A CLASSIFICATION OF STAR LOG SYMPLECTIC STRUCTURES ON A COMPACT ORIENTED SURFACE

MELINDA LANIUS

ABSTRACT. Given a compact oriented surface, we classify log Poisson bivectors whose degeneracy loci are locally modeled by a finite set of lines in the plane intersecting at a point. Further, we compute the Poisson cohomology of such structures and discuss the relationship between our classification and the second Poisson cohomology.

1. Introduction

As is well-known, two non-degenerate Poisson - i.e. symplectic - structures on a compact connected surface $S$ are the same precisely when they have the same de Rham cohomology class. In this work we give an analogous classification of a more general class of Poisson bivector on a surface. Since every bivector on a surface $S$ is Poisson, we organize Poisson structures on $S$ by their degeneracy loci and, in fact, Vladimir Arnol’d introduced a hierarchy for these degeneracies (Appendix 14 C [2]). Olga Radko provided a classification when the Poisson bivector degenerated linearly along a curve [14]. We will extend her classification to Poisson bivectors whose degeneracy loci are locally modeled by a finite set of lines in the plane intersecting at a point. Additionally, we will compute the Poisson cohomology of such structures $\pi$ and discuss how our classification relates to deformations of $\pi$.

1.1. Classification. Let $(S, \pi)$ be a non-degenerate Poisson structure on a surface $S$. The key idea to our approach is to not work with $\pi$, but instead to work with the corresponding symplectic structure $\omega$. Recall that $\pi$ induces a map $\pi^\sharp$ between $T^*S$ and $TS$. Because $\pi$ is non-degenerate, it admits an inverse.

$$T^*S \xrightarrow{\pi^\sharp} TS \xleftarrow{\omega^\flat = (\pi^\sharp)^{-1}}$$

This inverse map defines a symplectic form $\omega$ for $S$. Moser’s argument (see Theorem 2.70 in [3]), a fundamental technique in symplectic geometry, provides a classification: two symplectic forms $\omega_0$ and $\omega_1$ on a compact connected}

2010 Mathematics Subject Classification. 53D05, 53D17.
Key words and phrases. Poisson cohomology, log symplectic, $b$-symplectic.
surface $S$ are symplectomorphic if and only if they are cohomologous in degree 2 de Rham cohomology. The first main result of this paper is a classification similar in spirit to this symplectic case. We will elaborate on the meaning of certain terms in the exposition following the statement.

**Theorem 1.1.** (Classification of log symplectic surfaces). Let $(S, D)$ be a compact connected surface with star divisor $D$. Two log symplectic forms $\omega_0$ and $\omega_1$ on $(S, D)$ are symplectomorphic if and only if they are cohomologous in the degree 2 Lie algebroid cohomology of the $b$-tangent bundle.

To understand this theorem, let us begin by considering a class of Poisson structures that have some degeneracy. We are not able to immediately work with a corresponding form $\omega$ because the map $\pi^\sharp$ will not have an inverse. The trick is to find a class of Poisson structure where we can replace the tangent bundle with a Lie algebroid $\mathcal{A}$. We can view bivectors $\pi$ in this class as non-degenerate on $\mathcal{A}$ and define an associated symplectic form on $\mathcal{A}^*$. 

\[
\mathcal{A}^* \xrightarrow{\pi^\sharp} \mathcal{A}
\]

This isomorphism allows us to use Moser-type arguments and we can hope for a classification similar to the symplectic case.

In Section 3 we will provide a more detailed description of this method, but for now we offer a sampling of relevant recent literature: Victor Guillemin, Eva Miranda, and Ana Rita Pires [6] carried out this procedure for a class of structure called $b$-Poisson [5]. Geoffrey Scott proved versions of Moser’s theorem in what he named the $b^k$-setting and used it to establish this characterization for $b^k$-Poisson surfaces (See section 6.1 of [16] and in particular Theorem 6.7). In [7], we use Moser techniques to establish local normal forms for scattering-symplectic structures on manifolds of any even dimension. Eva Miranda and Arnau Planas [12] further use these Moser techniques to establish a version of Scott’s classification that is equivariant under the action of a group.

We will add to this collection of characterizations by carrying out the method with a class of Poisson structures which are called log symplectic. Work has been done in the setting where log symplectic bivectors are degenerate on a normal crossing divisor $D$, i.e. a set of smooth hypersurfaces $Z \subset M$ that intersect transversely, see [3] [6] [8] or [15]. Given a surface $S$, we will consider all log symplectic structures. In particular, we examine log Poisson bivectors that degenerate on a set $D$ of smooth hypersurfaces $Z \subset S$ whose intersections are modeled by a finite set of lines in the plane intersecting at a point. Such a set $D$ is called a *star divisor*.

To define the $b$-tangent bundle, the Lie algebroid we will use, is subtle and requires more work than in the case of single, or two intersecting, lines. We

---

1This is an alternative approach to Olga Radko’s work [14].
address the challenge of defining the $b$-tangent bundle in this more general setting by introducing an adapted atlas, which we call an *astral atlas*. The complete details of this construction are provided in Section 2.1.

**Figure 1.2.** Some local models

| Log symplectic |
|----------------|
| **b - symplectic** |
| $\pi = x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ |
| Linear degeneracy |
| $\pi = \lambda (x^2 - y^2) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ |
| Quadratic degeneracy |
| $\pi = \lambda (x^2 - y^3) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ |
| Cubic degeneracy |

**Normal crossing**

In Section 3 we classify log symplectic surfaces up to $b$-symplectomorphism. For simplicity, we state our results here using a fixed tubular neighborhood of $D$. A more detailed and general discussion can be found in Section 3.

We realize log symplectic Poisson bivectors as non-degenerate on Richard Melrose’s $b$-tangent bundle [11]. We will show, given a surface with star divisor $(S, D = \{Z_1, \ldots, Z_m\}))$, the second Lie algebroid cohomology of the $b$-tangent bundle is

$$^bH^2(S) \simeq H^2(S) \oplus \bigoplus_i H^1(Z_i) \oplus \bigoplus_{i<j} H^0(Z_i \cap Z_j).$$

Note that this result is quite surprising since the Lie algebroid cohomology is only noticing pairwise intersection of curves. This is the cohomology featured in the classification Theorem [11].

**Remark 1.3.** (*On the Arnol’d hierarchy of planar singularities*). As we briefly mentioned above, Arnol’d introduced a hierarchy of degeneracies for Poisson structures on a surface near a singular point (introduced in Appendix 14 C [2] without proofs). In dimension 2, because the Jacobi identity is trivial, all bivectors are Poisson. Consequently, any Poisson bivector $\pi$ is locally of
the form $\pi = f \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ for some smooth function $f$ and studying the local behavior of $\pi$ is closely related to understanding the degeneracy of the function $f$. Radko [14] classified all Poisson structures with degeneracy loci modeled on singularities of type $A_0$. We classify log symplectic structures that are locally modeled by the cubic degeneracy in Figure 1.2, a singularity of type $D_4^{0,0}$.

1.2. Poisson Cohomology. Next, we turn to Poisson cohomology. We state our result for log symplectic surfaces $(S, D, \pi)$ here using a fixed tubular neighborhood of $D$. A more detailed discussion is given in Section 4.

Theorem 1.4. (Poisson cohomology of a log Poisson structure). Given a log symplectic surface $(S, D = \{Z_1, \ldots, Z_k\}, \pi)$, i.e. a surface with a set $D$ of smooth hypersurfaces $Z_i \subset S$ whose intersections are modeled by a finite set of lines in the plane intersecting at a point, the Poisson cohomology in degrees 0 and 1 is

$$H^0_\pi(S) \simeq H^0(S) \quad \text{and} \quad H^1_\pi(S) \simeq H^1(S) \oplus \bigoplus_i H^0(Z_i).$$

The cohomology in degree 2 is a direct sum of the following vector spaces:

i) A single copy of $H^2(S)$.

ii) For each hypersurfaces $Z_i$, a copy of $H^1(Z_i)$.

iii) For each pair wise intersection of two hypersurfaces $Z_i, Z_j$ with $i < j$,

$$[H^0(Z_i \cap Z_j)]^2$$

iv) For each intersection of three hypersurfaces $Z_i, Z_j, Z_k$ with $i < j < k$,

$$[H^0(Z_i \cap Z_j \cap Z_k)]^3$$

v) For each intersection of four or more hypersurfaces $Z_{i_1}, Z_{i_2}, Z_{i_3}, \ldots, Z_{i_\ell}$ with $i_1 < i_2 < i_3 < \cdots < i_\ell$, a copy of

$$[H^0(Z_{i_1} \cap Z_{i_2} \cap \cdots \cap Z_{i_\ell})]^{\ell}.$$

1.3. Reconciling our classification and Poisson cohomology. Next we will discuss our process to prove Theorem 1.1 and its relation to Theorem 1.4. Previously, trying to relate this type of classification to Poisson cohomology was a ‘comparing apples and oranges’ situation. By computing Poisson cohomology in a new way using non-standard techniques, we have found a mathematically rigorous way to incorporate these two snapshots into a bigger picture.

We compute Poisson cohomology by constructing the rigged Lie algebroid $\mathcal{R}$ that we introduced in [7]. The Lie algebroid de Rham cohomology of $\mathcal{R}$ computes the Poisson cohomology of $S$. The $b$-de Rham forms, which we use to characterize deformations of log symplectic structures, are a sub complex of
\( \mathcal{R} \)-de Rham forms. This allows us to directly relate our classifying invariants, which are \( b \)-cohomology classes, to the second Poisson cohomology.

\[
\begin{array}{cccccc}
0 & \to & C^\infty(S) & \xrightarrow{\partial_x} & C^\infty(S, TS) & \xrightarrow{\partial_x} & C^\infty(S, \wedge^2 TS) & \to & 0 \\
0 & \to & C^\infty(S) & \xrightarrow{d} & \mathcal{R} \Omega^1(S) & \xrightarrow{d} & \mathcal{R} \Omega^2(S) & \to & 0 \\
0 & \to & C^\infty(S) & \xrightarrow{d} & b \Omega^1(S) & \xrightarrow{d} & b \Omega^2(S) & \to & 0
\end{array}
\]

Lichnerowicz complex: defines Poisson cohomology
\( \mathcal{R} \)-forms: computes Poisson cohomology
\( b \)-forms: classifies log symplectic structures

This enables us to easily assess the quality of our choice of subclass in our global perspective. For instance, if a generator of \( b H^2(S) \) were to vanish as a class in \( \mathcal{R} H^2(S) \), this would suggest that while there is no trivial deformation of \( \pi \) through log symplectic structures, there may be a trivial deformation through the class of all Poisson structures. However, this is not the case when viewing log-Poisson structures as log symplectic structures, meaning the \( b \)-tangent bundle was a ‘good’ choice of setting for classifying.

Additionally, in the case where we have a divisor with at least two hypersurfaces intersecting, the \( b \)-de Rham sub-complex does not provide all of the generators of Poisson cohomology. What is happening is that these additional generators cannot be viewed as Poisson structures on the \( b \)-tangent bundle. This is concretely illustrated through Example 4.5, which can be found in Section 4.

Acknowledgement: I am grateful to Pierre Albin for sharing many wonderful conversations over numerous cups of coffee and helping me to develop the ideas of this work. Travel support was provided by Pierre Albin’s Simon’s Foundation grant \#317883. I also greatly appreciate the time Ioan Mărcut spent chatting with me both in person and via emails about deformations. His insight greatly clarified the subject for me. To conclude, I appreciate the helpful suggestions of Rui Loja Fernandes, who read a draft of the introduction.
**Index of Notation and Common Terms**

- **$(S, D)$**  A surface $S$ and a divisor $D$.
- **$D$**  A divisor on a surface $S$, that is a set of smooth hypersurfaces, i.e. curves, $Z \subset S$.
- **$Z$ defining function**  A defining function for a hypersurface $Z \subset S$, usually denoted $x$. That is, $x \in C^\infty(S)$ such that
  
  $$Z = \{ p \in S : x(p) = 0 \}$$

  and $dx(p) \neq 0$ for all $p \in Z$.
- **$bTS$**  $b$-tangent bundle over a pair $(S, D)$, the vector bundle whose sections are the vector fields on $S$ that are tangent to $Z$ for all $Z \in D$.
- **$(\mathcal{A}, [\cdot, \cdot]_A, \rho_A)$**  A Lie algebroid over a surface $S$, that is, a triple consisting of a vector bundle $\mathcal{A} \to S$, a Lie bracket $[\cdot, \cdot]_A$ on the $C^\infty(S)$-module of sections $\Gamma(\mathcal{A})$, and a bundle map $\rho_A : \mathcal{A} \to TS$ such that
  
  $$[X, fY] = \mathcal{L}_{\rho_A(x)}f \cdot Y + f[X, Y]$$

  where $X, Y \in \Gamma(\mathcal{A})$, and $f \in C^\infty(S)$.
- **$A\Omega^k(S)$**  The set $\Gamma(\wedge^k \mathcal{A}^*)$ of smooth sections of the $k$-th exterior power of the dual bundle to $\mathcal{A}$, called the $A$-de Rham forms on $M$. This is a differential complex with differential operator defined by
  
  $$(d_A \beta)(\alpha_0, \alpha_1, \ldots, \alpha_k) =$$

  $$\sum_{i=0}^{k} (-1)^i \rho_A(\alpha_i) \cdot \beta(\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_k) +$$

  $$\sum_{0 \leq i < j \leq k} (-1)^{i+j} \beta([\alpha_i, \alpha_j]_A, \alpha_0, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_k)$$

  for $\beta \in A\Omega^k(M)$, and $\alpha_0, \ldots, \alpha_k \in \Gamma(\mathcal{A})$. This cohomology of this complex is called the Lie algebroid or $A$-de Rham cohomology.
- **$^0\mathcal{A}$**  An algebroid $\mathcal{A}$ rescaled by 0 over $(S, Z)$, i.e. the vector bundle whose sections are the $\mathcal{A}$-vector fields on $S$ that are zero at the hypersurface $Z$. 


2. Log symplectic surfaces

In this section we construct a setting where our Poisson structures of interest can be viewed as symplectic structures. This will allow us to establish log symplectic analogues of familiar results, such as Darboux and Moser’s theorem.

**Definition 2.1.** A *star divisor* $D$ on a surface $S$ is a finite collection

$$D = \{c_1, c_2, \ldots, c_n\}$$

of smooth closed curves that are pairwise transverse.

For any $p \in S$, the *degree* of $p$ is the number of curves of $D$ that contain $p$. In other words, given $p \in S$,

$$\text{deg}(p) = \# \{c_i \in D : p \in c_i\}.$$  

Note intersection points of $D$ are precisely points $p \in D$ where $\text{deg}(p) \geq 2$.

2.1. Astral atlas. We can equip a surface with a star divisor $(S, D)$ with an atlas $\{(U_\alpha, \phi_\alpha)\}$, which we call an astral atlas, that models $D$ as the intersection of lines at the origin in the xy-plane.

2.1.1. *Star metric.* Let $(S, D)$ be a surface with a star divisor $D$. A *star metric* on $S$ is a Riemannian metric $g$ such that for all intersection points $p \in S$, there exists a chart $(U, \phi)$ around $p$ such that each arc in $D \cap U$ is a geodesic. We will provide a sketch of Max Neumann-Coto’s construction of such a metric. A more detailed argument can be found in Lemma 1.2 of his paper [13].

We start with any Riemannian metric $g_S$ on our surface $S$ and a regular neighborhood $N$ of $D$. By regular neighborhood we mean a union of product neighborhoods $c_i \times (-\varepsilon, \varepsilon)$ such that the intersection of any two is either empty or a square. We put a flat metric $g_N$ on $N$ to make the rectangles and polygons Euclidean and so that each $c_i$ is a geodesic with respect to $g_N$.

**Figure 2.2.** Constructing $g$.

Let $\{p_1, \ldots, p_m\}$ be the intersection points of $(S, D)$. In any compact neighborhood of each $p_j$, there is a lower bound for the lengths of essential curves in $S \setminus N$. We scale $g_S$ by some positive real number $k$ so that the lengths of these essential curves with respect to $kg_S$ are greater than the lengths of each $c_i$ with respect to $g_N$.

Using appropriate bump functions, we construct a new metric on $S$ by patching $kg_S$ and $g_N$ together.
Intuitively, this new metric makes $S$ look like a mountain range around each intersection point $p_j$ with the curves in $D$ cutting out the ravines between peaks.

2.1.2. Star charts. A star metric allows us to construct charts that model the intersection of curves in $D$ as the intersection of lines in the plane.

**Lemma 2.3.** Let $(S, D)$ be a surface with a star divisor $D$. Let $g$ be a star metric on $(S, D)$. Fix an ordering $\{c_1, \ldots, c_n\}$ of the curves in $D$. At every point $p \in S$ of degree $\geq 2$, there exists a coordinate chart $(U, (x, y))$ centered at $p$ such that the following conditions are satisfied.

1. Given $(\alpha, \beta) = \min \{(i, j) \in \mathbb{N} \times \mathbb{N} : i < j \text{ and } p \in c_i, p \in c_j\}$, $c_\alpha \cap U = \{x = 0\}$ and $c_\beta \cap U = \{y = 0\}$.
2. For each $c_\ell$ with $p \in c_\ell$, there exist real numbers $A_\ell$ and $B_\ell$ such that $c_\ell \cap U = \{A_\ell x + B_\ell y = 0\}$.

If $p \in S$ is an intersection point, then charts centered at $p$ satisfying conditions [1] and [2] from Lemma 2.3 are called star charts. If $p \in S$ is not an intersection point, then any chart around $p$ not containing an intersection point is called a star chart. An astral atlas is an atlas consisting of star charts.

![Figure 2.4. Star charts of $(S, D)$](image-url)

**Proof.** Let $(S, D)$ be a surface with a star divisor $D$. Consider an atlas of Riemannian normal coordinate charts. Refine this to another atlas $\{(U, \phi)\}$ by taking charts that are also star charts.

Fix an ordering $\{c_1, \ldots, c_n\}$ on the elements of $D$. Given an intersection point $p \in S$ and a chart $(U, \phi)$, centered at $p$, from our atlas, we will take a linear isomorphism of this chart such that the two curves passing through $p$ with the smallest indices are mapped to $\{x = 0\}$ and $\{y = 0\}$ respectively. An astral atlas consists of charts that have been transformed according to this ordering on $D$. \qed
2.1.3. Transition functions between star charts. Let \((S, D, g)\) be a surface, star divisor, and star metric. Given an ordering on \(D\), let \(\{(U_\alpha, \phi_\alpha)\}\) be an astral atlas for the triple \((S, D, g)\). Let \((U, \phi) = (U, (x, y))\) and \((V, \rho) = (V, (\tilde{x}, \tilde{y}))\) be two star charts around an intersection point \(p \in S\). Further assume \(\text{deg}(p) \geq 3\).

Then \(\phi(U \cap D)\) is the intersection of at least 3 lines at a point. Let \(c_1, c_2, c_3\) denote the curves of minimal index passing through \(p\). Then \(c_1\) is locally given by \(\{x = 0\}\) and \(c_2\) is locally given by \(\{y = 0\}\). Further, \(c_3\) is defined by an equation of the form \(ax + by = 0\) for real numbers \(a, b\).

Consider the transition functions between \(\phi(U)\) and \(\rho(V)\).

In \(\rho(V)\), let \(\tilde{y}\) define the image of \(c_1\) and let \(\tilde{x}\) define the image of \(c_2\). Then \(\tilde{x} = f x\) and \(\tilde{y} = g y\) for positive functions \(f, g\).

![Transition functions between star charts at a point with degree at least three.](image)

Figure 2.5. Transition functions between star charts at a point with degree at least three.

Then \(ax + by = A\tilde{x} + B\tilde{y}\) for \(A, B \in \mathbb{R}\). After substituting, \(ax + by = Af x + Bg y\).

Thus, \(f = a/A\) and \(g = b/B\).

As a consequence, we conclude that around intersection points of \(\text{deg} \geq 3\), the class of transition functions that preserve star charts simply scale coordinates.

2.2. \textbf{b-tangent bundle over a star atlas.} Given a surface \(S\) with star divisor \(D\), we will use a star atlas to construct a vector bundle whose smooth sections are vector fields tangent to \(D\). We will begin by defining a vector bundle over each chart. Then, by arguing that all the transition maps are compatible with this structure, we will show that we have in fact constructed a rank 2 vector bundle. Let \(p_1, \ldots, p_m\) be all the points of \(S\) with degree
greater than 2. Consider the surface \( \widetilde{S} = S \setminus \{p_1, \ldots, p_m\} \). Then
\[
(\widetilde{S}, \widetilde{D}) = (S \setminus \{p_1, \ldots, p_m\}, D \cap \widetilde{S})
\]
is a surface with normal crossing divisor, that is, a surface with a set of smooth curves that intersect transversely. The b-tangent bundle \( b\mathcal{T}\widetilde{S} \), referred to as the log tangent bundle \( \log \mathcal{T}\widetilde{S} \) in [5], is the vector bundle whose smooth sections are the vector fields of \( \widetilde{S} \) that are tangent to \( \widetilde{D} \). In other words, the b-tangent bundle has smooth sections
\[
\left\{ u \in C^\infty(\widetilde{S}, \mathcal{T}\widetilde{S}) : u|_c \in C^\infty(c, Tc) \text{ for all } c \in \widetilde{D} \right\}.
\]

This will be our vector bundle away from \( p_1, \ldots, p_m \), the intersection points of degree greater than two. Let \( \{(U_\alpha, \phi_\alpha)\} \) be a star atlas of \((S, D)\).

Consider a degree \( k \) point \( p \in S \) where \( k \geq 3 \). Then \( p \) sits in some curves \( c_1, \ldots, c_k \). Let \((U, (x, y))\) be a star chart around \( p \).

Consider the vector fields, depicted in Figure 2.6,
\[
V = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad \text{and} \quad W = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.
\]
Away from \( c_1 \) and \( c_2 \), these generate \( T U \). Further, away from the origin, \( V \) is tangent to all lines passing through the origin.

**Figure 2.6.** Vector fields on \( TU \).

![Vector fields](image.png)

Let \( c_3, \ldots, c_k \) be locally defined by \( A_3x + B_3y, \ldots, A_kx + B_ky \) respectively, i.e. \( c_i \cap U = \{A_ix + B_iy = 0\} \). Then we define two vector fields
\[
V = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad \text{and} \quad \tilde{W} = \prod_{i=3}^{k} \left(A_ix + B_iy\right) \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}\right). \quad (2.1)
\]
In the following lemma we will show that $V$ and $\tilde{W}$ provide a basis for the space of smooth vector fields tangent to $D \cap U$.

**Lemma 2.7.** Let $(U, (x, y))$ be a star chart around a point $p \in S$ of degree at least 3. Any vector field $V \in C^\infty(U; TU)$ that is tangent to $D \cap U$ can be expressed as a $C^\infty(U)$-linear combination of the vector fields given in equation (2.1).

**Proof.** Given a star chart $(U, (x, y))$ centered at $p$, by definition the set $D \cap U$ is given by the lines $x = 0, y = 0, A_3x + B_3y = 0, \ldots, A_kx + B_ky = 0$. We want to find all vector fields that are tangent to these lines.

An arbitrary smooth vector field in $U$ has the form

$$V = a(x, y) \frac{\partial}{\partial x} + b(x, y) \frac{\partial}{\partial y}$$

for $a(x, y), b(x, y) \in C^\infty(U)$.

**Claim:** The functions $a$ and $b$ satisfy

$$a = x \left( \frac{h y \prod_{i=4}^{k} (A_i x + B_i y)}{A_3} + c_3 \right) \quad \text{and} \quad b = y \left( c_3 - \frac{h x \prod_{i=4}^{k} (A_i x + B_i y)}{B_3} \right)$$

for arbitrary smooth functions $h, c_3 \in C^\infty(U)$.

We will prove this claim by induction on the degree of $p$.

**Base case:** Assume $\deg p = 3$. Notice that $V$ is tangent to the line $x = 0$ if and only if

$$\left\langle V \cdot \frac{\partial}{\partial x} \right\rangle_{g_2} \bigg|_{\{x=0\}} = 0.$$

Equivalently, $V$ is tangent to $x = 0$ if and only if $a(x, y)|_{\{x=0\}} = 0$, i.e.

$$a(x, y) = a'(x, y)x$$

for $a'(x, y) \in C^\infty(U)$. Similarly, $V$ is tangent to the line $y = 0$ if and only if $b(x, y)|_{\{y=0\}} = 0$. That is,

$$b(x, y) = b'(x, y)y$$

for $b'(x, y) \in C^\infty(U)$. Because the gradient of a function is normal to its level sets, $V$ is tangent to $A_3x + B_3y = 0$ precisely when

$$\left\langle V \cdot \left( A_3 \frac{\partial}{\partial x} + B_3 \frac{\partial}{\partial y} \right) \right\rangle_{g_2} \bigg|_{\{A_3x+B_3y=0\}} = 0.$$

Equivalently, $V$ is tangent to $A_3x + B_3y = 0$ if and only if

$$A_3a(x, y) + B_3b(x, y) = c_3(x, y)(A_1x + B_1y)$$

(2.4)
for $c_3(x, y) \in \mathcal{C}^\infty(U)$.

We want to find all possible $a, b$ satisfying equations (2.2), (2.3), and (2.4). Substituting $a = a'x$ and $b = b'y$, we have

$$A_1a'x + B_1b'y = c_3(A_3x + B_3y).$$

Rearranging terms,

$$xA_3(a' - c_3) = yB_3(c_3 - b').$$

Notice that $y$ divides both the right and left hand sides, and thus

$$A_3(a' - c_3) = ey$$

for some $e \in \mathcal{C}^\infty(U)$. Further, $B_1(c_3 - b') = ex$. We can solve for $a'$ and $b'$ respectively:

$$a' = \frac{ey}{A_3} + c_3 \quad \text{and} \quad b' = c_3 - \frac{ex}{B_3}.$$

Thus, we have shown that if $V$ is tangent to the lines $x = 0, y = 0$, and $A_3x + B_3y = 0$, then

$$a = x \left( \frac{ey}{A_3} + c_3 \right) \quad \text{and} \quad b = y \left( c_3 - \frac{ex}{B_3} \right).$$

Note that for any smooth functions $c_3, e \in \mathcal{C}^\infty(U)$, we have

$$a = \mathcal{O}(x), \ b = \mathcal{O}(y), \ \text{and} \ A_3a + B_3b = \mathcal{O}(A_3x + B_3y).$$

Consequently, there are no restrictions on $c_3$ or $e$ and we have a characterization of all vector fields that are tangent to these three lines.

**Induction step:** Assume that the claim is true for a point $p$ satisfying $\deg p = n - 1$. We will verify the formula at a point $p$ of degree $n$.

The set $D \cap U$ is given by the lines

$$x = 0, y = 0, A_3x + B_3y = 0, \ldots, A_nx + B_ny = 0.$$  

We want to find all vector fields that are tangent to these lines. Since a vector field tangent to these lines is certainly also tangent to the lines

$$x = 0, y = 0, A_3x + B_3y = 0, \ldots, A_{n-1}x + B_{n-1}y = 0,$$

then we know that $a$ and $b$ at least satisfy

$$a = x \left( \frac{hy \prod_{i=4}^{n-1} (A_i x + B_i y)}{A_3} + c_3 \right), \ b = y \left( c_3 - \frac{hx \prod_{i=4}^{n-1} (A_i x + B_i y)}{B_3} \right).$$

Notice that $V$ is tangent to the line $A_nx + B_ny = 0$ precisely when

$$\left\langle \left( A_n \frac{\partial}{\partial x} + B_n \frac{\partial}{\partial y} \right) \right|_{\{A_n x + B_n y = 0\}} = 0.$$
Equivalently, $V$ is tangent to $A_n x + B_n y = 0$ if and only if

$$A_n a(x, y) + B_n b(x, y) = c_n(x, y)(A_n x + B_n y)$$

for $c_n(x, y) \in C^\infty(U)$.

By substituting in our expressions for $a$ and $b$ and rearranging terms, we have

$$c_3(A_n x + B_n y) + hxy \prod_{i=4}^{n-1} (A_i x + B_i y) \left( \frac{A_n}{A_3} - \frac{B_n}{B_3} \right) = c_n(A_n x + B_n y).$$

Note that $\frac{A_n}{A_3} - \frac{B_n}{B_3} = 0$ if and only if $A_n x + B_n y = 0$ defines the same line as $A_3 x + B_3 y = 0$. Notice that $A_n x + B_n y$ divides both the right hand side and the term $c_3(A_n x + B_n y)$, and thus $h = k(A_n x + B_n y)$ for some smooth function $k \in C^\infty(U)$.

Thus, we have shown if $V$ is tangent to the lines

$$x = 0, y = 0, A_3 x + B_3 y = 0, \ldots, A_n x + B_n y = 0,$$

then

$$a = x \left( ky \prod_{i=4}^{n} \frac{A_i x + B_i y}{A_3} + c_3 \right), \quad b = y \left( c_3 - \frac{kx \prod_{i=4}^{n} (A_i x + B_i y)}{B_3} \right).$$

Note that for any smooth functions $c_3, k \in C^\infty(U)$, we have

$$a = O(x), \quad b = O(y), \ldots, \quad A_n a + B_n b = O(A_n x + B_n y).$$

Consequently, there are no restrictions on $c_3$ or $k$ and we have a characterization of all vector fields that are tangent to a set of $n$ lines passing through the origin. By choosing $c_3 = 1, k = 0$ and $c_3 = A_3 x - B_3 y, k = 2A_3 B_3$, we have a local frame provided by the vector fields defined in equation 2.1.

As discussed in Section 2.1.3, the transition functions between star charts around points of degree greater than 2 simply scale $x$ and $y$. Thus this vector structure is preserved under the transition functions in an astral atlas, and we have constructed a vector bundle over $(S, D)$.

This vector bundle comes equipped with a Lie algebroid structure. The anchor map is inclusion into $TS$ and the Lie bracket is induced by the standard Lie bracket on $TS$.

Let us consider an explicit example of a $b$-tangent bundle over a star divisor.

**Example 2.8.** Let $\mathbb{R}^2$ be equipped with the standard Euclidean coordinates
and let star divisor $D$ consist of the lines 
\[ \{x = 0\}, \{y = 0\}, \{x + y = 0\}, \text{ and } \{x - y = 0\}. \]
The $b$-tangent bundle is generated by the vector fields 
\[ x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, (x - y)(x + y) \left( x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right) \]
and the $b$-cotangent bundle is generated by 
\[ \frac{dx}{x} + \frac{dy}{y}, \frac{1}{(x - y)(x + y)} \left( \frac{dx}{x} - \frac{dy}{y} \right). \]

**Definition 2.9.** A log symplectic structure on a surface $S$ with star divisor $D$ is a 2-form $\omega \in b\Omega^2(S) = C^\infty (S; \Lambda^2 (bT^*S))$ satisfying 
\[ d\omega = 0 \text{ and } \omega^n \neq 0. \]
The form $\omega$ induces a map $\omega^b$ between the $b$-tangent and $b$-cotangent bundles. 
\[ bTS \xrightarrow{\omega^b} bT^*S \]
The inverse map is induced by a bivector $\pi \in C^\infty (S; \Lambda^2 (bTS))$. This bivector is called a log Poisson structure on $(S, D)$.

Next, we provide an explicit example of a log symplectic structure on a compact surface.

**Example 2.10. A cubic log symplectic surface.**

Consider the sphere $S^2$ with spherical coordinates $0 \leq \theta < 2\pi$, $0 \leq \phi \leq \pi$.
We have a star divisor $D$ consisting of 
\[ \{\sin(\theta - \pi/6) = 0\}, \]
\[ \{\sin(\theta - \pi/2) = 0\}, \text{ and } \]
\[ \{\sin(\theta - 5\pi/6) = 0\}. \]
The form 
\[ \omega = \frac{d\theta \wedge d\phi}{\sin(\theta - \pi/6) \sin(\theta - \pi/2) \sin(\theta - 5\pi/6)} \]
is a log symplectic structure on $S^2$.

The associated Poisson bivector is given by
\[ \pi = \sin(\theta - \pi/6) \sin(\theta - \pi/2) \sin(\theta - 5\pi/6) \frac{\partial}{\partial \phi} \wedge \frac{\partial}{\partial \theta}. \]
3. Classification

The first step in classifying log symplectic surfaces is computing the Lie algebroid cohomology of the $b$-tangent bundle.

3.1. $b$-de Rham cohomology. Given a surface $S$ with a collection of transverse curves $\{Z_1, \ldots, Z_k\}$, the de Rham cohomology of the associated $b$-tangent bundle is

\[ bH^p(S) \simeq H^p(S) \oplus \bigoplus_i H^{p-1}(Z_i) \oplus \bigoplus_{i<j} H^{p-2}(Z_i \cap Z_j), \tag{3.1} \]

a fact originally pointed out in Appendix A.23 of [5]. Remarkably, this decomposition still holds for the more general case of a surface with star divisor!

**Theorem 3.1.** Let $(S, D)$ be a surface $S$ with star divisor $D = \{Z_1, \ldots, Z_k\}$. The Lie algebroid cohomology of the $b$-tangent bundle over $(S, D)$ is given by (3.1).

**Proof.** Let $(S, D)$ be a surface $S$ with star divisor $D$. We will compute the Lie algebroid cohomology of the $b$-tangent bundle over $(S, D)$ using a filtration $\mathcal{F}$ of cochain complexes.

**Constructing filtration $\mathcal{F}$.**

Given a subset $I \subseteq D$, let $b\Omega^*(S, I)$ denote the complex of b-de Rham forms of the $b$-tangent bundle over $(S, I)$. In this notation, we want to compute the cohomology of the complex $b\Omega^*(S, D)$.

Consider the sets

\[ \mathcal{F}_k := \text{v.s. Span} \left\{ \bigcup_{I \subseteq D, |I| = k} b\Omega^*(S, I) \right\}. \]

Notice that $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$ for all $k$. Each set $\mathcal{F}_k$ is a group under addition and has the structure of a module with scalars from the ring $\mathcal{F}_0 = C^\infty(S)$.

Note that $b\Omega^*(S, I) \subseteq b\Omega^*(S, D)$ for all $I \subseteq D$. Consequently, the modules $\mathcal{F}_k \subseteq b\Omega^*(S, D)$ for all $k$. It is easy to check that $\mathcal{F}_k$ is closed under the differential of $b\Omega^*(S, D)$. Thus we can define a differential $\mathcal{F}_k \xrightarrow{d} \mathcal{F}_{k+1}$ inherited from the differential on $b\Omega^*(S, D)$.

For each $k$, we have a chain complex:

\[ 0 \to \mathcal{F}_k \xrightarrow{d} \mathcal{F}_{k+1} \xrightarrow{d} \mathcal{F}_k \to 0. \]

We will employ the following filtration of cochain complexes:

\[ \mathcal{F} = \left\{ \Omega^*(S) = \mathcal{F}_0^* \subseteq \mathcal{F}_1^* \subseteq \mathcal{F}_2^* \subseteq \cdots \subseteq \mathcal{F}_{|D| - 1}^* \subseteq \mathcal{F}_{|D|}^* = b\Omega^*(S, D) \right\}. \]
In the next step, we consider various short exact sequences of complexes involving the terms in $\mathcal{F}$.

**A short exact sequence of complexes.**

The inclusions $\mathcal{F}^*_k \rightarrow \mathcal{F}^*_k$ of cochain complexes from filtration $\mathcal{F}$ fit into a short exact sequence of complexes

$$0 \rightarrow \mathcal{F}^*_{k-1} \rightarrow \mathcal{F}^*_k \rightarrow \mathcal{C}^*_k \rightarrow 0$$

where

$$\mathcal{C}^*_k = \mathcal{F}^*_k / \mathcal{F}^*_{k-1}.$$ 

The differential $\mathcal{d}d$ on $\mathcal{C}^*_k$ is induced by $d$ on $\mathcal{F}^*_k$: if $P$ is the projection $\mathcal{F}^*_k \rightarrow \mathcal{F}^*_k / \mathcal{F}^*_{k-1}$, then $\mathcal{d}d(\eta) = P(d(\theta))$ where $\theta \in \mathcal{F}^*_k$ is any form such that $P(\theta) = \eta$. Hence $(\mathcal{d}d)^2 = 0$ and $(\mathcal{C}^*_k, \mathcal{d}d)$ is a complex.

Given one of these short exact sequences, we will show that the boundary maps in the induced long exact sequences in cohomology are zero. This feature will facilitate our arrival at a nice description of $bH^p(S)$.

**Computing $H^p(\mathcal{F}^*_1)$.**

Let $k = 1$ and consider the short exact sequence

$$0 \rightarrow \mathcal{F}^*_0 \rightarrow \mathcal{F}^*_1 \rightarrow \mathcal{C}^*_1 \rightarrow 0$$

defined above. Note that $\mathcal{F}^*_0 = \Omega^*(S)$.

Let $D = \{Z_1, Z_2, \ldots, Z_\ell\}$ be our star divisor and consider defining functions $x_1, x_2, \ldots, x_\ell$ such that $Z_i = \{x_i = 0\}$. For each $i$, let $\chi_i : S \rightarrow \mathbb{R}$ be a smooth bump function supported in a neighborhood of $Z_i$ and with constant value one at $Z_i$.

An element $\mu \in \mathcal{F}^*_1$ can be expressed as

$$\mu = \sum_{i=1}^\ell \chi_i \cdot \frac{dx_i}{x_i} \land \alpha_i + \beta$$

where $\alpha_i \in \Omega^{p-1}(Z_i)$ and $\beta \in \Omega^p(S)$.

Given $\mu \in \mathcal{F}^*_1$, we write $\mathcal{R}(\mu) = \beta$ and $\mathcal{S}(\mu) = \mu - \mathcal{R}(\mu)$ for ‘regular’ and ‘singular’ parts. It is easy to see that $\mathcal{R}(d\mu) = d(\mathcal{R}(\mu))$ and $\mathcal{S}(d\mu) = d(\mathcal{S}(\mu))$. Thus we have a splitting

$$\mathcal{F}^*_1 = \Omega^*(S) \oplus \mathcal{C}^*_1$$

as complexes and $H^p(\mathcal{F}^*_1) \simeq H^p(S) \oplus H^p(\mathcal{C}^*_1)$. We are left to compute the cohomology group of the quotient complex $\mathcal{C}^*_1$. 
As mentioned above, \( P(\mu) \in \mathcal{G}_1^* \) is of the form
\[
P(\mu) = \sum_{i=1}^{\ell} \frac{dx_i}{x_i} \land \alpha_i
\]
for \( \alpha_i \in \Omega^{p-1}(Z_i) \). Then
\[
dP(\mu) = -\sum_{i=1}^{\ell} \frac{dx_i}{x_i} \land d\alpha_i.
\]

Note that as an element of \( \mathcal{F}_1^* \),
\[
\mu|_{Z_i} = \frac{dx_i}{x_i} \land \alpha_i.
\]

Thus, the condition that \( \mu|_{Z_i} = 0 \) implies that \( \alpha_i = 0 \). Consequently, given these choices of \( Z_i \) defining functions \( x_1, \ldots, x_\ell \), the set \( \ker(d : \mathcal{C}^p \to \mathcal{C}^{p+1}) \) can be identified with
\[
\bigoplus_{i=1}^{\ell} \{ \alpha_i \in \Omega^{p-1}(Z_i) : d\alpha_i = 0 \}.
\]
Further, the set \( \text{im}(d : \mathcal{C}^{p-1} \to \mathcal{C}^p) \) can be identified with
\[
\bigoplus_{i=1}^{\ell} \{ \alpha_i \in \Omega^{p-1}(Z_i) : \alpha_i = d\gamma_i, \gamma_i \in \Omega^{p-2}(Z_i) \}.
\]

For completeness, we next consider change of \( Z_i \) defining function. However, by a computation which can be found in example 2.11 of \cite{7}, these sets are invariant under change of \( Z_i \) defining functions. Thus
\[
H^p(\mathcal{G}_1^*) \simeq \bigoplus_i H^{p-1}(Z_i) \text{ and } H^p(\mathcal{F}_1^*) \simeq H^p(S) \oplus \bigoplus_i H^{p-1}(Z_i).
\]

**Computing \( H^p(\mathcal{F}_2^*) \).**

Let \( k = 2 \) and consider the short exact sequence
\[
0 \to \mathcal{F}_1^* \to \mathcal{F}_2^* \to \mathcal{C}_2^* \to 0.
\]

Let \( D = \{Z_1, Z_2, \ldots, Z_\ell\} \) be our star divisor. For each pair of curves \( Z_i, Z_j \) with \( i < j \), there is a pair of tubular neighborhoods \( \tau_i = Z_i \times (-\varepsilon, \varepsilon)_{x_{ij}} \) of \( Z_i \) and \( \tau_j = Z_j \times (-\varepsilon, \varepsilon)_{y_{ij}} \) of \( Z_j \) such that
\[
\left[ \frac{\partial}{\partial x_{ij}}, \frac{\partial}{\partial y_{ij}} \right] = 0.
\]

For existence of such neighborhoods see, for instance, section 5 of \cite{1}.

We begin by noting that \( \mathcal{F}_2^1 = \mathcal{F}_1^1 \).

For existence of such neighborhoods see, for instance, section 5 of \cite{1}.
Let $\chi_{i,j}$ be a smooth bump function supported near $Z_i \cup Z_j$ and with constant value one near $Z_i \cup Z_j$. An element $\mu \in \mathcal{F}_2^*$ can be expressed as

$$\mu = \sum_{i<j} \chi_{i,j} \cdot \frac{dx_{ij}}{x_{ij}} \wedge \frac{dy_{ij}}{y_{ij}} \alpha_{ij} + \beta$$

where $\alpha_{ij} \in \Omega^0(Z_i \cap Z_j)$ and $\beta \in \mathcal{F}^2_1$.

Because $\mathcal{F}^*_1$ is closed under $d$ and $d\mu = 0$ for all $\mu \in \mathcal{F}^*_2$, we have a splitting

$$\mathcal{F}_2^* = \mathcal{F}_1^* \oplus \mathcal{C}_2^*$$

as complexes and $H^p(\mathcal{F}_2^*) \simeq H^p(\mathcal{F}_1^*) \oplus H^p(\mathcal{C}_2^*)$. We are left to compute the cohomology group of the quotient complex $\mathcal{C}_2^*$. In this instance, $\mathcal{C}_1^* = 0$ and we are computing the cohomology of the sequence $0 \to \mathcal{C}_2^* \to 0$. Equivalently, we are left to identify the set $\mathcal{C}_2^*$.

Any $P(\mu) \in \mathcal{C}_2^*$ is of the form

$$P(\mu) = \sum_{i<j} \frac{dx_{ij}}{x_{ij}} \wedge \frac{dy_{ij}}{y_{ij}} \alpha_{ij}$$

for $\alpha_{ij} \in \Omega^0(Z_i \cap Z_j)$. In other words, each $\alpha_{ij}$ is a function defined on a set of discrete points.

Then the set $\mathcal{C}_2^*$ can be identified with

$$\bigoplus_{i<j} \{ \alpha_{ij} \in \Omega^{p-2}(Z_i \cap Z_j) \}.$$ 

By the computation which can be found in example 2.11 of [7], these sets are invariant under change of $Z_i$ and $Z_j$ defining functions. Thus

$$H^p(\mathcal{C}_2^*) \simeq \bigoplus_{i<j} H^{p-2}(Z_i \cap Z_j)$$

and

$$H^p(\mathcal{F}_2^*) \simeq H^p(S) \oplus \bigoplus_{i} H^{p-1}(Z_i) \oplus \bigoplus_{i<j} H^{p-2}(Z_i \cap Z_j).$$

In our final step, we will show that the cohomology of $\mathcal{C}_k^*$ vanishes for $k \geq 3$, and thus $H^p(\mathcal{F}_2^*)$ actually computes $bH^p(S)$.

**Computing $H^p(\mathcal{F}_k^*)$ for $k \geq 3$.**

Fix $k \geq 3$ and consider the short exact sequence

$$0 \to \mathcal{F}_{k-1}^* \to \mathcal{F}_k^* \to \mathcal{C}_k^* \to 0.$$ 

Given any $p \in S$ such that at least the $k$ curves in the set $I = \{Z_1, \ldots, Z_k\}$ intersect at $p$, there exists a star chart $(U_{p,I}, x, y)$ centered at $p$ such that

$$Z_1 = \{x = 0\}, Z_2 = \{y = 0\}, \ldots, Z_k = \{A_k x + B_k y = 0\}.$$
Let \( \chi_{p,I} \) be a smooth bump function supported in \( U_{p,I} \) and with constant value one in a neighborhood of \( p \).

Any element \( \mu \in \mathcal{F}^1_k \) can be expressed as

\[
\mu = \sum_{p, I} \chi_{p,I} \prod_{i=3}^{k} \frac{1}{(A_i x + B_i y)} \left( \frac{dx}{x} - \frac{dy}{y} \right) \wedge \alpha_{p,I} + \beta
\]

for \( \alpha_{p,I} \in \Omega^0(\{p\}) \) and \( \beta \in \mathcal{F}^1_{k-1} \). Any element \( \mu \in \mathcal{F}^2_k \) can be expressed as

\[
\mu = \sum_{p, I} \chi_{p,I} \prod_{i=3}^{k} \frac{1}{(A_i x + B_i y)} \frac{dx \wedge dy}{xy} \wedge \alpha_{p,I} + \beta
\]

for \( \alpha_{p,I} \in \Omega^0(\{p\}) \) and \( \beta \in \mathcal{F}^2_{k-1} \). Note that each \( \alpha_{p,I} \) is simply a real number.

Given \( \mu \in \mathcal{F}^\ell_k \), we write \( R(\mu) = \beta \) and \( S(\mu) = \mu - R(\mu) \) for 'regular' and 'singular' parts. One can check that \( R(d\mu) = d(R(\mu)) \) and \( S(d\mu) = d(S(\mu)) \).

Thus we have a splitting

\[
\mathcal{F}^* = \mathcal{F}^*_{k-1} \oplus \mathcal{C}^*_{k}
\]

as complexes and \( H^p(\mathcal{F}^*) \simeq H^p(\mathcal{F}^*_{k-1}) \oplus H^p(\mathcal{C}^*_{k}) \). We are left to compute the cohomology group of the quotient complex \( \mathcal{C}^*_{k} \).

We will first compute the cohomology at \( \mathcal{C}^1_k \). Any \( P(\mu) \in \mathcal{C}^1_k \) is of the form

\[
P(\mu) = \sum_{p, I} \chi_{p,I} \prod_{i=3}^{k} \frac{1}{(A_i x + B_i y)} \left( \frac{dx}{x} - \frac{dy}{y} \right) \wedge \alpha_{p,I}
\]

for \( \alpha_{p,I} \in \Omega^0(\{p\}) \). Then, as an element of \( \mathcal{C} \),

\[
dP(\mu) = \sum_{p, I} \chi_{p,I} \prod_{i=3}^{k} \frac{1}{(A_i x + B_i y)} \frac{dx \wedge dy}{xy} \wedge (k-2)\alpha_{p,I}.
\]

Thus \( \ker(d : \mathcal{C}^1_k \to \mathcal{C}^2_k) = \{\alpha_{p,I} = 0\} \) and \( H^1(\mathcal{C}^*_{k}) = 0 \) for all \( k \geq 3 \).

Next, we compute the cohomology at \( \mathcal{C}^2_k \). Any element \( P(\mu) \in \mathcal{C}^2_k \) is of the form

\[
P(\mu) = \sum_{p, I} \chi_{p,I} \prod_{i=3}^{k} \frac{1}{(A_i x + B_i y)} \frac{dx \wedge dy}{xy} \wedge \alpha_{p,I}
\]
for \( \alpha_p, \tau \in \Omega^0(\{ p \}) \). We have \( dP(\mu) = 0 \) so \( \ker(d : C^2_k \to 0) = C^2_k \). By equation (3.2), \( \im(d : C^1_k \to C^2_k) = C^2_k \). As desired, we have shown that \( H^2(C^*_k) = 0 \) for \( k \geq 3 \) and have completed the proof of the theorem. \( \square \)

**Remark 3.2. Spectral Sequences.** We can recontextualize the proof of Theorem 3.1 in the language of spectral sequences. We will employ the notation used in [4]. The differential complex \((b\Omega^*(S), d)\) is a filtered complex with filtration

\[ 0 \subseteq \Omega^*(S) = \mathcal{F}^*_0 \subseteq \mathcal{F}^*_1 \subseteq \mathcal{F}^*_2 \subseteq \cdots \subseteq \mathcal{F}^*_m = \Omega^*(S). \]

Then \( E^0_{d,k} = C^d_k \) and the associated graded complex of \((b\Omega^*(S), d)\) is

\[ \bigoplus_{k=1}^m C^*_k. \]

The computation above shows that the spectral sequence collapses at the first page and

\[ \bigoplus_{k=1}^m E^1_{d,k} = \bigoplus_{k=1}^m H^d(C^*_k) \simeq bH^d(S), \]

or that the cohomology of the associated graded complex computes the cohomology of \((b\Omega^*(S), d)\).

### 3.2. A log Darboux theorem.

Darboux’s theorem gives us a local description of Log Poisson structures which enables us to identify the rigged algebroid and compute Poisson cohomology. We establish this type of normal form theorem using a Moser-type argument in a neighborhood of \( p \in D \). Let \((S, D)\) be a surface \( S \) with star divisor \( D \) and assume \( p \in D \) has degree \( k \). In a star chart centered at \( p \), consider the log symplectic form

\[ \omega_0 = \left( \frac{1}{xy} \prod_{i=3}^k \frac{1}{(A_i x + B_i y)} + P \right) dx \wedge dy \quad (3.3) \]

where

\[ P = \sum_i \frac{\lambda_i}{x(A_i x + B_i y)} + \frac{\delta_i}{y(A_i x + B_i y)} + \sum_{i<j} \frac{\nu_{ij}}{(A_i x + B_i y)(A_j x + B_j y)} \]

for real numbers \( \lambda_i, \delta_i, \nu_{ij} \).

**Proposition 3.3.** Let \( \omega \) be a log symplectic form on \((S, D)\). Given a degree \( k \) point \( p \in D \), there exists a neighborhood \( U \) of \( p \) such that on \( U \) there is a \( b \)-symplectomorphism pulling \( \omega \) back to \((3.3)\).
Proof. Let \((S, D)\) be a surface \(S\) with star divisor \(D\) and assume \(p \in D\) has degree \(k\). Let \(\omega\) be a log symplectic structure on \(S\). The proof of Theorem 3.1 tells us that at any intersection point \(p \in D\), there exists a star chart \(U\) such that \(\omega\) is expressible as \(\omega_0 + \nu\) where \(\nu \in b\Omega^2(U)\) is an exact log 2-form and \(\omega_0\) is the log 2-form given by equation (3.3). Further,

\[
\omega|_p = \frac{1}{xy} \prod_{i=3}^{k} \frac{1}{(A_i x + B_i y)} dx \wedge dy
\]

and \(\nu|_p = 0\). We have that

\[
\omega - \omega_0 = \nu = d\gamma
\]

for some \(\gamma \in b\Omega^1(U)\). Note that \(\gamma\) is not a primitive for any terms of the form

\[
\frac{1}{xy} \prod_{i=3}^{k} \frac{c}{(A_i x + B_i y)} dx \wedge dy
\]

for \(c \in \mathbb{R}\). Consequently, when considered as a \(b\)-form, \(\gamma\) vanishes at the point \(p\) because it is not ‘maximally singular’.

Now we will proceed by the standard relative Moser argument (See [3] Sec. 7.3 for the smooth setting, and [16] Thm 6.4 for the \(b\)-symplectic version). Let \(\omega_t = (1 - t)\omega_0 + t\omega\). Then \(\frac{d\omega_t}{dt} = \omega - \omega_0 = d\gamma\). Because \(\gamma\) is a log one form, the vector field defined by \(i_v \omega_t = -\gamma\) is a log vector field and its flow fixes the divisor \(D \cap U\). Thus we can integrate \(\nu_t\) to an isotopy that fixes \(D \cap U\). This isotopy is the desired log-symplectomorphism. \(\square\)

3.3. Global Moser for log symplectic manifolds.

**Proposition 3.4.** Let \(S\) be a compact surface with a star divisor \(D\) and let \(\omega_0\) and \(\omega_1\) be two log symplectic forms on \((S, D)\). Suppose that there is a family of log symplectic forms \(\omega_t\) from \(\omega_0\) to \(\omega_1\) defined for \(0 \leq t \leq 1\). If the \(b\)-cohomology class \([\omega_t]\) is independent of \(t\), then there exists a family of diffeomorphisms

\[
\gamma_t : S \to S \text{ for } 0 \leq t \leq 1
\]

such that

\[
\gamma_t|_D \text{ is the identity map on } D \text{ and } \gamma_t^* \omega_t = \omega_0.
\]

Because the proof of Proposition 3.4 is very similar to the proof of Proposition 3.3 and follows closely the proofs of Theorem 38 in [6] or Theorem 6.5 in [16], we will omit the details. This log version of the global Moser theorem completes the proof of Theorem 1.1 and gives us a classification of log symplectic surfaces by \(b\)-de Rham cohomology classes.
4. Poisson Cohomology

To compute the Poisson cohomology of star log symplectic surfaces, we will use the method of rigged Lie algebroids that we introduced in [7]. In particular, we construct a Lie algebroid that is isomorphic to the Poisson Lie algebroid of $\pi$.

**Definition 4.1.** Let $(S, D, \omega)$ be a star log symplectic surface of the $b$-tangent bundle $(bTS, \rho : bTS \to TS)$ over $(S, D)$. The rigged Lie algebroid of $\omega$ is the vector bundle whose space of sections is

$$\{ u \in C^\infty(S; bTS) \mid i_u \omega \in \rho^*(T^*S) \} ,$$

i.e. the $b$-vector fields that when contracted into $\omega$ “smoothen” it into a smooth one form. Alternatively, the sections of the dual rigged bundle $\Gamma(R^*)$ are an extension to $S$ of the image $\omega^\flat(\Gamma(TS))$ away from $D$.

As discussed in [8], the rigged Lie algebroid is isomorphic to the Poisson Lie algebroid $T^*S$ with anchor map $\pi^\flat = (\omega^\flat)^{-1}$. However, completing the computation of cohomology using the de Rham cohomology of the rigged Lie algebroid is much more tractable.

**4.1. Constructing the rigged Lie algebroid.** Using Darboux coordinates for a log symplectic form $\omega$ on $(S, D)$, we will identify the rigged Lie algebroid. Recall that in a neighborhood of an intersection point $p$ of degree $k \geq 3$, $\omega$ can be expressed as

$$\omega_0 = \left( \frac{1}{xy} \prod_{i=3}^{k} \frac{1}{(A_i x + B_i y)} + P \right) dx \wedge dy$$

where

$$P = \sum_i \frac{\lambda_i}{x(A_i x + B_i y)} + \frac{\delta_i}{y(A_i x + B_i y)} + \sum_{i<j} \frac{\nu_{ij}}{(A_i x + B_i y)(A_j x + B_j y)}$$

for real numbers $\lambda_i, \delta_i, \nu_{ij}$. Then the space

$$\{ u \in C^\infty(S; bTS) \mid i_u \omega \in \rho^*(T^*S) \}$$

is locally generated by

$$xy \prod_{i=3}^{k} (A_i x + B_i y) \frac{\partial}{\partial x}, \quad xy \prod_{i=3}^{k} (A_i x + B_i y) \frac{\partial}{\partial y} .$$

These are local generators of the zero tangent bundle, that is the vector bundle whose space of sections are

$$\{ u \in C^\infty(S; TS) \mid u|_Z = 0 \text{ for all } Z \in D \}.$$
This vector bundle, first introduced by Rafe Mazzeo and Richard Melrose in
the context of manifolds with boundary [9, 10], is a Lie algebroid. We will use
anchor map the inclusion into the tangent bundle and Lie bracket induced by
the standard Lie bracket on \( TS \).

**Lemma 4.2.** The Poisson cohomology of a star log Poisson surface \((S, D, \pi)\)
is isomorphic to the de Rham cohomology \( 0H^\ast(S) \) of the zero tangent bundle
of \((S, D)\).

The details of the proof of this lemma can be found in Section 5 of [7]. Thus
to complete the proof of Theorem 1.4, it suffices to compute the de Rham
cohomology of the 0 tangent bundle of \((S, D)\).

### 4.2. 0-de Rham cohomology.

Given a surface \( S \) with a collection \( D \) of transverse curves \( \{Z_1, \ldots, Z_k\} \), the de Rham cohomology of the associated
0-tangent bundle is

\[
H^p(S) \oplus \bigoplus_i H^{p-1}(Z_i) \oplus \bigoplus_{i < j} (H^{p-2}(Z_i \cap Z_j) \oplus H^{p-2}(Z_i \cap Z_j; |N^*Z_i|^{-1} \otimes |N^*Z_j|^{-1}))
\]

which we originally computed in [3]. Unlike in the case of the \( b \)-tangent bundle,
this decomposition does not hold in the more general case when \( D \) is any star
divisor. In other words, the 0-de Rham cohomology perceives higher order
intersection of curves in \( D \) while \( b \)-de Rham does not.

**Theorem 4.3.** Let \((S, D)\) be a surface \( S \) with ordered star divisor
\[
D = \{Z_1, \ldots, Z_k\}.
\]

Given a star atlas with respect to this ordering, the Lie algebroid cohomology
of the 0-tangent bundle over \((S, D)\) is isomorphic to the expression (4.1) in
degree 0 and 1. In degree 2, the cohomology is a direct sum of the following
vector spaces:

- A single copy of \( H^2(S) \).
- Each hypersurfaces \( Z_i \) contributes \( H^1(Z_i) \).
- Each pair wise intersection of two hypersurfaces \( Z_i, Z_j \) with \( i < j \)
  contributes
  \[
  H^0(Z_i \cap Z_j) \oplus H^0(Z_i \cap Z_j; |N^*Z_i|^{-1} \otimes |N^*Z_j|^{-1}).
  \]
- Each intersection of three hypersurfaces \( Z_i, Z_j, Z_k \) with \( i < j < k \)
  contributes
  \[
  H^0(Z_i \cap Z_j \cap Z_k; |N^*Z_i|^{-1} \otimes |N^*Z_j|^{-1} \otimes |N^*Z_k|^{-1}) +
  H^0(Z_i \cap Z_j \cap Z_k; |N^*Z_i|^{-1} \otimes |N^*Z_k|^{-1}) \oplus
  H^0(Z_i \cap Z_j \cap Z_k; |N^*Z_j|^{-1} \otimes |N^*Z_k|^{-1}) \oplus
  H^0(Z_i \cap Z_j \cap Z_k; |N^*Z_i|^{-1} \otimes |N^*Z_j|^{-1}) \oplus
  H^0(Z_i \cap Z_j \cap Z_k; |N^*Z_i|^{-1} \otimes |N^*Z_k|^{-1}).
  \]
Each intersection of four or more hypersurfaces $Z_{i_1}, Z_{i_2}, Z_{i_3}, \ldots, Z_{i_\ell}$ with $i_1 < i_2 < i_3 < \cdots < i_\ell$ contributes

$$H^0(\bigcap_{i} Z_i; \bigotimes_i N^* Z_i^{-1}) \oplus H^0(\bigcap_{i \neq j} Z_i; \bigotimes_i N^* Z_i^{-1}) \oplus H^0(\bigcap_{i \neq j} Z_i; \bigotimes_i N^* Z_i^{-1}) \oplus H^0(\bigcap_{i \neq j} Z_i; \bigotimes_i N^* Z_i^{-1})$$

**Example 4.4.** Consider the log symplectic structure on the sphere introduced in Example 2.10. The theorem tells us that in fixed coordinates, we can identify $H^2_\pi(S^2) \simeq \mathbb{R}^{22}$.

Further, in the following example we demonstrate how many of the classes in $H^2_\pi(S)$ do not arise as bivectors on the $b$-tangent bundle.

**Example 4.5.** Consider $\mathbb{R}^2$ with standard coordinates equipped with Poisson bivector

$$\pi = (x + y)(x - y)x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$$

The degeneracy locus of $\pi$ is depicted in Figure 1.2. As described in Theorem 1.4, the second Poisson cohomology $H^2_\pi(\mathbb{R}^2)$ contains three copies of $H^0(\{x = 0\} \cap \{x - y = 0\} \cap \{x + y = 0\})$.

One of these vector spaces is generated by the bivector $\nu = x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$.

Since $\nu$ does not vanish at the lines $x + y = 0$ or $x - y = 0$, it does not correspond to an element of the $b$-de Rham sub-complex. Further, its corresponding element in the rigged algebroid $\mathcal{R}$ is not cohomologous to any $b$-de Rham form. Thus not all non-trivial deformations of $\pi$ near $\pi$ can be done through a path of log-symplectic structures.

**Proof.** Let $(S, D)$ be a surface $S$ with star divisor $D$. We will compute the Lie algebroid cohomology of the 0-tangent bundle over $(S, D)$ using a filtration $^0\mathcal{F}$ of cochain complexes.

**Constructing filtration $^0\mathcal{F}$.**

Given a subset $I \subseteq D$, let $^0\Omega^*(S, I)$ denote the complex of 0-de Rham forms of the 0-tangent bundle over $(S, I)$. In this notation, we want to compute the cohomology of the complex $^0\Omega^*(S, D)$.

Consider the sets

$$^0\mathcal{F}^*_k := \text{v.s. Span} \left\{ \bigcup_{I \subseteq D, |I| = k} ^0\Omega^*(S, I) \right\}.$$
Notice that $0\mathcal{F}^*_k \subseteq 0\mathcal{F}^*_k \oplus 1$ for all $k$. Each set $0\mathcal{F}^*_k$ is a group under addition and has the structure of a module with scalars from the ring $0\mathcal{F}^0_k = C^\infty(S)$.

Note that $0\Omega^*(S, I) \subseteq 0\Omega^*(S, D)$ for all $I \subseteq D$. Consequently, the modules $0\mathcal{F}^*_k \subseteq b\Omega^*(S, D)$ for all $k$. It is easy to check that $0\mathcal{F}^*_k$ is closed under the differential of $0\Omega^*(S, D)$. Thus we can define a differential $0\mathcal{F}^*_k \xrightarrow{d} 0\mathcal{F}^*_k \oplus 1$ inherited from the differential on $0\Omega^*(S, D)$.

For each $k$, we have a chain complex:

$$0 \rightarrow 0\mathcal{F}^*_0 \xrightarrow{d} 0\mathcal{F}^*_1 \xrightarrow{d} 0\mathcal{F}^*_2 \rightarrow 0$$

We will employ the following filtration of cochain complexes:

$$0\mathcal{F} = \{\Omega^*(S) = 0\mathcal{F}^*_0 \subseteq 0\mathcal{F}^*_1 \subseteq 0\mathcal{F}^*_2 \subseteq \cdots \subseteq 0\mathcal{F}^*_{|D|-1} \subseteq 0\mathcal{F}^*_{|D|} = 0\Omega^*(S, D)\}.$$

In the next step, we consider various short exact sequences of complexes involving the terms in $0\mathcal{F}$.

**A short exact sequence of complexes.**

The inclusions $0\mathcal{F}^*_{k-1} \rightarrow 0\mathcal{F}^*_k$ of cochain complexes from filtration $0\mathcal{F}$ fit into a short exact sequence of complexes

$$0 \rightarrow 0\mathcal{F}^*_{k-1} \rightarrow 0\mathcal{F}^*_k \rightarrow 0$$

where

$$0\mathcal{C}^*_k = 0\mathcal{F}^*_k / 0\mathcal{F}^*_{k-1}.$$  

The differential $0\mathcal{C}^*_k$ is induced by $d$ on $0\mathcal{F}^*_k$. If $P$ is the projection $0\mathcal{F}^*_k \rightarrow 0\mathcal{F}^*_k / 0\mathcal{F}^*_{k-1}$, then $0\mathcal{C}^*(\eta) = P(\eta)$ where $\eta \in 0\mathcal{F}^*_k$ is any form such that $P(\eta) = \eta$. Hence $(0\mathcal{C}^*)_2 = 0$ and $(0\mathcal{C}^*, 0\mathcal{C}^*)_2$ is a complex.

Given one of these short exact sequences, we will show that the boundary maps in the induced long exact sequences in cohomology are zero. This feature will facilitate our arrival at a nice description of $0H^p(S)$.

**Computing $H^p(0\mathcal{F}^*_1)$.**

Let $k = 1$ and consider the short exact sequence

$$0 \rightarrow 0\mathcal{F}^*_0 \rightarrow 0\mathcal{F}^*_1 \rightarrow 0\mathcal{C}^*_1 \rightarrow 0$$

defined above. Note that $0\mathcal{F}^*_0 = \Omega^*(S)$.

Let $D = \{Z_1, Z_2, \ldots, Z_\ell\}$ be our star divisor and consider defining functions $x_1, x_2, \ldots, x_\ell$ such that $Z_i = \{x_i = 0\}$. For each $i$, let $\chi_i : S \rightarrow \mathbb{R}$ be a smooth
bump function supported in a neighborhood of $Z_i$ and with constant value one near $Z_i$. An element $\mu \in \mathcal{F}_1^1$ can be expressed as

$$\mu = \sum_{i=1}^{\ell} \chi_i \cdot \left( \frac{dx_i}{x_i} \alpha_i + \frac{\beta_i}{x_i} \right) + \theta$$

where $\alpha_i \in C^\infty(Z_i)$, $\beta_i \in \Omega^1(Z_i)$, and $\theta \in \Omega^1(S)$.

Note that

$$d\mu = -\sum_{i=1}^{\ell} \chi_i \left( \frac{dx_i}{x_i} \wedge d\alpha_i - \frac{dx_i}{x_i^2} \beta_i \right) + d\chi_i \wedge \left( \frac{dx_i}{x_i} \alpha_i + \frac{\beta_i}{x_i} \right) + \frac{d\beta_i}{x_i} + d\theta.$$ 

An element $\mu \in \mathcal{F}_2^1$ can be expressed as

$$\mu = \sum_{i=1}^{\ell} \chi_i \cdot \frac{dx_i}{x_i^2} \wedge (\alpha_i + \beta_i x_i) + \theta$$

where $\alpha_i, \beta_i \in \Omega^1(Z_i)$ and $\theta \in \Omega^2(S)$. Note that $d\mu = 0$.

Given $\mu \in \mathcal{F}_1^p$, we write $\mathcal{R}(\mu) = \theta$ and $\mathcal{S}(\mu) = \mu - \mathcal{R}(\mu)$ for ‘regular’ and ‘singular’ parts. It is easy to see that $\mathcal{R}(d\mu) = d(\mathcal{R}(\mu))$ and $\mathcal{S}(d\mu) = d(\mathcal{S}(\mu))$. Thus we have a splitting

$$0_{\mathcal{F}_1^*} = \Omega^*(S) \oplus 0_{\mathcal{E}_1^*}$$

as complexes and $H^p(0_{\mathcal{F}_1^*}) \simeq H^p(S) \oplus H^p(0_{\mathcal{E}_1^*})$. We are left to compute the cohomology group of the quotient complex $0_{\mathcal{E}_1^*}$.

Any $P(\mu) \in 0_{\mathcal{E}_1^1}$ is of the form

$$P(\mu) = \sum_{i=1}^{\ell} \frac{dx_i}{x_i} \alpha_i + \frac{\beta_i}{x_i}$$

for $\alpha_i \in C^\infty(Z_i)$, $\beta_i \in \Omega^1(Z_i)$. Then

$$dP(\mu) = -\sum_{i=1}^{\ell} \frac{dx_i}{x_i} \wedge d\alpha_i - \frac{dx_i}{x_i^2} \beta_i + \frac{d\beta_i}{x_i}.$$

By inspecting the expression at each $Z_i$, this gives us kernel relations $d\alpha_i = 0$, $\beta_i = 0$. Given these choices of $Z_i$ defining functions $x_1, \ldots, x_\ell$, the ker$(d : \mathcal{C}_p^0 \to \mathcal{C}_p^{0+1})$ can be identified with

$$\bigoplus_{i=1}^{\ell} \{ \alpha_i \in C^\infty(Z_i) : d\alpha_i = 0 \}.$$

Thus

$$H^1(0_{\mathcal{E}_1^*}) \simeq \bigoplus_{i} H^0(Z_i).$$
For $H^2(\mathcal{O}_1^*)$, note that given $\mu \in 0^*F_1^2$ there exists $\tilde{\mu} \in 0^*F_1^1$ such that

$$P(\mu - d\tilde{\mu}) = \sum dx_i \wedge (\beta_i - d\alpha_i)$$

for $\beta_i \in \Omega^1(Z_i)$, $\alpha_i \in \Omega^0(Z_i)$, and that elements of this form are not in the image of $d$. Thus we can identify

$$H^2(\mathcal{O}_1^*) \simeq \bigoplus_i H^1(Z_i).$$

By a computation, which can be found in example 2.11 of [7], these sets are invariant under change of $Z_i$ defining function. Thus

$$H^p(\mathcal{O}_1^*) \simeq \bigoplus_i H^{p-1}(Z_i)$$

and

$$H^p(0^*F_1) \simeq H^p(S) \oplus \bigoplus_i H^{p-1}(Z_i).$$

**Computing $H^p(0^*F_2^*)$.**

Let $k = 2$ and consider the short exact sequence

$$0 \to 0^*F_1^* \to 0^*F_2^* \to 0^*F_2^* \to 0.$$

Let $D = \{Z_1, Z_2, \ldots, Z_\ell\}$ be our star divisor. For each pair of curves $Z_i, Z_j$ with $i < j$, there is a pair of tubular neighborhoods $\tau_i = Z_i \times (-\varepsilon, \varepsilon)_{x_i}$ of $Z_i$ and $\tau_j = Z_j \times (-\varepsilon, \varepsilon)_{y_j}$ of $Z_j$ such that

$$\left[ \frac{\partial}{\partial x_{ij}}, \frac{\partial}{\partial y_{ij}} \right] = 0.$$

For existence of such neighborhoods see, for instance, section 5 of [1].

Let $\chi_{ij}$ be a smooth bump function supported near $Z_i \cup Z_j$ and with constant value one near $Z_i \cup Z_j$. An element $\mu \in 0^*F_2^1$ can be expressed as

$$\mu = \sum_{i < j} \chi_{ij} \cdot \left( \frac{dx_{ij}}{x_{ij}y_{ij}}\alpha_{ij} + \frac{dy_{ij}}{x_{ij}y_{ij}}\beta_{ij} \right) + \theta$$

for $\alpha_{ij}, \beta_{ij} \in C^\infty(Z_i \cap Z_j)$, and $\theta \in 0^*F_1^1$. Note that $\alpha$ and $\beta$ are just constants. Thus

$$d\mu = \sum_{i < j} \frac{dx_{ij} \wedge dy_{ij}}{x_{ij}y_{ij}^2} \alpha_{ij} - \frac{dx_{ij} \wedge dy_{ij}}{x_{ij}^2y_{ij}} \beta_{ij} + d\theta.$$

An element $\mu \in 0^*F_2^2$ can be expressed as

$$\mu = \sum_{i < j} \frac{dx_{ij} \wedge dy_{ij}}{x_{ij}^2y_{ij}^2} (\alpha_{ij} + \beta_{ij}x_{ij} + \gamma_{ij}y_{ij} + \delta_{ij}x_{ij}y_{ij}) + \theta$$

for $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij} \in C^\infty(Z_i \cap Z_j)$, and $\theta \in 0^*F_1^1$. Note $d\mu = 0$. 


Given \( \mu \in \mathcal{F}_2^p \), we write \( \mathcal{R}(\mu) = \theta \) and \( \mathcal{S}(\mu) = \mu - \mathcal{R}(\mu) \) for ‘regular’ and ‘singular’ parts. It is easy to see that \( \mathcal{R}(d\mu) = d(\mathcal{R}(\mu)) \) and \( \mathcal{S}(d\mu) = d(\mathcal{S}(\mu)) \). Thus we have a splitting

\[
0 \mathcal{F}_2^* = 0 \mathcal{F}_1^* \oplus 0 \mathcal{C}_2^*
\]
as complexes and \( H^p(0 \mathcal{F}_2^*) \simeq H^p(0 \mathcal{F}_1^*) \oplus H^p(0 \mathcal{C}_2^*) \). We are left to compute the cohomology group of the quotient complex \( 0 \mathcal{C}_2^* \).

Any \( P(\mu) \in 0 \mathcal{C}_1^2 \) is of the form

\[
P(\mu) = \sum_{i<j} \frac{dx_{ij}}{x_{ij}y_{ij}} \alpha_{ij} + \frac{dy_{ij}}{x_{ij}y_{ij}} \beta_{ij}
\]
for \( \alpha_{ij}, \beta_{ij} \in C^\infty(Z_i \cap Z_j) \). Then

\[
dP(\mu) = \sum_{i<j} \frac{dx_{ij} \wedge dy_{ij}}{x_{ij}^2 y_{ij}} \alpha_{ij} - \frac{dx_{ij} \wedge dy_{ij}}{x_{ij}^2 y_{ij}} \beta_{ij}.
\]

Thus our kernel relations are \( \alpha_{ij} = 0 \) and \( \beta_{ij} = 0 \).

For \( H^2(0 \mathcal{C}_1^*) \), note that for \( \mu \in 0 \mathcal{F}_2^* \) there exists \( \tilde{\mu} \in 0 \mathcal{F}_2^1 \) such that

\[
P(\mu - d\tilde{\mu}) = \frac{dx_{ij}}{x_{ij}^2 y_{ij}} \wedge (\alpha_{ij} + \delta_{ij} x_{ij} y_{ij})
\]
for \( \alpha_{ij}, \delta_{ij} \in C^\infty(Z_i \cap Z_j) \). Further, there are no elements of this form in the image of \( d \). Thus, given these choices \( x_{ij}, y_{ij} \) of defining functions, we can identify

\[
H^2(0 \mathcal{C}_2^*) \simeq \bigoplus_{i<j} H^0(Z_i \cap Z_j) \oplus H^0(Z_i \cap Z_j).
\]

By a computation which can be found at the conclusion of the proof of Theorem 2.15 in [7], this cohomology can be identified independently of defining functions as

\[
H^2(0 \mathcal{C}_2^*) \simeq \bigoplus_{i<j} H^0(Z_i \cap Z_j) \oplus H^0(Z_i \cap Z_j; |N^*Z_i|^{-1} \otimes |N^*Z_j|^{-1}).
\]

Thus \( H^p(0 \mathcal{F}_2^*) \) is

\[
H^p(S) \oplus \bigoplus_i H^{p-1}(Z_i) \oplus \bigoplus_{i<j} H^{p-2}(Z_i \cap Z_j) \oplus H^{p-2}(Z_i \cap Z_j; |N^*Z_i|^{-1} \otimes |N^*Z_j|^{-1}).
\]

**Computing \( H^p(0 \mathcal{F}_k^*) \) for \( k \geq 3 \).**

Fix \( k \geq 3 \) and consider the short exact sequence

\[
0 \to 0 \mathcal{F}_{k-1}^* \to 0 \mathcal{F}_k^* \to 0 \mathcal{C}_k^* \to 0.
\]
Given any \( p \in S \) such that at least the \( k \) curves in the set \( I = \{Z_1, \ldots, Z_k\} \) intersect at \( p \), there exists a star chart with respect to \( I \) centered at \( p \), \((U_{p,I}, x, y)\), such that

\[
Z_1 = \{x = 0\}, Z_2 = \{y = 0\}, \ldots, Z_k = \{A_kx + B_ky = 0\}.
\]

Let \( \chi_{p,I} \) be a smooth bump function supported in \( U_{p,I} \) and with constant value one in a neighborhood of \( p \).

Let

\[
R := \prod_{i=3}^{k} (A_i x + B_i y).
\]

An element \( \mu \in 0F^1_k \) can be expressed as

\[
\mu = \sum_{p, I} \chi_{p,I} \cdot \left( \frac{dx}{xyR} \alpha_{p,I} + \frac{dy}{xyR} \beta_{p,I} \right) + \theta
\]

for \( \alpha_{p,I}, \beta_{p,I} \in \Omega^0(\{p\}) \) (i.e. constants) and \( \theta \in 0F^1_{k-1} \). Note

\[
d\mu = \sum_{p, I} \chi_{p,I} \cdot \left( \frac{dx \wedge dy}{x^2 y^2 R} \alpha_{p,I} + \frac{dx \wedge dy}{x^2 y^2 R} (\partial_y R) \alpha_{p,I} - \frac{dx \wedge dy}{x^2 y^2 R} \beta_{p,I} - \frac{dx \wedge dy}{x^2 y^2 R} (\partial_x R) \beta_{p,I} \right)
\]

\[
+ \sum_{p, I} d\chi_{p,I} \wedge \left( \frac{dx}{xyR} \alpha_{p,I} + \frac{dy}{xyR} \beta_{p,I} \right) + d\theta
\]

Any element \( \mu \in 0F^2_k \) can be expressed as

\[
\mu = \sum_{p, I} \chi_{p,I} \cdot \frac{dx \wedge dy}{x^2 y^2 R^2} \wedge (\alpha_{p,I} + \beta_{p,I}x + \gamma_{p,I}y + \delta_{p,I} xy) + \theta
\]

for \( \alpha_{p,I}, \beta_{p,I}, \gamma_{p,I}, \delta_{p,I} \in \Omega^0(\{p\}) \) and \( \theta \in 0F^2_{k-1} \). Note \( d\mu = 0 \).

Given \( \mu \in 0F^1_k \), we write \( \mathcal{K}(\mu) = \theta \) and \( \mathcal{G}(\mu) = \mu - \mathcal{K}(\mu) \) for ‘regular’ and ‘singular’ parts. One can check that \( \mathcal{K}(d\mu) = d(\mathcal{K}(\mu)) \) and \( \mathcal{G}(d\mu) = d(\mathcal{G}(\mu)) \). Thus we have a splitting

\[
0F^* = 0F^*_{k-1} \oplus 0C^*_k
\]

as complexes and \( H^p(0F^*_k) \simeq H^p(0F^*_{k-1}) \oplus H^p(0C^*_k) \). We are left to compute the cohomology group of the quotient complex \( 0C^*_k \).
We will first compute the cohomology at $0\mathcal{C}_k^1$. Any $P(\mu) \in 0\mathcal{C}_k^1$ is of the form

$$P(\mu) = \sum_{p, I \mid |I|=k} \chi_{p, I} \cdot \left( \frac{dx}{xyR} \alpha_{p, I} + \frac{dy}{xyR} \beta_{p, I} \right)$$

for $\alpha_{p, I}, \beta_{p, I} \in \Omega^0(\{p\})$. Then $dP(\mu)$ follows from our expression of $d\mu$ above. Thus we can identify the set $\text{ker}(d : \mathcal{C}_k^1 \to \mathcal{C}_k^2) = \{\alpha_{p, I} = 0, \beta_{p, I} = 0\}$ and $H^1(\mathcal{C}_k^1) = 0$ for all $k \geq 3$.

Next, we compute the cohomology at $\mathcal{C}_k^2$. Any element $P(\mu) \in \mathcal{C}_k^2$ is of the form

$$P(\mu) = \sum_{p, I \mid |I|=k} \chi_{p, I} \cdot \frac{dx \wedge dy}{x^2 y^2 R^2} \wedge (\alpha_{p, I} + \beta_{p, I} x + \gamma_{p, I} y + \delta_{p, I} x y)$$

for $\alpha_{p, I}, \beta_{p, I}, \gamma_{p, I}, \delta_{p, I} \in \Omega^0(\{p\})$. We have $dP(\mu) = 0$. Note that $d(0\mathcal{C}_k^1)$ has empty intersection with elements of the form

$$\frac{dx \wedge dy}{x^2 y^2 R^2} \wedge (\alpha_{p, I} + \beta_{p, I} x + \gamma_{p, I} y).$$

Using the expression of $d\mu$ for $\mu \in \mathcal{C}_k^1$ given above, we are left to consider when does the following equality hold at the point $p$:

$$\delta_{p, I} = (\partial_y P)\alpha_{p, I} - (\partial_x P)\beta_{p, I}$$

for $\alpha_{p, I}, \beta_{p, I}, \delta_{p, I} \in \Omega^0(\{p\})$. For this equation to hold true, $P$ would need to be a single line $P = Ax + By$. Thus when $k \geq 4$, the term

$$\sum_{p, I \mid |I|=k} \chi_{p, I} \cdot \frac{dx \wedge dy}{x^2 y^2 R^2} \wedge \delta_{p, I} x y$$

is not in the image of $d(0\mathcal{C}_k^1)$.

Let us consider the case where $k = 3$. In this instance, the numbers $\beta_{p, I} = 0$ and $\alpha_{p, I} = \frac{\delta_{p, I}}{B}$ provide a solution to the equality and

$$\sum_{p, I \mid |I|=k} \chi_{p, I} \cdot \frac{dx \wedge dy}{x^2 y^2 R^2} \wedge \delta_{p, I} x y$$

is in the image of $d(0\mathcal{C}_3^1)$. 
Thus, for these fixed $Z_i$ defining functions, we have identified

$$H^n(\mathcal{C}^*_3) \simeq \bigoplus_{p, |I|=3} (H^{n-2}(\{p\}))^3$$

and, for $k \geq 4$,

$$H^n(\mathcal{C}^*_k) \simeq \bigoplus_{p, |I|=k} (H^{n-2}(\{p\}))^4.$$

For completeness, we will consider what happens under change of defining functions. By a computation which can be found at the conclusion of the proof of theorem 2.15 in [7], the cohomology $H^p(\mathcal{C}^*_k)$ can be identified independently of defining functions as

$$\bigoplus_{i \in J} (H^{p-2}(\bigcap Z_j; \bigotimes_{i \in J \setminus \{i\}} |N^* Z_i|^{-1}) \oplus H^{p-2}(\bigcap Z_j; \bigotimes_{i \in J \setminus \{i\}} |N^* Z_i|^{-1})$$

when $k = 3$ and as

$$\bigoplus_{i \in J} (H^{p-2}(\bigcap Z_j; \bigotimes_{i \in J \setminus \{i\}} |N^* Z_i|^{-1}) \oplus H^{p-2}(\bigcap Z_j; \bigotimes_{i \in J \setminus \{i, i_2\}} |N^* Z_i|^{-1}))$$

when $k \geq 4$, where $J$ is a subset of integers $\{1, \ldots, k\}$ and $i_1, i_2$ are the smallest elements in $J$.

□

References

[1] Pierre Albin and Richard Melrose. Resolution of smooth group actions. *Contemp. Math* 535 (2011): 1-26.
[2] Vladimir Igorevich Arnol’d. Mathematical methods of classical mechanics. *Graduate texts in mathematics*, Volume 60, Springer-Verlag, 1989.
[3] Ana Cannas da Silva. Lectures on Symplectic Geometry, *Lecture Notes in Mathematics*, no. 1764, Springer-Verlag, 2001.
[4] Timothy Y. Chow. You could have invented spectral sequences. *Notices of the AMS* 53 (2006): 15-19.
[5] Marco Gualtieri, Songhao Li, Alvaro Pelayo, and Tudor Ratiu. The tropical momentum map: a classification of toric log symplectic manifolds. 2014. preprint arXiv:1407.3300
[6] Victor Guillemin, Eva Miranda, and Ana Rita Pires. Symplectic and Poisson geometry on b-manifolds. *Adv. Math.* 264 (2014), 864-896.
[7] Melinda Lanius. Symplectic, Poisson, and contact geometry on scattering manifolds. 2016. preprint arXiv:1603.02994
[8] ———. Poisson cohomology of a class of log symplectic manifolds. 2016. preprint arXiv:1605.03854
[9] Rafe Mazzeo. The Hodge cohomology of a conformally compact metric. *J. Diff. Geom.* 28 (1988), 309-339.
[10] Rafe Mazzeo and Richard Melrose. Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature. *J. Funct. Anal.* 108 (1987), 260-310.

[11] Richard Melrose. The Atiyah-Patodi-Singer Index Theorem. *Research Notes in Mathematics*, Book 4, A K Peters/CRC Press, 1993.

[12] Eva Miranda, and Arnau Planas. Equivariant classification of $b^m$-symplectic surfaces and Nambu structures. 2016. preprint arXiv:1607.01748

[13] Max Neumann-Coto. A characterization of shortest geodesics on surfaces. *Algebraic & Geometric Topology* 1.1 (2001): 349-368.

[14] Olga Radko. A classification of topologically stable Poisson structures on a compact oriented structure. *Journal of Symplectic Geometry* 1.3 (2002): 523-542.

[15] ———. Toward a classification of Poisson structures on surfaces. *Contemporary Mathematics* 315 (2002): 81-88.

[16] Geoffrey Scott. The Geometry of $b^k$ Manifolds. 2013. preprint arXiv:1304.3821v2.