Inverted catenoids, curvature singularities and tethered membranes

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Abstract

If a catenoid is inverted in any interior point, a deflated compact geometry is obtained which touches at two points (its poles). The catenoid is a minimal surface and, as such, is an equilibrium shape of a symmetric fluid membrane. The conformal symmetry of the Hamiltonian implies that inverted minimal surfaces are also equilibrium shapes. However, they exhibit curvature singularities at their poles. These singularities are associated with external forces pulling the poles together. Unlike the catenoid which is free of stress, there will be stress within the inverted shapes. If the surface area is fixed, reducing the external force induces a transition from a discocyte to a cup-shaped stomatocyte.

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1 Introduction

Biological membranes are fluid bilayers whose mechanical properties are described on mesoscopic scales by a bending energy quadratic in extrinsic curvature [1, 2, 3]. If bilayer asymmetry or constraints are ignored, this energy is invariant under the global conformal transformations of three-dimensional space [4, 5, 6]. Thus the bending energy is not only independent of the size of the membrane, it is unchanged under inversion in any point, \( x \to x/|x|^2 \). As a consequence, every equilibrium configuration has a counterpart related to it by inversion which is also an equilibrium. Remarkably, this apparently unnatural symmetry does have physical consequences.

In practice, what one does is pair inversions in special conformal transformations which can be treated perturbatively. The study of inversion in its own right, however, lies beyond the reach of perturbation theory. If the point of inversion lies on the surface, this point gets consigned to infinity; remote points on the surface, on the other hand get mapped into a neighborhood of the origin i.e. shapes do not simply get distorted, they may find themselves distorted beyond recognition; even their topology might change. Two shapes related by inversion will typically
describe very different physical conditions. Inversion has the potential to generate curvature singularities where none existed.

In this paper, we will examine various consequences for the physics of fluid membranes of this non-perturbative aspect of conformal symmetry. To keep things simple we will look only at axially symmetric shapes. If our objective had been to identify compact isolated geometries without internal or external constraints, this symmetry would be a little too confining: the only equilibrium geometries of this kind are spheres and Clifford torii with ratio of the wheel to tube radius of $\sqrt{2}$.

The catenoid is also an equilibrium geometry albeit an infinite one. Finite sections of catenoid play an important role as a geometrical component of fluid membranes [1]. Because mean curvature vanishes, the contribution made by the catenoid to the bending energy is entirely topological, depending only on the boundary behavior. The central region of the catenoid provides the neck geometry permitting membranes to bud without incurring the large energy penalties one might expect to be associated with the high curvatures which form in the neck region. While the curvatures may be large, the mean curvature and thus the added local energy vanishes. This is also the geometry involved in the assembly of topologically complicated membranes. Thus it should not be surprising that such structures are abundant in biological cells.

Spheres and torii are not very interesting under inversion. The catenoid, however, maps not only to a completely different equilibrium geometry, but to one that is also interesting physically. In particular, if the catenoid is inverted in its center of symmetry it transforms into a compact biconcave geometry: a discocyte. The two distal regions of the catenoid get mapped to the point of inversion. Thus the north and south poles of this geometry touch and, because the geometry possesses a tangent plane at these points, it is appropriate to think of the topology as spherical.

The problem is: how does one interpret such an equilibrium? The only regular equilibrium geometries with spherical topology are round spheres. Without constraints, one would thus expect this geometry to equilibrate by inflating into a sphere. Physically, the only way such a deflated equilibrium could exist is if there are external forces counteracting the tendency to inflate. Such forces will act as a source of stress in the membrane [7]. The appropriate way to quantify this connection is to describe the geometry in terms of a conserved stress tensor [8, 9].

Unlike a catenoid formed by a soap film spanning two rings which is under tension, a catenoid formed by an ideal fluid membrane is free of stress. However, the equilibrium described by its inverted counterpart is very different: it is under stress. As shown in [10], the source of this stress is a pair of equal and opposite localized external forces at the poles pulling them together.

External forces manifest themselves in curvature singularities. Despite the apparent smoothness of the inverted catenoid at the poles, there is a singularity lurking at these points. The
two principal curvatures diverge logarithmically as they are approached.

What happens when the point of inversion is changed is also interesting. By translating this point along the axis of symmetry we break the up-down symmetry. The discocyte morphs continuously into a stomatocyte. It is also possible to describe this transition analytically. Surface areas, however, are not preserved under inversion. To follow the transition in a physically meaningful way, the geometries should therefore be rescaled so that they possess the same area. One may then track the volume as a function of the external force tethering the poles together. The maximum external force consistent with this constraint occurs for the symmetric discocyte; a higher value presumably would rupture the membrane. For each value below this maximum, there is a unique axially symmetric geometry. As the force is reduced, the discocyte first inflates asymmetrically until a point is reached where it is identifiable as a stomatocyte; thereafter it deflates into an inverted sphere within a sphere. We have not conducted a complete stability analysis; however, there are reasons to expect this description to be stable with respect to small perturbations breaking the axial symmetry.

The paper is organized as follows: in section 2 we describe inverted minimal surfaces focusing on the stresses within them; in section 3 we provide a detailed physical interpretation of the inverted catenoid. Some of these results were announced in [10]. We finish with a brief discussion and an outline of our plans for future work.

2 Willmore energy and inverted minimal surfaces

A parametric description of an embedded two-dimensional surface in three-dimensional space is provided by the mapping \((u^1, u^2) \rightarrow X(u^1, u^2)\). The Willmore energy associated with this surface is given by

\[
H[X] = \int dA \left( K_{ab} - \frac{1}{2} g^{ab} K \right) \left( K_{ab} - \frac{1}{2} g_{ab} K \right),
\]

where \(g_{ab}\) is the metric, \(K_{ab}\) is the extrinsic curvature tensor, and \(K = g^{ab} K_{ab}\) is its trace (twice the mean curvature). \(dA\) is the induced element of area. This notation is summarized in Appendix I. \(H[X]\) is a measure of the energy associated with bending. It has the remarkable property that it is invariant with respect to the conformal transformations of the ambient space \([4]\): thus \(H[X]\) is invariant not only under the transformations of the surface induced by the Euclidean motions, translations and rotations, it is also invariant under transformations preserving angles: dilations \(X \rightarrow \lambda X\) and inversion,

\[
X \rightarrow \frac{X}{|X|^2}.
\]

Modulo a topological contribution proportional to the Gauss-Bonnet invariant \(H\) coincides
with the Helfrich energy

\[ H[X] = \frac{1}{2} \int dA K^2. \]  

(3)

There is only one local two-dimensional bending energy.

The equilibrium surfaces of the energy (1) satisfy the Euler-Lagrange equation

\[ - \nabla^2 K + \frac{1}{2} (K g_{ab} - 2 K_{ab}) K^{ab} K = 0, \]  

(4)

where \( \nabla^2 \) is the surface Laplacian compatible with the metric \( g_{ab} \). It is clear that minimal surfaces, satisfying \( K = 0 \) are solutions of Eq.(4). However, because of the conformal symmetry of the shape equation, these surfaces map under inversion to new solutions of Eq.(4). This is true not just infinitesimally but also for finite conformal transformations. In particular, it is true for inversion.

Let us first examine the behavior of minimal surfaces under inversion. In particular, let us first identify the equation satisfied by the transformed surface. It is well known that the principal curvatures transform under inversion by

\[ \bar{C}_I = |X|^2 \left( C_I - 2 \left( \frac{X \cdot n}{|X|^2} \right) \right). \]  

(5)

Thus the transformed mean curvature \( \bar{K} = C_1 + C_2 \) is given by

\[ \bar{K} = |X|^2 \left( K - 4 \left( \frac{X \cdot n}{|X|^2} \right) \right). \]  

(6)

We conclude from Eq.(6) that a surface satisfying

\[ K = 4 \frac{X \cdot n}{|X|^2} \]  

(7)

is mapped under inversion to a minimal surface, and conversely. It is clear that the only minimal surfaces mapping to minimal surfaces are the planes through the origin.\(^1\) If the origin itself lies on the surface, \( K \) will diverge there unless the surface aligns along \( X \) sufficiently fast as \( |X| \to 0 \). While this is true if the original minimal surface has planar ends it is not true if the ends behave like a catenoid. Inverted minimal surfaces typically possess curvature singularities.

Inverted minimal surfaces also satisfy Eq. (4). Thus any solution to Eq.(7) is also a solution to Eq.(4). It is straightforward, but instructive, to confirm this fact with an explicit calculation: we note that, modulo Eq. (7), we have

\[ \partial_a K = 4 \frac{1}{|X|^2} (K_{ab} - \frac{1}{2} g_{ab} K) (X \cdot e^b). \]  

(8)

\(^1\)Eq. (7) can also be written as \( n \cdot [\nabla^2 X + 4 X / |X|^2] = 0 \). Note, however, that there are no solutions consistent with peeling off \( n \). Surfaces satisfying this equation can also be interpreted as the stationary configurations of the functional \( H_1[X] = \int dA / |X|^4 \).
This makes use of the Weingarten equations, $\partial_a n = K_{ab} e^b$. Eq. (4) follows (except possibly at the origin) on using the Gauss equations, $\nabla_a e^b = -K_{ab} n$ as well as the contracted Codazzi-Mainardi equations,
\begin{equation}
\nabla_a K^{ab} - \nabla^b K = 0.
\end{equation}
Here $\nabla_a$ is the surface covariant derivative compatible with $g_{ab}$. Unlike minimal surfaces, the equilibrium of the inverted counterpart involves a non-trivial cancelation between the Laplacian of $K$ and the terms cubic in $K_{ab}$. If the origin lies on the surface, however, equation (7) will not generally hold at this point and there will be a geometrical singularity there. In the following section we will show how to interpret this apparent pathology physically in terms of a localized external force acting on the membrane at this point.

### 2.1 Conserved stress tensor and Noether charges for the inverted minimal surfaces

The physical interpretation of geometrical singularities in the transformed geometry at the origin of inversion is facilitated by identifying a stress tensor with the geometry. This tensor is given in terms of the extrinsic curvature tensor by [8, 9]
\begin{equation}
f^a = K(K^{ab} - \frac{1}{2}g^{ab}K) e^b - \partial^a K n.
\end{equation}
In equilibrium, $f^a$ is conserved so that $\nabla_a f^a = 0$. It is simple to check that the local conservation law reproduces the shape equation (4).

The transformed stress tensor is given by
\begin{equation}
\tilde{f}^a = |X|^4 \left(|X|^2 f^a + 4 \left(K^{ab} - \frac{1}{2}g^{ab}K\right) f^b_0\right),
\end{equation}
where
\begin{equation}
f^a_0 = (e^a \cdot X) n - (n \cdot X) e^a.
\end{equation}
The role of $f^a_0$ in the context of conformal symmetry has been noted elsewhere [16]. Curiously, it also provides an effective surface stress tensor for any external Laplace force [17].

In a minimal surface with $K = 0$ the stress vanishes. However, it does not vanish in its inverted counterpart satisfying Eq. (7). The corresponding stress tensor is given by
\begin{equation}
f^a = -\frac{4}{|X|^2} \left(K^{ab} - \frac{1}{2}g^{ab}K\right) f^b_0.
\end{equation}
It is also straightforward to demonstrate that $f^a$ given by Eq. (13) is conserved if $X \neq 0$: one uses Eq. (8) as well as the identity,
\begin{equation}
\nabla_b f_{0a} = g_{ab} n + K_b^c ((X \cdot e_a) e_c - (a \leftrightarrow c)) ,
\end{equation}
satisfied by $f_0$. 

As noted in [10], a curvature singularity at a point will be associated with a source of stress at that point. Let $\Gamma$ be any closed contour on the surface of the membrane encircling the point. Stoke’s theorem applied to the conservation law implies that the closed line integral

$$\oint ds \ l^a f_a$$

is a constant vector $\mathbf{F}$ along contours that are homotopically equivalent to $\Gamma$ on this surface [7]. Here $l = l^a e_a$ is the normal to $\Gamma$ tangent to the surface, and $ds$ is the element of arc-length along $\Gamma$. The set of values of $\mathbf{F}$ are the Noether charges associated with the translation invariance of the energy.

If the conservation law is valid everywhere, $\mathbf{F}$ must vanish on any topologically trivial loop. This is the case for a membrane with spherical topology. Thus, if $\mathbf{F} \neq 0$ on such a loop, there must be a source of stress within it [7]. In particular, a distributional source of stress manifest itself in a curvature singularity which is picked up by the line integral. $\mathbf{F}$ is a reparametrization invariant measure of the strength of singularity.

Note also that, in equilibrium, the total external force acting on the membrane must vanish. In an axially symmetric membrane the forces operating at the poles must be equal and opposite.

A concrete implementation of these ideas will be developed in the following section.

3 Inversion of catenoids

Conformal invariance permits a remarkably simple construction of a two-parameter family of axially symmetric solutions of Eq. (4). Begin with a catenoid which solves the shape equation trivially as a minimal surface with $K = 0$.

For a fixed axis of rotation, a catenoid $\Sigma_{(R_0, \xi_0)}$, is given as the level set $\Phi(R, Z) = 0$ of the function

$$\Phi(R, Z) = R - \frac{1}{R_0} \cosh(R_0 Z + \xi_0).$$

It is characterized by a scale $1/R_0$ and an offset $\xi_0/R_0$ along the axis. Thus any two catenoids are related by scaling and translation. They are the only non-trivial rotational minimal surfaces.

The image of the catenoid $\Sigma_{(R_0, \xi_0)}$ under inversion in the origin is the surface $\overline{\Sigma}_{(R_0, \xi_0)}$ described as the level set

$$\frac{R}{R^2 + Z^2} = \frac{1}{R_0} \cosh \left( \frac{R_0 Z}{(R^2 + Z^2)} + \xi_0 \right) = 0.$$  

Scaling the catenoid with an inverse length will give an inverted geometry with the ‘correct’ dimensions.
This is a simple transcendental equation in the variables $R$ and $Z$.

One particular solution, the symmetric biconcave discocyte with $\xi_0 = 0$, is illustrated in Fig.(1). The geometric profiles for several different values of $\xi_0$ are displayed in Fig.(2). Surface area is not preserved under inversion; thus, as we discuss below, these profiles have been rescaled so that they all possess the same surface area.

We note the following properties of these new surfaces:

- The geometry is bounded for all values of $\xi_0$. The circle $R = 1/R_0 \cosh \xi_0$ on the plane $Z = 0$ is mapped to the circle $R = R_0 / \cosh \xi_0$ on the same plane. The remote regions of the catenoid map to the point of inversion. Thus the north and south poles of this geometry touch. As we will see, they do so with a common tangent plane $Z = 0$. In this sense, the inverted surface does not self-intersect. The topology is spherical. However, there will be a singularity at the origin.

Generically, the behavior at the origin will not be so simple. Consider, for example, the fate of another simple geometry, the hyperboloid of revolution, under inversion in the origin. Asymptotically, the hyperboloid is conical. Because cones map to cones under inversion in the apex, the inverted geometry will be conical at the origin; it is not spherical. The existence of a tangent plane in the inverted catenoids is a consequence of the faster asymptotic growth rate (exponential) of the catenoid.

- When $\xi_0 = 0$, the surface is a biconcave discocyte possessing up-down symmetry. This symmetry is a consequence of the symmetry of the original catenoid with respect to the point of inversion.
Figure 2: Geometric profiles for $\xi_0 = 0, 1, 2, 3, 50, 100$. The surface is generated by rotating the profile about the vertical axis. The area is normalized.

- The symmetry is broken by translating the point of inversion along the axis. Beyond some critical value, $\xi_0^* = 1.86823$ discussed below, the lower concave region becomes vanishingly small: there is a transition from the symmetric biconcave discocyte to an asymmetrical stomatocyte.

- In the limit $\xi_0 \to \infty$, the stomatocyte degenerates into a completely deflated geometry consisting of a sphere within a sphere connected by an infinitesimal neck, a catenoid. It is possible to describe the complete sequence from discocyte to limiting deflated stomatocyte analytically.

It should be remarked that, technically, the lower concave region persists for all values of $\xi_0$; the ratio $\xi_0/R_0$ determines the position of the center along the axis; thus no matter how large a value we take for $\xi_0$, there always will be a region of the catenoid where $Z$ is negative; this region will map to negative $Z$ in the inverted geometry and will contain a concave component. Of course, as the profiles above clearly illustrate, while this concave region is always present, it becomes small in comparison with the overall size of the geometry. How small will be described quantitatively in our discussion of the isoperimetric ratio.

### 3.1 The isoperimetric ratio of the inverted shapes

To examine the geometry quantitatively, it is useful to introduce a parametric representation of the catenoid in terms of the angle that the tangent to the meridian makes with a plane of
constant $Z$:
\[
R(\Theta) = \frac{1}{R_0} \csc \Theta, \quad Z(\Theta) = \frac{1}{R_0} \ln \tan \Theta/2 + \frac{\xi_0}{R_0}.
\] (18)

where $\Theta$ lies in the interval $[0, \pi]$. This angle also provides a parametrization of the inverted catenoid $\bar{\Sigma}$:
\[
R(\Theta) = \frac{R_0 \sin \Theta}{1 + \sin^2 \Theta (\log \tan \Theta/2 + \xi_0)^2}, \quad Z(\Theta) = \frac{R_0 \sin^2 \Theta (\log \tan \Theta/2 + \xi_0)}{1 + \sin^2 \Theta (\log \tan \Theta/2 + \xi_0)^2}.
\] (19)

The interval of $\Theta$ does not change under inversion. A word of caution: $\Theta$ and its inverted counterpart $\bar{\Theta}$ are not the same angle i.e. $\Theta$ has nothing to do with the angles which are preserved under conformal transformation. While it might be more natural to parametrize the inverted surface by $\bar{\Theta}$, the advantage has to be weighed against the complicated form of the corresponding functional forms of $\bar{R}$ and $\bar{Z}$.

A straightforward calculation shows that the parametrization by $\Theta$ is isothermal. The metric on the inverted catenoid is described by the line element
\[
ds^2 = R_0^2 \Omega^2(\Theta) \left[ d\Theta^2 + \sin^2 \Theta \, d\varphi^2 \right].
\] (20)

where $\varphi$ is the polar azimuthal angle, and the conformal factor $\Omega$ is given by
\[
\Omega(\Theta)^{-1} = 1 + \sin^2 \Theta \left( \log \tan \frac{\Theta}{2} + \xi_0 \right)^2.
\] (21)

The area of the surface is given with respect to this parametrization by $A = 4\pi R_0^2 A(\xi_0)$ where
\[
A(\xi_0) = \frac{1}{2} \int_0^\pi d\Theta \sin \Theta \Omega(\Theta)^2,
\] (22)

with range $(0, \infty)$. The area of the surface is used to express $R_0$ as a function of $\xi_0$ for a fixed area $A$ as $R_0(\xi_0) = \sqrt{A / 4\pi A(\xi_0)}$. The volume enclosed by the surface is given by
\[
V = \frac{4\pi R_0^3}{3} V(\xi_0),
\] where
\[
V(\xi_0) = \frac{1}{2} \int_0^\pi d\Theta \sin^3 \Theta \left( 1 + \cos \Theta \left( \log \tan \frac{\Theta}{2} + \xi_0 \right) \right) \Omega^3(\Theta),
\] (23)

and thus the isoperimetric ratio $\nu$, defined by $\nu = V/V_A = 3\sqrt{4\pi V/A}$ where $V_A$ is the volume of a sphere of surface area $A$, is given by
\[
\nu(\xi_0) = V(\xi_0) / A(\xi_0)^{3/2}.
\] (24)

\[3\]More generally, $e_a \cdot k \neq e_a \cdot k$, where $k$ is a fixed unit vector.
The maximum value, \( v = 1 \) is attained by a sphere. Clearly the constraint that the poles touch introduces bending energy. As a consequence the value \( v = 1 \) is not attained in any equilibrium geometry with touching poles.

The isoperimetric ratio is plotted as a function of \( \xi_0 \) in figure 3(a). \( v(\xi_0) \) possesses three extrema, two maxima and one minimum. The maxima \( v_{\text{Max}} = 0.66564 \) occur when \( \xi_0^* = \pm 1.86823 \). The corresponding geometry marks the transition from discocyte to stomatocyte (depicted in figure 2). The shallow local minimum \( v_{\text{Min}} = 0.647639 \) occurs when \( \xi_0 = 0 \), the symmetric biconcave geometry. We observe, in particular, the interesting fact that the maximally inflated geometry is not the symmetric discocyte. \( v \) vanishes asymptotically in the limits \( \xi_0 \to \pm \infty \). This property is evident in the corresponding profile (see figure 2).

We observe, in particular, that \( v(\xi_0) \) may not be inverted for \( \xi_0 \) on the interval \( \xi_0 \in [0, \infty) \). This indicates that it cannot be used in place of \( \xi_0 \) as the 'order parameter' to describe the discocyte-stomatocyte transition. As we will see, the appropriate physical parameter is the Noether charge or external force \( F \). This physical interpretation will be developed in the following sections.

![Figure 3: (a) Isoperimetric ratio vs \( \xi_0 \); (b) Area ratio vs \( \xi_0 \).](image-url)

It was pointed out in the previous section that, from a technical point of view, the geometry remains biconcave for all values of \( \xi_0 \). We will now demonstrate that one of the concave regions becomes vanishingly small when \( \xi_0 \) is larger than the critical value \( \xi_0^* = 1.86832 \). To do this, we determine the fraction of surface area lying above \( Z = 0 \). We thus define the ratio

\[
\frac{A_0}{A} = \frac{1}{A(\xi_0)} \int_{\Theta_0(\xi_0)}^{\pi} d\Theta \sin \Theta \Omega(\Theta) ,
\]

where \( \Theta_0(\xi_0) = 2 \arctan(\exp(-\xi_0)) \) is the tangent angle along the meridian where the surface intersects \( Z = 0 \). For example, when \( \xi_0 = 0 \), \( \theta_0(0) = \frac{\pi}{2} \) and the area ratio is one half. Figure
3 (b) illustrates clearly the rapid approach of the ratio to 1 above the critical value, $\xi^*$. 

### 3.2 Curvature singularities

We expect a curvature singularity at the poles. This is because the only regular equilibrium geometries with spherical topology are the round spheres. We will now confirm that the geometries described by Eq. (17) are regular everywhere except at the poles where they display a (logarithmic) curvature singularity. The strength of this singularity will be related to the external force tethering the poles together.

In the case of the symmetric discocyte it is very simple to study this singular behavior analytically. To do this, note that, in the neighborhood of the origin, Eq. (17) is approximated by

$$
\frac{1}{R} \approx \frac{1}{2R_0} \exp \left( \frac{R_0 Z}{R^2} \right)
$$

which can be inverted for $Z$ as a function of $R$,

$$
\frac{Z}{R_0} \approx -\left( \frac{R}{R_0} \right) \log \left( \frac{R}{2R_0} \right)
$$

when $R \approx 0$. Whereas $Z, R$ vanishes at $R = 0$, it is clear that both the curvature along the meridian $C_\perp \approx Z, R R$, as well as that along the parallel, $C_\parallel \approx Z, R / R$, diverge logarithmically (see appendix II). Indeed, $C_\parallel$ and $C_\perp$ exhibit identical logarithmic divergences:

$$
C_\parallel, C_\perp \approx -\frac{2}{R_0} \log \left( \frac{R}{2R_0} \right)
$$

The poles are umbilical points of the geometry, albeit in a singular way.

We will now show that this qualitative behavior holds for all values of $\xi_0$. However, a different strategy becomes more appropriate when $\xi_0 \neq 0$. We have the following exact expressions for the principal curvatures with respect to the parametrization given by (19)

$$
C_\perp = \frac{\partial^2 \Theta \partial \Theta Z - \partial_\Theta Z \partial_\Theta R}{\left( (\partial_\Theta R)^2 + (\partial_\Theta Z)^2 \right)^{3/2}}
$$

$$
= -\frac{1}{R_0} \left( 3 + 2 \cos \Theta \left( \log \tan \frac{\Theta}{2} + \xi_0 \right) + \sin^2 \Theta \left( \log \tan \frac{\Theta}{2} + \xi_0 \right)^2 \right),
$$

and

$$
C_\parallel = -\frac{\partial_\Theta Z}{R \left( (\partial_\Theta R)^2 + (\partial_\Theta Z)^2 \right)^{1/2}}
$$

$$
= -\frac{1}{R_0} \left( 1 + 2 \cos \Theta \left( \log \tan \frac{\Theta}{2} + \xi_0 \right) - \sin^2 \Theta \left( \log \tan \frac{\Theta}{2} + \xi_0 \right)^2 \right).
$$
It is clear, by inspection, that the two principal curvatures diverge logarithmically at the poles \( \Theta = 0 \) and \( \Theta = \pi \). Note, however, that their difference remains finite with the value \( C_\perp - C_\parallel = -2/R_0 \) as consistency with Eq. (2) of [10] demands. The angular dependence of the curvatures for different values of \( \xi_0 \) is plotted in Figures 4 and 5.

Figure 4: \( C_\perp \) vs. \( \Theta \) for three different values of \( \xi_0 \): \( R_0 = 3, \xi_0 = 0 \) (left); \( R_0 = 3\sqrt{A(0)/A(2)}, \xi_0 = 2 \) (center); \( R_0 = 3\sqrt{A(0)/A(3)}, \xi_0 = 3 \) (right).

Figure 5: \( C_\parallel \) vs. \( \Theta \): \( R_0 = 3, \xi_0 = 0 \) (left); \( R_0 = 3\sqrt{A(0)/A(2)}, \xi_0 = 2 \) (center); \( R_0 = 3\sqrt{A(0)/A(3)}, \xi_0 = 3 \) (right).

Note, in particular, how sharp the divergence is in the neighborhood of the south pole. Note also that both curvatures diverge to infinity. This peculiarity is associated with the logarithm; it is not apparent visually in the corresponding profiles due to the existence of a tangent plane (figure 2).

The singularity at the poles indicates that these surfaces, unlike the catenoid, require external forces along the axis of symmetry to support them.

It should be pointed out that the characteristic logarithmic singularity we have examined is, in fact, predicted by the linear theory. Consider the linear approximation to the Monge representation of the surface in terms of its height \( h \) above a plane. If the gradients of \( h \) are small, the bending energy \( H \) can be expressed as

\[
H = \frac{1}{2} \int d^2 x \left( \nabla^2 h \right)^2 + \mathcal{O}(4)
\]  

(30)
where $\nabla_\perp$ is the gradient on the plane. The corresponding Euler-Lagrange equation is the biharmonic equation on the plane, $-(\nabla_\perp^2)^2 h = 0$. It is straightforward to show that the most general axially symmetric solution of this equation is given by

$$h = c_0 + c_1 R^2 \log R^2 + c_2 \log R + c_3 R^2$$

(31)

The linear Monge approximation to the inverted catenoids is given by solutions with $c_0, c_2 = 0$. Of course, it is beyond the scope of this approximation to model their global behavior.

### 3.3 Singularities as manifestations of sources of stress

Axial symmetry dictates that the force on the membrane must be directed parallel to the axis, $F = -2\pi c k$. We have shown recently that $c$ is the constant appearing in Eq. (2) of [10]. Thus using this equation and Eq. (7) we find that an inversion of the catenoid must satisfy the following equation

$$C_\perp - C_\parallel = \frac{c |X|^2}{2R^2}.$$  

(32)

If we now use the explicit expressions for the principal curvatures we identify $c = -4/R_0$. This is a reparametrization invariant measure of the singularity. It is also possible to see this directly using the expression given in Eq. (15): without loss of generality, choose the contour to be a closed circle of polar radius $R$ encircling the north pole. Now $l$ is the tangent to the meridian and $ds = Rd\phi$. We do not use Eq. (13) which is ill-defined at the pole. Using (10) and (28), and that on the north pole $n = -k$, we obtain

$$l_a f^a = \frac{1}{2} (C_\perp^2 - C_\parallel^2) l - (C_\perp + C_\parallel)' n \approx (C_\perp + C_\parallel)' k = \frac{4R_0}{R} k.$$  

(33)

As a consequence,

$$F = 8\pi R_0 k,$$

(34)

so that $c = -4/R_0$. The sign indicates that the force is always directed towards the interior.

### 3.4 The Phases of a tethered membrane

We now possess a physical interpretation of the inverted geometry as an equilibrium with a pair of equal and opposite localized external forces applied at the poles holding them together. This might be compression applied outside, or the tension supplied by an internal tether. The curvature singularity at the poles is a manifestation of local forces. It is unrelated to the
fact that they touch: they touch simply because all distant parts of the original catenoid get mapped to the origin, a peculiarity of the exact solution we have been handed. In a subsequent publication we will construct equilibrium configurations with a fixed non-vanishing interpolar distance.

The axial force $c$ may be interpreted as the order parameter describing the transition from discocytes to stomatocytes. The axial force and the reduced volume may be used to characterize axially symmetric compact shapes. The geometries with tethered poles that we have considered define a finite trajectory on this space illustrated in figure 6. Each point on this trajectory represents a specific geometry: the discocyte lies at one end (with the maximum force $c_1 = -4/3$), the limiting geometry consisting of a sphere within a sphere at the other (vanishing force $c_0 = 0$). The maximally inflated equilibrium geometry ($c_\ast = -0.711923$) provides a natural division of the trajectory into two phases: discocytes and stomatocytes.

4 Discussion

The invariance of the bending energy of two dimensional surfaces with respect to inversion in any point has been used to obtain exact results for a family of deflated biconcave equilibrium geometries with spherical topology and with tethered poles. The external force tethering the poles is imprinted as a curvature singularity on the membrane geometry. We end up with a geometry describing a completely different physical setup from the catenoid geometry we began with.

There is a, of course, a superficial similarly to the discocyte-stomatocyte transition of fluid
membranes induced by a change in the bilayer asymmetry. The quantitative study of the discocyte shape of a deflated fluid membrane and its stability has a long history dating back to Deuling and Helfrich’s pioneering work in the seventies [11] followed, after a long pause, by a series of groundbreaking studies by several groups (see, for example, [12] as well as [2, 13, 14] for reviews). The transition explored in this work is associated with a change in the bilayer asymmetry. An energy penalty or constraint on the area difference is introduced. These constraints break the conformal symmetry of the bending energy pointwise. In the inverted catenoid, on the other hand, there is no bilayer asymmetry. The transition is associated with a change in the localized external forces. The action of these forces breaks conformal symmetry only at isolated points. As a consequence, it is possible to constrain the isoperimetric ratio without sacrificing the conformal invariance of the energy. While it is tempting to emphasize the analogy between the two systems, it is probably a mistake to dwell on it.

These differences notwithstanding, the simple model we have described is relevant to the study of biological membranes in other ways. For the action of forces external to the membrane controlling its shape is the rule rather than the exception in biology. The cytoskeleton pushes and pulls on the plasma membrane [18]. Indeed, our toy model might after all be relevant to the stability of the biologically relevant biconcave discocyte shape assumed by a healthy erythrocyte which involves this interaction in an essential way.

It is also possible that our model is relevant for the theoretical study of the geometry of two very important organelles: the Golgi complex and the mitochondrion. It is known that the Golgi complex is not an isolated equilibrium structure [19]. It is stabilized against breakup by a complex network of microtubules that is dismantled and reassembled during mitosis. And the convoluted inner membrane of the mitochondrion is likely to be under compressive stress due to its confinement within the outer membrane [20].

In this context it is perhaps worth stressing that, in principle, it is always possible to reconstruct the distribution of external forces \( \{F_1, \ldots, F_N\} \) from a knowledge of the geometry in the neighborhood of an appropriate set of contours. The external forces leave their imprint on the local membrane geometry. As imaging techniques improve, this could become a very useful diagnostic tool in the study of intracellular membranes.

The solutions we have described are very special in that the poles touch. Intuitively, one would expect that solutions exist with poles held a finite distance apart. In the early nineties Ou-Yang Yhong-can, Naito and Okuda identified an exact analogue of such configurations for a model with a finite spontaneous curvature [22]. Generally, however, the effect of an external force will be to pull out two cylindrical tethers. Tether pulling in the context of a physically more realistic Hamiltonian has been the subject of some very nice recent theoretical analysis (see, for example, [23, 24, 25]). Such geometries will occur as solutions of the axially symmetric shape equation we derived in reference [10]. A detailed description of the solutions of this equation is, however, beyond the scope of this paper and will be addressed in a forthcoming paper. It will come as no surprise that conformal symmetry has a role to play in their characterization.
We have said nothing about the stability of the shapes we have discussed. Expanding the bending energy out to second order in deformations about a minimal surface, one can show that

\[ H = \frac{1}{2} \int dA \Phi (-\nabla^2 + \mathcal{R})^2 \Phi, \]  

(35)

where \( \Phi = \delta X \cdot n \) is the normal deformation and \( \mathcal{R} \) is the scalar curvature (twice the Gaussian curvature \( C_1C_2 \)). Remarkably, this expression depends only on intrinsic geometry. It should be compared with the corresponding expression for the area,

\[ H = \int dA \Phi (-\nabla^2 + \mathcal{R}) \Phi. \]  

(36)

In general, for any minimal surface \( \mathcal{R} < 0 \). Thus the catenoid spanning two rings becomes unstable as a minimal surface beyond a certain maximum separation. By contrast, note that the expression given by (35) is manifestly positive. Thus, if tension may be ignored, the same catenoid is stable as a Willmore surface. The conformal invariance of the energy should hold order by order in perturbations theory. This suggests that the inverted shapes are also stable. While one does need to confirm that there is no subtlety associated with singularities, conformal invariance clearly simplifies the analysis of stability.

So much for axially symmetric configurations. It is clearly possible to generate non-axially symmetric geometries held by pair of tethers by inverting these geometries in any point off the axis of symmetry. It appears reasonable to conjecture that all equilibrium shapes with a pair of tethers are generated by inversion.

It is also clear, however, that there is much more to this story. For the catenoid is but the simplest of a vast and growing number of interesting minimal surfaces. How do we describe equilibria held in place by three or more tethers? It is not unreasonable to guess that the inversion of non-axially symmetric minimal surfaces will play a role. We are currently examining the properties under inversion of the natural generalization of the catenoid, the n-noids of Jorge and Meeks [27]. This could provide a non-perturbative handle on yet another problem of interest in membrane biophysics: how does one identify the stable configurations assumed by proteins when the interaction between them is mediated by membrane curvature [28]?

**Appendix I**

The tangent vectors adapted to the parametrization are \( e_a = \partial_a X \) and \( n \) is the unit normal. The induced metric on the surface and the extrinsic curvature are given by \( g_{ab} = e_a \cdot e_b \) and \( K_{ab} = e_a \cdot \partial_b n \) respectively [29, 30, 31]. Indices are raised with the inverse metric \( g^{ab} \). The element of area is given by \( dA = \sqrt{\det g_{ab}} \, d^2u. \) \( K \) denotes (twice) the mean curvature: \( K = g^{ab} K_{ab} \)

**Appendix II**
The principal curvatures are given by

\[ C_\perp = \bar{\Theta}', \quad C_\parallel = \sin \bar{\Theta}/\bar{R} \]

where \( \bar{\Theta} \) denotes the angle that the tangent along the meridian makes with the planes of constant \( Z \), and the prime denotes a derivative with respect to arc-length along the meridian. The arc-length is given by

\[ ds^2 = (1 + Z^2_R) dR^2 \]

Thus \( \cos \bar{\Theta} = R' = -1/(1 + Z^2_R)^{1/2} \approx -1 \), \( \sin \bar{\Theta} = -Z' = Z_R/(1 + Z^2_R)^{1/2} \approx Z_R \), and \( \bar{\Theta}' \approx Z_{RR} \).

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