Robustifying Binary Classification to Adversarial Perturbation

Fariborz Salehi and Babak Hassibi

Abstract—Despite the enormous success of machine learning models in various applications, most of these models lack resilience to (even small) perturbations in their input data. Hence, new methods to robustify machine learning models seem very essential. To this end, in this paper we consider the problem of binary classification with adversarial perturbations. Investigating the solution to a min-max optimization (which considers the worst-case loss in the presence of adversarial perturbations) we introduce a generalization to the max-margin classifier which takes into account the power of the adversary in manipulating the data. We refer to this classifier as the "Robust Max-margin" (RM) classifier. Under some mild assumptions on the loss function, we theoretically show that the gradient descent iterates (with sufficiently small step size) converge to the RM classifier in its direction. Therefore, the RM classifier can be studied to compute various performance measures (e.g. generalization error) of binary classification with adversarial perturbations.

I. INTRODUCTION

Machine learning models have been very successful in many applications, ranging from spam detection, speech and visual recognition, to the analysis of genome sequencing and financial markets. Yet, despite this indisputable success, it has been observed that commonly used machine learning models (such as deep neural networks) are very instable in the presence of non-random perturbations [16], [1], [2]. The instability of machine learning models is a fundamental issue that needs to be addressed, especially when such models are used in sensitive applications such as autonomous systems. There have been many recent efforts to address this issue (a partial list of papers includes [9], [19], [13].) However, robustness comes at a cost and it is often the case that the adversarial training algorithms underperform on the clean data when compared with their (non-robust) counterparts. Understanding the tradeoffs (in accuracy) between the robust and standard models is an important problem the answer to which can help us find more efficient training methods. Recently, Javanmard et. al. [7] precisely characterized the tradeoff between standard and adversarial risks for the linear regression problem. They also analyze the performance of the resulting model under i.i.d. Gaussian training data. In this paper, we study the simple (yet fundamental) problem of binary classification where the goal is to find a classifier that has a high accuracy in predicting the binary labels when having feature vectors as its input. When the clean data is available, max-margin classifier [18] is the model of choice as maximizing the margin is interpreted as minimizing the risk of misclassification [3]. Recently, it was shown [14] that for a broad class of loss functions, including the well-known logistic loss, the gradient descent iterates converge to the max-margin classifier. More recently, the asymptotic performance of this classifier has been characterized in [10], [4], [12].

We consider the case where the training data is perturbed by an adversary and introduce the "Robust Max-margin" (RM) classifier as a generalization of max-margin to perturbed input data. We then consider the adversarial training method, in which the optimal parameter is a solution to a saddle-point optimization. We show that the gradient descent algorithm with properly-tuned step sizes converges in its direction to the RM classifier. A significant consequence of this result is that one can characterize various performance measures (e.g. generalization error) of adversarial training in binary classification by analyzing the performance of the RM classifier.

To the extent of our knowledge, this is the first work that introduces the robust max-margin classifier and proves the convergence of gradient descent iterates to this classifier. This paper was originally submitted on March 2020 to the Conference on Decision and Control (CDC). We should note that more recently in [6], the authors have shown similar results (referred to as the "robust separation") and analyze the performance of the resulting classifier under i.i.d. Gaussian training data. Their analysis on the performance of the resulting estimator is based on the Convex Gaussian Min-max Theorem [15], [17]. Similar analyses have been recently provided for the performance of max-margin classifiers as well as other generalized linear models [11], [10], [4], [5], [12].

The organization of the paper is as follows: In Section II we provide some background on the binary classification problem and how it connects with the max-margin
classifier. The mathematical setup for the problem of binary classification with perturbed training data is provided in Section III. The main result of the paper is presented in Section IV and the proofs are provided in Sections V and VI.

II. PRELIMINARIES

A. Notations

For any vector $w \in \mathbb{R}^p$, the binary classifier associated with $w$ is defined as: $C_w : \mathbb{R}^p \rightarrow \{-1, 1\}$, such that $C_w(x) = \text{Sign}(w^T x)$. $\mathbb{N}$ denotes the set of non-negative integers. For a vector $v$, $v^T$ denotes its transpose, and $\|v\|_p$ (for $p \geq 1$) is its $\ell_p$ norm, where we often omit the subscript for $p = 2$. $\sigma_{\text{max}}(M)$ denotes the maximum singular value of the matrix $M$. $0_d$ and $1_d$ respectively represent the all-one and all-zero vectors in dimension $d$. A function $f(.)$ is said to be $L$-smooth if its derivative, $f'(\cdot)$, is $L$-Lipschitz.

B. Background: binary classification with unperturbed data

Let $D = \{(x_i, y_i) : 1 \leq i \leq n\}$ denote a set of data points, where for $i = 1, \ldots, n$, $x_i \in \mathbb{R}^p$ is the feature vector, and $y_i \in \{-1, 1\}$ is the binary label. We assume that $D$ is linearly separable, i.e., there exist $w^* \in \mathbb{R}^p$ such that:

$$y_i = \text{Sign}(x_i^T w^*) , \quad \text{for } i = 1, 2, \ldots, n . \tag{1}$$

When the training data has no perturbation, one can attempt to find a classifier by minimizing the empirical loss on dataset $D$. In the setting of binary classification, the loss function is usually formed as,

$$\mathcal{L}(w) = \sum_{i=1}^{n} \ell(y_i x_i^T w) \tag{2}$$

where the function $\ell(\cdot) : \mathbb{R} \rightarrow \mathbb{R}_+$ is a decreasing function that approaches 0 as its input approaches infinity. A typical approach to find the minimizer of the loss function $\mathcal{L}(w)$ is through the iterative algorithms, such as the gradient descent (GD) algorithm. The convergence of the GD iterates on separable datasets has been studied in recent papers [8], [14], where it was shown, among others, that while the norm of the iterates approaches infinity, their direction would approach to the direction of the well-known $L_2$ max-margin classifier defined as,

$$w_M = \arg \min_{w \in \mathbb{R}^p} \|w\| \quad \text{s.t. } \quad y_i x_i^T w \geq 1 , \quad 1 \leq i \leq n . \tag{3}$$

In other words, their result states that for almost every $x \in \mathbb{R}^p$, $C_{w_t}(x) \rightarrow C_{w_M}(x)$ as $t$ grows, where $w_t$ denotes the result of GD after $t$ steps. The max-margin classifier (a.k.a. hard-margin SVM [3]) has been extensively studied in the machine learning community. This classifier simply maximizes the smallest distance of the data points to the separating hyperplane (referred to as the margin).

The abovementioned result, i.e., convergence of the GD iterates to the max-margin classifier, has significant consequences as the max-margin classifier can then be studied to compute various performance measures (such as the generalization error) of the resulting estimator. Very recently, researchers have exploited this result to accurately compute the generalization error of GD over the logistic loss [10].

III. BINARY CLASSIFICATION WITH ADVERSARIAL PERTURBATION

As explained earlier in Section II, understanding the behavior of machine learning models under perturbed input is very essential with the goal of improving the robustness of these models. Inspired by recent advances in understanding the behavior of machine learning models under adversarial perturbation, here we study the problem of binary classification with perturbed data.

We assume that the training data is a perturbed version of the underlying dataset, $D$. Let $D' = \{(x_i + z_i, y_i) : 1 \leq i \leq n\}$ denote the set of training data, where, for $i = 1, 2, \ldots, n$, $z_i \in S_i$ is the unknown perturbation, and the set $S_i$ consists of all the allowed perturbation vector. In the adversarial setting it is often assumed that the perturbation vectors, $\{z_i\}_{i=1}^n$, are chosen in such a way that the training algorithm is beguiled into generating a wrong solution.

Throughout this paper, we assume that the perturbation vectors have bounded norms by defining $B_i = \epsilon_i B_p$, where $B_p$ denotes the unit ball in $\mathbb{R}^p$, and $\epsilon_i \geq 0$, for $1 \leq i \leq n$, indicates the maximum allowed norm for the $i$-th perturbation vector, $z_i$. While the perturbation vectors are hidden to us, we assume having knowledge of $\{\epsilon_i\}_{i=1}^n$.

Note that the set of allowed perturbations can be different for different data points. This includes certain special cases such as: (1) only a subset of the data is perturbed ($\epsilon_i = 0$ if the $i$-th data point is not perturbed), and (2) all the data points have the same perturbation set, i.e., for some $\epsilon \geq 0$ we have $\epsilon_i = \epsilon$ for $1 \leq i \leq n$.

A. Saddle-point optimization

The parameters of the desired model are often derived by forming a loss function and solving an optimization problem to find a minimizer of the loss. In adversarial training, one should also consider the manipulative power of the adversary where the adversary attempts to misguide
the training algorithm. When the goal of a training algorithm is to minimize a loss function, one can view the adversary as an entity which attempts to maximize the loss. The following min-max optimization problem incorporates the contrary behaviors of the adversary and the training algorithm with respect to the loss function.

$$\min_{w \in \mathbb{R}^p} \max_{\mathbf{z}_i \in S_i, 1 \leq i \leq n} \mathcal{L}(w) := \sum_{i=1}^{n} \ell(y_i(x_i + \mathbf{z}_i)^T w).$$

(4)

In order to find a robust model, we should solve this saddle-point optimization. Under our assumptions on the perturbation sets, we can introduce the function $\mathcal{L}_\epsilon(w)$ which is the result of the inner maximization in (4), i.e.,

$$\mathcal{L}_\epsilon(w) = \sum_{i=1}^{n} \max_{\|\mathbf{z}_i\| \leq \epsilon_i} \ell(y_i(x_i + \mathbf{z}_i)^T w),$$

(5)

where $\epsilon = [\epsilon_1, \epsilon_2, \ldots, \epsilon_n]^T$. Therefore, the robust classifier is defined as a minimizer of $\mathcal{L}_\epsilon(w)$.

IV. MAIN RESULTS

In this section, we present the main results of the paper. First, in Section IV-A we introduce the Robust Max-margin (RM) classifier as an extension of the max-margin classifier when the training data is perturbed. Consequently, in Section IV-B we show that, under some conditions on the function $\ell(\cdot)$, gradient descent algorithm (with sufficiently small step size) would converge in its direction to the RM classifier.

A. Robust Max-margin (RM) Classifier

The max-margin classifier is a classifier that maximizes the minimum distance of the data points to the separating hyperplane (margin). In our setting where the training data is perturbed we should modify the notion of the margin to incorporate various perturbations across data points. More specifically, in order to get a robust classifier we would like the data points with higher perturbations to be farther away from the resulting separating hyperplane.

The **Robust Max-margin** classifier is defined as,

$$w^{(\epsilon)}_{RM} := \arg \min_{w \in \mathbb{R}^p} \|w\|$$

s.t. $y_i x_i^T w \geq 1 + \epsilon \|w\|, \quad 1 \leq i \leq n.$

(6)

As observed in the constraints of this optimization, the RM classifier enforces data points with higher perturbations to keep a larger distance from the separating hyperplane $\{\mathbf{x} : w_M^T \mathbf{x} = 0\}$. When the data is perturbed, we expect the RM classifier to outperform the max-margin classifier. Figure 1 depicts a comparison in generalization error between the max-margin and the RM classifier. Although for small perturbations, the two model behave the same, the RM classifier has a better performance as we increase the norm of perturbations.

Fig. 1: A comparison in generalization error (GE) between the max-margin and the robust max-margin.

While the separability of the data is necessary for the existence of the RM classifier, it is not sufficient. The following lemma provides a sufficient condition for its existence.

**Lemma 1:** The RM classifier exists when the data set $\mathcal{D} = \{(\mathbf{x}_i, y_i) : 1 \leq i \leq n\}$ is separable and,

$$\|\epsilon\|_{\infty} < \frac{1}{\|w_M\|_2},$$

(7)

where $w_M$ is the max-margin classifier.

**Proof:** The max-margin classifier, $w_M$, exists when $\mathcal{D}$ is linearly separable. Also, $w = \frac{1}{1 - \|w_M\|} w_M$ is a feasible point of the optimization. Therefore, the RM classifier exists and $\|w_M\| \leq \|w_{RM}\| \leq \|w\|$. When the perturbation sets are the same for different data points, one expects the RM classifier to be the same as the max-margin classifier.

**Lemma 2:** If $\epsilon = \epsilon \times 1_n$ for some $\epsilon \geq 0$, the RM classifier exists if and only if $\epsilon < \frac{1}{\|w_M\|}$. In this case,

$$w_{RM} = \frac{w_M}{1 - \epsilon \|w_M\|}.$$
Proof: Assume \( w_{RM} \) exists, then we have \( w = \frac{w_{RM}}{\|w_{RM}\|} \) satisfies the constraints in the optimization (3). Since \( w_M \) is the solution to this optimization, we have \( \|w_M\| \leq \|w\| \) which gives \( \epsilon \cdot \|w_M\| < 1 \). It is easy to check that \( w = \frac{w_M}{1-\epsilon \cdot \|w_M\|} \) is the solution to the optimization (6), as it satisfies the constraints and \( w_M \) is the optimal value of the optimization program (3). 

B. Convergence of GD Iterates

In this section, we present the main result of the paper that is the convergence of the gradient descent iterates to the RM classifier. As discussed earlier in Section III-A, the goal is to solve the following optimization problem:

\[
\min_{w \in \mathbb{R}^p} \mathcal{L}_\epsilon(w),
\]

where \( \mathcal{L}_\epsilon(\cdot) \) is defined in (3). Gradient descent (GD) is the common method of choice to find a minimizer of this optimization. Starting from an initialization, \( w_0 \in \mathbb{R}^p \), the GD iterates are generated through the following update rule:

\[
w_{t+1} = w_t - \eta \cdot \nabla \mathcal{L}_\epsilon(w_t), \quad t \in \mathbb{N},
\]

where \( \eta > 0 \) is the step size.

Our goal is to study the behavior of the GD iterates as \( t \) grows large. For our analysis, we need some assumptions to hold for the loss function \( \ell(\cdot) \).

Assumption 1: The function \( \ell : \mathbb{R} \to \mathbb{R}_+ \) is twice-differentiable, monotonically decreasing, and \( \beta \)-smooth. We note that the common choices of the loss function satisfy the conditions in Assumption 1. For instance, the logistic loss defined as \( \ell(u) = \log(1 + \exp(-u)) \) satisfies these conditions (with \( \beta = 1 \)). We first state the following lemma which provides some insights on the behavior of GD iterates, \( w_t \), as \( t \to \infty \).

**Lemma 3:** Consider the gradient descent iterates (10) with step size \( \eta < 2 \cdot \beta^{-1} \cdot (\sigma_{\max}(X) + \|\epsilon\|)^{-2} \), where \( X = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^{n \times p} \) is the data matrix, \( \mathcal{L}_\epsilon \) is defined in (3), and \( \ell(\cdot) \) satisfies Assumption 1. If the RM classifier exists, then, as \( t \to +\infty \), we have:

i. \( \|w_t\| \to +\infty \),

ii. \( \nabla \mathcal{L}_\epsilon(w_t) \to 0_p \), and,

iii. \( y_i x_i^T w_t - c_i \|w_t\| \to +\infty \), for \( i = 1, 2, \ldots, n \).

The proof of this lemma is provided in Section V. Lemma 3 provides useful insights on the behavior of the gradient descent iterates. With small enough step size, as \( t \) grows the norm of \( w_t \) becomes unbounded while making \( \mathcal{L}(w_t) \) closer to zero. Since \( w_t \) diverges, we focus our attention on its direction, i.e., the normalized vector \( \frac{w_t}{\|w_t\|} \). In fact, the classifier defined by \( w_t \), \( C_{w_t}(\cdot) \), only depends on its direction. Therefore, if \( w_{\infty} \) converges, we can claim that the classifiers generated by GD iterates converge.

Our main result in Theorem 1 states that the classifiers generated from the GD iterates converges to the RM classifier defined in Section IV-A. Before stating this result, we need the following definition which is a modified version of an assumption in [14].

**Definition 1:** A function \( f(u) \) has a tight exponential tail if there exist positive constants \( a, c, \tau, \mu \) such that for all \( u > \tau \):

\[
\left\{ \begin{array}{l}
f(u) \leq c(1 + \exp(-\mu \cdot u)) \exp(-a \cdot u), \\
f(u) \geq c(1 - \exp(-\mu \cdot u)) \exp(-a \cdot u).
\end{array} \right.
\]

**Theorem 1:** Let Assumption 1 holds and \(-\ell'(\cdot)\) has a exponential tail. Consider the gradient descent iterates in (10) with \( \eta < 2 \cdot \beta^{-1} \cdot (\sigma_{\max}(X) + \|\epsilon\|)^{-2} \). Then, for almost every dataset we have:

\[
\lim_{t \to \infty} \frac{\|w_t - w_{RM}\|}{\|w_{RM}\|} = 0.
\]

Therefore, the resulting classifier converges to the RM classifier.

**Remark 1:** The assumption on \(-\ell'(\cdot)\) having a tight exponential tail holds for common loss functions in binary classification. As an example, the derivative of the logistic function satisfies (11) with \( a = c = \mu = 1 \).

**Remark 2:** Theorem 1 states that while \( w_t \) diverges as \( t \) grows, its direction converges to the direction of the robust max-margin classifier. We should note that this convergence is quite slow. Figure 2 depicts the convergence of the direction of GD iterates to the RM classifier as \( t \to \infty \) where it can be observed the convergence becomes slow as \( t \) grows (the horizontal axis has a logarithmic scale). In our proof in Section VI, we theoretically establish that the rate of convergence is logarithmic.

V. PROOF OF LEMMA 3

In our proof we use the following lemma which characterizes the behavior of gradient descent iterates on smooth functions.

**Lemma 4 (Lemma 10 in [14]):** Let \( \mathcal{L}(w) \) be a \( \gamma \)-smooth non-negative objective. If \( \eta < \frac{\gamma}{2} \), then, for any starting point \( w(0) \), with the GD sequence

\[
w(t+1) = w(t) - \eta \nabla \mathcal{L}(w(t))
\]

we have that:

\[
\sum_{i=1}^{\infty} \|\nabla \mathcal{L}(w(u))\|^2 < +\infty.
\]

We also use the following corollary.
we can write,\
\[ \ell \]

First, we show that this sequence is increasing.

The distance between the max-margin and the RM classifier is
\[ \left\| \frac{w_M}{\|w_M\|} - \frac{w_{RM}}{\|w_{RM}\|} \right\| = 0.2192. \]

Corollary 1: For any positive constant \( C < \beta (\sigma_{\text{max}}(X) + \|\epsilon\|)^2 \), there exist \( R > 0 \), such that \( \|\nabla^2 \mathcal{L}_\epsilon(w)\| < C \) when \( \|w\| > R \).

The proof is straightforward, by computing the Hessian of \( \mathcal{L}_\epsilon(\cdot) \) and using the fact that \( \ell(\cdot) \) is twice-differentiable and \( \beta \)-smooth.

Since the function \( \ell(\cdot) \) is monotonically decreasing we can write,
\[ \mathcal{L}_\epsilon(w) = \sum_{i=1}^{n} \ell(y_i x_i^T w - \epsilon_i \|w\|) \]  \hspace{1cm} (13)

The gradient of the loss function can be computed as,
\[ \nabla \mathcal{L}_\epsilon(w) = \sum_{i=1}^{n} \ell'(y_i x_i^T w - \epsilon_i \|w\|)(y_i x_i - \epsilon_i \|w\|). \]  \hspace{1cm} (14)

Consider the sequence \( s_t := \frac{1}{\eta} w_{RM}^T x_t \) for \( t \in \mathbb{N} \).

First, we show that this sequence is increasing.

\[ s_t - s_{t+1} = w_{RM}^T \nabla \mathcal{L}_\epsilon(w_t) \]
\[ = \sum_{i=1}^{n} \ell'(y_i x_i^T w_t - \epsilon_i \|w_t\|)(y_i x_i - \epsilon_i \|w_t\|) \]
\[ \leq \sum_{i=1}^{n} \ell'(y_i x_i^T w_t - \epsilon_i \|w_t\|)(y_i x_i^T w_{RM} - \epsilon_i \|w_{RM}\|) \]
\[ \leq \sum_{i=1}^{n} \ell'(y_i x_i^T w_t - \epsilon_i \|w_t\|) < 0, \]

where for the first inequality we used the fact that \( \ell'(u) < 0 \) and Cauchy-Schwartz, and for the second inequality we used the constraints of the optimization \( \mathcal{L}_\epsilon(w) \).

Since \( \{s_t\}_{t \geq 0} \) is an increasing sequence in \( \mathbb{R} \) it either grows to \( +\infty \) or approaches a limit value. We analyze each of these cases separately.

Case 1: \( \lim_{t \to \infty} s_t = L < +\infty \)

When the sequence has a limit, we have \( \lim_{t \to \infty} s_t - s_{t+1} = 0 \). From the last inequality in (15), this implies that as \( t \to \infty \),
\[ \ell'(y_i x_i^T w_t - \epsilon_i \|w_t\|) \to 0, \text{ for } 1 \leq i \leq n. \]  \hspace{1cm} (16)

Since \( \ell'(u) \) is negative for \( u \in \mathbb{R} \), we must have
\[ y_i x_i^T w_t - \epsilon_i \|w_t\| \to +\infty, \text{ for } 1 \leq i \leq n, \]  \hspace{1cm} (17)

which is (iii). This also implies that \( \|w_t\| \to \infty \).

Finally, from (17) we have that \( \nabla \mathcal{L}_\epsilon(w_t) \to 0 \).

Case 2: \( \lim_{t \to \infty} s_t = +\infty \)

\[ \|w_t\| \geq \frac{1}{\eta} \|w_{RM}\| \] implies that \( \lim_{t \to \infty} \|w_t\| = +\infty \).

Using Corollary 1 for any constant \( C < \beta (\sigma_{\text{max}}(X) + \|\epsilon\|)^2 \), there exists a nonnegative integer \( t_0 \) such that the second derivative is bounded by \( C \) for any \( t > t_0 \).

Hence, we can use the result of Lemma 2 with \( \eta < 2 \cdot \beta^{-1} (\sigma_{\text{max}}(X) + \|\epsilon\|)^{-2} \) which gives \( \|\nabla \mathcal{L}_\epsilon(w_t)\| \to 0 \) as \( t \to +\infty \).

In order to show (iii), we use the last inequality in (15), as \( t \to \infty \) since \( w_{RM}^T \nabla \mathcal{L}_\epsilon(w_t) \to 0 \), we have:
\[ \ell'(y_i x_i^T w_t - \epsilon_i \|w_t\|) \to 0, \text{ for } 1 \leq i \leq n, \]  \hspace{1cm} (18)

which gives the desired result.

VI. PROOF OF THEOREM

For the RM classifier, we define the set of support vectors as:
\[ \mathcal{S} = S_{RM} := \{ i \in [n] : y_i x_i^T w_{RM} = 1 + \epsilon_i \|w_{RM}\| \}. \]  \hspace{1cm} (19)

First, we consider the KKT conditions for the optimization \( \mathcal{L}_\epsilon(w) \), which gives:
\[ w_{RM} = \sum_{i \in \mathcal{S}} \alpha_i (y_i x_i - \epsilon_i \hat{w}), \]  \hspace{1cm} (20)

where \( \hat{w} := \frac{w_{RM}}{\|w_{RM}\|} \) and \( \alpha_i \geq 0 \). It can be shown that when the data points are drawn from a continuous distribution, for almost every dataset the support vectors are linearly independent and \( \alpha_i \)'s are all positive (see also [8] and Appendix B in [14]). Given the fact that \( -\ell'(u) \) has an exponential tail, we assume \( \alpha, \gamma, \tau, \mu \) are positive constants such that:
\[ -\ell'(u) \leq \gamma (1 + \exp(-\mu \cdot u)) \exp(-\alpha \cdot u), \text{ and}, \]
\[ -\ell'(u) \geq \gamma (1 - \exp(-\mu \cdot u)) \exp(-\alpha \cdot u), \]  \hspace{1cm} (21)
for every $u \geq \tau$.

We define a vector $\tilde{w}$ such that:
\[
\exp (w^T (y_i x_i - \epsilon_i \tilde{w})) := \frac{\alpha_i}{\gamma \cdot \eta}, \quad \text{for } i = 1, 2, \ldots, n.
\]  
(22)

Recall that the gradient descent iterates are defined as,
\[
w_{t+1} - w_t = -\eta \nabla \mathcal{L}(w_t), \quad t \in \mathbb{N}.
\]  
(23)

Next, for $t \geq 0$ we define the residual vector $r_t \in \mathbb{R}^p$.
\[
r_t := w_t - \frac{1}{\alpha} \log(t)w_{RM} - \tilde{w}.
\]  
(24)

In our proof, we adopt a similar strategy as [14] and bound the norm of the residual vector $\|r(t)\|$ by a constant $C$ for every $t \geq 1$. Consider the following equation,
\[
\|r_{t+1}\|^2 - \|r_t\|^2 = \|r_{t+1} - r_t\|^2 + 2 r_t^T (r_{t+1} - r_t).
\]  
(25)

We bound each of the two terms in the RHS of (25). We start with bounding the first term in the (25). We have:
\[
\|r_{t+1} - r_t\|^2 = \|w_{t+1} - w_t - w_{RM} (\log(\frac{t+1}{t})/\alpha)\|^2 \\
\leq \eta^2 \|\nabla \mathcal{L}(w_t)\|^2 + (\alpha t)^{-2} \|w_{RM}\|^2 \\
+ 2(\eta/\alpha) \log(1+1/t)w_{RM}^T \nabla \mathcal{L}(w_t) \\
\leq \eta^2 \|\nabla \mathcal{L}(w_t)\|^2 + (\alpha t)^{-2} \|w_{RM}\|^2.
\]  
(26)

Where in the first inequality we replaced $w_{t+1} - w_t$ using the gradient descent iterates (23) along with $\log(1+u) \leq u$, and in the second inequality we exploited the inequality (15) that gives $\tilde{w}^T \nabla \mathcal{L}(w(t)) < 0$.

Since the norm of $w_t$ approaches infinity as $t$ grows, when $\eta < 2 \cdot \beta^{-1} \cdot (\sigma_{\text{max}}(X) + \|\epsilon\|)^{-2}$ we can use the result of Corollary 4 and Lemma 4 to have:
\[
\sum_{t=0}^{\infty} \|\nabla \mathcal{L}(w_t)\| < C_1,
\]  
(27)

for some constant $C_1 > 0$. Therefore, we can bound the sum over the first term in (25).
\[
\sum_{t \geq 1} \|r_{t+1} - r_t\|^2 \leq \eta^2 C_1 + \alpha^{-2} \|w_{RM}\|^2 \sum_{t \geq 1} t^{-2} < C_2.
\]  
(28)

Next, we will bound the second term in (25), i.e., $r_t^T (r_{t+1} - r_t)$. To do so, we first define the constant $\theta$ as follows:
\[
\theta := \min_{\epsilon_i \in S^c} y_i x_i , w_{RM} - \epsilon_i \|w_{RM}\| > 1,
\]  
(29)

where $S^c = [n] - S$ indicates the indices of non-support vectors. The following lemma provides an upper bound on $r_t^T (r_{t+1} - r_t)$ for $t \geq 1$.

Lemma 5: With the assumptions of Theorem 4, consider the gradient descent iterates $\{\tilde{w}_t\}_{t \in \mathbb{N}}$, the vector $r_t$ defined in (24), then, for constants $C \geq 0$ and $t_0 \in \mathbb{N}$, we have:
\[
r_t^T (r_{t+1} - r_t) \leq C t^{-\min(\theta, 1+\frac{1}{\alpha})}, \quad \forall t \geq t_0.
\]  
(30)

Using the result of Lemma 5 since $\theta > 1$ and $\mu/\alpha > 0$ we have:
\[
\sum_{t \geq t_0} r_t^T (r_{t+1} - r_t) \leq C \sum_{t \geq t_0} t^{-\min(\theta, 1+\frac{1}{\alpha})} < C_3.
\]  
(31)

Therefore, from (25), (28), and (31), we have,
\[
\|r_k\|^2 = \|r_0\|^2 + \sum_{t=1}^{k-1} \|r_{t+1} - r_t\|^2 < C_4, \quad \forall k \geq 1.
\]  
(32)

for a positive constant $C_4$. Consequently, from (24) we have,
\[
\|w_t - \frac{1}{\alpha} \log(t)w_{RM}\| \leq C_4 + \|w\|,
\]  
(33)

By some straightforward calculations we can get,
\[
\left\| \frac{w_t}{\|w_t\|} - \frac{w_{RM}}{\|w_{RM}\|} \right\|^2 \leq 2 \left[ \frac{\alpha(C_4 + \|w\|)}{\log(t)\|w_{RM}\|} \right]^2 \|w_{RM}\|,
\]  
(34)

which gives the desired result, i.e.,
\[
\lim_{t \to \infty} \left\| \frac{w_t}{\|w_t\|} - \frac{w_{RM}}{\|w_{RM}\|} \right\| = 0.
\]  
(35)

REFERENCES

[1] Battista Biggio, Igino Corona, Davide Maiorca, Blaine Nelson, Nedim Srndić, Pavel Laskov, Giorgio Giacinto, and Fabio Roli. Evasion attacks against machine learning at test time. In Joint European conference on machine learning and knowledge discovery in databases, pages 387–402. Springer, 2013.

[2] Nicholas Carlini and David Wagner. Towards evaluating the robustness of neural networks. In 2017 ieee symposium on security and privacy (sp), pages 39–57. IEEE, 2017.

[3] Corinna Cortes and Vladimir Vapnik. Support-vector networks. Machine learning, 20(3):273–297, 1995.

[4] Zeyu Deng, Abla Kammoun, and Christos Thrampouliidis. A model of double descent for high-dimensional binary linear classification. arXiv preprint arXiv:1911.05822, 2019.

[5] Melikasadat Emami, Mojtaba Sahrae-Arakani, Parthe Pandit, Sundeep Rangan, and Alyson K Fletcher. Generalization error of generalized linear models in high dimensions. arXiv preprint arXiv:2005.00180, 2020.

[6] Adel Javanmard and Mahdi Soltanolkotabi. Precise statistical analysis of classification accuracy for adversarial training. arXiv preprint arXiv:2010.11213, 2020.

[7] Adel Javanmard, Mahdi Soltanolkotabi, and Hamed Hassani. Precise tradeoffs in adversarial training for linear regression. arXiv preprint arXiv:2002.10477, 2020.

[8] Ziwei Ji and Matus Telgarsky. Risk and parameter convergence of logistic regression. arXiv preprint arXiv:1803.07300, 2018.
[9] Aleksander Madry, Aleksandar Makelov, Ludwig Schmidt, Dimitris Tsipras, and Adrian Vladu. Towards deep learning models resistant to adversarial attacks. arXiv preprint arXiv:1706.06083, 2017.

[10] Andrea Montanari, Feng Ruan, Youngtak Sohn, and Jun Yan. The generalization error of max-margin linear classifiers: High-dimensional asymptotics in the overparametrized regime. arXiv preprint arXiv:1911.01544, 2019.

[11] Fariborz Salehi, Ehsan Abbasi, and Babak Hassibi. The impact of regularization on high-dimensional logistic regression. In Advances in Neural Information Processing Systems, pages 11982–11992, 2019.

[12] Fariborz Salehi, Ehsan Abbasi, and Babak Hassibi. The performance analysis of generalized margin maximizer (gmm) on separable data. International Conference on Machine Learning (ICML), 2020.

[13] Ali Shafahi, Mahyar Najibi, Mohammad Amin Ghiasi, Zheng Xu, John Dickerson, Christoph Studer, Larry S Davis, Gavin Taylor, and Tom Goldstein. Adversarial training for free! In Advances in Neural Information Processing Systems, pages 3353–3364, 2019.

[14] Daniel Soudry, Elad Hoffer, Mor Shpigel Nacson, Suriya Gunasekar, and Nathan Srebro. The implicit bias of gradient descent on separable data. The Journal of Machine Learning Research, 19(1):2822–2878, 2018.

[15] Mihailo Stojnic. A framework to characterize performance of lasso algorithms. arXiv preprint arXiv:1303.7291, 2013.

[16] Christian Szegedy, Wojciech Zaremba, Ilya Sutskever, Joan Bruna, Dumitru Erhan, Ian Goodfellow, and Rob Fergus. Intriguing properties of neural networks. arXiv preprint arXiv:1312.6199, 2013.

[17] Christos Thrampoulidis, Samet Oymak, and Babak Hassibi. Regularized linear regression: A precise analysis of the estimation error. In Conference on Learning Theory, pages 1683–1709, 2015.

[18] V Vapnik. Estimation of dependences based on empirical data berlin, 1982.

[19] Weilin Xu, David Evans, and Yanjun Qi. Feature squeezing: Detecting adversarial examples in deep neural networks. arXiv preprint arXiv:1704.01155, 2017.