Kibble-Zurek behavior in disordered Chern insulators

Lara Ulčakar
Jozef Stefan Institute, Jamova 39, Ljubljana, Slovenia and
Faculty for Mathematics and Physics, University of Ljubljana, Jadranska 19, Ljubljana, Slovenia

Jernej Mravlje
Jozef Stefan Institute, Jamova 39, Ljubljana, Slovenia

Tomaž Rejec
Jozef Stefan Institute, Jamova 39, Ljubljana, Slovenia and
Faculty for Mathematics and Physics, University of Ljubljana, Jadranska 19, Ljubljana, Slovenia

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Even though no local order parameter in the sense of the Landau theory exists for topological phase transitions in Chern insulators, the highly non-local Berry curvature exhibits critical behavior near a quantum critical point. We investigate the critical properties of its real space analog, the local Chern marker, in weakly disordered Chern insulators. Due to disorder, inhomogeneities appear in the spatial distribution of the local Chern marker. Their size exhibits power-law scaling with the critical exponent matching the one extracted from the Berry curvature of a clean system. We drive the system slowly through such a quantum phase transition. The characteristic size of inhomogeneities in the non-equilibrium post-quench state obeys the Kibble-Zurek scaling. In this setting, the local Chern marker thus does behave in a similar way as a local order parameter for a symmetry breaking second order phase transition. The Kibble-Zurek scaling also holds for the inhomogeneities in the spatial distribution of excitations and of the orbital polarization.

The discovery of topological insulators has sparked great interest due to their novel properties that could furthermore be used for practical applications. Topological systems have been realized in solid state and in cold atoms. Recently, a lot of attention has been given to dynamical critical properties after quenches across topological phase transitions in topological insulators, superconductors and $p+ip$ superfluids. An important observation of interest to this paper was made in Refs. which showed that the number of excitations after a slow quench follows the Kibble-Zurek (KZ) scaling.

Developed by Kibble as a cosmological theory describing the formation of the early universe and applied to condensed matter systems by Zurek, the KZ mechanism describes non-equilibrium properties of a system that was driven in a finite time $\tau$ over a symmetry breaking second order phase transition. In equilibrium the relaxation time $\tau$ and the correlation length $\xi$ diverge as a function of the control parameter $u$ approaching the critical point $u_c$ by a power determined by critical exponents $\nu$ and $\nu_c \sim |u-u_c|^{-\nu}$ and $\xi \sim |u-u_c|^{-\nu_c}$ [50]. Because of the divergence the system evolves nonadiabatically across the critical point. In the case of phase transitions with spontaneous symmetry breaking that entail a degeneracy of the ground state, such a process produces regions corresponding to different choices of the ground state. The size of the regions is set by the equilibrium correlation length $\xi(t_F) \sim \tau^{\nu_c/(1+\nu_c)}$ at a ”freeze-out” time $t_F \sim \tau^{\nu_c/(1+\nu_c)}$, an approximate time at which the system stopped evolving adiabatically. The KZ scaling was observed experimentally in tunnel Josephson junctions [51, 52], multiferroics [53–55], ion Coulomb crystals [56, 57], Bose-Einstein condensates [58], and in a Rydberg atom quantum simulator [59].

Returning to topological insulators, the fact that the KZ scaling occurs in these systems is not expected on the first sight. Topological insulators have no spontaneously broken symmetry. They accordingly lack degeneracy of the ground state and have no local (Landau) order parameter. Instead, their phase is described by a quantized non-local order parameter, the topological invariant. Is it nevertheless possible to relate the KZ scaling to the freeze-out behavior and what are its manifestations in the real space? In Chern insulators, for example, the critical increase of a length scale has been noticed in the Berry curvature [60]. Some hope for establishing the analogy with systems with a local order parameter stems also from the discovery of the local Chern marker (LCM) that was introduced in Ref. as a local indicator of the topological phase in Chern insulators. Furthermore, in Ref. Caio et al. showed that in equilibrium, the LCM exhibits a length scale that grows as $|u-u_c|^{-\nu}$ close to a topological phase transitions. An open question is, how does the LCM behave during a slow quench?

In this paper we address the KZ mechanism in a Chern insulator. To reveal the KZ physics directly in real space, we calculate the LCM in the presence of weak disorder which breaks the translational symmetry. The disorder leads to the appearance of inhomogeneities in the LCM. We show that in the ground state these exhibit a length scale shown in Fig. [a] that grows as $|u-u_c|^{-1}$ as the topological transition is approached. Then we study a quench where we drive the system across a critical point...
The disorder is present on the staggered orbital binding energies \( u + \delta u(\mathbf{r}) \), where \( \delta u(\mathbf{r}) \) are uncorrelated and uniformly distributed on the interval \([-\delta u_0, \delta u_0]\). The disorder is assumed to be weak enough that the Anderson localization length is much longer than the system size. In absence of disorder, the QWZ model was experimentally realized in ultracold atoms.\(^{22, 64}\)

In a clean system, the QWZ Hamiltonian for a particular wavevector \( \mathbf{k} = (k_x, k_y) \) from the first Brillouin zone (BZ) is

\[
\hat{H}(\mathbf{k}) = (u + \cos k_x + \cos k_y)\hat{\sigma}_z + \sin k_x\hat{\sigma}_x + \sin k_y\hat{\sigma}_y. \quad (2)
\]

Its eigenstates \( |\psi_n(\mathbf{k})\rangle \), where the band index \( \alpha \) distinguishes the valence \((\alpha = v)\) from the conduction \((\alpha = c)\) band, can be used to calculate the Berry curvature of the valence band,

\[
\Omega(\mathbf{k}) = i\text{Tr}\{\hat{P}(\mathbf{k})[\partial_{k_x} \hat{P}(\mathbf{k}), \partial_{k_y} \hat{P}(\mathbf{k})]\} \quad (3)
\]

with \( \hat{P}(\mathbf{k}) = |\psi_n(\mathbf{k})\rangle \langle \psi_n(\mathbf{k})| \). In this work, we concentrate on the topological quantum phase transition that takes place at \( u_c = -2 \) where the topological invariant – the Chern number \( C = -\frac{1}{2\pi} \int_{\text{BZ}} d\mathbf{k} \Omega(\mathbf{k}) \) – changes its value from \( C = 0 \) at \( u < u_c \) to \( C = -1 \) at \( u_c < u < 0 \).

The Chern number can also be calculated from the real-space analog of the Berry curvature – the local Chern marker

\[
c(\mathbf{r}) = 2\pi i \sum_{\sigma} \langle \mathbf{r}, \sigma | \hat{P}[-i[\hat{x}, \hat{P}], -i[\hat{y}, \hat{P}]] | \mathbf{r}, \sigma \rangle \quad (4)
\]

as \( C = \lim_{N\to\infty} \frac{1}{N^2} \sum_{\mathbf{r}} c(\mathbf{r}) \). Here \( \hat{x} \) and \( \hat{y} \) are the position operators and \( \hat{P} = \sum_n |\Psi_n\rangle \langle \Psi_n| \) is the projector.
onto the subspace spanned by eigenstates $|\Psi_n\rangle$ of the valence band. In clean systems in the thermodynamic limit the LCM is uniform and equals the Chern number [61].

We calculated the LCM in the presence of a weak disorder $\delta u_0 = 0.05$. The real-space profiles of the LCM are shown in Fig. 1(a) for several values of $u$, ranging from deep in the trivial phase, across the critical point (note that the weak disorder does not significantly move the critical point [60]) to deep in the topological phase. The profiles are inhomogeneous and feature regions where the LCM deviates above (brown) and below (blue) the clean system value. While the amplitude of those deviations is proportional to $\delta u_0$, their size does not depend on the disorder strength (see Supplemental Material S1). The basic point is that, as the critical point is approached, the size of those regions grows. In the topological phase the deviations of the LCM from its clean system value are dominated by a contribution proportional to disorder. In Fig. 1(a-4) and Fig. 1(a-5), the disorder contribution is filtered out. Raw data is shown in Supplemental Material S1.

We measure the size of inhomogeneities $\xi_r$ by finding the distance where the autocovariance function of a LCM profile drops below zero (see Supplemental Material S2). Fig. 2 shows that $\xi_r$ exhibits a power-law scaling as $u$ approaches the critical point. Increasing the system size, the estimate of the scaling exponent approaches one. One can evaluate the correlation length also for a clean system by calculating the Berry curvature, which exhibits a peak at $k = 0$ [insets to Fig. 1(a)]. The width of the peak $\xi^{-1}_k$ shrinks on approaching the critical point with the correlation length exponent $\nu = 1$ [35] [40], thus in agreement with what we find from the analysis of the inhomogeneities in the LCM. Analogous results are obtained by measuring the radius of the peak in the LCM profile around a single impurity (see Supplemental material S3). All this confirms that, in the presence of disorder, the criticality of the correlation length in Chern insulators is observable in real space via the LCM.

**Quenches.**—We perform quenches in systems initially in the ground state, i.e., with the valence band filled and the conduction band empty, starting in the trivial regime at $u_0 = -2.5$, smoothly varying the parameter $u$ with time as $u(t) = u_0 + (u_1 - u_0) \sin^2(\pi t / \tau)$, and ending up at $t = \tau$ in the topological regime at $u_1 = -1.5$. We separately show that the length scale, predicted by the KZ mechanism, arises both in clean and in disordered systems.

**Quenches in clean systems.**—We extract $\xi_k$ after the quench from the Berry curvature of non-equilibrium post-quench states, calculated by replacing the pre-quench eigenstates $|\psi_n(k)\rangle$ in Eq. (3) with the corresponding time-evolved states. The post-quench form of the Berry curvature for different quench times is shown in Fig. 3(a). It exhibits a peak at $k = 0$ of the width proportional to $\tau^{-1/2}$, giving rise to a length scale proportional to $\tau^{1/2}$. Taking into account that $z \nu = 1$ [35], this result follows the KZ scaling. The KZ mechanism relates the characteristic length with the density of point defects $n \propto \xi^{-2}(t_F)$, making our results in agreement with earlier work [35] [38] [44] [45], which showed that excitation density is $n_{\text{exc}} = |u| |u_0| / 8\pi$.

It is interesting to look at how the Berry curvature evolves during the quench [insets to Fig. 1(b)]. Fig. 3(b) shows the evolution of $\xi_k$ of non-equilibrium states corresponding to different quench times (colored lines). This is compared to the critical behavior of $\xi_k$ of the corresponding instantaneous ground states (black line). Before entering the freeze-out zone (the shaded region for $\tau = 20$), the system evolves adiabatically and its $\xi_k$ is equal to the ground-state one. In the freeze-out zone, the system stops evolving adiabatically and its $\xi_k$ starts to deviate from the ground-state one: it increases linearly with a speed independent of the quench time. Only after the system exits the freeze-out zone, $\xi_k$ settles at an approximately constant value. Similar results were observed in quenches performed in a Rydberg atom quantum simulator [59]. This behavior goes beyond what is known in the literature as the “adiabatic-impulse approximation” which identifies the saturation of the length scale with the entry to the freeze-out zone [49].

**Quenches in disordered systems.**—The KZ length scale observed in a clean system manifests itself in a disordered system as the size of inhomogeneities in the post-quench LCM profile.

Similarly to the Berry curvature, the LCM profiles can also be calculated during a quench by replacing in Eq. (1) the projector $P$ onto the valence band with the projector onto the occupied subspace i.e, pre-quench eigenstates $|\Psi_n\rangle$ are replaced with the corresponding time-evolved states.

The evolution of the LCM profile during the quench
with $\tau = 20$ is shown in Fig. 3(b). Due to the conservation of the Chern number \[24, 35, 37, 67\], the clean system value of the LCM is zero throughout the quench. Let us now focus on deviations from this value. Prior to entering the freeze-out zone, the system evolves adiabatically and the LCM profiles match the ground-state ones, shown in Fig. 3(a). The growth of $\xi_r$ in the freeze-out zone, although present, lags behind that in instantaneous ground states. After the exit from the freeze-out zone, the profiles do not significantly change anymore. [The amplitude of the deviations of the LCM, however, grows strongly throughout the quench.] In Fig. 3(b) $\xi_r$ during quenches with different $\tau$ and in the corresponding ground states are shown with colored and black dots, respectively. They are seen to follow the behavior of $\xi_k$ (colored and black lines).

The post-quench inhomogeneities are larger for quenches performed more slowly (see Supplemental Material S1). Their size exhibits a power-law scaling with quench time as shown in Fig. 3(a). The scaling exponent is close to 1/2, which matches the KZ prediction.

Inhomogeneities appear also in the density of excitations $n_{exc}(r, t) = \sum_{\sigma}|\langle r, \sigma | P(t) \hat{P}_c(t) r, \sigma \rangle|^2$ where $\hat{P}_c(t)$ is the projector onto the instantaneous conduction band (see Supplemental Material S4), and in the deviation of the orbital polarization $p(r, t) = \sum_{\sigma}|\langle r, \sigma | \hat{P}_c(t) \hat{\sigma}_r r, \sigma \rangle|^2$ from the ground state one (see Supplemental Material S5). The profiles of the excitations and the polarization appear to be the same as those of the LCM. The scaling of $\xi_r$ of these quantities with quench time also conforms to the KZ prediction, as shown in Figs. 3(b) and 3(c).

$\xi_r$ in Fig. 3 saturates at a certain $\tau$. This is a finite-size effect: the value of $\tau$ where this happens increases with the system size $N$.

**Discussion.**—The scaling of $\xi_r$ can be explained by the Landau-Zener dynamics, taking additionally into account the effect of disorder on eigenstates of the post-quench Hamiltonian. Because the disorder is weak, these eigenstates maintain a definite value of the momentum magnitude $|k|$ up to a good approximation. The post-quench probability of an excitation is therefore given by the Landau-Zener formula $\exp(-\frac{2|k|\tau}{\pi})$. This provides a rough estimate for the highest momentum where excitations are present, $k_{max} = (\frac{1}{|\xi_r|})^{1/2}$.

In the presence of weak disorder, the eigenstates preserve the length scale $2\pi$. The post-quench projector to the occupied subspace $\hat{P}(\tau)$ is expected to contain within itself the finest of those length scales present, $\hat{P}(\tau) \sim \tau^{1/2}$. $\hat{P}(\tau)$ is an ingredient of expressions for all the quantities studied in this paper, which consequently after quench exhibit the $\tau^{1/2}$ scaling of $\xi_r$.

**Conclusions.**—We investigated the equilibrium and the dynamical critical properties in a Chern insulator as well as their relation via the KZ mechanism. We used a weak disorder to reveal the correlation length scale in the ground-state real-space profile of the LCM. After a quench the LCM exhibits inhomogeneities of a length scale that depends on the quench time as $\tau^{1/2}$. We followed the growth of the inhomogeneities during the quench and demonstrated that the KZ freeze-out mechanism applies. Through the lens of the LCM the critical behavior of weakly disordered Chern insulators is analogous to the one found in systems with spontaneously broken symmetries. The important difference is that the amplitude of the inhomogeneities vanishes with the vanishing disorder strength. The KZ scaling with $\tau$ can be experimentally tested by looking at the orbital polariza-
tion or (if the particle-hole symmetry would be broken, as is for instance in the Haldane model) also in the inhomogeneities of the charge density.

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SUPPLEMENTARY INFORMATION

S1. Additional LCM profiles

Fig. 5 shows that the size of inhomogeneities in the LCM does not depend on the weak disorder strength. On the other hand, the amplitude of deviations is proportional to $\delta u_0$. In Fig. 4 the ground-state LCM profiles in the topological phase are shown. They are dominated by a contribution proportional to the disorder. The corresponding profiles in Figs. 1(a-4) and 1(a-5) of the main article were obtained by filtering out this contribution with the Gaussian filter with the width $\sigma = 1$ (lattice spacing). For discussion, see Sec. S5. In Fig. 7 post-quench LCM profiles for various quench times are shown for the same disorder realization as in Fig. 1 of the main article.

![Fig. 5](image1)

**FIG. 5.** Ground-state LCM profiles at $u = -2.3$ with (a) $\delta u_0 = 0.05$ and (b) $\delta u_0 = 0.005$. $N = 70$.

S2. Estimation of the size of inhomogeneities

We estimate the size of the inhomogeneities in a real-space profile $A(\mathbf{r})$ from its position autocovariance function

$$R_{AA}(r) = \frac{\sum_{|\mathbf{r}|=r} \sum_{\mathbf{r}'} A(\mathbf{r}) A(\mathbf{r}+\mathbf{r}')}{\sum_{|\mathbf{r}|=r} \sum_{\mathbf{r}'} A(\mathbf{r}')^2}. \quad (5)$$

We identify the typical length scale $\xi_r$ in the LCM as the distance at which the autocovariance function crosses zero, $R_{cc}(\xi_r) = 0$. The autocovariance functions of the post-quench LCM profiles shown in Fig. 7 are plotted in Fig. 8 as a function of $r/\sqrt{\tau}$. Note that for slow enough quenches, the size of inhomogeneities $\xi_r$ scales as $r^{1/2}$.

S3. Ground-state LCM profiles around a single impurity

Here we study deviations $\delta c_1(\mathbf{r})$ of the LCM from its clean system value, induced by a single weak impurity with strength $\delta u$ at $\mathbf{r} = 0$. Only the ground state is
FIG. 8. Autocovariance functions of the post-quench LCM profiles for the disorder realization of Fig. 7(a). The profiles for \( \tau = 7, 20 \) and 100 are shown in Figs. 8(b), 8(c) and 8(d), respectively.

In Figs. 8(a) and 8(a), typical \( \delta c_1(r) \) profiles in the trivial and in the topological phases, respectively, are shown. In Figs. 8(b) and 8(b), \( \delta c_1(r) \) are plotted along the \( y = 0 \) line for several values of \( u \) in the trivial and in the topological phases, respectively. The position is rescaled as \( x|u - u_c|^{0.8} \), showing that the radius \( \xi_r \) of the region around the impurity where the LCM deviates below (for \( \delta u > 0 \)) the clean system value scales as a power law. The estimate of the scaling exponent on the trivial side approaches one as the system size is increased, see Fig. 9(c). Note, however, that \( \xi_r \) is not the only scale in which \( \delta c_1(r) \) can be described: it has also an internal structure.

A notable feature is that the behavior of \( \delta c_1(r) \) on the topological and on the trivial side are quite different. On the topological side, \( |\delta c_1(r)| \) is maximal at the position of the impurity. There it takes a value that is at least five times larger than the value on neighboring sites, whereas on the trivial side it has a minimum at the position of the impurity and takes a maximum on the neighboring sites. This distinction has important consequences for the behavior of the LCM in a disordered system. Namely, for a weak disorder one can write

\[
\delta c(r) \sim \int dr' \delta c_1(r-r') \delta u(r')
\]

where \( \delta c(r) \) is the deviation of the LCM in the disordered system from the clean system value and \( \delta u(r) \) the distribution of the disorder. \( \delta c_1(r) \) thus plays the role of the integration kernel through which the disorder is averaged. Now, on the topological side, because the largest contribution to \( \delta c_1(r) \) is local, one can expect that the \( \delta c(r) \) is dominated by a contribution directly proportional to disorder. This is indeed what one observes in Fig. S2.

S4. Real-space distribution of excitations

In this section we derive the real-space distribution of the excitations to the conduction band. Let us first consider a clean system. Let \( |\Psi_\alpha(k, t)\rangle = |k\rangle \otimes |\psi_\alpha(k, t)\rangle \) be the instantaneous eigenstates, i.e., the eigenstates of the Hamiltonian \( H(t) \), and \( |\Phi_\alpha(k, t)\rangle = |k\rangle \otimes |\varphi_\alpha(k, t)\rangle \) the states obtained by time-evolving the pre-quench eigenstates \( |\Psi_\alpha(k, 0)\rangle \) to time \( t \). Here \( |k\rangle = \frac{1}{N} \sum \xi e^{ikr}\rangle \) and \( \alpha = v, c \) denote the valence and the conduction bands, respectively. The total number of excitations
where the second line owes to the orthogonality of the plane waves. Recognizing \( \hat{P}(t) = \sum_k |\Psi_v(k, t)\rangle \langle \Psi_v(k, t)| \) as the projector onto the occupied subspace and \( \hat{P}_c(t) = \sum_k |\Psi_c(k, t)\rangle \langle \Psi_c(k, t)| \) as the projector onto the instantaneous conduction band, Eq. (6) may be written as a trace over the whole Hilbert space: 

\[
N_{\text{exc}}(t) = \text{Tr}\{\hat{P}(t)\hat{P}_c(t)\}. \tag{6}
\]

We calculate the trace in the real-space basis, 

\[
N_{\text{exc}}(t) = \sum_{r, \sigma} \langle r, \sigma |\hat{P}(t)\hat{P}_c(t)| r, \sigma \rangle. \tag{7}
\]

This expression can also be evaluated in a disordered system using 

\[
P(t) = \sum_{n \in v} |\Phi_v(n, t)\rangle \langle \Phi_v(n, t)| \quad \text{and} \quad P_c(t) = \sum_{n \in v} |\Phi_c(n, t)\rangle \langle \Phi_c(n, t)|
\]

where \( |\Phi_v(n, t)\rangle \) and \( |\Phi_c(n, t)\rangle \) are instantaneous eigenstates and time-evolved pre-quench eigenstates, respectively.

In Fig. 11 some post-quench profiles of excitations are shown.

![FIG. 11. Spatial distribution of excitations after quenches with (a) \( \tau = 20 \) and (b) \( \tau = 100 \). The disorder realization is the same as in Fig. 7.](image)

S5. Real-space distribution of the orbital polarization

We define the orbital polarization as the difference between the occupation of the orbital \( A \) and the occupation of the orbital \( B \):

\[
p(r, t) = \sum_{n \in v} |\langle r, A |\Phi_n(t)\rangle|^2 - |\langle r, B |\Phi_n(t)\rangle|^2 = \sum_{\sigma} \langle r, \sigma |\hat{P}(t)\hat{\sigma}_z| r, \sigma \rangle. \tag{8}
\]

Some post-quench profiles of the orbital polarization are shown in Fig. 12.

![FIG. 12. Deviation of the post-quench orbital polarization from the ground-state one for quenches with (a) \( \tau = 20 \) and (b) \( \tau = 100 \). The disorder realization is the same as in Fig. 7.](image)

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