New characterizations for the variation of the spectrum of an arbitrary matrix

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Abstract

The celebrated Hoffman–Wielandt theorem reveals the strong stability of the spectrum of a normal matrix under perturbations. Over the past decades, some analogs of the Hoffman–Wielandt theorem have been developed to characterize the stability of the spectrum of an arbitrary matrix. In this paper, we establish new perturbation bounds to characterize the variation of the spectrum of an arbitrary matrix. The counterparts of the existing results are also given, which are sharper than the existing ones. Moreover, our results have generalized some perturbation bounds for the spectrum of a normal matrix.

Keywords: Hoffman–Wielandt theorem, spectrum, perturbation

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1 Introduction

Let $\mathbb{C}^{m \times n}$ and $\mathbb{U}_n$ be the set of all $m \times n$ complex matrices and the set of all unitary matrices of order $n$, respectively. The identity matrix of order $n$ is denoted by $I_n$. For any $X \in \mathbb{C}^{m \times n}$, let $X^*$, $\|X\|_2$, and $\|X\|_F$ denote the conjugate transpose, the spectral norm, and the Frobenius norm of $X$, respectively. For any $M \in \mathbb{C}^{n \times n}$, its diagonal part, strictly lower triangular part, and strictly upper triangular part are denoted by $D(M)$, $L(M)$, and $U(M)$, respectively.

For any $M \in \mathbb{C}^{n \times n}$, we define

\[
\delta(M) := \left( \|M\|_F^2 - \frac{1}{n} |\text{tr}(M)|^2 \right)^{\frac{1}{2}},
\]

\[
s(M) := \max_{U \in \mathbb{U}_n} \left\{ s : U^*MU = \text{diag}(M_1, \ldots, M_s) \text{ and each } M_i \text{ is square} \right\},
\]

where $\text{tr}(M)$ denotes the trace of $M$. Obviously, $\delta(M) \leq \|M\|_F$ and $1 \leq s(M) \leq n$. Moreover, $\delta(M) = \|M\|_F$ if and only if $\text{tr}(M) = 0$, and $s(M) = n$ if and only if $M$ is normal, namely,
$MM^* = M^*M$. Let $A \in \mathbb{C}^{n \times n}$ and $\widetilde{A} = A + E \in \mathbb{C}^{n \times n}$ ($E \in \mathbb{C}^{n \times n}$ is a perturbation) have the spectra $\{\lambda_i\}_{i=1}^{n}$ and $\{\tilde{\lambda}_i\}_{i=1}^{n}$, respectively. For any permutation $\pi$ of $\{1, \ldots, n\}$, we define

$$D_2 := \left( \sum_{i=1}^{n} |\tilde{\lambda}_{\pi(i)} - \lambda_i|^2 \right)^{\frac{1}{2}}. \quad (1.3)$$

If both $A \in \mathbb{C}^{n \times n}$ and $\widetilde{A} = A + E \in \mathbb{C}^{n \times n}$ are normal, Hoffman and Wielandt [6] proved that there exists a permutation $\pi$ of $\{1, \ldots, n\}$ such that

$$D_2 \leq \|E\|_F, \quad (1.4)$$

which is the well known Hoffman–Wielandt theorem. Over the past decades, various analogs of the Hoffman–Wielandt theorem have been established to characterize the variation of the spectrum of a matrix (see, e.g., [9, 17, 2, 4, 7, 10, 16, 8, 12, 11, 13, 14, 18, 3]). The estimate (1.4) reveals the strong stability of the spectrum of a normal matrix. However, it may fail when $\widetilde{A}$ (or $A$) is non-normal. To remedy this limitation, Sun [17, Theorem 1.1] showed that

$$D_2 \leq \sqrt{n}\|E\|_F, \quad (1.5)$$

provided that $A \in \mathbb{C}^{n \times n}$ is normal and $\widetilde{A} = A + E \in \mathbb{C}^{n \times n}$ is non-normal. In view of the quantity $s(\cdot)$ defined in (1.2), Li and Sun [12, Theorem 2.3] refined the estimate (1.5) and derived that

$$D_2 \leq \sqrt{n - s(\widetilde{A}) + 1}\|E\|_F. \quad (1.6)$$

Recently, Xu and Zhang [18, Theorems 3.6 and 3.10] established that

$$D_2 \leq \sqrt{\|E\|_F^2 + (n - 1)\delta(E)^2}, \quad (1.7)$$

$$D_2 \leq \sqrt{\|E\|_F^2 + (n - s(\widetilde{A}))\delta(E)^2}. \quad (1.8)$$

It is easy to see that the estimates (1.7) and (1.8) are sharper than (1.5) and (1.6), respectively. For more theories on the variation of the spectrum of a normal matrix, we refer to the recent paper [18].

As is well known, for any $A \in \mathbb{C}^{n \times n}$, there is a nonsingular matrix $Q \in \mathbb{C}^{n \times n}$ such that

$$Q^{-1}AQ = \text{diag}(J_1, \ldots, J_p),$$

where each $J_i \in \mathbb{C}^{m_i \times m_i}$ ($\sum_{i=1}^{p} m_i = n$) is a Jordan block. For any $A \in \mathbb{C}^{n \times n}$ and $\widetilde{A} = A + E \in \mathbb{C}^{n \times n}$, using (1.5), Song [16, Theorem 2.1] derived that

$$D_2 \leq \begin{cases} \sqrt{n}(\sqrt{n - p} + 1)\|E_Q\|_F^2, & \text{if } \|E_Q\|_F < 1, \\ \sqrt{n}(\sqrt{n - p} + 1)\|E_Q\|_F, & \text{if } \|E_Q\|_F \geq 1, \end{cases} \quad (1.9)$$

where $m := \max_{1 \leq i \leq p} m_i$ and $E_Q := Q^{-1}EQ$. Some applications of the estimate (1.9) can be found, e.g., in [5, 1, 15]. On the basis of (1.6), Li and Chen [11, Theorem 2.1] restudied the variation of the spectrum of an arbitrary matrix and proved that

$$D_2 \leq \begin{cases} \sqrt{s_1(n - p + 1 + 2\sqrt{n - p}\|E_Q\|_F)}\|E_Q\|_F^2, & \text{if } \|E_Q\|_F < 1, \\ \sqrt{s_2(n - p + 2\sqrt{n - p} + \|E_Q\|_F)}\|E_Q\|_F^2, & \text{if } \|E_Q\|_F \geq 1, \end{cases} \quad (1.10)$$
Lemma 2.1. is invariant under a unitary similarity transformation.

In this section, we introduce some useful lemmas, which play a fundamental role in our analysis.

2 Preliminaries

Conclusions are given in Section 4.

Theoretical analysis suggests that our results (1.11) and (1.12) are sharper than (1.9) and (1.10), (see Remarks 3.1 and 3.4 for details. Furthermore, if the original matrix $A$ is normal, the estimates (1.11) and (1.12) will reduce to (1.7) and (1.8), respectively.

The rest of this paper is organized as follows. In Section 2, we introduce some auxiliary results, which play an important role in our analysis. In Section 3, we develop some new upper bounds to characterize the variation of the spectrum of an arbitrary matrix. Finally, some conclusions are given in Section 4.

2 Preliminaries

In this section, we introduce some useful lemmas, which play a fundamental role in our analysis.

The first lemma gives an upper bound for $\|L(M)\|_F^2 + \|U(M)\|_F^2$ (see [18, Lemma 3.1]), which is invariant under a unitary similarity transformation.

Lemma 2.1. Let $M \in \mathbb{C}^{n \times n}$. Then

$$\|L(M)\|_F^2 + \|U(M)\|_F^2 \leq \delta(M)^2,$$

(2.1)

where $\delta(M)$ is defined by (1.1).

From the inequality (2.1), we can readily observe that $M$ is a diagonal matrix if $\delta(M) = 0$. Indeed, $M$ must be a scalar matrix (i.e., $M = \mu I_n$ for some $\mu \in \mathbb{C}$), which can be seen from the following lemma.

Lemma 2.2. Let $M = (m_{ij}) \in \mathbb{C}^{n \times n}$. Then $\delta(M) = 0$ if and only if $M$ is a scalar matrix.

Proof. If $M = \mu I_n$, then $\|M\|_F^2 = n|\mu|^2$ and $\text{tr}(M) = n\mu$. By definition (1.1), we have $\delta(M) = 0$. Conversely, if $\delta(M) = 0$, then $\|M\|_F^2 = \frac{1}{n}|\text{tr}(M)|^2$, that is,

$$\sum_{i \neq j} |m_{ij}|^2 + \sum_{i=1}^n |m_{ii}|^2 = \frac{1}{n} \left| \sum_{i=1}^n m_{ii} \right|^2.$$

where $s_1 = n + 1 - s(T^{-1}Q^{-1} \tilde{A} Q T)$, $s_2 = n + 1 - s(Q^{-1} \tilde{A} Q)$, $T = \text{diag} \left( T_1, \ldots, T_p \right)$, $T_i = \text{diag} \left( 1, \varepsilon, \ldots, \varepsilon^{m_i-1} \right)$ with $\varepsilon = \|E_Q\|_F^2$.

In this paper, we establish some new perturbation bounds for the spectrum of an arbitrary matrix based on the estimates (1.7) and (1.8). For comparison, we here exhibit the counterparts of (1.9) and (1.10) (see Theorems 3.1 and 3.4), i.e.,

$$\|E_Q\|_F^2 \leq \sqrt{n(\sqrt{n-\rho} + \delta(E_Q))} \text{tr}((E_Q)^2) + \frac{1}{\sqrt{n}} \text{tr}(E_Q),$$

(1.11)

$$\|E_Q\|_F^2 \leq \sqrt{s_1(\sqrt{n-\rho} + \delta(E_Q)) \text{tr}((E_Q)^2) + \frac{1}{\sqrt{n}} \text{tr}(E_Q)},$$

(1.12)

Theoretical analysis suggests that our results (1.11) and (1.12) are sharper than (1.9) and (1.10), respectively; see Remarks 3.1 and 3.4 for details. Furthermore, if the original matrix $A$ is normal, the estimates (1.11) and (1.12) will reduce to (1.7) and (1.8), respectively.

The rest of this paper is organized as follows. In Section 2, we introduce some auxiliary results, which play an important role in our analysis. In Section 3, we develop some new upper bounds to characterize the variation of the spectrum of an arbitrary matrix. Finally, some conclusions are given in Section 4.
Due to
\[ \sum_{i=1}^{n} |m_{ii}|^2 \geq \frac{1}{n} \left( \sum_{i=1}^{n} |m_{ii}| \right)^2 \geq \frac{1}{n} \left( \sum_{i=1}^{n} m_{ii} \right)^2, \]
it follows that \( m_{ij} = 0 \) (\( \forall i \neq j \)) and \( m_{ii} = \text{const} \) (\( \forall i = 1, \ldots, n \)), i.e., \( M \) is a scalar matrix.

In order to analyze the variation of the spectrum of an arbitrary matrix, we need the following perturbation bounds for the spectrum of a normal matrix (see [18, Theorems 3.6, 3.10, and 4.2]).

**Lemma 2.3.** Let \( A \in \mathbb{C}^{n \times n} \) be a normal matrix with spectrum \( \{ \lambda_i \}_{i=1}^{n} \), and let \( \tilde{A} = A+E \in \mathbb{C}^{n \times n} \) with spectrum \( \{ \tilde{\lambda}_i \}_{i=1}^{n} \), where \( E \in \mathbb{C}^{n \times n} \) is a perturbation. Then there exists a permutation \( \pi \) of \( \{ 1, \ldots, n \} \) such that
\[
\mathbb{D}_2 \leq \sqrt{\|E\|_F^2 + (n-1)\delta(E)^2}, \\
(2.2)
\]
\[
\mathbb{D}_2 \leq \sqrt{\|E\|_F^2 + (n-s(A))\delta(E)^2}. \\
(2.3)
\]

In particular, if \( A \) is Hermitian, then there exists a permutation \( \pi \) of \( \{ 1, \ldots, n \} \) such that
\[
\mathbb{D}_2 \leq \sqrt{\|E\|_F^2 + \delta(E)^2}. \\
(2.4)
\]

For any \( A \in \mathbb{C}^{n \times n} \), there exists a nonsingular matrix \( Q \in \mathbb{C}^{n \times n} \) such that
\[
Q^{-1}AQ = \text{diag} \left( J_1, \ldots, J_p \right), \\
(2.5)
\]
where each \( J_i \in \mathbb{C}^{m_i \times m_i} \) (\( \sum_{i=1}^{p} m_i = n \)) is a Jordan block with the form
\[
J_i = \begin{pmatrix}
\lambda_i & 1 & 0 & \cdots & 0 \\
0 & \lambda_i & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_i & 1 \\
0 & 0 & \cdots & 0 & \lambda_i
\end{pmatrix}.
\]

Let \( 0 < \varepsilon \leq 1 \), and let
\[
T = \text{diag} \left( T_1, \ldots, T_p \right),
\]
where \( T_i = \text{diag} \left( 1, \varepsilon, \ldots, \varepsilon^{m_i-1} \right) \) for all \( i = 1, \ldots, p \). Then we have
\[
T^{-1}(Q^{-1}AQ)T = \text{diag} \left( T_1^{-1}J_1T_1, \ldots, T_p^{-1}J_pT_p \right) = \Lambda + \Omega, \\
(2.6)
\]
where \( \Lambda = \text{diag} \left( \lambda_1 I_{m_1}, \ldots, \lambda_p I_{m_p} \right) \) and \( \Omega = \text{diag} \left( \Omega_1, \ldots, \Omega_p \right) \) with
\[
\Omega_i = \begin{pmatrix}
0 & \varepsilon & 0 & \cdots & 0 \\
0 & 0 & \varepsilon & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \varepsilon \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix} \in \mathbb{C}^{m_i \times m_i} \quad \forall i = 1, \ldots, p.
\]

Under the above settings, we can show that the following lemma holds, which is the foundation of our analysis.
**Lemma 2.4.** Let $A \in \mathbb{C}^{n \times n}$ have the decomposition (2.5), and let $\tilde{A} = A + E$, where $E \in \mathbb{C}^{n \times n}$ is a perturbation. Let $\Lambda = \text{diag}(\lambda_1 I_{m_1}, \ldots, \lambda_p I_{m_p})$ and $T = \text{diag}(T_1, \ldots, T_p)$, where $T_i = \text{diag}(1, \varepsilon, \ldots, \varepsilon^{m_i-1})$ for all $i = 1, \ldots, p$. Then, for any $0 < \varepsilon \leq 1$, it holds that

$$||T^{-1}Q^{-1}\tilde{A}QT - \Lambda||_F^2 \leq \Phi(\varepsilon),$$  \hspace{1cm} (2.7)

where

$$\Phi(\varepsilon) := \varepsilon^{2(1-m)}\delta(E_Q)^2 + 2\varepsilon^2 n - p \delta(E_Q) + (n - p)\varepsilon^2 + \frac{1}{n} |\text{tr}(E)|^2$$

with $m = \max_{1 \leq i \leq p} m_i$.

**Proof.** From (2.6), we have that

$$T^{-1}Q^{-1}\tilde{A}QT - \Lambda = T^{-1}EQT + \Omega,$$

which yields

$$||T^{-1}Q^{-1}\tilde{A}QT - \Lambda||_F^2 = ||T^{-1}EQT||_F^2 + 2 \text{Re } \text{tr}(\Omega^*T^{-1}EQT) + ||\Omega||_F^2. \hspace{1cm} (2.8)$$

(i) Partitioning $E_Q$ as the block form $E_Q = (\hat{E}_{ij})_{p \times p}$ with $\hat{E}_{ij} \in \mathbb{C}^{m_i \times m_j}$, we have

$$||T^{-1}EQT||_F^2 = \sum_{i=1}^p \sum_{j=1}^p ||T_i^{-1}\hat{E}_{ij}T_j||_F^2.$$

Then

$$||T^{-1}EQT||_F^2 = \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^{m_i} \sum_{\ell=1}^{m_j} \varepsilon^{2(\ell-k)}||\hat{E}_{ij}k,\ell||^2$$

$$\leq \varepsilon^{2(1-m)} \sum_{i \neq j} ||\hat{E}_{ij}||_F^2 + \sum_{i=1}^p (||\mathcal{D}(\hat{E}_{ii})||_F^2 + \varepsilon^2 ||\mathcal{U}(\hat{E}_{ii})||_F^2 + \varepsilon^{2(1-m_i)} ||\mathcal{L}(\hat{E}_{ii})||_F^2)$$

$$\leq \varepsilon^{2(1-m)} \left( \sum_{i \neq j} ||\hat{E}_{ij}||_F^2 + \sum_{i=1}^p ||\mathcal{U}(\hat{E}_{ii})||_F^2 + \sum_{i=1}^p ||\mathcal{L}(\hat{E}_{ii})||_F^2 \right) + ||\mathcal{D}(E_Q)||_F^2$$

$$\leq \varepsilon^{2(1-m)} (||E_Q||_F^2 - ||\mathcal{D}(E_Q)||_F^2) + ||\mathcal{D}(E_Q)||_F^2$$

$$\leq \varepsilon^{2(1-m)} ||E_Q||_F^2 - (\varepsilon^{2(1-m)} - 1) ||\mathcal{D}(E_Q)||_F^2.$$

Since

$$||\mathcal{D}(E_Q)||_F^2 \geq \frac{1}{n} |\text{tr}(E)|^2,$$

we get

$$||T^{-1}EQT||_F^2 \leq \varepsilon^{2(1-m)} \delta(E_Q)^2 + \frac{1}{n} |\text{tr}(E)|^2. \hspace{1cm} (2.9)$$

(ii) It is easy to see that

$$\text{Re } \text{tr}(\Omega^*T^{-1}EQT) = \text{Re} \sum_{i=1}^p \text{tr}(\Omega_i^*T^{-1}_i\hat{E}_{ii}T_i) = \text{Re} \sum_{i=1}^p \sum_{j=2}^{m_i} \varepsilon(T_i^{-1}\hat{E}_{ii}T_i)_{j,j-1}.$$
Because \((T_i^{-1}\hat{E}_{ii}T_i)_{j-1,j} = \varepsilon(\hat{E}_{ii})_{j-1,j}\) for all \(i = 1, \ldots, p\) and \(j = 2, \ldots, m_i\), we obtain

\[
\text{Re tr}(\Omega^*T^{-1}E_QT) = \text{Re} \sum_{i=1}^{p} \sum_{j=2}^{m_i} \varepsilon^2(\hat{E}_{ii})_{j-1,j}
\leq \varepsilon^2 \sum_{i=1}^{p} \sum_{j=2}^{m_i} |(\hat{E}_{ii})_{j-1,j}|^{\frac{1}{2}}
\leq \varepsilon^2 \sqrt{n-p} \left( \sum_{i=1}^{p} \sum_{j=2}^{m_i} |(\hat{E}_{ii})_{j-1,j}|^{2} \right)^{\frac{1}{2}}
\leq \varepsilon^2 \sqrt{n-p} \left( \sum_{i=1}^{p} \|U(\hat{E}_{ii})\|^2_F \right)^{\frac{1}{2}}
\leq \epsilon^2 \sqrt{n-p} \|U(E_Q)\|_F.
\]

Due to \(\|U(E_Q)\|_F \leq \delta(E_Q)\) (see (2.1)), it follows that

\[
\text{Re tr}(\Omega^*T^{-1}E_QT) \leq \epsilon^2 \sqrt{n-p} \delta(E_Q). \tag{2.10}
\]

(iii) Furthermore, we can easily see that

\[
\|\Omega\|^2_F = (n-p)\epsilon^2. \tag{2.11}
\]

Combining (2.8), (2.9), (2.10), and (2.11), we can derive the inequality (2.7) immediately.

\[\square\]

3 Main results

In this section, we develop some new upper bounds to characterize the variation of the spectrum of an arbitrary matrix based on the estimate (2.7).

3.1 Complex eigenvalues case

In this subsection, we consider the (general) case that the eigenvalues of \(A \in \mathbb{C}^{n \times n}\) are complex.

Using (2.2) and (2.7), we can obtain the following estimate for \(D_2\), which is sharper than (1.9).

**Theorem 3.1.** Let \(A \in \mathbb{C}^{n \times n}\) have the decomposition (2.5), and let \(\hat{A} = A + E\), where \(E \in \mathbb{C}^{n \times n}\) is a perturbation. Assume that the spectra of \(A\) and \(\hat{A}\) are \(\{\lambda_i\}_{i=1}^{n}\) and \(\{\tilde{\lambda}_i\}_{i=1}^{n}\), respectively. Then there exists a permutation \(\pi\) of \(\{1, \ldots, n\}\) such that

\[
D_2 \leq \begin{cases} 
\sqrt{n(n-p) + 2\sqrt{n-p} \delta(E_Q) + \frac{\delta(E_Q)^2}{\|E_Q\|^2_F}} \|E_Q\|^2_F + \frac{1}{n} \|\text{tr}(E)\|^2, & \text{if } \|E_Q\|_F < 1, \\
\sqrt{n(\sqrt{n-p} + \delta(E_Q))^2 + \frac{1}{n} \|\text{tr}(E)\|^2}, & \text{if } \|E_Q\|_F \geq 1.
\end{cases} \tag{3.1}
\]

**Proof.** Observe that \(\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)\) is normal and the spectrum of \(T^{-1}Q^{-1}\hat{A}QT\) is \(\{\tilde{\lambda}_i\}_{i=1}^{n}\). In view of (2.2), we obtain

\[
D_2 \leq \sqrt{n\|T^{-1}Q^{-1}\hat{A}QT - \Lambda\|^2_F - \frac{n-1}{n} \|\text{tr}(E)\|^2}.
\]
By (2.7), we have
\[ D_2 \leq \sqrt{n\Phi(\varepsilon) - \frac{n-1}{n} |\text{tr}(E)|^2}. \]
Set
\[ \varepsilon = \begin{cases} \frac{1}{\|E_Q\|_F}, & \text{if } \|E_Q\|_F < 1, \\ 1, & \text{if } \|E_Q\|_F \geq 1. \end{cases} \]
The estimate (3.1) then follows immediately by using the following results:
\[ \Phi\left(\frac{1}{\|E_Q\|_F}\right) = \left(n - p + 2\sqrt{n - p}\delta(E_Q) + \frac{\delta(E_Q)^2}{\|E_Q\|_F^2}\right)\|E_Q\|_F^2 + \frac{1}{n} |\text{tr}(E)|^2, \]
\[ \Phi(1) = \left(\sqrt{n - p} + \delta(E_Q)\right)^2 + \frac{1}{n} |\text{tr}(E)|^2. \]
This completes the proof.

Remark 3.1. If \( \|E_Q\|_F < 1 \), then (3.1) reads
\[ D_2 \leq \sqrt{n\left(n - p + 2\sqrt{n - p}\delta(E_Q) + 1\right)\|E_Q\|_F^2 + \frac{1}{n} |\text{tr}(E)|^2}. \]
Due to
\[ \frac{\delta(E_Q)^2}{\|E_Q\|_F^2}\|E_Q\|_F^2 + \frac{1}{n} |\text{tr}(E)|^2 = \frac{n\|E_Q\|_F^2 - |\text{tr}(E)|^2}{\|E_Q\|_F^2}\|E_Q\|_F^2 + \frac{1}{n} |\text{tr}(E)|^2 \leq n\|E_Q\|_F^2, \]
it follows that
\[ D_2 \leq \sqrt{n\left(n - p + 2\sqrt{n - p}\delta(E_Q) + 1\right)}\|E_Q\|_F \leq \sqrt{n\left(\sqrt{n - p} + 1\right)}\|E_Q\|_F^{1/2}, \]
which is the first estimate in (1.9). On the other hand, if \( \|E_Q\|_F \geq 1 \), then (3.1) reads
\[ D_2 \leq \sqrt{n\left(\sqrt{n - p} + \delta(E_Q)\right)^2 + \frac{1}{n} |\text{tr}(E)|^2}. \]
Then
\[ D_2 \leq \sqrt{n\left(n - p + 2\sqrt{n - p}\delta(E_Q) + \|E_Q\|_F^2\right)} \leq \sqrt{n\left(\sqrt{n - p} + 1\right)}\|E_Q\|_F, \]
which is the second estimate in (1.9). Therefore, the estimate (3.1) is sharper than (1.9).

The following two estimates for \( D_2 \) are based on the different constraints for \( E_Q \).

**Theorem 3.2.** Under the assumptions of Theorem 3.1, there exists a permutation \( \pi \) of \( \{1, \ldots, n\} \) such that
\[
D_2 \leq \begin{cases} \sqrt{n\left(n - p + 2\sqrt{n - p}\delta(E_Q) + 1\right)}\delta(E_Q)\frac{1}{\|E_Q\|_F} + \frac{1}{n} |\text{tr}(E)|^2, & \text{if } \delta(E_Q) < 1, \\ \sqrt{n\left(\sqrt{n - p} + \delta(E_Q)\right)^2 + \frac{1}{n} |\text{tr}(E)|^2}, & \text{if } \delta(E_Q) \geq 1. \end{cases}
\]
Proof. Set
\[ \varepsilon = \begin{cases} \delta(E_Q)^{\frac{1}{m}}, & \text{if } \delta(E_Q) < 1, \\ 1, & \text{if } \delta(E_Q) \geq 1. \end{cases} \]

Direct computation yields
\[ \Phi\left( \delta(E_Q)^{\frac{1}{m}} \right) = (n - p + 2\sqrt{n-p} \delta(E_Q) + 1) \delta(E_Q)^{\frac{2}{m}} + \frac{1}{n} |\text{tr}(E)|^2. \]

Using the similar argument as in Theorem 3.1, we can derive the estimate (3.2). \qed

**Theorem 3.3.** Under the assumptions of Theorem 3.1, there exists a permutation \( \pi \) of \( \{1, \ldots, n\} \) such that
\[ \mathbb{D}_2 \leq \begin{cases} \sqrt{mn \left( \frac{n-p+2\sqrt{n-p} \delta(E_Q)}{m-1} \right)^{\frac{1}{m}}} \delta(E_Q)^{\frac{2}{m}} + \frac{1}{n} |\text{tr}(E)|^2, & \text{if } C_1 \text{ holds}, \\ \sqrt{n \left( \frac{\sqrt{n-p} + \delta(E_Q)}{2} \right)^{\frac{2}{m}}} + \frac{1}{n} |\text{tr}(E)|^2, & \text{if } C_2 \text{ holds}, \end{cases} \tag{3.3} \]

where the conditions \( C_1 \) and \( C_2 \) are given by
\[ C_1 : n - p + 2\sqrt{n-p} \delta(E_Q) > (m-1) \delta(E_Q)^2, \]
\[ C_2 : n - p + 2\sqrt{n-p} \delta(E_Q) \leq (m-1) \delta(E_Q)^2. \]

Proof. We first note that \( A \) is diagonalizable if and only if \( n = p \), or, equivalently, \( m = 1 \).

(i) If \( A \) is diagonalizable, then \( T = I_n, n = p, \) and \( m = 1 \). In this case, (2.7) reduces to
\[ \|Q^{-1}A - \Lambda\|_F^2 \leq \|E_Q\|_F^2. \]

Using (2.2), we obtain
\[ \mathbb{D}_2 \leq \sqrt{n \|Q^{-1}A - \Lambda\|_F^2} - \frac{n-1}{n} |\text{tr}(E)|^2 \leq \sqrt{n \|E_Q\|_F^2} - \frac{n-1}{n} |\text{tr}(E)|^2. \tag{3.4} \]

(ii) If \( A \) is not diagonalizable, then \( n > p \) and \( m > 1 \). Straightforward calculation yields
\[ \Phi'(\varepsilon) = 2\varepsilon \left( n - p + 2\sqrt{n-p} \delta(E_Q) - \frac{(m-1) \delta(E_Q)^2}{\varepsilon^{2m}} \right), \]

where \( \Phi'(\varepsilon) \) denotes the derivative of \( \Phi(\varepsilon) \) with respect to \( \varepsilon \). Evidently,
\[ \Phi'(\varepsilon) > 0, \quad \text{if } \varepsilon > \left( \frac{(m-1) \delta(E_Q)^2}{n-p+2\sqrt{n-p} \delta(E_Q)} \right)^{\frac{1}{2m}}, \]
\[ \Phi'(\varepsilon) < 0, \quad \text{if } 0 < \varepsilon < \left( \frac{(m-1) \delta(E_Q)^2}{n-p+2\sqrt{n-p} \delta(E_Q)} \right)^{\frac{1}{2m}}. \]

Then we set
\[ \varepsilon = \begin{cases} \left( \frac{(m-1) \delta(E_Q)^2}{n-p+2\sqrt{n-p} \delta(E_Q)} \right)^{\frac{1}{2m}}, & \text{if } n - p + 2\sqrt{n-p} \delta(E_Q) > (m-1) \delta(E_Q)^2, \\ 1, & \text{if } n - p + 2\sqrt{n-p} \delta(E_Q) \leq (m-1) \delta(E_Q)^2. \end{cases} \]
the estimates (3.1), (3.2), and (3.3) all reduce to

\[ \Phi \left( \frac{(m - 1)\delta(EQ)^2}{n - p + 2\sqrt{n - p}\delta(EQ)} \right)^{\frac{1}{m}} = m \left( \frac{n - p + 2\sqrt{n - p}\delta(EQ)}{m - 1} \right)^{1 - \frac{1}{m}} \delta(EQ)^{\frac{2}{m}} + \frac{1}{n} |\text{tr}(E)|^2. \]

The rest of the proof is similar to Theorem 3.1.

\[ \Box \]

**Remark 3.2.** We remark that (3.3) has contained the diagonalizable case (i). More specifically, if \( A \) is diagonalizable, then the condition \( C_2 \) is satisfied. From (3.3), we have that

\[ D_2 \leq \sqrt{n \delta(E)^2 + \frac{1}{n} |\text{tr}(E)|^2}, \]

which is consistent with (3.4).

**Remark 3.3.** If \( A \in \mathbb{C}^{n \times n} \) is normal, then \( Q \) can be chosen as a unitary matrix. In this case, the estimates (3.1), (3.2), and (3.3) all reduce to

\[ D_2 \leq \sqrt{n \delta(E)^2 + \frac{1}{n} |\text{tr}(E)|^2}, \]

which is exactly (2.2). In other words, these results have generalized the estimate (2.2).

Analogously, using (2.3), we can derive the following three estimates for \( D_2 \).

**Theorem 3.4.** Under the assumptions of Theorem 3.1, there exists a permutation \( \pi \) of \( \{1, \ldots, n\} \) such that

\[ D_2 \leq \begin{cases} \sqrt{s_1 \left( n - p + 2\sqrt{n - p}\delta(EQ) + \frac{\delta(EQ)^2}{\|EQ\|_F} \right)^2 \|EQ\|_F^2 + \frac{1}{n} |\text{tr}(E)|^2}, & \text{if } \|EQ\|_F < 1, \\ \sqrt{s_2 \left( \sqrt{n - p} + \delta(EQ) \right)^2 + \frac{1}{n} |\text{tr}(E)|^2}, & \text{if } \|EQ\|_F \geq 1, \end{cases} \]

where \( s_1 = n + 1 - s(T^{-1}Q^{-1}\tilde{AQ}T) \) with \( \varepsilon = \|EQ\|_F^\frac{1}{m} \) and \( s_2 = n + 1 - s(Q^{-1}\tilde{AQ}) \).

**Remark 3.4.** If \( \|EQ\|_F < 1 \), then (3.5) reads

\[ D_2 \leq \sqrt{s_1 \left( n - p + 2\sqrt{n - p}\delta(EQ) + \frac{\delta(EQ)^2}{\|EQ\|_F} \right)^2 \|EQ\|_F^2 + \frac{1}{n} |\text{tr}(E)|^2}. \]

Because

\[ s_1 \frac{\delta(EQ)^2}{\|EQ\|_F} \|EQ\|_F^\frac{2}{m} + \frac{1}{n} |\text{tr}(E)|^2 \leq s_1 \|EQ\|_F^\frac{2}{m} \left( \frac{s_1 - 1}{n} \right) |\text{tr}(E)|^2 \leq s_1 \|EQ\|_F^\frac{2}{m}, \]

it follows that

\[ D_2 \leq \sqrt{s_1 \left( n - p + 2\sqrt{n - p}\|EQ\|_F + 1 \right) \|EQ\|_F^\frac{1}{m}}, \]

which is the first estimate in (1.10). On the other hand, if \( \|EQ\|_F \geq 1 \), then (3.5) reads

\[ D_2 \leq \sqrt{s_2 \left( \sqrt{n - p} + \delta(EQ) \right)^2 + \frac{1}{n} |\text{tr}(E)|^2}. \]
Since
\[ n - p + 2\sqrt{n - p}\delta(E_Q) \leq (n - p + 2\sqrt{n - p})\|E_Q\|_F \quad \text{and} \quad s_2\delta(E_Q)^2 + \frac{1}{n}\|\text{tr}(E)\|^2 \leq s_2\|E_Q\|_F^2, \]
we obtain
\[ D_2 \leq \sqrt{s_2(n - p + 2\sqrt{n - p} + \|E_Q\|_F)} \|E_Q\|_F^{\frac{1}{2}}, \]
which is the second estimate in (1.10). In conclusion, the estimate (3.5) is sharper than (1.10).

**Theorem 3.5.** Under the assumptions of Theorem 3.1, there exists a permutation \( \pi \) of \( \{1, \ldots, n\} \) such that
\[
D_2 \leq \begin{cases} 
\sqrt{s_3(n - p + 2\sqrt{n - p}\delta(E_Q) + 1)\delta(E_Q) \frac{E_Q}{m} + \frac{1}{n}\|\text{tr}(E)\|^2}, & \text{if } \delta(E_Q) < 1, \\
\sqrt{s_2(\sqrt{n - p} + \delta(E_Q))^2 + \frac{1}{n}\|\text{tr}(E)\|^2}, & \text{if } \delta(E_Q) \geq 1, 
\end{cases} \tag{3.6}
\]
where \( s_3 = n + 1 - s(T^{-1}Q^{-1}\tilde{A}QT) \) with \( \varepsilon = \delta(E_Q)^\frac{1}{m} \) and \( s_2 = n + 1 - s(Q^{-1}\tilde{A}Q) \).

**Theorem 3.6.** Under the assumptions of Theorem 3.1, there exists a permutation \( \pi \) of \( \{1, \ldots, n\} \) such that
\[
D_2 \leq \begin{cases} 
\sqrt{\frac{m + 2s_2\sqrt{n - p}\delta(E_Q)}{m - 1}\delta(E_Q) \frac{E_Q}{m} + \frac{1}{n}\|\text{tr}(E)\|^2}, & \text{if } C_1 \text{ holds}, \\
\sqrt{s_2(\sqrt{n - p} + \delta(E_Q))^2 + \frac{1}{n}\|\text{tr}(E)\|^2}, & \text{if } C_2 \text{ holds,} 
\end{cases} \tag{3.7}
\]
where \( s_4 = n + 1 - s(T^{-1}Q^{-1}\tilde{A}QT) \) with \( \varepsilon = \left( \frac{(m - 1)\delta(E_Q)^2}{n - p + 2\sqrt{n - p}\delta(E_Q)} \right)^\frac{1}{m} \), \( s_2 = n + 1 - s(Q^{-1}\tilde{A}Q) \), and the conditions \( C_1 \) and \( C_2 \) are the same as in Theorem 3.3.

**Remark 3.5.** If \( A \in \mathbb{C}^{n \times n} \) is normal, then (3.5), (3.6), and (3.7) will reduce to
\[ D_2 \leq \sqrt{(n + 1 - s(\tilde{A}))\delta(E)^2 + \frac{1}{n}\|\text{tr}(E)\|^2}, \]
which is exactly the estimate (2.3).

**Example 3.1.** For any \( A \in \mathbb{C}^{n \times n} \), taking \( E = tI_n \) with \( 0 < |t| < \frac{1}{\sqrt{n}} \), we have that \( D_2 = \sqrt{|t|} \) for any permutation \( \pi \) of \( \{1, \ldots, n\} \). In this case, the aforementioned upper bounds for \( D_2 \) are listed as follows:

| Estimate | Upper bound | Estimate | Upper bound |
|----------|-------------|----------|-------------|
| (1.9)    | \( (\sqrt{n - p} + 1)n^{\frac{1}{m}}|t|^\frac{1}{m} \) | (1.10)   | \( \sqrt{s_1(n - p + 1 + 2|t|\sqrt{n^2 - np})n^{\frac{1}{m}}|t|^\frac{1}{m}} \) |
| (3.1)    | \( (n - p)n^{\frac{1}{m}}|t|^\frac{2}{m} + n|t|^2 \) | (3.5)    | \( \sqrt{s_1(n - p)n^{\frac{1}{m}}|t|^\frac{2}{m} + n|t|^2} \) |
| (3.2)    | \( n|t| \) | (3.6)    | \( \sqrt{n|t|} \) |
| (3.3)    | \( \sqrt{n|t|} \) | (3.7)    | \( \sqrt{n|t|} \) |
3.2 Real eigenvalues case

If the eigenvalues of $A \in \mathbb{C}^{n \times n}$ are all real, then we can get the following more accurate estimates by applying (2.4).

**Theorem 3.7.** Let $A \in \mathbb{C}^{n \times n}$ have the decomposition (2.5), and let $\tilde{A} = A + E$, where $E \in \mathbb{C}^{n \times n}$ is a perturbation. Let $\{\lambda_i\}_{i=1}^n$ and $\{\tilde{\lambda}_i\}_{i=1}^n$ be the spectra of $A$ and $\tilde{A}$, respectively. If $\lambda_i$ are all real, then there exists a permutation $\sigma$ of $\{1, \ldots, n\}$ such that

$$D_2 \leq \begin{cases} \sqrt{2 \left( n - p + 2 \sqrt{n - p} \delta(E_Q) + \frac{\delta(E_Q)^2}{\|E_Q\|_F^2} \right) + \frac{1}{n} |\text{tr}(E)|^2}, & \text{if } \|E_Q\|_F < 1, \\ \sqrt{2 \left( \sqrt{n - p} + \delta(E_Q) \right)^2 + \frac{1}{n} |\text{tr}(E)|^2}, & \text{if } \|E_Q\|_F \geq 1. \end{cases}$$

**Theorem 3.8.** Under the assumptions of Theorem 3.7, there exists a permutation $\sigma$ of $\{1, \ldots, n\}$ such that

$$D_2 \leq \begin{cases} \sqrt{2 \left( n - p + 2 \sqrt{n - p} \delta(E_Q) + 1 \right) \delta(E_Q) + \frac{1}{n} |\text{tr}(E)|^2}, & \text{if } \delta(E_Q) < 1, \\ \sqrt{2 \left( \sqrt{n - p} + \delta(E_Q) \right)^2 + \frac{1}{n} |\text{tr}(E)|^2}, & \text{if } \delta(E_Q) \geq 1. \end{cases}$$

**Theorem 3.9.** Under the assumptions of Theorem 3.7, there exists a permutation $\sigma$ of $\{1, \ldots, n\}$ such that

$$D_2 \leq \begin{cases} \sqrt{2m \left( \frac{n - p + 2 \sqrt{n - p} \delta(E_Q)}{m - 1} \right)^{1 - \frac{1}{m}}} \delta(E_Q) + \frac{1}{n} |\text{tr}(E)|^2, & \text{if } C_1 \text{ holds,} \\ \sqrt{2 \left( \sqrt{n - p} + \delta(E_Q) \right)^2 + \frac{1}{n} |\text{tr}(E)|^2}, & \text{if } C_2 \text{ holds}, \end{cases}$$

where the conditions $C_1$ and $C_2$ are the same as in Theorem 3.3.

**Remark 3.6.** If $A \in \mathbb{C}^{n \times n}$ is Hermitian, then $Q$ can be chosen as a unitary matrix. In this case, the estimates (3.8), (3.9), and (3.10) all reduce to (2.4). That is, these results have generalized the estimate (2.4).

**Remark 3.7.** Define

$$\kappa_2(Q) := \|Q^{-1}\|_2 \|Q\|_2 \quad \text{and} \quad D_\infty := \max_{1 \leq i \leq n} |\tilde{x}_{\sigma(i)} - \lambda_i|.$$ Because

$$\|E_Q\|_F \leq \min \left\{ \sqrt{\text{rank}(E)} \|E_Q\|_2, \kappa_2(Q) \|E\|_F \right\},$$

we can derive some deductive estimates for $D_2$. In addition, using $D_\infty \leq D_2$, we can obtain the corresponding estimates for $D_\infty$.

4 Conclusions

In this paper, we have developed some novel perturbation bounds for the spectrum of an arbitrary matrix, which include the counterparts of the existing ones. Theoretical analysis shows that these counterparts are sharper than the existing estimates. Furthermore, our results have generalized some perturbation bounds for the spectrum of a normal matrix.
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References

[1] E. Bänsch, P. Morin, and R. H. Nochetto. Preconditioning a class of fourth order problems by operator splitting. *Numer. Math.*, 118:197–228, 2011.

[2] R. Bhatia, F. Kittaneh, and R.-C. Li. Some inequalities for commutators and an application to spectral variation. II. *Linear Multilinear Algebra*, 43:207–219, 1997.

[3] Y. Chen, X. Peng, and W. Li. Relative perturbation bounds for eigenpairs of diagonalizable matrices. *BIT Numer Math*, to appear.

[4] S. C. Eisenstat and I. C. F. Ipsen. Three absolute perturbation bounds for matrix eigenvalues imply relative bounds. *SIAM J. Matrix Anal. Appl.*, 20:149–158, 1998.

[5] A. Galántai and C. J. Hegedűs. Perturbation bounds for polynomials. *Numer. Math.*, 109:77–100, 2008.

[6] A. J. Hoffman and H. W. Wielandt. The variation of the spectrum of a normal matrix. *Duke Math. J.*, 20:37–39, 1953.

[7] I. C. F. Ipsen. Relative perturbation results for matrix eigenvalues and singular values. *Acta Numer.*, 7:151–201, 1998.

[8] I. C. F. Ipsen. A note on unifying absolute and relative perturbation bounds. *Linear Algebra Appl.*, 358:239–253, 2003.

[9] W. Kahan. Spectra of nearly Hermitian matrices. *Proc. Amer. Math. Soc.*, 48:11–17, 1975.

[10] R.-C. Li. Relative perturbation theory: I. eigenvalue and singular value variations. *SIAM J. Matrix Anal. Appl.*, 19:956–982, 1998.

[11] W. Li and J.-X. Chen. The eigenvalue perturbation bound for arbitrary matrices. *J. Comput. Math.*, 24:141–148, 2006.

[12] W. Li and W. Sun. The perturbation bounds for eigenvalues of normal matrices. *Numer. Linear Algebra Appl.*, 12:89–94, 2005.

[13] W. Li and W. Sun. Combined perturbation bounds: I. eigensystems and singular value decompositions. *SIAM J. Matrix Anal. Appl.*, 29:643–655, 2007.
[14] W. Li and S.-W. Vong. On the variation of the spectrum of a Hermitian matrix. *Appl. Math. Lett.*, 65:70–76, 2017.

[15] R. Rehman and I. C. F. Ipsen. Computing characteristic polynomials from eigenvalues. *SIAM J. Matrix Anal. Appl.*, 32:90–114, 2011.

[16] Y. Song. A note on the variation of the spectrum of an arbitrary matrix. *Linear Algebra Appl.*, 342:41–46, 2002.

[17] J.-G. Sun. On the variation of the spectrum of a normal matrix. *Linear Algebra Appl.*, 246:215–223, 1996.

[18] X. Xu and C.-S. Zhang. New perturbation bounds for the spectrum of a normal matrix. *J. Math. Anal. Appl.*, 455:1937–1955, 2017.