Notes on $\omega$-graphs, reflexive $\omega$-graphs, their higher transformations, and $\omega$-operads

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Abstract

In this note we propose an $\omega$-operadical way to prove the existence of the $\omega$-graph of the $\omega$-graphs and the reflexive $\omega$-graph of the reflexive $\omega$-graphs.

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Introduction

In [2] we have proposed a unified technology to define the $\omega$-magma of the $\omega$-magmas and the reflexive $\omega$-magma of the reflexive $\omega$-magmas. Especially we have conjectured that, up to the contractibility of some specific $\omega$-operads of coendomorphisms, this technology can be used to define the strict $\omega$-category of the strict $\omega$-categories, but also the weak $\omega$-category of the weak $\omega$-categories. In [2] this technology uses the central notion of the standard action of the higher transformations.\footnote{That we call standard action for short.} However it is very important to notice that the discovery of this technology of the standard action\footnote{Which in fact can be generalised as we will see in a future paper in progress.} is completely independent with the notion of contractibility for the higher transformations that we have proposed in [2]: In particular it is the contiguity of these two ideas which, we conjectured, gives an $\omega$-operadical approach of the weak $\omega$-category of the weak $\omega$-categories.

In this note we just use this technology of the standard action to prove the existence of the $\omega$-graph of the $\omega$-graphs, and the reflexive $\omega$-graph of the reflexive $\omega$-graphs, where in that case no contractibility notion are involved. In fact this article can be considered as a result of [2], and has the goal to bring in light the power of this technology of the standard action, for these basic and simple higher structures which are respectively $\omega$-graphs and reflexive $\omega$-graphs. In addition the author wishes to convince the reader about the relevance of the technology
developed in [2] for higher category theory with \( \omega \)-operads.

A key ingredient for our purpose occur in the level of the pointed \( T \)-graphs (see [2]): We use a coglobular complex in \( T\text{-Gr}_{p,c} \)

\[
\begin{array}{cccc}
G^0 & \delta_0^1 & G^1 & \delta_1^2 & G^2 & \delta_2^{n-1} & G^{n-1} & \delta_{n-1}^n & G^n \\
\kappa_0^1 & & \kappa_1^2 & & \kappa_2^{n-1} & & \kappa_{n-1}^n & & \\
\end{array}
\]

which is more basic than the coglobular complex \( C^\bullet \) in \( T\text{-Gr}_{p,c} \)

\[
\begin{array}{cccc}
C^0 & \delta_0^1 & C^1 & \delta_1^2 & C^2 & \delta_2^{n-1} & C^{n-1} & \delta_{n-1}^n & C^n \\
\kappa_0^1 & & \kappa_1^2 & & \kappa_2^{n-1} & & \kappa_{n-1}^n & & \\
\end{array}
\]

which is used in [2], and called the coglobular complex for the higher transformations. The coglobular complex \( G^\bullet \) is build just by removing all cells "\( \mu_p^n \)" and "\( \nu_p^n \)" from it.

In this article we note \( \text{Set} \) the category of sets and \( \text{SET} \) the category of the sets and large sets.

1 \hspace{0.5em} \textbf{Standard actions of} \( \omega \)-\textbf{operads associated to a coglobular complex in} \( T\text{-CAT}_c \)

In this paragraph we recall briefly the section 2 of the article [2], with the slight modification in the level of the combinatoric used in \( T\text{-Gr}_{p,c} \) as in the last section. However it is necessary that the reader read also the section 1 of this article to understand well this section.

Consider a category of \( \omega \)-operads equipped with a specific structure or having a certain property that we call "\( P \)" such that if we call \( PT\text{-CAT}_c \) this category, then the forgetful functor \( U_P \) to \( T\text{-Gr}_{p,c} \) has a
left adjunction $F_P$. If we apply $F_P$ to the coglobular complex $G^\bullet$ (see the Introduction) we obtain a coglobular complex in $PT\text{-}\text{CAT}_c$

$$
\begin{array}{cccc}
B_0^P & \overset{\delta_1^1}{\longrightarrow} & B_1^P & \overset{\delta_1^2}{\longrightarrow} B_2^P & \cdots & B_{n-1}^P & \overset{\delta_1^{n-1}}{\longrightarrow} & B_n^P \\
\kappa_0^1 & & \kappa_1^1 & & \kappa_2^1 & & \cdots & \kappa_{n-1}^1 \\
\end{array}
$$

which is also a coglobular object $W_P$ of $T\text{-}\text{CAT}_c$. Thus we obtain the resulting standard action$^3$ of $T\text{-}\text{CAT}_1$

$$
\text{Coend}(W_P) \xrightarrow{\text{Coend}([A\text{lg}(\_)])} \text{Coend}(A_{P\text{op}}) \xrightarrow{\text{Coend}([\text{Ob}(\_)])} \text{Coend}(A_{0\text{op},P})
$$

where in particular $\text{Coend}(W_P)$ is the monochromatic $\omega$-operad of coendomorphism associated to this coglobular complex. These kind of standard action are similar to those in [2], but much more simpler combinatorically speaking, because based on the more basic diagram $G^\bullet$ in $T\text{-}\text{Gr}_{p,c}$.

As in [2] the main problem is to build a morphism of $\omega$-operads between the monochromatic $\omega$-operad $B_0^P$ (the "0-step" of the coglobular object $W_P$) and the monochromatic $\omega$-operad $\text{Coend}(W_P)$ (build with the whole coglobular object $W_P$). If this morphism exist then it shows that $\text{Coend}(A_{0\text{op}})$ is an algebra of $B_0^P$, and in that case we say that $B_0^P$ has the fractal property. If $B_0^P$ has this fractal property, it means that $P\omega$-categories, $P\omega$-functors, $P\omega$-natural transformations, $P\omega$-modifications, etc. form a $B_0^P$-algebra.

The paragraph below are devoted to give two examples of such $\omega$-operads with the fractal property. More precisely we are going to study the case where $P = \text{Id}$ (i.e we just work with $T\text{-}\text{CAT}_c$; compare with the beginning of the section 4 in [2]) and $P = \text{Id}_u$, i.e it is the property to have contractible units (for a given $\omega$-operad; compare with the section

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$^3$See paragraphs 1 and 2 of [2] for a complete description of this diagram. Here 1 design the terminal $\omega$-graph.
3.1 and the section 4 of [2]). In particular we are going to use the following free functors

\[ M : \mathcal{T} \mathcal{Gr}_{p,c} \longrightarrow \mathcal{T} \mathcal{CAT}_c, \quad \text{Id}_u : \mathcal{T} \mathcal{Gr}_{p,c} \longrightarrow \text{Id}_u \mathcal{T} \mathcal{CAT}_c \]

which are both defined in [2].

2 The coglobular complexes of the graphical \(\omega\)-operads

The category \(\omega\)-\(\mathcal{Gr}\) of the \(\omega\)-graphs has canonical higher transformations. First we are going to describe these higher transformations as presheaves on appropriate small categories \(\mathcal{G}_n\), and then see that they form an \(\omega\)-graph that we call the \(\omega\)-graph of the \(\omega\)-graphs. It is the combinatoric description of these small categories \(\mathcal{G}_n\) which allows to have a straightforward proof of the proposition 1, which basically says that these higher transformations are algebras for adapted 2-coloured \(\omega\)-operads (see below).

Consider the globe category \(\mathcal{G}\)

\[
\begin{tikzpicture}
  \node (0) at (0,0) {$0$};
  \node (1) at (1,0) {$1$};
  \node (2) at (2,0) {$2$};
  \node (n-1) at (n-1,0) {$n-1$};
  \node (n) at (n,0) {$n$};
  \node (n+1) at (n+1,0) {$n+1$};
  \node (n+2) at (n+2,0) {$n+2$};
  \node (n+3) at (n+3,0) {$n+3$};
  \draw[->] (0) -- node[above] {$s^1_0$} (1);
  \draw[->] (1) -- node[above] {$s^1_1$} (2);
  \draw[->] (2) -- node[above] {$\cdots$} (n-1);
  \draw[->] (n-1) -- node[above] {$s^1_{n-1}$} (n);
  \draw[->] (n) -- node[above] {$\cdots$} (n+1);
  \draw[->] (n+1) -- node[above] {$\cdots$} (n+2);
  \draw[->] (n+2) -- node[above] {$\cdots$} (n+3);
  \end{tikzpicture}
\]

subject to relations\(^4\) on cosources \(s^{n+1}_n\) and cotargets \(t^{n+1}_n\). For each each \(n \geq 1\) we are going to build similar categories \(\mathcal{G}_n\) in order that we will obtain a coglobular complex in \(\mathcal{CAT}\)

\[
\begin{tikzpicture}
  \node (G0) at (0,0) {$\mathcal{G}_0$};
  \node (G1) at (1,0) {$\mathcal{G}_1$};
  \node (G2) at (2,0) {$\mathcal{G}_2$};
  \node (Gn-1) at (n-1,0) {$\mathcal{G}_{n-1}$};
  \node (Gn) at (n,0) {$\mathcal{G}_n$};
  \draw[->] (G0) -- node[above] {$\delta^1_0$} (G1);
  \draw[->] (G1) -- node[above] {$\delta^1_1$} (G2);
  \draw[->] (Gn-1) -- node[above] {$\delta^1_{n-1}$} (Gn);
  \draw[->] (Gn) -- node[above] {$\delta^1_n$} (G_n+1);
  \end{tikzpicture}
\]

\(^4\)Which are describe in the section 1 of [4].
where $G_0$ is just the globe category $G$. Each category $G_n$ is called the $n$-globe category. The 1-globe category $G_1$ is given by the category

![1-globe category](image)

such that $\alpha_0^{n+1} \circ s_n^{m+1} = s_n^{m+1} \circ \alpha_0^n$, $\alpha_0^{n+1} \circ t_n^{m+1} = t_n^{m+1} \circ \alpha_0^n$.

The 2-globe category $G_2$ is given by the category

![2-globe category](image)

where in particular we have an arrow $\xi_1 : \bar{I} \to 0$. Arrows $s_0^{n+1}$, $t_0^{n+1}$, $\alpha_0^n$, $\beta_0^n$, and $\xi_1$ satisfy the following relations

- $\alpha_0^{n+1} \circ s_n^{m+1} = s_n^{m+1} \circ \alpha_0^n$, $\alpha_0^{n+1} \circ t_n^{m+1} = t_n^{m+1} \circ \alpha_0^n$,

- $\beta_0^{n+1} \circ s_n^{m+1} = s_n^{m+1} \circ \beta_0^n$, $\beta_0^{n+1} \circ t_n^{m+1} = t_n^{m+1} \circ \beta_0^n$,

- $\xi_1 \circ s_0^1 = \alpha_0^0$ and $\xi_1 \circ t_0^1 = \beta_0^0$.

More generally the $n$-globe category $G_n$ is given by the category

![n-globe category](image)
where in particular we have an arrow \( \xi_{n-1} : n - 1 \to 0 \), and also for each \( 1 \leq p \leq n - 2 \), we have arrows \( \beta_p \xrightarrow{\alpha_p} 0 \). Arrows \( s_{n+1}^n, t_{n+1}^n, \alpha_0^n, \beta_0^n, \alpha_p, \beta_p, \) and \( \xi_{n-1} \) satisfy the following relations

- \( \alpha_0^{n+1} \circ s_{n}^{m+1} = s_{n}^{m+1} \circ \alpha_0^n, \alpha_0^{n+1} \circ t_{n}^{m+1} = t_{n}^{m+1} \circ \alpha_0^n, \)

- \( \beta_0^{n+1} \circ s_{n}^{m+1} = s_{n}^{m+1} \circ \beta_0^n, \beta_0^{n+1} \circ t_{n}^{m+1} = t_{n}^{m+1} \circ \beta_0^n. \)

- \( \alpha_p \circ s_{p-1}^p = \beta_p \circ s_{p-1}^p = \alpha_{p-1} \) and \( \alpha_p \circ t_{p-1}^p = \beta_p \circ t_{p-1}^p = \alpha_{p-1} \), and we put \( \alpha_0 := \alpha_0^0 \) and \( \beta_0 := \beta_0^0 \).

- \( \xi_{n-1} \circ s_{n-2}^{m-1} = \alpha_{n-2} \) and \( \xi_{n-1} \circ t_{n-2}^{m-1} = \beta_{n-2} \).

The cosources and cotargets functors \( G_0 \xrightarrow{\delta_0^1} G_1 \) are such that \( \delta_0^1 \) sends \( G_0 \) to \( G_0 \), and \( \kappa_0^1 \) sends \( G_0 \) to \( G_0' \). The cosources and cotargets functors \( G_1 \xrightarrow{\delta_1^2} G_2 \) send \( G_0 \) to \( G_0 \), and \( G_0' \) to \( G_0' \). Also \( \delta_1^2 \) sends the symbols \( \alpha_0^n \) to the symbols \( \alpha_0^n \), and \( \kappa_1^2 \) sends the symbols \( \alpha_0^n \) to the symbols \( \beta_0^n \).

Now consider the case \( n \geq 3 \). The cosources and cotargets functors

\[
\begin{array}{ccc}
G_{n-1} & \xrightarrow{\delta_{n-1}^n} & G_n \\
\kappa_{n-1}^n & \xrightarrow{\kappa_{n-1}^n} & G_n
\end{array}
\]

are build as follows : First we remove the cell \( \xi_{n-1} \) and the cell \( \beta_{n-2} \) from \( G_n \), and we obtain the category \( G_{n-1}^{-} \). Clearly we have an isomorphism of categories \( G_{n-1}^{-} \simeq G_{n-1}^{-} \) (which sends \( \alpha_{n-2} \) to \( \xi_{n-2} \)), and also the embedding \( G_{n-1}^{-} \xrightarrow{\delta_{n-1}^n} G_n^{-} \). The composition of this embedding with the last isomorphism gives \( G_{n-1}^{-} \xrightarrow{\delta_{n-1}^n} G_n^{-} \). The cotarget functor \( \kappa_{n-1}^n \).
is built similarly: First we remove the cell $\xi_{n-1}$ and the cell $\alpha_{n-2}$ from $G_n$, and we obtain the category $G^+_{n-1}$. Clearly we have an isomorphism of categories $G^+_{n-1} \simeq G_{n-1}$ (which sends $\beta_{n-2}$ to $\xi_{n-2}$), and also the embedding $G^+_{n-1} \xrightarrow{\kappa_{n-1}^n} G_n$. The composition of this embedding with the last isomorphism gives $G_{n-1} \xrightarrow{\kappa_{n-1}^n} G_n$. It is easy to see that these functors $\delta_{n-1}^n$ and $\kappa_{n-1}^n$ verify the cosource cotarget conditions as for the globe category $G_0$ above. When we applied the contravariant functor $[-;Set]$ to the following coglobular complex in $\text{CAT}$

\[
\begin{array}{c}
\delta_0^1 & \delta_2^1 & \delta_1^2 & \delta_{n-1}^n \\
\kappa_0^n & \kappa_1^n & \kappa_{n-1}^n & \\
G_0^\text{op} & G_1^\text{op} & G_2^\text{op} & G_{n-1}^\text{op} & G_n^\text{op}
\end{array}
\]

it is easy to see that we obtain the $\omega$-graph of the $\omega$-graphs.

An object of the category of presheaves $[G_n^\text{op};Set]$ is called an $(n,\omega)$-graphs. For instance, if $n \geq 3$, the source functor $\sigma_{n-1}^n$ is described as follow: If $X : G_n^\text{op} \rightarrow Set$ is an $(n,\omega)$-graph, thus $X(\xi_{n-1}) : X(\bar{0}) \rightarrow X(n-1)$ is its underlying $(n-1)$-transformation, and $\sigma_{n-1}^n(X)(\xi_{n-2}) : X(\bar{0}) \rightarrow X(n-2)$ is the underlying $(n-2)$-transformation of $\sigma_{n-1}^n(X)$, and is defined by $\sigma_{n-1}^n(X)(\xi_{n-2}) = X(s_{n-1}^n) \circ X(\xi_{n-1})$. The target functors $\beta_{n-1}^n$ can be also describe easily.

---

5 For each $n \in \mathbb{N}$, $[G_n^\text{op};Set](0)$ means the set of objects of the presheaf category $[G_n^\text{op};Set]$.

6 Do not confuse the $(n,\omega)$-graphs with the $(\omega,n)$-graphs that we have defined in [3], which are completely different object. In [3], $(\omega,n)$-graphs are an important kind of $\omega$-graphs, which play a central role to define an algebraic approach of the weak $(\omega,n)$-categories.
Now let’s come back to the coglobular complex $G^\bullet$ in $\text{T-Gr}_{p,c}$ build in the Introduction. If we apply to it the free functor $M : \text{T-Gr}_{p,c} \longrightarrow \text{T-CAT}_c$ (see [1]) we obtain a coglobular complex in $\text{T-CAT}_c$

\[
\begin{array}{ccccccc}
& \delta_0^1 & \delta_1^2 & & & \ldots & \delta_{n-1}^n \\
\sigma_0^1 & B_0^1 & B_1^1 & & & \ldots & B_{n-1}^n \\
\kappa_1^2 & B_1^2 & B_2^2 & & & \ldots & B_n^n \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

which produces the following globular complex in $\text{CAT}$

\[
\begin{array}{ccccccc}
B_0^0 & \delta_0^1 & \delta_1^2 & & & \ldots & \delta_{n-1}^n \\
& \lhd & \lhd & & & \lhd & \lhd \\
\beta_0^1 & B_0^1 & B_1^1 & & & \ldots & B_{n-1}^n \\
& \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

and we have the easy proposition

**Proposition 1** The category $B_{0,1d}-\text{Alg}$ is the category $[G_{0,1}^{\text{op}};\text{Set}]$ of the $\omega$-graphs, $B_{1,1d}-\text{Alg}$ is the category $[G_{1,1}^{\text{op}};\text{Set}]$ of the $(1,\omega)$-graphs, and for each integer $n \geq 2$, $B_{n,1d}-\text{Alg}$ is the category $[G_{n,1}^{\text{op}};\text{Set}]$ of the $(n,\omega)$-graphs.

Lets note $B_{1d}^\bullet$ this coglobular object in $\text{T-CAT}_c$. Its standard action is given by the following diagram in $\text{T-CAT}_1$

\[
\begin{array}{ccccccc}
\text{Coend}(B_{1d}^\bullet) & \text{Coend}(\text{Alg}(\cdot)) & \text{Coend}(A_{1d}^{\text{op}}) & \text{Coend}(\text{Ob}(\cdot)) & \text{End}(A_{0,1d}) \\
\text{Coend}(B_{1d}^\bullet) & \text{Coend}(\text{Alg}(\cdot)) & \text{Coend}(A_{1d}^{\text{op}}) & \text{Coend}(\text{Ob}(\cdot)) & \text{End}(A_{0,1d}) \\
\end{array}
\]

It is a standard action of the higher transformations specific to the basic $\omega$-graph structure. The monochromatic $\omega$-operad $\text{Coend}(B_{1d}^\bullet)$ of coendomorphism plays a central role for $\omega$-graphs, and we call it the *white operad*. Also it is straightforward that $B_{1d}^0$ has the fractal property, because it is initial in the category $\text{T-CAT}_1$ of the category of $\omega$-operads, thus we have a unique morphism of $\omega$-operads

\[
\begin{array}{ccccccc}
B_{1d}^0 & \lhd_{1d} & \text{Coend}(B_{1d}^\bullet) \\
\end{array}
\]

If we compose it with the standard action of the $\omega$-graphs

\[
\begin{array}{ccccccc}
\text{Coend}(B_{1d}^\bullet) & \text{Coend}(\text{Alg}(\cdot)) & \text{Coend}(A_{1d}^{\text{op}}) & \text{Coend}(\text{Ob}(\cdot)) & \text{End}(A_{0,1d}) \\
\end{array}
\]
we obtain a morphism of $\omega$-operads

$$B^0_{Id} \xrightarrow{\phi} End(A_{0,Id})$$

which express an action of the $\omega$-operad $B^0_{Id}$ of the $\omega$-graphs on the globular complex $B^*\mathcal{A}lg(0)$ in $SET$ of the $(n, \omega)$-graphs ($n \in \mathbb{N}$), and thus gives an $\omega$-graph structure on the $(n, \omega)$-graphs ($n \in \mathbb{N}$).

3 The coglobular complexes of the reflexive graphical $\omega$-operads

By basing on the globe category $\mathcal{G}_0$ (see the section 2), we build the reflexive globe category $\mathcal{G}_{0,r}$ as follow: For each $n \in \mathbb{N}$ we add in $\mathcal{G}_0$ the formal morphism $\xrightarrow{n+1} n$ such that $1_{n+1}^{n} \circ s^{n+1}_n = 1_{n+1}^{n} \circ t^{n+1}_n = 1_n$. For each $0 \leq p < n$ we denote $1_n^p := 1_{p+1}^p \circ 1_{p+2}^p \circ \ldots \circ 1_{n-1}^p$.

For each each $n \geq 1$ we are going to build similar categories $\mathcal{G}_{n,r}$ in order that we will obtain a coglobular complex in $\mathcal{C}AT$

equipped with coreflexivity functors $i_{n+1}^n$. Each category $\mathcal{G}_{n,r}$ is called the reflexive $n$-globe category. It is build as the categories $\mathcal{G}_n$ ($n \geq 1$) where we just replace $\mathcal{G}_0$ and $\mathcal{G}'_0$ by $\mathcal{G}_{0,r}$ and $\mathcal{G}'_{0,r}$.

For each $n \geq 1$, the cosources and the cotargets functors

are build as those

$$\mathcal{G}_{n-1,r} \xrightarrow{\delta_{n-1}^n} \mathcal{G}_{n,r} \xleftarrow{\kappa_{n-1}^n} \mathcal{G}_n$$
of the section $\mathbb{E}$ where in addition $\delta_1^1$ sends for all $p \geq 0$, the reflexivity morphism $1_{p+1}^p$ to the reflexivity morphism $1_{p+1}^p$, and $\kappa_1^0$ sends the reflexivity morphism $1_{p+1}^p$ to the reflexivity morphism $1_{p+1}^p$. Also, if $n \geq 2$, $\delta_{n+1}^n$ and $\kappa_{n+1}^n$ send for all $p \geq 0$, the reflexivity morphism $1_{p+1}^p$ to the reflexivity morphism $1_{p+1}^p$, and the reflexivity morphism $1_{p+1}^p$ to the reflexivity morphism $1_{p+1}^p$. It is trivial to see that these functors $\delta_{n-1}^n$ and $\kappa_{n-1}^n$ verify the cosourse and cotarget conditions.

For each $n \geq 1$, the coreflexivity functor

$$G_{n,r} \xrightarrow{\iota_n^{n-1}} G_{n-1,r}$$

is built as follow: the coreflexivity functor $\iota_1^0$ sends, for all $q \geq 0$, the object $\bar{q}$ to $\bar{q}$, the object $\bar{q'}$ to $\bar{q'}$, the cosource morphisms $s_q^{q+1}$ and $s_{q'}^{q+1}$ to $s_q^{q+1}$, the cotarget morphisms $t_q^{q+1}$ and $t_{q'}^{q+1}$ to $t_q^{q+1}$, the functor morphisms $\alpha_q^0$ to $1_{\bar{q}}$. Also the coreflexivity functor $\iota_2^1$ sends, for all $q \geq 0$, the object $\bar{q}$ to $\bar{q}$, the object $\bar{q'}$ to $\bar{q'}$, the cosource morphism $s_q^{q+1}$ to the cosource morphism $s_q^{q+1}$, the cosource morphism $s_{q'}^{q+1}$ to the cosource morphism $s_{q'}^{q+1}$, the cotarget morphism $t_q^{q+1}$ to the cotarget morphism $t_q^{q+1}$, the cotarget morphism $t_{q'}^{q+1}$ to the cotarget morphism $t_{q'}^{q+1}$, the functor morphisms $\alpha_q^0$ to the functor morphisms $\alpha_q^0$, the functor morphisms $\beta_q^0$ to the functor morphisms $\beta_q^0$, the natural transformation morphism $\xi_1$ to $\alpha_0^0 \circ 1_{\bar{q}}$.

Also for each $n \geq 3$, the coreflexivity functor $\iota_{n}^{n-1}$ sends, for all $q \geq 0$, the object $\bar{q}$ to $\bar{q}$, the object $\bar{q'}$ to $\bar{q'}$, the cosource morphism $s_q^{q+1}$ to the cosource morphism $s_q^{q+1}$, the cosource morphism $s_{q'}^{q+1}$ to the cosource morphism $s_{q'}^{q+1}$, the cotarget morphism $t_q^{q+1}$ to the cotarget morphism $t_q^{q+1}$, the cotarget morphism $t_{q'}^{q+1}$ to the cotarget morphism $t_{q'}^{q+1}$, the functor morphisms $\alpha_q^0$ to the functor morphisms $\alpha_q^0$, the functor morphisms $\beta_q^0$ to the functor morphisms $\beta_q^0$. Also if $0 \leq p \leq n - 3$, it sends the $p$-transformation $\alpha_p$ to the $p$-transformation.
\(\alpha_p\), the \(p\)-transformation \(\beta_p\) to the \(p\)-transformation \(\beta_{p}^{1}\), the \((n-2)\)-transformation \(\alpha_{n-2}\) and \(\beta_{n-2}\) to the \((n-2)\)-transformation \(\xi_{n-2}\), and finally the \((n-1)\)-transformation \(\xi_{n-1}\) to \(\xi_{n-2} \circ 1_{n-1}^{n-2}\).

With this construction it is not difficult to show that functors \(i_{n}^{n-1}\) \((n \geq 1)\) verify the coreflexivity identities

\[i_{n}^{n-1} \circ \delta_{n-1}^{n} = 1_{G_{n-1}} = i_{n}^{n-1} \circ \kappa_{n-1}^{n}\]

When we applied the contravariant functor \([-; Set]\) to the following coglobular complex in \(\text{CAT}\)

\begin{equation*}
\begin{array}{cccccc}
G_{0,r}^{\text{op}} & \xrightarrow{i_{0}^{0}} & G_{1,r}^{\text{op}} & \xrightarrow{i_{1}^{1}} & G_{2,r}^{\text{op}} & \xrightarrow{i_{2}^{2}} & \cdots & \xrightarrow{i_{n-1}^{n-1}} & G_{n,r}^{\text{op}}
\end{array}
\end{equation*}

it is easy to see that we obtain the reflexive \(\omega\)-graph of the reflexive \(\omega\)-graphs

\begin{equation*}
\begin{array}{cccccc}
\cdots & \xrightarrow{\alpha_{n-1}^{n}} & [G_{n,r}^{\text{op}}; Set](0) & \xrightarrow{\beta_{n-1}^{0}} & [G_{n-1,r}^{\text{op}}; Set](0) & \xrightarrow{\sigma_{1}^{1}} & [G_{1,r}^{\text{op}}; Set](0) & \xrightarrow{\sigma_{0}^{0}} & [G_{0,r}^{\text{op}}; Set](0)
\end{array}
\end{equation*}

An object of the category of presheaves \([G_{n,r}^{\text{op}}; Set]\) is called a reflexive \((n, \omega)\)-graphs. For instance, if \(n \geq 3\), the reflexivity functor \(i_{n}^{n-1}\) can be described as follow : If \(X : G_{n-1}^{\text{op}} \to Set\) is an \((n-1, \omega)\)-graph, thus \(X(\xi_{n-2}) : X(0) \to X(n-2)\) is its underlying \((n-2)\)-transformation, and \(i_{n}^{n-1}(X)(\xi_{n-1}) : X(0) \to X(n-1)\) is the \((n-1)\)-transformation defined by : \(i_{n}^{n-1}(X)(\xi_{n-1}) = X(1_{n-1}^{n-2}) \circ X(\xi_{n-2})\).

\(^7\)By convention we put \(\alpha_{0} = \alpha_{0}^{0}\) and \(\beta_{0} = \beta_{0}^{0}\). In fact this convention is natural because in our point of view of \(n\)-transformations, 1-transformations are the usual natural transformations, and a 0-transformation must be seen a the underlying function \(F_{0}\) acting on the 0-cells of a functor \(F\).
Now consider the coglobular complex $G^\bullet$ in $\mathbb{T}$-Gr$^{p,c}$ as in the Introduction. If we apply the free functor $\text{Id}_u : \mathbb{T}$-Gr$^{p,c} \rightarrow \mathbb{T}$-$\text{CAT}_c$ (see [1]) to it we obtain a coglobular complex in $\mathbb{T}$-$\text{CAT}_c$

\[
\begin{align*}
B^0_{1\text{Id}_u} &\xrightarrow{\delta_0^1} B^1_{1\text{Id}_u} &\xrightarrow{\delta_1^2} B^2_{1\text{Id}_u} &\cdots &\xrightarrow{\delta_{n-1}^n} B^n_{1\text{Id}_u} \\
B^0_{1\text{Id}_u} &\xrightarrow{\kappa_0^1} B^1_{1\text{Id}_u} &\xrightarrow{\kappa_1^2} B^2_{1\text{Id}_u} &\cdots &\xrightarrow{\kappa_{n-1}^n} B^n_{1\text{Id}_u}
\end{align*}
\]

which produces the following globular complex in $\text{CAT}$

\[
\begin{align*}
B^0_{1\text{Id}_u} &\rightarrow \text{Alg} \xrightarrow{\beta_{n-1}^n} B^{n-1}_{1\text{Id}_u} &\rightarrow \text{Alg} \xrightarrow{\beta_{n-1}^n} B^1_{1\text{Id}_u} &\rightarrow \text{Alg} \xrightarrow{\beta_0^1} B^0_{1\text{Id}_u} &\rightarrow \text{Alg}
\end{align*}
\]

and we have the easy proposition

**Proposition 2** The category $B^0_{1\text{Id}_u}$-$\text{Alg}$ is the category $[G^\text{op}_{0,\omega}, \text{Set}]$ of the reflexive $\omega$-graphs, $B^1_{1\text{Id}_u}$-$\text{Alg}$ is the category $[G^\text{op}_{1,\omega}, \text{Set}]$ of the reflexive $(1, \omega)$-graphs, and for each integer $n \geq 2$, $B^n_{1\text{Id}_u}$-$\text{Alg}$ is the category $[G^\text{op}_{n,\omega}, \text{Set}]$ of the reflexive $(n, \omega)$-graphs.

\[\square\]

Let's note $B^\bullet_{1\text{Id}_u}$ this coglobular object in $\mathbb{T}$-$\text{CAT}_c$. According to the section [1] we obtain the following diagram in $\text{CAT}_1$

\[
\begin{align*}
\text{Coend}(B^\bullet_{1\text{Id}_u}) &\xrightarrow{\text{Coend}([\text{Alg}(\cdot)])} \text{Coend}(A^\text{op}_{1\text{Id}_u}) &\xrightarrow{\text{Coend}([\text{Ob}(\cdot)])} \text{Coend}(A^\text{op}_{0,1\text{Id}_u})
\end{align*}
\]

that we call the standard action of the reflexive $\omega$-graphs, thus which is a specific standard action. The monochromatic $\omega$-operad $\text{Coend}(B^\bullet_{1\text{Id}_u})$ of coendomorphism plays a central role for reflexive $\omega$-graphs, and we call it the blue operad. Also we have the following proposition

**Proposition 3** $B^0_{1\text{Id}_u}$ has the fractal property.

\[\square\]

**PROOF** The units of the $\omega$-operad $\text{Coend}(B^\bullet_{1\text{Id}_u})$ are given by identity morphisms $B^n_{1\text{Id}_u} \xrightarrow{1_{B^n_{1\text{Id}_u}}} B^n_{1\text{Id}_u}$. We are going to exhibit a morphism of $\omega$-operads

\[
B^n_{1\text{Id}_u} \xrightarrow{[1_{B^n_{1\text{Id}_u}} \cdot 1_{B^n_{1\text{Id}_u}}]_{n+1}} B^n_{1\text{Id}_u}
\]

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which is the contractibility of the unit $1_{B_{Idu}}$ with itself.

First consider the morphism of $G^{n+1} \xrightarrow{c_{n+1}^n} B_{Idu}^n$ of $\mathbb{T}$-$\mathcal{G}r_c$, which sends $u_m$ to $u_m$, $v_m$ to $v_m$, $\alpha_0^m$ to $\alpha_0^m$, $\beta_0^m$ to $\beta_0^m$, $\alpha_p$ to $\alpha_p$, $\beta_p$ to $\beta_p$, $\alpha_n$ to $\xi_n$, $\beta_n$ to $\xi_n$, and $\xi_{n+1}$ to $\gamma ([u_n; u_n]_n + 1 ; 1 \xi_n)$. This map $c_{n+1}^n$ equipped $B_{Idu}^n$ with an operation system of the type $G^{n+1}$ and $B_{Idu}^n$ has contractible units, so by the universality of the map $\eta_{n+1}$, we get a unique morphism of $\omega$-operads $[1_{B_{Idu}^n} ; 1_{B_{Idu}^n}]_{n+1}$.

This $(n+1)$-cell $[1_{B_{Idu}^n} ; 1_{B_{Idu}^n}]_{n+1}$ has arity the degenerate tree $1^n_{n+1}([n])$. Now we just need to prove that the following diagram commute serially, which shows that source and target of $[1_{B_{Idu}^n} ; 1_{B_{Idu}^n}]_{n+1}$ is the unit $1_{B_{Idu}^n}$.

But we have the following diagram which, on the left side commute serially, and on the right side commute.

The morphism $c_{n+1}^n$ is a morphism of $\mathbb{T}$-$\mathcal{G}r_{p,c}$, and also the morphisms $\delta_{n+1}^n$ and $\kappa_{n+1}^n$ on the bottom of this diagram. Their combinatorial descriptions show easily that we have the equalities $c_{n+1}^n \circ \delta_{n+1}^n = c_{n+1}^n \circ \kappa_{n+1}^n$.  

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\( \kappa_n^{n+1} = \eta_n \). So we have the equalities 
\[ [1_{B^i_{Idu}} : 1_{B^i_{Idu}}]_n^n \circ \delta_n^{n+1} = [1_{B^i_{Idu}} : 1_{B^i_{Idu}}]_n^n \circ \kappa_n^{n+1} = 1_{B^i_{Idu}}. \]
It shows that the \( \omega \)-operad \( \text{Coend}(B^i_{Idu}) \) has contractible units, and thus we have a unique morphism of \( \omega \)-operads

\[
B^0_{Idu} \xrightarrow{!_{Idu}} \text{Coend}(B^i_{Idu})
\]
which express the fractality of \( B^0_{Idu} \).

If we compose the morphism \(!_{Idu}\) with the standard action of the reflexive \( \omega \)-graphs

\[
\text{Coend}(B^i_{Idu}) \xrightarrow{\text{Coend}(\text{Alg}(\cdot))} \text{Coend}(A^0_{Idu}) \xrightarrow{\text{Coend}(\text{Ob}(\cdot))} \text{Coend}(A_{0,Idu})^{op}
\]
we obtain a morphism of \( \omega \)-operads

\[
B^0_{Idu} \xrightarrow{\varphi_r} \text{End}(A_{0,Idu})
\]
which express an action of the \( \omega \)-operad \( B^0_{Idu} \) of the reflexive \( \omega \)-graphs on the globular complex \( B^i_{Idu} \cdot \text{Alg}(0) \) in \( \text{SET} \) of the reflexive \((n, \omega)\)-graphs \((n \in \mathbb{N})\), and thus gives a reflexive \( \omega \)-graph structure on the reflexive \((n, \omega)\)-graphs \((n \in \mathbb{N})\).

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