SOME APPLICATIONS OF TRANSVERSALITY FOR INFINITE DIMENSIONAL MANIFOLDS

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Abstract. We present some transversality results for a category of Fréchet manifolds, the so-called $MC^k$-Fréchet manifolds. In this context, we apply the obtained transversality results to construct the degree of nonlinear Fredholm mappings by virtue of which we prove a rank theorem, an invariance of domain theorem and a Bursuk-Ulam type theorem.

This paper is devoted to the development of transversality and its applications to degree theory of nonlinear Fredholm mappings for non-Banachable Fréchet manifolds. The elaboration is mostly, but not entirely, routine; we shall discuss the related issues.

In attempting to develop transversality to Fréchet manifolds we face the following drawbacks which are related to lack of a suitable topology on a space of continuous linear maps:

1. In general, the set of isomorphisms between Fréchet spaces is not open in the space of continuous linear mappings.
2. In general, the set of Fredholm operators between Fréchet spaces is not open in the space of continuous linear mappings.

Also, a key point in the proof of an infinite dimensional version of Sard’s theorem is that a Fredholm mapping $\varphi$ near origin has a local representation of the form $\varphi(u,v) = (u, \eta(u,v))$ for some smooth mapping $\eta$; indeed, this is a consequence of an inverse function theorem.

To obtain a version of Sard’s theorem for Fréchet manifolds, based on the ideas of Müller [4], it was proposed by the author ([1]) to consider Fredholm operators which are Lipschitz on their domains. There is an appropriate metrizable topology on a space of Lipschitz linear mappings so that if we employ this space instead of a space of continuous linear mappings, the mentioned openness issues and the problem of stability of Fredholm mappings under small perturbation can be resolved. Furthermore, for mappings belong to a class of differentiability, bounded or $MC^k$-differentiability which is introduced in [4], a suitable version of an inverse function theorem is available, [4, Theorem 4.7].

An example of Lipschitz-Fredholm mapping of class $MC^k$ can be found in [3], where the Sard’s theorem [1, Theorem 4.3] is applied to classify all the holomorphic functions locally definable; this gives the additional motivation to study further applications of Sard’s theorem.

In this paper, first we improve the transversality theorem [2, Theorem 4.2] by considering all mappings of class $MC^k$, then use it to prove the parametric transversality theorem. Then, for Lipschitz-Fredholm mappings of class $MC^k$ we apply the transversality theorem to construct the degree (due to Cacciappoli, Shvarts and Smale), which is defined as the group of non-oriented cobordism class of $\varphi^{-1}(q)$ for some regular value $q$.

We then prove a rank theorem for Lipschitz-Fredholm mappings of class $MC^k$, and use it to prove an invariance of domain theorem and a Fredholm alternative theorem. Also, using the parametric transversality theorem we obtain a Bursuk-Ulam type theorem.

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1. BOUNDED FRÉCHET MANIFOLDS

In this section, we shall briefly recall the basics of $MC^k$-Fréchet manifolds for the convenience of readers, which also allows us to establish our notations for the rest of the paper. For more studies, we refer to [1, 2].

Throughout the paper we assume that $E, F$ are Fréchet spaces and $CL(E, F)$ is the space of all continuous linear mappings from $E$ to $F$ topologized by the compact-open topology. If $T$ is a topological space by $U \subseteq T$ we mean $U$ is open in $T$.

Let $\varphi : U \subseteq E \to F$ be a continuous map. If the directional (Gâteaux) derivatives

$$D \varphi(x)h = \lim_{t \to 0} \frac{\varphi(x + th) - \varphi(x)}{t}$$

exist for all $x \in U$ and all $h \in E$, and the induced map $D \varphi(x) : U \to CL(E, F)$ is continuous for all $x \in U$, then we say that $\varphi$ is a Keller’s differentiable map of class $C_1^k$. The higher directional derivatives and $C^k$-mappings, $k \geq 2$, are defined in the obvious inductive fashion.

To define bounded or $MC^k$-differentiability, we endow a Fréchet space $F$ with a translation invariant metric $\varrho$ defining its topology, and then introduce the metric concepts which strongly depend on the choice of $\varrho$. We consider only metrics of the following form

$$\varrho(x, y) = \sup_{n \in \mathbb{N}} \frac{1}{2^n} \left( \sup_{1 \leq q \leq n} \frac{\|x - y\|_{F,n}}{\|x - y\|_{F,n}} \right),$$

where $\|\|_{F,n}$ is a collection of seminorms generating the topology of $F$.

Let $\sigma$ be a metric that defines the topology of a Fréchet space $E$. Let $\mathbb{L}_{\sigma,\varrho}(E, F)$ be the set of all linear mappings $L : E \to F$ which are (globally) Lipschitz continuous as mappings between metric spaces $E$ and $F$, that is

$$Lip(L) := \sup_{x \in E \setminus \{0_F\}} \frac{\varrho(L(x), 0_F)}{\sigma(x, 0_F)} < \infty,$$

where $Lip(L)$ is the (minimal) Lipschitz constant of $L$.

The translation invariant metric

$$d_{\sigma,\varrho} : \mathbb{L}_{\sigma,\varrho}(E, F) \times \mathbb{L}_{\sigma,\varrho}(E, F) \longrightarrow [0, \infty), \ (L, H) \mapsto Lip(L - H)_{\sigma,\varrho},$$

on $\mathbb{L}_{\sigma,\varrho}(E, F)$ turns it into an Abelian topological group. We always topologize the space $\mathbb{L}_{\sigma,\varrho}(E, F)$ by the metric (1.1).

Let $\varphi : U \subseteq E \to F$ be a continuous map. If $\varphi$ is Keller’s differentiable, $D \varphi(x) \in \mathbb{L}_{\sigma,\varrho}(E, F)$ for all $x \in U$ and the induced map $D \varphi(x) : U \to \mathbb{L}_{\sigma,\varrho}(E, F)$ is continuous, then $\varphi$ is called bounded differentiable or $MC^1$ and we write $\varphi^{(1)} = \varphi'$. We define for $k > 1$ mappings of class $MC^k$, recursively.

An $MC^k$-Fréchet manifold is a Hausdorff second countable topological space modeled on a Fréchet space with an atlas of coordinate charts such that the coordinate transition functions are all $MC^k$-mappings. We define $MC^k$-mappings between Fréchet manifolds as usual. Henceforth, we assume that $M$ and $N$ are connected $MC^k$-Fréchet manifolds modeled on Fréchet spaces $(F, \varrho)$ and $(E, \sigma)$, respectively.

A mapping $\varphi \in \mathbb{L}_{\sigma,\varrho}(E, F)$ is called Lipschitz-Fredholm operator if its kernel has finite dimension and its image has finite co-dimension. The index of $\varphi$ is defined by

$$\text{Ind } \varphi = \dim \ker \varphi - \text{codim } \text{img } \varphi.$$
We denote by $\mathcal{LF}(E, F)$ the set of all Lipschitz-Fredholm operators, and by $\mathcal{LF}_i(E, F)$ the subset of $\mathcal{LF}(E, F)$ consisting of those operators of index $i$.

An $MC^k$-Lipschitz-Fredholm mapping $\varphi : M \to N$, $k \geq 1$, is a mapping such that for each $x \in M$ the derivative $D\varphi(x) : T_xM \to T_{\varphi(x)}N$ is a Lipschitz-Fredholm operator. The index of $\varphi$, denoted by Ind $\varphi$, is defined to be the index of $D\varphi(x)$ for some $x$ which does not depend on the choice of $x$, see [1, Definition 3.2].

Let $\varphi : M \to N$ be an $MC^k$-mapping. We denote by $T_x\varphi : T_xM \to T_{\varphi(x)}N$ the tangent map of $\varphi$ at $x \in M$. The corresponding value $f$ of the range $\text{Img} x$ called an embedding. A point $\varphi$ regular are called critical points and values, respectively.

Let $\varphi : M \to N$ be an $MC^k$-mapping, $k \geq 1$. We say that $\varphi$ is transversal to a submanifold $S \subseteq N$ and write $\varphi \pitchfork S$ if either $\varphi^{-1}(S) = \emptyset$, or if for each $x \in \varphi^{-1}(S)$

1. $(T_x\varphi)(T_xM) + T_{\varphi(x)}N = T_{\varphi(x)}N$,
2. $(T_x\varphi)^{-1}(T_{\varphi(x)}S)$ splits in $T_xM$.

In terms of charts, $\varphi \pitchfork S$ when $x \in \varphi^{-1}(S)$ there exist charts $(\phi, U)$ around $x$ and $(\psi, V)$ around $\varphi(x)$ such that

$$\psi : V \to V_1 \times V_2$$

is an $MC^k$-isomorphism on a product, with

$$\psi(\varphi(x)) = (0_E, 0_E), \quad \varphi(S \cap V) = V_1 \times \{0_E\}.$$  

Then the composite mapping

$$U \xrightarrow{\varphi} V \xrightarrow{\psi} V_1 \times V_2 \xrightarrow{\text{Pr}_{V_2}} V_2.$$  

is an $MC^k$-submersion, where $\text{Pr}_{V_2}$ is the projection onto $V_2$.

2. **Transversality theorems**

We generalize [2, Theorem 4.2] and [2, Corollary 4.1] for not necessarily Lipschitz-Fredholm mappings and finite dimensional submanifolds. We shall need the following version of the inverse function theorem for $MC^k$-mappings.

**Theorem 2.1.** [4, Theorem 4.7] Let $U \subseteq E$, $u_0 \in U$ and $\varphi : U \to E$ an $MC^k$-mapping, $k \geq 1$. If $\varphi'(u_0) \in \text{Aut}(E)$, then there exists $V \subseteq U$ of $u_0$ such that $\varphi(V)$ is open in $E$ and $\varphi|_V : V \to \varphi(V)$ is an $MC^k$-diffeomorphism.

**Proposition 2.1.** Let $\varphi : M \to N$ be an $MC^k$-mapping, $S \subseteq N$ an $MC^k$-submanifold and $x \in \varphi^{-1}(S)$. Then $\varphi \pitchfork S$ if and only if there are charts $(U, \phi)$ around $x$ with $\phi(x) = 0_E$ and $(V, \psi)$ around $y = \varphi(x)$ in $S$ with $\psi(y) = 0_F$ such that the following hold:

1. There are subspaces $E_1$ and $E_2$ of $E$, and $F_1$ and $F_2$ of $F$ such that $E = E_1 \oplus E_2$ and $F = F_1 \oplus F_2$. Moreover, $\psi(S \cap V) = F_1$ and

$$\phi(U) = E_1 + E_2 \subseteq E_1 \oplus E_2$$

$$\psi(V) = F_1 + F_2 \subseteq F_1 \oplus F_2,$$

where $0_E \in E_i \subseteq E_i$ and $0_F \in F_i \subseteq F_i$, $i = 1, 2$.  

(2) In the charts the local representative of $\varphi$ has the form
\[ \varphi_{\phi^I} = \varphi + \phi \circ \text{Pr}_{E_2}, \]
where $\varphi : E_1 + E_2 \to F_1$ is an $MC^k$-mapping, $\hat{\phi}$ is an $MC^k$-isomorphism of $E_2$ onto $F_2$ and $\text{Pr}_{E_2} : E \to E_2$ is the projection.

**Proof. Sufficiency:** Let $(U, \phi)$ and $(V, \psi)$ be charts that satisfy the assumptions we will prove $\varphi \cap S$.

In the charts, by using the identifications $T_x M \simeq E$, $T_y N \simeq F$, the tangent map $T_x \varphi : T_x M \to T_y N$ has the representation
\[ T_x \varphi = \varphi_{\phi^I}(0_E) : E \to F. \]
Also, we have the identification $T_y N \simeq F_1$.

Let $\text{Pr}_{F_2} : F \to F_2$ be the projection onto $F_2$. Since $\varphi_{\phi^I}(0_E) = (\varphi')' + \phi \circ \text{Pr}_{E_2}$ and $(\varphi')'(0) : E \to F_1$, it follows that for all $e \in E$, $e = e_1 + e_2 \in E_1 \oplus E_2$
\[ \varphi_{\phi^I}(0_E)e = (\varphi')'(0_E)e + \hat{\phi} \circ \text{Pr}_{E_2}. \]
Thus, $\text{Pr}_{F_2} \circ \varphi_{\phi^I}(0_E)e = \text{Pr}_{F_2} \circ \hat{\phi} \circ \text{Pr}_{F_2}(e)$ which means
\[ \text{Pr}_{F_2} \circ \varphi_{\phi^I}(0_E) = \text{Pr}_{F_2} \circ \hat{\phi} \circ \text{Pr}_{F_2}, \]
it is a surjective mapping of $E$ onto $F_2$.

Moreover, we have
\[ \ker(\text{Pr}_{F_2} \circ \varphi_{\phi^I}(0_E)) = \varphi_{\phi^I}(0_E)^{-1}(\text{Pr}_{F_2}(0_E)) = \varphi_{\phi^I}(0_E)^{-1}(F_1). \]
\[ = \{ e = e_1 + e_2 \in E_1 \oplus E_2 \mid (\varphi')'(0_E)e + \hat{\phi}(e_2) \in F_1 \} \]
\[ = \{ e = e_1 + e_2 \in E_1 \oplus E_2 \mid \hat{\phi}(e_2) = 0_F \} \]
\[ = \{ e = e_1 + e_2 \in E_1 \oplus E_2 \mid e_2 = 0_E \} \]
\[ = E_1. \]
Which is an $MC^k$-splitting in $E$ with a component $E_2$. From (3) it follows that
\[ \text{Pr}_{F_2} \circ \varphi_{\phi^I}(0_E) = \hat{\varphi} : \text{Pr}_{E_2} \to \text{Pr}_{F_2}, \]
which is an $MC^k$-isomorphism.

**Necessity:** Suppose $\varphi \cap S$. Since $S$ is an $MC^k$-submanifold of $N$ and $y = \varphi(x) \in S$, there is a chart $(W, w)$ around $y$ having the submanifold property for $S$ in $N$:
\[ w(W) = W_1 + W_2 \subset F_1 \oplus F_2 = F, \]
\[ w(S \cap W) = W_1 \subset F, \quad w(y) = 0_E. \]
Also, there is a chart $(X, x)$ around $x$ such that $x(x) = 0_F$, $\varphi(X) \subset W$ and
\[ \varphi_{\text{xxw}} : x(X) \subset E \to w(W) \subset F \]
is of class $MC^k$. It follows that $\varphi_{\text{xxw}}(0_E) \circ \text{Pr}_{F_2} : E \to F_2$ is an $MC^k$-submersion as $\varphi \cap S$.
That is, $\varphi_{\text{xxw}} \circ \text{Pr}_{F_2}$ and $E_1 := \varphi_{\text{xxw}}(0_E)^{-1}(F_1)$ splits in $E$ with the complement $E_2$ such that
\[ \eta := \text{Pr}_{F_2} \circ \varphi_{\text{xxw}}(0_E) |_{E_2} : E_2 \to F_2 \]
is an $MC^k$-isomorphism. Set $\tau := (\text{Pr}_{F_1} + \eta^{-1} \circ \text{Pr}_{F_2} \circ \varphi_{\text{xxw}}) : x(X) \to w(W)$, then $\tau$ is an $MC^k$-mapping and $\tau(0_E) = 0_E$ and $\tau'(0_E) = (\text{Pr}_{F_1} + \eta^{-1} \circ \text{Pr}_{F_2} \circ \varphi_{\text{xxw}})'(0_E) = \text{Pr}_{E_1} + \text{Pr}_{E_2} = \text{Id}_E$.
Because, for all $e = e_1 + e_2 \in E_1 \oplus E_2$ we have $(\varphi_{\text{xxw}})'(0_E)e = (\varphi_{\text{xxw}})'(0_E)e_1 + (\varphi_{\text{xxw}})'(0_E)e_2$. 
Therefore, \( \overline{\text{Theorem 2.2}} \) (Transversality Theorem) Assume 0 \( E \in \mathcal{X} \subseteq \mathfrak{M}(\mathcal{X}) \) is small enough. Let 

\[
\mathbf{x}_1 : \mathcal{X}_1 \to 0_E \in \mathcal{X}_2 \subseteq E
\]

be an \( MC^k \)-diffeomorphism such that

\[
\tau \circ \mathbf{x}_1^{-1} = \text{Id}_F. \tag{2.3}
\]

Thus,

\[
\text{Pr}_F \circ \tau \circ \varphi_{\mathbf{x}_2} = \eta \circ \text{Pr}_{E_2}. \tag{2.4}
\]

If, \( e = e_1 + e_2 \in \mathbf{x}_1(\mathcal{X}_1) \) and \( \mathbf{x}_1^{-1}(e) = \overline{e}_1 + \overline{e}_2 \), then by (2.3) and (2.4) we obtain

\[
\tau \circ \mathbf{x}_1^{-1}(e) = \tau(\overline{e}_1 + \overline{e}_2) = \overline{e}_1 + \eta^{-1} \circ \text{Pr}_F \circ \varphi_{\mathbf{x}_2}(\overline{e}_1 + \overline{e}_2) = e_1 + e_2.
\]

Therefore, \( \overline{e}_1 = e_1 \) and \( \tau^{-1} \circ \text{Pr}_F \circ \varphi_{\mathbf{x}_2} = e_2 \) and so

\[
\text{Pr}_F \circ \varphi_{\mathbf{x}_2}(\overline{e}_1 + \overline{e}_2) = \tau(e_2) = \tau \circ \text{Pr}_{E_2}(e_1 + e_2).
\]

This means, \( \text{Pr}_{E_2} \circ \varphi_{\mathbf{x}_2} \circ \mathbf{x}_1^{-1}(e) = \eta \circ \text{Pr}_{E_2}(e) \) for all \( e \in \mathbf{x}_1(\mathcal{X}_1) \). Now, define

\[
\phi := \mathbf{x}_1 \circ \mathbf{x}, \quad \mathcal{U} := \mathbf{x}^{-1}(\mathcal{X}),
\]

\[
\psi := \mathbf{w}, \quad \mathcal{V} := \text{small enough neighborhood of } y \text{ in } \mathcal{W}.
\]

Then, \( (\phi, \mathcal{U}) \) and \( (\psi, \mathcal{V}) \) are the desired charts. Indeed,

\[
\varphi_{\phi\psi} = \mathbf{w} \circ \varphi \circ (\mathbf{x}_1 \circ \mathbf{x}) = \mathbf{w} \circ \varphi \circ \mathbf{x}^{-1} \circ \mathbf{x}^{-1} = \varphi_{\mathbf{x}_2} \circ \mathbf{x}_1^{-1}.
\]

Thus,

\[
\varphi_{\phi\psi} = \text{Pr}_{F_1} \circ \varphi_{\mathbf{x}_2} \circ \mathbf{x}_1^{-1} + \text{Pr}_F \circ \varphi_{\mathbf{x}_2} \circ \mathbf{x}_1^{-1} = \overline{\varphi} + \hat{\phi} \circ \text{Pr}_{E_2},
\]

if we set \( \hat{\phi} := \eta \) and \( \overline{\varphi} := \text{Pr}_{E_1} \circ \varphi_{\mathbf{x}_2} \circ \mathbf{x}_1^{-1} \). \qed

**Theorem 2.2** (Transversality Theorem). Let \( \varphi : M \to N \) be an \( MC^k \)-mapping, \( k \geq 1 \), \( S \subset N \) an \( MC^k \)-submanifold and \( \varphi \pitchfork S \). Then, \( \varphi^{-1}(S) \) is either empty of \( MC^k \)-submanifold of \( M \) with

\[
(T_x\varphi)^{-1}(T_yS) = T_x(\varphi^{-1}(S)), \quad x \in \varphi^{-1}(S), \quad y = \varphi(x).
\]

If \( S \) has finite co-dimension in \( N \), then \( \text{codim}(\varphi^{-1}(S)) = \text{codim} S \). Moreover, if \( \text{dim } S = m < \infty \) and \( \varphi \) is an \( MC^k \)-Lipschitz-Fredholm mapping of index \( l \), then \( \text{dim } \varphi^{-1}(S) = l + m \).

**Proof.** Let \( x \in \varphi^{-1}(S) \), then by Proposition 2.1 there are chart \( (\phi, \mathcal{U}) \) around \( x \) and \( (\psi, \mathcal{V}) \) around \( y = \varphi(x) \) such that

\[
\phi(\mathcal{U}) = E_1 + E_2 \subseteq E_1 \oplus E_2,
\]

\[
\psi(\mathcal{V}) = F_1 + F_2 \subseteq F_1 \oplus F_2,
\]

\[
\varphi_{\phi\psi} = \overline{\varphi} + \hat{\phi} \circ \text{Pr}_{E_2}, \tag{2.5}
\]
where \( \varphi : E_1 + E_2 \to F_1 \) is an \( MC^k \)-mapping, \( \hat{\varphi} \) is an \( MC^k \)-isomorphism of \( E_2 \) onto \( F_2 \) and \( \Pr_{E_2} : E \to E_2 \) is the projection.

Let \( \hat{e} \in \varphi^{-1}(S) \cap \mathcal{U} \), then \( \hat{f} = \varphi(\hat{e}) \in S \cap \mathcal{V} \) and \( \psi(\varphi(\hat{e})) \in F_1 \subset F \). By (2.5), if \( \hat{e} = e_1 + e_2 \in E_1 + E_2 \) we have

\[
\varphi_{\hat{\varphi}}(\hat{e}) = \varphi_{\hat{\varphi}}(e_1 + e_2) = \varphi(e_1 + e_2) = \varphi_{\hat{\varphi}}(e_1) + \hat{\varphi}(e_2) \in E_1 \subset F_1.
\]

It follows \( \hat{\varphi}(e_2) = 0_{E}, e_2 = 0_{E} \), since \( \hat{\varphi}(e_2) \in F_2 \) and \( F_1 \cap F_2 = \{0_F\} \).

Thus, \( \hat{e} \in E_1 \) for all \( \hat{e} \in \varphi^{-1}(S) \cap \mathcal{U} \). Therefore, \( E_1 \subset \phi(\varphi^{-1}(S) \cap \mathcal{U}) \), since for each \( e_1 \in E_1 \) we have

\[
\varphi_{\hat{\varphi}}(e_1) = \varphi(e_1) + \hat{\varphi}(e_2) = \varphi(e_1) = \overline{\varphi}(e_1) \in E_1.
\]

Hence, \( \psi \circ \varphi \circ \varphi^{-1}(e_1) \in F_1 \) implies that \( \varphi \circ \varphi^{-1}(e_1) \in \psi^{-1}(F_1) = S \cap \mathcal{V} \) and so \( \varphi \circ \varphi^{-1}(e_1) \) which means \( \varphi^{-1}(e_1) \in \varphi(S) \cap \mathcal{V} \) that yields \( e_1 \in \psi(\varphi^{-1}(S) \cap \mathcal{V}) \). Therefore, for \( x \in \varphi^{-1}(F) \) there is a chart \( (\phi, \mathcal{U}) \) with \( \phi(\mathcal{U}) = E_1 + E_2 \subset E_1 \oplus E_2 \) and \( \phi(x) = 0_E, \phi(\varphi^{-1}(S) \cap \mathcal{V}) = E_1 \), which means \( \varphi^{-1}(F) \) is an \( MC^k \)-submanifold in \( M \).

In the charts, we have \( T_x \simeq E, T_y N \simeq F, T_x(\varphi^{-1}(S)) \simeq E_1 \) and \( T_y S \simeq F_1 \). From the proof of Proposition 2.1 we have

\[
\varphi_{\hat{\varphi}}(0_E)^{-1}(F_1) = E_1
\]

which yields \( (T_x \varphi)^{-1}(T_y S) = T_x(\varphi^{-1}(S)) \).

If \( S \) has finite co-dimension then \( F_2 \) has finite dimension and thus by Proposition 2.1,

\[
\text{codim(} \varphi^{-1}(S) \text{)} = \text{codim} \varphi^{-1}(S \cap \mathcal{V}) = \text{dim}(F_2) = \text{dim}(S).
\]

The proof of the last statement is standard. \( \square \)

As an immediate consequence we have:

**Corollary 2.1.** Let \( \varphi : M \to N \) be an \( MC^k \)-mapping, \( k \geq 1 \). If \( q \) is a regular value of \( \varphi \), then the level set \( \varphi^{-1}(q) \) is a submanifold of \( M \) and its tangent space at \( p = \varphi(q) \) is \( \ker T_{\varphi} \). Moreover, if \( q \) is a regular value of \( \varphi \) and \( \varphi \) is an \( MC^k \)-Lipschitz-Fredholm mapping of index \( l \), then \( \text{dim} \varphi^{-1}(S) = l \).

To prove the parametric transversality theorem we apply the following Sard’s theorem.

**Theorem 2.3.** [2, Theorem 3.2] If \( \varphi : M \to N \) is an \( MC^k \)-Lipschitz-Fredholm map with \( k > \max\{\text{Ind } \varphi, 0\} \). Then, the set of regular values of \( \varphi \) is residual in \( N \).

**Theorem 2.4** (The Parametric Transversality Theorem). Let \( A \) be a manifold of dimension \( n, S \subset N \) a submanifold of finite co-dimension \( m \). Let \( \varphi : M \times A \to N \) be an \( MC^k \)-mapping, \( k \geq \{1, n - m\} \). If \( \varphi \cap S \), then the set of all points \( x \in M \) such that the mappings

\[
\varphi_x : A \to N, \; (\varphi_x(\cdot) := \varphi(x, \cdot))
\]

are transversal to \( S \), is residual \( M \).

**Proof.** Let \( S = \varphi^{-1}(S), \Pr_M : M \times A \to M \) the projection onto \( M \) and \( \Pr_S \) be its restriction to \( S \). First, we prove that \( \Pr_S \) is an \( MC^k \)-Fredholm-Lipschitz mapping of index \( n - m \), i.e.,

\[
T_{(m, a)} \Pr_S : T_{(m, a)} S \to T_{m} M
\]

is a Lipschitz-Fredholm operator of index \( n - m \).

By Theorem 2.2 the inverse image \( S \) is an \( MC^k \)-submanifold of \( M \times A \), with model space \( S \), so that \( \Pr_S \) is an \( MC^k \)-mapping.
Let $\pi_M$ and $\pi_S$ be the local representatives of $\Pr_M$ and $\Pr_S$, respectively. We show that $\pi_M$ and consequently $\pi_S$ are Lipschitz-Fredholm operators of index $n - m$.

Finite dimensionality of $\mathbb{R}^n$ and closedness of $S$ implies that $K := S + \{0\} \times \mathbb{R}^n$ is closed in $E \times \mathbb{R}^n$. Also, codim $K$ is finite because it contains the finite co-dimensional subspace $S$. Therefore $K$ has a finite-dimensional complement $K_1 \subset E \times \{0\}$, that is $E \times \mathbb{R}^n = K \oplus K_1$. Let $K_2 := \mathbb{S} \cap \{0\} \times \mathbb{R}^n$. Since $K_2 \subset \mathbb{R}^n$ we can choose closed subspaces $S_1 \subset \mathbb{R}^n$ and $R_0 \subset \{0\} \times \mathbb{R}^n$ such that $S = S_1 \oplus K_1$ and $\{0\} \times \mathbb{R}^n = K_1 \oplus R_0$. Whence, $K = S_1 \oplus K_1 \oplus R_0$ and $E \times \mathbb{R}^n = S_1 \oplus K_1 \oplus R_0 \oplus K_2$.

The mapping $\pi_S |_{S_1 \oplus K_2}$ : $S_1 \oplus K_2 \to E$ is an isomorphism, $K_1 = \ker \pi_S$, and $\pi_M(K_2)$ is a finite dimensional complement to $\pi_M(S)$ in $\mathbb{R}^n$. Thus, $\pi_M$ is a Lipschitz-Fredholm operator and we have

$$\operatorname{Ind} \pi_M = \dim K_1 - \dim K_2 = \dim (K_1 \oplus R_0) - \dim (R_0 \oplus K_2).$$

Since, $K_1 \oplus R_0 = \{0\} \times \mathbb{R}^n$ and $R_0 \oplus K_2$ is a complement to $S$ in $E \times \mathbb{R}^n$ and therefore its dimension is $n$, so the index of $\pi_M$ is $n - m$.

Now, we prove that if $x$ is a regular value of $\Pr_S$ if and only if $\varphi_x \cap S$. From the definition of $\varphi$ we have $\forall (x, a) \in S$

$$T_{(x,a)} \varphi (T_x M \times T_a A) + T_{\varphi(x,a)} S = T_{\varphi(x,a)} N, \quad (2.6)$$

and

$$(T_{(x,a)} \varphi)^{-1} (T_{\varphi(x,a)} S) \text{ splits in } T_x M \times T_a A. \quad (2.7)$$

Since $A$ has finite dimension, it follows that the mapping $a \in A \mapsto \varphi(x, a)$ for a fixed $x \in M$ is transversal to $S$ if and only if

$$\forall (x, a) \in S, T_a \varphi_x (T_a A) + T_{\varphi(x,a)} S = T_{\varphi(x,a)} S. \quad (2.8)$$

Since $\Pr_S$ is a Lipschitz-Fredholm mapping, $\ker T \Pr_S$ splits at any point as its dimension is finite. Then $x$ is a regular value of $\Pr_S$ if and only if

$$\forall (x, a) \in S, \forall v \in T_x M, \exists u \in T_a A : T_{(v,u)} \varphi (v, u) \in T_{(x,a)} S. \quad (2.9)$$

Pick $x \in M$ and $a \in A$ such that $(x, a) \in S$ and let $w \in T_{(x,a)} S$. By (2.6) and (2.7) we obtain that there exist $v \in T_a A$, $x_1 \in T_x M$, $y_1 \in T_{(x,a)} S$ such that

$$T_{(x,a)} \varphi (v, x_1) + y_1 = w. \quad (2.10)$$

Then, there exists $x_2 \in T_x M$ such that $T_{(x,a)} \varphi (v, x_2) \in T_{\varphi(x,a)} S$. Hence,

$$w = T_{(x,a)} \varphi (v, x_1) - T_{(x,a)} \varphi (v, x_2) + T_{(x,a)} \varphi (v, x_2) + y_1$$

$$= T_{(x,a)} \varphi (0, x_1 - x_2) + T_{(x,a)} \varphi (v, x_2) + y_1$$

$$= T_{(x,a)} \varphi (0, u) + T_{\varphi(x,a)} S + y_2 = T_a \varphi_x (T_a A),$$

where $u = x_1 - x_2$ and $y_2 = T_{(x,a)} \varphi (v, x_2) + y_1 \in T_{\varphi(x,a)} S$. Thus, (2.8) holds.

Now we show that (2.8) implies (2.9). Pick $a \in A$, $x \in M$ such that $(x, a) \in S$. Let $v \in T_x M$, $a_1 \in T_a A$, $y_1 \in T_{\varphi(x,a)} S$ and set $w := T_{(x,a)} \varphi (v, x_1) + y_1$. By (2.8) there exist $a_2 \in T_a A$ and $y_2 \in T_{\varphi(x,a)} S$ such that $w = T_a \varphi_x (a_2) + y_2$. Then,

$$0_E = T_{(x,a)} \varphi (v, a_1) - T_0 \varphi_x (a_2) + y_1 - y_2 = T_{(x,a)} \varphi (v, a_1 - a_2) + y_1 - y_2,$$

so $T_{(x,a)} \varphi (v, a_1 - a_2) = y_2 - y_1 \in T_{\varphi(x,a)} S$ so (2.9) holds. Thus, we showed that if $x$ is a regular value of $\Pr_S$ if and only if $\varphi_x \cap S$. Since $\Pr_S : S \to M$ is a Lipschitz-Fredholm of
class $MC^k$ with the index $n - m$ and $\text{codim } S = \text{codim } S = m$ and $k > \{0, n - m\}$, the Sard’s theorem 2.3 concludes the theorem.

3. The degree of Lipschitz-Fredholm mappings

In this section we construct the degree of $MC^k$-Lipschitz-Fredholm mappings and apply it to prove an invariance of domain theorem, a rank theorem and a Bursuk-Ulam type theorem. The construction of the degree relies on the following transversality result.

**Theorem 3.1.** [2, Theorem 3.3] Let $\varphi : M \to N$ be an $MC^k$-Lipschitz-Fredholm mapping, $k \geq 1$. Let $\iota : A \to N$ be an $MC^1$-embedding of a finite dimension manifold $A$ with $k > \max\{\text{Ind } \varphi + \dim A, 0\}$. Then there exists an $MC^1$ fine approximation $g$ of $\iota$ such that $g$ is embedding and $\varphi \circ g$. Moreover, suppose $S$ is a closed subset of $A$ and $\varphi \cap \iota(S)$, then $g$ can be chosen so that $\iota = g$ on $S$.

We shall need the following theorem that gives the connection between proper and closed mappings.

**Theorem 3.2.** [5, Theorem 1.1] Let $A, B$ be Hausdorff manifolds, where $A$ is a connected infinite dimensional Fréchet manifold, and $B$ satisfies the first countability axiom, and let $\varphi : A \to B$ be a continuous closed non-constant map. Then $\varphi$ is proper.

Let $\varphi : M \to N$ be a non-constant closed Lipschitz-Fredholm mapping with index $l \geq 0$ of class $MC^k$ such that $k > l + 1$. If $q$ is a regular value of $\varphi$, then by Theorem 3.2 and Corollary 2.1 the preimage $\varphi^{-1}(q)$ is a compact submanifold of dimension $l$.

Let $\iota : [0, 1] \hookrightarrow N$ be an $MC^1$-embedding that connects two distinct regular values $q_1$ and $q_2$. By Theorem 3.1 we may suppose $\iota$ is transversal to $\varphi$. Thus, by Theorem 2.2 the preimage $M = \varphi^{-1}(\iota([0, 1]))$ is a compact $(l + 1)$-dimensional submanifold of $M$ such that its boundary, $\partial M$, is the disjoint union of $\varphi^{-1}(q_1)$ and $\varphi^{-1}(q_2)$, $\partial M = \varphi^{-1}(q_1) \cup \varphi^{-1}(q_2)$. Therefore, $\varphi^{-1}(q_1)$ and $\varphi^{-1}(q_2)$ are non-oriented cobordant which gives the invariance of the mapping. Following Smale [6] we associate to $\varphi$ a degree, denoted by $\deg \varphi$, defined as the non-oriented cobordism class of $\varphi^{-1}(q)$ for some regular value $q$. If $l = 0$, then $\deg \varphi \in \mathbb{Z}_2$ is the number modulo 2 of preimage of a regular value.

Let $O \subseteq M$. Suppose $\varphi : \overline{O} \to N$ is a non-constant closed continuous mapping such that its restriction to $O$ is an $MC^{k+1}$-Lipschitz-Fredholm mapping of index $k$, $k \geq 0$. Let $p \in N \setminus \varphi(\partial \overline{O})$ and let $p$ a regular value of $\varphi$ in the connected component of $N \setminus \varphi(\partial \overline{O})$ containing $p$, the existence of such regular value follows from Sard’s theorem 2.3. Again, we associate to $\varphi$ a degree, $\deg(\varphi, p)$, defined as non-oriented class of $k$-dimensional compact manifold $\varphi^{-1}(p)$. This degree does not depend on the choice of $p$.

The following theorem which presents the local representation of $MC^k$-mappings is crucial for the rest of the paper.

**Theorem 3.3.** [1, Theorem 4.2] Let $\varphi : U \subseteq E \to F$ be an $MC^k$-mapping, $k \geq 1$, $u_0 \in U$. Suppose that $D \varphi(u_0)$ has closed split image $F_1$ with closed topological complement $F_2$ and split kernel $E_2$ with closed topological complement $E_1$. Then, there are two open sets $\tilde{U}_1 \subseteq U$ and $V \subseteq F_1 \oplus E_2$ and an $MC^k$-diffeomorphism $\Psi : V \to \tilde{U}_1$, such that $(\varphi \circ \Psi)(f, e) = (f, \eta(f, e))$ for all $(f, e) \in V$, where $\eta : V \to E_2$ is an $MC^k$-mapping.

**Theorem 3.4** (Rank theorem for $MC^k$-mappings). Let $\varphi : U \subseteq E \to F$ be an $MC^k$-mapping, $k \geq 1$. Suppose $u_0 \in U$ and $D \varphi(u_0)$ has closed split image $F_1$ with closed complement $F_2$ and split kernel $E_2$ with closed complement $E_1$. Also, assume $D \varphi(U)(E)$ is closed in
F and 
\[ D \varphi(u)|_{E_1} : E_1 \rightarrow D \varphi(u)(E) \]
is an \(MC^k\)-isomorphism for each \(u \in U\). Then, there exist open sets \(U_1 \subseteq F_1 \oplus E_2, \ U_2 \subseteq E, \ V_1 \subseteq F, \) and \(V_2 \subseteq F\) and there are \(MC^k\)-diffeomorphisms 
\(\phi : V_1 \rightarrow V_2\) and \(\psi : U_1 \rightarrow U_2\) such that 
\[(\phi \circ \varphi \circ \psi)(f, e) = (f, 0), \ \forall (f, e) \in U_1.\]

\textbf{Proof.} By Theorem 3.3 there exists an \(MC^k\)-diffeomorphism \(\psi : U_1 \subseteq F_1 \oplus E_2 \rightarrow U_2 \subseteq E\) such that 
\[\varphi(f, e) = (\varphi \circ \psi)(f, e) = (f, \eta(f, e)),\]
where \(\eta : V \rightarrow E_2\) is an \(MC^k\)-mapping. Let \(\Pr_1 : F \rightarrow F_1\) be the projection. We obtain 
\[\Pr_1 \circ D \varphi(f, e)(w, v) = (w, 0), \] for \(w \in F_1\) and \(v \in E_2\) because 
\[D \varphi(f, e)(w, v) = (w, D \eta(f, e)(w, v)).\]
Hence, \(\Pr_1 \circ D \varphi(f, e)|_{F_1 \times \{0\}}\) is the identity mapping, \(\text{Id}_{F_1}\), on \(F_1\). Thereby,
\[D \varphi(f, e)|_{F_1 \times \{0\}} : F_1 \times \{0\} \rightarrow D \varphi(f, e)(F_1 \oplus E_2)\]
is one-to-one and therefore by our assumption 
\[D \varphi(f, e) \circ \Pr_1|_{D \varphi(f, e)(F_1 \oplus E_2)}\]
is the identity mapping. Suppose \((w, D \eta(f, e)(w, v)) \in D \varphi(f, e)(F_1 \oplus E_2)\), we obtain 
\[D \eta(f, e)v = 0 \] for all \(v \in E_2\), which means \(D_2 \eta(f, e) = 0\), since 
\[D \varphi(f, e)|_{F_1 \times \{0\}}(w, D \eta(f, e)(w, v)) = D \varphi(f, e)(w, 0)\]
\[= (w, D \eta(f, e)(w, 0))\]
\[= (w, D_1(f, e)w).\]
We have \(D^2 \varphi(f, e)v = (0, D_2 \eta(f, e)v)\), i.e., \(D^2 \varphi(f, e) = 0\) which means \(\varphi\) does not depend on the variable \(y \in E_2\). Let \(\Pr_2 : F_1 \oplus E_2 \rightarrow F_1\) be the projection and \(\varphi_f := \varphi(f, e) = (\varphi \circ \Pr_2)(f, e)\), so that \(\varphi_f : \Pr_2(U_1) \subseteq F_1 \rightarrow F\).

Let \(v_0 := (\Pr_2 \circ \psi^{-1})(u_0)\) and \(V \subseteq U\) be an open neighborhood of \(v_0\). Define the mapping 
\[\Phi : V \times F_2 \rightarrow F_1 \oplus F_2, \]
\[\Phi(f, e) = \varphi(f) + (0, e).\]
By the open mapping theorem 
\[D \Phi(v_0, 0) = (D \varphi(v_0), \text{Id}_{F_2}) : E \oplus F_2 \rightarrow F\]
is a linear \(MC^k\)-isomorphism, where \(\text{Id}_{F_2}\) is the identity mapping of \(F_2\). Now \(\Phi\) satisfies the inverse function theorem 2.1 at \(v_0\), therefore, there exist \(V_1, V_2 \subseteq F\) such that \((v_0, 0) \in V_2\) and \(\Phi(v_0, 0) = \varphi_{v_0}(v_0) \in V_1\) and an \(MC^k\)-diffeomorphism \(\phi : V_1 \rightarrow V_2\) such that \(\phi^{-1} = \Phi|_{V_1}\). Thus, for \((f, 0) \in V_2\) we have 
\[(\phi \circ \varphi)(f) = (\Phi \circ \Phi)(f, 0) = (f, 0),\]
and therefore, 
\[(\phi \circ \varphi \circ \psi)(f, e) = (f, 0), \ \forall (f, e) \in U_1.\]

\[\square\]

\textbf{Corollary 3.1} (Rank theorem for Lipschitz-Fredholm mappings). Let \(\varphi : M \rightarrow N\) be an \(MC^\infty\)-Lipschitz-Fredholm mapping of index \(k\) and \(\text{dimker } D \varphi(x) = m\), \(\forall x \in M\). Let \(C_1, C_2\) be topological complements of \(\mathbb{R}^m\) in \(E\) and \(\mathbb{R}^{m-k}\) in \(F\), respectively. Then, there exist
charts $\phi : U \subseteq M \to E = \mathbb{R}^m \oplus \mathbb{C}_1$ with $\phi(x) = 0_E$ and $\psi : V \subseteq N \to F = \mathbb{R}^{m-k} \oplus \mathbb{C}_2$ with $\psi(x) = 0_F$ such that

$$\psi \circ \phi \circ \phi^{-1}(f, 0) = (f, 0).$$

The following theorem gives the openness property of the set of Lipschitz-Fredholm mappings.

**Theorem 3.5.** [1, Theorem 3.2] The set $\mathcal{LF}(E, F)$ is open in $\mathcal{L}_{d,g}(E, F)$ with respect to the topology defined by the metric (1.1). Furthermore, the function $T \to \text{Ind} T$ is continuous on $\mathcal{LF}(E, F)$, hence constant on connected components of $\mathcal{LF}(E, F)$.

The proof of the following theorem is a minor modification of [7, Theorem 2].

**Theorem 3.6.** Let $\varphi : M \to N$ be a Lipschitz-Fredholm mapping of class $MC^k$, $k \geq 1$. Then, the set $\text{Sing}(\varphi) := \{m \mid D\varphi(m)\text{is not injective}\}$ is nowhere dense in $M$.

**Proof.** This is a local problem so assume $M$ is an open set in $E$ and $N$ is an open set in $F$. Let $s \in \text{Sing}(\varphi)$ be arbitrary and $U$ an open neighborhood of $s$ in $\text{Sing}(\varphi)$. For each $n \in \mathbb{N} \cup \{0\}$ define

$$S_n := \{m \in M \mid \dim D\varphi(m) \geq n\}.$$

Then, $M = M_0 \supset M_1 \supset \cdots$, therefore, is a unique $n_0$ such that $M = M_{n_0} \neq M_{n_0+1}$. Let $m_0 \in M_{n_0} \setminus M_{n_0+1}$ such that $\dim \ker D\varphi(m_0) = n_0$. By Theorem 3.5, there exists an open neighborhood $\mathcal{V}$ of $m_0$ in $U$ such that for all $v \in \mathcal{V}$ we have $\dim \ker D\varphi(v) \leq n_0$ and hence $\dim D\varphi(v) = n_0 \geq 1$. By Corollary 3.1, there is a local representative $\varphi$ around zero such that $\psi \circ \varphi \circ \phi^{-1}(f, e) = (f, 0)$ for $(f, e) \in C_1 \oplus \mathbb{R}^{n_0}$ which contradicts the injectivity of $\varphi$, therefore, $\text{Sing}(\varphi)$ contains a nonempty open set. The closedness of $\text{Sing}(\varphi)$ is obvious in virtue of Theorem 3.5. □

**Theorem 3.7** (Invariance of domain for Lipschitz-Fredholm mappings). Let $\varphi : M \to N$ be an $MC^k$-Lipschitz-Fredholm mapping of index zero, $k > 1$. If $\varphi$ is locally injective, then $\varphi$ is open.

**Proof.** Let $p \in U \subseteq M$ and $q = \varphi(p)$. The point $p$ has a connected open neighborhood $U \subseteq M$ such that $\varphi|_U : U \to N$ is proper and injective. Whence $q \notin \varphi(\partial U)$ and $\varphi(\partial U)$ is closed in $N$. Let $V$ be a connected component of $N \setminus \varphi(\partial U)$ containing $q$ which is its open neighborhood. Since $U$ is connected it implies that $\varphi(U) \subseteq V$. It follows from $\varphi(\partial U) \cap V = \emptyset$ that $\partial U \cap \varphi^{-1}(V) = U$ and so $\varphi|_U : U \to N$ is proper and injective. By Theorem 3.6 there is a point $x \in M$ such that the tangent map $T_x \varphi$ is injective and since $\text{Ind} \varphi = 0$ it is surjective too. Therefore, $y = \varphi(x)$ is a regular value with $\varphi^{-1}(y) = \{x\}$ and $\deg \varphi = 1$. It follows that $\varphi$ is surjective, because if it is not, then any point in $N \setminus \varphi(M)$ is regular and $\deg \varphi = 0$ which is contradiction. Then, $V = \varphi(U)$ is the open neighborhood of $q$. □

**Corollary 3.2** (Nonlinear Fredholm alternative). Let $\varphi : M \to N$ be an $MC^k$-Lipschitz-Fredholm mapping of index zero, $k > 1$. If $N$ is connected and $\varphi$ is locally injective, then $\varphi$ is surjective and finite covering mapping. If $M$ is connected and $N$ is simply connected, then $\varphi$ is a homeomorphism.

The following theorem is a generalization of the Borsuk-Ulam theorem, the proof is slight modification of the Banach case.

**Theorem 3.8.** Let $\varphi : \overline{U} \to F$ be a non-constant closed Lipschitz-Fredholm mapping of class $MC^2$ with index zero, where $U \subseteq F$ is symmetric. If $\varphi$ is odd and for $u_0 \in \overline{U}$ we have $u_0 \notin \varphi(\partial U)$. Then $\deg(\varphi, u_0) \equiv 1 \mod 2$. 


Proof. Since $D \varphi(u_0)$ is a Lipschitz-Fredholm mapping with index zero
\[ F = F_1 \oplus \ker \varphi = F_2 \oplus \text{Im} \varphi \]
and $\dim F_2 = \dim \ker \varphi$. The image $\varphi(\mathcal{U})$ is closed as $\varphi$ is closed, hence
\[ a = \varrho(\varphi(\mathcal{U}), u_0) > 0 \]
because $u_0 \notin \varphi(\partial \mathcal{U})$.

Let $\phi : F \to F$ be a global Lipschitz-compact linear operator with $\text{Lip}(\phi) < b$ for some $b > 0$. Define the mapping $\Phi^\phi : \overline{U} \to F$ by $\Phi^\phi(u) = \varphi(u) + \phi(u)$. Then $\Phi^\phi$ is a Lipschitz-Fredholm mapping of index zero. Suppose $b < a/k$ for some $k > 1$, then
\[ \varrho(\Phi^\phi(u), u_0) \geq \varrho(\varphi(u), u_0) - \text{Lip}(\phi) \varrho(U, u_0) > a - bk > 0, \quad \forall u \in \partial \mathcal{U}. \]
Therefore, $u_0 \notin \Phi^\phi(\partial \mathcal{U})$. We obtain \( \deg(\varphi, u_0) = \deg(\Phi^\phi, u_0) \) as the mapping
\[ \psi : [0, 1] \times \overline{U} \to F \]
defined by $(t, u) \to \varphi(u) + t\phi(u)$ is proper and $u_0 \notin \psi(\partial U)$ for all $t$. Considering the fact that $\psi(-u) = -\psi(u)$, we may use the perturbation by compact operators to find the degree of $\varphi$.

Let $C$ be a set of global Lipschitz-compact linear operators $\phi : F \to F$ with $\text{Lip}(\phi) < b < a/k$. Let $\hat{\phi} \in C$ be such that its restriction to $F_1$ equals $u_0$ and $\hat{\phi} \mid_{\ker D \varphi(u_0)} : \ker D \varphi(u_0) \to F_2$ is an $MC^1$-isomorphism. Therefore, $D \varphi(u_0) + \hat{\phi}$ and consequently $D \varphi(u_0)$ is an $MC^1$-isomorphism. Now define the mapping $\Psi : \mathcal{U} \times C \to F$ by $(u, \hat{\phi}) = \Phi^\phi(u)$. For sufficiently small $b$ the differential $D \Psi(u, \hat{\phi})(v, \psi) = (D \varphi(u) + \hat{\phi})v + \psi(u)$ is surjective at $u_0$ as $D \varphi(u_0)$ is an $MC^1$-isomorphism. Also, it is clear that it is surjective at the other points. Then, the mapping $\Psi$ satisfies the assumption of Theorem 2.4, therefore, $\Psi^{-1}(u_0)$ is a submanifold and the mapping $\Pi : \Psi^{-1}(u_0) \to C$ induced by the projection onto the second order is Lipschitz-Fredholm of index zero. By employing the local version of Sard’s theorem we may find a regular point $\hat{\phi}$ of $\Pi$, and from the proof of the Theorem 2.4 it follows that $u_0$ is a regular value of $\Phi^\phi$ and consequently $u_0$ is a regular value of $\varphi$. Thus, properness and $\varphi(-u) = -\varphi(u)$ imply that $\varphi^{-1}(u_0) = \{u_0, f_1, -f_1, \ldots, f_m, -f_m\}$ and therefore $\deg(\varphi, u_0) \equiv 1$ mod 2.

\[ \square \]

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