{(1)}, Y] \otimes X_{(2)},$$

for all $h \in A^o$.

Note that $\mathfrak{g}_R$ is not a crossed submodule of $\tilde{\mathfrak{g}}_\Gamma$; it is rather isomorphic to the quotient $\tilde{\mathfrak{g}}_\Gamma/k1_{A^o}$, via the projection $\varphi : \tilde{\mathfrak{g}}_\Gamma \to \mathfrak{g}_R, X \mapsto X - \varepsilon(X)1_{A^o}$. Thus, one has $x \triangleleft a = \varphi(x \leftarrow a), \quad \sigma(x \otimes y) = (\varphi \otimes \varphi) \hat{\sigma}(x \otimes y)$. Woronowicz ([Wor], Th. 5.3 and Th. 5.4) has shown that the triple $(\mathfrak{g}_R, \sigma, [, [\cdot, \cdot])]$ satisfies the axioms of a (left) quantum Lie algebra. This can be recovered by
lemma 2.3 for axioms 1-3, and antisymmetry (axiom 4) comes from (4.7) - after multiplication in $A^\circ$, it gives
\[ [x, y] = xy - m_{A^\circ} \circ \sigma(x \otimes y), \quad (x, y \in g_R). \] (4.9)

One can be more precise : $(g_R, \sigma, [\, , \,])$ is a quantum Lie algebra in $M^A$, but in general not in $C^A_\wedge$. The braiding $\sigma$ makes no problem since it is the crossed module braiding on $g_R$ in $C^A_\wedge$. The quantum Lie bracket is also a morphism in $M^A$ : this follows from the identity $[h, [x, y]] = [[h(1), x], [h(2), y]]$ for all $h, x, y \in A^\circ$. However, the quantum Lie bracket is not $A$-linear since $\Delta([x, y]) \neq [x(1), y(1)] \otimes [x(2), y(2)]$ in general.

Finally, we note from (4.7) that if $\gamma : g_R \to U(g_R)$ is the natural imbedding, then there is a unique algebra homomorphism $U(g_R) \to A^\circ$ such that $\gamma(x) \mapsto x$ for all $x \in g_R$. This algebra homomorphism needs not be injective nor surjective in general.

By lemma 2.4 there is an algebra isomorphism $S_R(g_R)/\langle 1_{A^\circ} - 1 \rangle \simeq U(g_R)$. (The role of $\gamma$ is played by $1_{A^\circ} \in g_R$, which is an element of degree 1 in the tensor algebra of $g_R$ and should not be confused with the unit element $1 \in \ker(T(g_R))$. Also by lemma 2.3 $U(g_R)$ can sometimes be itself a quantum symmetric algebra. We show that this happens when the calculus is inner.

Let $\sigma^t : \Gamma^\wedge_\oplus \to \Gamma^\wedge_\oplus$ be the crossed module braiding on $\Gamma_R \simeq (g_R)^*$. By definition, the quadratic extension of $\Gamma_R$ is
\[ \Gamma^\wedge, quad := T(\Gamma_R)/\langle \ker(id_\otimes^2 + \sigma^t) \rangle. \]

It is known that the crossed product $\Gamma^\wedge, quad = \Gamma^\wedge_\oplus \rtimes A$ has a structure of a graded differential Hopf algebra $[63]$ and maps onto Woronowicz’ external algebra $\Gamma^\wedge_{\text{Wor}}$. In some cases, it coincides with it. This is the case for instance for the standard quadratic extension of $\Gamma$.

Below we let $\Gamma_{\text{inv}} = \Gamma_R \cap \Gamma_R = \{ \eta \in \Gamma | \Delta_R(\eta) = 1 \otimes \eta, \Delta_R(\eta) = \eta \otimes 1 \}$.

**Theorem 4.2.** Let $(\Gamma, d)$ be a finite dimensional bicovariant FODC over some Hopf algebra $A$. If there exists $\theta \in \Gamma_{\text{inv}}$ such that $d(a) = a\theta - \theta a$ for all $a \in A$, then
\[ U(g_R) \simeq (\Gamma_R^\wedge, quad)^\dagger \] (4.10)
is isomorphic to the quadratic dual of $\Gamma_R^\wedge, quad$.

First, the hypothesis of the theorem can be interpreted as follows.

**Lemma 4.3.** There is a 1-1 correspondence between
1. $\theta \in \Gamma_{\text{inv}}$ such that $d(a) = a\theta - \theta a$ for all $a \in A$,
2. elements $\tilde{\theta} \in \ker \varepsilon_A \mod I_R$ satisfying $a\tilde{\theta} \equiv a \mod I_R$ for all $a \in \ker \varepsilon_A$ and $\text{ad}_L(\tilde{\theta}) \equiv 1 \otimes \tilde{\theta} \mod A \otimes I_R$,
3. subspaces $\mathcal{L}$ of $\tilde{g}_R$ satisfying $\tilde{g}_R = k1_{A^\circ} \otimes \mathcal{L}$, $[A^\circ, \mathcal{L}] \subset \mathcal{L}$ and $\Delta(\mathcal{L}) \subset A^\circ \otimes \mathcal{L}$.

It is given by
\[ \omega_R(\tilde{\theta}) = \theta, \quad \mathcal{L} = \{ x \in \tilde{g}_R | (1 - \tilde{\theta}, x) = 0 \} = (\Theta - \theta)^\perp. \] (4.11)
(The orthogonality is with respect to the bilinear form $(\, , \,)$ on $\Gamma \times \tilde{g}_R$).

**Proof.** The equivalence $\theta \Leftrightarrow \tilde{\theta}$ follows directly from the equivalence of sets $(1 - 3)$ and $(1' - 3')$ of axioms of a bicovariant FODC. Equivalence $\theta \Leftrightarrow \mathcal{L}$ : Let $\tilde{\theta}$ have the given properties, and set $I' = I_R \oplus k(1 - \tilde{\theta})$. The sum is direct since $\varepsilon_A(I_R) = 0$ and $\varepsilon_A(1 - \tilde{\theta}) = 1$. This implies that $\mathcal{L}$ has codimension 1 in $\tilde{g}_R$, with possible complement $k1_{A^\circ}$. $I'$ is obviously a left
ideal of $A$ closed under $\text{ad}_L$, therefore $\mathcal{L}$ is a left co-ideal of $A^\circ$, invariant under the left adjoint action of $A^\circ$. Conversely, let $\mathcal{L}$ have the given properties. Then $\mathcal{T} = \{a \in A|\langle a, \mathcal{L} \rangle = 0 \} \supset \mathcal{T}_f$ is a left ideal of $A$ closed under $\text{ad}_L$, and $\mathcal{T}_f = \mathcal{T}' \cap \ker \varepsilon_A$. Therefore $\mathcal{T}_f$ has codimension 1 in $\mathcal{T}'$, i.e. there exists $\xi \in A$, $\xi \notin \ker \varepsilon_A$, such that $\mathcal{T}' = \mathcal{T}_f + k\xi$. We normalize $\xi$ such that $\varepsilon_A(\xi) = 1$ and set $\theta = 1 - \xi \in \ker \varepsilon_A$. Since $\mathcal{T}_f \subset \mathcal{T}'$ are left ideals, one must have for all $a \in \ker \varepsilon_A$, $a\xi \in \mathcal{T}' \cap \ker \varepsilon_A = \mathcal{T}_f = \mathcal{T}'$ (i.e. $a\theta = a$ mod $\mathcal{T}_f$). Since $\mathcal{T}_f \subset \mathcal{T}'$ are closed under $\text{ad}_L$, one must have $\text{ad}_L(\xi) = a \otimes \xi + w$ for some $a \in A$ and $w \in A \otimes \mathcal{T}_f$. By the countax ideals and from $\varepsilon_A(\xi) = 1$, one has $a = (\text{id} \otimes \varepsilon_A)\text{ad}_L(\xi) = \varepsilon_A(\xi) = 1$.

Proof of the theorem. Define $\hat{\theta}$ and $\mathcal{L}$ as in the lemma. $\mathcal{L}$ satisfies $\hat{\mathcal{L}}_f = k1_{A^\circ} + \mathcal{L}$ and $\hat{\sigma}(\mathcal{L} \otimes \mathcal{L}) \subset \mathcal{L} \otimes \mathcal{L}$ (since $[A^\circ, \mathcal{L}] \subset \mathcal{L}$ and $\Delta(\mathcal{L}) \subset A^\circ \otimes \mathcal{L}$). Therefore, by lemma 2.4, $U(\hat{\mathcal{G}}_f) \simeq S_{\mathcal{L}}(\mathcal{G}_f) = T(\mathcal{G})/\langle \text{im}(\text{id}^{\mathcal{O}} - \sigma) \rangle$, whose quadratic dual is by definition $T(\mathcal{G}^*)/\langle \ker(\text{id}^{\mathcal{O}} + \sigma^*) \rangle$, i.e. $\Gamma^R_{\text{quad}}$.

Note that the simplest way to construct a bicovariant FODC is to pick some $\text{ad}_L$-invariant element $a$ and a left ideal $J$ of $A$, such that either $a$ or $J$ belongs to $\ker \varepsilon_A$, and set $\mathcal{T}_f = Ja$. In particular, we achieve the hypothesis of the theorem if we take $\mathcal{T}_f = \ker \varepsilon_A(1 - \theta)$ with $\theta \in \ker \varepsilon_A \text{ad}_L$-invariant. Another known construction of bicovariant FODC is by picking some central element $c$ of $A^\circ$. We identify when this calculus is inner.

Let $h \leftarrow a = \langle a, h(1) \rangle h(2)$ be the right (co)regular action of $a \in A$ on $h \in A^\circ$. One has:

Lemma 4.4. Let $c$ be central in $A^\circ$ and define $\mathcal{L}_c := c \leftarrow A$.

(i) $\mathcal{L}_c = k1_{A^\circ} + \mathcal{L}_c$ is the extended tangent space of a bicovariant FODC $\Gamma(c)$ over $A$. (ii) If $1_{A^\circ} \notin \mathcal{L}_c$, then $\Gamma(c)$ is inner, with differential implemented by a biinvariant element.

Proof. (i) is well-known but we need its proof for (ii). We apply lemma to $\xi(a, b) := \langle a \otimes b, \Delta(c) \rangle$. It obviously satisfies $m_{\mathcal{O}} \ast \xi = \xi \ast m$ and $\text{im}(\xi_1) \subset A^\circ$. So, for $h \in A^\circ$ and $a \in A$, one has $\text{Ad}_L(h \leftarrow a) = \text{Ad}_L(h(\xi_1(a))) = \xi_1(\text{Ad}_L(h(a))) = c \leftarrow \text{Ad}_L(h(a))$, i.e. $\mathcal{L}_c$ is a submodule of $A^\circ$ for $\text{Ad}_L$. Since $\mathcal{L}_c$ is a submodule for the right (co)regular action of $A$ by hypothesis, it is a left co-ideal of $A^\circ$. Therefore by lemma 4.1, $\mathcal{L}_c = k1_{A^\circ} + \mathcal{L}_c$ is an extended tangent space for $A$. Under the hypothesis of (ii), one $\xi(c) = k \otimes \mathcal{L}_c$, therefore by lemma 4.3 the element $\theta_c \in \Gamma_R(c)$ uniquely determined by $k(\Theta - \theta_c) = \mathcal{L}_c \cap \Gamma_R$ implements $\mathcal{D}$ and is biinvariant.

Lemma 4.5. The exists $\theta \in \Gamma_R$ (resp. $\theta \in \Gamma_{\text{inv}}$) such that $\mathcal{D}(a) = a\theta - \theta a$ for all $a \in A$ if and only if the imbedding $\Gamma_R \rightarrow \Gamma_R$ splits in $\mathcal{M} \quad (\text{resp.} \quad \mathcal{M}^{\circ})$.

Proof. Since $\Gamma_R/\Gamma_R \simeq k$ is trivial both as module and comodule, the imbedding $\Gamma_R \rightarrow \Gamma_R$ splits in $\mathcal{X} \mathcal{M}$ iff $\Gamma_R = \Gamma_R \otimes k\xi$ for some $\xi$ spanning the trivial $A$-module. Given such a $\xi$, normalized so that $\Theta - \xi \in \Gamma_R$, then $\theta = \Theta - \xi$ satisfies $\omega_R(a) = (a - \varepsilon(a)) \Theta = (a - \varepsilon(a)) \Theta$ for all $a$. Conversely, define $\xi = \Theta - \theta$. Finally, the imbedding $\Gamma_R \rightarrow \Gamma_R$ splits in $\mathcal{X} \mathcal{M}$ if it splits in both $\mathcal{X} \mathcal{M}$ and $\mathcal{M}^{\circ}$ (since $k$ is simple is both), so that the above $\theta$ and $\xi$ must be also left invariant.

From the lemma, we see that $\Gamma_{\text{inv}}$ (and all its quotients) is inner if and only if $A$ is semi-simple, hence finite dimensional. For infinite dimensional Hopf algebras, $\Gamma_{\text{inv}}$ cannot be inner, but all finite dimensional bicovariant FODC over $A$ are inner when the category $\mathcal{M}^{\circ}$ of finite dimensional left $A$-modules, or $\mathcal{X} \mathcal{M}^{\circ}$ of finite dimensional left crossed modules, is semi-simple. For $A = \mathcal{O}(G)$, the algebra of polynomial functions on some matrix group $G$, neither $\mathcal{M}^{\circ}$ nor $\mathcal{X} \mathcal{M}^{\circ}$ are semi-simple, and there are non inner differential calculi. For the quantizations $\mathcal{O}_q(G)$, $q$ not a root of unity, it is known that, at least for $G = SL(n)$ or $G = Sp(n)$, all finite dimensional bicovariant FODC are semi-simple and inner. This
is an indication that $A_{(f)}$ is semi-simple in this case, although there is apparently still no proof of this.

4.2 The co-quasitriangular case.

From now on, $(A,r)$ is a co-quasitriangular Hopf algebra. We work in the category $^A\mathcal{M}$ of left $A$-comodules. $A$ will always be a left $A$-comodule via the left adjoint coaction $\text{ad}_L(a) := \text{ad}_L(a)$ in the following. The analogues of the facts given in section 3.5 for a quasitriangular Hopf algebra $(H,\mathcal{R})$ are as follows. The (exact, monoidal) functor $\mathcal{F}_r : {}^A\mathcal{M} \to {}^A\mathcal{C}$ now sends a left $A$-comodule $(M,\delta_L)$ to the left crossed module $(M,\triangleright_r,\delta_L)$, where $a \triangleright_r m := (m^{(-1)}, r_2(a)) m^{(0)}$. For $(A,\text{ad}_L)$, this gives:

$$\begin{align*}
a \triangleright_r b &= (b^{(-1)}, r_2(a)) b^{(0)} = r(b_{(1)}, a_{(1)}) b_{(2)}, a_{(2)}. \quad (4.12) \\
\Psi_r(a \otimes b) &= a^{(-1)} \triangleright_r b \otimes a^{(0)} = (b^{(-1)}, r_2(a^{(-1)})) b^{(0)} \otimes a^{(0)} \\
&= b_{(3)} \otimes a_{(3)} r(b_{(2)}, a_{(1)}) r(b_{(1)}, S(a_{(5)}) r(b_{(4)}, a_{(2)}) r(b_{(5)}, a_{(4)}) \quad (4.13)
\end{align*}$$

The analogue of lemma 3.13 is:

**Lemma 4.6.** Let $r$ and $s$ be two co-quasitriangular structures on $A$.

The map $A \to A \otimes A^o, a \mapsto a^{(0)} \otimes a^{[1]} = a_{(2)} \otimes r_1(a_{(1)}) s_2(a_{(3)}),$ is a right coaction of $A^o$ on $A$. Let $a \xrightarrow{-r,s} b = b^{[0]} \langle a, b^{[1]} \rangle$ be the corresponding left action of $A$ on itself. Then $(A,\xrightarrow{-r,s},\text{ad}_L)$ is a left crossed $A$-module with braiding $\Xi_{r,s}$ below:

$$\Xi_{r,s}(a \otimes b) = a^{(-1)} \xrightarrow{-r,s} b \otimes a^{(0)} = b^{[0]} \otimes a^{(0)} \langle a^{(-1)}, b^{[1]} \rangle \quad (4.14)$$

Moreover, $(A,\xrightarrow{-r,s},\text{ad}_L)$ is in the image of the functor $\mathcal{F}_r$, for some co-quasitriangular structure $t$ on $A$, (if and) only if $s = r_{21}$. In particular, $\Psi_r = \Xi_{r,r_{21}}$.

Define the linear maps $m : A \otimes A \to A, \eta = \eta : k \to A, \mathcal{S} : A \to A$ by

$$\begin{align*}
m(a \otimes b) &= a \triangleright_r b := a_{(1)} (S(a_{(2)}) \triangleright_r b) = a_{(1)} b_{(2)} \langle b_{(1)} S(b_{(3)}), r_2 S(a_{(2)}) \rangle \quad (4.15) \\
\mathcal{S}(a) &= a_{(1)} \triangleright_r S(a_{(2)}) \quad (4.16)
\end{align*}$$

**Proposition 4.7.** (i) $A \equiv (A,\mathcal{M},\eta,\Delta,\varepsilon,\mathcal{S})$ is a Hopf algebra in $^A\mathcal{M}$, where the braiding is $\Psi_r$. It is $\Xi_{r,r}$-commutative, in the sense that $m \circ \Xi_{r,r} = m$. Its braided adjoint coaction $\text{ad}_L$, coincides with the adjoint coaction $\text{ad}_L$, and the crossed module braiding on $(A,\mathcal{M},\text{ad}_L)$ coincides with crossed module braiding on $(A,\mathcal{M},\text{ad}_L)$. (ii) $(A,\mathcal{M},\eta,\delta)$ is a braided Lie coalgebra in $^A\mathcal{M}$ for the braided Lie coproduct $\delta = \text{ad}_L$. It is counital with counit $\varepsilon_A$. Its canonical braiding $\Upsilon_A$ (see (3.31)) coincides with the crossed module braiding on $(A,\mathcal{M},\text{ad}_L)$. (iii) Let $\mathcal{I}$ be a submodule of $(A,\text{ad}_L)$. Then $(\mathcal{I},\text{ad}_L,\triangleright_r)$ is a crossed submodule of $A = (A,\text{ad}_L,\triangleright_r), i.e. A \triangleright_r \mathcal{I} \subset \mathcal{I}$, and the following are equivalent:

$$\begin{align*}
(a) & \quad A \mathcal{I} \subset \mathcal{I}, \quad (b) & \quad A \mathcal{I} \subset \mathcal{I}, \\
& \quad A \mathcal{I} \subset \mathcal{I} \quad (c) & \quad \mathcal{I} \subset A \subset \mathcal{I}
\end{align*}$$

**Proof.** (i) is well-known [Ma] although the version of $A$ given there is in the category of right $A$-comodules. The $\Xi_{r,r}$-commutativity of $A$ is the equivalent of the quasi-commutativity of $A$ ($m_{op} \star r = r \star m$). The new observation is that $\Xi_{r,r}$ is a crossed module braiding. Note that if $r_{21} = r$, then $\Xi_{r,r} = \Psi_r$, i.e. $A \equiv A$ is a commutative algebra in $^A\mathcal{M}$. Finally, the braided Hopf algebra $A$ is defined in [Ma] by the requirements that $\text{ad}_L = \text{ad}_L$ and that the crossed module braidings on $(A,\mathcal{M},\text{ad}_L)$ and $(A,\mathcal{M},\text{ad}_L)$ coincide, so these statements are just reminders.

(ii) Clearly, if $A$ is a braided Lie coalgebra, its canonical braiding (see (3.31)) is the crossed
module braiding on \((A, m, \text{ad}_L)\), i.e., on \((A, m, \text{ad}_L)\) by (i), that is: \(\Upsilon_A(a \otimes b) = a^{(1)} b^{(2)} \otimes a^{(0)}\).

According to (a dual version of) proposition \((3.12)\) we only need to check that axiom (C2) holds. It can be expressed as the equality \(\Psi_r \circ \Upsilon_A = (m \otimes \text{id}) \circ (\text{id} \otimes \text{ad}_L)\). Thus, we need to check that \(r(b^{(-2)}, a^{(-1)}) b^{(0)} \otimes b^{(0)} = a^{(0)} b^{(-1)} \otimes b^{(0)}\) for all \(a, b \in A\). This is implied by the equality \(r(b(1), a^{(-1)}) b(2) a^{(0)} = a \cdot b\), which holds since

\[
   a \cdot b = a(b(2)(b(1) S(b(3)), r_2 S(a(2)))) = a(b(2)(b(1), r_2 S(a(3)))) \langle S(b(3)), r_2 S(a(2)) \rangle = b(3) a(2) (b(1), r_2 S(a(3)))(b(2)) r_2(a(1)) = b(a(2) b(1), r_2(a(1) S b(3))) = r(b(1), a^{(-1)} b(2) a^{(0)}).
\]

The underlined terms are changed using \(S r_2 S = r_2\) and \(r_m \circ r = r \circ m\).

(iii) The first claim follows from the covariance of the functor \(F_r : A^M \to \mathcal{A}_C\). Then the equivalence of (a) and (b) follows directly from \(\text{ad}_L(I) \subset A \otimes I\) and the relation between \(\text{m}\) and \(\overline{\text{m}}\) (the reverse being \(ab = a(1) \cdot (a(2) \cdot r \cdot b)\)). The equivalence of (b) and (c) follows for the \(\Xi_{r, r}\)-commutativity property of \(\mathcal{A}\): If \(\text{ad}_L(I) \subset A \otimes I\), then \(\Xi_{r, r}(I \otimes A) \subset A \otimes I\); this is in fact an equality since \(\Xi_{r, r}\) is a crossed module braiding and the antipode of \(A\) is invertible. Therefore \(\Xi_{r, r}(I \otimes A) = A \otimes I\), and \(A_{/I} \otimes I = I_{/I}\).

\[\square\]

Remarks. (i) If \(r = R \in H \otimes H\), with \(H = A^0\), then the braided Hopf algebra \(A\) is dual to the braided Hopf algebra \(H\) of proposition \((3.14)\) in the sense that \(\langle \Psi_r(a \otimes b), x \otimes y \rangle = \langle a \otimes b, \Psi_r(x \otimes y) \rangle, \langle a \otimes b, \Delta(x) \rangle = \langle a, \Delta(x) \rangle\) for all \(a, b \in A, x, y \in H\), where the pairing between \(A \otimes A\) and \(H \otimes H\) is given by \(\langle a \otimes b, x \otimes y \rangle = \langle a, x \rangle \langle b, y \rangle\) (this is the opposite convention of \([\text{Ma}3]\))

(ii) Following [Schm], a central bicharacter on \(A\) is a convolution invertible map \(c : A \otimes A \to k\) such that \(c(ab, c) = c(a, c(b)) c(b, c(a)), c(a, bc) = c(a(b), c(b, c(a))), c(a, b(1)) = c(a, b(2)), c(a_1, b(a_2)) = a(a_1) c(a_2, b)\) for all \(a, b, c \in A\). If \(c\) is a central bicharacter, then \(c \circ r\) is also a co-quasitriangular structure on \(A\). But one easily checks that changing \(r\) in \(c \circ r\) does not affect \(m\).

(iii) If \(C\) is a subcoalgebra of \(A\), then \(C\) generates \(A\) as an algebra if and only if \(C\) generates \(A\) as an algebra (this follows from the reciprocal relations between \(m\) and \(\overline{m}\), and the fact that subcoalgebras are submodules of \((A, \text{ad}_L)\)).

(iv) The subspace \(A^\text{ad}_L\) = \(\{a \in A : \text{ad}_L(a) = 1 \otimes a\}\) is a commutative subalgebra of \(A\), and belongs to the center of \(A\). Indeed, for \(b \in A^\text{ad}_L\) and all \(a \in A\) one has \(b = a \cdot b = ab\) (the first equality comes from \(\Xi_{r, r}(b \otimes a) = a \otimes b\) and also \(\text{ad}_L(ab) = \text{ad}_L(a) \text{ad}_L(b), \text{ad}_L(a \cdot b) = \text{ad}_L(a) \text{ad}_L(b)\) (the last equality actually holds for all \(a, b \in A\) by axiom (C3))

\[\square\]

Theorem 4.8. Let \((A, r)\) be co-quasitriangular, \((\Gamma, d)\) be a bicovariant FODC over \(A\).

(i) \(\Gamma_R \simeq A / I_R\) is a counital braided Lie coalgebra in \(\mathcal{A}^M\), when regarded as \(A / I_R\).

(ii) If \((\Gamma, d)\) is finite dimensional, \(\tilde{\Gamma}_R\) is a unital braided Lie algebra in \(\mathcal{M}^A\). Moreover, \(U(\tilde{\Gamma}_R) \simeq B(\tilde{\Gamma}_R) / (L - 1)\) is a bialgebra in \(\mathcal{M}^A\).

(iii) If moreover there exists a biinvariant \(\theta \in \Gamma\) such that \(\theta \cdot a = a - a \cdot \theta a\) for all \(a \in A\), then \(U(\tilde{\Gamma}_R) \simeq B(\mathcal{L})\) as bialgebras, where \(\mathcal{L} = (\Theta - \theta)\) is a braided Lie subalgebra of \(\tilde{\Gamma}_R\).

\[\square\]

Proof. (i) By hypothesis, \(I_R \subset \ker \varepsilon_A\) satisfies \(\text{ad}_L(I_R) \subset A \otimes I_R\) and \(A I_R \subset I_R\). By the above proposition, this implies that \(I_R\) is a 2-sided ideal of \(A\), and a fortiori that \(\text{ad}_L(I_R) \subset A \otimes I_R + I_R \otimes A\). Therefore \(I_R\) is an ideal of \((A, m, \text{ad}_L)\) in the sense of definition \((3.14)\), and \(\Gamma_R \simeq A / I_R\) is a counital braided Lie coalgebra in \(\mathcal{A}^M\). (ii) According to (i) and lemma \((3.17)\), \(\tilde{\Gamma}_R\) is a braided Lie algebra in \(\mathcal{M}^A\) with braided Lie bracket \([, , ]\) = \(\text{ad}_L\) and coproduct \(\Delta\) adjoint (in the conventions of lemma \((3.17)\)) to the multiplication in \(A / I_R\). (iii) This is proposition \((3.8)\) and theorem \((3.12)\) put together.
4.3 A “quantum Lie functor”.

Let $D$ be the category of bicovariant first order differential calculi: its objects are triples $(A, \Gamma, d)$ where $A$ is a Hopf algebra and $(\Gamma, d)$ a bicovariant first order differential calculus over $A$. Morphisms are pairs $(\varphi^0, \varphi^1): (A, \Gamma_A, d_A) \to (B, \Gamma_B, d_B)$ such that $\varphi^0: A \to B$ is a Hopf algebra homomorphism and $\varphi^1: (\Gamma_A, d_A) \to (\Gamma_B, d_B)$ is a morphism of Hopf bimodules (over $A$) such that $\varphi^1 \circ d_A = d_B \circ \varphi^0$; equivalently, such that $\varphi^1 \circ \omega_{RA} = \omega_{RB} \circ \varphi^0$. Because of the surjectivity axiom (3) of a bicovariant FODC, $\varphi^1$, if it exists, is uniquely determined by $\varphi^0$. The condition of existence is easily seen to be that $\varphi^0(\mathcal{I}_{\Gamma_A}) \subset \mathcal{I}_{\Gamma_B}$. Let $CQT$ be the category of co-quasitriangular Hopf algebras: it consists of pair $(A, \mathfrak{r})$ where $A$ is a Hopf algebra and $\mathfrak{r}$ is a co-quasitriangular structure on $A$. Morphisms are Hopf algebra morphisms $\varphi: A \to B$ satisfying $r_B \circ \varphi = r_A$.

**Proposition 4.9.** There is an exact functor $L: CQT \to D$, which sends $(A, \mathfrak{r})$ to $(A, \Gamma(\mathfrak{r}), d)$ where $(\Gamma(\mathfrak{r}), d)$ is the bicovariant FODC over $A$ whose associated left ideal is $\mathcal{I}(\mathfrak{r}) := \ker \varepsilon_k A, \ker \varepsilon_A$ (the product in $\mathfrak{g} = A(\mathfrak{r})$).

**Proof.** $A$ is in particular a coalgebra in $A M$, i.e. $\text{ad}_L(a, b) = a^{-1} b^{(-1)} \otimes a^{(0)} b^{(0)}$ for all $a, b \in A$. Therefore, if $I$ and $J$ are any $\text{ad}_L$-invariant left ideals of $A$, their covariantized product $I \cdot J$ is also an $\text{ad}_L$-invariant ideal of $A$. By proposition $4.7$, $I$ and $J \cdot I$ are also left ideals of $A$. This holds in particular for $I = J = \ker \varepsilon_A$. Next one easily checks that if $\varphi: (A, r_A) \to (A, \mathfrak{r}_B)$ is a morphism in $CQT$, then the same map $\varphi$ is a morphism of $k$-algebras $A \to B$, and therefore satisfies $\varphi(\ker \varepsilon_A, \ker \varepsilon_A) \subset \ker \varepsilon_B, \ker \varepsilon_B$. This gives the functoriality property of $L$. Finally, if $\varphi$ is surjective (resp. injective), its restriction $\ker \varepsilon_A, \ker \varepsilon_A \to \ker \varepsilon_B, \ker \varepsilon_B$ is also surjective (resp. injective), proving exactness. Note that for the associated quantum Lie algebras, if $\varphi: A \to B$ is surjective (resp. injective), it means that $\mathfrak{g}_B$ imbeds in (resp. maps onto) $\mathfrak{g}_A$ as quantum Lie algebras in $M^A$. \hfill $\Box$

**Remarks :** If $(A, \varepsilon)$ is any augmented algebra, the space $(\ker \varepsilon/(\ker \varepsilon)^2)^*$ can be seen in either of the following ways:

$$(\ker \varepsilon/(\ker \varepsilon)^2)^* = \{ \chi \in A^0 | \chi(1) = 1, \chi((\ker \varepsilon)^2) = 0 \} = \text{Prim}_e(A^0) \simeq \text{Ext}_A^1(k_e, k_e)$$

where $\text{Prim}_e(A^0) = \{ \chi \in A^0 | \chi(ab) = \chi(a) \varepsilon(b) + \varepsilon(a) \chi(b) \}$ is the space of $\varepsilon$-primitive elements of the coalgebra $A^0$, and $\text{Ext}_A^1(k_e, k_e)$ parameterizes the exact sequences $0 \to k_e \to M \to k_e \to 0$ of $A$-modules ($k_e$ is the 1-dimensional $A$-module afforded by $\varepsilon: A \to k$, and the 2-dimensional module $M_\chi$ associated to $\chi \in \text{Prim}_e(A^0)$ has basis $v_0, v_1$ such that $av_0 = \varepsilon(a) v_0 + \chi(a) v_1$, $av_1 = \varepsilon(a) v_1$. Note that if $\chi \in \text{Prim}_e(A^0)$ and $a, b \in A$ satisfy $\varepsilon(a) = 1, \varepsilon(b) = 0$ and $ab = qba$ for some $q \in k$, then one must have $(1 - q) \chi(b) = 0$. Therefore, if $q \neq 1$, $\chi(b) = 0$. Alternatively, one has $(1 - q)b = -q(ba - 1) + (a - 1)b \in (\ker \varepsilon)^2$, therefore if $q \neq 1, b \in (\ker \varepsilon)^2$.\hfill $\diamond$

Until the end of this paragraph, $r$ is fixed and we write $\Gamma := \Gamma(\mathfrak{r})$ as above. Its tangent space $\mathfrak{g}_r = \text{Prim}_e(A^0)$ is the space of primitive elements of $(A^0)^*$. Note that if $A$ is commutative, $A = A$ (recall that if $A = O(G)$ is the algebra of polynomial functions on some algebraic group $G$, $\text{Prim}(A^0) = \text{Lie}(G)$ is the Lie algebra of $G$-if $G$ is finite, this is zero). In the general case, it is not obvious to determine $\text{Prim}_e((A^0)^*)$, but one can still describe some nice properties of $(\Gamma, d)$. First, $(\Gamma, d)$ is clearly never inner (unless it is zero). Moreover, $U(\mathfrak{g}_r)$ is a Hopf algebra in $M^A$ with Hopf structure $(A, \varepsilon, S)$ uniquely determined by

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x), \quad S(x) = -x$$ (4.17)
Indeed, the little coproduct $\delta$ on $\mathfrak{g}_R$ (see (4.8)) is dual to the multiplication on $\ker \varepsilon_A/(\ker \varepsilon_A \perp \ker \varepsilon_A)$ which is zero. Thus, $\Gamma$ has many properties of the standard differential calculus on Lie groups.

To complete the analogy, we show below that the action $\triangleright$ of $A$ on $\Gamma_R$ can be nicely “linearized”, and that the braiding $\sigma$ on $\mathfrak{g}_R$ in $\mathcal{C}_A^\dagger$ coincides with the braiding in $\mathcal{M}_A$, i.e. the category in which $\mathfrak{g}_R$ lives as a quantum Lie algebra (recall that they don’t in general).

Let $q = r_{21} \ast r : A \otimes A \rightarrow k$. The maps $q_1, q_2 : A \rightarrow A^\ast$ are given by $q_1 = r_2 \ast r_1$ and $q_2 = r_1 \ast r_2$. Recall from [RS] that $(A, r)$ is called co-triangular if $q = \varepsilon_A \otimes \varepsilon_A$ and co-factorizable if $q_1$ (equivalently $q_2$) is injective.

**Lemma 4.10.** (i) The factor crossed modules $(\ker \varepsilon_A, m, \text{ad}_L)/\mathcal{I}(r)$ and $(\ker \varepsilon_A, \triangleright_r, \text{ad}_L)/\mathcal{I}(r)$ are isomorphic. Therefore, the left action of $A$ on $(\Gamma_R, \triangleright, \Delta_L)$ and its crossed module braiding $\sigma^t$ are given by

$$a \triangleright \omega_R(b) = \omega_R(a \triangleright_r b) = r(b_{(1)}, a_{(1)}) \omega_R(b_{(2)}) \bar{r}(b_{(3)}, a_{(2)}),$$

$$\sigma^t \circ (\omega_R \otimes \omega_R) = (\omega_R \otimes \omega_R) \circ \Psi_r$$

where $\Psi_r$ is defined in (4.13). Moreover, for all $a \in A$, one has

$$\text{ad}_L(a) \equiv -1 \otimes a + \Delta(a) + (S \otimes \text{id})\Delta(a) \bmod \mathcal{I}(r) \otimes A.$$  

In particular, for all $x, y \in \mathfrak{g}_R$, one has

$$\langle a, [x, y] \rangle = \langle a_{(1)}, x \rangle \langle a_{(2)}, y \rangle - \langle \Psi_r(a_{(1)} \otimes a_{(2)}), x \otimes y \rangle$$

(ii) The $\Xi_{r, r}$-commutativity of $A$ implies that for all $\chi \in \text{Prim}_r((A^\ast)^o) = \mathfrak{g}_R$ and $a \in A$ one must have

$$q_1(a^{(-1)} \chi(a^{(0)})) = \chi(a) 1_{A^\ast}$$

(equivalently : $\text{ad}_L(a) - 1 \otimes a \in \ker q_1 \otimes A + \ker \varepsilon_A \otimes \ker \chi$.)

**Proof.** First one obviously has, for all $a, b \in A$, $a \triangleright_r b \equiv \varepsilon(a) b + (a - \varepsilon(a) 1) \varepsilon(b) \bmod (\ker \varepsilon_A \perp \ker \varepsilon_A)$. From this and the relation $ab = a_{(1)} \triangleright_r (a_{(2)} \triangleright_r b)$, we get

$$ab \equiv a \triangleright_r b + \varepsilon(b)(a - \varepsilon(a) 1) \bmod (\ker \varepsilon_A \perp \ker \varepsilon_A).$$

In particular, if $\varepsilon(b) = 0$, $ab \equiv a \triangleright_r b$ mod $(\ker \varepsilon_A \perp \ker \varepsilon_A)$, hence the crossed module isomorphism as stated. By the axiom $(1')$ of a bicovariant FODC, one has $a \triangleright \omega_R(b) = \omega_R(a(b - \varepsilon(b) 1)) = \omega_R(a \triangleright_r (b - \varepsilon(b) 1)) = \omega_R(a \triangleright_r b)$, where we have used (4.23) for the second equality and $\omega_R(a \triangleright_r 1) = \varepsilon(a) \omega_R(1) = 0$ for the third. (4.19) follows. Finally, for $a \in A$, one has by (4.23), $\text{ad}_L(a) = a_{(1)} S a_{(3)} \otimes a_{(2)} \equiv (a_{(1)} - \varepsilon(a_{(1)})) \otimes a_{(2)} + a_{(1)} \triangleright_r S a_{(3)} \otimes a_{(2)}$. Using $S(a) = S(a_{(1)}) \triangleright_r S(a_{(2)}) = S(S(a_{(1)}) \triangleright_r a_{(2)})$, we get $a_{(1)} \triangleright_r S a_{(3)} \otimes a_{(2)} = S((a_{(1)} S a_{(3)}) \triangleright_r a_{(4)}) \otimes a_{(2)} = (S \otimes \text{id})\Delta(a)$. (We have used that the braided antipode intertwines the coaction $\text{ad}_L$, and therefore also the action $\triangleright_r$.) For the next formula, we use that if $x \in \mathfrak{g}_R = \text{Prim}_r((A^\ast)^o)$, then $\langle 1, x \rangle = 0$ and $\langle S(a), x \rangle = -\langle a, x \rangle$.

(ii) By hypothesis, $\chi(a \triangleright_r b) = \chi(a) \varepsilon(b) + \varepsilon(a) \chi(b)$ for all $a, b \in A$. On the other hand, by the $\Xi_{r, r}$-commutativity of $A$, one also has (with the notations of lemma 4.6) $\chi(a, b) = \chi(b^{[0]} \varepsilon(a^{(0)}) \varepsilon(a^{(0)}) b^{[1]}).$ One then uses the identities $a^{(-1)} \varepsilon(a^{(0)}) = \varepsilon(a) 1$ and $\varepsilon(b^{[0]} b^{[1]}) = q_2(b)$ to get $\chi(a \triangleleft b) = \chi(b) \varepsilon(a) + q(a^{(-1)}, b) \chi(a^{(0)})$ for all $a, b \in A$. Comparing with the previous expression gives the claim.

Note that in terms of $d(a) = \omega_R(a_{(1)}) a_{(2)}$, (4.14) can be rewritten

$$a \cdot d(b) = r(b_{(1)}, a_{(1)}) d(b_{(2)} a_{(2)} \bar{r}(b_{(3)}, a_{(3)})).$$
We apply the lemma above to describe $\Gamma(r)$ when $(A, r)$ is of $GL(n)$ or $SL(n)$ type. In this case, the condition $m^{op} * r = r * m$ for $A$ (i.e. $m = m \circ \Xi_{r, r}$ for $A$) is almost the only set of relations.

Let $R = (R^i_j k_l)$ be a bi-invertible solution of the Yang-Baxter equation on $k^n \otimes k^n$. Recall that this means that the inverse $R^(-1)$, such that $R^i_k R^k_j \delta_{i l} = \delta_i^l \delta_j^k$, exists, as well as the second inverse $\widehat{R}$, such that $\widehat{R}^i_j R^k_j \delta_{i l} = \delta_i^k \delta_j^l$. Let $A(R) = A(C, r)$ be the FRT bialgebra defined by $R$ (see lemma [2]), where $C$ is the $n \times n$ matrix coalgebra with basis $t_{ij} (\Delta t_{ij} = t_{ik} \otimes t_{kj})$ and $r : C \otimes C \rightarrow k$ is the bilinear form such that $r(t_{ij}, t_{kl}) = R^i_j k_l$.

We assume that there exists a central grouplike $\det$ of degree $n$ (the quantum determinant) such that (1) $\det$ is not a zero divisor and there exists a (bi)algebra automomorphism $f$ of $A(R)$ such that $\det = f(\det)$ for all $a \in A(R)$, (2) there exists a matrix of elements $(\widehat{t}_{ij})$ of $A(R)$ such that $t_{i}^a t_{j}^b = \delta_{ij}^a \det$ for all $i, j$. Then the localization of $A(R)$ at $\det$, which we note $O_R(GL(n))$, has an antipode $S$ such that $S(t_{ij}^a) = \widehat{t}_{ij}^a \det = -f(\det)$. It is co-quasitriangular with $r = z r$ when restricted to $C \otimes C$, $z \in k^*$ is arbitrary (in our setting $z$ plays no role since in general the covariantized multiplication is invariant under the change $r \mapsto c * r$ for any central bicharacter $c$ on $A$). If $\det$ is central in $A(R)$ and if $(\det - 1)$ belongs to the left and right radicals of $r$, we set $O_q(SL(n)) = A(R)/(\det - 1)$. When $R = R_q$ is the Drinfeld-Jimbo $R$-matrix of type $A_n$, over $C$, we write $O_q(G)$ instead of $O_R(GL(n))$, $G = GL(n)$ or $G = SL(n)$, and $O(G) = O_{q=1}(G)$.

The left quantum trace of $O_R(GL(n))$, $G = GL(n)$ or $G = SL(n)$, is the element $\text{tr} = (\hat{R})_{ij}^{\mu} t_{ij}^\mu$. It is $ad_L$-invariant: $\text{tr}(ad_L(t_{\mu}^a)) = 1 \otimes \text{tr} t_{\mu}^a$. We shall assume that $\varepsilon(\text{tr}) \neq 0$ (recall that $\varepsilon(1) = 0$ can happen; for instance, for $O_q(SL(n))$, $\text{tr} = \sum_1^n q^{-2i} t_{ij}$ has counit zero when $q^n = 1$; likewise, for $O(G)$, $\text{tr} = tr$ has counit zero when $\text{char}(k)$ divides $n$).

Take first $A = O_R(GL(n))$, $G = GL(n)$. $\chi \in \mathfrak{g}_r = \text{Prim}_c((A)^\circ)$ is uniquely determined by its values on algebra generators of $A$, i.e. on $C$ and $\det^{-1}$. From $\det^{-1} = 1$, we get $\chi(\det^{-1}) = -\chi(\det)$. Therefore $\text{Prim}_c((A)^\circ)$ can be identified with the space of linear functionals on $C$ such that (4.22) holds for all $a \in A$ (since $\mathbb{m} = \mathbb{m} \circ \Xi_{r, r}$ are the only left relations to be checked). From the assumption $\varepsilon(\text{tr}) \neq 0$, $C$ decomposes as a direct sum of sub-comodules (for $ad_L$), $C = k \text{tr} \oplus C^+$, where $C^+ = C \cap \ker \varepsilon$. So we have a vector space decomposition

$$\mathfrak{g}_r = k z \oplus \mathfrak{g}^+_r$$

where $\mathfrak{g}^+_r = \{ \chi \in \mathfrak{g}_r | \chi(\text{tr}) = 0 \}$ and $k z = \{ \chi \in \mathfrak{g}_r : \chi(C^+) = 0 \}$. Clearly, whatever $R$ is, $k z \neq 0$, i.e. there exists a non zero functional $z$ on $C$ such that $z(C^+) = 0$.

One also easily checks that $\mathfrak{g}^+_r$ is a quantum Lie subalgebra of $\mathfrak{g}_r$. We would like to prove (when it makes sense, i.e. when $O_R(GL(n))$ has a Hopf algebra quotient $O_R(SL(n))$), that $\mathfrak{g}^+_r$ is the quantum Lie algebra of $O_R(SL(n))$, which we know imbeds into $\mathfrak{g}_r$ by the exactness of the functor $L$.

**Lemma 4.11.** Let $A = O_R(GL(n))$ as above. Assume that $\varepsilon(\text{tr}) \neq 0$, and that moreover (1) $\text{tr}$ is not a zero divisor in $A$, (2) $C^+$ contains no $ad_L$-invariant elements. Then for all $\chi \in \text{Prim}_c((A)^\circ)$, one has

$$\chi(\det) = \frac{n}{\varepsilon(\text{tr})} \chi(\text{tr}).$$

(In particular, $\chi(\det) = 0$ if and only if $\chi(\text{tr}) = 0$).

**Proof.** To prove this properly, one should take the general formula for $\det$ (see eg [DMMZ]) and reexpress $\det$ in terms of the covariantized product $\mathbb{m}$ of $A$. This is quite complicated and we use a trick that requires the listed assumptions, which are probably not necessary for a good proof.

For $a \in A$, let $a^{\circ n}$ be the $n$-th power of $a$ calculated in $A$. By iteration one checks that, for $\chi \in \text{Prim}_c((A)^\circ)$, $\chi(a^{\circ n}) = n \chi(a) \varepsilon(a)^{n-1}$ holds. From the decomposition $C = k \text{tr} \oplus C^+$,
from the fact that $\det \in C^n = C^n$, $\varepsilon(\det) = 1$, and that $tr$ is central in $A$, there must exists elements $\alpha_i \in (C^+)^\wedge$ such that

$$\varepsilon(tr)^n \det = (tr)^n + \sum_{i=1}^n (tr)^{n-1} \alpha_i,$$

i.e. $\det$ is a polynomial in $tr$ with coefficient $k$ for the leading one and in $C^+$ for the others. Since $\chi$ vanishes on $C^+, C^+ \subset \ker \varepsilon \ker \varepsilon$, we get $\varepsilon(tr)\chi(\det) = n \chi(tr) + \chi(\alpha_1)$. We need to show that $\alpha_1 = 0$. Since det is grouplike, it is $ad_L$-invariant. It is easy to see, using the property of the covariantized product and the $ad_L$-invariance of all powers of $tr$, that each of the terms of the above decomposition of det must be $ad_L$-invariant. Let $\tau = (tr)^{n-1}$. One must have $ad_L(\tau \alpha_1) = 1 \otimes (\tau \alpha_1) = (1 \otimes \tau)(1 \otimes \alpha_1)$. On the other hand, $ad_L(\tau \alpha_1) = ad_L(\tau)ad_L(\alpha_1) = (1 \otimes \tau)ad_L(\alpha_1)$. Since $1 \otimes \tau$ is not a zero divisor by hypothesis, this implies that $\alpha_1 \in C^+$ is $ad_L$-invariant, therefore zero by the second hypothesis.

The next proposition sums up the results for $O_R(G)$, $G = SL(n)$ or $G = GL(n)$. We assume that $R$ has all the good properties listed above : $O_R(G)$ are well defined Hopf algebras, $\varepsilon(tr) \neq 0$, and the statement of lemma 4.11 hold, whenever its hypothesis are necessary or not.

**Proposition 4.12.** (i) Assume $R_{21}R = 1$. For $G = GL(n)$, one has dim $g^R_\tau = n^2$, with basis $\{\chi^j_i : i, j = 1, ..., n\}$ such that $\langle a^*_b \chi^j_i \rangle = \delta^a_i \delta^b_j$. Let $\omega^a \beta = \omega_R(t^a \beta)$ and $\omega = (\omega^a \beta)$. The left crossed module structure of $\Gamma_R$ and the corresponding $\sigma^\omega$ are given by

$$t_1 \triangleright \omega_2 = R_{21}t_2 \omega_2, \quad \Delta_L(\omega^j_i) = t^a_i S^b_j \otimes \omega^a \beta, \quad \sigma^\omega(\omega_1 \otimes R_{21} \omega_2) = R_{21}t_2 \omega_2 \otimes \omega_1$$

Equivalentely, the first relation is $Rd(t_1)t_2 = t_2 d(t_1)R$. The quantum Lie algebra structure of $g^R_\tau$ is given by

$$\sigma(\chi^j_i \otimes \chi^k_l) = (\sigma(\chi^j_i \chi^k_l)) \omega^a \beta \chi^a \delta \chi^b \otimes \chi^c \delta \chi^d, \quad [\chi^j_i, \chi^k_l] = \delta_k^i \chi^j_1 - (\sigma(\chi^j_i \chi^k_l)) \omega^a \beta \chi^a \delta \chi^b$$

where $(\sigma(\chi^j_i \chi^k_l)) \omega^a \beta \chi^a \delta \chi^b = (R_{21})^{-1} \omega^a \beta R_{21}^{\alpha} \omega^c \delta (R_{21})^{-1} \omega^d \delta R_{21}$. The quantum Lie algebra for $SL(n)$ is the quantum Lie subalgebra $g^R_\tau = \{ \chi \in g^R \mid \chi(\lambda) = 0 \}$, of dimension $n^2 - 1$.
(ii) If $q$ is not a root of unity, the quantum Lie algebra of $O_q(SL(n))$ -obtained by the functor $L_\tau$- is zero.

**Proof.** (i) If $R_{21}R = 1$, one has $r_{21} \bullet r = \varepsilon_A \otimes \varepsilon_A$, therefore $q_1(a) = \varepsilon(a) A$ and (4.22) is trivially satisfied for all $a \in A$. So, the fact that $g^R_\tau \simeq C^*$ follows from the previous discussion and all formulas follow from corresponding ones in lemma 4.10 (for the formula of $\sigma$, one can use that $R_{21}^{-1} = \tilde{R} = R_{21}$).
(ii) For $A = O_q(SL(n))$, one has $\chi(\det) = \chi(1) = 0$ if $\chi \in \text{Prim}_e((A)^\wedge)$, which is equivalent to $\chi(tr) = 0$, i.e. its quantum Lie algebra can be identified with the space of functionals $\chi$ on $C$ satisfying $\chi(tr) = 0$ and (4.22). Recall that when $q$ is not a root of unity, $O_q(SL(n)$ is factorializable [HS], therefore $q_1$ is injective, and that $(C^+, ad_L)$ is a simple comodule. The first property and (4.22) tells that, if $\chi \neq 0$, ker $\chi \otimes C^+$ is a proper submodule of $(C^+, ad_L)$, which is simple. This is impossible therefore $\chi = 0$. Note that these arguments are also valid for $O_q(G)$, $G = SO(n)$ or $G = Sp(n)$, $q$ not a root of unity, i.e. their quantum Lie algebra is zero for these as well. For $q$ a root of unity, the comodule $(C^+, ad_L)$ remains simple, therefore the arguments are also valid provided $q_1$ is injective on the coefficient subalgebra of $(C^+, ad_L)$ -the smallest subcoalgebra $T$ of $O_q(G)$ such that $ad_L(C^+) \subset T \otimes C^+$. We do not know when this is true. 

$\square$
So to conclude, for the standard deformation of $O(G)$, the quantum Lie functor $L$ gives uninteresting results. However, $O(G)$ has some “softer” deformations, defined through triangular $R$-matrices, which behaves better (but which are less interesting in many other aspects). Such examples do exist: see e.g. [JC] and the references cited there for calculi that we cannot predict. Triangular $R$-matrices are known in higher dimension $\mathbb{E}\mathbb{H}$ but, up to our knowledge, associated Hopf algebras have not been studied yet.

4.4 Example: Finite groups.

We illustrate the results of theorem $L^8$ with the example of finite groups. The more interesting case of quantum groups is considered in the next section.

Let $G$ be a finite group with unit element $e$, $H = kG$ and $A = k(G)$ its dual with basis \{ $f_g : g \in G$ \} such that $f_g(g') = \delta_{g,g'}$. $A$ is co-quasitriangular with $\mathcal{r} = \varepsilon_A \otimes \varepsilon_A$ so that the braiding in $\mathcal{M}^A \simeq \mu^A$ is the usual flip, and $\mathcal{A} = A, \mathcal{H} = H$. It is well-known that bicovariant FODC over $A$ are in 1-1 correspondence with Ad-invariant subsets of $G$ not containing $e$, irreducible calculi corresponding to conjugacy classes. Since $A$ is semi-simple, they are all inner. For an Ad-invariant subset $C \subset G$, let $\theta_C = \sum_{g \in C} f_g$ and $c_C = \sum_{g \in C} g$ ($\theta_C \in A$ is ad$_L$-invariant and $c_C \in kG$ is central). The calculus $\Gamma_C$ corresponding to $C$ has associated ideal $\mathcal{I}_C = \ker \varepsilon_A(1 - \theta_C)$ with basis \{ $f_g : g \neq e, g \notin C$ \}, and extended tangent space $\tilde{\mathcal{C}}_C = k e \oplus c_C \hookrightarrow A$ with basis \{ $X_g = g : g \in C \cup \{e\}$ \}. The braided Lie algebra structure of $\tilde{\mathcal{C}}_C$ is given by

$$\Psi_r(X_g \otimes X_h) = X_h \otimes X_g, \quad \Delta(X_g) = X_g \otimes X_g, \quad \varepsilon(X_g) = 1, \quad [X_g, X_h] = X_{ghg^{-1}}.$$ 

Therefore the canonical braiding on $\tilde{\mathcal{C}}_C$ is given by

$$\Upsilon(X_g \otimes X_h) = X_{ghg^{-1}} \otimes X_g$$

and $B(\tilde{\mathcal{C}}_C)$ is the (usual) bialgebra generated by \{ $X_g : g \in C \cup \{e\}$ \} with the above coproduct and relations $X_g X_h = X_{ghg^{-1}} X_g$. Note that $\tilde{\mathcal{C}}_C = k e \oplus \mathcal{L}_C$ is indeed the trivial extension of the braided Lie subalgebra $\mathcal{L}_C := \{ X \in \tilde{\mathcal{C}}_C | (1 - \chi, X) = 0 \}$, with basis \{ $X_g : g \in C$ \}. The quantum Lie algebra $(\mathcal{C}_C, \sigma, [\, \cdot, \, \cdot\, ])$ of the differential calculus has basis \{ $x_g = g - e | g \in C$ \}; its structure maps and the coalgebra structure on $U(\mathcal{C}_C) \simeq B(\tilde{\mathcal{C}}_C)/(X_e - 1)$ are given by

$$[(x_g, x_h) = x_{ghg^{-1}} - x_h, \quad \Delta(x_g) = x_g \otimes 1 + 1 \otimes x_g + x_g \otimes x_g$$

$$\sigma(x_g \otimes x_h) = x_{ghg^{-1}} \otimes x_g, \quad \varepsilon(x_g) = 0.$$ 

Thus, $U(\mathcal{C}_C) \simeq B(\mathcal{L}_C)$ is generated by 1 and $x_g, g \in C$, with relations $x_g x_h - x_{ghg^{-1}} x_g = x_{ghg^{-1}} - x_h$. It is not quadratic with respect to the set of generators \{ $x_g$ \}, but it is with respect to the set \{ $X_g = 1 + x_g$ \}.

Remark. $U(\mathcal{C}_C) \simeq B(\mathcal{L}_C)$ has no antipode. One could think that by localizing at the multiplicative set generated by \{ $X_g : g \in C$ \} one would get a Hopf algebra with antipode $S(X_g) = (X_g)^{-1}$. This turns out to be wrong because the elements $X_g$ can be zero divisors. Example: let $G = S_3$ with Coxeter generators $s_1, s_2$ and relations $s_1^2 = e, s_1 s_2 s_1 = s_2 s_1 s_2$. Let $C$ be the class of transpositions and let $X_1 = X_{s_1}, X_2 = X_{s_2}, X_3 = X_{s_1 s_2 s_1}$ be the generators of $B(\mathcal{L}_C)$. The relations are:

$$X_i X_j = X_k X_i \quad ((i, j, k) \text{ any permutation of } (1, 2, 3))$$

Playing with these relations, one gets $X_i X_j^2 = X_j^2 X_i$ and $X_i X_j X_k = X_k X_i$, $(i, j, k)$ all distinct. Therefore $X_i^2$ is central in $B(\mathcal{L}_C)$ and $X_i(X_j^2 - X_k^2) = 0$. 

$\diamond$
5 Differential calculi and matrix braided Lie algebras on \( O_q(G) \)

In this section we apply the above general results for co-quasitriangular Hopf algebras to the standard \( q \)-deformations \( O_q(G) \) and their variants. These are characterised by the `quantum Killing form' \( \mathbf{q} = r_{21} * r \) being nontrivial and in this case there is a standard construction \cite{Ju, KS, Maj-98} for their bicovariant differential calculi going back to B. Jurco in an R-matrix setting. The corresponding braided Lie algebras in this case are the matrix ones in \cite{Maj-98}. For the general treatment we allow \( \mathbf{q} \) to be built in fact from pairs of coquasitriangular structures.

5.1 Construction of the calculi

Let \((A, r)\) be quasitriangular, \( r \) fixed. \( A \) also has a braided version in \( M^A \), which is the one usually appearing in the literature \cite{Maj}. We note it \( \mathbf{A}(r)^{\text{right}} \) to distinguish it from \( \mathbf{A} = \mathbf{A}(r)^{\text{left}} \) previously given. We note \( \mathbf{ad}_2(a) = a^{(0)} \otimes a^{(1)} \). Exactly as in \cite{Jr}, given arbitrary other co-quasitriangular structure \( s \) on \( A \), we have a left \( A^\circ \)-coalgebra \( \lambda_{r,s} : A \to A^\circ \otimes A \) given by \( a \mapsto a^{[-1]} \otimes a^{(0)} = s_2(a(1))r_1(a(3)) \otimes a(2) \). This gives a right crossed \( A \)-module \((A, \dot{\triangleright}_{r,s}, \mathbf{ad}_2)\) with braiding \( \Xi^{\text{right}}_{r,s} \) where

\[
\dot{\triangleright}_{r,s}(a \otimes b) = b(2) \otimes a(2) \langle Sb(1) b(3), s_2 a(1) \rangle r_1(a(3)) \Delta(a(b)).
\]

Note that \( \Xi^{\text{right}}_{r,s} \) can also be written

\[
s_21(a(1), b(1)) \Xi_{r,s}(X_{a(3)} \otimes X_{b(3)}) \mathbf{r}(a(3), b(2)) = s_{21}(a(1), b(2)) X_{b(1)} \otimes X_{a(2)} \mathbf{r}(a(3), b(3)).
\]

(5.1)

When \( s = r_{21} \), we write \( a \triangleright_r b = a \dot{\triangleright}_{r,s_1} b = b(2) \langle Sa(1) a(3), r_2 b(2) \rangle \) and \( \Psi^{\text{right}}_{r,s} = \Xi^{\text{right}}_{r,s_{21}} \), the braiding on \((A, \mathbf{ad}_2)\) in \( M^A \) thanks to \( r \). The multiplication in \( \mathbf{A}(r)^{\text{right}} \) is \( a \triangleright_r \mathbf{r}_{21}(b) \), and \( \mathbf{A}(r)^{\text{right}} \) is \( \Xi^{\text{right}}_{r,s} \)-commutative. We let

\[
\mathbf{q} = s_{21} \ast r.
\]

It still satisfies \( \mathbf{q} \ast m = m \ast \mathbf{q} \). One has \( \mathbf{q}_1 = s_2 \ast r_1 \), \( \mathbf{q}_2 = s_1 \ast r_2 \), and by straightforward applications of the properties of \( r \) and \( s \), one obtains for all \( a, b \in A \),

\[
\begin{align*}
\mathbf{q}_1(ab) &= s_2(b(1)) \mathbf{q}_1(a) r_1(b(2)), & \Delta(\mathbf{q}_1(a)) &= s_2(a(1)) r_1(a(3)) \otimes \mathbf{q}_1(a(2)). \\
\mathbf{q}_2(ab) &= s_1(a(1)) \mathbf{q}_2(b) r_2(a(2)), & \Delta(\mathbf{q}_2(a)) &= \mathbf{q}_2(a(2)) \otimes s_1(a(1)) r_2(a(3)).
\end{align*}
\]

(5.2, 5.3)

In addition, \( \varepsilon(\mathbf{q}_1(a)) = \varepsilon(a), \mathbf{q}_1(1) = 1 \) (\( i = 1, 2 \)). The following is well-known when \( \mathbf{r} = s \).

Lemma 5.1. (Intertwining properties of \( \mathbf{q} \)).

(i) \( \mathbf{q}_1 \) intertwines the left adjoint and coadjoint actions of \( A^\circ \), and \( \mathbf{q}_2 \) the right ones.

(ii) \( \mathbf{q}_1 : \mathbf{A}(s)^{\text{right}} \to A^\circ \) and \( \mathbf{q}_2 : \mathbf{A}(r)^{\text{left}} \to A^\circ \) are homomorphisms of algebras. In particular, \( \im \mathbf{q}_1 \) is a subalgebra of \( A^\circ \), and a left coideal by (5.2). Moreover, \( \mathbf{q}_1(A^{ad}) \) and \( \mathbf{q}_2(A^{ad}) \) belong to the center of \( A^\circ \).

Proof. (i) This is lemma \cite{Jr} applied to \( \xi = \mathbf{q} \). (ii) We prove it for \( \mathbf{q}_2 \) and \( \mathbf{A}(r)^{\text{left}} \). For \( a, b \in A \), using repeatedly that \( r_2 \) is an antialgebra map and a coalgebra map, we get

\[
\begin{align*}
\mathbf{q}_2(a \triangleright_r b) &= \mathbf{q}_2(a(1), b(2)) \langle b(1), S(b(3)), r_2 S(a(2)) \rangle \\
&= \langle b(1), r_2 S(a(4)) \rangle s_1(a(1)) \mathbf{q}_2(b(2)) r_2(a(2)) \langle S(b(3)), r_2 S(a(3)) \rangle \\
&= \langle b(1), r_2 S(a(4)) \rangle s_1(a(1)) r_2(a(2)) \langle b(2), r_2(a(3)) \rangle \mathbf{q}_2(b(3)) \\
&= \langle b(1), r_2 S(a(3)) r_2(a(2)) \rangle \mathbf{q}_2(a(1)) \mathbf{q}_2(b(2)) = \mathbf{q}_2(a) \mathbf{q}_2(b).
\end{align*}
\]
(The underlined term is transformed using $Sr_2S = r_2$ and $m * q = q * m$). Finally, if $\text{ad}_L(a) = 1 \otimes a$, then for all $h \in A^\circ$, $Ad_R^*(h)(a) = (1, h) a$, therefore $Ad_R h(q_2(a)) = \varepsilon(h) q_2(a)$ by (i), i.e., $q_2(a)$ is central in $A^\circ$. For $q_1$, the proof is analogous.

\[ \square \]

**Proposition 5.2.** Let $C_1$ be a subcoalgebra of $A$ containing $1_A$. Then $\tilde{g}(C_1, q) := q_1(C_1)$ is the extended tangent space of a bicovariant FODC over $(A, r)$, with associated left ideal

\[ I(C_1, q) = \{ a \in A | \forall c \in C_1, q(c, a) = 0 \} \supset \ker q_2. \]  

(ii) Conversely, let $(\Gamma, d)$ be a bicovariant FODC over $A$ with associated left ideal $I_\Gamma$. If $q_1$ is injective, then

\[ C_1 = \{ a \in A | q(a, I_\Gamma) = 0 \} \]

is a subcoalgebra of $A$ containing $1_A$. If moreover $\tilde{g}_r \subset \text{im} q_1$, then $\tilde{g}_r = q_1(C_1)$.

(iii) Assume $C_1 = k 1_A \oplus C$ for some subcoalgebra $C$ and that $q_1$ is injective on $C$. Then $\tilde{g}_r = 1_A \oplus \mathcal{L}$ where $\mathcal{L} = q_1(C)$ is a braided Lie subalgebra. The associated bicovariant FODC is inner, with tangent space $g_r = \{ x - x(x)1 : x \in \mathcal{L} \}$ and $U(g_r) \simeq B(\mathcal{L})$ as (quadratic) bialgebras.

Note that if we take $C_1 = A$, we get $I(C_1, q) = \ker q_2$ which is indeed a left crossed submodule of $(A, m, \text{ad}_L)$. It is the smallest ideal we can divide by through this construction, thus we should assume $q \not\in \varepsilon_A \otimes \varepsilon_A$.

**Proof.** (i) We check the properties (a), (b) and (c) of lemma 4.3 for $\tilde{g}_r = q_1(C_1)$. Since $1_A \in C_1$, one has $1_A \in q_1(C_1)$. Since $C_1$ is a subcomodule for $\text{ad}_L$, it is also a left submodule the left adjoint coaction of $A^\circ$ on $A$, therefore by the intertwining property of $q_1 : Ad_L h(q_1(c)) = q_1(Ad_L^*(h)(c)) \in q_1(C_1)$ for any $h \in A^\circ$ and $c \in C_1$. Finally, $q_1(C_1)$ is a left coideal by the left equality in (5.2).

(ii) For $a \in A$, $b \in I_\Gamma$ and $c \in C_1$, one has since $I_\Gamma$ is a left sided ideal of $A(r)^{\text{left}}$, $0 = \langle c, q_2(a, b) \rangle = \langle c, q_2(a) q_2(b) \rangle = \langle \Delta(c), q_2(a) \otimes q_2(b) \rangle$ (where $\Delta$ is the multiplication in $A(r)^{\text{left}}$). If $q_2$ is injective, $q_2(A)$ separates the elements of $A$, therefore $\Delta(C_1) \subset A \subset C_1$. Since $I_\Gamma$ is also a right ideal of $A(r)^{\text{left}}$ (proposition 4.7), one obtains likewise $\Delta(C_1) \subset C_1 \otimes C_1$, so $\Delta(C_1) \subset C_1 \otimes C_1$. By definition, $\tilde{g}_r = \{ x \in A^\circ | (I_\Gamma, x) = 0 \}$ therefore if $\tilde{g}_r \subset \text{im} q_1$, we immediately get $\tilde{g}_r = q_1(C_1)$ by definition of $C_1$.

(iii) $\Gamma$ is inner by lemma 4.3 and we apply theorem 4.8.

**Remark.** The above proposition is essentially well-known, but is slightly more general than analogous results in [KS], [Maj-98] because it can describe differential calculi which are not inner (the coalgebra imbedding $k1_A \hookrightarrow C_1$ might be non split in the non semi-simple case, for instance for quantum groups at roots of unity). It is shown however in [HS] that for the standard quantum groups $O_q(G)$, $G = SL(n)$ or $G = Sp(n)$, $q$ not a root of unity, any bicovariant FODC arises in this way for a uniquely determined subcoalgebra $C_1$ and for some pair of co-quasitriangular structures $(r, s)$, where $s = c * r$, $c$ a central bicharacter. 

The next proposition describes the braided and quantum Lie algebras $\tilde{g}_r$ and $g_r$ associated to $q$ and $C_1$. We assume that $q_1$ is injective on $C_1$, so that $\tilde{g}_r$ can be identified with $C_1$. (The given formulas are actually true in the non injective case, but should be considered with care).

**Proposition 5.3.** Let $C_1$ be a subcoalgebra of $A$ containing $1_A$ as before, and $\tilde{g}_r = q_1(C_1)$.

(i) For $c \in C_1$, let $X_c = q_1(c) \in \tilde{g}_r$. The right crossed module structure of $\tilde{g}_r$ over $A$ is given by (we write $c^{(0)} \otimes c^{(1)} := \text{ad}_R(c)$):

\[ X_c \sim c = X_{c^{(2)}} \circ (\text{ad}_L c^{(1)}) r_1(c^{(3)}) =: X_{c \rightarrow r, * a} \]

\[ \delta_R(X_c) = X_{c^{(2)}} \otimes S(c^{(1)}) c^{(3)} = X_{c^{(0)}} \otimes c^{(1)}. \]
The corresponding braiding \( \Psi = \Psi_r \) (in \( M^A \)) and \( \Upsilon = \tilde{\sigma} \) (in \( C^A \)) on \( \tilde{g}_R \) are:

\[
\Psi(X_a \otimes X_b) = X_{b(2)} \otimes X_{a(2)} \langle S(b_{(1)}) b_{(3)}, r_1 S(a_{(1)}) r_1(a_{(3)}) \rangle
\]

\[
\Upsilon(X_a \otimes X_b) = X_{b(2)} \otimes X_{a(2)} \langle S(b_{(1)}) b_{(3)}, s_2(a_{(1)}) r_1(a_{(3)}) \rangle
\]

The braided Lie algebra structure of \( \tilde{g}_R \) in \( M^A \) is given by:

\[
\Delta(X_c) = X_{c(i)} \otimes X_{c(j)}, \quad \varepsilon(c) = \varepsilon(c), \quad [X_a, X_b] = X_{b(0)} \otimes q(a, b^{(1)}).
\]

Let \( Y_c \) be the image of \( X_c \in \tilde{g}_R \subset A^o \) in \( B(\tilde{g}_R) \). The defining relations of \( B(\tilde{g}_R) \) can be written:

\[
s_{21}(a_{(1)}, b_{(1)}) Y_{b(2)} Y_{b(3)} r(a_{(3)}, b_{(2)}) = s_{21}(a_{(1)}, b_{(2)}) Y_{b(1)} Y_{a(2)} r(a_{(3)}, b_{(3)}).
\]

(ii) For \( c \in C \), let \( x_c = q_1(c) - \varepsilon(c) 1_{A^o} \in \tilde{g}_R \). The braiding \( \Psi_r \) (in \( M^A \)) and \( \sigma \) (in \( C^A \)) on \( \tilde{g}_R \) are given by the same formulas as above, with \( X \) replaced by \( x \); the quantum Lie bracket on \( \tilde{g}_R \) and the braided coproduct on \( U(\tilde{g}_R) \) are given by:

\[
\Delta(x_c) = x_c \otimes 1 + 1 \otimes x_c + x_{c(i)} \otimes x_{c(j)}, \quad [x_a, x_b] = q(a, b^{(1)}) x_{b(0)} - \varepsilon(a) x_b.
\]

Moreover, \( U(\tilde{g}_R) \) can also be seen as the algebra generated by the elements \( Y_c \) with relations (5.5) and \( Y_1 = 1 \).

Proof. This is a direct application of the definitions and the intertwining properties of \( q \).
The right action of \( A \) on \( \tilde{g}_R \) is given by (proposition [4.1]) \( X_c \leftarrow a = ((q_1(c))_{(1)}, a) (q_1(c))_{(2)} \).

\[
\langle a, s_2(c_{(1)}) r_{(c_{(3)})} \rangle q_1(c_{(2)}).
\]

Likewise, the coproduct \( \Delta \) on \( \tilde{g}_R \) is such that \( \langle a \otimes b, \Delta q_1(c) \rangle = \langle q_2(a) q_2(b), c \rangle + \langle a \otimes b, q_1(q_1(c)) \rangle \Delta(c) \).

Finally, the braided Lie bracket is \( [X_a, X_b] = Ad_{q_1(a)} q_1(b) \) and the braiding \( \Delta(x) \) is obtained directly from their definition:

\[
\Psi_r(X_a \otimes X_b) = \Delta(x_{a(0)} \otimes x_{b(0)}) r(a_{(1)}, b^{(1)}) \] and \( \sigma(x_{a(0)} \otimes x_{b(0)}) = x_{a(0)} \otimes x_{b} - b^{(1)} \).

Note that \( \Psi_r \) and \( \sigma \) almost coincide on \( \tilde{g}_R \) if \( q_1(C_1) = 1 \).

Moreover, \( U(\tilde{g}_R) \) can also be seen as the algebra generated by the elements \( Y_c \) with relations (5.5) and \( Y_1 = 1 \).

Remark. (i) Let \( c \) be a central bicharacter on \( A \) [degen]. If we take \( s = c * \Psi_r \), we get \( q = c_{21} \) which is non degenerate for \( s = s * r \), the right action of \( A \) on \( \tilde{g}_R \) = \( q_1(C_1) \),

and therefore also the left action on \( \tilde{g}_R \) depends on \( c \), but all the remaining defining structure maps of \( \tilde{g}_R \), i.e. \( \Psi, \Upsilon, [\cdot, \cdot] \) and \( \Delta \) do not as can be easily checked. Therefore \( c \) controls how \( \tilde{g}_R \) sits inside \( A^o \), and distinct \( c \)’s give different non isomorphic calculi, but the corresponding extended tangent spaces \( \tilde{g}_R \) = \( q_1(C_1) \) are isomorphic as abstract braided Lie algebras. In the following, we focus only on these, therefore we assume that \( s = r \) and write \( \Delta^{\text{right}} = \Delta^r \).

(ii) When \( 1 \rightarrow C_1 \) splits, which is the case we are interested in, one has \( \tilde{g}_R = k1_{A^o} \oplus \mathcal{L} \), \( \mathcal{L} = q_1(C) \). Then the three spaces \( \tilde{g}_R \), \( \mathcal{L} \) and \( C \) and identified, and \( U(\tilde{g}_R) \simeq B(\mathcal{L}) \) is generated by \( Y_c, c \in C \), with relations (5.5). According to the remarks following (4.9), there is an algebra homomorphism \( U(\tilde{g}_R) \simeq B(\mathcal{L}) \rightarrow A^o \) such that \( Y_c \rightarrow X_c \).

If \( s = r \) (or more generally if \( s = c * r \)), we see comparing (5.5) and the \( \varepsilon \)-right-commutativity of \( \Delta^{\text{right}} \).
that this homomorphism factors through a homomorphism $B(\mathcal{L}) \to \mathcal{A}_{\text{right}}^\oplus$, that is, it is the composition:

$$U(\mathfrak{g}_R) \simeq B(\mathcal{L}) \quad \xrightarrow{Y_c} \quad \mathcal{A}_{\text{right}}^\oplus \quad \xrightarrow{q_1} \quad \mathcal{A}^\oplus$$

(5.7)

(iii) If moreover $C$ is simple, let $\{\lambda^i_j : i, j = 1, \ldots, n\}$ be a basis such that $\Delta \lambda^i_j = \lambda^i_k \otimes \lambda^k_j$, let $\{X^i_j = X_{\lambda^i_j} = q_1(\lambda^i_j) : i, j = 1, \ldots, n\}$ be the corresponding basis of $\mathcal{L}$, and $Y^i_j = Y_{\lambda^i_j}$ the image of $X^i_j$ in $B(\mathcal{L})$. Note that one has $\Delta X^i_j = X^i_k \otimes X^k_j$ (coproduct in the braided Lie algebra $\mathcal{L}$) and $\Delta X^i_j = r_2(t^i_a) r_1(t^b_j) \otimes X^a_b$ (coproduct in $A^\ominus$). One defnes the tensors $R^i_j = r(\lambda^i_j, \lambda^k_j)$ and $R^{-1} = r(S\lambda^i_j, \lambda^k_j)$, $R^{-1} = r(\lambda^i_j, S\lambda^k_j)$, and $Q = R_2 R_1$. Then in the numerical suffix notation [RFT], the structure maps $\Psi$, $\Upsilon$, and $\hat{\cdot}$ of $\mathcal{L}$ are given by

$$\Psi(R_2 X_1 X_2^{-1} \otimes X_2) = X_2 \otimes R_2 X_1 X_2^{-1}$$

$$\Upsilon(R_2 X_1 R \otimes X_2) = X_2 \otimes R_2 X_1 R,$$

$$[R_2 X_1 R, X_2] = X_2 Q.$$

as in [Maj-94, Maj-93], i.e. we obtain the ‘matrix braided Lie algebras’ introduced there. Here the algebra $B(\mathcal{L})$ is abstractly generated by the $n^2$ elements $Y^i_j$ with relations

$$(R_2 Y_1 R) Y_2 = Y_2 (R_2 Y_1 R)$$

and coincides with the bialgebra of braided matrices $B(R)$ (a bialgebra in the category of right $A(R)$-comodules [Ma]). Finally, the quantum Lie bracket on $\mathfrak{g}_R$ is $[R_2 x_1 R, x_2] = x_2 Q - Q x_2$.

5.2 Example: $O_q(SL(n))$

Let $A = O_q(SL(n))$, $C$ its fundamental subcoalgebra, with basis $\{t^i_j : i, j = 1, \ldots, n\}$, $R$ its standard $R$-matrix, $r$ the unique co-quasitriangular structure on $A$ such that $r(t^i_j, t^k_l) = R^{-1}_{i k} R^l_{j l}$, det the quantum determinant. Recall that $A$ (resp. $\mathcal{A}_{\text{right}}^\ominus$) is the quotient of $A(R)$ (resp. $B(R)$) by the two-sided ideal generated by the central element det $-1$. Here $A(R)$ and $B(R)$ are the algebras generated by the matrix $t$ of elements $t^i_j$ and relations

$$A(R) : \quad R t_1 t_2 = t_2 t_1 R, \quad B(R) : \quad (R_2 t_1 R) t_2 = t_2 (R_2 t_1 R).$$

Consider the standard $n^2$-dimensional bicovariant FODC $\Gamma$ over $A$ corresponding to the subcoalgebra $C$. One has $U(\mathfrak{g}_R) \simeq B(R)$ by the previous section, and therefore there exists a central grouplike element (also written det) inside $U(\mathfrak{g}_R)$ such that $U(\mathfrak{g}_R)/(\text{det} - 1) \simeq \mathcal{A}_{\text{right}}^\ominus$. This gives the kernel of the first map in (5.7), for all values of $q$. If $q$ is not root of unity, the second is injective [RS, HS, BS]. Its image is described in [BS] proposition 5. With the definition of $U_q(sl(n))$ given in [BS], $q_1(A) = F_1(U_q(sl(n)))$ is the locally finite part of the left adjoint $U_q(sl(n))$-module. Therefore we have:

**Proposition 5.4.** Let $A = O_q(SL(n)) = A(R)/(\text{det} - 1)$, $q$ not a root of unity, and $\Gamma$ the standard $n^2$-dimensional bicovariant FODC over $A$. Then $U(\mathfrak{g}_R) \simeq B(R)$, and $B_q(sl(n)) = B(R)/(\text{det} - 1) \simeq \mathcal{A}_{\text{right}}^\ominus \simeq F_1(U_q(sl(n)))$ is a Hopf algebra in $M^A$.

This makes more precise the sense in which braided Lie algebras solve the ‘Lie algebra problem’ for quantum groups in [Maj-94]. It can also be viewed as the self-duality of the braided versions of quantum groups, i.e. the above quotient $B_q(sl(n))$ is isomorphic to a (braided) version of $U_q(sl(n))$ (with algebra the locally finite part) via the quantum Killing form [Ma].
The $n^2$-dimensional braided Lie algebra in this example can be denoted $sl_q(n)$ and the $n = 2$ case is computed explicitly in [Maj-94, Ex. 5.5]. The enveloping algebra $B(R) = BM_q(2)$ is the standard $2 \times 2$ braided matrix algebra [Maj]. The structural form of the general $B(R) = BM_q(n)$ and their homological properties appear in [LeB-94]. In particular, for generic $q$ it is known that they have the same Hilbert series as polynomials in $n^2$ variables.

We can give the the relations of $B(R)$ more explicitly as follows, in fact for the full multiparameter $SL(n)$-type family. In our conventions (which are slightly different from [LeB-94]) the R-matrix is

$$R^{i, j}_{k, l} = \delta^i_k \delta^j_l M_{ij} + \delta^i_k \delta^j_l L_{ij}, \quad M_{ij} = q^\delta_{ij} + \theta_{ji} q^\frac{r_{ij}}{q} + \theta_{ij} q^\frac{r_{ji}}{q}, \quad L_{ij} = \theta_{ji} (q - q^{-1}),$$

where $\theta_{ij}$ denotes the function which is 1 iff $i > j$ and otherwise zero, and $r_{ij} \neq 0$ are multiparameters defined for $i < j$ and constrained by $\prod_{i < j} \frac{r_{ij}}{q} = \prod_{i > j} \frac{r_{ij}}{q}$ for all $j$ as explained in [LeB-94]. The standard $O_q(SL(n))$ case is $r_{ij} = q$. We let $\mu = q - q^{-1}$. Let us also introduce the 'cocycle' defined for $i, j, k$ all distinct by

$$\sigma_{ijk} = \left( \frac{q R_{\sigma(k), \sigma(i)} R_{\sigma(j), \sigma(i)} R_{\sigma(k), \sigma(j)}}{R_{\sigma(i), \sigma(i)} R_{\sigma(k), \sigma(k)}} \right)^{-1} l(\sigma)$$

where $\sigma \in S_3$ is the unique permutation of $i, j, k$ such that $\sigma(i) > \sigma(j) > \sigma(k)$ and $l(\sigma)$ is its length. By convention, $\sigma_{ijk} = 0$ if the $i, j, k$ are not distinct. Finally, in order to make computations we need the matrices for $R^{-1}$ and $\tilde{R}$. Using that $R$ is q-Hecke, one can show that

$$R^{-1}(q, \{r_{ij}\}) = R(q^{-1}, \{r_{ij}^{-1}\}), \quad \tilde{R}^{i, j}_{k, l} = R^{-11}_{1} \sigma_{i, j} q^{2(l - j)}$$

which means that they have the same form as the above with $M_{ij}^{-1}$ in place of $M_{ij}$ and $-L_{ij}$ or $-L_{ij} q^{2(\mu - j)}$ respectively in place of $L_{ij}$. Let us denote by $\tilde{\sigma}_{ijk}$ the same expression as $\sigma_{ijk}$ but with $q, r_{ij}$ inverted.

**Lemma 5.5.**

$$M^{-1}_{ki} M_{jl} M^{-1}_{jk} = \sigma_{ijk} + q \delta_{ij} + q^{-1} (\delta_{ik} + \delta_{jk}) - (q + q^{-1}) \delta_{ij} \delta_{jk}$$

$$M^{-1}_{ki} M^{-1}_{jk} M_{ij} = \sigma_{ijk} \sigma_{ijl} + q^{-1} \sigma_{ijl} (\delta_{ki} + \delta_{kj}) + q \sigma_{ijk} (\delta_{li} + \delta_{lj})$$

$$+ \delta_{ij} (1 + \mu q \delta_{il} - q^{-1} \mu \delta_{jl}) + \delta_{iku} \delta_{jl} + \delta_{kl} \delta_{il} \delta_{lj}$$

if $k \neq l$ and 1 if $k = l$.

Finally, we obtain $\Upsilon$ from the explicit R-matrix formula for it in [Maj-94], namely

$$\Upsilon(X^i_j \otimes X^k_l) = R^{-1}_{m} a X^m_n \otimes \otimes a R_{a} b X^a c \otimes b R_{b} c X^b d \otimes c \otimes d \otimes k X^m_n \otimes X^o_p.$$}

This determines the relations of $B(R)$ as $\Upsilon$-commutative (one can also work from the ‘reflection’ form of the $B(R)$ relations as in [LeB-94] if one does not need $\Upsilon$ explicitly). Since we are dealing with an abstract braided Lie algebra, we do not distinguish between the $n^2$ basis elements $X^i_j$ of $L$ and their images in $B(L)$ as was done for instance in [5.7].

**Proposition 5.6.** The $\binom{n^2}{2}$ relations of the multiparameter $B(sl_{n,q}) = BM_q(n)$ may be listed for distinct $i, j, k, l$ as follows.

(i) For $i < k$: $X^i_i X^k_k = X^k_k X^i_i$. 

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(ii.a) For $k, l > i$: $X^i_jX^k_l = X^k_lX^i_j$.
(ii.b) For $i > k, l$: $X^i_jX^k_l = X^k_lX^i_j$.
(ii.c) For $l > i > k$: $X^i_jX^k_l - X^k_lX^i_j = -\mu \delta_b cX^i_jX^k_l$.
(ii.d) For $k > i > l$: $X^i_jX^k_l - X^k_lX^i_j = \mu \delta_b cX^k_lX^i_j$.
(ii.e) For $i < l$: $X^i_jX^k_l - q^{-2}X^k_lX^i_j = -q^{-1}\mu \sum_{a < c} X^i_aX^k_a$.
(ii.f) For $i > l$: $X^i_jX^k_l - X^k_lX^i_j = -q^{-1}\mu \sum_{a < c} X^i_aX^k_a$.
(ii.g) For $i > k$: $X^i_jX^k_l - X^k_lX^i_j = \mu q^{-1}\sum_{a < c} X^k_aX^i_a$.
(ii.h) For $i < k$: $X^i_jX^k_l - q^{2}X^k_lX^i_j = \mu q\sum_{a < c} X^k_aX^i_a$.

\[ (ii.a) \quad \text{For } i < k: \quad X^i_jX^k_l = X^k_lX^i_j \]
\[ (i) \quad \text{For } i < k: \quad X^i_jX^k_l = \sigma_{ijkl}X^i_jX^k_l = \mu \theta_{ijl}X^k_jX^i_l \sigma_{ijk}. \]

\[ (ii) \quad \text{For } i < k: \quad \sigma_{ijk}X^j_kX^l_i - qX^k_lX^i_j = \mu \sum_{a < c} X^k_aX^j_kX^l_i + \mu \theta_{ijl}X^k_jX^l_i. \]
\[ (iii) \quad \text{For } i < k: \quad \sigma_{ij}X^j_iX^k_l - qX^k_lX^i_j = \mu \sum_{a < c} X^k_aX^j_iX^k_l + \mu \theta_{ijl}X^k_jX^l_i. \]

\[ (ii.f) \quad \text{For } i < j: \quad X^i_jX^j_i - X^j_iX^i_j = \mu \sum_{a < c} X^j_aX^j_iX^i_j - \mu^{-1}\sum_{a < c} X^j_aX^i_j. \]

\[ + \mu^2 \sum_{b < j, a < i} X^b_aX^a_jX^2(a-i) - \mu^2 X^j_iX^j_i \sum_{a < i} X^a_aX^2(a-i). \]

\[ \text{Proof.} \quad \text{We write each } R \text{ as the sum of an } L \text{ and an } M \text{ term as explained above, giving } \]
16 terms for $\Upsilon$. We use a standard graphical notation to follow the values forced for the
summed indices by the $\delta$-functions in $L$ and $M$. We then use the above lemma to break
the down results further, to obtain:

$\Upsilon(X^i_j \otimes X^k_l) = L^i_j \otimes X^k_l (1 - \delta_{kl}) \left( \sigma_{ijkl}X^j_iX^k_l + q^{-1}(\delta_{ki} + \delta_{k}j)X^j_i + q(\delta_{li} + \delta_{lj})X^k_l \right)$

$= \mu \sum_{a < c} X^a_iX^a_jX^a_kX^a_l + q^{-1}X^j_iX^k_l - q^{-1}\mu \sum_{a < c} X^j_iX^k_l$.

\[ + q\mu \sum_{a < k} X^a_iX^a_jX^a_kX^a_l + q\mu \sum_{a < k} X^a_iX^a_jX^a_kX^a_l \]

$+ (q + q^{-1})X^i_jX^j_iX^k_lX^k_l + q^2(q - 1)X^i_jX^j_iX^k_lX^k_l$.

$= \mu \left( X^k_jX^i_jX^k_lX^k_l + \sum_{a < k} X^k_aX^a_jX^a_kX^a_l \right)$

$= q^{-1}\delta_{jk} + q^{-1}\delta_{jk} + q\delta_{ij} - (q + q^{-1})\delta_{ij}\delta_{jk}.$

$= X^i_jX^l_i \otimes X^a_aX^a_jX^a_kX^a_l + X^i_jX^l_i \otimes X^a_aX^a_jX^a_kX^a_l$.

Next we break up the $\binom{n^2}{2}$ relations into convenient special cases as stated. We then compute the relation $X^j_iX^k_l = L(X^j_i \otimes X^k_l)$ for $i, j, k, l$ according to the leading term on the left hand side in each of the cases stated. We then simplify the resulting set of equations. In some cases the simpler version arises from $X^k_lX^j_i = L(X^k_l \otimes X^j_i)$ instead.

Among other things, one may verify that $\text{tr}$ is known from general $R$-matrix methods for
braided matrices $\text{M}$ that the right-invariant $q$-trace element

$\text{tr} = \sum_i X^i_iX^{2i}.$
is central.

The braided Lie bracket of the multiparameter $sl_q(n)$ may be computed as $(\text{id} \otimes c)\Upsilon$ or directly from the R-matrix relations in $[Maj-94]$. In the $q$-Hecke case these reduce to

$$[X^i, X^j] = \delta^i_j X^i - mq^{-2} \delta^i_j X^i + \mu R^{-1} m R^n b R_j^c X^a X^m n$$

and the matrix coalgebra structure. Note also that $[\cdot, \cdot]$ necessarily closes on $\ker \epsilon$ which should be thought of as the infinitesimal elements of the braided Lie algebra (the classical model of a braided Lie algebra in $[Maj-94]$ is $L = k \oplus g$ with $g = \ker \epsilon$ a classical Lie algebra).

We write $sl_q(n) = \ker \epsilon$ inside $sl_q(n)$.

**Proposition 5.7.** Suppose that $\epsilon(\mathfrak{t}) \neq 0$, i.e. $q^{2n} \neq 1$. Then $sl_q(n) = k\mathfrak{t}\mathfrak{t} \oplus sl_q(n)$ in the braided category, where $sl_q(n) = \text{span}\{X^i, h_k \mid i \neq j; \; k = 1, \cdots, n - 1\}; \; \; h_i := X^i - X^{i+1}i+1$.

The $q$-Lie brackets are as follows. Let us define

$$H_i := \sum_{a<i} [a]_q h_a; \; \; [a]_q^2 := \frac{1 - q^{2a}}{1 - q^2}.$$  

(i) $[x, \mathfrak{t}\mathfrak{t}] = 0, \; \; [\mathfrak{t}\mathfrak{t}, \mathfrak{t}\mathfrak{t}] = \epsilon(\mathfrak{t}\mathfrak{t})\mathfrak{t}\mathfrak{t}, \; \; [\mathfrak{t}\mathfrak{t}, x] = \epsilon(\mathfrak{t})\lambda x, \; \; \forall x \in sl_q(n), \; \lambda = 1 + \mu^2$.

(ii.a) ‘Cartan’ relations

$$[h_i, h_i] = -\mu^2 q^2 H_i - \mu^2 q^{-2i} H_i + \mu^2 q^{-2i} [i + 1]_q h_i, \; \; [h_i, h_{i+1}] = \mu^2 q^{-2i} H_{i+1}$$

$$[h_i, h_{i-1}] = \mu^2 q^{-2(i-1)} H_i, \; \; \forall [i - j] > 1.$$  

(ii.b) ‘Weight’ relations for $k \neq 1$:

$$[h_i, X^k] = \mu X^k (q^{-2i}(q^{-1} \delta_{ki+1} - q \delta_{ki}) + q^{-1} \delta_{i} - q \delta_{i,i+1})$$

$$[X^k, h_i] = \mu X^k (q^{-1} \delta_{ki} - q \delta_{ki+1} + q^{-2i}(q^{-1} \delta_{i+1} - q \delta_{i}))$$

(ii.c) ‘Root’ relations for $i \neq j, k \neq 1$:

$$[X^i, X^j] = -q \mu^{-2} \delta_{kj} X^i + \mu \delta_{ij} \sigma_{ijk} X^k + q \mu \delta_{ij} \delta_{jk} q^{-2k} X^k - \mu^2 \delta_{ij} \delta_{jk} q^{-2(k-1)} H_k.$$  

\textbf{Proof.} Working from either $\Upsilon$ in the proof of the the preceding proposition or from the R-matrix formula, we obtain

$$[X^i, X^j] = \delta_{ij} X^i - q \mu \delta_{kj} X^i + \mu \delta_{il} (\sigma_{ijk} + q \delta_{ij} + q^{-1} (\delta_{ik} + \delta_{jk}) - (q + q^{-1}) \delta_{ij} \delta_{jk})$$

$$\quad - \mu^2 \delta_{ij} \delta_{jk} \sum_{a<k} X^a q^{2(a-k)} - \mu^2 \delta_{kl} \theta_{kl} X^i + \mu^2 \delta_{ji} \delta_{kl} X^k.$$  

One finds in particular that for all $k \neq l$,

$$[X^i, X^k] = X^k (1 - q \mu \delta_{kl} q^{-2i} + q^{-1} \mu \delta_{ik} + \mu^2 \theta_{ik}) - \mu^2 \delta_{ik} \sum_{a<k} q^{2(a-k)} X^a a - \mu^2 \theta_{ik} X^i$$

which gives the ‘Cartan’ relations after further computation. Similarly

$$[X^i, X^k] = X^k (1 + q \mu \delta_{il} - q \mu q^{-2i} \delta_{ik} + \mu^2 \theta_{il}),(39)$$
etc give the other relations. The $H_i$ arise from the summed $X^{a_i}$ terms written in terms of the $h_i$.

It is also possible to present the (ii.b) and (ii.c) relations above in terms of generators $X_i := X^{i+1}$, $Y_i := X^{i+1}$ with other ‘root vectors’ $X^j$ generated by repeated Lie brackets of these. Among these, we have (from the above):

\begin{align*}
[h_i, X_{j-1}] &= -q \mu q^{-2i} X_{j-1} = -q^{-2(i-1)} [X_{j-1}, h_i] \\
[h_i, X_i] &= q^{-1} \mu (1 + q^{-2i}) X_i = -q^{2(i+1)} [X_i+1, h_i] \\
[h_i, Y_{i-1}] &= q^{-1} \mu Y_{i-1} = -q^{2(i-1)} [Y_{i-1}, h_i] \\
[h_i, Y_i] &= -q \mu Y_i = -q^2 [Y_i, h_i] \\
[X_i, Y_j] &= \mu \delta_{ij} q^{-2i+1} (h_i - q \mu H_i) \\
[Y_j, X_i] &= -\mu \delta_{ij} q^{-2i+1} (q^2 h_i + q \mu H_i).
\end{align*}

The above results reduce for $n = 2$ to the computations in [Ma-94] Ex. 5.5, where $h = a - d$, $X = c$, $Y = b$ in the notation there. The classical $q \to 1$ limit should of course be taken after rescaling all the generators by $\mu^{-1}$. Similarly in $BM_q(n)$ we would obtain a commutative algebra (the coordinate algebra of the space of $n \times n$ matrices) without rescaling.

### 5.3 Generalised Lie algebras $sl_q(n)$

In this concluding section observe that a different quotient of the braided enveloping algebra or braided matrices $B(sl_q(n)) = BM_q(n)$ gives what could reasonably be called the enveloping algebra of ‘generalised Lie algebras’ $sl_q(n)$ of the type suggested by representation theory [LS, DG]. Indeed, we have already seen above that $sl_q(n) = k \oplus sl_q(n)$ where the $k$ is spanned by $tr$ and $sl_q(n) = \ker \epsilon$ cf. [Ma-94]. Since the braided Lie bracket restricted to $sl_q(n)$ is covariant it must also coincide with the ‘generalised Lie bracket’ defined via the $q$-deformed adjoint representation in the representation theory approach.

In fact we are in the situation of Section 3.3, i.e. the braided Lie algebra is split. As we explain now, this is a general feature of the setting of Section 5.1 with $C_1 = k \oplus C$ and $C$ simple (this includes in principle all simple FODC over standard quantum groups, although clearly the case $Q \mathfrak{g} SL(n)$ with its $n^2$-dimensional calculus is the most relevant). Let $tr = R^i a_i^j X_j$ be the right quantum trace of $C$. We have to assume that $\epsilon(tr) \neq 0$. It is $ad_R$-invariant ($ad_R(tr) = tr \otimes 1$), therefore for all $a \in A$ it satisfies $a \otimes tr \mapsto tr \otimes a$ for any braiding associated to a right crossed module $(A, ?, ad_R)$. By the intertwining properties of $q_1$, the element $c = q_1(tr)/\epsilon(tr) \in L$ is central in $A^c$, satisfies $\epsilon(c) = 1$, $\Psi(c \otimes c) = c \otimes c$, $\Psi(c \otimes -) = -c \otimes c$ and $\Psi(- \otimes c) = c \otimes -$. Finally, since $c$ is central in $A^c$ and since $L^+ := \ker \epsilon_L$ is simple for the left adjoint action (since $C = k^{tr} \oplus C^+$ is a semi-simple $A$-comodule for the adjoint coaction), one must have $[c, x] = \lambda x$ for all $x \in L^+$, where $\lambda$ is a constant, which we assume $\neq 0$. In this case the braided Lie algebra $L$ has a distinguished decomposition

$$L = k c \oplus L^+$$

and we have a decomposition of the canonical braiding $\Upsilon$ of $L$ as in Section 3.3, i.e.

$$\Upsilon(z \otimes c) = c \otimes z \quad \text{for all } z \in L,$$

and

$$\Upsilon(x \otimes y) = \omega(x \otimes y) + [x, y] \otimes c, \quad \Upsilon(c \otimes x) = \lambda x \otimes c + \rho(x)$$

for all $x, y \in L^+$, for uniquely determined maps $\omega : (L^+)^\otimes 2 \to (L^+)^\otimes 2$ and $\rho : L^+ \to (L^+)^\otimes 2$. Moreover, if $c$ is not a zero divisor then $(L^+, [\cdot, \cdot])$ is among other things a generalized Lie algebra in the sense of [LS] with generalized antisymmetrizer $\lambda^{-1}(id - \omega)$. The zero divisor condition holds in the multiparameter case by [LeB-94].
Next, we have for any split braided Lie algebra with $\mathcal{L}^+$ simple its reduced enveloping algebra $B_{\text{red}}(\mathcal{L}^+)$ as explained in Section 3.3. In our case of interest, it means

$$B_{\text{red}}(sl_q(n)) = B(\tilde{sl}_q(n))/\langle \text{tr} - \epsilon(\text{tr})\lambda \rangle.$$  

As explained in Section 3.3 the relations of $B_{\text{red}}(sl_q(n))$ contain the defining ‘antisymmetrizer’ relations of the ‘enveloping algebra’ of the algebra $U_{LS}(sl(n))$ (say) proposed in [LS] but in principle could contain further relations. One may check that at least for $n = 2$ the two constructions do coincide. This is the algebra

$$q^{-2}hX - Xh = \lambda(1 - q^{-4})X, \quad q^{2}hY - Yh = -\lambda q^{2}(1 - q^{-4})Y, \quad [X,Y] = \frac{q^{2} - 1}{q^{2} + 1}h + \lambda(1 - q^{-2})h$$

isomorphic after rescaling the generators to the Witten algebra $W_{q^{2}}(sl(2))$ as noted in [LeB-95]. Hence we propose (multiparameter) $B_{\text{red}}(sl_q(n))$ as a generalisation of the Witten algebra for $n \geq 2$ and the (multiparameter) braided matrices $BM_{q}(n)$ in Section 5.2 as its homogenisation.

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References

[BS] P. BAUMAN, F. SCHMIDT, Classification of bicovariant differential calculi over quantum groups (a representation-theoretic approach), Commun. Math. Phys. 194 (1998),71–86.

[BDMS] K. BRESSER, A. DIMAKIS, F. MUELLER-HOISSEN, A.SITARZ, Noncommutative geometry of finite groups, J. Phys. A 29 (1996) 2705-2736.

[Ber] D. BERNARD, Quantum Lie algebras and differential calculus on quantum groups Prog. Theor. Phys. Suppl. 102 (1990), 49-66; also : Phys. Lett. B 260 (1991), 389.

[Brz] T. BRZEZINSKI, Remarks on bicovariant differential calculi and exterior Hopf algebras, Lett. Math. Phys. 27 (4) (1993), 287–300.

[DG] G.W. DELIUS, M.D.GOULD, Quantum Lie algebras, their existence, uniqueness and $q$-antisymmetry, Commun. Math. Phys. 185(3) (1997), 709-722.

[DMMZ] E. DEMIDOV, Y.I. MANIN, E.E. MUKHIN, D.V. ZHDANOVICH, Non standard quantum deformations of $GL(n)$ and constant solutions of the Yang-Baxter equation, Progr. Theor. Phys. Suppl. 102 (1990), 203-218.

[Doi] Y. DOI, Braided bialgebras and quadratic bialgebras, Commun. Alg. 21(5) (1993), 1731–1749.

[Dri] V.G. DRINFEL’D, Quantum groups, Proc. Int. Congr. Math., ed A. Gleason, (Amer. Math. Soc., Providence, RI, Berkeley/Calif. 1986), Vol. 1 (1987), 798–820.

[EH] R. ENDELMAN, T.J. HODGES, Generalized Jordanian $R$-matrices of Cremmer-Gervais type, Lett. Math. Phys. 52(3) (2000), 225-237.

[FRT] L.D. FADDEEV, N.YU. RESHETYKHIN, L.A. TAKHTAJAN, Quantization of Lie groups and Lie algebras, Algebra and Analysis 1 (1987), 178–206.

[HS] I. HECKENBERGER, K. SCHMÜDGEN, Classification of Bicovariant Differential Calculi on the Quantum Groups $SL_q(n+1)$ and $Sp_q(2n)$, J. Reine Angew. Math. 502 (1998), 141-162.
[JC] A.D. Jacobs, J.F. Cornwell, Classification of bicovariant calculi on the Jordanian quantum groups $GL_{h,g}(2)$ and $SL_{h}(2)$ and quantum Lie algebras, *J. Phys. A* 31(44) (1998), 8869-8904.

[Ju] B. Jurčo, Differential calculi on quantized Lie groups, *Lett. Math. Phys.* 22 (1991) 177–186.

[KS] A. Klimyk, K. Schmüdgen, Quantum Groups and their representations, Texts and monographs in Physics, Springer (1997).

[LeB-S] L. Le Bruyn, S.P. Smith, Homogenized $sl(2)$, *Proc. Amer. Math. Soc.* 118(3) (1993), 725–730.

[LeB-VdB] L. Le Bruyn, M. Van der Bergh, On quantum spaces of Lie algebras, *Proc. Amer. Math. Soc.* 119(2) (1993), 407–414.

[LeB-94] L. Le Bruyn, Homological properties of braided matrices, *J. Algebra* 170(2) (1994), 596-607.

[LeB-95] L. Le Bruyn, Conformal $sl_2$ enveloping algebras. *Commun. Algebra* 23(4) (1995), 1325-1362.

[LS] V. Lyubashenko, A. Sudbery, Generalized Lie algebras of type $A_n$, *J. Math. Phys.* 39 (6) (1998), 3487-3504.

[Maj] S. Majid, Foundations of quantum group theory, Cambridge University Press, Cambridge, 1995.

[Maj-LN] S. Majid, Algebras and Hopf algebras in braided categories, *Lec. Notes Pure and Applied Maths* 158 (1994) 55-105, Marcel Dekker.

[Maj-93a] S. Majid, Braided Groups, *J. Pure Applied Algebra* 86 (1993) 187-221.

[Maj-93b] S. Majid, Transmutation theory and rank for quantum braided groups, *Math. Proc. Camb. Phil. Soc.* 113 (1993) 45-70.

[Maj-94] S. Majid, Quantum and braided Lie algebras, *J. Geom. Phys.* 13(4) (1994) 307-356.

[Maj-95] S. Majid, Solutions of the Yang-Baxter equations from braided Lie algebras and braided groups, *J. Knot Theory Ramifications* 4(4) (1995), 673–697.

[Maj-98] S. Majid, Classification of bicovariant differential calculi, *J. Geom. Phys.* 25(1-2) (1998), 119-140.

[RS] N.Ya. Reshetikhin, M.A. Semenov-Tian-Shansky, Quantum R-matrices and factorization problems, *J. Geom. Phys.* 5(4) (1988) 533-550.

[Ro] M. Rosso, Algèbres enveloppantes quantifiées, groupes quantiques compacts de matrices et calcul différentiel non commutatif. *Duke Math. J.* 61 (1), (1990) 11-40.

[Schbg] P. Schauenburg, Differential-graded Hopf algebras and quantum groups differential calculi, *J. Alg.* 180 (1996) 239–286.

[Schm] K. Schmüdgen, On coquasitriangular bialgebras, *Commun. Algebra* 27(10) (1999), 4919-4928.

[SchIII] A. Schüler, Differential Hopf Algebras on Quantum Groups of Type A, *J. Alg* 214 (1999), 479–518.

[Wam] M. Wambst, Quantum Koszul complexes for Majid’s braided Lie algebras, *J. Math. Phys.*, 35 (11) (1994), 6213–6223.

[Witt] E. Witten, Gauge theories, vertex modules and quantum groups, *Nucl. Phys B* 330 (1990), 285–346.

[Wor] S.L. Woronowicz, Differential calculus on compact matrix pseudogroups (quantum groups), *Comm. math. Phys.* 122 (1989) 125–170.