On a variance for twins of \( k \)-free numbers in arithmetic progressions *

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In this paper, we give a new upper bound of Barban-Davenport-Halberstam type for twins of \( k \)-free numbers in arithmetic progressions.

1 Introduction

In recent years, Brüdern and others [1,2] have made a breakthrough in the circle method. They give successful treatment, through the circle method, of binary additive problems involving \( k \)-free numbers. Their results depend upon the variance for \( k \)-free numbers (or twins of \( k \)-free numbers) in arithmetic progressions.

In this paper, we give new results on the variance for twins of \( k \)-free numbers in arithmetic progressions. Such result is analogous to the Barban-Davenport-Halberstam theorem for the primes in arithmetic progressions.

Let \( \mu_k(n) \) be the characteristic function of the \( k \)-free numbers,

\[
\mu_k(n) = \sum_{d^k|n} \mu(d),
\]

where \( \mu(n) \) is the Möbius function.

Let

\[
A_k(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a (\text{mod } q)}} \mu_k(n)\mu_k(n + 1)
\]

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\[ g(q, a) = \sum_{u,v=1}^{\infty} \mu(uv) \frac{u^k v^k}{u^k v^k} (q, u^k v^k) \quad (1.3) \]

By Lemma 3.1 in [2], we have
\[ A_k(x; q, a) = q^{-1} g(q, a) x + O(x^{\frac{2}{k} + \varepsilon}). \]

We consider the variance
\[ Y_k(x, Q) = \sum_{q \leq Q} \sum_{a=1}^{q} | A_k(x; q, a) - q^{-1} g(q, a) x |^2. \quad (1.4) \]

In [2], Brüdern, Perelli and Wooley obtained
\[ Y_k(x, Q) \ll x^{\frac{2}{k} + \varepsilon} Q^{2 - \frac{1}{k}} + x^{1 + \frac{1}{k} + \varepsilon}, \quad 1 < Q \leq x. \quad (1.5) \]

In this paper, we obtain the following

**THEOREM.** Suppose that \( 1 < Q \leq x \) and that \( k \) is an integer with \( k \geq 2 \). Then
\[ Y_k(x, Q) \ll x^{\frac{1}{k} + \varepsilon} Q^{2 - \frac{1}{k} - \varepsilon} + x^{1 + \frac{2}{k} \log Q} + x^{\frac{3}{2} + \frac{1}{2k} + \varepsilon}. \]

Actually, the first term is \( Q^{2 - \frac{1}{k}} \frac{x}{Q}^{\frac{1}{k} + \varepsilon} \). For \( k > 2 \), this result is superior to (1.5) when \( Q \gg x^{\frac{3}{4}} \).

We have
\[ Y_k(x, Q) = S_1(x, Q) - 2x S_2(x, Q) + x^2 S_3(x, Q) \quad (1.6) \]

where
\[ S_1(x, Q) = \sum_{q \leq Q} \sum_{a=1}^{q} A_k^2(x; q, a), \quad (1.7) \]

\[ S_2(x, Q) = \sum_{q \leq Q} \sum_{a=1}^{q} A_k(x; q, a) q^{-1} g(q, a), \quad (1.8) \]

\[ S_3(x, Q) = \sum_{q \leq Q} \sum_{a=1}^{q} q^{-2} g(q, a)^2. \quad (1.9) \]
We notice that the function $g(q,a)$ does not depend only on $q$ and $(a,q)$, unlike most sequences investigated before.

In section 2-3, we discuss $S_2(x,Q)$ and $S_3(x,Q)$, these depend on the solutions of the corresponding congruences. In section 4-7, we discuss $S_1(x,Q)$ by the Hardy-Littlewood method on the line of Vaughan\[9\]. In section 8, we discuss the singular series $\mathcal{S}(n)$. In section 9-10, we use the Hurwitz Zeta function to discuss the generating function of $\mathcal{S}(uv)$. In section 11, we use a trick to avoid the difficulty of calculating some constants and then we obtain the Theorem.

**Notation:** Let $k > 1$ denote a positive integer. Throughout, $\varepsilon$ is a sufficiently small positive number, the implicit constants in Vinogradov’s notation $\ll$, and in Landau’s $O$-notation, will depend at most on $k,\varepsilon$ unless it is pointed out depend upon the corresponding parameters. $n \equiv a (\text{mod } q)$ may be written as $n \equiv a(q)$. The greatest common divisor and the least common multiple of integers $a,b$ are denoted by $(a,b)$ and $[a,b]$ respectively; $\mu(n)$ denotes the Möbius function and $\tau(n)$ denotes the divisor function; $[w]$ denotes the integer part of $w$ and $e(\alpha) = \exp(2\pi i \alpha)$; $\sum_{a=1}^{\varphi(n)}'$ means $(a,q) = 1$. The letter $p$ denotes a prime number, and write $p^\ell \parallel n$ when $p^\ell \mid n$ but $p^{\ell+1} \nmid n$. Let $x$ denote a sufficiently large real number and $Q$ be a positive real number with $1 < Q \leq x$.

## 2 The formula for $S_2(x,Q)$

For fixed positive integers $u,v,q,r,s$, let

$$J_1 = \sum_{(u^k,q)\mid(a,(v^k,q))\mid[r^k|n,s^k]|n+1} \sum_{a=1}^{r^k} 1, \quad s_0 = \left(\frac{(u^k,q)}{(r^k,u^k,q)}, \frac{r^k}{(r^k,q)}\right).$$

**Lemma 2.1.** We have

$$J_1 = \frac{(r^k,u^k,q)s_0}{r^ks^k(u^k,q)(v^k,q)}(s^k, \frac{r^k(u^k,q)(v^k,q)}{(r^k,u^k,q)})x + O(1),$$

provided that

$$\left(\frac{(u^k,q)(r^k,q)}{(r^k,u^k,q)}, (v^k,q)\right) = \left(\frac{(u^k,q)}{(r^k,u^k,q)s_0}, s^k\right) = (r,s) = (q,u,v) = 1.$$
Proof. We use an idea of [4]. Let \( n = r^k b, \ a = (u^k, q)c, \) then

\[
J_1 = \sum_{c=1}^{q(u^k, q)-1} \sum_{r^k b \leq c} 1, \quad (r, s) = (q, u, v) = 1. \tag{2.3}
\]

Now,

\[
\frac{r^k}{(r^k, q)} b \equiv \frac{(u^k, q)c}{(r^k, q)} \pmod{\frac{q}{(r^k, q)}}, \quad (r^k, q) \mid (u^k, q)c.
\]

Since \( \frac{(u^k, q)(r^k, q)}{(r^k, u^k, q)} = [(u^k, q), (r^k, q)] \mid q, \) we write \( c = \frac{(r^k, q)}{(r^k, u^k, q)} d, \) then

\[
J_1 = \#\{1 \leq b \leq xr^{-k}, \ 1 \leq d \leq \frac{q(r^k, u^k, q)}{(u^k, q)(r^k, q)} : (v^k, q) \mid (u^k, q)\frac{(r^k, q)}{(r^k, u^k, q)} d + 1,
\]

\[
\frac{r^k}{(r^k, q)} b \equiv \frac{(u^k, q)d}{(r^k, u^k, q)} \pmod{\frac{q}{(r^k, q)}}, \quad s^k \mid r^k b + 1. \}
\]

We have

\[
\frac{r^k}{(r^k, q)} b \equiv \frac{(u^k, q)d}{(r^k, u^k, q)} \pmod{\frac{q}{(r^k, q)}},
\]

hence

\[
\frac{r^k}{(r^k, q)} (r^k, u^k, q) b \equiv d \pmod{\frac{q}{(r^k, q)} (r^k, u^k, q)},
\]

and

\[
\frac{(u^k, q)}{(r^k, u^k, q)} \mid \frac{r^k}{(r^k, q)} b, \tag{2.4}
\]

Write \( b = \frac{(u^k, q)}{(r^k, u^k, q)r_0} b_1, \) then

\[
J_1 = \#\{1 \leq \frac{(u^k, q)}{(r^k, u^k, q)r_0} b_1 \leq xr^{-k}, \ 1 \leq d \leq \frac{q(r^k, u^k, q)}{(u^k, q)(r^k, q)} : (v^k, q) \mid (u^k, q)\frac{(r^k, q)}{(r^k, u^k, q)} d + 1,
\]

\[
\frac{r^k}{(r^k, q)r_0} b_1 \equiv d \pmod{\frac{q(r^k, u^k, q)}{(r^k, q)(u^k, q)}}, \quad s^k \mid r^k \frac{(u^k, q)}{(r^k, u^k, q)r_0} b_1 + 1. \}
\]

We have

\[
\frac{(u^k, q)(r^k, q)}{(r^k, u^k, q)} d \equiv -1 \pmod{(u^k, q)}, \tag{2.5}
\]

and

\[
\frac{(u^k, q)(r^k, q)}{(r^k, u^k, q)} (u^k, q) = 1, \tag{2.6}
\]

\[
(v^k, q) \mid \frac{q(r^k, u^k, q)}{(u^k, q)(r^k, q)}.
\]

\[
\frac{r^k}{(r^k, q)} b \equiv \frac{(u^k, q)d}{(r^k, u^k, q)} \pmod{\frac{q}{(r^k, q)}},
\]

\[
\frac{(u^k, q)}{(r^k, u^k, q)} \mid \frac{r^k}{(r^k, q)} b, \tag{2.4}
\]

Write \( b = \frac{(u^k, q)}{(r^k, u^k, q)r_0} b_1, \) then

\[
J_1 = \#\{1 \leq \frac{(u^k, q)}{(r^k, u^k, q)r_0} b_1 \leq xr^{-k}, \ 1 \leq d \leq \frac{q(r^k, u^k, q)}{(u^k, q)(r^k, q)} : (v^k, q) \mid (u^k, q)\frac{(r^k, q)}{(r^k, u^k, q)} d + 1,
\]

\[
\frac{r^k}{(r^k, q)r_0} b_1 \equiv d \pmod{\frac{q(r^k, u^k, q)}{(r^k, q)(u^k, q)}}, \quad s^k \mid r^k \frac{(u^k, q)}{(r^k, u^k, q)r_0} b_1 + 1. \}
\]
\[
\frac{(u^k, q)}{(r^k, u^k, q)s_0}, s^k) = 1.
\] (2.8)

We write
\[d = d_1 + m(v^k, q), \ 1 \leq d_1 \leq (v^k, q), \ 0 \leq m < \frac{q(r^k, u^k, q)}{(v^k, q)(u^k, q)(v^k, q)},\]
then
\[
\frac{r^k}{(r^k, q)s_0}b_1 \equiv d_1 + m(v^k, q)(mod \ \frac{q(r^k, u^k, q)}{(r^k, q)(u^k, q)}).
\] (2.9)

\[
\frac{r^k}{(r^k, q)s_0}b_1 \equiv d_1(mod \ (v^k, q)).
\] (2.10)

If a prime \( p \ | \ (v, q), \) then \( p \ | \ v, \ p \ | \ q, \ p \ | \ u, \) again if \( p \ | \ \frac{r^k}{(r^k, q)s_0}, \) then \( p \ | \ r, \) hence, by (2.6) \( p \ | \ (\frac{(u^k, q)(r^k, q)}{(r^k, u^k, q)}, (v^k, q)) = 1, \) this is a contradiction.

We deduce that
\[
((v^k, q), \frac{r^k}{(r^k, q)s_0}) = 1.
\] (2.11)

Now the congruence (2.10) has a unique solution modulus \((v^k, q).\)

Let \( b_1 = b_0 + t(v^k, q), \ 1 \leq b_0 \leq (v^k, q), \) by (2.7) and (2.9)
\[
\frac{r^k}{(r^k, q)s_0}(v^k, q)t \equiv d_1 - \frac{r^k}{(r^k, q)s_0}b_0 + m(v^k, q)(mod \ \frac{q(r^k, u^k, q)}{(r^k, q)(u^k, q)}).
\] (2.12)

Also, we have
\[
\frac{r^k}{(r^k, u^k, q)s_0}(v^k, q)t \equiv -1 - \frac{r^k}{(r^k, u^k, q)s_0}b_0 (mod \ s^k).
\] (2.13)

By (2.7), (2.10) and (2.12),
\[
\frac{r^k}{(r^k, q)s_0}t \equiv (d_1 - \frac{r^k}{(r^k, q)s_0}b_0)(v^k, q)^{-1} + m(mod \ \frac{q(r^k, u^k, q)}{(r^k, q)(u^k, q)(v^k, q)}),
\] (2.14)
and
\[
d_1 \equiv \frac{r^k}{(r^k, q)s_0}b_0 (mod \ (v^k, q)).
\] (2.15)
By (2.5),
\[
\frac{(u^k, q)(r^k, q)}{(r^k, u^k, q)}d_1 \equiv -1 \mod (v^k, q)). \tag{2.16}
\]

By (2.14), we only want to count the number of \( t \) satisfying (2.13) and
\[
\frac{(u^k, q)}{(r^k, u^k, q)s_0}(b_0 + t(v^k, q)) \leq xr^{-k},
\]
that is
\[
t \leq \frac{x(r^k, u^k, q)s_0}{r^k(u^k, q)(u^k, q)} - \frac{b_0}{(v^k, q)}. \tag{2.17}
\]

By (2.5), (2.15) and (2.16),
\[
\frac{(u^k, q)r^k}{(r^k, u^k, q)s_0}b_0 \equiv -1 \mod (v^k, q)). \tag{2.18}
\]

Consequently, by (2.13) and (2.18),
\[
((v^k, q), s^k) \mid 1 + \frac{(u^k, q)r^k}{(r^k, u^k, q)s_0}b_0. \tag{2.19}
\]

By (2.3), (2.8) and (2.19), we have
\[
(s^k, r^k(u^k, q)(v^k, q)) \mid 1 + \frac{(u^k, q)r^k}{(r^k, u^k, q)s_0}b_0.
\]

By (2.13) and \((s, s_0) = 1\), we obtain
\[
\frac{r^k(u^k, q)(u^k, q)}{(r^k, u^k, q)}(s^k, r^k(u^k, q)(v^k, q))^{-1}l \equiv (-s_0 - \frac{r^k(u^k, q)}{(r^k, u^k, q)}b_0)(s^k, r^k(u^k, q)(v^k, q))^{-1} \mod s^k(s^k, \frac{r^k(u^k, q)(v^k, q)}{(r^k, u^k, q)}^{-1})
\]

Therefore, the number of \( t \) is
\[
\frac{x(r^k, u^k, q)s_0}{r^k(u^k, q)(v^k, q)}(s^k, \frac{r^k(u^k, q)(v^k, q)}{(r^k, u^k, q)}^{-1})^{-1} + O(1),
\]
that is
\[
J_1 = \frac{(r^k, u^k, q)s_0}{r^k s^k(u^k, q)(v^k, q)}(s^k, \frac{r^k(u^k, q)(v^k, q)}{(r^k, u^k, q)})x + O(1).
\]

The lemma follows.

Now, we give the formula for \( S_2(x, Q) \).
\[ S_2(x, Q) = \sum_{q \leq Q} q^{-1} \sum_{u,v=1}^{\infty} \frac{\mu(uv)}{u^kv^k} \left( \sum_{q=1}^{Q} \sum_{v \leq x, s \leq x+1} \mu_k(n) \mu_k(n+1) \right) \]

By Lemma 2.1,

\[
S_2(x, Q) = S_{2A}(x, Q) + S_{2B}(x, Q),
\]

where

\[
S_{2A}(x, Q) = \sum_{q \leq Q} q^{-1} \sum_{u,v=1}^{\infty} \frac{\mu(uv)}{u^kv^k} \left( \sum_{r^k \leq x} \frac{\mu(r) \mu(s)}{r^k, q} \right) x^{\frac{2}{k} \sum_{t|m^k} \mu^2(t) t^\tau(m) t^\tau(n)} q^{-1} (q, m^k),
\]

\[
S_{2B}(x, Q) \ll \sum_{q \leq Q} q^{-1} \sum_{u,v=1}^{\infty} \frac{\mu(uv)}{u^kv^k} \sum_{r^k \leq x} 1,
\]

and \( \sum_{r^k \leq x} \) means \( u, v, r, s, q \) satisfy (2.1) and (2.2).

We give upper bound for \( S_{2B}(x, Q) \) first.

\[
S_{2B}(x, Q) \ll x^{\frac{2}{k} \sum_{t|m^k} \mu^2(t) t^\tau(m) t^\tau(n)} q^{-1} (q, m^k) \ll x^{\frac{2}{k} \sum_{t|m^k} \mu^2(t) t^\tau(m) t^\tau(n)} \sum_{q \leq Q} q^{-1} (q, m^k) \ll x^{\frac{2}{k} \sum_{t|m^k} \mu^2(t) t^\tau(m) t^\tau(n)} \sum_{t|m^k} \sum_{q \leq Q} q^{-1},
\]

hence

\[
S_{2B}(x, Q) \ll x^{\frac{2}{k} \log Q}. \tag{2.21}
\]

By \( \left( \frac{u^k, q}{r^k, u^k, q} \right) = \left( \frac{r^k}{r^k, u^k, q} \right) = 1 \), we have

\[
s_0 = \frac{x^{\frac{2}{k} \log Q}}{\left( \frac{r^k}{r^k, u^k, q} \right)}. \tag{2.22}
\]

Since \( (u, v) = 1 \), the conditions (2.2) and (2.22) are equivalent to

\[
(r, v, q) = (s, u, q) = (s, r) = 1. \tag{2.23}
\]

By (2.2),

\[
S_{2A}(x, Q) = x \sum_{q \leq Q} q^{-1} \sum_{u,v=1}^{\infty} \frac{\mu(uv)}{u^kv^k} \sum_{r^k \leq x, s^k \leq x+1} \mu(r) \mu(s) r^{-k} s^{-k} \left( s^k, v^k, q \right) (r^k, u^k, q). \tag{2.24}
\]
The summations of $r, s$ can be completed to $\infty$.

\[
S_{2A}(x, Q) = c_1(Q)x + S_{2A_0}, \tag{2.25}
\]

where

\[
c_1(Q) = \sum_{q \leq Q} q^{-1} \sum_{u, v = 1}^{\infty} \frac{\mu(u)\mu(v)}{u^k v^k} \sum_{r, s = 1}^{\infty} \mu(r) \mu(s) r^{-k} s^{-k} (s^k, v^k, q)(r^k, u^k, q), \tag{2.26}
\]

and

\[
S_{2A_0} \ll x \sum_{q \leq Q} q^{-1} \sum_{u, v = 1}^{\infty} \frac{1}{u^k v^k} \left( \sum_{r, s \leq x} r^{-k} s^{-k} (s^k, v^k, q)(r^k, u^k, q) + \sum_{s^k > x, (r, s) = 1} r^{-k} s^{-k} (s^k, v^k, q)(r^k, u^k, q) \right) = S_{2Aa} + S_{2Ab}, \text{ say.}
\]

\[
S_{2Aa} \ll \sum_{q \leq Q} q^{-1} \sum_{u, v = 1}^{\infty} \frac{1}{u^k v^k} \sum_{s^k \leq x + 1} s^{-k} (s^k, v^k, q) \sum_{t | u, (t, s) = 1} \sum_{r, s > x} r^{-k}. \tag{2.26}
\]

Now

\[
\sum_{t | u, (t, s) = 1} t^{-k} + \sum_{t^k > x} t^{-k} \ll \sum_{t \leq x} t^{-k} (x^{\frac{1}{k}} t^{-\frac{1}{k}} - 1) + \sum_{t \leq x} t^{-k} + \sum_{t^k > x} t^{-k}.
\]

Hence

\[
S_{2Aa} \ll \sum_{q \leq Q} q^{-1} \sum_{u, v = 1}^{\infty} \frac{1}{u^k v^k} \sum_{s^k \leq x + 1} s^{-k} (s^k, v^k, q) (x^{\frac{1}{k}} - 1) \sum_{t \leq x} t^{-k} + \sum_{t^k > x} t^{-k}. \tag{2.26}
\]

Now, we estimate $S_{2Aa1}$.

\[
S_{2Aa1} \ll \sum_{q \leq Q} q^{-1} \sum_{u, v = 1}^{\infty} \frac{1}{u^k v^k} \sum_{s^k \leq x + 1} s^{-k} \sum_{t | u, (t, s) = 1} t^{-1} q^{-1} (s^k, v^k, q)(t^k, q).
\]

hence

\[
S_{2Aa1} \ll x^{\frac{1}{k}} \log Q. \tag{2.27}
\]
Secondly, as the estimate of $S_{2Aa1}$, we have

$$S_{2Aa2} \ll x \sum_{q \leq Q} q^{-1} \sum_{u,v=1}^{\infty} \frac{1}{u^k v^k} \sum_{s^k \leq x+1} s^{-k}(s^k, v^k, q) \sum_{t | u, (t,s) = 1} (t^k, q)t^{-k}$$

$$\ll x \sum_{u,v=1}^{\infty} \frac{1}{u^k v^k} \sum_{s^k \leq x+1} s^{-k} \sum_{t | u, (t,s) = 1} q^{-1}(s^k, v^k, q)(t^k, q)t^{-k}$$

$$\ll x \log Q \sum_{v=1}^{\infty} \frac{1}{v^k} \sum_{s^k \leq x+1} s^{-k}(s^k)(x^k)^{-2k+\varepsilon},$$

hence

$$S_{2Aa2} \ll x^{1-\varepsilon} \log Q.$$  \hspace{1cm} (2.28)

By (2.27) and (2.28),

$$S_{2Aa} \ll x^{\frac{1}{2}} \log Q + x^{\frac{1}{2} - 1+\varepsilon} \log Q.$$  \hspace{1cm} (2.29)

In the same manner we have

$$S_{2Ab} \ll x^{\frac{1}{2}} \log Q + x^{\frac{1}{2} - 1+\varepsilon} \log Q.$$  \hspace{1cm} (2.30)

By (2.29) and (2.30),

$$S_{2A0} \ll x^{\frac{1}{2}} \log Q + x^{\frac{1}{2} - 1+\varepsilon} \log Q.$$  \hspace{1cm} (2.31)

By (2.20), (2.21), (2.25) and (2.31), we obtain

$$S_2(x, Q) = c_1(Q)x + O(x^{\frac{2}{k}} \log Q),$$  \hspace{1cm} (2.32)

where $c_1(Q)$ is given by (2.26).

3. The formula for $S_3(x, Q)$

By (1.3) and (1.9),

$$S_3(x, Q) = \sum_{q \leq Q} \sum_{a=1}^{q} q^{-1} \sum_{u,v=1}^{\infty} \mu(uv) \sum_{t | u, (t,a) = 1} \mu(tk, q)tk$$

$$\sum_{r,s=1}^{\infty} \mu(r, q)rs.$$  \hspace{1cm} (3.1)
Let
\[ \Delta = \# \{1 \leq a \leq q : (u_k, q) | a, (r_k, q) | a, (v_k, q) | a + 1, (s_k, q) | a + 1 \}. \] (3.2)

Then
\[(u, s, q) = (v, r, q) = 1, \]
and
\[ \Delta = \# \{1 \leq m \leq \frac{q}{[(u_k, q), (r_k, q)]} : m[(u_k, q), (r_k, q)] + 1 \equiv 0 (mod [(v_k, q), (s_k, q)]) \}, \]
therefore,
\([[(u_k, q), (r_k, q)], [(v_k, q), (s_k, q)]) = 1, \]
and
\[ \Delta = \frac{q}{[(u_k, q), (r_k, q)][(v_k, q), (s_k, q)]} = \frac{q(r_k, u_k, q)(s_k, v_k, q)}{(u_k, q)(r_k, q)(v_k, q)(s_k, q)}. \] (3.3)

Hence, by (3.1), (3.2) and (3.3),
\[ S_3(x, Q) = c_1(Q), \] (3.4)
where \(c_1(Q)\) is given by (2.26).

### 4 Lemmas for \(S_1(x, Q)\)

As Vaughan had done in [9], our proof of the theorem uses the Hardy-Littlewood method (see [8]) and depends heavily on [2] where the bounds for
\[ S(\alpha) = S_k(\alpha) = \sum_{n \leq x} \mu_k(n) \mu_k(n + 1)e(n\alpha), \] (4.1)
and related expressions, are obtained.

Let \(R = x^{1/2 + \tau}\), where \(\tau = \tau_k\) is a sufficiently small positive number \((2\varepsilon < \tau < \frac{1}{10k})\), and let \(\mathfrak{M}\) denote the union of the intervals
\[ \mathfrak{M}(q, a) = \{\alpha : q\alpha - a \leq R^{-1} \}, \] (4.2)
with \(1 \leq a \leq q \leq x/R\) and \((q, a) = 1\), and \(\mathcal{M} = (R^{-1}, 1 + R^{-1}] \setminus \mathcal{M}\).

**Lemma 4.1.** We have

\[
\int_{\mathcal{M}} |S(\alpha)|^2 d\alpha \ll x^{1+\varepsilon} R^{1-\frac{1}{x}} + x^{2+\varepsilon} R^{2-3} + x^{\frac{4}{x+1}+1+\varepsilon} R^{-2}.
\]

Proof. This is Theorem 2 of [2] by taking \(Q = x/R\), we note that \(\mathcal{M} \subseteq (R^{-1}, 1 + R^{-1}]\), the interval \((Q^{-1}, 1 + Q^{-1}]\) of [2] may be replaced by \((R^{-1}, 1 + R^{-1}]\).

Let

\[
K(\alpha) = \sum_{u \leq Q} \sum_{v \leq x/u} e(uva).
\]

When \(Q \leq \sqrt{x}\) define

\[
K_q(\alpha) = \sum_{u \leq Q/q} \sum_{v \leq x/\alpha} e(quv\alpha),
\]

\[
H_q(\alpha) = \begin{cases} 
\sum_{u \leq Q/q} \sum_{v \leq x/u} e(uva), & (q > 1), \\
0, & (q = 1). 
\end{cases}
\]

When \(Q > \sqrt{x}\) define

\[
K_q(\alpha) = \sum_{u \leq \sqrt{x}/q} \left( \sum_{v \leq x/qu} e(quv\alpha) + \sum_{\sqrt{x} < v \leq \min(Q, x/\alpha)} e(quv\alpha) \right),
\]

\[
H_q(\alpha) = \begin{cases} 
\sum_{u \leq \sqrt{x}/q} \left( \sum_{v \leq x/u} e(uva) + \sum_{\sqrt{x} < v \leq \min(Q, x/u)} e(uva) \right), & (q > 1), \\
0, & (q = 1). 
\end{cases}
\]

We have the following results on these functions (see the section 2 of Vaughan [9]).

**Lemma 4.2.** Suppose that \(Q \leq x\) and \(q \in \mathbb{N}\). Then

\[
K(\alpha) = K_q(\alpha) + H_q(\alpha).
\]

Proof. This is Lemma 2.9 of [9].

**Lemma 4.3.** Suppose that \((a, q) = 1\) and \(|q\alpha - a| \leq q^{-1}\). Then

\[
K(\alpha) \ll (xq^{-1} + q) \log x.
\]
Proof. This is Lemma 2.10 of [9].

**LEMMA 4.4.** Suppose that \((a, q) = 1, q \in \mathbb{N}\) and \(\alpha = a/q + \beta\). Then
\[
K_q(\alpha) = K_q(\beta) \ll \sum_{u \leq \sqrt{x/q}} \frac{x}{qu + x\|qu\beta\|}.
\]
Moreover, if \(|\beta| \leq 1/(2\sqrt{x})\), then
\[
K_q(\alpha) \ll \frac{x \log x}{q + qx |\beta|}.
\]

Proof. This is Lemma 2.11 of [9].

**LEMMA 4.5.** Suppose that \((a, q) = 1, q \in \mathbb{N}\) and \(|q\alpha - a| \leq 1/(2\sqrt{x})\). Then
\[
H_q(\alpha) \ll (\min(Q, \sqrt{x}) + q) \log x.
\]

Proof. This is Lemma 2.12 of [9].

By Dirichlet’s theorem on diophantine approximation, Lemma 4.1 and Lemma 4.3,

\[
\sup_\mathbb{A} |K(\alpha)| \ll R \log x \quad (4.8)
\]

and

\[
\int_\mathbb{A} |S(\alpha)|^2 K(\alpha)d\alpha \ll x^{1+\varepsilon} R^{2-\frac{1}{2}} + x^{2+\varepsilon} R^{2-2} + x^{3+\varepsilon} R^{-1}
\ll x^{1+\frac{1}{k}} + x^{2+\frac{1}{2}+1-\frac{1}{2}}.
\]

(4.9)

By (1.2) and (1.7),

\[
S_1(x, Q) = \sum_{q \leq Q} \sum_{m, n \leq x \atop m \equiv n (\text{mod } q)} \mu_k(n)\mu_k(n+1)\mu_k(m)\mu_k(m+1)
\]

\[
= 2 \sum_{q \leq Q} \mu_k(n)\mu_k(n+1)\mu_k(m)\mu_k(m+1) + [Q] \sum_{n \leq x} \mu_k(n)\mu_k(n+1).
\]

Hence

\[
S_1(x, Q) = 2S_A(x, Q) + [Q] \sum_{n \leq x} \mu_k(n)\mu_k(n+1)
\]

where

\[
S_A(x, Q) = \sum_{q \leq Q} \sum_{m, n \leq x \atop m \equiv n (\text{mod } q)} \mu_k(n)\mu_k(n+1)\mu_k(m)\mu_k(m+1).
\]

(4.10)
By (4.1), (4.3) and (4.11),

\[
S_A(x, Q) = \int_{R^{-1}}^{1+R^{-1}} |S(\alpha)|^2 K(\alpha) \, d\alpha = \int_{\mathbb{R}} |S(\alpha)|^2 K(\alpha) \, d\alpha + \int_{\mathcal{M}} |S(\alpha)|^2 K(\alpha) \, d\alpha.
\]  \hspace{1cm} (4.12)

By (4.9) and (4.12),

\[
S_A(x, Q) = \int_{\mathbb{R}} |S(\alpha)|^2 K(\alpha) \, d\alpha + O(x^{\frac{1}{\theta} + \frac{2}{\theta} + \frac{4}{\theta + 2}}).
\]  \hspace{1cm} (4.13)

By (1.3)

\[
g(q, a) = \sum_{n=1}^{\infty} \mu(n) \left( \frac{n^k}{n^k} q \right) \psi_k(n; q, a),
\]  \hspace{1cm} (4.14)

where \( \psi_k(n; q, a) \) denotes the number of pairs \( u, v \) of natural numbers with \( uv = n \), \( (u, v) = 1 \), \( (u^k, q) | a \), \( (v^k, q) | a + 1 \). For fixed \( a, q \), the function \( \psi_k(n; q, a) \) is multiplicative in \( n \), hence

\[
g(q, a) = \prod_p \left( 1 - \frac{(p^k, q)}{p^k} \psi_k(p; q, a) \right).
\]

Let

\[
f(q) = \prod_{p | q} \left( 1 - \frac{2}{p^k} \right) \quad \text{and} \quad h(q, a) = \prod_{p | q} \left( 1 - \frac{(p^k, q)}{p^k} \psi_k(p; q, a) \right),
\]  \hspace{1cm} (4.15)

then, by (3.8) of [2],

\[
g(q, a) = g f(q) h(q, a),
\]  \hspace{1cm} (4.16)

where

\[
g = \prod_p \left( 1 - \frac{2}{p^k} \right),
\]  \hspace{1cm} (4.17)

By (3.9) and (3.10) of [2], we have

\[
h(q, a) = \prod_{(p^k, q) | a(a+1)} \left( 1 - \frac{(p^k, q)}{p^k} \right),
\]  \hspace{1cm} (4.18)

and for co-prime natural numbers \( q_1, q_2 \)

\[
h(q_1q_2, a) = h(q_1, a) h(q_2, a),
\]  \hspace{1cm} (4.19)
\( h(q, a) \) is a multiplicative function of \( q \).

We need the following sums of Gaussian type

\[
G(q, a) = \sum_{b=1}^{q} g(q, b)e\left(\frac{ab}{q}\right), \quad H(q, a) = \sum_{b=1}^{q} h(q, b)e\left(\frac{ab}{q}\right),
\]

(4.20)

which by (4.16) are related by

\[
G(q, a) = \rho f(q)H(q, a).
\]

(4.21)

As in [2], we introduce the following function

\[
H(q) = \sum_{a=1}^{q} \left| H(q, a) \right|^2.
\]

(4.22)

We define

\[
I(\beta) = \sum_{n \leq \beta} e(\beta n), \quad S^*(\alpha) = S^*(\alpha; q, a) = q^{-1}G(q, a)I(\alpha - \frac{a}{q}),
\]

(4.23)

\[
\Delta(\alpha) = \Delta(\alpha; q, a) = S(\alpha) - S^*(\alpha).
\]

(4.24)

Let

\[
\mathcal{M}(q, a; T) = \{ \alpha : \left| q\alpha - a \right| \leq T/x \},
\]

(4.25)

and \( \mathcal{M}(T) \) denote the union of the intervals \( \mathcal{M}(q, a; T) \) with \( 1 \leq a \leq q \leq T \) and \( (q, a) = 1 \), then

\[
\mathcal{M}(2T) \setminus \mathcal{M}(T) = \bigcup_{T < q \leq 2T} \bigcup_{a \leq q, (a, q) = 1} \{ \alpha : \left| q\alpha - a \right| \leq T/x \} \cup \bigcup_{q \leq T} \bigcup_{a \leq q, (a, q) = 1} \{ \alpha : T/x < \left| q\alpha - a \right| \leq 2T/x \}.
\]

and

\[
\mathcal{M} = \mathcal{M}(x/R) \subseteq \bigcup_{1 \leq 2^j \leq x/R} \mathcal{M}(2^{j+1}) \setminus \mathcal{M}(2^j).
\]

(4.26)

Hence, as the proof of Lemma 4.2 in [1],

\[
\int_{\mathcal{M}} \left| S^*(\alpha) \Delta(\alpha) \right| \, d\alpha \ll x \max_{1 \leq T \leq x/R} (U_1^*(T))^{1/2}(U_2^*(T))^{1/2} + \int_{\mathcal{M}(1)} \left| S^*(\alpha) \Delta(\alpha) \right| \, d\alpha,
\]

where

\[
U^*_1(T) = \sum_{T < q \leq 2T} q^{-1} \rho f(q) \sum_{a=1}^{q} \left| H(q, a) \right|^2,
\]

(4.27)

\[
U^*_2(T) = \sum_{q \leq T} q^{-1} \rho f(q) \sum_{a=1}^{q} \left| H(q, a) \right|^2.
\]

(4.28)
where

\[ U_1^r(T) = \int_{\mathfrak{M}(2T) \setminus \mathfrak{M}(T)} |S^*(\alpha)|^2 \, d\alpha, \quad U_2^r(T) = \int_{\mathfrak{M}(2T)} |\Delta(\alpha)|^2 \, d\alpha. \]  

(4.28)

**LEMMA 4.6.** Suppose that \( 1 \leq T \leq \frac{1}{2} \sqrt{x} \). Then

\[ \int_{\mathfrak{M}(T)} |S(\alpha) - S^*(\alpha)|^2 \, d\alpha \ll T^{3-\frac{3}{2}+\varepsilon} x^{\frac{3}{4}+1+\varepsilon} + x^{\frac{1}{2}+1+\varepsilon} T^2. \]

Proof. This is Lemma 5.1 of [2].

**LEMMA 4.7.** Suppose that \( 1 \leq 2T \leq \frac{1}{2} \sqrt{x} \). Then

\[ \int_{\mathfrak{M}(2T) \setminus \mathfrak{M}(T)} |S^*(\alpha)|^2 \, d\alpha \ll x T^{\frac{1}{2}+1+\varepsilon}. \]

Proof. This is Lemma 5.2 of [2].

**LEMMA 4.8.** We have

\[ \int_{\mathfrak{M}} |S(\alpha) - S^*(\alpha)|^2 \, d\alpha \ll x^{2+\varepsilon} R^{-3} + x^{4+1+\varepsilon} R^{-2}. \]

Proof. By taking \( T = x/R \) in Lemma 4.6, the lemma follows.

**LEMMA 4.9.** We have

\[ \sum_{q \leq x/R} \sum_{a=1}^q \int_{\mathfrak{M}(q,a)} |\Delta(\alpha; q, a)|^2 H_q(\alpha) \, d\alpha \ll x^{2+\varepsilon} R^{-3} + x^{4+1+\varepsilon} R^{-2}. \]

Proof. By Lemma 4.5 and Lemma 4.6, the lemma follows.

**LEMMA 4.10.** We have

\[ \int_{\mathfrak{M}(1)} |S^*(\alpha)\Delta(\alpha)| \, d\alpha \ll x^{\frac{2}{3}+\varepsilon}. \]

Proof. By Cauchy-Schwarz inequality and Lemma 4.6

\[ \int_{\mathfrak{M}(1)} |S^*(\alpha)\Delta(\alpha)| \, d\alpha \ll \int_{1-1/x}^{1+1/x} |S^*(\alpha)\Delta(\alpha)| \, d\alpha \ll \int_0^{x^{-1}} |S^*(\alpha)\Delta(\alpha)| \, d\alpha \ll (\int_0^{x^{-1}} |S^*(\alpha)|^2 \, d\alpha)^{1/2} (\int_0^{x^{-1}} |\Delta(\alpha)|^2 \, d\alpha)^{1/2} \ll \frac{1}{2}(x^{\frac{2}{3}}+x^{\frac{1}{2}+\varepsilon})^{1/2} \ll x^{\frac{2}{3}+\varepsilon}, \]
the lemma follows.

By (4.28), Lemma 4.6 and Lemma 4.7,

\[ U_1^*(T) \ll xT^{\frac{1}{k}-1+\varepsilon}, \quad U_2^*(T) \ll T^{3\frac{1}{k}x^{\frac{2}{k}}\frac{1}{k}-1+\varepsilon} + x^{\frac{2}{k+1}-1+\varepsilon}T^2. \quad (4.29) \]

Hence, by (4.26), (4.27), (4.29) and Lemma 4.10,

\[ \int_{\mathfrak{M}} |S^*(\alpha)\Delta(\alpha)| \, d\alpha \ll \log x \max_{1 \leq T \leq x/R} (xT^{\frac{1}{k}-1+\varepsilon})^{\frac{1}{2}} (T^{3\frac{1}{k}x^{\frac{2}{k}}\frac{1}{k}-1+\varepsilon} + x^{\frac{2}{k+1}-1+\varepsilon}T^2)^{\frac{1}{2}} + x^{\frac{2}{k+1}+\varepsilon}. \]

Combining these with Lemma 4.5,

\[ \sum_{q \leq x/R} \sum_{a=1}^{q} \int_{\mathfrak{M}(q,a)} |S^*(\alpha; q, a)\Delta(\alpha; q, a)H_q(\alpha)| \, d\alpha \]

\[ \ll x^{\frac{1}{k}+\frac{1}{2k}+\varepsilon}R^{\frac{1}{2k}} - 1 + x^{\frac{1}{k+1}+\frac{1}{2k}+\varepsilon}R^{-\frac{1}{2k}} - \frac{1}{2} + x^{\frac{1}{k}+\frac{1}{2k}+\varepsilon}. \quad (4.30) \]

**LEMMA 4.11.** We have

\[ \sum_{q \leq x/R} \sum_{a=1}^{q} \int_{\mathfrak{M}(q,a)} |\Delta(\alpha; q, a)|^2 |K(\alpha)| \, d\alpha \ll x^{1+\frac{1}{k} - \varepsilon} + x^{\frac{1}{2k+1}} x^{1-\frac{1}{k} - \varepsilon}. \]

Proof. We use the method of (4.8) in [9]. Let \( \mathfrak{M}_0(1, 1) = \{ \alpha : |\alpha - 1| \leq 1/x \} \) and for \( j \geq 1 \)

\[ \mathfrak{M}_j(q, a) = \begin{cases} \{ \alpha : 2^{j-1}x^{-1} \leq q\alpha - a \leq 2^jx^{-1} \}, & \text{when } q \leq 2^{j-1}; \\ \{ \alpha : q\alpha - a \leq 2^jx^{-1} \}, & \text{when } 2^{j-1} < q \leq 2^j. \end{cases} \]

\[ \mathfrak{M}_j = \bigcup_{1 \leq a \leq q \leq 2^j, \ (a,q)=1} \mathfrak{M}_j(q,a). \]

Choose \( J \) so that \( 2^{j-1} < x/R \leq 2^j \). For \( 1 \leq a \leq q \leq x/R \), \( (a,q)=1 \),

then

\[ \mathfrak{M}(q, a) \subseteq \bigcup_{q \leq 2^j} \mathfrak{M}_j(q,a), \quad \mathfrak{M} \subseteq \bigcup_{0 \leq j \leq J} \mathfrak{M}_j. \]

By Leemme 4.4, when \( \alpha \in \mathfrak{M}_j(q,a) \) with \( 1 \leq a \leq q \leq 2^j, (a,q)=1 \),

\[ K_q(\alpha) \ll 2^{-j}x \log x. \]
By Lemma 4.6 and (4.31),
\[ \sum_{q \leq x/R} \sum_{a=1}^q \int_{\mathfrak{M}(q,a)} |\Delta(\alpha; q, a)|^2 K_q(\alpha) \, d\alpha \ll x^{2+\varepsilon} \left( \frac{x}{R} \right)^{2-\varepsilon} + x^{1+\frac{4}{k}+\varepsilon} R^{-1}. \]

By Lemma 4.2, Lemma 4.9 and (4.32), the lemma follows.

By (4.23), (4.24) and Lemma 4.2, when \( \alpha \in \mathfrak{M}(q,a) \),
\[ |S(\alpha)|^2 K(\alpha) = |S^*(\alpha; q, a)|^2 K(\alpha) \]
\[ + 2(\Re(S^*(\alpha; q, a)\Delta(\alpha; q, a)))(K_q(\alpha) + H_q(\alpha)) + |\Delta(\alpha; q, a)|^2 K(\alpha). \]

By (4.13), (4.30) and Lemma 4.11,
\[ S_A(x, Q) = S_{A1}(x, Q) + S_{A2}(x, Q) + O(x^{1+\frac{1}{k-2}} + x^{1+\frac{4}{k}+\frac{1}{2}}), \]  
(4.33)

where
\[ S_{A1}(x, Q) = \sum_{q \leq x/R} \sum_{a=1}^q \int_{\mathfrak{M}(q,a)} |S^*(\alpha; q, a)|^2 K(\alpha) d\alpha, \]
(4.34)
\[ S_{A2}(x, Q) = 2 \sum_{q \leq x/R} \sum_{a=1}^q \int_{\mathfrak{M}(q,a)} (\Re(S^*(\alpha; q, a)\Delta(\alpha; q, a)) K_q(\alpha) d\alpha, \]
by (4.21) and (4.23)
\[ S_{A2}(x, Q) = 2 \sum_{q \leq x/R} q^{-1} f(q) \sum_{a=1}^q \int_{-1/qR}^{1/qR} (\Re(H(q,a)I(\beta)\Delta(\frac{\alpha}{q} + \beta; q, a))) K_q(\beta) d\beta. \]
(4.35)

5 The estimate of \( S_{A2}(x, Q) \)

**Lemma 5.1.** We have
\[ \sum_{b=1}^q h(q, b) \sum_{r \mid q} \mu(q) \frac{r}{q} g(r, b) = q^{-1} \sum_{b=1}^q h(q, b) \sum_{a=1}^q \frac{\epsilon(-ab)}{q} G(q, a). \]

Proof. By (4.20) and \( \sum_{a=1}^q \frac{\epsilon(\frac{an}{q})}{q} = \sum_{r \mid (q, n)} r \mu\left( \frac{n}{r} \right) \),
\[ q^{-1} \sum_{b=1}^q h(q, b) \sum_{a=1}^q \frac{\epsilon(-ab)}{q} G(q, a) \]
\[ = q^{-1} \sum_{b=1}^q h(q, b) \sum_{a=1}^q \frac{\epsilon(-ab)}{q} \sum_{t=1}^q g(q, t) e\left( \frac{at}{q} \right) \]
\[ = \sum_{r \mid (q, n)} r \mu\left( \frac{n}{r} \right) \sum_{a=1}^q \frac{\epsilon(-ab)}{q} g(q, a). \]
\[ q^{-1} \sum_{b=1}^{q} h(q, b) \sum_{r \mid q, r \mid t-b} r \mu \left( \frac{q}{r} \right) \sum_{t=1}^{q} g(q, t). \]

Hence

\[ q^{-1} \sum_{b=1}^{q} h(q, b) \sum_{a=1}^{q} G(q, a) c \left( \frac{-ab}{q} \right) = q^{-1} \sum_{r \mid q} r \mu \left( \frac{q}{r} \right) \sum_{b=1}^{q} h(q, b) \sum_{t=1}^{q} g(q, t). \quad (5.1) \]

By (4.14),

\[ \sum_{t=1}^{q} g(q, t) = \sum_{n=1}^{\infty} \mu(n) \frac{(n^k, q)}{n^k} \sum_{t=1}^{q} \psi_k(n; q, t), \quad (5.2) \]

where

\[ \psi_k(n; q, t) = \# \{ u, v \geq 1 : uv = n, (u, v) = 1, (u^k, q) \mid t, (v^k, q) \mid t + 1 \}. \]

Write \( t = b + rl, q = rm \), and \( l \) runs through complete residues modulus \( m \), we want to count the number of \( l \).

We have

\( (u^k, rm) \mid b + rl, (v^k, rm) \mid b + 1 + rl \).

Let

\[ (u^k, r) = r_1, (v^k, r) = r_2, (r_1, r_2) = 1, \]

then

\[ r_1 \mid b, r_2 \mid b + 1, \]

\[ (u^k, r_1, m) \mid \frac{b}{r_1} + \frac{r_1}{r_2} l, \quad (v^k, r_2, m) \mid \frac{b+1}{r_2} + \frac{r_1}{r_2} l. \]

These congruences have a unique solution of \( l \) modulus \( (u^k, r_1, m)(v^k, r_2, m) \), therefore the number of \( l \) is

\[ \frac{m}{(u^k, r_1r_2m)(v^k, r_1r_2m)} = \frac{mr_1r_2}{(u^k, r_1r_2m)(v^k, r_1r_2m)}. \]

Since \( (r_1, r_2) = (r_1, v) = (r_2, u) = 1 \), the number of \( l \) is

\[ \frac{mr_1r_2}{(u^k, r_1r_2m)(v^k, r_1r_2m)}. \]

Hence

\[ \sum_{t=1}^{q} g(q, t) = \sum_{n=1}^{\infty} \mu(n) \frac{(n^k, q)}{n^k} \sum_{u, v} \frac{mr_1r_2}{(n^k, r_1r_2m)}. \]
where \( \sum^\ast_{u,v} \) means \((u^k, r) = r_1 \mid b, (v^k, r) = r_2 \mid b + 1, uv = n, (u, v) = 1, u, v \geq 1, q = rm.\)

Let \( r = r_1r_2\Delta \), then

\[
(q, n^k) = (r_1r_2\Delta m, u^k v^k) = (r_1r_2\Delta m, u^k)(r_1r_2\Delta m, v^k) = (r_1\Delta m, u^k)(r_2\Delta m, v^k)
\]

\[
= r_1r_2(\Delta m, \frac{u^k}{r_1})(\Delta m, \frac{v^k}{r_2}).
\]

Since \((\frac{u^k}{r_1}, r_2\Delta) = (\frac{v^k}{r_2}, r_1\Delta) = 1\), we have \((\Delta, \frac{u^k v^k}{r_1r_2}) = 1\), hence

\[
(q, n^k) = r_1r_2(m, \frac{u^k}{r_1})(m, \frac{v^k}{r_2}) = (r_1r_2m, u^k)(r_1r_2m, v^k) = (r_1r_2m, (uv)^k).
\]

So, by (5.1) and (5.2),

\[
q^{-1} \sum_{b=1}^{q} h(q, b) \sum_{a=1}^{q} \, 'G(q, a)(\frac{-ab}{q}) = \sum_{r|q} \mu(\frac{q}{r}) \sum_{b=1}^{\frac{q}{r}} h(q, b) \sum_{n=1}^{\infty} \mu(n) \frac{1}{n^r} \sum_{u,v} \ast \frac{rm_{1}r_{2}}{q} \sum_{r|q} \mu(\frac{q}{r}) \sum_{b=1}^{\frac{q}{r}} h(q, b) \sum_{n=1}^{\infty} \mu(n) \frac{u^k v^k}{n^r} \psi(n; r, b),
\]

by (4.14), the lemma follows.

Write

\[
L = \sum_{a=1}^{q} \, 'H(q, a)\Delta(\frac{a}{q} + \beta; q, a),
\]

then, by (4.20) and (4.24),

\[
L = \sum_{b=1}^{q} h(q, b) \sum_{a=1}^{q} \, 'e(\frac{-ab}{q})\Delta(\frac{a}{q} + \beta; q, a) = \sum_{n \leq x} (\mu_k(n)\mu_k(n + 1) \sum_{b=1}^{\frac{q}{r}} h(q, b) \sum_{a=1}^{q} \, 'e(\frac{q}{r}(n - b)) - q^{-1} \sum_{b=1}^{\frac{q}{r}} h(q, b) \sum_{a=1}^{q} \, 'G(q, a)(\frac{-ab}{q}))e(\beta n).
\]

Write

\[
U(y) = \sum_{n \leq y} (\mu_k(n)\mu_k(n + 1) \sum_{b=1}^{\frac{q}{r}} h(q, b) \sum_{a=1}^{q} \, 'e(\frac{q}{r}(n - b)) - q^{-1} \sum_{b=1}^{\frac{q}{r}} h(q, b) \sum_{a=1}^{q} \, 'G(q, a)(\frac{-ab}{q})) = U_1(y) - U_2(y).
\]

and using \(e(n\beta) = e(x\beta) - 2\pi i\beta \int_{y}^{x} e(y\beta)dy\), we have

\[
L = U(x)e(x\beta) - 2\pi i\beta \int_{1}^{x} U(y)e(y\beta)dy.
\]

We estimate \(U_1(y)\) first.

Write

\[
E(x; q, a) = A_k(x; q, a) - q^{-1}g(q, ax).
\]

We have
\[
U_1(y) = U_1([y]) = \sum_{b=1}^{q} h(q, b) \sum_{n \leq [y]} \mu_k(n) \mu_k(n + 1) \sum_{r | (q, n - b)} r \mu\left(\frac{q}{r}\right)
\]
\[
= \sum_{r \mid q} r \mu\left(\frac{q}{r}\right) \sum_{b=1}^{q} h(q, b) \sum_{n \leq [y]} \mu_k(n) \mu_k(n + 1)
\]
\[
= \sum_{r \mid q} \mu_k\left(\frac{q}{r}\right) \sum_{b=1}^{q} h(q, b) \sum_{r \mid n - b} \mu_k(n) \mu_k(n + 1)
\]
and
\[
U_2(y) = \sum_{n \leq [y]} q^{-1} \sum_{b=1}^{q} h(q, b) \sum_{a=1}^{q} G(q, a) e\left(-\frac{ab}{q}\right)
\]
\[
= q^{-1} \sum_{b=1}^{q} h(q, b) \sum_{a=1}^{q} G(q, a) e\left(-\frac{ab}{q}\right) [y].
\]
We need an obvious result on \(Y_k(x, Q)\).

**Lemma 5.2.** For all \(Q, x\), we have

\[
Y_k(x, Q) \ll x^2 \log Q + Q^2.
\]

Proof. By (1.4) and \(g(q, a) \ll 1\),

\[
Y_k(x, Q) \ll \sum_{q \leq Q} \sum_{a=1}^{q} (x/q + 1)^2 \ll \sum_{q \leq Q} \sum_{a=1}^{q} (x^2 q^{-2} + 1) \ll x^2 \log Q + Q^2,
\]
the lemma follows.

By Lemma 5.1, we have

\[
U(y) = \sum_{r \mid q} r \mu\left(\frac{q}{r}\right) \sum_{b=1}^{q} h(q, b) E([y]; r, b).
\]

Hence

\[
L \ll \sum_{r \mid q} r \mu^2\left(\frac{q}{r}\right) \sum_{b=1}^{q} h(q, b) \sum_{r \mid q} r \mu\left(\frac{q}{r}\right) \sum_{b=1}^{q} h(q, b) \sum_{r \mid q} r \mu\left(\frac{q}{r}\right) \sum_{b=1}^{q} h(q, b) E([y]; r, b) | dy.
\]

By (4.35), Lemma 4.4 and \(I(\beta) \ll x/(1 + x | \beta |)\)

\[
S_{A2}(x, Q) \ll \sum_{q \leq x/R} q^{-1} \int_{0}^{1/qR} \frac{x}{\log x} (\sum_{r | q} r \mu^2\left(\frac{q}{r}\right) \sum_{b=1}^{q} h(q, b) \sum_{r \mid q} r \mu\left(\frac{q}{r}\right) \sum_{b=1}^{q} h(q, b) E([y]; r, b) | dy) d\beta \ll \int_{0}^{R-1} \frac{x^2 \log x}{(1 + x)^2} M(\beta) d\beta,
\]
where

\[
M(\beta) \ll \sum_{q \leq x/R} q^{-2} \left(\sum_{r | q} r \mu^2\left(\frac{q}{r}\right) \sum_{b=1}^{q} h(q, b) \sum_{r \mid q} r \mu\left(\frac{q}{r}\right) \sum_{b=1}^{q} h(q, b) E([y]; r, b) | dy\right).
\]

(5.5)
When \( r \mid q \), we have \( \sum_{b=1}^{q} \left| E(y; r, b) \right| = qr^{-1} \sum_{b=1}^{r} \left| E(y; r, b) \right| \), then

\[
\sum_{q \leq x/R} \sum_{r \mid q} \frac{1}{q} \left( \sum_{b=1}^{q} h(q, b) \right) \left| E([x]; r, b) \right| \ll \sum_{q \leq x/R} \sum_{r \mid q} \frac{1}{q} \left| E([x]; r, b) \right| \ll \log x \sum_{r \leq x/R} q^{-1} \sum_{b=1}^{r} \left| E([x]; r, b) \right|.
\]

By Cauchy inequality and (1.5),

\[
\sum_{q \leq x/R} \sum_{r \mid q} \frac{1}{q} \left( \sum_{b=1}^{q} h(q, b) \right) \left| E([x]; r, b) \right| \ll \log x \sum_{r \leq x/R} q^{-1} \sum_{b=1}^{r} \left| E([x]; r, b) \right| \ll \log^{3/2} x \left( \frac{2}{3} + \frac{2}{x^{3/1}} + \frac{2}{x^{2/1}} \right)^{1/2} \ll x^{1+\varepsilon} R^{1/2} + x^{2/3}.
\]

this contributes to \( S_{A2}(x, Q) \)

\[
\ll \left( x^{1+\varepsilon} R^{1/2} + x^{2/3} \right) \left( \int_{0}^{R^{-1}} \frac{x^{2 \log x}}{(1+x^{2})^{2}} d\beta \right)
\]

\[
\ll x \log x \left( x^{1+\varepsilon} R^{1/2} + x^{2/3} \right) \ll x^{2+\varepsilon} R^{1/2} + x^{1+\varepsilon}.
\]

For the second term of \( M(\beta) \), we have

\[
\sum_{q \leq x/R} \sum_{r \mid q} \frac{1}{q} \left( \sum_{b=1}^{q} h(q, b) \right) \left| E([y]; r, b) \right| dy
\]

\[
\ll \beta \int_{1}^{x/R} \log x \sum_{r \leq x/R} q^{-1} \sum_{b=1}^{r} \left| E([y]; r, b) \right| dy
\]

\[
\ll \beta \int_{1}^{x/R} \log x \sum_{r \leq x/R} q^{-1} \sum_{b=1}^{r} \left| E([y]; r, b) \right| dy + \beta \int_{x/R}^{x} \log x \sum_{r \leq x/R} q^{-1} \sum_{b=1}^{r} \left| E([y]; r, b) \right| dy.
\]

By Cauchy inequality and Lemma 5.2,

\[
\beta \int_{1}^{x/R} \log^{3/2} x \left( \sum_{r \leq x/R} \sum_{b=1}^{r} \left| E([y]; r, b) \right|^{2} \right)^{1/2} dy
\]

\[
\ll \beta \int_{1}^{x/R} \log^{3/2} x \left( y^{2} \log x + \left( \frac{2}{x} \right)^{2} \right)^{1/2} dy
\]

\[
\ll x^{2} R^{-2} \beta \log^{2} x,
\]

this contributes to \( S_{A2}(x, Q) \)

\[
\ll x^{2} R^{2} \beta \log^{2} x \int_{0}^{R^{-1}} \frac{x^{2 \log x}}{(1+x^{2})^{2}} / (1+x^{2})^{2} d\beta \ll x^{2} R^{2} \log^{4} x.
\]

By Cauchy inequality and (1.5),

\[
\beta \int_{x/R}^{x} \log x \sum_{r \leq x/R} q^{-1} \sum_{b=1}^{r} \left| E([y]; r, b) \right| dy
\]

\[
\ll \beta \int_{x/R}^{x} \log^{3/2} x \left( \frac{2}{x} + \frac{2}{x^{2/1}} + \frac{2}{x^{1/1}} \right)^{1/2} dy \ll \beta \left( x^{2+\varepsilon} R^{1/2} + x^{1+\varepsilon} \right),
\]

this contributes to \( S_{A2}(x, Q) \)
$$\ll x^{2+\varepsilon}R_1^{1-\frac{1}{k}} + x^{1+\frac{2}{k+1}+\varepsilon}.$$ 

Consequently,

$$S_{A2}(x, Q) \ll x^{2+\varepsilon}R_1^{1-\frac{1}{k}} + x^{1+\frac{2}{k+1}+\varepsilon} + x^2R^{-2}\log^4 x. \quad (5.6)$$

6 The estimate of $S_{A1}(x, Q)$

By (4.2), (4.23) and (4.34),

$$S_{A1}(x, Q) = \sum_{q \leq x} \sum_{a=1}^q \sum_{q \leq R, |a| \leq \frac{x}{q}} |G(q, a)|^2 \int_{1/qR \leq |\beta| \leq 1/2} q^{-2} |G(q, a)|^2 K(\frac{a}{q} + \beta) d\beta -$$

$$\sum_{q \leq x} \sum_{a=1}^q \sum_{q \leq R, |a| \leq \frac{x}{q}} |G(q, a)|^2 \int_{1/qR \leq |\beta| \leq 1/2} q^{-2} |G(q, a)|^2 K(\frac{a}{q} + \beta) d\beta$$

$$= S_B(x, Q) - S_C(x, Q), \quad (6.1)$$

say.

We consider $S_C(x, Q)$ first. By Lemma 4.4,

$$S_C(x, Q) \ll \sum_{q \leq x} q^{-2} \sum_{a=1}^q \int_{1/qR \leq |\beta| \leq 1/2} \min(x^2, \beta^{-2}) \frac{1}{u \leq \sqrt{2}} \frac{x}{u + x\|a + u \beta\|} d\beta$$

$$\ll \sum_{q \leq x} q^{-2} \sum_{a=1}^q \int_{1/qR \leq |\beta| \leq 1/2} \frac{x^{\beta^{-2}}}{u \leq \sqrt{2}} \frac{1}{u + x\|a + u \beta\|} d\beta$$

$$\ll \sum_{q \leq x} q^{-2} \sum_{a=1}^q \int_{u/qR \leq |\gamma| \leq 1/2} \frac{x^{\gamma^{-2}}}{u \leq \sqrt{2}} u^{-1} d\gamma.$$ 

Now,

$$ux \sum_{0 \leq |\beta| \leq u/2} \int_{j-1/2 \leq \gamma \leq j+1/2} \frac{x^{-2}}{u + x\|a + u \beta\|} d\gamma \ll ux \sum_{0 \leq |\beta| \leq u/2} \int_{-1/2 \leq \gamma \leq 1/2} \frac{x^{-2}}{u + x\|a + u \beta\|} d\gamma$$

$$\ll ux \sum_{0 \leq |\beta| \leq u/2} \frac{1}{u + x\|a + u \beta\|} d\beta \ll u \log x.$$ 

Hence

$$S_C(x, Q) \ll \sum_{q \leq x} q^{-2} \sum_{a=1}^q |G(q, a)|^2 \sum_{u \leq \sqrt{2}} u \log x$$

$$+ \sum_{q \leq x} q^{-2} \sum_{a=1}^q |G(q, a)|^2 \sum_{u \leq \sqrt{2}} \int_{u/qR \leq |\gamma| \leq 1/2} \frac{x^{\gamma^{-2}}}{u + x\|a + u \gamma\|} d\gamma$$

$$= S_{C1}(x, Q) + S_{C2}(x, Q), \text{ say.}$$

By Lemma 4.2 of [2], for any $Q \geq 1,$
\[ \sum_{Q < q \leq 2Q} q^{-2} f^2(q) H(q) \ll Q^{\frac{1}{k} - 1 + \varepsilon}. \]  

(6.2)

Hence

\[ S_{C1}(x, Q) \ll x \log x \sum_{q \leq x/R} q^{-2} f^2(q) H(q) \ll x \log x. \]  

(6.3)

**Lemma 6.1.** We have

\[ S_{C2}(x, Q) \ll x^{\frac{3}{2} + \frac{1}{2k} + 2\tau}. \]

Proof. We have

\[ S_{C2}(x, Q) \ll \sum_{q \leq x/R} q^{-2} \sum_{a=1}^q' |G(q, a)|^2 \sum_{u \leq \sqrt{x}} x \int_{u/q \leq |\gamma| \leq 1/2} \gamma^{-2} d\gamma \]

\[ \ll \sum_{q \leq x/R} q^{-2} \sum_{a=1}^q' |G(q, a)|^2 \sum_{u \leq \sqrt{x}} x \frac{uK}{u} \]

\[ \ll xR \log x \sum_{q \leq x/R} q^{-1} \sum_{a=1}^q' |G(q, a)|^2. \]

By partial summation and (6.2),

\[ S_{C2}(x, Q) \ll xR \log x \sum_{q \leq x/R} q^{-1} H(q) f(q)^2 \ll xR(\frac{x}{R})^{\frac{1}{k} + \varepsilon} \ll x^{\frac{3}{2} + \frac{1}{2k} + 2\tau}, \]

the lemma follows.

By Lemma 6.1 and (6.3),

\[ S_C(x, Q) \ll x^{\frac{3}{2} + \frac{1}{2k} + 2\tau}. \]

Hence

\[ S_{A1}(x, Q) = S_B(x, Q) + O(x^{\frac{3}{2} + \frac{1}{2k} + 2\tau}). \]

(6.4)

**7 The estimate of \( S_B(x, Q) \)**

We have

\[ \int_{-1/2}^{1/2} |I(\beta)|^2 K\left(\frac{\alpha}{q} + \beta\right) d\beta = \sum_{n_1, n_2 \leq x} \sum_{u v \leq x, u \leq Q \atop n_1 - n_2 + u v = 0} e(uv\frac{\alpha}{q}). \]

By (6.1),

\[ S_B(x, Q) = \sum_{q \leq x/R} q^{-2} \sum_{a=1}^q' |G(q, a)|^2 \sum_{u \leq Q} \sum_{v \leq x/u} e(uv\frac{\alpha}{q}) \sum_{m, n \leq x \atop n = m + uv} 1. \]
Hence

\[ S_B(x, Q) = \sum_{q \leq x/R} q^{-2} \sum_{a=1}^{q} |G(q, a)|^2 \sum_{u \leq Q} \sum_{v \leq x/u} e(\frac{aq}{q})[x - uv]. \] \tag{7.1}

By (6.2),

\[ \sum_{u \leq Q} \sum_{v \leq x/u} (x - uv) \sum_{q > x/R} q^{-2} \sum_{a=1}^{q} |G(q, a)|^2 \ll \sum_{u \leq Q} \sum_{v \leq x/u} (x / R)^{1/2 - 1 + \varepsilon} \ll (x / R)^{1/2 - 1 + \varepsilon} \sum_{u \leq Q} u \sum_{v \leq x/u} (x / u - v) \ll (x / R)^{1/2 - 1 + \varepsilon} \sum_{u \leq Q} u(x / u)^2 \ll x^{3/2 + \frac{1}{12} + 2\tau}, \]

and

\[ \sum_{q \leq x/R} q^{-2} \sum_{a=1}^{q} |G(q, a)|^2 \sum_{u \leq Q} \sum_{v \leq x/u} 1 \ll x \log x. \]

Hence

\[ S_B(x, Q) = \sum_{u \leq Q} \sum_{v \leq x/u} (x - uv)\mathcal{G}(uv) + O(x^{3/2 + \frac{1}{12} + 2\tau}) = S_E(x, Q) + O(x^{3/2 + \frac{1}{12} + 2\tau}), \] \tag{7.2}

where

\[ \mathcal{G}(n) = \sum_{q=1}^{\infty} \sum_{a=1}^{q} q^{-2} |G(q, a)|^2 e\left(\frac{an}{q}\right), \] \tag{7.3}

\[ S_E(x, Q) = \sum_{u \leq Q} \sum_{v \leq x/u} (x - uv)\mathcal{G}(uv). \] \tag{7.4}

### 8 The formula for \( \mathcal{G}(n) \)

Write

\[ J(q) = J(q; n) = \sum_{a=1}^{q} |H(q, a)|^2 e\left(\frac{an}{q}\right), \] \tag{8.1}

then, by (4.21) and (7.3),

\[ \mathcal{G}(n) = q^2 \sum_{q=1}^{\infty} q^{-2} f(q)^2 J(q). \]

By (6.2), \( \mathcal{G}(n) \) converges absolutely.

**Lemma 8.1.** For any natural numbers \( q_1, q_2, (q_1, q_2) = 1 \), we have
\[ J(q_1 q_2) = J(q_1) J(q_2), \]

the function \( J(q) \) is multiplicative.

Proof. By (4.19) and \( h(q, a + mq) = h(q, a) \), write \( a = a_1 q_2 + a_2 q_1, \ b = b_1 q_2 + b_2 q_1 \).

\[
J(q_1 q_2) = \sum_{a_1=1}^{q_1} \sum_{a_2=1}^{q_2} h(q_1, b_1 q_2) e\left(\frac{a_1 q_2 b_1}{q_1}\right) e\left(\frac{a_1 n}{q_1}\right) \left| \sum_{b_2=1}^{q_2} h(q_2, b_2 q_1) e\left(\frac{a_2 q_1 b_2}{q_2}\right) e\left(\frac{a_2 n}{q_2}\right) \right|^2 \]

\[
= \sum_{a_1=1}^{q_1} \sum_{b_1=1}^{q_1} \left| h(q_1, b_1 q_2) e\left(\frac{a_1 q_2 b_1}{q_1}\right)\right|^2 \sum_{a_2=1}^{q_2} h(q_2, b_2 q_1) e\left(\frac{a_2 q_1 b_2}{q_2}\right) e\left(\frac{a_2 n}{q_2}\right) \]

\[
= J(q_1) J(q_2),
\]

the lemma follows.

We need to give a formula for \( J(p^t) \), here \( p \) is a prime number and \( t \geq 1 \).

We have

\[
J(p^t) = \sum_{a=1}^{p^t} \left| H(p^t, a) \right|^2 e\left(\frac{an}{p^t}\right) - \sum_{a=1}^{p^t} \left| H(p^t, a) \right|^2 e\left(\frac{an}{p^t}\right) = J_1(p^t) - J_2(p^t), \tag{8.2}
\]

where

\[
J_1(p^t) = \sum_{a=1}^{p^t} \left| H(p^t, a) \right|^2 e\left(\frac{an}{p^t}\right),
\]

\[
J_2(p^t) = \sum_{a=1}^{p^t} \left| H(p^t, a) \right|^2 e\left(\frac{an}{p^t}\right).
\]

**Lemma 8.2.** When \( t > k \)

\[
J(p^t) = 0.
\]

Proof. When \( t > k \), by (4.20),

\[
J_1(p^t) = \sum_{a=1}^{p^t} \left| \sum_{b=1}^{p^t} h(p^t, b) e\left(\frac{ab}{p^t}\right) \right|^2 e\left(\frac{an}{p^t}\right)
\]

\[
= \sum_{b_1, b_2=1}^{p^t} h(p^t, b_1) h(p^t, b_2) \sum_{a=1}^{p^t} e\left(\frac{a(b_1 - b_2)}{p^t}\right) e\left(\frac{an}{p^t}\right)
\]

\[
= p^t \sum_{b_1, b_2=1}^{p^t} h(p^t, b_1) h(p^t, b_2) \sum_{b_1 - b_2 + n \equiv 0(p^t)}
\]

...
Also

\[ J(p^t) = \sum_{b_1, b_2=1}^{p^t} h(p^t, b_2) h(p^t, b_2) \sum_{a=1}^{p^t} e\left(\frac{a(b_1 - b_2 + n)}{p^t}\right) \]

\[ = \sum_{b_1, b_2=1}^{p^t} h(p^t, b_1) h(p^t, b_2) \sum_{a=1}^{p^t} e\left(\frac{a(b_1 - b_2 + n)}{p^t}\right) \]

\[ = p^{t-1} \sum_{b_1, b_2=1}^{p^t} h(p^t, b_1) h(p^t, b_2) . \]

By (3.9) of [2], for \( t > k \), one has \( h(p^t, a) = h(p^t, b) \) whenever \( a \equiv b \ (mod \ p^k) \). Hence

\[ J_2(p^t) = \sum_{b=1}^{p^t} h(p^t, b) h(p^t, b + n) \sum_{b_2=1}^{p^t} h(p^t, b_2) h(p^t, b_2 - b + n) = J_1(p^t), \]

therefore

\[ J(p^t) = J_1(p^t) - J_2(p^t) = 0, \]

the lemma follows.

From now on, we suppose \( 1 \leq t \leq k \).

By (4.7) of [2],

\[ h(p^t, a) = \begin{cases} 1 - p^{t-k}, & \text{if } p^t \mid a(a+1), \\ 1, & \text{otherwise.} \end{cases} \] (8.3)

For fixed \( p^t, n \), write

\[ \Phi_1 = \Phi_1(p^t) = \#\{b : p^t \mid b(b+1), \ b = 1, 2, ..., p^t\}, \]

\[ \Phi_2 = \Phi_2(p^t) = \#\{b : p^t \mid b(b+1), \ p^t \mid (b+n)(b+n+1), \ b = 1, 2, ..., p^t\}, \] (8.4)

\[ \Phi_3 = \Phi_3(p^t) = \#\{b : p^t \mid b(b+1), \ p^t \mid (b+n)(b+n+1), \ b = 1, 2, ..., p^t\}, \]

\[ \Phi_4 = \Phi_4(p^t) = \#\{b : p^t \mid b(b+1), \ b = 1, 2, ..., p^t\}, \]

\[ \Phi_5 = \Phi_5(p^t) = \#\{b_1, b_2 : p^t \mid b_1(b_1+1), \ p^t \mid b_2(b_2+1), \ b_1 - b_2 + n \equiv 0(p^{t-1}), \ b_1, b_2 = 1, 2, ..., p^t\}. \]
LEMMA 8.3. Suppose that $1 \leq t \leq k$. Then

$$J_1(p^t) = p^{2t} - 4p^{2t-k} + p^{3t-2k} \Phi_2.$$  

Proof. As in the proof of Lemma 8.1 and Lemma 8.2, by (8.3),

$$J_1(p^t) = p^t \sum_{b=1}^{p^t} h(p^t, b)h(p^t, b + n) = p^t (J_A + J_B),$$

where

$$J_A = \sum_{\substack{b=1 \in p^t \mid (b+1)}} (1 - p^t-k)h(p^t, b + n), J_B = \sum_{\substack{b=1 \in p^t \mid (b+1)}} h(p^t, b + n).$$

Firstly, we deal with $J_A$.

We note that $\Phi_1 = 2, \Phi_3 = 2 - \Phi_2, \Phi_4 = p^t - 2$.

By (8.3),

$$J_A = (1 - p^{t-k})^2 \Phi_2 + (1 - p^{t-k}) \sum_{\substack{b=1 \in p^t \mid (b+1)}} 1$$

$$= (1 - p^{t-k})^2 \Phi_2 + (1 - p^{t-k})(2 - \Phi_2) = 2(1 - p^{t-k}) + (1 - p^{t-k})(-p^{t-k}) \Phi_2,$$

hence

$$J_A = 2(1 - p^{t-k}) - p^{t-k}(1 - p^{t-k}) \Phi_2.$$  

Now, we deal with $J_B$. By (8.3),

$$J_B = (1 - p^{t-k}) \Phi_3 + \Phi_4 - \Phi_3 = p^t - 2 - p^{t-k} \Phi_3 = p^t - 2 - p^{t-k}(2 - \Phi_2),$$

hence

$$J_B = p^t - 2 - 2p^{t-k} + p^{t-k} \Phi_2.$$  

Therefore

$$J_1(p^t) = p^t (2(1 - p^{t-k}) - p^{t-k}(1 - p^{t-k}) \Phi_2 + p^t - 2 - 2p^{t-k} + p^{t-k} \Phi_2) = p^{2t} - 4p^{2t-k} + p^{3t-2k} \Phi_2,$$

the lemma follows.

LEMMA 8.4. We have

$$J_2(p) = p^2 (1 - 2p^{-k})^2.$$
Proof. By (4.20) and (8.2),
\[ J_2(p) = (\sum_{b=1}^{p} h(p, b))^2, \]
and
\[ \sum_{b=1}^{p} h(p, b) = \sum_{b=1}^{p} (1 - p^{1-k}) + \sum_{p|b(b+1)} 1 = 2(1 - p^{1-k}) + p - 2 = p - 2p^{1-k}. \]
Hence
\[ J_2(p) = p^2(1 - 2p^{-k})^2, \]
the lemma follows.

**Lemma 8.5.** Suppose that \(1 < t \leq k\). Then
\[ J_2(p^t) = p^{t-1}(p^{t+1} - 4p^{t-k+1} + p^{2t-2k}\Phi_5). \]

Proof. By (4.20) and (8.2),
\[ J_2(p^t) = p^{t-1}M, \]
where
\[ M = \sum_{b_1=1}^{p^t} h(p^t, b_1)h(p^t, b_2). \]
By (8.3), we obtain
\[ M = M_A + M_B, \]
where
\[ M_A = \sum_{b_1=1}^{p^t} \sum_{b_2=1}^{p^t} (1 - p^{t-k})h(p^t, b_2), \]
\[ M_B = \sum_{b_1=1}^{p^t} \sum_{b_2=1}^{p^t} h(p^t, b_2). \]
Again, by (8.3) and (8.5),
\[ M_A = M_{A1} + M_{A2}, \]
where
\[ M_{A1} = (1 - p^{t-k})^2\Phi_5, \]
\[ M_{A2} = (1 - p^{t-k})(\Phi_6 - \Phi_5), \]
and
\[ \Phi_6 = \#\{b_1, b_2 : p^t \mid b_1(b_1 + 1), b_1 - b_2 + n \equiv 0(p^{t-1}), b_1, b_2 = 1, 2, \ldots, p^t\} = 2p. \]

Hence
\[ M_{A2} = (1 - p^{t-k})(2p - \Phi_5), \]
and
\[ M_A = (1 - p^{t-k})^2\Phi_5 + (1 - p^{t-k})(2p - \Phi_5) = 2p(1 - p^{t-k}) - p^{t-k}(1 - p^{t-k})\Phi_5. \]

Write
\[ \Phi_7 = \#\{b_1, b_2 : p^t \mid b_1(b_1 + 1), b_1 - b_2 + n \equiv 0(p^{t-1}), b_1, b_2 = 1, 2, \ldots, p^t\}, \]
\[ \Phi_8 = \#\{b_1, b_2 : p^t \mid b_2(b_2 + 1), b_1 - b_2 + n \equiv 0(p^{t-1}), b_1, b_2 = 1, 2, \ldots, p^t\}, \]
then
\[ \Phi_7 = p(p^t - 2), \quad \Phi_8 = 2p - \Phi_5. \]

By (8.3),
\[ M_B = M_{B1} + M_{B2}, \]
where
\[ M_{B1} = (1 - p^{t-k})\Phi_8, \quad M_{B2} = \Phi_7 - \Phi_8. \]

Hence
\[ M_B = (1 - p^{t-k})\Phi_8 + \Phi_7 - \Phi_8 = p(p^t - 2) - p^{t-k}\Phi_8 = p(p^t - 2) - p^{t-k}(2p - \Phi_5). \]

We obtain
\[ M_B = p(p^t - 2) - 2p^{t-k+1} + p^{t-k}\Phi_5, \]
and
\[ M = 2p(1 - p^{t-k}) - p^{t-k}(1 - p^{t-k})\Phi_5 + p(p^t - 2) - 2p^{t-k+1} + p^{t-k}\Phi_5 \]
\[ = p^{t+1} - 4p^{t-k+1} + p^{2t-2k}\Phi_5, \]
the lemma follows.
LEMMA 8.6 We have

\[
\Phi_2 = \begin{cases} 
1, & \text{if } p^t \mid n(n+1), p^t \nmid n(n-1), \\
1, & \text{if } p^t \nmid n(n+1), p^t \mid n(n-1), \\
2, & \text{if } p^t \mid n(n+1), p^t \nmid n(n-1), \\
0, & \text{if } p^t \nmid n(n+1), p^t \nmid n(n-1). 
\end{cases}
\]

Proof. By (8.4),

\[
\Phi_2 = \#{\{b : p^t \mid b(b+1), p^t \mid (b+n)(b+n+1), b = 1, 2, ..., p^t\}},
\]

the condition \(p^t \mid b(b+1)\) means \(b = p^t, p^t - 1\). If \(p^t \mid n(n+1)\), then \(b = p^t\) satisfies the condition \(p^t \mid (b+n)(b+n+1)\); if \(p^t \mid n(n-1)\), then \(b = p^t - 1\) satisfies the condition \(p^t \mid (b+n)(b+n+1)\); If \(p^t \nmid n(n+1)\) and \(p^t \nmid n(n-1)\), then neither \(b = p^t\) nor \(b = p^t - 1\) satisfies the condition \(p^t \mid (b+n)(b+n+1)\). The lemma follows.

LEMMA 8.7 We have

if \(t = 1\), then \(\Phi_5 = 4\); if \(1 < t \leq k\), then

\[
\Phi_5 = \begin{cases} 
2, & \text{if } p^{t-1} \mid n, \\
2, & \text{if } p^{t-1} \nmid n, p^{t-1} \mid (n+1), p^{t-1} \mid (n-1), \\
1, & \text{if } p^{t-1} \nmid n, p^{t-1} \mid (n+1), p^{t-1} \nmid (n-1), \\
1, & \text{if } p^{t-1} \nmid n, p^{t-1} \nmid (n+1), p^{t-1} \mid (n-1), \\
0, & \text{if } p^{t-1} \nmid n, p^{t-1} \nmid (n+1), p^{t-1} \nmid (n-1). 
\end{cases}
\]

Proof. By (8.5),

\[
\Phi_5 = \#{\{b_1, b_2 : p^t \mid b_1(b_1 + 1), p^t \mid b_2(b_2 + 1), b_1 - b_2 + n \equiv 0(p^{t-1}), b_1, b_2 = 1, 2, ..., p^t\}},
\]

we need to count the number of \(b_1, b_2\) that satisfy the conditions \((\ast)\): \(p^t \mid b_1(b_1 + 1), p^t \mid b_2(b_2 + 1), b_1 - b_2 + n \equiv 0(p^{t-1})\).

We have \(b_1, b_2 = p^t, p^t - 1\).

If \(t = 1\), then \(\Phi_5 = 4\) is valid. We suppose that \(t > 1\).

If \(p^{t-1} \mid n\), then \(b_1 = b_2 = p^t\) and \(b_1 = b_2 = p^t - 1\) satisfy the conditions \((\ast)\);

If \(p^{t-1} \nmid n, p^{t-1} \mid (n+1)\), then \(b_1 = p^t, b_2 = p^t - 1\) and \(b_1 = p^t - 1, b_2 = p^t\) satisfy the conditions \((\ast)\);

If \(p^{t-1} \nmid n, p^{t-1} \nmid (n+1)\), then only \(b_1 = p^t, b_2 = p^t - 1\) satisfies the conditions \((\ast)\); if \(p^{t-1} \nmid n, p^{t-1} \nmid (n+1)\) then only \(b_1 = p^t - 1, b_2 = p^t\);
satisfies the conditions (*);
if \(p_{t-1} \nmid n, p_{t-1} \nmid (n + 1), p_{t-1} \nmid (n - 1)\) then there are not \(b_1, b_2\) satisfy the conditions (*).

The lemma follows.

**LEMMA 8.8** Suppose that \(1 \leq t \leq k\). Then

\[
J(p^t) = p^{3t-2k-1}(p\Phi_2 - \Phi_5).
\]

**Proof.** By (8.2), Lemma 8.3 and Lemma 8.4,

\[
J(p) = J_1(p) - J_2(p) = p^2 - 4p^{2-k} + p^{3-2k}\Phi_2 - p^2(1 - 2p^{-k})^2 = p^2 - 2k(p\Phi_2 - 4).
\]

We suppose \(1 < t \leq k\). By (8.2), Lemma 8.3 and Lemma 8.5,

\[
J(p^t) = J_1(p^t) - J_2(p^t) = p^{2t} - 4p^{2t-k} + p^{3t-2k}\Phi_2 - p^{t-1}(p^{t+1} - 4p^{t-k} + p^{2t-2k}\Phi_5)
\]

\[= p^{3t-2k-1}(p\Phi_2 - \Phi_5),\]

the lemma follows.

By Lemma 8.1, Lemma 8.2 and Lemma 8.8,

\[
S(n) = \varphi^2 \prod_p \left(1 + \sum_{t=1}^{k} p^{-2t} f(p^t) J(p^t) \right) = \varphi^2 \prod_p \left(1 + \sum_{t=1}^{k} p^{-2t} \left(\frac{p^k}{p^t-2}\right)^2 p^{3t-2k-1}(p\Phi_2 - \Phi_5) \right)
\]

\[= \varphi^2 \prod_p \left(1 + (p^k - 2)^{-2} \sum_{t=1}^{k} p^{-t-1}(p\Phi_2 - \Phi_5) \right),\]

that is

\[
S(n) = \varphi^2 \prod_p \Delta_p, \quad \text{(8.6)}
\]

where

\[
\Delta_p = 1 + (p^k - 2)^{-2} \sum_{t=1}^{k} p^{-t-1}(p\Phi_2 - \Phi_5). \quad \text{(8.7)}
\]

Write

\[
\Pi_1 = \prod_{p|n(n+1)} \Delta_p, \quad \Pi_{1A} = \prod_{p|n(n+1)} \Delta_p, \quad \Pi_{1B} = \prod_{p|n(n+1)} \Delta_p, \quad \Pi_1 = \Pi_{1A}\Pi_{1B}, \quad \text{(8.8)}
\]

\[
\Pi_2 = \prod_{p|n(n+1)} \Delta_p, \quad \Pi_{2A} = \prod_{p|n(n+1)} \Delta_p, \quad \Pi_{2B} = \prod_{p|n(n+1)} \Delta_p, \quad \Pi_2 = \Pi_{2A}\Pi_{2B}, \quad \text{(8.9)}
\]
then
\[ S(n) = \varrho^2 \Pi_{1A} \Pi_{1B} \Pi_{2A} \Pi_{2B}. \]  

(8.10)

For fixed \( n, p \), we write
\[ p^{\beta_1} \mid n, \ p^{\beta_2} \mid n + 1, \ p^{\beta_3} \mid n - 1. \]

We consider \( \Pi_{1A} \) first.

**LEMMA 8.9** We have
\[ \Pi_{1A} = \prod_{p \mid n} \Delta_p \prod_{p \mid n} \Delta_p \prod_{2 \mid n} \Delta_2. \]  

(8.11)

Proof. We have
\[ \Pi_{1A} = \prod_{p \mid n} \Delta_p \prod_{p \mid n} \Delta_p \prod_{2 \mid n} \Delta_2. \]

If \( p \mid n(n+1) \), \( p \mid n(n-1) \) and \( p \nmid n \), then \( p = 2, \beta_1 = 0, \ \min(\beta_2, \beta_3) = 1. \)

Hence, by Lemma 8.6 and Lemma 8.7,
\[ \Phi_2(2) = 2, \ \Phi_5(2) = 4, \]  
and then \( 2\Phi_2(2) - \Phi_5(2) = 0. \)

When \( 1 < t \leq k, 2 \nmid n, \)
\[ \Phi_2(2^t) = \begin{cases} 1, & \text{if } t \leq \max(\beta_2, \beta_3), \\ 0, & \text{if } t > \max(\beta_2, \beta_3). \end{cases} \]

\[ \Phi_5(2^t) = \begin{cases} 2, & \text{if } t = 2, \\ 1, & \text{if } 2 < t \leq \max(\beta_2, \beta_3) + 1, \\ 0, & \text{if } t > \max(\beta_2, \beta_3) + 1. \end{cases} \]

Write \( \beta = \max(\beta_2, \beta_3) \geq 2. \) We have
\[ 2\Phi_2(2^t) - \Phi_5(2^t) = \begin{cases} 0, & \text{if } t = 2, \\ 1, & \text{if } 2 < t \leq \beta, \\ -1, & \text{if } t = 1 + \beta, \\ 0, & \text{if } 1 + \beta < t \leq k. \end{cases} \]
Hence
\[ \Delta_2 = \begin{cases} 
1 + (2^k - 2)^{-2} \left( \sum_{3 \leq t \leq \beta} 2^{t-1} - 2^\beta \right), & \text{if } \beta < k, \\
1 + (2^k - 2)^{-2} \sum_{3 \leq t \leq k} 2^{t-1}, & \text{if } \beta \geq k.
\end{cases} \]

Then
\[ \Delta_2 = \begin{cases} 
1 - 4(2^k - 2)^{-2}, & \text{if } \beta < k, \\
1 + (2^k - 4)(2^k - 2)^{-2}, & \text{if } \beta \geq k.
\end{cases} \] (8.12)

We suppose that \( p \mid n(n+1) \), \( p \mid n(n-1) \) and \( p \mid n \), then \( \beta_2 = \beta_3 = 0 \).

By Lemma 8.6 and Lemma 8.7,
\[ p\Phi_2(p) - \Phi_5(p) = 2p - 4, \]
\[ \Phi_2(p^t) = \begin{cases} 
2, & \text{if } p^t \mid n, \\
0, & \text{if } p^t \nmid n,
\end{cases} \]
and for \( 1 < t \leq k \),
\[ \Phi_5(p^t) = \begin{cases} 
2, & \text{if } p^{t-1} \mid n, \\
0, & \text{if } p^{t-1} \nmid n.
\end{cases} \]

Hence
\[ p\Phi_2(p^t) - \Phi_5(p^t) = \begin{cases} 
2p - 2, & \text{if } t \leq \beta_1 \\
-2, & \text{if } t = 1 + \beta_1, \\
0, & \text{if } 1 + \beta_1 < t \leq k.
\end{cases} \]

Then
\[ \Delta_p = \begin{cases} 
1 + (p^k - 2)^{-2}(2p - 4) + (p^k - 2)^{-2} \left( \sum_{2 \leq t \leq \beta_1} p^{t-1} (2p - 2) - 2p^\beta_1 \right), & \text{if } \beta_1 < k, \\
1 + (p^k - 2)^{-2}(2p - 4) + (p^k - 2)^{-2} \left( \sum_{2 \leq t \leq k} p^{t-1} (2p - 2) \right), & \text{if } \beta_1 \geq k.
\end{cases} \]

Consequently
\[ \Delta_p = \begin{cases} 
1 - 4(p^k - 2)^{-2}, & \text{if } 1 \leq \beta_1 < k, \\
1 + 2(p^k - 2)^{-1}, & \text{if } \beta_1 \geq k.
\end{cases} \] (8.13)

By (8.11), (8.12) and (8.13), the lemma follows.

Secondly, we consider \( \Pi_{1B} \).

**Lemma 8.10** We have
\[ \Pi_{1B} = \prod_{p^k | n+1, p \nmid n-1} (1 - 4(p^k - 2)^{-2}) \prod_{p^k | n+1, p \nmid n-1} (1 + (p^k - 4)(p^k - 2)^{-2}). \]

Proof. We have \( p \mid n(n+1), p \nmid n(n-1), \beta_1 = \beta_3 = 0, \beta_2 \geq 1, \) and by Lemma 8.6 and Lemma 8.7,

\[ p\Phi_2(p) - \Phi_5(p) = p - 4, \]

for \( 1 < t \leq k, \)

\[ \Phi_2(p^t) = \begin{cases} 1, & \text{if } p^t \mid n+1, \\ 0, & \text{if } p^t \nmid n+1, \end{cases} \]

\[ \Phi_5(p^t) = \begin{cases} 1, & \text{if } p^{t-1} \mid n+1, \\ 0, & \text{if } p^{t-1} \nmid n+1, \end{cases} \]

then

\[ p\Phi_2(p^t) - \Phi_5(p^t) = \begin{cases} p - 1, & \text{if } t \leq \beta_2 \\ -1, & \text{if } t = 1 + \beta_2, \\ 0, & \text{if } 1 + \beta_2 < t \leq k, \end{cases} \]

and

\[ \Delta_p = \begin{cases} 1 + (p^k - 2)^{-2}(p - 4) + (p^k - 2)^{-2} \left( \sum_{2 \leq t \leq \beta_2} p^{t-1}(p - 1) - p^{\beta_2} \right), & \text{if } \beta_2 < k, \\ 1 + (p^k - 2)^{-2}(p - 4) + (p^k - 2)^{-2} \left( \sum_{2 \leq t \leq \beta_2} p^{t-1}(p - 1) \right), & \text{if } \beta_2 \geq k. \end{cases} \]

Consequently

\[ \Delta_p = \begin{cases} 1 - 4(p^k - 2)^{-2}, & \text{if } 1 \leq \beta_2 < k, \\ 1 + (p^k - 4)(p^k - 2)^{-2}, & \text{if } \beta_2 \geq k. \end{cases} \quad (8.14) \]

By (8.8) and (8.14), the lemma follows.

Now, we consider \( \Pi_2. \) By \( p \nmid n(n+1), \) Lemma 8.6 and Lemma 8.7,

\[ \Phi_2(p^t) = \begin{cases} 1, & \text{if } p^t \mid n-1, \\ 0, & \text{if } p^t \nmid n-1, \end{cases} \quad (8.15) \]

and for \( 1 < t \leq k \)

\[ \Phi_5(p^t) = \begin{cases} 1, & \text{if } p^{t-1} \mid n-1, \\ 0, & \text{if } p^{t-1} \nmid n-1. \end{cases} \quad (8.16) \]
LEMMA 8.11 We have

\[ \Pi_{2A} = \prod_{p \mid n(n+1)} (1 - 4p^3)^{-2} \prod_{p \mid n+1} (1 + p^3)^{-2}. \]

Proof. We have \( p \parallel n(n+1), p \mid (n-1), \beta_1 = \beta_2 = 0, \beta_3 \geq 1, \) and by Lemma 8.6, Lemma 8.7, (8.15) and (8.16),

\[ p\Phi_2(p) - \Phi_5(p) = p - 4, \]

and for \( 1 < t \leq k, \) by (8.15) and (8.16),

\[ p\Phi_2(p^t) - \Phi_5(p^t) = \begin{cases} 
  p - 1, & \text{if } t \leq \beta_3 \\
  -1, & \text{if } t = 1 + \beta_3, \\
  0, & \text{if } 1 + \beta_3 < t \leq k,
\end{cases} \]

and

\[ \Delta_p = \begin{cases} 
  1 + (p^3 - 2)^{-2}(p - 4) + (p^3 - 2)^{-2} \left( \sum_{2 \leq t \leq \beta_3} p^{t-1}(p - 1) - p^{\beta_3} \right), & \text{if } \beta_3 < k, \\
  1 + (p^3 - 2)^{-2}(p - 4) + (p^3 - 2)^{-2} \left( \sum_{2 \leq t \leq k} p^{t-1}(p - 1) \right), & \text{if } \beta_3 \geq k.
\end{cases} \]

Consequently

\[ \Delta_p = \begin{cases} 
  1 - 4(p^3 - 2)^{-2}, & \text{if } 1 \leq \beta_3 < k, \\
  1 + (p^3 - 4)(p^3 - 2)^{-2}, & \text{if } \beta_3 \geq k.
\end{cases} \] (8.17)

By (8.9) and (8.17), the lemma follows.

LEMMA 8.12 We have

\[ \Pi_{2B} = \prod_{p \mid n(n+1)(n-1)} (1 - 4p^3)^{-2}. \]

Proof. We have \( p \parallel n(n+1)(n-1), \beta_1 = \beta_2 = \beta_3 = 0, \) and by Lemma 8.6, Lemma 8.7, (8.15) and (8.16),

\[ p\Phi_2(p) - \Phi_5(p) = -4, \]

and for \( 1 < t \leq k, \)

\( \Phi_2(p^t) = 0, \Phi_5(p^t) = 0, \)

and

\[ p\Phi_2(p^t) - \Phi_5(p^t) = 0, \text{ and } \Delta_p = 1 - 4(p^3 - 2)^{-2}, \]

the lemma follows.

LEMMA 8.13 We have
\[ \mathcal{S}(n) = q^2 \prod_{p \nmid n} (1 - 4(p^k - 2)^{-2}) \prod_{p | n} (1 + 2(p^k - 2)^{-1}) \prod_{p | n} (1 + (2^k - 4)(2^k - 2)^{-2}) \times \]
\[ \prod_{p | n+1}^n (1 - 4(2^k - 2)^{-2}) \prod_{p | n+1}^n (1 - 4(p^k - 2)^{-2}) \prod_{p | n+1}^n (1 + (p^k - 4)(p^k - 2)^{-2}) \times \]
\[ \prod_{p | n(n+1)}^n (1 - 4(p^k - 2)^{-2}) \prod_{p | n(n+1)}^n (1 + (p^k - 4)(p^k - 2)^{-2}) \prod_{p | n(n+1)}^n (1 - 4(p^k - 2)^{-2}). \]

Proof. By Lemma 8.9, Lemma 8.10, Lemma 8.11, Lemma 8.12 and (8.10), the lemma follows.

**Lemma 8.14** We have
\[ \mathcal{S}(n) = q^2 \prod_{p \neq 2} (1 - 4(p^k - 2)^{-2}) \prod_{p | n}^n (1 + (2^k - 4)(2^k - 2)^{-2}) \prod_{p | n}^n (1 - 4(2^k - 2)^{-2}) \times \]
\[ \prod_{p | 2n}^n (1 - 4(2^k - 2)^{-2}) \prod_{p | 2n}^n (1 + 2(2^k - 2)^{-1}) \prod_{p | n+1}^n \frac{p^k - 2}{p^k - 4} \prod_{p | n+1}^n \frac{p^k - 3}{p^k - 4}, \]
\[ \prod_{p | n(n+1)(n-1)}^n (1 - 4(p^k - 2)^{-2}) = \prod_{p \neq 2} (1 - 4(p^k - 2)^{-2}) \prod_{p | n(n+1)(n-1)}^n (1 - 4(p^k - 2)^{-2})^{-1}, \]
the lemma follows at once.

### 9 The function \( G(u; s) \)

By (7.4),
\[ S_E(x, Q) = \frac{1}{2\pi i} \int_{(2)} \sum_{u \leq Q} \sum_{v=1}^\infty \mathcal{S}(uv)(uv)^{-s} \frac{x^{s+1}}{s(s+1)} ds, \]  
(9.1)

where \( \int_{(2)} \) means \( \int_{2-i\infty}^{2+i\infty} \).

Write
\[ G(u; s) = \sum_{v=1}^\infty \mathcal{S}(uv)u^{-s}, \quad \Re s > 1, \]
then
\[ S_E(x, Q) = \sum_{u \leq Q} \frac{1}{2\pi i} \int_{(2)} u^{-s}G(u; s) \frac{x^{s+1}}{s(s+1)} ds. \]  
(9.2)

Let
\[ G_1(u, s) = \sum_{v=1}^\infty \mathcal{S}(uv)u^{-s}, \quad G_2(u, s) = \sum_{v=1}^\infty \mathcal{S}(uv)u^{-s}, \]  
\[ 2^k | uv \]
\[ G_3(u, s) = \sum_{2|uv, s=1}^{\infty} G(u, s) = \sum_{2|uv, s=1}^{\infty} G(u, s), \]
\[ G_4(u, s) = \sum_{2|uv, s=1}^{\infty} G(u, s), \]
then
\[ G(u; s) = G_1(u, s) + G_2(u, s) + G_3(u, s) + G_4(u, s). \] (9.3)

As in [3], for integer \( a \), let
\[ \xi_k = e^{2\pi i/k}, \ 0 < a \leq 1, \ \zeta(s;\xi_k) = \sum_{m=1}^{\infty} \xi_k^m \zeta(s;\xi_k), \]
and for \( 0 < a \leq 1 \), let \( \zeta(s, a) = \sum_{n=0}^{\infty} (a + n)^{-s}, \Re s > 1 \).

When \( 1 \leq a \leq k \), we have \( \zeta(s, a) = k^{-s} \zeta(s, k) \).

For odd integer, we define the multiplicative functions \( g_1(d), g_2(d) \) as follows
\[ g_1(p^t) = \frac{2}{p^k - 4}, \ g_2(p^t) = \frac{1}{p^k - 4}, \ t \geq 1, p > 2, \] (9.4)
then
\[ \mu(d)^2 g_1(d) \ll \tau(d) d^{-k}, \ \mu(d)^2 g_2(d) \ll d^{-k}. \] (9.5)

For fixed \( u, d_1, d_2, d_3 \) and \( \Re s > 1 \), write
\[ H_1(s) = H_1(s; u, d_1, d_2, d_3) = \sum_{d_1|u, d_2|u, d_3|u} v^{-s}, \]
\[ H_2(s) = H_2(s; u, d_1, d_2, d_3) = \sum_{d_1|u, d_2|u+1, d_3|u-1} v^{-s}, \]
\[ H_3(s) = H_3(s; u, d_1, d_2, d_3) = \sum_{d_1|u, d_2|u+1, d_3|u-1} v^{-s}, \]
\[ H_4(s) = H_4(s; u, d_1, d_2, d_3) = \sum_{d_1|u+1, d_2|u, d_3|u} v^{-s}, \]
\[ H_5(s) = H_5(s; u, d_1, d_2, d_3) = \sum_{d_1|u-1, d_2|u, d_3|u} v^{-s}. \]

By Lemma 8.14 and (9.4),
\[ \mathcal{S}(n) = g^2 \prod_{p \neq 2} (1 - 4(pk - 2)^{-2}) \prod_{2^k | n-1} (1 + (2^k - 4)(2^k - 2)^{-2}) \times \]
\[ \prod_{2^k | n+1} (1 - 4(2^k - 2)^{-2}) \prod_{2^k | n} (1 - 4(2^k - 2)^{-2}) \prod_{2^k | n} (1 + 2(2^k - 2)^{-1}) \times \]
\[ \prod_{2^k | n-1, 2^k | n-2} (1 + 2(2^k - 2)^{-1}) \times \prod_{2^k | n-1, 2^k | n-2} (1 + 2(2^k - 2)^{-1}) \times \]
\[ \prod_{2^k | n-1, 2^k | n-2} (1 + 2(2^k - 2)^{-1}) \times \prod_{2^k | n-1, 2^k | n-2} (1 + 2(2^k - 2)^{-1}) \times \]
\[ \prod_{2^k | n-1, 2^k | n-2} (1 + 2(2^k - 2)^{-1}) \times \prod_{2^k | n-1, 2^k | n-2} (1 + 2(2^k - 2)^{-1}) \times \]
We only consider the case:

Proof. By (9.3) and (9.7), the lemma follows.

If we write

\[ h(d_1, d_2, d_3) = \mu(d_1 d_2 d_3)^2 g_1(d_1) g_2(d_2) g_2(d_3), \]

then, by (9.5),

\[ h(d_1, d_2, d_3) \ll \tau(d_1)d_1^{-k}d_2^{-k}d_3^{-k}, \tag{9.6} \]

and

\[
\mathcal{S}(n) = \sigma^2 \prod_{p \neq 2} (1 - 4(p^k - 2)^{-2}) \cdot \prod_{2^k | n+1 \text{ or } 2^k | n-1} (1 + (2^k - 4)(2^k - 2)^{-2}) \times \prod_{2^k | n-1, 2^k | n} (1 - 4(2^k - 2)^{-2}) \cdot \prod_{2^k | n} (1 - 4(2^k - 2)^{-2}) \cdot \prod_{2^k | n} (1 + 2(2^k - 2)^{-1}) \times \sum_{d_1,d_2,d_3} h(d_1, d_2, d_3). \tag{9.7} \]

**Lemma 9.1** For \(\Re s > 1\), we have

\[ G_1(u, s) = \sigma^2 \prod_{p \neq 2} (1 - 4(p^k - 2)^{-2})(1 + 2(2^k - 2)^{-2}) \cdot \sum_{(d_1,d_2,d_3,2)=1} h(d_1, d_2, d_3) H_1(s), \]

\[ G_2(u, s) = \sigma^2 \prod_{p \neq 2} (1 - 4(p^k - 2)^{-2})(1 - 4(2^k - 2)^{-2}) \cdot \sum_{(d_1,d_2,d_3,2)=1} h(d_1, d_2, d_3) (H_2(s) - H_1(s)), \]

\[ G_3(u, s) = \sigma^2 \prod_{p \neq 2} (1 - 4(p^k - 2)^{-2})(1 - 4(k^k - 2)^{-2}) \cdot \sum_{(d_1,d_2,d_3,2)=1} h(d_1, d_2, d_3) (H_3(s) - H_2(s) - H_4(s) - H_5(s)), \]

\[ G_4(u, s) = \sigma^2 \prod_{p \neq 2} (1 - 4(p^k - 2)^{-2})(1 + (2^k - 4)(2^k - 2)^{-2}) \cdot \sum_{(d_1,d_2,d_3,2)=1} h(d_1, d_2, d_3) (H_4(s) + H_5(s)), \]

\[ G(u; s) = \sigma^2 \prod_{p \neq 2} (1 - 4(p^k - 2)^{-2}) \cdot \sum_{(d_1,d_2,d_3,2)=1} h(d_1, d_2, d_3) (2^k + 1)(2^k - 2)^{-2} H_1(s) + (1 - 4(2^k - 2)^{-2}) H_3(s) + 2^k (2^k - 2)^{-2} H_4(s) + 2^k (2^k - 2)^{-2} H_5(s). \]

Proof. By (9.3) and (9.7), the lemma follows.

We only consider the case: \(\mu(d_1 d_2 d_3)^2 = 1\), \((d_2 d_3, u) = 1\). Write

\[
\frac{2^k d_1^k u}{(u, 2^k d_1^k)} b_1 \equiv 1 (mod \ d_2^k), \quad \frac{2^k d_1^k u}{(u, 2^k d_1^k)} b_2 \equiv 1 (mod \ d_3^k), \tag{9.8}
\]
\[
\begin{aligned}
\begin{cases}
d_k^3d_3^2 \equiv 1 \pmod{d_3^k}, \\
d_2^3d_2^2 \equiv 1 \pmod{d_3^k},
\end{cases}
\end{aligned}
\]  
(9.9)

\[
A_1 \equiv -b_1d_3^kd_3^2 + b_2d_2^kd_2^2 \pmod{d_2^kd_3^k}, \ 1 \leq A_1 \leq d_2^kd_3^k, (A_1, d_2d_3) = 1,
\]  
(9.10)

\[
\frac{d_1^ku}{(u, d_1^k)} b_5 \equiv 1 \pmod{d_2^k}, \quad \frac{d_1^ku}{(u, d_1^k)} b_6 \equiv 1 \pmod{d_3^k},
\]  
(9.11)

\[
A_3 \equiv -b_5d_3^kd_3^2 + b_6d_2^kd_2^2 \pmod{d_2^kd_3^k}, \ 1 \leq A_3 \leq d_2^kd_3^k, (A_3, d_2d_3) = 1,
\]  
(9.12)

\[
\frac{d_1^ku}{(u, d_1^k)} b_7 \equiv 1 \pmod{2^kd_2^k}, \quad \frac{d_1^ku}{(u, d_1^k)} b_8 \equiv 1 \pmod{d_3^k},
\]  
(9.13)

\[
\begin{aligned}
\begin{cases}
d_3^k(d_3^k) \equiv 1 \pmod{2^kd_2^k}, \\
2^kd_2^k(2^kd_3^k) \equiv 1 \pmod{d_3^k},
\end{cases}
\end{aligned}
\]  
(9.14)

\[
A_4 \equiv -b_7d_3^kd_3^2 + b_82^kd_2^k(2^kd_3^k) \pmod{2^kd_2^kd_3^k}, \ 1 \leq A_4 \leq 2^kd_2^kd_3^k, (A_4, 2d_2d_3) = 1,
\]  
(9.15)

\[
\frac{d_1^ku}{(u, d_1^k)} b_9 \equiv 1 \pmod{d_2^k}, \quad \frac{d_1^ku}{(u, d_1^k)} b_{10} \equiv 1 \pmod{2^kd_3^k},
\]  
(9.16)

\[
\begin{aligned}
\begin{cases}
2^kd_2^k(2^kd_3^k) \equiv 1 \pmod{d_2^k}, \\
d_3^kd_3^2 \equiv 1 \pmod{2^kd_3^k},
\end{cases}
\end{aligned}
\]  
(9.17)

\[
A_5 \equiv -b_92^kd_3^kd_3^2 + b_{10}d_3^kd_2^k \pmod{2^kd_2^kd_3^k}, \ 1 \leq A_5 \leq 2^kd_2^kd_3^k, (A_5, 2d_2d_3) = 1.
\]  
(9.18)

**Lemma 9.2** For $\Re s > 1$, we have

\[
H_1(s) = \frac{(u, 2kd_2^k)}{2^{kd_2^k}d_1^k} \zeta(s; A_1, d_2^kd_3^k), \quad H_3(s) = \frac{(u, d_3^k)}{d_1^k} \zeta(s; A_3, d_2^kd_3^k),
\]
Let use the following results of [3]. 

Proof. We have

\[ 2^k \mid uv, d_1^k \mid uv, d_2^k \mid uv + 1, d_3^k \mid uv - 1. \]

Write \( a = \frac{u^k d_1^k}{v}, v = av_1, \) then

\[ uav_1 \equiv -1(mod \ d_2^k), \ uav_1 \equiv 1 (mod \ d_3^k). \]

By (9.8) and (9.9),

\[
\begin{align*}
\left\{ \begin{array}{l}
v_1 \equiv -b_1 (mod \ d_2^k), \\
v_1 \equiv b_2 (mod \ d_3^k),
\end{array} \right.
\end{align*}
\]

hence, by (9.10), we obtain \( v_1 \equiv A_1 (mod \ d_2^k d_3^k) \) and

\[ H_1(s) = a^{-s} \sum_{v_1 \equiv A_1 (d_2^k d_3^k)} v_1^{-s} = a^{-s} \zeta(s; A_1, d_2^k d_3^k). \]

Similarly, the formulas for \( H_5(s), H_4(s) \) and \( H_5(s) \) follow from (9.11),(9.12),(9.13),(9.14),(9.15),(9.16),(9.17) and (9.18), and the lemma follows.

We use the following results of [3].

Let \( G(s) = -i(2\pi)^{s-1} \Gamma(1-s) \), then

\[
\zeta(s; a, k) = G(s)k^{-s}(e^{\frac{1}{2}i\pi s}(1 - s; \xi_k^a) - e^{-\frac{1}{2}i\pi s}(1 - s; \xi_{k}^{-a})).
\]

(9.19)

By Lemma 9.1 and Lemma 9.2(\( R_s > 1 \)),

\[
G(u; s) = g^2 \prod_{p \neq 2} \left( 1 - 4(p^k - 2)^{-2} \right) \sum_{(d_2 d_3, u) = 1 \atop (d_1 d_2 d_3, 2) = 1} h(d_1, d_2, d_3) \times
\]

\[
\left\{ 2^{k+1}(2^k - 2)^{-2}(u_2 d_2^k d_3^k)^s \zeta(s; A_1, d_2^k d_3^k) + (1 - 4(2^k - 2)^{-2}(u d_2^k d_3^k)^s \zeta(s; A_3, d_2^k d_3^k)
\right.
\]

\[
+ 2^k(2^k - 2)^{-2}(u d_2^k d_3^k)^s \zeta(s; A_3, 2^k d_2 d_3^k) + 2^k(2^k - 2)^{-2}(u d_2^k d_3^k)^s \zeta(s; A_5, 2^k d_2 d_3^k) \right\}. \]

(9.20)

We take (9.20) as the analytic continuation of \( G(u; s), R_s > \frac{1}{k} - 1 \), and \( G(u; s) \) has at most one simple pole at \( s = 1 \).

Write

\[
G_1(u, s) = \sum_{(d_2 d_3, u) = 1 \atop (d_1 d_2 d_3, 2) = 1} h(d_1, d_2, d_3) \frac{(u_2 d_2^k d_3^k)^s}{2^k d_2^k d_3^k} \zeta(s; A_1, d_2^k d_3^k),
\]

\[ H_4(s) = \frac{(u, d_2^k)^s}{d_2^k} \zeta(s; A_1, 2^k d_2 d_3^k), \quad H_5(s) = \frac{(u, d_3^k)^s}{d_3^k} \zeta(s; A_5, 2^k d_2 d_3^k). \]
\( G(3)(u, s) = \sum_{(d_2d_3,u)=1} h(d_1, d_2, d_3) \frac{(u,d_1^k)^s}{d_1^s} \zeta(s; A_3, d_2^k d_3^k), \)

\( G(4)(u, s) = \sum_{(d_2d_3,u)=1} h(d_1, d_2, d_3) \frac{(u,d_1^k)^s}{d_1^s} \zeta(s; A_4, 2^k d_2^k d_3^k), \)

\( G(5)(u, s) = \sum_{(d_2d_3,u)=1} h(d_1, d_2, d_3) \frac{(u,d_1^k)^s}{d_1^s} \zeta(s; A_5, 2^k d_2^k d_3^k), \)

then

\[
G(u; s) = \gamma^2 \prod_{p \neq 2} (1 - 4(p^k - 2)^{-2}) 2^{k+1}(2^k - 2)^{-2} G(1)(u, s)
+ \gamma^2 \prod_{p \neq 2} (1 - 4(p^k - 2)^{-2})(1 - 4(2^k - 2)^{-2}) G(3)(u, s)
+ \gamma^2 \prod_{p \neq 2} (1 - 4(p^k - 2)^{-2}) 2^k(2^k - 2)^{-2} G(4)(u, s)
+ \gamma^2 \prod_{p \neq 2} (1 - 4(p^k - 2)^{-2}) 2^k(2^k - 2)^{-2} G(5)(u, s).
\] (9.21)

10 The formula for \( S_E(x, Q) \)

Let \( T > 1 \), by (9.2),

\[
S_E(x, Q) = \sum_{u \leq Q} \frac{1}{2\pi i} \left( \int_{2+i\infty}^{2+iT} + \int_{2-i\infty}^{2-iT} + \int_{2+iT}^{2-iT} \right) u^{-s} G(u; s) \frac{x^{s+1}}{s(s+1)} ds.
\]

We have

\[
\sum_{u \leq Q} \frac{1}{2\pi i} \left( \int_{2+i\infty}^{2+iT} + \int_{2-i\infty}^{2-iT} \right) u^{-s} G(u; s) \frac{x^{s+1}}{s(s+1)} ds \ll \sum_{u \leq Q} u^{-2x^3 T^{-1}} \ll x^{3T-1},
\]

therefore

\[
S_E(x, Q) = \sum_{u \leq Q} \frac{1}{2\pi i} \int_{2-iT}^{2+iT} u^{-s} G(u; s) \frac{x^{s+1}}{s(s+1)} ds + O(x^{3T-1}) = A + O(x^{3T-1}),
\] (10.1)

where

\[
A = \sum_{u \leq Q} \frac{1}{2\pi i} \int_{2-iT}^{2+iT} u^{-s} G(u; s) \frac{x^{s+1}}{s(s+1)} ds.
\]

Let \( \frac{1}{k} - 1 + \tau \leq \sigma < 0 \), by residue theorem

\[
A = \sum_{u \leq Q} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+iT} u^{-s} G(u; s) \frac{x^{s+1}}{s(s+1)} ds + \sum_{u \leq Q} \frac{1}{2\pi i} \int_{2-iT}^{\sigma-iT} u^{-s} G(u; s) \frac{x^{s+1}}{s(s+1)} ds
+ \sum_{u \leq Q} \frac{1}{2\pi i} \int_{\sigma+i\infty}^{2+iT} u^{-s} G(u; s) \frac{x^{s+1}}{s(s+1)} ds + \text{Res}_{\sigma=0,1} \left( \sum_{u \leq Q} u^{-s} G(u; s) \frac{x^{s+1}}{s(s+1)} \right).
\]

**Lemma 10.1** For fixed \( \frac{1}{k} - 1 + \tau \leq \sigma \leq 2 \), \( |t| \geq 1 \) and \( s = \sigma + it \), we have the Stirling formula

\[
\Gamma(s) = \sqrt{2\pi e^{-\frac{1}{2}z}} |t|^{\frac{1}{2}z} e^{\frac{1}{2}z(\log |t| - 1)} e^{\frac{1}{2}z(\log |t|)} (1 + O(|t|^{-1})),
\]
where \( \lambda = 1 \) if \( t > 0 \), and \( \lambda = -1 \) if \( t < 0 \); and

\[
G(u; s) \ll |t| + |t|^{1-\sigma}, \quad \frac{1}{k} - 1 + \tau \leq \Re s \leq 2, \quad |t| \geq 1.
\]

Proof. The first part is the Corollary 3.3.3 of [5], or see 4.42 of [6].

By Theorem 7.1.1 of [5], for \(-1 < \Re s \leq 2, \ y \geq 0\)

\[
\zeta(s, a) = \sum_{0 \leq n \leq y} \frac{1}{(n+a)^s} + \frac{(y+a)^{1-s}}{s-1} + y^{\frac{1-\sigma}{2}} - s \int_{y}^{\infty} \frac{u-\frac{1}{2}}{(u+a)^{s+1}} \, du.
\]

When \( \Re s \geq \frac{1}{2} \), we choose \( y = 1 \), then

\[
\zeta(s, a) = a^{-s} + (1 + a)^{-s} + \frac{(1+a)^{1-s}}{s-1} - \frac{1}{2} (1 + a)^{-s} - s \int_{1}^{\infty} \frac{u-\frac{1}{2}}{(u+a)^{s+1}} \, du,
\]

hence

\[
\zeta(s, a) \ll a^{-\sigma} + |t|, \quad |t| \geq 1.
\]

When \( \frac{1}{k} - 1 < \Re s \leq \frac{1}{2} \), we choose \( y = |t| \geq 1 \), then

\[
\zeta(s, a) \ll a^{-\sigma} + \sum_{1 \leq n \leq |t|} \frac{1}{(n+a)^s} + |t|^{1-\sigma} + s \int_{|t|}^{\infty} \frac{u-\frac{1}{2}}{(u+a)^{s+1}} \, du,
\]

by partial integrations,

\[
|s \int_{|t|}^{\infty} \frac{u-\frac{1}{2}}{(u+a)^{s+1}} \, du| \ll |t|^{1-\sigma} + |t|^2 |t|^{-\sigma-1} \ll |t|^{1-\sigma}.
\]

We have

\[
\zeta(s, a) \ll a^{-\sigma} + |t| + |t|^{1-\sigma}, \quad \frac{1}{k} - 1 + \tau \leq \Re s \leq 2, \quad |t| \geq 1.
\]

By (9.6), (9.21) and \( 1 \leq A_1, A_3, A_4, A_5 \ll d_2 d_3 \),

\[
G(u; s) \ll 1 + |t| + |t|^{1-\sigma}, \quad \frac{1}{k} - 1 + \tau \leq \Re s \leq 2, \quad |t| \geq 1,
\]

the lemma follows.

By Lemma 10.1,

\[
\sum_{u \leq Q} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} u^{-s} G(u; s) \frac{s+1}{\Re(s+1)} \, ds, \quad \sum_{u \leq Q} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} u^{-s} G(u; s) \frac{s+1}{\Re(s+1)} \, ds \ll x^3 Q^{1-\sigma} T^{-\frac{1}{2}}.
\]

therefore

\[
A = \sum_{u \leq Q} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} u^{-s} G(u; s) \frac{s+1}{\Re(s+1)} \, ds
\]

\[
+ \text{Res}_{s=0,1} \left( \sum_{u \leq Q} u^{-s} G(u; s) \frac{s+1}{\Re(s+1)} \right) + O(x^3 Q^{1-\sigma} T^{-\frac{1}{2}}).
\]

(10.2)

Write

\[
B_1 = \sum_{u \leq Q} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} u^{-s} G(1)(u, s) \frac{s+1}{\Re(s+1)} \, ds,
\]

\[
B_2 = \sum_{u \leq Q} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} u^{-s} G(2)(u, s) \frac{s+1}{\Re(s+1)} \, ds,
\]

\[
B_3 = \sum_{u \leq Q} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} u^{-s} G(3)(u, s) \frac{s+1}{\Re(s+1)} \, ds,
\]

\[
B_4 = \sum_{u \leq Q} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} u^{-s} G(4)(u, s) \frac{s+1}{\Re(s+1)} \, ds,
\]

\[
B_5 = \sum_{u \leq Q} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} u^{-s} G(5)(u, s) \frac{s+1}{\Re(s+1)} \, ds,
\]
then, by (9.19),

\[
B_1 = \sum_{u \leq Q} \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} u^{-s} \sum_{(d_1, d_2, d_3) = 1} h(d_1, d_2, d_3) \frac{(u, 2^{k_2}d_1^k)^s}{2^{k_3}d_2^{k_3}} \zeta(s; A_1, d_2 d_3) \frac{x^{s+1}}{s(s+1)} \, ds
\]

\[
= \sum_{u \leq Q} \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} u^{-s} \sum_{(d_1, d_2, d_3) = 1} h(d_1, d_2, d_3) \frac{(u, 2^{k_2}d_1^k)^s}{2^{k_3}d_2^{k_3}} \zeta(s; A_1, d_2 d_3) \frac{x^{s+1}}{s(s+1)} \, ds = B_{11} + B_{12}, \text{ say.}
\]

We have

\[
\int_{\sigma-iT}^{\sigma+iT} u^{-s} \left( \frac{u, 2^{k_2}d_1^k}{2^{k_3}d_2^{k_3}} \right)^s G(s)(d_2 d_3)^{-ks} e^{\frac{1}{2}i\pi s} \zeta(1 - s; \zeta_{d_2 d_3}^{A_1}) \frac{x^{s+1}}{s(s+1)} \, ds
\]

\[
eq e^{\frac{1}{2}i\pi s} \zeta(1 - s; \zeta_{d_2 d_3}^{A_1}) \frac{x^{s+1}}{s(s+1)}
\]

\[
\frac{1}{(\sigma+it)(1+\sigma+it)} \, dt,
\]

and

\[
\int_{-T}^{-T} u^{-it} \left( \frac{u, 2^{k_2}d_1^k}{2^{k_3}d_2^{k_3}} \right)^{it} G(\sigma + it)(d_2 d_3)^{-k_t} e^{-\frac{1}{2}i\pi m_t \, m_{it \sigma}} \frac{x^{it}}{(\sigma+it)(1+\sigma+it)} \, dt
\]

\[
\ll (\int_{T_1}^{T_2} + \int_{-T_2}^{-T_1}) u^{-it} \left( \frac{u, 2^{k_2}d_1^k}{2^{k_3}d_2^{k_3}} \right)^{it} G(\sigma + it)(d_2 d_3)^{-k_t} e^{-\frac{1}{2}i\pi m_t \, m_{it \sigma}} \frac{x^{it}}{(\sigma+it)(1+\sigma+it)} \, dt + 1.
\]

Let \(1 \leq T_1 \leq T_2 \leq 2T_1\). By Lemma 10.1,

\[
\int_{T_1}^{T_2} u^{-it} \left( \frac{u, 2^{k_2}d_1^k}{2^{k_3}d_2^{k_3}} \right)^{it} G(\sigma + it)(d_2 d_3)^{-k_t} e^{-\frac{1}{2}i\pi m_t \, m_{it \sigma}} \frac{x^{it}}{(\sigma+it)(1+\sigma+it)} \, dt
\]

\[
= -i \int_{T_1}^{T_2} u^{-it} \left( \frac{u, 2^{k_2}d_1^k}{2^{k_3}d_2^{k_3}} \right)^{it} \left( 2\pi \right)^{it} e^{-\frac{1}{2}it(1-\sigma-\frac{1}{2})} e^{it(log t-1)} e^{-\frac{it}{2}(1-\sigma-\frac{1}{2})(1+O(t^{-1}))} \times
\]

\[
(d_2 d_3)^{-k_t} e^{-\frac{1}{2}i\pi m_t \, m_{it \sigma}} \frac{x^{it}}{(\sigma+it)(1+\sigma+it)} \, dt \ll T_1^{-\frac{3}{2}+\sigma},
\]

and

\[
\int_{T_1}^{T_2} u^{it} \left( \frac{u, 2^{k_2}d_1^k}{2^{k_3}d_2^{k_3}} \right)^{-it} G(\sigma - it)(d_2 d_3)^{k_t} e^{\frac{1}{2}i\pi m_t \, m_{it \sigma}} \frac{x^{-it}}{(\sigma-it)(1+\sigma-it)} \, dt
\]

\[
= -i(2\pi)^{\frac{1}{2}} e^\frac{i\pi}{4}(1-\sigma) \int_{T_1}^{T_2} u^{it} \left( \frac{u, 2^{k_2}d_1^k}{2^{k_3}d_2^{k_3}} \right)^{-it} \left( 2\pi \right)^{-it} t^{-\frac{1}{2}+\sigma} e^{it(log t-1)} (1+O(t^{-1})) \times
\]

\[
(d_2 d_3)^{k_t} e^{-\frac{1}{2}i\pi m_t \, m_{it \sigma}} \frac{x^{-it}}{(\sigma-it)(1+\sigma-it)} \, dt
\]

\[
= i(2\pi)^{\frac{1}{2}} e^\frac{i\pi}{4}(1-\sigma) \int_{T_1}^{T_2} u^{it} \left( \frac{u, 2^{k_2}d_1^k}{2^{k_3}d_2^{k_3}} \right)^{-it} \left( 2\pi \right)^{-it} t^{-\frac{1}{2}+\sigma} e^{it(log t-1)(d_2 d_3)^{k_t} m_{it \sigma} x^{-it}} dt + O(T_1^{-\frac{3}{2}+\sigma}).
\]

Write

\[
B = \int_{T_1}^{T_2} u^{it} \left( \frac{u, 2^{k_2}d_1^k}{2^{k_3}d_2^{k_3}} \right)^{-it} \left( 2\pi \right)^{-it} t^{-\frac{1}{2}+\sigma} e^{it(log t-1)(d_2 d_3)^{k_t} m_{it \sigma} x^{-it}} dt = \int_{T_1}^{T_2} t^{-\frac{1}{2}+\sigma} e^{itF(t)} \, dt,
\]

where
Theorem 6.4 of [7] and by partial integrations,

\[ \int_{T_1}^{T_2} e^{iF(t)} dt \ll T_1^{\frac{1}{2}} B \ll T_1^{1-\sigma}. \]

Hence

\[ \int_{-\infty}^{\infty} u^{-it} (u, 2^{k} d_1^k)^i G(\sigma + it) (d_2 d_3)^{-kti} e^{-\frac{1}{2} \pi t} m^{it} \times \frac{x^{it}}{(\sigma + it)(1 + \sigma + it)} dt \ll 1, \]

and, by (9.6),

\[ B_{11} \ll \sum_{\sigma \leq Q} x^{\sigma+1} u^{-\sigma} \ll x^{\sigma+1} Q^{1-\sigma}. \]

The same proof works for \( B_{12} \), then

\[ B_{12} \ll x^{\sigma+1} Q^{1-\sigma}, \]

we obtain

\[ \sum_{\sigma \leq Q} \frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} u^{-s} G(u,s) \frac{x^{s+1}}{s(s+1)} ds \ll x^{\sigma+1} Q^{1-\sigma}. \]

Similarly, we deduce

\[ \sum_{\sigma \leq Q} \frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} u^{-s} G(u,s) \frac{x^{s+1}}{s(s+1)} ds \ll x^{\sigma+1} Q^{1-\sigma}, \quad j = 3, 4, 5. \]

By (9.21),

\[ \sum_{\sigma \leq Q} \frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} u^{-s} G(u,s) \frac{x^{s+1}}{s(s+1)} ds \ll x^{\sigma+1} Q^{1-\sigma}. \]  \hspace{1cm} (10.3)

Let \( T \to \infty \), by (10.1), (10.2) and (10.3),

\[ S_E(x, Q) = \text{Res}_{s=0,1} (\sum_{\sigma \leq Q} u^{-s} G(u,s) \frac{x^{s+1}}{s(s+1)}) + O(x^{\sigma+1} Q^{1-\sigma}). \]  \hspace{1cm} (10.4)

We have

\[ \text{Res}_{s=1} (\sum_{\sigma \leq Q} u^{-s} G(u,s) \frac{x^{s+1}}{s(s+1)}) = c_2(Q) x^2, \]  \hspace{1cm} (10.5)
where \( c_2(Q) = \frac{1}{2} \sum_{u \leq Q} u^{-1} \text{Res}_{s=1} G(u; s) \), \( c_2(Q) \) is independence of \( x \).

We need the value \( \zeta(0; a, k) = \frac{1}{2} - \frac{a}{k}, \ 1 \leq a \leq k \).

\[
\text{Res}_{s=0}(\sum_{u \leq Q} u^{-s} G(u; s) \frac{x^{s+1}}{s(s+1)}) = x \sum_{u \leq Q} G(u; 0). \tag{10.6}
\]

**LEMMA 10.2** We have

\[
\sum_{u \leq Q} G_{(1)}(u, 0) = c_{31} Q + O(Q^{\frac{1}{2} + \varepsilon}), \quad \sum_{u \leq Q} G_{(3)}(u, 0) = c_{33} Q + O(Q^{\frac{1}{2} + \varepsilon}),
\]

\[
\sum_{u \leq Q} G_{(4)}(u, 0) = c_{34} Q + O(Q^{\frac{1}{2} + \varepsilon}), \quad \sum_{u \leq Q} G_{(5)}(u, 0) = c_{35} Q + O(Q^{\frac{1}{2} + \varepsilon}),
\]

where \( c_{31}, c_{33}, c_{34}, c_{35} \) are independence of \( Q \).

Proof. We only consider the case \((d_i, d_i) = 1, i \neq l\).

\[
\sum_{u \leq Q} G_{(1)}(u, 0) = \sum_{u \leq Q} \sum_{(d_1 d_2 d_3, u) = 1} h(d_1, d_2, d_3) \zeta(0; A_1, d_2^d d_3^k)
\]

\[
= \sum_{u \leq Q} \sum_{(d_1 d_2 d_3, u) = 1} h(d_1, d_2, d_3)\left(\frac{1}{2} - A_1 d_2^d d_3^k\right). \tag{10.7}
\]

We have

\[
\sum_{u \leq Q} 1 = Q \sum_{t \mid m} \mu(t) t^{-1} + O(\tau(m)),
\]

then

\[
\sum_{u \leq Q} h(d_1, d_2, d_3)\left(\frac{1}{2} - \sum_{(d_1 d_2 d_3, u) = 1} h(d_1, d_2, d_3) \sum_{u \leq Q} 1\right)
\]

\[
= \frac{1}{2} \sum_{(d_1 d_2 d_3, u) = 1} h(d_1, d_2, d_3)(Q \sum_{t \mid d_3} \mu(t) t^{-1} + O(\tau(d_2 d_3))),
\]

therefore, by (9.6),

\[
\sum_{u \leq Q} \sum_{(d_1 d_2 d_3, u) = 1} h(d_1, d_2, d_3)\left(\frac{1}{2} - \sum_{(d_1 d_2 d_3, u) = 1} \sum_{t \mid d_2 d_3} \mu(t) t^{-1} + O(1). \tag{10.8}
\]

Write

\[
C = \sum_{u \leq Q} A_1 = \sum_{u \leq Q} \{ -b_1 d_3^k d_3^k + b_2 d_2^k d_2^k \}
\]

where \( A_1 = \{ -b_1 d_3^k d_3^k + b_2 d_2^k d_2^k \} \) means \( 1 \leq A_1 \leq d_2^k d_3^k, (A_1, d_2 d_3) = 1 \).

Let \((u, 2^k d_1^k) = j, u = ju_1, \) then, by (9.8),(9.9) and (9.10),

\[
2^k d_1^k u_1 b_1 \equiv 1(\text{mod} \ d_3^k), \ 2^k d_1^k u_1 b_2 \equiv 1(\text{mod} \ d_3^k),
\]

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\[
C = \sum_{j_1=1}^{d_2^k} \sum_{j_2=1}^{d_3^k} \sum_{j=1}^{d_1^k} \{ -j_1 d_3^k \overline{d_3^k} + j_2 d_2^k \overline{d_2^k} \} \sum_{j|2^k d_1^k} D,
\]
where

\[
D = \# \{ u_1 : u_1 \leq Q_j^{-1}, (u_1, 2^k d_1^k j^{-1}) = 1, 2^k d_1^k u_1 j_1 \equiv 1 (\text{mod } d_2^k), 2^k d_1^k u_1 j_2 \equiv 1 (\text{mod } d_3^k) \}.
\]

\[
D = \sum_{t|2^k d_1^k j^{-1}} \mu(t) \sum_{u \leq Q_j^{-1} t^{-1}} \sum_{u \equiv Q_j^{-1} t^{-1} (\text{mod } d_2^k), u \equiv J (\text{mod } d_2^k d_3^k)} 1 = \sum_{t|2^k d_1^k j^{-1}} \mu(t) \sum_{u \leq Q_j^{-1} t^{-1}} \sum_{u \equiv Q_j^{-1} t^{-1} (\text{mod } d_2^k d_3^k)} 1,
\]
where

\[
J \equiv 2^k d_1^k t_1 j_1 d_3^k d_3^k + 2^k d_1^k t_2 d_2^k d_2^k (\text{mod } d_2^k d_3^k), \text{ and } 1 \leq J \leq d_2^k d_3^k.
\] (10.9)

We have

\[
\sum_{u \leq Q_j^{-1} t^{-1}} \sum_{u \equiv J (\text{mod } d_2^k d_3^k)} 1 = \begin{cases} Q_j^{-1} t^{-1} d_2^k d_3^k - O((Q_j^{-1} t^{-1} d_3^k d_3^k)^{1+\varepsilon}), & \text{if } d_2^k d_3^k \leq Q_j^{-1} t^{-1}, \\ 1, & \text{if } d_3^k d_3^k > Q_j^{-1} t^{-1} \geq J, \\ 0, & \text{if } d_2^k d_3^k \geq J > Q_j^{-1} t^{-1}. \end{cases}
\]

Therefore

\[
\sum_{u \leq Q_j^{-1} t^{-1}} \sum_{u \equiv J (\text{mod } d_2^k d_3^k)} h(d_1, d_2, d_3) A_{2^{-k} d_2^k} d_3^k
\]

\[
= \sum_{(d_1, d_2, d_3)=1} h(d_1, d_2, d_3) d_2^k d_3^k \sum_{j=1}^{d_1^k} \sum_{j_2=1}^{d_2^k} \sum_{j=1}^{d_3^k} \sum_{j_2=1}^{d_2^k} \sum_{j_1=1}^{d_1^k} \{ -j_1 d_3^k \overline{d_3^k} + j_2 d_2^k \overline{d_2^k} \} \sum_{j|2^k d_1^k t|2^k d_1^k j^{-1}} \mu(t) \sum_{u \equiv Q_j^{-1} t^{-1} (\text{mod } d_2^k d_3^k)} 1
\]

\[
= \sum_{(d_1, d_2, d_3)=1} h(d_1, d_2, d_3) d_2^k d_3^k \sum_{j=1}^{d_1^k} \sum_{j_2=1}^{d_2^k} \sum_{j=1}^{d_3^k} \mu(t) \sum_{d_3^k d_3^k \leq Q_j^{-1} t^{-1}} h(d_1, d_2, d_3) d_2^{-k} d_3^{-k} \times
\]

\[
\sum_{j=1}^{d_2^k} \sum_{j=1}^{d_2^k} \sum_{j=1}^{d_3^k} \sum_{j=1}^{d_3^k} \sum_{j=1}^{d_3^k} \{ -j_1 d_3^k \overline{d_3^k} + j_2 d_2^k \overline{d_2^k} \} (Q_j^{-1} t^{-1} d_2^{-k} d_3^{-k} + O((Q_j^{-1} t^{-1} d_3^{-k} d_3^{-k})^{1+\varepsilon}))
\]

\[
+ \sum_{(d_1, d_2, d_3)=1} h(d_1, d_2, d_3) d_2^{-k} d_3^{-k} \sum_{j=1}^{d_1^k} \sum_{j=1}^{d_3^k} \sum_{j=1}^{d_3^k} \sum_{j=1}^{d_3^k} \sum_{j=1}^{d_3^k} \{ -j_1 d_3^k \overline{d_3^k} + j_2 d_2^k \overline{d_2^k} \} (Q_j^{-1} t^{-1} d_2^{-k} d_3^{-k} + O((Q_j^{-1} t^{-1} d_3^{-k} d_3^{-k})^{1+\varepsilon}))
\]

By (9.6),

\[
\sum_{(d_1, d_2, d_3)=1} h(d_1, d_2, d_3) d_2^{-k} d_3^{-k} \times
\]

\[
\sum_{d_3^k d_3^k \leq Q_j^{-1} t^{-1}} \sum_{d_3^k d_3^k \leq Q_j^{-1} t^{-1}} \sum_{d_3^k d_3^k \leq Q_j^{-1} t^{-1}} \sum_{d_3^k d_3^k \leq Q_j^{-1} t^{-1}} h(d_1, d_2, d_3) d_2^{-k} d_3^{-k} \times
\]

\[
\sum_{j=1}^{d_3^k} \sum_{j=1}^{d_3^k} \sum_{j=1}^{d_3^k} \sum_{j=1}^{d_3^k} \sum_{j=1}^{d_3^k} \{ -j_1 d_3^k \overline{d_3^k} + j_2 d_2^k \overline{d_2^k} \} (Q_j^{-1} t^{-1} d_2^{-k} d_3^{-k} + O((Q_j^{-1} t^{-1} d_3^{-k} d_3^{-k})^{1+\varepsilon}))
\]
By (9.6) and (10.9), the formula is valid.

where $c \ll Q$.

Similarly, by the same method, we can obtain

$$
\sum_{u \leq Q} G_{(1)}(u, 0) = c_{31} Q + O(Q^{1/2+\varepsilon}),
$$

where $c_{31}$ is independence of $Q$.

Similarly, by the same method, we can obtain

$$
\sum_{u \leq Q} G_{(3)}(u, 0) = c_{33} Q + O(Q^{1/2+\varepsilon}), \quad \sum_{u \leq Q} G_{(4)}(u, 0) = c_{34} Q + O(Q^{1/2+\varepsilon}),
$$

where $c_{33}, c_{34}, c_{35}$ are independence of $Q$, the lemma follows.

Consequently, for $\frac{1}{k} - 1 + \tau \leq \sigma < 0$, by Lemma 10.2, (9.21), (10.4), (10.5) and (10.6),

$$
S_E(x, Q) = c_2(Q)x^2 + c_3xQ + O(x^{\sigma+1}Q^{1-\sigma} + xQ^{1/2+\varepsilon}),
$$

by choosing $\sigma = \frac{1}{k} - 1 + \tau < 0$, we deduce

$$
S_E(x, Q) = c_2(Q)x^2 + c_3xQ + O(x^{\frac{1}{k}+\tau}Q^{2-\frac{1}{k}-\tau} + xQ^{1/2+\varepsilon}), \quad (10.10)
$$

where $c_2(Q)$ is independence of $x$, and $c_3$ is independence of $x, Q$.  

11 Proof of the Theorem

When $k = 2$, the Theorem follows from Lemma 5.2. We suppose that $k > 2$.

We also note that $R = x^\frac{1}{2} + \tau$. By (4.33), (5.6), (6.4), (7.2), and (10.9),

$$S_1(x, Q) = 2c_2(Q)x^2 + 2c_3xQ + [Q] \sum_{n \leq x} \mu_k(n)\mu_k(n + 1) + O(x^{\frac{1}{k} + \gamma}Q^{2 - \frac{1}{k} - \gamma} + x^{\frac{3}{2} + \frac{1}{k} + 2\gamma})$$

(11.1)

By Theorem 1 of [4],

$$\sum_{n \leq x} \mu_k(n)\mu_k(n + 1) = \varrho x + O(x^{\frac{2}{k + 1}}),$$

where $\varrho$ is defined by (4.17).

Hence, by (1.5), (1.6), (2.32), (3.4), and (11.1),

$$Y_k(x, Q) = c_4(Q)x^2 + c_5xQ + O(x^{\frac{1}{k} + \gamma}Q^{2 - \frac{1}{k} - \gamma} + x^{\frac{3}{2} + \frac{1}{k} + 2\gamma}),$$

and

$$c_4(Q)x^2 + c_5xQ + O(x^{\frac{1}{k} + \gamma}Q^{2 - \frac{1}{k} - \gamma} + x^{\frac{3}{2} + \frac{1}{k} + 2\gamma}) \ll x^{\frac{2}{k} + \epsilon}Q^{2 - \frac{2}{k}} + x^{\frac{4}{k + 1} + \epsilon},$$

(11.2)

where $c_4(Q)$ is independence of $x$, and $c_5$ is independence of $x, Q$. (11.2) is valid for all $1 < Q \leq x$.

For fixed $Q$, we divide both sides of (11.2) by $x^2$ and let $x \to \infty$, we deduce $c_4(Q) = 0$ for all $Q > 1$.

hence

$$Y_k(x, Q) = c_5xQ + O(x^{\frac{1}{k} + \gamma}Q^{2 - \frac{1}{k} - \gamma} + x^{\frac{3}{2} + \frac{1}{k} + 2\gamma}),$$

and

$$c_5xQ + O(x^{\frac{1}{k} + \gamma}Q^{2 - \frac{1}{k} - \gamma} + x^{\frac{3}{2} + \frac{1}{k} + 2\gamma}) \ll x^{2 + \epsilon}Q^{2 - \frac{2}{k}} + x^{\frac{4}{k + 1} + \epsilon},$$

(11.3)

We divide both sides of (11.3) by $xQ$, by choosing $Q = x^{\frac{5}{6}}$ and let $x \to \infty$, we obtain $c_5 = 0$. 

Consequently,

\[ Y_k(x, Q) = O(x^{\frac{1}{k}+\tau}Q^{2-\frac{1}{k}-\tau} + x^{\frac{3}{2}+\frac{1}{2k}+2\tau}), \]

since \( \tau \) is a sufficiently small positive number, the Theorem follows at once.

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