On approximation of homeomorphisms of a Cantor set

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Abstract. We continue to study topological properties of the group Homeo(X) of all homeomorphisms of a Cantor set X with respect to the uniform topology τ, which was started in [B-K], [B-D-K 1; 2], [B-D-M], and [B-M]. We prove that the set of periodic homeomorphisms is τ-dense in Homeo(X) and deduce from this result that the topological group (Homeo(X), τ) has the Rokhlin property, i.e., there exists a homeomorphism whose conjugate class is τ-dense in Homeo(X). We also show that for any homeomorphism T the topological full group [T] is τ-dense in the full group [T].

1. Introduction

Many famous problems in ergodic theory involve the use of topologies on the group Aut(X, B, μ) of all measure-preserving transformations of a standard measure space. The first results on group topologies of Aut(X, B, μ) appeared in the paper [Hal 1]. Halmos introduced two topologies du and dw, which were called later the uniform and weak topologies, respectively. He defined the uniform topology du by saying that two automorphisms T and S are “close” to each other if the quantity μ({x ∈ X : Tx ≠ Sx}) is small enough. The weak topology is generated by the sets of the form N(T; E; ε) = {S ∈ Aut(X, B, μ) : μ(SE△TE) < ε}, where T ∈ Aut(X, B, μ) and E ∈ B. The use of these topologies turned out to be very fruitful and led to many outstanding results in ergodic theory (for references, see, for example, [B-K-M] and [C-F-S]). One of the most relevant results in the theory is the Rokhlin lemma [Ro] stating that the set of periodic automorphisms is du-dense in Aut(X, B, μ).

The idea of investigation of transformation groups by means of introducing various topologies into these groups was used in [B-D-K 1] and [B-D-K 2]. In the papers, the authors considered the groups Aut(X, B) of all automorphisms of a standard Borel space and the group Homeo(X) of all homeomorphisms of a Cantor set X with respect to the several topologies analogous to those in ergodic theory.

Following [B-D-K 2], we continue studying the group Homeo(X) of all homeomorphisms of a Cantor set X with respect to the topology τ (cf. Definition 1.1), which is obviously a direct analog of the topology du. We show that the set of all periodic homeomorphisms is τ-dense in Homeo(X)

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This result can be treated as a topological version of the Rokhlin lemma. As a corollary, we prove that the set of topologically free homeomorphisms is \( \tau \)-dense in \( \text{Homeo}(X) \) (Theorem 2.8). Recall that a homeomorphism is called \textit{topologically free} if the set of aperiodic points is dense.

In \([G-K]\), an interesting class of topological groups was defined: by definition, a topological group has the \textit{Rokhlin property} if it has an element whose conjugate class is dense. The authors raised the question: which groups possess this property. At the moment, there is an extensive list of groups that have the Rokhlin property. In particular, the group \( \text{Homeo}(X) \) and \( \text{Aut}(X) \) (Theorem 2.5). In [Gl-W], and the group \( \text{Aut}(X, B, \mu) \) with respect to the weak topology \([Hal2]\) have the Rokhlin property. See also the paper \([Ke-Ros]\) for a general approach to the study of groups with dense conjugate classes.

Motivated by this, we present a unique approach allowing us to show that the topological groups \( (\text{Aut}(X, B), \tau) \) and \( (\text{Homeo}(X), \tau) \) have the Rokhlin property (Theorem 2.5).

The other part of the paper is devoted to the study of full groups \( [T] \) of homeomorphisms \( T \in \text{Homeo}(X) \). Our motivation comes from the paper \([G-P-S]\), where full groups were indispensable in the study of orbit equivalence of Cantor minimal systems. It is worthwhile to investigate full groups and their dense subsets for any homeomorphism.

In this context, we show that for any \( T \in \text{Homeo}(X) \), the topological full group \( [[T]] \) is \( \tau \)-dense in the full group \( [T] \) (Theorem 2.1).

In the last section, we give a description of homeomorphisms from the topological full group \( [[T]] \) of aperiodic \( T \) (Theorem 3.3). We consider a subgroup \( \Gamma_Y \) of \( [[T]] \), which is an increasing union of permutation groups, and find a criterion when \( \Gamma_Y \) is \( \tau \)-dense in \( [[T]] \) (Theorem 3.4).

\textbf{Background.} Throughout the paper, \( X \) denotes a \textit{Cantor set} and \( B \) stands for the \( \sigma \)-algebra of Borel sets of \( X \). A one-to-one Borel map \( T \) of \( X \) onto itself is called an \textit{automorphism} of \( (X, B) \). Denote by \( \text{Aut}(X, B) \) the group of all automorphisms of \( (X, B) \) and by \( \text{Homeo}(X) \) the group of all homeomorphisms of \( X \).

Following \([B-K]\), recall the definition of the \textit{uniform topology} on \( \text{Aut}(X, B) \). Let \( \mathcal{M}_1(X) \) denote the set of all Borel probability measures on \( X \). For \( T, S \in \text{Aut}(X, B) \), denote \( E(T, S) = \{ x \in X : Tx \neq Sx \} \).

\textbf{Definition 1.1.} The uniform topology \( \tau \) on \( \text{Aut}(X, B) \) is defined by the base of neighborhoods \( \mathcal{U} = \{ U(T; \mu_1, \ldots, \mu_n; \varepsilon) \} \), where \( U(T; \mu_1, \mu_2, \ldots, \mu_n; \varepsilon) = \{ S \in \text{Aut}(X, B) : \mu_i(E(S, T)) < \varepsilon, \ i = 1, \ldots, n \} \). Here \( T \in \text{Aut}(X, B) \), \( \mu_1, \ldots, \mu_n \in \mathcal{M}_1(X) \), and \( \varepsilon > 0 \).

As \( \text{Homeo}(X) \) is a subgroup of \( \text{Aut}(X, B) \), we also denote by \( \tau \) the topology on \( \text{Homeo}(X) \) induced from \( (\text{Aut}(X, B), \tau) \).

Observe that \( \text{Aut}(X, B) \) and \( \text{Homeo}(X) \) are Hausdorff topological groups with respect to the uniform topology \( \tau \). More results related to topological properties of \( \text{Aut}(X, B) \) and \( \text{Homeo}(X) \) with respect to \( \tau \) can be found in \([B-D-K1]\), \([B-D-K2]\), \([B-D-M]\), \([B-K-M]\), \([B-M]\).
Let $T \in \text{Aut}(X, \mathcal{B})$. A point $x \in X$ is called periodic of period $n > 0$ if $T^n x = x$ and $T^i x \neq x$ for $i = 1, \ldots, n - 1$. If $T^n x \neq x$ for $n \neq 0$, the point $x$ is called aperiodic. We say that $T$ is aperiodic if it has no periodic points. Note that for any $T \in \text{Aut}(X, \mathcal{B})$ the set $X$ can be decomposed into a disjoint union of Borel sets $X = X_\infty \cup \bigcup_{n \geq 1} X_n$, where $X_n$ consists of all points of period $n$ and $X_\infty$ is formed by all aperiodic points. Notice that some $X_n$’s can be empty. Moreover, for every $X_n$, $n < \infty$, there exists a Borel set $X_n^0 \subset X_n$ such that $X_n = \bigcup_{i=0}^{n-1} T^i X_n^0$ is a disjoint union. We call \{X_\infty, X_1, X_2, \ldots\} the canonical partition of $X$ associated to $T$.

Recall that a finite family of disjoint Borel sets $\xi = \{A, TA, \ldots, T^{n-1}A\}$ is called a $T$-tower with the base $B(\xi) = A$ and the height $h(\xi) = n$. A partition $\Xi = \{\xi_1, \xi_2, \ldots\}$ of $X$ is called a Kakutani-Rokhlin (K-R) partition if every $\xi_i$ is a $T$-tower. For a K-R partition $\Xi$, we denote $\bigcup_{n \geq 1} B(\xi_i)$ by $B(\Xi)$ and call it the base of the K-R partition. Notice that for a K-R partition $\Xi$, one has that $T^{-1}B(\Xi) = \bigcup_{\xi \in \Xi} T^{h(\xi)}B(\xi)$.

For $T \in \text{Aut}(X, \mathcal{B})$, let $\text{Orb}_T(x) = \{T^n x : n \in \mathbb{Z}\}$ denote the $T$-orbit of $x$. With any homeomorphism $T \in \text{Homeo}(X)$, we can assign two full groups $[T]_C$ and $[T]_B$, where

$$[T]_C = \{S \in \text{Homeo}(X) : \text{Orb}_S(x) \subseteq \text{Orb}_T(x), x \in X\}$$

$$[T]_B = \{S \in \text{Aut}(X, \mathcal{B}) : \text{Orb}_S(x) \subseteq \text{Orb}_T(x), x \in X\}.$$

Here the subindices $C$ and $B$ stand for the cases of Cantor and Borel dynamics, respectively. Clearly, $[T]_C$ is a subgroup of $[T]_B$. Observe that if $S \in [T]_B$, then there is a Borel function $n_S : X \to \mathbb{Z}$ such that $Sx = T^{n_S(x)}x$ for all $x \in X$. The subgroup $[[T]] = \{S \in [T]_C : n_S$ is continuous\} is called the topological full group of $T$.

One of the main results in the approximation theory of Borel automorphisms is a Borel version of the Rokhlin lemma. The following $\tau$-version of the Rokhlin lemma was proved in [B-D-K, Proposition 3.6]. We also refer the reader to the works [N, Section 7] and [W, Section 4] for measure free versions of the result.

**Theorem 1.2.** Let $T$ be an aperiodic automorphism of $X$. Then there exists a sequence of periodic automorphisms $(P_n) \in \text{Aut}(X, \mathcal{B})$ such that $P_n \xrightarrow{\tau} T$, $n \to \infty$. Moreover, the automorphisms $P_n$ can be taken from $[T]_B$.

Denote by $\mathcal{P}_{\text{er}}$ the set of all homeomorphisms $P$ such that $P^n = \mathbb{I}$ for some $n \in \mathbb{N}$; and for $T \in \text{Homeo}(X)$, set $\mathcal{P}_{\text{er}}(T) = \mathcal{P}_{\text{er}} \cap [[T]]$.

2. **Rokhlin lemma**

In the section, we prove a topological version of the Rokhlin lemma, namely, we show that the set of periodic homeomorphisms is $\tau$-dense in $\text{Homeo}(X)$. Then, we deduce several corollaries of this result. In particular, we prove that the topological group $(\text{Homeo}(X), \tau)$ possesses the Rokhlin property and the topological full group $[[T]]$ is $\tau$-dense in $[T]_B$ for any $T \in \text{Homeo}(X)$.

**Theorem 2.1.** (1) The set $\mathcal{P}_{\text{er}}$ is $\tau$-dense in $\text{Homeo}(X)$.

(2) Let $T \in \text{Homeo}(X)$, then for any automorphism $S \in [T]_B$ and any
\(\tau\)-neighborhood \(U = U(S; \mu_1, \ldots, \mu_p, \varepsilon)\) of \(S\) there exists a periodic homeomorphism \(P \in [[T]]\) such that \(P \in U\).

**Proof.** Notice that statement (1) is an immediate corollary of (2). By Theorem \textbf{1.2}, it is enough to prove (2) for a periodic automorphism \(S\).

Let us sketch the main stages of the proof. (i) We find a finite number of disjoint \(S\)-towers consisting of closed sets and covering “almost” the entire space \(X\) with respect to the measures \(\mu_i\) such that on each level of these \(S\)-towers the automorphism \(S\) coincides with a power of \(T\). (ii) We extend the \(S\)-towers found in (i) to clopen ones constructed by powers of \(T\). (iii) Using the clopen towers, we define a periodic homeomorphism \(P\) which belongs to \(U(S; \mu_1, \ldots, \mu_p; \varepsilon)\).

(i) Let \(\Xi = \{X_1, X_2, \ldots\}\) be the canonical Borel partition of \(X\) associated to \(S\). Without loss of generality, we will assume that the sets \(X_i\) are not empty, \(i \in \mathbb{N}\).

We first find \(N \in \mathbb{N}\) such that
\[
\mu_j (X_1 \cup \ldots \cup X_N) > 1 - \frac{\varepsilon}{3} \quad \text{for all } j = 1, \ldots, p.
\]

For \(n \geq 1\), set \(Z_n = \{x \in X : Sx = T^i x \text{ for some } n \leq i \leq n\}\). For \(i \geq 1\) define the sets
\[
X_i^0(n) = \bigcap_{j=0}^{i-1} S^{-j} (S^j X_i^0 \cap Z_n),
\]
where \(X_i = X_i^0 \cup S X_i^0 \cup \ldots \cup S^{i-1} X_i^0\) is a disjoint union. Since \(S \in [T]_B\), we have that \(X_i = \bigcup_{n \geq 1} X_i(n)\), where \(X_i(n) = \bigcup_{j=0}^{i-1} S^j X_i^0(n)\). Then, find \(K \in \mathbb{N}\) such that
\[
\mu_j \left( \bigcup_{i=1}^{N} (X_i \setminus X_i(K)) \right) < \frac{\varepsilon}{3} \quad \text{for all } j = 1, \ldots, p.
\]

Denote by \(S_i\) the set of all maps from \(\{0, \ldots, i-1\}\) to \(\{-K, \ldots, K\}\). For \(\sigma \in S_i\), define the set
\[
X_i^0(K, \sigma) = \bigcap_{j=0}^{i-1} S^{-j} \left( \{x \in S^j X_i^0(K) : Sx = T^\sigma(j)x\} \right).
\]

Thus, we get a finite cover \(X_i^0(K) = \bigcup_{\sigma \in S_i} X_i^0(K, \sigma)\). Applying the standard argument, make the \(X_i^0(K, \sigma)\)'s disjoint and denote the obtained sets by \(X_i^0(K, \sigma)\) again. Some of the \(X_i^0(K, \sigma)\)'s can be empty, but, without loss of generality, we will assume they are not. Observe that \(S\) restricted to \(S^j X_i^0(K, \sigma)\) is equal to \(T^\sigma(j)\), for \(i \geq 1, j = 0, \ldots, i-1, \) and \(\sigma \in S_i\). This means that \(S\) is a homeomorphism on \(S^j X_i^0(K, \sigma)\).

For every \(X_i(K, \sigma) = \bigcup_{j=0}^{i-1} S^j X_i^0(K, \sigma)\), find a closed set \(A_i(\sigma) \subset X_i^0(K, \sigma)\) such that
\[
\mu_j \left( \bigcup_{i=1}^{N} \bigcup_{\sigma \in S_i} (X_i(K, \sigma) \setminus A_i(\sigma)) \right) < \frac{\varepsilon}{3} \quad \text{for } j = 1, \ldots, p,
\]
where \(A_i(\sigma) = \bigcup_{j=0}^{i-1} S^j A_i^0(\sigma)\).

(ii) Summing up the above, we get that \(\{A_i(\sigma) : 1 \leq i \leq N, \sigma \in S_i\}\) is a family of disjoint closed \(S\)-towers such that the automorphism \(S\) restricted
to $S^j A^0_i(\sigma)$ is equal to $T^{\sigma(j)}$. Furthermore, it follows from (2.1), (2.2), and (2.3) that

$$
\mu_j \left( \bigcup_{i=1}^{N} \bigcup_{\sigma \in S_i} A_i(\sigma) \right) > 1 - \varepsilon.
$$

As the closed $S$-towers $A_i(\sigma)$'s are disjoint, we can find clopen sets $A^0_i(\sigma) \supset A^0_i(\sigma)$ so that all the sets $A^0_i(\sigma)$ and $T^{\sigma(0)+\ldots+\sigma(j)} A^0_i(\sigma)$ are mutually disjoint for $i = 1, \ldots, N$, $j = 0, \ldots, i - 2$, and $\sigma \in S_i$.

(iii) Define the periodic homeomorphism $P$ as follows:

$$
P_x = \begin{cases} 
T^{\sigma(0)} x & \text{if } x \in \overline{A^0_i} \\
T^{\sigma(j+1)} x & \text{if } x \in T^{\sigma(0)+\ldots+\sigma(j)} \overline{A^0_i}(\sigma) \\
T^{-\sigma(0)-\ldots-\sigma(i-2)} x & \text{if } x \in T^{\sigma(0)+\ldots+\sigma(i-2)} \overline{A^0_i}(\sigma) \\
x & \text{if } 0 \leq j \leq i - 3
\end{cases} \quad \sigma \in S_i
$$

Clearly, $P$ is well-defined and belongs to $[[T]]$. By the definition of $P$, we have

$$
\{ x \in X \mid P x = S x \} \supset \bigcup_{i=1}^{N} \bigcup_{\sigma \in S_i} A_i(\sigma).
$$

Hence, we get by (2.4) that $P \in U(S; \mu_1, \ldots, \mu_p; \varepsilon)$. This completes the proof. □

**Remark.** After this work was submitted, B. Miller showed how using ideas of the proof above one can generalize Statement (2) of Theorem 2.1 to any countable group acting by homeomorphisms on a zero-dimensional Polish space [Mil].

**Rokhlin property**

We give several immediate corollaries of Theorem 2.1 which have the well-known analogs in ergodic theory.

**Corollary 2.2.** Let $T \in \text{Homeo}(X)$. Then, for every $\tau$-neighborhood $U$ of $T$, there exists a homeomorphism $P \in \text{Per}_0(T) \cap U$ whose associated canonical partition is clopen.

The next statement generalizes Theorem 4.5 of [B-K] proved originally for minimal homeomorphisms.

**Corollary 2.3.** Let $T \in \text{Homeo}(X)$. The topological full group $[[T]]$ of $T$ is $\tau$-dense in $[T]_C$.

As the group $\text{Homeo}(X)$ is not $\tau$-closed in $\text{Aut}(X, \mathcal{B})$, in [B-D-K 2] the authors brought up the question: how to describe the closure of $[[T]]$ in $(\text{Aut}(X, \mathcal{B}), \tau)$. They answered it for minimal homeomorphisms (see Theorem 2.8 of [B-D-K 2]) and we generalize it up to an arbitrary homeomorphism.

**Corollary 2.4.** Let $T \in \text{Homeo}(X)$. Then $[[T]] = [T]_C = [T]_B$. 
**Definition.** A topological group \( G \) possesses the **Rokhlin property** if the action of \( G \) on itself by conjugation is topologically transitive, i.e. there is an element of \( G \) whose conjugate class is dense.

The following proposition extends the list of topological groups that have the Rokhlin property. See also \([\text{Gl-W}]\) and \([\text{Ke-Ros}]\) for other examples.

**Theorem 2.5.** The topological groups \((\text{Aut}(X, \mathcal{B}), \tau)\) and \((\text{Homeo}(X), \tau)\) possess the Rokhlin property.

**Proof.** We prove this theorem for the group \((\text{Homeo}(X), \tau)\) only, for the other case the proof is similar.

Take a decomposition of the Cantor set \( X = \{x_0\} \cup \bigcup_{i \geq 1} X_i \) such that the \( X_i \)'s are non-empty clopen sets with \( \text{diam}(X_i \cup \{x_0\}) \to 0 \) as \( i \to \infty \). Let \( S \) be a homeomorphism such that \( Sx_0 = x_0 \) and \( S^i x = x \), \( S^j x \neq x \) for any \( x \in X_i \), \( j = 1, \ldots, i - 1 \). Our goal is to show that we can approximate any \( T \in \text{Homeo}(X) \) by elements from the conjugate class of \( S \). By Corollary 2.2, it suffices to approximate periodic homeomorphisms whose canonical partitions are clopen. Thus, suppose \( T \) has a clopen partition \( X = \bigcup_{i=1}^k Y_i \), where \( Y_i \) is the set of all points having \( T \)-period \( n_i \) for some \( n_i \geq 0 \). Observe that there exists a clopen set \( Y^0_i \) such that \( Y_i = \bigcup_{j=0}^{n_i-1} T^j Y^0_i \) is a disjoint union (see Lemma 3.2 of \([\text{B-D-K 2}]\)). Analogously, there exists a clopen set \( X^0_{n_i} \) with \( X_{n_i} = \bigcup_{j=0}^{n_i-1} S^j X^0_{n_i} \), a disjoint union.

Let \( U = U(T; \mu_1, \ldots, \mu_p; \varepsilon) \) be a \( \tau \)-neighborhood of \( T \). Take a non-empty clopen \( T \)-invariant set \( Z \) with \( \mu_i(Z) < \varepsilon \) for \( i = 1, \ldots, p \). Without loss of generality, we may assume that \( Y^0_i \setminus Z \) is not empty for \( i = 1, \ldots, k \). Let \( R_i \) be any homeomorphism from \( X^0_{n_i} \) onto \( Y^0_i \setminus Z \). Define a homeomorphism \( R \) as follows: let \( R \) be equal to \( T^j R_i S^{-j} \) whenever \( x \in S^j X^0_{n_i} \) for \( i = 1, \ldots, k \), \( j = 0, \ldots, n_i - 1 \) and let \( R \) map the rest of the space \( X \) onto \( Z \). It is not hard to check that \( RSR^{-1} \in U \).

In the setting of Borel dynamics, we need to produce a periodic transformation that has uncountably many orbits of any finite length. Then, the application of the Rokhlin lemma shows that its conjugate class is dense. \( \square \)

**Remark.** Let \( p \) be the topology on \( \text{Homeo}(X) \) generated by the metric \( D(T, S) = \sup_{x \in X} d(Tx, Sx) \), where \( d \) is a metric on \( X \) compatible with the topology. In \([\text{Gl-W}]\), it is shown that \((\text{Homeo}(X), p)\) has the Rokhlin property. Moreover, the elements whose conjugate classes are dense form a residual set with respect to \( p \).

**Topologically free homeomorphisms.**

It is interesting to compare the topological properties of the set \( \mathcal{A}p \) of all aperiodic homeomorphisms with respect to the both topologies \( \tau \) and \( p \). The following statement is proved in \([\text{B-D-K 2}]\), Theorem 2.1.

**Theorem 2.6.** The set \( \mathcal{A}p \) is dense in \((\text{Homeo}(X), p)\).

However, the situation in \((\text{Homeo}(X), \tau)\) is completely different. The set \( \mathcal{A}p \) is nowhere dense with respect to the topology \( \tau \). To see this, one can check that the set \( \mathcal{A}p \) is \( \tau \)-closed in \( \text{Homeo}(X) \). Then, the application of Theorem 2.1 implies the result.
The question we investigate in this section is “How can we extend the class of aperiodic homeomorphisms to produce a $\tau$-dense class?”. Apparently, the most natural extension of aperiodic homeomorphisms is the class of topologically free homeomorphisms.

**Definition.** It is said that a homeomorphism is *topologically free* if the set of all aperiodic points is dense.

In Theorem 2.8, we prove that the set of topologically free homeomorphisms is $\tau$-dense. To begin with, we need the following lemma on homeomorphism extensions proved in [Kn-R]. We will need the arguments used in its proof. Thus, we give a sketch of the proof, but without going into the details.

**Lemma 2.7.** Let $A$ and $B$ be closed nowhere dense subsets of Cantor sets $X$ and $Y$, respectively. Suppose there is a homeomorphism $h : A \to B$. Then $h$ can be extended to a homeomorphism $h^* : X \to Y$ such that $h^*|_A = h$.

**Sketch of the proof.** Find clopen sets $\{U_i\}$ and $\{V_j\}$ such that $X \setminus A = \bigcup_{i \geq 1} U_i$, $Y \setminus B = \bigcup_{j \geq 1} V_j$, and their diameters tend to zero. Find the points $a_i \in A$ such that $\text{dist}(U_i, A) = \text{dist}(U_i, a_i)$ and $b_j \in B$ with $\text{dist}(V_j, B) = \text{dist}(V_j, b_j)$.

Set $I = J = \mathbb{N}$. There exist injective functions $f : I \to J$ and $g : J \to I$ such that

$\text{dist}(U_i, a_i) > \text{dist}(V_{f(i)}, h(a_i))$ for $i \in I$,

$\text{dist}(V_j, b_j) > \text{dist}(U_{g(j)}, h^{-1}(b_j))$ for $j \in J$.

Applying the usual Schröder-Bernstein argument to $f$ and $g$, find disjoint sets $I' \cup I''$ and $J' \cup J''$ such that $f(I') = J'$ and $g(J'') = I''$.

Let $\phi$ be an arbitrary homeomorphism of $U' = \bigcup_{i \in I'} U_i$ onto $V' = \bigcup_{i \in I} V_i$ such that $\phi(U_i) = V_{f(i)}$. Analogously, let $\psi$ be a homeomorphism of $V'' = \bigcup_{j \in J''} V_j$ onto $U'' = \bigcup_{i \in I''} U_i$ such that $\psi(V_j) = U_{g(j)}$.

Define

$$h^*(x) = \begin{cases} \phi(x) & x \in U' \\ \psi^{-1}(x) & x \in U'' \\ h(x) & x \in A. \end{cases}$$

For the verification of continuity of $h^*$, we refer the reader to [Kn-R].

**Theorem 2.8.** The set of topologically free homeomorphisms is $\tau$-dense in $\text{Homeo}(X)$.

**Proof.** By Corollary 2.2, it suffices to approximate only homeomorphisms from $\mathcal{P}_{\text{Per}}$ that have clopen canonical partitions. Assume that $R$ belongs to $\mathcal{P}_{\text{Per}}$ and its canonical partition $X = X_{n_1} \cup \ldots \cup X_{n_m}$ is clopen. Recall that the set $X_{n_i}$ consists of all points with the period $n_i$. Consider a $\tau$-neighborhood $U = U(R; \mu_1, \ldots, \mu_k; \varepsilon)$. Since the $X_{n_i}$’s are $R$-invariant and clopen, we will prove the theorem under the assumption that $X = X_{n_i}$ for some $i$ and leave the generalization to the reader.

Suppose $X = \bigcup_{i=0}^{p-1} R^n F$ is a clopen partition and $R^p x = x$ for all $x \in X$. Using the standard Cantor argument, find a closed nowhere dense set $P \subset R^{p-1} F$ such that $\mu_i(P) > 1 - \varepsilon$ for $i = 1, \ldots, k$. Repeating the proof of Lemma 2.7, we extend the homeomorphism $R : P \to RP$ to a homeomorphism $T : R^{p-1} F \to F$ so that the homeomorphism $T^* \in \text{Homeo}(X)$
defined as $T^*|_{R^{p-1}F} = T|_{R^{p-1}F}$ and $T^* = P$ elsewhere is topologically free. To do this, it suffices to choose the functions $\psi$ and $\phi$ so that $\phi(x) \neq Rx$ and $\psi^{-1}(x) \neq Rx$ for $x \in R^{p-1}F \setminus P$. Since $E(T^*, R) = R^{p-1}F \setminus P$, we get that $T^* \in U$. 

3. Structure of homeomorphisms from topological full group

In this section, we discuss the structure of homeomorphisms from the topological full group $[[T]]$ for arbitrary aperiodic $T \in Homeo(X)$.

Consider a Cantor aperiodic system $(X, T)$. A Borel set $Y \subset X$ is called wandering if $T^nY \cap Y = \emptyset$ for all $n \geq 1$.

**Definition.** We say that a closed wandering set $Y$ is basic if every clopen neighborhood of $Y$ meets every $T$-orbit.

**Theorem 3.1.** Every Cantor aperiodic system has a basic set.

**Sketch of the proof.** Applying the argument developed in [B-D-M, Theorem 2], we can find a decreasing sequence of clopen sets $\{U_n\}$ such that: $U_{n+1} \subset U_n$; $T^n U_n \cap U_n = \emptyset$ for $i = 1, \ldots, n - 1$; and $U_n$ meets every $T$-orbit. Then $Y = \bigcap_n U_n$ is a basic set.

**Remark** For more results related to basic sets and their interaction with Bratteli diagrams, see [M].

Fix a triple $(X, T, Y)$, where $(X, T)$ is a Cantor aperiodic system and $Y$ is a basic set. Consider a clopen neighborhood $U$ of $Y$. It is not hard to check that for every $x \in U$, there is $n = n(x) > 0$ such that $T^n x \in U$. Therefore, it follows from the definition of a basic set that, by the first return function, we can construct a clopen K-R partition $\Xi$ of $X$ with the base $B(\Xi) = U$.

Take a decreasing sequence of clopen sets $\{U_n\}$ such that $Y = \bigcap_n U_n$. Constructing clopen K-R partitions for the $U_n$'s and refining them, we prove the following:

**Theorem 3.2.** Let $(X, T, Y)$ be a Cantor aperiodic system with a basic set $Y$. There exists a sequence of clopen K-R partitions $\{P_n\}$ of $X$ such that for all $n \geq 1$ (i) $P_{n+1}$ refines $P_n$; (ii) $h_{n+1} > h_n$, where $h_n$ is the height of the lowest $T$-tower in $P_n$; (iii) $B(P_n) \supset B(P_{n+1})$; (iv) the sequence $\{P_n\}$ generates the clopen topology of $X$; (v) $\bigcap_n B(P_n) = Y$.

We will follow here the method developed in [B-K] for minimal homeomorphisms (see also [K-W]). Let $\mathcal{P}$ be a clopen K-R partition with towers $\mathcal{P}(i)$, $i = 1, \ldots, k$. Define two partitions $\alpha = \alpha(\mathcal{P})$ and $\alpha' = \alpha'(\mathcal{P})$ of $\{1, 2, \ldots, k\}$. We say that $J$ is an atom of $\alpha$ if there exists a subset $J' \subset J$ such that

$$T(\bigcup_{i \in J} T^{h(i)-1} D_i) = \bigcup_{i' \in J'} D_{i'}$$

and for every proper subset $J_0$ of $J$, the $T$-image of $\bigcup_{i \in J_0} T^{h(i)-1} D_i$ is not a union of atoms from $\mathcal{P}$. It follows from (3.1) that $J'$ is uniquely defined by $J$ and $T$.

Let $S \in [[T]]$. Then, there are a finite set $K \subset \mathbb{Z}$ and clopen partition $\mathcal{E} = \{E_k : k \in K\}$ of $X$ such that $Sx = T^k x$ for $x \in E_k$ and $k \in K$. Denote by $\mathcal{E}(K)$ the clopen partition $\{S^k E_k : k \in K\}$. By Theorem 3.2, find a K-R
partition \( P = \{ P(i) : i = 1, \ldots, k \} \) with \( P(i) = \{ D_{0,i}, \ldots, D_{h(i)-1,i} \} \) and \( D_{j+1,i} = TD_{j,i} \) that refines \( E \) and \( E(K) \) and so that \( K \subset (-h, h) \), where \( h \) is the height of the lowest \( T \)-tower in \( P \).

Let \( F = \{(j, i) | i = 1, \ldots, k, j = 0, \ldots, h(i) - 1 \} \). Observe that for every pair \( (j, i) \in F \) there is a unique \( l = l(j, i) \in K \) such that

\[
S(D_{j,i}) = T^l D_{j,i}.
\]

Divide \( F = F(P) \) into three disjoint sets \( F_{in}, F_{top} \) and \( F_{bot} \) as follows:

(a) \( (j, i) \in F_{in} \) if \( S(D_{j,i}) \subset P(i) \), i.e. \( 0 \leq l + j \leq h(i) - 1 \);
(b) \( (j, i) \in F_{top} \) if \( S(D_{j,i}) \) goes through the top of \( P(i) \), i.e. \( l + j \geq h(i) \);
(c) \( (j, i) \in F_{bot} \) if \( S(D_{j,i}) \) goes through the bottom of \( P(i) \), i.e. \( l + j < 0 \),

where \( l \) is taken from (3.2).

Let \( \alpha \) and \( \alpha' \) be the partitions of \( \{1, \ldots, k\} \) defined by \( T \) and \( P \). For \( J \subset \{1, \ldots, k\} \), set \( h_J = \min \{ h(i) | i \in J \} \). For \( J \subset \alpha \) and \( J' \subset \alpha' \), let

\[
F_1(r, J) = \bigcup_{i \in J} D_{h(i) - h_J + r, i} \quad F_2(r', J') = \bigcup_{i' \in J'} D_{r, i'}
\]

where \( r = 0, \ldots, h_J - 1 \) and \( r' = 0, \ldots, h_{J'} - 1 \).

**Definition.** We say that \( S \in [[T]] \) belongs to \( \Gamma(P) \) if for each pair \( (j, i) \in F \) the following conditions hold:

(a) if \( (j, i) \in F_{top} \) and \( D_{j,i} \subset E_t \), then \( F_1(h_J - h(i) + j, J) \subset E_t \), where \( J \) is an atom of \( \alpha \) containing \( i \);
(b) if \( (j, i) \in F_{bot} \) and \( D_{j,i} \subset E_t \), then \( F_2(j, J') \subset E_t \), where \( J' \) is an atom of \( \alpha' \) containing \( i \).

Condition (a) means that whenever the set \( D_{j,i} \) goes through the top of \( P(i) \) under the action of \( S \), then the entire level \( F_1(r, J) \) containing \( D_{j,i} \) also goes through the top of \( P \). Similarly, one can clarify condition (b) by taking the \( D_{j,i} \)'s and levels \( F_2(j, J') \) containing them that go through the bottom of \( P \). Observe that if \( (j, i) \in F_{in} \), then the entire levels \( F_1(r, J) \) and \( F_2(r, J') \) containing \( D_{j,i} \) remain “within” \( P \).

Clearly, \( \Gamma(P) \) is a finite set. The following theorem reveals the structure of homeomorphisms from \([[T]]\) for an arbitrary aperiodic homeomorphism \( T \). Notice that this structure was found earlier for minimal homeomorphisms (see Theorem 2.2 in [B-K]). Since our proof is similar to that in [B-K Theorem 2.2], we omit it.

**Theorem 3.3.** Let \((X, T, Y)\) be a Cantor aperiodic system with a basic set \( Y \) and a sequence of \( K-R \) partitions \( \{P_n\} \) satisfy the conditions of Theorem 3.2. Then \([[T]] = \bigcup_n \Gamma(P_n) \) with \( \Gamma(P_n) \subset \Gamma(P_{n+1}) \).

**The subgroup** \( \Gamma_Y \)

Let \((X, T)\) be a Cantor aperiodic system with a basic set \( Y \). Define the subgroup \( \Gamma_Y \) of \([[T]]\) as follows: \( S \in \Gamma_Y \) if \( S \in \Gamma(P_n) \) (and hence \( S \in \Gamma(P_m) \), for \( m > n \)) implies that \( F(P_n) = F_{in} \). In other words, \( S \in \Gamma_Y \) if no level from \( P_n \) goes over the top as well as through the bottom under the action of \( S \). This means that \( S \) acts as a permutation on each \( T \)-tower from \( P_n \). Therefore, the group \( \Gamma_Y \) is an increasing union of permutation groups.

**Remark.** (1) Denote by \([[T]]_Y \) the subgroup of \([[T]] \) consisting of homeo-
morphisms that preserve the forward $T$-orbit of every $y \in Y$, i.e., $S \in [[T]]_Y$ if $S\{(T^n y : n \geq 0)\} = \{T^n y : n \geq 0\}$. Observe that $\Gamma_Y \subset [[T]]_Y$.

(2) The subgroup $[[T]]_Y$ is not $\tau$-dense in $[T]$. Therefore, so is $\Gamma_Y$. Indeed, take any $z \in T^{-1}Y$ and the Dirac measure $\delta_z$ supported by $\{z\}$. Consider $S \in U := U(T; \delta_z; 1/2)$. As $z \notin Y$ and $Sz = Tz \in Y$, $S$ does not preserve the forward $T$-orbit of $Tz$. Therefore, $U$ contains no elements from $[[T]]_Y$.

The fact that $\Gamma_Y$ is not $\tau$-dense in $[[T]]$ is mainly caused by the presence of discrete measures. We can partly overcome this obstacle by considering only continuous measures in the definition of the topology $\tau$. Denote by $\tau_0$ the topology defined by continuous measures as in Definition 1.1. One can check that $\tau_0$ is a Hausdorff group topology on $\text{Homeo}(X)$. The next theorem answers the question when $\Gamma_Y$ is $\tau_0$-dense in $[T]$.

**Theorem 3.4.** Suppose we have a Cantor aperiodic system $(X, T)$ with a basic set $Y$. Then the subgroup $\Gamma_Y$ is $\tau_0$-dense in $[T]$ if and only if the basic set $Y$ is at most countable.

**Proof.** (1) Assume that $Y$ is uncountable. Take any continuous measure $\mu$ supported by $T^{-1}Y$. Then for every $S \in U := U(T; \mu; 1/2)$ there is at least one $z \in T^{-1}Y$ such that $Sz = Tz$. This implies that $\{T^n(Tz) : n \geq 0\}$ is not $S$-invariant. Therefore, by (1) of the remark above we get that $\Gamma_Y \cap U = \emptyset$.

(2) Now, assume that $Y$ is countable. Observe that by Corollary 2.3 it is enough to approximate homeomorphisms from $[[T]]$ with elements of $\Gamma_Y$. Consider $R \in [[T]]$ and a $\tau_0$-neighborhood $U = U(R; \mu_1, \ldots, \mu_p; \varepsilon)$ of $R$. By definition of $R$, the sets $E_k = \{x \in X : Rx = T^k x\}$, $k \in K$, $|K| < \infty$, form a clopen partition of $X$. Let $k_0 = \sup\{|k| : k \in K\}$. As $Y$ is countable, $\mu(T^n Y) = 0$ for any continuous measure $\mu$ and integer $n$. Therefore, by Theorem 3.2 we can find a K-R partition $P_n$ such that $k_0 < 2h_n$, where $h_n$ is the height of the lowest $T$-tower in $P_n$, and

\[
\mu_j\left(\bigcup_{i=-k_0}^{k_0} T^i B(P_n)\right) < \varepsilon, \text{ for } j = 1, \ldots, p.
\]

Define a homeomorphism $S \in \Gamma_Y \cap U$ as follows:

Take a $T$-tower (say $\lambda = \{D, \ldots, T^h(\lambda)D\}$) from $P_n$. Consider an atom $T^D$ of $\lambda$. We have two possibilities:

(i) The $R$-orbit of the set $T^D$ does not leave the $T$-tower $\lambda$. In this case, we define the homeomorphism $S$ to be equal to $R$ on the $R$-orbit of $T^D$.

(ii) The set $T^D$ leaves $\lambda$ under the action of $R$. Then there exist integers $q < 0 < d$ such that $R^{d+1}T^D$ and $R^{d-1}T^D$ do not lie in $\lambda$ entirely, whereas the sets $R^j T^D$, $j = q, \ldots, d$ are contained in $\lambda$. In this case, we set $S = R$ on $R^i T^D$, $i = q, \ldots, d - 1$, and $S = R^{d+q}$ on $R^d T^D$.

Observe that the choice of $P_n$ guarantees that

\[
R^{d+1} T^D \subset \bigcup_{i=0}^{k_0} T^i D \cup \bigcup_{i=h(\lambda)-1-k_0}^{h(\lambda)-1} T^i D.
\]
Clearly, the homeomorphism $S$ constructed in (i) and (ii) is periodic, but it is not defined yet on the entire space. To expand its domain, we consider an atom $T^wD$ of $\lambda$ on which $S$ is not defined yet, and repeat (i) and (ii) with $T^wD$.

Repeating this procedure with every atom of $\lambda$, we define $S$ on $\lambda$. Moreover, $\lambda$ is $S$-invariant. By construction, $S$ coincides with $P$ everywhere, maybe, except for the set $\bigcup_{i=0}^{k_0} T^iD \cup \bigcup_{i=h(\lambda)-1-k_0}^{h(\lambda)-1} T^iD$.

To finish constructing $S$, we need to repeat the argument for every $T$-tower of $\mathcal{P}_n$.

The definition of $S$ implies that $S \in \Gamma_Y$ and

$$\{ x \in X : Sx \neq Rx \} \subset \bigcup_{i=-k_0}^{k_0} T^iB(\mathcal{P}_n).$$

Therefore, by (3.3) we have that $S \in U$. \hfill \Box

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References

[B-D-K 1] S. Bezuglyi, A.H. Dooley, and J. Kwiatkowski, Topologies on the group of Borel automorphisms of a standard Borel space, Topol. Methods in Nonlinear Anal., 27 (2006), 333-385.

[B-D-K 2] S. Bezuglyi, A.H. Dooley, and J. Kwiatkowski, Topologies on the group of homeomorphisms of a Cantor set, Topol. Methods in Nonlinear Anal., 27 (2006), 299-331.

[B-D-M] S. Bezuglyi, A.H. Dooley, and K. Medynets, The Rokhlin lemma for homeomorphisms of a Cantor set, Proc. Amer. Math. Soc. 133 (2005), 2957-2964.

[B-K] S. Bezuglyi, J. Kwiatkowski, The topological full group of a Cantor minimal system is dense in the full group, Topol. Methods in Nonlinear Anal., 16 (2000), 371 - 397.

[B-K-M] S. Bezuglyi, J. Kwiatkowski, and K. Medynets, Approximation in measurable, Borel, and Cantor dynamics, Contemp. Math. Amer. Math. Society, Volume 385, 2005.

[B-M] S. Bezuglyi, K. Medynets, Smooth automorphisms and path-connectedness in Borel dynamics, Indag. Math. 15, no. 4, (2004), 453-468.

[C-F-S] I. Cornfeld, S. Fomin, Ya. Sinai, Ergodic Theory, Grundlehren der mathematischen Wissenschaften 245, Springer-Verlag, 1982.

[G-K] E. Glasner and J. King, A zero-one law for dynamical properties, Topological Dynamics and Applications (A volume in honor of Robert Ellis), Contemp. Math., vol. 215, AMS, 1998, 215-242.

[G-P-S] T. Giordano, I. Putnam, and C. Skau, Full groups of Cantor minimal systems, Israel. J. Math., 111 (1999), 285 - 320.

[Gl-W] E. Glasner and B. Weiss, The topological Rohlin property and topological entropy, Amer. J. Math., 123 (2001), 1055 - 1070.

[Hal 1] P. Halmos, Approximation theories for measure-preserving transformations, Trans. Amer. Math. Soc., 55 (1944), 1-18.

[Hal 2] P. Halmos, Lectures on Ergodic Theory, The Mathematical Society of Japan, Publications of the Mathematical Society of Japan, no. 3, 1956, vi+99 pp.

[Kn-R] B. Knaster and M. Reichbach, Notion d’homogénéité des homéomorphies, Fund. Math., 40 (1953), 180-193.
[Ke-Ros] A. Kechris, C. Rosendal, Turbulence, amalgamation and generic automorphisms of homogenous structures, ArXiv:math.LO/0409567 v2, 30 September 2004.

[K-W] J. Kwiatkowski, M. Wata, Dimension and infinitesimal groups of Cantor minimal systems Topol. Methods Nonlinear Anal., 23, No.1, (2004), 161-202.

[M] K. Medynets, Cantor aperiodic systems and Bratteli diagrams, C. R., Math., Acad. Sci. Paris, 342, No. 1, (2006), 43-46.

[Mil] B. Miller, Density of topological full groups, preprint, 2006.

[N] M. Nadkarni, Basic Ergodic Theory, 2nd Edition, Birkhäuser, 1998.

[Ro] V.A. Rokhlin, Selected topics from the metric theory of dynamical systems (Russian), Usp. Mat. Nauk (N.S.), 4 (1949), no. 2, 57–128 (Engl. Transl. in ‘Amer. Math. Soc. Translations’, 49 (1966), 171 – 240).

[W] B. Weiss, Measurable dynamics, Contemp. Math., 26 (1984), 395-421.

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