Abstract. We consider the matrix least squares problem of the form \( \|AX - B\|_F^2 \) where the design matrix \( A \in \mathbb{R}^{N \times r} \) is tall and skinny with \( N \gg r \). We propose to create a sketched version \( \|\tilde{A}X - \tilde{B}\|_F^2 \) where the sketched matrices \( \tilde{A} \) and \( \tilde{B} \) contain weighted subsets of the rows of \( A \) and \( B \), respectively. The subset of rows is determined via random sampling based on leverage score estimates for each row. We say that the sketched problem is \( \epsilon \)-accurate if its solution \( \tilde{X}_{\text{opt}} = \arg \min X \|\tilde{A}X - \tilde{B}\|_F^2 \) satisfies \( \|AX - B\|_F^2 \leq (1 + \epsilon) \min X \|AX - B\|_F^2 \) with high probability. We prove that the number of samples required for an \( \epsilon \)-accurate solution is \( O\left(\frac{r}{\beta \epsilon} \right) \) where \( \beta \in (0, 1] \) is a measure of the quality of the leverage score estimates.

Key words. matrix sketching, leverage score sampling, randomized numerical linear algebra (RandNLA)

1. Introduction. Approximating the solution of an overdetermined system of linear equations is a fundamental problem in data science and statistics. This is often accomplished via the method of least squares which finds the matrix \( \arg \min X \|AX - B\|_F^2 \). Here we consider the problem of sketching this matrix least squares problem by row sampling according to probability distribution \( p \) and forming sketched matrices \( \tilde{A} \) and \( \tilde{B} \) which are weighted subsets of the rows of the original matrices. Our aim is for the solution to the sketched problem \( \tilde{X}_{\text{opt}} := \arg \min \|\tilde{A}X - \tilde{B}\|_F^2 \) to be \( \epsilon \)-accurate, i.e. whose residual satisfies the following property:

\[ \|\tilde{A}\tilde{X}_{\text{opt}} - \tilde{B}\|_F^2 \leq (1 + \epsilon) \min X \|AX - B\|_F^2 \]

We demonstrate that this occurs with probability \( 1 - \delta \) for matrix \( A \in \mathbb{R}^{N \times r} \) provided the number of sampled row is

\[ s = \left( \frac{r}{\beta} \right) \max \left\{ C \log(r/\delta), 1/(\delta \epsilon) \right\} \]

where \( C \) is a constant and \( \beta \in (0, 1] \) defines how well the probability distribution \( p \) approximates the leverage scores of the design matrix \( A \). We generally treat \( \delta \) as a constant and assume \( \epsilon \) is sufficiently small so that \( \epsilon^{-1} \geq C \delta \log(r/\delta) \). In this case, we can write \( s = O(r/(\beta \epsilon)) \).

This note provides a complete proof of this result with two motivations in mind. The first is that it provides the foundation for leverage-based sampling for low-rank tensor decomposition as described in [5]. To the best of our knowledge, the precise result stated in Theorem 6 of that paper was new and thus a condensed outline of the proof was provided in the appendix; this note provides the full proof. The second is that although many of the steps are from previous work or primarily extend results to the matrix case, we did not find a concise statement of...
the full logic of this style of sketching least squares results. We intend this note to provide such a reference.

2. **Preliminaries.** We begin by introducing the essential definitions for weighted row sampling and leverage scores. We then outline the structure of the proof in the next section.

**Weighted sampling.** Assuming we choose rows of a matrix according to some probability distribution, we consider how to weight the rows so that the subsampled norm is unbiased.

**Definition 2.1.** We say \( p \in [0,1]^N \) is a probability distribution if \( \sum_{i=1}^{N} p_i = 1 \).

**Definition 2.2.** For a random variable \( \xi \in [N] \), we say \( \xi \sim \text{multinomial}(p) \) if \( p \in [0,1]^N \) is a probability distribution and \( \Pr(\xi = i) = p_i \).

We can define a matrix that randomly samples rows from a matrix (or elements from a vector) with weights as follows. The following definition can be found, e.g., in \([6, \text{Defn. 16}]\) or \([3, \text{Alg. 1}]\).

**Definition 2.3.** We say \( S \in \mathbb{R}^{s \times N} \sim \text{randsample}(s, p) \) if \( s \in \mathbb{N}, p \in [0,1]^N \) is a probability distribution, and the entries on \( \Omega \) are defined as follows. Let \( \xi_j \sim \text{multinomial}(p) \) for \( j = 1, \ldots, s \); then

\[
S(j, i) = \begin{cases} \frac{1}{\sqrt{s p_i}} & \text{if } \xi_j = i, \\ 0 & \text{otherwise}, \end{cases} \quad \text{for all } (j, i) \in [s] \times [N].
\]

It is straightforward to show that such a sampling matrix is unbiased, so we leave the proof of the next lemma as an exercise for the reader.

**Lemma 2.4.** Let \( x \in \mathbb{R}^N \). Let \( p \in [0,1]^N \) be probability distribution such that \( p_i > 0 \) if \( x_i \neq 0 \) and let \( \Omega \sim \text{randsample}(s, p) \). Then \( \mathbb{E}\|Sx\|_2^2 = \|x\|_2^2 \).

**Leverage scores.** The distribution selected for \( p \) determines the quality of the estimate in a way that depends on the leverage scores of \( A \).

**Definition 2.5 (Leverage Scores [2]).** Let \( A \in \mathbb{R}^{N \times r} \) with \( N > r \), and let \( Q \in \mathbb{R}^{N \times r} \) be any orthogonal basis for the column space of \( A \). The leverage scores of the rows of \( A \) are given by

\[
\ell_i(A) = \|Q(i,:)\|_2^2 \quad \text{for all } i \in \{1, \ldots, N\}.
\]

The coherence is the maximum leverage score, denoted \( \mu(A) = \max_{i \in [N]} \ell_i(A) \).

The leverage scores indicate the relative importance of rows in the matrix \( A \). It is known that \( \ell_i(A) \leq 1 \) for all \( i \in [N] \), \( \sum_{i \in [N]} \ell_i(A) = r \), and \( \mu(A) \in [r/N, 1] \) \([6]\). The matrix \( A \) is called incoherent if \( \mu(A) \approx r/N \). Lastly, for any row sampling distribution \( p \) we can measure the discrepancy between it and the sampling distribution defined by the leverage scores via the misestimation factor \( \beta \in (0,1] \):

\[
\beta \leq \min_{i \in [N]} \frac{p_i r}{\ell_i(A)} \quad \text{for all } i \in [N].
\]
3. Outline of Proof. Consider the overdetermined matrix least squares problem defined by the design matrix \( A \in \mathbb{R}^{N \times r} \), with \( N > r \) and \( \text{rank}(A) = r \), and the matrix \( B \in \mathbb{R}^{N \times n} \). Define the optimal squared residual to be

\[
R^2 \triangleq \min_{X \in \mathbb{R}^{r \times n}} \|AX - B\|_F^2.
\]

The SVD of the design matrix is \( A = U_A \Sigma_A V_A^T \), so \( U_A \) is an orthonormal basis for the \( d \)-dimensional column space of \( A \). Let \( U_A^\perp \) be an orthonormal basis for the \((N-r)\)-dimensional subspace orthogonal to the column space of \( A \). We define \( B^\perp \) to be the projection of the columns of \( B \) onto this orthogonal subspace: \( B^\perp \triangleq U_A^\perp U_A^\perp^T B \). This matrix is important because the residual of the least squares problem is its Frobenius norm; \( X \) can be chosen so that each column in \( AX \) exactly matches the part of the corresponding column in \( B \) in the column space of \( A \) but cannot, by definition, match anything in the range spanned by \( U_A^\perp \):

\[
R^2 = \min_{X \in \mathbb{R}^{r \times n}} \|AX - B\|_F^2 = \|U_A^\perp U_A^\perp^T B\|_F^2 = \|B^\perp\|_F^2.
\]

Denoting the solution to the least squares problem by \( X_{\text{opt}} \) yields \( B = AX_{\text{opt}} + B^\perp \).

Now consider the sketching problem defined by a matrix \( S \in \mathbb{R}^{s \times N} \):

\[
(3.2) \min_{X \in \mathbb{R}^{r \times n}} \|SAX - SB\|_F^2.
\]

Following the technique in Drineas et al. \[4\], we split the proof into two parts. In section 4, we prove bounds on both the residual and the solution of the sketched system for a specific sketching matrix \( S \) that satisfies certain structural conditions. The proofs follow deterministically and do not consider the random aspect of the sketching matrix generation. In section 5, we then consider that \( S \) is drawn from a distribution over matrices \( D \), i.e., \( S \sim D \), and prove that the required structural conditions hold with high probability if the number of samples is large enough. Finally, the proof is completed by connecting these parts so that the bounds on the residual and solution hold with high probability.

4. Properties of sketching matrix under structural conditions. The main results mirror Lemma 1 and 2 in \[4\]. The structure is also similar to Theorem 23 in Woodruff \[6\], except that work uses CountSketch, a different type of sketching.

We begin by assuming that our design matrix satisfies two structural conditions:

\[
\text{(SC1)} \quad \sigma_{\text{min}}^2(SU_A) \geq 1/\sqrt{2}, \quad \text{and}
\]

\[
\text{(SC2)} \quad \|U_A^T S^T S B^\perp\|_F^2 \leq \epsilon R^2 / 2.
\]

We first consider bounds with no constraints on the matrix \( B \). The first result is analogous to \[4, Lemma 1\] except that we prove it for the matrix least squares case.

**Theorem 4.1.** For the overdetermined least squares problem (3.2), assume the sketch matrix \( S \) satisfies (SC1) and (SC2) for some \( \epsilon \in (0,1) \). Then the solution to the sketched problem, denoted \( \tilde{X}_{\text{opt}} \), satisfies the following two bounds:

\[
\|AX_{\text{opt}} - B\|_F^2 \leq (1 + \epsilon)\|AX_{\text{opt}} - B\|_F^2, \quad \text{and}
\]

\[
\|X_{\text{opt}} - \tilde{X}_{\text{opt}}\|_F^2 \leq \frac{\epsilon \|AX_{\text{opt}} - B\|_F^2}{\sigma_{\text{min}}^2(A)}.
\]
Proof. We begin by rewriting the sketched regression problem:

\[
\min_{\mathbf{X} \in \mathbb{R}^{d \times N}} \|\mathbf{SAX} - \mathbf{SB}\|^2_F = \min_{\mathbf{X} \in \mathbb{R}^{d \times N}} \|\mathbf{SA(X + X}_{\text{opt}} - X_{\text{opt}}) - \mathbf{S}(\mathbf{AX}_{\text{opt}} + \mathbf{B}^\perp)\|^2_F,
\]

\[
= \min_{\mathbf{X} \in \mathbb{R}^{d \times N}} \|\mathbf{SA(X - X}_{\text{opt}}) - \mathbf{SB}^\perp\|^2_F,
\]

\[
= \min_{\mathbf{Y} \in \mathbb{R}^{d \times N}} \|\mathbf{SU}_{\mathbf{A}}(\mathbf{Y - Y}_{\text{opt}}) - \mathbf{SB}^\perp\|^2_F.
\]

In the last line, we reparameterize the matrices \(\mathbf{X}\) and \(\mathbf{X}_{\text{opt}}\) in terms of the orthonormal basis \(\mathbf{U}_{\mathbf{A}}\) such that \(\mathbf{U}_{\mathbf{A}}\mathbf{Y} = \mathbf{AX}\) and the analogous relationships hold for \(\mathbf{X}_{\text{opt}}/\mathbf{Y}_{\text{opt}}\) and \(\mathbf{X}_{\text{opt}}/\mathbf{Y}_{\text{opt}}\). The solution, \(\mathbf{Y}_{\text{opt}}\), satisfies the normal equation, i.e.,

\[
(\mathbf{SU}_{\mathbf{A}})^{\text{T}}\mathbf{SU}_{\mathbf{A}}(\mathbf{Y}_{\text{opt}} - \mathbf{Y}_{\text{opt}}) = (\mathbf{SU}_{\mathbf{A}})^{\text{T}}\mathbf{SB}^\perp.
\]

By (SC1) we have that \(\sigma_i((\mathbf{SU}_{\mathbf{A}})^{\text{T}}\mathbf{SU}_{\mathbf{A}}) = \sigma_i^2(\mathbf{SU}_{\mathbf{A}}) \geq 1/\sqrt{2}\). Thus taking the norm squared of both sides, applying the structural conditions, and then the relation from the normal equation gives:

\[
\|\mathbf{Y}_{\text{opt}} - \mathbf{Y}_{\text{opt}}\|^2_F/2 \leq \|((\mathbf{SU}_{\mathbf{A}})^{\text{T}}\mathbf{SU}_{\mathbf{A}}(\mathbf{Y}_{\text{opt}} - \mathbf{Y}_{\text{opt}}))\|^2_F = \|((\mathbf{SU}_{\mathbf{A}})^{\text{T}}\mathbf{SB}^\perp\|^2_F.
\]

Finally we apply (SC2) to the right hand side of this inequality to obtain:

\[
\|\mathbf{Y}_{\text{opt}} - \mathbf{Y}_{\text{opt}}\|^2_F/2 \leq \|\mathbf{U}_{\mathbf{A}}^\text{T}\mathbf{S}\mathbf{SB}^\perp\|^2_F \leq \epsilon\mathbf{R}^2/2,
\]

\[
\rightarrow \|\mathbf{Y}_{\text{opt}} - \mathbf{Y}_{\text{opt}}\|^2_F \leq \epsilon\mathbf{R}^2.
\]

We can then immediately show that this result implies the desired result on the residual:

\[
\|\mathbf{B} - \mathbf{AX}_{\text{opt}}\|^2_F = \|\mathbf{B} - \mathbf{AX}_{\text{opt}} + \mathbf{AX}_{\text{opt}} - \mathbf{AX}_{\text{opt}}\|^2_F
\]

\[
= \|\mathbf{B} - \mathbf{AX}_{\text{opt}}\|^2_F + \|\mathbf{A}(\mathbf{X}_{\text{opt}} - \mathbf{X}_{\text{opt}})\|^2_F, \]

\[
\leq \mathbf{R}^2 + \epsilon\mathbf{R}^2 = (1 + \epsilon)\|\mathbf{B} - \mathbf{AX}_{\text{opt}}\|^2_F,
\]

where we have used in line 2 that the columns of \(\mathbf{B} - \mathbf{AX}_{\text{opt}} = \mathbf{B}^\perp\) are orthogonal to \(\mathbf{A}\) times any vector and in the third line that \(\mathbf{U}_{\mathbf{A}}\) is a matrix with orthonormal columns.

Lastly, to obtain the bound on the solution recall that \(\mathbf{A}(\mathbf{X}_{\text{opt}} - \mathbf{X}_{\text{opt}}) = \mathbf{U}_{\mathbf{A}}(\mathbf{Y}_{\text{opt}} - \mathbf{Y}_{\text{opt}})\). Taking the norm of both sides we have:

\[
\sigma^2_{\text{min}}(\mathbf{A})\|\mathbf{X}_{\text{opt}} - \mathbf{X}_{\text{opt}}\|^2_F \leq \|\mathbf{A}(\mathbf{X}_{\text{opt}} - \mathbf{X}_{\text{opt}})\|^2_F = \|\mathbf{U}_{\mathbf{A}}(\mathbf{Y}_{\text{opt}} - \mathbf{Y}_{\text{opt}})\|^2_F.
\]

Recall that we assume \(\text{rank}(\mathbf{A}) = d\) so that \(\sigma_{\text{min}}(\mathbf{A}) > 0\). We then apply (4.1) and rearrange to obtain the desired result:

\[
\|\mathbf{X}_{\text{opt}} - \mathbf{X}_{\text{opt}}\|^2_F \leq \frac{\|\mathbf{Y}_{\text{opt}} - \mathbf{Y}_{\text{opt}}\|^2_F}{\sigma^2_{\text{min}}(\mathbf{A})} \leq \frac{\epsilon^2\mathbf{R}^2}{\sigma^2_{\text{min}}(\mathbf{A})}.
\]
We can obtain a tighter bound on the solution matrix if we assume a constant fraction of the columns of \( \mathbf{B} \) is in the column space of \( \mathbf{A} \). This is typically a reasonable assumption for real-world least squares problems as the fit is only practically interesting if this is true.

**Theorem 4.2 ([4]).** For the overdetermined least squares problem (3.2), assume the sketch matrix \( \mathbf{S} \) satisfies (SC1) and (SC2) for some \( \epsilon \in (0, 1) \). Furthermore, assume that we have \( \| \mathbf{U}_\mathbf{A} \mathbf{U}_\mathbf{A}^\top \mathbf{B} \|_F \geq \gamma \| \mathbf{B} \|_F \) for some fixed \( \gamma \in (0, 1] \). Then the solution to the sketched problem, denoted \( \tilde{\mathbf{X}}_{\text{opt}} \), satisfies the following bound:

\[
\| \mathbf{X}_{\text{opt}} - \tilde{\mathbf{X}}_{\text{opt}} \|_F^2 \leq \epsilon^2 \kappa(\mathbf{A})^2 (\gamma^{-2} - 1) \| \mathbf{X}_{\text{opt}} \|_F^2,
\]

where \( \kappa(\mathbf{A}) \) denotes the condition number of the matrix \( \mathbf{A} \).

**Proof.** Start by bounding the residual squared using our assumption on \( \mathbf{B} \) as follows:

\[
\| \mathbf{AX}_{\text{opt}} - \mathbf{B} \|_F^2 = \| \mathbf{B} \|_F^2 - \| \mathbf{U}_\mathbf{A} \mathbf{U}_\mathbf{A}^\top \mathbf{B} \|_F^2 \\
\leq \gamma^{-2} \| \mathbf{U}_\mathbf{A} \mathbf{U}_\mathbf{A}^\top \mathbf{B} \|_F^2 - \| \mathbf{U}_\mathbf{A} \mathbf{U}_\mathbf{A}^\top \mathbf{B} \|_F^2 \\
= (\gamma^{-2} - 1) \| \mathbf{U}_\mathbf{A} \mathbf{U}_\mathbf{A}^\top \mathbf{B} \|_F^2 \\
= (\gamma^{-2} - 1) \| \mathbf{AX}_{\text{opt}} \|_F^2 \\
\leq \sigma_{\text{max}}^2(\mathbf{A})(\gamma^{-2} - 1) \| \mathbf{X}_{\text{opt}} \|_F^2.
\]

By the previous theorem, we have that

\[
\| \mathbf{X}_{\text{opt}} - \tilde{\mathbf{X}}_{\text{opt}} \|_F^2 \leq \frac{1}{\sigma_{\text{min}}^2(\mathbf{A})} \epsilon^2 \| \mathbf{AX}_{\text{opt}} - \mathbf{B} \|_F^2. 
\]

Plugging in the above inequality yields the desired result:

\[
\| \mathbf{X}_{\text{opt}} - \tilde{\mathbf{X}}_{\text{opt}} \|_F^2 \leq \frac{1}{\sigma_{\text{min}}^2(\mathbf{A})} \epsilon^2 \| \mathbf{AX}_{\text{opt}} - \mathbf{B} \|_F^2 \\
\leq \frac{\sigma_{\text{max}}^2(\mathbf{A})}{\sigma_{\text{min}}^2(\mathbf{A})} \epsilon^2 \| \mathbf{AX}_{\text{opt}} - \mathbf{B} \|_F^2 \\
= \epsilon^2 \kappa(\mathbf{A})^2 (\gamma^{-2} - 1) \| \mathbf{X}_{\text{opt}} \|_F^2.
\]

5. **Proof that sketching matrix meets structural conditions.** In this section, we show that the methodology for choosing the columns via the leverage-score-based sampling scheme yields the desired bounds. The first structural condition (SC1) can be shown as a corollary to the following result in Woodruff [6]:

**Lemma 5.1 ([6]).** Consider \( \mathbf{A} \in \mathbb{R}^{N\times r} \), its SVD \( \mathbf{U}_\mathbf{A} \mathbf{\Sigma}_\mathbf{A} \mathbf{V}_\mathbf{A}^\top \), and row leverage scores \( \ell_i(\mathbf{A}) \). Let \( \overline{\ell}(\mathbf{A}) \) be an overestimate of the leverage score such that for some positive \( \beta \leq 1 \), we have \( \ell_i(\mathbf{A}) \geq \beta \cdot \overline{\ell}(\mathbf{A}) \) for all \( i \in [N] \). Construct row sampling and rescaling matrix \( \mathbf{S} \in \mathbb{R}^{s\times N} \) by importance sampling according to the leverage score overestimates, \( \overline{\ell}(\mathbf{A}) \). If \( s > 144r \ln(2r/\delta)/(\beta \epsilon^2) \), then the following holds with probability at least \( 1 - \delta \) simultaneously for all \( i: 1 - \epsilon \leq \sigma_i^2(\mathbf{SU}_\mathbf{A}) \leq 1 + \epsilon \).

Fixing \( \epsilon = 1 - 1/\sqrt{2} \) in Lemma 5.1 yields the Corollary we require.

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1For completeness, we intend to include the proof of this lemma in a future version of this manuscript. It appears as Theorem 2.11 in [6].
Lemma 5.2. Consider $\mathbf{A} \in \mathbb{R}^{N \times r}$, its SVD $\mathbf{U}_\mathbf{A} \Sigma \mathbf{V}_\mathbf{A}^\top$, and row leverage scores $\ell_i(\mathbf{A})$. Let $\overline{\mathbf{f}}(\mathbf{A})$ be an overestimate of the leverage score such that for some positive $\beta \leq 1$, we have $p_i(\overline{\mathbf{f}}(\mathbf{A})) \geq \beta \cdot p_i(\ell_i(\mathbf{A}))$ for all $i \in [N]$. Construct row sampling and rescaling matrix $\mathbf{S} \in \mathbb{R}^{s \times N}$ by importance sampling according to the leverage score overestimates, $\overline{\mathbf{f}}(\mathbf{A})$. If $s > C r \ln(2r/\delta)/\beta$ with $C = 144/(1 - 1/\sqrt{\delta})^2$, then $\sigma_{min}(\mathbf{S} \mathbf{U}_\mathbf{A}) \geq 1/2$ with probability at least $1 - \delta$.

The second structural condition (SC2) can be proven using results for randomized matrix-matrix multiplication. Consider the matrix product $\mathbf{U}_\mathbf{A}^\top \mathbf{B}$. This projects the part of the columns of $\mathbf{B}$ outside of the column space of $\mathbf{A}$ onto the column space of $\mathbf{A}$ and thus by definition is equal to the all zeros matrix $\mathbf{0}_{r \times n}$ (we have assumed rank($\mathbf{A}$) = $r$). This condition requires us to bound how well the sampled product $\mathbf{U}_\mathbf{A}^\top \mathbf{S} \mathbf{B}$ approximates the original product. We can do this via the following lemma from Drineas, Kannan, and Mahoney [1].

Lemma 5.3 ([1]). Consider two matrices of the form $\mathbf{A} \in \mathbb{R}^{n \times m}$ and $\mathbf{B} \in \mathbb{R}^{n \times p}$ and let $s$ denote the number of samples. We form an approximation of the product $\mathbf{A}^\top \mathbf{B}$ as follows. Choose $s$ rows, denoted $\{\xi^{(1)}, \ldots, \xi^{(s)}\}$, according to the probability distribution defined by $p \in [0, 1]^n$ with the property that there exists $\beta > 0$ such that $p_k \geq \beta \|\mathbf{A}(k,:)\|^2 / \|\mathbf{A}\|_F^2$ for all $k \in [n]$. Then form the approximate product

$$
\frac{1}{s} \sum_{t=1}^s \frac{1}{p_{\xi^{(t)}}} \mathbf{A}(\xi^{(t)},:)^\top \mathbf{B}(\xi^{(t)},:) \triangleq (\mathbf{S} \mathbf{A})^\top \mathbf{S} \mathbf{B},
$$

where we define $\mathbf{S}$ to be the random row sampling and rescaling operator. We then have the following guarantee on the quality of the approximate product:

$$
\mathbb{E} \left[ \|\mathbf{A}^\top \mathbf{B} - (\mathbf{S} \mathbf{A})^\top \mathbf{S} \mathbf{B}\|_F^2 \right] \leq \frac{1}{\beta s} \|\mathbf{A}\|_F^2 \|\mathbf{B}\|_F^2.
$$

Proof. Fix $i, j$ to specify an element of the matrix product and let $\{\xi^{(1)}, \ldots, \xi^{(s)}\}$ be the indices of the sampled rows of $\mathbf{A}$ (and $\mathbf{B}$). We begin by calculating the expected value and variance of the corresponding element of the sampled matrix product, i.e., $[(\mathbf{S} \mathbf{A})^\top \mathbf{S} \mathbf{B}]_{ij}$. This can be written in terms of scalar random variables $X_t$ for $t = 1, \ldots, s$ as follows:

$$
X_t = \frac{\mathbf{A}(\xi^{(t)},i)^\top \mathbf{B}(\xi^{(t)},j)}{s p_{\xi^{(t)}}} \implies [(\mathbf{S} \mathbf{A})^\top \mathbf{S} \mathbf{B}]_{ij} = \sum_{t=1}^s X_t
$$

The expectation of $X_t$ and $X_t^2$ for all $t$ can be calculated as follows:

$$
\mathbb{E}[X_t] = \sum_{k=1}^n p_k \frac{\mathbf{A}_{ki} \mathbf{B}_{kj}}{s p_k} = \frac{1}{s} (\mathbf{A}^\top \mathbf{B})_{ij},
$$

$$
\mathbb{E}[X_t^2] = \sum_{k=1}^n p_k^2 \frac{\mathbf{A}_{ki}^2 \mathbf{B}_{kj}^2}{s^2 p_k} = \sum_{k=1}^n \frac{\mathbf{A}_{ki}^2 \mathbf{B}_{kj}^2}{s^2 p_k}.
$$

The relation between $X_t$ and $[(\mathbf{S} \mathbf{A})^\top \mathbf{S} \mathbf{B}]_{ij}$ gives $\mathbb{E} \left[ [(\mathbf{S} \mathbf{A})^\top \mathbf{S} \mathbf{B}]_{ij} \right] = \sum_{t=1}^s \mathbb{E}[X_t] = (\mathbf{A}^\top \mathbf{B})_{ij}$ and hence the estimator is unbiased. Furthermore, since the estimated matrix element is the
sum of $s$ independent random variables, its variance can be calculated as follows:

\[
\text{Var} \left[ \left( \sum_{t=1}^{s} X_t \right) \right] = \sum_{t=1}^{s} \left( \text{Var}[X_t] \right)
\]

\[
= \sum_{t=1}^{s} \left( \text{Var}[X_t] \right)
\]

Now we turn to the expectation we want to bound and apply these results:

\[
\mathbb{E} \left[ \|AB - (SA)^T SB\|_F^2 \right] = \sum_{i=1}^{m} \sum_{j=1}^{p} \mathbb{E} \left[ \left( \left( \sum_{t=1}^{s} X_t \right) \right) \right]
\]

\[
= \sum_{i=1}^{m} \sum_{j=1}^{p} \mathbb{E} \left[ \left( \left( \sum_{t=1}^{s} X_t \right) \right) \right]
\]

\[
= \sum_{i=1}^{m} \sum_{j=1}^{p} \text{Var} \left[ \left( \sum_{t=1}^{s} X_t \right) \right]
\]

where in the last line we have used that the Frobenius norm of a matrix is strictly positive.
Lastly, we use our assumption on the probabilities \( p_k \geq \frac{\beta \| A(k,:) \|_F^2}{\|A\|_F^2} \) to obtain the desired bound:

\[
\mathbb{E} \left[ \| A^T B - (SA)^T SB \|_F^2 \right] \leq \frac{1}{s} \sum_{k=1}^{n} \frac{\| A(k,:) \|_F^2 \| B(k,:) \|_F^2}{p_k},
\]

\[
\leq \frac{1}{s} \sum_{k=1}^{n} \left( \| A^T \|_F^2 \frac{\| A(k,:) \|_F^2 \| B(k,:) \|_F^2}{\beta \| A(k,:) \|_F^2} \right),
\]

\[
= \frac{1}{\beta s} \| A \|_F^2 \sum_{k=1}^{n} \| B(k,:) \|_F^2 = \frac{1}{\beta s} \| A \|_F^2 \| B \|_F^2.
\]

We can apply Lemma 5.3 to bound the probability of (SC2) holding.

**Lemma 5.4.** Consider full rank \( A \in \mathbb{R}^{N \times r} \), its SVD \( U_A \Sigma_A V_A^T \), and row leverage scores \( \ell_i(A) \). Define the probability distribution \( p \in [0, 1]^n \) and assume there exists \( \beta \in (0, 1) \) such that \( p_i \geq \beta \ell_i(A) / d \) for all \( i \in [N] \). Construct row sampling and rescaling matrix \( S \in \mathbb{R}^{S \times N} \) by importance sampling by the leverage score overestimates. Then provided \( s \geq \frac{2r}{\beta \delta} \), the property \( \| U_A S^T S B \|_F^2 \) bounds the probability \( \| \epsilon \|_F^2 \) with probability \( \delta \).

**Proof.** Apply Lemma 5.3 to obtain a bound on the expected value:

\[
\mathbb{E} \left[ \| U_A S^T S B \|_F^2 \right] = \mathbb{E} \left[ \| 0_{r \times n} - U_A S^T S B \|_F^2 \right],
\]

\[
= \mathbb{E} \left[ \| U_A B - U_A S^T S B \|_F^2 \right],
\]

\[
\leq \frac{1}{\beta s} \| U_A \|_F^2 \| B \|_F^2 = \frac{r}{\beta s} \| B \|_F^2 = \frac{r}{\beta s} \delta^2.
\]

Markov’s inequality states that for non-negative random variable \( X \) and scalar \( t > 0 \), we can bound the probability that \( X \geq t \) as \( \Pr[X \geq t] \leq \mathbb{E}[X] / t \). We can apply this inequality to bound the probability that the sketching matrix violates (SC2):

\[
\Pr_{S \sim D} \left[ \| U_A S^T S B \|_F^2 \geq \frac{\epsilon \| B \|_F^2}{2} \right] \leq \frac{2 \mathbb{E} \left[ \| U_A S^T S B \|_F^2 \right]}{\epsilon \| B \|_F^2} \leq \frac{2r}{\beta \delta e},
\]

where in the last step we have used our bound the expected value. Thus if we set the right-hand side equal to \( \delta \), we obtain that the probability that (SC2) holds is greater than or equal to \( 1 - \delta \) as desired. Solving for \( s \) yields that we thus must have \( s \geq \frac{2r}{\beta \delta e} \).

**6. Main Theorem.** We combine the above results to prove Theorem 6 in the [5], here written in the standard least squares notation.

**Theorem 6.1.** Consider the least squares problem \( \min_{X \in \mathbb{R}^{r \times n}} \| AX - B \|_2^2 \) where \( A \in \mathbb{R}^{N \times r} \) with \( r \ll N \) and \( \text{rank}(A) = r \) and \( B \in \mathbb{R}^{N \times n} \). Let \( p \in [0, 1]^N \) be a probability distribution and assume there exists a fixed \( \beta \in (0, 1) \) such that

\[
\beta \leq \min_{i \in [N]} \frac{p_i r}{\ell_i(A)} \text{ for all } i \in [N].
\]
For any $\epsilon, \delta \in (0, 1)$, set

$$s = (r/\beta) \max \{ C \log(r/\delta), 1/(\delta \epsilon) \} \quad \text{where} \quad C = 144/(1 - 1/\sqrt{2})^2,$$

and let $S = \text{randsample}(s, p)$. Define $X_{\text{opt}} \equiv \arg \min_{X \in \mathbb{R}^{r \times n}} \|AX - B\|_F^2$. Then $\tilde{X}_{\text{opt}} \equiv \arg \min_{X \in \mathbb{R}^{r \times n}} \|SAX - SB\|_F^2$ satisfies $\|A\tilde{X}_{\text{opt}} - B\|_F^2 \leq (1 + \epsilon)\|AX_{\text{opt}} - B\|_F^2$ with probability at least $1 - \delta$.

**Proof.** Applying Lemma 5.2, we have that (SC1) holds with probability $1 - \delta/2$ if $s = Cr \log(r/\delta)/\beta$. Applying Lemma 5.4, we have that (SC2) holds with probability $1 - \delta/2$ if $s = r/(\beta \delta \epsilon)$. Hence, a union bound says that (SC1) and (SC2) both hold with probability $1 - \delta$ if $s = (r/\beta) \max \{ C \log(r/\delta), 1/(\delta \epsilon) \}$. Combining this with Theorem 4.1 yields the result. ■

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