Universal breaking point asymptotic for energy spectrum of Riemann waves in weakly nonlinear non-dispersive media

Elena Kartashova\textsuperscript{a}, Efim Pelinovsky\textsuperscript{a,b}
\textsuperscript{a}Johannes Kepler University, Linz, Austria and
\textsuperscript{b}Institute of Applied Physics, Nizhny Novgorod, Russia

In this Letter we study the form of the energy spectrum of Riemann waves in weakly nonlinear non-dispersive media. For the simple wave equation with quadratic and cubic nonlinearity we demonstrate that the deformation of a Riemann wave over time yields an exponential energy spectrum which turns into a power law asymptotic for high wave numbers with a slope of approximately $-8/3$ at the last stage of evolution before the time of breaking $T$. We argue, that this is the universal asymptotic behavior of Riemann waves in any nonlinear non-dispersive medium at the point of breaking. We also demonstrate that additional weak dispersion or dissipation terms (yielding the Korteweg-de Vries or Burgers equation respectively) do not change the universal asymptotic appearing at time $T$ though no breaking occurs. Moreover, this universal asymptotic stays visible in the Fourier spectrum until times of order of $2T$ and more, for a wide range of smooth initial wave shapes. The results reported in this Letter may be used in various non-dispersive media, e.g. magneto-hydro dynamics, physical oceanography, nonlinear acoustics.

I. INTRODUCTION

Weakly nonlinear wave systems fall into two categories - dispersive and non-dispersive. In physical terms, for a dispersive wave system group velocity changes with frequency, for a non-dispersive wave system it does not. Writing the wave in the form of a sine wave $A e^{i(kx - \omega t)}$ we can introduce the notion of dispersion function $\omega(k)$, $\omega(k)$ being a real function, and define dispersive systems as $\omega_k \neq 0$ and non-dispersive systems as $\omega_k = 0$, [1].

One of the most important characteristics of a wave system, describing the wave field as a whole, is the distribution of energy over scales in Fourier space, i.e. the energy spectrum. For one initially excited dispersive wave with a dispersion function of the form $\omega(k) \sim k^\beta$ it is known, that the energy spectrum evolves from its initially exponential shape $e^\gamma k$, [2, 3], into a power law $k^\beta$ [3, 4], the point of transition from exponential to power law and the power law itself depending on the characteristic time and the dispersion function of the wave system, [5, 6].

The power law "tail" of the energy spectra can be described deterministically, [3] (in the wave systems with narrow frequency band excitation), or statistically, [4] (in the wave systems with distributed initial state). Basic model for describing a dispersive weakly nonlinear system is the nonlinear Schrödinger equation (NLS) and NLS-like equations modified to four and more nonlinear terms, e.g. [7, 8].

The class of the equations of the form

$$u_t + (\alpha u + \beta u^2)u_x + \gamma u_{xxx} = 0 \quad (1)$$

known as Korteweg-de Vries-like models (KdV-like) widely used in soliton theory [9], dispersionless shock waves [10–12] and soliton turbulence, e.g. [13–15]. Famous nonlinear Schrödinger equation (NLS) and NLS-like models can be obtained from (1) for narrow-band processes; accordingly the results obtained in [1, 3, 5–8] are also valid for KdV-models.

The dynamics of a nonlinear non-dispersive wave follow a simple model: one initially excited wave evolves into a shock wave and finally breaks, [1]. The evolution of an unidirectional nonlinear wave before breaking is described by one nonlinear equation, sometimes called the simple wave equation [1, 16], which reads

$$u_t + V(u)u_x = 0, \quad (2)$$

where $u$ is the wave function and $V(u)$ is a nonlinear local speed, $x$ is coordinate and $t$ is time. For example, if we regard surface gravity waves in shallow basin, $u$ is the water elevation and $V(u) = 3\sqrt{g(h + u)} - 2\sqrt{gh}$, where $g$ is acceleration due to gravity, $h$ is unperturbed water depth [17, 18]. A solution of (2) is called a Riemann wave.

Compare (1) and (2) we see that (2) can be regarded as a dispersionless limit of (1) and the nonlinearity $V(u)$ being approximated by two terms of its Taylor expansion: $V(u) = \alpha u + \beta u^2$ (any constant in the Taylor expansion of $V(u)$ can be removed by an appropriate change of variables).

The evolution of the wave field after breaking depends on the interplay of dispersion and dissipation. In essentially dissipative media, the shock wave persists and its amplitude spectrum is known to have the high-frequency asymptotic $k^{-1}$, [23, 28].

The dissipation - at least weak dissipation - can be accounted for by including a viscosity term $u_{xx}$ yielding, for quadratic nonlinear medium, the Burgers equation, [1]. It can then be reduced to the linear diffusion equation by the Hopf transformation and solved explicitly. The Burgers equation with small viscosity has been used for modeling one-dimensional turbulence in [19, 20, 28].
This may be seen the following way.

Solutions of (2) have the property, that any point of a given wave height $u(x)$ moves at constant speed. So we may write

$$u(x,t) = U[x - V(u)t],$$

(3)

where $U(x)$ is the initial wave profile. Now computing

$$u_x(x,t) = \frac{U_x}{(1 + tV_x)},$$

(4)

we see that steepness $u_x$ is growing with time if $V_x < 0$.

In a general hyperbolic model, if a wave reaches infinite steepness, wave breaking will occur. This is called a gradient catastrophe. The time of breaking $T$ may be computed as

$$T = 1/\max(-V_x).$$

(5)

It follows from (4) that the maximum value of wave steepness is proportional to

$$\max(u_x) \sim (T - t)^{-1}$$

(6)

The result of the wave deformation process depends on the local speed $V(u)$ which in turn depends on the form of nonlinearity. This is illustrated in Fig. 1 for quadratic and cubic nonlinearity. In both panels, the initial wave has the shape of a sine

$$U(x) = U_0 \sin(k_0x).$$

(7)

In the quadratic nonlinear media only one shock is formed within a wave period and media breaking occurs for wave height $u/U_0 = 0$ at the moment of time $T = (\alpha k_0 U_0)^{-1}$.

In the cubic nonlinear media two shocks within the wave period are formed, with opposite sign of slope, and breaking occurs for wave heights $u/U_0 = \pm \sqrt{2}/2$ simultaneously at the moment of time $T = (\beta k_0 U_0^3)^{-1}$.

III. FOURIER SPECTRA OF RIEMANN WAVES

The spatial spectrum of a wave $u(x,t)$ is defined as

$$S(k,t) = \int_{-\infty}^{+\infty} u(x,t) \exp(-ikx)dx.$$  

(8)

Next we compute the spatial spectrum of a Riemann wave in explicit form.

Having in mind (4) we perform the change of variables $X = x - V(u)t$ yielding

$$dx = \frac{dX}{1 - tV_x}.$$  

(9)

Now we can rewrite (8) as

$$S(k,t) = \int_{-\infty}^{+\infty} (1+V_X)U(X) \exp[-ik(X+Vt)]dX.$$  

(10)
By simple manipulation, this is transformed to

\[ S(k, t) = (-ik)^{-1} \int_{-\infty}^{+\infty} U_X \exp[-ik(X + tV)]dX. \] (11)

For one initially exited sine wave as given by (7), the integral in (11) may be computed analytically, both for quadratic and cubic nonlinearity, [22].

Using (1) with \( \alpha > 0, \beta = 0 \) we get quadratic nonlinearity, and the wave field may be represented by the Bessel-Fubini series well-known in nonlinear acoustics, [22]:

\[ u(x, t) = U_0 \sum_{n=1}^{\infty} A_n(\tau) \sin[nk_0x], \] (12)

where

\[ A_n(\tau) = \frac{2(-1)^{n+1}}{n\tau} J_n(n\tau), \quad \tau = t/T, \] (13)

with \( T = (\alpha U_0 k_0)^{-1} \) being the time of breaking, and \( J_n(z) \) being the Bessel function.

Fig. 2 depicts the time evolution of the energy spectrum given as values \( A_n^2 \) as a function of wave number \( k_n \) showing clearly: Up to the moment of time \( t = 3T/4 \) the spectrum in semi-logarithmic coordinates (upper panel) has the form of a straight line, which means it is exponential in linear coordinates. As time grows from \( t = 3T/4 \) to the moment of breaking \( t = T \), a power law spectrum is formed, with a slope of 2.67, which is close to \( 8/3 \). Using (1) with \( \alpha = 0, \beta > 0 \) we get cubic nonlinearity. The wave field may be represented by a Bessel-Fubuni-like series, [22], of the form

\[ u(x, t) = U_0 \sum_{n=0}^{\infty} \frac{1}{2n + 1} \{J_n[(n + 1/2)\tau] \sin[(2n + 1)(k_0x - \tau/2)] + J_{n+1}[(n + 1/2)\tau] \cos[(2n + 1)(k_0x - \tau/2)]\}, \] (14)

where again \( \tau = t/T \) and the time of breaking now is given by \( T = (\beta k_0 U_0^2)^{-1} \). Unlike the quadratic case where all Fourier harmonics are in phase in the cubic case we also have a phase shift of the Fourier harmonics, and the normalized amplitudes are

\[ A_n = \sqrt{J_n^2[(n + 1/2)\tau] + J_{n+1}^2[(n + 1/2)\tau]}. \] (15)

The energy spectrum is shown in Fig. 3. As in the case of a quadratic nonlinearity, it has exponential shape for small times (upper panel) and turns into a power law when time approaches the moment of breaking, \( t \to T \). The slope of the power law is 2.67 and again close to \( 8/3 \).

It is clearly seen from Fig. 4 (upper panel) that the energy spectrum has the exponential part, both for quadratic and cubic nonlinear media. The power law part of the energy spectrum is shown in Fig. 4, lower panel, for quadratic and cubic nonlinear media; the shape of the energy spectrum is practically undistinguishable for these two cases.
In the next two sections we present the results of numerical simulations which show the effect of dispersion (Sec.IV) and dissipation (Sec.V) on the formation of the universal asymptotic in the energy spectrum of Riemann waves. As a reference point we regard a particular case of (1) with $\alpha = 6$, $\beta = 0$, $\gamma = 0$:

$$u_t + 6 uu_x = 0. \quad (16)$$

This is the dispersionless Korteweg-de Vries equation (KdV) in canonic form. All numerical simulations presented in the next sections have been performed with an initially sinusoidal wave of the form (7), a tank length $L = 200$ and initial disturbance $U = U_0 \sin k_0 x$ such that the wave length equals the length of the tank, which means $k_0 = 0.0314$. The breaking time is $T = 26.5$.

Numerical simulations with other forms of smooth and physically relevant initial conditions - e.g. $U = U_0 \exp(-x^{2k_0})$ - have also been conducted; the results are quite similar to those with sinusoidal initial condition, and are not shown in the text below.

**IV. EFFECT OF DISPERSION**

Adding to (16) a dispersive term of the form $u_{xxx}$ we get

$$u_t + 6 uu_x + u_{xxx} = 0, \quad (17)$$

the classical KdV equation. The transformation of a sinusoidal wave into a group of solitons or cnoidal waves is a classical problem which has been studied in the frame of the KdV equation, with main interest on the evolution of the wave shape in physical space. Our interest is to examine the form of the energy spectrum in Fourier space and if something which resembles the universal asymptotic $-8/3$ may be observed under the influence of weak dispersion.

What happens in this case depends on the Ursell number, which is the ratio of the nonlinear term to the dispersive term:

$$Ur = \max_x (uu_x) / \max_x (u_{xxx}). \quad (18)$$

For shallow water waves the Ursell number can be computed as $Ur = A \cdot L^2 / h^3$ where $A$ and $L$ are the wave amplitude and wave length respectively, and $h$ is the depth of the water layer, $h \ll L$. Accordingly, the Ursell parameter changes with the wave length $L$.

For our numerical simulation, with $L$ chosen as $L = 200$, we get $Ur = L^2 / h^3 = 40000$ and the effect of dispersion should be small. Taking again initial conditions of the form (7), we observe the same time evolution as in the dispersionless KdV (16) at different times $t$ smaller than breaking time $T$; the results of our simulation are shown in Fig.5.

When $t$ goes beyond breaking time $T = 25.5$, the Fourier spectrum changes rapidly. Already at the moment of time $t = 27$ the formation of an isolated peak in the energy spectrum is visible as shown in Fig.6, lower panel; the shape of the surface elevation corresponding to this is shown in Fig.6, upper panel.

The further development is characterized by the formation of soliton-like disturbances in physical space (upper panels, Figs.7-9) leading to the appearance of narrow spectral peaks in the high-frequency range in Fourier.
space (lower panels, Figs. 7-9). The first spectral peak appears in the Fourier harmonic with $n = 34$ at the moment of time $t = 27$, as a soliton with soliton width 6 is the 34th harmonic of a wave with initial wavelength 200 ($200/6 \approx 34$). During time evolution this peak is downshifted to the left, so that at moment of time $t = 100$ the maximum of energy is at the harmonic with $n = 18$ which corresponds to a soliton width of $\approx 10.5$.

At a later time moment, new soliton-like disturbances become detectable in the Fourier spectrum, with essentially the same behavior. Thus the second peak appears at the harmonic with $n = 60$ and moves to the left, fi-
nally arriving at the harmonic with \( n = 40 \); the soliton amplitude is growing and the soliton width is decreasing.

Thus, the energy spectrum consists of two substantially different parts: the universal power law asymptotic \( k^{-8/3} \) at the initial stage after breaking and an exponential asymptotic of the newly appearing solitons (not shown in the figures).

The universal breaking time asymptotic \( k^{-8/3} \) stays unchanged up to the moment of time \( t \approx 2T \). After that, the power law part of the spectrum becomes shorter in Fourier space applying only to low frequencies and energy decays more slowly compared to the exponent \( -8/3 \); the "soliton"-part of the spectrum takes over from the right. For the chosen \( U_r \) (which is the relation between nonlinearity and dispersion) and time \( t \approx 4T \), three peaks in Fourier space are clearly visible in the figures. For bigger times the number of solitons in physical space is growing and they do not describe the front structure anymore as the width of the front becomes comparable to the initial wave length \( L \).

V. EFFECT OF DISSIPATION

The usual way of modeling dissipation is to add to (16) a term for dissipation \( -\nu u_{xx} \) with a small coefficient of viscosity \( \nu > 0 \). The resulting equation

\[
u_t + 6u_x - \nu u_{xx} = 0 \quad (19)\]

is the well known Burgers equation, which has a long time asymptotic \( k^{-2} \) for the energy spectrum, [28].

As it is shown above, if \( \nu = 0 \), the breaking time asymptotic is \( k^{-8/3} \). Our interest is to examine the form of the energy spectrum in Fourier space and if something which resembles the universal asymptotic \( -8/3 \) may be observed under the influence of dissipation.

With this aim we study the Burgers equation numerically, with initial condition (7) as above and the viscosity coefficient taken as \( \nu = 0.1 \). In Fig.10 the formation of the breaking time asymptotic \( k^{-8/3} \) is shown, depicting the spectrum at \( t = 10, 15, 22, 26 \) before breaking time \( T = 26, 5 \).

Beginning with \( t = 27 > T \), the appearance of a new asymptotic which decays more slowly than \( k^{-8/3} \) becomes evident; it is shown in Fig.11. With increase in time the wave shape turns into a triangle and its spectrum \( k^{-2} \) can be computed analytically, [28]. For \( t > 5T \) the initial wave is damped significantly and the spectrum becomes narrow again (not shown in figures).

VI. DISCUSSION AND CONCLUSIONS

The results presented in this Letter can be summarized as follows.

- We have found approximately the same power law of \( \approx k^{-8/3} \) for the energy spectrum of Riemann waves before breaking in a vicinity of the breaking point, in quadratic as well as in cubic media. As the quadratic and cubic nonlinearity are described by the first and second term of the Taylor expansion, this is an indication, but not a proof, that \( k^{-8/3} \) is the universal power law for Riemann waves with any nonlinearity.

- This may be re-formulated in the following way: any point of singularity in the (initially smooth) wave profile
of a Riemann wave before breaking may be described by a power law of \( x^{1/3} \). Indeed, the Fourier spectrum of the power function \( x^q \) is \( k^{-(1+q)} \), and the corresponding energy spectrum is \( k^{-2(1+q)} \). Taking \( q = 1/3 \) we get our energy spectrum of \( k^{-8/3} \).

The results of our numerical simulations are in contrast to the assumption normally found in literature [20, 28], that Riemann waves in media with quadratic nonlinearity before breaking have a wave profile of the form \( x^{1/2} \), yielding a power law for the energy spectrum of \( k^{-3} \). The difference can be seen clearly from Fig.11.

We have demonstrated that the universal power law asymptotic of \( k^{-8/3} \), inherited from the simple wave equation with quadratic nonlinearity

\[
  u_t + \alpha uu_x = 0.
\]  

"survives" for low-frequency range of harmonics in Fourier space with an additional dispersion term, yielding the KdV equation

\[
  u_t + 6uu_x + u_{xxx} = 0
\]

and with an additional dissipation term, yielding the Burgers equation

\[
  u_t + 6uu_x - 0.1u_{xx} = 0.
\]

In both cases the power law asymptotic in the low-frequency range changes in a similar way: the spectrum decay becomes slower.

A really surprising fact is that the universal power law asymptotic \( k^{-8/3} \) obtained for the breaking moment of time \( T \) in equation (20), is still observed both in the KdV and Burgers equation although breaking will not occur; moreover in the low frequency range it stays for times \( t \sim 2T \) and more.

In the high-frequency range the difference between effects of dispersion and dissipation becomes visible.

With dissipation taken into account, the well known power law asymptotic \( k^{-2} \) of the Burgers equation is formed.

With dispersion taken into account, a few soliton-like peaks in Fourier space are formed. The asymptotic of their envelope has exponential form which was not studied in detail yet. In our simulations the number of solitons at the wave front was growing rapidly increasing the front width to the extent that the model equation is not applicable any more.

We wanted to see how far the universal power asymptotic can be observed in other modifications of the simple wave equation. At our request, Karl Helfrich checked the shape of the energy spectrum in the results of his numerical simulations with the reduced Ostrovsky equation

\[
  (u_t + uu_x)_x = \gamma u
\]

with initial condition of the form (7). It turned out that the universal power asymptotic \( k^{-8/3} \) is observed both in rotational and non-rotational cases, i.e. for arbitrary \( \gamma \), [29]. The reduced Ostrovsky equation can be regarded as a modification of the Korteweg-de Vries equation, in which the usual dispersive term \( uu_{xxx} \) is replaced by the term \( \gamma u \), which represents the effect of background rotation. The Ostrovsky equation is an important theoretical tool for studying surface and internal solitary waves in the atmosphere and ocean taking into account the Earth’s rotation.

We found that in any weakly nonlinear medium which has been analyzed so far, non-dispersive or weakly dispersive, the time evolution of the energy spectrum goes through the same steps: an initially exponential spectrum in the vicinity of the breaking point turns into a power law for high frequencies. This enables us to estimate from the spectrum measured how far a wave system is away from breaking (smooth form of wave, the beginning of collapse or developed shock). This method may be most useful in many problems of nonlinear acoustics, physical oceanography, magnetohydrodynamics and laser optics, see e.g. [25–27].

We have demonstrated - though not proven - that the power law asymptotic \( k^{-8/3} \) of the energy spectrum of Riemann waves holds for the simple wave equation with arbitrary nonlinearity. The additional results we have found for modifications of the simple wave equation indicate, that the universal power law asymptotic is a manifestation of some fundamental properties which may be found in many wave systems and deserves further theoretical study. This is presently work in progress.

Acknowledgments. The authors acknowledge support by the Austrian Science Foundation (FWF) under projects P22943 and P24671. EP acknowledges also VolkswagenStiftung, RFBR grant 11-05-00216 and Federal Targeted Program "Research and educational personnel of innovation Russia" for 2009–2013.
[1] G. B. Whitham. *Linear and Nonlinear Waves* (Wiley Series in Pure and Applied Mathematics, 1999)
[2] E. Kartashova. *Nonlinear Resonance Analysis* (Cambridge University Press, 2010)
[3] E. Kartashova. *EPL* 97 (2012), 30004.
[4] V. E. Zakharov, V. S. L’vov and G. Falkovich. *Kolmogorov Spectra of Turbulence* (Series in Nonlinear Dynamics, Springer-Verlag, New York, 1992)
[5] E. Kartashova. *PRE* 86 (2012), 041129.
[6] E. Kartashova. *EPL* (To appear, 2013). See also E-print: arXiv:1302.5961.
[7] K. B. Dysthe. *Proc. R. Soc. A* 369 (1979), 105.
[8] S. J. Hogan. *Proc. R. Soc. A* 402 (1985), 359.
[9] R. Grimshaw, D. Pelinovsky, E. Pelinovsky, and T. Talipova. *Physica D* 159 (2001), 35.
[10] A. V. Gurevich, A. L. Krylov, and G.A. El. *JETP Lett.* 54 (1991), 102.
[11] P. D. Lax. *Commun. Pure Appl. Math.* 44 (1991): 1047.
[12] A. M. Kamchatnov, Y.-H. Kuo, T.-C. Lin, T.-L. Horng, S.-C. Guo, R. Clift, G.A. El, and R. H. J. Grimshaw. *Phys. Rev. E* 86 (2012), 036605.
[13] A. V. Gurevich, and K. P. Zybkin. *JETP* 88 (1999), 182.
[14] E. Pelinovsky, and A. Sergeeva (Kokorina). *Eur. J. Mech. B/Fluids* 25 (2006), 425.
[15] G. A. El, A. M. Kamchatnov, M. V. Pavlov, and S. A. Zykov. *J. Nonlin. Sci.* 21 (2011), 151.
[16] J. K. Engelbrecht, V. E. Fridman and E. N. Pelinovsky. *Nonlinear Evolution Equations* (Pitman Res. Not. Math. Ser. 180, London: Longman, 1988)