A Simple Third-Moment Reliability Index

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Abstract
The third-moment (3M) method has been considered to be one of the most reliable methods for calculating structural reliability with high computing efficiency. This paper proposes a new 3M reliability index that has a wider applicable range and less limitation. Numerical experiments show that the accuracy of the proposed method is higher than that of existing methods. In particular, when existing 3M methods are out of their respective applicable ranges, the proposed method can still be used to evaluate the failure probability. Therefore, the proposed method is deemed more applicable for practical engineering.

Keywords: third-moment method; structural reliability; failure probability; skewness

1. Introduction
When evaluating structural reliability, the most important step is to calculate the failure probability (i.e., the reliability index) of a structure; this is given as

\[ P_f = \Pr[Z = G(X) \leq 0] = \int_{G(X) \leq 0} f(x) \, dx \]  

(1)

where \( X \) is a vector of random variables representing uncertain structural quantities. \( f(x) \) is the probability density function (PDF) of the limit state function \( G(X) \). The domain of integration, \( G(X) \leq 0 \), denotes the failure set, and \( P_f \) is the failure probability. Difficulty in computing this probability has led to the development of various approximation methods, among which the first-order reliability method (FORM) (Shinozuka, 1983) is now used worldwide in engineering codes. Because of insufficient accuracy when applying FORM to nonlinear performance functions, the second-order reliability method (SORM) (Der Kiureghian et al., 1987) has been proposed to improve FORM. Although SORM is more accurate than FORM, it also requires the calculation of the design point and the curvature of failure of the limit state curve at the design point. However, when the PDFs of the basic random variables are unknown, neither FORM nor SORM are applicable. For such cases, sampling simulation methods (Nie and Ellingwood, 2005; Hurtado, 2007; Lu et al., 2008; Song and Lu, 2010) are known to be sufficiently accurate; however, when the performance function is complicated or high reliability is required, such methods are time consuming. Therefore, this paper proposes a novel third-moment (3M) method. Because the 3M method requires neither iteration nor computation of derivatives and has no shortcomings associated with the design point, it has been applied to various aspects of structural analysis, including corrosion probability of RC structures (Zhang et al., 2015), seismic reliability of structures (Liu et al., 2015), reliability of pavement structures (Mun, 2014), serviceability failure in braced excavations (Wang, 2014) etc. However, existing 3M methods do have some limitations: in some cases, due to the method's mathematical formula, existing 3M methods either cannot be used to calculate the failure probability or result in large errors.

By focusing on the problems in the mathematical formulae of existing 3M methods, namely, the inclusion of the square root, the unknown value of the denominator, and the logarithmic term in the approximation formula, the simple 3M method proposed in this paper has relatively high accuracy and less limitation. Several numerical examples verify the applicability of this proposed method.

2. Review of the Existing 3M Reliability Indices
2.1 Principle of the 3M Method
Without loss of generality, the limit state function \( Z \) can be standardized by
\[ Z_s = \frac{Z - \mu_G}{\sigma_G} \]  

(2)

where \( \mu_G \) and \( \sigma_G \) are the mean and standard deviation of \( G(X) \), respectively.

According to the definition of the probability, the failure probability can be expressed as

\[ P_f = P[G(X) \leq 0] = P[Z_s \sigma_G + \mu_G \leq 0] \]

\[ = P[Z_s \leq -\frac{\mu_G}{\sigma_G}] = P[Z_s \leq -\beta_{2M}] \]

(3)

where

\[ \beta_{2M} = \frac{\mu_G}{\sigma_G} \]  

(4)

is the second-moment (2M) reliability index.

Suppose the standardized variable \( Z_s \) can be expressed as a function of its third moment \( \alpha_{3G} \),

\[ Z_s = S(u, \alpha_{3G}) \]

(5)

where \( u \) is the standard normal variable, and \( \alpha_{3G} \) is the skewness of \( G(X) \) (Zhao and Lu, 2008).

Substituting Eq. (5) into Eq. (3), the failure probability can be expressed as

\[ P_f = P[z_s = S(u, \alpha_{3G}) \leq -\beta_{2M}] \]

(6)

if the inverse function of \( S \) is

\[ u = S^{-1}(Z_s) \]

(7)

According to Eqs. (6) and (7), it is not difficult to obtain

\[ P_f = P[u \leq S^{-1}(-\beta_{2M})] = \Phi[S^{-1}(-\beta_{2M})] \]

(8)

Therefore, the reliability index is expressed as

\[ \beta_{3M} = -\Phi^{-1}(P_f) = -S^{-1}(-\beta_{2M}) \]

(9)

For the first three moments of \( G(X) \) used in Eq. (9), the reliability index calculated by Eq. (9) is called the 3M reliability index—thus the name, the 3M reliability method. From Eq. (9), if the inverse function of \( S(u) \) is obtainable, the 3M reliability index can be given.

2.2 3M Reliability Index Based on 3P Lognormal Distribution

With the first three moments of the performance function \( z = G(X) \), assuming that \( Z_s \) obeys three-parameter (3P) lognormal distribution (Tichy 1994), the relationship between \( Z_s \) and \( u \) is given as

\[ Z_s = S(u) = u_b \left( 1 - \frac{1}{\sqrt{A}} \exp \left[ \text{Sign}(\alpha_{3G}) \sqrt{\ln(A)} u_b \right] \right) \]

(10)

where

\[ A = 1 + \frac{1}{u_b^2} \]

(11a)

\[ u_b = (a + b) + \sqrt{(a - b)^2 - \frac{1}{\alpha_{3G}}} \]

(11b)

\[ a = -\frac{3}{\alpha_{3G}} \left( \frac{1}{\alpha_{3G}} + \frac{1}{2} \right), \quad b = \frac{1}{2\alpha_{3G}^2} \sqrt{\alpha_{3G}^2 + 4} \]

(11c)

where \( \text{Sign}(x) \) is \(-1, 0, \) or 1 while \( x \) is negative, zero, or positive, respectively.

The relationship between \( u_b \) and \( \alpha_{3G} \) is given as

\[ \alpha_{3G} = \left( 3 + \frac{1}{u_b^2} \right) \frac{1}{u_b} \]

(12)

For small \( \alpha_{3G} \), i.e., \( \alpha_{3G} \leq 1 \), it has been derived that (Zhao and Ono, 2000a)

\[ Z_u = \frac{3}{\alpha_{3G}^2} \left[ 1 - \exp \left( \frac{\alpha_{3G}^2}{3} \left( u - \frac{\alpha_{3G}}{6} \right) \right) \right] \]

(13)

\[ u = \alpha_{3G} + \frac{3}{\alpha_{3G}} \ln \left( 1 - \frac{1}{3} \alpha_{3G} Z_u \right) \]

(14)

According to Eqs. (9) and (14), the 3M reliability index is obtained as

\[ \beta_{3M} = -\frac{\alpha_{3G}}{6} - \frac{3}{\alpha_{3G}} \ln \left( 1 - \frac{1}{3} \alpha_{3G} \beta_{2M} \right) \]

(15)

2.3 3M Reliability Index Based on 3P Square Normal Distribution

In another formula of the 3M reliability index, \( Z_s \) is assumed to obey 3P square normal distribution (Zhao and Ono, 2000b), the \( u-Z_u \) transformation is expressed as

\[ Z_u = S(u) = a_1 + a_2 u + a_3 u^2 \]

(16)

where

\[ a_3 = -a_1 = \pm \sqrt{2} \cos \left( \frac{\frac{\pi + |\theta|}{3}}{3} \right) \]

(17)

\[ a_2 = \sqrt{1 - 2a_3^2} \]

(18)

\[ \theta = \tan^{-1} \left( \frac{\sqrt{8 - \alpha_{3G}^2}}{\alpha_{3G}} \right) \]

(19)

For \(-1 < \alpha_{3G} < 1\), \( a_3 \) can be simplified as (Zhao and Ono, 2001)

\[ a_3 = \alpha_{3G}/6 \]

(20)

where the simplification error is less than 2\%. 

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Then, it can be derived that

\[ Z_u = \frac{1}{6} \alpha_M \left(1 - u^2\right) - u \]  
(21)

\[ u = \frac{\sqrt{9 + \alpha_M^2 - 6 \alpha_M Z_u} - 3}{\alpha_M} \]  
(22)

Using the relationship in Eq. (9), the 3M reliability index is obtained as (Zhao and Lu, 2006)

\[ \beta_{3M-2} = \frac{3 - \frac{9 + \alpha_M^2 - 6 \alpha_M \beta_{2M}}{\alpha_M}} \]  
(23)

\[ \frac{-120 r}{\beta_{2M}} \leq \alpha_M \leq \frac{40 r}{\beta_{2M}} \]  
(24)

where \( r \) is the allowable relative difference.

However, there are other limitations for the calculation of the reliability index for \( \beta_{3M-1} \) and \( \beta_{3M-2} \). First, for the antilogarithm of Eq. (15), the limitation is

\[ 1 - \frac{1}{3} \alpha_M \beta_{2M} > 0 \]  
\[ \alpha_M \beta_{2M} < 3 \]  
(25)

where \( \alpha_M \neq 0 \).

Furthermore, for the square root of the numerator and the variable of the denominator in Eq. (23), the following qualification should be observed:

\[ 9 + \alpha_M^2 - 6 \alpha_M \beta_{2M} \geq 0 \]  
\[ 6 \alpha_M \beta_{2M} \leq 9 + \alpha_M^2 \]  
\[ \beta_{2M} \leq \frac{9 + \alpha_M^2}{6 \alpha_M^2} \]  
(26)

where \( \alpha_M \neq 0 \) also. Fig.1. shows that the difference between the two reliability indices increases as \( |\alpha_M| \) increases; the further \( \alpha_M \) is from 0, the greater the error. In fact, in the case of negative \( \alpha_M \), the exact value calculated by Monte-Carlo (MC) simulation is in the middle of the two reliability indices calculated by \( \beta_{3M-1} \) and \( \beta_{3M-2} \). Furthermore, in the case of a positive \( \alpha_M \), the result of MC simulation is below the values of \( \beta_{3M-1} \) and \( \beta_{3M-2} \). Because the MC simulation can only be used for analysis of practical examples, different 3M reliability indices will be compared with MC simulation results in Section 4.

As shown in Figs. 2 and 3, the smaller the \( \beta_{2M} \), the smaller the difference between the two reliability indices and the more reliable the approximate value, where the dotted line is the 2M method. Figs. 2 and 3. show that the difference between the two reliability indices is greater when \( \alpha_M \) is positive compared with the case when \( \alpha_M \) is negative. In other words, the 3M method is more applicable when \( \alpha_M \) is negative.

According to the formulae mentioned above, one can see that:

1. If Eq. (15) cannot meet the limitation of Eqs. (24) and (25), the 3M reliability index cannot be defined.
2. If Eq. (23) cannot meet the limitation of Eqs. (24) and (26), the 3M reliability index cannot be defined.
3. If the value of \( \beta_{2M} \) or \( |\alpha_M| \) is large, their calculation error will be great.

**3. Proposed 3M Method**

**3.1 Proposed Formula**

In order to overcome the limitations and insufficient accuracy of current 3M reliability indices, the variation roles of the 3M reliability indices with respect to the 2M reliability index and the third moment of the performance function are thoroughly investigated.

Fitting the average value of \( \beta_{3M-1} \) and \( \beta_{3M-2} \), a simple 3M reliability index in the form of exponential function, using a trial and error method, is proposed as

\[ \beta_{3M-3} = \frac{1}{3} \beta_{2M} \left[ 2 + e^{\frac{-1}{2} \alpha_M \beta_{2M} \beta_{3M}} \right] \]  
(27)
where $\beta_{M3}$ is a fitting equation for the average value of $\beta_{M1}$ and $\beta_{M2}$.

In the case of $\alpha_{SG} \rightarrow 0$, Eq. (27) degenerates into

$$\beta_{M} = \beta_{M2}$$

(28)

Compared with $\beta_{M3}$, the proposed formula does not include any logarithmic term. Compared with $\beta_{M2}$, it does not include any square root. With the calculation error in an acceptable range, Eq. (27) is theoretically applicable for calculating the 3M reliability index in all cases except $\beta_{M2} \rightarrow 0$.

In the case of $\beta_{M2} \rightarrow 0$, $\lim \beta_{M3}$ is expressed as

$$\lim_{\beta_{M2} \rightarrow 0} \beta_{M3} = \begin{cases} 0, & \alpha_{SG} \geq 0 \\ \infty, & \alpha_{SG} < 0 \end{cases}$$

(29a)

$$\lim_{\beta_{M2} \rightarrow 0} \beta_{M3} = \begin{cases} 0, & \alpha_{SG} \leq 0 \\ -\infty, & \alpha_{SG} > 0 \end{cases}$$

(29b)

Thus the proposed Eq. (27) is not suitable for $\beta_{M2}$ which tends to 0. However, in engineering practice the reliability index is usually much larger than 0.

3.3 Adaptability of the Proposed Formula

Compared with $\beta_{M1}$ and $\beta_{M2}$, the proposed method $\beta_{M3}$ is accurate enough, as shown in Figs.1. - 3.

According to Figs.1, $\beta_{M1}$, $\beta_{M2}$, and $\beta_{M3}$ change with the value of $\alpha_{SG}$ for $\beta_{M2} = 1, 2, 3, 4$. As the values of $\alpha_{SG}$ and $\beta_{M2}$ increase, to compare with $\beta_{M1}$ and $\beta_{M2}$, the error of $\beta_{M3}$ becomes obvious. But Eq. (27) can still be used to calculate the reliability index, where other methods may not be within the applicable range, as shown in Figs.1. and 2.

The changes of $\beta_{M1}$, $\beta_{M2}$, and $\beta_{M3}$ with respect to $\beta_{M2}$ are depicted in Figs.2. and 3. for $\alpha_{SG} = 0.3, 0.6, 1.0$ and $-0.3, -0.6, -1.0$. One can see that the differences among the three 3M reliability indices are smaller when $\alpha_{SG}$ is negative than when it is positive. Because $\beta_{M1}$ and $\beta_{M2}$ are not accurate when $\alpha_{SG}$ is large (i.e., positive), it is too early to evaluate the accuracy of $\beta_{M3}$ based only on comparison with $\beta_{M1}$ and $\beta_{M2}$.

In Tables 1. and 2., the relative difference is given as $r = 2|\beta_{M} - \beta_{M3}|/(\beta_{M} + \beta_{M3})$, where $\beta_{M}$ is the average value of $\beta_{M1}$ and $\beta_{M2}$. It is shown that for the cases of $2 \leq \beta_{M2} \leq 4$ and $-1.0 \leq \alpha_{SG} \leq 0.3$, the relative difference remains at less than 5%. But $2 \leq \beta_{M2} \leq 4$ and $-1.0 \leq \alpha_{SG} \leq 0.3$ are not the applicable range of the proposed method. For the case of $\alpha_{SG} > 0.3$ or $\beta_{M2} > 4$, the relative difference is insignificant, because both $\beta_{M1}$ and $\beta_{M2}$ are inaccurate at this range. In other words, it is possible that the applicable range of the proposed method is wider than the existing methods, $\beta_{M1}$ and $\beta_{M2}$. Further comparison of the existing 3M methods and the MC simulation is given in Section 4.

3.3 The Degeneration Form of the Proposed Formula

In the case of $\alpha_{SG} \rightarrow 0$, with the aid of a first-order Taylor expansion of $e^\alpha$, $\beta_{M3}$ can be written as

$$\beta_{M3} = \frac{1}{3} \beta_{M2} \left[ 2 + e^{\frac{1}{3} \alpha_{SG} (\beta_{M2} - \frac{1}{\beta_{M2}})} \right]$$

(30)

$$\beta_{M3} = \beta_{M2} + \frac{1}{6} \alpha_{SG} (\beta_{M2}^2 - 1)$$

It has already been shown that both $\beta_{M1}$ and $\beta_{M2}$ have the same limit when $\alpha_{SG} \rightarrow 0$ (Zhao and Lu, 2006).

3.4 The Corresponding Relationship between $Z_u$ and $u$

According to Eqs. (7) and (9), the corresponding $u$ of Eq. (27) can be expressed as

$$u = -\frac{1}{3} Z_u \left[ 2 + e^{\frac{1}{3} \alpha_{SG} (Z_u - \frac{1}{Z_u})} \right]$$

(31)

According to Eqs. (5) and (31), $Z_u$ can be expressed as the inverse function of Eq. (31). Because the inverse function of Eq. (31) is nonexistent, the approximate function can be obtained and $Z_u$ can be expressed as

$$Z_u = \frac{3.8}{\alpha_{SG}} \left[ 1 - e^{-\frac{\alpha_{SG} u}{3.8}} \right]$$

(32)

In the case of $\alpha_{SG}$ equals $-1$ to 0.5 and $Z_u$ equals $-4.0$ to $-1.0$, the errors of Eqs. (31) and (32) are certified to be less than 7%, as shown in Fig.4.

| $\beta_{M2}$ | 2.0 | 3.0 | 4.0 | 5.0 |
|-------------|-----|-----|-----|-----|
| $\alpha_{SG} > 0$ | 0.43 | 0.6 | 0.73 | 0.31 | 0.41 | 0.48 | 0.23 | 0.31 | 0.35 | 0.19 | 0.24 | 0.29 |
| Relative difference | 2% | 5% | 10% | 2% | 5% | 10% | 2% | 5% | 10% | 2% | 5% | 10% |

Table 1. The Relative Difference between $\beta_{M1}$, $\beta_{M2}$, and $\beta_{M3}$ ($\alpha_{SG} > 0$)

| $\beta_{M2}$ | 2.0 | 3.0 | 4.0 | 5.0 |
|-------------|-----|-----|-----|-----|
| $\alpha_{SG} < 0$ | -0.8 | -1.72 | -4.18 | -1.6 | -2.21 | -15.74 | -0.81 | -1.13 | -1.56 | -0.57 | -0.79 | -1.0 |
| Relative difference | 2% | 5% | 10% | 2% | 5% | 10% | 2% | 5% | 10% | 2% | 5% | 10% |

Table 2. The Relative Difference between $\beta_{M1}$, $\beta_{M2}$, and $\beta_{M3}$ ($\alpha_{SG} < 0$)
4. Numerical Examples

In order to investigate the accuracy and availability of the proposed 3M reliability index, Eq. (27), six examples are examined under different conditions.

Example 1
First, the simple $R - S$ reliability model is considered.

$$G(X) = R - S$$

(33)

where $R$ and $S$ are the interior resistance and the exterior deterioration load effect, respectively.

The statistical parameters of random variables are listed in Table 3., where $\mu_R = 50 - 100$ and $\mu_S = 30$ are the means of $R$ and $S$, $\nu_R = 0.2$ and $\nu_S = 0.4$ are the coefficient of variation of $R$ and $S$, respectively. The exact results are obtained using MC simulation for $10^6$ samplings and the 2M results are given. The following six cases are investigated under the assumption that $R$ and $S$ obey different probability distributions (see Table 3.).

Table 3. The Probability Distribution Information of $R$ and $S$ in Different Cases

| Case number | $R$   | $S$   |
|-------------|-------|-------|
| 1           | Normal| Lognormal |
| 2           | Normal| Gamma |
| 3           | Normal| Gumbel |
| 4           | Lognormal| Normal |
| 5           | Lognormal| Weibull |
| 6           | Weibull| Normal |

For cases 1 - 6, as shown in Fig.5., one can see that as the value of $\mu_R$ increases, the results of each 3M method move further from the exact MC simulation. It is obvious that the 2M method is not accurate enough, while the proposed 3M method is in close agreement with the MC simulation in all cases. And it can be seen that the proposed 3M method has, either higher or at least the same accuracy as the existing methods.

Fig. 4. Comparison of Eqs. (31) and (32)

Fig. 5. Relationship between the Mean Value of $R$ and the Reliability Index for Example 1
Example 2
The analytical I-beam design problem, as shown in Fig. 6., is considered here.

The limit state function is given in terms of bending stress as

\[ G(X) = \sigma_{\text{max}} - S \]  \hspace{1cm} (34)

where

\[ \sigma_{\text{max}} = \frac{P(aL - a)d}{2LI} \]  \hspace{1cm} (35a)

\[ I = \frac{b_f d^3 - (b_f - t_w)(d - 2t_f)^3}{12} \]  \hspace{1cm} (35b)

The details of all the random variables are given in Table 4.

Table 4. Parameters of Random Variables for Example 2

| RVs | \( \mu \) | \( \nu \) | Distribution |
|-----|----------|----------|--------------|
| \( P \) | 14000 | 0.4 | Lognormal |
| \( L \) | 120 | 0.2 | Lognormal |
| \( a \) | 72 | 0.08 | Normal |
| \( S \) | 170000 | 0.03 | Normal |
| \( d \) | 2.3 | 0.02 | Normal |
| \( b_f \) | 2.3 | 0.02 | Normal |
| \( t_w \) | 0.16 | 0.03 | Normal |
| \( t_f \) | 0.16 | 0.03 | Normal |

In this example, the first three moments of \( G(X) \) can be obtained as \( \mu_\alpha = 455704, \sigma_\alpha = 249347, \) and \( \text{skewness} = 1.098. \) Reliability indices calculated by different methods are summarized in Table 5. The 2M is not applicable for an error of 24.04%, and the 3M-2 is out of its applicable range, which is expressed in Eq. (26). Furthermore for an error of 19.77%, the 3M-1 is considered inapplicable, as well. The 3M-3 is the only applicable method with an error as low as 5.11%.

Example 3
The proposed method is applicable in not only civil engineering but also other fields. The performance function of a shaft in a speed reducer can be defined as

\[ G(X) = S - \frac{32}{\pi D} \sqrt{\frac{F^2L^2}{16} + T^2} \]  \hspace{1cm} (36)

where \( X = \{ S, D, F, L, T \}, S (\text{MPa}) \) is the material strength, \( D (\text{mm}) \) is the diameter of the shaft, \( F (\text{N}) \) is the external force, \( T (\text{Nm}) \) is the external torque, and \( L (\text{mm}) \) is the length of the shaft. The performance function represents the difference between the strength and the maximum stress. The details of the random variables are given in Table 6.

The mean value, standard deviation, and skewness of \( G(X) \) are obtained as \( \mu_\alpha = 61.05, \sigma_\alpha = 17.30, \) and \( \text{skewness} = -1.898, \) respectively. Because \( |\alpha_{\text{skewness}}| \) is large in this example, none of the methods are accurate; however, the 3M-3 provides the highest accuracy, as shown in Table 7. Its error of 10.26% is much lower than the error of 20.52% for 3M-1 and 17.08% for 3M-2.

Table 6. Parameters of Random Variables for Example 3

| RVs | \( \mu \) | \( \nu \) | Distribution |
|-----|----------|----------|--------------|
| \( P \) | 0.2 | 17.0 | Normal |
| \( E \) | 5.11% | 0.18 | Lognormal |
| \( L \) | 27.82% | 70 | Normal |
| \( T \) | 50 | 2 | Lognormal |
| \( T \) | 60 | 2 | Normal |

Table 7. Comparison of Reliability Indices for Example 3

| Method | \( \beta \) | Error |
|--------|----------|-------|
| 2M | 1.828 | 24.04% |
| 3M-1 | 2.837 | 19.77% |
| 3M-2 | Out of range | — |
| 3M-3 | 2.449 | 5.11% |
| MC | 2.327 | — |

Example 4
A cantilever beam made of isotropic material, as shown in Fig. 7., is subjected to a distributed transverse load.

The performance function is the tip displacement, which is expressed as

\[ G(X) = \delta_{cr} - \delta = \delta_{cr} - \frac{Q}{8EI} \]  \hspace{1cm} (37)

where \( X = \{ \delta_{cr}, Q, L, E, I \}, \delta_{cr} = 10 \text{ mm} \) is the critical tip displacement (which is 1/500 of the length of the beam), \( Q \) is the constant distributed transverse load acting on the beam (\( \mu_\alpha = 5 \text{ N/mm}, \nu_\alpha = 0.3 \)), \( L \) is the length of the beam (\( \mu_\alpha = 5000 \text{ mm}, \nu_\alpha = 0.04 \)), \( E \) is the Young’s modulus of the beam material (\( \mu_\alpha = 73000 \text{ N/mm}^2, \nu_\alpha = 0.01 \)), and \( I \) is the moment of the cross-section (\( \mu_\alpha = 1.067 \times 10^6 \text{ mm}^4, \nu_\alpha = 0.001 \)). All of the random variables obey normal distribution.

In this example, the first three moments of \( G(X) \) can be obtained as \( \mu_\alpha = 4.94, \sigma_\alpha = 1.74, \) and \( \text{skewness} = -0.388. \) As shown in Table 8., the result of the 2M is far from MC simulation, while the errors of all three 3M methods are in acceptable range. Because it has the lowest error, 3M-3 is considered the most accurate method.
Table 8. Comparison of Reliability Indices for Example 4

| Method | $\beta$ | Error |
|--------|---------|-------|
| 2M     | 2.838   | 13.93%|
| 3M-1   | 2.482   | 0.54% |
| 3M-2   | 2.499   | 1.22% |
| 3M-3   | 2.476   | 0.30% |
| MC     | 2.469   |       |

Example 5

This example is a one-bay elastoplastic frame, as shown in Fig. 8. (Zhao and Ang, 2003),

where $M_1$, $M_s$, and $M_r$ are the member strengths and $S_1$ and $S_2$ are the loads. The mean values of the random variables are $\mu_M = 500$ kip, $\mu_S = 667$ kip, $\mu_S = 50$ kip, and $\mu_S = 100$ kip; the standard deviations are $\sigma_M = 75$ kip, $\sigma_M = 100$ kip, $\sigma_M = 15$ kip, and $\sigma_M = 10$ kip.

The performance functions that correspond to the six most likely failure modes obtained from stochastic limit analysis are listed as follows:

\[ g_1 = M_1 + 3M_2 + 2M_3 - 15S_1 - 10S_2 \]  
\[ g_2 = 2M_1 + 2M_2 - 15S_1 \]  
\[ g_3 = M_1 + M_2 + 4M_3 - 15S_1 - 10S_2 \]  
\[ g_4 = 2M_1 + 2M_2 + 3M_3 - 15S_1 \]  
\[ g_5 = M_1 + M_2 + 2M_3 - 15S_1 \]  
\[ g_6 = M_1 + 2M_2 + 3M_3 - 15S_1 \]  

The performance function of the series system can be expressed as the minimum of the performance functions that corresponds to all potential failure modes, which is

\[ G(X) = \min\{g_1, g_2, g_3, g_4, g_5, g_6\} \]  

Using the method found in Zhao and Ang, 2003, the first three moments of $G(X)$ can be obtained as $\mu_G = 1244.85$, $\sigma_G = 307.523$, and $\sigma_G = -0.307$. With different methods, the reliability indices are $\beta_{3M} = 4.048$, $\beta_{3M-1} = 3.438$, $\beta_{3M-2} = 3.480$, and $\beta_{3M-3} = 3.452$. Three 3M methods are in good agreement.

Different types of distribution of the random variables were also assumed. Assuming all the member strengths and loads are Weibull random variables, the results of the different 3M methods and the MC simulation (10⁶ samplings) are summarized in columns 3 - 5, respectively, in Table 9.

Table 9. Comparison of Reliability Indices for Example 5 with different Types of PDFs

| Method | Weibull | Gamma | Gumbel | Normal |
|--------|---------|-------|--------|--------|
| 2M     | (13.51%) | (13.73%) | (27.1%) | (4.5%) |
| 3M-1   | (5.46%)  | (4.90%)  | (12.46%) | (5.59%) |
| 3M-2   | (5.80%)  | (5.31%)  | (13.46%) | (5.59%) |
| 3M-3   | (5.49%)  | (4.14%)  | (12.74%) | (5.59%) |
| MC     | 3.382    | 3.521   | 3.139   | 3.805 |

Note: Percentage of error in the reliability index relative to that of the MC simulation is in parenthesis.

From Table 9., one can observe that each of the three 3M methods is in close agreement with 3M simulation, except for Gumbel distribution. In all cases, the proposed method is either more accurate than or as accurate as other methods.

Example 6

This example is a frame structure with two stories and two bays, as shown in Fig. 9.

The mean values of the probabilistic member strength are $\mu_M = \mu_M = \mu_M = \mu_M = -70$ ft kip, $\mu_M = 120$ ft kip, and $\mu_M = 90$ ft kip. The mean values of the probabilistic loads are $\mu_S = 5$ kip and $\mu_S = \mu_S = \mu_S = 10$ kip. The standard deviations of the member strength and loads are 0.15 and 0.25, respectively; the distributions of the member strength and loads are normal distribution and lognormal distribution, respectively.

The failure modes and corresponding performance functions are listed as follows:

\[ g_1 = 2M_1 + 2M_2 + 2M_3 - 15S_1 - 15S_2 \]  
\[ g_2 = M_6 + M_7 + 2M_8 - 10S_5 \]  
\[ g_3 = M_3 + 3M_5 - 10S_4 \]  
\[ g_4 = M_7 + 3M_8 - 10S_5 \]  
\[ g_5 = 2M_1 + 2M_2 + M_3 + M_5 - 15S_1 - 15S_2 \]  
\[ g_6 = M_6 + 3M_7 + 2M_8 - 15S_1 \]  

The performance function of the series system is the same as Eq. (39) in Example 5.
Using the method found in Zhao and Ang, 2003, the first three moments of $G(X)$ can be obtained as $\mu_G = 170.75$, $\sigma_G = 58.08$, and $\alpha_3 = -0.457$. With different methods, the results are listed in Table 10., which shows that the 2M result is far from MC simulation, while the error of 3M-1 and the proposed 3M-3 are accurate enough, with errors of less than 2%.

### Table 10. Comparison of Reliability Indices for Example 4

| Method | $\beta$   | Error |
|--------|-----------|-------|
| 2M     | 2.940     | 17.25%|
| 3M-1   | 2.506     | 1.33% |
| 3M-2   | 2.529     | 2.24% |
| 3M-3   | 2.501     | 1.13% |
| MC     | 2.473     | —     |

### 5. Conclusion

Based on existing methods for calculating the 3M reliability index, this paper proposes a new 3M method in which the following improvements are considered valuable:

1. The proposed method, with less mathematical limitation, is simpler for calculation of a 3M reliability index than other existing 3M methods.
2. Compared with other methods, the proposed method is accurate enough and its applicable range is much wider — especially in the case of negative $\alpha_3$. It is, therefore, considered applicable for cases out of the applicable range of existing methods.

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