Improvement in accuracy of point seismic sources wavefields numerical modeling

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Abstract. A new scheme of wave fields modelling from point sources in a homogeneous isotropic elastic medium is developed. To set the source the developed technique is based on the use of additional components of the finite-difference grid and the use of all the stress tensor components in the nodes. The results of the scheme employment in the two-dimensional homogeneous isotropic medium are presented; they demonstrate the numerical solutions’ improvement for the source being under consideration in the article.

1. Introduction

Numerical modelling of seismic wave propagation is necessary to address a number of seismic survey problems [1]. There are various approaches of modelling. The first one is the application of analytical approaches [2] (e.g. the Haskell-Thompson matrix method) is limited by horizontally layered model of the mediums. The problems of numerical methods for solving elastic equations in the frequency domain are the increased requirements to the memory and the complexity of algorithms paralleling [3]. The finite-difference method in the time domain is universal, easy to use in the processes of implementing and paralleling [4, 5, 6].

The so-called standard staggered grid scheme is also widely-used [4]. The storage of the various components of stresses and displacements in different nodes of stacks makes it possible to reduce the memory requirements. Nevertheless, this approach has some drawbacks - due to the fact that normal and shearing stresses are set in different nodes of the grid there may appear some problems with the accuracy of the nonisotropic seismic wave sources setting, determined by a seismic moments tensor [7, 8].

This work is aimed at improving the accuracy of the numerical solutions with the help of the additional grid nodes and with the increment of all stress tensor components. In this paper we deal with the two-dimensional homogeneous isotropic medium.

The results can be employed if there are sources of seismic wave appearing as a result of cracks’ disclosure and closure and the shifts along the cracks in case of the stress relaxation in the rock mass. It is a well-known fact that [9], such displacement discontinuities can be described by introducing the equivalent additives to the stress (seismic moment tensor). Modeling of this type of sources particularly important to deal with microseismic monitoring of geomechanical processes [10].

2. The finite difference scheme at the staggered grids

The modelling of seismic wave propagation is carried out in the framework of the linear elastic theory. In the case of two dimensions the system of equations is:
\[ \rho \frac{\partial^2 v_x}{\partial t^2} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z}, \]
\[ \rho \frac{\partial^2 v_z}{\partial t^2} = \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z}, \]
\[ \sigma_{xx} = (\lambda + 2\mu) \frac{\partial v_x}{\partial x} + \lambda \frac{\partial v_z}{\partial z}, \]
\[ \sigma_{zz} = (\lambda + 2\mu) \frac{\partial v_z}{\partial z} + \lambda \frac{\partial v_x}{\partial x}, \]
\[ \sigma_{xz} = \mu \left( \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right). \]

In which \( v_x, v_z \) - displacements, \( \sigma_{xx}, \sigma_{zz}, \sigma_{xz} \) - stresses, \( \rho \) - medium density, \( \lambda, \mu \) - the Lamé parameters.

The system of equations can be solved by the finite difference method using the scheme at the staggered grids [4]. Figure 1 shows the pattern of the second order finite-difference scheme at the staggered grids for the two-dimensional case. As it is shown in the figure, the components \( v_x, \sigma_{xx}, \sigma_{zz}, \sigma_{xz} \) are arranged in the half-integer grid nodes, and \( v_x \) - in the integral grid nodes. Calculation of the displacements components is performed on the entire time steps, the calculation of the stresses is performed on the half-integer time steps.

![Figure 1. The pattern of the finite difference scheme of the second order at the staggered grids.](image)

The point sources are perceived as elastic waves sources, determined by a symmetric seismic moment tensor \( M_{pq} \) [9]. The components of this tensor correspond to a double pair of forces in the \( p \) direction with the moment and with the arm in the \( q \) direction. This work [8] shows that in case of the finite difference scheme it is convenient to set the couples of forces by the stress tensor. Thus, the source of the shift type with a fracture plane parallel to \( xy \) and with shifts to \( x \) directions (see Figure 2), the seismic moment tensor will have only two non-zero components \( M_{13} \) and \( M_{31} \). To set the source with the seismic moment tensor \( M \) it is necessary to bring the increment to the stress component \( S_{13} \) in the corresponding grid node.
3. Description of the source setting scheme

The proposed method is based on finding the numerical solution in the form of a sum of the fields from the nodes, in which are caused the increments of the specified stress components are produced. The procedure involves the following steps:

1. The calculation of the fields set $U^n_S$ ($U^n_S$ - a field from the node with number $n$, with the stress increment $S$. For two-dimensional case illustrated in Figure 1 (a), $n$ goes from 1 to 9, $S$ takes values $S_{11}, S_{22}$ or $S_{12}$).

2. The construction $U_{sum} = \sum_{n=1}^{N} \left( \sum_{S} w^n_S U^n_S \right)$, in which $w^n_S$ - the weight with which $U^n_S$ is included in the solution.

3. Construction of the analytical solution for the selected source $U_{an}$.

4. Construction and minimization of the functional $\| U_{an} - U_{sum} \|^2$.

After the implementation of the above steps, we obtain a set of weights $w^n_S$ which minimizes $\| U_{an} - U_{sum} \|^2$, i.e. we obtain the numerical solution the closest to the analytical solution.

4. Example of the scheme implementation

Let us look at the example of the proposed scheme to increase the accuracy of the numerical solution, in which it will be used by 9 neighboring nodes of the grid (Figure 3).
As an example, we shall take the source, like the separation crack, which is set by a non-zero $M_{11}$. Let $U$ be the first component of the displacement field. Then, the analytical solution for the first displacement component $\mathbf{an}_U$ in the frequency domain has the following form:

$$
\mathbf{an}_U \mathbf{G} = \mathbf{an}_M \mathbf{G},
$$

in which Green’s function

$$
G_{11}(x, y) = \frac{1}{i8\rho} (A - 2B \gamma_1 \gamma_1'),
\quad A = \frac{H_0^{(2)}(q r)}{\alpha^2} + \frac{H_0^{(2)}(k r)}{\beta^2},
$$

$$
B = \frac{H_2^{(2)}(q r)}{\alpha^2} - \frac{H_2^{(2)}(k r)}{\beta^2},
\quad q = \omega / \alpha, \quad k = \omega / \beta, \quad r = \sqrt{x^2 + y^2}, \quad \gamma_1 = x / \sqrt{x^2 + y^2}, \quad H_n^{(2)} - \text{the second order Hankel function, } \alpha \text{ and } \beta \text{ longitudinal and shear waves velocities.}
$$

Numerical solution for this source will be sought in the form of a sum:

$$
U_{\text{sum}} = \sum_{n=1}^{9} \left( \sum_{S=S_1,S_1,S_2,S_2} w_S^n U_S^n \right)
$$

in which $U_S^n$ - the field obtained numerically from the node number $n$, wherein the stress $S$ increment is performed; index node runs the grid nodes close to the source 1,2..9 (Figure 1.); index $S$ runs the stresses with the increments $(S_{1,1}, S_{1,2}, S_{2,2})$; $w_S^n$ - weight coefficients with which field $U_S^n$ is included in $U_{\text{sum}}$.

The aim is to find the weights $w_S^n$ with which the norm $\|U_{an} - U_{\text{sum}}\|^2$ takes the minimum value. If we consider $L_2$ being the norm, then $\|U_{an} - U_{\text{sum}}\|^2$ is a second degree polynomial from the variables $w_S^n$, the number of which is equal to $9 \times S$. The polynomial type is:

$$
\|U_{an} - U_{\text{sum}}\|^2 = \sum_{n=1}^{9} \sum_{S=S_1,S_1,S_2,S_2} \left[ \sum_{i,j} (U_{1,i}^n)(U_{2,j}^n) \right] w_S^n w_S^n - 2\sum_{n=1}^{9} \sum_{S=S_1,S_1,S_2,S_2} \left[ \sum_{i,j} (U_{S,i}^n)(U_{an,j}) \right] w_S^n + \sum_{i,j} (U_{an,i}^n)
$$

in which $n_1, n_2$ run the nodes 1..9, index $S$ corresponds to the stress components, which get the increments; $i,j$ run all the computational grid nodes. To find the minimum point of the resulting
polynomial we shall differentiate it in accordance with each variable \( \frac{\partial}{\partial w_n} \|U_{an} - U_{sum}\|_2^2 = 0 \). So we obtain \( 9 \times S \) of the linear equations to find \( 9 \times S \) of the unknown \( w_S^0 \). We obtain the weights after finding the solutions to the linear equations system. These weights, the medium from the node \( n \) with the increment and the stress components with the \( S \) index are included in the numerical solution.

As a result, after the implementation of the scheme the following results are obtained. As we are searching \( U_{sum} \) through 9 nodes with the increment \( S_{11} \), the following weight coefficients are obtained: at the node 1, 3, 7, 9 – 0.6056; the node 2, 8 – 0.9366; 4, 6 – 0.6510; 5 – 1.9807, i.e. the received values are symmetrical to the central node. In this case, the relative error at the norm \( L_2 \) is 0.1333. When the source is set through one node the relative error is 0.1402. When we are searching for \( U_{sum} \) through 9 nodes and the sources not only from the increment \( S_{11} \), but also from \( S_{12} \) and \( S_{22} \) the relative error is reduced to 0.0746. Table 1 shows the relative errors in the various norms and with different methods of searching for the field \( U_{sum} \), which are calculated according to the formula

\[ e_r = \frac{\|U_{sum} - U_{an}\|_2}{\|U_{an}\|_2}. \]

**Table 1. Relative errors at various norms.**

| Method of the source setting                                           | Norm \( L_1 \) | Norm \( L_2 \) | Norm \( L_{\infty} \) |
|------------------------------------------------------------------------|-----------------|-----------------|---------------------|
| Through one node                                                       | 0.16            | 0.1402          | 0.1359              |
| Through 9 nodes (the sum of 9 fields from the sources with the component \( S_{11} \) increment) | 0.1545          | 0.133            | 0.1424              |
| Through 9 nodes from the sources with the components \( S_{11}, S_{22}, S_{12} \) increment | 0.0894          | 0.0746          | 0.077               |

The pictures below the diagrams comparing the analytical (red) and the numerical (blue) solutions of (blue) for the given source. In Figure 4, the source is set through one node with the component increment \( S_{11} \). In Figure 5 the source is set through 9 nodes with all the components increment. In the second case the solutions match is the best.

**Figure 4.** Displacement field \( v_x \) from the source. Blue – analytical solution, red – numerical solution determined through the setting of the source in 1 node.
5. Conclusion.
The results show that the proposed method of work helps to increase the accuracy of modelling of the wave fields from the point sources. For the further improvement of the accuracy it is possible to use a larger number of grid nodes to set the sources. Also it is possible to apply the same method for calculation of the weight coefficients and to widen it for the three-dimensional case.

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